Fermi arc induced vortex structure in Weyl beam shifts

Udvas Chattopadhyay,1 Li-Kun Shi,2 Baile Zhang,1,3 Justin C. W. Song,1,2 and Y. D. Chong1,3

1Division of Physics and Applied Physics, School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore 637371, Singapore

2Institute of High Performance Computing, A*STAR, Singapore 138632, Singapore

3Centre for Disruptive Photonic Technologies, Nanyang Technological University, Singapore 637371, Singapore

In periodic media, despite the close relationship between geometrical effects in the bulk and topological surface states, the two are typically probed separately. Here, we show that when beams in a Weyl medium reflect off an interface with a gapped medium, the trajectory is influenced by both bulk geometrical effects and the presence of Fermi arc surface states. The reflected beam experiences a displacement, analogous to the Goos-Hänchen or Imbert-Fedorov shifts, that forms a half-vortex in the two-dimensional surface momentum space. The half-vortex structure is centered at the point where the Fermi arc of the reflecting surface touches the Weyl cone, and the magnitude of the shift scales as the inverse square root of distance from the touching-point. This striking feature provides a way to use bulk transport to probe the topological characteristics of a Weyl medium.

Introduction.—One of the most interesting features of wave dynamics in both classical and quantum media is that wavepacket trajectories are not determined solely by the dispersion relation, but can also be influenced by the internal structure of the underlying wavefunctions. For instance, the equations of motion of a wavepacket in a periodic medium can include an “anomalous velocity” term tied to the Berry connection of the underlying Bloch functions [1–5]. Another example of nontrivial dynamics, originating from the field of optics, involves the displacement of a light beam as it reflects off a surface [6]. A lateral displacement is called a Goos-Hänchen (GH) shift [7], while a transverse one is called an Imbert-Fedorov (IF) shift [8], and both originate from the polarization degree of freedom of electromagnetic waves [10, 11]. In electronic systems such as strained graphene, similar shifts can be induced by pseudospin degrees of freedom, and are predicted to have observable effects on transport in heterojunction devices [12, 14].

Another phenomenon intimately linked to the internal structure of wavefunctions is the existence of topological surface states, which arise from subtle windings of the Bloch functions in momentum space [14]. For example, Weyl media in three dimensions (3D) feature linear band-crossing points called Weyl points that act as monopole sources of Berry curvature in momentum space [15], giving rise to a variety of non-trivial bulk dynamical effects [16–25]. At the same time, the net Berry flux between pairs of Weyl points guarantees the existence of “Fermi arcs” of surface states along any interface with a gapped medium [15]. Weyl media have been realized in multiple venues including photonic crystals and waveguide arrays [20, 28], materials such as TaAs [29, 32], as well as acoustic, mechanical, and electric metamaterials [33–35].

In spite of this bulk-edge correspondence, the non-trivial bulk dynamics of Weyl media (a band geometric property) and Fermi arc surface states (a topological property) have largely been probed separately. The former has been studied using spectroscopic tools such as angle-resolved photoemission [30], whereas the latter has been studied through bulk transport effects such as negative magnetoresistance [17, 22, 23] and anomalous Hall conductivity [15]. Another interesting bulk phenomenon found in Weyl media involves the GH and IF shifts experienced by a wavepacket as it undergoes partial reflection off a potential step in the bulk [20, 21, 24, 25]. These shifts are attributable to the Weyl medium’s spinor degree of freedom, and the direction of the IF shift has been shown to be determined by the sign of the Weyl point’s Berry flux. However, no direct connection to the Fermi arc surface states has been identified.

In this Letter, we show that when a beam in a Weyl medium reflects off an interface with a gapped medium, the displacement of the reflected beam exhibits an anomalous half-vortex structure in momentum space. As shown below, this effect provides a bulk probe of the topological Fermi arc at the reflecting surface. Previous studies into GH and IF shifts in Weyl media dealt with partial reflection off a potential step separating two Weyl media [20, 21, 25]. In contrast, we consider total reflection off an interface with a medium that is gapped (i.e., supporting no propagating waves at the operating energy). Unlike a potential step, such an interface features a Fermi arc that extends outward from the Weyl cone in the two-dimensional (2D) surface momentum space [15]. The real-space displacement of the reflected beam, $\Delta$, varies with the direction of the incident beam, which is characterized by its average in-plane momentum $K_\perp$. We show that $\Delta(\vec{K}_\perp)$ circulates around $\vec{K}_\parallel$, the point on the boundary of the Weyl cone touched by the Fermi arc, and its magnitude scales as an inverse square root, $|\Delta| \sim |\vec{K}_\perp - \vec{K}_\parallel|^{-1/2}$. This behavior is observed in two different Hamiltonian models with distinct boundary condition implementations, indicating that it is generic. In microwave photonic crystals [26], the magnitude of the predicted shift is several multiples of the lattice constant under realistic conditions. The displacement accumulates over successive reflections off two parallel surfaces, and thus affects the effective velocity of propagation [12] within a film of Weyl medium. To our knowledge, this constitutes the first prediction of the Fermi arc having...
an observable effect on beam trajectories in Weyl media, which may inspire further investigations into the physical effects of Fermi arcs.

**Plane waves in a Weyl medium.**—We consider the setup shown in Fig. 1(a), where a Weyl medium occupies the space $z > 0$ with a gapped medium in the space $z < 0$. A monochromatic beam of energy $E$ is incident from the Weyl medium, and reflects off the surface at $z = 0$. The reflected beam can experience a displacement, denoted by a vector $\Delta$ parallel to the $x$-$y$ plane. Note that the beam always undergoes total reflection, as the $z < 0$ medium does not support propagating modes.

