Chaotic Dynamics of Binary Systems *

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Abstract

We propose a theory of chaos for discrete systems, based on their representation
in a space of “binary histories”, $B^\infty$. We show that $B^\infty$ is a metrizable Cantor set
which embeds the attractor $\Lambda$, itself also a Cantor set.

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1. Introduction.

Most recent results on discrete systems concern states at or near equilibrium. Martinelli and Oliveri proved that Hamiltonian spin systems relax exponentially, under the strong mixing assumption \[^{[1]}\], and Comets and Neveu showed that fluctuations of thermodynamic variables in the Sherrington Kirkpatrick model are Gaussian processes \[^{[2]}\]. The relation of ground states to the vacuum of quantum field theories is well known \[^{[3]}\]; for example Cecotti and Vafa proved their equivalence in the case of the Ising models and supersymmetric field theories in dimension \(d = 2\) \[^{[4]}\].

Less is known about their behaviour far from equilibrium. Ising models can display interesting dynamical behaviour, as Neves and Schonmann showed in studies of the transition from a metastable to a stable equilibrium \[^{[5]}\] in \(d = 2\). Asymmetric Hopfield networks have transitions to disordered dynamical phases \[^{[6]}\]. Also, Cellular automata have non-trivial dynamics; soliton solutions were studied by Bobenko, Bordemann, Gunn and Pinkall \[^{[7]}\]. One of the more complex dynamical behaviours in Wolfram’s classification \[^{[8]}\] has been studied empirically as a form of “chaos” \[^{[9]}\], but it has never been entirely clear how this is related to chaotic dynamics in Euclidean space.

As is well-known, Markovian dynamics on a finite state space cannot be chaotic since every orbit must eventually fall onto a finite limit-cycle. However, one would like to have a framework for chaos which allows one to decide whether the complex dynamics of finite systems is \emph{approximately} chaotic. Unfortunately, for most finite systems there is no convenient quasi-representation in terms of real variables. There are different points of view on this problem, ranging from the fundamentalist, which concludes that a finite system
cannot be viewed as approximately chaotic, to the liberal, which reduces the definition of chaos to sensitive dependence on initial conditions and the exponential growth of the limit-cycle period with the size of the system.

Our feeling is that “chaos” should not be limited to real variables, as these are idealizations of a reality which could be viewed equally well in terms of finite state spaces. Indeed, the fact that most real variables have infinite algorithmic information\textsuperscript{10} is not satisfactory from a physicist’s point of view. Yet some form of idealization is necessary to define “chaos” rigorously.

Our purpose in this article is to propose a different idealization, inspired from symbolic dynamics\textsuperscript{11,12}, which is well suited to finite systems with a natural binary network representation (including Ising models, spin glasses and cellular automata).

Instead of specifying the coordinates of a point with infinite accuracy, we will assume that one is given the $N$–bit model of the system at every past tick of a clock, to the infinitely remote past: the state of a binary system is given by

$$S = \{S(0), S(-1), \ldots, S(-n), \ldots\},$$

where $S(-n)$ is a binary vector with components $S^i(-n) \in \{0, 1\}; i = 1, \ldots, N$. The space of such binary histories will be denoted by $B^\infty$ [Figure 1].

The approximation which makes this concept practical, akin to the 128–bit version of floating-point variables, is the truncation of the binary history to the $n$ most recent steps in the past. This truncation is valid if the difference between states with the same history for the first $n$ steps belongs to a small neighborhood of the origin. We will formalize this demand in the definition of the “semicausal topology”, which will be the keystone of our construction.
Thus, order $n$ Markov chains on finite spaces can have quasi-chaotic dynamics if $n$ is large, and be precisely chaotic in the $n \to \infty$ limit. Cellular automata and asymmetric spin glasses are $n = 1$ processes, so they do not provide good models of chaos. This fact manifests itself in the lack of an invariant attracting set where the dynamics is topologically transitive.

We begin by reviewing definitions for chaotic maps on real phase spaces \cite{[12]}. Let $f : \mathbb{R}^d \to \mathbb{R}^d$, $d \in \mathbb{Z}^+$ be a continuous map, such that $x_{n+1} = f(x_n)$.

