Representation Theory of Generalized Deformed Oscillator Algebras

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Abstract

The representation theory of the generalized deformed oscillator algebras (GDOA’s) is developed. GDOA’s are generated by the four operators \( \{1, a, a^\dagger, N\} \). Their commutators and Hermiticity properties are those of the boson oscillator algebra, except for \([a, a^\dagger]_q = G(N)\), where \([a, b]_q = ab - qba\) and \(G(N)\) is a Hermitian, analytic function. The unitary irreducible representations are obtained by means of a Casimir operator \(C\) and the semipositive operator \(a^\dagger a\). They may belong to one out of four classes: bounded from below (BFB), bounded from above (BFA), finite-dimensional (FD), unbounded (UB). Some examples of these different types of unirreps are given.

1 Introduction

Since the pioneering works of Arik and Coon [1], Kuryshkin, Biedenharn, and Macfarlane [2], many forms of deformed oscillator algebras (DOA’s) have been considered. They have played an important role in the construction of \(q\)-deformed Lie algebras and have found various applications to physical problems.

The necessity to introduce some order in the rich and varied choice of deformed commutation relations did however appear soon and various classification schemes were therefore proposed [3, 4, 5]. The representations of some forms of DOA’s have been investigated [6], which brought out the existence of additional non-Fock-space representations.

In this communication, the general representation theory of generalized DOA’s will be considered [7]. The representation theory of generalized deformed oscillator algebras is developed in Sec. 2, and illustrated on some examples in Sec. 3. Finally, Sec. 4 contains the conclusion.
2 Representation theory of generalized deformed oscillator algebras

1. A generalized deformed oscillator algebra (GDOA) is generated by the operators \(\{1, a, a^\dagger, N\}\) satisfying the Hermiticity conditions \((a^\dagger)^\dagger = a, N^\dagger = N,\) and the commutation relations

\[
\begin{align*}
[N, a] &= -a & [N, a^\dagger] &= a^\dagger, \\
[a, a^\dagger]_q &= G(N),
\end{align*}
\]

where \([a, b]_q = ab - q ba, q \in R,\) is a quommutator, and the deformation function \(G(N)\) is a Hermitian, analytic function.

The structure function, \(F(N)\), is defined as the solution of the functional equation

\[
F(x + 1) - q F(x) = G(x), \quad x \in R \quad \text{and} \quad F(0) = 0.
\]

2. Contrary to some definitions of GDOA’s where both a commutation and an anticommutation relations are assumed [4], we only keep a commuta tor. As explained in Refs. [8], this choice leads to the existence of a Casimir operator \(C\) that can be constructed by means of the structure function:

\[
C = q^{-N}(F(N) - a^\dagger a) = q^{-(N+1)}(F(N + 1) - aa^\dagger).
\]

This implies an algebraic relation between \(a^\dagger a, N\) and \(aa^\dagger, C\)

\[
a^\dagger a = F(N) - q^N C, \quad aa^\dagger = F(N + 1) - q^{N+1} C.
\]

Another useful Casimir operator is \(U = e^{i2\pi N}\).

3. The spectrum of \(N\) is discrete. This can be proved by the same technique as that used for parabosons in [4]. The Casimir operator \(U\) is unitary, so that in a given unirrep, it possesses a fixed eigenvalue of the form \(e^{i2\pi \nu}, \nu_0 \in R.\) On the other hand, the eigenvalues of \(U\) can be determined from those of the Hermitian operator \(N.\) The spectral mapping theorem leads to eigenvalues for \(U\) of the form \(e^{i2\pi x},\) where \(x \in R\) are the eigenvalues of \(N.\) The equivalence of the two expressions for the eigenvalues of \(U\) implies that \(x = \nu_0 + n, n \in Z,\) in a given unirrep.

