Test particles behavior in the framework of a lagrangian geometric theory with propagating torsion

G. Aprea, G. Montani, R. Ruffini
ICRA-International Center for Relativistic Astrophysics,
Dipartimento di Fisica, Università degli Studi di Roma “La Sapienza”,
Piazzale Aldo Moro 5, 00185 Roma, Italy.

April 10, 2003

Abstract

Working in the lagrangian framework, we develop a geometric theory in vacuum with propagating torsion; the antisymmetric and trace parts of the torsion tensor, considered as derived from local potential fields, are taken and, using the minimal action principle, their field equations are calculated. Actually these will show themselves to be just equations for propagating waves giving torsion a behavior similar to that of metric which, as known, propagates through gravitational waves. Then we establish a principle of minimal substitution to derive test particles equation of motion, obtaining, as result, that they move along autoparallels. We then calculate the analogous of the geodesic deviation for these trajectories and analyze their behavior in the nonrelativistic limit, showing that the torsion trace potential \( \phi \) has a phenomenology which is indistinguishable from that of the gravitational newtonian field; in this way we also give a reason for why there have never been evidence for it.

PACS number: 04.50.+h

Keywords: Torsion, Alternative theories of gravity.

1 Introduction

As well known, in non flat spaces the concept of parallel transport of vector fields needs the introduction of a connection which is also needed to define the covariant derivative. In fact, by means of a connection, we can define the equation of parallel transport in the following way: on a manifold \( M \), given a curve \( \gamma(t) \) passing for a point \( P \in M \), the parallel transported vector field of the vector field \( V^\alpha(P) \) along \( \gamma(t) \) is the solution of:

\[
\frac{dV^\alpha}{dt} = -C^\alpha_{\mu\nu} V^\nu \dot{\gamma}^\mu.
\]

In regard to the covariant derivative of a vector field \( V^\alpha(x) \), instead, it is defined in this way:

\[
D_\beta V^\alpha = \partial_\beta V^\alpha + C_{\beta\gamma}^\alpha V^\gamma
\]
The quantities $C_{\alpha\mu\nu}$ are just the coefficients of the affine connection and a space, endowed with them, is called an affine space and usually indicated by the symbol $L_4$; in general these coefficients are non tensorial quantities but that is not true for their antisymmetric part called torsion:

$$S_{\alpha\beta} = C_{[\alpha\beta]}^\alpha,$$

which indeed transforms like a tensor. Just because of this property, the presence of torsion in space-time denies the principle of equivalence of its importance; here we are not referring to the equivalence between inertial and gravitational mass, which is preserved since the theory remains geometric, but to the formulation of the equivalence principle [19] according to which, once defined an inertial 1 frame in a point, there the laws of physics are the same as those of special relativity. In the case of presence of torsion, the latter, being a tensor, can’t be set to zero by a convenient choice of the coordinates so, since we expect torsion to be source of some force, it is not possible to define an inertial frame in any point which is a necessary condition for the applicability of the principle.

In order to calculate lengths and angles, we also need to introduce in our affine space $L_4$ a metric and that can be done defining the square modulus of a vector $V^\alpha$ as:

$$\|V\|^2 = g_{\alpha\beta} V^\alpha V^\beta;$$

here $g_{\alpha\beta}$ is just the symmetric metric tensor. With the help of the tensor of non-metricity:

$$Q_{\alpha\beta\gamma} \equiv D_\alpha g_{\beta\gamma},$$

it is possible to establish a relation between the connection coefficients and torsion, metric, and nonmetricity:

$$C_{\alpha\beta\gamma} = \frac{1}{2} [\partial_\alpha g_{\beta\gamma} - \partial_\beta g_{\alpha\gamma} + \partial_\gamma g_{\alpha\beta}] + [S_{\alpha\beta\gamma} - S_{\beta\gamma\alpha} + S_{\gamma\alpha\beta}] +$$

$$+ \frac{1}{2} [Q_{\alpha\beta\gamma} - Q_{\gamma\beta\alpha} + Q_{\beta\gamma\alpha}] \equiv \Gamma_{\alpha\beta\gamma} + K_{\alpha\beta\gamma} + \frac{1}{2} [Q_{\alpha\beta\gamma} - Q_{\gamma\beta\alpha} + Q_{\beta\gamma\alpha}].$$

The new quantities we have introduced here are the Christoffel symbols $\Gamma_{\beta\alpha\gamma}$ and the contortion tensor $K_{\alpha\beta\gamma}$; the former are symmetric in the first two indices while the latter is antisymmetric in the last two. In the rest of this paper, in order to preserve lengths and angles under parallel displacement, we will assume the metric postulate according to which nonmetricity is vanishing; such a space is indicated by the symbol $U_4$:

| General linear space $L_4$ | $Q_{\alpha\beta\gamma}$ | Einstein-Cartan space $U_4$ | $S_{\alpha\beta\gamma}$ | Riemannian space $V_4$ |

