Research Article
Hypersurfaces with Null Higher Order Anisotropic Mean Curvature

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Given a positive function $F$ on $S^n$ which satisfies a convexity condition, for $1 \leq r \leq n$, we define for hypersurfaces in $\mathbb{R}^{n+1}$ the $r$th anisotropic mean curvature function $H_{r,F}$, a generalization of the usual $r$th mean curvature function. We call a hypersurface anisotropic minimal if $H_{r,F} = H_{r,1,F} = 0$, and anisotropic $r$-minimal if $H_{r+1,F} = 0$. Let $W$ be the set of points which are omitted by the hyperplanes tangent to $M$. We will prove that if an oriented hypersurface $M$ is anisotropic minimal, and the set $W$ is open and nonempty, then $x(M)$ is a part of a hyperplane of $\mathbb{R}^{n+1}$. We also prove that if an oriented hypersurface $M$ is anisotropic $r$-minimal and its $r$th anisotropic mean curvature $H_{r,F}$ is nonzero everywhere, and the set $W$ is open and nonempty, then $M$ has anisotropic relative nullity $n - r$.

1. Introduction

Let $F : S^n \rightarrow \mathbb{R}^n$ be a smooth function which satisfies the following convexity condition:

$$\left(D^2 F + FI\right) x > 0, \quad \forall x \in S^n,$$

where $S^n$ is the standard unit sphere in $\mathbb{R}^{n+1}, D^2 F$ denotes the intrinsic Hessian of $F$ on $S^n$, $I$ denotes the identity on $T_x S^n$, and $> 0$ means that the matrix is positive definite. We consider the map

$$\phi : S^n \rightarrow \mathbb{R}^{n+1},
\quad x \rightarrow F(x)x + \left(\text{grad}_{S^n} F\right)_x;$$

its image $W_F = \phi(S^n)$ is a smooth, convex hypersurface in $\mathbb{R}^{n+1}$ called the Wulff shape of $F$ (see [1–9]). When $F \equiv 1$, the Wulff shape $W_F$ is just $S^n$.

Now let $x : M \rightarrow \mathbb{R}^{n+1}$ be a smooth immersion of an oriented hypersurface. Let $N : M \rightarrow S^n$ denote its Gauss map. The map $v = \phi \circ N : M \rightarrow W_F$ is called the anisotropic Gauss map of $x$.

Let $S_F = -d\nu$. $S_F$ is called the $F$-Weingarten operator, and the eigenvalues of $S_F$ are called anisotropic principal curvatures. Let $\sigma_r$ be the elementary symmetric functions of the anisotropic principal curvatures $\kappa_1, \kappa_2, \ldots, \kappa_n$:

$$\sigma_r = \sum_{i_1 < i_2 < \ldots < i_r} \kappa_{i_1} \cdots \kappa_{i_r} \quad (1 \leq r \leq n).$$

We set $\sigma_0 = 1$. The $r$th anisotropic mean curvature $H_{r,F}$ is defined by $H_{r,F} = \sigma_r/C^n$, also see Reilly [10]. $H_{1,F} : = H_{1,1,F}$ is called the anisotropic mean curvature. When $F \equiv 1, S_F$ is just the Weingarten operator of hypersurfaces, and $H_{r,F}$ is just the $r$th mean curvature $H_r$ of hypersurfaces which has been studied by many authors (see [11–14]). Thus, the $r$th anisotropic mean curvature $H_{r,F}$ generalizes the $r$th mean curvature $H_r$ of hypersurfaces in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$.

We say that $x : M \rightarrow \mathbb{R}^{n+1}$ is anisotropic $r$-minimal if $H_{r+1,F} = 0$.

For $p \in M$, we define $v(p) = \dim \ker(S_F)$. We call $v = \min_{p \in M} v(p)$ the anisotropic relative nullity; it generalized the usual relative nullity.
For a smooth immersion $x : M \rightarrow Q^{n+1}_c$ of a hypersurface into an $(n+1)$-dimensional space form with constant sectional curvature $c$, we denote by
\[ W = Q^{n+1}_c - \bigcup_{p \in M} (Q^n_c)_p, \tag{4} \]
where for every $p \in M$, $(Q^n_c)_p$ is the totally geodesic hyper-surface of $Q^{n+1}_c$ tangent to $x(M)$ at $x(p)$. So, in the case of $c = 0$, $W$ is the set of points which are omitted by the hyperplanes tangent to $x(M)$.

