A Morley–Wang–Xu Element Method for a Fourth Order Elliptic Singular Perturbation Problem

Xuehai Huang · Yuling Shi · Wenqing Wang

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Abstract
A Morley–Wang–Xu (MWX) element method with a simply modified right hand side is proposed for a fourth order elliptic singular perturbation problem, in which the discrete bilinear form is standard as usual nonconforming finite element methods. The sharp error analysis is given for this MWX element method. And the Nitsche’s technique is applied to the MWX element method to achieve the optimal convergence rate in the case of the boundary layers. An important feature of the MWX element method is solver-friendly. Based on a discrete Stokes complex in two dimensions, the MWX element method is decoupled into one Lagrange element method of Poisson equation, two Morley element methods of Poisson equation and one nonconforming $P_1–P_0$ element method of Brinkman problem, which implies efficient and robust solvers for the MWX element method. Some numerical examples are provided to verify the theoretical results.

Keywords Fourth order elliptic singular perturbation problem · Morley–Wang–Xu element · Decoupling · Boundary layers · Fast solver

Mathematics Subject Classification 65N12 · 65N22 · 65N30 · 65F08

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Wanqing Wang
wangwenqing81@hotmail.com

Xuehai Huang
huang.xuehai@sufe.edu.cn

Yuling Shi
shiyuling@163.sufe.edu.cn

1 School of Mathematics, Shanghai University of Finance and Economics, Shanghai 200433, China
2 Department of Basic Teaching, Wenzhou Business College, Wenzhou 325035, China
1 Introduction

In this paper, we shall apply the Morley–Wang–Xu (MWX) element [28,40] to discretize the fourth order elliptic singular perturbation problem

\[
\begin{aligned}
\varepsilon^2 \Delta^2 u - \Delta u &= f \quad \text{in } \Omega, \\
u = \partial_n u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

(1.1)

where \( \Omega \subset \mathbb{R}^d \) with \( d \geq 2 \) is a convex and bounded polytope, \( f \in L^2(\Omega) \), \( n \) is the unit outward normal to \( \partial \Omega \), and \( \varepsilon \) is a real small and positive parameter.

The MWX element is the simplest finite element for fourth order problems, as it has the fewest degrees of freedom on each element. The generalization of the MWX element to any star-shaped polytope in any dimension is recently developed in the context of the virtual element in [13]. However it is divergent to discretize problem (1.1) by the MWX element in the following standard way when \( \varepsilon \) is very close to 0 [30,36]: find \( u_{h0} \in V_{h0} \) such that

\[
\varepsilon^2 (\nabla_h^2 u_{h0}, \nabla_h^2 v_h) + (\nabla_h u_{h0}, \nabla_h v_h) = (f, v_h) \quad \forall \; v_h \in V_{h0},
\]

(1.2)

where \( V_{h0} \) is the global MWX element space. To this end, a modified MWX element method was advanced in [37,42] to deal with this divergence by replacing \( (\nabla_h u_{h0}, \nabla_h v_h) \) with \( (\nabla \Pi_h u_{h0}, \nabla \Pi_h v_h) \), where \( \Pi_h \) is an interpolation operator from \( V_{h0} \) to some lower-order \( H^1 \)-conforming finite element space. Instead of introducing the interpolation operator, the combination of the MWX element and the interior penalty discontinuous Galerkin formulation [2] is proposed to discretize problem (1.1) in [43]. Both modified Morley element methods in [37,42,43] uniformly converge with respect to the parameter \( \varepsilon \).

Apart from the MWX element, there are many other \( H^2 \)-nonconforming elements constructed to design robust numerical methods for problem (1.1), including \( C^0 \) \( H^2 \)-nonconforming elements in [8–10,24,30,34,35,38,39,45] and fully \( H^2 \)-nonconforming elements in [14,15,35]. And a \( C^0 \) interior penalty discontinuous Galerkin (IPDG) method with the Lagrange element space was devised for problem (1.1) in [3,19]. We refer to [1,33,47] for the \( H^2 \)-conforming finite element methods of problem (1.1), which usually suffer from large number of degrees of freedom.

To design a simple finite element method for problem (1.1), we still employ the MWX element space and the standard discrete bilinear formulation as the left hand side of the discrete method (1.2) in this paper. We simply replace the right hand side \( (f, v_h) \) by \( (f, P_h v_h) \), where \( P_h \) is the \( H^1 \)-orthogonal projector onto the \( H^1 \)-conforming \( \ell \)th order Lagrange element space \( W_h \) with \( \ell = 1, 2 \). In a word, we propose the following robust MWX element method for problem (1.1): find \( u_{h0} \in V_{h0} \) such that

\[
\varepsilon^2 (\nabla_h^2 u_{h0}, \nabla_h^2 v_h) + (\nabla_h u_{h0}, \nabla_h v_h) = (f, P_h v_h) \quad \forall \; v_h \in V_{h0},
\]

(1.3)

The stiffness matrix of the discrete method (1.3) can be assembled in a standard way, which is sparser than that of the discrete method in [43]. After establishing the interpolation error estimate and consistency error estimate, the optimal convergence rate \( O(h) \) of the energy error is achieved. And the discrete method (1.3) possesses the sharp and uniform convergence rate \( O(h^{1/2}) \) of the energy error in consideration of the boundary layers.

An important feature of the discrete method (1.3) is solver-friendly. First the discrete method (1.3) is equivalent to finding \( w_h \in W_h \) and \( u_{h0} \in V_{h0} \) such that

\[
(\nabla w_h, \nabla \chi_h) = (f, \chi_h) \quad \forall \; \chi_h \in W_h, \quad (1.4)
\]

\[
\varepsilon^2 a_h(u_{h0}, v_h) + b_h(u_{h0}, v_h) = (\nabla w_h, \nabla v_h) \quad \forall \; v_h \in V_{h0}. \quad (1.5)
\]
Especially in two dimensions, thanks to the relationship between the Morley element space $V_{h0}$ and the vectorial nonconforming $P_1$ element space $V_{h0}^{CR}$ (cf. [18, Theorem 4.1]), the discrete method (1.5) can be decoupled into two Morley element methods of Poisson equation and one nonconforming $P_1$-$P_0$ element method of Brinkman problem, i.e., find $(z_h, \phi_h, p_h, w_h) \in V_{h0} \times V_{h0}^{CR} \times Q_h \times V_{h0}$ such that
\begin{align}
\text{(1.6a)} & \quad (\nabla \times z_h, \nabla \times u_h) = (\nabla w_h, \nabla \cdot v_h) \quad \forall v_h \in V_{h0}, \\
\text{(1.6b)} & \quad (\phi_h, \psi_h) + \varepsilon^2 (\nabla \times \phi_h, \nabla \times \psi_h) + (\nabla \cdot \psi_h, p_h) = (\nabla \times z_h, \nabla \cdot \psi_h) \quad \forall \psi_h \in V_{h0}^{CR}, \\
\text{(1.6c)} & \quad (\nabla \phi_h, \nabla \psi_h) = 0 \quad \forall \psi_h \in Q_h, \\
\text{(1.6d)} & \quad (\nabla \times u_h, \nabla \times \chi_h) = (\phi_h, \nabla \times \chi_h) \quad \forall \chi_h \in V_{h0}.
\end{align}

When $\varepsilon$ is small, the discrete method (1.5) can be easily solved by the conjugate gradient (CG) method with the auxiliary space preconditioner [46]. The decoupling (1.6a)–(1.6d) will induce efficient and robust solvers for the MWX element method (1.3) for large $\varepsilon$. The Lagrange element method of Poisson equation (1.4), and the Morley element methods of Poisson equation (1.6a) and (1.6d) can be solved by the CG method with the auxiliary space preconditioner, in which the $H^1$ conforming linear element discretization on the same mesh for the Poisson equation can be adopted as the auxiliary problem. And the algebraic multigrid (AMG) method is used to solve the auxiliary problem. As for the nonconforming $P_1$-$P_0$ element methods of Brinkman problem (1.6b)–(1.6c), we can use the block-diagonal preconditioner in [7,27,31] or the approximate block-factorization preconditioner in [11], which are robust with respect to the mesh size $h$. The resulting fast solver of the MWX element method (1.3) also works for the shape-regular unstructured meshes.

When $\varepsilon$ is close to zero, the uniform convergence rate $O(h^{l/2})$ of the energy error of the discrete method (1.3) is sharp but not optimal, where the optimal convergence rate should be $O(h^\ell)$ for $\ell = 1, 2$. To promote the convergence rate in the case of the boundary layers, we apply the Nitsche’s technique in [24] to the discrete method (1.3), i.e. impose the boundary condition $\partial_n u = 0$ weakly by the penalty technique [2]. The optimal error analysis is present for the resulting discrete method, whose convergence rate is uniform with respect to the perturbation parameter $\varepsilon$ when $\varepsilon$ approaches zero. Similarly, as (1.6a)–(1.6d), the discrete method with Nitsche’s technique on the boundary can also be decoupled into one Lagrange element method of Poisson equation, two Morley element methods of Poisson equation and one nonconforming $P_1$-$P_0$ element method with Nitsche’s technique of Brinkman problem, which is solver-friendly.

The rest of this paper is organized as follows. Some notations, connection operators and interpolation operators are shown in Sect. 2. In Sect. 3, we develop and analyze the MWX element method, and the MWX element method with Nitsche’s technique is devised and analyzed in Sect. 4. Section 5 focuses on the equivalent decoupling of the MWX element methods. Finally, some numerical results are given in Sect. 6 to confirm the theoretical results.

