Electron Fractionalization

S. A. Kivelson
Dept. of Physics, U.C.L.A., Los Angeles, CA 90095
(March 22, 2002)

A largely qualitative, and rather idiosyncratic discussion of electron fractionalization in condensed matter physics is presented, including some historical reflections and some speculations concerning future application of these ideas. Particular attention is paid to systems which exhibit spin-charge separation, i.e. the electron can decay into separate excitations which carry the electron spin and the electron charge; the soliton model of polyacetylene is treated as a paradigmatic example. This paper is based on a talk given at a symposium honoring A.J.Heeger, A.G.MacDiarmid, and H.Shirakawa, the winners of the year 2000 Nobel Prize in Chemistry.

I. SPIN-CHARGE SEPARATION IN THE 1DEG

As far as I know, the idea of electron fractionalization rose out of studies of the one-dimensional electron gas (1DEG). In particular, the idea of spin-charge separation was first explicitly treated by Luther and Emery in the context of a continuum (field theory) limit of the 1DEG. What they showed is that the Hamiltonian can be expressed as a sum

\[ H_{1DEG} = H_c[\phi_c] + H_s[\phi_s] + H_{\text{irr}}[\phi_c, \phi_s] \]  

where \( H_c \) and \( H_s \) are, respectively, the Hamiltonians which govern the dynamics of the spin and charge fields, \( \phi_c \) and \( \phi_s \), respectively, and \( H_{\text{irr}} \) (which they did not treat, explicitly) consists of terms that can be neglected in the long wave-length limit (irrelevant terms, in the renormalization group sense). Consequently, as far as the low energy physics is concerned, the spin and charge dynamics are completely decoupled from each other. Moreover, Luther and Emery showed that any space-time electronic correlation function can be expressed as the product of a correlator involving only the spin-fields and one involving only the charge fields.

This work is profound, and although its importance was not widely appreciated at the time, its influence has continued to grow ever since. Under many circumstances, the field theory methods are sufficiently powerful that all relevant correlation functions, even for finite temperature, frequency, or momentum, can be computed explicitly and exactly. In the last couple of years it has even become possible to compute various spectral functions exactly in cases in which the spectrum of either the charge or spin excitations is gapped, so that the bosonized field theory is importantly non-linear.

From a more modern perspective, this body of work concerns itself with the physics of a system at, or in the vicinity of a quantum critical point. An amazing thing about the 1DEG is that, due to the special character of quantum fluctuations in one dimension, there are quantum critical phases, not just critical points. The successfulness of this approach is related to the fact that critical theories are conformally invariant, and conformal invariance in one dimensional quantum or two dimensional classical statistical mechanics turns out to strongly constrain the nature of the correlation functions.

While the thrust of the present discussion concerns the broader implications of electron fractionalization, it is worth noting that in recent years, literal realizations of the 1DEG have become available for experimental study in nanowires, buckeytubes, and edge states in quantum Hall systems.

II. THE SSH MODEL OF SOLITONS IN POLYACETYLENE

The story of the soliton theory of polyacetylene in some sense starts with a beautiful result obtained in the context of relativistic quantum field theories by Jackiw and Rebbi. They noted that in certain field theories, especially in one spatial dimension, one can find solitons with fractional fermion number. However, the subject really got started some years later with the discovery of the conducting polymer, trans-polyacetylene, by Shirakawa, MacDiarmid, and Heeger. (There is no need to review the importance of this discovery here. However, readers who are unfamiliar with polyacetylene are referred to the Fig. 1 and its caption, for an introduction to this polymer.) In thinking about the remarkable properties of polyacetylene, Su, Schrieffer, and Heeger (SSH) discovered that solitons, which are precisely analogous to those of Jackiw and Rebbi, occur as the lowest energy electronic excitations in the simplest imaginable model (the SSH model) of a set of noninteracting electrons coupled to a lattice deformation (acoustic phonon). Here, however, the solitons have reversed charge-spin relations - they can be neutral with spin 1/2, or if spinless they have charge ±e.

Specifically, what SSH showed is that when an electron is added to an otherwise neutral polyacetylene chain, it can break up into two pieces, one of which carries the electron’s charge and the other its spin. This observation bears a clear family relation with the phenomenon of spin-charge separation in the 1DEG discussed above, but it is not identical. In the first place, undoped poly-

\[ H_c, H_s, \phi_c, \phi_s \]
acetylene is a semiconductor, with a moderately large \((2\Delta \sim 2\text{eV})\) gap, and so is nowhere near being quantum critical. The solitons in polyacetylene are not low energy excitations; the soliton creation energy is approximately \(E_s = 2\Delta / \pi\). Indeed, spin-charge separation does not occur in the SSH model in the same precise sense as it does in the 1DEG - for instance, there are substantial attractive interactions between the charged and neutral solitons, which lead to a bound-state (a polaron \([12]\)), with a binding energy \(E_p - 2E_s = -(2/\pi)(2 - \sqrt{2})\Delta\). In other words, spin charge coupling implies that the lowest energy excitation of the system with the quantum numbers of an electron is a quasi-particle with the same quantum numbers as the electron. In this sense, fractionalization is, in fact, a high energy feature of the spectrum of the SSH model. The lowest energy excitation made by adding two electrons to the system is a pair of charged solitons. (One can understand this effect qualitatively from the Hamiltonian in Eq. (1); under circumstances in which there is a gap to both spin and charge excitations, the renormalization group flows carry the system to a strong coupling fixed point where the terms in \(H_{\text{irr}}\) are no longer irrelevant, and in particular can lead to a short-range attraction between spin and charge solitons.)

