Geometric Schrödinger-Airy Flows on Kähler Manifolds

Xiaowei Sun and Youde Wang

Abstract

We define a class of geometric flows on a complete Kähler manifold to unify some physical and mechanical models such as the motion equations of vortex filament, complex-valued mKdV equations, derivative nonlinear Schrödinger equations etc. Furthermore, we consider the existence for these flows from $S^1$ into a complete Kähler manifold and prove some local and global existence results.

1 Introduction

Let $(N, J, h)$ be a complete Kähler manifold with complex structure $J$ and metric $h$. In this paper, we first introduce a class of new geometric flows from a circle $S^1$ or $\mathbb{R}$ into a complete Kähler manifold $N$. We will see that these flows are of strong physical and mechanical background and can be seen as the natural generalization or extension of some physical and mechanical models, for instance, the motion equations of vortex filament, complex-valued mKdV equations, Hirota equations, Schrödinger-Airy equations, derivative nonlinear Schrödinger equations etc. Therefore, to study these flows are of important physical and geometric significance. On the other hand, we may also provide some useful observation to these physical and mechanical equations from the view point of geometry. By virtue of the geometric observation we will employ some methods and techniques from geometric analysis to approach the existence problems for these flows and want to prove some results on the local and global existence for these geometric flows.

1.1 The definition of Schrödinger-Airy flows and background

For any smooth map $u(x, t)$ from $S^1 \times \mathbb{R}$ into $(N, J, h)$, Let $\nabla_x$ denote the covariant derivative $\nabla_{\partial_x}$ on the pull-back bundle $u^{-1}TN$ induced from the Levi-Civita connection $\nabla$ on $N$. For the sake of convenience, we always denote $\nabla_x u$ and $\nabla_t u$ by $u_x$ and $u_t$ respectively. The energy of a smooth map $v : S^1 \to N$ is defined as

$$E_1(v) \equiv \frac{1}{2} \int_{S^1} |v_x|^2 dx.$$ 

And the tension field of $v$ is written by $\tau(v) \equiv \nabla_x v_x$.

For the maps from a unit circle $S^1$ or a real line $\mathbb{R}$ into $N$, we define a class of geometric flows, which we would like to call geometric Schrödinger-Airy flow, as follows:

$$\frac{\partial u}{\partial t} = \alpha J_u \nabla_x u_x + \beta \left( \nabla_x^2 u_x + \frac{1}{2} R(u_x, J_u u_x) J_u u_x \right) + \gamma |u_x|^2 u_x,$$ (1.1)
where $\alpha$, $\beta$ and $\gamma$ are real constants, $R$ is the Riemannian curvature tensor on $N$ and $J_u \equiv J(u)$.

If $(N, J, h)$ is a locally Hermitian symmetric space, the geometric flow is an energy conserved system. Moreover, if $\gamma = 0$ it also preserves the following “pseudo-helicity” quantity

$$E_2(u) \equiv \int h(\nabla_x u_x, J u_x) dx.$$ 

In the case $\alpha = 1$ and $\beta = \gamma = 0$, the Schrödinger-Airy flow reduces to the Schrödinger flow from $S^1 \times \mathbb{R}$ or $\mathbb{R} \times \mathbb{R}$ into a Kähler manifold $(N, J, h)$ formulated by (see [3, 4, 12, 13, 19, 33])

$$\frac{\partial u}{\partial t} = J(u)\nabla_x u_x = J(u) \tau(u),$$

which is an infinite dimensional Hamilton system with respect to the energy functional. In particular, Rodnianski, Rubinstein and Staffilani in [36] established the global well-posedness of the initial value problem for the Schrödinger flow for maps from the real line into Kähler manifolds and for maps from the circle into Riemann surfaces.

If $\alpha = \gamma = 0$ and $\beta = 1$, (1.1) then reduces to the KdV geometric flow (see [38] for more details) on a Kähler manifold $(N, J, h)$ formulated by

$$\frac{\partial u}{\partial t} = \nabla_x^2 u_x + \frac{1}{2} R(u_x, J u_x) J u_x,$$

which is an infinite dimensional Hamilton system with respect to the pseudo-helicity functional.

The Schrödinger-Airy geometric flow (1.1) is a direct extension to a Kähler manifold of the following curve flow which is used to characterize the motion of vortex filament. The curve flow is about maps $u$ from $S^1 \times \mathbb{R}$ or $\mathbb{R} \times \mathbb{R}$ into Euclidean space $\mathbb{R}^3$ which satisfy the following evolution equation

$$\frac{\partial u}{\partial t} = \alpha u_s \times u_{ss} + \beta \left[ u_{sss} + \frac{3}{2} u_{ss} \times (u_s \times u_{ss}) \right].$$

(1.2)

More precisely, in [16], Fukumoto and Miyazaki discussed the motion of a thin vortex filament with axial velocity and reduced the equation of the vortex self-induced motion to a nonlinear evolution equation which can be formulated by using the Frenet frame of curve flow as

$$u_t = k b + \gamma \left( \frac{1}{2} k^2 t + k_s n + k \tau b \right).$$

(1.3)

Here $u = u(s, t)$ denotes an evolving filament curve from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}^3$ with arclength parameter $s$ and time $t$, $t$, $n$ and $b$ denote the unit tangent, normal and binormal vectors of the filament curve respectively; $k$ and $\tau$ denote the curvature and the torsion of the filament curve respectively, $k_s = \frac{dk}{ds}$ and $\gamma$ is a real constant. It is not difficult to see that by the Frenet-Serret formulas, (1.3) could also be reformulated as

$$u_t = u_s \times u_{ss} + \gamma \left[ u_{sss} + \frac{3}{2} u_{ss} \times (u_s \times u_{ss}) \right].$$

After rescaling with respect to $t$, the equation could be changed to (1.2). Thus, by Hasimoto transformation

$$\Psi = k \exp \left( i \int^s \tau ds' \right),$$

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the equation (1.2) would be reduced to the standard Schrödinger-Airy (or Hirota) equation (see [22, 38, 41])
\[ i\Psi_t + \alpha(\Psi_{ss} + \frac{1}{2} |\Psi|^2 \Psi) - i\beta(\Psi_{sss} + \frac{3}{2} |\Psi|^2 \Psi_s) = 0, \] (1.4)
which is a general model for propagation of pulses in an optical fiber. In the case \( \beta = 0 \), the equation (1.4) reduces to a cubic nonlinear Schrödinger equation and in the case \( \alpha = 0 \), (1.4) reduces to the modified KdV equation.

To see the inner relation between (1.1) and (1.2) more clearly, differentiating (1.2) with respect to \( s \) and and letting \( u(x, t) \equiv u_s(s, t) \) we obtain
\[ \frac{\partial u}{\partial t} = \alpha u \times u_{ss} + \beta \left[ u_{sss} + \frac{3}{2} (u_s \times (u \times u_s))_s \right]. \] (1.5)

One could verify that if given a smooth initial map \( u_0 = u(s, 0) \) into a unit sphere \( S^2 \), then the solution \( u \) to (1.5) will always be on \( S^2 \), i.e., the length of the tangent vector \( |u_s| \) is preserved (see [29]). So, the equation (1.5) describes the evolution of a geometric flow from \( \mathbb{R} \) or \( S^1 \) into \( S^2 \). Nishiyama and Tani have shown the time local and global existence of the initial value problem of (1.5) in [29] and [30] respectively.

It is not difficult to see that the Schrödinger-Airy geometric flow (1.1) is a natural generalization of (1.5). Indeed, for a map \( u(x) \) from \( S^1 \) or \( \mathbb{R} \) into a two dimensional standard unit sphere \( S^2 \),
\[ u \times : T_u S^2 \to T_u S^2 \]
is just the standard complex structure on \( S^2 \) and \( \tau(u) = \nabla_x u_x \) is the tangential part of \( u_{xx} \) on \( S^2 \). Meanwhile, a simple calculation shows that there hold
\[ R(u_x, J_{u_x})J_{u_x} = |u_x|^2 u_x, \]
\[ \nabla_x u_x = u_{xx} + (u_x, u_x)u, \]
\[ \nabla_x^2 u_x = \frac{d}{dx}(\nabla_x u_x) + (u_x, \nabla_x u_x)u = u_{xxx} + 3(u_x, u_{xx})u + |u_x|^2 u_x. \]
So,
\[ \nabla_x^2 u_x + \frac{1}{2} R(u_x, J_{u_x})J_{u_x} = u_{xxx} + \frac{3}{2} (u_x \times (u \times u_x))' \]
Hence, for the case \( N = S^2 \), the geometric Schrödinger-Airy flow (1.1) then reduces to
\[ \frac{\partial u}{\partial t} = \alpha J_u \nabla_x u_x + \beta \left( \nabla_x^2 u_x + \frac{1}{2} R(u_x, J_{u_x})J_{u_x} \right) \]
\[ = \alpha u \times u_{xx} + \beta \left[ u_{xxx} + \frac{3}{2} (u_x \times (u \times u_x))_x \right], \]
which is just the equation (1.5).
On the other hand, we recall that the derivative nonlinear Schrödinger (DNLS) equation

$$iq_t + q_{xx} = i(|q|^2q)_x, \quad x, t \in \mathbb{R} \quad (1.6)$$

where $q(x,t)$ is a complex-valued function, arises in the study of wave propagation in optical fibers [1] and in plasma physics [27] (see [8, 18, 26] for further references). Scattering and well-posedness for the Cauchy problem of this equation defined on $\mathbb{R}$ has been studied by many authors (see [39] and references therein). In particular, J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao [6, 7] showed that, under a smallness assumption on the $L^2$ norm of the initial data, this equation is globally well-posed in the energy space $H^1$.

While the propagation of nonlinear pulses in optical fibers is described to first order by the nonlinear Schrödinger equation, it is necessary when considering very short input pulses to include higher-order nonlinear effects. When the effect of self-steepening ($s \neq 0$) is included, the fundamental equation is

$$u_\tau = i\left(\frac{1}{2}u_{\xi\xi} + |u|^2u\right) - s(|u|^2u)_\xi \quad (1.7)$$

where $u$ is the amplitude of the complex field envelope, $\xi$ is a time variable, and $\tau$ measures the distance along the fiber with respect to a frame of reference moving with the pulse at the group velocity. Equation (1.6) is related to equation (1.7) by changing variables (see [26]):

$$u(\tau, \xi) = q(x,t) \exp\left(i\left(\frac{t}{4s^3} - \frac{x}{2s^2}\right)\right), \quad \tau = \frac{t}{2s^3}, \quad \xi = -\frac{x}{2s} + \frac{t}{2s^3}. \quad (\text{1.8})$$

It is an integrable equation and the initial value problem on the line can be analyzed by means of the inverse scattering transform as demonstrated by Kaup and Newell [25].

For a map $u(x)$ from $\mathbb{S}^1$ or $\mathbb{R}$ into a two dimensional standard unit sphere $S^2$, the equation

$$u_t = \alpha u \times u_{xx} + \gamma|u_x|^2u_x = \alpha J(u)\tau(u) + \gamma|u_x|^2u_x \quad (1.8)$$

is equivalent to the above derivative nonlinear Schrödinger equation. In fact, we could see this by adopting the generalized Hasimoto transformation used in [4].

Precisely, if we assume that $N$ is a compact closed Riemann surface and assume $u(x,t) \in H^2_0(\mathbb{R}, N)$ (the definition of $H^2_0(\mathbb{R}, N)$ is given in section 1.2) is a smooth solution of (1.8) on $\mathbb{R} \times [0, T)$ such that $u(x,t) \to Q \in N$ as $x \to \infty$. Let $\{e, Je\}$ denote the orthonormal frame for $u^{-1}TN$ constructed in the following manner: Fix a unit vector $e_0 \in T_QN$, and for any $t \in [0, T)$, let $e(x,t) = u(x,t)\lambda_0 e_0$ be the parallel translation of $e_0$ along the curve $u(\cdot,t)$, i.e., $\nabla_x e = 0$ and $\lim_{x \to \infty} e(x,t) = e_0$. In local conformal coordinates $z$, with $z(Q) = 0$, after fixing the coordinates of $e_0$ to be $\zeta_0 = \frac{1}{\lambda} e^{i\phi}$, the coordinates of the vector $e$ are given by $\zeta = \frac{1}{\lambda} e^{i\phi}$ where

$$\phi = \int_{-\infty}^{x} \text{Im}\left(\frac{\lambda_z}{\lambda} z_x\right) dx'$$

The expression of $\zeta$ and $\phi$ is derived by solving the equation $\nabla_x e = 0$, i.e.,

$$\zeta_x + 2(\log \lambda)z_x \zeta = 0.$$
Note that since \( \{e, Je\} \) is an orthonormal frame, we have that \( \nabla_t e = \eta Je \), where \( \eta \) is a real-valued function. Furthermore, in this frame the coordinates of \( u_t \) and \( u_x \) are given by two complex valued functions \( p \) and \( q \). We set as follows: first let

\[
\begin{align*}
  u_t &= p_1 e + p_2 Je, \\
  u_x &= q_1 e + q_2 Je,
\end{align*}
\]

where \( p_1, p_2, q_1, q_2 \) are real-valued functions of \((x, t)\), and let \( p = p_1 + ip_2 \) and \( q = q_1 + iq_2 \). Since \( \nabla_x e = 0 \), it is easy to see that \( \nabla_x u_x = q_{1x} e + q_{2x} Je \). Thus, by (1.8) we have

\[
p_1 e + p_2 Je = \alpha (-q_{2x} e + q_{1x} Je) + \gamma |q|^2 (q_1 e + q_2 Je),
\]

which is equivalent to

\[
p = i\alpha q_x + \gamma |q|^2 q. \tag{1.9}
\]

