Existence of solutions for first-order Hamiltonian stochastic impulsive differential equations with Dirichlet boundary conditions

Yu Guo§, Xiao-Bao Shu*, Qian bao Yin§

* College of Mathematics and Econometrics, Hunan University, Changsha, Hunan 410082, PR China.

Abstract: In this paper, we study the sufficient conditions for the existence of solutions of first-order Hamiltonian stochastic impulsive differential equations under Dirichlet boundary value conditions. By using the variational method, we first obtain the corresponding energy functional. And by using Legendre transformation, we obtain the conjugation of the functional. Then the existence of critical point is obtained by mountain pass lemma. Finally, we assert that the critical point of the energy functional is the mild solution of the first order Hamiltonian stochastic impulsive differential equation. Finally, an example are presented to illustrate the feasibility and effectiveness of our results.

Keywords: Random impulsive differential equation, Variational method, Critical point, Mountain pass lemma, Dirichlet boundary condition

1 Introduction

The phenomenon of random impulse exists widely in nature, and differential equation is one of the most powerful mathematical tools in scientific research. The content of this paper is the modern nonlinear methods and applications of stochastic impulsive differential equations. The purpose of this study is to explore the existence of solutions for a class of stochastic impulsive differential equations by using the critical point theory and variational method in modern mathematics. Stochastic impulsive differential equations are used in many fields, computer, finance, biomedicine, artificial intelligence, optimal control model [19, 20], and so on. Therefore, the study of impulsive differential equations has greatly practical significance. But in some practical cases, such as the mechanical problem [11, 15, 23], the impulse is random, so that means that the solution of stochastic impulsive differential equation is a random process, which is different from the corresponding fixed impulsive differential equation whose solution is a piecewise continuous function. Many scholars have studied fixed impulsive differential equations [ 4, 5, 9, 10, 14, 18, 21, 22, 25], while stochastic impulsive differential equations [6, 8, 17, 27, 29] are rarely involved. It is also of great significance to explore the use of nonlinear methods in the study of differential equations. This paper lays a foundation for the application of these methods in various fields of differential equation research in the future, and fills the gap in the field of studying stochastic impulse differential equation by nonlinear method.

Many scholars focus on the existence and multiplicity of solutions, and get most results of existence of solutions[7, 12, 13, 26, 28, 30, 31]. For example, Ravi P. Agarwal proved the multiplicity of second order...
impulsive differential equations in [26] by using Leggett Williams fixed point theorem:

\[
\begin{align*}
  y''(t) + \phi(t)f(y(t)) &= 0 \quad \text{for} \ t \in [0,1] \setminus \{t_1, \cdots, t_m\}, \\
  \Delta y(t_k) &= I_k(y(t^-_k)), \quad k = 1, \cdots, m, \\
  \Delta y'(t_k) &= J_k(y(t^-_k)), \quad k = 1, \cdots, m, \\
  y(0) &= y(1) = 0.
\end{align*}
\]

Here \( \Delta y(t_k) = y(t^+_k) - y(t^-_k) \) where \( y(t^+_k) \) (respectively \( y(t^-_k) \)) denote the right limit (respectively left limit) of \( y(t) \) at \( t = t_k \). Also \( \Delta y'(t_k) = y'(t^+_k) - y'(t^-_k) \). The upper and lower solution method is also used to study impulsive differential equations [30, 31]. In [30], Jianhua Shen and Weibing Wang established the existence condition of the solution by using the upper and lower solution method and Schauder’s fixed point theorem:

\[
\begin{align*}
  x''(t) &= f(t, x(t), x'(t)) \quad t \in J, t \neq t_k, \\
  \Delta x(t_k) &= I_k(x(t_k)), \quad k = 1, 2, \cdots, p, \\
  \Delta x'(t_k) &= J_k(x(t_k), x'(t_k)), \quad k = 1, 2, \cdots, p, \\
  g(x(0), x'(0)) &= 0, \quad h(x(1), x'(1)) = 0.
\end{align*}
\]

where \( J = [0,1], f : J \times \mathbb{R}^2 \rightarrow \mathbb{R} \) is continuous, \( I_k, J_k \in C(R) \) for \( 1 \leq k \leq p, 0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = 1, \Delta x(t_k) = x(t^+_k) - x(t^-_k) \) denotes the jump of \( x(t) \) at \( t = t_k \), \( x(t^+_k) \) and \( x(t^-_k) \) represent the right and left limits of \( x(t) \) at \( t = t_k \) respectively, and \( g, h : \mathbb{R}^2 \rightarrow \mathbb{R} \) are continuous. \( \Delta x'(t_k) = x'(t^+_k) - x'(t^-_k) \), where

\[
  x'(t_k^-) := \lim_{h \to 0^-} h^{-1}[x(t_k^- + h) - x(t_k^-)], \quad x'(t_k^+) := \lim_{h \to 0^+} h^{-1}[x(t_k^+ + h) - x(t_k^+)].
\]

Let \( J^* = J\setminus\{t_1, t_2, \cdots, t_p\}, PC(J) = \{u : J \rightarrow C(J^*), u(t^+_k), u(t^-_k) \text{ exist}, u(t_i) = u(t_i), i = 1, 2, \cdots, p\}, PC^1(J) = \{u \in PC(J) : \forall (t_i, t_{i+1}) \in C^1(t_i, t_{i+1}), u'(t^+_k), u'(t^-_k) \text{ exist}, u'(t_i) = u'(t_i), i = 1, 2, \cdots, p\} \)

Lijing Chen and Jitao Sun [31] discussed the nonlinear boundary value problem of first order impulsive functional differential equations by using the upper and lower solution method and monotone iterative technique:

\[
\begin{align*}
  x''(t) &= f(t, x(t), x(\theta(t))), \quad t \in J = [0,T], t \neq t_k, k = 1, 2, \cdots, p, \\
  \Delta x(t_k) &= I_k(x(t_k)), \quad k = 1, 2, \cdots, p, \\
  g(x(0), x(T)) &= 0,
\end{align*}
\]

where \( f \in C(J \times \mathbb{R}^2, \mathbb{R}), I_k \in C(R, \mathbb{R}), g \in C(R \times \mathbb{R} \times \mathbb{R}), \theta \in C(J, J), \Delta x(t_k) = x(t^+_k) - x(t^-_k) \), in which \( x(t^+_k), x(t^-_k) \) denote the right and left limits of \( x(t) \) at \( t = t_k, k = 1, 2, \cdots, p \), which are fixed such that \( 0 < t_1 < t_2 < \cdots < t_p < T \). Let \( J_0 = J\setminus\{t_1, t_2, \cdots, t_p\}, \tau = \max_k t_k - t_{k-1}, k = 1, 2, \cdots, p \), here \( t_0 = 0, t_{p+1} = T \). And \( PC(J) = \{u : J \rightarrow R | u \in C(J_0), u(t^+_k), u(t^-_k) \text{ exist}, u(t_i) = u(t_i), i = 1, 2, \cdots, p\} \)