We will study immersion with $W$ nonempty. In this direction, Hasanis and Koutroufiotis (see [15]) proved the following.

**Theorem 1.** Let $x : M \rightarrow Q^3_c$ be a complete minimal immersion with $c \geq 0$. If $W$ is nonempty, then $x$ is totally geodesic.

Later, in [16], Alencar and Frensel extended the result above assuming an extra condition. They proved the following.

**Theorem 2.** Let $x : M \rightarrow Q^3_c$ be an oriented, minimally immersed hypersurface. If $W$ is open and nonempty, then $x$ is totally geodesic.

In [17], Alencar and Batista studied hypersurfaces with null higher order mean curvature; they proved the following.

**Theorem 3.** Let $M$ be a complete and orientable Riemannian manifold and let $x : M \rightarrow Q^n_{c+1}$ be an isometric immersion with $H_{n+1} = 0$ and $H_{r} \neq 0$ everywhere, $r \geq 1$. If $W$ is open and nonempty, then the relative nullity $\nu = n - r$.

We note that, Alencar in [18] provides examples of nontotally geodesic minimal hypersurfaces in $\mathbb{R}^{2n}$, $n \geq 4$, with nonempty $W$; in [17], Alencar and Batista provides examples of $1$-minimal hypersurfaces in $\mathbb{H}^n$ with $H_{r} \neq 0$ everywhere in $\mathbb{R}^{2n}$, $n \geq 5$, with nonempty $W$ but $\nu \neq n - 1$. These examples show that it is necessary to add an extra hypothesis.

In this paper, we prove the anisotropic version of Theorems 2 and 3 for an immersion $x : M \rightarrow \mathbb{R}^{n+1}$. Explicitly, we prove the following two theorems.

**Theorem 4.** Let $x : M \rightarrow \mathbb{R}^{n+1}$ be an oriented, anisotropic minimally immersed hypersurface. If $W$ is open and nonempty, then $x(M)$ is a part of a hyperplane of $\mathbb{R}^{n+1}$.

**Theorem 5.** Let $x : M \rightarrow \mathbb{R}^{n+1}$ be an oriented immersed hypersurface with $H_{n+1} = 0$ and $H_{r} \neq 0$ everywhere, $r \geq 1$. If $W$ is open and nonempty, then the anisotropic relative nullity $\nu = n - r$.

### 2. Preliminaries

In this paper, we use the summation convention of Einstein and the following convention of index ranges unless otherwise stated:
\[ 1 \leq i, j, \ldots \leq n; \quad 1 \leq \alpha, \beta, \ldots \leq n + 1. \tag{5} \]

We define $F^* : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ to be
\[ F^*(y) = \frac{1}{F(z)} \sup \left\{ (y,z) \mid z \in \mathbb{R}^{n+1} \setminus \{0\} \right\}; \tag{6} \]
then $F^*$ is a Minkowski norm on $\mathbb{R}^{n+1}$. In fact, as proved in [19], $F^* : \mathbb{R}^{n+1} \setminus \{0\} \rightarrow \mathbb{R}$ is smooth and we have the following.

**Proposition 6.** (1) $F^*(y) > 0$, for all $y \in \mathbb{R}^{n+1} \setminus \{0\}$; (2) $F^*(ty) = tF^*(y)$, for all $y \in \mathbb{R}^{n+1}$, $t > 0$; (3) $F^*(y+z) \leq F^*(y) + F^*(z)$, for all $y, z \in \mathbb{R}^{n+1}$, and the equality holds if and only if $y = 0$, or $z = 0$ or $y = kz$ for some $k > 0$. (4) $W_F = \{ y \in \mathbb{R}^{n+1} \mid F^*(y) = 1 \}$.

We define
\[ \overline{g}_{\alpha\beta}(y) = \frac{1}{2} \frac{\partial^2 (F^*)^2}{\partial y^\alpha \partial y^\beta} (y), \tag{7} \]
\[ g_{\gamma}(X,Y) = \overline{g}_{\alpha\beta}(y) X^\gamma Y^\beta, \]
where $y \in \mathbb{R}^{n+1} \setminus \{0\}$ and $X = (X^1, X^2, \ldots, X^{n+1})$, $Y = (Y^1, Y^2, \ldots, Y^{n+1}) \in T_y \mathbb{R}^{n+1} \equiv \mathbb{R}^{n+1}$.

