Visible shapes of black holes M87* and SgrA*

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We review the physical origins for possible visible images of the supermassive black hole M87* in the galaxy M87 and SgrA* in the Milky Way Galaxy. The classical dark black hole shadow of the maximal size is visible in the case of luminous background behind the black hole at the distance exceeding the so-called photon spheres. The notably smaller dark shadow (dark silhouette) of the black hole event horizon is visible if the black hole is highlighted by the inner parts of the luminous accreting matter inside the photon spheres. The first image of the supermassive black hole M87*, obtained by the Event Horizon Telescope collaboration, shows the lensed dark image of the southern hemisphere of the black hole event horizon globe, highlighted by accreting matter, while the classical black hole shadow is invisible at all. A size of the dark spot on the Event Horizon Telescope (EHT) image agrees with a corresponding size of the dark event horizon silhouette in a thin accretion disk model in the case of either the high or moderate value of the black hole spin, $a \gtrsim 0.75$.

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I. INTRODUCTION

The enigmatic black holes are really black objects in the sky due to their physical properties. The famous quantum Hawking thermal radiation of black holes can be neglected in the case of numerous astrophysical black holes originated from the gravitational collapse of old massive stars. Nowadays, the only way to view black holes in the sky is a watching of black hole candidates highlighted by the surrounding matter. General relativity (Einstein’s theory of gravity) predicts the appearance of dark black hole images on the surrounding luminous background. The Event Horizon Telescope (EHT) collaboration presents the first image of the supermassive black hole M87* at 1.3 mm wavelength with an unprecedented high angular resolution \textsuperscript{1-6}. Indeed, this image is the direct experimental evidence of the black hole existence in the Universe besides the famous observations of gravitational waves from the coalescence of black holes by the LIGO collaboration.

The visible shapes of black hole images depend on the distribution of emitting matter around black holes. We describe below the possible visible shapes of black hole images of the central supermassive black hole M87* in the galaxy M87 and SgrA* in our native Milky Way Galaxy.

We show that the unique physical properties of the Kerr metric for rotating black hole \textsuperscript{7} provide two qualitatively different forms of the visible black images: the standard black hole shadow or the notably smaller event horizon

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shadow (or event horizon silhouette). A particular visible black image crucially depends on the prevalence of emitting matter outside or inside of the so-called “photon spheres” (see details and properties of photon spheres in Chapter III).

A standard black hole shadow is visible in the case of emitting matter placed outside the photon spheres (e.g., if there is a distant luminous background of extended hot gas clouds or luminous stars far outside the black hole). Meantime, a notably smaller event horizon silhouette is viewed, which is a shadow of the event horizon itself, in the case of emitting matter placed inside the photon spheres (e.g., if there is a highly luminous accreting matter in the vicinity of event horizon). To distinguish these two different black hole images, we will use in the following the term “classical black hole shadow” for black hole image in the case of emitting matter placed outside the photon spheres (e.g., if there is a luminous stationary background far behind the black hole).

We demonstrate below that on the first image of the supermassive black hole M87*, obtained by the Event Horizon Telescope collaboration, it is viewed namely the lensed dark image of the southern hemisphere of the black hole event horizon globe, highlighted by an accretion disk, while the classical black hole shadow is invisible at all.

The major scientific goal of the EHT collaboration is the registration of the supermassive black hole SgrA* image at the center of Milky Way [8–17]. This supermassive black hole is the nearest “dormant” or “sleeping” quasar with a very low radiation activity and the mass \( M = (4.3 \pm 0.3) \times 10^6 M_\odot \) [18–22]. Our native supermassive black hole SgrA* is evidently an object of intensive investigations [23–92]. The other goal of the EHT collaboration is the registration of the center of Milky Way [8–17]. This supermassive black hole is the nearest “dormant” or “sleeping” quasar with a highly luminous accretion disk. While the classical black hole shadow is invisible at all.

T elescope collaboration, it is viewed namely the lensed dark image of the southern hemisphere of the black hole event horizon itself. At last, in the most original Section IV we explain the significant features of the event horizon silhouette (the shadow of the distant luminous background, consisting of the extended hot gas clouds and bright stars. For this reason, the classical black hole shadow is complicated to observe in comparison with the event horizon silhouette (the shadow of the event horizon itself).

In standard astrophysical conditions, the brightness of the accreting disk greatly exceeds the corresponding one of the distant luminous background, consisting of the extended hot gas clouds and bright stars. For this reason, the classical black hole shadow is complicated to observe in comparison with the event horizon silhouette (the shadow of the event horizon itself).

In Section II we describe the general properties of the classical black hole shadow, when the black hole is highlighted by a distant luminous background. In Section III we elucidate the principal properties of photon spheres, which are crucial for understanding the possible forms of black hole images. At last, in the most original Section IV we explain the significant features of the event horizon silhouette (the event horizon shadow), produced by photons from the highly luminous accretion disk.

