BV right continuous solutions of differential inclusions involving time dependent maximal monotone operators

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Abstract

This paper is devoted to the study of evolution problems involving time dependent maximal monotone operators, which are right continuous and bounded in variation with respect to the Vladimirov’s pseudo distance. Several variants and applications are presented.

Keywords: Bounded variation, differential measure, maximal monotone operator, pseudo-distance, right continuous.

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1 Introduction

Let E be a separable Hilbert space, and I = [0, T] (T > 0). In this paper, we are mainly interested by the existence of bounded variation and right continuous (BVRC) solutions to evolution inclusion of the form

\[ -D u(t) \in A(t)u(t) + F(t, u(t)) \text{ a.e.,} \quad u(0) = u_0, \]  

(1.1)
governed by a time dependent maximal monotone operator A(t), in the vein of Kunze-Marques work [31], with a weakly compact, convex valued perturbation F : I × E ⇒ E. For this purpose, we consider the existence problem of BVRC solutions to (1.1) by assuming that t ↦ A(t) is of bounded variation and right continuous, in the sense that there exists a

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function \( r : I \to [0, \infty] \), which is right continuous on \([0, T]\) and nondecreasing with \( r(0) = 0 \) and \( r(T) < \infty \) such that
\[
\text{dis}(A(t), A(s)) \leq dr([s, t]) = r(t) - r(s), \quad 0 \leq s \leq t \leq T,
\]
where \( \text{dis}(\cdot, \cdot) \) is the pseudo-distance between maximal monotone operators introduced by Vladimirov [50]; see relation (2.1).

When \( C(t) \) is a closed convex moving set in \( E \), then with \( A(t) = \partial \delta_{C(t)} = N_{C(t)} \), we have
\[
\text{dis}(A(t), A(s)) = d_H(C(t), C(s)), \quad t, s \in I,
\]
here \( d_H \) denotes the Hausdorff distance. In this regard, our study extends some related results in the evolution problem governed by the convex sweeping process of the form
\[
-Du(t) \in N_{C(t)}(u(t)) + F(t, u(t)) \quad a.e., \quad u(0) = u_0.
\]

There is an intensive work concerning the existence of solutions to the sweeping process. This subject is vast; see [1, 37, 35, 34] and the references therein. However, there are a few results concerning the existence of BVRC solution to the sweeping process, see Benabdellah et al [9], Castaing-Marques [19], Adly et al [1, 2], Edmond-Thibault [25], Nacry et al [40].

Let \( \lambda \) be the Lebesgue measure on \( I \) and \( dr \) the Stieljes measure associated with \( r \). We set \( \nu := \lambda + dr \) and \( \frac{d\lambda}{d\nu} \) the density of \( \lambda \) with respect to the measure \( \nu \). A function \( u : I \to E \) is BVRC if \( u \) is of bounded variation and right continuous. By BVRC solution to (1.1) we mean that, giving \( u_0 \in D(A(0)) \), there exists a BVRC mapping \( u : I \to E \) such that
\[
\begin{align*}
&u(0) = u_0; \\
&u(t) \in D(A(t)) \quad \forall t \in I; \\
&-\frac{du}{d\nu}(t) \in A(t)u(t) + F(t, u(t)) \frac{d\lambda}{d\nu}(t) \quad \text{d}\nu - a.e. \, t \in I,
\end{align*}
\]

here \( \frac{du}{d\nu}(t) \) denotes the density of \( u \) relatively to \( \nu \). We aim to present the problem of existence and uniqueness of BVRC solutions to (1.1) according to the nature of the perturbation \( F \), and its applications such as Skorohod problem, relaxation, second order evolution, sweeping process. To the best of the author’s knowledge, this problem of existence and uniqueness of BVRC solution of (1.1) has never been considered in the literature before. The study of BV solutions of (1.1) and (1.2) has an increasing interest. Many attempts have been made to generalize evolution inclusion (1.2) containing deterministic or stochastic perturbations [24, 26]. Actually, (1.1) is the first study of convex, weakly compact valued perturbations of evolution inclusion governed by time dependent maximal monotone operators involving the existence of BVRC solutions. For some recent results dealing with existence of absolutely continuous, Lipschitz or continuous with bounded variation (BVC) solutions of differential inclusions governed by time dependent or time and state dependent maximal monotone operators, we refer to [3, 4, 5, 6, 10, 22, 29, 32, 44, 48].

The paper is organized as follows. In section 2, we recall some preliminary results needed later. In section 3, our main theorems state the existence of bounded variation and right
continuous solutions to the evolution inclusion \(1.1\) when \(t \mapsto A(t)\) is BVRC and the perturbation \(F : I \times E \rightrightarrows E\) is convex and weakly compact valued, separately scalarly upper semi continuous on \(E\) and measurable on \(I \times E\), with sharp application to the existence and uniqueness of BVRC solution to the evolution inclusion

\[-Du(t) \in A(t)u(t) + f(t, u(t)) \quad \text{a.e.,} \quad u(0) = u_0,\]

where \(f : I \times E \to E\) is single valued, Borel-measurable, and satisfies a Lipschitz condition. Here uniqueness is provided using a specific result due to Moreau \([34]\) on the property of BVRC mappings and some specific Gronwall type lemma. In section 4, we provide some applications to second order and fractional evolutions, Skorohod problem and relaxation problem.

The obtained results are quite new with remarkable corollaries in the setting of BVRC solutions, and they extend to time dependent BVRC maximal monotone operators some nice results of Adly et al \([1]\), Edmond-Thibault \([25]\) and Nacry et al \([40]\), dealing with the BVRC solutions, and they extend to time dependent BVRC maximal monotone operators some nice results of Tolstonogov \([47]\), dealing with the BVC sweeping process.

### 2 Notations and Preliminaries

In the whole paper \(I := [0, T] \ (T > 0)\) is an interval of \(\mathbb{R}\) and \(E\) is a separable Hilbert space with the scalar product \(\langle \cdot, \cdot \rangle\) and the associated norm \(\| \cdot \|\). \(\overline{B}_E\) denotes the unit closed ball of \(E\) and \(r \overline{B}_E\) its closed ball of center 0 and radius \(r > 0\). We denote by \(\mathcal{L}(I)\) the sigma algebra on \(I\), \(\lambda := dt\) the Lebesgue measure and by \(\mathcal{B}(E)\) (resp. \(\mathcal{B}(I)\)) the Borel sigma algebra on \(E\) (resp. on \(I\)). If \(\mu\) is a positive measure on \(I\), we will denote by \(L^p(I, E; \mu) \ p \in [1, +\infty[\) (resp. \(p = +\infty\)), the Banach space of classes of measurable functions \(u : I \to E\) such that \(t \mapsto ||u(t)||^p\) is \(\mu\)-integrable (resp. \(u\) is \(\mu\)-essentially bounded), equipped with its classical norm \(|| \cdot ||_p\) (resp. \(|| \cdot ||_\infty\)). We denote by \(\mathcal{C}(I, E)\) the Banach space of all continuous mappings \(u : I \to E\), endowed with the sup norm.

The excess between closed subsets \(C_1 \) and \(C_2 \) of \(E\), is defined by \(e(C_1, C_2) := \sup_{x \in C_1} d(x, C_2)\), and the Hausdorff distance between them is given by

\[d_H(C_1, C_2) := \max \left\{ e(C_1, C_2), e(C_2, C_1) \right\}.\]

The support function of \(S \subset E\) is defined by: \(\delta^*(a, S) := \sup_{x \in S} \langle a, x \rangle, \forall a \in E\).

If \(X\) is a Banach space and \(X'\) its topological dual, we denote by \(\sigma(X, X')\) the weak topology on \(X\), and by \(\sigma(X', X)\) the weak* topology on \(X'\).

Let \(A : E \rightrightarrows E\) be a set-valued map. We denote by \(D(A)\), \(R(A)\) and \(Gr(A)\) its domain, range and graph. We say that \(A\) is monotone, if \(\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0\) whenever \(x_i \in D(A)\), and \(y_i \in A(x_i), \ i = 1, 2\). In addition, we say that \(A\) is a maximal monotone operator of \(E\), if its graph could not be contained properly in the graph of any other monotone operator. By Minty’s Theorem, \(A\) is maximal monotone iff \(R(I_E + A) = E\), where \(I_E\) is the identity mapping of \(E\).

If \(A\) is a maximal monotone operator of \(E\), then for every \(x \in D(A)\), \(A(x)\) is nonempty closed and convex. We denote the projection of the origin on the set \(A(x)\) by \(A^0(x)\).
Let \( \eta > 0 \), then the resolvent and the Yosida approximation of \( A \) are the well-known operators defined respectively by \( J^A_\eta = (I_E + \eta A)^{-1} \) and \( A_\eta = \frac{1}{\eta}(I_E - J^A_\eta) \). These operators are single-valued and defined on all of \( E \), and we have \( J^A_\eta(x) \in D(A) \), for all \( x \in E \). For more details about the theory of maximal monotone operators we refer the reader to \([7, 11, 51]\).

Let \( A : D(A) \subset E \to 2^E \) and \( B : D(B) \subset E \to 2^E \) be two maximal monotone operators, then we denote by \( \text{dis}(A, B) \) the pseudo-distance between \( A \) and \( B \) defined by

\[
\text{dis}(A, B) = \sup \left\{ \frac{\langle y - y', x' - x \rangle}{1 + \|y\| + \|y'\|} : x \in D(A), y \in Ax, x' \in D(B), y' \in Bx' \right\}.
\] (2.1)

Our main results are established under the following hypotheses:

\( (H_1) \) There exists a function \( r : I \to [0, +\infty] \) which is right continuous on \([0, T] \) and nondecreasing with \( r(0) = 0 \) and \( r(T) < +\infty \) such that

\[
\text{dis}(A(t), A(s)) \leq dr([s, t]) = r(t) - r(s) \quad \text{for} \quad 0 \leq s \leq t \leq T.
\]

\( (H_2) \) There exists a nonnegative real constant \( c \) such that

\[
\|A^0(t, x)\| \leq c(1 + \|x\|) \quad \text{for} \quad t \in I, x \in D(A(t)).
\]

\( (H_3) \) \( \bigcup_{t \in I} D(A(t)) \) is ball compact, i.e., its intersection with any closed ball of \( E \) is compact.

For the proof of our main theorems we will need some elementary lemmas taken from reference \([31]\).

**Lemma 2.1** Let \( A \) be a maximal monotone operator of \( E \). If \( x \in D(A) \) and \( y \in E \) are such that

\[
\langle A^0(z) - y, z - x \rangle \geq 0 \quad \forall z \in D(A),
\]

then \( x \in D(A) \) and \( y \in A(x) \).

**Lemma 2.2** Let \( A_n \) \((n \in \mathbb{N})\) and \( A \) be maximal monotone operators of \( E \) such that \( \text{dis}(A_n, A) \to 0 \). Suppose also that \( x_n \in D(A_n) \) with \( x_n \to x \) and \( y_n \in A_n(x_n) \) with \( y_n \to y \) weakly for some \( x, y \in E \). Then \( x \in D(A) \) and \( y \in A(x) \).

**Lemma 2.3** Let \( A \) and \( B \) be maximal monotone operators of \( E \). Then

1) \( \eta > 0 \) and \( x \in D(A) \)

\[
\|x - J^B_\eta(x)\| \leq \eta\|A^0(x)\| + \text{dis}(A, B) + \sqrt{\eta(1 + \|A^0(x)\|)}\text{dis}(A, B).
\]

2) For \( \eta > 0 \) and \( x, x' \in E \)

\[
\|J^A_\eta(x) - J^A_\eta(x')\| \leq \|x - x'\|.
\]

**Lemma 2.4** Let \( A_n \) \((n \in \mathbb{N})\) and \( A \) be maximal monotone operators of \( E \) such that \( \text{dis}(A_n, A) \to 0 \) and \( \|A^0_n(x)\| \leq c(1 + \|x\|) \) for some \( c > 0 \), all \( n \in \mathbb{N} \) and \( x \in D(A_n) \). Then for every \( z \in D(A) \) there exists a sequence \((\zeta_n)\) such that

\[
\zeta_n \in D(A_n), \quad \zeta_n \to z \quad \text{and} \quad A^0_n(\zeta_n) \to A^0(z).
\] (2.2)
We finish this section by some types of Gronwall’s lemma, which are crucial for our purpose.

**Lemma 2.5** Let \((\alpha_i), (\beta_i), (\gamma_i)\) and \((a_i)\) be sequences of nonnegative real numbers such that 
\[ a_{i+1} \leq \alpha_i + \beta_i (a_0 + a_1 + \ldots + a_{i-1}) + (1 + \gamma_i)a_i \text{ for } i \in \mathbb{N}. \]
Then
\[ a_j \leq \left( a_0 + \sum_{k=0}^{j-1} \alpha_k \right) \exp \left( \sum_{k=0}^{j-1} (k\beta_k + \gamma_k) \right) \text{ for } j \in \mathbb{N}^*. \]

**Lemma 2.6** Let \(\mu\) be a positive Radon measure on \(I\). Let \(g \in L^1(I, \mathbb{R}; \mu)\) be a nonnegative function and \(\beta \geq 0\) be such that, \(\forall t \in I, 0 \leq \mu(\{t\})g(t) \leq \beta < 1\). Let \(\varphi \in L^\infty(I, \mathbb{R}; \mu)\) be a nonnegative function satisfying
\[ \varphi(t) \leq \alpha + \int_{[0,t]} g(s)\varphi(s)\mu(ds) \quad \forall t \in I, \]
where \(\alpha\) is a nonnegative constant. Then
\[ \varphi(t) \leq \alpha \exp \left( \frac{1}{1-\beta} \int_{[0,t]} g(s)\mu(ds) \right) \quad \forall t \in I. \]

Proof. This lemma is due to M.M. Marques. For a proof, see e.g. ([8], Lemma 2.1). □

**Lemma 2.7** Let \(\mu\) be a non-atomic positive Radon measure on the interval \(I\). Let \(c, p\) be nonnegative real functions such that \(c \in L^1(I, \mathbb{R}; \mu), p \in L^\infty(I, \mathbb{R}; \mu)\), and let \(\alpha \geq 0\). Assume that for \(\mu\)–a.e. \(t \in I\)
\[ p(t) \leq \alpha + \int_0^t c(s)p(s)\mu(ds). \]
Then, for \(\mu\)–a.e. \(t \in I\)
\[ p(t) \leq \alpha \exp \left( \int_0^t c(s)\mu(ds) \right). \]

The proof (see [8], Lemma 2.7; or [34], Lemma 4, taking \(\eta = 0\)) is not a consequence of the classical Gronwall lemma dealing with Lebesgue measure \(\lambda\) on \(I\). It relies on a deep result of Moreau-Valadier on the derivation of (vector) functions of bounded variation [39].

**Lemma 2.8** (Proposition 4.1 in [47]) Let \(m \in L^1(I, \mathbb{R}; \lambda)\) be a nonnegative function, and let \(x : I \rightarrow [0, +\infty[\) be a right continuous function of bounded variation. If
\[ \frac{1}{2} x^2(t) \leq \frac{1}{2} a^2 + \int_{[0,t]} m(s)x(s)ds, \quad t \in I, \quad a \geq 0, \]
then
\[ x(t) \leq a + \int_{[0,t]} m(s)ds, \quad t \in I. \]
3 Main results: Existence and uniqueness of BVRC solutions

We recall, unless stated, that in all the paper, $E$ is a separable real Hilbert space, $\lambda$ is the Lebesgue measure on $I$, $dr$ is the Stieljes measure associated with $r$, $\nu := \lambda + dr$ and $\frac{d\lambda}{d\nu}$ is the density of $\lambda$ with respect to the measure $\nu$.

In this section we are interested by the existence of bounded variation right continuous (shortly BVRC) solutions to the inclusion $(1.1)$. For the sake of completeness let us state and summarize some useful facts. We refer to [8, 43] for the proof.

**Theorem 3.1** Let $X$ be a separable Banach space, $(I, T_\mu, \mu)$ be a measure space, where $\mu$ is a positive Radon measure, and let $\Gamma : I \rightrightarrows X$ be a convex weakly compact valued multi-mapping, which is scalarly $T_\mu$-measurable and such that $\Gamma(t) \subset m(t)X$ for some nonnegative function $m \in L^1(I, \mathbb{R}; \mu)$. Let $S^1_I$ be the set of all $L^1(I, X; \mu)$-selections of $\Gamma$, i.e.,

$$S^1_I = \{ \phi \in L^1(I, X; \mu) : \phi(t) \in \Gamma(t) \forall t \in I \}.$$

Then the following properties hold.

(i) $S^1_I$ is convex weakly compact in $L^1(I, X; \mu)$.

(ii) Let $x_0 \in X$. For each $h \in S^1_I$, the mapping $t \in I \mapsto u_h(t) := x_0 + \int_{0}^{t} h(s) d\mu(s)$ is BVRC with $\frac{du_h}{d\nu} = h$ $\mu$-a.e., and the set $\{ u_h : h \in S^1_I \}$ is equi-right continuous with bounded variation, i.e., for all $h \in S^1_I$

$$\|u_h(t) - u_h(\tau)\| \leq \int_{\tau}^{t} m(s) d\mu(s) \text{ for all } 0 \leq \tau \leq t \leq T.$$

(iii) Let $(h_n)$ be a sequence in $S^1_I$, then by extracting a subsequence, that we do not relabel, $(h_n)$ converges weakly to some mapping $h \in S^1_I$, so that $(u_{h_n})$ pointwise converges weakly to the BVRC mapping $u_h$, with for all $t \in I$, $u_h(t) = x_0 + \int_{0}^{t} h(s) d\mu(s)$ and $\frac{du_h}{d\nu} = h$ $\mu$-a.e.

(iv) Assume further that for each $t \in I$, $\{ u_{h_n}(t) : n \in \mathbb{N} \}$ is relatively compact, then $(u_{h_n})$ pointwise converges strongly to $u_h$.