**Definition 1**: $f : \mathbb{R}^d \to \mathbb{R}^d$ has sensitive dependence on initial conditions on $\mathcal{A} \subset \mathbb{R}^d$ if $\exists \; \delta > 0 \; \exists \; \forall \; x \in \mathcal{A}$ and $\forall N(x)$ (neighborhood of $x$) $\exists \; y \in N(x)$ and $n \in \mathbb{N} \ni |f^n(x) - f^n(y)| > \delta$.

**Definition 2**: $f : \mathbb{R}^d \to \mathbb{R}^d$ is topologically transitive on $\mathcal{A} \subset \mathbb{R}^d$ if for any open sets $U$, $V \subset \mathcal{A}$ $\exists \; n \in \mathbb{Z} \ni f^n(U) \cap V \neq \emptyset$.

**Definition 3**: Let $\mathcal{A} \subset \mathbb{R}^d$ be a compact set. $f : \mathcal{A} \to \mathcal{A}$ is chaotic on $\mathcal{A}$ if $f$ has sensitive dependence on initial conditions and is topologically transitive on $\mathcal{A}$.

**Definition 4**: A closed and connected set $\mathcal{M} \subset \mathbb{R}^d$ is called a trapping region if $f(\mathcal{M}) \subset \mathcal{M}$.

**Definition 5**: The map $f$ has a chaotic attractor $\Lambda \subset \mathbb{R}^d$ if $\Lambda$ is a compact set on which $f$ is chaotic and exists a trapping region $\mathcal{M}$ such that $\Lambda = \bigcap_{n \geq 0} f^n(\mathcal{M})$.

Of the five definitions above, only the first uses the Euclidean metric explicitly. Since the property of “chaos” is topologically invariant, we will use a topological definition of “sensitivity to initial conditions”:

**Definition 1bis**: $f : \mathbb{R}^d \to \mathbb{R}^d$ has sensitive dependence on initial conditions on $\mathcal{M} \subset \mathbb{R}^d$,

\[ \mathbb{N} = \{0, 1, 2, \ldots\} \]
if there exists a field of neighborhoods \( \mathcal{N}(x) \), i.e., a function from the trapping region to the continuous topology on \( \mathbb{R}^d, \mathcal{N} : \mathcal{M} \to \tau(\mathcal{M}) \), such that

\[
\forall x \exists y \in \mathcal{N}(x) \cap \mathcal{M}, \ n \in \mathbb{N} \ni f^n(y) \notin \mathcal{N}(f^n(x))
\]

The metrical definition (1) is recovered if one requires that the neighborhoods \( \mathcal{N}(x) \) be \( \delta \)-balls centered at \( x \).

2. The Space of Binary Histories \( B^\infty \).

Let \( \mathbf{S} = (S_1(t), S_2(t), \ldots, S_N(t)) \) be an \( N \) - bit binary model of the state of the system at time \( t \). We denote the space of the \( 2^N \) possible binary states by \( B = \{ \mathbf{S} \} \) and the infinite set of binary histories of the system by

\[
B^\infty = \{ S = (S(0), S(-1), ...) \}.
\] (1)

We endow \( B^\infty \) with a topology such that near-neighbors in \( B^\infty \) have similar binary states in the recent past.

Definition 6: A semicausal topology on \( B^\infty \) with index \( \Delta \in \mathbb{N} \) is a topology generated by a base whose elements \( \mathcal{N}_n^\Delta(S), S \in B^\infty, n \in \mathbb{N} \) satisfy:

i) \( S' \in \mathcal{N}_n^\Delta(S) \implies \forall m < n, S'(-m) = S(-m) \).

ii) \( S'(-m) = S(-m) \forall m < n + \Delta \implies S' \in \mathcal{N}_n^\Delta(S) \).

Note that if \( S' \in \mathcal{N}_n^\Delta(S) \) it may or may not have the same binary states \( S'(-m) \) of \( S \) in the range \( n \leq m < n + \Delta \). One semicausal topology differs from another in which
differences are allowed between \( S'(-m) \) and \( S(-m) \) in this range, for \( S' \in N_n^\Delta (S) \) [Figure 2].

