Decomposition of the Hessian matrix for action at choreographic three-body solutions with figure-eight symmetry

Toshiaki Fujiwara, Hiroshi Fukuda
College of Liberal Arts and Sciences, Kitasato University

Hiroshi Ozaki
Laboratory of general education for science and technology, Faculty of Science, Tokai University
e-mail: fujiwara@kitasato-u.ac.jp, fukuda@kitasato-u.ac.jp and ozaki@tokai-u.jp

Abstract

We developed a method to calculate the eigenvalues and eigenfunctions of the second derivative (Hessian) of action at choreographic three-body solutions that have the same symmetries as the figure-eight solution. A choreographic three-body solution is a periodic solution to equal mass planar three-body problem under potential function $\sum_{i<j} U(r_{ij})$, in which three masses chase each other on a single closed loop with equal time delay.

We treat choreographic solutions that have the same symmetries as the figure-eight, namely, symmetry for choreography, for time reversal, and for time shift of half period. The function space of periodic functions are decomposed into five subspaces by these symmetries. Namely, one subspace of trivial oscillators with eigenvalue $4\pi^2/T^2 \times k^2$, $k = 0, 1, 2, \ldots$, four subspaces of choreographic functions, and four subspaces of “zero-choreographic” functions. Therefore, the matrix representation of the Hessian is also decomposed into nine corresponding blocks. Explicit expressions of base functions and the matrix representation of the Hessian for each subspaces are given.

The trivial eigenvalues with $k \neq 0$ are quadruply degenerated, while with $k = 0$ are doubly degenerated that correspond to the conservation of linear momentum in $x$ and $y$ direction. The eigenvalues in choreographic subspace have no degeneracy in general. In “zero-choreographic” subspace, every eigenvalues are doubly degenerated.
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1 Definitions and notations

Consider a planar three-body problem with equal masses \(m_\ell = 1\), \(\ell = 0, 1, 2\), defined by the Lagrangian

\[
L = \frac{1}{2} \sum_\ell \left( \frac{dq_\ell}{dt} \right)^2 + \sum_{i \neq j} V(|q_i - q_j|),
\]

\[(1)\]

\(q_\ell = (q^x_\ell, q^y_\ell) \in \mathbb{R}^2\). The potential \(V\) may be an extended Newton potential

\[
V_\alpha(|q_i - q_j|) = \begin{cases} 
\alpha^{-1}/|q_i - q_j|_\alpha & (\alpha \neq 0) \\
\log |q_i - q_j| & (\alpha = 0)
\end{cases}
\]

\[(2)\]

or a Lennard-Jones type potential

\[
V = \frac{a}{|q_i - q_j|_\alpha} - \frac{b}{|q_i - q_j|^2}.
\]

Choreographic solutions are periodic solutions to the equation of motion

\[
\frac{d^2q_\ell}{dt^2} = \frac{\partial}{\partial q_\ell} \sum_j V(|q_j - q_\ell|)
\]

\[(4)\]

that satisfy

\[(q_0, q_1, q_2) = (q(t), q(t + T/3), q(t - T/3)),\]

\[(5)\]

where \(T\) is the period.

Some choreographic solutions are known. The famous figure-eight solution was discovered numerically by C. Moore [1] in 1993. Then, A. Chenciner and R. Montgomery [2] in 2000 rediscovered this solution and gave a proof for existence. M. Šuvakov and V. Dmitrašinović in 2013 found many “slalom solutions”. L. Sbano [3] in 2004 found the figure-eight solution under the Lennard-Jones type potential. H. Fukuda, T. Fujiwara and H. Ozaki [7] in 2017 found many choreographic solutions under the Lennard-Jones type potential. Almost all these choreographic solutions have the same symmetries as the figure-eight solutions. Exceptional cases are \(k = \text{even slaloms}, \) where \(k\) is the “power of slalom” [6]. The symmetries that we use in this note will be shown in the section 3. Method in this note is applicable to the original figure-eight solution, \(k = \text{odd slaloms}, \) figure-eight solutions by Sbano et al., and Fukuda et al.

1.1 Definition of the Hessian

Consider the variation of action integral at a solution \(q_\ell(t)\) to second order of variation \(q_\ell(t) \rightarrow q_\ell(t) + \delta q_\ell(t),\)

\[
S[q + \delta q] = S[q] + \frac{1}{2} \int_0^T dt \left( \sum_\ell \left( \frac{d^2q_\ell}{dt^2} \right)^2 + \sum_{i \neq j} \delta q_i \left( \frac{\partial^2}{\partial q_\ell \partial q_j} \sum_j V \right) \delta q_j \right).
\]

\[(6)\]
Here, the first order term $\delta S$ is zero as $q_\ell$ is a solution. We restrict ourselves to consider the variational function $\delta q_\ell$ be a periodic function with period $T$. Then, partial integration makes

$$
\int_0^T dt \sum_{\ell} \left( \frac{d\delta q_\ell}{dt} \right)^2 = \int_0^T dt \sum_{\ell} \delta q_\ell \left( -\frac{d^2}{dt^2} \right) \delta q_\ell. \tag{7}
$$

Thus, the variation is given by

$$
S[q + \delta q] = S[q] + \frac{1}{2} \int_0^T dt \, \Psi \mathcal{H} \Psi, \tag{8}
$$

where

$$
\Psi = \begin{pmatrix}
\delta q_0^x \\
\delta q_0^y \\
\delta q_1^x \\
\delta q_1^y \\
\delta q_2^x \\
\delta q_2^y
\end{pmatrix}
$$

is $\mathbb{R}^6$ column vector and $\mathcal{H}$ is the Hessian that has the form

$$
\mathcal{H} = -\frac{d^2}{dt^2} + \mathcal{U} \Delta \mathcal{U}, \quad \mathcal{U} = \begin{pmatrix}
u_{12} & 0 & 0 \\
0 & u_{20} & 0 \\
0 & 0 & u_{01}
\end{pmatrix}.
\tag{10}
$$

Here, $u_{ij}$ is $2 \times 2$ matrix and $\Delta$ is $6 \times 6$ anti-symmetric matrix

$$
u_{ij} = \begin{pmatrix}u_{ij}^{xx} & u_{ij}^{xy} \\
u_{ij}^{yx} & u_{ij}^{yy}\end{pmatrix} , \quad \Delta = \begin{pmatrix}0 & E_2 & -E_2 \\
-E_2 & 0 & E_2 \\
E_2 & -E_2 & 0\end{pmatrix}, \quad E_2 = \begin{pmatrix}1 & 0 & 0 \\
0 & 0 & 1\end{pmatrix}, \tag{11}
$$

and $\mathcal{U}$ stands for the transpose of $\Delta$. The explicit expression of $u_{ij}$ for extended Newton potential, for example, is

$$
u_{ij} = \alpha + 2 \frac{1}{r_{ij}^{\alpha+2}} \left( (q_i^x - q_j^x)(q_i^y - q_j^y) + (q_i^y - q_j^y)^2 - (q_i^x - q_j^x)^2 \right) - \frac{1}{r_{ij}^{\alpha+2}} E_2, \tag{12}
$$

$$
u_{ij} = |q_i - q_j|.
$$

When there is no confusion, we will use the following abbreviated notations

$$
\Psi = \begin{pmatrix}\delta q_0 \\
\delta q_1 \\
\delta q_2\end{pmatrix} \quad \text{and} \quad \Delta = \begin{pmatrix}0 & 1 & -1 \\
-1 & 0 & 1 \\
1 & -1 & 0\end{pmatrix}, \tag{13}
$$

and similar. In this note, we consider the eigenvalue problem

$$
\mathcal{H} \Psi = \lambda \Psi. \tag{14}
$$
For linear operator $O$, we express its eigenvalue $O'$. If $O$ satisfies $O^2 = 1$, the eigenvalue is one of $O' = \pm 1$. While, if $O^3 = 1$ is satisfied, then $O' = 1, \omega, \omega^2$ with $\omega = (-1 \pm i\sqrt{3})/2$.

For any two functions $\Phi, \Psi$ and any operator $O$, we define and write the inner product

$$\langle \Phi | \Psi \rangle = \frac{2}{3T} \int_0^T dt \Phi(t)\Psi(t), \quad \langle \Phi | O | \Psi \rangle = \frac{2}{3T} \int_0^T dt \Phi(t)O\Psi(t).$$

The factor $2/(3T)$ is for later convenience. The inner product satisfy

$$\langle \Phi | \Psi \rangle = \langle \Psi | \Phi \rangle, \quad \langle \Phi | O | \Psi \rangle = \langle \Phi | O \Psi \rangle = \langle \Phi | O \Phi \Psi \rangle.$$  

For writing basis vectors, it is convenient to write $x$ and $y$ basis together in a matrix form,

$$\begin{pmatrix} x_1 & 0 \\ 0 & y_1 \\ x_2 & 0 \\ 0 & y_2 \\ x_3 & 0 \\ 0 & y_3 \end{pmatrix},$$

with

$$\frac{2}{3T} \int_0^T dt(x_1^2 + x_2^2 + x_3^2) = \frac{2}{3T} \int_0^T dt(y_1^2 + y_2^2 + y_3^2) = 1.$$

Then

$$a^x \begin{pmatrix} x_1 \\ 0 \\ x_2 \\ 0 \\ x_3 \\ 0 \end{pmatrix} + a^y \begin{pmatrix} 0 \\ y_1 \\ 0 \\ y_2 \\ 0 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 & 0 \\ 0 & y_1 \\ x_2 & 0 \\ 0 & y_2 \\ x_3 & 0 \\ 0 & y_3 \end{pmatrix} \begin{pmatrix} a^x \\ a^y \end{pmatrix}.$$  

If $x_k = y_k = z_k$, we simply write

$$\begin{pmatrix} z_1 \\ 0 \\ z_1 \\ 0 \\ z_2 \\ 0 \\ 0 \\ z_2 \\ 0 \\ z_3 \\ 0 \\ 0 \\ z_3 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}.$$  

It should be stressed that this is not a constraint but a matrix representation of two independent basis vectors. For example,

$$\Phi_n = \begin{pmatrix} \cos(n\nu t) \\ \cos(n\nu t + 2\pi/3) \\ \cos(n\nu t - 2\pi/3) \end{pmatrix}$$  

5
1.2 Decomposition of function space by orthogonal projection operator

A linear operator $\mathcal{P}$ is called “orthogonal projection operator” if it satisfies

$$\mathcal{P}(\mathcal{P} - 1) = 0 \text{ and } \mathcal{P}^* = \mathcal{P}$$  \hspace{0.5cm} (23)

This operator decomposes function space $W$ into two orthogonal subspaces $W_1$ and $W_0$.

$$W_1 = \{ f | f \in W, \mathcal{P}f = f \}, \quad W_0 = \{ f | f \in W, \mathcal{P}f = 0 \}, \quad W = W_1 \oplus W_0.$$  \hspace{0.5cm} (24)

The subspaces $W_1$ and $W_0$ are mutually orthogonal complements, because arbitrary elements $f \in W_1$ and $g \in W_0$ are orthogonal by $\langle g | f \rangle = \langle g | \mathcal{P}f \rangle = \langle \mathcal{P}g | f \rangle = 0$, and arbitrary element $\Psi \in W$ will be decomposed by the identities $\Psi = \mathcal{P}\Psi + (1 - \mathcal{P})\Psi$. Here, $\mathcal{P}\Psi \in W_1$ and $(1 - \mathcal{P})\Psi \in W_0$ since $\mathcal{P}(\mathcal{P}\Psi) = \mathcal{P}\Psi$ and $\mathcal{P}((1 - \mathcal{P})\Psi) = 0$. See figure 1.

