RESIDUAL-BASED A POSTERIORI ERROR ESTIMATES OF MIXED METHODS IN BIOT’S CONSOLIDATION MODEL

YUWEN LI, LUDMIL T. ZIKATANOV

Abstract. We present residual-based a posteriori error estimates of mixed finite element methods for the three-field formulation of Biot’s consolidation model. The error estimator is an upper and lower bound of the space time discretization error up to data oscillation. As a by-product, we also obtain new a posteriori error estimate of mixed finite element methods for the heat equation.

1. Introduction

The mathematical modeling of poro-elastic materials is aimed at describing the interactions between the deformation and fluid flow in a fluid-saturated porous medium. In this paper we provide a posteriori error estimators for the fully discrete, time dependent Biot’s consolidation model for poroelastic media. A pioneering model of poroelasticity in one-dimensional setting was given in [58]. Nowadays, the popular formulations are in three-dimensions and they follow the model by Maurice Biot in several works, e.g., [3, 4]. The system of partial differential equations describing the Biot’s consolidation model has a great deal of applications in geomechanics, petroleum engineering, and biomechanics.

The two-field formulation of Biot’s consolidation model is classical and has been investigated in e.g., [62, 57, 45, 18]. Three-field formulations, which include an unknown Darcy velocity, several conforming and non-conforming discretizations involving Stokes-stable finite-element spaces have been recently proposed as a viable approach for discretization of the Biot’s model. Various three field formulations were considered in [50, 51] with and a priori error estimates are presented in such a work. Recenly, three-field formulation using Stokes stable elements, based on displacement, pressure, and total pressure was proposed and analysed in [48]. A nonconforming discretization, which also provides element-wise mass conservation, is found in [30]. Parameter robust analysis using three field discontinuous Galerkin formulation is given in [29], where a general theory for the a priori error analysis was introduced. Other stable discretizations and solvers are presented in e.g., [38, 39, 55]. Readers are referred to [38] for parameter robust error analysis for four- and five-field formulations. Finite volume and finite difference discretizations have also been used in this field and we point to [23, 24, 47] for more results and references on such methods for Biot’s system. We note that our further considerations are restricted to the finite element method and we will not discuss finite difference and finite volume methods here.

Date: November 26, 2019.
2010 Mathematics Subject Classification. Primary 65N12, 65N15, 65N30.
There are a few works on a posteriori error control for the fully discretized time-dependent problem, see, e.g., [16, 17, 52, 60, 43, 36, 21, 19] for a posteriori error estimates of the primal formulation of the heat equation. A posteriori error estimation of the mixed formulation of the heat equation can be found in e.g., [7, 20, 37, 44]. In addition, equilibrated error estimators are derived in [18, 54, 35]. In addition, equilibrated error estimators are developed in [1] for the four- and five-field formulations. Comparing to the equilibrated error indicators, residual error estimators are simpler to implement and do not require solving auxiliary problems on local patches. Several space-time adaptive algorithms based on residual error estimators are proven to be convergent, see, e.g., [12, 34, 25].

A main result in our paper is the construction of the reliable a posteriori error estimator for the three field Biot’s system. To the best of our knowledge, there are no such error estimators for the mixed formulations of the Biot’s model using more than two fields. Formulations using more than two fields have conservation properties which makes them practically interesting, however, their analysis is more challenging. In this paper, we derive residual a posteriori error estimates for the three-field formulation and prove that the estimator is reliable, that is, it provides an upper bound of the space-time error in the natural variational norm. Since the three-field formulation directly approximates the flux $w \in H(\text{div}, \Omega)$, special attention must be paid to energy estimates and the residual in the dual space $H(\text{div}, \Omega)'$, which is a major obstacle in the construction of such error estimators. The analysis presented here with the help of regular decomposition and commuting quasi-interpolations, however, successfully tackles such problems, see Theorems 3.2 and 3.3 for details.

Another main result of this paper is the lower bound in Theorem 5.3. As far as we know, existing residual, equilibrated, and functional error estimators in Biot’s consolidation model are not shown to be lower bounds of the space-time discretization error. This is partly due to the complexity of the Biot’s model equations. Motivated by Verfürth’s technique introduced in [60], we split the residual and estimator into space and time parts. The temporal estimator can be controlled by the spatial estimator and discretization error, while the spatial estimator is in turn controlled by the finite element error and a small portion of the temporal estimator, where the “smallness” is due to a weight function in time. The details are given later in Section 5.

Since the three-field formulation of Biot’s consolidation model (2.2) contains the mixed formulation of the heat equation (2.3), we review existing a posteriori error estimates of mixed methods for the heat equation. Using a duality argument, [7] first obtained $L^2(0, T; H(\text{div}, \Omega)')$ a posteriori estimates of the flux variable and $L^\infty(0, T; L^2(\Omega))$ estimates of the potential in mixed methods for the heat equation. Using the idea of elliptic reconstruction proposed by [43], the works [37, 44] presented $L^2(0, T; L^2(\Omega))- and L^\infty(0, T; L^2(\Omega))-type a posteriori error estimates of the flux variable in mixed methods for the heat equation. However, there is no proof that the estimators proposed in [7, 37, 44] provide lower bounds of the discretization error. On the other hand, [20] presented an equilibrated estimator with a lower bound for the error in postprocessed potential based on the $L^2(0, T; H^1(\Omega)) \cap H^1(0, T; H^{-1}(\Omega))$-norm. Their estimator does not control the error in the flux variable. Comparing to the aforementioned error estimators, a
posteriori analysis in this paper indeed yields a new estimator for the mixed discretization of the heat equation that is both an upper and lower bound of the space-time error in the natural norm, see Section 5 for details.

The rest of this paper is organized as follows. In Section 2, we present preliminaries and derive energy estimate for the three field formulation of Biot’s consolidation model. Section 3 is devoted to a posteriori error estimates of a semi-discrete scheme (3.1). In Section 4, we develop a posteriori error estimator of the fully discrete scheme (4.1) and prove its reliability. In Section 5, we show that the error estimators are lower bounds of the space-time error and present a posteriori estimates of mixed methods for the heat equation.

2. Preliminaries and Energy estimates

Given a \( \mathbb{R}^d \)-valued function \( u \), the symmetric gradient \( \varepsilon \) and stress tensor \( \sigma \) are

\[
\varepsilon(u) := \frac{1}{2}(\nabla u + (\nabla u)^T), \quad \sigma(u) := 2\mu\varepsilon(u) + \lambda(\text{div } u)I,
\]

where \( \mu > 0, \lambda > 0 \) are Lamé coefficients, \( I \) is the \( d \times d \) identity matrix. Let \( \Omega \) be a Lipschitz domain in \( \mathbb{R}^d \) and \( T > 0 \) be the final time. The three-field formulation of the Biot’s consolidation model reads

\[
\begin{align*}
-\text{div } \sigma(u) + \alpha \nabla p &= f \text{ in } \Omega \times (0,T], \\
\partial_t(\beta p + \alpha \text{div } u) + \text{div } w &= g \text{ in } \Omega \times (0,T], \\
K^{-1}w + \nabla p &= 0 \text{ in } \Omega \times (0,T],
\end{align*}
\]

subject to the initial condition \( u(0) = u_0, p(0) = p_0 \) in \( \Omega \). For the simplicity of presentation, we consider homogeneous boundary conditions

\[
\begin{align*}
u &= 0 \text{ on } \Gamma_1 \times (0,T], \\
p &= 0 \text{ on } \Gamma_2 \times (0,T], \\
\sigma(u)n &= 0 \text{ on } \Gamma_2 \times (0,T], \\
(K\nabla p) \cdot n &= 0 \text{ on } \Gamma_1 \times (0,T],
\end{align*}
\]

where \( \partial\Omega = \Gamma_1 \cup \Gamma_2, \Gamma_1 \cap \Gamma_2 = \emptyset \), and \( n \) denotes the outward unit normal to \( \partial\Omega \). Note that the Neumann boundary condition for \( p \) imposes a boundary condition for \( w \). In addition, we assume \( \alpha, \beta \) are constants and \( K = K(x) \) is a time-independent and uniformly elliptic matrix-valued function, i.e.,

\[
C_1|\xi|^2 \leq \xi^T K(x) \xi \leq C_2|\xi|^2 \text{ for all } \xi \in \mathbb{R}^d \text{ and } x \in \Omega,
\]

where \( C_1, C_2 \) are positive constants. We introduce function spaces where we seek a weak solution to the system given in (2.1).

\[
\begin{align*}
V &= \{ v \in [H^1(\Omega)]^d : v = 0 \text{ on } \Gamma_1 \}, \quad Q = L^2(\Omega), \\
W &= \{ w \in [L^2(\Omega)]^d : \text{div } w \in L^2(\Omega), w \cdot n = 0 \text{ on } \Gamma_1 \},
\end{align*}
\]

Let \((\cdot,\cdot)\) denote the \( L^2(\Omega) \) inner product for scalar-, vector-, or matrix-valued functions. Next, we introduce the corresponding bilinear forms:

\[
\begin{align*}
a(u, v) &:= (\sigma(u), \varepsilon(v)), \\
b(v, q) &:= (\alpha \text{div } v, p), \\
c(p, q) &:= (\beta p, q), \\
d(z, q) &:= (\text{div } z, q), \\
e(w, z) &:= (K^{-1}w, z).
\end{align*}
\]

The norms associated with the bilinear forms given above are

\[
\|v\|_a^2 := a(v, v), \quad \|q\|^2 := c(q, q), \\
\|z\|_c^2 := e(z, z), \quad \|z\|_W^2 := \|z\|^2_e + \|\text{div } z\|,
\]
where $\| \cdot \|$ denotes the $L^2(\Omega)$ norm. For the spaces defined earlier we have the following correspondence with the norms: $V$ is equipped with the $\| \cdot \|_a$-norm, $Q$ is equipped with $\| \cdot \|_c$-norm, and $W$ is equipped with the $W$-norm. Because we are dealing with a time-dependent problem, we need the Hilbert-valued spaces of functions as follows: Given a Hilbert space $H$, we define

$$L^\infty(0, T; H) = \{ v : v(t) \in H \text{ for } t \in T, \text{ ess sup}_{0 \leq t \leq T} \| v(t) \|_H < \infty \},$$

$$L^2(0, T; H) = \{ v : v(t) \in H \text{ for } t \in T, \int_0^T \| v(t) \|^2_H dt < \infty \},$$

$$H^1(0, T; H) = \{ v \in L^2(0, T; H) : \partial_t v \in L^2(0, T; H) \},$$

see, e.g., [22] for more details. The variational formulation of (2.1) then is to find $F$ such that

$$\begin{align*}
&\partial_t u = f, \\
&c(\partial_t p, q) + b(\partial_t u, q) + d(w, q) = g, \\
&e(w, z) - d(z, p) = 0
\end{align*}$$

for all $v \in V$, $q \in Q$, and $z \in W$ a.e. $t \in (0, T]$. It can be observed that (2.2) with $u = v = 0$ reduces to the mixed formulation of the heat equation or time-dependent Darcy flow:

$$\begin{align*}
&c(\partial_t p, q) + d(w, q) = g, \\
&e(w, z) - d(z, p) = 0
\end{align*}$$

In the rest of this section, we establish an energy estimate of (2.2) which is the main tool for deriving a posteriori error estimates. The well-posedness of two-field formulation can be found in e.g., [62, 57]. For the three-field formulation we have the following result.

