Constructing Gravity Amplitudes from Real Soft and Collinear Factorisation

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Abstract

Soft and collinear factorisations can be used to construct expressions for amplitudes in theories of gravity. We generalise the “half-soft” functions used previously to “soft-lifting” functions and use these to generate tree and one-loop amplitudes. In particular we construct expressions for MHV tree amplitudes and the rational terms in one-loop amplitudes in the specific context of $\mathcal{N} = 4$ supergravity. To completely determine the rational terms collinear factorisation must also be used. The rational terms for $\mathcal{N} = 4$ have a remarkable diagrammatic interpretation as arising from algebraic link diagrams.

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I. INTRODUCTION

The $S$-matrix of a weakly coupled Quantum Field Theory is a key object which largely defines the theory and its interactions. The Feynman diagram approach is a very robust general method which, in principle, may be used to compute the $S$-matrix. Unfortunately this method is very complex, particularly in theories which contain large symmetries such as gauge theories and theories of gravity. Computing the $S$-matrix from the constraints it must satisfy is an old idea \cite{1} which has seen huge development in recent years using ideas based on unitarity \cite{2,3} and the factorisation properties of amplitudes \cite{4,7}.

In this article, we examine soft-factorisation on real kinematics in theories of gravity and introduce “soft-lifting” functions which allow us to express the $n$-graviton “Maximally-Helicity-Violating” (MHV) tree amplitudes as a “soft-lift” of either the three or four-point tree amplitudes. The soft-lift of the three point amplitudes gives an expression equivalent to previous forms but the soft-lift of the four-point amplitudes yields a novel expression for the MHV tree amplitude.

The same soft-lifting functions are key elements in the rational terms of $\mathcal{N}=4$ supergravity one-loop MHV amplitudes. In this one-loop example real soft-limits must be combined with information from the collinear limits to obtain the $n$-point expression. The expression for the $n$-point rational term was proposed in ref. \cite{8} where numerical checks were applied to it. Here we show that these rational terms have a remarkably simple interpretation in terms of one-loop link diagrams in the same way as has been found for the MHV tree amplitude \cite{9}.

Using the diagrammatic representation, we present an analytic proof that the rational expressions have the correct soft and collinear limits.

II. MHV TREE AMPLITUDES, HALF-SOFT FUNCTIONS AND TWISTOR LINK DIAGRAMS

MHV tree amplitudes for graviton scattering have been presented in a wide variety of forms. In this section we review a range of these and the relations between them. The original Berends, Giele and Kuijf (BGK) form of the MHV gravity amplitude \cite{10} is

$$M_n^{\text{tree}}(1^-, 2^-, 3^+, \ldots, n^+) = (-1)^n \langle 1 2 \rangle^8 \times \frac{[1 2] [n-2 n-1]}{[1 n-1]} \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} \langle i j \rangle \prod_{i=1}^{n-3} \prod_{j=i+2}^{n-1} \langle i j \rangle \prod_{l=3}^{n-3} \langle n |K_{l+1} n-1 l\rangle + \mathcal{P}_{(2,3,\ldots,n-2)} \right) \tag{2.1}$$

where $\mathcal{P}_{(2,3,\ldots,n-2)}$ indicates summing over the $(n-3)!$ permutations of legs $2, \ldots, n-2$. The tree amplitude has the, non-manifest, symmetry property that $M_n^{\text{tree}}(1^-, 2^-, 3^+, \ldots, n^+)/\langle 1 2 \rangle^8$ is crossing symmetric under all exchanges of legs including the two negative helicity legs. This fact may be proven (or is due to) by recognising the $n$-graviton tree amplitude is the same in pure gravity and $\mathcal{N}=8$ supergravity and then looking at the implications of the supersymmetric Ward identities. The argument for $\mathcal{N}=8$ supergravity follows from that for supersymmetric Yang-Mills \cite{11} using the $\mathcal{N}=8$ Ward

\footnote{The normalisation of the physical amplitude $M^{\text{tree}} = i(\kappa/2)^{n-2}M^{\text{tree}}, M^{1\text{-loop}} = i(\kappa/2)^n M^{1\text{-loop}}$. As usual we are using a spinor helicity formalism with the usual spinor products $(j l) \equiv (j^- l^+) = u_j u_k (k_l)$ and $[j l] \equiv (j^+ l^-) = \bar{u}_j u_k (k_l)$, and where $[i |K_{abc}] |j\rangle$ denotes $(i^+ |K_{abc}| j^+)$ with $K_{abc}^{\mu} = k_\mu + k_\nu + k_\sigma$ etc. Also $s_{ab} = (k_a + k_b)^2, t_{abc} = (k_a + k_b + k_c)^2$, etc.}
identities given, for example, in appendix E of ref. [12]. The BGK formulae was established using the Kawai, Llewellen and Tye (KLT) relations [13] which relate gravity amplitudes to products of Yang-Mills amplitudes for lower point functions and then by verifying its soft-factorisation properties [10]. Subsequently alternative proofs of the formulae have been presented [14]. MHV amplitudes are an important component of gravity theories and can be promoted to fundamental vertices to construct other tree amplitudes [15] as in Yang-Mills theories [10].

Gravity amplitudes have soft-limit singularities [10] as $k_n \to 0$,

$$M_n(\cdots, n-1, n^h) \to \text{Soft}_{n^+} \times M_{n-1}(\cdots, n-1)$$

where the positive helicity “soft factor” is given by

$$\text{Soft}_{n^+} = -\frac{1}{\langle a n \rangle \langle n b \rangle} \sum_{j \neq n, a, b} [j n] \langle a j \rangle \langle j b \rangle \langle j n \rangle.$$ (2.3)

with $\text{Soft}_{n^-}$ given by conjugation. The soft factor is independent of the choice of $a$ and $b$ $(\neq n)$ although this is not manifest. Recently it has also been suggested [17] that soft limits may be used to determine an $n$-point amplitude in terms of $n-1$ point amplitudes multiplied by a soft factor. This process, referred to as “inverse-soft”, has been applied to gravity tree amplitudes [9, 18, 19] and used, for example, to determine the MHV amplitudes in the form,

$$M_n^{\text{MHV}}(1, 2, \cdots n) = \sum_{i=1}^{n-2} G(n-1, n, \hat{i}) \times M_{n-1}^{\text{MHV}}(1, \cdots \hat{i}, \cdots \hat{n-1})$$

where $G$ denotes the soft-factor

$$G(a, b, i) = -\frac{\langle a i \rangle^2 [b i]}{\langle a b \rangle^2 \langle b i \rangle}.$$ (2.5)

In this expression the $n-1$-point amplitude is evaluated at a complex kinematic point where the legs $i$ and $n-1$ have been shifted:

$$\hat{k}_i = \lambda_i (\bar{\lambda}_i + \frac{\langle n n-1 \rangle}{\langle i n-1 \rangle} \bar{\lambda}_n), \quad \hat{k}_{n-1} = \lambda_{n-1} (\bar{\lambda}_{n-1} + \frac{\langle n i \rangle}{\langle n-1 i \rangle} \bar{\lambda}_n).$$ (2.6)

These complex momenta satisfy

$$\hat{k}_i + \hat{k}_{n-1} = k_i + k_{n-1} + k_n$$ (2.7)

Although this expression can be regarded as an inverse-soft relation it is very closely related to BCFW recursive expressions for gravity [6, 20]. Other related expressions for the MHV tree amplitude exist [21, 22].

