In this paper we present cuSten, a new library of functions to handle the implementation of 2D finite-difference/stencil programs in CUDA. cuSten wraps data handling, kernel calls and streaming into four easy to use functions that speed up development of numerical codes on GPU platforms. The paper also presents an example of this library applied to solve the Cahn-Hilliard equation utilising an ADI method with periodic boundary conditions.

Keywords: CUDA, Finite Difference, Library, PDEs, Stencil

I. INTRODUCTION

Many problems in Physics and Applied Mathematics can be expressed as systems of Partial Differential Equations (PDEs), examples of which include the Navier–Stokes [1], Euler [2,3], Black–Scholes [4] and Burgers [5] equations. In many situations analytic methods of solving a given system are not possible due to the complexity of the equations; an alternate approach is to solve the system numerically. To discretize the system numerically, several standard approaches exist, including the finite-difference, finite-volume, and finite-element methods. For definiteness, this work focus on the finite-difference method, however, it can be applied in any situation requiring stencil-based operations. The application of the finite-difference method turns the operators in the PDEs into expressions which can be input into a computer program. For high-resolution numerical simulations, numerical scheme allowing, it is desirable to solve these computational problems in parallel with multiple processors to reduce the time taken to find a solution, this has been traditionally tackled with the MPI or OpenMP libraries which allow for parallelisation across multiple CPU cores. More recently due to developments in technology, a reduction in cost compared to traditional multi-CPU platforms and increased performance, GPUs have become a common approach to parallelisation. Particularly the use of NVIDIA GPUs and their programming language CUDA have become prevalent. CUDA today includes many GPU versions of common numerical libraries such as the linear algebra package cuBLAS, the Fourier transform library cuFFT and cuSPARSE which provides the programmer with many common solution methods for sparse matrices.

There is a large field of literature associated with the implementation finite-difference methods using CUDA, a few examples include [6–9]. This literature commonly explains how to approach the problem but yet have no publicly available library or code that a programmer put into their own project and use, thus requiring them to rewrite the same code but yet have no publicly available library. Also libraries providing PDE solvers and other stencil based computations exist, such as [10] and indeed some approaches that can generate code for the programmer [11,12], but these libraries and approaches can be limiting due to investment cost in learning essentially a full software package.
or new method. The tool presented in this paper aims to address a gap in available software, introducing a simple set of four functions (three in many cases) for the programmer to implement their finite difference solver and these accessed much like cuBLAS or cuSPARSE giving freedom to the programmer to build the program as they choose but eliminating the need to worry about the finite difference implementation specifics. This tool allows a programmer to simply input their desired finite-difference stencil and the direction in which it should be applied and then the rest of the implementation, including the domain decomposition, boundary positioning and data handling are wrapped into functions that are easily called. This approach reduces the development time necessary for implementing new systems/solvers and provides a robust framework that does not involve a blackbox-approach to the solution from the programmer. Furthermore, the approach does not require a major overhead of time to invest in learning/implementing a new tool.

It is not intended that the code produced by this tool be the most efficient implementation of a given scheme versus a dedicated code for a specific problem. But it is intended that the development time of a code is drastically cut by removing the need for the programmer to do unnecessary work at development time. Common problems at development time include readjusting boundaries when changing finite difference schemes or ensuring the correct data has been loaded onto the GPU at the time of computation, both of these are dealt with by cuSten. 2D problems are the main focus of this new tool; 2D problems provide a testbed for the development numerical algorithms which can then be extended to 3D, where debugging, testing, and validation are more time-consuming. The extension of the present method to 3D is discussed in Section VI below. In terms of floating point precision this library focuses on the use of the double floating point type as in most application it is desirable to have 64 bit precision when solving PDEs, the source code is easily modified using a standard text editor with find and replace to change to other data types if so desired (this is discussed also in Section VI below).

II. SOFTWARE ARCHITECTURE

The library in this paper makes use of the CUDA programming language. For the sake of brevity we assume the reader is familiar the standard features of the language including kernels, shared memory etc. The tool is built on two main sets of code, one handling the creation and destruction of the cuSten_t data type which handles all of the programmer’s inputs (/src/struct) and the other handling the compute kernels (/src/kernels).

At the top level will be the main code solving whatever PDE is of concern to the programmer and the library is called through the header ‘cuSten.h’. The programmer provides the necessary memory to the library using Unified Memory along with the stencil details, these will be detailed in Section III. Unified Memory was chosen as it simplifies the handling of memory in the library and interfacing with the rest of programmer’s code, the unified address space also removes device RAM limitations.

