Poincaré Dual of D=4 N=2 Supergravity with Tensor Multiplets

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Abstract: We study, in an arbitrary even number $D$ of dimensions, the duality between massive $D/2$ tensors coupled to vectors, with masses given by an arbitrary number of “electric” and “magnetic” charges, and $(D/2-1)$ massive tensors. We develop a formalism to dualize the Lagrangian of $D = 4$, $N = 2$ supergravity coupled to tensor and vector multiplets, and show that, after the dualization, it is equivalent to a standard $D = 4$, $N = 2$ gauged supergravity in which the Special Geometry quantities have been acted on by a suitable symplectic rotation.
1. Introduction

It is a well known fact that orientifold compactifications of IIB string theory with fluxes turned on, give rise to a theory in which tensor multiplets are present, and are endowed with a mass; this fact has recently given rise to a renewed interest in theories describing tensor multiplets coupled to scalar and vector multiplets [1, 2, 3].

This same topic has been considered also in [4] where a theory describing, in $D$ even dimensions, a number of massive $D/2$–form tensors $B_I$ with masses given by “magnetic” and “electric” couplings has been dualized to a theory of massive $D/2−1$–dimensional form fields, generalizing the well-known duality in 4 dimensions between massive 2–form tensors and massive vectors. The “electric” and “magnetic” couplings which give mass to the tensors arise naturally from Fayet-Iliopoulos terms in the context of the dualization of $n_T$ massless scalars to 2–forms in $D = 4$, $N = 2$ supergravity [5, 6, 7]. In this light, the theory of [4] can be thought of as a generalization to an arbitrary number of dimensions of the linearized (without coupling to scalars) bosonic kinetic and mass terms for the $n_T$ 2–forms $B_I$ from $D = 4$, $N = 2$ supergravity coupled to $n_T$ tensor multiplets and $n_V$ vector multiplets [8].

Actually in [4] the proof of this generalized duality has been given only for two particular cases: that of a single tensor $B$, when $D = 4d$ and hence $B$ is a $2d$–form field with coupling constants $m$ and $e$, and that of a couple of tensors, $B_I$ with $I = 1, 2$ when $D = 4d + 2$ and hence the $B_I$‘s are $(2d + 1)$–form fields, taking the coupling constants to be $m^{1\Lambda} = m^{2\Lambda}$ and $e^{1\Lambda} = e^{2\Lambda}$, with $\Lambda = 1, 2$.

In both cases, the squared mass of the $(D/2−1)$–form after the dualization has been shown to depend on both “electric” and “magnetic” charges, via the relation:
\[
\mu = \sqrt{e^2 + m^2},
\]
(1.1)
so that no distinction appears between electric and magnetic charges. We will extend the analysis of [4] to the more general case in which an arbitrary number of “electric” and “magnetic” charges is present, in the linearized theory, showing that also in this case the Poincaré dual of a theory of \( n_T \) massive \( D/2 \)-form fields is a theory describing \( n_T \) massive \( D/2 - 1 \)-form fields. Moreover their square mass matrix will be given by the “sum of the squares” of the electric and magnetic charge matrices:

\[
(\mu^2)^{IJ} = (e^2 + m^2)^{IJ} \equiv (e^I_A \delta^{AB} e^J_B + m^{I\Lambda} \delta_{\Lambda\Sigma} m^{J\Sigma})
\]
(1.2)
thus generalizing relation (1.1).

The whole procedure requires the matrices \( e^I_A, m^{I\Lambda} \) and \( (e^2 + m^2)^{IJ} \) to be invertible, therefore from now on we will suppose that \( n_T = n_V = n \), so that all these matrices are square, and that \( \det e, \det m, \det (e^2 + m^2) \neq 0 \) in order to guarantee the existence of the inverses.

The paper is organized as follows: in section 2 we prepare the setting and write down the dual Lagrangian for a theory of \( n \) massive tensors in an arbitrary even number \( D \) of dimensions, distinguishing between \( D = 4d \) and \( D = 4d + 2 \) cases, generalizing the procedure of [4].

In section 3 we apply the whole formalism to the Lagrangian of \( D = 4, N = 2 \) supergravity coupled to \( n \) massive tensors and \( n \) vectors obtained in [6].

Finally, in section 4 we reinterpret the dual Lagrangian obtained with the dualization as a standard \( D = 4, N = 2 \) gauged supergravity, where the gauge is purely electric and the quantities of the special geometry can be obtained from the standard ones by means of a symplectic rotation.

We note that the fact that the \( N = 2 \) tensor coupled theory can be related to a standard \( N = 2 \) supergravity by means of duality transformations does not mean that they are the same theory. As a matter of fact we can show that they give rise to different \( N = 1 \) theory when a suitable truncation to \( N = 1 \) is performed (see reference [8]).

### 2. General dualization procedure for the linearized theory

In this section we generalize the procedure of reference [4] to obtain the dual Lagrangian of a theory of massive tensors in an arbitrary even number of dimensions with an arbitrary number of charges.

The linearized Lagrangian of a number \( n \) of massless vector fields and a number \( n \) of tensor fields, as coming from \( D = 4, N = 2 \) supergravity [5, 6], is\(^1\):

\[
\mathcal{L}^{T,V} = -\frac{1}{4} \delta^{IJ} \mathcal{H}_I \wedge \ast \mathcal{H}_J - e^I_A \left( \mathcal{F}^A + \frac{1}{2} m^{I\Lambda} B_J \right) \wedge B_I
\]

\(^1\)Note that, in order to make contact with [4] one has to redefine the “electric” and “magnetic” charges as:\( 2 e^I_A \rightarrow e^I_A, 2 m^{I\Lambda} \rightarrow m^{I\Lambda} \).
where $\delta$ is the usual Kronecker delta, $F^\Lambda$, $H_I$ are the field-strengths of the gauge vectors $A^\Lambda$ and the tensors $B_I$ respectively.