Another representation of the MHV tree amplitudes was given in [23] in terms of “half-soft” functions. These originally appeared in the box-coefficients of the one-loop MHV amplitude in $\mathcal{N} = 8$ supergravity and can be thought of as an off-shell version of the tree amplitude. The half-soft functions have the explicit form

$$h(a, \{1, 2, \cdots, m\}, b) = [1 2]_{\{1 2\}} \langle 2 3 \rangle \langle 3 4 \rangle \cdots \langle m-1, m \rangle \langle a 1 \rangle \langle a 2 \rangle \langle a 3 \rangle \cdots \langle a m \rangle \langle 1 b \rangle \langle m b \rangle + \mathcal{P}_{(2,3,\cdots,m)},$$ (2.8)
When the half-soft functions are at the maximum size for an \( n \)-point amplitude we have
\[
M_{n}^{\text{tree}}(1^{-}, 2^{-}, 3^{+}, \cdots, n^{+}) = (-1)^{n} \langle 1 2 \rangle^{6} \times h(1, \{3, 4, \cdots, n\}, 2) \tag{2.9}
\]
Note that this implies \( h(1, \{3, 4 \cdots, n\}, 2)/\langle 1 2 \rangle^{2} \) is completely crossing symmetric.

An alternative expression for the half-soft functions was also given in \[23\]
\[
h(a, \{1, 2, \cdots m\}, b) = \sum_{i_{1}, i_{2}, \cdots, i_{m}=0}^{m-2} \phi_{m}(i_{1}, i_{2}, i_{3}, \cdots i_{m}) \times \prod_{j=1}^{m}(\langle a j \rangle \langle j b \rangle)^{i_{j}-1} \tag{2.10}
\]
where the sum is restricted to \( \sum_{j} i_{j} = m - 2 \). The \( \phi_{m} \) are polynomial in the objects
\[
\hat{A}[a; b] = \frac{[a b]}{\langle a b \rangle} \tag{2.11}
\]
and are symmetric functions of their arguments. They are given recursively by
\[
\phi_{m}(i_{1}, i_{2}, i_{3}, \cdots 0) = \sum_{j=1}^{m-1} \phi_{m-1}(i_{1}, i_{2}, \cdots i_{j} - 1, \cdots i_{m-1}) \times \hat{A}[j; m] \tag{2.12}
\]
with the initial value
\[
\phi_{2}(0, 0) \equiv \hat{A}[1; 2] = \frac{[1 2]}{\langle 1 2 \rangle} \tag{2.13}
\]
In ref. \[23\] an interpretation of the \( \phi_{m} \) was given in terms of Young Tableaux.

In a more recent development \[9\] an equivalent expression for the tree amplitude was given
\[
M_{n}^{\text{MHV}} = (-1)^{n} \langle 1 2 \rangle^{6} \sum_{\text{trees}} \left( \prod_{\text{edges}: ab} \frac{[a b]}{\langle a b \rangle} \right) \left( \prod_{\text{vertices}: a} \langle a 1 \rangle \langle a 2 \rangle^{\text{deg}(a)-2} \right) \tag{2.14}
\]
The tree link diagrams are all the connected graphs which can be drawn between \( n - 2 \) labelled vertices representing the positive helicity legs. Vertices with any number of legs (or degree \( \text{deg}(a) \)) are allowed. For example, for the seven point amplitude we have the 125 diagrams obtained by the labellings of the topologies given in figure \[1\]

![Diagram](image_url)

**FIG. 1**: The topologies of the link diagrams for the seven point MHV amplitude. The vertices are labelled by the five positive helicity legs.
There is an obvious link diagram interpretation of the half-soft functions of maximum size following from (2.9) and (2.14). From (2.11) it can be seen that the same rules also generate the half-soft functions with restricted sets to positive helicity legs.

In the next section, we introduce “soft-lifting” functions which are generalisations of the half-soft functions. In some regards, we are following the spirit of “inverse soft” however we will be using real momenta. We apply these to evaluate MHV tree amplitudes and the rational parts of one-loop MHV amplitudes in \( N = 4 \) supergravity.

We find that we need to consider both collinear and soft limits in order to determine one-loop amplitudes. Gravity amplitudes are not singular in the collinear limit \( k_a \cdot k_b \rightarrow 0 \), but acquire collinear phase singularities which take a form that is specified in terms of amplitudes with one fewer external leg \[23\]. Specifically, if \( k_a \rightarrow zK \) and \( k_b \rightarrow (1-z)K \),

\[
M_n(\cdots, a^h, b^s) \xrightarrow{a||b} \sum_{h'} \text{Sp}^{h'h} M_{n-1}(\cdots, K^{h'}) + F_n
\]

where the \( h \)'s denote the various helicities of the gravitons and \( F_n \) is free of phase singularities. The “splitting functions” are \[23\]

\[
\text{Sp}^{++}_+ = -\frac{z^2 [a b]}{(1-z) (a b)}, \quad \text{Sp}^{++}_- = -\frac{[a b]}{z(1-z) (a b)}, \quad \text{Sp}^{+0} = 0.
\]

with the others obtained by conjugation.

### III. SOFT-LIFTING FUNCTIONS

Analyses of soft and collinear divergences have long been used as tools for constructing amplitudes \[10\] and indeed there are recent suggestions that gravity scattering amplitudes can be determined from their soft-behaviour alone \[17\]. In attempting to do this it is useful to define building blocks which have simple soft and collinear behaviour. For example, the “half-soft” functions of ref. \[12\] were used by the authors to construct the all-plus graviton one-loop amplitude. Here we generalise the half-soft functions to “Soft-Lifting Functions” which we will use in several constructions.

We define soft-lifting functions, \( \hat{S}^p[P^s; Q^p] \), where \( P^s = \{p_j\} \) and \( Q^p = \{q_k\} \) are disjoint sets of the positive helicity legs of length \( s \) and \( p \) respectively. When the set \( P^s \) is of length 1 the soft-lifting functions are the half-soft functions of ref. \[23\] up to a factor,

\[
\hat{S}^p[\{p_1\}; Q^p] \equiv (\langle m_1 p_1 \rangle^2 \langle m_2 p_1 \rangle^2) \times h(m_1, Q^p + p_1, m_2)
\]

From this we define the general soft-lifting function by

\[
\hat{S}^p[P^s; Q^p] = \sum_{\text{partitions}} \prod_{j=1}^{s} \hat{S}^p[\{p_j\}; Q_j]
\]

where the sum is over all partitions of the \( q_k \in Q^p \) into subsets \( Q_j \) where \( Q_1 \cup Q_2 \cdots \cup Q_s = Q^p \). The summation includes the terms where some of the \( Q_j \) are null. By definition, \( \hat{S}^0 = 1 \).

We use shortened notation \( \hat{S}^p[Q^p] \) for the special case of \( \hat{S}^p[P^s; Q^p] \) when \( P^s = P^+ - Q^p \) where \( P^+ \) is the full set of positive helicity legs (usually \{3, 4, \cdots , n\}). In this case \( s = n-2-p \).