2 Connection Operators and Interpolation Operators

2.1 Notation

Given a bounded domain $G \subset \mathbb{R}^d$ and a non-negative integer $m$, let $H^m (G)$ be the usual Sobolev space of functions on $G$, and $H^m (G; \mathbb{X})$ the Sobolev space of functions taking values in the finite-dimensional vector space $\mathbb{X}$ for $\mathbb{X}$ being $\mathbb{R}^d$ or $\mathbb{M}$, where $\mathbb{M}$ is the space of all $d \times d$ tensors. The corresponding norm and semi-norm are denoted respectively by $\| \cdot \|_{m,G}$
and $| \cdot |_{m,G}$. Let $(\cdot, \cdot)_G$ be the standard inner product on $L^2(G)$ or $L^2(G; \overline{\mathbb{X}})$. If $G$ is $\Omega$, we abbreviate $\| \cdot \|_{m,G}, | \cdot |_{m,G}$ and $(\cdot, \cdot)_G$ by $\| \cdot \|_m, | \cdot |_m$ and $(\cdot, \cdot)$, respectively. Let $H^0_0(G)$ be the closure of $C^\infty_0(G)$ with respect to the norm $\| \cdot \|_{m,G}$. Let $\mathbb{P}_m(G)$ stand for the set of all polynomials in $G$ with the total degree no more than $m$, and $\mathbb{P}_m(G; \mathbb{R}^d)$ the vectorial version of $\mathbb{P}_m(G)$. As usual, $|G|$ denotes the measure of a given open set $G$. For any finite set $S$, denote by $\#S$ the cardinality of $S$.

We partition the domain $\Omega$ into a family of shape regular simplicial grids $T_h$ (cf. [4,16]) with $h := \max_{K \in T_h} h_K$ and $h_K := \text{diam}(K)$. Let $\mathcal{F}_h$ be the union of all $(d-1)$-dimensional faces of $T_h$, $\mathcal{F}^i_h$ the union of all interior $(d-1)$-dimensional faces of the triangulation $T_h$, and $\mathcal{F}^\partial_h := \mathcal{F}_h \setminus \mathcal{F}^i_h$. Similarly, let $\mathcal{E}_h$ be the union of all $(d-2)$-dimensional faces of $T_h$, $\mathcal{E}^i_h$ the union of all interior $(d-2)$-dimensional faces of the triangulation $T_h$, and $\mathcal{E}^\partial_h := \mathcal{E}_h \setminus \mathcal{E}^i_h$. Set

$$
\mathcal{F}^i(K) := \{ F \in \mathcal{F}^i_h : F \subset \partial K \}, \quad \mathcal{F}^\partial(K) := \{ F \in \mathcal{F}^\partial_h : F \subset \partial K \},
$$

$$
\mathcal{E}(K) := \{ e \in \mathcal{E}_h : e \subset \partial K \}.
$$

For each $K \in T_h$, denote by $n_K$ the unit outward normal to $\partial K$. Without causing any confusion, we will abbreviate $n_K$ as $n$ for simplicity. For each $F \in \mathcal{F}_h$, denote by $h_F$ its diameter and fix a unit normal vector $n_F$ such that $n_F = n_K$ if $F \in \mathcal{F}^\partial_h$. In two dimensions, i.e. $d = 2$, we use $t_F$ to denote the unit tangential vector of $F$ if $F \in \mathcal{F}^\partial_h$, and abbreviate it as $t$ for simplicity. For $s \geq 1$, define

$$
H^s(T_h) := \{ v \in L^2(\Omega) : v|_K \in H^s(K) \quad \forall \ K \in T_h \}.
$$

For any $v \in H^s(T_h)$, define the broken $H^s$ norm and seminorm

$$
\| v \|_{s,h}^2 := \sum_{K \in T_h} \| v \|_{s,K}^2, \quad | v |_{s,h}^2 := \sum_{K \in T_h} | v |_{s,K}^2.
$$

For any $v \in H^2(T_h)$, introduce some other discrete norms

$$
\| v \|_{2,h}^2 := \| v \|_{2,h}^2 + \sum_{F \in \mathcal{F}^\partial_h} h_F^{-1} \| \partial_n v \|_{0,F}^2,
$$

$$
\| v \|_{2,h}^2 := \epsilon^2 \| v \|_{2,h}^2 + | v |_{1,h}^2, \quad \| v \|_{2,h}^2 := \epsilon^2 \| v \|_{2,h}^2 + | v |_{1,h}^2.
$$

For any face $F \in \mathcal{F}_h$, set

$$
\omega_F := \text{interior} \left( \bigcup_{F \subset \partial K} \overline{K} \right).
$$

For any simplex $K \in T_h$, denote

$$
T_K := \{ K' \in T_h : \overline{K'} \cap \overline{K} \neq \emptyset \}, \quad \omega_K := \text{interior} \left( \bigcup_{K' \in T_K} \overline{K'} \right),
$$

$$
\omega_K^2 := \text{interior} \left( \bigcup_{K' \in T_h} \overline{K'} \cap \overline{K} \neq \emptyset \right).
$$

Let $\nabla$ and $\text{div}$ be the gradient operator and divergence operator respectively. In two dimensions, let $\text{curl} v := (\partial_y v, -\partial_x v)^T$. Discrete differential operators $\nabla_h$, $\text{curl}_h$ and $\text{div}_h$ are defined as the elementwise counterparts of $\nabla$, $\text{curl}$ and $\text{div}$ associated with $T_h$ respectively. Throughout this paper, we also use “$\lesssim \cdots$” to mean that “$\leq C \cdots$”, where $C$ is a generic
positive constant independent of \( h \) and the parameter \( \varepsilon \), which may take different values at different appearances.

Moreover, we introduce averages and jumps on \((d - 1)\)-dimensional faces as in [25]. Consider two adjacent simplices \( K^+ \) and \( K^- \) sharing an interior face \( F \). Denote by \( n^+ \) and \( n^- \) the unit outward normals to the common face \( F \) of the simplices \( K^+ \) and \( K^- \), respectively. For a scalar-valued or vector-valued function \( v \), write \( v^+ := v|_{K^+} \) and \( v^- := v|_{K^-} \). Then define the average and jump on \( F \) as follows:

\[
\{v\} := \frac{1}{2}(v^+ + v^-), \quad [v] := v^+ n_F \cdot n^+ + v^- n_F \cdot n^-.
\]

Since \( n_F \cdot n^+ = -n_F \cdot n^- = \pm 1 \), it holds \( [v]|_F := (v^+ - v^-) n_F \cdot n^+ \). On a face \( F \) lying on the boundary \( \partial \Omega \), the above terms are defined by

\[
\{v\} := v, \quad [v] := v n_F \cdot n.
\]

Associated with the partition \( T_h \), the global Morley–Wang–Xu (MWX) element space \( \tilde{V}_h \) consists of all piecewise quadratic functions on \( T_h \) such that, their integral average over each \((d - 2)\)-dimensional face of elements in \( T_h \) are continuous, and their normal derivatives are continuous at the barycentric point of each \((d - 1)\)-dimensional face of elements in \( T_h \) (cf. [28,40,41]). And define

\[
V_h := \left\{ v \in \tilde{V}_h : \int_E v \, ds = 0 \quad \forall \ e \in E^3_h \right\},
\]

\[
V_{h0} := \left\{ v \in V_h : \int_F \partial_n v \, ds = 0 \quad \forall \ F \in F^3_h \right\}.
\]

Notice that we do not impose the boundary condition \( \int_F \partial_n v \, ds = 0 \) in the finite element space \( V_h \). Due to Lemma 4 in [40], we have

\[
\int_F [\nabla v_h] \, ds = 0 \quad \forall \ v_h \in V_h, \ F \in F^3_h, \quad (2.1)
\]

\[
\int_F [\nabla F v_h] \, ds = 0 \quad \forall \ v_h \in V_h, \ F \in F^3_h, \quad (2.2)
\]

\[
\int_F [\nabla v_h] \, ds = 0 \quad \forall \ v_h \in V_{h0}, \ F \in F^3_h, \quad (2.3)
\]

where the surface gradient \( \nabla F v_h := \nabla v_h - \partial_n F v_h n_F \).

### 2.2 Connection Operators

In this subsection we will introduce some operators to connect the Lagrange element space and the MWX element space for analysis. Let the Lagrange element space

\[
W_h := \left\{ v \in H^1_0(\Omega) : v|_K \in P_\ell(K) \quad \forall \ K \in T_h \right\}
\]

with \( \ell = 1 \) or 2. Define a connection operator \( E^L_h : V_h \to W_h \) with \( \ell = 2 \) as follows: Given \( v_h \in V_h \), \( E^L_h v_h \in W_h \) is determined by

\[
N(E^L_h v_h) := \frac{1}{|T_N|} \sum_{K \in T_N} N(v_h|_K)
\]

where \( \#T_N \) is the number of elements in \( T_N \).
for each interior degree of freedom $N$ of the space $W_h$, where $T_N \subset T_h$ denotes the set of simplices sharing the degree of freedom $N$. By the weak continuity of $V_{h0}$ and $V_h$ and the techniques adopted in [5,36], we have for any $s = 1, 2, 0 \leq m \leq s$ and $j = 0, 1, 2$ that
\[
|v_h - E_h^L v_h|_{m,K} \lesssim h^{s-m} |v_h|_{s,\omega_K} \quad \forall \ v_h \in V_{h0},
\]
\[
\|v_h - E_h^L v_h\|_{0,K} + h^K|v_h - E_h^L v_h|_{1,K} \lesssim h_K|v_h|_{1,\omega_K} \quad \forall \ v_h \in V_h,
\]
\[
|v_h - E_h^L v_h|_{j,K} \lesssim h^{2-j} (|v_h|_{2,\omega_K} + \sum_{K' \in T_K} \sum_{F \in F^3(K')} h_F^{-1/2} \|\partial_n v_h\|_{0,F}) \quad \forall \ v_h \in V_h
\]
for each $K \in T_h$. Then we get
\[
|v_h - E_h^L v_h|_{1,h} \lesssim h^{1-r} |v_h|_{1,h} |v_h|_{2,h}^{1-r} \quad \forall \ v_h \in V_{h0},
\]
\[
|v_h - E_h^L v_h|_{1,h} \lesssim h^{1-r} |v_h|_{1,h} ||v_h||_{2,h}^{1-r} \quad \forall \ v_h \in V_h
\]
with $0 \leq r \leq 1$.