Polyacetylene is a very complicated material, from a physicist’s viewpoint. It is rather disordered, even when undoped, and doping introduces all sorts of additional levels of randomness. It is moderately one-dimensional, with an in-chain bandwidth of order \(W \approx 10\text{eV}\) as compared to an interchain bandwidth of order one or two tenths of an \(eV\); this is not, however, sufficient to permit truly one dimensional physics to be manifest at very long distances, and clearly leads to rather strong soliton confinement. Nevertheless, many rather spectacular predictions of the soliton theory, both spectroscopic and dynamical, were confirmed by experiments performed with great vigor, determination, and creativity by a large community of scientists. I think by now there can be no question that the basic features of the soliton model of polyacetylene are not only right, in theory, but applicable to a range of experiments, as well \([13]\).

However, the real importance of the soliton model of polyacetylene was that it introduced a new paradigm into the field. It has led to ideas which have had a lasting impact on condensed matter physics:

- a) That there can exist quasi-particles with fractional quantum numbers (\(i.e.\) quantum numbers that are unrelated to those of an integer number of electrons and holes) that are robust entities, not just in low energy asymptopia, but in the real-world realm of materials physics;
- b) That such quasi-particles have a topological character, from which their stability derives;
- c) That these fractional quantum numbers are sharp quantum observables \([14]\), every bit as real as the charge of the electron.

A. Aside: the microscopic theory of polyacetylene

Before I leave the subject of polyacetylene, there is one further point, not directly related to fractionalization, that I will touch upon. While the character of the solitons, and the separation of charge and spin are robust consequences of the broken symmetry (dimerized) groundstate, there are many aspects of the SSH solution that are more microscopic, and model dependent. In part, it was the success of some of these more delicate aspects of the theory that led to the general acceptance of the soliton model.

In particular, there was a remarkable quantitative prediction made by SSH, concerning the magnitude of the dimerization in trans polyacetylene \([1\text{]}\). At the time of their first work on the subject, the in-chain bandwidth, \(W\), was known within 20% from various quantum chemical and band-structure calculations. Similarly, the spring constant \((K \approx 20\text{eV}/\text{Å}^2)\) of the \(\sigma\) bonds was well known to depend only on local chemistry, and so was known reliably. Thus, the only free parameter in the model was the electron-phonon coupling constant, \(\alpha\). SSH determined the value of this parameter, empirically, by fitting to the observed optical absorption gap, and were then able to predict the magnitude of the lattice dimerization, \(u = 0.04\text{Å}\), on the basis of this fit. Later quantitative measurements \([15]\) found \(u = 0.03 \pm 0.01\text{Å}\), thus confirming this prediction.

There are several things that are remarkable about this triumph of the SSH theory. The first is that this is a one dimensional system, so one might think that mean-field theory, which they employed, is unreliable. The neglect of electron-electron interactions seems, at first sight, to be equally suspect, both because these interactions are known, from quantum chemical studies of small polyenes, to be strong, and, as discussed above, interactions in the 1DEG are known to completely destroy Fermi-liquid behavior. However, this success was not an accident. The fact that the phonon frequencies, \(\hbar\omega_0\), are small on electronic scales, \(\hbar\omega_0/\Delta \sim 10^{-1}\), can easily be shown to justify the mean-field theory - indeed, effects of quantum lattice fluctuations were systematically studied and found to be quite mild \([16]\). The fact that a weak-coupling theory is reasonable is guaranteed by the fact that the gap is small compared to the bandwidth, \(\Delta/W \sim 10^{-1}\).

The proof that the effects of electron-electron interactions are largely perturbative was contained in some work I did with G.Zimanyi and A. Luther \([17]\) (ZKL), long after it ceased to be a hot issue. What ZKL (and, at about the same time, J.Voit, as well \([18]\) did was to treat
the problem of the 1DEG with both electron-electron (instantaneous) repulsions and electron-phonon induced (retarded) attractions using standard weak-coupling (one loop) renormalization group methods. What we found was that, even if at the microscopic level the electron-electron interactions are much stronger than the electron-phonon interactions, so long as $W/\hbar \omega_0$ and $W/\Delta$ are sufficiently large, the effective low energy theory is always dominated by the electron phonon interactions. The electron-electron interactions have the effect of renormalizing the effective electron-phonon coupling (curiously, they lead to a strong enhancement of $\alpha$), but other than that, have only perturbative effects on the low energy physics. Moreover, this effect is most pronounced when the electron density is commensurate, or nearly so, i.e. when there is roughly one electron per site.

