Hardness of Finding Independent Sets in 2-Colorable and Almost 2-Colorable Hypergraphs

Subhash Khot*  Rishi Saket†

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Abstract

This work studies the hardness of finding independent sets in hypergraphs which are either 2-colorable or are almost 2-colorable, i.e. can be 2-colored after removing a small fraction of vertices and the incident hyperedges. To be precise, say that a hypergraph is \((1-\varepsilon)\)-almost 2-colorable if removing an \(\varepsilon\) fraction of its vertices and all hyperedges incident on them makes the remaining hypergraph 2-colorable. In particular we prove the following results.

- For an arbitrarily small constant \(\gamma > 0\), there is a constant \(\xi > 0\), such that, given a 4-uniform hypergraph on \(n\) vertices which is \((1-\varepsilon)\)-almost 2-colorable for \(\varepsilon = 2^{-(\log n)^{\xi}}\), it is quasi-NP-hard to find an independent set of \(n/(2^{(\log n)^{1-\gamma}})\) vertices.

- For any constants \(\varepsilon, \delta > 0\), given as input a 3-uniform hypergraph on \(n\) vertices which is \((1-\varepsilon)\)-almost 2-colorable, it is NP-hard to find an independent set of \(\delta n\) vertices.

- Assuming the \(d\)-to-1 Games Conjecture the following holds. For any constant \(\delta > 0\), given a 2-colorable 3-uniform hypergraph on \(n\) vertices, it is NP-hard to find an independent set of \(\delta n\) vertices.

The hardness result on independent set in almost 2-colorable 3-uniform hypergraphs was earlier known only assuming the Unique Games Conjecture. In this work we prove the result unconditionally, combining Fourier analytic techniques with the Multi-Layered PCP of \(\text{DGKR03}\).

For independent sets in 2-colorable 3-uniform hypergraphs we prove the first strong hardness result, albeit assuming the \(d\)-to-1 Games Conjecture. Our reduction uses the \(d\)-to-1 Game as a starting point to construct a Multi-Layered PCP with the \textit{smoothness} property. We use analytical techniques based on the Invariance Principle of Mossel \(\text{[Mos10]}\). The smoothness property is crucially exploited in a manner similar to recent work of Håstad \(\text{[Hås12]}\) and Wenner \(\text{[Wen12]}\).

Our result on almost 2-colorable 4-uniform hypergraphs gives the first nearly polynomial hardness factor for independent set in hypergraphs which are (almost) colorable with constantly many colors. It partially bridges the gap between the previous best lower bound of \(\text{poly}(\log n)\) and the algorithmic upper bounds of \(\text{poly}^{\Omega(1)}\). This also exhibits a bottleneck to improving the algorithmic techniques for hypergraph coloring.

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* Department of Computer Science, University of Chicago, USA. email: khot@cs.nyu.edu
† IBM T. J. Watson Research Center, USA. email: rsaket@us.ibm.com

1 A problem is quasi-NP-hard if it admits a \(n^{\text{poly}(\log n)}\) time reduction from 3SAT.
1 Introduction

A $k$-uniform hypergraph consists of a set of vertices and a set of hyperedges, where each hyperedge is a subset of exactly $k$ vertices. For $k = 2$ this defines the usual notion of a graph. An independent set in a $k$-uniform hypergraph is a subset of vertices such that no hyperedge has all of its $k$ vertices from this subset. In other words, an independent set does not contain any hyperedge. The problem of finding independent sets of maximum size in (hyper)graphs is a fundamental one in combinatorial optimization. Note that the complement of an independent set is a vertex cover, i.e. a subset of vertices that contains at least one vertex from each hyperedge. Thus, finding a maximum sized independent set is same as finding a minimum vertex cover, an equally important problem in combinatorics. Throughout this paper, we shall frequently use the size of a set of vertices to mean its relative size, i.e. as a fraction of the total weight of the vertices.

The study of independent sets is closely related to that of hypergraph coloring. A hypergraph is $q$-colorable if its vertices can each be assigned one of $q$ distinct colors so that no hyperedge is monochromatic. The problem in hypergraph coloring is to determine the minimum possible value of $q$, which is known as the chromatic number of the hypergraph. Note that the color classes in a $q$-coloring form a partition of the vertices into $q$ disjoint independent sets. Thus, a $q$-colorable hypergraph has an independent set of size at least $1/q$. On the other hand, if a hypergraph does not have an independent set of size $1/q$ then it is not $q$-colorable either. Thus, the absence of large independent sets implies a large chromatic number.

This connection can also be studied with a relaxed notion of hypergraph coloring. Say that a hypergraph is almost $q$-colorable if there is a subset of vertices of size at most $\varepsilon$ such that removing this subset and all hyperedges containing a vertex from this subset makes the hypergraph $q$-colorable. Here $\varepsilon$ can be an arbitrarily small positive constant. It is easy to see that an almost $q$-colorable hypergraph contains $q$ pairwise disjoint independent sets containing within them at least $(1 - \varepsilon)$ fraction of vertices. Thus, there is at least one independent set of size $(1 - \varepsilon)/q$.

The problem of finding independent sets in (almost) $q$-colorable $k$-uniform hypergraphs is most interesting for small values of $q$ and $k$ and has been studied extensively from the complexity perspective in a sequence of works including [GHS02, Kho02a, Hol02, Kho02b, DRS05, BK09, BK10, GST11, KST12, Cha13]. For constant $q$ and $k$, the strongest hardness result in terms of the relative size of the independent set is by Khot [Kho02a] who showed the hardness of finding independent sets of size $(\log n)^{-\varepsilon}$ in 5-colorable 4-uniform hypergraphs on $n$ vertices. On the other hand, the best algorithms for these problems yield independent sets of size $n^{-\Omega(1)}$.

In this work we focus on the case of (almost) 2-colorable 3-uniform and 4-uniform hypergraphs. The motivation for our first result stems from the gap between the algorithmic and complexity results mentioned above. We prove the following.

**Theorem 1.1.** For any arbitrarily small constant $\gamma > 0$, there is a constant $\xi > 0$ such that given a 4-uniform hypergraph $G(V, E)$ on $n$ vertices such that removing $2^{-\left(\log n\right)^{\xi}}$ fraction of vertices and all hyperedges incident on them makes the remaining hypergraph 2-colorable, it is quasi-NP-hard to find an independent set in $G$ of size $n/\left(2^{\left(\log n\right)^{1-\gamma}}\right)$ vertices.

This is the first result showing an almost polynomial factor hardness for independent set in (almost) $q$-colorable $k$-uniform hypergraphs. While existing algorithms are for the case of exact colorability, they rely on the presence of a small number of pairwise disjoint independent sets covering almost all the vertices, and are also applicable to the case of almost colorability. Thus, the above result indicates a bottleneck in the improvement of existing algorithms. The hardness factor obtained is exponentially stronger than the previous lower bound of poly$(\log n)$ by Khot [Kho02a], albeit for the case of exact colorability.
Our next result is an analogue of the result of Bansal and Khot [BK09, BK10] who showed, assuming the Unique Games Conjecture (UGC), that it is NP-hard to find an independent set of $\delta$ fraction of vertices (for any constant $\delta > 0$) in an almost 2-colorable graph (i.e. almost bipartite graph). The related work of Guruswami and Sinop [GST11] showed a similar result for almost 2-colorable 3-uniform hypergraphs (with the hardness factor depending on the degree), assuming UGC. We show that it is possible to prove the result for 3-uniform hypergraphs without assuming UGC.

**Theorem 1.2.** For any constants $\varepsilon, \delta > 0$, given a 3-uniform hypergraph on $n$ vertices such that removing at most $\varepsilon$ fraction of vertices and the hyperedges incident on them makes the remaining hypergraph 2-colorable, it is NP-hard to find an independent set of $\delta n$ vertices.

The instances constructed in the Theorems 1.1 and 1.2 are degree regular, and thus also work for an alternate definition of almost colorability – which involves removing $\varepsilon$ fraction of the hyperedges instead of vertices – used in [GST11].

Our final result proves the first strong hardness factor for finding independent sets in 2-colorable 3-uniform hypergraphs, assuming the $d$-to-$1$ Games Conjecture of Khot [Kho02c].

**Theorem 1.3.** Assuming the $d$-to-1 Games Conjecture the following holds. For any constant $\delta > 0$, given a 2-colorable 3-uniform hypergraph on $n$ vertices, it is NP-hard to find an independent set of $\delta n$ vertices.

We note that Dinur, Regev and Smyth [DRS05] showed that 2-colorable 3-uniform hypergraphs are NP-hard to color with constantly many colors. However, their reduction produced instances with linear sized independent sets in the NO Case, and thus did not yield any hardness for finding independent sets in such hypergraphs. Our result therefore proves a stronger property, albeit assuming the conjecture.

In the remainder of this section we shall formally state the problems we study in this work, give an overview of previous related work and describe the techniques used in our results.

### 1.1 Problem Definition

Given a hypergraph $G$, let $\text{IS}(G)$ be the size of the maximum independent set in $G$ and let $\chi(G)$ be its chromatic number, i.e. the minimum number of colors required to color the hypergraph such that every hyperedge is non-monochromatic. We define the problem of finding independent sets in $q$-colorable hypergraphs as follows.

**ISCOLOR**($k, q, Q$) : Given a $k$-uniform hypergraph $G(V, E)$, decide between,

- **YES Case:** $\chi(G) \leq q$.
- **NO Case:** $\text{IS}(G) < \frac{|V|}{Q}$.

It is easy to see that if **ISCOLOR**($k, q, Q$) is NP-hard for some parameters $q, Q \in \mathbb{Z}^+$ then it is NP-hard to color a $q$-colorable $k$-uniform hypergraph with $Q$ colors. In this paper we also study a slight variant of this problem, in which the goal is to find independent sets in almost colorable hypergraphs. For parameters $k, q, Q$, and a parameter $\varepsilon > 0$ it is defined as follows.

**ISALMOSTCOLOR**$_\varepsilon(k, q, Q)$: Given a $k$-uniform hypergraph $G(V, E)$, decide between,

- **YES Case:** There is a subset of $(1 - \varepsilon)$ fraction of the vertices, such that for the $k$-uniform hypergraph $G'$ on this subset of vertices containing the hyperedges which lie completely inside it, $\chi(G') \leq q$. We also denote this by $\chi_\varepsilon(G) \leq q$. 

• NO Case: $\text{IS}(G) < \frac{|V|}{Q}$.

Note that the second property above, i.e. $\text{IS}(G) < \frac{|V|}{Q}$, implies that $\chi_\varepsilon(G) \geq Q - 1$ for sufficiently small $\varepsilon > 0$.

Using the above definitions the results of this paper can be concisely restated as follows. The number of vertices in the hypergraph is denoted by $n$.

Our Results

Theorem. (Theorem 1.1) For an arbitrarily small constant $\gamma > 0$, there is a constant $\xi > 0$ such that $\text{ISALMOSTCOLOR}_\varepsilon(4, 2, Q)$ is quasi-NP-hard, where $\varepsilon = 2^{-(\log n)\xi}$ and $Q = 2^{(\log n)^{1-\gamma}}$.

Theorem. (Theorem 1.2) For any constant $Q > 0$ and arbitrarily small constant $\varepsilon > 0$, $\text{ISALMOSTCOLOR}_\varepsilon(3, 2, Q)$ is NP-hard.

Theorem. (Theorem 1.3) Assuming the $d$-to-$1$ Games Conjecture the following holds. For any constant $Q > 0$, $\text{ISCOLOR}(3, 2, Q)$ is NP-hard.

1.2 Previous Work

The problem of finding independent sets in (almost) colorable graphs and hypergraphs has been studied extensively from algorithmic as well as complexity perspectives. On 2-colorable, i.e. bipartite graphs, the maximum independent set can be computed in polynomial time. On the other hand, a significant body of work – including [Wig83], [Blu94], [KMS98], [BK97], [ACC06], and [KT12] – has shown that a 3-colorable graph can be efficiently colored with $n^\alpha$ colors where the currently best value of $\alpha \approx 0.2038$ was shown in [KT12]. In particular, this shows that $\text{ISCOLOR}(2, 3, n^\alpha)$ can be efficiently solved. For 2-colorable 3-uniform hypergraphs Krivelevich et al. [KNS01] gave a coloring algorithm using $O(n^{1/5})$ colors, thus solving $\text{ISCOLOR}(3, 2, O(n^{1/5}))$. Chen and Frize [CF96] and Kelsen, Mahajan and Ramesh [KMH96] independently gave algorithms for coloring 2-colorable 4-uniform hypergraphs using $O(n^{3/4})$ colors, which implies an algorithm for $\text{ISCOLOR}(4, 2, O(n^{3/4}))$. In related work Chlamtac and Singh [CS08] gave an algorithm that on a 3-uniform hypergraph which has an independent set of $\gamma n$ vertices, efficiently computes an independent set of $n^{O(\gamma^2)}$ vertices. While the algorithmic approaches have studied the case of exactly colorable hypergraphs, they rely on the existence of disjoint independent sets and are also applicable to almost colorable hypergraphs.

Several hardness results for these problems have been obtained using either the PCP Theorem or well known conjectures as the starting point. Under standard complexity assumptions, Khot [Kho02a] showed the hardness of finding independent sets of size $(\log n)^{-\epsilon}$ in 5-colorable 4-uniform hypergraphs on $n$ vertices. Building upon similar work of Guruswami, Håstad and Sudan [GHS02], Holmerin [Hol02] showed that it is NP-hard to find an independent set of size $\delta$ in a 2-colorable 4-uniform hypergraph, where $\delta > 0$ is any constant. For 3-uniform hypergraphs which are 3-colorable, Khot [Kho02b] showed a hardness of finding independent sets of size $(\log \log n)^{-\epsilon}$. On 3-colorable graphs, assuming the so called Alpha Conjecture, Dinur et al. [DMR09] showed it is NP-hard to find independent sets of size $\delta$. Bansal and Khot [BK09] assumed the more well known Unique Games Conjecture to show that it is NP-hard to find independent sets of size $\delta$ in almost bipartite (i.e. almost 2-colorable) graphs. Guruswami and Sinop [GST11] showed a similar result for almost 2-colorable 3-uniform hypergraphs, the focus of their work being the case of bounded degree hypergraphs.
It is pertinent to note that while the algorithmic results have poly($n$) factors, the previous best inapproximability was a poly($\log n$) factor \cite{Kho02a}. Our result for independent set in almost 2-colorable 4-uniform hypergraphs – Theorem 1.1 – partially bridges this gap by showing an almost polynomial factor $2^{-(\log n)^{1-\varepsilon}}$, an exponential improvement over the previous lower bound.

