Efficient solvability of Hamiltonians and limits on the power of some quantum computational models

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We consider quantum computational models defined via a Lie-algebraic theory. In these models, specified initial states are acted on by Lie-algebraic quantum gates and the expectation values of Lie algebra elements are measured at the end. We show that these models can be efficiently simulated on a classical computer in time polynomial in the dimension of the algebra, regardless of the dimension of the Hilbert space where the algebra acts. Similar results hold for the computation of the expectation value of operators implemented by a gate-sequence. We introduce a Lie-algebraic notion of generalized mean-field Hamiltonians and show that they are efficiently (exactly) solvable by means of a Jacobi-like diagonalization method. Our results generalize earlier ones on fermionic linear optics computation and provide insight into the source of the power of the conventional model of quantum computation.

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Quantum models of computation are widely believed to be more powerful than classical ones. Although this has been shown to be true in a few cases, it is still important to determine when a quantum algorithm for a given problem is more resource efficient than any classical one, or, conversely, when a classical algorithm is just as efficient as any quantum counterpart. In general, one needs to know whether it is worth investing in building a quantum computer (QC) and what is required for success. In this paper, we show close connections between these issues and the efficient (or exact) solvability of Hamiltonians. In particular, we show that a class of quantum models we call generalized mean-field Hamiltonians (GMFHs) is efficiently solvable and furthermore does not provide a stronger-than-classical model of computation: A quantum device engineered to have dynamical gates generated by Hamiltonians from such a set cannot directly simulate universal efficient quantum computation and can be efficiently simulated by a classical computer (CC).

An algorithm is a sequence of elementary instructions that solves instances of a problem. It is said to be efficient if the resources required to solve problem instances of size $N$ are polynomial in $N$ (poly($N$)) resources. Typically, the size of a problem instance is the number of bits required to represent it, and the relevant resources are time and space. In the last few years it has been shown that many pure-state quantum algorithms can be efficiently simulated on a CC when the extent of entanglement is limited (e.g., 2, 3) or when the quantum gates available are far from allowing us to build a set of universal gates 4, 5, 6. Here, we focus on a Lie algebraic analysis to obtain other situations where quantum algorithms can be efficiently simulated by CCs. The so-called generalized coherent states (GCSs) 7 play a decisive role in our analysis.

The algorithms considered here make use of the Lie-algebraic model of quantum computing (LQC). An LQC algorithm begins with the specification of a semisimple, compact $M$-dimensional real Lie algebra $\hat{\mathfrak{h}}$ of skew-Hermitian operators acting on a finite-dimensional Hilbert space $\mathcal{H}$, with Lie bracket $[\hat{X}, \hat{Y}] := \hat{X}\hat{Y} - \hat{Y}\hat{X}$. Without loss of generality, the action is irreducible. The algorithm begins with a maximum-weight state $|hw\rangle$ in $\mathcal{H}$ and applies gates expressed as exponentials $e^{\hat{X}}$ for certain $\hat{X}$ in $\hat{\mathfrak{h}}$. The output of the algorithm is a noisy expectation of an operator in $\mathfrak{h}$ or in $e^{\hat{h}}$. LQC algorithms cannot trivially be classically simulated because of the possibility that the dimension of $\mathcal{H}$ is exponential in the specification complexity of $\hat{\mathfrak{h}}$ and $|hw\rangle$. In order to precisely define the model of LQC we require some results from the theory of Lie algebras. See 8 for a textbook covering the basic theory of Lie algebras.