In this section we label the two negative helicity legs as \( m_1 \) and \( m_2 \) (typically these are the legs 1 and 2).
The soft-lifting functions are polynomial in the “soft-components” \( A[p_i; q_i] \) and \( A[q_i; q_i] \)
where
\[
A[p; q] \equiv \hat{S}^1[\{p\}; \{q\}] = \frac{[q p] \langle pm_1 \rangle \langle pm_2 \rangle}{\langle q p \rangle \langle q m_1 \rangle \langle q m_2 \rangle} \quad q \neq m_1, m_2, p
\] (3.3)

In terms of the soft-components,
\[
\hat{S}^1[P^*; \{q_1\}] \equiv \sum_{p_j \in P^*} A[p_j; q_1]
\] (3.4)

Note that we could include \( m_1 \) and \( m_2 \) in the summation with no change in value since \( A[m_1; q] = A[m_2; q] = 0 \) so that \( \hat{S}^1[\{q_1\}] = \sum_{j \neq q_1} A[p_j; q_1] \). The \( \hat{S}^p \) are products of \( \hat{S}^1 \) with cycle terms removed,
\[
\hat{S}^p[P^*; Q^p] = \prod_{k=1}^{p} \hat{S}^1[P^* + Q^p - q_k; \{q_k\}] - \text{cycle terms.}
\] (3.5)

where a cycle term is a cyclic combination of \( A[q_i; q_j] \), that is terms of the form
\( A[q_1; q_2]A[q_2; q_3] \cdots A[q_n; q_1] \). The first few are given by,
\[
\hat{S}^2[P; \{q_1, q_2\}] = \hat{S}^1[P \cup \{q_2\}; q_1] \hat{S}^1[P \cup \{q_1\}; q_2] - A[q_1; q_2]A[q_2; q_1]
\]
\[
= \sum_{p_j \in P} A[p_j; q_1] \sum_{p_k \in P} A[p_k; q_2] + A[q_1; q_2] \sum_{p_j \in P} A[p_j; q_1] + A[q_2; q_1] \sum_{p_j \in P} A[p_j; q_2]
\]
\[
\hat{S}^3[P; \{q_1, q_2, q_3\}] = \hat{S}^1[P \cup \{q_2, q_3\}; \{q_1\}] \hat{S}^1[P \cup \{q_1, q_3\}; \{q_2\}] \hat{S}^1[P \cup \{q_1, q_2\}; \{q_3\}]
\]
\[
- \left( A[q_1; q_2]A[q_2; q_1] \hat{S}^1[P \cup \{q_1, q_2\}; \{q_3\}] + \mathcal{P}_{q_1q_2q_3} \right)
\]
\[
= \sum_{p_j \in P} A[p_j; q_1] \sum_{p_k \in P} A[p_k; q_2] \sum_{p_l \in P} A[p_l; q_3]
\]
\[
+ \left( \sum_{p_j \in P} A[p_j; q_1] \sum_{p_k \in P} A[p_k; q_2] (A[q_1; q_3] + A[q_2; q_3]) + \mathcal{P}_{q_1q_2q_3} \right)
\]
\[
+ \left( \sum_{p_j \in P} A[p_j; q_1] \left(A[q_1; q_2]A[q_1; q_3] + A[q_1; q_2]A[q_2; q_3] + A[q_3; q_2]A[q_1; q_3] \right) + \mathcal{P}_{q_1q_2q_3} \right)
\] (3.6)

We can also construct a diagrammatic representation of \( \hat{S}^p[P^*; Q^p] \). This representation
is based on tree structures emanating from “seed points” \( p_j \in P^* \). The rules for constructing
these trees are as follows: vertices \( q_k \) must be attached to either the \( p_j \) directly or to other \( q_k \) vertices. The links are directional, with a link from \( q_k \) to \( x \) giving a contribution of \( A[x; q_k] \).

The arrows on the links are necessary since \( A[p_j; q_k] \neq A[q_k; p_j] \). Closed loops are not allowed.
Individual \( p_j \) need not be connected to any \( q_k \). Each \( q_k \) has exactly one link starting from
it, but may have any number entering it. An example graph contributing to \( \hat{S}^8[P^4; Q^8] \) is
shown in fig. 2. This representation differs from that for the MHV tree and \( R_n \) rational term
but is closely related. Each \( q_k \)-dependent factor is \( \langle \langle 1 q_k \rangle \langle 2 q_k \rangle \rangle ^N \) where
\[
N = \text{number of incoming arrows} - \text{number of outgoing arrows} = \deg(q_k) - 2
\] (3.7)

since there is precisely one outgoing line. However the \( p_j \) dependent factor is just
So we can represent the soft-lifting function as

\[
\hat{S}[P^s; Q^r] = \sum_{\text{graphs}} \left( \prod_{\text{edges: } a \rightarrow b} A[a; b] \right) \left( \prod_{\text{vertices: } q_k} \frac{[a b]}{\langle a b \rangle} \right) \left( \prod_{\text{vertices: } p_j} \langle p_j 1 \rangle \langle p_j 2 \rangle \right)^{\deg(p_j)}
\]

\[
\hat{S}[P^s; Q^r] = \sum_{\text{graphs}} \left( \prod_{\text{edges: } ab} \frac{[a b]}{\langle a b \rangle} \right) \left( \prod_{\text{vertices: } q_k} \langle q_k 1 \rangle \langle q_k 2 \rangle \right)^{\deg(q_k) - 2} \left( \prod_{\text{vertices: } p_j} \langle p_j 1 \rangle \langle p_j 2 \rangle \right)^{\deg(p_j)}
\]

In the final form we have the vertex and link rules as for the MHV trees, but with a pre-factor and no connections between the \( p_j \).

In general \( \hat{S}^p[P^s; Q^p] \) satisfies the following iterative definition: setting

\[
T^p[P^s; Q^p] \equiv \prod_{k=1}^{p} \hat{S}^1[P^s; \{q_k\}]
\]

we have

\[
\hat{S}^p[P^s; Q^p] = \sum_{r=1}^{p} \sum_{Q_1 \subset Q^p; |Q_1| = r} T^r[P^s; Q_1] \times \hat{S}^{p-r}[Q_1; Q^p - Q_1]
\]

To see this from the diagrammatic representation, consider the subset of diagrams where each of the \( r \)-vertices of subset \( Q_1 \subset Q^p \) are attached directly to the \( p_j \in P^s \). They can be attached to any of the \( p_j \) and, summing over the possibilities, the links joining the \( q_k \) to the \( p_j \) give a factor of

\[
\sum_{\text{partitions: } \rho} \prod_{k=1}^{r} A[p(\rho, k), q_k] \equiv \prod_{k=1}^{r} \sum_{p_j \in P^s} A[p_j; q_k] = T[P^s; Q_1]
\]

where the sum is over all partitions of the \( q_k \in Q_1 \) amongst the \( p_j \in P^s \) and \( p(\rho, k) \) denotes which of the \( p_j \) vertices the \( q_k \) vertex is attached to in partition \( \rho \). In this subset of diagrams
the remaining \( q \in Q^p - Q_1 \) vertices are not attached directly the \( p_j \) vertices, so they must attach to the \( q_k \in Q_1 \) vertices in any combination. Using the diagrammatic representation these links yield a factor of \( \hat{S}^{m-r}[Q_1; Q^m - Q_1] \). Overall this subset of diagrams yields

\[
T^r[P^s; Q_1] \times \hat{S}^{m-r}[Q_1; Q^m - Q_1]
\]

Summing over \( r \) and all possible choices of \( Q_1 \) gives eq. (3.10).

We finish this section by noting the soft behaviour of the soft-lifting functions,

\[
\hat{S}^p[P^s; Q^p] \rightarrow - \text{Soft}_{q^+}(m_1, P^s + Q^p - q_1, m_2) \times \hat{S}^{p-1}[P^s, Q^p - q_1]
\]

\[
\hat{S}^p[Q^p] \rightarrow - \text{Soft}_{q^+} \times \hat{S}^{p-1}[Q^p - q_1]
\]

for \( q_1 \in Q^p \) with no soft-singularity if \( q_1 \notin Q^p \).