From \( \nabla_x u_t = \nabla_t u_x \), we have

\[
p_x = q_t + i\eta q. \tag{1.10}
\]

Combining (1.9) and (1.10), we obtain

\[
q_t = i\alpha q_{xx} + \gamma (|q|^2 q)_x - i\eta q. \tag{1.11}
\]

To get the expression of \( \eta \), we note that

\[
R(u_t, u_x) e = \nabla_t \nabla_x e - \nabla_x \nabla_t e = -\nabla_x (\eta Je) = -\eta_x Je.
\]

On the other hand

\[
R(u_t, u_x) e = (p_1 q_2 - p_2 q_1) R(e, Je) e = K(u) Im(p\bar{q}) Je,
\]

where \( K(u) = R(e, Je, e, Je) = h(e, R(e, Je) Je) \) is the Gaussian curvature of \( N \) at \( u(x, t) \). Thus we have

\[
\eta_x = -K(u) Im(p\bar{q}). \tag{1.12}
\]

Substituting (1.9) into (1.12) yields

\[
\eta_x = -K(u) Im(i\alpha q_x \bar{q}) = -\frac{\alpha}{2} K(u) (|q|^2)_x.
\]

Integrating this over \(( -\infty, x ] \) yields

\[
\eta(x, t) = -\frac{\alpha}{2} K(u) |q|^2 + \frac{\alpha}{2} \int_{-\infty}^{x} (K(u))_{x} (x', t) |q|^2 (x', t) dx' - \eta(-\infty, t). \tag{1.13}
\]

Thus, combining (1.11) and (1.13) yields

\[
q_t = i\alpha \left( q_{xx} + \frac{K(u)}{2} |q|^2 q \right) + \gamma (|q|^2 q)_x.
\]
Let $\Psi = q e^{i \int_0^t \eta(\tau, t') dt'}$ and we could easily see that

$$
\Psi_t = \frac{i\alpha}{2} \left( \Psi_{xx} + \frac{K(u)}{2} |\Psi|^2 \Psi \right) + \gamma \left( |\Psi|^2 \Psi \right)_x - \frac{i\alpha}{2} q \int_{-\infty}^x (K(u))_x(x', t)|\Psi|^2(x', t) dx' - i q \eta(\infty, t).
$$

(1.14)

It is easy to see that if $N$ is a Riemann surface with constant sectional curvature $K$, (1.14) is reduced to

$$
\Psi_t = \frac{i\alpha}{2} \left( \Psi_{xx} + \frac{K}{2} |\Psi|^2 \Psi \right) + \beta \left( \Psi_{xxx} + \frac{3K}{2} |\Psi|^2 \Psi_x \right) + \gamma \left( |\Psi|^2 \Psi \right)_x.
$$

(1.15)

This is why we call the geometric flow as geometric Schrödinger-Airy flow.

### 1.2 Main results and some notations

In this paper, we confine us to the case $\gamma = 0$. First, we discuss the local existence for the Cauchy problem of geometric Schrödinger-Airy flow from $S^1$ into a Kähler manifold $(N, J, h)$ defined by

$$
\begin{cases}
  u_t = \alpha J_u \nabla_x u_x + \beta \left( \nabla_x^2 u_x + \frac{1}{2} R(u_x, J_u u_x)J_u u_x \right), \quad x \in S^1; \\
  u(x, 0) = u_0(x),
\end{cases}
$$

(1.15)

where $\alpha, \beta$ are real positive constants. Furthermore, when $N$ is some kind of special locally Hermitian symmetric spaces, we could obtain some results on global existence of (1.15). The method we use here is the same as that we employed to discuss the KdV geometric flow in [38]. Remark that we only consider the case that the domain is $S^1$ in this paper and we could get similar results with those about the KdV flow.

Before stating our main results, we introduce several definitions on Sobolev spaces with vector bundle value. Let $(E, M, \pi)$ be a vector bundle with base manifold $M$. If $(E, M, \pi)$ is equipped with a metric, then we may define so-called vector bundle value Sobolev spaces as follows:
Definition 1.1. $H^m(M, E)$ is the completeness of the set of smooth sections with compact supports denoted by \{s \mid s \in C_0^\infty(M, E)\} with respect to the norm
\[
\|s\|_{H^m(M, E)}^2 = \sum_{i=0}^m \int_M |\nabla^i s|^2 dM.
\]
Here $\nabla$ is the connection on $E$ which is compatible with the metric on $E$.

Definition 1.2. Let $\mathbb{N}$ be the set of positive integers. For $m \in \mathbb{N} \cup \{0\}$, the Sobolev space of maps from $S^1$ into a Riemannian manifold $(N, h)$ is defined by
\[
H^{m+1}(S^1; N) = \{u \in C(S^1; N) \mid u_x \in H^m(S^1; TN)\},
\]
where $u_x \in H^m(S^1; TN)$ means that $u_x$ satisfies
\[
\|u_x\|_{H^m(S^1; TN)}^2 = \sum_{j=0}^m \int_{S^1} h(u(x))(\nabla_j u_x(x), \nabla_j u_x(x)) dx < +\infty.
\]
Similarly, we define

Definition 1.3. The Sobolev space of maps from $\mathbb{R}$ into a Riemannian manifold $(N, h)$ is defined by
\[
H^{m+1}(\mathbb{R}; N) = \{u \in C(\mathbb{R}; N) \mid u_x \in H^m(\mathbb{R}; TN)\},
\]
where $u_x \in H^m(\mathbb{R}; TN)$ means that $u_x$ satisfies
\[
\|u_x\|_{H^m(\mathbb{R}; TN)}^2 = \sum_{j=0}^m \int_{\mathbb{R}} h(u(x))(\nabla_j u_x(x), \nabla_j u_x(x)) dx < +\infty;
\]
and
\[
H_Q^{m+1}(\mathbb{R}; N) = \{u \in C(\mathbb{R}; N) \mid d_h(u(x), Q) \in L^2(\mathbb{R}), u_x \in H^m(\mathbb{R}; TN)\},
\]
where $d_h(u(x), Q)$ denotes the distance between $u(x)$ and $Q$.

We usually use $W^{k,p}(M, N)$ to denote the space of Sobolev maps from $M$ into $N$, and $W^{k,p}(M, \mathbb{R}^l)$ to denote the space of Sobolev functions.

Our main results are as follows:

Theorem 1.1. Let $(N, J, h)$ be a complete Kähler manifold. Then, the local solutions $u \in L^\infty([0, T], H^k(S^1, N))$ ($k \geq 4$) of the Cauchy problem (1.15) with the initial map $u_0 \in H^k(S^1, N)$ is unique. Moreover, the local solution is continuous with respect to the time variable, i.e., $u \in C([0, T], H^k(S^1, N))$.

Theorem 1.2. If $(N, J, h)$ is a noncompact complete Kähler manifold, then, for any integer $k \geq 4$ the Cauchy problem of (1.15) with the initial value map $u_0 \in H^k(S^1, N)$ admits a unique local solution $u \in C([0, T], H^k(S^1, N))$, where $T = T(N, \|u_0\|_{H^k})$. Moreover, if the initial value map $u_0 \in H^3(S^1, N)$ and $N$ is a complete Kähler manifold with $|\nabla^l R| \leq B_l$ ($l = 0, 1, 2, 3$) where $B_l$ are positive constants, then the Cauchy problem of (1.15) admits a local solution $u \in L^\infty([0, T], H^3(S^1, N))$, where $T = T(N, \|u_0\|_{H^3})$. 

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Theorem 1.3. Assume that \((N, J, h)\) is a complete locally Hermitian symmetric space satisfying
\[ h(R(Y, X)X, R(X, JX)JX) \equiv 0, \]
where \(R(\cdot, \cdot)\) is the Riemannian curvature operator on \(N\). Then for any integer \(k \geq 4\) the Cauchy problem (1.15) with the initial map \(u_0 \in H^k(S^1, N)\) admits a unique global solution \(u \in C([0, \infty), H^k(S^1, N))\).

Remark 1. If \(N = M_1 \times M_2 \times \cdots \times M_n\) is a product manifold where \((M_i, J_i, h_i)\) \((i = 1, 2, \cdots, n)\) are all manifolds satisfy the conditions in theorem 1.3, then the results in theorem 1.3 still hold true for \(N\).

Remark 2. We have shown in [38] that the identity on Riemannian curvature in Theorem 1.3 holds on Kähler manifolds with constant holomorphic sectional curvature, complex Grassmannians, the first class of bounded symmetric domains. The examples of Kähler manifolds with constant holomorphic sectional curvature are \(\mathbb{C}^k\), the flat complex torus \(\mathbb{C}T_l\), the complex projective spaces \(\mathbb{C}P^m\), complex hyperbolic spaces \(\mathbb{C}H^n\) and the compact quotient spaces of complex hyperbolic space modulo by a torsion free discrete subgroup of automorphism group of \(\mathbb{C}H^n\) etc.

Remark 3. It seems that the uniqueness results may be true for the local solution to the Cauchy problem of Schrödinger-Airy flow \(u \in L^\infty([0, T], H^3(S^1, N))\). If so, we can also improve the existence results and the regularity of solution. In particular, the uniqueness of solutions to the KdV flows from \(S^1\) does not depend on the the parabolic approximation. Maybe, one could find a different method to improve regularity.

As in [38], we still adopt the parabolic approximation and employ the geometric energy method developed in [12, 13] to show these local existence problems. To prove the global existence we need to exploit the following conservation laws \(E_1(u)\), \(E_3(u)\) and semi-conservation law \(E_4(u)\) where
\[
E_1(u) = \frac{1}{2} \int h(u_x, u_x) dx, \\
E_3(u) = \int h(\nabla_x u_x, \nabla_x u_x) dx - \frac{1}{4} \int h(u_x, R(u_x, J_u_x)J_u_x) dx, \\
E_4(u) = 2 \int h(\nabla_x^2 u_x, \nabla_x^2 u_x) dx - 3 \int h(\nabla_x u_x, R(\nabla_x u_x, u_x)u_x) dx - 5 \int h(\nabla_x u_x, R(\nabla_x u_x, J_u_x)J_u_x) dx.
\]

If \(N\) is a locally Hermitian symmetric space, for the smooth solution \(u\) to the Cauchy problem (1.15) we will establish the following in Section 3:
\[
\frac{d}{dt} E_1(u) = 0, \quad \frac{d}{dt} E_3(u) = \int h(R(\nabla_x u_x, u_x)u_x, R(u_x, J_u_x)J_u_x) dx, \\
\frac{d}{dt} E_4(u) \leq C(N, E_1(u_0), E_3(u_0))(1 + E_4).
\]
From (1.16) we deduce that when $N$ is a locally Hermitian symmetric space satisfying some geometric condition (see Corollary (3.5)), there holds true $E_3(u) = E_3(u_0)$. We utilize these conservation laws with respect to $E_1(u)$ and $E_3(u)$ to get a uniform a priori bound of $||\nabla_x u_x||_{L^2}$ independent of $T$. By virtue of (1.17), we obtain the global existence results.

This paper is organized as follows: In Section 2 we employ the geometric energy method to establish the local existence of Schrödinger-Airy geometric flow. We know that the conservation and semi-conservation laws mentioned before are crucial for us to establish the global existence of the Cauchy problem of Schrödinger-Airy geometric flow. We will give a detailed calculation in Section 3. The global existence of Schrödinger-Airy geometric flows on some special Kähler manifolds is proved in Section 4.

2 Local Existence and Uniqueness

In this section we establish the local existence and the uniqueness of solutions for the Cauchy problem of the Schrödinger-Airy flow (1.15) on a Kähler manifold $(N, J, h)$

$$\begin{cases}
  u_t = \alpha J_u \nabla_x u_x + \beta \left( \nabla_x^2 u_x + \frac{1}{2} R(u_x, J u_x) J u_x \right), & x \in S^1; \\
  u(x, 0) = u_0(x).
\end{cases}$$

We still use the approximate method as in [38] to show the local existence of (1.15). To this end, we discuss the following Cauchy problem:

$$\begin{cases}
  u_t = -\varepsilon \nabla_x^4 u_x + \alpha J_u \nabla_x u_x + \beta \left( \nabla_x^2 u_x + \frac{1}{2} R(u_x, J u_x) J u_x \right), & x \in S^1; \\
  u(x, 0) = u_0(x).
\end{cases} \tag{2.1}$$

where $\varepsilon > 0$ is a small positive constant.

