Random sampling and reconstruction in reproducing kernel subspace of mixed Lebesgue spaces

Prashant Goyal | Dhiraj Patel | S. Sivananthan

Department of Mathematics, Indian Institute of Technology Delhi, New Delhi 110016, India

Correspondence
S. Sivananthan, Department of Mathematics, Indian Institute of Technology Delhi, New Delhi 110016, India.
Email: siva@maths.iitd.ac.in

Communicated by: P. Agarwal

Funding information
Council of Scientific and Industrial Research, India; Department of Science and Technology, Government of India, Grant/Award Number: CRG/2019/002412

In this article, we consider the random sampling in the image space $V$ of an idempotent integral operator on mixed Lebesgue space $L^{p,q}(\mathbb{R}^{n+1})$. We assume some decay and regularity conditions on the integral kernel and show that the bounded functions in $V$ can be approximated by an element in a finite-dimensional subspace of $V$ on $C_{R,S} = \left[-\frac{R}{2}, \frac{R}{2}\right]^n \times \left[-\frac{S}{2}, \frac{S}{2}\right]$. Consequently, we show that the set of bounded functions concentrated on $C_{R,S}$ is totally bounded and prove with an overwhelming probability that the random sample set uniformly distributed over $C_{R,S}$ is a stable set of sampling for the set of concentrated functions on $C_{R,S}$. Further, we propose an iterative scheme to reconstruct the concentrated functions from their random measurements.

KEYWORDS
mixed Lebesgue space, random sampling, reconstruction algorithm, reproducing kernel space

MSC CLASSIFICATION
Primary 42A61, 42C15, 94A20, Secondary 26D15, 47B34

1 | INTRODUCTION

The sampling problem is motivated by the fundamental interest of digital electronics where one would like to convert an analog signal into a digital signal and vice versa for signal transmission or storage. Mathematically, given a function $f$, one wishes to discretize $f$ on some discrete sample set such that one can determine $f$ exactly from its discrete samples. The classical Shannon sampling theorem states that, if $f \in PW_{[-\frac{1}{2}, \frac{1}{2}]}(\mathbb{R})$, the space of functions in $L^2(\mathbb{R})$ whose Fourier transform supported in $[-\frac{1}{2}, \frac{1}{2}]$, then $f$ can be reconstructed uniquely and stably from its uniform sample values $\{f(k) : k \in \mathbb{Z}\}$ by using the formula

$$f(x) = \sum_{k \in \mathbb{Z}} f(k) \frac{\sin \pi(x - k)}{\pi(x - k)}.$$

It is one of the most fundamental theorems and basic tools in signal analysis. In particular, it serves as the theoretical basis of modern communication pulse code modulation. The classical historical review of sampling theory can be found in Butzer and Stens, Jerri, and Aldroubi and Gröchenig. Sampling problem is extensively studied in a general shift invariant space, function space with finite rate of innovation, and the image space of some idempotent integral operator on Euclidean space. In this article, we study the random sampling problem for the function space, which is image of an idempotent integral operator on mixed Lebesgue space $L^{p,q}(\mathbb{R}^{n+1})$. 

Math Meth Appl Sci. 2023;46:5119–5138. wileyonlinelibrary.com/journal/mma © 2022 John Wiley & Sons, Ltd.
Mixed Lebesgue space $L^{p,q}(\mathbb{R}^{n+1})$ is a generalization of Lebesgue space $L^p(\mathbb{R}^{n+1})$, which contains measurable functions with independent variables having different properties. In particular, time-varying functions belong to mixed Lebesgue space. The mixed Lebesgue space $L^{p,q}(\mathbb{R}^{n+1})$ is the collection of all measurable functions $f = f(x,y)$ such that

$$\|f\|_{L^{p,q}(\mathbb{R}^{n+1})}^q = \int_{\mathbb{R}} \left( \int_{\mathbb{R}^n} |f(x,y)|^p \, dx \right)^{\frac{q}{p}} \, dy < \infty, \text{ for } 1 \leq p, q < \infty. $$

In the early 1960s, Benedek and Panzone first introduced the mixed Lebesgue space by considering the space $L^{p}(X)$ of multi-variable measurable functions and norm on $L^p(X)$ defined as iterative $L^p$ norm of the function with each variable. Mixed Lebesgue space $L^{p,q}(\mathbb{R}^{n+1})$ inherits many properties of the standard $L^p(\mathbb{R}^{n+1})$, such as the Plancherel-Polya inequality, Leibniz rule, and some classical conjectures in harmonic analysis. The usefulness of mixed Lebesgue spaces motivates to study numerous other function spaces like Besov spaces, Sobolev spaces, and Bessel potential spaces with mixed norms. The classical spaces such as parabolic function space, Lorentz space, and Orlicz space were studied with mixed norms. Hart et al introduced the mixed-norm Hardy space $H^{p,q}(\mathbb{R}^{n+1})$ to improve the regularity of the bilinear operator. Further, due to the flexibility of independent variables of the function domain, interesting results were studied for the functions in mixed Lebesgue space in the context of partial differential equations. For example, to study the solution of partial differential equations involving both time and space variables, such as the heat or the wave equations. Functions in mixed Lebesgue space give precise information on the estimation of the parameters and induce better regularity for the solution of linear or non-linear equations. We refer to Huang and Yang for more details on mixed Lebesgue space.

The sampling problem draws attention towards higher dimensional time-varying signals. For example, video is a time-varying signal, and it can be viewed as a sequence of images. If the function $f(x,y,t)$ represents a black and white video signal, then for fix $t$, $f(\cdot,\cdot,t)$ denotes the still image/frame at time $t$, and for fix $(x,y)$, $f(x,y,\cdot)$ denotes the brightness at the point $(x,y)$. The mixed Lebesgue spaces $L^{p,q}(\mathbb{R}^{n+1})$ is ideal for modeling and measuring signals living in the time-space domain. The non-uniform sampling problem with mixed-norm studied for shift-invariant spaces, and image space of an idempotent integral operator.

A set $X = \{(x_i,y_j) : i \in \mathbb{Z}^n, j \in \mathbb{Z}\}$ is a stable sampling set for $V \subseteq L^{p,q}(\mathbb{R}^{n+1})$ if there exist $A,B > 0$ such that

$$A\|f\|_{L^{p,q}(\mathbb{R}^{n+1})} \leq \left( \sum_{j \in \mathbb{Z}} \left( \sum_{i \in \mathbb{Z}^n} |f(x_i,y_j)|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \leq B\|f\|_{L^{p,q}(\mathbb{R}^{n+1})} \forall f \in V. $$

This implies that the sampling set should be chosen appropriately so that $L^{p,q}$ norm is comparable to its discrete version. The Beurling density $D^-(X)$ of the sampling set $X$ characterizes the stability of the sampling set $X$ for Paley-Wiener space $PW_{[a,b]}(\mathbb{R})$. In particular, if $D^-(X) > 1$, then the sampling set $X$ is a stable set of sampling for $PW_{[a,b]}(\mathbb{R})$, where

$$D^-(X) = \lim_{r \to \infty} \inf_{y \in \mathbb{R}^n} \frac{\# [X \cap (y + [0,r])]}{r}. $$

Similar stable sampling condition is not true for $PW_S(\mathbb{R}^{n+1})$, where $S$ is a convex subset of $\mathbb{R}^{n+1}$; see Olevskii and Ulanovskii. The zero set of a function in $PW_S(\mathbb{R}^{n+1})$ is an analytic manifold. Therefore, finding a stable sampling set $X \subseteq \mathbb{R}^{n+1}$ that satisfying (1.1) is quite challenging. These drawbacks drove to study the sampling problem in the context of probabilistic framework. The recovery of signals from random measurement is one of the well-known methods used in compressed sensing and machine learning. Therefore, the random sampling theory is one of the most popular areas of research nowadays. Originally, the random sampling problem has been studied for multi-variable trigonometric polynomials and band-limited functions. In recent years, it was generalized to signals in a shift-invariant space, signals with finite rate of innovation, and signals in a reproducing kernel subspace of $L^p(\mathbb{R})$. The mixed Lebesgue space $L^{p,q}(\mathbb{R}^{n+1})$ inherits many properties of the classical $L^p(\mathbb{R}^{n+1})$ space; however, studying the random sampling problem in mixed Lebesgue space $L^{p,q}(\mathbb{R}^{n+1})$ brings various challenges due to the non-commutativity of the order of the integral. Recently, the random sampling problem studied for the shift-invariant subspace of mixed Lebesgue space $L^{p,q}(\mathbb{R}^{n+1})$ and established the sampling stability with high probability.
In random sampling, sample points are randomly taken according to a probability distribution. The stability of the random sample set over unbounded set \( \mathbb{R}^{n+1} \) is implausible in general. Bass and Gröchenig explained this issue by introducing the notion of relevant sampling.\(^{35,36} \) In this paper, we consider uniform probability distribution of the sampling set over the compact set \( C_{R,S} = \left[ -\frac{R}{2}, \frac{R}{2} \right] \times \left[ -\frac{S}{2}, \frac{S}{2} \right] \) and prove the random sampling result for the class of functions concentrated on \( C_{R,S} \). In particular, we estimate the probability bound such that the random sample set \( X = \{ (x_i, y_j) : x_i \in \mathbb{R}^n, y_j \in \mathbb{R}, 1 \leq i \leq l, 1 \leq j \leq m \} \) satisfy the following sampling inequality:

\[
A \| f \|_{L^p(\mathbb{R}^{n+1})} \leq \left( \frac{1}{m} \sum_{j=1}^{m} \left( \frac{1}{l} \sum_{i=1}^{l} |f(x_i, y_j)|^p \right)^{\frac{2}{p}} \right)^{\frac{1}{2}} \leq B \| f \|_{L^p(\mathbb{R}^{n+1})}
\]

for the function concentrated on \( C_{R,S} \). This implies the concentrated function can be recovered from its random samples with the estimated probability.