**IV. RELATIONSHIP TO MHV TREE AMPLITUDES**

Our first application of the soft-lifting functions is to re-express the \( n \)-point MHV amplitude as a “soft-lift” of the three-point tree amplitude \( M_{3}^{MHV}(1^-, 2^-, p^+) \), for any choice of positive helicity leg \( p \),

\[
M_{n}^{MHV} = (-1)^{n-3} \tilde{M}_{3}^{MHV}(1^-, 2^-, p^+) \times \hat{S}^{n-3}[P^+ - \{p\}]
\]

where

\[
\tilde{M}_{3}^{MHV}(1^-, 2^-, p^+) \equiv - \frac{(12)^6}{(1p)^2 (2p)^2}
\]

and \( P^+ = \{3, 4, \cdots n\} \) is the set of all positive helicity legs. This expression is just a relabelling of eq. (2.9) using eq. (3.1) in a suggestive form.

We can also soft-lift the four-point amplitude to a new expression for the \( n \)-point amplitude

\[
M_{n}^{MHV} = \frac{(-1)^{n-4}}{(n-3)} \sum_{\{p_1, p_2\} \subset P^+} \tilde{M}_{4}^{MHV}(1^-, 2^-, p_1^+, p_2^+) \hat{S}^{n-4}[P^+ - \{p_1, p_2\}]
\]

where

\[
\tilde{M}_{4}^{MHV}(1^-, 2^-, p_1^+, p_2^+) \equiv \langle 12 \rangle^6 \frac{[p_1 p_2]}{(p_1 p_2) (p_1 1) (p_1 2) (p_2 1) (p_2 2)}
\]

The \( \tilde{M}_{3}^{MHV} \) and \( \tilde{M}_{4}^{MHV} \) are not true amplitudes since their arguments are not constrained by momentum conservation. We have made specific (natural) choices for \( \tilde{M}_{3}^{MHV} \) and \( \tilde{M}_{4}^{MHV} \). These coincide with the on-shell amplitudes where \( \lambda_{1,2} \) are shifted as in eq. (2.9).

Expression (4.13) can be verified recursively by considering a BCFW shift \( [6] \) of the two negative helicity legs:

\[
\lambda_1 \rightarrow \lambda_1 + z\lambda_2, \quad \bar{\lambda}_2 \rightarrow \bar{\lambda}_2 - z\bar{\lambda}_1.
\]

Under this shift the amplitude has poles when \( \langle 1p_a \rangle \rightarrow 0 \), of the form

\[
\frac{[p_a 1]}{(1p_a) (2p_a)} \langle 12 \rangle^6 \tilde{M}_{n-1}^{MHV}(\hat{K}^-, \hat{2}^-, \cdots [p_a] \cdots)
\]

where \([p_a]\) denotes the absence of leg \( p_a \) from the argument list and

\[
\hat{K} = \lambda_{p_a} \frac{(2|k_1 + p_a|)}{(2p_a)}.
\]
Applying the shift to (4.3) we find poles for the same values of \( z \). The residues of these poles receive contributions from terms where leg \( p_n \) lies within \( \hat{M}_1 \) and from terms where it lies within the soft lifting function. In the latter case the residue can be expressed in terms of an \( n - 1 \) point version of (4.3), while for the former case we use the \( n - 1 \) point version of (2.9). Combining the two we find precisely the required complex factorisation of the tree amplitude (1.6). Since the expression matches the tree amplitude for \( n = 3, 4 \) this guarantees it is correct for all \( n \). (The large \( z \) behaviour also follows that of the gravity tree amplitude (2.9).)

Equations (4.1) and (4.3) look like the first two members of a chain of relationships of which eq. (2.4) looks like the last. This is particularly true if we sum over \( p \) in eq. (4.1) and divide by \( n \). However we have been unable to construct other possible terms in such a sequence.

V. APPLICATION: RATIONAL TERMS IN \( \mathcal{N} = 4 \) ONE-LOOP AMPLITUDES

A one-loop graviton scattering amplitude can receive contributions from a range of particle types circulating in the loop. We denote the contribution from a particle of spin-\( s \) to the graviton scattering amplitude by \( M_n^{[s]} \) (with \( M_n^{[0]} \) representing a real scalar). In a supergravity theory there can be contributions from minimally coupled matter multiplets. The contributions to graviton scattering amplitudes from the various supergravity multiplets are \[24\]

\[
M_n^{\mathcal{N}=8} = M_n^{[2]} + 8 M_n^{[3/2]} + 28 M_n^{[1]} + 56 M_n^{[1/2]} + 70 M_n^{[0]}
\]

\[
M_n^{\mathcal{N}=6,\text{matter}} = M_n^{[3/2]} + 6 M_n^{[1]} + 15 M_n^{[1/2]} + 20 M_n^{[0]}
\]

\[
M_n^{\mathcal{N}=4} = M_n^{[2]} + 4 M_n^{[3/2]} + 6 M_n^{[1]} + 4 M_n^{[1/2]} + 2 M_n^{[0]}
\]

\[
M_n^{\mathcal{N}=4,\text{matter}} = M_n^{[1]} + 4 M_n^{[1/2]} + 6 M_n^{[0]}
\]

\[
M_n^{\mathcal{N}=1,\text{matter}} = M_n^{[1/2]} + 2 M_n^{[0]}
\]

(5.1)

In terms of the supersymmetric matter contributions the one-loop \( \mathcal{N} = 4 \) supergravity amplitude is

\[
M_n^{\mathcal{N}=4} = M_n^{\mathcal{N}=8} - 4 M_n^{\mathcal{N}=6,\text{matter}} + 2 M_n^{\mathcal{N}=4,\text{matter}}
\]

(5.2)

Extensions to the basic \( \mathcal{N} = 4 \) theory can be obtained from variants of this formula \[25\]. For example the case of “type I” theory which is the dimensional reduction of \( \mathcal{N} = 1 \) ten dimensional supergravity is

\[
M_n^{\mathcal{N}=4s} = M_n^{\mathcal{N}=8} - 4 M_n^{\mathcal{N}=6,\text{matter}} + 8 M_n^{\mathcal{N}=4,\text{matter}}
\]

(5.3)

and if the supergravity is coupled to a \( \mathcal{N} = 4 \) gauge theory with gauge group \( G \),

\[
M_n^{\mathcal{N}=4s,G} = M_n^{\mathcal{N}=8} - 4 M_n^{\mathcal{N}=6,\text{matter}} + (8 + \dim G) M_n^{\mathcal{N}=4,\text{matter}}
\]

(5.4)

A general \( n \)-point one-loop amplitude in a massless theory such as gravity or QCD can be expanded in terms of loop momentum integrals, \( I_m[P^d(\ell)] \), where \( m \) denotes the number of vertices in the loop and \( P^d(\ell) \) is a polynomial of degree \( d \) in the loop momentum \( \ell \). For gravity we expect \( d = 2m \) since the three-point vertex is quadratic in momentum. While for supergravity theories the naive expectation would be \( d = 2m - r \) where \( r = 8, 6, 4, 2 \) for \( \mathcal{N} = 8, 6, 4, 1 \) respectively \[21, 26\]. However, the evidence from explicit calculations suggests an effective degree for loop momentum polynomial \[23, 27, 30\] of

\[
d_{\text{eff}} = (m + 4) - r
\]

