GENERIC REPRESENTATION THEORY OF FINITE FIELDS IN NONDESCRIPTING CHARACTERISTIC

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ABSTRACT. Let $\text{Rep}(\mathbb{F}; K)$ denote the category of functors from finite dimensional $\mathbb{F}$–vector spaces to $K$–modules, where $\mathbb{F}$ is a field and $K$ is a commutative ring. We prove that, if $\mathbb{F}$ is a finite field, and $\text{char} \mathbb{F}$ is invertible in $K$, then the $K$–linear abelian category $\text{Rep}(\mathbb{F}; K)$ is equivalent to the product, over all $n \geq 0$, of the categories of $K[GL_n(\mathbb{F})]$–modules.

As a consequence, if $K$ is also a field, then small projectives are also injective in $\text{Rep}(\mathbb{F}; K)$, and will have finite length. Even more is true if $\text{char} K = 0$: the category $\text{Rep}(\mathbb{F}; K)$ will be semisimple.

In a last section, we briefly discuss ‘$q = 1$’ analogues and consider representations of various categories of finite sets.

The main result follows from a 1992 result by L.G.Kovacs about the semigroup ring $K[M_n(\mathbb{F})]$.

1. INTRODUCTION

Let $\mathcal{V}(\mathbb{F})$ be the category of finite dimensional vector spaces over a finite field $\mathbb{F}$ of characteristic $p$, and let $K$ be a commutative ring, likely a field.

Then let $\text{Rep}(\mathbb{F}; K)$ denote the category whose objects are functors $F : \mathcal{V}(\mathbb{F}) \to K$–modules,

and whose morphisms are the natural transformations.

This is a $K$–linear abelian category in the usual way. For example,

$$0 \to F \to G \to H \to 0$$

is short exact in $\text{Rep}(\mathbb{F}; K)$ means that, for any $V \in \mathcal{V}(\mathbb{F})$, the sequence

$$0 \to F(V) \to G(V) \to H(V) \to 0$$

is a short exact sequence of $K$–modules.

Our papers [K1] – [K5] study the case when $K = \mathbb{F}$. Following terminology used in those papers, we refer $F \in \text{Rep}(\mathbb{F}; K)$ as a generic representation of the field $\mathbb{F}$. To explain this, note that there are evident connections with the representation theory of the general linear groups $GL_n(\mathbb{F})$, as $F \in \text{Rep}(\mathbb{F}; K)$ defines a family $\{ F(\mathbb{F}^n) | n = 0, 1, 2, \ldots \}$ of $K[GL_n(\mathbb{F})]$–modules.

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via evaluation. There is even more structure here, as \( F(\mathbb{F}^n) \) is a module for the semigroup ring \( K[M_n(\mathbb{F})] \), where \( M_n(\mathbb{F}) \) is the semigroup of all \( n \times n \) matrices over \( \mathbb{F} \). Indeed, as described in [K2], a generic representation \( F \) is roughly the same thing as a compatible sequence of \( K[M_n(\mathbb{F})] \)-modules for all \( n \).

There has been much success studying the case when \( K = \mathbb{F} \):

- Many extension groups \( \text{Ext}^*_{\text{Rep}(\mathbb{F}; \mathbb{F})}(F, G) \) have been calculated when \( F \) and \( G \) are classical functors. See, e.g., [FLS, FFSS, T].
- There is a deep connection between \( \text{Rep}(\mathbb{F}_p; \mathbb{F}_p) \) and the category of unstable modules over the mod \( p \) Steenrod algebra of algebraic topology. See [HLS, Sch, K1, K3].
- There are connections to both algebraic \( K \)-theory and the representation theory of algebraic groups. See, e.g., [FS, B].

By contrast, there has been relatively little said about the structure of \( \text{Rep}(\mathbb{F}; K) \) when \( K \) is a field of characteristic different than \( p \). Our main theorem here remedies this.

