A Unified Stochastic SIR Model

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Abstract
We propose a unified stochastic SIR model. The model is structural enough to allow for time-dependency, nonlinearity, demography and environmental disturbances. We present concise results on the existence and uniqueness of positive global solutions and investigate the extinction and persistence of the model. Examples and simulations are provided to illustrate the main results.

MSC: 60H10; 92D30; 93E15

Keywords: Stochastic SIR model, time-dependency, nonlinear transmission and recovery, positive global solution, extinction, persistence.

1 Introduction

Epidemiological compartment models have garnered much attention from researchers in an attempt to better understand and control the spread of infectious diseases. Mathematical analysis of such models aids decision-making regarding public health policy changes–especially in the event of an epidemic (e.g., COVID-19). One such compartmental model introduced by Kermack and McKendrick [9] in 1927 divides a population into three compartments–susceptible, infected, and recovered (SIR). The classical SIR model is as follows:

\[
\begin{aligned}
\frac{dX_t}{dt} &= -\beta X_t Y_t, \\
\frac{dY_t}{dt} &= (\beta X_t - \gamma) Y_t, \\
\frac{dZ_t}{dt} &= \gamma Y_t,
\end{aligned}
\]  

(1.1)
where \( \beta \) is the transmission rate and \( \gamma \) the recovery rate. Additionally, demography may be introduced to include birth rate \( \Lambda \) and mortality rate \( \mu \) as:

\[
\begin{align*}
\frac{dX_t}{dt} &= \Lambda - \mu X_t - \beta X_t Y_t, \\
\frac{dY_t}{dt} &= [\beta X_t - (\mu + \gamma)] Y_t, \\
\frac{dZ_t}{dt} &= \gamma Y_t - \mu Z_t.
\end{align*}
\]  

(1.2)

The basic SIR models (1.1) and (1.2) have many variations including the SIRD, SIRS, SIRV, SEIR, MSIR, etc. (cf. e.g., [1], [2] and [11]). Further, these deterministic models have been put into different stochastic frameworks, which makes the situation more realistic (cf. e.g., [3], [6], [8] and [12]). The existing models are often analyzed with a focus on specific diseases or parameters. Such studies have been successful in achieving new results; however, often it is the case that structural variability is lacking in these models. To overcome the drawbacks inherent in traditional approaches, we propose and investigate in this paper the unified stochastic SIR model:

\[
\begin{align*}
\frac{dX_t}{dt} &= b_1(t, X_t, Y_t, Z_t)dt + \sum_{j=1}^n \sigma_{1j}(t, X_t, Y_t, Z_t)dB_t^{(j)}, \\
\frac{dY_t}{dt} &= b_2(t, X_t, Y_t, Z_t)dt + \sum_{j=1}^n \sigma_{2j}(t, X_t, Y_t, Z_t)dB_t^{(j)}, \\
\frac{dZ_t}{dt} &= b_3(t, X_t, Y_t, Z_t)dt + \sum_{j=1}^n \sigma_{3j}(t, X_t, Y_t, Z_t)dB_t^{(j)}.
\end{align*}
\]  

(1.3)

Hereafter, \( \mathbb{R}_+ \) denotes the set of all positive real numbers, the drift functions \( b_i(t, x, y, z) \) and the diffusion functions \( \sigma_{ij}(t, x, y, z) \) are measurable on \( [0, \infty) \times \mathbb{R}^3_+ \), \( i = 1, 2, 3, \) \( j = 1, 2, \ldots, n \), and \( (B_t^{(j)})_{t \geq 0}, j = 1, 2, \ldots, n \), are independent standard one-dimensional Brownian motions.

We will show that the unified model (1.3) is structural in design that allows variability without sacrificing key results on the extinction and persistence of diseases. Namely, the model allows for time-dependency, nonlinearity (of drift and diffusion) and demography. Environmental disturbances can have profound effects on transmission, recovery, mortality and population growth. The above model encapsulates the stochastic perturbations driven by white noises \((B_t^{(j)})_{t \geq 0}\) with intensities \( \sigma_{ij}(t, X_t, Y_t, Z_t) \). An important structural feature we emphasize is time-dependency. Time-dependency can capture the progression of a disease insofar as mutations/transmissibility (e.g., Delta and Omicron variants of COVID-19, vaccination programs).

In the following sections, we establish results on the existence and uniqueness of positive global solutions, extinction and persistence of diseases, and provide illustrative examples and simulations. Section 2 is concerned with the model (1.3) for population proportions whereas Section 3 covers the model (1.3) for population numbers. Both approaches are commonly found in studies of SIR models; hence, the importance of investigation for a unifying model. At the time of writing this paper we are unaware of existing work on the unified SIR model and aim to add such a model to the existing literature; additionally, we will close the paper with proposed future work.
2 Model for population proportions

In this section, we let $X_t$, $Y_t$ and $Z_t$ denote respectively the proportions of susceptible, infected and recovered populations at time $t$. Define

$$\Delta := \{(x, y, z) \in \mathbb{R}_+^3 : x + y + z = 1\}.$$ 

We make the following assumptions.

