How hard is it to satisfy (almost) all roommates?*

Jiehua Chen    Danny Hermelin    Manuel Sorge    Harel Yedidsion
Ben-Gurion University of the Negev, Beer Sheva, Israel
jiehua.chen2@gmail.com, hermelin@bgu.ac.il, sorge@post.bgu.ac.il, yedidsio@post.bgu.ac.il

Abstract

The classical Stable Roommates problem (which is a non-bipartite generalization of the well-known Stable Marriage problem) asks whether there is a stable matching for a given set of agents (i.e. a partitioning of the agents into disjoint pairs such that no two agents induce a blocking pair). Herein, each agent has a preference list denoting who it prefers to have as a partner, and two agents are blocking if they prefer to be with each other rather than with their assigned partners.

Since stable matchings may not be unique, we study an NP-hard optimization variant of Stable Roommates, called Egalitarian Stable Roommates, which seeks to find a stable matching with a minimum egalitarian cost $\gamma$, (i.e. the sum of the dissatisfaction of the matched agents is minimum). The dissatisfaction of an agent is the number of agents that this agent prefers over its partner if it is matched; otherwise it is the length of its preference list. We also study almost stable matchings, called Min-Block Stable Roommates, which seeks to find a matching with a minimum number $\beta$ of blocking pairs. Our main result is that Egalitarian Stable Roommates parameterized by $\gamma$ is fixed-parameter tractable while Min-Block Stable Roommates parameterized by $\beta$ is W[1]-hard.

*This work is supported by the People Programme (Marie Curie Actions) of the European Union’s Seventh Framework Programme (FP7/2007-2013) under REA grant agreement number 631163.11 and Israel Science Foundation (grant no. 551145/14).
1 Introduction

This paper presents algorithms and hardness results for two variants of the Stable Roommates problem, a well-studied generalization of the classical Stable Marriage problem. Before going into describing our results, we give a brief background that will help motivate our work.

**Stable Marriage and Stable Roommates.** An instance of the Stable Marriage problem consists of two disjoint sets of $n$ men and $n$ women (collectively called agents), who are each equipped with his or her own personal strict preference list that ranks every member of the opposite sex. The goal is to find a bijection, or matching, between the men and the women that does not contain any blocking pairs. A blocking pair is a pair of man and woman who are not matched together but both prefer each other over their own matched partner. A matching with no blocking pairs is called a stable matching, and perfect if it is a bijection between all men and women.

Stable Marriage is a classical and fundamental problem in computer science and applied mathematics, and as such, entire books were devoted to it [21, 28, 42, 33]. The problem emerged from the economic field of matching theory, and it can be thought of as a generalization of the Maximum Matching problem when restricted to complete bipartite graphs. The most important result in this context is the celebrated Gale-Shapley algorithm [20]: This algorithm computes in polynomial time a perfect stable matching in any given instance, showing that regardless of their preference lists, there always exists a perfect stable matching between any equal number of men and women.

The Stable Marriage problem has several interesting variants. First, the preference list of the agents may be incomplete, meaning that not every agent is an acceptable partner to every agent of the opposite sex. In graph theoretic terms, this corresponds to the bipartite incomplete case. The preference lists could also have ties, meaning that two or more agents may be considered equally good as partners. Finally, the agents may not be partitioned into two disjoint sets, but rather each agent may be allowed to be matched to any other agent. This case corresponds to the case where the acceptability graph is not necessarily bipartite, and is referred to in the literature as the Stable Roommates problem.

While Stable Marriage and Stable Roommates seem very similar, there is quite a big difference between them in terms of their structure and complexity. For one, any instance of Stable Marriage always contains a stable matching (albeit perhaps not perfect), even if the preference lists are incomplete and with ties, and computing some stable matching in any Stable Marriage instance with $2n$ agents can be done in $O(n^2)$ time [20]. However, an instance of Stable Roommates may have no stable matchings at all, even in the case of complete preference lists. Moreover, even if a given Stable Roommates instance admits a solution, this solution may not be unique, and there might be other stable matchings with which the agents are more satisfied overall. Given these two facts, it makes sense to consider two types of optimization variants for Stable Roommates: In one type, one would want to compute a stable matching that optimizes a certain social

Optimization variants. As noted above, some Stable Roommates instances do not admit any stable matching at all, and in fact, empirical study suggests that a constant fraction of all sufficiently large instances will have no solution [41]. Moreover, even if a given Stable Roommates instance admits a solution, this solution may not be unique, and there might be other stable matchings with which the agents are more satisfied overall. Given these two facts, it makes sense to consider two types of optimization variants for Stable Roommates: In one type, one would want to compute a stable matching that optimizes a certain social
criteria in order to maximize the overall satisfaction of the agents. In the other, one would want to compute matchings which are as close as possible to being stable, where closeness can be measured by various metrics. In this paper, we focus on one prominent example of each of these two types—minimizing the egalitarian cost of a stable matching, and minimizing the number of blocking pairs in a matching which is close to being stable.

**Egalitarian optimal stable matchings.** Over the years several social optimality criteria have been considered, yet arguably one of the most popular of these is the egalitarian cost metric [37, 27, 32, 36]. Here, the dissatisfaction of an agent $x$ in a given stable matching depends on the rank of its partner $y$ in this matching, which is the number of agents that are strictly preferred over $y$ by $x$. The egalitarian cost of this matching is then the sum of the ranks of the partners of all agents. The corresponding Egalitarian Stable Marriage and Egalitarian Stable Roommates problems ask to determine whether there exists a stable matching with egalitarian cost at most $\gamma$, for some given cost bound $\gamma \in \mathbb{N}$ (Section 2 contains the formal definition).

When the input preferences do not have ties (but could be incomplete), Egalitarian Stable Marriage can be solved in $O(n^4)$ time [27]. For preferences with ties, Egalitarian Stable Marriage becomes NP-hard [32]. Thus, already in the bipartite case, it becomes apparent that allowing ties in preference lists makes the task of computing an optimal egalitarian matching much more challenging. Marx and Schlotter [36] showed that Egalitarian Stable Marriage is fixed-parameter tractable when parameterized by the parameter “sum of the lengths of all ties”.

For Egalitarian Stable Roommates, Feder [18] showed that the problem is NP-complete even if the preferences are complete and have no ties, and gave a 2-approximation algorithm for this case. Halldórsson et al. [22] showed inapproximability results for Egalitarian Stable Roommates, and Teo and Sethuraman [45] proposed a specific LP formulation for Egalitarian Stable Roommates and other variants. Cseh et al. [15] studied Egalitarian Stable Roommates for preferences with bounded length $\ell$ and without ties. They showed that the problem is polynomial-time solvable if $\ell = 2$, and is NP-hard for $\ell \geq 3$.

**Matchings with minimum number of blocking pairs.** For the case when no stable matchings exist, the agents may still be satisfied with a matching that is close to being stable. One very natural way to measure how close a matching is to being stable is to count the number of blocking pairs [29, 17]. Accordingly, the MIN-BLOCK STABLE ROOMMATES problem asks to find a matching with a minimum number of blocking pairs.

Figure 1: An example of three Stable Roommates instances, where $x \succ y$ means that $x$ is strictly preferred to $y$, and $x \sim y$ means that they are equally good and tied as a partner. The instance on the left is incomplete without ties and has exactly two stable matchings $\{(1, 2), (3, 4)\}$ and $\{(1, 4), (2, 3)\}$, both of which are perfect. The instance in the middle is incomplete with ties and has two stable matchings $\{(1, 3)\}$ and $\{(1, 2), (3, 4)\}$, the latter being perfect while the former not. The right instance is complete without ties and has no stable matchings at all.
Abraham et al. [6] showed that Min-Block Stable Roommates is NP-hard, and cannot be approximated within a factor of $n^{0.5-\varepsilon}$ unless P = NP, even if the given preferences are complete. They also showed that the problem can be solved in $n^{O(\beta)}$ time, where $n$ and $\beta$ denote the number of agents and the number of blocking pairs, respectively. This implies that the problem is in the XP class of parameterized complexity under its standard parameterization. Biró et al. [11] showed that the problem is APX-hard even if each agent has preference list of length at most 3, and presented a $(2\ell - 3)$-approximation algorithm for bounded list length $\ell$. Biró et al. [10] and Hamada et al. [23] showed that the related variant of Stable Marriage, where the goal is to find a matching with minimum blocking pairs among all maximum-cardinality matchings, cannot be approximated within $n^{1-\varepsilon}$ unless P = NP.

Our results. We analyze both Egalitarian Stable Roommates and Min-Block Stable Roommates from the perspective of parameterized complexity, under the natural parameterization of each problem (i.e. the egalitarian cost and number of blocking pairs, respectively). We show that while the former is (apart from a few variants) fixed-parameter tractable, the latter is W[1]-hard even when the preferences are complete and without ties. This shows a sharp contrast between the two problems: Computing an optimal egalitarian stable matching is a much easier task than computing a matching with minimum blocking pairs.

When no ties are present, an instance of the Egalitarian Stable Roommates problem has a lot of structure, and so we can apply a simple branching strategy for finding a stable matching with egalitarian cost of at most $\gamma$. When ties are present, the problem becomes much more challenging because several agents may be tied as a first ranked partner and it is not clear how to match them to obtain an optimal egalitarian stable matching. Moreover, we have to handle unmatched agents. When preference are complete or without ties, all stable matchings match the same (sub)set of agents and this subset can be found in polynomial time [21, Chapter 4.5.2]. Thus, unmatched agents do not cause any real difficulties. However, in the case of ties and with incomplete preferences, stable matchings may involve different sets of unmatched agents. Aiming at a socially optimal egalitarian stable matching, we consider the cost of an unmatched agent to be the length of its preference list [36] (also see Section 2 for the formal definition). Our first main result is given in the following theorem:

**Theorem 1.** Egalitarian Stable Roommates can be solved in $2^{O(\gamma^3)} \cdot n^3 \cdot (\log n)^3$ time (for preferences with ties), where $n$ denotes the number of agents and $\gamma$ denotes the egalitarian cost.

The general idea behind our algorithm is to apply random separation [12] to “separate” irrelevant pairs from the pairs that belong to the solution matching, and from some other pairs that would not block our solution. This is done in two phases, each involving some technicalities, but in total the whole separation can be computed in $f(\gamma) \cdot n^{O(1)}$ time. After the separation step, the problem reduces to Minimum-Weight Perfect Matching, and we can apply known techniques. We note that for the case where the preferences have no ties, a simple depth-bounded search tree algorithm suffices (Theorem 3).

Second, in Section 4, we show that Min-Block Stable Roommates is W[1]-hard with respect to the parameter $\beta$, the number of the blocking pairs, implying as a corollary a lower-bound on the running time of any algorithm. By adapting our reduction, we also answer in the negative an open question regarding the number of blocking agents proposed by Manlove [33, Chapter 4.6.5] (Corollary 1).

