Entangled multi-knot lattice model of anyon current

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We proposed an entangled multi-knots lattice model to explore the fractional statistics of anyons. This multi-knot lattice model provides a topological interpretation for the anyons in Ising model, transverse field Ising model, Kitaev honeycomb lattice model, and block spin-1 lattice model implementation of fractional quantum Hall system. The fusion rules of Abelian anyon and non-Abelian anyon are explicitly demonstrated by local braidings on crossing states of the multi-knot lattice. The eigenstate of these spin model are expressed into multi-layer knot lattice patter. The phase transition from disorder state to ordered state is classified by the change of topological Linking number, which revealed topological character of phaser transition. The physical implementation of non-Abelian anyons in multiknot lattice may provides another path to topological quantum computation.

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I. INTRODUCTION

Anyon as a type of exotic particle beyond fermion and boson statistics has potential application in fault-tolerant quantum computation [1]. It had attracted widespread interests since it provides a different path to build quantum computer [2]. A solid experimental manipulation of non-Abelian anyon is still a hard challenge. One promising candidate for experimental implementation of anyon is electron in magnetic field [2]. Exchanging two Abelian anyons generates an arbitrary phase upon the wavefunction \( \phi(r_1, r_2) = e^{i\theta}\phi(r_2, r_1) \), where \( \theta \in [0, 2\pi] \). The statistical phase can be controlled by the enclosed magnetic flux within the exchanging path loop. The interference fringes between Laughlin quasiparticles was controversial experimental signal of non-Abelian statistics [3] that is predicted by fractional quantum Hall theory [3][6]. While the experimental operation of Ising anyons suggested by conformal field theory of critical two-dimensional Ising model [7] still remains a difficult challenge so far [8].

Anyon in quantum lattice models is quasi-particle that bear non-trivial statistical factor, such as the vortexes excitation and plaquette excitation in Kitaev's toric code model [1], non-Abelian anyon in Kitaev honeycomb model [10], excitations in topological color codes model [11]. In the toric code model, the plaquette excitation and vertices excitation are a pair of Abelian anyons. The exchanging of the two anyons is performed on a loop of lattice squares. Non-Abelian anyon exists in the gapless phase of Kitaev honeycomb model [11]. Both the toric code model and honeycomb model are difficult to find a solid material correspondence in reality, even though the physic of kitave honeycomb model shows relevance to certain transition metal compounds, such as \((Na, Li)_2IrO_3\) iridates and \(RuCl_3\) [12].

Here we proposed another different approach to anyons: periodically entangled knots on lattice. The over-crossing point of many entangled knots are placed on periodical lattice, then the over-crossing states are mapped into spin state. In this model, anyons exist as running particles in these entangled wires. For the anyon knot model on square lattice, anyons are conventional positron(or electrons) and magnetic monopoles. For anyon knot model on honeycomb lattice, there are three color anyons: red, blue and yellow anyon as we...
named. These anyon knot model have exact correspondence with two dimensional Ising model, Kitaev honeycomb model as well as Heisenberg model. Each eigenstate of the anyon knot model corresponds to a knot configuration. Each knot configuration bear a topological invariant Jones polynomial, which are related to the non-Abelian Chern-Simons field theory \cite{13}.

\begin{equation}
L = \int_M Tr(\epsilon^{ijk} A_i(\partial_j A_k - \partial_k A_j) + \frac{2}{3} A_i[A_j, A_k]).
\end{equation}

Abelian Chern-simons field theory suggest that entangled many knot are classified by linking number, self-linking number as well as writhing number \cite{14}. The fusion rules of anyons has explicit demonstration in this anyon knot model with the assistance of braidings operations. Unlike the braidings operation in fractional quantum Hall states, here the braidings operation was implemented on spatial lattice. Here we showed the conventional Ising model also bear intrinsic topological order and topological phase transition. The variation of topological linking number can distinguish topological change from one state to another.

II. ANYONS OF ENTANGLED KNOT ON SQUARE LATTICE

A. Anyons in two states Ising model

A knot is a closed loop in three dimensional manifold that can map into a unit circle. We take M knots that project horizontal lines and N knots that project vertical lines and entangle them. Then these entangled (M+N) knots project a two dimensional lattice of over crossing points. If both the horizontal knots and vertical knots bends upward (downward), the base manifold of the over crossing lattice is a sphere in thermal dynamic limit (M,N \to \infty) (Fig. 1 (a)(c)). If the horizontal knots bend upward and the vertical knots bend downward, the knot lattice is equivalent to a torus (Fig. 1 (b)(d)).

The horizontal knot (black lines in Fig. 1 (a)(b)) could be implemented by electrical conductor wires, such as super-conducting wires. Only electrons or positron runs in the horizontal knot. While the vertical knot (purple lines in Fig. 1 (a)(b)) are currents for running magnetic monopoles with positive or negative magnetic charges. According to the electromagnetic induction effect, a running positron induce a circular magnetic field around the electric current. According to electromagnetic dynamics, a running magnetic monopole could also induce a circular electric field around the magnetic current. Then there exist an electromagnetic interaction between positrons and magnetic monopoles at each over crossing point. If the magnetic current lies in the same direction as the induced magnetic field by the electric current according to the right hand rule, then the energy of system would increase by one unit. This is case in Fig. 1 (e), where the upward magnetic current is above the left-moving electric current. On the contrary case (Fig. 1 (e)), both the magnetic current and electric current are slowed down, thus the energy of the system drops one unit. Thus each over crossing point can be mapped into an effective Ising spin with two states, \(|\uparrow\rangle\) and \(|\downarrow\rangle\). Under the action of effective Hamiltonian \(H_z = S_z\), the eigenvalues with respect to these two spin states are \(S = \pm 1\) (Fig. 1 (e)).

Every knot lattice configuration can be classified by a topological number called linking number, which is defined as the total number of positive crossing minus the total number of negative crossing, \(L_{\text{link}} = (N_+ - N_-)/2\). This linking number is equivalent to the total magnetization of spins in magnetic system,

\begin{equation}
L_{\text{link}} = M = \sum_i^n S_i^z.
\end{equation}

Thus total magnetization is a topological invariant for one knot lattice configuration in this lattice model, since every current segment is confined on local lattice site. Here Reidmeister move in knot theory is strictly confined within one lattice site. Total magnetization is not a knot variant, because different knot configuration may
share the same linking number, in physics language, different spin configurations may have the same magnetization value. Suppose the over-crossing state has probability to flip from +1 to −1 (or vice versa) under random cutting and reunite. A temperature $T$ can be defined as a number that is positively correlated to this flipping probability. Then different knot lattice configuration has different existence probability with respect to the total energy and temperature. We assume that the knot lattice with lower total energy has higher probability to exist at a fixed temperature, i.e., obeys the Maxwell distribution. Then this probability weight of a certain knot lattice configuration $A$ follows the same rule as spins in statistical mechanics,

$$p = \frac{1}{Z} e^{-\frac{H(A)}{k_B T}}, \quad Z = \sum_A e^{-\frac{H(A)}{k_B T}},$$

(3)

where $Z$ is the partition function which summarized the total probability of all possible configurations. As we know, each spin state flipping indicates a topological change of knot lattice, the entanglement between two knot either in crease or decreased by 1. Since partition function is the summation of all possible knot lattice configurations, it is a topological invariant under arbitrary flipping operations. When the knot lattice configuration is exposed to an external electric field (or magnetic field) that is perpendicular to the knot lattice plane, the electric current of positrons (or magnetic current of monopoles) likens to stay above (or below). The probability of a certain knot configuration is determined by its linking number, which is equivalent to the effective Hamiltonian

$$H_z = -\sum_i h_i S_i^z,$$

(4)

here $h_i$ represent the strength of external field. Obviously the ground state knot configuration corresponds to the highest linking number, $L = M \times N$. With the torus boundary condition, all the magnetic currents are above the electric currents at every over-crossing sites. The magnetic monopole generates electric field to propel the positrons in electric currents that pass through the inner zone enclosed by the magnetic loops. These magnetic loops can not separate from electric loops without cutting. For the spherical boundary condition, it is equivalent to making a copy of lattice and connect it with the original lattice on the boundary point by point, but magnetic current is below the electric current in the copied lattice. Thus it is equivalent to the case of torus boundary condition but with a larger linking number $L = 2M \times N$. The eigenvalue corresponds to the Hamiltonian $H_z = E_i = -h L_i$, where $L_i$ represents linking number with respect to the $i$th excited states. There are $(MN - 1)$-fold degenerated first excited state with respect to $L_1 = (MN - 2)$. The average Linking number reads,

$$\langle L_{\text{link}} \rangle = \frac{\sum_A (\sum_i S_i^z)^2 e^{-\frac{H(A)}{k_B T}}}{Z} = \frac{1}{k_B T} \frac{\partial}{\partial h} \ln[Z(h)],$$

(5)

where $Z(h)$ is the partition function for the free spin Hamiltonian $H$ with homogeneous external field $h$.

For a classical knot lattice of elastic wires, we introduce the coupling interaction between two nearest neighboring over-crossing points. Two neighboring sites with opposite over-crossing states generate a kink between them, which increased the elastic energy by one unit. While neighbors with the same over-crossing states bear smooth crossover between them, that decreased the elastic energy by one step. Thus the total energy of the knot lattice is

$$H_{\text{Ising}} = \sum_{(ij)} J_z S_i^z S_j^z, \quad S^z = \pm 1,$$

(6)

where $J_z < 0$. Hamiltonian is not a topological number. Its value changes under continuous transformation of knot lattice. In the ground state of this ferromagnetically coupled system, all over-crossings are oriented in the same say. The transition from disordered orientation to this uniform ordering occurs at critical temperature $T_c$. Since the topological linking number changes during this transition, we can also call this transition a topological phase transition.

The ground state of ferromagnetic Ising model has two-fold degeneracy, i.e., all spins either point up or down, $|g⟩ = |↑↑↑· · ·↑⟩ + |↓↓↓· · ·↓⟩$. For this knot lattice model, the ground state can be represented by two layers of multi-knot sphere or multi-knot lattice (Fig. 1(c)(d)). The first excited states is generated by flipping one spin, thus $|e⟩ = |↑↑↑· · ·↑⟩ + |↓↑↑· · ·↑⟩ + |↑↑↑· · ·↑⟩ + |↑↑↑· · ·↑⟩ + |↓↓↓· · ·↓⟩ + |↓↓↓· · ·↓⟩$. There must be $2MN$ layers of knot lattice in total to represent the first excited state (Fig. 1(c)(d)). Different knot configurations of eigenstate can transform into each other by braidings anyons, i.e., the positron and magnetic monopole.

The positron and magnetic monopole in this knot lattice model are dual anyon to each other. Each spin state is a collective wave function of two open fermionic strings of electric current or magnetic current. Since the two open strings are controlled by the four ending points at the middle point of each edge, the spin state here are

![FIG. 2: (a) Flipping $|↑⟩$ to $|↓⟩$ by braidings the positron ($e^+$) and magnetic monopole ($m^+$) twice in clockwise way. (b) Flipping $|↓⟩$ to $|↑⟩$ by braidings $e^+$ and $m^+$ in counterclockwise way.](image)
also collective wave function of four anyons on the interface (Fig. 2 (a) (b)). The $| \uparrow \rangle$ state can transform into $| \downarrow \rangle$ by braiding the positron ($e^+$) and magnetic monopole ($m^+$) twice in clockwise direction (Fig. 2 (a)),

$$| \downarrow \rangle = (R^{m^+ e^+}_0)^2 | \uparrow \rangle. \quad (7)$$

Since the spin state after braiding gains a phase factor $e^{i \pi}$, the statistical phase factor for $e^+$ and $m^+$ is $R^{m^+ e^+}_0 = e^{i \pi /2}$. Thus positron and magnetic monopole are dual anyons. Follow the same process, braiding $e^+$ and $m^+$ in counterclockwise direction leads to the same statistical phase (Fig. 2 (b)). This braiding operation generates an intermediate vacuum state. The two strings (the green arc in Fig. 2 (a)(b)) in this state are forbidden to touch each other, that is why they are fermionic strings $\psi_s$. Since monopole only runs in vertical current, while positron only run in the horizontal current, the two fermionic strings are effective converter that transform a monopole into a positron, or vice versa. The vacuum state physically implemented the fusion rules of anyons (Fig. 2 (a)(b)),

$$e \times \psi_s = m, \quad e \times m = \psi_s, \quad m \times \psi_s = e, \quad (8)$$

and the trivial fusion rules: $e \times e = I$, $m \times m = I$, $\psi_s \times \psi_s = I$. However this braiding operation is only focused on one spin in one layer, it is not the eigenstate of Ising model.

Braiding anyons in the eigenstate of Ising model leads to more complicate statistical behavior beyond an Abelian phase factor. We consider a synchronous braiding of two anyons in the bilayer knot configuration of Ising ground state (Fig. 1 (c)(d)). The two layers of knot lattice are conformal invariant, thus one can draw two lines out of their common center to locate the two anyons at the same projected position on the base manifold. Suppose the upper layer represent spin up $| \uparrow \uparrow \uparrow \cdots \uparrow \uparrow \rangle$, while the bottom layer represent spin down $| \downarrow \downarrow \downarrow \cdots \downarrow \downarrow \rangle$. If we flip one spin at the same site of the two layers, it would generate a quasi-particle in the first excited state. This spin flipping can be implemented by braiding operation. At a given lattice site, we braid anyons of ($| \uparrow \rangle + | \downarrow \rangle$) in the two layers synchronously (Fig. 3 (a)). Upon one braiding in clockwise direction, the upper layer spin transforms into vacuum state, while the fermionic string in the bottom layer becomes nontrivially entangled with one crossing. While the sum of these two knot configurations are not the eigenstate of the system. Further more, a one more braiding brings the spin up state in the upper layer to a spin down state, while the spin in the bottom layer now becomes entangled electric current and magnetic current with two crossings on one site. The sum of the two layer states is still not eigenstate of the system. That means these states are not physically accessible. In order to find the right knot configuration for the eigenstate, we have to introduce a Majorana fermionic ($\psi$) operation on the internal crossings within one site (Fig. 3 (a)). The output of this Majorana fermion is to flip the over-crossing, performing the same action as $S^+$ which is formulated into Jordan-Wigner transformation,

$$c_i = \prod_{j<i} S_j^z S_i^+, \quad c_i^\dagger = \prod_{j<i} S_j^z S_i^-, \quad \psi_i = (c_i + c_i^\dagger)/2. \quad (9)$$

Here the raising operator and lowering operator has the familiar form, $S^+ = (S^z + iS^y)/2$, $S^- = (S^z - iS^y)/2$. Here $c$ is conventional annihilation fermion, obviously $\psi$ is Majorana fermion, $\psi^\dagger = \psi$. In fact, the spin-string operator representation of fermions has a geometric implementation in this knot lattice. First cutting each isolated horizontal loops on a chosen edge and connecting one ending point at the cutting edge of one loop with the ending point of another loop, then the two loops fuse into one. Repeating this operation unites all the horizontal loops into one global loop. Thus Jordan-Wigner transformation has a natural implementation in this knot lattice model. We require that the flipping operation of spin operator $S^z$ only acts on spin states, i.e., $S^z | \uparrow \rangle = | \uparrow \rangle$ and $S^z | \downarrow \rangle = | \downarrow \rangle$, to avoid its undefined operation on the crossing of two fermionic strings. Here the Majorana operator $\psi$ acts on both the crossing of electric/magnetic current and two fermionic strings. After the first braiding, we require that the flipping operation of spin operator $S^z$ only acts on spin states, i.e., $S^z | \uparrow \rangle = | \uparrow \rangle$ and $S^z | \downarrow \rangle = | \downarrow \rangle$, to avoid its undefined operation on the crossing of two fermionic strings. Here the Majorana operator $\psi$ acts on both the crossing of electric/magnetic current and two fermionic strings.
ings operation $R_{C_i}^{m+} e^+$, there are two crossings appeared in the bottom layer, in order to bring it back to vacuum state, the Majorana operator could act either the first crossing or the second crossing to disentangle the two fermionic strings, then perform a Reidemeister move \cite{17} to reach the exact vacuum state as the upper layer (Fig. 3(b)). For the $| \downarrow \rangle$ acted by braid operator $R_{C_i}^{m+} e^+$ twice, two more Majorana fermion operators ($\psi_1 \psi_3$) have to be performed to map it back to eigenspace on different crossing points (Fig. 3(b)). Thus the magnetic monopole and positron obey non-Abelian fusion rules in the ground state of Ising model, \cite{2}

$$e \times m = I + \psi, \; \psi \times \psi = I, \; e \times \psi = m, \; m \times \psi = e. \tag{10}$$