Lagrangian (2.1) can be immediately generalized to an arbitrary (even) number $D$ of dimensions by promoting $B_I$ and $F^\Lambda$ to be $D/2$–form fields. As is well known, Lagrangian (2.1) and its generalization to higher dimensions are invariant under the following gauge transformations:

$$\delta A^\Lambda = -m^I A^\Lambda, \quad \delta B_I = d\Omega_I, \quad (2.2)$$

where $\Omega_I$ is a $D/2$–1-form field, if and only if the generalized tadpole cancellation condition:

$$\epsilon^I \delta A^\Lambda \pm \epsilon^I m^I A^\Lambda = 0 \quad (2.3)$$

holds. The minus sign holds in the $D = 4d$ case, when the $B_I$ are even forms (and hence they commute with each other), while the plus sign holds in the $D = 4d + 2$ case, when the $B_I$ are odd forms (anticommuting with each other).

It will prove convenient to add to (2.1) a total derivative:

$$-\frac{1}{2} \epsilon^I (m^{-1})_I^\Sigma F^\Lambda \wedge F^\Sigma, \quad (2.4)$$

in such a way that the topological term proportional to the “electric” charge $\epsilon^I$ can be written in a manifestly gauge invariant way:

$$-\frac{1}{2} \epsilon^I (m^{-1})_I^\Sigma (F^\Lambda + m^J A^\Lambda) \wedge (F^\Sigma + m^K B_K) \quad (2.5)$$

By means of a gauge transformation, the field-strengths of the vectors can be reabsorbed into the tensors:

$$\Omega_I = (m^{-1})_I^\Lambda A^\Lambda \Rightarrow A'^\Lambda \equiv A^\Lambda + \delta A^\Lambda = 0; \quad B'_I \equiv B_I + (m^{-1})_I^\Lambda F^\Lambda \quad (2.6)$$

and the Lagrangian can be written in the simpler form:

$$\pm \mathcal{L}^{T,V} = -\frac{1}{4} \delta^{IJ} H_I \wedge * H_J + \frac{1}{2} (m^2)^{IJ} B_I \wedge * B_J - \frac{1}{2} (em)^{IJ} B_I \wedge B_J \quad (2.7)$$

the two signs referring respectively to the case $D = 4d$ and $D = 4d + 2$. We have defined the following matrices:

$$\left( m^2 \right)^{IJ} \equiv \delta_{\Lambda\Sigma} m^I J^\Sigma \quad (2.8)$$

$$\left( em \right)^{IJ} \equiv \epsilon^I m^J A \quad (2.9)$$

\footnote{We adopt a mostly minus metric $(+,-,\ldots,-)$ so that, in order to have positive kinetic energy, when $D = 4d + 2$ an overall minus sign in front of the Lagrangian is needed.}
It should be noted that both matrices have a well-defined symmetry in the indices $I,J$: $(m^2)^{IJ}$ is symmetric by construction while $(em)^{IJ}$, due to the tadpole cancellation condition (2.3) is symmetric if $D = 4d$ and antisymmetric if $D = 4d + 2$.

Starting from the Lagrangian of equation (2.7), the process of dualization requires that $H_I$ and $B_I$ be considered as independent fields, enforcing the relation $H_I = dB_I$ by means of the equations of motion for a suitable Lagrange multiplier $\rho^I$ that should be added to the Lagrangian, which becomes:

$$ \pm \mathcal{L}^{T,V} = -\frac{1}{4} \delta^{IJ} H_I \wedge * H_J + \frac{1}{2} (m^2)^{IJ} B_I \wedge * B_J - \frac{1}{2} (em)^{IJ} B_J \wedge B_I + \rho^I \wedge (H_I - dB_I). $$

In order to get the dualized Lagrangian, at first one has to write down the equations of motion for the independent fields $B_I$ and $H_I$, then to solve these equations in terms of the dual field $\rho^I$ and finally to substitute the resulting expressions into the original Lagrangian. Because of the (anti)-symmetry properties of the matrix $(em)^{IJ}$ and the fact that the $B_I$ and $H_I$ fields are even or odd forms depending on the fact that $D = 4d$ or $D = 4d + 2$, it is more convenient to consider the two cases separately.

### 2.1 $D = 4d$

When $D = 4d$, the $B_I$’s are $2d$-forms, and hence they commute with all other $p$-form fields; the equations of motion are:

$$ \frac{1}{2} \delta^{IJ} * H_J + \rho^I = 0 $$

$$ (m^2)^{IJ} * B_J - (em)^{IJ} B_J + d\rho^I = 0 $$

Equation (2.11) is easily invertible in order to get $H_I$ in terms of $* \rho^I$, while from equation (2.12) and its Hodge dual it is possible to write $B_I$ in terms of both $d\rho^I$ and $*d\rho^I$ as:

$$ B_I = (e^2 + m^2)^{-1} * \rho^J + \left( \frac{e}{m} \right)^J_K d\rho^K $$

$$ * B_I = - (e^2 + m^2)^{-1} \left( d\rho^J - \left( \frac{e}{m} \right)^J_K *d\rho^K \right) $$

where the following matrices have been introduced:

$$ \left( \frac{e}{m} \right)^I_J \equiv e^I_L \delta^L \Sigma \left( m^{-1} \right)_{J \Sigma} $$

$$ (e^2)^{IJ} \equiv e^I_L \delta^L \Sigma e^J_{\Sigma} = (e^2)^{JI}. $$

As $(m^2)^{IJ}$, also $(e^2)^{IJ}$ is symmetric by construction. Using these symmetries, together with the following identities involving the product of two matrices:
which allow a number of simplifications, the dualized Lagrangian turns out to be:

\[ L^{\text{dual}} = \frac{1}{2} (e^2 + m^2)^{-1} \delta_{IJ} \rho^I \wedge * \rho^J + \frac{1}{2} (e^2 + m^2)^{-1} \left( \frac{e}{m} \right)_K^J \left( \frac{e}{m} \right)_L^K \rho^I \wedge \rho^J - \delta_{IJ} \rho^I \wedge * \rho^J. \]  

