ESTIMATES OF GREEN AND MARTIN KERNELS
FOR SCHRÖDINGER OPERATORS WITH SINGULAR
POTENTIAL IN LIPSCHITZ DOMAINS

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Abstract. Consider operators of the form \( L^{\gamma V} := \Delta + \gamma V \) in a bounded Lipschitz domain \( \Omega \subset \mathbb{R}^N \). Assume that \( V \in C^1(\Omega) \) satisfies \( |V(x)| \leq \bar{a} \delta(x)^{-2} \) for every \( x \in \Omega \) and \( \gamma \) is a number in a range \((\gamma_-, \gamma_+)\) described in the introduction. The model case is \( V(x) = \text{dist}(x, F)^{-2} \) where \( F \) is a closed subset of \( \partial \Omega \) and \( \gamma < c_H(V) = \) Hardy constant for \( V \). We provide sharp two sided estimates of the Green and Martin kernel for \( L^{\gamma V} \) in \( \Omega \). In addition we establish a pointwise version of the 3G inequality.

1. Introduction

Let \( \Omega \) be a bounded Lip domain in \( \mathbb{R}^N, N \geq 3 \). We study the operator
\[ L^{\gamma V} := \Delta + \gamma V \]
where \( V \in C^1(\Omega) \) and \( \gamma \) is a constant. We assume that the potential \( V \) satisfies the conditions:
\[ (1.1) \quad \exists \bar{a} > 0 : \quad |V(x)| \leq \bar{a} \delta(x)^{-2} \quad \forall x \in \Omega \]
\[ \delta(x) = \delta_\Omega(x) := \text{dist}(x, \partial \Omega) \]
and
\[ (1.2) \quad \gamma_- < \gamma < \gamma_+, \]
where
\[ (1.3) \quad \gamma_+ = \sup \{ \gamma : \exists u_\gamma > 0 \text{ such that } L^{\gamma V} u_\gamma = 0 \}, \]
\[ \gamma_- = \inf \{ \gamma : \exists u_\gamma > 0 \text{ such that } L^{\gamma V} u_\gamma = 0 \}. \]
By a theorem of Allegretto and Piepenbrink [18] or [14, Theorem 2.3], (1.3) is equivalent to,

$$
\begin{align*}
\gamma_+ &= \sup \{ \gamma : \int_\Omega |\nabla \phi|^2 \, dx \geq \gamma \int_\Omega \phi^2 V \, dx \quad \forall \phi \in H_0^1(\Omega) \}, \\
\gamma_- &= \inf \{ \gamma : \int_\Omega |\nabla \phi|^2 \, dx \geq \gamma \int_\Omega \phi^2 V \, dx \quad \forall \phi \in H_0^1(\Omega) \}.
\end{align*}
$$

If \( V \) is positive, \( \gamma_+ \) is the Hardy constant relative to \( V \) in \( \Omega \), denoted by \( c_H(V) \). Condition (1.1) and Hardy’s inequality imply that \( \gamma_+ > 0 \) and \( \gamma_- < 0 \). Clearly, if \( V > 0 \) then \( \gamma_- = -\infty \) and if \( V < 0 \) then \( \gamma_+ = \infty \). Finally, if \( \gamma \in (\gamma_-, \gamma_+) \) then there exists a Green function for \( L^{\gamma V} \) in \( \Omega \), denoted by, \( G_{\Omega}^{\gamma V} \). The subscript will be dropped, except when several domains are considered.

Assumptions (1.1) – (1.2) imply that \( -L^{\gamma V} \) is positive and its first eigenvalue \( \lambda_{\gamma V} \) is positive. The corresponding normalized eigenfunction is denoted by \( \varphi_{\gamma V} \). (The normalization is \( \varphi_{\gamma V}(x_0) = 1 \) where \( x_0 \) is a fixed reference point in \( \Omega \).)

The following result is due to Pinchover [17]. It is proved by adapting an argument from [15, Theorem 3].

**Lemma 1.1.** Assuming (1.1) – (1.2), there exists \( \epsilon > 0 \) such that the operator \( -(L^{\gamma V} + \epsilon \delta(x)^{-2}) \) has a positive supersolution.

For the convenience of the reader, a proof is provided in the next section.

This fact implies that \( L^{\gamma V}, \gamma \in (\gamma_-, \gamma_+) \), is weakly coercive in the sense of [1]. Therefore one may apply to it potential theoretic results of Ancona [1] and [3]. In particular one may apply to this operator the Boundary Harnack Principle [1], that plays a crucial role in the present work.

**Notation.** Let \( f, g \) be non-negative functions in a domain \( D \). The notation \( f \sim g \) in \( D \) means that there exist two positive constants \( c_1, c_2 \) – called similarity constants – such that

$$
c_1 f \leq g \leq c_2 f \quad \text{in } D.
$$

The notation \( f \precsim g \) in \( D \) means that there exist a positive constant \( c \) such that

$$
f \leq cg \quad \text{in } D.
$$

**Lemma 1.2.** Assume (1.1) – (1.2). For any \( x_0 \in \Omega \) and \( \epsilon > 0 \),

$$
G^{\gamma V} (\cdot, x_0) \sim \varphi_{\gamma V} (\cdot) \quad \text{in } \{ x \in \Omega : |x - x_0| > \epsilon \}.
$$

Of course, the similarity constants depend on \( x_0 \) and \( \epsilon \).
Remark. The fact that (1.5) holds for every potential satisfying (1.1) and (1.2) was pointed out to me by Alano Ancona.

For the convenience of the reader, a proof is provided in the next section.

It is well known that, in every compact set $F \subset \Omega$,
\begin{equation}
(1.6) \quad c_1(F)|x-y|^{2-N} \leq G^\gamma_V(x,y) \leq c_2(F)|x-y|^{2-N}.
\end{equation}
Sharp, two sided estimates of the Green kernel of the Laplacian, \textit{up to the boundary}, in Lipschitz domains, were obtained by Bogdan [13]. In smooth domains such estimates have been obtained in [6], when $V = \delta^{-2}$ (= the classical Hardy potential) and $\gamma \in (0, c_H(V))$. These estimates can be extended to a large class of potentials using results on comparison of Green functions for related operators. In the case of \textit{small perturbations} of the potential results of this type were obtained by Murata [11], [12] and Pinchover [15], [16]. See [14] for a survey of these and related papers. These results are obtained without any explicit assumptions on the domain, which may also be unbounded. However the assumptions on the operators – including the existence of a positive minimal Green function – and the definition of a ‘small’ perturbation reflect implicitly on the domain.

The results of Ancona [1] imply the existence of the Green function for a large class of potentials in bounded Lipschitz domains and even more general cases (e.g. John domains).

Combining the results of [16] with those of [1] one obtains for instance the following:

Assume that $\Omega$ is a bounded Lipschitz domains and that $V \in C^\alpha(\Omega)$, $\alpha \in (0,1]$, and $\gamma$ satisfy conditions (1.1) and (1.2). Let $W := \gamma V + V_0$ where $V_0 \in C^\alpha(\Omega)$ and
\begin{equation}
|V_0| \leq c\delta^{\epsilon-2} \quad \text{in } \Omega
\end{equation}
for some positive numbers $c, \epsilon$. Then
\[ G^\gamma_V \sim G^W_\Omega. \]
In particular, letting $V = 0$, we conclude that, for $V_0$ as above, the Bogdan estimates hold for the Green kernel of the operator $-\Delta + V_0$ in bounded Lipschitz domains.

This is also a consequence of the results of Ancona [3] in which the author established the equivalence of the Green functions for a pair of operators $L^V_i$, $i = 1, 2$ under very general conditions on $V_1 - V_2$. In fact the results of [3] apply to more general Schrödinger operators, where $\Delta$ is replaced by a linear second order elliptic operators whose coefficients
may be singular on \( \partial \Omega \). In this case the conditions are imposed on the weighted difference of the Schrödinger operators.

Sharp estimates have also been obtained for Hardy potentials in conical domains, possibly unbounded, e.g. [5], [10].

In the case of smooth domains and potentials with singularities in \( \Omega \), two sided estimates of the Green function have been obtained under very general conditions, see [7], [8], [19] and references therein. These estimates are sharp with respect to \( \ln G \).

In the present paper we derive sharp, up to the boundary, two-sided estimates of the Green kernel of \( L^\gamma V \) in bounded Lipschitz domains. Following are the main results.