In Jiang and Li,\(^{46} \) the random sampling problem is addressed for multiply generated shift-invariant space. In this article, we will discuss the random sampling problem for the image space of an idempotent integral operator on \( L^{p,q}(\mathbb{R}^{n+1}) \). As we mentioned in Example 1, with an appropriate choice of kernel, we can obtain the random sampling inequality for function space of finite rate of innovation, which is more general than multiply generated shift-invariant space. Moreover, the stability estimate of the random sampling operator in (4.1) does not depend on sample size. Further, we provide an iterative reconstruction algorithm for the function \( f \) concentrated on \( C_{R,S} \).

This article is divided into four sections. Section 2 includes the basic definitions, notations, and preliminary results. In Section 3, we show the finite-dimensional approximation of the reproducing kernel space and prove that the set of bounded function concentrated on \( C_{R,S} \) is totally bounded with respect to \( L^{\infty,\infty} \)-norm on \( C_{R,S} \). Section 4 is devoted to studying probabilistic results on random sampling. We propose an iterative scheme to reconstruct the concentrated function from random samples.

## 2 | PRELIMINARIES

In this section, we define reproducing kernel space \( V \). We assume some condition on integral kernel and discuss some preliminary properties of \( V \).

**Definition 1.** For \( 1 \leq p, q < \infty \), the mixed sequence space \( \ell^{p,q}(\mathbb{Z}^{n+1}) \) is a collection of all sequences \( \{ c = c(i,j) : i \in \mathbb{Z}^n, j \in \mathbb{Z} \} \) such that

\[
\| c \|_{\ell^{p,q}} = \left( \sum_{j \in \mathbb{Z}} \left( \sum_{i \in \mathbb{Z}^n} |c(i,j)|^p \right)^{\frac{2}{p}} \right)^{\frac{1}{2}} < \infty.
\]

It is easy to verify that \( \ell^{\infty,\infty}(\mathbb{Z}^{n+1}) = \ell^{\infty}(\mathbb{Z}^{n+1}) \) and \( L^{\infty,\infty}(\mathbb{R}^{n+1}) = L^{\infty}(\mathbb{R}^{n+1}) \).

For \( 1 \leq p, q < \infty \), we define the idempotent integral operator \( T \) on \( L^{p,q}(\mathbb{R}^{n+1}) \) with \( T^2 = T \) as

\[
T f(x, y) = \int_{\mathbb{R}} \int_{\mathbb{R}^n} K(x, y, s, t) f(s, t) dsdt,
\]

where the symmetric kernel \( K \) on \( (\mathbb{R}^n \times \mathbb{R}) \times (\mathbb{R}^n \times \mathbb{R}) \) has decay with

\[
|K(x, y, s, t)| \leq \frac{c}{(1 + \|x - s\|)^{\alpha} (1 + |y - t|)^{\beta}}, \quad (2.1)
\]

where \( c > 0, \alpha > \frac{n}{p'} + n + 2 + \frac{1}{q'}, \beta > \frac{1}{q'} + n + 2 + \frac{1}{q} \) with \( \frac{1}{p} + \frac{1}{p'} = 1, \frac{1}{q} + \frac{1}{q'} = 1 \) and \( \|x\|_1 = \sum_{i=1}^{n} |x(i)|, x = (x(1), x(2) \ldots x(n)) \in \mathbb{R}^n \).

In addition, assume that the kernel \( K \) satisfies the regularity condition

\[
\lim_{\varepsilon \to 0} \| w_\varepsilon(K) \|_W = 0, \quad (2.2)
\]
where \( w_r(K)(x, y, s, t) = \sup_{\|x', y', s', t'\|_w < 1} |K(x + x', y + y', s + s', t + t') - K(x, y, s, t)| \), and

\[
\|K\|_W = \max \left\{ \sup_{y \in \mathbb{R}} \|K_1(y, \cdot)\|_{L_1(\mathbb{R})}, \sup_{t \in \mathbb{R}} \|K_1(\cdot, t)\|_{L_1(\mathbb{R})} \right\},
\]

\[
K_1(y, t) = \max \left\{ \sup_{x \in \mathbb{R}^n} \|K(x, y, \cdot, t)\|_{L_1(\mathbb{R})}, \sup_{s \in \mathbb{R}^n} \|K(\cdot, y, s, t)\|_{L_1(\mathbb{R})} \right\}.
\]

Then, the range space \( V = \{ Tf : f \in L^p_q(\mathbb{R}^{n+1}) \} \) is the closed reproducing kernel subspace of \( L^p_q(\mathbb{R}^{n+1}) \); i.e., for \( x \in \mathbb{R}^{n+1} \), there exists \( C_x > 0 \) such that for all \( f \in V \),

\[
|f(x)| \leq C_x \|f\|_{L^p_q(\mathbb{R}^{n+1})}.
\]

For \( 0 < \delta < 1 \), the set of \( \delta \)-concentrated function on \( C_{R,S} \) is defined as follows:

\[
V^*(R, S, \delta) = \left\{ f \in V : \|f\|_{L^p_q(\mathbb{R}^{n+1})} \leq \|f\|_{L^p_q(C_{R,S})} \right\}.
\]

**Definition 2.** A countable set \( \Gamma = \{ \gamma_{i,j} = (x_i, y_j) : x_i \in \mathbb{R}^n, y_j \in \mathbb{R}, i \in \mathbb{Z}^n, j \in \mathbb{Z} \} \) is said to be relatively separated set with respect to both variables if

\[
A_\Gamma(\eta) = \sup_{(x, y) \in \mathbb{R}^{n+1}} \sum_{j \in \mathbb{Z}} \sum_{i \in \mathbb{Z}^n} \chi_{B((x_i, y_j); \eta)}(x, y) < \infty,
\]

for some \( \eta > 0 \). We say \( \eta \) is the gap for \( \Gamma \) with respect to \( \mathbb{R}^{n+1} \) if

\[
B_\Gamma(\eta) = \inf_{(x, y) \in \mathbb{R}^{n+1}} \sum_{j \in \mathbb{Z}} \sum_{i \in \mathbb{Z}^n} \chi_{B((x_i, y_j); \eta)}(x, y) > 1,
\]

where \( B((x, y); \eta) \) denotes the open ball of radius \( \eta \) center at \( (x, y) \) in \( \mathbb{R}^{n+1} \) with respect to \( \| \cdot \|_\infty \)-norm.

Following the work of Nashed and Sun,\(^9\) Jiang and Sun\(^24\) showed that when \( K \) satisfies the decay condition (2.1) and the regularity condition (2.2), there exist a relatively separate set \( \Gamma = \{ \gamma_{i,j} = (x_i, y_j) : x_i \in \mathbb{R}^n, y_j \in \mathbb{R}, i \in \mathbb{Z}^n, j \in \mathbb{Z} \} \) with positive gap \( \eta \left( < \frac{1}{n} \right) \), and two families \( \Phi = \{ \phi_f \}_{f \in \Gamma} \subseteq L^p_q(\mathbb{R}^{n+1}) \) and \( \Phi = \{ \phi_f \}_{f \in \Gamma} \subseteq L^{p', q'}(\mathbb{R}^{n+1}) \) such that \( f \in V \) can be reformulated by

\[
f = \sum_{f \in \Gamma} \langle f, \phi_f \rangle \phi_f.
\]

Moreover, the family \( \Phi \) is a \((p, q)-frame\) for \( V \); i.e., there exist positive constants \( A \) and \( B \) such that

\[
A \|f\|_{L^p_q(\mathbb{R}^{n+1})} \leq \|\langle f, \phi_f \rangle\|_{L^q_p(\Gamma)} \leq B \|f\|_{L^p_q(\mathbb{R}^{n+1})}, \forall f \in V.
\]

Let \( N \) be a positive real number. We define the finite dimensional subspace \( V_N \) of \( V \) as

\[
V_N = \left\{ \sum_{f \in \Gamma} c_f \phi_f : c_f \in \mathbb{R} \right\}.
\]
3 | FINITE-DIMENSIONAL ESTIMATE

In this section, we prove that on the compact domain $C_{R,S}$, a function in $V$ is approximated by a function in $V_N$. We also show that the set of bounded $\delta$-concentrated functions is totally bounded with respect to $\| \cdot \|_{L^p(C_{R,S})}$.

**Lemma 1.** For any given $\epsilon > 0$ and $f \in V$ with $\| f \|_{L^p(R^{n+1})} = 1$, we have

$$\left\| f - \sum_{\gamma \in \Gamma} \langle f, \phi_{\gamma} \rangle \phi_{\gamma} \right\|_{L^p(C_{R,S})} < \epsilon,$$

where for some $C > 0$,

$$N > R + S + \frac{2}{n} + BC4^{\frac{\gamma}{n}} \left( 2(4n^{-1} + S + 1)^{nq} + (4n^{-1} + R + 1)^{q} \right)^{\frac{1}{q}} R \delta e^{-\frac{1}{n\kappa}}. $$

**Proof.** The proof of this lemma is similar to that of lemma 2.1 from Patel.44 Given $f \in V$, we consider $f_N \in V_N$ by

$$f_N = \sum_{\gamma \in \Gamma} \langle f, \phi_{\gamma} \rangle \phi_{\gamma}. \quad (3.1)$$

