A quantum-network approach to spin interferometry driven by Abelian and non-Abelian fields

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Abstract. We present a theory of conducting quantum networks that accounts for Abelian and non-Abelian fields acting on spin carriers. We apply this approach to model the conductance of mesoscopic spin interferometers of different geometry (such as squares and rings), reproducing recent experimental findings in patterned InAsGa quantum wells subject to Rashba spin-orbit and Zeeman fields. Moreover, by introducing some additional field-texture engineering, we manage to single out a previously unnoticed spin-phase suppression mechanisms. We observe that our approach also applies to the study of complex networks and the spectral properties of closed systems.
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1. Introduction

Coherent spin transport in quantum electronics has attracted a great deal of attention that accompanied the development of mesoscopic physics along the last decades. This interest runs from the study of fundamental spin-based phenomena (such as weak antilocalization [1] and geometric/topological spin phases [2, 3]) to proposal for spintronic applications (such as spin field-effect transistors [4, 5, 6] and, more recently, spin-based platforms for topological quantum computing [7]). A common ingredient here is the role played by Abelian and non-Abelian phases produced by the carriers’ spin dynamics under the action of magnetic textures originated from either (i) purely magnetic sources, such as micromagnetic arrays leading to inhomogeneous Zeeman coupling [8, 9], (ii) purely electric sources, leading to spin-orbit interaction such as Rashba or Dresselhaus couplings, or (iii) hybrid sources, combining magnetic and spin-orbit fields.

The modelling of spin-dependent transport in mesoscopic systems can demand significant numerical efforts. The most popular technique has been the recursive construction of Green’s functions from tight-binding Hamiltonians [10], a very reliable method that sometimes turns out expensive in computational terms. More recently, the development of the open-source Kwant code [11], based on a wave-function scattering approach, represented a major step towards computational efficiency and stability with excellent results. Alternatively, particular systems characterized by an underlying network structure can be simulated by introducing more specific techniques based on a quantum-graph approach [12, 13, 14, 15, 16, 17, 18, 19, 20].
In this article we extend the quantum-network approach to model the transport properties of mesoscopic spin interferometers of different geometries, subjected to hybrid Zeeman and Rashba field textures. The geometric structure is a key feature, since the carriers’ spin dynamics and the corresponding spin-phase gathering are sensitive to it. The work is motivated by a series of experiments on InGaAs-based circuits performed by Nitta’s group [21, 22, 23, 24] showing evidence of electronic spin manipulation by geometric means in Aharonov-Casher interferometry. One experiment reveals the manipulation of the geometric spin phase independently of the dynamical spin phase in Rashba rings [22]. More recently, topological spin-phase transitions in polygonal Rashba circuits have been reported [24]. Here, we model those experimental settings and reproduce their results by means of a quantum-network model. Moreover, we take some steps forward and identify novel interferometric characteristics by proposing some field-texture engineering. We also validate the results of our quantum-network approach by performing corresponding numerical simulations on tight-binding models.

The paper is organized as follows. In Sec. 2 we take a quantum-network approach to model a one-dimensional (1D) circular loop as a regular polygon with a large number of vertices by following the method described in Ref. [19]. The method is generalized by introducing additional in-plane Zeeman fields beyond the perturbative approximation. This approach provides a full quantum mechanical solution for the propagation of spin carriers inside the polygonal structure: (1) all possible propagating paths are accounted and (2) there are no particular constraints imposed on the scattering matrix at the injector and collector nodes. In Sec. 3 we derive the conductance as a function of the Rashba spin-orbit coupling strength and the Zeeman field and compare the obtained results to the experimental data and perturbative methods. We point out a special condition for a magnetic field perpendicular to one of the wires and magnitude of Zeeman term proportional to Rashba spin-orbit coupling and study the effects on phase and transport properties. Sec. 4 is devoted to compare the results obtained by the quantum-network method with those obtained by using a numerical tight-binding approach, also for disordered systems. We present a short summary in Sec. 5 where we analyze the strength of the quantum network approach.

2. Model and Formalism

We will evaluate the transport properties of ring-like structures by employing the formalism of quantum networks (QNs). We will mainly focus on regular polygons and approximate a ring as being a polygon with a large number of edges — see Fig. 1. In general terms, a metric network (or graph), is a collection of nodes (or vertices), connected by edges (or 1D intervals) of specified lengths [25]. In the graph terminology, regular polygons are also known as \textit{2-regular graphs} [16]. A quantum network is a metric graph equipped with a Schrödinger operator [26, 27]. The wave function of the QN satisfies boundary conditions at the vertices, which ensure the continuity (uniqueness) of the wave function and the conservation of the probability current. The fulfilment
Figure 1. Sketch of the polygonal structures that we consider for the quantum transport: (a) square, (b) hexagon, (c) octagon, (d) ring. In panels (e) and (f) we show the square and the ring of the discretized tight-binding version that we use in Sec. 4.

of these boundary conditions guarantees that the resulting Schrödinger operator is Hermitian [13, 14, 27, 28]. The continuity condition implies that the wave function assumes a certain value at a vertex, regardless of the bond from which it is approached. The concept of extended normal derivative must be introduced to apply the second boundary condition, whose definition might vary depending on the differential operator, i.e., it depends on the presence of a magnetic field or a spin-orbit coupling (SOC). Current conservation implies that the sum of the outgoing extended derivatives at each vertex vanishes [13, 16].

In the following subsections, we consider a QN composed of single mode quantum wires (QWs) subject to Rashba SOC (RSOC) and to magnetic fields. In each case, a different Schrödinger operator (Hamiltonian) and the corresponding extended derivative are defined.

