Separation Distribution of Vacuum Bubbles in de Sitter Space

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Abstract

We compute the probability distribution of the invariant separation between nucleation centers of colliding true vacuum bubbles arising from the decay of a false de Sitter space vacuum. We find that even in the limit of a very small nucleation rate \( \Gamma H^{-5} \ll 1 \) the production of widely separated bubble pairs is suppressed. This distribution is of particular relevance for the recently proposed “colliding bubble braneworld” scenario, in which the value of \( \Omega_k \) (the contribution of negative spatial curvature to the cosmological density parameter) is determined by the invariant separation of the colliding bubble pair. We also consider the probability of a collision with a ‘third’ bubble.

I. INTRODUCTION

In Guth’s original proposal for old inflation [1], a false vacuum with the geometry of de Sitter space decays through the nucleation as the result of quantum tunnelling of critical bubbles filled with the true vacuum, presumably Minkowski space or very nearly so. The dynamics of false vacuum decay, which had been elucidated by a number of authors [2], may be characterized by the dimensionless parameter \( \Gamma/H^{d+1} \) (where \( H \) is the Hubble constant in the false vacuum and \( d \) is the total number of spatial dimensions, here \( d = 3 \)). The original ‘old’ inflation envisaged by Guth was unsuccessful because of the presence of two conflicting requirements. On the one hand, for inflation to solve the horizon problem, to smooth out whatever inhomogeneities may have existed initially, and to sweep away the primordial monopoles, \( \Gamma/H^{d+1} \) must be small. Yet for inflation to end through the percolation of a large number of small bubbles, as Guth had originally envisaged, \( \Gamma/H^{d+1} \) had to be large. Although this potential conflict is noted in Guth’s first paper, it was not until the more detailed work of Guth and Weinberg [3] that this incompatibility was quantitatively and definitively established.

Although this deficiency of ‘old inflation’ was remedied in the work of Linde [4] and of Albrecht and Steinhardt [5] in ‘new’ and ‘chaotic’ inflation by dispensing with bubbles

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altogether, with inflation ending through the ‘slow roll’ of a scalar field along a relatively flat potential, the idea of inflation with bubbles re-emerges in ‘open inflation’, where re-heating occurs through the evolution of the inflaton field within the bubble interior. In this case $\Gamma/H^{d+1}$ is so small that only a single bubble is contemplated.

More recently, the collision of an isolated pair of true vacuum bubbles filled with (4+1)-dimensional anti de Sitter space expanding within a false vacuum of (4+1)-dimensional de Sitter space has been proposed as a mechanism for setting up a Randall-Sundrum-like cosmology with well defined initial conditions. A successful braneworld cosmology must resolve the (4+1)-dimensional smoothness problem—that is, explain why the bulk geometry (and any additional fields in the bulk as well) respect the same three-dimensional spatial homogeneity and isotropy as the universe on the brane. It does not suffice to postulate an epoch of quasi-exponential expansion on the brane itself, because after such an inflationary epoch ends, through their gravitational coupling, imperfections in the bulk will inevitably induce inhomogeneities on the brane. Because anti de Sitter space, quite in contrast to de Sitter space, does not lose its hair—the amplitudes of perturbations remain constant rather than decay—some other mechanism to explain the spatial homogeneity and isotropy of the bulk, and to establish a preferred initial quantum state is required.

In the colliding bubble braneworld scenario, an initial (4+1)-dimensional de Sitter epoch sweeps away whatever inhomogeneities may have previously existed. After a modest number of expansion times, the quantum fluctuations about the classical vacuum rapidly approach the Bunch-Davies vacuum. Classically, the nucleation of a single true vacuum bubble breaks the $SO(5,1)$ symmetry of $dS^5$ to $SO(4,1)$, the subgroup of transformations that leave the nucleation center of the bubble invariant. For the case of two bubbles, this residual symmetry is further broken to $SO(3,1)$, the subgroup of those transformations that leave both nucleation centers invariant. The three-dimensional, completely spatial surface of bubble collision, from which our local brane endowed with a modified (3+1)-dimensional FLRW cosmology arises, is endowed with the geometry of $H^3$, of constant negative spatial curvature.

Owing to the negative spatial curvature of this geometry, the eventual fate of such a universe (in the absence of a non-vanishing cosmological constant or something similar) is that of an empty expanding universe whose expansion is predominantly driven by its spatial curvature, as in a Milne universe. However, because of the exponential suppression of the nucleation of bubbles, this dreary fate may lie far in our future. It is not difficult to envisage bubble separations so large that today the spatial curvature is still negligible. $\Omega_0$, the cosmological density parameter today (when $T_{rad} = 2.75 K$), is a monotonically increasing function of the bubble pair separation. The farther apart the bubbles, the larger $\rho_{coll}$ and the smaller the spatial curvature on the surface of collision. The radius of spatial curvature of the surface of collision is given by