(2.19)

As expected, (2.19) is the Lagrangian describing a theory of \( n \) massive \((2d-1)\) vector fields, whose mass matrix is \((\mu^2)^{IJ} = (e^2 + m^2)^{IJ} \).

2.2 \( D = 4d + 2 \)

In this case, the \( B_I \) fields are \( 2d + 1 \)-forms, hence anticommuting with all other odd forms; taking into account these different properties, we can obtain the equations of motion:

\[-\frac{1}{2} \delta_{IJ} * H^J + \rho^I = 0 \]  

(2.20)

\[(m^2)^{IJ} * B_J - (em)^{IJ} B_J - d\rho^I = 0 \]  

(2.21)

again, as in the previous case, the equations of motion for the \( H_I \) fields are easily inverted, while after some algebraic manipulation on the equations for the \( B_I \) and their Hodge dual, remembering that \((em)^{IJ}\) is now an antisymmetric matrix, and hence

\[ \left( \frac{e}{m} \right)_K^I \left( e^2 \right)_K^J = \left( \frac{e}{m} \right)_K^I \left( em \right)_K^L \left( \frac{e}{m} \right)_L^K = - \left( \frac{e}{m} \right)_K^J \left( e^2 \right)^{KI} \]  

(2.22)

we get the relations:

\[ B_I = (e^2 + m^2)^{-1} \rho^J + \left( \frac{e}{m} \right)_K^J d\rho^K \]  

(2.23)

\[ * B_I = (e^2 + m^2)^{-1} \rho^J + \left( \frac{e}{m} \right)_K^J d\rho^K \]  

(2.24)

Using (2.23) and (2.24) in (2.11), we get the following expression:

\[ L^{\text{dual}} = -\frac{1}{2} (e^2 + m^2)^{-1} \delta_{IJ} \rho^I \wedge * \rho^J - \frac{1}{2} (e^2 + m^2)^{-1} \left( \frac{e}{m} \right)_K^J \rho^I \wedge \rho^J + \delta_{IJ} \rho^I \wedge * \rho^J. \]  

(2.25)

which is the Lagrangian describing \( n \) massive vectors in \( D = 4d + 2 \) dimensions. We have therefore shown that, in an even number \( D \) of dimensions, a theory of \( n \) massive \( D/2 \)-form tensors is dual to a theory of \( n \) massive \( D/2 - 1 \)-form tensors.

\footnote{Here and in the next case, the topological term has been written for completeness, even if its presence does not modify the equations of motion.}
3. The $D = 4 \ N = 2$ Supergravity theory coupled to vector multiplets and scalar–tensor multiplets

The procedure seen in section 2 can be applied to the Lagrangian of the $D = 4 \ N = 2$ supergravity theory coupled to $n_V$ vector multiplets and $n_T$ scalar-tensor multiplets [6].

The first step is to write the Lagrangian of reference [6] in first-order formalism, which is more practical to perform the dualization. Actually we do not need all the Lagrangian, but only the terms coupled either to $H_I$ or $B_I$, which are collected in $\mathcal{L}_{\text{bos}}$ and $\mathcal{L}_{\text{Pauli}}$:

$$\mathcal{L}_{D=4}^{T, V} = \mathcal{L}_{\text{bos}} + \mathcal{L}_{\text{Pauli}}$$

$$\mathcal{L}_{\text{bos}} = -\frac{1}{2} \text{Im} N_{\Lambda \Sigma} \hat{F}^\Lambda \wedge * \hat{F}^\Sigma + \frac{1}{2} \text{Re} N_{\Lambda \Sigma} \hat{F}^\Lambda \wedge \hat{F}^\Sigma - \frac{1}{4} \mathcal{M}_{IJ} H_I \wedge * H_J +$$

$$+ H_I \wedge A^I - e^\Lambda \left( \hat{F}^\Lambda - \frac{1}{2} m^{J I} B_J \right) \wedge B_I$$

$$\mathcal{L}_{\text{Pauli}} = -\frac{i}{2} H_I \wedge * \left( U^{I A} \psi_A \gamma \gamma \zeta^a - U^{I A} \psi^A \gamma \gamma \zeta^a \right) +$$

$$- i H_I \wedge \Delta^I_{a \beta} \bar{\zeta}^a \gamma^\beta - H_I \wedge \left( U^{I A} \psi_A \zeta^a + U_{J A} \psi^A \zeta^a \right) +$$

$$- 2 i \text{Im} N_{\Lambda \Sigma} \hat{F}^\Lambda \wedge \left[ L^\Sigma \left( \hat{\psi}^A \gamma \gamma \psi^B \epsilon_{A B} \right) - \overline{L}^\Sigma \left( \psi_A \gamma \gamma \psi^B \epsilon_{A B} \right) \right] +$$

$$- 2 \text{Im} N_{\Lambda \Sigma} \hat{F}^\Lambda \wedge \left[ f^{I \Sigma} \left( \hat{\Lambda}_A \gamma \gamma \psi^B \epsilon_{A B} \right) + f^{I \Sigma} \left( \Lambda_A \gamma \gamma \psi^B \epsilon_{A B} \right) \right] +$$