(v) Assume that $\Gamma(t)$ is convex compact for each $t \in I$, then the set $\{ u_{h_n}(t) : n \in \mathbb{N} \}$ is relatively compact and $(u_{h_n})$ pointwise converges strongly to $u_h$.

Now, we proceed to state the main existence results. We begin with an existence of a second order BVRC solution to our evolution inclusion.

**Theorem 3.2** Let $f : I \longrightarrow E$ be a $\lambda$-measurable mapping such that $\|f(t)\| \leq M$, for all $t \in I$, for some nonnegative real constant $M$. Let for every $t \in I$, $A(t) : D(A(t)) \subset E \rightrightarrows E$ be a maximal monotone operator satisfying $(H_1)$, $(H_2)$ and $(H_3)$. Then for any $x_0 \in E$, $u_0 \in D(A(0))$, there exists a unique BVRC solution $(x, u) : I \rightarrow E \times E$ to the problem

$$
\begin{cases}
x(t) = x_0 + \int_{0}^{t} u(s) d\nu(s) & \forall t \in I; \\
u(0) = u_0; \\
u(t) \in D(A(t)) & \forall t \in I; \\
\frac{d\nu}{d\nu}(t) \in A(t)u(t) + f(t)\frac{d\lambda}{d\nu}(t) & \nu \text{-a.e. } t \in I.
\end{cases}
$$
with the estimate \( \frac{du}{d\nu}(t) \in K_B E \) \( \nu \)-a.e., where \( K \) is a positive constant.

Proof. We choose a sequence \((\varepsilon_n) n \subset [0, 1]\), which decreases to 0 as \( n \to \infty \) and partition \( 0 = t^0_n < t^1_n < \ldots < t^k_n = T \) of \( I \) such that

\[
|t^i_n - t^i_{n-1}| + dr([t^i_n, t^i_{n+1}]) \leq \varepsilon_n \quad \text{for } i = 0, \ldots, k_n - 1.
\] (3.1)

We set \( I^0_n = \{ t^0_n \} \) and \( I^n_i = [t^i_n, t^i_{n+1}] \) for \( i = 0, \ldots, k_n - 1 \).

Such a partition can be obtained by considering the measure \( \nu = dr + \lambda \) using the constructions developed in Castaing et al [19]. For \( i = 0, \ldots, k_n - 1 \), let

\[
\delta^i_n = dr([t^i_n, t^i_{n+1}]) = r(t^i_{n+1}) - r(t^i_n), \quad \eta^i_n = t^i_{n+1} - t^i_n, \quad \beta^i_n = \nu([t^i_n, t^i_{n+1}]).
\] (3.2)

Let us define, for every \( n \geq 1 \), sequences \((x^n_i)_{0 \leq i \leq k_n-1}\) and \((u^n_i)_{0 \leq i \leq k_n-1}\) such that \( x^0_n = x_0 \), \( u^0_n = u_0 \in D(A(0)) \), and for \( i = 0, \ldots, k_n - 1 \),

\[
u^n_{i+1} = J^1_n(u^n_i - \int_{t^i_n}^{t^i_{n+1}} f(s) d\lambda(s))
\] (3.3)

and

\[
x^n_{i+1} = x^n_i + \beta^n_{i+1} u^n_{i+1},
\] (3.4)

with \( J^1_n := J^{A_{\beta^i_n}} = (I_E + \beta^i_n A(t^i_{n+1}))^{-1} \).

Remark that by the definition of the resolvent we have \( u^n_{i+1} \in D(A(t^i_{n+1})) \) and

\[-\frac{1}{\beta^n_{i+1}} (u^n_{i+1} - u^n_i + \int_{t^i_n}^{t^i_{n+1}} f(s) d\lambda(s)) \in A(t^i_{n+1}) u^n_{i+1}.
\] (3.5)

For \( t \in [t^i_n, t^i_{n+1}], i = 0, \ldots, k_n - 1 \), set

\[
x_n(t) = x^n_i + \frac{\nu([t^i_n, t])}{\nu([t^i_n, t^i_{n+1}])} (x^n_{i+1} - x^n_i)
\] (3.6)

and

\[
v_n(t) = u^n_i + \frac{\nu([t^i_n, t])}{\nu([t^i_n, t^i_{n+1}])} (u^n_{i+1} - u^n_i + \int_{t^i_n}^{t^i_{n+1}} f(s) d\lambda(s)) - \int_{t^i_n}^{t} f(s) d\lambda(s),
\] (3.7)

so that \( v_n, x_n \) are of bounded variation and right continuous on \( I \), with \( v_n(t^i_n) = u^n_i \) and \( x_n(t^i_n) = x^n_i \).

**Step 1.** Let us show that the sequence \((v_n)\) of step approximations is uniformly bounded in norm and variation.
We have from (3.3), Lemma 2.3 (H1), (H2) and the boundedness of \( f \), for \( i = 0, \ldots, k_n - 1 \),

\[
\|u_{i+1}^n - u_i^n\| = \|J_{i+1}^n(u_i^n - \int_{t_i^n}^{t_{i+1}^n} f(s) d\lambda(s)) - u_i^n\| \\
\leq \|J_{i+1}^n(u_i^n - \int_{t_i^n}^{t_{i+1}^n} f(s) d\lambda(s)) - J_{i+1}^n(u_i^n)\| + \|J_{i+1}^n(u_i^n) - u_i^n\| \\
\leq \int_{t_i^n}^{t_{i+1}^n} \|f\| d\lambda(s) + \beta_{i+1}^n \|A^0(t_i^n, u_i^n)\| + \text{dis}(A(t_{i+1}^n), A(t_i^n)) \\
+ \sqrt{\beta_{i+1}^n (1 + \|A^0(t_i^n, u_i^n)\|) \text{dis}(A(t_{i+1}^n), A(t_i^n))} \\
\leq M \beta_{i+1}^n + (1 + c(1 + \|u_i^n\|)) \beta_{i+1}^n + \sqrt{(1 + c(1 + \|u_i^n\|))(\beta_{i+1}^n)^2} \\
\leq M \beta_{i+1}^n + (1 + c(1 + \|u_i^n\|)) \beta_{i+1}^n + (1 + c(1 + \|u_i^n\|)) \beta_{i+1}^n,
\]

that is,

\[
\|u_{i+1}^n - u_i^n\| \leq \left(2c\|u_i^n\| + 2(1 + c) + M\right) \beta_{i+1}^n. \tag{3.8}
\]

Then,

\[
\|u_i^n\| \leq (1 + 2c \beta_{i+1}^n) \|u_i^n\| + (2(1 + c) + M) \beta_{i+1}^n.
\]

By Lemma 2.5 we get

\[
\|u_i^n\| \leq \left(\|u_0\| + (2(1 + c) + M) \sum_{k=0}^{i-1} \beta_{k+1}^n\right) \exp\left(2c \sum_{k=0}^{i-1} \beta_{k+1}^n\right) \\
\leq \left(\|u_0\| + (2(1 + c) + M) \nu([0,T])\right) \exp\left(2c \nu([0,T])\right) =: K_1,
\]

and by (3.8)

\[
\|u_{i+1}^n - u_i^n\| \leq \left(2cK_1 + 2(1 + c) + M\right) \beta_{i+1}^n =: K_2 \beta_{i+1}^n.
\]

So that, if we set \( K = \max\{K_1, K_2\} \), we conclude that for \( 0 \leq i \leq k_n \), resp. \( i < k_n \):

\[
\|u_i^n\| \leq K, \text{ resp. } \|u_{i+1}^n - u_i^n\| \leq K \nu([t_i^n, t_{i+1}^n]). \tag{3.9}
\]

Now, for \( t \in [t_i^n, t_{i+1}^n] \), we have from (3.1), (3.7) and (3.9),

\[
\|v_n(t) - u_i^n\| = \left\|\frac{\nu(t, t_i^n)}{\nu(t_i^n, t_{i+1}^n)} (u_{i+1}^n - u_i^n) + \int_{t_i^n}^{t_{i+1}^n} f(s) d\lambda(s) - \int_{t_i^n}^{t} f(s) d\lambda(s)\right\| \\
\leq \|u_{i+1}^n - u_i^n\| + 2M \delta_{i+1}^n \leq (K + 2M) \epsilon_n =: M_1 \epsilon_n. \tag{3.10}
\]

Furthermore, from (3.9) and (3.10), for all \( n \in \mathbb{N} \),

\[
\|v_n(t)\| \leq K + M_1 \epsilon_n \leq K + M_1 =: M_2, \forall t \in I,
\]

that is

\[
\sup_n \|v_n\| = \sup_n \left(\sup_{t \in I} \|v_n(t)\|\right) \leq M_2. \tag{3.11}
\]
On the other hand, if we fix \( s \in [t^n_i, t^n_{i+1}] \) and \( t \in [t^n_j, t^n_{j+1}] \) with \( j > i \), we get by (3.11) and (3.10),

\[
\| v_n(t) - v_n(s) \| \leq \| v_n(t) - u^n_i \| + \| u^n_i - u^n_s \| + \| v_n(s) - u^n_s \|
\]

\[
\leq 2M_1\varepsilon_n + \sum_{k=0}^{j-i-1} \| u^n_{i+k+1} - u^n_{i+k} \|
\]

\[
\leq 2M_1\varepsilon_n + K \sum_{k=0}^{j-i-1} \nu([t^n_{i+k}, t^n_{i+k+1}]) = 2M_1\varepsilon_n + K\nu([t^n_{i}, t^n_{j}])
\]

\[
\leq 2M_1\varepsilon_n + K\nu([t^n_{i}, t^n_{j}]) \leq 2M_1\varepsilon_n + K(\nu([t^n_{i}, s]) + \nu([s, t]))
\]

\[
\leq 2M_1\varepsilon_n + K(\nu([t^n_{i}, t^n_{i+1}]) + \nu([s, t])).
\]

Finally, from (3.11), we obtain for \( n \in \mathbb{N} \) and \( 0 \leq s \leq t \leq T \),

\[
\| v_n(t) - v_n(s) \| \leq K\nu([s, t]) + (K + 2M_1)\varepsilon_n.
\] (3.12)

**Step 2.** Convergence of the sequences \((v_n)\) and \((x_n)\).

Let us define \( \theta_n : I \to I \) through

\[
\theta_n(t) = t^n_{i+1} \quad \text{for} \ t \in [t^n_i, t^n_{i+1}], \ i = 0, 1, \ldots, k_n - 1,
\]

and \( \theta_n(0) = 0 \). By (3.3), we know that for all \( t \in I \), \( v_n(\theta_n(t)) \in D(A(\theta_n(t))) \), using hypothesis \((H_3)\), we conclude that \((v_n(\theta_n(t)))\) is relatively compact.

On the other hand, observe that for all \( t \in I \), \( \nu([t, \theta_n(t)]) \to 0 \) as \( n \to \infty \). So that, from (3.12), \( \| v_n(\theta_n(t)) - v_n(t) \| \to 0 \). That is for all \( t \in I \), \((v_n(t))\) is also relatively compact.

Since the sequence \((v_n)\) of BVRC mappings is uniformly bounded in variation and in norm such that \((v_n(t)), t \in I\), is relatively compact, using Helly-Banach’s theorem [43], we may assume that there is a BV mapping \( u : I \to E \) such that \((v_n(t))\) converges strongly to \( u(t) \) for every \( t \in I \). In particular, \( u(0) = u_0 \) and by taking the limit in (3.12), we get

\[
\| u(t) - u(s) \| \leq K\nu([s, t]) \quad \text{for} \ 0 \leq s \leq t \leq T.
\]

It is clear that \( u \) is BVRC, and that, \( \|du\| \leq Kd\nu \) in the sense of the ordering of real measures and there exists a density \( u' \) of \( du \) with respect to \( d\nu \): \( du = u'\,d\nu \). Consequently, \( \|u'\|_\infty \leq K \), so that \( u' \in L^1(I, E; d\nu) \).

Moreover, since \( \|v_n(t) - u(t)\| \to 0 \), we obtain for all \( t \in I \),

\[
\|v_n(\theta_n(t)) - u(t)\| \leq \|v_n(t) - u(t)\| + \|v_n(t) - v_n(\theta_n(t))\| \to 0 \quad \text{as} \ n \to \infty.
\] (3.13)

Next remark that \( dv_n = v'_n\,d\nu \), where the density \( v'_n \) is given \( d\nu \)-almost everywhere by

\[
v'_n(t) = \frac{1}{\beta^n_{i+1}}(u^n_{i+1} - u^n_i + \int_{t^n_i}^{t^n_{i+1}} f(s)d\lambda(s)) - f(t)\frac{d\lambda}{d\nu}(t) \quad \text{for} \ t \in [t^n_i, t^n_{i+1}].
\] (3.14)

So that, by (3.9) and the boundedness of \( f \), we get

\[
\|v'_n\|_\infty \leq K + 2M.
\] (3.15)
Extracting a subsequence (not relabeled) we may assume that \((v'_n)\) converges weakly in \(L^1(I, H; d\nu)\) to some mapping \(w \in L^1(I, H; d\nu)\) with \(\|w(t)\| \leq K + 2M\ d\nu\)-a.e. Whence for \(t \in [0, T]\)

\[
\begin{align*}
  u(t) - u(0) &= \lim_{n \to \infty} (v_n(t) - v_n(0)) = \lim_{n \to \infty} dv_n([0, t]) \\
  &= \lim_{n \to \infty} \int_{[0, t]} v'_n d\nu = \lim_{n \to \infty} \int_{[0, t]} v'_n d\nu = \int_{[0, t]} w d\nu
\end{align*}
\]

that is

\[
(u' d\nu)([0, t]) = du([0, t]) = u(t) - u(0) = \int_{[0, t]} w d\nu = (w d\nu)([0, t]) \forall t \in I.
\]

Thus \(u' = w, d\nu\)-a.e. in \(I, i.e., (v'_n)\) converges weakly to \(u'\) in \(L^1(I, H; d\nu)\).

Now we note that

\[
x_n(t) = x_0 + \int_{[0, t]} v(\theta_n(s)) d\nu(s) \quad \forall t \in I.
\]

Indeed,

\[
x_0 + \int_{[0, t]} v(\theta_n(s)) d\nu(s) = x_0 + \int_{[0, t_1]} v(\theta_n(s)) d\nu(s) + \int_{[t_1, t_2]} v(\theta_n(s)) d\nu(s) + \cdots + \int_{[t_{i-1}, t_i]} v(\theta_n(s)) d\nu(s) + \int_{[t_i, t]} v(\theta_n(s)) d\nu(s)
\]

\[
= x_0 + \beta_1^n u_1^n + \beta_2^n u_2^n + \cdots + \nu([t_i^n, t]) u_{i+1}^n
\]

\[
= x_0 + \beta_2^n u_2^n + \cdots + \nu([t_i^n, t]) u_{i+1}^n
\]

\[
= x^n_0 + \nu([t^n_1, t]) \frac{x_{i+1}^n - x_i^n}{\beta_i^n} = x_n(t).
\]

As consequence we obtain from (3.9) and (3.13),

\[
\lim_{n \to \infty} x_n(t) = x_0 + \lim_{n \to \infty} \int_{[0, t]} v_n(\theta_n(s)) d\nu(s) = x_0 + \int_{[0, t]} u(s) d\nu(s) =: x(t),
\]

that is \(dx = u\ d\nu\)-a.e.

**Step 3.** We are going to show in this step that \(-u'(t) - f(t) \frac{d\lambda}{d\nu}(t) \in A(t) u(t)\) \(\nu\)-a.e. Referring to (3.5) and (3.14), there is a \(\nu\)-null set \(N_n\) such that

\[
-\frac{dv_n}{d\nu}(t) - f(t) \frac{d\lambda}{d\nu}(t) \in A(\theta_n(t)) v_n(\theta(t)) \quad \forall t \in I \setminus N_n,
\]

further

\[
v_n(\theta_n(t)) \in D(A(\theta_n(t))) \quad \forall t \in I.
\]
Since \(\text{dis}(A(\theta_n(t)), \theta(t)) \to 0\) as \(n \to \infty\), using Lemma 2.2, we conclude that \(u(t) \in D(A(t))\) for all \(t \in I\). Consequently, for our goal, using Lemma 2.4, it is enough to check that for \(v\) almost every \(t \in I\) and for all \(\gamma \in D(A(t))\),

\[
\left\langle \frac{d\zeta}{dv}(t) + f(t)\frac{d\lambda}{dv}(t), u(t) - \gamma \right\rangle \leq \left\langle A^0(t, \gamma), \gamma - u(t) \right\rangle.
\]

Indeed, since for all \(t \in I\), \(\text{dis}(A(\theta_n(t)), A(t)) \to 0\) as \(n \to \infty\) and since \((H_2)\) is satisfied, we may apply Lemma 2.4 to find a sequence \((\zeta_n)\) such that

\[
\zeta_n \in D(A(\theta_n(t))), \quad \zeta_n \to \gamma \quad \text{and} \quad A^0(\theta_n(t), \zeta_n) \to A^0(t, \gamma).
\]

Since \(A(t)\) is monotone, in particular by \((\text{3.13})\), for \(t \in I \setminus N_n\)

\[
\left\langle \frac{d\zeta}{dv}(t) + f(t)\frac{d\lambda}{dv}(t), v_n(\theta_n(t)) - \zeta \right\rangle \leq \left\langle A^0(\theta_n(t), \zeta_n), \zeta_n - v_n(\theta_n(t)) \right\rangle.
\]

Since \((v'_n)\) weakly converges to \(u'\) in \(L^1(I, E; dv)\), \((v'_n)\) Komlos converges \(\nu\)-a.e to \(u'\), then there is a negligible set \(N\) such that for \(t \notin N\)

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} v'_j(t) = u'(t) = \frac{du}{dv}(t).
\]

Whence, since

\[
\left\langle v'_n(t) + f(t)\frac{d\lambda}{dv}(t), u(t) - \gamma \right\rangle = \left\langle v'_n(t) + f(t)\frac{d\lambda}{dv}(t), v_n(\theta_n(t)) - \zeta_n \right\rangle
\]

\[
+ \left\langle v'_n(t) + f(t)\frac{d\lambda}{dv}(t), u(t) - v_n(\theta_n(t)) \right\rangle + \left\langle v'_n(t) + f(t)\frac{d\lambda}{dv}(t), \zeta_n - \gamma \right\rangle,
\]

then

\[
\frac{1}{n} \sum_{j=1}^{n} \left\langle v'_j(t) + f(t)\frac{d\lambda}{dv}(t), u(t) - \gamma \right\rangle = \frac{1}{n} \sum_{j=1}^{n} \left\langle v'_j(t) + f(t)\frac{d\lambda}{dv}(t), v_j(\theta_j(t)) - \zeta_j \right\rangle
\]

\[
+ \frac{1}{n} \sum_{j=1}^{n} \left\langle v'_j(t) + f(t)\frac{d\lambda}{dv}(t), u(t) - v_j(\theta_j(t)) \right\rangle + \frac{1}{n} \sum_{j=1}^{n} \left\langle v'_j(t) + f(t)\frac{d\lambda}{dv}(t), \zeta_j - \gamma \right\rangle,
\]

so that, by \((\text{3.19})\), \((\text{3.15})\) and since \(f\) is bounded, for \(t \in I \setminus (\cup_n N_n \cup N)\),

\[
\frac{1}{n} \sum_{j=1}^{n} \left\langle v'_j(t) + f(t)\frac{d\lambda}{dv}(t), u(t) - \gamma \right\rangle \leq \frac{1}{n} \sum_{j=1}^{n} \left\langle A^0(\theta_j(t), \zeta_j), \zeta_j - v_j(\theta_j(t)) \right\rangle
\]

\[
+ (K + 3M)\frac{1}{n} \sum_{j=1}^{n} \|u(t) - v_j(\theta_j(t))\| + (K + 3M)\frac{1}{n} \sum_{j=1}^{n} \|\zeta_j - \gamma\|.
\]

Passing to the limit when \(n \to \infty\), in this inequality, we get by \((\text{3.20})\), \((\text{3.18})\) and \((\text{3.13})\)

\[
\left\langle u'(t) + f(t)\frac{d\lambda}{dv}(t), u(t) - \gamma \right\rangle \leq \left\langle A^0(t, \gamma), \gamma - u(t) \right\rangle \quad \nu\text{-a.e.}
\]
The uniqueness of the solution is a consequence of the monotonicity of \( A(t) \). This completes our proof.

