The behaviour of charged spinning test particles moving along circular orbits in the equatorial plane of the Reissner-Nordström spacetime is studied in the framework of the Dixon-Souriau model completed with standard choices of supplementary conditions. The gravitomagnetic “clock effect”, i.e. the delay in the arrival times of two oppositely circulating particles as measured by a static observer, is derived and discussed in the cases in which the particles have equal/opposite charge and spin, the latter being directed along the $z$-axis.

Keywords: Spinning particles, Dixon-Souriau model.

1. Introduction

The Dixon-Souriau\textsuperscript{[1234]} equations of motion for a charged spinning test particle in a given gravitational and electromagnetic background are

\begin{align}
\frac{DP^\mu}{d\tau_U} &= -\frac{1}{2} R^\mu_{\nu\alpha\beta} U^\nu S^{\alpha\beta} + q F^\mu_{\nu\rho} U^\nu - \lambda S^{\rho\sigma} \nabla^\mu F_{\rho\sigma} \equiv F^{(\text{tot})\mu}, \\
\frac{DS^{\mu\nu}}{d\tau_U} &= P^{\mu} U^\nu - P^{\nu} U^\mu + \lambda [S^\rho_{\mu\nu} F_{\rho}^\nu - S^\rho_{\mu\rho} F_{\nu}^\rho] ,
\end{align}

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where $F^{\mu\nu}$ is the electromagnetic field, $P^\mu$ is the total 4-momentum of the particle, and $S^{\mu\nu}$ is the spin tensor (antisymmetric); $U$ is the timelike unit tangent vector of the “center of mass line” used to make the multipole reduction. As it has been shown by Souriau, the quantity $\lambda$ is an arbitrary electromagnetic coupling scalar constant. We note that the special choice $\lambda = -q/m$ (see Appendix A and 5) in flat spacetime, corresponds to the Bargman-Michel-Telegdi 6 spin precession law. Therefore, we will discuss the results of our analysis in this case too.

The test character of the particle under consideration refers to its mass and charge as well as to its spin, since all these quantities should not be large enough to affect the background metric. In what follows, with the magnitude of the spin of the particle, with the mass and with a natural lengthscale associated to the gravitational background we will construct a non-dimensional parameter as a smallness indicator, which we retain to the first order only so that the test character of the particle be fully satisfied.

In order to have a closed set of equations, Eqs. (1) and (2) must be completed with supplementary conditions (SC) whose standard choices in the literature are the

1. Corinaldesi-Papapetrou 7 conditions (CP): $S^{t\nu} = 0$,
2. Pirani 8 conditions (P): $S^{\mu\nu}U_\nu = 0$,
3. Tulczyjew 9 conditions (T): $S^{\mu\nu}P_\nu = 0$.

Finally, it is worth to stress that there is no agreement in the literature on whether this model can be equally valid for both macroscopic bodies and elementary particles. This is a long debated question and we will not enter this discussion. In fact, our main motivation here is to investigate how the presence of an electromagnetic structure both in the background and in the spinning particle affects previous results obtained for neutral spinning particles in the field of uncharged black holes 10 11.

2. Spinning particles in Reissner-Nordström spacetime

Let us consider the background of a static black hole of mass $M$ and charge $Q$, described by the Reissner-Nordström line element in standard spherical coordinates:

$$ds^2 = -\frac{\Delta}{r^2} dt^2 + \frac{r^2}{\Delta} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$  \hspace{1cm} (3)

where $\Delta = r^2 - 2Mr + Q^2$; the horizon radii are given by $r_\pm = M \pm \sqrt{M^2 - Q^2}$. The associated electromagnetic potential and field are

$$A = \frac{Q}{r} dt, \quad F = dA = -\frac{Q}{r^2} dt \wedge dr.$$  \hspace{1cm} (4)

Let us introduce an orthonormal frame adapted to the static observers

$$e_t = r\Delta^{-1/2} \partial_t, \quad e_r = \frac{\Delta^{1/2}}{r} \partial_r, \quad e_\theta = \frac{1}{r} \partial_\theta, \quad e_\phi = \frac{1}{r \sin \theta} \partial_\phi.$$  \hspace{1cm} (5)
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with dual

\[
\omega^t = \frac{\Delta^{1/2}}{r} \, dt, \quad \omega^r = r \Delta^{-1/2} dr, \quad \omega^\theta = r d\theta, \quad \omega^\phi = r \sin \theta d\phi, \tag{6}
\]

and let us assume that \( U \) is tangent to a (timelike) spatially circular orbit, with

\[
U = \Gamma[\partial_t + \zeta \partial_\phi] = \gamma[e^t + \nu e^\phi], \tag{7}
\]

where \( \zeta \) is the angular velocity with respect to infinity and \( \Gamma \) is a normalization factor

\[
\Gamma = \left(-g_{tt} - \zeta^2 g_{\phi\phi}\right)^{-1/2} \tag{8}
\]

which assures that \( U \cdot U = -1 \); here dot means scalar product with respect to the metric \( g \). The angular velocity \( \zeta \) is related to the local proper linear velocity \( \nu \) measured in the frame \( \{ \} \) by

\[
\zeta = \sqrt{-g_{tt} g_{\phi\phi}} \nu, \tag{9}
\]

so that \( \Gamma = \gamma / \sqrt{-g_{tt}} \). Here \( \zeta \) and therefore also \( \nu \) are assumed to be constant along the \( U \)-orbit. We limit our analysis to the equatorial plane \( (\theta = \pi/2) \) of the Reissner-Nordström solution; as a convention, the physical (orthonormal) component along \(-\partial_\theta\), perpendicular to the equatorial plane will be referred to as along the positive \( z \)-axis and will be indicated by \( \hat{z} \), when necessary (and so \( e^z = -e^\theta \)).

In the case of spinless and neutral particles \( (q = 0) \), particular attention is devoted to the timelike circular geodesics \( U_\pm \), such that \( \nabla_{U_\pm} U_\pm = 0 \), co-rotating \( (\zeta_+) \) and counter-rotating \( (\zeta_-) \) -with respect to the assumed positive (counterclockwise) variation of the \( \phi \)-angle- respectively. It results

\[
\zeta_\pm \equiv \pm \gamma g = \pm \frac{(Mr - Q^2)^{1/2}}{r^2}, \tag{10}
\]

so that

\[
U_\pm = \gamma_g [e^t \pm \nu_g e^\phi], \quad \nu_g = \left[\frac{Mr - Q^2}{\Delta}\right]^{1/2}, \quad \gamma_g = \left[\frac{\Delta}{r^2 - 3Mr + 2Q^2}\right]^{1/2}, \tag{11}
\]

with the timelike condition \( \nu_g < 1 \) satisfied if \( r > r_g^* = [3M + (9M^2 - 8Q^2)^{1/2}] / 2 \). At \( r = r_g^* \) one finds instead \( \nu_g = 1 \); clearly \( r_g^* \) marks the photon orbit in the Reissner-Nordström spacetime.