The first-order Hamiltonian system has not been involved by many people. In recent years, variational method has been used by many scholars to study the solutions of differential equation. In fact, it is very difficult to get a strong solution of a differential equation. The general method is to transform the differential equation into an integral equation, and then get its corresponding energy functional. In this way, we can use variational method and critical point theory to study differential equation. Many scholars have done a lot of work on differential equation by using variational method and critical point theory, such as [1, 13, 24, 29]. For the case of differential equations with fixed impulses see [2, 4, 9, 22, 14, 25]. For example, Jingli Xie, Jianli Liand Zhiguo Luo gives periodic and subharmonic solutions of second-order Hamiltonian equation with fixed pulse in [1] by using the linking theorem:

\[
\begin{align*}
  -q''(t) &= \nabla F(t, q(t)), \quad t \neq t_j, t \in R, \\
  \Delta q'(t_j) &= -g_j(q(t_j)), \quad j \in Z,
\end{align*}
\]

where \( q \in \mathbb{R}^N, \nabla F(t, q) = \text{grad}_q F(t, q), g_j(q) = \text{grad}_q G_j(q), G_j \in (\mathbb{R}^N, \mathbb{R}) \) for each \( j \in Z \), and the operator \( \Delta \) is defined as \( \Delta q(t_j) = \hat{q}(t^+_j) - \hat{q}(t^-_j) \), where \( \hat{q}(t^+_j), \hat{q}(t^-_j) \) denotes the right-hand (left-hand) limit of \( \hat{q} \) at
There exist an $m \in N$ and a $T > 0$ such that $0 = t_0 < t_1 < t_2 < \cdots < t_m = T, t_{j+m} = t_j + T,$ and $g_{j+m} = g_{j} : N \times R \times R^N \to R$ is $T$-periodic in its first variable and satisfies: $F(t, q)$ is measurable in $t$ for each $q \in R^N$ and continuously differentiable in $q$ for a.e. $t \in [0, T]$.

The stochastic pulse differential equation we study is more general than the fixed pulse differential equation above, and it has a wide range of applications, and can simulate the real life situation more. Inspired by [3, 32, 33], we obtain a class of first-order Hamiltonian systems and decide to study the existence of periodic solutions for first-order Hamiltonian systems with impulses under Dirichlet boundary value conditions.

$$u'(t) = A(t)u(t) = JD(t)u(t) = J\nabla H(t, u(t))$$

In the motion of Gas Block Simulation of Linear Convection, the circulation generated by the horizontal temperature gradient between the equator and the polar region is accompanied by the rising motion of the equatorial air and the sinking motion of the polar air. Because of the disturbance, the vertical displacement needs to consider the instantaneous impulse. Therefore, it is reasonable to add such an impulsive condition.

Hence, we consider the existence of solutions to the random pulse linear hamiltonian system boundary value problem:

$$\begin{cases}
u'(t) = A(t)u(t) &= JD(t)u(t) = J\nabla H(t, u(t)), \quad t \in [0, T]\setminus\{\xi_1, \xi_2, \cdots\}, \\
\Delta u(\xi_j) &= u(\xi_j^+) - u(\xi_j^-) = b_j(\tau_j), \quad \xi_j \in (0, T), \quad j = 1, 2, \cdots, \\
u(0) &= u(T) = 0,
\end{cases} \quad (1.1)$$

where $\forall t \in [0, T], A(t) \in M_{2n}(\mathbb{R})$ is a hamiltonian matrix. $H : [0, T] \times R^{2n} \to \mathbb{R}, (t, u) \to H(t, u)$ is a smooth Hamiltonian. $\nabla H$ is the gradient of $H$ with respect to $u$. $D \in gl(2n, \mathbb{R})$ is a symmetric matrix with respect to $t$ continuity. $gl(2n, \mathbb{R})$ is the set of all $2n \times 2n$ matrices in the field $\mathbb{R}$. $u(t) : [0, T] \times \Omega \to R^{2n}$ is a stochastic process. $\forall j \in N, \tau_j : \Omega \to F_j$, where $F_j := (0, d_j)$ is a random variable, with $0 < d_j < +\infty$, and $\forall i, j \in N, \tau_i, \tau_j$ are mutually independent when $i \neq j$, $b_j : F_j \to R^{2n}$. Set $\xi_{j+1} = \xi_j + \tau_j, \{\xi_j\}$ is a strictly increasing random variable sequence i.e. $0 < \xi_1 < \xi_2 < \cdots < \xi_k < \cdots < T \Rightarrow u(\xi_j^+) \leftarrow \lim_{t \to \xi_j^+} u(t), u(\xi_j^-) \leftarrow \lim_{t \to \xi_j^-} u(t)$ under the meaning of the sample orbit. The definition of this definition are reasonable because $\{\xi_k\}$ will become a series of fixed points under the realization of each sample orbit. We suppose that $\{N(t) : t \geq 0\}$ is the simple counting process generated by $\{\xi_k\}$, that is, $\{N(t) \geq n\} = \{\xi_n \leq t\}$, and denote $\psi_t$ the $\sigma$-algebra generated by $\{N(t), t \geq 0\}$. $J$ is the $2n \times 2n$ matrices,

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

where $I$ is the $n \times n$ identity matrix.

### 2 Preliminaries

Let $(\Omega, \psi, P)$ be a probability space. Let $L^q([0, T] \times \Omega, R)$ be the collection of all strongly measurable, $q$th-integrable, $\psi_t$-measurable $R$-valued random variables $x$ with norm $||x||_{L^q} = (E||x||^q)^{1/q}$, where the expectation $E$ is defined by $Ex = \int_\Omega x dP$. Let $PC([0, T]) := \{u(t) = u(t, \omega) \text{ random process}, u(\cdot, \omega) \text{ is a map from } [0, T] \text{ to } R^{2n} \text{ such that } u(t) \text{ is continuous on } [0, T] \setminus \{\xi_1, \xi_2, \cdots\} \text{ and } u(\xi_j^+) = u(\xi_j^-) \text{ exist}, j = 1, 2, \cdots; u(0) = u(T) = 0\}$. $PC([0, T])$ is a Banach space with norm $||u||_{PC} = \left(\max_{t \in [0, T]} E|u(t)|^2\right)^{1/2}$.

Define the Banach space $PC_1 = PC_1([0, T]) := \{u(t) = u(t, \omega) \text{ is random process}, u(\cdot, \omega) \text{ is a map from } [0, T] \text{ to } R^{2n} \text{ such that } u(t) \text{ and } u'(t) \text{ is continuous on } [0, T] \setminus \{\xi_1, \xi_2, \cdots\} \text{ and } u'(\xi_j^+) = u'(\xi_j^-) \text{ exist}, j = 1, 2, \cdots; u(0) = u(T) = 0\}$, with the norm $||u||_{PC_1} = \max\{||u||_{PC}, ||u'||_{PC}\}$.