$$- \frac{i}{4} \text{Im} N_{\Lambda \Sigma} \hat{F}^\Lambda \wedge \left[ \nabla_I f^{I \Sigma} \left( \hat{\Lambda}_A \gamma \gamma \lambda^B \epsilon_{A B} \right) - \nabla_I f^{I \Sigma} \left( \Lambda_A \gamma \gamma \lambda^B \epsilon_{A B} \right) \right] +$$

$$+ \frac{1}{2} \text{Im} N_{\Lambda \Sigma} \hat{F}^\Lambda \wedge \left[ L^{\Sigma} \zeta^a \gamma \gamma \zeta^\beta \epsilon_{a \beta} - \overline{L}^{\Sigma} \zeta^a \gamma \gamma \zeta^\beta \epsilon_{a \beta} \right]$$

(3.1)

where we have defined the following quantities:

$$\hat{F}^\Lambda \equiv F^\Lambda + m^{I A} B_I$$

(3.2)

$$A^I \equiv A^I_u d^u$$

(3.3)

$$\gamma \equiv \gamma^{\mu} d\gamma$$

(3.4)

$$\gamma^\beta \equiv \gamma^{\mu} d\gamma^\beta$$

(3.5)

$$\gamma^{\alpha \beta} \equiv (\ldots)^{\pm} \pm i^{\pm} (\ldots)$$

(3.6)

And the bilinear fermions $(\ldots)^{\pm}$ enjoy the following property:

$$(\ldots)^{\pm} \equiv \mp i^{\pm} (\ldots)$$

(3.7)

the Hodge dual operator $*$ being defined as:

$$*d\gamma^{\mu_1} \wedge \ldots \wedge d\gamma^{\mu_k} \equiv \frac{1}{(n-k)!} \sqrt{-g} e^{\mu_1 \ldots \mu_k \mu_{k+1} \ldots \mu_n} dx^{\mu_{k+1}} \wedge \ldots \wedge dx^{\mu_n}.$$
\[ h_{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} g_{uv} + \mathcal{M}_{IJK, A_{v}}^{I} A_{u}^{J} & \mathcal{M}_{IJK, A_{v}}^{K} \\ \mathcal{M}_{IJK, A_{v}}^{K} & \mathcal{M}_{IJK} \\ \end{pmatrix}. \] (3.10)

\( L^A \) and \( \mathcal{T}_i^A \) are the upper components of the symplectic sections of standard \( N = 2 \) supergravity, while \( \text{Im} \mathcal{N}_{\Lambda\Sigma} \) and \( \text{Re} \mathcal{N}_{\Lambda\Sigma} \) are the imaginary and real parts of the period matrix \( \mathcal{N}_{\Lambda\Sigma} \) of the special geometry; \( \mathcal{U}_{IA} \) and \( \Delta_I^{\alpha\beta} \) are the remnants of the quaternionic vielbein and symplectic connection in the dualized directions. \( (\zeta^\alpha) \) are the (anti)-chiral hyperinos, \( (\lambda^I_\bar{\alpha}) \) \( \lambda^A \) are the (anti)-chiral gauginos and \( (\psi^A) \) \( \psi_A \) are the (anti)-chiral gravitinos. Finally, \( \epsilon^{AB} \) is the totally antisymmetric tensor and \( \mathcal{C}^{\alpha\beta} \) is the charge conjugation matrix.

In order for the theory to be supersymmetric, as a consequence of the Ward identity of supersymmetry, the electric and magnetic charges must satisfy the constraint

\[ \epsilon^I_\Lambda m^{I\Lambda} - \epsilon^J_\Lambda m^{J\Lambda} = 0, \] (3.11)

which coincides with equation (2.3) in the case \( D = 4d \). Lagrangian (3.1) is invariant under the gauge transformations parametrized by a 1–form field \( \Omega_I \) already described in (2.2). As explained in the previous section, we can add to the Lagrangian a topological term which does not modify the equations of motion but provides a more compact expression for the terms of the type \( \mathcal{F} \wedge \mathcal{F} \).

The invariance of the action is now explicit, due to the fact that the fields \( B_I \) and \( \mathcal{F}^A \) appear only through either \( \hat{\mathcal{F}}^A \) or \( \mathcal{H}_I \), which are invariant under (2.2) by construction, since:

\[ \delta \hat{\mathcal{F}}^A = d \delta A^\Lambda + m^{I\Lambda} \delta B_I = - m^{I\Lambda} \Omega_I + m^{I\Lambda} \Omega_I = 0 \] (3.12)
\[ \delta \mathcal{H}_I = d \delta B_I = dd \Omega_I = 0 \] (3.13)

The effect of the gauge fixing of equation (2.6) is the replacement of \( \hat{\mathcal{F}}^A \) with \( m^{I\Lambda} B_I \).

