Quantum Group SU\(_q(2)\) and Pairing in Nuclei

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A scheme for treating the pairing of nucleons in terms of generators of Quantum Group SU\(_q(2)\) is presented. The possible applications to nucleon pairs in a single orbit, multishell case, pairing vibrations and superconducting nuclei are discussed. The formalism for performing BCS calculations with q-deformed nucleon pairs is constructed and the role played by deformation parameter q analyzed in the context of nucleons in a single orbit and for Sn Isotopes.

Pairing of nucleons manifests itself in the energy gap in even-even nuclei, odd-even staggering, moment of inertia of deformed nuclei, low lying 2\(^+\) states, ground state spins and decay properties of nuclei. Quasi spin operators, the generators of group SU(2) have been an interesting artefact for studying nucleon pairing since the time that Racah and Talmi [1] pointed out the group symmetries of the zero range pairing interaction model. In this talk, a more general scheme for treating the nuclear pairing problem in terms of generators of quantum group SU\(_q(2)\) is presented.

The quantum group SU\(_q(2)\), a q-deformed version of Lie algebra SU(2), has been studied extensively [2–4], and a q-deformed version of quantum harmonic oscillator developed [5,6]. The quantum group SU\(_q(2)\) is more general than SU(2) and contains the latter as a special case. The underlying idea in using the zero coupled nucleon pairs with q-deformations is that the commutation relations of nucleon pair creation and destruction operators are modified by the correlations as such are somewhat different in comparison with those used in deriving the usual theories. The q-deformed theories reduce to the corresponding usual theories in the limit q → 1. We first introduce the seniority scheme and the quasi-spin operators in section I. The q-deformed nucleon pairs are defined in section II which also contains a brief review of seniority scheme based on q-deformed nucleon pairs [7].

Section III contains the formulation of random phase approximation (RPA) with q-deformed nucleon pairs(boson approximation) for nuclei with no superconducting solution and RPA with q-deformed quasi-particle pairs (quasi-boson approximation) for superconducting nuclei [8]. In section IV, the formalism for BCS theory with q-deformed nucleon pairs is presented and it’s application to the case of Sn nuclei discussed [9]. The nucleon pairing in a single j shell has also been treated by Bonatsos et. al [10–12] by associating two Q-oscillators, one describing the J = 0 pairs and the other associated with J \(\neq\) 0 pairs. In their formalism, Q-oscillators involved reduce to usual harmonic oscillators as Q → 1 and the deformation is introduced in a way different from ours.

I. QUASI-SPIN OPERATORS AND SENIORITY SCHEME

Creation and destruction operators for a zero coupled nucleon pair in single particle orbit j are,

\[
Z_0 = \frac{1}{\sqrt{2}} (A^j \times A^j)^0 \quad \text{and} \quad \overline{Z}_0 = \frac{1}{\sqrt{2}} (B^j \times B^j)^0
\]

where

\[
A_{jm} = a_{jm}^\dagger; \quad B_{jm} = (-1)^{j+m} a_{j,-m}; \quad a_{jm}^\dagger a_{jm} + a_{jm} a_{jm}^\dagger = 1. \quad (1)
\]

With number operator defined as

\[
n_{op} = \sum_m a_{jm}^\dagger a_{jm}, \quad (2)
\]

and putting Ω = \(\frac{2j+1}{2}\), we can verify that

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\[ [Z_0, \overline{Z}_0] = \frac{n_{op}}{\Omega} - 1; \quad [n_{op}, Z_0] = 2Z_0; \quad [n_{op}, \overline{Z}_0] = -2\overline{Z}_0. \tag{3} \]

We can use quasi spin operators defined as
\[
S_{j+} = \sum_{jm>0} (-1)^{j-m} a_m^\dagger a_{j-m}^\dagger ; \quad S_{j-} = \sum_{jm>0} (-1)^{j-m} a_{j-m}^\dagger a_m
\tag{4}
\]
and for a single particle orbit \(j\) identify
\[
S_+ = \sqrt{\Omega} Z_0, \quad S_- = \sqrt{\Omega} \overline{Z}_0; \quad S_0 = \frac{\left(n_{op} - \Omega\right)}{2}.
\tag{5}
\]
The operators \(S_+, S_-\) and \(S_0\) are the generators of \(SU(2)\) and satisfy the commutation relations of angular momentum operators,
\[
[S_+, S_-] = 2S_0; \quad [S_0, S_\pm] = \pm S_\pm.
\tag{6}
\]
In seniority scheme an \(n\) nucleon state with \(v\) unpaired particles (\(v\) being the seniority) is represented by \(|n, v\rangle\), we have \(Z_0 \overline{Z}_0 |v, v\rangle = 0\). The states with \(p\) pairs of nucleons and \(n = v + 2p\) can be constructed as
\[
|n, v\rangle = N (Z_0)^p |v, v\rangle.
\]
Choosing a pairing Hamiltonian \(H = AZ_0 \overline{Z}_0\), the pairing energy in state \(|n, v\rangle\) is
\[
E(n, v) = \frac{A}{2(2j + 1)} (n - v) (2\Omega - n - v + 2).
\tag{7}
\]