$$R = H_{dS}^{-1} \tan[\sigma/2],$$

which diverges as $\sigma \to \pi$. Here $\sigma$ is the spatial geodesic separation between the nucleation centers in units where $H^{-1} = 1$, so that $\sigma = \pi$ corresponds to bubble separated exactly at the threshold of never striking each other. We observe that even though $\sigma$ for colliding bubbles is bounded from above by a finite limit, the surface of collision can be arbitrarily close to flat. The gamma factor $\gamma = 1/\sqrt{1 - (v/c)^2}$ of the colliding bubbles in the local
center-of-mass frame of the collision, which is equal to

$$\gamma_{\text{max}} = \frac{\sin[\sigma/2]}{\sin[r_c]}.$$  \hspace{1cm} (2)

where $r_c$ is the radius of the critical bubble (in units in which $H_{dS}^{-1} = 1$), however, is limited by a finite upper bound. Both effects drive $\Omega_0$ toward unity as $\sigma \rightarrow \pi$. The precise relationship between $\sigma$ and $\Omega_0$ depends on the details of the equation of state after the collision, which is unknown and model dependent.

In the colliding bubble scenario the value of $\Omega$ today is a random variable dependent on the interbubble separation which is stochastically determined. The object of this article is to study the distribution of the interbubble separations for the case of bubbles nucleating in de Sitter space.

II. BUBBLE COLLISION PROBABILITIES

We now compute the probability distribution for the interbubble separation of pairs of colliding bubbles. This separation $\sigma$ is defined in a coordinate free manner as the length of the (shortest) spacelike geodesic connecting the nucleation centers of the two colliding bubbles. We adopt units in which $H = 1$ and assume a spatially and temporally uniform nucleation rate $\Gamma$, the absence of correlations of the nucleations of nearby bubbles, and that bubbles cannot nucleate inside regions already within bubbles.

First a lightning review of the geometry of de Sitter space. Maximally extended $(d+1)$-dimensional de Sitter space $dS^{(d+1)}$ is most readily constructed as the embedding of the unit hyperboloid defined by

$$-T^2 + X_0^2 + \ldots + X_d^2 = 1$$  \hspace{1cm} (3)

in $(d+2)$-dimensional Minkowski space $M^{(d+2)}$ with the line element

$$ds^2 = -dT^2 + dX_0^2 + \ldots + dX_d^2.$$  \hspace{1cm} (4)

This space in its entirety may be covered by the coordinates

$$T = \sinh[\tau],$$

$$X_0 = \cosh[\tau] \, n_0,$$

$$\ldots$$

$$X_d = \cosh[\tau] \, n_d,$$  \hspace{1cm} (5)

where $(n_0, \ldots, n_d)$ are the coordinates of a point on the unit sphere $S^d$ embedded in $E^{(d+1)}$, with the line element

$$ds^2 = -d\tau^2 + \cosh^2[\tau]d\Omega_d^2.$$  \hspace{1cm} (6)

\footnote{Note that the notation here differs from that of ref. [13].}
Any two points with a spacelike separation and the property that their forward lightcones eventually intersect may be transformed to lie on the throat of the hyperboloid defined in eqn. (4) (i.e., the sphere $T = 0$). For the largest such spatial separation, $\sigma = \pi$, and two points are antipodal. This situation (with our idealization of zero critical bubble radius) corresponds to bubbles which almost but never collide. Larger spatial separations are also possible, but in this case no spacelike geodesics exist linking the two points.

As a technical device, it is useful to characterize the invariant separation between two points $P$ and $Q$ in terms of the $(d+2)$-dimensional Minkowski space inner product

$$I(P, Q) = -T_P T_Q \sum_{i=0}^{d} (X_i)_P \cdot (X_i)_Q. \quad (7)$$

It follows that

$$I(P, Q) = \cos[\sigma] \quad (8)$$

for spacelike separated points of the first kind and that

$$I(P, Q) < -1 \quad (9)$$

for those of the latter kind. (Timelike separations correspond to $I(P, Q) > +1$.)

![FIG. 1. The spatially flat coordinate slicing covers precisely half of maximally extended de Sitter space, as indicated in the Penrose-Carter conformal diagram above. The two vertical boundaries are identified and on the horizontal boundaries the conformal factor diverges.](image)

In terms of the flat coordinates, which cover precisely half of maximally extended de Sitter space, as indicated in Fig. 1, the line element is

$$ds^2 = -dt^2 + e^{2t} \cdot \left[dx_1^2 + \ldots + dx_d^2\right], \quad (10)$$

which after the change of variable to conformal time $\eta = -e^{-t}$, with $-\infty < \eta < 0$, becomes

$$ds^2 = \frac{1}{\eta^2} \cdot \left[-d\eta^2 + dx_1^2 + \ldots + dx_d^2\right]. \quad (11)$$