**Theorem 2.** Let $n$ denote the number of agents and $\beta$ denote the number of blocking pairs. Even when the input preferences are complete and without ties, Min-Block Stable Roommates is W[1]-hard with respect to $\beta$ and admits no $f(\beta) \cdot n^{\omega(\beta)}$-time algorithms unless the Exponential Time Hypothesis is false.
2 Definitions and notations

We now introduce necessary concepts and notation for the paper. Let $V = \{1, 2, \ldots, n\}$ be a set of $n$ agents with $n$ being even. Each agent $i \in V$ has a subset of agents $V_i \subseteq V$ which it finds acceptable as a partner and has a preference list $\succeq_i$ over $V_i$ (i.e. a transitive and complete binary relation on $V_i$). Here, $x \succeq_i y$ means that $i$ weakly prefers $x$ over $y$ (i.e. $i$ is better or as good as $y$). We use $\succ_i$ to denote the asymmetric part (i.e. $x \succeq_i y$ and $\neg(y \succeq_i x)$) and $\sim_i$ to denote the symmetric part of $\succeq_i$ (i.e. $x \succeq_i y$ and $y \succeq_i x$). For two agents $x$ and $y$, we call that $x$ most acceptable to $y$ if $x$ is a maximal element in the preference list of $y$. Note that an agent can have more than one most acceptable agent. For two disjoint subsets of agents $X \subseteq V$ and $Y \subseteq V$, $X \cap Y \neq \emptyset$, we write $X \succeq Y$ if for each pair of agents $x \in X$ and $y \in Y$ we have $x \succeq y$.

We use $\overline{V}$ to denote an arbitrary but complete preference list over $X$ without ties.

A preference profile $\mathcal{P}$ for $V$ is a collection $(\succeq_i)_{i \in V}$ of preference lists for each agent $i \in V$. A profile $\mathcal{P}$ may have the following properties: It is complete if for each agent $i \in V$ it holds that $V_i \cup \{i\} = V$; otherwise it is incomplete. We say that the profile $\mathcal{P}$ has ties if there is an agent $i \in V$ for which there are two agents $x, y \in V_i$ such that $x \sim_i y$ and we call that $x$ and $y$ are tied by $i$. We define for an instance $(V, \mathcal{P})$ a corresponding acceptability graph $G$, which has $V$ as its vertex set, and two agents are connected by an edge if each finds the other acceptable.

We assume without loss of generality that $G$ does not contain isolated vertices, meaning that each agent has at least one agent which it finds acceptable. We define the rank of an agent $i$ in the preference list of some agent $j$ as the number of agents $x$ that $j$ strictly prefers over $i$: $\text{rank}_j(i) := |\{x \mid x \succ_i j\}|$.

Given a preference profile $\mathcal{P}$ for a set $V$ of agents, a matching $M \subseteq E(G)$ is a subset of disjoint pairs of agents $\{x, y\}$ with $x \neq y$ (or edges in $E(G)$), where $E(G)$ is the set of edges in the corresponding acceptability graph $G$. We use the notion $V(M)$ to denote the set of agents that are assigned a partner by $M$, i.e. $V(M) := \{x, y \mid \{x, y\} \in M\}$, and if $V(M) = V$ we say that $M$ is perfect. For a pair $\{x, y\}$ of agents, if $\{x, y\} \in M$, then we denote $M(x)$ as the corresponding partner $y$; otherwise we call this pair unmatched. We write $M(x) = \perp$ if agent $x$ has no partner; that is, if agent $x$ is not involved in any pair in $M$.

Given a matching $M$ over $\mathcal{P}$, an unmatched pair $\{x, y\} \in E(G) \setminus M$ is blocking $M$ if the pair prefers to be matched to each other, i.e. it holds that $(M(x) = \perp \lor y \succ_x M(x)) \land (M(y) = \perp \lor x \succ_y M(y))$. A matching $M$ is stable if no unmatched pair is blocking $M$. Note that this stability concept is called weak stability when we allow ties in the preferences. We refer to the textbook by Gusfield and Irving [21], Manlove [33] for two other popular stability concepts for preferences with ties. The Stable Roommates problem is defined as follows:

**Stable Roommates**

**Input:** A preference profile $\mathcal{P}$ for a set $V = \{1, 2, \ldots, 2 \cdot n\}$ of $2 \cdot n$ agents.

**Question:** Does $\mathcal{P}$ admit a stable matching?

By our definition of stability, a stable matching for a Stable Roommates instance with complete preferences must assign a partner to each agent. This leads to the following observation.

**Observation 1.** Each stable matching in a Stable Roommates instance with complete preferences is perfect.

The two problems we consider in the paper are Egalitarian Stable Roommates and Min-Block Stable Roommates. The latter asks to determine whether a given preference profile $\mathcal{P}$ for a set of agents $V$ has a stable matching with at most $\beta$ blocking pairs. The former problem asks to find a stable matching with minimum egalitarian cost, where the egalitarian cost of a given matching $M$ is as follows: $\gamma(M) := \sum_{i \in V} \text{rank}_i(M(i))$, where we augment the definition rank with $\text{rank}_i(\perp) := |V_i|$. For example, the second profile in Figure 1 has two stable matchings $M_1 = \{(1,3)\}$ and $M_2 := \{(1,2), (3,4)\}$ with $\gamma(M_1) = 4$ and $\gamma(M_2) = 2$. 


The egalitarian cost, as originally introduced for the Stable Marriage problem, does not include the cost of an unmatched agent because the preference lists are complete. For complete preferences, a stable matching must assign a partner to each agent (Observation 1), meaning that our notion of egalitarian cost equals the one used in the literature. For preferences without ties, all stable matchings match the same subset of agents \( O \) (Chapter 4.5.2). Thus, the two concepts differ only by a fixed value which can be pre-determined in polynomial time \( O \). However, for incomplete preferences with ties, there seems to be no consensus on whether to “penalize” stable matchings by the cost of unmatched agents [13]. Our concept of egalitarian cost complies with Marx and Schlotter [36], but we tackle other concepts as well (Section 3.3).

3 Minimizing the egalitarian cost

In this section we give our algorithmic and hardness results for Egalitarian Stable Roommates. Section 3.1 treats the case when no ties are present, where we can use a straightforward branching strategy. In Section 3.2 we solve the case where ties are present. Herein, we need a more sophisticated approach based on random separation. Finally, in Section 3.3 we study variants of the egalitarian cost, differing in the cost assigned to unmatched agents.

3.1 Warm-up: Preferences without ties

By the stability concept, if there are two agents \( x \) and \( y \) that are each other’s most acceptable agents, i.e. \( \text{rank}_x(y) = \text{rank}_y(x) = 0 \), and consider no other agent as most acceptable, then any stable matching must contain the pair \( \{x, y\} \). Hence, we can safely add them to a solution matching. After we have matched all pairs of agents that each have cost zero, all remaining, unmatched agents induce cost at least one when they are matched. This leads to a simple depth-bounded branching algorithm. In terms of kernelization, we can “delete” any two agents that are most acceptable to each other and delete agents from some preference list that are ranked larger than \( \gamma \). This gives us a polynomial kernel.

**Theorem 3.** Let \( n \) denote the number of agents and \( \gamma \) denote the egalitarian cost. Egalitarian Stable Roommates without ties is solvable in \( O(\gamma! \cdot n^2) \) time and admits a size-\( O(\gamma^2) \) problem kernel.

**Proof Sketch.** Let \( V \) be the given set of agents. We aim to construct a stable matching of egalitarian cost at most \( \gamma \) for \( V \).

By [21], Chapter 4.5.2, when the input preferences do not have ties, all stable matchings match the same subset \( V' \subseteq V \) of agents, which can be computed in \( O(n^2) \) time. Thus, we can assume without loss of generality that we are going to search for a perfect stable matching \( M \) for \( V' \), which has egalitarian cost at most \( \gamma := \gamma - \sum_{v \in V \setminus V'} |V| \).

For the branching algorithm, we proceed as follows. Add to \( M \) all pairs \( \{x, y\} \) of agents that are each other’s most acceptable agents, i.e. \( \text{rank}_x(y) = \text{rank}_y(x) = 0 \). Afterwards, pick any unmatched agent \( u \in V' \) and let \( U \subseteq V' \) be the set of agents \( v \) which are acceptable to \( u \) such that \( \text{rank}_u(u) + \text{rank}_u(v) \leq \gamma \). Note that \( |U| \leq \gamma \) and that, clearly, \( u \) cannot be matched to any of his acceptable agents outside of \( U \). Branch into all possibilities to add \( \{u, v\} \) to \( M \) for \( v \in U \) and decrease the remaining budget \( \gamma \) accordingly: that is, make one recursive call for each possibilities. If afterwards \( \gamma > 0 \), then recurse with another yet unmatched agent. If \( \gamma = 0 \) or there is no unmatched agent anymore, then check whether the current matching \( M \) is stable in \( O(n^2) \) time. Accept if \( M \) is stable and otherwise reject.

Clearly, we can match all cost-zero pairs in \( O(n^2) \) time. The recursive procedure makes at most \( \gamma! \) recursive calls, in each of them, the budget is reduced by at least one, and, because of that, the available options to match the agent \( u \) above shrink by at least one. Hence, there are at most \( \gamma! \) calls, which is \( O(\gamma!) \). Each call can be carried out in \( O(n^2) \) time.
To obtain an \(O(\gamma^2)\)-size problem kernel, we proceed as follows with the set \(V'\). We compute a set \(U\) of agents and a set \(P\) of agent pairs that are not relevant to decide the instance. Then, we remove the agents in \(U\) and shrink the preference lists of the agents involved in \(P\) that have length more than \(\hat{\gamma}\), adjusting the preference lists of some other agents accordingly. Initially, let \(U := P := \emptyset\).

1. For each two agents \(x\) and \(y\) with \(\text{rank}_x(y) = \text{rank}_y(x) = 0\), add to \(U\) the two agents \(x, y\).

2. For each agent \(x\) in \(U\) and for each agent \(z\) that is acceptable to \(x\), introduce two new dummy agents \(x^1_z, x^2_z\). Replace in \(z\)'s preference list \(x\) with \(x^1_z\). The preference list of \(x^1_z\) is \(x^2_z \succ z\) and \(x^2_z\)'s only acceptable agent is \(x^1_z\). Note that in each stable matching, the dummy agents must be matched together, and the egalitarian cost induced by \(z\) and its partner is preserved. After the replacement, we remove from \(V'\) all agents in \(U\).

3. Replace the instance by a trivial no-instance if there are more than \(2 \cdot \hat{\gamma}\) non-dummy agents left. (Note that matching any non-dummy agent will induce cost at least one.)

4. For each agent \(x\) and each \(z\) of its acceptable agents with \(\text{rank}_x(z) > \hat{\gamma}\), add to \(P\) the ordered pair \((x, z)\).

5. For each ordered pair \((x, z)\) in \(P\), remove agent \(z\) in agent \(x\)'s preference list, and add two dummy agents \(x^1_z, x^2_z\) and modify \(z\)'s preference list as in the second step.