We introduce the Legendre transformation $H^*(t, \cdot)$ of $H(t, \cdot)$ and define it as

$$H^*(t, v) = (v, u) - H(t, u) \quad (2.1)$$
where
\[ v = \nabla H(t, u), \quad u = \nabla H^*(t, v) \]

**Definition 2.1 (Fréchet derivative)** E is Banach space, I : E → R is a functional on E, u ∈ E. If there is A(u) ∈ E*, such that
\[ I(u + \varphi) = I(u) + (A(u), \varphi) + \omega(u, \varphi) \]
where (A(u), \varphi) = A(u)\varphi, represents the value of functional A(u) at \varphi, \omega(u, \varphi) = o||\varphi||, i.e.
\[ \lim_{||\varphi|| \to 0} \frac{||\omega(u, \varphi)||}{||\varphi||} = 0, \]
The functional I is called Fréchet differentiable at u, A(u) is called the Fréchet derivative of I at u, then
\[ I(u + \varphi) = I(u) + (I'(u), \varphi) + o||\varphi|| \]
If I is Fréchet differentiable for any u, denote as I ∈ C^1(E, R).

**Lemma 2.1** If q > 2, α > 0 exists and The function H^* is differentiable in the domain of definition, with \( \frac{1}{p} + \frac{1}{q} = 1 \), \( M^* = \max_{||v||_{PC}=1} H^*(v) \), \( \alpha^* = (\alpha q)^{-\frac{q}{p}} / p \), make the following conditions hold
\[ qH(u) \leq (\nabla H(u), u) \]
\[ H(u) \leq \alpha||v||_{PC}^p \]
then, we have
\[ pH^* \geq (\nabla H^*(v), v) \]
\[ H^* \leq M^*||v||_{PC}^p, \quad ||v||_{PC} \geq 1 \]
\[ H^* \geq \alpha^*||v||_{PC}^p, \]

**Proof:** (i) We know that
\[ v = \nabla H(u) \Rightarrow (\nabla H(u), u) = (v, u) \geq qH(u) \]
\[ H^*(v) = (v, u) - H(u) \geq (1 - \frac{1}{q})(v, u) = \frac{1}{p}(\nabla H^*(v), v) \]

(ii) Fixed v, let \( f(w) = H^*(kw) \), by condition we have \( pf(w) \geq w f'(w) \), if \( w \geq 1 \), \( w^p f(1) \geq f(w) \), i.e. \( w^p H^*(v) \geq H^*(kw) \). When \( ||v||_{PC} \geq 1 \), then \( H^*(\frac{|v|_{PC}}{||v||_{PC}}) \geq ||v||_{PC}^p H^*(||v||_{PC}||v||_{PC}^q) \).

(iii) By \( H^*(v) = (v, u) - H(u) \) and \( H(u) \leq \alpha||u||_{PC}^p \), then
\[ H^*(v) \geq (v, u) - \alpha||u||_{PC}^p \]
\[ H^*(v) \geq \sup_u ((v, u) - \alpha||u||_{PC}^p) \]
\[ = ||v||_{PC}^p (\alpha q)^{-\frac{q}{p}} / p. \]

□

**Lemma 2.2** If u and v are two random processes, where the expectation E is defined as \( E(x) = \int_{\Omega} x \, dP \), then
\[ (E|u||v|)^2 \leq E|u|^2 E|v|^2 \]
Theorem 2.1

Define the function condition.

Lemma 2.3 (Embedding theorem) If $u(t)$ is a stochastic process and we denote by $PC_1 \to PC$ the embedding, then there is a constant $K$ such that

$$
\|u\|_{PC_1} \leq K \|\dot{u}\|_{PC}
$$

(2.2)

Theorem 2.1 (Mountain path lemma) $E$ is a Banach space, $\varphi \in C^1(E,R)$, if $\varphi$ satisfies

(i) $\varphi(0) = 0$, $\exists \rho > 0$, s.t. $\varphi_{\partial B_{\rho}(0)} \geq \alpha > 0$;

(ii) $\exists e \in E \setminus B_{\rho}(0)$, s.t. $\varphi(e) \leq 0$.

(iii) the $P - S.$ condition is fulfilled.

Then, $\varphi$ exists a critical point $u$ satisfying $\varphi'(u) = 0$ and $\varphi(u) = \max \{ \varphi(0), \varphi(e) \}$.

Remark 2.1 (P - S. condition) Suppose $\varphi \in C^1(E,R)$. If $\{\varphi(u_k)\}$ is bounded and $\{\varphi'(u_k)\} \to 0$ in $E^*$, when $k \to \infty$ implies that each $\{u_k\}$ is sequentially compact set in $E$. Then we call $\varphi$ satisfies $P - S.$ condition.

Now, we present some important conclusions that will be used in the next section.

Theorem 2.2 Define the function $\varphi \in PC_1$

$$
\varphi(u) = E \left[ \frac{1}{2} \int_0^T (J\dot{u},u(t))dt - \int_0^T H(t,u(t))dt - \frac{1}{2} \sum_{k=1}^{\infty} \left( \sum_{j=1}^{k} (Ju(\xi_j),b_j(\tau_j))I_A(\{\xi_j\}_{j=1}^{k}) \right) \right],
$$

(2.3)

where

$$
J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \quad I_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A, \end{cases}, \quad (u,v) := \sum_{i=1}^{2n} u(i)v(i)
$$

and $A$ is the set consisting of all sample orbits, and $\{\xi_i\}_{i=1}^{k}$ is a sample orbit.

If $u = Jv$, then

$$
\varphi(u) = E \left[ \frac{1}{2} \int_0^T (\dot{v}(t),u(t))dt - \int_0^T H(t,u(t))dt + \frac{1}{2} \sum_{k=1}^{\infty} \left( \sum_{j=1}^{k} (v(\xi_j),b_j(\tau_j))I_A(\{\xi_j\}_{j=1}^{k}) \right) \right]
$$

$$
= E \left[ \frac{1}{2} \int_0^T (\dot{v}(t),u(t))dt + \int_0^T (\ddot{v}(t),u(t))dt - \int_0^T H(t,u(t))dt \right]
$$

$$
+ \left[ \frac{1}{2} \sum_{k=1}^{\infty} \left( \sum_{j=1}^{k} (v(\xi_j),b_j(\tau_j))I_A(\{\xi_j\}_{j=1}^{k}) \right) \right]
$$

$$
= E \left[ \frac{1}{2} \int_0^T (J\ddot{v}(t),v(t))dt + \int_0^T (\ddot{v}(t),u(t))dt - \int_0^T H(t,u(t))dt \right]
$$

$$
+ \left[ \frac{1}{2} \sum_{k=1}^{\infty} \left( \sum_{j=1}^{k} (v(\xi_j),b_j(\tau_j))I_A(\{\xi_j\}_{j=1}^{k}) \right) \right]
$$
If $H^*$ in (2.1) is used to replace $(\dot{v}, u) - H(t, u)$, the conjugate action of $T$ periodic function space is obtained.

$$
\chi(v) = E \left[ \frac{1}{2} \int_0^T (J\dot{v}(t), v(t))dt + \int_0^T H^*(t, \dot{v}(t))dt + \frac{1}{2} \sum_{k=1}^\infty \left( \sum_{j=1}^k (v(\xi_j), b_j(\tau_j))I_A(\{\xi_j\}^{k}_{j=1}) \right) \right]
$$

(2.4)

We can prove that $\chi(v) \in C^1(PC_1, \mathbb{R})$ and $\forall v \in PC_1, \forall h \in PC_1$, 

$$
(\chi'(v), h) = E \left[ \frac{1}{2} \int_0^T (J\dot{v}(t), h(t))dt + \int_0^T (\nabla H^*(t, \dot{v}(t)) - \frac{1}{2} Jv(t), \dot{h}(t))dt \right]
$$

$$
+ \frac{1}{2} \sum_{k=1}^\infty \left( \sum_{j=1}^k (h(\xi_j), b_j(\tau_j))I_A(\{\xi_j\}^{k}_{j=1}) \right)
$$

(2.5)

Detailed proof of these will be given in Section 3.

