Universal property of quantum gravity implied by uniqueness theorem of Bekenstein-Hawking entropy

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Abstract. We have searched for a universal property of quantum gravity, where “universal” means to be independent of any existing model of quantum gravity (such as the super string theory, loop quantum gravity, causal dynamical triangulation, and so on). To do so, we have investigated carefully the basis of black hole thermodynamics. The uniqueness of Bekenstein-Hawking entropy lets us extract a reasonable universal property of quantum gravity from the conditions justifying Boltzmann formula. The universal property indicates a repulsive gravity at Planck length scale, otherwise stationary black holes can not be regarded as thermal equilibrium states of gravity.

1. Introduction
As mentioned in the abstract, we have investigated the basis of black hole thermodynamics in order to search for universal properties of quantum gravity, where “universal” means to be independent of any existing model of quantum gravity. The basic concept of black hole thermodynamics is the following consideration: Because a stationary black hole emits the thermal radiation (Hawking radiation), the black hole is in a thermal equilibrium state of gravity. And, usually, the microscopic origin of black hole is expected to be the quantum states of gravity. This basic concept depends on the thermal radiation theory, which indicates that the source of “Planck spectrum” is a thermal equilibrium object.

By accepting the above concept of black hole thermodynamics, it is reasonable to require that the quantum gravity satisfies the pre-supposition of thermal radiation theory, otherwise we can not regard black holes as thermal equilibrium states by observing the Hawking radiation. Here, the pre-supposition of thermal radiation theory is that the ordinary thermodynamics and quantum statistical mechanics work well on any thermal equilibrium object, at least in laboratory systems. Therefore, it is natural to require that the basic idea of statistical mechanics is applicable to thermal equilibrium states of not only laboratory systems but also quantum gravity (black holes). Here, the basic idea of statistical mechanics is, for example, the Boltzmann formula, $S_{BH} = \ln \Omega_g$, where $S_{BH}$ is the Bekenstein-Hawking entropy, and $\Omega_g$ is the number of states of black hole counted by the underlying quantum gravity.

From the above, we adopt the following two suppositions for our purpose:

Supposition 1: A stationary black hole is a thermal equilibrium state composed of micro-states of underlying quantum gravity. This is based on the Hawking radiation.

Supposition 2: Basic formalism of statistical mechanics (e.g., Boltzmann formula, grand partition function, and so on) is applicable to underlying quantum gravity so as to describe
the thermal equilibrium states (black holes). This is based on the pre-supposition of thermal radiation theory.

The issues for each supposition are as follows:

**Issue on supposition 1 (Uniqueness of BH entropy):** Note that, if the Boltzmann formula $S_{BH} = \ln \Omega_g$ holds for black holes, then it defines the Bekenstein-Hawking entropy $S_{BH}$ uniquely in microscopic point of view. This means that the Boltzmann formula can never work well unless the uniqueness of entropy is proved in macroscopic point of view of thermodynamics (without using any microscopic theory) [1, 2, 3]. In ordinary thermodynamics, the uniqueness of entropy is proven by some basic requirements of thermodynamics [1, 2]. However, as summarized in Section 2, some basic requirements of ordinary thermodynamics do not hold in black hole thermodynamics [4, 5, 6]. Then, it is not necessarily manifested whether the uniqueness of $S_{BH}$ holds in black hole thermodynamics. Hence, we have to prove the uniqueness of $S_{BH}$ before the discussion of Boltzmann formula.

**Issue on supposition 2 (Justification of Boltzmann formula):** Once the uniqueness of Bekenstein-Hawking entropy $S_{BH}$ is proved in macroscopic point of view, we can proceed to the investigation of Boltzmann formula. Note that the validity of Boltzmann formula in ordinary quantum statistical mechanics is justified by some intrinsic properties of Hilbert space of ordinary quantum mechanics. Those intrinsic properties appear in Ruelle-Tasaki theorem and Dobrushin theorem [3, 7, 8] as summarized in Section 3. Then, if $S_{BH}$ is given by the Boltzmann formula, we can expect that those theorems of ordinary quantum mechanics are related with some property of underlying quantum gravity. Some universal property of quantum gravity may be extracted from those theorems.

In Section 4 we have shown a conclusion about the universal property of quantum gravity.

