From Co-prime to the Diophantine Equation Based Sparse Sensing in Complex Waveforms

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Abstract—For frequency estimation, the co-prime sampling tells that in time domain, by two sub-Nyquist samplers with $M$ and $N$ down sampling rates, respectively, up to $O(MN)$ frequencies can be estimated based on autocorrelation, where $M$ and $N$ are co-prime. Similarly, in space domain for Direction-of-arrival (DOA) estimation, co-prime arrays are made of two uniform linear arrays and up to $O(MN)$ sources can be resolved with $O(M + N)$ sensors. In general, the idea behind co-prime sensing is the well-known Bazout’s Theorem. However, still from Bazout’s Theorem, in frequency estimation, the time delay is proportional to $MN$. Also, for DOA estimation, though with enhanced degrees of freedom, the sparsity of sensors is not arbitrary. In this letter, we restrain our focus on complex waveforms and present a framework under multiple samplers/sensors for both frequency and DOA estimation. We prove that there exist sampling schemes which can achieve arbitrary sparsity while the time delay is only proportional to the number of samples and resolution required. In the scenario of DOA estimation, we show there exists an array, of which the sparsity can be arbitrary with sufficiently many sensors.

Index Terms—Co-prime sampling, Sparse sampling, Linear Diophantine equation,

I. INTRODUCTION

The study on co-prime sampling, which was first proposed in [8] [20], spans almost last decade and has witnessed tremendous progress [13], [15], [18], [4], [24]. Such technique can be used in autocorrelation reconstruction based estimation by increasing the degrees of freedom while preserving sparsity either in time or space dimension. The key idea is that, for two sequences $\mathcal{M} = \{nMT, n = 1, 2, ..., N\}$ and $\mathcal{N} = \{mNT, m = 1, 2, ..., 2M\}$, where $M$ and $N$ are co-prime integers and $T$ is some fixed time or space unit, the difference of any elements between $\mathcal{M}$ and $\mathcal{N}$ will cover all consecutive multiples of $T$ starting from $-MN$ to $MN$. In the applications of frequency estimation, the two sequences $\mathcal{M}$ and $\mathcal{N}$ represent the signal samples in the time domain for two samplers, respectively, while for the case of Direction-of-arrival (DOA) estimation, $\mathcal{M}$ and $\mathcal{N}$ correspond to the locations of two uniform arrays. Such carefully designed difference can result in $O(MN)$ consecutive lags for autocorrelation estimation, which is usually used in spectral reconstruction methods, such as MUSIC and ESPRIT [22] etc. On the other hand, the sparsity parameters, $M$ and $N$, can be arbitrary once the co-prime property is preserved. Also more distinct lags bring more freedoms. Without dispute, the enhanced sparsity and freedoms will make breakthrough in the hardware limitations. Since such idea can be generalized to any integer Ring, apart from in one dimension, higher dimension cases have also been explored [21], [1], [12].

Though co-prime sensing has been well studied, two problems related remain open. First, in practice, enough samples are required in order to guarantee the estimation precision of an autocorrelation matrix. However, based on Bazout Theorem, for frequency estimation, the time delay between two snapshots to approximate autocorrelation with a fixed lag is exactly $MN$. When $M$ and $N$ are very large, it can be unacceptable. On the other hand, in DOA estimation, it is well known that the mutual coupling between elements in co-prime arrays is much less than that of nested arrays [9], [1], [14], or even super nested arrays [5], [6], [7]. However, the minimal distance between sensors in constructed arrays is still $T$, which equals the half of the wavelength in practice. Therefore, it is natural to ask whether there exists a sampling scheme in time domain with arbitrarily low sampling frequencies but the delay is only linear proportional to the sampling interval. Also for DOA, whether the minimal distance between sensors in co-prime arrays can be arbitrary large. In general, they would be very hard to solve whereas in this paper we give positive answers in a special case assuming the signals are complex.

The rest of the paper is organized as follows. Background of co-priming sampling is given in Section II. In Section III, we introduce the idea of Diophantine equations and propose a framework for frequency estimation in complex waveforms. In Section IV, we further extend this idea to space domain for the DOA estimations. Experimental results are provided in Section V with conclusions following at Section VI.