From the Euler’s theorem for homogeneous functions, we have
\[ \frac{\partial \overline{g}_{\alpha\beta}}{\partial y^\gamma} (z) z^\beta = \frac{1}{2} \frac{\partial^3 (F^*)^2}{\partial y^\alpha \partial y^\beta \partial y^\gamma} (z) z^\beta = 0, \tag{8} \]
where $z = (z^1, z^2, \ldots, z^{n+1}) \in \mathbb{R}^{n+1} \setminus \{0\}$. Thus,
\[ \frac{\partial g_{\gamma}(X,Z)}{\partial y^\gamma} = \overline{g}_{\alpha\beta}(z) \frac{\partial X^\alpha}{\partial y^\gamma} z^\beta + \overline{g}_{\alpha\gamma}(z) X^\alpha \frac{\partial z^\beta}{\partial y^\gamma}, \tag{9} \]
where $z = (z^1, z^2, \ldots, z^{n+1}) \in T_y \mathbb{R}^{n+1}$ is nonzero everywhere and $X = (X^1, X^2, \ldots, X^{n+1}) \in T_y \mathbb{R}^{n+1}$.

As $F^*$ is a Minkowski norm on $\mathbb{R}^{n+1}$, the following lemma holds (see [20, 21]).

**Lemma 7.** For any $y \in \mathbb{R}^{n+1} \setminus \{0\}$ and $u \in \mathbb{R}^{n+1}$ one has
\[ g_{\gamma}(y,z) \leq F^*(y) F^*(z), \tag{10} \]
and the equality holds if and only if there exists $t \geq 0$ such that $z = ty$.

Let $x : M \rightarrow \mathbb{R}^{n+1}$ be an oriented hypersurface in the Euclidean space $\mathbb{R}^{n+1}$. Let $v : M \rightarrow W_F$ denote its anisotropic Gauss map. Then for any $p \in M$, $v(p)$ is perpendicular to $x_c(T_p M)$ with respect to the inner product $g(v(p))$ and $F^*(v(p)) = 1$. Thus, we call $v(p)$ an anisotropic unit normal vector of $T_p M$.

### 3. A Connection on Hypersurfaces of Minkowski Space

Let $x : M \rightarrow \mathbb{R}^{n+1}$ be an oriented hypersurface in the Euclidean space $\mathbb{R}^{n+1}$ and denote $v : M \rightarrow W_F$ its anisotropic Gauss map.
Let $\nabla$ be the standard connection on the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. For vector fields $X, Y$ on $M$, we decompose $\nabla_X Y$ as the tangent part $\nabla_X Y$ and the anisotropic normal part $\Pi (X, Y)_\nu$ with respect to the inner product $g_\nu$. That is,

$$\nabla_X Y = \nabla_X Y + \Pi (X, Y)_\nu, \quad (11)$$

where $g_\nu(\nabla_X Y, \nu) = 0$.

We also have the Weingarten formula:

$$\nabla_X \nu = -S_F X, \quad g_\nu (S_F X, Y) = \Pi (X, Y), \quad (12)$$

where we have used (9).

It is easy to verify that $\nabla$ is a torsion free connection on $M$ and $\Pi$ is a symmetric second order covariant tensor field on $M$. We call $\Pi$ the anisotropic second fundamental form.

Let $\{e_1\}^n_{i=1}$ be a local frame of $M$ and $\{\omega^i\}^n_{i=1}$ its dual frame. Let $g_{ij} = g(e_i, e_j), \nabla e_i = \omega^j \otimes e_j, \Pi(e_i, e_j) = h_{ij}, h_{ij} = g^{ik} h_{ki}$, where $(g^{ik})$ is the inverse matrix of $(g_{ij})$. Then we have

$$d x = \omega^j e_j, \quad (13)$$

$$d e_i = \omega^j_i e_j + h_{ij} \omega^i_j, \quad (14)$$

$$d \nu = -h^i_j \omega^j_i e_j, \quad (15)$$

Differentiating (13) and using (14), we get

$$d \omega^i = \omega^i \wedge \omega^j, \quad (16)$$

$$h_{ij} = h_{ji}, \quad (17)$$

Differentiating (14) and using (14)-(15), we get

$$d h_{ij} = -h_{ik} h_{kj}^{;} \omega^k \wedge \omega^j, \quad (18)$$

where

$$d h_{ij} = -h_{ik} \omega^k \wedge h_{kj}^{;}, \quad (19)$$

and $R_{ik}^{;j} = -R_{ik}^{;j} = h_{ik} h_{kj}^{;} - h_{kj}^{;} h_{ik}^{;}$.