We start by introducing two different spin-dependent Hamiltonians for the QWs composing the QN. Each Hamiltonian will be solved by using a spinorial wave function ansatz of the following form

$$\Psi = e^{ikr} \begin{pmatrix} \chi_A \\ \chi_B \end{pmatrix},$$

where $\chi_{A/B}$ are the two components of the spinor, $k$ is the electronic momentum and $r$ is the local coordinate along the QW.

2.1. Case of RSOC & orbital magnetic field

Let us consider a QW that lies in the $xy$-plane; the wire points in the generic direction $\hat{\gamma} = (\cos \gamma, \sin \gamma, 0)$. The QW is subject to RSOC and to a magnetic field, perpendicular to the plane $xy$-plane, that interacts with the spin carriers through minimal coupling.
The wire Hamiltonian reads

\[ \hat{H} = \frac{1}{2m^*}(p + eA)^2 + \frac{\hbar k_R}{m^*} [(p + eA) \times \hat{z}] \cdot \sigma, \]

where \( k_R \) is the coupling constant of the RSOC (in inverse-length units), \( \hat{z} \) is the unit vector along the \( z \) axis, \( \sigma \) is the vector of the Pauli matrices describing the electron spin, and \( m^* \) is the electron effective mass. The RSOC strength \( k_R \) is related to the spin precession length \( L_{SO} \) by \( L_{SO} = \pi/k_R \). The wave function of the QW, fulfilling the Dirichlet boundary condition, can be written as a function of the values that the wave function takes at its vertices \( \alpha \) and \( \beta \):

\[ \Psi(r) = e^{-i\varphi(r)}e^{-i(\hat{\gamma} \times \hat{z}) \cdot \sigma k_R r} \begin{bmatrix} \sin(k(\ell - r))\Psi_\alpha + \sin(kr)e^{ikr(\ell)}e^{i(\hat{\gamma} \times \hat{z}) \cdot \sigma k_R \ell} \Psi_\beta \end{bmatrix}, \]

where \( \ell \) is the length of the quantum wire, \( r \) is the local coordinate along the QW measured from vertex \( \alpha \), and the momentum \( k \) is related to the energy \( \epsilon \) as \( k = \sqrt{2m\epsilon/\hbar^2 + k_R^2} \). The spinors \( \Psi_\alpha \) and \( \Psi_\beta \) are the values of the wave function at the vertices \( \alpha \) and \( \beta \), respectively. In Eq. (3), we have introduced a \( U(1) \) phase factor \( \varphi(r) \) related to the magnetic field via the vector potential \( A \):

\[ \varphi(r) = \frac{2\pi}{\phi_0} \int_\alpha^r dr' \cdot A(r'), \]

where \( \phi_0 = \hbar/e \) is the flux quantum. This phase eventually leads to the Aharonov-Bohm (AB) effect [29] in closed loops. The expression in Eq. (4) is proportional to the circulation of the vector potential between vertex \( \alpha \) and point \( r \) [27, 15]. The second phase factor in Eq. (3) is a SU(2) phase due to the RSOC. In closed loops it leads to the Aharonov-Casher (AC) effect [30] (electromagnetic dual of the AB effect) arising from the spin precession driven by the Rashba field [17, 18, 31].

The probability current corresponding to Hamiltonian (2) is

\[ j = -i\frac{\hbar}{2m^*} \left\{ \psi^\dagger \frac{\partial \psi}{\partial r} - \frac{\partial \psi^\dagger}{\partial r} \psi + 2i\frac{e}{\hbar} (\hat{\gamma} \cdot A) \psi^\dagger \psi + 2ik_R \psi^\dagger [(\hat{\gamma} \times \hat{z}) \cdot \sigma] \psi \right\}, \]

this expression hints that the probability current is not conserved by the continuity of the derivative of the wave function. However, the continuity of the extended derivative

\[ \frac{\partial}{\partial r} \rightarrow D = \frac{\partial}{\partial r} + i\frac{e}{\hbar} \hat{\gamma} \cdot A + ik_R (\hat{\gamma} \times \hat{z}) \cdot \sigma. \]

does ensure the conservation of the probability current.

Once the wave function of all QWs is written as in Eq. (3), the conservation of probability current using the extended derivative is imposed at the vertices. Solving the resulting equation system provides the value of the spinors at all the vertices \( \Psi_\alpha \), and \( \Psi(r) \) by extension.

2.2. Case of RSOC & Zeeman field

We now consider a QW subject to RSOC and an in-plane Zeeman field in the generic direction \( \hat{\alpha} \), where \( B = B(\cos \alpha, \sin \alpha, 0) \). The system Hamiltonian reads

\[ \hat{H} = \frac{p^2}{2m^*} + \frac{\hbar k_R}{m^*} (p \times z) \cdot \sigma + \mu B \cdot \sigma, \]
where $\mu$ is the Bohr magneton. Unlike the RSOC term, the Zeeman term does not depend on momentum and thus it breaks time-reversal symmetry (TRS).