Let $(x, y) \in C_{R,S}$; then, from Equations (2.3) and (3.1), we have

$$|f(x, y) - f_N(x, y)| = \left| \sum_{\gamma \in \Gamma} \langle f, \phi_{\gamma} \rangle \phi_{\gamma}(x, y) \right| \leq \sum_{|\gamma| > \frac{N}{2}} \sum_{|\lambda| > \frac{N}{2}} |b(\lambda, \kappa)\phi_{\lambda, \kappa}(x, y)| + \sum_{|\gamma| > \frac{N}{2}} \sum_{|\lambda| \leq \frac{N}{2}} |b(\lambda, \kappa)\phi_{\lambda, \kappa}(x, y)|$$

$$+ \sum_{|\gamma| \leq \frac{N}{2}} \sum_{|\lambda| \leq \frac{N}{2}} |b(\lambda, \kappa)\phi_{\lambda, \kappa}(x, y)|$$

$$=: I_1 + I_2 + I_3,$$

where $(\lambda, \kappa) = \gamma \in \Gamma$ and $b(\lambda, \kappa) = b(\gamma) = \langle f, \phi_{\gamma} \rangle$. We estimate bound of each $I_i$, $i = 1, 2, 3$ separately. Using Hölder's inequality for mixed sequence space and Equation (2.4) on $I_1$, we get

$$I_1 \leq \left( \sum_{|\lambda| > \frac{N}{2}} \left( \sum_{|\lambda| \leq \frac{N}{2}} |b(\lambda, \kappa)|^p \right)^{\frac{p'}{p}} \right)^{\frac{1}{p}} \left( \sum_{|\lambda| > \frac{N}{2}} \left( \sum_{|\lambda| \leq \frac{N}{2}} |\phi_{\lambda, \kappa}(x, y)|^{p'} \right)^{\frac{p}{p'}} \right)^{\frac{1}{p'}}$$

$$\leq B \left( \sum_{|\lambda| > \frac{N}{2}} \left( \sum_{|\lambda| \leq \frac{N}{2}} |\phi_{\lambda, \kappa}(x, y)|^{p'} \right)^{\frac{p}{p'}} \right)^{\frac{1}{p}},$$

where explicit expression of $\phi_{\lambda, \kappa}$ is given below (see Jiang & Sun24 for more details).

$$\phi_{\lambda, \kappa}(x, y) = \eta^{-\frac{1}{n}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} K(\lambda + z_1, \kappa + z_2, x, y) dz_1 dz_2.$$
Therefore, from (2.1), we have
\[
\phi_{\lambda, \kappa}(x, y) \leq c\eta^{q - \frac{n}{p} - \frac{n}{q}}\int_{[-\frac{n}{2}, \frac{n}{2}]^n} 1^{1 + |\kappa + z - y|} d\zeta d\zeta_2
\]
\[
\leq c\eta^{q - \frac{n}{p} - \frac{n}{q}}\int_{[-\frac{n}{2}, \frac{n}{2}]^n} 1^{1 + |\kappa + x - y|} d\zeta d\zeta_2
\]
\[
\leq c\eta^{q - \frac{n}{p} - \frac{n}{q}}\sum_{\|\lambda\| > \frac{N}{2}} A_\Gamma(\eta) dz
\]
\[
\leq 2n c\eta^{q - \frac{n}{p} - \frac{n}{q}}\sum_{\|\lambda\| > \frac{N}{2}} A_\Gamma(\eta) dz
\]
\[
\leq 4n c\eta^{q - \frac{n}{p} - \frac{n}{q}}\sum_{\|\lambda\| > \frac{N}{2}} A_\Gamma(\eta) dz
\]
\[
\leq 4n c\eta^{q - \frac{n}{p} - \frac{n}{q}}\sum_{\|\lambda\| > \frac{N}{2}} A_\Gamma(\eta) dz
\]
\[
\leq 4n c\eta^{q - \frac{n}{p} - \frac{n}{q}}\sum_{\|\lambda\| > \frac{N}{2}} A_\Gamma(\eta) dz
\]
where \(\omega_\alpha = (ap' - 1)(ap' - 2) \cdots (ap' - n)\).
Let $D_1 = cA_\Gamma(\eta)^{\frac{1}{p'} + \frac{1}{q'}}$, and using the fact that $\eta < \frac{2}{n}$, we have

$$I_1 \leq \frac{BD_1 4^{\frac{n}{p} + \frac{1}{p'}}}{\omega_{p'}^\frac{1}{p'} \omega_{\rho}^{\frac{1}{p}}} \frac{1}{\left(\frac{N-R}{2} - \frac{1}{n}\right)^{\frac{a-n}{p}} \left(\frac{N-S}{2} - \frac{1}{n}\right)^{\frac{\beta-1}{p'}}}.$$  

For $I_2$, similar lines of proof yield

$$\sum_{\|\xi\|_\infty \leq \frac{N}{2}} |\phi_{\xi,k}(x,y)|^{p'} \leq 2^n c^{p'} \eta^{\frac{n}{p'}} \left(1 + |\kappa - y| - \frac{\eta}{2}\right)^{\beta p'} \sum_{\|\xi\|_\infty \leq \frac{N}{2}} \int_{-\frac{N+S}{2}}^{\frac{N+S}{2}} \left(1 + ||z - x||_1 - \frac{na}{2}\right)^{ap} dz \leq \frac{2^n c^{p'} \eta^{\frac{n}{p'}}}{\left(1 + |\kappa - y| - \frac{\eta}{2}\right)^{\beta p'}} \int_{-\frac{N+S}{2}}^{\frac{N+S}{2}} A_\Gamma(\eta) \|z - x\|_1 - \frac{na}{2})^{ap} dz \leq \frac{4^n c^{p'} \eta^{\frac{n}{p'}}}{\left(1 + |\kappa - y| - \frac{\eta}{2}\right)^{\beta p'}} A_\Gamma(\eta) \left(\frac{N+\eta}{2}\right)^n \left(1 + |\kappa - y| - \frac{\eta}{2}\right)^{\beta p'}.$$  

Hence,

$$\left(\sum_{\|\xi\|_\infty \leq \frac{N}{2}} |\phi_{\xi,k}(x,y)|^{p'}\right)^{\frac{1}{p'}} \leq \frac{4^n c^{p'} \eta^{\frac{n}{p'}}}{\left(1 - \frac{na}{2}\right)^{ap'}} A_\Gamma(\eta)^{\frac{n}{p'}} \left(\frac{N+\eta}{2}\right)^{\frac{n}{p'}} \frac{1}{\left(1 + |\kappa - y| - \frac{\eta}{2}\right)^{\beta p'}}.$$  

Taking summation on $\kappa$, we get

$$\sum_{|\kappa| > \frac{N}{2}} \left(\sum_{\|\xi\|_\infty \leq \frac{N}{2}} |\phi_{\xi,k}(x,y)|^{p'}\right)^{\frac{1}{p'}} \leq \frac{4^n c^{p'} + 1}{\left(1 - \frac{na}{2}\right)^{ap'}} A_\Gamma(\eta)^{\frac{n}{p'}} \left(\frac{N+\eta}{2}\right)^{\frac{n}{p'}} \frac{B_\Gamma(\eta)}{\omega_{\rho} \left(\frac{N-S}{2}\right)^{\beta - n - 1}} \leq \frac{4^n c^{p'} + 1}{\left(1 - \frac{na}{2}\right)^{ap'}} A_\Gamma(\eta)^{\frac{n}{p'}} \left(\frac{N+\eta}{2}\right)^{\frac{n}{p'}} \frac{B_\Gamma(\eta)}{\omega_{\rho} \left(\frac{N-S}{2}\right)^{\beta - n - 1}},$$

whenever $N > S + \eta + 1$.

Put $D_2 = A_\Gamma(\eta)^{\frac{1}{p'} + \frac{1}{q'}}$. Then,

$$I_2 \leq \frac{BD_2 4^{\frac{n}{p} + \frac{1}{p'}} (4n^{-1} + S + 1)^n}{\omega_{p'}^\frac{1}{p'} \omega_{\rho}^{\frac{1}{p}}} \left(\frac{N-S}{2} - \frac{1}{n}\right)^{n - \frac{n}{p}} \left(\frac{N-S}{2} - \frac{1}{n}\right)^{\frac{\beta-1}{p'}}.$$  

In a similar way, we obtain

$$I_3 \leq \frac{BD_3 4^{\frac{n}{p} + \frac{1}{p'}} (4n^{-1} + R + 1)}{\omega_{p'}^{\frac{1}{p'}} \omega_{\rho}^{\frac{1}{p}}} \left(\frac{N-R}{2} - \frac{1}{n}\right)^{n - \frac{n}{p}} \left(\frac{N-R}{2} - \frac{1}{n}\right)^{\frac{\beta-1}{p'}}.$$  

where $D_3 = A_\Gamma(\eta)^{\frac{1}{p'} + \frac{1}{q'}}$ and $N > R + \eta + 1$.
Let \( C = \max\{D_1, D_2, D_3\} \). Therefore,

\[
\|f - f_N\|_{L^q(C_{k,s})}^q \leq \left( BC4^{\frac{n}{q} + \frac{1}{q'}} \right)^q \left( \frac{R^n S}{\omega_\beta \left( \frac{N-R}{2} - \frac{1}{n} \right)^{\frac{q}{2}}} \right) \left( \frac{R^n S(4n^{-1} + S + 1)^q}{\omega_\beta \left( \frac{N-R}{2} - \frac{1}{n} \right)^{\frac{q}{2}}} \right)
\]

\[
+ \frac{R^n S(4n^{-1} + R + 1)^q}{\omega_\beta \left( \frac{N-R}{2} - \frac{1}{n} \right)^{\frac{q}{2}}} + \frac{R^n S(4n^{-1} + R + 1)^q}{\omega_\beta \left( \frac{N-R}{2} - \frac{1}{n} \right)^{\frac{q}{2}}} + \left( \frac{N-R-S}{2} - \frac{1}{n} \right)^{-(n+2)q}
\]

whenever

\[
N > R + S + \frac{2}{n} + BC4^{\frac{n}{q} + \frac{1}{q'}} \left( 2(4n^{-1} + S + 1)^q + (4n^{-1} + R + 1)^q \right)^{\frac{1}{q}} R^n S^\frac{1}{q} e^{-\frac{1}{n+2}}.
\]

This completes the proof.

We recall the following result on estimation of covering number of closed and bounded ball in a finite dimensional normed space.

**Proposition 1** (Cucker & Zhou). Let \( Y \) be a Banach space of dimension \( s \) and \( B(0; r) \) denotes the closed ball of radius \( r \) center at origin in \( Y \). The minimum number of open balls of radius \( r_i \) to cover \( B(0; r) \) is bounded by \( \left( \frac{2r}{r_i} + 1 \right)^s \).

The following results for mixed Lebesgue space \( L^{p,q}(\mathbb{R}^{n+1}) \) is similar to the classical Lebesgue space \( L^p(\mathbb{R}^n) \); see Patel and Sampath.