As a by product of Theorem 3.2 we mention some useful application.

**Theorem 3.3** Under the hypotheses of Theorem 3.2, for any \( u_0 \in D(A(0)) \), there exists a unique BVRC solution \( u \) to the problem

\[
\begin{aligned}
    u(0) &= u_0; \\
    u(t) &\in D(A(t)) \quad \forall t \in I; \\
    \frac{du}{d\nu} &\in L^\infty(I, E; d\nu) \\
    -\frac{du}{d\nu} &\in A(t)u(t) + f(t)\frac{d\lambda}{d\nu}(t) \quad \nu - \text{a.e.} t \in I.
\end{aligned}
\]

In the following we apply our result and tools developed above to the sweeping process.

**Theorem 3.4** For every \( t \in I \), let us consider the closed convex valued mapping \( C : I \rightrightarrows E \) such that \( d_H(C(t), C(s)) \leq r(t) - r(\tau), \forall \tau \leq t \in I \) and \( C(t) \) is ball-compact. Let \( f : I \to E \) be a bounded \( \lambda \)-measurable mapping. Then for all \( u_0 \in C(0) \), there is a unique BVRC solution \( u(\cdot) \) to the evolution problem

\[
\begin{aligned}
    u(0) &= u_0; \\
    u(t) &\in C(t) \quad \forall t \in I; \\
    \frac{du}{d\nu} &\in L^\infty(I, E; d\nu); \\
    -\frac{du}{d\nu} &\in N_{C(t)}(u(t)) + f(t)\frac{d\lambda}{d\nu}(t) \quad \nu - \text{a.e.} t \in I,
\end{aligned}
\]

Proof. The proof is immediate by taking \( A(t) = \partial\delta_{C(t)} = N_{C(t)} \).

To be able to prove the main theorem of this section, let us begin by a compactness result.

**Lemma 3.1** Let for every \( t \in I \), \( A(t) : D(A(t)) \subset E \rightrightarrows E \) be a maximal monotone operator satisfying \( (H_1), (H_2) \) and \( (H_3) \). Let \( X : I \rightrightarrows E \) be a convex weakly compact valued measurable multi-mapping such that \( X(t) \subset M\overline{B}_E \), for all \( t \in I \), where \( M \) is a nonnegative constant. Then the BVRC solutions set \( X := \{ u_f : f \in S_X \} \), where \( S_X \) denotes the set of all \( L^1(I, E; \lambda) \)-selections of \( X \), to the evolution inclusion

\[
\begin{aligned}
    u(0) &= u_0 \in D(A(0)); \\
    u(t) &\in D(A(t)) \quad \forall t \in I; \\
    -\frac{du}{d\nu} &\in A(t)u(t) + f(t)\frac{d\lambda}{d\nu}(t) \quad d\nu - \text{a.e.} t \in I,
\end{aligned}
\]

is nonempty and sequentially compact with respect to the pointwise convergence on \( I \).
Proof. By Theorem 3.2 it is clear that $\mathcal{X}$ is nonempty. Now, for each $f \in S_{X}^{1}$, we have for almost every $t \in I$, $\|f(t)\| \leq M$, so that, by virtue of the estimation given in Theorem 3.2, the solutions set $\mathcal{X}$ is equi-BVRC. Namely
\[
u_{1}(t) = u_{0} + \int_{[0,t]} \frac{du_{f}}{d\nu}(s)d\nu(s) \quad \forall t \in I,
\]
with $\|\frac{du_{f}}{d\nu}(t)\| \leq K \nu$ a.e., where $K$ is a nonnegative constant which depends only on the data. Let for each $n \in \mathbb{N}$, $f_{n} \in S_{X}^{1}$ and let $u_{f_{n}}$ be the unique BVRC solution associated with $f_{n}$ to the inclusion
\[
in (P_{n}) \begin{cases}
u_{n}(0) = u_{0} \in D(A(0)); \nu \in I ;
\nu_{n}(t) = A(t)u_{f_{n}}(t) + f_{n}(t)\frac{d\lambda}{d\nu}(t) \quad \nu - a.e. t \in I.
\end{cases}
\]
By Theorem 3.1 we know that $S_{X}^{1}$ is convex and weakly compact in $L^{1}(I,E;\lambda)$, so we may assume that $(f_{n})$ weakly converges in $L^{1}(I,E;\lambda)$ to some mapping $f \in S_{X}^{1}$. Since $(u_{f_{n}})$ is a sequence in $\mathcal{X}$, by weak compactness, we may assume that $(\frac{du_{f_{n}}}{d\nu})$ weakly converges in $L^{1}(I,E;\nu)$ to $z \in L^{1}(I,E;\nu)$, with $\|z(t)\| \leq K \nu$ a.e., and since, by $(H_{3})$, for every $t \in I$, $(u_{f_{n}}(t))$ is relatively compact, we conclude by Theorem 3.1 that $(u_{f_{n}})$ converges pointwise strongly to a BVRC mapping $u$ where $u(t) = u_{0} + \int_{[0,t]} z(s)d\nu(s)$, for all $t \in I$, that is $\frac{du}{d\nu} = z$.

As $(\frac{du_{n}}{d\nu})$ weakly converges in $L^{1}(I,E;\nu)$ to $\frac{du}{d\nu}$, $(\frac{du_{n}}{d\nu})$ Komlos converges to $\frac{du}{d\nu}$. Similarly $(\frac{d\lambda}{d\nu})$ Komlos converges to $f\frac{d\lambda}{d\nu}$. As a consequence, by applying Komlos argument to the inclusion in $(P_{n})$, taking account that $u(t) \in D(A(t))$, we get finally
\[
u_{1}(t) = u_{0} + \int_{[0,t]} \frac{du}{d\nu}(s)d\nu(s) \quad \forall t \in I,
\]
with $u(0) = u_{0}$. By uniqueness of the solution, we conclude that $u = u_{f}$. Therefore $\mathcal{X}$ is sequentially compact with respect to the pointwise convergence on $I$. $\blacksquare$

**Theorem 3.5** Let for every $t \in I$, $A(t) : D(A(t)) \subseteq E \Rightarrow E$ be a maximal monotone operator satisfying $(H_{1})$, $(H_{2})$ and $(H_{3})$. Let $F : I \times E \Rightarrow E$ be a convex weakly compact valued multi-mapping satisfying:

1. for each $e \in E$, the scalar function $\delta^{*}(e,F(\cdot,\cdot))$ is $\mathcal{B}(I) \otimes \mathcal{B}(E)$-measurable;
2. for each $e \in E$ and for every $t \in I$, the scalar function $\delta^{*}(e,F(t,\cdot))$ is upper semicontinuous on $E$;
3. $F(t,x) \subseteq M(1 + \|x\|)\overline{B}_{E}$, for all $(t,x) \in I \times E$, where $M$ is a nonnegative constant.

Then the set of BVRC solutions to the inclusion
\[
in (P_{F}) \begin{cases}
u_{1}(0) = u_{0} \in D(A(0));
\nu_{1}(t) = A(t)u(t) \quad \forall t \in I ;
\frac{du}{d\nu} \in L^{\infty}(I,E;\nu);
\frac{du}{d\nu}(t) \in A(t)u(t) + F(t,u(t))\frac{d\lambda}{d\nu}(t) \quad \nu - a.e. t \in I,
\end{cases}
\]
is nonempty and sequentially compact with respect to the pointwise convergence on $I$. 

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Proof. Step 1. Let $u_o$ be the unique BVRC solution to the inclusion

$$\begin{cases}
  u_o(0) = u_0 \in D(A(0)); \\
  u_o(t) \in D(A(t)) \quad \forall t \in I; \\
  \frac{du_o}{dv}(t) \in A(t)u_o(t) \quad \nu - a.e. \ t \in I,
\end{cases}$$

and let $\alpha: I \to \mathbb{R}_+$ be the unique absolutely continuous solution to the ordinary differential equation

$$\dot{\alpha}(t) = M(1 + \alpha(t)) \quad \forall t \in I \quad \text{with} \quad \alpha(0) = \alpha_0 := \max_{t \in I} ||u_o(t)||,$$

that is, $\dot{\alpha}(t) = M(1 + \alpha_0) \exp(Mt)$. Since $\alpha \in L^\infty(I, \mathbb{R}; \lambda)$, the set

$$S := \{ h \in L^\infty(I, E; \lambda) : ||h(t)|| \leq \dot{\alpha}(t) \lambda - a.e. \},$$

is convex $\sigma(L^\infty(I, E; \lambda), L^1(I, E; \lambda))$-compact, and then it is also $\sigma(L^1(I, E; \lambda), L^\infty(I, E; \lambda))$-compact. For any $h \in S$, let us define

$$\Phi(h) = \left\{ f \in L^1(I, E; \lambda) : f(t) \in F(t, u_h(t)) \quad \lambda - a.e. \ t \in I \right\},$$

where $u_h$ is the unique BVRC solution to the inclusion

$$\begin{cases}
  u_h(0) = u_0 \in D(A(0)); \\
  u_h(t) \in D(A(t)) \quad \forall t \in I; \\
  -\frac{du_h}{dv}(t) - h(t)\frac{d\lambda}{dv}(t) \in A(t)u_h(t) \quad \nu - a.e. \ t \in I.
\end{cases}$$

In fact, for any $h \in S$, $\Phi(h)$ is the set of $L^1(E, B(I); \lambda)$-selections of the convex weakly compact valued scalarly $B(I)$-measurable mapping $t \mapsto F(t, u_h(t))$, by noting that $u_h$ is BVRC, then $u_h$ is Borel, i.e., $(B(I), B(E))$-measurable, hence by (1), $t \mapsto \delta^*(e, F(t, u_h(t)))$ is $B(I)$-measurable, and then $F(\cdot, u_h(\cdot))$ admits a Borel selection. This shows the non-emptiness of $\Phi(h)$. Using $(P_0)$ and $(P_h)$, we get by the monotonicity of $A(t)$

$$\left\langle \frac{du_h}{dv}(t) - \frac{du_o}{dv}(t) + h(t)\frac{d\lambda}{dv}(t), u_h(t) - u_o(t) \right\rangle \leq 0.$$

Since $h \in S$ it follows that

$$\left\langle \frac{du_h}{dv}(t) - \frac{du_o}{dv}(t), u_h(t) - u_o(t) \right\rangle \leq \dot{\alpha}(t)\frac{d\lambda}{dv}(t)\|u_h(t) - u_o(t)\|.$$

On the other hand, we know that $u_h$ and $u_o$ are BVRC and have the densities $\frac{du_h}{dv}$ and $\frac{du_o}{dv}$ relatively to $\nu$, by a result of Moreau concerning the differential measure [38], $\|u_h - u_o\|^2$ is BVRC and we have

$$d\|u_h - u_o\|^2 \leq 2\left\langle u_h(\cdot) - u_o(\cdot), \frac{du_h}{dv}(\cdot) - \frac{du_o}{dv}(\cdot) \right\rangle d\nu.$$
so that by integrating on \([0, t]\) and using the above estimate we get
\[
\frac{1}{2}\|u_h(t) - u_o(t)\|^2 \leq \int_0^t \dot{\alpha}(s)\|u_h(s) - u_o(s)\|ds.
\]
Thanks to Lemma 2.8, it follows that
\[
\|u_h(t) - u_o(t)\| \leq \int_0^t \dot{\alpha}(s)ds,
\]
so that for all \(t \in I\), we get
\[
\|u_h(t)\| \leq \|u_o(t)\| + \int_0^t \dot{\alpha}(s)ds \leq \alpha_0 + \int_0^t \dot{\alpha}(s)ds = \alpha(t).
\]
Whence, for any \(h \in S\) and for all \(f \in \Phi(h)\), we have by hypothesis (3), for \(\lambda\)-a.e \(t \in I\),
\[
\|f(t)\| \leq M(1 + \|u_h(t)\|) \leq M(1 + \alpha(t)) = \dot{\alpha}(t),
\]
that is \(\Phi(h) \subset S\), further \(\Phi(h)\) is convex. Clearly, if \(h\) is a fixed point of \(\Phi\) \((h \in \Phi(h))\), then \(u_h\)
is a BVRC solution of the inclusion under consideration. We show that \(\Phi : S \rightarrow S\) is a convex
\(\sigma(L^1(I, E; \lambda), \mathbb{L}^\infty(I, E; \lambda))\)-compact valued upper semicontinuous multi-mapping. By weak
compactness, it is enough to show that the graph of \(\Phi\) is sequentially weakly compact. Let \((h_n) \subset S\) a sequence, which \(\sigma(L^1(I, E; \lambda), \mathbb{L}^\infty(I, E; \lambda))\)-converges to \(\hat{h} \in S\), and let \((f_n) \subset S\)
such that \(f_n \in \Phi(h_n)\) and \((f_n) \sigma(L^1(I, E; \lambda), \mathbb{L}^\infty(I, E; d\lambda))\)-converges to \(\bar{f} \in S\). We need to
show that \(\bar{f} \in \Phi(\hat{h})\). By virtue of Lemma 3.1, we know that the set \(\mathcal{X} := \{u_h : h \in S\}\)
of solutions of \((P_h)\) is sequentially compact with respect to the pointwise convergence on \(I\).
Hence \((u_{h_n})\) converges pointwise to \(u_{\bar{h}} \in \mathcal{X}\). Since for a.e. \(t \in I\), \(f_n(t) \in F(t, u_{h_n}(t))\), the
inequality
\[
\langle 1_L(t)x, f_n(t) \rangle \leq \delta^*(\langle 1_L(t)x, F(t, u_{h_n}(t)) \rangle),
\]
holds for almost every \(t \in I\), for each \(L \in \mathcal{L}(I)\) and for each \(x \in E\). By integrating, we get
\[
\int_L \langle x, f_n(t) \rangle dt \leq \int_L \delta^*(x, F(t, u_{h_n}(t))) dt.
\]
By the weak convergence of \((f_n)\) and hypothesis (2), it follows that
\[
\int_L \langle x, \bar{f}(t) \rangle dt = \lim_{n \to \infty} \int_L \langle x, f_n(t) \rangle dt \leq \limsup_{n \to \infty} \int_L \delta^*(x, F(t, u_{h_n}(t))) dt
\]
\[
\leq \int_L \limsup_{n \to \infty} \delta^*(x, F(t, u_{h_n}(t))) dt \leq \int_L \delta^*(x, F(t, u_{\bar{h}}(t))) dt.
\]
Whence we get
\[
\int_L \langle x, \bar{f}(t) \rangle dt \leq \int_L \delta^*(x, F(t, u_{\bar{h}}(t))) dt
\]
for every \(L \in \mathcal{L}(I)\). Consequently \(\langle x, \bar{f}(t) \rangle \leq \delta^*(x, F(t, u_{\bar{h}}(t)))\) \(\lambda\)-a.e. From (23, Prop.
III.35), we get \(\bar{f}(t) \in F(t, u_{\bar{h}}(t))\) \(\lambda\)-a.e.
Applying Kakutani-Ky Fan fixed point theorem to the convex weakly compact valued upper semicontinuous multi-mapping $\Phi$ now shows that $\Phi$ admits a fixed point, $h \in \Phi(h)$, thus proving the existence of at least one BVRC solution to our inclusion (\text{I}_\nu).

\textbf{Step 2.} Compactness follows easily from the above arguments and the pointwise compactness of $X$ given in Lemma 3.1. \hfill \blacksquare

Let us mention a useful result, which leads us to several applications.