Completely neglected in the first formulation of the theory of General Relativity (GR) by Einstein, the introduction of torsion was later taken into consideration by Einstein himself [7], Eddington [6], Schrödinger [24] and principally Cartan [3][4][5], who had the idea of a theory in which torsion was connected with intrinsic angular momentum. Later this idea was shelved and only since the fifties the possibility of introducing torsion into GR had been revalued. Utiyama [30], Kibble [14] and Sciama [25][26], inspired by the work of Yang and Mills [31] on gauge theories, formulated a theory of gravitation as a gauge theory in which the presence of torsion was necessary; then by

1 In an inertial frame, a body at rest remain in such a state.
2 Although the tensorial formalism used, they are non tensorial quantities
the mid seventies Hehl et al. [8] managed to set up a Poincaré gauge theory of gravitation, the $U_4$ theory, with torsion corresponding to the rotation gauge potential. They assumed the geometric lagrangian density was the curvature scalar and that matter could be taken into account simply through the minimal coupling rules: $\eta_{\mu\nu} \rightarrow g_{\mu\nu}$, $\partial_{\mu} \rightarrow D_{\mu}$; in this theory intrinsic angular momentum creates torsion which can’t propagate through empty space; in brief, this is due to the fact that the field equations that relate torsion and spin are algebraic and don’t involve derivatives, allowing the substitution of torsion in the matter Lagrangian leading to an effective contact interaction. Since, in the first instance, it is reasonable to expect torsion to behave as any other interaction field, i.e. propagating into vacuum, this aspect of $U_4$ theory is unsatisfactory and in this paper a theory to overcome this problem is discussed. In the next section some assumptions are made about the form of the torsion tensor which are necessary to obtain a propagating torsion without changing the form of the action, namely the completely antisymmetric and trace part of the torsion tensor are considered derived from two local torsion potential. Then, in section 3 by the principle of least action we determine the field equations for these potentials which indeed reveal themselves to be wave equations. In the second part (sections 4 and 5) we discuss the problem of determining the equation of motion of test particles and establish a principle of minimal substitution which leads us to say that test particles move along autoparallels; finally we calculate the nonrelativistic limit of these trajectories and of the tidal effects and show that the torsion trace potential $\phi$ enters in all the equations in the same way as, in this limit, the gravitational potential does. Concluding remarks follow.

2 The form of the torsion tensor

As we have seen, torsion is a three indices tensor which is antisymmetric in the first two; according to the rules of group theory it can be decomposed in a completely antisymmetric part, a trace part and a third part with no special symmetry properties [9]. In our analysis we consider only the first two terms and, in addition, we assume that they can be derived from the exterior derivative of two potentials, in this way:

$$S_{[\mu\nu\lambda]} \rightarrow B_{\mu\nu\lambda} \equiv \partial_{[\mu} A_{\nu\lambda]} = \{D_{[\mu} A_{\nu\lambda]}\}$$ (7)

$$Tr(S_{\mu\nu\lambda}) - \frac{1}{3}((\partial_\mu \phi) g_{\nu\lambda} - (\partial_\nu \phi) g_{\mu\lambda})$$ (8)

$A_{\mu\nu}(x)$ is an antisymmetric tensor while $\phi(x)$ is a scalar. These potentials play a role analogous to that of metric in the symmetric part of the connection:

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2} [\partial_\alpha g_{\beta\gamma} - \partial_\beta g_{\alpha\gamma} + \partial_\gamma g_{\alpha\beta}].$$ (9)

By virtue of (9), we can also write the expression for the connection coefficients and contortion:

$$K_{\mu\nu\lambda} \equiv S_{\mu\nu\lambda} - S_{\nu\lambda\mu} + S_{\lambda\mu\nu} =$$

$$= \partial_{[\mu} A_{\nu\lambda]} + \frac{2}{3}((\partial_\mu \phi) g_{\nu\lambda} - (\partial_\nu \phi) g_{\mu\lambda})$$ (10)

$$C_{\mu\nu\lambda} = \Gamma_{\mu\nu\lambda} + K_{\mu\nu\lambda} =$$

$$= \Gamma_{\mu\nu\lambda} + \partial_{[\mu} A_{\nu\lambda]} + \frac{2}{3}((\partial_\lambda \phi) g_{\mu\nu} - (\partial_\nu \phi) g_{\mu\lambda})$$ (11)