**Lemma 2.** If \( f \in V(R, S, \delta) := \{ f \in V^*(R, S, \delta) : \|f\|_{L^p(C_{k,s})} = 1 \} \), then

\[
\|f\|_{L^{p,q}(\mathbb{R}^{n+1})} \leq D \|f\|_{L^p(C_{k,s})},
\]

where \( D = \frac{\sup_{\alpha \in C_{k,s}} \|K(x,y,\cdots)\|_{L^{p,q}(\mathbb{R}^{n+1})}}{(1-\delta)^{\frac{1}{q'}}} \).

**Lemma 3.** The set \( V(R, S, \delta) \) is totally bounded with respect to \( \|\cdot\|_{L^{p,q}(\mathbb{R}^{n+1})} \).

**Remark 1.**

(i) In the above lemma, we choose \( f_N \in V_N \) such that \( \|f - f_N\|_{L^{\infty}(C_{k,s})} \leq \frac{\epsilon}{2} \) and

\[
N = R + S + 2 + BC4^{\frac{n}{q} + \frac{1}{q'}} \left( 2(4n^{-1} + S + 1)^q + (4n^{-1} + R + 1)^q \right)^{\frac{1}{q}} R^n S^\frac{1}{q} e^{-\frac{1}{n+2}}.
\]

Then, dimension of \( V_N \) is bounded by

\[
N^{n+1}N_0(\Gamma) \leq 2^{n+1}N_0(\Gamma) \left[ (R + S + 2)^{n+1} + C_1 e^{-\frac{1}{n+2}} \right] := d_e.
\]
where \( N_0(\Gamma) = \sup_{k \in \mathbb{Z}^{n+1}} \left( k + \left[ -\frac{1}{2}, \frac{1}{2} \right]^{n+1} \right) \cap \Gamma \) and
\[
C_1 = \left( BC4n^{-\frac{5}{q}+\frac{1}{q+1}} \left( 2 \left(4n^{-1} + S + 1\right)^{nq} + \left( 4n^{-1} + R + 1 \right)^{q} \right) \right)^{\frac{1}{n+1}}.
\]

Note that the choice of \( N \) is not unique; one can also consider a different value of \( N \) according to the bound of \( N \).

(ii) Let \( A(\varepsilon) \) be a \( \varepsilon \)-net of the totally bounded set \( \mathcal{V}(R, S, \delta) \) and \( N(\varepsilon) \) be the number of element in \( A(\varepsilon) \). The set \( A(\varepsilon) \) is same as \( \frac{\varepsilon}{2} \) net of \( B(0; D + \frac{\varepsilon}{2}) \) in \( V_N \) with respect to \( \| \cdot \|_{L^{\infty}(C_{R,S})} \). Hence, Proposition 1 implies
\[
N(\varepsilon) \leq \exp \left( d \log \left( \frac{8D}{\varepsilon} \right) \right).
\]

4 RANDOM SAMPLING

In this section, we define independent random variable from uniformly distributed random sample set on \( C_{R,S} \). Using the Bernstein inequality on the independent random variables, we prove the random sampling result.

Let \( \{(x_i, y_j) : i, j \in \mathbb{N}\} \) be a sequence of independent and identically distributed random variables uniformly distributed over \( C_{R,S} \). For every \( f \in \mathcal{V} \), we define the random variable
\[
Z_{i,j}(f) = \| f(x_i, y_j) \| - \frac{1}{R^n S^q} \iint_{C_{R,S}} |f(x, y)| \, dx \, dy
\]
with expectation \( E(Z_{i,j}(f)) = 0 \).

To derive properties of \( Z_{i,j}(f) \), first, we recall the following results.

**Lemma 4** (Jiang & Li^6). For \( 1 \leq p, q \leq \infty \) and \( X = \{(x_i, y_j) : 1 \leq i \leq l, 1 \leq j \leq m\} \), we have
\[
\| \{ f(x_i, y_j) \} \|_{L^p(X)} \leq \sum_{i=1}^{m} \sum_{j=1}^{l} \| f(x_i, y_j) \|_{L^p(X)} \leq l^{\frac{1}{p}} m^{1-\frac{1}{p}} \| \{ f(x_i, y_j) \} \|_{L^p(X)},
\]
\[
\| f \|_{L^{1,q}(C_{R,S})} \leq R^{\frac{m}{2}} S^{1-\frac{1}{q}} \| f \|_{L^{p,q}(C_{R,S})},
\]
\[
\| f \|_{L^{p,q}(C_{R,S})} \leq R^{(p-1)n} S^{p-1} D^{p-1} \| f \|_{L^{1,q}(C_{R,S})}.
\]

**Lemma 5.** For \( f, g \in \mathcal{V} \) with \( \| f \|_{L^p(\mathbb{R}^{n+1})} = \| g \|_{L^p(\mathbb{R}^{n+1})} = 1 \), then the following inequalities holds:

a. \( \text{Var} \left( Z_{i,j}(f) \right) \leq \frac{1}{R^n S^q} \| f \|_{L^p(X)} \| \cdot \|_{L^p(\mathbb{R}^{n+1})} \|
\]

b. \( \| Z_{i,j}(f) \|_{L^\infty} \leq \sup_{(x,y) \in \mathbb{R}^{n+1}} \| K(x, y, \cdot, \cdot) \|_{L^p(\mathbb{R}^{n+1})} \|
\]
c. \( \text{Var} \left( Z_{i,j}(f) - Z_{i,j}(g) \right) \leq \frac{2}{R^n S^q} \| f - g \|_{L^\infty(\mathbb{R}^{n+1})} \|
\]
d. \( \| Z_{i,j}(f) - Z_{i,j}(g) \|_{L^\infty} \leq \| f - g \|_{L^\infty(\mathbb{R}^{n+1})} \|
\]

**Proof.**

a. Lemma 4 and reproducing properties of function in \( \mathcal{V} \) implies
\[ \text{Var} (Z_{i,j}(f)) = E \left( \left( |f(x_i, y_j)| - E \left( |f(x_i, y_j)| \right) \right)^2 \right) \]
\[ = E \left( |f(x_i, y_j)|^2 \right) - (E \left( |f(x_i, y_j)| \right))^2 \]
\[ \leq E \left( |f(x_i, y_j)|^2 \right) \]
\[ = \frac{1}{R^nS} \int_{C_{k,S}} |f(x, y)|^2 \, dx \, dy \]
\[ \leq \frac{1}{R^nS} \| f \|_{L^\infty(C_{k,S})} \| f \|_{L^1(C_{k,S})} \]
\[ \leq \frac{1}{R^nS} \sup_{(x,y) \in \mathbb{R}^{n+1}} \| K(x, y, \cdot, \cdot) \|_{L^{p'}(\mathbb{R}^{n+1})}. \]

b. Reproducing properties of function in \( V \) implies

\[ \| Z_{i,j}(f) \|_{\ell^\infty_\lambda} = \sup_{i,j \in \mathbb{N}} \left\{ \left| f(x_i, y_j) \right| - \frac{1}{R^nS} \int_{C_{k,S}} |f(x, y)| \, dx \, dy \right\} \]
\[ \leq \max \left\{ \| f \|_{L^\infty(C_{k,S})}, \frac{1}{R^nS} \| f \|_{L^1(C_{k,S})} \right\} \]
\[ \leq \sup_{(x,y) \in \mathbb{R}^{n+1}} \| K(x, y, \cdot, \cdot) \|_{L^{p'}(\mathbb{R}^{n+1})}. \]

c. Similar argument as in part (a) implies

\[ \text{Var} (Z_{i,j}(f) - Z_{i,j}(g)) = E \left( \left( |f(x_i, y_j)| - |g(x_i, y_j)| \right)^2 \right) - (E \left( |f(x_i, y_j)| - |g(x_i, y_j)| \right))^2 \]
\[ = \frac{1}{R^nS} \int_{C_{k,S}} |f(x, y)| - |g(x, y)| (|f(x, y)| + |g(x, y)|) \, dx \, dy \]
\[ \leq \frac{2}{R^nS} \| f - g \|_{L^\infty(C_{k,S})}. \]

d. Similar argument as in part (b) implies

\[ \| Z_{i,j}(f) - Z_{i,j}(g) \|_{\ell^\infty_\lambda} = \sup_{i,j \in \mathbb{N}} \left\{ \left| f(x_i, y_j) - g(x_i, y_j) \right| - \frac{1}{R^nS} \int_{C_{k,S}} (|f(x, y)| - |g(x, y)|) \, dx \, dy \right\} \]
\[ \leq \max \left\{ \| f - g \|_{L^\infty(C_{k,S})}, \frac{1}{R^nS} \| f - g \|_{L^1(C_{k,S})} \right\} \]
\[ \leq \| f - g \|_{L^\infty(C_{k,S})}. \]

This completes the proof.

In the rest of the article, we denote \( k = \sup_{(x,y) \in \mathbb{R}^{n+1}} \| K(x, y, \cdot, \cdot) \|_{L^{p'}(\mathbb{R}^{n+1})}. \) We recall the Bernstein's inequality for independent random variable with zero mean.

**Theorem 1** (Bennett\textsuperscript{37}). Let \( \{ Y_{i,j} : 1 \leq i \leq l, 1 \leq j \leq m \} \) be a sequence of independent random variable with expectation \( E(Y_{i,j}) = 0, \forall i, j. \) Assume that \( \text{Var} (Y_{i,j}) \leq \sigma^2 \) and \( |Y_{i,j}| \leq M \) for all \( i, j. \) Then, for \( \lambda \geq 0, \) we have

\[ P \left( \left| \sum_{j=1}^m \sum_{i=1}^l Y_{i,j} \right| \geq \lambda \right) \leq 2 \exp \left( -\frac{\lambda^2}{2lm\sigma^2 + \frac{\lambda}{2}M} \right). \]
Lemma 6. Let \((x_i, y_j) : i, j \in \mathbb{N}\) be a sequence of i.i.d random variables that are drawn uniformly from \(C_{R,S}\). Then,

\[
P\left( \sup_{f \in V(R,S,\delta)} \left| \sum_{j=1}^{m} \sum_{i=1}^{l} Z_{ij}(f) \right| \geq \lambda \right) \leq 3a \exp \left( -\frac{\lambda^2}{12lnR^{-\frac{n}{2}}S^{-\frac{1}{2}} + \lambda} \right),
\]

where \(a = \exp \left( G(R + S)^n + \eta \right) \) and \(b = \min \left\{ \sqrt{2C_2}, \frac{3}{4k} \right\} \) for some positive constant \(G, C_2\).

