Bethe Ansatz and exact form factors of the $O(N)$ Gross Neveu-model

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Abstract: We apply the algebraic nested $O(N)$ Bethe Ansatz to construct a general form factor formula for the $O(N)$ Gross-Neveu model. We examine this formula for several operators, such as the energy momentum, the spin-field and the current. We also compare these results with the $1/N$ expansion of this model and obtain full agreement. We discuss bound state form factors, in particular for the three particle form factor of the field. In addition for the two particle case we prove a recursion relation for the K-functions of the higher level Bethe Ansatz.

Keywords: Exact S-Matrix, Form Factors, Bethe Ansatz, Integrable Field Theories
1 Introduction

The $O(N)$ $\sigma$- and Gross-Neveu models are integrable and asymptotically free quantum field theories in 1+1 dimension. The S-matrices of these two models correspond to two solutions of the Yang-Baxter equation [1, 2]. In previous articles we constructed the $O(N)$ nested off-shell Bethe Ansatz [3, 4] and applied this technique to construct the exact form factors for the $O(N)$ $\sigma$ model [5]. Here we extend this work and construct the form factors for the $O(N)$ Gross-Neveu for arbitrary number of fundamental particles (for the two-particle case see [6]). The model exhibits a very rich bound state structure and kinks (see e.g. [7]), turning this study even more challenging.

Before we recall the S-matrix and all other details of this model we should mention that the integrable structure present in 1+1 dimension is now becoming relevant and actual in higher dimensional gauge theories under specific circumstances. Remarkably, in the articles [8–11] (see also references therein) a non-perturbative formulation of planar scattering in the $N = 4$ Supersymmetric Yang-Mills theory ($N = 4$ SYM) with the so called polygonal Wilson loops was proposed and a new decomposition of the Wilson loops in terms of the fundamental building blocks-Pentagon transitions was introduced. These transitions are directly related to the dynamics of the Gubser-Klebanov-Polyakov flux-tube [12], which can be computed exactly by exploring the integrability. In addition, three axioms about the transitions that single particles must satisfy were postulated and, interestingly, it is possible to verify that these axioms correspond to some deformations of the form factor equations in 1 + 1- dimensional integrable quantum field theories. Such exact and constructive developments in the $N = 4$ SYM theory opens, indeed, large perspectives in the view of using the exact integrability and the full machinery of the form factor program to get physical insights, specially in the case of non-trivial symmetry groups, such as $SU(N)$ and $O(N)$.

In this article we consider the $O(N)$-Gross-Neveu model for $N = \text{even}$. We do not use any Lagrangian to construct the model, nevertheless, we give the following motivation.

The $O(N)$-Gross-Neveu model describes the interaction of $N/2$ Dirac (or $N$ Majorana) fermions defined by the Lagrangian\(^{1}\) [13]

$$\mathcal{L}^{GN} = \sum_{\alpha=1}^{N/2} \bar{\psi}_\alpha i \gamma \partial \psi_\alpha + \frac{1}{2} g^2 \left( \sum_{i=1}^{N/2} \bar{\psi}_\alpha \psi_\alpha \right)^2. \quad (1.1)$$

It is known from semi-classical calculations [14] that there are bound states of two fundamental fermions in the scalar and the anti-symmetric tensor channel. Furthermore there are kinks such that the fundamental fermions are kink-kink bound states. The bootstrap program does not use the Lagrangian, but we are looking for an factorizing S-matrix of an $O(N)$-isovector $N$-plet of self conjugate fundamental fermions. However, now we assume bound states in the scalar and anti-symmetric tensor channel of two of them.

\(^{1}\)The Lagrangian (1.1) is invariant under $O(N)$ transformations of the vector of $N$ Majorana fermi fields $\psi_\alpha^{(i)}$ ($\alpha = 1, \ldots, N/2, i = 1, 2$), where $\psi_\alpha = \psi_\alpha^{(1)} + i \psi_\alpha^{(2)}$ [2].
In this article we construct the form factors of the model by using the solution of the $O(N)$ difference equation, derived previously [3, 5] by generalizing Tarasov’s methods [15] (see also [16]) of the algebraic Bethe Ansatz. Exact form factors for the energy-momentum, the spin-field and the current are computed and compared with the $1/N$ expansion of the $O(N)$ Gross-Neveu model. In the framework of 2-dimensional integrable QFTs the central problem is still the computation of the correlation functions or Wightman functions and the form factor program is exactly devoted to this purpose. The concept of a generalized form factor was introduced in [6, 17], where several consistency equations were formulated. Subsequently this approach was developed further and investigated in different models by Smirnov [18]. Generalized form factors are matrix elements of fields with many particle states. To construct these objects explicitly one has to solve generalized Watson’s equations which are matrix difference equations. To solve these equations the so called “off-shell Bethe Ansatz” is applied [3, 19–21]. The conventional Bethe Ansatz introduced by Bethe [22] is used to solve eigenvalue problems and its algebraic formulation was developed by Faddeev and coworkers (see e.g. [23]). The off-shell Bethe Ansatz has been introduced in [24] to solve the Knizhnik-Zamolodchikov equations which are differential equations. For other approaches to form factors in integrable quantum field theories see also [25–33]. The main result of this paper is the general form factor formula, written as an integral representation, which provides the solution of all form factors equations and whose main idea is briefly explained below. In the $O(N)$-Gross-Neveu model the fundamental particles form an isovector $N$-plet of $O(N)$. For a state of $n$ particles of kind $\alpha_i$ with rapidities $\theta_i$ and a local operator $O(x)$ the matrix element

$$\langle 0 | O(x) | \theta_1, \ldots, \theta_n \rangle^\alpha_n = e^{-ix(p_1 + \cdots + p_n)} F^O_\alpha(\theta)$$

defines a form factor which we write as (see [6])

$$F^O_\alpha(\theta) = K^O_\alpha(\theta) \prod_{1 \leq i < j \leq n} F(\theta_{ij})$$

(1.2)

where $F(\theta)$ is the minimal form factor function.

We propose the following Ansatz for the $K$-function in terms of a nested ‘off-shell’ Bethe Ansatz written as a multiple contour integral

$$K^O_\alpha(\theta) = N^O_n \int_{c^{(1)}_\alpha} dz_1 \cdots \int_{c^{(m)}_\alpha} dz_m \tilde{h}(\theta, z) p^O(\theta, z) \tilde{\Psi}_\alpha(\theta, z) .$$

(1.3)

Here $\tilde{h}(\theta, z)$ is a scalar function which depends only on the S-matrix. The dependence on the specific operator $O(x)$ is encoded in the scalar $p$-function $p^O(\theta, z)$ which is in general a simple function of $e^{\theta_i}$ and $e^{z_j}$. The state $\tilde{\Psi}_\alpha$ in (1.3) is a linear combination of the basic Bethe Ansatz co-vectors (see (2.22))

$$\tilde{\Psi}_\alpha(\theta, z) = L^\beta_\alpha(z) \tilde{\Phi}_\beta^\alpha(\theta, z)$$

(1.4)
where summation over all $\tilde{\beta} = (\tilde{\beta}_1, \ldots, \tilde{\beta}_m)$ is assumed. The $\tilde{\beta}$ form an $(N-2)$-plet of $O(N-2)$. For $L_{\tilde{\beta}}(z)$ we make again an Ansatz like (1.3). The nested off-shell Bethe Ansatz is obtained by iterating this procedure.

In the present paper we mainly consider the case where $\alpha$ correspond to the fundamental fermions of the $O(N)$-Gross-Neveu model Lagrangian (1.1). In forthcoming publications we will consider the kinks [34] and we will discuss, in particular, the $O(6)$-Gross-Neveu model in more detail [35].

The article is organized as follows. In Section 2 we recall some results and fix the notation concerning the $O(N)$ S-matrix, the monodromy matrix, etc. In Section 3 we discuss the generalized form factors formula for the $O(N)$-Gross-Neveu model. In Section 4 we apply the nested off-shell Bethe Ansatz to solve the $O(N)$ form factor equations. Section 5 is devoted to the computation of some examples. The appendices provide the more complicated proofs of the results we have obtained and further explicit calculations.

2 General settings

2.1 The $O(N)$-Gross-Neveu S-matrix

We consider the fundamental particles of the Lagrangian (1.1) which are fermions and transform as the vector representation of $O(N)$. The structure of the S-matrix\footnote{The S-matrix for the scattering of the fundamental fermions was calculated in [2].} is the same as that of the nonlinear $\sigma$-model [5]

$$S(\theta) = b(\theta)1 + c(\theta)P + d(\theta)K,$$  

(2.1)

however, here we are looking for a solution of the $O(N)$-Yang-Baxter equations with a bound state pole in the physical strip $0 < \text{Im} \theta < \pi$. Therefore, here “minimality” implies that the S-matrix for the scattering of two fundamental particles is of the form

$$S(\theta, N) = \frac{\sinh \theta + i \sin \pi \nu}{\sinh \theta - i \sin \pi \nu} S_{\text{min}}(\theta), \quad \text{with } \nu = \frac{2}{N - 2}. \quad (2.2)$$

This S-matrix was given by Zamolodchikov-Zamolodchikov [2]. The first factor in (2.2) is the sine-Gordon breather-breather [36] amplitude $S_{\text{GG}}^{bb}(\theta)$ and $S_{\text{min}}$ is the minimal $O(N)$ S-matrix which is the one of the nonlinear $\sigma$-model (see e.g. [5]). The position of the pole is dictated by the condition [37] that the pole has to be cancelled by a zero in the corresponding NLS-amplitude for the amplitude $S_+$. This condition\footnote{An additional pole in $S_+^{\text{GN}}$ would contradict positivity in the Hilbert space (for details see [37]).} fixes the pole and therefore the bound state mass spectrum

$$m_k = 2m \sin \frac{1}{2} k \pi \nu \quad (k = 1, 2, \ldots, N/2 - 2). \quad (2.3)$$

For each “principal” quantum number $k$ there exist particles $b_k^{(r)}$ which are anti-symmetric tensors of rank $r = k, k-2, \cdots \geq 0$, i.e. they transform according to the $r$-th fundamental representation of $O(N)$. These particles are bosons/fermions for $k$ even/odd. In addition
Figure 1. Particle spectrum of the $O(N)$-Gross-Neveu model for $N = 14$

there exist “kinks” of mass $m$ which transform as the two spinor representations of $O(N)$ (with positive or negative isotopic chirality).

Note the intimate connection between the spectrum of the GN-model, figure 1, and the Dynkin diagram figure 2. There exist exclusively such one-particle states which transform

Figure 2. Dynkin diagram for $O(N)$

according to one of the fundamental (or trivial) representations of $O(N)$.

Equation (2.1) writes in terms of the components as

$$S^{\delta\gamma}_{\alpha\beta}(\theta) = b(\theta)\delta^\delta_\alpha\delta^\gamma_\beta + c(\theta)\delta^\delta_\alpha\delta^\beta_\gamma + d(\theta)C^{\delta\gamma}C^{\alpha\beta}$$

with the rapidity difference $\theta$ of the particles and the “charge conjugation matrices”

$$C_{\alpha\beta} = \delta_{\alpha\bar{\beta}} \text{ and } C^{\alpha\beta} = \delta^{\alpha\bar{\beta}} \tag{2.4}$$

in the complex basis (see subsubsection 2.1.1). Crossing means

$$S^{\delta\gamma}_{\alpha\beta}(\theta) = C_{\alpha\gamma}S^{\alpha\delta}_{\beta\gamma}(i\pi - \theta)C^{\gamma\gamma} \tag{2.5}$$

or in terms of the amplitudes

$$b(\theta) = b(i\pi - \theta), \quad d(\theta) = c(i\pi - \theta). \tag{2.6}$$

The Yang-Baxter relation

$$S_{12}(\theta_{12})S_{13}(\theta_{13})S_{23}(\theta_{23}) = S_{23}(\theta_{23})S_{13}(\theta_{13})S_{12}(\theta_{12}) \tag{2.7}$$

implies [1] with $\nu = 2/(N - 2)$

$$c(\theta) = -\frac{i\pi\nu}{\theta}b(\theta), \quad d(\theta) = -\frac{i\pi\nu}{i\pi - \theta}b(\theta). \tag{2.8}$$
The three S-matrix eigenvalues are $S_\pm = b \pm c$ and $S_0 = b + c + Nd$ with

$$
(S_0, S_+, S_-) = \left( \frac{\theta + i\pi}{\theta - i\pi}, \frac{\theta - i\pi}{\theta + i\pi}, 1 \right) S_-. \tag{2.9}
$$

Unitarity reads

$$
S_{0,+,+}(\theta)S_{0,-,-}(-\theta) = 1. \tag{2.10}
$$

The highest weight amplitude is

$$
a(\theta) = S_+(\theta) = \exp \left( 2 \int_0^\infty \frac{dt}{t} \left( \frac{e^{-t(1-\nu)} - e^{-t}}{1 + e^{-t}} \right) \sinh t \frac{\pi}{i} \right) \tag{2.11}
$$

For later convenience we introduce

$$
\tilde{S}(\theta) = \frac{S(\theta)}{S_+(\theta)} = \tilde{b}(\theta) \mathbf{1} + \tilde{c}(\theta) \mathbf{P} + \tilde{d}(\theta) \mathbf{K} \tag{2.12}
$$

where $\nu$ is replaced by $\tilde{\nu} = 2/(N - 4)$.

2.1.1 Complex basis

For the Bethe Ansatz it is convenient to use instead of the real basis $|\alpha\rangle_r$, $(\alpha = 1, 2, \ldots, N)$ the complex basis

$$
|\alpha\rangle = \frac{1}{\sqrt{2}} \left( |2\alpha - 1\rangle_r + i|2\alpha\rangle_r \right), \quad |\bar{\alpha}\rangle = \frac{1}{\sqrt{2}} \left( |2\alpha - 1\rangle_r - i|2\alpha\rangle_r \right), \quad \alpha = 1, 2, \ldots, [N/2]
$$

Below we will use the notation

$$
|\theta\rangle_\alpha = |\alpha(\theta)\rangle, \quad |\theta\rangle_{\bar{\alpha}} = |\bar{\alpha}(\theta)\rangle
$$

for one particle and one antiparticle states with rapidity $\theta$. The weight vectors

$$
w = (w_1, \ldots, w_{[N/2]})
$$

of the one-particle states are given by

$$
w_k = \delta_{k\alpha} \nu \quad \text{for} \left| \alpha \right> \quad \text{and} \quad w_k = -\delta_{k\alpha} \quad \text{for} \left| \bar{\alpha} \right>$$
**Remark 1** For even \( N \) this means that we consider \( O(N) \) as a subgroup of \( U(N/2) \).

