Semilinear Dirichlet problem for subordinate spectral Laplacian

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Abstract

We study semilinear problems in bounded $C^{1,1}$ domains for non-local operators with a boundary condition. The operators cover and extend the case of the spectral fractional Laplacian. We also study harmonic functions with respect to the non-local operator and boundary behaviour of Green and Poisson potentials.

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1 Introduction

Let $D \subset \mathbb{R}^d$, $d \geq 2$, be a bounded $C^{1,1}$ domain, $f : D \times \mathbb{R} \to \mathbb{R}$ a function, and $\zeta$ a signed measure on $\partial D$. In this article we study the semilinear problem

$$\phi(-\Delta|_D)u(x) = f(x,u(x)) \quad \text{in } D,$$
$$\frac{u}{P_{D,\sigma}} = \zeta \quad \text{on } \partial D,$$

(1.1)

where $\phi : (0, \infty) \to (0, \infty)$ is a complete Bernstein function without drift satisfying a certain weak scaling conditions. The boundary condition will be described below whereas the operator $\phi(-\Delta|_D)$ can be written in its spectral form as well as a principal value integral:

$$\phi(-\Delta|_D)u(x) = \sum_{j=1}^{\infty} \phi(\lambda_j) \hat{u}_j \varphi_j = \text{P.V.} \int_D (u(x) - u(y)) J_D(x,y) \, dy + \kappa(x)u(x), \quad x \in D.$$

Here $(\lambda_j, \varphi_j)_{j \in \mathbb{N}}$ are eigenpairs of the Dirichlet Laplacian in $D$, and the singular kernel $J_D$ as well as the function $\kappa$ are completely determined by the function $\phi$. This said, $\phi(-\Delta|_D)$ is a non-local operator of elliptic type which in the case $\phi(\lambda) = \lambda^{\alpha/2}$, $\alpha \in (0, 2)$, is the spectral fractional Laplacian $(-\Delta|_D)^{\alpha/2}$. The operator $-\phi(-\Delta|_D)$ can be also viewed as the infinitesimal generator of the subordinate killed Brownian motion, where the subordinator has $\phi$ as its Laplace exponent.
The notion of the boundary condition is a bit abstract but, at this point, let us say that it can be understood as a limit at the boundary of \( u/P_\partial^\phi \sigma \) in the pointwise sense, or in the weak sense of (4.7), depending on the smoothness of the boundary datum, where \( P_\partial^\phi \sigma \) is a reference function defined as the Poisson potential of the \( d-1 \) dimensional Hausdorff measure on \( \partial D \).

Motivated by the recent articles [3, 8], see also the preprint [4], we consider solutions of (1.1) in the weak dual sense, see Definition 5.1, and we prove that the solutions have a special form of a sum of the Green and the Poisson potential, see Theorem 4.3.

Semilinear problems for the Laplacian have been studied for a long time now. In the monograph [32] it is said that this study is at least 50 years old now. However, the study of semilinear problems for non-local operators is quite recent and mostly oriented to the problems driven by the fractional Laplacian, see e.g. [22, 14, 1, 2, 5, 6, 7, 11, 21]. For more general operators than the fractional Laplacian see e.g. [24, 26] for linear problems, and [8] for semilinear problems. However, there are just a few articles discussing the semilinear Dirichlet problem for the spectral fractional Laplacian, see [19, 3]. To the best of our knowledge, this article is the first one to study semilinear problems for spectral-type operators more general than the spectral fractional Laplacian.

A typical difference between the local and the non-local setting is that in the non-local setting even solutions of the linear Dirichlet problem can explode at the boundary whereas in the local setting this does not happen. To be more precise, there exists a harmonic function with respect to \( \phi(-\Delta|_D) \) which explodes at the boundary, e.g. the reference function \( P_\partial^\phi \sigma \) is such one for which we prove the explosion rate, see (3.1).

The main goal of this article is to generalize results from [3] where the semilinear problem was studied for the spectral fractional Laplacian, and to generalize results from [8] to a slightly different type of a non-local operator in the special case of \( C^{1,1} \) bounded domain. To achieve this goal, we intensively use the potential-theoretic and analytic properties of the killed Brownian motion subordinated by a subordinator with the Laplace exponent \( \phi \), the process that gives \( \phi(-\Delta|_D) \) as its infinitesimal generator as it is shown in the article. Some of these properties are well known for a long time and belong to the general potential theory. However, some properties are pretty recently proved such as the sharp bounds for the potential kernel and the jumping kernel, the (boundary) Harnack principle, etc., see [30, 31].

Let us now describe the central results of the article which are given in Section 5. For the nonlinearity \( f \) in (1.1) in our results we assume that

\[
\text{(F). } f : D \times \mathbb{R} \to \mathbb{R} \text{ is continuous in the second variable, and there exist a locally bounded function } \rho : D \to [0, \infty] \text{ and a non-decreasing function } \Lambda : [0, \infty) \to [0, \infty) \text{ such that } |f(x, t)| \leq \rho(x)\Lambda(|t|), \ x \in D, \ t \in \mathbb{R}.
\]

First, in Proposition 5.4, we prove Kato’s inequality for \( \phi(-\Delta|_D) \) using which we develop a method of sub- and supersolution for \( \phi(-\Delta|_D) \) in Theorem 5.9. This theorem directly generalizes [3, Theorem 32] to our setting of more general non-local operators and also extends [8, Theorem 3.6] to slightly different non-local operators. In Theorem 5.10 we prove the existence of a solution when the nonlinearity \( f \) is non-positive and when the boundary measure \( \zeta \) is non-negative. This theorem comes as a generalization of [3, Theorem 8] to our setting of more general non-local operators. Moreover, we consider a more general boundary condition which can also be a measure whereas in [3, Theorem
only continuous functions where considered. The nonlinearity in our theorem is also slightly more general than the one in [3, Theorem 8]. A similar result in a different non-local setting can be found in [8, Theorem 3.10]. By the method of monotone iterations, in Theorem 5.14 we find a solution to the semilinear problem when both \( f \) and \( \zeta \) are non-negative. Finally, for a signed \( f \) and a signed \( \zeta \), in Theorem 5.16 we find a solution by the technique used in [11, Theorem 2.4]. After each theorem, we give a comment on the existence (and non-existence) of a solution in the spectral fractional Laplacian case for the power-like nonlinearity \( f \), see Remarks 5.13, 5.15 and 5.17.

Let us now give a short summary of the rest of the article. In Section 2 we introduce assumptions on \( \phi \) and recall the known results on the Green kernel. We connect the operator \( \phi(-\Delta|_D) \) to the subordinate killed Brownian motion as its infinitesimal generator, give a pointwise characterization of \( \phi(-\Delta|_D) \), and study the regularity of the Green potentials. The last part of the section deals with Poisson potentials and harmonic functions. In Proposition 2.19 we prove the existence of the Poisson kernel as a normal derivative of the Green function and in Theorem 2.23 we prove an integral representation formula for non-negative harmonic functions for \( \phi(-\Delta|_D) \). We finish the section by proving that harmonic functions are continuous, and by Theorem 2.27 in which we show that non-negative harmonic functions are those which satisfy the mean-value property with respect to the subordinate killed Brownian motion.

Section 3 deals with the boundary behaviour of potential integrals. Here we emphasize Theorem 3.6 which gives the boundary behaviour of the Green potentials. This theorem generalizes [19, Proposition 7] to our setting of more general non-local operators and more general functions. Furthermore, this theorem with Proposition 3.4 shows that in some cases the boundary condition (1.1) can be understood as a limit at the boundary in the pointwise sense. Finally, the section also contains Proposition 3.5 and Proposition 3.7 which show that the boundary condition in (1.1) can be viewed as a limit at the boundary in the weak sense.

Section 4 contains the basic properties of the linear Dirichlet problem where we prove that every weak solution of the Dirichlet problem is a sum of the Green and the Poisson potential, see Theorem 4.3.

Section 5 contains already described main results.

The article also contains the Appendix where we first provide a proof of the Green function sharp estimate in our setting, see Lemma A.1, modelled upon [30, Theorem 3.1]. We also give a technical proof of Theorem 3.6 modelled upon the proof of [8, Proposition 4.1], as well as prove Lemma A.5 which is an additional and a bit lengthy calculation providing an interpretation of the boundary condition. In Subsection A.4 we prove that the heat kernel of a killed Brownian motion upon exiting a \( C^{1,\alpha} \) domain is differentiable up to the boundary - a fact that appears to be known but for which we could not find an exact reference.

**Notation.** For an open set \( D \subset \mathbb{R}^d \): \( C(D) \) denotes the set of all continuous functions on \( D \), \( C^k(D) \) denotes \( k \)-times \((k \geq 1)\) continuously differentiable functions on \( D \), \( C^\infty(D) \) infinitely differentiable functions on \( D \), and \( C^\infty_c(D) \) infinitely differentiable functions with compact support on \( D \), where e.g. we write \( C^k(\overline{D}) \) for the set of functions in \( C^k(D) \) whose all derivatives of order less than \( k \) have a continuous extension to \( \overline{D} \). For \( \alpha \in (0, 1] \), by \( C^{1,\alpha}(\overline{D}) \) \((C^{1,\alpha}(D))\) we denote functions in \( C^1(\overline{D}) \) \((C^1(D))\) whose first partial derivatives
are uniformly Hölder continuous (locally Hölder continuous) in \( D \) with exponent \( \alpha \).

Further, \( L^1(D, \mu) \) is the set of all integrable functions on \( D \), and \( L^1_{\text{loc}}(D, \mu) \) the set of all locally integrable functions on \( D \), with respect to the measure \( \mu \) on \( D \). If \( \mu \) is the standard Lebesgue measure, we write \( L^1(D) \) and \( L^1_{\text{loc}}(D) \). The set \( L^2(D) \) denotes square-integrable functions with respect to the Lebesgue measure. The set \( H^1_0(D) \) denotes the closure of \( C^\infty_0(D) \) with respect to the Sobolev norm in the Sobolev space \( W^{1,2}(D) \) - the set of \( L^2(D) \) functions whose weak partial derivatives belong to \( L^2(D) \). The set \( B(\mathbb{R}^d) \) denotes Borel sets in \( \mathbb{R}^d \). The set \( M(\partial D) \) and \( M(D) \) denote Radon (signed) measures on \( \partial D \) and \( D \), respectively. We assume that all functions in the article are Borel functions, and all (signed) measures are Borel measures. Furthermore, in what follows when we say \( \nu \) is a measure, we mean that \( \nu \) is a non-negative measure on \( \mathbb{R}^d \). By \( |\nu| \) we denote the total variation of a signed measure \( \nu \), and the Dirac measure of a point \( x \in \mathbb{R}^d \) is denoted by \( \delta_x \).

The boundary of the set \( D \) is denoted by \( \partial D \). Notation \( U \subset D \) means that \( U \) is a nonempty bounded open set such that \( U \subset \overline{U} \subset D \) where \( \overline{U} \) denotes the closure of \( U \). By \( |x| \) we denote the Euclidean norm of \( x \in \mathbb{R}^d \) and \( B(x, r) \) denotes the ball around \( x \in \mathbb{R}^d \) with radius \( r > 0 \). For \( A, B \subset \mathbb{R}^d \) let \( \text{dist}(A, B) = \inf\{|x - y| : x \in A, y \in B\} \) and \( \text{diam}D = \sup\{|x - y| : x, y \in D\} \). Unimportant constants in the article will be denoted by small letters \( c, c_1, c_2, \ldots \), and their labeling starts anew in each new statement. By a big letter \( C \) we denote some more important constants, where e.g. \( C(a, b) \) means that the constant \( C \) depends only on parameters \( a \) and \( b \). All constants are positive finite numbers. For two positive functions \( f \) and \( g \) we write \( f \asymp g \) (\( f \lesssim g, f \gtrsim g \)) if there exist a finite positive constant \( c \) such that \( c^{-1}f \leq g \leq cf \) (\( f \leq cg, f \gtrsim cg \)). Finally, \( a \wedge b = \min\{a, b\} \) and \( a \vee b = \max\{a, b\} \).

## 2 Preliminaries

Let \( (W_t)_{t \geq 0} \) be a Brownian motion in \( \mathbb{R}^d \), \( d \geq 2 \), with the characteristic exponent \( \xi \mapsto |\xi|^2 \), \( \xi \in \mathbb{R}^d \). Let \( D \) be a non-empty open set, and \( \tau_D := \inf\{t > 0 : W_t \notin D\} \) the first exit time from the set \( D \). We define the killed process \( W^D \) upon exiting the set \( D \) by

\[
W^D_t := \begin{cases} 
W_t, & t < \tau_D, \\
\partial, & t \geq \tau_D,
\end{cases}
\]

where \( \partial \) is an additional point added to \( \mathbb{R}^d \) called the cemetery.

Let \( S \) be a subordinator independent of \( W \), i.e. \( S \) is an increasing Lévy process such that \( S_0 = 0 \), with the Laplace exponent

\[
\lambda \mapsto \phi(\lambda) = b\lambda + \int_0^\infty (1 - e^{-\lambda t})\mu(dt), \tag{2.1}
\]

where \( b \geq 0 \) and the measure \( \mu \) satisfies \( \int_0^\infty (1 + t)\mu(dt) < \infty \). The measure \( \mu \) is called the Lévy measure and \( b \) the drift of the subordinator.

The process \( X = ((X_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in D}) \) defined by \( X_t := W^D_{S_t} \) is called the subordinate killed Brownian motion. Here \( \mathbb{P}_x \) denotes the probability under which the process \( X \) starts from \( x \in D \), and by \( \mathbb{E}_x \) we denote the corresponding expectation.
2.1 Assumptions

The first assumption that we impose throughout the article concerns the set $D$. Although some results will be valid for general open sets, we always assume that $D$ is a bounded $C^{1,1}$ domain.

The second assumption concerns the Laplace exponent $\phi$, i.e. the subordinator $S$. A function of the form (2.1) is called a Bernstein function, see [37, Theorem 3.2], and such functions characterize subordinators, see [37, Chapter 5].

We impose the following assumption on $\phi$ throughout the article. (WSC). The function $\phi$ is a complete Bernstein function, i.e. the Lévy measure $\mu(dt)$ has a completely monotone density $\mu(t)$, and $\phi$ satisfies the following weak scaling condition at infinity: There exist $a_1, a_2 > 0$ and $\delta_1, \delta_2 \in (0, 1)$ satisfying

$$a_1 \lambda^{\delta_1} \phi(t) \leq \phi(\lambda t) \leq a_2 \lambda^{\delta_2} \phi(t), \quad t \geq 1, \lambda \geq 1. \quad (2.2)$$

The best-known subordinator with the property (WSC) is the $\alpha$-stable subordinator where $\phi(\lambda) = \lambda^{\alpha/2}$, for some $\alpha \in (0, 2)$, which satisfies exact (and even global) scaling condition (2.2). However, there are many other interesting subordinators that fall into our setting. For a short list of these, see e.g. [29, p. 3].

Allow us to give some comments on the assumptions above. Since we assume (WSC), the function $\phi^*(\lambda) := \frac{\lambda}{\phi(\lambda)}$ is a complete Bernstein function, too, see [37, Proposition 7.1], and $\phi^*$ is called the conjugate Bernstein function of $\phi$. We easily see that (2.2) also holds for $\phi^*$ but with different constants $a_1$, $a_2$, and $\delta_1$ and $\delta_2$, thus (WSC) also holds for $\phi^*$. By $\nu(dt) = \nu(t)dt$ we denote the Lévy measure of $\phi^*$.

In what follows we discuss properties of $\phi$ and the same will hold for $\phi^*$ or, to be more precise, for the counterparts of the function $\phi^*$. By a simple calculation, the scaling condition (2.2) implies that $b = 0$ in (2.1) and the well-known bound

$$\phi'(\lambda) \asymp \frac{\phi(\lambda)}{\lambda}, \quad \lambda \geq 1, \quad (2.3)$$

where, in fact, the upper bound holds for every Bernstein function and every $\lambda > 0$, and the lower bounds follows from (2.2). We use (2.3) many times throughout the article. The Lévy measure $\mu(dt)$ is infinite, see [37, p. 160], and the density $\mu(t)$ cannot decrease too fast, i.e. there is $c = c(\phi) > 1$ such that

$$\mu(t) \leq c\mu(t + 1), \quad t \geq 1,$$

see [28, Lemma 2.1]. Moreover, it holds that

$$\mu(t) \leq (1 - 2e^{-1})^{-1} \frac{\phi'(t-1)}{t^2}, \quad t > 0, \quad \text{and} \quad \mu(t) \geq c \frac{\phi(t-1)}{t^2}, \quad 0 < t \leq M, \quad (2.4)$$

for $M > 0$ and $c = c(\phi, M) > 0$, see [30, Eq. (2.13)] and [27, Proposition 3.3].

The potential measure $U$ of the subordinator $S$, defined by $U(A) := \int_0^\infty P(S_t \in A)dt$, $A \in B(\mathbb{R})$, has a decreasing density $u$ which satisfies $\int_0^1 u(t)dt < \infty$, see [37, Theorem 11.3]. In addition, it holds that

$$u(t) \leq (1 - 2e^{-1})^{-1} \frac{\phi'(t-1)}{t^2 \phi(t-1)^2}, \quad t > 0, \quad \text{and} \quad u(t) \geq c \frac{\phi'(t-1)}{t^2 \phi(t-1)^2}, \quad 0 < t \leq M, \quad (2.5)$$

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for $M > 0$ and $c = c(\phi, M) > 0$, see [30, Eq. (2.11)] and [27, Proposition 3.4]. The potential density of $\phi^*$ will be denoted by $v$.

It is worth noting that for a general Bernstein function a version of a global scaling condition holds
\[
1 \wedge \lambda \leq \frac{\phi(\lambda t)}{\phi(t)} \leq 1 \vee \lambda, \quad \lambda > 0, t > 0,
\] which we get directly from (2.1).

In [30, 31] important aspects of the potential theory of the process $X$ were developed such as the scale invariant Harnack principle and the boundary Harnack principle. Our assumption (WSC) implies (A1)-(A4) but not (A5) from [30, 31] so each time we use a result from [30, 31] we will explain how the assumption (A5) can be avoided.

In the article the case $d = 1$ is excluded since it would require a somewhat different potential theoretic methods.

### 2.2 Green function

Let us denote the transition density of the Brownian motion $W$ by
\[
p(t, x, y) = (4\pi t)^{-d/2}e^{-\frac{|x-y|^2}{4t}}, \quad x, y \in \mathbb{R}^d, \ t > 0.
\] Then the transition density of the killed Brownian motion $W^D$ is given by
\[
p_D(t, x, y) = p(t, x, y) - \mathbb{E}_x[p(t - \tau_D, W_{\tau_D}, y)\mathbf{1}_{\{\tau_D < t\}}], \quad x, y \in \mathbb{R}^d.
\] It is well known that $p_D(t, \cdot, \cdot)$ is symmetric and it seems that it is known that $p_D(\cdot, \cdot, \cdot) \in C^1((0, \infty) \times D \times D)$ since $D$ is a $C^{1,1}$ open domain. However, as we were unable to find an exact reference for the regularity up to the boundary of the transition density, we prove it in the Appendix in Lemma A.7. Furthermore, the following heat kernel estimate holds: There exist constants $T_0 = T_0(D) > 0$, $c_1 = c_1(T_0, D) > 0$, $c_2 = c_2(T_0, D) > 0$, $c_3 = c_3(D) > 0$, and $c_4 = c_4(D) > 0$ such that for all $x, y \in D$ and $t \in (0, T_0]$ it holds that
\[
\left[\frac{\delta_D(x)\delta_D(y)}{t} \wedge 1\right] \frac{1}{c_1t^{d/2}}e^{-\frac{|x-y|^2}{4t}} \leq p_D(t, x, y) \leq \left[\frac{\delta_D(x)\delta_D(y)}{t} \wedge 1\right] \frac{c_3}{t^{d/2}}e^{-\frac{c_4|x-y|^2}{t}}.
\] We note that the right hand side inequality in (2.9) holds for every $t > 0$. For the proofs see [38, Theorem 3.1 & Theorem 3.8], cf. [42, Theorem 1.1] and [18, Theorem 4.6.9]. Moreover, in [38, Remark 3.3] one can find an explanation why the lower bound in (2.9) cannot hold uniformly for every $t > 0$.

The semigroup $(P_t^D)_{t \geq 0}$ of the process $W^D$ is given by
\[
P_t^D f(x) = \int_D p_D(t, x, y)f(y)dy = \mathbb{E}_x[f(W_t); t < \tau_D] = \mathbb{E}_x[f(W_{\tau_D}^D)], \quad f \in L^\infty(D),
\] where $f(\partial) = 0$ for all Borel functions on $D$ by convention. It is well known that the semigroup $(P_t^D)_{t \geq 0}$ is strongly Feller since $D$ is $C^{1,1}$, i.e. $P_t^D(L^\infty) \subset C_D(D)$, and can be uniquely extended to a $L^2(D)$ semigroup. For details see e.g. [16, Chapter 2].
The potential kernel of $W^D$ (or the Green function of $W^D$) is defined as
\[ G_D(x,y) = \int_0^\infty p_D(t,x,y)dt, \quad x,y \in \mathbb{R}^d. \]

The kernel $G_D$ is symmetric, non-negative, finite off the diagonal and jointly continuous in the extended sense, see [16, Theorem 2.6], and it is the density of the mean occupation time for $W^D$, i.e. for $f \geq 0$ we have
\[ \int_D G_D(x,y)f(y)dy = \mathbb{E}_x\left[ \int_0^\infty f(W^D_t)dt \right], \quad x \in D. \]

Since $X$ is obtained by subordinating the killed Brownian motion $W^D$, it is well known that the $L^2(D)$ transition semigroup of $X$, denoted by $(Q^D_t)_t$, is given for $t > 0$ by
\[ Q^D_t f = \int_0^\infty P^D_s f \mathbb{P}(S_t \in ds), \quad f \in L^2(D), \]
see [37, Proposition 13.1]. Thus, $Q^D_t$ admits the density
\[ q_D(t,x,y) = \int_0^\infty p_D(s,x,y) \mathbb{P}(S_t \in ds). \]

The semigroup $(Q^D_t)_t$ is also strongly Feller since $(P^D_t)_t$ is, see [9, Proposition V.3.3]. The process $X$ has the potential kernel (i.e. the Green function of $X$) which is given by
\[ G^\phi_D(x,y) = \int_0^\infty q_D(t,x,y)dt = \int_0^\infty p_D(t,x,y)\mathbb{P}(S_t \in ds), \quad x,y \in \mathbb{R}^d. \]  

(2.11)

The kernel $G^\phi_D$ is symmetric, non-negative, and by the bound (2.9) finite off the diagonal. Moreover, $G^\phi_D$ is the density of the mean occupation time for $X$, i.e. for $f \geq 0$ we have
\[ \int_D G^\phi_D(x,y)f(y)dy = \mathbb{E}_x\left[ \int_0^\infty f(X_t)dt \right], \quad x \in D. \]

The closed form of $G^\phi_D$ is not known, but in [30, Theorem 3.1] the sharp estimate was obtained, i.e. we have
\[ G^\phi_D(x,y) \asymp \left( \frac{\delta_D(x)\delta_D(y)}{|x-y|^2} \wedge 1 \right) \frac{1}{|x-y|^d \phi(|x-y|^2)}, \quad x,y \in D, \]

(2.12)

where the constant of comparability depends only on $d$, $D$ and $\phi$. We note that the usage of the transience assumption (A5) from [30] in [30, Theorem 3.1] can be avoided, see Lemma A.1 in the Appendix for the details. Further, by using the upper bound of (2.9) and the bounds (2.12), we can repeat the proof of [30, Proposition 3.3] to get that $G^\phi_D$ is infinite on the diagonal and jointly continuous in the extended sense in $D \times D$.

