Discrete data assimilation via Ladyzhenskaya squeezing property in the 3D viscous primitive equations

Igor Chueshov

May 11, 2014

Abstract

We discuss the discrete data assimilation problem for the 3D viscous primitive equations arising in the modeling of large scale phenomena in oceanic dynamics. Our main result states possibility of asymptotically reliable prognosis based on a discrete sequence of finite number of scalar observations. Our method is quite general and can be applied to a wide class of dissipative systems.

Keywords: 3D viscous primitive equations, data assimilation, determining functionals.

2010 MSC: 35R60, 37N10, 76F25, 76F55

Introduction

Data assimilation problem is a question how to incorporate available observation data in computational schemes to improve quality of predicting of the future evolution of the corresponding dynamical system. This problem has a long history and was studied by many authors at different levels (see, e.g., the monographs [11, 16] and the references therein).

In this paper we consider the case when observations of the system are making in some sequence \(\{t_n\}\) of moments of time and use the same formulation of the data assimilation problem as in [14].

Our main goal is to demonstrate the role of the so-called Ladyzhenskaya squeezing property (which is valid for a wide class parabolic type PDEs, see [19, 20]) in the solving of data assimilation problems. As in [14] we also involve the notion of determining modes or, more generally, determining functionals. However our method is different from the approach developed in [14] for the 2D Navier–Stokes equations. Moreover, the method based on the squeezing property looks more general and can be applied to a wide class of dissipative systems admitting the Ladyzhenskaya property.

∗Department of Mechanics and Mathematics, Karazin Kharkov National University, Kharkov, 61022, Ukraine, e-mail: chueshov@karazin.ua
In this paper as a model we choose the system of the 3D viscous primitive equations which arise in geophysical fluid dynamics for modeling large scale phenomena in oceanic motions. In this case data assimilation problem is related with reliability of weather predictions.

The paper is organized as follows. In Section 1 we describe the model and quote its properties which need for data assimilation. The main technical tools are the Ladyzhenskaya squeezing property and the theory of determining functionals which we discuss shortly in Section 2. In Section 3 formulate the data assimilation problem, introduce the notion of asymptotically reliable prognosis and prove our main result concerning a finite number of scalar observations in a discrete sequence of times.

1 3D Primitive equations

The primitive equations are based on the so-called hydrostatic approximation of the 3D Navier-Stokes equations for velocity field $u$ and coupled to thermo- and salinity equations which are taken into account via small variation of density (or equivalently via buoyancy $b$), see, e.g., the survey [24] and the literature cited there. Below to simplify the presentation we consider periodic boundary conditions of the same type as in [23] and [24] (see also [8]). However, it should be noted that our results remains valid in the case of free type boundary conditions like in [3]. The case of mixed free-Dirichlet boundary conditions (see [18]) is more complicated and requires a separate consideration.

Let

$$\mathcal{O} = (0, L_1) \times (0, L_2) \times (-L_3/2, L_3/2) \subset \mathbb{R}^3.$$ 

We denote by $\bar{x} = (x, z) = (x_1, x_2, z)$ the spatial variable in $\mathcal{O}$ and suppose that $\nabla, \text{div}$ and $\Delta$ are the gradient, divergence and Laplace operators in the (horizontal) variable $x = (x_1, x_2)$.

After the reduction based on the hydrostatic relation we arrive (see, e.g., [24]) at the following equations for the horizontal fluid velocity

$$v_t + (v, \nabla)v - \left[ \int_0^z \text{div} v d\xi \right] \partial_z v - \nu [\Delta v + \partial_{zz} v] + f v^\perp = -\nabla \left[ p(x, t) + \int_0^z b d\xi \right] + G_f \quad \text{in} \quad \mathcal{O} \times (0, +\infty), \quad (1)$$

$$b_t + (v, \nabla)b - \left[ \int_0^z \text{div} v d\xi \right] \partial_z b - \nu [\Delta b + \partial_{zz} b] = G_b \quad \text{in} \quad \mathcal{O} \times (0, +\infty), \quad (2)$$

where $\nu > 0$ is the dynamical viscosity, $f$ is the Coriolis parameter and $v^\perp = (-v^2, v^1)$. The functions represents $G_f$ and $G_b$ are volume sources.
related to the fluid field and the buoyancy. As in [23, 24] the equations in (1) and (2) supplied with the conditions:

\[ \text{div} \int_{-L_3/2}^{L_3/2} v \, dz = 0; \quad v \text{ is periodic in } \bar{x} \text{ and even in } z, \quad \int_{\mathcal{O}} v d\bar{x} = 0; \quad (3) \]

\[ b \text{ is periodic in } \bar{x} \text{ and odd in } z. \quad (4) \]