Our intention is to restrict observables and Hamiltonians to operators in $\sqrt{-\hbar}\mathcal{H}$. The dimension of $\mathcal{H}$ may be exponential in $M$. Since we wish to implement computations with resources that are polynomial in $M$, our knowledge of $\hat{\mathfrak{h}}$ cannot involve explicit matrix representations of its operators. We therefore assume that $\hat{\mathfrak{h}}$ is specified as an abstract Lie algebra $\mathfrak{h}$ together with a “maximum weight” $w$ characterizing its action on $\mathcal{H}$. For computational purposes, we also use a small-dimensional faithful representation of $\mathfrak{h}$. To be specific, we use the adjoint representation, but for efficiency, one can choose the first fundamental representation instead. We use the following notational conventions: Objects with a “hat” (’) belong to the representation of $\hat{\mathfrak{h}}$ on $\mathcal{H}$. Objects with an “overline” (¯) belong to the chosen faithful representation. Lie algebraic objects with neither a hat nor an overline are associated with the abstract Lie algebra (representation unspecified). Implicit in these conventions are the representational isomorphisms $\hat{\mathfrak{h}} \rightarrow \mathfrak{h}$ and $\mathfrak{h} \rightarrow \hat{\mathfrak{h}}$.

For the purpose of efficient representation, it is convenient to work with the complexification $\mathbb{C}\mathfrak{h}$ of $\mathfrak{h}$ and use a Cartan-Weyl (CW) basis (see, for example, 9) for $\mathbb{C}\mathfrak{h}$. Thus, we assume a decomposition $\mathbb{C}\mathfrak{h} = \mathfrak{h}_D \oplus \mathfrak{h}^* \oplus \bar{\mathfrak{h}}^*$, where $\mathfrak{h}_D$ is
a Cartan subalgebra (CSA), and $\mathfrak{h}^+$ and $\mathfrak{h}^-$ are algebras of generalized raising and lowering operators, respectively. $\mathfrak{h}_D$ is linearly spanned by named elements $h_1, \ldots, h_r$, and $\mathfrak{h}^\pm$ by $e_{\alpha_j}^\pm, e_{\alpha_j}^\mp$. The $\alpha_j$ are linear functionals on $\mathfrak{h}_D$ called the positive roots of $\mathfrak{h}_D$. The abstract Lie algebra is specified by the identities $[h_k, h_l] = 0$, $[h_k, e_{\alpha_j}^+] = \pm c_{kj} e_{\alpha_j}^+$, $[e_{\alpha_j}^+, e_{\alpha_l}^-] = \sum_k b_{kj} h_k$, and for $j \neq k$, $[e_{\alpha_j}^+, e_{\alpha_k}^-] = c_{jk} e_{\alpha_j}^+ e_{\alpha_k}^-$. The bases of $\mathfrak{h}_D$ and $\mathfrak{h}^\pm$ may be chosen so that the “structure constants” $a_{kj}$, $b_{kj}$, and $c_{jk}$ are ratios of integers with poly($M$) digits. The structure constants do not uniquely specify the action of $\mathfrak{h}$ on $\mathcal{H}$. According to the representation theory of semisimple complex Lie algebras, this action is uniquely specified by its “maximum weight”, which is a linear functional $w$ on $\mathfrak{h}_D$ given by its values $w(h_k)$ on the distinguished basis of $\mathfrak{h}_D$. The $w(h_k)$ are integral and are the eigenvalues of $\mathfrak{h}_D$. The unique state $|\psi\rangle$ annihilated by $\mathfrak{h}^+$ generates the family of GCS of $\mathcal{H}$ with respect to $\mathfrak{h}$. The Hermitian inner product of $\mathcal{H}$ and the Hermitian transpose operation on $\mathcal{H}$ induce a corresponding Hermitian transpose operation on $\mathcal{C}\mathfrak{h}$. We assume that the CW basis is chosen so that the Hermitian transpose is given by $\hat{h}_k^\dagger = \hat{h}_k$ and $(e_{\alpha_j}^+)^\dagger = e_{\alpha_j}^-$. We also assume that the linear space on which $\mathfrak{h}$ acts is endowed with a Hermitian inner product for which the representation $\hat{\mathfrak{h}}$ is skew-Hermitian and the Hermitian transpose matches the one defined for $\mathfrak{h}$.