Not only does this result justify, a posteriori, the remarkable physical insight of SSH, it also explains why the empirically determined value of $\alpha$ is large compared to those found in microscopic, quantum chemical calculations: the empirical $\alpha$ is a renormalized coupling. It also explains the remarkable fact that when polyacetylene is “overdoped,” i.e. when the electron concentration deviates by more than about 6% from one $\pi$-electron per carbon, it behaves like a nearly non-interacting metal \[19\]; due to the strong doping dependence of the effective electron-phonon coupling, the expected Peierls instability of this quasi-one-dimensional metal is suppressed \[20\] to immeasurably small temperatures.

### III. FRACTIONALLY CHARGED LAUGHLIN QUASI-PARTICLES

From the first, the fact of electron fractionalization in the 1DEG and the SSH model of polyacetylene was recognized as incompatible with the conventional (Fermi liquid theory based) paradigms of condensed matter physics. However, it was widely believed at the time that this was still nothing more than a one-dimensional curiosity; there is a general theorem \[21\] to the effect that solitons in a local field theory in more than one dimension have infinite creation energy, unless coupled to a suitable gauge field. Thus, it was generally expected that: 1) No “real” (i.e. two or three dimensional) electronic system would ever exhibit true electron fractionalization. 2) The solitons of the one-dimensional theory would be confined into integer charged multiplets the moment interchain interactions were introduced. Both of these statements turn out to be wrong.

Almost immediately following the spectacular discovery of the fractional quantum Hall effect \[22\], Laughlin \[23\] wrote down his famous wavefunction, which captures the essential physics of this new state of matter. An essential feature of this physics is that the quasi-particle excitations of the fractional quantum Hall liquid are vortex-like excitations with fractional charge. The fact that the fractional quantum Hall effect necessarily implies the existence of fractionally charged quasi-particles can be seen straightforwardly from a slightly modified version \[24\] of the remarkable thought experiment introduced by Laughlin in his original paper:

A fractional quantum Hall liquid is an incompressible state (i.e. it has a gap, $\Delta$) with a quantized Hall conductance

$$\sigma_H = (e^2/h) \, \nu^*$$  \hspace{1cm} (2)

where $\nu^*$ is one of a set of discrete, rational fractions, of which the most prominent is $\nu^* = 1/3$. Consider taking a system in its ground state which exhibits the fractional quantum Hall effect, and piercing it with an infinitesimal hole through which a magnetic flux, $\Phi$, can be threaded. Because the system has a gap, this can be done adiabatically, so long as the rate at which the flux is threaded is slow compared to $\hbar/\Delta$. Once $\Phi$ reaches a magnetic flux quantum, $\Phi_0 = h c/e$, the flux can be effectively removed from the Hamiltonian by a gauge transformation; the resulting state is an eigenstate of the original Hamiltonian. This new eigenstate, however, has a well defined charge which has accumulated in the vicinity of the hole. To compute the induced charge, $Q$, consider measuring the time integrated flux of current through a large ring which encircles the hole

$$Q = \int dt \vec{\dot{Q}} = \int dt \oint d\vec{r} \times \vec{J}. \hspace{1cm} (3)$$

Now, the current density, $\vec{J}$, is the response, $J_j = \epsilon_{ij} \sigma_H E_j$ of the quantum Hall state to the electric field $\vec{E}(\vec{r}) = e^{-1} \dot{A} = i \dot{\Phi}/(2\pi r)$ produced by the time varying flux. (Here $\epsilon_{ij}$ is the Levi-Civita symbol.) Consequently,

$$Q = \sigma_H \Phi_0 = e \nu^*.$$ \hspace{1cm} (4)

If $\nu^*$ is fractional, there must exist quasi-particles with fractional charge $e^*$ such that $n e^* = Q$, where $n$ is an integer!

The fractionally charged quasi-particles are still topological, in a sense, but in a sense which escapes the general theorem. Their topological character is manifest as fractional statistics of the Laughlin quasi-particles under exchange. This was first recognized by Halperin \[25\], from a study of the analytic properties of multi-quasi-particle wavefunctions, and later derived in a more physical manner by Arovas, Wilczek, and Schrieffer \[26\], from a calculation of the Berry’s phase when two quasi-particles are adiabatically exchanged.

As the above argument suggests, both the fractional charge and the fractional statistics of the quasi-particles are, in a sense, implicit in the fractional quantum Hall effect itself \[27\]. The fractional quantization of the Hall conductance has been observed with an accuracy of
better than one part in $10^{-4}$. Experiments designed to directly measure the fractional charge of the quasi-particle have been carried out, with results consistent with expectations, although with nowhere the same level of accuracy. Experiments have been proposed to directly measure the fractional statistics, as well, although they will be hard. However, there is no question in anyone’s mind that the Laughlin quasi-particles exist, that they are as robust as the fractional quantum Hall state itself, and that they have the predicted fractional charge and statistics.