We also impose initial data:

\[ v(0) = v_0, \quad b(0) = b_0. \quad (5) \]

We note that the vertical component of the velocity field has the form

\[ w(\bar{x}, t) = -\int_0^z \text{div} v(x, \xi, t) \, d\xi \quad \text{for every } \bar{x} = (x, z) \in \mathcal{O} \]

and thus the full velocity field \((v^1, v^2, w)\) satisfies incompressibility condition.

We emphasize that the (surface) pressure \(p\) in (1) depends on 2D (horizontal) variable \(x\) only. Basing on this observation a new effective approach [3] for proving of the global well-posedness for problems like (1) and (2) have been implemented, see [3] and the discussion therein.

The system of the viscous 3D primitive equations was intensively studied for the different types of boundary conditions (see the literature cited in the survey [24]). The existence of week solutions was established in [21]; for global well-posedness of strong solutions we refer to [3] and also to [17, 18, 23]. The uniqueness of weak solutions is still unknown. The question on a global attractor for the viscous 3D primitive equations was considered in [15] (see also the papers [23] and [8] devoted to the periodic case).

We denote by \(\dot{\mathcal{H}}^s_{\text{per}}(\mathcal{O})\) the Sobolev space of order \(s\) consisting of periodic functions such that \(\int_{\mathcal{O}} f d\bar{x} = 0\) and introduce the following spaces:

\[ V_s = \left\{ v = (v^1, v^2) \in [\dot{\mathcal{H}}^s_{\text{per}}(\mathcal{O})]^2 : v^i \text{ is even in } z, \text{ div } \int_{-L_3/2}^{L_3/2} v \, dz = 0 \right\} \]

for \(s \geq 0\). We equip \(H \equiv V_0\) with \(L_2\)-norm \(\| \cdot \|\) and denote by \((\cdot, \cdot)\) the corresponding inner product. It is convenient to endow the spaces \(V_1\) and \(V_2\) with the norms \(\| \cdot \|_{V_1} = \| \nabla_{x,z} \cdot \|\) and \(\| \cdot \|_{V_2} = \| \Delta_{x,z} \cdot \|\). Here and below we use the notations \(\nabla_{x,z}\) and \(\Delta_{x,z}\) for gradient and Laplace operation in the 3D variable \((x, z)\).

We also introduce state spaces for the buoyancy variable by the formulas

\[ E_s = \left\{ b \in \dot{\mathcal{H}}^s_{\text{per}}(\mathcal{O}) : b \text{ is odd in } z \right\}, \quad s \geq 0. \]

We equip them with the standard Sobolev norms. We suppose \(W_s = V_s \times E_s\) with the corresponding (Hilbert) product norms.
As it was already mentioned, starting with [3] the global well-posedness of the equations in (1) and (2) was studied by many authors [17, 18, 23, 24]. The following result on well-posedness of strong solutions in the case of periodic boundary conditions was basically proved in [23] (see also [24] and Remark 2.2 in [8]).

**Proposition 1.1 ([23])** Let \( G_f \in V_0, G_b \in E_0, \) and \( U_0 = (v_0; b_0) \in W_1 \). Then problem (3)–(5) has a unique strong solution \((v(t); b(t))\):

\[
U(t; U_0) \equiv (v(t); b(t)) \in C([\mathbb{R}_+; W_1]) \cap L_2(0, T; W_2), \quad \forall T > 0.
\]

This solution generates a dynamical system \((S_t, W_1)\) with the evolution operator \(S_t\) defined by the relation \(S_t U_0 = U(t; U_0)\). The operator \(S_t\) satisfies the Lipschitz property:

\[
\|S_t U - S_t U_*\|_{W_1} \leq C_{T, \varrho} \|U - U_*\|_{W_1}, \quad t \in [0, T],
\]

for every \(T > 0\) and \(U, U_* \in B_1(\varrho) \equiv \{ U : \|U\|_{W_1} \leq \varrho \}\).