More over, the statistical factor of braidings two Ising anyons twice at one lattice site is no longer $e^{i \pi}$, it reads now

$$[R_{C_i}^{m+} e^+]^2 \begin{pmatrix} | \uparrow \rangle_i \\ | \downarrow \rangle_i \end{pmatrix} = \begin{pmatrix} e^{i \pi} & 0 \\ 0 & \psi_1 \psi_3 \end{pmatrix} \begin{pmatrix} | \uparrow \rangle_i \\ | \downarrow \rangle_i \end{pmatrix}. \tag{11}$$

Here $i$ represent the $i$th lattice site. The final state after this operation is the first excited state of ferromagnetic Ising model. The quasiparticle (or kink excitation around a lattice site) on the upper layer is a vacuum-like excitation, while the quasiparticle in the bottom layer is a Majorana fermion pair excitations. If the braidings operation was performed in counterclockwise direction, the two types of quasiparticles simply exchange their layer levels. Note that fermionic strings and unpaired Majorana fermion only exist for odd number of times of braidings (Fig. 4). For instance, three clockwise braidings on the eigenstate, ($| \uparrow \rangle + | \downarrow \rangle$), generates one Majorana fermion and one Majorana fermion pair,

$$[R_{C_i}^{m+} e^+]^3 \begin{pmatrix} | \uparrow \rangle_i \\ | \downarrow \rangle_i \end{pmatrix} = \begin{pmatrix} \psi_1 & 0 \\ 0 & \psi_1 \psi_3 \end{pmatrix} \begin{pmatrix} | \uparrow \rangle_i \\ | \downarrow \rangle_i \end{pmatrix}. \tag{12}$$

Further more, five clockwise braidings generate one Majorana fermion pair $\psi_1 \psi_3$ and a triplet cluster of Majorana fermions $\psi_1 \psi_3 \psi_5$ (Fig. 4). Thus the magnetic monopole and positron fused into a pair of Majorana fermion on the $S = +1$ sector, and fused into three Majorana fermions on the $S = -1$ sector. When the majorana fermion operators are acted on the even number of crossing sites, the overlapped vacuum state would flip a sign. The fusion rule for Magnetic monopole and positron passing through the two fermionic strings which is braided for $(2n+1)$ times is following,

$$e \times m = \psi_1 \psi_3 \psi_5 \ldots \psi_{2n-3} + \psi_1 \psi_3 \psi_5 \ldots \psi_{2n-1}$$

$$e \times \prod_{n=1}^{M} \psi_{2n-3} = m, \; m \times \prod_{n=1}^{M} \psi_{2n-3} = e. \tag{13}$$

This fusion rule is a natural output of multi-knot lattice model, but almost invisible in the conventional Ising model. Another main difference between this multi-knot lattice model and Ising model is the global electric field generated by the monopole current, which is confined in the plane pointing in the direction parallel to electric current. The same phenomena occurs for the electric current. Whenever the cutting of a electric loop occurs, the total magnetic flux loses one flux. All of the positrons along this electric current become static and lose its interaction with magnetic monopole at the crossing point. Thus there exist a topological correlation between Ising spin in each loop. The correlation length is proportional to the length of the loop.

**B. Anyons in block spin-1 Ising model**

The discussions above are focused on two-state Ising model, $S = \pm 1$. Thus there is no eigenstate for fermionic string are in knot configurations. In fact, the vacuum state generated by braidings operation can be naturally defined as spin zero state (Fig. 2 (a)(b)), i.e., $S^z = 0$. Then we arrive at a block spin-1 Ising model,

$$H_{spin1} = \sum_{(ij)} J_{ij} S_i^z S_j^z, \; S^z = (+1, 0, -1). \tag{14}$$

Block spin-1 Ising model is not exactly solved so far. For the ferromagnetic coupling $J_{ij} < 0$, the minimal energy state is magnetically ordered state, all spins points up or down. However the Hilbert space is highly enlarged due to the spin-zero state. Upon one braidings operation of $R_{C_i}^{m+} e^+$, the vacuum state and Majorana fermion state can coexist in the zero-energy space with a zero eigen-energy. More braidings operations can generates more Majorana fermions in other excited states. The knot configuration of the zero-energy states is made of many entangled loops with different sizes.

Even though the lattice model is confined in two dimensions, the knot configuration is in fact in three dimensional space. Then we can use topological quantum field theory and Jones polynomial \cite{13} to calculate the partition function of these links. We first choose the same lattice site $i$ in the multi-layer knot lattice representation of Hilbert space of block spin-1 Ising model. Then focus on the ground state layer and fix the knot configuration of the rest lattice sites except the lattice site...
i. Then for each fixed spin state, for instance $S_i^z = 0$, its partition function or Feynmann path integral in this layer could be computed as $Z(S_i^z = 0)_0$, here 0 means ground state. The partition function of $Z(S_i^z = +1)_0$ and $Z(S_i^z = -1)_0$ can be obtained following the same procedure. Repeating the same computation on all of the other knot lattice layers of eigen-states, it leads to the partition function, $Z(S_i^z = 0) = \sum_{r=0}^{M} Z(S_i^z = 0)_r$, $r$ represents eigen-energy levels. Then the partition function of the three states satisfy the familiar linear relation in topological quantum field theory and knot theory [13],

$$\alpha Z(S_i^z = +1) + \beta Z(S_i^z = 0) + \gamma Z(S_i^z = -1) = 0. \quad (15)$$

The three coefficients ($\alpha, \beta, \gamma$) in this Skine relation is computable for an explicit knot lattice state. Since partition function is a topological invariant, the linear combination of them is also topological invariant. Here the partition function depends on spin coupling strength and temperature. Combining the average linking number Eq. (5) with the Skine relation equation (15), the average linking number for vacuum state $S_i^z = 0$ reads,

$$\langle L_{link}(S(0)) \rangle = \frac{1}{k_B T} \frac{\partial \ln[I\{z Z(S(1)) - \gamma Z(S(-1))\}]}{\partial h}. \quad (18)$$

This equation offers one method for computing the average linking number in spin zero state as well as that of spin up state and spin down state. It is also useful for exploring topological phase transitions in quantum system of block spin-1 particles.

C. Quantum Hall effect of anyons in block spin-1 knot lattice

The knot lattice exposed to a homogeneous external magnetic field demonstrates fractional quantum Hall effect. Replacing the magnetic monopole and positron in the knot lattice (Fig. 1) with electrons is a natural implementation of two dimensional electron gas. A magnetic monopole with magnetic charge $Q_m$ could be placed at the center of the spherical lattice to exert a magnetic field perpendicular to the knot lattice plane (Fig. 6), then this knot lattice model of electron gas shows similar quantum Hall effect with quantized Hall resistance. Hall voltage can be defined on the four virtual edges in Fig. 1. The Hall resistance tensor are also defined on the projected two dimensional lattice expanded by horizontal current and vertical current. As all know, the Hall resistance increase as the magnetic field strength increases [20]. A serial plateaus show up for certain magnetic field strength.
\( R_{xy} = \frac{\hbar}{c^2} \frac{1}{\nu} \), \( \nu = \frac{q\rho}{B} \), \( (19) \)

\( \nu \) is the filling factor which counts how many electrons are filled into each magnetic flux quanta. \( (\nu = 1, 2, \ldots, n) \) corresponds to integral quantum Hall effect. \( \nu = n/(2\pi n \pm 1) \) leads to fractional quantum Hall effect \( (20) \). In this knot lattice model, if the electron current either oriented along horizontal direction or in vertical direction (Fig. 1), the Hall resistance term vanishes, because electrons have no chance to bend their velocity from X- to Y-direction. This knot lattice state only exist for a zero magnetic field, i.e., the external magnetic charge is \( Q_m = 0 \). Only a magnetic field can bend the electric current. A finite value of Hall resistance \( R_{xy} \) only exist for a knot lattice state with zero-crossing states, i.e., the vacuum knot configuration in Fig. 2 and red arcs showed in Fig. 4. Here we call these red arcs as turning arc. These turning arcs only appear for odd number of braidings operations (Fig. 4). Thus the braidings operator of Eq. (12) can be physically implemented by an effective magnetic field operator,

\[
[R^n_{	ext{CJ}} e^+] = \hat{B}. \tag{20}
\]

Wherever one unit of magnetic field \( \hat{B} \) is applied, the two anyons at the ending points of the electric current are exchanged in clockwise direction. The inverse operator of \( \hat{B} \), i.e., \( \hat{B}^{-1} \), braids the two anyons in counterclockwise direction (Fig. 2). Only if the magnetic field operators are performed by odd number of times on state \(| + 1 \rangle \) could generate turning arcs in vacuum state (Fig. 2, Fig. 4 and Fig. 3). However number of crossing points of these entangled turning arcs increase by one for each braidings operation. In order to unknot a pair of entangled arcs with \( 2n + 1 \) crossing points back to trivial vacuum state, there must be \( n \) Majorana fermion operators acting on the crossing points alternatively. This geometric operation is summarized as following algebra,

\[
\hat{B}^{2n+1} | + 1 \rangle = \prod_{i=1}^{n} \psi_i | 0 \rangle, \quad \hat{B}^{2n-1} | + 1 \rangle = \prod_{i=1}^{n} \psi_i | 0 \rangle,
\]

\[
\hat{B}^{2n} | 0 \rangle = \prod_{i=1}^{n} \psi_i | 0 \rangle, \quad [\hat{B}^{-1}]^{2n} | 0 \rangle = \prod_{i=1}^{n} \psi_i | 0 \rangle,
\]

\[
[\hat{B}^{-1}]^{2n+1} | - 1 \rangle = \prod_{i=1}^{n} \psi_i | 0 \rangle.
\]

\[
[\hat{B}^{-1}]^{2n-1} | + 1 \rangle = \prod_{i=1}^{n} \psi_i | 0 \rangle. \tag{21}
\]

These Majorana fermion operators can be implemented by electrons. They are filled into a magnetic flux bundle which is denoted by the number of operations of Magnetic field operator. Since the magnetic field is homogeneously distributed, all lattice sites are braided simultaneously. Thus the filling factor at a single lattice sites is the same as other lattice sites. Then we arrived at composite-fermions attached by magnetic field. The filling factor for these quasi-particles excited out of vacuum state is quite similar to that for fractional quantum Hall system \( (20) \),

\[
\nu = \frac{L_{\text{link}}}{N(B)} = \frac{N(\psi)}{N(B)} = \frac{n}{2n \pm 1}; \quad \nu = \frac{1}{2}. \tag{22}
\]

\( L_{\text{link}} \) is linking number, \( N(\psi) \) and \( N(B) \) are the number of Majorana fermion and the number of magnetic field operators correspondingly. This is a reasonable physical result, because the two entangled electric currents are equivalent to two solenoids which generates magnetic field passing through their interior. These highly entangled knot vacuum is a different implementation of composite fermions \( (25) \).

As for the state transitions from \(| + 1 \rangle \) to \(| - 1 \rangle \) (or vice versa) by a number of braidings operations of magnetic field operator, it is equivalent to let one electric current passing thorough the interior center line of the solenoid of the other electric current (Fig. 3 (a) (b)). In that case, magnetic flux exists by pairs. This state transition can exist at some lattice sites and can be generated from vacuum states which corresponds to resistance plateaus similar to quantum Hall effect. \( 2n \) number of braidings and passing through the interior center line of the solenoid \(| + 1 \rangle \) generates \( n \) Majorana fermions as its

FIG. 7: (a) Exchange two anyons within \(| \pm 1 \rangle \) states with three crossings. (b) The equivalent solenoid helix scheme for the same braidings operation with respect to (a). (c) The odd number of many writhing loops generated on square lattice. (d) One writhing loop generated on one arc. (e) The twisted free arcs generated by one braidings. (f) The output of one braidings on the crossing point of writhing loop leads to one untwisted free arc and one free loop.
eigen-excitation,
\[
\hat{B}^2|+1\rangle = |−1\rangle, \quad \hat{B}\hat{B}^{-1} = 1, \\
\hat{B}^4|+1\rangle = \psi_2|−1\rangle = \psi_1\psi_3|+1\rangle, \\
\hat{B}^6|+1\rangle = \psi_2\psi_4|−1\rangle = \psi_1\psi_3\psi_5|+1\rangle, \\
\hat{B}^{2n}|+1\rangle = \prod_{i=1}^{n-1} \psi_{2i}|−1\rangle = \prod_{i=1}^{n} \psi_{2i−1}|+1\rangle, \\
\hat{B}^{2n}|−1\rangle = \prod_{i=1}^{n} \psi_{2i−1}|−1\rangle = \prod_{i=1}^{n−1} \psi_{2i}|+1\rangle, \\
(\hat{B}^{-1})^{2n}|−1\rangle = \prod_{i=1}^{n} \psi_{2i−1}|−1\rangle = \prod_{i=1}^{n−1} \psi_{2i}|+1\rangle.
\]
so do the |−1⟩ state. The filling factor for the quasi-excitations from |+1⟩ to |−1⟩ (or vice versa) obeys the following filling factors,
\[
\nu = \frac{L_{\text{link}}}{N(B)} = \frac{N(\psi)}{N(B)} = \frac{n \pm 1}{2n}; \quad \nu = \frac{1}{2}.
\]  

The linking number \(L_{\text{link}}\) is an even or odd number of writhing loops. This integral or half-integral filling factor only exist at the eigen-energy level of vacuum state, |+1⟩ and |−1⟩. The quasi-particle excited out of these eigenstates obeys a similar fractional filling factor equations \([22, 24]\) by replacing \(N(\psi)\) with \((N(\psi) + W_w + T_i)\). The braiding algebra is independent of space scale, thus it is quite robust under renormalization. For example, if the two free turning arcs in the vacuum state generate writhing loops in the same way, then the total writhing number is an even serial, \(W_w = 0, 2, 4, 6, \ldots\). If the writhing number is zero, the even writhing number leads to the half filling serial for filling factors in eigen-energy levels, \(\nu = (2n + 1)/2\). For the quasi-particle or quasi-holes excited out of vacuum, even writhing number generates \(\nu = (n + W_w)/(2n ± 1)\). For instance, \(n = 1\) leads to the serial, \(\nu = (n + W_w)/(2n + 1) = 1/3, 1/5/3, 7/3, 3, \ldots\). \(n = 2\) leads to the serial, \(\nu = (n + W_w)/(2n − 1) = 2/3, 4/3, 8/3, 10/3, 4, \ldots\). Odd number of writhing loops only exist for the case that the two turning arc generates different number of writhing loops. The linking number of crossing current within one unit square could be equivalently viewed as the writhing number of a larger current crossing covering many unit squares. This equivalent induced the one-to-one mapping between integral filling factor and fractional filling factors. This knot lattice model offers a topological explanation to Jain’s composite fermion theory.