1. For any $T \in (0, \infty)$ and $N \in \mathbb{N}$, there exists $K_{T,N} > 0$ such that

$$\sum_{i=1}^{3} |b_i(t, x_1, y_1, z_1) - b_i(t, x_2, y_2, z_2)|^2 + \sum_{i=1}^{3} \sum_{j=1}^{n} |\sigma_{ij}(t, x_1, y_1, z_1) - \sigma_{ij}(t, x_2, y_2, z_2)|^2$$

$$\leq K_{T,N} |(x_1, y_1, z_1) - (x_2, y_2, z_2)|^2, \quad \forall t \in [0, T], (x_1, y_1, z_1), (x_2, y_2, z_2) \in \left[\frac{1}{N}, 1 - \frac{1}{N}\right]^3,$$

and

$$\sum_{i=1}^{3} |b_i(t, x, y, z)| + \sum_{i=1}^{3} \sum_{j=1}^{n} |\sigma_{ij}(t, x, y, z)| \leq K_{T,N}, \quad \forall t \in [0, T], (x, y, z) \in \left[\frac{1}{N}, 1 - \frac{1}{N}\right]^3.$$

Hereafter $| \cdot |$ denotes the Euclidean norm of $\mathbb{R}^3$.

2. For any $(x, y, z) \in \Delta$,

$$\sum_{i=1}^{3} b_i(t, x, y, z) = 0 \quad \text{and} \quad \sum_{i=1}^{3} \sigma_{ij}(t, x, y, z) = 0, \quad j = 1, 2, \ldots, n.$$

3. For any $T \in (0, \infty)$,

$$\inf_{t \in [0,T], (x,y,z) \in \Delta} \left\{ \frac{b_1(t, x, y, z)}{x} + \frac{b_2(t, x, y, z)}{y} + \frac{b_3(t, x, y, z)}{z} \right\} > -\infty,$$

and

$$\sum_{j=1}^{n} \sup_{t \in [0,T], (x,y,z) \in \Delta} \left\{ \frac{\sigma_{1j}(t, x, y, z)}{x} + \frac{\sigma_{2j}(t, x, y, z)}{y} + \frac{\sigma_{3j}(t, x, y, z)}{z} \right\} < \infty.$$

First, we prove the result on the existence and uniqueness of solution.

**Theorem 2.1** For any given initial value $(X_0, Y_0, Z_0) \in \Delta$, equation (1.3) has a unique strong solution taking values in $\Delta$. 

**Proof.** By Assumption 1, we have the existence and uniqueness of local strong solution to equation (1.3) on \([0, \tau]\), where \(\tau\) is the explosion time (cf. [10, Theorem 2.3.4]). By Assumption 2, we know that \(\frac{dX_t}{dt} + \frac{dY_t}{dt} + \frac{dZ_t}{dt} = 0\). Hence \((X_t, Y_t, Z_t) \in \Delta\) for \(t < \tau\) and

\[
\tau = \inf\{t > 0 : (X_t, Y_t, Z_t) \notin \Delta\} = \inf\{t > 0 : X_t = 0 \text{ or } Y_t = 0 \text{ or } Z_t = 0\}.
\]

We will show below that \(\tau = \infty\) a.s.. Assume the contrary that there exists \(T > 0\) such that

\[
P(\tau < T) > 0. \tag{2.1}
\]

Define

\[
V(x, y, z) = -\ln(xyz), \quad (x, y, z) \in (0,1)^3.
\]

By Itô’s formula, we obtain that for \(t < \tau\),

\[
dV(X_t, Y_t, Z_t) = -\left[\frac{b_1(t, X_t, Y_t, Z_t)}{X_t} + \frac{b_2(t, X_t, Y_t, Z_t)}{Y_t} + \frac{b_3(t, X_t, Y_t, Z_t)}{Z_t}\right] dt
- \sum_{j=1}^{n} \left[\frac{\sigma_{1j}(t, X_t, Y_t, Z_t)}{X_t} + \frac{\sigma_{2j}(t, X_t, Y_t, Z_t)}{Y_t} + \frac{\sigma_{3j}(t, X_t, Y_t, Z_t)}{Z_t}\right] dB_t^{(j)}
+ \frac{1}{2} \sum_{j=1}^{n} \left[\frac{\sigma_{1j}^2(t, X_t, Y_t, Z_t)}{X_t^2} + \frac{\sigma_{2j}^2(t, X_t, Y_t, Z_t)}{Y_t^2} + \frac{\sigma_{3j}^2(t, X_t, Y_t, Z_t)}{Z_t^2}\right] dt.
\]

Define

\[
M_1 := \inf_{t \in [0,T], (x,y,z) \in \Delta} \left\{\frac{b_1(t, x, y, z)}{x} + \frac{b_2(t, x, y, z)}{y} + \frac{b_3(t, x, y, z)}{z}\right\},
\]

\[
M_2 := \sum_{j=1}^{n} \sup_{t \in [0,T], (x,y,z) \in \Delta} \left\{\frac{|\sigma_{1j}(t, x, y, z)|}{x} + \frac{|\sigma_{2j}(t, x, y, z)|}{y} + \frac{|\sigma_{3j}(t, x, y, z)|}{z}\right\}.
\]

Then, we get

\[
E[V(X_{T\wedge\tau}, Y_{T\wedge\tau}, Z_{T\wedge\tau})] \leq V(X_0, Y_0, Z_0) - M_1 T + \frac{M_2^2 T}{2} < \infty,
\]

which contradicts with (2.1). Therefore, \(\tau = \infty\) a.s.. \(\square\)

**Remark 2.2** One can check that many existing models are special cases of our model, e.g., the semi-parametric SIR model introduced in [5] and the stochastic SIR model with vaccination considered in [4].