One subtlety, with potentially significant consequences, is swept under the rug in the above discussion. It is possible to find circumstances in which, depending on details of the electronic Hamiltonian, there can be more than one possible quantum Hall liquid with the same value of the Hall conductance. However, these different states generally will have quasi-particles with different quantum numbers, although always consistent with the constraint that there exist multiplets with charge $Q$ given in Eq. 1. The macroscopic distinctions between such states can be classified in terms of the quantum numbers of the quasi-particles. Alternatively, they can be classified in terms of certain topological properties of the ground state, namely the ground-state degeneracies on closed surfaces of varying connectivities. There is, of course, an intimate connection between these two approaches. It would seem that the former is more closely related to a conceivable experiment, although the topological structure of the ground state is also related to the character of the edge states produced in a system with boundaries.

\section*{IV. RVB AND SPIN-CHARGE SEPARATION IN TWO DIMENSIONS}

The quantum Hall system is still very special. It is two dimensional, but so is the electron gas in real MOSFETS and heterojunction devices, among other systems. However, the large magnetic field explicitly breaks time-reversal and reflection symmetry. Thus, although following Laughlin’s work, there was no denying that fractionally charged particles were a feature of the “real” world, they still occupied a small corner of that world.

Immediately following the discovery of high temperature superconductivity, Anderson proposed that the key to the problem lay in the occurrence of a never before documented state of matter, a spin-liquid or “resonating valence bond” (RVB) state, related to a state he originally proposed for quantum antiferromagnets on a triangular (or similarly frustrating) lattice. A spin-liquid, in this context, is defined to be an insulating state (with a charge gap) and an odd integer number of electrons per unit cell which breaks neither spin-rotational nor translational symmetry. Following Ander-

son, Jim Sethna, Dan Rokhsar and I showed that a consequence of the existence of such a spin-liquid state is that there exist quasi-particles with reversed charge spin relations, just like the solitons in polyacetylene: charge 0 spin 1/2 “spinons” and charge e spin 0 “holons.” Indeed, these quasi-particles were recognized as having a topological character analogous to that of the Laughlin quasi-particles in the quantum Hall effect.

There was a debate at the time concerning the proper exchange statistics, with proposals presented identifying the holon as a boson, a fermion, or a semion. I now believe that all sides of this debate were correct, in the sense that there is no universal answer to the question - depending on details of the Hamiltonian, it is possible to imagine transitions occurring between states in which the holon has different statistics. In fact, this debate represented the first steps in the theoretical exploration of the topological structure of spin liquid states. We will return to this in Sec. VI B, below.

The real question-mark hanging over the whole subject, as was pointed out most forcefully by Read and Sachdev, is whether, and under what circumstances a spin-liquid exists; indeed, they presented strong arguments that the most straightforward quantum disordering of an antiferromagnet will lead to a spin-Peierls state, rather than a spin-liquid. Moreover, for more than a decade, despite extensive effort, no one succeeded in producing a model system which could be convincingly shown to exhibit a spin-liquid ground-state.

In this context, I am happy to report that very recently, Moessner and Sondhi have managed to do just this! They have considered a model on a triangular lattice (thus returning very closely to the original proposal of Anderson) which is a bit of a caricature in the sense that the constituents are not single electrons, but rather valence bonds (hard-core dimers), much in the spirit pioneered by Pauling. However, the model is sufficiently well motivated, microscopically, and the spin-liquid character sufficiently robust, that I believe it is reasonable to declare victory. Moreover, the resulting spin-liquid state does not break time-reversal symmetry. This means that the point of principle has been established. Spin charge separation (and, presumably, still more exotic forms of electron fractionalization) can occur in more than one dimension, and in the absence of either explicit or spontaneous time-reversal symmetry breaking! It is now only a matter of time until this phenomenon is either confirmed in some existing material (maybe even the high temperature superconductors), or discovered in a new system.
V. FRACTIONALIZATION AT QUANTUM CRITICAL POINTS

Electron fractionalization of a sort can occur at quantum critical points in systems between their upper and lower critical dimensions, in the sense that an injected electron will, with probability one, decay into a multi-particle continuum \[5\]. However, this type of electron fractionalization is quite different from the electron fractionalization we have been discussing. In the first place, in more than one dimension, and in the limit \(T \to 0\), quantum critical phenomena typically occur at a critical point, as opposed to in a critical phase of matter. In the second place, there is no known quasi-particle description of the elementary excitations of such a quantum critical system \[56\]; in particular, there is no currently accepted \[53\] description of the critical state in terms of excitations with fractional quantum numbers.

However, there is another recent development which is exotic, but very intimately connected to the type of electron fractionalization discussed here. It was recognized \[54\] a couple of years ago that a certain class of layered (i.e. quasi-two dimensional) classical systems can exhibit a critical phase, a phase, moreover, in which at low energy, the couplings between the layers are negligible. Such “floating” phases may, in fact, have been seen in lamellar DNA-lipid bilayer complexes \[57\]. Moreover, a similar phenomenon can occur in the quantum theory of quasi-one dimensional systems \[58,59\]. In such systems, even though at a microscopic level there are weak but non-vanishing couplings between chains, all interchain couplings are irrelevant in the renormalization group sense; the low energy physics of the system is that of a set of decoupled 1DEGs!