Using the spinorial wave function ansatz in Eq. (1), the Hamiltonian can be cast into the following matrix:

$$\hat{H} = \left( \begin{array}{cc} \frac{\hbar^2 k^2}{2m} & \mathcal{M}^* \\
\mathcal{M} & \frac{\hbar^2 k^2}{2m^2} \end{array} \right),$$

(8)

where $\mathcal{M} = (\mu B \cos \alpha + \frac{\hbar^2 k_R k}{m} \sin \gamma) + i(\mu B \sin \alpha - \frac{\hbar^2 k_R k}{m} \cos \gamma)$. Hamiltonian (8) has the following eigenvalues and eigenvectors

$$\epsilon_\pm = \frac{\hbar^2 k^2}{2m^*} \pm |\mathcal{M}|,$$

(9)

$$|v_\pm\rangle = \frac{1}{\sqrt{2}} \left( e^{-i\theta/2} \pm e^{i\theta/2} \right),$$

(10)

where $\theta$ is the argument of $\mathcal{M}$. The eigenvectors $|v_\pm\rangle$ lie within the $xy$-plane and remain constant along the wire.

The energy spectrum of the quantum wire is given by Eq. (9), and consists of two energy bands due to the spin splitting induced by the RSOC and Zeeman interactions. The energy bands for the two limiting cases ($k_R = 0$ and $B = 0$) are shown in Fig. 2.

It is common knowledge that the dispersion relation in the presence of a Zeeman field is shifted vertically for opposite spins, while the RSOC shifts it horizontally [31]. In general, the interplay between both fields will result in a more complicated spectrum.

The method described in this section is valid when the Fermi energy $E_F$ lies above the pseudogap, so that there are four available scattering states for a given energy, two propagating forward and two backward. By pseudogap we mean the splitting induced by the Zeeman field, which is equal to $\Delta_B = 2\mu|B|$ at $k = 0$. For example, as we can see in the left panel of Fig. 2, for certain values of the energy, the only available states are those of the lower band. We label the momenta and arguments corresponding to the four states as $k_{f/b}^\pm$ and $\theta_{f/b}^\pm$, respectively. Here, the superscript f/b indicates whether the state propagates forward/backward, and the subscript $\pm$ indicates the energy band.

Any state with a given energy higher than the pseudogap can be written as a linear combination of these four available propagating states. A state with a defined propagation direction can be written as follows,

$$\Psi(r) = \hat{R}e^{i\bar{k}r}\Psi(0)$$

(11)

$$= \cos^{-1}\left( \frac{\Delta \theta}{2} \right) \left( \begin{array}{cc} \cos \frac{\Delta k r - \Delta \theta}{2} & i e^{i\bar{\theta}} \sin \frac{\Delta k r}{2} \\
e^{i\bar{\theta}} \sin \frac{\Delta k r}{2} & \cos \frac{\Delta k r + \Delta \theta}{2} \end{array} \right) e^{i\bar{k}r}\Psi(0).$$

(12)

For the sake of simplicity, the superscript f/b has been omitted in Eqs. (11-12). Here we have introduced the elements $\bar{k} = \frac{k_+ + k_-}{2}$, $\Delta k = k_+ - k_-$, $\bar{\theta} = \frac{\theta_+ + \theta_-}{2}$ and $\Delta \theta = \theta_+ - \theta_-$. Note that these four elements will be different for states propagating forward or backward in general. The matrix $\hat{R}$ is the spin evolution matrix, that relates
Figure 2. Energy spectrum as a function of momentum for $k_{SO} = 0$ (left) and $B = 0$ (right). In the left panel, $\Delta_B$ is the pseudo-gap opened by the Zeeman field at zero momentum. The four propagating states at fixed energy are labelled by $k_{\pm}^{f/b}$.

the value of the wave function at point $r$ to the initial value of the wave function at $r = 0$.

In general, the wave function of a QW will be a linear combination of the counter-propagating waves,

$$
\Psi(r) = \Psi^f(r) + \Psi^b(r)
= \hat{R}^f e^{i\bar{k}_fr} \Psi^f(0) + \hat{R}^b e^{i\bar{k}_br} \Psi^b(0).
$$

The spinors $\Psi^f(0)$ and $\Psi^b(0)$ are unknown constants that are fixed by applying the boundary conditions.

When the Zeeman field is zero, the forward and backward elements satisfy the relations $\bar{k}^b = -\bar{k}^f$, $\Delta\bar{k}^b = -\Delta\bar{k}^f$, $\bar{\theta}^b = \bar{\theta}^f + \pi$ and $\Delta\bar{\theta}^f = \Delta\bar{\theta}^b = 0$. After rearranging $\Psi^{f/b}(0)$ in terms of $\Psi(0)$ and $\Psi(\ell)$, the wave function can be written as

$$
\Psi(r) = \frac{\hat{R}(r)}{\sin(k\ell)}[\sin k(\ell - r)\Psi(0) + \sin (kr)\hat{R}^{-1}(\ell)\Psi(\ell)],
$$

where now the spin evolution matrix $\hat{R}$ coincides with the SU(2) phase factor in Eq. (3). When the Rashba term is zero and we have only the Zeeman term, which breaks TRS, we can still express the wave function with a structure similar to Eq. (15), with the spin rotation matrix that now reads:

$$
\hat{R}(r) = e^{i\bar{\alpha} \cdot \sigma \frac{\Delta\bar{\theta}^f}{2}}.
$$

Importantly we note that an energy band crossing occurs when the Zeeman field is perpendicular to the wire and the modulus is equal to the critical field $B = \frac{\hbar^2 k_m}{\mu}$. At this critical value, the effective magnetic field created by the RSOC cancels with the inplane magnetic field, so the only contribution to the energy comes from the kinetic term. This effect is very similar to the effect that arises in systems with Rashba and Dresselhaus SOCs when the strength of both SOCs are equal [32, 33, 34]. Both energy bands cross
for a certain momentum direction because the effective magnetic field vanishes. In this case, $M = 0$, so the angles $\theta_{\pm}$ for the two-fold degenerate solutions are not defined. A careful analysis shows that the spin evolution matrix is equal to the identity matrix. This comes as no surprise, since the SU(2) terms in the Hamiltonian cancel each other out. Therefore, the spatial evolution of the state is simply given by the dynamic phase factor $e^{i k r}$ that arises from the kinetic term.