II. BACKGROUND

Let us consider a complex waveform formed by $D$ components,

$$f(t) = \sum_{i=1}^{D} A_i e^{j(\omega_i t + \phi_i)} = \sum_{i=1}^{D} A_i e^{j\phi_i} e^{j\omega_i t}$$ (1)

where $\omega_i = 2\pi f_i T_s$ is the digital frequency; $A_i$, $f_i$ and $\phi_i$ are the amplitude, the frequency and the phase of the $i$th source, respectively, and $f_s = 1/T_s$ is the Nyquist sampling frequency. The phases $\phi_i$ are assumed to be random variables uniformly distributed in the interval $[0, 2\pi]$ and uncorrelated from each other. The two sampled streams with sampling interval $MT_s$ and $NT_s$, respectively, are expressed as

$$x_1[n_1] = f(Mn_1 T_s) + w_1(n_1)$$
$$x_2[n_2] = f(Nn_2 T_s) + w_2(n_2)$$ (2)
where \( w_1(n_1) \) and \( w_2(n_2) \) are uncorrelated zero mean white noise sequences with the same power \( \sigma^2 \). With Bézout's identity, there exist \( \{a_1 = \langle M^{-1} \rangle N, \beta_1 = \langle -N^{-1} \rangle M \} \), where \( \langle X \rangle_Y \) denotes the residue of \( X \) modulo \( Y \), such that \( a_1 M - \beta_1 N = 1 \). Thus we can always find some solutions to the following equation by selecting \( \alpha_k = ka_1 \) and \( \beta_k = k\beta_1 \),

\[
\alpha_k M - \beta_k N = k
\]

(3)

In [8], it is shown that for \( r \in \mathbb{Z} \), there exist \( \alpha_k^r \in \{rN, rN+1, \ldots, (r+2)N-1\} \) and \( \beta_k^r \in \{rM, rM+1, \ldots, (r+1)M-1\} \) such that

\[
\alpha_k^r M - \beta_k^r N = k
\]

(4)

for \( k = 0, 1, \ldots, MN \). On the other hand, it is not hard to observe that for \( k \neq 0 \), the autocorrelation of \( x[n] \) can be expressed as

\[
R_x[k] = \mathbb{E}_n \{x[n]x^*[n-k]\} = \sum_{i=1}^{D} A_i^2 e^{j\omega_i k}
\]

(5)

In order to estimate \( R_x[k] \), instead we use the average of the inner product of those pairs \( \{x_1[\alpha_k^r], x_2[\beta_k^r] \} \) due to equation (3), where \( x_1[\alpha_k^r] \) and \( x_2[\beta_k^r] \) are clearly two subsequences of \( x[n] \). By rewriting \( \mathbb{E}_n \{x[n]x^*[n-k]\} \) in the context of \( x_1[\alpha_k^r] \) and \( x_2[\beta_k^r] \), i.e., \( Er \{x_1[\alpha_k^r]x_2[\beta_k^r] \} \), we have the following shown in [20]. If \( L \) snapshots are used to estimate each \( \mathbb{E}_n \{x[n]x^*[n-k]\} \), the delay of co-prime sampling is \( O(LMNTa) \).

III. DIOPHANTINE EQUATION BASED SAMPLING

To remove the dependence on a factor of \( MN \) in time delay, a new structure is needed. Our construction centers around the idea to estimate the autocorrelation with higher-order statistics. Clearly higher-order statistics will be less robust than the lower one, while, as shown soon, much more flexibility will be provided. To be specific, we consider a class of generic Diophantine equation

\[
m_1 M_1 + m_2 M_2 + m_3 M_3 = k,
\]

(7)

instead of the special case with only two variables in (3).

Suppose there are three samplers with down sampling rate \( M_1, M_2 \) and \( M_3 \), respectively, where \( \text{gcd}(M_1, M_2, M_3) = 1 \). Here \( \text{gcd} \) denotes the greatest common divisor. Then there exist two groups of integers, \( \{a_1, a_2, a_3\} \) and \( \{b_1, b_2, b_3\} \), such that

\[
\begin{align*}
  a_1 M_1 + a_2 M_2 + a_3 M_3 &= 0 \\
  b_1 M_1 + b_2 M_2 + b_3 M_3 &= 1
\end{align*}
\]

