General Monogamy Relations for Multiqubit W-class States in terms of convex-roof extended negativity of assistance and squared Rényi-\(\alpha\) entanglement

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For multipartite entangled states, entanglement monogamy is an important property. We present some new analytical monogamy inequalities satisfied by the \(x\)-th power of the dual of convex-roof extended negativity (CREN), namely CREN of Assistance (CRENoA), with \(x \geq 2\) and \(x \leq 0\) for multiqubit generalized W-class states. We also provide the upper bound of the squared Rényi-\(\alpha\) entanglement (SR\(\alpha\)E) with \(\alpha\) in the region \([\sqrt{\frac{7}{12}} - 1)/2, (\sqrt{\frac{3}{3}} - 1)/2\] for multiqubit generalized W-class states.

I. INTRODUCTION

While classical correlation can be freely shared among parties in multi-party systems, quantum entanglement is restricted in its shareability. If a pair of parties are maximally entangled in multipartite systems, they cannot have any entanglement \([1, 2]\) nor classical correlations \([3]\) with the rest of the system. This restriction of entanglement shareability among multi-party systems is known as the monogamy of entanglement (MoE) \([4, 9]\).

The monogamy of entanglement (MoE) is one of the fundamental differences between quantum entanglement and classical correlations that a quantum system entangled with one of the other systems limits its entanglement with the remaining others. For example, MoE is a key ingredient to make quantum cryptography secure because it quantifies how much information an eavesdropper could potentially obtain about the secret key to be extracted \([10]\).

Coffman, Kundu, and Wootters established the first quantitative characterization of the MoE for the squared concurrence (SC) \([11, 15]\) in an arbitrary three-qubit quantum state. Another two well-known entanglement measures are convex-roof extended negativity (CREN) \([16]\) and Rényi-\(\alpha\) entanglement (RoE) \([17]\). CREN is a good alternative for MoE without any known example violating its property even in higher-dimensional systems and RoE is the generalization of entanglement of formation. Recently, the general monogamy relations for the \(x\)-th power of CREN has been shown to be a mixed state \(\rho_{A_1A_2...A_N}\) in a \(N\)-qubit system \([18]\).

\[\tilde{N}_x^{\alpha}(\rho_{A_1A_2...A_N}) \geq \sum_{i=1}^{N-1} \frac{\tilde{N}_x^{\alpha}(\rho_{A_iA_2...A_N})}{\tilde{N}_x^{\alpha}(\rho_{A_iA_2...A_N})},\] (1)

for \(x \geq 2\) and

\[\tilde{N}_x^{\alpha}(\rho_{A_1A_2...A_N}) < \sum_{i=1}^{N-1} \frac{\tilde{N}_x^{\alpha}(\rho_{A_iA_2...A_N})}{\tilde{N}_x^{\alpha}(\rho_{A_iA_2...A_N})},\] (2)

for \(x \leq 0\). Two years ago, Wei Song and Yan-Kui Bai showed the properties of the squared Rényi-\(\alpha\) entanglement (SR\(\alpha\)E) and proved that the lower bound of SR\(\alpha\)E in an arbitrary \(N\)-qubit mixed state \([19]\).

\[E_0^x(\rho_{A_1|A_2...A_N}) \geq E_0^x(\rho_{A_1A_2}) + ... + E_0^x(\rho_{A_1A_N}),\] (3)

where \(E_0^x(\rho_{A_1|A_2...A_N})\) quantifies the entanglement in the partition \(A_1|A_2...A_N\) and \(E_0^x(\rho_{A_1A_2})\) quantifies the one in two-qubit subsystem \(A_1A_2\) with the order \(\alpha \geq (\sqrt{7} - 1)/2\).

In this paper, we show the general monogamy relations for the \(x\)-th power of CRENoA of generalized multiqubit W-class states. This part provides a more efficient way for MoE. We also prove that the SR\(\alpha\)E with the order \(\alpha\) ranges in the region \([\sqrt{\frac{7}{12}} - 1)/2, (\sqrt{\frac{3}{3}} - 1)/2\] also obeys a general monogamy relation for arbitrary generalized multiqubit W-class states.