Theorem 1.2 unconditionally proves the hardness result for independent set in almost 2-colorable 3-uniform hypergraphs, which was earlier known only assuming the Unique Games Conjecture. We also show – in Theorem 1.3 – the first inapproximability for the case of 2-colorable 3-uniform hypergraphs assuming the $d$-to-1 Games Conjecture.

In the rest of this section we give an informal overview of the techniques used to proves our results.

1.3 Overview of Techniques

The results of this work follow a common template of reductions from an instance of a NP-hard constraint satisfaction problem – the so called Outer Verifier – via its combination with a proof encoding – the Inner Verifier. However, the techniques used to prove Theorems 1.1, 1.2 and 1.3 are somewhat varied and we describe them separately.

Almost 2-Colorable 4-Uniform Hypergraphs

The goal of this result is to prove an almost polynomial hardness factor for independent set in almost 2-colorable 4-uniform hypergraphs. To accomplish this, the size of the hardness reduction needs to be bounded. Thus, one cannot use Long Codes which have an unmanageable blowup for our purpose. Instead, we use Hadamard Codes which are exponentially shorter and have been used in previous works \cite{KP06, KS08a} for a similar reason. The Hadamard Code $H^v$ of an element $v \in \mathbb{F}[2]^m$ is indexed by all $x \in \mathbb{F}[2]^m$ such that $H^v(x) := x \cdot v \in \mathbb{F}[2]$. The “gadget” used for the reduction is as follows.

Consider the following 4-uniform hypergraph. The vertex set is $\mathbb{F}[2]^m$. Let $e_1 \in \mathbb{F}[2]^m$ be the element which has 1 in the first coordinate and 0 everywhere else. For any $x, y, z \in \mathbb{F}[2]^m$, add a hyperedge between the elements $x, y, x + z$ and $y + z + e_1$, where the addition is done in the vector space $\mathbb{F}[2]^m$. This is (essentially) a 4-uniform hypergraph. Consider any element $v \in \mathbb{F}[2]^m$ such that $v_1 = 1$. It is easy to see that $H^v(x) + H^v(x + z) + H^v(y) + H^v(y + z + e_1) = 1$, and thus the coloring to $\mathbb{F}[2]^m$ given by the value of $H^v$ is a valid 2-coloring of this hypergraph. On the other hand it can be shown that any independent set $S \subseteq \mathbb{F}[2]^m$ of size $\delta 2^m$ can be decoded into a list of elements $v$ such that $v_1 = 1$. This analysis uses only some basic tools from Fourier Analysis.

The above gadget can be combined with a parallel repetition of an appropriate linear constraint system. In our case, we choose a specialized instance of MAX-3LIN constructed by Khot and Ponnuswami \cite{KP06}. The main idea in this combination is to do the folding only over the homogeneous constraints and use the non-homogeneous constraints to play the role of $e_1$ in the above gadget. The almost polynomial hardness factor is obtained by an appropriate number of rounds of parallel repetition which is afforded by the parameters of the MAX-3LIN instance used in the reduction.

Almost 2-Colorable 3-Uniform Hypergraphs

This reduction uses as the Outer Verifier a layered constraint satisfaction problem, referred to as the Multi-Layered PCP. This PCP was used earlier by Khot \cite{Kho02b} for similar results for 3-Colorable 3-Uniform Hypergraphs and by Dinur, Guruswami, Khot and Regev \cite{DGKR03} and Sachdeva and Saket \cite{SS11} in their
hardness results for hypergraph vertex cover. Due to some fundamental limitations of existing techniques, the use of this PCP is necessitated for proving results for independent sets in 3-uniform hypergraphs.

The Inner Verifier uses a biased Long Code encoding similar to the reductions of Dinur, Khot, Perkins and Safra [DKPS10], Khot and Saket [KS12] and Sachdeva and Saket [SS13]. The following gadget encapsulates the Inner Verifier. Consider the biased Long Code $H = \left\{ 1, 2, * \right\}^m$. The associated measure is induced by sampling each coordinate independently to be 1 or 2 with probability $\frac{1-\varepsilon}{2}$ and * with probability $\varepsilon$. Let $\mathcal{H}_0, \mathcal{H}_1, \ldots, \mathcal{H}_d$ be $d + 1$ identical copies of $\mathcal{H}$. A vertex weighted 3-uniform hypergraph is constructed by taking the union of the $d + 1$ Long Codes with weights given by the measure. Consider $x \in \mathcal{H}_0$ and $y, z \in \mathcal{H}_k$ ($1 \leq k \leq d$), such that for any $i \in [m]$ the tuple $(x_i, y_i, z_i)$ is not $(1, 1, 1)$ or $(2, 2, 2)$. Add a hyperedge between $x$, $y$ and $z$ for all such choices. It is easy to see that for any $j \in [m]$, removing all the vertices $x$ such that $x_j = *$ and all hyperedges incident on these vertices makes the hypergraph 2-colorable by coloring the rest of the vertices $y$ according to whether $y_j = 1$ or 2. On the other hand, using Russo’s Lemma and Friedgut’s Junta Theorem one can show that if there is an independent set $I$ which has at least $\delta$ fraction of measure from each of the $d + 1$ Long Codes, then it can be decoded into a distinguished coordinate $\ell \in [m]$. This Inner Verifier is robust enough to be combined with the Multi-Layered PCP to yield the desired result.

The hardness factor obtained, however, is much weaker than in the previous reduction, due to our use of Long Codes and also due to the structure of the Multi-Layered PCP.

1.3.1 2-Colorable 3-Uniform Hypergraphs

For independent set in 2-colorable 3-uniform hypergraphs, the existing PCP techniques seem insufficient to yield the desired results. Thus, we rely on the $d$-to-1 Games Conjecture of Khot [Kho02c]. This conjecture was earlier used to establish hardness results for independent sets in 4-colorable graphs [DMR09]. Our use of this conjecture is similar to that of O’Donnell and Wu [OW09] who showed an optimal $\frac{5}{8} + \varepsilon$ factor hardness for a satisfiable instance of MAX-3CSP. In a recent work Hästad [Hås12] showed the same result unconditionally. We also make use of certain techniques used in [Hås12].

The Outer Verifier in our reduction is a multi-layered PCP constructed using the $d$-to-1 games problem. The construction of this PCP ensures a smoothness property which has been used in several previous works [Kho02b, KS06, KS08b, GRSW12] including the above mentioned work of Hästad [Hås12] and a related work of Wenner [Wen12]. The Inner Verifier yields a 3-uniform hypergraph with hyperedges corresponding to a 3-query PCP test over Long Codes which is in a same vein as the test used in [OW09] and [Hås12]. The analysis is based in large part on the Invariance Principle of Mossel [Mos10], the application of which follows an approach used by O’Donnell and Wu [OW09], while avoiding certain complications they face. The smoothness property is crucial for the analysis and is leveraged in a manner similar to [Hås12].

Organization of Paper. The next section contains the known PCP constructions which shall be the starting points in our reductions for Theorems 1.1 and 1.2. We shall also state the $d$-to-1 Games Conjecture that we shall require for proving Theorem 1.3 and describe the smooth layered PCP we construct based on this assumption, a sketch of the construction being deferred to Section A.

Sections 3, 4 and 5 contain the hardness reduction and proofs for Theorems 1.1, 1.2 and 1.3 respectively along with a description of the mathematical tools needed to complete the analyses.
2 Preliminaries

In this section we shall describe some useful results in PCPs and hardness of approximation along with the description of the $d$-to-1 Games Conjecture.

For proving Theorem 1.1 we shall begin with the following theorem of Khot and Ponnuswami [KP06] on the hardness of a specific gap version of MAX-3LIN with a desirable setting of the parameters. An instance of MAX-3LIN consists of a system of linear equations over $\mathbb{F}[2]$ where each equation has exactly 3 variables, the goal being to find an assignment to the variables satisfying the maximum number of equations. The instance is said to be $d$-regular if each variable occurs in exactly $d$ equations.

Theorem 2.1. [KP06] Given a 7-regular instance $A$ of MAX-3LIN over $\mathbb{F}[2]$ on $n$ variables, unless $\text{NP} \subseteq \text{DTIME}(2^{O(\log^2 n)})$, there is no polynomial time algorithm to distinguish between the following two cases,

- **YES Case.** There is an assignment to the variables of $A$ that satisfies $1 - c(n) := 1 - 2^{-\Omega(\sqrt{\log n})}$ fraction of the equations (completeness).
- **NO Case.** No assignment to the variables of $A$ satisfies more than $1 - s(n) := 1 - \Omega(\log^{-3} n)$ fraction of the equations (soundness).

The usefulness of the above theorem is due to the fact that the completeness is very close to 1, while the soundness is bounded away from 1 to allow for poly$(\log n)$ rounds of parallel repetition.

The rest of this section describes PCP constructions – required for Theorems 1.2 and 1.3 – which are somewhat more complicated.

2.1 Multi-Layered PCP

The Multi-Layered PCP described here was constructed by Dinur, Guruswami, Khot and Regev [DGKR03] who also proved its useful properties. An instance $\Phi$ of the Multi-Layered PCP is parametrized by integers $L, R > 1$. The PCP consists of $L$ sets of variables $V_1, \ldots, V_L$. The label set (or range) of the variables in the $i$th set $V_i$ is a set $\pi_i$ such that $|\pi_i| = R^O(L)$. For any two integers $1 \leq l < l' \leq L$, the PCP has a set of constraints $\Phi_{l,l'}$ in which each constraint depends on one variable $v \in V_l$ and one variable $v' \in V_{l'}$. The constraint (if it exists) between $v \in V_l$ and $v' \in V_{l'}$ is denoted and characterized by a projection $\pi_{v \rightarrow v'} : R_l \rightarrow R_{l'}$. A labeling to $v$ and $v'$ satisfies the constraint $\pi_{v \rightarrow v'}$ if the projection (via $\pi_{v \rightarrow v'}$) of the label assigned to $v$ coincides with the label assigned to $v'$.

The following useful ‘weak-density’ property of the Multi-Layered PCP was defined in [DGKR03], which (roughly speaking) states that any significant subset of variables induces a significant fraction of the constraints between some pair of layers.

Definition 2.2. An instance $\Phi$ of the Multi-Layered PCP with $L$ layers is weakly-dense if for any $\delta > 0$, given $m \geq \lceil \frac{2}{\delta} \rceil$ layers $l_1 < l_2 < \cdots < l_m$ and given any sets $S_i \subseteq V_{l_i}$ for $i \in [m]$ such that $|S_i| \geq \delta |V_{l_i}|$, there always exist two layers $l_{i'}$ and $l_{i''}$ such that the constraints between the variables in the sets $S_{i'}$ and $S_{i''}$ is at least $\frac{2}{\delta^2}$ fraction of the constraints between the sets $V_{l_{i'}}$ and $V_{l_{i''}}$.

The following inapproximability of the Multi-Layered PCP was proven by Dinur et al. [DGKR03] based on the PCP Theorem ([AS98], [ALM98]) and Raz’s Parallel Repetition Theorem ([Raz98]).

Theorem 2.3. There exists a universal constant $\gamma > 0$ such that for any parameters $L > 1$ and $R$, there is a weakly-dense $L$-layered PCP $\Phi = \cup \Phi_{l,l'}$ such that it is NP-hard to distinguish between the following two cases:
In addition we make the assumption that the pre-image of every singleton is of size exactly $d$. We now state the $d$-to-1 Games Conjecture.

**Conjecture 2.5.** ($d$-to-1 Games Conjecture [Kho02c]) There is a fixed positive integer $d$ such that for any $\zeta > 0$, there exist integers $k$ and $m$ such that given a $d$-to-1 Game instance $\mathcal{L}$ with label sets $[k]$ and $[m]$ it is NP-hard to distinguish between the following two cases:

- **YES** Case. There is a labeling to the variables that satisfies all the constraints.
- **NO** Case. Any labeling to the variables satisfies at most $\zeta$ fraction of constraints.

In addition we make the assumption$^3$ that the instance $\mathcal{L}$ is bi-regular, i.e. for any variable $v \in \mathcal{V}$ the number of constraints containing $v$ is the same, and similarly for any variable $u \in \mathcal{U}$ the number of constraints containing $u$ is the same.

Using Conjecture 2.5, we have the following layered PCP with an additional smoothness property.

### 2.3 Smooth $d$-to-1 Multi-Layered PCP

The following is an analogue of the Multilayered PCP based on the $d$-to-1 conjecture and also incorporating the smoothness property. We shall refer to it as the Smooth $d$-to-1 MLPCP.

An instance $\Phi$ of the Smooth $d$-to-1 MLPCP is parametrized by integers $d, L, R, T > 1$. The PCP consists of $L$ sets of variables $V_1, \ldots, V_L$. The label set (or range) of the variables in the $l$th set $V_l$ is a set $R_l$ where $|R_l| = R^{O(T L)}$. For any two integers $1 \leq l < l' \leq L$, the PCP has a set of constraints $\Phi_{l,l'}$ in which each constraint depends on one variable $v \in V_l$ and one variable $v' \in V_{l'}$. The constraint (if it exists) between $v \in V_l$ and $v' \in V_{l'}$ ($l < l'$) is denoted and characterized by a projection $\pi_{v \rightarrow v'} : R_l \to R_{l'}$. The projection $\pi_{v \rightarrow v'}$ has the property that for every $j \in R_{l'}$, $|\pi_{v \rightarrow v'}^{-1}(j)| = d^{l-l'}$. A labeling to $v$ and $v'$ satisfies the constraint $\pi_{v \rightarrow v'}$ if the projection (via $\pi_{v \rightarrow v'}$) of the label assigned to $v$ coincides with the label assigned to $v'$.

We have a similar weak density property as in the previous section.

