Instantons and BPS Wilson loops

Massimo Bianchi, Michael B. Green and Stefano Kovacs

Department of Applied Mathematics and Theoretical Physics
CMS, Wilberforce Road, Cambridge CB3 0WA, UK
E-mail: M.Bianchi@damtp.cam.ac.uk, M.B.Green@damtp.cam.ac.uk, S.Kovacs@damtp.cam.ac.uk

ABSTRACT: The one-instanton contribution to a circular BPS Wilson loop in $\mathcal{N} = 4$ $SU(2)$ Yang–Mills theory is evaluated in semiclassical approximation. This article amplifies part of a talk given by MBG at the Strings 2001 conference, Mumbai, India (January 5-10, 2001). The results are preliminary and a more complete exposition will be contained in a forthcoming paper.

KEYWORDS: AdS/CFT; superstrings; conformal field theory; Wilson loop.

*On leave of absence from Dipartimento di Fisica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica 1, 00133 Roma, ITALY
... and having heard that when a man in a forest thinks he is going in a straight line, in reality he is going in a circle, I did my best to go in a circle, hoping in this way to go in a straight line. For I stopped being half-witted and became sly, whenever I took the trouble. And my head was a storehouse of useful knowledge. And if I did not go rigorously in a straight line, with my system of going in a circle, at least I did not go in a circle, and that was something. From ‘Molloy’ by Samuel Beckett

1. Introduction

The correspondence between Yang–Mills theory and string theory embodied in the AdS/CFT conjecture has proved a fertile framework for stimulating ideas in quantum gravity as well as in conformal field theory. There is by now much evidence for the validity of the conjecture based on perturbative and non-perturbative calculations. This includes the detailed correspondence between the effects of D-instantons that appear in the derivative expansion (the $\alpha'$ expansion) of the type IIB superstring theory and Yang–Mills instantons of $\mathcal{N} = 4$ $SU(N)$ supersymmetric Yang-Mills theory in the large $N$ limit. This agreement is particularly intriguing since the instanton calculations are semi-classical (weak Yang–Mills coupling) while, when the low energy supergravity approximation to string theory is used, the AdS/CFT correspondence should only work at strong coupling.

Tests of the AdS/CFT correspondence involve comparison of observables in string theory with corresponding observables in Yang–Mills theory. Although a great
deal of effort has been devoted to comparison of amplitudes in type IIB supergravity with correlation functions of gauge-invariant composite operators in the Yang–Mills theory, the properties of Wilson loops are less well established. A Wilson loop in the large-$N$ gauge theory should be described by a minimal surface embedded in $AdS_5$ with a boundary that is a curve on the four-dimensional boundary. The work to be described below considers the effect of instantons on a circular BPS Wilson loop of the $\mathcal{N} = 4$ Yang–Mills theory. This talk will only sketch the calculation, details of which will appear shortly \[1\]. For our purposes the theory will be considered with euclidean time, which means that $AdS_5$ may be considered to be a ball $B_5$ with boundary $S^4$.

In a gauge theory with extended supersymmetry it is natural to consider a special class of Wilson loops that satisfy a BPS condition \[3, 4, 6\]. In $\mathcal{N} = 4$ supersymmetric Yang-Mills theory the BPS Wilson loop is a functional of a closed curve, $C$, that is defined by \[3, 4, 5, 6\]

$$\langle W(C) \rangle = \frac{1}{N} \langle \text{Tr} \mathcal{P} \exp \left\{ i \int_C (A_\mu \hat{x}^\mu + [\bar{\theta}_A \hat{x}_\mu \bar{\theta}^\mu \lambda^A + \theta^A \hat{y}_i \bar{\Gamma}_i^A \lambda^B + h.c.] + i \varphi_i \hat{y}^i) ds \right\} \rangle ,$$

(1.1)

where $\lambda^A$ and $\varphi_i$ are the four Weyl fermion and six real scalar fields in the $\mathcal{N} = 4$ supermultiplet and $\bar{\Gamma}_i^A$ are $SO(6)$ gamma matrices. The trace is in the fundamental representation of $SU(N)$ and the symbol $\mathcal{P}$ indicates path ordering of the gauge group contractions around the loop. The loop can be parameterized by $x^\mu = x^\mu(s)$ with $s \in [0, 1]$ and $x^\mu(0) = x^\mu(1)$. The six coordinates $y^i$ are conjugate to the central charges of the $\mathcal{N} = 4$ superalgebra. In the following we shall be concerned with the special class of loops with $\theta^A(s) = 0$. In this case if $|\hat{x}| = |\hat{y}|$ the expression (1.1) satisfies a BPS condition, which means that it is invariant under half of the 32 superconformal symmetries. More generally, the BPS condition allows $\theta^A$ to be freely shifted by any Killing spinor $\kappa^A$. This can be seen by exhibiting the $\kappa$ symmetry of the Wilson loop \[5\]. The residual supersymmetries are defined by the relation between the spinor parameters $\kappa^A_\alpha(x)$ and $\bar{\kappa}^\alpha_A(x)$ (which are linear functions of $x^\mu$),

$$\hat{x}_\mu \sigma^\mu \bar{\kappa}_A = \hat{y}_i \bar{\Gamma}_i^A \kappa^B .$$

(1.2)

The expression (1.1) can be obtained by viewing it as the holonomy of an infinitely massive W boson arising from the spontaneous symmetry breaking $SU(N + 1) \rightarrow SU(N) \times U(1)$.