The embedding of these coordinates into $M^{(d+1)}$ is given by
\[ T = \sinh[t] + e^t \frac{x^2}{2}, \]
\[ X_0 = \cosh[t] - e^t \frac{x^2}{2}, \]
\[ X_i = e^t x_i, \] (12)

where \( \mathbf{x} \cdot \mathbf{y} = x_1y_1 + \ldots + x_dy_d. \) The invariant separation defined above is given by

\[ I(P, Q) = \frac{\eta_P^2 + \eta_Q^2 - (\mathbf{x}_P - \mathbf{x}_Q)^2}{2\eta_P\eta_Q}. \] (13)

We now turn to the issue of initial conditions. It is necessary to specify an initial surface over which there is everywhere false vacuum. As has been previously pointed out in the literature, an unstable de Sitter vacuum extended infinitely far into the past of maximally extended de Sitter space does not make any sense. Bubbles formed during the contracting phase eventually percolate. However, if consideration is restricted to a subregion of de Sitter space over which there is only expansion (such as that covered by the flat coordinates), then, if \( \Gamma \) is sufficiently small, inflation is eternal in the forward time direction (i.e., the bubbles never percolate), and presumably any conclusions drawn are insensitive to the character of this initial surface.

We employ the flat coordinates, assuming a false vacuum everywhere on the surface \( t = t_- \) and considering only bubbles that nucleate for \( t \) in the interval \( t_- < t < t_+ \). (Eventually we take the double limit \( t_- \to -\infty, t_+ \to +\infty. \))

![Diagram](image)

FIG. 2. We take as given that a bubble has nucleated at the point \( P \). The surface \( \Sigma \) indicates the surface of possible nucleation centers separated by the same proper distance \( \sigma \) from \( P \). For a bubble that nucleates at \( Q \) on \( \Sigma \) to collide with the bubble that nucleated at \( P \) (rather than with some other intermediate bubble), it is necessary that no bubble nucleated in the singly shaded region.

The probability density for a bubble to nucleate at time \( t \) among all the bubble nucleating in the time interval \( t_- < t < t_+ \) is given by the normalized density

\[ \frac{dP_{bn}}{dt}(t) = e^{\Gamma V(t)} \times \left[ \int_{t_-}^{t_+} dt' e^{\Gamma V(t')} \right]^{-1} \] (14)

where \( V(t) \) is the volume of the past light cone of the nucleation center lying above the initial surface \( t = t_- \). Here we have considered an infinitesimally thin tube of constant co-moving d-volume, whose physical d-volume scales as \( e^{dt} \).
Next we consider bubble collisions. At first we restrict ourselves to the simplest case of bubble in (1+1)-dimensional de Sitter space to avoid the complications arising from the fact that in higher dimensions a bubble can collide with many other bubbles. However, for all volumes and measures we retain the general formulae, valid for all \( d \). In (1+1)-dimensions, each bubble collides with precisely one bubble from the left and one bubble from the right. Once the (1+1)-dimensional case has been thoroughly examined, we generalize to arbitrary dimension in the small \( \Gamma \) limit.

We now calculate the conditional probability

\[
\frac{dP_{bc}(\sigma|t)}{d\sigma} = \int_{\Sigma} d\Sigma \ \Gamma \ e^{-\Gamma V_{\text{excl}}(\Sigma)}.
\]

The various volumes are indicated in Fig. 2. \( P \) is the nucleation center of the bubble that has by assumption nucleated at time \( t \). The presence of this bubble implies that no other bubble has nucleated in the part of its past lightcone to the future of the initial surface (i.e., the doubly shaded region of the figure). \( \Sigma \) is the locus of all points a proper distance \( \sigma \) to the right of \( P \) and \( d\Sigma \) is the volume element on this surface. \( Q \) is a point on this surface, a potential nucleation center from a bubble a proper distance \( \sigma \) from \( P \) that may collide with the bubble emanating from \( P \). However, for such a collision to occur, it is necessary that no bubble nucleate in the lightly shaded region, of volume \( V_{\text{excl}} \). It is necessary to exclude the case of a bubble nucleating in the volume \( V_{\text{excl}} \), for any such bubble would collide first with the bubble emanating from \( P \). The probability of no nucleation in the lightly shaded region is \( \exp(-\Gamma V_{\text{excl}}) \), hence the appearance of this factor in eqn. (15). If we remove the upper cut-off, taking \( t_+ \rightarrow +\infty \), the integral of the density from \( \sigma = 0 \) to \( \sigma = \pi \) becomes equal to unity—in other words, a collision occurs with unit probability.