II. MONOGAMY OF CONCURRENCE AND CONVEX-ROOF EXTENDED NEGATIVITY

Given a bipartite pure state \(|\psi\rangle_{AB}\) in a \(d \times d\) \((d \leq d')\) quantum system, its concurrence, \(C(|\psi\rangle_{AB})\) is defined as \([20]\)

\[C(|\psi\rangle_{AB}) = \sqrt{2[1 - Tr(\rho^2_A)]},\] (4)

where \(\rho_A\) is reduced density matrix by tracing over the subsystem \(B\), \(\rho_A = Tr_B(|\psi\rangle_{AB}\langle\psi|)\) (and analogously for \(\rho_B\)). For any mixed state \(\rho_{AB}\), the concurrence is given by the minimum average concurrence taken over all decompositions of \(\rho_{AB}\), the so-called convex roof

\[C(\rho_{AB}) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle),\] (5)

where the convex roof is notoriously hard to evaluate and therefore it is difficult to determine whether or not an arbitrary state is entangled.

Similarly, the concurrence of assistance (CoA) of any mixed state \(\rho_{AB}\) is defined as \([21]\)

\[C_\alpha(\rho_{AB}) = \max_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle),\] (6)

where the maximum is taken over all possible pure state decompositions \(\{p_i, |\psi_i\rangle\}\) of \(\rho_{AB}\).

Another well-known quantification of bipartite entanglement is convex-roof extended negativity (CREN). For a bipartite mixed state \(\rho_{AB}\), CREN is defined as

\[\tilde{N}(\rho_{AB}) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i N(|\psi_i\rangle),\] (7)

where the minimum is taken over all possible pure state decompositions \(\{p_i, |\psi_i\rangle\}\) of \(\rho_{AB}\).
Similar to the duality between concurrence and CoA, we can also define a dual to CREN, namely CRENOA, by taking the maximum value of average negativity over all possible pure state decomposition, i.e.

$$\tilde{N}_a(\rho_{AB}) = \max_{\{p_i, |\psi_i\rangle\}} \sum_i p_i N(|\psi_i\rangle), \quad (8)$$

where the maximum is taken over all possible pure state decompositions \(\{p_i, |\psi_i\rangle\}\) of \(\rho_{AB}\).

In the following we study the monogamy property of the CRENOA for the \(n\)-qubit generalized W-class states \(|\psi\rangle \in H_A_1 \otimes H_A_2 \otimes ... \otimes H_{A_n}\) defined by

$$|\psi\rangle = a|000...\rangle + b_1|011...0\rangle + ... + b_n|000...1\rangle, \quad (9)$$

with \(|a|^2 + \sum_{i=1}^{n} |b_i|^2 = 1\).

**Lemma 1.** For \(n\)-qubit generalized W-class states \(|\psi\rangle\), we have

$$\tilde{N}(\rho_{A_1A_i}) = \tilde{N}_a(\rho_{A_1A_i}), \quad (10)$$

where \(\rho_{A_1A_i} = Tr_{A_2...A_{i-1}A_{i+1}...A_n}(|\psi\rangle\langle\psi|)\).

**Proof.** We assume \(\rho_{A_1A_i} = |x\rangle_{A_1A_i} \langle x| + |y\rangle_{A_1A_i} \langle y|\),

where

$$|x\rangle_{A_1A_i} = a|00\rangle_{A_1A_i} + b_1|10\rangle_{A_1A_i} + b_i|01\rangle_{A_1A_i},$$