Now we consider the extinction and persistence of diseases. Namely, we investigate whether a disease will extinct with an exponential rate or will be persistent in mean. The system (1.3) is called persistent in mean if

\[
\lim inf_{t \to \infty} \frac{1}{t} \int_0^t Y_s ds > 0 \text{ a.s.}
\]
Theorem 2.3 Let \((X_t, Y_t, Z_t)\) be the solution to equation (1.3) with \((X_0, Y_0, Z_0) \in \Delta\). We assume that

\[
\limsup_{t \to \infty} \sup_{(x,y,z) \in \Delta} |\sum_{j=1}^n \sigma_{2j}(t, x, y, z)| < \infty. \tag{2.2}
\]

(i) If

\[
\alpha := \limsup_{t \to \infty} \sup_{(x,y,z) \in \Delta} \left[ \frac{b_2(t, x, y, z)}{y} - \frac{\sum_{j=1}^n \sigma_{2j}^2(t, x, y, z)}{2y^2} \right] < 0, \tag{2.3}
\]

then

\[
\limsup_{t \to \infty} \frac{\ln Y_t}{t} \leq \alpha \text{ a.s.}. \tag{2.4}
\]

(ii) If there exist positive constants \(\lambda_0\) and \(\lambda\) such that

\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t \left[ \lambda_0 Y_s + \frac{b_2(s, X_s, Y_s, Z_s)}{Y_s} - \frac{\sum_{j=1}^n \sigma_{2j}^2(s, X_s, Y_s, Z_s)}{2Y_s^2} \right] ds \geq \lambda, \tag{2.5}
\]

then

\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t Y_s ds \geq \frac{\lambda}{\lambda_0} \text{ a.s.}. \tag{2.6}
\]

(iii) If there exist positive constants \(\lambda_0\) and \(\lambda\) such that

\[
\liminf_{t \to \infty} \inf_{(x,y,z) \in \Delta} \left[ \lambda_0 y + \frac{b_2(t, x, y, z)}{y} - \frac{\sum_{j=1}^n \sigma_{2j}^2(t, x, y, z)}{2y^2} \right] \geq \lambda, \tag{2.7}
\]

then

\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t Y_s ds \geq \frac{\lambda}{\lambda_0} \text{ a.s.}. \tag{2.8}
\]

Proof. (i) By Itô’s formula, we have that

\[
\ln Y_t = \ln Y_0 + \int_0^t \left[ \frac{b_2(s, X_s, Y_s, Z_s)}{Y_s} - \frac{\sum_{j=1}^n \sigma_{2j}^2(s, X_s, Y_s, Z_s)}{2Y_s^2} \right] ds + \int_0^t \sum_{j=1}^n \sigma_{2j}(s, X_s, Y_s, Z_s) \frac{dB_s^{(j)}}{Y_s}. \tag{2.8}
\]

By (2.2) and the strong law of large numbers for martingales (cf. [10, Theorem 1.3.4]), we get

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t \sum_{j=1}^n \sigma_{2j}(s, X_s, Y_s, Z_s) \frac{dB_s^{(j)}}{Y_s} = 0 \text{ a.s.}. \tag{2.9}
\]
Then, (2.4) holds by (2.3), (2.8) and (2.9).

(ii) By (2.5) and (2.8), we obtain that for any \( \eta \in (0, \lambda) \) there exists \( T_\eta > 0 \) such that

\[
\ln Y_t \geq (\lambda - \eta)t - \lambda_0 \int_0^t Y_s ds + \ln Y_0 + \int_0^t \sum_{j=1}^n \frac{\sigma_2 j(s, X_s, Y_s, Z_s)}{Y_s} dB_s^{(j)}, \quad t \geq T_\eta.
\]

Therefore, (2.6) holds by (2.9) and [7, Lemma 5.1].

(iii) Obviously, condition (2.7) implies condition (2.5). Hence, the assertion is a direct consequence of assertion (ii).

\[\square\]

**Remark 2.4** If we take the following assumption of our model:

\[
b_2(t, x, y, z) = b_{2,1}(t, x, y, z) - b_{2,2}(t, x, y, z),
\]

where \( b_{2,i}(t, x, y, z) \geq 0 \) for any \( (t, x, y, z) \in [0, \infty) \times \mathbb{R}_+^3 \), \( i = 1, 2 \), then condition (2.3) can be strengthened to

\[
\alpha^* := \limsup_{t \to \infty} \sup_{(x,y,z) \in \Delta} \left[ \frac{b_{2,1}^2(t, x, y, z)}{2 \sum_{j=1}^n \sigma_2^2 j(t, x, y, z)} - \frac{b_{2,2}(t, x, y, z)}{y} \right] < 0. \tag{2.10}
\]

In fact, we have \( \alpha \leq \alpha^* \) and hence condition (2.10) implies that

\[
\limsup_{t \to \infty} \frac{\ln Y_t}{t} \leq \alpha^* \text{ a.s.}
\]

Throughout this paper, we denote by \( L^\infty_+[0, \infty) \) the set of all bounded, non-negative, measurable functions on \([0, \infty)\). For \( f \in L^\infty_+[0, \infty) \), define

\[
\overline{f} := \sup_{t \in [0, \infty)} f(t), \quad \underline{f} := \inf_{t \in [0, \infty)} f(t).
\]