**Theorem 1.1.** Let \( \mathbb{F} \) be a finite field of characteristic \( p \). If \( p \) is invertible in a commutative ring \( K \), there is a natural equivalence of \( K \)-linear abelian categories

\[
\text{Rep}(\mathbb{F}; K) \simeq \prod_{n=0}^{\infty} K[GL_n(\mathbb{F})]\text{-modules}.
\]

Some structural results about \( \text{Rep}(\mathbb{F}; K) \) are immediate corollaries.

**Corollary 1.2.** If \( K \) is a field of characteristic different than \( p \), then all projectives in \( \text{Rep}(\mathbb{F}; K) \) are also injective, and indecomposable projectives have only finitely many composition factors.

**Corollary 1.3.** If \( K \) is a field of characteristic 0, then \( \text{Rep}(\mathbb{F}; K) \) is semisimple.

We remark that in \( \text{Rep}(\mathbb{F}; \mathbb{F}) \), all non-constant indecomposable projective functors have an infinite number of composition factors. These corollaries show how different the situation is in non-describing characteristic.

Here is an example of the the computational implications of our theorem.

**Example 1.4.** Let \( Gr(V) \) be the set of all subspaces of a finite dimensional \( \mathbb{F} \)-vector space \( V \), and let \( K[Gr] \in \text{Rep}(\mathbb{F}; K) \) be the functor sending \( V \) to \( K[Gr(V)] \), the \( K \)-module spanned by this set. Suppose \( p \) is invertible in \( K \). Under the correspondence of the main theorem, \( K[Gr] \) corresponds to the sequence of trivial modules for the groups \( GL_n(\mathbb{F}) \). We conclude that there is an isomorphism of graded \( K \)-algebras

\[
\text{Ext}^*_{\text{Rep}(\mathbb{F}; K)}(K[Gr], K[Gr]) \simeq \prod_{n=0}^{\infty} H^*(GL_n(\mathbb{F}); K).
\]

It is interesting to note that much about the right side of this isomorphism was computed by Quillen [Q].
The theorem will be proved in §2. In some sense, the proof is formal, starting from a result of L.G.Kovács [Ko] about the semigroup ring \( K[M_n(\mathbb{F})] \).

In §3 we discuss Kovács’ theorem and some related formulations of both his work and our main theorem.

In §4 we briefly discuss ‘\( q = 1 \)’ analogues and consider representations of various categories of finite sets.

Acknowledgements Steven Sam has recently verified a long conjectured structural property of \( \text{Rep}(\mathbb{F}; \mathbb{F}) \) – it is generated by Noetherian projectives – and his proof works without change for \( \text{Rep}(\mathbb{F}; K) \) for arbitrary Noetherian rings \( K \). It was talks with Sam about generic representation theory that inspired the project here.

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2. Proof of the main theorem

2.1. A theorem of L.G.Kovács. The key input of our proof is an elegant 1992 result of L.G.Kovács [Ko]. This followed a related announcement by Faddeev [F], and related results by Okniński and Putcha [OP]. Kovács also thanks W. D. Munn for conversations about this topic, and mentions that Munn had a proof (apparently unpublished) of precisely the result we state below.

To state this, let \( \text{Sing}_n(\mathbb{F}) \subset M_n(\mathbb{F}) \) denote the set of singular matrices. The \( K \)-linear span of this set \( K[\text{Sing}_n(\mathbb{F})] \) is a two sided ideal in the semigroup ring \( K[M_n(\mathbb{F})] \), and the quotient identifies with \( K[GL_n(\mathbb{F})] \). Thus one has a short exact sequence

\[
0 \rightarrow K[\text{Sing}_n(\mathbb{F})] \rightarrow K[M_n(\mathbb{F})] \rightarrow K[GL_n(\mathbb{F})] \rightarrow 0.
\]

**Theorem 2.1.** [Ko] If \( p \) is invertible in \( K \), \( K[\text{Sing}_n(\mathbb{F})] \) contains an idempotent \( e^S_n \) that serves as a unit. Thus the short exact sequence (2.1) splits as a sequence of unital \( K \)-algebras.