We could imbed $N$ into a Euclidean space $\mathbb{R}^n$ for some large positive integer $n$. Then $N$ could be regarded as a sub-manifold of $\mathbb{R}^n$ and $u : S^1 \times \mathbb{R} \to N \subset \mathbb{R}^n$ could be represented as $u = (u^1, \ldots, u^n)$ with $u^i$ being globally defined functions on $S^1$ so that the Sobolev-norms of $u$ make sense. We have

$$||u||_{W^{m,2}}^2 = \sum_{i=0}^m ||D^i u||_{L^2}^2,$$

where $D$ denotes the covariant derivative for functions on $S^1$. The equation (2.1) then becomes a fourth order parabolic system in $\mathbb{R}^n$. In the appendix of [38], we have shown that the parabolic equation admits a local solution $u_\varepsilon \in C([0, \infty), W^{k,2}(S^1, N))$ if the initial value map $u_0 \in W^{k,2}(S^1, N)$ where $k \geq 3$.

Thus, to prove the local existence of (1.15), we just need to find a uniform positive lower bound $T$ of $T_\varepsilon$ and uniform bounds for various norms of $u_\varepsilon(t)$ in suitable spaces for $t$ in the time interval $[0, T)$. Once we get these bounds it is clear that $u_\varepsilon$ subconverge to a strong solution of (1.15) as $\varepsilon \to 0$. 

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Throughout this paper, we simply denote $h(X, Y)$ by $\langle X, Y \rangle$ for all $X, Y \in \Gamma(u^{-1}TN)$. Note that if $X \in \Gamma(u^{-1}TN)$ we have in local coordinates

$$\left(\nabla_{x}X\right)^{\alpha} = \frac{\partial X^{\alpha}}{\partial x} + \Gamma_{\beta\gamma}^{\alpha}(u) \frac{\partial u^{\beta}}{\partial x} X^{\gamma}$$

and for $X = u_{x}$ we have

$$\left(\nabla_{t}u_{x}\right)^{\alpha} = \frac{\partial^{2}u^{\alpha}}{\partial t \partial x} + \Gamma_{\beta\gamma}^{\alpha}(u) \frac{\partial u^{\beta}}{\partial t} \frac{\partial u^{\gamma}}{\partial x}.$$ 

It is easy to see that $\nabla_{t}u_{x} = \nabla_{x}u_{t}$. Now let $u = u_{c}$ be a solution of (2.1). We have the following results.

**Lemma 2.1.** (i) Assume that $N$ is a complete Kähler manifold with uniform bounds on the curvature tensor $R$ and its covariant derivatives of any order (i.e., $|\nabla^{l}R| \leq B_{l}$, $l = 0, 1, 2, \ldots$), and $u_{0} \in H^{k}(S^{1}, N)$ with an integer $k \geq 3$. Then there exists a constant $T = T(||u_{0}||_{H^{3}})$, independent of $\varepsilon \in (0, 1)$, such that if $u \in C([0, T_{\varepsilon}), H^{k}(S^{1}, N))$ is a solution of (2.1) with $\varepsilon \in (0, 1)$, then $T(||u_{0}||_{H^{3}}) \leq T_{\varepsilon}$ and $||u(t)||_{H^{m+1}} \leq C(||u_{0}||_{H^{m+1}})$ for any integer $2 \leq m \leq k-1$.

(ii) Assume that $N$ is a complete Kähler manifold and $u_{0} \in H^{k}(S^{1}, N)$ with an integer $k \geq 5$. Then there exists a constant $T = T(||u_{0}||_{H^{5}}) > 0$, independent of $\varepsilon \in (0, 1)$, such that if $u \in C([0, T_{\varepsilon}), H^{k}(S^{1}, N))$ is a solution of (2.1) with $\varepsilon \in (0, 1)$, then $T(||u_{0}||_{H^{5}}) \leq T_{\varepsilon}$ and $||u(t)||_{H^{m+1}} \leq C(||u_{0}||_{H^{m+1}})$ for any integer $2 \leq m \leq k-1$.

**Proof.** First fix a $k \geq 3$ and let $m$ be any integer with $2 \leq m \leq k-1$. We may assume that $u_{0}$ is $C^{\infty}$ smooth. Otherwise, we always choose a sequence of smooth functions $\{u_{0}^{l}\}$ such that $u_{0}^{l} \to u_{0}$ with respect to the norms $|| \cdot ||_{H^{k}}$ where $k \geq 3$.

As $N$ may not be compact we let $\Omega \triangleq \{p \in N : \text{dist}_{N}(p, u_{0}(S^{1})) < 1\}$, which is an open subset of $N$ with compact closure $\bar{\Omega}$. Let

$$T' = \sup \{t > 0 : u(S^{1}, t) \subset \Omega\}.$$ 

Now prove that for $k = 3$,

$$\frac{d}{dt}||u_{x}||_{H^{2}}^{2} \leq C(\Omega, \alpha, \beta) \sum_{l=2}^{4} ||u_{x}||_{H^{2}}^{2l},$$

for all $t \in [0, T_{\varepsilon}]$.

We differentiate each term in $||u_{x}||_{H^{2}}^{2}$ with respect to $t$. We have

$$\frac{d}{dt} \int |u_{x}|^{2}dx = 2 \int \langle u_{x}, \nabla_{t}u_{x} \rangle dx \quad = \quad 2 \int \langle u_{x}, \nabla_{x}u_{t} \rangle dx - 2 \int \langle \nabla_{x}u_{x}, u_{t} \rangle dx.$$ 

Substituting (2.1) yields

$$\frac{d}{dt} \int |u_{x}|^{2}dx$$

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Thus substituting Eq. (2.1) into the equality there will appear three parts: the first part that contains $\varepsilon$, the Schrödinger part which contains $\alpha$ and the KdV part which contains $\beta$. We need to estimate each terms in these three parts. When dealing with terms from the KdV part, we will use the results in [38] directly since the calculations are the same.

Now we consider $\int |\nabla_x u_x|^2 dx$. Differentiating it with respect to $t$ yields

$$\frac{d}{dt} \int |\nabla_x u_x|^2 dx = 2 \int (\nabla_x u_x, \nabla_t \nabla_x u_x) dx$$

$$= 2 \int (\nabla_x u_x, \nabla_x^2 u_x) dx + 2 \int (\nabla_x u_x, R(u_t, u_x)u_x) dx$$

$$= 2 \int (\nabla_x u_x, u_t) dx + 2 \int (u_t, R(\nabla_x u_x, u_x) u_x) dx.$$  

Thus substituting Eq. (2.1) yields

$$\frac{d}{dt} \int |\nabla_x u_x|^2 dx$$

$$= -2\varepsilon \int |\nabla_x^3 u_x|^2 dx - 2\varepsilon \int (\nabla_x^3 u_x, R(\nabla_x u_x, u_x) u_x) dx$$

$$+ \alpha \left( 2 \int (\nabla_x^3 u_x, J \nabla_x u_x) dx + 2 \int (J \nabla_x u_x, R(\nabla_x u_x, u_x) u_x) dx \right)$$

$$+ \beta \left( 2 \int (\nabla_x^3 u_x, \nabla_x \nabla_x u_x) dx + \int (\nabla_x^3 u_x, R(u_x, J u_x) u_x) dx \right)$$

$$+ 2 \int (\nabla_x^2 u_x, R(\nabla_x u_x, u_x) u_x) dx$$

$$= \int (R(u_x, J u_x) J u_x, R(\nabla_x u_x, u_x) u_x) dx.$$  

(2.4)

For the second term of the right hand side, integrating by parts yields

$$- 2\varepsilon \int (\nabla_x^2 u_x, R(\nabla_x u_x, u_x) u_x) dx$$

$$= 2\varepsilon \int (\nabla_x^2 u_x, (\nabla_x R)(\nabla_x u_x, u_x) u_x) dx + 2\varepsilon \int (\nabla_x^2 u_x, R(\nabla_x^2 u_x, u_x) u_x) dx$$

$$+ 2\varepsilon \int (\nabla_x^2 u_x, R(\nabla_x u_x, u_x) \nabla_x u_x) dx.$$
\begin{equation}
\leq C(\Omega) \left( \int |\nabla_x^2 u_x||\nabla_x u_x||u_x|^3 dx + \int |\nabla_x^2 u_x||u_x|^2 dx + \int |\nabla_x^2 u_x||\nabla_x u_x||u_x|^2 dx \right). \tag{2.5}
\end{equation}

For the Schrödinger part on the right of (2.4), we have
\begin{align*}
\alpha & \left( 2 \int \langle \nabla_x^2 u_x, J\nabla_x u_x \rangle dx + 2 \int \langle J\nabla_x u_x, R(\nabla_x u_x, u_x) u_x \rangle dx \right) \\
& = -2\alpha \int \langle \nabla_x^2 u_x, J\nabla_x^2 u_x \rangle dx - 2\alpha \int \langle \nabla_x u_x, R(\nabla_x u_x, u_x) J u_x \rangle dx \\
& \leq C(\Omega, \alpha) \int |\nabla_x u_x|^2 |u_x|^2. \tag{2.6}
\end{align*}

For the KdV part in (2.4), after integrating by parts we have
\begin{align*}
\beta & \left( 2 \int \langle \nabla_x^2 u_x, \nabla_x^3 u_x \rangle dx + \int \langle \nabla_x^3 u_x, R(u_x, J u_x) J u_x \rangle dx \\
& + 2 \int \langle \nabla_x^2 u_x, R(\nabla_x u_x, u_x) u_x \rangle dx \\
& + \int \langle R(u_x, J u_x) J u_x, R(\nabla_x u_x, u_x) u_x \rangle dx \right) \\
& \leq C(\Omega) \left( \int |\nabla_x^2 u_x||\nabla_x u_x||u_x|^2 dx + \int |\nabla_x^2 u_x||u_x|^4 dx + \int |\nabla_x u_x||u_x|^5 dx \right). \tag{2.7}
\end{align*}

Using Hölder inequality and interpolation inequalities, Eqs. (2.4)–(2.7) yield
\begin{equation}
\frac{d}{dt} \int |\nabla_x u_x|^2 dx + 2\varepsilon \int |\nabla_x^2 u_x|^2 dx \leq C(\Omega) ||u_x||_{H^2}^4. \tag{2.8}
\end{equation}

Next we compute \( \int |\nabla_x^2 u_x|^2 dx \). We have
\begin{align*}
\frac{d}{dt} \int |\nabla_x^2 u_x|^2 dx & = 2 \int \langle \nabla_x \nabla_x^2 u_x, \nabla_x^2 u_x \rangle dx \\
& = 2 \int \langle \nabla_x \nabla_x^2 u_x, \nabla_x^2 u_x \rangle dx + 2 \int \langle R(u_t, u_x) \nabla_x u_x, \nabla_x^2 u_x \rangle dx \\
& = -2 \int \langle \nabla_x^2 u_x, \nabla_x^3 u_x \rangle dx - 2 \int \langle \nabla_x^3 u_x, R(u_t, u_x) u_x \rangle dx \\
& + 2 \int \langle \nabla_x^2 u_x, R(u_t, u_x) \nabla_x u_x \rangle dx \\
& = -2 \int \langle u_t, \nabla_x^5 u_x \rangle dx - 2 \int \langle u_t, R(\nabla_x^3 u_x, u_x) u_x \rangle dx \\
& + 2 \int \langle u_t, R(\nabla_x^2 u_x, \nabla_x u_x) u_x \rangle dx. \tag{2.9}
\end{align*}

Substituting (2.1) into (2.9) while noting that
\[ \int \langle J\nabla_x u_x, \nabla_x^3 u_x \rangle dx = \int \langle J\nabla_x^3 u_x, \nabla_x u_x \rangle dx = 0; \]
we have
\[
\frac{d}{dt} \int |\nabla^2 u_x|^2 dx + 2\varepsilon \int |\nabla^4 u_x|^2 dx
= 2\varepsilon \int \langle \nabla^3 u_x, R(\nabla^3 u_x, u_x) \rangle dx - 2\varepsilon \int \langle \nabla^3 u_x, R(\nabla^2 u_x, \nabla u_x) \rangle dx + I_0,
\]
\[\tag{2.10}\]
where
\[
I_0 = -\alpha \left( 2 \int \langle J \nabla u_x, R(\nabla^3 u_x, u_x) \rangle dx - 2 \int \langle J \nabla u_x, R(\nabla^2 u_x, \nabla u_x) \rangle dx \right)
- \beta \left( \int \langle \nabla^3 u_x, (u_x, Ju_x)J u_x \rangle dx + 2 \int \langle \nabla^2 u_x, R(\nabla^3 u_x, u_x) \rangle dx \right.
+ \int \langle (u_x, Ju_x)J u_x, R(\nabla^2 u_x, \nabla u_x) \rangle dx - 2 \int \langle \nabla^2 u_x, R(\nabla^2 u_x, \nabla u_x) \rangle dx
- \left. \int \langle (u_x, Ju_x)J u_x, R(\nabla^2 u_x, \nabla u_x) \rangle \right). \]

