PROOF OF SOME HYPERGEOMETRIC CONGRUENCES VIA
THE WZ METHOD

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Abstract. In this paper, we establish the following two congruences:

\[
\sum_{k=0}^{(p+1)/2} (3k - 1) \frac{(-1)^2 (\tfrac{1}{2})_k^4}{k!^3} \equiv p - 6p^3 \left( -\frac{1}{p} \right) + 2p^3 \left( -\frac{1}{p} \right) E_{p-3} \quad (\text{mod } p^4),
\]

\[
\sum_{k=0}^{p-1} (3k - 1) \frac{(-1)^2 (\tfrac{1}{2})_k^4}{k!^3} \equiv p - 2p^3 \quad (\text{mod } p^4),
\]

where \( p > 3 \) is a prime, \( E_{p-3} \) is the \((p-3)\)-th Euler number and \((-)\) is the Legendre symbol.

The first congruence modulo \( p^3 \) was conjectured by Guo and Schlosser recently.

1. Introduction

In 1997, Van Hamme \cite{21} conjectured that for any odd prime \( p \) one have

\[
\sum_{k=0}^{(p-1)/2} (-1)^k (4k + 1) \frac{(\tfrac{1}{2})_k^3}{k!^3} \equiv p \left( -\frac{1}{p} \right) \quad (\text{mod } p^3),
\]

where \((x)_k = x(x+1) \cdots (x+k-1)\) is the Pochhammer symbol and \((-)\) is the Legendre symbol. This is a \( p \)-adic analogue of the following Ramanujan-type formula for \( 1/\pi \) due to Bauer \cite{1}:

\[
\sum_{k=0}^{\infty} (-1)^k (4k + 1) \frac{(\tfrac{1}{2})_k^3}{k!^3} = \frac{2}{\pi}.
\]

(1.1) was later confirmed by Mortenson \cite{13} in 2008. In 2018, Guo \cite{3} gave a \( q \)-analogue of (1.1) as follows:

\[
\sum_{k=0}^{(p-1)/2} (-1)^k q^{2k} [4k + 1] \frac{(q; q^2)_k^3}{(q^2; q^2)_k^3} \equiv [p]q^{(p-1)^2/4} (-1)^{(p-1)/2} \quad (\text{mod } [p]^3),
\]

where \((x; q)_n = (1 - x)(1 - xq) \cdots (1 - xq^{n-1})\) denotes the \( q \)-Pochhammer symbol and \([n] = 1 + q + \cdots + q^{n-1}\) denotes the \( q \)-integer. In 2012, Sun obtained the following refinement of (1.1)...
modulo $p^4$:
\[
\sum_{k=0}^{(p-1)/2} (-1)^k (4k + 1) \left(\frac{1}{2}\right)_k^3 \equiv p \left(\frac{-1}{p}\right) + p^3 E_{p-3} \pmod{p^4},
\]
where $E_n$ is the $n$-th Euler number defined by
\[
\sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = \frac{2e^x}{e^{2x} + 1} \quad \text{for} \quad |x| < \frac{\pi}{2}.
\]

Recently, Guo, Schlosser and Zudilin studied the $q$-congruences concerning basic hypergeometric series systematically. For example, in 2019, Guo and Zudilin [8] developed the so-called $q$-microscope method to prove a series of basic hypergeometric congruences. For more details about their work one may consult [4, 5, 6, 7, 8].

In 2011, Sun [18, Conj. 5.1(ii)] conjectured that for any prime $p > 3$ one have
\[
\sum_{k=0}^{(p-1)/2} (-1)^k (4k + 1) \left(\frac{1}{2}\right)_k^3 \equiv p \left(\frac{-1}{p}\right) + p^3 E_{p-3} \pmod{p^4}.
\]
This was confirmed by Mao and Zhang in [11]. In 2019, Guo and Schlosser [6 (6.5)] obtained a $q$-analogue of (1.3) modulo $p^3$. Analogously, they proposed the following conjectural congruence:
\[
\sum_{k=0}^{(p+1)/2} (3k - 1) \left(\frac{-1}{2}\right)_k^2 \left(\frac{1}{2}\right)_k^3 \frac{4^k}{k!^3} \equiv p \pmod{p^3},
\]
where $p$ is an odd prime.