Collecting all the fermionic bilinears, it is possible to rewrite the Lagrangian in a more compact expression:

\[ \mathcal{L}_{D=4}^{T,V} = - \frac{1}{4} \mathcal{M}^{IJK} \mathcal{H}_I \wedge ^* \mathcal{H}_J + \mathcal{H}_I \wedge A^I + \mathcal{H}_I \wedge S^I - \frac{1}{2} (m^2)^{IJ} B_I \wedge ^* B_J + \]
\[ -\frac{1}{2} (\tilde{e}m)^{IJ} B_I \wedge B_J + B_I \wedge T^I + \rho^I \wedge (\mathcal{H}_I - dB_I) \] (3.14)

where \( (m^2)^{IJ} \) is defined as in (2.8) with the matrix \( \text{Im} \mathcal{N} \) instead of \( \delta \). Notice that \( \text{Im} \mathcal{N} \) is negative definite, and hence the sign of the term proportional to \( (m^2)^{IJ} \) is different from that in (2.10). We have also introduced the following quantity:

\[ \tilde{\epsilon}^I_\Lambda = \epsilon^I_\Lambda - \text{Re} \mathcal{N}_{\Lambda\Sigma} m^{I\Sigma} \] (3.15)

and \( (\tilde{e}m)^{IJ} \) is defined as in (2.9) replacing \( \epsilon^I_\Lambda \) with \( \tilde{\epsilon}^I_\Lambda \). The \( T^I \) are the Pauli terms arising from the coupling of the fermions to \( \hat{\mathcal{F}} \), while \( S^I \) is the remnant of the quaternionic Pauli terms coupled to the differential \( dq^I \) of the axionic coordinates which have been dualized.
Notice that $(\tilde{e}m)^{IJ}$ is symmetric in the indices $I,J$ because of the tadpole cancellation condition and of the symmetry of the real part of the period matrix $\mathcal{N}_{\Lambda\Sigma}$.

From Lagrangian (3.14) it is easy to write down the equations of motion for the fields $\mathcal{H}_I$ and $B_I$:

$$
\frac{\delta \mathcal{L}}{\delta \mathcal{H}_I} = 0 \quad \Rightarrow \quad \frac{1}{2} \mathcal{M}^{IJ} * \mathcal{H}_J = S^I + A^I - \rho^I \quad (3.16)
$$

$$
\frac{\delta \mathcal{L}}{\delta B_I} = 0 \quad \Rightarrow \quad (m^2)^{IJ} * B_J + (\tilde{e}m)^{IJ} B_J = T^I - \delta \rho^I \quad (3.17)
$$

and their Hodge dual:

$$
\frac{1}{2} \mathcal{M}^{IJ} * \mathcal{H}_J = * (S^I + A^I - \rho^I) \quad (3.18)
$$

$$(m^2)^{IJ} * B_J + (\tilde{e}m)^{IJ} * B_J = * \delta \rho^I - * T^I \quad (3.19)
$$

equations (3.16) and (3.18) can be immediately inverted to obtain the expression for $\mathcal{H}_I$ (respectively $* \mathcal{H}_I$) in terms of $* \rho^I$ ($\rho^I$) while the usual linear combination of equations (3.17) and (3.19) gives the expression for $B_I$ ($B_I$) in terms of $\delta \rho^I$ and $* \delta \rho^I$:

$$
\mathcal{H}_I = 2 \mathcal{M}_{IJ} * (S^J + A^J - \rho^J) \quad (3.20)
$$

$$
* \mathcal{H}_I = 2 \mathcal{M}_{IJ} (S^J + A^J - \rho^J) \quad (3.21)
$$

$$
B_I = (\tilde{e}^2 + m^2)^{-1} \left[ * T^J - \left( \frac{\tilde{e}}{m} \right)_I^J T^K \right] - \left( \frac{\tilde{e}}{m} \right)_I^J * \delta \rho^K \quad (3.22)
$$

$$
* B_I = (\tilde{e}^2 + m^2)^{-1} \left[ \delta \rho^J + \left( \frac{\tilde{e}}{m} \right)_I^J * \delta \rho^K \right] - \left( \frac{\tilde{e}}{m} \right)_I^J * T^K \quad (3.23)
$$

Notice that there are again some sign differences with equations (2.13), (2.14) due to the new definition of $(m^2)^{IJ}$, $(\tilde{e}^2)^{IJ}$ and $(\frac{\tilde{e}}{m})^J_I$ with Im $\mathcal{N}$ instead of $\delta$. By substituting (3.22–3.23) in (3.14), one gets the dualized Lagrangian:

$$
\mathcal{L}_{Dual} = \mathcal{M}_{IJ} ({}^*A^I \wedge A^J + \mathcal{M}_{IJ} * \rho^I \wedge \rho^J - 2 \mathcal{M}_{IJ} * A^I \wedge \rho^J) + \\
+ 2 \mathcal{M}_{IJ} ({}^*A^I \wedge * \rho^J) \wedge S^J + \mathcal{M}_{IJ} * S^I \wedge S^J + \\
- \frac{1}{2} (\tilde{e}^2 + m^2)^{-1} d * \rho^I \wedge d \rho^J + \frac{1}{2} (\tilde{e}^2 + m^2)^{-1} \left( \frac{\tilde{e}}{m} \right)_I^K \left( \frac{\tilde{e}}{m} \right)_J^K d \rho^I \wedge d \rho^J + \\
+ (\tilde{e}^2 + m^2)^{-1} * T^I \wedge d \rho^J - (\tilde{e}^2 + m^2)^{-1} \left( \frac{\tilde{e}}{m} \right)_I^K T^K \wedge d \rho^J + \\
- \frac{1}{2} (\tilde{e}^2 + m^2)^{-1} * T^I \wedge T^J + \frac{1}{2} (\tilde{e}^2 + m^2)^{-1} \left( \frac{\tilde{e}}{m} \right)_I^K T^K \wedge T^J \quad (3.24)
$$

It should be noticed that, in the absence of Fayet-Iliopoulos terms ($e^I_{IA} = m^I_{IA} = 0$), the equations of motion for $\mathcal{H}_I$ would have given the standard $N = 2$ Lagrangian if $dq^I = - \rho^I$, 


where the minus sign is due to the opposite sign of the Lagrange multiplier term in the standard Lagrangian, when dualizing $\nabla q^I$ (see reference [5]) with respect to that in (3.14). This Lagrangian describes a system made up by $n$ massive vectors, coupled to scalars ($A^I$) and fermions ($T^I$). Together with all the undualized terms of $N = 2\ D = 4$ Lagrangian of [6], it describes a supergravity theory with massive vectors, which cannot be identified with a standard $N = 2\ D = 4$ supergravity, due both to the presence of a number of massive vectors, and to the fact that the vector- and hyper-multiplets are not written in the standard way.