The formal specification of an LQC algorithm requires the structure constants of an abstract CW decomposition of $\mathfrak{h}$ and the weight coefficients $w(h_k)$ determining $|\psi\rangle$. The specification complexities of $\mathfrak{h}$ and $|\psi\rangle$ are the number of bits required to represent the numerators and denominators of the structure constants and the $w(h_k)$. Thus they are polynomial in the dimension of $\mathfrak{h}$ and $\log \max(w(h_k))$. The gates of the algorithm may be unitary exponentials $e^{iX}$, with $X$ a member of the CW basis. The gate’s resource requirement is the number of bits required to represent $t$ plus $|t|$. More generally, we can allow as gates any $e^{iH}$ with $H \in \mathfrak{h}$, where the resource requirement is given by the specification complexity of $e^{iH}$ (defined below). There are several alternatives for how the algorithm’s output is obtained. We consider two. In the first, the output is obtained by measuring the expectation of an operator $A \in \mathcal{C}\mathfrak{h}$. In the second, it is obtained from the absolute value of the expectation of an operator $\hat{U} \in \mathcal{C}\mathfrak{h}$. The resource cost of making the measurement is proportional to the sum of number of bits of precision and the specification complexity of $\hat{A}$ or $\hat{U}$. The specification complexity of $\hat{A}$ is that of $\hat{A}$ (the corresponding operator in the abstract Lie algebra $\mathfrak{h}$) and is given by the number of bits used to represent the coefficients of $\hat{A}$ when expressed in the CW basis. If $\hat{U}$ is of the form $e^{iH}$ with $H \in \mathcal{C}\mathfrak{h}$, its specification complexity is that of $H$ plus $\max(|H_{\alpha_j}|)$ where $H_{\alpha_j}$ ranges over the coefficients of $H$ expressed in the CW basis. Our assumption about the resource cost of measurement (i.e., the number of bits $b$ of precision) makes the LQC model just defined very powerful but physically unreasonable. In particular, an LQC algorithm gives exponentially better precision than an algorithm of similar resource cost for the standard quantum computational model. The standard quantum algorithm would need to be repeated exponentially many times in $b$ to return an expectation value with $b$ bits of precision. In the standard quantum computational model, the hypothetical ability to determine expectation values with $b$ bits of precision using resources polynomial in $b$ implies the ability to efficiently solve problems in #P, the class of problems associated with the ability to count the number of solutions to NP-complete problems such as satisfiability. This is a consequence of more general results in [10].

A natural question is when and how LQC can efficiently simulate, or be simulated by, standard quantum or classical computation. The measurement models we introduced for LQC have the same form as many typical problems in physics, which involve the evaluation of correlation functions

$$
\langle W \rangle = \text{Tr}[\rho \hat{W}],
$$

where $\rho = \sum_{s=1}^L p_s |\phi_s\rangle \langle \phi_s|$. $|\phi_s\rangle$ is the density operator of the system ($p_s > 0$; $\sum_s p_s = 1$), $|\phi_s\rangle$ are pure states, and $\hat{W}$ is a Hermitian or unitary operator acting on $\mathcal{H}$. In general, the dimension $d$ of $\mathcal{H}$ increases exponentially in the problem size $N$, where the problem size is determined by quantities such as the volume or number of particles of the system. An algorithm to evaluate $\langle W \rangle$ with accuracy $\epsilon$ is efficient if the amount of resources required is bounded by polylog($d$) + poly($1/\epsilon$).

An efficient quantum algorithm to evaluate Eq. (1) exists if the state $\rho$ (or a good approximation to it) can be efficiently prepared on a QC and if $\hat{W}$ can be efficiently measured by using, for example, the indirect techniques described in Refs. [11, 12]. Unfortunately, known classical algorithms for this purpose typically require resources polynomial in the dimension $d$, which can be exponential in the problem size $N$. However, if the problem can be specified Lie algebraically, this classical complexity can be greatly reduced and exponential rather than polynomial accuracy is efficiently achievable.