Among other things, this means that precisely the same form of one dimensional spin-charge separation discussed in the first section occurs in such systems, even though at a microscopic level there are finite higher dimensional couplings. For the models studied to date, the floating phases occur only in an extreme region of parameter space \[60\]. It may be that this implies that floating phases are very rare, indeed. However, now that we know they exist in principle, we can begin looking for them. Already, there is evidence \[61\] that such a floating phase occurs in one particular member of the family of high temperature superconductors.

VI. WHERE ELSE MIGHT ELECTRON FRACTIONALIZATION OCCUR?

I think this is the tip of the iceberg. The Fermi liquid based view of the electronic properties of solids has been very successful as a basis for understanding the essential physics of a wide range of conventional solids, including metals, semiconductors, and superconductors. It runs into difficulty in highly correlated solids, where so called “non Fermi liquid” (NFL) behavior is observed; here, even the given name admits to our complete lack of any successful theory of what causes this behavior. At best, the Landau quasi-particles are very strongly interacting in such solids. More likely, in many cases no such quasi-particle description is possible, and in some cases, electron fractionalization provides the correct framework for thinking about these systems.

A. Electron fractionalization as intermediate scale physics

While the conditions for true electron fractionalization, in the sense that it is really possible to have arbitrarily widely separated excitations with fractional quantum numbers, may be rather restrictive, the notion may be much more widely applicable.

A Fermi liquid (as contrasted with a Boltzman liquid) is only precisely defined in the limit that the temperature, \(T \to 0\); however, systems with a true Fermi liquid ground-state are certainly rare, at best, and may well (due to the famous Kohn-Luttinger theorem \[65\]) be nonexistent. Never-the-less, over a broad intermediate range of temperatures such that \(T_F \gg T \gg T_c\), the Fermi liquid description is both qualitatively and quantitatively valid. (Here \(T_F\) is the Fermi temperature and \(T_c\) is some ordering temperature, for instance to a superconducting state.)

Similarly, there occur circumstances in which a fractionalized description of the spectrum of a system over an intermediate range of temperatures, frequencies, and length scales may be valid, even though in assymptopia it breaks down. This is certainly true of a typical quasi-one dimensional system where, at temperatures above any ordering temperature, the system can be treated as one dimensional, with the effects of interchain coupling making only perturbative corrections to the 1D physics. Even below \(T_c\), the physics of the 1DEG is manifest at all but the lowest frequencies. Recently, considerable progress \[66\] has been made in obtaining an understanding of the consequences of electron fractionalization for various electronic spectroscopies of correlated electronic systems, and on the nature \[67\] of the crossover from a fractionalized spectrum above \(T_c\) to a spectrum with a well defined quasi-particle piece below \(T_c\). In many respects, these spectra account \[68,69\] well for properties of the observed spectra (especially those obtained in angle resolved photoemission \[70\]) in the high temperature superconductors, including many features that simply cannot be understood in the context of Fermi liquid theory. It has also been suggested \[71\], that spin-charge separation at intermediate scales may generally be the basis of an electronic mechanism of high temperature superconductivity.
B. Topological order and electron fractionalization

We now address the problem of classifying phases in which true electron fractionalization occurs, e.g. in which spinons are deconfined. It is now clear from the work of Wen [22] and Senthil and Fisher [14] that the best macroscopic characterization of fractionalized phases in two or more dimensions, given that they frequently possess no local order parameter, is topological. Specifically, a fractionized phase exhibits certain predictable ground-state degeneracies on various closed surfaces - degeneracies which Senthil and Fisher have given a physical interpretation in terms of “vison expulsion.” Unlike the degeneracies associated with conventional broken symmetries, these degeneracies are not lifted by small external fields which break either translational or spin rotational symmetry. It has even been shown [22,14] (funny as this may sound) that topological order is amenable to experimental detection. Once topological classification is accepted, the one-to-one relation between spin-liquids and electron fractionalization, implied in our previous discussion, is eliminated. Indeed, it is possible to imagine ordered (broken symmetry) states, proximate to a spin liquid phase, which will preserve the ground-state degeneracies of the nearby spin liquid, and hence will exhibit spin-charge separation.

C. Electron fractionalization and quantum computing

One of the most exciting areas where electron fractionalization may play an important role is in the developing field of quantum computing. It was recently shown [22] that many of the vexing problems of decoherence, which are barriers to construction of a functioning quantum computer, can be avoided if the quantum states in question have an appropriate topological character. Certain types of electron fractionalization, including those which occur in some complicated quantum Hall states [22] and in the RVB spin-liquid [74], have topological structure that could be useful for this purpose. Here, even if (as seems likely) rather special circumstances are required to obtain true electron fractionalization, it may be worthwhile to seek artificial methods for achieving those special circumstances.