6. Remove all dummy agents \(x^1_z\) from the instance whose preference lists now consist of only dummy agents.

After these changes, at most \(\hat{\gamma}\) non-dummy agents remain, each of which has at most \(\hat{\gamma}\) acceptable agents. To bound the number of dummy agents, note that exactly half of the dummy agents appear in some non-dummy agent’s preference list. The remaining half of the dummy agents have only one acceptable agent in their respective preference lists. Hence, there are at most \(2 \cdot \hat{\gamma}^2\) dummy agents. In total, the corresponding acceptability graph has \(O(\hat{\gamma}^2)\) edges, meaning that the instance has size \(O(\hat{\gamma}^2) = O(\gamma^2)\).

### 3.2 Preferences with ties

When the preferences may contain ties, we can no longer assume that if two agents are each other’s most acceptable agents, denoted as a good pair, then a minimum egalitarian cost stable matching would match them together. This is because their match could force other pairs to be matched together that have large cost. Nevertheless, a good pair will never block any other pair, i.e. no agent in a good pair will form with an agent in some other pair a blocking pair. It is straightforward to see that each stable matching must contain a maximal set of disjoint good pairs. However, it may also contain some other pairs which have non-zero cost. We call such pairs costly pairs. Aiming to find a stable matching \(M\) with egalitarian cost at most \(\gamma\), it turns out that we can also identify in \(f(\gamma) \cdot n^{O(1)}\) time a subset \(S\) of costly pairs, which contains all costly pairs of \(M\) and contains no two pairs that may induce a blocking pair. It hence suffices to find a minimum-cost maximal matching in the graph induced by \(S\) and the good pairs. The crucial idea is to use the random separation technique \cite{12} to highlight the difference between the matched costly pairs in \(M\) and the unmatched costly pairs. This enables us to ignore the costly pairs, which pairwise block each other or are blocked by some pair in \(M\), and obtain the desired subset \(S\).

**Perfectness.** Before we get to the algorithm, we show that we can focus on the case where our desired stable matching is perfect, i.e., each agent is matched, even when the input preferences are incomplete. (Note that the case with complete preferences is covered by Observation \[1\].)
We show this by introducing dummy agents to extend each non-perfect stable matching to a perfect one, without disturbing the egalitarian cost.

Lemma 1. **Egalitarian Stable Roommates for n agents and egalitarian cost γ** is $O(\gamma \cdot n^2)$-time reducible to **Egalitarian Stable Roommates for n agents and egalitarian cost γ** with an additional requirement that the stable matching should be perfect.

**Proof.** Let $(V, P, \gamma)$ be an instance of **Egalitarian Stable Roommates**. Construct another instance $(V', P', \gamma')$ of **Egalitarian Stable Roommates** as follows. Define $k = \gamma$ if $\gamma$ is even and $k = \gamma - 1$ otherwise. Introduce a set $A := \{a_1, \ldots, a_k\}$ of $k$ agents, and let $V' := V \cup A$. Let $V^*$ consists of all agents in $V$ that each have at most $\gamma$ acceptable agents. To obtain $P'$, define the preference list of each agent in $V \setminus V^*$ to be the same as in $P$. All agents in $A$ have the same set of acceptable agents, namely $A \cup V^*$, which are tied as most acceptable. Consistently, for each agent $b \in V^*$, the preference list of $b$ in $P'$ is $L_b \succ A$, where $L_b$ is the preference list of $b$ in $P$. This completes the construction of $(V', P', \gamma)$. It can clearly be carried out in $O(\gamma \cdot n^2)$ time.

We claim that $(V, P, \gamma)$ admits a stable matching $M$ with egalitarian cost at most $\gamma$ if and only if $(V', P', \gamma)$ admits a perfect stable matching $M'$ with egalitarian cost at most $\gamma$.

For the “only if” part, let $V_{\perp} \subseteq V$ be the agents left unmatched by $M$. Observe that $|V_{\perp}| \leq \gamma$ as each unmatched agent has at least one acceptable agent without loss of generality and thus contributes at least one unit to the egalitarian cost. Moreover, since $|V|$ is even, $|V_{\perp}|$ is even. Construct a matching $M'$ for $(V', P', \gamma)$ with $M \subseteq M'$ by matching each agent in $V_{\perp}$ to a unique agent in $A$. Match the remaining, so far unmatched, agents in $A$ among themselves. Note that this is possible because both $V_{\perp}$ and $A$ are even.

Observe that $M'$ is perfect. Matching $M'$ is also stable: No agent in $V \setminus V_{\perp}$ is involved in a blocking pair according to $M$ and each agent in $V_{\perp}$ is matched to some agent in $A$, and each agent in $A$ is matched to one of his most acceptable agents. It remains to determine the egalitarian cost of $M'$.

Note that each agent in $A$ costs zero in the egalitarian cost of $M'$ because they are matched with their most acceptable agents. Hence, the only difference between $\gamma(M)$ and $\gamma(M')$ may arise from the cost of the agents in $V_{\perp}$. Let $b \in V_{\perp}$ and let $\ell$ be the number of agents acceptable to $b$ according to $P$. By our egalitarian cost definition and by the preference lists of $b$ and $M(b)$ in $P$, it is easy to see that the cost of $b$ for $M$ is the same as the cost of $\{b, M'(b)\}$ for $M'$ by our egalitarian cost definition. Hence, indeed $M'$ is a perfect stable matching and has egalitarian cost at most $\gamma$.

For the “if” part, let $M'$ be a perfect stable matching of egalitarian cost at most $\gamma$ for $P'$. Obtain a matching $M$ for $P$ by taking $M = \{p \in M' \mid p \subseteq V\}$. Observe that no two agents $a, b$ that are both unmatched with respect to $M$ are acceptable to each other as otherwise, they would prefer to be with each other rather than their partners in $M'$ and thus form a blocking pair for $M'$ (note that the partners assigned to $a$ and $b$ by $M'$ are in $A$ and hence in the last position of the preference lists of $a$ and $b$ according to $P'$).

We claim that $M$ is stable for $(V, P, \gamma)$. Assume towards a contradiction that $\{a, b\} \subseteq V$ is blocking $M$. This implies that $a$ and $b$ are acceptable to each other, and at least one of the agents $a$ and $b$ is matched in $M$. Furthermore, either $a$ or $b$ needs to be unmatched by $M$ as, otherwise, $\{a, b\}$ is a blocking pair for $M'$—a contradiction. Say in $M$, $a$ is unmatched but $b$ is matched. However, then $a$ would prefer $b$ over any agent in $A$, meaning that $\{a, b\}$ is a blocking pair for $M'$—a contradiction.

Finally, by an analogous reasoning as given for the “only if” part, we can obtain that the egalitarian costs of $M$ and $M'$ remain the same, which is at most $\gamma$.

Lemma 1 allows us in a subprocedure of our main algorithm to compute a minimum-cost perfect matching in polynomial time instead of a minimum-cost maximal matching (which is NP-hard).
The algorithm. As mentioned, our algorithm is based on random separation [12]. We apply it already in derandomized form using Naor et al.'s construction of universal sets [38]. A family \( \mathcal{F} \) of subsets of some universe \( U \) with \( n \) elements is called an \((n,d)\)-universal set if for each subset \( A \subseteq U \) of cardinality \( d \) we have that the family \( \mathcal{F}_A := \{ A \cap S \mid S \in \mathcal{F} \} \) contains all \( 2^d \) possible subsets of \( A \), i.e. \( \forall A' \subseteq A : A' \in \mathcal{F}_A \). Naor et al. [38] showed that an \((n,d)\)-universal set of cardinality \( 2^d \cdot d^{O(\log d)} \cdot \log n \) can be computed in \( 2^d \cdot d^{O(\log d)} \cdot n \cdot \log n \) time.

In the remainder of this section, we prove our main result:

Theorem 1. **Egalitarian Stable Roommates can be solved in** \( 2^{O(\gamma^3)} \cdot n^3 \cdot (\log n)^3 \) **time (for preferences with ties), where** \( n \) **denotes the number of agents and** \( \gamma \) **denotes the egalitarian cost.**

For brevity we denote by a **solution** a stable matching \( M \) with egalitarian cost at most \( \gamma \). By Lemma 1 we assume that each solution is perfect. Let \( \mathcal{P} \) be a preference profile for a set \( V \) of agents, possibly incomplete and containing ties. We say that two pairs \( p, p' \in \binom{V}{2} \) **block** each other if, assuming both pairs \( p \) and \( p' \) were matched, they would induce a blocking pair. Formally, \( p \) and \( p' \) **block** each other if \( v \succ u \quad u' \quad v' \), where \( p := \{u, v\} \) and \( p' := \{u', v'\} \). In this case, we also say that \( p \) and \( p' \) **block** each other due to the blocking pair \( \{u, u'\} \) with \( u \in p \) and \( u' \in p' \). Our goal is to construct a graph with vertex set \( V \) which contains all matched “edges” of the solution and some other edges for which no two edges in this graph are blocking each other. Pricing the edges with their corresponding cost, by Lemma 1 it is then enough to find a minimum-cost perfect matching. The graph is constructed in three phases. We give an outline in Algorithm 1. In the first phase, we start with the acceptability graph of our profile \( \mathcal{P} \) and remove all edges whose “costs” each exceed \( \gamma \). In the second and the third phases, we remove all edges that block each other while keeping a stable matching with minimum egalitarian cost intact.

We introduce some more additional necessary notation. Let \( G \) be the acceptability graph corresponding to \( \mathcal{P} \) with vertex set \( V \), which also denotes the agent set, and with edge set \( E \). We call an edge \( e := \{x, y\} \) a **zero edge** if it has cost zero, i.e. \( \text{rank}_x(y) + \text{rank}_y(x) = 0 \), otherwise it is a **costly edge** if the cost does not exceed \( \gamma \). We ignore all edges with cost exceeding \( \gamma \). We introduce two subsets \( E^\text{zero} \) and \( E^\text{exp} \) such that \( E^\text{zero} \) consists of all zero edges \( e \), i.e. \( E^\text{zero} := \{ \{x, y\} \in E \mid \text{rank}_x(y) + \text{rank}_y(x) = 0 \} \), and \( E^\text{exp} \) consists of all costly edges, i.e. \( E^\text{exp} := \{ \{x, y\} \in E \mid 0 < \text{rank}_x(y) + \text{rank}_y(x) \leq \gamma \} \). Two edges \( e \) and \( e' \) are **blocking each other** if the corresponding pairs block each other.

**Algorithm 1:** Constructing a perfect stable matching of egalitarian cost at most \( \gamma \).

**Input:** A set of agents \( V \), a preference profile \( \mathcal{P} \) over \( V \), a budget \( \gamma \in \mathbb{N} \).