(5.5)
with \( r = 4 \) for \( \mathcal{N} = 4 \), \( r = 7 \) for \( \mathcal{N} = 6 \) and \( r = 8 \) for \( \mathcal{N} = 8 \). Performing a Passarino-Veltman \[31\] reduction on the loop momentum integrals yields an amplitude (to \( O(\epsilon) \) in the dimensional reduction parameter \( \epsilon \)),

\[
A_n^{1\text{loop}} = \sum_{i \in C} c_i I_4^i + \sum_{j \in D} d_j I_3^j + \sum_{k \in E} e_k I_2^k + R_n, \tag{5.6}
\]

where \( c_i, d_i, e_i \) and \( R_n \) are rational functions and the \( I_4, I_3, \) and \( I_2 \) are scalar box, triangle and bubble functions respectively. The mathematical form of these integral functions depends on whether the momenta flowing into a vertex are null (massless) or not (massive). For integrals with the \( d_{\text{eff}} \) of eq. (5.5), the expansion of (5.6) simplifies: \( \mathcal{N} = 8 \) amplitudes contain only box integral contributions \[23, 27\], \( \mathcal{N} = 6 \) amplitudes contain only box and triangle contributions, while for \( \mathcal{N} = 4 \) since \( d_{\text{eff}} = m \) the amplitude has the full spectrum of integral functions and rational terms. While this coincides with the power counting for Yang-Mills, supersymmetry imposes other simplifications on the \( \mathcal{N} = 4 \) supergravity amplitudes, in particular the vanishing of the “all-plus” and “single-minus” one-loop amplitudes which simplifies the factorisation structure.

In terms of the expansion (5.6), the \( n \)-graviton MHV amplitude for a \( \mathcal{N} = 4 \) matter multiplet is \[28\]

\[
M_n^{\mathcal{N}=4,\text{matter}}(1^-, 2^-, 3^+, \ldots, n^+) = \frac{(-1)^n}{8} (12)^8 \sum_{2 < a < b < n} \binom{\langle a \rangle \langle 2a \rangle \langle 1 b \rangle \langle 2 b \rangle}{\langle a b \rangle^2 (12)^2} h(a, M, b) h(b, N, a) \text{tr}^2[a M b N] I_4^{\mathcal{M} \mathcal{N}, \text{trunc}} + \sum_{1 \in A, 2 \in B} e_{A,B} I_2(K_2^{A}) + R_n, \tag{5.7}
\]

The summation over boxes is over subsets \( M, N \) such that \( 1 \in M, 2 \in N \) and \( M \cup N \cup \{a, b\} = \{1, \ldots n\} \). The summation over bubbles is over subsets \( A \) and \( B \) where \( 1 \in A, 2 \in B \), each contain at least one positive helicity leg and \( A \cup B = \{1, \ldots n\} \). The truncated box-functions are the specific combinations of the scalar box and triangle functions

\[
I_4^{\text{trunc}} = I_4 + \sum_j \tilde{b}_{ij} I_3^j \tag{5.8}
\]

which are IR and UV finite (see, for example, the appendix of ref. \[32\] for explicit expressions). Using truncated box functions automatically implements the constraints from IR and UV singularities \[33\] with the single remaining constraint \( \sum e_{A,B} = 0 \).

\[\text{FIG. 3: The box and bubble functions appearing in the } \mathcal{N} = 4 \text{ MHV one-loop amplitude}\]
The coefficients of the scalar bubbles, $e_{A,B}$, are presented explicitly in ref. [32] using canonical forms [34]. The precise form of these does not impact on the rational terms (unlike the box coefficients) so we do not present them here.

The $n$-point rational term, which completes the $n$-point amplitude $M(1^-, 2^-, 3^+, \ldots n^+)$ was proposed in ref. [3] and shown numerically to have the correct soft and collinear limits for $n \leq 10$,

$$R_n = (-1)^n \langle 1 2 \rangle^4 \left( \frac{R_0}{2} + \sum_{r=3}^{n-2} R_r \right)$$  \hspace{1cm} (5.9)

In the above

$$R_0^0 = \sum_{a,b \in P^+} R_n^{0,a,b}$$  \hspace{1cm} (5.10)

where

$$R_n^{0,a,b} = \sum_{1 \in M, 2 \in N} \frac{[a b]^2}{[a b]^2} h(a, M, b) h(b, N, a) \times \left( \langle 1 a \rangle \langle 2 a \rangle \langle 1 b \rangle \langle 2 b \rangle \right)^2$$  \hspace{1cm} (5.11)

and there is a contribution for each box integral function present in the amplitude. The $R_n^{0,a,b}$ contain spurious quadratic singularities, $(a b)^{-2}$, which are necessary to cancel those in the box integral contributions [32] and take the form

$$c_{box} \times \frac{s_{ab}^2}{2 \text{tr}(a M b N)^2}$$  \hspace{1cm} (5.12)

Note that this $R_n^{0,a,b}$-term taken together with the corresponding box integral contribution has no phase-singularity as $a$ and $b$ become collinear.

The remaining $R_n^r$ are

$$R_n^r = \sum_{P^r \subset P^+ \atop |P^r|=r} C_r[P^r] \times \hat{S}^{n-2-r} [P^+ - P^r], \quad r = 3, \ldots, n-2$$  \hspace{1cm} (5.13)

The sum is over all subsets $P^r$ of $P^+$ of length $r$ (of which there are $(n-2)!/r!/(n-2-r)!$) and $P^+ - P^r$ are the remaining positive helicity legs. The $C_r$ are

$$C_r[P^r] = \sum_{\text{perms}} \frac{[p_1 p_2] [p_2 p_3] \ldots [p_r p_1]}{\langle p_1 p_2 \rangle \langle p_2 p_3 \rangle \ldots \langle p_r p_1 \rangle}$$  \hspace{1cm} (5.14)

where the sum over permutations is over the $(r-1)!/2$ cyclically independent choices of orderings of $\{p_1, \ldots, p_r\}$ (we take the cycle $(p_1, p_2, \ldots, p_n, p_1)$ to be equivalent to the complete reversal $(p_n, p_{n-1}, \ldots, p_2, p_1)$). This implies the definition

$$C_2[\{p_1, p_2\}] = \frac{1}{2} \frac{[p_1 p_2]^2}{\langle p_1 p_2 \rangle^2}$$  \hspace{1cm} (5.15)