\textbf{Corollary 3.1} Let for every $t \in I$, $A(t) : D(A(t)) \subset E \Rightarrow E$ be a maximal monotone operator satisfying $(H_1)$, $(H_2)$ and $(H_3)$. Let $f : I \times E \to E$ satisfying:

(i) $f(\cdot, x)$ is $\mathcal{B}(I)$-measurable on $I$, for all $x \in E$.

(ii) $\|f(t, x) - f(t, y)\| \leq M \|x - y\|$ for all $(t, x, y) \in I \times E \times E$.

(iii) $\|f(t, x)\| \leq M(1 + \|x\|)$ for all $(t, x) \in I \times E$, for some nonnegative constant $M$.

Assume further that there is $\beta \in ]0, 1[$ such that $\forall t \in I$, $0 \leq 2M \frac{d\lambda(t)}{dv(t)}(\{t\}) \leq \beta < 1$. Then there is a unique BVRC solution to the problem

$$
\begin{cases}
    u(0) = u_0 \in D(A(0)); \\
    u(t) \in D(A(t)) \quad \forall t \in I; \\
    \frac{du}{dv} \in L^\infty(I, E; dv); \\
    -\frac{du}{dv}(t) - f(t, u(t)) \frac{d\lambda}{dv}(t) = A(t)u(t) + f(t, u(t)) \frac{d\lambda}{dv}(t) \quad \nu - a.e. \ t \in I.
\end{cases}
$$

Proof. Existence follows from Theorem 3.3. We need only to prove the uniqueness.

Suppose that there are two BVRC solutions $u$ and $v$ to the problem under consideration, that is

$$
-\frac{du}{dv}(t) - f(t, u(t)) \frac{d\lambda}{dv}(t) \in A(t)u(t),
$$

$$
-\frac{dv}{dv}(t) - f(t, v(t)) \frac{d\lambda}{dv}(t) \in A(t)v(t).
$$

By the monotonicity of $A(t)$ we get

$$
\left\langle \frac{dv}{dv}(t) - \frac{du}{dv}(t), v(t) - u(t) \right\rangle \leq 0.
$$

By hypothesis (ii)

$$
\left\langle \frac{dv}{dv}(t) - \frac{du}{dv}(t), v(t) - u(t) \right\rangle \leq \left\langle \frac{d\lambda}{dv}(t)f(t, u(t)) - \frac{d\lambda}{dv}(t)f(t, v(t)), v(t) - u(t) \right\rangle
\leq M \frac{d\lambda}{dv}(t)\|v(t) - u(t)\|^2.
$$

On the other hand, we know that $u$ and $v$ are BVRC and have the densities $\frac{du}{dv}$ and $\frac{dv}{dv}$ relatively to $\nu$, by a result of Moreau concerning the differential measure \cite{38}, $\|v - u\|^2$ is BVRC and we have

$$
d\|v - u\|^2 \leq 2 \left\langle v(\cdot) - u(\cdot), \frac{dv}{dv}(\cdot) - \frac{du}{dv}(\cdot) \right\rangle d\nu,
$$

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so that by integrating on $[0,t]$ and using the above estimate we get
\[
\|v(t) - u(t)\|^2 = \int_{[0,t]} \frac{d\|u - v\|^2}{d\nu} d\nu(s) \leq \int_{[0,t]} 2\langle v(s) - u(s), \frac{dv}{d\nu}(s) - \frac{du}{d\nu}(s) \rangle d\nu(s)
\]
\[
\leq \int_{[0,t]} 2M \frac{d\lambda}{d\nu}(s) \|v(s) - u(s)\|^2 d\nu(s).
\]
According to the assumption $0 \leq 2M \frac{d\lambda}{d\nu}(t) d\nu(\{t\}) \leq \beta < 1$ and using Lemma 2.6 we deduce from the last inequality that $u = v$ in $I$. This completes the proof.

**Remark 3.1** Actually Corollary 3.1 is an extension of Theorem 4.1 in Adly et al (\[1\]) dealing with the BVRC solution for convex sweeping process.

An inspection of the proofs of our preceding theorems, allows us to state the following existence result with a fairly general perturbation taking the form $f + F$.

**Theorem 3.6** Let for every $t \in I$, $A(t) : D(A(t)) \subset E \Rightarrow E$ be a maximal monotone operator satisfying $(H_1)$, $(H_2)$ and $(H_3)$. Let $f : I \times E \to E$ be such that for every $x \in E$ the mapping $f(\cdot, x)$ is $\mathcal{B}(I)$-measurable on $I$ and such that
\begin{enumerate}[(i)]  
  \item $\|f(t,x)\| \leq m$, $\forall (t,x) \in I \times E$, for some nonnegative constant $m$;
  \item $\|f(t,x) - f(t,y)\| \leq m\|x - y\|$, $\forall (t,x,y) \in I \times E \times E$.
\end{enumerate}

Let $F : I \times E \Rightarrow E$ be a convex compact valued multi-mapping satisfying:
\begin{enumerate}[(i)]  
  \item for each $e \in E$, the scalar function $\delta^*(e,F(\cdot,\cdot))$ is $\mathcal{B}(I) \otimes \mathcal{B}(E)$-measurable;
  \item for each $e \in E$ and for every $t \in I$, the scalar function $\delta^*(e,F(t,\cdot))$ is upper semicontinuous on $E$;
  \item $F(t,x) \subset M(1 + \|x\|)B_E$, $\forall (t,x) \in I \times E$, for some nonnegative constant $M$.
\end{enumerate}
Assume further that there is $\beta \in [0,1]$ such that $\forall t \in I$, $0 \leq 2m \frac{d\lambda}{d\nu}(t) d\nu(\{t\}) \leq \beta < 1$. Then for $u_0 \in D(A(0))$, there is a BVRC mapping $u : I \to H$ satisfying
\[
(P_{f,F}) \begin{cases} 
  u(0) = u_0 \in D(A(0)); \\
  u(t) \in D(A(t)) \quad \forall t \in I; \\
  \frac{du}{d\nu} \in L^\infty(I,E; d\nu); \\
  -\frac{du}{d\nu}(t) \in A(t)u(t) + (f(t,u(t)) + F(t,u(t)))\frac{d\lambda}{d\nu}(t) \quad \nu - a.e. \quad t \in I.
\end{cases}
\]

Proof. **Step 1.** We proceed as in the proof of Theorem 3.5. Let $u_0$ be the unique BVRC solution of problem $(P_0)$. Let $\alpha : I \to \mathbb{R}_+$ be the unique absolutely continuous solution of the ordinary differential equation
\[
\dot{\alpha}(t) = M(1 + \alpha(t) + mT) \quad \forall t \in I \quad \text{with} \quad \alpha(0) = \alpha_0 := \sup_{t \in I} \|u_0(t)\|.
\]
Since $\dot{\alpha} \in L^\infty(I,\mathbb{R}; \lambda)$, the set
\[
S := \{h \in L^\infty(I,E; \lambda) : \|h(t)\| \leq \dot{\alpha}(t) \ a.e. \},
\]
is clearly convex $\sigma(L^1(I,E;\lambda),L^\infty(I,E;\lambda))-\text{compact}$. For any $h \in S$, let us define
\[
\Phi(h) = \left\{ \psi \in L^1(I,E;\lambda) : \psi(t) \in F(t,u_h(t)) \lambda - \text{a.e.} \right\},
\]
where $u_h$ is the unique BVRC solution to the inclusion
\[
(P_{f,h}) \begin{cases} 
  u_h(0) = u_0 \in D(A(0)); \\
  u_h(t) \in D(A(t)) \forall t \in I; \\
  -\frac{du_h}{d\nu}(t) \in A(t)u_h(t) + (f(t,u_h(t)) + h(t)) \frac{d\lambda}{d\nu}(t) \nu - \text{a.e.} t \in I.
\end{cases}
\]
The existence and uniqueness of such a solution is granted by Corollary 3.1. In fact, to see that, let us set for any $h \in S$, $g(t,x) = f(t,x) + h(t)$ for all $(t,x) \in I \times E$, then $g$ satisfies:
\[
\|g(t,x)\| \leq m + \hat{\alpha}(t) \leq m + \|\hat{\alpha}\|_\infty \forall (t,x) \in I \times E,
\]
and
\[
\|g(t,x) - g(t,y)\| \leq \|x - y\| \forall (t,x,y) \in I \times E \times E.
\]
Furthermore, by the arguments given in the proof of the pointwise compactness of the set $\mathcal{X}$ given in Lemma 3.1, it is not difficult to show that the set $\tilde{\mathcal{X}} = \{ u_h : h \in S \}$, where $u_h$ is the unique BVRC solution to problem $(P_{f,h})$, is sequentially compact with respect to the pointwise convergence on $I$.

Now, using $(P_0)$, $(P_{f,h})$, the monotonicity of $A(t)$, Moreau’s inequality for BVRC functions and the arguments of the proof of Theorem 3.5 for all $t \in I$, one has the estimation
\[
\frac{1}{2}\| u_h(t) - u_0(t) \|^2 \leq \int_0^t \| f(s,u_h(s)) + h(s) \| \| u_h(s) - u_0(s) \| ds.
\]
Thanks to Lemma 2.8 it follows that
\[
\| u_h(t) - u_0(t) \| \leq \int_0^t \| f(s,u_h(s)) + h(s) \| ds \leq \int_0^t (m + \hat{\alpha}(s)) ds,
\]
so that for all $t \in I$, one gets
\[
\| u_h(t) \| \leq \alpha_0 + \int_0^t (\hat{\alpha}(s) + m) ds = \alpha(t) + mt \leq \alpha(t) + mT. \tag{3.22}
\]
Consequently, for any $h \in S$ and for all $\psi \in \Phi(h)$, we have by hypothesis $(jjj)$ and $(3.22)$,
\[
\| \psi(t) \| \leq M(1 + \| u_h(t) \|) \leq M(1 + \alpha(t) + mT) = \hat{\alpha}(t) \lambda - \text{a.e.,}
\]
that is $\Phi(h) \subset S$, and by hypothesis $(j)$, it is clear that for any $h \in S$, $\Phi(h)$ is nonempty, further it is a convex set. In fact, $\Phi(h)$ is the set of $L^1_E(I,B(I);\lambda)$-selections of the convex weakly compact valued measurable multi-mapping $t \mapsto F(t,u_h(t))$. Clearly, if $h$ is a fixed point of $\Phi$ ($h \in \Phi(h)$), then $u_h$ is a BVRC solution to the inclusion $(P_{f,E})$. So that, we finish our proof as in Theorem 3.5 by using the same arguments and the pointwise compactness of the set $\tilde{\mathcal{X}}$. 


4 Applications

4.1 Second order BVRC evolution inclusion

Theorem 4.1 Let for every \( t \in I \), \( A(t) : D(A(t)) \subset E \Rightarrow E \) be a maximal monotone operator satisfying \((H_1), (H_2)\) and \((H_3)\) \( D(A(t)) \subset X(t) \subset \gamma(t) E \) for all \( t \in I \), where \( X : I \Rightarrow E \) is a convex compact valued Lebesgue-measurable multi-mapping and \( \gamma : I \rightarrow \mathbb{R} \) is a nonnegative \( L^1(I, \mathbb{R}; \lambda) \)-integrable function.

Let \( f : I \times E \times E \rightarrow E \) be such that for every \( x, y \in E \), the mapping \( f(\cdot, x, y) \) is \( \mathcal{B}(I) \)-measurable and for every \( t \in I \), the mapping \( f(t, \cdot, \cdot) \) is continuous on \( E \times E \) and satisfying

(i) \( \|f(t, x, y)\| \leq M \forall (t, x, y) \in I \times E \times E \),
(ii) \( \|f(t, z, x) - f(t, z, y)\| \leq M \|x - y\| \forall (t, z, x, y) \in I \times E \times E \),

for some nonnegative constant \( M \).

Assume further that there is \( \beta \in ]0, 1[ \) such that \( \forall t \in I \), \( 0 \leq 2M \frac{dA}{dv}(t)dv(\{t\}) \leq \beta < 1 \). Then for \( x_0 \in E \), \( u_0 \in D(A(0)) \) there is an absolutely continuous mapping \( x : I \rightarrow E \) and a BVRC mapping \( u : I \rightarrow E \) with density \( \frac{du}{dv} \) with respect to \( \nu \) satisfying

\[
\begin{align*}
&\begin{cases}
x(t) = x_0 + \int_0^t u(s)ds & \forall t \in I; \\
u(0) = u_0; \\
u(t) \in D(A(t)) & \forall t \in I; \\
\frac{du}{dv} \in L^\infty(I, E; dv); \\
- \frac{du}{dv}(t) \in A(t)u(t) + f(t, x(t), u(t))\frac{d\lambda}{dv}(t) & \nu-a.e. t \in I.
\end{cases}
\end{align*}
\]

Proof. Let

\( \mathcal{X} := \{ u_\psi \in \mathcal{C}(I, E) : u_\psi(t) = x_0 + \int_0^t \psi(s)ds, t \in I, \psi \in S_{X}^1 \} \).

Then \( \mathcal{X} \) is a convex compact subset of \( \mathcal{C}(I, E) \) using the compactness of the convex compact valued integral \( \int_0^t X(s)ds \) (cf [14]). For any \( h \in \mathcal{X} \), there is a unique BVRC solution to

\[
\begin{align*}
&\begin{cases}
u_h(0) = u_0; \\
u_h(t) \in D(A(t)) & \forall t \in I; \\
- \frac{du_h}{dv}(t) \in A(t)u_h(t) + f(t, h(t), u_h(t))\frac{d\lambda}{dv}(t) & \nu-a.e. t \in I.
\end{cases}
\end{align*}
\]

with \( u_h(t) = u_0 + \int_{0}^{t} \frac{du_h}{dv}(s)dv(s) \) for all \( t \in I \) and \( \|\frac{du_h}{dv}(t)\| \leq K \) \( \nu \)-a.e. Existence and uniqueness of such a solution is ensured by Corollary 3.1. Indeed, for any fixed \( h \in \mathcal{X} \), the mapping \( f_h(t, x) = f(t, h(t), x) \) satisfies \( \|f_h(t, x)\| \leq M \) for all \( (t, x) \in I \times E \), \( \|f_h(t, x) - f_h(t, y)\| \leq M \|x - y\| \) for all \( (t, x, y) \in I \times E \times E \), while the estimate of the velocity is given in the proof of Theorem 3.2. Now for each \( h \in \mathcal{X} \), let us consider the mapping

\[ \psi(h)(t) := x_0 + \int_0^t u_h(s)ds \quad \forall t \in I. \]
Then it is clear that \( \psi(h) \in \mathcal{X} \) because by \((H_3)\)'s, \( u_h(t) \in D(A(t)) \subset X(t) \) for all \( t \in I \).

Our aim is to prove that \( \psi : \mathcal{X} \to \mathcal{X} \) is continuous in order to obtain the existence theorem by a fixed point approach. This need a careful look using the estimate of the BVRC solution given above. It is enough to show that, if \( (h_n) \) converges uniformly to \( h \) in \( \mathcal{X} \), then the sequence \( (u_{h_n}) \) of BVRC solutions associated with \( (h_n) \) of problems

\[
\begin{cases}
  u_{h_n}(0) = u_0; \\
  u_{h_n}(t) \in D(A(t)) \quad \forall t \in I; \\
  -\frac{du_{h_n}}{d\nu}(t) \in A(t)u_{h_n}(t) + f(t, h_n(t), u_{h_n}(t)) \frac{d\lambda}{d\nu}(t) \nu - a.e. t \in I.
\end{cases}
\]

pointwise converges to the BVRC solution \( u_h \) associated with \( h \) of problem

\[
\begin{cases}
  u_h(0) = u_0; \\
  u_h(t) \in D(A(t)) \quad \forall t \in I; \\
  -\frac{du_h}{d\nu}(t) \in A(t)u_h(t) + f(t, h(t), u_h(t)) \frac{d\lambda}{d\nu}(t) \nu - a.e. t \in I.
\end{cases}
\]

As \( (u_{h_n}) \) is uniformly bounded and bounded in variation since \( \|u_{h_n}(t) - u_{h_n}(\tau)\| \leq K(\nu([\tau, t])) \), for \( \tau \leq t \) with \( (u_{h_n}(t)) \subset D(A(t)) \subset X(t) \subset \gamma(t)\mathcal{B}_E \), for all \( t \in I \), it is relatively compact, by Theorem 3.1 and the Helly principle [43], we may assume that \( (u_{h_n}) \) pointwise converges to a BV mapping \( u(\cdot) \). Now, since for all \( t \in I \), \( u_{h_n}(t) = u_0 + \int_{[0,t]} \frac{du_h}{d\nu}(s) d\nu(s) \) and \( \frac{du_h}{d\nu}(s) \in \mathcal{K}_E \nu-a.e. \), we may assume that \( \frac{du_h}{d\nu}(\cdot) \) converges weakly in \( L^1(I, E; \nu) \) to \( w \in L^1(I, E; \nu) \) with \( w(t) \in \mathcal{K}_E \nu-a.e. \), so that

\[
\lim_{n \to \infty} u_{h_n}(t) = u_0 + \int_{[0,t]} w(s) d\nu(s) \quad \forall t \in I.
\]

By identifying the limits, we get

\[
u-a.e.
\]

\[
\lim_{n \to \infty} f(t, h_n(t), u_{h_n}(t)) = f(t, h(t), u(t)) \quad \forall t \in I.
\]

As consequence \( (f(\cdot, h(\cdot), u_h(\cdot)) \frac{d\lambda}{d\nu}(\cdot)) \) pointwise converges to \( f(\cdot, h(\cdot), u(\cdot)) \frac{d\lambda}{d\nu}(\cdot) \). Since \( \frac{du_h}{d\nu} \) weakly converges to \( \frac{du}{d\nu} \) in \( L^1(I, E; \nu) \), we may assume that it Komlos converges to \( \frac{du}{d\nu} \).