The main goal of this paper is to show (1.4) by establishing the following generalization.

**Theorem 1.1.** For any prime $p > 3$, we have
\[
\sum_{k=0}^{(p+1)/2} (3k - 1) \left(\frac{-1}{2}\right)_k^2 \left(\frac{1}{2}\right)_k^3 \frac{4^k}{k!^3} \equiv p - 6p^3 \left(\frac{-1}{p}\right) + 2p^3 \left(\frac{-1}{p}\right) E_{p-3} \pmod{p^4}.
\]

**Remark 1.1.** One may check (1.4) for $p = 3$ directly. Thus (1.5) is actually an extension of (1.4).

Note that for $k \geq (p + 3)/2$ we have
\[
\left(\frac{-1}{2}\right)_k^2 \left(\frac{1}{2}\right)_k \equiv 0 \pmod{p^3}.
\]
Thus we have
\[
\sum_{k=0}^{p-1} (3k - 1) \left(\frac{-1}{2}\right)_k^2 \left(\frac{1}{2}\right)_k \frac{4^k}{k!^3} \equiv \sum_{k=0}^{(p+1)/2} (3k - 1) \left(\frac{-1}{2}\right)_k^2 \left(\frac{1}{2}\right)_k \frac{4^k}{k!^3} \pmod{p^3}.
\]
However, the above congruence does not hold modulo $p^4$. Below we state our second result.
Theorem 1.2. For any prime \( p > 3 \), we have
\[
\sum_{k=0}^{p-1} (3k - 1) \left( \frac{(-\frac{1}{2})^k}{k!} \right) \frac{4^k}{k!^3} \equiv p - 2p^3 \pmod{p^4}.
\]
(1.6)

We shall prove these two theorem by WZ method and Mathematica Package Sigma. One may refer to [14] and [15] for the usage of these tools respectively.

2. Proofs of Theorems 1.1 and 1.2

Our proofs are based on the following Wilf-Zeilberger pair (WZ pair) which can be verified directly.

Lemma 2.1. Set
\[
\begin{align*}
F(n, k) &= (6n^2 - 5n + 1 - 4nk + 2k) \left( \frac{(-\frac{1}{2} - k)_n^2}{n!} \frac{(-\frac{1}{2})_n}{4^n} \right), \\
G(n, k) &= 4(-2n + 1)n^3 \left( \frac{(-\frac{1}{2} - k)_n^2}{(3 + 2k - 2n)^2} \frac{(-\frac{1}{2})_n}{4^n} \right).
\end{align*}
\]

Then we have
\[
F(n, k + 1) - F(n, k) = G(n + 1, k) - G(n, k)
\]
for any nonnegative integers \( n \) and \( k \).

Remark 2.1. Finding a new WZ pair is not easy. We have never seen the above WZ pair in any former literature.

Lemma 2.2. Let \( n \) be a positive integer. Then we have
\[
\begin{align*}
\sum_{k=1}^{n} \left( \frac{(-n)_k(1+n)_k}{(1)_k^2} \right) H_k^{(2)} &= -2(-1)^n \sum_{k=1}^{n} \left( \frac{-1}{k^2} \right)_n^2, \\
\sum_{k=1}^{n} \left( \frac{(-n)_k(1+n)_k}{(1)_k^2} \right) H_k^2 &= 4(-1)^n H_n^2 + 2(-1)^n \sum_{k=1}^{n} \left( \frac{-1}{k^2} \right)_n^2, \\
\sum_{k=1}^{n} \left( \frac{(-n)_k(1+n)_k}{(1)_k^2} \right) H_k &= 2(-1)^n H_n,
\end{align*}
\]

where \( H_n^m = \sum_{k=1}^{n} 1/k^m \) is the \( n \)-th harmonic number of order \( m \).