Equations (13-14) are the key step to generalize the methods that study QNs when an inplane Zeeman field is applied. Boundary conditions are then applied at the vertices of the wire in order to obtain $\Psi^U/b(0)$, and $\Psi(r)$ by extension. The Zeeman term does not contribute any additional term to the extended derivative, so it is given by (6) with $A = 0$.

2.3. Formalism for quantum transport

To study the transport properties of the QNs, we attach semi-infinite input and output leads to the vertices of the network [15, 17, 19]. Each lead consists of a quantum wire with two spin channels. The leads are not subjected to any interaction, so they are characterized at zero temperature by the Fermi energy and a wave vector $k$. We assume that the leads are connected to uncorrelated reservoirs, so that there are no phase relationships among electrons in different channels [35].

In a system with $N_{\text{in}}$ ($N_{\text{out}}$) input (output) channels, if an electron is injected through input channel $\sigma$ with wavenumber $k$, the wave function alongside the channels can be written as

$$\Psi_{\text{in},\sigma'}(r) = e^{i k r} \delta_{\sigma' \sigma} + r_{\sigma' \sigma} e^{-i k r},$$

$$\Psi_{\text{out},\sigma'}(r) = t_{\sigma' \sigma} e^{i k r},$$

where $r$ is the position measured from the edge, and it is negative for input leads and positive for output leads. Here $r_{\sigma' \sigma}$ and $t_{\sigma' \sigma}$ are the channel-resolved reflection and transmission coefficients respectively, so that $\sum_{\sigma'} N_{\text{in}} |r_{\sigma' \sigma}|^2 + \sum_{\sigma'} N_{\text{out}} |t_{\sigma' \sigma}|^2 = 1$. The indices $\sigma$ and $\sigma'$ specify both the lead and the spin state of the channel. We define the total transmission and reflection coefficients of a channel $\sigma'$ as

$$T_{\sigma'} = \sum_{\sigma} |t_{\sigma' \sigma}|^2, \quad R_{\sigma'} = \sum_{\sigma} |r_{\sigma' \sigma}|^2.$$

where the sum runs over the input channels. The total transmission (reflection) is given by the sum of the transmission (reflection) coefficients of the output (input) channels,

$$T = \sum_{\sigma'} T_{\sigma'} = \sum_{\sigma' \sigma} |t_{\sigma' \sigma}|^2,$$

$$R = \sum_{\sigma'} R_{\sigma'} = \sum_{\sigma' \sigma} |r_{\sigma' \sigma}|^2.$$

The zero-temperature conductance $G$ based on the Landauer formula reads [36]:

$$G = \frac{e^2}{h} \text{Tr} \left[ tt^\dagger \right] = \frac{e^2}{h} T.$$
It is clear from the previous expression that the conductance is bounded by the number of input channels, such that $G \leq N_{in}e^2/h$.

The derivatives of the wave function in the leads must also be taken into account when imposing the conservation of probability current. In an isolated quantum network, by imposing the continuity of the wave function and the conservation of the probability current we obtain a set of linear homogeneous equations where the variables are the values of the wave function at the vertices. This allows us to study the spectral properties of the quantum network via a secular equation [27, 16].

When adding the external leads, the energy of the system is fixed by the Fermi energy of the leads. The transmission and reflection coefficients can be written in terms of the values of the wave function at the contacts. Due to the first term in the rhs of Eq. (17), the set of equations becomes inhomogeneous, with a unique solution for $T$ and $R$.

If there is no Zeeman field, the wave function of the network is described by the values it takes at the vertices $\Psi_\alpha$ [see Eqs. (3)]. For each input (output) lead there are two reflection (transmission) coefficients, one per spin channel. In order to satisfy the single-valuedness of the wave function at the vertices connected to external leads, one can write the reflection and transmission coefficients of the leads as a function of $\Psi_\alpha$. The number of variables of the problem is then equal to the number of vertices $V$. At each vertex the sum of the outgoing extended derivatives must be equal to zero, so there are $V$ equations that impose the continuity of probability current. These equations fix the values of $\Psi_\alpha$, and consequently the reflection/transmission coefficients.

When the Zeeman field is finite, the wave function is described by spinors $\Psi^{f}_{\alpha,\beta}(0)$ and $\Psi^{b}_{\alpha,\beta}(0)$, which specify the wave function of a bond at one of its endpoints [see Eq. (14)]. The subscripts indicate that the QW is connected to vertices $\alpha$ and $\beta$. Together with the reflection/transmission coefficients, there are $2N + N_{ext}$ unknown variables, where $N$ is the number of edges of the quantum network and $N_{ext}$ is the number of input/output leads. For a vertex $\alpha$ connected to $N_{\alpha}$ edges, we can write $N_{\alpha} - 1$ equations that impose the single-valuedness of the wave function. The total number of equations that verify the continuity of the wave function at the edges of the QN are $2N + N_{ext} - V$. In addition, at each vertex the sum of the outgoing extended derivatives must be equal to zero. In total there are $2N + N_{ext}$ equations that fix the values of the spinors and the transmission/reflection coefficients.