**Theorem 2.3** If the random impulsive differential equation (1.1) has a mild solution $u = \nabla H^*(t, \dot{v}(t))$, then $v$ is a critical point of $\chi(v)$ i.e. $(\chi'(v), h) = 0, \forall h \in PC_1$. And if $v \in PC_1, v$ is a critical point of $\chi(v)$, then $u = \nabla H^*(t, \dot{v}(t))$ is a mild solution of (1.1).

Proof: Suppose $0 = t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} = T$, where $t_1, t_2, \cdots, t_k$ is a sample orbit. Let $u \in PC_1$ is a mild solution of (1.1). If $u \in C^1([0, T] \setminus \{t_1, t_2, \cdots, t_k\}) \cap PC_1$ then $u$ is the solution of (1.1) and satisfies

$$
\dot{u}(t) = J\nabla H(t, u(t)).
$$

From hypothesis $u = Jv$ and conjugate symmetry, we have

$$
\begin{align*}
J\dot{v}(t) &= J\nabla H(t, u(t)) \\
\dot{v}(t) &= \nabla H(t, u(t)) \\
u(t) &= \nabla H^*(t, \dot{v}(t)).
\end{align*}
$$

Let’s take the inner product by $h(t) \in PC_1 \cap C^1([0, T] \times \Omega, \mathbb{R}^{2n})$ of both sides, then integration from 0 to $T$, we get

$$
\int_0^T (u(t), \dot{h}(t))dt = \int_0^T (\nabla H^*(t, \dot{v}(t)), \dot{h}(t))dt.
$$

That means

$$
\frac{1}{2} \int_0^T (u(t), \dot{h}(t))dt = \int_0^T (\nabla H^*(t, \dot{v}(t)) - \frac{1}{2} u(t), \dot{h}(t))dt,
$$

(2.6)

where

$$
\int_0^T (u(t), \dot{h}(t))dt = \sum_{j=0}^{t_{j+1}} (u(t), \dot{h}(t))dt
$$

$$
= \sum_{j=0}^{t_{j+1}} \left[ (u(t), h(t))|^{t_{j+1}}_{t_j} - \int_{t_j}^{t_{j+1}} (\dot{u}(t), h(t))dt \right]
$$

$$
= - \sum_{j=1}^{k} (h(t_j), \Delta u(t_j)) - \int_0^T (\dot{u}(t), h(t))dt.
$$

(2.7)

Then we put (2.7) into (2.6) and consider the impulsive condition in (1.1), we obtain that

$$
\frac{1}{2} \int_0^T (\dot{u}(t), h(t))dt + \int_0^T (\nabla H^*(t, \dot{v}(t)) - \frac{1}{2} u(t), \dot{h}(t))dt + \frac{1}{2} \sum_{i=1}^{k} (h(t_j), b_j(\tau_j)) = 0.
$$

(2.8)
By $u(t) = Jv(t)$
\[
\frac{1}{2} \int_0^T (J\dot{v}(t), h(t)) dt + \int_0^T (\nabla H^*(t, \dot{v}(t)) - \frac{1}{2} Jv(t), \dot{h}(t)) dt + \frac{1}{2} \sum_{j=1}^k (h(t_j), b_j(\tau_j)) = 0. \quad (2.9)
\]

Thus we know $\langle \chi'(v), h \rangle = 0$, i.e. $v$ is a critical point of $\chi(v)$.

On the other hand, if $v \in PC_1$ is a critical point of $\chi$, i.e. $\langle \chi'(v), h \rangle = 0, \forall h \in PC_1$,
\[
\frac{1}{2} \int_0^T (J\dot{v}(t), h(t)) dt + \int_0^T (\nabla H^*(t, \dot{v}(t)) - \frac{1}{2} Jv(t), \dot{h}(t)) dt + \frac{1}{2} \sum_{j=1}^k (h(t_j), b_j(\tau_j)) = 0 \quad (2.10)
\]

by $u = Jv$
\[
\frac{1}{2} \int_0^T (\dot{u}(t), h(t)) dt + \int_0^T (\nabla H^*(t, \dot{v}(t)) - \frac{1}{2} u(t), \dot{h}(t)) dt + \frac{1}{2} \sum_{j=1}^k (h(t_j), b_j(\tau_j)) = 0, \forall h \in S. \quad (2.11)
\]

Since $h \in C^1$, we know $h(t_j^+) = h(t_j^-), j = 1, 2, 3, \ldots$, and $h(0) = h(T) = 0$. We will prove $u = Jv$ is the solution of (2.11):
\[
\frac{1}{2} \sum_{i=0}^k \int_{t_i}^{t_{i+1}} (\dot{u}(t), h(t)) dt + \int_0^T (\nabla H^*(t, \dot{v}(t)) - \frac{1}{2} u(t), \dot{h}(t)) dt + \frac{1}{2} \sum_{j=1}^k (h(t_j), b_j(\tau_j)) = 0.
\]

For the convenience, let $t_0 = 0, t_{k+1} = T$ and $h(t_0) = h(t_{k+1}) = 0$,
\[
\frac{1}{2} \sum_{i=0}^k \int_{t_i}^{t_{i+1}} (\dot{u}(t), h(t)) dt + \int_0^T (\nabla H^*(t, \dot{v}(t)) - \frac{1}{2} u(t), \dot{h}(t)) dt + \frac{1}{2} \sum_{j=1}^k (h(t_j), b_j(\tau_j)) = 0.
\]
\[
\Rightarrow \int_0^T (\dot{h}, u - \nabla H^*(t, \dot{v}(t))) dt + \frac{1}{2} \sum_{j=1}^k (h(t_j), \Delta u(t_j) - b_j(\tau_j)) = 0 \quad (2.12)
\]

Set
\[
\delta_j(t) = \begin{cases} 1, & \text{if } t = t_j, \\ 0, & \text{if } t \neq t_j, \end{cases}
\]
then (2.12) can be written as
\[
\int_0^T \left[ (\dot{h}, u - \nabla H^*(t, \dot{v}(t))) + \frac{1}{2} \sum_{j=1}^k (h(t_j), \Delta u(t_j) - b_j(\tau_j)) \delta_j(t) \right] dt = 0 \quad (2.13)
\]
thus $u \in PC_1$ is the mild solution of the equation
\[
\begin{align*}
u &= \nabla H^*(t, \dot{v}(t)), & t &\in [0,T] \setminus \{t_1, t_2, \ldots, t_k\} \\
\dot{u}(t) &= J\nabla H(t, u(t)), & t &\in [0,T] \setminus \{t_1, t_2, \ldots, t_k\}
\end{align*}
\]
and (2.13) imply that the random impulse condition $\Delta u(t_j) = b_j(\tau_j), j = 1, 2, \ldots, k$ hold.
Thus $u \in PC_1$ is a mild solution of (2.11).