To take advantage of the removal of RAM limitations the library allows the programmer to divide their domain into ‘tiles’ such that each tile will fit into the device RAM. Each tile is a chunk of the total domain in the y direction to ensure the memory is contiguous. The tiles are loaded onto the GPU in time for the kernel to be launched such that there are no GPU page faults. The programmer also has the option to unload the tiles onto host RAM after the computation is completed on a given tile, this can be for IO or if the programmer needed to free device RAM for the next tile or a new task. This system of loading/unloading data and computation is implemented as a pipeline using separate streams for data loading/unloading and kernel launches ensuring that everything overlaps. Events are used to ensure the data has been loaded prior to the launch of a kernel.

The programmer has the choice of supplying a standard linear stencil or a function pointer with additional input coefficients to the library, examples of which are discussed in Sections IV A and IV B respectively. Within the compute kernel blocks of data with suitable boundary halos are loaded into shared memory. The stencil or function is then applied to the block with each thread calculating the output for its position. When this has completed the data is then output as blocks into the memory provided by the programmer for output, the same memory cannot be used for both as all the blocks require overlapping data and thus cannot use already output values.

III. SOFTWARE FUNCTIONS

The programmer can use up to four functions for the application of any given finite difference stencil, in most cases only three are required. The possible stencil directions include x, y and xy, where xy allows for cross derivatives which require that diagonal information is available for the stencil to be completed. Each direction then comes with a periodic and non-periodic boundary option along with a choice between supplying just a set of weights (example in IV A) which are applied linearly or a function pointer (examples in Section IV B and VB) that can be used to apply
more sophisticated schemes. The naming convention for the functions available in the library is

custen[Create/Destroy/Swap/Compute]2D[X/Y/XY][p/np][BLANK/Fun]

The descriptions for the options are as follows:

Create: This will take the programmer inputs such as the stencil size, weights, number of tiles to use etc. and return the cuSten_t ready for use later in the code.

Destroy: This will undo everything in create, freeing pointers and streams etc. To be used when the programmer has finished using the current stencil, for example at the end of a program.

Swap: This will swap all relevant pointers, in other words swap the input and output data pointers around so the stencil can be applied to the updated stencil after timestepping. The need for this function is generally dependent on the overall numerical scheme a programmer is using, it is not needed in a situations.

Compute: This will run the computation applying the stencil to the input data and outputting it to the appropriate output pointer.

X: Apply the stencil in the x direction.

Y: Apply the stencil in the y direction.

XY: Apply the stencil in the xy direction simultaneously (for situations with cross derivatives etc.). The library will account for corner halo data in this situation.

p: Apply the stencil with periodic boundary conditions.

np: Apply the stencil with non-periodic boundary conditions, this leaves suitable boundary cells untouched for the programmer to then apply their own boundary conditions.

Fun: Version of the function to be used if supplying a function pointer, otherwise leave blank.

The functions are then called in order of Create, Compute, Swap (if necessary) and then Destroy at the end of the program. Complete usage examples are found in the next section with further examples found in examples/src. The complete API can be found in the Doxygen documentation, see README on how to generate this.

IV. EXAMPLES

In this section we provide an overview of using library. We present three examples. The first is an implementation using linear stencil weights. The second involves a function pointer instead. The third example is at the level of a detailed physics problem (advection in Fluid Mechanics), and is included here to demonstrate to the user how to modify the source code as necessary. These three examples (and more) can be found in examples/src. The README provides compilation details. In all examples in the repository we take derivatives of various trigonometric functions as these are easy to benchmark against in periodic and non-periodic domains.

A. Standard Weights

We present here the example 2d_x_np.cu, it is recommended to have this example open in a text editor to follow along. In this example we implement an 8th order accurate central difference approximation to the second derivative of \( \sin(x) \) in the x direction. The domain has 1024 points in x and 512 points in y, set by nx and ny respectively with the domain size lx set to \( 2\pi \).