For a generic vertex $\alpha$, the continuity of the probability current reads

$$\sum_{(\alpha,\beta)} D\Psi_{\alpha,\beta}(r)|_{r=0} = 0,$$

where the sum $\sum_{(\alpha,\beta)}$ runs over all vertices $\beta$ which are connected to $\alpha$.

Equation (23) can be expressed in terms of $\Psi^{f}_{\alpha,\beta}(0)$ and $\Psi^{b}_{\alpha,\beta}(0)$ using Eq. (14). In this case, the equation for the internal vertices is

$$\sum_{\delta \in \{f,b\}} \sum_{(\alpha,\beta)} M_{\alpha,\beta}^{\delta} \Psi^{\delta}_{\alpha,\beta}(0) = 0.$$

(24)
In the case where the QW is subject to RSOC and a Zeeman field,

\[ M_{\alpha,\beta}^\delta = i\alpha + \frac{\Delta k^\delta}{2} \tan \frac{\Delta \beta}{2} \sigma_z \]

\[ + i \left( \frac{\Delta k^\delta}{2} \frac{1}{\cos \frac{\Delta \beta}{2}} \hat{\theta} + k_R (\hat{\gamma} \times \hat{z}) \right) \cdot \sigma. \] (26)

Consider a quantum network with a single input and output leads. If an electron with spin \( \sigma \) is injected along the input lead, the equations for the external vertices read

\[ \sum_{\delta \in \{f, b\}} \sum_{\langle \alpha, \beta \rangle} M_{\alpha,\beta}^\delta \Psi_{\alpha,\beta}^\delta(0) = i k \chi_{\sigma} - i k \sum_{\sigma'} r_{\sigma'\sigma} \chi_{\sigma'} \] (27)

\[ \sum_{\delta \in \{f, b\}} \sum_{\langle \alpha, \beta \rangle} M_{\alpha,\beta}^\delta \Psi_{\alpha,\beta}^\delta(0) = -i k \sum_{\sigma'} t_{\sigma'\sigma} \chi_{\sigma'} \] (28)

Coefficients \( r_{\sigma'\sigma} \) and \( t_{\sigma'\sigma} \) can be expressed as a linear combination of \( \Psi_{\alpha,\beta}^\delta(0) \) by applying the continuity of the wave function. Together with the equations that impose the continuity of the wave function of the internal edges, we obtain an inhomogeneous system of linear equations with 4N variables (2 per spinor): the inhomogeneous term arises due to the first term on the rhs of Eq. (27). The system can be solved numerically to obtain the value of the spinor of each bond at the local coordinate \( r = 0 \). Once \( \Psi_{\alpha,\beta}^\delta(0) \) are obtained, it is straightforward to compute \( r_{\sigma'\sigma} \) and \( t_{\sigma'\sigma} \). Furthermore, Eq. (14) provides the value of the wave function at any given point. In the case where the Zeeman term is zero, the boundary conditions for the leads reduce to the known cases in Ref. [15, 17, 18]

\[ M_{\alpha,\alpha} \Psi_{\alpha} + \sum_{\langle \alpha, \beta \rangle} M_{\alpha,\beta} \Psi_{\beta} = 0. \] (29)

3. Results

In this section we study the transport properties of different polygons using the formalism we have introduced in the previous section. We consider a series of regular polygons of constant perimeter \( P \) with an even number of vertices, each polygon is connected to an input and output field-free leads at opposite vertices — see Fig. 1. We evaluate the system conductance from the transmission probability making use of the Landauer-Büttiker formalism [37] as dictated by Eq. (22). Taking the number of edges to infinity, the series of regular polygons converges to a circle, so we recover the conductance for a ring [19] — this coincides with the one obtained analytically [38].

The average momentum of incoming electrons must fulfill certain limits. As first, the Fermi wavelength of the electron must be much smaller than the natural length of the system. Smaller wave numbers are more prone to decoherence [22]. In our system, the upper bound for the wavelength is given by the perimeter of the polygon:

\[ k \gg \frac{2\pi}{P}. \] (30)
Figure 3. Average conductance $\langle G \rangle_k$ in units of $2e^2/h$ for various polygons: (a) square, (b) hexagon, (c) octagon and (d) ring, subject to an orbital magnetic field $\phi/\phi_0$ and RSOC $k_R P/(2\pi)$.

The wavelength is also limited by the size of the edges of the polygon. On the one hand, the wavelength of the electron must be smaller than the length of the edges of the polygon so that multiple wavelengths fit in one edge. Otherwise, the electron would not “see” the edge. On the other hand, in the specific case of the ring, the requirement is just the opposite, the wavelength must be much greater than the length of the edges of the polygon, so that each edge is hardly noticed by the electron and the polygon can really be treated as a ring:

$$k \ll \frac{2\pi N}{P} = \frac{2\pi}{L}, \quad (31)$$

where $N$ is the number of edges of the polygon used to simulate the ring, and $L$ is the length of each edge. Obviously, the greater the number of edges, the more will the results for the polygon resemble those of the ring. The spin precession length $L_{SO}$ should also be much larger than the length of the edges for the same reason.

3.1. Case of RSOC & orbital magnetic field

We first study the transport properties of polygons subject to orbital magnetic field and RSOC. The average conductance for multiple polygons averaged over a range of values of $k_{in} \langle G \rangle_k$ for unpolarized electrons is shown in Fig. 3. The conductance is given in units of $2e^2/h$. We have performed an energy average to smooth the energy-dependent oscillations [38, 17]. The results for the ring were obtained by using a polygon with 100 edges.