**Theorem 1** With $\rho$ as defined following Eq. (1), if $|\phi_s\rangle = e^{iA_s}|\psi\rangle$ are GCS’s of $\hat{A}_s (\hat{A}_s \in \mathfrak{h})$ and $\hat{W} \in \mathcal{C}\mathfrak{h}$, then $\langle \hat{W} \rangle$ can be classically computed to accuracy $\epsilon$ in time polynomial in $\log(1/\epsilon)$ and the sum of the specification complexities of $\mathfrak{h}$, $|\psi\rangle$, $\hat{W}$, $A_s$, and $p_s$.

**Proof:** We have $\langle \hat{W} \rangle = \sum_{s=1}^L p_s \langle \hat{W} \rangle_{|\psi_s\rangle}$, where $\hat{W}_s = e^{-A_s} \hat{W} e^{A_s}$. In the CW basis,

$$
\hat{W}_s = \sum_{k=1}^r u_k^s \hat{h}_k + \sum_{j=1}^l v_j^s e_{\alpha_j}^+ + v_j^s e_{\alpha_j}^-.
$$

where $u_k^s, v_j^s \in \mathbb{C}$. To obtain these coefficients, we can compute $\hat{W}_s$ in the adjoint representation: $\hat{W}_s = e^{-A_s} \hat{W} e^{A_s} = \sum_{k=1}^r u_k^s \hat{h}_k + \sum_{j=1}^l v_j^s e_{\alpha_j}^+ + v_j^s e_{\alpha_j}^-$. To compute the $u_k^s$ and
$\psi_j^n$ to accuracy $\delta$ requires computing the matrix exponentials $e^{\pm A_n}$, and matrix multiplication followed by an expansion of the resulting matrix in terms of the CW basis. The matrix exponentials can be obtained to accuracy $\delta^r$ (in the 2-norm) in time polynomial in $\log(1/\delta^r)$ and the maximum of the entries of $A_n$ by direct series expansion or other, more efficient methods [13]. Matrix multiplication and basis expansion increase the 2-norm error by at most a constant factor, so that the $u_k^n$ and $v_k^n$ can be efficiently obtained to the desired accuracy.

Using the property that the $\hat{e}_\alpha^n$ either map $\{|hw\rangle\}$ to an orthogonal state or annihilate it, we rewrite Eq. (1) as

$$\langle \hat{W} \rangle = \sum_{s=1}^L p_s \sum_{k=1}^r u_k^s w(h_k)$$

and this sum can be evaluated efficiently with respect to the given specification complexities.

The following variant of Thm. 1 holds for $\hat{W} = e^{\hat{H}}$ with $\hat{H} \in \mathcal{C}h$.

**Theorem 2** If $|\phi_\alpha\rangle = e^{\hat{A}} |hw\rangle$ ($\hat{A}_n \in \mathfrak{h}$) are GCS’s of $\mathfrak{h}$ and $\hat{W} = e^{\hat{H}}$ with $\hat{H} \in \mathcal{C}h$, then $|\langle \hat{W} \rangle|^2$ can be classically computed to accuracy $\epsilon$ in time polynomial in $\log(1/\epsilon)$ and the sum of the specification complexities of $\mathfrak{h}$, $|hw\rangle$, $\hat{W}$, $\hat{A}_n$, and $p_s$.