The first assumption we make is that the spectrum of \(N\) is nondegenerate. Then, we suppose the existence of a normalized simultaneous eigenvector, \(|c, \nu_0\rangle,\) of \(C\) and \(N:\)

\[
\begin{align*}
C|c, \nu_0\rangle &= c|c, \nu_0\rangle, \\
N|c, \nu_0\rangle &= \nu_0|c, \nu_0\rangle, \\
\langle c, \nu_0|c, \nu_0\rangle &= 1.
\end{align*}
\]

4. By the repeated action of \(a\) and \(a^\dagger\) on this vector, it is possible to construct new eigenvectors of \(C\) and \(N.\) With

\[
\begin{align*}
|c, \nu_0 + n\rangle &= (a^\dagger)^n |c, \nu_0\rangle \quad \text{if} \quad n > 0 \\
&= (a)^{-n} |c, \nu_0\rangle \quad \text{if} \quad n < 0,
\end{align*}
\]
we get
\[ C |c, \nu_0 + n\rangle = c |c, \nu_0 + n\rangle, \tag{11} \]
\[ N |c, \nu_0 + n\rangle = (\nu_0 + n) |c, \nu_0 + n\rangle, \tag{12} \]
if \( |c, \nu_0 + n\rangle \) is non-vanishing.

These vectors are also eigenvectors of the semi-positive operators \( a^\dagger a \) and \( aa^\dagger \):
\[ a^\dagger a |c, \nu_0 + n\rangle = \lambda_n |c, \nu_0 + n\rangle, \tag{13} \]
\[ aa^\dagger |c, \nu_0 + n\rangle = \mu_n |c, \nu_0 + n\rangle, \tag{14} \]
with, according to (5),
\[ \lambda_n = F(\nu_0 + n) - q^{\nu_0 + n} c, \tag{15} \]
\[ \mu_n = \lambda_{n+1}. \tag{16} \]

The existence of \( |c, \nu_0\rangle \) and the positiveness of \( a^\dagger a \) impose a condition on \( \lambda_0 \):
\[ \lambda_0 = F(\nu_0) - q^{\nu_0} c \geq 0. \tag{17} \]

As long as they exist, the vectors \( |c, \nu_0 + n\rangle \) can be normalized:
\[ |c, \nu_0 + n\rangle \equiv \left( \prod_{i=1}^{n} \lambda_i \right)^{-1/2} (a^\dagger)^n |c, \nu_0\rangle \text{ if } n = 0, 1, 2, \ldots \tag{18} \]
\[ \equiv \left( \prod_{i=0}^{\lfloor n \rfloor - 1} \lambda_{-i} \right)^{-1/2} (a)^{-n} |c, \nu_0\rangle \text{ if } n = 0, -1, -2, \ldots. \tag{19} \]

In the representation spanned by these vectors, the matrix elements of the GDOA’s generators are
\[ \langle c, \nu_0 + m| C |c, \nu_0 + n\rangle = c \delta_{m,n}, \tag{20} \]
\[ \langle c, \nu_0 + m| N |c, \nu_0 + n\rangle = (\nu_0 + n) \delta_{m,n}, \tag{21} \]
\[ \langle c, \nu_0 + m| a |c, \nu_0 + n\rangle = \sqrt{\lambda_n} \delta_{m,n-1}, \tag{22} \]
\[ \langle c, \nu_0 + m| a^\dagger |c, \nu_0 + n\rangle = \sqrt{\lambda_{n+1}} \delta_{m,n+1}. \tag{23} \]

The Fock-space representations are characterized by \( c = \nu_0 = 0 \).

5. The existence condition of \( |c, \nu_0 + n\rangle \) (and thus of \( |c, \nu_0 + n\rangle \)) derives from the unitary condition for the representation. The latter imposes the positivity of the \( a^\dagger a \) eigenvalues, as expressed in the following proposition:

**Proposition**: If there exists some \( n_1 \in \{-1, -2, -3, \ldots\} \) such that \( \lambda_{n_1} < 0 \), and \( \lambda_n \geq 0 \) for \( n = 0, -1, \ldots, n_1 + 1 \), then an irreducible representation of a deformed oscillator algebra can be unitary only if \( \lambda_n = 0 \) for some \( n_1 \in \ldots. \)
\[\{0, -1, \ldots, m_1 + 1\}.\] If there exists some \(m_2 \in \{2, 3, 4, \ldots\}\) such that \(\lambda_{m_2} < 0\), and \(\lambda_n \geq 0\) for \(n = 0, 1, \ldots, m_2 - 1\), then it can be unitary only if \(\lambda_n = 0\) for some \(n_2 \in \{1, 2, \ldots, m_2 - 1\}\).

