Systematic search for successful lepton mixing patterns with nonzero $\theta_{13}$

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We perform a systematic search for simple but viable lepton mixing patterns. Our main criterion is that the mixing matrix can be parameterized by three rotation angles, which are simple fractions of $\pi$. These simple rotation angles possess exact expressions for their sines and cosines, and often arise in the flavor symmetry models. All possible parameterizations of the mixing matrix are taken into account. In total, twenty successful mixing patterns are found to be consistent with the latest neutrino oscillation data (including the recent T2K results) in the CP conserving case, whereas fifteen mixing patterns are allowed in the maximal CP violating case. Potential radiative corrections to the constant mixing patterns are also calculated by solving the renormalization group equations.

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I. INTRODUCTION

Recent solar, atmospheric, reactor and accelerator neutrino experiments have provided us with compelling evidence that neutrinos are massive and lepton flavors are mixed. In the framework of three-flavor neutrino oscillations, the mixing is described by a $3 \times 3$ unitary matrix $V$, which is usually parameterized \cite{1} by three mixing angles ($\theta_{12}$, $\theta_{23}$, and $\theta_{13}$) and three CP violating phases out of which one is the Dirac phase ($\delta$) and the other two are the Majorana phases ($\rho$ and $\sigma$). In the standard parameterization advocated by the Particle Data Group \cite{2} and in Refs. \cite{3}, the lepton mixing matrix reads

$$
V = \begin{pmatrix} 
    c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\sigma} \\
    -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\
    s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} 
\end{pmatrix} \begin{pmatrix} 
    e^{i\rho} & 0 & 0 \\
    0 & e^{i\sigma} & 0 \\
    0 & 0 & 1 
\end{pmatrix}, \quad (1)
$$

where $s_{ij} \equiv \sin \theta_{ij}$ and $c_{ij} \equiv \cos \theta_{ij}$ (for $ij = 12, 23, 13$). If neutrinos are Dirac particles, the phases $\rho$ and $\sigma$ will be irrelevant and can be rotated away through a redefinition of the neutrino fields. The latest global analysis of current neutrino oscillation data yields \cite{4}

$$
31.0^\circ < \theta_{12} < 37.1^\circ , \\
35.7^\circ < \theta_{23} < 53.1^\circ , \\
4.1^\circ < \theta_{13} < 12.9^\circ , \quad (2)
$$

at 3\sigma C.L., and the best-fit values of three mixing angles are $\theta_{12} = 34.0^\circ$, $\theta_{23} = 40.4^\circ$ and $\theta_{13} = 9.1^\circ$. Driven in particular by the latest T2K results \cite{5}, $\theta_{13} = 0^\circ$ is currently disfavored at the more than 3\sigma level.

So far it is still unclear how to theoretically understand the observed lepton mixing. One tentative way is to start with experimental values of leptonic mixing angles and conjecture a simple constant mixing pattern, which may turn out to be suggestive of the underlying symmetry of lepton mixing. In fact, several interesting constant mixing patterns have been suggested along with the progress in neutrino oscillation experiments, and shown to be derivable from the flavor symmetries. For instance, the democratic \cite{6}, bi-maximal \cite{7}, tri-bimaximal \cite{8}, hexagonal \cite{9} or both golden ratio \cite{10} patterns can be realized in models with different discrete flavor symmetries, such as $S_3$, $A_4$, $S_4$, $A_5$, dihedral groups, etc., see \cite{11} for recent reviews, and \cite{12} for a summary of proposed mixing scenarios.

Note that all the aforementioned constant mixing patterns lead to a vanishing $\theta_{13}$, which mainly due to long-baseline data is now disfavored at 3\sigma C.L. \cite{4}. While more statistics and complementary measurements from reactor neutrino experiments will tell us whether $\theta_{13}$ is indeed as large as it currently appears to be, it is without doubt timely to consider the ways to cope with a sizable $\theta_{13}$. Indeed, after the results of T2K \cite{5} were released, several possible ways, with a large range in what regards the level of sophistication, to realize a relatively large $\theta_{13}$ have been discussed in Refs. \cite{13}. Earlier analyses can be found for instance in Refs. \cite{14, 15}.

In this work, we search in a systematic way for successful mixing patterns with initial nonzero $\theta_{13}$. Simple requirements are the starting point of our analysis: (i)
the mixing matrix is a product of three rotations; (ii) the angles associated with the rotations are simple fractions of π, such that the resulting sines and cosines are given by exact expressions. In particular the latter criterion is reminiscent of the results of many flavor symmetry models. We find in total twenty viable mixing patterns in the CP conserving case, while fifteen different feasible patterns exist in the CP violating case. Furthermore, leaving the CP phase arbitrary gives in total 66 successful mixing patterns.

The precision era which neutrino physics has recently entered requires that the renormalization effects should be taken into account. If the underlying flavor symmetry works at some high-energy scale, the mixing angles will receive radiative corrections and deviate from the predictions of a given mixing pattern when running from the symmetry scale to the low-energy scale. We will therefore consider the renormalization group equation (RGE) effects on our scenarios.

