Abstract. We establish a Dwyer–Kan equivalence of relative categories of combinatorial model categories, presentable quasicategories, and other models for locally presentable (∞,1)-categories. This implies that the underlying quasicategories of these relative categories are also equivalent.
1 Introduction

Combinatorial model categories and presentable quasicategories are the two most used formalisms for locally presentable (∞, 1)-categories. It has long been conjectured that these formalisms should be equivalent in a certain sense, see, for example, Problems 8 and 11 on Hovey’s algebraic topology problem list [1999.e]. Theorem 1.1 is a solution to (one precise formulation of) these problems.

Partial results in this direction existed for a long time, see, in particular, the work of Dugger [2000.b] and Lurie [2017]. For example, we know that the underlying quasicategory of a combinatorial model category is presentable, and up to an equivalence of quasicategories, every presentable quasicategory arises in this manner. Likewise, the underlying functor of quasicategories of a left Quillen functor is a left adjoint functor between presentable quasicategories, and up to an equivalence of functors, every left adjoint functor between presentable quasicategories arises in such a manner.

However, locally presentable (∞,1)-categories can themselves be organized into an (∞,1)-category, so it is natural to inquire whether the resulting (∞,1)-categories of combinatorial model categories and presentable quasicategories are equivalent. In this article, we formalize these (∞,1)-categories as relative categories and prove the following result.

Theorem 1.1. The following relative categories are Dwyer–Kan equivalent. In particular, their underlying quasicategories and homotopy (2,1)-categories are equivalent.

- The relative category \( \text{CMC} \) of combinatorial model categories, left Quillen functors, and left Quillen equivalences.
- The relative category \( \text{CRC} \) of combinatorial relative categories, homotopy cocontinuous relative functors, and Dwyer–Kan equivalences.
- The relative category \( \text{PrL} \) of presentable quasicategories, left adjoint functors, and equivalences.

These equivalences are implemented in two flavors:

- Working in the Zermelo–Fraenkel set theory, we have a Dwyer–Kan equivalence of relative categories \( \text{CMC} \) (Definition 3.7), \( \text{CRC} \) (Definition 4.1), \( \text{PrL} \) (Definition 5.1), and \( \text{PrL}' \) (Definition 5.3).
- Assuming the existence of a strongly inaccessible cardinal \( U \), we have a Dwyer–Kan equivalence of relative categories \( \text{CMC}_U \) (Definition 3.3), \( \text{CRC}_U \) (Definition 4.4), \( \text{PrL}_U \) (Definition 5.2), and \( \text{PrL}'_U \) (Definition 5.4).

Furthermore, these equivalences are compatible with each other, as explained in the proof. Used in 1.0*, 1.2, 1.4*.

Proof. Combine Theorem 8.7 and Theorem 9.10 to establish the Dwyer–Kan equivalences \( \text{CMC} \to \text{CRC} \) (Definition 6.1) and \( \text{PrL} \to \text{PrL} \) (Definition 9.1). We also have Dwyer–Kan equivalences \( \text{CMC}_U \to \text{CMC}_U \) (Proposition 3.10), \( \text{CRC}_U \to \text{CRC}_U \) (Proposition 4.5), \( \text{PrL}_U \to \text{PrL}_U \) (Proposition 5.4), where \( \text{CMC}_U \) (Definition 3.8), \( \text{CRC}_U \) (Definition 4.3), \( \text{PrL}_U \) (Definition 5.2) are certain full subcategories of \( \text{CMC} \), \( \text{CRC} \) and \( \text{PrL} \). Restricting the Dwyer–Kan equivalences \( \text{CMC} \to \text{PrL} \) to the corresponding full subcategories establishes Dwyer–Kan equivalences \( \text{Co} \) and \( \text{N} \) to the corresponding full subcategories establishes Dwyer–Kan equivalences \( \text{Co} \) and \( \text{N} \) to the corresponding full subcategories establishes Dwyer–Kan equivalences \( \text{Co} \) and \( \text{N} \) to the corresponding full subcategories establishes Dwyer–Kan equivalences \( \text{Co} \) and \( \text{N} \).

As shown in Proposition 8.8 and Proposition 9.11, these equivalences are compatible with certain naturally defined relative functors \( \text{Co} \) and \( \text{N} \). As shown in Proposition 8.8 and Proposition 9.11, these equivalences are compatible with certain naturally defined relative functors \( \text{Co} \) and \( \text{N} \). As shown in Proposition 8.8 and Proposition 9.11, these equivalences are compatible with certain naturally defined relative functors \( \text{Co} \) and \( \text{N} \).

The following theorem is a solution to (one precise formulation of) Problem 9 on Hovey’s list [1999.c].

Theorem 1.2. The following relative categories are Dwyer–Kan equivalent (and hence also equivalent to the categories in Theorem 1.1).

- The relative category \( \text{CMC} \) of combinatorial model categories and left Quillen functors.
- Left proper combinatorial model categories and left Quillen functors.
- Simplicial combinatorial model categories and simplicial left Quillen functors.
- Simplicial left proper combinatorial model categories and simplicial left Quillen functors.

In all four cases, weak equivalences are given by left Quillen equivalences. Used in 1.4*

Proof. See Proposition 8.3

Theorem 1.3. The following relative categories are Dwyer–Kan equivalent.

- Cartesian combinatorial model categories, left Quillen functors, and left Quillen equivalences.
- Same as the previous item, but additionally required to be simplicial (with simplicial left Quillen functors), or left proper, or both.
- Cartesian closed presentable quasicategories.
Proof. See Proposition 8.4.

We also compare the resulting constructions to derivators. Derivators by their nature are not fully homotopy coherent, so some truncation must be performed. Given a relative category $C$ we can extract from it its homotopy $(2,1)$-category by taking the hammock localization $\mathcal{H}_C$ of $C$ and replacing each simplicial hom-object $\mathcal{H}_C(X,Y)$ with its fundamental groupoid.

**Theorem 1.4.** The following $(2,1)$-categories are equivalent.
- The homotopy $(2,1)$-category of presentable quasicategories.
- The homotopy $(2,1)$-category of combinatorial model categories.
- The homotopy $(2,1)$-category of combinatorial left proper model categories.
- The $(2,1)$-category of presentable derivators, left adjoints, and isomorphisms.

**Proof.** The equivalence of the first and second $(2,1)$-categories follows from Theorem 1.1. The equivalence of the second and third $(2,1)$-categories follows from Theorem 1.2. For the equivalence of the third and fourth $(2,1)$-categories, see Renaudin [2006, Theorem 3.4.4].

**Remark 1.5.** The Dwyer–Kan equivalence between $\text{CRC}$ and $\text{PrL}$ shown in Theorem 9.10 works abstractly with any pair of Quillen equivalent models for $(\infty,1)$-categories, since all what is used in Theorem 9.10 is a Quillen equivalence $\mathcal{N} \dashv \mathcal{K}$ together with a fibrant replacement functor $\mathcal{R}$ and a cofibrant replacement functor (i.e., the identity functor for the Joyal model structure, but a nontrivial functor for other models). In particular, the same proof establishes Dwyer–Kan equivalences between appropriate versions of relative categories of complete Segal spaces, Segal categories, simplicial categories, etc. We do not include proofs in this paper because doing so would require us to develop notions of homotopy colimits, homotopy ind-completions, and homotopy local presentability in each of these settings, and then show their compatibility with each other. However, one can also transport these notions from a model where they are already developed (such as quasicategories) along derived Quillen equivalences connecting quasicategories to whatever models we are interested in. Indeed, this is essentially how we defined objects, morphisms, and weak equivalences of $\text{CRC}$. With this convention, Theorem 9.10 immediately yields Dwyer–Kan equivalences of the relative subcategories of complete Segal spaces, Segal categories, simplicial categories, marked simplicial sets, quasicategories, relative categories, and other models of $(\infty,1)$-categories, once we replace $\text{CRC}$ with an analogously defined relative category where we take as objects the relevant model of a homotopy locally presentable category and as morphisms the relevant model of a homotopy cocontinuous functor.

### 1.6. Previous work

Lurie [2017, Proposition A.3.7.6] shows that any presentable quasicategory is equivalent to the homotopy coherent nerve of the category of bifibrant objects of a combinatorial simplicial model category. In the same proposition, he shows that the underlying quasicategory of a model category of simplicial presheaves is a presentable quasicategory. Combined with Dugger [2000, Propositions 3.2 and 3.3], this shows that the underlying quasicategory of a combinatorial model category is a presentable quasicategory. This is essentially how we defined objects, morphisms, and weak equivalences of $\text{CRC}$. With this convention, Theorem 9.10 immediately yields Dwyer–Kan equivalences of the relative subcategories of complete Segal spaces, Segal categories, simplicial categories, marked simplicial sets, quasicategories, relative categories, and other models of $(\infty,1)$-categories, once we replace $\text{CRC}$ with an analogously defined relative category where we take as objects the relevant model of a homotopy locally presentable category and as morphisms the relevant model of a homotopy cocontinuous functor.

Renaudin [2006, Theorem 3.4.4] proves that the functor from the localization of the 2-category of combinatorial left proper model categories at left Quillen equivalences to the 2-category of presentable derivators, left adjoints, and modifications is an equivalence of 2-categories. Arlin [2016, Theorems 4.1, 5.1, 6.4] establishes analogous results for quasicategories. Low [2013, Theorem 4.15] establishes an equivalences of bicategories of complete Segal spaces and quasicategories, based on the work of Riehl–Verity [2013, A.4] on the 2-category of quasicategories. Szumiło [2014, B] establishes an equivalence between the fibration categories of cocomplete quasicategories and cofibration categories.

Rezk–Schwede–Shipley [2000, Theorem 1.1] show that any left proper cofibrantly generated model category that satisfies a certain realization axiom introduced there, is Quillen equivalent to a simplicial model category. In Theorem 1.2 they show that the existence of a Quillen equivalence between such model categories is equivalent to the existence of a left Quillen equivalence between $\text{CRC}$ and $\text{PrL}$.
categories implies the existence of a simplicial Quillen equivalences. Dugger [1998, Theorem 1.2] proves that a left proper combinatorial model category is Quillen equivalent to a simplicial left proper combinatorial model category. Dugger [2000, Corollary 1.2] drops the left properness assumption from the previous theorem.

1.7. Prerequisites

We assume familiarity with basics of the following topics from homotopy theory. Appropriate references will be given throughout the text.

- Locally presentable and accessible categories, including regular cardinals, $\lambda$-filtered colimits, $\lambda$-accessible categories, $\lambda$-accessible functors, locally $\lambda$-presentable categories, $\lambda$-presentable objects (denoted by $K_\lambda$), $\lambda$-ind-completions (denoted by $\text{Ind}^\lambda$), the sharp ordering of regular cardinals (denoted by $\kappa \triangleleft \lambda$). See Gabriel–Ulmer [1971], Makkai–Paré [1989, b], and Adámek–Rosicky [1994].

- Simplicial homotopy theory, including simplicial sets, simplicial maps, simplicial weak equivalences, and the simplicial Whitehead theorem (Proposition 2.46). See Goerss–Jardine [1999, a] and Dugger–Isaksen [2002, b].

- Model categories, including model structures, left Quillen functors, projective model structures on presheaves, Reedy model structures, left Bousfield localizations. See Hovey [1999, b], Hirschhorn [2003], and Barwick [2007, b].

- Relative categories, including relative functors, simplicial categories, hammock localizations (denoted by $\mathcal{H}$). See Dwyer–Kan [1980, a, 1980, b, 1980, d] and Barwick–Kan [2010].

- Quasicategories, including the Joyal model structure, limits and colimits in quasicategories, $\lambda$-ind-completions of quasicategories, $\lambda$-presentable quasicategories. See Joyal [2002, a], Lurie [2017, 2021], and Cisinski [2019, a].

1.8. Further directions

We expect the methods developed in this paper to be applicable to other similar statements, some of which are indicated in Conjecture 1.9 and Conjecture 1.10. Considerations of length prevent us from including proofs in this article.

Conjecture 1.9. The relative functor from the relative category of combinatorial symmetric monoidal model categories to the relative category of closed symmetric monoidal presentable quasicategories that sends a monoidal model category to its underlying symmetric monoidal quasicategory is a Dwyer–Kan equivalence of relative categories. In particular, the underlying quasicategories are also equivalent. Used in 1.8.

Nikolaus–Sagave [2015, c, Theorem 1.1, Theorem 2.8] show that the underlying symmetric monoidal quasicategory functor is homotopy essentially surjective and homotopy full on 1-morphisms.

Conjecture 1.10. Fix a combinatorial symmetric monoidal model category $V$. The relative functor from the relative category of combinatorial $V$-enriched model categories to the relative category of presentable $V$-enriched quasicategories that sends an enriched model category to its underlying enriched quasicategory is a Dwyer–Kan equivalence of relative categories. In particular, the underlying quasicategories are also equivalent. Used in 1.8.

Haugseng [2013, a, Theorem 5.8] shows that the underlying quasicategory of the relative category of $V$-enriched small categories, $V$-enriched functors, and Dwyer–Kan equivalences is equivalent to the quasicategory of $V$-enriched small quasicategories, where $V$ is the underlying symmetric monoidal quasicategory of $V$.

1.11. Acknowledgments

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2 Preliminaries

In this paper, we adopt a convention that a category need not be small or locally small.

Definition 2.1. A category is given by a class $O$ of objects, a class $M$ of morphisms, together with source, target, identity, and composition maps that satisfy the usual axioms. In particular, morphisms $X \to Y$ between objects $X, Y$ in a category $C$ can form a proper class $C(X,Y)$. A locally small category is a category $C$ such that for any objects $X, Y \in C$, the class $C(X,Y)$ is a set. A small category is a category $C$ such that the class of objects $O$ and the class of morphisms $M$ are both sets. An essentially small category is a category $C$ that is equivalent to a small category. Used in $\square$.

Definition 2.2. Suppose $\lambda$ is a regular cardinal. A $\lambda$-small set is a set $X$ of cardinality strictly less than $\lambda$. A $\lambda$-small category is a small category $C$ whose set of morphisms is a $\lambda$-small set.

2.3. Accessible categories

We now review some of the more specialized definitions from Low [2014]. A good example of a category $C$ to keep in mind is the category of $\lambda$-presentable objects in some locally presentable category, which is an essentially small category, in fact a small category according to Remark 2.12. Thus, Definition 2.4 can be seen as defining analogues of the usual notions (like that of a $\kappa$-presentable object, $\kappa$-accessible category, locally $\kappa$-presentable category) in the setting of small categories whose objects are limited in size by some larger regular cardinal $\lambda$. This relationship is further clarified by Proposition 2.5.

Definition 2.4. (Low [2014, Definition 1.2].) Given regular cardinals $\kappa \leq \lambda$, a $(\kappa, \lambda)$-presentable object $A \in C$ such that the functor $C(A, -): C \to \text{set}$ preserves $\lambda$-small $\kappa$-filtered colimits. The full subcategory of $(\kappa, \lambda)$-presentable objects in $C$ is denoted by $K_{\kappa, \lambda}(C)$. A $\kappa$-presentable object is an object that is $(\kappa, \lambda)$-presentable for all regular cardinals $\lambda$. The full subcategory of $\kappa$-presentable objects in $C$ is denoted by $K_\kappa(C)$. A $(\kappa, \lambda)$-accessibly generated category (Low [2014, Definition 3.2]) is an essentially small category that admits $\lambda$-small $\kappa$-filtered colimits and every object is the colimit of some $\lambda$-small $\kappa$-filtered diagram of $(\kappa, \lambda)$-presentable objects.

Proposition 2.5. (Low [2014, Theorem 3.11].) If $\kappa \leq \lambda$ are regular cardinals, then for an idempotent-complete essentially small category $C$ the following conditions are equivalent:

- $C$ is a $(\kappa, \lambda)$-accessibly generated category.
- $\text{Ind}^\lambda(C)$ is a $\kappa$-accessible category.
- $C$ is equivalent to $K_\lambda(D)$ for some $\kappa$-accessible category $D$.

Used in $\square$.

Corollary 2.6. If $\kappa \leq \lambda$ are regular cardinals, then for an idempotent-complete essentially small category $C$ the following conditions are equivalent:

- $C$ is a $(\kappa, \lambda)$-accessibly generated category that admits $\lambda$-small colimits;
- $\text{Ind}^\lambda(C)$ is a locally $\kappa$-presentable category;
- $C$ is equivalent to $K_\lambda(D)$ for some locally $\kappa$-presentable category $D$.

Used in $\square$.

Definition 2.7. (Low [2014, Definition 2.2].) Given a regular cardinal $\kappa$, a strongly $\kappa$-accessible functor is a functor between $\kappa$-accessible categories that preserves $\kappa$-filtered colimits and $\kappa$-presentable objects. Used in $\square$.

Proposition 2.8. (Adámek–Rosický [1994, Theorem 2.19].) Every accessible functor $F$ is strongly $\lambda$-accessible for arbitrarily large regular cardinals $\lambda$; if $\kappa$ is a regular cardinal, there is a regular cardinal $\lambda \geq \kappa$ such that $F$ is strongly $\lambda$-accessible. Used in $\square$.

Proposition 2.9. Given a regular cardinal $\kappa$ and $\kappa$-accessible categories $C$ and $D$, the functors $\text{Ind}^\kappa$ and $K_\kappa$ induce an equivalence of groupoids between the groupoid of functors $K_\kappa(C) \to K_\kappa(D)$ and the groupoid of strongly $\kappa$-accessible functors $C \to D$. Given a regular cardinal $\kappa$ and locally $\kappa$-presentable categories $C$ and $D$, the functors $\text{Ind}^\kappa$ and $K_\kappa$ induce an equivalence of groupoids between the groupoid of functors $K_\kappa(C) \to K_\kappa(D)$ that preserve $\kappa$-small colimits and the groupoid of strongly $\kappa$-accessible left adjoint functors $C \to D$. Used in $\square$.
2.10. Size aspects

In this article we use the Zermelo–Fraenkel set theory with the axiom of choice. In particular, we do not assume any large cardinal axioms, since we intend the results of this paper to be usable in papers that do not assume any additional axioms.

One subtlety that emerges from this decision is that the three main relative categories of this paper (combinatorial model categories, combinatorial relative categories, and presentable quasicategories) must be defined with more care than usual, since such categories typically have a proper class of objects, so cannot themselves be elements of a class or objects in a category.

