Motion of a charged particle in the magnetic dipole field

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This note discusses the motion of a charged particle in the magnetic dipole field and a modified Barut’s lepton mass formula. It is shown that a charged particle in the magnetic dipole field has no bound states, which means that Barut’s lepton mass formula may have no physical basis.

Historically, the subquark theories (including the topics such as the symmetries and structures in leptons and quarks) captured particular attention of many researchers [1–9]. The occurrence of the leptonic fermion chain (e, μ, τ, ...) is a novel phenomenon for which we have so far no theory to interpret the origin of the generation of leptons. Regarding this subject, the fundamental problems are as follows: why does the generation phenomenon exist? what causes the generation number is not up to three [10]? Studying the mass formula for leptons may provide clue to the physicists on how the fundamental mechanism involved works in the above-mentioned problems. For this aim, several authors probed the lepton and quark mass spectra [11–14].

Of all the above investigations, the most important work that deserves emphasis here is that of Barut [12], who proposed a mass formula for muon on the basis of magnetic self-interaction of the electron [11]. He believed that the radiative effects give an anomalous magnetic moment to the electron which implies an extra magnetic energy [11,12]. By using the quantization formulation according to Bohr-Sommerfeld procedure, Barut obtained the magnetic energy of a system consisting of a charge and a magnetic moment as

\[ E_n = \lambda n^4 \]

with \( n \) being a principal quantum number. The lepton mass formula suggested by Barut is [12]

\[ m_n = m_e \left( 1 + \frac{3}{2\alpha} \sum_{l=0}^{n} l^4 \right) \]

where \( m_e \) stands for the electron mass and the integer \( n \) represents the generation label of charged leptons (for instance, \( n = 0, 1, 2 \) corresponds to the electron, muon (μ) and tau (τ) particle, respectively). It should be noted that, as far as the known fermion chain (e, μ, τ) is concerned, this mass formula of charged leptons agrees with experimental values extremely well.

Barut argued that although the Bohr-Sommerfeld quantization is approximative, the final result might be exact as was the case in Bohr-Sommerfeld derivation of the Balmer formula [12]. Here, however, I will show that this viewpoint may be not the true case. In this note, by considering the motion of a charged particle in the field of a magnetic dipole, it will be shown that the charged particle near a magnetic dipole may have no bound states. This, therefore, implies that Barut’s magnetic energy formula \( E_n = \lambda n^4 \) may be not valid.

In history, Dirac investigated the motion of an electron in the field of a magnetic monopole and showed that the electron cannot be bound to the monopole field [15]. Even though Dirac did not take into account the electron spin (which will give rise to a magnetic moment) in his analysis, the above conclusion is still valid if one treats the same problem by taking into consideration the electron spin (and hence its spinning magnetic moment) [16]. In what follows I will deal with the problem of the stationary Klein-Gordon equation governing the motion of a scalar particle (or an electron without taking account of its spin and spinning magnetic moment) in the presence of a magnetic dipole field.

It is well known that the three-dimensional magnetic vector potential of a magnetic dipole field written in a polar coordinate system \((r, \theta, \varphi)\) is

\[ A_\varphi = \frac{r \sin \theta}{r^2 + a^2} \frac{\mu_0 I a^2}{4} \]

where the coefficient is \( \lambda = \frac{\mu_0 I a^2}{4} \). Here, \( \mu_0, a \) and \( I \) denote the vacuum permeability, ring radius and electric current of the magnetic dipole. Note that here it is assumed that a circular ring carrying a current \( I \) forms a magnetic dipole. If the radius \( a \) is much less than the spatial scale, then the

*Since the mathematical procedure in this note is trivial, it will be submitted nowhere else for publication, just uploaded at the e-print archives.