FIG. 1. Schematic Representation of Solitons in Polyacetylene: The figure represents three polyacetylene chains with a soliton (S) anti-soliton (S̄) pair on the middle one. In terms of structure, the vertices indicate the position of C atoms and the ends of the vertical bonds indicate H positions; the double bonds are slightly shorter than the single bonds. Note that there are two degenerate patterns of alternating single and double bonds (dimerization) in the ground-state; we talk of a given pattern breaking the translational symmetry of the polymer, but in fact, because of the zig-zag, it is actually a reflection plane symmetry (about a C site) which is being broken, or alternatively a screw symmetry. A strong coupling charactarature of the electronic state can be deduced from the figure as follows: Associate with each C atom the two core 1S electrons. The thin lines represent a pair of electrons in a bonding σ orbital, and the heavy lines a pair of π electrons (that is, electrons in an out-of-plane 2P C orbital) in a bonding, or valence bond, state. The local neutrality of the pristine (outer) polyacetylene chains is easily seen from the fact that there are 2 core electrons, 3 σ electrons and 1 π electron per C, and 1 σ electron per H. From this it is clear that S and S̄ each have charge +e and, since all electrons are paired, spin 0. There is also, clearly, a non-bonding π orbital left over on the central C in each soliton. If this orbital were singly occupied, the soliton would have charge 0 and spin 1/2, while if it were doubly occupied, it would have charge −e and spin 0. Although in reality the π electrons, in particular, are rather extended, so that the π electron density on the short bonds is only a few percent larger than that on the long bonds, because the spectrum is gapped, the quantum numbers of the solitons are unchanged upon adiabatic continuity from the strong coupling limit, described above. All the fancy topological theorems that have been applied to this problem are no better than this simple derivation, often called “the Schrieffer counting argument.” The fact that interchain couplings inevitably cause soliton confinement can also be seen from the figure. Between the solitons, the dimerization on neighboring chains has the same sense, while beyond them, it has the negative sense. Since interchain couplings favor one relative phase or the other by a given energy per unit length of chain, there is manifestly a confining potential which grows linearly with the separation between solitons, if the interchain couplings are not zero.