In order to recover a more usual Lagrangian, we make use of the St"uckelberg mechanism, which allows one to rewrite the mass terms as kinetic terms for a number of scalars coupled to vectors, with a gauge invariance in order to keep the correct number of degrees of freedom fixed.

The St"uckelberg mechanism works as follows: at first it has to be noted that the kinetic term for the vectors is invariant under gauge transformations:

$$\delta \rho^I = d\Omega^I$$  \hspace{1cm} (3.25)

where $\Omega^I$ is a 0–form gauge parameter. On the contrary the mass terms, in which the bare field $\rho^I$ is present, are not invariant under (3.27).

In order to extend this invariance to the whole Lagrangian, we can think of the mass terms as being already gauge-fixed, as in equation (2.6), so that it is possible to recover the gauge invariance introducing a new 0–form field $\phi^I$ together with a massless vector $A^\Lambda$, whose gauge transformations leave $\rho^I$ invariant. Summarizing we can write $\rho^I$ as:

$$-\rho^I \equiv k^I_\Lambda A^\Lambda + d\phi^I$$  \hspace{1cm} (3.26)

where now all is invariant under the following gauge transformations:

$$\delta A^\Lambda = d\Omega^\Lambda$$ \hspace{1cm} (3.27)

$$\delta \phi^I = -k^I_\Lambda \Omega^\Lambda$$ \hspace{1cm} (3.28)

since the $k^I_\Lambda$ are constant. The r.h.s. of equation (3.26) can also be interpreted as the covariant derivative of the scalar field $\phi^I$, where the gauge fields are the vectors $A^\Lambda$, and $k^I_\Lambda$ have the role of Killing vectors.

After this redefinition, the Lagrangian reads:

$$\mathcal{L} = \mathcal{L}_{\text{scal}}^{\text{kin}} + \mathcal{L}_{\text{vect}}^{\text{kin}} + \mathcal{L}_{\text{scal}}^{\text{Pauli}} + \mathcal{L}_{\text{vect}}^{\text{Pauli}} + \mathcal{L}_{\text{inv}}^{A^f} + \mathcal{L}_{\text{non inv}}^{A^f}$$  \hspace{1cm} (3.29)

where:

$$\mathcal{L}_{\text{scal}}^{\text{kin}} = \mathcal{M}_{IJ} \left( *A^I \wedge A^J + *\nabla \phi^I \wedge \nabla \phi^J + 2* A^I \wedge \nabla \phi^J \right)$$  \hspace{1cm} (3.30)

$$\mathcal{L}_{\text{scal}}^{\text{Pauli}} = 2\mathcal{M}_{IJ} * A^I \wedge S^J + 2\mathcal{M}_{IJ} * \nabla \phi^I \wedge S^J$$  \hspace{1cm} (3.31)

$$\mathcal{L}_{\text{inv}}^{A^f} = \mathcal{M}_{IJ} S^I \wedge S^J$$  \hspace{1cm} (3.32)
\[ L_{\text{kin}} = -\frac{1}{2} (\tilde{e}^2 + m^2)^{-1}_{IJ} k^I_A k^J_S F^A \wedge * F^\Sigma + \frac{1}{2} (\tilde{e}^2 + m^2)^{-1}_{IK} \left( \frac{\tilde{e}}{m} \right)^K_J k^I_A k^J_S F^A \wedge F^\Sigma \] 

\[ L_{\text{Pauli}} = - (\tilde{e}^2 + m^2)^{-1} k^J_T \wedge * F^A + (\tilde{e}^2 + m^2)^{-1} \left( \frac{\tilde{e}}{m} \right)^K_J k^I_A T^I \wedge F^A \] 

(3.34)

\[ L_{\text{non inv}} = -\frac{1}{2} (\tilde{e}^2 + m^2)^{-1}_{IJ} T^I \wedge * T^J + \frac{1}{2} (\tilde{e}^2 + m^2)^{-1} \left( \frac{\tilde{e}}{m} \right)^K_J T^I \wedge T^J \] 

(3.35)

where as usual \( F^A \equiv dA^A \).

The kinetic scalar terms of equation (3.30), together with the \( g_{uv}^* dq^u \wedge dq^v \) term of [5, 6] give back the standard kinetic term for the hyperscalars, provided we identify:

\[ \nabla \phi^I \equiv \nabla q^I. \] 

(3.36)

The Pauli terms of equation (3.31) can be divided into two sets: those proportional to the “rectangular” vielbein \( U_{I\alpha} \), and those proportional to the connection \( \Delta^I_{\alpha \beta} \). The former join the analogue terms proportional to \( P_{u\alpha} \) in [5, 6], giving rise to the standard \( N = 2 \) Pauli terms coupling the hyperscalars to the hyperinos.

The latter is necessary in order to reconstruct the standard covariant derivative of the hyperinos. Indeed the kinetic term of the hyperinos in the Lagrangian of [6] has, as pointed out in [6] in the ungauged case, the following symplectic connection term:

\[ i\zeta^u dq^u \Delta_{u\alpha}^\beta \zeta_\beta + \text{c.c.} \] 

(3.37)

where \( \Delta_{u\alpha}^\beta = \hat{\Delta}_{u\alpha}^\beta - A^I_u \hat{\Delta}^I_{\alpha \beta} \), the hatted quantities being the quaternionic ones.