We may combine eqns. (14) and (15) to obtain the probability density properly averaged over the nucleation time for the first bubble \( t \) for collisions of bubbles separated by a proper distance \( \sigma \)

\[
\frac{dP_{av}(\sigma)}{d\sigma} = \int_{t_-}^{t_+} dt \ \frac{dP_{bc}(\sigma|t)}{d\sigma} \frac{dP_{bn}(t)}{dt}.
\]

Particularizing to (1+1) dimensions and replacing the proper time \( t \) with the conformal time \( \eta \), we rewrite eqn. (15) as

\[
\frac{dP}{d\cos[\bar{\sigma}]} = \Gamma \int_{-\infty}^{0} \frac{d\eta}{\eta^2} \int_{0}^{\infty} dx \ \delta\left(\cos[\sigma] - \cos[\bar{\sigma}]\right) e^{-\Gamma V_{\text{excl}}}.
\]

Without loss of generality we may set the coordinates of \( P \) to \( \eta_P = -1, x_P = 0 \). We eliminate the integration over \( x \) by considering

\[
f(x) = \cos[\sigma] = \frac{\eta^2 + \eta_P^2 - (x - x_P)^2}{2\eta \eta_P} = \frac{x^2 - \eta^2 - 1}{2\eta},
\]

so that eqn. (17) becomes
\[
\Gamma \int_{-\infty}^{0} \frac{d\eta}{\eta^2} \left| \frac{\partial f}{\partial x} \right|^{-1} e^{-\Gamma V_{\text{excl}}(x(\eta), \eta)}
= \Gamma \int_{-\infty}^{0} \frac{d\eta}{\eta^2} \left( \frac{-\eta}{x} \right) e^{-\Gamma V_{\text{excl}}(x(\eta), \eta)}
= \Gamma \int_{-\infty}^{0} \frac{d\eta}{\eta^2} \frac{1}{\sqrt{1 + 2\eta \cos[\bar{\sigma}] + \eta^2}} e^{-\Gamma V_{\text{excl}}(x(\eta), \eta)}. \tag{19}
\]

We now turn to the computation of \(V_{\text{excl}}(x, \eta)\). As can readily be seen from Fig. 2, this volume depends only on the point of intersection \((\bar{x}, \bar{\eta})\) of the left-moving null geodesic through \((x, \eta)\) with the right-moving null geodesic emanating from \(P\). It follows that

\[
\bar{\eta} = \frac{1}{2} \left[ \eta + x - 1 \right]. \tag{20}
\]

We compute

\[
V_{\text{excl}}(\bar{\eta}) = \int_{-\infty}^{-1} \frac{d\eta}{\eta^2} 2(1 - |\bar{\eta}|) + \int_{-1}^{-|\bar{\eta}|} \frac{d\eta}{\eta^2} 2(-|\bar{\eta}| - \eta) = 2 \ln \left( \frac{1}{|\bar{\eta}|} \right). \tag{21}
\]

Consequently, eqn. \((19)\) becomes

\[
\frac{dP}{d\cos[\bar{\sigma}]} = \Gamma \int_{-\infty}^{0} \frac{d\eta}{|\eta|} \frac{1}{\sqrt{1 + 2\eta \cos[\bar{\sigma}] + \eta^2}} \exp \left[ +2\Gamma \log \left( \frac{(1 - \eta - \sqrt{1 + 2\eta \cos[\bar{\sigma}] + \eta^2})}{2} \right) \right]
= \Gamma \int_{0}^{+\infty} \frac{d\eta}{\eta} \frac{1}{\sqrt{1 - 2\eta \cos[\bar{\sigma}] + \eta^2}} \left[ \frac{1 + \eta - \sqrt{1 - 2\eta \cos[\bar{\sigma}] + \eta^2}}{2} \right]^{-2\Gamma}. \tag{22}
\]

For small \(\Gamma\), the dominant contribution to the integral lies in the neighborhood of \(\eta = 0\). Therefore, we may approximate the above expression as

\[
\Gamma \int_{0}^{+\infty} \frac{d\eta}{\eta} \frac{1}{\sqrt{1 - 2\eta \cos[\bar{\sigma}] + \eta^2}} \left[ \frac{\eta(1 + \cos[\bar{\sigma}])}{2} \right]^{2\Gamma}
\approx \Gamma \left( \frac{1 + \cos[\bar{\sigma}]}{2} \right)^{2\Gamma} \int_{0}^{1} d\eta \eta^{(2\Gamma-1)}
\approx \frac{1}{2} \left( \frac{1 + \cos[\bar{\sigma}]}{2} \right)^{2\Gamma} \approx \frac{1}{2}. \tag{23}
\]

In other words, for \(\Gamma \to 0^+\), we obtain a uniform distribution in \(\cos \sigma\), or

\[
\frac{dP}{d\sigma} = \frac{\sin[\sigma]}{2}. \tag{24}
\]

We now generalize to \(d > 1\). Since any given bubble suffers an indeterminate number of collisions with other bubbles, it is not at the outset clear how to weight the various collisions to obtain a sensible distribution for \(dP/d\sigma\). We first define the distribution in a
rather artificial way, later showing that the result obtained is more generally valid in the small $\Gamma$ limit.