The eigenmodes of the bulk Weyl medium are described by a $2 \times 2$ Hamiltonian $H_w = \sum_j v_j k_j \sigma_j$, where for each direction $j \in \{1, 2, 3\}$, $v_j$ is the phase velocity, $k_j$ is the wavenumber, and $\sigma_j$ is a Pauli matrix. (In quantum mechanical contexts, we set $\hbar = 1$ for brevity.) The Weyl point possesses a chirality invariant $C = \text{sgn}(v_x v_y v_z)$, which can only be altered by annihilation with another Weyl point [15]. We henceforth take $v_x = v_y = v_z = v$, so that $C = \text{sgn}(v)$. For given $\vec{k}$, the modal eigenenergy (or eigenfrequency) is $E = v |\vec{k}|$, and the wavefunction is a superposition of two basis wavefunctions with coefficients given by the spinor components of the envelope function $\psi(\vec{k}, \vec{r}) = \Psi(\vec{k}) \exp (i \vec{k} \cdot \vec{r})$, where $\Psi(\vec{k})$ is an eigenvector of $H_w(\vec{k})$ and $\vec{r} \equiv (x, y, z)$.

For incident and reflected plane waves (not beams), the eigenvectors are taken to be

$$
\psi_i = \frac{1}{\sqrt{1 + \eta^2}} \begin{bmatrix} 1 \\ \eta e^{i \alpha} \end{bmatrix}, \quad \psi_r = \frac{e^{i \phi}}{\sqrt{1 + \eta^2}} \begin{bmatrix} 1 \\ \eta e^{i \alpha} \end{bmatrix},
$$

where

$$
\alpha = \tan^{-1} \left( \frac{k_y}{k_x} \right), \quad \eta = \sqrt{\frac{E - v k_x^2}{E + v k_x^2}},
$$

and $k_z = \pm \sqrt{(E/v)^2 - |\vec{k}_z|^2}$.

The $k_z < 0$ ($k_z > 0$) branch is chosen for the incident (reflected) wave, and $\vec{k}_z$ is the projection of $\vec{k}$ onto the $k_x$-$k_y$ plane. The phase factor $e^{i \phi}$ is a reflection coefficient, determined by the boundary condition at $z = 0$.

From the assumption that the real-space Hamiltonian is Hermitian, it can be shown [39,41] that the boundary of a Weyl medium is characterized by a single real angular parameter $\theta_b \in [0, 2\pi]$, such that

$$
\left[ \begin{bmatrix} 1 & e^{-i \theta_b(\vec{k}_\perp)} \end{bmatrix} \right] \psi_{\text{tot}} \bigg|_{z=0} = 0,
$$

where $\psi_{\text{tot}}$ denotes the sum of the incident and reflected envelope functions. Note that $\theta_b$ may vary with $\vec{k}_\perp$. Using Eq. (3), we derive

$$
e^{i \phi} = \sqrt{\frac{1 + \eta^2}{1 + \eta^2}} \begin{bmatrix} 1 + \eta e^{i \alpha} e^{-i \theta_b} \\ 1 + \eta e^{i \alpha} e^{-i \theta_b} \end{bmatrix}.
$$

Eq. (3) also yields surface states, which have the form $\psi_s \propto e^{-\kappa z} [e^{-i \theta_b} - 1]^T$ for $z > 0$, where $\kappa = k_x \sin \theta_b - k_y \cos \theta_b \geq 0$. Such surface states are found to exist along a “Fermi arc” [15] parameterized by

$$E = -k_x \cos \theta_b - k_y \sin \theta_b.
$$

In this model, the Fermi arc extends to infinity, since there is only a single Weyl cone.

The resulting reflection phase behavior is displayed in Fig. 1(b)–(c), for two representative cases where $\theta_b$ is a constant independent of $\vec{k}_\perp$. The color map gives the values of $\phi$ within the circular domain $|\vec{k}_\perp| \leq E/v \equiv K_W$, which is a section of the Weyl cone. The Fermi arc lies outside the cone and touches its boundary tangentially. The touching-point, and the orientation of the Fermi arc, depend on the choice of $\theta_b$. We observe that $\phi$ winds by $2\pi$ during a half-encirclement around the Fermi arc touching-point.

These features can be understood by considering the Weyl cone section’s boundary, $|\vec{k}_\perp| = K_W$, which is parameterized by the polar angle $\alpha$ [Eq. (2)]. As we approach the boundary from the inside (i.e., real $k_x \to 0$), the expressions for $\psi_i$ and $\psi_r$ in Eq. (1) become linearly dependent, so that $\psi_{\text{tot}} \propto (1 + e^{i \phi}) \psi_f$ and $\eta \to 1$. Hence, the boundary condition (3) can be satisfied in two ways: (i) $\phi = \pi$, so that $\psi_{\text{tot}}$ vanishes, or (ii) $\alpha - \theta_b(\vec{k}_\perp) = \pi \mod 2\pi$; so that the vectors in Eq. (3) are orthog-
Beams in Weyl media.—A monochromatic beam of energy $E$ can be described as a superposition of planar eigenmodes, with wavenumbers $\vec{k}$ constrained by $E = v|\vec{k}|$. The envelope function has the form

$$\Psi(\vec{r}, \vec{k}_\perp) = \int d^2k_\perp g(k_x - K_x) g(k_y - K_y) \psi(\vec{k}, \vec{r})$$

where $\vec{k}_\perp \equiv (k_x, k_y)$ denotes the in-plane wavevector of each eigenmode, $\vec{K}_\perp$ is the central (average) value for the in-plane wavevector of the beam, $g(k_j - K_j)$ is a $k$-space envelope function, and $\psi(\vec{k}, \vec{r})$ is the envelope function for a planar eigenmode. We let each $g$ be a Gaussian function of unit area, zero mean, and standard deviation $\sigma_k$. For each $E$, the choice of sign for $k_z$ is determined by the beam direction.

We plug the incident and reflected eigenmodes [Eq. (1)] into Eq. (6), and expand the wavevector-dependent functions $\phi$ and $\alpha$ keeping only lowest-order terms, so as to compare the centers of the incident and reflected beams in the $z = 0$ plane. This results in the following formula for the displacement of the reflected beam:

$$\Delta(\vec{K}_\perp) = \left[ -\nabla_{k_\perp} \phi + \frac{\eta_\perp^2 - 1}{\eta_\perp^2 + 1} \nabla_{k_\perp} \alpha \right]_{\vec{k}_\perp = \vec{K}_\perp}$$

Here, $\nabla_{k_\perp}$ denotes the two-dimensional in-plane $k$-space derivative, and $\vec{K}_\perp$ is the mean wave-vector for the incident beam. Note that although $\phi$, $\eta_\perp$, and $\alpha$ are based on a specific eigenmode gauge choice [Eq. (1)], the shift $\Delta$ is gauge-independent. Details of the calculation are given in the Supplemental Material [12].