The line element of the classical Kerr metric [7, 104, 160–167], describing the rotating black hole in standard Boyer–Lindquist coordinates \((t, r, \theta, \phi)\) [161], is

\[
ds^2 = -e^{2\nu}dt^2 + e^{2\psi}(d\phi - \omega dt)^2 + e^{2\mu_1}dr^2 + e^{2\mu_2}d\theta^2,
\]

where

\[
e^{2\nu} = \frac{\Sigma \Delta}{A}, \quad e^{2\psi} = \frac{A \sin^2 \theta}{\Sigma}, \quad e^{2\mu_1} = \frac{\Sigma}{\Delta}, \quad e^{2\mu_2} = \Sigma, \quad \omega = \frac{2Mar}{A}, \quad \Delta = r^2 - 2Mr + a^2, \quad \Sigma = r^2 + a^2 \cos^2 \theta, \quad A = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta.
\]

In these equations \( M \) — black hole mass, \( a = J/M \) — black hole specific angular momentum (spin), \( \omega \) — frame-dragging angular velocity. We use the units with the gravitational constant \( G = 1 \) and the velocity of light \( c = 1 \). For simplification of formulas in the following, we often use the dimensional values for space distances \( r \Rightarrow \eta r/M \), for time intervals \( t \Rightarrow t/M \) and etc. In other words, we will measure the radial distances in units \( GM/c^2 \) and time intervals in units \( GM/c^2 \). We also will use the dimensionless value for black hole spin \( a = J/M^2 \leq 1 \), by supposing that \( 0 \leq a \leq 1 \). The black hole event horizon radius \( r_h \) is the largest root of the quadratic equation \( \Delta = 0 \):

\[
r_h = 1 + \sqrt{1 - a^2},
\]

There are four integrals of motion for test particles in the Kerr metric: \( \mu \) — test particle mass, \( E \) — particle total energy, \( L \) — particle azimuth angular momentum and \( Q \) — Carter constant, related with the non-azimuth angular momentum of the test particle and with non-equatorial motion [162]. The corresponding first-order differential
equations of motion for test particle are \[104, 160, 162-167\]:

\[
\frac{d\Sigma}{d\tau} = \pm \sqrt{R(r)},
\]

(5)

\[
\frac{d\Sigma}{d\tau} = \pm \sqrt{\Theta(\theta)},
\]

(6)

\[
\Sigma \frac{d\phi}{d\tau} = L \sin^{-2} \theta + a(\Lambda^{-1} P - E),
\]

(7)

\[
\Sigma \frac{dt}{d\tau} = a(L - aE \sin^2 \theta) + (r^2 + a^2)\Lambda^{-1} P.
\]

(8)

Here \(\tau\) — the proper particle time or affine parameter along the trajectory of massless \((\mu = 0)\) particle. In these equations the effective radial potential \(R(r)\) governs the radial motion in these equations:

\[
R(r) = P^2 - \Delta [\mu^2 r^2 + (L - aE)^2 + Q],
\]

(9)

where \(P = E(r^2 + a^2) - aL\), and, respectively, the effective polar potential \(\Theta(\theta)\) defines the polar motion of test particles:

\[
\Theta(\theta) = Q - \cos^2 \theta[a^2(\mu^2 - E^2) + L^2 \sin^{-2} \theta].
\]

(10)

In particular, the zeros of these potentials define the turning points \(dR/d\tau = 0\) and \(d\Theta/d\tau = 0\) in the radial and polar directions, respectively.

All trajectories of massive test particles \((\mu \neq 0)\) depend on three parameters (constants of motion or orbital parameters): \(\gamma = E/\mu, \lambda = L/E\) and \(q = \sqrt{Q}/E\). Respectively, the corresponding trajectories of massless particles \((\mu \neq 0)\) depend only on two parameters: \(\lambda = L/E\) and \(q = \sqrt{Q}/E\). From equations of motion (5)–(8) it follows that Carter constant \(Q \geq 0\) for all particle trajectories reaching the space infinity at \(r = \infty\). At finite distances from Kerr black hole there are “vortex” orbits of test particles with \(Q < 0\) \[168\]. The vortex orbits are beyond the scope of this article because we are interested mainly in the photon trajectories with \(Q \geq 0\), reaching a distant observer very far from the black hole, formally at \(r = \infty\).