It is easy to check that the basis sets \( N_n^\Delta (S) \) satisfy the

**Property 1:** \( N_{n+\Delta}^\Delta (S) \subset N_n^\Delta (S') \forall S' \in N_{n+\Delta}^\Delta (S) \).

**Definition 7:** The causal topology is a semicausal topology with \( \Delta = 0 \). The base elements \( N_0^\Delta (S) \) are uniquely defined by \( i \) and \( ii \) above.

From here on we will assume that \( B^\infty \) is equipped with a semicausal topology. To simplify the notation we will drop the \( \Delta \) in the notation and write the basis elements simply as \( N_n (S) \). The following proposition is easy to verify:

**Property 2:** \( B^\infty \) is a boolean algebra (a ring with an idempotent cross operation) under the logical operations

\[
XOR = +, \quad (2)
\]
\[
AND = \times \quad (3)
\]

performed on each bit in the infinite binary chain \( S \). The “zero” is the element

\[
0 = (0, 0, 0, 0, ...), \quad (4)
\]

and the identity is the element

\[
e = (1, 1, 1, 1, ...). \quad (5)
\]

The addition operation is nilpotent

\[
S + S = 0 \quad (6)
\]

and the multiplication is idempotent

\[
S \times S = S. \quad (7)
\]
Definition 8: The sequence \( \{S_k\} \), \( k \in \mathbb{N} \) is a Cauchy net if

\[
\forall n \in \mathbb{N} \ \exists k_0 \in \mathbb{N} \ \exists S_k - S_{k'} \in N_n(0) \ \forall k, k' > k_0,
\]

see Ref. [13].

Theorem 1: \( \mathcal{B}^\infty \) is complete.

Proof: We need to show that any Cauchy net of elements in \( \mathcal{B}^\infty \) converges to an element in \( \mathcal{B}^\infty \). Let \( \{S_k\} \) be a Cauchy net. From definition 8, \( S_{k'}(-m) = S_k(-m) \ \forall m < n \) and \( k, k' > k_0 \). Let us construct \( S \) from \( m = 0 \) down to \( m = n - 1 \) such that \( S(-m) = S_k(-m) \ \forall m < n \); by incrementing \( n \) this construction leads to a unique binary history \( S \). The sequence \( \{S_k\} \) converges to \( S \) which by construction is of the form (1), so that \( S \in \mathcal{B}^\infty \). \( \blacksquare \)

Theorem 2: \( \mathcal{B}^\infty \) is Hausdorff.

Proof: Let \( S \neq S' \), then \( \exists n \in \mathbb{N} \ \exists S(-n) \neq S'(-n) \). The neighborhoods \( N_{n+1}(S) \) and \( N_{n+1}(S') \) are disjoint. \( \blacksquare \)

Theorem 3: \( \mathcal{B}^\infty \) is perfect.

Proof: A set is perfect if it is closed and every point is an accumulation point. \( \mathcal{B}^\infty \) is closed because it is the total space so it is both open and closed. Let \( S \in \mathcal{B}^\infty \) and consider the sequence \( \{S_k\} \) given by

\[
S_k(-n) = S(-n), \forall n < k
\]

and

\[
S_k(-n) = S(-n) + 1, \forall n \geq k,
\]
where 1 is given by (5). This sequence converges to $S$ and by construction $S_k \neq S \forall k$. Thus, $S$ is an accumulation point.

Lemma 1: If $S' \notin N_n (S)$ then $N_n (S) \cap N_m (S') = \emptyset \forall m \geq n + \Delta$.

Proof: From definition 6, if $S' \notin N_n (S)$ then $S' (-k) \neq S (-k)$ for some $k < n + \Delta$. Let $S_0 \in N_m (S')$ then $S_0 (-k) = S' (-k) \forall k < n + \Delta \leq m$, which implies that $S_0 \notin N_n (S)$.

Lemma 2: The complement $N_n^c (S)$ of any neighborhood is an open set.

