Notes on the stability threshold for radially anisotropic polytropes

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ABSTRACT
We discuss some contradictions found in the literature concerning the problem of stability of collisionless spherical stellar systems, which are the simplest anisotropic generalization of the well-known polytrope models. Their distribution function \( F(E, L) \) is a product of power-law functions of the energy \( E \) and angular momentum \( L \), i.e. \( F \propto L^{-\gamma} \exp(-E)^q \). On the one hand, calculation of the growth rates in the framework of linear stability theory and \( N \)-body simulations shows that these systems become stable when the parameter \( s \) characterizing the velocity anisotropy of the stellar distribution is lower than some finite threshold value, \( s < s_{\text{crit}} \). On the other hand, Palmer & Papaloizou showed that the instability remains up to the isotropic limit \( s = 0 \).

Using our method of determining the eigenmodes for stellar systems, we show that the growth rates in weakly radially anisotropic systems are indeed positive, but decrease exponentially as the parameter \( s \) approaches zero, i.e. \( \gamma \propto \exp(-s_s/s) \). In fact, for systems with a finite lifetime this means stability.

Key words: Galaxy: centre – galaxies: kinematics and dynamics.

1 INTRODUCTION
Stability properties of stellar spherical clusters determine a set of dynamically allowable equilibrium configurations. The presence of an instability can, for example, lead to ellipsoidal deformation.

For a long time, it was believed that any stellar spherical clusters, excluding pathological models, were stable. This belief appeared after the classical works by Antonov (1960, 1962) devoted to isotropic systems, and was reiterated in following papers concerning some particular anisotropic systems (e.g. Mikhailovsky, Fridman & Epel'baum 1970; Doremus, Feix & Baumann 1971). Some time was therefore required to realize the very possibility of instability of spheres, first suggested by Polyachenko & Shukhman (1972), until Merritt & Aguilar (1985), Barnes, Goodman & Hut (1986) and May & Binney (1986) made it widely known.

Radial orbit instability is suppressed for sufficiently rounded orbits, or when the kinetic energy stored in transverse directions \( T \perp \) becomes sufficiently high. Polyachenko & Shukhman (1981) proposed a global anisotropy parameter as a ratio of the radial to transverse kinetic energy of the system, \( \xi \equiv 2T_r/T \perp \). For the Idlis model (Idlis 1956), they found the stability boundary to be \( \xi = 1.59 \). Later Fridman & Polyachenko (1984), using this and two other families of models, proposed a hypothesis that the global anisotropy parameter can give a general stability criterion for anisotropic systems: the system is stable if \( \xi < 1.7 \pm 0.25 \).

Later studies of spherical models by means of linear stability analysis (Saha 1991; Weinberg 1991; Bertin et al. 1994) and \( N \)-body simulations (Merritt & Aguilar 1985; Barnes et al. 1986; Merritt 1987; Dejonghe & Merritt 1988; Meza & Zamorano 1997) used a variety of models, including radially anisotropic on the periphery and isotropic in the centre and vice versa. In these works, the stability boundaries in terms of the global anisotropy parameter fall in the broad range \( 1.2 \leq \xi < 2.9 \). Thus, the hypothesis regarding universal stabilization in a narrow region of \( \xi \) was proven false. Note, however, that in each case a certain value of the stability boundary corresponding to a radially anisotropic system (\( \xi > 1 \)) was found. Such a boundary was also found for so-called generalized polytropes, with distribution function (DF)

\[
F(E, L) = C(s, q) L^{-\gamma} \exp(-E)^q ,
\]

1 Results of stability analysis of one of the three families of models described in Fridman & Polyachenko (1984) were reconsidered later in Polyachenko (1987)’s report. The reconsidered stability boundary \( 2T_r/T \perp \) fell between 2.05 and 2.10, instead of the previous boundary \( 2T_r/T \perp = 1.62 \), i.e. the systems proved to be more stable than was supposed before. The same boundary for this family was obtained later by Dejonghe & Merritt (1988) with the help of \( N \)-body simulations. A new (corrected) boundary falls slightly outside the interval suggested by Fridman & Polyachenko (1984).
Radially anisotropic polytropes

which becomes stable at $\xi \approx 1.4$ (Fridman & Polyachenko 1984; Barnes et al. 1986). Here $E$ and $L$ denote the energy and the angular momentum of a star, $C(s, q)$ is the normalizing constant and $s$ and $q$ are parameters of the model. The additive constant in gravitational potential $\Phi_q(r)$ is chosen in such a way that $\Phi_q(R) = 0$, where $R$ is the radius of the system.

Generalized polytropes are the simplest generalization of isotropic polytrope models (the latter correspond to $s = 0$). Polytrope models are classical ones in terms of stability theory of both gaseous and collisionless gravitating systems. One can recall, for example, the work of Antonov (1962), in which the stability of polytrope models with decreasing $DF$ was shown. Models with increasing $DF$ can be unstable, but they give an exotic mass distribution with density increasing outwards, thus describing unrealistic stellar systems.