If \(G_f \in V_{m-1}, G_b \in E_{m-1}\) and \(U_0 = (v_0; b_0) \in W_m\) for some \(m \geq 2\) then (see [23, 24]) the solution \(U\) lies in the class \(C([\mathbb{R}_+; W_m]) \cap L_2^{loc}(\mathbb{R}_+; W_{m+1})\). This observation makes it possible to use smooth approximations of solutions in the calculations with multipliers (see, e.g., [8]).

For our goal the following assertion is important.

**Proposition 1.2** Let the hypotheses of Proposition 1.1 be in force. Then

- There exist positive constants \(a_0\) and \(a_1\) such that
  \[
  \|S_t U\| \leq e^{-a_0 t} \|U\| + a_1 K_G, \quad t \geq 0, \quad U \in W_1,
  \]
  \[
  \text{where } K_G^2 = \|G_f\|^2 + \|G_b\|^2.
  \]

- If we assume in addition that
  \[
  G_f \in V_1, \quad G_b \in E_1 \quad \text{and also} \quad (\partial_z G_f; \partial_z G_b) \in \left[L_6(\mathcal{O})\right]^3,
  \]
  then for every \(\varrho > 0\) and \(0 < \alpha \leq \beta < +\infty\) there exists the constant \(C(\alpha, \beta, \varrho) > 0\) such that
  \[
  \|S_t U\|_{W_2} \leq C(\alpha, \beta, \varrho) \quad \text{for every } t \in [\alpha, \beta], \quad \|U\| \leq \varrho.
  \]

**Proof.** The first statement is achieved by the standard multipliers \(v\) and \(b\) applied to (1) and (2), see [23, 24], for instance.

The second statement is a more complicated and based mainly on the calculations given in [8] and [23]. The corresponding argument involves the splitting of the system into 2D Navier-Stokes type equations coupled with 3D Burgers type model (see [8] and also [23, 24]) and consists of several
steps based on the application of the same multipliers as in [3, 23, 24]. The spatial periodicity of the system allows us to use freely higher order multipliers like $\Delta^2_{x,z}v$ and $\Delta^2_{x,z}b$. For some related details we refer to the paper [8] which contains a very similar argument in the proof of Theorem 3.1 on the existence of a smooth absorbing set. \[\square\]

Proposition 1.2 implies that the system $(S_t, W_1)$ possesses an absorbing set which is bounded in $W_2$. More precisely we have the following assertion.

**Corollary 1.3 (Smooth Absorbing Ball)** Let (7) be in force. Then there exists $K > 0$ such that the ball

$$\mathcal{B} \equiv \mathcal{B}_2(K) = \{ U \in W_2 : \|U\|_{W_2} \leq K \}$$

is absorbing for the dynamical system $(S_t, W_1)$ generated by problem (7)-(5), i.e., for any bounded set $B$ in $W_1$ there is $t_B$ such that

$$S_t B \subset \mathcal{B} \quad \text{for all } t \geq t_B.$$ 

**Proof.** It follows from (6) that

$$\|S_t U\| \leq 1 + a_1 K_G \quad \text{for all } t \geq t_B,$$

and thus by (8) we have that

$$\|S_{t+1} U\|_{W_2} \leq K \equiv C(1,1,1 + a_1 K_G) \quad \text{for all } t \geq t_B,$$

i.e. the ball $\mathcal{B}$ possesses the desired property. \[\square\]

We can also prove the Lipschitz property in $H$ provided one of two solutions belongs to $W_2$.

**Proposition 1.4** Let $U_1(t)$ and $U_2(t)$ be two strong solutions to (1)-(5). Assume that $\|U_1(t)\|_{W_2} \leq R$ for all $t \in [0,T]$ for some $T > 0$. Then

$$\|U_1(t) - U_2(t)\| \leq C_T(R)\|U_1(0) - U_2(0)\|, \quad \forall t \in [0,T]. \quad (9)$$

**Proof.** We note (see, e.g., [24]) that problem (1)-(5) can be written in the form

$$\partial_t U + \nu AU + B(U, U) + CU = G, \quad U(0) = U_0,$$

where $A$ is a positive self-adjoint operator in $H$ generated by the bilinear form

$$a(U, U_s) = \int_{\mathbb{O}} [\nabla_{x,z} v \cdot \nabla_{x,z} v_s + \nabla_{x,z} b \cdot \nabla_{x,z} b_s] \, dx \, dz, \quad (10)$$

where $U = (v; b)$ and $U_s = (v_s; b_s)$ are from $W_1$, $C$ is a bounded skew-symmetric operator, and $B(U, U)$ is a quadratic operator possessing the properties

$$B(U^*, U), U) = 0, \quad B(U, U^*), U) \leq C\|U\|_{W_1}^{3/2}\|U\|^{1/2}\|U^*\|_{W_2} \quad (11)$$
for every $U^* \in W_2$ and $U \in W_1$. Thus for the difference $V = U_1(t) - U_2(t)$ we have that
\[
\frac{1}{2} \partial_t \|V(t)\|^2 + \nu a(V(t), V(t)) + B(V, U_1), V = 0
\]
which, via (11) and Gronwall's lemma, implies the relation in (9). □