It is convenient to introduce the Lie relative curvature \( \kappa_{(lie)} \) of each orbit

\[
k_{(lie)} = -\partial_\phi \ln \sqrt{g_{\phi\phi}} = -\frac{\Delta^{1/2}}{r^2} = -\frac{\zeta_g}{\nu_g}, \tag{12}
\]

as well as a Frenet-Serret (FS) intrinsic frame along \( U \), defined by

\[
E^t = U, \quad E^r = e^r, \quad E^\theta = e^\theta, \quad E^\phi = \gamma [\nu e^t + e^\phi], \tag{13}
\]
satisfying the following system of evolution equations
\[
\frac{DU}{d\tau} = a(U) = \kappa E_r,
\frac{DE_r}{d\tau} = \kappa U + \tau_1 E_\phi,
\frac{DE_\phi}{d\tau} = -\tau_1 E_r,
\frac{DE_\zm}{d\tau} = 0,
\]
(14)
where
\[
\kappa = k_{(\text{lie})} \gamma^2 [\nu^2 - \nu_g^2],
\tau_1 = -\frac{1}{2\gamma^2} \frac{dk}{d\nu} = -k_{(\text{lie})} \frac{\gamma^2}{\gamma_g} \nu;
\]
(15)
in this case the second torsion \(\tau_2\) is identically zero. The dual of (13) is given by
\[
\hat{\Omega}_t = -U^\flat,
\hat{\Omega}_r = \omega^r,
\hat{\Omega}_\zm = \omega^\zm,
\hat{\Omega}_\phi = \gamma [ -\nu \omega^t + \omega^\phi ],
\]
(16)
where the symbol \(\flat\) denotes the completely covariant form of a generic tensor.

In the spinless case, Eqs. (1) and (2) reduce to the well known equations of motion for a charged particle in an external electromagnetic field:
\[
ma(U)^\mu = qF^\mu\nu U^\nu,
\]
(17)
where \(a(U) = \nabla_U U\) is the particle’s 4-acceleration. Since only the radial component survives, Eq. (17) writes explicitly as
\[
0 = m\gamma [\nu^2 - \nu_g^2] + \frac{\nu_g qQ}{\zeta_g r^2}.
\]
(18)
This equation gives the values of the linear velocity \(\nu = \pm \nu_0^\pm\) which are compatible to a given \(qQ\) on a circular orbit with radius \(r\):
\[
\nu_0^\pm = \left\{ \nu_g^2 \left[ 1 - \frac{1}{2\zeta_g} \frac{q^2 Q^2}{r^4} \right] \pm \frac{\nu_g qQ}{\zeta_g r^2} \left[ \frac{1}{4} \nu_g^2 q^2 Q^2 r^4 + \frac{1}{\gamma_g^2} \right]^{1/2} \right\}^{1/2},
\]
(19)
and so \(\nu_0^\pm = (1 - \nu_0^\pm)^{-1/2}\), where the parameter \(\tilde{q} = q/m\) has been introduced. The selection of the \(\pm\) sign inside the square root in Eq. (19) should be done properly. To clarify this point, let us introduce the limiting value of the parameter \(\tilde{q}\) corresponding to a particle at rest (i.e. \(\nu = 0\) in Eq. (19))
\[
\tilde{q}_\text{lim} = \nu_g \zeta_g r^2 Q = \frac{Mr - Q^2}{Q\sqrt{\Delta}}.
\]
(20)
By introducing this quantity into Eq. (19) one gets the equivalent relation
\[
\frac{\tilde{q}}{\tilde{q}_\text{lim}} = \gamma \left( 1 - \frac{\nu^2}{\nu_g^2} \right) \equiv f_{\nu_g}(\nu),
\]
(21)
whose solution (19) can be conveniently rewritten as
\[
\nu_0^\pm = \nu_g \left[ \Lambda \pm \frac{\tilde{q}}{\tilde{q}_\text{lim}} (\Lambda^2 - \nu_g^2 \Xi) \right]^{1/2},
\]
(22)
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where

\[
\Lambda = 1 - \frac{\nu_g^2}{2} \left( \frac{\tilde{q}}{\tilde{q}_{\text{lim}}} \right)^2, \quad \Xi = 1 - \left( \frac{\tilde{q}}{\tilde{q}_{\text{lim}}} \right)^2.
\] (23)

It is evident that for all the radii \( r > r^*_g \), i.e. in the region where \( \nu_g < 1 \), the function \( f_{\nu_g}(\nu) \) has a local maximum at \( \nu = 0 \) (where \( \tilde{q}/\tilde{q}_{\text{lim}} = 1 \)); moreover, \( f_{\nu_g}(\pm \nu_g) = 0 \). In this case (\( r > r^*_g \)) the solutions of Eq. (18) correspond to \( \nu = \pm \nu_g^0 \). Differently, for all the radii \( r_+ < r < r^*_g \), i.e. in the region where \( \nu_g > 1 \), the function \( f_{\nu_g}(\nu) \) has a local minimum at \( \nu = 0 \), and there are no values for \( \nu \in (-1, 1) \) such that \( f_{\nu_g}(\nu) = 0 \). In this case (\( r_+ < r < r^*_g \)) the solutions of Eq. (18) correspond to \( \nu = \pm \nu_g^+ \). The special cases \( r = r^*_g (\nu_g = 1) \) and \( r = r_+ (\nu_g \to \infty) \) correspond to \( f_1(\nu) = 1/\gamma \) and so \( \nu = \pm [1 - (\tilde{q}/\tilde{q}_{\text{lim}})^2]^{1/2} \), and \( f_\infty(\nu) = \gamma \) and so \( \nu = \pm [1 - (\tilde{q}/\tilde{q}_{\text{lim}})^{-2}]^{1/2} \), respectively. The situation is summarized in Fig. 1, where the ratio \( \tilde{q}/\tilde{q}_{\text{lim}} \) is plotted as a function of the linear velocity \( \nu \) for fixed values of the background parameters. Drawing horizontal lines (i.e. \( \tilde{q}/\tilde{q}_{\text{lim}} = \) fixed lines) in the figure allows to visualize graphically the solutions \( \nu = \pm \nu_0^\pm \) in the various cases described above.