The introduction of torsion potentials for the antisymmetric part of torsion is already present in the literature [3] and has its main motivation just in obtaining a torsion

\[3\]See, for example, [11]
propagating into vacuum. As far as the expression (8) for the trace part is concerned, it is worth noting that the same expression is present in [12] but in a different contest; in fact, in Hojman et al’s article it was introduced to get a coupling of torsion to electromagnetic field which didn’t break gauge symmetry. Really there is another way available to obtain propagation; it consists, in analogy to Yang Mills theory, in introducing square terms in curvature and torsion in the Einstein-Hilbert action. This approach is discussed, among others, in [27], [20], [28], [29], [18], [2]; here we make the different choice of using torsion potentials which, we believe, has these advantages: 1) we can preserve the simplicity of the Einstein-Hilbert action with the minimal substitution \( \Gamma_{\alpha\beta\gamma} \rightarrow G_{\alpha\beta\gamma} + K_{\alpha\beta\gamma} \); 2) we have put both riemannian connection and torsion on the same level since as the former is derived from metric, the latter is derived from potentials; 3) in the limit of small and slow varying \( \phi \) the action (13) is equivalent to the low energy limit of string theory lagrangian, as already mentioned in [11] (and reference therein), suggesting torsion potentials to be a necessary ingredient in more general theory.

### 3 Field equations

According to what we have said in the previous sections our Lagrangian density is of the Hilbert-Einstein form \(^4\):

\[
A = -\frac{1}{2k} \int dx \sqrt{-g} \, R(x) \equiv -\frac{1}{2k} \int dx \sqrt{-g} \, g^{\beta\gamma} \left( \partial_{\alpha} C_{\beta\gamma}^{\delta} - \partial_{\beta} C_{\alpha\gamma}^{\delta} - C_{\alpha\gamma}^{\eta} C_{\beta\eta}^{\delta} + C_{\beta\gamma}^{\eta} C_{\alpha\eta}^{\delta} \right).
\]

We will obtain the field equations with the least action principle calculating the variations with respect to the metric and both the torsion potentials. In order to simplify the variational calculation we proceed now in splitting the action in its riemannian part plus torsion-depending terms; with the help of \([11]\), we get \(^5\):

\[
A = -\frac{1}{2k} \int dx \sqrt{-g} \left( \frac{1}{2} R(x) - B^{\alpha\beta} B_{\alpha\beta} - \frac{2}{3} \left( \partial_{\alpha} \phi \right)^2 \right);
\]

Variations respect to \( g_{\alpha\beta} \), \( A_{\alpha\beta} \) and \( \phi \) yield:

\[
\begin{align*}
-\frac{1}{2} \gamma_{\alpha\beta} & = -\frac{1}{2} g^{\alpha\beta} B^{\mu\nu\sigma} B_{\mu\nu\sigma} + 3 B^{\mu\nu\sigma} B_{\mu\nu\sigma} + \frac{8}{3} \left( \frac{1}{2} g^{\alpha\beta} \left( \partial_{\mu} \phi \right)^2 - g^{\alpha\beta} g^{\beta\sigma} \left( \partial_{\mu} \phi \right) \left( \partial_{\nu} \phi \right) \right) = 0, \\
\frac{1}{D_{\mu}} B^{\mu\alpha\beta} & = 0, \\
\frac{1}{D_{\mu}} g^{\mu\nu} \partial_{\nu} \phi & = 0.
\end{align*}
\]

In the first equation \([15]\) there is the Riemannian Einstein tensor as in GR; moreover in this case we have four terms all quadratic in the torsion potentials. By virtue of this

\(^4\)Here \( k \) is a constant related to Newton’s gravitational constant \( G = 6.670 \times 10^{-8} \text{cm}^3\text{g}^{-1}\text{s}^{-2} \) via:

\[
\frac{1}{k} = \frac{c^4}{8\pi G},
\]

\(^5\)The symbol “\( \{ \} \)” stands for riemannian
if we are interested in solving it at first order for little values of the torsion potentials we can neglect those quadratic terms and fall back in the GR field equations; we can solve them and get the metric to be put in equations (16) and (17) to find, at the first order, the torsion potentials. As for equations (16) and (17), it can be seen that the goal of a propagating torsion has been reached, in fact we have two second order PDE for both the potentials. Equation (18) can be simplified using its invariance under the gauge transformation:

\[ A_{\alpha\beta} \to A'_{\alpha\beta} = A_{\alpha\beta} + \frac{1}{\partial_\alpha Y_\beta} - \frac{1}{\partial_\beta Y_\alpha}; \] (18)

in fact, if we choose \( Y \) such that

\[ \frac{1}{\partial_\alpha A'_{\alpha\beta}} = 0. \] (19)

after some calculation involving commutation rules for covariant derivatives, it can be put in the form:

\[ \frac{1}{\partial_\mu D_\mu A'_{\alpha\beta}} - R_{\mu\alpha\beta} A'_{\mu\rho} + R_{\alpha\beta} A'_{\mu\rho} + R_{\mu\rho} A'_{\alpha\beta} = \Delta_{DR}(A') = 0. \] (20)

Here \( \Delta_{DR} \) is the De Rham operator which generalize the Laplacian operator in non flat spaces. It is easy to show that a field \( A' \) obeying (20) and (19) has only one freedom degree, or, in other words, it can have only one direction of polarization in the iperplane normal to its propagation direction. As far as equation (17) is concerned, it is worth noting that it is a massless Klein-Gordon field equation so that we can consider the potential \( \phi \) as a geometrical manifestation of this field.