$$|y\rangle_{A_1A_i} = \sqrt{\sum_{k \neq i} |b_k|^2} |00\rangle_{A_1A_i}. $$

From the HJW theorem in Ref. [22], for any pure-state decomposition of \(\rho_{A_1A_i} = \sum_{h=1}^{r} |\phi_h\rangle_{A_1A_i} \langle \phi_h|\), one has \(|\phi_h\rangle_{A_1A_i} = u_{h1}|x\rangle_{A_1A_i} + u_{h2}|y\rangle_{A_1A_i}\) for some \(r \times r\) unitary matrices \(u_{h1}\) and \(u_{h2}\) for each \(h\). Consider the normalized bipartite pure state \(|\tilde{\phi}_h\rangle_{A_1A_i} = |\phi_h\rangle_{A_1A_i}/\sqrt{p_h}\) with \(p_h = |\langle \phi_h|\phi_h\rangle|\). In Ref. [24], for any bipartite pure state \(|\psi\rangle\), one has

$$C(|\psi\rangle) = \mathcal{N}(|\psi\rangle).$$

and combining with the Lemma 1 in Ref. [23], for \(|\tilde{\phi}_h\rangle_{A_1A_i}\), we have

$$\mathcal{N}(|\tilde{\phi}_h\rangle_{A_1A_i}) = \frac{2}{p_h} |uh_1|^2 |b_1|^2 |b_i|.$$ 

Then combing (7) and (8), we can obtain

$$\tilde{N}(\rho_{A_1A_i}) = \min_{\{p_h, |\phi_h\rangle_{A_1A_i}\}} \sum_h p_h \mathcal{N}(|\tilde{\phi}_h\rangle_{A_1A_i}).$$

**Theorem 1.** For the \(n\)-qubit generalized W-class states \(|\psi\rangle\), the CRENOA satisfies

$$\tilde{N}_a(\rho_{A_1|A_j_1...A_{j_m-1}}) \geq \frac{x}{2x - 1} \sum_{i=1}^{m-1} \tilde{N}_a^{\pi}(\rho_{A_iA_{j_i}}), \quad (11)$$

where \(x \geq 2\) and \(\rho_{A_1A_2...A_{j_{m-1}}}\) is the \(m\)-qubit, \(2 \leq m \leq n\), reduced density matrix of \(|\psi\rangle\).

**Proof.** For the \(n\)-qubit generalized W-class state \(|\psi\rangle\), according to the definitions of \(\tilde{N}(\rho)\) and \(\tilde{N}_a(\rho)\), one has

$$\tilde{N}_a(\rho_{A_1|A_j_1...A_{j_m-1}}) \geq \tilde{N}(\rho_{A_1|A_j_1...A_{j_m-1}}).$$

When \(x \geq 2\), we have

$$\tilde{N}_a^{\pi}(\rho_{A_1|A_j_1...A_{j_m-1}}) \geq \left(\frac{\tilde{N}(\rho_{A_1|A_j_1...A_{j_m-1}}) + \tilde{N}(\rho_{A_1|A_j_1...A_{j_m-1}})}{2}\right)^x$$

$$= \frac{1}{2^x} \tilde{N}_a^{\pi}(\rho_{A_1|A_j_1...A_{j_m-1}}) \left(1 + \frac{\tilde{N}(\rho_{A_1|A_j_1...A_{j_m-1}})}{\tilde{N}(\rho_{A_1|A_j_1...A_{j_m-1}})}\right)^x \geq \frac{1}{2^x} \tilde{N}_a^{\pi}(\rho_{A_1|A_j_1...A_{j_m-1}}) \left(1 + \frac{\tilde{N}(\rho_{A_1|A_j_1...A_{j_m-1}})}{\tilde{N}_a^{\pi}(\rho_{A_1|A_j_1...A_{j_m-1}})}\right)^x$$

$$= \frac{1}{2^x} \tilde{N}_a^{\pi}(\rho_{A_1|A_j_1...A_{j_m-1}}) + \frac{x}{2^x} \tilde{N}_a^{\pi}(\rho_{A_1|A_j_1...A_{j_m-1}})$$

Here we have used in the first inequality the inequality \(a^x \geq (\frac{a+b}{2})^x\) for \(a \geq b > 0\) and \(x \geq 0\). The second inequality is due to \((1 + t)^x \geq 1 + xt^x\) for \(x \geq 1\) and \(1 \geq t \geq 0.\)