In what follows, we would like to use the functor $K_\lambda$ to construct small categories. A priori, if $C$ is a locally presentable category, then $K_\lambda(C)$ is an essentially small category that is not necessarily small. We circumvent this problem by observing that the categories $C$ we are interested in are constructed as categories of presheaves of sets or simplicial sets on a small category, or reflective localizations thereof. Accordingly, we identify a full and essentially surjective subcategory $\text{Set}$ of the category of sets such that for any regular cardinal $\lambda$, the full subcategory of $\text{Set}$ consisting of $\lambda$-small sets in $\text{Set}$ is a small category.

**Definition 2.11.** Denote by $\text{Set}$ the full subcategory of the category of sets and maps of sets on objects given by sets whose rank does not exceed their cardinality. Used in §6.1, §6.6, §6.8, §6.11, §6.14, §6.22, §6.23, §6.27.

**Remark 2.12.** Locally presentable categories can be equivalently defined as accessible reflective localizations of categories of presheaves of sets on small categories. If we interpret sets as objects of $\text{Set}$ (Definition 2.11), then for any locally presentable category $C$ and any regular cardinal $\lambda$, the category $K_\lambda(C)$ is a small category. This convention will be used throughout this article. Used in §2.4, §2.10*, §2.12, §2.13, §2.17*, §2.20, §2.21, §7.1, §7.5*.

**Remark 2.13.** Given a small category $C$ and a regular cardinal $\lambda$, we can construct the $\lambda$-ind-completion $\text{Ind}^\lambda(C)$ of $C$ as follows. Take the category of presheaves of sets on $C$ (Definition 2.11) and extract the full subcategory of presheaves that are small $\lambda$-filtered colimits of representable functors. Using Definition 2.11, the resulting category $\text{Ind}^\lambda(C)$ is locally presentable in the sense of Remark 2.12. In particular, for any regular cardinal $\mu \geq \lambda$, the category $\text{Ind}^\lambda_\mu(C) := K_\mu(\text{Ind}^\lambda(C))$ is a small category. We refer to $\text{Ind}^\lambda_\mu(C)$ as the $(\lambda, \mu)$-ind-completion of $C$. By Kashiwara–Schapira [2006.a, §6.1] (whose proof for $\text{Ind}^\lambda$ extends formally to the $\mu$-small case), the construction $\text{Ind}^\lambda_\mu$ is a pseudofunctor with the expected universal property: for every small category $D$ that admits $\mu$-small $\lambda$-filtered colimits, the canonical restriction functor

$$\text{Cat}(\text{Ind}^\lambda_\mu(C), D) \to \text{Cat}(C, D)$$

becomes an equivalence of categories if on the left side we take functors that preserve $\mu$-small $\lambda$-filtered colimits. Used in §6.1, §6.4, §6.8, §6.14, §6.22, §6.24, §6.27.

We conclude this section by examining the 1-categorical analogue of the three main relative categories of this article: $\text{CMC}$ (Definition 3.7), $\text{CRC}$ (Definition 4.1), and $\text{PrL}$ (Definition 5.1). Informally, we want to define the category $\text{LPC}$ of locally presentable categories and left adjoint functors.

Except for trivial cases, a left Quillen equivalence between combinatorial model categories never admits an inverse that is a left Quillen functor. Thus, when we later define $\text{CMC}$ in Definition 3.7 we are naturally forced to use the notion of a category equipped with a subcategory of weak equivalences, i.e., a relative category (Barwick–Kan [2010]). Furthermore, the objects we are interested in have a 2-categorical nature and we must take into account the notion of an equivalence between morphisms, e.g., left adjoint functors can be naturally isomorphic. A common approach to this is to use 2-categories, defined by Bénabou in 1965, which would lead us to develop a theory of relative 2-categories. However, relative categories themselves can encode higher homotopy groups for hom-objects by virtue of using appropriately chosen weak equivalences. Thus, we stay in the realm of 1-categories and encode all structures as relative categories.

The relative category $\text{LPC}$ can be informally described as the relative category of locally presentable categories, left adjoint functors, and equivalences of categories. This naive definition does not make sense in the usual ZFC set theory without large cardinal axioms because proper classes (such as the class of objects of a nonterminal locally presentable category) cannot be elements of other classes. We circumvent the problem by observing that a locally presentable category or a left adjoint functor between locally presentable categories can be specified using sets only, without referring to classes. The two fundamental facts that we need are as follows (see Corollary 2.6, Proposition 2.8, and Proposition 2.9 for a precise formulation):
• Any locally presentable category is the \( \lambda \)-ind-completion of a small category \( C \) that admits \( \lambda \)-small colimits, for some regular cardinal \( \lambda \).
• Any left adjoint functor between locally presentable categories is the \( \mu \)-ind-completion of a functor between small categories, for some (possibly larger) regular cardinal \( \mu \).

**Definition 2.14.** The relative category \( \text{LPC} \) of locally presentable categories is defined as follows. Objects are pairs \((\lambda, C)\), where \( \lambda \) is a regular cardinal and \( C \) is a small category that admits \( \lambda \)-small colimits. Morphisms \((\lambda, C) \to (\mu, D)\) exist if \( \lambda \leq \mu \), in which case they are functors \( C \to D \) that preserve \( \lambda \)-small colimits. Morphisms are composed by composing their underlying functors. Weak equivalences are morphisms \((\lambda, C) \to (\mu, D)\) such that the functor \( C \to D \) is equivalent to the canonical inclusion \( C \to \text{Ind}^\lambda_\mu(C) \).

**Remark 2.13.** Used in \( \text{LPC} \).

In particular, for \( \lambda = \mu \), a weak equivalence \((\lambda, C) \to (\mu, D)\) is simply an equivalence of categories, since the \((\lambda, \lambda)\)-ind-completion is equivalent to the idempotent completion and categories that admit \( \lambda \)-small colimits are automatically idempotent complete.

### 2.15. Mapping spaces of locally presentable categories

The sole purpose of this subsection is to establish **Proposition 2.17** and supporting definitions, which are only used in **Proposition 2.22**. The latter proposition is not used anywhere else in the paper and only serves a clarifying role. Thus, the reader may want to skip directly to §2.18 if the definition of \( \text{LPC} \) appears to be sufficiently motivated.

A priori, introducing weak equivalences in a category like \( \text{LPC} \) may produce nontrivial higher homotopy groups (e.g., \( \pi_2 \)) in the hom-objects of the hammock localization. **Proposition 2.17** shows that this is not the case and, furthermore, hom-objects are weakly equivalent to nerves of categories one can naturally expect as mapping objects in \( \text{LPC} \). This shows that our definition of the relative category \( \text{LPC} \) is reasonable.

We remark that the description of the mapping object as the nerve of the category of zigzags could be promoted to a genuine zigzag of weak equivalences of \((\infty, 1)\)-categories. Since zigzags cannot be composed in a strictly associative way, we would need to encode such categories as Segal categories (Dwyer–Kan–Smith [1989, §7]). Although this is possible, the resulting writeup would need several additional pages, while being at best tangential to the main theorems of this article, so we do not include it here and are content to prove a statement for an individual hom-space.

**Definition 2.16.** Given objects \( X, Y \in \text{LPC} \), the simplicial set \( \text{LPC}_{\text{Set}}(X, Y) \) is defined as the nerve of the following category. Objects are zigzags \( X \to Z \leftarrow Y \) in the category \( \text{LPC} \), where \( Y \to Z \) is a weak equivalence. Morphisms \((X \to Z \leftarrow Y) \to (X \to Z' \leftarrow Y)\) are weak equivalences \( Z \to Z' \) that make the triangles \( XZZ' \) and \( YZZ' \) commute. We have a canonical map

\[
\text{LPC}_{\text{Set}}(X, Y) \to \mathcal{H}_{\text{LPC}}(X, Y)
\]

that interprets zigzags and their morphisms as hammocks.

**Proposition 2.17.** The canonical map

\[
\text{LPC}_{\text{Set}}(X, Y) \to \mathcal{H}_{\text{LPC}}(X, Y)
\]

(Definition 2.16) is a simplicial weak equivalence. Furthermore, both simplicial sets are homotopy 1-truncated, i.e., are disjoint unions of spaces of the form \( K(\pi, 1) \).

**Proof.** We show that the relative category \( \text{LPC} \) satisfies the conditions of Dwyer–Kan [1980, Proposition 8.1], so by Dwyer–Kan [1980, Proposition 8.2] it admits a homotopy calculus of left fractions, which implies (Dwyer–Kan [1980, Proposition 7.2]) that the nerve of the category of zigzags is weakly equivalent to the corresponding hom-object in the hammock localization, as desired.

Indeed, weak equivalences in \( \text{LPC} \) satisfy the 2-out-of-3 property and \( \text{LPC} \) admits a functorial commutation of weak equivalences past morphisms, as exhibited by the following construction. Suppose \( s: (X', C') \leftarrow (\lambda, C) \) is a weak equivalence and \( u: (\lambda, C) \to (\mu, D) \) is a morphism in \( \text{LPC} \). We need to construct a functorial weak equivalence \( t: (\mu, D) \to (\nu, \text{Ind}^\nu_\mu D) \) and a functorial morphism \( w: (X', C') \to (\nu, \text{Ind}^\nu_\mu D) \) such that...

\[•\]
Continuing Definition 2.11, given an inaccessible cardinal, we define the category $\mathbf{Set}_{\lambda}$ as the full subcategory of the category of $\lambda$-small sets, where $\lambda$ is a strongly inaccessible cardinal. A $\lambda$-small set is a set of rank less than $\lambda$. Using $\lambda$-small sets on a $\lambda$-ind-completion of groupoids, we define the category $\mathbf{Grpd}$. A 2-filtered 2-category is a weak equivalence; then extend this pseudofunctor to a pseudonaturally equivalent strict functor using the construction of Power [1989, §4.2]. This construction is applicable for any object $(\mu, D)$ in $\mathbf{PC}$ the class of morphisms with codomain $(\mu, D)$ is a set thanks to the conventions of Definition 2.11.

Finally, to prove that every hom-object $[\mathbf{PC}_{\lambda}, X, Y]$ is homotopy 1-truncated, i.e., weakly equivalent to the nerve of a groupoid, we argue in a manner similar to Dwyer–Kan [1980, proof of Proposition 7.3], except that in our case we use 2-filtered 2-categories instead of filtered categories. Consider the (2,1)-category $Y/W$, defined as follows. Objects are weak equivalences $Y \to Z$. Morphisms $y: (Y \to Z) \to (Y' \to Z')$ are weak equivalences $y: Z \to Z'$ that make the triangle $YZZ'$ commute. 2-morphisms $y \to y'$ are natural isomorphisms. The resulting (2,1)-category $Y/W$ is a 2-filtered 2-category (Dubuc–Street [2006, Definition 2.1]).

By construction, the hom-object $[\mathbf{PC}_{\lambda}, X, Y]$ is the nerve of the 2-colimit of the diagram $Y/W \to \mathbf{Grpd}$ that sends an object $(Y \to Z) \in Y/W$ to the groupoid $\mathbf{Grpd}(X, Z)$ of morphisms $X \to Z$ in $\mathbf{PC}$ and their natural isomorphisms, a morphism $y: (Y \to Z) \to (Y' \to Z')$ to the induced functor of groupoids

$$\mathbf{Grpd}(X, y): \mathbf{Grpd}(X, Z) \to \mathbf{Grpd}(X, Z'),$$

and a 2-morphism $y \to y'$ to the induced isomorphism of functors $\mathbf{Grpd}(X, y) \to \mathbf{Grpd}(X, y')$. A 2-filtered 2-colimit of groupoids is also a homotopy filtered colimit, which means that the simplicial set $[\mathbf{PC}_{\lambda}, X, Y]$ is a homotopy filtered colimit of nerves of groupoids, hence is itself weakly equivalent to the nerve of a groupoid.

2.18. **Universes**

Although we do not assume any large cardinal axioms for our main results, we find it useful to formulate explicit comparison results for existing definitions of the categories $\mathbf{CMC}$, $\mathbf{CRM}$, $\mathbf{CRC}$, $\mathbf{PrL}$, $\mathbf{CRC}$, $\mathbf{CMC}$, $\mathbf{CMC}$ that use large cardinals. For the purposes of formulating these three comparison results, it suffices to assume the existence of a strongly inaccessible cardinal, i.e., a Grothendieck universe.

We start with the simpler definition, assuming the existence of a strongly inaccessible cardinal $U$. The following definition collects the pertinent adjustments to the notions of category theory that rely on the distinction between sets and classes.

**Definition 2.19.** Suppose $U$ is a strongly inaccessible cardinal. A $U$-small set is a set of rank less than $U$. A $U$-small class is a set whose elements are $U$-small sets. A $U$-small category is a category whose classes of objects and morphisms are $U$-small sets. A locally $U$-small category is a category whose classes of objects and morphisms are $U$-small classes, and hom-classes between any pair of objects are $U$-small sets. Using $U$-small categories, we define $U$-small limits and colimits (defined using $U$-small diagrams), $U$-complete and $U$-cocomplete categories (defined using $U$-small limits and colimits), presheaves of $U$-small sets on a $U$-small category (which themselves form a locally $U$-small category, and also force all notions of categories defined in the remainder of this definition to be locally $U$-small), $U$-$\Lambda$-presentable objects (defined using $U$-small $\Lambda$-filtered colimits), $U$-$\lambda$-ind-completions (defined by closing representables under $U$-small $\lambda$-filtered colimits in presheaves of $U$-small sets), $U$-accessible categories (defined using $U$-$\lambda$-ind-completions of $U$-small categories), $U$-accessible functors (defined using $U$-$\lambda$-ind-completions of functors between $U$-small categories), locally $U$-presentable categories (defined using $U$-cocomplete $U$-accessible categories), cofibrant $U$-generation (defined using $U$-small sets), and $U$-combinatorial model categories (defined using cofibrantly $U$-generated locally $U$-presentable model categories).

**Definition 2.20.** Continuing Definition 2.11, given an inaccessible cardinal $U$, we define the category $\mathbf{Set}_U$ as the full subcategory of the category of $U$-small sets consisting of sets whose rank does not exceed their cardinality.
Remark 2.21. Continuing Remark 2.12, given a locally \( U \)-presentable category \( C \), the category \( K^U_\kappa(C) \) is essentially \( U \)-small, but not necessarily \( U \)-small. However, once we adopt the convention that locally \( U \)-presentable categories are defined as accessible reflective localizations of categories of presheaves valued in the category \( \text{Set}_U \) (Definition 2.20) on a \( U \)-small category, then \( K^U_\kappa(C) \) is a \( U \)-small category for any regular cardinal \( \kappa < U \). Thus, \( K^U_\kappa \) is a functor from the category of locally \( U \)-presentable categories and strongly \( U \)-\( \kappa \)-accessible left adjoint functors (see Definition 2.7 with conventions of Definition 2.19) to the category of \( U \)-small categories with \( \kappa \)-small colimits and functors that preserve \( \kappa \)-small colimits. Used in 2.22, 2.27*

Remark 2.22. Continuing Remark 2.13, we define the \( U \)-\( \lambda \)-ind completion of a small category \( C \) using the conventions of Remark 2.21. The resulting category \( \text{Ind}^U_\lambda(C) \) is locally \( U \)-presentable in the sense of Remark 2.21 and \( \text{Ind}^U_\lambda(C) = K_\mu(\text{Ind}^\lambda(C)) \) is a \( U \)-small category for any regular cardinal \( \mu \). Used in 2.22*

We now use these definitions to define a simpler version \( \text{LPC}_U \) of the relative category \( \text{LPC} \) (Definition 1.14) whose objects are actual categories, as opposed to pairs \((\lambda, C)\) that we used for \( \text{LPC} \). The price to pay is that the relative category \( \text{LPC}_U \) depends on the strongly inaccessible cardinal \( U \) in an essential way. Warning 2.24.

Definition 2.23. Assuming \( U \) is a strongly inaccessible cardinal, the relative category \( \text{LPC}_U \) of locally \( U \)-presentable categories is defined as follows. Objects are locally \( U \)-presentable categories. Morphisms are left adjoint functors. Weak equivalences are equivalences of categories. Used in 2.24, 2.25, 2.26

Warning 2.24. Suppose \( U < U' \) are strongly inaccessible cardinals. If a category is locally \( U \)-presentable and locally \( U' \)-presentable, then it is a preorder. Furthermore, \( \text{LPC}_{U'} \) is not equivalent to \( \text{LPC}_U \). In particular, the naive candidates for such equivalences do not work. For example, if we try to write down a functor \( \text{LPC}_{U'} \to \text{LPC}_U \) that sends a locally \( U' \)-presentable category \( C \) to \( K^U_\kappa(C) \) (i.e., \((U, U')\)-\( \kappa \)-presentable objects in \( C \) as in Definition 2.4), this construction is only applicable to strongly \( U \)-accessible left adjoint functors, not all left adjoint functors. Likewise, if we try to write down a functor \( \text{LPC}_U \to \text{LPC}_{U'} \) that sends a locally \( U \)-presentable category \( C \) to \( \text{Ind}^U_\lambda(C) \) (i.e., its \((U, U')\)-ind-completion as in Remark 2.13), this construction only produces strongly \( U \)-accessible functors between locally \( U \)-presentable categories, so does not give a full functor. This explains why in Proposition 2.27, both relative categories must depend on \( U \). Used in 2.22*

In order to compare the relative categories \( \text{LPC}_U \) (Definition 2.14) and \( \text{LPC}_U \) (Definition 2.23), we must take Warning 2.24 into account and modify the relative category \( \text{LPC}_U \) (Definition 2.14) to ensure that the resulting relative category \( \text{LPC}_U \) can be weakly equivalent to the category \( \text{LPC}_U \). (Definition 2.23).

Definition 2.25. Given a strongly inaccessible cardinal \( U \), the relative category \( \text{LPC}_U \) is defined as the full subcategory of \( \text{LPC}_U \) (Definition 2.14) on objects \((\lambda, C)\), where \( \lambda < U \) and \( C \) is a \( U \)-small category. Used in 2.25*

In order to compare the relative categories \( \text{LPC}_U \) (Definition 2.24) and \( \text{LPC}_U \) (Definition 2.23), we define a comparison functor between them.