**Proof.** In the first part of the proposition, we must have \(|c, \nu_0 + m_1\rangle = 0\) as otherwise \(a^\dagger a\) would have a negative eigenvalue. This implies that \(a|c, \nu_0 + m_1 + 1\rangle = 0\), which can be achieved in two ways, either \(|c, \nu_0 + m_1 + 1\rangle = 0\), or \(|c, \nu_0 + m_1 + 1\rangle \neq 0\) and \(\lambda_{n_1} = 0\) in the former case, we can proceed in the same way and find that at least one of the conditions \(\lambda_{m_1 + 2} = 0, \ldots, \lambda_{n_2} = 0\), \(|c, \nu_0 - 1\rangle = 0\) must be satisfied. But the last one is equivalent to \(\lambda_0 = 0\), since \(|c, \nu_0\rangle \neq 0\) by hypothesis. This concludes the proof of the first part of the proposition. The second part can be demonstrated in a similar way by using \(aa^\dagger\), and \(\mu_n = \lambda_{n+1}\) instead of \(a^\dagger a\), and \(\lambda_n\).

6. This leads to four classes of unirreps:

- If there exists some \(n_1 \in \{0, -1, -2, \ldots\}\) such that \(\lambda_{n_1} = 0\) and \(\lambda_n > 0\) for all \(n \in \{n_1 + 1, n_1 + 2, \ldots\}\), then \(|c, \nu_0 + n_1\rangle\) is annihilated by \(a\), and one gets a bounded from below (BFB) unirrep. By repeating the unirrep construction beginning with \(|c, \nu_0 + n_1\rangle \equiv |c, \nu_0\rangle\) and replacing \(\nu_0\) by \(\tilde{\nu}_0 = \nu_0 + n_1\), \(\lambda_n\) by \(\tilde{\lambda}_n = \lambda_{n+1}\), \(\tilde{\lambda}_0 = 0\), the vector space of the BFB unirrep is spanned by the vectors defined in (18). The eigenvalue of the Casimir operator \(C\) is \(c = q^{-\tilde{\nu}_0}F(\tilde{\nu}_0)\).

- If there exists some \(n_2 \in \{1, 2, 3, \ldots\}\) such that \(\lambda_{n_2} = 0\) and \(\lambda_n > 0\) for all \(n \in \{n_2 - 1, n_2 - 2, \ldots\}\), then \(|c, \nu_0 + n_2 - 1\rangle\) is destroyed by \(a^\dagger\), and one gets a bounded from above (BFA) unirrep. By repeating the unirrep construction beginning with \(|c, \nu_0 + n_2 - 1\rangle \equiv |c, \nu_0\rangle\) and replacing \(\nu_0\) by \(\tilde{\nu}_0 = \nu_0 + n_2 - 1\), \(\lambda_n\) by \(\tilde{\lambda}_n = \lambda_{n+1} - 1\), \(\tilde{\lambda}_1 = 0\), the vector space of the BFA unirrep is spanned by the vectors defined in (19), with \(c = q^{-\tilde{\nu}_0}F(\tilde{\nu}_0)\).

- If there exists some \(n_1 \in \{0, -1, -2, \ldots\}\) and \(n_2 \in \{1, 2, 3, \ldots\}\) such that \(\lambda_{n_1} = \lambda_{n_2} = 0\) and \(\lambda_n > 0\) for all \(n \in \{n_1 + 1, n_1 + 2, \ldots, n_2 - 1\}\), then \(|c, \nu_0 + n_1\rangle \langle c, \nu_0 + n_2 - 1\rangle\) is annihilated by \(a\) \((a^\dagger)\), and one gets a finite-dimensional (FD) unirrep with dimension \(d = p + 1 = n_2 - n_1\). By repeating the unirrep construction beginning with \(|c, \nu_0 + n_1\rangle \equiv |c, \nu_0\rangle\) and changing \(\nu_0\) into \(\tilde{\nu}_0 = \nu_0 + n_1\), \(\lambda_n\) into \(\tilde{\lambda}_n = \lambda_{n+1}\), \(\tilde{\lambda}_0 = 0\), the vector space of the FD unirrep is spanned by the vectors defined in (18) with \(n \leq p\). The eigenvalue of \(C\) is \(c = q^{-\tilde{\nu}_0}F(\tilde{\nu}_0) = q^{-\tilde{\nu}_0}F(\tilde{\nu}_0 + p + 1)\).