If the Hessian $\mathcal{H}$ is commutative with $\mathcal{P}$, matrix representation of the Hessian is also decomposed into two blocks for $W_1$ subspace and $W_0$ subspace.
the matrix elements of $\mathcal{H}$ between $W_1$ subspace and $W_0$ subspace are zero,
\[ \langle g|\mathcal{H}|f \rangle = \langle g|\mathcal{H}|Pf \rangle = \langle \mathcal{P}g|\mathcal{H}|f \rangle = 0 \text{ for } f \in W_1 \text{ and } g \in W_0. \quad (25) \]

In the following sections, we will find some orthogonal projection operators that are commutable each other and $\mathcal{H}$. These projection operators decompose the function space into small subspaces and make the matrix representation of $\mathcal{H}$ small blocks.

## 2 Four zero eigenvalues and quadruply degenerated trivial eigenvalues

In this section, we describe 4 zero eigenvalues and quadruply degenerated trivial eigenvalues, which always exist for the Hessian (10).

Corresponding to the conservation law, the linear momentum for $x$ and $y$ direction, angular momentum and the energy, the Hessian always has 4 zero eigenvalues for variations $\delta q_\ell = \delta q(t)$, $\delta q(t), \delta q(t)$ and $dq_\ell/dt$.

Since the function space is composed by all periodic functions with period $T$, it contains a trivial eigenfunction $\Psi(t) = \delta q(t)\delta q(t)\delta q(t)$ that describes three bodies move coherently. Since, $\Delta \Psi = 0$ for this function, the eigenvalue problem is reduced to the simple form
\[ H\Psi = -\frac{d^2}{dt^2}\Psi = \lambda \Psi, \quad (26) \]
that yields quadruply degenerated eigenvalue $\lambda = 4\pi^2/T^2 \times k^2$. Namely $\delta q = \delta q, \delta q, \delta q, \delta q$ for positive integers $k$. The two trivial functions for $k = 0$ are already counted for translational function in the previous paragraph.

Let $\sigma$ be the cyclic permutation operator $\sigma \delta q_0, \delta q_1, \delta q_2 = \delta q_1, \delta q_2, \delta q_0$, or in the matrix form,
\[ \sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } \sigma = \sigma^{-1}. \quad (27) \]

Since $\sigma^3 = 1$, it defines the orthogonal projection operator
\[ \mathcal{P}_\sigma = \frac{1}{3}(1 + \sigma + \sigma^2), \quad \mathcal{P}_\sigma(\mathcal{P}_\sigma - 1) = 0. \quad (28) \]

This operator splits the function space into subspace with the eigenvalue $\mathcal{P}_\sigma' = 1$ or $\mathcal{P}_\sigma' = 0$. The subspace whose element has $\mathcal{P}_\sigma' = 1$ is the trivial subspace. The complemental subspace is composed of functions with $\mathcal{P}_\sigma' = 0$. We call this subspace "zero-centre-of-mass" subspace, because each function in this subspace keeps the center of mass at the origin,
The figure-eight solution belongs to zero-center-of-mass subspace.

Now, let us proceed to the symmetry of the Hessian at choreographic solutions with figure-eight symmetry.

3 Symmetries of the figure-eight solution

The figure-eight solution with proper \( x, y \) coordinates and origin of time has the following symmetries,

- **Choreographic symmetry:**
  \[
  \begin{pmatrix}
  q_0(t + T/3) \\
  q_1(t + T/3) \\
  q_2(t + T/3)
  \end{pmatrix}
  =
  \begin{pmatrix}
  q_1(t) \\
  q_2(t) \\
  q_0(t)
  \end{pmatrix},
  \tag{30}
  \]

- **Time reversal symmetry:**
  \[
  \begin{pmatrix}
  q_0(-t) \\
  q_1(-t) \\
  q_2(-t)
  \end{pmatrix}
  =
  - \begin{pmatrix}
  q_0(t) \\
  q_1(t) \\
  q_2(t)
  \end{pmatrix},
  \tag{31}
  \]

and

- **Time shift in \( T/2 \):**
  \[
  \begin{pmatrix}
  q_0(t + T/2) \\
  q_1(t + T/2) \\
  q_2(t + T/2)
  \end{pmatrix}
  =
  - \begin{pmatrix}
  \mu q_0(t) \\
  \mu q_1(t) \\
  \mu q_2(t)
  \end{pmatrix},
  \mu \begin{pmatrix}
  x \\
  y
  \end{pmatrix}
  =
  \begin{pmatrix}
  -x \\
  y
  \end{pmatrix}.
  \tag{32}
  \]

3.1 Choreographic symmetry

Let \( R^{1/3} \) be a time displacement operator \( R^{1/3} \delta q(t) = \delta q(t + T/3) \) and \( \mathcal{C} = \sigma^{-1} R^{1/3} \). Then the choreographic symmetry in (30) is equivalent to

\[
\mathcal{C} \begin{pmatrix}
  q_0(t) \\
  q_1(t) \\
  q_2(t)
  \end{pmatrix}
  =
  \begin{pmatrix}
  q_0(t) \\
  q_1(t) \\
  q_2(t)
  \end{pmatrix}.
  \tag{33}
  \]

Since the function space has period \( T \),

\[
\int_0^T dt f(t) g(t + T/3) = \int_0^T dt f(t - T/3) g(t).
  \tag{34}
  \]

Namely, \( R^{1/3} = R^{-1/3} \) and \( \mathcal{C} = \mathcal{C}^{-1} \). Since the operator \( \mathcal{C} \) satisfies \( \mathcal{C}^3 = 1 \), it defines the orthogonal projection operator

\[
\mathcal{P}_\mathcal{C} = \frac{1}{3} (1 + \mathcal{C} + \mathcal{C}^2), \quad \mathcal{P}_\mathcal{C} (\mathcal{P}_\mathcal{C} - 1) = 0.
  \tag{35}
  \]

Thus the operator \( \mathcal{P}_\mathcal{C} \) decomposes the function space \( W \) into two orthogonal subspaces; the choreographic subspace \( W_\mathcal{C} \) that made of functions that belong
Figure 2: Decomposition of function space $W = W_C \oplus W_z$ by orthogonal projection operator $P_C$. We call a function $\Psi$ “non-choreographic” function if $(1 - P_C)\Psi \neq 0$. While, we call the subspace $W_z$ “zero-choreographic” subspace.

To the eigenvalue $P_C' = 1$ (i.e. $C' = 1$) and its orthogonal complement. We call this complement “zero-choreographic” subspace $W_z$.

$$W_c = \{ \Psi | P_C \Psi = \Psi \}, \quad W_z = \{ \Psi | P_C \Psi = 0 \}, \quad W = W_c \oplus W_z$$ (36)

The figure-eight solution belongs to choreographic subspace. In this note, we call the subspace $W_z$ “zero-choreographic” subspace to avoid confusion with “non-choreographic” function. A function is called “non-choreographic”, if it is not choreographic. See figure 2.

The eigenvalue of $H$ in zero-choreographic subspace is doubly degenerated. Because $H$ and $C$ commute, if $\Psi_\lambda \neq 0$ belongs to the eigenvalue $H' = \lambda$ then $\Psi_\lambda$ and $\Phi_\lambda = (C - C^2)\Psi_\lambda/\sqrt{3}$ belong to the same eigenvalue. Since $\Psi_\lambda$ satisfies $(1 + C + C^2)\Psi_\lambda = 0$, $\langle \Psi_\lambda | \Phi_\lambda \rangle = 0$ and $\langle \Phi_\lambda | \Phi_\lambda \rangle = \langle \Psi_\lambda | \Psi_\lambda \rangle \neq 0$. Actually,

$$\langle \Psi_\lambda | \Phi_\lambda \rangle = \langle \Psi_\lambda | (C - C^2)\Psi_\lambda \rangle/\sqrt{3} = (\langle \Psi_\lambda | C\Psi_\lambda \rangle - \langle C\Psi_\lambda | \Psi_\lambda \rangle) /\sqrt{3} = 0,$$

$$\langle \Phi_\lambda | \Phi_\lambda \rangle = \langle \Psi_\lambda | (C^2 - C)(C - C^2)\Psi_\lambda \rangle /3 = \langle \Psi_\lambda | 1 | \Psi_\lambda \rangle.$$ (37)

Therefore $\Psi_\lambda$ and $\Phi_\lambda$ are orthogonal bases of two dimensional subspace for each $\lambda$. This proves the statement of this paragraph.

3.2 Time reversal symmetry

Let $\Theta$ be the time reversal operator $\Theta(\delta q_k(t)) = \delta q_k(-t)$, $\tau$ be the operator for exchange of the second and third row,

$$\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$ (38)
Table 1: Zero-center-of-mass subspace is decomposed into $2^3$ subspaces by three projection operators $P_C$, $P_T$ and $P_M$. Correspondence between the name of subspaces and the eigenvalues are shown. The last $\pm$ sign correspond to two eigenvalue of $P_M' = 1$, 0 respectively.

| $P_C'$ | $P_T'$ | $P_M'$ |
|-------|-------|-------|
| 1     | 1     | 1     |
| 0     | 0     | 0     |
| choreographic cos $\pm$ | zero-choreographic cos $\pm$ | choreographic sin $\pm$ | zero-choreographic sin $\pm$

and let $T = \tau \Theta$. Then the time reversal symmetry in (31) for the figure-eight solution is

$$T \begin{pmatrix} q_0(t) \\ q_1(t) \\ q_2(t) \end{pmatrix} = - \begin{pmatrix} q_0(t) \\ q_1(t) \\ q_2(t) \end{pmatrix}.$$  

(39)

Since, $T^2 = 1$, this operator defines the orthogonal projection operator

$$P_T = \frac{1}{2}(1 + T), \ P_T(P_T - 1) = 0.$$  

(40)

This operator decomposes the function space into two subspaces with $P_T' = 1$ (i.e. $T' = 1$) and with $P_T' = 0$ ($T' = -1$). The figure-eight solution belongs to the subspace with $T' = -1$.