**Theorem 1.3.** Assume (1.1) – (1.2) and \( N \geq 3 \).

Then, for every \( b > 0 \) there exists a constant \( C(b) \), depending also on \( N, r_0, \kappa, \bar{a} \), such that: if \( x, y \in \Omega \) and

\[
|x - y| \leq \frac{1}{b} \min(\delta(x), \delta(y))
\]

then

\[
\frac{1}{C(b)} |x - y|^{2-N} \leq G^\gamma_{\Omega}(x, y) \leq C(b) |x - y|^{2-N}.
\]

In the next theorems, \( C \) stands for a constant depending only on \( r_0, \kappa, \bar{a} \) and \( N \).

**Theorem 1.4.** Assume (1.1) – (1.2) and \( N \geq 3 \).

If \( x, y \in \Omega \) and

\[
\max(\delta(x), \delta(y)) \leq \frac{r_0}{10 \kappa}
\]

\[
\min(\delta(x), \delta(y)) \leq \frac{|x - y|}{16(1 + \kappa)^2}
\]

then

\[
\frac{1}{C} |x - y|^{2-N} \frac{\varphi^\gamma V(x) \varphi^\gamma V(y)}{\varphi^\gamma V(x_y) \varphi^\gamma V(y_x)} \leq G^\gamma_{\Omega}(x, y)
\]

\[
\leq C |x - y|^{2-N} \frac{\varphi^\gamma V(x) \varphi^\gamma V(y)}{\varphi^\gamma V(x_y) \varphi^\gamma V(y_x)}.
\]

The points \( x_y, y_x \) depend on the pair \((x, y)\). If

\[
\hat{r}(x, y) := |x - y| \vee \delta(x) \vee \delta(y) \leq r_0/10 \kappa
\]
they can be chosen arbitrarily in the set

\[(1.12)\]

\[A(x, y) := \{ z \in \Omega : \frac{1}{2} \, \delta(x) \leq \delta(z) \leq 2 \, \delta(x) \} \cap B_{4 \delta(x, y)}(x) \]

Otherwise set \(x_y = y_x = x_0\) where \(x_0\) is a fixed reference point.

Remark. There exists a constant \(C\) such that for any two points \(x, y \in \Omega\) and any \(P, Q \in A(x, y)\),

\[
\frac{1}{C} \phi \gamma V(P) \leq \phi \gamma V(Q) \leq C \phi \gamma V(P).
\]

This is a consequence of the strong Harnack inequality (see Lemma 3.2 below) and the fact that, under condition \((1.10)\), \(\delta(x, y) \sim |x - y|\).

The same observation is valid if \(A(x, y)\) is replaced by

\[(1.13)\]

\[A_b(x, y) := \{ z \in \Omega : \frac{1}{b} \, \delta(x, y) \leq \delta(z) \leq b \, \delta(x, y) \} \cap B_{4 \delta(x, y)}(x)
\]

where \(b\) is a number in \((1, \frac{r_0}{10 \kappa})\) and \(C\) is a constant depending on \(b\). Consequently Theorem 1.4 remains valid if \(A(x, y)\) is replaced by \(A_b(x, y)\) and \(C\) by \(C_b\), i.e. a constant depending on \(b\).

Let \(K_\Omega \gamma V\) denote the Martin kernel of \(L \gamma V\) in \(\Omega\). As a consequence of the previous result we obtain

**Theorem 1.5.** Assume \((1.1) - (1.2)\) and \(N \geq 3\).

If \(x \in \Omega\), \(y \in \partial \Omega\) and \(|x - y| < \frac{r_0}{10 \kappa}\) then

\[
(1.14) \quad \frac{\phi \gamma V(x)}{C \phi \gamma V(x_y)^2} |x - y|^{2-N} \leq K_\Omega \gamma V(x, y) \leq C \frac{\phi \gamma V(x)}{\phi \gamma V(x_y)^2} |x - y|^{2-N},
\]

where \(x_y\) is an arbitrary point in \(A(x, y)\).

**Definition 1.6.** Let \(\zeta \in \partial \Omega\). A unit vector \(\nu\) in \(\mathbb{R}^N\) is an inner pseudo normal at \(\zeta\) if

\[
a_\nu(\zeta) := \limsup_{x \in \partial \Omega; \ x \to \zeta} \frac{\langle x - \zeta, \nu \rangle}{|x - \zeta|} < 1.
\]

Let \(\lambda \in (0, 1)\). The vector \(\nu\) is a \(\lambda\) - inner normal at \(\zeta\) if \(a_\nu(\zeta) < \lambda\).

Another consequence of the previous estimates is the following version of the 3G inequality.

**Theorem 1.7.** Assume \((1.1) - (1.2)\) and \(N \geq 3\). In addition assume that there exist numbers \(A \in (0, 1)\) and \(b_1 > 1\) such that, if \(x, y \in \Omega\)
satisfy (1.9) and \( x, y \) lie on a \( \lambda \)–inner normal at \( \zeta \) for a point \( \zeta \in \partial \Omega \) and some \( \lambda < \Lambda \) then

\[
(1.15) \quad b_1 \delta(y) \leq \delta(x) \implies \varphi_{\gamma V}(y) \leq C(b_1) \varphi_{\gamma V}(x).
\]

Under these assumptions, if \( x, y, z \) are three distinct points in \( \Omega \) then,

\[
(1.16) \quad \frac{G_{\gamma V}(x, y)G_{\gamma V}(y, z)}{G_{\gamma V}(x, z)} \leq C'(b_1) \left( |x - y|^{2-N} + |y - z|^{2-N} \right).
\]

Remark. The 3G inequality, in various forms, has been studied in numerous papers. In [3] it was established with respect to a larger class of potentials – in particular, without assuming (1.15) – but with certain restrictions on the configuration of the three points.

2. Notations and preliminaries

We start with the proof of two auxiliary lemmas stated in the introduction.

Proof of Lemma [1.1] Let \( V_1, V_2 \in C^2(\Omega) \) and assume that \( V_i \) satisfies (1.1) and that \( \gamma_-(V_i) < 1 < \gamma_+(V_i), \ i = 1, 2 \). The latter assumption implies the existence of a positive eigenfunction, say \( u_i \), of \(-L_{V_i}\), with positive eigenvalue. Let \( a \in (0, 1) \) and put,

\[
u := u_1^a u_2^{1-a}, \quad W := a V_1 + (1 - a) V_2.
\]

A straight forward computation yields:

\[
(2.1) \quad -L^W(u) = -\Delta u - Wu = f_1 + f_2
\]

where

\[
f_1 = -\frac{au}{u_1} L_{V_1} u_1 - \frac{(1-a)u}{u_2} L_{V_2} u_2 \geq 0
\]

\[
f_2 = a(1-a)u \left[ \frac{\nabla u_1}{u_1} - \frac{\nabla u_2}{u_2} \right]^2 \geq 0.
\]

Thus \( u \) is a positive \( L^W \) superharmonic function.

Now we apply this result to the following case:

\( V_1 = \gamma V \) where \( V \) and \( \gamma \) satisfy (1.1) and (1.2),

\( V_2 = \frac{c_H}{2} \delta^{-2} \) where \( c_H \) is the classical Hardy constant in \( \Omega \).
It follows that for every $a \in (0, 1)$, the function $v = \varphi_{\lambda V}^a \varphi_1^{1-a} L^W$ superharmonic in $\Omega$, i.e.,

$$-(\Delta + a \gamma V)v \geq \epsilon \delta^{-2}, \quad \epsilon := \frac{(1-a)c_H}{2}.$$ 

As this result is valid for any $\gamma \in (\gamma_-, \gamma_+)$ and any $a \in (0, 1)$, Lemma 1.1 holds.

Proof of Lemma 1.2. Put $V_1 = \gamma V$ and $V_2 = \gamma V + \lambda \gamma V$. Then $\varphi_{\gamma V}$ is a minimal positive ground state solution of $L_{V_2}$. Evidently $W := V_2 - V_1$ is a small perturbation of $V_1$. Therefore (1.5) is a consequence of Theorem 3.1 and Lemma 3.6 of [15].

