The heat kernel of the asymmetric quantum Rabi model

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Abstract

In this paper we derive an explicit formula for the heat kernel of the asymmetric quantum Rabi model, a symmetry breaking generalization of the quantum Rabi model (QRM). The method described here is the extension of a recently developed method for the heat kernel of the QRM that uses the Trotter–Kato product formula instead of path integrals or stochastic methods. In addition to the heat kernel formula, we give applications including the explicit formula for the partition function and the Weyl law for the distribution of the eigenvalues, obtained from the corresponding spectral zeta function.

Keywords: asymmetric quantum Rabi model, heat kernel, partition function, spectral zeta

1. Introduction

The quantum Rabi model (QRM) is widely recognized as one of the fundamental models in quantum optics and the study of its properties has been in the spotlight of theoretical and experimental physics for a number of years [1–4]. One of the motivations is the potential application of the QRM to the development quantum computation and quantum information sciences. In parallel, there is a growing interest in the mathematical study of the properties of the QRM, including the study of solutions and dynamics [5, 6], large spectral asymptotics [7], symmetry and degeneracy for asymmetric QRMs [8, 9], entanglement properties [10], Floquet analysis [11], spectral zeta functions [12, 13] and algebro-geometric analysis [14, 15].

In [16] the explicit (or analytical) formula for the heat kernel of the QRM was obtained based on the Trotter–Kato product formula. The derivation of the formula involves the use of several techniques not common to this type of computation such as the use of harmonic analysis on finite groups or an extensive combinatorial (graph theoretical) analysis (see section 4 for an overview of the method). The resulting heat kernel formula is given as a power series with coefficients consisting of multiple integrals, which are interpreted as orbits of the action of
the infinite symmetric group on the inductive limit of the groups $\mathbb{Z}_2^n$ ($n \geq 0$), or as a as type of discrete path integral [13]. We note that conventional approaches based on Feynman integrals or Feynman–Kac formulas have not produced fully explicit formulas for the heat kernel of the QRM.

A careful examination of the method in [16] reveals that it may be generalized to systems other than the QRM Hamiltonian. Therefore, the understanding of the scope and limitations of the method is a significant topic of research both in theoretical physics and mathematical. In this paper, as an starting point for this program, we extend the computation of the heat kernel to one of the simplest, yet significant, generalizations of the QRM, the asymmetric quantum Rabi model (AQRM). The asymmetric model is considered to provide a more realistic description of the circuit QED experiments employing flux qubits in comparison with the QRM, and since it is ubiquitous in modern solid devices, the study of its spectrum is both physically and mathematically relevant [17–19].

The Hamiltonian of the AQRM is given by

$$H_R^{\varepsilon} = \omega a^\dagger a + \Delta \sigma_z + g \left( a + a^\dagger \right) \sigma_x + \varepsilon \sigma_y,$$

(1)

where, as usual,

$$\sigma_z = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

(2)

are the Pauli matrices, $a^\dagger$ and $a$ are the creation and annihilation operators of the bosonic mode with frequency $\omega$ (in this paper we set $\omega = 1$), i.e. $[a, a^\dagger] = 1$, $\varepsilon \in \mathbb{R}$ and $\Delta, g > 0$. The QRM Hamiltonian is recovered by taking $\varepsilon = 0$.

The AQRM was introduced as a ‘symmetry breaking’ generalization of the QRM in [2], where the proof of the exact solvability of both models was presented. We remark that it is the $\mathbb{Z}_2$-symmetry in the QRM Hamiltonian that allows the presence of degeneracies in the spectrum in a natural way. In fact, the presence of the bias parameter $\varepsilon \neq 0$ breaks the natural symmetry of the QRM and makes the spectrum of the AQRM multiplicity free. Indeed, for the AQRM there is no obvious way to define subspaces whose respective spectral graphs intersect creating degeneracies in the spectrum of the AQRM. It was shown in [8] (based on previous works [20, 21]) that degeneracies may appear for certain choices of parameters $g$ and $\Delta$ when the parameter $\varepsilon$ is half-integer. In [22] the symmetry operator for small values of half-integer $\varepsilon$ was obtained explicitly along with a general method of computation for the arbitrary half-integer $\varepsilon$ case (see also [23]). A more systematic approach to the half-integer case is given in [15, 24].

Recall that heat kernel $K_R^{(\varepsilon)}(x, y, t)$ of the AQRM is the integral kernel corresponding to the operator $e^{-itH_R^{\varepsilon}}$ (one-parameter semigroup), that is, $K_R^{(\varepsilon)}(x, y, t)$ satisfies

$$e^{-itH_R^{\varepsilon}} \phi(x) = \int_{-\infty}^{\infty} K_R^{(\varepsilon)}(x, y, t) \phi(y) \ dy$$

(3)

for a compactly supported smooth function $\phi : \mathbb{R} \to \mathbb{C}^2$. Equivalently, $K_R^{(\varepsilon)}(x, y, t)$ is the matrix-valued function satisfying the heat equation

$$\frac{\partial}{\partial t} K_R^{(\varepsilon)}(x, y, t) = -H_R^{\varepsilon} K_R^{(\varepsilon)}(x, y, t)$$

(4)

for $t > 0$ and $\lim_{t \to 0} K_R^{(\varepsilon)}(x, y, t) = \delta(y - x)I_2$ for $x, y \in \mathbb{R}$. It is worth noting that the heat kernel actually controls the time evolution for the AQRM since the change of variable $t \mapsto it$ in (4) gives the time propagator, that is, the integral kernel for the fundamental solution of the Schrödinger equation corresponding to (1).
The formula developed in this paper (formula (6) in theorem 2) shows that the heat kernel of the AQRM is also given as uniformly convergent series on the parameter $\Delta$ with coefficients consisting on multiple integrals. Remarkably, the factors corresponding to the bias parameter $\varepsilon$ on each coefficient in (6) appear independent of the system parameters $g, \Delta$.

The main contribution of this paper is the development of the theoretical tools for the generalization of the method of computation for the heat kernel of the AQRM. In particular, we reformulate the harmonic analysis on finite groups to resolve certain difficulties that arise since the non-commutative part is no longer a diagonal matrix (see section 4.2). On the other hand, the scalar part of the computation remain largely the same as the case of the QRM, so in order to keep the exposition short and to avoid repeating the computation of the heat kernel, we focus on the new features and we refer the reader to [16] for some of the details. The difficulties that appear in the computation of the heat kernel of the AQRM actually appear in the case of more general models, making the result of this paper a significant step towards the full understanding and generalization of the method (see remark 4.1 for discussion to the case of the Dicke model).

Two applications of the heat kernel formula are the formula for the time propagator, which is obtained by meromorphic continuation of the heat kernel to the imaginary line, and the explicit formula for the partition function of the AQRM (corollary 2.3). The propagator formula is expected to provide more robust computations for time evolution for the AQRM (see [25]), but explicit numerical studies are yet to be performed.

In addition, the explicit formulas of the heat kernel and partition function of the AQRM open the way for the study of the spectrum of the AQRM using spectral zeta function methods. The general idea behind spectral zeta functions is that the eigenvalues of a system may be approached more easily by studying certain symmetric functions defined on them [26, 27]. The spectral zeta function of a physical model also allows the study of the spectrum from the viewpoint of number theory with applications both to number theory and physics, including zeta regularization and the computation of Casimir energy [28].

The (Hurwitz-type) spectral zeta function $\zeta^{(c)}(s; \tau)$ of the AQRM is given by the Dirichlet series

$$\zeta^{(c)}_{R}(s; \tau) = \sum_{j=1}^{\infty} \left( \lambda^{(c)}_{j} + \tau \right)^{-s},$$

for $\Re(s) > 1$. Here, $\lambda^{(c)}_{j}$ are the (ordered) eigenvalues of (1). In this paper, the meromorphic continuation is obtained by identifying the spectral zeta function of the AQRM as the Mellin transform of the partition function of the AQRM and then changing the path of integration in an appropriate way.

One of the hallmark applications of spectral zeta functions methods is the Weyl law for the distribution of the eigenvalues of the AQRM. Concretely, the Weyl law describes the asymptotics of the number of eigenvalues of a system that are smaller than a certain bound. In the case of the AQRM, we prove the Weyl law and show that it does not depend on the system parameters, including the bias parameter $\varepsilon$ (see theorem 3.3).

In [8] it was shown that the $G$-function of the AQRM is essentially given by the spectral determinant, a generalization of the characteristic polynomial of a matrix. In particular, this shows that there is a fundamental relation between the spectral zeta function and the $G$-function that describes the analytic solvability of the AQRM. The proof given in [8] was under the assumption of the meromorphic continuation of the spectral zeta function of the AQRM (by the method of [12]), which was not yet proved at the time. We complete the proof in this paper

$$\zeta^{(c)}_{R}(s; \tau) = \sum_{j=1}^{\infty} \left( \lambda^{(c)}_{j} + \tau \right)^{-s},$$
by giving the meromorphic continuation of (5) using the explicit formula for the partition function (see theorem 3.2).

Let us describe the structure of this paper. First, in section 2 we present the main results of the paper, that is, the explicit formulas for the heat kernel and partition function of the AQRM. In section 3 we study the spectral zeta function and its properties, including the meromorphic continuation and the Weyl law. The rest of the paper is devoted to the proof of the heat kernel formula. In section 4 we give a summary of the general method of computation. In section 4.1 we give some general remarks on the AQRM and make the initial computations with the Trotter-Kato product formula to obtain a limit formula that resembles a Riemann sum. Next, in section 4.2 using Fourier analysis in the family of finite groups $\mathbb{Z}_N$ we transform certain sums in the limit into multiple Riemann integrals allowing us to obtain a second limit expression that can be evaluated as a Riemann integral. The evaluation of the Riemann sum completes the proof of the explicit formula for the heat kernel and the partition function is obtained directly as a corollary.

2. Main results

In this section we give the main results of this paper, the explicit formulas for the heat kernel and the partition function for the AQRM. The proof of the formulas is the main contribution of this paper and are given in detail in section 4.