If the external magnetic field distribution is not homogeneous but with a fluctuating strength distribution within the scale of a few lattice sites, the filling factor would be more continuous. Even though it is quite hard to confine a magnetic field in a nanoscale circle, inhomogeneous doping of magnetic particle still provides a possible way. In this case, the crossing states in different block squares would be acted by different times of
braidings operators at the same time and generate Majorana fermions at different locations simultaneously. In that case, the renormalized filling factor is computed by direct summations,
\[ \nu = \frac{\sum_i (L_{ink} + W_\omega + T_1)}{\sum_i N(B)}. \] (26)

The linking number is \( L_{ink} = N_i(\psi) \). These filling factors are mostly likely showed as plateau in Hall resistance, but fits in the continuous straight lines. The writhing free loop in the counting above is equivalent to a free fermions. While free loop without self-crossing is simply vacuum. It takes four fermionic turning arcs to form a vacuum loop. Thus the vacuum loop behaves as boson. Even number of electrons are trapped in vacuum loop and can not move freely in the whole space. If the whole lattice is covered by free vacuum loops without any exception point, the total linking number of this insulating state is zero. The Hall resistance of this insulating phase depend the oddness or eveness of the length of the virtual edge. If the total number of columns is even, then the local in- and out-current pair cancelled each other to reach a zero Hall current. If the total number of columns is odd, there always exists one unpaired in- or out-current. The Hall resistance is determined by the filling factors above.

The quantum Hall effect for the knot lattice model of electrons is intrinsically originated from the Chern-Simons field theory: the non-Abelian Chern simons action is a topological invariant of many entangled knots \[13\], while the Abelian Chern-Simons theory determines the topologically quantized Hall resistance in fractional quantum Hall effect \[10\]. Here the Hamiltonian of electrons moving along the tangential vector of the horizontal loop or vertical loop can be formulated as the same for quantum Hall systems \[20\],
\[ H_{qh} = \sum_i \hbar \omega \left[ (P_{i,x} + a_{c,x,i})^2 + (P_{i,y} + a_{c,y,i})^2 \right] + V, \]
\[ P_{i,x} = -i \partial_{i,x} + A_{i,x}, \quad P_{i,y} = -i \partial_{i,y} + A_{i,y}, \] (27)

here the potential term \( V = V_i + V_{ij} \). \( V_i = e^2/4d \) is the on-site repulsive interaction at each over crossing point with \( d \) the distance between the upper current and lower current. \( V_{ij} \) is column interaction between electrons. Since the on-site distance between electron is much smaller than the distance between electron at different lattice site, then \( V_i \gg V_{ij} \). The on-site repulsion dominates the dynamics. The gauge field vectors, \( A_{i,x} \) and \( A_{i,y} \), is induced by magnetic field and obeys symmetric gauge \( (A_{i,x} = eBy/2, A_{i,y} = eBx/2) \). Since electron are still moving in continuous channels, the continuous Hamiltonian theory for fractional quantum Hall effect also work here. Note here the on-site repulsion potential \( V_i \) has periodical distribution. \( a_{c,x} \) is the Chern-Simons gauge field. 2p flux quanta is attached to electron under Chern-Simons transformation. In the knot lattice model, 2p flux quanta is attached to Each ending point is attached by a flux quanta which is assigned by Chern-Simons field. The collective wave function of electron currents in knot lattice can be described by an extended Laughlin wave function,
\[ \Psi_{2n+1} = \prod_a \prod_{i < j} (z_{i,a} - z_{j,b})^{2n+1}(z_{i,a} - z_{i,b})^{2n+1}f(z), \]
\[ f(z) = \exp \left( -\sum_{i,a} |z_{i,a}|^2/4l^2 \right). \] (28)

\( z_{i,a} \) represent the coordinate of the four ending points at the middle of each unit square, which is the ending point of two crossing strings. Here \( a = |0\rangle \) represent the vacuum state. \( b = | - 1 \rangle \) or \( b = | + 1 \rangle \) represent the spin-up state or spin-down state. This Laughling wave function indicates the two fermionic strings in the same unit square cell obey fractional statistics. The composite fermions in different unit square cell also obey fractional statistics. For instance, suppose the ending point at the \( th \) unit cell (the blue square in Fig. \( a \) exchanges its position with one point in the \( th \) unit cell (the red triangle in Fig. \( b \). It takes three braidings to bring blue square to the position of red triangle, but takes only two inverse braidings to bring the red triangle to the home of blue square (Fig. \( c \). As a result, there is only one braidings survived at the \( th \) unit cell. Thus braidings two fermions in different unit cell obeys the same statistics as the that within one unit cell.

The collective wave function of other filling factors can be constructed by Jain’s composite fermions theory \[22\]. In fact, since these electrons are always running in closed loop which bears non-zero vorticity. Each loop of electric current also generate a magnetic field. The Abelian Chern-Simons action in fact counts the total helicity of these entangled knots. When the knot configuration fluctuates from one pattern to another, some knot inevitably will be cut and reunite, this induces some opposite magnetic fields against the external magnetic field due to the Lenz’s law in electromagnetism theory. Thus the fractional quantum Hall effect can still exist in this knot lattice model even if there is no external magnetic field.

D. Topological insulator model and quantum spin Hall model on square knot lattice

The presence of magnetic field in square knot lattice breaks time reversal symmetry. Without external magnetic field but introducing spin-orbital coupling into the Hamiltonian results in quantum anomalous Hall effect \[19\]. In fact, the knot configurations provide a geometric representation of the fermion filling states on square lattice. On the square knot lattice Fig. \( 1 \), we define the crossing state that blue wire is below the black wire as the zero filling state of fermionic operator, i.e., \( |0\rangle = |S = -1 \rangle \). The output of annihilation fermion operator \( c_i \) on \( |0\rangle \) is zero. The creation fermion operator, \( c_i^\dagger \),
generates one fermion out of zero filling state, \( c_i^\dagger|0\rangle = |1\rangle \). The one fermions state is defined as the opposite crossing state of \(|0\rangle\), i.e., \(|1\rangle = |S = +\rangle \) (Fig. 1 (e)). For spinless particle, Pauli principal forbids two fermions at the same site, thus \( c_i^\dagger c_i^\dagger|1\rangle = 0 \). The annihilation operator brings the one fermions state to vacuum, \( c_i|0\rangle = |0\rangle \).

If the fermions bear intrinsic spin state, each current at the crossing point has to be oriented as the four crossing states (Fig. 1 (e)) to match the coupling between different spin states. We first focus on spinless fermions in the following. For a given knot lattice configuration like Fig. 1 (a) (b), a natural operation Hamiltonian on this state is the topological insulator model [26],

\[
H_{ah} = \sum_i \frac{1}{2} \left[ (c_i^\dagger S_z c_i + c_i^\dagger S_z c_i + 2mc_i^\dagger S_z c_i) \right] - \sum_i \frac{\epsilon_i^2}{2} \left[ (c_i^\dagger S_z c_i + c_i^\dagger S_y c_i + e_{x_i} n_{e_x} c_i^\dagger c_j + e_{y_i} n_{e_y} c_i^\dagger c_j) \right] + h.c. \tag{29}
\]

The spin 1/2 operator \( S_x \) (or \( S_y \)) is global operator which only exist on the horizontal (or vertical) current. The \( S_z \) is the local magnetization at every lattice sites as well as the hopping current segment connecting two neighboring lattice sites. The fourier transformation of fermion operators on this knot lattice is naturally anisotropic,

\[
c_{k_x}^\dagger = \frac{1}{\sqrt{N}} \sum_n e^{-ik_x n_{e_x} c_{j+ne_x}}^\dagger, \\
c_{k_y} = \frac{1}{\sqrt{N}} \sum_n e^{ik_y n_{e_y} c_{j+ne_y}} \tag{30}
\]

This two-band topological insulator model reduces to diagonal Hamiltonian in momentum space reads,

\[
H_{ah} = \sum_k [\sin(k_x) c_{k_x}^\dagger c_{k_x} + \sin(k_y) c_{k_y}^\dagger c_{k_y} S_z + mc_{k_x}^\dagger c_{k_x} + \cos(k_x) c_{k_x}^\dagger c_{k_y} + \cos(k_y) c_{k_y}^\dagger c_{k_y} S_z]. \tag{31}
\]

Non-zero Chern numbers exist for the energy spectrum of this model with respect to different polarization degrees [26]. The Chern number in momentum space is not solely determined by the real space topology, since the same knot configuration acted by different Hamiltonian maps out different energy spectrum. Different Hamiltonian organizes the knot lattice layers in different way. However topological physics in real space area still encoded in momentum space. The linking number is defined as total number of positive crossing minus the total number of negative crossings. While here the negative crossing is represented by zero filling state, \( n_i = c_i^\dagger c_i = 0 \). The positive crossing is counted by the one fermions filling state, \( n_i = 1 \). Thus the Linking number in this fermion-spin coupling model is in fact the total number of fermions,

\[
L_{link} = N_c = \sum_i c_i^\dagger c_i. \tag{32}
\]

The total number of fermions is a topological number in this knot square lattice. The diagonal electronic conductance of this knot lattice model is naturally quantized by the number of unbroken channels in X-direction or Y-direction. The global spin component \( S_z \) is coupled to running fermions in X-loops \( \sum_k \sin(k_x) n_{k_x} S_z \). \( S_y \) is coupled to running fermions in Y-loops \( \sum_k \sin(k_y) n_{k_y} \). However the \( S_z \) component is not only coupled to on-site occupation, but also coupled to the fermion current in X- and Y-loops. As along as the fermion numbers in X- or Y-loops are not zero, they will contribute to the \( S_z \) component.

The real space Hamiltonian Eq. (29) assigned a spin \( S_z \) component on each crossing point. The loop currents along X-direction carry \( S_x \). The loop currents along Y-direction carry \( S_y \). The evolution of each spin component is governed by Heisenberg equation,

\[
\partial_t S_z = \frac{\hbar}{\gamma} (\sum_{k_x} \sin(k_x) n_{k_x} S_y - \sum_{k_y} \sin(k_y) n_{k_y} S_x) \tag{33}
\]

In the continuum limit, the right hand side of Eq. (33) is equivalent to Rashba spin-orbital coupling. For a constant polarization \( \partial_t S_z = 0 \), then \( \sum_{k_x} \sin(k_x) n_{k_x} \langle S_y \rangle = \sum_{k_y} \sin(k_y) n_{k_y} \langle S_z \rangle \). In the classical representation of spin, \( S_x \) can be expressed as the projection of a total spin,

\[
\langle S_x \rangle = S \cos(\theta), \quad \langle S_y \rangle = S \sin(\theta). \tag{34}
\]

Then the stable configuration of spin components obeys equation,

\[
\sum_{k_x} \sin(k_x) n_{k_x} \sin(\theta) = \sum_{k_y} \sin(k_y) n_{k_y} \cos(\theta). \tag{35}
\]

The orientation of spin in plane is labeled by the projection angle \( \theta \). Obviously \( \langle S_y \rangle \) increase when \( \langle S_x \rangle \) decreases. In order to fulfill the balance Eq. (35), the total number of fermions in X-loops has to be reduced, in the meantime, the total number of fermions in the Y-loop must increase. Since the total fermions number is a conserved number. Thus there must exist turning arcs in the knot square lattice to fuse the X-loop into Y-loop, driving the fermions from X-loop into Y-loop. In this sense, the output effect of spin-orbital coupling is equivalent to an external magnetic field. Since the global spin plays the same action at every lattice site, the turning arc shows up around every lattice site.

The Hall conductance of this two band model is quantized by the first Chern number in momentum space [26]. As all know, the energy function derived in real space model is exactly the same as its equivalent model in momentum space, even though sometimes it is quite difficult to get the formulation of energy spectrum in real space. Fourier transformation does not change the intrinsic topology of the energy manifold, except a coordinate transformation from space index into wave vector index. The momentum space is the reciprocal space of
real space. However the wave vector in thermal dynamics limit turns into a continuum variable. While the space index for the Hamiltonian in real space is still discrete. If we consider a finite system with finite particle number and lattice numbers, a knot in real space can still map into a knot in momentum space, maybe it is expressed into different geometry, but its topology should remains the same. As an explicit example to support above conjecture, we study an extreme simple case, that both the fermion occupation in X-loop and Y-loop are single occupied, $n_{k_x} = n_{k_y} = 1$. For a general consideration, the lattice constant $a_x$ in X-loop is controlled at a different value from that of Y-loops, i.e., $a_x \neq a_y$. A vector in real space, $R = n_xa_x + n_ya_y$, is dual vector of the reciprocal vector in momentum space, $K = k_xb_x + k_yb_y$, here ($b_x = 2\pi/a_x, b_y = 2\pi/a_y$). The two dual vectors obey the unit relation, $a_i \cdot b_j = \delta_{ij}$. The coupling current of the three spin components are listed as following:

$$
\langle S_x \rangle : I_x = \sum_{k_x} \sin(k_x),
$$
$$
\langle S_y \rangle : I_y = \sum_{k_y} \sin(k_y),
$$
$$
\langle S_z \rangle : I_z = \sum_k [m + \cos(k_x) + \cos(k_y)].
$$

(36)

In order to encode the lattice constants into the equations above, we reformulate the wave vector components as

$$
k_x = 2\pi \omega_x k, \quad k_y = 2\pi \omega_y k;
$$

where $\omega$ is the spatial oscillation frequency of the current defined by $(\omega_x = 1/a_x, \omega_y = 1/a_y)$, which counts how many unit lattice length is covered by one wavelength. For a general consideration, a phase factor $\phi_i$ is introduced into each current, then the three coupling fermion current reads,

$$
I_x = \sum_k \sin(2\pi \omega_x k + \phi_x),
$$
$$
I_y = \sum_k \sin(2\pi \omega_y k + \phi_y),
$$
$$
I_z = \sum_k [m + \cos(2\pi \omega_x k + \phi_x) + \cos(2\pi \omega_y k + \phi_y)].
$$

(38)

The three current functions above in fact define a Fourier knot in momentum space [34]. One familiar example of Fourier knot for physicist is Lissajous knots [34], which is usually visualized by oscilloscope. Inputting two sinusoidal electric currents into the vertical and horizontal channels at the same time, the oscilloscope displays different closed loops with respect different to frequency ratios.

The fourier knot is not always a closed knot. Closed knot only appears if the wavelength ratio $(\omega_x : \omega_y)$ is a rational number. For instance, if $(\omega_x : \omega_y = 1 : 1, \phi_x = \phi_y = 0)$, the Lissajous curve is a circle and project a straight line in $k_x - k_y$ plane. The knot for $(\omega_x : \omega_y = 1 : 2$ is wave circle in three dimensions but project a cross shape in $k_x - k_y$ plane (Fig. 8). For higher number of ratios, the Lissajous knot demonstrate more fluctuations (Fig. 8). The fourier knot configuration of the original fermion current $I_1$ is independent of the value of polarization $m$. Different $m$ simply shift the whole knot upward or downward. However this is not the case for a normalized fermion current.

The topological Chern number is defined by normalized fermion current in momentum space [26][14]. Here we choose the similar formulation of three fermion current as Ref. [26] for simplicity but with adjustable frequency, $(I_x = \sin(\omega_x k), I_y = \sin(\omega_y k), I_z = [m + \cos(\omega_x k) + \cos(\omega_y k)])$, then use the total energy spectrum, $E(k) = \sqrt{I_x^2 + I_y^2 + I_z^2}$, to normalize the fermion currents,

$$
n_x = \sin(\omega_x k)/E, \quad n_y = \sin(\omega_y k)/E,
$$
$$
n_z = [m + \cos(\omega_x k) + \cos(\omega_y k)]/E.
$$

(39)

This normal fermion current also demonstrate three dimensional knot in momentum space. We first study the special case, $\omega_x : \omega_y = 1 : 1$. The knot configuration show different geometry with respect different polarization degree $m$. For $m = 0$, it is two parallel lines instead of a closed loop. For $0 < m < 3.5$, the normal current is an upward parabola with double wells instead of a closed loop. For $3.5 < m$, the normal current is a closed loop, which approaches to a downward parabola shape for $m \to \infty$. For $-3.5 < m < 0$, the normal current is an downward parabola with double wells. For $m < -3.5$, the normal current is also a closed loop but approaches to an upward parabola.