### 3.3 Time shift symmetry in $T/2$

Let $R^{1/2}$ be a time displacement operator $R^{1/2}\delta q(t) = \delta q(t + T/2)$, $\mu$ be the mirror operator with respect to $y$ axis $\mu(\delta q_x^k, \delta q_y^k) = (\delta q_x^k, -\delta q_y^k)$ and $M = \mu R^{1/2}$. Then the time shift symmetry in $T/2$ is

$$M \begin{pmatrix} q_0(t) \\ q_1(t) \\ q_2(t) \end{pmatrix} = \begin{pmatrix} q_0(t) \\ q_1(t) \\ q_2(t) \end{pmatrix}.$$  

(41)

Since $M$ satisfies $M^2 = 1$, the eigenvalues are $\pm 1$, and the orthogonal projection operator $P_M$ is

$$P_M = \frac{1}{2}(1 + M), \ P_M(P_M - 1) = 0.$$  

(42)

This operator decompose the function space into two subspaces with $P_M' = 1$ (i.e. $M' = 1$) and $P_M' = 0$ ($M' = -1$). The figure-eight solution belongs to the subspace with $P_M' = 1$.

### 3.4 Summary of symmetries for the figure-eight solution

In the section 2 and previous subsections, we introduced operators $\sigma$, $C$, $T$, $M$ and corresponding projection operators $P_{\sigma}$, $P_C$, $P_T$, $P_M$. All these projection operators commute each other. Therefore, these operators decompose the
Figure 3: Decomposition of matrix representation of the Hessian, whose total size is $(2 \times 18N)^2$. Here, each · stands $(2 \times N)^2$ diagonal matrix, and * stands $(2 \times N)^2$ matrix. So, the size of each 4 choreographic blocks are $(2 \times N)^2$, and zero-choreographic blocks are $(2 \times 2N)^2$. All no marked elements are 0.

function space into $2^4$ subspaces. Actually, we don’t divide the trivial subspace $P'_\sigma = 1$. The zero-centre-of-mass subspace $P'_\sigma = 0$ will be divided into $2^3$ subspaces. See table 1. The figure-eight solution belongs to the subspace $P'_\sigma = 0$ and $P'_C = 1$ ($C' = 1$) and $P'_T = 0$ ($T' = -1$) and and $P'_M = 1$ ($M' = 1$).

The eigenspace of $H$ is also divided into these subspaces, because all these projection operators commute with $H$.

In the next section, we decompose the function space by these four symmetries.

4 Subspaces and base functions

Before decompose the function space, we count the degree of this space. To make the degree finite, we introduce a cutoff in Fourier series, $1, \cos(\nu t), \cos(2\nu t), \cos(3\nu t), \ldots, \cos(3N\nu t)$ and $\sin(\nu t), \sin(2\nu t), \sin(3\nu t), \ldots, \sin(3N\nu t)$, $\nu = 2\pi/T$ and $N = 2^M$ with some integer $M$. Then the degrees are 3 for three bodies, 2 for $x$ and $y$ components, $6N + 1$ for Fourier components. So, total degree is $6(6N + 1) \sim 2 \times 18N$. We will decompose this space. See figure 3. In this figure, trivial subspace is not decomposed. The compliment, zero-centre-of-mass subspace, will be decomposed into $2^4$ subspaces.

Although we are considering the function space of $\mathbb{R}^6$, it is convenient to
consider the function space of $\mathbb{C}^6$ in the following two subsections. This is because operator $\mathcal{O}$ with $\mathcal{O}^3 = 1$ has the eigenvalue only $\mathcal{O}' = 1$ in $\mathbb{R}$. This makes our decomposition not straightforward. It has three eigenvalues $\mathcal{O}' = 1, \omega, \omega^2$, $\omega = (-1 + i\sqrt{3})/2$ in $\mathbb{C}$ instead. We will use $e^{-i3N\nu t}, \ldots, e^{-i\nu t}, 1, e^{i\nu t}, \ldots, e^{i3N\nu t}$ instead of sin and cos for the Fourier series. We again have $6N + 1$ Fourier components.

### 4.1 Decomposition by $\mathcal{P}_\sigma$

Since $\sigma^3 = 1$, the eigenvalue of $\sigma$ are $\sigma' = 1, \omega, \omega^2$ with $\omega = (-1 + i\sqrt{3})/2$. Then, the basis for $\mathcal{P}_\sigma = 1$ are trivial,

$$\sigma' = 1 \leftrightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} f(t)$$  \hspace{1cm} (43)

with arbitrary periodic function $f(t)$.

While function for $\mathcal{P}' = 0$ are zero-centre-of-mass,

$$\sigma' = \omega \leftrightarrow \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix} f(t), \text{ or } \sigma' = \omega^2 \leftrightarrow \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix} f(t).$$  \hspace{1cm} (44)

Therefore $1/3$ of whole basis are trivial and $2/3$ are zero-centre-of-mass. Namely, $2(6N + 1)$ basis are trivial. We don’t decompose them more.

In the following sections, we will decompose $4(6N + 1)$ basis by other projection operators.

### 4.2 Decomposition by $\mathcal{P}_c$

Since $\{R^{1/3}\}^3 = 1$, the eigenvalues of $R^{1/3} = 1, \omega, \omega^2$. The eigenfunctions are $e^{i3k\nu t}, e^{i(3k+1)\nu t}$ and $e^{i(3k+2)\nu t}$ with $k \in \mathbb{Z}$ respectively. More precisely,

$$\begin{align*}
\{R^{1/3}\}' = 1 & \leftrightarrow e^{i(-3N)\nu t}, e^{i(-3N+3)\nu t}, \ldots, e^{i(-3)\nu t}, 1, e^{i3\nu t}, e^{i6\nu t}, \ldots, e^{i(3N)\nu t}, \\
\{R^{1/3}\}' = \omega & \leftrightarrow e^{i(-3N+1)\nu t}, e^{i(-3N+4)\nu t}, \ldots, e^{i(-2)\nu t}, e^{i4\nu t}, e^{i6\nu t}, \ldots, e^{i(3N-2)\nu t}, \\
\{R^{1/3}\}' = \omega^2 & \leftrightarrow e^{i(-3N+2)\nu t}, e^{i(-3N+5)\nu t}, \ldots, e^{-i\nu t}, e^{i2\nu t}, e^{i5\nu t}, \ldots, e^{i(3N-1)\nu t}. \\
\end{align*}$$  \hspace{1cm} (45)

Each subspace has $2N + 1$, $2N$ and $2N$ functions.

Then, choreographic basis belong to $\mathcal{P}_c' = 1 \leftrightarrow \mathcal{C}' = (\sigma^{-1})'(R^{1/3})' = 1 \leftrightarrow (\sigma', (R^{1/3})') = (\omega, \omega)$ or $(\omega^2, \omega^2)$. Namely, the following $8N$ basis are choreographic,

$$\begin{align*}
\begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix} e^{i(3k+1)\nu t} \text{ and } \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix} e^{i(3k+2)\nu t}. \\
\end{align*}$$  \hspace{1cm} (46)
It should be stressed again that
\[
\begin{pmatrix}
1 \\ \omega \\ \omega^2
\end{pmatrix}
\]
represents two independent basis
\[
\begin{pmatrix}
1 \\ 0 \\ \omega \\
0 \\ 1 \\ \omega^2 \\
\omega^2 \\ 0 \\ \omega
\end{pmatrix}
, \quad \begin{pmatrix}
1 \\ \omega \\ \omega^2 \\
0 \\ 1 \\ 0 \\
\omega^2 \\ 0 \\ \omega^2
\end{pmatrix}.
\]
So,
\[
\begin{pmatrix}
1 \\ \omega \\ \omega^2
\end{pmatrix} e^{i(3k+1)\nu t}, \quad k = -N, -N + 1, \ldots, -1, 1, \ldots N - 1
\]
stands for \(2 \times 2N\) base functions.

Zero-choreographic basis \(P'_C = 0\) are composed of \(C' = (\sigma^{-1})'(R_{1/3}')' = \omega\) or \(\omega^2\). Namely, \(C' = (\sigma^{-1})'(R_{1/3}')' = \omega \leftrightarrow (\sigma', (R_{1/3}')') = (\omega, \omega^2)\) or \((\omega^2, 1) \leftrightarrow
\[
\begin{pmatrix}
1 \\ \omega \\ \omega^2
\end{pmatrix} e^{i(3k+2)\nu t} \quad \text{and} \quad \begin{pmatrix}
1 \\ \omega \\ \omega^2
\end{pmatrix} e^{i3k\nu t},
\]
and \(C' = (\sigma^{-1})'(R_{1/3}')' = \omega^2 \leftrightarrow (\sigma', (R_{1/3}')') = (\omega, 1), (\omega^2, \omega) \leftrightarrow
\[
\begin{pmatrix}
1 \\ \omega \\ \omega^2
\end{pmatrix} e^{3k\nu t} \quad \text{and} \quad \begin{pmatrix}
1 \\ \omega \\ \omega^2
\end{pmatrix} e^{i(3k+1)\nu t}.
\]

So, there are \(16N + 4\) zero-choreographic basis.

Thus, total \(4(6N+1)\) basis are split into \(8N\) choreographic basis and \(16N+4\) zero-choreographic basis.

### 4.3 Decomposition by \(P_T\)

Now consider the action \(T = \tau \Theta\) to the above basis (46), (50) and (51). The operator \(\tau\) exchange \(\omega\) and \(\omega^2\) and \(\Theta\) exchange \(e^{ik\nu t}\) and \(e^{-ik\nu t}\). As the result, \(T = \tau \Theta\) exchange a base and its complex conjugate. Therefore, the projection operator \(P_T = (1 + T)/2\) picks its real part, while \(1 - P_T = (1 - T)/2\) picks its imaginary part.