Definition 2.26. Assuming \( U \) is a strongly inaccessible cardinal, the relative functor

\[
\text{Ind}_U: \text{LPC}_U \to \text{LPC}_U
\]

between the relative categories \( \text{LPC}_U \) (Definition 2.25) and \( \text{LPC}_U \) (Definition 2.23) is defined as follows.

The functor \( \text{Ind}_U \) sends an object \((\lambda, C) \in \text{LPC}_U \) to the category \( \text{Ind}_U(\lambda, C) \) constructed as follows. Objects are pairs

\[
(F: (\kappa, B) \to (\lambda, C), X \in \text{Ind}^\lambda_U(B)),
\]

where \( F \) is a morphism in \( \text{LPC}_{U'} \). Morphisms

\[
(F: (\kappa, B) \to (\lambda, C), X \in \text{Ind}^\lambda_U(B)) \to (F': (\kappa', B') \to ((\lambda, C), X' \in \text{Ind}^\lambda_U(B'))
\]

are given by morphisms

\[
h: \text{Ind}^\lambda_U(F)(X) \to \text{Ind}^{\lambda, \lambda'}(F')(X')
\]
in the category \( \text{Ind}_U^\lambda(C) \). Here

\[
\text{Ind}_U^{\kappa,\lambda}(F): \text{Ind}_U^\kappa(B) \to \text{Ind}_U^\lambda(C), \quad \text{Ind}_U^{\kappa',\lambda}(F'): \text{Ind}_U^\kappa(B') \to \text{Ind}_U^\lambda(C)
\]

are the induced \( U \)-cocontinuous functors constructed using the formula for pointwise left Kan extensions; the latter construction picks out one specific functor with the desired universal property. Morphisms are composed in the obvious way, with the composition being strictly associative and unital. By construction, the resulting category \( \text{Ind}_U^\lambda(C) \) is equivalent to its full subcategory on objects with \( F = \text{id}_{(\lambda, C)} \), which is equivalent to \( \text{Ind}_U^\lambda(C) \) and therefore is an object in \( \text{LPC} \).

The functor \( \text{Ind}_U \) sends a morphism \( G: (\lambda, C) \to (\mu, D) \) to the functor

\[
\text{Ind}_U(\lambda, C) \to \text{Ind}_U(\mu, D), \quad (F, X) \mapsto (G \circ F, X), \quad h \mapsto \beta \circ \text{Ind}_U^{\lambda,\mu}(G)(h) \circ \alpha,
\]

where

\[
\alpha: \text{Ind}_U^{\kappa,\lambda}(G \circ F)(X) \to \text{Ind}_U^{\kappa,\mu}(G)(\text{Ind}_U^{\kappa,\lambda}(F)(X)), \quad \beta: \text{Ind}_U^{\kappa,\mu}(G)(\text{Ind}_U^{\kappa',\lambda}(F')(X')) \to \text{Ind}_U^{\kappa',\mu}(G \circ F')(X')
\]

are the canonical isomorphisms. This yields a (strict) functor \( \text{LPC} \to \text{LPC} \) because the composition of \( \alpha \) and \( \beta \) for the same data of \( (F, X) \) yields the identity map on \( \text{Ind}_U^{\kappa,\mu}(G)(\text{Ind}_U^{\kappa,\lambda}(F)(X)) \).

Finally, the functor \( \text{Ind}_U \) is a relative functor, which follows immediately from the fact that the category \( \text{Ind}_U(\lambda, C) \) is equivalent to its full subcategory generated by objects with \( F = \text{id}_{(\lambda, C)} \), which itself is equivalent to \( \text{Ind}_U(\lambda, C) \), so a weak equivalence \( (\lambda, C) \to (\mu, \text{Ind}_U^\lambda(C)) \) is sent to the equivalence \( \text{Ind}_U^\lambda(C) \to \text{Ind}_U^\mu(\text{Ind}_U^\lambda(C)). \)

The following proposition and its proof serve as a base for the three comparison results: [Proposition 3.10, Proposition 4.3, Proposition 5.4].

**Proposition 2.27.** Assuming \( U \) is a strongly inaccessible cardinal, the relative (strict) functor

\[
\text{Ind}_U: \text{LPC} \to \text{LPC}
\]

of [Definition 2.26](#) is a [Dwyer–Kan equivalence](#) of relative categories. [Used in 2.27](#) [12].

**Proof.** [Dwyer–Kan equivalences](#) of relative categories are stable under filtered colimits. We introduce filtrations on \( \text{LPC} \) and \( \text{LPC} \) that are respected by the functor \( \text{Ind}_U \). We then show that \( \text{Ind}_U \) induces a [Dwyer–Kan equivalence](#) on each step of the filtration.

Fix a regular cardinal \( \nu \). Define \( \text{LPC}^\nu \) as the full subcategory of \( \text{LPC} \) consisting of objects \( (\lambda, C) \) for which \( \lambda \leq \nu \). Define \( \text{LPC}^\nu \) as the full subcategory of \( \text{LPC} \) consisting of objects given by locally \( U,\nu \)-presentable categories and morphisms given by strongly \( U,\nu \)-accessible functors [Definition 2.7](#). By construction and [Proposition 2.4](#), the functor \( \text{Ind}_U \) restricts to a functor

\[
\text{Ind}_{U,\nu}: \text{LPC}^{\nu} \to \text{LPC}^{\nu}.
\]

To show that \( \text{Ind}_U \) is a [Dwyer–Kan equivalence](#), it suffices to construct a pseudoinverse (strict) functor

\[
\text{K}_{U,\nu}: \text{LPC}^{\nu} \to \text{LPC}^{\nu}
\]

together with pseudonatural equivalences

\[
\eta: \text{id}_{\text{LPC}^{\nu}} \to \text{K}_{U,\nu} \circ \text{Ind}_{U,\nu}
\]

and

\[
\varepsilon: \text{id}_{\text{LPC}^{\nu}} \to \text{Ind}_{U,\nu} \circ \text{K}_{U,\nu}.
\]
Indeed, the functors $\text{Ind}_{U,\nu}$ and $K_{U,\nu}$ induce the corresponding functors on hammock localizations. Furthermore, using the Gray tensor product $\otimes$, the pseudonatural equivalences $\eta$ and $\varepsilon$ are converted (Johnson–Yau [2013.a, Corollary 12.2.30]) into strict 2-functors $\text{LPC}_{U,\nu} \otimes \{0 \to 1\} \to \text{LPC}_{U,\nu}$ and we can conclude by observing that the hammock localization of $\text{LPC}_{U,\nu} \otimes \{0 \to 1\}$ (and likewise for $\text{LPC}_{U,\nu} \otimes \{0 \to 1\}$) is weakly equivalent to the hammock localization of $\text{LPC}_{U,\nu} \otimes \{0\} \equiv \text{LPC}_{U,\nu}$, with the weak equivalence implemented using the maps $\{0\} \to \{0 \to 1\}$ and $\{0 \to 1\} \to \{0\}$.

The functor $K_{U,\nu}$ sends $C \in \text{LPC}_{U,\nu}$ to the object $(\nu, K_{U,\nu}(C)) \in \text{LPC}_{U,\nu}$, which is well-defined by Remark 2.21. (At this point, the presence of a filtration is crucial: without having $\nu$ at our disposal, we would not be able to define the first component of an object in $\text{LPC}_{U,\nu}$ in a functorial way.) The functor $K_{U,\nu}$ sends a functor $F: C \to D$ in $\text{LPC}_{U,\nu}$ to the restriction $(\nu, K_{U,\nu}(C)) \rightarrow (\nu, K_{U,\nu}(D))$, which is well-defined because $F$ is a strongly $U$-accessible functor.

The pseudonatural weak equivalence $\eta: \text{id}_{\text{LPC}_{U,\nu}} \to K_{U,\nu} \circ \text{Ind}_{U,\nu}$ is given on an object $(\lambda, C) \in \text{LPC}_{U,\nu}$ by the embedding

$$(\lambda, C) \rightarrow (\nu, K_{U,\nu}(\text{Ind}_{U,\nu}(\lambda, C)))$$

induced by the Yoneda embedding functor

$$C \rightarrow \text{Ind}_{U,\nu}(\lambda, C), \quad X \mapsto (\text{id}_{\lambda, C}, Y(X)).$$

This morphism is indeed a weak equivalence, since taking $U$-ind-completions on both sides yields a functor equivalent to the identity functor $\text{Ind}_{U}(\lambda, C) \rightarrow \text{Ind}_{U}(\lambda, C)$.

The lax naturality constraint for $\eta$ is a natural isomorphism between two compositions in the following diagram:

$$(\lambda, C) \rightarrow (\nu, K_{U,\nu}(\text{Ind}_{U,\nu}(\lambda, C))) \downarrow \downarrow (\mu, D) \rightarrow (\nu, K_{U,\nu}(\text{Ind}_{U,\nu}(\mu, D))),$$

which boils down to constructing a natural isomorphism for the two compositions in

$$C \rightarrow \text{Ind}_{U}(\lambda, C) \downarrow \downarrow \text{Ind}_{U}(\nu, (\lambda, C)),
\text{Ind}_{U}(\mu, D) \rightarrow \text{Ind}_{U}(\mu, D),$$

which is constructed using the universal property of $\text{Ind}_{U}$ (Remark 2.13). The latter property also implies the lax unity and lax naturality properties for pseudonatural transformations.

The pseudonatural weak equivalence $\varepsilon: \text{id}_{\text{LPC}_{U,\nu}} \to \text{Ind}_{U,\nu} \circ K_{U,\nu}$ is given on an object $D \in \text{LPC}_{U,\nu}$ by the restricted Yoneda embedding

$$D \rightarrow \text{Ind}_{U,\nu}(K_{U,\nu}(D)) = \text{Ind}_{U}(\nu, K_{U,\nu}(D)), \quad d \mapsto (\text{id}_{\nu, K_{U,\nu}(D)}, Y(d)).$$
(Recall that by Remark 2.22, the right side is implemented as $U$-small $\nu$-filtered colimits of representable presheaves on $K_{U,\nu}(D)$, which allows us to make sense of the restricted Yoneda embedding.) Since $D$ is locally $U,\nu$-presentable, this morphism is indeed an equivalence of categories.

The lax naturality constraint for $\varepsilon$ is a natural isomorphism between two compositions in the following diagram:

$$
\begin{array}{ccc}
D & \longrightarrow & \text{Ind}_{U,\nu}(K_{U,\nu}(D)) \\
\downarrow & & \downarrow \\
E & \longrightarrow & \text{Ind}_{U,\nu}(K_{U,\nu}(E)).
\end{array}
$$

Such an isomorphism is constructed by the universal property of $\text{Ind}_{U,\nu}$. The latter property also implies the lax unity and lax naturality properties for pseudonatural transformations. 

The following proposition compares the hammock localizations of $\mathbf{LPC}_U$ $\mathbf{LPC}_U$, and $\mathbf{LPC}_U$ with the naive 2-categorical versions of these categories. It is not used in the remainder of the article. Recall the notation and conventions of §2.1. Again, we choose to compare only the individual hom-spaces, without attempting to assemble them into a weak equivalence of Segal categories.

**Proposition 2.28.** Given any objects $X, Y \in \mathbf{LPC}_U$ (respectively $\mathbf{LPC}_U$), the following canonical simplicial maps are weak equivalences:

$$
\mathbf{LPC}_U^{\mathbf{Set}}(X, Y) \to \mathbf{H}_{\mathbf{LPC}_U}(X, Y), \quad \mathbf{LPC}_U^{\mathbf{Set}}(X, Y) \to \mathbf{H}_{\mathbf{LPC}_U}(X, Y).
$$

The first map is constructed like in Proposition 2.17 but using only objects of $\mathbf{LPC}_U$. In the second map, $\mathbf{LPC}_U^{\mathbf{Set}}(X, Y)$ is the nerve of the groupoid whose objects are $U$-cocontinuous functors $X \to Y$ and morphisms are natural isomorphisms of such functors. 

**Proof.** The proof for the first inclusion is identical to that of Proposition 2.17, but using only objects of $\mathbf{LPC}_U$. For $\mathbf{LPC}_U$, we use the same reasoning to show that the map from an analogous simplicial set $\mathbf{LPC}_U^{\mathbf{Set}}(X, Y)$ (whose vertices are zigzags $X \to Z \leftarrow Y$) into $\mathbf{H}_{\mathbf{LPC}_U}(X, Y)$ is a simplicial weak equivalence. Observe that the natural map

$$
\mathbf{LPC}_U^{\mathbf{Set}}(X, Y) \to \mathbf{LPC}_U^{\mathbf{Set}}(X, Y)
$$

is given by the nerve of the functor that converts a morphism $X \to Y$ into a zigzag $X \to Y \leftarrow Y$, where the map $Y \to Y$ is the identity. Thus functor is faithful (by construction), and full because any zigzag $X \to Y \leftarrow Z$ (where the second map is an equivalence of categories) is weakly equivalent to a zigzag $X \to Z \leftarrow Z$, where the second map is identity. 

### 2.29. Relative categories

Consistent with our convention for categories, we do not require relative categories or simplicial categories to be locally small. In particular, in a simplicial category $C$ the hom-object $C(X,X')$ for any objects $X, X' \in C$ can have a proper class $C(X,X')_n$ of $n$-simplices for any $n \geq 0$. Thus, a simplicial category is a category enriched in simplicial classes.

The notion of a Dwyer–Kan equivalence of simplicial categories (Dwyer–Kan [1980.c, §2.4]) continues to make sense for simplicial categories that are not locally small: a simplicial functor $F : C \to D$ is a Dwyer–Kan equivalence if any object in $D$ is homotopy equivalent to $F(X)$ for some object $X \in C$ and for any object $X, X'$, the induced map $C(X,X') \to D(F(X), F(X'))$ is a simplicial weak equivalence of simplicial classes. The latter can be defined, for example, by adopting the statement of the simplicial Whitehead theorem (for nonfibrant simplicial sets) as a definition.

The hammock localization construction of Dwyer–Kan [1980.b, §2.1] continues to make sense for relative categories that are not small or locally small.

**Remark 2.30.** Recall that for a relative category $\mathcal{C}$ with objects $X, Y \in \mathcal{C}$, the simplicial class $\mathbf{H}_{\mathcal{C}}(X, Y)$ is constructed as the filtered colimit

$$
\colim_n N(C_{X,Y}^Z),
$$

where
where $Z$ runs over all possible zigzag types (a finite sequence like $←←→←→→←→$, which freely generates a category, turned into a relative category by declaring all left-pointing arrows $←$ to be weak equivalences), $N$ denotes the nerve functor, and $\mathcal{C}_X^Z$ is the category of relative functors $Z \to C$ that map the leftmost and rightmost objects of $Z$ to $X$ and $Y$ respectively. The resulting simplicial category is known as the **hammock localization** of $C$.

**Definition 2.31.** A **Dwyer–Kan equivalence** of relative categories is a relative functor whose hammock localization is a Dwyer–Kan equivalence of simplicial categories. Used in 2.27*, 8.6*.

**Definition 2.32.** A relative functor $F : C \to D$ is **homotopically essentially surjective** if any object in $D$ is weakly equivalent to an object in the image of $F$. A relative functor $F : C \to D$ is **homotopically fully faithful** if for any objects $X, X' \in C$ the induced simplicial map $H_C(X, X') \to H_D(F(X), F(X'))$ is a simplicial weak equivalence. Used in 2.33, 2.34, 7.12*, 8.1*, 8.5, 8.6.

**Proposition 2.33.** A relative functor that is homotopically essentially surjective and homotopically fully faithful is a Dwyer–Kan equivalence. Used in 8.1*.

**Remark 2.34.** In the context of Dugger [2000.b, Definition 3.1], homotopically surjective functors used in that definition coincide with homotopically essentially surjective functors.

**Definition 2.35.** (Barwick–Kan [2010, §3.3.]) A **homotopy equivalence of relative categories** is a relative functor $F : C \to D$ such that there is a relative functor $G : D \to C$ together with zigzags of natural weak equivalences $\eta : \text{id}_C \to G \circ F$ and $\varepsilon : F \circ G \to \text{id}_D$. Every homotopy equivalence of relative categories is a Dwyer–Kan equivalence because its hammock localization is a Dwyer–Kan equivalence of simplicial categories, but the converse need not hold.

The literature on homotopy limits and colimits in relative categories is sparse. While Dwyer–Hirschhorn–Kan–Smith [2004, Chapter VIII] do provide an account of homotopy limits and colimits in relative categories whose class of weak equivalences satisfies the 2-out-of-6 property, it does not readily extend to all relative categories.

Instead, we apply the right Quillen equivalence from small relative categories to simplicial sets equipped with the Joyal model structure, and then use the theory of limits and colimits in quasicategories.

**Notation 2.36.** Recall (Barwick–Kan [2010, Corollary 6.11(i)]) that the model category of small relative categories is Quillen equivalent to the Joyal model structure on simplicial sets via a right Quillen equivalence, which we denote by $\mathcal{N} : \text{RelCat} \to \mathcal{hSet}^\text{Joyal}$. The fibrant replacement functor on $\text{RelCat}$ will be denoted by $\mathcal{R} : \text{RelCat} \to \text{RelCat}$. The corresponding left Quillen equivalence will be denoted by $\mathcal{K} : \mathcal{hSet}^\text{Joyal} \to \text{RelCat}$. The fibrant replacement functor on $\text{RelCat}$ will be denoted by $\mathcal{R} : \text{RelCat} \to \text{RelCat}$.

**Definition 2.37.** Suppose $D$ is a small category. A small relative category $C$ admits $D$-indexed homotopy colimits if the small quasicategory $\mathcal{N}\mathcal{R}C$ admits $D$-indexed quasicategorical colimits. A small relative category $C$ admits $D$-indexed homotopy limits if the small quasicategory $\mathcal{N}\mathcal{R}C$ admits $D$-indexed quasicategorical limits.

**Remark 2.38.** A Dwyer–Kan equivalence $F : C \to C'$ of relative categories induces an equivalence $\mathcal{N}\mathcal{R}(F) : \mathcal{N}\mathcal{R}C \to \mathcal{N}\mathcal{R}C'$ of quasicategories. Thus, $C$ admits $D$-indexed homotopy (co)limits if and only if $C'$ does.