The normal fermion current for other frequency ratios shows more fluctuating knots in momentum space. Different value of magnetization $m$ classified the knots or unknots into different zones. For $m = 0$, there only exist parabola curves instead of closed loops. For most cases, the total number of these parabola curves equals to the sum of the two frequency number. As showed in Fig. 9, there are two branches for $\omega_x : \omega_y = 1 : 1$, three branches for $1 : 2$ and five branches for $1 : 4$. However, this rule does not hold for all cases, there are only two branches for $1 : 3$, six branches for $3 : 5$, and ten branches for $5 : 7$, but only 12 branches for $7 : 9$. We are not fully aware of
the physics reason for this serial. However, the output of extremely high frequency ratios are the same, it all leads to two separated band with different edge branches (Fig. 9, 1:100, m=0).

The Lissajous curves for a magnetization $m < 3.5$ are always a collection of parabola curves for arbitrary frequency ratio $\omega_x : \omega_y$. For example, the output curve of $\omega_x : \omega_y = 1 : 2$ with $m = 2$ is two upward parabola curves with double wells, which turn into downward parabola curves for $m = -2$ (Fig. 9). While the ratio $1 : 3$ with $m = 2$ generates one parabola curve with three wraps (Fig. 9). $\omega_x : \omega_y = 1 : 6$ results in four deformed upward parabola curves (Fig. 10). The high frequency ratio with $m = 2$ finally converges to a cup-like network with four touching point on the bottom (Fig. 9, (1:100, m=2)).

Closed Lissajous knot only exist for $m > 3.5$. The $1 : 1$ case is a loop without self-writhing loops. While $1 : 2$ and $1 : 3$ generate a loops with two writhing loops and three writhing loops correspondingly (Fig. 9). For high magnetization $m = 40$, the Lissajous knot form a ten-like knot with many writing loops (Fig. 10, 3:7, m=40), which turns into a tent-like network cage, as showed by the case of $3 : 100, m = 4$ (Fig. 10). These Lissajous knots approaches to tent-like parabola network for high magnetization $m \rightarrow \infty$, one example is showed in (9, 1:1, m=100).

The topological quantum field theory of non-Abelian Chern-Simons action provides a topological invariant for many entangled knots. While Fourier knot is only a special case of general link with many entangle knot. Thus we could introduce the Chern-Simons action [13] [14] to quantify the Fourier knots here. Another topological invariant for the collection of all of these knots is partition function. The partition function for knots in real space share the same formulation as that for the knot in momentum space. These topological invariants are global topological invariant, sometimes the local topological invariant for a special eigenstate is really relevant to experiment measurement. For instance, the ground state and the first excited state is the most concern for physicists. For any knot configuration of ground state, Euler number is always a topological invariant. The Euler number of a closed curve which is homotopic to circle is always zero. The Euler characteristic number for a continuous closed curve can be computed by Morse theorem For a given knot in momentum space, there always exist some critical points at which fermion current satisfies $\partial n_x / \partial k_x = 0$. For instance, the $\infty$ shaped knot in $k_x - k_y$ plane (Fig. 8) has 6 critical points. The local curve of two critical points on the upper boundary is approximated by $n(k_x) = -ak_x^2$, and $n(k_x) = ak_x^2$ is for two points on the bottom boundary. The critical point on the left boundary is $n(k_y) = bk_y^2$, and $n(k_y) = -b^2$ for the right one. Then the Euler characteristic number is computed by the morse theorem [12],

$$\chi = \sum_{q=0}^{M} (-1)^q c_q,$$

(40)

$q$ is an index counting the independent directions in which the current decreases. For the $\infty$ shaped knot, there are two points with $q = 1$ above, two points with $q = 0$ on the bottom, one $q = 0$ for the left boundary and one $q = 1$ for the right boundary. $c_q$
counts the total number of points with index q. The Euler characteristic number of this ∞ shaped knot is $\chi = (-1)^03 + (-1)^13 = 0$. For the Lissajous knot, the number of critical points of the horizontal critical point to that of the vertical point is directly readable by the frequency ration on oscilloscope, $c'_q : c''_q = \omega_x : \omega_y$. For the magnetization $m > 3.5$ or $m < -3.5$, the fermion currents are closed curves, thus the Euler number is zero.

The fermions current for the other two cases is not homotopic to knot anymore. In that case, Euler-Poincare equation is more effective for computing the topological numbers \( \chi \),

$$\chi = \sum_{q=0}^{M} (-1)^q b_q, \quad (41)$$

Here \( b_q \) is the Betti number, which counts the number of the \( q \) dimensional simplex. For instance, the 0 dimensional simplex is a point. 1 dimensional simplex is a line segment. 2 dimensional simplex is a 2D surface. For the zero magnetization case, \( m = 0 \), the fermion current in different frequency ration are composed of curves. In that case, these curve approaches to parabola curve for an infinite wave vector. For this knot lattice system, the momentum wave vectors has a cut-off at the unit lattice space, \( k_x = 1/a_x \) and \( k_y = 1/a_y \). In that case, there always exist two ending points (\( b_0 = 2 \)) and one line (\( b_1 = 1 \)) for each branch, thus the Euler number is $\chi = (-1)^02 + (-1)^11 = 1$. This Euler number has the same value for the parameter range, $0 < m < 3.5$ or $-3.5 < m < 0$, but the fermion current for this case has only one band. This is because a finite magnetization \( m \) breaks time reversal symmetry. Similarly, the fermion current for \( m > 3.5 \) or \( m < -3.5 \) also has only one band, as shown by the tent-like cage (\( 3 : 100, m = 4 \) in (Fig. 10)). In fact, the Euler number of a two dimensional simplex is equivalent to the first Chern number on manifold in its continuum limit. However the Euler number above are computed on one dimensional curves instead of two dimensional surface.

The energy spectrum (Fig. 10) shows closed knots in momentum space only exist for a gapped two band model. If the two bands form a periodic closed spectrum loops, its corresponding current knot in momentum space are collections of parabola with double wells. The two bands intersecting each other for zero magnetization \( m = 0 \). Its corresponding current knot is a pair of separated flat branches. For a larger magnetization \( m = 40 \), the two band in spectrum becomes almost flat band with fine wavy structure which is induced by the interference between the waves with two frequencies, 3 and 7. A reversed magnetization induced a phase flip of the spectrum wave. In the mean time, the corresponding knot in momentum space switched to opposite direction. For a larger frequency ratio, \( 3 : 100 \), the spectrum wave becomes a modulated composite wave with many fine wave in each macroscopic wave section (Fig. 10), the corresponding knot in momentum space is a tent-like network (Fig. 11 (a)). For a finite lattice, these exist a gapless edge state on the boundary.

Topological insulator model only consider the coupling between the orbital of spinless fermion and a global spin operator \( \hat{\sigma}_x \). Thus the knot square lattice implementation of topological insulator model only incorporate undirected fermion current in X- and Y-loop. \( |0 \rangle \) state was defined as vertical current above horizontal current, while the opposite setup defines \( |1 \rangle \). The spin-up and spin-down fermion have a natural implementation by directed fermion currents in the knot lattice. For charged spin fermions in the loop channel, an electric field confined in the x-y plane could induce quantum spin Hall effect \([31]\). Here electric field is oriented along the y-axis to drive electrons running in the Y-loops. Then the spin up state \( |↑0 \rangle \) is defined as a state that the positive Y-current is above the X-current, \( |↑0 \rangle \) corresponds to that the Y-current is below the X-current (Fig. 11 (a)). The negative Y-current defines the spin-down states correspondingly (Fig. 11). The direction of spin is perpendicular to the electric current and electric field, following the equation \( J_j = \sigma_s \epsilon_{ijk} E_k \). The action of spin fermion operators obey the following rules, \( c_1 |↑0 \rangle = |↑1 \rangle \), \( c_1 |↓1 \rangle = 0 \), \( c_1 |↑0 \rangle = |↑1 \rangle \). The operation of spin-down fermion operator follows similar rules. The effective Hamiltonian for this spin Hall system is composed of two topological insulator model Eq. \( [29] \), but incorporate an opposite Y-current,

$$H_s(k) = \begin{pmatrix} H_T(k) & 0 \\ 0 & H^*_T(-k) \end{pmatrix} \quad (42)$$

Here the Hamiltonian \( H_s(k) \) share the same formulation as topological insulator model Hamiltonian Eq. \( [31] \) in momentum space. This Hamiltonian reduces to the effective Hamiltonian of quantum spin Hall insulator near the \( \Gamma \) point \([32]\). The spin Hall current also turns from Y-loop into X-loop due to spin-orbital coupling interaction (Fig. 11 (a)). The gapless edge current runs along the
The Bardeen-Cooper-Schrieffer (BCS) fermion pairing model is a successful model for conventional superconductor 27. Electrons with opposite spin and momentum are coupled into pairs as the main carrier of superconductor current. A self-consistent construction of fermions pairing in knot square lattice has only two possible configurations (Fig. 12 (a)). An alternative distribution of the |+1⟩ and |−1⟩ crossing states construct one stable fermions pairing state. Flipping |+1⟩ to |−1⟩ (or vice versa) on the whole lattice is another equivalent pairing state (Fig. 12 (a)). Thus the fermion pairing on knot square lattice has two fold degeneracy. The fermion pairing Hamiltonian in the two dimensional bulk area reads,

$$H_{\text{bulk}}^{xy} = -V \sum_{ij} [c_{i,j}^\dagger c_{i-1,j}^\dagger c_{i,j+1}^\dagger c_{i-1,j+1} - c_{i,j+1}^\dagger c_{i,j}^\dagger c_{i+1,j}^\dagger c_{i-1,j}] + \epsilon_{ij} c_{i,j}^\dagger c_{i,j}. \quad (43)$$

The Fourier transformation of fermions have separate wave vectors in X-loops and Y-loops correspondingly, $c_{i,s} = \frac{1}{\sqrt{N}} \sum_k e^{i k x_i} c_{k,s}$, and $c_{i,s} = \frac{1}{\sqrt{N}} \sum_k e^{i k y_i} c_{k,s}$. Substituting this Fourier transformation into the Hamiltonian Eq. (43) in real space leads to

$$H_{\text{bulk}}^{xy}(k) = \sum_k [\epsilon_k c_k^\dagger c_k - \sum_{k,k'} V c_k^\dagger c_{-k}^\dagger c_k c_{-k'}^\dagger]. \quad (44)$$

This Hamiltonian bears the same structure as the BCS Hamiltonian in momentum space. Following the usual mean field approach, we define the same energy gap function for exciting a Cooper pairing, $\Delta = \sum_k V c_k c_{-k}$. Usually this energy gap is a complex function, $\Delta = \Delta_1 + i \Delta_2$, $\Delta_{ij} = \Delta_1 - i \Delta_2$. The bulk pairing Hamiltonian can be formulated as a fermion spinor $\psi_k^T = [c_k^\dagger, c_{-k}^\dagger]^T$ coupled to a pseudo-spin vector, $\vec{\sigma}$,

$$H_{\text{bulk}}^{xy}(k) = \psi_k^\dagger [\epsilon_k \mathbf{I} - \Delta_1 S_x + \Delta_2 S_y] \psi_k. \quad (45)$$

Here $\mathbf{I}$ is a 2 by 2 unit matrix. $S_x$ and $S_y$ are conventional Pauli matrices. For different pairing states 27, this pairing Hamiltonian defines different Fourier knot in momentum. For instance, the p-wave pairing gap function defines a typical Fourier knot 28,

$$I_x = \Delta_1 = \sum_k \sin(2\pi \omega_x k + \phi_x),$$

$$I_y = \Delta_2 = \sum_k \sin(2\pi \omega_y k + \phi_y). \quad (46)$$

However here there is no $\sigma_z$ component. Thus the pairing gap function defines two dimensional Lissajous curves in momentum space. The fermion pairing state here is in fact single plaquette state in each unit square. It is an antiferromagnetic order state for the block Ising spin $|+1⟩$ and $|−1⟩$ (Fig. 12 (a)) in real space. The self-consistent pairing state is the two fold degenerated ground state of the Ising Hamiltonian for coupling block spin, $H_{\text{pair}} = J S_i^x S_i^y$, with $J > 0$. The eigenstate of $S_i^z$ is the block spin states showed in Fig. 12(a). Two neighboring unit squares with same block spin can not match each other self-consistently. Two fermions with opposite spins would collide each other on the square boundary where no channel exist for them to continue the current without turning back. While inside each unit square, the two incoming fermions of X-loop could tunnel up into the Y-channel and split up to fit into the continuous current loops without frustration. This convective fermion...
The energy spectrum of this mean-field Hamiltonian is
e defined a Fourier knot in three dimensional momentum space. For a periodically located multi-layer knot lattice, the paring Hamiltonian $H_{\text{bulk}}$ maps into four fermion knot interaction in momentum space by Fourier transformation. Because the fermion pair only moves upward along Z-axis, the time reversal symmetry is broken. The tunneling Hamiltonian under mean field approximations reads,

$$H_{\text{bulk}} = -V \sum_{i,j} [\delta_{i,j} (\varepsilon_{i,j} c_{i,j}^\dagger c_{i,j} + h.c.)]$$

(47)

Here the bottom line (upper line) is denoted by $b$ ($u$). A normal state is not fermion pairing state, the local crossing state maybe randomly distributed over the whole lattice. Then we have to use a multi-layer of knot lattice to represent the superposition of quantum states. For an odd number of Y-loops or X-loops on the boundary moves to the opposite direction of the bottom end, the spin results in contradiction. However, if one block spin orientation of electric field, $\vec{E}/|\vec{E}| = (\vec{J} \times \vec{s})/|\vec{J}| |\vec{s}|$, here $\vec{J}$ is the fermion current vector, $\vec{s}$ is the spin vector. While for an even number of Y-loops, the Y-current fuses into X-current by flipping the orientation of spins in a certain way so that the electric orientation also flips (Fig. 12 (c)), i.e., $\vec{E} \rightarrow -\vec{E}$. The effective Hamiltonian of the edge current for both the two cases reads,

$$H_{\text{edge}} = t_{ij} \sum_{\langle ij \rangle} (c_{i}^\dagger - e_{x} c_{i}^\dagger I + e_{x} c_{i}^\dagger c_{i}^\dagger).$$

(52)

The equivalent Hamiltonian in momentum space is $H_{\text{edge}} = 2t_{ij} \sin(k_{x})c_{i}^\dagger c_{i}^\dagger$, which reduces to a gapless dispersion near $k_{x} \rightarrow 0$. The edge current on the upper boundary moves to the opposite direction of the bottom current, so does the left edge current and right edge current. This phenomena holds both for the even number and odd number of Y-loops (Fig. 12 (b) (c)).

FIG. 13: (a) The vortex with opposite chirality locate on different square sublattice. The vacuum vortex lines entangle with other vacuum arcs. (b) The unoriented Seifert surface for fermions pairing state.