Thus, the basis for \((P'_\sigma, P'_C, D'_T) = (0, 1, 1)\) are
\[
\begin{pmatrix}
\cos((3n + 1)\nu t) \\
\cos((3n + 1)\nu t + 2\pi/3) \\
\cos((3n + 1)\nu t - 2\pi/3)
\end{pmatrix} = \begin{pmatrix}
\cos((3n + 1)\nu t) \\
\cos((3n + 1)\nu (t + T/3)) \\
\cos((3n + 1)\nu (t - T/3))
\end{pmatrix}
\]

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or
\[
\begin{pmatrix}
\cos((3n + 2)\nu t) \\
\cos((3n + 2)\nu t - 2\pi/3) \\
\cos((3n + 2)\nu t + 2\pi/3)
\end{pmatrix} =
\begin{pmatrix}
\cos((3n + 2)\nu t) \\
\cos((3n + 2)\nu (t + T/3)) \\
\cos((3n + 2)\nu (t - T/3))
\end{pmatrix}.
\] (53)

Explicitly,
\[
\begin{pmatrix}
\cos(\nu t) \\
\cos(\nu (t + T/3)) \\
\cos(\nu (t - T/3))
\end{pmatrix},
\begin{pmatrix}
\cos(2\nu t) \\
\cos(2\nu (t + T/3)) \\
\cos(2\nu (t - T/3))
\end{pmatrix},
\begin{pmatrix}
\cos(4\nu t) \\
\cos(4\nu (t + T/3)) \\
\cos(4\nu (t - T/3))
\end{pmatrix},
\begin{pmatrix}
\cos(5\nu t) \\
\cos(5\nu (t + T/3)) \\
\cos(5\nu (t - T/3))
\end{pmatrix},
\ldots,
\begin{pmatrix}
\cos((3N - 2)\nu t) \\
\cos((3N - 2)\nu (t + T/3)) \\
\cos((3N - 2)\nu (t - T/3))
\end{pmatrix},
\begin{pmatrix}
\cos((3N - 1)\nu t) \\
\cos((3N - 1)\nu (t + T/3)) \\
\cos((3N - 1)\nu (t - T/3))
\end{pmatrix}.
\] (54)

Let \(k_n = \{1, 2, 4, 5, 7, 8, 10, \ldots, 3N - 2, 3N - 1\}\), namely, the series of positive integers without multiple of 3. This series is given by
\[
k_n = \frac{1}{4} \left(6n - (-1)^n - 3\right), \quad n = 1, 2, 3, \ldots, 2N - 1, 2N.
\] (55)

There are \(4N\) basis for \((P'_\sigma, P'_C, T') = (0, 1, 1)\), and \(4N\) basis for \((P'_\sigma, P'_C, T') = (0, 1, -1)\). We write them
\[
(P'_\sigma, P'_C, T') = (0, 1, 1) \leftrightarrow cc_n =
\begin{pmatrix}
\cos(k_n \nu t) \\
\cos(k_n \nu (t + T/3)) \\
\cos(k_n \nu (t - T/3))
\end{pmatrix}
\] (56)

and
\[
(P'_\sigma, P'_C, T') = (0, 1, -1) \leftrightarrow cs_n =
\begin{pmatrix}
\sin(k_n \nu t) \\
\sin(k_n \nu (t + T/3)) \\
\sin(k_n \nu (t - T/3))
\end{pmatrix},
\] (57)

for \(n = 1, 2, 3, \ldots, 2N - 1, 2N\).
For zero-choreographic subspace, \((C', T') = (0, 1)\) basis are
\[
\begin{pmatrix}
1 \\
-1/2 \\
-1/2
\end{pmatrix}, \begin{pmatrix}
\cos(\nu t) \\
\cos(\nu t - 2\pi/3) \\
\cos(\nu t + 2\pi/3)
\end{pmatrix}, \begin{pmatrix}
\cos(2\nu t) \\
\cos(2\nu t + 2\pi/3) \\
\cos(2\nu t - 2\pi/3)
\end{pmatrix}, \begin{pmatrix}
\cos(3\nu t) \\
\cos(3\nu t - 2\pi/3) \\
\cos(3\nu t - 2\pi/3)
\end{pmatrix}, \begin{pmatrix}
\cos(4\nu t) \\
\cos(4\nu t - 2\pi/3) \\
\cos(4\nu t + 2\pi/3)
\end{pmatrix}, \begin{pmatrix}
\cos(5\nu t) \\
\cos(5\nu t + 2\pi/3) \\
\cos(5\nu t - 2\pi/3)
\end{pmatrix}, \begin{pmatrix}
\cos(6\nu t) \\
\cos(6\nu t - 2\pi/3) \\
\cos(6\nu t + 2\pi/3)
\end{pmatrix},
\]
\[
\ldots, \begin{pmatrix}
\cos((3N - 2\nu t) \\
\cos((3N - 2\nu t - 2\pi/3) \\
\cos((3N - 2\nu t + 2\pi/3)
\end{pmatrix}, \begin{pmatrix}
\cos((3N - 1\nu t) \\
\cos((3N - 1\nu t + 2\pi/3) \\
\cos((3N - 1\nu t - 2\pi/3)
\end{pmatrix}, \begin{pmatrix}
\cos(3N\nu t) \\
\cos(3N\nu t - 2\pi/3) \\
\cos(3N\nu t + 2\pi/3)
\end{pmatrix}
\]
\]

Using series \(\ell_n = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots, 3N - 2, 2N - 1, 3N, 3N\}\) which is generated by
\[
\ell_n = \frac{1}{8}(6n + (1 - i)(-i)^n + (1 + i)^n - 3)
\]
for \(n = 1, 2, 3, \ldots, 4N + 1\), the basis are
\[
(P'_\sigma, P'_C, T') = (0, 0, 1) \leftrightarrow zc_n = \begin{pmatrix}
\cos(\ell_n \nu t) \\
\cos(\ell_n \nu t - (-1)^n 2\pi/3) \\
\cos(\ell_n \nu t + (-1)^n 2\pi/3)
\end{pmatrix},
\]
for \(n = 1, 2, 3, \ldots, 4N + 1\). There are \(4N + 1\) basis. Similarly,
\[
(P'_\sigma, P'_C, T') = (0, 0, -1) \leftrightarrow zs_n = (-1)^{n+1} \begin{pmatrix}
\sin(\ell_n \nu t) \\
\sin(\ell_n \nu t - (-1)^n 2\pi/3) \\
\sin(\ell_n \nu t + (-1)^n 2\pi/3)
\end{pmatrix},
\]
for \(n = 1, 2, 3, \ldots, 4N + 1\). There are again \(4N + 1\) basis. The reason of the factor \((-1)^{n+1}\) will be shown soon.

We can easily check the relation
\[
C \begin{pmatrix}
zc_n \\
zs_n
\end{pmatrix} = \begin{pmatrix}
-1/2 & \sqrt{3}/2 \\
-\sqrt{3}/2 & -1/2
\end{pmatrix} \begin{pmatrix}
zc_n \\
zs_n
\end{pmatrix}.
\]
Namely, the operator \(C\) makes \(2\pi/3\) rotation in the subspace spanned by \(zc_n\) and \(zs_n\) for each \(n\). To prove this relation, it is useful to note that the behavior of \(n, \ell_n\) and \(\ell_n \nu T/3 = 2\pi \ell_n /3\) are summarized in the following table.

\[
\begin{array}{cccc}
n \mod 4 & 1 & 2 & 3 \\
\ell_n \mod 3 & 1 & 2 & 0 \\
2\pi \ell_n /3 \mod 2\pi & 2\pi/3 & -2\pi/3 & 0
\end{array}
\]
So, to prove the relation in (62), it is sufficient to prove the four cases $n = 1, 2, 3, 4$. The factor in (61) is chosen to satisfy (62).

Now, consider a function $\Psi_+$ in $(\mathcal{P}_oC', \mathcal{P}_r') = (0, 1)$ and $\Psi_-$ in $(\mathcal{P}_oC', \mathcal{P}_r') = (0, -1)$,

$$\Psi_+ = \sum_n z c_n a_n, \quad \Psi_- = \sum_n z s_n a_n$$  \hspace{1cm} (64)

with the same coefficients $a_n$. Then, we have

$$\mathcal{C} \left( \begin{array}{c} \Psi_+ \\ \Psi_- \end{array} \right) = \left( \begin{array}{cc} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{array} \right) \left( \begin{array}{c} \Psi_+ \\ \Psi_- \end{array} \right).$$  \hspace{1cm} (65)

So, if $\Psi_+$ belongs to $\mathcal{H}' = \lambda$ then $\Psi_-$ belongs to the same $\lambda$, because $\mathcal{C}(\mathcal{H} - \lambda)\Psi_+ = 0$ implies

$$0 = (\mathcal{H} - \lambda)\mathcal{C}\Psi_+ = (\mathcal{H} - \lambda) \left( -\frac{1}{2}\Psi_+ + \frac{\sqrt{3}}{2}\Psi_- \right) = \frac{\sqrt{3}}{2}(\mathcal{H} - \lambda)\Psi_-.$$  \hspace{1cm} (66)

Inverse is also true. This proves explicitly the double degeneracy of zero-choreographic subspace.

### 4.4 Decomposition by $\mathcal{P}_M$

For integer $m$, $\cos(m\nu(t + T/2)) = \cos(m\nu t + m\pi) = (-1)^m \cos(m\nu t)$ and $\sin(m\nu(t + T/2)) = (-1)^m \sin(m\nu t)$. So, for $x$ and $y$ components,

$$\mathcal{M}' = 1 \leftrightarrow \left( \begin{array}{cc} \cos((2m + 1)\nu t) & 0 \\ 0 & \cos(2m'\nu t) \end{array} \right): \left( \begin{array}{cc} \sin((2m + 1)\nu t) & 0 \\ 0 & \sin(2m'\nu t) \end{array} \right)$$  \hspace{1cm} (67)

and

$$\mathcal{M}' = -1 \leftrightarrow \left( \begin{array}{cc} \cos(2m\nu t) & 0 \\ 0 & \cos((2m' + 1)\nu t) \end{array} \right): \left( \begin{array}{cc} \sin(2m\nu t) & 0 \\ 0 & \sin((2m' + 1)\nu t) \end{array} \right).$$  \hspace{1cm} (68)

Decompose $k_n$

$$k_n = \{1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, \ldots, 3N - 2, 3N - 1\}$$  \hspace{1cm} (69)

into odd set and even set,

$$k_n^o = \{1, 5, 7, 11, 13, 17, \ldots, 3N - 1\} = \frac{1}{2}(6n - (-1)^n - 3), \quad n = 1, 2, 3, \ldots, N$$  \hspace{1cm} (70)

and

$$k_n^e = \{2, 4, 8, 10, 14, 16, \ldots, 3N - 2\} = \frac{1}{2}(6n - (-1)^n - 3), \quad n = 1, 2, 3, \ldots, N.$$  \hspace{1cm} (71)
Since \( k_n \) \( \mod 3 = \{1, 2, 1, 2, \ldots \} \) and \( k_n' \) \( \mod 3 = \{2, 1, 2, 1 \ldots \} \),

\[
\sin(k_n'2\pi/3) = (-1)^{n+1} \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin(k_n2\pi/3) = (-1)^n \frac{\sqrt{3}}{2}.
\] \hspace{1cm} (72)

Similarly,

\[
\ell_n = \{0, 1, 2, 3, 4, 5, 6, 7, \ldots, 3N - 2, 3N - 1, 3N, 3N\} \quad \text{into odd set and even set,}
\]

\[
\ell_n' = \{1, 3, 3, 5, 7, 9, 9, 11, 13, 15, 15, \ldots, 3N - 1\}
\]

\[
= \frac{1}{4} \left(6n + (-1)^{n-1}(-i)^{n} - (1 + i)^{n} - 3 \right), \quad n = 1, 2, 3, \ldots, 2N
\] \hspace{1cm} (74)

and

\[
\ell_n'' = \{0, 2, 4, 6, 8, 10, 12, \ldots, 3N - 2, 3N, 3N\}
\]

\[
= \frac{1}{4} \left(6n + (-1)^{n} + (1 - i)^{n} + (1 + i)^{n} - 3 \right), \quad n = 1, 2, 3, \ldots, 2N + 1.
\]

To make the number of elements of \( \ell_n' \) and \( \ell_n'' \) equal, we just omit the last element of \( \ell_n'' \). So, we use

\[
\ell_n'' = \{0, 2, 4, 6, 8, 10, 12, \ldots, 3N - 2, 3N\}
\]

\[
= \frac{1}{4} \left(6n + (-1)^{n} + (1 - i)^{n} + (1 + i)^{n} - 3 \right), \quad n = 1, 2, 3, \ldots, 2N.
\] \hspace{1cm} (75)