Theorem 2.1. The heat kernel $K^z_{\bar{r}}(x,y,t)$ of the AQRM is given by the uniformly convergent series

$$K^z_{\bar{r}}(x,y,t) = K_0(x,y,g,t) \left[ \sum_{\lambda=0}^{\infty} (t\Delta)^\lambda e^{-2\xi \left( \cosh \left( \frac{t}{\lambda} \right) \right) (-1)^{1+\lambda}} \right. $$

$$\times \int_{0\leq \mu_1 \leq \cdots \leq \mu_\lambda \leq 1} \exp \left( 4g^2 \frac{\cosh(t(1-\mu_\lambda))}{\sinh(t)} \left( \frac{1+(-1)^{\lambda}}{2} \right) + \xi_\lambda(\mu_\lambda,t) \right) $$

$$\times \left[ (-1)^{\lambda} \cosh \left( \frac{1}{-\sinh} \right) \right] \left[ \theta_\lambda(x,y,\mu_\lambda,t) + \varepsilon(\eta_\lambda(\mu_\lambda,t) + t) \right] d\mu_\lambda \right],$$

with $\mu_0 = 0$, $\mu_\lambda = (\mu_1, \mu_2, \ldots, \mu_\lambda)$ the integration variables and $d\mu_\lambda = d\mu_1 d\mu_2 \ldots d\mu_\lambda$ for $\lambda \geq 1$. Here,

$$K_0(x,y,g,t) = \frac{e^{\nu t}}{\sqrt{\pi} \left( 1 - e^{-2t} \right)} \exp \left( - \frac{1 + e^{-2t}}{2(1 - e^{-2t})} (x^2 + y^2) + \frac{2e^{-t}xy}{1 - e^{-2t}} \right)$$

and the functions $\theta_\lambda(x,y,\mu_\lambda,t)$, $\xi_\lambda(\mu_\lambda,t)$ and $\eta_\lambda(\mu_\lambda,t)$ are given by

$$\theta_\lambda(x,y,\mu_\lambda,t) = \frac{2\sqrt{2} e^{\nu t}}{1 - e^{-2t}} \left( x e^{-1} + e^{-1} - 2y \right) \left( \frac{1 - (-1)^\lambda}{2} \right) - \sqrt{2} g(x-y) \frac{1 + e^{-t}}{1 - e^{-2t}}$$

$$+ \frac{2\sqrt{2} e^{\nu t}}{1 - e^{-2t}} (-1)^\lambda \sum_{\gamma=0}^{\lambda} \left( (-1)^\gamma [x(e^{\gamma(1-\mu_\gamma)} + e^{\mu_\gamma - 1})] - y(e^{-\mu_\gamma} + e^{\mu_\gamma}) \right)$$

\[\quad\]
\[ \xi_{\lambda}(\mu_{\lambda}, t) = \frac{2g^2 e^{-t}}{1 - e^{-2t}} \left(e^{\frac{1}{2}(1 - \mu_{\lambda})} - e^{\frac{1}{2}\mu_{\lambda}(1 - 1)}\right)^2 (-1)^{\lambda} \sum_{\gamma=0}^{\lambda} (-1)^{\gamma} (e^{-\mu_{\gamma}} + e^{\mu_{\gamma}}) \]
\[ - \frac{2g^2 e^{-t}}{1 - e^{-2t}} \sum_{0 \leq \alpha < \beta \leq \lambda - 1} \left((e^{(\mu_{\beta} + 1)} + e^{\mu_{\beta} + 1}) - (e^{(\mu_{\beta})} + e^{(\mu_{\beta} + 1)})\right) \]
\[ \times \left((e^{\mu_{\alpha}} + e^{-\mu_{\alpha}}) - (e^{\mu_{\alpha} + 1} + e^{-\mu_{\alpha} + 1})\right), \]
\[ \eta_{\lambda}(\mu_{\lambda}, t) = -2t(-1)^{\lambda} \sum_{\gamma=1}^{\lambda} (-1)^{\gamma} \mu_{\gamma}, \] \tag{8}

where we use the convention \( \mu_0 = 0 \) whenever it appears in the formulas above.

In the theorem above, for \( \lambda = 0 \), to unify the notation we use the formal expression
\[ \int \cdots \int f(\mu_0) \, d\mu_0 = f(x), \] \tag{9}
for any function \( f \).

We note that since the action of the parameter \( \varepsilon \) in the Hamiltonian (1) is just the displacement \( \varepsilon \sigma_z \), in the heat kernel formula (6) its contribution (that is, the function \( \eta_{\lambda}(\mu_{\lambda}, t) \) in (8)) appears linearly inside the exponential.

It is also worth noting that in the coefficients of (6) and in the functions (8) the system parameters \( g, \Delta, \varepsilon \) do not appear mixed. In particular, setting \( \varepsilon = 0 \) we recover immediately the formula for the heat kernel of the QRM given in [16].

Directly from the analytical formula (6) we verify that the heat kernel is well-behaved with respect to the spatial variables. The proof of the following proposition may be adapted directly from that of the QRM, given in [13].

**Proposition 2.2.** Let \( K_R^{(\varepsilon)}(x,y,t) \) be written in matrix form, that is,
\[ K_R^{(\varepsilon)}(x,y,t) = \begin{pmatrix} k_{11}(x,y,t; g, \Delta, \varepsilon) & k_{12}(x,y,t; g, \Delta, \varepsilon) \\ k_{21}(x,y,t; g, \Delta, \varepsilon) & k_{22}(x,y,t; g, \Delta, \varepsilon) \end{pmatrix}. \] \tag{10}
Then, for fixed \( g, \Delta, t > 0 \) and \( \varepsilon \in \mathbb{R} \), there are positive constants \( a, b \) such that
\[ |k_{i,j}(x,y,t; g, \Delta, \varepsilon)| \leq ae^{-b(x^2 + y^2)}, \] \tag{11}
for \( i, j \in \{1, 2\} \).

Proposition 2.2 shows that \( K_R^{(\varepsilon)}(x,y,t) \) is a continuous function with respect to the spatial variables \( x, y \). Similar results may be obtained with respect to the time variable \( t \) and the system parameters.

Another application of the formula (6) is the formula for the integral kernel of the time-evolution operator (i.e. the time or wave propagator) of the AQRM. It is enough to consider the meromorphic continuation and the change of variable \( t \to it \) in (6). We refer the reader to [13] for the technical details for the case of the QRM which can be easily extended to cover the AQRM (see also proposition 3.1 below).

### 2.1. Partition function

Next, we give the explicit formula of the partition function of the AQRM. While we do not use the physical interpretation in this paper, we recall that, in general, the partition function of
a system describes the statistical properties of the system in thermodynamic equilibrium as a function of temperature and other parameters.

First, we denote by

$$\lambda_1^{(c)} < \lambda_2^{(c)} < \lambda_3^{(c)} < \ldots < \lambda_n^{(c)} < \ldots \ (\rightarrow \infty)$$

(12)

the eigenvalues of the AQRM. Then, the partition function $Z^{(c)}_R(\beta)$ is defined by

$$Z^{(c)}_R(\beta) = \sum_{n=1}^{\infty} e^{-\beta \lambda_n^{(c)}} = \text{Tr} \left[e^{-\beta H^c}\right].$$

(13)

where $\text{Tr}$ denotes the operator trace.

The explicit formula for the partition function then is obtained directly from (6) in an elementary way. The proof is given in section 4.2.

**Corollary 2.3.** The partition function $Z^{(c)}_R(\beta)$ of the AQRM is given by

$$Z^{(c)}_R(\beta) = \frac{2e^{2\beta}}{1-e^{-\beta}} \left[ \text{ch}(\epsilon \beta) \sum_{\lambda=1}^{\infty} (\beta \Delta)^{2\lambda} \prod_{0 \leq \mu_1 \leq \ldots \leq \mu_{2\lambda} \leq 1} \Theta_{2\lambda}(g, \beta, \mu_{2\lambda}) \text{ch} \left(\epsilon \beta \left(1 - 2 \sum_{\gamma=1}^{2\lambda} (-1)^\gamma \mu_\gamma\right)\right) \right] \mu_{2\lambda},$$

(14)

where

$$\Theta_{2\lambda}(g, \beta, \mu_{2\lambda}) = \exp \left(-2g^2 \coth \left(\frac{\beta}{2}\right) + 4g^2 \frac{\text{sh}(\beta (1 - \mu_{2\lambda}))}{\text{sh}(\beta)} \right) + \xi_{2\lambda}(\mu_{2\lambda}, \beta) + \psi_{2\lambda}^- (\mu_{2\lambda}, \beta)$$

(15)

with

$$\psi_{2\lambda}^- (\mu_{2\lambda}, t) = \frac{4g^2}{\text{sh}(t)} \left[ \sum_{\gamma=0}^{\lambda} (-1)^\gamma \text{sh} \left(t \left(\frac{1}{2} - \mu_\gamma\right)\right) \right]^2$$

(16)

for $\lambda \geq 1$ and $\mu_{2\lambda} = (\mu_1, \mu_2, \ldots, \mu_\lambda)$ and where $\mu_0 = 0$.

In particular, from the explicit formula (14) it is immediate to verify that

$$Z_{R}^{(\epsilon^c)}(\beta) = Z_{R}^{(c)}(\beta),$$

(17)

reflecting the well-known fact that the spectrum of $H^\epsilon_R$ is equal to the spectrum of $H^{\epsilon^c}_R$ (see e.g. proposition 5.2 of [8]).

In the next section we give applications of the explicit formulas of the partition function by means of the spectral zeta function associated to the AQRM.

### 3. Applications to the spectrum of the AQRM via spectral zeta functions

In this section, we prove the meromorphic continuation for the spectral zeta function of the AQRM necessary for the applications to the analysis of the spectrum. In mathematical physics,
zeta function methods have been used to obtain significant results including the computation of Casimir energy (Casimir effect), zeta-function regularization and to establish Weyl laws, that is, the asymptotic behavior of spectral counting functions (see [27, 28]). We note that the additional terms in the partition function (13), with respect that of the QRM, do not change the convergence properties in a significant way. For a more detailed exposition, we refer the reader to [13] (appendix A).

The (Hurwitz-type) spectral zeta function $\zeta_R^{(c)}(s; \tau)$ is defined by the Dirichlet series

$$\zeta_R^{(c)}(s; \tau) = \sum_{j=1}^{\infty} \left( \lambda_j^{(c)} + \tau \right)^{-s}. \quad (18)$$

We verify (see e.g. [12] for the QRM case) that (18) is absolutely convergent for $s > 1$ for $\tau \in \mathbb{C} - \text{Spec}(H_R)$ with $\Re(\tau) > \Delta + g^2 + |\varepsilon|$.