We order the states as: 1, 2, \ldots, \bar{2}, \bar{1}. Then the charge conjugation matrix in the complex basis is of the form

\[
C^{\delta \gamma} = \delta^{\delta \bar{\gamma}}, \quad C_{\alpha \bar{\beta}} = \delta_{\alpha \bar{\beta}} \tag{2.14}
\]

\[
C = \begin{pmatrix}
0 & \cdots & 0 & 1 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 1 & 0 \\
\vdots & \ddots & \ddots & \vdots \\
1 & \cdots & 0 & 0
\end{pmatrix}.
\]

The annihilation-creation matrix in (2.1) may be written as

\[
K_{\alpha \bar{\beta}}^{\delta \gamma} = C^{\delta \gamma} C_{\alpha \bar{\beta}}.
\]

### 2.1.2 Bound states

Following [7, 37, 38] we write for the fundamental fermions \( \alpha, \beta, \beta', \alpha' \)

\[
i \text{Res}_{\eta} (\sigma S)_{\alpha \beta}^{\beta' \alpha'} (\theta) = \sum_{\gamma} \Gamma_{\alpha \beta}^{\gamma \beta'} \Gamma_{\gamma \alpha}^{\beta} : i \text{Res}_{\eta} \Gamma_{\alpha \beta}^{\gamma \beta'} : = \Gamma_{\alpha \beta}^{\gamma \beta'} \tag{2.15}
\]

where \( \sigma = -1 \) is the statistics factor and the dual intertwiner is defined by the crossing relation

\[
\Gamma_{\gamma}^{\beta \alpha} = C_{\gamma \gamma'} \Gamma_{\alpha \beta'}^{\gamma'} C_{\beta' \beta} C_{\alpha' \alpha} : \tag{2.16}
\]

with the charge conjugation matrix \( C \). Here we have for \( \eta = \pi \nu \)

\[
i \text{Res}_{\eta} (\sigma S)_{\alpha \beta}^{\beta' \alpha'} (\theta) = -i \text{Res}_{\eta} \left( \frac{\sinh \theta + i \sin \pi \nu}{\sinh \theta - i \sin \pi \nu} \right) \left( S_{\text{min}} \right)_{\alpha \beta}^{\beta' \alpha'} (\theta) = 2 \tan \pi \nu \left( S_{\text{min}} \right)_{\alpha \beta}^{\beta \alpha'} (i \pi \nu)
\]

### 2.2 \( O(N) \) nested “off-shell” Bethe Ansatz

The “off-shell” Bethe Ansatz is used to construct vector valued functions which have symmetry properties according to a representation of the permutation group generated by a factorizing S-matrix. In addition they satisfy matrix differential [39] or difference [19] equations. For the application to form factors we use the co-vector version \( K_{1 \ldots n} (\theta) \in V_{1 \ldots n} = \ldots \)
\( (\otimes_{i=1}^{n} V)^\dagger \), \((\theta_i \in \mathbb{C}, i = 1, \ldots, n)\). We write the components of the co-vector \( K_{\alpha} \) as \( K_{\alpha}^\dagger \), where \( \alpha = (\alpha_1, \ldots, \alpha_n) \) is a state of \( n \) particles. Solutions of the \( O(N) \) equations

\[
K_{i,j}(\ldots, \theta_i, \theta_j, \ldots) = K_{j,i}(\ldots, \theta_j, \theta_i, \ldots) \tilde{S}_{ij}(\theta_{ij})
\]

where constructed in [3, 4] (see also [19, 21]). These equations are equivalent to the form factor equations (i) and (ii) (see (3.3) and (3.4)). The solutions have been constructed in terms of a nested \( O(N) \) “off-shell” Bethe Ansatz in [3, 4]. Here we need special solutions which satisfy in addition the form factor equations (iii) and (iv) (see (3.5)).

We consider a state with \( n \) particles and write the off-shell Bethe Ansatz co-vector valued function as

\[
K_\alpha(\theta) = \int_{C_\alpha^{(1)}} \cdots \int_{C_\alpha^{(m)}} k(\theta, \tilde{z}) \tilde{\Psi}_\alpha(\theta, \tilde{z})
\]

where \( \alpha = (\alpha_1, \ldots, \alpha_n) \), \( \theta = (\theta_1, \ldots, \theta_n) \) and \( \tilde{z} = (z_1, \ldots, z_m) \). This Ansatz transforms the complicated matrix equations (3.3)-(3.5) to simple equations for the scalar function \( k(\theta, \tilde{z}) \) (see [21] and below). The integration contour \( C_\alpha \) will be specified in section 4. The state \( \tilde{\Psi}_\alpha \) in (2.17) is the linear combination (1.4) of the basic Bethe Ansatz co-vectors (2.22).

As usual in the context of the algebraic Bethe Ansatz [23, 40] the basic Bethe Ansatz co-vectors \( \Phi^j_\alpha \) (for any fixed \( j \)) are obtained from the monodromy matrix. We consider a state with \( n \) particles and as is usual in the context of the algebraic Bethe Ansatz we define [23, 40] the monodromy matrix by

\[
\tilde{T}_{1 \ldots n, 0}(\theta, \theta_0) = \tilde{S}_{10}(\theta_1 - \theta_0) \cdots \tilde{S}_{n0}(\theta_n - \theta_0) = \left\{ \begin{array}{c} \cdots \\ 1 \\ \vdots \\ n \\ 0 \end{array} \right. .
\]

It is a matrix acting in the tensor product of the “quantum space” \( V^{1 \ldots n} = V_1 \otimes \cdots \otimes V_n \) and the “auxiliary space” \( V_0 \). All vector spaces \( V_i \) are isomorphic to a space \( V \) whose basis vectors label all kinds of particles. Here \( V \cong \mathbb{C}^N \) is the space of the vector representation of \( O(N) \).

Suppressing the indices \( 1 \ldots n \) we write the monodromy matrix in the complex basis as (following the notation of Tarasov [15])

\[
\tilde{T}_\alpha^{\alpha'} = \begin{pmatrix} \tilde{A}_1 & (\tilde{B}_1)^{\tilde{a}} \\ (\tilde{C}_1)^{\tilde{a}} & (\tilde{A}_2)^{\tilde{a}} & (\tilde{B}_2)^{\tilde{a}} \\ \tilde{C}_2 & (\tilde{C}_3)^{\tilde{a}} & \tilde{A}_3 \end{pmatrix}
\]

where \( \alpha, \alpha' \) assume the values 1, 2, \ldots, \( \bar{2} \), \( I \) corresponding to the basis vectors of the auxiliary space \( V \cong \mathbb{C}^N \) and \( \tilde{a}, \tilde{a}' \) assume the values 2, \ldots, \( \bar{2} \) corresponding to the basis vectors
of \( \hat{V} \cong \mathbb{C}^{N-2} \). We will also use the notation \( \hat{A} = \hat{A}_1, \hat{B} = \hat{B}_1, \hat{C} = \hat{C}_1 \) and \( \hat{D} = \hat{A}_2 \) which is an \((N-2) \times (N-2)\) matrix in the auxiliary space. As usual the Yang-Baxter algebra relation for the S-matrix yields the typical TTS-relation which implies the basic algebraic properties of the sub-matrices \( \hat{A}_i, \hat{B}_i, \hat{C}_i \).

The reference co-vector is defined as usual by

\[
\Omega \hat{B}_i = 0
\]

with the solution

\[
\Omega_\alpha = \delta^1_{\alpha_1} \cdots \delta^1_{\alpha_n}.
\] (2.20)

It satisfies

\[
\Omega \tilde{T}(\theta, z) = \Omega \begin{pmatrix} a_1(\theta, z) & 0 & 0 \\ * & a_2(\theta, z) & 0 \\ * & * & a_3(\theta, z) \end{pmatrix},
\] (2.21)

\[
a_1(\theta, z) = 1, \ a_2(\theta, z) = \prod_{k=1}^n \tilde{b}(\theta_i - z), \ a_3(\theta, z) = \prod_{k=1}^n \left( \tilde{b}(\theta_i - z) + \tilde{d}(\theta_i - z) \right).
\]

The basic Bethe Ansatz co-vectors in (2.17) are defined as (for more details see [3])

\[
\Phi_{\beta_1}(\theta, z) = \left( \Pi_{\beta_1}(z) \Omega \tilde{T}_{1}^{\beta_m}(\theta, z_m) \cdots \tilde{T}_{1}^{\beta_1}(\theta, z_1) \right)_\alpha = \begin{array}{c}
\beta_1 \\
\vdots \\
\beta_m \\
\end{array}
\begin{array}{c}
1 \\
\cdots \\
1 \\
\end{array}
\begin{array}{c}
z_i \\
\theta_1 \\
\theta_n \\
\end{array}
\begin{array}{c}
\alpha_1 \\
\cdots \\
\alpha_n \\
\end{array}
\] (2.22)

The matrix \( \Pi_{\beta_1}(z) \) intertwines between the S-matrix \( S \) of \( O(N) \) and the of \( \hat{S} \) of \( O(N-2) \)

\[
\tilde{S}_{ij}(z_{ij} \nu/\nu) \Pi_{...ij...}(z) = \Pi_{...ji...}(z) \tilde{S}_{ij}(z_{ij}).
\] (2.23)

This matrix \( \Pi \) is necessary\(^4\) because for the next level Bethe Ansatz the S-matrix \( \hat{S}(\theta) \) for \( O(N-2) \) has to be used. The co-vectors (2.22) are generalizations of vectors introduced by Tarasov [15] for a 3-state model, the Korepin-Izergin model. The following relations for special components of \( \Pi \) will be used below (for more details see [3–5])

\[
\Pi_{\beta_1}(z) = \begin{cases} 
0 & \text{for } \beta_1 = 1, \ \text{or } \beta_m = 1 \\
\delta_{\beta_1}^{\beta_1} \Pi_{\beta_2 \cdots \beta_m} & \text{for } \beta_1 \neq 1 \\
\Pi_{\beta_1 \cdots \beta_{m-1} \beta_m} & \text{for } \beta_m \neq 1.
\end{cases}
\] (2.24)

In particular for \( m = 2 \)

\[
\Pi_{\beta_1 \beta_2}(z) = \delta_{\beta_1}^{\beta_1} \delta_{\beta_2}^{\beta_2} + f(z_2) \delta_{\beta_1}^{\beta_1} \delta_{\beta_2}^{\beta_1} \delta_{\beta_2}^{\beta_2}, \ f(z) = \frac{i \pi \nu}{z + i \pi (1 - \nu)}.
\] (2.25)

\(^4\)This matrix \( \Pi \) is trivial for the \( SU(N) \) Bethe Ansatz because the \( SU(N) \) S-matrix does not depend on \( N \) for a suitable normalization and parametrization.
Remark 2 If the co-vector valued function $L_{\beta}(z)$ in (1.4) satisfies equation (4.10), the state $\tilde{\Psi}_{\alpha}(\theta,z)$ is a symmetric function of the $z_i$.

It is well known (see [3]) that the ‘off-shell’ Bethe Ansatz states are highest weight states if they satisfy certain matrix difference equations. For $n$ particle states the $O(N)$ weights are

$$(w_1, \ldots, w_{[N/2]}) = (n - n_1, \ldots, n_{[N/2]} - n_+ - n_+, n_+ - n_+)$$

where $n_1 = m, n_2, \ldots$ are the numbers of $\tilde{T}$ operators in (2.22) and the higher levels of the nesting. In particular $n_\pm$ are the numbers of positive/negative chirality spinor $C$-operators.

For the on-shell Bethe Ansatz for $N$ even see also [41]. As is well known (see e.g. [42–44] and references therein), the various levels of the nested Bethe Ansatz correspond to the nodes of the Dynkin diagram of the corresponding Lie algebra $D_{N/2}$ for $N = \text{even}$ (see Fig. 2). In [5] we used for the $O(N)$ $\sigma$-model the group isomorphism $O(4) \simeq SU(2) \otimes SU(2)$ to start the nesting procedure with form factors of the $SU(2)$ chiral Gross-Neveu model [45]. For the $O(N)$ Gross-Neveu model it is also possible to use the group isomorphism $O(6) \simeq SU(4)$ to start the nesting with form factors of the $SU(4)$ chiral Gross-Neveu model [45]. This will be performed in detail in a separate paper [35].

3 Generalized form factors

For a state of $n$ particles of kind $\alpha_i$ with rapidities $\theta_i$ and a local operator $O(x)$ we define the form factor functions $F_{\alpha_1 \ldots \alpha_n}(\theta_1, \ldots, \theta_n)$, or using a short hand notation $F_{\underline{n}}^O(\theta)$, by

$$\langle 0 | O(x) | \theta_1, \ldots, \theta_n \rangle_{\underline{n}} = e^{-ix(p_1 + \cdots + p_n)} F_{\underline{n}}^O(\theta), \quad \text{for } \theta_1 > \cdots > \theta_n.$$  \hfill (3.1)

where $\underline{n} = (\alpha_1, \ldots, \alpha_n)$ and $\theta = (\theta_1, \ldots, \theta_n)$. For all other arrangements of the rapidities the functions $F_{\underline{n}}^O(\theta)$ are given by analytic continuation. Note that the physical value of the form factor, i.e. the left hand side of (3.1), is given for ordered rapidities as indicated above and the statistics of the particles. The $F_{\underline{n}}^O(\theta)$ are considered as the components of a co-vector valued function $F_{\underline{1} \ldots \underline{n}}^O(\theta) \in V_{\underline{1} \ldots \underline{n}} = (V^{1 \ldots n})^\dagger$ which may be depicted as

$$F_{\underline{1} \ldots \underline{n}}^O(\theta) = \begin{bmatrix} O \\ \theta_1 | \cdots | \theta_n \end{bmatrix}.$$ \hfill (3.2)

Now we formulate the main properties of form factors in terms of the functions $F_{\underline{1} \ldots \underline{n}}^O$. They follow (see [20]) from general LSZ-assumptions and “maximal analyticity”, which means that $F_{\underline{1} \ldots \underline{n}}^O(\theta)$ is a meromorphic function with respect to all $\theta$’s and in the ‘physical’ strips $0 < \text{Im} \theta_{ij} < \pi$ ($\theta_{ij} = \theta_i - \theta_j, i < j$) there are only poles of physical origin as for example bound state poles. The generalized form factor functions satisfy the following
3.1 Form factor equations

The co-vector valued auxiliary function $F_{\alpha}^O(\theta)$ is meromorphic in all variables $\theta_1, \ldots, \theta_n$ and satisfies the following relations:

(i) The Watson’s equations describe the symmetry property under the permutation of both, the variables $\theta_i, \theta_j$ and the spaces $i,j = i+1$ at the same time

$$F_{..ij...}(\ldots, \theta_i, \theta_j, \ldots) = F_{..ji...}(\ldots, \theta_j, \theta_i, \ldots) (\sigma S)_{ij}(\theta_{ij})$$

for all possible arrangements of the $\theta$'s. The factor $\sigma$ is the statistics factor of the particles $i,j$ which is $\sigma_{ij} = -1$ for the fundamental fermions.

(ii) The crossing relation implies a periodicity property under the cyclic permutation of the rapidity variables and spaces

$$\langle p_1 \mid O(0) \mid p_2, \ldots, p_n \rangle_{\text{in,conn.}}^{\text{out,1}} = \sigma_{1\bar{1}}^{C^{11}} \sigma_{O1} F_{1\ldots n}(\theta_1+i\pi, \theta_2, \ldots, \theta_n) = F_{2\ldots n1}(\theta_2, \ldots, \theta_n, \theta_1-i\pi) C^{1\bar{1}}$$

The charge conjugation matrix $C^{1\bar{1}}$ is given by (2.14).

(iii) There are poles determined by one-particle states in each sub-channel. In particular the function $F_{2\bar{2}}^O(\theta)$ has a pole at $\theta_{12} = i\pi$ such that

$$\text{Res}_{\theta_{12}=i\pi} F_{1\ldots n}^O(\theta_1, \ldots, \theta_n) = 2i C_{12} F_{3\ldots n}^O(\theta_3, \ldots, \theta_n) (1-\sigma_2^O (\sigma S)_{2n} \ldots (\sigma S)_{23})$$

where $\sigma_2^O$ is the statistics factor of the operator $O$ with respect to the particle 2.

(iv) If there are also bound states in the model the function $F_{2\bar{2}}^O(\theta)$ has additional poles. If for instance the particles 1 and 2 form a bound state (12), there is a pole at $\theta_{12} = i\eta$, $(0 < \eta < \pi)$ such that

$$\text{Res}_{\theta_{12}=i\eta} F_{12\ldots n}^O(\theta_1, \theta_2, \ldots, \theta_n) = F_{(12)\ldots n}^O(\theta_{(12)}, \ldots, \theta_n) \sqrt{2} \Gamma^{(12)}_{12}\gamma$$

where the bound state intertwiner $\Gamma^{(12)}_{12}$ and the values of $\theta_1, \theta_2, \theta_{(12)}$ and $\eta$ are given in [7, 37, 38].