The remaining part of this work is organized as follows. In Sec. III we recall all different parameterizations of a 3×3 unitary matrix, and fix our notations. Then, in Sec. IV we summarize the feasible mixing patterns which are compatible with experimental data and in particular predict a nonzero θ13. Sec. V is devoted to a general discourse on the radiative corrections to the constant mixing patterns. Finally, we conclude in Sec. VI.

II. PARAMETERIZATIONS OF LEPTON MIXING MATRIX

First of all, let us review the classification of possible parameterizations of a lepton mixing matrix 16. Since the Majorana phases can always be recast into a diagonal matrix on the right-hand side of V, they have no influence on our results and will be ignored for now. If the leptonic CP violation is absent, V is simply a 3 × 3 orthogonal matrix and can be written as a product of three rotation matrices with three different rotation angles (θ1, θ2, θ3), i.e.

\[ V = R_{12}(\theta_1)R_{23}(\theta_2)R_{13}(\theta_3), \]

where \( ij, kl, mn = 12, 23, 13 \)

\[ R_{12}(\theta) = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \]

\[ R_{23}(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix}, \]

\[ R_{13}(\theta) = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}. \]

Note that the order of the rotations is not specified. The Dirac-type CP violating phase \( \phi \) can be included in the above parameterization by replacing the entry “1” with a phase factor \( e^{-i\phi} \) in the second rotation matrix on the right-hand side of Eq. 3. Taking \( R_{12}(\theta) \) in Eq. (4) for example, we have

\[ R_{12}(\theta, \phi) = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & e^{-i\phi} \end{pmatrix}. \]

Although there are several different ways to introduce the CP violating phase, the choice in Eq. 6 is advantageous in the sense that the phase parameter \( \phi \) is always located in a 2×2 submatrix of V, in which each element is a sum of two terms with the relative phase \( \phi \).

In Ref. 16 it was shown that only nine distinct parameterizations exist, namely

\[ P_1 : \quad V = R_{12}(\theta_1)R_{23}(\theta_2, \varphi)R_{12}^{-1}(\theta_3), \]
\[ P_2 : \quad V = R_{23}(\theta_1)R_{12}(\varphi, \theta_2)R_{23}^{-1}(\theta_3), \]
\[ P_3 : \quad V = R_{23}(\theta_1)R_{13}(\theta_2, \varphi)R_{13}^{-1}(\theta_3), \]
\[ P_4 : \quad V = R_{12}(\theta_1)R_{13}(\theta_2, \varphi)R_{23}^{-1}(\theta_3), \]
\[ P_5 : \quad V = R_{13}(\theta_1)R_{12}(\varphi, \theta_2)R_{13}^{-1}(\theta_3), \]
\[ P_6 : \quad V = R_{12}(\theta_1)R_{23}(\varphi, \theta_2)R_{13}^{-1}(\theta_3), \]
\[ P_7 : \quad V = R_{23}(\theta_1)R_{12}(\theta_2, \varphi)R_{13}(\theta_3), \]
\[ P_8 : \quad V = R_{13}(\theta_1)R_{12}(\theta_2, \varphi)R_{23}(\theta_3), \]
\[ P_9 : \quad V = R_{13}(\theta_1)R_{23}(\theta_2, \varphi)R_{12}(\theta_3). \]

Here \( R_{ij}^{-1}(\theta) = R_{ij}(-\theta) \). Three of the nine parameterizations belong to the class \( ij = mn \neq kl \) and six to the class \( ij \neq kl = mn \). Note that \( P_3 \) is just the standard (“PDG”) parameterization in Eq. 11, up to a simple phase redefinition.

The effects of CP violation are usually characterized by the Jarlskog invariant \( J_{CP} \), which is defined as

\[ \text{Im} \left[ V_{\alpha i} V_{\beta j}^* V_{\alpha j}^* V_{\beta i} \right] = J_{CP} \sum_{\gamma = e, \mu, \tau} \sum_{k = 1, 2, 3} (\epsilon_{ijk} \epsilon_{\alpha \beta \gamma}). \]

It is straightforward to verify that the Jarlskog invariant is given by

\[ J_{CP} = s_1 c_1 s_2 c_2 s_3 c_3 \sin \varphi, \]
for \( P_1, P_2 \) and \( P_5 \), and

\[ J_{CP} = s_1 c_1 s_2 c_2 s_3 c_3 \sin \varphi, \]
for \( P_3, P_4, P_6, P_7, P_8 \) and \( P_9 \), where \( s_i = \sin \theta_i \) and \( c_i = \cos \theta_i \) for \( i = 1, 2, 3 \).

It is worthwhile to remark that although these nine parameterizations are mathematically equivalent, one of them may turn out to be more useful than the others for a specific problem. For instance, three mixing angles in the standard parameterization \( P_3 \) can be unambiguously extracted from neutrino oscillation experiments, which is not the case for the other parameterizations. The flavor symmetry behind the observed lepton mixing pattern may be manifest in a certain parameterization, which is currently unknown to us, so we consider all possibilities in Eq. 6 when searching for viable constant lepton mixing patterns.
III. SIMPLE ROTATION ANGLES AND VIABLE MIXING PATTERNS

One immediate question arises: which values of the rotation angles should we use to account for the observed lepton mixing? To make the mixing patterns simple and suggestive of flavor symmetries, we set the following criteria: (i) three rotation angles are simple fractions of $\pi$, i.e. $\pi/n$ with $n$ being an integer; (ii) the choice of $n$ is governed by the requirement that the sines and cosines of the rotation angles possess exact expressions, i.e. they are expressible as simple terms involving only square roots. Although these two criteria are essentially set to make the lepton mixing matrix as simple as possible, they might be realized in the flavor symmetry models (e.g., the dihedral group $D_n$ [18]), where the lepton mixing patterns can be intimately related to Clebsch–Gordan coefficients favoring small integers and their square roots.