**Theorem 2.1.** Let $u_0 \in V$, $p_0 \in Q$, $f \in H^1(0, T; V')$, and $g \in L^2(0, T; Q)$. Then the variational formulation (2.2) admits a unique weak solution

$$(u, p, w) \in H^1(0, T; V) \times H^1(0, T; Q) \times L^2(0, T; W).$$

We skip the proof of Theorem 2.1 as it directly follows from the energy estimates in Lemma 2.2 and a standard argument using a Galerkin method in space, in the same fashion as for the linear parabolic equation (see, e.g., [22]). For the purpose of a posteriori error estimation, we consider a more general variational problem: Find $\tilde{u} \in H^1(0, T; V)$, $\tilde{p} \in H^1(0, T; Q)$, $\tilde{w} \in L^2(0, T; W)$, such that

$$\begin{align*}
&a(\tilde{u}, v) - b(v, \tilde{p}) = (F_1, v), \\
&c(\partial_t \tilde{p}, q) + b(\partial_t \tilde{u}, q) + d(\tilde{w}, q) = (F_2, q), \\
&e(\tilde{w}, z) - d(z, \tilde{p}) = (F_3, z)
\end{align*}$$

where $F_1 \in H^1(0, T; V')$, $F_2 \in L^2(0, T; Q')$, $F_3 \in H^1(0, T; W')$ are time-dependent bounded linear functionals living in dual spaces. At each time $t \in [0, T]$, the dual
norms are given by
\[
\|F_1\|_r = \|F_1\|_{W'} := \sup_{v \in V', \|v\| = 1} \langle F_1, v \rangle,
\]
\[
\|F_2\|_r = \|F_2\|_{Q'} := \sup_{q \in Q', \|q\| = 1} \langle F_2, q \rangle,
\]
\[
\|F_3\|_r = \|F_3\|_{W'} := \sup_{z \in W', \|z\| = 1} \langle F_3, z \rangle.
\]

Norms of \( \partial_t F_1 \in V' \) and \( \partial_t F_3 \in W' \) are defined in a similar fashion. Given \( t \in [0, T] \) and an interval \( I \subseteq [0, T] \), we make use of the norms
\[
\| (\tilde{u}, \tilde{p}, \tilde{w}) (t) \|_2^2 := \| \tilde{u}(t) \|_a^2 + \| \tilde{p}(t) \|_c^2 + \| \tilde{w}(t) \|_e^2,
\]
\[
\| (\tilde{u}, \tilde{p}, \tilde{w}) \|_{L^2(I, X)}^2 := \int_I \left( \| \partial_t \tilde{u} \|_a^2 + \| \partial_t \tilde{p} \|_c^2 + \| \partial_t \tilde{w} \|_e^2 \right) ds.
\]

The following energy estimate is crucial to a posteriori error estimation of numerical methods for (2.2).

**Lemma 2.2.** There exists a constant \( C_{\text{stab}} \) dependent only on \( \mu, \alpha, \beta, K, \Omega \) such that for all \( t \in (0, T] \),
\[
\| (\tilde{u}, \tilde{p}, \tilde{w}) (t) \|_2^2 + \| (\tilde{u}, \tilde{p}, \tilde{w}) \|_{L^2(0, t, X)}^2 \leq C_{\text{stab}} \left\{ \| (\tilde{u}, \tilde{p}, \tilde{w}) (0) \|_2^2 \right. \\
+ \left. \int_0^t \| F_2 \| \, ds \right\}^2.
\]

**Proof.** Setting \( v = \partial_t \tilde{u}, z = \tilde{w}, q = \tilde{p} \) in (2.4) yields
\[
(2.5) \quad \frac{1}{2} \frac{d}{dt} \| \tilde{u} \|_a^2 + \frac{1}{2} \frac{d}{dt} \| \tilde{p} \|_c^2 + \| \tilde{w} \|_e^2 = \langle F_1, \partial_t \tilde{u} \rangle + \langle F_2, \tilde{p} \rangle + \langle F_3, \tilde{w} \rangle.
\]

On the other hand, differentiating (2.4a) and (2.4c) with respect to time \( t \) gives
\[
\begin{align*}
\alpha (\partial_t \tilde{u}, v) - b(v, \partial_t \tilde{p}) &= \langle \partial_t F_1, v \rangle, \\
\epsilon (\partial_t \tilde{w}, z) - d(z, \partial_t \tilde{p}) &= \langle \partial_t F_3, z \rangle.
\end{align*}
\]

Taking as test functions \( v = \partial_t \tilde{u}, z = \tilde{w} \) in the equations above and using (2.4b) with \( q = \partial_t \tilde{p} \), we have
\[
(2.6) \quad \| \partial_t \tilde{u} \|_a^2 + \| \partial_t \tilde{p} \|_c^2 + \frac{1}{2} \frac{d}{dt} \| \tilde{w} \|_e^2 = \langle \partial_t F_1, \partial_t \tilde{u} \rangle + \langle F_2, \partial_t \tilde{p} \rangle + \langle \partial_t F_3, \tilde{w} \rangle.
\]

Using (2.5), (2.6), the Cauchy–Schwarz and Young’s inequalities, we obtain
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \| \tilde{u} \|_a^2 + \frac{1}{2} \frac{d}{dt} \| \tilde{p} \|_c^2 + (1 - \delta) \| \tilde{w} \|_e^2 &\leq (1 - \delta) \| \tilde{w} \|_e^2 + \frac{1}{2} \| \partial_t \tilde{u} \|_a^2 + \frac{1}{2} \| \partial_t \tilde{p} \|_c^2 + \frac{1}{2} \| \tilde{w} \|_e^2 \\
&\leq G + \| F_2 \|_e \| \tilde{p} \|_c + \delta \| \tilde{w} \|_e^2,
\end{align*}
\]
where \( \delta > 0 \) and
\[
G = \| F_1 \|_a^2 + \| \partial_t F_1 \|_a^2 + \frac{1}{2} \| F_2 \|_c^2 + \frac{\delta^{-1}}{2} \| F_3 \|_c^2 + \frac{\delta^{-1}}{2} \| \partial_t F_3 \|_c^2.
\]

Integrating the previous inequality yields
\[
(2.7) \quad \frac{1}{2} \| (\tilde{u}, \tilde{p}, \tilde{w}) (t) \|_2^2 + \int_0^t \left( \frac{1}{2} \| \partial_t \tilde{u} \|_a^2 + \frac{1}{2} \| \partial_t \tilde{p} \|_c^2 + (1 - \delta) \| \tilde{w} \|_e^2 \right) ds
\]
\[
\leq \frac{1}{2} \| (\tilde{u}, \tilde{p}, \tilde{w}) (0) \|_2^2 + \int_0^t \left( G + \| F_2 \|_e \| \tilde{p} \|_c + \delta \| \tilde{w} \|_e^2 \right) ds.
\]
Recall that \( \| \bar{p} \|_{L^\infty(0,t;Q)} := \max_{0 \leq s \leq t} \| \bar{p}(s) \|_c \). In particular, (2.7) implies that
\[
\frac{1}{2} \| \bar{p}(s) \|_c^2 \leq \frac{1}{2} \| (\tilde{u}, \bar{p}, \tilde{w})(0) \|_c^2 \\
+ \int_0^t (G + \delta \| \text{div} \tilde{w} \|^2) \, ds + \| \bar{p} \|_{L^\infty(0,t;Q)} \int_0^t \| F_2 \|_c \, ds
\]
for all \( 0 \leq s \leq t \). Hence a combination of the previous estimate with
\[
\| \bar{p} \|_{L^\infty(0,t;Q)} \int_0^t \| F_2 \|_c \, ds \leq \frac{1}{4} \| \bar{p} \|_{L^\infty(0,t;Q)}^2 + \left( \int_0^t \| F_2 \|_c \, ds \right)^2
\]
shows that
\[
\frac{1}{4} \| \bar{p} \|_{L^\infty(0,t;Q)}^2 \leq \frac{1}{2} \| (\tilde{u}, \bar{p}, \tilde{w})(0) \|_c^2 \\
+ \int_0^t (G + \delta \| \text{div} \tilde{w} \|^2) \, ds + \left( \int_0^t \| F_2 \|_c \, ds \right)^2.
\]
Using (2.7) and (2.8) and a Young’s inequality, we obtain
\[
\frac{1}{2} \| (\tilde{u}, \bar{p}, \tilde{w})(t) \|_c^2 + \int_0^t \left( \frac{1}{2} \| \partial_t \tilde{u} \|_c^2 + \frac{1}{2} \| \partial_t \bar{p} \|_c^2 + (1 - \delta) \| \tilde{w} \|_c^2 \right) \, ds
\[
\leq \| (\tilde{u}, \bar{p}, \tilde{w})(0) \|_c^2 + 2 \int_0^t (G + \delta \| \text{div} \tilde{w} \|^2) \, ds + 2 \left( \int_0^t \| F_2 \|_c \, ds \right)^2,
\]
Let \( C \) be a generic constant dependent only on \( \beta, \mu, \Omega \). It follows from (2.4b) with \( q = \text{div} \tilde{w} \) that
\[
\| \text{div} \tilde{w} \|_c^2 \leq C (\| F_2 \|_c^2 + \| \partial_t \bar{p} \|_c^2 + \| \partial_t \tilde{u} \|_c^2).
\]
The temporal derivative of (2.4c) implies that
\[
\| \partial_t \tilde{w} \|_c^2 \leq C \left( \| \partial_t F_3 \|_c^2 + \| \partial_t \tilde{w} \|_c^2 \right).
\]
The inf-sup condition of \( d \), (2.4a) and (2.4c) imply that
\[
\| \bar{p} \|_c + \| \tilde{u} \|_a \leq C \left( \| F_1 \|_c + \| F_3 \|_c + \| \tilde{w} \|_c \right)
\]
Combining (2.9)–(2.12) and choosing sufficiently small \( \delta > 0 \) completes the proof. \( \Box \)