We note that the operator $A$ generated by form (10) has a discrete spectrum. This means that there exists an orthonormal basis $\{e_k\}$ in $H$ such that
\[
Ae_k = \lambda_k e_k, \quad 0 < \lambda_1 \leq \lambda_2 \leq \ldots, \quad \lim_{k \to +\infty} \lambda_k = +\infty.
\]
We denote by $P_N$ the orthoprojector onto $\text{Span}\{e_1, \ldots, e_N\}$ and $Q_N = I - P_N$.

The following Ladyzhenskaya squeezing property (see [19, 20]) of the evolution operator $S_t$ is the main ingredient of our further data assimilation considerations.

**Proposition 1.5 (Squeezing property)** Let (7) be in force. Then for every $q < 1$, $0 < \alpha \leq \beta < +\infty$ and $L$ there exists $N_* = N(\alpha, \beta, L, q)$ such that
\[
\|Q_N [S_tU - S_tU_*]\|_{W_1} \leq q \|U - U_*\|_{W_1}, \quad \forall t \in [\alpha, \beta], \quad \forall N \geq N_*,
\]
for any $U$ and $U_*$ from the set
\[
\mathcal{D} = \{U \in W_2 : \|S_tU\|_{W_2} \leq L \text{ for all } t \in [0, \beta]\}.
\]

**Proof.** The same type argument as in the proof of Theorem 3.5 in [8] leads to the desired result. □

## 2 Observation/measurement functionals

To describe observation/measurement procedure we use a finite family $\mathcal{L}$ of linear continuous functionals $\{l_j : j = 1, \ldots, N\}$ on the phase space. If $U$ is a phase vector which corresponds to some state of the system, then, similar to [14], we can treat the values $\{l_j(U) : j = 1, \ldots, N\}$ as a set of observation data. Our task is now to determinate the state $U$ with the help of observation functionals $\{l_j\}$. Therefore to describe admissible observations it is natural to involve well-developed theory of determining functionals. This theory starts with the pioneering paper [13] on determining modes and was developed by many authors for different classes of PDE systems and different families of functionals (see the recent discussion in [12]). For a general theory of the determining functionals we refer to [4], see also [7, 9, 10] for a development of this theory based on the notion of the completeness defect. The concept of completeness defect which was introduced in [5, 6] seems a convenient tool in characterization of observation functionals.
Definition 2.1 Let $V$ and $H$ be reflexive Banach spaces and $V$ is continuously and densely embedded into $H$. The \textit{completeness defect} of a set $\mathcal{L}$ of linear functionals on $V$ with respect to $H$ is the value

$$
\epsilon_{\mathcal{L}}(V, H) = \sup \{ \| w \|_H : w \in V, l(w) = 0, l \in \mathcal{L}, \| w \|_V \leq 1 \} .
$$

(13)

It is obvious that $\epsilon_{\mathcal{L}_1}(V, H) \geq \epsilon_{\mathcal{L}_2}(V, H)$ provided $\text{Span}\mathcal{L}_1 \subset \text{Span}\mathcal{L}_2$. In addition, $\epsilon_{\mathcal{L}}(V, H) = 0$ if and only if the class of functionals $\mathcal{L}$ is complete in $V$; this means that the property $l(w) = 0$ for all $l \in \mathcal{L}$ implies $w = 0$. We can also generalize the notion of the completeness defect by considering some seminorms $\mu_V$ in (13) instead of the norm $\| \cdot \|_H$ (see, e.g., [10]).

Below we use the so-called interpolation operators which are related with the set of functionals given. To describe their properties we need the following notion.

Definition 2.2 Let $V \subset H$ be separable Hilbert spaces and $R$ be a linear operator from $V$ into $H$. As in [1] the value

$$
\epsilon^H_V(R) = \sup \{ \| u - Ru \|_H : \| u \|_V \leq 1 \} \equiv \| I - R \|_{V \to H}
$$

is said to be the \textit{global approximation error} in $H$ arising in the approximation of elements $v \in V$ by elements $Rv$. Here and below $\| \cdot \|_{V \to H}$ denotes the operator norm for linear mappings from $V$ into $H$.