Differentiating (15) and using (14), we get

$$d h_{ij} = h_{i,j}^{;} - h_{j,i}^{;} \omega^i \wedge \omega^j, \quad (20)$$

Note that $h_{ij}^{;}$ is the matrix of the $F$-Weingarten operator $S_F = -d \nu$, its eigenvalues are called the anisotropic principal curvatures, and we denote them by $\kappa_1, \ldots, \kappa_n$.

We have $n$ invariants, the elementary symmetric function $\sigma_r$ of the anisotropic principal curvatures:

$$\sigma_r = \sum_{i_1 < \ldots < i_r} \kappa_{i_1} \ldots \kappa_{i_r} \quad (1 \leq r \leq n). \quad (21)$$

For convenience, we set $\sigma_0 = 1$. The $r$th anisotropic mean curvature $H_{r,F}$ is defined by

$$H_{r,F} = \frac{\sigma_r}{C_n}, \quad C_n = \frac{n!}{r!(n-r)!}. \quad (22)$$

Using the characteristic polynomial of $S_F$, $\sigma_r$ is defined by

$$\det (t I - S_F) = \sum_{r=0}^n (-1)^r \sigma_r t^{n-r}. \quad (23)$$

So, we have

$$\sigma_r = \frac{1}{r!} \sum_{i_1, \ldots, i_r} \delta_{i_1 \ldots i_r}^{j_1 \ldots j_r} h_{j_1}^{;} \ldots h_{j_r}^{;}, \quad (24)$$

where $\delta_{i_1 \ldots i_r}^{j_1 \ldots j_r}$ is the usual generalized Kronecker symbol; that is, $\delta_{i_1 \ldots i_r}^{j_1 \ldots j_r}$ equals $+1$ (resp., $-1$) if $i_1 \ldots i_r$ are distinct and $(j_1 \ldots j_r)$ is an even (resp., odd) permutation of $(i_1 \ldots i_r)$ and in other cases it equals zero.

**Definition 8.** Let $f : M \to \mathbb{R}$ be a smooth function. One defines the gradient (with respect to the induced metric $g_\nu$ on $M$) $\text{grad } f$ of the function $f$ by

$$g_\nu (\text{grad } f, X) = X (f), \quad (25)$$

where $X$ is any smooth vector field on $M$.

Define $f_i$ by $df = f_i \omega^i$; then

$$\text{grad } f = g_i \omega^i, \quad (26)$$

We define

$$d V = |e_1, \ldots, e_n| \omega^1 \wedge \ldots \wedge \omega^n, \quad (27)$$

where $|e_1, \ldots, e_n|$ is the determinant of the matrix $(e_1, \ldots, e_n, \nu)$. Then $d V$ is a volume element on $M$.

**Definition 9.** Let $X$ be a smooth vector field on $M$. One defines the divergence (with respect to the volume element $dV$) $\text{div } X$ by $d \langle i(X) dV \rangle = \langle dV(X, Y) \rangle, \forall Y \in \mathcal{X}(M).$

**Lemma 10.** Let $X = X^i e_i$; then $\text{div } X = X^i$, where

$$d X^i + X^j \omega^i_j = X^i \omega^i_j. \quad (29)$$

**Proof.** By (14), (15), we get

$$d |e_1, \ldots, e_n| = \omega^i_j |e_1, \ldots, e_n, | \omega^i_j. \quad (30)$$

From the definition of $i(X)$, we have

$$i (X) dV = \sum_i (-1)^{i+1} X_i |e_1, \ldots, e_n, | \omega^1 \wedge \ldots \wedge \omega^n. \quad (31)$$
So,
\[
d (i(X) dV) = \sum_i (-1)^{i+1} (dX^i) \wedge [e_1, \ldots, e_{n}, y] \omega^i \\
\wedge \cdots \wedge \omega^n \\
+ \sum_{j=2} (-1)^{j+1} X^j \left( d [e_1, \ldots, e_{n}, y] \right) \\
\wedge \omega^1 \wedge \cdots \wedge \omega^j \wedge \cdots \wedge \omega^n \\
+ \sum_{j=i+1} (-1)^{j+1} X^j \left( d [e_1, \ldots, e_{n}, y] \right) d\omega^j \wedge \omega^i \\
\wedge \omega^1 \wedge \cdots \wedge \omega^j \wedge \cdots \wedge \omega^n \\
= X^i_j dV.
\] (32)