II. NUCLEON PAIRS WITH Q-DEFORMATION

To construct nucleon pairs with q-deformation, we next examine the generators of quantum group \(SU_q(2)\). The operators \(S_+(q)\), \(S_-(q)\), and \(S_0(q)\) satisfy the commutation relations
\[
[S_+(q), S_-(q)] = \{2S_0(q)\}_q; \quad [S_0(q), S_\pm(q)] = \pm S_\pm(q),
\tag{8}
\]
where
\[
\{x\}_q = \frac{(q^x - q^{-x})}{(q - q^{-1})}.
\tag{9}
\]
Expressing the creation and annihilation operators for \(q\)-deformed nucleon pair as
\[
Z_0(q) = \frac{1}{\sqrt{\Omega}} S_+(q); \quad \overline{Z}_0(q) = \frac{1}{\sqrt{\Omega}} S_-(q),
\tag{10}
\]
the commutation relations for \(q\)-deformed nucleon pair creation and destruction operators are found to be
\[
[Z_0(q), \overline{Z}_0(q)] = \frac{n_{op} - \Omega}{\Omega}_q, \quad [n_{op}, Z_0(q)] = 2Z_0(q); \quad [n_{op}, \overline{Z}_0(q)] = -2\overline{Z}_0(q).
\tag{11}
\]
The pairing Interaction Hamiltonian is now written as \(H(q) = AZ_0(q) \overline{Z}_0(q)\). Using \(q = e^\tau, (\tau \neq 0)\) the pairing energy in seniority scheme for q-deformed pairs is
\[
E_q(n, v) = \frac{2A \sinh (p\tau) \sinh [(\Omega - v - p + 1)\tau]}{(2j + 1) \sinh^2(\tau)}.
\tag{12}
\]
Application to various isotopes in single particle orbits \(1f_{\pm}^2\), and \(1g_{\pm}^2\) have shown a good agreement with experimental ground state energies for small values of deformation parameter. The formalism for realizing a multishell calculation was also developed and applied to Calcium isotopes. It was found that in general weakly interacting heavily deformed nucleon pairs reproduced the spectra very similar to that produced by strongly interacting weakly deformed nucleon pairs. However, depending upon the distance from the closed shell, the energy spectra could shrink or expand with increase in deformation.
III. RPA WITH Q-DEFORMED NUCLEON PAIRS AND Q-DEFORMED QUASI-PARTICLE PAIRS

Using the q-deformed pair creation and destruction operators of Eq. (11) we derived the Random Phase Approximation equations for the pairing vibrations of nuclei. For nuclei with no superconducting solution, the boson creation operator that links the ground state of the nucleus $|A, 0\rangle$ to the excited eigen state $\nu$ of the $A + 2$ nucleon system with $J^\pi = 0^+$ is defined as

$$R^\nu_+ = \sum_{m} X^\nu_m \left( \frac{S_{m+}^+(q)}{\sqrt{\{\Omega_m\}_q}} \right) - \sum_{i} Y^\nu_i \left( \frac{S_{i+}^+(q)}{\sqrt{\{\Omega_i\}_q}} \right)$$

such that

$$|A + 2, \nu\rangle = R^\nu_+ |A, 0\rangle \ , \ \ \ \ R^\nu_+ |A, 0\rangle = 0 .$$

We use the indices $mn(ij)$ for single-particle(hole) levels and $R^\nu_+ = (R^\nu_+)^\dagger$.