In the following we shall use the simplified notation \( \nu_0 \) for the quantity \( \nu_0^\pm \) defined in Eq. (19) (or, equivalently, Eq. (22)), with the prescription to select the \( \pm \) sign according to the above discussion.

Fig. 1. The ratio \( \tilde{q}/\tilde{q}_{\text{lim}} \) is plotted as a function of \( \nu \) for the choice of the background parameter \( Q/M = .6 \), so that \( r_g^*/M \approx 2.737 \). The figure shows three different behaviours: the lower curve corresponds to \( r/M = 8 \) (and so \( r > r_+^* \)), the upper curve to \( r/M = 2.1 \) (and so \( r < r_+^* \)), while the intermediate curve corresponds to \( r = r_+^* \). The boundary of the dashed region corresponds to the outher horizon value \( r^*/M = 1.8 \). The corresponding solutions \( \nu = \pm \nu_0^\pm \) are given by the intersections of each curve with the horizontal line obtained by fixing a particular value of the parameter \( \tilde{q}/\tilde{q}_{\text{lim}} \). As an example, the dashed horizontal line allows to visualize the solutions \( \nu = \pm \nu_0^- \) corresponding to the choice \( \tilde{q}/\tilde{q}_{\text{lim}} = -1 \).
Let us now turn to spinning particles. To study the motion of a charged and spinning test particle on circular orbits let us consider first the evolution equation of the spin tensor (2). By contracting both sides of Eq. (2) with $U_\nu$, one obtains the following expression for the total 4-momentum

$$P_\mu = -(U \cdot P)U^\mu - U_\nu \frac{DS_\mu \nu}{d\tau_U} + \lambda [S^{\mu \nu} F_\rho \nu - S^{\nu \rho} F_\mu \nu]U_\nu \equiv mU^\mu + P_s^\mu, \quad (24)$$

where $m = -U \cdot P$ reduces to the ordinary mass in the case in which the particle is not spinning, and $P_s$ is a 4-vector orthogonal to $U$. The easiest way to satisfy the force equation (1) is to look for solutions for which the mass $m$ and the frame components of the spin tensor are all constant along the orbit.

From these assumptions and Eq. (24), Eq. (2) implies

$$S_{\hat{t} \hat{t}} = 0, \quad S_{\hat{r} \hat{t}} = 0, \quad \left[ \frac{\nu_2}{\zeta_2 g} + \frac{\nu_2}{\zeta_2 g} \lambda \frac{Q}{r^2} \right] S_{\hat{t} \hat{t}} + S_{\hat{t} \hat{t}} \nu = 0. \quad (25)$$

From Eqs. (13)–(16) it follows that

$$\frac{DS}{d\tau_U} = m_s [\Omega \wedge U], \quad (26)$$

where

$$m_s \equiv ||P_s|| = \gamma \frac{\zeta_2}{\nu_2} \left[ -\nu_2 S_{\hat{r} \hat{t}} + \nu S_{\hat{t} \hat{r}} \right] - \frac{\lambda}{r^2} \frac{Q}{\nu_2} S_{\hat{r} \hat{r}}, \quad (27)$$

hence $P_s$ can be written as

$$P_s = m_s \Omega \hat{\phi}. \quad (28)$$

From Eqs. (24) and (28) and provided $m + \nu m_s \neq 0$, the total 4-momentum $P$ can be written in the form $P = \mu U_p$, with

$$U_p = \gamma_p [\epsilon_\hat{t} + \nu_\epsilon \epsilon_\hat{r}], \quad \nu_p = \frac{\nu + m_s / m}{1 + \nu m_s / m}, \quad \mu = \frac{\gamma}{\gamma_p} (m + \nu m_s), \quad (29)$$

where $\gamma_p = (1 - \nu_\epsilon^2)^{-1/2}$; $U_p$ is a timelike unit vector and $\mu$ has the property of a physical mass.

Let us now consider the equation of motion (1). The total force acting on the particle is equal to:

$$F^{(\text{tot})} = \gamma \left\{ \zeta_2 \nu S_{\hat{r} \hat{t}} + \frac{1}{r^2} \left[ \frac{2Mr - 3Q^2}{r^2} + \frac{\zeta_2 Q}{\nu_2 r^2} \right] S_{\hat{t} \hat{r}} + \frac{qQ}{r^2} \right\} e_{\hat{r}} - \frac{\gamma \nu}{r^2} S_{\hat{t} \hat{t}} \epsilon_{\hat{r}}, \quad (30)$$

while the term on the left hand side of Eq. (1) can be written, from Eqs. (24) and (28), as

$$\frac{DP}{d\tau_U} = ma(U) + m_s \frac{DE_{\hat{t}}}{d\tau_U}, \quad (31)$$

where $a(U)$ and $DE_{\hat{t}} / d\tau_U$ are given in Eq. (14), and the quantities $\mu, m, m_s$ are all constant along the world line of $U$. The term $a(U)$ is the acceleration of the center
of mass line $U$, while the term $DE_\phi^U/d\tau_U$ represents the first torsion of $U$, so that $m_sDE_\phi^U/d\tau_U$ is a spin-orbit coupling force.

Since $DP/d\tau_U$ is directed radially as from Eqs. (14) and (31), Eq. (1) requires that $S_{\theta\phi}^U = 0$ (and therefore also $S_{t\theta}^U = 0$ from Eq. (25)); hence Eq. (1) can be written as

$$m\kappa - m_s\tau_1 - F_{i}^{(tot)} = 0,$$

or, more explicitly,

$$0 = m\gamma[\nu^2 - \nu_g^2] + m_s\frac{\gamma^\nu}{\gamma_g} + \frac{\nu_g}{\zeta_g}[Q S_{t}^U + \frac{1}{r^2} \frac{2Mr - 3Q^2}{r^2} + \lambda \zeta_g Q] S_{t}\hat{r} + \frac{Q}{r^2}.$$

This equation establishes the relations among the particle’s spin, the linear velocity $\nu$ and the charge $q$ for a (timelike) spatially equatorial circular orbit to exist as such.