- If \(\lambda_n > 0\) for all \(n \in \mathbb{Z}\), one gets an unbounded (UB) unirrep on a vector space spanned by the vectors defined in (18) and (19) and \(\nu_0 \in [0, 1]\) (the representations corresponding to values of \(\nu_0\) differing by integer steps are equivalent).
3 Examples

To illustrate the general representation theory of GDOA’s, we consider the unirreps of three popular GDOA’s derived from the boson algebra by the recursive minimal-deformation procedure \([5]\).

Note that we consider the case \(q < 0\). This case is omitted in most studies, because the corresponding algebras are considered as deformations of the fermion oscillator algebra. For \(q\) negative, the parameter \(\nu_0\) is restricted to integer values so that \(q^{\nu_0}\) is well defined.

3.1 The Arik-Coon oscillator algebra

It is characterized by \(G(N) = 1\) \([1]\), so that

\[
F(N) = \frac{q^N - 1}{q - 1} = [N]_q,
\]

which gives

\[
C = \frac{1 - q^{-N}}{q - 1} - q^{-N}a^\dagger a,
\]

and

\[
\lambda_n = \left(\frac{1}{q - 1} - c\right)q^{\nu_0+n} - \frac{1}{q - 1}.
\]

The different classes of unirreps of this GDOA are listed in table 1. The UB unirreps, which diverge for \(q \to 1^-\), are referred to as classically singular representations \([10]\).

3.2 The Chaturvedi-Srinivasan oscillator algebra

It is characterized by \(q = 1\) on the left-hand side of (2), and \(G(N) = q^N\) on the right-hand side, i.e., \([a, a^\dagger] = q^N\) \([11]\). \(F(N)\) is therefore solution of

\[
F(x + 1) - F(x) = G(x), \quad x \in R \quad \text{and} \quad F(0) = 0,
\]

and

\[
C = F(N) - a^\dagger a.
\]

We obtain

\[
F(N) = [N]_q,
\]

again, but now

\[
C = [N]_q - a^\dagger a,
\]

so that

\[
\lambda_n = \frac{q^{\nu_0+n} - 1}{q - 1} - c.
\]

The unirreps of this algebra are given in table 2. The classically singular representations are now the UB unirreps with \(q > 1\), contrary to what happens for the Arik-Coon oscillator.
Table 1: Unirrep classification for the Arik-Coon oscillator algebra. The cases where \( q = 1 \) and \( q = -1 \) correspond to the boson and fermion oscillator algebras, respectively.

| \( q \) | Type | Characterization |
|--------|------|------------------|
| \( q > 1 \) | BFB  | \( \tilde{\nu}_0 \in R, c = q^{-\tilde{\nu}_0 |\tilde{\nu}_0|}, \tilde{\lambda}_n = [n]_q \) |
| \( q = 1 \) | BFB  | \( \tilde{\nu}_0 \in R, c = \tilde{\nu}_0, \tilde{\lambda}_n = n \) |
| \( 0 < q < 1 \) | BFB  | \( \tilde{\nu}_0 \in R, c = q^{-\tilde{\nu}_0 |\tilde{\nu}_0|}, \tilde{\lambda}_n = [n]_q \) |
|       | UB   | \( 0 \leq \tilde{\nu}_0 < 1, c \leq (q-1)^{-1}, \tilde{\lambda}_n = [\tilde{\nu}_0 + n]_q - cq^\tilde{\nu}_0 + n \) |
| \( -1 < q < 0 \) | BFB  | \( \tilde{\nu}_0 \in Z, c = q^{-\tilde{\nu}_0 |\tilde{\nu}_0|}, \tilde{\lambda}_n = [n]_q \) |
|       | UB   | \( 0 \leq \tilde{\nu}_0 < 1, c = (q-1)^{-1}, \tilde{\lambda}_n = (1-q)^{-1} \) |
| \( q = -1 \) | FD   | \( \tilde{\nu}_0 \in 2Z, p = 1, c = 0, \tilde{\lambda}_n = (1 - (-1)^n)/2 \) |
|       | FD   | \( \tilde{\nu}_0 \in 2Z + 1, p = 1, c = -1, \tilde{\lambda}_n = (1 - (-1)^n)/2 \) |
|       | UB   | \( 0 \leq \tilde{\nu}_0 < 1, c = -1/2, \tilde{\lambda}_n = 1/2 \) |
| \( q < -1 \) | BFA  | \( \tilde{\nu}_0 \in Z, c = q^{-\tilde{\nu}_0 - 1 |\tilde{\nu}_0 + 1|}, \tilde{\lambda}_n = [n-1]_q \) |
|       | UB   | \( 0 \leq \tilde{\nu}_0 < 1, c = (q-1)^{-1}, \tilde{\lambda}_n = (1-q)^{-1} \) |