The series with \( \ell_n'' \nu t \) in (60) is

\[
\begin{pmatrix}
\cos(\nu t) \\
\cos(\nu t - 2\pi/3) \\
\cos(\nu t + 2\pi/3)
\end{pmatrix}
\begin{pmatrix}
\cos(3\nu t) \\
\cos(3\nu t - 2\pi/3) \\
\cos(3\nu t + 2\pi/3)
\end{pmatrix}
\begin{pmatrix}
\cos(3\nu t) \\
\cos(3\nu t + 2\pi/3) \\
\cos(2\nu t - 2\pi/3)
\end{pmatrix}
\begin{pmatrix}
\cos(3\nu t) \\
\cos(3\nu t + 2\pi/3) \\
\cos(2\nu t - 2\pi/3)
\end{pmatrix}
\begin{pmatrix}
\cos(5\nu t) \\
\cos(5\nu t + 2\pi/3) \\
\cos(5\nu t - 2\pi/3)
\end{pmatrix}
\begin{pmatrix}
\cos(9\nu t) \\
\cos(9\nu t - 2\pi/3) \\
\cos(9\nu t + 2\pi/3)
\end{pmatrix}
\begin{pmatrix}
\cos(9\nu t) \\
\cos(9\nu t + 2\pi/3) \\
\cos(9\nu t - 2\pi/3)
\end{pmatrix}
\begin{pmatrix}
\cos(11\nu t) \\
\cos(11\nu t + 2\pi/3) \\
\cos(11\nu t - 2\pi/3)
\end{pmatrix}
\] \hspace{1cm} (76)

\[
\ldots
\]

\[
= \begin{pmatrix}
\cos(\ell_n'' \nu t) \\
\cos(\ell_n'' \nu t + (-1)^{(n+1)/2}2\pi/3) \\
\cos(\ell_n'' \nu t - (-1)^{(n+1)/2}2\pi/3)
\end{pmatrix}
\] \hspace{1cm} \text{for } n = 1, 2, 3, \ldots, 2N.

\]
Similarly, the series with $\ell_n^\nu t$ in (60) is
\[
\begin{pmatrix}
1 \\
-1/2 \\
-1/2
\end{pmatrix}, \begin{pmatrix}
\cos(2\nu t) \\
\cos(2\nu t + 2\pi/3) \\
\cos(2\nu t - 2\pi/3)
\end{pmatrix}, \begin{pmatrix}
\cos(4\nu t) \\
\cos(4\nu t - 2\pi/3) \\
\cos(4\nu t + 2\pi/3)
\end{pmatrix}, \begin{pmatrix}
\cos(6\nu t) \\
\cos(6\nu t - 2\pi/3) \\
\cos(6\nu t + 2\pi/3)
\end{pmatrix}, \\
\begin{pmatrix}
\cos(6\nu t) \\
\cos(8\nu t) \\
\cos(6\nu t - 2\pi/3)
\end{pmatrix}, \begin{pmatrix}
\cos(8\nu t) \\
\cos(8\nu t + 2\pi/3) \\
\cos(8\nu t - 2\pi/3)
\end{pmatrix}, \begin{pmatrix}
\cos(10\nu t) \\
\cos(10\nu t - 2\pi/3) \\
\cos(10\nu t + 2\pi/3)
\end{pmatrix}, \begin{pmatrix}
\cos(12\nu t) \\
\cos(12\nu t - 2\pi/3) \\
\cos(12\nu t + 2\pi/3)
\end{pmatrix}, \ldots
\]
\[
= \begin{pmatrix}
\cos(\ell_n^\nu t) \\
\cos(\ell_n^\nu t - (-1)^{(n+1)/2}2\pi/3) \\
\cos(\ell_n^\nu t + (-1)^{(n+1)/2}2\pi/3)
\end{pmatrix}
\text{ for } n = 1, 2, 3, \ldots, 2N.
\tag{77}
\]

4.5 Summary for the base functions

4.5.1 Choreographic subspace

Using $k_n^\sigma$ and $k_n^\varepsilon$,
\[
(P'_\sigma, P'_C, T', M') = (0, 1, 1, 1) \leftrightarrow cc_n^+ = \begin{pmatrix}
\cos(k_n^\sigma \nu t) & 0 \\
0 & \cos(k_n^\varepsilon \nu t) \\
\cos(k_n^\sigma \nu (t + T/3)) & 0 \\
\cos(k_n^\varepsilon \nu (t + T/3)) & 0 \\
\cos(k_n^\sigma \nu (t - T/3)) & 0 \\
\cos(k_n^\varepsilon \nu (t - T/3)) & 0
\end{pmatrix}, \\
(78)
\]

and
\[
(P'_\sigma, P'_C, T', M') = (0, 1, 1, -1) \leftrightarrow cc_n^- = \begin{pmatrix}
\cos(k_n^\sigma \nu t) & 0 \\
0 & \cos(k_n^\varepsilon \nu t) \\
\cos(k_n^\sigma \nu (t + T/3)) & 0 \\
\cos(k_n^\varepsilon \nu (t + T/3)) & 0 \\
\cos(k_n^\sigma \nu (t - T/3)) & 0 \\
\cos(k_n^\varepsilon \nu (t - T/3)) & 0
\end{pmatrix}, \\
(79)
\]

for $n = 1, 2, 3, \ldots, N$.

Similarly,
\[
(P'_\sigma, P'_C, T', M') = (0, 1, -1, 1) \leftrightarrow cs_n^+ = \begin{pmatrix}
\sin(k_n^\sigma \nu t) & 0 \\
\sin(k_n^\sigma \nu (t + T/3)) & 0 \\
\sin(k_n^\varepsilon \nu (t + T/3)) & 0 \\
\sin(k_n^\sigma \nu (t - T/3)) & 0 \\
\sin(k_n^\varepsilon \nu (t - T/3)) & 0
\end{pmatrix}, \\
(80)
\]

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and

$$(\mathcal{P}_\sigma', \mathcal{P}_c', \mathcal{T}', \mathcal{M}') = (0, 1, -1, -1) \leftrightarrow c_{s_n}^- = \begin{pmatrix}
\sin(k_n^c \nu t) & 0 \\
0 & \sin(k_n^c \nu t) \\
\sin(k_n^c \nu (t + T/3)) & 0 \\
0 & \sin(k_n^c \nu (t + T/3)) \\
\sin(k_n^c \nu (t - T/3)) & 0 \\
0 & \sin(k_n^c \nu (t - T/3))
\end{pmatrix}$$

(81)

for $n = 1, 2, 3, \ldots, N$.

It may useful to define a function

$$\Phi_c(f, k^x, k^y, n) = \begin{pmatrix}
f(k_n^x \nu t) & 0 \\
0 & f(k_n^x \nu t) \\
f(k_n^x \nu (t + T/3)) & 0 \\
0 & f(k_n^x \nu (t + T/3)) \\
f(k_n^x \nu (t - T/3)) & 0 \\
0 & f(k_n^x \nu (t - T/3))
\end{pmatrix}.$$  

(82)

Then,

$$cc_n^+ = \Phi_c(\cos, k^o, k^x, n), \quad cc_n^- = \Phi_c(\cos, k^o, k^x, n),$$

$$cs_n^+ = \Phi_c(\sin, k^o, k^x, n), \quad cs_n^- = \Phi_c(\sin, k^o, k^x, n)$$

(83)

### 4.5.2 Zero-choreographic subspace

$$(\mathcal{P}_\sigma', \mathcal{P}_c', \mathcal{T}', \mathcal{M}') = (0, 0, 1, 1) \leftrightarrow$$

$${\check{z}}c_n^+ = \begin{pmatrix}
\cos(\ell_n^x \nu t) & 0 \\
0 & \cos(\ell_n^x \nu t) \\
\cos(\ell_n^x \nu t + (-1)^{[(n+1)/2]} 2\pi/3) & 0 \\
0 & \cos(\ell_n^x \nu t - (-1)^{[(n+1)/2]} 2\pi/3) \\
\cos(\ell_n^x \nu t - (-1)^{[(n+1)/2]} 2\pi/3) & 0 \\
0 & \cos(\ell_n^x \nu t + (-1)^{[(n+1)/2]} 2\pi/3)
\end{pmatrix},$$

(84)

and

$$(\mathcal{P}_\sigma', \mathcal{P}_c', \mathcal{T}', \mathcal{M}') = (0, 0, 1, -1) \leftrightarrow$$

$${\check{z}}c_n^- = \begin{pmatrix}
\cos(\ell_n^x \nu t) & 0 \\
0 & \cos(\ell_n^x \nu t) \\
\cos(\ell_n^x \nu t - (-1)^{[(n+1)/2]} 2\pi/3) & 0 \\
0 & \cos(\ell_n^x \nu t + (-1)^{[(n+1)/2]} 2\pi/3) \\
\cos(\ell_n^x \nu t + (-1)^{[(n+1)/2]} 2\pi/3) & 0 \\
0 & \cos(\ell_n^x \nu t - (-1)^{[(n+1)/2]} 2\pi/3)
\end{pmatrix}.$$  

(85)
Then, the base functions are $z_s^\Psi_n$ for $zs^-(\nu t)\leftrightarrow 0$

$$zs^+_n = (-1)^{n+1} \begin{pmatrix} \sin(\ell_n^c \nu t) & 0 \\ 0 & \sin(\ell_n^e \nu t) \\ \sin(\ell_n^e \nu t - (-1)^{(n+1)/2}2\pi/3) & 0 \\ 0 & \sin(\ell_n^e \nu t + (-1)^{(n+1)/2}2\pi/3) \end{pmatrix},$$

and

$$zs^-_n = (-1)^{n+1} \begin{pmatrix} \sin(\ell_n^c \nu t) & 0 \\ 0 & \sin(\ell_n^e \nu t) \\ \sin(\ell_n^e \nu t - (-1)^{(n+1)/2}2\pi/3) & 0 \\ 0 & \sin(\ell_n^e \nu t + (-1)^{(n+1)/2}2\pi/3) \end{pmatrix}. \quad (86)$$

for $n = 1, 2, 3, \ldots, 2N$. Similarly,

$$(P'_\sigma, P'_C, T', M') = (0, 0, -1, 1) \leftrightarrow

zs^+_n = (-1)^{n+1} \begin{pmatrix} \sin(\ell_n^c \nu t) & 0 \\ 0 & \sin(\ell_n^e \nu t) \\ \sin(\ell_n^e \nu t + (-1)^{(n+1)/2}2\pi/3) & 0 \\ 0 & \sin(\ell_n^e \nu t - (-1)^{(n+1)/2}2\pi/3) \end{pmatrix},$$

and

$$(P'_\sigma, P'_C, T', M') = (0, 0, -1, -1) \leftrightarrow

zs^-_n = (-1)^{n+1} \begin{pmatrix} \sin(\ell_n^c \nu t) & 0 \\ 0 & \sin(\ell_n^e \nu t) \\ \sin(\ell_n^e \nu t - (-1)^{(n+1)/2}2\pi/3) & 0 \\ 0 & \sin(\ell_n^e \nu t + (-1)^{(n+1)/2}2\pi/3) \end{pmatrix}. \quad (87)$$