**Output:** A stable matching of egalitarian cost at most \( \gamma \) if it exists.

```plaintext
/* Phase 1 */
1 (V, E) ← The acceptability graph of \( \mathcal{P} \)

2 \( E^\text{zero} ← \{ \{x, y\} \in E \mid \text{rank}_x(y) + \text{rank}_y(x) = 0 \} \)   // The set of zero edges in \( E \)

3 \( E^\text{exp} ← \{ \{x, y\} \in E \mid 1 \leq \text{rank}_x(y) + \text{rank}_y(x) \leq \gamma \} \)   // The set of costly edges in \( E \)

4 \( E_1 ← E^\text{zero} \cup E^\text{exp} \)

/* Phase 2 */
5 \( E^\text{exp} ← \{|E^\text{exp}|, \gamma + \gamma^3\}\)-universal set over the universe \( E^\text{exp} \)

6 for \( E' \in E^\text{exp} \) do

7 Apply Reduction rules 1 and 2 to \( E_1 \) to obtain \( E_2 \)

8 /* Phase 3 */

9 for \( V' \in C \) do

10 Apply Reduction rules 3 and 4 to \( E_1 \) to obtain \( E_3 \)

11 \( M ← \text{Minimum-cost perfect matching in the graph } (V, E_3) \)

12 if \( M \neq \perp \) and \( M \) has cost at most \( \gamma \) then return \( M \)

13 reject
```

8
Phase 1. We construct a graph \( G_1 = (V, E_1) \) from \( G \) with vertex set \( V \) and with edge set \( E_1 := E^\text{zero} \cup E^\text{exp} \). If \( n \) is the number of agents, i.e. \(|V| = n\), we can compute \( G_1 \) in \( O(k \cdot n^2) \) time. The following is easy to see.

Lemma 2. If \( \mathcal{P} \) has a stable matching \( M \) with egalitarian cost at most \( k \), then \( M \subseteq E_1 \).

Observe also that a zero edge cannot block any other edge because the agents in a zero edge already obtain their most acceptable agents. Thus, we have the following.

Lemma 3. If two edges in \( E_1 \) block each other, then they are both costly edges.

Phase 2. In the second phase, comprising lines 8 to 10 in Algorithm 1 we remove from \( G_1 \) some of the costly edges that block each other. Note that by Lemma 3, no zero edge is blocking any other edge. For technical reasons, we now distinguish two types of costly edges: We say that a costly edge \( e \) with \( e := \{u, v\} \) is critical for its endpoint \( u \) if the largest possible rank of \( v \) over all linearizations of the preference list of \( u \) exceeds \( \gamma \), i.e. \(|\{x \in V_u \setminus \{v\} \mid x \succeq_u v\}| > \gamma\)

Otherwise, edge \( e \) is harmless for \( u \). If an edge is critical for at least one endpoint, then we call it critical and otherwise harmless. Observe that an edge \( e \) which is critical for its endpoint \( u \) could still belong to a solution. If two edges \( e, e' \) block each other due to \( u \in e, u' \in e' \), and if \( e' \) is harmless for \( u' \), then we say that \( e \) harmlessly blocks \( e' \). Note that blocking is symmetric while harmlessly blocking is not.

Intuitively, we now want to distinguish the solution edges from all edges blocked by the solution and there is a “small” number of harmless edges blocked by the solution, so we can easily distinguish between them. For the critical edges, we do not have such a bound; we deal with the critical edges blocked by the solution in Phase 3 in some other way.

Lemma 4. Let \( M \) be a stable matching with egalitarian cost at most \( \gamma \). In \( G_1 \), at most \( k^3 \) edges are harmlessly blocked by the edges in \( M \).

Proof. By Lemma 3 if an edge \( e' \) from \( G_1 \) is blocked by an edge \( e \) in \( M \), then both edges \( e' \) and \( e \) are costly edges. Pick a costly edge \( e \) in \( M \), denote its endpoints by \( u \) and \( v \), and let \( F_e \) denote the set of all edges that are harmlessly blocked by \( e \). Recall that each edge \( e' := \{u', v'\} \in F_e \) that is blocked by \( e \) induces a blocking pair consisting of some endpoint of \( e \), say \( u \), and some endpoint of \( e' \), say \( u' \). By the definition of blocking pairs, it follows that \( \text{rank}_u(u') \leq \text{rank}_u(v) - 1 \). Thus, since \( e \) has cost at most \( \gamma \), meaning that \( \text{rank}_u(v) + \text{rank}_v(u) \leq \gamma \), there are at most \( \gamma - 2 \) different endpoints of the costly edges in \( F_e \), which each form with an endpoint of \( e \) a blocking pair. Since each edge \( e' \) in \( F_e \) is harmlessly blocked it follows that \( |\{x \in V_{u'} \setminus \{v'\} \mid x \succeq_{u'} v'\}| \leq \gamma \). Thus, \( F_e \) has at most \( \gamma \) edges which shares the same endpoint \( u' \) and could be harmlessly blocked by \( e \) due to \( u' \) and one of the endpoints of \( e \). In total, we obtain that \( |F_e| \leq \gamma \cdot (\gamma - 2) \). Since \( M \) has egalitarian cost at most of \( \gamma \), it has at most \( \gamma \) costly edges that could block some edges. Hence, there are in total \( \gamma \cdot (\gamma - 2) \cdot \gamma < k^3 \) edges harmlessly blocked by \( M \).

In order to identify and delete all harmless edges that are blocked by some costly edge in a solution \( M \) we apply random separation. Compute a \((|E^\text{exp}|, \gamma + \gamma^3)\)-universal set \( \mathcal{F}^\text{exp} \) over the universe \( E^\text{exp} \). For each element of \( \mathcal{F}^\text{exp} \), perform all the computations below (in this phase and in Phase 3). Let \( M' := M \cap \mathcal{F}^\text{exp} \) be the set of all costly edges in \( M \) and let \( B_M \) be the set of all edges harmlessly blocked by some edge in \( M \). By the definition of costly edges and by Lemma 4 it follows that \( |M' \cup B_M| \leq \gamma + \gamma^3 \). By the properties of universal sets, \( \mathcal{F}^\text{exp} \) contains a member \( E' \) that “separates” \( M' \) from \( B_M \), i.e. \( M' \subseteq E' \) and \( B_M \subseteq E^\text{exp} \setminus E' \). Call such member good for \( M \). Formally, we call a member \( E' \) good if there is a stable matching \( \gamma \) with egalitarian cost at most \( \gamma \), such that each costly edge in \( M \) belongs to \( E' \), and each edge that is harmlessly blocked by \( M \) belongs to \( E^\text{exp} \setminus E' \). By the property of universal sets, if there is a solution \( M \), then \( \mathcal{F}^\text{exp} \) contains a member \( E' \) which is good for \( M \). In the following we present two data reduction rules that delete edges and show their correctness. By correctness we mean that, if some member \( E'' \in E^\text{exp} \) is good, then a corresponding solution is still present after the edge deletion.
Recall that the goal was to compute a graph that contains all edges from a solution and some other edges such that no two edges in the graph block each other. Observe that, we can ignore the edges in \( E^{\text{exp}} \setminus E' \), because, if \( E' \) is good, then it contains all costly edges in the corresponding solutions. This implies the correctness of the following reduction rule.

**Reduction rule 1.** Remove all edges in \( E^{\text{exp}} \setminus E' \) from \( E_1 \).

Apply also the following reduction rule.

**Reduction rule 2.** If there are two edges \( e, e' \in E' \) that mutually harmlessly block each other, then remove both \( e \) and \( e' \) from \( E_1 \).

*Proof of the correctness of Reduction rule 2.* By the definition of \( E' \) being good, no edges in the solution \( M \) harmlessly block any edge in \( E' \). Since \( M \subseteq E' \), this implies that no two edges in \( M \) could harmlessly block each other. Thus, we can safely delete \( e \) and \( e' \). \( \square \)

Let \( G_2 = (V, E_2) \) be the graph obtained from \( G_1 \) by exhaustively applying Reduction rules 1 and 2. By the definition of \( E' \) being good and by the correctness of Reduction rules 1 and 2, we have the following.

**Lemma 5.** If there is a stable matching \( M \) with egalitarian cost at most \( \gamma \), then \( E^{\text{exp}} \) contains a member \( E' \) such that the edge set \( E_2 \) of \( G_2 \) defined for \( E' \) contains all edges of \( M \).

Furthermore, by Lemma 3 and since all pairwise harmlessly blocking edges are deleted by Reduction rule 2, we have the following.

**Lemma 6.** If two edges in \( G_2 \) block each other due to a corresponding blocking pair \( \{u, u'\} \), then one of the edges is critical for \( u \) or \( u' \).

**Phase 3.** In lines 8 to 12 in Algorithm 1 we remove from \( G_2 \) the remaining critical edges that are blocked by the solution \( M \) or blocking each other. While we cannot upper-bound the number of critical edges blocked by \( M \) by a function in \( \gamma \), we can upper-bound the number of agents that are involved in some blocking pair between \( M \) and critical edges by a function in \( \gamma \). We introduce one more notion regarding such agents. Consider an arbitrary matching \( N \) (i.e. a set of disjoint pairs of agents) of \( G_2 \). Let \( e \in N \) and \( e' \in E_2 \setminus N \) be two edges that induce a blocking pair \( \{u, u'\} \) where \( u \in e \) and \( u' \in e' \). Then, we say that \( u' \) is a *culprit* of \( N \) or simply *culprit* if \( N \) is clear from the context. Similarly to the proof of Lemma 4, we obtain the following upper bound on the number of culprits with respect to a solution.

**Lemma 7.** Each stable matching \( M \) with egalitarian cost at most \( \gamma \) has at most \( \gamma^2 \) culprits.

*Proof.* Clearly, there are at most \( \gamma \) costly edges in \( M \). Each edge that is blocked by an edge in \( M \) is costly and is blocked by a costly edge in \( M \). Pick a costly edge \( e \) in \( M \) and consider the set \( F_e \) of all edges blocked by \( e \). Recall that each edge \( e' \) blocked by \( e \) would induce a blocking pair \( \{u, u'\} \) with \( u \in e \) and \( u' \in e' \). By the same reasoning as used in Lemma 4, we obtain that \( F_e \) has at most \( \gamma - 2 \) endpoints. Hence, there are in total \( \gamma \cdot (\gamma - 2) < \gamma^2 \) culprits. \( \square \)

Due to Lemma 7 the number of culprits plus the number of agents incident with some costly edge from the solution is upper-bounded by \( \gamma^2 + 2 \cdot \gamma \). We now use a universal set to find at least one subset that partitions the set of consisting culprits and the agents in costly edges from \( M \) into two subsets \( A \cup B \) with the following property: \( A \) consists of the vertices that are incident with zero edges or harmless edges from \( M \) and \( B \) consists of the vertices that are incident with critical edges from \( M \). Let \( n = |V| \) be the number of agents. Compute an \((n, 2 \cdot \gamma + \gamma^2)\)-universal set \( C \) on the set \( V \) of agents. We call a subset \( V' \subseteq C \) *good* if there is a solution \( M \subseteq E_2 \) such that for each agent \( v \in V \) that is a culprit (of \( M \)) or incident with some costly edge in \( M \) the following holds:
- If \( v \) is incident with a zero edge or an edge in \( M \) that is harmless for \( v \), then \( v \in V' \),

- and \( v \in V \setminus V' \) otherwise, i.e. if \( v \) is incident with an edge in \( M \) that is critical for \( v \), then \( v \in V \setminus V' \).