Proof. We divide the proof in four parts.

Step 1. Let \(f \in V(R,S,\delta)\). From Lemma 3, we can construct a sequence \(\{f_r\}_{r \in \mathbb{N}}\) such that \(f_r \in A(2^{-r})\) and \(\|f - f_r\|_{L^\infty(C_{R,S})} < 2^{-r}\). Then, the random variable \(Z_{ij}(f)\) can be written as

\[
Z_{ij}(f) = Z_{ij}(f_1) + \sum_{i=2}^{\infty} (Z_{ij}(f_r) - Z_{ij}(f_{r-1})),
\]

due to the fact that the partial sum \(s_{r'}(f) = Z_{ij}(f_1) + \sum_{r=2}^{r'} (Z_{ij}(f_r) - Z_{ij}(f_{r-1})) = Z_{ij}(f_{r'})\) and

\[
\|Z_{ij}(f) - Z_{ij}(f_{r'})\|_{L^\infty} \leq \|f - f_{r'}\|_{L^\infty(C_{R,S})} \to 0 \text{ when } r' \to \infty.
\]

We consider the event

\[
E = \left\{ \sup_{f \in V(R,S,\delta)} \left| \sum_{j=1}^{m} \sum_{i=1}^{l} Z_{ij}(f) \right| \geq \lambda \right\}, \quad E_1 = \left\{ \exists f_1 \in A \left( \frac{1}{2} \right) : \left| \sum_{j=1}^{m} \sum_{i=1}^{l} Z_{ij}(f_1) \right| \geq \frac{\lambda}{2} \right\},
\]

and for \(r \geq 2\), we define

\[
E_r = \left\{ \exists f_r \in A(2^{-r}) \text{ and } f_{r-1} \in A(2^{-r+1}) \text{ with } \|f_r - f_{r-1}\|_{L^\infty(C_{R,S})} \leq 3 \cdot 2^{-r} : \left| \sum_{j=1}^{m} \sum_{i=1}^{l} Z_{ij}(f_r) - Z_{ij}(f_{r-1}) \right| \geq \frac{\lambda}{2r^2} \right\}.
\]

If the independent random sample set \(\{(x_i, y_j) : i, j \in \mathbb{N}\}\) satisfies

\[
\sup_{f \in V(R,S,\delta)} \left| \sum_{j=1}^{m} \sum_{i=1}^{l} Z_{ij}(f) \right| \geq \lambda,
\]

then at least one of the event \(E_r, r \geq 1\) holds. Therefore, \(E \subseteq \bigcup_{r=1}^{\infty} E_r\).

Step 2. Theorem 1 for the random variable \(\{Z_{ij}(f_1) : 1 \leq i \leq l, 1 \leq j \leq m\}\) and Lemma 5 imply

\[
P \left( \frac{\lambda}{2} \right) \leq 2 \exp \left( -\frac{\lambda^2}{2lnR^{-\frac{n}{2}}S^{-\frac{1}{2}}k + \frac{1}{3}k\lambda} \right)
\]

Hence,

\[
P(E) \leq 2N \left( \frac{1}{2} \right) \exp \left( -\frac{3}{4k} \frac{\lambda^2}{6lnR^{-\frac{n}{2}}S^{-\frac{1}{2}} + \lambda} \right).
\]
Step 3. We can estimate the probability of the event $\mathcal{E}_r$ in the similar way as in previous step. Theorem 1 and Lemma 5 imply

$$P \left( \sum_{j=1}^{m} \sum_{i=1}^{l} \left| Z_{ij}(f_r) - Z_{ij}(f_{r-1}) \right| \geq \frac{\lambda}{2r^2} \right) \leq 2 \exp \left( - \frac{\lambda^2}{4r^4} \left( 4lmR^{-\frac{s}{r}} S^{-\frac{1}{r}} \| f_r - f_{r-1} \|_{L^\infty(C_{r,k})} + \frac{2}{3} \frac{\lambda}{2r} \| f_r - f_{r-1} \|_{L^\infty(C_{r,k})} \right) \right) \leq 2 \exp \left( - \frac{2^r}{4r^4} \frac{\lambda^2}{12lmR^{-\frac{s}{r}} S^{-\frac{1}{r}} + \lambda} \right).$$

From Remark 1, we have

$$N(c) \leq \exp \left( 2^{n+1} N_0(\Gamma) \left[ (R + S + 2)^{n+1} + C_1 e^{-\frac{n+1}{4}} \right] \log \left( \frac{8D}{c} \right) \right).$$

Therefore,

$$N(2^{-r}) \leq \exp \left( 2^{n+1} N_0(\Gamma) \left[ (R + S + 2)^{n+1} + C_1 e^{-\frac{n+1}{4}} \right] \left[ (r + 3) \log 2 + \log D \right] \right)$$

$$N(2^{-r+1}) \leq \exp \left( 2^{n+1} N_0(\Gamma) \left[ (R + S + 2)^{n+1} + C_1 e^{-\frac{n+1}{4}} \right] \left[ (r + 2) \log 2 + \log D \right] \right).$$

Hence,

$$N(2^{-r})N(2^{-r+1}) \leq \exp \left( 2^{n+1} N_0(\Gamma) \left[ (R + S + 2)^{n+1} + C_1 e^{-\frac{n+1}{4}} \right] \left[ (2r + 5) \log 2 + 2 \log D \right] \right).$$

Therefore,

$$P(\mathcal{E}_r) \leq 2 \exp \left( 2^{n+1} N_0(\Gamma) \left[ (R + S + 2)^{n+1} + C_1 e^{-\frac{n+1}{4}} \right] \left[ (2r + 5) \log 2 + 2 \log D \right] - \frac{2^r}{4r^4} \frac{\lambda^2}{12lmR^{-\frac{s}{r}} S^{-\frac{1}{r}} + \lambda} \right)$$

$$\leq 2 \exp \left[ 2^r \left( 2^{n+1} N_0(\Gamma) \left[ (R + S + 2)^{n+1} 2^{-\frac{n+1}{4}} + C_1 2^{-\frac{r}{2} \log(D+3)} \right] \times \left[ (2r + 5) \log 2 + 2 \log D \right] - \frac{2^r}{4r^4} \frac{\lambda^2}{12lmR^{-\frac{s}{r}} S^{-\frac{1}{r}} + \lambda} \right) \right]$$

$$\leq 2 \exp \left[ 2^r \left( 2^{n+1} N_0(\Gamma) \left[ (R + S + 2)^{n+1} (4 + \log D) + C_1 (4(n + 2)(n + 3) + \log D) \right] - \frac{2^r}{4r^4} \frac{\lambda^2}{12lmR^{-\frac{s}{r}} S^{-\frac{1}{r}} + \lambda} \right) \right].$$

Let

$$C_2 = \frac{2^{\frac{1}{2}} \log(2)^4}{(n + 3)^4},$$

$$C_3 = 2^{n+1} N_0(\Gamma) \left[ (R + S + 2)^{n+1} (4 + \log D) + C_1 (4(n + 2)(n + 3) + \log D) \right],$$

$$M = \frac{\lambda^2}{12lmR^{-\frac{s}{r}} S^{-\frac{1}{r}} + \lambda}.$$
Step 4. Since \( \mathcal{E} \subseteq \bigcup_{r=1}^{\infty} \mathcal{E}_r \), we get \( P(\mathcal{E}) \leq \sum_{r=1}^{\infty} P(\mathcal{E}_r) \). For \( u, v > 0 \), the series \( \sum_{r=2}^{\infty} e^{-uv} \leq \frac{1}{u \log u} e^{-uv} \). Therefore,

\[
\sum_{r=2}^{\infty} P(\mathcal{E}_r) \leq \frac{2^{\frac{1}{n+1}}(n+3)}{(n+2)(C_2M - C_3) \log 2} \exp \left(-2^{\frac{1}{n+1}}(C_2M - C_3)\right)
\]

\[
\leq \frac{6}{C_2M - C_3} \exp \left(-\sqrt{2}(C_2M - C_3)\right).
\]

We choose \( \lambda \) large enough such that \( (C_2M - C_3) \geq 6 \). Hence,

\[
\sum_{r=2}^{\infty} P(\mathcal{E}_r) \leq \exp \left( \sqrt{2}C_3 \right) \exp \left(-\sqrt{2}C_2\frac{\lambda^2}{12lmR^n \gamma S^{-\frac{1}{\gamma}} + \lambda}\right).
\]

Let \( a_1 = \max \left\{ \exp \left( \sqrt{2}C_3 \right), N \left( \frac{1}{2} \right) \right\} \) and \( b = \min \left\{ \sqrt{2}C_2 \frac{\lambda}{4n}, \frac{\lambda}{\alpha} \right\} \). Then,

\[
P(\mathcal{E}) \leq 3a_1 \exp \left(-b \frac{\lambda^2}{12lmR^n \gamma S^{-\frac{1}{\gamma}} + \lambda}\right).
\]

To compute the bound of the constant \( a_1 \), consider

\[
\exp \left( \sqrt{2}C_3 \right) = \exp \left( 2^{n+1}N_0(\Gamma) \left[(R + S + 2)^{n+1}(4 + \log D) + C_1 (4(n + 2)(n + 3) + \log D)\right]\right)
\]

\[
\leq \exp \left( 2^{n+1}N_0(\Gamma) \left[(R + S + 2)^{n+1} + C_1 \right] (4(n + 2)(n + 3) + \log D)\right),
\]

\[
N \left( \frac{1}{2} \right) \leq \exp \left( 2^{n+1}N_0(\Gamma) \left[(R + S + 2)^{n+1} + 2C_1 \right] \log 16D\right)
\]

\[
\leq \exp \left( 2^{n+1}N_0(\Gamma) \left[(R + S + 2)^{n+1} + 2C_1 \right] (4(n + 2)(n + 3) + \log D)\right),
\]

and \( C_1 = \left( BC4^{\frac{n+1}{\gamma}} \right)^{n+1} \left( 2(4n^{-1} + S + 1)^{pq} + (4n^{-1} + R + 1)^q \right)^{\frac{1}{\gamma}} \).