4 Test particles paths

The problem of the determination of the equation of motion of a test particle can be approached in a number of ways; one is that proposed by Papapetrou [21] which consists in obtaining the equation of motion from the conservation law of the energy-momentum tensor. According to us this approach has some unsatisfactory aspects: the first is that we can have some ambiguity on the right way to get the conservation law of the energy-momentum tensor since we can start both using Noether theorem and Ricci identities, but, in spaces with torsion, the results can be different [10]; in addition, once we have the conservation law, we must have the expression of the energy-momentum tensor which is rather difficult especially in the case of presence of nonriemannian quantities as torsion; in this case, in fact, a anti-symmetric part of the energy-momentum tensor, probably depending on spin, is involved and it is not clear either whether it is correct to give a semiclassical expression of it, being spin a purely quantum quantity, or its explicit form. Than we have the approach of Hojman [13] which consists in defining all the possible scalar quantities that can be in the test particle action; then the action is build up and the equation of motion are obtained by variations with respect to the particle coordinates. According to us this approach has one of the same unsatisfactory aspects of the previous, since, taking into account the spin of the test particle, again we need to have a semiclassical expression for the spin depending part of the test particle action although spin should be treated only in quantum mechanics. Another way to get the test particle equation of motion is to make use of the principle of the shortest path which assumes that a test particle moves from a point A to another B in space in a way such that its trajectory has the least length among all the curves joining A with B. Although this method seems simple and nice, it is completely regardless of the presence of torsion because the last,
not contributing to the length of a curve, neither appears in the equation of motion of any test particle. According to us, instead, the presence of a tensorial quantity as torsion, which has a role in the parallel transport of vector fields in space, should have some effects on the motion and so we assume as the correct method to have the equation of motion in curved space from the knowledge of that in flat space the following minimal substitution rule:

\[
\text{Ordinary derivative } \left( \frac{d}{d\tau} \right) \rightarrow \text{Covariant derivative } \left( D \frac{d}{d\tau} \right),
\]

According to this rule the equation of motion in curved space is obtained from the analogous one of special relativity

\[
\frac{du^\alpha}{d\tau} = 0,
\]

where \( u^\alpha \) is the 4-velocity via the (21):

\[
\frac{D u^\alpha}{d\tau} = 0 \quad (23)
\]

which can be rewritten as:

\[
\frac{du^\alpha}{d\tau} = -C_{\alpha \mu \nu} u^\mu u^\nu = -\Gamma_{\mu \nu}^{\alpha} u^\mu u^\nu - g^{\alpha \lambda} \frac{2}{3} ((\partial_\lambda \phi) g_{\mu \nu} - (\partial_\nu \phi) g_{\mu \lambda}) u^\mu u^\nu. \quad (24)
\]

This is the autoparallel equation which, together with the geodesic equation, is a special curve that can be defined in non flat spaces; while the latter is the shortest curve joining two points, the former is the curve whose tangent vector is parallel transported along it. The autoparallel curve is the simplest generalization of the flat space equation of motion (22) which is suitable to take into account of torsion or other nonriemannian quantities.

It is worth noting that it is also possible to introduce a new action principle such that, starting from the action

\[
A^M = -\frac{M}{2} \int_{\tau_1}^{\tau_2} d\tau \ x^2,
\]

where \( \tau \) is the proper time, we can have autoparallels as the right trajectories. This approach is proposed in [15], [16], [17] and here we summarize it briefly. The key point is that a spacetime with torsion, which can be obtained from a flat spacetime by a nonholonomic mapping, is affected by a closure failure of parallelograms; as a consequence, variations of test particle trajectories cannot be performed as in the usual way in flat spacetime, i.e. keeping \( \delta x^a(\tau) \) vanishing at endpoints. In fact, the variations \( \delta^S q^\mu(\tau) \), images of \( \delta x^a(\tau) \) under a nonholonomic mapping, are generally not closed; so they can be chosen to be zero at the initial point but then they are nonvanishing at the final point, this failure being proportional to torsion. The variational calculation, then, can be done as follows; first we rewrite explicitly the relation between the old \((x^a(\tau))\) and the new \((q^\mu(\tau))\) paths in integral form:

\[
q^\mu(\tau) = q^\mu(\tau_1) + \int_{\tau_1}^{\tau} d\tau' e_a^\mu(q(\tau')) \dot{x}^a(\tau'),
\]

where \( e_a^\mu(q(\tau')) \) is the nonholonomic mapping; then we calculate the variation associated to \( q^\mu(\tau) \):

\[
\delta^S q^\mu(\tau) = \int_{\tau_1}^{\tau} d\tau' \left[ \left( \delta^S e_a^\mu(q(\tau')) \right) \dot{x}^a(\tau') + e_a^\mu(q(\tau')) \delta^S \dot{x}^a(\tau') \right]
\]

\[\text{6} \]In this context it is obvious how to include nonmetricity.
After having introduced a further quantity called auxiliary nonholonomic variation:

\[ \delta q^\nu(\tau) \equiv e^\mu_\nu(q(\tau)) \delta x^\mu(\tau), \]  