(v) Naturally, since we are dealing with relativistic quantum field theories we finally have

$$F_{1\ldots n}^O(\theta_1+\mu, \ldots, \theta_n+\mu) = e^{s\mu} F_{1\ldots n}^O(\theta_1, \ldots, \theta_n)$$

if the local operator transforms under Lorentz transformations as $O \to e^{s\mu}O$ where $s$ is the “spin” of $O$. 

The property (i) - (iv) may be depicted as

\begin{align*}
(\text{i}) \quad \mathcal{O} & \cdot \mathcal{O} = \mathcal{O} \\
(\text{ii}) \quad \mathcal{O} \otimes \mathcal{O} & = \mathcal{O} \\
(\text{iii}) \quad \frac{1}{2i} \text{Res}_{\theta_{12}=\pi} \mathcal{O} & = \mathcal{O} \\
(\text{iv}) \quad \frac{1}{\sqrt{2}} \text{Res}_{\theta_{12}=\pi} \mathcal{O} & = \mathcal{O}
\end{align*}

where \( \times \) denotes the statistics factor \( \sigma_1^\mathcal{O} \) of the operator \( \mathcal{O}(x) \) with respect to particle 1. The statistics factors in (ii) and (iii) are not arbitrary, but consistency and crossing (2.5) implies that both are the same and that the for anti-particle \( \sigma_1^\mathcal{O} \sigma_1^\mathcal{O} = 1 \) holds (see also [21]).

As was shown in [20, 38] the form factor equations follow from general LSZ-assumptions and “maximal analyticity”.

We will now provide a constructive and systematic way of how to solve the form factor equations for the co-vector valued function \( F_{\mathcal{O},n} \), once the scattering matrix is given.

**Minimal form factors:** The solutions of Watson’s and the crossing equations (i) and (ii) for two particles

\begin{align*}
F(\theta) &= S(\theta) F(-\theta) \\
F(i\pi - \theta) &= F(i\pi - \theta)
\end{align*}

with no poles in the physical strip \( 0 \leq \text{Im} \theta \leq \pi \) and at most a simple zero at \( \theta = 0 \) are the minimal form factors [6]

\begin{align*}
F_{\text{min}}^+(\theta) &= \exp \int_0^\infty dt \frac{e^{-t(1-\nu)} - e^{-t}}{t \sinh t} \left( 1 - \cosh t \left( 1 - \frac{\theta}{i\pi} \right) \right) \\
F_{\text{min}}^-(\theta) &= \frac{\cosh \frac{1}{2} (i\pi - \theta) \Gamma^2 (\frac{1}{2} + \frac{1}{2} \nu)}{\Gamma (1 + \frac{1}{2} \nu - \frac{1}{2i\pi} \theta) \Gamma (\frac{1}{2} \nu + \frac{1}{2i\pi} \theta)} F_{\text{min}}^+(\theta) \\
F_{\text{min}}^0(\theta) &= \frac{2 \tanh \frac{1}{2} (i\pi - \theta)}{i\pi - \theta} F_{\text{min}}^-(\theta)
\end{align*}

They belong to the S-matrix eigenvalues \( S_+ (\theta) \), \( S_- (\theta) \) and \( S_0 (\theta) \) of (2.9). For the construction of the off-shell Bethe Ansatz the minimal solution of the form factor equation (i) (see (3.3)) for the highest weight eigenvalue of the \( O(N) \) S-matrix

\begin{align*}
F(\theta) &= \sigma S_+ (\theta) F(-\theta) = -a(\theta) F(-\theta)
\end{align*}
is essential. We define

\[ F(\theta) = c \cosh \frac{1}{2} (i\pi - \theta) F^\text{min}_+ (\theta) \]
\[ = c \exp \left( \int_0^\infty \frac{dt}{t \sinh t} \frac{1 + e^{-t(1-\nu)}}{1 + e^{-t}} \left( 1 - \cosh t \left( -\frac{\theta}{i\pi} \right) \right) \right) \]
\[ F(\theta) = \frac{G \left( \frac{1}{2} \frac{\theta}{i\pi} \right) G \left( 1 - \frac{1}{2} \frac{\theta}{i\pi} \right) G \left( \frac{1}{2} - \frac{1}{2} \nu \right) G \left( \frac{3}{2} - \frac{1}{2} \nu - \frac{1}{2} \frac{\theta}{i\pi} \right)}{G \left( \frac{1}{2} + \frac{1}{2} \frac{\theta}{i\pi} \right) G \left( \frac{3}{2} - \frac{1}{2} \nu \right) G \left( 1 - \frac{1}{2} \nu + \frac{1}{2} \frac{\theta}{i\pi} \right) G \left( 2 - \frac{1}{2} \nu - \frac{1}{2} \frac{\theta}{i\pi} \right)} \]

where \( G(z) \) is Barnes G-function, which satisfies

\[ G(1+z) = \Gamma(z) G(z) \]}

For convenience we have introduced the constant \( c \) (see (4.6))

\[ c = G^2 \left( \frac{1}{2} \right) G^2 \left( 1 - \frac{1}{2} \nu \right) G^{-2} \left( \frac{3}{2} - \frac{1}{2} \nu \right) . \]

The full 2-particle form factors are

\[ F_{+,0}(\theta) = -\cos \frac{1}{2} \pi\nu \sinh \frac{1}{2} (\theta - i\pi\nu) \sinh \frac{1}{2} (\theta + i\pi\nu) F^\text{min}_{+,0}(\theta) . \]

They are non-minimal solutions of (3.8) containing the bound state pole at \( \theta = i\pi\nu \) (see (2.16) of [6]).

### 4 \( O(N) \) form factors and Bethe Ansatz

#### 4.1 The fundamental Theorem

We write the general form factor \( F^O_{1...n}(\theta) \) for \( n \)-fundamental particles following [6] as (1.2) where \( F(\theta) \) is the minimal form factor function (3.13). The K-function \( K^O_{1...n}(\theta) \) contains the entire pole structure and is determined by the form factor equations (i) - (iii). We propose the Ansatz (1.3) for the K-function in terms of a nested ‘off-shell’ Bethe Ansatz (2.17)

\[ K^O_x(\theta) = N_m^O \int_{c^{(1)}} dz_1 \cdots \int_{c^{(m)}} dz_m \tilde{h}(\theta, z) \tilde{p}^O(\theta, z) \tilde{\Psi}_{\alpha}(\theta, z) \]

written as a multiple contour integral. The scalar function \( \tilde{h}(\theta, z) \) depends only on the S-matrix and not on the specific operator \( O(x) \)

\[ \tilde{h}(\theta, z) = \prod_{i=1}^n \prod_{j=1}^m \tilde{\phi}(\theta_i - z_j) \prod_{1 \leq i < j \leq m} \tau(z_i - z_j) . \]

---

\( ^5 \)The minus sign in (3.12) and the factor \( \cosh \frac{1}{2} (i\pi - \theta) \) is due to fermionic statistics of the fundamental particles (see also eq. 4.14 of [20]).

\( ^6 \)Private communication: Karol K. Kozlowski pointed out to one of the authors (M.K.), that the minimal form factors may be expressed in terms of Barnes G-function.
Figure 3. The integration contours $\mathcal{C}_n^{(e)}$ and $\mathcal{C}_n^{(o)}$. The bullets refer to poles of the integrand resulting from $\tilde{\phi}(\theta_i - z_j)$ and the small open circles refer to poles originating from $\tilde{S}(\theta_i - z_j)$.

The functions $\tilde{\phi}$ and $\tau$ satisfy the shift equations

\begin{align*}
\tilde{\phi}(\theta - 2\pi i) &= -b(\theta)\tilde{\phi}(\theta) \quad (4.2) \\
\tau(z - 2\pi i)/b(2\pi i - z) &= \tau(z)/b(z) \quad (4.3)
\end{align*}

which are related to the form factor equation (ii) or (3.4) [3–5]. Here for the $O(N)$ Gross-Neveu model

\begin{equation}
\tau(z) = \frac{1}{\tilde{\phi}(-z)\tilde{\phi}(z)} \quad (4.4)
\end{equation}

where $\tilde{\phi}(\theta)$ is

\begin{equation}
\tilde{\phi}(\theta) = \Gamma \left( 1 - \frac{1}{2}\nu + \frac{1}{2\pi i}\theta \right) \Gamma \left( -\frac{1}{2\pi i}\theta \right). \quad (4.5)
\end{equation}

The form factor equation (iii) or (3.5) (as will be discussed in appendix A) requires that

\begin{equation}
F(\theta)F(\theta + i\pi)\tilde{\phi}(-\theta - i\pi + i\pi\nu)\tilde{\phi}(\theta) = 1. \quad (4.6)
\end{equation}

The function (4.5) satisfies this relation. Notice that the equations (4.5) and (4.6) also determine the normalization constant $c$ in (3.13) and (3.14).

The integration contours $\mathcal{C}_n^{(j)}$ depend on whether $j$ is even or odd, they are depicted in Fig. 3. The dependence on the specific operator $O(x)$ is encoded in the scalar p-function $p^O(\theta, z)$ which is in general a simple function of $e^{\theta_1}$ and $e^{z_1}$ (see below section 5). By means of the Ansatz (1.2) and (1.3) we have transformed the complicated form factor equations (i) - (v) (which are in general matrix equations) into much simpler scalar equations for the scalar p-function.

**Theorem 3** We make the following assumptions:

1. The p-function $p^O(\theta, z)$ satisfies the equations

\begin{align*}
(i') & \quad p^O(\theta, z) \text{ is symmetric under } \theta_i \leftrightarrow \theta_j \\
(ii'_1) & \quad p^O(\theta, z) = \sigma^O(-1)^m p^O(\theta_1 + 2\pi i, \theta_2, \ldots, z) \\
(ii'_2) & \quad p^O(\theta, z) = (-1)^n p^O(\theta, z_1 + 2\pi i, z_2, \ldots) \\
(iii') & \quad p^O(\theta, z) = p^O(\theta, z)
\end{align*}

(4.7)
where in (iii') \( \theta_{12} = i\pi, \ z_1 = \theta_1 - i\pi \nu \) and \( z_2 = \theta_2 \). The short notations \( \tilde{\theta} = (\theta_3, \ldots, \theta_n) \) and \( \tilde{z} = (z_3, \ldots, z_m) \) are used.

2. The higher level function \( L_2(z) \) in (1.4) satisfies (i) - (iii) of (4.10) - (4.12) for \( k = 1 \)

3. A suitable choice of the normalization constants in (1.3).

Then the co-vector valued function \( F_2(\theta) \) given by the Ansatz (1.2) and the integral representation (1.3) satisfies the form factor equations (i) - (v) of (3.3) - (3.7).

The proof of this theorem can be found in appendix A.

### 4.2 Higher level off-shell Bethe Ansatz

For convenience we use the variables \( u, v \) with \( \theta = i\pi \nu_k u, \ z = i\pi \nu_k v \) and \( \nu_k = 2/(N-2k-2) \). Let \( S^{(k)}(u) \) be the \( O(N-2k) \) S-matrix with

\[
\tilde{S}^{(k)}(u) = S^{(k)}/S^{(k)}_+ = \tilde{b}(u)1 + \tilde{c}(u)P + \tilde{d}_k(u)K
\]

(4.8)

We define

\[
K^{(k)}_2(u) = \tilde{N}_{m_k} \int_{C_\alpha} dv_1 \cdots \int_{C_\alpha} dv_{m_k} \tilde{h}(u, v) p^{(k)}(u, v) \tilde{\Psi}^{(k)}_2(u, v)
\]

(4.9)

\[
\tilde{\Psi}^{(k)}_2(u, v) = L^{(k)}_2(v) (\tilde{\Phi}^{(k)})^{(k)}_2(u, v), \quad L^{(k)}_2(v) = K^{(k+1)}_2(v)
\]

with \( u = u_1, \ldots, u_{n_k}, \ v = v_1, \ldots, v_{m_k} \) and \( m_k = n_{k+1} \).

The equations (i) - (iv) for \( k > 0 \) read in terms of these variables as

(i) The symmetry property under the permutation of both, the variables \( u_i, u_j \) and the spaces \( i, j = i + 1 \) at the same time

\[
K^{(k)}_{\ldots ij\ldots}(\ldots, u_i, u_j, \ldots) = K^{(k)}_{\ldots ji\ldots}(\ldots, u_j, u_i, \ldots) \tilde{S}^{(k)}_{ij}(u_{ij})
\]

(4.10)

for all possible arrangements of the \( u \)’s.

(ii) The periodicity property under the cyclic permutation of the rapidity variables and spaces

\[
K^{(k)}_{1\ldots n_k}(u_1 + 2/\nu, u_2, \ldots, u_{n_k})C^{11} = K^{(k)}_{2\ldots n_k1}(u_2, \ldots, u_{n_k}, u_1)C^{11}
\]

(4.11)

with the charge conjugation matrix \( C^{11} \).

(iii) The function \( K^{(k)}_{1\ldots n_k}(u) \) has a pole at \( u_{12} = 1/\nu_k \) such that

\[
\text{Res}_{u_{12}=1/\nu_k} K^{(k)}_{1\ldots n_k}(u_1, \ldots, u_{n_k}) = \prod_{j=3}^{n_k} (\tilde{\phi}(u_{i1} + 1)\tilde{\phi}(u_{i2})C_{12}K^{(k)}_{3\ldots n_k}(u_3, \ldots, u_{n_k})
\]

(4.12)
There are also bound states in the model, therefore the function \( F_{\alpha}^{(k)}(u) \) has an additional pole. If for instance the particles 1 and 2 form a bound state (12), there is a pole at \( u_{12} = 1 \) such that

\[
\text{Res}_{u_{12}=1} F_{12\ldots n}^{(k)}(u_1, u_2, \ldots, u_n) = F_{(12)\ldots n}^{(k)}(u_{(12)}, \ldots, u_n) \sqrt{2} \Gamma_{12}^{(12)}
\]

where the bound state intertwiner \( \Gamma_{12}^{(12)} \) and the values of \( u_1, u_2, u_{(12)} \) are given in [7, 37, 38].

These equations are similar to the form factor equations (i) - (iv) of (3.3)-(3.6) for \( O(N - 2k) \). However, there are two differences: 1) the shift in (ii)\((k)\) is not the one of \( O(N - 2k) \) but that of \( O(N) \), 2) in (iii)\((k)\) there is only one term on the right hand side.

The p-function \( p^{(k)}(u, v) \) must satisfy the equations

(i)’ \( p^{(k)}(u, v) \) is symmetric under \( u_i \leftrightarrow u_j, \ v_i \leftrightarrow v_j \)

(ii)’ \( p^{(k)}(u, v) = (-1)^{m_p} p^{(k)}(u_1 + 2/v, u_2, \ldots, v) = (-1)^{n_k} p^{(k)}(u, v_1 + 2/v, v_2, \ldots) \)

(iii)’ \( p^{(k)}(u, z) = p^{(k)}(\tilde{u}, \tilde{z}) \) for \( u_{12} = 1/v_k, \ v_1 = u_1 - 1 \) and \( v_2 = u_2 \).

The short notations \( \tilde{u} = (u_3, \ldots, u_{n_k}) \) and \( \tilde{v} = (v_3, \ldots, v_{m_k}) \) are used. Below we will replace \( p^{(k)}(u, v) \) by 1 which will not change the results, if the \( p^{(k)} \) satisfy the conditions (4.14).

Lemma 4 The vector valued function \( K_{\alpha}^{(k)}(u) \) of (4.9) for \( 0 < k < \frac{1}{2}(N - 4) \) satisfies the equations (i)\((k)\) and (ii)\((k)\). In particular, it satisfies the residue relation (iii)\((k)\), if the corresponding relation is satisfied for \( K_{\beta}^{(k+1)}(u) \) and for a suitable choice of the normalization constants in (4.9). The numbers \( m_k = n_{k+1} \) are given by the numbers of particles \( n = n_0 \) and the weights of the operator \( \mathcal{O} \)

\[
w^\mathcal{O} = (w_1, \ldots, w_{|N/2|}) = (n_0 - n_1, \ldots, n_{|N/2|} - 2 - n_-, n_-, n_+) \quad (4.15)
\]

The proof of this lemma can be found in appendix C.1.