To define interpolation operators later, we also need another two connection operators $E_h : W_h \rightarrow V_h$ and $E_{h0} : W_h \rightarrow V_{h0}$. For any $v_h \in W_h$, $E_h v_h \in V_h$ is determined by
\[
\int_F E_h v_h \, ds = \int_F v_h \, ds \quad \forall \ e \in \mathcal{E}_h^j,
\]
\[
\int_F \partial_n F(E_h v_h) \, ds = \int_F \{\partial_n F v_h\} \, ds \quad \forall \ F \in \mathcal{F}_h.
\]
And $E_{h0} v_h \in V_{h0}$ is determined by
\[
\int_F E_{h0} v_h \, ds = \int_F v_h \, ds \quad \forall \ e \in \mathcal{E}_h^j,
\]
\[
\int_F \partial_n F(E_{h0} v_h) \, ds = \int_F \{\partial_n F v_h\} \, ds \quad \forall \ F \in \mathcal{F}_h^i.
\]

### 2.3 Interpolation Operators

Let $I_h^{SZ}$ be the Scott–Zhang interpolation operator [32] from $H_0^1(\Omega)$ onto $W_h$ with $\ell = 2$. For any $1 \leq s \leq 3$ and $0 \leq m \leq s$, it holds (cf. [32, (4.3)])
\[
|v - I_h^{SZ} v|_{m,K} \lesssim h^{s-m} |v|_{s,\omega_K} \quad \forall \ v \in H_0^1(\Omega) \cap H^s(\Omega), \quad K \in T_h.
\]
(2.6)

Then define two quasi-interpolation operators $I_h : H_0^1(\Omega) \rightarrow V_h$ and $I_{h0} : H_0^1(\Omega) \rightarrow V_{h0}$ as
\[
I_h := E_h I_h^{SZ}, \quad I_{h0} := E_{h0} I_h^{SZ}.
\]

Next we will derive the error estimates of the interpolation operators $I_h$ and $I_{h0}$ following the argument in [24].

**Lemma 2.1** Let $2 \leq s \leq 3$ and $0 \leq m \leq s$. We have
\[
|v - I_{h0} v|_{m,h} \lesssim h^{s-m} |v|_{s} \quad \forall \ v \in H^s(\Omega) \cap H_0^1(\Omega),
\]
(2.7)
\[
|v - I_{h0} v|_{1,h} \lesssim |v|_1 \quad \forall \ v \in H_0^1(\Omega),
\]
(2.8)
\[
|v - I_h v|_{m,h} \lesssim h^{s-m} |v|_{s} \quad \forall \ v \in H^s(\Omega) \cap H_0^1(\Omega),
\]
(2.9)
\[
|v - I_h v|_{1,h} \lesssim |v|_1 \quad \forall \ v \in H_0^1(\Omega).
\]
(2.10)
Proof We only prove the inequalities (2.9)–(2.10). The inequalities (2.7)–(2.8) can be achieved by the same argument. Take any $K \in T_h$. By the definition of $E_h$, we have

$$
\int_F (I_h^{SZ} v - E_h I_h^{SZ} v) s ds = 0 \quad \forall \ e \in E(K),
$$

$$
\int_F \delta_{n_F}( (I_h^{SZ} v - E_h I_h^{SZ} v) ) s ds = \frac{n_F \cdot n_K}{2} \int_F ||\delta_{n_F}(I_h^{SZ} v)\|| s ds \quad \forall \ F \in \mathcal{F}^i(K),
$$

$$
\int_F \delta_{n_F}( (I_h^{SZ} v - E_h I_h^{SZ} v) ) s ds = 0 \quad \forall \ F \in \mathcal{F}^\partial(K).
$$

Applying the inverse inequality, scaling argument and Cauchy–Schwarz inequality, it follows

$$
|I_h^{SZ} v - E_h I_h^{SZ} v|_{m,K} \lesssim h_K^{d/2-m} \sum_{F \in \mathcal{F}^i(K)} \left| \int_F \delta_{n_F}(I_h^{SZ} v) s ds \right|
$$

$$
\lesssim h_K^{3/2-m} \sum_{F \in \mathcal{F}^i(K)} \left\| \delta_{n_F}(I_h^{SZ} v - v) \right\|_{0,F}, \tag{2.11}
$$

which together with the trace inequality and (2.6) implies

$$
|I_h^{SZ} v - E_h I_h^{SZ} v|_{m,K} \lesssim h_K^{s-m} |v|_{s,\omega_K^2}.
$$

Employing (2.6), we get

$$
|v - I_h v|_{m,K} \leq |v - I_h^{SZ} v|_{m,K} + |I_h^{SZ} v - E_h I_h^{SZ} v|_{m,K} \lesssim h_K^{s-m} |v|_{s,\omega_K^2},
$$

which indicates (2.9).

Next we obtain from (2.11), the inverse inequality and (2.6) that

$$
|I_h^{SZ} v - E_h I_h^{SZ} v|_{1,K} \lesssim h_K \sum_{F \in \mathcal{F}^i(K)} \left\| \delta_{n_F}(I_h^{SZ} v) \right\|_{0,F}^2 \lesssim |I_h^{SZ} v|_{1,\omega_K} \lesssim |v|_{1,\omega_K^2}^2.
$$

Finally we achieve (2.10) from the last inequality, triangle inequality and (2.6).

Let $u^0 \in H_0^1(\Omega)$ be the solution of the Poisson equation

$$
\begin{cases}
-\Delta u^0 = f & \text{in } \Omega, \\
u^0 = 0 & \text{on } \partial\Omega.
\end{cases} \tag{2.12}
$$

Since the domain $\Omega$ is convex, we have the following regularities [22,24,30]

$$
|u|_2 + \varepsilon |u|_3 \lesssim \varepsilon^{-1/2} \|f\|_0, \tag{2.13}
$$

$$
|u - u^0|_1 \lesssim \varepsilon^{1/2} \|f\|_0, \tag{2.14}
$$

$$
\|u^0\|_2 \lesssim \|f\|_0. \tag{2.15}
$$

Lemma 2.2 Assume $u \in H_0^2(\Omega) \cap H^3(\Omega)$ and $u^0 \in H_0^1(\Omega) \cap H^s(\Omega)$ with $2 \leq s \leq 3$. We have

$$
||u - I_h u||_{s,h} \lesssim (\varepsilon h + h^2) |u|_3, \tag{2.16}
$$

$$
||u - I_h u||_{s,h} \lesssim \varepsilon^{-1/2} h^{1-r} \|f\|_0 + h^{s-1} |u^0|_s \tag{2.17}
$$

with $0 \leq r \leq 1$. 


Applying (2.9) and (2.13) again, we obtain

\[ |u - u^0 - I_h(u - u^0)|_{1,h} \lesssim h |u - u^0|_2. \]

Due to (2.10), it follows

\[ |u - u^0 - I_h(u - u^0)|_{1,h} \lesssim |u - u^0|_1. \]

Combining the last two inequalities, we get from (2.13)–(2.15) that

\[ |u - u^0 - I_h(u - u^0)|_{1,h} \lesssim h^{1-r} |u - u^0|_1 |u - u^0|_2^{1-r} \lesssim \epsilon^{r-1/2} h^{1-r} \| f \|_0. \]

Then using the triangle inequality and (2.9), we acquire

\[ |u - I_h u|_{1,h} \leq |u - u^0 - I_h(u - u^0)|_{1,h} + |u^0 - I_h u^0|_{1,h} \lesssim \epsilon^{r-1/2} h^{1-r} \| f \|_0 + h^{s-1} |u^0|_s. \]  

(2.18)

Applying (2.9) and (2.13) again, we obtain

\[ \epsilon |u - I_h u|_{2,h} \lesssim \epsilon |u|_2 \lesssim \epsilon^{1/2} \| f \|_0, \quad \epsilon |u - I_h u|_{2,h} \lesssim \epsilon h |u|_3 \lesssim \epsilon^{-1/2} h \| f \|_0. \]

Thus

\[ \epsilon |u - I_h u|_{2,h} \lesssim \epsilon^{r-1/2} h^{1-r} \| f \|_0. \]  

(2.19)

By the trace inequality, (2.9) and (2.13),

\[ \sum_{F \in \mathcal{F}_h^0} h_F^{-1} \| \partial_h (u - I_h u) \|_{0,F}^2 \lesssim \sum_{K \in T_h} (h_K^{-2} |u - I_h u|_{1,K}^2 + |u - I_h u|_{2,K}^2) \lesssim h^{2-2r} |u|_2^2 |u|_3^2 |u|_{2,h}^2 \lesssim \epsilon^{2r-3} h^{2-2r} \| f \|_0^2. \]

Finally we derive (2.17) from (2.18)–(2.19) and the last inequality.

\[ \square \]

**Lemma 2.3** Assume \( u \in H_\Omega^0 (\Omega) \cap H^3(\Omega) \) and \( u^0 \in H_\Omega^1 (\Omega) \cap H^2(\Omega) \). We have

\[ \| u - I_{h_0} u \|_{\varepsilon,h} \lesssim (\varepsilon h + h^2) |u|_3, \]

(2.20)

\[ \| u - I_{h_0} u \|_{\varepsilon,h} \lesssim h^{1/2} \| f \|_0. \]  

(2.21)

**Proof** The inequality (2.20) is the immediate result of (2.7). Applying the multiplicative trace inequality (cf. [22, Theorem 1.5.1.10]), (2.6) and (2.13)–(2.15),

\[ \sum_{F \in \mathcal{F}_h^0} h_F \| \partial_h (I_h^{SZ}(u - u^0) - (u - u^0)) \|_{0,F}^2 \lesssim h |u - u^0|_1 |u - u^0|_2 \lesssim h \| f \|_0^2. \]

Using (2.6) and (2.15) again,

\[ \sum_{F \in \mathcal{F}_h^0} h_F \| \partial_h (I_h^{SZ} u - u) \|_{0,F}^2 \lesssim \sum_{F \in \mathcal{F}_h^0} h_F \| \partial_h (I_h^{SZ}(u - u^0) - (u - u^0)) \|_{0,F}^2 + \sum_{F \in \mathcal{F}_h^0} h_F \| \partial_h (I_h^{SZ} u^0 - u^0) \|_{0,F}^2 \lesssim h \| f \|_0^2 + h^2 |u^0|_2^2 \lesssim h \| f \|_0^2. \]
By the definitions of $I_{h0}$ and $I_h$, it follows

$$|I_h u - I_{h0} u|^2_{1,h} \lesssim \sum_{F \in \mathcal{F}_h} h_F \| \partial_n (I_h^{SZ} u) \|^2_{0,F} = \sum_{F \in \mathcal{F}_h} h_F \| \partial_n (I_h^{SZ} u - u) \|^2_{0,F} \lesssim h \| f \|^2_0.$$ 

On the other side, we get from (2.6) and (2.13) that

$$|I_h u - I_{h0} u|^2_{2,h} \lesssim \sum_{F \in \mathcal{F}_h} h_F^{-1} \| \partial_n (I_h^{SZ} u) \|^2_{0,F} = \sum_{F \in \mathcal{F}_h} h_F^{-1} \| \partial_n (I_h^{SZ} u - u) \|^2_{0,F}$$

$$\lesssim h |u|_2 |u|_3 \lesssim \varepsilon^{-2} h \| f \|^2_0.$$ 

Thus we obtain from the last two inequalities that

$$\| I_h u - I_{h0} u \|_{\varepsilon,h} \lesssim h^{1/2} \| f \|_0,$$

which combined with (2.17) indicates (2.21). \hfill \Box

## 3 Morley–Wang–Xu Element Method

We will propose an MWX element method for the fourth order elliptic singular perturbation problem (1.1) in this section.