We will further restrict our attention to circular Wilson loops of radius $R$ centred at the origin in the $(x^1, x^2)$ plane defined by

$$(x^1)^2 + (x^2)^2 = R^2, \quad x^3(s) = 0, \quad x^0(s) = 0, \quad \hat{y}^i(s) = |\hat{x}| n^i ,$$

(1.3)

\[1\]The conference talk also reviewed results by MBG and SK concerning $1/N$ corrections to instanton induced correlation functions and discussed their AdS correspondence \[3\].
where \( n^i \) is a unit vector in the six ‘internal’ \( SU(4) \) directions. The superconformal invariance of the theory guarantees that the Wilson loop, \( \langle W(R) \rangle \), is a constant that is independent of \( R \). Some perturbative contributions to Wilson loops of this kind (namely, the ‘rainbow’ diagrams) have been calculated to all orders in the coupling constant \( g \) and argued to be exact, at least in the large \( N \) limit. A clever anomaly argument in \([3]\) uses the fact that a straight line Wilson loop has no perturbative contributions to determine the expression for a circular Wilson loop. The conformal transformation that maps the line to a circle is anomalous and generates a nontrivial expression for \( \langle W(R) \rangle \). Although this has been conjectured to be correct not only to all orders in perturbation theory but also to all orders in the \( 1/N \) expansion it has been argued that the \( 1/N \) corrections are not universal \([9]\). The result is consistent with expectations based on the AdS/CFT correspondence. Our aim is to compute the one-instanton contribution to the circular Wilson loop. We will work to lowest order in the Yang–Mills coupling constant, \( g_{YM} \) (the semiclassical approximation) and take the gauge group to be \( SU(2) \), although it is very straightforward to generalize this to \( SU(N) \) for arbitrary \( N \). The very notion of a ‘straight line Wilson loop’ with sensible boundary conditions is problematic in the context of the instanton calculation so we will consider the circular case directly.

A single instanton in the \( \mathcal{N} = 4 \) \( SU(2) \) Yang–Mills theory has eight bosonic moduli and sixteen fermionic moduli. The bosonic moduli correspond to broken translation, dilation and gauge orientation symmetries. The fermionic moduli are in one-to-one correspondence with the broken supersymmetries and conformal supersymmetries. In the following we will evaluate the integral over these supermoduli in the background of a Wilson loop. Ignoring the three gauge moduli, which are irrelevant since the Wilson loop is gauge invariant, the bosonic collective coordinates \( (x^\mu_0, \rho_0) \) parametrize a copy of \( AdS_5 \). The fermionic collective coordinates \( (\eta^A, \bar{\xi}^A) \) can be packaged into the chiral spinor,

\[
\zeta^A(x) = \eta^A + x_\mu \sigma^\mu \bar{\xi}^A,
\]

which (up to rescalings) is also a Killing spinor of \( AdS_5 \), which indicates a holographic connection between the Yang–Mills instanton and the D-instanton of the IIB string theory in \( AdS_5 \times S^5 \) \([10, 11]\).

The value of the Wilson loop in this background is given, to leading order in the coupling constant, by substituting the classical instanton solution into \([12]\) and integrating over the supermoduli with the appropriate invariant measure. The ‘classical’ solution must include not only the standard instanton solution but also the dependence of the fields on fermionic moduli. In the instanton background the fields \( \varphi^i \) and \( A_\mu \) have zero modes that depend on the fermionic moduli which are induced by the presence of Yukawa couplings in the usual manner. The multiplet of these zero modes can be generated from the classical instanton profile of the vector potential by successive application of the supersymmetries that are broken by the instanton.
Thus, two transformations generate
\[ \hat{\varphi}^{ia} = \frac{1}{2} F_{\mu\nu}^{a} \hat{\varphi}^i_{[AB} \sigma^{\mu\nu} \zeta^{B]} , \] (1.5)
where the hat indicates the value of a field containing fermionic collective coordinates induced by the instanton background and \( F_{\mu\nu}^{a} \) is the anti self-dual field strength of the classical instanton solution.\(^2\) Performing another two supersymmetry transformations gives
\[ \hat{A}_\mu = \frac{1}{4!} \varepsilon_{ABCD} \zeta^A \sigma_{\mu\nu} \zeta^B \, D^\nu (F_{\lambda\kappa}^{a} \zeta^C \sigma^{\lambda\kappa} \zeta^D) , \] (1.6)
which is a potential that corresponds to a self-dual field strength, \( \hat{F}_{\mu\nu}^{a+} \). The calculation of the Wilson loop to leading order in \( g_{YM} \) involves substituting the expressions (1.5) and (1.6) into (1.1). The sixteen fermionic integrals are then saturated by expanding the exponential to extract the terms with sixteen powers of \( \zeta \). This involves dealing with the complicated combinatorics of terms with \( n \hat{\varphi}'s \) and \( m \hat{A}'s \) with \( 2n + 4m = 16 \) that arise from expanding the exponent in (1.1).