Proof: Due to lemma 1, $N_n^c (S)$ can be expressed by

$$N_n^c (S) = \bigcup_{S' \in N_n^c (S)} N_{n+\Delta} (S'),$$

which is a union of open sets, then $N_n^c (S)$ is an open set.

Theorem 4: $\mathcal{B}^\infty$ is totally disconnected.

Proof: We must show that the connected component $\mathcal{C} (S)$ of each $S \in \mathcal{B}^\infty$ consists of just the point $S$. By contradiction: Let $S' \in \mathcal{C} (S)$ with $S' \neq S$. Then $S$ and $S'$ differ in at least one binary state: $S (-n) \neq S' (-n)$. Then $S' \notin N_{n+1} (S)$. Now,

$$\mathcal{B}^\infty = N_{n+1} (S) \cup N_{n+1}^c (S)$$

is a separation of $\mathcal{B}^\infty$ because by lemma 2 bought of them are disjoint non-empty open sets. This implies that $\mathcal{C} (S) \subset N_{n+1} (S)$ but $S' \notin N_{n+1} (S)$ and we have a contradiction.
Theorem 5: $\mathcal{B}^\infty$ is compact.

Before proving the theorem let us first prove three lemmas.

Lemma 3: The number of distinct sets $\mathcal{N}_n(S) \forall S \in \mathcal{B}^\infty$ and $n$ fixed, is finite.

Proof: For any $S \in \mathcal{B}^\infty$, $\mathcal{N}_n(S)$ is totally defined by specifying the first $n + \Delta$ binary states and the list of which differences are allowed between $S'(-m)$ and $S(-n')$ in the range $n \leq m < n + \Delta$, in all a finite amount of information for any fixed $n$.

Lemma 4: Let $\mathcal{V} = \{U_i\}$ be an open covering of $\mathcal{B}^\infty$ such that there exists a $n_0 \in \mathbb{N}$ such that $\forall S \in \mathcal{B}^\infty \exists m \leq n_0 \ni \mathcal{N}_m(S) \subset U_i$ for some $U_i \in \mathcal{V}$. Then there exists a finite subcovering of $\mathcal{V}$.

Proof: $\forall S \in \mathcal{B}^\infty$ let $U_\alpha \in \mathcal{V}$ be such that $S \in U_\alpha$ and $\mathcal{N}_m(S) \subset U_\alpha$ with the minimum possible $m$. By lemma 3, the number of distinct $\mathcal{N}_m(S)$ with $m \leq n_0$ is finite, so the set of such base elements $\{\mathcal{N}_m(S)\}$, is a finite covering. Since for each $\mathcal{N}_m(S)$ there is an associated $U_\alpha$ and $\mathcal{N}_m(S) \subset U_\alpha$, $\{U_\alpha\}$ is a finite subcovering of $\mathcal{B}^\infty$.

Lemma 5: If $g : \mathcal{N}_n(S) \rightarrow \mathbb{N}$ is a non-bounded function, then there exists $\mathcal{N}_m(S') \subset \mathcal{N}_n(S)$, with $\mathcal{N}_m(S') \neq \mathcal{N}_n(S)$, such that $g : \mathcal{N}_m(S') \rightarrow \mathbb{N}$ is non-bounded.

Proof: The neighborhood $\mathcal{N}_n(S)$ can be expressed as a finite union of neighborhoods in the following way: let $m \geq n + \Delta$, the set

$$\mathcal{W} = \{S' \in \mathcal{N}_n(S) \mid S'(-k) = S(-k) \forall k \geq m + \Delta\}$$