The results presented in Fig. 3 are a perfect example for highlighting the difference between an Abelian and a non-Abelian gauge field, due to the orbital magnetic field and the RSOC, respectively. The conductance shows periodic AB oscillations, where the period is the flux quantum $\phi_0$ for all the polygons. The maxima correspond to the constructive quantum interference of the electrons travelling through different paths. For example, in the Rashba field free limit, the constructive interference occurs for integer multiples of the flux quantum. The AB phase acquired by an electron when moving around the polygon is $2\pi \phi/\phi_0$. Adding the contributions of all the possible
paths gives rise to the interference pattern. The phase acquired by a particle moving through the shortest possible paths (clockwise and counterclockwise paths) will have the same magnitude, but opposite sign. For integer multiples of the flux quantum, the phase difference between the two paths is an integer multiple of $2\pi$ resulting in a constructive interference. However, the conductance is not exactly $2e^2/h$. The longer paths will also contribute to the transmission amplitude, each of which have a different dynamical phase. On the other side, for half integer multiples of the flux quantum, the contribution of the two opposite paths to the transmission amplitude are in counterphase, so the conductance drops to zero.

The RSOC modifies the phase acquired by the electrons when travelling through the polygon; this leads to a shift of the position of the conductance maxima with respect to the magnetic flux. At $k_R P = 2\pi$, the conductance maxima appear at half integer multiples of $\phi_0$, while the conductance vanishes for integer multiples of the flux quantum. The additional phase arises from the electron spin precession around the effective magnetic field arising from the SO coupling, i.e. it acquires a non-Abelian SU(2) phase resulting in the Aharonov-Casher effect [30].

The oscillations of the conductance show a quasiperiodic dependence on the RSOC. There are broader and narrower maxima for different values of $k_R P$ associated to two different frequencies that give rise to the oscillatory pattern. The periodicity of the broader maxima is related to the length of the edges of the polygon, where the period is $N\pi$. This period tends to infinity as the number of edges tends to infinity, so there are no broad maxima for the ring, apart from the one located at the origin.

The periodicity of the narrow maxima is related to the length of the perimeter, therefore, it has a weaker dependence on the number of edges of the polygon. The quasiperiod ranges from $4\pi$ for the case of the square to $2\pi$ as the number of edges and the RSOC strength increase. Oscillations of period $2\pi$ are identified with the adiabatic limit: when the dimensionless RSOC tends to infinity, we obtain the adiabatic limit in which the spin is aligned with the effective magnetic field during transport, and Berry phases arise [19, 38]. Adiabatic spin transport is never really achieved in polygons, where vertices act as spin-scattering centers due to the abrupt change of direction of the RSOC at the vertices of polygons.

### 3.2. Case of RSOC & Zeeman field

In this section we study the interplay between Zeeman and AC phases using the method described in Sec. 2.2. The quantum wires of the network are subject to RSOC and an inplane Zeeman field, which breaks TRS. The conductance for different polygons is shown in Fig. 4. The dashed line represents the critical line, the points where the applied Zeeman field is equal in magnitude to the effective magnetic field created by the RSOC, $B_{SO} = \frac{\hbar k_{th}}{m_f}$, which can be obtained from direct comparison between the second and third terms in Eq. (7). The dimensionless RSOC and Zeeman couplings are given by $k_R P/(2\pi)$ and $m\mu B P/(2\pi k h^2)$, respectively.
Figure 4. Average conductance $\langle G \rangle_k$ in units of $2e^2/h$ for $\alpha = 0$ for various polygons: (a) square, (b) hexagon, (c) octagon and (d) ring. The dashed line corresponds to the critical line, where $B_{SO} = B$.

For the ring, as the applied Zeeman field is increased, the AC oscillations shift to weaker values of the SO field. These results are consistent with the ones obtained by Nitta et al. experimentally in InGaAs-based quantum ring arrays [22]. In the experiment, the periodicity of the magnetoresistance oscillations is halved due to the Altshuler-Aronov-Spivak (AAS) effect, the AB effect in the time-reversal paths [39]. We recover this result in the presence of disorder — see Sec. 3.3.

The RSOC field $B_{SO}$ is radial to the ring, while the in-plane Zeeman field is homogeneous. Both fields lie on the $xy$ plane, so the solid angle subtended by the magnetic field in parameter space corresponding to the Berry phase depends on whether the total magnetic field encircles the origin or not [40]. For simplicity, the Zeeman field is applied along the direction of the positive $x$-axis. For $B < B_{SO}$, the solid angle is $\Omega = 2\pi$, corresponding to a Berry phase $\pi$. However, for $B > B_{SO}$ the solid angle vanishes, so that the Berry phase is 0.

The topology of the field texture, which coincides with the spin-eigenstate texture in the adiabatic limit, changes when the critical line is traversed. The results in Fig. 4 show that the effects of this transition manifest in the conductance oscillations even in a nonadiabatic regime — small $k_R P/(2\pi)$.

Strongly non-adiabatic spin textures allow for a transition of the topological properties of the spin eigenmodes when an in-plane Zeeman field is applied. These properties are characterized by the winding parity of the spin eigenmodes [24]. As it has been stated, for the ring this topological transition occurs when the applied Zeeman field is equal to the effective SO field. However, the AC oscillations for polygons show a sign reversal for much lower values of the Zeeman field.