(28)

which, in contrast to \( \delta^S q^\nu(\tau) \), vanishes at endpoints and forms closed paths in q-space, we derive the relation

\[ \frac{d}{d\tau} \delta^S q^\nu(\tau) = \left( \delta^S e^\nu_\mu(q(\tau)) \right) \dot{x}^\mu(\tau) + e^\mu_\nu(q(\tau)) \delta \dot{x}^\nu(\tau) = \]  

\[ = \left[ \delta^S e^\nu_\mu(q(\tau)) \right] \ddot{x}^\mu(\tau) + e^\mu_\nu(q(\tau)) \frac{d}{d\tau} \left[ e^\nu_\sigma(q(\tau)) \delta q^\sigma(\tau) \right], \]  

(29)

which, after the substitutions:

\[ \delta^S e^\mu_\nu = -C^\lambda_\mu_\nu \delta^S e^\lambda_\nu + \frac{d}{d\tau} e^\nu_\lambda \equiv C^\lambda_\mu_\nu q^\lambda e^\nu_\mu, \]  

(30)

can be rewritten:

\[ \frac{d}{d\tau} \delta^S q^\nu = -C^\lambda_\mu_\nu \delta^S q^\lambda q^\nu + C^\lambda_\mu_\nu q^\lambda \delta q^\nu + \frac{d}{d\tau} \delta q^\nu, \]  

(31)

or, better,

\[ \frac{d}{d\tau} \delta^S b^\mu = -C^\lambda_\mu_\nu \delta^S b^\lambda q^\nu + 2S^\lambda_\mu_\nu q^\nu \delta q^\nu, \]  

(32)

where we have introduced \( \delta^S b^\mu \), the difference between \( \delta^S q^\nu \) and \( \delta q^\nu \). Now we can calculate the variation of the action analogous of (25) under a nonholonomic variation \( \delta^S q^\nu = \delta q^\nu + \delta^S b^\nu \):

\[ \delta^S A^M = \delta^S \left( -\frac{M}{2} \int_{\tau_1}^{\tau_2} d\tau g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu \right) = -M \int_{\tau_1}^{\tau_2} d\tau \left( g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu + \frac{1}{2} \partial_\mu g_{\lambda\nu} \delta^S q^\lambda \dot{q}^\nu \right); \]  

(33)

using \( [\delta^S, d/d\tau] = 0 \) which follow from (26), we can partially integrate the \( \delta^S q \)-term and, by the identity \( \partial_\mu g_{\lambda\nu} \equiv C_{\mu\nu\lambda} + C_{\mu\lambda\nu} \), get:

\[ \delta^S A^M = -M \int_{\tau_1}^{\tau_2} d\tau \left[ -g_{\mu\nu} \left( \ddot{q}^\mu + \Gamma^\mu_{\lambda\nu} \dot{q}^\lambda \dot{q}^\kappa \right) \right] \delta q^\mu + \left( g_{\mu\nu} \dot{q}^\mu \frac{d}{d\tau} \delta^S q^\nu + C_{\mu\lambda\nu} \delta^S b^\lambda q^\nu \right). \]  

(34)

Now it is straightforward to obtain first the equation of motion in absence of torsion \( (\delta^S b^\mu(\tau) \equiv 0) \):

\[ \ddot{q}^\nu + \Gamma^\nu_{\lambda\kappa} \dot{q}^\lambda \dot{q}^\kappa = 0, \]  

(35)

that is the geodesic equation; then we can consider all the affine connection and get, with the help of (29), the autoparallel equation:

\[ \ddot{q}^\nu + C^\mu_{\lambda\kappa} \dot{q}^\lambda \dot{q}^\kappa = 0. \]  

(36)

As for the possibility to get the autoparallel motion from the energy-momentum tensor conservation, we now give a possible modification of the test particle action such that this result could be partially obtained. We assume the test particle action to be:

\[ A^M = \int d\tau \; u^\mu u^\nu e^{-\frac{\phi}{2}} g_{\mu\nu}; \]  