By the characterization of Bernstein functions the conjugate Bernstein function $\phi^*$ generates a subordinator $(T_t)_{t \geq 0}$, see [37, Chapter 5]. From the previous subsection it follows that $(T_t)_{t \geq 0}$ has a potential measure which also has the decreasing density which...
we denote by $V(dt) = v(t)dt$, see [37, Theorem 11.3 & Corollary 11.8]. We define the potential kernel generated by $\varphi^*$ with
\[
G_D^\varphi(x,y) = \int_0^\infty p_D(t,x,y)v(t)dt, \quad x,y \in \mathbb{R}^d.
\] (2.13)

Since $\varphi^*$ satisfies (WSC), $G_D^\varphi$ is also symmetric, finite off the diagonal, jointly continuous in extended sense $D \times D$ and satisfies the sharp bound (2.12) where $\phi$ is replaced by $\varphi^*$. Of course, the kernel $G_D^\varphi$ can be viewed as the potential kernel of the process $(W^D_t)_{t \geq 0}$.

The kernels $G_D$, $G_D^\phi$ and $G_D^{\varphi^*}$ are also connected by the following well-known factorization.

Lemma 2.1. For $x, y \in D$ it holds that
\[
\int_D G_D^\phi(x,\xi)G_D^{\varphi^*}(\xi,y)d\xi = G_D(x,y).
\] (2.14)

Proof. The claim follows from [37, Proposition 14.2(ii)] where we set $\gamma = \delta_y$. \hfill \Box

2.3 Operator $\phi(-\Delta|_D)$

Let $\{\varphi_j\}_{j \in \mathbb{N}}$ be a Hilbert basis of $L^2(D)$ consisting of eigenfunctions of the Dirichlet Laplacian $-\Delta|_D$, associated to the eigenvalues $\lambda_j$, $j \in \mathbb{N}$, i.e. $\varphi_j \in H^1_0(D) \cap C^\infty(D) \cap C^{1,1}(\overline{D})$ and
\[
-\Delta|_D \varphi_j = \lambda_j \varphi_j, \quad \text{in } D,
\] (2.15)

see [12, Theorem 9.31] and [23, Section 8.11]. Here (2.15) can be viewed in various equivalent ways, e.g. as a distributional or a pointwise relation. Also, $\Delta|_D$ in (2.15) can be viewed as the $L^2(D)$-infinitesimal generator of the semigroup $(P^D_t)_t$, i.e.
\[
\Delta|_D u = \lim_{t \to 0} \frac{P^D_t u - u}{t}, \quad u \in \mathcal{D}(\Delta|_D),
\]

where $\mathcal{D}(\Delta|_D)$ is the domain of the generator $\Delta|_D$ and the limit is taken with respect to $L^2(D)$ norm. We note that $\mathcal{D}(\Delta|_D)$ is a class of functions $f \in H^1_0(D)$ such that $\Delta f$ exists in the weak distributional sense and belongs to $L^2(D)$, see [16, Theorem 2.13]. For more details, see [16, Chapter 2] and [12, Chapter 9]. Note that since $\varphi_j$ is an eigenfunction, we have
\[
P^D_t \varphi_j = e^{-\lambda_j t} \varphi_j,
\] (2.16)

see [36, Lemma 7.10]. Further, since we assume that $D$ is $C^{1,1}$, it is well known that $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$, and by the Weyl’s asymptotic law we have
\[
\lambda_j \asymp j^{2/d}, \quad j \in \mathbb{N}.
\] (2.17)

Also, we choose the basis $\{\varphi_j\}_{j \in \mathbb{N}}$ such that $\varphi_1 > 0$ in $D$, see [12, Chapter 9]. Hence, another very important sharp estimate for $\varphi_1$ holds:
\[
\varphi_1(x) \asymp \delta_D(x), \quad x \in D.
\] (2.18)
The interior estimate is trivial since \( \varphi_1 \) is smooth and positive. The boundary bound follows from Hopf’s lemma, see e.g. [20, Hopf’s lemma in Section 6.4.2].

Consider the Hilbert space

\[
H_D(\phi) := \left\{ v = \sum_{j=1}^\infty \hat{u}_j \varphi_j \in L^2(D) : \|v\|_{H_D(\phi)}^2 := \sum_{j=0}^\infty \phi(\lambda_j)^2 |\hat{u}_j|^2 < \infty \right\}.
\]

The spectral operator \( \phi(-\Delta|_D) : H_D(\phi) \rightarrow L^2(D) \) is defined as

\[
\phi(-\Delta|_D)u = \sum_{j=1}^\infty \phi(\lambda_j) \hat{u}_j \varphi_j, \quad u \in H_D(\phi). \tag{2.19}
\]

Note that \( H_D(\phi) \hookrightarrow L^2(D) \) and we will show in the next proposition that \( C_{C^\infty(D)}^\infty(\varphi) \subset H_D(\phi) \), see (2.26). Now it is obvious that \( \phi(-\Delta|_D) \) is an unbounded operator, densely defined in \( L^2(D) \) and has the bounded inverse \([\phi(-\Delta|_D)]^{-1} : L^2(D) \rightarrow H_D(\phi) \) given by

\[
[\phi(-\Delta|_D)]^{-1}u = \sum_{j=1}^\infty \frac{1}{\phi(\lambda_j)} \hat{u}_j \varphi_j, \quad u \in L^2(D). \tag{2.20}
\]

In the next proposition we prove that a potential relative to \( G_D^\phi \) is the inverse of \( \phi(-\Delta|_D) \). The proof is similar to [3, Lemma 9] but we give the complete proof for the reader’s convenience since some elements of the proof will be important in what follows.

**Proposition 2.2.** Let \( f \in L^2(D) \). For a.e. \( x \in D \) it holds that \( G_D^\phi(x,\cdot)f(\cdot) \in L^1(D) \) and

\[
[\phi(-\Delta|_D)]^{-1}f(x) = \int_D G_D^\phi(x,y)f(y)dy. \tag{2.21}
\]

**Proof.** First we prove (2.21) for \( f = \varphi_1 \geq 0 \). Fubini’s theorem yields

\[
\int_D G_D^\phi(x,y)\varphi_1(y)dy = \int_0^\infty u(t) \int_D p_D(t,x,y)\varphi_1(y)dydt = \int_0^\infty e^{-\lambda_1t}\varphi_1(x)u(t)dt = \frac{1}{\phi(\lambda_1)} \varphi_1(x) \quad \text{for a.e. } x \in D, \tag{2.22}
\]

where in the second equality we used (2.16), in the third [37, Eq. (5.20)], and in the last equality (2.20). By the elliptic regularity there exist constants \( C = C(d,D) \) and \( k = k(d) \) such that \( \|\nabla \varphi_j\|_{L^\infty(D)} \leq (C\lambda_j)^k \|\varphi_j\|_{L^2(D)} = (C\lambda_j)^k \), see (A.35). Recall that \( \varphi_j \in C^{1,1}(\overline{D}) \) and that \( \varphi_j \) vanishes on the boundary so the mean value theorem implies

\[
\frac{\varphi_j}{\delta_D} \leq (C\lambda_j)^k. \tag{2.23}
\]

Since \( \varphi_1 \propto \delta_D \), by the previous inequality, Fubini’s theorem, and the same calculations as in (2.22), we have that (2.21) holds for every \( \varphi_j, j \in \mathbb{N} \). By linearity the same is true for the linear span of \( \{\varphi_j : j \in \mathbb{N}\} \).
Let
\[ \mathcal{G} f(x) := \int_D G_D^\phi(x, y) f(y) dy, \quad (2.24) \]
for \( f \in L^2(D) \) and \( x \in D \) such that the integral exists. In what was proved, \( \mathcal{G} f(x) \) is well defined for every \( f \in \text{span}\{\varphi_j : j \in \mathbb{N}\} \) and a.e. \( x \in D \). Moreover, from \( \mathcal{G}\varphi_j = \frac{1}{\phi(\lambda_j)} \varphi_j = [\phi(-\Delta|_D)]^{-1} \varphi_j \) it follows that for \( f \in \text{span}\{\varphi_j : j \in \mathbb{N}\} \) we have
\[ \|\mathcal{G} f\|_{H_D(\phi)}^2 = \|f\|_{L^2(D)}^2. \quad (2.25) \]

Hence, the map \( f \mapsto \mathcal{G} f \) uniquely extends to a linear isometry from \( L^2(D) \) to \( H_D(\phi) \) which coincides with \( [\phi(-\Delta|_D)]^{-1} \). Further, a consequence of (2.22) is that \( G_D^\phi(x, \cdot) \in L^1(D) \) for a.e. \( x \in D \) since by Fubini’s theorem
\[ \int_D \left( \int_D G_D^\phi(x, y) dy \right) \varphi_1(x) dx = \frac{1}{\phi(\lambda_1)} \int_D \varphi_1(y) dy < \infty. \]

Next we prove that (2.21) holds a.e. in \( D \) for \( f = \psi = \sum_{j=1}^{\infty} \hat{\psi}_j \varphi_j \in C_c^\infty(D) \). Take the approximating sequence \( f_n = \sum_{j=1}^{n} \hat{\psi}_j \varphi_j, n \in \mathbb{N} \), and note that \( \mathcal{G} f_n = [\phi(-\Delta|_D)]^{-1} f_n \to [\phi(-\Delta|_D)]^{-1} f = \mathcal{G} f \) in \( L^2(D) \) since \( f_n \to f \) in \( L^2(D) \). Moreover, by integrating by parts \( m \in \mathbb{N} \) times we get
\[ \hat{\psi}_j = \int_D \psi(x) \varphi_j(x) dx = \frac{(-1)^m}{\lambda_j^m} \int_D \Delta^m \psi(x) \varphi_j(x) dx, \]
which implies
\[ \|\hat{\psi}_j\| \leq \frac{\|\Delta^m \psi\|_{L^2(D)}}{\lambda_j^m} =: C(m, \psi) \frac{1}{\lambda_j^m}. \quad (2.26) \]

Hence, by using (2.17), (2.23), and (2.26) for large enough \( m \in \mathbb{N} \), it follows that \( f_n \) converges uniformly in \( D \) to \( f = \psi \). This implies that \( \mathcal{G} f_n = \int_D G_D^\phi(\cdot, y) f_n(y) dy \to \int_D G_D^\phi(\cdot, y) f(y) dy \) a.e. in \( D \) since \( G_D^\phi(x, \cdot) \in L^1(D) \) for a.e. \( x \in D \). Thus, by uniqueness of the limit \( \mathcal{G} f = [\phi(-\Delta|_D)]^{-1} f = \int_D G_D^\phi(\cdot, y) f(y) dy \) a.e. in \( D \).

Take now \( f \in L^2(D) \), and let \( (f_n)_n \subset C_c^\infty(D) \) which converges to \( f \) in \( L^2(D) \). Hence, \( \mathcal{G} f_n = [\phi(-\Delta|_D)]^{-1} f_n \to [\phi(-\Delta|_D)]^{-1} f = \mathcal{G} f \) in \( L^2(D) \). On the other hand,
\[ \int_D \left| \int_D G_D^\phi(x, y)(f_n(y) - f(y)) dy \right| \varphi_1(x) dx \leq \frac{1}{\phi(\lambda_1)} \int_D \varphi_1(y) |f_n(y) - f(y)| dy \leq \frac{1}{\phi(\lambda_1)} \|f_n - f\|_{L^2(D)} \to 0, \]
which shows that \( G_D^\phi(\cdot, y) f(y) \in L^1(D) \) a.e. in \( D \) and by taking the subsequence we get
\[ \mathcal{G} f_n = \int_D G_D^\phi(\cdot, y) f_n(y) dy \to \int_D G_D^\phi(\cdot, y) f(y) dy \quad \text{a.e. in } D, \]
thus \( [\phi(-\Delta|_D)]^{-1} f = \int_D G_D^\phi(\cdot, y) f(y) dy \) a.e. in \( D \). \(
\)
In what follows, for the operator $G$ from the proof of the previous lemma we will write
\[
G^\phi_D f(x) := \int_D G^\phi_D(x, y)f(y)dy = \mathbb{G}f(x), \quad x \in D. \tag{2.27}
\]

**Remark 2.3.** Proposition 2.2 implies that $G^\phi_D(L^2(D)) = H_D(\phi)$ and that
\[
\phi(- \Delta|_D)(G^\phi_D f) = f, \quad f \in L^2(D).
\]

By the general theory of semigroups, this means that $-\phi(- \Delta|_D)$ defined by (2.19) is the infinitesimal generator of the semigroup $(Q^D_t)_t$ and that $H_\phi(D)$ is the domain of $-\phi(- \Delta|_D)$, see e.g. [41, 34]. In particular, $\mathcal{D}(\Delta|_D) \subset \mathcal{D}(\phi(- \Delta|_D)) = H_\phi(D)$ by [37, Theorem 13.6].

Further, we note that $C^{1,1}(\overline{D}) \subset \mathcal{D}(\Delta|_D)$. Indeed, since the first partial derivatives of a $C^{1,1}(\overline{D})$ function are Lipschitz functions, the first partial derivatives have the second partial derivatives almost everywhere. Furthermore, these second partial derivatives are in $L^\infty(D)$ since the first ones satisfy the Lipschitz property uniformly in $D$. By [16, Theorem 2.13] we get $C^{1,1}(\overline{D}) \subset \mathcal{D}(\Delta|_D)$. Hence, by the first part of this remark, we have $C_c^\infty(D) \subset C^{1,1}(\overline{D}) \subset \mathcal{D}(\Delta|_D) \subset \mathcal{D}(\phi(- \Delta|_D)) = H_\phi(D)$.

For sufficiently regular functions $\phi(- \Delta|_D)u$ can be expressed pointwisely. At this point we only consider $u \in C^{1,1}(D) \cap \mathcal{D}(\Delta|_D)$ but later on in Proposition 2.15 we will prove the pointwise representation of $\phi(- \Delta|_D)$ for $u \in C^{1,1}(D) \cap H_\phi(D)$.

**Lemma 2.4.** Let $u \in C^{1,1}(D) \cap \mathcal{D}(\Delta|_D)$. Then for a.e. $x \in D$
\[
\phi(- \Delta|_D)u(x) = P.V. \int_D [u(x) - u(y)]J_D(x, y)dy + \kappa(x)u(x), \tag{2.28}
\]

where
\[
J_D(x, y) := \int_0^\infty p_D(t, x, y)\mu(t)dt, \quad \kappa(x) := \int_0^\infty \left(1 - \int_D p_D(t, x, y)dy\right)\mu(t)dt.
\]

In particular, (2.28) holds for $u \in C_c^\infty(D)$.

**Remark 2.5.** The function $J_D$ is called the jumping density and the function $\kappa$ is called the killing function of the process $X$. Obviously, $J_D$ is non-negative and symmetric. It is also finite off the diagonal and satisfies $\int_D (1 \wedge |x - y|^2)J_D(x, y)dy < \infty$ since the following estimate holds
\[
J_D(x, y) \asymp \left(\delta_D(x)\delta_D(y) \wedge 1\right)\frac{\phi(|x - y|^2)}{|x - y|^d}, \quad x, y \in D. \tag{2.29}
\]

Here the constant of comparability depends only on $d$, $D$ and $\phi$ and the proof of (2.29) is essentially the same as the proof of (2.12). By applying comments given for the proof of [30, Proposition 3.5] and using similar manipulations as in the proof of Lemma A.1 to avoid using (A5) from [30], we easily obtain (2.29), so we skip the proof.

The killing function $\kappa$ is continuous and $\kappa \in L^1(D, \delta_D(x)dx)$. Indeed, since the semigroup $P^D_t$ is strongly Feller, $1 - P^D_t1(x) = \mathbb{P}_x(\tau_D \leq t)$ is continuous in $x$. Further,
for $\varepsilon > 0$ such that $\varepsilon < 2\delta_D(x)$ it holds that $\mathbb{P}_x(\tau_D \leq t) \leq \mathbb{P}_x(\tau_B(x,\varepsilon) \leq t) = \mathbb{P}_0(\tau_B(0,1) \leq \frac{t}{\varepsilon}) \leq c_1(\varepsilon)(1 + t)$, where the last inequality follows by e.g. [25, Theorem 1]. Now the dominated convergence theorem yields the continuity of $\kappa$. Finally, $\int_D \kappa(x) \varphi_1(x)dx = \phi(\lambda_1) \int_D \varphi_1(x)dx$ by (2.16), so (2.18) yields $\kappa \in L^1(D, \delta_D(x)dx)$.

**Proof of Lemma 2.4.** It is known that for all $u \in D(\Delta|_D)$ it holds that

$$\phi(- \Delta|_D)u = \int_0^\infty (u - \rho_u(t))\mu(t)dt$$

(2.30)

see [37, Theorem 13.6], since $D(\Delta|_D) \subset D(-\phi(- \Delta|_D)) = H_0(D)$ by Remark 2.3. The rest of the proof is dedicated to showing that the right hand sides of (2.30) and (2.28) are equal.

Let $u \in C^{1,1}(D) \cap D(\Delta|_D)$ and $x \in D$. First, we show that the principal value integral in (2.28) is well defined. Indeed, fix $\delta > 0$ such that $\delta < (1 + \delta_D(x)/4)$ and let $\varepsilon > 0$ such that $\varepsilon < \delta$. We have

$$\int_{D \setminus B(x,\varepsilon)} (u(x) - u(y)) J_D(x,y)dy$$

$$= \int_{D \setminus B(x,\varepsilon)} (u(x) - u(y) + \nabla u(x) \cdot (y - x)1_{B(x,\delta)}(y)) J_D(x,y)dy$$

$$- \int_{B(x,\delta) \setminus B(x,\varepsilon)} \nabla u(x) \cdot (y - x) J_D(x,y)dy$$

$$= I_1 - I_2.$$  

By a $C^{1,1}$ version of Taylor’s theorem we have

$$|u(x) - u(y) + \nabla u(x) \cdot (y - x)1_{B(x,\delta)}(y)| \leq c_1 (1 + |x - y|^2),$$

(2.31)

where $c_1 > 0$ depends on $\delta$ and $\|u\|_{C^{1,1}(B(x,\delta_D(x)/2))}$. Hence, the integral $I_1$ is finite and converges as $\varepsilon \to 0$ by dominated convergence theorem.

For the second integral, by Fubini’s theorem and (2.8), we have

$$I_2 = \int_0^\infty \int_{B(x,\delta) \setminus B(x,\varepsilon)} \nabla u(x) \cdot (y - x)p(t, x, y)dy\mu(t)dt$$

$$- \int_0^\infty \int_{B(x,\delta) \setminus B(x,\varepsilon)} \nabla u(x) \cdot (y - x)E_x[p(t - \tau_D, W_{\tau_D}, y)1_{\{\tau_D < t\}}]dy\mu(t)dt$$

$$=: J_1 - J_2.$$  

The integral $J_1$ is zero for all $\varepsilon < \delta$ since the kernel $p(t, x, y)$ is symmetric in $y$ around $x$, and since the region of integration is symmetric around $x$. For the integral $J_2$ note that $|\nabla u(x) \cdot (y - x)| \leq c_2 \delta$, $y \in B(x, \delta)$, where $c_2 = c_2(u) = \max_{B(x,\delta_D(x)/2)} |\nabla u(x)|$, i.e. $c_2$ depends on local properties of $u$ around $x$. Also,

$$p(t - \tau_D, W_{\tau_D}, y)1_{\{\tau_D < t\}} \leq \frac{4\pi}{{t - \tau_D}^{d/2}} e^{-\frac{\delta_D(x)^2}{{t - \tau_D}}}1_{\{\tau_D < t\}} \leq c_3(1 + t), \quad y \in B(x, \delta),$$

(2.32)
where \( c_3 = c_3(d, \delta_D(x)) > 0 \). Thus,
\[
\int_{B(x,\varepsilon) \setminus B(x,\delta)} |\nabla u(x) \cdot (y - x)| \, \mathbb{E}_x[p(t - \tau_D, W_{\tau_D}, y)1_{\{\tau_D < t\}}] \, dy \leq c_4 \delta^{d+1}(1 \wedge t), \quad t > 0,
\]
where \( c_4 = c_4(d, D, u, \delta_D(x)) > 0 \). In other words, we showed that
\[
|I_2| \leq c_6 \delta^{d+1},
\]
where \( c_6 = c_6(d, D, u, \delta_D(x), \mu) > 0 \). Moreover, the bounds (2.32) and (2.33) imply that the integral \( J_2 \) converges as \( \varepsilon \to 0 \) by the dominated convergence theorem. Hence, \( I_2 \)
converges as \( \varepsilon \to 0 \). Finally, this means that the principal value integral in (2.28) is well defined.

Now we prove (2.28). For the fixed \( \delta > 0 \) from above, by using (2.30) we have
\[
\phi(-\Delta|_D)u(x) = \int_0^\infty \left( u(x) - u(y) \right) P^D_t \mathbf{1}(x) + u(x) P^D_t \mathbf{1}(x) - P^D_t u(x) \right) \mu(t) \, dt
\]
where the change of the order of integration, as well as taking the limit outside the
Integral, was justified by (2.31), (2.32) and (2.33).

**Remark 2.6.** Lemma 2.4 suggest the pointwise definition of the operator \( \phi(-\Delta|_D) \), i.e. we define
\[
\phi_p(-\Delta|_D)u(x) = \text{P.V.} \int_D [u(x) - u(y)] J_D(x, y) \, dy + \kappa(x)u(x),
\]
for every function \( u \) and \( x \in D \) for which (2.34) is well defined. E.g. this is true for every
\( x \in D \) if \( u \in C^{1,1}(D) \cap L^1(D, \delta_D(x) \, dx) \) by the proof of Lemma 2.4 and the bound (2.29).

To conclude the subsection, we bring the well-known factorization of the Dirichlet
Laplacian \( -\Delta|_D \) which is closely related to Lemma 2.1. Since \( \phi^* \) satisfies (**WSC**),
the operator \( \phi^*(-\Delta|_D) \) can be defined in the same way as \( \phi(-\Delta|_D) \), and the same properties
hold for \( \phi^*(-\Delta|_D) \). In what follows, such comments on the objects defined relative to \( \phi 
and relative to \( \phi^* \) will be skipped.

**Lemma 2.7.** For \( \psi \in C^\infty_c(D) \), it holds that
\[
\phi(-\Delta|_D) \cap \phi^*(-\Delta|_D) \psi = \phi^*(-\Delta|_D) \cap \phi(-\Delta|_D) \psi = (-\Delta|_D) \psi, \quad a.e. \; in \; D.
\]
Further, \( (-\Delta|_D) \psi = -\Delta \psi \).

**Proof.** Recall that the operator \( \Delta|_D \) is the infinitesimal generator of the semigroup \( (P^D_t)_t \)
which on \( C^\infty_c(D) \) functions acts like the standard Laplacian \( \Delta \). Hence, the claim follows
from [37, Corollary 13.25] since \( C^\infty_c(D) \subset \mathcal{D}(\Delta|_D) \).
2.4 Green potentials

In this subsection we prove some useful identities related to the Green potentials, develop some integrability conditions and prove two regularity properties for $G^\phi_D f$.