There are the integral form \[104, 160, 162, 166\] of equations of test particle motion in the Kerr Metric, which are useful for numerical calculations:

\[
\int \frac{dr}{\sqrt{R(r)}} = \int \frac{d\theta}{\sqrt{\Theta(\theta)}},
\]

(11)

\[
\tau = \int \frac{r^2}{\sqrt{R(r)}} \, dr + \int \frac{a^2 \cos^2 \theta}{\sqrt{\Theta(\theta)}} \, d\theta,
\]

(12)

\[
\phi = \int \frac{aP}{\Delta \sqrt{R(r)}} \, dr + \int \frac{L - aE \sin^2 \theta}{\sin^2 \theta \sqrt{\Theta(\theta)}} \, d\theta,
\]

(13)

\[
t = \int \frac{(r^2 + a^2)P}{\Delta \sqrt{R(r)}} \, dr + \int \frac{(L - aE \sin^2 \theta)a}{\sqrt{\Theta(\theta)}} \, d\theta.
\]

(14)

In these equations the effective potentials \(R(r)\) and \(\Theta(\theta)\) are defined in equations (9) and (10). These specific integrals in (11)–(14) are the contour (path) integrals along the particle trajectory. These contour integrals are monotonic growing along the particle trajectory: the integrands in these contour integrals do not change their signs in transition through the radial and polar turning points. In particular, the contour integrals along a particle trajectory in (11) come to the ordinary ones if there are no radial and polar turning points along the particle trajectory

\[
\int_{r_0}^{\sigma} \frac{dr}{\sqrt{R(r)}} = \int_{\theta_0}^{\theta_0} \frac{d\theta}{\sqrt{\Theta(\theta)}},
\]

(15)

where \(r_0\) and \(\theta_0\) — the initial (starting) particle radial and polar angle coordinates. Respectively, in the case of a trajectory with only one turning point \(\theta_{min}(\lambda, q)\) (the extreme point in the polar effective potential \(\Theta(\theta)\)), the contour integrals in (11) are written through the ordinary integrals in the form

\[
\int_{r_0}^{\sigma} \frac{dr}{\sqrt{R(r)}} = \int_{\theta_{min}}^{\theta_0} \frac{d\theta}{\sqrt{\Theta(\theta)}} + \int_{\theta_{min}}^{\theta_0} \frac{d\theta}{\sqrt{\Theta(\theta)}}.
\]

(16)
FIG. 1. The classical shadow (lighter blue disk) of the Schwarzschild black hole \((a = 0)\), highlighted by a distant luminous background. It is shown a typical 3D photon trajectory (multicolored 3D curve), which is starting from the distant background. Then, this trajectory is winding multiply near the radius of circular photon orbit around the black hole event horizon \(g_{\text{ob}}\) (darker blue sphere) at the return radius \(r_{\text{min}} = r_{\text{ph}} = 3\). The finishing point of this photon trajectory is the north polar point on the outline of the black hole shadow, viewed by a distant observer. Inside the black hole shadow is shown a fictitious image (dark blue disk with a radius \(r = 2\)) of the lensed image of the black hole event horizon in the imaginary Euclidean space (in the absence of gravity). The circular orbits of photons, producing the outline of shadow, are placed on the purple photon sphere with the radius \(r = 3\).

Also, the contour integrals in (11) in the case of the trajectory with two turning points \(\theta_{\text{min}}(\lambda, q)\) and \(r_{\text{min}}(\lambda, q)\) (the extreme point in the radial effective potential \(R(r)\)), are written through the ordinary integrals in the form

\[
\int_{r_{\text{min}}}^{r_{\text{ph}}} \frac{dr}{\sqrt{R(r)}} + \int_{r_{\text{min}}}^{r_{\text{ph}}} \frac{dr}{\sqrt{R(r)}} = \int_{\theta_{\text{min}}}^{\theta_{\text{ph}}} \frac{d\theta}{\sqrt{\Theta(\theta)}} + \int_{\theta_{\text{min}}}^{\theta_{\text{ph}}} \frac{d\theta}{\sqrt{\Theta(\theta)}}.
\]

(17)

II. CLASSICAL BLACK HOLE SHADOW: BLACK HOLE HIGHLIGHTING BY DISTANT LUMINOUS BACKGROUND

The classical black hole shadow is a capture photon cross-section in the black hole gravitational field. It is observable if there is a distant luminous background behind the black hole at the distance, exceeding the corresponding radius of the photon spheres (see definition and specific features of photon spheres in Section III). The classical black hole shadow is investigated in details in numerous works [100–105, 169–258].

The observed outline (contour) of the classical black hole shadow, projected on the celestial sphere, is defined by the photon orbits with a constant radius, \(r = r_{\text{ph}} = \text{const}\), named either the spherical photon orbits or photon spheres. In a general case of the rotating Kerr black hole (with a black hole spin \(a \neq 0\)) the photons on spherical orbits are moving in the azimuth and polar (latitude) directions on the surface of a constant radius \(r = r_{\text{ph}}\) by oscillating in the polar direction between the minimum \(\theta_{\text{min}}\) and maximum \(\theta_{\text{max}} = \pi - \theta_{\text{min}}\) polar angles (see. definitions and details in Chapter III).