Unified memory is allocated with dataOld set to the input \( \sin(x) \) and answer set to \( -\sin(x) \), dataNew is zeroed to ensure correct output. We choose to implement this scheme on compute device 0 by setting deviceNum and implement the scheme using a single tile, setting numTiles to 1. The stencil is then implemented by setting the parameters numSten, numStenLeft and numStenRight along with providing an array of the stencil weights the same length as numSten. numSten is the total number of points in the stencil, in this case 9, while numStenLeft/Right are the number of points in the left and right of the stencil, both 4 in this case. A cuSten_t named xDirCompute is then declared and fed along with the above parameters into custenCreate2DXnp, this then equips cuSten_t with the necessary information. The ordering of parameters to be fed into custenCreate2DXnp can be found in both the Doxygen documentation and cuSten/src/struct/cuSten_struct_functions.h.

The computation is run using custenCompute2DXnp(&xDirCompute, 1) where the 1 indicates we wish to load the data back to the host memory after the computation is completed, 0 if you wish to leave it in device memory. Finally the result is output along with the expected answer to stdout, the 4 cells on either side in the x direction will be 0.0 due to the boundary, these would then be set by the programmer using suitable boundary conditions in a full solver. Then the custenDestroy2DXnp function is called to destroy the cuSten_t. Memory is then freed in the usual manner.
B. Function Pointer

Now we present the function pointer version of the above example, named 2d_x_np_fun.cu, again is it recommended to have a text editor open with the code to follow along. Many of the parameters are the same as before except this time we remove the weights and replace them with coefficients that are then fed into the function pointer by the library.

The function pointer in this case implements a standard second-order accurate central-difference approximation to the second derivative of \( \sin(x) \). We supply numSten, numStenLeft and numStenRight as before but now we also need numCoe to specify how many coefficients we need in our function pointer.

Our function pointer is of type devArg1X, where the 1 indicates how many input data sets are required. Each thread in a block will call the function and it returns the desired output value for that thread, each index in the array has one thread assigned to it. The inputs are pointers to the input data, the coefficients and the index location in the stencil

\[
\text{CentralDifference}(\text{double* data, double* coe, int loc})
\]

The central-difference scheme is implemented in a standard way with indexing done relative to loc, the coefficient in this case is set to \( 1.0/\Delta x^2 \) as is standard. A key point to notice it that the programmer must allocate memory for the function pointer on the device, this can be seen on line 131 and 132 of the example code prior to calling the Create function.

The rest of the access to the API is then the same as before except some of the inputs change and there is a ‘Fun’ at the end of each function name, for example custenCreate2DXnpFun. We will see later in Section V how function pointers provide us with a powerful tool to apply stencils to non-linear quantities, in particular we will see this with the cubic term of the Cahn–Hilliard equation to which we wish to apply a Laplacian.

C. Advection

The library also comes with an extra variant of the above functions 2d_xyADVWENO_p in which a 2D periodic advection WENO scheme has been implemented by modifying the 2DXYp source code. This is included as an example to show the user how to modify the source code as necessary to more specific needs or in situations where the function pointer may not meet requirements, for example in this situation where extra data needed to be input in the form of \( u \) and \( v \) velocities. The files can be found in the cuSten/src folder with how its called in examples/src/2d_xyWENOADV_p.cu.

A brief overview of the modifications made to the 2DXYp code are as follows:

- The stencil dimensions are now set automatically when the creation function is called.
- The \( u \) and \( v \) velocities were linked to the cuSten type with appropriate tiling.
- Additional asynchronous memory copies were included in the memory loading portion of the code to ensure the velocities are present on the device at the required time.
- The corner data copying to shared memory blocks was removed from the kernel as it is no longer required.
- The standard stencil compute was removed and replaced with a device function call to a WENO solver, details of the solver can be found in [2].

V. CUCAHNPENTADI

In this section we show how the cuSten library can be used as part of a larger solver that the authors developed using the cuPentBatch [13] solver, a batched pentadiagonal matrix solver. The equation we wished to solve was the 2D Cahn–Hilliard equation. The Cahn–Hilliard equation models phase separation in a binary liquid: when a binary fluid in which both components are initially well mixed undergoes rapid cooling below a critical temperature, both phases spontaneously separate to form regions rich in the fluid’s component parts. The regions expand over time in a phenomenon known as coarsening [14]. The equation is extremely well studied and is a popular model in polymer physics and interfacial flows.