$^3$It is not known whether this assumption can be made WLOG. However, all known Label Cover constructions are bi-regular which makes the assumption, in the authors’ opinion, a reasonable one.
Definition 2.6. An instance \( \Phi \) of Smooth \( d \)-to-1 MLPCP with \( L \) layers is weakly-dense if for any \( \delta > 0 \), given \( m \geq \lceil \frac{2}{\delta} \rceil \) layers \( l_1 < l_2 < \cdots < l_m \) and given any sets \( S_i \subseteq V_{l_i} \), for \( i \in [m] \) such that \( |S_i| \geq \delta |V_{l_i}| \); there always exist two layers \( l_\nu \) and \( l_{\nu'} \) such that the constraints between the variables in the sets \( S_\nu \) and \( S_{\nu'} \) is at least \( \frac{\delta^2}{4} \) fraction of the constraints between the sets \( V_{l_\nu} \) and \( V_{l_{\nu'}} \).

We also have the smoothness property as defined below.

Definition 2.7. An instance \( \Phi \) of Smooth \( d \)-to-1 MLPCP with \( L \) layers and parameter \( T \) has the smoothness property if for any two layers \( l < l' \), and variable \( v \in V_l \) and two distinct labels \( i, j \in R_l \),

\[
\Pr_{v' \in N(v) \cap V_{l'}} \left[ \pi_{v \to v'}(i) = \pi_{v \to v'}(j) \right] \leq \frac{1}{T},
\]

where the probability is taken over a random variable in \( V_{l'} \) which has a constraint with \( v \).

The following inapproximability of the Smooth \( d \)-to-1 MLPCP essentially follows from combining Conjecture 2.5 with the layered construction of [Kho02b]. A sketch of the construction is provided in Section A.

Theorem 2.8. Assuming Conjecture 2.5 the following holds. There exists a universal constant positive integer \( d \) such that for any arbitrarily small constant \( \zeta > 0 \), there exists a positive integer \( R \), such that for every \( L, T > 1 \), there is a weakly-dense and smooth \( L \)-layered PCP with parameters \( d, T, R \), \( \Phi = \cup \Phi_{l,l'} \), such that it is NP-hard to distinguish between the following two cases:

- **YES** Case. There exists an assignment of labels to the variables of \( \Phi \) that satisfies all the constraints.
- **NO** Case. For every \( 1 \leq l < l' \leq L \), not more than \( \zeta \) fraction of the constraints in \( \Phi_{l,l'} \) can be satisfied by any assignment.

3 Independent Set in Almost 2-Colorable 4-Uniform Hypergraphs

This section presents a hardness reduction from Theorem 2.1 to an instance of \( \text{ISALMOSTCOLOR}_{\varepsilon}(4, 2, Q) \). The reduction employs an Inner Verifier based on Hadamard Codes. The Hadamard Code of an element \( v \in \mathbb{F}[2]^m \) is a \( \mathbb{F}[2] \)-valued code indexed by the elements of \( \mathbb{F}[2]^m \) and its value at \( x \in \mathbb{F}[2]^m \) is the dot-product \( x \cdot v \in \mathbb{F}[2] \).

3.1 Hardness Reduction

Let \( \mathcal{A} \) be the \( \text{MAX}-3\text{LIN} \) instance given by Theorem 2.1. The reduction begins with choosing a positive integer \( r \) which we shall set later. In the first part of the reduction we shall construct an Outer Verifier which shall be an \( r \)-round parallel repetition of a verifier-prover game obtained from the instance \( \mathcal{A} \).

3.1.1 Outer Verifier

Let \( \Phi_r \) be the collection of all blocks of \( r \) variables each from \( \mathcal{A} \), and \( \Psi_r \) be the collection of all blocks of \( r \) equations each.

Consider the following 2-prover 1-round game 2P1R(\( \mathcal{A}, r \)):

1. The Verifier chooses one block \( W \) uniformly at random from \( \Psi_r \). From each equation in \( W \), the verifier chooses one out of the three variables at random to construct a block \( U \) of \( \Phi_r \).
2. The Verifier sends $U$ to Prover-1 and $W$ to Prover-2 and expects from each prover an assignment to all the variables that it received.

3. The Verifier accepts if the assignment given by Prover-2 satisfies all equations of $W$ and is consistent with the assignment given to the variables of $U$ by Prover-1.

The Parallel Repetition Theorem of Raz [Raz98] and its subsequent strengthening by Holenstein [Hol09] and Rao [Rao08] imply the following.

**Theorem 3.1.** The 2 prover 1 round game $2P1R(A, r)$, where $A$ is an instance on $n$ variables given by Theorem 271 has the following properties:

- **YES Case.** If $A$ is a YES instance then the Verifier accepts with probability at least $(1 - c(n))^r$.
- **NO Case.** If $A$ is a NO instance then the Verifier accepts with probability at most $(1 - s(n)^\kappa)^r/\kappa$ for some universal constant $\kappa > 1$.

For the rest of the reduction we shall assume that none of the blocks $W$ or $U$ contain a repeated variable. This omits only a tiny fraction of blocks which does not change any parameter noticeably.

### 3.1.2 Inner Verifier

Consider a block $W$ of $r$ equations. It contains $3r$ distinct variables say $x_1, x_2, \ldots, x_{3r-1}, x_{3r}$. We may assume without loss of generality that the $i$th equation consists of the variables $x_{3i-2}, x_{3i-1},$ and $x_{3i}$, for $i = 1, \ldots, r$. We shall now associate an element of $\mathbb{F}[2]^{3r+1}$ with each of the $r$ equations of $W$. Note that the $(3r+1)$th coordinate is extra and added to help with ensuring consistency.

Suppose that the $i$th (for some $i \in [r]$) equation is of the form $x_{3i-2} + x_{3i-1} + x_{3i} = 0$, then let $h_i \in \mathbb{F}[2]^{3r+1}$ be such that the dot-product $h_i \cdot x = x_{3i-2} + x_{3i-1} + x_{3i}$ for any $x \in \mathbb{F}[2]^{3r+1}$. Otherwise, if the $i$th equation is of the form $x_{3i-2} + x_{3i-1} + x_{3i} = 1$, then let $h_i$ be such that $h_i \cdot x = x_{3i-2} + x_{3i-1} + x_{3i} + x_{3r+1}$. Our assumption that the block $W$ does not contain a repeated variable implies that the set of elements $\{h_i\}_{i=1}^r$ is linearly independent. Let $H_W$ be the $r$ dimensional space spanned by $\{h_i\}_{i=1}^r$. For completing the reduction we also define an element $h_W \in \mathbb{F}[2]^{3r+1}$ so that $h_W \cdot x = x_{3r+1}$ for any $x \in \mathbb{F}[2]^{3r+1}$.

Let $\overline{C}[W]$ be a $\{0, 1\}$ code indexed by the elements of $\mathbb{F}[2]^{3r+1}/H_W$, i.e. the set of cosets of the subspace $H_W$ in the space $\mathbb{F}[2]^{3r+1}$. Since $H_W$ is a $r$ dimensional subspace, the size of the code $\overline{C}[W]$ is $2^{2r+1}$. We say that $\overline{C}[W]$ is folded over $H_W$. It is easy to see that any $\overline{C}[W] : \mathbb{F}[2]^{3r+1}/H_W \mapsto \{0, 1\}$ can be unfolded into $C[W] : \mathbb{F}[2]^{3r+1} \mapsto \{0, 1\}$ such that, $C[W](x+y) = \overline{C}[W](x+H_W)$ for any $x \in \mathbb{F}[2]^{3r+1}$ and $y \in H_W$. For notational convenience we shall represent the coset $x + H_W$ simply by $x$, and this shall be clear from the context.

Ideally, $C[W]$ is supposed to be the Hadamard Code of a satisfying assignment to the variables in $W$ with the $(3r+1)$th coordinate set to 1 $\in \mathbb{F}[2]$, so that the code $\overline{C}[W]$ is well defined and folded over the subspace $H_W$.

We are now ready to define the vertices and hyperedges of the instance $G(V, E)$ of ALMOSTCOLHYP(2, 4).

**Vertices.** The vertex set $V$ consists of all the locations of $\overline{C}[W]$ for each $W \in \Psi_r$, i.e. each block $W$ of $r$ equations.

**Hyperedges.** Consider any choice of $U \in \Phi_r$ and $W \in \Psi_r$ by the verifier in the game $2P1R(A, r)$ in Step 1. Let $U$ and $W' \in \Psi_r$ be another choice with the same block of $r$ variables $U$. Let $\pi_W : \mathbb{F}[2]^{3r+1} \mapsto \mathbb{F}[2]^r$ be a projection onto the coordinates of the $r$ variables of $U$ from the block of $(3r+1)$ coordinates corresponding
to the $3r$ variables of $W$ and the extra coordinate as defined above. The extra coordinate plays no part in this projection. We shall also use the notation $\pi^{-1} : \mathbb{F}[2]^r \rightarrow \mathbb{F}[2]^{3r+1}$, which extends a vector by filling in zeros in the rest of the coordinates. Similarly, $\pi_W$, be the projection for $W$. Let $\overline{C}[W]$ and $\overline{C}[W']$ be the codes of $W$ and $W'$. For all such choices of $U$, $W$ and $W'$ do the following.

1. For all choices of elements $x, y \in \mathbb{F}[2]^{3r+1}$ and $z \in \mathbb{F}[2]^r$ such that $z \neq 0$, do step 2.

2. Add a hyperedge between the vertices (or locations of the codes): $\overline{C}[W](x), \overline{C}[W](x + \pi_W^{-1}(z) + h_W), \overline{C}[W'](y)$ and $\overline{C}[W'](y + \pi_W^{-1}(z))$. It is easy to see that since $z \neq 0$ the four vertices chosen above are distinct.

This completes the hardness reduction and we move to its analysis.

### 3.2 YES Case

In the YES Case the instance $A$ has an assignment $\sigma^*$ to its variables that satisfies $(1 - c(n))$ fraction of its equations. Call the equations satisfied by $\sigma^*$ as good. Similarly, call a block of $r$ equations as good if all of its equations are good. Clearly, at least $(1 - c(n))^r$ fraction of the blocks are good.

For any good block $W$ let $C[W] : \mathbb{F}[2]^{3r+1} \rightarrow \mathbb{F}[2]$ be the Hadamard Code of the assignment $\sigma^*(W) \in \mathbb{F}[2]^{3r}$ to the variables in $W$, concatenated with a 1 in the $(3r + 1)$th coordinate. Let us denote this concatenated vector as $(\sigma^*(W), 1)$. In other words, $C[W](x) = (\sigma^*(W), 1) \cdot x \in \mathbb{F}[2]$, for $x \in \mathbb{F}[2]^{3r+1}$. Since $\sigma^*$ satisfies all equations in $W$, it is easy to see that it is invariant over the cosets of $H_W$, i.e.,

$$C[W](x + y) = C[W](x) \quad \text{for} \quad x \in \mathbb{F}[2]^{3r+1} \text{ and } y \in H_W.$$ 

Thus this can be folded into the code $\overline{C}[W]$ by defining $\overline{C}[W](x + H_W) = C[W](x)$. As before, we shall use $\overline{C}[W](x)$ to represent the value over the coset $x + H_W$.

The above defines a 2-coloring of the locations of the codes of all good blocks depending on its value in $\mathbb{F}[2]$. We shall show that any hyperedge completely induced by these locations is non-monochromatic.

Consider a choice of $U, W$ and $W'$ in the construction of the hyperedges where $W$ and $W'$ are good blocks. Let $x, y$ and $z$ be chosen as in Step 1. We shall show that,

$$\overline{C}[W](x) + \overline{C}[W](x + \pi_W^{-1}(z) + h_W) + \overline{C}[W'](y) + \overline{C}[W'](y + \pi_W^{-1}(z)) = 1, \quad (1)$$

which implies that the corresponding hyperedge is non-monochromatic. To see this, observe that the LHS of the above equation is,

$$(\sigma^*(W), 1) \cdot x + (\sigma^*(W), 1) \cdot (x + \pi_W^{-1}(z) + h_W) + (\sigma^*(W'), 1) \cdot y + (\sigma^*(W'), 1) \cdot (y + \pi_W^{-1}(z))$$

$$(\sigma^*(W), 1) \cdot (\pi_W^{-1}(z)) + (\sigma^*(W'), 1) \cdot (\pi_W^{-1}(z)) + (\sigma^*(W), 1) \cdot h_W$$

$$= (\pi_W(\sigma^*(W))) + \pi_W(\sigma^*(W')) \cdot z + 1$$

$$= 1, \quad (2)$$

where the second last equation follows from the definition of $h_W$ and last equation follows from the fact that $\sigma^*$ is a global assignment so its projection onto $U$ from $W$ or $W'$ is the same.

Thus, after removing a $1 - (1 - c(n))^r$ fraction of vertices corresponding to the blocks which are not good and all hyperedges incident on them, the rest of the hypergraph is 2-colorable.
### 3.3 NO Case

Let $\mathcal{I}$ be an independent set in $G$. For every block $W$ of $r$ equations, let $\mathcal{C}[W]$ be the indicator of $\mathcal{I}$ restricted to the locations of the code $\mathcal{C}[W]$. Here, $\mathcal{C}[W]$ is a thought of as a $\{0, 1\}$ real valued code.