Then we have

$$\tilde{N}_a^{\pi}(\rho_{A_1|A_j_1...A_{j_m-1}}) \geq \frac{x}{2^x - 1} \tilde{N}_a^{\pi}(\rho_{A_1|A_j_1...A_{j_m-1}}).$$

Proof. Similar to the proof of Theorem 1, for $y \leq 0$, we get

$$\tilde{N}_n^y(a_{A_1|A_2...A_{j-1}}) \leq \frac{y}{2^y-1} \tilde{N}_n^y(a_{A_1|A_2...A_{j-1}})$$

Combining with Lemma 1, we have

$$\tilde{N}_n^y(a_{A_1|A_2...A_{j-1}}) \leq \frac{y}{2^y-1} \tilde{N}_n^y(a_{A_1|A_2...A_{j-1}})$$

The second inequality is due to the monogamy relation for the $x$-th power of CREN (2).

Theorem 2. For the $n$-qubit generalized W-class state $|\psi\rangle \in H_{A_1} \otimes H_{A_2} \otimes ... \otimes H_{A_n}$ with $\tilde{N}(a_{A_1|A_2...A_{j-1}}) \neq 0$ for $1 \leq i \leq m-1$, we have

$$\tilde{N}_n^y(a_{A_1|A_2...A_{j-1}}) < \frac{y}{2^y-1} \tilde{N}_n^y(a_{A_1|A_2...A_{j-1}}) \quad \text{(12)}$$

where $y \leq 0$ and $a_{A_1|A_2...A_{j-1}}$ is the $m$-qubit reduced density matrix as in Theorem 1.

Combining with Lemma 1, we have

$$\tilde{N}_n^y(a_{A_1|A_2...A_{j-1}}) \leq \frac{y}{2^y-1} \tilde{N}_n^y(a_{A_1|A_2...A_{j-1}})$$

where $y \leq 0$ and $a_{A_1|A_2...A_{j-1}}$ is the $m$-qubit reduced density matrix as in Theorem 1.

As an example, consider the 5-qubit generalized W-class state (9) with $a = b_2 = \frac{1}{\sqrt{10}}$, $b_1 = \frac{1}{\sqrt{15}}$, $b_3 = \sqrt{\frac{2}{15}}$, $b_4 = \sqrt{\frac{2}{15}}$. We have

The optimal lower bounds can be obtained by varying the parameter $x$, see Fig. 1.

Fig. 1: solid red line is the lower bound of $\tilde{N}_n(a_{A_1|A_2A_3})$ and solid blue line is the lower bound of $\tilde{N}_n(a_{A_1|A_2A_3})$ as functions of $x \geq 2$ from our result, red dashed line is the lower bound of $C_n(a_{A_1|A_2A_3})$ and blue dashed line is the lower bound of $C_n(a_{A_1|A_2A_3})$ as functions of $x \geq 2$ from [23].

III. MONOGAMY OF RÉNYI-$\alpha$ ENTANGLEMENT

Rényi-$\alpha$ entanglement (R$\alpha$E) is well-defined entanglement measure which is the generalization of entanglement of formation. For a bipartiure pure state $|\psi\rangle_{AB}$, the R$\alpha$E is defined as

$$E_\alpha(|\psi\rangle_{AB}) := S_\alpha(\rho_A) = \frac{1}{1-\alpha} \log_2(\text{tr} \rho_A^\alpha) \quad \text{(13)}$$

where the Rényi-$\alpha$ entropy is $S_\alpha(\rho_A) = [\log_2(\sum_i \lambda_i^\alpha)]/(1-\alpha)$ with $\alpha$ being a nonnegative real number and $\lambda_i$ being the eigenvalue of reduced density matrix $\rho_A$. For a bipartite mixed state $\rho_{AB}$, the R$\alpha$E is defined via the convex-roof extension

$$E_\alpha(\rho_{AB}) = \min \sum_i p_i E_\alpha(|\psi_i\rangle_{AB}) \quad \text{(14)}$$

where the minimum is taken over all possible pure state decompositions of $\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB} \langle \psi_i|$. In particular, for a two-qubit mixed state, the R$\alpha$E with $\alpha \geq 1$ has an analytical formula which is expressed as a function of the SC [24]

$$E_\alpha(\rho_{AB}) = f_\alpha \left[ C^2(\rho_{AB}) \right] \quad \text{(15)}$$

with $x \geq 2$. The optimal lower bounds can be obtained by varying the parameter $x$, see Fig. 1.