Summarizing, from the equations of motions (1) and (2) and before imposing supplementary conditions, the spin tensor turns out to be completely determined by two components only, namely $S_{t}\hat{r}$ and $S_{r}\hat{\phi}$, related by Eq. (33):

$$S = \omega^\nu \wedge [S_{t}\omega^t + S_{r}\omega^\phi].$$

It is useful to introduce together with the quadratic invariant

$$s^2 = \frac{1}{2} S_{\mu\nu}S^{\mu\nu} = -S_{tt}^2 + S_{r\phi}^2,$$

another frame adapted to $U_p$ given by

$$E_p^0 = U_p, \quad E_p^1 = e_r, \quad E_p^2 = \gamma_p(\nu_p e_i + e_\phi), \quad E_p^3 = e_z,$$

whose dual frame is denoted by $\Omega^a$. To discuss the features of the motion we need to supplement Eq. (33) with further conditions. We shall do this in the next section following the standard approaches existing in the literature.

2.1. The Corinaldesi-Papapetrou (CP) supplementary conditions

The CP supplementary conditions require $S_{t}\hat{r} = 0$, so that

$$S = s \omega^\nu \wedge \omega^\phi,$$

and Eq. (33) gives the spin needed to have a circular orbit with given $\nu$ and $q$:

$$\hat{s} = \frac{1}{M\gamma\nu} \left[ \gamma(\nu^2 - \nu_g^2) + \frac{\nu_g}{\gamma_g} Q^2 \right] \left[ \gamma\nu_g \zeta_g (\nu^2 - \nu_g^2) + \frac{\lambda}{\gamma_g} Q \right]^{-1};$$

here $\hat{s} = \pm |s| = \pm |s|/(mM)$ denotes the non-dimensional signed magnitude of the spin per unit (bare) mass $m$ of the test particle and unit mass $M$ of the black hole.
The behaviour of the spin parameter $\hat{s}$ as a function of $\nu$ is shown in Fig. 2 for the choice $\lambda = -\tilde{q}$ of the electromagnetic coupling scalar (we assume this form for $\lambda$ also in the figures Figs. 3 and 4), and fixed values of the parameters $\tilde{q}, Q/M$ and $r/M$. As expected, Eq. (38) shows that $\hat{s}$ vanishes for $\nu = \pm \nu_0$, with $\nu_0$ given by Eq. (19), whenever they exist.

Solving Eq. (38) for $\nu$ in the limit of small spin, namely if $\hat{s} \ll 1$, we have to first order in $\hat{s}$

$$\nu \simeq \pm \nu_0 + \mathcal{N}'^{(CP)} \hat{s}, \quad \mathcal{N}'^{(CP)} = \frac{MQ}{r^2} \left[ 1 + \frac{\gamma_2^2}{\gamma_g^2} \right]^{-1} \left[ \lambda - \nu_g^2 (\lambda + \tilde{q}) \right].$$

The corresponding angular velocity $\zeta$ and its reciprocal are

$$\zeta \simeq \pm \zeta_0 + \frac{\zeta_0 \nu_0}{\nu} \mathcal{N}'^{(CP)} \hat{s}, \quad \frac{1}{\zeta} \simeq \pm \frac{1}{\zeta_0} - \frac{\nu_0}{\zeta_0} \mathcal{N}'^{(CP)} \hat{s},$$

where $\zeta_0 = \nu_0 \gamma_g/\nu_g$. It is worth to note that the choice $\lambda = -\tilde{q}$ simplifies the coefficient $\mathcal{N}'^{(CP)}$ in Eq. (39) as

$$\mathcal{N}'^{(CP)} = -\frac{MQ}{r^2} \left[ 1 + \frac{\gamma_2^2}{\gamma_g^2} \right]^{-1} \tilde{q}.$$

The total 4-momentum $P$ is given by (29) with

$$m_\nu = -M \hat{s} \left[ \gamma_0 \nu_0 \gamma_g + \lambda Q/M \right];$$

and reduces to

$$\nu_p \simeq \nu - \frac{M}{\gamma_0^2} \left[ \lambda Q/M + \gamma_0 \nu_0 \gamma_g \right] \hat{s},$$

if $\lambda = -\tilde{q}$.

### 2.2. The Pirani (P) supplementary conditions

The P supplementary conditions require $S_{\dot{r}t} + S_{\dot{r}\phi} = 0$ ($S^{\mu\nu} U_\nu = 0$) or

$$S = s \omega^i \wedge \Omega^\phi, \quad \Omega^\phi = \gamma [\nu \omega^i + \omega^\phi],$$

so that $(S_{\dot{r}t}, S_{\dot{r}\phi}) = (-s \gamma \nu, s \gamma)$ and Eq. (33) gives

$$\dot{s} = \frac{1}{M \nu} \left[ \gamma (\nu^2 - \nu_g^2) + \frac{\nu_0}{\gamma_0} \frac{\tilde{q} Q}{r^2} \right] \left( \frac{\gamma_0}{\gamma_g} \left( \frac{\gamma_0^2}{\gamma_g^2} (\nu^2 - \nu_g^2) + \nu_g^2 \right) \right)
+ \left( \frac{\nu_0}{\gamma_0} \right) \frac{2 M r - 3 Q^2}{\nu_0^2 r^4} - \lambda \frac{Q}{\nu_0} \left( \frac{\gamma_0^2}{\gamma_g^2} - 2 \right)^{-1},$$

(46)
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Fig. 2. In the case of CP supplementary conditions, and for the choice $\lambda = -\tilde{q}$ of the electromagnetic coupling scalar, the spin parameter $\hat{s}$ is plotted as a function of the linear velocity $\nu$, for $Q/M = .6$, $\tilde{q} = .1, 1, 2, 20$ and $r/M = 8$, from (a) to (d) respectively. With this choice of the parameter values, from Eq. (20) we deduce that the critical value for the charge to mass ratio of the particle is $\tilde{q}_{\text{lim}} \approx 1.831$ and $r_{g\ast}/M \approx 2.737$; as a consequence, the spin $\hat{s}$ vanishes for $\nu = \pm \nu_0$ in cases (a) and (b) only, with $\nu_0 \approx 0.387$ and $\nu_0 \approx 0.274$ respectively. In the cases (c) and (d) the solution $\nu_0$ does not exist. In addition, from Eq. (38) we have that $\hat{s}$ diverges for (a) $\nu \approx 0.446$, (b) $\nu \approx 0.698$, (c) $\nu \approx 0.824$ and (d) $\nu \approx 0.996$. However, in these plots as well as in the next Figs. 3 and 4 large values of $\hat{s}$ have not a direct physical interpretation in the framework of the Dixon-Souriau model: in such a case, in fact, the spinning particle loses its test character.
Fig. 3. In the case of Pirani’s supplementary conditions, and for the choice $\lambda = -\tilde{q}$ of the electromagnetic coupling scalar, the spin parameter $\hat{s}$ is plotted as a function of the linear velocity $\nu$, for $Q/M = .6$, $\tilde{q} = .1, 1.5, 2, 20$ and $r/M = 8$, from (a) to (d) respectively. As in the CP case, since the critical value for the charge to mass ratio of the particle is $\tilde{q}_{\text{lim}} \approx 1.831$ and $r^*_g/M \approx 2.737$ for this choice of the parameter values, the spin $\hat{s}$ vanishes for $\nu = \pm \nu_0$ in cases (a) and (b) only, with $\nu_0 \approx 0.387$ and $\nu_0 \approx 0.175$ respectively. Moreover, from Eq. (46) we have that $\hat{s}$ diverges in case (d) only, at $\nu \approx 0.631$. Note that the plots (c) and (d) have a physical meaning only for $\nu \approx \pm 1$, where the spin parameter $\hat{s}$ is small enough.
where, as before, the spin per unit mass has been introduced. The behaviour of the spin parameter $\hat{s}$ as a function of $\nu$ is shown in Fig. 3, with $\lambda = -\tilde{q}$ and fixed values of the parameters $\tilde{q}$, $Q/M$ and $r/M$.