Up to $n = 12$, one has the following nine rotation angles satisfying the above criteria

$$\vartheta \in \left\{ \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{5}, \frac{\pi}{6}, \frac{\pi}{8}, \frac{\pi}{10}, \frac{\pi}{12} \right\},$$  

for which the values of $\sin^2 \vartheta$ and $\cos^2 \vartheta$ have been listed in Table I. Note that for $n > 12$, there also exist simple angles whose sines and cosines have exact expressions. In general, one can always implement the relation

$$\cos \vartheta = \sqrt{\frac{1 - \cos \vartheta}{2}},$$

to calculate $\cos(\pi/2n)$ if $\cos(\pi/n)$ is already known. However, it should be rather difficult to relate the expressions with three or more square roots to a flavor symmetry, and in addition the resultant lepton mixing pattern becomes very complicated. Furthermore, as $n$ increases, $\vartheta$ becomes smaller, and because the smallest mixing matrix element $V_{\text{e}3}$ is generally related to the smallest rotation angle, the resulting mixing matrix ends up not being in agreement with experimental data.

The lepton mixing matrix $V$ can be obtained by inserting the rotation angles $\vartheta_i = \pi/n_i$ (for $i = 1, 2, 3$) into Eq. (6), where $n_i \in \{1, 2, 3, 4, 5, 6, 8, 10, 12\}$. For later convenience, we denote the mixing pattern constructed through the parameterization $P_j$ (for $j = 1, 2, \ldots, 9$) with rotation angles $\vartheta_i = \pi/n_i$ as $V = P_j(n_1, n_2, n_3)$ for a given CP violating phase $\varphi$. Comparing between the standard parameterization defined in Eq. (11) and $P_j(n_1, n_2, n_3)$, one can immediately extract the three “standard” lepton mixing angles

$$\sin^2 \theta_{12} = \frac{|V_{\text{e}2}|^2}{1 - |V_{\text{e}3}|^2},$$
$$\sin^2 \theta_{23} = \frac{|V_{\text{e}3}|^2}{1 - |V_{\text{e}3}|^2},$$
$$\sin^2 \theta_{13} = |V_{\text{e}3}|^2,$$

which allow us to confront the lepton mixing matrix $P_j(n_1, n_2, n_3)$ with neutrino oscillation data and thus find out the viable patterns. Note that with our set of rotation angles we can not generate a viable pattern with the standard parameterization $P_3$ in Eq. (11), because $|V_{\text{e}3}| = \sin \theta_{13}$ in this case and the smallest possible angle in our scenario is $\pi/12$, implying a too large value of $\sin^2 \theta_{13} = 0.07$. The largest allowed value of $\theta_{13}$ approximates to $\pi/14$, which does not possess an exact expression for its sine or cosine.

There exist nine distinct parameterization schemes and $9 \times 9 \times 9$ different combinations of rotation angles in each scheme, so we are left with $9^9 = 6561$ possible mixing matrices for a fixed CP violating phase $\varphi$. We have numerically studied all these possibilities for the CP conserving case $\varphi = 0$ and the maximal CP violating case $\varphi = \pi/2$ by comparing the predicted mixing angles with the 3$\sigma$ global-fit data given in Eq. (2). The viable cases are listed in Table II. We find that 20 mixing patterns are allowed in the CP conserving case, while only 15 patterns are compatible with experimental data in the maximal CP violating case. All in all, the feasible patterns for different CP violating phases are different. Only $P_7(5, 5, 12)$ is allowed for both $\varphi = 0$ and $\varphi = \pi/2$, showing the importance of the CP violating phase in searching for successful lepton mixing patterns. We should note here an important difference between the standard parameterization $P_3$ in Eq. (11) and the others. In the standard parameterization, a choice of three angles can directly be confronted with experiments, namely knowing $|V_{\text{e}2}|$, $|V_{\text{e}3}|$, and $|V_{\mu 3}|$, fixes the mixing angles and leaves the phase undetermined. Consider now $P_2$, whose explicit form is

$$V = \begin{pmatrix} c_2 & s_2 c_3 & -s_2 s_3 \\ -c_1 s_2 & c_1 c_2 c_3 + s_1 s_2 e^{-i\varphi} & -c_1 c_2 s_3 + s_1 c_3 e^{-i\varphi} \\ s_1 s_2 & -s_1 c_2 c_3 + c_1 s_2 e^{-i\varphi} & s_1 c_2 s_3 + c_1 c_3 e^{-i\varphi} \end{pmatrix}.$$ 

Since $|V_{\text{e}2}|$, $|V_{\text{e}3}|$, and $|V_{\mu 3}|$ are known experimentally, two angles $(\vartheta_2$ and $\vartheta_3)$, as well as some combination of $\vartheta_1$ and the CP phase $\varphi$, can be determined. On the other hand, choosing three angles, as we have done in this paper, fixes the CP phase $\varphi$ at the same time. This is true for all the alternative parameterizations $P_{1,2,4,5,6,7,8,9}$ of the lepton mixing matrix.