For the following proposition, recall that we require model categories to have finite limits and finite colimits, but not necessarily small limits or colimits.
Definition 2.39. Suppose D is a small category and C and C’ are small relative categories that admit D-indexed homotopy colimits (respectively limits). A relative functor F: C → C’ preserves D-indexed homotopy colimits if the functor 
\[ \mathcal{N}(F): \mathcal{N}(C) \to \mathcal{N}(C') \]
preserves D-indexed quasicategorical colimits. A relative functor F: C → C’ preserves D-indexed homotopy limits if the functor \( \mathcal{N}(F) \) preserves D-indexed quasicategorical limits. Used in 3.4, 3.5, 3.6.

2.40. Simplicial Whitehead theorem

Definition 2.41. Denote by sSet\(^{-}\) the category of functors \{0 → 1\} → sSet. Objects are simplicial maps (depicted vertically) and morphisms are commutative squares, where the two vertical maps are the source and target. Equip sSet\(^{-}\) with the projective model structure. Used in 2.42, 2.43, 2.44.

Remark 2.42. In the model category sSet\(^{-}\) (Definition 2.41), projectively cofibrant objects are simplicial maps that are cofibrations. Projective cofibrations are commutative squares where the top map and pushout product of left and top maps is a cofibration of simplicial sets. Fibrant objects are simplicial maps whose domain and codomain is a Kan complex.

Proposition 2.43. Fix a simplicial model category M, such as sSet\(^{-}\) (Definition 2.41). Suppose \( \alpha \) is a cofibration between cofibrant objects in M and \( \Omega \) is a fibrant object in M. The map of sets \( \text{hom}(\alpha, \Omega) \) is surjective if and only if the map of sets \( \pi_0 \text{Map}(\alpha, \Omega) \) is surjective. Here hom denotes mapping sets in M and \( \text{R Map} \) denotes derived mapping simplicial sets in M. Used in 2.44.

Proof. Since \( \alpha \) is a cofibration between cofibrant objects and the object \( \Omega \) is fibrant, the simplicial map \( \text{Map}(\alpha, \Omega) \) is a fibration between fibrant objects in simplicial sets. A fibration of simplicial sets is surjective on 0-simplices if and only if the induced map on \( \pi_0 \) is a surjection. Thus, the map of sets \( \text{hom}(\alpha, \Omega) \) is surjective if and only if the map of sets \( \pi_0 \text{Map}(\alpha, \Omega) \) is surjective. The latter map is isomorphic to \( \pi_0 \text{Map}(\alpha, \Omega) \) because \( \alpha \) is a cofibration between cofibrant objects and \( \Omega \) is fibrant. □

Proposition 2.44. Fix a simplicial model category M, such as sSet\(^{-}\) (Definition 2.41). Suppose \( \alpha \) and \( \beta \) are weakly equivalent cofibrations between cofibrant objects in M and \( \Omega \) is a fibrant object in M. Then the map of sets \( \text{hom}(\alpha, \Omega) \) is a surjection of sets if and only if \( \text{hom}(\beta, \Omega) \) is a surjection of sets. Here hom denotes mapping sets in M. Used in 2.44.

Proof. By Proposition 2.43, the map \( \text{hom}(\alpha, \Omega) \) is surjective if and only if \( \pi_0 \text{Map}(\alpha, \Omega) \) is surjective. Likewise, the map \( \text{hom}(\beta, \Omega) \) is surjective if and only if \( \pi_0 \text{Map}(\beta, \Omega) \) is surjective. Since \( \alpha \) is weakly equivalent to \( \beta \), the map of sets \( \pi_0 \text{Map}(\alpha, \Omega) \) is isomorphic to the map of sets \( \pi_0 \text{Map}(\beta, \Omega) \), which proves the lemma. □

Definition 2.45. Denote by \( \iota \) and \( \lambda \) the weakly equivalent projective cofibrations between projectively cofibrant objects in sSet\(^{-}\), given by the following commutative squares:

\[
\begin{array}{ccc}
\text{sph} & \longrightarrow & \text{ndisk} \\
\downarrow & & \downarrow_{\simeq} \\
\text{odisk} & \longrightarrow & \text{relh}
\end{array}
\]

\[
\begin{array}{ccc}
\partial \Delta^n & \longrightarrow & \Delta^n \\
\downarrow & & \downarrow \\
\Delta^n & \longrightarrow & \Delta^n \times \Delta^1 \sqcup_{\partial \Delta^n \times \Delta^1} \partial \Delta^n.
\end{array}
\]

Here \( \text{sph} \) means “sphere”, \( \text{odisk} \) means “old disk”, \( \text{ndisk} \) means “new disk”, \( \text{relh} \) means “relative homotopy”. The idea is that \( \text{relh} \) expresses a relative homotopy from the old disk \( \text{odisk} \) to the new disk \( \text{ndisk} \) relative boundary \( \text{sph} \), the sphere. We also set \( \text{disks} = \text{odisk} \sqcup_{\text{sph}} \text{ndisk} \), both disks combined, which is the boundary of \( \text{relh} \). Used in 2.47.

The following lemma reformulates a criterion due to Kan [1957, Theorem 7.2], originally due to Whitehead [1943 Theorem 1] in the case of topological spaces.

14
**Proposition 2.46.** (Dugger–Isaksen [2002].) A simplicial map \( p \) between Kan complexes is a weak equivalence if and only if the map of sets

\[
\text{hom}(\lambda, p): \text{hom}(\text{dom} \lambda, p) \to \text{hom}(\text{dom} \lambda, p)
\]

is a surjection. Here \( \text{hom} \) denotes mapping sets in the category \( sSet \). Used in 1.7*, 2.46*.

**Corollary 2.47.** Suppose \( \iota \) is a projective cofibration between projectively cofibrant objects in \( sSet \) that is weakly equivalent to the map \( \lambda \) in **Definition 2.45**. Then a simplicial map \( p \) between Kan complexes is a weak equivalence if and only if the map of sets

\[
\text{hom}(\iota, p): \text{hom}(\text{codom} \iota, p) \to \text{hom}(\text{dom} \iota, p)
\]

is a surjection of sets. Here \( \text{hom} \) denotes mapping sets in the category \( sSet \). Used in 2.48.

**Remark 2.48.** Expanding the statement of **Corollary 2.47**, a map \( p: A \to B \) of Kan complexes is a weak equivalence if and only if for any commutative square

\[
\begin{array}{ccc}
sph & \xrightarrow{a} & A \\
\downarrow & & \downarrow p \\
odisk & \xrightarrow{b} & B
\end{array}
\]

we can find maps \( d: \text{ndisk} \to A \) and \( e: \text{relh} \to B \) that make the following diagram commute:

\[
\begin{array}{ccc}
sph & \xrightarrow{a} & A \\
\downarrow & & \downarrow p \\
odisk & \xrightarrow{b} & B \\
\downarrow & & \downarrow \pi \\
\text{ndisk} & \xrightarrow{d} & A \\
\downarrow & & \downarrow p \circ \gamma \\
\text{relh} & \xrightarrow{e} & B \\
\end{array}
\]

**Corollary 2.49.** Denote by \( \Lambda \) the simplicial subset of \( \Delta^2 \) generated by the 1-simplices \( 0 \to 2 \) and \( 1 \to 2 \). A simplicial map \( p: A \to B \) is a simplicial weak equivalence whenever for any \( k \geq 0, n \geq 0 \), and a commutative square

\[
\begin{array}{ccc}
Sd^k \partial \Delta^n & \xrightarrow{\alpha} & A \\
\downarrow & & \downarrow p \\
Sd^k \Delta^n & \xrightarrow{\beta} & B
\end{array}
\]

we can construct maps

\[
\gamma: Sd^k \Delta^n \to A, \quad \Gamma: \Lambda \times Sd^k \partial \Delta^n \to A, \quad \Pi: \Delta^2 \times Sd^k \partial \Delta^n \to B
\]

such that the map \( \Gamma \) is a simplicial homotopy from \( \beta \) to \( p \circ \gamma \), the map \( \Pi \) restricts to \( p \circ \pi \) on \( \Lambda \times Sd^k \partial \Delta^n \), the map \( \pi \) restricts to \( \alpha \) on \( 0 \times Sd^k \partial \Delta^n \), the restrictions of \( \pi \) to \( 1 \times Sd^k \partial \Delta^n \) and \( \gamma \) to \( Sd^k \partial \Delta^n \) coincide, and the restrictions of \( \Pi \) to \( (0 \to 1) \times Sd^k \partial \Delta^n \) and \( \Gamma \) to \( \Delta^1 \times Sd^k \partial \Delta^n \) coincide. Used in 2.50, 8.6*.

**Remark 2.50.** We illustrate **Corollary 2.49** with the following diagrams, where the left diagram depicts \( A \) and the right diagram depicts \( B \). We depict only a single radius connecting a point \( \alpha \) (respectively \( p \circ \alpha \)) on the sphere \( Sd^k \partial \Delta^n \) to the center of \( Sd^k \Delta^n \), represented by \( \cdots \):

![Diagram](image-url)
Thus, the left diagram depicts a sphere (represented by the single vertex $\alpha$) being filled by a disk (represented by the bottom chain of morphisms going to $\cdots$), whereas the right diagram takes the image of the left diagram under $p$, and then homotopes it relative boundary to the map $\beta$, using the indicated triangle $\Pi$ together with a finite collection of squares that look like $\Gamma$. Used in 8.6.

3 Combinatorial model categories

See Beke [2000.d], Barwick [2007.b], Lurie [2017, Appendix A], Low [2014.a] for the background on combinatorial model categories. We recall some basic definitions to fix terminology.

Definition 3.1. A model structure on a category $C$ is a pair of weak factorization systems $(C, AF), (AC, F)$ such that the class $W = AF \circ AC$ satisfies the 2-out-of-3 property. A model category is a model structure on a category that admits finite limits and finite colimits. A model structure is cofibrantly generated if both weak factorization systems $(C, AF), (AC, F)$ are generated by a set of morphisms. A combinatorial model category is a cofibrantly generated model structure on a locally presentable category. A left Quillen functor between model categories is a functor that preserves elements of $C$ (cofibrations) and $AC$ (acyclic cofibrations) and all small colimits that exist in $C$.

We now review some of the more specialized definitions from Low [2014.a]. Similar notions can be found in Chorny–Rosický [2011.b].

Definition 3.2. (Low [2014.a, Definition 5.11].) Given regular cardinals $\kappa \triangleleft \lambda$, a $(\kappa, \lambda)$-miniature model category is a model category $M$ such that there exist $\lambda$-small sets of morphisms in $K_\kappa^\lambda(M)$ (Definition 2.4) that cofibrantly generate the model structure of $M$ and the underlying category of $M$ satisfies the following properties:

- $M$ is a $(\kappa, \lambda)$-accessibly generated category (Definition 2.4);
- $M$ has finite limits and $\lambda$-small colimits;
- Hom-sets in $K_\kappa^\lambda(M)$ are $\lambda$-small.

Used in 3.4, 3.7, 6.2*, 6.4*, 7.5, 7.8, 8.3, 8.3*, 8.6*.

Definition 3.3. (Low [2014.a, Definition 5.1].) Given regular cardinals $\kappa \triangleleft \lambda$, a strongly $(\kappa, \lambda)$-combinatorial model category is a combinatorial model category $M$ such that there exist $\lambda$-small sets of morphisms in $K_\kappa(M)$ that cofibrantly generate the model structure of $M$ and the underlying category of $M$ satisfies the following properties:

- $M$ is a locally $\kappa$-presentable category;
- $K_\kappa(M)$ is closed under finite limits in $M$;
- Hom-sets in $K_\kappa(M)$ are $\lambda$-small.

Used in 3.4, 3.5, 7.7, 8.6*.

Proposition 3.4. (Low [2014.a, Proposition 5.12, Theorem 5.14(i)].) The functor $K_\lambda$ and pseudofunctor $\text{Ind}^\lambda$ establish a correspondence between $(\kappa, \lambda)$-miniature model categories and strongly $(\kappa, \lambda)$-combinatorial model categories. In particular, we have equivalences of model categories $C \rightarrow \text{Ind}^\lambda(K_\lambda(C))$ and $D \rightarrow K_\lambda(\text{Ind}^\lambda(D))$ that preserve and reflect weak equivalences, fibrations, and cofibrations. This correspondence preserves left Quillen equivalences. Furthermore, any combinatorial model category is a strongly $(\kappa, \lambda)$-combinatorial model category for some regular cardinals $\kappa \triangleleft \lambda$ (Low [2014.a, Proposition 5.6]). Any strongly $(\kappa, \lambda)$-combinatorial model category is a strongly $(\kappa, \mu)$-combinatorial model category for any $\mu \triangleright \lambda$ (Low [2014.a, Remark 5.2]). Used in 3.6, 7.7, 7.12*, 8.6*.

The following proposition follows from Proposition 2.8.
Proposition 3.6. (Low [2014,a, Proposition 5.6, Lemma 2.5].) For any left Quillen functor $F: C \to D$ between combinatorial model categories and any regular cardinal $\kappa$, there is a regular cardinal $\lambda \geq \kappa$ such that $F$ is a left $\lambda$-Quillen functor (Definition 3.5). Used in $\ref{prop:prop3.6}, \ref{prop:prop3.7}$. 

The relative category $\textbf{CMC}$ can be informally described as follows. Objects are combinatorial model categories. Morphisms are left Quillen functors. Weak equivalences are left Quillen equivalences. To avoid size issues, we follow $\S\ref{section:2.18}$.

Definition 3.7. The relative category $\textbf{CMC}$ is defined as follows. Objects are pairs $(\lambda, C)$, where $\lambda$ is a regular cardinal and $C$ is a small $(\kappa, \lambda)$-miniature model category (Low [2014,a, Definition 5.11]), where $\kappa$ is some regular cardinal such that $\kappa \prec \lambda$. Morphisms $(\lambda, C) \to (\mu, D)$ exist if $\lambda \lessdot \mu$, in which case they are left Quillen functors $C \to D$. Weak equivalences are generated as a subcategory by morphisms $(\lambda, C) \to (\mu, D)$ for which $\lambda = \mu$ and $C \to D$ is a left Quillen equivalence, together with morphisms $(\lambda, C) \to (\mu, D)$ for which the left Quillen functor $C \to D$ exhibits $D$ as the $(\lambda, \mu)$-ind-completion of $C$ (Low [2014,a, Theorem 5.14]). Used in $\ref{prop:prop1.1}, \ref{prop:prop3.10}, \ref{prop:prop3.7}, \ref{prop:prop3.8}, \ref{prop:prop3.9}, \ref{prop:prop3.10}, \ref{prop:prop6.0}, \ref{prop:prop6.1}, \ref{prop:prop6.2}, \ref{prop:prop6.3}, \ref{prop:prop6.4}, \ref{prop:prop7.0}, \ref{prop:prop7.7}, \ref{prop:prop7.11}, \ref{prop:prop7.12}, \ref{prop:prop8.1}, \ref{prop:prop8.2}, \ref{prop:prop8.3}, \ref{prop:prop8.4}, \ref{prop:prop8.5}, \ref{prop:prop8.6}, \ref{prop:prop8.7}$.

While we do not assume any large cardinal axioms for the main results of this paper, we can ask whether in presence of a strongly inaccessible cardinal our definition of $\textbf{CMC}$ is equivalent to the more obvious definition of $\textbf{CMC}$ that uses universes. This is answered in the affirmative by the following definitions and proposition.

Definition 3.8. Given a strongly inaccessible cardinal $U$, the relative category $\textbf{CMC}_U$ is defined as the full subcategory of $\textbf{CMC}$ (Definition 3.7) on objects $(\lambda, C)$, where $\lambda < U$ and $C$ is $U$-small. Used in $\ref{prop:prop1.1}, \ref{prop:prop3.10}$. 

Definition 3.9. Given a strongly inaccessible cardinal $U$, the relative category $\textbf{CMC}_U'$ is the relative category of $U$-combinatorial model categories, left Quillen functors, and left Quillen equivalences. Used in $\ref{prop:prop1.1}, \ref{prop:prop2.18}, \ref{prop:prop3.10}$. 

Proposition 3.10. Assuming Definition 2.19, we have a Dwyer–Kan equivalence of relative categories

$$\textbf{CMC}_U \to \textbf{CMC}_U'$$

of Definition 3.8 and Definition 3.9. Used in $\ref{prop:prop1.1}, \ref{prop:prop2.26}, \ref{prop:prop4.5}, \ref{prop:prop5.4}, \ref{prop:prop8.7}$. 

Proof. The functor

$$\text{Ind}_U: \textbf{LPC}_U \to \textbf{LPC}_U'$$

is constructed by promoting the functor

$$\text{Ind}_U: \textbf{CMC}_U \to \textbf{CMC}_U'$$

of Proposition 2.27 to a functor

$$\text{Ind}_U: \textbf{CMC}_U \to \textbf{CMC}_U'$$

as described in Low [2014,a, Theorem 5.14]. By definition of $\textbf{CMC}$ (Definition 3.7), this functor preserves weak equivalences, so it is a relative functor.

In complete analogy to Proposition 2.27, we introduce filtrations on $\textbf{CMC}_U$ and $\textbf{CMC}_U'$ (indexed by a regular cardinal $\nu$) that are respected by the functor $\text{Ind}_U$ and show that $\text{Ind}_U$ induces a homotopy equivalence of relative categories for each step of the filtration.

Fix a regular cardinal $\nu$. Define $\textbf{CMC}_{U,\nu}$ as the full subcategory of $\textbf{CMC}_U$ consisting of objects $(\lambda, C)$ for which $\lambda \leq \nu$. Define $\textbf{CMC}_{U,\nu}'$ as the full subcategory of $\textbf{CMC}_U'$ consisting of objects given by strongly $U$-$(\kappa, \nu)$-combinatorial model categories (for some regular cardinal $\kappa \lessdot \nu$) and morphisms given by left $U$-$\nu$-Quillen functors. By construction and Low [2014,a, Theorem 5.14], the functor $\text{Ind}_U$ restricts to a functor

$$\text{Ind}_{U,\nu}: \textbf{CMC}_{U,\nu} \to \textbf{CMC}_{U,\nu}'$$

We now show that $\text{Ind}_{U,\nu}$ is a homotopy equivalence of relative categories.
The inverse functor is

$$K_{U,ν}: \text{CMC}_{U,ν} \to \text{CMC}_{U,ν}$$

which is well-defined by Low [2014.a, Proposition 5.12]. The functor $K_{U,ν}$ sends an object $C \in \text{CMC}_{U,ν}$ to the object $(ν, K^U_ν(C)) \in \text{CMC}_{U,ν}$ and a functor $F: C \to D$ in $\text{CMC}_{U,ν}$ to the restriction

$$(ν, K^U_ν(C)) \to (ν, K^U_ν(D)),$$

which is well-defined because $F$ is a left $U$-$ν$-Quillen functor.