The next lemma says that the definition of the Green potential $G^\phi_D f$ in (2.27) makes sense for $f \in L^1(D, \delta_D(x)dx)$, too, and that the operator $f \mapsto G^\phi_D f$ is bounded from $L^1(D, \delta_D(x)dx)$ to itself.

**Lemma 2.8.** It holds that

$$G^\phi_D \delta_D(x) \asymp \delta_D(x), \quad x \in D,$$

where the constant of comparability depends only on $d$, $D$ and $\phi$. Further, if $\lambda \in \mathcal{M}(D)$ such that $\int_D \delta_D(x)|\lambda|(dx) < \infty$ then

$$x \mapsto G^\phi_D \lambda(x) := \int_D G^\phi_D(x,y)\lambda(dy) \in L^1(D, \delta_D(x)dx),$$

and there is $C = C(d, D, \phi) \geq 1$ such that $\|G^\phi_D \lambda\|_{L^1(D, \delta_D(x)dx)} \leq C \int_D \delta_D(x)|\lambda|(dx)$.

**Proof.** Recall that $\varphi_1(x) \asymp \delta_D(x)$, $x \in D$, by (2.18), thus by (2.22)

$$G^\phi_D \delta_D(x) \asymp G^\phi_D \varphi_1(x) = \frac{1}{\phi(\lambda_1)} \varphi_1(x) \asymp \delta_D(x), \quad x \in D.$$

The second and the third claim follow from Fubini’s theorem and (2.35). \qed

**Corollary 2.9.** There is $C = C(d, D, \phi) > 0$ such that for every $f \in L^1(D, \delta_D(x)dx)$ it holds that $\|G^\phi_D f\|_{L^1(D, \delta_D(x)dx)} \leq C \|f\|_{L^1(D, \delta_D(x)dx)}$.

**Remark 2.10.** Let us note that by using (2.12) it easily follows that $G^\phi_D f \in L^\infty(D)$ for $f \in L^\infty(D)$.

2.4.1 Operator $\phi(- \Delta|_D)$ revisited

In the next lemma we prove the boundary estimate of $\phi(- \Delta|_D)\psi$ for $\psi \in C^\infty_c(D)$ which will allow us to define the operator $\phi(- \Delta|_D)$ in the distributional sense.

**Lemma 2.11.** For $\psi \in C^\infty_c(D)$ there is $C_1 = C_1(d, D, \phi, \psi) > 0$ such that

$$|\phi(- \Delta|_D)\psi(x)| \leq C_1 \delta_D(x), \quad x \in D.$$

In addition, if $\psi \geq 0$, $\psi \neq 0$, then there is $C_2 = C_2(d, D, \phi, \psi) > 0$ such that

$$\phi(- \Delta|_D)\psi(x) \leq -C_2 \delta_D(x), \quad x \in D \setminus \text{supp } \psi.$$

**Proof.** Let $\psi \in C^\infty_c(D)$ and note that $\phi(\lambda) \leq (1 \wedge \lambda)$ by (2.6). Thus, from (2.17), (2.23), and (2.26) for large enough $m \in \mathbb{N}$, we have

$$\left|\frac{\phi(- \Delta|_D)\psi(x)}{\delta_D(x)}\right| \leq \sum_{j=1}^\infty |\hat{\psi}_j| \phi(\lambda_j) \left\|\frac{\varphi_j}{\delta_D}\right\|_{L^\infty(D)} \leq C_1(d, D, \phi, \psi).$$
For the other bound let \(x^* = \arg \max_{x \in D} \psi(x)\), and let \(r > 0\) such that \(B(x^*, 2r) \subset \text{supp} \psi\) and \(\psi \geq c > 0\) on \(B(x^*, 2r)\). For \(x \in D \setminus \text{supp} \psi\), by using the representation (2.28) and the bound (2.29), we have

\[
\phi(-\Delta|_D)\psi(x) = -\int_{\text{supp} \psi} \psi(y) J_D(x, y) dy \leq -\int_{B(x^*, r)} c_1 \delta_D(x) dy \leq -C_2 \delta_D(x), \quad (2.39)
\]

where \(C_2 = C_2(d, D, \psi, \phi) > 0\).

**Definition 2.12.** For \(f \in L^1(D, \delta_D(x)dx)\) we define the distribution \(\widetilde{\phi}(-\Delta|_D)f\) in \(D\) by

\[
\langle \widetilde{\phi}(-\Delta|_D)f, \psi \rangle := \int_D f(x)\phi(-\Delta|_D)\psi(x)dx, \quad \psi \in C_c^\infty(D).
\]

**Remark 2.13.** Sometimes for \(\widetilde{\phi}(-\Delta|_D)f\) we say \(\phi(-\Delta|_D)f\) in the distributional sense. Notice that Lemma 2.11 implies that the integral defining \(\phi(-\Delta|_D)f\) is well defined.

By following the calculations from [10, Section 3], we get that for \(f \in C^{1,1}(D) \cap L^1(D, \delta_D(x)dx)\) we have \(\widetilde{\phi}(-\Delta|_D)f = \phi_p(-\Delta|_D)f\).

The next proposition says that the relation from Remark 2.3 can be also extended to \(\widetilde{\phi}(-\Delta|_D)\).

**Proposition 2.14.** Let \(\mu \in \mathcal{M}(D)\) such that \(\int_D \delta_D(x)|\mu|(dx) < \infty\). Then \(\widetilde{\phi}(-\Delta|_D)G_D^\phi \mu = \mu\).

**Proof.** Let \(\psi \in C_c^\infty(D)\) and recall that \(\phi(-\Delta|_D)\psi \in L^2(D)\) which follows by taking \(m \in \mathbb{N}\) large enough in (2.26). Hence, by Proposition 2.2 we have a.e. in \(D\)

\[
\psi = [\phi(-\Delta|_D)]^{-1}(\phi(-\Delta|_D)\psi) = G_D^\phi(\phi(-\Delta|_D)\psi).
\]

Thus, by using Lemma 2.8 and Lemma 2.11, Fubini’s theorem gives us

\[
\langle \widetilde{\phi}(-\Delta|_D)G_D^\phi \mu, \psi \rangle = \langle G_D^\phi \mu, \phi(-\Delta|_D)\psi \rangle
\]

\[
= \int_D \left( \int_D G_D^\phi(x, y) \mu(dy) \right) \phi(-\Delta|_D)\psi(x) dx
\]

\[
= \int_D \left( \int_D G_D^\phi(x, y) \phi(-\Delta|_D)\psi(x) dx \right) \mu(dy) = \int_D \psi(y) \mu(dy).
\]

The following proposition connects the spectral, the distributional, and the pointwise definition of \(\phi(-\Delta|_D)\) for nice enough functions.

**Proposition 2.15.** If \(u \in C^{1,1}(D) \cap H_\phi(D)\), then

\[
\phi(-\Delta|_D)u = \widetilde{\phi}(-\Delta|_D)u = \phi_p(-\Delta|_D)u
\]

holds a.e. in \(D\).

**Proof.** Let \(u \in C^{1,1}(D) \cap H_\phi(D)\). Recall that \(H_\phi(D) = G_D^\phi(L^2(D)) \subset L^2(D) \subset L^1(D, \delta_D(x)dx)\), so \(u = G_D^\phi h\) for some \(h \in L^2(D)\), and \(\phi(-\Delta|_D)u = h\). However, \(u \in C^{1,1}(D)\) so \(\widetilde{\phi}(-\Delta|_D)u = \phi_p(-\Delta|_D)u\) by Remark 2.13, and \(\phi(-\Delta|_D)u = h\) by Proposition 2.14.
2.4.2 Regularity of Green potentials

In the two following claims we deal with the regularity properties of \( G^\phi_D f \). The first claim says that Green potentials are continuous and this fact is rather simple to see and prove. We also prove that the Green potential of a \( C^\infty_c(D) \) function is a \( C^{1,1}(\overline{D}) \) function, i.e. we prove a smoothness result for a specific class of functions.

**Proposition 2.16.** If \( f \in L^1(D, \delta_D(x)dx) \cap L^\infty_{\text{loc}}(D) \), then \( G^\phi_D f \in C(D) \).

**Proof.** Let \( x \in D \), \( \eta \in (0, \delta_D(x)/2) \) and \( (x)_n \subset D \) such that \( x_n \to x \) and \( |x_n - x| < \eta/2 \), \( n \in \mathbb{N} \). We have

\[
|G^\phi_D f(x_n) - G^\phi_D f(x)| \leq \int_D |G^\phi_D(x_n, y) - G^\phi_D(x, y)||f(y)|dy
\]

\[
\leq \int_{D \cap B(x, \eta)^c} |G^\phi_D(x_n, y) - G^\phi_D(x, y)||f(y)|dy
\]

(2.40)

\[
+ \int_{B(x, \eta)} G^\phi_D(x_n, y)|f(y)|dy
\]

(2.41)

\[
+ \int_{B(x, \eta)} G^\phi_D(x, y)|f(y)|dy.
\]

(2.42)

The first integral (2.40) goes to 0 as \( n \to \infty \) by the dominated convergence theorem since \( G^\phi_D \) is continuous, \( f \in L^1(D, \delta_D(x)dx) \), and since the bound (2.12) holds.

For the integrals (2.41) and (2.42) note that \( M := \sup_{y \in B(x, \delta_D(x)/2)} |f(y)| < \infty \) since \( f \in L^\infty_{\text{loc}}(D) \). Further, by (2.12) for all \( w \in B(x, \eta/2) \) we have

\[
\int_{B(x, \eta)} G^\phi_D(w, y)|f(y)|dy \leq c_1 M \int_{B(w, \frac{\eta}{2})} \frac{1}{|w - y|^d \phi(|w - y|^{-2})}dy
\]

\[
\leq c_2 M \int_0^{\frac{\eta}{r}} \frac{dr}{r \phi(r^{-2})} \leq c_3 M \int_0^{\frac{\eta}{r}} \frac{\phi'(r^{-2})}{r^3 \phi(r^{-2})} = c_3 M, \quad (2.43)
\]

where in the last equality we used the substitution \( t = \phi(r^{-2}) \) and \( c_3 = c_3(d, D, \phi) > 0 \). Thus, the second and the third integral can be made arbitrarily small. \( \square \)

**Remark 2.17.** From Proposition 2.16 it follows that

\[
\lim_{\xi \to x} \int_D |G^\phi_D(\xi, y) - G^\phi_D(x, y)||f(y)|dy = 0,
\]

(2.44)

uniformly on compact subsets of \( D \).

Indeed, fix a compact set \( K \subset D \) and \( \varepsilon > 0 \). First choose \( \eta > 0 \) from Proposition 2.16 such that \( \text{dist}(K, \partial D) > 2\eta \) and \( (c_3 M)/\phi(\frac{4}{9\eta^2}) < \varepsilon/3 \), where \( M = \sup_{y \in K + B(0, \eta)} |f(y)| \), see (2.43). Thus, we tamed the integrals (2.41) and (2.42). For the integral (2.40) notice that the convergence \( \lim_{\xi \to x} G^\phi_D(\xi, y) = G^\phi_D(x, y) \) is uniform in \( x \in K \) and \( y \in D \cap B(x, \eta)^c \) since \( G^\phi_D \) is jointly continuous and since \( G^\phi_D \) continuously vanishes at the boundary by (2.12). Hence, (2.44) holds uniformly on compact sets.

**Proposition 2.18.** If \( f \in C^\infty_c(D) \), then \( G^\phi_D f \in C^{1,1}(\overline{D}) \).
Proof. By Proposition 2.2 we have $G_D^\phi f = \sum_{j=1}^{\infty} \frac{1}{\phi(\lambda_j)} \hat{f}_j \varphi_j$ a.e. in $D$. However, $G_D^\phi f \in C(D)$ by Proposition 2.16. Also, recall that there is $c_1 = c_1(m, f) > 0$ such that $|\hat{f}_j| \leq c_1 \lambda_j^{-m}$, $j \in \mathbb{N}$, by (2.26), hence in the light of (A.33) and (A.34), for large enough $m \in \mathbb{N}$ we have

$$\left\| \sum_{j=1}^{\infty} \frac{1}{\phi(\lambda_j)} \hat{f}_j \varphi_j \right\|_{C^{1,1}(D)} \leq \sum_{j=1}^{\infty} \frac{c_2}{\phi(\lambda_j) \lambda_j^m} (1 + \lambda_j)^{d/4 + 1} < \infty$$

by (2.17) and by (2.6), where $c_2 = c_2(d, D, m, f) > 0$.

In other words, $G_D^\phi f = \sum_{j=1}^{\infty} \frac{1}{\phi(\lambda_j)} \hat{f}_j \varphi_j$ everywhere in $D$ and $G_D^\phi f \in C^{1,1}(D)$.

2.5 Poisson kernel and harmonic functions

Recall that the Poisson kernel of the Brownian motion (i.e., of the Dirichlet Laplacian) can be defined as

$$P^\phi_D(x, z) = -\frac{\partial}{\partial n} G_D(x, z), \quad x \in D, z \in \partial D,$$

since we assume that $D$ is a $C^{1,1}$ bounded domain, see [20, Section 2.2.4]. Here $\frac{\partial}{\partial n}$ denotes the derivate in the direction of the outer normal. In this subsection we study the Poisson kernel of the process $X$ which we define as the normal derivative of the Green kernel of the process $X$ and we study harmonic functions relative to $\phi(-\Delta|_D)$, or, as we show at the end of the subsection, relative to $X$.

Proposition 2.19. The function

$$P^\phi_D(x, z) := -\frac{\partial}{\partial n} G_D^\phi(x, z), \quad x \in D, z \in \partial D,$$

(2.46)

is well defined and $(x, z) \mapsto P^\phi_D(x, z) \in C(D \times \partial D)$. Moreover,

$$P^\phi_D(x, z) \asymp \frac{\delta_D(x)}{|x - z|^{d+2 \phi(|x - z|^{-2})}}, \quad x \in D, z \in \partial D,$$

(2.47)

where the constant of comparability depends only on $d, D$ and $\phi$. Finally, it holds that

$$\int_D G_D^{\phi^*}(x, \xi) P_D^\phi(\xi, z) d\xi = P_D(x, z), \quad x \in D, z \in \partial D.$$

(2.48)

Proof. Let $x \in D$ and $z \in \partial D$. For $y \in D$ we have

$$\frac{G_D^\phi(x, y)}{\delta_D(y)} = \int_0^\infty \frac{1}{\delta_D(y)} p_D(t, x, y) u(t) dt.$$

In what follows, we always consider $y \in D$ which is in the direction of the normal derivative in $z$, close enough to $z$ so that $\delta_D(x) \leq 2|x - y|$.
Recall that $p_D \in C^1((0, \infty) \times \overline{D} \times \overline{D})$ since $D$ is $C^{1,1}$, see Lemma A.7, hence $-\frac{\partial}{\partial n} p_D(t, x, z) = \lim_{y \to z} \frac{p_D(t, x, y) - p_D(t, x, z)}{\delta(y)}$ exists. Further, from (2.9), there exist constants $c_1, c_2 > 0$ (depending on $D$) such that for all $t > 0$, and $x, y \in D$ we have

$$\frac{p_D(t, x, y)u(t)}{\delta(y)} \leq c_1 \frac{\delta_D(x)}{t^{d/2+1}} e^{-\frac{c_2|x-y|^2}{t}}u(t) \leq c_1 \frac{\delta_D(x)}{t^{d/2+1}} e^{-\frac{c_2\delta_D(x)^2}{t}}u(t). \quad (2.49)$$

Recall that $u$ is decreasing and that $\int_0^1 u(t) dt < \infty$, hence the right hand side of (2.49) is in $L^1((0, \infty), dt)$. By using the dominated convergence theorem we conclude that $P_D^\phi(x, z)$ is well defined and

$$P_D^\phi(x, z) = \lim_{y \to z} \frac{G_D^\phi(x, y)}{\delta(y)} = -\int_0^\infty \frac{\partial}{\partial n} p_D(t, x, z) u(t) dt. \quad (2.50)$$

Moreover, (2.47) immediately follows from the definition of $P_D^\phi$ and (2.12).

Now we show that $P_D^\phi$ is jointly continuous on $D \times \partial D$. Let $(x_n)_n \subset D$ such that $x_n \to x \in D$ and such that $\delta_D(x_n) \geq \delta_D(x)/2$. Also, take $(z_n)_n \subset \partial D$ such that $z_n \to z \in \partial D$. By taking the limit $y \to z$ in the first inequality in (2.49) without the term $u(t)$, we obtain for all $n \in \mathbb{N}$ and all $t \in (0, \infty)$

$$0 \leq -\frac{\partial}{\partial n} p_D(t, x_n, z_n) = c_1 \frac{\delta_D(x_n)}{t^{d/2+1}} e^{-\frac{c_2|x_n-z_n|^2}{t}} \leq c_1 \frac{\delta_D(x_n)}{t^{d/2+1}} e^{-\frac{c_2\delta_D(x_n)^2}{t}}, \quad (2.50)$$

which also holds for $z$ instead of $z_n$. Since $\frac{\partial}{\partial n} p_D(t, x, z) \in C((0, \infty) \times \overline{D} \times \partial D)$, see Lemma A.7, by using the dominated convergence theorem with the bound derived from (2.50) we get

$$|P_D^\phi(x, z) - P_D^\phi(x_n, z_n)| \leq \int_0^\infty \left| \frac{\partial}{\partial n} p_D(t, x_n, z_n) - \frac{\partial}{\partial n} p_D(t, x, z) \right| u(t) dt \to 0, \quad \text{as } n \to \infty.$$

We are left to prove (2.48). Obviously, Lemma 2.1 implies

$$-\frac{\partial}{\partial n} \left( \int_D G_D^\phi(x, \xi) G_D^\phi(\xi, \cdot) d\xi \right)(z) = P_D^\phi(x, z), \quad x \in D, z \in \partial D.$$

We need to justify that the normal derivative can go inside the integral. To this end, let $x \in D$, $z \in \partial D$, and $\varepsilon > 0$ such that $\delta_D(x) > 3\varepsilon$. Again, we only consider $y \in D$ which is in the direction of the normal derivative. For $|z - y| \leq \varepsilon/2$ we have

$$\int_D G_D^\phi(x, \xi) G_D^\phi(\xi, y) \delta_D(y) d\xi = \int_{D \cap B(z, \varepsilon)} G_D^\phi(x, \xi) G_D^\phi(\xi, y) \delta_D(y) d\xi + \int_{D \cap B(z, \varepsilon)} G_D^\phi(x, \xi) G_D^\phi(\xi, y) \delta_D(y) d\xi =: I_1 + I_2.$$

For the integral $I_1$ by the sharp bounds (2.12) we have

$$\frac{G_D^\phi(\xi, y)}{\delta_D(y)} \lesssim \frac{\delta_D(\xi)}{|\xi - y|^{d+2}\phi(|\xi - y|^{-2})}. \quad (2.51)$$
Thus, if \( \xi \in D \cap B(z, \varepsilon)c \), we have \( G^\phi_D(\xi, y) \leq c_3 \delta_D(\xi) \), where \( c_3 = c_3(\phi, D, d, \varepsilon) > 0 \).

Further, \( G_D^\phi \delta_D \asymp \delta_D \) by Lemma 2.8, hence the integral \( I_1 \) converges to

\[
\int_{D \cap B(y, \varepsilon)c} G_D^\phi(x, \xi) P_D^\phi(\xi, z) d\xi,
\]
as \( y \to z \).

The integral \( I_2 \) we break into two additional integrals

\[
I_2 = \int_{D \cap B(z, \varepsilon)} G_D^\phi(x, \xi) \frac{G_D^\phi(\xi, y)}{\delta_D(y)} d\xi
\]

\[
\leq \int_{B(y, \frac{\delta_D(y)}{2})} G_D^\phi(x, \xi) \frac{G_D^\phi(\xi, y)}{\delta_D(y)} d\xi + \int_{D \cap B(y, \frac{\delta_D(y)}{2}) \cap B(y, 2\varepsilon)} G_D^\phi(x, \xi) \frac{G_D^\phi(\xi, y)}{\delta_D(y)} d\xi
\]

\[
=: J_1 + J_2.
\]

Recall that \( 3\varepsilon \leq \delta_D(x) \) so \( \frac{1}{2}|x - z| \leq |x - \xi| \leq 2|x - z| \) for all \( \xi \in B(y, 2\varepsilon) \). Hence, (2.12) applied on \( G_D^\phi \) implies

\[
G_D^\phi(x, \xi) \leq c_4 \delta_D(\xi), \quad \xi \in B(y, 2\varepsilon) \cap D, \tag{2.52}
\]

where \( c_4 = c_4(d, D, \phi^*, |x - z|) > 0 \) and is independent of \( \varepsilon \) in the sense if \( \varepsilon \to 0 \), the constant \( c_4 \) remains the same.

For \( J_1 \) note that \( \delta_D(\xi) \leq \frac{3}{2} \delta_D(y) \) for \( \xi \in B(y, \delta_D(y)/2) \) so by using the bounds (2.12) and (2.52) we have

\[
J_1 \leq c_6 \int_{B(y, \frac{\delta_D(y)}{2})} \frac{\delta_D(\xi)}{\delta_D(y) |\xi - y|^d \phi(|\xi - y|^{-2})} \leq \frac{1}{c_6} \int_{B(y, \frac{\delta_D(y)}{2})} \frac{1}{|\xi - y|^d \phi(|\xi - y|^{-2})}
\]

\[
\leq c_7 \int_0^{\delta_D(y)/2} \frac{\phi'(r^{-2})}{r^3 \phi(r^{-2})^2} dr \leq c_8 \frac{1}{\phi(4/\delta_D(y)^2)},
\]

where \( c_8 \) is independent of \( y \) and \( \varepsilon \). In the second to last inequality we used (2.3) and for the last one we used the substitution \( t = \phi(r^{-2}) \).

For \( J_2 \) note that \( \delta_D(\xi) \leq \delta_D(y) + |y - \xi| \leq 3|\xi - y| \), for \( \xi \in B(y, \delta_D(y)/2)c \), hence by the sharp bounds (2.12) we have

\[
J_2 \leq c_9 \int_{B(y, \frac{\delta_D(y)}{2}) \cap B(y, 2\varepsilon)} \frac{1}{|\xi - y|^d \phi(|\xi - y|^{-2})} \leq c_{10} \int_{\delta_D(y)/2}^{2\varepsilon} \frac{\phi'(r^{-2})}{r^3 \phi(r^{-2})^2} dr \leq c_{11} \frac{1}{\phi(\frac{1}{4\varepsilon^2})},
\]

where \( c_{11} \) is independent of \( y \) and \( \varepsilon \). Hence, for sufficiently small \( \varepsilon \) the integral \( I_2 \) can be made sufficiently small. Thus, (2.48) holds.

\[ \square \]

**Remark 2.20.** We emphasize that the assumption of the regularity of \( \partial D \) was essential in the proof of the previous proposition. Prior to the proof, regularity was used for obtaining the sharp bounds for \( G_D^\phi \) and \( G_D^\phi^* \) and for proving the regularity of \( p_D \). This led to showing the well-definiteness of \( P_D^\phi \), to the sharp bounds for \( P_D^\phi \), and to the identity (2.48). In the remainder of the section, the regularity of \( \partial D \) will be also heavily used but we omit comments like this one from now on.
Now we deal with harmonic functions with respect to the operator $\phi(-\Delta|_D)$. Our first goal is to show the integral representation of positive harmonic functions which we show in Theorem 2.23. After that, in Theorem 2.25 we show the continuity of harmonic functions and at the end of the subsection we connect harmonic functions with functions that satisfy a certain mean-value property with respect to $X$, see Theorem 2.27.