We continue by recommending a few simple but interesting mixing patterns and discuss their implications for the leptonic mixing angles and leptonic CP violation:
(1) Pattern $P_1(12, 4, 4)$ with $\varphi = 0$ – The lepton mixing matrix takes the form

$$V = \begin{pmatrix}
\frac{\sqrt{3}-1}{\sqrt{3}+1} + \frac{\sqrt{3}}{4}\sqrt{3} - \frac{\sqrt{3}-1}{\sqrt{3}+1} & \frac{\sqrt{3}-1}{\sqrt{3}+1} & \frac{\sqrt{3}-1}{\sqrt{3}+1} \\
\frac{\sqrt{3}-1}{\sqrt{3}+1} & \frac{\sqrt{3}-1}{\sqrt{3}+1} & \frac{\sqrt{3}-1}{\sqrt{3}+1} \\
\frac{\sqrt{3}-1}{\sqrt{3}+1} & \frac{\sqrt{3}-1}{\sqrt{3}+1} & \frac{\sqrt{3}-1}{\sqrt{3}+1}
\end{pmatrix},$$

(13)

which leads to

$$\sin^2 \theta_{13} = \frac{1}{8} \left(2 - \sqrt{3}\right),$$

$$\sin^2 \theta_{23} = \frac{2 + \sqrt{3}}{6 + \sqrt{3}},$$

$$\sin^2 \theta_{12} = \frac{1}{2} - \frac{\sqrt{3}}{6 + \sqrt{3}},$$

or explicitly $\theta_{13} \approx 10.5^\circ$, $\theta_{23} \approx 44.0^\circ$ and $\theta_{12} \approx 34.3^\circ$. Such a mixing pattern is in excellent agreement with neutrino oscillation data. The predictions for $\theta_{12}$ and $\theta_{23}$ fall into the $\sigma$ ranges, while that for $\theta_{13}$ is even slightly larger than the best-fit value but well within the $2\sigma$ range.

(2) Pattern $P_1(12, 6, 4)$ with $\varphi = 0$ – The lepton mixing matrix takes the form

$$V = \begin{pmatrix}
\frac{3+\sqrt{3}}{4\sqrt{2}} & \frac{3\sqrt{3}-1}{8} & \frac{3-\sqrt{3}}{4}
\end{pmatrix},$$

(15)

which leads to

$$\sin^2 \theta_{13} = \frac{3}{32} \left(2 - \sqrt{3}\right),$$

$$\sin^2 \theta_{23} = \frac{14 + 3\sqrt{3}}{26 + 3\sqrt{3}},$$

$$\sin^2 \theta_{12} = \frac{14 - \sqrt{3}}{26 + 3\sqrt{3}},$$

or explicitly $\theta_{13} \approx 9.1^\circ$, $\theta_{23} \approx 51.7^\circ$ and $\theta_{12} \approx 32.1^\circ$. Note that the prediction for $\theta_{13}$ in this mixing pattern is almost the best-fit value.

(3) Pattern $P_1(12, 4, 6)$ with $\varphi = \pi/2$ – The lepton mixing matrix takes the form

$$V = \begin{pmatrix}
\frac{\sqrt{3+1}-\sqrt{3}(3+\sqrt{3})}{2\sqrt{2}} & \frac{3+\sqrt{3}(3+\sqrt{3})}{3+\sqrt{3}(3+\sqrt{3})} & \frac{3}{4}
\end{pmatrix},$$

(17)

which leads to

$$\sin^2 \theta_{13} = \frac{1}{8} \left(2 - \sqrt{3}\right),$$

$$\sin^2 \theta_{23} = \frac{2 + \sqrt{3}}{6 + \sqrt{3}},$$

$$\sin^2 \theta_{12} = \frac{10 - \sqrt{3}}{24 + 4\sqrt{3}},$$

or explicitly $\theta_{13} \approx 10.5^\circ$, $\theta_{23} \approx 44.0^\circ$ and $\theta_{12} \approx 31.1^\circ$. In addition, the Jarlskog invariant is $J_{CP} = \sqrt{6}/64 \approx 3.8\%$.

(4) Pattern $P_1(12, 4, 6)$ with $\varphi = \pi/2$ – The lepton mixing matrix takes the form

$$V = \begin{pmatrix}
\frac{3+\sqrt{3}}{4\sqrt{2}} & \frac{3\sqrt{3}-1}{8} & \frac{3-\sqrt{3}}{4}
\end{pmatrix},$$

(19)

which leads to

$$\sin^2 \theta_{13} = \frac{1}{8} \left(2 - \sqrt{3}\right),$$

$$\sin^2 \theta_{23} = \frac{4}{6 + \sqrt{3}},$$

$$\sin^2 \theta_{12} = \frac{10 - \sqrt{3}}{24 + 4\sqrt{3}}.$$
or explicitly $\theta_{13} \approx 10.5^\circ$, $\theta_{23} \approx 46.0^\circ$ and $\theta_{12} \approx 31.1^\circ$. In addition, the Jarlskog invariant is $J_{\text{CP}} = \sqrt{6}/64 \approx 3.8\%$, which is the same as that in the previous case. Note that the mixing patterns in Eqs. (17) and (19) differ only in the predictions of $\theta_{23}$, i.e. $\theta_{23} = 44.0^\circ$ in the former case while $\theta_{23} = 46.0^\circ$ in the latter.