2. Symmetries of the super Wilson Loop

2.1 The bosonic model

Even in the absence of fermionic moduli it is difficult to evaluate the classical instanton contribution to the Wilson loop by direct integration over the bosonic moduli. However, it is possible to proceed by making use of the symmetries of the problem. It is straightforward to see that the presence of the circular loop partially breaks the conformal \( SO(4,2) \) symmetry to a residual unbroken \( SO(2,2) \) subgroup. The instanton calculation will require continuation to euclidean signature in which case the residual symmetry is an \( SO(3,1) \) subgroup of \( SO(5,1) \).

The fact that the instanton is invariant in form under these transformations, up to irrelevant gauge transformations, allows us to make use of this symmetry in order to map an arbitrary instanton to one that is located at the centre of the loop. The expression for the integrand of the loop then becomes abelian and it is easy to evaluate the integral explicitly. Of course, such a classical calculation is not directly relevant to pure Yang–Mills theory since in that case the classical conformal symmetry is broken by quantum fluctuations that require renormalization. In this toy example the integral diverges since there is no suppression of instantons with arbitrarily small scale size or ones that are located arbitrarily far from the loop. Nevertheless, it proves useful to study this example since it will later be generalized to the case of the superconformal \( \mathcal{N} = 4 \) theory.

\(^2\)We may ignore the change in the classical field equations due to the presence of the Wilson loop source since such changes enter at higher order in \( g_{YM} \).
The infinitesimal conformal transformations,
\[ \delta x^\mu = a^\mu + \omega^{\mu\nu} x_\nu + \lambda x^\mu + x^2 b^\mu - 2b \cdot x x^\mu , \]  
are generated by \((P_\mu, J_{\mu\nu}, D, K_\mu)\), where \((a^\mu, \omega^{\mu\nu}, \lambda, b^\mu)\) are constant parameters. In order to streamline the discussion of the conformal properties it proves very convenient to make use of Dirac’s formalism \[12\] for representing the conformal group by extending four-dimensional Minkowski space-time to six dimensions with signature \((4, 2)\) and coordinates \(X^M (M = 0, \ldots, 5)\), where \(X^0\) and \(X^4\) are time-like. In this way, \(SO(4, 2)\) can be represented linearly by rotations and boosts on \(X^M\). In order to find a representation of \(SO(4, 2)\) in five dimensions the flat six-dimensional coordinates are taken to satisfy the invariant constraint,
\[ X^2 \equiv \eta_{MN} X^M X^N \equiv \eta_{\mu\nu} X^\mu X^\nu + (X^4)^2 - (X^5)^2 = C^2 , \]  
where \(C\) is an arbitrary constant scale that drops out of the conformally invariant theory. For now we are using the six-dimensional metric \(\eta_{MN} = \text{diag}(+ - - - + +)\) and the four-dimensional metric \(\eta_{\mu\nu} = \text{diag}(+ - - -)\), although later we will consider the Wick rotation of \(X^0\) that is relevant for the instanton calculation. The constraint is solved in terms of five-dimensional coordinates that define \(AdS_5\) with scale \(C\). It will prove notationally convenient to choose the arbitrary scale to be the radius of the Wilson loop, \(C = R\), from now on. A conventional parameterization of \(AdS_5\) in terms of \(x^\mu\) and \(\rho\) is obtained by the identifications
\[ X^\mu = R \frac{x^\mu}{\rho} , \quad X^4 = \frac{1}{2} \left( \rho + \frac{R^2 - x^2}{\rho} \right) , \quad X^5 = \frac{1}{2} \left( \rho - \frac{R^2 + x^2}{\rho} \right) , \]  
which represents an \(AdS_5\) hypersurface in \(\mathbb{R}^6\). Lorentz transformations of the six-dimensional coordinates generated by the generalized angular momenta,
\[ L_{MN} = X_M \partial_N - X_N \partial_M , \]  
are isomorphic to \(SO(4, 2)\) transformations acting on \(AdS_5\), with the identifications of the fifteen generators,
\[ J_{\mu\nu} = L_{\mu\nu} , \quad D = L_{45} , \quad P_\mu = L_{4\mu} + L_{5\mu} , \quad K_\mu = L_{4\mu} - L_{5\mu} . \]  
Furthermore the trivial ‘flat’ six-dimensional integration measure is equivalent, after the constraint, to the \(AdS_5\) measure,
\[ R^{-4} \int \delta(X^2 - R^2) d^6 X = \int \frac{d^4 x d\rho}{\rho^5} . \]  
It is easy to verify that the boundary of \(AdS_5\) is mapped into itself under the \(SO(4, 2)\) transformations and that the standard four-dimensional action of \(SO(4, 2)\) results from the limit in which the radial variable goes to the boundary, \(\rho \to 0\) \[12\].
Whereas a generic Wilson loop breaks the \( SO(4, 2) \) conformal symmetry completely, we will make use of the fact that a circular Wilson loop leaves an \( SO(2, 2) \) subgroup unbroken. This follows from the fact that a loop in the \((x^1, x^2)\) plane on the boundary of \( AdS_5 \) at \( \rho = 0, |x_l| = R, x_t = 0 \) is mapped into itself under \( SO(2, 2) \) transformations, where \( x_l = (x_1, x_2) \) and \( x_t = (x_0, x_3) \) are longitudinal and transverse vectors. These \( SO(2, 2) \) transformations are described in the six-dimensional formalism as those that leave invariant the quadratic form,