5 Examples

In this section, to illustrate our general results we present some simple examples.

5.1 Current

The classical Noether current (real basis) \( J_\mu^{\alpha\beta} = \bar{\psi}^{\alpha} \gamma_\mu \psi^{\beta} \) transforms as the antisymmetric tensor representation of \( O(N) \) and has therefore the weights \( w^f = (w_1, \ldots, w_{|N/2|}) = (1, 1, 0, \ldots, 0) \) (see [3, 4]) which implies with (4.15) that the numbers \( n_i \) of integrations in the various levels of the off-shell Bethe Ansatz satisfy

\[ n - 2 = n_1 - 1 = n_2 = \cdots = n_{|N/2| - 2} = n_- + n_+ \]
The conservation law $\partial_\mu J^\mu = 0$ implies that there exists a pseudo-potential $J(x)$ with
\[ J^\alpha_\mu (x) = \epsilon_{\mu\nu} \partial^\nu J^\alpha_\beta (x). \]

For the form factors we have
\[ F^J_{\alpha\beta}(\theta) = -i \epsilon_{\mu\nu} \sum_{\nu} p^\nu_{\nu} F^J_{\alpha\beta}(\theta). \] (5.1)

Because the Bethe Ansatz yields highest weight states we obtain the matrix elements of the highest weight component of $J^\alpha_\beta$ which means in the complex basis $J(x) = J^{12}(x)$.

We propose the form factors of the operator $J(x)$ (for $n = m + 1 = n_1 + 1 = n_2 + 2$ even)
\[ \langle 0 | J(0) | \theta \rangle_\alpha = F^J_{\alpha\beta}(\theta) = \prod_{i<j} F^J_{\alpha\beta}(\theta) K^J_{\alpha\beta}(\theta) \]
\[ K^J_{\alpha\beta}(\theta) = N^J_{\mu} \int_{C^{(1)}} dz_1 \cdots \int_{C^{(m)}} dz_m \hat{h}(\theta, z) p^J(\theta, z) \tilde{\Psi}_{\alpha}(\theta, z) \]
with the p-function
\[ p^J(\theta, z) = e^{\frac{1}{2} \left( \sum_{i=1}^{n} \epsilon_i - \sum_{j=1}^{m} \epsilon_j (2) - \frac{1}{2} n_2 \pi \delta \right)} / \sum_{i=1}^{n} e^{\theta_i} - e^{\frac{1}{2} \left( \sum_{i=1}^{n} \epsilon_i - \sum_{j=1}^{m} \epsilon_j (2) - \frac{1}{2} n_2 \pi \delta \right)} / \sum_{i=1}^{n} e^{-\theta_i} \] (5.2)
which satisfies (4.7) with
charge $Q^J = 0$
weight vector $w^J = (1, 1, 0, \ldots, 0)$
statistics factor $\sigma^J = 1$
spin $s^J = 0$, $s^J = 1$

For example for 2-particle form factor we obtain (see appendix B)
\[ F^{J^{\alpha\beta}}_{\alpha_1\alpha_2}(\theta_1, \theta_2) = \text{im} \left( \delta^\alpha_{\alpha_1} \delta^\beta_{\alpha_2} - \delta^\beta_{\alpha_1} \delta^\alpha_{\alpha_2} \right) \frac{1}{\cosh \frac{1}{2} \theta_{12}} F_-(\theta) \] (5.3)
\[ F^{J^{\alpha\beta}}_{\alpha_1\alpha_2}(\theta_1, \theta_2) = i \left( \delta^\alpha_{\alpha_1} \delta^\beta_{\alpha_2} - \delta^\beta_{\alpha_1} \delta^\alpha_{\alpha_2} \right) \bar{v}(\theta_1) \gamma_\mu u(\theta_2) F_-(\theta) \] (5.4)
with $F_-(\theta)$ of (3.10), $\bar{v}(\theta_1) \gamma^\pm u(\theta_2) = \pm 2me^{\pm \frac{1}{2}(\theta_1 + \theta_2)}$ and (5.1). This result agrees with [6].

5.2 Field

The fundamental field $\psi^\alpha(x)$ in the Lagrangian (1.1) transforms as the vector representation of $O(N)$ and has therefore the weights $w = (w_1, \ldots, w_{[N/2]}) = (1, 0, \ldots, 0)$ (see [3, 4]) which implies with (4.15) that the numbers $n_i$ of integrations in the various levels of the off-shell Bethe Ansatz satisfy
\[ n - 1 = n_1 = n_2 = \cdots = n_{[N/2]-2} = n_- + n_+, n_- = n_+ \]
Because the Bethe Ansatz yields highest weight states we obtain the matrix elements of the highest weight component of $\psi^\alpha$ which means in the complex basis $\alpha = 1$. We use the short notation $\psi = \psi^1$. For convenience we multiply the field with the Dirac operator and take

$$\chi(x) = i(-i\gamma\partial + m)\psi(x). \quad (5.5)$$

We propose for the $n$-particle form factors ($n = m + 1$ odd) for the spinor components $\chi^{(\pm)}$

$$\langle 0 | \chi^{(\pm)}(0) | \theta \rangle_\alpha = F_\alpha^{(\pm)}(\theta) \prod_{i<j} F(\theta_{ij}) K^{(\pm)}_\alpha(\theta) \quad (5.6)$$

$$K^{(\pm)}_\alpha(\theta) = N_\alpha^\chi \int_{C_\alpha^{(e)}} \int_{C_\alpha^{(m)}} dz_1 \ldots \int_{C_\alpha^{(m)}} dz_m \tilde{h}(\theta, z)p^{\chi^{(\pm)}}(\theta, z) \tilde{\Psi}_\alpha(\theta, z) \quad (5.7)$$

with the p-function (for $n = m + 1 = \text{odd} > 1$)

$$p^{\chi^{(\pm)}}(\theta, z) = \exp \left( \mp \frac{1}{2} \left( \sum_{j=1}^{n} \theta_j - \sum_{j=1}^{m} z_j - \frac{1}{2} m \pi i \nu \right) \right) \quad (5.8)$$

which solves (4.7) with

- charge $Q^\psi = 1$
- weight vector $w^\psi = (1, \ldots, 0)$
- statistics factor $\sigma^\psi = -1$
- spin $s^\psi = \frac{1}{2}$

The one particle form factor is trivial

$$\langle 0 | \psi(0) | \theta \rangle_\alpha = F_\alpha^{\psi}(\theta) = \delta_1^1 u(\theta).$$

The three particle form factor is obtained by the Ansatz (5.6), the integral representation (5.7) and the state (1.4) for $n = 3, m = 2$

$$\langle 0 | \chi(0) | \theta \rangle_\alpha = F_\alpha^{\chi}(\theta) = F(\theta_{12})F(\theta_{13})F(\theta_{23}) K^{\chi}_\alpha(\theta) \quad (5.9)$$

with

$$\tilde{h}(\theta, z) = \prod_{i=1}^{3} \frac{\tilde{\phi}(\theta_i - z_1)\tilde{\phi}(\theta_i - z_2)}{\tilde{\phi}(z_{12})\tilde{\phi}(-z_{12})},$$

$$p^{\chi^{\pm}}(\theta, z) = e^{\mp \frac{1}{2}(\theta_1 + \theta_2 + \theta_3 - z_1 - z_2 - i\pi \nu)},$$

$$\tilde{\Psi}_\alpha(\theta, z) = L_{\beta_1 \beta_2}^\chi(z) \left( \prod_{\beta_1 \beta_2} \frac{\tilde{\chi}^{\beta_1 \beta_2}(\theta, z_2)\tilde{\chi}^{\beta_1}(\theta, z_1)}{\tilde{\chi}^{\beta_1 \beta_2}(\theta, z_1)\tilde{\chi}^{\beta_1}(\theta, z_2)} \right)_\alpha.$$
We could not perform the integrations\(^7\) in (5.9) for general \(N\), but we expand the exact expression in \(1/\!\!\!N\)-expansion to compare the result with the \(1/\!\!\!N\)-expansion of the \(O(N)\) Gross-Neveu model in terms of Feynman graphs.

**1/\!\!\!N expansion:** We obtain the 3-particle form factor of \(\chi(x)\) up to \(O(N^{-2})\) as (see appendix E.1)

\[
F_{\alpha\beta\gamma}^{\chi} = \frac{8\pi m}{N} \left( \delta^\alpha_\gamma C_{\alpha\beta} \cosh \frac{1}{2} \theta_{12} u(\theta_3) - \delta^\gamma_\beta C_{\alpha\gamma} \cosh \frac{1}{2} \theta_{13} u(\theta_2) + \delta^\alpha_\beta C_{\beta\gamma} \cosh \frac{1}{2} \theta_{23} u(\theta_1) \right)
\]

(5.11)

which agrees with the \(1/\!\!\!N\) expansion using Feynman graphs (see appendix E.2).

**Bound state form factor of \(\psi\):** We discuss the bound state fusion of 2 fundamental fermions \(f + f \to b_2\), a boson of mass \(m_2\) (see (2.3)). Writing (5.5) as

\[
\psi(x) = (i\gamma\partial + m)\tilde{\chi}(x),\quad \tilde{\chi}(x) = -i \left( \begin{array}{c} \Box + m^2 \\ 1 \end{array} \right)^{-1} \chi(x),
\]

we apply the form factor equation (iv), i.e. (3.6)

\[
\Res_{\theta_{12}=i\pi\nu} F^O_{123}(\theta_1, \theta_2, \theta_3) = F^O_{(12)3}(\theta_{(12)}, \theta_3) \sqrt{2} \Gamma_{12}^{(12)}
\]

to the operator \(O = \tilde{\chi}\) \(^8\)

\[
\Res_{\theta_{12}=i\pi\nu} F^{\tilde{\chi}(\pm)}_{111}(\theta) = F^{\tilde{\chi}(\pm)}_{b_2(0,2)}(\theta_0, \theta_3) \sqrt{2} \Gamma_{11}^{b_2(0,2)}.
\]

The result may be written as

\[
F^{\tilde{\chi}(\pm)}_{b_2(0,2)}(\theta_0, \theta_3) = e^{\mp \frac{1}{2} \theta_0} \left( e^{\pm \frac{1}{2} i\pi\nu} f_{13}(\theta_03) + e^{\mp \frac{1}{2} i\pi\nu} f_{32}(\theta_03) \right)
\]

\[
+ e^{\mp \frac{1}{2} \theta_0} \left( e^{\mp \frac{1}{2} i\pi\nu} f_{11}(\theta_03) + e^{\pm \frac{1}{2} i\pi\nu} f_{22}(\theta_03) \right).
\]

where the functions \(f_{ij}\) may be calculated in terms of hypergeometric functions \(3F_2\) (for more details see appendix D). For example \(f_{13}\) is plotted for \(N = 12\) in Fig. 4.

The pole at \(\theta = \frac{3}{2}i\pi\nu\) (here \(x = 0.3\)) belongs to the bound state fusion \(b^{(r)}_2 + f \to b^{(r\pm)}_3\), a fermion of mass \(m_3\) (see (2.3)). The pole at \(\theta = i\pi \left( 1 - \frac{1}{2} \nu \right)\) (here \(x = 0.9\)) belongs to the bound state fusion \(b^{(r)}_2 + f \to f\), which is again the fundamental fermion. These are examples of the general “bootstrap principal” [38].

**5.3 Energy momentum**

The energy momentum tensor is in terms of fields is

\[
T^{\mu\nu}(x) = \frac{1}{2} i \bar{\psi} \gamma^\mu \partial^\nu \psi - g^{\mu\nu} \mathcal{L}
\]

\(^7\)Doing one integral we obtain a generalization of Meijer’s G-functions. The second integration does not yield known functions (to our knowledge). One could, of course, apply numerical integration techniques and determine the asymptotic behavior for large \(\theta\)’s which is under investigation [46].

\(^8\)Strictly speaking \(F^{\tilde{\chi}}_{111} \pm F^{\tilde{\chi}}_{111}\) give \(F^{\tilde{\chi}}_{b_{2(0,2)}1}\).
Figure 4. Plot of the bound state form factor function $f_{13}(x)$, ($\theta = i\pi x$) for $N = 12$ ($\nu = 1/5$).

with the trace

$$T^\mu_\mu(x) = m\bar{\psi}\psi.$$ 

Because $T^{\mu\nu}$ is an $O(N)$ iso-scalar we have the weights $w = (w_1, \ldots, w_{[N/2]}) = (0, \ldots, 0)$ (see [3, 4]) which implies that

$$n = n_1 = \cdots = n_{[N/2]} - 2 = n_- + n_+, n_- = n_+$$

We write the energy momentum tensor in terms of an energy momentum potential (see e.g. [5])

$$T^{\mu\nu}(x) = R^{\mu\nu}(i\partial_x)T(x)$$

$$R^{\mu\nu}(P) = -P^\mu P^\nu + g^{\mu\nu}P^2.$$ 

$$T^\mu_\mu(x) = P^2 T(x)$$

For $\bar{\psi}\psi$ we propose the $n$-particle form factor as

$$\langle 0 | \bar{\psi}\psi(0) | \theta \rangle_\alpha = F^\psi_\alpha(\theta) = N^\psi_n \prod_{i<j} F(\theta_{ij}) K_{\alpha}^{\psi}(\theta)$$

$$K_{\alpha}^{\psi}(\theta) = \int_{c_1^{(1)}} dz_1 \cdots \int_{c_m^{(m)}} dz_m \hat{h}(\theta, z)p^{\bar{\psi}\psi}(\theta, z) \tilde{\Psi}_\alpha(\theta, z)$$

(5.12)

with $m = n$ even and

$$\hat{h}(\theta, z) = \prod_{i=1}^n \prod_{j=1}^m \hat{\phi}_i(\theta_i - z_j) \prod_{1 \leq i < j \leq m} \tau_{ij}(z_{ij}),$$

$$p^{\bar{\psi}\psi}(\theta, z) = 1$$

(5.13)

$$\tilde{\Psi}_\alpha(\theta, z) = L_\beta(z) \left( \prod_{j=1}^n \tilde{\phi}_j(z) \Omega \tilde{T}_{\beta 1}^{\beta 1}(\theta, z_{1m}) \cdots \tilde{T}_{\beta 1}^{\beta 1}(\theta, z_{11}) \right) \alpha$$

We do not calculate the integrals in (5.12) for general $N$, but we derive the 2 particle form factor following [6] (see appendix C.2).\footnote{We also have used mathematica for a numerical check of 5.12 for $n = 2$.}
\[ F_{\alpha_1 \alpha_2}^{\bar{\psi} \psi}(\theta) = \langle 0 | \bar{\psi}(0) | p_1, p_2 \rangle^{\text{in}}_{\alpha_1 \alpha_2} = C_{\alpha_1 \alpha_2} \bar{v}(\theta_1) u(\theta_2) F_0(\theta_{12}) \]  

(5.14)

and

\[ F_{\alpha_1 \alpha_2}^{T_{\mu\nu}}(\theta) = \langle 0 | T^{\mu\nu}(0) | p_1, p_2 \rangle^{\text{in}}_{\alpha_1 \alpha_2} = C_{\alpha_1 \alpha_2} \bar{v}(\theta_1) \gamma^\mu u(\theta_2) \frac{1}{2} (p_1^\mu - p_2^\mu) F_0(\theta_{12}) \]

\[ = C_{\alpha_1 \alpha_2} \bar{v}(\theta_1) u(\theta_2) m \frac{(p_1 - p_2)^\mu (p_1^\nu - p_2^\nu)}{(p_1 - p_2)^2} F_0(\theta_{12}) \]

\[ F_{\alpha_1 \alpha_2}^T(\theta) = \langle 0 | T(0) | p_1, p_2 \rangle^{\text{in}}_{\alpha_1 \alpha_2} = C_{\alpha_1 \alpha_2} \frac{\bar{v}(\theta_1) u(\theta_2)}{4m \cosh \frac{1}{2} \theta_{12}} F_0(\theta_{12}) \]

with \( F_0(\theta) \) given by (3.11) and (3.15).