(8)

and the signs of \( (a_i) \) are not the same, nor do \( (b_i) \). Here we use the fact that [3] has solutions if \( \text{gcd}(M_1, M_2, M_3) = 1 \). Then for each \( k, l \in \mathbb{Z} \),

\[
(kb_1 + la_1)M_1 + (kb_2 + la_2)M_2 + (kb_3 + la_3)M_3 = k
\]

(9)

For \( k = 1, 2, \ldots, K \) and \( l = 1, 2, \ldots, L \), if we have both \( (kb_1 + la_1) \) and \( (kb_3 + la_3) \) to be positive and \( (kb_2 + la_2) \) to be negative, we construct the sequence \( \{\hat{x}_1, \hat{x}_2, \hat{x}_3\} = \langle -(kb_2 + la_2)\rangle \hat{x}_3 \right\} \langle kb_3 + la_3\rangle \rangle \), where \( \hat{x}_i[n] = f(M_i n T_a) + w_i(n) \) denotes the sample stream with a down sampling rate \( M_i \). In [20], we show that \( \mathbb{E}_n \{\hat{x}_1[kb_1 + la_1] \hat{x}_2[-(kb_2 + la_2)] \hat{x}_3[kb_3 + la_3]\} = \sum_{i=1}^{D} A_i^2 e^{j\omega_i k} \) once \( a_1 M_1 (\omega_i - \omega_v) + a_2 M_2 (\omega_v - \omega_u) \neq 0 \) for \( i \neq v \neq u \in \{1, 2, \ldots, D\} \), which fails with negligible probability. The results of spectral estimation using the constructed sequence will be the same as that of \( x[n] \).

Furthermore, if all the above assumptions hold, the time delay of the proposed scheme is determined by the maximum value of \( \{kb_i + la_i\} \). In the following, we show the validity by giving a specific construction, where \( b_i \) and \( a_i \) can further be independent of \( M_i \), the down sampling rate.

Theorem 1. There exist constants \( c_1 \) and \( c_2 \) such that the time delay of the proposed scheme is upper bounded by \( (c_1 K + c_2 L)M N \), where \( K \) is the number of consecutive lags, \( L \) is the number of snapshots required to estimate \( \mathbb{E}_n \{\hat{x}_1[kb_1 + la_1] \hat{x}_2[-(kb_2 + la_2)] \hat{x}_3[kb_3 + la_3]\} \). Here \( M = \max \{M_1, M_2, M_3\} \).

Proof. Consider a set of integers, say, \( \{2, 3, 5\} \). Clearly \( \text{gcd}(2, 3, 5) = 1 \). Now we try to find out two integer groups \( \{a_1, a_2, a_3\} \) and \( \{b_1, b_2, b_3\} \) such that

\[
\begin{align*}
  2a_1 + 3a_2 + 5a_3 &= 0 \\
  a_1 + a_2 + a_3 &= 0 \\
  2b_1 + 3b_2 + 5b_3 &= 1 \\
  b_1 + b_2 + b_3 &= 0
\end{align*}
\]

(11)

We choose \( \{a_1 = 2, b_1 = 1\} \), \( \{a_2 = -3, b_2 = -2\} \), \( \{a_3 = 1, b_3 = 1\} \) as a solution of (11). Due to \( \sum_i a_i = 0 \) and \( \sum_i b_i = 0 \), clearly they are solutions to

\[
(kb_1 + la_1)(2 + \Gamma) + (kb_2 + la_2)(3 + \Gamma) + (kb_3 + la_3)(5 + \Gamma) = k
\]

(12)

for any \( \Gamma \in \mathbb{Z} \). Because \( k \in \{1, 2, \ldots, K\} \) and \( l \in \{1, 2, \ldots, L\} \), we have \( \max_{k, l} |kb_1 + la_1| \leq 2K + 3L \). Also \( (kb_1 + la_1) \) and \( (kb_2 + la_2) \), i.e., \( (k + 2\Gamma) \) and \( (k - \Gamma) \), respectively, are always positive while \( (kb_2 + la_2) \) is negative. Thus, the solution is satisfied and the total time delay is upper bounded by \( (2K + 3L)(5 + \Gamma)T_a \).