From Fig. 1, one gets that the optimal lower bounds of $\tilde{N}_n(a_{A_1|A_2A_3})$ and $\tilde{N}_n(a_{A_1|A_2A_3})$ are 0.203 and 0.385, respectively, attained at $x = 2$ while the lower bounds of each in terms of CoA are given by 0.249 and 0.471 [23]. One can see that choosing CRENNoA as a mathematical characterization of MoE is better than choosing CoA for $x \geq 2$. 

Recently, Wang et al further proved that the formula in (14) holds for the order $\alpha \geq (\sqrt{7} - 1)/2 \approx 0.823$ [25].

From Theorem 2 in Ref. [19], one has that for a bipartite $2 \otimes d$ mixed state $\rho_{AC}$, the Rényi-$\alpha$ entanglement has an analytical expression

$$E_\alpha(\rho_{AC}) = f_\alpha \left[ C^2(\rho_{AC}) \right] \quad \text{(17)}$$
where the order \( \alpha \) ranges in the region \([((7\sqrt{7} - 1)/2, (\sqrt{13} - 1)/2)]\).

**Theorem 3.** For the \( n \)-qubit generalized W-class states \( |\psi\rangle \in H_{A_1} \otimes H_{A_2} \otimes \cdots \otimes H_{A_n} \), we have

\[
E_\alpha(|\psi\rangle_{A_1A_2\ldots A_n}) \leq \sum_{i=2}^{n} E_\alpha(\rho_{A_1A_i}),
\]

(18)

where \( \rho_{A_1A_i}, 2 \leq i \leq n, \) is the 2-qubit reduced density matrix of \( |\psi\rangle \) and the order \( \alpha \) ranges in the region \([((7\sqrt{7} - 1)/2, (\sqrt{13} - 1)/2)]\).

**Proof.** For the \( n \)-qubit generalized W-class states \( |\psi\rangle \), we have

\[
E_\alpha(|\psi\rangle_{A_1A_2\ldots A_n}) = f_\alpha \left( C^2(|\psi\rangle_{A_1A_2\ldots A_n}) \right)
= f_\alpha \left( \sum_{i=2}^{n} C^2(\rho_{A_1A_i}) \right)
\leq \sum_{i=2}^{n} f_\alpha(C^2(\rho_{A_1A_i}))
= \sum_{i=2}^{n} E_\alpha(\rho_{A_1A_i}),
\]

where \( f_\alpha(x) = \frac{1}{1-\alpha} \log_2 \left[ \left( \frac{1}{2} - \frac{x}{2} \right)^\alpha + \left( \frac{1}{2} + \frac{x}{2} \right)^\alpha \right] \). We have used in the first and last equalities that the entanglement of formation obeys the relation (19). The second equality is due to the fact that \( C^2(|\psi\rangle_{A_1A_2\ldots A_n}) = \sum_{i=2}^{n} C^2(\rho_{A_1A_i}) \). The inequality is due to the fact that the Rényi-\( \alpha \) entanglement \( E_\alpha \left( C^2 \right) \) with \( \alpha \in \left[ \frac{7\sqrt{7} - 1}{2}, \frac{\sqrt{13} - 1}{2} \right] \) is monotonic increasing and concave as a function of the squared concurrence \( C^2[19] \).

Next we will present an upper bound of \( \text{SR}_\alpha \text{E} \). Before giving the result, we consider the following lemma.