**Proof:** We can expand $|\langle \hat{W} \rangle|^2$ as

$$|\langle \hat{W} \rangle|^2 = \sum_{s,s'} p_s p_{s'} \langle \phi_\alpha | \hat{W}^\dagger | \phi_{s'} \rangle \langle \phi_{s'} | \hat{W} \rangle$$

$$= \sum_{s,s'} p_s p_{s'} \text{tr} \hat{O}_{s,s'}$$

$$|hw\rangle \langle hw| e^{-\hat{A}^*} e^{\hat{H}^*} e^{\hat{A}} |hw\rangle \langle hw| e^{-\hat{A}} e^{\hat{H}} e^{\hat{A}^*} |hw\rangle \langle hw|$$

$\hat{O}_{s,s'}$ is proportional to $|hw\rangle \langle hw|$ and its trace is the constant of proportionality. We can express $\text{tr} |hw\rangle \langle hw|$ as a limit of operators in $\mathcal{C}h$. Let $L = \sum_{k=1}^r w(h_k) h_k$ and define $\omega$ by $L |hw\rangle = \omega |hw\rangle$. Then $\langle \psi | L | \psi \rangle < \omega$ for $\langle \psi \rangle \neq |hw\rangle$, from which it follows that $\text{tr} |hw\rangle \langle hw| \rightarrow \lim_{t \rightarrow \infty} e^{-t \omega} e^{t L}$ because the eigenvalues of $L$ are integral, convergence is exponentially fast in $t$. Let

$$E(t) = \sum_{s,s'} p_s p_{s'} e^{-3 \omega t} e^{t L} e^{-\hat{A}^*} e^{\hat{H}^*} e^{\hat{A}} e^{t L} e^{-\hat{A}} e^{\hat{H}} e^{\hat{A}^*} e^{t L}.$$

$E(t)$ is positive definite Hermitian and converges to $\hat{O}_{s,s'}$ as $t \rightarrow \infty$. For a given $t$, we can compute $\hat{E}(t)$ by computing exponentials and multiplying matrices in the adjoint representation. Observe that the maximum eigenvalue $\kappa(t)$ of $E(t)$ converges exponentially fast to $|\langle \hat{W} \rangle|^2$. To compute $\kappa(t)$ we first determine $\hat{Q}(t)$ such that $\hat{E}(t) = \hat{e}^{\hat{Q}(t)}$. With the assumed Hermitian inner product on the adjoint representation, $\hat{E}(t)$ is positive definite. Thus, there is a unique Hermitian $\hat{Q}(t)$ satisfying $\hat{E}(t) = \hat{e}^{\hat{Q}(t)}$, and $\hat{Q}(t)$ is necessarily in $\text{SU}(L)$. The operator $\hat{Q}(t)$ can be obtained via any conventional efficient diagonalization procedure for non-negative definite matrices.

We can then use an efficient Jacobi-like diagonalization procedure [15] to obtain unitary operators $\hat{U}(t) \in \mathcal{U}$ and $\hat{q}(t) \in \mathcal{H}_D$ such that $\hat{U}(t) \hat{q}(t) \hat{U}(t)^\dagger = \hat{Q}(t)$. The maximum eigenvalue of $\hat{E}(t)$ is given by the exponential of the maximum eigenvalue of $\hat{q}(t)$. At this point we require a number of results from the representation theory of Lie algebras. For example, see [8]. The element $\hat{q}(t)$ induces an alternative order on the roots, according to which a root $\alpha_j$ is positive if $\alpha_j(\hat{q}(t))$ is positive. (To remove degeneracies, it may be necessary to slightly perturb $\hat{q}(t)$.) For this ordering, we determine simple roots $\beta_k$ and corresponding members $h_k^n \in \mathcal{H}_D$ such that $h_k^n$ is isomorphic to $h_k$ via a member of the Weyl group. We can expand $\hat{q}(t) = \sum_j q_j h_j^n$. Uniqueness of maximal weights in representations of Lie algebras implies that the maximum eigenvalue of $\hat{q}(t)$ is given by $w(\hat{q}(t)) = \sum_j w(h_k^n) q_j$.