In the region of absolute convergence, we have a Mellin transform representation of (18) as

$$\zeta_R^{(c)}(s; \tau) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} t^{s-1} Z_R^{(c)}(t) e^{-\tau t} dt. \quad (19)$$

In order to obtain useful applications of the spectral zeta function we need to extend the domain of definition of (18) via meromorphic continuation. The relation between the partition function and the spectral zeta function in (19) allows us to give a proof of the meromorphic continuation. We note that other methods may also be used for proving the meromorphic continuation (for instance the parametrix method used in [12, 29] for other models) but the explicit formulas allows us to give the contour integral expression (21) suitable for further applications.

Let us define the function $\Omega^{(c)}(t)$ as the denominator of (14), that is,

$$\Omega^{(c)}(t) = (1 - e^{-t}) Z_R^{(c)}(t), \quad (20)$$

and describe its complex analytic properties. The additional terms depending on $\varepsilon$ are on (14) easily bounded and the proposition below follows as in the case of QRM (see [13]).

**Proposition 3.1.** The series defining the function $\Omega^{(c)}(t)$ is uniformly convergent in compacts in the complex domain $D$ consisting a union of a half plane $\Re(t) > 0$ and a disc centered at origin with radius $r < \pi$. In particular, $\Omega^{(c)}(t)$ is a holomorphic function in the region $D$.

The main result of this section is the path integral expression for the spectral zeta function that gives the meromorphic continuation to the complex plane with a simple pole at $s = 1$.

**Theorem 3.2.** For $\tau > \Delta + g^2 + |\varepsilon|$, we have

$$\zeta_R^{(c)}(s; \tau) = -\frac{\Gamma(1-s)}{2\pi i} \int_{\infty}^{0+} (-w)^{s-1} \frac{\Omega^{(c)}(w) e^{-\tau w}}{1 - e^{-w}} dw. \quad (21)$$

Here the contour integral is given by the path which starts at $\infty$ on the real axis, encircles the origin (with a radius smaller than $2\pi$) in the positive direction and returns to the starting point and it is assumed $|\arg(-w)| \leq \pi$. This gives the meromorphic continuation of (18) to the whole plane where the only singularity is a simple pole with residue 2 at $s = 1$. 
Proof. By the hypothesis and the complex analytic properties of $\Omega^{(\varepsilon)}(t)$ given in proposition 3.1, it is legitimate to change the path of integration of (19) to obtain

$$\int_0^{(0+)} \frac{(-w)^{s-1} \Omega^{(\varepsilon)}(w) e^{-\tau w}}{1 - e^{-w}} \, dw = \left\{ e^{\pi i (s-1) \mu} - e^{-\pi i (s-1) \mu} \right\} \int_0^\infty \frac{\rho^{s-1} \Omega^{(\varepsilon)}(\rho) e^{-\rho \tau}}{1 - e^{-\rho}} \, d\rho,$$

and the formula (21) follows directly. The computation of the residue at $s = 1$ is straightforward and we omit it (see theorem 3.2 in [13]).

3.1. Weyl law for the AQRM

As an application of the spectral zeta methods, we give the Weyl law for the distribution of the spectrum of the AQRM. Concretely, it describes the asymptotic growth of the spectral counting function.

Definition 3.1. The spectral counting function $N^{(\varepsilon)}_R$ is defined as

$$N^{(\varepsilon)}_R(T) = \# \{ \lambda \in \text{Spec}(H^{\varepsilon}_R) \mid \lambda \leq T \}.$$

for $T > 0$.

We recall that formulas for the asymptotic behavior of spectral counting functions are usually called Weyl laws for his work on the eigenvalue asymptotics of the Laplace operator in a bounded domain (see e.g. [30]). One of the methods to establish Weyl laws is the use of Tauberian theorems (see e.g [31]) and the value of the residue of the corresponding spectral zeta function (see [12, 29, 32])

Theorem 3.3. We have

$$N^{(\varepsilon)}_R(T) \sim 2T,$$

as $T \to \infty$.

In particular, the distribution of eigenvalues does not depend on the bias parameter $\varepsilon$.

Proof. We follow the method of [29]. Note that for $a > 0$ we have

$$e^{-\mu a} = \int_0^\infty e^{-\mu t} \delta(t-a) \, dt.$$

Then, for $s > 1$ we obtain

$$\int_0^\infty e^{-\mu t} N^{(\varepsilon)}_R(e^t) \, dt = \zeta^{(\varepsilon)}_R(s; \tau) + f(s),$$

where $f(s)$ is certain holomorphic function in $s > 0$ (given in terms of a finite number of eigenvalues). Thus, by theorem 3.2, we have

$$\int_0^\infty e^{-\mu t} N^{(\varepsilon)}_R(e^t) \, dt = \frac{2}{s-1} + h(s),$$

for a holomorphic function $h(s)$ in $s > 0$ and the result follows from the Wiener–Ikehara theorem (see e.g. corollary 8.7 in [31]).

This result provides asymptotic evidence for the conjecture of Braak for the distribution of energy levels of the QRM and the corresponding extension for the AQRM (see [2]).
3.2. Spectral determinant and G-function

Another application of the meromorphic continuation is the result given in [8] on the spectral determinant of the AQRM. The spectral determinant is a generalization of the characteristic polynomial for operators, that is, it is an entire function that vanishes exactly at \( \tau = \lambda^2 \text{Spec}(H^e_R) \). The meromorphic continuation at the point \( s = 0 \) allows us to relate via the zeta-regularized product the spectral determinant and the spectral zeta function.

The spectral determinant of AQRM is defined as

\[
\det(\tau - H^e_R) = \exp \left( -s \frac{d}{ds} \zeta_{H^e_R}(s, \tau) \bigg|_{s=0} \right). \tag{24}
\]

In [8] it was shown that spectral determinant (24) is essentially equivalent (up to a non-vanishing factor) to the \( G \)-function used in the studies of exact-solvability of the AQRM [20]. However, the result was conditional on the meromorphic continuation of the spectral zeta function of the AQRM to \( s = 0 \), which was not proved at the time.

**Corollary 3.4 ([8]).** There exists an entire non-vanishing function \( c^e(\tau; g, \Delta) \) such that

\[
\det(\tau - g^2 - H^e_R) = c^e(\tau; g, \Delta) \mathcal{G}^e(\tau; g, \Delta). \tag{25}
\]

Here, \( \mathcal{G}^e(\tau; g, \Delta) \) is the generalized \( G \)-function of the AQRM defined as

\[
\mathcal{G}^e(\tau; g, \Delta) = G^e(x; g, \Delta) \Gamma(\varepsilon - x)^{-1} \Gamma(-\varepsilon - x)^{-1}, \tag{26}
\]

where \( G^e(x; g, \Delta) \) is the usual \( G \)-function of the AQRM.

**Proof.** By theorem 3.2 the spectral zeta function \( \zeta_{H^e_R}(s; \tau) \) is holomorphic at \( s = 0 \) and the spectral determinant \( \det(\tau - g^2 - H^e_R) \) is well-defined by (24). Then the result follows by the Weierstrass factorization theorem applied to both \( \det(\tau - g^2 - H^e_R) \) and \( \mathcal{G}^e(x; g, \Delta) \).

Note that different from the usual \( G \)-function \( G^e(x; g, \Delta) \) is an entire function which zeros correspond to the eigenvalues of AQRM, not only the regular spectrum. We also remark that the function \( \mathcal{G}^e(x; g, \Delta) \) was originally considered for numerical computations in [20] in a truncated form.

A deeper investigation of the relation between the exact solvability of interaction models and the meromorphic continuation of the corresponding spectral zeta functions is outside of the scope of the present paper.

4. Proof of the heat kernel formula

Before proceeding to the proof of theorem 2.1, we give a brief description of the general method of the computation of the heat kernel developed in [16], dividing the process into a number of steps to simplify the discussion.

The first step is to write the Hamiltonian (1) as

\[
H^e_R = H_1 + H_2, \tag{27}
\]

in such a way that each \( H_i \) for \( i = 1, 2 \) satisfies the hypothesis of the Trotter–Kato product formula (see e.g [33, 34]), and such that the heat kernel can be explicitly computed. In the case of the AQRM, by means of a Bogoliubov transformation the choice of operators is natural. In particular, \( H_1 \) is a type of non-commutative quantum harmonic oscillator.

The Trotter–Kato product formula then gives

\[
e^{-tH^e_R} = \lim_{N \to \infty} \left( e^{-tH_1/N} e^{-tH_1/N} \right)^N, \tag{28}
\]
and the second step is to find an expression for the integral kernel of the operator

\[ \left( e^{-dt} u - dt \right)^N \] (29)

for arbitrary \( N \in \mathbb{Z}_{\geq 1} \). To do this we have to consider both the scalar and matrix-valued part of the integral kernel of (29) by separate. In the case of the AQRM, the scalar value part is identical to that of the QRM (and is evaluated by Gaussian integration) but the matrix-value part requires a different method for the computations.

Once the integral kernel of (29) is computed, the sum (37) defining the heat kernel is rearranged by using the commutation structure of the matrices involved in the Hamiltonian (1). In a way that resembles a Riemann-sum (64). However, at this point it cannot be evaluated directly as a Riemann integral due to irregular oscillation of the signs appearing in certain sums in (64).

To overcome this problem, in the next step we exploit the non-commutative structure arising from the matrices of (1) to use Fourier analysis in finite groups \( \mathbb{Z}_2^k \), for \( k \geq 1 \). In particular, using Parseval identity in the dual stage (Fourier transformed) we control, with respect to certain vector invariant, the oscillation of the signs in the equation (64). This step may be interpreted as a transformation into a type of radial function (see the comments following definition 3.6 in [16]). For the AQRM, the Fourier analysis is considerably more complicated than the case of the QRM and requires a different approach since the intermediate expressions cannot be computed directly (see propositions 4.4 and 4.5).