**Lemma 2.** [19] Let \( \psi_{A_1\ldots A_n} \) be a generalized W-class state in (11). For any \( m \)-qubit subsystems \( A_1A_2\cdots A_{m-1} \) of \( A_1\cdots A_n \) with \( 2 \leq m \leq n - 1 \), the reduced density matrix \( \rho_{A_1A_2\cdots A_{m-1}} \) of \( \psi_{A_1\cdots A_n} \) is a mixture of a \( m \)-qubit generalized W-class state and vacuum.

In the following, we assume the \( m \)-qubit subsystems \( A_1A_2\cdots A_{m-1} \) of \( A_1\cdots A_n \) with \( 2 \leq m \leq n - 1 \) is exactly \( A_1\cdots A_m \). Then we can have the result below.

**Theorem 4.** For the \( n \)-qubit generalized W-class states \( |\psi\rangle \in H_{A_1} \otimes H_{A_2} \otimes \cdots \otimes H_{A_n} \), we have

\[
E_\alpha^2(\rho_{A_1A_2A_3\ldots A_n}) \leq (n - 1) \sum_{i=2}^{n} E_\alpha^2(\rho_{A_1A_i}),
\]

(19)

where \( E_\alpha^2(\rho_{A_1A_2A_3\ldots A_n}) \) quantifies the entanglement in the partition \( A_1|A_2\ldots A_n \) and \( E_\alpha^2(\rho_{A_1A_i}) \) quantifies the one in two-qubit subsystem \( A_1A_i \) with the order \( \alpha \in \left[ \frac{7\sqrt{7} - 1}{2}, \frac{\sqrt{13} - 1}{2} \right] \).

**Proof.** We first consider the monogamy relation in an \( n \)-qubit pure state \( \psi_{A_1A_2\cdots A_n} \). Thus we can obtain

\[
E_\alpha^2(|\psi\rangle_{A_1A_2\ldots A_n}) \leq \left( \sum_{i=2}^{n} E_\alpha(\rho_{A_1A_i}) \right)^2
\leq \left( \sum_{i=2}^{n} \rho_{A_1A_i} \right)^2 \sum_{i=2}^{n} E_\alpha^2(\rho_{A_1A_i})
= (n - 1) \left( \sum_{i=2}^{n} E_\alpha^2(\rho_{A_1A_i}) \right),
\]

where in the first inequality we have used Theorem 3 and \( a^2 \leq b^2 \) for \( 0 \leq a \leq b \), and in the second inequality we have used the Cauchy-Schwarz inequality.

Next from Lemma 2, we consider \( \rho_{A_1\cdots A_n} \) is a mixture of a \( n \)-qubit generalized W-class state and vacuum. Then since we have the pure decomposition of \( \rho_{A_1\cdots A_n} \),

\[
\rho_{A_1A_2\ldots A_n} = \sum_{j} p_j |\psi_j\rangle_{A_1A_2\ldots A_n} \langle \psi_j|,
\]

Thus, we can obtain

\[
E_\alpha^2(\rho_{A_1A_2\ldots A_n}) = \sum_{j} p_j E_\alpha(|\psi_j\rangle_{A_1A_2\ldots A_n})^2
\leq \left( \sum_{j} p_j \right)^2 \left( \sum_{i=2}^{n} E_\alpha(\rho_{A_1A_i}) \right)^2
= \left( \sum_{i=2}^{n} E_\alpha(\rho_{A_1A_i}) \right)^2
\leq \left( \sum_{i=2}^{n} \rho_{A_1A_i} \right)^2 \left( \sum_{j} p_j E_\alpha(\rho_{A_1A_i}) \right)^2
= (n - 1) \left( \sum_{i=2}^{n} E_\alpha^2(\rho_{A_1A_i}) \right),
\]

where in the first inequality we have used Theorem 3 and \( a^2 \leq b^2 \) for \( 0 \leq a \leq b \), and in the second inequality we have used the Cauchy-Schwarz inequality. The last equality is due to \( \sum_j p_j = 1 \).