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above-mentioned magnetic vector potential can be reduced to \( A_\varphi = \frac{\lambda e}{\hbar c} \). For convenience, in this note we consider only the 2D case (\( i.e., \theta = \frac{\pi}{2} \)) of the stationary Klein-Gordon equation, which is of the form

\[
\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \psi}{\partial \rho} \right) + \frac{1}{\rho^2} \left( \frac{\partial}{\partial \varphi} - \frac{ie}{\hbar} \rho A_\varphi \right)^2 \psi = \left[ -\frac{E^2}{\hbar^2 c^2} + \left( \frac{mc^2}{\hbar} \right)^2 \right] \psi
\]

(2)
in the 2D polar coordinate system \((\rho, \varphi)\). Substitution of \( \psi = R(\rho) \exp[im\varphi] \) into (2) yields

\[
\left( \frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} - \frac{m^2}{\rho^2} \right) R + \frac{\eta}{\rho^3} R + \frac{\sigma}{\rho^4} R + \xi R = 0
\]

(3)
with \( \xi = \frac{E^2}{\hbar^2 c^2} - \left( \frac{m c^2}{\hbar} \right)^2 \), \( \eta = \frac{2 \omega m \hbar}{\hbar} \) and \( \sigma = -\frac{\omega^2 \hbar^2}{\hbar} \). Similar to the discussion of 2D hydrogen atom, we insert

\[
R(\rho) = \rho^{(|m|)} \exp(-\beta \rho) u(\rho) \quad \text{with} \quad \beta = \sqrt{-\xi}
\]

(4)
into Eq.(3), and obtain

\[
\frac{d^2 u}{d\rho^2} + \left[ 2 \left( \left| \frac{m}{\rho} \right| - \beta \right) \frac{d u}{d\rho} + \frac{1}{\rho^2} \frac{d u}{d\rho} \right] + \left[ \frac{\eta}{\rho^3} - \frac{\beta}{\rho^4} \right] u = 0.
\]

(5)

On substituting \( u = \sum_{\nu=0}^{\infty} b_{\nu} \rho^{\nu+2} \) into Eq.(5), making use of \( \frac{d^2}{d\rho^2} = \sum_{\nu=0}^{\infty} b_{\nu} (s+\nu)(s+\nu-1) \rho^{s+\nu-2} \), \( \frac{d}{d\rho} = \sum_{\nu=0}^{\infty} b_{\nu} (s+\nu) \rho^{s+\nu-1} \), and equating the coefficients of the various powers of \( \rho \) to zero we get the recurrence relation

\[
b_{\nu+1}(s+\nu+1)(s+\nu) - 2\beta b_{\nu}(s+\nu) + (2|m| + 1)b_{\nu+1}(s+\nu+1) - \beta(2|m| + 1)b_{\nu} + \eta b_{\nu+2} + \sigma b_{\nu+3} = 0.
\]

(6)
For simplicity, we assume that in Eq.(6) the \( \sigma \)- term is much less than the \( \eta \)- term, so the former one can be ignored in what follows. Thus the recurrence relation (6) without the \( \sigma \)- term is rewritten as

\[
b_{\nu+1}(s+\nu+1)(s+\nu+2|m| + 1) = \beta(2s + 2\nu + 2|m| + 1)b_{\nu} - \eta b_{\nu+2}.
\]

(7)
When \( \nu \) tends to \(+\infty\), \( b_{\nu+1} \rightarrow \frac{2\beta b_{\nu} - \eta b_{\nu+2}}{q_{\nu+1}} \). This, therefore, implies that \( b_{\nu+1} \rightarrow \frac{2\beta b_{\nu} - \eta b_{\nu+2}}{q_{\nu+1}} \) and the behavior of \( u \) converges like \( \exp(2\beta \rho) \) for \( \nu \rightarrow \infty \), which is divergent at \( \rho \rightarrow \infty \). So, the maximal \( \nu \) in the series of the function \( u \) should be a finite integer rather than an infinite one. In the following we will discuss the recurrence relation from the various aspects:

**A.**
We set \( q_{\nu} = (s + \nu + 1)(s + \nu + 2|m| + 1) \), \( p_{\nu} = 2s + 2\nu + 2|m| + 1 \), and from Eq.(7) we can obtain

\[
q_{\nu} b_{\nu+1} = \beta p_{\nu} b_{\nu} - \eta b_{\nu+2},
q_{\nu+1} b_{\nu+2} = \beta p_{\nu+1} b_{\nu+1} - \eta b_{\nu+3}.
\]

(8)
It follows from Eqs.(8) that

\[
\left( q_{\nu} + \eta \frac{p_{\nu+1}}{q_{\nu+1}} \right) b_{\nu+1} = \beta p_{\nu} b_{\nu} + \eta^2 \frac{b_{\nu+3}}{q_{\nu+2}}.
\]

(9)
If it is required that \( b_{\nu+1} = 0 \) (and hence \( b_{\nu+n} = 0, n > 1 \)), then only the case of \( \beta p_{\nu} = 0 \) satisfies this requirement, since \( b_{\nu+3} = 0 \). This, therefore, means that \( \beta = 0 \). Thus \( \xi = 0 \). But unfortunately the solution corresponding to such an \( \xi \) is not a bound one.

**B.**
If we assume that \( b_{\nu} = 0, b_{\nu+3} = 0 \), then it follows from Eqs.(8) that

\[
q_{\nu} b_{\nu+1} + \eta b_{\nu+2} = 0,
\beta p_{\nu+1} b_{\nu+1} - q_{\nu+1} b_{\nu+2} = 0.
\]

(10)
If one requires Eqs.(10) to possess nonvanishing \( b_{\nu+1}, b_{\nu+2} \), then he can arrive at
\[ q_\nu q_{\nu+1} + \eta\beta p_{\nu+1} = 0, \]  
(11)

namely, the determinant of the coefficient matrix of Eqs.(10) is vanishing. This, however, means that \( \beta \leq 0 \) and that it will cause the function \( u \) to be divergent at large spatial scale \( (r \to \infty) \).

C.

According to Eqs.(8), we can obtain

\[ q_\nu b_{\nu+1} = \beta p_\nu b_\nu - \eta b_{\nu+2}, \]
\[ q_{\nu+1} b_{\nu+2} = \beta p_{\nu+1} b_{\nu+1} - \eta b_{\nu+3}, \]
\[ q_{\nu+2} b_{\nu+3} = \beta p_{\nu+2} b_{\nu+2} - \eta b_{\nu+4}. \]

(12)

If, for example, \( b_{\nu+3} = 0, p_{\nu+2} = 0 \), it follows that \( b_{\nu+4} = 0 \), and consequently \( b_{\nu+5} = 0, b_{\nu+6} = 0 \). So, we have

\[ q_\nu b_{\nu+1} = \beta p_\nu b_\nu - \eta b_{\nu+2}, \quad q_{\nu+1} b_{\nu+2} = \beta p_{\nu+1} b_{\nu+1}. \]

(13)

Under the condition \( b_\nu \neq 0 \), it follows that

\[ \left( q_\nu + \frac{\eta p_{\nu+1}}{q_{\nu+1}} \right) b_{\nu+1} = \beta p_\nu b_\nu. \]

(14)

So, we have the following recurrence relations

\[ b_{\nu+1} = \frac{\beta p_\nu}{q_\nu + \frac{\eta p_{\nu+1}}{q_{\nu+1}}} b_\nu, \quad b_{\nu+2} = \frac{\beta^2 p_{\nu+1} p_\nu}{q_\nu q_{\nu+1} + \eta p_{\nu+1}} b_\nu, \quad \ldots \]

(15)

It is apparently seen that all the coefficients \( b_{\nu+n} \) can be expressed in terms of \( b_\nu \) and \( \beta \). Unfortunately, it is not possible for us to obtain \( \beta \).