Generalized polytropes are more versatile. The global anisotropy parameter for (1) can be obtained in a simple form:

$$\xi = \frac{2}{2 - s}. \tag{2}$$

Note that for this model the (local) anisotropy parameter $\beta(r)$ (Binney 1980) does not depend on the radius: $\beta = 1 - 1/\xi = s/2$. It is possible to evaluate expressions for the radial and transverse kinetic energy when $s < 2$; the limit $s \to 2$ corresponds to a system in which almost all orbits are radial. Since the $s = 0$ case is stable (considering only realistic models with $q > 0$) and the case $s \to 2$ is unstable due to radial orbit instability, there should be a critical value of the parameter $s = s_{\text{crit}}$ that divides stable and unstable systems.

Using the matrix method for spheres (Polyachenko & Shukhman 1981), which is analogous to the Kalnajs matrix method for discs (Kalnajs 1977), it was found that growth rates of the instability became small for $s \lesssim 0.6$, almost independent of parameter $q$ (Fridman & Polyachenko 1984). Thus, the critical parameter for generalized polytropes is $s_{\text{crit}} \approx 0.6$ (or $\xi \approx 1.4$). A similar result was obtained by $N$-body simulations (Barnes et al. 1986).

Palmer & Papaloizou (1987, henceforth PP87) have investigated the same models using approximate equations for unstable modes with low growth rates. They showed that instability must persist even for models arbitrary close to the isotropic limit $s = 0$; this seemingly contradicts the previous results mentioned above.

Solutions of the approximate equation form an infinite number of unstable modes with decreasing growth rates (eigenvalues), which accumulate near zero frequency $\omega = 0$. These eigenvalues correspond to eigenfunctions with different numbers of nodes; the nodeless eigenfunction gives the largest growth rate. As was argued in PP87, the largest eigenvalue cannot be caught by the approximate equation since it is too large to fulfil the assumed condition.

Guided by PP87’s models B and C, which correspond to $s = 2/3$ and $s = 1/3$ ($q = 1$), we have calculated the growth rates near the isotropic limit using the approximate equation. The result was paradoxical: the largest eigenvalue kept on growing to infinity as $s \to 0$, while one would expect that all modes, including the nodeless one, would decrease to zero.

This paper pursues two goals. First, we try to reconcile the results obtained by the matrix and $N$-body methods on one hand and the results of PP87 on the other. Secondly, we clarify the paradox regarding the applicability of the approximate equation of PP87. For simplicity, we shall assume models with $s < 1$ only. This condition provides sufficiently smooth gravitational potential in the centre and linear dependence of the precession velocity on angular momentum for nearly radial orbits (for more details, see below). In Section 2, we derive an approximate equation for modes with low growth rates from the full integral equations for spheres obtained by Polyachenko, Polyachenko & Shukhman (2007). It coincides in all but one important detail with the approximate integro-differential equation of PP87 (the equivalence of the two equations is demonstrated in the Appendix). Numerical results are given in Section 3 and our conclusions in Section 4.

2 THE INTEGRAL EQUATION FOR MODES WITH LOW GROWTH RATES

Traditional linear stability theories employ matrix methods, expanding the perturbed potential and density in series using special biorthonormal sets of basis functions (Kalnajs 1977; Polyachenko & Shukhman 1981). As a result, one obtains a set of integral equations that incorporates the mode frequency in a complicated non-linear manner. Thus each frequency is to be obtained separately by, for example, Cauchy integration in the complex plane.

Recently we have proposed an alternative method for calculation of eigenmodes (Polyachenko 2004, 2005; Polyachenko et al. 2007). The advantages of our method are (i) a linear form of the equation for eigenmodes and (ii) the absence of the necessity to construct the bi-orthonormal set (which must be customized for a particular model of unperturbed density to reduce the number of basis functions included in the expansion of perturbed potential and density). The alternative method is the most adequate one to derive an approximate integral equation similar to that used in PP87. We start with the full integral equation for perturbations proportional to a spherical harmonic with index $l$:

$$\phi_{l_1, l_2}(E, L) = \frac{4\pi G}{2l + 1} \int_{\Omega_1} D_l^2 \frac{\partial F}{\partial E} \frac{F(E', L')}{\omega - \Omega_{l_1, l_2}(E', L')} \, dE' \, dL', \tag{3}$$