For simplicity set for all \( t \in I \), \( g_n(t) = f(t, h_n(t), u_{h_n}(t)) \frac{d\lambda}{d\nu}(t) \) and \( g(t) = f(t, h(t), u_h(t)) \frac{d\lambda}{d\nu}(t) \). There is a \( \nu \)-negligible set \( N \) such that for \( t \in I \setminus N \)

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \left( \frac{du_h}{d\nu}(t) + g_j(t) \right) = \frac{du}{d\nu}(t) + g(t).
\]
Let $\eta \in D(A(t))$. From
\[
\left\langle \frac{du_{h_n}}{d\nu}(t) + g_n(t), u(t) - \eta \right\rangle = \left\langle \frac{du_{h_n}}{d\nu}(t) + g_n(t), u_{h_n}(t) - \eta \right\rangle + \left\langle \frac{du_{h_n}}{d\nu}(t) + g_n(t), u(t) - u_{h_n}(t) \right\rangle,
\]
let us write
\[
\frac{1}{n} \sum_{j=1}^{n} \left\langle \frac{du_{h_n}}{d\nu}(t) + g_j(t), u(t) - \eta \right\rangle = \frac{1}{n} \sum_{j=1}^{n} \left\langle \frac{du_{h_n}}{d\nu}(t) + g_j(t), u_{h_j}(t) - \eta \right\rangle \\
+ \frac{1}{n} \sum_{j=1}^{n} \left\langle \frac{du_{h_n}}{d\nu}(t) + g_j(t), u(t) - u_{h_j}(t) \right\rangle,
\]
so that
\[
\frac{1}{n} \sum_{j=1}^{n} \left\langle \frac{du_{h_n}}{d\nu}(t) + g_j(t), u(t) - \eta \right\rangle \leq \frac{1}{n} \sum_{j=1}^{n} \left\langle A^0(t, \eta), \eta - u_{h_j}(t) \right\rangle + (K + M) \frac{1}{n} \sum_{j=1}^{n} \|u(t) - u_{h_j}(t)\|.
\]
Passing to the limit when $n \to \infty$, this last inequality gives immediately
\[
\left\langle \frac{du}{d\nu}(t) + g(t), u(t) - \eta \right\rangle \leq \left\langle A^0(t, \eta), \eta - u(t) \right\rangle \text{ a.e.}
\]
On the other hand, since for all $t \in I$, $u_{h_n}(t) \in D(A(t))$, then $u(t) \in \overline{D(A(t))}$. As a consequence, by Lemma 2.1 $u(t) \in D(A(t))$ and
\[
-\frac{du}{d\nu}(t) \in A(t)u(t) + g(t) = A(t)u(t) + f(t, h(t), u(t)) \frac{d\lambda}{d\nu}(t) \text{ } \nu - \text{a.e.,}
\]
with $u(0) = u_0 \in D(A(0))$, so that by uniqueness $u = u_h$. Consequently, for all $t \in I$,
\[
\psi(h_n)(t) - \psi(h)(t) = \int_0^t (u_{h_n}(s) - u_h(s))ds,
\]
and since $(u_{h_n}(s) - u_h(s)) \to 0$ and is pointwise bounded; $\|u_{h_n}(s) - u_h(s)\| \leq 2\gamma(s)$, we conclude by Lebesgue theorem, that
\[
\sup_{t \in I} \|\psi(h_n)(t) - \psi(h)(t)\| \leq \int_0^T \|u_{h_n}(s) - u_h(s)\|ds \to 0,
\]
so that $\psi(h_n) \to \psi(h)$ in $C(I, E)$. Whence $\psi : \mathcal{X} \to \mathcal{X}$ is continuous and so has a fixed point, say $h = \psi(h) \in \mathcal{X}$, that means
\[
\begin{cases}
  h(t) = \psi(h)(t) = x_0 + \int_0^t u_h(s)ds \text{ } \forall t \in I; \\
  u_h(0) = u_0; \\
  u_h(t) \in D(A(t)) \text{ } \forall t \in I; \\
  -\frac{du_h}{d\nu}(t) \in A(t)u_h(t) + f(t, h(t), u_h(t)) \frac{d\lambda}{d\nu}(t) \text{ } \nu - \text{a.e. } t \in I.
\end{cases}
\]
Corollary 4.1. Let $C : I \rightarrow E$ be a convex closed valued multi-mapping such that

(i) $d_H(C(t), C(s)) \leq |r(t) - r(\tau)|$, for all $\tau, t \in I$;

(ii) $C(t) \subset X(t) \subset \gamma(t)B_E$ for all $t \in I$, where $X : I \rightarrow E$ is a convex compact valued Lebesgue-measurable multi-mapping and $\gamma : I \rightarrow \mathbb{R}$ is a nonnegative $L^1(I, \mathbb{R}; \lambda)$-integrable function.

Let $f : I \times E \times E \rightarrow E$ satisfying all the hypotheses in Theorem 4.1. Assume further that there is $\beta \in [0, 1]$ such that $\forall t \in I$, $0 \leq 2M \frac{d\mu}{d\nu}(t) d\nu(\{t\}) \leq \beta < 1$.

Then, for $u_0 \in C(0)$, $x_0 \in E$, there is an absolutely continuous mapping $x : I \rightarrow E$ and a BVRC mapping $u : I \rightarrow E$ with density $\frac{du}{d\nu}$ w.r.t $\nu$ satisfying

\[
\begin{aligned}
  x(t) &= x_0 + \int_0^t u(s) ds \quad \forall t \in I; \\
  u(0) &= u_0; \\
  u(t) &\in C(t) \quad \forall t \in I; \\
  \frac{du}{d\nu} &\in L^\infty(I, E; d\nu); \\
  -\frac{d\beta}{d\nu}(t) &\in N_{C(0)}(u(t)) + f(t, x(t), u(t)) \frac{d\lambda}{d\nu}(t) \quad \nu - a.e. t \in I.
\end{aligned}
\]

4.2 A new application to Skorohod problem

We present some new versions of the Skorohod problem in Castaing et al. [7] dealing with the sweeping process associated with an absolutely continuous (or continuous) closed convex moving set $C(t)$ in $E = \mathbb{R}^d$. Although we deal with deterministic case, it is a step forward to Skorohod problem in the stochastic setting, see the recent articles by Castaing et al. [7, 18, Rascu [41] and Maticiu et al. [36], for references on this stochastic subject.

Theorem 4.2. Let $E = \mathbb{R}^d$ and let for every $t \in I = [0, 1]$, $A(t) : D(A(t)) \subset E \rightarrow E$ be a maximal monotone operator satisfying $(H_1)$ and $(H_2)$. Suppose that $b : I \times E \rightarrow E$ is a $(B(I) \otimes B(E), B(E))$-measurable mapping satisfying:

(j) $\|b(t, x)\| \leq M$, for all $(t, x) \in I \times E$, for some nonnegative constant $M$.

(jj) For all $t \in I$, $b(t, \cdot)$ is continuous on $E$.

Let $y_0 \in D(A(0))$. Then there exist a BVRC mapping $X : I \rightarrow E$ and a BVRC mapping $Y : I \rightarrow E$ satisfying

\[
\begin{aligned}
  &X(0) = Y(0) = y_0; \\
  &X(t) = \int_0^t b(s, X(s)) ds + Y(t) \quad \forall t \in I; \\
  &Y(t) \in D(A(t)) \quad \forall t \in I; \\
  &-\frac{dY}{d\nu}(t) \in A(t)Y(t) + \left( \int_0^t b(s, X(s)) ds \right) \frac{d\lambda}{d\nu}(t) \quad \nu - a.e. t \in I.
\end{aligned}
\]

Proof. Let us set for all $t \in I$

\[
X^0(t) = y_0, \quad h^1(t) = \int_0^t b(s, X^0(s)) ds,
\]

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then $h^1$ is Lipschitz continuous with $\|h^1(t)\| \leq M$ for all $t \in I$. By Theorem 3.3 there is a unique BVRC mapping $Y^1 : I \to E$ solution of the problem

$$
\begin{cases}
Y^1(0) = y_0; \\
Y^1(t) \in D(A(t)) \forall t \in I; \\
- \frac{dY^1}{d\nu}(t) \in A(t)Y^1(t) + h^1(t)\frac{d\lambda}{d\nu}(t) \nu \text{ - a.e. } t \in I
\end{cases}
$$

with

$$Y^1(t) = y_0 + \int_{[0,t]} \frac{dY^1}{d\nu}(s) \forall t \in I$$

and $\|\frac{dY^1}{d\nu}\| \leq K \nu - a.e.$, where $K$ is a positive constant depending on the data. Set for all $t \in I$

$$X^1(t) = h^1(t) + Y^1(t) = \int_0^t b(s,X^0(s))ds + Y^1(t),$$

so that $X^1$ is BVRC.

Now we construct $X^n$ by induction as follows. Let for all $t \in I$

$$h^n(t) = \int_0^t b(s,X^{n-1}(s))ds.$$

Then $h^n$ is Lipschitz continuous with $\|h^n(t)\| \leq M$ for all $t \in I$. By Theorem 3.3 there is a unique BVRC mapping $Y^n : I \to E$ solution of the problem

$$
\begin{cases}
Y^n(0) = y_0; \\
Y^n(t) \in D(A(t)) \forall t \in I; \\
- \frac{dY^n}{d\nu}(t) \in A(t)Y^n(t) + h^n(t)\frac{d\lambda}{d\nu}(t) \nu \text{ - a.e. } t \in I
\end{cases}
$$

with

$$Y^n(t) = y_0 + \int_{[0,t]} \frac{dY^n}{d\nu}(s) \forall t \in I$$

and $\|\frac{dY^n}{d\nu}\| \leq K \nu - a.e.$ Set for all $t \in I$

$$X^n(t) = h^n(t) + Y^n(t) = \int_0^t b(s,X^{n-1}(s))ds + Y^n(t),$$

so that $X^n$ is BVRC, and

$$- \frac{dY^n}{d\nu}(t) \in A(t)Y^n(t) + \left(\int_0^t b(s,X^{n-1}(s))ds\right)\frac{d\lambda}{d\nu}(t) \nu \text{ - a.e.} \quad (4.1)$$

As $(Y^n)$ is equi-BVRC, and for all $t \in I$, $(Y^n(t)) \subset D(A(t))$, we may assume that $(Y^n)$ converges pointwise to a BVRC mapping $Y : I \to E$. Using the estimate $\|\frac{dY^n}{d\nu}\| \leq K \nu$-a.e., we may also assume that $(\frac{dY^n}{d\nu})$ weakly converges in $L^1(I;E;\nu)$ to $\frac{dY}{d\nu}$, and by hypothesis (j),
(b(\cdot, X^{n-1}(\cdot)))$ weakly converges to $Z \in L^1(I, E; \lambda)$. Hence $\int_0^t b(s, X^{n-1}(s)) ds \to \int_0^t Z(s) ds$ for each $t \in I$. So we get

$$\lim_{n \to \infty} X^n(t) = \lim_{n \to \infty} \left( \int_0^t b(s, X^{n-1}(s)) ds + Y^n(t) \right) = \int_0^t Z(s) ds + Y(t) =: X(t).$$

As $(X^n(\cdot))$ pointwise converges to $X(\cdot)$ on $I$, $(b(\cdot, X^{n-1}(\cdot)))$ is uniformly bounded and, by hypothesis $(jj)$, it pointwise converges to $b(\cdot, X(\cdot))$. Then by Lebesgue’s theorem

$$\lim_{n \to \infty} \int_0^t b(s, X^{n-1}(s)) ds = \int_0^t b(s, X(s)) ds.$$

By identifying the limits we have

$$X(t) = \int_0^t b(s, X(s)) ds + Y(t) \ \forall t \in I.$$

From (4.1), repeating the argument involving Komlos techniques we get $Y(t) \in D(A(t))$, for all $t \in I$, and

$$-\frac{dY}{d\nu}(t) \in A(t)Y(t) + \int_0^t b(s, X(s)) ds \frac{d\lambda}{d\nu}(t) \ \nu - a.e. t \in I.$$

The proof is therefore complete. \[\square\]

**Theorem 4.3** Let $I := [0, T]$ and let for every $t \in I$, $A(t) : D(A(t)) \subset \mathbb{R}^e \to \mathbb{R}^e$ be a time dependent maximal monotone operator satisfying (H1), with $r$ continuous instead of right continuous, and (H2).

Let $z \in C^{1-\text{var}}(I, \mathbb{R}^d)$ the space of continuous functions of bounded variation defined on $I$ with values in $\mathbb{R}^d$. Let $\mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ the space of linear mappings $f$ from $\mathbb{R}^d$ to $\mathbb{R}^e$ endowed with the operator norm

$$\|f\|_{\mathcal{L}} := \sup_{x \in \mathbb{R}^d} \|f(x)\|_{\mathbb{R}^e}.$$

Let us consider a class of continuous integrand operator $b : I \times \mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ satisfying for a nonnegative constant $M$

(a) $\|b(t, x)\|_{\mathcal{L}} \leq M$, for all $(t, x) \in I \times \mathbb{R}^e$.

(b) $\|b(t, x) - b(t, y)\|_{\mathcal{L}} \leq M\|x - y\|_{\mathbb{R}^e}$, for all $(t, x, y) \in I \times \mathbb{R}^e \times \mathbb{R}^e$

Let $a \in D(A(0))$. Then there exist a BVC function $X : I \to \mathbb{R}^e$ and a BVC function $Y : I \to \mathbb{R}^e$ satisfying

$$\begin{cases} 
X(0) = Y(0) = a; \\
X(t) = \int_0^t b(\tau, X(\tau)) d\tau + Y(t) \ \forall t \in I; \\
-\frac{dY}{d\nu}(t) \in A(t, Y(t)) + \left( \int_0^t b(\tau, X(\tau)) d\tau \right) \frac{d\lambda}{d\nu}(t) \ \nu - a.e. t \in I.
\end{cases}$$

where $\int_0^t b(\tau, X(\tau)) d\tau$ denotes the Riemann-Stieltjes integral of the continuous function $b(\cdot, X(\cdot))$ with respect to $z$. 24
Proof. Let us set, for all \( t \in I \),

\[
X^0(t) = a, \quad h^1(t) = \int_0^t b(\tau, a) d\tau
\]

then by Proposition 2.2 in [27], we have

\[
\left\| \int_0^t b(\tau, a) d\tau \right\|_{L^\infty} \leq \| b(\cdot, a) \|_{\infty} \cdot |z|_{1-\text{var};[0,t]}.
\] (4.2)

Moreover

\[
\int_0^t b(\tau, a) d\tau - \int_0^s b(\tau, a) d\tau = \int_s^t b(\tau, a) d\tau
\] (4.3)

so that by condition (a)

\[
\| h^1(t) - h^1(s) \| \leq M |z|_{1-\text{var};[s,t]} \quad \forall 0 \leq s \leq t \leq T,
\]

and in particular

\[
\| h^1(t) \| \leq M |z|_{1-\text{var};[0,t]} \leq M |z|_{1-\text{var};[0,T]} =: L \quad \forall t \in I.
\] (4.5)

By Theorem \[\text{5.3}\] there is a unique BVC function \( Y^1 : I \rightarrow \mathbb{R}^e \) such that

\[
- \frac{dY^1}{d\nu}(t) \in A(t, Y^1(t)) + h^1(t) \frac{d\lambda}{d\nu}(t) \quad \nu - \text{a.e. } t \in I
\]

with \( \| \frac{dY^1}{d\nu}(t) \| \leq K, \nu \text{ a.e.} \) where \( K \) is a nonnegative constant, which depends only on the data. Set for all \( t \in I \)

\[
X^1(t) = h^1(t) + Y^1(t) = \int_0^t b(\tau, X^0(\tau)) d\tau + Y^1(t),
\]

so that \( X^1 \) is BVC with

\[
- \frac{dY^1}{d\nu}(t) \in A(t, Y^1(t)) + \left( \int_0^t b(\tau, X^0(\tau)) d\tau \right) \frac{d\lambda}{d\nu}(t) \quad \nu - \text{a.e.}
\]

Now, we construct \( X^n \) by induction as follows. Let for all \( t \in I \)

\[
h^n(t) = \int_0^t b(\tau, X^{n-1}(\tau)) d\tau.
\]

Then, by Proposition 2.2 in [27] and assumption (a), we have the estimate

\[
\| h^n(t) - h^n(s) \| \leq M |z|_{1-\text{var};[s,t]} \quad \forall 0 \leq s \leq t \leq T
\] (4.4)

and in particular

\[
\| h^n(t) \| \leq M |z|_{1-\text{var};[0,t]} \leq M |z|_{1-\text{var};[0,T]} =: L \quad \forall 0 \leq t \leq T.
\] (4.5)
By Theorem 3.3 there is a unique BVC function \( Y^n : I \rightarrow \mathbb{R}^e \) such that

\[- \frac{dY^n}{d\nu}(t) \in A(t,Y^n(t)) + h^n(t)\frac{d\lambda}{d\nu}(t) \quad \nu \text{-a.e. } t \in I\]

with \( \| \frac{dY^n}{d\nu}(t) \| \leq K, \nu\text{-a.e.} \), for some nonnegative constant \( K \), which does not depend on \( n \) since \( L \) is an upper bound for all \( \| h^n(t) \| \). Let us set

\[ X^n(t) = h^n(t) + Y^n(t) = \int_0^t b(\tau, X^{n-1}(\tau))d\tau + Y^n(t) \quad \forall t \in I,\]

so that \( X^n \) is BVC and

\[- \frac{dY^n}{d\nu}(t) \in A(t,Y^n(t)) + \left( \int_0^t b(\tau, X^{n-1}(\tau))d\tau \right)\frac{d\lambda}{d\nu}(t), \quad \nu \text{-a.e. } t \in I. \quad (4.6)\]

with \( \| \frac{dY^n}{d\nu}(t) \| \leq K, \nu\text{-a.e.} \). We note that \( (Y^n) \) is uniformly bounded and equicontinuous, then we may assume that it converges uniformly to a continuous mapping \( Y : I \rightarrow \mathbb{R}^e \) and \( \left( \frac{dY^n}{d\nu} \right) \) weakly converges in \( L^1(I,\mathbb{R}^e;\nu) \) to \( \frac{dY}{d\nu} \) with \( \| \frac{dY}{d\nu} \| \leq K, \nu\text{-a.e.} \). Now, from (4.4) and (4.5) we know that \( (h^n) \) is bounded and equicontinuous. By Ascoli theorem, we may assume that \( (h^n) \) converges uniformly to a continuous mapping \( h \). Hence \( X^n(t) = h^n(t) + Y^n(t) \) converges uniformly to \( X(t) := h(t) + Y(t) \). So, \( (b(\cdot, X^{n-1}(\cdot))) \) converges uniformly to \( b(\cdot, X(\cdot)) \) using the Lipschitz condition (b). So that, using Proposition 2.7 in [27], \( \left( \int_0^t b(\tau, X^{n-1}(\tau))d\tau \right) \) converges uniformly to \( \int_0^t b(\tau, X(\tau))d\tau \). By identifying the limits we have

\[ X(t) = \int_0^t b(\tau, X(\tau))d\tau + Y(t) \quad \forall t \in I.\]