**Definition 2.21.** A function $h \in L^1(D, \delta_D(x)dx)$ is called harmonic in $D$ if $\tilde{\phi}(-\Delta|_D)h = 0$ in $D$.

First, we present a connection between harmonic functions and classical harmonic functions.

**Proposition 2.22.** A function $h \in L^1(D, \delta_D(x)dx)$ is harmonic in $D$ if and only if $G^\phi_D h$ is a classical harmonic function in $D$. In particular, for every $z \in \partial D$, the function $x \mapsto P^\phi_D(x, z)$ is harmonic in $D$.

**Proof.** The first part of the claim follows by the following calculation. Take $\psi \in C^\infty_c(D)$. Then by using Lemma 2.7, Proposition 2.2, and Fubini’s theorem we have

$$
\int_D h(x)\phi(-\Delta|_D)\psi(x)dx = \int_D h(x) \left[ \phi^*(-\Delta|_D)^{-1} \circ (-\Delta)\psi(x) \right] dx
$$

$$
= \int_D h(x)G^\phi_D((-\Delta)\psi)(x)dx
$$

$$
= -\int_D G^\phi_D h(x) \Delta\psi(x)dx,
$$

i.e. $h$ is harmonic if and only if $G^\phi_D h$ is a classical harmonic function in $D$.

If $z \in \partial D$, then $P^\phi_D(\cdot, z) \in L^1(D, \delta_D(x)dx)$ by the bound $(2.47)$, see also the beginning of the proof of Theorem 2.23 with $\zeta = \delta_z$. The second claim now follows from $(2.48)$ and the fact that the kernel $P^\phi_D(\cdot, z)$ is a classical harmonic function. \hfill $\square$

**Theorem 2.23.** If a non-negative function $h \in L^1(D, \delta_D(x)dx)$ is harmonic in $D$, then there exists a finite non-negative measure $\zeta \in \mathcal{M}(\partial D)$ such that

$$
h(x) = \int_{\partial D} P^\phi_D(x, z)\zeta(dz), \quad \text{for a.e. } x \in D. \tag{2.53}
$$

Moreover, there is $C = C(d, D, \phi) > 0$ such that

$$
\|h\|_{L^1(D, \delta_D(x)dx)} \leq C\|\zeta\|_{\mathcal{M}(\partial D)}. \tag{2.54}
$$

Conversely, every function of the form $(2.53)$ is harmonic in $D$.

**Proof.** Let $h$ be represented as $(2.53)$. Since $P^\phi_D(x, \cdot) \in C(\partial D)$ for fixed $x \in D$ by Proposition 2.19, hence bounded, the function $h$ is well defined. Further, since $\delta_D(x) \leq |x - z|, z \in \partial D$, from $(2.47)$ and Fubini’s theorem we get

$$
\int_D h(x) \delta_D(x)dx \leq c_1 \int_{\partial D} \int_D \frac{1}{|x - z|^{d+2}} \frac{\phi(|x - z|^{-2})}{\phi(\delta(x)^2)} dx \zeta(dz)
$$

$$
\leq c_1 \int_{\partial D} \int_{B(z, \text{diam}D)} \frac{1}{|x - z|^{d+2}} \phi(|x - z|^{-2}) dx \zeta(dz)
$$

$$
\leq c_2 \int_{\partial D} \frac{\zeta(dz)}{\phi(\text{diam}D^{-2})} < \infty,
$$

20
where \( c_2 = c_2(d, D, \phi) > 0 \), i.e. \( h \in \mathcal{L}^1(D, \delta_D(x)dx) \) and \( \|h\|_{\mathcal{L}^1(D, \delta_D(x)dx)} \leq C\|\zeta\|_{\mathcal{M}(\partial D)} \). Take now \( \psi \in \mathcal{C}_c^\infty(D) \). Fubini’s theorem and Proposition 2.22 yield
\[
\int_D P^\phi_D(x)\phi(-\Delta|_D)\psi(x)dx = \int_{\partial D} \left( \int_D P^\phi_D(x, z)\phi(-\Delta|_D)\psi(x)dx \right) \zeta(dz) = 0,
\]
i.e. \( h \) is harmonic in \( D \).

Conversely, let \( h \) be a non-negative harmonic function in \( D \). Then \( G^\phi_D h \) is a classical non-negative harmonic function in \( D \) by Proposition 2.22. By the representation of non-negative classical harmonic functions there is a non-negative finite measure \( \zeta \in \mathcal{M}(\partial D) \) such that
\[
G^\phi_D h(x) = \int_{\partial D} P^\phi_D(x, z)\zeta(dz), \quad \text{for a.e. } x \in D. \tag{5.55}
\]
Applying (2.48) to the right hand side of (5.55) we get
\[
\int_D G^\phi_D(x, \xi)h(\xi)d\xi = \int_D G^\phi_D(x, \xi) \left[ \int_{\partial D} P^\phi_D(\xi, z)\zeta(dz) \right] d\xi, \quad \text{for a.e. } x \in D. \tag{5.56}
\]
By using Proposition 2.14 in (5.56) we obtain
\[
h(\xi) = \int_{\partial D} P^\phi_D(\xi, z)\zeta(dz), \quad \text{for a.e. } \xi \in D.
\]

Motivated by the previous theorem, we introduce the definition of the Poisson integral.

**Definition 2.24.** For a finite signed measure \( \zeta \in \mathcal{M}(\partial D) \) we define the Poisson integral of \( \zeta \) by
\[
P^\phi_D \zeta(x) := \int_{\partial D} P^\phi_D(x, z)\zeta(dz), \quad x \in D.
\]

Note that the finiteness of the (signed) measure \( \zeta \) in the previous definition is a necessary and sufficient condition for for the integral defining \( P^\phi_D \zeta \) to be finite, see (2.47).

If \( \zeta \in \mathcal{L}^1(\partial D) \), we slightly abuse the notation in Definition 2.24 where we set \( P^\phi_D \zeta(x) = \int_{\partial D} P^\phi_D(x, z)\zeta(z)\sigma(dz) \), where \( \sigma \) is the \( d-1 \) dimensional Hausdorff measure on \( \partial D \). Since the set \( D \) is \( C^{1,1} \), the measure \( \sigma \) is finite so we can define the Poisson integral of \( \sigma \)
\[
P^\phi_D \sigma(x) = \int_{\partial D} P^\phi_D(x, z)\sigma(dz), \quad x \in D, \tag{5.57}
\]
which will be of great importance for the boundary condition of the semilinear problem.

We finish the subsection with two properties of harmonic functions of the form \( P^\phi_D \zeta \).

**Theorem 2.25.** A non-negative harmonic function in \( D \) is continuous in \( D \) (after a modification on the Lebesgue null set). Furthermore, for every finite (signed) measure \( \zeta \in \mathcal{M}(\partial D) \), we have \( P^\phi_D \zeta \in C(D) \).
Proof. Let $h \in L^1(D, \delta_D(x)dx)$ be a non-negative harmonic function in $D$. By Theorem 2.23 there exists a finite non-negative measure $\zeta \in \mathcal{M}(\partial D)$ such that $h = P^\phi_D \zeta$ a.e. in $D$. In Proposition 2.19 it was proved that the function $P^\phi_D (\cdot, \cdot)$ is continuous in the first variable and that the sharp bounds (2.47) hold, so we can use the dominated convergence theorem to get $P^\phi_D \zeta \in C(D)$.

In the theory of Markov processes, harmonicity of a function is considered relative to the process itself, i.e. it is said that a function $f : D \to [-\infty, \infty]$ is harmonic in $D$ with respect to $X$ if for every $U \subseteq D$ and $x \in U$

$$h(x) = \mathbb{E}_x [h(X_{\tau^D_X})]$$

holds, where $\tau^D_X = \inf \{t > 0 : X_t \notin U \}$ and where we implicitly assume $\mathbb{E}_x [||h(X_{\tau^D_X})||] < \infty$ for every $x \in U \subseteq D$. The relation (2.58) is often referred to as the mean-value property of the function $f$ with respect to $X$. In order not to confuse, if $f$ is harmonic in $D$ with respect to $X$, we will say that $f$ satisfies the mean-value property with respect to $X$. We note that $\mathbb{E}_x [||h(X_{\tau^D_X})||] < \infty$ for every $x \in U \subseteq D$ implies that $f \in L^1(D, \delta_D(x)dx)$, see the proof of [30, Lemma 3.6] where instead of the inequality $U^{D,B}(x,y) \leq G_X(x,y)$ use $U^{D,B}(x,y) \leq G^\phi_D(x,y)$.

The connection between non-negative functions that satisfy the mean-value property with respect to $X$ and non-negative functions that satisfy the mean-value property with respect to $W^D$ is known due to [39, Theorem 3.6] which we cite in the next claim.

**Theorem 2.26.** If a non-negative function $h$ satisfies the mean-value property in $D$ with respect to $X$, then $s := G^\phi_D h$ satisfies the mean-value property in $D$ with respect to $W^D$. Conversely, if a non-negative function $s$ satisfies the mean-value property in $D$ with respect to $W^D$, then

$$h(x) := \int_0^\infty (s(x) - P^D_t s(x)) \nu(t) dt = \phi^*_p(\Delta|_D)s(x), \quad x \in D,$$

satisfies the mean-value property in $D$ with respect to $X$, $h$ is continuous and $G^\phi_D h = s$.

**Proof.** Everything follows from [39, Theorem 3.6] except the second equality in (2.59). To finish the proof, it follows from the proof of [39, Lemma 3.4] that

$$|s(x) - P^D_t s(x)| \leq c(1 + t), \quad x \in K,$$

where $K$ is any compact subset of $D$ and $c = c(d, D, s|_K) > 0$. Also, $s \in C^\infty(D)$ since it is a classical harmonic function so by the same calculations as in Lemma 2.4 we get that

$$\int_0^\infty (s(x) - P^D_t s(x)) \nu(t) dt = \phi^*_p(\Delta|_D)s.$$

The following theorem says that non-negative harmonic functions and non-negative functions with the mean-value property with respect to $X$ are essentially the same.
Theorem 2.27. If a non-negative function \( h \in L^1(D, \delta_D(x)dx) \) is harmonic in \( D \), then (after a modification on the Lebesgue null set) \( h \) satisfies the mean-value property with respect to \( X \). Conversely, if \( h \geq 0 \) satisfies the mean-value property with respect to \( X \), then \( h \) is harmonic in \( D \).

Proof. Let \( h \geq 0 \) be harmonic in \( D \). Theorem 2.23 implies that we can modify \( h \) such that \( h = P^\phi_D \zeta \) in the whole \( D \) for some non-negative and finite \( \zeta \in \mathcal{M}(\partial D) \). This also means that \( h \in C(D) \) by Theorem 2.25. Since \( G^\phi_D h = P^\phi_D \zeta \) in \( D \) by (2.48), the claim follows from Theorem 2.26 because \( P^\phi_D \zeta \) is a (smooth) classical harmonic function, hence it satisfies the mean-value property with respect to \( W^D \).

Conversely, if \( h \geq 0 \) satisfies the mean-value property with respect to \( X \), then \( G^\phi_D h \) satisfies the mean-value property with respect to \( W^D \) by Theorem 2.26. By the classical theory of harmonic functions, \( G^\phi_D h \) is a classical harmonic function in \( D \). Proposition 2.22 now implies that \( h \) is harmonic in \( D \).

3 Boundary behaviour of potential integrals

In this section we study the boundary behaviour of Poisson and Green integrals which will serve as a foundation for the understanding of the boundary condition of the (semi)linear problem and the understanding of the connection between weak and distributional solutions in the next section. However, these problems are also interesting in themselves. We emphasize that the essential assumption for all the results in this section is that \( D \) is a \( C^{1,1} \) bounded domain and the regularity of \( \partial D \) is heavily used in every proof in this section. First we give a sharp bound for \( P^\phi_D \sigma \),

Lemma 3.1. It holds that

\[
P^\phi_D \sigma(x) \asymp \frac{1}{\delta_D(x)^2 \phi(\delta_D(x)^{-2})}, \quad x \in D,
\]

where the constant of comparability depends only on \( d, D \) and \( \phi \).

Proof. In Proposition 2.19 we have proved that

\[
P^\phi_D(x, z) \asymp \frac{\delta_D(x)}{|x-z|^{d+2\phi(|x-z|^{-2})}}, \quad x \in D, z \in \partial D,
\]

where the constant of comparability depends only on \( d, D \) and \( \phi \). Also, in the following calculations, it is easy to check that every comparability constant remains to depend only on \( d, D \) and \( \phi \).

For the upper bound, note that \( \delta_D(x) \leq |x-z|, \ z \in \partial D \) so by using (2.6) we have \( \delta_D(x)^2 \phi(\delta_D(x)^{-2}) \leq |x-z|^2 \phi(|x-z|^{-2}) \), thus

\[
P^\phi_D \sigma(x) \asymp \int_{\partial D} \frac{\delta_D(x)}{|x-z|^{d+2\phi(|x-z|^{-2})}} \sigma(dz) \leq \frac{1}{\delta_D(x)^2 \phi(\delta_D(x)^{-2})},
\]

since \( \int_{\partial D} |x-z|^{-d} \asymp \delta_D(x)^{-1}, \ x \in D \).
For the lower bound fix $x \in D$ and choose $\Gamma = \{ z \in \partial D : |x - z| \leq 2\delta_D(x) \}$. Recall that $\phi$ is increasing so
\[
P_D^D\sigma(x) \succ \int_{\partial D} \frac{\delta_D(x)}{|x - z|^{d+2}\phi(|x - z|^{-2})}\sigma(dz) \geq \frac{1}{4\delta_D(x)^2\phi(\delta_D(x)^{-2})}\delta_D(x) \int_{\Gamma} \frac{\sigma(dz)}{|x - z|^d},
\]
since $\int_{\Gamma} |x - z|^{-d} \succ \delta_D(x)^{-1}$, $x \in D$, by reducing to the flat case, see Lemma A.2. □

**Remark 3.2.** For the classical Poisson kernel $P_D$, defined in (2.45), it is well known that $P_D(x, z) \succ \frac{\delta_D(x)}{|x - z|^d}$, for $x \in D$ and $z \in \partial D$. Moreover, since $P_D$ is the density of $W_{\tau_D}$, we have $P_D\sigma(x) = E_x[1(W_{\tau_D})] = 1$. In particular, by the sharp bound (3.1) and by the scaling condition (2.2), $P_D^D\sigma$ explodes when approaching the boundary of $D$ whereas $P_D\sigma$ obviously does not.

**Remark 3.3.** In what follows we will need the following inequality
\[
P_D^D(x, z) \succ \frac{\delta_D(x)}{|x - z|^d}, \quad x \in D,
\]
which holds by the sharp bounds (2.47) and (3.1), and since by (2.6) it holds that $\delta_D(x)^2\phi(\delta_D(x)^{-2}) \leq |x - z|^2\phi(|x - z|^{-2})$, for $x \in D$ and $z \in \partial D$.

The two following propositions deal with the boundary behaviour of Poisson integrals. They generalize [3, Proposition 25 & Theorem 26] to our more general non-local setting.

**Proposition 3.4.** Let $\zeta \in C(\partial D)$. It holds
\[
\lim_{D \ni x \to z \in \partial D} \frac{P_D^D(\zeta(x))}{P_D^D\sigma(x)} = \zeta(z)
\]
uniformly on $\partial D$.

**Proof.** Note that $\zeta$ is uniformly continuous since $D$ is bounded and let $M = 2\sup_{z \in \partial D} |\zeta(z)|$. For $\varepsilon > 0$ choose $\eta > 0$ such that if $y, z \in \partial D$ and $|y - z| < \eta$, then $|\zeta(y) - \zeta(z)| \leq \varepsilon$. For $z \in \partial D$ let $\Gamma_z = \{ y \in \partial D : |y - z| < \eta \}$. Now if $|x - z| \leq \frac{\eta}{2}$, then by using (3.2) we have
\[
\left| \frac{P_D^D\zeta(x)}{P_D^D\sigma(x)} - \zeta(z) \right| \leq \frac{1}{P_D^D\sigma(x)} \int_{\partial D} P_D^D(x, y) |\zeta(y) - \zeta(z)| \sigma(dy)
\]
\[
\leq c_1\delta_D(x) \int_{\Gamma_z} \frac{|\zeta(y) - \zeta(z)|}{|x - y|^d} \sigma(dy) + c_1\delta_D(x) \int_{\partial D \setminus \Gamma_z} \frac{|\zeta(y) - \zeta(z)|}{|x - y|^d} \sigma(dy)
\]
\[
\leq c_2\varepsilon + c_1\delta_D(x)M\sigma(\partial D) \left( \frac{\eta}{2} \right)^{-d},
\]
where in the last inequality for the first term we used $\delta_D(x) \succ \int_{\partial D} |x - y|^{-d}\sigma(dy)$, hence $c_2 = c_2(d, D, \phi) > 0$. Now the claim follows by taking $x$ close enough to $z$. □
**Proposition 3.5.** For any \( \zeta \in L^1(\partial D) \) and any \( \varphi \in C(\Omega) \) it holds that

\[
\frac{1}{t} \int_{\delta_D(x) \leq t} \frac{P_D^\phi \zeta(x)}{P_D^\phi \sigma(x)} \varphi(x) dx \xrightarrow{t \downarrow 0} \int_{\partial D} \varphi(y) \zeta(y) d\sigma(y).
\]

**Proof.** We can repeat the proof of [3, Theorem 26] almost to the letter. Indeed, take \( \varphi \in C(D) \) and note the \( h_1 \) of [3] is our \( P_D^\phi \sigma \), and \( \varphi \) of [3] is our \( \varphi \). We repeat the proof up to the definition of

\[
\Phi(t, y) := \frac{1}{t} \int_{\delta_D(x) < t} \frac{P_D^\phi(x, y)}{P_D^\phi \sigma(x)} \varphi(x) dx.
\]

Now we use Remark 3.3 and the boundedness of \( \varphi \) to obtain

\[
|\Phi(t, y)| \leq c_1 \frac{\|\varphi\|_{L^\infty(D)}}{t} \int_{\delta_D(x) < t} \frac{\delta_D(x)}{|x-y|^d} dx \leq c_2,
\]

by the reduction to the flat boundary, see [3, Lemma 40], where \( c_2 = c_2(\phi, D, d, \varphi) > 0 \). The rest of the proof is now the same as in [3]. \( \square \)

Now we turn to the boundary behaviour of Green integrals. Here the pointwise limits are harder to get and we must assume some kind of uniformity of the integrating function.

**Theorem 3.6.** Let \( U : (0, \infty) \to [0, \infty) \) such that

1. **(U1) integrability condition holds**

\[
\int_0^1 U(t) t dt < \infty; \quad \text{(3.3)}
\]

2. **(U2) almost non-increasing condition holds**, i.e. there exists \( C > 0 \) such that

\[
U(t) \leq CU(s), \quad 0 < s \leq t \leq 1; \quad \text{(3.4)}
\]

3. **(U3) reverse doubling condition holds**, i.e. there exists \( C > 0 \) such that

\[
U(t) \leq CU(2t), \quad t \in (0, 1); \quad \text{(3.5)}
\]

4. **(U4) boundedness away from zero holds**, i.e. \( U \) is bounded from above on \([c, \infty)\) for each \( c > 0 \).

Then \( U(\delta_D) \in L^1(D, \delta_D(x) dx) \) and

\[
G_D^\phi(U(\delta_D))(x) \asymp \frac{1}{\delta_D(x) \phi(\delta_D(x))^{-2}} \int_0^{\delta_D(x)} U(t) t dt + \delta_D(x) \int_{\delta_D(x)}^{\text{diam}D} \frac{U(t)}{t^2 \phi(t^{-2})} dt.
\]

In particular,

\[
\lim_{D \ni x \to z \in \partial D} \frac{G_D^\phi(U(\delta_D))(x)}{P_D^\phi \sigma(x)} = 0. \quad \text{(3.7)}
\]

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This theorem generalizes [19, Proposition 7] to more general non-local operators and more general functions since in [19] this result was proved in the case of the spectral fractional Laplacian and for functions of the form $U(t) = t^\beta$.

**Proof of Theorem 3.6.** The proof of this claim is very similar to the proof of [8, Proposition 4.1] so the details can be found in the Appendix in Section A.2.

The following proposition appears as [3, Theorem 27] for the case of the spectral fractional Laplacian but in our more general setting the proof gets a little more complicated, cf. [3, Eq. (46)] and (3.10).

**Proposition 3.7.** Let $\lambda \in \mathcal{M}(D)$ such that $\int_D \delta_D(x) |\lambda|(dx) < \infty$. Then

$$\frac{1}{t} \int_{\{\delta_D(x) \leq t\}} \frac{G_D^\phi(x)}{P_D^\phi \sigma(x)} \varphi(x) dx \xrightarrow{t \downarrow 0} 0, \quad \varphi \in C(\overline{D}). \quad (3.8)$$

**Proof.** Without loss of generality we may assume that $\lambda$ is a non-negative measure. It is enough to prove that (3.8) holds for $\varphi \equiv 1$. By using Fubini’s theorem it follows that

$$\frac{1}{t} \int_{\{\delta_D(x) \leq t\}} \frac{G_D^\phi(x)}{P_D^\phi \sigma(x)} dx = \int_D \left( \frac{1}{t} \int_{\{\delta_D(y) \leq t\}} \frac{G_D^\phi(x,y)}{P_D^\phi \sigma(x)} dx \right) \lambda(dy). \quad (3.9)$$

Lemma A.4 for $U \equiv 1 \geq 1/P_D^\phi \sigma$ and Lemma A.5 imply that there is $C = C(d,D,\phi) > 0$ such that

$$\int_{\{\delta_D(x) \leq t\}} \frac{G_D^\phi(x,y)}{P_D^\phi \sigma(x)} dx \leq \begin{cases} Ct \delta_D(y), & \delta_D(y) < \frac{t}{2}, \\ C \tilde{f}(y,t), & \delta_D(y) \geq \frac{t}{2}, \end{cases} \quad (3.10)$$

where $0 \leq \tilde{f}(y,t) \leq t \delta_D(y)$ in $\{\delta_D(y) \geq \frac{t}{2}\}$ and $f(y,t)/t \to 0$ as $t \to 0$ for every $y \in D$. Hence, (3.9) and (3.10) imply

$$\frac{1}{t} \int_{\{\delta_D(x) \leq t\}} \frac{G_D^\phi(x)}{P_D^\phi \sigma(x)} dx \leq C \int_{\{\delta_D(y) < \frac{t}{2}\}} \delta_D(y) \lambda(dy) + C \int_{\{\delta_D(y) \geq \frac{t}{2}\}} \frac{\tilde{f}(y,t)}{t} \lambda(dy)$$

from which the claim of the lemma follows by using the dominated convergence theorem.

\[\Box\]

### 4 Linear Dirichlet problem

In this section we deal with a linear Dirichlet problem for $\phi(-\Delta_D)$ and develop some basic properties of a weak solution to the problem. At the end of the section, we connect the weak formulation of the problem with the distributional.