For the first term of (2.10), integrating by parts yields
\[
2\varepsilon \int \langle \nabla^3 u_x, R(\nabla^3 u_x, u_x) \rangle dx
= -2\varepsilon \int \langle \nabla^3 u_x, \nabla R(\nabla^2 u_x, u_x) \rangle dx - 2\varepsilon \int \langle \nabla^3 u_x, R(\nabla^2 u_x, \nabla u_x) \rangle dx
- 2\varepsilon \int \langle \nabla^3 u_x, R(\nabla^2 u_x, \nabla u_x) \rangle dx. \tag{2.11}\]
Thus, for any \(\delta > 0\), (2.11) together with the second term of (2.10) yields
\[
2\varepsilon \int \langle \nabla^3 u_x, R(\nabla^3 u_x, u_x) \rangle dx - 2\varepsilon \int \langle \nabla^3 u_x, R(\nabla^2 u_x, \nabla u_x) \rangle dx \leq \varepsilon \delta \int |\nabla^3 u_x|^2 dx + 4\varepsilon \delta \int |\nabla^3 u_x|^2 dx
+ \frac{\varepsilon}{2\delta} \left( \int |(\nabla R) \nabla^2 u_x, u_x \rangle^2 dx + \int |R(\nabla^2 u_x, u_x)u_x \rangle^2 dx \right.
+ \int |R(\nabla^2 u_x, \nabla u_x)w |^2 dx + 2 \int |R(\nabla^2 u_x, \nabla u_x)w |^2 dx \right)
\leq \varepsilon \delta \int |\nabla^3 u_x|^2 dx + 4\varepsilon \delta \int |\nabla^3 u_x|^2 dx + \frac{C(\Omega)}{2\delta} \int |\nabla^2 u_x|^2 |u_x|^4 + |\nabla u_x|^2 |u_x|^2) \int dx
\leq \varepsilon \delta \int |\nabla^3 u_x|^2 dx + 4\varepsilon \delta \int |\nabla^3 u_x|^2 dx + \frac{C(\Omega)}{2\delta} (||u_x||_{L^2}^4 + ||u_x||_{L^2}^6 + ||u_x||_{H^2}^6 + ||u_x||_{H^2}^8), \tag{2.12}\]
where we used the following interpolation inequalities
\[
||u_x||_{L^\infty} \leq C(\Omega)(||\nabla u_x||_{L^2}^2 + ||u_x||_{L^2}^2)^{\frac{1}{2}} ||u_x||_{L^2}^2; \]
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\[ ||\nabla_x u_x||_{L^\infty} \leq C(\Omega)(||\nabla_x^2 u_x||_{L^2} + ||\nabla_x u_x||_{H^2})^\frac{1}{2}||\nabla_x u_x||_{L^2}^\frac{1}{2}.\]

For the left part \( I_0 \) of (2.10), after integrating by parts repeatedly (see [38] for detail), we could obtain that

\[
I_0 \leq C(\Omega, \alpha) \left( \int |\nabla_x^2 u_x| |\nabla_x u_x|^2 |u_x| dx + \int |\nabla_x^2 u_x| |u_x|^3 dx \right) + C(\Omega, \beta) \left( \int |\nabla_x^2 u_x|^2 |\nabla_x u_x| |u_x| dx + \int |\nabla_x^2 u_x| |\nabla_x u_x|^3 |u_x| dx \right) + \int |\nabla_x^2 u_x| |\nabla_x u_x||u_x|^4 dx \right). \quad (2.13)
\]

Thus, by Hölder inequality and interpolation inequalities, we obtain the desired bound

\[ I_0 \leq C(\Omega, \alpha, \beta) \sum_{l=2}^{4} ||u_x||_{H^l}^2. \]

Hence we have

\[
\frac{d}{dt} \int |\nabla_x^2 u_x|^2 |dx + 2\varepsilon \int |\nabla_x^4 u_x|^2 |dx 
\leq \varepsilon \delta \int |\nabla_x^4 u_x|^2 |dx + 4\varepsilon \delta \int |\nabla_x^3 u_x|^2 |dx + C(\Omega, \alpha, \beta)\left( \frac{1}{2\delta} + 1 \right) \sum_{l=2}^{4} ||u_x||_{H^l}^2. \quad (2.14)
\]

In view of (2.3), (2.8), and (2.14), we have

\[
\frac{d}{dt} ||u_x||_{H^2}^2 + (2 - \delta)\varepsilon \int |\nabla_x^4 u_x|^2 |dx + (1 - 4\delta)\varepsilon \int |\nabla_x^3 u_x|^2 |dx 
\leq C(\Omega, \alpha, \beta)\left( \frac{1}{2\delta} + 1 \right) \sum_{l=2}^{4} ||u_x||_{H^l}^2. \]

Let \( \delta = \frac{1}{8} \) and we get the desired inequality (2.2).

For \( 3 \leq m \leq k - 1 \), by the similar process, we could get the following inequality

\[
\frac{d}{dt} ||u_x||_{H^m}^2 \leq C(\Omega, ||u_x||_{H^{m-1}}, \alpha, \beta)||u_x||_{H^m}^2, \quad (2.15)
\]

where \( C(\Omega, ||u_x||_{H^{m-1}}, \alpha, \beta) \) depends on \( \alpha, \beta, ||u_x||_{H^{m-1}} \) and the bounds on the curvature \( R \) and its covariant derivatives \( \nabla^l R \) with \( l \leq m + 1 \) on \( \Omega \subset N \). We omit the details of the proof.

Let \( f(t) = ||u_x||_{H^2}^2 + 1 \), then we have

\[
\frac{df}{dt} \leq C(\Omega, \alpha, \beta)f^4, \quad f(0) = ||u_0||^2 + 1. \quad (2.16)
\]
It follows from (2.16) that there exists constants $T_0 > 0$ and $C_0 > 0$ such that

$$
||u_x||_{H^2} \leq C_0, \quad t \in [0, \min(T_0, T')].
$$

Now let $T = \min(T_0, T')$. If $m = 3$, by the Gronwall inequality, we can obtain from (2.15):

$$
||u_x||_{H^3} \leq C_1(\Omega, T, ||u_{0x}||_{H^3}, \alpha, \beta), \quad \text{for all} \quad t \in [0, T].
$$

Then by induction we have that there exists a constant $C_{m-2}(\Omega, ||u_{0x}||_{H^m}) > 0$, such that for any $3 \leq m \leq k - 1$

$$
\text{ess sup}_{t \in [0,T]} ||u_x||_{H^m} \leq C_{m-2}(\Omega, ||u_{0x}||_{H^m}, \alpha, \beta). \quad (2.17)
$$

Since $\Omega$ is compact, consequently $||u(t)||_{L^\infty(S^1)}$ is uniformly bounded for $t \in [0, T]$.

If $N$ is of uniform bounds on the curvature tensor and its derivatives of any order, it is easy to see from the above arguments that $T = T_0$ since the coefficients of the above differential inequalities depend only on the bounds on Riemann curvature tensor $R$ and its covariant derivatives $\nabla^i R$ of some order on $N$. That is $T = T(S, ||u_0||_{H^3})$ depends only on $N, u_0$, not on $0 < \varepsilon < 1$.

Now we consider the case $N$ is a noncompact, complete Kähler manifold without the bounded geometry assumptions. Note that a positive lower bound of $T'$ can also be derived from (2.17) when $k \geq 5$. Indeed, it is easy to see from the approximate equation of Schrödinger-Airy flow and the interpolation inequality that (2.17) implies

$$
\text{ess sup}_{t \in [0,T]} ||u_t||_{L^2(S^1, T N)} \leq C(\Omega, ||u_{0x}||_{H^3}).
$$

On the other hand, from the approximate equation of Schrödinger-Airy flow we have

$$
\nabla_x u_t = -\varepsilon \nabla^2_x u_x + \alpha J \nabla^2_x u_x + \beta \left( \nabla^3_x u_x + \frac{1}{2} \nabla_x (R(u_x, Ju_x) Ju_x) \right).
$$

Hence, when $k \geq 5$ we infer from (2.17) and the interpolation inequality that

$$
\text{ess sup}_{t \in [0,T]} ||u_t||_{H^1(S^1, T N)} \leq C(\Omega, ||u_{0x}||_{H^4}, \alpha, \beta).
$$

However, by the interpolation inequality, for some $0 < a < 1$ there holds

$$
||u_t(s)||_{L^\infty} \leq C ||u_t(s)||_{H^1}^a ||u_t(s)||_{L^2}^{1-a}.
$$

This implies that, for some $M > 0$, there holds true

$$
\text{ess sup}_{t \in [0,T]} ||u_t||_{L^\infty} \leq M.
$$

Thus we have

$$
\sup_{x \in S^1} d_N(u(x, t), u_0(x)) \leq Mt, \quad \text{for} \quad t < T.
$$
If $T' > T_0$ we get the lower bound, so we may assume that $T' \leq T_0$. Then letting $t \to T'$ in the above inequality we get $\mathcal{M}T' \geq 1$. Therefore, if we set $T = \min\{\frac{1}{\mathcal{M}}, T_0\}$, then the desired estimates hold for $t \in [0, T]$.

It is easy to find that the solution to (2.1) with $\varepsilon \in (0, 1)$ must exists on the time interval $[0, T]$. Otherwise, we always extend the time interval of existence to cover $[0, T]$, i.e., we always have $T_\varepsilon \geq T$. Thus we complete the proof of this lemma.

**Lemma 2.2.** If $(N, J, h)$ is a complete Kähler manifold with uniform bounds on the curvature tensor and its covariant derivatives of any order (i.e., $|\nabla^l R| \leq B_l$, $l = 0, 1, 2, \cdots$), then, for any integer $k \geq 3$ the Cauchy problem of (1.15) with the initial value map $u_0 \in H^k(S^1, N)$ admits a local solution $u \in L^\infty([0, T], H^k(S^1, N))$, where $T = T(N, \|u_0\|_{H^3})$.

Before proving Lemma 2.2, we remark that in [13], Ding and Wang have shown that the $H^m$ norm of section $\nabla u$ defined in section 1.3 is equivalent to the usual Sobolev $W^{m+1, 2}$ norm of the map $u$. Precisely, we have

**Lemma 2.3.** ([13]) Assume that $N$ is a compact Riemannian manifold with or without boundary and $m \geq 1$. Then there exists a constant $C = C(N, m)$ such that for all $u \in C^\infty(S^1, N)$,

$$\|Du\|_{W^{m-1, 2}} \leq C \sum_{i=1}^m \|\nabla u\|_{H^{m-1, 2}}^i$$

and

$$\|\nabla u\|_{H^{m-1, 2}} \leq C \sum_{i=1}^m \|Du\|_{W^{m-1, 2}}^i.$$  

**Proof of Lemma 2.2.** We assume that $N$ is compact and embed $N$ into $\mathbb{R}^n$. If $u_0 : S^1 \to N$ is $C^\infty$, then from Lemma 2.1 we have that the Cauchy problem (2.1) admits a unique smooth solution $u_\varepsilon$ which satisfies the estimates in Lemma 2.1. Hence by Lemma 2.1 and Lemma 2.3 we have that for any integer $p > 0$ and $\varepsilon \in (0, 1)$:

$$\sup_{t \in [0, T]} \|u_\varepsilon\|_{W^{p, 2}(N)} \leq C_p(N, u_0),$$

(2.18)

where $C_p(N, u_0)$ does not depend on $\varepsilon$. Hence, by sending $\varepsilon \to 0$ and applying the embedding theorem of Sobolev spaces to $u$, we have $u_\varepsilon \to u \in C^p(S^1 \times [0, T])$ for any $p$. It is easy to check that $u$ is a solution to the Cauchy problem (2.1).

If $u_0 : S^1 \to N$ is not $C^\infty$, but $u_0 \in W^{k, 2}(S^1, N)$, we may always select a sequence of $C^\infty$ maps from $S^1$ into $N$, denoted by $u_{i0}$, such that

$$u_{i0} \to u_0 \quad \text{in} \quad W^{k, 2}, \quad \text{as} \quad i \to \infty.$$ 

Thus following from Lemma 2.3 we have

$$\|\nabla_x u_{i0}\|_{H^{k-1}} \to \|\nabla_x u_0\|_{H^{k-1}}, \quad \text{as} \quad i \to \infty.$$
Thus there exists a unique, smooth solution $u_i$, defined on time interval $[0, T_i]$, of the Cauchy problem \((2.1)\) with $u_0$ replaced by $u_0$. Furthermore, from the arguments in Lemma 2.1, we could obtain that if $i$ is large enough, then there exists a uniform positive lower bound of $T_i$, denoted by $T$, such that the following holds uniformly with respect to large enough $i$:

$$\sup_{t \in [0, T]} \|\nabla u_i(t)\|_{H^{k-1}} \leq C(T, \|u_{0x}\|_{H^{k-1}}).$$

Hence from Lemma 2.3 we deduce

$$\sup_{t \in [0, T]} \|Du_i(t)\|_{W^{k-2,2}} \leq C(T, \|u_{0x}\|_{W^{k-2,2}}), \tag{2.19}$$

and by \((2.1)\) we have

$$\frac{du_i}{dt} \in L^2([0, T], W^{k-3,2}(S^1, N)).$$

By Sobolev theorem, it is easy to see that $u_i \in C^{0, \frac{1}{2}}([0, T], W^{k-3,2}(S^1, N))$.