Acknowledgements: Substantial improvements in this manuscript were made in response to comments by M.P.A.Fisher, E.Fradkin, C.Nayak and S.Sachdev.
[1] For a review, see V. J. Emery in Highly Conducting 1D Solids, eds. J. T. Devreese, et al (Plenum, New York, 1979).
[2] V.J.Emery and A. Luther, Phys. Rev. Lett. 33, 589 (1974).
[3] Some very exciting recent papers in which the correlation functions of the IDEG in a massive phase are computed exactly using recently developed Bethe ansatz form factor methods, include D.Orgad, Philos. Mag. B 81, 375 (2001) and Davide Controzzi, Fabian H.L. Essler, and Alexei M. Tsvelik, cond-mat/0005349.
[4] A. Belavin, A.M. Polyakov, A. Zamolodchikov, and A.Luther, Phys. Rev. B 24, 6447 (1982). Aspects of Symmetry (Cambridge University Press, Cambridge, 1997) Chapter 11.
[5] See, for example, M. Grayson, D.C.Tsui, L.N.Pfeiffer, and K.W.West, Phys. Rev. Lett. 64, 1764-1767 (2000).
[6] See, for example, Z.Yao, H.W.Ch.Postma, L.Balents, and C.Dekker, Nature 402, 273-6 (1999) and references therein.
[7] See, for example, M. Grayson, D.C. Tsui, L.N.Pfeiffer, K.W. West, A.M.Chang, Phys. Rev. Lett. 86, 2645-8 (2001) and references therein.
[8] R. Jackiw and C. Rebbi, Phys. Rev. D 13, 3398 (1976).
[9] W.P. Su, J.R.Schrieffer, and A.J.Heeger, Phys. Rev. Lett. 42, 1698 (1979); Phys. Rev. B22, 2099 (1980).
[10] Independently, and at about the same time, S.A.Brazovskii, JETP Lett. 28, 606 (1978) had found the same soliton, studying a continuum version of the SSH model, and M.J.Rice Phys. Lett. 71, 152 (1979), from a somewhat more phenomenological viewpoint, had obtained similar results, as well.
[11] H. Takayama, Y.R. Lin-Liu, and K. Maki, Phys. Rev. B21, 2388 (1980).
[12] W-P. Su, Solid State Commun. 35, 899 (1980); D.K.Campbell, A.R.Bishop and K.Fesser, Phys. Rev. B26, 6862 (1982); S.A.Brasovski and N.Kirova, JETP Lett. 33, 4 (1981).
[13] For a critical review, see A.J.Heeger, S.A.Kivelson, J.R. Schrieffer, and W-P. Su, Rev. Mod. Phys. 60, 781 (1988).
[14] S.A.Kivelson and J.R.Schrieffer, Phys. Rev. B24, 6447 (1982); J.S.Bell and R.Rajaraman, Phys. Lett. 116, 151 (1982); R.Jackiw, A.K.Kerman, I.Klebanov, and G.Semenoff, Nucl. Phys. B 225, 223 (1983). For a beautiful recent review of this issue, see R. Rajaraman, cond-mat/0103366.
[15] C.R.Fincher, C.E.Chen, A.J.Heeger, A.G.MacDiarmid, and J.Hastings, Phys. Rev. Lett. 48, 100 (1982).
[16] My favorite paper on this subject is J.E.Hirsch and E. Fradkin, Phys. Rev. Lett. 49, 402 (1982) and E.Fradkin and J.E.Hirsch, Phys. Rev. B27, 1680 (1983). This paper combines powerful analytical theory with serious numerical experiment; it was one of the earliest applications of quantum monte-carlo methods to systems of interacting fermions.
[17] G.T.Zimanyi, S.A.Kivelson, and A.Luther, Phys. Rev. Lett. 60, 2089 (1988).
[18] J. Voit, Phys. Rev. Lett. 64, 323 (1990).
[19] S.A.Kivelson and A.J.Heeger, Synth. Met. 22, 371 (1988).
[20] S.A.Kivelson and M.I.Salkola, Synth. Met. 44, 281 (1991).
[21] This is referred to as Derrick’s theorem. For a clear discussion, see S. Coleman, Aspects of Symmetry (Cambridge University Press, Cambridge, 1990), pg. 194.
[22] D.C.Tsui, H.L.Stormer, and A.C.Gossard, Phys. Rev. Lett. 48, 1559 (1982).
[23] R.B.Laughlin Phys. Rev. Lett. 50, 1395 (1983).
[24] A.Karlhede, S.A.Kivelson, and S.L.Sondhi, in Correlated Electron Systems, ed. by V.J.Emery (World Scientific, Singapore, 1992) pgs. 242-347.
[25] B.I.Halperin, Phys. Rev. Lett. 52, 1583 (1984).
[26] D.P.Arovaz, J.R.Schrieffer, and W.Filczek, Phys. Rev. Lett. 53, 722 (1984).
[27] S.A.Kivelson, Phys. Rev. Lett. 65, 3369 (1990).
[28] R.B.Laughlin, Rev. Mod. Phys. 71, 863 (2000).
[29] R.B.Laughlin, Science 242, 525-33 (1988).
[30] V.J.Goldman and B. Su, Science 267, 1010 (1995). See also J.A.Simmons, H.P.Wei, L.W.Engel, D.C. Tsui, M.Shayegan, Phys. Rev. Lett. 63, 1721 (1989); R. de-Picciotto, M. Reznikov, M. Heiblum, and V. Umansky, G. Bunin, and D.Mahalu, Nature 389, 162-164, (1997); Saminadayar, D.C.Glattli, Y.Jin, and B. Etienne, Phys. Rev. Lett. 79, 2526-2529, (1997).
[31] C.Chamon, D. E. Freed, S. A. Kivelson, S. L. Sondhi, and X. G. Wen, Phys. Rev. B55, 2331-43 (1997).
[32] X.G.Wen, Int. J. Mod. Phys. B 4, 239-71 (1990).
[33] Stacked layers of weakly coupled 2DEGs can exhibit an interesting extension of quantum Hall physics to three dimensions. Under conditions in which each isolated layer would be in a fractional quantum Hall liquid state, with a correspondingly gapped spectrum, weak coupling between layers has only perturbative effects on the state. Consequently, such a system will still have quasi-particles with fractional charge, even though it is three dimensional. The nature of infinite layer quantum Hall states is explored in X.Qui, R.Joynt, and A.H.MacDonald, Phys. Rev. Lett. 713-746 (2001).
[34] J.B.Bednorf and K.A.Mueller, Z. Phys. B 64, 189 (1986).
[35] P.W.Anderson, Science 235, 1169 (1987).
[36] P.W.Anderson, Mat. Res. Bull 8, 153 (1973) and P.Fazekas and P.W.Anderson, Phil. Mag. 30, 23 (1974).
[37] For a recent scholarly overview of the field, and extensive references, see R.Moessner, S.L.Sondhi, and E. Fradkin, cond-mat/0103396 and E. Fradkin, Field Theories of Condensed Matter Systems (Addison-Wesley, 1991).
[38] S.A.Kivelson, D.S.Rokhsar, and J.P.Sethna, Phys. Rev. B35, 865 (1987).
[39] P.W.Anderson, G. Baskaran, Z. Zou, and T. Hsu, Phys. Rev. Lett. 58, 2790-2793 (1987).
[40] N.Read and B.Chakraborty, Phys. Rev. B40, 7133 (1989).
[41] V. Kalmeyer and R.B.Laughlin, Phys. Rev. Lett. 59, 2095 (1987); R.B.Laughlin, Phys. Rev. Lett. This proposal concerning the nature of the putative spin-liquid was, in some ways, the most straightforward, as it stems from a direct analogy with the known physics of the fractional quantum Hall effect. While no one, to my knowledge, has yet to construct an explicit model which possesses a spin-
liquid ground-state which spontaneously breaks time-reversal and chiral symmetry, I think there is renewed reason for optimism that such models can be found.