1/N expansion: For \( N \to \infty \) we obtain

\[ F_{\alpha_1 \alpha_2}^{\bar{\psi} \psi}(\theta) = C_{\alpha_1 \alpha_2} \bar{v}(\theta_2) u(\theta_1) \frac{2 \coth \frac{1}{2} \theta_{12}}{\theta_{12} - i\pi} + O(1/N) . \]

This result agrees with the one obtained by computing Feynman graphs. This calculation is similar to that, which was done in [6] for the \( O(N) \) Gross-Neveu model up to \( O(1/N^2) \). Note that the leading term for \( N \to \infty \) is not the free value.

Conclusions:

In this article we have enlarged our \( O(N) \) Bethe Ansatz knowledge of the \( O(N) \) Gross-Neveu model, which exhibits a very rich bound state structure and, consequently, creates a rich form factor hierarchy. We have computed the form factors for the fundamental Fermi field, which transforms as a vector representation of \( O(N) \). Then we have also constructed the form factors for the Noether current and the energy-momentum tensor. In addition for the two particle case we have proved the recursion relation for the higher level \( K \)-function. Finally we have checked our results against the usual \( 1/N \) expansion and found full agreement. In a forthcoming paper we will investigate the kink form factors, possibly proving a kink field equation. Moreover, we will perform a detailed analysis of the \( O(6) \) Gross-Neveu model, a starting point in the nesting procedure.

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Appendices

A Proof of the main theorem 3

The co-vector valued function \( K^{O}_2(\theta) \) given by the integral representation (1.3) can be written as a sum of “Jackson-type Integrals” as investigated in [3] because of the identity

\[
\int_{C_a} dz \Gamma(a - z) f(z) = 2\pi i \sum_{l=-\infty}^{\infty} \Gamma(a - l) f(z + l) \tag{A.1}
\]

where the \( C_a \) encircles the poles of \( \Gamma(a - z) \) anti-clockwise. For these expressions symmetry properties and matrix difference equation have been proved in [3] which imply the form factor equations (i) and (ii). We have to prove, that the assumptions of theorem 3 picks those solutions of (i) and (ii), which in addition satisfy the residue relation (iii)

\[
\text{Res} \left|_{\theta_{12}=i\pi} \right. K^{O}_{1...n}(\theta_1, \ldots, \theta_n) = 2i C_{12} F^{O}_{3...n}(\theta_3, \ldots, \theta_n) (1 - \sigma^O_2 (\sigma S)_{2n} \ldots (\sigma S)_{2n}) \tag{A.2}
\]

and (iv)

\[
\text{Res} \left|_{\theta_{12}=i\pi} \right. K^{O}_{12...n}(\theta_1, \theta_2, \ldots, \theta_n) = F^{O}_{(12)...n}(\theta_{(12)}, \ldots, \theta_n) \sqrt{2} \Gamma_{12}^{(12)}. \tag{A.3}
\]

**Proof.** The K-function \( K^{O}_{1...n}(\theta) \) defined by (1.2) contains the entire pole structure and is determined by the form factor equations (i) - (iii) which read in terms of \( K^{O}_{1...n}(\theta) \) as

\[
K^{O}_{ij...}(\ldots, \theta_i, \theta_j, \ldots) = K^{O}_{ji...}(\ldots, \theta_j, \theta_i, \ldots) \tilde{S}_{ij}(\theta_{ij}) \tag{A.2}
\]

\[
K^{O}_{1...n}(\theta_1 + 2\pi i, \theta_2, \ldots, \theta_n) \sigma^O_1 C_{11} = K^{O}_{2...n1}(\theta_2, \ldots, \theta_n, \theta_1) C_{11} \tag{A.3}
\]

\[
\text{Res} \left|_{\theta_{12}=i\pi} \right. K^{O}_{1...n}(\theta) = \frac{2i}{F(i\pi)} C_{12} \prod_{i=3}^{n+1} \tilde{\phi}(\theta_{1i} + i\pi \nu) \tilde{\phi}(\theta_{2i}) K^{O}_{3...n}(\theta_3, \ldots, \theta_n) (1 - \sigma^O_2 S_{2n} \ldots S_{23}) \tag{A.4}
\]

where (4.6) has been used.

(i) Follows as in [3]

(ii) Follows as in [3], however, here (4.7) is responsible for the statistics of the operator \( \sigma^O_1 \) in (A.3).

(iii) The residue of

\[
K^{O}_{1...n}(\theta) = N^{O}_{n} \int_{C_2^{(1)}} dz_1 \cdots \int_{C_2^{(m)}} dz_m \tilde{h}(\theta, z) p^{O}(\theta, z) \tilde{\Psi}_{1...n}(\theta, z) \tag{A.5}
\]

can be written as a sum of two terms

\[
\text{Res} \left|_{\theta_{12}=i\pi} \right. K^{O}_{1...n}(\theta) = \left( \frac{1}{\text{Res} \left|_{\theta_{12}=i\pi} \right.} + \frac{2}{\text{Res} \left|_{\theta_{12}=i\pi} \right.} \right) K^{O}_{1...n}(\theta). \tag{A.6}
\]

This is because for each \( z_j \) integration with \( j \) even the contours will be “pinched” at two points (see Fig. 3):
(1) \( z_j = \theta_2 \approx \theta_1 - i\pi \)

(2) \( z_j = \theta_1 - 2\pi i \approx \theta_2 - i\pi \)

In appendix C.1 we prove for general level \( k \) of the off-shell Bethe Ansatz the residue formulas. The general result imply for \( k = 0 \) that the contribution from the pinching (1) gives

\[
\text{Res}_{\theta_{12}=i\pi} K_{1\ldots n}(\theta) = \frac{2i}{F(i\pi)} C_{12} \prod_{i=3}^{n} \tilde{\phi}(\theta_{11} + i\pi \nu) \tilde{\phi}(\theta_{12}) K_{3\ldots n}(\theta_3, \ldots, \theta_n) \quad \text{(A.6)}
\]

for a suitable choice of the normalization constants in (A.5). Therefore we have proved

\[
\text{Res}_{\theta_{12}=i\pi} F_{1\ldots n}(\theta_1, \ldots, \theta_n) = 2i C_{12} F_{3\ldots n}(\theta_3, \ldots, \theta_n).
\]

To investigate \( \text{Res}_{\theta_{12}=i\pi} F_{1\ldots n}(\theta) \) due to the pinching at \( z_j = \theta_1 - 2\pi i \approx \theta_2 - i\pi \) we use (ii) and (i) to write

\[
F_{1\ldots n}(\theta) \sigma_1^O = C_{11} F_{2\ldots n}(\theta_2, \ldots, \theta_n, \theta_1 - 2\pi i) C^{11}
\]

\[
= C_{11} F_{21\ldots n}(\theta_2, \theta_1 - 2\pi i, \ldots, \theta_n) C^{11} (\sigma S)_{1n} \ldots (\sigma S)_{13}
\]

We use the result for \( \text{Res}_{\theta_1=\theta_2+i\pi} \) and obtain with

\[
\text{Res}_{\theta_1=\theta_2+i\pi} F_{1\ldots n}(\theta) \sigma_1^O = - \text{Res}_{\theta_2=(\theta_1-2\pi i)+i\pi} C_{11} F_{21\ldots n}(\theta_2, \theta_1 - 2\pi i, \ldots, \theta_n) C^{11} (\sigma S)_{1n} \ldots (\sigma S)_{13}
\]

\[
= -C_{11} 2i C_{21} F_{3\ldots n}(\theta_3, \ldots, \theta_n) C^{11} (\sigma S)_{1n} \ldots (\sigma S)_{13}
\]

\[
= -2i C_{12} F_{3\ldots n}(\theta_3, \ldots, \theta_n) \sigma_2^O (\sigma S)_{2n} \ldots (\sigma S)_{23} \sigma_1^O
\]

using \( \sigma_1^O \sigma_1^O = 1 \).

(iv) Because there are bound states we also have to discuss the form factor equation (iv) (3.6)

\[
\text{Res}_{\theta_{12}=i\pi \nu} F_{12\ldots n}(\theta_1, \theta_2, \tilde{\theta}_1) = F_{12\ldots n}(\theta_{12}, \tilde{\theta}_1) \sqrt{2} \Gamma_{12}^{(12)}
\]

The bound state form factor \( F_{12\ldots n}(\theta_{12}, \tilde{\theta}) \) is then obtained from the residue

\[
\text{Res}_{\theta_{12}=i\pi \nu} K_{12\ldots n}(\theta) = \text{Res}_{\theta_{12}=i\pi \nu} N_{12}^{(n)} \int_{\mathbb{C}(1)}^{\theta_{12}} dz_1 \ldots \int_{\mathbb{C}(m)}^{\theta_{12}}\tilde{h}(\theta, \tilde{\theta}) p^{(\theta, \tilde{\theta})} \tilde{\Psi}_{1\ldots n}(\theta, \tilde{\theta})
\]

Similar as in the proof of (iii) the residue is obtained from pinching at:

\( z_j = \theta_1 - i\pi \nu \approx \theta_2 \) for \( \mathcal{C}^{(o)} \) and \( z_j = \theta_2 \approx \theta_1 - i\pi \nu \) for \( \mathcal{C}^{(e)} \).

Here we will not perform the lengthy calculations and write the complicated result, but in appendix D we will calculate the bound state form factors for the examples of section 5.
B Two-particle current form factor

Derivation of (5.3) and (5.4):

**Proof.** The two-particle K-function of the current is

\[ K^{J}_{2}(\theta) = N_{2}^{J} \int_{c_{\alpha}} dz \; \tilde{h}(\theta, z) p^{J}(\theta, z) \tilde{\Psi}_{\alpha}(\theta, z) \]

with the p-function (5.2) for \( n = 2 \) and \( m = 1 \)

\[ p^{J}(\theta, z) = \frac{e^{\frac{i}{2}(\theta_{1} + \theta_{2})}}{e^{\theta_{1}} + e^{\theta_{2}}} + \frac{e^{-\frac{i}{2}(\theta_{1} + \theta_{2})}}{e^{-\theta_{1}} + e^{-\theta_{2}}} = \frac{1}{\cosh \frac{1}{2}(\theta_{1} + \theta_{2})} \]  

(B.1)

and the Bethe state

\[ \tilde{\Psi}_{\alpha}(\theta, z) = \delta_{\alpha_{1}}^{2} \delta_{\alpha_{2}}^{1} \tilde{c}(\theta_{1} - z) + \delta_{\alpha_{1}}^{1} \delta_{\alpha_{2}}^{2} \tilde{h}(\theta_{1} - z) \tilde{c}(\theta_{2} - z) \]

Doing the integral we obtain

\[ K^{J}_{12}(\theta_{12}) = N_{2}^{J} \int_{c_{\alpha}} dz \; \tilde{h}(\theta, z) p^{J}(\theta, z) \tilde{\Psi}_{21}(\theta, z) = -N_{2}^{J} 8\pi^{3} 4^{\nu} \Gamma(1 - \nu) c \frac{F_{-}(\theta)}{F(\theta)} \]  

(B.2)

and \( K^{J}_{12}(\theta) = -K^{J}_{21}(\theta) \).

We use again the variables \( u = \theta/(i\pi \nu) \) and \( v = z/(i\pi \nu) \), consider the component \( K^{J}_{21}(\theta) \) and calculate the integral

\[ I = \frac{1}{2\pi i} \int_{c_{\alpha}} dv \; I(u, v) \]

\[ I(u, v) = \tilde{h}(u, v) \tilde{\Psi}(u, v), \tilde{\Psi}(u, v) = \tilde{c}(u_{1} - v) \tilde{c}(u_{2} - v), \tilde{\Psi}(u, v) = \tilde{c}(u_{1} - v). \]

Writing the integrals in terms of sums over residues we obtain (see Fig. 3)

\[ I = I_{1} + I_{2} = \sum_{l=0}^{\infty} s_{1}(u_{1}, u_{2}, l) + \sum_{l=0}^{\infty} s_{2}(u_{1}, u_{2}, l) \]

(B.3)

\[ s_{l}(u_{1}, u_{2}, l) = \text{Res}_{v=v_{0}(u_{1}, l)} I(u_{1}, u_{2}, v), v_{0}(u, l) = u - 1 + 2l/\nu \]

Using the Gauss formula

\[ _{2}F_{1}(a, b; c; 1) = \sum_{n=0}^{\infty} \frac{\Gamma(a + n) \Gamma(b + n)}{\Gamma(a) \Gamma(b) \Gamma(c + n) \Gamma(1 + n)} = \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)} \]

(B.4)

we get

\[ I = I_{1} + I_{2} \]

\[ = 2\nu \pi \sqrt{\pi} \frac{\Gamma(-\frac{1}{2} \nu) \Gamma(\frac{1}{2} \nu + \frac{1}{2}) \cos \frac{1}{2} \pi \nu}{\sin \frac{1}{2} \pi \nu (u_{12} + 1) \sin \frac{1}{2} \pi \nu (u_{12} - 1)} \frac{1}{\Gamma(1 + \frac{1}{2} \nu - \frac{1}{2} \nu u_{12})} \]

which agrees with (B.2) taking (B.1) into account.
Therefore using (3.13) and (3.15) we finally obtain with the normalization constant
\[ N^J_2 = \frac{1}{8} \frac{4^{-\nu}}{\epsilon \pi^3 \Gamma(1-\nu)} \] (B.5)
for the pseudo-potential $J^{\alpha\beta}(x)$ the two-particle form factors (5.3) and (5.4) for the current. The normalization is chosen such that the form factor agrees for $F_{-}(\theta) \to F_{-}(i\pi) = 1$ with the free field expression.

\[ \square \]

C  Higher level K-functions

C.1  Proof of lemma 4

Lemma 4 also holds for $k = 0$ in (4.12), if Res is replaced by Res as explained in appendix A. For convenience we use here the variables $u$ and $v$ with $\theta = i\pi \nu_k u$, $z = i\pi \nu_k v$ and $\nu_k = 2/(N - 2k - 2)$ (for the S-matrix see (4.8)).