### 2. Uniqueness theorem of Bekenstein-Hawking entropy

In the rigorous axiomatic formulation of ordinary thermodynamics [1, 2], the basic axioms consist of not only the “four laws of thermodynamics” but also some more requirements. One of the axioms of ordinary thermodynamics modified in black hole thermodynamics is the classification requirement of state variables, all thermodynamic state variables are classified into two categories: extensive variables (e.g., energy, entropy, and so on) and intensive variables (e.g., temperature, pressure, and so on). Here, the extensivity and intensivity are defined by the scaling behaviour as follows: Under the scaling of “system’s size” $V \rightarrow \alpha V$, and $N \rightarrow \alpha N$ (where $V$ is volume, and $N$ is mol-number for a gas in a container), the extensive variable $X$ is scaled as $X \rightarrow \alpha X$, and the intensive variable $Y$ is invariant $Y \rightarrow Y$. This classification is used in showing many theorems of ordinary thermodynamics including the uniqueness theorem of entropy.

On the other hand, in the framework of black hole thermodynamics, e.g., for Schwarzschild black hole of radius $R_{BH}$, some representative state variables are as follows: Bekenstein-Hawking entropy $S_{BH} = \pi R_{BH}^2$, Hawking temperature $T_{BH} = (4\pi R_{BH})^{-1}$ and the internal energy $E_{BH} = R_{BH}/2$ (in units $c = 1, G = 1, \hbar = 1$). For the Schwarzschild black hole, the basic scaling is given by the scaling of length size $R_{BH} \rightarrow \lambda R_{BH}$, and we find the scaling behaviours as [5].

$$S_{BH} \rightarrow \lambda^2 S_{BH}, \quad T_{BH} \rightarrow \lambda^{-1} T_{BH}, \quad \text{and} \quad E_{BH} \rightarrow \lambda E_{BH}.$$  

This implies that the state variables in black hole thermodynamics should be classified not into two categories, but into three categories: The extensive variables (e.g., $S_{BH}$), intensive variables (e.g., $T_{BH}$), and energy variables (e.g., $E_{BH}$). In this classification, the energy variables, which are extensive in ordinary thermodynamics, form one independent category in black hole thermodynamics, since the scaling behaviour of energy is different from that of
extensive variables. Furthermore, the scaling behaviour of extensive and intensive variables are different from that of ordinary thermodynamics. But, the scaling behaviour of [extensive variable] × [intensive variable] (e.g., $T_{BH} S_{BH}$) is the same with that of energy variables [4, 6].

Because the classification of state variables of black holes are changed, the uniqueness of $S_{BH}$ is not manifested. However, as shown in [4], the uniqueness of $S_{BH}$ can be proved with modifying the proof of uniqueness in ordinary thermodynamics. The statement of theorem is:

**Uniqueness theorem of entropy:** Let $K$ be an extensive variable, which increases along irreversible adiabatic processes. Then, there exist two constants $a$ ($>0$) and $b$ such that $K = a S_{BH} + b$, where $S_{BH}$ is the Bekenstein-Hawking entropy. (See [4] for the proof.)

Once this theorem is proved, we can proceed to the investigation of Boltzmann formula. In this sense, we can regard the conclusion in Section 4 as a result implied by the uniqueness of $S_{BH}$.

3. Conditions justifying Boltzmann formula

Consider the ordinary quantum mechanics. For simplicity, consider the system of $N$ identical particles in a region of volume $V$ with no external field, and:

- Interaction potential: $\Phi(\vec{x}_1, \cdots, \vec{x}_N) = \sum_{j=1}^{\infty} \phi^{(j)}(\vec{x}_{i_1}, \cdots, \vec{x}_{i_j})$, where $i_j \in (1, \cdots, N)$.
- Energy eigenvalue: $E_k(V, N)$, where $k = 0, 1, 2, \cdots$, and $E_k \leq E_{k+1}$ (“=” is for degeneracy).
- Number of states: $\Omega(V, N; U) := \text{“Number of eigen states satisfying } E_k \leq U \text{”} = \max_k$.

First, let us show the Ruelle-Tasaki theorem, which clarifies the sufficient conditions for the validity of Boltzmann formula [4, 7, 3]:

**Ruelle-Tasaki theorem:** Suppose the following two conditions of $\Phi(\vec{x}_1, \cdots, \vec{x}_N)$:

**Condition A:** Arbitrary $j$-particle interaction $\phi^{(j)}$, becomes negative for sufficiently large distribution of $j$ particles. That is, there exists a constant $r_A (>0)$, such that $\phi^{(j)}(\vec{x}_{i_1}, \cdots, \vec{x}_{i_j}) \leq 0$, for $r_A \leq \min_{k=1, \cdots, j} |\vec{x}_{i_k} - \vec{x}_{i_l}|$.