Interpolating the spaces $L^\infty([0, T], W^{k,2}(S^1, N))$ and $C^{0, \frac{1}{2}}([0, T], W^{k-3,2}(S^1, N))$ yields that

$$u_i \in C^{0, \gamma}(0, T], W^{k-6\gamma,2}(S^1, N)) \quad \text{for} \quad \gamma \in (0, \frac{1}{2}). \tag{2.20}$$

Therefore when letting $\gamma$ small while using Rellich’s theorem and the Ascoli-Arzela theorem, from \((2.19)\) and \((2.20)\) we obtain that there exists

$$u \in L^\infty([0, T], W^{k,2}(S^1, N)) \cap C([0, T], W^{k-1,2}(S^1, N))$$

such that

$$u_i \to u \quad \text{[weakly*]} \quad \text{in} \quad L^\infty([0, T], W^{k,2}(S^1, N)), \quad u_i \to u \quad \text{in} \quad C([0, T], W^{k-1,2}(S^1, N))$$

upon extracting a subsequence and re-indexing if necessary.

It remains to verify that $u$ is a strong solution to \((2.1)\). We need to check that for any $v \in C^\infty(S^1 \times [0, T], \mathbb{R}^n)$ there holds

$$\int_0^T \int_{S^1} \langle u_t, v \rangle \, dx \, dt = \alpha \int_0^T \int_{S^1} \langle J_u \nabla_x u_x, v \rangle \, dx \, dt$$

$$+ \beta \left( \int_0^T \int_{S^1} \langle \nabla_x^2 u_x, v \rangle \, dx \, dt + \frac{1}{2} \int_0^T \int_{S^1} \langle R_u(u_x, J_u u_x) J_u u_x, v \rangle \, dx \, dt \right).$$

First we always have that for each $u_i$

$$\int_0^T \int_{S^1} \langle u_{it}, v \rangle \, dx \, dt = \alpha \int_0^T \int_{S^1} \langle J_u \nabla_x u_{ix}, v \rangle \, dx \, dt$$

$$+ \beta \left( \int_0^T \int_{S^1} \langle \nabla_x^2 u_{ix}, v \rangle \, dx \, dt + \frac{1}{2} \int_0^T \int_{S^1} \langle R_u(u_{ix}, J_u u_{ix}) J_u u_{ix}, v \rangle \, dx \, dt \right).$$

For each $y \in N \subset \mathbb{R}^n$, let $P(y)$ be the orthogonal projection from $\mathbb{R}^n$ onto $T_y N$, we have

$$\nabla_x u_x = P(u) u_{xx},$$

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\[ \nabla^2_x u_x = P(u)(D(P(u)u_{xx})) = P(u)(P(u))_x u_{xx} + P(u)u_{xxx}, \]  
\[ \nabla^2_x u_{ix} = P(u_i)(P(u_i))_x u_{ixx} + P(u_i)u_{ixxx}. \]  

Hence we have

\[
\int_0^T \int_{S^1} |(J_u \nabla_x u_x, v) - (J_{u_i} \nabla_x u_{ix}, v)| dx dt 
\leq \int_0^T \int_{S^1} |(J_u - J_{u_i}) P(u)u_{xx}, v)| dx dt + \int_0^T \int_{S^1} |J_{u_i}(P(u) - P(u_i)) u_{xx}, v)| dx dt 
+ \int_0^T \int_{S^1} |J_{u_i} P(u_i)(u_{xx} - u_{ixx}), v)| dx dt; \tag{2.23}
\]

and

\[
\int_0^T \int_{S^1} |\langle \nabla^2_x u_x, v \rangle - \langle \nabla^2_x u_{ix}, v \rangle| dx dt 
\leq \int_0^T \int_{S^1} |\langle (P(u) - P(u_i))u_{xx}, v \rangle| dx dt + \int_0^T \int_{S^1} |\langle P(u_i)(u_{xx} - u_{ixx}), v \rangle| dx dt 
+ \int_0^T \int_{S^1} |\langle (P(u)(P(u))_x - P(u_i)(P(u_i)))_x u_{xx}, v \rangle| dx dt 
+ \int_0^T \int_{S^1} |\langle P(u_i)(P(u_i))_x (u_{xx} - u_{ixx}), v \rangle| dx dt. \tag{2.24}
\]

Moreover,

\[
\int_0^T \int_{S^1} |\langle R_u(u_x, J_{u}u_{x})J_u u_x, v \rangle - \langle R_{u_i}(u_{ix}, J_{u_i}u_{ix})J_{u_i} u_{ix}, v \rangle| dx dt 
\leq \int_0^T \int_{S^1} |R_u(u_x, J_u u_x)J_u u_x - R_{u_i}(u_{ix}, J_{u_i}u_{ix})J_{u_i} u_{ix}| v| dx dt. \tag{2.25}
\]

Since \( N \) is compact, it is obviously that

\[ \|R(\cdot)\|_{L^\infty(N)} < \infty \quad \text{and} \quad \|P(\cdot)D(P(\cdot))\|_{L^\infty(N)} < \infty. \]

Hence we obtain that each term on the right hand side of (2.23)–(2.25) converges zero as \( i \) goes to infinity. This implies that

\[
\lim_{i \to \infty} \int_0^T \int_{S^1} \langle J_u \nabla_x u_{ix}, v \rangle dx dt = \int_0^T \int_{S^1} \langle J_u \nabla_x u_x, v \rangle dx dt; 
\]

\[
\lim_{i \to \infty} \int_0^T \int_{S^1} \langle \nabla^2_x u_{ix}, v \rangle dx dt = \int_0^T \int_{S^1} \langle \nabla^2_x u_x, v \rangle dx dt; 
\]

\[
\lim_{i \to \infty} \int_0^T \int_{S^1} \langle R_{u_i}(u_{ix}, J_{u_i}u_{ix})J_{u_i} u_{ix}, v \rangle dx dt = \int_0^T \int_{S^1} \langle R_u(u_x, J_u u_x)J_u u_x, v \rangle dx dt. 
\]

On the other hand, we also have

\[
\lim_{i \to \infty} \int_0^T \int_{S^1} \langle u_{it}, v \rangle dx dt = - \int_0^T \int_{S^1} \langle u, v_t \rangle dx dt + \int_0^T \langle (u(T), v(T)) - \langle u_0, v(0) \rangle \rangle dx dt. 
\]
Thus, from the above equalities we have
\[
\alpha \int_0^T \int_{S^1} \langle J_u \nabla_x u_x, v \rangle \, dx dt + \beta \left( \int_0^T \int_{S^1} \langle \nabla_x^2 u_x, v \rangle \, dx dt + \frac{1}{2} \int_0^T \int_{S^1} \langle R(u_x, J_u x)J_u x, v \rangle \, dx dt \right)
\]
\[= - \int_0^T \int_{S^1} \langle u, v \rangle \, dx dt + \int_{S^1} \langle (u(T), v(T)) - (u_0, v(0)) \rangle \, dx dt.
\] (2.26)

Note that \( \nabla_x u_x \in L^2(S^1 \times [0, T], \mathbb{R}^n) \), thus (2.26) implies \( u_t \in L^2(S^1 \times [0, T], \mathbb{R}^n) \). Therefore for any smooth function \( v \) we always have
\[
\int_0^T \int_{S^1} \langle u_t, v \rangle \, dx dt = \alpha \int_0^T \int_{S^1} \langle J_u \nabla_x u_x, v \rangle \, dx dt
\]
\[+ \beta \left( \int_0^T \int_{S^1} \langle \nabla_x^2 u_x, v \rangle \, dx dt + \frac{1}{2} \int_0^T \int_{S^1} \langle R(u_x, J_u x)J_u x, v \rangle \, dx dt \right),
\]
which means that \( u \) is a strong solution of (2.1).

It is easy to see that if \( N \) is noncompact manifold with bounded geometry and the domain is \( S^1 \), we could find a compact subset of \( N \), denoted by \( \Omega \), such that \( u_0(S^1) \subset \Omega \subset \mathbb{R}^n \). Then we could repeat the same process as in the case \( N \) is compact (also see [?]), then we could obtain the same results.

Hence we complete the proof of Lemma 2.2. Moreover, when \( k \geq 4 \), Theorem 1.1 below asserts that the solutions are unique.

Now we prove Theorem 1.1 and show the uniqueness of the solutions.

**Proof of Theorem 1.1.** Without loss of generality, we may assume that \( N \) is compact, since \( u(x, t) \in L^\infty([0, T], H^4(S^1, N)) \) implies that \( \{u(x, t) : (x, t) \in S^1 \times [0, T]\} \subset N \). We regard \( N \) as a submanifold of \( \mathbb{R}^n \). Let \( u, v : S^1 \times [0, T] \to \mathbb{R}^n \) be two solutions of (2.1) such that \( u(x, 0) = v(x, 0) = u_0 \) and \( u, v \in L^\infty([0, T], W^{k,2}(S^1, N)) \) for \( k \geq 4 \). Let \( w = u - v \) and it makes sense as a \( \mathbb{R}^n \)-valued function. It is worthy to point out that both the curvature tensor \( R \) and the complex structure \( J \) here should be regarded as operators on \( \mathbb{R}^n \), such that \( R(u)(u_x, J_u x)J_u x - R(v)(v_x, J_v x)J_v x \) makes sense in \( \mathbb{R}^n \).

By the discussion, in Lemma 2.3, for the solution \( u \) of (1.15), we have
\[
u_t = \alpha J_u P(u)u_{xx} + \beta \left( P(u)u_{xxx} + P(u)(P(u))_xu_{xx} + \frac{1}{2} R(u)(u_x, J_u u_x)J_u u_x \right).
\]
Hence we have
\[
w_t = \alpha \left( J_u P(u)u_{xx} - J_v P(v)v_{xx} \right)
\]
\[+ \beta \left( P(u)w_{xxx} + [P(u) - P(v)]v_{xxx}\right.
\]
\[+ P(u)(P(u))_xw_{xx} + [P(u)(P(u))_x - P(v)(P(v))_x]v_{xx}
\]
\[+ \frac{1}{2} (R(u)(u_x, J_u u_x)J_u u_x - R(v)(v_x, J_v v_x)J_v v_x) \right).
\] (2.27)
We could show that there exists a constant $C$ which only depends on $N$, $\|u\|_{W^{4,2}}$, $\|v\|_{W^{4,2}}$, $\alpha$, and $\beta$ such that

$$
\frac{d}{dt}\|w\|^2_{W^{1,2}} \leq C(N, \|u\|_{W^{4,2}}, \|v\|_{W^{4,2}}, \alpha, \beta)\|w\|^2_{W^{1,2}}.
$$

Then by Gronwall’s inequality we could obtain that $w \equiv 0$ which yields the uniqueness of the solution. We omit the details about the proof of (2.28) since the process is almost the same with that in \[38\].

Thus it suffices to show that $\nabla^{k-1}u_x \in C([0, T]; L^2(S^1, T N))$ for $k \geq 4$. In the proof of Theorem 2.2 we have seen that the solution $u \in L^\infty([0, T], H^k(S^1, N)) \cap C([0, T], H^{k-1}(S^1, N))$, thus by the discussion about (2.18), (2.19) and the equation of Schrödinger-Airy flow, we could easily get that

$$
\frac{d}{dt}\|\nabla^{k-1}u_x\|^2_{L^2} \leq C,
$$

which implies that

$$
\|\nabla^{k-1}u_x(t, x)\|^2_{L^2(S^1, T N)} \leq \|\nabla^{k-1}u_x(0, x)\|^2_{L^2(S^1, T N)} + Ct.
$$

Hence we obtain

$$
\limsup_{t \to 0} \|\nabla^{k-1}u_x(t, x)\|^2_{L^2(S^1, T N)} \leq \|\nabla^{k-1}u_x(0, x)\|^2_{L^2(S^1, T N)}.
$$

On the other hand, $u \in L^\infty([0, T], H^k(S^1, N)) \cap C([0, T], H^{k-1}(S^1, N))$ implies that, with respect to $t$, $\nabla^{k-1}u_x(t, x)$ is weakly continuous in $L^2(S^1, T N)$, we have

$$
\|\nabla^{k-1}u_x(0, x)\|^2_{L^2(S^1, T N)} \leq \liminf_{t \to 0} \|\nabla^{k-1}u_x(t, x)\|^2_{L^2(S^1, T N)}.
$$

Thus,

$$
\lim_{t \to 0} \|\nabla^{k-1}u_x(t, x)\|^2_{L^2} = \|\nabla^{k-1}u_x(0, x)\|^2_{L^2},
$$

which implies that $\nabla^{k-1}u_x(t, x)$ is continuous in $L^2(S^1, T N)$ at $t = 0$. Now by the uniqueness of $u(t, x)$, we get that $\nabla^{k-1}u_x(t, x)$ is continuous at each $t \in [0, T]$, i.e. $u \in C([0, T], H^k(S^1, N))$ for all $k \geq 4$. However, if $k \leq 3$, we could not get the continuity of $\|u\|_{H^k}$ about $t$ on $[0, T]$ without the uniqueness of $u$. Thus we complete the proof of Theorem 1.1.