### 3.1 Morley–Wang–Xu Element Method

To present the MWX element method, we need the $H^1$-orthogonal projection $P_h : H^1(T_h) \to W_h$: given $v_h \in H^1(T_h)$, $P_h v_h \in W_h$ is determined by

$$(\nabla P_h v_h, \nabla \chi_h) = (\nabla v_h, \nabla \chi_h) \quad \forall \chi_h \in W_h.$$ 

It is well-known that for $s \geq 1$ (cf. [4,16])

$$|v - P_h v|_1 \lesssim h^{\min\{s-1,\ell\}} \| v \|_s \quad \forall \ v \in H^{s}_{0}(\Omega) \cap H^{\ell}(\Omega). \quad (3.1)$$

We propose the following MWX element method for problem (1.1): find $u_{h0} \in V_{h0}$ such that

$$\varepsilon^2 a_h(u_{h0}, v_h) + b_h(u_{h0}, v_h) = (f, P_h v_h) \quad \forall \ v_h \in V_{h0}, \quad (3.2)$$

where

$$a_h(u_{h0}, v_h) := (\nabla^2_h u_{h0}, \nabla^2_h v_h), \quad b_h(u_{h0}, v_h) := (\nabla_h u_{h0}, \nabla_h v_h).$$

We use the simplest MWX element to approximate the exact solution in the discrete method (3.2). Compared to the standard nonconforming finite element method, we only replace the right hand side term $(f, v_h)$ by $(f, P_h v_h)$, thus the MWX element method (3.2) possesses a sparser stiffness matrix than that of the discrete method in [43].

### 3.2 Error Estimates

Applying the Cauchy–Schwarz inequality and (2.20)–(2.21), we have following error estimates for $I_{h0}$.
Lemma 3.1 Assume \( u \in H^2_0(\Omega) \cap H^3(\Omega) \) and \( u^0 \in H^1_0(\Omega) \cap H^2(\Omega) \). We have for any \( v_h \in V_{h0} \) that
\[ \varepsilon^2 a_h(I_{h0}u - u, v_h) + b_h(I_{h0}u - u, v_h) \lesssim (eh + h^2)\|u\|_{3,h}, \]  
(3.3)
\[ \varepsilon^2 a_h(I_{h0}u - u, v_h) + b_h(I_{h0}u - u, v_h) \lesssim h^{1/2}\|f\|_0 \|v_h\|_{\varepsilon,h}. \]  
(3.4)

Lemma 3.2 Assume \( u \in H^2_0(\Omega) \cap H^3(\Omega) \) and \( u^0 \in H^1_0(\Omega) \cap H^2(\Omega) \). We have for any \( v_h \in V_{h0} \) that
\[ \varepsilon^2 a_h(u, v_h) + \varepsilon^2(\nabla^2u, \nabla E_h^L v_h) \lesssim \varepsilon^{-1/2}h^{1-r}\|f\|_0 \|v_h\|_{\varepsilon,h} \]  
with \( 0 \leq r \leq 1 \).

Proof We get from integration by parts and (2.3) that
\[ a_h(u, v_h) + (\nabla^2u, \nabla_h v_h) = \sum_{K \in \mathcal{T}_h} ((\nabla^2u)n, \nabla_h v_h)_{\partial K} = \sum_{F \in \mathcal{F}_h} ((\nabla^2u)n_F - Q_F^L ((\nabla^2u)n_F), \nabla_h v_h)_F \]  
\[ = \sum_{F \in \mathcal{F}_h} ((\nabla^2u)n_F - Q_F^L ((\nabla^2u)n_F), \nabla_h v_h)_F \]  
where \( Q_F^L \) is the \( L^2 \)-orthogonal projection onto the constant space on face \( F \). By the error estimate of \( Q_F^L \) (cf. [4,16]) and the inverse inequality, we have
\[ a_h(u, v_h) + (\nabla^2u, \nabla_h v_h) \lesssim h^{1-r} |v_h|_{1,h} |v_h|_{2,h} |u|_3. \]  
On the other side, it follows from (2.4) that
\[ (\nabla^2u, \nabla (E_h^L v_h - v_h)) \lesssim \|u\|_3 \|E_h^L v_h - v_h\|_{1,h} \lesssim h^{1-r} |v_h|_{1,h} |v_h|_{2,h} |u|_3. \]  
Combining the last two inequalities gives
\[ a_h(u, v_h) + (\nabla^2u, \nabla E_h^L v_h) \lesssim h^{1-r} |v_h|_{1,h} |v_h|_{2,h} |u|_3. \]  
By the definition of \( \|v_h\|_{\varepsilon,h} \), we achieve
\[ \varepsilon^2 a_h(u, v_h) + \varepsilon^2(\nabla^2u, \nabla E_h^L v_h) \lesssim \varepsilon^{1+r}h^{1-r}|u|_3\|v_h\|_{\varepsilon,h}. \]  
Finally we derive (3.5) from (2.13).

Lemma 3.3 Assume \( u \in H^2_0(\Omega) \cap H^3(\Omega) \) and \( u^0 \in H^1_0(\Omega) \cap H^s(\Omega) \) with \( 2 \leq s \leq 3 \). It holds for any \( v_h \in V_{h0} \) that
\[ b_h(u, v_h - E_h^L v_h) - (f, P_h v_h - E_h^L v_h) \lesssim (\varepsilon^{-1/2}h^{1-r}\|f\|_0 + h^{\min[s-1,\ell]}\|u^0\|_s)\|v_h\|_{\varepsilon,h} \]  
(3.6)
with \( 0 \leq r \leq 1 \).

Proof Since \( P_h v_h - E_h^L v_h \in H^1_0(\Omega) \), we get from (2.12), integration by parts and the definition of \( P_h \) that
\[ (f, P_h v_h - E_h^L v_h) = (\nabla u^0, \nabla (P_h v_h - E_h^L v_h)) = (\nabla u^0, \nabla P_h v_h) - (\nabla u^0, \nabla E_h^L v_h) \]
Using the triangle inequality and (3.3)–(3.4), we get

\[ \nabla (P_h u^0, \nabla v_h) - \nabla (u^0, \nabla E_h^L v_h) = (\nabla (u^0 - P_h u^0, \nabla v_h) + (\nabla (u^0, \nabla (v_h - E_h^L v_h)). \]

Thus we have

\[ b_h(u, v_h - E_h^L v_h) - (f, P_h v_h - E_h^L v_h) = (\nabla (u - u^0, \nabla v_h) + (\nabla (u^0, \nabla (v_h - E_h^L v_h)). \]

Adopting the Cauchy–Schwarz inequality and (2.14), it holds

\[ b_h(u, v_h - E_h^L v_h) - (f, P_h v_h - E_h^L v_h) \leq |u - u^0| \| v_h - E_h^L v_h \|_{1,h} + |u^0 - P_h u^0| \| v_h \|_{1,h} \leq \varepsilon^{1/2} \| f \|_0 \| v_h - E_h^L v_h \|_{1,h} + |u^0 - P_h u^0| \| v_h \|_{1,h}. \]

Hence we obtain

\[ |v_h - E_h^L v_h|_{1,h} \lesssim \varepsilon^{r-1} h^{1-r} \| v_h \|_{\varepsilon,h}. \]

Hence we acquire from (3.5) and (3.6) that

\[ \| u - u_h \|_{\varepsilon,h} \lesssim \varepsilon^{r-1/2} h^{1-r} \| f \|_0 + h^{\min[s-1,\varepsilon]} \| u^0 \|_s + h(\varepsilon + h)| u |_3, \]  \hspace{1cm} (3.7) \]

\[ \| u - u_h \|_{\varepsilon,h} \lesssim h^{1/2} \| f \|_0. \]  \hspace{1cm} (3.8) \]

\[ \| u^0 - u_h^0 \|_{\varepsilon,h} \lesssim (\varepsilon^{1/2} + h^{1/2}) \| f \|_0 \]  \hspace{1cm} (3.9) \]

with \( 0 \leq r \leq 1. \)

**Theorem 3.4** Let \( u \in H^2_0(\Omega) \) be the solution of problem (1.1), and \( u_{h0} \in V_{h0} \) be the discrete solution of the MWX element method (3.2). Assume \( u \in H^3(\Omega) \) and \( u^0 \in H^s(\Omega) \cap H^3(\Omega) \) with \( 2 \leq s \leq 3 \). We have

\[ \| u - u_h \|_{\varepsilon,h} \lesssim \varepsilon^{r-1/2} h^{1-r} \| f \|_0 + h^{\min[s-1,\varepsilon]} \| u^0 \|_s + h(\varepsilon + h)| u |_3, \]  \hspace{1cm} (3.7) \]

\[ \| u - u_h \|_{\varepsilon,h} \lesssim h^{1/2} \| f \|_0. \]  \hspace{1cm} (3.8) \]

\[ \| u^0 - u_h^0 \|_{\varepsilon,h} \lesssim (\varepsilon^{1/2} + h^{1/2}) \| f \|_0 \]  \hspace{1cm} (3.9) \]

with \( 0 \leq r \leq 1. \)