4. \( L_{r,F} \) Operator for Hypersurfaces

We introduce the Newton transformation defined by
\[
P_r = \sigma_r I - \sigma_{r-1} S_F + \cdots + (-1)^r S_F^r, \quad r = 0, \ldots, n; \quad (33)
\]
then
\[
P_0 = I, \quad P_n = 0, \quad P_r = \sigma_r I - P_{r-1} S_F. \quad (34)
\]

Lemma 11. The matrix of \( P_r \) is given by:
\[
(P_r)_{ij}^j = \frac{1}{r!} \delta^{i-j}_{1-j} h_1^j \cdots h_r^j. \quad (35)
\]

Proof. We prove Lemma II inductively. For \( r = 0 \), it is easy to check that (35) is true.

We can check directly
\[
\delta^{i-j}_{1-j} = \begin{vmatrix} 
\delta_{i_1}^{j_1} & \delta_{i_2}^{j_2} & \cdots & \delta_{i_{r-1}}^{j_{r-1}} & \delta_{i_r}^{j_r} \\
\delta_{i_1}^{j_1} & \delta_{i_2}^{j_2} & \cdots & \delta_{i_{r-1}}^{j_{r-1}} & \delta_{i_r}^{j_r} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\delta_{i_1}^{j_1} & \delta_{i_2}^{j_2} & \cdots & \delta_{i_{r-1}}^{j_{r-1}} & \delta_{i_r}^{j_r} \\
\delta_{i_1}^{j_1} & \delta_{i_2}^{j_2} & \cdots & \delta_{i_{r-1}}^{j_{r-1}} & \delta_{i_r}^{j_r} \\
\end{vmatrix}.
\] (36)

Assume that (35) is true for \( r = k \), we only need to show that it is also true for \( r = k + 1 \). For \( r = k + 1 \), using (24) and (36), we have
\[
\text{RHS of (35)} = \frac{1}{(k+1)!} \sum_{i_{k+1}} \delta^{i_{k+1}} h_1^{i_{k+1}} \cdots h_r^{i_{k+1}}
\]
\[
= \frac{1}{(k+1)!} \sum_{i_{k+1}} \left( \delta^{i_{k+1}} h_1^{i_{k+1}} - \delta^{i_{k+1}} h_1^{i_{k+1}} \cdots h_r^{i_{k+1}} + \cdots \right) h_1^{i_{k+1}} \cdots h_r^{i_{k+1}}
\]
\[
= \sigma_k h_i^{i_{k+1}} - \frac{1}{(k+1)!} \sum_{i_{k+1}} \delta^{i_{k+1}} h_1^{i_{k+1}} \cdots h_r^{i_{k+1}} + \cdots
\]
\[
= \delta^{i_{k+1}} - \frac{1}{(k+1)!} \sum_{i_{k+1}} \left( P_{k+1}^{i_{k+1}} \right)^j h_j^{i_{k+1}}
\] (37)

Lemma 12. For each \( r \), one has
(a) \( (P_r)_{ij}^j = 0 \);
(b) Trace\((P_r S_F^r) = (r + 1) \sigma_r \);
(c) Trace\((P_r) = (n - r) \sigma_r \);
(d) Trace\((P_r S_F^r) = \sigma_r \sigma_{r+1} - (r + 2) \sigma_{r+2} \).

Proof. (a) Noting \( (j, j) \) is skew symmetric in \( \delta^{i-j}_{1-j} \) and \( (j, j) \) is symmetric in \( h_1^j \cdots h_r^j \) (from (19)), we have
\[
\sum_j (P_r)_{ij}^j = \frac{1}{(r-1)!} \sum_{i_{r-1}, \ldots, i_1} \delta^{i-j}_{1-j} h_1^j \cdots h_r^j = 0. \quad (38)
\]
(b) Using (35) and (24), we have
\[
\text{Trace} (P_r S_F^r) = \sum_{ij} (P_r)_{ij}^j h_j^i
\]
\[
= \frac{1}{r!} \sum_{i_{r-1}, \ldots, i_1} \delta^{i-j}_{1-j} h_1^j \cdots h_r^j \quad (39)
\]
\[
= (r + 1) \sigma_r. \quad (40)
\]
(c) Using (b) and the definition of \( P_r \), we have
\[
\text{Trace} (P_r) = \text{tr} (\sigma_r I) - \text{tr} (P_{r-1} S_F) = n \sigma_r - r \sigma_r = (n - r) \sigma_r. \quad (41)
\]
(d) Using (b) and the definition of \( P_{r+1} \), we have
\[
\text{Trace} (P_r S_F^r) = \text{Trace} (\sigma_{r+1} S_F) - \text{Trace} (P_{r+1} S_F)
\]
\[
= \sigma_r \sigma_{r+1} - (r + 2) \sigma_{r+2}. \quad (42)
\]
Remark 13. When $F = 1$, Lemma 12 was a well-known result (e.g., see Barbosa and Colares [22], or Reilly [23]).