**Proof of Theorem 1.2.** To show the existence of the Cauchy problem (1.15) with an initial map $u_0 \in H^4(S^1, N)$, we need to consider the following Cauchy problems:

$$
\begin{align*}
&u_t = \alpha J \nabla u_x + \beta \left( \nabla_x^2 u_x + \frac{1}{2} R(u_x, J_u u_x) J_u u_x \right), \quad x \in S^1; \\
&u(x, 0) = u_0(x).
\end{align*}
$$

(2.29)

Here $u_0 \in C^\infty(S^1, N)$ and $\|u_0 - u_0\|_{H^4} \to 0$. By Lemma 3.1 we know that for each $i$ and any $k \geq 5$, (2.29) admits a local solution $u^i \in L^\infty([0, T_i^{\max}], H^k(S^1, N))$, where $T_i^{\max} = T_i^{\max}(S^1, \|u_0\|_{H^5})$ is the maximal existence interval of $u^i$. 20
As $N$ is not compact we let $\Omega_i \triangleq \{ p \in N : \text{dist}_N(p, u_0^i(S^1)) < 1 \}$, which is an open subset of $N$ with compact closure $\overline{\Omega}_i$. Denote $\Omega_\infty \triangleq \{ p \in N : \text{dist}_N(p, u_0^\infty(S^1)) < 1 \}$ and $\Omega_0 \triangleq \{ p \in N : \text{dist}_N(p, \Omega_\infty) < 1 \}$.

Since $\| u_0^i - u_0 \|_{H^4} \to 0$, then $\Omega_i \subset \subset \Omega_0$ as $i$ is large enough. Let $T'_i = \sup \{ t > 0 : u^i(S^1, t) \subset \Omega_i \}$.

By the same argument as in Lemma 3.1 we can show that for all $t \in [0, T_i]$ holds true
\[
\frac{d}{dt} \| u^i_x \|_{H^2}^2 \leq C(\Omega_0) \sum_{l=2}^{4} \| u^i_x \|_{H^2}^2.
\]
If we let $f^i(t) = \| u^i_x \|_{H^2}^2 + 1$, then we have
\[
\frac{df^i}{dt} \leq C(\Omega_0)(f^i)^4, \quad f^i(0) = \| u^i_{0x} \|^2 + 1.
\]
It follows from the above differential inequality that there holds true
\[
f^i(t) \leq \left( \frac{(f^i(0))^3}{1 - 3(f^i(0))^3 C(\Omega_0) t} \right)^{\frac{1}{3}},
\]
as
\[
t < \frac{1}{3(f^i(0))^3 C(\Omega_0)}.
\]
Then, there exists constants
\[
T_0^i = T_0^i(\Omega_0, \| u^i_{0x} \|_{H^2}) = \frac{1}{4(f^i(0))^3 C(\Omega_0)} > 0,
\]
\[
C_0^i = \frac{1}{4} f^i(0) > 0
\]
and $C_{k-2}^i = C_{k-2}(\| u^i_{0x} \|_{H^k}) > 0$ such that
\[
\| u^i_x \|_{H^2} \leq C_0^i, \quad t \in [0, \min(T_0^i, T'_i)].
\]
and for $k \geq 3$
\[
\| u^i_x \|_{H^k} \leq C_{k-2}^i, \quad t \in [0, \min(T_0^i, T'_i)].
\]
Since $\| u_0^i - u_0 \|_{H^4} \to 0$, when $i$ is large enough we have
\[
T_0^i = \frac{1}{4(\| u^i_{0x} \|_{H^2}^2 + 1 + \delta_0) C(\Omega_0)} < T_0^i,
\]
where $\delta_0$ is a small positive number. It is easy to see that, as $i$ is large enough,
\[
C_0^i \leq C_0(\| u^i_{0x} \|_{H^2}) + \delta_0 \quad \text{and} \quad C_1^i \leq C_1(\| u^i_{0x} \|_{H^3}) + \delta_0. \quad (2.30)
\]
In fact, we always have $T_i^{\max} > \min(T_0, T'_i)$ when $i$ is large enough. Otherwise, by Lemma 3.1 we can find a time-local solution $u_1$ of (1.15) and $u_1$ satisfies the initial value condition

$$u_1(x, T_i^{\max} - \epsilon) = u(x, T_i^{\max} - \epsilon),$$

where $0 < \epsilon < T_i^{\max}$ is a small number. Then by the local existence theorem, $u_1$ exists on the time interval $(T_i^{\max} - \epsilon, T_i^{\max} - \epsilon + \eta)$ for some constant $\eta > 0$. The uniform bounds on $\|u_x\|_{H^2}$ and $\|\nabla_x^n u_x\|_{L^2}$ (for all $m > 2$) implies that $\eta$ is independent of $\epsilon$. Thus, by choosing $\epsilon$ sufficiently small, we have

$$T_i^e = T_i^{\max} - \epsilon + \eta > T_i^{\max}.$$  

By the uniqueness result, we have that $u_1(x, t) = u(x, t)$ for all $t \in [T_i^{\max} - \epsilon, T_i^e)$. Thus we get a solution of the Cauchy problem (1.15) on the time interval $[0, T_e)$, which contradicts the maximality of $T_i^{\max}$.

Now we need to show that $T_i^e$ have a uniform lower bound as $i$ is large enough. For each large enough $i$, if $T_i^e \geq T_0$ we obtain the lower bound. Otherwise, by the same argument as in Lemma 3.1 we have

$$T_i^e \geq \frac{1}{M_i},$$

where

$$M_i = \sup_{[0, \min(T_0, T_i^e)]} \|u_i\|_{L^\infty} \leq C \sup_{[0, \min(T_0, T_i^e)]} \|u_i\|_{H^1} \sup_{[0, \min(T_0, T_i^e)]} \|u_i\|_{L^2}^{1-a} \equiv M_i.$$  

It should be pointed out that to derive the estimates $L^\infty$ estimates on $\|u_i(s)\|_{L^2}$ and $\|u_i(s)\|_{H^1}$ we need only to have $u_i \in L^\infty([0, \min(T_0, T_i^e)], H^4(S^1, N))$, since the equation of KdV flow is a third-order dispersive equation. It is not difficult to see from (2.30) that there exists a positive constant $M(\Omega_0, ||u_0x||_{H^3})$ such that, as $i$ is large enough,

$$M_i(\Omega_0, ||u_0x||_{H^3}) \leq M(\Omega_0, ||u_0||_{H^3}),$$

since $\|u_i - u_0\|_{H^4} \to 0$.

Let $T^* = \min(T_0, \frac{1}{M_i})$. As $i$ is large enough, we always have $u_i \in L^\infty([0, T^*], H^4(S^1, N))$. By letting $i \to \infty$ and taking the same arguments as in Theorem 1.1, we know there exists $u \in L^\infty([0, T^*], H^4(S^1, N))$ such that

$$u_i \to u \ [\text{weakly}^* \text{ in } L^\infty([0, T^*], H^4(S^1, N))$$

and $u$ is a local solution to (2.1). Theorem 1.1 tells us that the local solution is unique and the local solution is continuous with respect to $t$, i.e., $u \in C([0, T^*], H^4(S^1, N))$. Thus, we complete the proof of the theorem. \hfill \Box

### 3 Conservation Laws

In this section we show the conservation laws $E_1(u)$, $E_2(u)$, $E_3(u)$ and the semi-conservation law $E_4(u)$ introduced in the first section with some special assumptions about $N$. These will help us to obtain the global existence of the KdV geometric flow in the next section.
First, we recall that a complex manifold $N$ of real dimension $2n$ and integrable complex structure $J$ is said to be Kähler if it possesses a Riemannian metric $h$ for which $J$ is an isometry, as well as a symplectic form $\omega$ satisfying the compatibility condition $\omega(X,Y) = g(JX,Y)$ for all tangent vector fields $X, Y \in \Gamma(TN)$ which denotes the space of smooth sections on $TN$.

Now we derive some conservation laws of the new geometric flow (1.15) on a locally Hermitian symmetric space. The computational process about the KdV part is the same with that in [38] thus we omit many details and use the results in [38] directly.

**Lemma 3.1.** Assume $N$ is a locally Hermitian symmetric space. If $u : S^1 \times (0,T) \to N$ is a smooth solution of the Cauchy problem of the Hirota geometric flow (1.15), then

$$ \frac{dE_1}{dt} = \frac{1}{2} \frac{d}{dt} \int |u_x|^2 dx = 0, $$

in other words, $E_1(u) = E_1(u_0)$ for all $t \in (0,T)$.

**Proof.** With the assumption of $N$, we have $\nabla R = 0$. Similar with the computation before, we get

$$ \frac{dE_1}{dt} = \frac{1}{2} \frac{d}{dt} \int |u_x|^2 dx = -\int \langle \nabla_x u_x, u_t \rangle dx $$

$$ = -\alpha \int \langle \nabla_x u_x, J \nabla_x u_x \rangle dx - \beta \left( \int \langle \nabla_x u_x, \nabla_x^2 u_x \rangle dx + \frac{1}{2} \int \langle \nabla_x u_x, R(u_x, J u_x) J u_x \rangle dx \right). $$

(3.1)

Obviously, by the antisymmetry of $J$, the first part on the right of (3.1) vanishes. The second part vanishes too, since in [38] we have proved that $E_1$ is preserved by KdV geometric flow with the same assumptions. Precisely, we have

$$ \beta \left( \int \langle \nabla_x u_x, \nabla_x^2 u_x \rangle dx + \frac{1}{2} \int \langle \nabla_x u_x, R(u_x, J u_x) J u_x \rangle dx \right) $$

$$ = \frac{\beta}{2} \int \nabla_x (\langle \nabla_x u_x, \nabla_x u_x \rangle) dx + \frac{\beta}{8} \int \nabla_x (\langle u_x, R(u_x, J u_x) J u_x \rangle) dx $$

$$ - \frac{\beta}{8} \int \langle u_x, (\nabla_x R)(u_x, J u_x) J u_x \rangle dx $$

$$ = 0.$$

The last equality holds since $\nabla_x R = 0$. Hence we have

$$ \frac{dE_1}{dt} = 0.$$

This completes the proof. \qed

**Lemma 3.2.** Assume $N$ is a is a locally Hermitian symmetric space. If $u : S^1 \times (0,T) \to N$ is a smooth solution of the Cauchy problem of the geometric Schrödinger-Airy flow (1.15), the pseudo-helicity of $u$

$$ E_2(u) \equiv \int \langle \nabla_x u_x, J u_x \rangle dx $$

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is preserved too, i.e.
\[
\frac{dE_2}{dt} = 0.
\]

**Proof.** Differentiating \(E_2\) with respect to \(t\), we have
\[
\frac{dE_2}{dt} = \int \langle \nabla_t \nabla_x u_x, J u_x \rangle dx + \int \langle \nabla_x u_x, J \nabla_t (u_x) \rangle dx
\]
\[
= \int \langle \nabla_x^2 u_x, J u_x \rangle dx + \int \langle R(u_t, u_x) u_x, J u_x \rangle dx + \int \langle \nabla_x u_x, J \nabla_x u_t \rangle dx
\]
\[
= 2 \int \langle u_t, J \nabla_x^2 u_x \rangle dx - \int \langle u_t, R(u_x, J u_x) u_x \rangle dx.
\]

Substituting (1.15) yields,
\[
\frac{dE_2}{dt} = \alpha \left( 2 \int \langle J \nabla_x u_x, J \nabla_x^2 u_x \rangle dx - \int \langle J \nabla_x u_x, R(u_x, J u_x) u_x \rangle dx \right)
\]
\[
+ \beta \left( \int \langle R(u_x, J u_x, J u_x, J u_x) u_x \rangle dx - \int \langle \nabla_x^2 u_x, R(u_x, J u_x) u_x \rangle dx \right)
\]
\[
- \frac{1}{2} \int \langle R(u_x, J u_x) J u_x, R(u_x, J u_x) u_x \rangle dx \right).
\]

We have known that KdV flow on any Kähler manifold preserves \(E_2\) (see [38]). Thus, if \(N\) is a locally Hermitian symmetric space, the same computation yields
\[
\frac{dE_2}{dt} = \alpha \left( 2 \int \langle J \nabla_x u_x, J \nabla_x^2 u_x \rangle dx - \int \langle J \nabla_x u_x, R(u_x, J u_x) u_x \rangle dx \right)
\]
\[
- \beta \left( \int \langle R(u_x, J u_x, J u_x, J u_x) u_x \rangle dx - \int \langle \nabla_x^2 u_x, R(u_x, J u_x) u_x \rangle dx \right)
\]
\[
- \frac{1}{4} \int \nabla_x (\langle u_x, R(u_x, J u_x) J u_x \rangle) \rangle dx = 0.
\]

This completes the proof. \(\square\)

Now we prove the third conservation law \(E_3\) with some special conditions about \(N\).
To begin with, we compute \(\frac{dE_3}{dt}\) as before where
\[
E_3(u) = \int \| \nabla_x u_x \|^2 dx - \frac{1}{4} \int \langle u_x, R(u_x, J u_x) J u_x \rangle dx.
\]

For the first term of \(E_3\), we have
\[
\frac{d}{dt} \int \| \nabla_x u_x \|^2 dx = 2 \int \langle \nabla_x^3 u_x, u_t \rangle dx + 2 \int \langle u_t, R(\nabla_x u_x, u_x) u_x \rangle dx.
\]

Substituting \(u_t\) we get
\[
\frac{d}{dt} \int \| \nabla_x u_x \|^2 dx = \alpha \left( 2 \int \langle \nabla_x^3 u_x, J \nabla_x u_x \rangle dx + 2 \int \langle J \nabla_x u_x, R(\nabla_x u_x, u_x) u_x \rangle dx \right)
\]
\[
+ \beta \left( 2 \int \langle \nabla_x^3 u_x, \nabla_x^2 u_x \rangle dx + 2 \int \langle \nabla_x^2 u_x, R(\nabla_x u_x, u_x) u_x \rangle dx \right)
\]
Lemma 3.3. Let $N$ be a locally Hermitian symmetric space. Then, for a smooth solution $u : S^1 \times (0, T) \to N$ to Hirota geometric flow (1.15) on $N$, we have

$$\frac{dE_3}{dt} = \beta \int \langle R(\nabla_x u_x, u_x), R(u_x, Ju_x) Ju_x \rangle dx.$$
Thus we obtain that $E_3(u)$ would be conserved under some additional conditions about $N$. More precisely, the following lemma helps us get the conservation law.