The shape of a classical black hole shadow in the Kerr metric is defined analytically in the parametric form
FIG. 2. The classical shadow (closed magenta region) of the extreme Kerr black hole ($a = 1$), highlighted by the distant luminous background and viewed by a distant observer in the black hole equatorial plane. A closed purple region is an envelope of photon spheres for photons with $\lambda \leq 0$, while a closed green region is an envelope of photon spheres for photons with $\lambda \geq 0$. Multicolored curves are the examples of photon trajectories, producing an outline (boundary) of the classical black hole shadow, with orbital parameters, respectively, $(\lambda, q) = (2, 0)$ — corotating skimming photon in the equatorial plane, $(\lambda, q) = (-7, 0)$ — counter-rotating photon in the equatorial plane and $(\lambda, q) = (0, \sqrt{11 + 8\sqrt{2}})$ — fully spherical photon orbit. A blue disk with a radius $r = 1$ inside the black hole shadow is the observed position of the black hole event horizon in the imaginary Euclidean space (in the absence of gravity). A magenta arrow is the black hole rotation axes.

$$(\lambda, q) = (\lambda(r), q(r)),$$

namely (see, e.g., [102, 104]):

$$\lambda = \frac{(3 - r)r^2 - a^2(r + 1)}{a(r - 1)}, \quad q^2 = \frac{r^3[4a^2 - r(r - 3)^2]}{a^2(r - 1)^2},$$

where $r$ is a radius of photon sphere and $\lambda$ and $q$ are, respectively, the horizontal and vertical impact parameters of photons on the celestial sphere, viewed by a static distant observer in the black hole equatorial plane. In this equation $r$ is a radius of the photon sphere for given impact parameters $\lambda$ and $q$. Strictly speaking, equations (18) with $q \geq 0$ reproduce only upper half of the shadow. Lower half of shadow is a mirror reflection of the upper one with respect to the black hole equatorial plane (due to the Kerr metric reflection symmetry over the equatorial plane).

James Bardeen named the classical black hole shadow of the Kerr black hole as the “viewed boundary” of the black hole in his pioneering work [102]. For a more general modern definition of the classical black hole shadow see, e.g., [259–261].

The photon spheres are reduced to photon circles with radius $r_{\text{ph}} = 3$ in the simplest limiting case of the Schwarzschild black hole ($a = 0$). The corresponding radius of the classical black hole shadow in the Schwarzschild black hole case is $r_{\text{ph}} = 3\sqrt{3}$.

Figure 1 shows the 3D illustration of the classical black hole shadow formation in the Schwarzschild black hole case ($a = 0$), when the luminous background is placed behind the black hole at the distance, exceeding the size of photon sphere (purple sphere with a radius $r_{\text{ph}} = 3$). Throughout this paper we use the standard (dimensionless) Boyer–Lindquist coordinates $(t, r, \theta, \phi)$ in all 3D Figures similar to the Figure 1.
FIG. 3. Direct images and also the first and second light echoes of the lensed compact star (luminous probe) in discrete times at the circular equatorial orbit with radius $r = 20$ around a near extreme black hole SgrA* viewed by a distant observer. All multiple images of this star are placed outside the classical black hole shadow (the closed black region). A dashed magenta circle is the projection of the black hole event horizon on the celestial sphere in the imaginary Euclidean space (in the absence of gravity).

In the limiting case of extreme Kerr black hole ($a = 1$) the expressions (18) for classical black hole shadow are simplified:

$$\lambda = -r^2 + 2r + 1, \quad q^2 = r^3(4 - r).$$  (19)

It must be noted, that these limiting formulas do not produce the closed form of the outline (boundary) for the classical black hole shadow due to the nonuniform nature of the limit $a \to 1$ in the Boyer-Lindquist coordinates. It must be added the vertical line ($r = 1, 0 \leq q \leq \sqrt{3}$) to close the outline of shadow \[102\]. The inverted forms of expressions (19) are

$$r_{ph}(\lambda) = 1 + \sqrt{2 - \lambda}, \quad q_{ph}(\lambda)^2 = (1 + \sqrt{2 - \lambda})^3(3 - \sqrt{2 - \lambda}).$$  (20)

See in Figure[2] the corresponding 3D illustration of the classical black hole shadow formation in the extreme Kerr black hole case ($a = 1$). As in Figure[1] the luminous background is placed behind the black hole at the distance, exceeding the size of all photon spheres. A closed purple region is an envelope of all photon spheres for photons with $\lambda < 0$ and, respectively, a closed green region is an envelope of all photon spheres for photons with $\lambda > 0$). Multicolored curves in this Figure are the examples of photon trajectories, producing an outline (boundary) of the classical black hole shadow, with orbital parameters, respectively, $$(\lambda, q) = (2, 0)$$ — corotating skimming photon in the equatorial plane, $$(\lambda, q) = (-7, 0)$$ — counter-rotating photon in the equatorial plane and $$(\lambda, q) = (0, \sqrt{11 + 8 \sqrt{2}})$$ — fully spherical photon orbit. A blue disk with a radius $r = 1$ inside the black hole shadow is the observed position of the black hole event horizon in the imaginary Euclidean space (in the absence of gravity). A magenta arrow is the black hole rotation axes.