Integration in (3) is over the curved triangle in the phase plane $(E', L')$: $\Phi_q(0) < E' < 0$, $0 < L' < \Omega_{\text{circ}}(E')$; $\Omega_{\text{circ}}(E)$ is the angular momentum on the circular orbit with energy $E$; $\omega$ is the eigenfrequency; $\Omega_{l_1, l_2}(E, L) \equiv l_1 \Omega_1(E, L) + l_2 \Omega_2(E, L)$; $\Omega_{\text{circ}}$ are the orbital frequencies; $D_l^2 F(E', L') = \Omega_{l_1, l_2}(E', L') \left( \frac{\partial F}{\partial E} + l_1^2 \frac{\partial F}{\partial l} \right)$; the coefficients $D_l^2$ are equal to zero for odd $l - k$, otherwise

$$D_l^2 = \frac{1}{2E} \frac{1}{(l + k)!} \left( \frac{k}{l - k} \right) \left( \frac{l}{l + k} \right)^2.$$

Here $l_1$ and $l_2$ are indices of expansion over angular variables $w_1$ and $w_2$ in the action-angle formalism (Landau & Lifshitz 1976),

$$\delta \Phi(l_1, I_1, l_2, w_1, w_2) = \sum_{l_1, l_2} \delta \Phi(l_1, l_2) \exp[i(l_1 w_1 + l_2 w_2)],$$

conjugated to the action variables $I_1 (i = 1, 2, 3)$:

$$I_1 = \int \rho_{\text{max}} \sqrt{2E - 2\Phi_q(r) - L^2/r^2} \, dr, \quad I_2 = L, \quad I_3 = L_{\circ}.$$

Due to degeneracy in the azimuthal number $m$, the eigenfrequency $\omega$ can be calculated only for axially symmetric perturbations $\delta \Phi(r, \theta; t) = \chi(r) P(t) e^{-i\omega t}$. The kernel of the integral equation is

$$\Pi_{l_1, l_2, l_3}(E, L, E', L') = \int dw_1 \int dw'_1 \int F_{l_1} \left[ r(E, L, w_1), r'(E', L', w'_1) \right] \cos \Theta_{l_1, l_2, l_3}(E, L, w_1) \cos \Theta_{l_1, l_2, l_3}(E', L', w'_1), \tag{4}$$

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where $\Theta_{l_1 l_2}(E, L; w_1) = (\Omega_{l_1 l_2}/\Omega_1) w_1 - l_2 \delta \Phi(E, L; w_1)$.

$$\delta \Phi(E, L; w_1) = L \int_{\max(E, L)}^r \frac{dx}{x \sqrt{(2E - 2\Phi_0(x)) x^2 - L^2}}.$$  

$F_{l}(r, r') = r_{l} / r_{l+1}^{(s)}$, $r_{\min} = \min(r, r')$, $r_{\max} = \max(r, r')$.

Finally, the eigenfunctions $\phi_{l_1 l_2}(E, L)$ are connected to the radial part of the perturbed potential as follows:

$$\phi_{l_1 l_2}(E, L) = \frac{1}{\pi} \int_0^\pi \cos \Theta_{l_1 l_2}(E, L; w_1) \chi_{l_1 l_2}(E, L; w_1) \, dw_1.$$  

To obtain the approximate equation by PP87 from (3) (or rather its full equivalent, in the form of the integral equation in $E$-space), one should make the following two simplifications: (i) low frequencies $\omega = \gamma v$ and even spherical numbers $l$; (ii) domination of nearly radial orbits.

The denominators of resonance terms $l_{l_1 l_2}$ contain construction proportional to the precession rate:

$$\Omega_{l_1 l_2}(E', L') = l_{l_1 l_2} (\Omega_2 - \Omega_1) \equiv l_{l_2} \Omega_{\text{pp}}(E', L'),$$  

which is small for nearly radial orbits,2

$$\Omega_{\text{pp}}(E, L) = \sigma(E) L.$$  

Dropping the non-resonance terms and denoting $\phi_{l_1 l_2}(E, 0)$ by $\Phi(E)$, from (3) one has

$$\Phi(E) = -\frac{8\pi G}{2l + 1} \sum_{l_{l_2} \leq l_{l_2}} D^2_{l_{l_2}} \int \frac{dE'}{v(E')} \frac{dE'}{v(E')} \Phi(E') \times \Pi(E'; E') \Omega_{\text{pp}}(E', L') \frac{dF}{dL} \left( \frac{v(E')}{\sigma(E)} \right)^2$$  

For a DF in the form $F(E, L) = g(E) L^{-s}$, the approximate equation reads

$$\Phi(E) = \frac{8\pi G}{2l + 1} \sum_{l_{l_2} \leq l_{l_2}} D^2_{l_{l_2}} \int \frac{dE'}{v(E')} \Phi(E') \times \left( g(E') \sigma(E') \right) \Pi(E'; E') \frac{s L_{l_1 l_2}^2 dL}{1 + 2L^2},$$  

where $v(E) \equiv \Omega_{l_1 l_2}(E, 0)$ and $\Pi(E, L)$ is the result of reduction of $\Pi_{l_1 l_2 l_2}(E, L; E', L')$ for the radial orbits.