3 Main Results

**Theorem 3.1** When $b_j(\tau_j)$ satisfy the following assumptions respectively

\[(H_1)\text{ Let } B = E \left( \sum_{k=1}^{\infty} \sum_{j=1}^{k} |b_j(\tau_j)| I_A(\{\xi_j\}_{j=1}^{k}) \right) < +\infty.\]

Then the $\chi(v)$ defined in (2.4) fulfills $\chi \in C^1(\text{PC}_1, \mathbb{R})$ and satisfies (2.5).
Proof: We divide the proof into several parts. 1. Let \( J_1(v) = \frac{1}{2} \int_0^T E(J \dot{v}(t), v(t))dt \), and we will prove that \( J_1(v) \in C^1(PC_1, \mathbb{R}) \).

\[ J_1(v + h) = J_1(v) + \frac{1}{2} \int_0^T E(J \dot{v}, h)dt + \frac{1}{2} \int_0^T E(J \dot{h}, v)dt + \frac{1}{2} \int_0^T E(J \dot{h}, h)dt. \]

Since

\[
\left( \frac{1}{2} \int_0^T E(J \dot{h}, h)dt \right)^2 \leq \frac{1}{2} \int_0^T E|J \dot{h}||h|dt^2 \\
\leq T \frac{1}{4} \int_0^T (E|\dot{h}||h|)^2 dt \\
\leq T \frac{1}{4} \int_0^T E|\dot{h}|^2 E|h|^2 dt \\
\leq T \frac{1}{4} \int_0^T \max_{t \in [0, T]} E|\dot{h}|^2 \max_{t \in [0, T]} E|h|^2 dt \\
= T^2 \frac{1}{4} \|h\|^2_{PC}|\dot{h}|^2_{PC} \\
\leq T^2 \frac{1}{4} \|h\|^2_{PC},
\]

then we have

\[
\lim_{\|h\|_{PC_1} \to 0} \frac{\frac{1}{2} \int_0^T E(J \dot{h}, h)dt}{\|h\|_{PC_1}} = 0.
\]

It follows that

\[
(J'_1(v), h) = \frac{1}{2} \int_0^T E(J \dot{v}, h)dt + \frac{1}{2} \int_0^T E(J \dot{h}, v)dt.
\]

For fixed \( v \), \( J'_1(v) \) is a linear functional with respect to \( h \). By Cauchy-Schwarz inequality, we have

\[
\left( \frac{1}{2} \int_0^T E(J \dot{v}, h)dt \right)^2 \leq \frac{1}{2} \int_0^T E|J \dot{v}||h|dt^2 \\
= T \frac{1}{4} \int_0^T (E|\dot{v}||h|)^2 dt \\
\leq T \frac{1}{4} \int_0^T E|\dot{v}|^2 E|h|^2 dt \\
\leq T \frac{1}{4} \int_0^T \max_{t \in [0, T]} E|\dot{v}|^2 \max_{t \in [0, T]} E|h|^2 dt \\
= T^2 \frac{1}{4} \|h\|^2_{PC}|\dot{v}|^2_{PC} \\
\leq T^2 \frac{1}{4} \|h\|^2_{PC},
\]

then, we have \( \left( \frac{1}{2} \int_0^T E(J \dot{h}, v)dt \right)^2 \leq T^2 \frac{1}{4} \|\dot{h}\|^2_{PC_1}|v|^2_{PC_1} \)

\[
|\langle J'_1(v), h \rangle| \leq T \frac{1}{2} \|h\|_{PC_1}\|v\|_{PC_1} + T \frac{1}{2} \|v\|_{PC_1}\|h\|_{PC_1} \\
= (T\|h\|_{PC_1})\|v\|_{PC_1},
\]
Step 1: We will prove that $J_1^1(v)$ is a bounded functional in $PC_1$.

2. Let $J_2(v) = \int_0^T E[H^*(t, \dot{v}(t))] dt$, and we will prove that $J_2(v) \in C^1(PC_1, \mathbb{R})$. 
\forall v, h \in PC_1$, we have

$$J_2(v + h) = J_2(v) + \int_0^T E(u, \dot{h}(t)) dt$$

$$= J_2(v) + \int_0^T E(\nabla H^*(t, \dot{v}(t)), \dot{h}(t)) dt$$

therefore when $v$ fixed, $J_2(v)$ is a linear functional w.r.t. $h$.

$$(J_2'(v), h) = \int_0^T E(\nabla H^*(t, \dot{v}(t)), \dot{h}(t)) dt$$

$$\leq \left( \int_0^T E \left| \nabla H^*(t, \dot{v}(t)) \right|^2 \right)^{\frac{1}{2}} \leq T \left( \sum_{k=1}^{\infty} \sum_{j=1} E(h(\xi_j), b_j(\tau_j)) I_A(\{\xi_j\}_{j=1}^1) \right)^{\frac{1}{2}}$$

By $u = Jv = \nabla H^*(t, \dot{v}(t))$, bring in the above formula, we have

$$(J_2'(v), h) \leq \left( \int_0^T E (|Jv|^2) |h(t)|^2 dt \right)^{\frac{1}{2}}$$

$$\leq T ||h||_{PC_1} ||Jv||_{PC_1}$$

where $\|v\|_{PC_1}$ is independent of $h$, therefore $J_2'(v)$ is a bounded functional in $PC_1$.

3. Let $J_3(v) = \sum_{k=1}^{\infty} \sum_{j=1} E(v(\xi_j), b_j(\tau_j)) I_A(\{\xi_j\}_{j=1}^k)$, and we will prove that $J_3(v) \in C^1(PC_1, \mathbb{R})$.
\forall v, h \in PC_1$, we have

$$J_3(v + h) = J_3(v) + \sum_{k=1}^{\infty} \sum_{j=1} E(h(\xi_j), b_j(\tau_j)) I_A(\{\xi_j\}_{j=1}^k).$$

then

$$(J_3'(v), h) = \sum_{k=1}^{\infty} \sum_{j=1} E(h(\xi_j), b_j(\tau_j)) I_A(\{\xi_j\}_{j=1}^k).$$

when $v$ fixed, $J_3'(v)$ is a linear functional w.r.t. $h$. By the property of hamiltonian matrix, we have

$$|(J_3'(v), h)| \leq \left( \sum_{k=1}^{\infty} \sum_{j=1} E(h(\xi_j)^2 \|E(b_j(\tau_j))^2 I_A(\{\xi_j\}_{j=1}^k) \right)^{\frac{1}{2}} \leq B ||h||_{PC_1},$$

thus $J_3'(v)$ is a bounded functional in $PC_1$.