Next we introduce some notations that will be used throughout the paper. Given $r, \rho$ positive denote

$$T^0(r, \rho) = \{ \xi = (\xi_1, \xi') \in \mathbb{R} \times \mathbb{R}^{N-1} : |\xi'| < r, |\xi_1| < \rho \}.$$ 

If $\xi = \xi^P$ is centered at $P$ we denote by $T^P_{\xi^P}(r, \rho)$ the cylinder $T^0(r, \rho)$ in this set of coordinates. However, as a rule we shall drop the subscript $\xi^P$.

Since $\Omega$ is a bounded Lipschitz domain there exists $\kappa \geq 1$ and $r_0 > 0$ such that, for every $P \in \partial \Omega$, there exists an Euclidean set of coordinates $\xi = \xi^P$, centered at $P$, and a $\kappa$-Lipschitz function $f^P : \mathbb{R}^{N-1} \mapsto \mathbb{R}$ such that $f^P(0) = 0$ and

$$(2.2) \quad T^P(r_0, 10\kappa r_0) \cap \Omega = \{(\xi_1, \xi') : |\xi'| < r_0, f^P(\xi') < \xi_1 < 10\kappa r_0 \}$$

in the set of coordinates $\xi^P$. Any set of coordinates centered at $P$ such that (2.2) holds is called an admissible set of coordinates at $P$ and $T^P(r, \rho)$, $r \in (0, r_0)$ and $\rho \in (0, 10\kappa r_0]$ is called a standard cylinder at $P$. The couple $(\kappa, r_0)$ is called the Lipschitz characteristic of $\Omega$. It is not unique, but will be kept fixed throughout the paper.

For $r \leq r_0, \rho \leq 10\kappa r_0$ we denote

$$(2.3) \quad \omega^P(r, \rho) := T^P(r, \rho) \cap \Omega$$

where $T^P(r, \rho)$ is a standard cylinder at $P$. If $\xi \in \omega^P(r_0, 10\kappa r_0)$, $\xi_1 > 0$ and $|\xi'|/\xi_1 < \kappa/2$ we say that the unit vector in the direction $\overrightarrow{P\xi}$ is an approximate normal at $P$. This vector is denoted by $n(P, \xi)$.

The boundary Harnack principle (briefly BHP) due to [1] plays a crucial role in the paper. For easy reference, we state it below.
Theorem 2.1. Let $P \in \partial \Omega$ and let $T^P(r, \rho)$ be a standard cylinder at $P$. There exists a constant $c$ depending only on $N, \bar{a}$ and $\rho$ such that whenever $u$ is a positive $L^N$ harmonic function in $\omega^P(r, \rho)$ that vanishes continuously on $\partial \Omega \cap T^P(r, \rho)$ then

\[ c^{-1} r^{N-2} G^V(\omega, A') \leq \frac{u(x)}{u(A)} \leq cr^{N-2} G^V(\omega, A'), \quad \forall x \in \Omega \cap T^P(r, \rho/2) \]

where $A = (\rho/2)(1,0,...,0)$, $A' = (2\rho/3)(1,0,...,0)$ in the corresponding set of local coordinates $\xi^p$.

In particular, for any pair $u,v$ of positive $L^N$ harmonic functions in $\omega^P(r, \rho)$ that vanish on $\partial \Omega \cap T^P(r, \rho)$:

\[ u(x)/v(x) \leq Cu(A)/v(A), \quad \forall x \in \Omega \cap T^P(r/2, \rho/2) \]

where $C = c^2$.

Remark. (i) Inequality (2.4) implies that (in the notation of the theorem)

\[ \frac{1}{c} r^{2-N} \leq G^V(\omega, A') \leq cr^{2-N}. \]

(ii) Inequality (2.4) remains valid for any $A, A'$ such that $A = (a_1 \rho, 0,...,0)$, $A' = (a_2 \rho, 0,...,0)$ and $0 < a_1 < a_2 < 1$. In this case, the constant $c$ depends also on $a_1/a_2$.

3. PROOF OF THEOREM 1.3

The proof is based on several lemmas in which we assume, without further mention, that conditions (1.1) – (1.2) are satisfied.

Notation (i) Put

\[ \Gamma_y(x) := a_N |x - y|^{2-N} \]

where $a_N$ is the constant such that $-\Delta \Gamma_y = \delta_y$.

(ii) Denote by $G^V_y$ the function $x \mapsto G^V(x, y)$.

(iii) For every $b > 1$ and $z \in \Omega$, put

\[ B^+_b = \{ x \in \Omega : |x - z| < \frac{\delta(z)}{b} \}, \]

\[ S^+_b = \{ x \in \Omega : |x - z| = \frac{\delta(z)}{b} \}. \]
Lemma 3.1. Let $z \in \Omega$, $\delta(z) < r_0/4$. For every $b > 1$ there exists a constant $c^* = c^*(b) > 0$, independent of $z$, such that

$$\frac{1}{c^*} \delta(z)^{2-N} \leq G^{\gamma V}_\Omega(x, z) \leq c^* \delta(z)^{2-N} \quad \forall x \in S_b^z$$

(3.3)

Proof. First we prove,

Assertion 1. For every $z$ as above, there exists a constant $c_1$ depending on $b$ but independent of $z$ and a point $\zeta \in S_b^z$ such that

$$\frac{1}{c_1} \delta(z)^{2-N} \leq G^{\gamma V}_\Omega(\zeta, z) \leq c_1 \delta(z)^{2-N}.$$  

(3.4)

Let $Q \in \partial \Omega$ be a point such that $|Q - z| = \delta(z)$. Let $T^Q(r_0, 10\kappa r_0)$ be a standard cylinder at $Q$ associated with a local set of coordinates $\xi^Q$.

Let $P$ be the point on $\partial \Omega$ such that $(\xi^Q)'(P) = (\xi^Q)'(z)$. Then $\xi^P := \xi^Q - \xi^Q(P)$ is a local set of coordinates at $P$ and $T^P(r_0/2, 5r_0\kappa)$ is a standard cylinder at $P$ relative to $\xi^P$. (Recall that $\delta(z) < r_0/4$.)

Let $\zeta$ be the point of intersection of the segment $[P, z]$ with the sphere $S_b^z$. We apply Theorem BHP in $T^P(2\delta(z), 20\kappa \delta(z))$ when $A' = z$ and $A = \zeta$. This is possible because

$$\delta(z) \leq |Pz| < |PQ| + |Qz| \leq (\kappa + 1)\delta(z),$$

$$\frac{b-1}{b} \delta(z) \leq |P\zeta| = |Pz| - \delta(z)/b \leq (\kappa + 1 - \frac{1}{b})\delta(z)$$

and consequently,

$$1 - \frac{1}{b} \leq \frac{|P\zeta|}{|Pz|} \leq 1 - \frac{1}{b(\kappa + 1)}.$$  

Thus (3.4) - with a constant $c_1$ depending on $b$ but independent of $z$ - is a consequence of (2.6) and the remark following it.

The sphere $S_b^z$ can be covered by $c'(N)$ balls of radius $r' = \delta(z)/4b$ centered on the sphere. If $x \in S_b^z$ then $\delta(x) \geq \frac{b-1}{b} \delta(z) \geq 4(b - 1)r'$.

Therefore by the classical Harnack inequality, there exists a constant $C'(N)$ (independent of $z$) such that

$$\sup_{x \in S_b^z} (G^{\gamma V}_\Omega)(x, z) \leq C'(N) \inf_{x \in S_b^z} (G^{\gamma V}_\Omega)(x, z).$$

(3.5)

This inequality and Assertion 1 imply (3.3).