Let $U, W$ and $W'$ be the choices in the construction of the hyperedges. For all choices of $x, y$ and $z$ in Step 1 of the construction, we have:

$$\mathcal{C}[W](x) \cdot \mathcal{C}[W](x + \pi^{-1}_W(z) + h_W) \cdot \mathcal{C}[W'](y) \cdot \mathcal{C}[W'](y + \pi^{-1}_{W'}(z)) = 0. \quad (3)$$

As mentioned earlier, we can unfold the codes into $C[W]$ and $C[W']$ to rewrite the above as,

$$C[W](x) \cdot C[W](x + \pi^{-1}_W(z) + h_W) \cdot C[W'](y) \cdot C[W'](y + \pi^{-1}_{W'}(z)) = 0. \quad (4)$$

For convenience of notation, we shall refer to $C[W]$ as $A$ and $C[W']$ as $B$. Doing the usual Fourier expansion and using standard tools from Fourier Analysis over folded codes (refer to Section 3B for an overview) we get the following.

$$\sum_{\alpha, \alpha', \beta, \beta' \in \mathbb{F}[2]^{r+1}} \hat{A}_\alpha \chi_\alpha(x) \hat{A}_{\alpha'} \chi_{\alpha'}(x + \pi^{-1}_W(z) + h_W) \hat{B}_\beta \chi_\beta(y) \hat{B}_{\beta'} \chi_{\beta'}(y + \pi^{-1}_{W'}(z)) = 0$$

$$\Rightarrow \sum_{\alpha, \alpha', \beta, \beta' \in \mathcal{H}_W, \beta, \beta' \in \mathcal{H}_{W'}} \hat{A}_\alpha \hat{A}_{\alpha'} \hat{B}_\beta \hat{B}_{\beta'} \chi_{\alpha + \alpha'}(h_W) \chi_{\pi_W(\alpha')} \chi_{\beta + \beta'}(y) \chi_{\pi_{W'}(\beta')}(z) = 0 \quad (5)$$

The above is true for all $x, y$ and $z$ such that $z \neq 0$ which are independent of each other. Thus, for a fixed value of $x$ and $y$, the expectation of the LHS of Equation (5) over all $z \in \mathbb{F}[2]^r$ is equal to $2^{-r}$ times its value at $z = 0$. Observing that in the expectation over all $z \in \mathbb{F}[2]^r$ only terms satisfying $\pi_W(\alpha') = \pi_W(\beta')$ survive, we obtain,

$$\sum_{\alpha, \alpha', \beta, \beta' \in \mathcal{H}_W, \beta, \beta' \in \mathcal{H}_{W'}} \hat{A}_\alpha \hat{A}_{\alpha'} \hat{B}_\beta \hat{B}_{\beta'} \chi_{\alpha + \alpha'}(h_W) \chi_{\pi_W(\alpha')} \chi_{\beta + \beta'}(y) = 2^{-r} \sum_{\alpha, \alpha' \in \mathcal{H}_W, \beta, \beta' \in \mathcal{H}_{W'}} \hat{A}_\alpha \hat{A}_{\alpha'} \hat{B}_\beta \hat{B}_{\beta'} \chi_{\alpha + \alpha'}(h_W) \chi_{\beta + \beta'}(y). \quad (6)$$

Taking a further expectation over $x$ and $y$, we observe that the only terms that survive on the LHS are those in which $\alpha = \alpha'$, $\beta = \beta'$ and $\pi_W(\alpha) = \pi_W(\beta)$, while the terms that survive on the RHS have $\alpha = \alpha'$ and $\beta = \beta'$. Thus we obtain,

$$\sum_{\alpha, \beta \in \mathcal{H}_W, \beta \in \mathcal{H}_{W'}} \hat{A}_\alpha^2 \hat{B}_\beta^2 \chi_{\alpha}(h_W) = 2^{-r} \sum_{\alpha \in \mathcal{H}_W, \beta \in \mathcal{H}_{W'}} \hat{A}_\alpha^2 \hat{B}_\beta^2 \chi_{\alpha}(h_W) \leq 2^{-r}, \quad (7)$$

where the last inequality is because the sum of squares of the Fourier coefficients is at most 1. Now, $\chi_{\alpha}(h_W) = -1$ if $\alpha \cdot h_W = 1$ and 1 otherwise. Thus,

$$\sum_{\alpha \in \mathcal{H}_W, \beta \in \mathcal{H}_{W'}} \hat{A}_\alpha^2 \hat{B}_\beta^2 \geq 2^{-r} \sum_{\alpha \in \mathcal{H}_W, \beta \in \mathcal{H}_{W'}} \hat{A}_\alpha^2 \hat{B}_\beta^2 - 2^{-r} \quad (8)$$

$$\geq \hat{A}_0^2 \hat{B}_0^2 - 2^{-r}. \quad (9)$$
The above gives a strategy for the provers of 2P1R($A, r$). Suppose Prover-1 receives a block of variables $U$ and Prover-2 receives a block of equations $W$.

Strategy of Prover-2: It chooses a vector $\alpha \in \mathbb{F}[2]^{3r+1}$ satisfying: (i) $\alpha \perp H_W$, and (ii) $\alpha \cdot h_W = 1$ with probability $\tilde{A}_0^2$, where $A = C[W]$. Since $\alpha$ satisfies (i) and (ii), the first $3r$ coordinates give a satisfying assignment to the variables in $W$. This assignment is returned to the Verifier by Prover-2.

Strategy of Prover-1: It chooses a block $W'$ from the choice of the verifier of 2P1R($A, r$) conditioned on the block of variables picked being $U$. It then chooses $\beta \in \mathbb{F}[2]^{3r+1}$ satisfying: $\beta \perp H_W$, with probability $\tilde{B}_\beta^2$ where $B = C[W']$. The assignment to the variables in $U$ contained in the first $3r$ coordinates of $\beta$ is returned to the Verifier.

Suppose that the independent set $I$ contains $\delta$ fraction of the vertices of $G$, i.e. locations of the codes. Recall that we set the value of the code $\overline{C}[W]$ to be the indicator of $I$ restricted to its locations. Thus, for at least $\delta/2$ fraction of the blocks $W$, $E_x[C[W](x)] \geq \delta/2$. Call such blocks as heavy.

Conditioned on the block of variables $U$, let $p_U$ be the fraction of choices of block of equations $W$ by the verifier 2P1R($A, r$) such that $W$ is heavy. From the above $E_U[p_U] \geq \delta/2$, by the regularity of $A$. Thus, the probability that both $W$ and $W'$ are heavy – where $W'$ is obtained from the strategy of Prover-1 – is $E_U[p_U^2] \geq E_U[p_U]^2 \geq \delta^2/4$. Noting that the weight of $\overline{C}[W]$ is same as that of $C[W]$ which is given by the empty coefficient of the Fourier expansion, we obtain that the verifier accepts with probability at least,

$$\frac{\delta^2}{4} \left( \frac{\delta^2}{4} - 2^{-r} \right) \geq \left( \frac{\delta^2}{4} - 2^{-r} \right)^2.$$

From Theorem 3.1 this implies that $\delta^2/4 \leq (1 - s(n)^{\kappa(r/2\kappa)} + 2^{-r}$.

**Setting the Parameters.** We set $r = \log^\ell n$ for a large enough constant $\ell$. The size of the hypergraph is $N = 2^{\Theta(\log n)}$. In the YES case, the number of vertices to be removed is at most $c(n)r \leq 2^{(\log N)^\xi}$ for some positive constant $\xi$ (depending on $\ell$). Further, we obtain that $(1 - s(n)^{\kappa(r/2\kappa)} + 2^{-r} \leq 2^{(\log N)^{1-\gamma}}$ for an arbitrarily small $\gamma$ by an appropriately large choice of the constant $\ell$. The above analysis yields a bound of $2^{-(\log N)^{1-\gamma}}$ on the relative size of the largest independent set in the NO case, for arbitrarily small $\gamma > 0$.

### 4 Independent Set in Almost 2-colorable 3-uniform hypergraphs

We first need a few useful definitions and results for our analysis which follows a pattern similar to previous works [DKPS10, KS12, SS13] and we shall use their notation.

#### 4.1 Preliminaries

A family $F \subseteq \{*, 1, 2\}^m$ is called monotone if for any $F \in F$ and $F' \in F$ obtained by changing a * to either 1 or 2 in any coordinate, $F' \in F$. For a parameter $p \in [0, 1]$, define the measure $\mu_p$ on $\{1, 2, *\}^m$ by $\mu_p(F) = p^m - m'(1 - p)^m'$, where $m'$ is the number of coordinates of $F$ with * in them, for any $F \in \{1, 2, *\}^m$. In other words $\mu_p$ is the product measure assigning in each coordinate a measure $1 - p$ to * and $\frac{p}{2}$ to each of 1 and 2. The measure of a family $F \subseteq \{1, 2, *\}^m$ is $\mu_p(F) = \sum_{F \in F} \mu_p(F)$.

A set $C \subseteq [m]$ is a $(\delta, p)$-core for a family $F$, if there exists a family $F'$ such that $\mu_p(F \triangle F') \leq \delta$ and $F'$ depends only on the coordinates in $C$. Let $t \in (0, 1)$ be a given parameter and $C \subseteq [m]$. A core-family $[F]^C_C$ is a family on the set of coordinates $C$ which resembles $F$ restricted to $C$. Formally,

$$[F]^t_C \overset{\text{def}}{=} \left\{ F \in \{*, 1, 2\}^C \left| \Pr_{F' \in \mu_p^C \setminus C} [(F, F') \in F] > t \right. \right\}.$$
where \((F, F')\) is an element in \(\{*, 1, 2\}^m\) by combining \(F\) on coordinates in \(C\) and \(F'\) on \([m]\setminus C\). The influence of a coordinate \(i \in [m]\) for a family \(F\) under the measure \(\mu_p\) is defined as follows:

\[
\Inf_i^p(F) := \mu_p\left(\{F : F \mid_{i=\ast} \not\in F \text{ and } F \mid_{i=j} \in F \text{ for some } j \in \{1, 2\}\right),
\]

where \(F \mid_{i=\ast}\) is an element identical to \(F\) except on the \(i\)th coordinate where it is \(*\), and \(F \mid_{i=r}\), for \(r \in \{1, 2\}\) is similarly defined. The average sensitivity of \(F\) at \(p\) is the sum of influence of all coordinates:

\[
as_p(F) := \sum_{i=1}^{m} \Inf_i^p(F).
\]

Let \(D^p\) be a distribution on \(\{*, 1, 2\}^2\) defined by first sampling \((1, 2)\) and \((2, 1)\) uniformly with probability \(1/2\) each and then changing each coordinate to \(*\) independently with probability \(1 - p\). It is easy to see that both the marginals of \(D^p\) are identical to \(\mu_p\).

### 4.1.1 Useful Results

The following variant of Russo’s Lemma was proved in [DKPST10] (as Lemma 1).

**Lemma 4.1 (Russo’s Lemma [Rus82]).** Let \(F \subseteq \{*, 1, 2\}^m\) be monotone, then \(\mu_p(F)\) is increasing with \(p\). In fact,

\[
\frac{1}{2} \cdot \as_p(F) \leq \frac{d\mu_p(F)}{dp} \leq \as_p(F).
\]

The following corollary follows from the above and is proved in [SS13].

**Corollary 4.2.** For a monotone family \(F \subseteq \{*, 1, 2\}^m\),

1. For any \(p' \geq p\), \(\mu_{p'}(F) \geq \mu_p(F)\).
2. For any \(\varepsilon > 0\), there is a \(p' \in [1 - \varepsilon, 1 - \varepsilon/2]\) such that \(\as_{p'}(F) \leq \frac{4}{\varepsilon}\).

The following is a generalization of Friedgut’s Junta Theorem which is proved in [ST11].

**Theorem 4.3** (Friedgut’s Theorem [Fri98 ST11]). Fix \(\delta > 0\). Let \(F \subseteq \{*, 1, 2\}^m\) be monotone with \(a = \as_p(F)\), for \(p \in [0, 1]\). There exists a function \(C_{\text{Friedgut}}(p, \delta, a) \leq c_p^{a/\delta}\), for a constant \(c_p\) depending only on \(p\), so that \(F\) has a \((\delta, p)\)-core \(C\) of size \(|C| \leq C_{\text{Friedgut}}(p, \delta, a)\).

The above theorem shall be used along with the following generalization of Lemma 3.1 in [DS05] proved in [SS13].

**Proposition 4.4.** If \(C\) is a \((\delta, p)\)-core of \(F\), then \(\mu_p^C\left(\left|F\right|C^{3/4}\right) \geq \mu_p(F) = 3\delta\).

Using the above one can prove the following lemma.

**Lemma 4.5.** For a fixed parameter \(p \in (0, 1)\) and a positive constant \(\delta\), given a monotone family \(F \subseteq \{*, 1, 2\}^m\) such that \(\mu_p(F) \geq \delta\), there exists a subset \(S \subseteq [m]\) such that \(|S| \leq c_p^{16/(1-p)\delta}\), for some constant \(c_p\) depending only on \(p\), and two elements \(F, F' \in F\) such that for all \(j \not\in S\), \((F(j), F'(j))\) is not \((1, 1)\) or \((2, 2)\).

**Proof.** We first choose \(\bar{c}_p = \max\{c_{p'} \mid p' \in [1 - \varepsilon, 1 - \varepsilon/2]\}\) where \(p := 1 - \varepsilon\). By Corollary 4.2 there is a \(p' \in [1 - \varepsilon, 1 - \varepsilon/2]\) such that \(a := \as_{p'}(F) \leq 4/\varepsilon = 4/(1-p)\). Using Theorem 4.3, one
can obtain a \((\delta/4, \mu')\)-core \(S\) of \(\mathcal{F}\) of size \(|S| \leq c_{16/\delta(1-p)}^3\). By Proposition 4.4 and using the fact that 

\[ \mu_{\mu'}(\mathcal{F}) \geq \mu_\mu(\mathcal{F}) \geq \delta, \]

we get that,

\[ \mu_{\mu'}^S(\mathcal{F}^{3/4}_S) \geq \delta - 3\delta/4 = \delta/4 > 0, \]

where \([\mathcal{F}^{3/4}_S]\) is the core-family with respect to the measure \(\mu_{\mu'}\). Choose an element \(\overline{F} \in [\mathcal{F}^{3/4}_S]\)

Probabilistically construct \(F, F' \in \mathcal{F}\) as follows. For \(j \in S\), set \(F(j)\) and \(F'(j)\) to the corresponding value \(\overline{F}(j)\). For \(j \notin S\) independently sample \((F(j), F'(j))\) from \(D\mu'\). Since the marginals of \(D\mu'\) are distributed as \(\mu_{\mu'}\), by the definition of a core-family, we have,

\[ \Pr[F \in \mathcal{F} \text{ and } F' \in \mathcal{F}] \geq 1 - 2 \left( \frac{1}{4} \right) \geq \frac{1}{2}. \]

Moreover, since \((1, 1)\) and \((2, 2)\) do not lie in the support of \(D\mu'\), the elements \(F, F'\) satisfy the condition of the lemma.

\[ \square \]

4.2 Hardness Reduction

Let \(\delta, \varepsilon > 0\) be parameters that we shall set later. We begin with an instance \(\Phi\) of the Multi-Layered PCP from Theorem 2.3. The number of layers \(L\) of \(\Phi\) is chosen to be \(\lceil 32\delta^{-2} \rceil\). The parameter \(R\) shall be set later to be large enough. In the following paragraphs we describe the construction of a weighted 3-uniform hypergraph \(G\) with vertex set \(V\) a hyperedge set \(E\) and a weight function \(\text{wt}\) on the vertices, as an instance of \(\text{ISALMOSTCOLOR}_\varepsilon(3, 2, 1/\delta)\).