Fig. 2 shows the streamline plot of vector field $\Delta$ as a function of $\vec{K}_\perp$. We find that $\Delta$ has a half-vortex structure in the vicinity of the Fermi arc touching-point $\vec{K}_{fa}$, regardless of the value of $\theta$. The direction of $\Delta$ winds by $\pi$ during a half-encirclement of $\vec{K}_{fa}$, which means that the shift is not generically purely lateral (GH-like) or transverse (IF-like), but depends on the orientation of the incident beam relative to the Fermi arc. The magnitude of the shift scales as

$$|\Delta| \sim |\vec{K}_\perp - \vec{K}_{fa}|^{-1/2}$$

This scaling is observed numerically for different directions stretching away from $\vec{K}_{fa}$, and we are able to show that it comes from the $-\nabla_{k_\perp} \phi$ term in Eq. (7); details are given in the Supplemental Material [12]. The analysis also shows that the vortex at the $k$-space origin [Fig. 1(b)–(c)] does not yield a vortex in $\Delta$, because the two terms in Eq. (7) have similar magnitudes and opposite vorticities and thus cancel. Near the Weyl cone, however, the $\nabla_{k_\perp} \alpha$ term is negligible and only the $-\nabla_{k_\perp} \phi$ term contributes.

The left panels in Fig. 2 also show $\Delta$ winding around a point $\vec{K}_{op}$ opposite to the Fermi arc touching-point. Around this point, the magnitude of the beam shift scales as $|\Delta| \sim |\vec{K} - \vec{K}_{op}|^{1/2}$ (i.e., it vanishes rather than diverging at $\vec{K}_{op}$), which may be difficult to observe.

**Paired Weyl cones.**—To show that the above results are not model-specific, we consider an alternative model described by the quadratic Hamiltonian [13, 14]

$$H = v \left[ \frac{-k_y}{\beta(k_x^2 - m) + ik_z} \right]$$

This has dispersion $E = \pm v \sqrt{\beta^2 (k_x^2 - m)^2 + k_y^2 + k_z^2}$, and exhibits either paired Weyl points or a complete bandgap, depending on the choice of $m$. In the region $z > 0$, we set $m = m_0 > 0$, so that there is a pair of Weyl points at $\vec{k} = [\pm \sqrt{m_0}, 0, 0]$; there are two Weyl cones for small $|E|$, which merge into a single band as $|E|$ increases.
In the region \( z < 0 \), we set \( m = -m_1 < 0 \), so that there is a complete band gap in the range \(|E| < v|\beta|m_1|\). We choose \( E \) so that it lies within this gap.

The calculation of the beam shift proceeds along similar lines. The form of the planar eigenmodes in the \( z > 0 \) region is similar to Eq. (1), and evanescent for \( z < 0 \). Instead of using the boundary equation (3), we require the components of \( \mathbf{v}_{\text{out}} \) to be continuous at \( z = 0 \). The Fermi arc is found to be the line segment \(|k_x| < \sqrt{m_0}, k_y = E/v\). Finally, we construct beams similar to (6) and find the reflected beam displacement \( \Delta \).

The resulting plots of \( \Delta \) versus \( \mathbf{K} \perp \) are shown in Fig. 3. The behavior is highly similar to the single-cone case, despite key differences in the calculation—not only in the Hamiltonian, but also how the boundary conditions were implemented. There are now two Weyl cones, each with its own Fermi arc touching-point, and the \( \Delta \) vector field has a half-vortex structure in the vicinity of each touching-point. This behaviour persists even for larger values of \( E \) such that the Weyl cones merge into a single band, and a numerical fit shows the same inverse square root scaling around the touching-points. The winding direction is opposite in the two Weyl cones, consistent with the fact that they have opposite chiralities.

Discussion.—The easiest way to observe the predicted beam shift may be to use a classical Weyl medium, such as a microwave-scale photonic crystal of the sort implemented by Lu et al. [26]. In a microwave experiment, a phase array can be used to generate the incident beam, and a simple metal surface can serve as the reflecting surface [45]. In Ref. [26], the lattice constant is 13.4 mm and the Weyl points occur at frequency \( f \approx 11.3 \text{GHz} \), with \( v \approx 7 \times 10^7 \text{ms}^{-1} \). If we operate 5% above the Weyl point frequency (\( \delta f \approx 0.57 \text{GHz} \)), the Weyl cone section has radius \( K_W \approx 50 \text{m}^{-1} \). Employing a beam of momentum-space width \( \sigma_k \approx 5 \text{m}^{-1} \) (real-space width \( \approx 100 \text{mm} \)), with incident beam direction such that \(|\mathbf{K} \perp - \mathbf{K}_\alpha| \approx 15 \text{m}^{-1} \), allows the envelope to have negligible overlap with the boundary of the Weyl cone. The model of Eqs. (1)–(7) then predicts \(|\Delta| \approx 46 \text{mm} \), which should be easily observable.

In solid state systems, the structure of GH- and IF-like shifts of electron wavepackets provide a means of using bulk dynamics to probe the topological Fermi arc states in Weyl semimetals and related materials. For example, in a Weyl semimetal thin film bounded above and below by insulating media, a beam or traveling wavepacket will undergo repeated reflections off the two parallel surfaces; a straightforward calculation shows that the shifts accumulate rather than canceling out [12], producing an anomalous boost to the in-plane motion. In much the same way that GH-shifts have been predicted to directly contribute to two-terminal conductance in graphene \( p-n \) interfaces [12], the unusual Fermi arc induced shifts in Weyl media discussed in this paper may be detectable via the transport characteristics of Weyl thin films.