For example, the five point [28] and six-point expressions are

$$R_5 = - \langle 1 2 \rangle^4 \left( \frac{R_0}{2} + \frac{[3 4] [4 5] [5 3]}{\langle 3 4 \rangle \langle 4 5 \rangle \langle 5 3 \rangle} \right)$$

$$R_6 = \langle 1 2 \rangle^4 \left( \frac{R_0}{2} + \frac{[3 4] [4 5] [5 3]}{\langle 3 4 \rangle \langle 4 5 \rangle \langle 5 3 \rangle} \sum_{j \neq 6} \frac{[6 j] [1 j] [2 j]}{\langle 6 j \rangle \langle 1 6 \rangle \langle 2 6 \rangle} + \{6 \leftrightarrow 3, 4, 5\} \right)$$

$$+ \left( \frac{[3 4] [4 5] [5 6] [6 3]}{\langle 3 4 \rangle \langle 4 5 \rangle \langle 5 6 \rangle \langle 6 3 \rangle} + \frac{[3 4] [4 6] [6 5] [5 3]}{\langle 3 4 \rangle \langle 4 6 \rangle \langle 6 5 \rangle \langle 5 3 \rangle} + \frac{[3 5] [5 4] [4 6] [6 3]}{\langle 3 5 \rangle \langle 5 4 \rangle \langle 4 6 \rangle \langle 6 3 \rangle} \right)$$  \hspace{1cm} (5.16)
We will show analytically that $R_n$ has the correct soft and collinear limits. While our expression for $R_n$ is conjectural for $n > 5$, experience strongly suggests that expressions with the correct soft and collinear factorisations are likely to be correct. An important step, which we require to establish the collinear limits of $R_n$, is the identification

$$\frac{R_n^0}{2} = \sum_{\{p_1, p_2\} \subset P^+} C_2[\{p_1, p_2\}] \times \hat{S}^{n-4}[\{p_1, p_2\}; P^+ - \{p_1, p_2\}] \quad (5.17)$$

which we would naturally label as $R_n^2$. To do so we use an identity for the quadratic product of half-soft functions,

$$\sum_M h(a, M + c, b)h(b, N + d, a) = \sum_M h(c, M + a, d)h(d, N + b, c) \quad (5.18)$$

where the summation over $M$ is over all subsets of \{1, 2, \ldots, n\} - \{a, b, c, d\} and $N = \{1, 2, \ldots, n\} - \{a, b, c, d\} - M$. We have verified this identity at specific kinematic points for $n \leq 12$. Using this identity and the definition of the soft-lifting function eq.\(5.12\) we have

$$R_n^{0,p_1,p_2} = \frac{[p_1 p_2]^2}{\langle p_1 p_2 \rangle^2} \sum_M h(p_1, M + 1, p_2)h(p_2, N + 2, p_1) \times (\langle 1 p_1 \rangle \langle 2 p_1 \rangle \langle 1 p_2 \rangle \langle 2 p_2 \rangle)^2$$

$$= \frac{[p_1 p_2]^2}{\langle p_1 p_2 \rangle^2} \sum_M h(1, M + p_1, 2)h(2, N + p_2, 1) \times (\langle 1 p_1 \rangle \langle 2 p_1 \rangle \langle 1 p_2 \rangle \langle 2 p_2 \rangle)^2$$

$$= \frac{[p_1 p_2]^2}{\langle p_1 p_2 \rangle^2} \hat{S}^{n-4}[\{p_1, p_2\}; P^+ - \{p_1, p_2\}] = 2C_2[\{p_1, p_2\}] \times \hat{S}^{n-4}[\{p_1, p_2\}; P^+ - \{p_1, p_2\}] \quad (5.19)$$

Consequently, we can rewrite $R_n$ in the unified form

$$R_n = (-1)^n \langle 1 2 \rangle^4 \sum_{r=2}^{n-2} R_r^c \quad (5.20)$$

We now make the observation that this expression for the rational term $R_n$, remarkably, also has an algebraic diagrammatic expression akin to that for the MHV tree amplitude \[9\]. First, consider the cycle term $C_r$,

$$C_r[\{p_1, \ldots, p_r\}] = \sum_{\text{permutations}} \frac{[p_1 p_2] [p_2 p_3] \cdots [p_r p_1]}{\langle p_1 p_2 \rangle \langle p_2 p_3 \rangle \cdots \langle p_r p_1 \rangle} \quad (5.21)$$

where the sum is over all permutations of the $r$ positive helicity legs in the cycle. Each term in the sum may be interpreted as a one-loop link graph of the type shown in fig.\(4\) where all $r$ positive helicity legs lie in the loop. In general $R_r^c$ contains $C_r$ factors multiplied by soft-lifting functions. Previously we saw how the soft-lifting factors could be expressed in terms of link tree diagrams emanating from seed points \(3.8\). This motivates expressing $R_n$ as

$$R_n^{MHV} = (-1)^n \langle 1 2 \rangle^4 \sum_{\text{one-loop}} \left( \prod_{\text{edges :} a b} \frac{[a b]}{\langle a b \rangle} \right) \left( \prod_{\text{vertices :} a} (\langle a 1 \rangle \langle a 2 \rangle)^{\text{deg}(a)-2} \right) \quad (5.22)$$
where the sum is over all distinct, connected, one-loop link graphs involving vertices labelled by the \( n - 2 \) positive helicity legs. Terms within \( R_n^r \) correspond to graphs with \( r \) vertices in the loop.

\[ \sum \text{other terms} \]

**FIG. 4:** The loop part of a link diagram producing a term in \( R_n^r \). The \( r \) positive helicity legs of \( P^r \) lie in the loop.

The connection between fig. 4 and the corresponding term in \( C_r[P^r] \) is fairly clear when we note that \( \text{deg}(a) = 2 \) for each vertex and the sum over permutations is simply the sum over diagrams. In the \( R_n^r \) term the \( C_r \) is multiplied by \( \hat{S}^{n-2-r}[P^+ - P^r] \). The individual terms in the soft-lifting function correspond to individual diagrams of the type of fig. 2. Multiplying the two factors, individual terms will correspond exactly to diagrams where the trees of fig. 2 are attached to the loop of fig. 4 as in fig. 5.

\[ \text{Diagram} \]

**FIG. 5:** A link diagram corresponding to a term in \( C_{n-10} \times \hat{S}^8[P^{n-10}; Q^8] \)

We now present a diagrammatic proof that the rational terms have the correct soft and collinear limits. For the \( \mathcal{N} = 4 \) MHV amplitude several of these vanish since the one-loop amplitude with a single negative helicity leg vanishes in a supersymmetric theory. Consequently, there is no soft-singularity when one of the negative helicity legs vanishes since the target amplitude vanishes. Similarly, in the \( m^-b^+ \) collinear limit there is no \( S^{++} \) term. In the \( a^+b^+ \) collinear limit only the \( S^{++} \) splitting function is non-vanishing. Furthermore the MHV amplitude has no multi-particle poles for real momenta. Although the power-counting of \( \mathcal{N} = 4 \) supergravity is the same as Yang-Mills, the lack of amplitudes with a single negative helicity leg gives the MHV amplitude a simpler factorisation structure.

The soft and collinear limit constraints apply to whole amplitudes, but for our amplitude the integral function contributions and the rational terms have most of the appropriate limits independently. The exception to this is in the \( a^+b^+ \) limit where the \( C_2[\{a,b\}] \) terms combine with the box integral contributions [32].
A. Diagrammatic Proof: \( a^+ b^+ \)-Collinear Limit

We can use the diagrammatic representation to give a proof that \( R_n \) has the correct \( a^+, b^+ \) collinear limit. Consider the limit of the \( n \)-point amplitude when \( k_a \to zK, k_b \to \bar{z}K \), with \( \bar{z} = 1 - z \). Consider a diagram contributing to the \( n - 1 \) point amplitude and focus on the vertex labelled \( K \) as in fig. 6. This vertex may, or may not lie within a loop.

![Image](6.png)

**FIG. 6**: The \( K \)-vertex in a link diagram for the \( n - 1 \) point amplitude. The vertex may or may not be part of the loop.

In general this vertex will have \( n_K > 0 \) legs attached to it and will have an associated vertex factor
\[
\langle 1 K \rangle \langle 2 K \rangle^{n_K - 2} \tag{5.23}
\]
The \( n \)-point diagrams which contribute to this vertex in the collinear limit must have legs \( a \) and \( b \) connected by a single link. These are of the form show in fig. 7. The link contributes
\[
\frac{[a b]}{\langle a b \rangle} \to -z\bar{z}S^{++} \tag{5.24}
\]
with the other links smoothly going to those involving \( K \) in fig. 6.