\textbf{Vertices}. Consider a variable \(v \in V_l\), i.e. in the \(l\)th layer of \(\Phi\). Let a \textit{Long Code} \(\mathcal{H}^v\) be a copy of the set \(\{1, 2, *\}^{R_l} \) equipped with the measure \(\mu_\mu\) where \(p := 1 - \varepsilon\). The set of vertices \(\mathcal{H} := \bigcup_{1 \leq i \leq L} \bigcup_{v \in V_l} \mathcal{H}^v\).

The weight of any \(x \in \mathcal{H}^v\) is,

\[ \text{wt}(x) = \frac{\mu_\mu(x)}{L|V_l|}. \]

Thus, the total weight of the vertices corresponding to any layer of the PCP is \(1/L\), which is equally distributed over the Long Codes of all the variables in that layer.

\textbf{Hyperedges}. For all variables \(v \in V_l\) and \(u \in V_l'\) \((l < l')\) such that there is a constraint \(\pi_{v\rightarrow u}\) between them, add a hyperedge between all \(x \in \mathcal{H}^u\) and \(y, z \in \mathcal{H}^v\) which satisfy the following condition: For any \(j \in R_l\) and \(i = \pi_{v\rightarrow u}(j) \in R_l',\) the tuple \((x_i, y_j, z_j)\) is not \((1, 1, 1)\) or \((2, 2, 2)\).

4.3 YES Case

In the YES case, there is an assignment \(\sigma\) of labels to the variables of \(\Phi\) that satisfies all the constraints. Construct a partition of \(\mathcal{H}\) into disjoint subsets \(\mathcal{H}_1, \mathcal{H}_2\) and \(\mathcal{H}_*\) as follows. For any variable \(v\) of \(\Phi\), add \(x \in \mathcal{H}^v\) to \(\mathcal{H}_{\pi_\sigma(v)}\). It is easy to see that \(\text{wt}(\mathcal{H}_*) = \varepsilon\) and \(\text{wt}(\mathcal{H}_1) = \text{wt}(\mathcal{H}_2) = \left(\frac{1-\varepsilon}{2}\right)\).

Furthermore, Let \(v, u\) be variables such that there is a constraint \(\pi_{v\rightarrow u}\) between them. Suppose there is a hyperedge between \(x \in \mathcal{H}^u\) and \(y, z \in \mathcal{H}^v\). Since \(\sigma\) is a satisfying assignment, \(\pi_{v\rightarrow u}(\sigma(v)) = \sigma(u)\). By the construction of the hyperedges, this implies that the tuple \((x_{\sigma(v)}, y_{\sigma(u)}, z_{\sigma(u)})\) is not \((1, 1, 1)\) or \((2, 2, 2)\), and thus the hyperedge \((x, y, z)\) is not contained in \(\mathcal{H}_1\) or in \(\mathcal{H}_2\). Therefore, removing the set of vertices \(\mathcal{H}_*\) of weight \(\varepsilon\) and the hyperedges incident on it makes the hypergraph 2-colorable.
4.4 NO Case

In the NO Case assume that there is a maximal independent set $I \subseteq \mathcal{H}$ of weight $\text{wt}(I) \geq \delta$. From the construction of the hyperedges, it is easy to see that any maximal independent set is monotone. Let $I^v := I \cap \mathcal{H}^v$ for any variable $v$ of $\Phi$. Thus, each $I^v$ is a monotone family.

Consider the set of variables

$$U := \left\{ u \in V \mid \mu_p(I^u) = \frac{\text{wt}(I^u)}{\text{wt}(\mathcal{H}^u)} \geq \frac{\delta}{2} \right\}.$$ 

By averaging, it is easy to see that,

$$\sum_{u \in U} \text{wt}(I^u) \geq \frac{\delta}{2}.$$ 

Another averaging shows that for at least $\frac{\delta}{4} L \geq \frac{\delta}{3}$ layers $l$, at least $\frac{\delta}{4}$ fraction of variables in layer $l$ belong to $U$. Applying the weak density property we obtain two layers $l < l'$ such that at least $\frac{\delta^2}{8}$ fraction of the constraints between $V_l$ and $V_{l'}$ are between the variables in $U_l := U \cap V_l$ and $U_{l'} := U \cap V_{l'}$. The following key lemma follows from Lemma 4.5.

**Lemma 4.6.** For any variable $v \in U_l$ there is a subset $S^v \subseteq R_l$ of size $|S^v| \leq t(\varepsilon, \delta) := c_\varepsilon \frac{1}{\delta}$ for some constant $c_\varepsilon > 0$ depending on $\varepsilon$, and elements $y^v, z^v \in I^v$ such that for all $j \in R_l \setminus S^v$, the tuple $(y_j^v, z_j^v)$ is not $(1, 1)$ or $(2, 2)$.

Note that in the above, if $S^v$ is empty then $y^v$ and $z^v$ will trivially ensure a hyperedge in $I$, so we may assume it is non-empty.

Using the above lemma we can now define the labeling for each of the variables in $U_l$ and $U_{l'}$.

**Labeling for** $v \in U_l$: Choose a label $\rho(v) \in R_l$ uniformly at random from $S^v$.

**Labeling for** $u \in U_{l'}$: This choice is made depending on the labeling of variables in $U_l$. Let $N(u) \subseteq V_l$ be all the variables in $V_l$ which have a constraint with $u$. Choose a label $\lambda(u)$ defined below,

$$\lambda(u) := \arg\max_{a \in R_{l'}} \left| \{ v \in N(u) \cap U_l \mid \pi_{u \rightarrow v}(\rho(v)) = a \} \right|.$$ 

In other words, $\lambda(u)$ is the label in $R_{l'}$ which is the projection of the maximum number of labels of the neighbors of $u$ in $U_l$.

For the rest of the analysis we shall focus on one variable $u \in U_{l'}$ and its neighborhood in $U_l, N(u) \cap U_l$. To complete the analysis we need the following lemma proved in [DGKR03].

**Lemma 4.7.** Let $A_1, A_2, \ldots, A_N$ be a collection of $N$ sets, each of size at most $T \geq 1$. If there are not more than $D$ pairwise disjoint subsets in the collection then there must exist an element which belongs to at least $\frac{N}{D T^2}$ sets.

Consider the collection $\{\pi_{u \rightarrow v}(S^v) \mid v \in N(u) \cap U_l\}$. Each subset in this collection is of size at most $t(\varepsilon, \delta)$. Each such subset $\pi_{u \rightarrow v}(S^v)$ rules out $I^u$ containing any element $x^u$ with * in all coordinates corresponding to $\pi_{u \rightarrow v}(S^v)$. Otherwise, $(x^u, y^v, z^v)$ would be a hyperedge in $I$. Suppose there are $r$ pairwise disjoint subsets in this collection. This would reduce the measure $\mu_p(I^u)$ by a factor of $\left(1 - \varepsilon^{t(\varepsilon, \delta)}\right)^r$. However, $\mu_p(I^u) \geq \frac{\delta}{2}$. Thus, $r$ is at most $\log \left(\frac{0.5}{\delta} \right) \log \left(1 - \varepsilon^{t(\varepsilon, \delta)}\right)$ by Lemma 4.7, there is an element $a$ contained in at least

$$\log \left(1 - \varepsilon^{t(\varepsilon, \delta)}\right) \frac{1}{(t(\varepsilon, \delta) \log \left(\frac{0.5}{\delta}\right))}.$$ 

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fraction of the subsets in the collection \( \{ \pi_{v \to u}(S'') \mid v \in N(u) \cap U_1 \} \). This implies that in expectation, over the choice of \( \{ \rho(v) \mid v \in N(u) \cap U_1 \} \), \( \pi_{v \to u}(\rho(v)) = a \) for at least,

\[
\xi(\varepsilon, \delta) := \frac{\log \left( 1 - e^{t(\varepsilon, \delta)} \right)}{t(\varepsilon, \delta)^2 \log \left( \frac{4}{\delta^2} \right)},
\]

fraction of \( N(u) \cap U_1 \). Thus, in expectation the labelings \( \rho \) and \( \lambda \) satisfy \( \xi(\varepsilon, \delta) \left( \frac{\delta^2}{\Pi} \right) \) fraction of the constraints between the layers \( l \) and \( l' \). Choosing the parameter \( R \) of \( \Phi \) to be small enough gives a contradiction.

5 Independent Set in 2-Colorable 3-Uniform Hypergraphs

We begin with a few useful definitions and results, which can also be found in greater detail in [Mos10]. The correlation between two correlated probability spaces is defined as follows.

Definition 5.1. Suppose \( (\Omega^{(1)} \times \Omega^{(2)}, \mu) \) is a finite correlated probability space with the marginal probability spaces \( (\Omega^{(1)}, \mu) \) and \( (\Omega^{(2)}, \mu) \). The correlation between these spaces is,

\[
\rho(\Omega^{(1)}, \Omega^{(2)}; \mu) = \sup \left\{ \left| E_\mu[fg] \right| \mid f \in L^2(\Omega^{(1)}, \mu), g \in L^2(\Omega^{(2)}, \mu), E[f] = E[g] = 0; E[f^2], E[g^2] \leq 1 \right\}.
\]

Let \( (\Omega_i^{(1)} \times \Omega_i^{(2)}, \mu_i)_{i=1}^n \) be a sequence of correlated spaces. Then,

\[
\rho\left( \prod_{i=1}^n \Omega_i^{(1)}; \prod_{i=1}^n \Omega_i^{(2)}; \prod_{i=1}^n \mu_i \right) \leq \max_i \rho(\Omega_i^{(1)}, \Omega_i^{(2)}; \mu_i).
\]

Further, the correlation of \( k \) correlated spaces \( (\prod_{j=1}^k \Omega^{(j)}, \mu) \) is defined as follows:

\[
\rho(\Omega^{(1)}, \Omega^{(2)}, \ldots, \Omega^{(k)}; \mu) := \max_{1 \leq i \leq k} \rho\left( \prod_{j=1}^{i-1} \Omega^{(j)} \times \prod_{j=i+1}^k \Omega^{(j)}; \mu \right).
\]

Lemma 5.2. Let \( (\Omega^{(1)} \times \Omega^{(2)}, \mu) \) be two correlated spaces such that the probability of the smallest atom in \( (\Omega^{(1)} \times \Omega^{(2)}, \mu) \) is at least \( \alpha \in (0, 1/2] \). Define a bipartite graph between \( \Omega^{(1)} \) and \( \Omega^{(2)} \) with an edge between \( (a, b) \in \Omega^{(1)} \times \Omega^{(2)} \) if \( \mu(a, b) > 0 \). If this graph is connected then,

\[
\rho(\Omega^{(1)}, \Omega^{(2)}; \mu) \leq 1 - \alpha^2/2.
\]

We shall also refer to the following Gaussian stability measures in our analysis.

Definition 5.3. Let \( \Phi : \mathbb{R} \to [0, 1] \) be the cumulative distribution function of the standard Gaussian. For a parameter \( \rho \), define,

\[
\underline{\Gamma}_\rho(\mu, \nu) = \Pr[X \leq \Phi^{-1}(\mu), Y \geq \Phi^{-1}(1 - \nu)],
\]

\[
\overline{\Gamma}_\rho(\mu, \nu) = \Pr[X \leq \Phi^{-1}(\mu), Y \leq \Phi^{-1}(\nu)],
\]

where \( X \) and \( Y \) are two standard Gaussian variables with covariance \( \rho \).

The Bonami-Beckner operator is defined as follows.
Lemma 5.7. First restate Lemma 6.2 of [Mos10].

Then there exists an absolute constant \( C \) such that for

\[
\gamma = C \frac{(1 - \rho) \nu}{\log (1/\nu)},
\]

and \( k \) functions \( \left\{ f_j \in L^2(\prod_{i=1}^{n} \Omega_i^{(j)}) \right\}_{j=1}^{k} \), the following holds,

\[
\left| \mathbb{E} \left[ \prod_{j=1}^{k} f_j \right] - \mathbb{E} \left[ \prod_{j=1}^{k} T_{1-\gamma} f_j \right] \right| \leq \nu \sum_{j=1}^{k} \sqrt{\text{Var}[f_j]} \sqrt{\text{Var} \left[ \prod_{j' \neq j}^{k} T_{1-\gamma} f_{j'} \prod_{j' > j}^{k} f_{j'} \right]}.
\]
Our analysis shall also utilize the following bi-linear Gaussian stability bound from \cite{Mos10} to locate influential coordinates.

**Theorem 5.8.** Let $(\Omega_i^{(1)} \times \Omega_i^{(2)}, \mu_i)$ be a sequence of correlated spaces such that for each $i$, the probability of any atom in $(\Omega_i^{(1)} \times \Omega_i^{(2)}, \mu_i)$ is at least $\alpha \leq 1/2$ and such that $\rho(\Omega_i^{(1)}, \Omega_i^{(2)}; \mu_i) \leq \rho$ for all $i$. Then there exists a universal constant $C$ such that, for every $\nu > 0$, taking

$$\tau = \exp \left( C \frac{\log(1/\alpha) \log(1/\nu)}{\nu(1-\rho)} \right),$$

for functions $f : \prod_{i=1}^n \Omega_i^{(1)} \mapsto [0,1]$ and $g : \prod_{i=1}^n \Omega_i^{(2)} \mapsto [0,1]$ that satisfy,

$$\max_i \min(\Inf_i(f), \Inf_i(g)) \leq \tau,$$

for all $i$, we have,

$$\sum_i \rho(E[f]_i, E[g]_i) - \nu \leq E[f g] \leq \prod_i \rho(E[f]_i, E[g]_i) + \nu.$$

Before describing the hardness reduction we define the following useful distribution and state its properties.