Proof. These identities were found by Sigma. Here we just prove the third one as an example. Set
\[
S_n := \sum_{k=1}^{n} \left( \frac{(-n)_k(1+n)_k}{(1)_k^2} \right) H_k.
\]

Step 1: Load Sigma in Mathematica and input \( S_n \);
Step 2: Use the command \texttt{GenerateRecurrence} to find that $S_n$ satisfies

$$(1 + n)S_n + (3 + 2n)S_{n+1} + (n + 2)S_{n+2} = 0;$$

Step 3: Use the command \texttt{SolveRecurrence} to solve the above recurrence relation and obtain a particular solution $\{0\}$ and the basic system of solutions $\{-(-1)^n, (-1)^nH_n\}$ of the homogeneous version;

Step 4: Use the command \texttt{FindLinearCombination} to get another form of $S_n$ as follows

$$S_n = 2(-1)^nH_n.$$  

\[\square\]

Remark 2.2. The identities in Lemma 2.2 may appeared in [2].

Lemma 2.3. \[16\] \[17\] For any prime $p > 3$ we have

$$H_{p-1} \equiv 0 \pmod{p^2}, \quad H_{(p-1)/2} \equiv -2q_p(2) + p q_p^2(2) \pmod{p^2},$$

$$H_{p-1}^{(2)} \equiv H_{(p-1)/2}^{(2)} \equiv 0 \pmod{p}, \quad H_{[p/4]} \equiv 4(-1)^{(p-1)/2}E_{p-3} \pmod{p},$$

where $q_p(2) = (2^{p-1} - 1)/p$ denotes the Fermat quotient.

Lemma 2.4. For any prime $p > 3$, we have

$$\sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k^2} \equiv 2(-1)^{(p-1)/2}E_{p-3} \pmod{p}.$$  

Proof. By Lemma 2.3,

$$\sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k^2} \equiv \sum_{k=1}^{(p-1)/2} \frac{1 + (-1)^k}{k^2} = \sum_{k=1}^{(p-1)/2} \frac{1}{k^2} 2 \sum_{k=1}^{[p/4]} \frac{1}{k^2} \equiv 2(-1)^{(p-1)/2}E_{p-3} \pmod{p}.$$  

This proves Lemma 2.4  

Recall that the Bernoulli numbers $B_0, B_1, \ldots$ are defined by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \quad (0 < |x| < 2\pi).$$

The Bernoulli polynomials are given by

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k} \quad (n = 0, 1, 2, \ldots).$$

The congruences in the following lemma have been proved by different authors.
Lemma 2.5. Let \( p > 3 \) be a prime. Then
\[
\sum_{k=0}^{(p-1)/2} \left( \frac{2k}{k} \right)^2 \frac{1}{16^k} \equiv (-1)^{(p-1)/2} + p^2 E_{p-3} \pmod{p^3}, \quad (\text{Sun} \ [18])
\]
\[
\sum_{k=1}^{p-1} \frac{2k}{k} \equiv 0 \pmod{p^2} \pmod{p^2}, \quad (\text{Sun and Tauraso} \ [20])
\]
\[
\sum_{k=1}^{(p-1)/2} \frac{2k}{k} \equiv (-1)^{(p+1)/2} \frac{8}{3} p E_{p-3} \pmod{p^2}, \quad (\text{Sun} \ [18])
\]
\[
\sum_{k=1}^{p-1} \frac{2k}{k^2} \equiv \frac{1}{2} \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{3} \right) \pmod{p}, \quad (\text{Mattarei and Tauraso} \ [9])
\]
\[
\sum_{k=1}^{p-1} \frac{2k}{k} H_k \equiv \frac{1}{3} \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{3} \right) \pmod{p}. \quad (\text{Mao and Sun} \ [10])
\]