**Example 2.5** (a) Let \( \beta, \gamma, \xi, \sigma_1, \sigma_2, \varphi_1, \varphi_2, \varphi_3 \in L^\infty_+[0, \infty) \). Define

\[
\varphi(t, x, y) = \varphi_1(t)x + \varphi_2(t)y + \varphi_3(t)xy, \quad (t, x, y) \in [0, \infty) \times \mathbb{R}_+^2.
\]

We consider the system

\[
\begin{aligned}
&dX_t = -\frac{\beta(t)X_t^{\xi(t)}Y_t}{1+\varphi(t,X_t,Y_t)} dt - \frac{\sigma_1(t)X_tY_t}{1+\varphi(t,X_t,Y_t)} dB_t^{(1)}, \\
&dY_t = \left[ \frac{\beta(t)X_t^{\xi(t)}Y_t}{1+\varphi(t,X_t,Y_t)} - \gamma(t)Y_t \right] dt + \frac{\sigma_1(t)X_tY_t}{1+\varphi(t,X_t,Y_t)} dB_t^{(1)} + \sigma_2(t)Y_tZ_t dB_t^{(2)}, \\
&dZ_t = \gamma(t)Y_t dt - \sigma_2(t)Y_tZ_t dB_t^{(2)}.
\end{aligned} \tag{2.11}
\]

Suppose that \( \xi \geq 1 \).
We have $\frac{dX}{dt} + \frac{dY}{dt} + \frac{dZ}{dt} = 0$. Hence, by Theorem 2.1, equation (2.11) has a unique strong solution taking values in $\Delta$. If 
\[ \beta < \gamma, \]
then by Theorem 2.3(i) we obtain that the disease extincts with exponential rate \[ -\alpha \geq \gamma - \beta. \]

Additionally, a key feature of the system (2.11) to note is that the transmission function is in the form of power function which differs from the often seen bilinear form.

(b) Let $\beta, \gamma_1, \gamma_2, \alpha \in L^\infty_+[0, \infty)$ We consider the system
\[
\begin{align*}
&\frac{dX}{dt} = -\beta(t)X(t)Y(t)dt - \sigma(t)X(t)Y(t)Z(t)dB_t, \\
&\frac{dY}{dt} = [\beta(t)X(t) - \gamma_1(t) + \gamma_2(t)Z(t)]Y(t)dt + 2\sigma(t)X(t)Y(t)Z(t)dB_t, \\
&\frac{dZ}{dt} = [\gamma_1(t) - \gamma_2(t)Z(t)]Y(t)dt - \sigma(t)X(t)Y(t)Z(t)dB_t.
\end{align*}
\]

We have $\frac{dX}{dt} + \frac{dY}{dt} + \frac{dZ}{dt} = 0$. Hence, by Theorem 2.1, equation (2.12) has a unique strong solution taking values in $\Delta$. Assuming that $\gamma_1 < \gamma_2 \leq \beta$, $\sigma^2 < \frac{\gamma_2 - \gamma_1}{2}$. Then, condition (2.7) is satisfied. Therefore, by Theorem 2.3(iii), we obtain that the disease is persistent and
\[ \liminf_{t \to \infty} \frac{1}{t} \int_0^t Y_s ds \geq \frac{\gamma_2 - \gamma_1 - 2\sigma^2}{\gamma_2} a.s.. \]

### 3 Model for population numbers

In this section, we let $X_t, Y_t$ and $Z_t$ denote respectively the numbers of susceptible, infected and recovered individuals at time $t$. First, we make the following assumption.

1. For any $T \in (0, \infty)$ and $N \in \mathbb{N}$, there exists $K_{T,N} > 0$ such that
\[
\begin{align*}
&\sum_{i=1}^3 |b_i(t, x_1, y_1, z_1) - b_i(t, x_2, y_2, z_2)|^2 + \sum_{i=1}^3 \sum_{j=1}^n |\sigma_{ij}(t, x_1, y_1, z_1) - \sigma_{ij}(t, x_2, y_2, z_2)|^2 \\
&\leq K_{T,N} |(x_1, y_1, z_1) - (x_2, y_2, z_2)|^2, \quad \forall t \in [0, T], (x_1, y_1, z_1), (x_2, y_2, z_2) \in \left[ \frac{1}{N}, N \right]^3,
\end{align*}
\]
and
\[
\begin{align*}
&\sum_{i=1}^3 |b_i(t, x, y, z)| + \sum_{i=1}^3 \sum_{j=1}^n |\sigma_{ij}(t, x, y, z)| \leq K_{T,N}, \quad \forall t \in [0, T], (x, y, z) \in \left[ \frac{1}{N}, N \right]^3.
\end{align*}
\]
By this assumption, we have the existence and uniqueness of local strong solution to equation (1.3) on \([0, \tau]\), where \(\tau\) is the explosion time. Next, we make the following assumption.