Note that \( e^S_n \) as in this theorem is necessarily unique. One easily sees that such an \( e^S_n \) satisfies the following properties (see §3.1).

- \( e^S_n \) is central in \( K[M_n(\mathbb{F})] \).
- \( e^S_n \) is fixed under the transpose automorphism of \( K[M_n(\mathbb{F})] \).
- \( e^S_n \) is fixed under conjugation by any element of \( GL_n(\mathbb{F}) \).

In §3.1 we will explicitly describe \( e^S_n \) when \( \mathbb{F} = \mathbb{F}_2 \).

For the rest of the section, we assume that \( p = \text{char} \mathbb{F} \) is invertible in \( K \).

We name the complementary idempotent:

**Definition 2.2.** Let \( e^C_n = 1 - e^S_n \in K[M_n(\mathbb{F})] \).
$e_n^G$ is a central idempotent satisfying the following properties.

- $e_n^G K[M_n(F)]e_n^G \simeq K[GL_n(F)]$ as algebras.
- $e_n^G \cdot [A] = 0$ for all $A \in \text{Sing}_n(F)$.

### 2.2. An old family of projective generators.

**Definition 2.3.** Let $P_n \in \text{Rep}(F; K)$ be defined by letting $P_n(V) = K[\text{Hom}(F^n, V)]$.

Yoneda’s lemma tells us:

- There is a natural isomorphism $\text{Hom}_{\text{Rep}(F; K)}(P_n, F) \simeq F(F^n)$ for all $F \in \text{Rep}(F; K)$.

Let $P = \{P_n \mid n = 0, 1, 2, \ldots\}$. As every object in $\mathcal{V}(F)$ is isomorphic to $F^n$ for some $n$, we deduce:

- $\mathcal{P}$ is family of projective generators for $\text{Rep}(F; K)$.

### 2.3. A new family of projective generators.

The algebra $\text{End}_{\text{Rep}(F; K)}(P_n)$ identifies with $K[M_n(F)]$, and we make the following definition.

**Definition 2.4.** Let $P_n^G = P_n \cdot e_n^G$, a direct summand of the projective $P_n$.

Let $gr_k(n)$ be the number of $k$-dimensional subspaces of $F^n$. (If $F = F_q$, this is commonly denoted $\left[\begin{array}{c} n \\ k \end{array}\right]_q$.)

**Proposition 2.5.** There is an isomorphism of functors

$$\bigoplus_{k=0}^n gr_k(n)P_n^G \simeq P_n.$$ 

Thus $\mathcal{P}^G = \{P_n^G \mid n = 0, 1, 2, \ldots\}$ is a set of projective generators for $\text{Rep}(F; K)$.

We defer the proof to later in the section.

**Proposition 2.6.**

$$\text{Hom}_{\text{Rep}(F; K)}(P_n^G, P_m^G) = \begin{cases} K[GL_n(F)] & \text{if } m = n \\ 0 & \text{if } m \neq n. \end{cases}$$

We also defer the proof of this.

Since $\text{End}_{\text{Rep}(F; K)}(P_n^G) = K[M_n(F)]$, the two propositions combine to prove what was stated as the main theorem in [Ko].

**Corollary 2.7.** If $p$ is invertible in $K$, there is an isomorphism of $K$-algebras

$$K[M_n(F)] \simeq \prod_{k=0}^n M_{gr_k(n)}(K[GL_k(F)]).$$
2.4. **Proof of Theorem 1.1.** The main theorem will follow immediately from the last two propositions, using the ‘multiobject’ version of classic Morita equivalence.

One has the following general situation. Suppose $\mathcal{Q}$ is a set of objects in a $K$–linear abelian category $\mathcal{A}$. Let $\text{End}(\mathcal{Q})$ be the full subcategory of $\mathcal{A}$ whose set of objects is $\mathcal{Q}$. Then let $\text{End}(\mathcal{Q})$–mod denote the category of contravariant $K$–linear functors from $\text{End}(\mathcal{Q})$ to $K$–mod.