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Supplemental Material for
Fermi arc induced vortex structure in Weyl beam shifts

In this supplement, we discuss the winding behavior of the planar reflection coefficient $\phi$; derive analytical formulas for the shift of Gaussian beams in a Weyl medium, and compare them to numerical results; derive the eigenvectors, reflection coefficients, and beam shifts in the quadratic Hamiltonian model; and derive the accumulation of beam shifts in a thin film geometry. Unless otherwise specified, it is assumed that $v = 1$ and that $\theta_b$ is a constant.

A. Reflection phase winding around Fermi arc touching-point

As mentioned in the main text, the non-trivial behavior of the reflection phase near the Fermi arc touching-point can be understood by considering the limiting case $k^\pm_z \to 0$. For $k^\pm_z = 0$, the incident and reflected waves are linearly dependent and the total (envelope) wavefunction is

$$
\psi_{\text{tot}} = \frac{1}{\sqrt{2}} \left( 1 + e^{i\phi} \right) \begin{bmatrix} 1 \\ e^{i\alpha} \end{bmatrix}.
$$

(S1)

The boundary condition (Eq. 3 of main text) then implies

$$
(1 + e^{i\phi})(1 + e^{i(\alpha - \theta_b)}) = 0,
$$

(S2)

which can be satisfied by (i) $\phi = \pi$ and (ii) $\alpha - \theta_b = \pi \mod 2\pi$. Fig. S1 shows $\phi$ as a function of $\alpha$ parametrizing circular patches of different radius close to the boundary. As the boundary $|\vec{K}_\perp| = E$ is approached, $\phi$ becomes ill-defined and $\alpha = \theta_b - \pi$ defines the Fermi arc touching-point. This is consistent with the behavior shown in Fig. 1(b)–(c) of the main text.

To analytically derive the direction in which $\phi$ winds, consider momenta close to the boundary, such that $k^\pm_z = \pm \delta k_z$ where $\delta k_z$ is a small positive number. Let $q = \delta k_z/E$, and expand $\eta_-$ up to linear order in $q$:

$$
\eta_- = \sqrt{1 + q} \approx 1 + q, \quad \eta_-^2 \approx 1 + 2q.
$$

(S3)

Then the reflection coefficient is

$$
e^{i\phi} = \frac{1 + \eta_- e^{i\beta}}{\eta_- + e^{i\beta}} = -1 - \frac{2iq \sin \beta}{1 + \cos \beta} + O(q^2),
$$

(S4)

FIG. S1: Reflection phase $\phi$ versus $\alpha$ (the polar angle in the two-dimensional $k$-space of the incident beam), for $E = 1$, $\theta_b = \pi$, and different values of $|\vec{K}_\perp|$ close to the Weyl cone boundary. The Fermi arc touching-point occurs at $\alpha = 0$. Exactly at the boundary ($|\vec{K}_\perp| = E$), Eq. S2 states that $\phi = 0$ for all $\alpha \neq 0$ and is undefined at $\alpha = 0$. 
where $\beta = \alpha - \theta_b$. Hence, to leading order,

$$\sin \phi \approx -\frac{2q \sin \beta}{1 + \cos \beta}. \quad (S5)$$

At exactly $\beta = \pm \pi$, the denominator diverges and the approximation breaks down. For small angle deviations, $\beta = \pi + \delta \beta$, we find that $\sin \phi \approx 2q \sin(\delta \beta)$, so that $\delta \phi$ switches sign with $\delta \beta$. Note that $q > 0$ for the upper cone, and $q < 0$ for the lower cone, so the two cones have opposite windings.

### B. Simple derivation of gaussian beam shifts

Let $f$ be a function of the in-plane momenta $\vec{k}_\perp = (k_x, k_y)$. We take a gaussian beam of the form

$$\Psi = \frac{1}{2\pi \Delta_x \Delta_y} \int_{-\infty}^{\infty} dk_x dk_y \ e^{-\frac{(k_x-K_x)^2}{\Delta_x^2}} \ e^{-\frac{(k_y-K_y)^2}{\Delta_y^2}} \ e^{if(x,y,k_x,k_y)}e^{ik_x x+ik_y y}. \quad (S6)$$

To find the center of this beam, expand $f(x, y, k_x, k_y)$ about the mean wave-vector $\vec{K}_\perp = (K_x, K_y)$ as

$$f(x, y, k_x, k_y) \approx f(x, y, K_x, K_y) + \frac{\partial f}{\partial k_x} \bigg|_{\vec{K}_\perp} (k_x - K_x) + \frac{\partial f}{\partial k_y} \bigg|_{\vec{K}_\perp} (k_y - K_y). \quad (S7)$$

Here, only terms up to first order are retained. The integral (S6) can be written as

$$\Psi = C \int_{-\infty}^{\infty} dk_x \xi(k_x) e^{ik_x x} \int_{-\infty}^{\infty} dk_y \xi(k_y) e^{ik_y y} \quad (S8)$$

where

$$C = \frac{\exp[i f(x, y, K_x, K_y)]}{2\pi \Delta_x \Delta_y}, \quad \xi(k_x) = \exp(-\frac{(k_x - K_x)^2}{2\Delta_x^2}) \exp(i \frac{\partial f}{\partial k_x} \bigg|_{\vec{K}_\perp} (k_x - K_x)). \quad (S9)$$

Evaluating the integrals, we obtain

$$\Psi = \exp \left( i f + i K_x x + i K_y y \right) \exp \left[ \frac{\Delta_x^2}{2} \left( x + \frac{\partial f}{\partial k_x} \right)^2 \right] \exp \left[ -\frac{\Delta_y^2}{2} \left( y + \frac{\partial f}{\partial k_y} \right)^2 \right]. \quad (S10)$$

We find that the probability amplitude $|\Psi|^2$ is a gaussian in real space, centered at

$$\vec{R} = \left[ -\frac{\partial f}{\partial k_x}, -\frac{\partial f}{\partial k_y} \right]_{\vec{K}_\perp}. \quad (S11)$$