Proof of Theorem 1.4. It is easy to see that
\[
(3n - 1) \left( -\frac{1}{2} \right)^2 n \left( \frac{1}{2} \right) n \frac{4^n}{n!^3} = -(6n^2 - 5n + 1) \left( -\frac{1}{2} \right)^3 \frac{4^n}{n!^3} = F(n, 0),
\]
where \( F(n, k) \) is defined in Lemma 2.1. Thus by Lemma 2.1 we have
\[
\sum_{n=0}^{(p+1)/2} (3n - 1) \left( -\frac{1}{2} \right)^2 n \left( \frac{1}{2} \right) n \frac{4^n}{n!^3} = - \sum_{n=0}^{(p+1)/2} F(n, 0) = -F \left( \frac{p+1}{2}, 0 \right) - \sum_{n=0}^{(p-1)/2} F(n, 0)
\]
\[
= - F \left( \frac{p+1}{2}, 0 \right) - \sum_{n=0}^{(p-1)/2} \sum_{k=0}^{(p-3)/2} (F(n, k) - F(n, k+1)) = - \sum_{n=0}^{(p-1)/2} F \left( n, \frac{p-1}{2} \right)
\]
\[
= - F \left( \frac{p+1}{2}, 0 \right) - \sum_{n=0}^{(p-1)/2} \sum_{k=0}^{(p-3)/2} (G(n, k) - G(n+1, k)) = - \sum_{n=0}^{(p-1)/2} F \left( n, \frac{p-1}{2} \right)
\]
\[
= - F \left( \frac{p+1}{2}, 0 \right) + \sum_{k=0}^{(p-3)/2} G \left( \frac{p+1}{2}, k \right) - \sum_{n=0}^{(p-1)/2} F \left( n, \frac{p-1}{2} \right). \quad (2.1)
\]

Below we first consider
\[
F \left( \frac{p+1}{2}, 0 \right) \pmod{p^4}
\]
. It is routine to check that
\[
\left( 6 \left( \frac{p+1}{2} \right)^2 - 5 \left( \frac{p+1}{2} \right) + 1 \right) / (p+1)^3 \equiv \frac{1}{2} p - \frac{3}{2} p^3 \pmod{p^4}.
\]
Thus we have
\[
F \left( \frac{p + 1}{2}, 0 \right) \equiv - (2p - 6p^3) 4^{(p-1)/2} \frac{(\frac{1}{2})^3}{(1)_{(p-1)/2}^3} = - (2p - 6p^3) \frac{p-1)}{4^{p-1}} \pmod{p^3}.
\]

Recall Morley’s congruence (cf. [12])
\[
\left( \frac{p - 1}{p - 1} \right) \equiv (-1)^{(p-1)/2} 4^{p-1} \pmod{p^3}.
\]

Thus we have
\[
F \left( \frac{p + 1}{2}, 0 \right) \equiv - (2p - 6p^3) (-1)^{(p-1)/2} 16^{p-1} \equiv - (2p - 6p^3) (-1)^{(p-1)/2} (1 + pq_p(2))^4
\]
\[
\equiv (-1)^{(p+1)/2} (2p - 6p^3 + 8p^2 q_p(2) + 12p^3 q_p^2(2)) \pmod{p^4}.
\]

Now we consider
\[
\sum_{k=0}^{(p-3)/2} G \left( \frac{p + 1}{2}, k \right) \pmod{p^4}.
\]

By the definition of \( G(n, k) \) we have
\[
\sum_{k=0}^{(p-3)/2} G \left( \frac{p + 1}{2}, k \right) = -p \sum_{k=0}^{(p-3)/2} \frac{\left( -\frac{1}{2} - k \right)^2}{(1)_{(p-1)/2}^3} \frac{(\frac{1}{2})_{k+1}^2}{4^{(p+1)/2}}.
\]

It is easy to check that
\[
\left( -\frac{1}{2} - k \right)_{(p-1)/2} = (2 - p) \left( -\frac{1}{2} \right)_{(p-1)/2} \frac{(\frac{1}{2})_{k+1}^2}{(1 - \frac{p}{2})_{k+1}^2}.
\]