\[
U(x, \rho) = (X^4)^2 - (X^l)^2 \equiv X_L^2,
\]

where \( X^l = (X^4, X^t) \). After imposing the constraint (2.2) the quantity

\[
V(x, \rho) \equiv X_T^2 = U - R^2
\]

is also invariant, where \( X_T = (X^0, X^3, X^5) \). For fixed \( U \) (2.7) defines a four-dimensional hyperbolic surface in \( AdS_5 \) once the constraint (2.2) is imposed. The four-dimensional surfaces of constant \( U \) foliate \( AdS_5 \) in such a manner that they all meet on the circle at the boundary, \( \rho = 0, x_t = 0, |x_l| = R \). The minimal value of \( U = R^2 \) is achieved on a degenerate surface that is two-dimensional. This surface is, in fact, the surface of minimal area in \( AdS_5 \) that bounds the loop of radius \( R \) on the boundary that was described in \[3, 6\]. It follows from (2.7) that a general point in \( AdS_5 \) has the same value of \( U \) as a point at the centre of the loop with \( \tilde{x}^\mu = 0 \) and \( \tilde{\rho}(x, \rho) \) determined by

\[
U(\tilde{x} = 0, \tilde{\rho}) = \frac{(R^2 + \tilde{\rho}^2)^2}{4\tilde{\rho}^2} = \frac{(R^2 - x_l^2 - x_t^2 + \rho^2)^2}{4\rho^2} + \frac{R^2 x_l^2}{\rho^2} = U(x, \rho).
\]

We will later interpret \( AdS_5 \) as the moduli space of an \( SU(2) \) instanton.

### 2.2 The \( \mathcal{N} = 4 \) supersymmetric theory

We are now interested in extending this discussion to the \( \mathcal{N} = 4 \) supersymmetric theory which is invariant under the supergroup \( SU(2, 2|4) \). The bosonic part of this group is \( SO(4, 2) \times SU(4) \). The fifteen transformations of the \( SU(4) \approx SO(6) \) R-symmetry group (with parameters \( \omega^{ij} \)) have the form, \( \delta y^i = \omega^{ij} y_j \). In addition, there are four Poincaré supersymmetries with generators \((Q^A, Q_{\dot{A}}^A)\), and four superconformal symmetries with generators \((S_{Aa}, S_{\dot{A}a})\).

We can extend Dirac’s six-dimensional representation of \( SO(4, 2) \) to \( SU(2, 2|4) \) by including an appropriate set of fermionic coordinates. There is a wide variety of possible choices for such coordinates but we will choose a ‘chiral’ representation that is particularly well adapted for the instanton calculation and requires euclidean signature. Recall that we wish to integrate over the sixteen components of the broken supersymmetries and conformal supersymmetries, \( \eta^A_\dot{A} \) and \( \bar{\xi}^{\dot{a}A} \). Therefore, in addition to the six coordinates \( X^M \), we now include a quartet of Grassmann spinors, \( \Theta^A_{\dot{a}} \), where
a is a four-component spinor index. This will be interpreted as a chiral spinor in the six-dimensional description. The corresponding $\mathcal{N} = 4$ instanton superspace will be chosen to be the supercoset

$$\mathcal{N} = 4 \quad \frac{SU(2, 2|4)}{\text{Span}\{SO(4, 1) \times SO(5); Q^A_b, S^B_a\}},$$

(2.10)

where the generators in $SO(6)/SO(5)$ will play no role in the following and the remaining coordinates are simply the supermoduli associated with the instanton.

Using a standard super-coset construction \[4\] it is straightforward to exhibit the action of $SU(2, 2|4)$ on these coordinates \[4\]. These transformations may be written compactly in a six-dimensional notation by the identification

$$\Theta^A_a = (n^A_a + x \cdot \sigma_{\alpha \dot{\alpha}} \tilde{\xi}_A^{\alpha \dot{\alpha}}, \tilde{\xi}_A^{\alpha \dot{\alpha}}),$$

(2.11)

where $a = (\alpha, \dot{\alpha})$ is a spinor index of $SO(4, 2)$ (or $SO(5, 1)$ in the euclidean theory) and takes values from 1 to 4. Our notation anticipates the fact that we will identify the sixteen fermionic variables with the sixteen collective coordinates of the instanton that are associated with the broken superconformal symmetries. The 32 supercharges can be packaged into six-dimensional chiral spinors $Q^A_a = (Q^A_a, \bar{S}^A_a)$ and $\bar{Q}_a = (\bar{Q}_a, S^A_a)$ and represented in terms of the supercoordinates by

$$Q^A_a = \partial_\Theta^A_a, \quad \bar{Q}_a = \Theta^A_a \Theta_a^B \partial_\Theta^A_B + \frac{1}{4} \hat{\Gamma}^{MNB}_a \Theta_a^B L_{MN},$$