![Image](7.png)

**FIG. 7**: The corresponding link diagrams in the \( n \)-point amplitude

In these diagrams the same set of initial legs must be attached to vertices \( a \) and \( b \). All possible combinations are present. In general \( n_a \) of the original legs will be attached to vertex
$a$ and $n_b$ to vertex $b$ with $n_a + n_b = n_K$. There will be $n_k! / n_a! / (n_K - n_a)!$ independent choices of which legs are attached. In the collinear limit the factors associated with the two vertices give

$$\left(\langle 1 a \rangle \langle 2 a \rangle \right)^{n_a-1} \left(\langle 1 b \rangle \langle 2 b \rangle \right)^{n_b-1} \frac{a|b}{z^{n_a-1} \bar{z}^{n_b-1}} \left(\langle 1 K \rangle \langle 2 K \rangle \right)^{n_a+n_b-2}$$  \hspace{1cm} (5.25)

Summing over all contributions gives the descendant diagram times a factor of

$$\sum_{n_a=0}^{n_K} \frac{n_K!}{n_a!(n_K - n_a)!} z^{n_a} \bar{z}^{n_K-n_a} \times -S_{-}^{++} = (z + \bar{z})^{n_K} \times -S_{-}^{++} = -S_{-}^{++}$$  \hspace{1cm} (5.26)

Note that this proof relies upon the fact that the loop-diagrams shown in fig. 8 do not contribute to the $a^+ b^+$ collinear limit but instead cancel against the box integral contributions in this limit [32].

Figure 8: These link diagrams do not contribute to $R_{n-1}^2$ in the $a^+ b^+$ collinear limit but cancel against terms arising from the box integral contribution.

**B. Diagrammatic Proof: Soft Limit**

We can also use the diagrammatic representation to show that $R_n$ has the correct soft limit as $k_n \to 0$. Consider a link-diagram of the $n - 1$ point amplitude and examine the set of $n$-point diagrams which might give this in the soft limit. In eq. (5.22) there is no soft-singularity from the links and the only possible singularity is from the vertex contribution when $\text{deg}(n) = 1$. These factors arise where a link from vertex $n$ is added to the $n - 1$ point diagram as in figure 9.

Figure 9: The diagrams contributing to the right-hand diagram in the soft limit of leg $n$. 

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In the soft limit, the diagram where leg $n$ is attached to vertex $j$ gives a contribution of
\[- \frac{[n \ j] \langle 1 \ j \rangle \langle 2 \ j \rangle}{\langle j \ n \rangle \langle 1 \ n \rangle \langle 2 \ n \rangle} \times (n - 1)\text{-point diagram} \tag{5.27}\]

Summing the different contributions gives the correct soft factor of eq. (2.3) with $a, b = 1, 2$.

C. Diagrammatic Proof: $m_1^{-}n^{+}$-Collinear Limit

Finally we consider the limit of the $n$-point amplitude when $k_{m_1} \to zK$, $k_n \to \bar{z}K$, with $\bar{z} = 1 - z$. Examining, eq. (5.22), we see that contributions only arise in this limit when leg $n$ is an isolated leg, much as the soft-leg is isolated in fig. 9. Each diagram gives a $\langle m_1 \ n \rangle^{-1}$ singularity but summing over the diagrams reduces this to a collinear phase singularity as required. The diagram where vertex $n$ is attached to vertex $j$ gives the descendant diagram times a factor of
\[- \frac{[n \ j] \langle m_1 \ j \rangle \langle m_2 \ j \rangle}{\langle j \ n \rangle \langle m_1 \ n \rangle \langle m_2 \ n \rangle} \tag{5.28}\]

Using the Schouten identity,
\[\langle m_1 \ j \rangle \langle n \ j \rangle = \langle m_1 \ X \rangle \langle n \ X \rangle + \langle m_1 \ n \rangle \langle j \ X \rangle \langle n \ j \rangle \langle n \ X \rangle \tag{5.29}\]

The second term cancels the singularity and these terms do not contribute. Summing over the remaining contribution from each diagram we have
\[\sum_{j \in P^{+\!-\!n}} \frac{[n \ j] \langle m_1 \ X \rangle \langle m_2 \ j \rangle}{\langle n \ X \rangle \langle m_1 \ n \rangle \langle m_2 \ n \rangle} = - \frac{\langle m_1 \ X \rangle}{\langle n \ X \rangle \langle m_1 \ n \rangle \langle m_2 \ n \rangle} \sum_{j \in P^{+\!-\!n}} [n \ j] \langle j \ m_2 \rangle \]
\[= - \frac{\langle m_1 \ X \rangle}{\langle n \ X \rangle \langle m_1 \ n \rangle \langle m_2 \ n \rangle} \sum_{j=m_1,m_2,n} [n \ j] \langle j \ m_2 \rangle \]
\[= - \frac{\langle m_1 \ X \rangle}{\langle n \ X \rangle \langle m_1 \ n \rangle \langle m_2 \ n \rangle} [n \ m_1] \langle m_1 \ m_2 \rangle \tag{5.30}\]

Now in the collinear limit we can see this has the collinear phase singularity
\[- \frac{m_1 \parallel}{(1 - z)} \frac{z [n \ m_1]}{\langle n \ m_1 \rangle} = - \frac{1}{z^2} S^+_{++} \tag{5.31}\]

Consequently, after including the pre-factor $\langle m_1 \ m_2 \rangle^4$, $R_n^r$ reduces to $R_{n-1}^r$ times a soft factor in the collinear limit,
\[(-1)^n \langle m_1 \ m_2 \rangle^4 R_n^r \to S^+_{++} \times (-1)^{n-1} \langle K \ m_2 \rangle^4 R_{n-1}^r. \tag{5.32}\]

This completes the proof that the expression for the rational term satisfies all real factorisation constraints. Although this is short of a full proof, experience strongly suggests that the expression is correct. Explicit computations using string based rules confirm the expression for $n = 4, 5$ [24, 28]. Confirmation of the result beyond $n = 5$ will probably require development of complex factorisation techniques [33] or use of the gravity-gauge relations [36, 37].
VI. CONCLUSIONS

We have shown how soft-lifting functions can be used to generate $n$-point MHV tree amplitudes and the rational pieces of the one-loop MHV $n$-graviton amplitude in $\mathcal{N} = 4$ supergravity. In the latter case we have explicitly checked that our ansatz has the correct behaviour in both the soft and collinear limits. In some ways this is implementing the ideas of the “inverse-soft” computations but we have applied it to one-loop computation and used real-soft factorisation functions. In this context soft-factorisation is not enough to generate the $n$-point term and we must also appeal to collinear factorisation properties.

Using the same functional rules as for the tree expression of \[9\], our analytic expression is in one-one correspondence with the set of one-loop connected link diagrams. The soft-lifting functions themselves have a diagrammatic representation akin to that for the MHV tree amplitudes.

In the language of the soft-lifting functions we might think of lifting some other low-point "seed" expression to obtain the corresponding $n$-point contribution. For example, the existence of seeds for next-to-MHV $\mathcal{N} = 4$ and $\mathcal{N} = 1$ MHV amplitudes would provide fascinating generalisations of this process.