We claim that the necessary steps can be implemented with polynomial resources in the dimension of the Lie algebra and the number of digits of precision of $\kappa(t)$. The matrix and root manipulations can be implemented efficiently, but with respect to the precision of entries of the matrix. It is necessary to realize that unless the weight $w$ is sufficiently small, $\hat{E}(t)$ converges to 0 exponentially fast in $t$. However, because the $|w(h_k^n)|$ are polynomial in $M$, the rate of convergence to 0 is bounded by $e^{-\text{poly}(M)}$. To compute $\kappa(t)$ to a desired number $P$ of digits of precision, it suffices to compute in the low dimensional matrix representation with a precision of $\text{poly}(M) + \text{poly}(P)$ digits, which can be done with polynomial resources. The relevant Weyl group transformations can be done efficiently by use of one of the constructive proofs of the transitivity of the Weyl group. See, for example, the proof of Thm. 2.63 in [16].

Important special cases motivating these results are fermionic linear optics quantum computation (and equivalent matchgate models introduced by Valiant), which is efficiently classically simulatable [4, 13], and models that also include linear fermionic operators $(so(2N + 1))$ for which an extension of the canonical Bogoliubov mapping exists [19]. Natural bosonic analogues of the fermionic results also exist. As a result it is possible to efficiently simulate quantum computational models in which coherent states are acted on by linear optical circuits, and measured via homodyne detection [6], and of models with initial multimode squeezed states and squeezing gates as well as linear ones [21]. Like LQC with the second measurement strategy, these involve the efficient simulation, in the dimension of a Lie algebra, of a computational model in which coherent states of a Lie group with gates generated by the algebra constitute the initial states and computation. However, in the bosonic case the relevant algebra is not semisimple, and the relevant irreps are infinite-dimensional.

We can now address the important question of the classical simulatability of LQCs.

**Theorem 3** For both LQC measurement schemes, the result of an LQC algorithm $\hat{A}$ can be obtained by use of classical
computation in time polynomial in the specification complexity of $A$.

Proof: The action of the gates of the algorithm result in the state $|\phi\rangle = \prod_{m=1}^{t} e^{\hat{A}_m}|\text{hw}\rangle$ where $\hat{A}_m \in \mathfrak{h}$. Let

$$\langle \hat{W} \rangle = (\text{hw}) \prod_{m=1}^{t} e^{-\hat{A}_m} \hat{W} \prod_{m=1}^{t} e^{\hat{A}_m}|\text{hw}\rangle.$$ (5)

The result of the algorithm is $\langle \hat{W} \rangle$ if $\hat{W} \in \mathfrak{h}$, or $|\langle \hat{W} \rangle|$ if $\hat{W} \in e^{\mathfrak{h}}$. The result can be computed by generalizing the algorithms given in the proofs of Thms. 1 and 2. All that is required is to compute the full product $\prod_{m=1}^{t} e^{\hat{A}_m}$ instead of the single exponential required for Thms. 1 and 2. The complexity of the method is polynomial with respect to the specification complexity of $A$.

The complexity of our algorithm for computing the correlation function is bounded by $\log(1/\epsilon)$, and the number of terms that arise is bounded by polynomial in $q$. This is similar to the procedure of Wick's theorem. After this transformation, terms that retain some lowering or raising operators contribute nothing to the correlation functions. The remaining terms' contribution is easily computed from $\langle \text{hw}|\hat{h}_k|\text{hw}\rangle = w(h_k)$. The contribution to the complexity of the procedure of the formal multiplication and standardization procedure grows exponentially in $q$. The number of terms that arise is bounded by poly($M)^q$, so that for fixed $q$, the complexity remains polynomial in the given specification complexities. Further details are available in 22.