As mentioned before, the CP violating phase plays an important role in searching for successful lepton mixing patterns. In this regard, we have so far focused on the special cases with $\varphi = 0$ and $\varphi = \pi/2$. Choosing an arbitrary CP violating phase, i.e. $\varphi \in [0, \pi]$, leads to the 66 viable patterns in Fig. 1. One can immediately construct the lepton mixing matrix from the $P_j(n_1, n_2, n_3)$ notation. It is amazing that from the 6561 possible mixing patterns, only about 1% are compatible with current oscillation data.

**IV. RADIATIVE CORRECTIONS**

Now we proceed to consider possible radiative corrections to the mixing patterns in Table 11. As argued in Sec. I, the mixing patterns under consideration may arise from certain flavor symmetries preserved at high-energy scales, such as the grand unification (e.g. $\Lambda \sim 10^{16}$ GeV) or the seesaw scale (e.g. $\Lambda \sim 10^{14}$ GeV), whereas the leptonic mixing parameters are determined or constrained in neutrino oscillation experiments at low energies. The gap between the high-energy predictions and the low-energy measurements is bridged by the RG evolution, which may significantly change the model predictions. On the other hand, the RG running effects could also serve as an explanation for the discrepancy between the flavor symmetric mixing pattern and the observed one.

The RGEs for leptonic mixing parameters have been derived within various theoretical frameworks 19. In the supersymmetric theories with large tan $\beta$, it has been found that the RG evolution may lead to significant modifications to the mixing parameters, in particular the solar mixing angle $\theta_{12}$ (see e.g. Ref. 20 and references therein). To be explicit, we write down the RGEs for three leptonic mixing angles in the approximation of $\tau$-lepton dominance (i.e., $Y^\nu_\tau Y^\nu_\ell \approx \text{diag}(0, 0, y^\nu_\tau)$) in view of

$$y^\nu_\ell \ll y^\nu_{\ell'} \ll y^\nu_\tau \quad [21],$$

$\dot{\theta}_{12} \approx - C y^2 \sin 2\theta_{12} c_{12}^2 s_{23} \frac{c^2_{12}}{8 \pi^2 \Delta m^2_{\text{sol}}/\Delta m^2_{\text{atm}}} \left[ m_2^2 + m_3^2 + 2 m_1 m_2 c_{2(\rho - \sigma)} \right],$

$\dot{\theta}_{13} \approx + C y^2 s^2 \sin 2\theta_{13} c_{13}^2 c_{23} \frac{c^2_{13}}{2 \pi^2 \Delta m^2_{\text{atm}} (1 + \zeta)} \left[ m_1 c_{2(\rho + \delta)} - (1 + \zeta) m_2 c_{2(\rho + \delta)} - \zeta m_3 c_3 \right],$

$\dot{\theta}_{23} \approx - C y^2 s^2 \cos 2\theta_{23} c_{23}^2 \frac{c^2_{23}}{8 \pi^2 \Delta m^2_{\text{atm}}} \left[ c_{12}^2 (m_2^2 + m_3^2 + 2 m_1 m_3 c_{2(\rho - \sigma)}) + s_{12}^2 (m_1^2 + m_2^2 + 2 m_1 m_2 c_{2(\rho - \sigma)} (1 + \zeta)^{-1}) \right],$

where $\dot{\theta}_{ij} \equiv \frac{d\theta_{ij}}{dt}$ with $t = \ln(\mu/\mu_0)$, $\zeta \equiv \Delta m^2_{\text{sol}}/\Delta m^2_{\text{atm}}$ with $\Delta m^2_{\text{sol}} \equiv m_3^2 - m_2^2 \approx 7.6 \times 10^{-5} \text{ eV}^2$ and $|\Delta m^2_{\text{atm}}| \equiv |m_3^2 - m_2^2| \approx 2.3 \times 10^{-3} \text{ eV}^2$ at the low-energy scale, and $y_\ell$ denotes the Yukawa coupling of tau lepton. In the standard model (SM) $C = -3/2$ while in the minimal supersymmetric standard model (MSSM) $C = 1$. Terms of $O(\theta_{13})$ have been safely neglected in Eq. [21], where we have defined $c_{2(\rho - \sigma)} \equiv \cos 2(\rho - \sigma)$, $c_{2\rho} \equiv \cos 2\rho$ and so on.