We start by postulating as given the nucleation of a bubble, taken without loss of generality to be situated at $\eta = -1$, $x = 0$, and compute the distribution for the first other bubble to collide with this bubble. (This may seem a bit artificial because a preferred time direction has been singled out.) Defined in this way, eqn. (19) becomes

$$V_{\text{excl}}(|\bar{\eta}|) = \frac{d}{d-1} \int_{|\bar{\eta}|}^{1} \frac{d\eta}{\eta_{d+1}} \left[ (\eta + 1 - 2|\bar{\eta}|)^{d} - (1 - \eta)^{d} \right] + \int_{1}^{+\infty} \frac{d\eta}{\eta_{d+1}} \left[ (\eta + 1 - 2|\bar{\eta}|)^{d} - (\eta - 1)^{d} \right]$$

(25)

where $\Omega_{d-1}$ is the area of the $(d-1)$-unit sphere. In the small $\Gamma$ limit, we are only interested in the most singular part of $V_{\text{excl}}$ as $|\bar{\eta}| \to 0^+$. In the limit, the second integral in the expression above is finite, and consequently can be ignored. The first integral is dominated by the contribution in the neighborhood of the lower limit of integration, which may be approximated as

$$V_{\text{excl}} \approx \frac{d}{d-1} \int_{|\bar{\eta}|}^{1} \frac{d\eta}{\eta^{d+1}} 2(\eta - |\bar{\eta}|) \approx \frac{2\Omega_{d-1}|\bar{\eta}|^{-d+1}}{d(d-1)}.$$

(26)

We obtain (setting $\bar{\Gamma} = \Gamma\Omega_{(d-1)})$

$$\frac{dP(\sigma)}{d\cos[\sigma]} = \Gamma\Omega_{d-1} \int_{0}^{+\infty} \frac{d\eta}{\eta^{d}} \sqrt{1 - 2\eta \cos[\sigma] + \eta^2} \exp \left[ -2\Gamma\Omega_{d-1}|\bar{\eta}|^{-d+1} \right]$$

$$\approx \bar{\Gamma} \int_{0}^{+\infty} \frac{d\eta}{\eta^{d}} \exp \left[ -2\bar{\Gamma} \left( \eta(1 + \cos[\sigma]) \right)^{-d+1} \right]$$

$$\approx \frac{\bar{\Gamma}}{(d-1)} \int_{0}^{+\infty} dz \exp \left[ -2\bar{\Gamma} \left( \frac{(1 + \cos[\sigma])}{2} \right)^{-d+1} z \right]$$

$$= \frac{d}{2} \left( \frac{1 + \cos[\sigma]}{2} \right)^{d-1} = \frac{d}{2} \cos^{2} \left[ \frac{\sigma}{2} \right]^{d-1},$$

(27)

so that

$$\frac{dP(\sigma)}{d\sigma} = \frac{d}{2} \sin \sigma \left( \frac{1 + \cos[\sigma]}{2} \right)^{d-1}.$$

(28)

We observe a power law suppression for large bubbles.

We now explain why the distribution above is more broadly valid, when multiple collisions have been taken into account. As before, we consider a bubble whose nucleation center has been placed at $\eta = -1$, $x = 0$. We consider an infinitesimal region on the surface of the bubble from $(\bar{\eta}, 1 + \bar{\eta})$ and $(\bar{\eta} + d\bar{\eta}, 1 + \bar{\eta} + d\bar{\eta})$ and some interval of solid angle as well, and consider, given that some exterior bubble first strikes this bubble in that patch, what is the distribution of $\sigma$. Computing this distribution involves extending the patch radially outward and into the past along the relevant null geodesics. Let $\xi$ range from 0 to $+\infty$. Then
\[ \eta = \bar{\eta} - \xi, \]
\[ r = 1 + \bar{\eta} + \xi. \]  
(29)

It follows that
\[ \cos[\sigma] = \frac{r^2 - \eta^2 - 1}{2\eta} = \frac{\bar{\eta} + \xi}{\bar{\eta} - \xi}. \]  
(30)

Therefore, \( \xi = -\bar{\eta}\tan^2[\sigma/2] \). For small \( \Gamma \), we may take \( \bar{\eta} \) to be very close to zero, because all but a few bubbles nucleate very long after the nucleation of the first bubble. Using the fact that
\[ \frac{dP}{d\cos[\sigma]} \sim \frac{dV}{d\cos[\sigma]} \]  
(31)

we obtain
\[ dV \sim d\xi \left( \frac{r^{d-1}}{\eta^{d+1}} \right) \approx \frac{d\xi}{\eta^{d+1}} = \frac{d\xi}{(\bar{\eta} - \xi)^{d+1}} \sim \left( \cos^2 \left[ \frac{\sigma}{2} \right] \right)^{d-1} d(\cos[\sigma]) \]  
(32)

confirming the result in eqn. (28). By means of this last calculation, all bubble collisions for which a portion of the bubble collision surface is tangent to one of the \( t = (\text{constant}) \) surfaces are counted and given equal weight.

### III. COLLISIONS WITH A THIRD BUBBLE

In the colliding bubble braneworld scenario the universe that arises from the collision of two bubbles has a hyperbolic geometry that would be infinite in extent in the absence of any collision with a third bubble. However, it is inevitable that sooner or later collisions with third bubbles occur, and such collisions truncate the extent of this hyperbolic universe. Physically, the relevant question is whether such a cosmological scenario is likely to contain a hyperbolic patch large enough to contain the entire universe observable to us today, for the consequences of a collision within our past lightcone with a third bubble would hardly be subtle. At a very minimum, such a collision would create an \( O(1) \) perturbation in \( \Delta T/T \) of the CMB anisotropy. More dire consequences are also possible.