In the mathematical framework, a single scalar concentration field \( C(x, t) \) characterizes the binary mixture. As such, a concentration level \( C = \pm 1 \) indicates phase separation of the mixture into one or other of its component
parts, while $C = 0$ denotes a perfectly mixed state. The free energy for the mixture can be modeled as $F[C] = \int_{\Omega} \left[ \frac{1}{2} (C^2 - 1)^2 + \frac{(1/2) \gamma |\nabla C|^2 \right] d^n x$, where the first term promotes demixing and the second term smooths out sharp gradients in transition zones between demixed regions; also, $\gamma$ is a positive constant, $\Omega$ is the container where the binary fluid resides, and $D$ is the dimension of the space. The twin constraints of mass conservation and energy minimization suggest a gradient-flow dynamics for the evolution of the concentration: $\partial_t C = \nabla \cdot \left[ D(C) \nabla (\delta F/\delta C) \right]$, where $\delta F/\delta C$ denotes the functional derivative of the free energy and $D(C) \geq 0$ is the mobility function, assumed for simplicity in this work to be a positive constant. As such, the basic model equation reads

$$\frac{\partial C}{\partial t} = D \nabla^2 \left( C^3 - C - \gamma \nabla^2 C \right), \quad x \in \Omega, \quad t > 0. \quad (1a)$$

The initial condition is given as

$$C(x, t = 0) = f(x), \quad x \in \Omega. \quad (1b)$$

### A. Discretisation

For simplicity, we focus on the case where $\Omega = (0, 2\pi)^D$, with periodic boundary conditions applied in each of the $D$ spatial dimensions. The method of solution we choose is based on the ADI method presented in [15] for the linear hyperdiffusion equation – we extend that scheme here and apply it to the nonlinear Cahn–Hilliard equation as follows:

$$L_xw = -\frac{2}{3} (C^n - C^{n-1}) - \frac{2}{3} \Delta t \nabla^4 C^{n+1} + \frac{2}{3} D \Delta t \nabla^2 \left( C^3 - C \right)^n \quad (2a)$$

$$L_yv = w \quad (2b)$$

$$C^{n+1} = \tilde{C}^{n+1} + v, \quad (2c)$$

Where $L_x = I + \frac{1}{3} D \gamma \Delta t \partial_{xxx}$ and similarly for $L_y$. In Equation (2), each one of the matrix inversions is solved using cuPentBatch as per the method presented in [13] and we transpose the matrix when changing from the $x$ direction to $y$ direction sweep to ensure the data is in the proper interleaved format. To deal with the periodic element of the inversion the method is the same as in Reference [13, 16].

To recover the initial $n - 1$ time step required by the above method we use a standard ADI method for the initial step which is given by:

$$\frac{C^{n+1/2}_{ij} - C^n_{ij}}{\Delta t/2} = -\Delta x^{-4} \left[ \delta_x^2 C^{n+1/2}_{ij} + 2 \delta_x \delta_y C^n_{ij} + \delta_y^2 C^n_{ij} \right] + D(\delta_x + \delta_y)(C^3 - C)^n, \quad (3a)$$

$$\frac{C^{n+1/2}_{ij} - C^{n+1/2}_{ij}}{\Delta t/2} = -\Delta y^{-4} \left[ \delta_x^2 C^{n+1/2}_{ij} + 2 \delta_x \delta_y C^{n+1/2}_{ij} + \delta_y^2 C^{n+1}_{ij} \right] + D(\delta_x + \delta_y)(C^3 - C)^{n+1/2}, \quad (3b)$$

where we have used the compact notation in $x$ (similarly in $y$)

$$\delta_x \phi_i = \phi_{i+1} - 2 \phi_i + \phi_{i-1}, \quad (4a)$$

$$\delta_y \phi_i = \phi_{i+1} - 4 \phi_i + 6 \phi_{i-1} - 4 \phi_{i-2} + \phi_{i-3}, \quad (4b)$$

and $\Delta = \Delta x = \Delta y$ for a uniform grid. These stencils are applied similarly in the full scheme above.

### B. Application of cuSten

The code for the example can be found with the repository in the cuCahnPentADI folder, supplied also in this folder is a makefile to compile the files and a python script to analyse the results which we present in Section [14]. cuSten is applied for all of the finite-difference elements of the code excluding the matrix inversion where we use cuPentBatch. Between lines 148 and 190 we can see an application of a more sophisticated function pointer than presented in Section [13]. Here we apply the Laplacian to the right-hand-side (RHS) non-linear term $C^3 - C$. The
coefficients are declared twice, once on lines 438 - 472 for the initial standard ADI method and then again on lines 756 - 773 for the full scheme. This shows a clear example of ease of use of the function pointers and the easy swap in/out of values. Note how the indexing starts from the top left of the stencil and sweeps left to right in i, row by row in j for indexing.