The algorithms given above can also be used to analyze certain interacting physical models. We use the term GMFH 1 for Hamiltonians belonging to $\sqrt{-\mathfrak{h}}$ for $\mathfrak{h}$ in a sequence of semisimple compact Lie algebras of dimension $M \leq \log(d)$ acting on $d$-dimensional Hilbert spaces. A GMFH is necessarily specified in terms of a basis of $\mathfrak{h}$ that can be efficiently transformed to a CW basis. An example of a GMFH is given by the $N$ spin-1/2 Ising model in a transverse magnetic field $H_I = \sum_{j=1}^{N} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^z)$, where $H_I$ is an element of the Lie algebra $so(2N)$, with dimension $M = 2N^2 - N \equiv \log(d)$, where $d = 2^N$. Interestingly, this model can be exactly solved and, as we will show, this result can be extended to any GMFH. We say that a Hamiltonian acting on a $d$ dimensional Hilbert space can be efficiently (exactly) solved when any one of its eigenvalues and a description of the corresponding eigenstate can be obtained and represented in exponential precision on a CC. In general, this definition makes sense when we focus on Hamiltonians describing the interactions of $N$-body systems, where $d$ increases exponentially with $N$.

Theorem 4 GMFHs can be efficiently solved.

Proof: Let $\hat{H}_{MF}$ be a GMFH in $\sqrt{-\mathfrak{h}}$ given in terms of a CW basis of $\mathfrak{h}$ as in Eq. 2. We show that to solve $\hat{H}_{MF}$ it suffices to diagonalize it according to

$$\hat{H}_D = \hat{U} \hat{H}_{MF} \hat{U}^\dagger = \sum_{k=1}^{r} \varepsilon_k \hat{h}_k,$$ (6)

with $\varepsilon_k \in \mathbb{R}$ and $\hat{U} \in e^{\mathfrak{h}}$ unitary. The eigenvalues of $\hat{H}_{MF}$ are shared with those of $\hat{H}_D$. A description of the corresponding eigenspaces consists of an eigenspace of $\hat{H}_D$ transformed by $\hat{U}^\dagger$, where $\hat{U}$ may be described by a sequence of LQC gates. According to the representation theory of Lie algebras, the eigenspaces of $\hat{H}_D$ consist of weight states of $\mathfrak{h}$, which can be obtained from the highest weight state by applying lowering operators. They are characterized by linear functionals $\lambda$ on $H_D$ of the form $\lambda(h_k) = w(h_k) - \sum_j n_j \alpha_j(h_k)$, where the $n_j$ are non-negative integers. Which choices of $n_j$ correspond to weight states is readily determined from the representation theory of Lie algebras. Once we have expanded $\hat{H}_D = \sum_k \varepsilon_k \hat{h}_k$, the eigenvalue corresponding to $\lambda$ is readily computed as $\lambda(\hat{H}_D) = \sum_k \varepsilon_k \lambda(h_k)$. 

The meaning of Thm. 3 can be expressed in terms of generalized entanglement. In a Lie algebraic framework, a GU state is a GCS of a semisimple compact Lie algebra. Thus, Thm. 3 states that if a quantum algebraic framework, a GU state is a GCS of a semisimple Lie algebra. Therefore, higher-order correlation functions can also be computed efficiently, provided the order is not too large.
To efficiently diagonalize $\hat{H}_{MF}$ and obtain a specification of $\hat{U}$, we compute in the adjoint representation and apply a generalization of the Jacobi method \cite{15} to $H_{3MF}$. It yields an exponentially converging diagonalization and an expression for $\hat{U}$ in terms of a sequence of exponentials of members of the $su(2)$ subalgebras of $\mathbb{C}h$ generated by the pairs $c^+_\alpha\gamma_j$. This suffices for our purposes.