Concretely, to obtain an estimate of the likelihood of a collision with a third bubble within our past lightcone, we consider how the infinite \( H^{(d-1)} \) surface of collision of the two bubbles is truncated by collisions with other bubbles. We take as given the initial collision of the two bubbles, assuming at least that the point \( \xi = 0 \) on \( H^{(d-1)} \) has been spared from a collision with a third bubble. Here \( ds^2 = R^2 (d\xi^2 + \sinh^2[\xi]d\Omega_{(d-1)}^2) \) is the metric on \( H^{(d-1)} \). We calculate the probability distribution for the radius \( \xi \) of the largest sphere \( B_\xi = \{ (\xi, \theta, \phi) | \xi < \xi \} \subset H^{(d-1)} \) such that no bubble has nucleated within the past lightcone of \( B_\xi \). As before we employ the coordinates \( ds^2 = (1/\eta^2)(-d\eta^2 + d\mathbf{x}^2) \) for the larger \((d + 1)\)-dimensional de Sitter space into which \( H^{(d-1)} \) is embedded. If the integration over the volume contained within a past lightcone of a spacetime point is taken all the way back to \( \eta = -\infty \), the resulting integral diverges logarithmically as \( \ln(|\eta_{\text{min}}|) \). However, since by assumption no bubble has nucleated in the past lightcone of the origin \( O \) at \( \xi = 0 \), we are
interested in the volume \( V_\xi \) that lies between the past lightcone of \( B_\xi \) and the past lightcone of the origin \( O \), and the integral defining \( V_\xi \) is convergent. One obtains
\[
\frac{dP}{d\xi} = \Gamma \frac{dV}{d\xi} \exp[-\Gamma V(\xi)].
\] (33)

For a typical value of \( \xi \), \( \Gamma V(\xi) \approx 1 \).

We first consider the simplest case with \( d = 2 \), later straightforwardly generalizing to \( d > 2 \). We define the hyperboloid \( H^{(d-1)} \) as the locus of points a proper time \( \tau \) to the future of its focus, where proper time is calculated using the de Sitter metric and \( \sinh[\tau] = R \).

Without loss of generality we fix the focus to the point \( \eta = -1, x = 0 \). From the relation for the invariant separation in de Sitter space, it follows that the hyperboloid is defined by the expression
\[
1 + \eta^2 - x^2 = 1 + \bar{\eta}^2 - 2\bar{\eta} = \cosh[\tau] = \sqrt{1 + R^2}.
\] (34)

Here \((\bar{\eta}, 0)\) is the spacetime position vector of the origin of the hyperboloid \( O \). It follows that \( R = (1 - \bar{\eta}^2)/(2\bar{\eta}) \).

We find it convenient to parameterize the hyperboloid according to
\[
\eta(\theta) = -\sqrt{1 + R^2} + R \sec[\theta],
\]
\[
r(\theta) = R \tan[\theta],
\] (35)

where \( 0 \leq \theta < \theta_{\text{max}} \), such that \( \tan[\theta_{\text{max}}] = 1/R \).

As \( R \to \infty \), \( \theta_{\text{max}} \to 0 \).

In terms of the distance in the larger de Sitter space, the line element along the hyperboloid gives
\[
\frac{ds}{d\theta} = \frac{R \sec[\theta]}{\sqrt{1 + R^2 - R \sec[\theta]}},
\] (36)

or in terms of the natural dimensionless distance on \( H^{(d-1)} \)
\[
\frac{d\xi}{d\theta} = \frac{\sec[\theta]}{\sqrt{1 + R^2 - R \sec[\theta]}}.
\] (37)

To compute \( dV/d\xi \) in eqn. (33) at a certain value of \( \xi \), we use the relation
\[
\frac{dV}{d\xi} = \frac{d\theta}{d\xi} \frac{dV}{d\theta}
\] (38)

and compute the infinitesimal volume \((dV/d\theta)d\theta\) swept out by the displacement \( d\theta \) as the past lightcone at \( \theta \) is displaced to \( (\theta + d\theta) \). In the underlying Minkowski space, with metric \( ds^2 = -d\eta^2 + dx^2 \), the cone at \( P(\theta) = (\eta(\theta), r(\theta), 0) \) suffers a rigid displacement by
\[
\frac{\partial}{\partial \theta} = R \sec[\theta] \cdot \left[ \tan[\theta] \frac{\partial}{\partial \eta} + \sec[\theta] \frac{\partial}{\partial r} \right].
\] (39)