3. Error estimator for the semi-discrete problem

Let \( \mathcal{T}_h \) be a conforming simplicial triangulation of \( \Omega \) that is aligned with \( \Gamma_1 \) and \( \Gamma_2 \). The mesh \( \mathcal{T}_h \) is shape-regular in the sense that
\[
\max_{K \in \mathcal{T}_h} \frac{r_K}{\rho_K} := \tilde{C}_{\text{shape}} < \infty,
\]
where \( r_K, \rho_K \) are radii of circumscribed and inscribed spheres of \( K \). Let \( V_h \subset V \), \( W_h \subset W \), \( Q_h \subset Q \) be suitable finite element spaces based on \( \mathcal{T}_h \). In particular, we choose \( V_h \times Q_h \) to be a stable mixed element pair for the Stokes equation, and \( W_h \times Q_h \) to be a stable mixed element pair for the mixed formulation of Poisson’s equation. It has been shown in e.g., [29, 55] that this choice leads to stable space discretization. For example, \( V_h \times Q_h \) can be chosen to be the \( (P_1 + \text{face bubble functions}) \times P_1 \) element (see [26]) and \( W_h \times Q_h \) can be the lowest order Raviart–Thomas (see [53]) or Brezzi–Douglas–Marini element (see [5]). Let \( \mathcal{F}(\mathcal{T}_h) \)
denote the collection of faces in $\mathcal{T}_h$. Let $\mathcal{P}_k(K)$ denote the space of polynomials of degree no greater than $k$ on $K$, and

$$V_{h,1} = \{ v \in V : v|_K \in [\mathcal{P}_1(K)]^d \text{ for all } K \in \mathcal{T}_h \},$$

$$B_h = \{ v \in V : v|_K \in \text{span}\{ \phi_F n_F \}_{F \subseteq \partial K} \text{ for all } K \in \mathcal{T}_h \}.$$

Here, $\phi_F$ is the face bubble function on the face $F \in \mathcal{F}(\mathcal{T}_h)$, i.e., $\phi_F = \prod_{z_j \in F} \lambda_j$ where $\lambda_j$ is the barycentric coordinate corresponding to the vertex $z_j$ in the face $F$. The triple $V_h \times Q_h \times W_h$ can be chosen as $V_h \times Q_h^0 \times W_h^0$, where

$$\bar{V}_h = V_{h,1} \oplus B_h,$$

$$Q_h^0 = \{ q \in L^2(\Omega) : q|_K \in \mathcal{P}_0(K) \text{ for all } K \in \mathcal{T}_h \},$$

$$W_h^0 = \{ z \in W : z|_K \in [\mathcal{P}_0(K)]^d + \mathcal{P}_0(K)x \text{ for all } K \in \mathcal{T}_h \}.$$

Here $x = (x_1, x_2, \ldots, x_d)^T$ is the linear position vector. In general, we assume the inclusion $W_h^0 \subseteq W_h$.

The semi-discrete version of (2.2) is to find $u_h \in H^1(0, T; V_h), p_h \in H^1(0, T; Q_h)$, and $w_h \in L^2(0, T; W_h)$ such that $u_h(0) = u_0^h, p_h(0) = p_0^h$ and

\begin{align}
(3.1a) & \quad a(u_h, v) - b(v, p_h) = (f, v), \quad v \in V_h, \\
(3.1b) & \quad c(\partial_t p_h, q) + b(\partial_t u_h, q) + d(w_h, q) = (g, q), \quad q \in Q_h, \\
(3.1c) & \quad e(w_h, z) - d(z, p_h) = 0, \quad z \in W_h.
\end{align}

Here $u_0^h \in V_h, p_0^h \in Q_h$ are some finite element approximation to $u_0$ and $p_0$. In this section, we derive a posteriori error estimation for the semi-discrete method (3.1). To this end, let

\begin{align}
(3.2) & \quad e_u = u - u_h, \quad e_w = w - w_h, \quad e_p = p - p_h.
\end{align}

It follows from (2.2) that the errors satisfy

\begin{align}
(3.3a) & \quad a(e_u, v) - b(v, e_p) = \langle r_1, v \rangle, \quad v \in V, \\
(3.3b) & \quad c(\partial_t e_p, q) + b(\partial_t e_u, q) + d(e_w, q) = \langle r_2, q \rangle, \quad q \in Q, \\
(3.3c) & \quad e(e_w, z) - d(z, e_p) = \langle r_3, z \rangle, \quad z \in W,
\end{align}

where the residuals $r_1(t) \in V'$, $r_2(t) \in Q'$, $r_3(t) \in W'$ are defined by

\begin{align*}
\langle r_1, v \rangle & \quad = (f, v) - a(u_h, v) + b(v, p_h), \\
\langle r_2, q \rangle & \quad = (g, q) - c(\partial_t p_h, q) - b(\partial_t u_h, q) - d(w_h, q), \\
\langle r_3, z \rangle & \quad = -e(w_h, z) + d(z, p_h).
\end{align*}

With the help of Lemma 2.2, we immediately obtain the following corollary.

**Corollary 3.1.** For the errors defined in (3.2) and $t \in (0, T]$, we have

\begin{align*}
&\| (e_u, e_p, e_w) (t) \|_2^2 + \| (e_u, e_p, e_w) \|_{L^2(0, T; X)}^2 \leq C_{\text{stab}} \{ \| (e_u, e_p, e_w) (0) \|_2^2 \}
\end{align*}

\begin{align*}
+ \left( \int_0^T \| r_2 \| ds \right)^2 + \int_0^T \left( \| r_1 \|_T^2 + \| r_2 \|_T^2 + \| r_3 \|_T^2 + \| \partial_t r_3 \|_T^2 \right) ds.
\end{align*}

For a $\mathbb{R}^d$-valued function $\mathbf{v} = (v_i)_{1 \leq i \leq d}$ and a scalar-valued function $v$, let

$$\text{curl } \mathbf{v} = (\partial_{x_3} v_2 - \partial_{x_2} v_3, \partial_{x_1} v_3 - \partial_{x_3} v_1, \partial_{x_3} v_1 - \partial_{x_1} v_3)$$

when $d = 3$,

$$\text{curl } \mathbf{v} = (-\partial_{x_1} v, \partial_{x_2} v, \partial_{x_3} v)^T,$$

rot $\mathbf{v} = \partial_{x_1} v_2 - \partial_{x_2} v_1$ when $d = 2$. 

\[\]
Let $\mathcal{N}_h^0$ denote the lowest order Nédélec edge element space (see [46])

$$\mathcal{N}_h^0 = \{ v \in [L^2(\Omega)]^3 : \text{curl} \, v \in [L^2(\Omega)]^3, \, v \times n = 0 \text{ on } \Gamma_1, \}
$$

and $V_h$ denote the scalar linear element space

$$V_h = \{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_1 \}.$$  

For each $K \in \mathcal{T}_h$, let $h_K = \| K \|^{\frac{1}{2}}$ denote the size of $K$, $\| \cdot \|_K$ denote the $L^2$-norm on $K$, and $\| \cdot \|_{\partial K}$ denote the $L^2$-norm on $\partial K$. To estimate the norms of $\mathcal{H}_3, \partial_r \mathcal{H}_3 \in \mathcal{W}'$, we need the following theorem, which is a combination of the $H^1$-regular decomposition (see e.g., [27, 49, 14]) and bounded quasi-interpolation operators which commute with the exterior differentiation (see [56, 14]).

**Theorem 3.2.** Let $\Omega \subset \mathbb{R}^d$ where $d = 3$ (resp. $d = 2$). There exist quasi-interpolations $\Pi_h : [L^2(\Omega)]^3 \rightarrow \mathcal{W}_h^0$ (resp. $\Pi_h : [L^2(\Omega)]^2 \rightarrow \mathcal{W}_h^0$), and $J_h : [L^2(\Omega)]^3 \rightarrow \mathcal{N}_h^0$ (resp. $J_h : L^2(\Omega) \rightarrow V_h$) such that $\Pi_h \text{curl} = \text{curl} J_h$. In addition, for any $z \in \mathcal{W}$, there exist $\phi \in \mathcal{V}$ (resp. $\phi \in H^1(\Omega)$, $\phi|_{\Gamma_1} = 0$) and $\phi \in \mathcal{V}$, such that

$$z = \text{curl} \phi + \phi, \quad \Pi_h z = \text{curl} J_h \phi + \Pi_h \phi,$$

and

$$\sum_{K \in \mathcal{T}_h} h_K^{-2} \| \phi - J_h \phi \|_K^2 + h_K^{-2} \| \phi - \Pi_h \phi \|_K^2
$$

$$+ h_K^{-1} \| \phi - J_h \phi \|_{\partial K}^2 + h_K^{-1} \| \phi - \Pi_h \phi \|_{\partial K}^2 \leq C_{\text{reg}} \| z \|_{\mathcal{W}},$$

where $C_{\text{reg}}$ depends only on $K$, $\Omega$, $\Gamma_1$, $\mathcal{C}_{\text{shape}}$.

Theorem 3.2 or its variants are widely used in the a posteriori error estimation of stationary problems based on $H(\text{div})$ or $H(\text{curl})$, see, e.g., [9, 56, 31, 14, 11, 42, 41, 28].