The natural weak equivalences

$$\text{id}_{\text{CMC}_{U,ν}} \to K_{U,ν} \circ \text{Ind}_{U,ν}, \quad \text{id}_{\text{CMC}_{U,ν}} \to \text{Ind}_{U,ν} \circ K_{U,ν}$$

are inherited from the proof of Proposition 2.27 and are weak equivalences because their underlying functors are equivalences of categories and the model structures coincide by Low [2014.a, Proposition 5.12 and Theorem 5.14].

4 Combinatorial relative categories

Definition 4.1. The relative category $\text{CRC}$ is defined as follows. Objects are pairs $(λ, C)$, where $λ$ is a regular cardinal and $C$ is a small relative category that admits $λ$-small homotopy colimits (Definition 2.37). Morphisms $(λ, C) \to (μ, D)$ exist if $λ \leq μ$, in which case they are relative functors $C \to D$ that preserve $λ$-small homotopy colimits (Definition 2.33). Weak equivalences $(λ, C) \to (μ, D)$ are generated as a subcategory by morphisms $(λ, C) \to (μ, D)$ for which $λ = μ$ and $C \to D$ is a Dwyer–Kan equivalence, together with morphisms $(λ, C) \to (μ, D)$ for which the functor $F: C \to D$ exhibits $D$ as the $(λ, μ)$-ind-completion of $C$, namely, the functor $\mathcal{N}(F)$ exhibits $\mathcal{N}(F)$ as the quasicategory of $μ$-presentable objects in the $λ$-ind-completion of the quasicategory $\mathcal{N}(F)$ (see Definition 5.1 and Notation 2.36).

Remark 4.2. In Definition 4.1, we created the class of weak equivalences in $\text{CRC}$ using the functor $\mathcal{N}: \text{CRC} \to \text{PrL}$ of Definition 9.1, where the weak equivalences of $\text{PrL}$ are given in Definition 5.1. A more natural way to introduce weak equivalences in $\text{CRC}$ is to define homotopy $μ$-presentable objects and homotopy $λ$-ind-completions of relative categories directly, without referring to quasicategories. Such an approach would produce exactly the same class of weak equivalences. However, it would require us to introduce all the relevant definitions and show their compatibility with analogous quasicategorical definitions, further adding to the length of this article, whereas in the quasicategorical context the necessary results are already available in Cisinski [2019.a, Chapter 7]. Thus, we bypass the issue by transferring weak equivalences from $\text{PrL}$.

Definition 4.3. Given a strongly inaccessible cardinal $U$, the relative category $\text{CRC}_U$ is defined as the full subcategory of $\text{CRC}$ (Definition 4.1) on objects $(λ, C)$, where $λ < U$ and $C$ is $U$-small.

Definition 4.4. Given a strongly inaccessible cardinal $U$, the relative category $\text{CRC}_U$ is the relative category of locally $U$-small relative categories $C$ such that $\mathcal{N}(C) \in \text{PrL}_U$, with morphisms given by relative functors $F$ such that $\mathcal{N}(F)$ is a morphism in $\text{PrL}_U$, and weak equivalences being Dwyer–Kan equivalences of relative categories.

Proposition 4.5. There is a Dwyer–Kan equivalence of relative categories

$$\text{CRC}_U \to \text{CRC}_U$$

of Definition 4.3 and Definition 4.4. Used in 1.1, 4.5.

Proof. The proof is very similar to the proofs of Proposition 2.27 and Proposition 3.10. We only briefly indicate the necessary modifications. Instead of $\text{Ind}_{\text{PrL}}: \text{PrL}_U \to \text{PrL}_U$, we use the functor

$$\text{RInd}_U = \text{Reedy} \circ \text{MInd}_U: \text{CRC}_U \to \text{CRC}_U.$$
where $\text{Reedy}_\nu$ is as in Definition 6.3 and $\text{MInd}_\nu$ is as in Definition 7.7. The functor $\text{RInd}_{\nu}$ indeed lands in $\text{CRC}_{\nu}$ by Proposition 7.10 (taking $\nu = U$ and $C = \text{CRC}_{\nu}$ there).

Given a regular cardinal $\nu$, we take $\text{CRC}_{\nu}$ to be the full subcategory of $\text{CRC}_{\nu}$ consisting of objects $(\lambda, C)$ with $\lambda \leq \nu$ and $\text{CRC}_{\nu, \nu}$ to be the subcategory of $\text{CRC}_{\nu}$ consisting of objects $C$ such that $\mathcal{N}\mathcal{R}C$ is a $U\nu$-presentable quasicategory and morphisms $F$ such that $\mathcal{N}\mathcal{R}F$ is a strongly $U\nu$-accessible left adjoint functor of quasicategories.

The homotopy inverse to $\text{RInd}_{\nu}$ is given by the functor $\text{RK}_{\nu}^U$ that sends $C \in \text{CRC}_{\nu}$ to $(\nu, C_\nu)$, where $C_\nu$ is the full subcategory of $C$ on whose objects whose images in $\mathcal{N}\mathcal{R}C$ are $U\nu$-presentable objects (in the quasicategorical sense). By Proposition 7.10, we have a natural weak equivalence $(\kappa, E) \to (\nu, \text{RK}^U_{\nu}(\kappa, E))$ (where $(\kappa, E) \in \text{CRC}_{\nu}$) that sends $e \in E$ to $n \mapsto (\Delta^n \otimes y(e))$, where $y$ is the Yoneda embedding. We also have a natural weak equivalence $C \to \text{RInd}_{\nu}(\nu, \text{RK}^U_{\nu}(C))$ for $C \in \text{CRC}_{\nu}$, which sends $X \in C$ to $n \mapsto \Delta^n \otimes (A \mapsto \mathcal{H}_C(A, X))$. \[\square\]

5 Presentable quasicategories

Recall that a quasicategory is presentable if it is accessible (Lurie 2017, Definition 5.4.2.1) and admits small colimits (Joyal 2002, Definition 4.5). The relative category $\mathcal{P}rL$ can be informally described as the relative category of presentable quasicategories, left adjoint functors, and equivalences. To avoid size issues, we follow §2.18.

Definition 5.1. The relative category $\mathcal{P}rL$ is defined as follows. Objects are pairs $(\lambda, C)$, where $\lambda$ is a regular cardinal and $C$ is a small quasicategory that admits $\lambda$-small colimits. Morphisms $(\lambda, C) \to (\mu, D)$ exist if $\lambda \leq \mu$, in which case they are functors $C \to D$ that preserve $\lambda$-small colimits. Weak equivalences are morphisms $(\lambda, C) \to (\mu, D)$ such that $C \to D$ exhibits $D$ as the $(\lambda, \mu)$-ind-completion of $C$, i.e., the quasicategory of $\mu$-presentable objects (Lurie 2017, Definition 5.3.4.5) in the $\lambda$-ind-completion of $C$ (Lurie 2017, Definition 5.3.5.1).

Definition 5.2. Given a strongly inaccessible cardinal $U$, the relative category $\mathcal{P}rL_U$ is defined as the full subcategory of $\mathcal{P}rL$ (Definition 5.1) on objects $(\lambda, C)$, where $\lambda < U$ and $C$ is $U$-small.

Definition 5.3. Given a strongly inaccessible cardinal $U$, the relative category $\mathcal{P}rL_U$ is the relative category of $U$-presentable quasicategories, left adjoint functors, and equivalences of quasicategories, i.e., weak equivalences in the Joyal model structure.

Proposition 5.4. Assuming Definition 2.10, we have a Dwyer–Kan equivalence

$$\text{QInd}_{\nu} : \mathcal{P}rL_U \to \mathcal{P}rL_{\nu}$$

of relative categories $\mathcal{P}rL_U$ (Definition 5.2) and $\mathcal{P}rL_{\nu}$. We only briefly indicate the necessary modifications. Instead of $\text{Ind}_{\nu} : \mathcal{L}P_{\nu} \to \mathcal{L}P_{\nu}$ we use the functor

$$\text{QInd}_{\nu} = \mathcal{N} \circ \mathcal{R} \circ \text{Reedy} \circ \text{MInd}_{\nu} \circ \mathcal{K} : \mathcal{P}rL_{\nu} \to \mathcal{P}rL_{\nu},$$

where $\mathcal{K}$, $\mathcal{N}$, and $\mathcal{R}$ are as in Notation 2.36. $\text{Reedy}_{\nu}$ is as in Definition 6.3 and $\text{MInd}_{\nu}$ is as in Definition 7.7. This somewhat cumbersome and roundabout definition of $\text{QInd}_{\nu}$ is explained by the fact that we need a (strict) relative functor, whereas the familiar quasicategorical constructions of ind-completions only provide homotopy coherent functors.

Given a regular cardinal $\nu$, we take $\mathcal{P}rL_{\nu, \nu}$ to be the full subcategory of $\mathcal{P}rL_{\nu}$ consisting of objects $(\lambda, C)$ with $\lambda \leq \nu$ and $\mathcal{P}rL_{\nu, \nu}$ to be the subcategory of $\mathcal{P}rL_{\nu}$ consisting of $U\nu$-presentable quasicategories and strongly $U\nu$-accessible left adjoint functors of quasicategories.

The homotopy inverse to $\text{QInd}_{\nu}$ is given by the functor $\text{QK}^U_{\nu}$ that sends $C \in \mathcal{P}rL_{\nu}$ to $(\nu, C_\nu)$, where $C_\nu$ is the full subcategory of $C$ on $U\nu$-presentable objects (in the quasicategorical sense). We have a natural weak equivalence $(\kappa, E) \to (\nu, \text{QK}^U_{\nu}(\kappa, E))$ (where $(\kappa, E) \in \mathcal{P}rL_{\nu}$) given by the quasicategorical variant of the Yoneda embedding. We also have a natural weak equivalence $C \to \text{QInd}_{\nu}(\nu, \text{QK}^U_{\nu}(C))$ for $C \in \mathcal{P}rL_{\nu}$, given by the quasicategorical variant of the restricted Yoneda embedding. \[\square\]
Remark 5.5. Assuming Definition 2.19, we can turn $\PrL$, into a simplicial category by declaring the hom-object $\PrL_A(B, C)$ to be the simplicial subset of the maximal Kan subcomplex in the mapping simplicial set $B^A$, comprising connected components of left adjoint functors. The homotopy coherent nerve of this simplicial category is precisely the quasicategory $\PrL$ constructed by Lurie [2017, Definition 5.5.3.1].

6 From combinatorial model categories to combinatorial relative categories

In this section we define two weakly equivalent Dwyer-Kan equivalences $\text{CMC} \xrightarrow{\text{Cof}} \text{CRC}$. The first equivalence, $\text{Cof}$, is defined in a straightforward way by restricting to full subcategories of cofibrant objects. The second equivalence, $\text{Reedy}$, is defined by taking the relative category of cosimplicial resolutions, i.e., Reedy cofibrant cosimplicial objects whose cosimplicial structure maps are weak equivalences. This enables us to construct left Quillen equivalences from simplicial presheaves on such categories of diagrams to the original model category, replicating a construction of Dugger [2000].

Definition 6.1. The relative functor

$$\text{Cof} : \text{CMC} \rightarrow \text{CRC}$$

is defined as follows. An object $(\lambda, C) \in \text{CMC}$ is sent to $(\lambda, \text{Cof}(C))$, where $\text{Cof}(C)$ is the relative category of cofibrant objects in $C$ with induced weak equivalences. A morphism $(\lambda, C) \rightarrow (\mu, D)$ is sent to the morphism $(\lambda, \text{Cof}(C)) \rightarrow (\mu, \text{Cof}(D))$ given by the restriction and coarsening of $F$. Used in 1.1*, 6.0*, 6.1, 6.2, 6.2*, 6.3, 6.4*, 7.12*, 8.2, 8.7, 8.7*, 8.8, 8.8*.

Proposition 6.2. Definition 6.1 is correct. Used in 6.4.

Proof. Given an object $(\lambda, M) \in \text{CMC}$, we have to show that $(\lambda, \text{Cof}(M)) \in \text{CRC}$. That is, if $M$ is a small $(\kappa, \lambda)$-miniature model category, we have to show that the small relative category $\text{Cof}(M)$ admits $\lambda$-small homotopy colimits. By Definition 2.37, this means that the small quasicategory $N\mathcal{R}M(\text{Cof}(M))$ admits $\lambda$-small colimits. Since $\text{Cof}(M) \rightarrow M$ is a Dwyer-Kan equivalence, it suffices to show that the small quasicategory $N\mathcal{R}M$ admits $\lambda$-small colimits. The small quasicategory $N\mathcal{R}M$ is a localization of $M$ with respect to its class of weak equivalences in the sense of Cisinski [2019.a, Definition 7.1.2], denoted by $L(M)$ there. By Cisinski [2019.a, Remark 7.9.10], the small quasicategory $L(M) \simeq N\mathcal{R}M$ admits $\lambda$-small colimits.

Given a morphism $(\lambda, M) \rightarrow (\mu, N)$ in $\text{CMC}$, we have to show that the functor $N\mathcal{R}M \rightarrow N\mathcal{R}N$ preserves $\lambda$-small colimits. The latter functor is an induced functor between localizations of $M$ and $N$ in the sense of Cisinski [2019.a, Definition 7.1.2], denoted by $L(M) \rightarrow L(N)$ there. By Cisinski [2019.a, Remark 7.9.10], the map $L(M) \rightarrow L(N)$ preserves $\lambda$-small colimits.

The functor $\text{Cof}$ preserves weak equivalences in $\text{CMC}$. Indeed, the latter are generated by left Quillen equivalences and ind-completions. $\text{Cof}$ maps left Quillen equivalences $(\lambda, M) \rightarrow (\lambda, N)$ to homotopy equivalences of relative categories (Definition 2.35); the homotopy inverse is given by the right derived functor of the right adjoint, composed with a cofibrant replacement functor. $\text{Cof}$ maps a morphism $(\lambda, M) \rightarrow (\mu, \text{Ind}_{\lambda}^\mu(M))$ to the morphism

$$(\lambda, \text{Cof}(M)) \rightarrow (\mu, \text{Cof}(\text{Ind}_{\lambda}^\mu(M))),$$

which is weakly equivalent to the morphism

$$(\lambda, M) \rightarrow (\mu, \text{Ind}_{\lambda}^\mu(M)).$$

Taking $N = \text{Ind}_{\lambda}^\mu(M)$, we can identify the latter morphism with

$$(\lambda, K_{\lambda}(N)) \rightarrow (\mu, K_{\mu}(N)).$$

The left Quillen functor $K_{\lambda}(N) \rightarrow K_{\mu}(N)$ preserves weak equivalences. Furthermore, in the model category $N$ the $\lambda$-filtered or $\mu$-filtered colimits are also homotopy colimits. Thus, $K_{\lambda}(N)$ respectively $K_{\mu}(N)$ comprise the homotopy $\lambda$-presentable respectively homotopy $\mu$-presentable objects in $N$. Hence, applying the functor $N\mathcal{R}M$ (equivalently, $N\mathcal{R}$ or simply $N$) yields a functor of quasicategories that is equivalent to the inclusion $K_{\lambda}(N) \rightarrow K_{\mu}(N)$.
**Definition 6.3.** The relative functor

\[ \text{Reedy} : \text{CMC} \to \text{CRC} \]

is defined as follows. An object \((\lambda, C)\) is sent to the pair \((\lambda, \text{Reedy}(C))\), where \(\text{Reedy}(C)\) is the small relative category of cosimplicial resolutions in \(C\), i.e., Reedy cofibrant cosimplicial objects in \(C\) whose cosimplicial structure maps are weak equivalences. We equip \(\text{Reedy}(C)\) with degreewise weak equivalences. A morphism \((\lambda, C) \to (\mu, D)\) is sent to the morphism

\[ (\lambda, \text{Reedy}(C)) \to (\mu, \text{Reedy}(D)) \]

given by the relative functor \(\text{Reedy}(C) \to \text{Reedy}(D)\) itself induced by the left Quillen functor \(C \to D\).

The natural weak equivalence

\[ \text{ev}_0 : \text{Reedy} \to \text{Cof} \]

(\(\text{Cof}\) was introduced in Definition 6.1) sends an object \((\lambda, C) \in \text{CMC}\) to the morphism

\[ \text{ev}_0(\lambda, C) : (\lambda, \text{Reedy}(C)) \to (\lambda, \text{Cof}(C)) \]

induced by the relative functor

\[ \text{Reedy}(C) \to \text{Cof}(C) \]

that evaluates a Reedy cofibrant cosimplicial diagram at the simplex \([0] \in \Delta\). Used in 4.5*, 5.4*, 6.0*, 6.3, 6.4*, 7.9, 7.10*, 7.11, 7.12, 7.12*, 8.1, 8.1*, 8.2, 8.5, 8.5*, 8.6, 8.6*, 8.7*, 9.11*.

**Proposition 6.4.** Definition 6.3 is correct.

*Proof.* Given an object \((\lambda, M) \in \text{CMC}\), we have to show that \((\lambda, \text{Reedy}(M)) \in \text{CRC}\). That is, if \(M\) is a small \((\kappa, \lambda)\)-miniature model category, we have to show that the small relative category \(\text{Reedy}(M)\) admits \(\lambda\)-small homotopy colimits. Since we have a Dwyer–Kan equivalence \(\text{Reedy}(M) \to \text{cof}(M)\), it suffices to recall (Proposition 6.2) that \(\text{cof}(M)\) admits \(\lambda\)-small homotopy colimits.