Lemma 14. One has

$$ (\sigma_r)_k = \sum_{i,j} (P_{r-1})_{i,j}^j h_{i,j}^k. \quad (42) $$

Proof. From the definition of $\sigma_r$, we have the following calculation:

$$ (\sigma_r)_k = \frac{1}{r!} \sum_{i_1,\ldots,i_r, j_1,\ldots,j_r} \delta_{i_1,j_1}^1 \cdots \delta_{i_r,j_r}^r (h_{i_1}^1 \cdots h_{i_r}^r)_k \\
= \frac{1}{(r-1)!} \sum_{i_1,\ldots,i_{r-1}, j_1,\ldots,j_{r-1}} \delta_{i_1,j_1}^1 \cdots h_{i_r}^r \cdots h_{i_k}^k \\
= \sum_{i,j} (P_{r-1})_{i,j}^j h_{i,j}^k. \quad (43) $$

We define an operator $L_{r,F} : C^\infty(M) \to C^\infty(M)$ by

$$ L_{r,F} (f) = \text{div} \left( P_r \nu \gamma \right). \quad (44) $$

In the sequel, we will need the following lemma. Item (a) is essentially the content of Lemma 1.1 and Equation (1.3) in [24], while item (b) is quoted as Proposition 1.5 in [25].

Lemma 15. Let $x : M \to \mathbb{R}^{n+1}$ be an oriented hypersurface, and $0 \leq r \leq n-1$, $p \in M$.

(a) If $\sigma_{r+1}(p) = 0$, then $P_r$ is semidefinite at $p$;
(b) if $\sigma_{r+1}(p) = 0$ and $\sigma_{r+2}(p) \neq 0$, then $P_r$ is definite at $p$.

Another important result is as follows (see [26]).

Lemma 16. Let $x : M \to \mathbb{R}^{n+1}$ be an oriented hypersurface, and $p \in M$.

(a) For $1 \leq r \leq n$, one has $H_{r+1,F}^2 \geq H_{r,F} H_{r+1,F}$. Moreover, if equality happens for $r = 1$ or for some $1 < r < n$, with $H_{r+1,F} \neq 0$ in this case, then $p$ is an anisotropic umbilical point (i.e. $\kappa_1(p) = \kappa_2(p) = \cdots = \kappa_n(p)$);
(b) if, for some $1 \leq r < n$, one has $H_{r,F} = H_{r+1,F} = 0$, then $H_{r+2,F} = 0$ for all $r \leq j \leq n$. In particular, at most $r - 1$ of the anisotropic principal curvatures are different from zero.

The result below is standard, so we omit the proof.

Lemma 17. Let $x : M \to \mathbb{R}^{n+1}$ be an oriented hypersurface. The operator $L_{r,F}$ associated to the immersion $x$ is elliptic if and only if $P_r$ is positive definite.

Definition 18. Let $f : M \to \mathbb{R}$ be a smooth function. The Laplacian $\Delta f$ is defined by $\Delta f := L_{0,F} f = \text{div} (\text{grad} f)$.