Now we are in a position to prove a posteriori error estimator of the system given in (3.1). For each face $F$, let $n_F$ be a unit normal to $F$ where $n_F$ is chosen to be outward pointing when $F$ is a boundary face. For each interior $F \in \mathcal{F}(\mathcal{T}_h)$ shared by $K_1$, $K_2 \in \mathcal{T}_h$ and a piecewise $H^1$-function $\chi$, let $[\chi] := \chi|_{K_1} - \chi|_{K_2}$ denote the jump across $F$, where $n_F$ is pointing from $K_1$ to $K_2$. For boundary face $F \subset \Gamma_2$ that is a face of $K \in \mathcal{T}_h$, let $[\chi] := \chi|_{K}$. For any boundary face $F \subset \Gamma_1$, we set $[\chi] : = 0$. Regarding the mesh $\mathcal{T}_h$, we use the following error indicators

$$\mathcal{E}_h^1(u_h, p_h, f) := \sum_{K \in \mathcal{T}_h} \{ h_K^2 \| f + \text{div} \sigma(u_h) - \alpha \nabla p_h \|_K^2
$$

$$+ \sum_{F \in \mathcal{F}(\mathcal{T}_h), F \subset \partial K} h_K \| [\sigma(u_h) - \alpha p_h I n_F] \|_F^2 \},
$$

$$\mathcal{E}_{\mathcal{T}_h}(\partial_t u_h, \partial_t p_h, w_h, g) := \sum_{K \in \mathcal{T}_h} \| g - \beta \partial_t p_h - \text{div} \partial_t u_h - \text{div} w_h \|_K^2.
$$

Another error estimator is

$$\mathcal{E}_h^3(p_h, w_h) := \sum_{K \in \mathcal{T}_h} \{ h_K^2 \| K^{-1} w_h + \nabla p_h \|_K^2 + h_K^2 \| \text{curl}(K^{-1} w_h) \|_K^2
$$

$$+ \sum_{F \in \mathcal{F}(\mathcal{T}_h), F \subset \partial K} h_K \| (K^{-1} w_h) \times n_F \|_F^2 + h_K \| [p_h] \|_{1,F}^2 \} \text{ when } d = 3,$$
and when \(d = 2\) (for a two dimensional problem)
\[
\mathcal{E}_h^3(p_h, w_h) := \sum_{K \in \mathcal{T}_h} \left\{ h_K^2 \| \mathbf{K}^{-1} \mathbf{w}_h + \nabla p_h \|_{K}^2 + h_K^2 \| \text{rot}(\mathbf{K}^{-1} \mathbf{w}_h) \|_{K}^2 \right\} \\
+ \sum_{F \in \mathcal{F}(\mathcal{T}_h), F \subset \partial K} h_K \left\{ \| \mathbf{K}^{-1} \mathbf{w}_h \cdot \mathbf{t}_F \|_{F}^2 + h_K \| p_h \|_{F}^2 \right\},
\]
where \(\mathbf{t}_F\) is a unit tangent vector to \(F\). The next theorem presents a posteriori error estimates of the semi-discrete method (3.1).

**Theorem 3.3.** When \(d = 2\) or 3, there exists a constant \(C_{\text{rel}}\) dependent only on \(\mu, \alpha, \beta, \mathbf{K}, \Omega, \Gamma_1\) and the shape regularity of \(\mathcal{T}_h\), such that
\[
\left\| (e_u, e_p, e_w)(t) \right\|^2 + \left\| (e_u, e_p, e_w) \right\|^2_{L^2(0, T; X)} \leq C_{\text{rel}} \left\{ \left\| (e_u, e_p, e_w)(0) \right\|^2 + \right. \\
+ \left. \int_0^t \mathcal{E}_h^2\left( \partial_t u_h, \partial_t p_h, w_h, g \right) d\tau \right\}^2 + \int_0^t \left( \mathcal{E}_h^1(u_h, p_h, f) + \mathcal{E}_h^1(\partial_t u_h, \partial_t p_h, \partial_t f) \right) \\
+ \mathcal{E}_h^2(\partial_t u_h, \partial_t p_h, w_h, g) + \mathcal{E}_h^3(p_h, w_h) + \mathcal{E}_h^3(\partial_t p_h, \partial_t w_h) d\tau \right\}.
\]

**Proof.** We focus on the case \(d = 3\) since the proof when \(d = 2\) is similar. In the proof, we use \(C\) to denote generic constant dependent only on \(\mu, \alpha, \beta, \mathbf{K}, \Omega, \Gamma_1\). In view of Corollary 3.1, it remains to estimate the norm of each residual. Let \(I_h : \mathbf{V} \rightarrow \mathbf{V}_h\) denote the Clément interpolation (see [13, 61]). Thanks to (3.1a), it holds that for each \(v \in \mathbf{V}\),
\[
\langle r_1, v \rangle = (f, v - I_h v) - a(u_h, v - I_h v) + b(v - I_h v, p_h).
\]

Element-wise integration by parts leads to
\[
\langle r_1, v \rangle = (f, v - I_h v) + \sum_{K \in \mathcal{T}_h} - \int_K \sigma(u_h) : \varepsilon(v - I_h v) + \int_K \text{div}(\alpha(v - I_h v)) p_h
\]
\[
= \sum_{K \in \mathcal{T}_h} \int_K (f + \text{div}(\sigma(u_h)) - \alpha \nabla p_h) \cdot (v - I_h v)
\]
\[
+ \sum_{F \in \mathcal{F}(\mathcal{T}_h), F \subset \partial K} \int_F \left( -\sigma(u_h) + \alpha p_h \mathbf{n}_F \right) \cdot (v - I_h v).
\]

Combining the previous equation with the Cauchy–Schwarz inequality and shape-regularity of \(\mathcal{T}_h\), we obtain
\[
\langle r_1, v \rangle \leq C\mathcal{E}_h^1(u_h, p_h, f)^{\frac{1}{2}} \left( \sum_{K \in \mathcal{T}_h} h_K^{-2} \| v - I_h v \|_{K} \right)^{\frac{1}{2}} + h_K^{-1} \| v - I_h v \|_{\partial K} \leq C\| v \|_{H^1(\Omega)}^{\frac{1}{2}}.
\]

It then follows from (3.6), the well-known approximation result
\[
\sum_{K \in \mathcal{T}_h} h_K^{-2} \| v - I_h v \|_{K}^2 + h_K^{-1} \| v - I_h v \|_{\partial K} \leq C\| v \|_{H^1(\Omega)}^{\frac{1}{2}}
\]
and the Korn’s inequality (cf. [33])
\[
\| v \|_{H^1(\Omega)} \leq C\| v \|_{a}, \quad v \in \mathbf{V},
\]
that
\[
\| r_1 \| \leq C\mathcal{E}_h^1(u_h, p_h, f)^{\frac{1}{2}}.
\]

Similarly (3.5) implies that for each \(v \in \mathbf{V}\),
\[
\langle \partial_t r_1, v \rangle = (\partial_t f, v - I_h v) - a(\partial_t u_h, v - I_h v) + b(v - I_h v, \partial_t p_h).
\]
Then the next estimate

\begin{equation}
\| \partial_t r_1 \| \leq C \mathcal{E}_{T_h}^3 (\partial_t u_h, \partial_t p_h, \partial_t f).
\end{equation}

can be proved in the same way as (3.8). The norm of \( r_2 \) is trivially estimated by

\begin{equation}
\| r_2 \| \leq C \| g - \beta \partial_t p_h - \text{div} \partial_t u_h - \text{div} w_h \|.
\end{equation}

To estimate \( \| r_3 \| \), we use (3.1c) to obtain

\begin{equation}
(r_3, z) = -\varepsilon (w_h, z - \Pi_h z) + d(z - \Pi_h z, p_h).
\end{equation}

Due to Theorem 3.2, there exists \( \varphi \in V \) and \( \phi \in V \) such that

\begin{equation}
z - \Pi_h z = \text{curl}(\varphi - J_h \varphi) + \phi - \Pi_h \phi,
\end{equation}

where \( \varphi \) and \( \phi \) satisfy (3.4). Using (3.11), (3.12), and element-wise integration by parts, we arrive at

\begin{align}
(r_3, z) &= -(K^{-1} w_h, \text{curl}(\varphi - J_h \varphi)) - (K^{-1} w_h, \varphi - \Pi_h \phi) + (\text{div}(\varphi - \Pi_h \phi), p_h) \\
&= \sum_{K \in T_h} - \int_K \text{curl}(K^{-1} w_h) \cdot (\varphi - J_h \varphi) - \int_K (K^{-1} w_h + \nabla p_h) \cdot (\phi - \Pi_h \phi) \\
&+ \sum_{F \in F(T_h)} \int_F -[(K^{-1} w_h) \times n_F] \cdot (\varphi - J_h \varphi) + \int_F [p_h](\phi - \Pi_h \phi) \cdot n_F.
\end{align}

It then follows from the previous equation, the Cauchy–Schwarz inequality, and (3.4) that

\begin{equation}
\langle r_3, z \rangle \leq C \mathcal{E}_{T_h}^3 (p_h, w_h)^{\frac{1}{2}} \| z \|_W.
\end{equation}

Similarly, it holds that

\begin{equation}
\| \partial_t r_3 \| \leq C \mathcal{E}_{T_h}^3 (\partial_t p_h, \partial_t w_h)^{\frac{1}{2}}.
\end{equation}

Combining (3.8)–(3.10), (3.13), (3.14) completes the proof. \( \square \)

4. Fully discrete method

Let \( 0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T \) and \( \tau_n = t_n - t_{n-1} \) for \( n = 1, 2, \ldots, N \). Let \( T^n_h \) be a conforming simplicial triangulation of \( \Omega \) aligned with \( \Gamma_1 \) and \( \Gamma_2 \). Let \( V^n_h, Q^n_h, W^n_h \) be finite element subspaces of \( V, Q, W \) described in Section 3 based on grid \( T^n_h \), respectively. \( \{ T^n_h \}_{n=0}^N \) is uniformly shape-regular w.r.t. \( n \), that is,

\[ \max_{0 \leq n \leq N} \max_{K \in T^n_h} \frac{r_K}{\rho_K} := C_{\text{shape}} < \infty. \]

Given a sequence \( \{ \chi^n \}_{n=0}^N \), we define the backward difference as

\[ \delta_t \chi^n = \frac{\chi^n - \chi^{n-1}}{\tau_n}, \]

and the continuous linear interpolant as

\[ \chi^T(t) = \frac{t - t_{n-1}}{\tau_n} \chi^n + \frac{t_n - t}{\tau_n} \chi^{n-1}, \quad t \in [t_{n-1}, t_n]. \]
It is noted that $\partial_t \chi^r = \delta_t \chi^n$ over $[t_{n-1}, t_n]$. Let $u_h^n, p_h^n, w_h^n$ be suitable approximations to $u(0), p(0), w(0)$, and $f^n = f(t_n), g^n = g(t_n)$. The fully discrete scheme for (2.2) is to find $u_h^n \in V^n_h, p_h^n \in Q^n_h, w_h^n \in W^n_h$ with $n = 1, 2, \ldots, N$, such that

\begin{align}
(4.1a) & \quad a(u_h^n, v) - b(v, p_h^n) = \langle f^n, v \rangle, \quad v \in V^n_h, \\
(4.1b) & \quad c(\delta_t p_h^n, q) + b(\delta_t u_h^n, q) + d(w_h^n, q) = \langle g^n, q \rangle, \quad q \in Q^n_h, \\
(4.1c) & \quad e(w_h^n, z) - d(z, p_h^n) = 0, \quad z \in W^n_h.
\end{align}