Again, by Lemma 7 the culprits of \( M \) and the agents incident with some costly edge in \( M \) form a set of cardinality at most \( \gamma^2 + 2 \cdot \gamma \). Thus, by a similar reasoning as given for Phase 2 and by the properties of universal sets, if there is a solution \( M \subseteq E_2 \), then \( C \) contains a member \( V' \) that is good.

We describe two reduction rules to delete some edges from \( G_2 \) and show their correctness, i.e. if some vertex subset \( V' \in C \) is good, then the rules do not delete any edge of a corresponding solution.

**Reduction rule 3.** For each agent \( y \in V' \), delete all incident edges that are critical for \( y \).

*Proof of the correctness of Reduction rule 3.* Assume that \( V' \) is good and let \( M \subseteq E_2 \) be a corresponding solution. If agent \( y \) is incident with an edge in \( M \), then this edge is either a zero edge or an edge which is harmless for \( v \). Thus, no deleted edge is contained in \( M \). \( \square \)

After having exhaustively applied Reduction rule 3 we use the following reduction rule. Recall that by Lemma 6 if two edges in \( E_2 \) block each other, then one of them is critical.

**Reduction rule 4.** If \( E_2 \) contains two edges \( e \) and \( e' \) that induce a blocking pair \{\( u, u' \)\} with \( u \in e \) and \( u' \in e' \) such that \( e \) is critical for \( u \), then remove \( e' \) from \( E_2 \).

*Proof of the correctness of Reduction rule 4.* Assume that \( V' \) is good and let \( M \subseteq E_2 \) be a corresponding solution. Towards a contradiction assume that \( e' \in M \). Hence, \( u \) is a culprit of \( M \). Since Reduction rule 3 has been applied exhaustively, all critical edges that are incident with agents in \( V' \) are deleted. Thus, by the definition of \( V' \) being good, it follows that \( u \in V \setminus V' \). Since \( u \) is a culprit, it is incident with an edge \{\( u, w \)\} in \( M \) that is critical for \( u \), i.e. \(|\{x \in V_u \setminus \{w\} \mid x \succeq_u w\}| > \gamma \). Let \( v \) denote the other endpoint of \( e \) different from \( u \). It follows that \( v \sim_u w \). Since \( e \) and \( e' \) block each other, \( u' \succ_u v \). It follows that also \( u' \succ_u w \). This implies that the matched edge \( e' \) blocks another matched edge in \( M \)—a contradiction to \( M \) being stable. \( \square \)

Let \( G_3 = (V, E_3) \) be the graph obtained after having exhaustively applied Reduction rules 3 and 4 to \( G_2 \). As mentioned, by the properties of universal sets, if there is a solution contained in \( E_2 \), then the above constructed universal set \( C \) contains a good subset. Thus, by the correctness of Reduction rules 3 and 4 we have the following.

**Lemma 8.** If there is a stable matching \( M \subseteq E_2 \) with egalitarian cost at most \( \gamma \), then the constructed universal set \( C \) contains a good subset \( V' \subseteq V \) such that the edge set \( E_3 \) of \( G_3 \) defined for \( V' \) contains all edges of \( M \).

Furthermore, since for each member \( V' \in C \), we delete all edges that pairwisely block each other, each perfect matching in \( G_3 \) induces a stable matching. We thus have the following.

**Lemma 9.** If \( G_3 \) admits a perfect matching \( M \) with edge cost at most \( \gamma \), then \( M \) corresponds to a stable matching with egalitarian cost at most \( \gamma \).

*Proof.* Since the costs of the edges in \( G_3 \) are exactly the egalitarian cost induced when put into a matching, the egalitarian cost of \( M \) is at most \( \gamma \). Towards a contradiction assume that a perfect matching \( M \) in \( G_3 \) has two blocking edges \( e \) and \( e' \). By Lemma 6 one of \( e \) and \( e' \) is critical for its endpoint in the corresponding blocking pair. Since Reduction rule 4 does not apply to \( G_3 \) anymore, the other edge is not present in \( E_3 \), a contradiction. \( \square \)
Thus, to complete Algorithm \(1\) in line \(11\) we compute a minimum-cost perfect matching in \(G_3\) and output yes, if it has egalitarian cost at most \(\gamma\).

Summarizing, by Lemma \(1\) if there is a stable matching of egalitarian cost at most \(\gamma\), then it is perfect and thus, by Lemmas \(2\) \(5\) and \(8\) there is a perfect matching in \(G_3\) of cost at most \(\gamma\).

Hence, if our input is a yes-instance, then Algorithm \(1\) accepts by returning a desired solution. Furthermore, if it accepts, then by Lemma \(9\) the input is a yes-instance.

As to the running time, different from Algorithm \(1\) before any other computation, we first compute the universal set \(\mathcal{F}^{exp}\) from Phase 2 in \(O^{\gamma^3} \cdot n^2 \cdot \log n\) time and the universal set \(\mathcal{C}\) from Phase 3 in \(O^{\gamma^2} \cdot n \cdot \log n\) time. Note that we can reuse the second set during the course of the algorithm. The remaining computation time can be estimated as follows. First, we compute \(G_1\) in \(O(\gamma \cdot n^2)\) time by checking for each pair of agents, whether they are most acceptable to each other or whether their sum of ranks is at most \(\gamma\). Then, in Phase 2, we consider all possible \(2^{O(\gamma^2)} \cdot \log n\) members over the universal set and for each member we need the following computation time. We first compute \(G_2\) in \(O(\gamma \cdot n^2)\) time (note that \(O(\gamma)\) time is enough to check whether two given edges block each other, assuming the preference lists are ordered). Then we consider all possible \(2^{O(\gamma^2)} \log n\) partitions in the universal set \(\mathcal{C}\) and for each of them compute \(G_3\). Computing \(G_3\) can be done in \(O(\gamma \cdot n^2)\) time by similar reasoning as before. Finally, the minimum-weight perfect matching can be found in \(O(n^3 \log n)\) time \(14\). Thus, overall the running time is \(2^{O(\gamma^3)} \cdot n^3 \cdot (\log n)^3\).

### 3.3 Variants of the egalitarian cost for unmatched agents

As discussed in Section \(1\) and in Section \(2\) when the input preferences are incomplete, a stable matching may leave some agents unmatched. In the absence of ties, all stable matchings mismatch the same set of agents \(21\), Chapter 4.5.2. Hence, whether an unmatched agent should infer any cost is not relevant in terms of complexity. However, when preferences are incomplete and with ties, stable matchings may involve different sets of matched agents. It turns out that the cost of unmatched agents changes the parameterized complexity dramatically.

In this section, we consider two variants of assigning costs to unmatched agents: zero cost or a constant fixed cost, and we show that for both cost variants, seeking for an optimal egalitarian stable matching is parameterized tractable. (Note that the cost \(\infty\), that is, allowing only perfect matchings, is already covered in Section \(3.2\))

**Unmatched agents have cost zero.** By the definition of stability, if two agents are acceptable to each other, but their corresponding dissatisfaction is large, then either they are matched, contributing a large, or possibly too large, portion to the egalitarian cost, or one of the two agents must be matched with someone else. This can be used to model a choice of truth value for a variable in a reduction from satisfiability problems.

**Theorem 4.** If the cost of the unmatched agents is zero, then Egalitarian Stable Roommates with incomplete preferences and ties is NP-complete even in the case where the egalitarian cost is zero and each agent has at most three acceptable agents.

**Proof.** We reduce from the NP-complete 3SAT variant in which each literal appears exactly twice and each clause has exactly three literals \(8\). Let \(\phi\) be a corresponding boolean formula in conjunctive normal form with variable set \(X\) and clause set \(\mathcal{C}\).

The reduction proceeds as follows. For each variable \(x_i \in X\), we introduce 3 agents \(a_i^0\), \(a_i^{\text{true}}\), and \(a_i^{\text{false}}\), and 12 agents, denoted as \(b_i^j\), \(c_i^j\), and \(d_i^j\) for all \(j \in \{1, 2\}\) and \(s \in \{\text{true}, \text{false}\}\). The agents for one specific variable \(x_i\) are shown in the variable gadget in Figure \(2\). The preference list of an agent \(v\) in the variable gadget is defined as follows: Each agent that is adjacent to \(v\) with a solid line is a most acceptable agent of \(v\) (and vice versa); they have rank 0 in each other’s preference list. Each agent that is adjacent to \(v\) with a dashed line has rank 1. For each clause \(C_j\) (note that it contains three literals), we construct a clause gadget that consists of
two agents $u_i^*$ and $w_i^*$, and of nine agents, denoted as $x_{i,1}$, $y_{j,i}$, and $z_{j,i}$ for all $i \in \{1, 2, 3\}$. The agents for clause \( C_j \) are shown in the variable gadget on the left in Figure 2. We define their preference lists in the same way as for the variable gadgets.

To combine the clause and variable gadgets, for each variable \( x_i \in X \) and for each clause \( C_j \in C \) we do the following: If \( x_i \) appears positively in a clause \( C_j \in C \), then we pick two agents \( c_{i,s}^\text{true}, d_{i,s}^\text{false} \), in \( x_i \)'s variable gadget and two agents \( y_{j,r}, z_{j,r} \) in \( C_j \)'s clause gadget that have not been used for combining before, and identify \( c_{i,s}^\text{true} = z_{j,r} \) and \( d_{i,s}^\text{false} = y_{j,r} \). Analogously, if \( x_i \) appears negatively in \( C_j \), then we identify \( c_{i,s}^\text{false} = z_{j,r} \) and \( d_{i,s}^\text{false} = y_{j,r} \).

This completes the construction, which can clearly be done in polynomial time. Observe that each agent has at most three acceptable agents.

To see that a satisfying assignment for \( \phi \) induces a stable matching with egalitarian cost 0, construct a matching \( M \) as follows. For each variable \( x_i \) that is assigned to true, let \( \{a_i^*, a_i^\text{true}\} \in M \) and, for each \( s \in \{1, 2\} \), let \( \{c_{i,s}^\text{true}, d_{i,s}^\text{false}\}, \{b_{i,s}^\text{true}, c_{i,s}^\text{false}\} \in M \). Accordingly, for each variable \( x_i \) that is assigned to false, let \( \{a_i^*, d_i^\text{false}\} \in M \) and, for each \( i \in \{1, 2\} \) let \( \{b_{i,s}^\text{true}, c_{i,s}^\text{false}\}, \{c_{i,s}^\text{true}, d_{i,s}^\text{false}\} \in M \). Observe that, if an agent from the variable gadget is matched, then it is matched with one of its most acceptable agents. Furthermore, if an agent (except \( d_{i,s}^\text{false/true} \)) is not matched, then all of its acceptable agents are matched. For each clause \( C_j \), pick a literal that satisfies it, say it is the literal \( x_1 \), and let \( \{w_1^*, x_{j,2}\}, \{w_1^*, x_{j,3}\} \in M \). Again, each matched agent is with one of its most acceptable agents, hence the egalitarian cost is 0.