Hence, we have \( a_1 \leq \exp \left( G(R + S)^{n+1} \right) = a, \) where \( R, S \geq 1 \) and

\[
G = 2^{n+1}N_0(\Gamma) \left[(4(n + 2)(n + 3) + \log D) \left(1 + 2 \left(3BC4^{\frac{n+1}{\gamma}} \right)^{n+1}\right)\right).
\]

This complete the proof.

**Theorem 2.** Let \( 1 \leq p, q < \infty \). Suppose \( \{x_i, y_j\} : 1 \leq i \leq l, 1 \leq j \leq m\) is an independent random variables that are uniformly distributed over the cube \( C_{R,S} \). Then, for any \( 0 < \mu < 1 \), the sampling inequality

\[
\frac{(1 - \mu)(1 - \delta)^{pq}D^{1-pq}}{R^{np} S^q} \|f\|_{L^p(\mathbb{R}^{n+1})} \leq \left( \frac{1}{m} \sum_{i=1}^{m} \left( \frac{1}{l} \sum_{j=1}^{l} |f(x_i, y_j)|^p \right)^{\frac{1}{p}} \right)^{\frac{1}{p}} \leq D(1 - \delta)^{\frac{1}{p}} \|f\|_{L^p(\mathbb{R}^{n+1})}
\]

holds for all \( f \in V^*(R, S, \delta) \) with probability at least

\[
1 - 3a \exp \left(-b \frac{1}{12R^{2np}S^q + D^{1-pq}R^{np} S^q}\right).
\]

Here, the positive constants \( a \) and \( b \) are same as in Lemma 6.
Proof. Let \( \{(x_i, y_j) : 1 \leq i \leq l, 1 \leq j \leq m\} \) be an i.i.d. random samples uniformly drawn from \( C_{R,S} \). Then, the event \( \mathcal{E} \) define in Lemma 6 is the complement of the event

\[
\mathcal{E}^c = \left\{ \sup_{f \in V(R,S,\delta)} \left| \sum_{j=1}^m \sum_{i=1}^l Z_{ij}(f) \right| \leq \lambda \right\}.
\]

The event \( \mathcal{E}^c \) is equivalent to

\[
\left| \sum_{j=1}^m \sum_{i=1}^l |f(x_i, y_j)| \right| \leq \frac{lm}{R^n S} \int \int_{C_{R,S}} |f(x, y)| \, dx \, dy \leq \lambda \frac{lm}{R^n S} \int \int_{C_{R,S}} |f(x, y)| \, dx \, dy - \lambda
\]

\[
\leq \sum_{j=1}^m \sum_{i=1}^l |f(x_i, y_j)| \leq \frac{lm}{R^n S} \int \int_{C_{R,S}} |f(x, y)| \, dx \, dy + \lambda.
\]

Lemma 4 and the choice \( \lambda = \frac{\mu lm(1 - \delta)^{pq} D^{-pq}}{R^n S} \) yields the inequality

\[
\frac{1}{l \gamma} m \frac{1}{l}(1 - \delta)^{pq} D^{1-pq} \frac{1}{R^n S} (1 - \mu) \leq \left( \sum_{j=1}^m \left( \sum_{i=1}^l |f(x_i, y_j)|^p \right)^{\frac{q}{p}} \right) \leq \left( \frac{1}{m} \sum_{j=1}^m \left( \frac{1}{l} \sum_{i=1}^l |f(x_i, y_j)|^p \right)^{\frac{1}{q}} \right)^{\frac{q}{p}}
\]

for every \( f \in V(R, S, \delta) \) with probability at least

\[
P(\mathcal{E}^c) = 1 - P(\mathcal{E}) \geq 1 - 3a \exp \left( \frac{-b \mu^2 lm(1 - \delta)^{2pq} D^{2(1-pq)}}{12R^{2n-p} S^{2q-1} + \mu D^{1-pq}(1 - \delta)^{pq} R^n S} \right)
\]

\[
\geq 1 - 3a \exp \left( \frac{-b \mu^2 lm(1 - \delta)^{2pq} D^{2(1-pq)}}{12R^{2n-p} S^{2q} + \mu D^{1-pq} R^n S} \right).
\]

In particular, random samples \( X = \{(x_i, y_j) : i, j \in \mathbb{N}\} \) i.i.d. uniformly distributed over \( C_{R,S} \) satisfy

\[
\frac{(1 - \mu)(1 - \delta)^{pq} D^{1-pq}}{R^n S} \|f\|_{L^p(S^{n+1})} \leq \left( \frac{1}{m} \sum_{j=1}^m \left( \frac{1}{l} \sum_{i=1}^l |f(x_i, y_j)|^p \right)^{\frac{1}{q}} \right)^\frac{q}{p}
\]

(4.2)

for every \( f \in V^*(R, S, \delta) \) with probability at least \( 1 - 3a \exp \left( -b \frac{\mu^2 lm(1 - \delta)^{2pq} D^{2(1-pq)}}{12R^{2n-p} S^{2q} + \mu D^{1-pq} R^n S} \right) \).

The set \( V^*(R, S, \delta) \) is a subset of reproducing kernel space \( V \), and for every \( (x, y) \in \mathbb{R}^{n+1} \), we have

\[
|f(x, y)| \leq \|K(x, y, \cdot, \cdot)\|_{L^p(\mathbb{R}^{n+1})} \|f\|_{L^p(\mathbb{R}^{n+1})}, \quad \forall f \in V.
\]

Hence, for each \( f \in V^*(R, S, \delta) \),

\[
\left( \frac{1}{m} \sum_{j=1}^m \left( \frac{1}{l} \sum_{i=1}^l |f(x_i, y_j)|^p \right)^{\frac{1}{q}} \right)^\frac{q}{p} \leq D(1 - \delta)^{\frac{1}{q}} \|f\|_{L^p(\mathbb{R}^{n+1})},
\]

(4.3)

Therefore, (4.2) and (4.3) imply the random sampling inequality (4.1) with probability at least \( 1 - 3a \exp \left( -b \frac{\mu^2 lm(1 - \delta)^{2pq} D^{2(1-pq)}}{12R^{2n-p} S^{2q} + \mu D^{1-pq} R^n S} \right) \).

This complete the proof.
Example 1. For \((x, y) \in \mathbb{R}^n \times \mathbb{R}\), we define

\[
\varphi(x, y) = \varphi(x(1), \ldots, x(n), y) = \left(\frac{3}{2}\right) \frac{3}{\sqrt{n^2 - 3n - 2}} \max \left\{ 1 - 3 \sum_{i=1}^{n} |x(i)| - 3|y|, 0 \right\}
\]

with \(\text{supp}(\varphi) \subseteq \left[\frac{1}{3}, 1\right]^{n+1}\). We follow the ideas of Li et al.\(^4\) to derive a reconstruction algorithm for the recovery of functions in various reproducing kernel subspaces of Lebesgue functions from their random sample values. Thus, if the random sample set satisfies above inequality, then the sample set is a stable set of sampling for the set \(V^s(R, S, \delta)\). Further, there exists \(C_n > 0\) such that

\[
|\varphi(x, y)| \leq C_n e^{-\|x\|^2 + |y|^2},
\]

and

\[
|K(x, y, s, t)| \leq \left(\frac{3}{2}\right) \frac{3C_n}{\sqrt{n^2 - 3n - 2}} \sum_{(a, b) \in \Gamma} \frac{e^{-\|x-a\|^2} e^{-\|y-b\|^2}}{\sqrt{\pi}} \leq \left(\frac{3}{2}\right) \frac{3C_n}{\sqrt{n^2 - 3n - 2}} \sum_{(a, b) \in \Gamma} \frac{e^{-\frac{1}{2} \left(3|x-a|^2 - 2||x||_2||y||_2 + 3||y||^2\right)}}{\sqrt{\pi}} \leq \left(\frac{1}{2}\right) \frac{C_n S_n}{\sqrt{\pi}} \int \left\{ \frac{n}{2} e^{-\|x-a\|^2} \right\}.
\]

where \(S_n\) represents the surface area of the unit sphere in \(\mathbb{R}^n\). This implies the kernel \(K\) satisfy the decay condition (2.1).

For given \(c > 0\), the sampling inequality (4.1) holds for all \(f \in V^s(R, S, \delta)\) with probability at least \(1 - \epsilon\) if

\[
3a \exp\left(-b \frac{\mu^2 l m(1 - \delta)^{2pq} D^{(1-pq)}}{12R^{2np} S^{2q} + D^{1-pq} R^{np} S^q}\right) < \epsilon.
\]

For small \(\delta, b = \frac{2^{7pq} (1 - \log 2)^4}{3^{(n+3)^4}} > \frac{1}{2^{5(n+3)^4}}\). Hence,

\[
\exp\left(G(R + S)^{n^2+n} - \frac{1}{2^{10(n+3)^4}} \times \frac{\mu^2 l m(1 - \delta)^{2pq} D^{(1-pq)}}{12R^{2np} S^{2q} + D^{1-pq} R^{np} S^q}\right) < \epsilon^{-\frac{3}{2}}.
\]

Thus,

\[
\lim_{n \to \infty} \frac{2^{10(n+3)^4} \left(G(R + S)^{n^2+n} + \log \left(\frac{3}{\epsilon}\right)\right)}{\mu^2 l m(1 - \delta)^{2pq} D^{(1-pq)}} < \frac{12R^{2np} S^{2q} + D^{1-pq} R^{np} S^q}{\mu^2 l m(1 - \delta)^{2pq} D^{(1-pq)}}.
\]

Thereby, if the random sample set satisfy above inequality, then the sample set is a stable set of sampling for the set \(V^s(R, S, \delta)\) with high probability.