For the particular case of Eq. (1) of the main text, the two components of the incident beam are centered at $(0, 0)$ and $(-\partial_{k_x} \alpha, -\partial_{k_y} \alpha)_{\vec{K}_\perp=K_\perp}$ and the center of the beam is calculated by taking weighted averages over the pre-factors:

$$\vec{R}_i = \left( \frac{-\eta^2}{1 + \eta^2} \nabla_{\vec{K}_\perp} \alpha \right)_{\vec{K}_\perp=K_\perp} \quad (S12)$$

where, as in the main text,

$$\alpha = \tan^{-1} \left( \frac{k_y}{k_x} \right), \quad \eta_{\pm} = \sqrt{\frac{E - vk_z^2}{E + vk_z^2}}, \quad (S13)$$

$$k_z^\pm = \pm \sqrt{(E/v)^2 - |\vec{k}_\perp|^2}. \quad (S14)$$

Similarly, the reflected beam is centered at

$$\vec{R}_r = \left( \frac{1}{1 + \eta^2} \nabla_{\vec{K}_\perp} \phi - \frac{\eta^2}{1 + \eta^2} \nabla_{\vec{K}_\perp} (\alpha + \phi) \right)_{\vec{K}_\perp=K_\perp}. \quad (S15)$$

The shift is given by their difference:

$$\Delta(\vec{R}_\perp) = \left( -\nabla_{\vec{K}_\perp} \phi + \left( \frac{\eta^2}{1 + \eta^2} - \frac{\eta^2}{1 + \eta^2} \right) \nabla_{\vec{K}_\perp} \alpha \right)_{\vec{K}_\perp=K_\perp} \quad (S16)$$

Using the equality $\eta_+ = 1/\eta_-$ yields Eq. (6) of the main text.
C. Non-Gaussian beams and gauge invariance

In this section, we show that the same beam shift formulas can be derived even if the Gaussian envelope approximation is relaxed. This derivation also shows that even though the reflection coefficient $\phi$ depends on the gauge choice used in the definition of the eigenfunctions [e.g., Eq. (1) of the main text], the beam shift $\Delta$ is a physical quantity that is gauge invariant. The beam shifts are derived in terms of geometrical connections defined using the spinors of the incident and reflected beams.

Let $g(\vec{k}_\perp)$ be a real valued function peaked at $\vec{k}_\perp = \vec{K}_\perp$. The incident and reflected beam can be written as

$$\Psi_i(\vec{r}, \vec{K}_\perp) = \int d\vec{k}_\perp g(\vec{k}_\perp - \vec{K}_\perp) \psi_i(\vec{k}_\perp) e^{i\vec{k}_\perp \cdot \vec{r} + i\eta_+ z}$$

(S17)

$$\Psi_r(\vec{r}, \vec{K}_\perp) = \int d\vec{k}_\perp g(\vec{k}_\perp - \vec{K}_\perp) r(\vec{k}_\perp) \psi_r(\vec{k}_\perp) e^{i\vec{k}_\perp \cdot \vec{r} + i\eta_+ z}.$$  

(S18)

Here, $r(\vec{k}_\perp) = e^{i\phi(\vec{k}_\perp)}$ is the reflection coefficient. To track the peak of the wavepacket, we calculate the probability amplitude at the interface $z = 0$:

$$\left| \Psi_i(\vec{r})(\vec{k}_\perp, \vec{r}) \right|^2 = \int d\vec{k}_\perp d\vec{k}_\perp' g_F(\vec{k}_\perp, \vec{k}_\perp', \vec{K}_\perp) e^{i\theta_r(\vec{r})(\vec{k}_\perp, \vec{k}_\perp')}$$

(S19)

where

$$g_F(\vec{k}_\perp, \vec{k}_\perp', \vec{K}_\perp) = g(\vec{k}_\perp, \vec{k}_\perp') g(\vec{k}_\perp, \vec{K}_\perp'),$$

(S20)

$$\theta_i(\vec{k}_\perp, \vec{k}_\perp') = -i \log \left( \psi_i(\vec{k}_\perp) | \psi_i(\vec{k}_\perp') \right) + (\vec{k}_\perp - \vec{k}_\perp') \cdot \vec{r}$$

(S21)

$$\theta_r(\vec{k}_\perp, \vec{k}_\perp') = -i \log \left( \psi_r(\vec{k}_\perp) | \psi_r(\vec{k}_\perp') \right) + (\vec{k}_\perp - \vec{k}_\perp') \cdot \vec{r} + \phi(\vec{k}_\perp) - \phi(\vec{k}_\perp').$$

(S22)

We have used Dirac’s bra-ket notation to express the various $k$-space integrals. The peak of the probability amplitude in real space is the stationary point $\vec{R}$ determined by

$$\nabla_{\vec{k}_\perp} \theta_i(\vec{r})(\vec{k}_\perp, \vec{R}_i) |_{\vec{R}_i} = 0$$

(S23)

This gives the peaks of the incident and reflected beams:

$$\vec{R}_i = \vec{K}_\perp$$

(S24)

$$\vec{R}_r = \vec{K}_\perp - \nabla_{\vec{k}_\perp} \phi(\vec{k}_\perp) |_{\vec{K}_\perp}.$$  

(S25)

Here,

$$\vec{A}_i(\vec{k}_\perp) = i \left( \psi_i(\vec{k}_\perp) | \nabla_{\vec{k}_\perp} \psi_i(\vec{k}_\perp) \right)$$

(S26)

is the Berry connection for the incident (reflected) wave. By explicit calculation using the eigenstates of Weyl Hamiltonian, we obtain

$$\vec{A}_i(\vec{k}_\perp) = -\frac{\eta^2}{\eta^2_\perp + 1} \nabla_{\vec{k}_\perp} \alpha$$

(S27)

$$\vec{A}_r(\vec{k}_\perp) = -\frac{\eta^2}{\eta^2_\perp + 1} \nabla_{\vec{k}_\perp} \alpha.$$  

(S28)

The shift is then given by

$$\Delta(\vec{K}_\perp) = \vec{R}_r - \vec{R}_i = \vec{A}_r(\vec{K}_\perp) - \vec{A}_i(\vec{K}_\perp) - \nabla_{\vec{k}_\perp} \phi(\vec{k}_\perp) |_{\vec{K}_\perp}$$

(S29)

$$= \frac{\eta^2 - 1}{\eta^2_\perp + 1} \nabla_{\vec{k}_\perp} \alpha - \nabla_{\vec{k}_\perp} \phi.$$  

(S30)

We have used the fact that $\eta_- = 1/\eta_+$. 