(2.12)

which satisfy the $SU(2, 2|4)$ superalgebra

$$\{Q^A_a, \bar{Q}^B_b\} = \frac{1}{4} \delta^B_a \hat{\Gamma}^{MNA}_b J_{MN} + \frac{1}{4} \delta^a_b \hat{\Gamma}^{ij}_A T_{ij},$$

(2.13)

where $J_{MN} = L_{MN} + S_{MN}$ are the generators of $SO(4, 2)$ and $T_{ij}$ are the generators of $SO(6)$. More explicitly,

$$L_{MN} = X_M \partial_N - X_N \partial_M, \quad S_{MN} = \Theta^A_a \hat{\Gamma}^{MNB}_{a} \partial_\Theta^A_b, \quad T_{ij} = \Theta^A \hat{\Gamma}^{ij}_A B \partial_\Theta^B_a.$$

(2.14)

The superconformal transformations of the coordinates generated by these charges are

$$\delta \Theta^A_a = \epsilon^A_a, \quad \bar{\delta} \Theta^A_a = \Theta^B_a \Theta^B_a \epsilon^B_b,$$

$$\delta X^M = 0, \quad \bar{\delta} X^M = \frac{1}{2} \bar{\epsilon}_A \hat{\Gamma}^{MN} \Theta^A X_M.$$

(2.15)

The presence of the Wilson loop breaks this symmetry to a subgroup of $SU(2, 2|4)$ that leaves the loop invariant up to reparametrizations. Since we have fixed a direction $(n^i)$ in the internal space, the loop is only invariant under an $SO(5) \approx Sp(4)$ subgroup of the $SU(4) \approx SO(6)$ R-symmetry group. It will prove convenient to
define the $Sp(4)$ singlet $\Omega_{AB} = n_i \hat{F}^i_{AB}$. We now want to find the fermionic part of the superconformal group that leaves the loop invariant. Since the bosonic symmetry that preserves the loop, $SO(2,2) \times Sp(4)$, is the bosonic part of the supergroup $OSp(2,2|4)$ (which is a subgroup of $SU(2,2|4)$) this is a natural candidate for the invariance group of the loop. To verify that this is indeed the case we need to identify the Killing spinors that satisfy the relation (1.2). It turns out that these correspond to symmetries generated by the linear combinations

\[ G_A = \sigma^{12} Q_A + \frac{1}{R} \Omega_{AB} S^B \]  

(2.16)

and its conjugate, $\bar{G}^A$. The index $A$ is raised and lowered by the symplectic metric $\Omega_{AB}$. In a compact notation the surviving supersymmetry generators $G$ and $\bar{G}$ can be packaged into $G^a_A$, where $a$ is an index in the $(2,2)$ of $SO(2,2)$ and the supersymmetry algebra becomes

\[ \{G^a_A, G^b_B\} = \Omega_{AB} J^{ab} + T_{(AB)} H^{ab}, \]  

(2.17)

where the six generators of $SO(2,2)$, $(\Pi^+, J_{xy}, \Pi^-, J_{zt})$ have been assembled into $J^{ab}$ and $H^{ab}$ is the symmetric $SO(2,2)$ invariant metric. The remaining commutation relations of the $OSp(2,2|4)$ algebra are

\[ [J_{ab}, J_{cd}] = H_{bc} J_{ad} + \text{perms}, \quad [T_{AB}, T_{CD}] = \Omega_{BC} T_{AD} + \text{perms}, \]

\[ [T_{AB}, J_{ab}] = 0, \quad [J_{ab}, G_{Ac}] = H_{bc} G_{Aa} - H_{ac} G_{Ab}, \]

\[ [T_{AB}, G_{Ca}] = \Omega_{BC} G_{Ac} + \Omega_{AC} G_{Bc}. \]  

(2.18)

3. One-instanton contribution to the Wilson loop

We are now in a position to consider the integral over the instanton supermoduli that enters into the expression for the Wilson loop. We have been unable to find any analytic treatment of instanton contributions to Wilson loops in the literature, even for the purely bosonic theory, although there have been many approximate and numerical treatments in the context of (lattice) QCD.

3.1 The bosonic model

We will again begin by first considering the toy conformally invariant model in which there are no fermions and only the bosonic moduli arise. The expression for the classical instanton contribution to the standard Wilson loop of pure $SU(2)$ Yang–Mills theory (ignoring overall constants) is given by an integral over the instanton moduli,

\[ \langle W(R) \rangle_{\text{bos}} = \int d^6 X_0 \delta(X_0 \cdot X_0 - R^2) \text{Tr} P e^{i \int C A \cdot \dot{x} ds} \]

\[ = \int d^6 X_0 \delta(X_0 \cdot X_0 - R^2) \text{Tr} P \exp \left( i \int_C \frac{\eta_{\mu \nu}(x^\mu - x_0^\mu) \sigma_\alpha}{\rho_0^2 + |x - x_0|^2} \dot{x}^\nu ds \right) \]  