The following assertion (see [6, 7] for the proof) shows that the completeness defect provides us with a bound from below for the best possible global approximation error.

Theorem 2.3 Let $V$ and $H$ be the separable Hilbert spaces such that $V$ is compactly and densely embedded into $H$. Let $\mathcal{L}$ be a set of linear functionals on $V$. Then we have the following relations

$$
\epsilon_{\mathcal{L}}(V, H) = \min \{ \epsilon^H_V(R) : R \in \mathcal{R}_{\mathcal{L}} \},
$$

where $\mathcal{R}_{\mathcal{L}}$ is the family of linear bounded operators $R : V \to H$ and such that $Rv = 0$ for all $v \in \mathcal{L}^\perp = \{ v \in V : l(v) = 0, l \in \mathcal{L} \}$. Moreover, we have that

$$
\epsilon_{\mathcal{L}}(V, H) = \epsilon^H_V(I - Q_{\mathcal{L}}) = \sup \{ \| Q_{\mathcal{L}}u \|_H : \| u \|_V \leq 1 \},
$$

(14)

where $Q_{\mathcal{L}}$ is the orthoprojector in $V$ onto $\mathcal{L}^\perp$.

One can show (see [7]) that any operator $R \in \mathcal{R}_{\mathcal{L}}$ has the form

$$
Rv = \sum_{j=1}^{N} l_j(v)\psi_j, \quad \forall v \in V,
$$

(15)

where $\{ \psi_j \}$ is an arbitrary finite set of elements from $V$. This why $\mathcal{R}_{\mathcal{L}}$ is called the set of interpolation operators corresponding to the set $\mathcal{L}$. An
operator $R \in \mathcal{R}_L$ is called Lagrange interpolation operator, if it has form (15) with \{\psi_j\} such that $l_k(\psi_j) = \delta_{kj}$. In the case of Lagrange operators we have that $R^2 = R$, i.e., $R$ is a projector.

We also note that the operator $Q_L$ in (14) has the following structure $Q_L = I - P_L$ with $P_L v = \sum_{j=1}^{N} (\xi_j, v) \xi_j, \forall v \in V$, where $\{\xi_j\}$ is the orthonormal basis in the orthogonal supplement $\mathcal{M}_L$ to the annihilator $L^\perp$ in $V$. We call $P_L$ the optimal interpolation operator corresponding to the set $L$.

Our main example is related with the eigen-basis of the operator $A$ defined by the form (10).

Example 2.4 (Modes) Denote by $L$ the set of functionals $L = \{l_j(u) = (u, e_j) : j = 1, 2, \ldots, N\}$, where $\{e_k\}$ are eigenfunctions of the operator $A$ given by the form (10), see (12). The optimal interpolation operator $P_L$ is Lagrange in this case and has the form

$$P_L v = \sum_{j=1}^{N} (e_j, v) e_j, \forall v \in W_1.$$  \hfill (16)

Moreover, $\epsilon_L(W_i, H) = \epsilon_i^H(P_L) = \lambda_{N+1}^{-i/2}, i = 1, 2$. Thus the completeness defect and the global approximation error $\|I - P_L\|_{W_i \rightarrow H}$ can be made small after an appropriate choice of $N$.

3 Discrete data assimilation

We consider the discrete data assimilation problem in the sense due to [14]. The paper [14] is focused on the case where the measurement data is taken at a sequence of discrete times $t_n$ in contrast with the papers [2, 22] which consider continuous data assimilation. All these papers deal with for the incompressible two-dimensional Navier–Stokes equations.

Following the idea presented in [14] we accept the following definition.