**Example.** The fermionic Hamiltonians $\dot{\hat{H}}_{MF} = \sum_{i,j=1}^N \gamma_i^T \gamma_j - \delta_{ij}/2 + u_{ij} c^+_i c^+_j$, where the operator $c^+_i (c^+_j)$ creates (annihilates) a spinless fermion at the $i$th site, belong to a representation of the Lie algebra $so(2N)$ of dimension $M = 2N^2 - N \leq \text{polylog}(d)$. A faithful representation of $so(2N)$ is given by $c^+_ic^+_j \leftrightarrow T_{i,j} - T_{N+j,N+i}$, $c^+_ic^+_j \leftrightarrow T_{i,N+j} - T_{j,N+i}$, and $c^+_j c^+_j \leftrightarrow T_{N+i,j} - T_{N+j,i}$, where the $2N \times 2N$ matrices $T_{kk}$ have $+1$ in the $k$th row and $k$th column, and zeros otherwise. Theorem 5 is the result equivalent to the one given by the Bogoliubov transformation \cite{24}, where the Hamiltonian maps as $\dot{\hat{H}}_{MF} \rightarrow \hat{H}_D = \sum_{k=1}^n \epsilon_k (\gamma_k^T \gamma_k - 1/2)$, where the operator $\gamma_k^T \gamma_k$ creates (annihilates) a fermionic quasiparticle in the $k$th mode.

Although LQC algorithms and GMFHs can be efficiently simulated or solved on a CC, it may still be useful to implement the algorithms or simulate GMFHs with QCs. In particular, there may be problems where a key component is expressed in terms of LQC or GMFHs but a more complex quantum computation is required to determine the information of interest. One case of interest is where the LQC or GMFH component requires preparing a GCS. One way for such a GCS to arise is as the ground state of a GMFH. According to the next theorem, GCSs are efficiently preparable on a QC that has efficient access to the LQC initial state and gates.

**Theorem 5** Let a GCS $|\phi\rangle$ of $\dot{\hat{H}}$ be specified as the ground state of a Hamiltonian $\dot{\hat{H}} \in \sqrt{-\hat{H}}$. Then $|\phi\rangle$ can be prepared by use of resources polynomial in the specification complexity of $H$ on a QC with the ability to initialize $|hw\rangle$ and efficiently apply LQC gates.

**Proof:** It suffices to determine a $\hat{U} \in e^{\hat{H}}$ expressed as a polynomial product of LQC gates such that $\hat{H} = \hat{U} H_D U^\dagger$ with $H_D \in \hat{H}_D$ such that $D$ induces the root order associated with the CW basis. (See the proof of Thm. 1 for how an element of $\hat{H}_D$ induces a root order.) The state $|\phi\rangle$ is then obtained as $\hat{U}|hw\rangle$ and hence is efficiently preparable using LQC operations. To determine $\hat{U}$ we can first use the generalization of the Jacobi method as discussed previously. This yields an element of $\hat{H}_D$ that does not necessarily induce the desired root order. To complete the determination of $\hat{U}$ requires using a sequence of Weyl reflections to obtain the desired root order. The sequence may be obtained using the method mentioned at the end of the proof of Thm 1.

Our results provide analogues of the Gottesman-Knill theorem \cite{4} (cf. also \cite{23, 26}) concerning the efficient simulatability of Clifford-group computational models, and of results on the simulatability of certain multimode coherent-state and squeezed-state computational models \cite{7}. One might hope for a treatment, perhaps based on Lie groups and groups of Lie type, that will unify these results, specifically those based on (1) finite dimensional semisimple Lie algebras, (2) Bosonic linear optics with homodyne detection (tied to an infinite-dimensional irreducible representation of a solvable Lie algebra) and possibly squeezing (involving a nilpotent Lie algebra), and (3) Clifford groups and semigroups. Our results cast additional light on why quantum computers may be more powerful than classical computers. It is a crucial fact that the generators of its gate-set, though their number can be chosen to grow polynomially, generate an exponentially large Lie algebra acting on an exponentially large Hilbert space. If the growth of the dimension of the generated Lie algebra is polynomial, a computation with this gate set using compatible state preparations and measurements can be simulated with polynomial efficiency on a classical computer by working in a low-dimensional faithful representation of the Lie algebra. What other algebraically constrained models of quantum computation are efficiently classically simulatable? Such structures may underlie the efficient solvability of further classes of Hamiltonians of condensed matter models, which go beyond the GMFHs, such as those solvable via a Bethe-type Ansatz.

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