In the following, we give an iterative scheme for the reconstruction of \(\delta\)-concentrated functions on \(C_{R,S}\). The iterative reconstruction method from non-uniform sampling is presented in various reproducing kernel subspaces of Lebesgue space.\(^9\)\(^1\)\(^4\) We follow the ideas of Li et al.\(^4\) to derive reconstruction algorithm for the recovery of functions in \(V^s(R, S, \delta)\) from their random sample values.
Before going into further details on reconstruction scheme, we first define covering radius for countable set $X$ and give probabilistic bound on covering radius.

**Definition 3.** Let $X = \{ (x_i, y_j) : 1 \leq i \leq l, 1 \leq j \leq m \}$ be a finite subset of $C_{R,S}$. A positive constant $\theta_X(l, m)$ is called covering radius for $C_{R,S}$ if

$$\theta = \theta_X(l, m) := \inf \left\{ r > 0 : C_{R,S} \subseteq \bigcup_{j=1}^{m} \bigcup_{i=1}^{l} B(x_i, y_j; r) \right\}.$$  

Observe that, covering radius $\theta$ of $X$ satisfies $\theta \leq \eta$, where $\eta$ denotes the gap of $X$ with respect to $C_{R,S}$. Following the idea of Bass and Gröchenig and Li et al. we prove the following probabilistic bound on covering radius.

**Lemma 7.** Let $X = \{ (x_i, y_j) : 1 \leq i \leq l, 1 \leq j \leq m \}$ be an i.i.d. random variables uniformly distributed over $C_{R,S}$. For a given $0 < \sigma < 1$,

$$P(\theta_X(l, m) > \sigma) \leq \left( \frac{4}{\sigma} \right)^{n+1} R^n S \exp \left( -lm \frac{\sigma^{n+1}}{2^{n+1} R^n S} \right).$$

**Proof.** Let $\Gamma \subseteq C_{R,S} \ni \frac{\sigma}{2}$ net of $C_{R,S}$. Therefore, the collection $\{ B \left( \gamma; \frac{\sigma}{2} \right) : \gamma \in \Gamma \}$ is an open cover of $C_{R,S}$, and the cardinality of $\Gamma$ is bounded by

$$\#\Gamma \leq \left( \frac{2R}{\sigma} + 1 \right)^n \left( \frac{2S}{\sigma} + 1 \right) \leq \left( \frac{4}{\sigma} \right)^{n+1} R^n S.$$  

Moreover, for any $\gamma \in \Gamma$ and $0 < r \leq 1$, there exists $z_\gamma \in B(\gamma; r)$ such that

$$B \left( z_\gamma; \frac{r}{2} \right) \subseteq B(\gamma; r) \cap C_{R,S}.$$  

If the event $\mathcal{E} := \{ \theta_X(l, m) > \sigma \}$ is true, then there exists a point $\gamma_0 \in \Gamma$ such that $B \left( \gamma_0; \frac{\sigma}{2} \right) \cap X = \phi$. Therefore,

$$P(\theta_X(l, m) > \sigma) \leq P \left( B \left( \gamma; \frac{\sigma}{2} \right) \cap X = \phi \text{ for some } \gamma \in \Gamma \right) \leq \sum_{\gamma \in \Gamma} P \left( B \left( \gamma; \frac{\sigma}{2} \right) \cap X = \phi \right) \leq \sum_{\gamma \in \Gamma} \left( 1 - \frac{\text{vol} \left( B \left( \gamma; \frac{\sigma}{2} \right) \cap C_{R,S} \right)}{R^n S} \right)^{lm} \leq \left( \frac{4}{\sigma} \right)^{n+1} R^n S \left( 1 - \frac{\sigma^{n+1}}{2^{n+1} R^n S} \right)^{lm} P(\theta_X(l, m) > \sigma) \leq \left( \frac{4}{\sigma} \right)^{n+1} R^n S \exp \left( -lm \frac{\sigma^{n+1}}{2^{n+1} R^n S} \right).$$

This complete the proof.

The regularity condition (2.2) of the kernel $K$ implies, for $\frac{1}{\|K\|_w} - \delta > 0$, there exists $\epsilon_0 > 0$ such that

$$\|w_\epsilon(K)\|_w < \frac{1}{\|K\|_w} - \delta, \text{ whenever } 0<\epsilon<\epsilon_0.$$
In particular, for a given finite sample set \( X \subseteq C_{R,S} \), if the covering radius of \( X \) satisfies \( \theta < \epsilon_0 \), then \((\|w_0(K)\|_W + \delta) \|K\|_W < 1\).

**Theorem 3.** Let \( \delta < \frac{1}{\|K\|_W} \). If \( X = \{(x_i, y_j) : 1 \leq i \leq l, 1 \leq j \leq m\} \) is an i.i.d. random variables uniformly distributed over \( C_{R,S} \), then \( f \in V^*(R, S, \delta) \) can be uniquely reconstructed from its random sample \( \{f(x_i, y_j) : 1 \leq i \leq l, 1 \leq j \leq m\} \) with the probability at least

\[
1 - (4\epsilon_0^{-1})^{n+1} R^n S \exp \left(-l m \frac{\epsilon_0^{n+1}}{2^{n+1} R^n S}\right).
\]

The iterative scheme for reconstruction of \( f \) is given by

\[
f_0 = S_X f, \quad f_r = f_0 + f_{r-1} - S_X f_{r-1}, \quad r \geq 1,
\]

where \( S_X f = \sum_{j=1}^{m} \sum_{i=1}^{l} f(x_i, y_j) T \beta_{i,j} \) and \( \{\beta_{i,j}\} \) is the partition of unity over \( C_{R,S} \). Further, \( f_r \) converges to \( f \) exponentially, and if \( \theta \) denotes the covering radius of \( X \), then

\[
\|f_r - f\|_{L^p(\mathbb{R}^{n+1})} \leq 1 + \|K\|_W \left(\|w_0(K)\|_W + \delta\right) \frac{1}{1 - \|K\|_W (\|w_0(K)\|_W + \delta)} ||f||_{L^p(\mathbb{R}^{n+1})}^{n+1}
\]

**Proof.** Let \( X = \{(x_i, y_j) : 1 \leq i \leq l, 1 \leq j \leq m\} \) be an i.i.d. random variable uniformly distributed over \( C_{R,S} \) with covering radius \( \theta \). Then, the collection \( \{B_0(x_i, y_j) : 1 \leq i \leq l, 1 \leq j \leq m\} \) is a cover for \( C_{R,S} \) and there exists a partition of unity \( \{\beta_{i,j} : 1 \leq i \leq l, 1 \leq j \leq m\} \) over \( C_{R,S} \); i.e., on \( C_{R,S} \) the collection \( \{\beta_{i,j}\} \) satisfies

(i) \( \text{supp} (\beta_{i,j}) \subseteq B_0 (x_i, y_j) \), (ii) \( 0 \leq \beta_{i,j} \leq 1 \), and (iii) \( \sum_{j=1}^{m} \sum_{i=1}^{l} \beta_{i,j} \equiv 1 \).

Now, we consider the iteration where

\[
f_0 = S_X f, \quad f_r = f_0 + f_{r-1} - S_X f_{r-1}, \quad r \geq 1.
\]

For every \((x, y) \in \mathbb{R}^{n+1}\), we consider \( Q_X f(x, y) = \sum_{j=1}^{m} \sum_{i=1}^{l} f(x_i, y_j) \beta_{i,j}(x, y) \); then, \( S_X = T Q_X \). Hence, for \( f \in V^*(R, S, \delta) \), we have

\[
\|f - S_X f\|_{L^p(\mathbb{R}^{n+1})} = \|T f - T Q_X f\|_{L^p(\mathbb{R}^{n+1})}
\]

\[
\leq \|K\|_W \left(\|f - Q_X f\|_{L^p_0(C_{R,S})} + \|f\|_{L^p_0(C_{R,S})}\right).
\]

For \((x, y) \in C_{R,S}\)

\[
\|f(x, y) - Q_X f(x, y)\| \leq \sum_{j=1}^{m} \sum_{i=1}^{l} |f(x, y) - f(x_i, y_j)| \beta_{i,j}(x, y) \leq w_0(f)(x, y),
\]

\[
\|f - Q_X f\|_{L^p_0(C_{R,S})} \leq \|w_0(K)\|_W \|f\|_{L^p_0(\mathbb{R}^{n+1})}
\]

as \( w_0(f)(x, y) \leq \int_{\mathbb{R}^n} w_0(K)(x, y, s, t) f(s, t) ds dt \). Therefore,

\[
\|f - S_X f\|_{L^p(\mathbb{R}^{n+1})} \leq \|K\|_W (\|w_0(K)\|_W + \delta) \|f\|_{L^p(\mathbb{R}^{n+1})}.
\]

From the definition of iteration algorithm, we have

\[
f_r - f_{r-1} = (I - S_X) f_0, \quad \text{and} \quad f_r = S_X f + \sum_{j=1}^{r} (T - S_X)^j S_X f.
\]
where \( I \) is an identity operator. Now, we define \( R = I + \sum_{j=1}^{\infty} (T - S_X)^j \), and if the covering radius of \( X \) satisfy \( \theta \leq \frac{\epsilon_0}{2} \), then \( R \) is a bounded operator, and

\[
\|Rf\|_{L^p(\mathbb{R}^{n+1})} \leq \|f\|_{L^p(\mathbb{R}^{n+1})} + \sum_{j=1}^{\infty} \|\mathbf{I} - S_X\| \|Tf\|_{L^p(\mathbb{R}^{n+1})}
\]
\[
\leq \left( 1 + \frac{\|K\|}{1 - \|K\| (\|w_0(K)\| + \delta)} \right) \|f\|_{L^p(\mathbb{R}^{n+1})}.
\]

Therefore, \( R \) is pseudo-inverse of pre-contraction operator \( S_X \); i.e., \( RS_Xf = S_XRf = f \). This implies

\[
\|f_r - f\|_{L^p(\mathbb{R}^{n+1})} = \left\| \sum_{j=r+1}^{\infty} (I - S_X)^j S_X f \right\|_{L^p(\mathbb{R}^{n+1})}
\]
\[
\leq \sum_{j=r+1}^{\infty} \|I - S_X\| \|S_X f\|_{L^p(\mathbb{R}^{n+1})}
\]
\[
\leq \frac{1 + \|K\| (\|w_0(K)\| + \delta)}{1 - \|K\| (\|w_0(K)\| + \delta)} (\|K\| (\|w_0(K)\| + \delta))^{r+1} \|f\|_{L^p(\mathbb{R}^{n+1})}.
\]

If \( \theta \leq \frac{\epsilon_0}{2} \), then \( \|K\| (\|w_0(K)\| + \delta) < 1 \), and therefore, \( \|f_r - f\|_{L^p(\mathbb{R}^{n+1})} \to 0 \) exponentially as \( r \to \infty \). This together with Lemma 7 implies that \( f \in V^*(R, S, \delta) \) can be reconstructed from random sample values with the probability at least

\[
1 - (4\epsilon_0^{-1})^{n+1} R^n S \exp \left( -l \frac{m_0^n}{2^{n+1} R^n S} \right).
\]

This complete the proof.