The gravitational lensing by black holes provides, in general, the infinite number of images \[262, 266\]. Christopher Cunningham and James Bardeen elaborated the very usable classification scheme for multiple images (or light echoes) \[262, 263\], based on the number of intersections the black hole equatorial plane by photon on its way from the initial emission point to a distant observer. An astrophysical example of the stationary luminous background is shown in Figure[3] demonstrating direct images and also the first and second light echoes of the lensed images of a compact star (luminous probe) in discrete times at the circular equatorial orbit with radius $r = 20$ around a near extreme black hole. The orbital radius of this star exceeds the corresponding photon spheres. Therefore, this star plays a role of the distant stationary background and all its multiple images are placed outside the classical black hole shadow (see details in \[267\] and animation of numerical modeling in \[268\]).
FIG. 4. The radii of photon spheres depend on one of the photon orbit parameters, \( \lambda \) or \( q \). At \( a = 1 \) the photon spheres (red curve) are placed at radii \( r_{ph} = 1 + \sqrt{2 - \lambda} \), \( \sqrt{2} = (1 + \sqrt{2 - \lambda})(3 - \sqrt{2 - \lambda}) \). The corresponding photon spheres exist at the radial interval \( 1 \leq r_{ph} \leq 4 \). At \( a = 0 \) the photon sphere is reduced to the photon circle with radius (green semicircle) with \( \lambda^2 + q^2 = 27 \). A blue curve corresponds to the radii of photon spheres at \( a = 0.6 \).

III. PHOTON SPHERES

Among the others, a striking feature of the Kerr metric is the existence of relativistic spherical orbits for massive and massless particles moving on the sphere \( r = \text{const} \) and oscillating in polar (latitude) direction between the turning points. From differential equations of motion in the Kerr metric (5)–(8) it follows that parameters of spherical orbits are defined by the common solutions of equations \( R(r) = dR(r)/dr = 0 \), where the radial effective potential \( R(r) \) is from (9). Correspondingly, the polar turning points \( \theta_{\min} \) and \( \theta_{\max} = \pi - \theta_{\min} \) are defined by zeros of effective polar potential \( \Theta(\theta) \) from (10). Spherical orbits were described in the pioneering work by Daniel Wilkins [168] (see also [269–275]). The spherical orbits are reduced to the circular ones in the limiting case of equatorial orbits with \( q = 0 \) [160].

The spherical orbits of photons are naturally named as “photon spheres”. The outline (contour) of classical black hole shadow is defined namely by photon spheres according to expressions (18).

Figure 4 shows the radii of photon spheres, depending on one of the photon orbit parameters \( \lambda \) or \( q \) according to expressions (18). At \( a = 1 \) the photon spheres (red curve) are placed at radii \( r_{ph} = 1 + \sqrt{2 - \lambda} \) at the radial interval \( 1 \leq r_{ph} \leq 4 \). At \( a = 0 \) the photon sphere are reduced to photon circles with a radius \( r_{ph} = 3 \) (green semicircle) with \( \lambda^2 + q^2 = 27 \). A blue curve corresponds to the radii of photon spheres at \( a = 0.6 \).

A turning point in polar direction \( \theta_{\min} \) on the spherical photon trajectory defined by the condition \( \Theta(\theta) = 0 \). According to the Cunningham–Bardeen classification scheme of the multiple lensed images [262, 263], the photons, providing the prime image of the emitting source, do not intersect the black hole equatorial plane on their way from the source to a distant observer.

In the Schwarzschild black hole case \( (a = 0) \) the corresponding turning point \( \theta_{\min} = \arccos(q/(3 \sqrt{3}) \). In the Kerr case \( (a \neq 0) \) the polar turning point (if it exists) is placed at

\[
\cos^2 \theta_{\min} = \frac{4a^2q^2 + (q^2 + \lambda^2 - a^2)^2 - (q^2 + \lambda^2 - a^2)^2}{2a^2}.
\]

This expression for \( \theta_{\min} \) is used in the numerical solution of integral equations (15)–(17).

Figure 5 shows some 3D examples of photon spheres (spherical photon trajectories) around the extreme Kerr black hole. These photon oscillates in polar direction between \( \theta_{\min} \) and \( \theta_{\max} = \pi - \theta_{\min} \). The multicolored curves in this Figure are the examples of spherical photon trajectories with orbital parameters, respectively, \( (r, \lambda, q) = (1, 2, 0) \) — co-rotating photon in the equatorial plane, \( (r, \lambda, q) = (1 + \sqrt{2}, 0, \sqrt{11 + 8 \sqrt{2}}) \) — fully spherical photon orbit,
FIG. 5. Examples of photon spheres in the case of the extreme Kerr black hole \((a = 1)\). The multicolored curves correspond to spherical photon trajectories with orbital parameters, respectively, \((r, \lambda, q) = (1, 2, 0)\) — co-rotating photon in the equatorial plane, \\
\((r, \lambda, q) = (1 + \sqrt{2}, 0, \sqrt{11 + 8 \sqrt{2}})\) — fully spherical photon orbit, \((r, \lambda, q) = (1 + 2 \sqrt{2}, -6, \sqrt{16 \sqrt{2} - 13})\) and \((r, \lambda, q) = (4, -7, 0)\) — counter-rotating photon in the equatorial plane.