$\square$
Lemma 3.2. Let $F \in C(\Omega)$ be a positive function satisfying the strong Harnack inequality. Let $b, b_0$ be two numbers such that $0 < b < b_0$, let $x, y \in \Omega$ and put $r = |x - y|$. Suppose that

$$r < \frac{r_0}{10\kappa b_0}, \quad br \leq \min(\delta(x), \delta(y)) \leq \max(\delta(x), \delta(y)) \leq (b_0 + 1)r,$$

where $(\kappa, r_0)$ is the Lipschitz characteristic of $\Omega$ (see Section 2). Then there exists a constant $c^*$, independent of $x, y$ (but depending on $N, \kappa, r_0, b, b_0$ and the Harnack constants for $F$) such that

$$\frac{1}{c^*} F(x) \leq F(y) \leq c^* F(x).$$

Proof. Let $X \in \partial \Omega$ be a point such that $|x - X| = \delta(x)$. Let $\xi^X$ be an admissible set of local coordinates at $X$ associated with the cylinder $T^X(r_0, 10\kappa r_0)$ (see (2.2)). Put

$$r_1 = |\xi^X_1(x - y)|, \quad r' = |(\xi^X)'(x - y)|.$$

Let $Y \in \partial \Omega$ be the point such that $(\xi^X)'(Y) = (\xi^X)'(y)$ and let $\xi^Y$ be the set of coordinates centered at $Y$ given by

$$\xi^Y = \xi^X - \xi^X(Y).$$

Denote

$$\tilde{\delta}(z) := \xi^X_1(z) - f^X((\xi^X)'(z)) \quad \forall z \in T^X(r_0, 10\kappa r_0) \cap \Omega.$$

Note that $\tilde{\delta}(z)$ is simply the distance between $z$ and $\partial \Omega$ measured along the line through $z$ parallel to the $\xi^X_1$ axis. The definition of standard cylinder (see (2.2)) implies that $\tilde{\delta}(z) > 0$ for every $z$ as above. In addition,

$$\frac{\tilde{\delta}(z)}{\sqrt{1 + \kappa^2}} \leq \delta(z) \leq \tilde{\delta}(z).$$

The right inequality is trivial and the left follows from the Lipschitz property of $\partial \Omega$ and the previous remarks on $\tilde{\delta}$.

Put $d = \delta(x)$ and

$$J = (\partial \Omega \cap T^X(r_0/2, 10\kappa r_0)) + (d, 0, \ldots, 0)$$

$$= \{ z : \xi^X_1(z) = f^X((\xi^X)'(z)) + d, \quad |(\xi^X)'(z)| \leq r_0 / 2 \}.$$

Note that $x \in J$ (but $y$ need not be in $J$) and

$$\tilde{\delta}(z) = d \quad \forall z \in J.$$
Denote by $y^*$ the $\xi_1^X$-projection of $y$ on $J$:

$$\xi_1^X(y^*) = f^X((\xi^X)'(y)) + d, \quad (\xi^X)'(y^*) = (\xi^X)'(y).$$

If $[x, y]$ is parallel to the $\xi_1$ axis, it is easy to see that (3.7) holds. Therefore we may assume that $x \neq y^*$.

Let $\Pi$ denote the plane containing $x, y$ that is parallel to the $\xi_1^X$ axis. Then $y^* \in \Pi$ and we denote by $I_J(x, y^*)$ the closed section of the curve $\Pi \cap J$ with end points $x, y^*$. Let $I(x, y)$ be the curve connectiong $x, y$ given by,

$$I(x, y) = I_J(x, y^*) \cup [y^*, y].$$

If $y \neq y^*$ and $\theta$ is the angle between $I_J(x, y^*)$ and $[y^*, y]$ then $|\cot \theta| \leq \frac{1}{k}$. Therefore $I(x, y)$ is a Lipschitz curve.

By definition, $\delta(x) \leq d = \tilde{\delta}(x)$. Therefore, by (3.6), and (3.8),

$$br \leq d \leq \delta(x)\sqrt{1 + \kappa^2} \leq (b_0 + 1)r\sqrt{1 + \kappa^2}.$$

By (3.8) and (3.10),

$$\frac{d}{\sqrt{1 + \kappa^2}} \leq \delta(z) \leq d \quad \forall z \in J.$$

Hence,

$$\delta(J) := \min_J \delta(z) \geq \frac{br}{\sqrt{1 + \kappa^2}}.$$

For every $z \in [y^*, y]$, $(\xi^X)'(z) = (\xi^X)'(Y)$. Therefore $\tilde{\delta}(z)$ lies between $\delta(y)$ and $\delta(y^*)$ for every $z \in [y^*, y]$. Since $\delta(y) \geq \delta(y)$ and, by (3.10), $\delta(y^*) = d$ it follows that

$$\min_{[y^*, y]} \tilde{\delta}(z) \geq \min(d, \delta(y)).$$

Hence, by (3.6), (3.8) and (3.12),

$$\min_{[y^*, y]} \delta(z) \geq \frac{br}{\sqrt{1 + \kappa^2}}.$$

The curve $I_J(x, y^*)$ is given by,

$$\{\xi^X(t) : \xi_1^X(t) = f^X((\xi^X)'(t)) + d, \quad (\xi^X)'(t) = (1 - t)(\xi^X)'(x) + t(\xi^X)'(y), \quad t \in (0, 1)\}$$

Therefore

$$|I_J(x, y^*)| \leq \int_0^1 |\nabla f^X \frac{d(\xi^X)'}{dt}| dt \leq \kappa |(\xi^X)'(x - y)| = \kappa r'.$$
Furthermore, by (3.6), (3.8) and (3.12)

$$|y - y^*| = |	ilde{\delta}(y) - \tilde{\delta}(y^*)| = |\tilde{\delta}(y) - d| \leq \max(\tilde{\delta}(y), d)$$

$$\leq \max(\tilde{\delta}(y)\sqrt{1 + \kappa^2}, d) \leq (b_0 + 1)\sqrt{1 + \kappa^2} r.$$ 

Thus the curve $I(x, y)$ has total length no larger than $C_{\hat{b}_0}r$ where $C_{\hat{b}_0} = \kappa + (b_0 + 1)\sqrt{1 + \kappa^2}$.

Let $P_0, \ldots, P_m$ be distinct points on $I(x, y), P_0 = x, P_m = y$ and let $D_i$ be the open ball of radius $s := br/4\sqrt{1 + \kappa^2}$ centered at $P_i, i = 0, \ldots, m$. We assume that the points $P_i$ are so distributed that $D_i \cap D_{i+1} \neq \emptyset, D_i \cap D_j = \emptyset$ if $|i - j| > 1$.

By (3.14) and (3.15), $\delta(P_i) \geq 2s$. Since the total length of $I_J(x, y^*) \cup [y^*, y]$ is not larger than $C_{\hat{b}_0}r$, the number of points $m + 1$ needed in order to achieve such a configuration depends only on $b, b_0$ and $\kappa$. Therefore, as $F$ satisfies the strong Harnack inequality, (3.7) follows.

**Lemma 3.3.** Let $N \geq 3$. Assume that there exists $b_0 > 1$ such that the statement of Theorem 1.3 is valid when $b > b_0$. Then it is also valid when $b \in (0, b_0]$.

If $0 < b \leq b_0, x, y \in \Omega$ and

$$r := |x - y| < r_0/10\kappa b_0, \quad br \leq \min(\delta(x), \delta(y))$$

then

$$\frac{1}{c^*}r^{2-N} \leq C_{\Omega}^{\gamma V}(x, y) \leq c^*r^{2-N},$$

where $c^*(b, b_0)$ is a constant independent of $x, y$.

**Proof.** If $\delta(y) > (b_0 + 1)r$ then $\delta(x) > b_0r$ so that (1.7) holds for some $b > b_0$ and therefore (1.8) holds by assumption. The statement is symmetric in $(x, y)$ so that we may assume:

$$\max(\delta(x), \delta(y)) \leq (b_0 + 1)r.$$ 

As shown in the proof of Lemma 3.2 (3.19) and (3.21) imply that the points $x, y$ can be joined by a Lipschitz curve $I(x, y) := I_J(x, y^*) \cup [y^*, y]$ (notation as in that lemma) such that:

$$\text{length } I(x, y) \leq C_{\hat{b}_0}r, \quad C_{\hat{b}_0} = \kappa + (b_0 + 1)\sqrt{1 + \kappa^2}$$

$$\min_{z \in I(x, y)} \delta(z) \geq br\sqrt{1 + \kappa^2}.$$ 

Let $t := br/4b_0\sqrt{1 + \kappa^2}$. Then

$$\delta(z) > 2b_0t, \quad \forall z \in I(x, y).$$
Since \( r > t \), \( x \not\in B_t(y) \). Let \( \eta \) be the closest point to \( x \) among all points \( z \in I(x,y) \) such that \(|z - y| = 2t\). By the assumption of the lemma and (3.23),
\[
1 \leq G^\gamma_V(\eta) \leq c t^{2-N}.
\]
Let \( I(x,\eta) \) denote the part of \( I(x,y) \) connecting \( x \) and \( \eta \). By Lemma \ref{lemma:boundary} applied to the function \( F := G^\gamma_V \) in the domain \( \Omega \setminus B_t(y) \) we obtain
\[
1 \leq G^\gamma_V(x) \leq c G^\gamma_V(\eta) \leq c' G^\gamma_V(x).
\]
Here \( c' \) depends on \( b,b_0,\kappa \) and also on the constant associated with the strong Harnack inequality for \( G^\gamma_V \) in the domain \( \Omega \setminus B_t(y) \). This constant is independent of \( y \) and, for balls \( B(t,z) \), it is independent of \( z \in I(x,\eta) \) provided that (3.19) and (3.21) hold. This is a simple consequence of the Boundary Harnack principle. Finally (3.24) and (3.25) imply (3.20).

\[\square\]

The next result is classical. We list it for easy reference.