It may useful to define a function

$$\Psi_z(f, \ell^x, \ell^y, \epsilon, n) = \begin{pmatrix} f(\ell_n^x \nu t) & 0 \\ 0 & f(\ell_n^y \nu t) \\ f(\ell_n^x \nu t + \epsilon(-1)^{(n+1)/2}2\pi/3) & 0 \\ 0 & f(\ell_n^y \nu t - \epsilon(-1)^{(n+1)/2}2\pi/3) \end{pmatrix}. \quad (88)$$

Then, the base functions are

$$zs^+_n = \Psi(\cos, \ell^c, \ell^c, 1, n), \quad zs^-_n = \Psi(\cos, \ell^c, \ell^c, -1, n),$$

$$zs^+_n = (-1)^{n+1}\Psi(\sin, \ell^c, \ell^c, 1, n), \quad zs^-_n = (-1)^{n+1}\Psi(\sin, \ell^c, \ell^c, -1, n). \quad (89)$$
5 Matrix elements of Hessian

5.1 Definition of matrix elements

Let \( \phi_n \) be a 6 \( \times \) 2 matrix that describes bases and \( a_n \) be a 2 column vector that describes degrees of freedom. For,

\[
\phi_n = \begin{pmatrix}
\delta q^x_{0n} & 0 \\
0 & \delta q^y_{0n} \\
\delta q^x_{1n} & 0 \\
0 & \delta q^y_{1n} \\
\delta q^x_{2n} & 0 \\
0 & \delta q^y_{2n}
\end{pmatrix}
\quad \text{and} \quad
a_n = \begin{pmatrix}
a^x_n \\
a^y_n
\end{pmatrix},
\]

(90)

the product \( \phi_n a_n \) represents the vector

\[
\phi_n a_n = \begin{pmatrix}
\delta q^x_{0n} a^x_n \\
\delta q^y_{0n} a^y_n \\
\delta q^x_{1n} a^x_n \\
\delta q^y_{1n} a^y_n \\
\delta q^x_{2n} a^x_n \\
\delta q^y_{2n} a^y_n
\end{pmatrix}.
\]

(91)

Note that we have only two degrees of freedom \( a^x_n \) and \( a^y_n \) for given \( n \).

Now, consider the eigenvalue problem

\[
\mathcal{H} \sum_{n=1,2,3,\ldots} \phi_n a_n = \lambda \sum_{n=1,2,3,\ldots} \phi_n a_n.
\]

(92)

Substituting the expression

\[
\sum_{n=1,2,3,\ldots} \phi_n a_n = (\phi_1, \phi_2, \phi_3, \ldots) \begin{pmatrix}
a_1 \\
a_2 \\
a_3 \\
\vdots
\end{pmatrix}
\]

(93)

into the eigenvalue problem, and multiply \((\phi_1, \phi_2, \phi_3, \ldots)\) from the left and taking the inner product, we get the eigenvalue problem in a matrix representation

\[
M_{\mathcal{H}} \begin{pmatrix}
a_1 \\
a_2 \\
a_3 \\
\vdots
\end{pmatrix} = \lambda \begin{pmatrix}
a_1 \\
a_2 \\
a_3 \\
\vdots
\end{pmatrix}.
\]

(94)

Where \( M_{\mathcal{H}} \) is the matrix representation of the Hessian,

\[
M_{\mathcal{H}} = \begin{pmatrix}
\langle \phi_1 | \mathcal{H} | \phi_1 \rangle & \langle \phi_1 | \mathcal{H} | \phi_2 \rangle & \langle \phi_1 | \mathcal{H} | \phi_3 \rangle & \cdots \\
\langle \phi_2 | \mathcal{H} | \phi_1 \rangle & \langle \phi_2 | \mathcal{H} | \phi_2 \rangle & \langle \phi_2 | \mathcal{H} | \phi_3 \rangle & \cdots \\
\langle \phi_3 | \mathcal{H} | \phi_1 \rangle & \langle \phi_3 | \mathcal{H} | \phi_2 \rangle & \langle \phi_3 | \mathcal{H} | \phi_3 \rangle & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

(95)

The inner product is defined by (15).
5.2 Preparations

Since we are considering the Hessian at a choreography \( q_{0}(t) = q(t) \), \( q_{1}(t) = q(t + T/3) \) and \( q_{2}(t) = q(t - T/3) \), the function \( u_{ij} \) satisfy \( u_{20}(t) = u_{12}(t + T/3) \) and \( u_{01}(t) = u_{12}(t + T/3) \). Let \( u_{12}(t) = u(t) \), then \( u_{20}(t) = R^{1/3} u(t) R^{-1/3} \) and \( u_{01}(t) = R^{-1/3} u(t) R^{1/3} \). Then, \( i\Delta U \Delta \) in (10) is

\[
\begin{align*}
\begin{pmatrix}
1 & 0 & 0 \\
0 & R^{-1/3} & 0 \\
0 & 0 & R^{1/3}
\end{pmatrix}
\begin{pmatrix}
\cos(k\nu t) \\
\cos(k\nu(t + T/3)) \\
\cos(k\nu(t - T/3))
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & u & 0 \\
0 & 0 & u
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & R^{-1/3} & 0 \\
0 & 0 & R^{1/3}
\end{pmatrix}
\Delta.
\end{align*}
\]

In the following subsections, we use the notation

\[
u = \begin{pmatrix} u^{xx} & u^{xy} \\ u^{yx} & u^{yy} \end{pmatrix}.
\]

We will use the lower case letters \( a, b, \ldots \) to express \( x \) or \( y \).

5.3 Elements for choreographic subspace

For the matrix elements for choreographic subspace, consider

\[
\begin{align*}
\begin{pmatrix}
1 & 0 & 0 \\
0 & R^{-1/3} & 0 \\
0 & 0 & R^{1/3}
\end{pmatrix}
\begin{pmatrix}
\cos(k\nu t) \\
\cos(k\nu(t + T/3)) \\
\cos(k\nu(t - T/3))
\end{pmatrix}
&= \begin{pmatrix}
1 & 0 & 0 \\
0 & R^{-1/3} & 0 \\
0 & 0 & R^{1/3}
\end{pmatrix}
\begin{pmatrix}
\cos(k\nu(t + T/3)) - \cos(k\nu(t - T/3)) \\
\cos(k\nu(t - T/3)) - \cos(k\nu t) \\
\cos(k\nu t) - \cos(k\nu(t + T/3))
\end{pmatrix} \\
&= (\cos(k\nu(t + T/3)) - \cos(k\nu(t - T/3))) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
&= -2 \sin(2\pi k/3) \sin(k\nu t)^3(1, 1, 1).
\end{align*}
\]

Similarly,

\[
\begin{align*}
\begin{pmatrix}
1 & 0 & 0 \\
0 & R^{-1/3} & 0 \\
0 & 0 & R^{1/3}
\end{pmatrix}
\begin{pmatrix}
\sin(k\nu t) \\
\sin(k\nu(t + T/3)) \\
\sin(k\nu(t - T/3))
\end{pmatrix}
&= (\sin(k\nu(t + T/3)) - \sin(k\nu(t - T/3)))^3(1, 1, 1) \\
&= 2 \sin(2\pi k/3) \cos(k\nu t)^3(1, 1, 1).
\end{align*}
\]

Therefore, \( ab \) component of \( \langle \Phi_{c}(\cos, k^{x}, k^{y}, m)|i\Delta U \Delta|\Phi(k^{x}, k^{y}, n) \rangle \) is

\[
\langle \Phi_{c}(\cos, k^{x}, k^{y}, m)|i\Delta U \Delta|\Phi(k^{x}, k^{y}, n) \rangle^{ab}
= \sin(2\pi k^{a}/3) \sin(2\pi k^{b}/3) \times \frac{8}{T} \int_{0}^{T} dt \ u^{ab}(t) \sin(k^{a}_{m} \nu t) \sin(k^{b}_{n} \nu t)
\]

(100)
The relation (72) yields
\[
\sin(2\pi k^a_m/3) \sin(2\pi k^c_m/3) = \sin(2\pi k^c_m/3) \sin(2\pi k^a_m/3) = (-1)^{m+n} \frac{3}{4},
\]
(101)
\[
\sin(2\pi k^a_m/3) \sin(2\pi k^c_m/3) = -(-1)^{m+n} \frac{3}{4}.
\]
So, we finally get the matrix elements
\[
\langle cc^+_m | \Delta U | cc^+_m \rangle
= (-1)^{m+n} \frac{6}{T} \int_0^T dt \begin{pmatrix}
  u^{xx} \sin(k^a_m \nu t) \sin(k^c_m \nu t) & -u^{xy} \sin(k^a_m \nu t) \sin(k^c_m \nu t) \\
  -u^{xy} \sin(k^c_m \nu t) \sin(k^a_m \nu t) & u^{yy} \sin(k^a_m \nu t) \sin(k^c_m \nu t)
\end{pmatrix},
\]
(102)
\[
\langle cc^+_m | \Delta U | cc^-_m \rangle
= (-1)^{m+n} \frac{6}{T} \int_0^T dt \begin{pmatrix}
  u^{xx} \sin(k^c_m \nu t) \sin(k^a_m \nu t) & -u^{xy} \sin(k^c_m \nu t) \sin(k^a_m \nu t) \\
  -u^{xy} \sin(k^a_m \nu t) \sin(k^c_m \nu t) & u^{yy} \sin(k^c_m \nu t) \sin(k^a_m \nu t)
\end{pmatrix},
\]
(103)
\[
\langle cs^+_m | \Delta U | cs^+_m \rangle
= (-1)^{m+n} \frac{6}{T} \int_0^T dt \begin{pmatrix}
  u^{xx} \cos(k^a_m \nu t) \cos(k^c_m \nu t) & -u^{xy} \cos(k^a_m \nu t) \cos(k^c_m \nu t) \\
  -u^{xy} \cos(k^c_m \nu t) \cos(k^a_m \nu t) & u^{yy} \cos(k^c_m \nu t) \cos(k^a_m \nu t)
\end{pmatrix},
\]
(104)
and
\[
\langle cs^+_m | \Delta U | cs^-_m \rangle
= (-1)^{m+n} \frac{6}{T} \int_0^T dt \begin{pmatrix}
  u^{xx} \cos(k^c_m \nu t) \cos(k^a_m \nu t) & -u^{xy} \cos(k^c_m \nu t) \cos(k^a_m \nu t) \\
  -u^{xy} \cos(k^a_m \nu t) \cos(k^c_m \nu t) & u^{yy} \cos(k^a_m \nu t) \cos(k^c_m \nu t)
\end{pmatrix}.
\]
(105)
The differences between the above elements are only the place of \( k^a \) and \( k^c \).