In the limit of small $\hat{s}$, Eq. (46) gives

$$\nu \simeq \pm \nu_0 + \mathcal{N}^{(P)} \hat{s},$$

$$\mathcal{N}^{(P)} = -M \left[ 1 + \frac{\gamma_0^2}{\gamma_0^2} \left\{ \left[ 2Q \left( \lambda - \tilde{q} \right) + \frac{2z_{\nu_0} Q}{\tilde{q}} \right] r^2 + \frac{3Mr - 4Q^2}{r^4} \right\} \right], \quad (47)$$

to first order in $\hat{s}$. The corresponding angular velocity $\zeta$ and its reciprocal are

$$\zeta \simeq \pm \zeta_0 + \frac{\zeta_0}{\nu_0} \mathcal{N}^{(P)} \hat{s}, \quad \frac{1}{\zeta} \simeq \pm \frac{1}{\zeta_0} - \frac{\zeta_0}{\nu_0} \mathcal{N}^{(P)} \nu_0 \hat{s}. \quad (48)$$

The total 4-momentum $P$ is given by (29) with

$$m_\nu = M \hat{s} \gamma \left[ \frac{\zeta_0}{\nu_0} \gamma (\nu - \nu_0) - \lambda \frac{Q}{r^2} \right]; \quad (49)$$

to first order in $\hat{s}$,

$$\nu_p \simeq \nu - \frac{M Q}{\gamma_0 \rho_0} \left( \lambda + \tilde{q} \right) \hat{s}. \quad (50)$$

Note that the above expression becomes simply $\nu_p \simeq \nu$ with the choice $\lambda = -\tilde{q}$ for the electromagnetic coupling scalar, causing the coupling effect between the particle’s spin and the background electric field to disappear. Thus, this choice makes vanishing the difference between the center mass line $U$ and the $U$-orbit, at least to first order in $\hat{s}$, reproducing the corresponding result obtained in the Schwarzschild case.

2.3. The Tulczyjew (T) supplementary conditions

The T supplementary conditions require $S_{rf} = S_{rf\tilde{g}} = 0$ ($S = S^{\mu \nu} \nu_\nu = 0$), or

$$S = s \omega^f \wedge \Omega^p \phi, \quad \Omega^p \phi = \gamma_p \left[ -\nu_p \omega^f + \omega^\phi \right] \quad (51)$$

so that $(S_{rf}, S_{rf\tilde{g}}) = (-s_{\gamma_p \nu_p}, s_{\gamma_p})$ and Eq. (33) gives

$$\hat{s} = -\frac{1}{M \gamma_p} \left[ \gamma (\nu - \nu_0)^2 + \frac{\nu_0}{\zeta_0} \frac{Q}{r^2} \right] \left\{ \gamma \nu \frac{\zeta_0}{\nu_0} \left[ \frac{\gamma_0^2}{\gamma_0^2} (\nu_0 - \nu_0^2) + \nu_0^2 \right] \right.$$

$$\left. + \frac{\nu_0}{\zeta_0} \frac{2Mr - 3Q^2}{r^4} \nu_p - \frac{\lambda Q}{r^2} \left( \frac{\gamma_0^2}{\gamma_0^2} \nu - \nu_p \right) \right\}^{-1}. \quad (52)$$

Recalling its definition (27), $m_\nu$ becomes

$$\frac{m_\nu}{m} = M \hat{s} \gamma_p \left[ \frac{\zeta_0}{\nu_0} \gamma (\nu_0 - \nu_0^2) - \lambda \frac{Q}{r^2} \right], \quad (53)$$
and using Eq. (29) for $\nu_p$, we obtain

$$\dot{s} = \frac{\nu_g}{M\lambda (1 - \nu_s)} \left( \nu - \nu_p \right) \left( \frac{\gamma (\nu \nu_p - \nu_q^2) - \nu_g \lambda Q}{\zeta g} \right);$$  \hspace{1cm} (54)

this condition must be considered together with Eqs. (52) and (54) and solving with respect to $\nu_p$ we have that

$$\nu_p^{(\pm)} = \frac{1}{2} \left\{ \nu - \frac{\lambda Q}{\gamma^2 r^2} \left( 3\zeta g - \gamma \nu_q q^2 \right) \right\} \left[ \frac{\nu_g}{\gamma} \left( \frac{\zeta^2 g + 2MR - 3Q^2}{r^4} \right) \right] \pm \sqrt{\Psi},$$

$$\Psi = \frac{\nu^2}{\gamma^2} \nu_g^2 \lambda^2 q^2 Q^4 - 2\nu_g \nu_q \frac{\lambda}{\gamma} \left[ \frac{\nu^2}{\gamma^2} q + (3\nu^2 - 4) \frac{\lambda}{\gamma^2} \right] \frac{Q^3}{r^6}$$

$$+ \left\{ \zeta g \left[ \frac{\nu^2}{\gamma^2} q + (9\nu^2 - 8\nu_g^2) \frac{\lambda^2}{\gamma^2} \right] - 2\nu_g \frac{\lambda}{\gamma} \left[ \nu^2 (1 + \nu_q^2) - 4\nu_g^2 \right] \right\}$$

$$- \frac{\nu^2}{\zeta g} \left( 2 - \nu^2 \right) \frac{2MR - 3Q^2}{r^4} \} \frac{Q^2}{r^4} - \nu_g \zeta g \left\{ \left[ \zeta g^2 (2\nu^2 - \nu_q^2 - 1) \right] \right. $$

$$+(\nu^2 (1 + \nu_g^2) - 2\nu_q^2) \frac{2MR - 3Q^2}{r^4} \} - \frac{\lambda}{\gamma^2} \left[ \zeta g^2 (5\nu^2 - 4\nu_g^2) \right]$$

$$+(3\nu^2 - 2\nu_g^2) \frac{2MR - 3Q^2}{r^4} \} \frac{Q}{r^2} + \frac{1}{\gamma^2} \left\{ 4\zeta g^2 \left[ \zeta g^2 \nu^4 - \nu_g^4 \frac{2MR - 3Q^2}{r^4} \right] \right. $$

$$+ \nu^2 \nu_g^2 \left[ \frac{13M^2}{r^6} - \frac{12Q^2}{3MR - 2Q^2} \right] \right\}. \hspace{1cm} (55)$$