is finite and

$$\mathcal{N}_n(S) = \bigcup_{S' \in \mathcal{W}} \mathcal{N}_m(S') .$$

Then $g$ must be non-bounded in at least one of the $\mathcal{N}_m(S')$. ■
Proof of theorem 5: By theorem 2, $\mathcal{B}^\infty$ is Hausdorff then by the lemma 4 it remains to prove that for any covering $\mathcal{V} \ni n_0 \in \mathbb{N} \ni \forall S \in \mathcal{B}^\infty \exists m \leq n_0 \ni \mathcal{N}_m (S) \subset U_i$ for some $U_i \in \mathcal{V}$. By contradiction let us suppose that there does not exist such $n$. Then there exists a covering $\mathcal{V}$ such that $\forall n \in \mathbb{N} \ni \exists S \in \mathcal{B}^\infty \ni \mathcal{N}_m (S) \subset U_i$ for some $U_i \in \mathcal{V}$ implies $m > n$. This defines a function

$$g : \mathcal{B}^\infty \rightarrow \mathbb{N}$$

given by $g (S) = n$, which is not bounded. Then by lemma 5, there exists a nested sequence of neighborhoods

$$\mathcal{N}_{n_1} (S_1) \supset \mathcal{N}_{n_2} (S_2) \supset \ldots \supset \mathcal{N}_{n_k} (S_k) \supset \ldots$$

with $n_1 < n_2 < \ldots$, such that $g$ is non-bounded $\forall \mathcal{N}_{n_k} (S_k)$. The sequence $\{S_k\}$ is a Cauchy net, therefore by theorem 1 it converges to an element of $\mathcal{B}^\infty$, but that element is not covered by $\mathcal{V}$ since $g \rightarrow \infty$, and therefore we have a contradiction.

From theorems 3, 4 and 5 we have the following

Corollary: $\mathcal{B}^\infty$ is a Cantor set.

3. Chaotic dynamics in $\mathcal{B}^\infty$.

Let us introduce the following notation: by $S (n) \in \mathcal{B}^\infty$ with $n \in \mathbb{Z}$ we understand

$$S (n) = (S (n), S (n-1), S (n-2), \ldots). \quad (8)$$
A binary dynamical system in $\mathcal{B}^\infty$ is a map $F$ that is induced by non-vanishing continuous functions $F_i : \mathcal{B}^\infty \to \mathbb{R} - \{0\}$, $i = 1, \ldots, N$, such that

$$S_i (n + 1) = \Theta \circ F_i (S(n)), \quad (9)$$

where $\Theta (x) = 0, 1$ for $x \leq 0$, $x > 0$ respectively. The map $F : \mathcal{B}^\infty \to \mathcal{B}^\infty$ is given by

$$F(S(n)) = S(n + 1). \quad (10)$$

We should stress that due to the fact that $\mathcal{B}^\infty$ is totally disconnected it is not difficult to construct continuous non-vanishing functions $F_i$ that change sign.

**Property 3**: The dynamical map defined by (10) is continuous.

**Proof**: $F_i : \mathcal{B}^\infty \to \mathbb{R} - \{0\}$ is continuous and $\Theta : \mathbb{R} - \{0\} \to \mathbb{Z}_2$ is also continuous, so the composition $\Theta \circ F_i$ is continuous.\[\blacksquare\]

Now we will extend the definitions 1 to 5 of Sec. 1 in a natural way, so they fit into the dynamics generated by $F : \mathcal{B}^\infty \to \mathcal{B}^\infty$.

**Definition 9**: The map $F : \mathcal{B}^\infty \to \mathcal{B}^\infty$ has sensitive dependence on initial conditions on $A \subset \mathcal{B}^\infty$ if $\exists n \in \mathbb{N} \exists S \in A$ and $\forall N_m(S) \exists S' \in N_m(S) \cap A$ and $k \in \mathbb{N} \exists F^k(S') \notin N_n(F^k(S)).$

**Definition 10**: $F : \mathcal{B}^\infty \to \mathcal{B}^\infty$ is topologically transitive on $A \subset \mathcal{B}^\infty$ if for any open sets $U, V \subset A \exists n \in \mathbb{Z} \exists F^n(U) \cap V \neq \emptyset$. In the last expression, if $F$ is noninvertible we understand the set $F^{-k}(U)$ as the set of all points $S \in \mathcal{B}^\infty$ such that $F^k(S) \in U$.

**Definition 11**: Let $A \subset \mathcal{B}^\infty$ be a compact set. $F : A \to A$ is chaotic on $A$ if $F$ has sensitive dependence on initial conditions and is topologically transitive on $A$.