**Proof.** The relations (i)$^{(k)}$ and (ii)$^{(k)}$ follow as above in the proof of theorem 3 from the results of [3]. To prove (iii)$^{(k)}$ we calculate
\[ \text{Res}_{u_{12} = 1/\nu_k} K^{(k)}_{1,...,n_k}(u) = \text{Res}_{u_{12} = 1/\nu_k} \tilde{N}^{(k)}_{m_k} \int_{C_u^e} dv_1 \ldots \int_{C_u^e} dv_{m_k} \tilde{h}(u,v) \tilde{\Psi}^{(k)}_{1,...,n_k}(u,v). \] (C.1)

The contours $C_u^e$ will be “pinched” at $v_j = u_2 \approx u_1 - 1/\nu_k$.\(^{10}\) Due to symmetry it is sufficient to determine the contribution from $v_2$ and multiply the result by $[m_k/2]$. The contribution to residue is given by the $v_2$ integration $\int_{u_2} dv_2 \ldots$ along small circle around $v_2 = u_2$ (see figure 3). The S-matrix $\tilde{S}^{(k)}(u_2 - v_2)$ yields the permutation operator $\tilde{S}^{(k)}(0) = P$ and the S-matrix $\tilde{S}^{(k)}(u_1 - v_2)$ yields $K$ after taking Res$_{u_{12} = 1/\nu_k} \tilde{S}^{(k)}(u_{12})$. In the representation of the Bethe state (2.22)
\[ \left( \tilde{\Phi}^{(k)} \right)_{\alpha}^{\beta}(u,v) = \left( \Pi^{(k)} \right)_{\beta}^{\alpha}(u,v) \Omega^{(k)}(\tilde{T}^{(k)})_{k+1}^{\beta_m}(u,v_m) \ldots \tilde{T}^{(k)}_{k+1}(u,v_1) \]
we may move for generic values of the other $v_2$ the operator $\left( \tilde{T}^{(k)} \right)_{k+1}^{\beta}(u,v_2)$ to the left by means of the $TTS = STT$ commutation rule (14) of [3] and (2.23) (using the short notation $\tilde{T}(v_i) = \left( \tilde{T}^{(k)} \right)_{k+1}^{\beta}(u,v_i)$)
\[ \Pi^{(k)} \Omega^{(k)} \tilde{T}(v_m) \ldots \tilde{T}(v_2) \tilde{T}(v_1) = \tilde{S}^{(k+1)}(v_32) \ldots \tilde{S}^{(k+1)}(v_m2) \Pi^{(k)} \Omega^{(k)} \tilde{T}(v_2) \tilde{T}(v_m) \ldots \tilde{T}(v_1). \]

Because of (2.24) $\left( \Pi^{(k)} \right)_{...}^{\beta_m} \Pi^{(k)} \Omega^{(k)} \tilde{T}(v_2) = 0$. However, the pole of $\left( L^{(k)} \right)_{1,...,n_k}(v)$ (see e.g. (4.12)) at $v_2 = 1/\nu_k + 1$ will produce a singular contribution from the $v_1$-integration $\int_{v_1 = 1} dv_1 \ldots$ (which is a part of $\int_{C_u^{(v)}} dv_1 \ldots$). We have a 0/0 situation which we can resolve as follows:

\(^{10}\)For $k = 0$ there is a second pinching point at $v_j = u_1 - 2/\nu \approx u_2 - 1/\nu_k$ as explained in appendix A.
We take \( i = 1 \) and multiply the result by \([(m_k + 1)/2]\) and shift again \( \tilde{T}(u_1) \) and \( \tilde{T}(v_2) \) through all the other \( \tilde{T}(v_i) \) as above \((\tilde{T}^i = \tilde{T}(v_i) = \left(\tilde{T}(k)\right)^{\dagger}_{k+1} (u, v_i))\)

\[
\begin{align*}
\Pi_{1\ldots m_k}^{(k)} \left( \Omega_k \tilde{T}^{m_k} \ldots \tilde{T}^{2} \tilde{T}^{1} \right) \overline{\alpha} \\
= \Pi_{1\ldots m_k}^{(k)} \tilde{\mathcal{S}}_{21}^{(k)} \ldots \tilde{\mathcal{S}}_{m_k+1}^{(k)} \tilde{\mathcal{S}}_{32}^{(k)} \ldots \tilde{\mathcal{S}}_{m_k}^{(k)} \left( \Omega_k \tilde{T}^{1} \tilde{T}^{2} \tilde{T}^{m_k} \ldots \tilde{T}^{3} \right) \overline{\alpha} \\
= \tilde{\mathcal{S}}_{21}^{(k+1)} \ldots \tilde{\mathcal{S}}_{m_k+1}^{(k+1)} \tilde{\mathcal{S}}_{32}^{(k+1)} \ldots \tilde{\mathcal{S}}_{m_k}^{(k+1)} \Pi_{3\ldots m_k}^{(k)} \left( \Omega_k \tilde{T}^{1} \tilde{T}^{2} \tilde{T}^{m_k} \ldots \tilde{T}^{3} \right) \overline{\alpha}.
\end{align*}
\]

Applying this to \( L_{1\ldots m_k}^{(k)}(\bar{u}, \bar{v}) \) using (4.10) for higher levels we get

\[
\tilde{\Psi}_\alpha^{(k)}(u, v) = L_{3\ldots m_k}^{(k)}(\bar{u}, \bar{v}) \Pi_{3\ldots m_k}^{(k)} \left( \Omega_k \tilde{T}^{1} \tilde{T}^{2} \tilde{T}^{m_k} \ldots \tilde{T}^{3} \right) \overline{\alpha}.
\]

For \( u_{12} \approx 1/\nu_k, \; v_1 \approx u_1 - 1 \) and \( v_2 \approx u_2 \) (i.e. \( v_{12} \approx 1/\nu_k - 1 = 1/\nu_{k+1} \)) we may replace inside \( \tilde{\Psi}_\alpha^{(k)} \) the S-matrices

\[
\begin{align*}
\tilde{\mathcal{S}}^{(k)}(u_1 - v_1) &\rightarrow \tilde{c}(u_1 - v_1) P, \\
\tilde{\mathcal{S}}^{(k)}(u_2 - v_1) &\rightarrow 1, \\
\tilde{\mathcal{S}}^{(k)}(u_1 - v_2) &\rightarrow \tilde{d}_k(u_{12}) K, \\
\tilde{\mathcal{S}}^{(k)}(u_2 - v_2) &\rightarrow P.
\end{align*}
\]

For the first relation it has been used that \( \tilde{b}(u) \approx -\tilde{c}(u) \approx 1/(u - 1) \) (for \( u \rightarrow 1 \)) and the only nonvanishing contribution from \( \tilde{\mathcal{S}}^{(k)}(u_1 - v_1) \) is \( \tilde{b}(u_1 - v_1)(1 - P + \text{const} K) \beta^1_{\beta^1} = \tilde{c}(u_1 - v_1)P \beta^1_{\beta^1} \) because \( (1 - P)_{\alpha^1_{\beta^1}} = K_{\alpha^1_{\beta^1}} = 0 \) and moreover \( (\Pi^{(k)})_{\alpha^k_{\beta^k}} = 0 \) and \( (\Pi^{(k)})_{\alpha^k_{\beta^k+1}} = 0 \) hold (see (2.24)). Therefore we may replace (see Fig. C.3 for \( k = 0 \))

\[
\tilde{\Psi}_\alpha^{(k)}(u, v) \rightarrow \tilde{c}(u_1 - v_1) \tilde{d}_k(u_{12}) L_{3\ldots m_k}^{(k)}(\bar{u}, \bar{v}) \tilde{C}^{21}
\]

\[
\times C_{\alpha} \prod_{j=3}^{m_k} \frac{1}{a_k(u_1 - v_j) a_k(u_2 - v_j)} \Pi_{3\ldots m_k}^{(k)} \left( \Omega_k \tilde{T}^{m_k} \ldots \tilde{T}^{3} \right) \overline{\alpha}.
\]

(C.2)

---

(C.3)
Note that unitarity and crossing imply for $u_{12} = 1/\nu_k$

$$C_{12}S^{(k)}(u_1 - v_{m_k}) \ldots S^{(k)}(u_1 - v_3)S^{(k)}(u_2 - v_{m_k}) \ldots S^{(k)}(u_2 - v_3) = C_{12}S^{(k)}(u_1 - v_{m_k}) \ldots S^{(k)}(u_1 - v_3)S^{(k)}(u_2 - v_{m_k}) \ldots S^{(k)}(u_2 - v_3) \frac{a_k(u_1 - v_{m_k}) \ldots a_k(u_1 - v_3)a_k(u_2 - v_{m_k}) \ldots a_k(u_2 - v_3)}{a_k(u_1 - v_3)a_k(u_2 - v_{m_k}) \ldots a_k(u_2 - v_3)u_{12}}.$$  

We calculate for $v_{12} \rightarrow 1/\nu_{k+1}$

$$L_{3..m_k21}^{(k)}(\tilde{\nu}) \hat{C}^{21} = \left( \prod_{v_i=1/\nu_{k+1}} \text{Res} \tilde{d}_{k+1}(v) \right) \prod_{j=3}^{m_k} a_k(v_{1j})a_k(v_{2j}) \tilde{d}(v_{j1} + 1)\tilde{d}(v_{j2})L_{3..n}^{(k)}(\tilde{\nu}).$$

It has been used that from (4.12) and (4.10) follows

$$\prod_{j=3}^{m_k} \tilde{d}(v_{j1} + 1)\tilde{d}(v_{j2})L_{3..n}^{(k)}(\tilde{\nu}) \hat{C}_{12}^{21} = \text{Res}_{v_{12} = 1/\nu_{k+1}} L_{12..m}^{(k)}(\tilde{\nu}) \hat{C}_{12}^{21} = \left( \prod_{v_i=1/\nu_{k+1}} \text{Res} \tilde{d}_{k+1}(v) \right) \prod_{j=3}^{m_k} a_k(v_{1j})a_k(v_{2j}) \tilde{d}(v_{j1} + 1)\tilde{d}(v_{j2})L_{3..n}^{(k)}(\tilde{\nu}).$$

where again unitarity and crossing for $v_{12} = 1/\nu_{k+1}$ has been used. We write

$$\tilde{h}(\nu, \tilde{\nu}) = \prod_{i=1}^{n} \prod_{j=1}^{m_k} \tilde{d}(u_i - v_j) \prod_{1 \leq i < j \leq m_k} \tau_{ij}(v_{ij})$$

$$= \left( \prod_{i=1}^{2} \prod_{j=1}^{2} \tilde{d}(u_i - v_j) \right) \left( \prod_{j=3}^{m_k} \tilde{d}(u_1 - v_j)\tilde{d}(u_2 - v_j) \right) \left( \prod_{i=3}^{n_k} \tilde{d}(u_i - v_1)\tilde{d}(u_i - v_2) \right)$$

$$\times \tau(v_{12}) \prod_{j=3}^{m_k} \tau(v_{j1})\tau(v_{j2}) \tilde{h}(\tilde{\nu}, \tilde{\nu})$$

and obtain finally

$$\text{Res}_{u_{12} = 1/\nu_k} K_{\tilde{\nu}}^{(k)}(u) = \tilde{N}_{m_k}^{(k)}(m_k - 1)$$

$$\times \left( \text{Res}_{v_{12} = 1/\nu_k} \tilde{d}_{k+1}(v) \right) \left( \prod_{j=1}^{2} \tilde{d}(u_i - v_j) \right) \left( \prod_{i=3}^{n_k} \tilde{d}(u_1 - v_1)\tilde{d}(u_2 - v_2) \right) \left( \prod_{i=3}^{n_k} \tilde{d}(u_i - v_1)\tilde{d}(u_i - v_2) \right) \left( \prod_{j=3}^{m_k} \tilde{d}(v_{j1} + 1)\tilde{d}(v_{j2}) \right) \frac{1}{\tilde{N}_{m_k-2}} C_{\tilde{\nu}}^{(k)}(\tilde{\nu})$$

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It has been used that for \( u_{12} = 1/\nu k \), \( v_{12} = 1/\nu k^+1 \), \( u_2 = v_2 \), \( u_1 = v_2 + 1/\nu k = v_1 + 1 \)

\[
\left( \frac{a_{k+1}(v_{1j})a_{k+1}(v_{2j})}{a_k(u_{1} - v_{1j})a_k(u_{2} - v_{2j})} \tilde{\phi}(v_{1j} + 1)\tilde{\phi}(v_{2j}) \right) \left( \tilde{\phi}(u_1 - v_j)\tilde{\phi}(u_2 - v_j)\tau(v_{1j})\tau(v_{2j}) \right) = 1
\]

which can be shown by means of (4.2) and the formulas

\[
a_k(u_1)a_k(u_2) = \tilde{b}(-u_2)/\tilde{b}(u_1)
\]

\[
\tilde{b}(u)\tilde{\phi}(u) = -\tilde{\phi}(1 - u).
\]

The final result is that equation (4.12) holds for a suitable choice of the normalization constants in (4.9).

\section{Two-particle higher level K-functions}

For the examples of section 5 we need higher level K-functions. In particular \( K_{\alpha_1\alpha_2}^{(k)}(\theta_1, \theta_2) \) (level \( k = 0, 1, 2, \ldots \)) belonging to \( O(N - 2k) \) in the iso-scalar two-particle channel (with weights \( w = (0, \ldots, 0) \)). The K-function of a iso-scalar operator is of the form

\[
K_{\alpha_1\alpha_2}^{(k)}(u_1, u_2) = C_{\alpha_1\alpha_2}^{(N-2k)}K(u_{12}, k)
\]

(C.4)

where \( C_{\alpha_1\alpha_2}^{(N-2k)} \) is the \( O(N - 2k) \) charge conjugation matrix\(^{11}\). For convenience we use here the parameterization \( \theta = i\pi \nu_k u \), \( z = i\pi \nu_k v \), \( (\nu_k = 2/(N - 2k - 2)) \) and the S-matrix \( \tilde{S}^{(k)}(u) \) (see (4.8)). From

\[
w = (w_1, \ldots, w_{N/2}) = (0, \ldots, 0) = (n - n_1, \ldots, n_{N/2} - 2 - n_2, n_{N/2} - 1)
\]

follows that for all levels \( n_k = 2 \).

**Lemma 5** The vector valued functions \( K_{\alpha_1\alpha_2}^{(k)}(u_1, u_2) \) with

\[
K(u, k) = \frac{\Gamma \left( 1 - \frac{1}{2}\nu - \frac{1}{2}\nu u \right) \Gamma \left( -\frac{1}{2}\nu + \frac{1}{2}\nu u \right)}{\Gamma \left( \frac{1}{2} - \frac{1}{2}k\nu - \frac{1}{2}\nu u \right) \Gamma \left( \frac{1}{2} - \frac{1}{2}k\nu + \frac{1}{2}\nu u \right)}
\]

(C.5)

satisfy for \( k = 0, 1, 2, \ldots < N/2 - 2 \) the recursion relation

\[
K_{\alpha}^{(k)}(u) = N^{(k)} \int_{c_\alpha} dv_1 \int_{c_\alpha} dv_2 \tilde{h}(u,v) L_{\beta}^{(k)}(v, k) \tilde{\Phi}_{\beta}(u, v)
\]

(L.6)

\[
L_{\beta}^{(k)}(v, k) = K^{(k+1)}_{\beta}(v) = C_{\beta}^{(N-2k-2)}K(v_{12}, k + 1)
\]

\[
\tilde{h}(u,v) = \prod_{i=1}^{2} \left( \tilde{\phi}(u_i - v_1)\tilde{\phi}(u_i - v_2) \right) \frac{1}{\tilde{\phi}(v_{12})\tilde{\phi}(v_{12})}
\]

\[
\tilde{\Phi}_{\beta}(u, v) = \left( \prod_{i=1}^{2} (\nu)\Omega \tilde{T}_{1}^{\beta_2}(u, v_2)\tilde{T}_{1}^{\beta_1}(u, v_1) \right)
\]

(C.6)

and the normalization

\[
N^{(k)} = \frac{\Gamma \left[ \frac{1}{2} - k\nu \right] \Gamma \left( 1 - \frac{1}{2}k\nu - \frac{1}{2}\nu \right)}{8\pi^2 \Gamma \left( \frac{1}{2} - \frac{1}{2}k\nu \right) \Gamma \left( \frac{1}{2} - \frac{1}{2}k\nu + \frac{1}{2}\nu \right) \Gamma \left( -\frac{1}{2}\nu \right)}
\]

(C.7)

\(^{11}\)In the real basis this would be \( \delta_{\alpha_1\alpha_2} \).
Proof. The function (C.5) satisfies

\[(i): \quad K(u,k) = K(-u,k) \tilde{S}^{(k)}_0(u)\]
\[(ii): \quad K(1/\nu - u,k) = K(1/\nu + u,k)\]  

(C.8)

with the scalar eigenvalue of \( \tilde{S}^{(k)}(u) = S^{O(N-2k)}(u) \) (see (2.9))

\[\tilde{S}^{(k)}_0(u) = \frac{u + 1/\nu_k u + 1}{u - 1/\nu_k u - 1} = \frac{u + (1/\nu - k) u + 1}{u - (1/\nu - k) u - 1}.\]

The minimal solution (with no poles in the physical strip \( 0 \leq \text{Re} u \leq 1/\nu \)) is

\[K_m(u,k) = \frac{1}{\Gamma(\frac{\alpha}{2} - \frac{1}{2} k \nu - \frac{1}{2} \nu u) \Gamma(\frac{\beta}{2} (1 - \nu k) + \frac{1}{2} \nu u) \Gamma(1 + \frac{1}{2} \nu - \frac{1}{2} \nu u) \Gamma(\frac{1}{2} \nu + \frac{1}{2} \nu u)}\]

(C.9)

whereas \( K(u,k) \) has the bound state pole at \( u = 1 \) (\( \theta = i \pi \nu_k \)).