(4.1) is based on the implicit Euler time discretization. Since we aim to use Lemma 2.2, let $u_h^n(t), p_h^n(t), w_h^n(t)$ be the continuous linear interpolants of $u_h^n, p_h^n, w_h^n$ defined above. Let

\begin{align}
E_u &= u - u_h^n, \quad E_p = p - p_h^n, \quad E_w = w - w_h^n.
\end{align}

Rewriting (2.2), we obtain the error equation

\begin{align}
(4.3) & \quad a(E_u, v) - b(v, E_p) = \langle R_1, v \rangle, \quad v \in V, \\
& \quad c(\delta_t E_p, q) + b(\delta_t E_u, q) + d(E_w, q) = \langle R_2, q \rangle, \quad q \in Q, \\
& \quad e(E_w, z) - d(z, E_p) = \langle R_3, z \rangle, \quad z \in W,
\end{align}

where residuals $R_1(t) \in V', R_2(t) \in Q', R_3(t) \in W'$ for each $t \in [0, T]$ are

\begin{align}
\langle R_1, v \rangle & := \langle f, v \rangle - a(u_h^n, v) + b(v, p_h^n), \\
\langle R_2, q \rangle & := \langle g, q \rangle - c(\delta_t p_h^n, q) - b(\delta_t u_h^n, q) - d(w_h^n, q), \\
\langle R_3, z \rangle & := -e(w_h^n, z) + d(z, p_h^n).
\end{align}

Similarly to the error analysis of the semi-discrete problem, it suffices to analyze the norm of the above three residuals. However, the mesh used in the fully discrete scheme is allowed to change at different time levels. Given two triangulations $T_{h_1}$ and $T_{h_2}$ of $\Omega$, let $T_{h_1} \lor T_{h_2}$ denote the minimal common refinement, i.e., $T_{h_1} \lor T_{h_2}$ is the coarsest conforming triangulation that is a refinement of both $T_{h_1}$ and $T_{h_2}$. Similarly, let $T_{h_1} \land T_{h_2}$ denote the maximal common coarsening, i.e., $T_{h_1} \land T_{h_2}$ is the finest conforming triangulation that is a coarsening of both $T_{h_1}$ and $T_{h_2}$. The operations $\land$ and $\lor$ on triangulations are widely used in adaptivity literature, see, e.g., [8, 15].

To handle simultaneously $(u_h^n, p_h^n, w_h^n)$ and $(u_h^{n-1}, p_h^{n-1}, w_h^{n-1})$, we assume for $T_h^n, T_h^{n-1}$ with each $1 \leq n \leq N$, the maximal common coarsening $T_h^n \land T_h^{n-1}$ and the minimal common refinement $T_h^n \lor T_h^{n-1}$ exist. This assumption is true when $\{T_h^n\}_{h \to 0}$ are newest vertex bisection refinement of the same macrotriangulation, cf. [36, 8]. In addition, there is a uniform bound on the ratio of the sizes of elements in $K \in T_h^n \lor T_h^{n-1}$ and of elements $K' \in T_h^n \land T_h^{n-1}$ contained in $K$, that is,

\begin{align}
(4.5) & \quad \sup_{1 \leq n \leq N} \sup_{K' \subset K, K' \in T_h^n \lor T_h^{n-1}} \frac{h_K}{h_{K'}} := C_{\text{ratio}} < \infty.
\end{align}

Similar assumptions are made in a posteriori error estimates of the heat equation [60]. Within the interval $[t_{n-1}, t_n]$, we split the residuals into

\begin{align}
(4.6a) & \quad R_1 = f - f_h^n + S_1^n + T_1^n, \\
(4.6b) & \quad R_2 = g - g_h^n + S_2^n + T_2^n, \\
(4.6c) & \quad R_3 = S_3^n + T_3^n.
\end{align}
where the spatial residuals \( S^n_1 \in V' \), \( S^n_2 \in Q' \), \( S^n_3 \in W' \) are defined as
\[
\langle S^n_1, v \rangle := \langle f^n_h, v \rangle - a(u^n_h, v) + b(v, p^n_h),
\]
\[
\langle S^n_2, q \rangle := \langle g^n_h, q \rangle - c(\delta^n p^n_h, q) - b(\delta^n u^n_h, q) - d(w^n_h, q),
\]
\[
\langle S^n_3, z \rangle := -e(w^n_h, z) + d(z, p^n_h),
\]

the temporal residuals \( T^n_1(t) \in V' \), \( T^n_2(t) \in Q' \), \( T^n_3(t) \in W' \) for \( t \in [t_{n-1}, t_n] \) are
\[
\langle T^n_1, v \rangle := a(u^n_h - u^n_{t_h}, v) - b(v, p^n_h - p^n_{t_h}),
\]
\[
\langle T^n_2, q \rangle := d(w^n_h - w^n_{t_h}, q),
\]
\[
\langle T^n_3, z \rangle := e(w^n_h - w^n_{t_h}, z) - d(z, p^n_h - p^n_{t_h}).
\]

By (4.4), the temporal derivatives of \( R_1 \) and \( R_3 \) over \([t_{n-1}, t_n]\) are
\[
(4.7a) \quad \langle \partial_t R_1, v \rangle = \langle \partial_t f - \delta t f^n_{h}, v \rangle + \langle \delta_t S^n_1, v \rangle,
\]
\[
(4.7b) \quad \langle \partial_t R_3, v \rangle = \langle \delta_t S^n_3, v \rangle.
\]

Recall error estimators in Section 3. We use the following fully discrete spatial error indicators
\[
E^n_1 := E^n_{T_h}(u^n_h, p^n_h, f^n_h), \quad E^n_{1,t} := E^n_{T_h \cup T_{h-1} \cup T_{h-2}}(\delta_t u^n_h, \delta_t p^n_h, \delta_t f^n_h),
\]
\[
E^n_2 := E^n_{T_h}(\delta_t u^n_h, \delta_t p^n_h, w^n_h, g^n_h),
\]
\[
E^n_3 := E^n_{T_h}(p^n_h, w^n_h), \quad E^n_{3,t} := E^n_{T_h \cup T_{h-1} \cup T_{h-2}}(\delta_t p^n_h, \delta_t w^n_h).
\]

Throughout the rest, we say \( A \lesssim B \) provided \( A \leq CB \), where \( C \) is a constant dependent only on \( \mu, \alpha, \beta, K, \Omega, \Gamma_1, C_{\text{shape}}, C_{\text{ratio}} \). Since the spatial residuals are time-independent, their norms can be estimated as in the proof of Theorem 3.3.

**Lemma 4.1.** For \( 1 \leq n \leq N \), it holds that
\[
(4.8a) \quad \| S^n_1 \|^2 \lesssim E^n_1,
\]
\[
(4.8b) \quad \| \delta_t S^n_1 \|^2 \lesssim E^n_{1,t},
\]
\[
(4.8c) \quad \| S^n_2 \|^2 \lesssim E^n_2,
\]
\[
(4.8d) \quad \| S^n_3 \|^2 \lesssim E^n_3,
\]
\[
(4.8e) \quad \| \delta_t S^n_3 \|^2 \lesssim E^n_{3,t}.
\]

**Proof.** For \( v \in V \), let \( v_h \) be the Clément interpolant on \( T^n_h \). It follows from (4.1a) and element-wise integration by parts that
\[
\langle S^n_1, v \rangle = \langle S^n_1, v - v_h \rangle = \sum_{K \in T_h} \int_K (f^n_h + \text{div } \sigma(u_h) - \alpha \nabla p_h) \cdot (v - v_h)
\]
\[
- \sum_{F \in F(T_h)} \int_F (|\sigma(u_h) - \alpha p_h I| n_F) \cdot (v - v_h).
\]
Using (4.9) and the same analysis of estimating \( \| r_1 \| \) in Theorem 3.3, we obtain
\[
\| S^n_1 \| = \sup_{v \in V, \| v \|_{a = 1}} \langle S^n_1, v \rangle \lesssim (E^n_1)^{1/2}.
\]

For any \( v \in V \), let \( v_h \) be the Clément interpolant on \( T^n_h \). Using (4.5), (4.1a), and integrating by parts over \( T^n_h \cup T^{n-1}_h \), and following again the same
analysis of estimating $\|r_1\|$, in Theorem 3.3, we obtain a similar estimate:
\[
\|\delta_i S^\alpha_i\| = \sup_{v \in V, \|v\|_\alpha = 1} (\delta_i S^\alpha_i, v) = \sup_{v \in V, \|v\|_\alpha = 1} \langle \delta_i S^\alpha_i, v - \tilde{v}_h \rangle
\]
\[
= \sup_{v \in V, \|v\|_\alpha = 1} (\delta_i f^\alpha_i, v - \tilde{v}_h) - a(\delta_i u^\alpha_i, v - \tilde{v}_h) + b(v - \tilde{v}_h, \delta_i p^\alpha_i) \lesssim (E^\alpha_i(t))^2.
\]
The remaining estimates can be proved in the same way.

Proof. Applying Lemma 2.2 to (4.3) yields $\|f - f_h\|_\ell$ and $\|\partial_t f - \delta_t f_h\|_\ell = \|\partial_t f - \delta_t f_h\|_\ell$. We present the first main result of this paper in the following theorem.