Now consider boundary conditions of the form
\[
M(\vec{k}_\perp)[\psi_i(\vec{k}_\perp) + r(\vec{k}_\perp)\psi_r(\vec{k}_\perp)] = 0,
\]  
where \(M(\vec{k}_\perp)\) is a \(2 \times 2\) matrix characterizing the boundary. The reflection phase can be written as
\[
\phi(\vec{k}_\perp) = i \log[M\psi_r(\vec{k}_\perp)] - i \log[M\psi_i(\vec{k}_\perp)].
\]  
The derivative is
\[
\nabla \phi = i \frac{\nabla M\psi_r}{M\psi_r} - i \frac{\nabla M\psi_i}{M\psi_i}.
\]  
We define two new objects
\[
\psi^\theta_{i(r)} = M\psi_{i(r)},
\]  
which are normalized as \(\langle \psi^\theta_{i(r)}(\vec{k}_\perp) | \psi^\theta_{i(r)}(\vec{k}_\perp) \rangle = 1\). Then Eq. \(\text{(S33)}\) reads
\[
\nabla \phi = \left\langle \psi^\theta_{i(r)}(\vec{k}) | i \nabla_E \psi^\theta_{i(r)}(\vec{k}) \right\rangle - \left\langle \psi^\theta_{i(r)}(\vec{k}) | i \nabla_E \psi^\theta_{i(r)}(\vec{k}) \right\rangle,
\]  
which is the difference between two geometric (non-Berry) connections. These geometric connections are related to the parallel transport determined by the boundary condition.

Although both \(\vec{R}_i\) and \(\vec{R}_r\) are gauge dependent (which amounts to a different choice of origin for the individual waves), their difference is gauge invariant. When
\[
\psi_{i(r)}(\vec{k}_\perp) \to \psi_{i(r)}(\vec{k}_\perp) \exp(i\chi_{i(r)}(\vec{k}_\perp)),
\]  
we find that
\[
\vec{A}_{i(r)}(\vec{k}_\perp) \to \vec{A}_{i(r)}(\vec{k}_\perp) - \nabla_{\vec{k}_\perp} \chi(\vec{k}_\perp)
\]  
\[
\nabla_{\vec{k}_\perp} \phi(\vec{k}_\perp) \to \nabla_{\vec{k}_\perp} \phi(\vec{k}_\perp) - \nabla_{\vec{k}_\perp} \chi_r(\vec{k}_\perp) + \nabla_{\vec{k}_\perp} \chi_i(\vec{k}_\perp).
\]  
Hence, the shift given by Eq. \(\text{(S30)}\) is gauge invariant.

The shift itself is given by
\[
\vec{\Delta}(\vec{k}_\perp) = \left[ \frac{\eta^2 - 1}{\eta^2 + 4} \nabla_{\vec{k}_\perp} \alpha(\vec{k}_\perp) - \nabla_{\vec{k}_\perp} \phi(\vec{k}_\perp) \right]_{\vec{k}_\perp = \vec{k}_\perp}.
\]  
The reflection coefficient is given by
\[
r = e^{i\phi} = -\frac{1 + \eta_- e^{i\beta}}{\eta_- + e^{i\beta}},
\]  
where \(\beta(\vec{k}_\perp) = \alpha(\vec{k}_\perp) - \theta_b\). Therefore,
\[
\nabla \phi = -\frac{i}{r(\vec{k}_\perp)} \nabla r(\vec{k}_\perp)
\]  
\[
= -\frac{i}{r} \frac{ie^{i\beta}(\eta_-^2 - 1)(\nabla \beta) + (e^{2i\beta} - 1)(\nabla \eta_-)}{(\eta_- + e^{i\beta})^2}
\]  
\[
= \frac{(\eta_-^2 - 1)(\nabla \beta) + 2 \sin \beta \nabla \eta_-}{1 + \eta_-^2 + 2 \eta_- \cos \beta},
\]  
and
\[
\nabla \eta_- = -\frac{E}{\eta(E + k_z^2/|\vec{k}_\perp|^2)} \frac{\vec{k}_\perp}{|\vec{k}_\perp|^2}
\]  
\[
\nabla \beta = \nabla \alpha = (-k_y, k_x)/|\vec{k}_\perp|^2.
\]  
Putting everything together, we obtain the explicit formula
\[
\vec{\Delta}(\vec{k}_\perp) = \left[ \frac{\eta^2(\vec{k}_\perp) - 1}{\eta^2(\vec{k}_\perp) + 1} + \frac{k_z(\vec{k}_\perp)}{E + |\vec{k}_\perp| \cos \beta(\vec{k}_\perp)} \right] \frac{1}{|\vec{k}_\perp|^2} \frac{-k_y, k_x}{|\vec{k}_\perp|^2} + \frac{E \sin \beta(\vec{k}_\perp)}{|\vec{k}_\perp|E + |\vec{k}_\perp|^2 \cos \beta(\vec{k}_\perp)} \frac{\vec{k}_\perp}{|\vec{k}_\perp|^2}.
\]
FIG. S2: (a) Zoomed-in image of the shift vector $\Delta$ close to the Fermi arc touching-point. (b) Log-log plot of the magnitude of the shift vector in different directions away from the Fermi arc touching-point. The straight lines show numerical least squares fits, which indicate that the magnitudes indeed scale as $|\Delta| \sim |\delta \vec{K}_{\perp}|^{-1/2}$.

FIG. S3: Momentum dependence of the vector fields (a) $-\nabla \phi$ and (b) $\nabla \alpha$, for $\theta_b = \pi/2$. The half-vortex structure near the Fermi arc touching-point is evidently due to the gradient of $\phi$. Note that $-\nabla \phi$ and $\nabla \alpha$ circulate in opposite directions around the origin.