Once the oscillation of the signs is controlled, we transform certain sums of the type (64) over the groups \( \mathbb{Z}_2^k \) into iterated integrals using standard Riemann-Stieltjes theory. Finally, we rewrite the limit expression (64) into a Riemann-type sum (102) that can be evaluated in a straightforward way completing the computation of the heat kernel.

**Remark 4.1.** Let us consider the computation in the case of the Dicke model to describe the difficulties arising in the computation of the heat kernel. For \( M \in \mathbb{Z}_{\geq 1} \), Dicke model is the model with Hamiltonian

\[ H_D^{(M)} = \omega a^\dagger a + \sum_{i=1}^{M} \frac{1}{2} \Delta_i g_i \sigma_z^{(i)} + \frac{(a + a^\dagger)}{\sqrt{M}} \sum_{i=1}^{M} g_i \sigma_z^{(i)} \] (30)

with \( g_i, \Delta_i > 0 \) and where \( \sigma_z^{(i)} = I_2 \otimes I_2 \otimes \ldots \otimes I_2 \otimes \sigma_z \otimes I_2 \otimes \ldots \otimes I_2 \) with \( \sigma_z \) in the \( i \)th position and similar for \( \sigma_z^{(i)} \) (see e.g. [4]).

Following the method described in this section, the heat kernel \( K_M(x,y,t) \) of the Dicke model is given by

\[ K_M(x,y,t) = \lim_{N \to \infty} \sum_{s \in \mathbb{Z}_2^N} G_N^{(M)}(u, \Delta, s) I_N^{(M)}(x,y,u,s), \] (31)

where \( G_N^{(M)}(u, \Delta, s) \) is the matrix part and \( I_N^{(M)}(x,y,u,s) \) is the scalar part. The scalar part is obtained essentially by multivariate Gaussian integration similar to the QRM case.

To transform (31) into a Riemann-sum for the evaluation of the heat kernel it is necessary to introduce Fourier analysis on the groups \( (\mathbb{Z}_2^M)^k \) and to generalize the combinatorial considerations given in [16]. In the language of the group actions, the combinatorial part is equivalent to considering the action of the group \( \mathcal{S}_M \times \mathcal{S}_\infty \) on \( (\mathbb{Z}_2^M)^k \). Here \( \mathcal{S}_M \) is the symmetric group and \( \mathcal{S}_\infty \) is the infinite symmetric group.

The combinatorial part is of considerably difficulty and requires the introduction of several new techniques while the Fourier analysis part generalizes directly from the case of the AQRM.
(see section 4.2), making the study of the AQRM heat kernel relevant also as an intermediate step for the generalization of the method. The computation of the heat kernel for a more general type of models, including the Dicke model and the two-photon Rabi model, is the subject of a forthcoming paper of the author with Masato Wakayama.

4.1. Hamiltonian factorization and initial considerations

First, we write the Hamiltonian $H_R$ as the sum of two simpler Hamiltonians whose heat kernels can be easily computed. Clearly, we can write

$$H_R = b^\dagger b - g^2 + \Delta \sigma_z + \varepsilon \sigma_x,$$

with $b = b(g) = a + g \sigma_y$. We note that the operator $b^\dagger b$ may be regarded as a non-commutative (or displaced) version of the quantum harmonic oscillator since the operators $b, b^\dagger$ satisfy $[b, b^\dagger] = I_2$. Thus, the spectrum of $b^\dagger b - g^2$ is given by

$$\text{Spec} \left( b^\dagger b - g^2 \right) = \left\{ n - g^2 | n \in \mathbb{Z}_{\geq 0} \right\},$$

where each eigenvalue has multiplicity 2.

The operators $H_1 = b^\dagger b - g^2$ and $H_2 = \Delta \sigma_z + \varepsilon \sigma_x$ satisfy the conditions of the Trotter-Kato product formula and we have

$$e^{-itH_R} = e^{-(b^\dagger b - g^2 + \Delta \sigma_z + \varepsilon \sigma_x)} = \lim_{N \to \infty} \left( e^{-\left( b^\dagger b - g^2 \right) / N} e^{-i(\Delta \sigma_z + \varepsilon \sigma_x) / N} \right)^N,$$

in the strong operator topology (see [16] for the detailed discussion on the convergence for the case of the QRM which applies to this case with no changes).

The next step is to compute the integral kernel of the operator $e^{-\left( b^\dagger b - g^2 \right)} e^{-i(\Delta \sigma_z + \varepsilon \sigma_x)}$. This is done in the standard way by using the Schwartz kernel and the Mehler’s formula for the quantum harmonic oscillator (see e.g. [33]).

**Proposition 4.1.** The integral kernel $D(x, y, t)$ for $e^{-\left( b^\dagger b - g^2 \right)} e^{-i(\Delta \sigma_z + \varepsilon \sigma_x)}$ is given by

$$D(x, y, t) = \frac{u^2}{\sqrt{\pi (1 - u^2)}} \exp \left( -\frac{1 - u}{1 + u} \left( \frac{(x + y)^2 + 8g^2}{4} - \frac{1 + u}{1 - u} \frac{(x - y)^2}{4} \right) \right) \times \exp \left( \frac{1 - u}{1 + u} \frac{\sqrt{2}g(x + y)\sigma_z}{\sigma_x} \right) u^{\Delta \sigma_z + \varepsilon \sigma_x},$$

with $u = e^{-t}$.

Next, we explicitly compute the integral kernel $D_N(x, y, t)$ of the operator

$$\left( e^{-\left( b^\dagger b - g^2 \right)} e^{-i(\Delta \sigma_z + \varepsilon \sigma_x)} \right)^N$$

given by

$$D_N(x, y, t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} D(x, v_1, t) D(v_1, v_2, t) \cdots D(v_{N-1}, y, t) dv_{N-1} dv_{N-2} \cdots dv_1.$$

By using the elementary identity

$$e^{\alpha \sigma_x} = \cosh(\alpha) I - \sinh(\alpha) J = \frac{1}{2} (I + J) e^{\alpha} + \frac{1}{2} (I - J) e^{-\alpha},$$

we get
valid for arbitrary \( \alpha \in \mathbb{C} \), to expand the matrix terms in (35) we rewrite \( D_N(x,y,t) \) as the sum

\[
D_N(x,y,t) = \sum_{s \in \mathbb{Z}_2^N} G^{(e)}_N(u, \Delta, s) I_N(x,y,u,s),
\]

(39)

for a \( 2 \times 2 \) matrix-valued function \( G^{(e)}_N(u, \Delta, s) \) and a scalar function \( I_N(x,y,u,s) \) for which we compute explicit expressions below. The group structure of

\[
\mathbb{Z}_2^N = \{0, 1\}^N,
\]

(40)

appearing in (39) as a set, is fundamental for the computation of the heat kernel (see section 4.2 below).

Since the function \( I_N(x,y,u,s) \) in (39) does not depend on the bias parameter \( \varepsilon \), it has the same expression as in the QRM case. In particular, we observe that all the integrals in (37) are contained in \( I_N(x,y,u,s) \) and these are evaluated by multivariate Gaussian integration.

Let we recall the notation used in the expression of \( I_N(x,y,u,s) \). For \( s \in \mathbb{Z}_2^N \) and \( i,j \in \{1, 2, \ldots, N\} \), define

\[
\eta_i(s) = (-1)^{s(i)} + (-1)^{s(i+1)},
\]

(41)

\[
\Lambda^{(j)}(u) = u^{j-1} \left( 1 - u^{2(N-j)+1} \right), \quad \Omega^{(i,j)}(u) = u^{i-j} \left( 1 - u^2 \right) \left( 1 - u^{2(N-j)} \right).
\]

(42)

**Theorem 4.2** *(theorem 2.5 of [16])*: For \( N \in \mathbb{Z}_{\geq 1} \), we have

\[
I_N(x,y,u,s) = \frac{1}{K_0(x,y,g,u^N)} \exp \left( \frac{\sqrt{2} g (1-u)}{1-a^{2N}} \sum_{j=1}^{N} (-1)^{s(j)} (x\Lambda^{(j)}(u) + y\Lambda^{(N-j+1)}(u)) \right) \times \exp \left( \frac{g^2 (1-u)^2}{2(1+u^2)(1-u^{2N})} \sum_{i=1}^{N-1} \eta_i(s)^2 \Omega^{(i,i)}(u) + 2 \sum_{i<j} \eta_i(s) \eta_j(s) \Omega^{(i,j)}(u) \right) - \frac{2Ng^2(1-u)}{1+u}.
\]

(43)

To simplify later computations, we set

\[
\tilde{I}_N(x,y,u,s) = I_N(x,y,u,s) / K_0(x,y,g,u^N).
\]

(44)

In general, throughout the computation of the heat kernel, we see that the computations related to the scalar parts of the heat kernel do not largely differ to the symmetric case.

### 4.1.1 Non-commutative part

In contrast with the scalar part \( I_N(x,y,u,s) \) given in theorem 4.2, since the matrix \( \Delta \sigma_z + \varepsilon \sigma_x \) is not diagonal, the analysis of the non-commutative part is more involved than in the QRM case. In this section, we first obtain an explicit expression of \( G^{(e)}_N(u,s) \) and give the expression of the heat kernel as the limit of a Riemann-type sum.

Let

\[
I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},
\]

(45)

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It is important to note that for the case

\[ G_N^{(e)}(u, s) = \frac{1}{2\pi} \prod_{\epsilon=1}^{N} \left[ I + (-1)^{1-s(\epsilon)} J \right] u^{\Delta \sigma_i + \epsilon \sigma_i}. \tag{46} \]

To simplify the computations, we consider the diagonalization of the matrix

\[ M = \Delta \sigma_z + \epsilon \sigma_x. \tag{47} \]

Concretely, we verify that

\[ CMC^{-1} = \begin{bmatrix} -\mu & 0 \\ 0 & \mu \end{bmatrix}, \tag{48} \]

with \( \mu = \sqrt{\epsilon^2 + \Delta^2} \) and

\[ C = \begin{bmatrix} \Delta - \mu & \epsilon \\ \Delta + \mu & \epsilon \end{bmatrix}. \tag{49} \]

For the computation of the heat kernel of the AQRM, the parameter \( \mu \) plays a similar role than the parameter \( \Delta \) in the computation of the QRM heat kernel.