Furthermore, for each agent \( v \) from the clause gadget, \( v \) is either matched (with egalitarian cost 0), or each agent acceptable to \( v \) is matched to one of its most acceptable agents. Hence, there are no blocking pairs. We conclude that \( M \) is a stable matching and has egalitarian cost 0.

Now let \( M \) be a stable matching of egalitarian cost 0. Note that, in the variable gadget of each variable \( x_i \), agent \( a_i^* \) is matched to either \( a_i^\text{true} \) or \( a_i^\text{false} \). This is because \( M \) being stable implies that at least one of \( a_i^\text{true} \) and \( a_i^\text{false} \) has to be matched and \( a_i^* \) is the only agent acceptable to them that has cost 0. For each variable \( x_i \), assign to \( x_i \) true if \( \{a_i^*, a_i^\text{true}\} \in M \) and false otherwise. We claim that each clause \( C_j \in C \) is satisfied in this way. To see this, consider \( C_j \)'s clause gadget and observe that one of the triple \( x_{j,1}, x_{j,2}, \) and \( x_{j,3} \) is not matched in \( M \) because there are only two agents that are most acceptable to the triple. Say \( x_{j,1} \) is not matched (the other cases are symmetric). This implies that \( p = \{y_{j,1}, z_{j,1}\} \in M \), which corresponds to a literal as by our construction. Assume that this pair \( p \) equals a pair \( \{a_i^\text{true}, c_{i,s}^\text{true}\} \) for some variable \( x_i \) (i.e., it occurs positively in \( C_j \)). This is without loss of generality, due to symmetry. Since \( \{a_i^\text{true}, c_{i,s}^\text{true}\} \) should be neither a member of nor blocking \( M \), it follows that \( a_i^\text{true} \) is matched to \( a_i^* \), meaning that variable \( x_i \) is set to true. Thus, clause \( C_j \) is satisfied, as claimed.

Unmatched agents have some constant positive cost. If the unmatched agents have some constant positive cost \( c \), then it is easy to see that \( \text{Egalitarian Stable Roommates} \) belongs to \( \text{XP} \).

**Proposition 1.** If the cost of each unmatched agent is some positive constant, then \( \text{Egalitarian Stable Roommates} \) belongs to \( \text{XP} \).
Itarian Stable Roommates with incomplete preferences and with ties can be solved in \( n^\gamma \cdot 2^{O(\gamma^3)} \cdot n^3 \cdot (\log n)^3 \) time, where \( \gamma \) is the egalitarian cost.

**Proof.** Let \( c \) be the cost for an unmatched agent. The algorithm is as follows. Guess, by trying all possibilities, a subset \( A \) of at most \( \gamma/c \) unmatched agents. Remove \( A \) from the set of agents \( V \) and all preference lists and modify the preference lists of the remaining agents as follows. For each agent \( u \in V \setminus A \) who was acceptable to some agent \( a \in A \), remove from \( u \)'s preference list all agents \( b \) for which \( u \) strictly preferred \( a \) over \( b \). In the remaining instance, search for a perfect stable matching of egalitarian cost at most \( \gamma - c|A| \). This can be done in \( 2^{O(\gamma^3)} \cdot n^3 \cdot (\log n)^3 \) time using Theorem 4. If such a matching exists, accept and otherwise reject.

We cannot substantially improve on the above algorithm in general, however. Indeed, we can use the same idea as in the reduction for Theorem 4 and utilize the fact that, when there are ties, an agent can select his partner from an unbounded number of agents with the same cost, in order to obtain a polynomial-time parameterized reduction from the \([\mathbf{W}[1]]\)-complete Independent Set problem (parameterized by the size of the independent set solution) to Egalitarian Stable Roommates.

**Theorem 5.** Let \( n \) denote the number of agents and \( \gamma \) denote the egalitarian cost. If the cost of each unmatched agent is some positive constant, then Egalitarian Stable Roommates with incomplete preferences and ties is \([\mathbf{W}[1]]\)-hard with respect to \( \gamma \). It does not admit an \( f(\gamma) \cdot n^{o(\gamma)} \)-time algorithm unless Exponential Time Hypothesis is false.

**Proof.** We reduce from Independent Set, which, given an \( n \)-vertex graph \( G = (U, E) \) with vertex set \( U \) and edge set \( E \), and a number \( k \in \mathbb{N} \), asks whether \( G \) admits a \( k \)-vertex independent set, a vertex subset \( U' \subseteq U \) pairwise non-adjacent vertices. Let \( c \) be the cost an unmatched agent, which is positive; without loss of generality, we assume that \( c \) is also a positive integer. To construct an instance of Egalitarian Stable Roommates, set the egalitarian cost to \( c \cdot k \) and construct a preference profile as follows. Set the set \( V \) of agents to \( U \cup A \cup D \), where \( U \) is a set of vertex agents, \( A = \{a_1, a_2, \ldots, a_{n-k}\} \) is a set of \( n-k \) selector agents, and \( D = \{d_{1,1}^u, d_{1,2}^u| i \in \{1, 2, \ldots, c\} \land u \in U \} \) is a set of \( 2 \cdot c \cdot n \) dummy agents. Note that we use \( U \) for both the vertex set and the set of vertex agents, and we will make it clear whether we mean the vertices or their corresponding vertex agents.

For each \( i \in \{1, 2, \ldots, n-k\} \) the set of acceptable agents to \( a_i \) is precisely \( U \) and each pair of acceptable agents are tied, that is, they have rank 0 in \( a_i \)'s preference list. For each vertex \( u \in U \), the set of the acceptable agents of the corresponding agent \( u \) is \( \mathbb{A} \cup \{d_{1,1}^u, d_{1,2}^u \cup \cdots \cup d_{1,c}^u \} \cup N(u) \). The preference list of \( u \) is \( A \succ d_{1,1}^u \succ d_{1,2}^u \succ \cdots \succ d_{1,c}^u \succ N(u) \), i.e. \( A \) are all tied with rank 0, followed by \( c \) dummy agents, and finally the agents from the neighborhood \( N(u) \) are all tied with rank \( c + 1 \). Finally, for each \( u \in U \) and \( i \in \{1, 2, \ldots, c\} \), agent \( d_{1,i}^u \) is the only acceptable agent of \( d_{2,i}^u \), and \( d_{1,i}^u \)'s preference list is \( d_{2,i}^u \succeq u \). This completes the construction, which can clearly be done in polynomial time. If the reduction is correct, then it implies the result, because Independent Set is well-known to be \([\mathbf{W}[1]]\)-hard and, moreover, an \( f(k) \cdot n^{o(k)} \)-time algorithm for Independent Set would contradict the Exponential Time Hypothesis.

To see that a size-\( k \) independent set \( U' \) induces a stable matching of egalitarian cost \( c \cdot k \), match each agent \( a_i \) with a distinct vertex in \( U \setminus U' \) and, for each \( u \in U \) and each \( i \in \{1, 2, \ldots, c\} \), match \( d_{1,i}^u \) with \( d_{2,i}^u \). Observe that the thus constructed matching \( M \) has egalitarian cost \( c \cdot k \), because, apart from the \( k \) unmatched agents in \( U' \), each agent is matched with one of its most acceptable agents. To see that \( M \) is also stable, it suffices to show that no pair of agents \( u, v \in U' \) induces a blocking pair. This clearly holds, since \( u \) and \( v \) are not adjacent in \( G \) and thus not acceptable to each other. Thus, \( M \) is a stable matching of egalitarian cost at most \( c \cdot k \).

To see that a stable matching \( M \) of egalitarian cost \( c \cdot k \) induces a size-\( k \) independent set \( U' \) for \( G \), let \( U' \) be the set of agents in \( U \), which are not assigned to some selector agent \( a_i \) by \( M \).
as a partner. Observe that each selector agent $a_i$ is matched to some agent in $U$. Otherwise, there is some agent $b \in U$ which is either unmatched or matched to some other agent in $U$ or some dummy agent. Hence $\{b, a_i\}$ would form a blocking pair. Thus, each selector agent $a_i$ is matched to some agent in $U$, meaning that $|U'| = k$. Observe also that each agent $d_{2i'}$ is matched to $d_{2i'}$ because, otherwise, they would form a blocking pair. Hence, the only possible matches for agents in $U'$ are their neighbors in $G$. However, if two agents in $U'$ are matched, then $M$'s egalitarian cost is in total larger than $c \cdot k$, a contradiction. Thus, no two agents in $U'$ are matched together, and since $M$ is stable, no two agents in $U'$ are acceptable to each other. Thus, $U'$ is of size $k$ and induces an independent set, as required.  

4 Minimizing the number of blocking pairs

It is known that MIN-BLOCK STABLE ROOMMATES is NP-complete even for complete preferences without ties and can be solved in $O(n^{2(\beta+2)})$ time, where $n$ is the number of agents and $\beta$ is the number of blocking pairs a matching is allowed to have \[6\]. The NP-hardness reduction is from the problem of computing a minimum maximal matching in a graph. However, the reduction is not a parameterized reduction and it is also not clear how to adapt it to obtain a parameterized reduction. We show in this section that MIN-BLOCK STABLE ROOMMATES is W[1]-hard with respect to the parameter “the number $\beta$ blocking pairs”. That is, unless FPT = W[1], there is no $f(\beta) \cdot n^{O(1)}$-time algorithm. Furthermore, assuming the Exponential Time Hypothesis, our reduction implies that there is no $f(\beta) \cdot n^{o(\beta)}$-time algorithm. By adapting our reduction, we also answer an open question by Manlove \[33, Chapter 4.6.5\], showing that minimizing the number $\eta$ of blocking agents is NP-hard and W[1]-hard with respect to $\eta$.

First, we discuss a gadget that induces many blocking pairs if an agent is not matched to one of a specified set of agents. Abraham et al. \[6\] used a similar idea in their NP-hardness reduction. Consider the following profile with three pair-wise disjoint sets of agents $U, A, R$ with $A := \{a^0, a^1, \ldots, a^{2k+2}\}$. The preference orders of the agents in $A$ are as follows:

**Construction 1.**

\[
\begin{align*}
\text{agent } a^0: & \quad \overrightarrow{U} \succ a^1 \succ a^2 \succ \cdots \succ a^{2k+2} \succ \overrightarrow{R} \\
\text{agent } a^i: & \quad a^{i+1} \succ a^{i+2} \succ \cdots \succ a^{2k+2} \succ a^0 \succ a^1 \succ \cdots \succ a^{i-1} \succ \overrightarrow{U} \succ \overrightarrow{R}, \quad \forall i \text{ with } 1 \leq i \leq 2k + 2.
\end{align*}
\]

By the above preferences, one can verify that if a matching does not match $a^0$ to one of the agents from $U$, then it will have at least $k + 1$ blocking pairs. However, there is a matching $M$ for which no blocking pair involves any agent of $A$.