**Distribution $D_{\delta,r}$**

We define the probability measure $D_{\delta,r}$ over the random variables $(X, Y = \{Y_i\}_{i=1}^r, Z = \{Z_i\}_{i=1}^r)$, where $X, Y_i, Z_i \in \{-1,1\}$. A tuple $(X, Y, Z)$ is sampled from $D_{\delta,r}$ by first choosing $X, Y_1, \ldots, Y_r \in \{-1,1\}$ independently and uniformly at random, and setting each $Z_i = -Y_i$. Finally, with probability $\delta$, $j \in [r]$ is chosen u.a.r and $Y_j$ and $Z_j$ are both set to $-X$. Let $X, Y$ and $Z$ define the correlated probability spaces $\Omega^{(1)}, \Omega^{(2)}$ and $\Omega^{(3)}$ respectively with the joint probability measure $D_{\delta,r}$. Note that the marginal probability spaces $\Omega^{(2)}, D_{\delta,r}$ and $\Omega^{(3)}, D_{\delta,r}$ are identical. Also, for $i \neq j \in [r]$, $Y_i$ is independent of $Y_j$ and $Z_j$. It is easy to see the following lemma.

**Lemma 5.9.** For any probability $\delta$ and integer $r > 0$,

(i) The minimum probability of an atom in $D_{\delta,r}$ is at least $\xi := \frac{\delta}{2^r}$.  
(ii) $\rho(\Omega^{(1)} \times \Omega^{(2)}; D_{\delta,r}) \leq \delta$.  
(iii) $\rho(\Omega^{(1)} \times \Omega^{(2)}, \Omega^{(3)}; D_{\delta,r}) \leq 1 - \xi^2/2 = \frac{\delta^2}{2^{2r}}$.  
(iv) $\rho(\Omega^{(2)}, \Omega^{(3)}; D_{\delta,r}) \leq 1 - \xi^2/2$.  
(v) $\rho(\Omega^{(1)} \times \Omega^{(2)}, \Omega^{(3)}; D_{\delta,r}) \leq 1 - \xi^2/2$.

**Proof.** The first part can be computed by observing that the atom in $D_{\delta,r}$ with minimum probability is the one in which there is a $j \in [r]$ such that $Y_j = Z_j$, and this atom has probability $\xi$ as defined. The second part is immediate since $X$ is independent of $(Y, Z)$ with probability $1 - \delta$. The third and fourth parts follow from (i) and by showing that Lemma 5.2 is applicable, which can be inferred in a manner similar to the proof of connectedness in \cite{OW09}. We omit the details here. The fifth part follows from Definition 5.1. \hfill \square

In the rest of this section we shall sometimes omit writing the joint distribution along with $\Omega^{(1)}, \Omega^{(2)}$ and $\Omega^{(3)}$, as it will be clear from the context.

### 5.1 Hardness Reduction

We begin with an instance $\Phi$ from Theorem \ref{thm:2.8} with the number of layers $L = \lceil 32 e^{-2} \rceil$, for a parameter $\varepsilon > 0$ which denotes the size of the independent set in the NO Case.
5.1.1 Construction of $G(H, E)$

We continue with the construction of the instance $G(H, E)$, a 3-uniform hypergraph. The construction uses a parameter $\delta$ which we shall fix later.

**Vertices.** Consider a variable $v$ of $\Phi$ in layer $l$. Let $H^v$ be a copy of $\{-1, 1\}^{R_1}$. The vertex set $H := \bigcup_{l \in [L]} \bigcup_{v \in V_l} H^v$. The weight of a vertex $x \in H^v$ for $v \in V_l$ is $2^{-R_1} \nu/(L|V_l|)$. Thus, the total weight of all the vertices corresponding to a particular layer is $1/L$.

**Hyperedges.** Consider two variables $v \in V_l$ and $u \in V_{l'}$ with a constraint $\pi_{v \rightarrow u}$ between them. Note that for every $i \in R_l$, $|\pi_{v \rightarrow u}(i)| = d^{l' - l}$. For convenience, we let $r = d^{l' - l}$, and dropping the subscript we shall refer to the projection simply as $\pi$. Let $x \in H^v$ and $y, z \in H^v$ be chosen by sampling $(x_i, y_{\pi^{-1}(i)}, z_{\pi^{-1}(i)})$ from $(\Omega(1) \times \Omega(2) \times \Omega(3); D_{\delta_r})$ independently for each $i \in R_l$. Let $D^{vu}$ denote the probability distribution of the choice of $(x, y, z)$. For all such $(x, y, z)$ in the support of $D^{vu}$ add a hyperedge between these three vertices $x, y$ and $z$.

5.2 YES Case

In the YES Case, let $\sigma$ be the labeling to the variables that satisfies all constraints in $\Phi$. For every vertex $x \in H^v$ for a variable $v$ in layer $l$, color $x$ with $x_{\sigma(v)}$. It is easy to see from the above construction of the hyperedges that this is a valid 2-coloring of the hypergraph.

5.3 NO Case

Suppose that there is an independent set of $\varepsilon > 0$ fraction of vertices. For a variable $v$ of $\Phi$, let $f_v$ be the indicator of the independent set in the long code $H^v$. Let the heavy variables $v$ be such that $\mathbb{E}[f_v] \geq \frac{\varepsilon}{2}$. After averaging and arguments analogous to those in Section 5.3 we obtain two layers $l < l'$ such that the heavy variables in these two layers induce at least $\varepsilon^2/16$ fraction of constraints between these two layers. As before, we set $r = d^{l' - l}$. Also, we shall denote $R_{l'}$ by $R_1$ and $R_l$ by $R_2$.

We need to show that,

$$\mathbb{E}_{v,u} \left[ \mathbb{E}_{(x,y,z) \sim D^{vu}} \left[ f_u(x) f_v(y) f_v(z) \right] \right] > 0,$$

(9)

where the outer expectation is over pairs of heavy variables $v \in V_l$ and $u \in V_{l'}$ which share a constraint. The analysis consists of two main steps. In the first step we show that unless $f_u$ and $f_v$ share influential coordinates, one can re-randomize the $x$ variable to be independent in the inner expectation of Equation (9). However, the notion of influence of $f_v$ used in this step depends on the choice of $u$.

The second step shows that for a non-trivial fraction of heavy neighbors $u$ of $v$, the notion of influence used in the first step can be made independent of $u$. In addition it shows that for these $u$, the marginal expectation $\mathbb{E}[f_v(y) f_u(z)]$ induced by $D^{vu}$ is bounded away from zero. This step crucially uses the smoothness property of the PCP.

5.3.1 Making $x$ independent

Let us fix a pair of heavy vertices $v, u$ which share a constraint $\pi$. For convenience we shall think of the distribution $D^{vu}$ being on $\otimes_{i \in R_l} (x_i, y_{\pi^{-1}(i)}, z_{\pi^{-1}(i)})$, where each $(x_i, y_{\pi^{-1}(i)}, z_{\pi^{-1}(i)})$ is sampled independently from $(\Omega(1) \times \Omega(2) \times \Omega(3); D_{\delta_r})$. We represent the space of $(x_i, y_{\pi^{-1}(i)}, z_{\pi^{-1}(i)})$ by the correlated space $(\Omega(1) \times \Omega(2) \times \Omega(3); D_{\delta_r})$, which is an independent copy of $(\Omega(1) \times \Omega(2) \times \Omega(3))$. Thus, the space of $\otimes_{i \in R_l} (x_i, y_{\pi^{-1}(i)}, z_{\pi^{-1}(i)})$ is $\prod_{i \in R_l} (\Omega(1) \times \Omega(2) \times \Omega(3))$. The $i$th coordinate influence of a function $f$
on $\prod_{i \in R_1} \Omega_1^{(2)} = \prod_{i \in R_1} \Omega_1^{(3)}$ is denoted by $\mathcal{I}_{i_1}(f_v)$. The probability measure on all these spaces is induced by $T^{i_1}$. 

Using the above and since the functions $f_u$ and $f_v$ are all in the range $[0, 1]$ we have the following lemma which follows from Lemma 5.9 and Lemma 5.7.

**Lemma 5.10.** There is a universal constant $C$ such that for an arbitrarily small choice of $\nu > 0$, letting $\gamma = C \frac{\nu^2}{\log(1/\nu)}$, the following holds,

\[
\begin{align*}
|E[f_u(x)f_v(y)f_v(z)] - E[T_{1-\gamma}f_u(x)T_{1-\gamma}f_v(y)T_{1-\gamma}f_v(z)]| &\leq \nu, \\
|E[f_v(y)f_v(z)] - E[T_{1-\gamma}f_v(y)T_{1-\gamma}f_v(z)]| &\leq \nu,
\end{align*}
\]

where the $T_{1-\gamma}$ is the Bonami-Beckner operator over $\{-1, 1\}^{R_1} = \prod_{i \in R_1} \Omega_1^{(1)}$ and $T_{1-\gamma}$ is the Bonami-Beckner operator over the space $\prod_{i \in R_1} \Omega_i^{(2)} = \prod_{i \in R_1} \Omega_i^{(3)}$. To be precise, $T_{1-\gamma}$ resamples from each $\Omega_i^{(2)}$ independently with probability $\gamma$. Note that $T_{1-\gamma}$ depends on the constraint $\pi$ and hence on the choice of $u$.

Using a value of $\gamma$ which we shall obtain from the above lemma, consider the function $F(y, z) = T_{1-\gamma}f_v(y)T_{1-\gamma}f_v(z)$ over the space $\prod_{i \in R_1} (\Omega_i^{(2)} \times \Omega_i^{(3)})$. For the time being let $f'$ denote $T_{1-\gamma}f_v$ and $f_i'$ denote the function $f'$ depending only on the $i$th space $\Omega_i^{(2)} = \Omega_i^{(3)}$ where the fixing of the rest of the coordinates will be clear from the context. Thus, $F(y, z) = f'(y)f'(z)$. The $i$th influence of $F$ in the space $\prod_{i \in R_1} (\Omega_i^{(2)} \times \Omega_i^{(3)})$ can be written as:

\[
\mathcal{I}_{i_1}(F) = \frac{1}{2} \mathbb{E}_{(y_{\pi^{-1}(j)}, z_{\pi^{-1}(j)}) \sim (\Omega_j^{(2)} \times \Omega_j^{(3)})} \left[ \mathbb{E}_{((Y_1, Z_1), (Y_2, Z_2)) \sim (\Omega^{(2)} \times \Omega^{(3)})^2} \left[ (f'_i(Y_1) f'_i(Z_1) - f'_i(Y_2) f'_i(Z_2))^2 \right] \right]
\]

The following inequality was proved in Lemma 4 of the work of Samorodnitsky and Trevisan [ST09].

**Lemma 5.11.** Let $a_1, a_2, b_2, b_2 \in [-1, 1]$. Then, $(a_1a_2 - b_1b_2)^2 \leq 2((a_1 - b_1)^2 + (a_2 - b_2)^2)$.

Using the above lemma we obtain the following bound.

**Lemma 5.12.** From the definitions used above,

\[
\mathcal{I}_{i_1}(F) \leq 4\mathcal{I}_{i_1}(f').
\]

**Proof.** Using Lemma 5.11 and the fact that $f'$ is bounded in $[0, 1]$ we can upper bound $\mathcal{I}_{i_1}(F)$ in Equation (12) by

\[
\begin{align*}
\frac{1}{2} \mathbb{E}_{(y_{\pi^{-1}(j)}, z_{\pi^{-1}(j)}) \sim (\Omega_j^{(2)} \times \Omega_j^{(3)})} \left[ \mathbb{E}_{((Y_1, Z_1), (Y_2, Z_2)) \sim (\Omega^{(2)} \times \Omega^{(3)})^2} \left[ (f'_i(Y_1) - f'_i(Y_2))^2 + (f'_i(Z_1) - f'_i(Z_2))^2 \right] \right] \\
= 4\mathcal{I}_{i_1}(f').
\end{align*}
\]

\[
\square
\]
We also have the following lemma.

**Lemma 5.13.** Let Inf\(_j\) be the \(j\)th coordinate influence over the space \((-1, 1)\) equipped with the uniform measure. Then, for \(i \in R_1\), \(\text{Inf}_i(f') \leq r \sum_{j \in \pi^{-1}(i)} \text{Inf}_j(f')\).

**Proof.** By the definition of influence, the LHS of the assertion can be written as,

\[
\frac{1}{2} E_{y \mid \pi^{-1}(i) \leftarrow \Omega_1^{(2)}} \left[ E_{(Y^0, Y^r) \leftarrow (-1, 1)^2} \left[ (f'_1(Y^0) - f'_1(Y^r))^2 \right] \right].
\]

Order the coordinates in \(\pi^{-1}(i)\) as \(1, \ldots, r\) and define (depending on the choice of \(Y^0\) and \(Y^r\)) a sequence \(Y^1, \ldots, Y^{r-1}\) where \(Y^k\) contains the value of the first \(r - k\) coordinates from \(Y^0\) and the rest from \(Y^r\). Letting \(R'_2 := R_2 \setminus \pi^{-1}(i)\), the above expression can be rewritten as,

\[
\frac{1}{2} E_{y \mid R'_2 \leftarrow (-1, 1)^{R'_2}} \left[ E_{(Y^0, Y^r) \leftarrow (-1, 1)^2} \left[ \left( \sum_{k=0}^{r-1} (f'_1(Y^k) - f'_1(Y^{k+1})) \right)^2 \right] \right]
\]

\[
\leq \frac{1}{2} E_{y \mid R'_2 \leftarrow (-1, 1)^{R'_2}} \left[ E_{(Y^0, Y^r) \leftarrow (-1, 1)^2} \left[ \sum_{k=0}^{r-1} (f'_1(Y^k) - f'_1(Y^{k+1}))^2 \right] \right]
\]

\[
= r \sum_{j \in \pi^{-1}(i)} \text{Inf}_j(f'),
\]

where we used Cauchy-Schwarz to obtain the first inequality.

The following lemma uses the above analysis to show that \(x\) can be made independent of \(y\) and \(z\) without incurring much loss, unless \(f_u\) and \(f_v\) have matching influential coordinates.