**Condition B:** Interaction potential $\Phi$ is bounded below. That is, there exists a constant $\phi_B (>0)$, such that $\Phi(\vec{x}_1, \cdots, \vec{x}_N) \geq -N \phi_B$.

Then, the limit $\sigma(\varepsilon, \rho) := \lim_{\mathrm{t.l.}} \frac{\ln \Omega(V, N; U)}{V}$ exists uniquely, where $\mathrm{lim}$ is thermodynamic limit defined by $V \rightarrow \infty$ with fixing $\rho := N/V$, and $\varepsilon := U/V$ at constant values.

By this theorem, the thermodynamics limit of $\ln \Omega(V, N; U)$ is defined well. Furthermore, it can be also proved that $\sigma(\varepsilon, \rho)$ is concave about its arguments $(\varepsilon, \rho)$ and monotonically increasing about $\varepsilon$, and remains constant along reversible adiabatic processes [3]. These behaviours are some characteristic properties of entropy already known in thermodynamics. Then, in statistical mechanics, it is assumed that $\ln \Omega(V, N; U)$ is equal to the entropy (the Boltzmann formula).

Obviously, the above conditions A and B are the sufficient conditions for the validity of Boltzmann formula. Furthermore, a system, which holds Boltzmann formula but violates conditions A or B, has not been found experimentally. Hence, at least in laboratory systems, it is reasonable to require that any physical system satisfies the conditions A and B.

Next, let us show the Dobrushin theorem, which clarifies the necessary condition for the existence of thermal equilibrium states [4, 7, 8]:

**Dobrushin theorem:** Consider the case satisfying the following two pre-suppositions:

**Pre-supposition C:** The $j(\neq 2)$-particle interactions disappear, and the total interaction potential is a sum of two-particle interactions $\Phi(\vec{x}_1, \cdots, \vec{x}_N) = \sum_{1 \leq i < j \leq N} \phi^{(2)}(\vec{x}_i, \vec{x}_j)$.

**Pre-supposition D:** $\phi^{(2)}(\vec{x}, \vec{y}) \propto |\vec{x} - \vec{y}|^{-\alpha}$, ($\alpha > 0$) for sufficiently large $|\vec{x} - \vec{y}|$. 

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Under these pre-suppositions, if the ground partition function can be defined uniquely (i.e., if thermal equilibrium states exist), then the following inequality holds:

\[ I^2_V := \frac{1}{V^2} \int \int_V d^3x_1 d^3x_2 \phi^{(2)}(\vec{x}_1, \vec{x}_2) \geq 0. \tag{1} \]

Obviously, \( I^2_V \geq 0 \) is the necessary condition for the existence of thermal equilibrium states. Note that the pre-suppositions C and D are natural when we consider the gravity.

4. Conclusion

Our supposition 2 in Section 1 implies that the ordinary quantum mechanics and quantum gravity shares the same properties, which justify the validity of Boltzmann formula and the existence of thermal equilibrium states. The sufficient conditions A, B and the necessary condition \( I^2_V \geq 0 \) are expected to be such shared properties \(^1\). Note that these three conditions (A, B and \( I^2_V \geq 0 \)) are for the interaction potential.

On the other hand, it is not clear whether the full quantum gravity is expressed by using the interaction potential \( \Phi \) or Hamiltonian. However, the semi-classical expression of quantum gravity should be expressed by an effective Lagrangian using an interaction potential.

Hence, our suggestion is:

**Implication from the above:** The interaction potential in the effective Lagrangian of quantum gravity should satisfy the conditions A, B and \( I^2_V \geq 0 \), in order to regard a stationary black hole as a thermal equilibrium state of quantum gravity, whose entropy is given by the Boltzmann formula. The typical form of the effective potential is shown in Figure 1. This means that **the semi-classical correction to Einstein-Hilbert action expresses a repulsive gravity at a short length scale (which may be Planck scale)**.

In this paper, any existing model of quantum gravity is not used. (See [4] for detail.) Therefore, this implication can be regarded as a universal property of quantum gravity.

![Figure 1. Typical form of the effective potential of quantum gravity at semi-classical level. Thermal equilibrium states of gravity (black holes) may not exist unless a repulsive region appears, where \( r_A \) may be Planck scale.](image)

**References**

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\(^1\) So far, as a fact, no counter-example to conditions A and B is found experimentally. Then, it may be reasonable to extend this fact to quantum gravitating systems, so that quantum gravity also satisfies those conditions.