**Lemma 10.** Let $P$ be a preference profile with complete preferences and with agent set $U \cup A \cup R$ such that $A = \{a^0, a^1, \ldots, a^{2k+2}\}$ and the preference lists of $A$ obey Construction \[1\]. The following holds.

1. Any matching $M$ with $M(a^0) \notin U$ has at least $k + 1$ blocking pairs.
2. The matching $M$ with $M(a^0) \in U$ and $M(a^i) = a^{i+k+1}$, $1 \leq i \leq k + 1$, does not have any blocking pair that involves $A \setminus \{a^0\}$.

**Proof.** To show the first statement, we first partition the set $A$ into two subsets $X \cup Y$ where $X$ denotes the subset of agents $a^j$ with $M(a^j) \notin A$, and $Y$ denotes the subset of agents $a^j$ with $M(a^j) \in A$. Note that $|X| \geq 1$ because $|A|$ is odd. Since $|X| + |Y| = 2 \cdot k + 3$ and $|Y| \leq 2k + 2$, it follows that $|X| + |Y|/2 \geq 2 \cdot k + 3 - |Y|/2 \geq k + 2$. We show that the number of blocking pairs that $M$ has is at least $|X| - 1 + |Y|/2 \geq k + 1$.

As mentioned, $X$ contains at least one agent $a^l$ with $M(a^l) \notin A$. For each further agent $a^j \in X \setminus \{a^l\}$ we have one blocking pair $\{a^l, a^j\}$, because each agent in $A \setminus \{a^0\}$ prefers all other agents in $A$ to agents in $U \cup R \cup \{\perp\}$ and $M(a^0) \notin U$. Thus, there are at least $|X| - 1$ blocking pairs.
We claim that for each agent \( a^j \in Y \), at least one of \( a^i \) and \( M(a^j) \in A \) forms a blocking pair with \( a^j \). Let \( a^j := M(a^j) \). Agent \( a^i \) prefers both of \( a^j, a^j' \) to \( M(a^j) \) since \( M(a^j) \notin A \) and \( M(a^j) \notin U \). Without loss of generality, let \( j \) be the index encountered first out of \( j, j' \), when we successively decrease \( i \), where we continue with \( k + 2 \) when we reach \( 0 \). Agent \( a^j \) prefers \( a^j \) to \( a^j \) and thus \( \{a^j, a^j\} \) is a blocking pair. Hence, there are at least \( |Y|/2 \) blocking pairs each different from the \( |X| - 1 \) blocking pairs we have found before.

Thus we have shown that there are at least \( |X| - 1 + |Y|/2 \geq k + 1 \) blocking pairs in total.

As for the second statement, suppose towards a contradiction that \( M \) has a blocking pair \( \{a^i, u\} \) with \( 1 \leq i \leq 2k + 2 \). Then, by the preference lists of \( A \), it must hold that \( u \in A \setminus \{a^0\} \). Without loss of generality, let \( u = a^j \) with \( i < j \). If \( 1 < j < k + 1 \), then \( M(a^j) = a^{j+k+1} \). However, the preference list of \( a^i \) implies that \( a^{j+k+1} \) over \( a^j \)—a contradiction to \( \{a^i, a^j\} \) being a blocking pair. If \( j > k + 1 \), then \( M(a^j) = a^{j-k-1} \). Since \( \{a^i, a^j\} \) is a blocking pair, it must hold that \( a^i \) prefers \( a^j \) to \( M(a^j) = a^{j-k-1} \). This implies that \( i < j - k - 1 \leq k + 1 \) as \( i < j \), and that \( M(a^j) = a^{j+k+1} \). By the preference list of \( a^i \), we have that \( a^i \) prefers \( M(a^j) = a^{j+k+1} \) to \( a^j \) because \( k + 2 \leq i + k + 1 < j \)—a contradiction to \( \{a^i, a^j\} \) being a blocking pair. Summarizing, no blocking pair of \( M \) involves an agent \( a^i \) with \( 1 \leq i \leq 2k + 2 \).  

Using Lemma 10 we can prove our second main result (Theorem 2), by providing a parameterized reduction from the \( W[1] \)-complete problem \( \text{Independent Set} \) parameterized by the size of the independent set 11 (also see the definition in the proof of Theorem 3).

**Theorem 2.** Let \( n \) denote the number of agents and \( \beta \) denote the number of blocking pairs. Even when the input preferences are complete and without ties, \( \text{Min-Block Stable Roommates} \) is \( W[1] \)-hard with respect to \( \beta \) and admits no \( f(\beta) \cdot n^{o(\beta)} \)-time algorithms unless the \( \text{Exponential Time Hypothesis} \) is false.

**Proof.** Given an \( \text{Independent Set} \) instance \( (G = (V, E), k) \), we construct a \( \text{Min-Block Stable Roommates} \) instance with the following groups of agents: \( U, U^*, A_i, B_i, i \in \{1, 2, \ldots, k\} \), where \( U \) and \( U^* \) correspond to the vertex set \( V \), i.e. for each vertex \( v_i \in V \), we construct two vertex agents \( u_i, u_i^* \) with \( u_i \in U \) and \( u_i^* \in U^* \). The sets \( A_i := \{a_0^i, a_1^i, \ldots, a_2^{k+2}_i\} \) (resp. \( B_i := \{b_0^i, b_1^i, \ldots, b_2^{k+2}_i\} \)) consists of \( 2k + 3 \) agents each, such that the preference lists of \( A_i \) (resp. \( B_i \)) obey Construction 1 regarding \( U, A_i, R_i \) (resp. \( U, B_i, R_i^* \)) with \( R_i = U^* \cup \bigcup_{j=1, j\neq i}^{k} A_j \cup B_j \) (resp. \( R_i^* = U \cup \bigcup_{j=1, j\neq i}^{k} A_j \cup B_j \)) being the set of remaining agents. For each \( v_i \in V \), the preference lists of \( u_i \) and \( u_i^* \) are as follows:

- **agent \( u_i \):** \( u_i^* > N(u_i) > a_0^i > a_1^i > \cdots > a_k^i > R \)
- **agent \( u_i^* \):** \( u_i > b_0^i > b_1^i > \cdots > b_k^i > R \),

where \( N(u_i) \) is the set of agents that correspond to the neighbors of \( u_i \). The notions \( \overrightarrow{N(u_i)} \) and \( \overrightarrow{R} \) denote an arbitrary but fixed order of the agents in \( N(u_i) \) and of the remaining agents that are not stated explicitly, respectively.

We now show that \( (G = (V, E), k) \) is a yes-instance of \( \text{Independent Set} \) if and only if the constructed profile admits a matching with at most \( \beta := k \) blocking pairs.

For the “if” part, suppose that \( M \) is a matching with at most \( k \) blocking pairs. From the first statement of Lemma 10 it follows that for each \( i, 1 \leq i \leq n \), agent \( a_i^j \) from \( A_i \) must be matched to an agent from \( U \) and agent \( b_0^i \) must be matched to an agent from \( U^* \), as otherwise there will be at least \( k + 1 \) blocking pairs. We show that \( V' := \{v_i \mid M(u_i) \in \{a_0^i \mid 1 \leq i \leq k\}\} \) is an independent set of size \( k \). Clearly, \( |V'| = k \). Suppose for the sake of a contradiction that \( V' \) contains two adjacent vertices \( v_i, v_j \) with \( \{v_i, v_j\} \in E \). Thus, \( \{u_i, u_j\} \) constitutes a blocking pair because \( u_i, u_j \) are not matched to each other and by construction \( u_i \) prefers \( u_j \) to \( M(u_i) \) and \( u_j \) prefers \( u_i \) to \( M(u_j) \). However, for each \( u_i \in U', \{u_i, u_i^*\} \) is a blocking pair, meaning that
we have in total at least \( k + 1 \) blocking pairs—a contradiction. Hence, indeed \( V' \) is a \( k \)-vertex independent set in \( G \).

For the “only if” part, let \( V' \) be an independent set of size \( k \). First, without loss of generality, assume that \( V' = \{ v_1, v_2, \ldots, v_k \} \) (otherwise rename the vertices accordingly). We claim that the following matching \( M \) has at most \( k \) blocking pairs: for each \( v_i \in V \), let \( M(u_i) = a_i^0 \) and \( M(u_i) = b_i^0 \) if \( v_i \in V' \); otherwise \( M(u_i) = u_i^* \); for each \( j \) with \( 1 \leq j \leq k + 1 \), let \( M(a_i^j) = a_i^{j+k+1} \) and \( M(b_i^j) = b_i^{j+k+1} \).

We show that the pairs \( \{u_i, u_i^*\} \) with \( v_i \in V' \) are the only blocking pairs. First, by the second statement of Lemma 10 for each \( i \) with \( 1 \leq i \leq k \), no blocking pair of \( M \) involves an agent from \( A_i \cup B_i \setminus \{a_i^0, b_i^0\} \). Second, for each \( u_i \in U \) with \( p = \{u_i, u_i^*\} \in M \), it is obvious that no unmatched pair is blocking \( p \).

Third, for each \( u_i \in U \) with \( p = \{u_i, a_i^0\} \) and thus \( p^* = \{u_i^*, b_i^0\} \), no other unmatched pair except pair \( \{u_i, u_i^*\} \) is blocking \( p \) or \( p^* \). To show this, we need to show that for each \( x \neq u_i^* \), if \( u_i \) prefers \( x \) over \( a_i^0 \), then \( x \) prefers \( M(x) \) over \( u_i \), and that for each \( x \neq u_i, u_i^* \) prefers \( x \) over \( b_i^0 \), then \( x \) prefers \( M(x) \) over \( u_i^* \). For the first case, if \( u_i \) prefers \( x \) over \( a_i^0 \), then \( x \) cannot be some agent from \( N(u_i) \) since \( V' \) is an independent set and \( x \) cannot be \( a_i^0 \) because otherwise \( i' < i \), implying that \( a_i^0 \) prefers \( u_i \) to \( u_i^* \). By the preferences of \( u_i \), no other agent \( x \neq u_i^* \) can build with \( u_i \) a blocking pair. Analogously, we can show that no other agent \( x \neq u_i \) can form with \( u_i^* \) a blocking pair. Altogether, we thus showed that \( M \) has at most \( k \) blocking pairs, as required.