**Definition 12**: A closed subset $\mathcal{M} \subset \mathcal{B}^\infty$ is called a trapping region if $F(\mathcal{M}) \subset \mathcal{M}$.
Property 4: \( F^n (M) \) is compact and closed \( \forall n \in \mathbb{N} \).

Proof: Since every closed subset of a compact set is compact, it follows that \( M \) is compact and since \( F \) is continuous \( F^n (M) \) is compact. Since \( B^\infty \) is Hausdorff every compact subset of it is closed, so \( F^n (M) \) is closed \[^{15}\].

Definition 13: The map \( F : B^\infty \rightarrow B^\infty \) has an attractor \( \Lambda \subset B^\infty \) if there exists a trapping region \( M \) such that

\[
\Lambda = \bigcap_{n \geq 0} F^n (M).
\]

Property 5: \( \Lambda \) is compact and closed.

Proof: \( \Lambda \) is an intersection of closed sets, so it is closed. Since every closed subset of a compact space \( B^\infty \) is compact, it follows that \( \Lambda \) is compact.

Property 6: The restriction of \( F \) to \( \Lambda \) is completely defined by specifying \( \Lambda \) itself, as follows.

For any point \( S \in \Lambda \) and \( n \geq 1 \), let \( S_n \in \Lambda \) be such that \( S_n (-n - k) = S (-k) \), \( k \in \mathbb{N} \). Then \( F (S) = (S_n (-n + 1), S_n (-n), ...) \).

Proof: Since \( S \in \Lambda \), for any \( n \in \mathbb{N} \) there is an element \( S_n \in \Lambda \) such that \( F^n (S) = S_n \). By (9) \( F \) acts as a down-shift map for all but the most recent slice, therefore \( S_n (-n - k) = S (-k) \), \( k \in \mathbb{N} \).

Definition 14: \( \Lambda \) is called a chaotic attractor if \( F \) is chaotic on \( \Lambda \).

Theorem 6: If \( \Lambda \) is a chaotic attractor then it is perfect.

Proof: By property 5, \( \Lambda \) is closed, it remains to prove that every point in \( \Lambda \) is an accumulation point of \( \Lambda \). By contradiction, let \( S_0 \in \Lambda \) be an isolated point, then there exists
\( n \in \mathbb{N} \ni N_n(S_0) \cap \Lambda = \{S_0\} \) then, by topological transitivity and the fact that \( F^{-1} \) exists, \( \Lambda \) consists of an isolated orbit (the orbit of \( S_0 \)) but this violates sensitivity to initial conditions on \( \Lambda \), so \( \Lambda \) could not be a chaotic attractor.

**Theorem 7**: If \( \Lambda \) is a chaotic attractor then it is a Cantor set.

**Proof**: The theorem follows directly from property 5, theorem 6 and the fact that a subset of a totally disconnected set is also totally disconnected.

### 4. Metric in \( \mathcal{B}^\infty \).

The space \( \mathcal{B}^\infty \) is metrizable as we will see below. This enforces its utility in using it to model chaotic dynamical systems.

**Definition 15**: A metric \( d(S_1, S_2) \) is called *semicausal* if it induces a semicausal topology.

**Definition 16**: A metric over \( \mathcal{B}^\infty \) is *bounded* if \( \exists M \in \mathbb{R} \ni \forall S_1, S_2 \in \mathcal{B}^\infty \, d(S_1, S_2) < M \).