As obvious from the above beta functions of the mixing angles, $\theta_{12}$ receives typically larger RG corrections than $\theta_{23}$ and $\theta_{13}$, whose corrections are of the same order. Furthermore, when running from a high-energy scale to the electroweak scale $\Lambda_{\text{EW}} = 10^2$ GeV, the radiative corrections to $\theta_{12}$ are typically (see below) positive in the MSSM, i.e. $\theta_{12}(\Lambda) < \theta_{12}(\Lambda_{\text{EW}})$. In contrast, in the SM $\theta_{12}$ receives only negative corrections because of the sign flip in $C$. However, the RG effects in the SM are generally small due to the absence of tan $\beta$ enhancement. As for $\theta_{23}$, the RG corrections could be either positive or negative, depending mainly on the model and the neutrino mass ordering. In the normal mass ordering ($m_1 < m_2 < m_3$) both typically decrease in the SM and increase in the MSSM, whereas for the inverted mass ordering ($m_3 < m_1 < m_2$) the behavior is opposite. Finally, nonzero $\theta_{13}$ can run in both directions, and receives corrections of the same order as $\theta_{23}$.

When searching for viable lepton mixing patterns, we have ignored the Majorana phases, which indeed do not change the mixing angles. However, they may play a significant role in the RG evolution of mixing angles, in particular for $\theta_{12}$. In the CP conserving limit with $\rho = \sigma = 0$, $\theta_{12} \propto (m_1 + m_2)^2/\Delta m^2_{\text{sol}}$, which may strongly boost the RG running in the nearly degenerate or inverted mass ordering. On the contrary, in the limit of $\rho - \sigma = \pi/2$, $\theta_{12} \propto (m_2 - m_1)/(m_2 + m_1)$, which is always smaller than one and hence suppresses the RG effects on $\theta_{12}$. In the latter case, the running mainly comes from the next-to-leading-order terms of $\theta_{13}$, and thus $\theta_{12}$ may run to a slightly smaller value. Since both $\rho$ and $\sigma$ are entirely unconstrained in oscillation experiments, we shall allow them to freely vary between 0 and $\pi$.

We have numerically solved the full set of RGEs for leptonic mixing angles. More explicitly, the lepton mixing patterns in Table 11 are assumed at a cutoff scale

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1 In the flavor symmetry models, the corrections to fermion masses and mixing patterns may also originate from the flavor symmetry breaking and the higher-dimensional operators. Since the significance of these potential corrections is highly model-dependent, we shall concentrate on the generic RGE corrections in the current study.
FIG. 1: Viable patterns for an arbitrary CP violating phase $\varphi$. The red, green, and blue bars denote the allowed ranges of $\varphi$, for which the predicted mixing angles fall into their 1$\sigma$, 2$\sigma$, and 3$\sigma$ intervals, respectively. Patterns with $\varphi = 0$ are written in blue, while patterns with $\varphi = \pi/2$ are written in red. The pattern accommodating both $\varphi = 0$ and $\varphi = \pi/2$ is highlighted in green.
\[ \theta \]

in the MSSM (red bars for the normal mass ordering and blue bars for the inverted mass ordering). The initial values are labeled as black diamonds. Note that \( \theta = 0 \) is assumed and \( \tan \beta = 10 \) is adopted in the MSSM. Furthermore, we allow the lightest neutrino mass \( m_1 \) (or \( m_2 \)) to vary in the range of \((0 \ldots 0.1)\) eV in the MSSM, whereas \( m_3 \) (or \( m_2 \)) varies in the range of \((0 \ldots 0.2)\) eV in the SM. The vertical lines correspond to the best-fit value and the 1\( \sigma \), 2\( \sigma \) and 3\( \sigma \) intervals. The mixing scenarios written in red are not valid without RG corrections.

\[ \Lambda = 10^{10} \text{ GeV}, \] and then we evolve the mixing parameters down to the electroweak scale \( \Lambda_{\text{EW}} \) in order to compare them with experimental data. Note that the dependence of RG corrections on the cutoff scale is logarithmic and therefore the precise value of \( \Lambda \) is not quite relevant. Other physical parameters, e.g. gauge couplings and fermion masses, are taken from Ref. [22]. In the case with \( \varphi = 0 \), we show in Fig. 2 the RG evolution of \( \theta_{12} \) in the MSSM with \( \tan \beta = 10 \) as well as in the SM. One can observe that in the MSSM some viable mixing patterns can in principle receive RG corrections so large that they are no longer valid, for instance \( P_{3}(10, 4, 8) \) in the inverted mass ordering. However, as mentioned above, by choosing \( \rho - \sigma = \pi/2 \) one can suppress the running of \( \theta_{12} \), a situation with interesting consequences for neutrinoless double beta decay, as discussed in Ref. [14]. Note that, although the predictions on \( \theta_{12} \) are same for both \( P_{3}(3, 5, 12) \) and \( P_{4}(8, 6, 6) \), they suffer from different RG corrections, reflecting the importance of \( \theta_{23} \) in the RG evolution [cf. Eq. (21)]. In the SM, the RG corrections are in general small, and all the viable mixing patterns remain valid at the electroweak scale. In Fig. 3 we have also shown twelve new mixing scenarios, which generate too small or large \( \theta_{12} \) when not corrected by RG effects, but can enter the allowed range after RG corrections. These cases are indicated by writing them in red (see below).