5.4 Elements for zero-choreographic subspace

To calculate the \( ab \) element of \( \langle zc^+_m | \Delta U | zc^+ \rangle \), let \( \epsilon^x = (-1)^{(n+1)/2} \) and \( \epsilon^y = -(-1)^{(n+1)/2} \). Then
\[
\begin{pmatrix}
  1 & 0 & 0 \\
  0 & R^{-1/3} & 0 \\
  0 & 0 & R^{1/3}
\end{pmatrix} \Delta \begin{pmatrix}
  \cos(\ell^a_n \nu t) \\
  \cos(\ell^a_n \nu t + 2\pi \epsilon^a_n / 3) \\
  \cos(\ell^a_n \nu t - 2\pi \epsilon^a_n / 3)
\end{pmatrix}
\]
(106)
\[
= \sqrt{3} \epsilon^a_n \begin{pmatrix}
  \sin(\ell^a_n \nu t) \\
  \sin(\ell^a_n \nu (t - T/3) - \pi \epsilon^a_n / 3) \\
  \sin(\ell^a_n \nu (t + T/3) + \pi \epsilon^a_n / 3)
\end{pmatrix}.
\]
Therefore,
\[
\langle z^+_{m} | \Delta \mathcal{U} | z^+_{n} \rangle^{ab} = e^{a}_{m} e^{b}_{n} \frac{1}{T} \int_{0}^{T} dt \ u^{ab}(t) \left( \cos \left( (\ell^{a}_{m} - \ell^{b}_{n}) \nu t \right) - \cos \left( (\ell^{a}_{m} + \ell^{b}_{n}) \nu t \right) \right) \\
+ \cos \left( (\ell^{a}_{m} - \ell^{b}_{n}) \nu (t - T/3) - (\ell^{a}_{m} - \ell^{b}_{n}) \pi/3 \right) - \cos \left( (\ell^{a}_{m} + \ell^{b}_{n}) \nu (t - T/3) - (\ell^{a}_{m} + \ell^{b}_{n}) \pi/3 \right) \\
+ \cos \left( (\ell^{a}_{m} - \ell^{b}_{n}) \nu (t + T/3) + (\ell^{a}_{m} - \ell^{b}_{n}) \pi/3 \right) - \cos \left( (\ell^{a}_{m} + \ell^{b}_{n}) \nu (t + T/3) + (\ell^{a}_{m} + \ell^{b}_{n}) \pi/3 \right) \right) \\
= e^{a}_{m} e^{b}_{n} \frac{3}{T} \int_{0}^{T} dt \ u(t) L(\ell^{a}, \ell^{b}, \epsilon_{m}, \epsilon_{b}, m, n). \tag{107}
\]

The term \( L(\ell^{a}, \ell^{b}, \epsilon_{m}, \epsilon_{b}, m, n) \) turns out to be
\[
L(\ell^{a}, \ell^{b}, \epsilon_{m}, \epsilon_{b}, m, n) = (-1)^{m+n} \cos \left( ((-1)^{m} \ell^{a}_{m} - (-1)^{n} \ell^{b}_{n}) \nu t \right). \tag{108}
\]

Using \((-1)^{[(n+1)/2]} = 1, -1, 1, 1, -1, 1, 1, \ldots = (-1)^{[n/2]}, \)

Finally, we get
\[
\langle n_{m}^{+} | \Delta \mathcal{U} | n_{n}^{+} \rangle = (-1)^{[m/2]+[n/2]} \frac{3}{T} \\
\times \int_{0}^{T} dt \left( u^{xx}(t) \cos \left( ((-1)^{m} \ell^{a}_{m} - (-1)^{n} \ell^{b}_{n}) \nu t \right) - u^{xy}(t) \cos \left( ((-1)^{m} \ell^{a}_{m} - (-1)^{n} \ell^{b}_{n}) \nu t \right) \right) \\
- u^{yx}(t) \cos \left( ((-1)^{m} \ell^{a}_{m} - (-1)^{n} \ell^{b}_{n}) \nu t \right) + u^{yy}(t) \cos \left( ((-1)^{m} \ell^{a}_{m} - (-1)^{n} \ell^{b}_{n}) \nu t \right) \right). \tag{109}
\]

Exchanging \( \ell^{a} \leftrightarrow \ell^{b}, \) we get \( \langle n_{m}^{-} | \Delta \mathcal{U} | n_{n}^{-} \rangle, \)
\[
\langle n_{m}^{-} | \Delta \mathcal{U} | n_{n}^{-} \rangle = (-1)^{[m/2]+[n/2]} \frac{3}{T} \\
\times \int_{0}^{T} dt \left( u^{xx}(t) \cos \left( ((-1)^{m} \ell^{a}_{m} - (-1)^{n} \ell^{b}_{n}) \nu t \right) - u^{xy}(t) \cos \left( ((-1)^{m} \ell^{a}_{m} - (-1)^{n} \ell^{b}_{n}) \nu t \right) \right) \\
- u^{yx}(t) \cos \left( ((-1)^{m} \ell^{a}_{m} - (-1)^{n} \ell^{b}_{n}) \nu t \right) + u^{yy}(t) \cos \left( ((-1)^{m} \ell^{a}_{m} - (-1)^{n} \ell^{b}_{n}) \nu t \right) \right). \tag{110}
\]

Similarly,
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & R^{-1/3} & 0 \\
0 & 0 & R^{1/3}
\end{pmatrix}
\Delta 
\begin{pmatrix}
\sin(\ell^{a}_{m} \nu t) \\
\sin(\ell^{a}_{m} \nu t + 2\pi \epsilon_{m}^{a}/3) \\
\sin(\ell^{a}_{m} \nu t - 2\pi \epsilon_{m}^{a}/3)
\end{pmatrix}
\]
\[
= \sqrt{3} \epsilon_{m}^{a} 
\begin{pmatrix}
\cos(\ell^{a}_{m} \nu t) \\
- \cos(\ell^{a}_{m} \nu (t - T/3) - \pi \epsilon_{n}^{a}/3) \\
- \cos(\ell^{a}_{m} \nu (t + T/3) + \pi \epsilon_{n}^{a}/3)
\end{pmatrix}. \tag{111}
\]
Therefore,
\[
\langle z_{s_m^+}^{+}|\Delta U\Delta|z_{s_n^+}^{+}\rangle^{ab} = (-1)^{m+n} c_m^a c_n^b \frac{1}{T} \int_0^T dt \ u^{ab}(t) \left( \cos \left( (\ell_m^a - \ell_n^b) \nu t \right) + \cos \left( (\ell_m^a + \ell_n^b) \nu t \right) \right) + \cos \left( (\ell_m^a - \ell_n^b) \nu(t - T/3) - (\ell_m^a + \ell_n^b) \nu t \right) + \cos \left( (\ell_m^a + \ell_n^b) \nu(t + T/3) + (\ell_m^a + \ell_n^b) \nu t \right) + \cos \left( (\ell_m^a + \ell_n^b) \nu(t - T/3) - (\ell_m^a - \ell_n^b) \nu t \right)
\]
\[
= (-1)^{m+n} c_m^a c_n^b \frac{3}{T} \int_0^T dt \ u(t) L'(\ell^a, \ell^b, \epsilon^a, \epsilon^b, m, n).
\] (112)

The term \( L'(\ell^a, \ell^b, \epsilon^a, \epsilon^b, m, n) \) turns out to be
\[
L'(\ell^a, \ell^b, \epsilon^a, \epsilon^b, m, n) = \cos \left( ((-1)^m \ell_m^a - (-1)^n \ell_n^b) \nu t \right).
\] (113)

Thus, we directly get
\[
\langle z_{s_m^+}^{+}|\Delta U\Delta|z_{s_n^+}^{+}\rangle = \langle z_{c_m^+}^{+}|\Delta U\Delta|z_{c_n^+}^{+}\rangle,
\]
\[
\langle z_{s_m}^{-}|\Delta U\Delta|z_{s_n}^{-}\rangle = \langle z_{c_m}^{-}|\Delta U\Delta|z_{c_n}^{-}\rangle.
\] (114)

This is a direct proof of the double degeneracy of \( \mathcal{H} \) in \( \mathcal{P}_c' = 0 \) subspace.

5.5 Summary for the matrix elements

Let \( \hat{a}^{ab}(k) \) be
\[
\begin{pmatrix}
\hat{u}^{xx}(k) \\
\hat{u}^{yy}(k)
\end{pmatrix} = \frac{1}{T} \int_0^T dt \begin{pmatrix}
u^{xx}(t) & \nu^{xy}(t) \\
\nu^{yx}(t) & \nu^{yy}(t)
\end{pmatrix} \cos(kv t),
\] (115)

\( k = 0, 1, 2, \ldots, 6N - 1, N = 2^M. \)

5.5.1 Choreographic subspace

Let \( C(k^x, k^y, \epsilon, m, n) \) be
\[
C(k^x, k^y, \epsilon, m, n) = 3(-1)^{m+n} \begin{pmatrix}
\hat{u}^{xx}(k^x + \epsilon k^x) & -\hat{u}^{xy}(k^x + \epsilon k^y) \\
-\hat{u}^{yx}(k^y + \epsilon k^x) & \hat{u}^{yy}(k^y + \epsilon k^y)
\end{pmatrix}.
\] (116)

Subspace: \( (\mathcal{P}_c', \mathcal{P}_T', \mathcal{P}_A') = (0, 1, 1), \)
name: Choreographic cos+,
symbol: \( cc^\updownarrow \),
range: \( m, n = 1, 2, 3, \ldots, N, \)
matrix element:
\[
\langle cc_m^+|\Delta U\Delta|nc_n^+\rangle = C(k^a, k^c, -1, m, n) - C(k^a, k^c, 1, m, n).
\] (117)
Subspace: \((P'_c, P'_T, P'_M) = (0, 1, -1)\),
name: Choreographic \(\cos^−\),
symbol: \(cc^−\),
rang: \(m, n = 1, 2, 3, \ldots, N\),
matrix element:
\[
\langle cc_m | \Delta U | nc_n^− \rangle = C(k^e, k^o, -1, m, n) - C(k^e, k^o, 1, m, n).
\] (118)

Subspace: \((P'_c, P'_T, P'_M) = (0, -1, 1)\),
name: Choreographic \(\sin^+\),
symbol: \(cs^+\),
rang: \(m, n = 1, 2, 3, \ldots, N\),
matrix element:
\[
\langle cs_m^+ | \Delta U | ns_n^+ \rangle = C(k^o, k^e, -1, m, n) + C(k^o, k^e, 1, m, n).
\] (119)

Subspace: \((P'_c, P'_T, P'_M) = (0, -1, -1)\),
name: Choreographic \(\sin^−\),
symbol: \(cs^−\),
rang: \(m, n = 1, 2, 3, \ldots, N\),
matrix element:
\[
\langle cs_m^− | \Delta U | ns_n^− \rangle = C(k^e, k^o, -1, m, n) + C(k^e, k^o, 1, m, n).
\] (120)