Due to a singularity of the DF, the integral in (7) diverges when $\gamma = 0$ and is large when $\gamma$ is small. This justifies the omission of non-resonance terms, and sets constraints on the maximum value of $\gamma$.

It is clear that the main contribution to the integral comes from a narrow region $L \sim \gamma / \sigma$; thus one can change the variable of integration and replace the upper boundary by infinity:

$$\int_0^{L_{l_1 l_2}^2} \frac{dL}{\gamma^2 + k^2 \sigma^2 L^2} = \frac{\gamma^{-1}(k \sigma)^{-1}}{2} I(s),$$  

where

$$I(s) \equiv \int_0^\infty \frac{x^{s-1}}{1 + x^2} dx = \frac{\pi}{2 \sin(\frac{\pi s}{2})}.$$  

By an appropriate change of the eigenfunction, one can reduce the problem to the integral equation

$$\lambda(s) \Psi(E) = \int_0^\pi dE' R_{\lambda}(E, E') \Psi(E'),$$  

with

$$\lambda = \gamma^s,$$  

and the positively defined symmetric kernel function

$$R_{\lambda}(E, E') = \alpha(s) \sqrt{h(E) h(E')} Q(E, E'),$$  

where $h(E) = g(E) \sigma^{-1} v(E)$, $Q(E, E') = \int_{\max(E)}^\pi \int_{\max(E')}^\pi \frac{dr dr' F(r, r')}{\sqrt{2E - 2\Phi_0(r')} \sqrt{2E - 2\Phi_0(r)}}.$$  

In the Appendix we show that (10) in $E$-space is fully equivalent to the approximate integral equation in $r$-space obtained by PP87.

The kernel (12) defines a self-adjoint Hilbert–Schmidt operator in infinite-dimensional space, so the eigenvalues $\lambda_n (n = 1, 2, \ldots)$ must have an accumulation point, $\lim_{n \to \infty} \lambda_n = 0$. The existence of arbitrary small eigenvalues is crucial for PP87 in demonstrating the instability of singular generalized polytopes with $s > 0$.

Note that in the limit $s \ll 1$, equation (10) with kernel (12) looks unnatural. Let us consider explicitly the case $s = 0$. The integral equation and the kernel then read as

$$A \Psi(E) = \int_{E_0}^E dE' R_{\lambda}(E, E') \Psi(E'),$$  

$$R_{\lambda}(E, E') = \alpha(0) \sqrt{\frac{g_0(E) g_0(E') v(E) v(E')}{\sigma(E) \sigma(E')}} Q(E, E'),$$  

with $A \equiv \lambda(0)$ and $\alpha(0) = [32\pi G/(2l + 1)] \sum_{k=1}^l D^2_{l_{l_2}}$. For example, in units in which $4\pi G = 1$, for $l = 2$ one has $\alpha(0) = 3/5$. The norm of the kernel is of order unity and thus the first several eigenvalues, corresponding to eigenfunctions with few nodes, must be of order unity. It needs to be emphasized that there is no small parameter left in problem (15); the only small parameter $s$ in the isotropic limit has disappeared from the equations.

We have calculated several of the largest eigenvalues $\Lambda_s$ for spherical indices $l = 2, 4, 6$. The results are summarized in Table 1. The eigenvalues exceeding 1 are emphasized by boldface. In particular, for $l = 2$ two eigenvalues are greater than 1. This means that an arbitrary small anisotropy (or arbitrary small s) will produce exponentially high growth rates:

$$\gamma_n = \Lambda_n^{-1} \propto \exp \left( \frac{1}{s} \ln \Lambda_n \right), \quad n = 0, 1.$$  

Table 1. The largest six eigenvalues $\Lambda_s$ (i.e. $\lambda_n$ at $s = 0$) for quadrupole $l = 2$ and the next two spherical harmonics ($l = 4$ and $l = 6$) for parameter $q = 1.0$.

| $l$ \( \times \) $n$ | $1 \times 0$ | $1 \times 1$ | $2 \times 1$ | $2 \times 2$ | $2 \times 3$ | $2 \times 4$ | $2 \times 5$ |
|------------------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 2                | 3.7851    | 1.0301    | 0.4762    | 0.2637    | 0.1621    | 0.1068    |           |
| 4                | 1.3838    | 0.4194    | 0.2215    | 0.1370    | 0.0921    | 0.0654    |           |
| 6                | 0.7029    | 0.2209    | 0.1232    | 0.0803    | 0.0566    | 0.0419    |           |

2For generalized polytopes, the gravitational potential behaves like $\Phi_0(r) \propto r^{-\gamma}$ near the centre. Thus, for the case of interest $s < 1$, the gravitational force $-\Phi_0'(r)$ is non-singular at the centre and hence the precession velocity of nearly radial orbits is indeed linear with respect to the angular momentum, $\Omega_{\text{pp}}(E, L) \approx \sigma(E) L$ (see e.g. Touma & Tremaine 1997). Note that for singular $\Phi_0'(r)$ (say, $\Phi_0(r) \propto r^p$, with $p < 1$) the dependence of precession rate on $L$ is not linear, i.e. $\Omega_{\text{pp}} \propto L^p$.  