For completeness, we also show the RG corrections to \( \theta_{13} \) in Fig. 3 As expected, no visible effects can be seen in the SM, while in the MSSM, the RG running may lead to tiny deviations (less than one degree) from their initial values. Even in this case, the values of \( \theta_{13} \) from some mixing patterns listed in Table III may exceed their 3\( \sigma \) ranges, e.g., the mixing pattern \( P_{4}(10, 4, 4) \). Furthermore, the RG effects on \( \theta_{23} \) are of similar size to those on \( \theta_{23} \) [cf. Eq. (21)], and therefore no significant radiative

FIG. 2: RG correction to \( \theta_{12} \) in the SM (green bars), and in the MSSM (red bars for the normal mass ordering and blue bars for the inverted mass ordering). The initial values are labeled as black diamonds. Note that \( \varphi = 0 \) is assumed and \( \tan \beta = 10 \) is adopted in the MSSM. Furthermore, we allow the lightest neutrino mass \( m_1 \) (or \( m_3 \)) to vary in the range of \((0 \ldots 0.1)\) eV in the MSSM, whereas \( m_3 \) (or \( m_2 \)) varies in the range of \((0 \ldots 0.2)\) eV in the SM. The vertical lines correspond to the best-fit value and the 1\( \sigma \), 2\( \sigma \) and 3\( \sigma \) intervals. The mixing scenarios written in red are not valid without RG corrections.

FIG. 3: The RG corrections to \( \theta_{13} \) in the SM (green bars), and in the MSSM (red bars for the normal mass ordering and blue bars for the inverted mass ordering). The initial values are labeled as black diamonds. The input parameters are the same as in Fig. 2. Note that the corrections in the SM appear to be invisible. The mixing scenarios written in red are not valid without RG corrections.
corrections can be acquired.

Now we consider the maximal CP violating case with \( \varphi = \pi/2 \). The RG corrections to \( \theta_{12} \) are depicted in Fig. 4. Similar to the CP conserving case, sizable RG effects can be present. In the MSSM and in the case of the inverted mass ordering, one can observe that for instance \( P_5(12, 5, 6) \) can become incompatible with experimental data, and similarly for \( P_6(12, 4, 6) \) in the SM. On the other hand, there are eleven new mixing schemes which become valid only after sizable RG corrections.

Let us turn to the mixing patterns which are not compatible with data at the cutoff scale but can be modified to the proper parameter ranges with the help of the RG running. This is in particular possible for the mixing patterns with both \( \theta_{23} \) and \( \theta_{13} \) in the currently-favored ranges but a smaller \( \theta_{12} \), since \( \theta_{12} \) may be lifted up into the correct parameter interval in the MSSM. For example, the mixing patterns with \( \theta_{12} = 30^\circ \), or \( \sin \theta_{12} = 1/2 \), could be of particular interest \(^2\). We have also performed a systematic search for such mixing patterns in our scenario. As mentioned above, in the case of \( \varphi = 0 \) (\( \varphi = \pi/2 \)), there exist twelve (eleven) new patterns. Now we shortly discuss three of them:

1. **Pattern \( P_5(4, 4, 3) \) with \( \varphi = 0 \)** – The lepton mixing matrix is

\[
V = \begin{pmatrix}
\sqrt{\frac{1}{2} - \frac{\sqrt{3}}{4}} & \frac{1}{2} & -\sqrt{\frac{1}{2} - \frac{\sqrt{3}}{4}} \\
\frac{\sqrt{3}}{4} & \frac{1}{2} & -\frac{\sqrt{3}}{4} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{pmatrix},
\]

which leads to

\[
\sin^2 \theta_{13} = \frac{1}{16} (5 - 2\sqrt{6}), \\
\sin^2 \theta_{23} = \frac{6}{11 + 2\sqrt{6}}, \\
\sin^2 \theta_{12} = \frac{4}{11 + 2\sqrt{6}},
\]

or explicitly \( \theta_{13} \approx 4.6^\circ \), \( \theta_{23} \approx 38.0^\circ \) and \( \theta_{12} \approx 30.1^\circ \).

Evolving the mixing angles through the full set of RGEs in the MSSM with \( \tan \beta = 10 \), we obtain, at the electroweak scale, \( \theta_{12} \in [30.1^\circ, 36.9^\circ] \) for the normal mass ordering and \( \theta_{12} \in [30.1^\circ, 36.9^\circ] \) for the inverted mass ordering [cf. Fig. 2]. Thus the mixing angles become consistent with the experimental data with the help of radiative corrections.

2. **Pattern \( P_2(3, 6, 12) \) with \( \varphi = 0 \)** – The lepton mixing matrix takes the form

\[
V = \begin{pmatrix}
\frac{\sqrt{2}}{2} & \frac{\sqrt{2} - 1}{4} & \frac{\sqrt{2} - 1}{4} \\
\frac{\sqrt{2} - 1}{4} & \frac{3(\sqrt{2} + \sqrt{6})}{16} & \frac{3(\sqrt{2} + \sqrt{6})}{16} \\
\frac{\sqrt{2} - 1}{4} & \frac{3(\sqrt{2} + \sqrt{6})}{16} & \frac{3(\sqrt{2} + \sqrt{6})}{16}
\end{pmatrix},
\]

which leads to

\[
\sin^2 \theta_{13} = \frac{1}{16} (2 - \sqrt{3}), \\
\sin^2 \theta_{23} = \frac{9(2 + \sqrt{3})}{4(14 + \sqrt{3})}, \\
\sin^2 \theta_{12} = \frac{2 + \sqrt{3}}{14 + \sqrt{3}},
\]

or explicitly \( \theta_{13} \approx 7.4^\circ \), \( \theta_{23} \approx 46.9^\circ \) and \( \theta_{12} \approx 29.1^\circ \).