In view of the above discussions, it can be easily seen that the series \( u \) cannot terminate for a positive \( \beta \). Thus a charged particle is never bound to the magnetic dipole field. This, therefore, means that Barut’s lepton mass formula [12] lacks a power physical basis. Even though my result is obtained only by considering the motion of a charged scalar particle in a magnetic field of a magnetic doublet, it may be concluded that this result may also hold for a charged spinning particle (e.g., electron) in the presence of the magnetic dipole field, by analogy with Dirac and Chandra’s work [15,16].

A great majority of the lepton mass formulae have two disadvantages: (i) they cannot agree with experimental results very well; (ii) the generation number of fermions in these mass formulae are often infinite and/or they cannot interpret the finite-generation-number phenomenon. Although Barut’s formula is in good agreement with experiments (agreeing to the experimental results about one part in \( 10^3 \)), it cannot explain that why the generation number is finite [This can be seen from Eq.(1), where the generation label \( n \) can take the arbitrary integers]. In this note, I will put forward a potential mass formula for the charged leptons, which may overcome the two above-mentioned disadvantages. Such a mass formula is written as follows\(^1\)

\[ m_n = \left( 1 + \frac{1}{2\alpha} \sum_{l=0}^{n} C_{3+l}^{2l} \right) m_e \quad \text{with} \quad C_{3+l}^{2l} = \frac{3!}{l!(3-l)!}. \]

(16)

Here \( n \) denotes the generation label of various charged leptons. It should be emphasized that the complete agreement exists between this lepton mass spectrum (16) and experiments to an incredibly high degree of accuracy (i.e., agreeing to the experiments about one part in \( 10^3 \)).

Another interesting lepton mass formula suggested by us is written in the following form [17]

\[^1\]This lepton mass formula was suggested in Nov.1999 - Mar.2000. In fact, such an expression for the leptonic mass spectrum may be considered the modified version of Barut’s lepton mass formula (1).
\[ m_n = C_3^n \left( \frac{1}{2} \right)^n \left( \frac{1}{\alpha} \right) m_e \quad \text{with} \quad C_3^n = \frac{3!}{n!(3-n)!}, \quad \text{(17)} \]

which also agrees reasonably well with experimental values for the masses of e, \( \mu \) and \( \tau \).

Apparently, both the above two lepton mass spectra (16) and (17) accommodate four generations of fermions, since the generation label \( n \) can take only 0, 1, 2 and 3, which correspond to electron, muon, tau and \( f \) particle\(^2\).

Additionally, I can also suggest an other mass formula for charged leptons\(^3\)

\[ m_n = C_3^n C_2^n \left( \frac{1}{4\alpha} \right) m_e \quad \text{with} \quad C_2^n = \frac{2!}{n!(2-n)!}, \quad \text{(18)} \]

which accommodates only three generations of fermions. This formula agrees with experimental values to less than 1%.

All the above mass spectra (16)-(18) for the charged leptons are established based on both the experimental values and the mathematical structures of the so-called supersymmetric Pegg-Barnett oscillator [17]. It is believed that the investigation of the leptonic mass spectra may provide us with an insight into the generation problem (generation phenomenon, generation origin and generation structures) of leptons and quarks. We hold that these leptonic mass spectra presented in this note deserve further investigation both theoretically and experimentally.

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\[ 2 \text{The mass of the charged lepton corresponding to } n = 3, \text{ which we call, for brevity, } f \text{ lepton, is respectively } 3.4 \text{ and } 2.6 \text{ Gev in Eq.(16) and (17). Note that here the name of the } f \text{ lepton (should such exist) is inspired by considering that the } f \text{ lepton may be the } fourth \text{ (and even the } final) \text{ generation of leptons in accordance with the mass spectra (16) and (17).} \]

\[ 3 \text{This mass formula was proposed in the spring of 2000. It has not yet been published elsewhere.} \]

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