In the following, we give a general framework of multiple samplers. Given \( N \) positive integers, denoted by \( M_1, M_2, \ldots, M_N \), a distributed co-prime sampling can be naturally constructed by selecting any three of them and implementing by the above scheme. To provide a concrete strategy to efficiently make use of the triple cross difference between samples collected from each sampler, let us revisit the idea we apply before. For one such subgroup, say, \( \{M_1, M_2, M_3\} \), \( i_1, i_2, i_3 \in \{1, 2, \ldots, N\} \), we still try to construct two special solutions, \( \{a_{i_1}, a_{i_2}, a_{i_3}\} \) and \( \{b_{i_1}, b_{i_2}, b_{i_3}\} \), such that

\[
\begin{align*}
  a_{i_1} M_{i_1} + a_{i_2} M_{i_2} + a_{i_3} M_{i_3} &= 0 \\
  a_{i_1} + a_{i_2} + a_{i_3} &= 0 \\
  b_{i_1} M_{i_1} + b_{i_2} M_{i_2} + b_{i_3} M_{i_3} &= 1 \\
  b_{i_1} + b_{i_2} + b_{i_3} &= 0
\end{align*}
\]

(13)

which can be simplified to find out \( a_{i_1}, a_{i_2}, b_{i_1}, b_{i_2} \),

\[
\begin{align*}
  a_{i_1} (M_{i_1} - M_{i_2}) + a_{i_2} (M_{i_2} - M_{i_3}) &= 0 \\
  b_{i_1} (M_{i_1} - M_{i_3}) + b_{i_2} (M_{i_3} - M_{i_2}) &= 1
\end{align*}
\]

(14)
To be specific, we set \( M_1, M_2, \ldots, M_N \) are the sequence of consecutive numbers starting from 1 to \( N \) shifted by some integer \( \Gamma \), i.e., \( M_i = i + \Gamma \). To lighten the expression, the following results are presented in an asymptotic sense of \( N \).

For any \( M_i \), we consider the following sequence

\[
M_1 - M_i, M_2 - M_i, \ldots, M_N - M_i
\]

and try to estimate the number of co-prime pairs among them. Since the numbers in (15) are still consecutive, among which the number of primes is upper bounded by \( \pi(N) \), where \( \pi(N) \) denotes the number of primes no bigger than \( N \). Thus, by randomly picking any two of them, the probability of the two picked numbers which are co-prime is lower bounded by

\[
\frac{\pi(N)}{N(N-1)} < \frac{1}{\pi^2}
\]

where \( p_j \) is the \( j \)-th prime in the natural order and the above inequality follows from the density of primes [2]. Here we use the fact that if we randomly select two numbers from \( \mathbb{Z} \), the probability that they both share a prime factor \( p_j \) is \( \frac{1}{p_j} \). Therefore, we can totally find \( \frac{\pi(N)}{N(N-1)} \) many subsets of size three such that they are co-prime and also there exists a solution satisfying (13). Here it is because (13) has solutions equivalent to that \((M_i - M_1) \) and \((M_i - M_2)\) are co-prime. Moreover, from (14), it is clear that both \(|a_i|\) and \(|b_i|\), if exist, are upper bounded by \( 2(N-1) \). Furthermore, without loss of generality, we assume \( M_i > M_1 > M_2 > M_3 \) and then, in (15), \( M_i - M_1 > 0 \) while \( M_3 - M_2 < 0 \). Hence, we can especially set \( a_i = M_1 - M_2, a_3 = M_1 - M_3, b_1 = (M_1 - M_2)^{-1}(M_1 - M_3) \) and \( b_2 = (M_1 - M_2)^{-1}(M_1 - M_3) \), which all are positive. Thus both \( a_i \) and \( b_i \) should be negative due to the restriction in (13). Therefore, for \( k = 1, 2, \ldots, K \) and \( l = 1, 2, \ldots, L \), (\( k|a_i + la_i \)) are always negative whereas (\( k|b_i + la_i \)) and (\( k|b_i + la_i \)) are positive. When we apply Theorem 1 on each sequence \( \{\hat{x}_1[kb_i + la_i], \hat{x}_2[kb_i + la_i]\} \) satisfying (13), the following theorem can be derived.

**Theorem 2.** For arbitrary \( N \), there exists a distributed co-prime sampling scheme which can provide at least \( L \frac{1}{\pi^2} N(N - 1)(N - 2) \) many virtual samples for estimating \( \{\hat{x}_1[kb_i + la_i], \hat{x}_2[kb_i + la_i]\} \) with time delay upper bounded by \( 2(N - 1)(K + L)MT_s \).