It is easy to see that $\Delta$ is an elliptic differential operator.
Lemma 20. For $0 \leq r \leq n-1$, one has the following
\[ L_{r,F}u = -g_r(\nabla \sigma_{r+1}, x) - (r + 1) \sigma_{r+1} - (\sigma_1 \sigma_{r+1} - (r+2) \sigma_2). \]  
(52)

Remark 21. Recall $\sigma_1 = nH_F$ and $\|I\|^2 = \sigma_1^2 - 2\sigma_2$; let $r = 0$ in (52); we get
\[ \Delta u = -n(H_F + g_r(\text{grad} H_F, x)) - \|I\|^2 u. \]  
(53)

5. Proof of Theorems 4 and 5

We fix a point $o \in W$ as the origin of $\mathbb{R}^{n+1}$. Without loss of generality, we assume, for each $p \in M$, $\nu(p)$ is the anisotropic unit normal vector of $x(M)$ at $x(p)$ such that \( \langle x(p), \nu(p) \rangle \nu(p) > 0 \) (otherwise we consider the function $-u$ instead). This gives an orientation to $M$; indeed, the component of the position vector $x$ perpendicular (with respect to the inner product $g_2$) to $M$ defines a never zero, anisotropic normal, vector field on $M$, such that the support function $u = \langle x(p), \nu(p) \rangle \nu(p)$ is positive on $M$.

5.1. Proof of Theorem 4. Since $x$ is anisotropic minimal, from (53) we get
\[ \Delta u = -\|I\|^2 u \leq 0, \quad \text{on } M. \]  
(54)

Let $u_* = \inf_M u$. We claim that $u_*$ is attained at some point $x_0 \in M$. Consider a sequence $\{x_k\} \subset M$ such that $u(x_k) \to u_*$ as $k \to +\infty$. To each $x_k$ we associate $y_k = u(x_k)\nu(x_k)$; then $y_k \in T_{x_k}^* M$. Since $\|y_k\|_{2^{n+1}} = u(x_k)\|\nu(x_k)\|_{2^{n+1}}$ is bounded, there exists a subsequence, which again we call $\{y_k\}$, such that $y_k \to y_0$ for some $y_0 \in \mathbb{R}^{n+1}$. Since $\bigcup_{p \in M} T_p^* M$ is closed and $\{y_k\} \subset \bigcup_{p \in M} T_p^* M$ we deduce that $y_0 \in T_{x_0}^* M$ for some $x_0 \in M$. Thus, by the continuity of $F^*$ and Lemma 7,
\[ u_* = \lim_{k \to -\infty} u(x_k) = \lim_{k \to +\infty} F^*(y_k) \]  
(55)

\[ = F^*(y_0) \geq g(x_0)(y_0, \nu(x_0)) = u(x_0), \]
so $u^* = u(x_0)$ as needed. Now, from the usual maximum principle $u$ is constant, $u = u_* = u(x_0) > 0$. From (54) we then have $\Delta u \equiv 0$ and $x$ is totally geodesic.

5.2. Proof of Theorem 5. Since $H_{r+1,F} = 0$, from Lemma 20 we get
\[ L_{r,F}u = (r + 2) \sigma_{r+1}u. \]  
(56)

Using Lemma 15(a) we have that $P_r$ is semidefinite. Since $H_{r,F}$ does not vanish, we have that $H_{r,F}$ is positive or negative, because $c(r)H_{r,F} = \text{Trace}(P_r)$, where $c(r) = (n-r)C_n$. Now we use Lemma 16 and obtain the following:
\[ 0 = H_{r+1,F}^2 \geq H_{r,F}H_{r+2,F}. \]  
(57)

Using the information above, we claim that $H_{r+2,F} \equiv 0$.

Case (i) $(H_{r,F} > 0)$. In this case, $P_r$ is positive definite, and $L_{r,F}$ is elliptic by Lemma 17. Using (57) we conclude that $H_{r+2,F} \leq 0$. Whereas from (56) we have
\[ L_{r,F}u \leq 0. \]  
(58)

Following exactly the proof as in Theorem 4, we conclude that $u$ is constant, $u = u_* = u(x_0) > 0$. From (56) we then have $H_{r+2,F} \equiv 0$.

Case (ii) $(H_{r,F} < 0)$. In this case, $P_r$ is negative definite, and $-L_{r,F}$ is elliptic by Lemma 17. Using (57) we conclude that $H_{r+2,F} \geq 0$. Whereas from (56) we have
\[ -L_{r,F}u \leq 0. \]  
(59)

Now, following exactly the proof as in Theorem 4, we conclude that $u$ is constant, $u = u_* = u(x_0) > 0$. From (56) we then have $H_{r+2,F} \equiv 0$.

Thus we conclude that $H_{r+2,F} \equiv 0$. Now, we use Lemma 16(b) to conclude that $H_{j,F} = 0$ for $j \geq r + 1$ and so that $v \geq n - r$. Since $H_{r,F}$ does not change sign we have that $v = n - r$.

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