\((r, \lambda, q) = (1 + 2 \sqrt{2}, -6, \sqrt{16 \sqrt{2} - 13})\) and \((r, \lambda, q) = (4, -7, 0)\) — counter-rotating photon in the equatorial plane.

Figure 6 shows 3D green regions, which are the envelopes of all photon spheres with \(\lambda \geq 0\) in the case of Kerr black holes with spin \(a = 1\) and \(a = 0.95\). In the extreme black hole case with \(a = 1\) the green part of the event horizon globe is a region for the very specific photon spheres, which are called “the skimming photons” [168]. The orbit parameters of skimming photons are \(\lambda = 2\) and \(0 \leq q \leq \sqrt{3}\). These skimming photons move both in the azimuth and latitude direction on the sphere with radius \(r = 1\) by oscillating in polar directions with \(0 \leq \cos[\theta_{\min}] \leq \sqrt{2} \sqrt{3} - 3\), as it follows from equations (19) and (21).

IV. EVENT HORIZON SILHOUETTE: BLACK HOLE HIGHLIGHTING BY ACCRETION DISK

In the general case of a static distant observer, placed at the given radius \(r_0 \gg r_h\) (e.g., practically at the space infinity), at the given polar angle \(\theta_0\) and at the given azimuth \(\phi_0\), it must be used the horizontal impact parameter \(\alpha\) and vertical impact parameter \(\beta\) on the celestial sphere (see details in [102, 262, 263]):

\[
\alpha = -\frac{\lambda}{\sin \theta_0}, \quad \beta = \pm \sqrt{\Theta(\theta_0)},
\]

where the effective polar potential \(\Theta(\theta)\) is from Equation (10).

In the simplest case of the spherically symmetric Schwarzschild black hole \((a = 0)\) the boundary of the event horizon image (the boundary of the dark event horizon silhouette), viewed by a distant observer (which is placed at \(\theta_0 = \pi/2\)), is defined by solution of the integral equation

\[
\int_2^\infty \frac{dr}{\sqrt{R(r)}} = 2 \int_{\theta_{\min}}^{\pi/2} \frac{d\theta}{\sqrt{\Theta(\theta)}},
\]
FIG. 6. Envelopes of photon spheres with $\lambda \geq 0$ (closed green regions) in the case of Kerr black holes with spin $a = 1$ (left panel) and $a = 0.95$ (right panel). In the case of extreme Kerr black hole ($a = 1$) there are skimming photons, which are moving both in the azimuth and latitude directions on the sphere with radius $r = 1$ by oscillating in polar directions with $0 \leq \cos[\theta_{\text{min}}] \leq 2 \sqrt{3} - 3$.

where $\theta_{\text{min}}$ is a turning point in polar direction on the photon trajectory for the direct image of the small accreting fragment (probe), defined by the condition $\Theta(\theta) = 0$. According to the Cunningham–Bardeen classification scheme of the multiple lensed images \cite{262, 263}, the photons, providing the prime image of the emitting source do not intersect the black hole equatorial plane on their way from the source to a distant observer. The event horizon radius of the Schwarzschild black hole is $r_h = 2$, and turning point $\theta_{\text{min}} = \arccos(q/\sqrt{q^2 + \lambda^2})$. An integral in the right-hand-side of equation \cite{23} in this case is equal $\pi/\sqrt{q^2 + \lambda^2}$. In result, the numerical solution of the integral equation \cite{23} gives for the radius of the event horizon image (silhouette) the value $r_{\text{eh}} = \sqrt{q^2 + \lambda^2} \approx 4.457$. This radius is notably smaller than the corresponding radius of the black hole shadow $r_{\text{sh}} = 3 \sqrt{3} \approx 5.2$.

The supermassive black hole SgrA* at the center of the Milky Way galaxy has a mass $M = (4.3 \pm 0.3) \times 10^6 M_\odot$, i.e., three orders of magnitude less than in the case of M87*, but at the same time the black hole SgrA* is placed three orders of magnitude closer than the black hole M87*. So the event horizons of these two black holes have approximately the same angular sizes accessible for observations by the EHT. Rotation axis orientation of the black hole SgrA* most probably coincides with the rotation axis of the Milky Way galaxy \cite{276}. For concreteness, we suppose that distant observer is placed near the equatorial plane of the black hole SgrA* at $\cos \theta_0 = 0.1$ or $\theta_0 \approx 84.24^\circ$.