The global pairing state has only two consistent patterns which place the two local pairing patterns alternatively on the whole lattice, any local flipping of one block spin results in contradiction. However, if one block spin of current crossing is braided into a vacuum state, the rest pairing states can still exist consistently. In other words, breaking a local Cooper pair on one lattice does not destroy the super-conducting states on other lattice sites. For the block spin state $|+1\rangle$, there are two braiding...
operators to break the Cooper pair, one is counterclockwise braiding turning an angle of $\pi/4$ away from X-loop, which is denoted as $B_{\n\n}$. It results in a vacuum state with positive chirality following the right hand rule, | + 0). The other braiding operators turn an angle of $3\pi/4$ away from X-loop, that we denote as $B_{\n\n}$. This clockwise braiding disentangles the crossing of pairing current into a vacuum state with negative chirality, | − 0). The clockwise braiding $B_{\n\n}$ turns | − 1) into negative vacuum state | − 0). While counterclockwise braiding $B_{\n\n}$ braids | − 1) into positive vacuum state | + 0). Four neighboring vacuum states with the same chirality can form a minimal loop. Electrons run around a magnetic flux through this minimal loop (Fig. 13 (a)). Each minimal loop represents a vortex in superconductor. This knot square lattice for fermion pairing pattern is equivalent to the overlap of two square sub-lattices, which is denoted by the black discs and white circles correspondingly in Fig. 13 (a). These two sublattices are the dual lattice of the unit cell, on which the magnetic flux with opposite chirality are distributed. Negative chiral vortex around a negative flux only exist at the black sites, while positive vortex only sit on white sites. Each flux site is surrounded by four unit cells. Since each braiding over an unit cell only generates two arcs with the same chirality, opposite vortex loops cannot coexist as the nearest neighbors. However, a vacuum current can separate two vortex loops to prevent them from annihilation (Fig. 13 (a)).

Each vortex loop confines two electrons. One electron vanishes from unit cell at site (i, j) and generated at site (i − 1, j − 1), this process only draws a half circle. In the mean time, another electron must annihilate at (i − 1, j − 1) and generates at (i, j). The fermion pair current is topologically quantized. The complete vortex loop carries an integral winding number $W = \pm 1$. While the half vortex loop carries a half winding number $W = \pm 1/2$. A quarter arc carries a fractional winding number, $W = \pm 1/4$. The complete vortex loop carries two electric charges. A quarter arc has only half charge, $e/2$. If more braiding operations are performed over the two vacuum arc within one unit cell, the vortex would entangle with supercurrent of fermion pair, or two vortex lines may also entangle each other (Fig. 13 (a)). This nontrivial entanglement is characterized by linking number. Helicity is an effective topological quantity of entangled vortex lines,

$$H_{helicity} = \int(dx^3) \cdot (\nabla \times J), \quad (53)$$

where $J$ is the familiar supercurrent of fermion pairs,

$$J = \frac{ie\hbar}{2m} \left( \Delta^* \hat{D} \Delta - \Delta \hat{D} \Delta^* \right), \quad (54)$$

here $\Delta$ is the gap function for fermion pairs. Since Cooper pair is composite boson of electron pair with opposite spins, $\Delta$ is also a bosonic operator after the second quantization. $\hat{D} = \nabla + i2\pi\phi_0$ is the covariant derivative. The spin of electron flips by the accumulated phase factor of magnetic flux (Fig. 12 (a)), $\phi_0 = \oint A dx = \oint \nabla \times \vec{J} dx^2 = \frac{1}{2}\oint \nabla \times \vec{J} dx^2$. The helicity Eq. (53) is actually equivalent to abelian Chern-simons action, which is topological number of many knots [14]. If there are 2N crossing points between vortex lines over the whole vortex lattice, it takes $N$ Majorana fermions to bring entangled vacuum state to free vortex state. Similar to fractional quantum Hall state, fractional filling state also exist for two entangled vortex in superconductor.

Beside the linking number of entangled knots, Euler characteristic is also an effective topological characterization of the knot square lattice of fermion pairing model. The Euler Characteristic for an oriented, connected compact surface is $\chi(S) = 2(1 - g)$, where $g$ is the number of genus (holes). It is in fact the first Chern number. Seifert surface has to be constructed in order to compute the Chern number of a knot. For an arbitrary knot, Seifert’s algorithm first color the projected plaquettes alternatively into a check board state, then lift up the plaquette with same color and view the plaquette with opposite color as hole. Then at each crossing point, each in-arrow must connect to its nearest neighboring out-arrow. Then the knot is decomposed into oriented loops. Filling these loops with color to generate a disk and then connect them by twisted mobius strip whose boundary projects the original crossing states in two dimensional plane, this surface is so called Seifert surface [30]. To apply the Seifert algorithm on this knot square lattice, we first choose the plaquette on the white sublattice as filled surface (represented by the green spherical surface), while those on the black sublattice are holes (represented by the white blank zone in Fig. 13 (b)). Then connect the in-arrow to out-arrow at every crossing site. It finally reduced to periodically distributed vortex loops (The green spheres in Fig. 12 (b)). Then we fill these loops and connect them by Mobius strip in such a way that the original current crossing are the projection of the two edges of Mobius strip (Fig. 12 (b)), which are represented by the green belt that connects two green spheres in Fig. 13 (b). Then the knot square lattice of pairing states map into a lattice of periodically distributed Mobius strips (Fig. 13 (b)). However this Seifert surface is still not an oriented manifold due to the opposite arrows of fermion pairing in this special model. In order to construct a closed current loop, we introduce two more currents that connect the two in-arrow to the two out-arrow in the middle waist of the Mobius strip (Fig. 12 (b)). Then it leads to an oriented current network on a unoriented Seifert surface. In fact, the Euler characteristic for this case can not distinguish an oriented surface from an unoriented one. The Seifert surface in Fig. 12 (b) has an Euler characteristic of $\chi = 2(1 - 5) = -8$, since it has five genus. For a square lattice with $(2N + 1) \times (2N + 1)$ unit squares, the total number of genus is $g = \frac{(2N + 1)^2 + 1}{2}$. The corresponding Euler Characteristic is $\chi = 1 - (2N + 1)^2$. 

$$\chi = 1 - (2N + 1)^2.$$
The Euler characteristic is an effective topological number for classifying different fermion paring patterns on square lattice. The ferromagnetic ordering of current crossing is the minimal self-consistent lattice model for fermions pairings. Strip ordering and ferromagnetic ordering could show up if a local crossing flips under thermal fluctuation. In that case, two out-arrows (or in-arrows) may collide on the boundary of unit cell, on which there is no outgoing current exist. In order to construct self-consistent continuous current loops, extra current perpendicular to the two out-arrows on the boundary have to be introduced (the red arrows in Fig. 14 (b) (c) (d)). For the ferromagnetic order state, each one of the original four unit cells is divided into four small unit cells with half lattice constant (Fig. 14(c) (d)). Thus the total number of unit cells increased to 16. While the current of strip ordering on the boundary between $|+1\rangle$ and $|-1\rangle$ are consistent, any extra current would results in contradiction (Fig. 14(b)). Thus extra current are only introduced on the boundary between $|+1\rangle$ and $|-1\rangle$ or $|+1\rangle$ (or $|-1\rangle$) (Fig. 14(b)). The corresponding two dimensional surface with respect to different ordering state can also be constructed following Seifert algorithm. The Seifert surface of $(2N+1)\times(2N+1)$ knot square lattice has a Euler characteristic $\chi=1-(2N+1)^2$ genus, while this topological number increases to $\chi=1-4(2N+1)^2$ for a homogeneous ferromagnetic ordering state. The strip ordering has an intermediate Euler characteristic number. The ferromagnetic ordering of current crossing in real space actually can be mapped into separated current loops by Reidemeister moves. While the antiferromagnetic ordering of current crossing requires the maximal number of flipping operations on certain crossing points to map it into separated loops. The total number of flipping operations to map many entangled knots into free loops can be used to quantify the topological entanglement of a link. In this sense, the antiferromagnetic ordering of block spin for fermions pairing has the maximal topological entanglement. Thus superconductor should has maximal topological entanglement.

The Seifert surface provides mathematical mapping of knot square lattice into two dimensional surface with genus. If we consider the physical implementation of Seifert algorithm, the filled plaquette is physically implemented by magnetic flux in vortex loops. The connection of an in-arrow to an out-arrow is physically implementable by local braidings operation on each crossing point, which can be realized by physical magnetic field in plane. However in this knot square lattice, the filled vortex loop can not connect to its four neighbors at the same time to construct a two dimensional surface with many holes. A vortex either exist as an isolated loop or only connect to one of its four nearest neighboring to construct a dimerized lattice of vortex pairs (Fig. 15). Only vortex with the same chirality can form a dimer by coupling to the crossing supercurrents (Fig. 15 (a) (c)) or vacuum state (Fig. 15 (b)). Different dimer lattice have different Euler characteristic number. We make an inverse filling of the Seifert surface of dimer lattice. Then each filled vortex loop becomes a hole of continuous surface. For a lattice with $2N\times 2N$ unit cells, there exist $N^2$ isolated vortex loops, and $N^2/2$ possible vortex dimers. The Euler characteristic of isolated vortex lattice is $\chi=2(1-N^2)$. The vacuum vortex dimer phase has an Euler characteristic $\chi=2(1-N^2/2)$, so does the positive and negative vortex dimer phase. Euler characteristic can not distinguish a Mobius hole from a trivial hole.

These vortex loops behave as fermion in the vortex dimer lattice, which is originated from the fermionic string arc in this knot lattice model. Each fermionic string arc can be represented by a Grassmann number. The Grassmann number obeys the following algebra, $\eta_i\eta_j=-\eta_j\eta_i$, $\eta_i\eta_i=0$. We place a Grassmann number $\eta_i$ at the center of each vortex loop. These vortex loops are regularly distributed on square lattice. For a dimer lattice of the same chiral vortex loops, the total number of all different covering patterns can be computed by Kasteleyn matrix [29], which is equivalent to the square root of the determinant of quadratic fermion action, $S=\sum_{i<j}M_{ij}\eta_i\eta_j$. The total number of all different covering patterns equals to the partition function.

**FIG. 14:** (a) The current crossing states for anti-ferromagnetic ordering, $|\psi\rangle=|\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\rangle$. (b) The current crossing states for strip ordering, $|\psi\rangle=|\uparrow\uparrow\uparrow\downarrow\downarrow\rangle$. (c) (d) The current crossing states for ferromagnetic ordering, $|\psi\rangle=|\downarrow\downarrow\downarrow\rangle$ and $|\psi\rangle=|\uparrow\uparrow\uparrow\uparrow\uparrow\rangle$.

**FIG. 15:** (a)The dimer lattice of negative vortex pairs. (b) The dimer lattice of vortex pair coupled to vacuum. (c) The dimer lattice of positive vortex pairs.
of this fermion action,

\[ Z = \int d\eta \exp \sum_{i<j} M_{ij} \eta_i \eta_j = \pm \sqrt{\det[M]} \]  \hspace{1cm} (55)

Note here the vacuum vortex dimer admit a self-consistent coexistence with the positive vortex dimer or negative vortex dimer. But the positive vortex dimer and negative vortex dimer can not perfectly coexist without introducing geometric frustrations (Fig. 18 (a)). The effective action for vortex dimers on square lattice bears the same formulation as classical dimers \[30\] but with rotated wave vectors,

\[ S_0 = \sum_k [i \sin(k_x + k_y) - \sin(k_x - k_y)] \eta_k \eta_{-k}. \]  \hspace{1cm} (56)

This effective action holds for a lattice covered by pure dimers. If we consider the hybrid dimer lattice covered by positive vortex dimer and vacuum vortex dimer (or negative vortex dimer and vacuum dimer) together, then the total number of all possible covering patterns is the product of two equal partition function for pure dimers, \( Z^2 \). In that case, the effective action for vortex dimer lattice is

\[ S_h = \sum_k [i \sin(k_x + k_y) - \sin(k_x - k_y)] \eta_k \eta_{-k}^{\dagger}. \]  \hspace{1cm} (57)

The gap closing points are periodically distributed in momentum space. This dimer counting does not take into account of internal state of dimer. If the vortex lines between two vortices were braided for for many times (Fig. 18 (a)), then the vortex dimer couples to anyons with the same formulation as classical dimers \[30\] but with rotated wave vectors, 

\[ \hat{\psi}_i = e^{i/4} \sum_{x,y} \hat{A}_{ij} \hat{d}_{ij} \hat{\eta}_{ij}. \]  \hspace{1cm} (60)

The path integral of gauge potential \( \exp[\int A_{\mu} d\xi^\mu] = \exp[i\phi] \) is only carried out along the fermionic string arcs. Then the string operator is the product of six Grassmann operators,

\[ \hat{\eta}_a = \eta_{i,j+1} \eta_{i-1,j+1} \eta_{i,j-1}, \hspace{0.5cm} \hat{\eta}_b = \eta_{i,j-1} \eta_{i+1,j-1} \eta_{i+1,j+1}, \]  \hspace{1cm} (61)

where \( \hat{\eta}_a \) is the product of three fermion arc around the first vortex of dimer, and \( \hat{\eta}_b \) represent the second vortex inside the vortex dimer. The ordered Grassmann string operator switches to negative if the order of six operators is reversed. The string operator of an isolated vortex is equivalent to a Wilson loop operator, it is the product of four Grassmann operators, which is equivalent to boson. The effective Hamiltonian for a vortex dimer coupled to fermion pairing state is

\[ H_{\text{vor}} = -\sum_{i,j} U \hat{L}_{ij} c_{i,j}^\dagger c_{i,j+1}^\dagger c_{i,j+1}^\dagger c_{i,j+1}^\dagger e^{i/4} f A_{\mu} d\xi^\mu. \]  \hspace{1cm} (62)

In the mean field approximation, the corresponding Hamiltonian in momentum space becomes a dressed fermion pairing Hamiltonian after integration of Grassmann variable

\[ H_{\text{vor}}(k) = L(k) \psi^\dagger_k [\Delta_1 S_x + \Delta_2 S_y + \Delta N(k) S_z] \psi_k. \]  \hspace{1cm} (63)

This original fermion pairing gap could close at the gapless point of vortex dimer spectrum. This induces the Fermi arc in momentum space. The energy current of each spin component defines the location of a point of knot in momentum space. The current knot square lattice is still in superconducting state in the presence of vortex. If the two crossing super-current between two vortex are braided more than three times in the same direction, a Majorana fermion will be generated to raise the local energy, but does not break the pairing states. The fractional statistics of this Majorana can be described by Laughlin wave function. This superconducting resonance vortex dimer (SRVD) state may shed light on pseudogap state of unconventional superconductor \[37\].

III. ANYONS OF QUANTUM KNOT LATTICE MODEL ON HONEYCOMB LATTICE

A. The knot lattice model of transverse field Ising chian model

Kitaev honeycomb lattice model includes the coupling terms of three spin components, it is a typical quantum
FIG. 16: One layer of the multilayer knot patterns for effective mapping of Ising model and Kitaev model on honeycomb lattice.

FIG. 17: (a) Three color anyon currents entangled each other to implement the local tribein for Ising model on honeycomb lattice. The three braidings satisfy Yang-Baxter equation. (b) Three color anyon currents entangled each other to implement coupling style in Kitaev honeycomb model. (c) The fusion rule of the three color anyons into two color Majorana fermions in the Hilbert section of $S^z$, and the fusion of two Majorana fermions. (d) The fusion rule of the three color anyons into three color Majorana fermions in the Hilbert section of $S^z$.