It is worthwhile to mention that following our idea that for any given \( N \) integers, the number of subgroups where solutions \( a_i \) and \( b_i \) satisfying (13) exist is upper bounded by

\[
\left( \frac{N}{3} \right) - \left( \frac{N - e}{3} \right) - \left( \frac{N - N_e}{3} \right) < \frac{N(N - 1)(N - 2)}{8}
\]

Here we use the fact that when \( M_1, M_2 \) and \( M_3 \) are all even or odd integers, (14) is not solvable. Therefore, the proposed construction is close to the optimal.

**IV. DIRECTION OF ARRIVAL ESTIMATION WITH MULTIPLE COPRIME ARRAYS**

As mentioned earlier, another important application of co-prime sensing is to provide enhanced freedoms for DOA estimation, which has applications [23][17][19]. Consider a linear array of \((M_1 + 2M_2 - 1)\) sensors, of which the positions are given by \( \{M_1 m_1 d, m_1 = 0, 1, \ldots, M_2\} \cup \{M_2 m_2 d, m_2 = 0, 1, \ldots, 2M_2 - 1\} \). Here \( d = \frac{\lambda}{2} \) and \( \lambda \) corresponds to the wavelength. As indicated by Bazout Theorem, the difference set \( \{m_1M_1 - m_2M_2\} \) will enumerate all consecutive integers from \(-M_1M_2\) to \(M_1M_2\), which can further provide \(2M_1M_2 + 1\) freedoms. In general, there are two primary concerns in DOA estimation. The first is the number of consecutive lags. As shown in [8], both the resolution and the freedoms are proportional to the number of longest consecutive integer sequence generated. Second, a larger minimal distance between sensors will always be desirable in order reduce coupling.

Let \( a_i(\theta_i) = e^{j(2\pi/\lambda)d_i \sin \theta_i} \) be the element of the steering vector corresponding to \( \theta_i \), where \( d_i \) is the location of \( l \)-th sensor. Assuming \( f_c \) to be the center frequency of the band of interest, for narrow-band sources centered at \( f_i + f_c \), \( i = 1, 2, \ldots, D \), the received signal at the \( l \)-th sensor is expressed by

\[
\hat{x}_l(t) = \sum_{i=1}^{D} a_i(\theta_i) s_i(t) e^{2\pi f_i t}
\]

Though consecutiveness can be relaxed by only requiring distinct lags with sparse sensing techniques [10].
where $s_1(t)$ is a narrow-band source. When a slow-fading channel is considered, we assume $s_i(t)$ as some constant $s_i$ in a coherence time block \cite{3}. With the similar idea we use in frequency estimation, when $t_1 + t_3 = t_2$,

$$
\mathbb{E}_{t_1, t_2}[x_{t_1}(t_1) \cdot x_{t_2}^*(t_2) \cdot x_{t_3}(t_3)] = \sum_{i=1}^{D} s_i^3 e^{i(2\pi/\lambda)(d_{i_1} - d_{i_2} + d_{i_3}) \sin \theta_i},
$$

where $t_1, t_2, t_3 \in \{1, 2, ..., N\}$. Similarly, $\mathbb{E}_{t_1, t_2}[x_{t_1}(t_1) \cdot x_{t_2}(t_2) \cdot x_{t_3}^*(t_3)] = \sum_{i=1}^{D} s_i^3 e^{i(2\pi/\lambda)(-d_{i_1} + d_{i_2} - d_{i_3}) \sin \theta_i}$. With the above assumptions, it suffices to consider the DOA estimation in complex waveforms. Similarly, we consider the triple difference rather than the pairwise one.

**Theorem 3.** By assigning $M_1 = qp_1$, $M_2 = qp_2$ and $M_3 = p_1 p_2$ such that $q, p_1$ and $p_2$ are relatively co-prime integers, we select the following locations for sensors $\{m_1 M_1, m_2 M_2, m_3 M_3\mid m_1 \in \{0, 1, ..., 2p_2 - 1\}, m_2 \in \{0, 1, ..., p_1 - 1\}, m_3 \in \{0, 1, ..., q - 1\}\}$ in three uniform subarrays. Then the linear difference set $\{(\pm m_1 M_1 - m_2 M_2) \pm m_3 M_3\}$ will contain consecutive integers starting from $-p_1p_2q$ to $p_1p_2q$.