**Remark 4.2.** It is important to note that for the case \( \epsilon = 0 \), the expression (48) may not appear to be valid. The problem is that in the entries of the matrix \( C^{-1} \), the parameter \( \epsilon \) appears in the denominator. However, this is not a problem since in this case the matrix \( \Delta \sigma_z \) is already a diagonal matrix (so in this case we take \( C = I \)). Moreover, the formula (48) agree with the case \( \epsilon = 0 \) using a limit interpretation.

Next, we define auxiliary functions to give the explicit expression of \( G_N^{(e)}(u, s) \). For \( v, w \in \{0, 1\} \), the function \( h_{v, w}(\tau) \) is given by

\[
h_{v, w}(\tau) = \left( \frac{1 + (-1)^{v+w}}{2} \right) (1 + \tau) + \left( \frac{(-1)^v + (-1)^w}{2} \right) \frac{\epsilon}{\mu} (1 - \tau) + \left( \frac{1 - (-1)^{v+w}}{2} \right) \frac{\Delta}{\mu} (1 - \tau). \tag{50} \]

**Example 4.1.** The function \( h_{v, w}(\tau) \) has a simple form for fixed values of \( v, w \in \{0, 1\} \). Namely,

\[
h_{0,0}(\tau) = \frac{1}{\mu} (\mu (1 + \tau) + \epsilon (1 - \tau)), \quad h_{1,1}(\tau) = \frac{1}{\mu} (\mu (1 + \tau) - \epsilon (1 - \tau))
\]

\[
h_{0,1}(\tau) = h_{1,0}(\tau) = \frac{\Delta}{\mu} (1 - \tau). \tag{51} \]

Note also that the function \( h_{v, w}(\tau) \) is invariant under permutation of the variables \( v \) and \( w \).

On the other hand, for \( s \in \mathbb{Z}_2^k \), we define the matrix-valued function

\[ M_k(s) = M_{ij}, \tag{52} \]

where \( s(1) = i \) and \( s(k) = j \) for \( i, j = 0, 1 \), and

\[ M_{00} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, M_{01} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}, M_{10} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, M_{11} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \tag{53} \]

**Proposition 4.3.** For \( s \in \mathbb{Z}_2^k \), we have

\[
G_k^{(e)}(u, s) = \prod_{i=1}^{k-1} h_{s(i), s(i+1)} (u^{2\mu}) \left( u^{k-1} \right) \mu u^{\Delta \sigma_z + \epsilon \sigma_x}, \tag{54} \]

\[ u^{k-1} = u^{k-1} \mu u^{\Delta \sigma_z + \epsilon \sigma_x}, \]
Proof. For \( v, w, \rho \in \{0, 1\} \), it is enough to verify

\[
M_{v,w} C^{-1} u^{-\mu \sigma} C \left( I + (-1)^{1-\rho} J \right) = \frac{h_{w,\rho} (u^{2\mu})}{u^\rho} M_{v,w},
\]

with \( C \) defined in (49). The proof is then completed by induction.

As in the case of QRM, we give a special definition for the scalar part of \( G_{k}^{(c)}(u,s) \) for later use.

**Definition 4.1.** For \( k \geq 1 \), the function \( g_k(u,s) \) is given by

\[
g_k(u,s) = \prod_{i=1}^{k-1} h_{\rho(i),\rho(i+1)} \left( u^{2\mu} \right).
\]

Let us summarize the computation up to this point and write the limit expression corresponding to the heat kernel. By definition, the heat kernel of the AQRM is given by

\[
K_{R}^{(c)}(x,y,t) = \lim_{N \to \infty} \sum_{s \in Z_N^2} G_N^{(c)} \left( u^{1/N}, s \right) I_N \left( x, y, u^{1/N}, s \right).
\]

In order to evaluate this limit, we transform (57) into a Riemann-type sum that we can later evaluate. A key observation is that the matrices in (54) depend only on the first and last entry of the vectors of \( Z_N^2 \), thus it is convenient to define subsets of \( Z_N^2 \) where the first and last entries of its elements are fixed.

**Definition 4.2.** Let \( N \in \mathbb{Z}_{\geq 1} \) and \( i, j \in \mathbb{Z}_2 \). The subset \( C^{(N)}_{i,j} \subset \mathbb{Z}_2^2 \) is given by

\[
C^{(N)}_{i,j} = \{ s \in \mathbb{Z}_2^2 | s(1) = i, s(N) = j \}.
\]

In practice, we consider partitions of \( \mathbb{Z}_2^N \) with fixed tails of ones or zeros and the restriction of the function \( I_N(x, y, u^{1/N}, s) \) defined in (44) to these subsets. For instance, for \( k \geq 1 \), we write

\[
I_N(x, y, u, s \oplus 0_{N-k+1}) = f_{0}^{(k,N)}(x, y, u, g) R_{0}^{(k,N)}(u, g, s),
\]

\[
I_N(x, y, u, s \oplus 1_{N-k+1}) = f_{1}^{(k,N)}(x, y, u, g) R_{1}^{(k,N)}(u, g, s),
\]

with functions \( f_{\mu}^{(k,N)}(x, y, u, g) \) and \( R_{\mu}^{(k,N)}(u, g, s) \) for \( \mu \in \{0, 1\} \) defined below. Note that \( s \in C^{(k-1)}_{i,1} \) and in the second line \( s \in C^{(k-1)}_{i,0} \) for \( i = 0, 1 \).

**Definition 4.3.** For \( k \geq 1 \), the function \( f_{\mu}^{(k,N)}(x, y, u) \) is given by

\[
f_{\mu}^{(k,N)}(x, y, u) = \exp \left( (-1)^{\mu} \sqrt{2g} \frac{(1-u)}{1-u^{2N}} \sum_{j=k}^{N} \left( x \Lambda^{(j)}(u) + y \Lambda^{(N-j+1)}(u) \right) \right) \times \exp \left( \frac{2g^2 (1-u)^2}{(1+u)^2 (1-u^{2N})} \sum_{i=k}^{N-1} \Omega^{(i)}(u) + \sum_{i=k}^{N-1} \sum_{j=i+1}^{N-1} \Omega^{(ij)}(u) \right)
\]

\[
-2Ng^2 \frac{(1-u)}{1+u},
\]

(61)
while $R_{\mu}^{(k,N)}(x,y,u,\bar{s})$ is given, for $\bar{s} \in Z_{k-1}$, by

$$R_{\mu}^{(k,N)}(x,y,u,\bar{s}) = \exp \left( \frac{\sqrt{2}g(1-u)}{1-u^{2N}} \sum_{j=1}^{k-1} (-1)^{j} (x\Lambda^{(j)}(u) + y\Lambda^{(N-j+1)}(u)) \right)$$

$$\times \exp \left( \frac{g^{2}(1-u)^{2}}{2(1+u)^{2}(1-u^{2N})} \left[ \sum_{j=1}^{k-2} \eta_{j}(\bar{s})^{2}\Omega^{(i,j)}(u) + 2 \sum_{j=1}^{k-2} \sum_{j+i+1}^{k-2} \eta_{j}(\bar{s})\eta_{j+i}(\bar{s}) \Omega^{(i,j)}(u) + 4(-1)^{n} \sum_{i=1}^{k-2} \sum_{j=k}^{k-2} \eta_{j}(\bar{s})\Omega^{(i,j)}(u) \right] \right).$$

(62)

With these preparations (see also definition 2.1 in [16] for details), we see that the limit expression (57) is equal to the sum of the limits

$$\frac{1}{2} \lambda_{0}(x,y,g,u) \lim_{N \to \infty} \left( \frac{h_{0,0}(u^{\bar{\pi}})}{2u^{\pi}} \right)^{N-1} f_{0}^{(1,N)}(x,y,u^{\bar{\pi}},g)M_{0,0}$$

$$+ \left( \frac{h_{1,1}(u^{\bar{\pi}})}{2u^{\pi}} \right)^{N-1} f_{1}^{(1,N)}(x,y,u^{\bar{\pi}},g)M_{1,1}$$

(63)

and

$$\lambda_{0}(x,y,g,u) \lim_{N \to \infty} \left( \frac{h_{0,1}(u^{\bar{\pi}})}{2u^{\pi}} \right)^{N-1} \sum_{i=0}^{N} \sum_{j=k+2}^{N} \left( \frac{h_{i,j}(u^{\bar{\pi}})}{2u^{\pi}} \right)^{N-k}$$

$$\times f_{i}^{(k,N)}(x,y,u^{\bar{\pi}},g)M_{ij} \sum_{s \in C_{j}^{k-1}} g_{k-1}(u^{\bar{\pi}},s)R_{\mu}^{(k,N)}(u^{\bar{\pi}},s) \frac{M_{i,j}(\Delta)}{u^{\Delta}}.$$  

(64)

Note that since $h_{0,0}(\tau) \neq h_{1,1}(\tau)$ for $\epsilon \neq 0$ (see example 4.1), the expression of the limit for the AQRM splits according to the tail of zeros or ones. Actually, this difference turns out to be only apparent and the limit expression simplifies later.

Next, we use the harmonic analysis in the groups $Z_{N}$ to transform the sum above into one that can be evaluated as a Riemann sum by controlling the oscillation of the signs in (43). The main tool is Parseval identity on the abelian groups $Z_{2}^{k-1}$ applied to the sum

$$\sum_{s \in C_{j}^{k-1}} g_{k-1}(u^{\bar{\pi}},s)R_{\mu}^{(k,N)}(u^{\bar{\pi}},s),$$

(65)

by taking advantage of the fact that the set $C_{j}^{(k-1)}$ may be equipped with a $Z_{2}^{k-1}$ abelian group structure by ignoring the first and last entries of the vectors in $C_{j}^{(k-1)}$.

**Remark 4.3.** The use of the Fourier transform also allows to identify the limit (64) as an orbit sum over the action of the infinite symmetric group $S_{\infty}$ on the (induced limit of) finite groups $Z_{N}$ for $N \geq 0$. We refer the reader to [13] for the description of this interpretation for the case of the QRM.
**4.2. Fourier analysis and limit evaluation**

In order to evaluate sums of the type of (65), we need to describe the Fourier transform of $g_k(u, s)$ and $R_w^{(k,N)}(u, s)$. This is a necessary step in order to use Parseval identity and ultimately rewrite the limit expression (64) into one that can be evaluated as a Riemann integral. A good reference for the Fourier transform on finite groups (and particularly in the abelian group $\mathbb{Z}_2^k$) can be found in [35].