(3.1)
where $\eta^a_{\mu\nu}$ is the standard 'tHooft eta symbol. Since the form of the instanton solution is invariant under $SO(4,2)$ transformations a particular one-instanton configuration can be transformed into an equivalent one by acting with an element of $SO(2,2)$, which maps the loop onto itself. In this way an arbitrary instanton with moduli $(x_0, \rho_0)$ can be mapped into the special configuration in which it is at the centre of the loop with moduli $(\tilde{x}_0 = 0, \tilde{\rho}_0(x_0, \rho_0))$. Then the path ordered exponential becomes simple since $\sigma_a \eta^a_{\mu\nu} x^\mu \dot{x}^\nu ds = R^2 \sigma^3 d\phi$, where $0 \leq \phi \leq 2\pi$ is the angle around the loop, which has been taken to lie in the $(x_1, x_2)$ plane. In this way the exponent is abelianized and the integration over $\phi$ followed by the trace simply leads to

$$
\langle W(R) \rangle_{bos} = \int d^6X \delta(X^2 - R^2) W_B(|X_T|),
$$

(3.2)

(dropping the subscript 0 from the collective coordinates, $X^M$) where the bosonic Wilson loop density is a scalar function of the $SO(2,2)$ invariant which can be written as

$$
W_B(|X_T|) = \cos \left( \frac{\pi |X_T|}{\sqrt{R^2 + X^2_T}} \right)
$$

(3.3)

after using the constraint $X^2_L = X^2_T + R^2$. These expressions are to be interpreted after a Wick rotation of $X^0$.

The specific $SO(2,2)$ group element that generates this transformation is of the form $\exp(a^l \Pi^l_+ + a^t \Pi^t_0)$ where the parameters $a^l(x_0^0, \rho_0)$ and $a^t(x_0^0, \rho_0)$ are functions of the collective coordinates that can be determined by elementary group theory. The integral (3.2) is highly divergent, which was anticipated in this toy model. However, the expectation is that the supersymmetry of the $\mathcal{N} = 4$ theory will lead to compensating cancellations.

### 3.2 Superinvariants

We saw earlier in (2.7) how to define quadratic invariants, $U \equiv X^2_L$ and $V \equiv X^2_T = U - R^2$, for the group $SO(2,2)$. Clearly any function of $V$, such as the bosonic Wilson loop density $W_B(|X_T|)$ (3.3) is also invariant. We can extend this to an invariant of $OSp(2,2|4)$ by solving the equation

$$
(\epsilon^A_a Q^a_A + \epsilon^a_A Q_A^A) W(X, \Theta) = 0
$$

(3.4)

with the condition $W(X, 0) = W_B(|X_T|)$. The restriction to $OSp(2,2|4)$ is built into the condition

$$
\epsilon^a_A = \Omega_{AB} H^{ab} \epsilon^B_b.
$$

(3.5)

$Sp(4)$ and $SO(2,2)$ indices are raised and lowered by the symplectic metric $\Omega_{AB} = \eta^i_i \Gamma^i_{AB}$ and the symmetric tensor $H^{ab} = \Gamma^{ab}_{412}$, respectively. Equation (3.4) imposes
the condition that the loop is BPS so it is annihilated by the fermionic generators of $OSp(2,2|4)$. It can be written as

$$\frac{\partial W(X, \Theta)}{\partial \Theta^A} + \Omega_{AB} H^{ab} \left[ \Theta^B \Theta^C \frac{\partial W(X, \Theta)}{\partial \Theta^C} + \frac{1}{4} \Gamma^{MNE} \Theta^E L_{MN} W(X, \Theta) \right] = 0, \quad (3.6)$$

which can be solved in a standard fashion, giving

$$W(X, \Theta) = P \exp \left( - \int_0^1 \frac{du}{4u} f^{MN}(u\Theta)L_{MN} \right) W_B(|X_T|), \quad (3.7)$$

where $P$ indicates that the exponential is to be path ordered. The matrix function $f^{MN}(u\Theta)$ is defined by

$$f^{MN}(\Theta) = \bar{\Theta}^A \Gamma^{MN} \Theta^A + \bar{\Theta}^A \Theta^B \bar{\Theta}^B \Gamma^{MN} \Theta^A \quad (3.8)$$

where as in (3.3) $\bar{\Theta}_A^a = \Omega_{AB} H^{ab} \Theta_b^B$. Two important features of $f^{MN}$ are worth stressing. Firstly it is at most quartic in $\Theta$, all higher order terms being zero after (anti)symmetrization of the indices implied by the structure of the contractions. Secondly it only has non-zero entries for the nine elements of the coset$^3 SO(4,2)/SO(2,2)$. In other words the only $L_{MN}$’s that enter (3.7) are those of the form $L_{ij}$ where the index $i = 1, 2, 3$ labels the directions in $X_L$ and $j = 1, 2, 3$ labels the directions in $X_T$.

It is notable that the dependence on $\Theta$ in (3.7) enters via the exponential prefactor that acts simply as a rotation on the bosonic coordinates, $X^M$. This means that the invariant $W(X, \Theta)$ is simply obtained by transforming the coordinates $X^M$ in $W_B(|X_T|)$ by a $\Theta$-dependent rotation,

$$X^M \rightarrow \tilde{X}^M = R_{MN}^M(\Theta) X^N, \quad (3.9)$$

where the rotation matrix, $R(\Theta)$, is given by the six-dimensional (fundamental) representation of the path-ordered exponential in (3.7). In fact, a naive argument at this point would say that all of the dependence on $\Theta$ can be eliminated by simply changing integration variables from $X^M$ to $\tilde{X}^M$. Since the jacobian for this change of variables is trivial, the resulting Wilson loop would clearly vanish after the Grassmann integrations. However, this ignores a crucial subtlety – the original bosonic integration is divergent and has to be regulated near the boundary of $AdS_5$. In the presence of a regulator, no matter how it is defined, the change of variables $X \rightarrow \tilde{X}$ introduces dependence on $\Theta$ into the boundary conditions so the Grassmann integral no longer vanishes automatically.