As \( \int_0^t b(\tau, X^{n-1}(\tau))d\tau \rightarrow \int_0^t b(\tau, X(\tau))d\tau \) uniformly on \( I \), \( \left( \frac{dY^n}{d\nu} \right) \) weakly converges in \( L^1(I,\mathbb{R}^e;\nu) \) to \( \frac{dY}{d\nu} \). From (4.6), \( \left( \frac{dY^n}{d\nu} \right) \) converges weakly in \( L^1(I,\mathbb{R}^e;\nu) \) to \( t \mapsto \frac{dY}{d\nu}(t) + \left( \int_0^t b(\tau, X^{n-1}(\tau))d\tau \right)\frac{d\lambda}{d\nu}(t) \) converges weakly in \( L^1(I,\mathbb{R}^e;\nu) \) to \( t \mapsto \frac{dY}{d\nu}(t) + \left( \int_0^t b(\tau, X(\tau))d\tau \right)\frac{d\lambda}{d\nu}(t) \), from (4.6) we get

\[- \frac{dY}{d\nu}(t) - \left( \int_0^t b(\tau, X(\tau))d\tau \right)\frac{d\lambda}{d\nu}(t) \in A(t,Y(t)) \quad \nu \text{-a.e. } t \in I\]

by repeating the Komlos argument given in the proofs of the above theorems. The proof is therefore complete. 

\[\text{4.3 A relaxation problem.}\]

In the same vein we present a new existence of BVRC solution dealing with a BVRC perturbation. Let \( C : I \Rightarrow \overline{B_E} \) be a convex closed valued mapping with bounded right continuous retraction, in the sense that there is a bounded and right continuous function \( \rho : I \rightarrow [0, +\infty] \) such that \( e(C(t), C(\tau)) \leq \rho(t) - \rho(\tau), \) for all \( t, \tau \in I \) \( (\tau \leq t) \). Suppose further that \( \text{Gr}(C) \in \mathcal{B}(I) \otimes \mathcal{B}(E) \). We denote by

\[ S_C^{BVRC} := \{ u : I \rightarrow E : u \text{ is BVRC, } u(t) \in C(t) \ \forall t \in I \}; \]
\( S_C^\infty := \{ u \in L^\infty(I, E; \lambda) : u(t) \in C(t) \ \forall t \in I \} \).

By Valadier \cite{Valadier1982}, these sets are nonempty and \( cl(S_{BVRC}^C) = S_C^\infty \), here \( cl \) denotes the closure with respect to the \( \sigma(L^\infty, L^1) \)-topology. Shortly, \( S_{BVRC}^C \) is dense in \( S_C^\infty \) with respect to this topology. Then we have the following.

**Theorem 4.4** Let for every \( t \in I \), \( A(t) : D(A(t)) \subset E \Rightarrow E \) be a maximal monotone operator satisfying (H1), (H2) and (H3). Let \( a : E \to \mathbb{R} \) be a mapping such that
(i) \( |a(x)| \leq M \), for all \( x \in E \), for some constant \( M > 0 \).
(ii) \( |a(x) - a(y)| \leq M \|x - y\| \), for all \( x, y \in E \).
Assume further that there is \( \beta \in ]0, 1[ \) such that \( \forall t \in I \), \( 0 \leq 2M \frac{d\lambda}{d\nu}(t) d\nu(\{t\}) \leq \beta < 1 \). Then for \( u_0 \in D(A(0)) \) the following hold:
(1) the set \( S_C^\infty \) of BVRC solutions to the inclusion
\[
\begin{cases}
    u(0) = u_0; \\
    u(t) \in D(A(t)) \ \forall t \in I; \\
    \frac{du}{d\nu}(t) \in L^\infty(I, E; \nu); \\
    \frac{d}{d\nu}(t) \in A(t)u(t) + a(u(t))h(t)\frac{d\lambda}{d\nu}(t) \ \nu \text{ -- a.e., } h \in S_C^\infty
\end{cases}
\]
is nonempty and compact with respect to the topology of pointwise convergence on \( I \).
(2) The set \( S_{BVRC}^C \) of BVRC solutions to the inclusion
\[
\begin{cases}
    u(0) = u_0; \\
    u(t) \in D(A(t)) \ \forall t \in I; \\
    \frac{du}{d\nu}(t) \in L^\infty(I, E; \lambda) \\
    \frac{d}{d\nu}(t) \in A(t)u(t) + a(u(t))h(t)\frac{d\lambda}{d\nu}(t) \ \nu \text{ -- a.e., } h \in S_{BVRC}^C
\end{cases}
\]
is nonempty and is dense in the compact set \( S_C^\infty \).

Proof. We first note that if \( h \in S_C^\infty \) and \( v : I \to E \) is BVRC, then \( t \mapsto a(v(t))h(t) \) belongs to \( L^\infty(I, E; \lambda) \). Let us also note that the mapping \( f_h : I \times E \to E \) defined by \( f_h(t, x) := a(x)h(t) \) satisfies the conditions \( \|f_h(t, x)\| \leq M \) for all \( (t, x) \in I \times E \), and \( \|f_h(t, x) - f_h(t, y)\| \leq M \|x - y\| \) for all \( (t, x, y) \in I \times E \times E \), for any \( h \in S_C^\infty \). By Corollary 3.1, for each \( h \in S_C^\infty \) (resp. \( h \in S_{BVRC}^C \)), there is a unique BVRC solution \( v_h \) to the inclusion
\[
\begin{cases}
    v_h(0) = u_0; \\
    v_h(t) \in D(A(t)) \ \forall t \in I; \\
    \frac{dv_h}{d\nu}(t) \in L^\infty(I, E; \nu); \\
    -\frac{dv_h}{d\nu}(t) \in A(t)v_h(t) + a(v_h(t))h(t)\frac{d\lambda}{d\nu}(t) \ \nu \text{ -- a.e. } t \in I.
\end{cases}
\]
So the BVRC solutions sets are given by:
\[ S_{C}^{\infty} = \{ v_h : h \in S^{\infty}_C \} , \text{ and } S_{C}^{BVRC} = \{ v_h : h \in S^{BVRC}_C \} . \] Let \( (h_n) \subset S^{\infty}_C \). As it is shown in the proof of Theorem 3.3, the sequence \((v_{h_n})\) of BVRC solutions is equi-BVRC, namely
\[ v_{h_n}(t) = u_0 + \int_{[0,t]} \frac{dv_{h_n}}{dv}(s)ds \quad \forall t \in I, \quad \| \frac{dv_{h_n}}{dv}(s) \| \leq K \quad \nu - a.e. \]

By weak compactness we may ensure that \((\frac{dv_{h_n}}{dv}(\cdot)) \rightarrow w\) weakly in \( L^1(I, E; \nu) \) with \( \| w(t) \| \leq K \) \( \nu\)-a.e., so that \( v_{h_n}(t) \rightarrow v(t) = u_0 + \int_{[0,t]} w(s)dv(s)ds \), for all \( t \in I \) with \( \frac{dv}{dv}(s) = w(s) \). Further it is clear that \( a(v_{h_n}(t)) \rightarrow a(v(t)) \). Now we prove

**Main fact:** \( a(v_{h_n}(\cdot))h_n(\cdot) \rightarrow a(v(\cdot))h(\cdot) \) **weakly in** \( L^1(I, E; \lambda) \). Indeed, let \( g \in L^\infty(I, E; \lambda) \).

We have for all \( t \in I \)
\[ \langle g(t), a(v_{h_n}(t))h_n(t) \rangle = \langle a(v_{h_n}(t))g(t), h_n(t) \rangle. \]

It is clear that \( k_n(t) := a(v_{h_n}(t))g(t) \) and \( k(t) := a(v(t))g(t) \) satisfy \( \| k_n(t) \| \leq M \| g(t) \| \)
and \( \| k(t) \| \leq M \| g(t) \| \) for all \( t \in I \) with \( k_n(t) \rightarrow k(t) \). As \((h_n)\) weakly converges to \( h \) in \( L^1(I, E; \lambda) \) we get \( \lim_{n \rightarrow \infty} \langle k_n, h_n \rangle = \langle k, h \rangle \[3\] that is
\[ \lim_{n \rightarrow \infty} \int_0^T \langle g(t), a(v_{h_n}(t))h_n(t) \rangle dt = \int_0^T \langle g(t), a(v(t))h(t) \rangle dt. \]

This shows the required fact. From \( \frac{dv_{h_n}}{dv}(\cdot) + a(v_{h_n}(\cdot))h_n(\cdot) \frac{d\lambda}{dt}(\cdot) \rightarrow \frac{dv}{dv}(\cdot) + a(v(\cdot))h(\cdot) \frac{d\lambda}{dt}(\cdot) \) weakly \( L^1(I, E; \nu) \) and the inclusion
\[ -\frac{dv_{h_n}}{dv}(t) - a(v_{h_n}(t))h_n(t) \frac{d\lambda}{dt}(t) \in A(t)v_{h_n}(t) \quad \nu - a.e., \]

by repeating the convergence limit involving Komlos argument given in the proof of Theorem 3.3 we get
\[ -\frac{dv}{dt}(t) - a(v(t))h(t) \frac{d\lambda}{dt}(t) \in A(t)v(t) \quad \nu - a.e. \ t \in I. \]

By uniqueness we have \( v = v_h \). We conclude that the mapping \( \phi : h \mapsto v_h \) from the compact metrizable set \( S^{\infty}_C \subset L^{\infty}(I, E; \lambda) \) to \( S^{\infty}_C \) endowed with the topology of pointwise convergence is continuous. Hence \( \{ v_h : h \in S^{\infty}_C \} \) endowed with the topology of pointwise convergence is compact. Since \( S^{BVRC}_C \) is dense in \( S^{\infty}_C \), we conclude that \( \{ v_h : h \in S^{BVRC}_C \} \) is dense in \( \{ v_h : h \in S^{\infty}_C \} \).

**Theorem 4.5** Let for \( t \in I, A(t) : D(A(t)) \subset E \Rightarrow E \) be a maximal monotone operator satisfying \((H_1), (H_2)\) and \((H_3)\). Let \( Ext(B_E) \) be the set of extreme points of \( B_E \). Let us denote
\[ \mathcal{M}_{B_E} := \{ u \in L^\infty(I, E; \lambda) : u(t) \in B_E \quad \forall t \in I \} \]

\[ \text{1Here one may invoke a general fact, that on bounded subsets of } L^\infty \text{ the topology of convergence in measure coincides with the topology of uniform convergence on uniformly integrable sets, i.e. on relatively weakly compact subsets, alias the Mackey topology. This is a lemma due to Grothendieck [28] Ch.5 §4 no 1 Prop. 1 and exercise} \text{ (see also [19] for a more general result concerning the Mackey topology for bounded sequences in } L^\infty). \]
\[ M_{\text{Ext}(\mathcal{P}_E)} := \{ u \in L^\infty(I, E; \lambda) : u(t) \in \text{Ext}(\mathcal{P}_E) \quad \forall t \in I \}. \]

Let \( a : E \to \mathbb{R} \) be a mapping such that for some constant \( M > 0 \),
(i) \(|a(x)| \leq M \), for all \( x \in E \),
(ii) \(|a(x) - a(y)| \leq M \|x - y\| \), for all \( x, y \in E \).
Assume further that there is \( \beta \in ]0, 1[ \) such that \( \forall t \in I \), \( 0 \leq 2M \frac{d\lambda}{d\nu}(t)d\nu\{t\} \leq \beta < 1 \). Then for \( u_0 \in D(A(0)) \), the following hold:
(1) the set \( \mathcal{S}_{M_{\text{Ext}(\mathcal{P}_E)}} \) of BVRC solutions of the inclusion
\[
\begin{cases}
    u(0) = u_0; \\
    u(t) \in D(A(t)) \quad \forall t \in I; \\
    \frac{du}{d\nu}(t) \in L^\infty(I, E; \nu); \\
    -\frac{du}{d\nu}(t) \in A(t)u(t) + a(u(t))h(t)\frac{d\lambda}{d\nu}(t) \quad \nu-a.e., \quad h \in M_{\text{Ext}(\mathcal{P}_E)}
\end{cases}
\]
is nonempty and compact for the topology of pointwise convergence on \( I \).
(2) The set \( \mathcal{S}_{M_{\text{Ext}(\mathcal{P}_E)}} \) of BVRC solutions of the inclusion
\[
\begin{cases}
    u(0) = u_0; \\
    u(t) \in D(A(t)) \quad \forall t \in I; \\
    \frac{du}{d\nu}(t) \in L^\infty(I, E; \nu); \\
    -\frac{du}{d\nu}(t) \in A(t)u(t) + a(u(t))h(t)\frac{d\lambda}{d\nu}(t) \quad \nu-a.e., \quad h \in M_{\text{Ext}(\mathcal{P}_E)}
\end{cases}
\]
is nonempty and is dense in the compact set \( \mathcal{S}_{M_{\text{Ext}(\mathcal{P}_E)}} \).

Proof. The proof is similar to the proof of Theorem 4.4 with appropriated modifications. By using the fact that \( M_{\text{Ext}(\mathcal{P}_E)} \) is dense in \( M_{\mathcal{P}_E} \), we conclude that the set \( \mathcal{S}_{M_{\text{Ext}(\mathcal{P}_E)}} \) is dense in the set \( \mathcal{S}_{M_{\mathcal{P}_E}} \) by virtue of Ljapunov theorem e.g [13].

Remark 4.1 Similar results concerning sweeping process by convex closed moving sets, can be easily deduced from Theorem 4.4 and Theorem 4.5.

4.4 Fractional evolutions

Let \( I := [0, 1] \), we investigate in the sequel a fractional order evolution inclusion problem \( \mathcal{P}_{f,A}(D^\alpha) \) coupled with a time dependent maximal monotone operator \( A(t) \) with perturbation \( f \) in \( E \), of the form
\[ D^\alpha h(t) + \lambda D^{\alpha-1}h(t) = u(t) \quad t \in I; \quad (4.7) \]
\[ I^{\beta}_{0+} h(t)|_{t=0} := \lim_{t \to 0} \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} h(s)ds = 0; \quad (4.8) \]
\[ h(1) = I^{\gamma}_{0+} h(1) = \int_0^1 \frac{(1-s)^{\gamma-1}}{\Gamma(\gamma)} h(s)ds; \quad (4.9) \]
\[-\frac{du}{dt}(t) \in A(t)u(t) + f(t, h(t), u(t)) \quad \text{a.e. } t \in I,\]

where \( \alpha \in [1, 2], \beta \in [0, 2 - \alpha], \lambda \geq 0, \gamma > 0 \) are given constants, \( D^\alpha \) is the standard Riemann-Liouville fractional derivative, \( \Gamma \) is the gamma function and \( f : I \times E \times E \to E \) is a single valued mapping.

**Definition 4.1 (Fractional Bochner-integral)** Let \( \zeta : I \to E \) and \( a \in I \). The fractional Bochner-integral of order \( \alpha > 0 \) of the function \( \zeta \) is defined by

\[(I^\alpha_a \zeta)(t) := \int_a^t \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} \zeta(s)ds \quad \forall t > a.\]

We refer to [30, 33, 42] for the general theory of Fractional Calculus and Fractional Differential Equations.

We denote by \( W^\alpha_{B; E}(I) \) the space of all continuous functions in \( C(I, E) \) such that their Riemann-Liouville fractional derivative of order \( \alpha - 1 \) are continuous and their Riemann-Liouville fractional derivative of order \( \alpha \) are Bochner-integrable.

**Definition 4.2 (Green function [16])** Let \( \alpha \in [1, 2], \omega \in [0, 2 - \alpha], k \geq 0, \varsigma > 0 \) and \( G : I \times I \to \mathbb{R} \) be the function defined by

\[G(t, s) = \varphi(s)I_{0+}^{\alpha-1}(\exp(-\lambda t)) + \begin{cases} \exp(\lambda s)I_{s+}^{\alpha-1}(\exp(-\lambda t)) & \text{if } 0 \leq s \leq t \leq 1, \\ 0 & \text{if } 0 \leq t \leq s \leq 1, \end{cases}\]

where

\[\varphi(s) = \frac{\exp(\lambda s)}{\mu_0} \left[ \left(I_{s+}^{\alpha-1+\gamma}(\exp(-\lambda t))\right)(1) - \left(I_{s+}^{\alpha-1}(\exp(-\lambda t))\right)(1) \right]\]

with \( \mu_0 = \left(I_{0+}^{\alpha-1}(\exp(-\lambda t))\right)(1) - \left(I_{0+}^{\alpha-1+\gamma}(\exp(-\lambda t))\right)(1) \).