**Definition 4.4.** Let $\rho = (\rho_1, \rho_2, \ldots, \rho_k) \in \mathbb{Z}_2^k$. The function $|\cdot| : \mathbb{Z}_2^k \to \mathbb{C}$ is given by

$$|\rho| = \sum_{i=1}^{k} \rho_i.$$  

(66)

Let $j_1 < j_2 < \ldots < j_{|\rho|}$ the position of the ones in $\rho$, that is, $\rho_{\overline{j}} \equiv 1$ for all $i \in \{1, 2, \ldots, |\rho|\}$ and if $\rho_i = 1$ then $i \in \{j_1, j_2, \ldots, j_{|\rho|}\}$. The function $\varphi_\rho : \mathbb{Z}_2^k \to \mathbb{C}$ is given by

$$\varphi_\rho(\rho) = \sum_{i=1}^{|\rho|} (-1)^{|\rho|+1-i} j_{|\rho|+1-i} = j_{|\rho|} \rho_1 + \ldots + (-1)^{|\rho|-1} j_{|\rho|} \rho_i,$$

(67)

and $\varphi_\rho(0) = 0$ where $\mathbb{Z}_2$ is the identity element in $\mathbb{Z}_2^k$. For $k = 0$, define $\varphi_\rho(\rho) = |\rho| = 0$ where $\rho$ is the unique element of $\mathbb{Z}_2^0$.

In the case of the function $g_k(u, s)$ we define a special notation to simplify the computation of the Fourier transform.

**Definition 4.5.** Let $v, w \in \{0, 1\}$. Then, for $s \in \mathbb{Z}_2^k$ with $k \geq 1$, define the function $g_k^{(v,w)}(u, s)$ by

$$g_k^{(v,w)}(u, s) = h_{v,x(1)}(u^{2^\mu}) h_{x(1),w}(u^{2^\mu}) \prod_{j=1}^{k-1} h_{x(0),s(x(1))}(u^{2^\mu}).$$

(68)

In addition, for $\rho \in \mathbb{Z}_2^k$, define

$$g_k^{(v,w)}(u, s) = h_{v, w}(u^{2^\mu}).$$

(69)

For $s \in C_{v,w}^{(k+2)}$, we have

$$2^{k+2} u^{(k+1)\mu} g_{k+2}(u, s) = g_k^{(v,w)}(u, s),$$

(70)

where for $s \in \mathbb{Z}_2^{k+2}$, $\bar{s} \in \mathbb{Z}_2^k$ is the projection of $s$ obtained by removing the first and last component.

Next, we obtain the Fourier transform of $g_k^{(v,w)}(u, s)$. In contrast with the QRM case, it is not possible to obtain a simple formula except for the case $\varepsilon = 0$.

**Proposition 4.4.** For $\rho \in \mathbb{Z}_2^k$, we have

$$\left[ g_k^{(v,0)}(\rho), g_k^{(v,1)}(\rho) \right] = \left[ h_{0,v}(u^{2^\mu}), h_{1,v}(u^{2^\mu}) \right] \prod_{i=1}^{k} B(\rho_i),$$

(71)

where the matrix-valued function $B(s)$, for $s \in \{0, 1\}$, is given by

$$B(s) = \begin{bmatrix} h_{0,0} (u^{2^\mu}) & h_{0,1} (u^{2^\mu}) \\ (-1)^y h_{1,0} (u^{2^\mu}) & (-1)^y h_{1,1} (u^{2^\mu}) \end{bmatrix}.$$
Proof. The case \( k = 0 \) is trivial. For \( k \geq 1 \), let \( \rho = (\rho_1, \rho_2, \ldots, \rho_k) \in \mathbb{Z}_2^k \) and \( \delta = (\rho_1, \rho_2, \ldots, \rho_{k-1}) \in \mathbb{Z}_2^{k-1} \), then we have

\[
\left[ g_k^{(v,0)}(\rho), g_k^{(v,1)}(\rho) \right] = \left[ \sum_{s \in \mathbb{Z}_2^k} g_{k+1}^{(v,0)}(s) \chi_{\rho}(s), \sum_{s \in \mathbb{Z}_2^k} g_{k+1}^{(v,1)}(s) \chi_{\rho}(s) \right]
\]

\[
= \left[ h_{0,0}\left(u^{2\mu}\right) g_{k-1}^{(v,0)}(\delta) + (-1)^{\rho_1} h_{0,1}\left(u^{2\mu}\right) g_{k-1}^{(v,1)}(\delta), h_{0,1}\left(u^{2\mu}\right) g_{k-1}^{(v,0)}(\delta) + (-1)^{\rho_1} h_{1,1}\left(u^{2\mu}\right) g_{k-1}^{(v,1)}(\delta) \right]
\]

and the result follows by induction.

Note that (71) is equivalent to

\[
\overline{g_k^{(v,w)}}(\rho) = \left[ h_{0,0}\left(u^{2\mu}\right), h_{1,1}\left(u^{2\mu}\right) \right] \prod_{i=1}^{k-1} B(\rho_i) \left[ h_{0,w}\left(u^{2\mu}\right), (-1)^{\rho_1} h_{1,w}\left(u^{2\mu}\right) \right].
\]

Next, we consider the sum

\[
\sum_{\rho \in \mathbb{Z}_2^k} g_k^{(v,w)}(\rho) \prod_{i=1}^{k} (A_i)^{\rho_i},
\]

for \( A_i \in \mathbb{C} \). In this case, while the result resembles the case of the QRM (see proposition 3.13 of [16]), the method of proof is different. In particular, the matrices (72) in the expression of \( g_k^{(v,w)}(\rho) \) correspond to the recurrence relation appearing in the QRM case in the proof of proposition 3.13 of [16].

In the next proposition, for a vector \( \rho = (\rho_1, \rho_2, \ldots, \rho_k) \in \mathbb{Z}_2^k \) with \( |\rho| = \ell \), the numbers \( j_1 < j_2 < \ldots < j\ell \) correspond to the positions of the ones in the vector \( \rho \).

Proposition 4.5. For \( A_i \in \mathbb{C} \) (\( i \leq k \)), we have

\[
\sum_{\rho \in \mathbb{Z}_2^k} g_k^{(v,w)}(\rho) \prod_{i=1}^{k} (A_i)^{\rho_i} = \sum_{\ell=0}^{k} h_{\ell,\ell+w}(u^{2\mu}) \left( h_{0,1}(u^{2\mu}) \right) \left( h_{1-w,1-w}(u^{2\mu}) \right)^{k-\ell} \prod_{i=1}^{j_\ell} \left( 1 + (-1)^{w+\ell+i} A_i \right) \prod_{i=j_{\ell+1}}^{j_{\ell+1}} \left( 1 + (-1)^{w+\ell+i} A_i \right)
\]

where \( j_0 = 0 \) and \( j_{\ell+1} = k \). The function \( \alpha(\ell) \) is given by

\[
\alpha(\ell) = k - \left[ \frac{\ell}{2} \right] - \varphi(\ell).
\]

Proof. First, let us define the function

\[
F_k(\tau) = \sum_{\rho \in \mathbb{Z}_2^k} \prod_{i=1}^{k} A_i^{\rho_i} B(\rho_i).
\]
Clearly, we have
\[ F_k(\tau) = F_{k-1}(\tau)(B(0) + A_iB(1)), \tag{79} \]
and, by the elementary identity,
\[ (B(0) + A_iB(1)) = D(A_i)B(0), \tag{80} \]
with
\[ D(x) = \begin{bmatrix} 1 + x & 0 \\ 0 & 1 - x \end{bmatrix}, \tag{81} \]
it follows that (78) is given by
\[ F_k(\tau) = \prod_{i=1}^{k} D(A_i)B(0). \tag{82} \]
Next, we write (72) as
\[ B(0) = \frac{1}{\mu}(aI + bJ + cK), \tag{83} \]
with
\[ a = \mu \left( 1 + u^{2\mu} \right), \quad b = \Delta \left( 1 - u^{2\mu} \right), \quad c = \varepsilon \left( 1 - u^{2\mu} \right), \tag{84} \]
and
\[ K = \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{85} \]
It follows that (82) is given by
\[ F_k(\tau) = \frac{1}{\mu^k} \prod_{i=1}^{k} D(A_i)(aI + bJ + cK), \tag{86} \]
and using the commutation relations
\[ ID(x) = D(x)I, \quad JD(x) = D(-x)J, \quad KD(x) = D(x)K, \tag{87} \]
we obtain
\[ F_k(\tau) = \frac{1}{\mu^k} \sum_{s \in \mathbb{Z}_k^{k+1}} (bJ)^{[\rho]}(aI - bK)^{k-|\rho|}D(aI + bK)^{\alpha(s)} \prod_{i=0}^{k} \left( \prod_{n=\rho+i+1}^{\ell} (-1)^{\rho+i} A_n \right), \tag{88} \]
with \( \alpha(s) \) given by \( \alpha((0)) = 1, \alpha((1)) = 0 \) and recursively as
\[ \alpha(s \oplus (0)) = \alpha(s) + 1, \]
\[ \alpha(s \oplus (1)) = k - |s| - \alpha(s) \tag{89} \]
for \( s \in \mathbb{Z}_k^{k+1} \). We verify directly that \( \alpha(s) = k - \left| \frac{|s|}{2} \right| - \varphi(s) \) by comparing the recurrence relations (89) with (77). Next, we write
\[ \sum_{\rho \in \mathbb{Z}_k^{k+1}} \delta^{(\varphi, \omega)}(\rho) \prod_{i=1}^{k} \left( A_i \right)^{\alpha_i} \left[ B_{\omega,\omega}(u^{2\mu}), h_{1,\omega}(u^{2\mu}) \right] \prod_{\rho \in \mathbb{Z}_k^{k+1}} \left[ B(\rho)A_0 \delta_{0}(w) \right] \left( B_1(w) \right), \tag{90} \]
and the result is obtained by substituting equation (88) in the left-hand side of (76) and multiplying the matrices.