Hence we obtain
\[
\sum_{k=0}^{(p-3)/2} G \left( \frac{p + 1}{2}, k \right) = -p(p - 2) \frac{(\frac{1}{2})^2}{(1)_{(p-1)/2}^3} \frac{(\frac{1}{2})_{(p+1)/2}^2}{4^{(p+1)/2}} \sum_{k=0}^{(p-3)/2} \frac{(\frac{1}{2})_{k+1}^2}{(1 - \frac{p}{2})_{k+1}^2}.
\]

Now by Morley’s congruence (2.2) again we get
\[
(p - 2)^2 \frac{(\frac{1}{2})^2}{(1)_{(p-1)/2}^3} \frac{(\frac{1}{2})_{(p+1)/2}^2}{4^{(p+1)/2}} = -2 \cdot 4^{(p-1)/2} \cdot \frac{(\frac{1}{2})_{(p-1)/2}^3}{(1)_{(p-1)/2}^3} = -2 \frac{(p-1)^3}{4^{p-1}}
\]
\[
\equiv 2(-1)^{(p+1)/2} 16^{p-1} = 2(-1)^{(p+1)/2} (1 + 4pq_p(2) + 6p^3 q_p^2(2)) \pmod{p^3}.
\]

On the other hand, we have
\[
\sum_{k=0}^{(p-3)/2} \frac{(\frac{1}{2})_{k+1}^2}{(1 - \frac{p}{2})_{k+1}^2} = \sum_{k=1}^{(p-1)/2} \frac{(\frac{1}{2})_k^2}{(1 - \frac{p}{2})_k^2}.
\]
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\[
\sum_{k=1}^{(p-1)/2} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2} \equiv \sum_{k=1}^{(p-1)/2} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2} \left(1 + pH_k + \frac{p^2}{2} H_k^2 + \frac{p^2}{4} H_k^{(2)}\right) \quad (\text{mod } p^3)
\]

\[
\sum_{k=1}^{(p-1)/2} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2} \left(1 + pH_k + \frac{p^2}{2} H_k^2 + \frac{p^2}{4} H_k^{(2)}\right) \quad (\text{mod } p^3)
\]

by noting that

\[
\left(1 - \frac{p}{2}\right)_k \equiv (1)_k \left(1 - \frac{p}{2} H_k + \frac{p^2}{4} \sum_{1 \leq i < j \leq k} \frac{1}{ij}\right)
\]

\[
(1)_k \left(1 - \frac{p}{2} H_k + \frac{p^2}{8} (H_k^2 - H_k^{(2)})\right) \quad (\text{mod } p^3).
\]

Clearly, for \(k = 1, \ldots, (p-1)/2\) we have

\[
\left(\frac{1}{2}\right)_k \equiv \left(\frac{1-p}{2}\right)_k \left(\frac{1+p}{2}\right)_k.
\]

Thus by Lemma 2.2 we have

\[
\sum_{k=1}^{(p-1)/2} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2} H_k \equiv 2(-1)^{(p-1)/2} H_{(p-1)/2} \quad (\text{mod } p^2),
\]

\[
\sum_{k=1}^{(p-1)/2} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2} H_k^2 \equiv 4(-1)^{(p-1)/2} H_{(p-1)/2}^2 + 2(-1)^{(p-1)/2} \sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k^2} \quad (\text{mod } p),
\]

\[
\sum_{k=1}^{(p-1)/2} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2} H_k^{(2)} \equiv -2(-1)^{(p-1)/2} \sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k^2} \quad (\text{mod } p).
\]