Definition 3.1 Let $U(t) = S_t U_0$ be a solution to (11)–(15) with initial data $U_0$ at time $t_0$. Let $L = \{l_j\}$ be a finite family of functionals on $H$ (each functional $l_j$ is interpreted as a single observational measurement). Let $R_L$ be some Lagrange interpolation operator related with $L$ such that the sequence $\{r_k^n \equiv R_L U(t_n)\}$ represents the (joint) observational measurements of the reference solution $U(t)$ at a sequence $\{t_n\}$ of times, we call the sequence $\{r_k^n\}$ observation values. Now we can construct prognostic values at time $t_n$ by the formula

$$u_n = (1 - R_L) S_{t_n - t_{n-1}} u_{n-1} + r_k^n, \quad n = 1, 2, \ldots, \hfill (17)$$
where $u_0$ is (unknown) vector which, according to [14], corresponds to an initial guess of the reference solution $U(t_0)$. We can also define the prognostic (piecewise continuous) trajectory as

$$u(t) = S_{t-t_n}u_n \quad \text{for} \quad t \in [t_n, t_{n+1}), \quad n = 0, 1, 2, \ldots \quad (18)$$

We say that the prognosis is **asymptotically reliable** at a sequence of times $t_n$ if

$$\|U(t_n) - u_n\|_{W_1} \to 0 \quad \text{as} \quad n \to +\infty.$$  

Our goal is to find conditions on $R_L$, $t_n$ and $\eta$ which guarantee that the prognosis based on a finite number of single observations is asymptotically reliable.

We assume that $0 < \alpha \leq t_{n+1} - t_n \leq \beta < +\infty$ for some positive $\alpha$ and $\beta$.

The following assertion gives us a dissipativity property for prognostic values which is important for our application of the Ladyzhenskaya squeezing property.

**Lemma 3.2** Assume that $\|U(t)\|_{W_2} \leq K$ for all $t \geq t_0$. Let

$$\|R_L\|_{W_2 \rightarrow H} \leq c_0 \quad \text{and} \quad \|1 - R_L\|_{H \rightarrow H} \leq c_1 \quad \text{with} \quad c_1 < e^{-a_0\alpha},$$

where $a_0$ is the constant in (6). Then there exists $n_* > 0$ such that

$$\|u_n\| \leq 1 + q_* \quad \text{for all} \quad n \geq n_*,$$

(19)

where $q_* = (a_1K_G + c_0K)(1 - c_1e^{-a_0\alpha})^{-1}$. If we assume in addition that $\|1 - R_L\|_{W_2 \rightarrow W_2} \leq c_2$, then

$$\|u_n\|_{W_2} \leq g \equiv c_2C(\alpha, \beta, 1 + q_* + (1 + c_2)K \quad \text{for all} \quad n \geq m_* \equiv 1 + n_*,$$

(20)

where $C(\alpha, \beta, g)$ is the constant from (8).

**Proof.** One can see from Proposition 1.2 that

$$\|u_n\| \leq c_1e^{-a_0\alpha}\|u_{n-1}\| + a_1K_G + c_0K, \quad n = 1, 2, \ldots$$

This implies that

$$\|u_n\| \leq q_*^n\|u_0\| + g_*, \quad n = 1, 2, \ldots$$

where $q_* = c_1e^{-a_0\alpha}$. This yields (19).

To prove (20) we note that

$$\|u_n\|_{W_2} \leq c_2\|S_{t_n-t_{n-1}}u_{n-1}\|_{W_2} + (1 + c_2)K, \quad n = 1, 2, \ldots$$

Hence (20) follows from (8) and (19). \qed

In the case of (spectral) modes we have the following assertion.
Corollary 3.3  Let $\mathcal{L}$ be the same as in Example 2.4. If we take in (17) $R_L$ to be the interpolation operator given by (16), then there exist positive constants $C(K_G, K, \alpha, \beta)$ and $m_\ast$ such that

$$
\|u_n\|_{W_2} \leq q \equiv C(K_G, K, \alpha, \beta) \quad \text{for all } n \geq m_\ast.
$$

Proof. In this case $c_0 = \lambda_1^{-1}$ and $c_1 = c_2 = 1$. □

Now we are in position to obtain the main result.

Theorem 3.4  Assume that $\mathcal{L}$ is a finite family of functionals on $H$ and there is a Lagrange interpolation operator $R_L$ possessing the properties:

$$
\|1 - R_L\|_{H \rightarrow H} \leq c_1 \quad \text{and} \quad \|1 - R_L\|_{W_2 \rightarrow W_2} \leq c_2
$$

(21)

with the constants $c_1$ and $c_2$ independent of $\mathcal{L}$ such that $c_1 < e^{a_0 \alpha}$, where $a_0$ is the constant in (6). Then there exists $\epsilon_\ast > 0$ such that under the condition $\epsilon(W_1, H) \leq \epsilon_\ast$ the prognosis in (17) is asymptotically reliable for every $u_0 \in H$.