Adding to (3.37) the terms in (3.31) we get:

\[ dq^u \hat{\Delta}_{u\alpha}^\beta - dq^u A^I_u \hat{\Delta}^I_{\alpha \beta} + dq^u A^I_u \hat{\Delta}^I_{\alpha \beta} + \nabla q^I \hat{\Delta}_{I\alpha}^\beta = \nabla q^I \hat{\Delta}_{u\alpha}^\beta \] 

(3.38)

Where \( \hat{\Delta}_{u\alpha}^\beta \) is the standard \( N = 2 \) gauged supergravity \( \text{Sp}(2n_H) \) connection of the quaternionic manifold. Finally the 4-fermion terms of (3.32) are the same appearing as a result of the dualization of the scalars into tensors in the Lagrangian of reference [6], with the opposite sign: the total contribution of both terms is vanishing, that is they are only rearrangements of the bilinear fermions caused by the dualization procedure.

Let us now turn to the last three terms of (3.23): they have the expected form to be interpreted as standard \( N = 2 \) supergravity Lagrangian terms, except for their coupling matrices. The vector kinetic terms of (3.33) have the correct form, provided we make the following redefinition of the period matrix:

\[ (\tilde{e}^2 + m^2)^{-1}_{IJ} k^I_A k^J_S = (\text{Im} \mathcal{N}')_{A \Sigma} \] 

(3.39)

\[ (\tilde{e}^2 + m^2)^{-1}_{IK} \left( \frac{\tilde{e}}{m} \right)^K_J k^I_A k^J_S = (\text{Re} \mathcal{N}')_{A \Sigma} \] 

(3.40)

The Pauli terms in (3.34) can be treated collectively, indeed let us write \( T^I \) and its Hodge dual as:
\[ T^I = \text{Im} N_{\Delta \Sigma} m^I \left[ X^\Sigma_a \chi^a - \overline{X}^\Sigma_a \chi^+ \right] \]  
\[ \ast T^I = i \text{Im} N_{\Delta \Sigma} m^I \left[ X^\Sigma_a \chi^a - \overline{X}^\Sigma_a \chi^+ \right] \]  
(3.41)  

where \( \chi^a \) (\( \chi^a = (\chi^a)^* \)) is a fermion bilinear and \( X^\Lambda_a \) is the upper part of a symplectic vector:

\[ X^\Lambda_a \equiv \left\{ L^\Lambda_A, \overline{X}^\Lambda_a \equiv \left\{ f^\Lambda_i \right\} \right. \]  
(3.43)


gathering the common factors, they read:

\[ L_{\text{Pauli}}^{\text{vec}} = (\hat{e}^2 + m^2)^{-1} F^A_{\mu \nu} \left[ -i \text{Im} N_{\Delta \Pi} m^{J \Pi} \delta^K_J \left( \frac{\hat{e}^K}{m} \right) j \text{Im} N_{\Delta \Pi} m^{J \Pi} \right] X^\Delta_a \chi^a - \text{c.c.} \]

\[ = (\hat{e}^2 + m^2)^{-1} F^A_{\mu \nu} \left[ (\hat{e}^K - N_{\Delta \Sigma} m^{K \Sigma}) X^\Delta_a \chi^a - \text{c.c.} \right] \]

(3.44)

so that we are forced to introduce the new symplectic vectors:

\[ L^\Lambda_i = (k^{-1})^A_I \left( \epsilon^I_{\Pi} - N_{\Pi \Sigma} m^{I \Sigma} \right) L^\Pi \]

(3.45)

\[ \overline{f}^\Lambda_i = (k^{-1})^A_I \left( \epsilon^I_{\Pi} - N_{\Pi \Sigma} m^{I \Sigma} \right) \overline{f}^\Pi_i \]

(3.46)

in terms of which (3.34) reads:

\[ L_{\text{Pauli}}^{\text{vec}} = F^\Lambda_{\mu \nu} \left( \text{Im} N' \right)_{\Delta \Sigma} \left[ 4 L^I_{\Sigma} \left( \psi^A_{\mu} \psi^B_{\nu} \epsilon_{AB} \right) - 4 i f^I_{\gamma \delta} \left( \overline{\chi}^A_{\mu} \gamma^\mu \psi^B_{\nu} \epsilon_{AB} \right) \right] + \]

\[ + \frac{1}{2} \nabla_i f^I_{\gamma \delta} \overline{X}^A_{\gamma \mu \nu} \chi^B_{\epsilon} \epsilon_{AB} - L^I_{\Sigma} \zeta_{\gamma \mu \nu} \chi^A_{\rho} \epsilon^{\rho \beta} + \text{c.c.} \]

(3.47)

Note that \( L' \) and \( \overline{f} \) can be shown to be covariantly holomorphic as following from the Special Geometry, so that the redefinitions (3.45), (3.46) are consistent.

Let us now turn to the 4–fermion terms; it should be noticed that, together with the terms of (3.35), there are the non-invariant terms already present in [10] in order to have a supersymmetric theory; these terms can collectively be written (see [11]):

\[ \mathcal{L}^{4f} = \frac{i}{4} \text{Im} N_{\Delta \Sigma} X^\Lambda_a \chi^A_b \chi^a \chi^b - \text{c.c.} \]  
(3.48)

while those coming from dualization turn out to be:

\[ \mathcal{L}_{\text{non-inv}}^{4f} = - \frac{1}{2} \left( \hat{e}^2 + m^2 \right)^{-1} (i \delta^K_J \text{Im} N_{\Delta \Pi} m^{J \Pi} \Delta - \left( \frac{\hat{e}^K}{m} \right) J \text{Im} N_{\Delta \Pi} m^{J \Pi}) \text{Im} N_{\Delta \Sigma} m^{I \Lambda} \cdot \left[ X^\Sigma_a \chi^a + \overline{X}^\Sigma_a \chi^a \right] \left[ X^\Pi_b \chi^b + \overline{X}^\Pi_b \chi^b \right] \]

(3.49)
summing (3.48) and (3.49), after some algebraic manipulation we get the following expression:

\[
\mathcal{L}^{4f} = \frac{i}{4} \text{Im} \mathcal{N}'_{\Lambda \Sigma} X^\Lambda_a X^\Sigma_b \chi^a \chi^b + \text{c.c.}
\]  

(3.50)

which is the expected 4-fermions Lagrangian in terms of the new symplectic sections and period matrix.