Combining these with (2.4)–(2.6) and in view of Lemmas 2.3, 2.4 and 2.5 we arrive at

\[
\sum_{k=0}^{(p-3)/2} G \left(\frac{p+1}{2}, k\right)
\]

\[
\equiv 2p(-1)^{(p-1)/2}(1 + 4pq_p(2) + 6q^2_p(2))
\]

\[
\times \left((-1)^{(p-1)/2} + 2p^2 E_{p-3} - 1 - 4(-1)^{(p-1)/2} pq_p(2) + 10(-1)^{(p-1)/2} p^2 q^2_p(2)\right)
\]

\[
\equiv 2p - 2(-1)^{(p-1)/2} p - 8(-1)^{(p-1)/2} p^2 q_p(2) - 12(-1)^{(p-1)/2} p^3 q^2_p(2)
\]

\[
+ 4(-1)^{(p-1)/2} p^3 E_{p-3} \quad (\text{mod } p^4).
\]

(2.7)

Finally, we consider

\[
\sum_{n=0}^{(p-1)/2} F \left(n, \frac{p-1}{2}\right) \quad (\text{mod } p^4).
\]
By the definition of $F(n, k)$ we get that
\[
\sum_{n=0}^{(p-1)/2} F\left(n, \frac{p-1}{2}\right) = p + \sum_{n=1}^{(p-1)/2} (6n^2 - 3n - 2np + p)\left(\frac{-1}{2}\right)_n^2 \left(\frac{1}{2}\right)_n 4^n \left(\frac{1}{n}\right)_n^3
\]
\[
= p - p^2 \sum_{n=1}^{(p-1)/2} (6n^2 - 3n - 2np + p)\left(\frac{1}{2}\right)_n 4^n \left(\frac{1}{n}\right)_n \left(1 - pH_n\right)(n + p)(3n - p)
\]
\[
\equiv p - 3p^2 \sum_{n=1}^{(p-1)/2} \left(\frac{1}{2}\right)_n 4^n \left(\frac{1}{n}\right)_n - \frac{p^3}{2} \sum_{n=1}^{(p-1)/2} \left(\frac{1}{2}\right)_n 4^n \left(\frac{1}{n}\right)_n^2
\]
\[
+ \frac{3}{4} p^3 \sum_{n=1}^{(p-1)/2} \left(\frac{1}{2}\right)_n 4^n H_n \left(\mod p^4\right).
\]
(2.8)

Note that
\[
\left(\frac{1}{2}\right)_n \equiv \frac{(2n)}{4^n}.
\]

Thus by Lemma 2.5 we have
\[
\sum_{n=0}^{(p-1)/2} F\left(n, \frac{p-1}{2}\right) \equiv p + 2(-1)^{(p-1)/2} p^3 E_{p-3} \left(\mod p^4\right).
\]
(2.9)

Substituting (2.3), (2.7) and (2.9) into (2.1) we arrive at
\[
\sum_{n=0}^{(p+1)/2} (3n - 1)\left(\frac{-1}{2}\right)_n^2 \left(\frac{1}{2}\right)_n 4^n \left(\frac{1}{n}\right)_n^3
\]
\[
\equiv (-1)^{(p-1)/2} (2p - 6p^3 + 8p^2 q_p(2) + 12p^3 q_p^2(2)) + 2p - 2(-1)^{(p-1)/2} p - 8(-1)^{(p-1)/2} p^2 q_p(2)
\]
\[
- 12(-1)^{(p-1)/2} p^3 q_p^2(2) + 4(-1)^{(p-1)/2} p^3 E_{p-3} - p - 2(-1)^{(p-1)/2} p^3 E_{p-3}
\]
\[
\equiv p - 6(-1)^{(p-1)/2} p^3 + 2(-1)^{(p-1)/2} p^3 E_{p-3} \left(\mod p^4\right).
\]

This coincides with (1.5) since $(-1)^{(p-1)/2} = \left(\frac{-1}{p}\right)$.

The proof of Theorem 1.2 is now complete.