**Lemma 3.4.** For every \( y \in \Omega \),
\[
\lim_{x \to y} \frac{G^\gamma_V}{\Gamma_y} = 1.
\]

**Lemma 3.5.** Assume \( N > 3 \). Let \( \gamma \in (\gamma_-,\gamma_+) \) and \( y \in \Omega \). For every \( x \in \Omega \), denote
\[
g_y(x) := |x - y|^{3-N} \frac{\varphi^\gamma_V(y)}{\delta(y)}
\]
Then, there exists a number \( b_0 > 1 \) dependent on \( \gamma V \) and \( \bar{a} \), but not on \( y \), such that, for every \( b > b_0 \) there exists a constants \( c > 0 \), dependent on \( b \) but not on \( y \), such that
\[
-L^\gamma_V(\varphi^\gamma_Y \Gamma_y + cg_y) \geq \varphi^\gamma_Y \delta_y \text{ in } B^y_b.
\]

**Proof.** Let \( b > 1 \). By the strong Harnack inequality,
\[
\sup_{B^y_b} \varphi^\gamma_V = \inf_{B^y_b} \varphi^\gamma_V = c'(b) < \infty.
\]
A straightforward computation yields,
\[
-L^\gamma_V(\varphi \Gamma_y) = \varphi(x)\delta_y - 2\Gamma_y(x) \cdot \nabla \varphi(x)
\]
\[
- \Gamma_y(x)(\Delta \varphi(x) - \gamma V \varphi(x))
\]
\[
= \varphi(x)\delta_y - 2\Gamma_y(x) \cdot \nabla \varphi(x) + \lambda_1 \Gamma_y(x) \varphi(x).
\]
By interior elliptic estimates (see e.g. [9, Theorem 6.2]) and (3.28),

$$\left|\nabla \varphi(\xi)\right| \leq C_1 \frac{1}{\delta(\xi)} \sup_{|\xi - x| < \delta(\xi)/b} \varphi(x) \leq C_2 \frac{\varphi(\xi)}{\delta(\xi)} \forall \xi \in \Omega.$$  

(3.30)

The constant $C_2$ is independent of $y$. By (3.28) and (3.30),

$$|2\nabla \Gamma_y(x) \cdot \nabla \varphi(x)| \leq C_0 |x - y|^{1-N} \frac{\varphi(y)}{\delta(y)} =: h_y(x) \quad \forall x \in B_y^y.$$  

(3.31)

The constant $C_0$ is independent of $y$. By (3.29),

$$- L^V(\varphi \Gamma_y) \geq \varphi \delta_y - h_y, \quad \forall x \in B_y^y.$$  

(3.32)

Denote

$$L_\mu := \Delta + \frac{\mu}{\delta^2}$$  

and let $\mu = \gamma \bar{a}$ for $\bar{a}$ as in (1.1). Then for any positive function $f \in L^1_{loc}(\Omega)$ and any $\gamma > 0$

$$- L_\mu f \leq - L^V f, \quad - L_- f \leq - L^V f$$  

(3.34)

The second inequality is valid because we assume $|V| \leq \bar{a}\delta^{-2}$.

If $f_y(x) := |x - y|^{3-N}$, a simple calculation yields

$$- L_\mu f_y = (N - 3 - \mu) \frac{|x - y|^2}{\delta(x)^2} |x - y|^{1-N}.$$  

(3.35)

For $x \in B_y^y$, $|x - y| \leq \delta(y)/b$ and $(1 - \frac{1}{b})\delta(y) < \delta(x)$ so that

$$|x - y|^2/\delta(x)^2 < (b - 1)^{-2}.$$  

Therefore,

$$- L_\mu f_y \geq \ell |x - y|^{1-N}, \quad \ell := N - 3 - \frac{\mu}{(b-1)^2}.$$  

(3.36)

If $N > 3$, let $\mu = \gamma \bar{a}$ and let $b_0$ be sufficiently large so that $\ell > 0$ for $b \geq b_0$. Note that the choice of $b_0$ does not depend on $y$.

Pick a constant $c$ such that

$$c\ell > C_0, \quad C_0 \text{ as in (3.31)}.$$  

Then by (3.29), (3.31), (3.32), (3.34) and (3.36),

$$- L^V(\varphi \Gamma_y + cg_y) \geq \varphi \delta_y - h_y - L_\mu(cg_y) \geq \varphi \delta_y$$  

in $B_y^y$ for $b \geq b_0$. This proves (3.27).
Completion of proof of Theorem 1.3.

The case $N > 3$. In $B^y_b$: $|x - y|^{3-N}/\delta(y) < |x - y|^{2-N}/b$. Therefore, using (3.28),

$$
(3.38) \quad \varphi_{\gamma V}(y) \Gamma_y \leq (\varphi_{\gamma V} \Gamma_y + c g_y) \\
\leq (\varphi_{\gamma V}(x) + (c/b)\varphi_{\gamma V}(y)) \Gamma_y \leq (c'(b) + c/b)\varphi_{\gamma V}(y) \Gamma_y \quad \text{in } B^y_b.
$$

For $a \in \mathbb{R}$ and $c$ as in (3.27)

$$
(3.39) \quad F_{a,y} := a \varphi_{\gamma V}(y) G^\gamma_{y} - (\varphi_{\gamma V} \Gamma_y + c g_y).
$$

Note that, as a distribution, $\varphi_{\gamma V} \delta_y = \varphi_{\gamma V}(y) \delta_y$. Therefore, by (3.27), for every $a \in (0,1)$,

$$
(3.40) \quad -L^V(F_{a,y}) \leq (a-1)\varphi_{\gamma V}(y) \delta_y < 0 \quad \text{in } B^y_b \setminus \{y\}.
$$

By Lemma 3.4 for every $a \in (0,1)$ there exists $\beta_y \in (b, \infty)$ such that

$$
(3.41) \quad F_{a,y} \leq 0 \quad \text{in } B^y_{\beta_y}, \quad \beta_y < \beta.
$$

Furthermore, by Lemma 3.1, if $a$ is sufficiently small (depending on $b$ but not on $y$) then

$$
(3.42) \quad F_{a,y} \leq 0 \quad \text{on } \partial B^y_b.
$$

Hence for $a$ and $\beta$ as above

$$
(F_{a,y})_+ = 0 \quad \text{on } \partial B^y_b \cup \partial B^y_{\beta_y}
$$

and, by (3.40), $(F_{a,y})_+$ is $L^V$ subharmonic in $B^y_b \setminus B^y_{\beta_y}$. Consequently $(F_{a,y})_+ = 0$ in this domain. As $\beta$ can be chosen arbitrarily large it follows that $(F_{a,y})_+ = 0$ in $B^y_b \setminus \{y\}$. Therefore, by (3.28), (3.38), (3.39),

$$
(3.43) \quad G^\gamma_{y} \leq C_b \Gamma_y \quad \text{in } B^y_b,
$$

for $b > b_0$ and $C_b$ independent of $y$. Finally applying Lemma 3.3 we conclude that (3.43) holds for every $b > 0$.