**Lemma 4.1 [19]** Let \( G \) be the above defined function. Then

(i) \( G(\cdot, \cdot) \) satisfies the following estimate

\[|G(t, s)| \leq \frac{1}{\Gamma(\alpha)} \left( \frac{1 + \Gamma(\gamma + 1)}{\mu_0 \Gamma(\alpha) \Gamma(\gamma + 1)} + 1 \right) =: M_G.\]

(ii) If \( u \in W^\alpha_{B; E}(I) \) satisfies boundary conditions (4.7), (4.8) and (4.9), then

\[u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t G(t, s) \left(D^\alpha u(s) + \lambda D^{\alpha-1}u(s)\right)ds \quad \forall t \in I.\]
Let \( \zeta \in L^1(I, E; \lambda) \) and let \( u_\zeta : I \to E \) be the function defined by

\[
u_\zeta(t) := \int_0^1 G(t, s)\zeta(s)ds \quad \forall t \in I.
\]

Then

\[
(I_{0+}^\beta u_\zeta)(t)|_{t=0} = 0 \quad \text{and} \quad u_\zeta(1) = (I_{0+}^\gamma u_\zeta)(1).
\]

Moreover, \( u_\zeta \in W_{B,E}^{\alpha,1}(I) \) and we have

\[
(D^\alpha u_\zeta)(t) + \lambda (D^{\alpha-1} u_\zeta)(t) = \zeta(t) \quad \forall t \in I.
\]

From Lemma 4.1 we summarize a crucial fact.

**Lemma 4.2** \([16]\) Let \( \zeta \in L^1(I, E; \lambda) \). Then the boundary value problem

\[
\begin{aligned}
D^\alpha u(t) + \lambda D^{\alpha-1} u(t) &= \zeta(t) \quad \forall t \in I \\
(I_{0+}^\beta u(t))|_{t=0} &= 0, \quad u(1) = (I_{0+}^\gamma u)(1)
\end{aligned}
\]

has a unique \( W_{B,E}^{\alpha,1}(I) \)-solution defined by

\[
u(t) = \int_0^1 G(t, s)\zeta(s)ds \quad \forall t \in I.
\]

**Theorem 4.6** \([16]\) Let \( X : I \rightrightarrows E \) be a convex compact valued measurable multi-mapping such that \( X(t) \subset \overline{\gamma B_E} \) for all \( t \in I \), where \( \overline{\gamma} \) is a nonnegative constant. Then the \( W_{B,E}^{\alpha,1}(I) \) solutions set of problem

\[
\begin{aligned}
D^\alpha u(t) + \lambda D^{\alpha-1} u(t) &= \zeta(t) \quad \text{a.e.} \ t \in I, \quad \zeta \in S_X^\gamma; \\
(I_{0+}^\beta u(t))|_{t=0} &= 0, \quad u(1) = (I_{0+}^\gamma u)(1)
\end{aligned}
\]

is a convex compact subset of \( C(I, E) \).

Now we present our existence theorem for problem \( P_{f,A}(D^\alpha) \).

**Theorem 4.7** Let for every \( t \in I \), \( A(t) : D(A(t)) \subset E \rightrightarrows E \) be a maximal monotone operator satisfying \((H_1), (H_2)\) and \((H_3)\) \( D(A(t)) \subset X(t) \subset \overline{\gamma B_E} \) for all \( t \in I \), where \( X : I \rightrightarrows E \) is a convex compact valued measurable multi-mapping and \( \overline{\gamma} \) is a nonnegative constant.

Let \( f : I \times E \times E \to E \) be such that for every \( x, y \in E \), the mapping \( f(\cdot, x, y) \) is \( B(I) \)-measurable and for every \( t \in I \), the mapping \( f(t, \cdot, \cdot) \) is continuous on \( E \times E \) and satisfies for some nonnegative constant \( M \)

(i) \( ||f(t, x, y)|| \leq M \) for all \( (t, x, y) \in I \times E \times E \),

(ii) \( ||f(t, z, x) - f(t, z, y)|| \leq M||x - y|| \) for all \( (t, z, x, y) \in I \times E \times E \times E \).

Assume further that there is \( \bar{\beta} \in [0, 1] \) such that \( \forall t \in I, \ 0 \leq 2M \frac{d\lambda}{dt}(t)d\nu(\{t\}) \leq \bar{\beta} < 1. \)
Then there is a $W^{\alpha,1}_{B,E}(I)$ mapping $h : I \rightarrow E$ and a BVRC mapping $u : I \rightarrow E$ satisfying the coupled system

$$
\begin{aligned}
D^\alpha h(t) + \lambda D^{\alpha-1} h(t) &= u(t) \quad \forall t \in I; \\
(I_0^\alpha h)(t)|_{t=0} = 0, \quad h(1) = (I_0^\alpha h)(1); \\
u(t) &\in D(A(t)) \quad \forall t \in I; \\
- \frac{du}{d\nu}(t) &\in A(t)u(t) + f(t,h(t),u(t)) \frac{d\lambda}{d\nu}(t) \quad \nu - a.e. t \in I.
\end{aligned}
$$

Proof. Let us consider

$$X := \{ u : I \rightarrow E : u(t) = \int_0^1 G(t,s)\zeta(s)ds, \ \zeta \in S^1_{X} \}.$$

From Theorem 4.6, $X$ is a convex compact subset of $C(I,E)$. For any $h \in X$, there is a unique BVRC solution to the problem

$$
\begin{aligned}
u_h(0) &= u_0; \\
u_h(t) &\in D(A(t)) \quad \forall t \in I; \\
- \frac{du}{d\nu}(t) &\in A(t)u_h(t) + f(t,h(t),u_h(t)) \frac{d\lambda}{d\nu}(t) \quad \nu - a.e. t \in I.
\end{aligned}
$$

with $u_h(t) = u_0 + \int_{0,t} \frac{du}{d\nu}(s)d\nu(s)$ for all $t \in I$ and $\| \frac{du}{d\nu}(t) \| \leq K \ \nu$-a.e. Existence and uniqueness of such a solution is ensured by Corollary 3.1. Indeed, for any fixed $h \in X$, the mapping $f_h(t,x) = f(t,h(t),x)$ satisfies $\|f_h(t,x)\| \leq M$ for all $(t,x) \in I \times E$, $\|f_h(t,x) - f_h(t,y)\| \leq M\|x-y\|$ for all $(t,x,y) \in I \times E \times E$, while the estimate of the velocity is given in the proof of Theorem 3.2.

For any $h \in X$, we consider the mapping $\psi(h)(t) := \int_0^1 G(t,s)u_h(s)ds$, for all $t \in I$. Then, it is clear that $\psi(h) \in X$ because by $(H_3)$, $u_h(t) \in D(A(t)) \subset X(t) \subset \overline{\gamma B}_E$ for all $t \in I$. Our aim is to prove that $\psi : X \rightarrow X$ is continuous in order to obtain the existence theorem by a fixed point approach. This need a careful look using the estimate of the BVRC solution given above. It is enough to show that, if $(h_n)$ converges uniformly to $h$ in $X$, then the sequence $(u_{h_n})$ of BVRC solutions associated with $(h_n)$ of problems

$$
\begin{aligned}
u_{h_n}(0) &= u_0; \\
u_{h_n}(t) &\in D(A(t)) \quad \forall t \in I; \\
- \frac{du_{h_n}}{d\nu}(t) &\in A(t)u_{h_n}(t) + f(t,h_n(t),u_{h_n}(t)) \frac{d\lambda}{d\nu}(t) \quad \nu - a.e. t \in I.
\end{aligned}
$$

pointwise converge to the BVRC solution $u_h$ associated with $h$ of the problem

$$
\begin{aligned}
u_h(0) &= u_0; \\
u_h(t) &\in D(A(t)) \quad \forall t \in I; \\
- \frac{du_h}{d\nu}(t) &\in A(t)u_h(t) + f(t,h(t),u_h(t)) \frac{d\lambda}{d\nu}(t) \quad \nu - a.e. t \in I.
\end{aligned}
$$

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This fact is ensured by repeating the machinery given in the proof of Theorem 4.1. We have, by using (ii) of Lemma 4.1:

\[
\|\psi(h_n)(t) - \psi(h)(t)\| = \| \int_0^1 G(t,s)u_{h_n}(s)ds - \int_0^1 G(t,s)u_h(s)ds \| \\
= \| \int_0^1 G(t,s)(u_{h_n}(s) - u_h(s))ds \|
\]

\[
\leq \int_0^1 M_G\|u_{h_n}(s) - u_h(s)\|ds
\]

since \((u_{h_n}(s) - u_h(s)) \to 0\) and is pointwise bounded; \(\|u_{h_n}(s) - u_h(s)\| \leq 2\tilde{\gamma}\), we conclude that

\[
\sup_{t \in I}\|\psi(h_n)(t) - \psi(h)(t)\| \leq \int_0^1 M_G\|u_{h_n}(s) - u_h(s)\|ds \to 0,
\]

using the Lebesgue dominated convergence theorem, so that \(\psi(h_n) - \psi(h) \to 0\) in \(C(I,E)\). That is, \(\psi\) has a fixed point say \(h = \psi(h) \in \mathcal{X}\). It follows that

\[
\begin{cases}
h(t) = \psi(h)(t) = \int_0^1 G(t,s)u_h(s)ds \quad \forall t \in I; \\
u_h(t) \in D(A(t)) \quad \forall t \in I; \\
-\frac{du_h}{d\nu}(t) \in A(t)u_h(t) + f(t,h(t),u_h(t))\frac{d\lambda}{d\nu}(t) \quad \nu - \text{a.e. } t \in I.
\end{cases}
\]

Coming back to Lemma 4.2 and applying the above notations, this means that we have just shown that there exist a mapping \(h \in W^{\alpha,1}_{B,E}(I)\) and a BVRC mapping \(u_h : I \to E\) satisfying

\[
\begin{cases}
D^\alpha h(t) + \lambda D^{\alpha - 1}h(t) = u_h(t) \quad \forall t \in I; \\
(I_0^\beta h)(t)|_{t=0} = 0, \quad h(1) = (I_0^\beta h)(1) \\
u_h(t) \in D(A(t)) \quad \forall t \in I \\
-\frac{du_h}{d\nu}(t) \in A(t)u_h(t) + f(t,h(t),u_h(t))\frac{d\lambda}{d\nu}(t) \quad \nu - \text{a.e. } t \in I.
\end{cases}
\]

**Corollary 4.2** Let \(C : I \to E\) be a convex compact valued multi-mapping such that

(i) \(d_H(C(t),C(s)) \leq |r(t) - r(s)|\), for all \(\tau, t \in I\); 
(ii) \(C(t) \subseteq X(t) \subseteq \gamma B_E\) for all \(t \in I\) where \(X : I \to E\) is a convex compact valued measurable multi-mapping and \(\gamma\) is a nonnegative constant.

Let \(f : I \times E \times E \to E\) satisfying all the hypotheses in Theorem 4.7. Assume further that there is \(\tilde{\beta} \in [0,1]\) such that \(\forall t \in [0,T], 0 \leq 2M \frac{d\lambda}{d\nu}(\{t\}) \leq \tilde{\beta} < 1\). Then there is a \(W^{\alpha,1}_{B,E}(I)\) mapping \(h : I \to E\) and a BVRC mapping \(u : I \to E\) satisfying the coupled system

\[
\begin{cases}
D^\alpha h(t) + \lambda D^{\alpha - 1}h(t) = u(t) \quad \forall t \in I; \\
(I_0^\beta h)(t)|_{t=0} = 0, \quad h(1) = (I_0^\beta h)(1) \\
u_h(t) \in C(t) \quad \forall t \in I \\
-\frac{du_h}{d\nu}(t) \in N_{C(I)}(u(t)) + f(t,h(t),u(t))\frac{d\lambda}{d\nu}(t) \quad \nu - \text{a.e. } t \in I.
\end{cases}
\]

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Comments. Some comments are in order. An easy comparison between Theorem 4.1 and Theorem 4.7 shows that Theorems 4.1 deals with second order equation coupled with a time-dependent maximal monotone BVRC in variation evolution, while Theorem 4.7 deals with fractional order equation coupled with a time-dependent maximal monotone BVRC in variation evolution, in particular, coupled with the sweeping process associated with a convex compact valued BVRC mapping; see Corollary 4.1 and Corollary 4.2 respectively. Actually, there is a pioneering work in Castaing et al [17] dealing with existence of AC solutions for a system of fractional order (second order) equation coupled with a time and state dependent absolutely continuous maximal monotone operator evolution. A large synthesis of the study of dynamical systems coupled with monotone set-valued operators is given in Brogliato et al [12].

4.5 Second order BVRC evolution inclusion with a time and state dependent maximal monotone operator

Let $I = [0,T]$ and let $E$ be a separable Hilbert space. We begin with a useful lemma that is inspired from [17].

Lemma 4.3 Let for every $(t,x) \in I \times E$, $A_{(t,x)} : D(A_{(t,x)}) \subset E \to 2^E$ be a maximal monotone operator satisfying:

$(H_1)$ $\text{dis}(A_{(t,x)}, A_{(\tau,y)}) \leq r(t) - r(\tau) + \|x - y\|$ for all $0 \leq \tau \leq t \leq T$ and for all $(x,y) \in E \times E$, where $r : I \to [0, +\infty]$ is nondecreasing and right continuous on $I$.
$(H_2)$ $\|A_{(t,x)}^0(y)\| \leq c(1 + \|x\| + \|y\|)$ for all $(t,x,y) \in I \times E \times D(A_{(t,x)})$, for some nonnegative constant $c$.

Let $X : I \Rightarrow E$ be a convex compact valued Lebesgue-measurable multi-mapping such that $X(t) \subset \gamma(t)\overline{B}_E$ for all $t \in I$, where $\gamma$ is a nonnegative Lebesgue-integrable function and let

$$X := \{u_\zeta \in C(I,E) : u_f(t) = x_0 + \int_0^t \zeta(s)ds, \forall t \in I, \zeta \in S^1_X\}.$$

Then the following hold.
(a) $X$ is a convex compact subset of $C(I,E)$ and equi-absolutely continuous.
(b) For any $h \in X$, the operator $A_{(t,h(t))}$ is equi-BVRC in variation, in the sense that there is a non decreasing right continuous mapping $\rho : I \to [0, +\infty]$ such that

$$\text{dis}(A_{(t,h(t))}, A_{(\tau,h(\tau))}) \leq \rho(t) - \rho(\tau) \quad \forall \tau, t \in I \ (\tau \leq t).$$

Further, $\|A_{(t,h(t))}^0(y)\| \leq d(1 + \|y\|)$ for all $y \in D(A_{(t,h(t))})$ where $d$ is a nonnegative constant.

Proof. the proof of (a) is classical using the compactness of the convex compact valued integral $\int_0^t X(s)ds$ (cf [14]). It is obvious that for any $\zeta \in S^1_X$, $\|u_\zeta(t) - u_\zeta(\tau)\| \leq \int_\tau^t \gamma(s)ds, \ 0 \leq \tau \leq t \leq T$.

(b) The time-dependent maximal monotone operator $A_{(t,h(t))}$ is BVRC in variation. Indeed,
for all \(0 \leq \tau \leq t \leq T\), we have by (\(H_1\))

\[
\text{dis}(A_{(t,h(t))}, A_{(r,h(r))}) \leq r(t) - r(\tau) + \|h(t) - h(\tau)\| \leq r(t) - r(\tau) + \int_{\tau}^{t} \gamma(s) ds = \rho(t) - \rho(\tau),
\]

where \(\rho(t) = r(t) + \int_0^t \gamma(s) ds\), for all \(t \in I\). So \(\rho\) is nondecreasing right continuous on \(I\).

Furthermore, by (\(H_2\)) we have

\[
\|A^0_{(t,h(t))} y\| \leq c(1 + \|h(t)\| + \|y\|) \leq d(1 + \|y\|) \tag{4.11}
\]

for all \(y \in D(A_{(t,h(t))})\), where \(d\) is a nonnegative generic constant, because \(h\) is uniformly bounded.

\[\square\]

**Theorem 4.8** Let for every \((t,x) \in I \times E\), \(A_{(t,x)} : D(A_{(t,x)}) \subset E \rightarrow 2^E\) be a maximal monotone operator satisfying (\(H_1\)), (\(H_2\)) and

(\(H_3\)) \(D(A_{(t,x)}) \subset X(t) \subset \gamma(t) B_E\) for all \((t,x) \in I \times E\), where \(X : I \equiv E\) is a convex compact valued Lebesgue-measurable multi-mapping and \(\gamma\) is a nonnegative \(L^1(I; \mathbb{R}; \lambda)\)-integrable function.

Set for all \(t \in I\) \(\rho(t) = r(t) + \int_0^t \gamma(s) ds\) and \(\mu = \lambda + d\rho\) and let \(f : I \times E \times E \rightarrow E\) be such that for every \(x, y \in E\), the mapping \(f(\cdot, x, y)\) is \(\mathcal{B}(I)\)-measurable and for every \(t \in I\), the mapping \(f(t, \cdot, \cdot)\) is continuous on \(E \times E\) and satisfying for some nonnegative constant \(M\)

(i) \(\|f(t,x,y)\| \leq M\) for all \((t,x,y) \in I \times E \times E\);

(ii) \(\|f(t,z,x) - f(t,z,y)\| \leq M\|x - y\|\) for all \((t,z,x,y) \in I \times E \times E \times E\).

Assume further that there is \(\beta \in [0, 1]\) such that \(\forall t \in I, 0 \leq 2M \frac{d\rho}{dt}(t) d\mu(\{t\}) \leq \beta < 1\).