The equations of motion are set up for $R^\nu_+$ using single-particle plus pairing Hamiltonian and the RPA equations for the system obtained using the commutation relations of eq. (8). The dispersion relation

$$\frac{1}{G} = \sum_{n} \frac{\{\Omega_n\}_q}{(2\epsilon_n - \hbar\omega_\nu)} - \sum_{j} \frac{\{\Omega_j\}_q}{(2\epsilon_j - \hbar\omega_\nu)}$$

along with the normalization condition easily yields a graphical solution. A similar procedure is followed for constructing the solution for two-hole phonon states such that

$$|A - 2, \mu\rangle = R^\mu_- |A, 0\rangle \ , \ \ \ \ R^\mu_- |A, 0\rangle = 0 ,$$

where

$$R^\mu_- = \sum_{m} X^\mu_m \left( \frac{S_{m-}^-(q)}{\sqrt{\{\Omega_m\}_q}} \right) - \sum_{i} Y^\mu_i \left( \frac{S_{i-}^-(q)}{\sqrt{\{\Omega_i\}_q}} \right) .$$

The two phonon states

$$|A, \nu, \mu\rangle = R^\nu_+ R^\mu_- |A, 0\rangle$$

are the excited $0^+$ states of the nucleus with excitation energy

$$E(0^+) = \hbar\omega_\nu + \hbar\omega_\mu .$$

The q-deformed RPA when applied to study the pairing vibrational states in the nucleus $^{208}$Pb showed that for $\tau = 0.405$ the experimental excitation energy of the double pairing vibration state and the transfer cross section for two neutron transfer are well reproduced [8].

For superconducting nuclei, one has to construct the quasi-boson creation and destruction operators from q-deformed quasi-particle pair creation and annihilation operators. The set of coupled equations

$$(\hbar\omega_\nu - 2E_m)X^\nu_m = -G \sqrt{\{\Omega_m\}_q} \sum_{p} \sqrt{\{\Omega_p\}_q} \left[ X^\nu_p \left( u^2_v u^2_p + v^2_m v^2_p \right) - Y^\nu_p \left( u^2_m v^2_p + v^2_m u^2_p \right) \right]$$

and

$$(\hbar\omega_\nu + 2E_m)Y^\nu_m = G \sqrt{\{\Omega_m\}_q} \sum_{p} \sqrt{\{\Omega_p\}_q} \left[ Y^\nu_p \left( u^2_v u^2_p + v^2_m v^2_p \right) - X^\nu_p \left( u^2_m v^2_p + v^2_m u^2_p \right) \right]$$

can be solved using standard procedure to furnish the roots $E = \hbar\omega_\nu$. For testing the formalism, q-deformed boson and quasi boson approximation calculations for 20 nucleons in two shells were performed, and compared with exact shell model results. The deformed boson approximation results for $\tau = i0.104$ and deformed quasiboson approximation energies for $\tau = 0.15$ overlap the exact calculation results in a wide region away from the phase transition region. The deformation effectively results in including the correlations left out in normal approximate treatments. One can expect, therefore, that in a realistic calculation deformation parameter can be used as a quantitative measure of correlations left out in an approximate treatment in comparison with the exact results.
IV. GAP EQUATION IN QBCS AND THE GROUND STATE ENERGY

Pairing effect cannot be interpreted as a contribution to an average static potential (as in Hartree Fock) or contribution to average vibrating single-particle potential (as in RPA). It is analogous to Superconductivity in metals. In 1959 Belyaev [14] successfully applied to nuclei the Bardeen-Cooper-Schrieffer (BCS) theory originally formulated to explain superconductivity in metals [15]. In view of the usefulness of formulating the nucleon pairing problem in terms of the generators of SU_q(2), we are encouraged to formulate a q-deformed version of BCS theory or qBCS. Following the idea of building correlations into the theory by using pair generators satisfying \( q \)-commutation relations, we next present the \( q \)-analog of BCS theory (\( q \)BCS) for nuclei. The formalism when applied to the case study of \(^{114-124}\text{Sn}\) nuclei elucidates the role played by \( q \)-deformation in these nuclei.

For \( N \) nucleons in \( m \) single particle orbits, we consider the trial wave function,

\[
\Psi = \Phi_{j_1} \Phi_{j_2} \cdots \Phi_{j_m}
\]

where for the orbit \( j \),

\[
\Phi_j = u_j \sum_{n=0}^{\Omega_j} \left( \frac{v_j}{u_j} \right)^n \left[ \frac{\Omega_j!}{n!(\Omega_j-n)!} \right]^{\frac{1}{2}} |n\rangle ; \Omega_j = \frac{2j+1}{2}
\]

and

\[
|n\rangle = \left[ \frac{\{\Omega_j-n\}!_q}{\{n\}_q! \{\Omega_j\}_q!} \right]^{\frac{1}{2}} (S_{j+}(q))^n |0\rangle
\]

is the normalized wave function for \( n \) zero coupled nucleon pairs with \( q \)-deformation occupying single particle orbit \( j \).