By substituting $\nu_p = \nu_p^{(\pm)}$ for instance into $\nu$ and $\dot{s}$, we obtain a relation between $\nu$ and $\dot{s}$. The reality condition of (55) requires that $\nu$ takes values outside the interval $(\nu_-, \nu_+)$, where $\nu_\pm$ are the roots of the equation $\Psi(\nu) = 0$, depending on the parameters $q$, $Q/M$ and $r/M$.

The behaviour of the spin parameter $\dot{s}$ as a function of $\nu$ is shown in Fig. 4 for $\lambda = -\tilde{q}$ and fixed values of the parameters $q$, $Q/M$ and $r/M$. This plot shows that there exists a range of velocities $\nu$ which is forbidden for physical $U$-orbits. A natural explanation of this is the lack of centrifugal and electric forces strong enough to balance the spin force.

To first order in $\dot{s}$ we have

$$\nu \simeq \pm \nu_0 + \mathcal{N}^{(T)} \dot{s}, \hspace{1cm} \mathcal{N}^{(T)} \equiv \mathcal{N}^{(P)}; \hspace{1cm} (56)$$

therefore, the angular velocity $\zeta$ and its reciprocal coincide with the corresponding ones derived in the case of P supplementary conditions (see Eq. (15)). From the
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In the case of Tulczyjew’s supplementary conditions, and for the choice $\lambda = -\tilde{q}$ of the electromagnetic coupling scalar, the spin parameter $\hat{s}$ is plotted as a function of the linear velocity $\nu$, for $Q/M = .6$, $\tilde{q} = .1, 1.5, 2, 10$ and $r/M = 8$, from (a) to (d) respectively. The plots have two branches corresponding to $\nu = \nu_{p}^{(+)}$ (solid) and to $\nu = \nu_{p}^{(-)}$ (dashed). In cases (a) and (b) the spin $\hat{s}$ vanishes for $\nu = \pm \nu_{0}$, with $\nu_{0} \approx 0.387$ and $\nu_{0} \approx 0.175$ respectively, being the charge to mass ratio of the particle less than the critical value $\tilde{q}_{lim} \approx 1.831$ while the timelike geodesics exist for $r > r_{g}^{+}$ ($r_{g}^{+}/M \approx 2.737$). The shaded regions contain the forbidden values of $\nu$ ($\tilde{\nu}_{\pm} \approx \pm 0.280$ in (a) and $\tilde{\nu}_{\pm} \approx \pm 0.069$ in (b)); at the boundary of both forbidden regions the spin is finite ($\hat{s} \approx \pm 0.10.28$ in (a) and $\hat{s} \approx \pm 0.30.28$ in (b)). In cases (c) and (d) $\hat{s} = 0$ corresponds to values of $\nu$ such that $\nu_{p}^{(\pm)} = \pm 1$, as expected from Eqs. (54) and (52); moreover, the reality condition of (55) is satisfied for all values of $\nu$, so that no forbidden region is present; however, in contrast with cases (a) and (b) there are some intervals of values of the linear velocity $\nu$ where the timelike condition $|\nu_{p}^{(\pm)}| \leq 1$ for either $\nu_{p}^{(+)}$ or $\nu_{p}^{(-)}$ is not satisfied.
preceding approximate solution for $\nu$ we also have that

$$\nu_p^{(\pm)} \simeq \nu - \frac{M Q}{\gamma_0 q^2} (\lambda + \tilde{q}) \hat{s} ,$$

and the total 4-momentum $P$ is given by Eq. (29) with $\nu_p = \nu_p^{(\pm)}$. As in the P case, the choice $\lambda = -\tilde{q}$ for the electromagnetic coupling scalar simplifies the above solution as $\nu_p \simeq \nu$.

3. Charged spinning test particles at rest

Let us consider the case of a charged spinning test particle at rest. It is enough to put $\nu = 0$ in the equation of motion (33), obtaining

$$0 = -m \nu_g^2 + \frac{\nu_g q Q}{\tilde{r}^2} \left[ \frac{2Mr - 3Q^2}{\nu_g r^2} \right] S_{\tilde{r} \tilde{r}} + \frac{qQ}{\tilde{r}^2} .$$

The total 4-momentum $P$ is given by Eq. (29) with

$$\nu_p = \frac{m_s}{m} = -\frac{1}{m} \left[ \nu_g \tilde{q} + \lambda \frac{Q}{r^2} \right] S_{\tilde{r} \tilde{r}} .$$

In this case ($\nu = 0$) the CP and P supplementary conditions coincide, and we have the following result:

a) CP, P: $S_{\tilde{t} \tilde{r}} = 0$;

Eq. (58) becomes

$$0 = -m \nu_g^2 + \frac{\nu_g q Q}{\tilde{r}^2} ,$$

giving the well known equilibrium condition for a spinless charged test particle

$$\tilde{q} = \frac{Mr - Q^2}{Q \sqrt{\Delta}} ,$$

specifying the position at which the particle can be held at rest, for fixed values of the mass to charge ratios of Reissner-Nordström source and particle. Note that equilibrium cannot exist if the particle and black hole have opposite charge, as well as either $q$ or $Q$ are zero. The only nonvanishing component of the spin tensor $S_{\tilde{r} \tilde{\phi}} = s$ remains arbitrary, as from Eq. (58), giving no contribution to the equilibrium condition of the test particle. The total 4-momentum $P$ is given by Eq. (29) with

$$\nu_p = -\left[ \nu_g \tilde{q} + \lambda \frac{Q}{r^2} \right] M \hat{s} ,$$

from Eq. (59).
b) T: $S_{\hat{r}} + S_{\hat{r}}u_p = 0$; 
from Eq. (59) we have 

$$
\nu_p(\pm) = \pm \sqrt{\frac{3}{2}} \left\{ 1 - \left[ 1 - 4M^2s^2 \left( \nu_g \zeta_g + \lambda Q r^2 \right)^2 \right]^{1/2} \right\}, \quad (63)
$$