2. There is an invariant subset \(\Gamma \subset \mathbb{R}^3_+\) of the system (1.3), i.e., \((X_t, Y_t, Z_t) \in \Gamma\) for any \((X_0, Y_0, Z_0) \in \Gamma\). Moreover, for any \(T \in (0, \infty)\),

\[
\sup_{t \in [0, T], (x,y,z) \in \Gamma} \left\{ \frac{(x-1)b_1(t, x, y, z)}{x} + \frac{(y-1)b_2(t, x, y, z)}{y} + \frac{(z-1)b_3(t, x, y, z)}{z} \right\} < \infty,
\]

and

\[
\sum_{j=1}^{n} \sup_{t \in [0, T], (x,y,z) \in \Gamma} \left\{ \frac{|\sigma_{1j}(t, x, y, z)|}{x} + \frac{|\sigma_{2j}(t, x, y, z)|}{y} + \frac{|\sigma_{3j}(t, x, y, z)|}{z} \right\} < \infty.
\]

Now we present the result on the existence and uniqueness of solution.

**Theorem 3.1** For any given initial value \((X_0, Y_0, Z_0) \in \Gamma\), equation (1.3) has a unique strong solution taking values in \(\Gamma\).

**Proof.** By Assumption 1, we have the existence and uniqueness of local strong solution to equation (1.3) on \([0, \tau]\), where \(\tau\) is the explosion time. We will show below that \(\tau = \infty\) a.s.. Define

\[
\tau_N = \inf \left\{ t \in [0, \tau) : (X_t, Y_t, Z_t) \notin \left[ \frac{1}{N}, N \right]^3 \right\}, \quad N \in \mathbb{N},
\]

and

\[
\tau_\infty = \lim_{N \to \infty} \tau_N.
\]

We have that \(\tau_\infty \leq \tau\) so it suffices to show \(\tau_\infty = \infty\) a.s.. Hence assume the contrary that there exist \(\varepsilon > 0\) and \(T > 0\) such that

\[
\mathbb{P}(\tau_\infty < T) > \varepsilon,
\]

which implies that

\[
\mathbb{P}(\tau_N < T) > \varepsilon, \quad \forall N \in \mathbb{N}.
\]  

(3.1)

Define

\[
V(x, y, z) = (x - 1 - \ln x) + (y - 1 - \ln y) + (z - 1 - \ln z), \quad (x, y, z) \in (0, \infty)^3.
\]
By Itô’s formula, we obtain that for $t \leq \tau_N$,

$$
    dV(X_t, Y_t, Z_t) = [b_1(t, X_t, Y_t, Z_t) + b_2(t, X_t, Y_t, Z_t) + b_3(t, X_t, Y_t, Z_t)] \, dt
    + \sum_{j=1}^n \left[ \sigma_{1j}(t, X_t, Y_t, Z_t) + \sigma_{2j}(t, X_t, Y_t, Z_t) + \sigma_{3j}(t, X_t, Y_t, Z_t) \right] dB_t^{(j)}
    - \left[ \frac{b_1(t, X_t, Y_t, Z_t)}{X_t} + \frac{b_2(t, X_t, Y_t, Z_t)}{Y_t} + \frac{b_3(t, X_t, Y_t, Z_t)}{Z_t} \right] dt
    - \sum_{j=1}^n \left[ \frac{\sigma_{1j}(t, X_t, Y_t, Z_t)}{X_t} + \frac{\sigma_{2j}(t, X_t, Y_t, Z_t)}{Y_t} + \frac{\sigma_{3j}(t, X_t, Y_t, Z_t)}{Z_t} \right] dB_t^{(j)}
    + \frac{1}{2} \sum_{j=1}^n \left[ \frac{\sigma_{1j}^2(t, X_t, Y_t, Z_t)}{X_t^2} + \frac{\sigma_{2j}^2(t, X_t, Y_t, Z_t)}{Y_t^2} + \frac{\sigma_{3j}^2(t, X_t, Y_t, Z_t)}{Z_t^2} \right] dt.
$$

Define

$$
    M_1 := \sup_{t \in [0, T], (x,y,z) \in \Gamma} \left\{ \frac{(x-1)b_1(t, x, y, z)}{x} + \frac{(y-1)b_2(t, x, y, z)}{y} + \frac{(z-1)b_3(t, x, y, z)}{z} \right\},
    M_2 := \sum_{j=1}^n \sup_{t \in [0, T], (x,y,z) \in \Gamma} \left\{ \frac{\left| \sigma_{1j}(t, x, y, z) \right|}{x} + \frac{\left| \sigma_{2j}(t, x, y, z) \right|}{y} + \frac{\left| \sigma_{3j}(t, x, y, z) \right|}{z} \right\}.
$$

Then, we get

$$
    \mathbb{E}[V(X_{T \wedge \tau_N}, Y_{T \wedge \tau_N}, Z_{T \wedge \tau_N})] \leq V(X_0, Y_0, Z_0) + M_1 T + \frac{M_2^2 T}{2} < \infty, \ \forall N \in \mathbb{N}.
$$

However, by (3.1), we get

$$
    \mathbb{E}[V(X_{T \wedge \tau_N}, Y_{T \wedge \tau_N}, Z_{T \wedge \tau_N})] > \varepsilon \left[ \left( \frac{1}{N} - 1 + \ln N \right) \wedge (N - 1 - \ln N) \right] \to \infty \ \text{as} \ N \to \infty.
$$

We have arrived at a contradiction. Therefore, $\tau = \infty$ a.s..<br>

Similar to Theorem 2.3, we can prove the following result on the extinction and persistence of diseases.