5 | CONCLUSION

We considered the random sampling problem for a class of functions, which is an image of an idempotent integral operator and localized on a finite cuboid. We established a stability estimate for the sampling operator on the considered class of functions with overwhelming probability. The reconstruction algorithm shows that the function concentrated on cuboid recovers completely from their random information, and the iterative scheme converges exponentially with high probability.

ACKNOWLEDGEMENTS

We acknowledge the anonymous reviewers for carefully reading the article and providing us with helpful suggestions and feedback that helped to improve the quality of the paper. P. Goyal and S. Sivananthan acknowledge the Department of Science and Technology, Government of India, for the financial support through Project No. CRG/2019/002412. D. Patel acknowledges Council of Scientific and Industrial Research for the financial support.

CONFLICT OF INTEREST

This work does not have any conflict of interest.

ORCID

Prashant Goyal https://orcid.org/0000-0002-7294-855X
Dhiraj Patel https://orcid.org/0000-0002-4583-3896
S. Sivananthan https://orcid.org/0000-0002-9038-2721
REFERENCES

1. Butzer PL, Stens RL. Sampling theory for not necessarily band-limited functions: a historical overview. *SIAM Rev.* 1992;34(1):40-53.
2. Jerri AJ. The Shannon sampling theorem—its various extensions and applications: A tutorial review. *Proc. IEEE.* 1977;65(11):1565-1596.
3. Aldroubi A, Gröchenig K. Nonuniform sampling and reconstruction in shift-invariant spaces. *SIAM Rev.* 2001;43(4):585-620.
4. Aldroubi A, Gröchenig K. Beurling-Landau-type theorems for non-uniform sampling in shift invariant spline spaces. *J Fourier Anal Appl.* 2000;6(1):93-103.
5. Aldroubi A, Sun Q, Tang W-S. Nonuniform average sampling and reconstruction in multiply generated shift-invariant spaces. *Constr Approx.* 2004;20(2):173-189.
6. Vetterli M, Marziliano P, Blu T. Sampling signals with finite rate of innovation. *IEEE Trans Sig Process.* 2002;50(6):1417-1428.
7. Sun Q. Nonuniform average sampling and reconstruction of signals with finite rate of innovation. *SIAM J Math Anal.* 2007;38(5):1389-1422.
8. Cheng C, Jiang Y, Sun Q. Sampling and Galerkin reconstruction in reproducing kernel spaces. *Appl Comput Harmonic Anal.* 2016;41(2):638-659.
9. Nashed MZ, Sun Q. Sampling and reconstruction of signals in a reproducing kernel subspace of $L^p(\mathbb{R}^d)$. *J Funct Anal.* 2010;258(7):2422-2452.
10. Nashed MZ, Sun Q, Xiao J. Convolution sampling and reconstruction of signals in a reproducing kernel subspace. *Proc Am Math Soc.* 2013;141(6):1995-2007.
11. Kumar A, Patel D, Sampath S. Sampling and reconstruction in reproducing kernel subspaces of mixed Lebesgue spaces. *J Pseudo-Differ Oper Appl.* 2020;11(2):843-868.
12. Benedek A, Panzone R. The space $L^p$, with mixed norm. *Duke Math J.* 1961;28(3):301-324.
13. Torres RH, Ward EJ. Leibniz’s rule, sampling and wavelets on mixed Lebesgue spaces. *J Fourier Anal Appl.* 2015;21(5):1053-1076.
14. Hart J, Torres R, Wu X. Smoothing properties of bilinear operators and Leibniz-type rules in Lebesgue and mixed Lebesgue spaces. *Trans Am Math Soc.* 2018;370(12):8581-8612.
15. Córdoba A, Crespo E. Radial multipliers and restriction to surfaces of the Fourier transform in mixed-norm spaces. *Math Zeitschrift.* 2017;286(3):1479-1493.
16. Besov OV, Il’in VP, Nikol’skii SM. *Integral Representations of Functions and Imbedding Theorems*, Vol. 2: John Wiley & Sons; 1979.
17. Fernandez DL. Lorentz spaces, with mixed norms. *J Funct Anal.* 1977;25(2):128-146.
18. Goranda RA. Parabolic function spaces with mixed norm. *Trans Am Math Soc.* 1978;246:451-461.
19. Milman M. A note on $L(p, q)$ spaces and Orlicz spaces with mixed norms. *Proc Am Math Soc.* 1981;83(4):743-746.
20. Brezis H. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer Science & Business Media; 2010.
21. Huang L, Yang D. On function spaces with mixed norms—a survey. *J Math Study.* 2021;54:262-336.
22. Lu Y, Xian J. Non-uniform random sampling and reconstruction in signal spaces with finite rate of innovation. *Acta Applicandae Mathematicae.* 2021;54:262-336.
23. Zhao J, Kostić M, Du W-S. On generalizations of sampling theorem and stability theorem in shift-invariant subspaces of Lebesgue and Wiener amalgam spaces with mixed-norms. *Symmetry.* 2021;13(2):331.
24. Jiang Y, Sun W. Adaptive sampling of time-space signals in a reproducing kernel subspace of mixed Lebesgue space. *Banach J Math Anal.* 2020;14(3):821-841.
25. Beurling A. *The Collected Works of Arne Beurling*, Vol. 2: Birkhäuser, Boston, MA; 1989.
26. Landau HJ. Necessary density conditions for sampling and interpolation of certain entire functions. *Acta Math.* 1967;117:37-52.
27. Ortega-Cerdà J, Seip K. *Fourier frames*. 2002;2002:789-806.
28. Olevskii AM, Ulanovskii A. *Functions With Disconnected Spectrum in Lorentz spaces, with mixed norms*. 2013;141(6):1995-2007.
29. Ortega-Cerdà J, Seip K. *Fourier frames*. 2002;2002:789-806.
30. Poggio T, Shenkel CR. On the mathematical foundations of learning. *Am Math Soc.* 2001;13(2):1-49.
31. Poggio T, Shenkel CR. On the mathematical foundations of learning. *Am Math Soc.* 2002;39(1):1-49.
32. Bass RF, Gröchenig K. Random sampling of multivariate trigonometric polynomials. *SIAM J Math Anal.* 2005;36(3):773-795.
33. Bass RF, Gröchenig K. Random sampling of bandlimited functions. *Israel J Math.* 2010;177(1):1-28.
34. Bass RF, Gröchenig K. Relevant sampling of bandlimited functions. *Illinois J Math.* 2013;57(1):43-58.
35. Yang J, Wei W. Random sampling in shift invariant spaces. *J Math Anal Appl.* 2013;398(1):26-34.
36. Führ H, Xian J. Relevant sampling in finitely generated shift-invariant spaces. *J Approx Theory.* 2019;240:1-15.
37. Yang J. Random sampling and reconstruction in multiply generated shift-invariant spaces. *Anal Appl.* 2019;17(02):323-347.
38. Li Y, Chen J. Reconstruction from convolution random sampling in local shift invariant spaces. *Inverse Problems.* 2019;35(12):125008.
39. Jiang Y, Li W. Convolution random sampling in multiply generated shift-invariant spaces of $L^p(\mathbb{R}^d)$. *Ann Funct Anal.* 2021;12(1):1-22.
40. Li Y, Xian J. Non-uniform random sampling and reconstruction in signal spaces with finite rate of innovation. *Acta Applicandae Mathematicae.* 2020;169(1):247-277.
41. Jiang Y, Zhao J. Random sampling and reconstruction of signals with finite rate of innovation. *Bull Korean Math Soc.* 2022;59(2):285-301.
42. Patel D, Sampath S. Random sampling in reproducing kernel subspaces of $L^p(\mathbb{R}^d)$. *J Math Anal Appl.* 2020;491(1):124270.
45. Li Y, Sun Q, Xian J. Random sampling and reconstruction of concentrated signals in a reproducing kernel space. *Appl Comput Harmonic Anal*. 2021;54:273-302.
46. Jiang Y, Li W. Random sampling in multiply generated shift-invariant subspaces of mixed Lebesgue spaces $L^p_q (\mathbb{R} \times \mathbb{R}^d)$. *J Comput Appl Math*. 2021;386:113237.
47. Bennett G. Probability inequalities for the sum of independent random variables. *J Am Stat Assoc*. 1962;57(297):33-45.
48. Aldroubi A, Feichtinger H. Exact iterative reconstruction algorithm for multivariate irregularly sampled functions in spline-like spaces: the $L^p$–theory. *Proc Am Math Soc*. 1998;126(9):2677-2686.

**How to cite this article**: Goyal P, Patel D, Sivanathan S. Random sampling and reconstruction in reproducing kernel subspace of mixed Lebesgue spaces. *Math Meth Appl Sci*. 2023;46(5):5119-5138. doi:10.1002/mma.8821