**Theorem 4.2.** For $1 \leq i \leq n$, let $f^i_h$ be the $L^2$-projection of $f^i$ onto $V^i_h$, $g^i_h$ the $L^2$-projection onto $Q^i_h$. There exists a constant $C_{drel}$ dependent only on $\mu$, $\alpha$, $\beta$, $K$, $\Omega$, $C_{shape}, C_{ratio}$ such that for $n = 1, 2, \ldots, N$, the error defined in (4.2) satisfy
\[
\|E, E_p, E_w(t_n)\|^2 + \|E, E_p, E_w\|^2_{L^2(0, t_n, X)}
\]
\[
\leq C_{drel} \left\{ \|E, E_p, E_w(0)\|^2 + \left( \sum_{i=1}^n \tau_i \tilde{E}_{time}^i + \tau_i (E^i_2)^2 + \tilde{E}_{data}^i \right) \right. 
\]
\[
+ \left. \sum_{i=1}^n \tau_i \tilde{E}_{time}^i + \tau_i \tilde{E}_{space}^i + \tilde{E}_{data}^i \right\},
\]
where
\[
\tilde{E}_{time}^i = \| \text{div}(w^i_h - w^{i-1}_h) \|_\alpha, \quad \tilde{E}_{data}^i = \int_{t_{i-1}}^{t_i} \| g - g^i_h \| dt,
\]
\[
E_{time}^i = \| u^i_h - u^{i-1}_h \|^2 + \| p^i_h - p^{i-1}_h \|^2 + \| w^i_h - w^{i-1}_h \|^2_\ell,
\]
\[
E_{space}^i = E_{1, b}^i + E_{3}^i + E_{1}^i + E_{3, t}^i,
\]
\[
E_{data}^i = \int_{t_{i-1}}^{t_i} (\| f - f^i_h \|^2 + \| \partial_t f - \delta_t f^i_h \|^2 + \| g - g^i_h \|^2) dt.
\]

**Proof.** Applying Lemma 2.2 to (4.3) yields
\[
\|E, E_p, E_w(t_n)\|^2 + \|E, E_p, E_w\|^2_{L^2(0, t_n, X)}
\]
\[
\leq C_{stab} \left\{ \|E, E_p, E_w(0)\|^2 + \left( \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \| R_2 \| dt \right)^2 \right. 
\]
\[
+ \left. \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \| R_1 \|^2 + \| \partial_t R_1 \|^2 + \| R_2 \|^2 + \| R_3 \|^2 + \| \partial_t R_3 \|^2 \right) \right\} dt.
\]
For any $v \in V$, the continuity of $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ implies that
\[
\| T^i_1 \|_\ell = \sup_{v \in V, \|v\|_\alpha = 1} \langle T^i_1, v \rangle \lesssim \| u^i_h - u^i_{h-1} \|_\alpha + \| p^i_h - p^i_{h-1} \|_c.
\]
A combination of (4.6a), (4.8a) and (4.10) then shows that when $t \in [t_{i-1}, t_i]$,
\[
\| R^i_1 \|, \lesssim \| f - f^i_h \|_\ell + \| u^i_h - u^i_h \|_\alpha + \| p^i_h - p^i_h \|_c + (E^i_1)^{\frac{1}{2}}.
\]
For $t \in [t_{i-1}, t_i]$, it is readily checked that
\[
\| u^i_h - u^i_h \|_\alpha = \frac{t_i - t}{\tau_i} \| u^i_h - u^{i-1}_h \|_\alpha, \quad \| p^i_h - p^i_h \|_c = \frac{t_i - t}{\tau_i} \| p^i_h - p^{i-1}_h \|_c.
\]
Integrating (4.11) over \([t_{i-1}, t_i]\) and using (4.12), we obtain
\[
\int_{t_{i-1}}^{t_i} \| R_1 \|^2 dt \lesssim \int_{t_{i-1}}^{t_i} \| f - f_h^i \|^2 dt + \tau_i \| u_h^{i-1} \|^2 + \tau_i \| p_h^{i-1} \|^2 + \tau_i E_{1, t}^i.
\]
On the other hand, using (4.7a) and (4.8b), we obtain for \( t \in [t_{i-1}, t_i], \)
\[
\| \partial_t R_1 \| \lesssim \| \partial_t f - \delta_t f_h^i \| + (E_{1, t}^i)^{1/2}.
\]
Similarly, using (4.6), (4.7b), Lemma 4.1, and (4.12), one can estimate \( R_3, \partial_t R_3, R_2 \) and obtain the following bounds
\[
\int_{t_{i-1}}^{t_i} \| R_3 \|^2 dt \lesssim \tau_i \| w_h^{i-1} - w_h^{i-2} \|^2 + \tau_i \| p_h^{i-1} - p_h^{i-2} \|^2 + \tau_i E_{3, t}^i,
\]
(4.15)
\[
\int_{t_{i-1}}^{t_i} \| \partial_t R_3 \|^2 dt \lesssim \tau_i E_{3, t}^i,
\]
and
\[
\int_{t_{i-1}}^{t_i} \| R_2 \|^2 dt \lesssim \int_{t_{i-1}}^{t_i} \| g - g_h^{i-1} \| dt + \tau_i \| \text{div}(w_h^{i-1} - w_h^{i}) \| + \tau_i (E_2^i)^{1/2},
\]
(4.16)
\[
\int_{t_{i-1}}^{t_i} \| R_2 \|^2 dt \lesssim \int_{t_{i-1}}^{t_i} \| g - g_h^{i-1} \|^2 dt + \tau_i \| \text{div}(w_h^{i-1} - w_h^{i}) \|^2 + \tau_i E_{2, t}^i.
\]
Combining (4.13)–(4.16) completes the proof.

**Remark 4.3.** The first two terms in \( E_{\text{data}}^i \) can be further estimated by
\[
\| f - f_h^i \| \lesssim \sum_{K \in T_h^i} h_K^2 \| f - f_h^i \|^2_K,
\]
\[
\| \partial_t f - \delta_t f_h^i \| \lesssim \sum_{K \in T_h^i \cup T_h^{i-1}} h_K^2 \| \partial_t f - \delta_t f_h^i \|^2_K.
\]

5. Lower bound

In this section, we show that \( \tau_n E_{\text{time}}^n \) and \( \tau_n E_{\text{space}}^n \) are lower bounds of the space-time discretization error of the fully discrete scheme (4.1). First we present a lemma comparing the spatial residual with the spatial error indicators. Since the spatial error estimators and residuals are time-independent, the proof follows from the well-known Verfürth bubble function technique for a posteriori error estimates for stationary Stokes and Poisson’s equations, see, e.g., [59, 2, 14]. Throughout the rest, \( C \) is a generic constant that depends only on \( \lambda, \mu, \alpha, \beta, K, \Omega, \Gamma_1, C_{\text{shape}}, C_{\text{ratio}} \). Hence the constant in lower bounds may not be locking free.

**Lemma 5.1.** Let \( \lambda, \mu, K \) be piecewise constants over \( T_h^n \). For \( 1 \leq n \leq N \), it holds that
\[
\begin{align*}
E_{1}^n &\lesssim C \| S^n_h \|^2, \\
E_{1,t}^n &\lesssim C \| \delta_t S^n_h \|^2, \\
E_{2}^n &\lesssim \| S^n_h \|^2, \\
E_{3}^n &\lesssim \| S^n_h \|^2, \\
E_{3,t}^n &\lesssim \| \delta_t S^n_h \|^2.
\end{align*}
\]
Proof. To prove (5.1a), it suffices to find \( v \in V \), such that
\[
\mathcal{E}_1^n \leq C(S^n_1, v), \quad \|v\|^2 \leq C\mathcal{E}_1^n.
\]
For each \( K \in T_h \), let \( R_K = (f_k^n + \text{div} \sigma(u_h) - \alpha \nabla p_h)|_K \) and let \( \psi_K \) denote the volume bubble function supported on \( K \). For each \( F \in F(T_h) \), let \( J_F = -[\sigma(u_h) - \alpha p_h |n_F, \) and \( \psi_F \) denote the face bubble function supported on \( \Omega_F \), union of neighboring elements of \( F \). \( \psi_K \) and \( \psi_F \) are scaled so that their maximum is 1. \( v \) is defined as
\[
v = \gamma_1 \sum_{K \in T_h} h_K^2 R_K \psi_K + \gamma_2 \sum_{F \in F(T_h)} h_F J_F \psi_F,
\]
where \( h_F \) is the diameter of \( F \), \( \gamma_1, \gamma_2 \) are undetermined constants. Using the Cauchy–Schwarz inequality and finite overlapping of supports of \( \{\psi_K\} \) and \( \{\psi_F\} \), one case easily show that \( \|v\|^2 \leq C\mathcal{E}_1^n \). On the other hand, (4.9) implies
\[
\langle S^n_1, v \rangle \geq C\mathcal{E}_1^n \text{ then follows from Young's inequality and suitable}\ \gamma_1, \gamma_2, \text{see Lemma} 5.1 \text{ of [60] for details. Other lower bounds are proved in the same way.} \quad \Box
\]

Remark 5.2. Based on the \( \| \cdot \|_q \)-norm, it seems that the dependence on \( \lambda \) in lower bounds (5.1a) and (5.1b) cannot be avoided. To obtain an error estimator that is a robust lower bound, one can apply the analysis here to the four- or five-field formulation [38, 1] in Biot’s consolidation model.

We present the second main result in the following theorem. Similar technique in the proof was used in [60] for proving the lower bound in a posteriori error estimation for the primal formulation of the heat equation.

Theorem 5.3. Let \( \lambda, \mu, K \) be piecewise constants on \( T^n_h \). For \( n = 1, 2, \ldots, N \),
\[
\tau_n \mathcal{E}^n_{\text{time}} + \tau_n \mathcal{E}^n_{\text{space}} \leq C \left\{ \|E_u, E_p, E_w\|_{L^2(\tau_{n-1}, \tau_n; X)} \right\} + \int_{\tau_{n-1}}^{\tau_n} \left\{ \|f - f_h^n\|^2 + \|g - g_h^n\|^2 \right\} dt.
\]