**Proof of Theorem 1.2.** As in the proof of Theorem 1.1 by Lemma 2.1 we obtain
\[
\sum_{n=0}^{p-1} (3n - 1)\left(\frac{-1}{2}\right)_n^2 \left(\frac{1}{2}\right)_n 4^n \left(\frac{1}{n}\right)_n^3 = -\sum_{n=0}^{p-1} F(n, 0) = \sum_{k=0}^{(p-3)/2} G(p, k) - \sum_{n=0}^{p-1} F\left(n, \frac{p-1}{2}\right).
\]
(2.10)
In light of Lemma 2.3, we obtain

\[ \sum_{n=0}^{p-1} F \left( n, \frac{p-1}{2} \right) \equiv p - \frac{3}{4} p^2 \left( \sum_{n=1}^{p-1} \frac{(\frac{1}{2})_n}{(1)_n} 4^n \right) - \frac{p^3}{2} \left( \sum_{n=1}^{p-1} \frac{(\frac{1}{2})_n}{(1)_n} 4^n \right) \]

\[ + \frac{3}{4} p^3 \sum_{n=1}^{p-1} \frac{(\frac{1}{2})_n}{(1)_n} 4^n H_n \mod p^4. \]

Via a similar discussion as in the computation of (2.8) we arrive at

Therefore by Fermat’s little theorem we have

\[ \sum_{n=1}^{p-1} \frac{(\frac{1}{2})_n}{(1)_n} 4^n \equiv -2 \mod p. \]

Below we evaluate \( \sum_{k=0}^{(p-3)/2} G(p, k) \mod p^4. \) Note that

\[ \left( -\frac{1}{2} - k \right)_{p-1} = \frac{(-\frac{1}{2})_{p-1} (\frac{3}{2})_k}{(\frac{5}{2} - p)_k} \]

and \( p \nmid (3/2 - p)_k \) for \( k = 1, \ldots, (p - 5)/2. \) It is easy to see that

\[ \sum_{k=0}^{(p-3)/2} G(p, k) = \sum_{k=0}^{(p-3)/2} (-2p + 1) \left( p - \frac{3}{2} \right) \left( -\frac{1}{2} - k \right)_{p-1} (\frac{1}{2})_{p-1} 4^p \]

\[ = \sum_{k=0}^{(p-5)/2} \frac{-2p + 1}{(p - \frac{3}{2})^2} \left( \frac{3}{2} \right)_{p-1} (\frac{5}{2} - p)_k \mod p^4. \]

In light of Lemma 2.3 we obtain

\[ \left( -\frac{1}{2} \right)_p = -\frac{1}{2} \cdot \frac{1}{2} \cdots (p - \frac{3}{2}) = -\frac{(2p - 1)!}{2^{2p-1}(2p - 1)(p - 1)!} \]

\[ = -p \frac{1}{2^{2p-1}(2p - 1)} \equiv -p \frac{1}{2^{2p-1}(2p - 1)} \mod p^4. \]

Therefore by Fermat’s little theorem we have

\[ \sum_{k=0}^{(p-5)/2} \frac{-2p + 1}{(p - \frac{3}{2})^2} \left( \frac{3}{2} \right)_{p-1} (\frac{5}{2} - p)_k \]

\[ \equiv 2p^3 \sum_{k=0}^{(p-5)/2} \frac{1}{(2k + 3)^2} = 2p^3 \left( \sum_{k=0}^{(p-3)/2} \frac{1}{(2k + 1)^2} - 1 \right) = 2p^3 \left( H_p^{(2)} - \frac{1}{4} H_{p-1}^{(2)} - 1 \right) \]

\[ \equiv -2p^3 \mod p^4. \]
On the other hand, with the help of (2.13), we arrive at
\[ G \left( p, \frac{p - 3}{2} \right) = (-2p + 1) \frac{(1 - \frac{1}{2})_p^2 \left( -\frac{1}{2} \right)_p 4^p}{(1)^3_{p-1}} \equiv -p(-2p + 1) \frac{(1)^3_{p-1} 4^p}{(1)^3_{p-1} 2^{2p-1}(2p - 1)} = 2p \pmod{p^4}. \] (2.15)