**Definition 4.1.** Let $\lambda \in \mathcal{M}(D)$ and $\zeta \in \mathcal{M}(\partial D)$ such that

$$\int_D \delta_D(x) |\lambda|(dx) + |\zeta|(\partial D) < \infty. \quad (4.1)$$
We say that $u \in L^1_{loc}(D)$ is a weak solution to the problem
\[
\begin{aligned}
\phi(-\Delta)|_D u &= \lambda, \quad \text{in } D, \\
\frac{u}{P^\phi_{\sigma,\phi}} &= \zeta, \quad \text{on } \partial D,
\end{aligned}
\] (4.2)
if for every $\psi \in C_c^\infty(D)$ it holds that
\[
\int_D u(x)\psi(x)dx = \int_D G^\phi_D(\psi(x)\lambda(x)) - \int_{\partial D} \frac{\partial}{\partial n} G^\phi_D(\psi(z)\zeta(z)).
\] (4.3)
If in (4.3) we have $\leq$ ($\geq$) instead of the equality and the inequality holds for every non-negative $\psi \in C_c^\infty(D)$, then we say $u$ is a weak subsolution (supersolution) to the problem (4.2).

Remark 4.2. (a) Let $\psi \in C_c^\infty(D)$. From the calculations in the proof of Proposition 2.19, see also (2.46) and (2.47), it follows that $\frac{\partial}{\partial n} G^\phi_D(\psi(z))$ is well defined and
\[
-\frac{\partial}{\partial n} G^\phi_D(\psi(z)) = \int_D P^\phi_D(y,z)\psi(y)dy, \quad z \in \partial D,
\] holds, hence $\frac{\partial}{\partial n} G^\phi_D(\psi) \in L^\infty(\partial D)$. Moreover, Lemma 2.8 implies that $|G^\phi_D(\psi(x))| \lesssim \delta_D(x)$, thus the condition (4.1) ensures that the integrals in (4.3) are well defined.

(b) If $u$ is a solution to the linear problem (4.2), then by using Fubini’s theorem in (4.3) we get that
\[
u = G^\phi_D \lambda + P^\phi_D \zeta, \quad \text{a.e. in } D. \tag{4.4}
\]
This implies that $u \in L^1(D,\delta_D(x)dx)$. Indeed, $G^\phi_D \lambda \in L^1(D,\delta_D(x)dx)$ by Lemma 2.8, and $P^\phi_D \zeta \in L^1(D,\delta_D(x)dx)$ by (2.54).

Conversely, the function defined in (4.4) is the solution of linear problem (4.2) which we also get by using Fubini’s theorem in (4.3).

The following theorem summarizes the previous remark.

**Theorem 4.3.** Let $\lambda \in M(D)$ and $\zeta \in M(\partial D)$ such that (4.1) holds. Then the linear problem (4.2) has a unique weak solution $u$ for which it holds that $u \in L^1(D,\delta_D(x)dx)$ and
\[
u(x) = G^\phi_D \lambda(x) + P^\phi_D \zeta(x), \quad \text{for a.e. } x \in D.
\]
Furthermore, there is $C = C(d,D,\phi) > 0$ such that
\[
\|u\|_{L^1(D,\delta_D(x)dx)} \leq C \left( \int_D \delta_D(x)|\lambda|(dx) + |\zeta|(\partial D) \right). \tag{4.5}
\]

In the next corollary we bring a version of a maximum principle for the weak solution.

**Corollary 4.4.** Let $\lambda \in M(D)$ and $\zeta \in M(\partial D)$ such that (4.1) holds. If $\lambda \geq 0$ and $\zeta \geq 0$, then the unique solution $u$ of the linear problem (4.2) satisfies $u \geq 0$ a.e. in $D$. 27
Now we connect the weak and the distributional formulation of the Dirichlet problem. First, we define the distributional solution.

**Definition 4.5.** We say that \( u \in L^1(D, \delta_D(x)dx) \) is a distributional solution of (4.2) if for every \( \psi \in C_\infty_c(D) \) it holds that
\[
\int_D u(x)\phi(-\Delta|_D)\psi(x)dx = \int_D \psi(x)\lambda(dx),
\]
and if for every \( \varphi \in C(\partial D) \) it holds that
\[
\lim_{t \downarrow 0} \frac{1}{t} \int_{\{\delta_D(x) \leq t\}} \frac{u(x)}{P_D^\phi \sigma(x)} \varphi(x)dx = \int_{\partial D} \varphi(z)\zeta(dz).
\]  

**Proposition 4.6.** Let \( \lambda \in M(D) \) and \( \zeta \in L^1(\partial D) \) such that (4.1) holds. Then the weak solution of (4.2) is also a distributional solution of (4.2).

**Proof.** The weak solution is given by \( u = G_D^\phi \lambda + P_D^\phi \zeta \) so the relation (4.6) follows from Proposition 2.14 and Theorem 2.23. The boundary condition (4.7) follows from Proposition 3.5 and Proposition 3.7.

## 5 Semilinear Dirichlet problem

In this section we study the following semilinear problem.

**Definition 5.1.** Let \( f : D \times \mathbb{R} \to \mathbb{R} \) and \( \zeta \in M(\partial D) \) such that \(|\zeta|(|\partial D|) < \infty\). We say that \( u \in L^1_{loc}(D) \) is a weak solution to the problem
\[
\begin{cases}
\phi(-\Delta|_D)u(x) = f(x, u(x)), & \text{in } D, \\
\frac{u}{P_D^\phi \sigma} = \zeta, & \text{on } \partial D,
\end{cases}
\]  
if
\[
\int_D u(x)\psi(x)dx = \int_D G_D^\phi \psi(x)f(x, u(x))dx - \int_{\partial D} \frac{\partial}{\partial n} G_D^\phi \psi(z)\zeta(dz), \quad \psi \in C_\infty_c(D). 
\]  

If in the equation above we have \( \leq (\geq) \) instead of the equality and the inequality holds for every non-negative \( \psi \in C_\infty_c(D) \), then we say \( u \) is a weak subsolution (supersolution) to (5.1).

Note that if \( u \) is a solution to the semilinear problem (5.1), then it is implicitly assumed that \( x \mapsto f(x, u(x)) \in L^1(D, \delta_D(x)dx) \) since only then the first integral in (5.2) is well defined. For the sake of brevity, we will frequently use the notation \( f_u(x) := f(x, u(x)) \), \( x \in D \), which is also known as the Nemytskii operator. Further, in the same way as in the linear case, we can see that if \( u \) is a weak solution of (5.1), then by Fubini’s theorem used in (5.2) we get
\[
u = G_D^\phi f_u + P_D^\phi \zeta.
\]

Conversely, if \( u \) satisfies (5.3), then \( u \) is a weak solution of (5.1).

In the following subsection we prove Kato’s inequality in our setting. This will help us to obtain existence and uniqueness results for various nonlinearities \( f \) in the semilinear problem, which we do in the final subsection of the article.
5.1 Kato’s Inequality

The proof of Kato’s inequality in our setting, i.e. Proposition 5.4, is motivated by the proofs of Kato’s inequality found in [3, 15] for the case of the spectral fractional Laplacian and the fractional Laplacian, respectively. First we need a lemma.

**Lemma 5.2.** Let \( w \) be the weak solution to the linear problem

\[
\begin{cases}
\phi(-\Delta|_D) u = h, & \text{in } D, \\
\frac{\partial u}{\partial \nu} = 0, & \text{on } \partial D,
\end{cases}
\]

for \( h \in L^1(D, \delta_D(x)\,dx) \). Let \( \Lambda \in C^2(\mathbb{R}) \) be a convex function such that \( \Lambda(0) = 0 \), and such that \( |\Lambda'| \leq C \) for some \( C > 0 \). Then

\[
\int_D \Lambda(w(x))\phi(-\Delta|_D)\psi(x)\,dx \leq \int_D \Lambda'(w(x))h(x)\psi(x)\,dx, \quad \psi \in C_c^\infty(D),
\]

and

\[
\Lambda(w) \leq G_D^\phi [\Lambda'(w)h] \quad \text{a.e. in } D.
\]

**Proof.** Recall that \( w = G_D^\phi h \in L^1(D, \delta_D(x)\,dx) \).

Let \( h \in C_c^\infty(D) \). Then by Proposition 2.18 we have \( w = G_D^\phi h \in C^{1,1}(\overline{D}) \) from which we can calculate \( \phi(-\Delta|_D)w \) and \( \phi(-\Delta|_D)\Lambda(w) \) pointwisely, see Proposition 2.15. We have

\[
\phi(-\Delta|_D)[\Lambda \circ w](x) = \text{P.V.} \int_D [\Lambda(w(x)) - \Lambda(w(y))] J_D(x,y) \, dy + \kappa(x) \Lambda(w(x))
\]

\[
= \Lambda'(w(x)) \text{P.V.} \int_D [w(x) - w(y)] J_D(x,y) \, dy + \kappa(x) \Lambda(w(x))
\]

\[
- \text{P.V.} \int_D \left( [w(x) - w(y)]^2 J_D(x,y) \int_0^1 \Lambda''(w(x) + t[w(y) - w(x)]) (1-t) \, dt \right) \, dy
\]

\[
\leq \Lambda'(w(x)) \phi(-\Delta|_D)w(x),
\]

where we have used that \( \Lambda'' \geq 0 \) in \( \mathbb{R} \) and that \( \Lambda(t) \leq t \Lambda'(t) \), which follows from \( \Lambda(0) = 0 \) and the fact that \( \Lambda' \) is non-decreasing. Integrating the previous inequality with respect to \( \psi(x)\,dx \), where \( 0 \leq \psi \in C_c^\infty(D) \), we get (5.4). Furthermore, since both \( w \) and \( \Lambda(w) \) are in \( C^{1,1}(\overline{D}) \), both sides of the previous inequality are in \( L^2(D) \), see Remark 2.3, so we can apply Proposition 2.2 to get

\[
\Lambda(w) = G_D^\phi [\phi(-\Delta|_D)\Lambda(w)] \leq G_D^\phi [\Lambda'(w)h] \quad \text{a.e. in } D,
\]

i.e. (5.5) holds.

Let \( h \in L^1(D, \delta_D(x)\,dx) \) and \( (h_n)_n \subset C_c^\infty(D) \) such that \( h_n \to h \) in \( L^1(D, \delta_D(x)\,dx) \) and a.e. in \( D \). By Corollary 2.9 we have \( w_n := G_D^\phi h_n \to w \) in \( L^1(D, \delta_D(x)\,dx) \) so by considering a subsequence we may assume that \( w_n \to w \) a.e., too. From the first part of the proof we know

\[
\int_D \Lambda(w_n(x))\phi(-\Delta|_D)\psi(x)\,dx \leq \int_D \Lambda'(w_n(x))h_n(x)\psi(x)\,dx
\]

and

\[
\Lambda(w_n) \leq G_D^\phi [\Lambda'(w_n)h_n] \quad \text{a.e. in } D,
\]
for all \( n \in \mathbb{N} \) and all \( 0 \leq \psi \in C_c^\infty(D) \).

Now we will take \( n \) in (5.6) and (5.7) to infinity. Recall that \( |\phi(-\Delta_D)|\psi| \leq C_1 \delta_D \) by Lemma 2.11. Also, since \( |\Lambda'| \leq C \), we have \( |\Lambda(t) - \Lambda(s)| \leq C|t - s| \). By using these two facts and the fact that both \( w_n \to w \) and \( h_n \to h \) in \( L^1(D, \delta_D(x)dx) \), both sides of (5.6) converge. Hence, by taking the limit in (5.6) we obtain

\[
\int_D \Lambda(w(x))\phi(-\Delta_D)\psi(x)dx \leq \int_D \Lambda'(w(x))h(x)\psi(x)dx.
\]

Before we take the limit in equality (5.7), note that \( \Lambda \in C^2(\mathbb{R}) \) so \( \Lambda(w_n) \to \Lambda(w) \) and \( \Lambda'(w_n) \to \Lambda'(w) \) a.e. in \( D \). Further, again by \( |\Lambda'| \leq C \) and the fact that \( h_n \to h \) in \( L^1(D, \delta_D(x)dx) \) we have

\[
\left| G_D^\phi \left[ \Lambda'(w_n)h_n \right] - G_D^\phi \left[ \Lambda'(w)h \right] \right| \leq G_D^\phi \left[ |\Lambda'(w) - \Lambda'(w_n)||h| \right] + G_D^\phi \left[ |\Lambda'(w_n)||h - h_n| \right] \to 0, \quad n \to \infty,
\]

where the first term goes to zero by the dominated convergence theorem, and the second by the continuity of \( G_D^\phi \) acting on \( L^1(D, \delta_D(x)dx) \), i.e. by Lemma 2.8. This calculation justifies taking the limit in (5.7) to get

\[
\Lambda(w) \leq G_D^\phi \left[ \Lambda'(w)h \right] \quad \text{a.e. in } D.
\]

\[ \square \]

**Remark 5.3.** For \( h \in L^\infty(D) \) the inequalities (5.4) and (5.5) hold for every convex function \( \Lambda \in C^2(\mathbb{R}) \) such that \( \Lambda(0) = 0 \) since the assumption \( |\Lambda'| \leq C \) was used only as a technical tool to justify the usage of the dominated convergence theorem for general \( h \in L^1(D, \delta_D(x)dx) \).

In the next proposition we prove Kato’s inequality which says that we can take \( \Lambda(t) = t^+ = t \vee 0 \) in Lemma 5.2.

**Proposition 5.4** (Kato’s inequality). Let \( w \) be the weak solution to the linear problem

\[
\begin{align*}
\phi(-\Delta_D)u &= h, \quad \text{in } D, \\
\frac{\partial u}{\partial \nu_{D,s}} &= 0, \quad \text{on } \partial D,
\end{align*}
\]

for \( h \in L^1(D, \delta_D(x)dx) \). Then for every \( \psi \in C_c^\infty(D) \), \( \psi \geq 0 \), it holds that

\[
\int_D w(x)^+ \phi(-\Delta_D)\psi(x)dx \leq \int_{\{w > 0\}} h(x)\psi(x)dx. \tag{5.8}
\]

Moreover, it holds that

\[
w^+ \leq G_D^\phi \left[ 1_{\{w > 0\}}h \right], \quad \text{a.e. in } D. \tag{5.9}
\]
Proof. First, let us prove (5.8). Set \( \Lambda(t) = t \vee 0 \) and \( w = G_D^\phi h \) where \( h \in L^1(D, \delta_D(x)dx) \). Also, for every \( n \in \mathbb{N} \) let \( \Lambda_n : \mathbb{R} \to \mathbb{R} \) be defined by

\[
\Lambda_n(t) = \begin{cases} 
0, & t \leq 0 \\
\frac{n^2t^3}{6}, & t \in (0, \frac{1}{n}] \\
\frac{1}{3n} - t + nt^2 - \frac{nt^3}{6}, & t \in (\frac{1}{n}, \frac{2}{n}] \\
t - \frac{1}{n}, & t > \frac{2}{n}.
\end{cases}
\]

(5.10)

We have that \( \Lambda_n \in C^2(\mathbb{R}) \), \( 0 \leq \Lambda_n \leq \Lambda \), and \( 0 \leq \Lambda'_n \leq 1 \) in \( \mathbb{R} \). Also, \( \Lambda_n \to \Lambda \) and \( \Lambda'_n \to 1 \) \((0, \infty)\) in \( \mathbb{R} \) as \( n \to \infty \). Thus, Lemma 5.2 yields

\[
\int_D \Lambda_n(w(x))\phi(-\Delta|_D)\varphi(x)dx \leq \int_D \Lambda'_n(w(x))h(x)\varphi(x)dx
\]

(5.11)

and the relation (5.8) follows from (5.11) by using the dominated convergence theorem.

Let us now turn to (5.9). Consider again the sequence \( \Lambda_n \) defined above. Lemma 5.2 yields

\[
\Lambda_n(w) \leq G_D^\phi [\Lambda'_n(w)h], \quad \text{a.e. in } D \text{ and for all } n \in \mathbb{N}.
\]

(5.12)

Again, by taking \( n \to \infty \) and by using the dominated convergence theorem we get

\[
w^+ \leq G_D^\phi [1_{\{w \geq 0\}}h], \quad \text{a.e. in } D.
\]

\[\square\]

Remark 5.5. By modifying the proof of the previous proposition we also get

\[
\int_D w(x)^+\phi(-\Delta|_D)\psi(x)dx \leq \int_{\{w \geq 0\}} h(x)\psi(x)dx,
\]

(5.13)

and

\[
w^+ \leq G_D^\phi [1_{\{w \geq 0\}}h], \quad \text{a.e. in } D.
\]

(5.14)

Indeed, in the proof we only need to change \( \Lambda_n \) to \( \tilde{\Lambda}_n \in C^2(\mathbb{R}) \) such that \( \tilde{\Lambda}_n(t) = \Lambda_n(t + \frac{2}{n}) - \frac{1}{n} \). For \( \tilde{\Lambda}_n \) it holds that

\[-\frac{1}{n} \leq \tilde{\Lambda}_n \leq \Lambda, \quad 0 \leq \tilde{\Lambda}'_n \leq 1, \quad \lim_n \tilde{\Lambda}_n = \Lambda, \quad \text{and} \quad \lim_n \tilde{\Lambda}'_n = 1_{(0, \infty)} \]

in \( \mathbb{R} \). By repeating the procedure in the proof of the previous proposition we get the claim.

Remark 5.6. Note that Kato’s inequality was proved only for weak solutions of linear problems with a zero boundary condition whereas the classical Kato’s inequality holds for subsolutions even if the considered linearity is a measure, see [13]. To the best of our knowledge it is not clear whether the inequality (5.8) holds for subsolutions since the non-local nature of the operator \( \phi(-\Delta|_D) \) causes problems in the calculations in Proposition 5.4. Even in simpler non-local cases as in [3] and [15] Kato’s inequality was proved only for solutions, see [3, Lemma 31] and [15, Proposition 2.4].
In the next corollary we bring a simple consequence of Kato’s inequality which is the fact interesting in itself.

**Corollary 5.7.** Let \( u \) and \( v \) be weak solutions of (5.1). Then \( \max\{u, v\} \) is a subsolution of (5.1).

**Proof.** Applying Proposition 5.4 to the \( w := u - v \) and \( h(x) := f(x, u(x)) - f(x, v(x)) \) we get

\[
\int_D w^+(x) \phi(-\Delta|_D) \psi(x) dx \leq \int_{u>v} [f(x, u(x)) - f(x, v(x))] \psi(x) dx, \quad \psi \in C_c^\infty(D), \psi \geq 0.
\]

Since \( \max\{u, v\} = v + (u - v)^+ = v + w^+ \) we have for all non-negative \( \psi \in C_c^\infty(D) \)

\[
\int_D \max\{u, v\}(x) \phi(-\Delta|_D) \psi(x) dx
\]

\[
\leq \int_D f(x, v(x)) \psi(x) dx - \int_{\partial D} \frac{\partial}{\partial n} G_D^\phi \psi(z) \zeta(dz)
\]

\[
+ \int_{u>v} [f(x, u(x)) - f(x, v(x))] \psi(x) dx
\]

\[
= \int_{u\leq v} f(x, v(x)) \psi(x) dx - \int_{\partial D} \frac{\partial}{\partial n} G_D^\phi \psi(z) \zeta(dz)
\]

\[
+ \int_{u>v} f(x, u(x)) \psi(x) dx
\]

\[
= \int_D f(x, \max\{u, v\}(x)) \psi(x) dx - \int_{\partial D} \frac{\partial}{\partial n} G_D^\phi \psi(z) \zeta(dz).
\]

\( \square \)

### 5.2 Semilinear problem

In this subsection we prove the existence and uniqueness results for the semilinear problem (5.1). As such, the subsection is central to the article.

For the nonlinearity \( f \) in the following problems we will almost always assume that the following condition holds.

\[ (F). \quad f : D \times \mathbb{R} \to \mathbb{R} \text{ is continuous in the second variable, and there exist a locally bounded function } \rho : D \to [0, \infty] \text{ and a non-decreasing function } \Lambda : [0, \infty) \to [0, \infty) \text{ such that } |f(x, t)| \leq \rho(x)\Lambda(|t|), \ x \in \bar{D}, \ t \in \mathbb{R}. \]

From now on, the function \( f \) will be solely used as a nonlinearity in the semilinear problem and the functions \( \rho \) and \( \Lambda \) are solely used as the functions in the condition (F) for \( f \).

Our first result is the uniqueness theorem for general nonlinearity \( f \) which is non-increasing in the second variable.
Proposition 5.8. If the nonlinearity $f$ in (5.1) is non-increasing in the second variable, then the weak solution of (5.1), if it exists, is unique (up to the modification on the Lebesgue null set).

**Proof.** Let $u$ and $v$ be two solutions of (5.1). Then $w := u - v$ solves the linear problem

$$
\begin{aligned}
\phi(-\Delta|_D)w(x) &= f(x,u(x)) - f(x,v(x)), & \text{in } D, \\
\frac{w}{P^\sigma_D} &= 0, & \text{on } \partial D.
\end{aligned}
$$

By Kato’s inequality (5.9), since $f$ is non-increasing in the second variable, we have

$$w^+ \leq G^\phi_D \left[ 1_{\{u > v\}} \cdot (f_u - f_v) \right] \leq 0. \quad (5.15)$$

Thus, $u \leq v$ a.e. in $D$. Reversing the roles of $u$ and $v$ we get $u \geq v$ a.e. in $D$, hence $u = v$ a.e. in $D$. \qed

The next theorem, Theorem 5.9, deals with a semilinear problem with a zero boundary condition and it is a generalization of [3, Theorem 32] to our setting of more general non-local operators. Theorem 5.9 will be of great importance for a general semilinear problem (with a non-zero boundary condition), and it is, in fact, the cornerstone of the proof of Theorem 5.10. A somewhat similar role to a semilinear problem in slightly different non-local setting plays [8, Theorem 3.6].

**Theorem 5.9.** Let $f$ satisfy (F). Assume that there exist a supersolution $\overline{u}$ and a sub-solution $\underline{u}$ to the semilinear problem

$$
\begin{aligned}
\phi(-\Delta|_D)u(x) &= f(x,u(x)), & \text{in } D, \\
\frac{u}{P^\sigma_D} &= 0, & \text{on } \partial D,
\end{aligned}
$$

of the form $\underline{u} = G^\phi_D h \underline{h}$ and $\overline{u} = G^\phi_D \overline{h}$ such that $\underline{u} \leq \overline{u}$, $\overline{h}(x) \leq f(x,\underline{u}(x))$ and $f(x,\overline{u}(x)) \leq \overline{h}(x)$ a.e. in $D$, and such that $
\overline{u}, \underline{u} \in L^1(D, \delta_D(x)dx) \cap L^\infty_{\text{loc}}(D)$. Further, assume that $\rho \Lambda(\underline{u} \vee \overline{u}) \in L^1(D, \delta_D(x)dx)$.

Then there exist weak solutions $u_1, u_2 \in L^1(D, \delta_D(x)dx)$ of (5.16) such that every solution of (5.16) with property $\underline{u} \leq u \leq \overline{u}$ satisfies

$$\underline{u} \leq u_1 \leq u \leq u_2 \leq \overline{u}.\quad (5.16)$$

Furthermore, every weak solution $u$ of (5.16) with property $\underline{u} \leq u \leq \overline{u}$ is continuous after the modification on a Lebesgue null set.