Figure 4 shows the conductance for a Zeeman field direction $\alpha = 0$. However, the conductance pattern shows a strong dependence of $\alpha$ for intermediate values of $B$. While the ring has a continuous rotational symmetry, polygons only remain invariant under rotations of an integer multiple of $2\pi/N$. In addition, they have $N$ symmetry planes, as well as inversion symmetry. The input and output leads break the rotational symmetry of polygons. Numerical calculations show that the only remaining symmetry planes
Figure 5. Average conductance $\langle G \rangle_k$ of a square for a Zeeman field with an orientation of $\alpha = 0$ (left) and $\alpha = \pi/4$ (right).

are the horizontal and vertical planes. Therefore, the conductance will be the same for angles $\alpha, \alpha' = -\alpha$ and $\alpha'' = \pi - \alpha$. Introducing elastic disorder to the system removes these symmetry planes, although the inversion center remains, so that the conductance is the same for $\alpha$ and $\alpha''' = \alpha + \pi$.

The conductance of a square for two orientations of the Zeeman field is shown in Fig. 5. For small values of the Zeeman field strength, the RSOC field contributes the most to the total magnetic field, so the oscillation pattern is almost identical in both cases. In the small RSOC limit, the conductance is also very similar for both orientations: for $k_R = 0$, the total magnetic field is equal to the applied Zeeman field, so it is homogeneous along the ring. The only effect on the energy spectrum is the splitting of the energy bands into two parabolas with opposite spin direction (see Fig. 2). The phase acquired by the electrons following the two shortest possible paths (the direct clockwise and counterclockwise paths) is the same, although the contribution of the longer paths results in the interference effect that gives rise to the oscillatory effect.

The main difference between the two panels in Fig. 5 is found around the critical line, where the oscillations of the conductance for $\alpha = \pi/4$ are smoothed. As explained in Sec. 2.2, the energy bands of a wire become degenerate when the Zeeman and RSOC strength are the same and the Zeeman field is perpendicular to one of the edges of the polygon. The RSOC field is perpendicular to the motion of the electron, so in some edges, it cancels with the applied Zeeman field. For $\alpha = \pi/4$, the Zeeman field is perpendicular to two opposite edges of the square, as it can be seen in Fig. 6. Therefore, electrons with any spin orientation propagating along these edges will not precess, and the interference effect between the upper and lower paths will be mitigated. In addition, the total magnetic field along the remaining two edges is the same, so the phase acquired
by electrons following the direct clockwise and counterclockwise paths is the same. Higher-order contributions, due to paths that go several times around the polygons, are responsible for the non constant conductance along the critical line. The conductance

\[
\text{SOI} \quad \text{SOI + B}
\]

direct	
path
TR	
ACW
TR	
CW

Figure 6. Total magnetic field in a square for direct clockwise and counterclockwise (first row) paths and time-reversed (second & third rows) paths when \(B_{SO} = B\) and \(\alpha = \pi/4\). The Green arrow represents the Zeeman field.

along the critical line for multiple values of \(\alpha\) is shown in Fig. 7. For the square, as \(\alpha\) approaches the \(\alpha = \pi/4\) direction, the oscillations become less pronounced. Moreover, the oscillations show a more regular pattern, with a steadily increasing amplitude.

Similar effects of mitigation of the spin procession due to RSCO are present in polygons with larger number of edges as well. However, the oscillatory pattern for the corresponding values of \(\alpha\) do not resemble the regular pattern obtained for the square. As the number of edges of the polygon increases, the length of the edges become smaller, so the edge where the spin does not process has a smaller contribution towards the

\[
\text{SOI} \quad \text{SOI + B}
\]

Figure 7. Average conductance \(\langle G \rangle_k\) as a function of the direction of the Zeeman field \(\alpha\) along the critical line for the square without disorder (a) and with disorder (b). The disorder is implemented so as explained in Sec. 3.3. Disorder decreases the average conductance due to the random value of the dynamical phase. The conductance traces are vertically shifted of a constant 0.2 factor for clarity.
interference.

3.3. Disordered case

In this section, we show how disorder is affecting the results we have presented in the previous sections. In general, disorder is inevitable in nanostructures and we need to account for its effects. Within the QN formalisms, disorder can be introduced in several ways, e.g. randomly distributed pointlike scatterers, or more generally, random elastic scattering matrix along the edges. We implement disorder following the scheme proposed in Refs. [15, 17, 18, 19]: to simulate arbitrary shifts of the wave function phase we implement random fluctuation of the length of each edge while keeping the perimeter $P$ of the polygon constant. We introduce therefore a length fluctuation parameter $\delta \ell$.

Contrary to the standard Anderson-like disorder, this model of elastic disorder will not produce a wave function localization in single loop, but will lead to a reduction of the amplitude of oscillations of the average conductance. The main effect of disorder is to double the period of the oscillations of the average conductance of the system, introducing the so-called Altshuler-Aronov-Spivak oscillations [39], in addition to a lifting of the complete destructive interference [18, 15, 19]. These quantum oscillations are dominated by the time-reversal paths [22, 24]. For the sake of simplicity, in the following we present results for the case of disorder only considering the polygonal structures that are experimentally more relevant: the square and the ring — see Fig. 8 for the case of AB and AC interference, and Fig. 9 for the case of AC and Zeeman field.

We observe from Fig. 8 how for the same strength of disorder, the ring geometry is more resilient to show the doubling of the frequency with respect to magnetic flux and RSOC. The results in the presence of disorder in Fig. 9 are more similar to the one obtained considering only the time-reversed paths in Ref. [24].