(37)
then we can calculate its variations with respect to $g_{\mu\nu}$ and $\phi$:

$$
g_{T\mu\nu} = \frac{\delta A^M}{\delta g_{\mu\nu}} = \int d\tau \frac{u^\mu u^\nu}{\sqrt{-g}} e^{-\frac{1}{4}\phi} \delta (x - x_0),
$$

$$
\phi_T = \frac{\delta A^M}{\delta \phi} = -\frac{1}{4} \int d\tau \frac{u^\mu u^\nu}{\sqrt{-g}} e^{-\frac{1}{4}\phi} g_{\mu\nu} \delta (x - x_0),
$$

from comparison with

$$
\delta A = \int d^4x \sqrt{-g} (g_{T\mu\nu} \delta g_{\mu\nu} + \phi_T \delta \phi).
$$

Now, following the same procedure of Hammond [11], we consider the motion of a test particle in a background geometry which is perturbed in a negligible way by it and start from the identity:

$$
(\sqrt{-g} g_{T\mu\nu})_\nu = \sqrt{-g} g_{T\mu\nu} - \sqrt{-g} \Gamma^\alpha_{\mu\beta} g_{T\alpha\beta};
$$

now we integrate it over a small volume $dV$ where the only appreciable energy momentum tensor is that of the test particle and, discarding all surface terms, with the help of the conservation law in our case:

$$
g_{T\mu\nu} = \frac{8}{3} \partial^\mu \phi \phi_T,
$$

we get:

$$
\frac{1}{w^0} \frac{d}{d\tau} \int dV(\sqrt{-g} g_{T(\mu0)}) = \frac{8}{3} \partial^\mu \phi \int dV(\sqrt{-g} \phi_T) - \Gamma^\alpha_{\mu\beta} \int dV(\sqrt{-g} g_{T(\alpha\beta)}).
$$

This, with the help of [35], can be rewritten in the form:

$$
\frac{du^\alpha}{d\tau} = -\Gamma^\alpha_{\mu\nu} u^\mu u^\nu - \frac{2}{3} g^\alpha\beta (\partial_\beta \phi) g_{\mu\nu} u^\mu u^\nu.
$$

If we multiply both members of this equation by $u_\alpha$ we get:

$$
0 = u_\alpha \partial^\alpha \phi.
$$

Now if we consider our autoparallel equation [21] we see that it is in accordance with these last two equations [35], [11], meaning that the Papapetrou motion is included as a special case.

## 5 Autoparallels, Autoparallel deviation and their nonrelativistic limit

At the end of the last section we have concluded, on the basis of our minimal substitution rule, that test particles follow autoparallel trajectories whose equation we recall now:

$$
\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau} - K^\alpha_{\beta\gamma} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau}.
$$

It is easy to see that in this expression the antisymmetric part of torsion contribution vanishes; it only contributes as a source for the metric through [35]. In this section we will study the nonrelativistic limit of autoparallels and in addition we will calculate the analogous of the geodesic deviation and we will see the role of torsion in the tidal forces.
5.1 Nonrelativistic limit of autoparallels

In order to calculate the nonrelativistic limit of (45), we make the following assumptions:

a) 3-velocity much smaller than c so that we can put $u^i \simeq v^i$;
   
   b) the gravitational field is static and weak, i.e. the quantities $h_{\alpha\beta} = g_{\alpha\beta} - \eta_{\alpha\beta}$ (here $\eta_{\alpha\beta}$ is the Minkovsky metric) are very small;
   
   c) the torsion potential $\phi$ is static and weak.

By virtue of these assumptions, since we want to keep only first order terms, we will eliminate all those terms containing $\phi$, $h_{\alpha\beta}$ e $v^i$ more than once. After some calculations we obtain:

$$\frac{d^2 x_i}{dt^2} = -\kappa \frac{\partial h_{00}}{2} - \frac{2}{3} \partial_i \phi.$$  \hspace{1cm} (46)

Now we recall that in GR we had:

$$\frac{d^2 x_i}{dt^2} = -\kappa \frac{\partial h_{00}}{2},$$  \hspace{1cm} (47)

that allowed us to identify $h_{00}$ with the gravitational potential $\Phi$:

$$\frac{\kappa}{2} h_{00} = \Phi.$$  \hspace{1cm} (48)

As we can see from (49) the “force” due to the torsion potential is present in the same form of the gravitational field $h_{00}$; in addition, as for the order we are interested in, and reminding of the supposed field’s static nature, equation (17) for the field $\phi$ reduces to:

$$\Delta \phi (\vec{x}) = 0,$$  \hspace{1cm} (49)

which is the same of the gravitational field:

$$\Delta h_{00} (\vec{x}) = 4\pi \rho.$$  \hspace{1cm} (50)