A functor

$$\Theta : \mathcal{A} \to \text{End}(\mathcal{Q})$$–mod

is defined as follows. Given $A \in \mathcal{A}$ and $Q \in \mathcal{Q}$, let

$$\Theta(A)(Q) = \text{Hom}_\mathcal{A}(Q, A).$$

The functor $\Theta$ is always the right half of an adjoint pair $(\Psi, \Theta)$ where $\Psi(\mathcal{M}) = \mathcal{M} \otimes_{\text{End}(\mathcal{Q})} Q$, suitably interpreted.

The multiobject Morita theorem goes as follows.

**Lemma 2.8.** If $\mathcal{Q}$ is set of small projective generators for $\mathcal{A}$, then $\Psi$ and $\Theta$ are inverse equivalences of $K$–linear abelian categories.

Theorem 1.1 now follows by letting $\mathcal{A} = \text{Rep}(\mathbb{F}; K)$ and $\mathcal{Q} = \mathcal{P}^G$. Proposition 2.5 tells us that $\mathcal{P}^G$ is a set of small projective generators for $\text{Rep}(\mathbb{F}; K)$, so that

$$\text{Rep}(\mathbb{F}; K) \simeq \text{End}(\mathcal{P}^G)$$–mod,

and then Proposition 2.6 shows that

$$\text{End}(\mathcal{P}^G)$$–mod $\simeq \prod_{n=0}^{\infty} K[GL_n(\mathbb{F})]$–mod.

It remains to prove the two propositions.

2.5. **Proof of Proposition 2.5.** Let $\text{Inj}(W, V)$ and $\text{Surj}(W, V)$ be the sets of injective and surjective linear maps in $\text{Hom}(W, V)$. Then $K[\text{Inj}(\mathbb{F}^k, V)]$ is a free right $K[GL_k(\mathbb{F})]$–module and $K[\text{Surj}(W, \mathbb{F}^k)]$ is a free left $K[GL_k(\mathbb{F})]$–module.

**Lemma 2.9.** Composition of linear maps induces an isomorphism of $K$–modules

$$\bigoplus_{k=0}^{n} K[\text{Inj}(\mathbb{F}^k, V)] \otimes_{K[GL_k(\mathbb{F})]} K[\text{Surj}(\mathbb{F}^n, \mathbb{F}^k)] \cong K[\text{Hom}(\mathbb{F}^n, V)].$$

Assuming $p$ is invertible in $K$, we can use our idempotents $e^G_k \in K[M_k(\mathbb{F})]$ to write this isomorphism in a way that exhibits naturality in $V$.

The inclusion $\text{Inj}(\mathbb{F}^k, V) \subset \text{Hom}(\mathbb{F}^k, V)$ induces an isomorphism of right $K[GL_k(\mathbb{V})]$–modules

$$K[\text{Inj}(\mathbb{F}^k, V)] \cong K[\text{Hom}(\mathbb{F}^k, V)]e^G_k$$
Corollary 2.10. There is an isomorphism of functors
\[
\bigoplus_{k=0}^{n} K[\text{Hom}(F^k, V)] e_n^G \otimes_{K[GL_k(F)]} K[\text{Surj}(F^n, F^k)] \sim \to K[\text{Hom}(F^n, V)].
\]

Otherwise said, we have verified the next corollary.

Corollary 2.10. There is an isomorphism of functors
\[
\bigoplus_{k=0}^{n} P_k^G \otimes_{K[GL_k(F)]} K[\text{Surj}(F^n, F^k)] \sim \to P_n.
\]

Now we decompose \(K[\text{Surj}(F^n, F^k)]\). The kernel (a.k.a. nullspace) of any surjection \(A : F^n \to F^k\) will be a subspace of \(F^n\) of codimension \(k\), and this kernel is invariant under left multiplication by elements in \(GL_k(F)\). Thus we have the following.