As an example, we still consider the 5-qubit generalized W-class states (9) with \( a = b_2 = \frac{1}{\sqrt{10}}, b_1 = \frac{1}{\sqrt{5}}, b_3 = \sqrt{\frac{2}{5}} \), \( b_4 = \sqrt{\frac{3}{5}} \). We have

\[
E_\alpha^2(\rho_{A_1A_2A_3}) \leq 2 \left( E_\alpha^2(\rho_{A_1A_2}) + E_\alpha^2(\rho_{A_1A_3}) \right)
\]
and

\[
E_\alpha^2(\rho_{A_1A_2A_3A_4}) \leq 3 \left( E_\alpha^2(\rho_{A_1A_2}) + E_\alpha^2(\rho_{A_1A_3}) + E_\alpha^2(\rho_{A_1A_4}) \right)
\]

where

\[
E_\alpha(\rho_{A_1A_2}) = \frac{1}{1 - \alpha} \log_2 \left[ \left( \frac{1 - \frac{\sqrt{24}}{\sqrt{75}}}{2} \right)^\alpha + \left( 1 + \frac{\sqrt{24}}{\sqrt{75}} \right)^\alpha \right].
\]
\[ E_\alpha(\rho_{A_1|A_3}) = \frac{1}{1 - \alpha} \log_2 \left( \frac{1 - \sqrt{\frac{217}{225}}}{2} + \frac{1 + \sqrt{\frac{217}{225}}}{2} \right)^\alpha \]

and

\[ E_\alpha(\rho_{A_1|A_4}) = \frac{1}{1 - \alpha} \log_2 \left( \frac{1 - \sqrt{\frac{93}{74}}}{2} + \frac{1 + \sqrt{\frac{93}{74}}}{2} \right)^\alpha \]

with the order \( \alpha \in [(\sqrt{7} - 1)/2, (\sqrt{13} - 1)/2] \). See Fig. 2.

![Graph](image)

Fig. 2: solid red line is the upper bound of \( E_\alpha^2(\rho_{A_1|A_2,A_4}) \) and solid blue line is the upper bound of \( E_\alpha^2(\rho_{A_1|A_2,A_3,A_4}) \) as functions of \( \alpha \) when \( \alpha \) ranges in the region \( [(\sqrt{7} - 1)/2, 0.99] \). When \( \alpha \) ranges in the region \([1.001, (\sqrt{13} - 1)/2]\), as the red dashed line and the blue dashed line show, we do not have an upper bound of \( E_\alpha^2(\rho_{A_1|A_2,A_3}) \) and \( E_\alpha^2(\rho_{A_1|A_2,A_3,A_4}) \) in this example.

From Fig. 2, one gets that the optimal upper bounds of \( E_\alpha^2(\rho_{A_1|A_2,A_4}) \) and \( E_\alpha^2(\rho_{A_1|A_2,A_3,A_4}) \) are 0.02334 and 0.24211 attained at \( \alpha = 0.971 \) when \( \alpha \in [(\sqrt{7} - 1)/2, 0.99] \). This upper bounds can be easily generalized to arbitrary \( n \)-qubit generalized W-class states \( |\psi\rangle \in H_{A_1} \otimes H_{A_2} \otimes \ldots \otimes H_{A_n} \).

IV. CONCLUSIONS AND REMARKS

We have investigated the monogamy relations of multi-qubit generalized W-class states in terms of CRENoA and SRoE. We have proved that the monogamy inequality of \( x \)-th power for CRENoA when \( x \geq 2 \) and \( x \leq 0 \). Our result shows that choosing CRENoA as a mathematical characterization of the monogamy of entanglement is better than choosing CoA for \( x \geq 2 \). We also show the monogamy inequality for SRoE when \( \alpha \) ranges in the region \( [(\sqrt{7} - 1)/2, (\sqrt{13} - 1)/2] \). We can find the optimal upper bound for \( E_\alpha^2(\rho_{A_1|A_2,A_3,A_4}) \) when the order \( \alpha \in [(\sqrt{7} - 1)/2, (\sqrt{13} - 1)/2] \) by using our approach in Theorem 4. It is still an open problem to be answered that whether there exists the monogamy inequality for SRoE when \( \alpha \geq (\sqrt{13} - 1)/2 \) in generalized W-class states.

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