It remains to compute the Fourier transform of the function $R_{_{\mu}}^{(\nu,w)}(s)$. Here, since there parameter $\varepsilon$ does not appear in (62) there is no change from the computation in the QRM case. Let us just introduce some of the basic notations and refer to the appropriate results when needed. We write the function $R_{_{\mu}}^{(v,w)}(s)$

$$R_{_{\mu}}^{(v,w)}(s) = \exp \left( a_{0}^{(\mu)} + \sum_{\chi \in \mathbb{S}_{k-3}} a_{\chi}^{(\mu)} \chi(s) \right),$$

(91)

where the coefficients $a_{\chi}^{(\mu)}$ are implicitly defined (see lemma 3.5 of [16] for the explicit form). Then, the Fourier transform of $R_{_{\mu}}^{(v,w)}(s)$ is given by

$$\tilde{R}_{_{\mu}}^{(v,w)}(s) = 2^{k-3} \exp \left( a_{0}^{(\mu)} \right) \left( \sum_{\xi \in \mathbb{Z}_{2}^{k-3}} C_{\xi,\chi_{\rho}} \right).$$

(92)

where

$$C_{\chi_{\rho}} = \sum_{r \in \{0,1\}^{\ell}} T^{(r)}(a) .$$

(93)

Here, for $a \in \mathbb{C}^{\ell}$ and an index vector $r \in \{0,1\}^{\ell}$ we define

$$T^{(r)}(a) = \prod_{i=1}^{\ell} \left[ \cosh(a_{i})^{(1-\tau_{i})} \sinh(a_{i})^{\tau_{i}} \right].$$

(94)

where $a_{i}$ (resp. $r_{i}$) denotes the $i$th component of $a$ (resp. $r$).

**Remark 4.4.** Equation (94) is a consequence of the fact that all characters of the group $\mathbb{Z}_{2}^{k}$ are real characters. For more general models, the computation of the heat kernel may require a different approach depending on the group that appears in place of $\mathbb{Z}_{2}^{k}$.

We now return to the evaluation of the heat kernel. Let $\eta \in \{0,1\}$, then, by Parseval identity and (92), we have

$$\sum_{s \in \mathbb{C}_{1}^{(k-1)}} g_{k-1}(s) R_{_{\mu}}^{(k,N)}(s) = \frac{1}{2^{k-1} \mu(k-2)^{\ell}} \sum_{s \in \mathbb{Z}_{2}^{k-3}} g_{k-3}^{(v,w)}(s) R_{_{\mu}}^{(v,w)}(s)$$

$$= \frac{1}{2^{k-1} \mu(k-2)^{\ell}} \sum_{\rho \in \mathbb{Z}_{2}^{k-3}} g_{k-3}^{(v,w)}(\rho) R_{_{\mu}}^{(v,w)}(\rho)$$

$$= \frac{1}{2^{k-1} \mu(k-2)^{\ell}} \exp \left( a_{0}^{(\eta)} \right) \sum_{\rho \in \mathbb{Z}_{2}^{k-3}} g_{k-3}^{(v,w)}(\rho) \sum_{r \in \{0,1\}^{\ell}} T^{(r)}(a^{(\eta)}) .$$

(95)
notice that the extra $2^{(k-1)}$ in the denominator (compared to the QRM case) corresponds to the factors $h_{r,w}(u^{2\mu})$ inside the matrices $B(\rho)$ in (76). Next, we have

$$
\sum_{\rho \in \mathbb{Z}_2^{k-3}} g_{k-3}^{(v,w)}(\rho) \sum_{r \in V_0^{(k-3)}} T^{(\sigma,\rho)}(a^{(q)}) = \sum_{r \in V_0^{(k-3)}} T^{(r)}(a^{(q)})
$$

$$
\times \sum_{\rho \in \mathbb{Z}_2^{k-3}} g_{k-3}^{(v,w)}(\rho) \prod_{i=1}^{k-3} (\tanh(q^{(n)})^{1-n_i})^\rho_i
$$

(96)

and we apply proposition 4.5, to get

$$
\sum_{r \in V_0^{(k-3)}} T^{(r)}(a^{(q)}) \sum_{\rho \in \mathbb{Z}_2^{k-3}} \alpha(\rho) \prod_{i=0}^{\ell-1} \prod_{n=\ell+1}^{\ell+k-\ell} (1 + (-1)^{\mu+n} A_{\alpha}^{(r)})
$$

$$
= \sum_{\ell=0}^{k-3} h_{v,\ell+w}(u^{2\mu})(h_{1-w,1-w}(u^{2\mu}))^{\ell} (h_{1-w,1-w}(u^{2\mu}))^{\ell}
$$

$$
\times \sum_{\rho \in \mathbb{Z}_2^{k-3}} \alpha(\rho) \sum_{r \in V_0^{(k-3)}} T^{(r)}(a^{(q)}) \prod_{i=0}^{\ell-1} \prod_{n=\ell+1}^{\ell+k-\ell} (1 + (-1)^{\mu+n} A_{\alpha}^{(r)})
$$

(97)

and by using lemma 3.14 of [16], we obtain

$$
\sum_{\rho \in \mathbb{Z}_2^{k-3}} h_{r,|\rho|+w}(u^{2\mu}) (h_{1-w,1-w}(u^{2\mu}))^{\ell} \left( \frac{h_{0,1}(u^{2\mu})}{h_{w,w}(u^{2\mu})} \right)^{|\rho|}
$$

$$
\times \left( \frac{h_{w,w}(u^{2\mu})}{h_{1-w,1-w}(u^{2\mu})} \right)^{\alpha(\rho)} \exp \left( \sum_{m=0}^{k-3-|\rho|} \sum_{j=1}^{\ell} (-1)^{|\rho|+w} d_{\rho}(m) + \sum_{n=1}^{\ell+k-\ell} \rho_i \right)
$$

(98)

Summing up, we see that (95) is equal to

$$
\sum_{s \in \mathbb{C}^{k-1}} R_{0,s}^{(1,N)}(s)
$$

$$
= \frac{1}{2^{2(k-1)}(k-2\mu)} (h_{1-w,1-w}(u^{2\mu}))^{k-3} \sum_{\rho \in \mathbb{Z}_2^{k-3}} h_{r,|\rho|+w}(u^{2\mu}) \left( \frac{h_{0,1}(u^{2\mu})}{h_{w,w}(u^{2\mu})} \right)^{|\rho|}
$$

$$
\times \left( \frac{h_{w,w}(u^{2\mu})}{h_{1-w,1-w}(u^{2\mu})} \right)^{\alpha(\rho)} \exp \left( a_{0,1}^{(q)} + \sum_{m=0}^{k-3-|\rho|} \sum_{j=1}^{\ell} (-1)^{|\rho|+w} d_{\rho}(m) + \sum_{n=1}^{\ell+k-\ell} \rho_i \right)
$$

(99)
4.2.1. Second limit expression and final computation. In the next step, we reformulate the limit expression (64) of the heat kernel using the Fourier analysis developed in the previous section. As a result, in the dual setting the sign changes in (64) are controlled when we fix the length of the vectors in the Fourier transformed expression (99), allowing us to evaluate the limit (64) as a Riemann integral.

First, by an elementary computation we see that

$$\log \left( \frac{h_{i,i} \left( e^{-\sqrt{N} \tau} \right)}{2e^{-\sqrt{N} \tau}} \right) = (-1)^i \frac{t_e}{N} + O \left( \frac{1}{N^2} \right),$$

(100)

for \( i = 0, 1 \) and thus we have the limits

$$\lim_{N \to \infty} \left( \frac{h_{0,0} \left( u \frac{\tau}{\sqrt{N}} \right)}{2u \frac{\tau}{\sqrt{N}}} \right)^{N-1} = u^{-\varepsilon}, \quad \lim_{N \to \infty} \left( \frac{h_{1,1} \left( u \frac{\tau}{\sqrt{N}} \right)}{2u \frac{\tau}{\sqrt{N}}} \right)^{N-1} = u^\varepsilon.$$

(101)

Using these limits, we rewrite the limit expression (64) using (99) as

$$\begin{align*}
K_0(x,y,g,u) & \left\{ e^{-2u \frac{\tau}{\sqrt{N}} - i(x-y)} \begin{bmatrix} \cosh & -\sinh \\ -\sinh & \cosh \end{bmatrix} \right( \sqrt{2g(x+y)} \frac{1-e^{-\sqrt{N} \tau}}{1+e^{-\sqrt{N} \tau}} + \varepsilon) \\
& + \frac{u^\varepsilon - \varepsilon}{2} \lim_{N \to \infty} \left( \frac{h_{0,1} \left( u \frac{\tau}{\sqrt{N}} \right)}{2u \frac{\tau}{\sqrt{N}}} \right)^{N-1} \sum_{k \geq 2} \left( \frac{N}{2} \right)^{N-1} \left( \begin{array}{c} \alpha_{|\rho|+1} \left( u \frac{\tau}{\sqrt{N}} \right) \\
-\beta_{|\rho|+1} \left( u \frac{\tau}{\sqrt{N}} \right) \end{array} \right) \\
& \times \left( \frac{h_{0,0} \left( u \frac{\tau}{\sqrt{N}} \right)}{h_{0,0} \left( u \frac{\tau}{\sqrt{N}} \right)} \right)^{\rho} \left( \begin{array}{c} \alpha_{\rho} \left( u \frac{\tau}{\sqrt{N}} \right) \\
-\beta_{\rho} \left( u \frac{\tau}{\sqrt{N}} \right) \end{array} \right) \exp \left( a_{(0)}^{(0)} + \sum_{m=0}^{k-4} \sum_{j=1}^{m-3} (-1)^{j+1} a_{m,j}^{(0)} + \sum_{m=0}^{k-4} \sum_{j=1}^{m-3} (-1)^{j+1} a_{m,j}^{(1)} \right) \right) \\
& + \frac{u^{-\varepsilon} - \varepsilon}{2} \lim_{N \to \infty} \left( \frac{h_{1,0} \left( u \frac{\tau}{\sqrt{N}} \right)}{2u \frac{\tau}{\sqrt{N}}} \right)^{N-1} \sum_{k \geq 2} \left( \frac{N}{2} \right)^{N-1} \left( \begin{array}{c} \alpha_{|\rho|+1} \left( u \frac{\tau}{\sqrt{N}} \right) \\
-\beta_{|\rho|+1} \left( u \frac{\tau}{\sqrt{N}} \right) \end{array} \right) \\
& \times \left( \frac{h_{0,0} \left( u \frac{\tau}{\sqrt{N}} \right)}{h_{1,1} \left( u \frac{\tau}{\sqrt{N}} \right)} \right)^{\rho} \left( \begin{array}{c} \alpha_{\rho} \left( u \frac{\tau}{\sqrt{N}} \right) \\
-\beta_{\rho} \left( u \frac{\tau}{\sqrt{N}} \right) \end{array} \right) \exp \left( a_{(0)}^{(1)} + \sum_{m=0}^{k-4} \sum_{j=1}^{m-3} (-1)^{j+1} a_{m,j}^{(1)} + \sum_{m=0}^{k-4} \sum_{j=1}^{m-3} (-1)^{j+1} a_{m,j}^{(1)} \right) \right)
\end{align*}$$