Then, for any \((x_0, u_0) \in E \times D(A_{(0,x_0)})\) there exist an absolutely continuous mapping \(x : I \rightarrow E\) and a BVRC mapping \(u : I \rightarrow E\) with density \(\frac{du}{dt}\) with respect to \(\mu\), such that

\[
\begin{align*}
x(t) &= x_0 + \int_0^t u(s) ds \quad \forall t \in I; \\
x(0) &= x_0, \quad u(0) = u_0; \\
u(t) &\in D(A_{(t,x(t))}) \quad \forall t \in I; \\
-\frac{du}{dt}(t) &\in A_{(t,x(t))} u(t) + f(t, x(t), u(t)) \frac{dx}{dt}(t) \quad \mu - a.e. t \in I.
\end{align*}
\]

Proof. Let

\[
\mathcal{X} := \{\zeta \in \mathcal{C}(I,E) : \zeta(t) = x_0 + \int_0^t \zeta(s) ds, \forall t \in I, \zeta \in S^1_X\}.
\]

Then by Lemma \(\ref{lem:3}\, (a)\), \(\mathcal{X}\) is a convex compact subset of \(\mathcal{C}(I,E)\) and is equi-absolutely continuous. For each \(h \in \mathcal{X}\), by Lemma \(\ref{lem:3}\,(b)\), the time-dependent maximal monotone operator \(A_{(t,h(t))}\) satisfies relations \(\ref{eq:10}\) and \(\ref{eq:11}\). By applying Corollary \(\ref{cor:3}\) with \(\nu\) replaced by \(\mu\), for any \(h \in \mathcal{X}\), there is a unique BVRC solution of the problem

\[
\begin{align*}
u_h(0) &= u_0; \\
u_h(t) &\in D(A_{(t,h(t))}) \quad \forall t \in I; \\
-\frac{du_h}{d\mu}(t) &\in A_{(t,h(t))} u_h(t) + f(t, h(t), u_h(t)) \frac{d\lambda}{d\mu}(t) \quad \mu - a.e. t \in I,
\end{align*}
\]

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with \( u_h(t) = u_0 + \int_{[0,t]} \frac{du_h}{d\mu}(s) d\mu(s) \) for all \( t \in I \) and \( \| \frac{du_h}{d\mu}(t) \| \leq K \) \( \mu \)-a.e. Indeed, for any fixed \( h \in \mathcal{X} \), the mapping \( f_h(t, x) = f(t, h(t), x) \) satisfies \( \| f_h(t, x) \| \leq M \) for all \( (t, x) \in I \times E \), \( \| f_h(t, x) - f_h(t, y) \| = \| f(t, h(t), x) - f(t, h(t), y) \| \leq M \| x - y \| \) for all \( (t, x, y) \in I \times E \times E \), while the estimate of the velocity is given the proof of Theorem 5.2. Now for each \( h \in \mathcal{X} \), let us consider the mapping

\[
\psi(h)(t) := x_0 + \int_0^t u_h(s) ds \quad \forall t \in I.
\]

From (\( H_3 \)), it is clear that \( \Phi(h) \in \mathcal{X} \). Our aim is to prove that \( \psi : \mathcal{X} \to \mathcal{X} \) is continuous in order to obtain the existence theorem by a fixed point approach. It is enough to show that, if \( (h_n) \) converges uniformly to \( h \) in \( \mathcal{X} \), then the sequence \( (u_{h_n}) \) of BVRC solutions associated with \( (h_n) \) of problems

\[
\begin{cases}
  u_{h_n}(0) = u_0; \\
  u_{h_n}(t) \in D(A(t,h_n(t))) \quad \forall t \in I; \\
  -\frac{du_{h_n}}{d\mu}(t) \in A(t,h_n(t)) u_{h_n}(t) + f(t, h_n(t), u_{h_n}(t)) \frac{d\lambda}{d\mu}(t) \mu-a.e. t \in I.
\end{cases}
\]

pointwise converge to the BVRC solution \( u_h \) associated with \( h \) of the problem

\[
\begin{cases}
  u_h(0) = u_0; \\
  u_h(t) \in D(A(t,h(t))) \quad \forall t \in I; \\
  -\frac{du_h}{d\mu}(t) \in A(t,h(t)) u_h(t) + f(t, h(t), u_h(t)) \frac{d\lambda}{d\mu}(t) \mu-a.e. t \in I.
\end{cases}
\]

As \( (u_{h_n}) \) is uniformly bounded and bounded in variation since \( \| u_{h_n}(t) - u_{h_n}(\tau) \| \leq K(\mu(\tau, t)) \), for \( \tau \leq t \) and \( u_{h_n}(t) \in D(A(t,h_n(t))) \subset X(t) \subset \gamma(t) \overline{B}_E \), for all \( t \in I \), and hence it is relatively compact, by Theorem 5.1, we may assume that \( (u_{h_n}) \) pointwise converges to a BV mapping \( u(\cdot) \). Now, since for all \( t \in I \), \( u_{h_n}(t) = u_0 + \int_{[0,t]} \frac{du_{h_n}}{d\mu} d\mu \) and \( \frac{du_{h_n}}{d\mu}(s) \in K\overline{B}_E \mu\text{-a.e.} \) we may assume that \( \frac{du_{h_n}}{d\mu} \) converges weakly in \( L^1(I, E; \mu) \) to \( w(t) \in L^1(I, E; \mu) \) with \( w(t) \in K\overline{B}_E \mu\text{-a.e.} \), so that

\[
\lim_{n \to \infty} u_{h_n}(t) = u_0 + \int_{[0,t]} w(s) d\mu(s) \quad \forall t \in I.
\]

By identifying the limits, we get

\[
u(t) = u_0 + \int_{[0,t]} w(s) d\mu(s) \quad \forall t \in I,
\]

with \( \frac{du}{d\mu} = w \). Whence, using the hypothesis on \( f \), we obtain

\[
\lim_{n \to \infty} f(t, h_n(t), u_{h_n}(t)) = f(t, h(t), u(t)) \quad \forall t \in I.
\]

As consequence, \( (f(\cdot, h_n(\cdot), u_{h_n}(\cdot)) \frac{du_{h_n}}{d\mu}(\cdot)) \) pointwise converges to \( f(\cdot, h(\cdot), u(\cdot)) \frac{du}{d\mu}(\cdot) \). Since \( \frac{du_{h_n}}{d\mu} \) weakly converges to \( \frac{du}{d\mu} \) in \( L^1(I, E; \mu) \), we may assume that it Komlos converges to \( \frac{du}{d\mu} \).
For simplicity, set for all $t \in I$, $z_n(t) = f(t, h_n(t), u_{h_n}(t)) \frac{dh_n}{dt}(t)$ and $z(t) = f(t, h(t), u_{h}(t)) \frac{dh}{dt}(t)$. Hence $(\frac{du_n}{dt} + g_n(t))$ Komlos converges to $\frac{du}{dt} + g(t)$. Further, we note that $u(t) \in D(A(t, h(t)))$ for all $t \in I$. Indeed we have $\text{dis}(A(t, h_n(t)), A(t, h(t))) \leq \|h_n(t) - h(t)\| \to 0$ and it is clear that $(y_n = A^0(t, h_n(t))u_{h_n}(t))$ is bounded, hence relatively weakly compact. By applying Lemma 2.2 to $u_{h_n}(t)$ and to a convergent subsequence of $(y_n)$ we conclude that $u(t) \in D(A(t, h(t)))$.

Now, apply Lemma 2.4 to $A(t, h_n(t))$ and $A(t, h(t))$ to find a sequence $(\eta_n)$ such that such that $\eta_n \in D(A(t, h_n(t))), \eta_n \to \eta, A^0(t, h_n(t))\eta_n \to A^0(t, h(t))u(t)$. From the inclusion

$$- \frac{du_n}{dt}(t) \in A(t, h_n(t))u_{h_n}(t) + z_n(t) \quad \mu - a.e.,$$

we get by the monotonicity of $A(t, x)$,

$$\left\langle \frac{du_n}{dt}(t) + z_n(t), u_{h_n}(t) - \eta_n \right\rangle \leq \left\langle A^0(t, h_n(t))\eta_n, \eta_n - u_{h_n}(t) \right\rangle \quad \mu - a.e.. \tag{4.13}$$

On the other hand, since

$$\left\langle \frac{du_n}{dt}(t) + z_n(t), u(t) - \eta \right\rangle = \left\langle \frac{du_n}{dt}(t) + z_n(t), u_{h_n}(t) - \eta_n \right\rangle + \left\langle \frac{du_n}{dt}(t) + z_n(t), u(t) - u_{h_n}(t) - (\eta - \eta_n) \right\rangle,$$

we can write

$$\frac{1}{n} \sum_{j=1}^{n} \left\langle \frac{du_n}{dt}(t) + z_j(t), u(t) - \eta \right\rangle = \frac{1}{n} \sum_{j=1}^{n} \left\langle \frac{du_n}{dt}(t) + z_j(t), u_{h_j}(t) - \eta_j \right\rangle$$

$$+ \frac{1}{n} \sum_{j=1}^{n} \left\langle \frac{du_n}{dt}(t) + z_j(t), u(t) - u_{h_j}(t) \right\rangle + \sum_{j=1}^{n} \left\langle \frac{du_n}{dt}(t) + z_j(t), \eta_j - \eta \right\rangle,$$

so that

$$\frac{1}{n} \sum_{j=1}^{n} \left\langle \frac{du_n}{dt}(t) + z_j(t), u(t) - \eta \right\rangle \leq \frac{1}{n} \sum_{j=1}^{n} \left\langle A^0(t, h_j(t))\eta_j, \eta_j - u_{h_j}(t) \right\rangle + (K + M) \frac{1}{n} \sum_{j=1}^{n} \|u(t) - u_{h_j}(t)\|.$$

$$+(K + M) \frac{1}{n} \sum_{j=1}^{n} \|\eta_j - \eta\|.$$

Passing to the limit when $n \to \infty$, this last inequality gives immediately

$$\left\langle \frac{du}{dt}(t) + z(t), u(t) - \eta \right\rangle \leq \left\langle A^0(t, h(t))\eta, \eta - v(t) \right\rangle \quad \mu - a.e.$$

As a consequence, by Lemma 2.1 we get $- \frac{du}{dt}(t) \in A(t, h(t))u(t) + z(t) \quad \mu - a.e.$ with $u(t) \in D(A(t, h(t)))$ for all $t \in I$, so that by uniqueness $u = u_h$. That is, for all $t \in I$,

$$\psi(h_n)(t) - \psi(h)(t) = \int_{0}^{t} (u_{h_n}(s) - u_{h}(s)) ds,$$
and since \((u_n(s) - u_h(s)) \to 0\) and is pointwise bounded; \(\|u_n(s) - u_h(s)\| \leq 2\gamma(s)\), we conclude by Lebesgue dominated convergence theorem, that

\[
\sup_{t \in I} \|\psi(h_n)(t) - \psi(h)(t)\| \leq \int_0^T \|u_n(s) - u_h(s)\| ds \to 0,
\]

so that \(\psi(h_n) - \psi(h) \to 0\) in \(C(I, E)\). Since \(\psi : X \to X\) is continuous it has a fixed point, say \(h = \psi(h) \in X\), that means

\[
\begin{cases}
  h(t) = \psi(h)(t) = x_0 + \int_0^t u_h(s) ds \quad \forall t \in I; \\
  u_h(0) = u_0; \\
  u_h(t) \in D(A(t, h(t))) \quad \forall t \in I; \\
  -\frac{du_h}{d\mu}(t) \in A(t, h(t))u_h(t) + f(t, h(t), u_h(t)) \frac{d\lambda}{d\mu}(t) \quad \mu - a.e. t \in I.
\end{cases}
\]

As a result, we have the following corollary.

**Corollary 4.3** Let \(C : I \times E \to E\) be a convex compact valued multi-mapping such that

(i) \(d_H(C(t, x), C(t, y)) \leq |r(t) - r(t)| + \|x - y\|\), for all \(t \in I\) and for all \((x, y) \in E \times E\).

(ii) \(C(t, x) \subset X(t) \subset \gamma(t)E\) for all \((t, x) \in I \times E\), where \(X : I \to E\) is a convex compact valued Lebesgue-measurable multi-mapping and \(\gamma : I \to \mathbb{R}\) is a nonnegative \(L^1(I, \mathbb{R}; \lambda)\)-integrable function.

Let for all \(t \in I\), \(\rho(t) = r(t) + \int_0^t \gamma(s) ds\) and \(\mu = \lambda + d\rho\) and let \(f : I \times E \times E \to E\) be such that for every \(x, y \in E\), the mapping \(f(\cdot, x, y)\) is \(B(I)\)-measurable and for every \(t \in I\), the mapping \(f(t, \cdot, \cdot)\) is continuous on \(E \times E\) and satisfying for some nonnegative constant \(M\)

(i) \(\|f(t, x, y)\| \leq M\) for all \((t, x, y) \in I \times E \times E\),

(ii) \(\|f(t, z, x) - f(t, z, y)\| \leq M\|x - y\|\) for all \((t, z, x, y) \in I \times E \times E\).

Assume further that there is \(\beta \in [0, 1]\) such that \(\forall t \in I, 0 \leq 2M \frac{d\rho}{d\lambda}(t)d\mu\{t\}) \leq \beta < 1\).

Then, for any \((x_0, u_0) \in E \times C(0, x_0)\) there exist an absolutely continuous \(x : I \to E\) and a BVRC \(u : I \to E\) with density \(\frac{dx}{dt}\) with respect to \(\mu\), such that

\[
\begin{cases}
  x(t) = x_0 + \int_0^t u(s) ds \quad \forall t \in I; \\
  x(0) = x_0, u(0) = u_0; \\
  u(t) \in C(t, x(t)) \quad \forall t \in I; \\
  -\frac{du}{d\tau}(t) \in N_C(t, x(t))u(t) + f(t, x(t), u(t)) \frac{d\lambda}{d\mu}(t) \quad \mu - a.e. t \in I.
\end{cases}
\]

To finish the paper we develop a control problem where the controls are BVRC mappings.

The tools we give have some importance since they allow to treat some second order or some coupled system with a time and state dependent operator \(A(t, x)\).

**Theorem 4.9** Let for every \((t, x) \in I \times E\), \(A(t, x) : D(A(t, x)) \subset E \to 2^E\) be a maximal monotone operator satisfying \((H_1)\), \((H_2)\) and \((H_3)'\). For any bounded set \(B \subset E\), \(\bigcup_{x \in B} D(A(t, x))\) is relatively compact.
Let \( X := \{ u : I \rightarrow Q : \| u(t) - u(s) \| \leq \rho(t) - \rho(s), \tau, t \in I (\tau \leq t) \} \), where \( Q \) is compact subset of \( E \) and \( \rho : I \rightarrow [0, +\infty] \) is nondecreasing and right continuous. Let \( \mu = \lambda + d r + d \rho \) where \( d r \) and \( d \rho \) are the Stieljes measures associated with \( r \) and \( \rho \). Let \( f : I \times E \times E \rightarrow E \) be such that for every \( x, y \in E \), the mapping \( f(\cdot, x, y) \) is \( B(I) \)-measurable and for every \( t \in I \), the mapping \( f(t, \cdot, \cdot) \) is continuous on \( E \times E \) and satisfying for some nonnegative constant \( M \),

\[
(i) \| f(t, x, y) \| \leq M \text{ for all } (t, x, y) \in I \times E \times E,
(ii) \| f(t, z, x) - f(t, z, y) \| \leq M \| x - y \| \text{ for all } (t, z, x, y) \in I \times E \times E \times E.
\]

Assume further that there is \( \beta \in [0, 1] \) such that \( \forall t \in I, 0 \leq 2M \frac{d \rho}{d \mu} (t) d \mu (\{ t \}) \leq \beta < 1 \).

Then the following hold:

(a) \( X \) is sequentially compact with respect to the pointwise convergence.
(b) For each \( h \in X \), the maximal monotone operator, \( t \mapsto A_{(t, h(t))} \) is equi-BVRC in variation,
\[
\text{dist}(A_{(t, h(t))}, A_{(\tau, h(\tau))}) \leq r(t) - r(\tau) + \rho(t) - \rho(\tau) = d(r + \rho)([\tau, t]) \quad (\tau \leq t).
\]
(c) For each \( x_0 \in Q \), for each \( h \in X \) with \( h(0) = x_0 \), there is a unique BVRC mapping \( u_h : I \rightarrow E \) satisfying
\[
- \frac{d u_h}{d \mu} (t) \in A_{(t, h(t))} u_h(t) + f(t, h(t), u_h(t)) \frac{d \lambda}{d \mu}(t) \quad \mu \text{ a.e. } t \in I.
\]
(d) the mapping \( h \mapsto u_h \) from \( X \) to \( B^{1-\text{var}}(I, E) \)\(^2\) is continuous for the pointwise convergence, i.e. if \( h_n \rightarrow h \) then \( u_{h_n} \rightarrow u_h \) pointwise.

Proof. (a) The pointwise compactness follows from the argument in the proof of Theorem 3.1 using the Helly theorem. (b) is obvious using assumption (\( H_1 \)). (c) Since \( A_{(t, h(t))} \) is equi-BVRC in variation, and \( D(A_{(t, h(t))}) \) is included in a compact set for all \( h \in X \) by assumption (\( H_3 \)'), by applying Corollary 3.1 with \( \nu \) replace by \( \mu \), there is a unique BVRC solution \( u_h \) to
\[
- \frac{d u_h}{d \mu} (t) \in A_{(t, h(t))} u_h(t) + f(t, h(t), u_h(t)) \frac{d \lambda}{d \mu}(t) \quad \mu \text{ a.e. } t \in I.
\]
(d) follows from the machinery given via Komlos convergence by noting the estimation \( \frac{d u_h}{d \mu} (t) \in L^B_{\mu} \), where \( L \) is a nonnegative generic constant.

5 Conclusion

We have established existence of BVRC solutions for evolution inclusions governed by time dependant maximal monotone operators. Our results contain novelties with sharp applications. However, there remain several issues that need full developments, for instance, the existence of BVRC solution with different types of perturbation, e.g. Skorohod problem and

\( B^{1-\text{var}}(I, E) \) denotes the space of bounded variation \( E \)-valued mappings.
when the perturbation is unbounded closed valued and Lipschitz. These considerations lead to several new research for related problems, for instance, the differential (equation) (variational inequalities) (fractional inclusion) coupled with a **time and state dependent BVRC in variation** maximal monotone operator. Actually, we are able to solve some mentioned problems by combining some techniques in Castaing et al [17] with those given here.

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