[42] S.A.Kivelson, Phys. Rev. B\textbf{39}, 259-264 (1989).
[43] This is discussed in some generality by E.Demler, H.-Y.Kee, Y-B. Kim, C.Nayak and T.Senthil, manuscript in preparation.
[44] T. Senthil, and M.P.A.Fisher, Phys. Rev. Lett.\textbf{86}, 292-5, (2000) and Phys. Rev. B\textbf{60}, 1654-1667, (1999).
[45] L. Balents,M.P.A.Fisher, and C.Nayak, Phys. Rev. B\textbf{56}, 270 (1997).
[46] S.A.Kivelson, Phys. Rev. B\textbf{39}, 259-264 (1989).
[47] This is discussed in some generality by E.Demler, H.-Y.Kee, Y-B. Kim, C.Nayak and T.Senthil, manuscript in preparation.
[48] T. Senthil, and M.P.A.Fisher, Phys. Rev. Lett.\textbf{86}, 292-5, (2000) and Phys. Rev. B\textbf{63}, 134521 (2001).
[49] Related results were obtained some time ago by S.Sachdev, Phys. Rev. B\textbf{45}, 1237712396 (1992) and by G. Misguich, C. Lhuillier, B. Bernu, and C. Waldtmann, Phys. Rev. B\textbf{60}, 1064 (1999).
[50] D.S.Rokhsar and S.A.Kivelson, Phys. Rev. Lett.\textbf{61}, 2376 (1988).
[51] See, e.g. M. Vojta, Y. Zhang and S. Sachdev, Phys. Rev. Lett.\textbf{85}, 4940 (2000) Int. J. Mod. Phys. B 14, 3719 (1999).
[52] K.G. Wilson, Phys. Rev. D \textbf{7}, 2911-26 (1973).
[53] For a comprehensive review, see S.Sachdev, \textit{Quantum Phase Transitions} (Cambridge University Press, Cambridge, 1999).
[54] S. Chakravarty, B. I.Halperin, D. Nelson, Phys. Rev. B\textbf{39}, 2344 (1989).
[55] There is, however, an interesting speculation by B. A. Bernevig, D. Giuliano, and R. B. Laughlin, cond-mat/0004291, that such a description is possible.
[56] L. Golobovic and M. Golobovic, Phys. Rev. Lett.\textbf{80}, 4341 (1998); C.S.O’Hern and T.C.Lubensky, \textit{ibid}, \textbf{80}, 4345 (1998); C.S.O’Hern, J.Toner, and T.C.Lubensky, \textit{ibid}, \textbf{83}, 2745 (1999).
[57] For a recent discussion of the relevant theory and experiment, see L. Golubovic, T. C. Lubensky, C. S. O’Hern, cond-mat/0003128, and references therein.
[58] S. A. Kivelson, E. Fradkin and V. J. Emery, Nature \textbf{393}, 550 (1998).
[59] V.J.Emery, E.Fradkin, S.A.Kivelson, and T.C.Lubensky, Phys. Rev. Lett.\textbf{85}, 2160 (2000).
[60] A. Vishwanath, D. Carpentier, Phys. Rev. Lett.\textbf{86}, 676 (2001).
[61] R. Mukhopadhyay, C.L.Kane, and T.C.Lubensky, cond-mat/0102163.
[62] S. L. Sondhi, K. Yang, Phys. Rev. B\textbf{63}, 54430 (2001).
[63] There are some subtleties and caveats associated with this statement. For a discussion of the current state of the theory of this issue, see Ref. [61].
[64] J.W. Zhou et al, Science \textbf{286}, 269 (1999); T.Noda et al, Science \textbf{286}, 265 (1999).
[65] For a review, see R.Shankar, Rev. Mod. Phys. \textbf{66}, 129-92 (1994).
[66] E. W. Carlson, D. Orgad, S. A. Kivelson, and V. J. Emery, Phys. Rev. B\textbf{62}, 3422 (2000).
[67] D. Orgad, S. A. Kivelson, E. W. Carlson, V. J. Emery, X. J. Zhou, and Z. X. Shen, Phys. Rev. Lett.\textbf{86}, 4362 (2001)
[68] C. Lannert, M. P. A. Fisher, and T. Senthil, cond-mat/0101249.
[69] J. Zaanen, Z. Nussinov, cond-mat/0006193.
[70] See, for example, T. Valla et al, Science \textbf{285}, 2110 (1999); A. V. Fedorov, Phys. Rev. Lett. \textbf{82}, 2179 (1999); D. L. Feng \textit{et. al.,} Science \textbf{289}, 277 (2000); H. Ding \textit{et al,} cond-mat/0006143, X. J. Zhou \textit{et. al.,} Science \textbf{286}, 5438 (1999); X. J. Zhou \textit{et. al,} cond-mat/0009002 and Ref. [67].