Using a variational approach with the single particle plus pairing Hamiltonian for \( q \)-deformed pairs given by

\[
H = \sum_r \varepsilon_r n_{r\text{op}} - G \sum_{rs} S_{r+}(q) S_{s-}(q)
\]

where \( r, s \equiv j_1, j_2, \ldots, j_m \), (23)

and the gap parameter defined as,

\[
\Delta(q) = G \left\langle \Psi \left| \sum_r S_{r+}(q) \right| \Psi \right\rangle = \sum_r \Delta_r(q)
\]

we obtain the occupancies,

\[
v_j^2 = 0.5 \left( 1 - \frac{\varepsilon'_j - \lambda}{\sqrt{\left( \varepsilon'_j - \lambda \right)^2 + \left( \Delta(q) \frac{\{\Omega_j\}_q}{\Omega_j} \right)^2}} \right),
\]

gap parameter

\[
\Delta(q) = \sum_j G \{\Omega_j\}_q 0.5 \left( 1 - \frac{(\varepsilon'_j - \lambda)^2}{(\varepsilon'_j - \lambda)^2 + \left( \Delta(q) \frac{\{\Omega_j\}_q}{\Omega_j} \right)^2} \right)^{\frac{1}{2}}
\]

and consequently the gap equation

\[
\frac{G}{2} \sum_j \frac{\{\Omega_j\}_q^2}{\sqrt{(\varepsilon'_j - \lambda)^2 \Omega_j^2 + \left( \Delta(q) \{\Omega_j\}_q \right)^2}} = 1.
\]
includes 1 d examined the heavy Sn isotopes with more so after the observation of heaviest doubly magic nucleus examined in Ref. [9]. There has been an increased interest in the experimental and theoretical study of Sn isotopes, relation between the pairs determined by a characteristic q valence nucleons occupying degenerate 1 d measured by D is concerned. The ground state binding energies are however lowered by the deformation. The pairing correlations, ∆ in Ref. [9]. The results of q N to reproduce parameter takes some typical successively increasing values varying from 1 d in Sn isotopes. It is immediately seen that for the choice ε q = 1 d the conventional BCS theory. The underlying bcs theory. To get more clues as to whether it is possible to replace the pairing interaction by a suitable commutation going to zero for successively lower values of coupling strength, for example Sn 70 that is conventional BCS theory the pairing correlation vanishes for Sn 0 that is conventional BCS theory. We may infer that the ground state zero seniority state, which is linked to the j-value for the system at hand, real nuclei have also been examined in Ref. [16,17]. We examined the heavy Sn isotopes with N = 14, 16, 18, 20, 22, and 24 neutrons outside 50 Sn core. The model space includes 1d 2, 0g 2, 2s 2, 1d 2, and 0h 2 single particle orbits, with excitation energies 0.0, 0.22, 1.90, 2.20, and 2.80 MeV respectively. The pairing correlation function D = ∆(q)/G as a function of G for the cases where deformation parameter takes some typical successively increasing values varying from 1.0 to 1.7 shows some interesting features. In 50 70, pairing correlations are found to increase as q increases while the pairing strength G is kept fixed. For q = 1.0 that is conventional BCS theory the pairing correlation vanishes for G < G c (∼ 0.065 MeV) as expected. We find D going to zero for successively lower values of coupling strength, for example G c ∼ 0.04 MeV for q = 1.3 as the deformation q of zero coupled pairs increases. We may infer that the qBCS takes us beyond BCS theory.

We include the effect of terms containing u_j v_j^2 left out earlier, we now replace the chemical potential λ by

$$\lambda(q) = \lambda + \frac{Gv_j^2 \{\Omega_j\}_q}{\Omega_j} \left( \{\Omega_j\}_q - \Omega_j + 1 \right).$$

The ground state BCS energy, ⟨Ψ | H | Ψ⟩ is

$$E_{\text{bcs}}(q) = \sum_{j=1}^{m} \left( 2\epsilon_j \Omega_j v_j^2 - G v_j^2 \{\Omega_j\}_q \left( \{\Omega_j\}_q - \Omega_j + 1 \right) - \frac{(\Delta(q))^2}{G} \right)$$

We notice that in a very natural way, the SU_q(2) symmetry introduces in the interaction energy, a q dependence which is linked to the j-value of the orbit occupied by the zero coupled nucleon pairs.