Proof. First by (2.2) and definitions of residuals in (4.4), we have
\[
(5.2a) \quad \|R_1\| \lesssim \|E_u\|_a + \|E_p\|_c,
(5.2b) \quad \|R_2\| \lesssim \|\partial_t E_p\|_c + \|\partial_t \text{ div} E_u\|_c + \|\text{ div} E_w\|,
(5.2c) \quad \|R_3\| \lesssim \|E_w\|_c + \|E_p\|_c,
(5.2d) \quad \|\partial_t R_1\| \lesssim \|\partial_t E_u\|_a + \|\partial_t E_p\|_c,
(5.2e) \quad \|\partial_t R_3\| \lesssim \|\partial_t E_w\|_c + \|\partial_t E_p\|_c.
\]
Consider the bilinear form
\[
B(w, p; z, q) = c(w, z) - d(z, p) + d(w, q)
\]
of the mixed formulation of the elliptic equation. Due to the inf-sup condition of \( B \), there exist \( z \in W \) and \( q \in Q \) with \( \|z\|_W = 1, \|q\|_c = 1 \) such that
\[
\|w^n_h - w^n_i\|_W + \|p^n_h - p^n_i\| \lesssim B(w^n_h - w^n_i, p^n_h - p^n_i, z, q)
\]
\[
= (T^n_2, q) + (T^n_3, z), \quad t \in [t_{n-1}, t_n].
\]
Using the previous estimate, the triangle and Cauchy–Schwarz inequalities, we obtain
\[
\|w^n_t - u^n_t\| W + \|p^n_h - p^n_h\|
\]
\[
(5.3)
\]
where \( C \) is independent of \( n \), \( h \), \( p \). - Setting \( v = (u^n_h - u^n_h)/\|u^n_h - u^n_h\|a \) in the definition of \( T^n_1 \) we have
\[
\|u^n_h - u^n_h\|a = (T^n_1, v) + b(v, p^n_h - p^n_h)
\]
\[
(5.4)
\]
where \( C \) is independent of \( n \), \( h \), \( p \). - A combination of (5.3), (5.4), (4.12) and Lemma 4.1 shows that
\[
\tau_n E^n_{time} \leq \int_{t_{n-1}}^{t_n} \|f - f^n_h\|^2 + \|g - g^n_h\|^2 + \|R_1\|^2
\]
\[
+ \|R_2\|^2 + \|R_3\|^2 dt + \tau_n (E^n_1 + E^n_2 + E^n_3).
\]
(5.5) - It remains to estimate \( E^n_1, E^n_2, \) and \( E^n_3 \). Let
\[
\phi(t) = (\alpha + 1) \left( \frac{t - t_{n-1}}{\tau_n} \right)^\alpha,
\]
where \( \alpha > 0 \) is a constant that will be specified later. It follows from Lemma 5.1 and the triangle inequality that
\[
\tau_n (E^n_1 + E^n_2 + E^n_3) = \int_{t_{n-1}}^{t_n} \phi(t) (E^n_1 + E^n_2 + E^n_3) dt
\]
\[
(5.6)
\]
where the generic constant \( C \) is independent of \( \alpha \). For any \( v \in V \) with \( \|v\|_a = 1 \), direct calculation shows that
\[
\int_{t_{n-1}}^{t_n} \phi(t) \langle T^n_1, v \rangle^2 dt = \int_{t_{n-1}}^{t_n} \phi(t) \langle a(u^n_h - u^n_h, v) - b(v, p^n_h - p^n_h) \rangle^2 dt
\]
\[
= \{a(u^n_h - u^n_h, v) - b(v, p^n_h - p^n_h)\}^2 \int_{t_{n-1}}^{t_n} \phi(t) \left( \frac{t_n - t}{\tau_n} \right)^2 dt
\]
\[
\leq C \int_{t_{n-1}}^{t_n} \phi(t) (\|T^n_1\|^2 + \|T^n_2\|^2 + \|T^n_3\|^2) dt.
\]
(5.7) - Hence
\[
\int_{t_{n-1}}^{t_n} \phi(t) \langle T^n_1, v \rangle^2 dt \leq \sup_{v \in V, \|v\|_a = 1} \int_{t_{n-1}}^{t_n} \phi(t) \langle T^n_1, v \rangle^2 dt \leq C \tau_n E^n_{time} F(\alpha).
\]
Similarly, we have
\begin{equation}
\tau_n \leq C(\alpha + 1) \int_{t_n}^{t_{n+1}} \left( \|f_n\|^2 + \|f - f_n\|^2 + \|g - g_n\|^2 + \|R_1\|^2 + \|R_2\|^2 + \|R_3\|^2 \right) dt.
\end{equation}

Combining (5.6), (5.7) and (5.8), we obtain
\begin{equation}
\tau_n (E_1^n + E_2^n + E_3^n) \leq C(\alpha + 1) \int_{t_n}^{t_{n+1}} \left( \|f_n\|^2 + \|f - f_n\|^2 + \|g - g_n\|^2 + \|R_1\|^2 + \|R_2\|^2 + \|R_3\|^2 \right) dt.
\end{equation}

Note that $F(\alpha) \to 0$ as $\alpha \to \infty$. It then follows from (5.5) and (5.9) with sufficiently large $\alpha$ that
\begin{equation}
\tau_n E_{\text{time}}^n \leq C \int_{t_n}^{t_{n+1}} \left( \|f_n\|^2 + \|f - f_n\|^2 + \|g - g_n\|^2 + \|R_1\|^2 + \|R_2\|^2 + \|R_3\|^2 \right) dt.
\end{equation}

In the end, using Lemma 5.1 and (4.7a), (4.7b), we have
\begin{equation}
E_{1,t}^n + E_{3,t}^n \leq C(\|\delta t S^n_t\|^2 + \|\delta t S^n_3\|^2) \leq C(\|\partial_t f - \delta t f\|^2 + \|\partial_t R_1^n\|^2 + \|\partial_t R_3^n\|^2).
\end{equation}

Combining (5.10), (5.11), (5.12), and (5.2) finishes the proof. \hfill \Box

In practice, one can use $E_{\text{time}}^n$ and $E_{\text{data}}^n$ to adjust the time step size and $E_{\text{space}}^n$ to refine and coarsen the temporal mesh. Readers are referred to e.g., [61] for adaptive algorithms based on space-time error estimators and [12, 34, 25] for convergence analysis of adaptive methods for the heat equation.

In the end, we present a new error estimator of mixed methods for (2.3). The fully discrete scheme (4.1) with $u_h = v = 0$ reduces to
\begin{align}
\tag{5.13a}
& c(\delta t p^n_h, q) + d(w^n_h, q) = (g^n, q), \quad q \in Q_h^n,
\tag{5.13b}
& e(w^n_h, z) - d(z, p^n_h) = 0, \quad z \in W_h^n,
\end{align}

which is a discretization of the heat equation or time-dependent Darcy flow (2.3). Therefore, the a posteriori analysis for (4.1) directly applies to (5.13). Here $W_h^n \times Q_h^n$ is the Raviart–Thomas and Brezzi–Douglas–Marini mixed element space. Given an interval $I \subseteq [0, T]$, we define the norm
\[ \|(q, z)\|_{L^2(I, Y)}^2 := \int_I (\|q\|^2 + \|\partial_q q\|^2 + \|z\|^2_{W} + \|\partial_z z\|^2_{W}) ds. \]

Going through the proof of Theorems 4.2 and 5.3, $R_1$ disappears when deriving the upper and lower bounds for the error of (5.13). Therefore we obtain the following a posteriori error estimates.
Corollary 5.4. For $n = 1, 2, \ldots, N$, the error of (5.13) satisfy
\[
\|E_p(t_n)\|_c^2 + \|E_w(t_n)\|_c^2 + \|(E_p, E_w)\|_{L^2(0, t_n; Y)}^2 \\
\lesssim \eta_{\text{init}} + \left( \sum_{i=1}^n \tau_i \tilde{E}^i_{\text{time}} + \tau_i \tilde{E}^i_{\text{space}} \right) + \tilde{E}^i_{\text{data}}^2 \\
+ \sum_{i=1}^n \left( \tau_i \tilde{E}^i_{\text{time}} + \tau_i \tilde{E}^i_{\text{space}} + \int_{t_{i-1}}^{t_i} \|g - g_h\|_c^2 dt \right),
\]
where
\[
\eta_{\text{init}} = \|p(0) - p_h^0\|_c^2 + \|w(0) - w_h^0\|_c^2, \\
\tilde{E}^i_{\text{time}} = \|p_h^i - p_h^{i-1}\|_c^2 + \|w_h^i - w_h^{i-1}\|_c^2, \\
\tilde{E}^i_{\text{space}} = \|g_h^i - \beta \delta t p_h^i - \text{div} w_h^i\|_2^2, \\
\eta_{\text{space}} = \tilde{E}^i_{\text{space}} + \mathcal{E}_{\mathcal{T}_h}^2(p_h^i, w_h^i) + \mathcal{E}_{\mathcal{T}_h^1 \cup \mathcal{T}_h^1}^2(\delta t p_h^i, \delta_t w_h^i).
\]

In addition, let $K$ be piecewise constant on $\mathcal{T}_h^n$. It holds that
\[
\tau_n \tilde{E}^n_{\text{time}} + \tau_n \tilde{E}^n_{\text{space}} \lesssim \|(E_p, E_w)\|_{L^2(t_{n-1}, t_n; Y)}^2 + \int_{t_{n-1}}^{t_n} \|g - g_h\|_c^2 dt.
\]

6. Conclusion

In this paper, we obtain a two-sided residual a posteriori error estimator for the three-field mixed method in Biot’s consolidation model. The same analysis naturally generalizes to mixed methods for the five-field formulation based on weakly symmetric stress tensor (see [38]). Combining our analysis with a posteriori error estimation of mixed methods for elasticity using strong symmetric stress (see e.g., [6, 10, 40]), one can obtain two-sided residual estimator for the four-field formulation.

Acknowledgements

The work of Zikatanov was supported in part by NSF grants DMS-1720114 and DMS-1819157.