A left Quillen functor \(M \to N\) induces a left Quillen functor \(M^\Delta \to N^\Delta\) between the corresponding Reedy model categories of cosimplicial objects. Therefore, it induces a relative functor \(\text{Reedy}(M) \to \text{Reedy}(N)\) between the corresponding relative categories of cofibrant objects. If \((\lambda, M) \to (\mu, N)\) is a morphism, then the relative functor \(\text{Reedy}(M) \to \text{Reedy}(N)\) is weakly equivalent to the relative functor \(\text{cof}(M) \to \text{cof}(N)\), which preserves \(\lambda\)-small homotopy colimits (Definition 2.3). Thus, a morphism \(g : (\lambda, M) \to (\mu, N)\) in \(\text{CMC}\) is sent to a morphism \(h\) in \(\text{CRC}\). Furthermore, if \(g\) is a weak equivalence, then so is \(h\) by the 2-out-of-3 property.

Finally, the natural transformation \(\text{ev}_0(\lambda, C) : (\lambda, \text{Reedy}(C)) \to (\lambda, \text{Cof}(C))\) is a weak equivalence: its weak inverse is a natural transformation that sends \(X \in \text{Cof}(C)\) to the Reedy cofibrant resolution of the constant cosimplicial object on \(X\).
7 From combinatorial relative categories to combinatorial model categories

In this section we introduce and study constructions that allow us to pass from the relative category $\text{CRC}$ to the relative category $\text{CMC}$. The primary source of difficulty is the fact that the regular cardinal $\lambda$ may increase in an uncontrolled fashion. This prevents us from defining a relative functor $\text{CRC} \to \text{CMC}$. Instead, we provide an ad hoc construction for every small subcategory of $\text{CRC}$.

Definition 7.1. A simplicial set is a simplicial object in the category $\text{Set}$ of Definition 2.11. The category of simplicial sets is denoted by $\text{sSet}$. A simplicial presheaf is a presheaf of simplicial sets. Used in 2.16, 2.17, 2.17*, 2.28, 2.28*, 2.36, 7.2*.

Definition 7.2. Given a small relative category $C$, the model category $\text{sPSh}(C)$ of simplicial presheaves on $C$ is defined as follows. Its underlying category is the category of simplicial presheaves on the underlying category of $C$. Its model structure is the left Bousfield localization of the projective model structure at morphisms of simplicial presheaves that are representable by a weak equivalence in $C$. Used in 7.2*, 7.3, 7.4, 7.5, 7.6, 7.10*, 7.12*, 8.4*.

The fibrant objects in $\text{sPSh}(C)$ are precisely the relative functors $C^{op} \to \text{Set}_{\text{kan}}$.

Definition 7.3. Given a relative functor $F: C \to D$ between small relative categories, the left Quillen functor $\text{sPSh}(F): \text{sPSh}(C) \to \text{sPSh}(D)$ is the unique (up to a unique isomorphism) simplicial left Quillen functor that restricts to $F$ on representable presheaves. Used in 7.4.

Definition 7.4. The relative functor $\text{sPSh}: \text{RelCat} \to \text{CombModCat}$ is constructed by applying the strictification construction of Power [1989, §4.2] to the pseudofunctor $\text{sPSh}$ constructed in Definition 7.2 and Definition 7.3. Used in 7.5, 7.7.

Remark 7.5. Definition 7.4 contains a considerable abuse of notation: the category $\text{CombModCat}$ is supposed to have combinatorial model categories as objects, which is not possible since combinatorial model categories of simplicial presheaves have a proper class of objects. However, we only need the functor $\text{sPSh}$ to construct the functor $\text{MInd}$ (Definition 7.6), itself used to construct the functor $\text{MInd}_{\nu}$ (Definition 7.7) landing in miniature model categories (Definition 3.2), which do form a relative category. Thus, the functor $\text{MInd}$ is well-defined and the abuse of notation is harmless. Used in 7.6.

We now introduce the small model category $\text{MInd}(\lambda, C)$, which models the homotopy $\lambda$-ind-completion $\text{Ind}^\lambda C$ of a small relative category $C$. The 1-categorical construction that we imitate here presents $\text{Ind}^\lambda C$ by the category of functors $C^{op} \to \text{Set}$ that preserve $\lambda$-small limits, provided that $C$ admits $\lambda$-small colimits. The latter category can be encoded in turn as the reflective localization of the category of functors $C^{op} \to \text{Set}$ at morphisms of the form $\text{colim}_I Y \circ D \to Y(\text{colim}_I D)$ for small diagrams $D: I \to C$. In the model-categorical setting, reflective localizations become left Bousfield localizations and we use quasicategories to define the class of localizing morphisms to avoid developing the relevant machinery of homotopy colimits directly for relative categories.

Definition 7.6. Given an object $(\lambda, C) \in \text{CRC}$ the model category $\text{MInd}(\lambda, C)$ is defined as the left Bousfield localization of $\text{sPSh}(C)$ at the set of maps of the form $\eta_D$ (constructed in the next paragraph) for a set of representatives $D$ of weak equivalence classes of diagrams $D: I \to \text{sPSh}(C)$ of weakly representable presheaves, where $I$ is a $\lambda$-small relative category. Since $C$ is a small relative category, such representatives form a set.

The morphism $\eta_D$ is constructed as follows. Consider the adjunction of quasicategories

$$
\begin{array}{ccc}
\text{L} & \text{N} & \\
\text{sPSh}(C) & \text{MSet} & \text{N} & \text{RelCat},
\end{array}
$$

22
where \( Y: C \to \mathbf{sPSh}(C) \) is the Yoneda embedding functor and \( L \) is the left adjoint of \( N \mathbf{R}Y \). Suppose \( I \) is a \( \lambda \)-small relative category and \( D: I \to \mathbf{sPSh}(C) \) is a relative functor.

Consider the induced diagram of quasicategories

\[
N \mathbf{R}(D): N \mathbf{R}I \to N \mathbf{R}\mathbf{sPSh}(C).
\]

The unit map \( \alpha_D \) of the object

\[
\text{colim}(N \mathbf{R}(D)) \in N \mathbf{R}\mathbf{sPSh}(C)
\]

has the form

\[
\alpha_D: \text{colim}(N \mathbf{R}(D)) \to N \mathbf{R}Y(L(\text{colim}N \mathbf{R}(D))).
\]

Denote by \( \eta_D \) some morphism in \( \mathbf{sPSh}(C) \) whose image in \( N \mathbf{R}\mathbf{sPSh}(C) \) is equivalent to \( \alpha_D \). This completes the construction of \( \eta_D \) and the definition of \( \mathbf{MInd}(\lambda, C) \).

Given a morphism \( F: (\lambda, C) \to (\mu, D) \) in \( \mathbf{CRC} \), the left Quillen functor

\[
\mathbf{MInd}(F): \mathbf{MInd}(\lambda, C) \to \mathbf{MInd}(\mu, D)
\]

coinsides with \( \mathbf{sPSh}(F) \) as a functor. In particular, \( \mathbf{MInd} \) is itself a functor, keeping in mind \( \text{Remark 7.5} \).

**Definition 7.7.** Given an object \( (\lambda, C) \in \mathbf{CRC} \) and a regular cardinal \( \nu \geq \lambda \) such that \( \mathbf{MInd}(\lambda, C) \) is a strongly \( (\kappa, \nu) \)-combinatorial model category (Definition 3.3), the small model category \( \mathbf{MInd}(\lambda, C) \) is defined as the model category \( K_\nu(M\mathbf{Ind}(\lambda, C)) \) (Low [2014.a, Proposition 5.12]), which is guaranteed to be small by \( \text{Remark 2.12} \).

If \( \mathbf{MInd}(\lambda, C) \) and \( \mathbf{MInd}(\mu, D) \) are defined and \( \mathbf{MInd}(F) \) is a left \( \nu \)-Quillen functor (Definition 3.5), then we denote by

\[
\mathbf{MInd}(F): \mathbf{MInd}(\lambda, C) \to \mathbf{MInd}(\mu, D)
\]

the functor \( K_\nu(M\mathbf{Ind}(F)) \).

By \( \text{Definition 7.4} \), \( \mathbf{MInd} \) is a relative functor from the relative category of strongly \( (\kappa, \nu) \)-combinatorial model categories and left \( \nu \)-Quillen functors to the relative category \( \mathbf{CMC} \) if we decorate the resulting objects and morphisms with \( \nu \) as the first component.

**Proposition 7.8.** Given an object \( (\lambda, C) \in \mathbf{CRC} \), there are arbitrarily large regular cardinals \( \mu \geq \lambda \) such that the small model category \( \mathbf{MInd}(\lambda, C) \) is defined and is a \( (\kappa, \mu) \)-miniature model category for some regular cardinal \( \kappa \).

Given a morphism \( (\lambda, C) \to (\mu, D) \) in \( \mathbf{CRC} \), there are arbitrarily large regular cardinals \( \nu \) such that the left Quillen functor \( \mathbf{MInd}(\lambda, C) \to \mathbf{MInd}(\mu, D) \) is defined.

**Proof.** Apply Proposition 3.4 and Proposition 3.6.

**Definition 7.9.** Suppose \( \iota: C \to \mathbf{CRC} \) is an inclusion of a small full subcategory \( C \) and \( \nu \) is a regular cardinal such that \( \mathbf{MInd}_\nu \) is defined for all objects and morphisms of \( C \). (Such a regular cardinal always exists by Proposition 7.8.) The natural transformation

\[
\eta: \iota \to \mathbf{MInd}_\nu \circ \iota
\]

sends an object \( (\kappa, E) \in C \) to the morphism

\[
(\kappa, E) \to (\nu, \mathbf{Reedy}(\mathbf{MInd}_\nu(\kappa, E)))
\]

induced by the canonical functor

\[
E \to \mathbf{Reedy}(\mathbf{MInd}_\nu(\kappa, E)), \quad X \mapsto (n \mapsto \Delta^n \otimes Y(X)).
\]
Proposition 7.10. The natural transformation of Definition 7.9 is a natural weak equivalence.

Proof. Compose the morphism

\[(\kappa, E) \to (\nu, \text{Reedy} \circ \text{MInd}_\nu(\kappa, E))\]

with the weak equivalence

\[\text{ev}_0 : (\nu, \text{Reedy} \circ \text{MInd}_\nu(\kappa, E)) \to (\nu, \text{MInd}_\nu(\kappa, E)).\]

It remains to show that the composition

\[(\kappa, E) \to (\nu, \text{MInd}_\nu(\kappa, E))\]

is a weak equivalence.

By Cisinski [2019,a, Remark 7.9.10], the functor \(\text{NR} \) applied to the projective model structure on simplicial presheaves on \(E\) yields a quasicategory equivalent to the quasicategory of presheaves on the nerve of \(E\). By Cisinski [2019,a, Proposition 7.11.4], the quasicategory \(\text{NR} \circ \text{PSh}(E)\) is equivalent to the reflective localization of the quasicategory of presheaves on the nerve of \(E\) with respect to weak equivalences of \(E\). The latter localization is itself equivalent to the quasicategory of presheaves on \(\text{NR} \circ E\). Furthermore, by the same proposition, the left Bousfield localization \(\text{MInd}_\nu(\kappa, E)\) (Definition 7.6) of \(\text{PSh}(E)\) is equivalent to the reflective localization of presheaves on \(\text{NR} \circ E\) at morphisms constructed in Definition 7.6. The latter localization is itself equivalent to the category of presheaves on \(\text{NR} \circ E\) that (as functors from \((\text{NR} \circ E)/D3/D4\) to spaces) preserve \(\kappa\)-small limits. This is precisely the \(\kappa\)-ind-completion of the quasicategory \(\text{NR} \circ E\), which shows that \((\kappa, E) \to (\nu, \text{MInd}_\nu(\kappa, E))\) is a weak equivalence by definition of \(\text{CRC} \).

Definition 7.11. Suppose \(\iota : C \to \text{CMC}\) is an inclusion of a small full subcategory \(C\) into the relative category \(\text{CMC}\) (Definition 3.7) and \(\nu\) is a regular cardinal such that \(\text{MInd}_\nu(\lambda, \text{Reedy} \circ \iota)\) (Definition 6.3) is defined for all objects and morphisms of the diagram \(\text{Reedy} \circ \iota\) (Definition 6.3). The natural transformation

\[\text{Re} \circ \text{MInd}_\nu \circ \text{Reedy} \circ \iota \to \iota\]

sends an object \((\lambda, M) \in \text{CMC}\) to the morphism

\[(\nu, \text{MInd}_\nu(\lambda, \text{Reedy})(M)) \to (\nu, \text{Ind}_\nu^\lambda(M))\]

given by the left Quillen functor

\[\text{Re} \circ \text{MInd}_\nu(\lambda, \text{Reedy})(M) \to \text{Ind}_\nu^\lambda(M)\]

induced by the functor

\[\Delta^{op} \times \text{Reedy}(M) \to M\]

that sends \(([n], X) \to X_n\).

Proposition 7.12. The natural transformation

\[\text{Re} \circ \text{MInd}_\nu \circ \text{Reedy} \circ \iota \to \iota\]

of Definition 7.11 is a natural weak equivalence.

Proof. To show that for any \((\lambda, M) \in \text{CMC}\) the left adjoint functor

\[\text{Re} \circ \text{MInd}_\nu(\lambda, \text{Reedy})(M) \to \text{Ind}_\nu^\lambda(M)\]

is a left Quillen equivalence, it suffices to show that the left adjoint functor

\[\text{Re} \circ \text{MInd}_\nu(\lambda, \text{Reedy})(M) \to \text{Ind}_\nu^\lambda(M)\]

is a left Quillen equivalence. Used in 7.11, 8.2, 8.3.
(defined in the same way as \( \operatorname{Re} \)) is a left \( \nu \)-Quillen equivalence (Definition 3.3), after which we can pass to the subcategories of \( \nu \)-presentable objects to recover \( \operatorname{Re} \).

Given a model category \( \mathcal{M} \), consider the left adjoint functor

\[
\operatorname{RE} : \sPSh(\operatorname{Reedy}(\mathcal{M})) \to \operatorname{Ind}^{\lambda} \mathcal{M}
\]

that sends \( ([n], X) \mapsto X_n \). This functor is a left Quillen functor because the image of some generating projective cofibration \( (\partial \Delta^n \to \Delta^n) \otimes X \) is precisely the \( n \)-th latching map of \( X \), which is a cofibration by definition of a Reedy cofibrant cosimplicial object. Likewise, the image of some generating projective acyclic cofibration \( (\Lambda^k_n \to \Delta^n) \otimes X \) is a weak equivalence. Finally, a weak equivalence \( X \to X' \) of representable presheaves is sent to the morphism \( X_0 \to X'_0 \) in \( \operatorname{Ind}^{\lambda} \mathcal{M} \), which is a weak equivalence by definition of \( \operatorname{Reedy}(\mathcal{M}) \).

Next, observe that the left Quillen functor \( \operatorname{RE} \) factors through the localization

\[
\sPSh(\operatorname{Reedy}(\mathcal{M})) \to \operatorname{Min}(\lambda, \operatorname{Reedy}(\mathcal{M})).
\]

Indeed, suppose \( D : I \to \sPSh(\operatorname{Reedy}(\mathcal{M})) \) is a \( \lambda \)-small diagram of weakly representable simplicial presheaves and consider the morphism \( \eta_D \) constructed (Definition 7.6). To show that the left derived functor of \( \operatorname{RE} \) sends \( \eta_D \) to a weak equivalence in \( \operatorname{Ind}^{\lambda} \mathcal{M} \), pass to the setting of quasicategories by restricting to cofibrant objects and applying the functor \( \mathcal{N} \mathcal{R} \), which yields the functor of quasicategories

\[
\mathcal{N} \mathcal{R}(\operatorname{Cof} \circ \operatorname{RE})(\mathcal{M})) \to \mathcal{N} \mathcal{R}((\operatorname{Min}(\lambda, \operatorname{Reedy}(\mathcal{M})))).
\]

By Definition 7.6, the image of \( \eta_D \) in the quasicategory \( \mathcal{N} \mathcal{R}(\operatorname{Cof} \circ \operatorname{RE})(\mathcal{M})) \) is equivalent to the unit map

\[
\alpha_D : \colim(\mathcal{N}(\mathcal{R}(D))) \to \mathcal{N}(\mathcal{R}(\mathcal{L}(\colim(\mathcal{N}(\mathcal{R}(D)))))
\]

and the functor \( \mathcal{N} \mathcal{R}(\operatorname{Cof} \circ \operatorname{RE}) \) is equivalent to \( \mathcal{L} \). By the triangle identity for quasicategorical adjunctions, the map \( \mathcal{L}(\alpha_D) \) is equivalent to the identity map on the object \( \mathcal{L}(\colim(\mathcal{N}(\mathcal{R}(D)))) \) in the quasicategory \( \mathcal{N} \mathcal{R}(\operatorname{Reedy}(\mathcal{M})) \), which shows that the left derived functor of \( \operatorname{RE} \) sends the map \( \eta_D \) to a weak equivalence in \( \operatorname{Ind}^{\lambda} \mathcal{M} \).

The functor \( \operatorname{RE} \) is homotopically essentially surjective (Definition 2.32). Indeed, given any object \( X \in \mathcal{M} \), take the Reedy cofibrant resolution \( R \) of the constant cosimplicial object on \( X \). Then \( \operatorname{RE}(Y(R)) = R_0 \in \mathcal{M} \subseteq \operatorname{Ind}^{\lambda} \mathcal{M} \), so every object in \( \mathcal{M} \subseteq \operatorname{Ind}^{\lambda} \mathcal{M} \) is weakly equivalent to an object in the image of the left derived functor of \( \operatorname{RE} \). Since the latter image is closed under small \( \lambda \)-filtered homotopy colimits in \( \operatorname{Ind}^{\lambda} \mathcal{M} \), its closure under weak equivalences must coincide with \( \operatorname{Ind}^{\lambda} \mathcal{M} \).

The right adjoint of \( \operatorname{RE} \) is the functor

\[
R : \operatorname{Ind}^{\lambda} \mathcal{M} \to \operatorname{Min}(\lambda, \operatorname{Reedy}(\mathcal{M})), \quad X \mapsto (([n], R) \mapsto M(R_n, X)).
\]

The functor \( R \) preserves \( \lambda \)-filtered colimits, hence its right derived functor preserves \( \lambda \)-filtered homotopy colimits.