Additionally, if the nonlinearity $f$ is non-increasing in the second variable, the weak solution of (5.16) is unique.

**Proof.** Step 1: existence of a solution to (5.16). Define the function $F : D \times \mathbb{R} \to \mathbb{R}$ by

$$
F(x,t) = \begin{cases} 
  f(x,u(x)), & t < \underline{u}(x), \\
  f(x,t), & \underline{u} \leq t \leq \overline{u}, \\
  f(x,\overline{u}(x)), & \overline{u}(x) < t,
\end{cases}
$$

where $f(x,u(x)) = f(x,\underline{u}(x))$ for $u(x) \leq \underline{u}(x)$, and $f(x,\overline{u}(x)) = f(x,\overline{u}(x))$ for $\overline{u}(x) \leq u(x)$.
and denote by $F_v(x) := F(x, v(x))$. Note that since $f$ is continuous in the second variable, so is $F$. Further, $|F_v| \leq \rho \Lambda([|u| \vee |\pi|])$, hence $F_v \in L^1(D, \delta_D(x)dx)$, for all $v \in L^1(D, \delta_D(x)dx)$.

Also, the mapping $v \mapsto F_v$ is continuous from $L^1(D, \delta_D(x)dx)$ to $L^1(D, \delta_D(x)dx)$. Indeed, take $v_n \to v$ in $L^1(D, \delta_D(x)dx)$ and let $(v_{n_k})_k$ be a subsequence of $(v_n)_n$ which converges to $v$ a.e. By Lemma A.6 the family $\{F_{v_{n_k}} : k \in \mathbb{N}\}$ is uniformly integrable with respect to the measure $\delta_D(x)dx$, hence by Vitali’s theorem [35, Theorem 16.6], we get $F_{v_{n_k}} \to F_v$ in $L^1(D, \delta_D(x)dx)$ because $F$ is continuous in the second variable. However, the limit does not depend on the subsequence $(v_{n_k})_k$ so $v \mapsto F_v$ is continuous.

Next we prove that the operator $K : L^1(D, \delta_D(x)dx) \to L^1(D, \delta_D(x)dx)$ defined by

$$Kv(x) = \int_D G_D^\phi(x, y)F(y, v(y))dy, \quad x \in D,$$

is compact. Since $v \mapsto F_v$ is continuous in $L^1(D, \delta_D(x)dx)$, Corollary 2.9 implies that $K$ is continuous $L^1(D, \delta_D(x)dx)$, too. To have compactness, we are left to prove that $K$ maps bounded sets to relatively compact sets. To this end, take a bounded sequence $(v_n)_n \subset L^1(D, \delta_D(x)dx)$. Recall that $|F_{v_n}| \leq \rho \Lambda([|u| \vee |\pi|])$, and notice that $\rho \Lambda([|u| \vee |\pi|]) \in L^1(D, \delta_D(x)dx) \cap L_{\text{loc}}^\infty(D)$ since $\pi, u \in L^1(D, \delta_D(x)dx) \cap L_{\text{loc}}^\infty(D)$, so $(Kv_n)_n$ are pointwise bounded by Proposition 2.16 and equicontinuous by Remark 2.17. By Arzelà-Ascoli theorem, there is a subsequence $(Kv_{n_k})_k$ of $(Kv_n)_n$ which converges pointwise to some $u \in C(D) \cap L^1(D, \delta_D(x)dx)$. Since $Kv_n = G_D^\phi F_{v_n}$, Lemma 2.8 implies that $\{Kv_{n_k} : k \in \mathbb{N}\}$ is uniformly integrable with respect to the measure $\delta_D(x)dx$ since $\{F_{v_{n_k}} : k \in \mathbb{N}\}$ is. However, $Kv_{n_k} \to u$ pointwise so by Vitali’s theorem [35, Theorem 16.6] we have $Kv_{n_k} \to u$ in $L^1(D, \delta_D(x)dx)$.

This means that $K$ is compact so by Schauder’s fixed point theorem there is $u \in L^1(D, \delta_D(x)dx)$ such that $Ku = u$ in $D$, i.e. $u$ solves

$$\begin{cases}
\phi(-\Delta_D)u(x) = F(x, u(x)), & \text{in } D, \\
\dfrac{\partial u}{\partial \nu_D} = 0, & \text{on } \partial D.
\end{cases}$$

We need to prove that $\underline{u} \leq u \leq \overline{u}$ in $D$ which would mean that $u$ also solves (5.16). For this step we will use Kato’s inequality.

More precisely, applying Proposition 5.4 to $w = u - \overline{u} = G_D^\phi(F_u - \overline{h})$ we get

$$\begin{equation}
(u - \overline{u})^+ \leq G_D^\phi \left[1_{\{|u > \overline{u}|\}} \cdot (F_u - \overline{h})\right] \leq G_D^\phi \left[1_{\{|u > \overline{u}|\}} \cdot (f\overline{u} - f\overline{u})\right] = 0, \tag{5.17}
\end{equation}$$

where the second inequality holds since $F(x, u(x)) = f(x, \overline{u}(x))$ on $\{u \geq \overline{u}\}$ and since we assume $f(x, \overline{u}) \leq \overline{h}$ a.e. in $D$. This means $u \leq \overline{u}$ a.e. in $D$. Similarly we get that $\underline{u} \leq u$ a.e. in $D$. Hence, we found a solution to the problem (5.16).

Step 2: finding the maximal and the minimal solution. We adapt a method from [17, Theorem 1.3] which uses Zorn’s lemma.

Let $\mathcal{P} := \{u \in L^1(D, \delta_D(x)dx) : \underline{u} \leq u \leq \overline{u} \text{ and } u \text{ solves (5.16)}\}$. Let $\{u_i\}_{i \in I}$ be a totally ordered subset of $\mathcal{P}$. Since $u_i \in \mathcal{P}$ and since we have $\rho \Lambda([|u| \vee |\pi|]) \in L^1(D, \delta_D(x)dx) \cap L_{\text{loc}}^\infty(D)$, it follows that $\{u_i\}_{i \in I}$ is equicontinuous in $D$. In fact, by Remark 2.17 the set $\{u_i\}_{i \in I}$ is equicontinuous on every compact subset of $D$. Hence, the function $u := \sup_{i \in I} u_i$ is continuous and $u$ can be approximated by $\{u_i\}_{i \in I}$ uniformly.
on compact subsets of $D$. Moreover, $D$ is $\sigma$-compact so we can choose an increasing sequence $(u_n)_n \subset \{ u_i \}_{i \in \mathbb{Z}}$ such that $\lim_n u_n(x) = u(x)$ for all $x \in D$.

By the dominated convergence theorem, since $|f_{u_n}| \leq \rho(\|u\| \vee |\|\|)$, it easily follows by the continuity of $f$ in the second variable that $u = \lim_n u_n = \lim_n G^\phi_D(f_{u_n}) = G^\phi_D(f_u)$, i.e. $u \in \mathcal{P}$. Now Zorn’s lemma implies that there exists the maximal solution $u_2$ of (5.16). We find the minimal solution $u_1$ in the same way.

**Step 3: continuity of solutions.** We prove that every solution of (5.16) with property $\underline{u} \leq u \leq \overline{u}$ is continuous up to the modification. Indeed, every solution satisfies $u = G^\phi_D f_u$ a.e. in $D$. Furthermore, since $\underline{u} \leq u \leq \overline{u}$ and $\rho(\|u\| \vee |\|\|) \in L^1(D, \delta_D(x)dx) \cap L^\infty(D)$, we have $G^\phi_D f_u \in C(D)$ by Proposition 2.16. Finally, $\tilde{u} := G^\phi_D f_u$ is a continuous modification of $u$, hence $f_u = f_{\tilde{u}}$ a.e. in $D$, hence $\tilde{u} = G^\phi_D f_{\tilde{u}}$ in $D$. 

**Step 4: uniqueness of solution.** In the case when $f$ is non-increasing in the second variable, uniqueness follows from Proposition 5.8. 

By using the previous theorem, a method of sub- and super-solutions and the approximation of harmonic functions, we solve a semilinear problem that deals with a non-positive nonlinearity $f$ and a non-negative boundary condition $\zeta$. Theorem 5.10 generalizes [3, Theorem 8] to our setting of more general non-local operators. Moreover, we consider a more general boundary condition which can also be a measure, whereas in [3, Theorem 8] only continuous functions were considered. The nonlinearity in our theorem is also slightly more general than the one in [3, Theorem 8]. A similar result in a slightly different non-local setting can be found in [8, Theorem 3.10].

**Theorem 5.10.** Let $f : D \times \mathbb{R} \rightarrow (-\infty, 0]$ such that $f(x, 0) = 0$, $x \in D$, and such that $f$ satisfies (F). Further, let $\zeta \in \mathcal{M}(\partial D)$ be a finite non-negative measure such that

$$\rho(\|\|\|) \in L^1(D, \delta_D(x)dx).$$

Then the problem

$$\begin{cases}
\phi(- \Delta|_D) u(x) = f(x, u(x)), & \text{in } D, \\
\frac{\partial}{\partial \nu} G^\phi_D u = \zeta, & \text{on } \partial D,
\end{cases}$$

(5.18)

has a weak solution $u \in C(D) \cap L^1(D, \delta_D(x)dx)$.

Additionally, if $f$ is non-increasing in the second variable, the continuous weak solution of (5.18) is unique.

**Proof.** Let $(\tilde{f}_k)_k$ be a non-negative sequence of bounded functions such that $G^\phi_D \tilde{f}_k \uparrow P^\phi_D \zeta$ in $D$. This sequence exists by [8, Appendix A.1] since the semigroup $(Q^D_t)_t$ is strongly Feller, $G^\phi_D \delta_D \asymp \delta_D$ by Lemma 2.8, and since $P^\phi_D \zeta$ is a continuous function with the mean-value property with respect to $X$, see Theorem 2.25 and Theorem 2.27.

We build a sequence of solutions to the following semilinear problems

$$\begin{cases}
\phi(- \Delta|_D) u(x) = f(x, u(x)) + \tilde{f}_k, & \text{in } D, \\
\frac{\partial}{\partial \nu} G^\phi_D u = 0, & \text{on } \partial D.
\end{cases}$$

(5.19)

For every $k \in \mathbb{N}$, a subsolution to (5.19) is $\underline{u} = 0$ since $f(x, 0) = 0$ and since $\tilde{f}_k \geq 0$. A supersolution to (5.19) is $\overline{u} = G^\phi_D \tilde{f}_k$ because $f$ is non-positive. Note that both $\underline{u}$ and
\( \overline{u} \) are bounded functions, so it is trivial to check that the assumptions of Theorem 5.9 are satisfied. Hence, for every \( k \in \mathbb{N} \) there is a solution \( u_k \geq 0 \) to (5.19) which is also continuous in \( D \) and satisfies

\[
 u_k = G^\phi_D f_{u_k} + G^\phi_D \tilde{f}_k, \quad \text{in } D. \tag{5.20}
\]

Now we find an appropriate subsequence of \((u_k)_k\) which converges to a solution of (5.18). Since \( G^\phi_D \tilde{f}_k \) is continuous and increases to the continuous function \( P^\phi_D \zeta \), by Dini’s theorem the convergence is locally uniform so the usual \( 3\varepsilon \)-argument gives equicontinuity of the family \((G^\phi_D \tilde{f}_k)_k\). Also, since \( |f_{u_k}| \leq \rho \Lambda(P^\phi_D \zeta) \), equicontinuity of \((G^\phi_D (f_{u_k}))_k\) follows by Proposition 2.16 and Remark 2.17. Hence, Arzelà-Ascoli theorem gives us a subsequence, denoted again by \((u_k)_k\), which converges to a continuous function \( u \).

Now we show that \( u \) is a solution of (5.18). Obviously, since \( u = \lim_{k \to \infty} u_k \) and \( 0 \leq u_k \leq G^\phi_D \tilde{f}_k \leq P^\phi_D \zeta < \infty \), \( u \) is non-negative and finite. Further, \( G^\phi_D \tilde{f}_k \uparrow P^\phi_D \zeta \), so we are left to prove that \( G^\phi_D f_{u_k} \to G^\phi_D f_u \). However, this is easy since \( |f_{u_k}| \leq \rho \Lambda(P^\phi_D \zeta) \), so continuity of \( f \) in the second variable and the dominated convergence imply \( G^\phi_D f_{u_k} \to G^\phi_D f_u \).

Uniqueness, if \( f \) is non-increasing in the second variable, follows from Proposition 5.8.

\[ \square \]

**Remark 5.11.** Applying Zorn’s lemma argument from the proof of Theorem 5.9 we get that for the problem (5.18) there exists a minimal solution \( u_1 \) and a maximal solution \( u_2 \) such that for every solution \( u \) of (5.18) we have

\[
 0 \leq u_1 \leq u \leq u_2 \leq P^\phi_D \zeta, \quad \text{in } D.
\]

We say that \( \Lambda : [0, \infty) \to [0, \infty) \) satisfies the doubling condition if there exists \( C > 0 \) such that

\[
 \Lambda(2t) \leq C \Lambda(t), \quad t \geq 1. \tag{5.21}
\]

If \( \Lambda \) is non-decreasing, the condition (5.21) implies that for every \( c_1 > 1 \) there is \( c_2 = c_2(C, c_1) > 0 \) such that

\[
 \Lambda(c_1 t) \leq c_2 \Lambda(t), \quad t \geq 1. \tag{5.22}
\]

**Corollary 5.12.** Let \( f : D \times \mathbb{R} \to (-\infty, 0] \) such that \( f(x, 0) = 0 \), \( x \in D \). Let \( f \) also satisfy (F) such that \( \Lambda \) satisfies the doubling condition (5.21).

If \( \rho \Lambda \left( \frac{1}{\delta_D (\delta_D)} \right) \in L^1(D) \delta_D(x)dx \), then the problem

\[
 \begin{aligned}
 \phi(-\Delta_D) u(x) &= f(x, u(x)), & \text{in } D, \\
 \frac{u}{P^\phi_D} &= \zeta, & \text{on } \partial D,
 \end{aligned} \tag{5.23}
\]

has a continuous weak solution for every non-negative function \( \zeta \in C(D) \). Additionally, if \( f \) is non-increasing in the second variable, the continuous weak solution is unique.

In particular, if \( f(x, t) = -|t|^p \), then the equation (5.23) has a unique continuous weak solution for \( p < \frac{1}{1-\gamma_1} \), where \( \delta_1 \) comes from (2.2).
Proof. Note that for $\zeta \in C(D)$ we have $P_D^\phi \zeta \leq c_1 \frac{1}{\delta_D^2 \phi(\delta_D^{-2})}$ by Lemma 3.1 since $\zeta$ is bounded on $\partial D$. Thus, from the doubling condition we have
\[
\rho \Lambda(P_D^\phi \zeta) \leq c_2 \rho \Lambda \left( \frac{1}{\delta_D^2 \phi(\delta_D^{-2})} \right) \in L^1(D, \delta_D(x)dx)
\]
so we can apply Theorem 5.10 to get the claim.

In the special case $f(x,t) = -|t|^p$ we have $\rho \equiv 1$ and $\Lambda(t) = t^p$ so (2.2) and the reduction to the flat case give us
\[
\int_D \rho \Lambda \left( \frac{1}{\delta_D^2 \phi(\delta_D^{-2})} \right) \delta_D dx \geq \int_D \frac{\delta_D}{\delta_D^2 \phi(\delta_D^{-2})} \delta_D dx \\geq 1^{-p+2p\delta_1} dt
\]
which is finite if $p < \frac{1}{1-s}$.

Remark 5.13. Assume that we are in the spectral fractional Laplacian case in the previous corollary, i.e. $\phi(\lambda) = \lambda^s$, for some $s \in (0,1)$. Then we can find a solution of (5.23) for $f(x,t) = -|t|^p$ and for every non-negative $\zeta \in C(\partial D)$ if $p < \frac{1}{1-s}$ since $\delta_1 = s$ in this case.

Conversely, if $f(x,t) = -|t|^p$ for $p \geq \frac{1}{1-s}$, and we additionally demand that the boundary condition holds pointwisely for a non-negative $\zeta \in C(\partial D)$ such that $\zeta \neq 0$, then the problem (5.23) does not have a solution. Indeed, assume that $u$ is a solution to (5.23) and that the boundary condition holds pointwisely. Then $u \gtrsim \delta_D^{2s-2}$ near $z \in \partial D$ such that $\zeta(z) > 0$ since $P_D^\phi \zeta \gtrsim \delta_D^{2s-2}$ near such $z$, see Proposition 3.4. Thus, $|u|^p \not\in L^1(D, \delta_D(x)dx)$ since $p \geq \frac{1}{1-s}$, i.e. $G_D^\phi f_u = \infty$ in $D$ by Lemma 2.8, which is a contradiction.

One of the weaknesses of Theorem 5.9 is that one has to have a supersolution and a subsolution which are strictly Green potentials, i.e. a supersolution and a subsolution cannot consist of Poisson integrals which are annihiled by $\phi(-\Delta|_D)$, since only then we may use Kato’s inequality (5.9). However, in some cases we can exploit some other methods for obtaining a solution to a semilinear problem. For example, in the next theorem we deal with a non-negative nonlinearity $f$ and a non-negative boundary condition $\zeta$ and we use a method of monotone iterations to obtain a solution.

Theorem 5.14. Let $f : D \times \mathbb{R} \to [0,\infty)$ satisfy (F), and let $f$ be a non-decreasing function in the second variable. Let $\zeta$ be a non-negative finite measure on $\partial D$ such that
\[
G_D^\phi(\rho \Lambda(2P_D^\phi \zeta)) \leq P_D^\phi \zeta, \quad \text{in } D.
\]
Then there is a continuous non-negative solution to
\[
\begin{cases}
\phi(-\Delta|_D)u(x) = f(x,u(x)), & \text{in } D, \\
\frac{\partial}{\partial D^\sigma} = \zeta, & \text{on } \partial D.
\end{cases}
\]

Proof. We use a method of monotone iterations. Let $u_0 = 0$, and define for $n \geq 1$
\[
u_n = G_D^\phi(f_{u_{n-1}}) + P_D^\phi \zeta.
\]
Since $f$ is non-negative and non-decreasing in the second variable, it follows that $(u_n)_n$ is non-negative and non-decreasing, too. However, by induction it is easy to see that $0 \leq u_n \leq 2P_D^\phi \zeta$. Indeed, for $u_0$ this fact is trivial, and for $n \geq 1$ by (5.24) we have

$$u_n = G_D^\phi (f_{u_{n-1}}) + P_D^\phi \zeta \leq G_D^\phi (\rho \Lambda (2P_D^\phi \zeta)) + P_D^\phi \zeta \leq 2P_D^\phi \zeta.$$ 

This means that $u = \lim_{n \to \infty} u_n$ is well defined. Since $f$ is continuous in the second variable by (F) and since the integrability condition (5.24) holds, by the dominated convergence theorem we get

$$u = G_D^\phi f_u + P_D^\phi \zeta,$$

i.e. we found a solution to (5.25).

For the continuity of $u$, note that since $u \leq 2P_D^\phi \zeta$, the condition (5.24) implies that $f_u \in L^1(D, \delta_D(x)dx)$ in the following way

$$\int_D f_u(x) \delta_D(x)dx \leq \int_D f_u(x) G_D^\phi \delta_D(x)dx = \int_D G_D^\phi (f_u(x)) \delta_D(x) \leq \int_D P_D^\phi \zeta(x) \delta_D(x)dx < \infty.$$ 

Also, the bound $u \leq 2P_D^\phi \zeta$ implies $f_u \in L^\infty_{loc}(D)$. Now Proposition 2.16 and Theorem 2.25 give $u \in C(D)$. 

\[\square\]

Remark 5.15. If we are in the spectral fractional Laplacian case in the previous theorem, i.e. if $\phi(\lambda) = \lambda^s$ for some $s \in (0,1)$, then there exists a solution of (5.25) for any non-negative $\zeta \in C(\partial D)$ and for the nonlinearity $f(x,t) = m|t|^p$, where $m > 0$ is sufficiently small and $p < \frac{1}{1-s}$. Indeed, in this case $P_D^\phi \zeta \lesssim \delta_D(x)^{2-2s}$, and $(P_D^\phi \zeta)^p \in L^1(D, \delta_D(x)dx)$ if $p < \frac{1}{1-s}$. Obviously, we chose the parameter $m > 0$ so small so that (5.24) holds.

Conversely, if $p \geq \frac{1}{1-s}$, then the problem (5.25) does not have a solution for $f(x,t) = m|t|^p$ for any $m > 0$ and for any non-negative $\zeta \in C(\partial D)$ such that $\zeta \neq 0$. Indeed, assume that $u$ solves (5.25). Then $u \geq P_D^\phi \zeta$ since $f \geq 0$ and $P_D^\phi \zeta \gtrsim \delta_D^{2-2s}$, near $z \in \partial D$ such that $\zeta(z) > 0$, see Proposition 3.4. Hence for $p \geq \frac{1}{1-s}$ the function $(P_D^\phi \zeta)^p \notin L^1(D, \delta_D(x)dx)$ which implies $u = G_D^\phi f_u + P_D^\phi \gtrsim G_D^\phi (\rho \Lambda (\delta_D^p \zeta^p)) = \infty$ in $D$, by Lemma 2.8.

To obtain a solution to a semilinear problem with an unsigned nonlinearity $f$ and an unsigned boundary condition $\zeta$ we need some stronger assumptions on the nonlinearity $f$. The following theorem is in the spirit same as [11, Theorem 2.4] and [8, Corollary 3.8] which were proved in a different non-local setting.

Theorem 5.16. Let $f : D \times \mathbb{R} \to \mathbb{R}$ satisfy (F) and let $\zeta$ be a finite measure on $\partial D$. Assume that $G_D^\phi \rho \in C_0(D)$ and $G_D^\phi (\rho \Lambda (2P_D^\phi |\zeta|)) \in C_0(D)$. Assume additionally that:

(a) $\Lambda$ is sublinearly increasing, i.e. $\lim_{t \to \infty} \Lambda(t)/t = 0$, or

(b) $m > 0$ is sufficiently small.

Then the semilinear problem

$$\begin{cases}
\phi(-\Delta_D)u(x) = m f(x,u(x)), & \text{in } D, \\
uP_D^\phi = \zeta, & \text{on } \partial D. 
\end{cases}$$

(5.26)

has a weak continuous solution $u$ such that $|u| \leq C + P_D^\phi |\zeta|$, for some constant $C \geq 0$.

If, in addition, $f$ is non-increasing in the second variable, $u$ is a unique weak solution to (5.26).
Proof. The proof follows the proof of [11, Theorem 2.4] and we repeat the main steps for the reader’s convenience.

Define the operator $T$ on $C_0(D)$ by

$$Tv(x) = \int_D G_D^\phi(x,y)m f(y,v(y) + P_D^\phi(\zeta))dy, \quad v \in C_0(D), \ x \in D.$$  

Our goal is to get a fixed point of the operator $T$ from which we will extract a solution to (5.26).