5.5.2 Zero-choreographic subspace

Let \(Z(\ell^x, \ell^y, m, n)\) be
\[
Z(\ell^x, \ell^y, m, n) = 3(-1)^{[m/2]+[n/2]}
\times \left(\tilde{u}^{xy}((-1)^m \ell_x^m - (-1)^n \ell_y^m) - \tilde{u}^{yx}((-1)^m \ell_y^m - (-1)^n \ell_x^m) \right).
\] (121)

Subspace: \((P'_c, P'_T, P'_M) = (0, \pm 1, 1)\),
name: Zero-choreographic \(\cos^+\) and \(\sin^+,\)
symbol: \(zc^+\) and \(zs^+,\)
rang: \(m, n = 1, 2, 3, \ldots, 2N\),
matrix element:
\[
\langle zc_m^+ | \Delta U | zc_n^+ \rangle = \langle zs_m^+ | \Delta U | zs_n^+ \rangle = Z(\ell^o, \ell^o, m, n).
\] (122)

Subspace: \((P'_c, P'_T, P'_M) = (0, \pm 1, -1)\),
name: Zero-choreographic \(\cos^−\) and \(\sin^−\),
symbol: \(zc^−\) and \(zs^−\),
rang: \(m, n = 1, 2, 3, \ldots, 2N\),
matrix element:
\[
\langle zc_m^− | \Delta U | zc_n^− \rangle = \langle zs_m^− | \Delta U | zs_n^− \rangle = Z(\ell^o, \ell^o, m, n).
\] (123)
6 Discussions

6.1 Scaling and coupling dependence of eigenvalues

Consider the eigenvalue problem $\mathcal{H}\Psi = \lambda\Psi$, namely,

$$\left( -\frac{d^2}{dt^2} + t\Delta U \Delta \right)\Psi = \lambda\Psi$$

(124)

for extended Newton potential in (2).

The eigenvalues scale $\lambda \rightarrow \lambda/\mu^2$ for the scaling of $t \rightarrow \mu t$ and $q \rightarrow \mu^{2/(2+\alpha)}q$.

Therefore, it is useful to consider the scale invariant eigenvalue $\tilde{\lambda}$ defined by

$$\tilde{\lambda} = \frac{\lambda}{4\pi^2/T^2}.$$  (125)

Here, the factor $4\pi^2$ is for later convenience.

Consider a system under a homogeneous potential with coupling constant $g^2 > 0$, described by

$$L = \frac{1}{2}\sum\left(\frac{dq_i}{dt}\right)^2 + g^2\sum V_\alpha(|q_i - q_j|).$$  (126)

The Lagrangian in (1) has $g^2 = 1$. Then the corresponding eigenvalue problem is

$$\left( -\frac{d^2}{dt^2} + g^2 t\Delta U \Delta \right)\Psi = \lambda\Psi.$$  (127)

The eigenvalue for the same period $T$ is the same for $g^2 = 1$, because a scale transformation $t \rightarrow t$ and $q \rightarrow g^2/(2+\alpha)q$ makes $g^2 \rightarrow 1$. Therefore, the scale invariant eigenvalue $\tilde{\lambda}$ is also invariant for changing coupling constant $g^2 > 0$.

6.2 Numerical calculation

In this subsection, we describe a detail to calculate the eigenvalues and eigenfunctions of the Hessian for choreographic sine plus subspace to which the figure-eight solution belongs.

We introduced a cut-off of the Fourier series $1, \cos(\nu t), \cos(2\nu t), \ldots, \cos(3N\nu t)$ and $\sin(\nu t), \sin(2\nu t), \ldots, \sin(3N\nu t)$ with sufficient large $N = 2^M$. Then, the largest number of $k^0_n$ and $k^c_n$ are

$$k^0_n = \{1, 5, 7, \ldots, k^0_N = 3N - 1\}, \quad k^c_n = \{2, 4, 8, \ldots, k^c_N = 3N - 2\}. \quad (128)$$

Using the equations

$$2\sin(k\nu t)\sin(k'\nu t) = -\cos((k + k')\nu t) + \cos((k - k')\nu t),$$
$$2\cos(k\nu t)\cos(k'\nu t) = \cos((k + k')\nu t) + \cos((k - k')\nu t). \quad (129)$$

the matrix elements (102), (103), (104) and (105) are expressed by the Fourier integral

$$\tilde{u}(k) = \frac{1}{T} \int_0^T u(t)\cos(k\nu t). \quad (130)$$
The largest \( k \) to calculate the matrix elements in (104) for (128) is \( k_{\text{max}} = 2k_o^N = 6N - 2 < 6N - 1 \). We calculate the Fourier integral by Fast Fourier Transformation,

\[
\hat{u}(k) = \text{Re} \left( \frac{1}{N} \sum_{0 \leq s \leq 6N - 1} u \left( \frac{T_s}{N} \right) \exp(2\pi iks/N) \right). \tag{131}
\]

By Nyquist-Shannon-Someya sampling theorem, we have to take the sampling number larger than \( 2k_{\text{max}} \) to calculate \( \hat{u}(k) \) for \( 0 \leq k \leq k_{\text{max}} \) correctly.

Then, the elements (104) are

\[
\langle cs_m^+ | \Delta \Delta | cs_n^+ \rangle = \begin{cases} (-1)^{m+n+3} \left( \hat{u}_{xx}(k_m^o + k_n^o) + \hat{u}_{xx}(k_m^o - k_n^o) - \hat{u}_{xy}(k_m^o + k_n^e) - \hat{u}_{xy}(k_m^o - k_n^e) \right) \\ - \hat{u}_{xy}(k_m^e + k_n^o) - \hat{u}_{xy}(k_m^e - k_n^o) + \hat{u}_{yy}(k_m^e + k_n^e) + \hat{u}_{yy}(k_m^e - k_n^e) \end{cases}. \tag{132}
\]

Then, make the Hessian, \( H_{mn} = \langle cs_m^+ | -d^2/dt^2| cs_n^+ \rangle + \langle cs_m^+ | \Delta \Delta | cs_n^+ \rangle \), and solve the eigenvalue problem \( H\Phi = \lambda\Phi \).

Inversely, once we get the eigenvalue \( \lambda \) and the eigenfunction

\[
\Phi = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_N \end{pmatrix}, \tag{133}
\]

the eigenfunction \( \Psi \) is given by

\[
\Psi = \sum_{1 \leq n \leq N} cs_n^+ a_n = \begin{pmatrix} \delta q(t) \\ \delta q(t + T/3) \\ \delta q(t - T/3) \end{pmatrix}, \tag{134}
\]

where

\[
\delta q(t) = \sum_{1 \leq n \leq N} \begin{pmatrix} \sin(k_m^o \nu t) & 0 \\ 0 & \sin(k_m^e \nu t) \end{pmatrix} \begin{pmatrix} a_n^x \\ a_n^y \end{pmatrix} = \sum_{1 \leq n \leq N} \begin{pmatrix} \sin(k_m^o \nu t) a_n^x \\ \sin(k_m^e \nu t) a_n^y \end{pmatrix}. \tag{135}
\]

Each component of the right hand side is a Fourier series. Using a series \( b_k \), defined by

\[
b_k = a_n \text{ for } n = 1, 2, 3, \ldots, N, \text{ all other } b_k = 0, \tag{136}
\]

the series is given by

\[
\delta q(t) = \sum_{1 \leq n \leq N} \sin(k_m \nu t) a_n = \sum_{0 \leq k \leq 3N - 1} \sin(k \nu t) b_k = -\text{Im} \left( \sum_{0 \leq k \leq 3N - 1} b_k \exp(-2\pi i kT) \right). \tag{137}
\]

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We can use Inverse Fast Fourier Transform to calculate the right hand side numerically. Namely, for discrete time \( t = Ts/(3N) \), and \( s = 0, 1, 2, \ldots, 3N - 1 \),

\[
\delta q \left( \frac{Ts}{3N} \right) = -\text{Im} \left( \sum_{0 \leq k \leq 3N-1} b_k \exp(-2\pi iks/(3N)) \right). \quad (138)
\]

6.3 Application to the figure-eight solution

Lower 5 scale invariant eigenvalues \( \tilde{\lambda} = \lambda/(4\pi^2/T^2) \) for Newton potential (\( \alpha = 1 \)) are listed in the equation (139).

| \( cc^+ \)   | \( cc^- \)   | \( cs^+ \)   | \( cs^- \)   | \( zc^+ \) and \( zs^+ \) | \( zc^- \) and \( zs^- \) |
|----------------|----------------|----------------|----------------|-----------------------------|-----------------------------|
| 1.14719 × 10^{−10} | 4.04112        | 16.5764       | 26.117         | 50.469                     |
| 1.93391         | 7.26091        | 19.289        | 27.3167        | 50.6671                    |
| 2.32495         | 6.14928        | 18.5694       | 27.9174        | 50.9476                    |
| −6.81259 × 10^{−10} | 3.51412        | 16.4395       | 25.7645        | 50.272                     |
| 0.00174304     | 0.946257       | 5.02738       | 8.91162        | 12.5666                    |
| −0.0744809     | 1.24605        | 4.92153       | 9.14293        | 11.6371                    |

(139)

Tiny eigenvalues of order \( 10^{-10} \) in \( cc^+ \) and \( cs^- \) are zero eigenvalues that correspond to the conservation of energy and angular momentum respectively.

This calculation is done with \( N = 2^9 \). (For definition of \( N \), see section 4 and figure 3).

6.4 A question for eigenvalue problem by differential equations

The eigenvalue problem \( H\Psi = \lambda\Psi \) can be solved by directly solving the differential equation (10) and (14) with periodic boundary condition. Mathematica provides “NDEigensystem” function for this purpose. This function gives us an easy method to calculate the eigenvalues and eigenfunctions in a moderate precision. It is good idea to use this function if you need a moderate precision.

Actually, we used this function twice. The first time is to get rough estimates of eigenvalues and eigenfunctions at an early stage of our calculations. The second time is to check our program developed here at the last stage.

In these calculations, we eliminated the variable \( \delta q_2 \) by \( \delta q_2 = -\delta q_0 + \delta q_1 \) to eliminate the trivial subspace. And we impose periodic boundary condition \( \delta q_k(t + T) = \delta q_k(t) \). Equivalently, we restricted the function space to zero-centre-of-mass and periodic. It works fine.

We have one question. Can we restrict the function space more and more in this direct calculation by imposing suitable boundary conditions? For example, we can restrict the function space to choreographic subspace by imposing the boundary conditions

\[
\delta q_1(t) = \delta q_0(t + T/3), \quad \delta q_2(t) = \delta q_1(t + T/3), \quad \delta q_0(t) = \delta q_2(t + T/3). \quad (140)
\]

The question is how to realize this boundary condition in numerical calculations, for example in “NDEigensystem”? And how to realize boundary conditions
for choreographic cos + in calculations? A strong motivation for this note is to classify eigenvalues and eigenvectors in suitable subspace defined by the symmetry. If we can directly solve the eigenvalue equation in each subspace with realizing suitable boundary conditions, this will provide a concise and simple method. It will be a future investigation.

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