**Theorem 3.2** Let $(X_t, Y_t, Z_t)$ be the solution to equation (1.3) with $(X_0, Y_0, Z_0) \in \Gamma$. We assume that

$$
    \limsup_{t \to \infty} \sup_{(x,y,z) \in \Gamma} \frac{\left| \sum_{j=1}^n \sigma_{2j}(t, x, y, z) \right|}{y} < \infty.
$$

(i) If

$$
    \alpha := \limsup_{t \to \infty} \sup_{(x,y,z) \in \Gamma} \left[ \frac{b_2(t, x, y, z)}{y} - \frac{\sum_{j=1}^n \sigma_{2j}^2(t, x, y, z)}{2y^2} \right] < 0, \quad (3.2)
$$
then

\[ \limsup_{t \to \infty} \frac{\ln Y_t}{t} \leq \alpha \text{ a.s.}. \]

(ii) If there exist positive constants \( \lambda_0 \) and \( \lambda \) such that

\[ \liminf_{t \to \infty} \frac{1}{t} \int_0^t \left[ \lambda_0 Y_s + \frac{b_2(s, X_s, Y_s, Z_s)}{Y_s} - \frac{\sum_{j=1}^n \sigma_{2j}^2(s, X_s, Y_s, Z_s)}{2Y_s^2} \right] ds \geq \lambda, \]

then

\[ \liminf_{t \to \infty} \frac{1}{t} \int_0^t Y_s ds \geq \frac{\lambda}{\lambda_0} \text{ a.s..} \]

(iii) If there exist positive constants \( \lambda_0 \) and \( \lambda \) such that

\[ \liminf_{t \to \infty} \inf_{(x, y, z) \in \Gamma} \left[ \lambda_0 y + \frac{b_2(t, x, y, z)}{y} - \frac{\sum_{j=1}^n \sigma_{2j}^2(t, x, y, z)}{2y^2} \right] \geq \lambda, \]

then

\[ \liminf_{t \to \infty} \frac{1}{t} \int_0^t Y_s ds \geq \frac{\lambda}{\lambda_0} \text{ a.s..} \]

Example 3.3 Let \( \Lambda, \mu, \beta, \gamma, \varepsilon, \sigma \in L^\infty[0, \infty) \). We consider the system

\[
\begin{align*}
\dot{X}_t &= [\Lambda(t) - \mu(t)X_t - \beta(t)X_tY_t]dt - \sigma(t)X_tY_tdB_t, \\
\dot{Y}_t &= [\beta(t)X_tY_t - (\mu(t) + \gamma(t) + \varepsilon(t))Y_t]dt + \sigma(t)X_tY_tdB_t, \\
\dot{Z}_t &= [\gamma(t)Y_t - \mu(t)Z_t]dt.
\end{align*}
\]

(3.3)

Suppose that

\[ \mu > 0. \]

By (3.3), we get

\[ d(X_t + Y_t + Z_t) \leq [\Lambda - \mu(X_t + Y_t + Z_t)]dt, \]

which implies that

\[ \Gamma := \left\{ (x, y, z) \in \mathbb{R}^3_+ : x + y + z \leq \frac{\Lambda}{\mu} \right\} \]

is an invariant set of the system (3.3). Hence, equation (3.3) has a unique strong solution taking values in \( \Gamma \) by Theorem 3.1.

Define

\[ \alpha^* := \sup_{x \in (0, \frac{\Lambda}{\mu})} \left[ \beta x - (\mu + \gamma + \varepsilon) - \frac{\sigma^2 x^2}{2} \right]. \]
Then, we have

condition (3.2)

\[ \Leftrightarrow \alpha = \limsup_{t \to \infty} \sup_{x \in (0, \frac{X}{\mu})} \left[ \beta(t) x - (\mu(t) + \gamma(t) + \varepsilon(t)) - \frac{\sigma^2(t)x^2}{2} \right] < 0 \]

\[ \Leftrightarrow \alpha \leq \alpha^* < 0 \]

\[ \Leftrightarrow \alpha^* = \max \left\{ \frac{\beta X}{\mu} - (\mu + \gamma + \varepsilon) - \frac{\sigma^2X^2}{2\mu^2}, \frac{\beta^2}{2\sigma^2} - (\mu + \gamma + \varepsilon) \right\} < 0 \]

\[ \Leftrightarrow \left\{ \begin{array}{ll} \alpha^* = \frac{\beta X}{\mu} - (\mu + \gamma + \varepsilon) - \frac{\sigma^2X^2}{2\mu^2} < 0, & \text{if } \sigma^2 \leq \frac{\mu \beta}{X}, \\ \alpha^* = \frac{\beta^2}{2\sigma^2} - (\mu + \gamma + \varepsilon) < 0, & \text{if } \sigma^2 > \frac{\mu \beta}{X}. \end{array} \right. \]

Thus, by Theorem 3.2(i), we obtain that if

\[ \sigma^2 \leq \frac{\mu \beta}{X} \quad \text{and} \quad \tilde{R}_0 := \frac{\beta X}{\mu(\mu + \gamma + \varepsilon)} - \frac{\sigma^2X^2}{2\mu^2(\mu + \gamma + \varepsilon)} < 1, \]

then the disease extincts with exponential rate

\[ -\alpha \geq (\mu + \gamma + \varepsilon) \left( 1 - \tilde{R}_0 \right); \]

if

\[ \sigma^2 > \max \left\{ \frac{\mu \beta}{X}, \frac{\beta^2}{2(\mu + \gamma + \varepsilon)} \right\}, \]

then the disease extincts with exponential rate

\[ -\alpha \geq (\mu + \gamma + \varepsilon) - \frac{\beta^2}{2\sigma^2}. \]

This result generalizes the result given in [7, Theorem 2.1].