The one-loop result is quite remarkable: although we might expect tree relations to extend to the integrands of loop expressions, there is no reason to expect them to survive the integration. Indeed, it is not evident why the one-loop link diagrams should give the rational terms of the $\mathcal{N} = 4$ theory rather than other supergravity theories. The original tree diagrams had an understanding in twistor space \[38\] but it is opaque as to why this would extend to $\mathcal{N} = 4$ one-loop amplitudes. It would be interesting to consider $\mathcal{N} = 4$ MHV amplitudes beyond one-loop however despite spectacular recent progress \[39, 40\] there are very few explicit supergravity calculations beyond one-loop.

[1] R.J. Eden, P.V. Landshoff, D.I. Olive, J.C. Polkinghorne, The Analytic S Matrix, (Cambridge University Press, 1966).
[2] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B 425 (1994) 217 hep-ph/9403226.
[3] Z. Bern, L. J. Dixon, D. C. Dunbar, D. A. Kosower, Nucl. Phys. B435 (1995) 59 hep-ph/9409265.
[4] Z. Bern, L. J. Dixon and D. A. Kosower, Nucl. Phys. B 513 (1998) 3 hep-ph/9708239.
[5] R. Britto, F. Cachazo and B. Feng, Nucl. Phys. B 725 (2005) 275 hep-th/0412103.
[6] R. Britto, F. Cachazo, B. Feng and E. Witten, Phys. Rev. Lett. 94 (2005) 181602 hep-th/0501052.
[7] Z. Bern and G. Chalmers, Nucl. Phys. B 447, 465 (1995) hep-ph/9503236.
[8] D. C. Dunbar, J. H. Ettle and W. B. Perkins, Phys. Rev. Lett. 108, (2012) 061603 [1111.1153 [hep-th]].
[9] D. Nguyen, M. Spradlin, A. Volovich and C. Wen, JHEP 1007 (2010) 045 arXiv:0907.2276 [hep-th].
[10] F. A. Berends, W. T. Giele and H. Kuijf, Phys. Lett. B 211, 91 (1988).
[11] M.T. Grisaru, H.N. Pendleton and P. van Nieuwenhuizen, Phys. Rev. D15:996 (1977); M.T. Grisaru and H.N. Pendleton, Nucl. Phys. B124:81 (1977); S.J. Parke and T. Taylor, Phys. Lett. 157B:81 (1985).
[12] Z. Bern, L. J. Dixon, D. C. Dunbar, M. Perelstein and J. S. Rozowsky, Nucl. Phys. B 530
(1998) 401 [hep-th/9802162].
[13] H. Kawai, D. C. Lewellen and S. H. H. Tye, Nucl. Phys. B 269 (1986) 1.
[14] L. J. Mason and D. Skinner, Commun. Math. Phys. 294 (2010) 827 [arXiv:0808.3907 [hep-th]].
[15] N. E. J. Bjerrum-Bohr, D. C. Dunbar, H. Ita, W. B. Perkins and K. Risager, JHEP 0601 (2006) 009 [hep-th/0509016].
[16] F. Cachazo, P. Svrček and E. Witten, JHEP 0409, 006 (2004) [hep-th/0403047].
[17] N. Arkani-Hamed, F. Cachazo, C. Cheung and J. Kaplan, JHEP 1003 (2010) 020 [arXiv:0907.5418 [hep-th]].
[18] Z. Bern, C. Boucher-Veronneau and A. J. Larkoski, JHEP 1109 (2011) 130 [arXiv:1108.5385 [hep-th]].
[19] M. Bullimore, JHEP 1101 (2011) 055 [arXiv:1008.3110 [hep-th]].
[20] J. Bedford, A. Brandhuber, B. J. Spence, G. Travaglini, Nucl. Phys. B712 (2005) 59 [hep-th/0412108];
F. Cachazo and P. Svrcek, [hep-th/0502160].
[21] A. Hodges, [arXiv:1108.2227 [hep-th]].
[22] J. J. Heckman and H. Verlinde, [arXiv:1112.5210 [hep-th]], [arXiv:1112.5209].
[23] Z. Bern, L. J. Dixon, M. Perelstein and J. S. Rozowsky, Nucl. Phys. B 546, 423 (1999) [hep-th/9811140].
[24] D. C. Dunbar and P.S. Norridge, Nucl. Phys. B 433, 181 (1995) [hep-th/9408014].
[25] D. C. Dunbar, B. Julia, D. Seminara and M. Trigiante, JHEP 0001, 046 (2000) [hep-th/9911158].
[26] Z. Bern, D.C. Dunbar and T. Shimada, Phys. Lett. B 312, 277, (1993) [hep-th/9307001].
[27] N. E. J. Bjerrum-Bohr, D. C. Dunbar, H. Ita, W. B. Perkins and K. Risager, JHEP 0612 (2006) 072 [arXiv:hep-th/0610043];
M. B. Green, J. H. Schwarz and L. Brink, Nucl. Phys. B 198 (1982) 474;
Z. Bern, N. E. J. Bjerrum-Bohr, D. C. Dunbar, JHEP 0505 (2005) 056. [hep-th/0501137];
N. E. J. Bjerrum-Bohr, D. C. Dunbar and H. Ita, Phys. Lett. B 621 (2005) 183 [hep-th/0503102].
[28] Z. Bern, J. J. Carrasco and H. Johansson, Phys. Rev. D 78 (2008) 085010 [arXiv:0803.3998 [hep-th]].
[29] Z. Bern, J. J. Carrasco, D. Forde, H. Ita and H. Johansson, Phys. Rev. D 77 (2008) 025010 [arXiv:0707.1035 [hep-th]].
[30] N. E. J. Bjerrum-Bohr and P. Vanhove, JHEP 0804 (2008) 065 [arXiv:0802.0868 [hep-th]].
[31] G. Passarino and M. Veltman, Nucl. Phys. B 160, 151, (1979).
[32] D. C. Dunbar, J. H. Ettle and W. B. Perkins, Phys. Rev. D 84 (2011) 125029 [arXiv:1109.4827 [hep-th]].
[33] D. C. Dunbar and P. S. Norridge, Class. Quant. Grav. 14 (1997) 351 [hep-th/9512084].
[34] D. C. Dunbar, W. B. Perkins and E. Warrick, JHEP 0906 (2009) 056 [arXiv:0903.1751 [hep-ph]].
[35] D. C. Dunbar, J. H. Ettle and W. B. Perkins, JHEP 1006 (2010) 027 [arXiv:1003.3398 [hep-th]].
[36] Z. Bern, J. J. M. Carrasco and H. Johansson, Phys. Rev. D 78 (2008) 085011 [arXiv:0805.3993 [hep-ph]], Phys. Rev. Lett. 105 (2010) 061602 [arXiv:1004.0476 [hep-th]]; Z. Bern, T. Dennen, Y. -t. Huang, M. Kiermaier, Phys. Rev. D 82 (2010) 065003.
[37] Z. Bern, C. Boucher-Veronneau and H. Johansson, Phys. Rev. D 84 (2011) 105035 [arXiv:1107.1935 [hep-th]].
[38] N. Arkani-Hamed, F. Cachazo, C. Cheung and J. Kaplan, JHEP 1003 (2010) 110
[39] Z. Bern, S. Davies, T. Dennen and Y.-t. Huang, arXiv:1202.3423 [hep-th].
[40] C. Boucher-Veronneau and L. J. Dixon, JHEP 1112 (2011) 046 arXiv:1110.1132 [hep-th].