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However, the growth rates of other modes are exponentially small:
\[
\gamma_n(s) = \Lambda_n^{1/2} \propto \exp\left(\frac{1}{s} \ln \frac{1}{\Lambda_n}\right), \quad n \geq 2.
\] (18)

Note that (17) contradicts equation (6), in which the kernel becomes zero at \( s = 0 \) due to the term \( \partial F/\partial L \). The inconsistency evidently comes from changing the upper limit of integration in (8) to infinity: for \( s = 0 \) this integral turns into \( \int_0^{L_{\text{max}}} L \, dL / (\gamma^2 + k^2 \sigma^2 L^2) \) and diverges if \( L_{\text{max}} \to \infty \). However, such a form of the integrand is valid for nearly radial orbits only. Additionally, we have expanded the integration region up to infinite angular momentum. This is justified for systems mainly populated by nearly radial orbits, but not for nearly isotropic ones.

To cope with anomalously growing modes (17), one can take into account the finite value of \( L_{\text{max}} \) in (8). Changing the variable of integration, \( L = [\gamma/(k\sigma)] \), the integral (8) can be reduced to
\[
\int_0^{L_{\text{max}}} \frac{L^{-3/2} dL}{\gamma^2 + k^2 \sigma^2 L^2} = \gamma^{-1}(k\sigma)^{-2} \int_0^{\infty} \frac{\Gamma(\gamma)}{x^{\gamma+1}} \, dx,
\]
Since \( \sigma L_{\text{circ}} \sim \Omega_2[E, L_{\text{circ}}(E)] - \frac{\Omega_1(E)}{L_{\text{circ}}(E)} = \Omega - \frac{\Omega}{2} \), where \( \Omega \) and \( \Omega \) are the circular and radial frequencies, one can replace the upper boundary of integration by \( \Omega / \gamma \), with some characteristic frequency \( \Omega / \gamma \sim \Omega_0 \), \( \Omega_0 = (GM)/[2R^3] \). Then instead of (10), we have
\[
\dot{I}(s, \Omega / \gamma) = \pi \left( \frac{1}{\sin \left( \frac{\pi}{\Omega} \right)} \right) \left[ 1 - \exp\left( -s \ln \frac{\Omega}{\gamma} \right) \right].
\] (19)

(20)

For very small \( s, s \ll \ln \Omega/\gamma \), the expression gives \( \dot{I}(s, \Omega / \gamma) \approx \ln \Omega / \gamma \), for \( s \gg \ln \Omega / \gamma \) it coincides with the old expression (9).

The corrected approximate integral equation can be written in the same form as the old one (10), (12), but instead of (11) one should write a new relation between \( \lambda \) and \( \gamma \):
\[
\lambda = \frac{\gamma^2}{1 - \gamma / \Omega \gamma}, \quad \gamma = (1 + \lambda / \Omega \gamma)^{1/2}.
\] (21)

This relation clearly provides exponentially small growth rates for all modes at \( s \to 0 \). In this limit, the spectrum of eigenvalues \( \lambda_n(s) \) remains the same, but now
\[
\gamma_n(s) \propto \left( \frac{\Lambda_n}{1 + \Lambda_n} \right)^{1/2} \exp\left[ -\frac{1}{s} \ln \left( 1 + \frac{1}{\Lambda_n} \right) \right].
\] (22)

asymptotically tends to zero when \( s \to 0 \).

3 UNSTABLE MODES OF GENERALIZED POLYTROPES

For numerical evaluation of the growth rates of anisotropic polytropes, one usually introduces the following units:
\[
A(s, q) = 1, \quad \Psi(0) = 1.
\] (23)
The dependence of characteristic frequency $\Omega_D$ on parameter $s$ for $q = 1$ (solid line) and $q = 0.7$ (dashed line).