4. Symplectic rotation

The final expression for the dual Lagrangian is therefore that of a standard \(N = 2\) gauged supergravity [10], provided the formulae (3.39–3.40) and (3.45–3.46) define respectively a period matrix and the upper components of a symplectic section.

Let us consider the transformation which, acting on the charges vector, rotates it into a vector of purely electric charges, that is:

\[
\begin{pmatrix} m^I \Lambda \\ e^I \Lambda \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ k^I \Lambda \end{pmatrix}.
\]

(4.1)

This transformation is performed by means of the matrix \(S\):

\[
S = \begin{pmatrix} \left( k^{-1}\right)_I^\Sigma e^I \Lambda - \left( k^{-1}\right)_I^\Sigma m^I \Lambda \\ 0 \end{pmatrix} \frac{1}{k^I \Sigma (e^{-1})^I \Lambda}.
\]

(4.2)

where we have supposed that the matrix \(k^I \Lambda\) is invertible, as well as \(m^I \Lambda\) and \(e^I \Lambda\). The effect of \(S\) on the period matrix turns out to be:

\[
\mathcal{N}'_{\Lambda \Sigma} = k^I \Lambda \left( e^{-1}\right)^I \Sigma \mathcal{N}_{\Pi \Delta} \left[ \left( k^{-1}\right)_I^\Sigma e^I \Lambda - \left( k^{-1}\right)_I^\Sigma m^I \Lambda \mathcal{N}_{\Pi \Delta} \right]^{-1}
\]

(4.3)

According to equation (4.3) the real and the imaginary part of the rotated period matrix are:

\[
\begin{align*}
(\text{Im} \mathcal{N}')_{\Lambda \Sigma} & = (e^2 + m^2)^{-1}_I k^I \Lambda k^I \Sigma \\
(\text{Re} \mathcal{N}')_{\Lambda \Sigma} & = (e^2 + m^2)^{-1}_K k^I \Lambda k^I \Sigma
\end{align*}
\]

(4.4)

(4.5)

This same transformation will act also on the symplectic sections according to:

\[
\begin{pmatrix} L^\Lambda \\ M'^\Lambda \end{pmatrix} \equiv \begin{pmatrix} \left( k^{-1}\right)_I^\Sigma e^I \Lambda - \left( k^{-1}\right)_I^\Sigma m^I \Lambda \\ 0 \end{pmatrix} \frac{1}{k^I \Sigma (e^{-1})^I \Lambda} \left( \begin{pmatrix} L^\Lambda \\ M_\Lambda \end{pmatrix} \right) = \begin{pmatrix} \left( k^{-1}\right)_I^\Lambda \left( e^I \Lambda - \mathcal{N}_{\Lambda \Sigma} m^I \Sigma \right) L^\Lambda \\ k^I \Lambda k^I \Sigma \left( e^{-1}\right)^I \Lambda M_\Lambda \end{pmatrix}
\]

(4.6)

and similarly for the symplectic section \(\frac{L^I \Lambda}{k'}_{\Lambda \Sigma} F^\Lambda F^\Sigma\). Therefore, modulo a total derivative term \((e^{-1})^I \Lambda (m^{-1})_I \Sigma F^\Lambda F^\Sigma\) which does not modify the equations of motion, the transformation \(S\) acting on the symplectic sections and period matrix reproduces the expressions of equations (3.39–3.40) and (3.45–3.46); this implies that the effect of the introduction
of the Fayet-Iliopoulos terms $m^I$ and $e^I_A$ in the Lagrangian coupled to tensor multiplets corresponds, after the dualizations of the massive tensors to massive vectors, and the subsequent reinterpretation of these latter as massless vectors and scalars via the St"uckelberg mechanism, to a purely electric gauging of the undualized standard Lagrangian, together with a rotation of all the symplectic sections and the period matrix with the matrix $S$.

5. Conclusions

In this paper we have shown that, in an even number $D$ of dimensions, an arbitrary number $n$ of $D/2$ massive tensor fields, with masses given by both electric and magnetic couplings, are dual to $n$ massive $(D/2 - 1)$ tensors, whose masses are only of the electric type and are given by relation (1.2). The formalism developed to show this duality has been applied to the Lagrangian of the $N = 2$ $D = 4$ supergravity coupled to $n$ tensor and $n$ vector multiplets, showing that it is dual to a standard $N = 2$ $D = 4$ gauged supergravity where only electric charges are present.

The Lagrangian obtained in section 3 is related to the standard formulation of $N = 2$ $D = 4$ gauged supergravity $\Box$ by a symplectic rotation, defined in (4.2) acting on all the Special Geometry quantities. Since however a symplectic rotation is not a symmetry of the theory, unless in the generical symplectic matrix $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ the blocks $B$ and $C$ are set to 0 $\Box$, the two theories related by the symplectic rotation (4.2) are not the same theory, as is evident by looking at their $N = 1$ truncations which give rise to different theories $\Box$.

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