From parts 1.-3. we can conclude that $\chi \in C^1(PC_1, \mathbb{R})$ and satisfies (2.5).

\[ \text{\Box} \]

\textbf{Theorem 3.2} When impulse satisfies assumption (H1), $H(u)$ and $\nabla H(u)$ satisfy the following assumptions respectively

(H2) $H(u) \leq \alpha \|u\|^q_{PC}, (q > 2, \frac{1}{p} + \frac{1}{q} = 1)$ holds, where $\alpha$ is a constant.

(H3) $|\nabla H(u)| \geq qH(u)$

Then $\chi(v)$ satisfies P. S. condition.

\textbf{Proof:} Step 1: We will prove that $\{v_k\}$ is a bounded set in $PC_1$ provided $\{\chi(v_k)\}$ is a bounded set and $\{\chi'(v_k)\} \to 0$ in $PC_1^*$ as $k \to \infty$. 


Theorem 3.3 Suppose that \( H(u) \) and \( \nabla H^*(u) \) satisfies:

\( H1 \) \( B = E \left( \sum_{k=1}^{\infty} \sum_{j=1}^{k} b_j(\tau_j) I_A(\{\xi_j\}_{j=1}^{k}) \right) < +\infty \)
\( H2 \) \( qH(u) \leq (\nabla H(\{u\}_{i=1}^{\infty}), u) \leq 0, \forall (t, u) \in [0, T) \times R^{2n}, \)
\( H3 \) \( H(u) \leq \tilde{\alpha} ||u||_{PC}^p, q > 2, \alpha > 0, \frac{1}{p} + \frac{q}{2} = 1, \tilde{\alpha}^* = (\tilde{\alpha}q)^{-\frac{\tilde{\alpha}}{p}}(1 - \frac{\tilde{\alpha}}{2})\tilde{\alpha}^* > \frac{1}{4}B. \)

Then \( \chi(v) \in (PC, R) \) and \( \chi(v) \) satisfies \( P - S \) condition in \( PC_1 \). By Mountain pass lemma, we can get \( \chi \) has a critical point, i.e. Equation (1.1) has at least a mild solution in \( PC_1 \).
Proof: (1) By hypothesis (H1), and using similar approach with Theorem 3.1, we can prove that \(\chi(v) \in C^1(PC_1, R)\).

(2) Next we prove \(\chi(v)\) satisfies \(P - S\). condition on \(E\).

1) We will prove if \(\{\chi(v_k)\}\) is a bounded set and \(\chi'(v_k) \rightarrow 0\) \((k \rightarrow \infty)\) in \(PC_1^*\), then \(v_k\) is a bounded set in \(PC_1\).

\[
\chi(v) = \frac{1}{2} \int_0^T E(J\dot{v}(t), v(t))dt + \int_0^T EH^*(t, \dot{v}(t))dt + \frac{1}{2} \sum_{k=1}^\infty \left( \sum_{j=1}^k E(v(\xi_j), b_j(\tau_j))I_A(\{\xi_j\}^k_{j=1}) \right)
\]

\[
(\chi'(v), h) = \frac{1}{2} \int_0^T E(J\dot{v}(t), h(t))dt + \int_0^T E(\nabla H^*(t, \dot{v}(t)) - \frac{1}{2} Jv(t), \dot{h}(t))dt + \frac{1}{2} \sum_{k=1}^\infty \left( \sum_{j=1}^k E(h(\xi_j), b_j(\tau_j))I_A(\{\xi_j\}^k_{j=1}) \right)
\]

then

\[
\chi(v_k) - (\chi'(v_k), v_k) = \int_0^T EH^*(t, \dot{v}_k(t))dt - \int_0^T E(\nabla H^*(t, \dot{v}_k(t)) - \frac{1}{2} Jv_k(t), \dot{v}_k(t))dt
\]

\[
= \int_0^T EH^*(t, \dot{v}_k(t))dt - \int_0^T E(\frac{1}{2} \nabla H^*(t, \dot{v}_k(t)), \dot{v}_k(t))dt
\]

\[
\geq \int_0^T EH^*(t, \dot{v}_k(t))dt - \frac{p}{2} \int_0^T EH^*(t, \dot{v}_k(t))dt
\]

\[
= (1 - \frac{p}{2}) \int_0^T EH^*(t, \dot{v}_k(t))dt
\]

\[
\geq (1 - \frac{p}{2}) \frac{\alpha^*}{||\dot{v}_k||}_{PC} ||v_k||_{PC}^p
\]

then if \(\{v_k\}\) is unbounded, by \(||v||_{PC} \leq K||\dot{v}||_{PC}\), then \(\{\dot{v}_k\}\) is unbounded, then it’s in contradiction with \(\chi(v_k)\) is bounded with \(\chi'(v_k) \rightarrow 0\) \((k \rightarrow \infty)\).

2) We will prove \(\{v_k\}\) is a bounded set in \(PC_1\), \(\chi'(v_k) \rightarrow 0\) \((k \rightarrow \infty)\) in \(PC_1^*\), then \(\{v_k\}\) is a sequential compact set in \(PC_1\).

We consider that, \(H(t, u)\) is a smooth Hamiltonian and due to \(\{v_k\}\) is a bounded set in \(PC_1\), then there is \(\{v_{k_j}\} \subset \{v_k\}\) satisfies

\(v_{k_j} \rightarrow v\) is uniform convergence in \((0, T)\);

\(v_{k_j} \rightarrow v\) is weak convergence in \(PC_1\).

\[
\Rightarrow ||v_{k_i} - v_{k_j}||_{PC} \leq K||\dot{v}_{k_i} - \dot{v}_{k_j}||_{PC} = ||\nabla H(Jv_{k_i} - Jv_{k_j})||_{PC} \rightarrow 0
\]

From this, when \(i, j \rightarrow \infty\), \(||v_{k_i} - v_{k_j}||_{PC} \rightarrow 0\), we have \(\{v_{k_j}\}\) is a Cauchy sequential compact in \(PC_1\), and by the completeness of \(PC_1\), we further get uki \(\{v_{k_j}\}\) convergent in \(PC_1\). Then \(\{v_k\}\) is a sequential compact in \(PC_1\).

By 1), 2) we have \(\chi(v)\) satisfies \(P - S\) condition on \(PC_1\).