Substituting (2.14) and (2.15) into (2.12) we have
\[ \sum_{k=0}^{(p-3)/2} G(p, k) \equiv 2p - 2p^3 \pmod{p^4}. \] (2.16)

Combining (2.16) and (2.11) we finally obtain
\[ \sum_{n=0}^{p-1} (3n - 1) \frac{(-\frac{1}{2})^2 n \left( \frac{1}{2} \right)_n 4^n}{n!^3} \equiv p - 2p^3 \pmod{p^4}. \]

Now the proof of Theorem 1.2 is complete. □

References

[1] G. Bauer, *Von den coefficienten der Reihen von Kugelfunctionen einer variabeln*, J. Reine Angew. Math. 56 (1859), 101–121.
[2] H. W. Gould, *Combinatorial Identities*, Morgantown Printing and Binding Co., West Virginia, 1972.
[3] V.J.W. Guo, *A q-analogue of a Ramanujan-type supercongruence involving central binomial coefficients*, J. Math. Anal. Appl. 458 (2018), 590–600.
[4] V.J.W. Guo and M. J. Schlosser, *Some new q-congruences for truncated basic hypergeometric series*, Symmetry 11 (2019), Art. 268.
[5] V.J.W. Guo and M. J. Schlosser, *Proof of a basic hypergeometric supercongruence modulo the fifth power of a cyclotomic polynomial*, J. Difference Eq. Appl. 25 (2019), 921–929.
[6] V.J.W. Guo and M. J. Schlosser, *Some q-supercongruences from transformation formulas for basic hypergeometric series*, Constr. Approx., to appear.
[7] V.J.W. Guo and M. J. Schlosser, *Some new q-supercongruences for truncated basic hypergeometric series: even powers*, Results Math., to appear.
[8] V.J.W. Guo and W. Zudilin, *A q-microscope for supercongruences*, Adv. Math. 346 (2019), 329–358.
[9] S. Mattarei and R. Tauraso, *Congruences for central binomial sums and finite polylogarithms*, J. Number Theory 133 (2013), 131–157.
[10] G.-S. Mao and Z.-W. Sun, *Two congruences involving harmonic numbers with applications*, Int. J. Number Theory 12 (2016), 527–539.
[11] G.-S. Mao and T. Zhang, *Proof of Suns conjectures on super congruences and the divisibility of certain binomial sums*, Ramanujan J. 50 (2019), 1–11.
[12] F. Morley, *Note on the congruence 2^{4n} \equiv (-1)^n(2n)!/(n!)^2, where 2n+1 is a prime*, Ann. Math. 9 (1895), 168–170.
[13] E. Mortenson, *A p-adic supercongruence conjecture of van Hamme*, Proc. Amer. Math. Soc. 136 (2008), 4321–4328.
[14] M. Petkovšek, H. S. Wilf and D. Zeilberger, *A = B*, A K Peters, Wellesley, 1996.
[15] C. Schneider, *Symbolic summation assists combinatorics*, Sém. Lothar. Combin. 56 (2007), Article B56b.
[16] Z.-H. Sun, *congruences concerning Bernoulli numbers and Bernoulli Polynomials*, Discrete Appl. Math. 105 (2000), 193–223.

[17] Z.-H. Sun, *Congruences involving Bernoulli and Euler numbers*, J. Number Theory 128 (2008), no. 2, 280–312.

[18] Z.-W. Sun, *Super congruences and Euler numbers*, Sci. China Math. 54 (2011), 2509–2535.

[19] Z.-W. Sun, *A refinement of a congruence result by van Hamme and Mortenson*, Illinois J. Math. 56 (2012), 967–979.

[20] Z. W. Sun and R. Tauraso, *New congruences for central binomial coefficients*, Adv. in Appl. Math. 45 (2010), 125–148.

[21] L. van Hamme, *Some conjectures concerning partial sums of generalized hypergeometric series*, Lect. Notes Pure Appl. Math. 192 (1997), 223–236.

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