We parameterize the surface of the cone using \( w, w_{\text{min}} \leq w \leq +\infty \), and \( \psi, -\pi < \psi < +\pi \), where \( w_{\text{min}} = -\eta(\theta) \),

\[
\text{10}
\]
\[ \eta = -w, \]
\[ r = (w - w_{\min}) \cos[\psi], \]
\[ x = (w - w_{\min}) \sin[\psi]. \tag{40} \]

The volume swept out by a surface element \((dw)(d\psi)\) is given by the determinant
\[
\frac{dV}{d\psi dw d\theta} = R \sec[\theta](w - w_{\min}) \begin{vmatrix} \tan[\theta] & \sec[\theta] & 0 \\ -1 & \cos[\psi] & \sin[\psi] \\ 0 & -\sin[\psi] & \cos[\psi] \end{vmatrix}
\]
\[ = R \sec[\theta](w - w_{\min}) \left[ \tan[\theta] + \sec[\theta] \cos[\psi] \right] \]
\[ = R \sec^2[\theta](w - w_{\min}) \left[ \cos[\psi] - \cos[\pi/2 + \theta] \right]. \tag{41} \]

Since we are interested only in the volume swept out in the forward direction, we restrict the integration to the subregion over which this determinant is positive—in other words, over the range \(|\psi| < \pi/2 + \theta\). Including the conformal factor \(w^{-3}\) to reflect de Sitter rather than Minkowski volume, we obtain
\[
\frac{dV}{d\theta} = 2 R \sec^2[\theta] \int_{|\eta(\theta)|}^{+\infty} \frac{dw}{w^3} (w - |\eta(\theta)|) \int_{0}^{\pi/2 + \theta} d\psi \left[ \cos[\psi] - \cos[\pi/2 + \theta] \right]
\]
\[ = \frac{R \sec^2[\theta]}{\sqrt{1 + R^2} - R \sec[\theta]} \left[ \cos[\theta] + \left( \frac{\pi}{2} + \theta \right) \sin[\theta] \right], \tag{42} \]

which in light of eqn. (37) becomes
\[ \frac{dV}{d\xi} = R \left[ 1 + \left( \frac{\pi}{2} + \theta \right) \tan[\theta] \right]. \tag{43} \]

We are concerned with the limit \(\Gamma \ll 1\), in which the values of \(\theta\) of interest lie very near \(\theta_{\max}\). In this limit, \(r \to 1\) and \(\tan \theta \to 1/R\). For large \(R\), \(\theta_{\max}\) is small, and \(dV/d\xi \approx R\), giving \(V(\xi) \approx R \xi\). Since \(\Gamma\) is exponentially small, the requirement that \(\Gamma V(\xi) = O(1)\), where here \(\langle \xi \rangle\) is the expectation value of \(\xi\), implies that \(\langle \xi \rangle\) is exponentially large.

To generalize to higher dimensions, we must modify eqn. (12) by changing the conformal factor from \(w^{-3}\) to \(w^{-(d+1)}\) and replacing the factor of \(\Omega_0 = 2\), with \(\Omega_{(d-2)}\) where \(\Omega_1 = 2\pi\), \(\Omega_2 = 4\pi\), .... Moreover, \(dr\) becomes \(r^{d-1}dr\), where now the position of the origin for \(r\) does matter. With these modifications, eqn. (12) is now
\[
\frac{dV}{d\theta} = \Omega_{(n-2)} R \sec^2[\theta] \int_{|\eta(\theta)|}^{+\infty} \frac{dw}{w^{d+1}} (w - |\eta(\theta)|) \int_{0}^{\pi/2 + \theta} d\psi \left[ \cos[\psi] - \cos[\pi/2 + \theta] \right] \left[ r(\theta) + (w - \eta(\theta)) \cos[\psi] \right]^{d-2}
\]
\[ = \Omega_{(d-2)} R \sec^2[\theta] \frac{1}{|\eta(\theta)|} \int_{1}^{+\infty} \frac{du}{u^{d+1}} (u - 1) \int_{0}^{\pi/2 + \theta} d\psi \left[ \frac{r(\theta)}{|\eta(\theta)|} + (u - 1) \cos[\psi] \right]^{d-2}. \tag{44} \]