(3) At last, we verify whether \(\chi(v)\) fuills the conditions of Mountain path lemma.

a) It is obvious that \(\chi(0) = 0\). Since

\[
\chi(v) = \frac{1}{2} \int_0^T E(J\dot{v}(t), v(t))dt + \int_0^T EH^*(t, \dot{v}(t))dt + \frac{1}{2} \sum_{k=1}^\infty \left( \sum_{j=1}^k E(v(\xi_j), b_j(\tau_j))I_A(\{\xi_j\}^k_{j=1}) \right)
\]

\[
= \frac{1}{2} \int_0^T E(\dot{v}(t), Jv(t))dt + \int_0^T EH^*(t, \dot{v}(t))dt + \frac{1}{2} \sum_{k=1}^\infty \left( \sum_{j=1}^k E(v(\xi_j), b_j(\tau_j))I_A(\{\xi_j\}^k_{j=1}) \right)
\]
The main result could have many applications, now, we give an example to illustrate this theorem. We have a critical point, i.e. equation (1.1) has weak solutions in \( \chi \). This completes the proof.

\[
\chi(v) = \frac{1}{2} \int_0^T E(\dot{v}(t), \nabla H^*(t, \dot{v}(t))) dt + \frac{1}{2} \sum_{k=1}^{\infty} \left( \sum_{j=1}^{k} E(v(\xi_j), b_j(\tau_j)) I_A(\{\xi_j\}_{j=1}^k) \right)
\]

\[
\geq (1 - \frac{p}{2}) \int_0^T E H^*(t, \dot{v}(t)) dt + \frac{1}{2} \sum_{k=1}^{\infty} \left( \sum_{j=1}^{k} E(v(\xi_j), b_j(\tau_j)) I_A(\{\xi_j\}_{j=1}^k) \right)
\]

\[
\geq (1 - \frac{p}{2}) \delta^* T ||v||_{PC}^p - \frac{1}{2} B ||v||_{PC}^p
\]

and \( K^p ||v||_{PC}^p \geq ||v||_{PC}^p \), then

\[
\chi(v) \geq \frac{1}{2} \delta^* T ||v||_{PC}^p - \frac{1}{2} B ||v||_{PC}^p, 1 < p < 2.
\]

Take \( \rho = (\frac{K^p}{p})^{\frac{1}{2-p}} > 0, u \in \partial B_\rho(0) \), then \( ||v||_{PC} = \rho \), then

\[
\chi(v) \geq (1 - \frac{p}{2}) \delta^* T \frac{K^p}{p} ||v||_{PC}^p - \frac{1}{2} B ||v||_{PC}^p
\]

\[
\geq ||v||_{PC} ((1 - \frac{p}{2}) \delta^* T \frac{K^p}{p} ||v||_{PC}^{p-1} - \frac{1}{2} B)
\]

\[
= ||v||_{PC} ((1 - \frac{p}{2}) \delta^* - \frac{1}{2} B) > 0
\]

Here we used the assumption (H3).

b) Now, we consider \( v_1(t) = (\cos \frac{2\pi t}{\rho}) e + (\sin \frac{2\pi t}{\rho}) J e \), where \( ||e||_{PC} = ||e||_{PC} \geq \max(1, \rho) \), then

\[
\int_0^T (Jv_1(t), v_1(t)) = - \left( \frac{T^2}{2\pi} \right) ||e||^2_{PC} = - \left( \frac{T^2}{2\pi} \right) ||e||^2_{PC}
\]

\[
\chi(v_1) \leq - \left( \frac{T^2}{4\pi} \right) ||e||^2_{PC} + TM^* ||e||^2_{PC} + \frac{1}{2} B ||e||_{PC}
\]

since \( p < 2 \) and when \( ||e||_{PC} \to +\infty \) we have \( \chi(v_1) \to -\infty \), there is always an \( e \) such that \( \chi(v_1) < 0 \), by \( ||e||_{PC} \geq \rho \), so \( v_1 \in E \setminus B_\rho(0) \).

From a),b) and \( \chi(v) \in C^1(\mathcal{P}C_1, \mathbb{R}) \) satisfies \( P - S \) condition, by Mountain path lemma, we can obtain \( \chi \) has a critical point, i.e. equation (4.1) has weak solutions in \( PC_1 \). This completes the proof.

\[\square\]

### 4 Example

The main result could have many applications, now, we give an example to illustrate this theorem. We consider the following random impulsive differential equation with boundary value problems.

\[
\begin{cases}
  u'(t) = 10 |u|^8 J u(t), & t \in [0, T] \setminus \{\xi_1, \xi_2, \cdots\},
  \\
  \Delta u(\xi_j) = u(\xi_j^+) - u(\xi_j^-) = \frac{1}{\tau_j}, & \xi_j \in (0, T), \ j = 1, 2, \cdots,
  \\
  u(0) = u(T) = 0,
\end{cases}
\]

(4.1)
Its Hamilton quantity is $H(u) = |u|^{10}$. By transforming it into a polar equation we know that (4.1) period is $\pi/5|u|^{-8}$.

Let $\tau_j \sim U(0, \frac{1}{2^j})$, then the probability density function of $\tau_j$ is

$$p(x) = \begin{cases} 
2^j & x \in (0, \frac{1}{2^j}), \\
0 & x \notin (0, \frac{1}{2^j}),
\end{cases} \quad (4.2)$$

Set $\xi_0 = 0, \xi_{j+1} = \xi_j + \tau_j$. Obviously, $\{\xi_j\}$ is a process with independent increments and the impulsive moments $\xi_j$ form a strictly increasing sequence. And for every $j \in \mathbb{N}$,

$$\xi_j < \xi_{j+1} < \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{j+1}} < 1 \quad (4.3)$$

So in this example, $b_j(\tau_j) = \frac{4}{2^j} \tau_j$, and $\tau_j$ is a random variable defined from $\Omega$ to $F_j = (0, d_j) = (0, \frac{1}{2^j})$. Suppose $\tau_j$ and $\tau_i$ are independent of each other when $i \neq j$, $u(\xi_j^+) := \lim_{t \to \xi_j^+} u(t), u(\xi_j^-) := \lim_{t \to \xi_j^-} u(t)$.

$$\sum_{j=1}^{k} |b_j(\tau_j)| = \sum_{j=1}^{k} \frac{j}{4^j} \tau_j \leq \frac{4^{-k}}{9} (-3k + 4k^{k+1} - 4)$$

So, we have proved that $B = E \left( \sum_{k=1}^{\infty} \sum_{j=1}^{k} |b_j(\tau_j)| I_A(\{\xi_j\}_{j=1}^{k}) \right) < \frac{4}{9}$. 

$$(\nabla u, u) = (10|u|^8 u, u) = 10|u|^{10} \geq 10|u|^{10}$$

$$H(u) = |u|^{10} \leq ||u||_{L^C}^{10}$$

then we have $q = 10, p = \frac{10}{4}, \hat{\alpha} = 1$, then $\hat{\alpha}^* \approx 0.696$, then $(1 - q)\hat{\alpha}^* \geq \frac{4}{9}B$.

So, the equation (4.1) meets all the conditions of the theorem (3.3). By Mountain pass lemma, we can get equation (4.1) has at least a mild solution in $PC_1$.