D. Inverse square root scaling of the beam displacement

Fig. S2 shows the inverse square-root scaling of the magnitude of the shift near the Fermi arc touching-point $\vec{K}_{fa}$. As mentioned in the main text, the vortex structure near the Fermi arc touching-point is due to the gradient of the reflection phase, as explicitly shown in Fig. S3. The scaling can be verified by expanding the gradient of reflection phase about the Fermi arc touching-point. The latter is derived by requiring the penetration constant of the Fermi arc surface state to vanish, which yields

$$\vec{K}_{fa} = \left(-E \cos(\theta_b), -E \sin(\theta_b) \right).$$  

We define $\vec{k}_{\perp} = \vec{K}_{fa} + \delta \vec{k}$, where $\delta \vec{k} = (\delta k_x, \delta k_y)$ is the wave-vector measured from the Fermi arc touching-point. The shift $\Delta$, given by (S46), is to be expanded up to linear order in $\delta \vec{k}$.

We can expand $k_z$ in linear order as follows:

$$k_z \approx -\sqrt{2E[\delta k_x \cos(\theta_b) + \delta k_y \sin(\theta_b)]} = -Eq$$  

where

$$q = \sqrt{\frac{2[\delta k_x \cos(\theta_b) + \delta k_y \sin(\theta_b)]}{E}}.$$  

(S49)
And from the definition of \( \alpha \) we have
\[
\cos(\alpha) \approx \frac{p \sin(\theta_b)}{E} - \cos(\theta_b), \quad \sin(\alpha) \approx \frac{p \cos(\theta_b)}{E} - \sin(\theta_b),
\]  
(S50)

where
\[
p = \delta k_x \sin(\theta_b) - \delta k_y \cos(\theta_b).
\]  
(S51)

and this gives
\[
\sin \beta \approx -\frac{p}{E}, \quad \cos \beta \approx -1
\]  
(S52)

where \( \beta = \alpha - \theta_b \). Now, from (S43) and (S46) we have
\[
\nabla \phi \approx -\frac{-q}{(1 - \sqrt{1 - q^2}) E^2 (1 - q^2)} (-\delta k_y + E \sin \theta_b, \delta k_x - E \cos \theta_b)
\]  
\[\quad - \frac{\beta}{E^2 (\sqrt{1 - q^2} - (1 - q^2))} \frac{1}{E q} (\delta k_x - E \cos \theta_b, \delta k_y - E \sin \theta_b).
\]  
(S53)

which expanding up to linear order in \( q \) simplifies to
\[
\nabla \phi \approx -\frac{2}{E^2} \left(\frac{1}{1 + q} - \delta k_y + E \sin \theta_b, \delta k_x - E \cos \theta_b\right) \frac{2p}{E^3 q^3}(\delta k_x - E \cos \theta_b, \delta k_y - E \sin \theta_b)
\]  
(S54)

Noting that \( p/q^3 \sim 1/q \), it is readily seen that the leading order term scales as \( 1/q \).

E. Derivation of the shift for the quadratic Hamiltonian

The quadratic Hamiltonian
\[
H = v \begin{bmatrix} -k_y & \gamma(k_x^2 - m) - i k_z \\ \gamma(k_x^2 - m) + i k_z & k_y \end{bmatrix}
\]
(S55)

has dispersion relation
\[
E = \pm v \sqrt{(k_x^2 - m)^2 + k_y^2 + k_z^2}.
\]  
(S56)

This can exhibit either paired Weyl points or complete band-gap, depending on the choice of \( m \). In the region \( z > 0 \), we set \( m = m_0 > 0 \), so that there is a pair of Weyl points at \( k = [ \pm \sqrt{m_0}, 0, 0] \); for small values of \( |E| \), there are two distinct Weyl cones which merge as \( |E| \) increases. In the region \( z < 0 \), we set \( m = -m_1 < 0 \) so that there is a complete band gap in the range \( |E| < \gamma |m_1| \). In the Weyl medium, for incident and reflected plane waves, the eigenvectors reads
\[
\Psi_i = \frac{1}{\sqrt{1 + \eta^2}} \left[ \begin{array}{c} e^{-i\alpha_-} \\ \eta \end{array} \right], \quad \Psi_r = \frac{e^{i\phi}}{\sqrt{1 + \eta^2}} \left[ \begin{array}{c} e^{-i\alpha_+} \\ \eta \end{array} \right],
\]  
(S57)

where
\[
\alpha_\pm = \tan^{-1} \left( \frac{k_x^\pm}{\gamma(k_x^2 - m_0)} \right), \quad \eta = \frac{E + v k_y}{E - v k_y},
\]  
(S58)

where as before \( k_z < 0 (k_z > 0) \) branch chosen for the incident (reflected) wave.

The reflection amplitude \( e^{i\phi} \) is calculated by matching the total incident and reflected envelope functions at \( z = 0 \) plane with the evanescent wave in the band-gap medium given by
\[
\psi_i = \frac{t}{N_t} \left[ \begin{array}{c} E/v - k_y \\ \gamma(k_x^2 + m_1) + \kappa \end{array} \right] e^{ik_z x + ik_y y + \kappa z},
\]  
(S59)
where $\kappa = \sqrt{\gamma^2(k_x^2 + m_1)^2 + k_y^2 - (E/v)^2}$ is the inverse decay length of the evanescent field, $t$ is the transmission coefficient and $N_\ell$ is a normalization factor. Equating $\psi_i + \psi_r = \psi_t$ at $z = 0$ for all $x$ and $y$, the reflection coefficient $e^{i\phi}$ is calculated:

$$e^{i\phi} = \frac{\eta(E - vk_y) - [v\gamma(k_x^2 + m_1) + vk]\eta e^{-i\alpha} - \eta(E - vk_y)}{e^{-i\alpha + [v\gamma(k_x^2 + m_1) + vk] - \eta(E - vk_y)}}$$

(S60)

The spatial shift in the reflected beam can be calculated in a manner similar to the previous case, which gives

$$\Delta(\vec{K}_\perp) = \left(-\nabla_{\vec{k}_\perp} \phi - \frac{2}{\eta^2 + 1} \nabla_{\vec{k}_\perp} \alpha_\perp\right).$$

(S61)

The results are shown in Fig. 3 of the main text.