**Proof.** We consider a three-level Bazout Theorem. For the equation $mM - mN = k$, we decompose $N$ into a product of two co-prime numbers $p_1, p_2$ and assign $M$ to be $q$. Then, we use $\pm (m_1 p_1 - m_2 p_2)$ to construct $n$ for $n \in \{-p_1p_2, ..., p_1p_2\}$, which further yields $\{\pm (m_1 M_1 - m_2 M_2)\} = \{nq, n = -p_1p_2, ..., p_1p_2\}$. Thus, applying Bazout Theorem again on $\{(\pm m_1 M_1 - m_2 M_2) \pm m_3 M_3\}$ will contain consecutive integers starting from $-p_1p_2q$ to $p_1p_2q$.

From Theorem\cite{3} it shows that with $p_1 + 2p_2 + q - 1$ sensors, at least $2p_1p_2q + 1$ freedoms can be provided. Comparing to conventional co-prime array based DOA estimation, given $M_1 + M_2 - 2$ sensors, the corresponding freedoms are $2M_1M_2 + 1 \leq (\frac{M_1+M_2}{2})^2 + 1$. Moreover, the minimal distance among those sensors is $\text{min}\{q, p_1, p_2\}$ since the locations of any two sensors share at least one common divisor from $\{q, p_1, p_2\}$. Thus, with sufficiently many sensors, the sparsity of proposed array can be arbitrarily large.

On the other hand, to estimate the autocorrelation at lag $k$, we will find the snapshots at time $t_1, t_2$, and $t_3$, where $t_1 + t_3 = t_2$, from the three uniform subarrays, respectively. Therefore, assuming that each sensor has collected $L$ snapshots, by the above grouping strategy, we can reconstruct $L(L-1)/2$ samples for autocorrelation estimation at each lag, in comparison to $L$ samples in co-prime arrays. Thus, more samples can compensate for precision downside in DOA estimation since the third-order statistics applied in proposed Diophantus arrays \cite{19} is less robust than the second-order based estimation in co-prime arrays.

**V. SIMULATION**

We present two numerical simulation in Fig. 1, which compares the performance of the proposed Diophantus Equation based sparse reconstruction in the application of frequency and DOA estimation with that of traditional Multiple Signal Classification (MUSIC). For frequency estimation, we randomly generate $D = 5$ and $10$ frequencies, respectively and set $K = L$. The proposed method is used to estimate the frequencies with three samplers of down sampling rate $M_1 = 2 \times 10^6$, $M_2 = 3 \times 10^6$ and $M_3 = 5 \times 10^6$, and we average the root mean square error (RMSE) on 100 times independent Monte-carlo runs with signal-to-noise ratio (SNR) ranges from $-10$ to $10$dB. To evaluate the performance of the proposed method, we choose MUSIC as a baseline to estimate frequencies and the results are shown in Fig. 1(a). As expected, a small compromise in accuracy exists for proposed strategy since we use three channels instead of two. However, the time delay of co-prime sampling is around $10^6$ times longer than ours.

In the case of DOA estimation, we randomly generate $D = 3$ and $10$ independent sources, respectively. For proposed Diophantus array, we select $p_1 = 4, p_2 = 3$ and $q = 5$ and thus totally 14 sensors are used. We still use MUSIC algorithm as the baseline with $L = 18$ and $L = 50$ snapshots. As analyzed before, for each lag $k$, we can find $O(L^2)$ samples. The simulations are run 100 times and the averaged RMSEs are shown in Fig. 1(b). We can see that the proposed strategy in some cases is even with better performance than MUSIC. Furthermore, our method provides up to 149 freedoms compared with 57 in a co-prime array. Also the minimal distance between sensors is $3d$, comparing to $d$ in an existing nested or co-prime array.

**VI. CONCLUSION**

In this letter, we generalize the co-prime based sparse sensing based on the idea of Diophantine equations to deal with complex waveforms. The proposed scheme establishes a new tradeoff which provides more flexibility in the parameter selection and the sparsity requirement. Experimental results also support the theory.

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