**Definition 17**: A metric on \( \mathcal{B}^\infty \) is *local in time* if there exist real numbers \( a > 1 \) and \( w > 0 \) such that

\[
d(S_1, S_2) \leq wa^{-n}
\]

implies that

\[
S_1(-m) = S_2(-m)
\]

\( \forall m \leq n \).
The following is an example of a semicausal metric in $B^\infty$:

$$d(S_1, S_2) = \sqrt{\sum_{n=0}^{\infty} \frac{1}{2^n} \frac{d_n(S_1 + S_2)}{1 + d_n(S_1 + S_2)}}$$  \hspace{1cm} (11)

where

$$d_n(S) \equiv \sum_{i=1}^{N} S^i(-n)$$

and the plus sign is defined by (2). Note that $d_n(S_1 + S_2)$ is the Hamming distance of the binary models $S_1(-n)$ and $S_2(-n)$. It is easy to check that (11) satisfies the following properties:

i) $d(S_1, S_2) \leq \sqrt{\frac{2N}{1+N}} \forall S_1, S_2 \in B^\infty$, so it is bounded.

ii) $d$ is semicausal with $\Delta = 2$.

iii) $d$ is local in time.

It is possible to define a “dot” product in $B^\infty$ given by

$$S \cdot S' = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{d_n(S \times S')}{{1 + d_n(S \times S')}}$$

where the “cross” product is defined by (3). Using equations (6) and (7) the metric (11) can be written as

$$d(S_1, S_2) = \sqrt{(S_1 + S_2) \cdot (S_1 + S_2)}.$$
5. Conclusion.

From an initial ansatz, to replace the usual idealization of physical states as “points” on a differentiable manifold by another idealization as infinite “binary histories”, we proceeded to define a class of topologies which make the truncation to finite binary histories a valid approximation, in the same sense that the continuous topology on $\mathbb{R}^d$ allows one to approximate a real coordinate by a finite string of digits or bits. With this topology and the natural Boolean algebra structure, the space of binary histories was shown to have several interesting properties, including those of Cantor sets: it is compact, totally disconnected and yet every point is an accumulation point.

Continuous dynamical maps on the space of binary histories can lead to attracting sets within $\mathcal{B}^\infty$, in which case an attractor is defined in the usual way. The dynamical map is said to be chaotic on the attractor if it is sensitive to initial conditions and topologically transitive.

It is remarkable that the dynamics on the attractor is uniquely specified by the attractor itself (Property 6): given any initial state on the attractor,

$$ S = \{S(0), S(-1), \cdots \} \in \Lambda. $$

one finds, for each $n$, one and only one state $S_n \in \Lambda$ such that

$$ S_n = \{S_n(0), S_n(-1), S_n(-n+1), S(0), S(-1), \cdots \}. $$

Indeed, since $S \in \Lambda$ and $F(\Lambda) = \Lambda$, one has $S_n \in \Lambda$; $S_n$ is unique because $S_n = F(S_{n-1}), S_n' = F(S_{n-1}) \Rightarrow S_n = S_n'$. The dynamics is then given by $F(S) = S_1, F(F(S)) = S_2$, etc. In other words, to determine $F(S)$ it is sufficient to scan $\Lambda$ in search of
the state $S_1$ which is equal to $S$ downshifted one step in time, with the extra binary model $S_1(0)$ on top; in this sense the dynamical map is related to a shift map, as in symbolic dynamics.

The relation to symbolic dynamics is suggestive, and one might wish to regard our formalism as its generalization. However, there are some non-trivial differences between chaos on $B^\infty$ and the shift map on the space of itineraries $\Sigma^N$ of Ref. [12]. First of all, points of $B^\infty$ represent the history of the binary system towards the past only, whereas itineraries extend also to $t \rightarrow +\infty$. Secondly, we are describing chaos on a proper subset $\Lambda$ of a larger binary space, $B^\infty$, so that $B^\infty$ plays the role of “embedding space”. In contrast, the chaotic dynamics on all of $\Sigma^N$ is homeomorphic to that which takes place on a Cantor subset $\Xi$ of the attractor, that is $\Xi \subset \Lambda \subset \mathbb{R}^d$. In other words one needs to appeal to a homeomorphism to $\mathbb{R}^d$ for the embedding space. The two differences, one related to causality and the other to the intrinsic nature of the embedding space $B^\infty$, allowed us to develop the formalism of chaos on binary systems without ever invoking differentiable manifolds, thereby lending support to our claim that this formalism can be regarded as a different representation of reality based on binary histories rather than real variables.