being $S_{\hat{r}} = s\gamma p$; the solution $\nu_p(\pm)$ corresponds to negative (positive) values of the quantity $s(\nu_g \zeta_g + \lambda Q r^2)$. By substituting Eq. (63) into Eq. (58), with $S_{\hat{t}} = s\gamma p\nu_p$, we get 

$$
0 = -m\nu_g^2 + \nu_g qQ r^{-2} - \nu_g m \frac{2Mr - 3Q^2}{r^2} + \lambda \frac{\zeta_g Q}{r^2} \left[ \nu_g \zeta_g + \lambda \frac{Q}{r^2} \right]^{-1} \cdot \left\{ 1 - \left[ 1 - 4M^2s^2 \left( \nu_g \zeta_g + \lambda \frac{Q}{r^2} \right)^2 \right]^{1/2} \right\}, \quad (64)
$$

which gives the equilibrium positions for the particle with given $q$, $Q$ and $\hat{s}$. In contrast with the CP, P cases, equilibrium is possible even if either $q$ or $Q$ are zero. Note that to first order in $\hat{s}$ the above equation reduces again to the equilibrium condition (61).

4. Clock-effect for charged spinning test particles

As we have seen in all cases examined above, charged spinning test particles move on circular orbits on the equatorial plane of the Reissner-Nordström spacetime which, to first order in the spin parameter $\hat{s}$, are close to a geodesic (as expected): 

$$
\frac{1}{\zeta(\text{SC}, \pm, \pm)} = \pm \frac{1}{\zeta_0} \pm M|\hat{s}|J_{\text{SC}}, \quad J_{\text{SC}} = -\frac{\nu_g N^{(\text{SC})}}{\zeta_0 M^2}, \quad (65)
$$

where both quantities $\zeta_0$ and $J_{\text{SC}}$ are functions of the charge $q$ of the particle, and, thus, depend on its sign. Eq. (65) identifies these orbits according to the chosen supplementary condition, the signs in $1/\zeta_0$ corresponding to co/counter-rotating orbits while the signs in front of $\hat{s}$ refer to a positive or negative spin direction along the $z$-axis; for instance, the quantity $\zeta_{(P,+,+)}$ denotes the angular velocity of $U$, derived under the choice of Pirani’s supplementary conditions and corresponding to a co-rotating orbit $+$ with spin-down $-$ alignment, etc. Therefore one can measure the difference in the arrival times after one complete revolution with respect to a static observer. If we consider a pair of particles with equal charge of definite sign, the coordinate time difference is simply 

$$
\Delta t_{(+,+,+)} = 2\pi \left( \frac{1}{\zeta(\text{SC}, +, +)} + \frac{1}{\zeta(\text{SC}, -, +)} \right) = 4\pi M|\hat{s}|J_{\text{SC}}, \quad (66)
$$

and analogously for $\Delta t_{(+, -, -, +)}$. 
These formulas can be further generalized taking into account the possibility to consider rotating particles of opposite charge. By considering explicitly the dependence on the sign of \( q \), Eq. (65) can be rewritten as

\[
\frac{1}{\zeta_{SC,\pm,\pm,\pm}} = \pm \frac{1}{\zeta_0^{(\pm)}} \pm M|\delta|J_{SC}^{(\pm)}. \tag{67}
\]

As we shall see soon, the electric interaction contributes significantly to the clock effect according to the relative signs of the particle charges.

In the case \( \lambda = -\tilde{q} \) for the electromagnetic coupling scalar, the terms \( J_{SC} \) defined in Eq. (65) become

\[
J_{CP}^{(\pm)} = \nu_0 \frac{1}{\zeta_g} \frac{1}{\nu_0} \left[ 1 + \frac{\gamma_0^2}{\gamma g} \right]^{-1} \tilde{q} \frac{Q}{r^2},
\]

\[
J_{P}^{(\pm)} \equiv J_{T}^{(\pm)} = -\nu_0 \frac{1}{\zeta_g} \frac{1}{\nu_0} \left[ 2 \frac{\tilde{q} Q}{r^2} - \frac{1}{\nu_0} \frac{\gamma_0^2}{\gamma g} \frac{1}{\nu_g} 3Mr - 4Q^2 \right], \tag{68}
\]

where \( \nu_0 \) is further depending on \( q \) (more precisely, it is a function of the parameter \( \tilde{q} \equiv \pm|\tilde{q}| \), and so the coefficients \( J_{SC}^{(\pm)} \) as well). Therefore, in the case of a co-rotating (+), spin up (+), positive charged (+) particle and a counter-rotating (−), spin down (−), negative charged (−) particle, for instance, the coordinate time difference is given by:

\[
\Delta t_{(+,+,+;-,_-)} = 2\pi \left( \frac{1}{\zeta_{SC,+,+,+}} + \frac{1}{\zeta_{SC,-,-,-}} \right)
= 2\pi \left\{ \frac{1}{\zeta_0^{(+)}} - \frac{1}{\zeta_0^{(-)}} + M|\delta| \left[ J_{SC}^{(+)} - J_{SC}^{(-)} \right] \right\}, \tag{69}
\]

with a natural extension of the used notation for \( \Delta t \). Analogously for other choices of charge sign and spin directions. Note that no clock effect is found when the CP supplementary conditions are imposed if either \( q = 0 \) or \( Q = 0 \), in complete agreement with the results obtained in the Schwarzschild case 10 for neutral particles.