In the case of the modes described in Example 2.4 there exists $N_\ast$ such that the prognosis (17) is asymptotically reliable with $R_L = P_L$, where $P_L$ is given by (16) with some $N \geq N_\ast$.

Proof. We obviously have that

$$
U(t_n) - u_n = (1 - R_L)[S_{t_n-t_{n-1}}U(t_{n-1}) - S_{t_n-t_{n-1}}u_{n-1}], \quad n \geq m_\ast.
$$

In the case of modes we have that $I - R_L = Q_N$. Therefore using Corollary 3.3 by Proposition 1.5 we can choose $N_\ast$ such that and thus

$$
\|U(t_n) - u_n\|_{W_1} = q\|U(t_{n-1}) - u_{n-1}\|_{W_1}, \quad n \geq m_\ast,
$$

with $q < 1$. This implies

$$
\|U(t_n) - u_n\|_{W_1} \rightarrow 0 \quad \text{as } n \rightarrow +\infty
$$

with exponential speed. Therefore the statement of the theorem is valid in the case of modes.

It is obvious that under conditions (21) the hypotheses of Lemma 3.2 are in force. Therefore in the general case Proposition 1.5 implies that

$$
\|S_{\Delta_n}U(t_{n-1}) - S_{\Delta_n}u_{n-1}\|_{W_1}
\leq q_N\|U(t_{n-1}) - u_{n-1}\|_{W_1} + \lambda_1^{1/2}\|S_{\Delta_n}U(t_{n-1}) - S_{\Delta_n}u_{n-1}\| (22)
$$

for $n \geq m_\ast$ with $\Delta_n = t_n - t_{n-1}$, where $q_N < 1$ can be chosen as small as we need at the expense of $N$. By Proposition 1.4 we have that

$$
\|S_{\Delta_n}U(t_{n-1}) - S_{\Delta_n}u_{n-1}\| \leq C_\beta(q)\|U(t_{n-1}) - u_{n-1}\|, \quad n \geq m_\ast.
$$
Since \( l_j(U(t_{n-1})) = l_j(u_{n-1}) \), this gives
\[
\| S_{\Delta_n}U(t_{n-1}) - S_{\Delta_n}u_{n-1} \| \leq \epsilon(W_1,H)C_\beta(\theta)\| U(t_{n-1}) - u_{n-1} \|_{W_1}, \quad n \geq m_*.
\]
Thus (22) yields
\[
\| U(t_n) - u_n \|_{W_1} \leq \tilde{q}\| U(t_{n-1}) - u_{n-1} \|_{W_1}
\]
for \( n \geq m_* \), where
\[
\tilde{q} = \| I - R_L \|_{W_1 \to W_1} \left[ qN + \lambda^{1/2}\epsilon(W_1,H)C_\beta(\theta) \right].
\]
By the operators interpolation from condition (21) we have that
\[
\| I - R_L \|_{W_1 \to W_1} \leq \sqrt{c_1c_2}.
\]
Hence we can choose \( N \) and \( \epsilon(W_1,H) \) such that \( \tilde{q} < 1 \). Therefore the prognosis is asymptotically reliable with exponential speed. \( \square \)

We conclude our considerations with several remarks.

**Remark 3.5** As an example of set \( \mathcal{L} \) functionals \( \{l_j\} \) satisfying (21) we can consider *generalized modes* which are defined by the formulas:
\[
l_j(u) = (Ke_j,u), \quad j = 1, \ldots, N,
\]
where \( \{e_j\} \) is the eigen-basis of the operator \( A \) and \( K \) is a linear invertible self-adjoint operator in \( H \) with maps \( W_2 \) into itself and is bounded in both spaces \( H \) and \( W_2 \). In this case the operator \( R_L \) has the form (15) with \( \psi_j = K^{-1}e_j \). One can see that we can apply Theorem 3.4 with \( \alpha \) greater than \( \frac{1}{q_0}\ln(1 + \| K \|_{H \to H}) \).

**Remark 3.6** Under the conditions of Theorem 3.4 we also have that
\[
\lim_{t \to +\infty} \| U(t) - u(t) \|_{W_1} = 0
\]
for the prognostic trajectory given by (18). Thus the prognosis is also reliable in the sense used in [14].