$^3$In euclidean signature $SO(5,1)/SO(3,1)$
3.3 The $\mathcal{N} = 4$ supersymmetric Yang–Mills theory

The expression for the Wilson loop in the $\mathcal{N} = 4$ $SU(2)$ theory is given by the integral (up to an overall constant)

$$\langle W(R) \rangle = \int d^6 X \delta(X_1^2 - X_2^2 - R^2) \int d^{16} \Theta W(X, \Theta),$$

(3.10)

with $W(X, \Theta)$ given by (3.7). The integral over the supermoduli can be evaluated by expanding the exponential in a power series in $\Theta$ in order to determine the $\Theta^{16}$ term. Defining

$$\Phi = \Theta_A \Gamma^{ij} \Theta^A L_{ij}, \quad A = \Theta^B \Theta_B \Gamma^{ij} \Theta^A L_{ij},$$

(3.11)

the $\Theta^{16}$ terms in the expansion of the exponent in (3.7) can be written schematically as

$$\frac{1}{8!} \Phi^8 + \frac{1}{6!} \Phi^6 A + \frac{1}{4! 2! 4!} \Phi^4 A^2 + \frac{1}{2! 3! 8} \Phi^2 A^3 + \frac{1}{4! 16} A^4.$$  

(3.12)

The precise coefficients and contractions can be evaluated taking into account the fact that the operators $A$ and $\Phi$ do not commute. The terms in this expansion are in one-to-one correspondence with the terms in the expansion of the original Wilson loop (1.1) in powers of $\hat{\varphi}$ and $\hat{A}$ (defined in (1.5) and (1.6), respectively).

However, it is considerably more laborious to evaluate the action of these terms on the function $W_B(\|X_T\|)$ that enters into (3.7) Part of the problem is the evaluation of the Grassmann integration over the sixteen fermionic collective coordinates. The method we have used is to decompose the integration into product of integrals over two eight-component $SO(8)$ spinors defined by

$$\hat{\theta} = (\Theta^1, \Theta^2), \quad \check{\theta} = (\Theta^3, \Theta^4).$$

(3.13)

Then use can be made of the standard expression

$$\int d^8 \theta \gamma^{m_1 n_1} \theta \gamma^{m_2 n_2} \theta = t_8^{m_1 n_1 \ldots m_4 n_4},$$

(3.14)

where $m_i, n_i = 1, \ldots 8$ and $t_8$ is a well-known $SO(8)$ invariant tensor [13]. In this way, after many algebraic manipulations, the Grassmann integrations in (3.10) can be performed and result in tensors that contract into products of even powers of the angular momentum generators, $L_{ij}$. It is easy to see that terms with an odd number of $L$’s vanish due to the (angular) bosonic integrations and therefore can be ignored. This however leaves a huge number of terms, each of which consists of a tensor with eight, twelve or sixteen indices contracting into a product of four, six or eight $L$’s which act on $W_B(\|X_T\|)$. In order to evaluate these expressions we have made extensive use of the algebraic software package REDUCE. The result has the
\[ F(|X_T|) \equiv \int d^{10}\Theta \left[ \frac{1}{8!} \Phi^8 + \frac{1}{4!2!4} \Phi^4 A^2 + \frac{1}{4!16} A^4 \right] W_B(|X_T|) \]

\[ = \sum_{n=1}^{8} \sum_{k=0}^{[\frac{n+2}{2}]} C_{n+2-2k}^{(n)} |X_T|^{n-2k} R^{2k} \frac{\partial^n W_B(|X_T|)}{\partial |X_T|^n}. \quad (3.15) \]

Although we have not yet completed the evaluation of the coefficients \( C_m^{(n)} \) we can consider general features that arise after substituting (3.15) into the expression (3.7) for the Wilson loop. The bosonic integral superficially seems to be divergent, at least term by term in the expansion (3.12). Such divergences arise at the boundary of moduli space. However, we have also seen that the fermionic integrations naively lead to a vanishing result if the subtleties of the boundary of moduli space are ignored. We therefore regulate the divergences by cutting off the integration near the boundary. This is achieved by requiring that \( X_4 < \Lambda \) so that the bosonic integral becomes

\[ \langle W(R) \rangle_\Lambda = \int_\Lambda dX_4 \int d^2 X t d^3 X_T \delta(X_4^2 - X_T^2 - X_T^2 - R^2) F(|X_T|) \]

\[ = 4\pi^2 \int_0^{\Lambda^2 - R^2} d|X_T|^2 |X_T|^2 (\Lambda - \sqrt{X_T^2 + R^2}) F(|X_T|). \quad (3.16) \]

As before, this expression is to be evaluated with the choice of euclidean signature for \( X^\mu \) so that \( X_T^2 > 0 \). The integration is now performed by substituting the expression for \( F(|X_T|) \) from (3.13) and using the explicit expression for \( W(|X_T|) \) in (3.3). The integral in (3.16) is at most linearly divergent rather than having the quartic divergence implied by naive dimensional analysis as in the bosonic model.