(102)

where

$$\alpha_\tau (\tau) = \frac{1}{2} (h_{0,1}(\tau) - h_{1,0}(\tau)), \quad \beta_\tau (\tau) = \frac{1}{2} (h_{0,0}(\tau) + h_{1,1}(\tau)).$$

(103)

Next, we simplify the expressions appearing in the limit (102) to complete the computation of the heat kernel. Note that since

$$\lim_{N \to \infty} h_{\nu, \omega} \left( u \frac{\tau}{\sqrt{N}} \right) = 1, \quad \lim_{N \to \infty} h_{\nu, -1- \omega} \left( u \frac{\tau}{\sqrt{N}} \right) = 0,$$

(104)

we have

$$\lim_{N \to \infty} \frac{1}{u \frac{\tau}{\sqrt{N}}} \left[ \begin{array}{c} \alpha_{|\rho|+1} \left( u \frac{\tau}{\sqrt{N}} \right) \\
-\beta_{|\rho|+1} \left( u \frac{\tau}{\sqrt{N}} \right) \end{array} \right] = \left[ \begin{array}{c} (-1)^{|\rho|+1} \\
1 \end{array} \right].$$

(105)
and

$$\lim_{N \to \infty} \frac{1}{u^{\pi \ell}} \left[ -\alpha_1 \left( \frac{u^\pi}{w} \right) - \alpha_2 \left( \frac{u^\pi}{w} \right) \right] = \left[ (-1)^{|\rho|+1} - (-1)^{|\rho|} \right]. \quad (106)$$

and at the limit we see that these are the matrices that appear in the final formula (6).

Next, we consider the limit of the quotients of the functions $h_{w,\ell}(u)$ appearing in (102). For $\ell \in \mathbb{R}$ fixed and $w \in \{0, 1\}$, we see directly that

$$\left( \frac{h_{w,\ell} \left( e^{-i \frac{2\pi}{N}} \right)}{h_{1-w,1-w} \left( e^{-i \frac{2\pi}{N}} \right)} \right)^{\ell} = e^{(-1)^{\ell^2} \ell \varepsilon / N} + O \left( \frac{1}{N} \right), \quad (107)$$

In particular, for $w \in \{0, 1\}$ we have

$$\lim_{N \to \infty} \left( \frac{h_{w,\ell} \left( u^\pi \right)}{h_{1-w,1-w} \left( u^\pi \right)} \right)^{\ell} = 1. \quad (108)$$

Similarly, for $\ell \in \mathbb{R}$ we observe that

$$\left( \frac{h_{0,\ell} \left( u^\pi \right)}{h_{1-w,1-w} \left( u^\pi \right)} \right)^{\ell} = \frac{(\ell \Delta N)}{1} \left( e^{(-1)^{\ell^2} \ell \varepsilon / N} + O \left( \frac{1}{N} \right) \right). \quad (109)$$

These limits may be verified by considering the power series expansion of the logarithm as in (100).

Let us now consider the expressions appearing in the inner sums in the limit (102). For instance, for $w \in \{0, 1\}$ and fixed $|\rho| = \ell$, from the expression

$$\left( \frac{h_{0,\ell} \left( u^\pi \right)}{h_{1-w,1-w} \left( u^\pi \right)} \right)^{\ell} = \left( \frac{h_{w,\ell} \left( u^\pi \right)}{h_{1-w,1-w} \left( u^\pi \right)} \right)^{\alpha(\rho)}, \quad (110)$$

a direct computation gives

$$\left( \frac{\ell \Delta N}{N} \right)^{\ell} e^{-\frac{(16)^{\ell^2}}{N}} e^{(-1)^{\ell^2} \ell \varepsilon / N} + O \left( \frac{1}{N} \right). \quad (111)$$

In this case, since the function $\varphi(\rho)$ appears in the part corresponding to the $\varepsilon$, we need to evaluate it using multiple Riemann integrals (see section 3.4 of [16]). Concretely, we see that

$$\sum_{s \in C^{(l-1)}_{\rho}=\ell} \left( \frac{h_{1,\rho} \left( u^\pi \right)}{h_{0,0} \left( u^\pi \right)} \right)^{\alpha(\rho)} g_{e-1} (s) R_{N}^{(k,N)} (s) = \exp \left( -2 (-1)^{\gamma} e^{\frac{1}{N}} \right)$$

$$\times \sum_{s \in C^{(l-1)}_{\rho}=\ell} \exp \left( (-1)^{\gamma} \frac{2a(\rho) \varepsilon}{N} \right) \exp \left( \frac{h_{1,\rho} \left( u^\pi \right)}{h_{0,0} \left( u^\pi \right)} \right)^{\alpha(\rho)} + \sum_{m=0}^{k-4k} \sum_{j=1}^{k-4k-m} (-1)^{k+|\rho|+w} h_{m} \left( u^\pi \right) \alpha(m) \alpha(m+j) \right). \quad (112)$$
up to $O\left(\frac{1}{N}\right)$ terms. Then, the sum in (112) over $g^{(k-1)}_{\nu} \simeq \mathbb{Z}^{k-3}_2$ with fixed $|\rho| = \ell$ is interpreted as a sum over $j_1 < j_2 < \ldots < j_{k-3} \leq k-3$, that is, over the position of the ones in the vectors $\rho \in \mathbb{Z}^{k-3}_2$.

For $\lambda \geq 1$, define the functions

$$f_{\lambda}^{(\eta)}(z_1, z_2, \ldots, z_\lambda; a \hat{z}) = (-1)^{\eta+1} \frac{2^{\gamma} \gamma}{1 - a} \sum_{\gamma=1}^{\lambda} (-1)^{\gamma-1} \left[ u^{2\gamma} \left( 1 - u^{\gamma(2z+i+\gamma)} \right) \left( 1 - u^{\gamma(2z+\gamma)} \right) \right]$$

$$- yu \left( 1 - u^{\gamma(2z+\gamma)} \right) \left( 1 - u^{\gamma(2z+2\gamma)} \right)$$

$$- \frac{2 \gamma^2}{1 - a^2} u^{\hat{z}} (1 - a^{1-\hat{z}})^2 \sum_{\gamma=1}^{\lambda} (-1)^{\gamma-1} \left( 1 - u^{\gamma(2z+i+\gamma)} \right) \left( 1 - u^{\gamma(2z+\gamma)} \right)$$

$$- \frac{2 \gamma^2}{1 - a^2} \sum_{0 \leq \alpha \leq \beta < b \mod 2} u^{\frac{\gamma(2z+i+\alpha)}{\beta - \alpha + 1}} \left( 1 - u^{\gamma(2z+i+b+\gamma)} \right) \left( 1 - u^{\gamma(2z+i+\gamma)} \right)$$

$$\times \left( 1 - u^{\frac{\gamma(2z+i+\alpha)}{\beta - \alpha + 1}} \right) \left( 1 - u^{\frac{\gamma(2z+i+\beta+\gamma)}{\beta - \alpha + 1}} \right),$$

and

$$g_{\lambda}^{(\eta)}(z_1, z_2, \ldots, z_\lambda; \frac{t}{N}) = (-1)^{\eta+1} \frac{2 \gamma}{N} \sum_{\gamma=1}^{\lambda} (-1)^{\gamma-1} z_{\lambda+1-\gamma}$$

where as before, we set $z_0 = 0$ and $z_{\lambda+1} = k - 2$. Notice that for fixed $\lambda$, $f_{\lambda}^{(\eta)}(z; a \hat{z})$ and $g_{\lambda}^{(\eta)}(z_1, z_2, \ldots, z_\lambda; \frac{t}{N})$ are smooth functions on $z_i$, with $i = 1, 2, \ldots, \lambda$, for any $u \in (0, 1)$.

To transform the sum (112) into a multiple integral we need the following lemma. The proof follows the case of the QRM with no significant changes and we refer the reader to section 3.4 of [16] for the details.

**Lemma 4.6.** For fixed $\lambda \geq 1$ and $a \in \mathbb{Z}_{\geq 1}$ with $a \leq N$, we have

$$\sum_{1 \leq i_1 < i_2 < \ldots < i_\lambda} e_{\lambda}^{(\eta)}(i_1, \ldots, i_\lambda; a \hat{z}) + g_{\lambda}^{(\eta)}(i_1, \ldots, i_\lambda; \hat{z})$$

$$= \int_{0}^{a} \int_{0}^{z_1} \ldots \int_{0}^{z_{\lambda}} e_{\lambda}^{(\eta)}(z_1, \ldots, z_{\lambda}; a \hat{z}) + g_{\lambda}^{(\eta)}(z_1, \ldots, z_{\lambda}; \hat{z}) \, dz + O\left(a^\lambda \right).$$

With these preparations, the computation of the heat kernel of the AQRM can be completed by partitioning the sums in (102) according to the norm of $\rho$. The limit of a generic matrix component in (102) after the simplifications of this section is given by
The result is obtained by substituting (118) in (117), using the elementary identity
\[
\int_{-\infty}^{\infty} e^{-\alpha x^2} \cosh(x\eta) \, dx = \sqrt{\frac{\pi}{\alpha}} e^{\frac{\eta^2}{4\alpha}}.
\] (119)
valid for \( \alpha > 0 \) and \( \eta, \in \mathbb{R} \), for the integration.
Data availability statement
No new data were created or analysed in this study.

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