**Proof** Let \( v_h = I_{h0} u - u_{h0}. \) We obtain from (1.1) that

\[ -\varepsilon^2 (\nabla \nabla^2 u, \nabla E_h^L v_h) + (\nabla u, \nabla E_h^L v_h) = (f, E_h^L v_h). \]

Then it follows from (3.2) that

\[ \varepsilon^2 a_h(u - u_{h0}, v_h) + b_h(u - u_{h0}, v_h) = \varepsilon^2 a_h(u, v_h) + b_h(u, v_h) - (f, P_h v_h). \]

Hence we acquire from (3.5) and (3.6) that

\[ \varepsilon^2 a_h(u - u_{h0}, v_h) + b_h(u - u_{h0}, v_h) \lesssim (\varepsilon^{r-1/2} h^{1-r} \| f \|_0 + h^{\min[s-1,\varepsilon]} \| u^0 \|_s) \| v_h \|_{\varepsilon,h}. \]

Using the triangle inequality and (3.3)–(3.4), we get

\[ \varepsilon^2 a_h(I_{h0} u - u_{h0}, v_h) + b_h(I_{h0} u - u_{h0}, v_h) \]
\[ \lesssim \left( \varepsilon^{r-1/2} h^{1-r} \| f \|_0 + h^{\min[s-1, \ell]} \| u^0 \|_s + h (\varepsilon + h) |u|_3 \right) \| v_h \|_{\varepsilon,h}, \]
\[ \varepsilon^2 a_h(I_h u - u_0, v_h) + b_h(I_h u - u_0, v_h) \lesssim h^{1/2} \| f \|_0 \| v_h \|_{\varepsilon,h}. \]

Thus
\[ \| I_h u - u_0 \|_{\varepsilon,h} \lesssim \varepsilon^{r-1/2} h^{1-r} \| f \|_0 + h^{\min[s-1, \ell]} \| u^0 \|_s + h (\varepsilon + h) |u|_3, \]
\[ \| I_h u - u_0 \|_{\varepsilon,h} \lesssim h^{1/2} \| f \|_0. \]

Finally we get (3.7)–(3.8) by combining the last three inequalities and (2.20)–(2.21). The estimate (3.9) is a direct result of (3.8) and (2.13)–(2.15). \( \square \)

4 Imposing Boundary Condition Using Nitsche’s Method

In consideration of the boundary layers of problem (1.1), we will adjust the MWX element method (3.2) by using Nitsche’s method to impose the boundary condition \( \partial_n u = 0 \) weakly in this section, as in [24].

4.1 Discrete Method

Through applying the Nitsche’s technique, the MWX element method with weakly imposing the boundary condition is to find \( u_h \in V_h \) such that

\[ \varepsilon^2 \tilde{a}_h(u_h, v_h) + \tilde{b}_h(u_h, v_h) = (f, P_h v_h) \quad \forall \ v_h \in V_h, \quad (4.1) \]

where

\[
\tilde{a}_h(u_h, v_h) := (\nabla_h^2 u_h, \nabla_h^2 v_h) - \sum_{F \in \mathcal{F}_h^\partial} (\partial_{nn}^2 u_h, \partial_n v_h)_F - \sum_{F \in \mathcal{F}_h^\partial} (\partial_n u_h, \partial_{nn}^2 v_h)_F
\]

\[ + \sum_{F \in \mathcal{F}_h^\partial} \sigma h_F (\partial_n u_h, \partial_n v_h)_F \]

with \( \sigma \) being a positive real number.

**Lemma 4.1** There exists a constant \( \sigma_0 > 0 \) depending only on the shape regularity of \( T_h \) such that for any fixed number \( \sigma \geq \sigma_0 \), it holds

\[ \| v_h \|_{2,h}^2 \lesssim \tilde{a}_h(v_h, v_h) \quad \forall \ v_h \in V_h. \quad (4.2) \]

**Proof** By the trace inverse inequality [6,44], there exists a constant \( C_{\text{inv}} > 0 \) such that

\[ \| \partial_{nn}^2 v_h \|_{0,F} \leq C_{\text{inv}} h_F^{-1/2} |v_h|_{2,K} \quad \forall \ F \subset \partial K, \ K \in T_h. \]

Applying the Cauchy–Schwarz inequality, it follows

\[ 2 \sum_{F \in \mathcal{F}_h^\partial} (\partial_{nn}^2 v_h, \partial_n v_h)_F \leq 2 \sum_{F \in \mathcal{F}_h^\partial} \| \partial_{nn}^2 v_h \|_{0,F} \| \partial_n v_h \|_0,F \]

\[ \leq 2 C_{\text{inv}} |v_h|_{2,h} \left( \sum_{F \in \mathcal{F}_h^\partial} h_F^{-1} \| \partial_n v_h \|_{0,F}^2 \right)^{1/2}. \]
Hence

\[
\tilde{a}_h(v_h, v_h) = |v_h|_{2,h}^2 - 2 \sum_{F \in \mathcal{F}_h^3} (\partial_{nn}^2 v_h, \partial_n v_h)_F + \sum_{F \in \mathcal{F}_h^3} \frac{\sigma}{h_F} \|\partial_n v_h\|_{0,F}^2 \geq \frac{1}{2} |v_h|_{2,h}^2 + (\sigma - 2C_{\text{inv}}^2) \sum_{F \in \mathcal{F}_h^3} h_F^{-1} \|\partial_n v_h\|_{0,F}^2.
\]

The proof is finished by choosing \(\sigma_0 = 2C_{\text{inv}}^2 + 1\). \(\square\)

We refer to [17,29] for the explicit estimation of the penalty parameter \(\sigma_0\).

By (4.2), we have

\[
\|v_h\|_{\varepsilon,h}^2 \lesssim \varepsilon^2 \tilde{a}_h(v_h, v_h) + b_h(v_h, v_h) \quad \forall \, v_h \in V_h.
\]

It is obvious that

\[
\varepsilon^2 \tilde{a}_h(\chi_h, v_h) + b_h(\chi_h, v_h) \lesssim ||| \chi_h |||_{\varepsilon,h} ||| v_h |||_{\varepsilon,h} \quad \forall \, \chi_h, v_h \in V_h.
\]

The last two inequalities indicate the wellposedness of the MWX element method (4.1).

### 4.2 Error Estimates

In this subsection we will present the error analysis for the discrete method (4.1).

**Lemma 4.2** Assume \(u \in H^2_0(\Omega) \cap H^3(\Omega)\) and \(u^0 \in H^1_0(\Omega) \cap H^s(\Omega)\) with \(2 \leq s \leq 3\). We have

\[
\varepsilon^2 \tilde{a}_h(I_h u - u, v_h) + b_h(I_h u - u, v_h) \lesssim (\varepsilon h + h^2) |u|_3 ||| v_h |||_{\varepsilon,h} \quad \forall \, v_h \in V_h,
\]

\[
\varepsilon^2 \tilde{a}_h(I_h u - u, v_h) + b_h(I_h u - u, v_h) \lesssim (\varepsilon^{-r/2} h^{1-r} \|f\|_0 + h^{s-1} ||| u^0 |||_s) ||| v_h |||_{\varepsilon,h} \quad \forall \, v_h \in V_h
\]

with \(0 \leq r \leq 1\).

**Proof** According to the trace inequality and (2.9), it follows

\[
\sum_{F \in \mathcal{F}_h^3} h_F \|\partial_{nn}^2(I_h u - u)\|_{0,F}^2 \lesssim h^2 |u|^2_3.
\]

Then we get from (2.13) that

\[
- \sum_{F \in \mathcal{F}_h^3} (\partial_{nn}^2(I_h u - u), \partial_n v_h)_F \leq \sum_{F \in \mathcal{F}_h^3} \|\partial_{nn}^2(I_h u - u)\|_{0,F} \|\partial_n v_h\|_{0,F} \lesssim ||| v_h |||_{2,h} \left( \sum_{F \in \mathcal{F}_h^3} h_F \|\partial_{nn}^2(I_h u - u)\|_{0,F}^2 \right)^{1/2} \lesssim \varepsilon^{-2} ||| v_h |||_{\varepsilon,h} \min\{\varepsilon^{-1/2} h \|f\|_0, \varepsilon h |u|_3\}.
\]
Using the inverse inequality, (2.9) and (2.13), we obtain
\[- \sum_{F \in \mathcal{F}_h^0} (\partial^2_{nn}(I_h u - u), \partial_n v_h)_F \leq \sum_{F \in \mathcal{F}_h^0} \|\partial^2_{nn}(I_h u - u)\|_{0,F} \|\partial_n v_h\|_{0,F} \\lesssim \|v_h\|_{1,h} \left( \sum_{F \in \mathcal{F}_h^0} h_F^{-1} \|\partial^2_{nn}(I_h u - u)\|_{0,F}^2 \right)^{1/2} \\lesssim \|v_h\|_{1,h} \|u\|_3 \lesssim \varepsilon^{-3/2} \|\psi_h\|_{\varepsilon,h} \|f\|_0.\]
Hence it follows from the last two inequalities that
\[- \varepsilon^2 \sum_{F \in \mathcal{F}_h^0} (\partial^2_{nn}(I_h u - u), \partial_n v_h)_F \lesssim \|\psi_h\|_{\varepsilon,h} \min\{\varepsilon^{r-1/2} h^{1-r} \|f\|_0, \varepsilon h|u|_3\}. \quad (4.6)\]
Since
\[\varepsilon^2 \tilde{a}_h(I_h u - u, v_h) + b_h(I_h u - u, v_h) \lesssim \|\tilde{a}_h u - u\|_{\varepsilon,h} \|\psi_h\|_{\varepsilon,h} - \varepsilon^2 \sum_{F \in \mathcal{F}_h^0} (\partial^2_{nn}(I_h u - u), \partial_n v_h)_F,\]
we acquire (4.4) from (2.16) and (4.6), and (4.5) from (2.17) and (4.6). \qed