4. Validation of the quantum network method within the tight-binding approach

In this section we present results of a fully numerical approach, using a tight-binding model. Besides verifying the quantum network approach, we can also investigate the effect of the coupling of the first mode to higher order modes. We will use Kwant, a python package facilitating tight-binding transport simulations [11]. The typical system we run simulations on are shown in Fig. 1(e) and 1(f). We choose the units such that $\hbar = m^*_e = e = 1$. The Hamiltonian for a square lattice with RSOC [41] is

$$H_{\text{RSOC}} = \frac{i\lambda}{2a} \sum_i \left( c_i^\dagger \sigma_x c_{i+\hat{y}} - c_i^\dagger \sigma_y c_{i+\hat{x}} \right) + \text{H.c.}$$

(32)

where $\lambda$ is the RSOC strength, $a$ is the lattice size, $c_i^\dagger/c_i$ are the creation/annihilation operators, and $\sigma_i$, $i \in \{x,y,z\}$ are the Pauli matrices. The orbital magnetic field is implemented by adding a phase to all the hopping terms in a cut of the lower part of
Figure 8. Average conductance $\langle G \rangle_{k,\delta \ell}$ as a function of the orbital magnetic field and of the RSOC in the presence of disorder for the cases of the square (upper row) and of the ring (lower row) obtained with the quantum network method. The three columns refer to different strengths of the length fluctuations: $\delta \ell = 5\%, 10\%$ and $20\%$.

the path $t \rightarrow t e^{i\phi}$. Following as much as possible the notation of the analytical section, the Hamiltonian for the Zeeman field is written as

$$H_{\text{Zeeman}} = \frac{\mu |B|}{2a^2} \sum_i c_i^\dagger c_i (\sigma_x \cos \alpha + \sigma_y \sin \alpha).$$

(33)

We stick to rather narrow systems here, in order for the system to contain only a small number of energy bands (few modes). For calculating the average conductance we integrate over a few energy cycles at energies at which the system only hosts one single mode. The results can be seen in the left panel of Fig. 10. We find good qualitative agreement with the analytical results. Deviations from the analytical results can come from multiple sources. In the simulations we have to make a compromise between taking a thin wire, which can host only few modes, and a thicker wire, through which the electrons can move more easily without scattering. Results presented here have been run for an approximate width $W = 3$ of the system (leads and edges). Wider systems seem to lose contrast, probably due to coupling to higher modes. In narrower systems the lattice spacing has to be reduced substantially, especially in the case of the ring shaped system, likely due to the irregular shape.

For simulations with disorder we use Anderson Hamiltonian, meaning each lattice site $i$ is subjected to an onsite energy $\varepsilon$ which is randomly chosen from an interval
Figure 9. Average conductance $\langle G \rangle_{k, \delta \ell}$ as a function of the in-plane Zeeman term and of the RSOC in the presence of disorder for the cases of the square (upper row) and of the ring (lower row) obtained with the quantum network method. The three columns refer to different strengths of the length fluctuations: $\delta \ell = 5\%$, $10\%$ and $20\%$.

The Hamiltonian can be written as

$$H_{\text{Anderson}} = \frac{U_0}{2a^2} \sum_i \varepsilon_i c_i ^\dagger c_i$$

where $a$ is the lattice constant. In our comparison between disordered systems, one has to take into account that Anderson disorder also results in localization of the wave function, whereas variation of the path length does not do this. One can therefore not expect a direct correspondence between the analytical and numerical results, but only a qualitative tendency to manifesting the same behavior. As can be observed in the right panel of Fig. 10, the conductance decreases substantially, because the disorder strength we used is quite strong $U_0 = 0.8t$, which is $20\%$ of $4t$. We changed the color scaling so as to have more contrast in the range in which the conductance varies. Here we have averaged over 20 disorder configurations. Despite this lowering of the conductance in general, we do observe the same pattern as we observe in the case of the disordered quantum networks. For weaker disorder strengths, the conductance sticks to higher values, but the period doubling is not clearly observed.
5. Conclusions and Outlook

In this work, we have presented a generalization of the quantum network method to model 1D networks subject to hybrid field textures produced by the combined action of Zeeman and Rashba spin-orbit couplings. Contrary to previous studies [22], this method does not rely on perturbation theory, so it is valid for systems with high Zeeman fields. In addition, while other methods rely on spin-related phases accumulated by the carriers between input and output leads by following geometric paths with a finite number of windings, this method calculates the exact wave function of the electron along the edges. More specifically, we have studied the interplay of the non-Abelian phases introduced by the Aharonov-Casher effect and the in-plane Zeeman field in 1D polygons, in addition to the customary Abelian phase produced by the flux associated to a perpendicular magnetic field. We have considered polygonal structures with an increasing number of vertices in order to simulate a ring structure by sending to infinity the number of vertices. We have shown that in the case of the Rashba spin-orbit interaction, there is a double dependence of the conductance both on the perimeter of the system and the number of vertices itself. In the case of the interplay between the two non-Abelian phases due to the in-plane Zeeman field and the Rashba spin-orbit interaction, our results for the conductance are in agreement with the experimental observations [22, 24]. Furthermore, we have validated our network approach comparing our results with the ones obtained by a standard tight-binding approach. This method can also be used to obtain the spectrum and the eigenfunctions of the isolated polygons via the corresponding secular
equation [27, 16], which can be used to calculate the geometric and dynamical phases. The advantage of this method compared to a standard tight-binding one is that it would allow to account for more complex structures, which can become very computationally demanding within the tight-binding approach. The quantum network approach has been successfully employed for investigating the chaotic properties of complex networks, results obtained are in full agreement with random matrix theory [16]. The theory presented here offers an extension to this line of research.

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