FIG. 18: The two entangled loops as eigenstate representation of one dimensional transverse Ising model.

spin model that bears non-Abelian anyons. If other spin components are introduced into the two-state Ising model or block spin-1 Ising model, then the spin $S^z$ component would be non-commutative with the other spin components. Quantum dynamics make the implementation of three spin components by knot lattice more complicate. We first take one dimension Ising spin chain model to explore the knot lattice configuration for entangled quantum states. The transverse field Ising chain model includes both $S^z$ and $S^y$,

$$H_t = - \sum_{(ij)} (J_z S_i^z S_j^z + h_i S_i^z). \quad (64)$$

Replacing $S^y_i$ with $S^y_i$ leads to an equivalent quantum model under duality transformation. The knot representation of this model is two loops periodically entangled with each other (Fig. 16). For a vanished transverse field $h_i = 0$, the spin chain reduced to classical Ising model. For the antiferromagnetic coupling $J_z > 0$, the ground state is two loops with the maximal winding numbers, one loop wraps around the other at every lattice site (Fig. 18 (a)). For ferromagnetic coupling $J_z < 0$, its ground state is two separated loops, i.e., One loop is above the other everywhere on the whole chain of lattice (Fig. 18 (d)). A weak transverse field acts as a perturbation on the ground state by flipping the crossing states from over-crossing to under crossing (or vice versa) upon certain number of lattice sites instead of the whole lattice. While if coupling strength between the nearest neighboring spin drops to zero, $J_z = 0$, then the eigenstate of the spin chain is determined by the eigenstate of spin operator $S^z$ alone, $(| \uparrow \rangle \pm | \downarrow \rangle)$, i.e., $S^z(| \uparrow \rangle \pm | \downarrow \rangle) = \pm 1(| \downarrow \rangle \pm | \uparrow \rangle)$. 

\[ \]
The eigenstate knot lattice of transverse Ising model is a bilayer knot lattice, the bottom layer carrying $| \uparrow \rangle$ plus (or minus for a positive $h$) the upper layer carrying $| \downarrow \rangle$. If $S^i_\alpha$ is replaced by $S^i_\beta$ in Hamiltonian Eq. (41), a phase factor $e^{i\pi/2}$ must be added to the upper knot lattice layer with $| \downarrow \rangle$ (Fig. 23(d)) in order to fulfill the eigen-equation of $S^i_\beta$, i.e., $S^i_\beta(| \uparrow \rangle \pm i| \downarrow \rangle) = \pm (| \uparrow \rangle \pm i| \downarrow \rangle)$. The transverse Ising model admits a quantum phase transition when the neighboring coupling strength equals to the transverse field. While adding $e^{i\pi/2}$ to the transverse Ising model in two dimensional lattice (Fig. 3 (a)).

One spin flipping is realized by at least two braidings in the same direction. However single braidings could result in vacuum states, that is not the eigenstate of two-states Ising spin. Thus we extend the two-states Ising spin to block spin-1 operator, $S_\alpha^i$ has three eigenstates $| \pm 1 \rangle$ and $| 0 \rangle$. For a spin chain oriented along X-direction, there are two independent braidingss, one is $B_x$, which generates one up and one down Majorana fermion arc (the green arcs in Fig. 18). The other is $B_y$, generating a left and a right Majorana fermion arc. These braidingss are implementable by Magnetic field. If the spin chain is acted only by $B_y$, the resulting vacuum state is composed of $N$ unit circles for $N$ lattice sites. This is the dimerized insulator states (Fig. 18 (b)). For the other global braidings operation of $B_x$, the generated vacuum state is two separated loops (Fig. 18 (c)). Particles can run through the whole lattice but split into two bands. If there are only two $B_y$ braidingss on the left and the right edge correspondingly, and $B_x$ braidingss over the whole inner section, the two separated loops can fuse into one. This leads to the gapless vacuum state. Different vacuum states can be distinguished by topological numbers.

Similar to the two dimensional block spin-1 Ising model, this block spin-1 Ising chain also have non-Abelian anyons in the eigenenergy level. The superposition of vacuum state and Majorana fermion state is naturally embedded in the Hilbert space. The eigenstate in vacuum state is a natural quantum qubit for topological quantum computation [1]. Repeating the braidingss operations drives the Majorana fermion to jump from bottom layer to upper layer (or vice versa). The topological correlation for this block spin-1 chain is easier to calculate than the two-dimensional lattice model. The Jones polynomial for a given quantum state here is exactly computable by Skein relation, which view the vacuum state as the superposition of $| \uparrow \rangle$ and $| \downarrow \rangle$. Thus every quantum state is associated with a Jones polynomial. However different state may share the same Jones Polynomial, we may call this topological degeneracy, which fulfill the requirement for topological quantum computation.

B. The knot lattice of spin 1/2 Ising model on Honeycomb lattice

The multi-knot lattice model can also constructed honeycomb lattice to implement Ising model, Kitaev model
as well as other spin-spin coupled quantum models. Since there are three currents intersect at one lattice site, we abandon the magnetic monopoles and positrons to avoid local frustrations. We introduce three color anyons, the red ($\sigma_1$), yellow ($\sigma_2$) and blue anyons ($\sigma_3$), a pair of anyons is running in two currents that is oriented by one arm of the local tribein which has three arms separated by $2\pi/3$. One possible implementation of $\sigma_1$ is an electron attached by a flux phase factor $e^{i\phi_1}$, so does $\sigma_2$ and $\sigma_3$. An over-crossing occurs on each arm of the tribein which is enclosed by an hexagonal unit cell (Fig. 14). The sequenced three braidings at the three intersecting squares obey Yang-Baxter equation (Fig. 17 (a) (b)),

$$R^v\sigma R^w\sigma R^z\sigma = R^z\sigma R^w\sigma R^v\sigma R^w\sigma.$$  

(67)

Here $R^v\sigma$ are the braiding operators on the over crossing points. Thus a rotated tribein or its mirror configuration is equivalent to the original coupling tribein. We first input Ising coupling between neighboring unit cells,

$$H_h = \sum_i J_z(S_{i,e_x}^z S_{i+e_x}^z + S_{i, e_y}^z S_{i+e_y}^z + S_{i, e_z}^z S_{i+e_z}^z).$$  

(68)

The local crossing state of each arm in $i$th unit cell is only acted by $S_i^z$. The conventional Ising spin on lattice is represented by single $\uparrow$ or $\downarrow$. While here every unit cell has three internal spin components that has the freedom to point up or down. In order to implement classical Ising model, we only choose two block spin configurations, $|\uparrow\uparrow\rangle_i$ and $|\downarrow\downarrow\rangle_i$.

$$\text{The knot configuration in Fig. 17 (a) corresponds to |\uparrow\uparrow\uparrow\rangle_i.}$

Flipping all the three crossing states of Fig. 17 (a) results in $|\downarrow\downarrow\downarrow\rangle_i$, which indicates the generation of statistical phase $\exp[i\pi]$. Each spin flip is performed by braiding one pair of anyons twice. Three spin flips requires 6 times of braiding anyons on the three arms. Thus each anyon carries a statistical phase factor, $\exp[i\pi/6]$.

The coupling interaction between crossing points within the hexagon unit cell can also be introduced to construct an Ising model on kagome lattice [2]. The three frustrated spins within the triangle lattice either generate three kinks or one kink. The block up-spin/down-spin above represents three kinks running in clockwise/counterclockwise direction. The single kink state exist for the case that one current is above the other two which always have one current lifted in opposite direction. These frustrated states are generated by spin flipping upon $|\downarrow\rangle_i$ or $|\uparrow\rangle_i$.

$$|\uparrow\uparrow\rangle_i, |\downarrow\uparrow\rangle_i, |\downarrow\downarrow\rangle_i, |\uparrow\downarrow\rangle_i, |\uparrow\uparrow\downarrow\rangle_i, |\downarrow\uparrow\downarrow\rangle_i.$$  

(69)

They are in fact frustrated triple spin states on triangle lattice, obeying the following Hamiltonian,

$$H_f = \sum_i J_z(S_{i, e_x}^z S_{i+e_x}^z + S_{i, e_y}^z S_{i+e_y}^z + S_{i, e_z}^z S_{i+e_z}^z),$$  

(70)

The coupling term within the unit cell has an equivalent formulation by the sum of three arm spins,

$$H_f = \sum_i J_z(S_{i, e_x}^z S_{i+e_x}^z + S_{i, e_y}^z S_{i+e_y}^z + S_{i, e_z}^z S_{i+e_z}^z + S_{i, e_z}^z S_{i+e_y}^z)$$  

$$+ \sum_i \frac{J_z}{2}[(S_{i, e_x}^z + S_{i, e_y}^z + S_{i, e_z}^z)^2 - 3].$$  

(71)

Obviously the ground state requires $(S_{i, e_x}^z + S_{i, e_y}^z + S_{i, e_z}^z) = \pm 1$, since the sum of the three arm spins can not reach zero. Thus the frustrated spin configurations Eq. (69) is the ground state. The first excited state of single hexagonal unit cell is $E = 3J_z$ corresponds the three kink states. $E = -J_z$ corresponds the 6 degenerated single kink states (Eq. (69)). Only flipping one spin could transform the ground state with single kink to the first excited states with three kinks. One spin flip contributes a $\exp[i\pi/2]$ phase factor to the block spin. Here a spin flipping is performed by braidings two anyons twice. Thus each anyon carries an Abelian phase factor $\exp[i\pi/2]$. Each braiding costs an energy unit $\pm 2J_z$.

For single triangular unit cell, braiding anyon within the ground state shows other statistical phases. For instance, in order to map $|\uparrow\uparrow\rangle_i$ to $|\downarrow\downarrow\rangle_i$, only the spin in $e_x$ is flipped by braiding anyons twice, however the the collective wave function acquires a phase $\exp[2i\pi]$. Thus the anyon in $e_y$ carries a phase $\exp[i\pi]$. While mapping state $|\uparrow\uparrow\rangle_i$ to $|\downarrow\downarrow\rangle_i$ requires three spin flips which is carried out by 6 braidings on local tribein. Thus each anyon carries a statistical phase $\exp[i\pi/3]$. The statistical factors of anyons are not uniformly distributed along different mapping paths from one eigenstate to another. Thus we call the anyons within these frustrated states as non-Abelian anyons.

These highly degenerated frustrate states have an exact one-to-one mapping to multi-layer knot lattice configurations similar to what we showed before. Thus each state could be labeled by a topological linking number of those entangled loops. This topological linking number is in fact equivalent to the total magnetization of spins. One special topological character of these frustrated state is there exist at least one current that can be disentangled from the other two without cutting at each local triangle. Repeating this operation for all of the other 5 triangles around the hexagon plaquette, we can always find a free loop current around the local hexagon plaquette. Anyon keeps running in this loop around the hexagon plaquette without losing energy or charges, but generates a magnetic flux passing through the center of hexagon plaquette. As showed in last section, electrons running in knot lattice model bears quantum Hall effect. For this two-state Ising spin on Kagome lattice, it only implements the quantum Hall effect with even number of magnetic fluxes. The two states Ising spin $S_i^z = \pm 1$ on this kagome lattice must be replaced by three states Ising spin $S_i^z = \pm 1, 0$ in order to implement the fractional quantum Hall effect.
C. The knot lattice of spin 1/2 Kitaev honeycomb lattice model

This knot lattice model can also be constructed to explore the non-Abelian anyon in the gapless phase of Kitaev honeycomb model [10]. The knot patterns for constructing honeycomb lattice is showed in Fig. 16, here the local triebin is acted by different spin operators on each arm,

\[ H_k = \sum_i (J_x S_i^x S_{i+1}^x + J_y S_i^y S_{i+1}^y + J_z S_i^z S_{i+1}^z). \]  

(72)

Different ferromagnetic coupling strengths are assigned in each direction of local tribeins, i.e., \( J_l < 0, (l = x, y, z). \) Here \((S^x, S^y, S^z)\) are spin-1/2 operators. In the knot lattice model (Fig. 16), we introduce three color anyons, the red \((\sigma_R)\), yellow \((\sigma_Y)\) and blue anyons \((\sigma_B)\), to implement this anisotropic coupling type.

The knot pattern generated by the red and blue anyon is only acted by spin operator \( S^x \). \( S^y \) acts on the red and yellow anyon. While \( S^z \) acts on yellow and blue anyon (Fig. 17 (a)). The Yang-Baxter equation still holds for this knot lattice model (Fig. 17 (a)). Extending this anisotropic coupling triebin over the whole honeycomb lattice, it naturally leads to an equivalent model to Kitaev model on honeycomb lattice, except here every local coupling triebin has been rotated for consistence (Fig. 16).

The three color anyons carry topological color charges. A running red anyon generates yellow and blue field to drive the motion of yellow anyon and blue anyon correspondingly. In gapped phase of this knot lattice model, these anyons obey Abelian anyon fusion rules (Fig. 17 (b) (c)),

\[ \sigma_R \times \sigma_Y = \psi_B, \quad \sigma_Y \times \sigma_B = \psi_R, \quad \sigma_B \times \sigma_R = \psi_Y. \]  

(73)

When the three triebi all fuse into Majorana fermions, a triplet of Majorana fermion is left in the middle, \( \psi_R \psi_Y \psi_B \), which is a fermionic cluster (Fig. 17 (c)).

The knot lattice model has three conserved plaquette operators around three adjacent hexagon plaquette (Fig. 16),

\begin{align*}
W_R &= S_1^z S_2^z S_3^z S_4^z S_5^z S_6^z, \\
W_B &= S_1^y S_2^y S_3^y S_4^y S_5^y S_6^y, \\
W_Y &= S_1^y S_2^y S_3^y S_4^y S_5^y S_6^y.
\end{align*}

(74)

here \((1, 2, 3, 4, 5, 6)\) indicate the six lattice sites around each hexagon. \( W_B \) draws a closed loop around the red loops in Fig. 16, so does the blue loops and yellow loops for \( W_B \) and \( W_Y \) on other hexagon plaquette. The plaquette operator is in fact Wilson loop, which can be viewed as generalized magnetic flux. A conserved color anyon runs in each Wilson loop. These Wilson loop operators commute with Kitaev Hamiltonian, \([W_p, H_k] = 0, (p = R, B, Y)\). Thus Wilson loop and the knot lattice share the same ground state. The minimum energy state of the knot lattice model is reached at vortex-free state, i.e., \( W_p = +1 \). A vortex is generated in the plaquette if \( W_p = -1 \).

The knot lattice pattern of the ground state in each plaquette is consist of \((2^6 - 1)\) layers of link lattice, since each Wilson loop operator is constructed actually by 6 effective spin operators. The eigenstate of Wilson loop operator is superposition state of 6 spins values, which is either \(| \downarrow \rangle \) or \(| \uparrow \rangle \). The ground state of this Wilson loop is a spin liquid state, which does not show magnetically ordered state. We denote the spin configuration at the ground state of Wilson loop operator as,

\[ |g\rangle = \sum_{q=1}^6(\pm 1)_q |g\rangle = +1 |g\rangle, \quad p = R, B, Y. \]  

(75)

For example, the eigenstate of Wilson loop operator, \( W_R \), is the superposition state of every possible spin configuration with even number of \(| \downarrow \rangle \),

\[ |g\rangle = | \downarrow\uparrow\downarrow\downarrow\uparrow\downarrow\rangle + | \downarrow\downarrow\downarrow\downarrow\uparrow\downarrow\rangle + \cdots, \]

\[ W_R |g\rangle = +1 |g\rangle. \]  

(76)

Flipping one spin at any one of the six sites generates a plaquette vortex. For the other two plaquette operators, \( W_B \) and \( W_Y \), the spin configuration of ground state is consists of a bilayer knot lattice, so that the spin operator fulfills the eigen-equation at every lattice site,

\begin{align*}
S_i^x (| \uparrow \rangle_1^i + | \downarrow \rangle_1^i) &= +1 (| \uparrow \rangle_1^i + | \uparrow \rangle_1^i), \\
S_i^y (| \uparrow \rangle_1^i + i | \downarrow \rangle_1^i) &= +1 (| \uparrow \rangle_1^i + i | \downarrow \rangle_1^i), \\
S_i^z (| \uparrow \rangle_1^i + | \uparrow \rangle_1^i) &= +1 (| \uparrow \rangle_1^i + | \uparrow \rangle_1^i). \quad (77)
\end{align*}

One exemplar bilayer knot pattern that fulfills the equations above is the sum of two layers (Fig. 19). The \( e^{i\pi/2} \) phase factor on the \( S_i^y \) sector is assigned by the its eigenstate construction Eq. (77). While the \( e^{i\pi/2} \) phase
factor on $S^z$ sector is assigned by the commutator relation of spin operators, $S^a S^b = i\epsilon_{abc} S^c$. For the negative eigenvalue of $S_y^2$, $S_y^2|\frac{1}{2}\rangle = -1|\frac{1}{2}\rangle$, we introduce a minus between the two layers. The $S^x$ and $S^y$ sector state spontaneously flipped a sign. To avoid the vanishing of eigenstate of $S_z^2$, we add a $e^{i\pi}$ phase upon the the sector of $S_z^2$ in the second layer. At the same time, to satisfy the equation $S^a S^b|g\rangle = i\epsilon_{abc} S^c|g\rangle$, the product of the eigenvalues of $S^x$ and $S^y$ must be negative to be consistent with the eigenvalue of $S_z^2$. Thus we add a $e^{i\pi}$ phase shift to the sector of $S_y^2$. Repeating this knot lattice construction process for the six spins of the plaquette, we can derive the superposition state of 62 layers of knot plaquette as the eigenstate of the Wilson loop operator.

braiding the color anyons in the knot lattice with respect to the eigenstate of Kitaev model follows the same fusion Eq. (11) as that for Ising model. Unlike the bilayer configuration of Ising ground state, the ground state of one plaquette in Kitaev honeycomb model has 62 layers. Braiding two anyons across the 62 layers synchronously in clockwise (or counterclockwise) direction would generate either a vacuum state or a Majorana fermion state in each layer following the basic fusion rules,

$$\sigma_B \times \sigma_R = I + \psi_B, \quad \sigma_R \times \sigma_V = I + \psi_B,$$

$$\sigma_V \times \sigma_B = I + \psi_B.$$

The anyon braiding in $S^x$ and $S^y$ sector carries phase factors due to commutator relations of spin operators and eigenstate construction.