Lemma 2.11. There is an isomorphism of left \(K[GL_k(F)]\)-modules
\[
K[\text{Surj}(F^n, F^k)] \cong \bigoplus_W K[GL_k(F)],
\]
where the sum runs over \(W < F^n\) of codimension \(k\).

As \(\{W < F^n \mid \text{codim } W = k\}\) has cardinality \(gr_k(n)\), the last lemma and corollary imply Proposition 2.5.

2.6. Proof of Proposition 2.6. Recall that each \(e_n^G\) is a central idempotent in \(K[M_n(F)]\) satisfying the following two properties:

- \(e_n^G K[M_n(F)] e_n^G \cong K[GL_n(F)]\) as algebras.
- \([B] \cdot e_n^G = 0\) for all \(B \in \text{Sing}_n(F)\).

By construction, we have isomorphisms
\[
\text{Hom}_{\text{Rep}(F;K)}(P_m^G, P_n^G) = e_m^G K[M_{m,n}(F)] e_n^G,
\]
where we have identified \(\text{Hom}(F^n, F^m)\) with \(M_{m,n}(F)\), the set of \(m \times n\) matrices over \(F\).

Thus our first property says that \(\text{End}_{\text{Rep}(F;K)}(P_m^G, P_n^G) \cong K[GL_n(F)]\).

If \(m < n\), then every element in \(e_m^G K[M_{m,n}(F)]\) can be written as a linear combination of terms of the form \([AB]\), where \(A \in M_{m,n}(F)\) and \(B \in \text{Sing}_n(F)\). The second property above tells us that for such \(A\) and \(B\), \([AB] \cdot e_n^G = [A]([B] \cdot e_n^G) = 0\), so that \(\text{Hom}_{\text{Rep}(F;K)}(P_m^G, P_n^G) = 0\).

There is a similar proof when \(m > n\); this case also follows from the one just proved by using transpose of matrices.

3. Further remarks and related results

3.1. On Kovács’ theorem. We make a few observations as an aid to those readers who might wish to better understand Kovács’ proof that \(K[\text{Sing}_n(F)]\) contains a unit. These are also explicitly or implicitly said in \(Ko\).

Let \(e_{n-1} = [I_{n-1}] \in K[\text{Sing}_n(F)]\) where \(I_{n-1}\) is the \(n \times n\) matrix which has 1’s on the first \((n - 1)\) diagonal entries and is zero elsewhere.
Lemma 3.1. An element \( e \in K[\text{Sing}_n(F)] \) is two sided unit if and only if it satisfies the following two properties:

- \( e \) is invariant under the conjugation action of \( \text{GL}_n(F) \) on \( K[\text{Sing}_n(F)] \).
- \( ee_{n-1} = e_{n-1} \).

Proof. Suppose that \( e \) satisfies the two properties. Any \( A \in \text{Sing}_n(F) \) admits a decomposition of the form \( A = BC \), where \( B \) is an idempotent of rank \( n-1 \) and so is conjugate to \( I_{n-1} \). The two properties imply that \( e[B] = [B] \), and thus \( e[A] = (e[B])[C] = [B][C] = [A] \). In other words, \( e \) is a left unit for \( K[\text{Sing}_n(F)] \).

Taking the transpose is an antiautomorphism of \( K[\text{Sing}_n(F)] \), so \( e^T \) is a right unit for \( K[\text{Sing}_n(F)] \). But then \( e = e^T \) is a two sided unit.

The other implication of the lemma is immediate. \( \square \)

Elements in \( K[\text{Sing}_n(F)]^{\text{GL}_n(F)} \) are linear combinations of orbit sums. Kovács discovers that one only needs to use conjugacy classes of matrices he terms ‘semi-idempotent’: matrices \( E \) that act as the identity on the image of \( E^N \) for \( N >> 0 \). Combined with the last lemma, one is well on one’s way to finding the unit \( e \in K[\text{Sing}_n(F)] \).