## 5 Conclusion

In this paper, we study the existence of solutions for first-order impulsive Hamiltonian systems (1.1) by using the critical point theory and the variational method. First, we prove that the conjugate action (2.4) of the functional is Fréchet differentiable. Then we prove that the critical point of (2.4) is a mild solution of Hamiltonian system (1.1). Then, it is proved that the conjugate action of the functional (2.4) satisfies the P. - S. condition under the given conditions. Finally, through mountain pass lemma, it is found that the critical point of (2.4) is a mild solution of Hamiltonian system (1.1).

In our future research, we will continue to study Hamiltonian systems with random impulses. For example, we will study some properties of second-order Hamiltonian systems with impulses.

## References

[1] Xie, J., Li, J. & Luo, Z. Periodic and subharmonic solutions for a class of the second-order Hamiltonian systems with impulsive effects. Bound Value Probl 2015, 52 (2015).

[2] Jianshe Yu, Honghua Bin, Zhiming Guo. Periodic Solutions for Discrete Convex Hamiltonian Systems via Clarke Duality[EB/OL]. Beijing: Sciencepaper Online [2005-12-12].

[3] Li, J., Nieto, J. J., & Shen, J. (2007). Impulsive periodic boundary value problems of first-order differential equations. Journal of Mathematical Analysis and Applications, 325(1), 226-236.

[4] Z.H. Zhang, R. Yuan, An application of variational methods to Dirichlet boundary value problem with impulses, Nonlinear Analysis: Real World Applications 11 (2010) 155-162.
[5] X. Xian, D. O'Regan, R. P. Agarwa, Multiplicity results via topological degree for impulsive boundary value problems under non-well-ordered upper and lower solution conditions, Hindawi Publishing Corporation Boundary Value Problems Volume 2008, Article ID 197205, 21 pages doi:10.1155/2008/197205.

[6] S.J. Wu, X.Z. Meng, Boundedness of nonlinear differential systems with impulsive effect on random moments, Acta Mathematicae Applicatae Sinica, English Series Vol.20, No.1 (2004) 147-154.

[7] Sun, JT, Chen, HB, Nieto, JJ, Otero-Novoa, M: The multiplicity of solutions for perturbed second-order Hamiltonian systems with impulsive effects. Nonlinear Anal. 72 (2010) 4575-4586.

[8] S.J. Wu, X.L. Guo, S.Q. Lin, Existence and uniqueness of solutions to random impulsive differential systems, Acta Mathematicae Applicatae Sinica, English Series Vol.22, No.4 (2006) 627-632.

[9] Y. Tian, W.G. Ge, Applications of variational methods to boundary value problem for impulsive differential equations, Proc.Edinburgh Math.Soc. 51 (2008) 509-527.

[10] X.-B. Shu, Y. Shi, A study on the mild solution of impulsive fractional evolution equations, Appl. Math. Comput. 273 (2016) 465-476.

[11] A.F.B.A. Prado, Bi-impulsive control to build a satellite constellation, Nonlinear Dyn. Syst. Theory 5 (2005) 169-175.

[12] Sun, JT, Chen, HB, Nieto, JJ: Infinitely many solutions for second-order Hamiltonian system with impulsive effects. Math. Comput. Model. 54 (2011) 544-555.

[13] Zhou, JW, Li, YK: Existence of solutions for a class of second order Hamiltonian systems with impulsive effects. Nonlinear Anal. 72 (2010) 1594-1603.

[14] Juan J. Nieto, Variational formulation of a damped Dirichlet impulsive problem, Applied Mathematics Letters 23 (2010) 940-942.

[15] X. Liu, A.R. Willms, Impulsive controllability of linear dynamical systems with applications to maneuvers of Spacecraft, Math. Problems Engineer. 2 (1996) 277-299.

[16] J.L. Li, Juan J. Nieto, J.H. Shen, Impulsive periodic boundary value problems of first-order differential equations, J. Math. Anal. Appl. 325 (2007) 226-236.

[17] S. Li, L. Shu, X. Shu, F. Xu, Existence and Hyers-Ulam stability of random impulsive stochastic functional differential equations with finite delays, Stochastics, 91 (6) (2019) 857-872.

[18] Y. Guo, X.-B. Shu, Y. Li, F. Xu, The existence and Hyers-Ulam stability of solution for an impulsive Riemann-Liouville fractional neutral functional stochastic differential equation with infinite delay of order $1 < \beta < 2$, Bound. Value Probl. 2019 (59) (2019) 1-C18.

[19] Z. Guan, G. Chen, T. Ueta, On impulsive control of a periodically forced chaotic pendulum system, IEEE Trans. Automat. Control 45 (2000) 1724-1727.

[20] P.K. George, A.K. Nandakumaran, A. Arapostathis, A note on controllability of impulsive systems, J. Math. Anal. Appl. 241 (2000) 276-283.

[21] S. Deng, X. Shu, J. Mao, Existence and exponential stability for impulsive neutral stochastic functional differential equations driven by fBm with noncompact semigroup via Monch fixed point, J. Math. Anal. Appl. 467 (1) (2018) 398-420.

[22] P. Chen, X.H. Tang, Existence and multiplicity of solutions for second-order impulsive differential equations with Dirichlet problems, Applied Mathematics and Computation 218 (2012) 11775-11789.
[23] T.E. Carter, Optimal impulsive space trajectories based on linear equations, J. Optim. Theory Appl. 70 (1991) 277-297.

[24] Ambrosetti, A., Mancini, G. Solutions of minimal period for a class of convex Hamiltonian systems. Math. Ann. 255 (1981) 405–421.

[25] Juan J. Nieto, D. O’Regan, Variational approach to impulsive differential equations, Nonlinear Analysis: Real World Applications 10 (2009) 680-690.

[26] Agarwal, RP, O’Regan, D: A multiplicity result for second order impulsive differential equations via the Leggett Williams fixed point theorem. Appl. Math. Comput. 161 (2005) 433-439.

[27] P.P Niu, X.-B. Shu, Y.J. Li, The Existence and Hyers-Ulam stability for second order random impulsive differential equation, Dynamic Systems and Applications, 28, No. 3 (2019), 673-690.

[28] Nieto, JJ, O’Regan, D: Variational approach to impulsive differential equations. Nonlinear Anal., Real World Appl. 10 (2009) 680-690.

[29] S.J. Wu, Y.R. Duan, Oscillation stability and boundedness of second-order differential systems with random impulses, Computers and Mathematics with Applications 49 (2005) 1375-1386.

[30] Shen, JH, Wang, BW: Impulsive boundary value problems with nonlinear boundary conditions. Nonlinear Anal., Real World Appl. 69 (2008) 4055-4062.

[31] Chen, LJ, Sun, JT: Nonlinear boundary value problem of first order impulsive functional differential equations. J. Math. Anal. Appl. 318 (2006) 726-741.

[32] J. Mawhin and M. Willem, Critical Point Theory and Hamiltonian Systems, Springer-Verlag, New York, 1989.

[33] J. Q. Liu and Z. Q. Wang, Remarks on subharmonics with minimal periods of Hamiltonian systems, Nonl. Anal. T. M. A., Vol.20, 7(1993), 803-821.