**Lemma 5.14.** There is a universal constant \(C\) such that for an arbitrarily small constant \(\nu > 0\), and

\[
\gamma = \frac{\nu \xi^2}{2 \log(1/\nu)}, \quad \tau = \frac{\nu C \log(1/\xi) \log(1/\nu)}{\nu(1+\nu)}\]

unless there is \(i \in R_1\) such that,

\[
\min(\text{Inf}_i(T_{1-\gamma}f_u), 4r \sum_{j \in \pi^{-1}(i)} \text{Inf}_j(\overline{T}_{1-\gamma}f_v)) \geq \tau,
\]

we have,

\[
E[f_u(x) f_v(y) f_v(z)] \geq \prod_{i} E[f_u], E[f_v(y) f_v(z)] - \nu) - 2\nu.
\]

**Proof.** Suppose that there exists no \(i \in R_1\) as in the condition of the lemma. Using Lemmas 5.12 and 5.13 our supposition implies that there exists no \(i \in R_1\) such that,

\[
\min(\text{Inf}_i(T_{1-\gamma}f_u), \text{Inf}_i(F)) \geq \tau,
\]

where \(F(y, z)\) was defined as \(T_{1-\gamma}f_v(y) \overline{T}_{1-\gamma}f_v(z)\). Using Theorem 5.8 and Lemma 5.9 the above implies,

\[
E[T_{1-\gamma}f_u(x)T_{1-\gamma}f_v(y) \overline{T}_{1-\gamma}f_v(z)] \geq \prod_{i} E[T_{1-\gamma}f_u(x)], E[T_{1-\gamma}f_v(y) \overline{T}_{1-\gamma}f_v(z)] - \nu
\]

\[
= \prod_{i} (E[f_u(x)], E[\overline{T}_{1-\gamma}f_v(y) \overline{T}_{1-\gamma}f_v(z)]) - \nu.
\]

Using Lemma 5.10 the above implies that

\[
E[f_u(x) f_v(y) f_v(z)] \geq \prod_{i} (E[f_u(x)], E[f_v(y) f_v(z)] - \nu) - 2\nu.
\]

\[\Box\]
Note that there are two issues that are left to resolve. Firstly, we need to lower bound \(E[f_v(y)f_v(z)]\). Secondly, \(\text{Inf}_{i}(\overline{T}_{1-\gamma}(f_v))\) depends on the choice of \(u\). We shall identify a significant fraction of heavy neighbors \(u\) of \(v\), for which the expectation is bounded as well as \(\text{Inf}_{i}(\overline{T}_{1-\gamma}(f_v)) \approx \text{Inf}_{i}(T_{1-\gamma}(f_v))\), the latter being independent of \(u\). For this we shall utilize the smoothness property of the PCP.

### 5.3.2 Identifying good neighbors \(u\)

Let us first set a parameter \(s\) as,

\[
s := \max \left( \frac{r}{\xi} \ln \left( \frac{1}{\nu} \right), \frac{r}{2\gamma} \ln \left( \frac{32r^2}{\tau} \right) \right).
\]

Let the Efron-Stein decomposition of \(f_v\) with respect to \((-1, 1)^{R_2}\) be,

\[
f_v = \sum_{\alpha \subseteq R_2} \hat{f}_{v,\alpha} \chi_{\alpha}.
\]

It can be seen (see [Hås12]) that the Efron-Stein decomposition of \(f_v\) with respect to \(\prod_{\alpha \subseteq R_1} \Omega_{\alpha}^{(2)}\) is,

\[
f_v = \sum_{\beta \subseteq R_1} f^\beta_v,
\]

where,

\[
f^\beta_v = \sum_{\alpha \subseteq R_2 \atop \pi(\alpha) = \beta} \hat{f}_{v,\alpha} \chi_{\alpha}
\]

We say that a subset \(\alpha\) is shattered by \(\pi = \pi_{v \rightarrow u}\) if \(|\pi(\alpha)| = |\alpha|\). Using this we decompose \(f_v\) into three functions, depending on the choice of \(u\), as follows

\[
\begin{align*}
f_1 &= \sum_{\alpha : |\alpha| \geq s} \hat{f}_{v,\alpha} \chi_{\alpha} \\
f_2 &= \sum_{\alpha : |\alpha| < s \atop \alpha \text{ not shattered}} \hat{f}_{v,\alpha} \chi_{\alpha} \\
f_3 &= \sum_{\alpha : |\alpha| < s \atop \alpha \text{ shattered}} \hat{f}_{v,\alpha} \chi_{\alpha}
\end{align*}
\]

To identify the good neighbors of \(v\), we need the following key lemma.

**Lemma 5.15.** With expectation taken over a random neighbor \(u \in V_v\) which shares a constraint with \(v\), \(E[\|f_2\|_2] \leq (s/\sqrt{T})\). Here \(T\) is the smoothness parameter from Theorem 2.8.

**Proof.** For a given \(\alpha \subseteq R_2\) such that \(|\alpha| < s\), the probability (over \(u\)) that it is not shattered is at most

\[
\sum_{i \neq j \in \alpha} \Pr[\pi_{v \rightarrow u}(i) = \pi_{v \rightarrow u}(j)] \leq \frac{s^2}{T}.
\]

Since, \(\sum \hat{f}_{v,\alpha}^2 \leq 1\), we obtain that,

\[
E[\|f_2\|_2] \leq (E[\|f_2\|_2^2])^{1/2} \leq \frac{s}{\sqrt{T}}.
\]

\(\square\)
The above lemma implies that for at least $1 - (s^2/T)^{1/4}$ fraction of the neighbors $u \in V_v$ of $v$, $\|f_2\|_2 \leq (s^2/T)^{1/4}$. Call such neighbors $u$ of $v$ which satisfy this bound as good.

**Lower bounding $E[f_v(y)f_v(z)]$**

We first set $\eta = \frac{2\epsilon}{T}$. It is easy to see that for any $j \in R_2$, $E[y_jz_j] = -\frac{1}{2} \left(1 - \frac{2}{T}\right) + \frac{\eta}{2} = 1 + \eta$. We shall first lower bound $E[f_v(y)T_{1-\eta}f_v(-y)]$. We shall need the following lemma from [MOR+06] which is obtained using the reverse hypercontractive inequality over the boolean domain.

**Lemma 5.16.** Let $A, B \subseteq \{-1, 1\}^n$ have relative densities,

$$\frac{|A|}{2^n} = e^{-a^2/2} \quad \frac{|B|}{2^n} = e^{-b^2/2},$$

and let $y \in \{-1, 1\}$ be uniform and $y'$ be a $\rho$-correlated copy of $y$, i.e. $E[y_iy'_i] = \rho$, independently for each $i \in [n]$, for some $\rho > 0$. Then,

$$\Pr[y \in A, y' \in B] \geq \exp \left[-\frac{1}{2} \cdot \frac{a^2 + b^2 + 2\rho ab}{1 - \rho^2}\right]. \quad (24)$$

Since $f_v$ is an indicator function let $A = \{y \mid f_v(y) = 1\}$. As $v$ was chosen to be heavy, we have $E[f_v] \geq \frac{\xi}{2}$. Let $B = -A$, i.e. $B = \{-y \mid y \in A\}$. It is easy to see that

$$E[f_v(y)T_{1-\eta}f_v(-y)] = \Pr[y \in A, y' \in B], \quad (25)$$

where $y'$ is a $1 - \eta$ correlated copy of $y$. Using Lemma 5.16 we obtain,

$$E[f_v(y)T_{1-\eta}f_v(y)] \geq \left(\frac{\xi}{2}\right)^{4/\eta}. \quad (26)$$

The following two lemmas decompose two expectations we are interested in.

**Lemma 5.17.** Using the decompositions above,

$$|E[f_v(y)T_{1-\eta}f_v(-y)] - E[f_3(y)T_{1-\eta}f_3(-y)]| \leq 2\|f_2\|_2 + 2\nu. \quad (27)$$

**Proof.** By Lemma 5.5 and Equation (18), we have

$$|E[f_v(y)T_{1-\eta}f_1(-y)]| \leq \|f_v\|_2 \|f_1\|_2 (1 - \eta)^3 \leq \nu,$$

by our setting of $s$ and since $\|f_v\|_2, \|f_1\|_2 \leq 1$. Furthermore,

$$|E[f_v(y)T_{1-\eta}f_2(-y)]| \leq \|f_v\|_2 \|f_2\|_2 \leq \|f_2\|_2.$$

We can repeat the above with $E[f_v(y)T_{1-\eta}f_3(-y)]$ using the fact that $\|T_{1-\eta}f_3(-y)\|_2 \leq 1$ to obtain the lemma. \qed

**Lemma 5.18.** Using the decompositions above and having $(y, z)$ sampled from $\prod_{i \in R_1} (\Omega_i^{(2)} \times \Omega_i^{(3)}); D^{vu}$,

$$|E[f_v(y)f_v(z)] - E[f_3(y)f_3(z)]| \leq 2\|f_2\|_2 + 2\nu. \quad (28)$$

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Proof. Using the bound (iv) of Lemma 5.9, the decomposition in Equations (19) and (20), and Lemma 5.5 we obtain,

\[ |E[f_v(y)f_1(z)]| \leq \|f_v\|_2 \|f_1\|_2 (1 - \eta)^{s/r} \leq \nu, \]

by our setting of \( s \). The rest of the proof is analogous to Lemma 5.17.

Note that \( y_i \) is independent of \( y_j \) and \( z_j \) for \( i \neq j \in R_2 \). Also, when sampling \( z \) given \( y \) the coordinates in a shattered subset \( \alpha \) are flipped independently with probability \( 1 - \frac{1}{2} \).

Thus,

\[ E[f_3(y)f_3(z)] = \sum_{\alpha:|\alpha| < s} \hat{T}_{v,\alpha}^2 (-1 + \eta)^{|\alpha|} = E[f_3(y)T_{1-\eta}f_3(-y)]. \]

From the above analysis, Lemma 5.15 and Equation (26), we have that for all good neighbors \( u \) of \( v \),

\[ E[f_v(y)f_v(z)] \geq \left( \frac{\varepsilon}{2} \right)^{4/\eta} - 4 \left( \frac{s^2}{T} \right)^{1/4} - 4\nu, \quad (29) \]

where \( y \) and \( z \) are sampled according to \( D^{vu} \).

**Showing** \( \text{Inf}_i(T_{1-\gamma}f_v) \approx \text{Inf}_i(T_{1-\gamma}f_v) \)

Recall that \( T_{1-\gamma} \) is the Bonami-Beckner operator on the space \( \prod_{i \in R_1} \Omega_i^{(2)} \) and \( T_{1-\gamma} \) is over \( \{-1, 1\}^{R_2} \) equipped with the uniform measure. Let \( h = T_{1-\gamma}f_v \) and \( g = T_{1-\gamma}f_v \). Define the functions \( h_i := T_{1-\gamma}f_i \) and \( g_i := T_{1-\gamma}f_i \) for \( i = 1, 2, 3 \).

Since the operators \( T_{1-\gamma} \) and \( T_{1-\gamma} \) are contractions, by Lemma 5.15 we have that for good neighbors \( u, \|h_2\|_2, \|g_2\|_2 \leq (s^2/T)^{1/4} \). Also, by Lemma 5.5 and Efron-Stein decompositions of \( f_v \) (Equations (18), (19) and (20)), we obtain: \( \|h_1\|_2 \leq (1 - \gamma)^{s/r} \) and \( \|g_1\|_2 \leq (1 - \gamma)^s \). By our setting of \( s \), we get \( \|h_1\|_2^2, \|g_1\|_2^2 \leq \frac{32r^2}{s^2} \).

For a subset \( \alpha \) which is shattered, it is easy to see that that \( \hat{h}_\alpha = \hat{g}_\alpha = \hat{T}_{v,\alpha}(1 - \gamma)^{|\alpha|} \). Using the definition of influence over the domain \( \{-1, 1\}^{R_2} \) we obtain the following lemma.

**Lemma 5.19.** For any \( i \in R_2 \),

\[ |\text{Inf}_i(T_{1-\gamma}f_v) - \text{Inf}_i(T_{1-\gamma}f_v)| \leq 2 \left( \frac{s^2}{T} \right)^{1/4} + \frac{\tau}{16r^2}. \]

**Choice of Parameters.** Given \( \varepsilon > 0 \), fix \( \delta \in (0, 1/2) \), which also fixes \( \eta \). The choice of \( L \) made at the beginning of Section 5.1 is fixed and therefore the maximum possible value of \( r \) is also fixed. Choose \( \nu \) small enough so that

\[ \varGamma_{\delta} \left( \frac{\varepsilon}{2}, \frac{\varepsilon}{2}, \frac{\varepsilon}{2}, \frac{\varepsilon}{2}, \frac{\varepsilon}{2}, \frac{\varepsilon}{2}, \frac{\varepsilon}{2}, \frac{\varepsilon}{2}, \frac{\varepsilon}{2}, \frac{\varepsilon}{2}, \frac{\varepsilon}{2}, \frac{\varepsilon}{2}, \frac{\varepsilon}{2}, \frac{\varepsilon}{2}, \frac{\varepsilon}{2} \right) - 2\nu > 0. \quad (30) \]

This also fixes the choice of \( \gamma \) and \( \tau \) by Lemma 5.14 and the choice of \( s \) as defined above. Then choose \( T \) to be large enough so that

\[ 4 \left( \frac{s^2}{T} \right)^{1/4} \leq \min \left\{ \frac{1}{2} \left( \frac{\varepsilon}{2} \right)^{4/\eta}, \frac{\varepsilon^2}{128} \right\}, \]

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and,
\[
2 \left( \frac{s^2}{T} \right)^{1/4} \leq \frac{\tau}{16r^2}.
\]
The above setting implies that for all good neighbors \( u \) of \( v \),
\[
E[f_u(y)f_v(z)] \geq \frac{1}{2} \left( \frac{\varepsilon^2}{2} \right)^{4/\eta} - 4\nu,
\]
and for any \( i \in R_2 \), using Lemma 5.19
\[
|\text{Inf}_i(T_{1-\gamma}f_v) - \text{Inf}_i(T_{1-\gamma}f_u)| \leq \frac{\tau}{8r^2}.
\]
Using Equations (30), (31) and (32) along with Lemma 5.14 for a heavy and good neighbor \( u \) of \( v \) yields an \( i^* \) such that,
\[
\min(\text{Inf}_{i^*}(T_{1-\gamma}f_u), 4r \sum_{j \in \pi^{-1}(i^*)} \text{Inf}_j(T_{1-\gamma}f_v)) \geq \frac{\tau}{2}.
\]

**Labeling.** The labeling to a heavy variable \( u \in V_l \) is given by choosing a label \( i \in R_1 \) independently with probability proportional to \( \text{Inf}_i(T_{1-\gamma}f_u) \). The label of a heavy variable \( v \in V_l \) is similarly assigned given by choosing \( j \in R_2 \) independently with probability proportional to \( \text{Inf}_i(T_{1-\gamma}f_v) \). Note that the sum of all influences of \( T_{1-\gamma}f_u \) is bounded by \( 1/\gamma \).