The Bethe state in (C.6) is

\[\Phi^{(k)}(\beta_2, u, v) = \left( \Pi^{(k)}(\beta_2, u, v) \Omega \tilde{T}_1^{\beta_2}(u, v_2) \tilde{T}_1^{\beta_1}(u, v_1) \right)_{\alpha}^{(k)}\]

\[= (\Pi^{(k)})^{\beta_1 \beta_2}_{\beta_1 \beta_2} v_2 f_k(v_{12}) \tilde{C}^{\beta_1 \beta_2}_{\beta_1 \beta_2} \delta^{k+1}_0 \delta^{k+1}_1, \quad f_k(v) = \frac{1}{v + 1/\nu_k - 1},\]

which may be depicted (for \( k = 0 \)) as

\[
\Phi^{(k)}_{\alpha \beta}(u, v, w) = \begin{cases} 
\Pi_{\alpha \beta}^{k+1} & v_1, v_2 \\
1 & u_1, u_2 
\end{cases}, \quad \Pi_{\alpha \beta}^{k+1}(u_1, u_2) = \begin{cases} 
\alpha & \beta \\
\beta & \alpha 
\end{cases} + f(u_{12}) \begin{cases} 
1 & 1 \\
1 & 1 
\end{cases} \]

(C.10)

Because of (C.4) it is sufficient to consider only one component of (C.6) and for convenience we take \( \Phi^{(k)}_{\alpha \beta} \) with \( \alpha = k + 1, k + 1 \) and define

\[\tilde{\Phi}_k(u, v) = \left( \Pi^{(k)}(\beta_1 \beta_2, u, v) \right)_{\alpha \beta}^{(k)} \]

\[= \left( \Pi^{(k)}(\beta_1 \beta_2) \right)_{\beta_1 \beta_2}^{(N-2k-2)} \tilde{\Phi}^{(k)}(\beta_1 \beta_2, k+1, u, v)\]

\[= \left( \Pi^{(k)}(\beta_1 \beta_2) \right)_{\beta_1 \beta_2}^{(N-2k-2)} \left( \tilde{\Phi}_1(\beta_1 \beta_2, u, v_2) d_k(u_1 - v_1) + f_k(v_1 - v_2) \left( \tilde{\Phi}_1(\beta_1 \beta_2, u_1 - v_1) + d_k(u_1 - v_1) \right) \right)\]

\[= (N - 2k - 2) \left( \frac{\delta_k}{(u_1 - v_1 - 1)(u_1 - v_2 - 1)(v_1 - v_2 + 1/\nu_k - 1)} \right)\]

where (for \( k = 0 \)) \( \Phi^{(k)}_{\alpha \beta}(u, v, w) \) may be depicted as

\[
\begin{align*}
&\begin{cases} 
\beta_1 & \beta_2 \\
v_1 & 1 \\
u_2 & 1 
\end{cases} + f(v_{12}) \begin{cases} 
\beta_1 & \beta_2 \\
v_1 & 1 \\
u_2 & 1 
\end{cases} + f(v_{12}) \begin{cases} 
\beta_1 & \beta_2 \\
v_1 & 1 \\
u_2 & 1 
\end{cases}
\end{align*}
\]
Then because \( L^{(c)}(v) \tilde{\Phi}^{(c)} \frac{J}{k+1,k+1} (u,v) = K(v_{12},k+1) \tilde{\Phi}_k(u,v) \)

\[
K_{k+1,k+1}^{(k)} (u) = N^{(k)} \int_{c^{(e)}_c} dv_1 \int_{c^{(e)}_c} dv_2 \tilde{h}(u,v) K(v_{12},k+1) \tilde{\Phi}_k(u,v)
= N^{(k)} (2\pi i)^2 (N - 2k - 2) J(u_{12})
\] (C.11)

where

\[
J(u_{12}) = \frac{1}{(2\pi i)^2} \int_{c^{(e)}_c} dv_1 \int_{c^{(e)}_c} dv_2 J(u,v) \tag{C.12}
\]

\[
J(u,v) = \tilde{\phi}(u_1 - v_1) \tilde{\phi}(u_1 - v_2) \tilde{\phi}(u_1 - v_2) \tilde{\phi}(u_2 - v_1) \phi(u_2 - v_2) \phi(v_{12})
\]

\[
\varphi(v) = \frac{(1 - v) K(v,k + 1)}{\phi(v) \phi(-v) (v + 1/\nu - k - 1)}.
\]

Because the function \( J(u) \) satisfies (C.8) it is proportional to \( K(u,k) \) if there are no zeroes and exactly one pole at \( u = 1 \) in \( 0 \leq \text{Re} u \leq 1/\nu^2 \). Finally we obtain

\[
J(u) = J_{11} + J_{12} + J_{21} + J_{22} = \text{const.} K(u,k)
\] (C.13)

\[
\text{const.} = \frac{2^{-\nu - 1} \Gamma(-\frac{1}{2} \nu) \Gamma\left(\frac{1}{2} - \frac{1}{2} \nu + \frac{1}{2} \nu u\right) \Gamma(1 - \frac{1}{2} \nu u)}{\Gamma\left(-\frac{1}{2} \nu + 1 - \frac{1}{2} \nu u\right) \Gamma\left(\frac{3}{2} - \frac{1}{2} \nu u\right)}
\]

where \( \text{const.} \) is calculated by taking the residue at \( u = 1 \) on both sides of (C.13).

Finally we turn to (C.6). By (C.11) and (C.4) we have

\[
K(u,k) = K_{k+1,k+1}^{(k)} (u) = N^{(k)} (2\pi i)^2 (N - 2k - 2) J(u_{12})
\] (C.14)

Therefore the normalization is given by

\[
1 = N^{(k)} (2\pi i)^2 (N - 2k - 2) \text{const.}
\]

or (C.7).

In particular for \( k = 0 \) and \( k = 1 \)

\[
K(u) = K(u,0) = -\frac{2}{\pi} \cos \frac{1}{2} \pi \nu u \frac{1}{\nu u - 1} \Gamma\left(-\frac{1}{2} \nu + \frac{1}{2} \nu u\right) \Gamma(1 - \frac{1}{2} \nu - \frac{1}{2} \nu u)
\]

\[
L(u) = K(u,1) = \frac{\Gamma(1 - \frac{1}{2} \nu - \frac{1}{2} \nu u) \Gamma\left(-\frac{1}{2} \nu + \frac{1}{2} \nu u\right)}{\Gamma\left(1 + \frac{1}{2} \left(1 - \nu\right) - \frac{1}{2} \nu u\right) \Gamma\left(\frac{3}{2} - \frac{1}{2} \nu u\right)}
\]

The function \( L(u) \) is that of (5.10) and it is used to calculate the 2-particle form factor on the energy momentum (5.14) and also to calculate the 3-particle form factor of the field 5.9. In particular the 2-particle K-function of the scalar operator \( \bar{\psi} \psi \) is up to a constant equal to \( K(u) \). With the normalization in (5.12)

\[
N_{\bar{\psi} \psi}^2 = 2m/\left(\pi^2 \nu^2 c \Gamma^2 \left(\frac{1}{2} (1 - \nu)\right)\right)
\]

we obtain (5.14)

\[
F_{\alpha_1 \alpha_2}^\bar{\psi} \psi (\theta) = C_{\alpha_1 \alpha_2} \tilde{v}(\theta_1) u(\theta_2) F_0(\theta_{12})
\]

which agrees with the result of [6]. The normalization is chosen such that the form factor agrees for \( \theta \to i\pi \) with the free field expression.

\[\text{This is suggested by numerical calculations using mathematica.}\]
D Bound state form factors

We discuss the form factor equation (iv)

\[ \text{Res}_{\theta_{12}=i\eta} F_{12}^{\gamma}(\theta_1, \theta_2, \ldots, \theta_n) = F_{(12)\ldots n}^{\gamma}(\theta_{(12)}, \ldots, \theta_n) \sqrt{2} \Gamma_{12}^{(12)} \]

for the examples of section (5). Of course, one may easily calculate the residues for two-particle form factors for the pseudo-potential \( J^{\alpha\beta}(x) \) (5.3) and \( \bar{\psi}\psi(x) \) (5.14) directly, however we will check here whether the general pinching procedure of appendix A will give the same result. In addition we obtain the bound state form factor of the three-particle form factor for the field.

**Two-particle current form factor:** By the form factor equation (iv) (3.6) the two-particle bound state form factor for the pseudo-potential \( J^{\alpha\beta}(x) \) is

\[ \text{Res}_{\theta_{12}=i\eta} F_{12}^{J^{\alpha\beta}}(\theta_1, \theta_2) = F_{(12)\ldots n}^{J^{\alpha\beta}}(\theta_{(12)}, \ldots, \theta_n) \sqrt{2} \Gamma_{12}^{(12)}, \theta_{(12)} = \frac{1}{2}(\theta_1 + \theta_2) \]

where the bound state intertwiner \( \Gamma_{12}^{(12)} \) is given by (2.15) and (2.16).

In appendix B we calculated the two-particle form factor for the pseudo-potential \( J^{\alpha\beta}(x) \) in terms of the integral

\[ I(u_{12}) = \frac{1}{2\pi i} \oint_{C_{2}^{(o)}} dv I(u, v) \]

\[ I(u, v) = \tilde{h}(u, v)\tilde{\Psi}(u, v), \quad \tilde{h}(u, v) = \tilde{\phi}(u_1 - v)\tilde{\phi}(u_2 - v), \quad \tilde{\Psi}(u, v) = \tilde{c}(u_1 - v). \]

with the result

\[ I(u) = \frac{2^{\nu} \sqrt{\pi} \Gamma(-\frac{1}{2} \nu) \Gamma\left(\frac{1}{2} \nu + \frac{1}{2}\right) \cos \frac{1}{2} \pi \nu \sin \frac{1}{2} \pi \nu (u + 1) \sin \frac{1}{2} \pi \nu (u - 1) \Gamma\left(\frac{1}{2} \nu + \frac{1}{2} \nu u\right)\Gamma\left(1 + \frac{1}{2} \nu - \frac{1}{2} \nu u\right)}{\sin \frac{1}{2} \pi \nu (u + 1) \sin \frac{1}{2} \pi \nu (u - 1) \Gamma\left(\frac{1}{2} \nu + \frac{1}{2} \nu u\right)\Gamma\left(1 + \frac{1}{2} \nu - \frac{1}{2} \nu u\right)} \]

and the residue

\[ \text{Res}_{u=1} I(u) = -\left(\Gamma\left(-\frac{1}{2} \nu\right)\right)^2. \]

In appendix A we remarked that the residue is obtained from pinching at:

\[ z = \theta_1 - i\pi \nu \approx \theta_2 \quad \text{for} \quad C^{(o)} \]

\[ \text{Res}_{u_{12}=1} I(u_{12}) = \text{Res}_{u_{12}=1} \frac{1}{2\pi i} \oint_{u_{12}-1} dv \tilde{\phi}(u_1 - v)\tilde{\phi}(u_2 - v)\tilde{c}(u_1 - v) \]

\[ = \text{Res}_{u_{12}=1} \tilde{\phi}(1)\tilde{\phi}(-u_{12} + 1) = -\left(\Gamma\left(-\frac{1}{2} \nu\right)\right)^2 \]

which means that the pinching procedure gives the same result as the direct calculation.
Two-particle form factor of $\bar{\psi}\psi$: By the form factor equation (iv) (3.6) the two-particle bound state form factor for $\bar{\psi}\psi$ is

$$\text{Res}_{\theta_{12} = i\pi \nu} F_{12}^{\bar{\psi}\psi}(\theta_1, \theta_2) = F_{(12)}^{\bar{\psi}\psi}(\theta_{12}) \sqrt{2 \Gamma_{12}^{(12)}}.$$  

In appendix C.2 we calculated the two-particle form factor for $\bar{\psi}\psi(x)$ in terms of the integral

$$J(u_{12}) = \frac{1}{(2\pi i)^2} \int_{C_{2}^{(o)}} dv_1 \int_{C_{2}^{(c)}} dv_2 J(u, v),$$

$$J(u, v) = \tilde{\phi}(u_1 - v_1)\tilde{c}(u_1 - v_1)\tilde{\phi}(u_1 - v_2)\tilde{c}(u_1 - v_2)\tilde{\phi}(u_2 - v_1)\tilde{\phi}(u_2 - v_2)\varphi(v_{12})$$

$$\varphi(v) = \frac{(1 - v) K(v, k + 1)}{\phi(v)\phi(-v) (v + 1/\nu - k - 1)},$$

with the result

$$J = \frac{c_2}{\cos \pi \nu} K(u, 0) = \frac{c_2}{\cos \pi \nu} \Gamma \left(1 - \frac{1}{2}\nu - \frac{1}{2}\nu u\right) \Gamma \left(-\frac{1}{2}\nu + \frac{1}{2}\nu u\right) \Gamma \left(1 + \frac{1}{2} - \frac{1}{2}\nu u\right) \Gamma \left(\frac{1}{2} + \frac{1}{2}\nu u\right)$$

and the residue is

$$\text{Res}_{u=1} J(u) = \frac{c_2}{\cos \pi \nu} \text{Res}_{u=1} K(u, 0) = \frac{4}{(1 - \nu) \pi} \left(\Gamma \left(-\frac{1}{2}\nu\right)\right)^2$$

In appendix A we remarked that the residue is obtained from pinching at: $z = \theta_1 - i\pi \nu \approx \theta_2$ for $C^{(o)}$ and $z_j = \theta_2 \approx \theta_1 - i\pi \nu$ for $C^{(c)}$, therefore (see (C.12))

$$\text{Res}_{u_{12}=1} J = \text{Res}_{u_{12}=1} \frac{1}{(2\pi i)^2} \left( \oint_{C_{2}^{(o)}} dv_1 \int_{C_{2}^{(c)}} dv_2 - \int_{C_{2}^{(o)}} dv_1 \oint_{C_{2}^{(c)}} dv_2 \right) J(u, v) = R_1 + R_2$$

$$R_1 = - \sum_{l_2=0}^{\infty} \text{Res}_{u_{12}=1} (s_{11}(u_1, u_2, 0, l_2) + s_{12}(u_1, u_2, 0, l_2))$$

It turns out that $s_{12}$ gives no contribution and

$$R_1 = - \text{Res}_{u_{12}=1} \sum_{l_2=0}^{\infty} s_{11}(u_1, u_2, 0, l_2) = 2 \frac{\left(\Gamma \left(-\frac{1}{2}\nu\right)\right)^2}{\pi (1 - \nu)}$$

such that again

$$\text{Res}_{u_{12}=1} J = \text{Res}_{u_{12}=1} (J_{11} + J_{22}) = 4 \frac{\left(\Gamma \left(-\frac{1}{2}\nu\right)\right)^2}{\pi (1 - \nu)}$$

which means that the pinching procedure gives the same result as the direct calculation.