5.2 Deviation of autoparallels

Following standards calculations [22] and reminding that now test particles move along autoparallels, if we take two of them initially very close each other, we find their relative acceleration to be:

$$\frac{D^2 s^\alpha}{d\tau^2} = -R^\alpha_{\beta\gamma\delta} s^\delta \ u^\beta \ u^\gamma +$$

$$-\frac{2}{3} \left\{ g^{\alpha\beta} \delta_{\beta} (\partial_\gamma \phi) + g^{\alpha\beta} g_{\beta\gamma} (\partial_\delta \phi) \right\} \left( \frac{ds^\gamma}{d\tau} u^\beta + \frac{ds^\beta}{d\tau} u^\gamma \right) +$$

$$-\frac{2}{3} \left\{ \delta_{\beta} (\partial_\gamma \phi) + g^{\alpha\beta} g_{\beta\gamma} \right\} s^\delta \ u^\beta \ u^\gamma.$$  \hspace{1cm} (51)

Here $s^\alpha$ is an infinitesimal vector representing the relative displacement between the two particles. Now we can substitute our expression for the contortion tensor (10) and get:

$$\frac{D^2 s^\alpha}{d\tau^2} = -R^\alpha_{\beta\gamma\delta} s^\delta \ u^\beta \ u^\gamma +$$

$$-\frac{2}{3} \left\{ g^{\alpha\beta} \delta_{\beta} (\partial_\gamma \phi) + g^{\alpha\beta} g_{\beta\gamma} (\partial_\delta \phi) \right\} \left( \frac{ds^\gamma}{d\tau} u^\beta + \frac{ds^\beta}{d\tau} u^\gamma \right) +$$

$$-\frac{2}{3} \left\{ \delta_{\beta} (\partial_\gamma \phi) + g^{\alpha\beta} g_{\beta\gamma} \right\} s^\delta \ u^\beta \ u^\gamma.$$  \hspace{1cm} (52)
This equation represents the generalization to a theory with torsion of the geodesic deviation of GR:

\[
\frac{D^2 s^\alpha}{d\tau^2} = -R_{\alpha\beta\gamma\delta} s^\beta u^\gamma u^\delta.
\]  

(54)

It is easy to see that the difference between the two expressions consists in two linear terms in torsion and torsion derivative, the first of which is multiplied by a term of relative velocity. Once again we see completely antisymmetric part of torsion not contributing, if not as a source in the field equation (15), to this expression.

To calculate the nonrelativistic limit of (53), we will make the same assumptions b), c) made in the last paragraph plus:

a) involved velocities are much smaller than c and:

\[
\frac{dx^\alpha}{d\tau} \simeq (1, 0, 0, 0);
\]

d) \(s^0 = \frac{ds^0}{d\tau} = 0\), which simply means that the particles accelerations are compared at the same time.

In this case, by virtue of our assumptions, we will keep only terms containing at most, as factors multiplied by \(s^i\), one between \(h_{i\beta} \) and \(\phi\). Then, (38) reduces to:

\[
\frac{d^2 s^i}{dt^2} \simeq -R_{j00}^i s^j - \frac{2}{3} \eta^{ij} (\partial_k \partial^0 \phi) s^k;
\]

(55)

so the tidal field is:

\[
G^i = -R_{j00}^i s^j - \frac{2}{3} \eta^{ij} (\partial_k \partial^0 \phi) s^k.
\]

(56)

Now, from GR, it is known that in the nonrelativistic limit:

\[
R_{j00}^i = \partial_i \partial^0 \Phi,
\]

(57)

where \(\Phi\) is the gravitational potential.

Taking that into account we can rewrite the final expression for the tidal field:

\[
G_i = - (\partial_i \partial^0 \Phi) s_j - \frac{2}{3} (\partial_i \partial^0 \phi) s_j.
\]

(58)

So, in the nonrelativistic limit, torsion produces a tidal forces effect analogous to the one produced by the gravitational field.

It must be noted, now, that since the fields \(h_{00}\) and \(\phi\), in the nonrelativistic limit, both obey a Poisson PDE (19 and 56) and enter in equations (10) and 38 in the same way, it is impossible to distinguish the effect of the torsion field from that of the gravitational one unless we know exactly the source and the initial condition for the latter; this, together with the small intensity of the torsion forces, makes them even more difficult to be detected.
6 Concluding remarks

We now want to summarize briefly the main results stated here. First we have exposed a formulation of a geometric theory of the GR type in the vacuum which is able to predict propagating torsion; than, in order to determine the equation of motion of test particle in presence of this new geometric quantity, we have established a principle of minimal substitution \[ \text{(21)} \] which implied autoparallels were the right trajectories. Finally we have determined the analogous of the geodesic equation for the autoparallels and calculated the nonrelativistic limit of both this deviation \[ \text{(58)} \] and the autoparallels \[ \text{(46)} \] showing that in those expressions, and also in the field equation \[ \text{(49)} \], in this limit, the field \( \phi \) enters in the same way as the gravitational field \( h_{00} \) making itself difficult to be detectable.