By (3.3), we get

\[ X_t + Y_t = X_0 + Y_0 + \int_0^t [\Lambda(s) - \mu(s)X_s - (\mu(s) + \gamma(s) + \varepsilon(s))Y_s]ds. \]

Since

\[ X_t + Y_t \leq \frac{\Lambda}{\mu}, \quad \forall t \geq 0, \quad (3.4) \]

we get

\[ \lim_{t \to \infty} \frac{1}{t} \int_0^t [\Lambda(s) - \mu(s)X_s - (\mu(s) + \gamma(s) + \varepsilon(s))Y_s]ds = 0, \]

which implies that

\[ \lim_{t \to \infty} \frac{1}{t} \int_0^t X_s ds \geq \frac{\Lambda}{\mu} - \lim_{t \to \infty} \frac{\mu + \gamma + \varepsilon}{\mu t} \int_0^t Y_s ds. \quad (3.5) \]
Suppose
\[ \tilde{R}_0 := \frac{\beta \Lambda}{\mu (\mu + \gamma + \varepsilon)} - \frac{\sigma^2 \Lambda^2}{2 \mu^2 (\mu + \gamma + \varepsilon)} > 1. \]

Then, by (3.3)–(3.5), we get
\[ \liminf_{t \to \infty} \frac{1}{t} \int_0^t \left[ \frac{\beta (\mu + \gamma + \varepsilon)}{\mu} \cdot Y_s + \frac{b_2(s, X_s, Y_s, Z_s)}{Y_s} - \frac{\sigma^2(s) X_s^2}{2} \right] ds \geq \frac{\beta \Lambda}{\mu} - (\mu + \gamma + \varepsilon) - \frac{\sigma^2 \Lambda^2}{2 \mu^2}. \]

Therefore, by Theorem 3.2(ii), we obtain that the disease is persistent and
\[ \liminf_{t \to \infty} \frac{1}{t} \int_0^t Y_s ds \geq \frac{\mu (\tilde{R}_0 - 1)}{\beta} \ a.s.. \]

This result generalizes the result given in [7, Theorem 3.1].

4 Simulations

We now present simulations corresponding to Examples 2.5 and 3.3. Simulations are completed using the Euler-Maruyama scheme with a time step \( \Delta t = 0.001 \) which run from time \( t = 0 \) to \( t = 1000 \). We include both the stochastic and deterministic results to demonstrate the effect of noise on such systems. Time \( t \) is not given with a specific calendar time unit but we may consider that \( t \) represents days, weeks or years, etc.

(i) We assume that the system (2.11) in Example 2.5(a) has initial values \((X_0, Y_0, Z_0) = (0.8, 0.19, 0.01)\) and set the parameters as follows:

| \( f(t) \) | \( f \) | \( \tau \) |
|-------------|-------|-------|
| \( \beta(t) = 0.5 + 0.2 \sin(\frac{t}{30}) \) | 0.3 | 0.7 |
| \( \gamma(t) = 0.85 + 0.1 \cos(\frac{t}{30}) \) | 0.75 | 0.95 |
| \( \xi(t) = 1.2 + 0.05 \cos(t) \) | 1.15 | 1.25 |
| \( \varphi_i(t) = 0.1 + 0.001 \cos(t), i = 1, 2 \) | 0.099 | 0.101 |
| \( \varphi_3(t) = 0.2 + 0.002 \cos(t) \) | 0.198 | 0.202 |
| \( \sigma_1(t) = 2 + 0.1 \cos(t) \) | 1.9 | 2.1 |
| \( \sigma_2(t) = 2 + 0.1 \sin(t) \) | 1.9 | 2.1 |

In Figure 1 below, it is illustrated that the extinction of the disease occurs at an exponential rate. In accordance with Example 2.5(a), the disease will extinct with exponential rate \(-\alpha \geq \gamma - \beta = 0.05\).
(ii) We assume that the system \((2.12)\) in Example 2.5(b) has initial values \((X_0, Y_0, Z_0) = (0.57, 0.32, 0.11)\) and parameters:

| \(f(t)\)            | \(\bar{f}\) | \(\bar{f}\) |
|----------------------|-------------|-------------|
| \(\beta(t) = 1.1 + 0.1 \cos(\frac{t}{200})\) | 1           | 1.2         |
| \(\gamma_1(t) = 0.34 + 0.03 \cos(\frac{t}{50})\) | 0.31        | 0.37        |
| \(\gamma_2(t) = 1.01 + 0.2 \sin(\frac{t}{150})\) | 0.81        | 1.21        |
| \(\sigma(t) = 0.341 + 0.05 \sin(t) + \cos(t)\) | \(0.341 - 0.05 \sqrt{2}\) | \(0.341 + 0.05 \sqrt{2}\) |

We achieve results which illustrate persistence of the disease, as is displayed in Figure 2 below. Furthermore, we have \(\lambda = 0.81\), \(\lambda = 0.10099\) and

\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t Y_s ds \geq \frac{0.10099}{0.81} = 0.1247.
\]

Figure 1: E-M scheme simulation of system \((2.11)\).