**Lemma 3.4.** ([38]) Assume $(N, J, h)$ is one of the following three kinds of manifolds: Kähler manifolds with constant holomorphic sectional curvature $K$, complex Grassmannians $G_{p,q}(\mathbb{C})$ and a complex hyperbolic Grassmannians $D_{m,l}(\mathbb{C})$. Then there always holds true

$$\langle R(Y, X)X, R(X, JX)JX \rangle \equiv 0,$$

for any tangent vector fields $X$ and $Y$ on $N$.

**Corollary 3.5.** If $N = M_1 \times M_2 \times \cdots \times M_n$ is a product manifold where $(M_i, h^i)$ $(i = 1, 2, \cdots, n)$ is a locally Hermitian symmetric space satisfying $h^i(R^i(Y, X)X, R^i(X, JX)JX) \equiv 0$ where $R^i$ is the Riemann curvature on $M_i$, then for a smooth solution $u : S^1 \times (0, T) \to N$ to Hirota geometric flow (1.15) on $N$, $E_3(u)$ is preserved, i.e.,

$$\frac{dE_3}{dt} = \frac{d}{dt} \left( \int |\nabla_x u_x|^2 dx - \frac{1}{4} \int \langle u_x, R(u_x, Ju_x)Ju_x \rangle dx \right) = 0.$$

The proof is easy and standard. We omit the detail.

Next we prove the semi-conservation law about $E_4$, where

$$E_4(u) = 2 \int |\nabla_x^2 u_x|^2 dx - 3 \int \langle \nabla_x u_x, R(\nabla_x u_x, u_x)u_x \rangle dx\]

$$- 5 \int \langle \nabla_x u_x, R(\nabla_x u_x, Ju_x)Ju_x \rangle dx.$$

**Lemma 3.6.** With the same assumptions as in Lemma 3.1, we have

$$\frac{dE_4}{dt} \leq C(E_4 + 1),$$

where $C$ is a constant depends on $N$, $E_1(u_0)$ and $||\nabla_x u_x||_{L^2}$.

*Proof.* For simplicity, we still denote $E_4(u) \triangleq A_1 F_1 + A_2 F_2 + A_3 F_3$, as before, where $A_1 = 2, A_2 = -3, A_3 = -5$,

$$F_1 \triangleq \int |\nabla_x^2 u_x|^2 dx,$$

$$F_2 \triangleq \int \langle \nabla_x u_x, R(\nabla_x u_x, u_x)u_x \rangle dx,$$

and

$$F_3 \triangleq \int \langle \nabla_x u_x, R(\nabla_x u_x, Ju_x)Ju_x \rangle dx$$

respectively.
We first consider $F_1$. Differentiating it with respect to $t$, we have
\[
\frac{dF_1}{dt} = \frac{d}{dt} \int |\nabla_x^2 u_x|^2 dx
\]
\[
= -2 \int \langle u_t, \nabla_x^5 u_x \rangle dx - 2 \int \langle u_t, R(\nabla_x^3 u_x, u_x) \rangle dx 
+ 2 \int \langle u_t, R(\nabla_x^2 u_x, \nabla_x u_x) \rangle dx.
\]
(3.6)

After substituting (1.15) into above, we get two parts in the equality, i.e. the Schrödinger part and the KdV part. Here we mainly deal with the first part. For the KdV part, we use the results in [38]. Then we have
\[
\frac{dF_1}{dt} = \frac{d}{dt} \int |\nabla_x^2 u_x|^2 dx = \alpha I_{11} + \beta I_{12},
\]
(3.7)
where
\[
I_{11} = -2 \int \langle J \nabla_x u_x, \nabla_x^2 u_x \rangle dx - 2 \int \langle J \nabla_x u_x, R(\nabla_x^3 u_x, u_x) \rangle dx 
+ 2 \int \langle J \nabla_x u_x, R(\nabla_x^2 u_x, \nabla_x u_x) \rangle dx;
\]
and $I_{12}$ is the KdV part:
\[
I_{12} = 15 \int \langle \nabla_x^2 u_x, R(\nabla_x^2 u_x, J u_x) \rangle \nabla_x u_x dx + 9 \int \langle \nabla_x^2 u_x, R(\nabla_x^2 u_x, \nabla_x u_x) \rangle dx 
+ 2 \int \langle R(u_x, J u_x) J u_x, R(\nabla_x^2 u_x, \nabla_x u_x) \rangle dx 
+ 2 \int \langle R(\nabla_x u_x, J u_x) J u_x, R(\nabla_x^2 u_x, u_x) \rangle dx 
+ \int \langle R(u_x, J u_x) J \nabla_x u_x, R(\nabla_x^2 u_x, u_x) \rangle dx
\]
(3.8)

Obviously, the first term of $I_{11}$ vanishes since
\[
\int \langle J \nabla_x u_x, \nabla_x^5 u_x \rangle dx = \int \langle J \nabla_x^3 u_x, \nabla_x^3 u_x \rangle dx = 0.
\]
For the second term of $I_{11}$, we have
\[
-2 \int \langle J \nabla_x u_x, R(\nabla_x^3 u_x, u_x) \rangle dx = 2 \int \langle \nabla_x^3 u_x, R(\nabla_x u_x, J u_x) \rangle dx
\]
\[
= -2 \int \langle \nabla_x^2 u_x, R(\nabla_x^2 u_x, J u_x) \rangle dx - 2 \int \langle \nabla_x^2 u_x, R(\nabla_x u_x, J \nabla_x u_x) \rangle dx 
- 2 \int \langle \nabla_x^2 u_x, R(\nabla_x u_x, J u_x) \rangle \nabla_x u_x dx.
\]
For the third term of $I_{11}$, it is easy to check that
\[
2 \left\langle J \nabla_x u_x, R(\nabla_x^2 u_x, \nabla_x u_x)u_x \right\rangle dx = -2 \left\langle \nabla_x^2 u_x, R(\nabla_x u_x, J u_x)\nabla_x u_x \right\rangle dx
\]
Hence
\[
I_{11} = -2 \left\langle \nabla_x^2 u_x, R(\nabla_x^2 u_x, J u_x)u_x \right\rangle dx - 2 \left\langle \nabla_x^2 u_x, R(\nabla_x u_x, J \nabla_x u_x)u_x \right\rangle dx - 4 \left\langle \nabla_x^2 u_x, R(\nabla_x u_x, J u_x)\nabla_x u_x \right\rangle dx.
\]
(3.9)

Next we compute $\frac{dF_2}{dt}$. From [38] we have
\[
\frac{dF_2}{dt} = \frac{d}{dt} \int \left\langle \nabla_x u_x, R(\nabla_x u_x, u_x)u_x \right\rangle dx
\]
\[
= 2 \int \left\langle u_t, R(\nabla_x^3 u_x, u_x)u_x \right\rangle dx - 2 \int \left\langle u_t, R(\nabla_x^2 u_x, \nabla_x u_x)u_x \right\rangle dx + 10 \int \left\langle u_t, R(\nabla_x^2 u_x, u_x)\nabla_x u_x \right\rangle dx + 2 \int \left\langle u_t, R(\nabla_x u_x, u_x)u_x, u_x \right\rangle dx.
\]
Thus, substituting (1.15) yields
\[
\frac{dF_2}{dt} = \frac{d}{dt} \int \left\langle \nabla_x u_x, R(\nabla_x u_x, u_x)u_x \right\rangle dx = \alpha I_{21} + \beta I_{22},
\]
(3.10)

where
\[
I_{21} = 2 \int \left\langle J \nabla_x u_x, R(\nabla_x^3 u_x, u_x)u_x \right\rangle dx - 2 \int \left\langle J \nabla_x u_x, R(\nabla_x^2 u_x, \nabla_x u_x)u_x \right\rangle dx
\]
\[
+ 10 \int \left\langle J \nabla_x u_x, R(\nabla_x^2 u_x, u_x)\nabla_x u_x \right\rangle dx
\]
\[
+ 2 \int \left\langle J \nabla_x u_x, R(\nabla_x u_x, u_x)u_x, u_x \right\rangle dx;
\]

and $I_{22}$ is the KdV part
\[
I_{22} = 6 \int \left\langle \nabla_x^2 u_x, R(\nabla_x^2 u_x, u_x)\nabla_x u_x \right\rangle dx
\]
\[
- 2 \int \left\langle R(u_x, J u_x)J u_x, R(\nabla_x^2 u_x, \nabla_x u_x)u_x \right\rangle dx
\]
\[
+ 4 \int \left\langle R(u_x, J u_x)J u_x, R(\nabla_x^2 u_x, u_x)\nabla_x u_x \right\rangle dx
\]
\[
- \int \left\langle R(u_x, J u_x)J \nabla_x u_x, R(\nabla_x^2 u_x, u_x)u_x \right\rangle dx
\]
\[
- 2 \int \left\langle R(\nabla_x u_x, J u_x)J u_x, R(\nabla_x^2 u_x, u_x)u_x \right\rangle dx
\]
\[
+ 2 \int \left\langle R(\nabla_x u_x, u_x)u_x, R(\nabla_x^2 u_x, u_x)u_x \right\rangle dx
\]
\[
+ \int \left\langle R(u_x, J u_x)J u_x, R(\nabla_x u_x, u_x)u_x \right\rangle dx.
\]
(3.11)
For the first term of $I_{21}$, integrating by parts yields

$$
2 \int (J \nabla_x u_x, R(\nabla^3_x u_x, u_x) u_x) dx = -2 \int (\nabla^2_x u_x, R(\nabla_x u_x, J u_x) u_x) dx
$$

$$
= 2 \int (\nabla^2_x u_x, R(\nabla^2_x u_x, J u_x) u_x) dx + 2 \int (\nabla^2_x u_x, R(\nabla_x u_x, J \nabla_x u_x) u_x) dx
$$

$$
+ 2 \int (\nabla^2_x u_x, R(\nabla_x u_x, J u_x) \nabla_x u_x) dx.
$$

Moreover,

$$
10 \int (J \nabla_x u_x, R(\nabla^2_x u_x, u_x) \nabla_x u_x) dx = -10 \int (\nabla^2_x u_x, R(\nabla_x u_x, J \nabla_x u_x) u_x) dx.
$$

Hence we have

$$
I_{21} = 2 \int (\nabla^2_x u_x, R(\nabla^2_x u_x, J u_x) u_x) dx - 8 \int (\nabla^2_x u_x, R(\nabla_x u_x, J \nabla_x u_x) u_x) dx
$$

$$
+ 4 \int (\nabla^2_x u_x, R(\nabla_x u_x, J u_x) \nabla_x u_x) dx
$$

$$
- 2 \int (\nabla_x u_x, R(\nabla_x u_x, u_x) u_x J u_x) dx.
$$

(3.12)

Similarly, we deduce:

$$
\frac{dF_3}{dt} = \frac{d}{dt} \int (\nabla_x u_x, R(\nabla_x u_x, J u_x) J u_x) dx
$$

$$
= 2 \int (u_t, R(\nabla^3_x u_x, J u_x) J u_x) dx + 6 \int (u_t, R(\nabla^2_x u_x, J \nabla_x u_x) J u_x) dx
$$

$$
+ 2 \int (u_t, R(\nabla^2_x u_x, J u_x) \nabla_x u_x) dx + 2 \int (u_t, R(\nabla_x u_x, J \nabla_x u_x) J \nabla_x u_x) dx
$$

$$
+ 2 \int (u_t, R(\nabla_x u_x, J u_x) J u_x, u_x) dx
$$

$$
= \alpha I_{31} + \beta I_{32}.
$$

(3.13)

Here $I_{32}$ is the KdV part:

$$
I_{32} = 6 \int (\nabla^2_x u_x, R(\nabla^2_x u_x, J \nabla_x u_x) J u_x) dx
$$

$$
- \int (R(\nabla_x u_x, J u_x) J \nabla_x u_x, R(\nabla^2_x u_x, J u_x) J u_x) dx
$$

$$
- 2 \int (R(\nabla_x u_x, J u_x) J u_x, R(\nabla^2_x u_x, J u_x) J u_x) dx
$$

$$
+ 2 \int (R(u_x, J u_x) J u_x, R(\nabla^2_x u_x, J \nabla_x u_x) J u_x) dx
$$

$$
+ \int (R(u_x, J u_x) J u_x, R(\nabla_x u_x, J \nabla_x u_x) J \nabla_x u_x) dx
$$

$$
+ 2 \int (R(\nabla_x u_x, J u_x) J u_x, R(\nabla^2_x u_x, u_x) u_x) dx
$$

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Hence we obtain
\[ I_{31} = 2 \int \langle J \nabla_x u_x, R(\nabla_x^2 u_x, J u_x) \rangle dx + 6 \int \langle J \nabla_x u_x, R(\nabla_x^2 u_x, J \nabla_x u_x) \rangle dx + 2 \int \langle J \nabla_x u_x, R(\nabla_x u_x, u_x) u_x \rangle dx. \]