Example 3.2. We show how this all works when \( n = 2 \) and \( F = F_2 \), the field with two elements. There are three relevant orbit sums, and we find ourselves looking for \( a, b, c \in \mathbb{Z} [\frac{1}{2}] \) such that

\[
e = a \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\} + b \left\{ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\} + c \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\]

satisfies \( e \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \).

This leads to the system of equations

\[
1 = 2a \\
0 = a + b \\
0 = 2a + b + c,
\]

which has solution \( a = 1/2, b = -1/2, \) and \( c = -1/2 \).

3.2. Split recollement. As described in \([K2]\), for any fields \( K \) and \( F \), one has a recollement diagram:

\[
K[\text{GL}_n(F)] \text{–mod} \quad \Downarrow q \quad \uparrow l \\
\text{K[GL}_n(F)] \quad \text{K[M}_n(F)] \text{–mod} \quad \Downarrow r \quad \text{K[M}_n^{-1}(F)] \text{–mod}.
\]

In this diagram, the functors \( i \) and \( e \) are exact, with left adjoints \( q \) and \( l \), and right adjoints \( p \) and \( r \). The natural maps \( l(e(N)) \to N \) and \( N \to e(r(N)) \)
are both isomorphisms for all $K[M_{n-1}(F)]$–modules $N$.

Explicitly, $i$ is the pullback of modules via the quotient map $K[M_n(F)] \to K[GL_n(F)]$, while

$$e(M) = e_{n-1}M,$$

$$l(N) = K[M_n(F)]e_{n-1} \otimes_{K[M_{n-1}(F)]} N,$$

and

$$r(N) = \text{Hom}_{K[M_{n-1}(F)]}(e_{n-1}K[M_n(F)], N).$$

Kovács’ theorem is easily seen to be equivalent to the statement that, when $F$ is a finite field of of characteristic $p$ and $\frac{1}{p} \in K$, $l(N) \simeq r(N)$ and thus are both exact, and similarly $q(M) \simeq p(M)$. It follows that the assignment $M \mapsto (e(M), q(M))$ induces an equivalence of $K$–linear abelian categories

$$K[M_n(F)]–\text{mod} \simeq K[M_{n-1}(F)]–\text{mod} \times K[GL_n(F)]–\text{mod}.$$ 

3.3. Splitting the rank filtration. Similarly, if $e_n(F) = F(\mathbb{F}^n)$,

$$e_n : \text{Rep}(\mathbb{F}; K) \to K[M_n(F)]–\text{mod}$$

is an exact functor with left and right adjoints $l_n$ and $r_n$ defined by letting

$$l_n(M) = P_n \otimes_{K[M_n(F)]} M$$

and

$$r_n(M)(V) = \text{Hom}_{K[M_n(F)]}(K[\text{Hom}(V, \mathbb{F}^n)], M).$$

The counit of the $(l_n, e_n)$ adjunction takes the form

$$P_n \otimes_{K[M_n(F)]} F(\mathbb{F}^n) \to F,$$

and we let $F^n \subseteq F$ be the image of this map.

This defines a canonical increasing rank filtration of a generic representation $F$:

$$F^0 \subseteq F^1 \subseteq F^2 \subseteq \cdots \subseteq \bigcup_{n=0}^{\infty} F^n = F.$$

A reinterpretation of our main theorem goes as follows. If $\frac{1}{p} \in K$, then $l_n(M) \simeq r_n(M)$, both $l_n$ and $r_n$ are exact, and the rank filtration splits: there is a natural decomposition

$$F \simeq \bigoplus_{k=0}^{\infty} F^k/F^{k-1}.$$
4. **Generic representations of finite sets, and a \( q = 1 \) analogue.**

One might wonder to what extent the analogues of our results here are true if one considers representations of finite sets, rather than finite dimensional vector spaces over \( \mathbb{F}_q \).