Applying the same argument as in Lemma 3.2, from (2.1)–(2.2) and (2.5) we obtain the following estimate
\[\varepsilon^2 \tilde{a}_h(u, v_h) + \varepsilon^2 (\text{div} \nabla^2 u, \nabla E_h^L v_h) \lesssim \varepsilon^{r-1/2} h^{1-r} \|f\|_0 \|\psi_h\|_{\varepsilon,h} \quad (4.7)\]
for any \(v_h \in V_h\). And applying the same argument as in Lemma 3.3, we acquire
\[b_h(u, v_h - E_h^L v_h) - (f, P_h v_h - E_h^L v_h) \lesssim (\varepsilon^{r-1/2} h^{1-r} \|f\|_0 + h^\min\{s-1,\ell\} \|u^0\|_s) \|\psi_h\|_{\varepsilon,h} \quad \forall \ v_h \in V_h. \quad (4.8)\]

**Theorem 4.3** Let \(u \in H_0^2(\Omega)\) be the solution of problem (1.1), and \(u_h \in V_h\) be the discrete solution of the MWX element method (4.1). Assume \(u^0 \in H_0^1(\Omega) \cap H^s(\Omega)\) with \(2 \leq s \leq 3\). We have
\[\|u - u_h\|_{\varepsilon,h} \lesssim \varepsilon^{r-1/2} h^{1-r} \|f\|_0 + h^\min\{s-1,\ell\} \|u^0\|_s, \quad (4.9)\]
\[\|u^0 - u_h\|_{\varepsilon,h} \lesssim \varepsilon^{1/2} \|f\|_0 + h^\min\{s-1,\ell\} \|u^0\|_s, \quad (4.10)\]
with \(0 \leq r \leq 1\).

**Proof** Let \(v_h = I_h u - u_h\). Adopting the similar argument as in the proof of Theorem 3.4, we get from (4.7) and (4.8) that
\[\varepsilon^2 \tilde{a}_h(u - u_h, v_h) + b_h(u - u_h, v_h) \lesssim (\varepsilon^{r-1/2} h^{1-r} \|f\|_0 + h^\min\{s-1,\ell\} \|u^0\|_s) \|\psi_h\|_{\varepsilon,h}.\]
Together with (4.5), we have
\[\varepsilon^2 \tilde{a}_h(I_h u - u_h, v_h) + b_h(I_h u - u_h, v_h) \lesssim (\varepsilon^{r-1/2} h^{1-r} \|f\|_0 + h^\min\{s-1,\ell\} \|u^0\|_s) \|\psi_h\|_{\varepsilon,h}.\]
Then we obtain from (4.3) that
\[\|I_h u - u_h\|_{\varepsilon,h} \lesssim \varepsilon^{r-1/2} h^{1-r} \|f\|_0 + h^\min\{s-1,\ell\} \|u^0\|_s.\]
Therefore we conclude (4.9) from (2.17), and (4.10) from (4.9) and (2.13)–(2.15). \qed
Remark 4.4 When $\varepsilon$ approaches zero, the uniform convergence rate $O(h^{1/2})$ of $\|u - u_{h0}\|_{e,h}$ in (3.8) is sharp but not optimal. As a comparison, the convergence rate $O(h^\ell)$ of $\|u - u_{h}\|_{e,h}$ with $\ell = 1, 2$ in Theorem 4.3 is optimal. Taking the limit of (3.7) would also give the optimal order, but $|u|_3$ is not uniform in $\varepsilon$.

5 Equivalent Formulations

In this section, we will show some equivalent solver-friendly formulations of the MWX element methods (3.2) and (4.1).

Lemma 5.1 The MWX element method (3.2) is equivalent to finding $w_h \in W_h$ and $u_{h0} \in V_{h0}$ such that

$$
(\nabla w_h, \nabla \chi_h) = (f, \chi_h) \quad \forall \chi_h \in W_h, \quad (5.1)
$$

$$
\varepsilon^2 a_h(u_{h0}, v_h) + b_h(u_{h0}, v_h) = (\nabla w_h, \nabla v_h) \quad \forall v_h \in V_{h0}. \quad (5.2)
$$

Proof By the definition of the $H^1$-orthogonal projection $P_h$ and (5.1) with $\chi_h = P_h v_h$, the right hand side of (5.2), it follows

$$(\nabla w_h, \nabla v_h) = (\nabla w_h, \nabla P_h v_h) = (f, P_h v_h).$$

Therefore the MWX element method (3.2) is equivalent to the discrete method (5.1)–(5.2). \qed

If the fourth order elliptic singular perturbation problem (1.1) is equipped with non-homogeneous boundary conditions, the discrete boundary condition of $w_h$ should agree with the value of $u$ on the boundary $\partial \Omega$.

Similarly, we have the equivalent formulation of the MWX element method (4.1).

Lemma 5.2 The MWX element method (4.1) is equivalent to finding $w_h \in W_h$ and $u_h \in V_h$ such that

$$
(\nabla w_h, \nabla \chi_h) = (f, \chi_h) \quad \forall \chi_h \in W_h, \quad (5.3)
$$

$$
\varepsilon^2 \tilde{a}_h(u_h, v_h) + b_h(u_h, v_h) = (\nabla w_h, \nabla v_h) \quad \forall v_h \in V_h. \quad (5.4)
$$

In two dimensions, we can further decouple the discrete methods (5.2) and (5.4) into the discrete methods of two Poisson equations and one Brinkman problem. To this end, define the vectorial nonconforming $P_1$ element space

$$
V_{h}^{CR} := \{ v \in L^2(\Omega; \mathbb{R}^2) \colon v|_K \in P_1(K; \mathbb{R}^2) \text{ for each } K \in T_h, \int_F [v] ds = 0 \text{ for each } F \in \mathcal{F}_h, \text{ and } \int_F v \cdot n ds = 0 \text{ for each } F \in \mathcal{F}_h^0 \},
$$

$$
V_{h0}^{CR} := \{ v \in V_{h}^{CR} \colon \int_F v ds = 0 \text{ for each } F \in \mathcal{F}_h^0 \}.
$$

And let $Q_h \subset L^2_0(\Omega)$ be the piecewise constant space with respect to $T_h$, where $L^2_0(\Omega)$ is the subspace of $L^2(\Omega)$ with vanishing mean value.

Due to Theorem 4.1 in [18], we have the following relationship between Morley element spaces and vectorial Crouzeix-Raviart element spaces

$$
\text{curl}_h V_{h0} = \{ v_h \in V_{h0}^{CR} : \text{div}_h v_h = 0 \}, \quad (5.5)
$$
\textbf{Lemma 5.3} In two dimensions, the discrete method (5.2) can be decoupled into two Morley element methods of Poisson equation and one nonconforming \( P_1-P_0 \) element method of Brinkman problem, i.e., find \((z_h, \phi_h, p_h, w_h) \in V_h \times V_h^{CR} \times Q_h \times V_h\) such that

\[(\text{curl}_h z_h, \text{curl}_h v_h) = (\nabla w_h, \nabla v_h) \quad \forall \ v_h \in V_h, \quad (5.7a)\]

\[(\phi_h, \psi_h) + \varepsilon^2 (\nabla_h \phi_h, \nabla_h \psi_h) + (\text{div}_h \psi_h, p_h) = (\text{curl}_h z_h, \psi_h) \quad \forall \ \psi_h \in V_h^{CR}, \quad (5.7b)\]

\[(\text{div}_h \phi_h, q_h) = 0 \quad \forall \ q_h \in Q_h, \quad (5.7c)\]

\[(\text{curl}_h u_{h0}, \text{curl}_h \chi_h) = (\phi_h, \text{curl}_h \chi_h) \quad \forall \ \chi_h \in V_h. \quad (5.7d)\]

\textbf{Proof} Thanks to (5.5), it follows from (5.7c)–(5.7d) that

\[\phi_h = \text{curl}_h u_{h0} \quad \text{and} \quad u_{h0} \in V_{h0}.\]

For any \( v_h \in V_{h0} \), it is apparent that

\[(\nabla_h \text{curl}_h u_{h0}, \nabla_h \text{curl}_h v_h) = a_h(u_{h0}, v_h).\]

Then replacing \( \phi_h \) with \( \text{curl}_h u_{h0} \) and \( \psi_h \) with \( \text{curl}_h v_h \) in (5.7b), we achieve

\[\varepsilon^2 a_h(u_{h0}, v_h) + b_h(u_{h0}, v_h) = (\text{curl}_h z_h, \text{curl}_h v_h),\]

which combined with (5.7a) induces (5.2). \( \square \)

Similarly, we get the decoupling of the discrete method (5.4) based on (5.6).

\textbf{Lemma 5.4} In two dimensions, the discrete method (5.4) can be decoupled into two Morley element methods of Poisson equation and one nonconforming \( P_1-P_0 \) element method of Brinkman problem, i.e., find \((z_h, \phi_h, p_h, w_h) \in V_h \times V_h^{CR} \times Q_h \times V_h\) such that

\[(\text{curl}_h z_h, \text{curl}_h v_h) = (\nabla w_h, \nabla v_h) \quad \forall \ v_h \in V_h, \quad (5.8a)\]

\[(\phi_h, \psi_h) + \varepsilon^2 c_h(\phi_h, \psi_h) + (\text{div}_h \psi_h, p_h) = (\text{curl}_h z_h, \psi_h) \quad \forall \ \psi_h \in V_h^{CR}, \quad (5.8b)\]

\[(\text{div}_h \phi_h, q_h) = 0 \quad \forall \ q_h \in Q_h, \quad (5.8c)\]

\[(\text{curl}_h u_{h0}, \text{curl}_h \chi_h) = (\phi_h, \text{curl}_h \chi_h) \quad \forall \ \chi_h \in V_h. \quad (5.8d)\]

where

\[c_h(\phi_h, \psi_h) := (\nabla_h \phi_h, \nabla_h \psi_h) - \sum_{F \in \mathcal{F}_h^0} (\partial_n(\phi_h \cdot t), \psi_h \cdot t) F - \sum_{F \in \mathcal{F}_h^0} (\phi_h \cdot t, \partial_n(\psi_h \cdot t)) F + \sum_{F \in \mathcal{F}_h^0} \frac{\sigma}{h_F} (\phi_h \cdot t, \psi_h \cdot t) F.\]

Combining Lemmas 5.1 and 5.3 yields an equivalent discrete method of the MWX element method (3.2). And combining Lemmas 5.2 and 5.4 yields an equivalent discrete method of the MWX element method (4.1).