Figure 2: E-M scheme simulation of system \((2.12)\).
The remaining simulations are concerned with Example 3.3, system (3.3). We assume that the total population is 95 million and set the initial values using proportional values for convenience. The initial condition is set to \((X_0, Y_0, Z_0) = \left(\frac{298}{95}, \frac{300}{95}, \frac{5}{95}\right)\). In the following simulations, the parameters will change to demonstrate their effects on a system with unchanging initial condition. The first two simulations illustrate extinction of the disease and the final simulation will illustrate persistence of the disease. We initially set the parameters as follows:

| \(f(t)\) | \(\beta(t)\) | \(\gamma(t)\) | \(\varepsilon(t)\) | \(\sigma(t)\) | \(\Lambda(t)\) | \(\mu(t)\) |
|-----------|-------------|-------------|-------------|-------------|-------------|-------------|
| \(\beta(t) = 0.8 + 0.1 \sin\left(\frac{t}{250}\right)\) | 0.7 | 0.9 | 2.3 | 0.32 | ≥ 0.36 | 11.94 |
| \(\gamma(t) = 2 + 0.1 \cos\left(\frac{t}{300}\right)\) | 1.9 | 2.1 | 0.34 | 0.1296 | < 0.1418 | 12.06 |
| \(\varepsilon(t) = 2.5 + 0.2 \sin(t)\) | 2.3 | 2.7 | 0.34 | 0.1296 | < 0.1418 | 1.9 |
| \(\sigma(t) = 0.34 + 0.01 (\sin\left(\frac{t}{75}\right) + \cos(t)\) | ≥ 0.32 | ≤ 0.36 | 2 |
| \(\Lambda(t) = 12 + 0.06 \sin(t)\) | 11.94 | 12.06 | 12 |
| \(\mu(t) = 1.94 + 0.04 \cos(t)\) | 1.9 | 1.98 | 1 |

It is important to note that since the initial condition is unchanging this forces two parameters, namely \(\Lambda(t)\) and \(\mu(t)\), to remain unchange for these simulation purposes. Moreover, we have that

\[
\Gamma = \left\{(x, y, z) \in \mathbb{R}^3_+ : x + y + z \leq \frac{\Lambda}{\mu} = 6.3474\right\}
\]

as the invariant set for the system (3.3). That is, this system has a unique strong solution taking values in \(\Gamma\) per Theorem 3.1. Given these parameters and following Example 3.3, we have

\[
\tilde{R}_0 = \frac{\beta \Lambda}{\mu(\mu + \gamma + \varepsilon)} - \frac{\sigma^2 \Lambda^2}{2\mu^2(\mu + \gamma + \varepsilon)} \leq 0.5983 < 1 \text{ and } \sigma^2 < 0.1296 < 0.1418 = \frac{\mu\beta}{\Lambda}.
\]

As demonstrated below in Figure 3, the disease will go extinct with exponential rate

\[-\alpha \geq (\mu + \gamma + \varepsilon) \left(1 - \tilde{R}_0\right) \geq 2.45.\]
We now make only the alteration of a single parameter in the system (3.3). Assume that $\sigma(t)$ has the following form:

\[
\sigma(t) = 1.34 + 0.01(\sin(t) + \cos(t))
\]

This alteration yields

\[
\sigma^2 \geq 1.7424 > 0.1418 = \max \left\{ \frac{\mu \beta}{\Lambda}, \frac{\beta^2}{2(\mu + \gamma + \varepsilon)} \right\}.
\]

Thus, we have the scenario in which the disease goes extinct with exponential rate

\[
-\alpha \geq (\mu + \gamma + \varepsilon) - \frac{\beta^2}{2\sigma^2} \geq 5.87.
\]

Moreover, if we compare Figure 4 to the above Figure 3 we notice the disease appears to go extinct at a faster rate which is as expected given the above results.

![Figure 4: E-M scheme simulation 2 of system (3.3) illustrating disease extinction.](image)

Now assume that the parameters for the system (3.3) have been modified such that

\[
\beta(t) = 3.8 + 0.1 \sin(t) \quad \gamma(t) = 1 + 0.1 \cos(t) \quad \sigma(t) = 0.76 + 0.01(\sin(t) + \cos(t))
\]

This modification yields

\[
\tilde{R}_0 = \frac{\beta \Lambda}{\mu(\mu + \gamma + \varepsilon)} - \frac{\sigma^2 \Lambda^2}{2\mu^2(\mu + \gamma + \varepsilon)} \geq 1.73 > 1.
\]
In Figure 5 below, we see such a modification yields disease persistence as opposed to disease extinction achieved in the previous two simulations for the system (3.3).

![Figure 5: E-M scheme simulation 3 of system (3.3) illustrating disease persistence.](image)

## 5 Conclusion and future work

In this paper, we propose and investigate the unified stochastic SIR model given by the system (1.3). We have presented two forms of the model—one for population proportions and the other for population numbers. For both forms of the model, we have given results on the extinction and persistence of diseases; moreover, we have shown that these results still hold with time-dependent, nonlinear parameters and multiple noise sources. Notably, we give examples and simulations that agree with the theoretical results and illustrate the impact that noise has on a given SIR model system.

In a further investigation of this work, we plan to introduce jumps and periodicity into the system (1.3) and obtain results on the extinction and persistence of diseases. This investigation will begin immediately with the concluding of this paper.

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