3.3 The Tamm-Dancoff oscillator algebra

It is characterized by \( G(N) = q^N \) [12], so that

\[
F(N) = q^{N-1}N \quad \text{and} \quad C = q^{-1}N - q^{-N}a^\dagger a ,
\]

and

\[
\lambda_n = q^{\nu_0 + n - 1}(\nu_0 + n - qc).
\]

Only one class of unirreps exists for this GDOA, as showed in table 3.

4 Conclusion

We developed the representation theory of GDOA’s. The classification of their unirreps can be most easily performed in terms of the eigenvalues of the Casimir operators \( U \) and \( C \). We showed that the spectrum of the number operator \( N \) is discrete. Under the assumption that it is nondegenerate, the unirreps can fall into one out of four classes (BFB, BFA, FD, UB), bosonic, and fermionic or parafermionic Fock-space representations occurring as special cases.
Table 2: Unirrep classification for the Chaturvedi-Srinivasan oscillator algebra.

| \( q \) | Type | Characterization |
|---|---|---|
| \( q > 1 \) | BFB | \( \tilde{\nu}_0 \in R, c = [\tilde{\nu}_0]_q, \tilde{\lambda}_n = q^{\tilde{\nu}_0} [n]_q \) |
| | UB | \( 0 \leq \tilde{\nu}_0 < 1, c \leq -(q-1)^{-1}, \tilde{\lambda}_n = [\tilde{\nu}_0 + n]_q - c \) |
| \( 0 < q < 1 \) | BFB | \( \tilde{\nu}_0 \in R, c = [\tilde{\nu}_0]_q, \tilde{\lambda}_n = q^{\tilde{\nu}_0} [n]_q \) |
| | UB | \( 0 \leq \tilde{\nu}_0 < 1, c \leq -(q-1)^{-1}, \tilde{\lambda}_n = [\tilde{\nu}_0 + n]_q - c \) |
| \( -1 < q < 0 \) | BFB | \( \tilde{\nu}_0 \in \mathbb{Z}, c = [\tilde{\nu}_0]_q, \tilde{\lambda}_n = q^\tilde{\nu}_0 [n]_q \) |
| \( q = -1 \) | FD | \( \tilde{\nu}_0 \in \mathbb{Z}, p = 1, c = 0, \tilde{\lambda}_n = (1 - (-1)^n) / 2 \) |
| | UB | \( \tilde{\nu}_0 = 0, c < 0, \tilde{\lambda}_n = -c + (1 - (-1)^n) / 2 \) |
| \( q < -1 \) | BFA | \( \tilde{\nu}_0 \in \mathbb{Z} + 1, c = [\tilde{\nu}_0 + 1]_q, \tilde{\lambda}_n = q^{\tilde{\nu}_0+1} [n - 1]_q \) |

Table 3: Unirrep classification for the Tamm-Dancoff oscillator algebra.

| \( q \) | Type | Characterization |
|---|---|---|
| \( 0 < q \neq 1 \) | BFB | \( \tilde{\nu}_0 \in R, c = q^{-1} \tilde{\nu}_0, \tilde{\lambda}_n = q^{\tilde{\nu}_0+n-1} [n]_q \) |

We provided examples for each of these classes, although in the FD case, only two-dimensional unirreps were encountered. Higher-dimensional FD unirreps do however arise for some known deformed oscillator algebras [13].

Applications of deformed oscillator algebras have been restricted up to now to their Fock-space representations. Whether non-Fock-space representations, such as those considered here, may have some useful applications remains an interesting open question.

Note: During the Colloquium, Michèle Irac-Astaud pointed out to us the existence of some previous related works on the classification of GDOA’s unirreps [14], confirming the present results.

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