\textbf{Theorem 5.5} In two dimensions, the MWX element method (3.2) can be decoupled into (5.1) and (5.7a)–(5.7d). That is, the MWX element method (3.2) can be decoupled into one Lagrange element method of Poisson equation, two Morley element methods of Poisson equation and one nonconforming \( P_1-P_0 \) element method of Brinkman problem.
Theorem 5.6 In two dimensions, the MWX element method (4.1) can be decoupled into one Lagrange element method of Poisson equation (5.3), two Morley element methods of Poisson equation and one nonconforming $P_1-P_0$ element method of Brinkman problem (5.8a)–(5.8d).

The decoupling of the fourth order elliptic singular perturbation problem (1.1) into two Poisson equations and one Brinkman problem in the continuous level have been developed in [12,20]. The decoupling of the Morley element method of the biharmonic equation into two Morley element methods of Poisson equation and one nonconforming $P_1-P_0$ element method of Stokes equation was firstly discovered in [26].

When $\varepsilon$ is very small, the stiffness matrix of the MWX element method (3.2) is very close to the stiffness matrix of the MWX element method for the Poisson equation, which can be efficiently solved by the CG method with AMG as the preconditioner.

When $\varepsilon \approx 1$, the equivalences in Theorems 5.5–5.6 will induce efficient and robust Poisson based solvers for the MWX element method (3.2) and (4.1). The Lagrange element methods of Poisson equation (5.1) and (5.3), the Morley element methods of Poisson equation (5.7a), (5.7d), (5.8a) and (5.8d) can be solved by CG method with the auxiliary space preconditioner [46], in which the $H^1$ conforming linear element discretization on the same mesh for the Poisson equation can be adopted as the auxiliary problem. And the AMG method is used to solve the auxiliary problem. If $\ell = 1$, the Lagrange element methods of Poisson equation (5.1) and (5.3) are the linear Lagrange element methods, which can be solved efficiently by CG method using the classical AMG method as the preconditioner.

As for the nonconforming $P_1-P_0$ element methods of Brinkman problem (5.7b)–(5.7c) and (5.8b)–(5.8c), we can use the block-diagonal preconditioner in [7,27,31] or the approximate block-factorization preconditioner in [11], which are robust with respect to the parameter $\varepsilon$ and mesh size $h$. We will adopt the following approximate block-factorization preconditioner in the numerical part

$$\begin{pmatrix} A_h - B_h^T \tilde{M}_h^{-1} B_h & B_h^T \\ B_h & -\tilde{M}_h \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ \tilde{M}_h^{-1} B_h & -I \end{pmatrix}^{-1} \begin{pmatrix} A_h & -B_h^T \\ 0 & \tilde{M}_h \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ \tilde{M}_h^{-1} B_h & -I \end{pmatrix} \begin{pmatrix} A_h & -B_h^T \\ 0 & \tilde{M}_h \end{pmatrix}^{-1},$$

where $\begin{pmatrix} A_h & B_h^T \\ B_h & \tilde{M}_h \end{pmatrix}$ is the stiffness matrix of the Brinkman equation (5.7b)–(5.7c), and $\tilde{M}_h = \alpha \varepsilon^{-2} M_h$ with $\alpha > 0$ and $M_h$ being the mass matrix for the pressure. Since all these solvers are based on the solvers of the Poisson equation and we use AMG method to solve the discrete methods of the Poisson equation, the designed fast solver of the MWX element method (3.2) [the MWX element method (4.1)] based on the discrete method (5.1) and (5.7a)–(5.7d) [the discrete method (5.3) and (5.8a)–(5.8d)] also works for the shape-regular unstructured meshes.

6 Numerical Results

In this section, we will provide some numerical examples to verify the theoretical convergence rates of the MWX element method (3.2) and (4.1), and test the efficiency of a solver based on the decoupled method (5.7a)–(5.7d). Let $\Omega$ be the unit square $(0, 1)^2$, and we use the uniform triangulation of $\Omega$. All the experiments are implemented with the scikit-fem library.
Table 1 Error $\|u - u_{h0}\|_{\varepsilon, h}$ of the MWX method (3.2) for Example 6.1 with different $\varepsilon$ and $h$

| $\varepsilon$ | $h$ | $2^{-1}$ | $2^{-2}$ | $2^{-3}$ | $2^{-4}$ | $2^{-5}$ | $2^{-6}$ | $2^{-7}$ | $2^{-8}$ |
|--------------|-----|---------|---------|---------|---------|---------|---------|---------|---------|
| $1$          |     | 1.232E+01 | 7.584E+00 | 3.839E+00 | 1.896E+00 | 9.433E-01 | 4.710E-01 | 2.354E-01 | 1.177E-01 |
|              | $10^{-1}$ | 1.617E+00 | 1.024E+00 | 4.386E-01 | 1.977E-01 | 9.539E-02 | 4.723E-02 | 2.356E-02 | 1.177E-02 |
|              | $10^{-2}$ | 1.148E+00 | 7.291E-01 | 2.383E-01 | 1.820E-02 | 6.062E-03 | 2.537E-03 | 1.200E-03 | 1.200E-03 |
|              | $10^{-3}$ | 1.143E+00 | 7.260E-01 | 2.371E-01 | 1.665E-02 | 4.202E-03 | 1.057E-03 | 2.761E-04 | 2.761E-04 |
|              | $10^{-4}$ | 1.143E+00 | 7.260E-01 | 2.371E-01 | 1.666E-02 | 4.205E-03 | 1.055E-03 | 2.641E-04 | 2.641E-04 |
|              | $10^{-5}$ | 1.143E+00 | 7.260E-01 | 2.371E-01 | 1.666E-02 | 4.205E-03 | 1.055E-03 | 2.642E-04 | 2.642E-04 |

[23]. The source code in this section can be downloaded from https://github.com/YerbaPage/FEM_forth_perturb.

Example 6.1 We first test the MWX element method (3.2) with the exact solution

$$u(x, y) = \sin^2(\pi x) \sin^2(\pi y).$$

The right hand side $f$ is computed from (1.1). Notice that the solution $u$ does not have boundary layers. Take $\ell = 1$.

The energy error $\|u - u_{h0}\|_{\varepsilon, h}$ with different $\varepsilon$ and $h$ is shown in Table 1. From Table 1 we observe that $\|u - u_{h0}\|_{\varepsilon, h} = O(h)$ for $\varepsilon = 1, 10^{-1}, 10^{-2}$, which agrees with the theoretical convergence result (3.7) with $r = 0$. While numerically $\|u - u_{h0}\|_{\varepsilon, h} = O(h^2)$ for $\varepsilon = 10^{-3}, 10^{-4}, 10^{-5}$ in Table 1, which is superconvergent and one order higher than the theoretical convergence result (3.7) with $r = 1$.

Then examine the efficiency of solvers for the MWX element method (3.2). The stop criterion in our iterative algorithms is the relative residual is less than $10^{-8}$. And the initial guess is zero. First, we solve the MWX element method (5.1)–(5.2) using the CG method with AMG method (AMG-CG) as the preconditioner. According to the iteration steps listed in the third column of Tables 3 and 4, equation (5.1) is highly efficiently solved by the AMG-CG solver. From Table 2 we can see that the AMG-CG solver is very efficient for $\varepsilon = 10^{-3}, 10^{-4}, 10^{-5}$, and the iteration steps of the AMG-CG solver is also acceptable for $\varepsilon = 10^{-2}, 10^{-3}$.

However the AMG-CG solver deteriorates for $\varepsilon = 1, 10^{-1}$, which means the AMG-CG solver doesn’t work for large $\varepsilon$. To deal with this, we adopt a solver for the MWX element method (5.2) based on the equivalent decoupling (5.7a)–(5.7d). To be specific, we adopt the AMG-CG solver to solve equations (5.7a) and (5.7d), and the GMRES method with the preconditioner (5.9), in which the parameter $\alpha = 2$, the restart in the GMRES is 20, and AMG is used to approximate the inverse of $A_h$. By the iteration steps listed in Tables 3 and 4, the AMG-CG solver is highly efficient for solving equations (5.7a) and (5.7d). And the preconditioned GMRES algorithm is also very efficient and robust for the nonconforming $P_1-P_0$ element method of Brinkman problem (5.7b)–(5.7c) for $\varepsilon = 1, 10^{-1}$.
### Table 2
Iteration steps of the MWX methods (3.2) for Example 6.1 with different \( \epsilon \) and \( h \)

| \( \epsilon \) | \( h \) | \( 2^{-1} \) | \( 2^{-2} \) | \( 2^{-3} \) | \( 2^{-4} \) | \( 2^{-5} \) | \( 2^{-6} \) | \( 2^{-7} \) | \( 2^{-8} \) |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 6 | 12 | 24 | 49 | 104 | 241 | > 1000 |
| \( 10^{-1} \) | 1 | 5 | 9 | 18 | 39 | 84 | 183 | 484 |
| \( 10^{-2} \) | 1 | 3 | 5 | 7 | 8 | 15 | 30 | 62 |
| \( 10^{-3} \) | 1 | 3 | 5 | 6 | 7 | 15 | 59 |
| \( 10^{-4} \) | 1 | 3 | 5 | 6 | 7 | 8 | 10 | 10 |
| \( 10^{-5} \) | 1 | 3 | 5 | 6 | 7 | 8 | 10 | 10 |
| #dofs | 25 | 81 | 289 | 1089 | 4225 | 16641 | 66049 | 263169 |