(iii) on the diagonal $\varepsilon = \varepsilon' \ll 1$ with $s = 0$,

$$Q(\varepsilon, \varepsilon') \simeq \sqrt{\frac{3}{2}} \left( \frac{1}{2} + G \right) \varepsilon^{-1/2} = 1.734 \varepsilon^{-1/2},$$

where $G = 0.915965594 \ldots$ is Catalan’s constant;

(iv) for $\varepsilon' \ll \varepsilon \ll 1$ and arbitrary $s$,

$$Q(\varepsilon, \varepsilon') = \frac{5 \sqrt{\pi}}{12p} \frac{\Gamma \left( \frac{1}{2p} \right)}{\Gamma \left( \frac{1}{p} + \frac{1}{2p} \right)} K^{-1/p} e^{-1/2} (\varepsilon')^{1/p-1/2}. \quad (29)$$

The integral equation (10) has been solved for modes with spherical harmonics $l = 2, 4, 6$. First of all, we were interested in models with small parameter $s \ll 1$ to find out how small the growth rates of weakly anisotropic systems are. Also, we have calculated growth rates for models B ($s = 2/3, q = 1$) and C ($s = 1/3, q = 1$) of PP87. In the numerical work, we restricted ourselves to models within the range $0 \leq s \leq 0.8$.

Fig. 1 shows the behaviour of the characteristic frequency $\Omega_D(s) = (GM/2R^3)^{1/2}$ for $q = 1$ and $q = 0.7$. One can see that the dependence is weak and $\Omega_D \sim 0.1$ in the considered range of parameter $s$.

Fig. 2 shows the growth rates $\sigma_s$ in units of characteristic frequency $\Omega_D$ versus parameter $s$ for $q = 1$. Dashed lines show growth rates obtained from (10) with relation (11); this case is equivalent to the approximate equation used by PP87. For $s = 1/3$ and $s = 2/3$, our growth rates agree satisfactorily with the one obtained in the cited paper. Numbers denote modes ($0$ – nodeless mode, $1$ – mode with one node, etc.). The growth rates of the first two modes increase violently as the model approaches the isotropic limit; the other modes decrease exponentially to zero.

Solid lines show the growth rates versus parameter $s$ obtained from (21). Its behaviour complies with our intuitive expectations that the unstable modes should be stabilized in the isotropic limit.

A region near $s \approx 0$ is magnified in Fig. 3. Due to fast (exponential) decrease of the growth rates, they become negligibly small at $s = 0.01$.

Fig. 4 shows several first eigenfunctions $\Psi_n(E)$ ($n = 0, 1, 2, 3$) of integral equation (10) for the model $s = 1/3, q = 1$ and corresponding radial parts of the potential $\chi_n(r)$:

$$\chi_n(r) = \int_{-1}^{0} dE \sqrt{h(E)} \Psi_n(E) \int_{0}^{\max(E)} \frac{dr' \mathcal{F}_n(r, r')}{\sqrt{2E - 2\Phi_0(r')}} \quad (30)$$

(for derivation of this relation, see the Appendix). In all cases, the eigenfunctions have an equal number of nodes coinciding with $n$. The form of the radial parts of the potential $\chi_n(r)$ is in qualitative agreement with the approximate form $\chi_n(r) \sim r^n$ for $n = 0.08$. 

Figure 1. The dependence of characteristic frequency $\Omega_D$ on parameter $s$ for $q = 1$ (solid line) and $q = 0.7$ (dashed line).

Figure 3. The same as in Fig. 2, on a log–log axis.

Figure 4. The first four eigenfunctions of quadrupole harmonics $l = 2$: $\Psi_n(E)$ (upper) and the radial part of the potential $\chi_n(r)$ (lower). The normalization of the eigenfunction is arbitrary.
Figure 5. The dependence of the growth rates $\gamma_n$ from (20) of the first seven unstable modes on parameter $s$ for the model $q = 0.7$. Crosses mark the growth rates obtained by Fridman & Polyachenko (1984). The characteristic frequency is taken as $\Omega = 0.02$.

agreement with the results presented in fig. 1 of PP87, which are solutions of the integral equation (2.25).

It is interesting to compare the growth rates obtained with our approximate integral equation with independent calculations of generalized polytropes. In Fig. 5, we show the dependence of growth rates for first seven modes on the parameter $s$ for $q = 0.7$. Crosses mark an ‘experimental’ curve from Fridman & Polyachenko (1984). The curve breaks at $s \approx 0.6$, $\gamma \approx 0.004$ because of the accuracy of the matrix method employed. The numbers agree poorly with each other, so that it is impossible to join the curve marked by crosses with a new curve of the principal mode $n = 0$.

The validity of the approximate integral equation is restricted to very small growth rates. For modes with sufficiently high $n$, this restriction does not play any role, since their growth rates are small for all $s$, but it is important for modes with few nodes, especially for the nodeless one. Its growth rate increases with anisotropy, and presumably at $s \lesssim 0.15$ it reaches the upper boundary. This might be the reason for the evident discrepancy between the two curves. In Fig. 5, the region of uncertainty $0.15 \lesssim s < 0.6$ is shown by a dashed line.

4 CONCLUSIONS

(1) It is difficult to reconcile the results of linear stability analysis (Fridman & Polyachenko 1984) and $N$-body experiments (Barnes et al. 1986) with the results by PP87 for two reasons.