The linear terms for the RHS are all implemented using standard weighted schemes, noting the change in stencil shapes between the initial step (5x3 and 3x5) and the full scheme (5x5), this is one of the key advantages of the library of changing stencil shape and the boundaries are dealt with (in this case periodic). The additional static functions at the start of the file apply the time stepping parts of the algorithm and combination of terms to set the full RHS. Output is done using the standard HDF5 library, this is required for the cuPentBatchADI program but not the cuSten library itself.

\[ s(t) = \frac{1}{1 - \langle C^2 \rangle} \]  
\[ k_1(t) = \frac{\int d^n k \hat{C}^2}{\int d^n k \langle |k|^{-1} |\hat{C}|^2 \rangle} \]

FIG. 1. Plot showing \( s(t) \) and \( k_1 \) as functions of \( t \). We can see the clear \( t^{1/3} \) behaviour as expected in each.

C. Numerical Results

In order to analyse the performance of the code we use two standard tests to quantify the coarsening rate [17]. First we have the quantity \( s(t) \) which can be defined as

\[ s(t) = \frac{1}{1 - \langle C^2 \rangle} \]  \hspace{1cm} (5)

Where \( \langle \cdot \rangle \) denotes the spacial average, which we calculate by a simple integration over the domain using Simpson’s rule. Secondly we plot \( 1/k_1(t) \), which also captures the growth in length scales, where \( k_1 \) can be defined as

\[ k_1(t) = \frac{\int d^n k |\hat{C}|^2}{\int d^n k |k|^{-1} |\hat{C}|^2} \]  \hspace{1cm} (6)

with the hat denoting the Fourier Transform. We run the simulation to a final time \( T = 100 \) with \( n_x = n_y = 1024 \) points. The initial conditions are a random uniform distribution of values between \(-0.1\) and \(0.1\), we have set the coefficients \( D \) and \( \gamma \) to 0.6 and 0.01 respectively. The initial condition is chosen to mimic a ‘deep quench’, where the system is cooled suddenly below the critical temperature, which allows for phase separation to occur spontaneously [18]. The quantities \( s(t) \) and \( 1/k_1(t) \) are plotted in Figure as a function of \( t \) with a reference line of \( t^{1/3} \) included as both should scale proportionally to this. We can see clear match between our two quantities and \( t^{1/3} \). Plots of various
timesteps are included in Finite-size effects spoil the comparison between numerics and theory towards the end of the computation, as by that time the \((C = \pm 1)\)-regions fill out the computational domain. Figure 2 to illustrate the behaviour of the solution in space and time: the system clearly evolves into extended regions where \(C = \pm 1\), which grow over time, consistent with Figure 1 and the established theory [19].

![Contour plot of solution at various times](image)

**FIG. 2.** Contour plot of solution at various times

VI. DISCUSSION AND CONCLUSIONS

A. Possible Future Extensions to the Library

As previously mentioned the current library is limited to 2D uniform grids with double precision. Future areas of expansion could include moving the current library functions into C++ templates, this would allow for easier generalisation to other data types without the current need for find and replace to be done manually. Expansion to 3D and non-uniform grids is less trivial. 3D would require a different approach to loading data than currently implemented as data will not be contiguous in RAM in the z direction, a more sophisticated loading scheme with pointers would be required. For non-uniform grids additional data would need to be loaded into memory, it is likely in this situation that a hybrid of modifying the code such as in the WENO example to have extra data available (\(u\) and \(v\) velocities in the case of WENO, coordinate transformations in the case of a non-uniform grid) and using function pointers would be the best approach to make to the existing source.
B. MPI

The design of the library lends itself to an MPI domain decomposition to be used in a hybrid code with the cuSten library. Each MPI process could be assigned to a GPU using the deviceNum parameter, then the user would apply the non periodic versions of the stencils along with using MPI to swap the boundary halos. Memory exchange is simplified in MPI due to the use of Unified Memory, the required data will be copied directly between GPU devices. This allows for the application of this library in much larger solvers which require more than just a single GPU.