To obtain the estimate from below, we consider the Green kernel of $L^\gamma V$ in $B^y_b$. Clearly

$$
(3.44) \quad G^\gamma_{\Omega_b} < G^\gamma_{B^y_b} \quad \text{in } B^y_b.
$$

We blow up the ball $B^y_b$ by the transformation: $\xi = b(x - y)/\delta(y)$ which maps $B^y_b$ to the unit ball $|\xi| < 1$. Under this transformation $L^\gamma V$ becomes,

$$
\tilde{L}^W = \Delta_{\xi} + W, \quad \text{where } |W| \leq \bar{a}(b - 1)^{-2}
$$
and $\tilde{\alpha}$ as in (1.1). If $\tilde{G}^W$ denotes the Green kernel of $\tilde{L}^W$ in $|\xi| < 1$ then,

$$\tilde{G}_0^W(\xi) = \left(\frac{b}{\delta(y)}\right)^{2-N}G_{B^y_b}(x,y).$$

It is known that,

$$\frac{1}{c} |\xi|^{2-N} \leq \tilde{G}_0^W(\xi) \leq c|\xi|^{2-N}, \quad |\xi| < 1/2$$

where $c$ depends only on the bound for $|W|$. Therefore,

$$\frac{1}{C} |x-y|^{2-N} \leq G_{B^y_b}(x,y) \leq C|x-y|^{2-N}, \quad |x-y| < \delta(y)/2b$$

where $C$ depends only on $\tilde{\alpha}$ and $b$. This inequality and (3.44) imply (3.45)

$$G_{\gamma V}^y \geq C'_b \Gamma_y \text{ in } B^y_{2b},$$

where $C'_b$ is independent of $y$.

The case $N = 3$. If $N = 3$, $g_y = \phi_{\gamma V}(y)/\delta(y)$ and $\Gamma_y(x) = a_3|x-y|^{-1}$. As before we choose the constant $c$ in (3.27) as follows:

$$c = C_0/\ell = -C_0(b-1)^2/\mu,$$

where $C_0$ is the constant in (3.31). Since $c < 0$ the completion of the proof requires certain modifications.

The constant $c'(b)$ in (3.28) decreases as $b$ increases and $c'(b) \downarrow 1$ as $b \uparrow \infty$. Choose $b_0$ such that

$$1 \leq c'(b) \leq 2 \quad \forall b > b_0.$$

Put

$$D^y_b := \{x : |x-y| < \frac{a_3\mu}{4C_0b^2}\delta(y)\}.$$

If $x \in D^y_b$ then,

$$\frac{C_0b^2}{\mu\delta(y)} < a_3 \frac{|x-y|^{-1}}{4} = \frac{1}{4} \Gamma_y(x).$$

If, in addition, $b > b_0$ then,

$$\phi_{\gamma V}(x)\Gamma_y(x) + cg_y(x) > \phi_{\gamma V}(x)\Gamma_y(x) - \frac{C_0b^2}{\mu\delta(y)} \phi_{\gamma V}(y)$$

(3.47)

$$\geq \phi_{\gamma V}(y)\left(\frac{1}{c'(b)}\Gamma_y(x) - \frac{C_0b^2}{\mu\delta(y)}\right) \geq \frac{1}{4} \Gamma_y(x)\phi_{\gamma V}(y).$$

Let $F_{a,y}$ be as in (3.39). In view of (3.47) we can proceed as before and – replacing $B^y_b$ by $D^y_b$ – we obtain,

(3.48) $G_{\gamma V}^y \leq C_b \Gamma_y \text{ in } D^y_b, \quad b > b_0$
where \( C_b \) is independent of \( y \). Applying Lemma 3.3 we conclude that this inequality holds for every \( b > 0 \).

Finally the proof of inequality (3.45) applies, without modification, to the case \( N = 3 \). \( \square \)

4. Proof of Theorem 1.4

The core of the proof is in the following result.

**Lemma 4.1.** Let \( x, y \) be points in \( \Omega \) such that

\[
|x - y| < r_0/4\kappa, \quad \max(\delta(x), \delta(y)) < \frac{|x - y|}{b}, \quad b = 16(\kappa + 1)^2.
\]

Then there exists a constant \( C' \) depending only on \( r_0, \kappa \) and \( \bar{a} \) such that (1.11) holds.

**Proof.** Let \( X, Y \in \partial \Omega \) be points such that

\[
|x - X| = \delta(x), \quad |y - Y| = \delta(y).
\]

Let \( \xi^X \) be an admissible set of local coordinates at \( X \) associated with the cylinder \( T^X(r_0, 10\kappa r_0) \) (see (2.2)). Put

\[
r = |x - y|, \quad r_1 = |\xi^X_1(x - y)|, \quad r' = |(\xi^X)'(x - y)|
\]

and similarly \( R = |X - Y| \) etc.

The relation

\[
\overrightarrow{xy} = \overrightarrow{xX} + \overrightarrow{XY} + \overrightarrow{Yy}
\]

together with (4.2) and (4.1) yields

\[
\max(|r - R|, |r' - R'|, |r_1 - R_1|) \leq 2r/b.
\]

In particular \( R \neq 0 \). By assumption, (see (2.2)),

\[
R_1 = |\xi^X_1(X) - \xi^X_1(Y)| = |f^X((\xi^X)'(X)) - f^X((\xi^X)'(Y))| \leq \kappa R'.
\]

Hence, by (4.3)

\[
r_1 \leq \kappa R' + 2r/b \leq \kappa(r' + 2r/b) + 2r/b = \kappa r' + 2(\kappa + 1)r/b.
\]

Therefore, with \( b \) as in (4.1),

\[
r^2 \leq (\kappa r' + 2(\kappa + 1)r/b)^2 + (r')^2 \\
\leq (1 + \kappa^2)(r')^2 + 4\kappa(1 + \kappa)r^2/b + 4(\kappa + 1)^2(r/b)^2 \\
\leq (1 + \kappa^2)(r')^2 + \frac{r^2}{2}.
\]

Thus,

\[
r' \leq r \leq \beta r' \quad \text{where} \quad \beta := \sqrt{2(1 + \kappa^2)}.
\]
It follows that
\[ (4.6) \quad \beta / b < 1 / 16, \quad \max(|x - X|, |y - Y|) \leq \frac{r}{b} < \frac{r'}{16}. \]
Hence, by (4.3) and (4.4)
\[ (4.7) \quad R' \leq R \leq \sqrt{1 + \kappa^2 R'}, \quad 7r'/8 \leq R' \leq 9r'/8.9r_1/8. \]
Therefore,
\[ (4.8) \quad x \in T^X(3R'/8, 6\kappa R'), \quad y \in T^Y(3R'/8, 6\kappa R') \]
where \( T^Y \) is expressed in the coordinates \( \xi^Y := \xi^X - \xi^X(Y) \). Recall,
\[ (4.9) \quad |\xi^X(Y)| = |X - Y| = R, \quad |(\xi^X)'(Y)| = R', \quad |\xi_1^X(Y)| = R_1 \leq \kappa R'. \]
Furthermore, as \( R' \geq 7r'/8 \), we have
\[ (4.10) \quad x \notin T^Y(3R'/4, 6\kappa R'), \quad y \notin T^X(3R'/4, 6\kappa R'). \]
However \( T^X(3R'/4, 6\kappa R') \cap T^Y(3R'/4, 6\kappa R') \cap \Omega \neq \emptyset \). In fact, if \( \Pi_1(X, Y) \) is the half plane whose boundary is the \( \xi_1^X \) axis and contains the point \( Y \) then,
\[ (4.11) \quad \Pi_1(X, Y) \cap [\xi_1^X = 4\kappa R'] \cap [||(\xi^X)'| = R'/2] \]
\[ \subset T^X(3R'/4, 6\kappa R') \cap T^Y(3R'/4, 6\kappa R') \cap \Omega. \]
The intersection on the left hand side consists of a single point \( S \) where
\[ \xi_1^X(S) = 4\kappa R', \quad |(\xi^X)'(S)| = R'/2. \]
Applying the BHP theorem in \( T^X(3R'/4, 6\kappa R') \) when \( A_X := S \) and \( A'_X \) is defined by \( \xi^X(A'_X) := (5\kappa R', 0) \) yields,
\[ (4.12) \quad \frac{G^Y(x, x_0)}{G^Y(S, x_0)} \sim \frac{G^X(x, A'_X)}{G^X(S, A'_X)} \]
\[ \frac{G^Y(x, x_0)}{G^Y(S, x_0)} \sim \frac{G^Y(x, y)}{G^Y(S, y)} \quad \forall x \in T^X(3R'/8, 3\kappa R'). \]
The second relation is valid because \( y \notin T^X(3R'/4, 6\kappa R') \).
By Proposition 3.1 \( G^Y(S, A'_X) \sim (R')^{2-N} \sim r^{2-N}. \) (The relation \( r \sim R' \) follows from (4.5) and (4.7)). Moreover, \( G^X(x, x_0) \sim \varphi_Y(x) \). Hence,
\[ (4.13) \quad G^Y(x, A'_X) \sim \frac{\varphi_Y(x)}{\varphi_Y(S)} r^{2-N} \]
\[ G^Y(x, y) \sim \frac{\varphi_Y(x)}{\varphi_Y(S)} G^Y(S, y) \quad \forall x \in T^X(3R'/8, 3\kappa R'). \]
In these relations the similarity constants depend only on \( x_0, r_0, \kappa \) and \( \bar{a} \).
Next we apply the BHP theorem in $T^Y(3R'/4, 6\kappa R')$ when $A'_Y := S$. Since $\xi^Y(S) = \xi^X(S) - \xi^X(Y)$ and, by (4.4), $|\xi^X(Y)| = R_1 \leq \kappa R'$, it follows that