References

1. Elyes Ahmed, Florin Adrian Radu, and Jan Martin Nordbotten, Adaptive poromechanics computations based on a posteriori error estimates for fully mixed formulations of Biot’s consolidation model, Comput. Methods Appl. Mech. Engrg. 347 (2019), 264–294. MR 3899054
2. A. Alonso, Error estimators for a mixed method, Numer. Math. 74 (1996), no. 4, 385–395.
3. Maurice A. Biot, General theory of three-dimensional consolidation, Journal of Applied Physics 12 (1941), no. 2, 155–164.
4. ________, Theory of elasticity and consolidation for a porous anisotropic solid, Journal of Applied Physics 26 (1955), no. 2, 182–185.
5. Franco Brezzi, Jim Douglas Jr., and L. D. Marini, Two families of mixed finite elements for second order elliptic problems, Numer. Math. 2 (1985), no. 47, 217–235.
6. Carsten Carstensen, Dietmar Gallistl, and Joscha Gedicke, Residual-based a posteriori error analysis for symmetric mixed Arnold-Winther FEM, Numer. Math. 142 (2019), no. 2, 205–234.
7. J. M. Cascón, L. Ferragut, and M. I. Asensio, Space-time adaptive algorithm for the mixed parabolic problem, Numer. Math. 103 (2006), no. 3, 367–392. MR 2221054
8. J. Manuel Cascon, Christian Kreuzer, Ricardo H. Nochetto, and Kunibert G. Siebert, Quasi-optimal convergence rate for an adaptive finite element method, SIAM J. Numer. Anal. 46 (2008), no. 5, 2524–2550.
9. J. Manuel Cascon, Ricardo H. Nochetto, and Kunibert G. Siebert, Design and convergence of AFEM in $H(\text{div})$, Math. Models Methods Appl. Sci. 17 (2007), no. 11, 1849–1881. MR 2372340
10. Long Chen, Jun Hu, Xuehai Huang, and Hongying Man, Residual-based a posteriori error estimates for symmetric conforming mixed finite elements for linear elasticity problems, Sci. China Math. 61 (2018), no. 6, 973–992.
11. Long Chen and Yongke Wu, Convergence of adaptive mixed finite element methods for the Hodge Laplacian equation: without harmonic forms, SIAM J. Numer. Anal. 55 (2017), no. 6, 2905–2929.
12. Zhiming Chen and Jia Feng, An adaptive finite element algorithm with reliable and efficient error control for linear parabolic problems, Math. Comp. 73 (2004), no. 247, 1167–1193. MR 2061143
13. Ph. Clément, Approximation by finite element functions using local regularization, Rev. Française Automat. Informat. Recherche Opérationnelle Sé. 9 (1975), no. R–2, 77–84. MR 0400739
14. Alan Demlow and Anil N. Hirani, A posteriori error estimates for finite element exterior calculus: the de Rham complex, Found. Comput. Math. 14 (2014), no. 6, 1337–1371.
15. Kenneth Eriksson and Claes Johnson, Adaptive finite element methods for parabolic problems. I. A linear model problem, SIAM J. Numer. Anal. 28 (1991), no. 1, 43–77. MR 1083324
16. Alexandre Ern and Sébastien Meunier, A posteriori error analysis of Euler-Galerkin approximations to coupled elliptic-parabolic problems, M2AN Math. Model. Numer. Anal. 42 (2009), no. 2, 353–375.
17. Alexandre Ern, Iain Smears, and Martin Vohralík, Equilibrated flux a posteriori error estimates in $L^2(\mathcal{H}^1)$-norms for high-order discretizations of parabolic problems, IMA J. Numer. Anal. 39 (2019), no. 3, 1158–1179. MR 3984054
18. Alexandre Ern and Martin Vohralík, A posteriori error estimation based on potential and flux reconstruction for the heat equation, SIAM J. Numer. Anal. 48 (2010), no. 1, 198–223. MR 2608366
19. Alexandre Ern and Martin Vohralík, Polynomial-degree-robust a posteriori estimates in a unified setting for conforming, nonconforming, discontinuous galerkin, and mixed discretizations, SIAM J. Numer. Anal. 53 (2015), no. 2, 1058–1081.
20. Lawrence C. Evans, Partial differential equations, second ed., Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 2010. MR 2597943
21. F. J. Gaspar, F. J. Lisbona, and P. N. Vabishchevich, A finite difference analysis of Biot's consolidation model, Appl. Numer. Math. 44 (2003), no. 4, 487–506. MR 1957690 (2003m:74034)
22. Alexandre Ern and Martin Vohralík, Staggered grid discretizations for the quasi-static Biot’s consolidation problem, Appl. Numer. Math. 56 (2006), no. 6, 888–898. MR 2222126 (2006m:74039)
23. Vivette Girault and Pierre-Arnaud Raviart, Finite element methods for Navier-Stokes equations, Springer Series in Computational Mathematics, vol. 5, Springer-Verlag, Berlin, 1986, Theory and algorithms. MR 851383
24. R. Hiptmair, Finite elements in computational electromagnetism, Acta Numer. 11 (2002), 237–339. MR 2009375
25. Michael Holst, Yuwen Li, Adam Mihalik, and Ryan Szypowski, Convergence and optimality of adaptive mixed methods for Poisson’s equation in the FEbc framework, J. Comp. Math. (2019).
26. Qingguo Hong and Johannes Kraus, Parameter-robust stability of classical three-field formulation of Biot’s consolidation model, Electron. Trans. Numer. Anal. 48 (2018), 202–226.
30. Xiaozhe Hu, Carmen Rodrigo, Francisco J. Gaspar, and Ludmil T. Zikatanov, *A nonconforming finite element method for the Biot’s consolidation model in poroelasticity*, Journal of Computational and Applied Mathematics **310** (2017), 143 – 154.

31. Jianguo Huang and Yifeng Xu, *Convergence and complexity of arbitrary order adaptive mixed element methods for the poisson equation*, Sci. China Math. **55** (2012), no. 5, 1083–1098.

32. Dongho Kim, Eun-Jae Park, and Boyoon Seo, *Space-time adaptive methods for the mixed formulation of a linear parabolic Problem*, J. Sci. Comput. **74** (2018), no. 3, 1725–1756. MR 3767825

33. Vladimir Alexandrovich Kondratiev and Olga Arsenievna Oleinik, *On Korn’s inequalities*, C. R. Acad. Sci. Paris Sér. I Math. **308** (1989), no. 16, 483–487. MR 995908

34. Christian Kreuzer, Christian A. Mörler, Alfred Schmidt, and Kunibert G. Siebert, *Design and convergence analysis for an adaptive discretization of the heat equation*,IMA J. Numer. Anal. **32** (2012), no. 4, 1375–1403. MR 2991832

35. Kundan Kumar, Svetlana Matculevich, Jan Nordbotten, and Sergey Repin, *Guaranteed and computable bounds of approximation errors for the semi-discrete Biot problem*, arXiv e-prints (2018), arXiv:1808.08036.

36. Omar Lakkis and Charalambos Makridakis, *Elliptic reconstruction and a posteriori error estimates for fully discrete linear parabolic problems*, Math. Comp. **75** (2006), no. 256, 1627–1658. MR 2240628

37. Mats G. Larson and Axel Målqvist, *A posteriori error estimates for mixed finite element approximations of parabolic problems*, Numer. Math. **118** (2011), no. 1, 33–48. MR 2793901

38. Jeonghun J. Lee, *Robust error analysis of coupled mixed methods for Biot’s consolidation model*, J. Sci. Comput. **69** (2016), no. 2, 610–632. MR 3551338

39. Jeonghun J. Lee, Kent-Andre Mardal, and Ragnar Winther, *Parameter-robust discretization and preconditioning of Biot’s consolidation model*, SIAM J. Sci. Comput. **39** (2017), no. 1, A1–A24. MR 3590654

40. Yuwen Li, *Quasi-optimal adaptive hybridized mixed finite element methods for linear elasticity*, arXiv e-prints (2019), arXiv:1909.02551.

41. , *Quasi-optimal adaptive mixed finite element methods for controlling natural norm errors*, arXiv e-prints (2019), arXiv:1907.03852.

42. Yuwen Li, *Some convergence and optimality results of adaptive mixed methods in finite element exterior calculus*, SIAM J. Numer. Anal. **57** (2019), no. 4, 2019–2042. MR 3995302

43. Charalambos Makridakis and Ricardo H. Nochetto, *Elliptic reconstruction and a posteriori error estimates for parabolic problems*, SIAM J. Numer. Anal. **41** (2003), no. 4, 1585–1594. MR 2034895

44. Sajid Memon, Neela Nataraj, and Amiya Kumar Pani, *An a posteriori error analysis of mixed finite element Galerkin approximations to second order linear parabolic problems*, SIAM J. Numer. Anal. **50** (2012), no. 3, 1367–1393. MR 2970747

45. Márcio A. Murad, Vidar Thomée, and Abimael F. D. Loula, *Asymptotic behavior of semidiscrete finite-element approximations of Biot’s consolidation problem*, SIAM J. Numer. Anal. **33** (1996), no. 3, 1065–1083. MR 1393902

46. J.-C. Nédélec, *Mixed finite elements in $\mathbb{R}^3$*, Numer. Math. **35** (1980), no. 3, 315–341.

47. Jan Martin Nordbotten, *Stable cell-centered finite volume discretization for Biot equations*, SIAM Journal on Numerical Analysis **54** (2016), no. 2, 942–968.

48. Ricardo Oyarzúa and Ricardo Ruiz-Baier, *Locking-free finite element methods for poroelasticity*, SIAM J. Numer. Anal. **54** (2016), no. 5, 2951–2973.

49. J. E. Pasciak and J. Zhao, *Overlapping schwarz methods in $H(curl)$ on polyhedral domains*, J. Numer. Math. **10** (2002), no. 3, 221–234.

50. Phillip Joseph Phillips and Mary F. Wheeler, *A coupling of mixed and continuous Galerkin finite element methods for poroelasticity. I. The continuous in time case*, Comput. Geosci. **11** (2007), no. 2, 131–144. MR 2327964

51. , *A coupling of mixed and continuous Galerkin finite element methods for poroelasticity. II. The discrete-in-time case*, Comput. Geosci. **11** (2007), no. 2, 145–158. MR 2327966

52. Marco Picasso, *Adaptive finite elements for a linear parabolic problem*, Comput. Methods Appl. Mech. Engrg. **167** (1998), no. 3-4, 223–237. MR 1673951

53. P-A. Raviart and J. M. Thomas, *A mixed finite element method for 2nd order elliptic problems*, Mathematical aspects of finite element methods (Proc. Conf., Consiglio Naz. delle
Ricerche (C.N.R.), Rome, 1975), 1977, pp. 292–315. Lecture Notes in Math., Vol. 606. MR 0483555

54. Rita Riedlebeck, Daniele A. Di Pietro, Alexandre Ern, Sylvie Granet, and Kyrylo Kazymyrenko, Stress and flux reconstruction in Biot’s poro-elasticity problem with application to a posteriori error analysis, Comput. Math. Appl. 73 (2017), no. 7, 1593–1610. MR 3622156

55. C. Rodrigo, X. Hu, P. Ohm, J. H. Adler, F. J. Gaspar, and L. T. Zikatanov, New stabilized discretizations for poroelasticity and the Stokes’ equations, Comput. Methods Appl. Mech. Engrg. 341 (2018), 467–484.

56. Joachim Schöberl, A posteriori error estimates for Maxwell equations, Math. Comp. 77 (2008), no. 262, 633–649. MR 2373173

57. R. E. Showalter, Diffusion in poro-elastic media, J. Math. Anal. Appl. 251 (2000), no. 1, 310–340. MR 1790411

58. K. Terzaghi, Theoretical soil mechanics, Wiley: New York, 1943.

59. Rüdiger Verfürth, A posteriori error estimators for the Stokes equations. II. Nonconforming discretizations, Numer. Math. 60 (1991), no. 2, 235–249. MR 1133581

60. ______, A posteriori error estimates for finite element discretizations of the heat equation, Calcolo 40 (2003), no. 3, 195–212. MR 2025602

61. Rüdiger Verfürth, A posteriori error estimation techniques for finite element methods, Numerical Mathematics and Scientific Computation, Oxford University Press, Oxford, 2013. MR 3059294

62. Alexander Ženíšek, The existence and uniqueness theorem in Biot’s consolidation theory, Apl. Mat. 29 (1984), no. 3, 194–211. MR 747212

Department of Mathematics, Pennsylvania State University, University Park, PA 16802.

E-mail address: yuuli@psu.edu, ltyl@psu.edu