In this paper we use the (motivated by astrophysics) model of the black hole illuminated (highlighted) by a thin accretion disk in the black equatorial plane. In this model the outline (contour) of the dark event horizon silhouette (or, simply, the black hole silhouette) is defined by the highly red-shifted photons, emitted near the black hole event horizon by the thin accreting disk and registered by a distant observer. We calculate numerically the form of the black hole silhouette, which does not depend on the emission pattern of the thin accretion disk but completely determined by the black hole gravitational field. It appears that the form of the dark event horizon silhouette in this model is in good agreement with the form of the dark spot on the image of the supermassive black hole M87* obtained by the
FIG. 7. A visible dark silhouette of the northern hemisphere of event horizon (black region) illuminated by a thin accretion disk in the equatorial plane of the black hole with the spin $a = 0.9982$, corresponding to the orientation of supermassive black hole SgrA* with respect to a distant observer. The outline (contour) of this silhouette is defined by the highly red-shifted photons, emitted near the black hole event horizon by the thin accreting disk and registered by a distant observer. It is shown the photon trajectory (multicolored 3D curve), producing the north pole point on the outline of the event horizon dark silhouette.

EHT collaboration.

Figure 7 from [277] shows a visible dark silhouette of the northern hemisphere of the black hole event horizon illuminated by a thin accretion disk in the equatorial plane of the black hole with the spin $a = 0.9982$, corresponding to the orientation of the supermassive black hole SgrA*. It is shown the typical photon trajectory (multicolored 3D curve), with parameters $\lambda = 0.063$ and $q = 0.121$, emitted by the hot accreting matter in the black hole equatorial plane at the radius $r = 1.01 r_h$ and reaching a distant observer near the external boundary (contour) of the dark silhouette of the northern hemisphere of the event horizon globe.

It is necessary to note that the motion of the accreting matter in the region inside the photon spheres and adjoining the event horizon is non-stationary. The accreting matter in this region is falling into the black hole along the spiraling down trajectories (see more details in [244, 245, 277]).

Figure 8 shows the possible forms of the dark event horizon silhouette (event horizon image) of the supermassive
FIG. 8. The dark silhouettes of the northern hemisphere of the event horizon (black region) in the case of supermassive black hole SgrA* ($\theta_0 = 17^\circ$), projected inside the boundary of classical black hole shadow (closed purple curves) for the values of black hole spin, respectively, $a = 0.9982$ (left), 0.65 (middle) and 0 (right).

FIG. 9. Two 3D photon trajectories, starting from the different points of the circle with a radius $r = 1.01 r_h$ at the thin accretion disk (orange oval) in the equatorial plane of rotating black hole with the spin $a = 0.9982$ and reaching a distant observer near the outer contour of the event horizon silhouette (dark gray region). The closed dark red curve is a projection of the classical black hole shadow contour on the celestial sphere.
black hole SgrA* for three values of the black hole spin \(a\). Note, that in the case of rotation axis orientation of the black hole SgrA* relative to a distant observer, the contour of the northern hemisphere of the lensed event horizon globe (black region) projected inside the boundary of classical black hole shadow (closed purple region).

A space orientation of the supermassive black hole M87* and its equatorial accretion disk relative to a distant observer at the Earth (or at the near-Earth space orbit) is shown in Figure 9. The dashed magenta circle in this Figure and in all other similar ones corresponds to the black hole event horizon image in the imaginary Euclidean space. There are also shown two 3D photon trajectories, starting from the different points of the circle with a radius \(r = 1.01 \, r_h\) at the thin accretion disk (light green oval) in the equatorial plane of rotating black hole with the spin \(a = 0.9982\) and reaching a distant observer near the outer contour of the event horizon silhouette (dark gray region). Parameters of these two photon trajectories are \(\lambda_1 = -0.047\) and \(q_1 = 2.19\) and, respectively, \(\lambda_2 = -0.029\) and \(q_2 = 1.52\).

The corresponding forms of the dark silhouette of the supermassive black holes SgrA* and M87*, highlighted by thin accretion disks, are shown, respectively in Figures 8 and 10 for three different values of spin \(a\).

Note, that the dark event horizon silhouettes, similar to ones in Figures 7–10 were reproduced during many years in numerical modeling of accretion disks with the inner edge at the black hole event horizon (see, e. g., [292, 293]).

Figure 11 demonstrates a numerical model for the gravitational lensing of compact star, falling into the fast rotating black hole SgrA* (\(a = 0.9982\)) and observed in discrete time intervals by a distant static observer, placed a little bit above the equatorial plane. Falling star has a zero azimuth angular momentum and moves in the black hole equatorial plane. Images of this star are projected on the celestial sphere inside the classical black hole shadow (a big closed light purple region), when this star is approaching the black hole event horizon, and then are multiply winding up very near to the black hole equatorial parallel \(\theta = \pi/2\) on the lensed event horizon globe. It is shown the first circle of this multiple winding. The brightness of the lensed star is exponentially faded in time during successive windings (see animation in [291]). The closed blue curves are meridians and parallels on the reconstructed image of the lensed event horizon globe (for details see [244, 245, 277]).