There are various categories of finite sets one might consider, two of which are the categories of based finite sets, \( \Gamma \), and unbased finite sets, \( \text{Fin} \). The analogues of our results don’t hold for either of these, as the next examples show.

If \( C \) is a small category, we let \( \text{Rep}(C; K) \) denote the category of covariant functors from \( C \) to \( K \text{-mod} \).

**Example 4.1.** It is an observation of Pirashvili [Pi] that \( \text{Rep}(\Gamma; K) \) is equivalent to \( \text{Rep}(\text{Epi}; K) \), where \( \text{Epi} \) is the category of finite sets and epimorphisms. We let \( P_1^{\text{Epi}}(S) = K[\text{Epi}(n, S)] \), where \( n = \{1, \ldots, n\} \). Then there is an inclusion \( P_1^{\text{Epi}} \to P_2^{\text{Epi}} \) which is never split. Thus \( \text{Rep}(\Gamma; K) \) is not semisimple, even when \( K \) is a field of characteristic 0.

A similar thing happens if one considers the category of contravariant functors \( \text{Rep}(\Gamma^{\text{op}}; K) \).

**Example 4.2.** (We thank Steven Sam for showing us this example.) Let \( P_1 \in \text{Rep}(\text{Fin}; K) \) be defined by \( P_1^\text{Fin}(S) = K[S] \) for all finite sets \( S \), and, by abuse of notation, let \( K \) denote the functor with constant value \( K \). There is a natural transformation \( \epsilon : P_1^\text{Fin} \to K \) defined by letting \( \epsilon([s]) = 1 \) for all \( s \in S \). This is an epimorphism which is never split, even when \( K \) is a field of characteristic 0.

Dually, there is a non-split natural inclusion \( K \to K^S \) of contravariant functors of \( \text{Fin} \).

Now let \( \text{Inj}_* \) denote the subcategory of based sets \( \Gamma \) having the same objects – finite based sets \( (S, s_0) \) – and with morphisms equal to based maps \( f : (S, s_0) \to (T, t_0) \) that are one-to-one on the complement of \( f^{-1}(t_0) \).

If we let \( \text{Inj} \) denote the category of finite sets and injections, and \( \text{Iso} \) denote the category of finite sets and bijections, then one notes that there is a decomposition

\[
\text{Inj}_* = \text{Inj} \circ \text{Iso} \circ \text{Inj}^{\text{op}}
\]

of the sort discussed in [Si] or [II1, II2]. The main theorem of any of these applies to show the following theorem.

**Theorem 4.3.** For any commutative ring \( K \), there are natural equivalences of \( K \)-linear abelian categories

\[
\text{Rep}(\text{Inj}_*; K) \simeq \text{Rep}(\text{Iso}; K) \simeq \prod_{n=0}^{\infty} K[\Sigma_n]-\text{mod}.
\]

Explicitly, see [Si] Theorem 2.5], or apply [II2 Theorem 7.1] to either [II2 Example 3.15] or [II2 Example 3.19].
Furthermore, the construction in [H2] yields the following analogue of Corollary 2.7. Let $R_n = \text{End}_\text{Inj}^*_n(n_*)$, where $n_* = \{0, 1, \ldots, n\}$ with base-point 0. Then $R_n$ is the $n$th symmetric inverse semigroup, and is also called the rook monoid in [So].

**Corollary 4.4.** For all commutative rings $K$, there is an isomorphism of $K$–algebras

$$K[R_n] \simeq \prod_{k=0}^{n} M_{\binom{n}{k}}(K[\Sigma_k]).$$

(Further discussion about this theorem, corollary, and generalizations may appear in a short note.)

**Remark 4.5.** We note that our category $\text{Inj}^*_n$ is the same as the category called ‘FI#’ in [CEF], and Corollary 4.4 is a strengthening to all rings $K$ of the main result in [So].

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