**Remark 3.7** The number of functionals which provides an asymptotically reliable prognosis according to Theorem 3.4 is finite. However the estimates for this number which follows from the statement of theorem are not optimal and even not constructive. The derivation of optimal bounds for the numbers requires more careful analysis of constants related to dissipativity and squeezing properties of individual trajectories. We refer to [14] for more constructive approach based on the multipliers technique and developed in the case of the 2D Navier–Stokes equations for the reference solution from the global attractor.
References

[1] J.-P. Aubin, Approximation of Elliptic Boundary-Value Problems, Wiley, New York, 1972.

[2] A. Azouani, E. Olson, E. S. Titi, Continuous data assimilation using general interpolant observables, Preprint [arXiv:1304.0997] (2013).

[3] C. Cao, E.S. Titi, Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics, Annals of Math., 166 (2007), 245–267.

[4] B. Cockburn, D.A. Jones, E. S. Titi, Estimating the number of asymptotic degrees of freedom for nonlinear dissipative systems. Math. Comp., 66 (1997), 1073–1087.

[5] I. Chueshov, On the finiteness of the number of determining elements for von Karman evolution equations, Math. Meth. Appl. Sci., 20 (1997), 855–865.

[6] I. Chueshov, Theory of functionals that uniquely determine asymptotic dynamics of infinite-dimensional dissipative systems, Russian Math. Surv., 53 (1998), 731–776.

[7] I. Chueshov, Introduction to the Theory of Infinite-Dimensional Dissipative Systems. University Lectures in Contemporary Mathematics, Acta, Kharkov, 2002 (from the Russian edition (Acta, 1999)); see also http://www.emis.de/monographs/Chueshov/.

[8] I. Chueshov, A squeezing property and its applications to a description of long time behaviour in the 3D viscous primitive equations. Preprint arXiv:1211.4408 (2012); to be published in Proc. Royal Soc. Edinburgh, Ser.A.

[9] I. Chueshov, I. Lasiecka, Long-Time Behavior of Second Order Evolution Equations with Nonlinear Damping. Memoirs of AMS, vol.195, no. 912, AMS, Providence, RI, 2008.

[10] I. Chueshov, I. Lasiecka, Von Karman Evolution Equations. Springer, New York, 2010.

[11] Data Assimilation. Making Sense of Observations, (Eds: W.Lahoz, B. Khattatov, R. Ménard), Springer, New York, 2010.

[12] C. Foias, M. S. Jolly, R. Kravchenko, E. S. Titi, Navier-Stokes equations, determining forms, determining modes, inertial manifolds, dissipative dynamical systems, arXiv:1208.5134v1 (2012).
[13] C. Foias, G. Prodi, Sur le comportement global des solutions non-stationnaires des équations de Navier-Stokes en dimension 2. Rend. Sem. Mat. Univ. Padova, 39 (1967), 1–34.

[14] K. Hayden, E. Olson and E.S. Titi, Discrete data assimilation in the Lorenz and 2D Navier-Stokes equations, Physica D, 240 (2011), 1416–1425.

[15] N. Ju, The global attractor for the solutions to the 3d viscous primitive equations, Disc. Cont. Dyn. Sys., 17 (2007), 159–179.

[16] E. Kalnay, Atmospheric modeling, data assimilation and predictability. Cambridge University Press, 2003.

[17] G. M. Kobelkov, Existence of a solution “in the large” for the 3D large-scale ocean dynamics equations. C. R. Acad. Sci. Paris, Ser.I, (2006) 343, 283–286.

[18] I. Kukavica, M. Ziane, On the regularity of the primitive equations of the ocean, Nonlinearity, 20 (2007), 2739–2753.

[19] O. A. Ladyzhenskaya, Finding minimal global attractors for the Navier-Stokes equations and other partial differential equations, Russian Math. Surveys, 42 (1987), 27–73.

[20] O. A. Ladyzhenskaya, Attractors for Semigroups and Evolution Equations. Cambridge University Press, 1991.

[21] J. L. Lions, R. Temam, S. Wang, On the equations of the large scale ocean, Nonlinearity, 5 (1992), 1007–1053.

[22] E. Olson and E.S. Titi, Determining modes for continuous data assimilation in 2-D turbulence, Journal of Statistical Physics, 113 (2003), 799–840.

[23] M. Petcu, On the three dimensional primitive equations, Adv. Dif. Eq., 11 (2006), 1201–1226.

[24] M. Petcu, R. Temam, M. Ziane, Some mathematical problems in geophysical fluid dynamics, in: Special Volume on Computational Methods for the Atmosphere and the Oceans, in: Handbook of Numerical Analysis, vol. 14, Elsevier, 2008, pp. 577–750.