3-particle form factor of $\psi$: We discuss the bound state fusion of 2 fundamental fermions $f + f \rightarrow b_2$. We write (5.5) as

$$\psi(x) = (i\gamma \partial + m) \bar{\chi}(x), \quad \bar{\chi}(x) = -i \left(\Box + m^2\right)^{-1} \chi(x),$$

- 32 -
and apply the form factor equation (3.6) to $\tilde{\chi}^{13}$

$$\text{Res}_{\theta_{12} = i\pi \nu} F_{111}^\chi(\theta) = F_{b_{21}}^\chi(\theta_0, \theta_3) \sqrt{2r_{111}}. $$

The component $K_{111}$ of the K-function (similar as for $\tilde{\psi}_3$ in appendix C.2) can be written in terms of

$$J^X(u) = \frac{1}{(2\pi i)^2} \int_{C_{u,2}} dv_1 \int_{C_{u,1}} dv_2 J^X(u, v) p^X(u, v)$$

$$J^X(u, v) = \left( \prod_{i=1}^{3} \prod_{j=1}^{2} \tilde{\phi}(u_i - v_j) \right) \tilde{b}(u_1 - v_1) \tilde{b}(u_1 - v_2) \tilde{c}(u_2 - v_1) \tilde{c}(u_2 - v_2) \varphi(v_{12})$$

$$\varphi(v) = \frac{(1 - v) K(v, 1)}{\tilde{\phi}(v) \tilde{\phi}(-v) (v + 1/\nu - 1)}$$

In appendix A we remarked that the residue is obtained from pinching at:

$z_1 = \theta_1 - i\pi \nu \approx \theta_2 (v_1 = u_1 - 1 \approx u_2)$ for $C^{(c)}$ and $z_2 = \theta_2 \approx \theta_1 - i\pi \nu (v_2 = u_2 \approx u_1 - 1)$ for $C^{(c)}$. Therefore the bound state form factor is obtained from

$$\text{Res}_{u_{12} = \nu} J^X(u) = \text{Res}_{u_{12} = \nu} \frac{1}{(2\pi i)^2} \left( \int_{C_{u,1}} - \int_{C_{u,2}} \right) dv_1 dv_2 J^X(u, v) p^X(u, v)$$

The integrals may be calculated in terms of hypergeometric functions $3F_2$. We obtain

$$F_{b_{21}}^\chi(\theta_0, \theta_3) = e^{\pm \frac{1}{2} \theta_0} \left( e^{\pm \frac{1}{2} i\pi \nu} f_{13}(\theta_0, \theta_3) + e^{\mp \frac{1}{2} i\pi \nu} f_{32}(\theta_0, \theta_3) \right)$$

$$+ e^{\pm \frac{1}{2} \theta_3} \left( e^{\mp \frac{1}{2} i\pi \nu} f_{11}(\theta_0, \theta_3) + e^{\pm \frac{1}{2} i\pi \nu} f_{22}(\theta_0, \theta_3) \right)$$

where $f_{13}(\theta_0, \theta_3)$ and $f_{12}(\theta_0, \theta_3)$ are the results from the integrations

$$\int_{C_{u_1}} dv_1 \int_{C_{u_2}} dv_2 \ldots \text{ and } \int_{C_{u_1}} dv_1 \int_{C_{u_2}} dv_2 \ldots \text{ respectively.}$$

For example up to a constant (see Fig. 4)

$$f_{13}(u) = \frac{1}{\Gamma(1 - \frac{1}{4} \nu - \frac{1}{4} \nu u)} \Gamma(-\frac{1}{4} \nu + \frac{1}{4} \nu u) \Gamma(-\frac{1}{4} \nu + \frac{1}{4} \nu u)$$

$$\Gamma\left( \frac{3}{2} \right) \Gamma\left( \frac{3}{4} + \frac{1}{4} \nu u \right) \Gamma\left( \frac{3}{4} - \frac{1}{4} \nu u \right) \cot \frac{1}{2} \pi \nu (u - \frac{1}{2}) \cot \frac{1}{2} \pi \nu (u + \frac{1}{2})$$

$$\times 3F_2\left( -\frac{1}{4} \nu + 1, -\frac{1}{4} \nu + \frac{1}{4} \nu u; 1, \frac{1}{4} \nu + 1; \frac{3}{4} \nu u \right) F_b(u)$$

where $F_b(u)$ is the minimal highest weight form factor function in the $b_{21}^{(r)} + f$ sector

$$F_b(\theta) = \text{const.} \left( \sin \frac{1}{2} \theta \right) \frac{F_{b_{11}}^{\min}(\theta + \frac{1}{2} i\pi \nu) F_{b_{11}}^{\min}(\theta - \frac{1}{2} i\pi \nu)}{\Gamma(1 + \frac{1}{4} \nu - \frac{\theta}{2 i\pi}) \Gamma(\frac{3}{4} \nu + \frac{\theta}{2 i\pi})}$$

or explicitly in terms of $(u = \theta/(i\pi \nu))$

$$F_b(u) = \frac{(\sin \frac{1}{2} \theta) G\left( \frac{1}{2} \nu + \frac{1}{2} \nu u \right) G\left( -\frac{3}{4} \nu - \frac{1}{4} \nu u \right) G\left( 1 + \frac{1}{4} \nu - \frac{1}{2} \nu u \right) G\left( \frac{1}{2} - \frac{3}{4} \nu + \frac{1}{2} \nu u \right)}{G\left( \frac{1}{2} + \frac{1}{4} \nu + \frac{1}{2} \nu u \right) G\left( 2 - \frac{3}{4} \nu - \frac{1}{2} \nu u \right) G\left( \frac{3}{2} + \frac{1}{4} \nu - \frac{1}{2} \nu u \right) G\left( 1 - \frac{3}{4} \nu + \frac{1}{2} \nu u \right)}.$$
It satisfies Watsons equation

\[ \frac{F_b(\theta)}{F_b(-\theta)} = a(\theta + \frac{1}{2}i\pi \nu)a(\theta - \frac{1}{2}i\pi \nu)\frac{\theta + \frac{1}{2}i\pi \nu}{\theta - \frac{1}{2}i\pi \nu} = a_b(\theta) \]

where \(a_b(\theta)\) is the highest weight scattering amplitude in the \(b_2^\pm + f\) sector.

E 1/N expansion

E.1 1/N expansion of the exact 3-particle field form factor

For \(\chi^\delta(x) = i(-i\gamma_\partial + m)\psi^\delta(x)\) we derive for the highest weight component \(\chi(x) = \chi^1(x)\)

\[ F_{111}^\chi(\theta) = \frac{8\pi m}{N} \left( \frac{\cosh \frac{1}{2}(\theta_{12})}{\theta_{12} - i\pi} u(\theta_3) - \frac{\cosh \frac{1}{2}(\theta_{13})}{\theta_{13} - i\pi} u(\theta_2) \right) + O(N^{-2}) \]  

(E.1)

which is equivalent to (5.11).

**Proof.** The p-function of \(\chi(x)\) for three particles and \(\nu = 0\) is

\[ p^{\chi(\pm)} = \exp \left( \mp \frac{1}{2} \left( \theta_1 + \theta_2 + \theta_3 - z_1 - z_2 \right) \right) \]  

We have to consider (up to const.)

\[ K_{111}^{\chi(\pm)}(\theta) = \int_{\mathcal{C}_z} dz_1 \int_{\mathcal{C}_z} dz_2 \prod_{i=1}^3 \left( \tilde{\phi}(\theta_i - z_1) \tilde{\phi}(\theta_i - z_2) \right) \frac{1}{\phi(z_{12}) \phi(-z_{12})} p^{\chi(\pm)}(z) \tilde{\Psi}_{111}(\theta, z). \]

This formula is similar as (C.6) for \(k = 0\) (which correspond to the operator \(\bar{\psi}\psi\)), only we have here to add the factor \(\left( \tilde{\phi}(\theta_3 - z_1) \tilde{\phi}(\theta_3 - z_2) \right)\) and to replace the p-function \(p^{\chi(\pm)}\).

Therefore we get using (C.14) for small \(\nu\) (up to constants)

\[ K_{111}^{\chi(\pm)}(\theta) = K(\theta_{12}, 0) \frac{\exp \left( \mp \frac{1}{2} \theta_3 \right)}{\sinh \frac{1}{2} \theta_{13} \sinh \frac{1}{2} \theta_3} + (2 \leftrightarrow 3) \]

\[ = \frac{\cosh \frac{1}{2}(\theta_{12})}{(\theta_{12} - i\pi) \sinh \frac{1}{2} \theta_{12} \sinh \frac{1}{2} \theta_{13} \sinh \frac{1}{2} \theta_3} \exp \left( \mp \frac{1}{2} \theta_3 \right) + (2 \leftrightarrow 3) + O(\nu) \]

\[ = \frac{1}{\theta_{12} - i\pi} \coth \frac{1}{2} \theta_{12} \frac{\exp \left( \mp \frac{1}{2} \theta_3 \right)}{\sinh \frac{1}{2} \theta_{13} \sinh \frac{1}{2} \theta_3} + (2 \leftrightarrow 3) + O(\nu) \]

\[ F_{111}^{\chi}(\theta) = \frac{\cosh \frac{1}{2}(\theta_{12})}{\theta_{12} - i\pi} u(\theta_3) - \frac{\cosh \frac{1}{2}(\theta_{13})}{\theta_{13} - i\pi} u(\theta_2) + O(\nu) \]

which is (E.1) up to a constant. The normalization is obtained by the form factor equation (iii)

\[ \text{Res}_{\theta_{12}=\pi} F_{111}^{\psi}(\theta) = 2i \left( 1 - a(\theta_{23}) \right) F_{1}^{\psi}(\theta_3) \]

\[ = \frac{4\pi}{N} \left( \frac{1}{\sinh \theta_{23}} - \frac{1}{\theta_{23}} \right) u(\theta_3) + O(N^{-2}) \]
where
\[ F^{\psi}_{\alpha \beta \gamma}(\theta) = \frac{i(\gamma(p_1 + p_2 + p_3) + m)}{8m^2 \cosh \frac{1}{2} \theta_{12} \cosh \frac{1}{2} \theta_{13} \cosh \frac{1}{2} \theta_{23}} F^{\psi \chi}_{\alpha \beta \gamma}(\theta). \]

It has been used that
\[ K(\theta, 0) = -2i\pi \frac{\cosh \frac{1}{2} \theta}{(z - i\pi) \sinh \frac{1}{2} \theta} + O(\nu) \]
\[ \tilde{\phi}(\theta) = \frac{-i\pi}{\sinh \frac{1}{2} \theta} + O(\nu) \]
\[ F(\theta) = -i \sinh \frac{1}{2} \theta + O(\nu) \]
\[ a(\theta) = 1 + \nu i \pi \left( \frac{1}{\sinh \theta} - \frac{1}{\theta} \right) + O(\nu^2). \]

E.2 1/N perturbation theory

Introducing the auxiliary field \( \sigma(x) \) the Lagrangian (1.1) may be written as
\[ \mathcal{L}^{GN} = \bar{\psi}(i\gamma \partial - \sigma) \psi - \frac{1}{2g^2} \sigma^2 \]
and the Green’s functions in 1/N expansions are obtained from the expansion of
\[ Z(\xi, \bar{\xi}) = \int d\sigma \exp \left( iA_{\text{eff}}(\sigma) - \bar{\xi} S \xi \right) \]
\[ A_{\text{eff}}(\sigma) = -i \frac{1}{2} N \ln(i\gamma \partial - \sigma) - \int d^2 x \frac{1}{2g^2} \sigma^2 \]
with the \( \sigma \) propagator [2, 6]
\[ \bar{\Delta}_\sigma(k) = \left( \frac{1}{N} \int \frac{d^2 p}{(2\pi)^2} \text{tr} \left( \frac{1}{\gamma p - m} \left( \frac{1}{\gamma(p + k) - m} - \frac{1}{m} \right) \right) \right)^{-1} = -\frac{4\pi i}{N} \tanh \frac{1}{2} \phi \]
where \( k^2 = -4m^2 \sinh^2 \frac{1}{2} \phi \). This propagator together with the simple vertex of Fig. 5 yield

Figure 5. The elementary vertex for the \( O(N) \) Gross-Neveu model. With respect to isospin the vertex is proportional to the unit matrix.

the Feynman rules which allow to calculate general vertex functions in the 1/N-expansion. For example the four point vertex function is
\[ \bar{\Gamma}^{(4)}_{AB\alpha \beta}(p_3, -p_4, p_1, p_2) = \delta^\alpha_\beta \delta^\gamma_\delta C^{DC}_{AB}(p_2 - p_3) - \delta^\gamma_\delta \delta^\alpha_\beta C^{CD}_{AB}(p_3 - p_1) \quad (E.2) \]
where \( A, B, C, D \) are spinor indices, \( \alpha \beta \gamma \delta \) are isospin indices and \( G \) is given by the Feynman graph of Fig. 6. Taking into account the contributions from the propagator we obtain
\[ G_{AB}^D(k) = \begin{array}{c}
A \\
\downarrow \\
\begin{array}{c}
p_1 \\
\downarrow \\
k
\downarrow \\
p_2 \\
B
\end{array}
\end{array} \]

Figure 6. The four point vertex

\[ G(k) = -1 \otimes 1 \tilde{\Delta}_\sigma(k) = \frac{4\pi i}{N} 1 \otimes 1 \tanh \frac{1}{2} \phi \ . \quad (E.3) \]

where the tensor product structure of the spinor matrices is obvious from Fig. 6.

3-particle form factor of the fundamental fermi field: We now calculate the three particle form factor of the fundamental fermi field in $1/N$-expansion in lowest nontrivial order. For convenience we multiply the field with the Dirac operator and take

\[ \chi \delta D(x) = i(-i\gamma \partial + m) D \psi^{D'}(x) \]

and define

\[ \chi \delta D(0) | p_1, p_2 \rangle_{\alpha \beta}^{in} = F^{\gamma D}_{\alpha \beta}(\theta_3; \theta_1, \theta_2). \]

By means of LSZ-techniques one can express the connected part in terms of the 4-point vertex function (E.2) in lowest order

\[ F^{\chi \delta D}_{\alpha \beta}(\theta_3; \theta_1, \theta_2) = \bar{u}_C(p_3) \left\{ \delta_{\alpha \delta} \delta_{\beta \gamma} G_{AB}^D(p_2 - p_3) - \delta_{\alpha \gamma} \delta_{\beta \delta} G_{AB}^D(p_3 - p_1) \right\} u^A(p_1) u^B(p_2) \]

given by the Feynman graphs of Fig. 7

Figure 7. The connected part of the three particle form factor of the fundamental fermi field in $1/N$-expansion.

\[ F^{\chi \delta D}_{\alpha \beta}(\theta_3; \theta_1, \theta_2) = \bar{u}_C(p_3) \left\{ \delta_{\alpha \delta} \delta_{\beta \gamma} G_{AB}^D(p_2 - p_3) - \delta_{\alpha \gamma} \delta_{\beta \delta} G_{AB}^D(p_3 - p_1) \right\} u^A(p_1) u^B(p_2) \]

where $G$ is given by Fig. 6 and eq. (E.3) and the spinors by $u_{\pm}(p) = \sqrt{m} e^{\mp \theta/2}$. It turns out that for $p_1, p_2$ and $p_3$ on-shell several terms vanish or cancel and we obtain up to order $1/N$ using $\bar{u}(\theta_1) u(\theta_2) = 2m \cosh \frac{1}{2} \theta_{12}$

\[ F^{\chi \delta D}_{\alpha \beta}(\theta_3; \theta_1, \theta_2) = \frac{i\pi}{N} 8m \left\{ \delta_{\alpha \delta} \delta_{\beta \gamma} \frac{\sinh \frac{1}{2} \theta_{23}}{\theta_{23}} u^D(p_1) - \delta_{\alpha \gamma} \delta_{\beta \delta} \frac{\sinh \frac{1}{2} \theta_{13}}{\theta_{13}} u^D(p_2) \right\} . \quad (E.5) \]

By crossing ($\theta_3 \rightarrow \theta_3 + i\pi$) this gives $F^{\chi \delta D}_{\alpha \beta \gamma}$ and agrees with the $1/N$ expansion of the exact form factor (5.11).
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