where \(u = w/|\eta(\eta)|\).
For the dimension of greatest interest \((d = 4)\), it is not possible to evaluate
\[
V(\theta) = \int_0^\theta d\theta \frac{dV}{d\theta}
\]  
(45)
with the integrand defined in eqn. \((44)\). However, for \(\Gamma \ll 1\), we are interested in \(\theta\) very near \(\theta_{\text{max}}\), so that \(r(\theta)/|\eta(\theta)| \gg 1\). Hence, we may drop all but the leading term in powers of \(r(\theta)/|\eta(\theta)|\) in eqn. \((44)\), which will then reduce to
\[
\frac{dV}{d\theta} \approx R \sec^2[\theta] \frac{r^2(\theta)}{|\eta(\theta)|^3} \left[ \cos[\theta] + \left( \frac{\pi}{2} + \theta \right) \sin[\theta] \right].
\]  
(46)
As \(\theta \to \theta_{\text{max}}\), \(r \to 1\) and
\[
\frac{1}{|\eta(\theta)|} = \frac{\cos[\theta] \sin[\theta_{\text{max}}]}{\cos[\theta] - \cos[\theta_{\text{max}}]} \approx \frac{\cos[\theta_{\text{max}}]}{(\theta_{\text{max}} - \theta)},
\]  
(47)
so that
\[
V \approx \left( R + \frac{\pi}{2} + \theta_{\text{max}} \right) \cos^2[\theta_{\text{max}}] \int d\theta \frac{1}{(\theta_{\text{max}} - \theta)^3}
\approx \left( R + \frac{\pi}{2} + \theta_{\text{max}} \right) \left( \frac{R^2}{1 + R^2} \right) \left[ \frac{1}{2(\theta_{\text{max}} - \theta)^2} + O \left( \frac{1}{(\theta_{\text{max}} - \theta)} \right) \right].
\]  
(48)
From eqn. \((37)\) it follows that
\[
\frac{d\xi}{d\theta} \approx \frac{1}{(\theta_{\text{max}} - \theta)}.
\]  
(49)
Therefore,
\[
\xi \approx -\ln[\theta_{\text{max}} - \theta].
\]  
(50)
and
\[
(\theta_{\text{max}} - \theta) \approx \exp[-\xi]
\]  
(51)
and
\[
V \approx \frac{1}{2} \left( R + \frac{\pi}{2} + \theta_{\text{max}} \right) \left( \frac{R^2}{1 + R^2} \right) \exp[+2\xi].
\]  
(52)
Given that \([6], [13]\)
\[
\Gamma \approx \exp[-A(m_4\ell)^2]
\]  
(53)
where \(A = O(1)\), we find that for \((m_4\ell) \gg 1\),
\[
\langle \xi \rangle \approx \frac{A}{2}(m_4\ell)^2
\]  
(54)
where we have suppressed all logarithmic corrections. This result implies that for large \((m_4\ell)\), the \(SO(3,1)\) symmetric regions where the bubbles collide are composed of patches of a size almost always containing several curvature lengths, suggests that one is unlikely to observe a collision with a third bubble.
IV. DISCUSSION

The object of this study was twofold: first, to determine whether in the limit $\Gamma \to 0$ it is possible to obtain, with a non-negligible probability, bubble separations very near but just short of the maximal separation $\sigma = \pi$, where the bubbles just barely strike each other; and, second, to determine whether collisions with a third bubble constrain or rule out the colliding bubble braneworld scenario.

With respect to the first question, we found that as $\Gamma \to 0^+$, the probability distribution $dP/d\sigma$ approaches a limit independent of $\Gamma$, given in eqn. (28), having a power law suppression of values of $\sigma$ near $\pi$. Consequently, no matter how small $\Gamma$, $R$ (the curvature radius of the surface of collision $H^3$) is typically of order $H_{\text{dS}}^{-1}$, the curvature radius of the de Sitter space into which the bubble expands. To obtain a value of $\Omega_0$ close enough to one today, $H_{\text{dS}}^{-1}$ must be sufficiently large. As discussed in ref. [13], estimating $R_{\text{min}}$ depends on the unknown details of the equation of state on the brane arising from the collision, although in a rather insensitive way. For purposes of illustration, we adopt the assumption of a radiation equation of state in the aftermath of the collision, obtaining $R \gtrsim 10^{24} \ell (m_4 \ell)^{-2/5}$. Here $\ell$ is the AdS curvature radius inside the bubble. The largest admissible value $\ell \approx 1$ mm gives $R \gtrsim 10^{10}$ cm. While this large curvature is somewhat of an embarrassment, the five-dimensional bulk cosmological constant problem implied here is orders of magnitude milder than the four-dimensional effective cosmological constant today. One might hope to evade this restriction by appealing to the anthropic principle. In other words, one would have $H_{\text{dS}}^{-1}$ small so that most bubble collisions would result in empty and uninhabitable universes. We, however, would descend from the collision of one of those rare pairs for which $\Omega_0$ is not minuscule. Such a scenario, however, would predict a distribution of $\Omega_0$ seen by us peaked near the lowest acceptable value, much as in Weinberg’s proposed anthropic resolution of the cosmological constant problem [17], and this prediction is at variance with observation.

With respect to the second question, we found that collisions with third bubbles are rare, allowing the surface of collision to span several curvature radii, a size in excess of that of the universe observable to us today. Our analysis, however, considered only bubbles that collide with the surface of collision. Bubbles that collide from the side with one of the two bubbles from which our universe originated do lie in our past lightcone and in principle could affect us. In the idealised colliding bubble scenario such a collision simply results in another universe like ours; however, the possibility that some of the debris from the collision escapes into the bubble interior and subsequently propagates to our universe cannot be ruled out. We have not considered such possible collisions.

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\[ Note that the directions of the inequalities in eqns. (66), (69), and (70) of ref. [13] should be reversed. \]
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