Finally, Figure 12 shows the superposition of the image of M87*, obtained by the EHT with both the contours of classical black hole shadows (purple closed curves) and the event horizon silhouettes (dark regions) for different values of the black hole spin, \(a = 0.998\) (left panel), \(a = 0.75\) (central panel) and \(a = 0\) (right panel). A white dashed circle with a radius 21 \(\mu\)as corresponds to \(\approx 5.5MG/c^2\). This circle was used by the EHT collaboration to model a bright crescent for the reconstruction of the M87* image [1]. The angular size of the M87* gravitational radius \(MG/c^2\) corresponds to \(\theta = 3.8 \, \mu\)as. The modeled dark spot on the right panel in the \(a = 0\) case produces an excessively large dark spot in comparison with one on the EHT image. We note that a size of the dark spot on the EHT image agrees with a corresponding size of the dark event horizon silhouette in a thin accretion disk model in the case of either the high or moderate value of the black hole spin, \(a \geq 0.75\). Meantime, the EHT image details are blurred due to the observation resolution, and the direct identification of the edge of the bright crescent with the shadow or the silhouette can easily be misleading.

In the Figure 12 a position angle of the large-scale jet \(PA = 288^\circ\) and the viewing angle between the jet axis and line-of-sight \(\theta_0 = 17^\circ\) according to [292, 294]. We are especially grateful the authors of paper [293] for pointing out the wrong angle of the large-scale jet from M87* \(PA = 215^\circ\), which we used for superposition of the EHT image of supermassive black hole M87* with the simulated event horizon silhouettes [294]. All model images, projected on the celestial sphere, must be rotated counter-clockwise to the angle 73° to correct this error in [294]. Additionally,
FIG. 11. Numerical modeling of the gravitational lensing of the compact star, falling into the fast rotating black hole SgrA* ($a = 0.9982$) and observed in discrete time intervals by a distant static observer, placed a little bit above the equatorial plane. The falling star has a zero azimuth angular momentum and moves in the black hole equatorial plane. An yellow curve is a viewed trajectory of this star. The images of this star are projected on the celestial sphere inside the classical black hole shadow (a big closed light purple region), when this star is approaching the black hole event horizon, and then start to multiply winding up around the black hole very near to the black hole equatorial parallel $\theta = \pi/2$ on the lensed event horizon globe. It shown the first circle of this multiple winding. A brightness of the lensed star is exponentially faded in time during successive windings. The closed blue curves are meridians and parallels on the reconstructed image of the lensed event horizon globe.

our claim in [294] on the preferable value of spin $a = 0.75$ for M87* is dubious. It seems that the dark silhouette of the event horizon on the image of M87* is either heavily shaded or accretion disk around this black hole is not thin.

V. DISCUSSIONS

The classical black hole shadow is visible if the emission of luminous matter outside the photonic spheres dominates (e.g., if there is a distant luminous background of extended hot gas clouds or luminous stars far beyond the black hole). On the contrary, the much smaller event horizon silhouette is visible if the emission of luminous matter within the photonic spheres dominates (e.g., if there is a highly luminous accreting matter in the vicinity of the event horizon).

The classical black hole shadow is complicated to observe with the present state of art either due to the low luminosity of the distant background far behind the black hole or due to the extremely high accretion activity of the black hole, which completely dilute the emission from distant background. The basic necessary requirement for observation of the black hole silhouette is, of course, the stability of the basic image structure among different days
FIG. 12. Superposition of the M87* image, obtained by the EHT, with both the contours of classical black hole shadows (purple closed curves) and the event horizon silhouettes (dark regions) for different values of the black hole spin, $a = 0.998$ (left panel), $a = 0.75$ (central panel) and $a = 0$ (right panel). Magenta arrows — a black hole rotation axis. Small dashed ring — a black hole event horizon projection on the celestial sphere in the imaginary Euclidean space (in the absence of gravity). The modeled dark spot on the right panel in the $a = 0$ case produces an excessively large dark spot in comparison with one on the EHT image. At the same time, a size of the dark spot on the EHT image agrees with a corresponding size of the dark event horizon silhouette in a thin accretion disk model in the case of either the high (left panel) or moderate (central panel) value of the black hole spin, $a \gtrsim 0.75$.

of observations according to [1].

Photons, emitted near the event horizon by the luminous matter falling into the black hole, undergo extremely high red-shift by reaching a distant observer. Therefore, the registration accuracy of the event horizon silhouette strongly depends on the angular resolution and sensitivity of the used telescope.

A size of the dark spot on the EHT image agrees with a corresponding size of the dark event horizon silhouette in a thin accretion disk model (see Figure 12) in the case of either the high or moderate value of the black hole spin, $a \gtrsim 0.75$. The form of the dark event horizon silhouette does not depend on the emission pattern of the thin accretion disk but completely determined by the Kerr black hole gravitational field.

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