Let $r_p = \sup_{x \in D} G_D^\phi \rho(x) < \infty$ and $r_\zeta = \sup_{x \in D} G_D^\phi(\rho\Lambda(2P_D^\phi|\zeta|))(x) < \infty$. Let $C \geq 0$ and define $K := \{v \in C_0(D) : \|v\|_\infty \leq C\}$. It is easy to show that for $a, b > 0$ we have $\Lambda(a + b) \leq \Lambda(2a) + \Lambda(2b)$.

$$|f(y,v(y) + P_D^\phi(\zeta)(y))| \leq \rho(y)\Lambda(|v(y)| + P_D^\phi|\zeta|(y)) \leq \rho(y)\Lambda(2C) + \rho\Lambda(2P_D^\phi|\zeta|(y)), \quad v \in K,$$

so $Tv \in C_0(D)$ by the upper bound and the same calculations as in Proposition 2.16. Moreover,

$$\|Tv\|_\infty = \sup_{x \in D} \left| \int_D G_D^\phi(x,y)m f(y,v(y) + P_D^\phi(\zeta))(y)dy \right| \leq \sup_{x \in D} \int_D G_D^\phi(x,y)m(\rho(y)\Lambda(2C) + \rho\Lambda(2P_D^\phi|\zeta|(y)))dy \leq m(r_p\Lambda(2C) + r_\zeta).$$

If $m$ is sufficiently small or $\Lambda$ sublinearly increases, there is $C > 0$ such that $m(r_p\Lambda(2C) + r_\zeta) \leq C$. Fix this $C$. We will now use Schauder’s fixed point theorem on $T$. By the choice of $C$, we have $T[K] \subseteq K$. Also, $T$ is a continuous operator on $K$. This is proved by assuming the opposite as in the proof of [8, Theorem 3.6 (iii)] for the operator defined in equation (3.14) therein, see also [8, Eq. (3.15)]. Further, the family $\{Tv : v \in K\}$ is equicontinuous in $D$ by the inequality

$$|Tv(x) - Tv(\xi)| \leq \int_D |G_D^\phi(x,y) - G_D^\phi(\xi,y)|m(\rho(y)\Lambda(2C) + \rho\Lambda(2P_D^\phi|\zeta|(y)))dy, \quad v \in K,$$

and by the Remark 2.17. Arzelà-Ascoli theorem implies that $T[K]$ is precompact in $K$, thus, by Schauder’s fixed point theorem there exists $u_0 \in K$ such that $Tu_0 = u_0$. To finish the proof, notice that the function

$$u(x) := u_0(x) + P_D^\phi(\zeta)(x) = \int_D G_D^\phi(x,y)m f(y,u(y))dy + P_D^\phi(\zeta)(x)$$

solves (5.26), and it holds that $u \in C(D)$ and $|u| \leq C + P_D^\phi|\zeta|$. \hfill \Box

Remark 5.17. In the spectral fractional case where $\phi(\lambda) = \lambda^s$, for some $s \in (0, 1)$, when $\zeta \in C(\partial D)$, we have a solution of (5.26) for the nonlinearity $f$ which satisfies $|f(x, t)| \lesssim |t|^p$ if $p < \frac{s}{1-s}$. Indeed, in that case $P_D^\phi|\zeta| \lesssim \delta_D^{2s-2}$, hence $(P_D^\phi|\zeta|)^p \in L^1(D, \delta_D(x)dx)$ and $G_D^\phi\left((P_D^\phi|\zeta|)^p\right) \in C_0(D)$ by Theorem 3.6, or see [19, Proposition 7]. Note that the range $p < \frac{s}{1-s}$ is worse than the one for Corollary 5.12 and Theorem 5.14, see Remarks 5.13 and 5.15.
A Appendix

A.1 Green function estimate

Lemma A.1. Under assumption (WSC) it holds that

\[ G_D^\phi(x, y) \asymp \left( \frac{\delta_D(x)\delta_D(y)}{|x-y|^2} \wedge 1 \right) \frac{1}{|x-y|^d \phi(|x-y|^2)}, \quad x, y \in D, \tag{A.1} \]

where the constant of comparability depends only on \( d, D \) and \( \phi \).

Proof. We slightly modify the proof of [30, Theorem 3.1] where the claim was proved under assumptions (A1)-(A5) from [30]. Since (WSC) implies (A1)-(A4) from [30], we show that assumption (A5), which assumes that \( \int_0^1 \phi(\lambda)^{-1}d\lambda < \infty \), can be dropped in our setting. To shorten the proof, we note that every constant of comparability in the proof will depend at most on \( d, D \) and \( \phi \).

The lower bound proved in [30] does not use (A5) so we need to modify just the calculations for the upper bound.

Similarly as in [30], let us define

\[ I_1(r) := \int_0^r \left( \frac{\delta_D(x)\delta_D(y)}{t} \wedge 1 \right) t^{-d/2} e^{-\frac{c}{r^2}} u(t)dt, \]
\[ I_2(r) := \int_{2r}^{2\text{diam}D} \left( \frac{\delta_D(x)\delta_D(y)}{t} \wedge 1 \right) t^{-d/2} e^{-\frac{c}{r^2}} u(t)dt, \]
\[ L := \int_{2\text{diam}D}^\infty e^{-\lambda_1 t} \delta_D(x)\delta_D(y)u(t)dt, \]

where \( \lambda_1 \) is the first eigenvalue of \( -\Delta|_D \), see Subsection 2.3, and the constant \( c \) is the constant \( c_4 \) from (2.9). In addition to the bounds (2.9), there is another one for all big enough \( t > 0 \):

\[ p_D(t, x, y) \asymp e^{-\lambda_1 t} \varphi_1(x)\varphi_1(y) \asymp e^{-\lambda_1 t} \delta_D(x)\delta_D(y), \quad x, y \in D, \quad t \geq \text{diam} D, \]

see [18, Theorem 4.2.5] and [38, Remark 3.3]. Hence,

\[ G_D^\phi(x, y) = \int_0^\infty p_D(t, x, y)u(t)dt = \left( \int_0^{|x-y|^2} + \int_{|x-y|^2}^{2\text{diam}D} + \int_{2\text{diam}D}^\infty \right) p_D(t, x, y)u(t)dt \]
\[ \lesssim I_1(|x-y|) + I_2(|x-y|) + L. \tag{A.2} \]

Obviously,

\[ L \leq \delta_D(x)\delta_D(y) \int_0^\infty e^{-\lambda_1 t} u(t) = \frac{\delta_D(x)\delta_D(y)}{\phi(\lambda_1)} \lesssim \left( \frac{\delta_D(x)\delta_D(y)}{|x-y|^2} \wedge 1 \right) \frac{1}{|x-y|^d \phi(|x-y|^2)}, \]

since \(|x-y|^{-d} \phi(|x-y|^2)^{-1}\) explodes at \( x = y \) by (WSC).
For $I_1$ we imitate the calculations for [30, Eq. (3.7)]. Since $u(t) \lesssim \frac{\phi'(t^{-1})}{t^2\phi(t^{-1})}$ by (2.4) for all $t > 0$, and since $t \mapsto \phi'(t^{-1})/\phi(t^{-1})^2$ increases, by the change of variables $c r^2/t = s$ we have

$$I_1(r) \lesssim \int_0^{r^2} \left( \frac{\delta_D(x)\delta_D(y)}{t} \wedge 1 \right) t^{-d/2}e^{-\frac{s^2}{2}} \frac{\phi'(t^{-1})}{t^2\phi(t^{-1})^2} dt$$

$$\leq \frac{\phi'(r^{-2})}{\phi(r^{-2})^2} \int_0^{r^2} \left( \frac{\delta_D(x)\delta_D(y)}{t} \wedge 1 \right) t^{-d/2-2}e^{-\frac{s^2}{2}} dt$$

$$\lesssim \left( \frac{\delta_D(x)\delta_D(y)}{r^2} \wedge 1 \right) \frac{\phi'(r^{-2})}{r^{d+2}\phi(r^{-2})^2} \int_c^\infty s^{d/2+1}e^{-s} ds \lesssim \left( \frac{\delta_D(x)\delta_D(y)}{r^2} \wedge 1 \right) \frac{1}{r^d\phi(r^{-2})},$$

where the last inequality follows from (2.3).

The calculation for $I_2$ is slightly different than the one for [30, Eq. (3.8)]. Note that $u(t) \lesssim \frac{\phi'(t^{-1})}{t^2\phi(t^{-1})^2} \lesssim \frac{1}{t\phi(t^{-1})} \lesssim \frac{r^{d-1}}{r^{d}\phi(r^{-2})}$, for $r^2 \leq t \leq (2\text{diam}D)^2$, where in the last approximate inequality we used ???. Hence

$$I_2(r) \lesssim \frac{1}{r^{2d+2}\phi(r^{-2})^2} \int_0^{(2\text{diam}D)^2} \left( \frac{\delta_D(x)\delta_D(y)}{t} \wedge 1 \right) r^{d-1-d/2}e^{-\frac{s^2}{2}} dt$$

$$\leq \left( \frac{\delta_D(x)\delta_D(y)}{r^2} \wedge 1 \right) \frac{1}{r^{2d+2}\phi(r^{-2})^2} \int_r^{(2\text{diam}D)^2} r^{d-2-d/2} dt \lesssim \left( \frac{\delta_D(x)\delta_D(y)}{r^2} \wedge 1 \right) \frac{1}{r^d\phi(r^{-2})}.$$

The claim now follows from (A.2).

\[\square\]

### A.2 Boundary behaviour of some integrals

**Lemma A.2.** For $\Gamma = \{ y \in \partial D : |x-y| \leq 2\delta_D(x) \}$ it holds that

$$\int_{\Gamma} |x-y|^{-d} \propto \delta_D(x)^{-1}, \quad x \in D.$$

**Proof.** Since $D$ is a $C^{1,1}$ set, for small enough $\delta_D(x)$ the boundary part $\Gamma$ can be described as $\Gamma = \{ q \in \mathbb{R}^{d-1} : |\delta_D(x)-f(q)| + |q| \leq 4\delta_D(x)^2 \}$, for some $C^{1,1}$ function $f$ on $\mathbb{R}^{d-1}$ such that $f(0) = 0$ and $\nabla f(0) = 0$, whereas $x$ can be viewed as $x = (0, \ldots, 0, \delta_D(x))$. Hence

$$\int_{\Gamma} \frac{\delta_D(x)}{|x-y|^d} d\sigma(y) \propto \int_{\{q \in \mathbb{R}^{d-1} : |\delta_D(x)-f(q)| + |q| \leq 4\delta_D(x)^2 \}} \frac{1}{(|f(q)|^2 + |q|^2)^{d/2}} dq$$

$$\propto \int_{\{z \in \mathbb{R}^{d-1} : |1-f(\delta_D(x)z)| \leq 4\delta_D(x)^2 \}} \frac{1}{(|1-f(\delta_D(x)z)|^2 + |z|^2)^{d/2}} dz,$$

where we first used that $|\nabla f|$ is bounded by the Lipschitz property and then the substitution $q = \delta_D(x)z$. Since $f \in C^{1,1}(\mathbb{R}^{d-1})$ such that $f(0) = 0$ and $\nabla f(0) = 0$, by the dominated convergence theorem the last integral converges to

$$\int_{\{z \in \mathbb{R}^{d-1} : |z|^2 \leq 4 \}} \frac{1}{(1+|z|^2)^{d/2}} dz < \infty$$
as \( \delta_D(x) \to 0 \).

The two following lemmas are in the spirit the same as [8, Lemma A.4 & Lemma A.5]. These lemmas from [8] lead to a result that is an analogue of Theorem 3.6 in the case of subordinate Brownian motions, see [8, Proposition 4.1].

Let \( \epsilon = \epsilon(D) > 0 \) be such that the map \( \Phi : \partial D \times (-\epsilon, \epsilon) \to \mathbb{R}^d \) defined by \( \Phi(y, \delta) = y + \delta \mathbf{n}(y) \) defines a diffeomorphism to its image, cf. [4, Remark 3.1]. Here \( \mathbf{n} \) denotes the unit interior normal. Without loss of generality assume that \( \epsilon < \text{diam}(D)/20 \).

**Lemma A.3.** Let \( \eta < \epsilon \) and assume that conditions \((\text{U1})-(\text{U4})\) hold true. Then for any \( x \in D \) such that \( \delta_D(x) < \eta/2 \),

\[
G_D^\phi U(D)1_{(\delta_D < \eta)}(x) \asymp \frac{1}{\delta_D(x)^2 \phi(\delta_D(x)^{-2})} \int_0^{\delta_D(x)} U(t) t \, dt + \delta_D(x) \int_0^{\eta} U(t) t \, dt + \delta_D(x) \int_{3\delta_D(x)/2}^{\eta} \frac{U(t)}{t^2 \phi(t^{-2})} \, dt.
\]

In particular, \( G_D^\phi U(D)1_{(\delta_D < \eta)}(x) < \infty \) if and only if the integrability condition \((3.3)\) holds true. Moreover, all comparability constants depend only on \( d \), \( D \) and \( \phi \) and are independent of \( \eta \).

**Proof.** Fix some \( r_0 < \epsilon \) and fix \( x \in D \) as in the statement. Define

\[
\begin{align*}
D_1 &= B(x, \delta_D(x)/2) \\
D_2 &= \{ y : \delta_D(y) < \eta \} \setminus B(x, r_0) \\
D_3 &= \{ y : \delta_D(y) < \delta_D(x)/2 \} \cap B(x, r_0) \\
D_4 &= \{ y : 3\delta_D(x)/2 < \delta_D(y) < \eta \} \cap B(x, r_0) \\
D_5 &= \{ y : \delta_D(x)/2 < \delta_D(y) < 3\delta_D(x)/2 \} \cap (B(x, r_0) \setminus B(x, \delta_D(x)/2)).
\end{align*}
\]

Thus we have that

\[
G_D^\phi U(D)1_{(\delta_D < \eta)}(x) = \sum_{j=1}^5 \int_{D_j} G_D^\phi(x, y) U(\delta_D(y)) \, dy =: \sum_{j=1}^5 I_j.
\]

**Estimate of \( I_1 \):** We show that

\[
I_1 \asymp \frac{U(\delta_D(x))}{\phi(\delta_D(x)^{-2})} \lesssim \frac{1}{\delta_D(x)^2 \phi(\delta_D(x)^{-2})} \int_0^{\delta_D(x)} U(t) t \, dt.
\]

Indeed, let \( y \in D_1 \). Then \( \delta_D(y) > \delta_D(x)/2 > |y - x| \) implying that

\[
G_D^\phi(x, y) \asymp \frac{1}{|x - y|^{d+\phi(|x - y|^{-2})}}.
\]

Further, by using first \((3.4)\) and then \((3.5)\) we have that

\[
U(\delta_D(y)) \asymp U(\delta_D(x)).
\]
Finally, by (3.4) we get
\[
\frac{1}{\delta_D(x)^2 \phi(\delta_D(x)^{-2})} \int_0^{\delta_D(x)} U(t) t \, dt \gtrsim \frac{U(\delta_D(x))}{\delta_D(x)^2 \phi(\delta_D(x)^{-2})} \int_0^{\delta_D(x)} t \, dt \times \frac{U(\delta_D(x))}{\phi(\delta_D(x)^{-2})}.
\]

**Estimate of I_2:** Next, we show that
\[
I_2 \asymp \delta_D(x) \int_0^\eta U(t) t \, dt. \tag{A.8}
\]

Let \( y \in D_2 \). Then \( r_0 < |y - x| < \text{diam}(D) \) so that \( |y - x| \asymp 1 \). This implies that \( G^\phi_D(x, y) \asymp \delta_D(x) \delta_D(y) \). Therefore
\[
I_2 \asymp \delta_D(x) \int_{D_2} U(\delta_D(y)) \delta_D(y) \, dy \asymp \delta_D(x) \int_{\delta_D(y) < \eta} U(\delta_D(y)) \delta_D(y) \, dy \asymp \delta_D(x) \int_0^\eta U(t) t \, dt,
\]
where the last approximate equality follows by the co-area formula.

In estimates for \( I_3 \), \( I_4 \) and \( I_5 \) we will use the change of variables formula based on a diffeomorphism \( \Phi : B(x, r_0) \rightarrow B(0, r_0) \) satisfying
\[
\Phi(D \cap B(x, r_0)) = B(0, r_0) \cap \{ z \in \mathbb{R}^d : z \cdot e_d > 0 \}, \quad \Phi(y) \cdot e_d = \delta_D(y) \quad \text{for any } y \in B(x, r_0), \quad \Phi(x) = \delta_D(x) e_d,
\]
cf. [4, p. 38]. For the point \( z \in \mathbb{R}_+^d = \{ z \in \mathbb{R}^d : z \cdot e_d > 0 \} \) we will write \( z = (\tilde{z}, z_d) \).

Several times we also use the following integrals:
\[
\int_0^a \frac{s^{d-2}}{(1 + s)^d} \, ds = \frac{(1 + 1/a)^{1-d}}{(d - 1)}, \quad a > 0, \tag{A.9}
\]
\[
\int_0^a \frac{s^{d-2}}{(1 + s)^{d+2}} \, ds = \frac{a^{d-1}}{d+1} \left( 2a(1 + a) + d + 2ad + d^2 \right), \quad a > 0. \tag{A.10}
\]

**Estimate of I_3:** We prove that
\[
I_3 \asymp \frac{1}{\delta_D(x)^2 \phi(\delta_D(x)^{-2})} \int_0^{\delta_D(x)} U(t) t \, dt. \tag{A.11}
\]
To see this, take \( y \in D_3 \). Then \( \delta_D(y) \leq \delta_D(x)/2 \) implying \( |x - y| \geq \delta_D(x)/2 \), and thus
\[
G^\phi_D(x, y) \asymp \frac{\delta_D(x) \delta_D(y)}{|x - y|^{d+2} \phi(|x - y|^{-2})}. \tag{A.12}
\]
Therefore, by repeating the first five lines of the calculations in [8, p. 34 for the integral $I_3$] with our bound (A.12) we get

$$I_3 \asymp \int_0^{\rho_0/\delta_D(x)} s^{d-2} \int_0^{1/2} \frac{U(\delta_D(x)h)h}{(1-h+s)^{d+2} \phi(\delta_D(x)^{-2}(1+h+s)^{-2})} dh \, ds \times \int_0^{\rho_0/\delta_D(x)} s^{d-2} \int_0^{1/2} \frac{U(\delta_D(x)h)h}{(1+s)^{d+2} \phi(\delta_D(x)^{-2}(1+s)^{-2})} dh \, ds,$$

(A.13)

where the last line comes from $\frac{1}{2} \leq h \leq 1$. Further, for $\phi$ it holds that

$$(1+s)^{-2} \phi(\delta_D(x)^{-2}) \leq \phi(\delta_D(x)^{-2}(1+s)^{-2}) \leq \phi(\delta_D(x)^{-2}),$$

(A.14)

see (2.6). Since we have

$$\int_0^{\rho_0/\delta_D(x)} s^{d-2} \frac{ds}{(1+s)^d} \asymp 1, \quad \int_0^{\rho_0/\delta_D(x)} s^{d-2} \frac{ds}{(1+s)^{d+2}} \asymp 1,$$

(A.15)

by (A.9) and (A.10), by applying the inequalities (A.14) to (A.13) and by using (A.15) we obtain

$$I_3 \asymp \frac{1}{\phi(\delta_D(x)^{-2})} \int_0^{1/2} U(\delta_D(x)h)h \, dh \times \frac{1}{\delta_D(x)^2 \phi(\delta_D(x)^{-2})} \int_0^{\delta_D(x)/2} U(t) \, dt.$$

This proves (A.11) since the almost non-increasing condition (3.4) implies

$$\int_0^{\delta_D(x)/2} U(t) \, dt = \int_0^{\delta_D(x)} U(t/2) t \, dt \geq \int_0^{\delta_D(x)} U(t) \, dt.$$

**Estimate of $I_4$:** By applying the same calculations as in $I_3$, we show that

$$I_4 \asymp \delta_D(x) \int_0^\gamma \frac{U(t)}{t^2 \phi(t^{-2})} \, dt.$$ 

(A.16)

Let $y \in D_4$. Then $|x-y| \geq \delta_D(x)/2$ and $|x-y| \geq \delta_D(y)/3$, hence $C^\phi_D(x,y)$ is of the form (A.12). By following the computation in [8, p. 35 for the integral $I_4$] with our bound (A.12), we arrive at

$$I_4 \asymp \int_0^{\rho_0/\delta_D(x)} s^{d-2} \int_0^{\eta/\delta_D(x)} \frac{U(\delta_D(x)h)h}{(h-1+s)^{d+2} \phi(\delta_D(x)^{-2}((h-1)+s)^{-2})} dh \, ds \times \int_0^{\rho_0/\delta_D(x)} s^{d-2} \int_0^{\eta/\delta_D(x)} \frac{U(\delta_D(x)h)h}{(h+s)^{d+2} \phi(\delta_D(x)^{-2}(h+s)^{-2})} dh \, ds \times \int_0^{\rho_0/\delta_D(x)} s^{d-2} \int_0^{\eta/\delta_D(x)} \frac{U(\delta_D(x)h)h}{h^2(1+s)^{d+2} \phi((\delta_D(x)h)^{-2}(1+s)^{-2})} dh \, ds,$$

(A.17)
where the second line comes from \( \frac{1}{3}h \leq h - 1 \leq h \). By applying (A.14) in (A.17), since the relations (A.15) also hold for \( r_0/(\delta_D(x)h) \) instead of \( r_0/\delta_D \), we get

\[
I_4 \leq \int_{3/2}^{\eta/\delta_D(x)} \frac{U(\delta_D(x)h)}{h^2 \phi((\delta_D(x)h)^{-2})} dh \\
\leq \delta_D(x) \int_0^{\eta/2} \frac{U(t)}{t^2 \phi(t^{-2})} dt.
\]

**Estimate of** \( I_5 \): Under the almost non-increasing condition (3.4) and the doubling condition (3.5) we show that

\[
I_5 \lesssim \frac{U(\delta_D(x))}{\phi(\delta_D(x)^{-2})} \lesssim \frac{1}{\delta_D(x)^2 \phi(\delta_D(x)^{-2})} \int_0^{\delta_D(x)} U(t) t dt.
\]  

(A.18)

Indeed, let \( y \in D_5 \). Then \( |x - y| > \delta_D(x)/2 > \delta_D(y)/3 \), hence \( G_D^\phi(x, y) \) is of the form (A.12). The estimate (A.7) also holds since \( \delta_D(y) \asymp \delta_D(x) \). Therefore

\[
I_5 \asymp \delta_D(x) \int_{D_5} \frac{U(\delta_D(\eta)) \delta_D(\eta)}{|x-y|^{d+2} \phi(|x-y|^{-2})} dy \\
\lesssim \frac{U(\delta_D(x))}{\phi(\delta_D(x)^{-2})} \int_{D_5} \frac{1}{|x-y|^d} dy,
\]

where the last line comes from \( |x-y|^2 \phi(|x-y|^{-2}) < \phi'(|x-y|^{-2}) \geq \phi'(4\delta_D(x)^{-2}) \sim \delta_D(x)^2 \phi(\delta_D(x)^{-2}) \) since \( \phi' \) decreases. To end the calculations, it was shown in [4, p. 42] that the last integral is comparable to 1. This proves the first approximate inequality in (A.18), while the second was already proved in the estimate of \( I_1 \).

The proof is finished by noting that \( I_1 + I_5 \lesssim I_3 \).