First, the approximate integral equation derived in PP87 on which their analysis is based is applicable to very small growth rates only. In the isotropic limit corresponding to $s \to 0$, these growth rates are exponentially small, i.e. $\gamma \propto \exp(-a_s s)$. The estimates show that even for $s \approx 0.5$ the allowed growth rates are much less than $0.01\Omega$, which is definitely below any reasonable accuracy of the matrix method and $N$-body experiments.

Secondly, strictly speaking the approximate integral equation is not applicable to unstable modes having eigenfunctions with just a few nodes, including the principal mode with maximum growth rate, since for $n$ not too large (say, $n = 0, 1, 2$) the eigenvalues $\lambda_n$ are not very small even in the limit $s \to 0$. However, the principal mode plays a major role in determining the stability boundary (see Fridman & Polyachenko 1984; Barnes et al. 1986).

(2) The growth rates are formally accurate when they are small. However, it is likely that once $\sigma_0 \equiv \gamma_0 \Omega_\Omega < 1$ they give reasonable estimates of actual growth rates in practice (PP87). Then, since isotropic systems with decreasing DF are stable (Antonov 1960, 1962), all modes should become stable when $s$ approaches zero and the valid approximate equation must describe all modes correctly. This fact is in contradiction to our solution of PP87’s approximate equation (2.25) (the equivalent of our equation (10), in which the relation $\lambda = \gamma'$ is assumed). This solution demonstrates explicitly that for the quadrupole harmonic $l = 2$ there are two modes with exponentially increasing growth rates for $s \to 0$.

The reason for such a discrepancy arises from the behaviour of the terms of the approximate integral equation in the limit $s \to 0$. Using the notation of PP87, the growth rates can be expressed in the form

$$\gamma' = \frac{A_3}{A_1 - A_2},$$

where $A_i$ are some averages (quadratic forms) of the positively defined operators. In particular, $A_3$ denotes the average of the integral operator defined by (12). $A_1$ denotes the omitted non-resonance part and $A_1$ denotes the operator that is the left side of the radial Poisson equation. Since both terms $A_1$ and $A_3$ are retained in the limit when the system becomes isotropic, Palmer & Papaloizou infer that instability exists no matter how weak the divergence in DF as $L \to 0$.

In this paper we argue that the term $A_3$ must vanish when $s \to 0$ in order to comply with stabilization of isotropic models. Using our alternative method of determining the unstable eigenmodes based on the solution of a linear eigenvalue problem, we derived the appropriate integral equation. This equation gives a set of unstable modes, all of which become stable in the isotropic limit $s \to 0$.

To summarize, our considerations prove that the instability growth rates of all modes in unbounded models at $L = 0$ do not indeed vanish unless the models are isotropic, but become exponentially small. In actuality this means stability if we take into account the finite lifetimes of real astronomical objects. Additionally, the most probable distributions are non-singular ones. We therefore have to infer that stable distributions generally become unstable at some finite value of radial anisotropy, i.e. a finite anisotropy threshold exists, where the width of the threshold depends on a particular model.