The knot lattice in real space also map out knot configuration in momentum space, since the energy of system is independent of representation parameters. The fermions operator on knot lattice have periodical distribution in real space, thus it has a natural mapping into momentum space by Fourier transformation. Then the huge number of knot lattice layers in which electronic wave propagates could reduce to a statistical distribution of oscillating frequency in wave vector space. This distribution is summarized by the spectrum of interacting system. The spin 1/2 Kitaev Hamiltonian was mapped into quadratic fermion Hamiltonian, which is equivalent to complex fermion pairing Hamiltonian [10]. In Kitaev’s Majorana fermion pairing representation of one spin operator, $[S^a = ib^a c, (a = x, y, z)]$, the spin-spin coupling Hamiltonian is mapped into a fermion coupling Hamiltonian in momentum space [10],

$$H_{kk} = \frac{1}{2} \sum_{k,\alpha,\beta} iA_{\alpha,\beta}(k)(\psi_{-k,\alpha}\psi_{k,\beta}),$$

(79)

here $\alpha, \beta$ represent the local tribe $[e_x, e_y, e_z]$ inside each unit cell. The spectrum matrix has an off-diagonal representation,

$$A(k) = \begin{pmatrix} 0 & f(k) \\ -f(k)^* & 0 \end{pmatrix},$$

(80)

where the spectrum function $f(k) = 2(J_x e^{ikr_1} + J_y e^{ikr_2} + J_z).$ $r_1$ are the translational invariant unit vectors on the hexagonal lattice, which points to the the next nearest neighboring unit cell, $r_1 = (\frac{1}{2}, \frac{\sqrt{3}}{2}), r_2 = (-\frac{1}{2}, \frac{\sqrt{3}}{2}).$ The Energy spectrum is $E(k) = \sqrt{f(k)^2 + f(k)}$. The Kitaev Hamiltonian in momentum space can also decompose into fermion-spin coupling formulation,

$$H_{kk}(k) = \frac{1}{2} \sum_{k,\alpha,\beta} i(I_y S_y + I_x S_x)_{\alpha,\beta}(\psi_{-k,\alpha}\psi_{k,\beta}),$$

(81)

Here $S_x$ and $S_y$ are conventional Pauli matrices. The currents along the direction of $x$-spin and $y$-spin components are

$$I_y = J_x \cos(\frac{k_x}{2} + \frac{\sqrt{3}k_y}{2}) + J_y \cos(-\frac{k_x}{2} + \frac{\sqrt{3}k_y}{2}) + J_z,$$

$$I_x = J_x \sin(\frac{k_x}{2} + \frac{\sqrt{3}k_y}{2}) + J_y \sin(-\frac{k_x}{2} + \frac{\sqrt{3}k_y}{2}).$$

(82)

These two current depicts the Fourier knot in momentum space, which usually express the variable $k_x$ by spatial frequency and a free parameter, i.e., $(k_x = \omega_x k, k_y = \omega_y k)$. These two currents are actually the projection of the energy spectrum to $X$-spin and $Y$-spin, $E(k) = \sqrt{I_x^2 + I_y^2}$. In the exact solutions by Jordan-Wigner transformation, $I_y = \epsilon \rho$ represents the kinetic energy of free fermion. $I_x = \Delta$ is the energy gap for exciton pair. The energy spectrum $E(k) = \sqrt{\epsilon^2 + \Delta^2}$ has a similar form as p-wave BCS pairing model. If we consider the evolution of a time dependent knot lattice state with eigenergy $E(k)$, its final state oscillates as a matter wave, $|\psi_I\rangle = A \exp[i E t |\psi_i\rangle].$ Higher $E(k)$ indicates a faster oscillation with higher energy. In the mean time, the energy current also oscillates in space. $\omega_x$ counts the number of oscillations in unit space along X-direction. Since the Kitaev model in momentum space is an effective coupling model between fermion and Pseudospin vector. When the Pseudospin vector rotates in momentum space, the fermion current in the color channels also fluctuates following time dependent Heisenberg
In the gapless phase, the Majorana fermion in each unit hexagon cell has a constant existence probability even though it still oscillates from site to site. The flipping probability of current crossing state of the multi-layer knot lattice has a static value. The gapless spectrum exists when the three coupling coefficients satisfy the following inequalities [10],

\[ |J_x| < |J_y| + |J_z|, \quad |J_y| < |J_z| + |J_x|, \]
\[ |J_z| < |J_x| + |J_y|. \]

(83)

In this case, the three coupling bonds of local tribein reaches a balance, then noncommutative characteristic of three component spin operators dominates the system. The gapless phase is an ordered phase of spin liquid and shows up after the quantum entropy generated by oscillating Majorana fermions is suppressed. The gapless phase in Kitaev model corresponds to a vanishing spectrum of fermions pairing system, which is in fact the locations of vortex in momentum space.

The Fourier knots of current and its corresponding normalized current also depicts a complicated knot which winds into a closed half-moon shape (Fig. 22 (c)). Thus the energy current of gapless phase has an Euler characteristic \( \chi \neq 0 \), while the gapped phase has \( \chi = 0 \), and frequency ratio, \( \omega_x : \omega_y = 1 : 100 \) (Fig. 22 (c)). The corresponding normalized currents, \( n_x = I_x/E(k) \), \( n_y = I_y/E(k) \), depicts an open band (Fig. 22 (f)). In the meantime, the oscillating two bands touch each other periodically along \( k = 0 \) (Fig. 22 (d)). For the gapped phase, the energy current draws serial knots which pass the interior of a circular area without forbidden hole (Fig. 22 (b)). It winds \( \omega_y \) closed loops for frequency ratio \( \omega_x : \omega_y = 1 : \omega_y \). Trefoil knot also appears for frequency ratio \( \omega_x : \omega_y = 3 : 1 \) (Fig. 21 (b)). The energy spectrum shows two separated bands with many oscillations enveloped in one wave package (Fig. 22 (a)). The energy current knot turns into a filled disc at the high frequency ratio of \( \omega_x : \omega_y = 1 : 100 \) (Fig. 22 (c)). The corresponding normalized current also depicts a complicate knot which winds into a closed half-moon shape (Fig. 22 (c)). Thus the energy current of gapless phase has an Euler characteristic \( \chi = 0 \), while the gapped phase has a nonzero Euler characteristic, \( \chi = 2 \). This knot lattice implementation of Kitaev model has a straightforward extension into knot lattice in three dimensions by anyonic loop model [22].

D. The gapless edge current of non-Abelian anyons in a finite knot lattice

For a finite lattice without periodic boundary condition, the boundary coupling types have to be carefully arranged in order to map the Kitaev model consistently. Because there are always three color strings branches on
the upper boundary or bottom boundary, there is no ways to connect the strings with the same color witho
out over crossing or under crossing other current. The
only consistent connection pattern is fusing color anyon
into colored Majorana fermions (Fig. 20). Two edge cur-
rents of mixed anyons and Majorana fermions exist both
on the upper boundary and bottom boundary (Fig. 20).
However, if the lattice structure is not perfect hexago-
nal lattice, instead it contains certain plaquette formed
by odd number of edges, then the anyons in the edge
current does not obey Abelian fusion rules. There must
exist some fermionic vacuum state and unpaired Major-
ana fermions. For instance, the minimal extension of
spin 1/2 kitaev model on a lattice with odd plaquette is
tetrahedron lattice which is constructed by four triangles
in Fig. 23 (a)). If extra over-crossings are intro-
duced for consistence, the tetrahedron lattice model fi-

corns (Fig. 23 (a)), If extra over-crossings are intro-
duced for consistence, the tetrahedron lattice model fi-
nally turns into honeycomb lattice model again. If vac-

to the other state, and one vacuum states to the other. (b)
The eigenstate of knot lattice for the gapless edge current
fermion runs in one direction, either in clockwise direc-
tion or in counterclockwise direction.

Each spin operator in kitaev honeycomb model is ex-
pressed by a pair of Majorana fermion \[ \sigma^\alpha = i b^\beta_{c_j} \].
In the vacuum state of this knot lattice model, \( b^\beta_{\alpha} \) re-

The effective Majorana fermion Hamiltonian of the gapless current reads (Fig. 23 (b)),

\[
H_t = J_x S_{e_x}^x S_{e_x}^x + J_y S_{e_y}^y S_{e_y}^y + J_z S_{e_z}^z S_{e_z}^z
+ J_2 S_{e_2}^x S_{e_2}^x + J_y S_{e_y}^y S_{e_y}^y + J_z S_{e_z}^z S_{e_z}^z.
\]

The tribein of each lattice cannot and operate sites without introducing extra cross-

If extra over-crossings are intro-
duced for consistence, the tetrahedron lattice model fi-
nally turns into honeycomb lattice model again. If vac-

to the other state, and one vacuum states to the other. (b)
The eigenstate of knot lattice for the gapless edge current

Thus Abelian anyons runs in the inner land currents
and obeys the conventional fusion rule Eq. (73). Non-Abelian anyons runs on the edge and generate
Majorana fermions and fermionic vacuum (grey bonds in
Fig. 23 (b)) simultaneously. There are three unpaired
Majorana fermions running in the edge loop. Odd loops
breaks the time-reversal symmetry under the transforma-
tion \( TS_i T^{-1} = -S_i \). Thus the three running Majorana
Here $b$ is a general Majorana fermion. $\hat{\phi}$ is a fermionic vacuum operator. One usual construction of Majorana fermion by conventional fermion operator is

$$b_j = (c_j + c_j^\dagger), \quad b_j = (c_j - c_j^\dagger)/i. \quad (89)$$

The ground state of this Majorana fermion is $|g\rangle = (|0\rangle - c_j^\dagger|0\rangle)/\sqrt{2}$, i.e., $b_j|g\rangle = -|g\rangle$. While the vacuum fermion has no conventional construction. The only self-consistent construction of vacuum fermion and Majorana fermion simultaneously is based on knot lattice chain model (Fig. 18). The creation operator can be defined as product of two clockwise braidings operators (Fig. 24). While the annihilation operator is the product of two counterclockwise braidings operators. The vacuum operator is also product of braidings operators (Fig. 24 (a)),

$$c_j = B_{n\cup,j}^2, \quad c_j = B_{n\cup,j}^2, \quad \hat{\phi} = B_{n\cup,j}B_{t\cup,j},$$

$$c_j = B_{t\cup,j}^2, \quad c_j = B_{t\cup,j}^2, \quad \hat{\phi} = B_{n\cup,j}B_{t\cup,j}. \quad (90)$$

The creation and annihilation operator have two different representations by braidings operators in different direction, so does the vacuum operator (Fig. 23 (a)). We first focus one operator set for simplicity. In order to match the conventional symbol of quantum operators, we replace the symbol ($\cup$) and ($\cap$) with ($-$ and $+$),

$$c_j = (B_{n,j})^2, \quad c_j = (B_{n,j}^+)^2, \quad \hat{\phi} = B_{n,j}B_{n,j},$$

$$c_j = (B_{t,j}^+)^2, \quad c_j = (B_{t,j}^+)^2, \quad \hat{\phi} = B_{n,j}B_{t,j}. \quad (91)$$

The indices $n$ and $t$ indicate normal and tangential direction along which the braidings is performed. Then the fermion chain model of non-Abelian anyon are expressed by pure braidings operators,

$$H^c = \sum_{(j)} J_x \hat{\phi}_{3j+1}[(B_{n,3j+1}^-)^2 + (B_{n,3j+1}^+)^2] + J_y \hat{\phi}_{3j+2}[(B_{n,3j+2}^-)^2 + (B_{n,3j+2}^+)^2] + J_z \hat{\phi}_{3j+3}[(B_{n,3j+3}^-)^2 + (B_{n,3j+3}^+)^2], \quad (92)$$

where fermionic vacuum operator is

$$\hat{\phi}_j = [B_{n,j}^- B_{n,j}^- - B_{n,j}^+ B_{n,j}^+]. \quad (93)$$

The non-commutative characteristic of braidings operators are is consistent with non-Abelian anyon statistics. The eigenstate of this non-Abelian anyon Hamiltonian can be constructed by the sum of two knot chains (Fig. 24 (b)), i.e., $H^c|\psi\rangle = \pm|\psi\rangle$. For an isolated crossing state, single braidings operator is neither fermion nor boson. Since exchanging two pairs of double braidings operator contributes $\exp[i\pi]$, each braidings is equivalent to an anyon which bears an effective statistical factor $\exp[i\pi/2]$. However if there exist many connected neighboring zero energy states and only a few crossing states, the statistical phase factor from collective spin up to spin down is not $\exp[i\pi]$ any more. In that case, the statistical phase is determined by the filling factor,

$$\nu = \frac{L_{link}}{N(B)}. \quad (94)$$

$N(B)$ is total number of braidings to transform one collective state to another one. This number has a straightforward counting from a Hamiltonian in real space. An equivalent counting can also be performed in momentum space under Fourier transformation,

$$B_{n,3j}^- = \frac{1}{\sqrt{N}} \sum_{k_x} e^{-ik_x}B_n^-(k_x),$$

$$B_{n,3j}^+ = \frac{1}{\sqrt{N}} \sum_{k_x} e^{ik_x}B_n^+(k_x). \quad (95)$$

The anyon fermion pair model is mapped into a coupling model of four braidings operators. The braidings operator in momentum space braids different energy levels. The spin currents with opposite in quantum spin Hall effect can not cross each other in energy spectrum, this defines an over-crossing state or under-crossing state. The level crossing avoidance point defines different vacuum state. Higher excited knot states of energy levels in momentum space also exist for non-Abelian anyon models.