A. Single orbit with 2Ω degenerate states and Sn nuclei

For N nucleons in a single orbit with an occupancy of 2Ω, the ground state wave function is Ψ = Φ_j and E_{bcs}(q) is

$$E_{\text{bcs}}(q) = \varepsilon_j N - G \frac{N}{4\Omega} \left( 2 \{\Omega_j\}_q - N + \frac{N}{\Omega} \right).$$

The exact energy of the N nucleon zero seniority state,

$$E_{\text{exact}} = \varepsilon_j N - G \frac{N}{4} (2\Omega_j - N + 2),$$

can be reproduced (E_{bcs}(q) = E_{\text{exact}}) by choosing q value and the pairing strength G' such that

$$G = \frac{G' \Omega_j (2\Omega_j - N + 2)}{\{\Omega_j\}_q \left( 2 \{\Omega_j\}_q - N + \frac{N}{\Omega} \right)},$$

for the choice ε_j = 0.0 .

In Ref. [6] the single orbit limit of qBCS is applied to nuclear sdg major shell with Ω = 16, and 4,10,14,20,24,30 valence nucleons occupying degenerate 1d 2, 0g 2, 2s 2, 1d 2, and 0h 2 orbits. The intensity of pairing strength required to reproduce E_{\text{exact}} decreases with increasing q and ultimately G → 0 for all cases. It is also found that the strongly coupled zero coupled pairs of BCS theory may well be replaced by weakly coupled q-deformed zero coupled pairs of qBCS theory. To get more clues as to whether it is possible to replace the pairing interaction by a suitable commutation relation between the pairs determined by a characteristic q value for the system at hand, real nuclei have also been examined in Ref. [6]. There has been an increased interest in the experimental and theoretical study of Sn isotopes, more so after the observation of heaviest doubly magic nuclei 106 Sn 50 in nuclear fragmentation reactions [14,15,16,18]. We examined the heavy Sn isotopes with N = 14, 16, 18, 20, 22, and 24 neutrons outside 50 Sn core. The model space includes 1d 2, 0g 2, 2s 2, 1d 2, and 0h 2 single particle orbits, with excitation energies 0.0, 0.22, 1.90, 2.20, and 2.80 MeV respectively. The pairing correlation function D = ∆(q)/G as a function of G for the cases where deformation parameter takes some typical successively increasing values varying from 1.0 to 1.7 shows some interesting features. In 50 70, pairing correlations are found to increase as q increases while the pairing strength G is kept fixed. For q = 1.0 that is conventional BCS theory the pairing correlation vanishes for G < G c (∼ 0.065 MeV) as expected. We find D going to zero for successively lower values of coupling strength, for example G_c ∼ 0.04 MeV for q = 1.3 as the deformation q of zero coupled pairs increases. We may infer that the qBCS takes us beyond BCS theory.

The sets of G, q values that reproduce the empirical ∆ for 50 70 are next used to calculate the gap parameter ∆ and the ground state BCS energy E_N, for even isotopes 114-124 Sn and compared with the experimental values of ∆ in Ref. [6]. The results of qBCS for Sn isotopes are not much different from BCS as far as the Gap parameter ∆ is concerned. The ground state binding energies are however lowered by the deformation. The pairing correlations, measured by D = ∆(q)/G, are seen to increase as q increases (for q real) while the pairing strength G is kept fixed, in Sn isotopes. It is immediately seen that q parameter is a very good measure of the pairing correlations left out in the conventional BCS theory. The underlying q-deformed nucleon pairs show increasingly strong binding as the value
of $q$ is increased. It opens the possibility of obtaining the exact correlation energies by choosing appropriately the combination of $G, q$ values.

The results of our study of $q$BCS are consistent with our earlier conclusions [7,8] that the $q$-deformed pairs with $q > 1$ ($q$ real) are more strongly bound than the pairs with zero deformation and the binding energy increases with increase in the value of parameter $q$. In contrast by using complex $q$ values one can construct zero coupled deformed pairs with lower binding energy in comparison with the no deformation zero coupled nucleon pairs [8]. In general the pairing correlations in $N$ nucleon system, measured by $D = \Delta(q)/\sqrt{G}$, increase with increasing $q$ (for $q$ real) and $q$BCS takes us beyond the BCS theory. The formalism can be tested for several other systems, for example metal grains, where cooper pairing plays an important role.

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