$$3\kappa R' \leq \xi^Y_1(S) \leq 5\kappa R', \quad |(\xi^Y)'(S)| = R'/2.$$  

We choose $A_Y$ so that $\xi^Y(A_Y) = (2\kappa R', 0)$. As in the first relation of (4.13), we have

$$G^\gamma V(y, S) \sim \frac{\varphi^\gamma V(y)}{\varphi^\gamma V(A_Y)} \rho^{2-N} \quad \forall y \in T^Y(3R'/8, \frac{3\kappa}{2} R').$$

Combining (4.14) and the second relation of (4.13) we obtain,

$$G^\gamma V(x, y) \sim \frac{\varphi^\gamma V(x)}{\varphi^\gamma V(A_X) \varphi^\gamma V(A_Y)} \rho^{2-N}.$$  

Here we used the symmetry of $G^\gamma V$ and substituted $S = A_X$. Again the similarity constants depend only on $x_0, r_0, \kappa$ and $\bar{a}$.

Note that $A_X, A_Y$ are points lying 'above' $X$ and $Y$ respectively, i.e. on an approximate normal from the respective boundary point, at a distance proportional to $|x - y|$ which in turn is proportional to $|X - Y|$. Applying Lemma 3.2 to $\varphi^\gamma_V$ we see that (4.15) remains valid if $A_X, A_Y$ are replaced by any two points in $A(x, y)$. The similarity constant is independent of $r$, but depends on the proportionality constants mentioned above and therefore on $x_0, r_0, \kappa$ and $\bar{a}$.

\[ \square \]

**Lemma 4.2.** Let $x, y \in \Omega$ satisfy,

$$\min(\delta(x), \delta(y)) \leq \frac{|x - y|}{b} \leq \max(\delta(x), \delta(y)), \quad b = 16(\kappa + 1)^2.$$  

Then there exists a constant $C'$ depending only on $r_0, \kappa$ and $\bar{a}$ such that (1.11) holds.

**Proof.** We assume,

$$\delta(x) \leq \frac{|x - y|}{b} < \delta(y).$$

Denote by $X$ the point on $\partial\Omega$ such that $|x - X| = \delta(x)$. Let $\xi^X$ be an admissible set of local coordinates at $X$ associated with the cylinder $T^X(r_0, 10\kappa r_0)$ (see (2.2)). Put

$$r = |x - y|, \quad r_1 = |\xi^X_1(x - y)|, \quad r' = |(\xi^X)'(x - y)|.$$  

Let $Y \in \partial\Omega$ be the point such that $(\xi^X)'(Y) = (\xi^X)'(y)$ and let $R = |X - Y|$ etc. We consider the following two cases separately:
(a) \( r > 2r_1 \) (b) \( r \leq 2r_1 \).

Case (a). The Lipschitz property of \( \partial \Omega \) implies

\[
R_1 \leq \kappa R', \quad R' < R < \sqrt{1 + \kappa^2 R'}.
\]

Assumption (4.17) together with the definition of \( Y \) imply,

\[
|\xi^X(x)| \leq r/b, \quad |R' - r'| = |(\xi^X)'(X - x)| < r/b.
\]

As \( r - r_1 < r' \), (a) implies that \( r' < r < 2r' \) and consequently, by (4.19),

\[
r'(1 - 2/b) < R' < r'(1 + 2/b)
\]

These in turn imply that \( |\xi^X(x)| \leq \frac{2}{32} R' < R'/32 \). Therefore (4.8), (4.10) and (4.11) hold and the continuation of the proof is the same as in the proof of Lemma 4.1.

Case (b). Let \( X \) and \( \xi^X \) be as in part (a). By assumption \( r_1 = \xi^X_1(y) - \xi^X_1(x) > r/2 \) and by construction \( \xi^X_1(x) = \delta(x) < r/b \). Consequently

\[
r/2 + \delta(x) < \xi^X_1(y) = r_1 + \xi^X_1(x) \leq r(1 + \frac{1}{b}).
\]

Moreover, as \( (\xi^X)'(x) = 0 \),

\[
(\xi^X)'(y) = r' < (\sqrt{3}/2)r
\]

We apply Theorem BHP in the standard cylinder \( T^X(r, 10\kappa r) \). Let \( A' = y \) and let \( A \in \Omega \) be the point \( \xi^X(A) = (r/4, 0) \). Put \( v = G_{\gamma^V}(\cdot, y) \) and \( w = G_{\gamma^V}(\cdot, x_0) \) where \( x_0 \) is a reference point in \( \Omega \) such that \( \delta(x_0) > r_0 \). Then, by BHP,

\[
\frac{v(z)}{w(z)} \sim \frac{v(A)}{w(A)} \quad \forall z \in T^X(r/2, 5\kappa r).
\]

Recall that \( v(A) = G_{\gamma^V}(A, A') \sim r^{2-N} \) and \( w \sim \varphi_{\gamma^V} \) in \( T^X(r, 10\kappa r) \). Therefore (4.23) implies:

\[
1 \frac{\varphi_{\gamma^V}(x)}{C \varphi_{\gamma^V}(A)} r^{2-N} \leq G_{\gamma^V}(x, y) \leq C \frac{\varphi_{\gamma^V}(x)}{\varphi_{\gamma^V}(A)} r^{2-N}.
\]

The constant \( C \) and all the similarity constants depend only on \( x_0, \kappa, N \) and \( \bar{a} \).

Since \( \delta(y) \geq r(1/2 - 1/b) \) while \( \delta(x) \leq r/b \) it follows that

\[
|x - y| \sim \delta(y) \sim r.
\]

Therefore, by Lemma 3.2, \( \varphi_{\gamma^V}(A) \sim \varphi_{\gamma^V}(x_y) \) for \( x_y \in A(x, y) \) so that (4.24) is equivalent to (1.11).
5. Theorems 1.5 and 1.7

5.1. Proof of Theorem 1.5. It is well-known that, under the assumptions of the theorem,

\[(5.1) \quad K^{\gamma V}(x, y) = \lim_{z \to y} \frac{G^{\gamma V}(x, z)}{G^{\gamma V}(x_0, z)}.\]

Using the estimate of the Green function (1.11) we obtain (for \(z\) near to \(y\)),

\[
\frac{G^{\gamma V}(x, z)}{G^{\gamma V}(x_0, z)} \sim |x - z|^{2-N} \frac{\varphi^{\gamma V}(x) \varphi^{\gamma V}(z)}{\varphi^{\gamma V}(x_z) \varphi^{\gamma V}(z_x) \varphi^{\gamma V}(z)} = |x - z|^{2-N} \frac{\varphi^{\gamma V}(x)}{\varphi^{\gamma V}(x_z) \varphi^{\gamma V}(z_x)},
\]

where \(x_z\) and \(z_x\) can be chosen arbitrarily from the set \(A(x, z)\) (see (1.12)). As \(z \to y\) we may replace these points by a point \(x_y \in A(x, y)\). This yields (1.14).

\[\square\]

5.2. Proof of Theorem 1.7.

In this section \(b\) stands for a number that may vary in the interval \([b_0, 2b_0]\), \(b_0 = \max(b_1, 16(1 + \kappa)^2)\) and \(b_1\) as in (1.15).