The apparent linear divergence has a coefficient that should vanish although the explicit expression for \( F(|X_T|) \) does not make this at all apparent. The absence of this linear divergence requires

\[ \sum_{n=1}^{8} (-)^n (n+2)! C_{n+2}^{(n)} = 0, \quad \sum_{n=1}^{8} (-)^n n! C_n^{(n)} = 0. \quad (3.17) \]

While the second of these conditions can be seen to hold by a simple argument, the first condition depends on details of the coefficients \( C_m^{(n)} \), which we are currently evaluating (and will be presented in [ ]). Assuming that the linear divergence is indeed absent, as it must be, we can show that a subleading logarithmic divergence is automatically absent since the coefficients also satisfy the sum rule

\[ \sum_{n=1}^{8} (-)^n (n+3)! C_{n+2}^{(n)} = 0. \quad (3.18) \]

The remaining integral is finite and its value is given by letting \( \Lambda \to \infty \) in (3.16) which gives

\[ \langle W(R) \rangle = -4\pi^2 \int_0^{\infty} dX_T X_T^2 \sqrt{X_T^2 + R^2} F(|X_T|). \quad (3.19) \]
It turns out that this expression reduces to purely boundary terms which combine into the result

$$
\langle W(R) \rangle = \frac{\pi^4}{2^4 \cdot 4!} \left( \frac{g_{YM}^2}{8\pi^2} \right)^4 e^{2\pi i \tau} \sum_{n=2}^{8} \sum_{k=2}^{n} (-)^n \frac{(n+3)!}{k+3} c_{n+2}^{(n)},
$$

(3.20)

where the overall normalization has been reinstated and takes into account the standard $SU(2)$ instanton measure and $\tau$ is the complexified Yang–Mills coupling constant, $\tau = \vartheta_{YM}/2\pi + 4\pi i/g_{YM}^2$. When combined with the complex conjugate contribution due to an anti-instanton the result is real but $\vartheta_{YM}$-dependent and independent of $R$.

4. Discussion

We have seen that the value of the Wilson loop in an instanton background can be obtained by direct integration over the supermoduli. The calculation made use of the residual $OSp(2,2|4)$ symmetry of the configuration. The expression for the instanton contribution has the form of the integral of a divergence over $AdS_5$ and therefore the result can be written as a surface term on the $\rho = 0$ boundary. More specifically, the support for this integral arises from $\rho \sim 0$, $|x_l| \sim R$ and $x_t \sim 0$, which means it comes from small instantons touching the loop. This is just the region in which intuitive arguments would suggest the instantons could contribute. Although we have not yet completed the calculation of the exact value of the loop there is no reason to expect it to vanish. Assuming that it is nonzero, it would be interesting to identify a topological origin for the coefficient characterizing the result. The generalization of the semiclassical approximation to gauge group $SU(N)$ and arbitrary instanton number in the large $N$ limit is straightforward, given the results of [14, 2] concerning the ADHM construction in this case.

It is far less trivial to make the comparison with the type IIB supergravity implied by the AdS/CFT correspondence. The familiar problem is that the supergravity limit corresponds to the limit of large 'tHooft coupling, $\lambda = g_{YM}^2 N >> 1$. The semiclassical instanton contributions to certain correlation functions of gauge invariant local operators [1, 14] do, in fact, precisely match the corresponding D-instanton contributions. Presumably, this agreement means that the leading instanton contributions to these correlation functions are independent of the coupling constant. However, in the case of the Wilson loop we know that the perturbative contribution results in the behaviour, $e^\lambda = e^{\sqrt{g_{YM}^2 N}}$ and a similar behaviour probably also multiplies the instanton contribution. Clearly, this factor reduces to a trivial constant in the semiclassical ($g_{YM} \rightarrow 0$) limit and is therefore not seen in this approximation.

The presence of a nonzero instanton contribution to the Wilson loop should come as no surprise. After all, the $\mathcal{N} = 4$ theory possesses the highly nontrivial Montonen–Olive $SL(2,\mathbb{Z})$ duality symmetry and a separate Wilson loop can be defined for each
test dyonic particle carrying any of the infinite set of electric and magnetic charges that are related by this symmetry. Although we have not investigated this in detail, it is implausible that these relationships can be satisfied without the characteristic dependence on $\vartheta_{YM}$ that enters into the instanton terms. The precise way in which S-duality is implemented still remains to be understood.

As emphasized by Shenker [15], matrix models quite generally have instantons that are related to the presence of D-instantons in string theory. Although the work of [4, 8] suggests that there is a connection between the Wilson loop and a gaussian matrix model, the analysis of [3] clearly shows that matching the large $N$ results does not exclude the presence of a non-trivial potential in the matrix model. If this turns out to be the case, the instanton effects discussed in this paper would naturally be identified with those of the matrix model.

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