**Lemma A.4.** Let \( \eta < \epsilon \) and assume that conditions (U1)-(U4) hold true. There exists \( c(\eta) > 0 \) such that for any \( x \in D \) satisfying \( \delta_D(x) \geq \eta/2 \),

\[
G_D^\phi(U(\delta_D)) \mathbf{1}_{(\delta_D < \eta)}(x) \leq c(\eta).
\]  

(A.19)

**Proof.** Fix \( x \in D \) as in the statement and define

\[
D_1 = \{ y : \delta_D(y) < \eta/4 \}, \\
D_2 = \{ y : \eta/4 \leq \delta_D(y) < \eta \}.
\]

Then

\[
G_D^\phi(U(\delta_D)) \mathbf{1}_{(\delta_D < \eta)}(x) = \sum_{j=1}^2 \int_{D_i} G_D^\phi(x, y) U(\delta_D(y)) dy =: \sum_{j=1}^2 J_j.
\]

**Estimate of** \( J_1 \): We show that

\[
J_1 \lesssim \frac{1}{\eta^2 \phi(\eta^{-2})} \int_0^{\eta} U(t) t dt.
\]  

(A.20)

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Let $y \in D_1$. Then $\delta_D(y) < \eta/4 \leq \delta_D(x)/2$, hence by using $|x - y| \geq \delta_D(x) - \delta_D(y)$ we have that $|x - y| > \delta_D(y)$ and $|x - y| > \delta_D(x)/2$. This implies that $G_D^\phi(x, y)$ satisfies (A.12). Therefore,

$$J_1 \lesssim \delta_D(x) \int_{D_1} \frac{U(\delta_D(y)) \delta_D(y)}{|x - y|^{d+2} \phi(|x - y|^{-2})} dy \lesssim \frac{\delta_D(x)}{\eta^2 \phi(\eta^{-2})} \int_{D_1} U(\delta_D(y)) \delta_D(y) \frac{1}{|x - y|^{d}} dy,$$

since on $D_1$ we have $|x - y| \geq \eta/4$, hence $|x - y|^2 \phi(|x - y|^{-2}) \gtrsim \eta^2 \phi(\eta^{-2})$ by (2.6).

By using the co-area formula we get (below $dy$ denotes the $d-1$ dimensional Hausdorff measure on $\{\delta_D(y) = t\}$)

$$J_1 \lesssim \frac{\delta_D(x)}{\eta^2 \phi(\eta^{-2})} \int_0^{\eta/4} U(t) t \int_{\delta_D(y) = t} \frac{1}{|x - y|^{d}} dy \ dt. \quad \text{(A.21)}$$

The inner integral in (A.21) is estimated as follows: For $\delta_D(y) = t$ it holds that $|x - y| \geq \delta_D(x) - t$, hence $|x - y|^{-d} \leq (\delta_D(x) - t)^{-d}$. The $d - 1$ dimensional Hausdorff measure of $\{\delta_D(y) = t\}$ is larger than or equal to the $d - 1$ dimensional Hausdorff measure of the sphere around $x$ of radius $\delta_D(x) - t$ which is comparable to $(\delta_D(x) - t)^{d-1}$. This implies that the inner integral is estimated from above by a constant times $(\delta_D(x) - t)^{-1}$. Thus

$$J_1 \lesssim \frac{1}{\eta^2 \phi(\eta^{-2})} \delta_D(x) \int_0^{\eta/4} U(t) t (\delta_D(x) - t)^{-1} dt.$$

If $t < \eta/4$, then $t < \delta_D(x)/2$, implying $\delta_D(x)/2 < \delta_D(x) - t < \delta_D(x)$. Therefore,

$$J_1 \lesssim \frac{1}{\eta^2 \phi(\eta^{-2})} \int_0^{\eta} U(t) t dt.$$

**Estimate of $J_2$:** It holds that

$$J_2 \leq U(\eta/4). \quad \text{(A.22)}$$

Let $y \in D_2$. By the almost non-increasing condition (3.4) we have $U(\delta_D(y)) \leq c_1 U(\eta/4)$, hence

$$J_2 \lesssim \int_{\eta/4 < \delta_D(y) < \eta} \frac{U(\delta_D(y))}{|x - y|^{d} \phi(|x - y|^{-2})} dy \lesssim U(\eta/4) \int_{\eta/4 < \delta_D(y) < \eta} \frac{1}{|x - y|^{d} \phi(|x - y|^{-2})} dy$$

$$\leq U(\eta/4) \int_{B(x, 2diam D)} \frac{1}{|x - y|^{d} \phi(|x - y|^{-2})} dy \lesssim U(\eta/4).$$

The last estimate uses the fact that the integral is not singular.

By putting together estimates for $J_1$ and $J_2$, we see that there exists $c_2 > 0$ such that

$$G_D^\phi(U(\delta_D)1_{(\delta_D < \eta)})(x) \leq c_2 \left( \frac{1}{\eta^2 \phi(\eta^{-2})} \int_0^{\eta} U(t) t dt + U(\eta/4) \right) =: c(\eta).$$

\[\square\]

**Proof of Theorem 3.6.** Fix some $\eta < \epsilon$ and treat it as a constant. Note that on $\{\delta_D(y) \geq \eta\}$ it holds that $U$ is bounded (by the assumption (U4)). Therefore

$$G_D^\phi(U(\delta_D)1_{(\delta_D \geq \eta)})(x) \lesssim G_D^\phi \delta_D(x) \asymp \delta_D(x), \quad x \in D,$$

(A.23)
by Lemma 2.8. For the lower bound of this term note that on \( \{ \delta_D(x) \geq \eta/2 \} \) we have
\[
G_D^\phi(U(\delta_D)1_{(\delta_D \geq \eta)})(x) \geq \int_{B(x, \eta/4)} \frac{1}{|x-y|^d \phi(|x-y|^2)} dy \asymp \frac{1}{\phi(16/\eta^2)} \geq 1,
\] (A.24)
and on \( \{ \delta_D(x) \leq \eta/2 \} \) we have
\[
G_D^\phi(U(\delta_D)1_{(\delta_D \geq \eta)})(x) \geq \delta_D(x) \int_{\delta_D(y) \geq \eta} \frac{\delta_D(y)}{|x-y|^{d+2} \phi(|x-y|^2)} dy \geq \delta_D(x) .
\] (A.25)
Since \( \delta_D(x) \asymp 1 \) on \( \{ \delta_D(x) \geq \eta/2 \} \), we have just obtained \( G_D^\phi(U(\delta_D)1_{(\delta_D \geq \eta)})(x) \asymp \delta_D(x) \)
in \( D \). Further, by Lemma A.4, if \( \delta_D(x) \geq \eta/2 \), then \( G_D^\phi(U(\delta_D)1_{(\delta_D \leq \eta)})(x) \leq c(\eta) \). Hence,
\[
G_D^\phi(U(\delta_D))(x) \asymp 1, \quad \delta_D(x) \geq \eta/2.
\]
Since for \( \delta_D(x) \geq \eta/2 \) the right-hand side of (3.6) is also comparable to 1, this proves the claim for the case \( \delta_D(x) \geq \eta/2 \).

Assume now that \( \delta_D(x) < \eta/2 \). By Lemma A.3 and (A.23) we have that (3.6) holds where we clearly replaced \( \eta \) of (A.3) by \( \text{diam} D \) since we treat \( \eta \) as a constant.

Finally, we prove that \( G_D^\phi(U(\delta_D))(x)/P_D^\phi \sigma(x) \to 0 \) as \( x \to \partial D \). It is obvious that \( P_D^\phi \sigma \) annihilates the first and the second term of (3.6). For the third term, note that on \( \{ t \geq \delta_D(x) \} \) we have \( t^2 \phi(t^{-2}) \geq \delta_D(x)^2 \phi(\delta_D(x)^{-2}) \) and \( U(t) \delta_D(x) \leq U(t) t \). By applying the dominate convergence theorem we obtain
\[
\frac{\delta_D(x)}{\text{diam} D} \int_{\delta_D(x)}^{\text{diam} D} \frac{U(t)}{t \phi(t^{-2})} dt \leq \int_{\delta_D(x)}^{\text{diam} D} U(t) \delta_D(x) dt \to 0,
\]
as \( \delta_D(x) \to 0 \).

\[\square\]

**Lemma A.5.** Let \( t < \epsilon \). There exists \( C = C(d, D, \phi) > 0 \) such that for \( \delta_D(x) \geq \frac{1}{2} \) it holds that
\[
\int_{\delta_D(y) \leq t} \frac{G_D^\phi(x,y)}{P_D^\phi \sigma(y)} dy \leq C \tilde{f}(x,t),
\]
where \( 0 \leq \tilde{f}(x,t) \leq t \delta_D(x) \) on \( \{ \delta_D(x) \geq t/2 \} \) and \( \tilde{f}(x,t)/t \to 0 \) as \( t \to 0 \) for every fixed \( x \in D \).

**Proof.** We need a little adaptation of Lemma A.4. We break the set \( D_2 \) into three pieces. Fix \( r_0 < \epsilon \) and \( x \in D \) as in the statement. Define
\[
D_1 = \{ y : \delta_D(y) < t/4 \},
D_2 = \{ y : t/4 \leq \delta_D(y) < t \} \cap B(x, t/4),
D_3 = \{ y : t/4 \leq \delta_D(y) < t \} \cap B(x, t/4)^c \cap B(x, r_0),
D_4 = \{ y : t/4 \leq \delta_D(y) < t \} \cap B(x, r_0)^c.
\]

Then
\[
\int_{\delta_D(y) \leq t} \frac{G_D^\phi(x,y)}{P_D^\phi \sigma(y)} dy = \sum_{i=1}^{4} \int_{D_i} \frac{G_D^\phi(x,y)}{P_D^\phi \sigma(y)} dy = \sum_{i=1}^{4} J_i.
\]
Estimate of $J_1$: We prove

$$J_1 \lesssim t^2.$$  \hfill \text{(A.26)}

Let $y \in D_1$. Then $\delta_D(y) < t/4 \leq \delta_D(x)/2$, hence $|x-y| > \delta_D(x)/2 > \delta_D(y)$. This implies that $G_D^\phi(x,y)$ satisfies (A.12), and $G_D^\phi(x,y)/P^\phi_0 \sigma(y) \lesssim \delta_D(x) \delta_D(y)/|x-y|^d$. Therefore, by using the co-area formula in the second comparison, we get

$$J_1 \lesssim \delta_D(x) \int_{D_1} \frac{\delta_D(y)}{|x-y|^d} \approx \delta_D(x) \int_0^{t/4} h \left( \int_{\delta_D(y)=h} \frac{\sigma(dy)}{|x-y|^d} \right) dh.$$  

The inner integral is estimated as before, see the paragraph under (A.21), i.e. the inner integral is bounded from above by a constant times $(\delta_D(x) - h)^{-1}$. Thus

$$J_1 \lesssim \delta_D(x) \int_0^{t/4} \frac{h}{\delta_D(x) - h} dh.$$  

However, when $h < t/4$ we have $\frac{1}{4} \delta_D(x) \leq \delta_D(x) - h \leq \delta_D(x)$, therefore

$$J_1 \lesssim \int_0^{t/4} h \, dh \lesssim t^2.$$  

In the following integral estimates we have $y \in D$ such that $t/4 \leq \delta_D(y) \leq t$ so

$P^\phi_0 \sigma(y) \asymp \frac{1}{t \phi(t^{-2})}$.

Estimate of $J_2$: We prove

$$J_2 \lesssim t^2.$$  \hfill \text{(A.27)}

On $D_2$ we obviously have $G_D^\phi(x,y) \asymp \frac{1}{|x-y|^d \phi(|x-y|^{-2})}$, hence

$$J_2 \lesssim t^2 \phi(t^{-2}) \int_{B(x,t/4)} \frac{1}{|x-y|^d \phi(|x-y|^{-2})} dy \asymp t^2 \phi(t^{-2}) \frac{1}{\phi(t^{-2})} \asymp t^2.$$  

Estimate of $J_3$: We prove that $J_3 \lesssim f(x,t)$ for a function $f$ which satisfies $0 \leq f(x,t)/t \leq \delta_D(x)$ and $f(x,t)/t \to 0$ as $t \to 0$ for every fixed $x \in D$.

To this end, since $y \in D_3$, hence $|x-y| \geq t/4$, it holds that $G_D^\phi(x,y) \asymp \frac{\delta_D(x) t}{|x-y|^d + \phi(|x-y|^{-2})}$.

Hence,

$$J_3 \asymp t^3 \phi(t^{-2}) \delta_D(x) \int_{D_3} \frac{1}{|x-y|^{d+2} \phi(|x-y|^{-2})} dy =: f(x,t).$$  \hfill \text{(A.28)}

Since $|x-y| \geq t/4$, we have $|x-y|^2 \phi(|x-y|^{-2}) \geq t^2 \phi(t^{-2})$ by (2.6), hence

$$f(x,t)/t \lesssim \delta_D(x) \int_{D_3} \frac{1}{|x-y|^d} dy.$$  \hfill \text{(A.29)}

Also, by reducing to the flat case we have

$$\int_{D_3} \frac{1}{|x-y|^d} dy \asymp \int_{t/4}^{t/4} \int_{t/4}^{r-t} \frac{r^{d-2}}{(|\delta_D(x)-h|+r)^d} dh \, dr \times \int_{t/4}^{t/4} \int_{(t-\delta_D(x))/r}^{(t-\delta_D(x))/r} \frac{1}{r (|\rho|+1)^d} d\rho \, dr.$$  

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Since $\rho \mapsto 1/(|\rho| + 1)$ is bell-shaped, and the inner interval $[t/4 - \delta_D(x)/r, (t - \delta_D(x))/r]$ has fixed length, the inner integral is maximal when the inner interval is symmetric (which is when $\delta_D(x) = \frac{t}{8}$), thus, we get

$$
\int_{D_3} \frac{1}{|x-y|^d} \lesssim \int_{t/4}^{r_0} \int_{-3t/(8r)}^{3t/(8r)} \frac{1}{r(|\rho| + 1)^d} d\rho dr
= 2 \int_{t/4}^{r_0} \int_{0}^{3t/(8r)} \frac{1}{r(\rho + 1)^d} d\rho dr.
$$

Further, $1 \leq \rho + 1 \leq 3t/(8r) + 1 \leq 3$ so we get

$$
\int_{D_3} \frac{1}{|x-y|^d} \lesssim \int_{t/4}^{r_0} \frac{t}{r^2} dr \lesssim 1.
$$

Inserting the bound (A.30) into (A.29), we get that $0 \leq f(x,t)/t \lesssim \delta_D(x)$ where the constant of comparability depends only on $d$, $D$ and $\phi$.

Further, if we fix $x$ and let $t \to 0$, then it is clear that $1_{D_3} \to 0$, and that $|x-y| - d-2\phi(|x-y|^{-2})^{-1} \leq c$ for every $y \in D_3$ for all small enough $t > 0$. Hence, $f(x,t)/t \to 0$ as $t \to 0$.

**Estimate of $J_4$:** We prove

$$
J_4 \lesssim t^3 \phi(t^{-2}) \delta_D(x).
$$

For $y \in D_4$ we have $G^\phi_D(x,y) \lesssim \frac{\delta_D(x)}{|x-y|^{d+2\phi(|x-y|^{-2})}} \lesssim \delta_D(x)t$ since $r_0 \leq |x-y| \leq \text{diam} D$. Hence, $J_4 \lesssim t^3 \phi(t^{-2}) \delta_D(x)$.

To finish the proof, note that we can take $\bar{f}(x,t) = c(t^2 + f(x,t) + t^3 \phi(t^{-2}) \delta_D(x))$ for some constant $c = c(d, D, \phi) > 0$.

**A.3 Uniform integrability of some classes of functions**

**Lemma A.6.** Let $f : D \times \mathbb{R} \to \mathbb{R}$ be continuous in the second variable and $u_1, u_2 \in L^1(D, \delta_D(x)dx)$ such that $u_1 \leq u_2$. Assume that for every $u \in L^1(D, \delta_D(x)dx)$ such that $u_1 \leq u \leq u_2$ a.e. in $D$ it holds that $x \mapsto f(x,u(x)) \in L^1(D, \delta_D(x)dx)$. Then the family

$$
\mathcal{F} := \{f(\cdot,u(\cdot)) \in L^1(D, \delta_D(x)dx) : u_1 \leq u \leq u_2 \text{ a.e. in } D\}
$$

is uniformly integrable in $D$ with respect to the measure $\delta_D(x)dx$, hence bounded in $L^1(D, \delta_D(x)dx)$.

**Proof.** Before we start the proof, we refer the reader to [35, Chapter 16] for details on the uniform integrability. Also, the proof is motivated by the proof of a similar claim which can be found in [33, Section 2].

Suppose that the family $\mathcal{F}$ is not uniformly integrable. Then there is $\varepsilon > 0$, a sequence $(v_n)_n \subset L^1(D, \delta_D(x)dx)$ such that $u_1 \leq v_n \leq u_2$ a.e. in $D$, and a sequence $(E_n)_n$ consisting of measurable subsets of $D$ with property

$$
\int_{E_n} |f(x,v_n(x))| \delta_D(x)dx \geq \varepsilon, \quad n \in \mathbb{N}.
$$
Now use \([33, \text{Lemma 2.1}]\) with \(w_n(\cdot) = |f(\cdot, v_n(\cdot))|\delta_P(\cdot)/\varepsilon \in L^1(D)\) to extract a subsequence \((v_n)_k\) of \((v_n)_n\) and disjoint sets \(F_k \subset E_n\) such that
\[
\int_{F_k} |f(x, v_n(x))|\delta_D(x)dx \geq \frac{\varepsilon}{2}, \quad k \in \mathbb{N}.
\]
To finish the proof, define
\[
v(x) = \begin{cases} v_n(x), & x \in F_k, \\ u_1(x), & x \in \cap_{k=1}^\infty F_k^c. \end{cases}
\]
We have \(u_1 \leq v \leq u_2\) in \(D\), hence \(v \in L^1(D, \delta_D(x)dx)\). Further,
\[
\int_D |f(x, v(x))|\delta_D(x)dx = \sum_{k=1}^\infty \int_{F_k} |f(x, v_n(x))|\delta_D(x)dx = \infty,
\]
which is a contradiction. \(\square\)

### A.4 Regularity of transition densities

The following result on the regularity up to the boundary of the transition kernel of the killed Brownian motion appears to be well known, but we were unable to find an exact reference. In the article we assumed that \(D\) is a \(C^{1,1}\) bounded domain, but this result we give for a slightly more general open set since the claim is important in itself.

**Lemma A.7.** Let \(D\) be an open bounded \(C^{1,\alpha}\) domain for some \(\alpha \in (0, 1]\). For the transition density \(p_D(\cdot, \cdot, \cdot)\) of the killed Brownian motion upon exiting the set \(D\) it holds that \(p_D \in C^1((0, \infty) \times \overline{D} \times \overline{D})\).

**Remark A.8.** Moreover, we will see in the proof of the previous lemma that \(p_D\) is somehow independently regular, variable by variable. E.g. we can differentiate \(p_D(t, x, y)\) in \(x\) up to the boundary, then differentiate the obtained function in \(y\) up to the boundary, and then differentiate in \(t\) as many times as we want. This can be done up to \(C^{1,\alpha}(\overline{D})\) regularity in the second and the third variable and up to \(C^\infty(0, \infty)\) regularity in the first variable.

**Proof of Lemma A.7.** Note that \(p_D(t, x, y) \leq p(t, x, y)\) everywhere by \((2.8)\) so for fixed \(t > 0\) and \(x \in D\) we have that the mapping \(y \mapsto p_D(t, x, y)\) is in \(L^\infty(D) \subset L^2(D)\). Hence, by the spectral representation of \(L^2(D)\) functions we have
\[
p_D(t, x, y) = \sum_{j=1}^\infty e^{-\lambda_j t} \varphi_j(x)\varphi_j(y), \quad (A.32)
\]
where we have used \((2.16)\).

Now we show that the sum in \((A.32)\) converges uniformly and is bounded in a certain strong sense. First note that \(\varphi_j \in C^{1,\alpha}(\overline{D})\) by \([23, \text{Theorem 8.34}]\). Furthermore, by \([23, \text{Theorem 8.33}]\) the following estimate holds
\[
\|\varphi_j\|_{C^{1,\alpha}(D)} \leq c_1(1 + \lambda_j)\|\varphi_j\|_{L^\infty(D)}, \quad (A.33)
\]
where $\| \cdot \|_{C^{1,\alpha}(D)}$ is the standard $C^{1,\alpha}(D)$ Hölder norm and $c_1 = c_1(d, D) > 0$. Also, the eigenvalues satisfy the well known estimate

$$\| \varphi_j \|_{L^\infty(D)} \leq c_2 \lambda_j^{d/4} \| \varphi_j \|_{L^2(D)} = c_2 \lambda_j^{d/4},$$

(A.34)

see e.g. [40, Theorem 1.6], where $c_2 = c_2(d) > 0$. In particular, this inequality and the inequality in (A.33) imply

$$\| \nabla \varphi_j \|_{L^\infty(D)} \leq C_1 (1 + \lambda_j) \| \varphi_j \|_{L^\infty(D)} \leq c_3 (1 + \lambda_j)^{d/4 + 1},$$

(A.35)

for $c_3 = c_3(d, D) > 0$. Also note that since $\varphi_j$ vanishes on the boundary, by the mean-value theorem for every $x \in D$ there is some $\tilde{x}$ between $x$ and the closest boundary point to $x$ such that

$$\left| \frac{\varphi_j(x)}{\delta_D(x)} \right| = \| \nabla \varphi_j(\tilde{x}) \| \leq \| \nabla \varphi_j \|_{L^\infty(D)}.$$

Hence, for the sum in (A.32) the following uniform bound holds

$$\sum_{j=1}^\infty e^{-\lambda_j t} |\varphi_j(x)| |\varphi_j(y)| \leq \sum_{j=1}^\infty e^{-\lambda_j t} \| \varphi_j \|_{L^\infty(D)} \| \varphi_j \|_{L^\infty(D)}$$

$$\leq c_2 \sum_{j=1}^\infty e^{-\lambda_j t} \lambda_j^{d/2} < \infty,$$

(A.36)

$$\sum_{j=1}^\infty e^{-\lambda_j t} |\varphi_j(x)| \left| \frac{\varphi_j(y)}{\delta_D(y)} \right| \leq \sum_{j=1}^\infty e^{-\lambda_j t} \| \varphi_j \|_{L^\infty(D)} \| \nabla \varphi_j \|_{L^\infty(D)}$$

$$\leq c_4 \sum_{j=1}^\infty e^{-\lambda_j t} (1 + \lambda_j)^{d/2 + 1} < \infty,$$

(A.37)

where $c_4 = c_4(d, D) > 0$ and the sums converge by Weyl’s law, see (2.17). Similar bounds hold if we take the derivate by the variable $t$ or by the variable $y$.

Since $\varphi_j \in C^{1,\alpha}(D)$ and since the bounds (A.36) and (A.37) hold, we can pass the needed limits inside the sum (A.32) to get $p_D \in C^1((0, \infty) \times \overline{D} \times \overline{D})$.

In addition, since the bounds (A.33)-(A.35) hold, we can pass the limits inside the sum in the representation (A.32) to get that the density $p_D$ is regular, variable by variable up to $C^{1,\alpha}(\overline{D})$ regularity in the second and the third variable and up to $C^\infty(0, \infty)$ regularity in the first variable, see Remark A.8.

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