References

[1] Aubin, T., *A course in differential geometry*, Graduates studies in mathematics, vol. 27, American Mathematical Society, 2000.
[2] Blagojevic, M., *Gravitation and Gauge Symmetries*, Institute of Physics Publishing, Bristol, UK (2002).
[3] Cartan, É., *Sur une généralization de la notion de courbure de Riemann et les espaces à torsion*, C. R. Acad. Sci. (Paris) 174, (1922), 593.
[4] Cartan, É., *Sur les variétés à connexion affine et la théorie de la relativité généralisée I, I (suite)*, Ann. Ec. Norm. Sup. 40 (1923) 325; 41 (1924), 1.
[5] Cartan, É., *Sur les variétés à connexion affine et la théorie de la relativité généralisée II*, Ann. Ec. Norm. Sup. 42 (1925), 17.
[6] Eddington, A., S., *A generalization of Weyl's theory of electromagnetic and gravitational fields*, Proc. R. Soc. Lond. A99 (1921), 104.
[7] Einstein, A., *The meaning of Relativity*, (Princeton University Press, Princeton, NY), 5th Ed., 1956.
[8] Hehl, F., W., von der Heide, P., Kerlick, G., D., Nester, J., M., *General Relativity with spin and torsion: Foundations and prospects*, Rev. Mod. Phys. 48 (1976), 393.
[9] Hehl, F., W., McCrea, J., D., Mielke, E., W., Ne’eman, Y., *Metric-affine gauge theory of gravity: Field equations, Noether identities, world spinors, and breaking of dilation invariance*, Phys. Rep. A192 (1994), 122.
[10] Hehl, F., W., McCrea, J., D., *Bianchi identities and the automatic conservation of energy-momentum and angular momentum in general-relativistic field theories*, Found. Phys. 16 (1986), 267.
[11] Hammond, R.T., *Spin, Torsion, Forces*, Gen. Rel. Grav. 26 (1994), 247.
[12] Hojman, S., Rosenbaum, M., Ryan, M., P., *Propagating torsion and gravitation*, Phys. Rev. D 19 (1979), 430.
[13] Hojman, S., *Lagrangian theory of the motion of spinning particles in torsion gravitational theories*, Phys. Rev. D 18 (1978), 2741.
[14] Kibble, T.W.B., *Lorentz invariance and the gravitational field* J. Math. Phys. 2 (1961), 212.
[15] Kleinert, H., Fiziev, P., *New action principle for classical particle trajectories in spaces with torsion*, Europhys. Lett. 35, 241 (1996) [hep-th/9503074].
[16] Kleinert, H., Pelster, A., *Autoparallels from a new action principle*, Gen. Rel. Grav. 31 (1999), 1439 [gr-qc/9605028].
[17] Kleinert, H., *Nonholonomic Mapping Principle for Classical and Quantum Mechanics in Spaces with Curvature and Torsion*, (gr-qc/0203029).

[18] Kuhfuss, R., Nitsch, J., *Propagating modes in gauge field theories of gravity*, Gen. Rel. Grav. **18** (1964), 1207.

[19] Misner, C.W., Thorne, K.S., Wheeler, J.A., *Gravitation*, (Freeman, San Francisco), 1973.

[20] Neville, D.E., *Spin-2 propagating torsion*, Phys. Rev. D **23** (1981), 1244.

[21] Papapetrou, A., *Spinning test-particles in general relativity*, Proc. Roy. Soc. Lond. **A209** (1948), 248.

[22] Ohanian, H., C., Ruffini, R., *Gravitazione e Spazio-Tempo*, (Zanichelli, Bologna) (1997), 291.

[23] Schouten, J.A., *Ricci calculus*, (Springer, Berlin), 2nd ed., 1954.

[24] Schrödinger, E., Proc. R. Irish Acad. **A49** (1943), 43, 135.

[25] Sciama, D.W., *On the analogy between charge and spin in general relativity*, Recent Developments in General Relativity, (Pergamon+PWD, Oxford), 1962, 415.

[26] Sciama, D.W., *The physical structure of general relativity*, Rev. Mod. Phys. **36** (1964), 463 and 1103.

[27] Sezgin, E., van Nieuwenhuizen, P. *New ghost free gravity lagrangians with propagating torsion*, Phys. Rev. D **21** (1980), 3269.

[28] Sezgin, E., *Class of ghost free gravity lagrangians with massive or massless propagating torsion*, Phys. Rev. D **24** (1981), 1677.

[29] Tseytlin, A.A., *On the Poincare and the de Sitter gauge theories of gravity with propagating torsion*, Phys. Rev. D **26** (1982), 3327.

[30] Utiyama, R., *Invariant theoretical interpretation of interaction*, Phys. Rev. **101** (1956), 1597.

[31] Yang, C., N., Mills, R., L., *Conservation of isotopic spin and isotopic gauge invariance*, Phys. Rev. **96** (1954), 191.