### Table 3
Iteration steps of the decoupled methods (5.7a)–(5.7d) for Example 6.1 with \( \epsilon = 1 \) and different \( h \)

| \( h \) | #dofs | Equation (5.1) | Equation (5.7a) | Equations (5.7b)–(5.7c) | Equation (5.7d) |
|---|---|---|---|---|---|
| \( 2^{-1} \) | 24 | 1 | 1 | 16 | 1 |
| \( 2^{-2} \) | 112 | 1 | 1 | 16 | 1 |
| \( 2^{-3} \) | 480 | 4 | 5 | 34 | 5 |
| \( 2^{-4} \) | 1984 | 6 | 7 | 34 | 7 |
| \( 2^{-5} \) | 8064 | 6 | 9 | 41 | 9 |
| \( 2^{-6} \) | 32512 | 7 | 11 | 43 | 11 |
| \( 2^{-7} \) | 130560 | 7 | 14 | 44 | 14 |
| \( 2^{-8} \) | 523264 | 9 | 17 | 46 | 17 |
| \( 2^{-9} \) | 2095104 | 9 | 20 | 50 | 21 |
| \( 2^{-10} \) | 8384512 | 12 | 27 | 55 | 27 |

### Table 4
Iteration steps of the decoupled methods (5.7a)–(5.7d) for Example 6.1 with \( \epsilon = 10^{-1} \) and different \( h \)

| \( h \) | #dofs | Equation (5.1) | Equation (5.7a) | Equations (5.7b)–(5.7c) | Equation (5.7d) |
|---|---|---|---|---|---|
| \( 2^{-1} \) | 24 | 1 | 1 | 26 | 1 |
| \( 2^{-2} \) | 112 | 1 | 3 | 35 | 3 |
| \( 2^{-3} \) | 480 | 4 | 5 | 39 | 5 |
| \( 2^{-4} \) | 1984 | 6 | 7 | 50 | 7 |
| \( 2^{-5} \) | 8064 | 6 | 9 | 57 | 9 |
| \( 2^{-6} \) | 32512 | 7 | 11 | 74 | 11 |
| \( 2^{-7} \) | 130560 | 7 | 14 | 74 | 14 |
| \( 2^{-8} \) | 523264 | 9 | 17 | 78 | 17 |
| \( 2^{-9} \) | 2095104 | 9 | 20 | 83 | 21 |
| \( 2^{-10} \) | 8384512 | 12 | 27 | 83 | 27 |
In summary, to efficiently solve the MWX method (3.2) we can employ the AMG-CG solver for small $\varepsilon$ and the GMRES method with the preconditioner (5.9) for large $\varepsilon$.

**Example 6.2** Next we verify the convergence of the MWX methods (3.2) and (4.1) for problem (1.1) with boundary layers. Let the exact solution of the Poisson equation (2.12) be

$$u^0(x, y) = \sin(\pi x) \sin(\pi y).$$

Then the right hand term for both problems (1.1) and (2.12) is set to be
Finally we verify the convergence of the MWX method (3.2), equivalently
\[ \| u_N - u_h \|_0 = 6.441E-02, \]
\[ \| u_N - u_h \|_{1,h} = 4.834E-01, \]
\[ \| u_N - u_h \|_{2,h} = 6.368E+00, \]
\[ \| u_N - u_h \|_{ \epsilon,h} = 4.190E-01. \]

The explicit expression solution \( u \) for problem (1.1) with this right hand term is unknown. The solution \( u \) possesses strong boundary layers when \( \varepsilon \) is very small. Here we choose \( \varepsilon = 10^{-6} \). We use the discrete solution of the MWX element method (4.1) with \( h = 2^{-7} \), denoted by \( u_h^N \), as the approximation of \( u \). Errors \( \| u_N - u_h \|_0, \| u_N - u_h \|_{1,h}, \| u_N - u_h \|_{2,h} \) and \( \| u_N - u_h \|_{ \epsilon,h} \) of the discrete method (3.2) for \( \ell = 1 \) and \( \ell = 2 \) are present in Tables 5 and 6 respectively, from which we can see that \( \| u_N - u_h \|_0 = O(h^{1.5}), \| u_N - u_h \|_{1,h} = O(h^{0.6}), \| u_N - u_h \|_{2,h} = O(1), \) and \( \| u_N - u_h \|_{ \epsilon,h} \) is close to \( O(h^{0.56}) \). The numerical convergence rate of error \( \| u_N - u_h \|_{ \epsilon,h} \) coincides with (3.8).

After applying the Nitsche’s technique with the penalty constant \( \sigma = 5 \), errors \( \| u_N - u_h \|_0, \| u_N - u_h \|_{1,h}, \| u_N - u_h \|_{2,h} \) and \( \| u_N - u_h \|_{ \epsilon,h} \) of the discrete method (4.1) for \( \ell = 1 \) and \( \ell = 2 \) are present in Tables 7 and 8 respectively. When \( \ell = 1 \), numerically \( \| u_N - u_h \|_0 = O(h^{2.2}), \| u_N - u_h \|_{1,h} = O(h^{1.67}), \| u_N - u_h \|_{2,h} = O(h^{0.66}), \) and \( \| u_N - u_h \|_{ \epsilon,h} = O(h^{1.62}) \). All these convergence rates are optimal. And the convergence rates of \( \| u_N - u_h \|_{1,h} \) and \( \| u_N - u_h \|_{2,h} \) are almost half order higher than the optimal rates, as indicated by (4.9). For \( \ell = 2 \), it is observed from Table 8 that \( \| u_N - u_h \|_0 = O(h^{3}), \| u_N - u_h \|_{1,h} = O(h^2), \| u_N - u_h \|_{2,h} = O(h) \) and \( \| u_N - u_h \|_{ \epsilon,h} = O(h^2) \). Again all these convergence rates are optimal, and the convergence rate of \( \| u_N - u_h \|_{ \epsilon,h} \) is in coincidence with (4.9).

**Example 6.3** Finally we verify the convergence of the MWX method (3.2), equivalently (5.1)–(5.2), for a problem with a corner singularity on a non-convex domain. Let \( \Omega \) be the
Table 10 Errors of the discrete method (3.2) for Example 6.3 with different $h$ when $\varepsilon = 1$ and $\ell = 2$

| $h$ | $2^{-1}$ | $2^{-2}$ | $2^{-3}$ | $2^{-4}$ | $2^{-5}$ | $2^{-6}$ | $2^{-7}$ |
|-----|----------|----------|----------|----------|----------|----------|----------|
| $\|u - u_h\|_0$ | 1.732E–02 | 5.491E–03 | 1.740E–03 | 5.600E–04 | 1.863E–04 | 6.481E–05 | 2.364E–05 |
| $|u - u_h|_{1,h}$ | 7.554E–02 | 3.140E–02 | 1.183E–02 | 4.228E–03 | 1.494E–03 | 5.297E–04 | 1.897E–04 |
| $|u - u_h|_{2,h}$ | 6.127E–01 | 4.013E–01 | 2.548E–01 | 1.605E–01 | 1.010E–01 | 6.367E–02 | 4.013E–02 |
| $\|u - u_h\|_{\varepsilon,h}$ | 5.403E–01 | 3.708E–01 | 2.432E–01 | 1.563E–01 | 9.956E–02 | 6.314E–02 | 3.994E–02 |

Table 11 Errors of the discrete method (3.2) for Example 6.3 with different $h$ when $\varepsilon = 10^{-6}$ and $\ell = 1$

| $h$ | $2^{-1}$ | $2^{-2}$ | $2^{-3}$ | $2^{-4}$ | $2^{-5}$ | $2^{-6}$ | $2^{-7}$ |
|-----|----------|----------|----------|----------|----------|----------|----------|
| $\|u - u_h\|_0$ | 2.030E–02 | 3.950E–03 | 7.150E–04 | 1.310E–04 | 2.500E–05 | 5.090E–06 | 1.110E–06 |
| $|u - u_h|_{1,h}$ | 1.220E–01 | 4.120E–02 | 1.400E–02 | 4.790E–03 | 1.660E–03 | 5.760E–04 | 2.010E–04 |
| $|u - u_h|_{2,h}$ | 9.630E–01 | 6.430E–01 | 4.340E–01 | 2.950E–01 | 2.010E–01 | 1.380E–01 | 9.530E–02 |
| $\|u - u_h\|_{\varepsilon,h}$ | 1.020E–01 | 3.730E–02 | 1.330E–02 | 4.660E–03 | 1.630E–03 | 5.710E–04 | 2.000E–04 |

Table 12 Errors of the discrete method (3.2) for Example 6.3 with different $h$ when $\varepsilon = 10^{-6}$ and $\ell = 2$

| $h$ | $2^{-1}$ | $2^{-2}$ | $2^{-3}$ | $2^{-4}$ | $2^{-5}$ | $2^{-6}$ | $2^{-7}$ |
|-----|----------|----------|----------|----------|----------|----------|----------|
| $\|u - u_h\|_0$ | 3.772E–03 | 6.095E–04 | 9.987E–05 | 1.644E–05 | 2.719E–06 | 4.546E–07 | 7.773E–08 |
| $|u - u_h|_{1,h}$ | 3.749E–02 | 1.171E–02 | 3.720E–03 | 1.181E–03 | 3.744E–04 | 1.185E–04 | 3.743E–05 |
| $|u - u_h|_{2,h}$ | 5.349E–01 | 3.439E–01 | 2.206E–01 | 1.408E–01 | 8.952E–02 | 5.674E–02 | 3.589E–02 |
| $\|u - u_h\|_{\varepsilon,h}$ | 3.371E–02 | 1.110E–02 | 3.620E–03 | 1.165E–03 | 3.716E–04 | 1.180E–04 | 3.735E–05 |

$L$-shaped domain $(-1, 1)^2 \setminus [0, 1) \times (-1, 0]$ and the exact solution be (cf. [21])

$$u(r, \varphi) = r^{5/3} \sin \left( \frac{5\varphi}{3} \right).$$

The exact solution $u$ contains a singularity at the origin of $\Omega$, and we only have $u \in H^{8/3-\varepsilon}(\Omega)$ for any $\varepsilon > 0$.

After imposing appropriate inhomogeneous boundary condition, errors $\|u - u_h\|_0$, $|u - u_h|_{1,h}$, $|u - u_h|_{2,h}$ and $\|u - u_h\|_{\varepsilon,h}$ of the discrete method (3.2) for $\varepsilon = 1$ and $\varepsilon = 10^{-6}$
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