Note that the first term of \( I_{31} \):
\[ 2 \int \langle J \nabla_x u_x, R(\nabla_x^2 u_x, J u_x) \rangle dx = 2 \int \langle \nabla_x^2 u_x, R(\nabla_x u_x, u_x) \rangle dx \]
\[ = -2 \int \langle \nabla_x^2 u_x, R(\nabla_x^2 u_x, u_x) \rangle dx - 2 \int \langle \nabla_x^2 u_x, R(\nabla_x u_x, u_x) \nabla_x u_x \rangle dx. \]

For the second term of \( I_{31} \) we have:
\[ 6 \int \langle J \nabla_x u_x, R(\nabla_x^2 u_x, J \nabla_x u_x) \rangle dx = 6 \int \langle \nabla_x^2 u_x, R(\nabla_x u_x, u_x) \rangle \nabla_x u_x \rangle dx. \]

Hence we obtain
\[ I_{31} = -2 \int \langle \nabla_x^2 u_x, R(\nabla_x^2 u_x, u_x) \rangle dx + 4 \int \langle \nabla_x^2 u_x, R(\nabla_x u_x, u_x) \rangle dx \]
\[ -2 \int \langle \nabla_x u_x, R(\nabla_x u_x, u_x) \rangle dx. \] (3.15)

In view of (3.7)–(3.15) we have
\[ \frac{dE_4}{dt} = \sum_{i=1}^{3} A_i \frac{dF_i}{dt} = \alpha \sum_{i=1}^{3} I_{1i} + \beta \sum_{j=1}^{3} I_{2j} \]
\[ = 3(3A_1 + 2A_2) \beta \int \langle \nabla_x^2 u_x, R(\nabla_x^2 u_x, u_x) \nabla_x u_x \rangle dx \]
\[ + 3(5A_1 + 2A_3) \beta \int \langle \nabla_x^2 u_x, R(\nabla_x^2 u_x, J u_x) \nabla_x u_x \rangle dx \]
\[ + 2(A_1 - A_2 + A_3) \beta \int \langle R(\nabla_x u_x, J u_x), R(\nabla_x^2 u_x, u_x) u_x \rangle dx \]
\[ - 2(A_1 - A_2 + A_3) \alpha \int \langle \nabla_x^2 u_x, R(\nabla_x^2 u_x, u_x) u_x \rangle dx \]
\[ - 2(A_1 + 4A_2 + A_3) \alpha \int \langle \nabla_x^2 u_x, R(\nabla_x u_x, J \nabla_x u_x) \rangle \rangle dx \]
\[ - 4(A_1 - A_2) \alpha \int \langle \nabla_x^2 u_x, R(\nabla_x u_x, J u_x) \rangle \nabla_x u_x \rangle dx \]
\[ - 2A_2 \alpha \int \langle \nabla_x u_x, R(\nabla_x u_x, u_x) u_x \rangle \rangle dx \]
\[ - 2A_3 \alpha \int \langle \nabla_x u_x, R(\nabla_x u_x, J u_x) \rangle \rangle dx \]
\[ + 2(A_1 - A_2)\beta \int \langle R(u_x, J u_x) J u_x, R(\nabla^2_x u_x, \nabla u_x)u_x \rangle dx \]
\[ + (A_1 + 4A_2)\beta \int \langle R(u_x, J u_x) J u_x, R(\nabla^2_x u_x, u_x) \nabla u_x)\rangle dx \]
\[ + (A_1 - A_2)\beta \int \langle R(u_x, J u_x) J \nabla u_x, R(\nabla^2_x u_x, u_x) \rangle dx \]
\[ + 2A_2 \beta \int \langle R(\nabla_x u_x, u_x)u_x, R(\nabla^2_x u_x, u_x) \rangle dx \]
\[ + A_2 \beta \int \langle R(u_x, J u_x) J u_x, R(\nabla_x u_x, u_x, u_x) \rangle dx \]
\[ - A_3 \beta \int \langle R(u_x, J u_x) J \nabla u_x, R(\nabla^2_x u_x, J \nabla u_x) \rangle dx \]
\[ - 2A_3 \beta \int \langle R(\nabla_x u_x, J u_x) J u_x, R(\nabla^2_x u_x, J \nabla u_x) \rangle dx \]
\[ + 2A_3 \beta \int \langle R(u_x, J u_x) J u_x, R(\nabla_x u_x, J \nabla u_x) \rangle \nabla u_x \rangle dx \]
\[ + A_3 \beta \int \langle R(u_x, J u_x) J u_x, R(\nabla_x u_x, J \nabla u_x) \rangle \nabla u_x \rangle dx. \]  (3.16)

Since \( A_1 = 2, A_2 = -3 \) and \( A_3 = -5 \), the first four terms with higher order derivatives vanish. Let’s denote the remaining terms of (3.16) by \( G \). It is easy to see that

\[ |G| \leq C(N)\alpha \int (|\nabla^2_x u_x| |\nabla_x u_x|^2 |u_x| + |\nabla_x u_x|^2 |u_x|^4) dx \]
\[ + C(N)\beta \int (|\nabla^2_x u_x| |\nabla_x u_x|^4 + |\nabla_x u_x|^3 |u_x|^3 + |\nabla_x u_x|^7) dx \]
\[ \leq C(N, \alpha, \beta) \left( \|u_x\|_{L^\infty} (\int |\nabla^2_x u_x|^2 dx)^{\frac{1}{2}} (\int |\nabla_x u_x|^4 dx)^{\frac{1}{4}} + \|u_x\|_{L^2} \right) \]
\[ + \|u_x\|_{L^\infty}^2 (\int |\nabla^2_x u_x|^2 dx)^{\frac{1}{4}} (\int |\nabla_x u_x|^2 dx)^{\frac{1}{4}} \] 
\[ + \|u_x\|_{L^\infty}^4 \int |\nabla_x u_x|^3 dx + \|u_x\|_{L^\infty}^6 (\int |\nabla_x u_x|^2 dx)^{\frac{1}{2}} (\int |u_x|^2 dx)^{\frac{1}{2}} \].

By the interpolation inequality for sections on vector bundles (see [13] for details):

\[ \|u_x\|_{L^\infty} \leq C(N)(\|\nabla_x u_x\|^2_{L^2} + \|u_x\|^2_{L^2})^{\frac{1}{2}} \|u_x\|^2_{L^2} \]
\[ \leq C(N, \|\nabla_x u_x\|_{L^2}, E_1(u_0)) \]
\[ \|\nabla_x u_x\|^3_{L^3} \leq C(N)(\|\nabla^2_x u_x\|^2_{L^2} + \|\nabla_x u_x\|^2_{L^2})^{\frac{3}{2}} \|\nabla_x u_x\|^\frac{3}{2}_{L^2} \]
\[ \leq C(N, \|\nabla_x u_x\|_{L^2})(1 + \|\nabla^2_x u_x\|^2_{L^2}) \]
\[ \|\nabla_x u_x\|^4_{L^4} \leq C(N)(\|\nabla^2_x u_x\|^2_{L^2} + \|\nabla_x u_x\|^2_{L^2})^{\frac{4}{3}} \|\nabla_x u_x\|^\frac{4}{3}_{L^2} \]

we have

\[ |G| \leq C(1 + \|\nabla^2_x u_x\|^2_{L^2}), \]
which implies
\[
\frac{dE_4}{dt} \leq C(1 + \int |\nabla_x^2 u_x|^2 dx) \\
\leq C(1 + E_4).
\]
where \( C = C(N, ||\nabla_x u_x||_{L^2}, E_1(u_0), \alpha, \beta) \) only depends on \( N, \alpha, \beta, E_1(u_0) \) and \( ||\nabla_x u_x||_{L^2} \). This completes the proof. \( \square \)

4 Global existence

In this section we will prove Theorem 1.3. Since \( u_0 \in H^4(S^1, N) \), we can always choose a sequence of smooth maps \( u_{0i} \in C^\infty(S^1, N) \) such that, as \( i \to \infty \),
\[
||u_{0i} - u_0||_{H^4} \to 0.
\]
From the previous arguments in Theorem 1.3, we know that the Cauchy problem (1.15) with the initial map \( u_{0i} \) admits a unique smooth local solution \( u^i \) such that
\[
u^i \in C([0, T(N, ||u_{0i}||_{H^4})], H^k(S^1, N)))
\]
for any \( k \geq 4 \). Obviously, we can see easily that \( T(N, ||u_{0i}||_{H^4}) \) have a uniform lower bound. Hence, letting \( i \to \infty \), we obtain the local solution to the Cauchy problem of the Schrödinger-Airy flow with the initial map \( u_0 \in H^4(S^1, N) \). So, to prove Theorem 1.3, we only need to consider the case \( u_0 \) is a smooth map from \( S^1 \) into \( N \).

Let \( u \) be the local smooth solution of (1.15) which exists on the maximal time interval \([0, T)\). We only need to consider the case where \( T < \infty \).

From Lemma 3.1, we know that the energy is preserved by the solution \( u \), i.e.
\[
E_1(u(t)) = E_1(u_0), \quad \text{for any} \quad t \in [0, T).
\]
Moreover, by the assumptions on \( N \) given in the theorem and Corollary 3.5 we know that \( E_3 \) is preserved, that is
\[
E_3(u) = \int |\nabla_x u_x|^2 dx - \frac{1}{4} \int \langle u_x, R(u_x, Ju_x)Ju_x \rangle dx
\]
is a constant \( E_3(u_0) \). Thus we have
\[
||\nabla_x u_x||_{L^2}^2 = E_3(u_0) + \frac{1}{4} \int \langle u_x, R(u_x, Ju_x)Ju_x \rangle dx \\
\leq E_3(u_0) + C(N) \int |u_x|^4 dx \\
\leq C(N, E_1(u_0), E_3(u_0)) \tag{4.1}
\]
where we used the interpolation inequality
\[
||u_x||_{L^4}^4 \leq \left( ||\nabla_x u_x||_{L^2}^2 + ||u_x||_{L^2}^2 \right) ||u_x||_{L^2}^2
\]
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\[
\leq \frac{1}{2} ||\nabla_x u_x||^2_{L^2} + C(E_1(u_0)).
\]

Thus from Lemma 3.6, we have that
\[
\frac{dE_4}{dt} \leq C(N, E_1(u_0), E_3(u_0))(1 + E_4).
\]

By Gronwall inequality, we get that \(E_4(u(t))\) is uniformly bounded on \([0, T]\). Hence, we obtain
\[
2||\nabla_x^2 u_x||^2_{L^2} = E_4(u) + 3 \int \langle \nabla_x u_x, R(\nabla_x u_x, u_x)u_x \rangle dx
\]
\[
+ 5 \int \langle \nabla_x u_x, R(\nabla_x u_x, J_u_x)J_u_x \rangle dx
\]
\[
\leq C(N, E_4(u_0)) + C(N)||u_x||_{L^\infty}||\nabla_x u_x||^2_{L^2}.
\]

In view of (3.17), (4.1) and the boundedness of \(E_4\), we see that \(||\nabla_x^2 u_x||_{L^2}\) is uniformly bounded on \([0, T]\). Hence we have
\[
\sup_{t \in [0, T]} ||u_x||_{H^2} \leq C(N, E_1(u_0), E_3(u_0), E_4(u_0)).
\]

It follows from the proof of Theorem 2.2 that for \(m > 2\)
\[
\sup_{t \in [0, T]} ||\nabla_x^m u_x||_{L^2} \leq C(N, E_1(u_0), ||\nabla_x u_0||_{L^2}, ||\nabla_x^2 u_0||_{L^2}, \ldots, ||\nabla_x^m u_0||_{L^2}).
\]

Thus, if \(T\) is finite, we can find a time-local solution \(u_1\) of (1.15) and \(u_1\) satisfies the initial value condition
\[
u_1(x, T - \epsilon) = u(x, T - \epsilon),
\]
where \(0 < \epsilon < T\) is a small number. Then by the local existence theorem, \(u_1\) exists on the time interval \((T - \epsilon, T - \epsilon + \eta)\) for some constant \(\eta > 0\). The uniform bounds on \(||u_x||_{H^2}\) and \(||\nabla_x^m u_x||_{L^2}\) (for all \(m > 2\)) implies that \(\eta\) is independent of \(\epsilon\). Thus, by choosing \(\epsilon\) sufficiently small, we have
\[
T_1 = T - \epsilon + \eta > T.
\]

By the uniqueness result, we have that \(u_1(x, t) = u(x, t)\) for all \(t \in [T - \epsilon, T_1]\). Thus we get a solution of the Cauchy problem (1.15) on the time interval \([0, T_1]\), which contradicts the maximality of \(T\). □

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Xiaowei Sun
School of Applied Mathematics,
Central University of Finance and Economics,
Beijing 100081, P.R. China.
Email: sunxw@cufe.edu.cn

Youde Wang
Academy of Mathematics and Systems Science
Chinese Academy of Sciences,
Beijing 100190, P.R. China.
Email: wyd@math.ac.cn