C. Concluding Remarks

In this paper we have shown how cuSten can be used to simplify the implementation of finite difference programs in CUDA. cuSten has a lightweight interface with a minimal learning curve required to implement the functions as part of a wider project. The library has been benchmarked in a Cahn–Hilliard solver and numerous examples are provided to show potential users how to use the functionality provided. It has wide ranging applications in finite-difference solver development and in further areas requiring stencil based operations such as image processing and optimisation problems.

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VII. BIBLIOGRAPHY

[1] Charles R Doering and John D Gibbon. Applied analysis of the Navier-Stokes equations, volume 12. Cambridge University Press, 1995.
[2] Stanley Osher and Ronald Fedkiw. Level set methods and dynamic implicit surfaces, volume 153. Springer Science & Business Media, 2006.
[3] Jan S Hesthaven. Numerical methods for conservation laws: From analysis to algorithms, volume 18. SIAM, 2018.
[4] Paul Wilmott, Sam Howison, and Jeff Dewynne. The Mathematics of Financial Derivatives: A Student Introduction. Cambridge University Press, 1995.
[5] Gerald Beresford Whitham. Linear and nonlinear waves, volume 42. John Wiley & Sons, 2011.
[6] Paulius Micikevicius. 3d finite difference computation on gpus using cuda. In Proceedings of 2nd Workshop on General Purpose Processing on Graphics Processing Units, ACM International Conference Proceeding Series, pages 79–84. ACM, 2009.
[7] David Micha and Dimitri Komatitsch. Accelerating a three-dimensional finite-difference wave propagation code using gpu graphics cards. Geophysical Journal International, 182(1):389–402, 2010.
[8] Andreas Schfer and Dietmar Fey. High performance stencil code algorithms for gpgpus. Procedia Computer Science, 4:2027 – 2036, 2011. Proceedings of the International Conference on Computational Science, ICCS 2011.
[9] Dheevatsa Mudigere. Data access optimized applications on the gpu using nvidia cuda. Master’s Thesis, Technische Universität München, 2009.
[10] Andreas Schäfer and Dietmar Fey. Libgeodecomp: A grid-enabled library for geometric decomposition codes. In Proceedings of the 15th European PVM/MPI Users’ Group Meeting on Recent Advances in Parallel Virtual Machine and Message Passing Interface, pages 285–294, Berlin, Heidelberg, 2008. Springer-Verlag.
[11] Yongpeng Zhang and Frank Mueller. Auto-generation and auto-tuning of 3d stencil codes on gpu clusters. In Proceedings of the Tenth International Symposium on Code Generation and Optimization, CGO ’12, pages 155–164, New York, NY, USA, 2012. ACM.
[12] Justin Holewinski, Louis-Noël Pouchet, and P. Sadayappan. High-performance code generation for stencil computations on gpu architectures. In Proceedings of the 26th ACM International Conference on Supercomputing, ICS ’12, pages 311–320, New York, NY, USA, 2012. ACM.
[13] Andrew Gloster, Lennon O’Naraigh, and Khang Ee Pang. cuPentBatch – A batched pentadiagonal solver for NVIDIA GPUs. arXiv e-prints, page arXiv:1807.07382, Jul 2018.
[14] J. W. Cahn and J. E. Hilliard. Free energy of a nonuniform system. i. interfacial energy. J. Chem. Phys, 28:258–267, 1957.
[15] Richard M. Beam and R. F. Warming. Alternating direction implicit methods for parabolic equations with a mixed derivative. *Society for Industrial and Applied Mathematics. SIAM Journal on Scientific and Statistical Computing*, 1(1):131–29, 03 1980. Copyright - Copyright] 1980 Society for Industrial and Applied Mathematics; Last updated - 2012-02-18.

[16] I. M. Navon. Pent: A periodic pentadiagonal systems solver. *Communications in Applied Numerical Methods*, 3(1):63–69, 1987.

[17] Aurore Naso and Lennon Nraigh. A flow-pattern map for phase separation using the navierstokescahnhilliard model. *European Journal of Mechanics - B/Fluids*, 72:576 – 585, 2018.

[18] J. Zhu, L. Q. Shen, J. Shen, V. Tikare, and A. Onuki. Coarsening kinetics from a variable mobility cahn–hilliard equation: Application of a semi-implicit fourier spectral method. *Phys. Rev. E*, 60:3564–3572, 1999.

[19] I. M. Lifshitz and V. V. Slyozov. The kinetics of precipitation from supersaturated solid solutions. *J. Chem. Phys. Solids*, 19:35–50, 1961.