Suppose \( u \) is a good and heavy neighbor of a heavy variable \( v \). Then analysis above along with Lemma 5.14 and Equation 33 implies that the labeling strategy will succeed for \( v \) and \( u \) with probability \( \frac{\tau}{2} \). Additionally, from the above analysis, at least \( \frac{\varepsilon^2}{125} \) fraction of constraints between layers \( l \) and \( l' \) are between heavy variables \( u \in V_l \) and \( v \in V_{l'} \) such that \( u \) is a good neighbor of \( v \). Thus, the probabilistic labeling strategy satisfies in expectation \( \frac{\varepsilon^2}{2048r^2} \) fraction of constraints. By choosing the soundness \( \zeta \) to be small enough we obtain a contradiction.

**References**

[ACC06] S. Arora, E. Chlamtac, and M. Charikar. New approximation guarantee for chromatic number. In *Proceedings of the ACM Symposium on the Theory of Computing*, pages 215–224, 2006.

[ALM+98] S. Arora, C. Lund, R. Motwani, M. Sudan, and M. Szegedy. Proof verification and the hardness of approximation problems. *Journal of the ACM*, 45(3):501–555, 1998.

[AS98] S. Arora and S. Safra. Probabilistic checking of proofs: A new characterization of NP. *Journal of the ACM*, 45(1):70–122, 1998.

[BK97] A. Blum and D. R. Karger. An \( \tilde{O}(n^{3/14}) \)-coloring algorithm for 3-colorable graphs. *Information Processing Letters*, 61(1):49–53, 1997.

[BK09] N. Bansal and S. Khot. Optimal long code test with one free bit. In *Proceedings of the Annual Symposium on Foundations of Computer Science*, pages 453–462, 2009.

[BK10] N. Bansal and S. Khot. Inapproximability of hypergraph vertex cover and applications to scheduling problems. In *Proceedings of the International Colloquium on Automata, Languages and Programming*, pages 250–261, 2010.
A. Blum. New approximation algorithms for graph coloring. *Journal of the ACM*, 41(3):470–516, 1994.

H. Chen and A. M. Frieze. Coloring bipartite hypergraphs. In *Proc. IPCO*, pages 345–358, 1996.

S. O. Chan. Approximation resistance from pairwise independent subgroups. In *Proceedings of the ACM Symposium on the Theory of Computing*, pages 447–456, 2013.

E. Chlamtac and G. Singh. Improved approximation guarantees through higher levels of SDP hierarchies. In *Proc. APPROX-RANDOM*, pages 49–62, 2008.

I. Dinur, V. Guruswami, S. Khot, and O. Regev. A new multilayered PCP and the hardness of hypergraph vertex cover. In *Proceedings of the ACM Symposium on the Theory of Computing*, pages 595–601, 2003.

I. Dinur, S. Khot, W. Perkins, and M. Safra. Hardness of finding independent sets in almost 3-colorable graphs. In *Proceedings of the Annual Symposium on Foundations of Computer Science*, pages 212–221, 2010.

I. Dinur, E. Mossel, and O. Regev. Conditional hardness for approximate coloring. *SIAM Journal of Computing*, 39(3):843–873, 2009.

I. Dinur, O. Regev, and C. D. Smyth. The hardness of 3-uniform hypergraph coloring. *Combinatorica*, 25(5):519–535, 2005.

I. Dinur and S. Safra. On the hardness of approximating minimum vertex cover. *Annals of Mathematics*, 165(1):439–485, 2005.

E. Friedgut. Boolean functions with low average sensitivity depend on few coordinates. *Combinatorica*, 18(1):27–35, 1998.

V. Guruswami, J. Håstad, and M. Sudan. Hardness of approximate hypergraph coloring. *SIAM Journal of Computing*, 31(6):1663–1686, 2002.

V. Guruswami, P. Raghavendra, R. Saket, and Y. Wu. Bypassing UGC from some optimal geometric inapproximability results. In *Proceedings of the Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 699–717, 2012.

V. Guruswami and A. Sinop. The complexity of finding independent sets in bounded degree (hyper)graphs of low chromatic number. In *Proceedings of the Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1615–1626, 2011.

J. Håstad. On the NP-hardness of Max-Not-2. In *Proc. APPROX-RANDOM*, pages 170–181, 2012.

J. Holmerin. Vertex cover on 4-regular hyper-graphs is hard to approximate within $2 - \epsilon$. In *Proceedings of the Annual IEEE Conference on Computational Complexity*, 2002.

T. Holenstein. Parallel repetition: Simplification and the no-signaling case. *Theory of Computing*, 5(1):141–172, 2009.
[Kho02a] S. Khot. Hardness results for approximate hypergraph coloring. In *Proceedings of the ACM Symposium on the Theory of Computing*, pages 351–359, 2002.

[Kho02b] S. Khot. Hardness results for coloring 3-colorable 3-uniform hypergraphs. In *Proceedings of the Annual Symposium on Foundations of Computer Science*, pages 23–32, 2002.

[Kho02c] S. Khot. On the power of unique 2-prover 1-round games. In *Proceedings of the ACM Symposium on the Theory of Computing*, pages 767–775, 2002.

[KMS96] P. Kelsen, S. Mahajan, and R. Hariharan. Approximate hypergraph coloring. In *Proc. SWAT*, pages 41–52, 1996.

[KMS98] D. R. Karger, R. Motwani, and M. Sudan. Approximate graph coloring by semidefinite programming. *Journal of the ACM*, 45(2):246–265, 1998.

[KNS01] M. Krivelevich, R. Nathaniel, and B. Sudakov. Approximating coloring and maximum independent sets in 3-uniform hypergraphs. *Journal of Algorithms*, 41(1):99–113, 2001.

[KP06] S. Khot and A. K. Ponnuswami. Better inapproximability results for MaxClique, Chromatic Number and Min-3Lin-Deletion. In *Proceedings of the International Colloquium on Automata, Languages and Programming*, pages 226–237, 2006.

[KS06] S. Khot and R. Saket. A 3-query non-adaptive PCP with perfect completeness. In *Proceedings of the Annual IEEE Conference on Computational Complexity*, pages 159–169, 2006.

[KS08a] S. Khot and R. Saket. Hardness of minimizing and learning DNF expressions. In *Proceedings of the Annual Symposium on Foundations of Computer Science*, pages 231–240, 2008.

[KS08b] S. Khot and R. Saket. On hardness of learning intersection of two halfspaces. In *Proceedings of the ACM Symposium on the Theory of Computing*, pages 345–354, 2008.

[KS12] S. Khot and R. Saket. Hardness of finding independent sets in almost $q$-colorable graphs. In *Proceedings of the Annual Symposium on Foundations of Computer Science*, pages 380–389, 2012.

[KT12] K. Kawarabayashi and M. Thorup. Combinatorial coloring of 3-colorable graphs. In *Proceedings of the Annual Symposium on Foundations of Computer Science*, pages 68–75, 2012.

[MOR+06] E. Mossel, R. O’Donnell, O. Regev, J. E. Steif, and B. Sudakov. Non-interactive correlation distillation, inhomogeneous Markov chains, and the reverse Bonami-Beckner inequality. *Israel Journal of Mathematics*, 154:299–336, 2006.

[Mos10] E. Mossel. Gaussian bounds for noise correlation of functions. *GAFA*, 19:1713–1756, 2010.

[OW09] R. O’Donnell and Y. Wu. Conditional hardness for satisfiable 3-CSPs. In *Proceedings of the ACM Symposium on the Theory of Computing*, pages 493–502, 2009.

[Rao08] A. Rao. Parallel repetition in projection games and a concentration bound. In *Proceedings of the ACM Symposium on the Theory of Computing*, pages 1–10, 2008.

[Raz98] R. Raz. A parallel repetition theorem. *SIAM Journal of Computing*, 27(3):763–803, 1998.
A Construction of Smooth \(d\)-to-1 MLPCP

The construction of the Smooth \(d\)-to-1 Multi-Layered PCP \(\Phi\) closely follows the construction used \([\text{Kho02b}]\). We shall only give the construction.

We begin with an instance \(L\) of the \(d\)-to-1 Game given by Conjecture 2.5 with the variable sets \(U, V\) and label sets \([k]\) and \([m]\). For convenience we refer to the variables in \(V\) as \(V\)-variables and those in \(U\) as \(U\)-variables.

The variables of \(\Phi\) in the \(l\)th layer are sets of \((TL + L - l)\ \forall\)-variables and \((l - 1)\ \forall\)-variables. The label set \(R_l\) of layer \(l\) is the set of all \((TL + L - 1)\)-tuples of labelings to \(TL + L - l\ \forall\)-variables and \(l - 1\ \forall\)-variables.

There is a constraint between a variable \(v\) in layer \(l\) and a variable \(u\) in layer \(l'\) of \(\Phi\) if replacing \((l - l')\ \forall\)-variables \(q_1, \ldots, q_{l-l'}\) from the set associated with \(v\), with \(\forall\)-variables \(p_1, \ldots, p_{l-l'}\) such that \(p_r\) has a constraint with \(q_r\) in \(L\) for \(r = 1, \ldots, l - l'\), yields the set associated with \(u\). The constraint \(\pi_{v\to u}\) is projection which checks the consistency of the labels, according to whether the variables of \(L\) common to both \(u\) and \(v\) are assigned identically and the assignments to \(p_1, \ldots, p_{l-l'}\) and \(q_1, \ldots, q_{l-l'}\) are consistent. It is easy to see that \(\pi_{v\to u}^{-1}(i) = d^{l-l'}\) for any \(i \in R_v\).

The proof of weak density follows from the bi-regularity property of \(\Phi\) in a manner identical to the proof in \([\text{DGKR03}]\). The proof of soundness is identical to the proof in \([\text{Kho02b}]\). The proof of hardness in Theorem 2.3 follows from standard arguments as given in \([\text{DGKR03}]\). We omit these proofs.

B Fourier Analysis

We will be working over the field \(\mathbb{F}[2]\). Define the following homomorphism \(\phi\) from \((\mathbb{F}[2], +)\) to the multiplicative group \(\{-1, 1\}\), by \(\phi(a) := (-1)^a\). We now consider the vector space \(\mathbb{F}[2]^m\) for some positive
integer \( m \). We define the ‘characters’ \( \chi_{\alpha} : \mathbb{F}[^2]^m \mapsto \{-1, 1\} \) for every \( \alpha \in \mathbb{F}[^2]^m \) as,

\[
\chi_{\alpha}(f) := \phi(\alpha \cdot f), \quad f \in \mathbb{F}[^2]^m
\]

where ‘\( \cdot \)’ is the inner product in the vector space \( \mathbb{F}[^2]^m \). The characters \( \chi_{\alpha} \) satisfy the following properties,

\[
\begin{align*}
\chi_0(f) &= 1 \quad \forall f \in \mathbb{F}[^2]^m \\
\chi_{\alpha}(0) &= 1 \quad \forall \alpha \in \mathbb{F}[^2]^m \\
\chi_{\alpha + \beta}(f) &= \chi_{\alpha}(f)\chi_{\beta}(f) \\
\chi_{\alpha}(f + g) &= \chi_{\alpha}(f)\chi_{\alpha}(g)
\end{align*}
\]

and,

\[
E_{f \in \mathbb{F}[^2]^m} [\chi_{\alpha}(f)] = \begin{cases} 
1 & \text{if } \alpha = 0 \\
0 & \text{otherwise}
\end{cases}
\]

The characters \( \chi_{\alpha} \) form an orthonormal basis for \( L^2(\mathbb{F}[^2]^m) \). We have,

\[
\langle \chi_{\alpha}, \chi_{\beta} \rangle = \begin{cases} 
1 & \text{if } \alpha = \beta \\
0 & \text{otherwise}
\end{cases}
\]

where,

\[
\langle \chi_{\alpha}, \chi_{\beta} \rangle := E_{f \in \mathbb{F}[^2]^m} [\chi_{\alpha}(f)\chi_{\beta}(f)].
\]

Let \( A : \mathbb{F}[^2]^m \mapsto \mathbb{R} \) be any real valued function. Then the Fourier expansion of \( A \) is given by,

\[
A(x) = \sum_{\alpha \in \mathbb{F}[^2]^m} \hat{A}_{\alpha}\chi_{\alpha}(x),
\]

where,

\[
\hat{A}_{\alpha} = E_{x \in \mathbb{F}[^2]^m} [A(x)\chi_{\alpha}(x)].
\]

A useful equality is:

\[
\hat{A}_0 = E_{x \in \mathbb{F}[^2]^m} [A(x)].
\]

**Folding**

The following lemma gives a property of the Fourier coefficients of any homogeneously folded function.

**Lemma B.1.** Let \( A : \mathbb{F}[^2]^m \mapsto \mathbb{R} \) be any function such that \( A(x + y) = A(x) \) for some \( y \in \mathbb{F}[^2]^m \) and all \( x \in \mathbb{F}[^2]^m \). Then if \( \hat{A}_{\alpha} \neq 0 \), then \( \alpha \cdot y = 0 \).

**Proof.** Assume \( \hat{A}_{\alpha} \neq 0 \). By definition and using the folding property,

\[
\hat{A}_{\alpha} = E_{x \in \mathbb{F}[^2]^m} [A(x)\chi_{\alpha}(x)]
= E_{x \in \mathbb{F}[^2]^m} [A(x + y)\chi_{\alpha}(x + y)]
= E_{x \in \mathbb{F}[^2]^m} [A(x)\chi_{\alpha}(x + y)]
= E_{x \in \mathbb{F}[^2]^m} [A(x)\chi_{\alpha}(x)]\chi_{\alpha}(y)
= \hat{A}_{\alpha}\chi_{\alpha}(y).
\]

Thus, if \( \hat{A}_{\alpha} \neq 0 \), then \( \chi_{\alpha}(y) = 1 \). Thus, \( \phi(\alpha \cdot y) = 1 \). This implies that \( \alpha \cdot y = 0 \). \( \square \)