One could go one step further and suggest that other theories of physics could be rewritten by thinking of $B^\infty$ as the space of physical states and call “real” the elements $S \in B^\infty$ rather than the coordinates on differentiable manifolds. Of course that would probably turn out to be rather inconvenient for most systems; we are only making this outrageous suggestion to emphasize that the mathematical constructions which best represent reality are nothing but those which make reality look simple; in this sense surely $B^\infty$ is a more appropriate framework to describe the physical reality of binary systems than differentiable manifolds!
The fact that $B^\infty$ is a Cantor set is perhaps not surprising in light of the analogy to the symbolic dynamics of chaotic maps $f : \mathbb{R}^d \to \mathbb{R}^d$. If one regards an $N$-bit binary vector $S(0)$ as defining a partition of phase space into $2^N$ disjoint subsets $O_\alpha$, $\alpha = 1, 2, \cdots, 2^N$, then the specification of two consecutive binary vectors, \{S(0), S(−1)\} defines a finer partition into subsets $O_\alpha \cap f(O_\beta)$. For example the $O_\alpha$ might be thought of as a tiling of phase space, in the sense of Berend and Radin [16].

Repeating the procedure to finer and finer partitions one obtains the image of $B^\infty$ in real phase space as the infinite intersection set

$$\bigcup_{\{\alpha, \beta, \gamma, \cdots\}} (O_\alpha \cap f(O_\beta) \cap f^2(O_\gamma) \cap \cdots),$$

much like the textbook construction of the Cantor set from the intersections of the intervals $I = (0, 1)$, $f(I) = (0, \frac{1}{3}) \cup (\frac{2}{3}, 1)$, etc. Note that one could replace an $N$-bit description of the state for two consecutive time steps by a $2N$-bit description for a single time step, based on a partition of phase space into $2^{2N}$ disjoint cells $O_\alpha \cap f(O_\beta)$; this indicates that there is an exact renormalization group transformation relating refinement in space with extension of binary histories towards the past; this may be an interesting line of investigation to pursue which might be expected to raise issues of universality along the lines of Feigenbaum’s work [17].

One rather common example of truncated binary histories is the case of computer models for chaotic time series prediction. In the so-called “method of delays” [18], the coordinates of a point in phase space are taken to be the delayed measurements of a single variable, so that the state vector is given by $x = \{x(0), x(−1), x(−2), \cdots, x(−T + 1)\}$. Here, $T$ is the embedding dimension and each coordinate is represented as a 128-bit binary word. The Euclidean metric in $\mathbb{R}^T$ induces a semicausal metric on the space of binary histories.
A priority in the continuation of this work is to further elucidate the connection between chaos on binary systems and real chaos. One notes first of all that there cannot be a homeomorphism between the embedding space $\mathcal{B}^\infty$ and $\mathbb{R}^d$: one is a Cantor set and the other a differentiable space! The reason why real and binary embedding spaces cannot be homeomorphic is that given any map from the space of binary histories to real phase space, there is a continuous curve in the latter which takes one across the boundary which separates binary histories beginning with distinct binary vectors, $S'(0) \neq S(0)$, so the map is discontinuous at the boundary. This fact should not be regarded as a serious problem, since chaos is defined on the attractor and there is no impediment to a homeomorphism from the attractor $\Lambda \subset \mathcal{B}^\infty$ to a Cantor subset of $\mathbb{R}^d$, or for that matter from $\mathcal{B}^\infty$ to a larger Cantor subset of $\mathbb{R}^d$. The formal connection between chaos on the space of binary histories and real chaos is the subject of ongoing research.

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Figure Captions

[1] A state in the space of binary histories, $S \in \mathcal{B}^\infty$, is a succession of $N$-bit binary words giving an approximate description, or “model” of the system at times $t = 0, -1, -2, \ldots$.

[2] A state $S'$ in the neighborhood $\mathcal{N}^\Delta_n(S)$ has the same binary words as $S$ for the slices $t = 0, -1, \cdots, -n$ and can have any binary word at all beyond $t = -n - \Delta$. In between these two bounds, the differences which are allowed for $S' \in \mathcal{N}^\Delta_n(S)$ characterize the particular semicausal topology.