F. Consecutive reflections in a thin film geometry

Consider a film of Weyl medium of thickness $2L$, within the space $|z| < L$ bounded above and below by a gapped medium. In this geometry, a beam in the Weyl medium will reflect repeatedly off the two parallel surfaces, much like a bouncing-ball trajectory within a waveguide, as shown in Fig. S4(a).

As the beam reflects consecutively off the bottom and top surfaces of the film, the beams displacements accumulate instead of cancelling. Fig. S4(b)–(c) plots the numerically-calculated beam shift over two consecutive reflections, using the quadratic Weyl Hamiltonian with continuity boundary conditions at $|z| = L$. The shift from the upper surface can be found by replacing $k_{\pm} \rightarrow k_{\mp}$ and $\kappa \rightarrow -\kappa$ in the previous equations.

In explicit terms, we take the Hamiltonian

$$H = \begin{bmatrix} -k_y & (k_x^2 - m) - ik_z \\ (k_x^2 - m) + ik_z & k_y \end{bmatrix}, \quad m = \begin{cases} m_0 > 0, \ |z| \leq L \text{ (Weyl medium)} \\
-m_1, \ |z| > L \text{ (gapped medium)} \end{cases}$$

(S62)

Inside the Weyl semimetal, we look for states $\propto e^{i\vec{k}_\perp \cdot \vec{r}_\perp + ik_z}$ with $\vec{k}_\perp = (k_x, k_y)$ and $\vec{r}_\perp = (x, y)$. We focus on a fixed energy $E$, whose bulk states read

$$\psi_w(\vec{k}_\perp, \vec{r}_\perp, z) = \frac{1}{\sqrt{1 + \eta^2}} \left[e^{i\alpha_\perp} \eta \right] e^{i\vec{k}_\perp \cdot \vec{r}_\perp + ik_z}$$

$$\eta = \sqrt{\frac{E + k_y}{E - k_y}}$$

$$\alpha_\perp = \arctan \left(\frac{k_z}{k_x^2 - m_0}\right)$$

$$k_z^\pm = \pm \sqrt{E^2 - (k_x^2 - m_0)^2 - k_y^2}.$$
Here, $\psi_{\pm}^w(\vec{k}_\perp, \vec{r}_\perp, z)$ propagates to the upper/lower surface $z = \pm L$. We assume that the energy lies within the gap of the external medium. In the $z > L$ ($z < -L$) region, the solution has the decaying form

$$
\psi_{\pm}^{evs}(\vec{k}_\perp, \vec{r}_\perp, z) \propto \left[ \frac{E - k_y}{k^2_x + m_1 \mp \kappa} \right] e^{ik_x x + ik_y \mp \kappa z}.
$$

This wave function is not normalized, and due to energy conservation,

$$
\kappa = \sqrt{(k^2_x + m_1)^2 - (k^2_y - m_0)^2}.
$$

At the interface $z = \pm L$, the incident wave $\psi_{\pm}^w$, the reflected wave $\psi_{\mp}^w$, and the evanescent wave $\psi_{\pm}^{evs}$ must match:

$$
\psi_{\pm}^w(\vec{k}_\perp, \vec{r}_\perp, \pm L) + r_{\pm}(\vec{k}_\perp)\psi_{\mp}^w(\vec{k}_\perp, \vec{r}_\perp, \pm L) = t_{\pm}(\vec{k}_\perp)\psi_{\pm}^{evs}(\vec{k}_\perp, \vec{r}_\perp, \pm L),
$$

where $r_{\pm}(\vec{k}_\perp)$ and $t_{\pm}(\vec{k}_\perp)$ are reflection and transmission coefficients at $z = \pm L$. By multiplying $(k^2_x + m_1 \mp \kappa, -E + k_y)$ to both sides of the equation, we obtain the total reflection coefficient

$$
r_{\pm}(\vec{k}) = -\frac{e^{i\alpha_{\pm}(k^2_x + m_1 \mp \kappa)} - \sqrt{E^2 - k_y^2}}{e^{i\alpha_{\pm}(k^2_x + m_1 \mp \kappa)} - \sqrt{E^2 - k_y^2}} \equiv -\exp[i\phi_{\pm}(\vec{k}_\perp)].
$$

For wave packets composed by $\psi_{\pm}^w$, the shift between the total reflected wave and the incident wave in real space at the interface $z = \pm L$ is

$$
\vec{\Delta}_{\pm}(\vec{k}_\perp) = \vec{A}_\mp(\vec{k}_\perp) - \vec{A}_\pm(\vec{k}_\perp) - \nabla_{\vec{k}_\perp} \phi_{\pm}(\vec{k}_\perp)
$$

where $\vec{A}_{\pm}(\vec{k}_\perp) = i\langle \psi_{\pm}^w(\vec{k}_\perp) | \nabla_{\vec{k}_\perp} | \psi_{\mp}^w(\vec{k}_\perp) \rangle$ is the Berry connection for the incident/reflected state. Since $\vec{A}_+(\vec{k}_\perp) - \vec{A}_-(\vec{k}_\perp)$ and $\vec{A}_-(\vec{k}_\perp) - \vec{A}_+(\vec{k}_\perp)$ cancel, after two consecutive reflections between opposite surfaces $z = \pm L$, the total shift is

$$
\vec{\Delta}_{tot}(\vec{k}_\perp) = \vec{\Delta}_+(\vec{k}_\perp) + \vec{\Delta}_-(\vec{k}_\perp) = -\nabla_{\vec{k}_\perp} \phi_+(\vec{k}_\perp) - \nabla_{\vec{k}_\perp} \phi_-(\vec{k}_\perp).
$$

The resulting map of the shift is shown in Fig. S4.