Step 1. Suppose that each of the pairs \((x, y), (y, z), (x, z)\) satisfies (1.7) for some fixed \(b > 0\). Then, by Theorem 1.3 inequality (1.16) reduces to

\[(5.2) \quad (|x - y||y - z|)^{2-N} \leq C_b|x - z|^{2-N}(|x - y|^{2-N} + |y - z|^{2-N}).\]

This inequality is easily verified. By the triangle inequality,

\[
\frac{1}{2} |x - y| \leq \max(|x - z|, |y - z|).
\]

If \(|x - y| \leq 2|x - z|\) then \((|x - z||x - y|)^{2-N} \leq 2^{N-2}\) and (5.2) follows. If \(2|x - z| < |x - y|\) then \(|x - y| \leq 2|y - z|\) so that \((|x - y||y - z|)^{2-N} \leq 2^{N-2}\) and again (5.2) follows.

Step 2. Assume that the pair \(x, z\) satisfies (1.7). Then, by Theorem 1.3

\[G(x, z) \sim |x - z|^{2-N}.\]

Therefore, by Theorem 1.4 and inequality (5.2), (1.16) reduces to

\[(5.3) \quad \varphi^{\gamma V}(y)^2 \varphi^{\gamma V}(x) \varphi^{\gamma V}(z) \leq \varphi^{\gamma V}(x_y)^2 \varphi^{\gamma V}(y_z)^2.\]
where \( x_y \) is an arbitrary point in \( A(x, y) \) and \( y_z \in A(y, z) \). We proceed to prove (5.3).

**Case 2a.** Assume that the pair \( x, y \) satisfies (1.7). Then \( x, y \in A_b(x, y) \) and we may choose \( x_y = x \) as well as \( y_z = y \). Consequently, \( (5.4) \)

\[ \varphi_{\gamma V}(x)\varphi_{\gamma V}(y) \sim \varphi_{\gamma V}(x_y)^2. \]

By the same reasoning, if the pair \( y, z \) too satisfies (1.7) then \( (5.5) \)

\[ \varphi_{\gamma V}(z)\varphi_{\gamma V}(y) \sim \varphi_{\gamma V}(y_z)^2 \]

and (5.3) holds.

Now we have to verify (5.3) when \( y, z \) does not satisfy (1.7), i.e., \( (5.6) \)

\[ \min(\delta(y), \delta(z)) \leq \frac{1}{b}|y - z| \]

for some \( b \geq 16(1 + \kappa)^2 \). We shall show that (5.3) implies \( (5.7) \)

\[ \varphi_{\gamma V}(y)\varphi_{\gamma V}(z) \lesssim \varphi_{\gamma V}(y_z)^2. \]

We verify this inequality in each of the cases:

\[ \begin{cases} 
(i) & \delta(z) \leq \frac{|y - z|}{b} \leq \delta(y) \\
(ii) & \delta(y) \leq \frac{|y - z|}{b} \leq \delta(z) \\
(iii) & \max(\delta(y), \delta(z)) \leq \frac{1}{b}|y - z| 
\end{cases} \]

If (i) holds then \( y \in A_b(y, z) \). Further we choose a point \( \zeta \in A_b(y, z) \) such that:

\[ \delta(\zeta) = \hat{r}(y, z) = |y - z| \vee \delta(y) \]

and the pair \( z, \zeta \) lies on a \( \lambda \) pseudo-normal. Therefore we may choose \( y_z = y \) as well as \( y_z = \zeta \). By (1.15), \( \varphi_{\gamma V}(z) \lesssim \varphi_{\gamma V}(\zeta) \) and (5.7) holds.

Clearly, the same conclusion holds if (ii) holds.

Finally if (iii) holds we choose \( \zeta \in A_b(y, z) \) as above and \( \eta \in A_b(y, z) \) in the same way except that now the pair \( y, \eta \) lies on a \( \lambda \) pseudo-normal. By (1.15),

\[ \varphi_{\gamma V}(z) \lesssim \varphi_{\gamma V}(\zeta), \quad \varphi_{\gamma V}(y) \lesssim \varphi_{\gamma V}(\eta). \]

Choosing once \( y_z = \zeta \) and once \( y_z = \eta \) we obtain (5.7).

**Case 2b** Assume that the pair \( (y, z) \) satisfies (5.6) and the pair \( x, y \) satisfies a similar inequality:

\[ \min(\delta(x), \delta(y)) \leq \frac{1}{b}|x - y|. \]

Then as shown in Case 2a:

\[ \varphi_{\gamma V}(y)\varphi_{\gamma V}(z) \lesssim \varphi_{\gamma V}(y_z)^2, \quad \varphi_{\gamma V}(x)\varphi_{\gamma V}(y) \lesssim \varphi_{\gamma V}(x_y)^2 \]

which implies (5.3).
Step 3. It remains to consider the case when the pair $x, z$ does not satisfy (1.7), i.e.,

\begin{equation}
\min(\delta(x), \delta(z)) \leq \frac{1}{b}|x - z|
\end{equation}

for some $b \geq 32(1 + \kappa)^2$. Then, by Theorem 1.4 and inequality (5.2), (1.16) reduces to

\begin{equation}
\varphi_{\gamma V}(y)^2 \varphi_{\gamma V}(xz)^2 \lesssim \varphi_{\gamma V}(xy)^2 \varphi_{\gamma V}(yz)^2.
\end{equation}

Case 3a. Assume that,

\begin{equation}
\max(\delta(x), \delta(z)) \leq \frac{1}{b}|x - z|.
\end{equation}

By the triangle inequality,

\[ \max(|x - y|, |y - z|) > \frac{1}{2}|x - z|. \]

Without loss of generality we assume that the maximum is $|x - y|$ so that

\begin{equation}
|x - y| > \frac{1}{2}|x - z| \geq \frac{b}{2} \max(\delta(x), \delta(z)).
\end{equation}

Let $x_y \in A_b(x, y)$ and $x_z \in A_b(x, z)$ be points lying on a $\lambda$ pseudo normal such that

\[ \delta(x_y) = 2|x - y|, \quad \delta(x_z) = \frac{2}{b}|x - z|. \]

(Note that $x$ or $y$ need not be in $A_b(x, y)$.) In view of (5.11) such a choice is possible and by (1.15)

\begin{equation}
\varphi_{\gamma V}(xz) \lesssim \varphi_{\gamma V}(xy)
\end{equation}

Let $y_z \in A(y, z)$ be a point such that:

(i) if $\delta(y) > |y - z|/b$ then $y_z = y$. (Note that, as $\delta(z) \leq (1 + b)\delta(y)$, $\delta(y) > \hat{r}(y, z)/(b + 1)$).

(ii) if $\delta(y) \leq |y - z|/b$ then $\delta(y_z) = |y - z|$ and $y, y_z$ lie on a $\lambda$ pseudo normal.

In either case, using (1.15), we obtain

\begin{equation}
\varphi_{\gamma V}(y) \lesssim \varphi_{\gamma V}(yz)
\end{equation}

Inequality (5.9) follows from (5.12) and (5.13).

Case 3b. Assume that:

\begin{equation}
\delta(x) \leq \frac{1}{b}|x - z| \leq \delta(z).
\end{equation}

Then $\delta(z) \geq \frac{1}{b}\hat{r}(x, z)$ so that $z \in A_b(x, z)$. Therefore we may and shall choose $x_z = z$. 


Next we choose $y, z \in A_b(y, z)$ as follows: If $\delta(y) \geq \frac{1}{b} (\delta(z) \lor |y - z|)$ then $\delta(y) \geq \frac{1}{b} \hat{r}(y, z)$ and we choose $y, z$ such that $\delta(y_z) = \delta(y)$ and $z, y_z$ lie on a $\lambda$ pseudo normal.

If $\delta(y) \leq \frac{1}{b} (\delta(z) \lor |y - z|)$ then $\delta(y) \leq \frac{1}{b} \hat{r}(y, z)$. In this case we choose a point $y_z \in A_b(y, z)$ such that $\delta(y_z) = \hat{r}(y, z)$ and $z, y_z$ lie on a $\lambda$ pseudo normal. In either case, by (1.15),

\[ \varphi_{\gamma V}(z) \lesssim \varphi_{\gamma V}(y_z). \]

Since, by choice, $z = x_z$ we have $\varphi_{\gamma V}(x_z) \lesssim \varphi_{\gamma V}(y_z)$. In order to establish (5.9), it remains to show that,

\[ \varphi_{\gamma V}(y) \lesssim \varphi_{\gamma V}(x_y) \]

where $x_y$ is a point in $A(x, y)$. This is proved in the same way as (5.13) replacing $y, y_z$ by $y, x_y$.

\[ \square \]

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