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3 For convenience, we have changed the signs of all three of PP87’s operators $C_i$ to make them positively defined.
Since
\[ v_r = \pm \sqrt{2E - 2\Phi_0(r) - L^2/r^2}, \quad v_\theta = \pm \frac{L}{r} \sqrt{1 - \sin^2 \theta_0/\sin^2 \theta} \]
and
\[ v_\phi = \frac{L \sin \theta_0}{r \sin \theta}, \]
we obtain for the volume element in the velocity space
\[
dv = dv_r dv_\theta dv_\phi \]
\[
= 4L dL dE d(\sin \theta_0) \]
\[
= r^2 \sqrt{2E - 2\Phi_0(r) - L^2/r^2} \sqrt{\cos^2 \theta_0 - \cos^2 \theta} \]
(The factor 4 appears here because we have to take into account particles having velocities \(v_r\) and \(v_\theta\) of both signs.) To integrate over \(\theta\) in (A3), it is convenient to go to integration over \(w_2\). To this end, we have
\[ \sin \theta d\theta = \cos \theta_0 \sin(w_2 - \partial S_1/\partial I_2) dw_2 \]
and
\[
\int_0^\pi P_\ell \cos \theta \, e^{i\ell w_2} \sin \theta \, d\theta = \frac{1}{2} \int_0^{2\pi} P_\ell \cos \theta \, e^{i\ell w_2} dw_2. \quad (A6)
\]
As a result, we obtain
\[
\Pi(\ell) = \frac{-1}{(4\ell + 1)(2\pi)} \sum_{l_2} \sum_{l_1} \int_0^\pi \int \frac{4L dL dE}{4E - 2\Phi_0(r) - L^2/r^2} \]
\[
\times \phi_{\ell l_1} (E, L) \, e^{i\ell(w_1 + w_2) / \Omega_{l_1 l_2}} \frac{\partial F / \partial I_1 + I_2 \partial F / \partial I_2}{\omega - \Omega_{l_1 l_2}} \]
\[
\int_0^{2\pi} P_\ell \cos \theta_0 \, e^{i\ell w_2} dw_2, \quad (A7)
\]
Taking into account that the integral over \(z\) is
\[
\int_0^1 dz P^{l_2}_{\ell_2}(0) P^{l_1}_{\ell_1}(z) P^{l_2}_{\ell_2}(0) P^{l_1}_{\ell_1}(z) = \frac{2}{2l_1 + 1} D^{l_2}_{l_1},
\]
we find the final expression for the radial part of the perturbed density \(\Pi(\ell)\):
\[
\Pi(\ell) = -\frac{4\pi}{L} \sum_{l_2} \sum_{l_1} D^{l_2}_{l_1} \int_0^{2\pi} \int L dL dE \]
\[
\times \phi_{l_1 l_2} (E, L) \cos \Theta_{l_1 l_2} \frac{\Omega_{l_1 l_2} \partial F / \partial E + I_1 \partial F / \partial L}{\sqrt{2E - 2\Phi_0(r) - L^2/r^2}} \]
\[
\Omega_{l_1 l_2}, \quad (A8)
\]
where (see Section 2)
\[
\phi_{l_1 l_2} (E, L) = \frac{1}{2\pi} \int_0^{2\pi} \cos \Theta_{l_1 l_2} (E, L, w_1) \chi(r, E, L, w_1) \, dw_1
\]
\[
= \frac{\Omega_{l_1 l_2}}{\pi} \int_0^{2\pi} \frac{dF(r) \cos \Theta_{l_1 l_2}}{\sqrt{2E - 2\Phi_0(r) - L^2/r^2}}.
\]
Keeping in the sum over \(l_1\), the resonance summand \(l_1 = -\frac{1}{2} l_2\) and supposing growth rate to be small, we retain the contribution with \(\partial F / \partial L\) only:
\[
\Pi(\ell) \approx \frac{8\pi}{L} \sum_{l_2} D^{l_2}_{l_1} \int_0^{2\pi} \int L dL dE \]
\[
\times \phi_{l_1 l_2} (E, L) \frac{\partial F / \partial L}{\sqrt{2E - 2\Phi_0(r) - L^2/r^2}} \Omega_{l_1 l_2} \cos \Theta_{l_1 l_2} \]
\[
\times \frac{\Omega_{l_1 l_2} \partial F / \partial L}{\sqrt{2E - 2\Phi_0(r) - L^2/r^2}} \cos \Theta_{l_1 l_2}. \quad (A9)
\]
Since the leading contribution to the integral over \( L \) comes from small \( L \), we substitute \( L = 0 \) where possible and suppose \( \Omega_{\pi} = \sigma(E)L \). Then \( \Theta_{i_2 j_2}(E, 0) = i_2 \pi \), and finally

\[
\Pi(r) = -\frac{8\pi s}{\gamma s} \int \frac{1}{r^2} \sum_{l_2=2}^{l} D_{l}^{k}\int \frac{g(E) \sigma_{i}^{-1}(E) dE}{\sqrt{2E - 2\Phi_0(r)}} = \frac{v(E)}{\pi} \int_{0}^{r_{\max}(E)} \frac{dr' \chi(r')}{\sqrt{2E - 2\Phi_0(r')}}.
\]

Substitution in the r.h.s. of (A1) yields the integro-differential equation (2.25) of PP87 in \( r \)-space:

\[
\frac{d}{dr} r \frac{d\chi(r)}{dr} - l(l + 1) \chi(r) = -\frac{4\pi G}{\gamma s} \int_{0}^{R} K(r, r') \chi(r') dr',
\]

with the kernel

\[
K(r, r') = \frac{4\pi s}{\sin(\pi s/2)} \sum_{l_2=2}^{l} D_{l}^{k} \frac{g(E) \sigma_{i}^{-1}(E) v(E) dE}{\sqrt{2E - 2\Phi_0(r)}}.
\]

Now it is easy to demonstrate that (A11) is equivalent to the integral equation (10) in \( E \)-space. We write (A10) in the form

\[
\Pi(r) = -\frac{4\pi s}{\sin(\pi s/2) \gamma s} \int \frac{1}{r^2} \sum_{l_2=2}^{l} D_{l}^{k} \int \frac{g(E) \sigma_{i}^{-1}(E) dE}{\sqrt{2E - 2\Phi_0(r)}} \Phi(E),
\]

where

\[
\Phi(E) = \frac{v(E)}{\pi} \int_{0}^{r_{\max}(E)} \frac{dr' \chi(r')}{\sqrt{2E - 2\Phi_0(r')}}.
\]

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