The gapless edge current is actually one dimensional Majorana fermion chain including vacuum state. One dimensional quantum chain of ordinary fermions with long range hopping terms is a more general case of non-
Abelian anyon.

$$\hat{H}_L = \sum_{i=1}^{m} \left( \sum_{j=1,\alpha} e^{i\phi} t_{j,\alpha} c_{i,\alpha}^\dagger c_{i+j,\alpha} \right).$$  \hspace{1cm} (96)

Here $m$ is the distance number that counts the number of lattice site between two lattice sites that is connected by continuous long range channels. In the presence of the long range hopping Hamiltonian Eq. (96), long loops covering many sites would show up in the eigenstates. For the exemplar strip ordering phase in Fig. 25 the long range hopping Hamiltonian reads,

$$\hat{H}_L = \sum_{j=1,\alpha} e^{i\phi} t_{j,\alpha} c_{i,j,\alpha}^\dagger c_{i+j,\alpha}.$$  \hspace{1cm} (97)

Suppose there exist a string of gapless modes covering four neighboring unit cells (Fig. 25 (a)). Under one braiding operation at first sites (Fig. 25 (a)), it generates one $|+1\rangle$ state, this knot maps to a trivial circle by Reidemeister moves. The output state under two braidings is $|+1, -1\rangle$ (Fig. 25 (b)), this knot has a linking number $L_{link} = 1$, which results in a filling factor $\nu = 1/2$. The output knot under three same braidings is $|+1, +1, +1\rangle$, it take one flipping on 2nd or two flips on 1st and 3rd sites to map into trivial knot state, thus the filling factor here is 1/3 or 2/3. The strip knot has eight alternative crossings (Fig. 25(d)) after eight braidings, which has a linking number of $L_{link} = 4$. In order to map the maximal linked state into trivial circle, it takes four flipping operations at the 2nd, 4th, 6th, and 8th crossing sites. Then the ferromagnetic phase of this strip loop could be map into trivial circle by Reidemeister move. This defines a collective vacuum state (Fig. 25 (e)). Each flipping operation is equivalent to a Majorana fermion operator. The four flipping costs eight braiding operators. Thus the filling factors of this collective vacuum state is $\nu = 4/8 = 1/2$. This is a local filling factor. It turns into a global filling factor for a periodic distribution of strip knots over a two dimensional knot lattice (Fig. 25). If the strip vacuum state covers 2n unit cells, the filling factor could reach $(n \pm 1)/2n$. In this sense, the fractional filling state is essentially a topological finite size effect.

\section*{E. knot lattice model for quantum Hall states}

Haldane model on honeycomb lattice demonstrates quantum Hall effect without Landau levels \cite{44}. The honeycomb knot lattice can also construct the eigenstate of Haldane model (Fig. 16, Fig. 26). The honeycomb lattice is composed of two triangular sub-lattice. In Haldane model, besides the hopping of fermions between nearest neighbors, the hopping of fermions among the next near-

![FIG. 26: One of many layers of honeycomb knot lattice implementation of Haldane model with the next nearest neighboring hopping terms for integral quantum Hall effect.](image)

est neighbor sites are also introduced \cite{44},

$$H_{hal} = \sum_{\langle ij \rangle \alpha} t_{1} c_{i,\alpha}^\dagger c_{j,\alpha} + \sum_{\langle \langle ij \rangle \alpha} e^{i\phi} t_{2} c_{i,\alpha}^\dagger c_{j,\alpha} + h.c. \hspace{1cm} (98)$$

The phase factor is $\phi = 2\pi (2\phi_0 + \phi_0)/\phi_0$. If the next nearest neighbor sites are directly connected by continuous current channels (For instance, knot lattice in Fig. 26), the next nearest neighbor hopping terms in Hamiltonian Eq. (98) can be directly performed on knot lattice to flip the local crossing state. However every lattice site must be an over-crossing state or under crossing state, in order to keep continuous channel connecting any next nearest neighbors. If there exist some block spin $|0\rangle$ states on certain lattice sites, there would not exist a minimal channel to connect certain next nearest neighbors (Fig. 17 (c)). Thus a superposition state of many honeycomb knot lattice with only crossing states can construct the eigenstate of Haldane model. Since the braiding operator is only allowed to perform on two states, $|+1\rangle$ or $|-1\rangle$, the denominator of filling factor can only be even number. Opposite writhing current (Fig. 4) can implement the two opposite magnetic field on renormalized sublattice. The Hall resistance in integral quantum Hall effect is quantized by first Chern number of Berry phase. The first Chern number is equivalent to Euler characteristic on a discrete lattice. In order to map the Chern number to Euler characteristic, the energy current in momentum space must be triangularized into grid. How to derive an exact mathematical mapping is still an unsolve problem in mathematics.

As showed in last section, fractional filling factor in quantum Hall effect is essentially quantized by the linking number of crossing current. The block vacuum state in block spin-1 knot lattice is inevitable to produce fractional quantum Hall states. As we already seen in knot square lattice model, more braiding within one unit square inevitably creates more lattice sites within the unit square. This means fractional Hall effect only exist for long range hopping dynamics to derive higher link-
The three spin components, space can be decomposed as a function of Fourier 
ated during this braidings process. Thus the filling fac-
creases from three to zero. Three crossings were elimi-
immunity states, the corresponding writhing number de-
to construct Haldane model (Fig. 26). It takes 
lattices sites, the next nearest neighboring hopping still 

The Hall conductivity \( \sigma \) and on-site energy \( M \) model in momentum space with respect different phases \( \phi \) and on-site energy \( M = 1 \). The frequency ratio 
for the figures above is \( \omega_z : \omega_y = 1:100 \) and \( t_2 = -2 \).

The energy spectrum of Haldane model in momentum 
leap is \( \nu \) \( = 0 \) \( (\nu = 0) \) \( \equiv \). This integer measures the sign of 
the winding number of knot in momentum space. The 
circular trefoil knot in momentum space for \( \phi = -3.14 \)
and \( \phi = -1 \) turn into the opposite direction of that for 
positive phases, i.e., \( \phi = 3.14 \) and \( \phi = 1 \) (Fig. 27 (b-
c)). The phase \( \phi \) only modifies the initial phase of the 
oscillating energy spectrum wave, but does not change 
the gap characteristic between two bands, except here 
the energy gap reaches the maximal value at zero phase 
(Fig. 27 (a)). The trefoil knot in momentum space here 
is the dual mapping of trefoil knot in real space (Fig. 26).

The Haldane model is a topological band model with 
Chern number \( C = 1 \). A topological flat band model with 
Chern number \( C = 2 \) could be constructed on square 
lattice with the third nearest neighboring hopping \[ 49 \]. 
This square knot lattice is a natural implementation of 
Kane-Mele model \[ 45 \], which induces spin-orbital cou-
ing number. Since fractional filling factor only depends 
on topological braiding operations, it is independent of 
specific lattice structure. Each lattice site must admits a 
block spin 1 state, i.e., two crossing states \( (| + 1 \rangle, | - 1 \rangle) \) 
and two vacuum states \( (| 0 \rangle, | n \rangle) \). A global vacuum 
state is a set of minimal loop around the center of each 
unit cell, there is no next nearest neighboring hopping. 
Global vacuum state does not match Haldane model with 
next nearest neighbors hopping terms. If all of the vac-

For those channels turning left (right) to reach the next 
neighbors, \( \mu_{ij} = +1 (\mu_{ij} = -1) \). The eigenstate 
of kane-mele model can be constructed by multi-layer 
knot configurations without block vacuum state. In order 

to implement higher order of fractional quantum Hall 
states, the kane-mele model must be extended to include 
long range hopping terms. A multilayer generalization of 
Haldane model with the third nearest neighboring hop-
pings is topological flat band model with arbitrary Chern 
numbers \[ 49 \]. The same physics hold for the Haldane model on honeycomb lattice with the third nearest neigh-

\( \phi \) \( > 0 (\phi < 0) \) \[ 44 \]. The integer here \( \nu \) \( = +1 \) \( (\nu = -1) \) for 

\[ H_3 = H_{hal} + \sum_j \sum_{m=1,\alpha} t_{3} \epsilon_{j\alpha} c_{j+c+m,\alpha}. \]  

FIG. 27: (a)(b) The energy spectrum and knot of Haldane model in momentum space with respect different phases \( \phi \) and on-site energy \( M = 1 \). (c) The knot of Haldane model in momentum space on-site energy \( M = 0 \). The energy gap increases from three to zero. Three crossings were eliminated during this braiding process. Thus the filling factor of the vacuum excitation is one.

The energy spectrum of Haldane model in momentum space \[ 44 \] can be decomposed as a function of Fourier knot drawn by the three energy current with respect to the three spin components,

\[ I_x = \cos(-\sqrt{3}\pi \omega_x k + \pi \omega_y k) + \cos(\sqrt{3}\pi \omega_x k + \pi \omega_y k) \]
\[ I_y = \sin(-\sqrt{3}\pi \omega_x k + \pi \omega_y k) + \sin(\sqrt{3}\pi \omega_x k + \pi \omega_y k) \]
\[ I_z = [M - 2t_2 \sin(\phi)] \sin(-\sqrt{3}\pi \omega_x k + 3\pi \omega_y k) \]
\[ + \sin(\sqrt{3}\pi \omega_x k + 3\pi \omega_y k). \]

The Hall conductivity \( \sigma^{xy} \) is quantized at the Fermi level, \( \nu(e^2/h) \). The integer here \( \nu = +1 (\nu = -1) \) for

\[ \phi > 0 (\phi < 0) \] \[ 44 \].

FIG. 28: The square knot lattice of double currents for implementation of Haldane model and Kane-meile model.

Three layers of knot lattice construction of Haldane model on honeycomb lattice generates a topological band model with Chern number \( C = 3 \) \[ 49 \]. Note inter-layer hopping does not exist in this construction. A straightforward construction of topological flat band model with Chern number \( C = N \) is consists of \( N \) layers Haldane
model with the third nearest neighboring hopping \[^{[49]}\].

One exemplar knot lattice for fractional quantum Hall state of \(1/3\) filling is topological flat band model with Chern number \(C = 2\). The local knot pattern on two dimensional square lattice visualize hopping distance of particles. For instance, the next nearest neighboring hopping on square lattice is represented by red knot cross (Fig. 29). The next-next-nearest-neighboring hopping runs along the L-shaped channel. The Fourth neighboring hopping terms run along the big blue knot cross channel (Fig. 29). The vacuum state within one unit cell acts as a cut off of a long knot pattern. To construct knot lattice with respect to fractional quantum Hall state, a combination of these basic knot elements must be arranged periodically on the whole lattice. Similar to the one dimensional strip knot, the initial state of the big crossing-knot could be set to vacuum state or ferromagnetic state. The filling factor for a given knot state is determined by its linking number and the number of braidings to reach that state from initial state. Fractional quantum Hall is a natural existence in this knot square lattice since electron runs in the channel that always turns from X-direction into Y-direction or vice versa. The long range hopping terms in Hamiltonian also expands a the energy spectrum of free particle into higher dimensions. Each energy level in the conventional model containing only nearest neighbor hopping terms now expands to many hyperfine levels. Thus the conventional fully filled energy levels now become partially filled, which results in fractional filling factors. Fractional filling states bears a topological origin, thus it does not depends on specific lattice structure. A knot lattice for the topological flat band model with Chern number \(C = 3\) can also be constructed on triangular lattice (Fig. 30).

The minimal vacuum loop excitation in square knot lattice model is different from that of honeycomb knot lattice. There are four Majorana arcs surrounding one lattice site to form the minimal loop in vacuum excitation of the square knot lattice (Fig. 29). Thus each vacuum loop excitation behaves as Boson. While the minimal vacuum loop around each center of hexagon unit cell is formed by three Majorana fermion arcs, thus the vacuum loop behaves as fermion (Fig. 67 (d)). Every pair of Majorana fermion arcs indicates the existence of one magnetic flux. Similar to block spin-1 Ising model, if the crossing state is braided for many times, it would generate extra crossing points within one unit cell. Then Majorana fermion operator as a crossing state flipper have to be introduced to bring the local crossing back to Hilbert space.

Spin-spin coupling models with long range hopping interaction may also bear fractional quantum Hall effect. Long range spin-spin coupling term could be added into Kitaev honeycomb model Eq. (72).

\[
\hat{H}_L = \sum_{j=1}^{m} J_{\alpha} S_{i,j}^{\alpha} S_{i+1,j}^{\alpha}.
\]

The block spin-1 Kitaev honeycomb may also demonstrate fractional quantum hall effect. Most crossing lines become turning arcs that transform one type of anyon into another. The conserved plaquette operators still exist for this block spin-1 Kitaev Hamiltonian due to the same commutator relation of spin operators. The three plaquette operators, \([W_p, (p = R, B, Y)]\), bear the same formulation as Eq. (74). The ground state of this block spin-1 plaquette operator is the same as that of spin-1/2 Kitaev model. The first excited state is determined by \(W_p = 0\), \((p = R, B, Y)\). If any one of the six unit cell of hexagon plaquette joins in the vacuum state of zero crossing state, then it generates one plaquette excitation that we called vacuum excitation. There exist 62 possible vacuum excitations in total. The topological operation of these vacuum excitation is to disentangle two neighboring loops. The vacuum excitation is generated by braidings two anyons once (Fig. 67 (d)). braidings anyons twice upon the ground along the same direction leads the pla-
quette into the second excited states. The three states of $S = \pm 1, 0$ are eigenstate of $S^z$ sector. While in the $S^x$ or $S^y$ sector, even number of braidings plus Majorana fermion flipping does not generate the eigenstate of $S^z$. In that case, the bilayer knot of eigenstates for $S^x$ or $S^y$ have to be acted synchronously.

A more direct implementation of fractional quantum Hall effect on honeycomb knot lattice is to introduce an external magnetic field by placing one Dirac magnetic monopole at the center of a sphere lattice, or one magnetic monopole ring around the center line of a torus lattice. Then the input electrons carrying different gauge phases flow into different knot channels. Its Hall resistance bears the fundamental character of fractional quantum Hall effect. The partition function of the knot lattice is still a topological invariant. However every hexagon unit cell was attached by a complex coefficient. The total linking number of the knot lattice is now a complex number as well. To compute the partition function, one could start with the vacuum state of uniform loop state, i.e., there is no crossing at any lattice site. Then using Slein relation and Jones polynomial repeatedly leads to the final partition function.

IV. CONCLUSION AND OUTLOOK

The entangled multiknot lattice provides a geometric representation of eigenstate of quantum many body system. For instance, the over crossing and under crossing of two lines in knot lattice can represent the spin up and spin down states in Ising model. Each knot lattice configuration represents a possible state of Hamiltonian. Start from this basic correspondence relation, the spin configuration in the Hilbert space of classical Ising model, quantum Ising model, Kitaev honeycomb model and Heisenberg spin model can be represented by the superposition of many layers of multi-knots lattice. The key difference is the spin in conventional spin models is a single particle state, while the single spin state in multi-knot lattice model is a collective wave function of two crossing strings. The fractional statistics of anyons in the multi-knot lattice is explicitly demonstrated by single spin state. This operation has a spontaneous extension to three dimensional(3D) spin models, such as 3D Ising model, 3D Kitaev type model and 3D Heisensberg model. There are many different ways to map a spin operator into a fermion operator, such as Jordan-Wigner transformation, then the interacting fermions system can also be represented by multi-knot lattice model which can extend to topological insulator models. Then the topological linking number provide a new topological number to quantify different quantum phases. For the phase transition in two dimensional Ising model, the linking number already shows significant different values in the disorder phase from that in the magnetic ordered phase. In fact, this multi-knot lattice bears a more general topological function, which is defined by the non-Abelian Chern simons action. The abelian Chern-Simons action is a topological number of many one dimensional knots, which includes self-linking number, writhing number and twisting numbers. It is a a also a topological action of three dimensional knot. A periodical lattice of three dimensional knot can be extended to implement four dimensional quantum Hall effect. Another more realistic implementation of knot lattice model is a network of optical fibers, in which spinning photons demonstrates optical spin Hall effect.

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