Observe that the reduction is both a polynomial and parameterized reduction. Furthermore, INDEPENDENT SET cannot be solved in \( f(k) \cdot |V|^o(k) \)-time with \( |V| \) and \( k \) being the number of vertices and the size of an independent set, unless the Exponential Time Hypothesis fails \([35]\). Since the above reduction sets the number \( \beta \) of blocking pairs to be \( k \), it follows that MIN-BLOCK STABLE ROOMMATES also does not admit an \( f(\beta) \cdot n^{o(\beta)} \)-time algorithm with \( n \) and \( \beta \) being the number of agents and the number of blocking pairs, unless the Exponential Time Hypothesis fails.

Interestingly, the construction above shows that it is sometimes beneficial not to match two agents (e.g. \( u_i, u_i^* \)) although they are most acceptable to each other, in order to avoid introducing too many blocking pairs in the gadgets from Construction \([1]\). Moreover, the reduction given in the proof of Theorem \([2]\) shows that the lower-bound on the number \( \beta \) of blocking pairs given by Abraham et al. \([6, Lemma 4]\) is tight. A slight modification of the reduction for Theorem \([2]\) answers an open question by Manlove \([32, Chapter 4.6.5]\) pertaining to the complexity of the following related problem. In MIN-BLOCKING AGENTS STABLE ROOMMATES, we are given a preference profile and an integer \( \eta \), and we want to know whether there is a matching with at most \( \eta \) blocking agents. Herein, an agent is blocking if it is involved in a blocking pair.

**Corollary 1.** MIN-BLOCKING AGENTS STABLE ROOMMATES is NP-hard and \( W[1] \)-hard with respect to the number \( \eta \) of blocking agents.

**Proof sketch.** We use the same reduction as in the proof of Theorem \([2]\) except when using Construction \([1]\) we introduce \( 4 \cdot k^2 + 2 \cdot k + 2 \) agents instead of \( 2 \cdot k + 3 \) and we set the number \( \eta \) of blocking agents to be \( 2 \cdot k \). By the same proof as for Lemma \([10]\) if \( M(a_i^0) \notin U \), then there are at least \( 2k^2 + k \) blocking pairs, which must involve at least \( 2 \cdot k + 1 \) agents. Since at most \( 2 \cdot k \) blocking agents are allowed, the same proof as in Theorem \([2]\) yields that, if the MIN-BLOCK AGENTS STABLE ROOMMATES instance is a yes-instance, then the INDEPENDENT SET instance is also a yes-instance. In the other direction, observe that the matching constructed from a size-\( k \) independent set in Theorem \([2]\) induces exactly \( 2 \cdot k = \eta \) blocking agents.
5 Conclusion and outlook

We showed that Egalitarian Stable Roommates and Min-Block Stable Roommates, though both NP-hard in the classical complexity point of view, behave completely differently in a parameterized perspective. In particular, we showed that Egalitarian Stable Roommates is fixed-parameter tractable with respect to the egalitarian cost $\gamma$ while Min-Block Stable Roommates is W[1]-hard with respect to the number $\beta$ of blocking pairs.

Our work leads to two open questions. First, we showed that for preferences without ties, Egalitarian Stable Roommates admits a size-$O(\gamma^2)$ kernel. It would thus be interesting to see whether Egalitarian Stable Roommates also admits a polynomial kernel when ties are present. Second, it would be interesting to settle the parameterized complexity of Min-Block Stable Roommates for the combined parameter $\beta$ and the maximum length of a preference list. Note that our parameterized reduction for Theorem 2 produces an instance with unbounded preference list length.

References

[1] URL http://www.nrmp.org. National Resident Matching Program website.

[2] URL http://www.nes.scot.nhs.uk/sfas. Scottish Foundation Allocation Scheme website.

[3] A. Abdulkadiroğlu, P. A. Pathak, and A. E. Roth. The Boston public school match. American Economic Review, 95(2):368–371, 2005.

[4] A. Abdulkadiroğlu, P. A. Pathak, and A. E. Roth. The New York City high school match. American Economic Review, 95(2):364–367, 2005.

[5] D. Abraham, N. Chen, V. Kumar, and V. S. Mirrokni. Assignment problems in rental markets. In Internet and Network Economics, Second International Workshop, pages 198–213, 2006.

[6] D. J. Abraham, P. Biró, and D. Manlove. “Almost stable” matchings in the roommates problem. In Approximation and Online Algorithms, Third International Workshop, pages 1–14, 2005.

[7] M. Băicu and M. Balinski. Student admissions and faculty recruitment. Theoretical Computer Science, 322(2):245–265, 2004.

[8] P. Berman, M. Karpinski, and A. D. Scott. Approximation hardness of short symmetric instances of MAX-3SAT. Technical Report 49, Electronic Colloquium on Computational Complexity, 2003.

[9] P. Biró and S. Kiselgof. College admissions with stable score-limits. Central European Journal of Operations Research, 23(4):727–741, 2015.

[10] P. Biró, D. Manlove, and S. Mittal. Size versus stability in the marriage problem. Theoretical Computer Science, 411(16-18):1828–1841, 2010.

[11] P. Biró, D. Manlove, and E. McDermid. “Almost stable” matchings in the Roommates problem with bounded preference lists. Theoretical Computer Science, 432:10–20, 2012.

[12] L. Cai, S. M. Chan, and S. O. Chan. Random Separation: A New Method for Solving Fixed-Cardinality Optimization Problems. In Proceedings of the Second International Workshop on Parameterized and Exact Computation (IWPEC ’06), pages 239–250. Springer, 2006.
[13] Y. Chen and T. Sönmez. Improving efficiency of on-campus housing: An experimental study. *American Economic Review*, 92(5):1669–1686, 2002.

[14] W. Cook and A. Rohe. Computing Minimum-Weight Perfect Matchings. *INFORMS Journal on Computing*, 11(2):138–148, 1999.

[15] Á. Cseh, R. W. Irving, and D. F. Manlove. The stable roommates problem with short lists. In *Proceedings of the 9th International Symposium on Algorithmic Game Theory (SAGT '16)*, pages 207–219, 2016.

[16] R. G. Downey and M. R. Fellows. *Fundamentals of Parameterized Complexity*. Springer, 2013.

[17] K. Eriksson and O. Häggström. Instability of matchings in decentralized markets with various preference structures. *International Journal of Game Theory*, 36(3):409–420, Mar 2008.

[18] T. Feder. A new fixed point approach for stable networks and stable marriages. *Journal of Computer and System Sciences*, 45(2):233–284, Oct. 1992.

[19] A. Gai, D. Lebedev, F. Mathieu, F. de Montgolfier, J. Reynier, and L. Viennot. Acyclic preference systems in P2P networks. In *Proceedings of the 13th International Euro-Par Conference*, pages 825–834, 2007.

[20] D. Gale and L. S. Shapley. College admissions and the stability of marriage. *The American Mathematical Monthly*, 120(5):386–391, 2013.

[21] D. Gusfield and R. W. Irving. *The Stable marriage problem–Structure and algorithms*. Foundations of computing series. MIT Press, 1989.

[22] M. M. Halldórsson, R. W. Irving, K. Iwama, D. Manlove, S. Miyazaki, Y. Morita, and S. Scott. Approximability results for stable marriage problems with ties. *Theoretical Computer Science*, 306(1-3):431–447, 2003.

[23] K. Hamada, K. Iwama, and S. Miyazaki. An improved approximation lower bound for finding almost stable maximum matchings. *Information Processing Letters*, 109(18):1036–1040, 2009.

[24] A. Hylland and R. Zeckhauser. The efficient allocation of individuals to positions. *Journal of Political Economy*, 87(2):293–314, 1979.

[25] R. W. Irving. Optimal stable marriage. In *Encyclopedia of Algorithms*, pages 1470–1473. Springer, 2016.

[26] R. W. Irving. Stable marriage. In *Encyclopedia of Algorithms*, pages 2060–2064. Springer, 2016.

[27] R. W. Irving, P. Leather, and D. Gusfield. An efficient algorithm for the ‘optimal’ stable marriage. *Journal of the ACM*, 34(3):532–543, 1987.

[28] D. Knuth. *Mariages Stables*. Les Presses de L’Université de Montréal, 1976.

[29] E. Kujansuu, T. Lindberg, and E. Mäkinen. The Stable Roommates problem and chess tournament pairings. *Divulgaciones Matemáticas*, 7(1):19–28, 1999.

[30] D. Lebedev, F. Mathieu, L. Viennot, A. Gai, J. Reynier, and F. de Montgolfier. On using matching theory to understand P2P network design. In *Proceedings of the International Network Optimization Conference (INOC 2007)*, pages 1–7, 2007.
[31] D. Manlove. Hospitals/residents problem. In M. Kao, editor, *Encyclopedia of Algorithms*. Springer, 2008.

[32] D. Manlove, R. W. Irving, K. Iwama, S. Miyazaki, and Y. Morita. Hard variants of stable marriage. *Theoretical Computer Science*, 276(1-2):261–279, 2002.

[33] D. F. Manlove. *Algorithmics of Matching Under Preferences*, volume 2 of *Series on Theoretical Computer Science*. WorldScientific, 2013.

[34] D. F. Manlove and G. O’Malley. Paired and altruistic kidney donation in the UK: Algorithms and experimentation. *ACM Journal of Experimental Algorithms*, 19(1), 2014.

[35] Marek Cygan, Fedor V. Fomin, Łukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michal Pilipczuk, and Saket Saurabh. *Parameterized Algorithms*. Springer, 2015.

[36] D. Marx and I. Schlotter. Parameterized complexity and local search approaches for the stable marriage problem with ties. *Algorithmica*, 58(1):170–187, 2010.

[37] D. G. McVitie and L. B. Wilson. The Stable Marriage problem. *Communications of the ACM*, 14(7):486–490, July 1971.

[38] M. Naor, L. J. Schulman, and A. Srinivasan. Splitters and near-optimal derandomization. In *Proceedings of 36th Annual IEEE Symposium on Foundations of Computer Science*, pages 182–191, 1995.

[39] M. Niederle and A. E. Roth. Market culture: How norms governing exploding offers affect market performance. Working Paper 10256, National Bureau of Economic Research, February 2004.

[40] B. Pittel and R. W. Irving. An upper bound for the solvability of a random stable roommates instance. *Random Structures and Algorithms*, 5(3):465–487, 1994.

[41] E. Ronn. NP-complete stable matching problems. *Journal of Algorithms*, 11(2):285–304, 1990.

[42] A. E. Roth and M. A. O. Sotomayor. *Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis*. Cambridge University Press, 1992. Part of Econometric Society Monographs.

[43] A. E. Roth, T. Sönmez, and M. U. Ünver. Pairwise kidney exchange. *Journal of Economic Theory*, 125(2):151–188, 2005.

[44] A. E. Roth, T. Sönmez, and M. U. Ünver. Efficient kidney exchange: Coincidence of wants in markets with compatibility-based preferences. *American Economic Review*, 97(3):828–851, 2007.

[45] C. Teo and J. Sethuraman. On a cutting plane heuristic for the stable roommates problem and its applications. *European Journal of Operational Research*, 123(1):195–205, 2000.