Again we evolve the mixing angles via the RGEs in the MSSM with \( \tan \beta = 10 \), and obtain \( \theta_{12} \in [29.2^\circ, 37.7^\circ] \) in the normal mass ordering case while \( \theta_{12} \in [29.1^\circ, 38.8^\circ] \) in the inverted mass ordering [cf. Fig. 2]. These values of \( \theta_{12} \) are well compatible with the 3\sigma range in Eq. (2).

3. **Pattern \( P_2(4, 6, 12) \) with \( \varphi = \pi/2 \)** – The lepton mixing matrix takes the form

\[
V = \begin{pmatrix}
\frac{\sqrt{2}}{4} & \frac{\sqrt{2} + 1}{4\sqrt{2}} & -\frac{\sqrt{2} - 1}{4\sqrt{2}} \\
\frac{\sqrt{2} + 1}{4\sqrt{2}} & \frac{3 + \sqrt{3} - 2(\sqrt{3} - 1)}{8} & -\frac{3 - \sqrt{3} + 2(\sqrt{3} + 1)}{8} \\
-\frac{\sqrt{2} - 1}{4\sqrt{2}} & -\frac{3 - \sqrt{3} + 2(\sqrt{3} + 1)}{8} & \frac{3 + \sqrt{3} - 2(\sqrt{3} - 1)}{8}
\end{pmatrix},
\]

\(^2\) Another example of this kind is the so-called tetra-maximal mixing pattern with \( \theta_{12} \approx 30.4^\circ \) (\( \tan \theta_{12} = 2 - \sqrt{2} \)) \(^2\).
which leads to
\[
\sin^2 \theta_{13} = \frac{1}{16} (2 - \sqrt{3}), \\
\sin^2 \theta_{23} = \frac{1}{2}, \\
\sin^2 \theta_{12} = \frac{2 + \sqrt{3}}{14 + \sqrt{3}},
\]

or explicitly \( \theta_{13} \approx 10.5^\circ, \theta_{23} = 45.0^\circ \) and \( \theta_{12} \approx 29.1^\circ \). In addition, the Jarlskog invariant is \( J_{\text{CP}} = \sqrt{3}/64 \approx 2.7\% \). Similar to the above two patterns, the RG running in the MSSM leads to \( \theta_{12} \in [29.0^\circ, 36.6^\circ] \) for the normal mass ordering and \( \theta_{12} \in [29.0^\circ, 37.1^\circ] \) for the inverted mass ordering (cf. Fig. 4), which are in agreement with the experimental data.

In general, the evaluation of RGEs may play a crucial role in searching for realistic mixing patterns. Especially in the supersymmetric case, a larger \( \tan \beta \) typically leads to more significant RG corrections. In this sense, the mixing patterns with smaller \( \theta_{12} \) could be more favorable for a larger \( \tan \beta \). It is also worthwhile to stress that the RGEs under discussion are given in the effective theory approach, which is essentially the same for different kinds of flavor or seesaw models. In the realistic flavor symmetry models, the flavons might induce additional contributions to the RGEs. If a seesaw model is considered, the RG running between the seesaw thresholds may also lead to remarkable modifications, and thus should be carefully treated. A thorough survey on the RGEs in a specific model and on the different values of \( \tan \beta \) is beyond the scope of this work, and we refer the readers to Refs. [19–21] for more detailed discussions.

V. CONCLUSIONS

Motivated by the recent indications of a nonzero \( \theta_{13} \), we have performed a systematic search for simple but viable lepton mixing patterns by setting two criteria: (i) the lepton mixing matrix is parameterized by three rotation angles, which are simple fractions of \( \pi \); (ii) the sines and cosines of these rotation angles possess exact expressions. In total, we have found 20 viable mixing patterns in the CP conserving limit, while 15 viable patterns exist in case of maximal CP violation. Moreover, in the most general cases with the CP phase unconstrained, only 66 mixing patterns out of 6561 combinations are found to be compatible with current data.

Furthermore, radiative corrections to the mixing patterns have been calculated by solving the RGEs of leptonic mixing parameters. We have shown that the RG running can induce sizable corrections to the lepton mixing patterns, which eventually could render some patterns to be unsuccessful in describing lepton mixing. On the other hand, we have also pointed out some interesting mixing patterns, which are incompatible with current oscillation data at the high-energy scale but become viable at the low-energy scale after the RG corrections are properly taken into account.

We hope that the successful constant mixing patterns found in this work can be helpful in searching for the underlying flavor symmetries and shed some light on the final solution to the flavor puzzle. At least, they could serve as a useful phenomenological description of lepton mixing.

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