The regular cardinal \( \nu \) satisfies the conditions of Dugger [2000, Proposition 3.2], so the functor \( \operatorname{RE} \) is a left Quillen equivalence once we show that the derived unit map of any object \( P \in \operatorname{Min}(\lambda, \operatorname{Reedy}(\mathcal{M})) \) is a weak equivalence. Since the left derived functor of \( \operatorname{RE} \) and the right derived functor of \( R \) preserve \( \lambda \)-filtered homotopy colimits, it suffices to establish the case when \( P \) is a \( \lambda \)-small homotopy colimit of representable presheaves in \( \operatorname{Min}(\lambda, \operatorname{Reedy}(\mathcal{M})) \). By construction of \( \operatorname{Min}(\lambda, \operatorname{Reedy}(\mathcal{M})) \), any such homotopy colimit is weakly equivalent to the representable presheaf of some \( Q \in \operatorname{Reedy}(\mathcal{M}) \). Without loss of generality we can assume \( Q \) to be (the representable presheaf of) a Reedy bifibrant cosimplicial object in \( \mathcal{M} \). Now \( \operatorname{RE}(Y(Q)) = Q_0 \) is bifibrant in \( \operatorname{Ind}^{\lambda} \mathcal{M} \), so the derived unit map of \( Q \) is simply the ordinary unit map of \( Q \). Its codomain is

\[
R(\operatorname{RE}(Y(Q))) = R(Q_0) = (([n], R) \mapsto M(R_n, Q_0)).
\]

Observe that the simplicial set \( ([n], R) \mapsto M(R_n, Q_0) \) is weakly equivalent to the derived mapping simplicial set from \( R_\nu \) to \( Q_\nu \), since \( R \) is a cosimplicial resolution of \( R \). Thus, the simplicial presheaf \( R(\operatorname{RE}(Y(Q))) \) is weakly equivalent to the representable presheaf of \( Q_0 \), hence also to the representable presheaf of \( Q \).
8 Equivalence of combinatorial model categories and combinatorial relative categories

Theorem 8.1. The relative functor
\[ \text{Reedy} : \text{CMC} \to \text{CRC} \]
(Definition 6.3) is a Dwyer–Kan equivalence of relative categories. Used in 7.7.

Proof. The functor Reedy is homotopically essentially surjective by Proposition 8.5 and homotopically fully faithful by Proposition 8.6, so by Proposition 2.33 it is a Dwyer–Kan equivalence of relative categories. □

Somewhat more generally, we have the following result.

Theorem 8.2. Suppose \( \Lambda : C \to \text{CMC} \) is a relative functor such that the construction of \( \text{MInd}_\nu \) (Definition 7.7) as well as the constructions of Definition 7.9 and Definition 7.11 lift through \( \Lambda \), and Proposition 7.10 and Proposition 7.12 continue to hold for these lifts. Then the relative functor \( \text{Reedy} \circ \Lambda \) (and hence also \( \text{Cof} \circ \Lambda \)) is a Dwyer–Kan equivalence of relative categories. In particular, the relative functor \( \Lambda \) itself is a Dwyer–Kan equivalence of relative categories.

More generally, suppose \( \Lambda : C \to \text{CMC} \) is a relative functor and \( \Sigma : D \to \text{CRC} \) is a relative inclusion such that the functor \( \text{Reedy} \circ \Lambda \) factors through the image of \( \Sigma \) once we restrict them to the image of \( \Sigma \), and Proposition 7.10 and Proposition 7.12 continue to hold for these lifts. Then the relative functor \( \text{Reedy} \circ \Lambda : C \to D \) is a Dwyer–Kan equivalence of relative categories.

Proof. Proposition 8.3 and Proposition 8.6 continue to hold in this generality, since the indicated properties of \( \text{MInd}_\nu \) and the natural transformations of Definition 7.9 and Definition 7.11 lift through \( \Lambda \) once we restrict them to the image of \( \Sigma \), and Proposition 7.10 and Proposition 7.12 continue to hold for these lifts.

Proposition 8.3. Theorem 8.2 is applicable to the following choices of \( C \), constructed exactly like \( \text{CMC} \) (Definition 3.7), but with the indicated changes to objects and morphisms:
- left proper miniature model categories and left Quillen functors;
- simplicial miniature model categories and simplicial left Quillen functors;
- left proper miniature simplicial model categories and simplicial left Quillen functors;

Here a \( (\kappa, \lambda) \)-miniature simplicial model category is a \( (\kappa, \lambda) \)-miniature model category (Definition 3.3) enriched over the cartesian model category of \( \lambda \)-small simplicial sets. Used in 1.2.

Proof. This is an immediate consequence of the construction of \( \text{MInd}_\nu \) as a left Bousfield localization of the category of simplicial presheaves on a small category. We remark that the notions of left properness and simpliciality for miniature model categories match the same notions for combinatorial model categories: see Low [2014.a, Remark 5.17] for left proper model categories and Low [2014.a, Remark 5.19] for simplicial model categories.

Proposition 8.4. Theorem 8.2 is applicable to the following choices of \( C \) and \( D \), constructed exactly like \( \text{CMC} \) (Definition 3.7) and \( \text{CRC} \) (Definition 4.1), but with the indicated changes to objects and morphisms:
- For \( C \), we take cartesian combinatorial model categories, which we can require to be left proper, or cartesian, or both.
- For \( D \), we take relative categories \( (\lambda, C) \) such that the category \( C \) admits finite products and the quasicategory \( \mathcal{N} \mathcal{R}C \) is cartesian closed.

Furthermore, the relative categories \( C \) and \( D \) are Dwyer–Kan equivalent to the full subcategory of \( \text{PrL} \) (Definition 5.1) on cartesian closed presentable quasicategories. Used in 1.3.

Proof. Given \( (\lambda, C) \in \text{CRC} \), the model category \( \text{MInd}_\nu (\lambda, C) \) is cartesian whenever \( C \) has finite products (which ensures the pushout product axiom for cofibrations in \( \text{sPSh}(C) \)) and the morphisms used for the left Bousfield localization of \( \text{sPSh}(C) \) are closed under derived pushout products. By Cisinski [2019.a, Proposition 7.11.4] this is true whenever the quasicategory \( \mathcal{N} \mathcal{R}C \) is a reflective localization of the quasicategory of presheaves on a small quasicategory with respect to a set of morphisms that are closed under pushout products. This is true for any cartesian closed quasicategory in \( \text{PrL} \).
Proposition 8.5. The relative functor

\[\text{Reedy}_{\mathcal{CMC}} : \mathcal{CRC} \rightarrow \mathcal{CRC}\]

(Definition 6.3) is a homotopically essentially surjective relative functor of relative categories. \(\text{Used in } 6.2\).

Proof. Given an object \((\lambda, C) \in \mathcal{CRC}\) Proposition 7.16 supplies (for a sufficiently large regular cardinal \(\mu \geq \lambda\)) a weak equivalence

\[
(\lambda, C) \rightarrow (\mu, \text{Reedy}_{\mathcal{CMC}} \text{Min}^\bullet_{\mathcal{CMC}} (\lambda, C)),
\]

which establishes the homotopy essential surjectivity of the relative functor \(\text{Reedy}_{\mathcal{CMC}}\).

Proposition 8.6. The relative functor

\[\text{Reedy}_{\mathcal{CMC}} : \mathcal{CRC} \rightarrow \mathcal{CRC}\]

(Definition 6.3) is a homotopically fully faithful relative functor of relative categories: for any objects \((\lambda, C), (\mu, D) \in \mathcal{CMC}\) the induced map

\[
\mathcal{H}_{\mathcal{CMC}}((\lambda, C), (\mu, D)) \rightarrow \mathcal{H}_{\mathcal{CRC}}(\text{Reedy}_{\mathcal{CMC}}(\lambda, C), \text{Reedy}_{\mathcal{CMC}}(\mu, D))
\]

is a simplicial weak equivalence. \(\text{Used in } 8.1^*, 8.2^*\)

Proof. We invoke a variant of the simplicial Whitehead theorem (Corollary 2.49). Suppose we are given a commutative square

\[
\begin{array}{ccc}
\text{Sd}^k \partial \Delta^n & \xrightarrow{\alpha} & \mathcal{H}_{\mathcal{CMC}}((\lambda, C), (\mu, D)) \\
\downarrow & & \downarrow \mathcal{H}_{\text{Reedy}_{\mathcal{CMC}}}
\end{array}
\]

\[
\begin{array}{ccc}
\text{Sd}^k \Delta^n & \xrightarrow{\beta} & \mathcal{H}_{\mathcal{CRC}}(\text{Reedy}_{\mathcal{CMC}}(\lambda, C), \text{Reedy}_{\mathcal{CMC}}(\mu, D)),
\end{array}
\]

where \(\text{Sd}\) denotes the barycentric subdivision functor. Denote by \(\Lambda\) the simplicial subset of \(\Delta^2\) generated by the 1-simplices \(0 \rightarrow 2\) and \(1 \rightarrow 2\). We construct maps

\[
\gamma : \text{Sd}^k \Delta^n \rightarrow \mathcal{H}_{\mathcal{CMC}}((\lambda, C), (\mu, D)),
\]

\[
\Gamma : \Delta^1 \times \text{Sd}^k \Delta^n \rightarrow \mathcal{H}_{\mathcal{CRC}}(\text{Reedy}_{\mathcal{CMC}}(\lambda, C), \text{Reedy}_{\mathcal{CMC}}(\mu, D)),
\]

\[
\pi : \Lambda \times \text{Sd}^k \partial \Delta^n \rightarrow \mathcal{H}_{\mathcal{CMC}}((\lambda, C), (\mu, D)),
\]

\[
\Pi : \Delta^2 \times \text{Sd}^k \partial \Delta^n \rightarrow \mathcal{H}_{\mathcal{CRC}}(\text{Reedy}_{\mathcal{CMC}}(\lambda, C), \text{Reedy}_{\mathcal{CMC}}(\mu, D))
\]

such that the map \(\Gamma\) is a simplicial homotopy from \(\beta\) to \(\mathcal{H}_{\text{Reedy}_{\mathcal{CMC}}} \circ \gamma\), the map \(\Pi\) restricts to \(\mathcal{H}_{\text{Reedy}_{\mathcal{CMC}}} \circ \pi\) on \(\Lambda \times \text{Sd}^k \partial \Delta^n\), the map \(\pi\) restricts to \(\alpha\) on \(0 \times \text{Sd}^k \partial \Delta^n\), the restrictions of \(\pi\) to \(1 \times \text{Sd}^k \partial \Delta^n\) and \(\gamma\) to \(\text{Sd}^k \partial \Delta^n\) coincide, and the restrictions of \(\Pi\) to \((0 \rightarrow 1) \times \text{Sd}^k \partial \Delta^n\) and \(\Gamma\) to \(\Delta^1 \times \text{Sd}^k \partial \Delta^n\) coincide. The maps \(\Gamma\), \(\gamma\), \(\Pi\), and \(\pi\) are constructed in the remainder of the proof. All conditions required for \(\Gamma\), \(\gamma\), \(\Pi\), and \(\pi\) will be satisfied automatically by construction. We refer the reader to Remark 2.50 for a pictorial representation of the maps \(\Gamma\), \(\gamma\), \(\Pi\), and \(\pi\).

Reduction to a fixed zigzag type. Recall (Remark 2.30) that for a relative category \(\mathcal{C}\) with objects \(X, Y \in \mathcal{C}\), the simplicial set \(\mathcal{H}_\mathcal{C}(X, Y)\) is constructed as the filtered colimit

\[
\colim_Z N(C^Z_{X, Y}),
\]

where \(Z\) runs over all possible zigzag types (a finite sequence like \(\leftarrow \rightarrow \rightarrow \rightarrow \rightarrow \leftarrow\), turned into a relative category by declaring all arrows \(\leftarrow\) to be weak equivalences), \(N\) denotes the nerve functor, and \(C^Z_{X, Y}\) is the category of relative functors \(Z \rightarrow \mathcal{C}\) that map the leftmost and rightmost objects of \(Z\) to \(X\) and \(Y\) respectively.

All simplicial sets involved in the simplicial Whitehead theorem have finitely many nondegenerate simplices, hence are compact objects in the category of simplicial sets. Thus, the all maps to \(\mathcal{H}\) in the simplicial Whitehead theorem factor through some fixed term of the colimit, indexed by a fixed zigzag type \(Z\). From now on, we work with this fixed zigzag type \(Z\). Now, maps of simplicial sets \(S \rightarrow N(C^Z_{X, Y})\) can be identified
with functors \( \pi_{\leq 1} \, S \to C_{X,Y}^\mathcal{F} \), where \( \pi_{\leq 1} \) denotes the fundamental category functor. The latter functors can themselves be identified with relative functors \( Z \times \pi_{\leq 1} \, S \to C \) that are constant functors valued in \( X \) respectively \( Y \) when restricted to the leftmost respectively rightmost object of \( Z \). From now on, we interpret existing simplicial maps and construct new simplicial maps to \( \mathcal{H} \) in this form, as diagrams given by relative functors \( Z \times \pi_{\leq 1} \, S \to C \). Since the value of such a diagram on the leftmost and rightmost vertex of \( Z \) is prescribed, in the remainder of the proof we construct relative functors \( Z \times \pi_{\leq 1} \, S \to C \) as follows: we pick some interior vertex \( z \in Z' \), construct a functor \( \pi_{\leq 1} \, S \to C \), establish naturality with respect to morphisms in \( Z \), and verify the fact that left-pointing maps are sent to weak equivalences.

**Selection of the regular cardinal \( \nu \).** We now define the regular cardinal \( \nu \) that will be used in constructions of the maps \( \Gamma \), \( \gamma \), \( \Pi \), and \( \pi \). Apply the functor \( \text{MInd} \) (Definition 7.6) to all vertices and edges of the diagram \( \beta \). This produces a commutative diagram of combinatorial model categories. Choose a regular cardinal \( \nu \) such that all vertices in this diagram are strongly \((\kappa, \nu)\)-combinatorial model categories for some \( \kappa \leq \nu \) (Definition 3.3) and all edges in this diagram are left \( \nu \)-Quillen functors (Definition 3.3). Since \( \text{Sd}^k \Delta^n \) has only finitely many nondegenerate vertices and edges, such a regular cardinal \( \nu \) exists by Proposition 3.3 and Proposition 3.6.

**Construction of the map \( \gamma \).** Apply the functor \( \text{MInd}_{\nu} \) (Definition 7.7) to the diagram \( \beta \). The choice of \( \nu \) guarantees that \( \text{MInd}_{\nu} \) is defined for all objects and morphisms of \( \beta \). The resulting model categories are \((\kappa, \nu)\)-miniature model categories by Proposition 7.8, so we can interpret the result as a map

\[
\delta: \text{Sd}^k \Delta^n \to \mathcal{H}_{\text{CMC}}((\nu, \text{MInd}_{\nu}(\lambda, \text{Reedy}(C))), (\nu, \text{MInd}_{\nu}(\mu, \text{Reedy}(D))))
\]

Define \( Z' \to Z \) as \( \leftarrow \), i.e., the zigzag type \( Z' \) is obtained from \( Z \) by attaching 4 additional morphisms as indicated. From now on, we will be constructing simplicial maps of zigzag type \( Z' \). Where necessary, existing maps of zigzag type \( Z \) are silently promoted to the zigzag type \( Z' \) by adding identity morphisms. Now produce a map

\[
\gamma: \text{Sd}^k \Delta^n \to \mathcal{H}_{\text{CMC}}((\lambda, C), (\mu, D))
\]

by attaching to every zigzag in \( \delta \) the weak equivalences

\[
(\lambda, C) \to (\nu, \text{Ind}_{\nu}^\lambda(C)) \leftrightarrow (\nu, \text{MInd}_{\nu}(\lambda, \text{Reedy}(C))), \quad (\nu, \text{MInd}_{\nu}(\mu, \text{Reedy}(D))) \to (\nu, \text{Ind}_{\nu}^\mu(D)) \leftrightarrow (\mu, D).
\]

Here the left Quillen functor \( \text{MInd}_{\nu}(\lambda, \text{Reedy}(C)) \to \text{Ind}_{\nu}^\lambda(C) \) is a left Quillen equivalence by Proposition 7.12.

**Construction of the map \( \Gamma \).** The map

\[
\Gamma: \Delta^1 \times \text{Sd}^k \Delta^n \to \mathcal{H}_{\text{CRG}}(\text{Reedy}^\lambda(\lambda, C), \text{Reedy}^\mu(\mu, D))
\]

is a simplicial homotopy from \( \beta \) to \( \mathcal{H}_{\text{Reedy}} \circ \gamma \) constructed as a natural transformation of diagrams of zigzag type \( Z' \), i.e., a functor

\[
Z \times \pi_{\leq 1} \, (\text{Sd}^k \Delta^n) \to \text{CRC}^{\pi_{\leq 1}(\Delta^1)}.
\]

First, promote \( \beta \) to the zigzag type \( Z' \) by precomposing with the relative functor \( Z' \to Z \) that collapses the outer two vertices on each side. This amounts to attaching to every zigzag in \( \beta \) the identity maps

\[
(\lambda, \text{Reedy}^\lambda(\lambda, C)) \to (\lambda, \text{Reedy}^\mu(\lambda, C)) \leftrightarrow (\lambda, \text{Reedy}^\mu(\lambda, C)), \quad (\mu, \text{Reedy}^\lambda(\mu, D)) \to (\mu, \text{Reedy}^\mu(\mu, D)) \leftrightarrow (\mu, \text{Reedy}^\mu(\mu, D)),
\]

ensuring that both \( \beta \) and \( \mathcal{H}_{\text{Reedy}} \circ \gamma \) have the same zigzag type \( Z' \).

Now we construct \( \Gamma \) as a natural weak equivalence from the diagram of \( \beta \) to the diagram of \( \mathcal{H}_{\text{Reedy}} \circ \gamma \). Following the tactic outlined in the paragraph on reduction to a fixed zigzag type, we work with a fixed interior vertex \( z \in Z' \) and construct a natural transformation of functors \( \pi_{\leq 1}(\text{Sd}^k \Delta^n) \to \text{CRC} \).

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If the vertex \( z \) belongs to \( Z \subset Z' \), the value of \( \Gamma \) on some object \( W \in \pi_{\leq 1}(Sd^k \Delta^n) \) with \( \beta(W) = (\kappa, E) \in CRC \) is given by the weak equivalence (Proposition 7.10) in \( CRC \)

\[
\Gamma_{\kappa, E}: (\kappa, E) \to (\nu, \text{Reedy MInd}_\nu(\kappa, E))
\]

whose underlying relative functor

\[
E \to \text{Reedy MInd}_\nu(\kappa, E)
\]

sends an object \( X \in E \) to the Reedy cofibrant cosimplicial diagram \( n \mapsto \Delta^n \otimes Y(X) \).