Finally, we remark that, in contrast with the case of an uncharged black hole, a non-zero clock effect appears even for spinless charged particles in a Reissner-Nordström spacetime. In fact, in this case the time delay measured after a complete revolution between a co-rotating (+) positive charged (+) particle and a counter-rotating (−), spin down (−), negative charged (−) particle is simply given by:

\[
\Delta t_{(+,+,+;-,_-)} = 2\pi \left[ \frac{1}{\zeta_0^{(+)}} - \frac{1}{\zeta_0^{(-)}} \right]. \tag{70}
\]

As an example, let us suppose that the two particles move on a circular orbit with radius \( r > r^*_g \), to which correspond the solutions \( \pm \nu_0^{(-)} \), according to the discussion made in Section 2. The difference in the arrival times (70) is, thus, given by

\[
\Delta t_{(+,+,+;-,_-)} = \frac{2\pi}{\zeta_g} \left[ 1 + \frac{r_g^2}{4} \left( \frac{\tilde{q}}{\tilde{q}_{lim}} \right)^4 \right]^{-1/2} \Xi^{-1/2} \left\{ \Lambda + \frac{|\tilde{q}|}{\tilde{q}_{lim}} (\Lambda^2 - \nu_g^2 \Xi)^{1/2} \right\}^{1/2},
\]
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\[ - \left[ \Lambda - \frac{|\tilde{q}|}{\tilde{q}_{\text{lim}}} \left( \Lambda^2 - \nu g \Xi \right)^{1/2} \right]^{1/2}, \]  

from Eq. (22). Finally, it is interesting to consider the limit of Eq. (71) for small values of the ratio $|\tilde{q}|/\tilde{q}_{\text{lim}}$:

\[ \Delta t_{(+,+,-,-)} \simeq \frac{2\pi}{\zeta g} \left( \frac{|\tilde{q}|}{\tilde{q}_{\text{lim}}} \right), \tag{72} \]

to first order in $|\tilde{q}|/\tilde{q}_{\text{lim}}$.

5. Conclusions

Charged spinning test particles on circular motion around a Reissner-Nordström black hole have been discussed in the framework of the Dixon-Souriau approach supplemented by standard conditions, generalizing the corresponding analysis previously done in the Schwarzschild case. In the limit of small spin, the orbit of the particle is close to a circular geodesic and the difference in the angular velocities with respect to the geodesic value can be of arbitrary sign, corresponding to the two spin-up and spin-down orientations along the $z$-axis. For co-rotating and counter-rotating both spinning or even spinless charged test particles a nonzero gravitomagnetic clock effect appears, just as in the Schwarzschild case.

Appendix A. The Dixon-Souriau model

The model proposed by Souriau to describe the motion of a charged spinning test particle in the presence of a gravitational as well as an electromagnetic field arises from the main idea to condense along a single curve (the particle worldline $U$) a small extended body undergoing the laws of the electrodynamics of continuous media, that is

\[ \nabla_\mu J^\mu = 0, \quad \nabla_\mu T^{\mu\nu} + F_{\mu\nu}J^\mu = 0, \tag{A.1} \]

where $J^\mu$ and $T^{\mu\nu}$ are the current density and the energy-momentum tensor respectively. This mathematical procedure carries out through a variational method allows to define four physical quantities, the total 4-momentum $P$, the (antisymmetric) spin tensor $S$, the charge $q$ and the (antisymmetric) electromagnetic moment tensor $\mathcal{M}$, satisfying the following “universal” equations

\[ \frac{dP^\mu}{dU} = -\frac{1}{2} R_{\nu\alpha\beta\gamma} U^\nu S^{\alpha\beta} + q F_{\nu\gamma} U^\gamma - \frac{1}{2} \mathcal{M}^{\rho\sigma} \nabla_\mu F_{\rho\sigma}, \]

\[ \frac{dS^{\mu\nu}}{dU} = P^{\mu} U^{\nu} - P^{\nu} U^{\mu} + \mathcal{M}^{\mu\rho} F_{\rho\nu} - \mathcal{M}^{\nu\rho} F_{\rho\mu}, \tag{A.2} \]

where the charge $q$ is constant along $U$.

This model needs to be completed by further conditions. First of all, let us choose the simplest form for the electromagnetic moment tensor $\mathcal{M}$, that is let us
assume that it is proportional to the spin tensor $S$ by the electromagnetic coupling scalar $\lambda$:
\[
\mathcal{M}^{\mu\nu} = \lambda S^{\mu\nu}.
\] (A.3)

The value of $\lambda$ can be fixed by taking the flat spacetime limit of the model (see [5]), which gives exactly the Bargman-Michel-Telegdi equations [6] as well as other general relativistic corrections:
\[
ma(U) = qE(U),
\]
\[
\nabla_{(\text{fw})}S(U) = \lambda B(U) \times_U S(U),
\] (A.4)

where $a(U) = \nabla_U U$ is the acceleration of the orbit, $E(U)$, $B(U)$ are the electric and magnetic part of the electromagnetic field respectively
\[
E(U)^\alpha = F_{\alpha\beta}U^\beta, \quad B(U)^\alpha = \frac{1}{2} \eta(U)^{\alpha\beta\gamma}F_{\beta\gamma},
\] (A.5)

and $S(U)$ is the magnetic part of the spin tensor
\[
S(U)^\alpha = \frac{1}{2} \eta(U)^{\alpha\beta\gamma}S_{\beta\gamma};
\] (A.6)

here $\nabla_{(\text{fw})}S(U) = P(U)\nabla_U S(U)$ denotes the Fermi-Walker temporal derivative of $S(U)$ along $U$, being $P(U)^\alpha = \delta^\alpha_\mu + U^\mu U_\alpha$ the projector into the local rest space of $U$ (LRS$_U$), and $\eta(U)^{\alpha\beta\gamma} = U^\rho \eta_{\alpha\beta\gamma}$ is the only spatial field resulting from the measurement of the unit (oriented) volume 4-form $\eta$, which defines the spatial cross product $\times_U$ as well as the dual operation on the LRS$_U$. If the charged particle were an electron, from these relations we have that $\lambda = -q/(2m)$ or $\lambda = -\mu_B g$, where $\mu_B = q/(2m)$ is the Bohr magneton and $g = 2$ the Landé factor for the electron itself. Souriau proposed a more general expression for this factor, completing the scheme given by Eqs. (A.2) and (A.3) by the $T$ supplementary conditions $S^{\mu\nu}P_\nu = 0$, that is
\[
\lambda = \frac{f'(\alpha)}{f(\alpha)}(P \cdot U),
\] (A.7)

where $f$ is an arbitrary positive function, and $P_\mu P^\mu = f(\alpha)$. In this case the 4-velocity $U$ results parallel to
\[
\omega P^\mu + S^{\mu\nu} \left\{ [g - f'(\alpha)] F_{\nu\rho} P^\rho + \frac{1}{2} f'(\alpha) S^{\rho\sigma} \nabla_\nu F_{\rho\sigma} + \frac{1}{2} R_{\alpha\beta\nu\rho} S^{\alpha\beta} P^\rho \right\},
\] (A.8)

with
\[
\alpha = S^{\mu\nu}F_{\mu\nu}, \quad \omega = f(\alpha) + \frac{q}{2} \alpha + \frac{1}{4} R_{\lambda\mu\nu\rho} S^{\lambda\mu} S^{\nu\rho}.
\] (A.9)
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