If the vertex \( z \) does not belong to \( Z \subset Z' \), then it is one of the two interior vertices added to the zigzag \( Z \).

Suppose \( z \) is adjacent to the leftmost vertex of \( Z' \) (corresponding to \( (\lambda, C) \)); the other case (corresponding to \( (\mu, D) \)) is treated symmetrically. The resulting morphism does not depend on the choice of \( W \in \pi_{\leq 1}(Sd^k \Delta^n) \) and is given by the weak equivalence

\[
(\lambda, \text{Reedy}(C)) \to (\nu, \text{Reedy} \text{Ind}_\nu(\lambda, C))
\]

induced by the relative functor

\[
\text{Reedy}(C) \to \text{Reedy} \text{Ind}_\nu(\lambda, C)
\]

obtained by applying \( \text{Reedy} \) to the canonical inclusion

\[
C \to \text{Ind}_\nu(\lambda, C).
\]

This completes the construction of \( \Gamma \).

**Construction of the maps \( \pi \) and \( \Pi \).** Next, we construct the maps

\[
\pi: \Lambda \times Sd^k \partial \Delta^n \to H^{\text{CMC}}(\lambda, C), (\mu, D)), \quad \Pi: \Delta^2 \times Sd^k \partial \Delta^n \to H^{\text{CRC}}(\text{Reedy}(\lambda, C), \text{Reedy}(\mu, D))
\]

using similar techniques. As before, fix some interior vertex \( z \in Z' \) and construct functors

\[
\pi_{\leq 1}(Sd^k \partial \Delta^n) \to \text{CMC}_{\leq 1}^{\pi}, \quad \pi_{\leq 1}(Sd^k \partial \Delta^n) \to \text{CRC}_{\leq 1}^{\pi} \Delta^2.
\]

If the vertex \( z \) belongs to \( Z \subset Z' \), the value of \( \pi \) on some object \( W \in \pi_{\leq 1}(Sd^k \partial \Delta^n) \) with \( \alpha(W) = (\kappa, M) \) is given by the following object in \( \text{CMC}_{\leq 1}^{\pi} \):

\[
\begin{array}{c}
\downarrow \iota \\
(\kappa, M)
\end{array}
\]

\[
\begin{array}{c}
(\nu, \text{Ind}_\nu(\kappa, M))
\downarrow \text{ev}
\end{array}
\]

\[
(\nu, \text{MInd}_\nu(\kappa, \text{Reedy}(M))),
\]

where the map \( \iota \) is the canonical inclusion and the map \( \text{ev} \) is defined on representables via the formula \( \text{ev}(\Delta^n \otimes R) = R_n \), where \( R \in \text{Reedy}(M) \).

Likewise, the map \( \Pi \) is given by the following object in \( \text{CRC}_{\leq 1}^{\pi} \Delta^2 \):

\[
\begin{array}{c}
\downarrow \text{Reedy}(\iota)
\end{array}
\]

\[
\begin{array}{c}
(\nu, \text{Reedy}(\kappa, M))
\end{array}
\]

\[
\begin{array}{c}
(\nu, \text{Reedy}(\text{MInd}_\nu(\kappa, \text{Reedy}(M))),
\end{array}
\]

\[
\begin{array}{c}
\downarrow \Gamma_{\kappa, \text{Reedy}(M)}
\end{array}
\]

\[
(\nu, \text{Reedy}(\text{MInd}_\nu(\kappa, \text{Reedy}(M))),
\]

\[
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\]
where the map $\Gamma_{\kappa, \text{Reedy}(M)}$ was defined in the previous part of the proof: it sends $R \in \text{Reedy}(M)$ to the Reedy cofibrant object $n \mapsto \Delta^n \otimes Y(R)$.

If the vertex $z$ does not belong to $Z \subset Z'$, then it is one of the two interior vertices added to the zigzag $Z$. Suppose $z$ is adjacent to the leftmost vertex of $Z'$ (corresponding to $(\lambda, C)$); the other case (corresponding to $(\mu, D)$) is treated symmetrically. The resulting object in $\text{CMC}^{\leq 1}_U$ does not depend on the choice of $W \in \pi_{\leq 1}(Sd^k \partial \Delta^n)$ and is given by the following diagram:

\[ \begin{array}{ccc}
(\nu, \text{Ind}_{\nu}^\lambda(C)) & \xrightarrow{id} & (\nu, \text{Ind}_{\nu}^\lambda(C)) \\
\downarrow & & \downarrow \\
(\lambda, \text{Reedy}(C)) & \xrightarrow{\Gamma_{\lambda, \text{Reedy}(C)}} & (\lambda, \text{Reedy}(C))
\end{array} \]

where $\iota$ denotes the canonical inclusion. Likewise, the map $\Pi$ is given by the following object in $\text{CRC}^{\leq 1}_U$:

\[ \begin{array}{ccc}
(\nu, \text{Reedy}(\text{Ind}_{\nu}^\lambda(C))) & \xrightarrow{\text{Reedy}(\iota)} & (\nu, \text{Reedy}(\text{Ind}_{\nu}^\lambda(C))) \\
\downarrow & & \downarrow \\
(\lambda, \text{Reedy}(C)) & \xrightarrow{\Gamma_{\lambda, \text{Reedy}(C)}} & (\lambda, \text{Reedy}(C))
\end{array} \]

where the map $\Gamma_{\lambda, \text{Reedy}(C)}$ was defined in the previous part of the proof: evaluate $\text{Reedy}$ on the canonical inclusion $C \rightarrow \text{Ind}_{\nu}^\lambda(C)$.

**Theorem 8.7.** The relative functor $\text{Cof} : \text{CMC} \rightarrow \text{CRC}$ (Definition 6.1) is a Dwyer–Kan equivalence of relative categories. Used in $\dagger$.

**Proof.** The relative functor $\text{Cof}$ is weakly equivalent to the relative functor $\text{Reedy}$ (Definition 6.3) via the natural weak equivalence $\text{Reedy} \rightarrow \text{Cof}$ of Definition 6.3. By Theorem 8.1, $\text{Reedy}$ is a Dwyer–Kan equivalence, hence so is $\text{Cof}$.

The following proposition is not used in the remainder of the article. It shows that the more straightforward way to define a Dwyer–Kan equivalence $\text{CMC}_U \rightarrow \text{CRC}_U$ is weakly equivalent to the functor $\text{Cof}$ under the Dwyer–Kan equivalences $\text{CMC}_U \rightarrow \text{CMC}_U$ (Proposition 3.10) and $\text{CRC}_U \rightarrow \text{CRC}_U$ (Proposition 4.3).

**Proposition 8.8.** Suppose $U$ is a strongly inaccessible cardinal. Consider the functor $\text{Cof}_U : \text{CMC}_U \rightarrow \text{CRC}_U$ that sends an object $M \in \text{CMC}_U$ to the relative category of cofibrant objects in $M$ and a left Quillen functor $M \rightarrow N$ in $\text{CMC}_U$ to the induced functor between the categories of cofibrant objects. The functors

$\text{Cof}_U \circ \text{Ind}_U : \text{CMC}_U \rightarrow \text{CRC}_U$

and

$\text{RInd}_U \circ \text{Cof}_U : \text{CMC}_U \rightarrow \text{CRC}_U$

are naturally weakly equivalent. Used in $\dagger$.

**Proof.** The natural weak equivalence is given by the morphism

$\text{Cof}_U(\text{Ind}_U(\lambda, M)) \rightarrow \text{RInd}_U(\text{Cof}_U(\lambda, M))$

that sends a cofibrant object $A \in \text{Ind}_U(\lambda, M)$ to $n \mapsto \Delta^n \otimes (B \mapsto \mathcal{E}_M(B, A))$.
9 Equivalence of combinatorial relative categories and presentable quasicategories

Definition 9.1. The relative functor
\[ \mathcal{N} : \text{CRC} \to \text{PrL} \]
between the relative categories \( \text{CRC} \) (Definition 4.1) and \( \text{PrL} \) (Definition 5.1) is defined as follows. An object \( (\lambda, C) \) is sent to \( (\lambda, \mathcal{N}C) \). A morphism \( (\lambda, C) \to (\mu, D) \) given by a relative functor \( F : C \to D \) is sent to the morphism
\[ (\lambda, \mathcal{N}C) \to (\mu, \mathcal{N}D) \]
given by the functor
\[ \mathcal{N}F : \mathcal{N}C \to \mathcal{N}D. \]
The relative functors \( \mathcal{N} \) and \( \mathcal{R} \) are defined in Notation 2.36. Used in 9.1, 9.2, 9.3, 9.4.

Proposition 9.2. This definition is correct.

Proof. If \( (\lambda, C) \in \text{CRC} \) then the small quasicategory \( \mathcal{N}C \) admits \( \lambda \)-small colimits by Definition 2.37. Likewise, if \( (\lambda, C) \to (\mu, D) \) is a morphism in \( \text{CRC} \), the functor \( \mathcal{N}C \to \mathcal{N}D \) preserves \( \lambda \)-small colimits by Definition 2.37. We now show that \( \mathcal{N} \) preserves weak equivalences by establishing this claim separately for each generating class. If \( (\lambda, C) \to (\mu, D) \) is a weak equivalence such that \( \lambda = \mu \) and \( C \to D \) is a Dwyer–Kan equivalence, then \( \mathcal{N}C \to \mathcal{N}D \) is a Dwyer–Kan equivalence between fibrant objects, and \( \mathcal{N}C \to \mathcal{N}D \) is an equivalence of quasicategories because \( \mathcal{N} \) is a right Quillen functor. For weak equivalences \( (\lambda, C) \to (\mu, D) \) of the second generating class (i.e., involving ind-completions), applying \( \mathcal{N} \) produces a weak equivalence in \( \text{PrL} \) by definition of \( \text{CRC} \) (Definition 4.1), since we defined the second generating class there as the preimage of corresponding weak equivalences in \( \text{PrL} \). }

Definition 9.3. The relative functor
\[ \mathcal{K} : \text{PrL} \to \text{CRC} \]
between the relative categories \( \text{PrL} \) (Definition 5.1) and \( \text{CRC} \) (Definition 4.1) is defined as follows. An object \( (\lambda, C) \) is sent to \( (\lambda, KC) \), where \( K \) is the functor from Notation 2.36. A morphism \( (\lambda, C) \to (\mu, D) \) given by a map of simplicial sets \( F : C \to D \) is sent to the morphism \( (\lambda, KC) \to (\mu, KD) \) given by the functor \( KC \to KD \). Used in 9.1, 9.2, 9.3, 9.4.

Proposition 9.4. This definition is correct.

Proof. Suppose \( (\lambda, C) \in \text{PrL} \). Since the functor \( C \to \mathcal{N}KC \) is an equivalence, by Definition 2.37 the small relative category \( KC \) admits \( \lambda \)-small homotopy colimits.

Suppose \( (\lambda, C) \to (\mu, D) \) is a morphism in \( \text{PrL} \). Since the morphism \( \mathcal{N}KC \to \mathcal{N}KD \) is weakly equivalent to \( C \to D \), by Definition 2.37 the relative functor \( KC \to KD \) preserves \( \lambda \)-small homotopy colimits.

We show that \( \mathcal{K} \) preserves weak equivalences by establishing this claim separately for each generating class. If \( (\lambda, C) \to (\mu, D) \) is a weak equivalence such that \( \lambda = \mu \) and \( C \to D \) is an equivalence of quasicategories, then the relative functor \( KC \to KD \) is a Dwyer–Kan equivalence because \( K \) is a left Quillen functor and all simplicial sets are cofibrant in the Joyal model structure. If \( (\lambda, C) \to (\mu, D) \) is a weak equivalence such that \( C \to D \) exhibits \( D \) as the quasicategory of \( \mu \)-presentable objects in the \( \lambda \)-ind-completion of \( C \), the morphism \( (\lambda, KC) \to (\mu, KD) \) is a weak equivalence in \( \text{CRC} \) if its image under \( \mathcal{N} \) is a weak equivalence in \( \text{PrL} \). The resulting morphism \( (\lambda, \mathcal{N}KC) \to (\mu, \mathcal{N}KD) \) is weakly equivalent to the original morphism \( (\lambda, C) \to (\mu, D) \) via the derived unit map, which completes the proof. }

Definition 9.5. The natural transformation
\[ \eta : \text{PrL} \to \mathcal{N} \circ \mathcal{K} \]
from the identity functor on the relative category \( \text{PrL} \) (Definition 5.1) to the composition of relative functors \( \mathcal{N} \) (Definition 9.1) and \( \mathcal{K} \) (Definition 9.3) is constructed as follows. Given \( (\lambda, C) \in \text{PrL} \), we send it to the map
\[ (\lambda, C) \to (\lambda, \mathcal{N}KC) \]
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given by composing the unit map $C \to \mathcal{N} K C$ with the map $\mathcal{N} K C \to \mathcal{N} \mathcal{R} K C$. The relative functors $\mathcal{N}$, $\mathcal{K}$, and $\mathcal{R}$ are defined in Notation 2.36.

**Proposition 9.6.** This definition is correct and $\eta$ is a natural weak equivalence. Used in 9.10.

*Proof.* Suppose $(\lambda, C) \to (\mu, D)$ is a morphism in $\text{PrL}_\mathcal{K}$. We must show that the square

$$
\begin{array}{ccc}
(\lambda, C) & \longrightarrow & (\mu, D) \\
\downarrow & & \downarrow \\
(\lambda, \mathcal{N} \mathcal{R} K C) & \longrightarrow & (\mu, \mathcal{N} \mathcal{R} K D)
\end{array}
$$

commutes, which follows from the commutativity of the following diagram:

$$
\begin{array}{ccc}
C & \longrightarrow & D \\
\downarrow & & \downarrow \\
\mathcal{N} K C & \longrightarrow & \mathcal{N} K D \\
\downarrow & & \downarrow \\
\mathcal{N} \mathcal{R} K C & \longrightarrow & \mathcal{N} \mathcal{R} K D.
\end{array}
$$

The top square commutes because the unit is a natural transformation. The bottom square commutes because $\mathcal{R}$ is a functor and the fibrant replacement map $\text{id} \to \mathcal{R}$ is a natural transformation.

Finally, $\eta$ is a weak equivalence because $\mathcal{K}$ and $\mathcal{N}$ form a Quillen equivalence, so the derived unit map of $\mathcal{K} \dashv \mathcal{N}$ is a weak equivalence.

**Definition 9.7.** The relative endofunctor $\mathcal{R} : \mathcal{CRC} \to \mathcal{CRC}$

on the relative category $\mathcal{CRC}$ (Definition 4.1) is constructed as follows. An object $(\lambda, C) \in \mathcal{CRC}$ is sent to $(\lambda, \mathcal{R} C)$, where $\mathcal{R}$ is the functor from Notation 2.36. A morphism $(\lambda, C) \to (\mu, D)$ given by a relative functor $F : C \to D$ is sent to the morphism $(\lambda, \mathcal{R} C) \to (\mu, \mathcal{R} D)$ given by the relative functor $\mathcal{R} F : \mathcal{R} C \to \mathcal{R} D$. Used in 9.8.

**Definition 9.8.** The zigzag $\varepsilon$ of natural transformations

$$
\mathcal{K} \circ \mathcal{N} \to \mathcal{R} \leftarrow \text{id}_{\mathcal{CRC}}
$$

between functors $\mathcal{K} \circ \mathcal{N}$ (Definition 9.3, Definition 9.1), $\mathcal{R}$ (Definition 9.7), and $\text{id}_{\mathcal{CRC}}$ is constructed as follows. Given $(\lambda, C) \in \mathcal{CRC}$ we send it to the zigzag

$$
(\lambda, \mathcal{K} \mathcal{N} \mathcal{R} C) \to (\lambda, \mathcal{R} C) \leftarrow (\lambda, C),
$$

where the first map is the counit of $\mathcal{R} C$ and the second map is the fibrant replacement map.

**Proposition 9.9.** This definition is correct and $\varepsilon$ is a zigzag of natural weak equivalences. Used in 9.10.

*Proof.* The naturality of the first transformation follows from the naturality of counit maps and the naturality of the second transformation follows from the naturality of the fibrant replacement map $\text{id} \to \mathcal{R}$. The counit map $\mathcal{K} \mathcal{N} \mathcal{R} C \to \mathcal{R} C$ is the derived counit map of a Quillen equivalence, hence is a weak equivalence. The fibrant replacement map is a weak equivalence by definition.

**Theorem 9.10.** The functor $\mathcal{N} : \mathcal{CRC} \to \text{PrL}_\mathcal{R}$ (Definition 9.1) is a Dwyer–Kan equivalence. Used in 1.1.

*Proof.* Combine Proposition 9.6 and Proposition 9.9.

The following proposition is not used in the remainder of the article. It shows that the more straightforward way to define a Dwyer–Kan equivalence $\mathcal{N}_U : \mathcal{CRC}_U \to \text{PrL}_U$ is weakly equivalent to the functor $\mathcal{N}$ under the Dwyer–Kan equivalences $\text{RInd}_U : \mathcal{CRC}_U \to \mathcal{CRC}_U$ (Proposition 4.5) and $\text{QInd}_U : \text{PrL}_U \to \text{PrL}_U$ (Proposition 5.4).

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Proposition 9.11. Suppose $U$ is a strongly inaccessible cardinal. Consider the functor

$$N_U = N: CRC_U \to \mathcal{P}L_U.$$  

The functors

$$N_U \circ RInd_U : CRC_U \to \mathcal{P}L_U$$

and

$$QInd_U \circ N: CRC_U \to \mathcal{P}L_U$$

are connected by a zigzag of natural weak equivalences. Used in 1.1*. 

Proof. The natural weak equivalence that we need has the form

$$N_U(RInd_U(\lambda, C)) \to QInd_U(N(\lambda, C)).$$

Unfolding the definitions, we need a natural weak equivalence

$$N(R(Reedy(MInd_U(\lambda, C)))) \to N(R(Reedy(MInd_U(\lambda, K\mathcal{V}RC)))).$$ 

Such a natural weak equivalence is induced by the zigzag $K\mathcal{V}RC \rightarrow RC \leftarrow C$ of Proposition 9.9.

10 References

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