Novel Conditions on the Non-Normal Cayley Graphs of Valency Six

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Abstract
A Cayley graph \( X = \text{Cay}(G, S) \) on a group \( G \) is said to be normal if the right regular representation \( R(G) \) of \( G \) is normal in the full automorphism group \( \text{Aut}(X) \). In this paper, two novel conditions are outlined to identify the non-normal Cayley graphs of valency six. As an application, some non-normal Cayley graphs of valency six on \( A_4 \) and \( A_5 \) are obtained.

Keywords: Automorphism Groups, Cayley Graph, Normal Cayley Graph

1. Introduction
Let \( X \) be a finite simple undirected graph, we use \( V(X) \), \( E(X) \), \( A(X) \) and \( \text{Aut}(X) \) to denote its vertex set, edge set, arc set and full automorphism group, respectively. For every \( u,v \in V(X) \), \( \{u, v\} \) is the edge incident to \( u \) and \( v \) in \( X \). A graph is called vertex-transitive if its automorphism group is transitive on the vertex set. A graph is called edge-transitive if its automorphism group is transitive on the edge set. Similarly an arc-transitive graph is a graph whose automorphism group is transitive on the arc set. Throughout this paper the symmetric group of degree \( n \) and the alternating group of degree \( n \) are denoted by \( S_n \) and \( A_n \), respectively.

Let \( G \) be a permutation group on a set \( A \) and \( \alpha \in A \). Denote by \( G_\alpha \) the stabilizer of \( \alpha \) in \( G \), that is, the subgroup of \( G \) fixing the point \( \alpha \). Permutation group \( G \) is semiregular on \( A \) if \( G_\alpha = 1 \) for every \( \alpha \in A \) and regular if \( G \) is transitive and semiregular. Let \( G \) be a finite group and let \( S \) be a subset of \( G \) such that \( 1 \in S \) and \( S^{-1} = S \). The Cayley graph \( X = \text{Cay}(G, S) \) on \( G \) with respect to \( S \) is defined as the graph with a vertex set \( V(X) = G \) and an edge set \( E(X) = \{\{g, h\} | g, h \in G, gh^{-1} \in S\} \). A Cayley graph \( \text{Cay}(G, S) \) is connected if and only if \( G = \langle S \rangle \). Let \( A = \text{Aut}(\text{Cay}(G, S)) \). It is obvious that \( R(G) \) are contained in \( A \). Also is regular on the set \( V(X) \). Thus a Cayley graph is vertex transitive. If \( A_1 \) denotes the stabilizer of the vertex 1 in \( A \) then \( \text{Aut}(G, S) = \{\alpha \in \text{Aut}(G) | S^\alpha = S\} \) is a subgroup of \( A_1 \). A Cayley graph \( \text{Cay}(G, S) \) is said to be normal if \( R(G) \) is normal in \( A \).

A lot of study has been done in normality of Cayley graphs. For example, normality of Cayley graphs of order \( p^2 \) and \( 2p \) has been determined by Dobson\(^1\) and Du\(^2\), respectively. Disconnected normal Cayley graphs are highlighted by Wang\(^3\). Further, Preager\(^4\) has developed a perspective which identifies \( \text{Cay}(G, S) \) is normal if \( N_{A_1}(R(G)) \) is transitive on edges and \( \text{Cay}(G, S) \) is a connected cubic Cayley graph on a non-abelian simple group. Also vast majority of normal connected cubic Cayley graphs on non-abelian simple groups are specified by Fang\(^5\).

In 2005, Feng and Xu\(^6\) proved that every connected tetravalent Cayley graph on a regular \( p \)-group is normal when \( p \neq 2, 5 \). One year later, normality of tetravalent Cayley graphs on dihedral groups have been discussed by Wang and Xu\(^7\).

In 2007, normality of the connected Cayley graph of valency 5 on \( A_5 \) has been determined by Feng and Zhou\(^8\), although in \(^9\) the normality of the connected Cayley graphs of valency 3 and 4 on \( A_4 \) has been proved by Xu and Xu. For more results on the normality of Cayley graphs, we refer the reader to \(^{1-3}\).

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In this paper, we have presented two main theorems with new conditions in order to ease the identification of non-normal Cayley graphs of valency 6.

2. Preliminaries

First we will give some preliminary results which use in the next.

Let \( X = \text{Cay}(G, S) \) be a Cayley graph of \( G \) with respect to \( S \) and \( \text{Aut}(G, S) = \left\{ a \in \text{Aut}(G) | S^a = S \right\} \). Set \( A := \text{Aut}(X) \) and denote by \( A_1 \) the stabilizer of the vertex 1 in \( A \). The following proposition is basic.

**Lemma 2.1 [6, Proposition 1.1]**

As the above notations:

(i) \( \text{Aut}(X) \) contains the right regular representation \( R(G) \) of \( G \) and so \( X \) is vertex-transitive.

(ii) \( X \) is undirected if and only if \( S^{+} = S \). Hence, all Cayley(di) graphs are vertex-transitive.

(iii) \( X \) is connected if and only if \( G = \langle S \rangle \).

**Lemma 2.2 [6, Proposition 1.2]**

We have:

(i) \( N_{A}(R(G)) = R(G) \text{Aut}(G, S) \),

(ii) \( A = R(G) \text{Aut}(G, S) \) if and only if \( R(G) \) is normal in \( G \).

**Lemma 2.3 [11, Proposition 1.5]**

The Cayley (di) graph is normal if and only if \( A_1 = \text{Aut}(G, S) \).

3. Discussion of Main Theorems

Now two sufficient conditions are given on the non-normal Cayley graphs of valency 6 for a finite group.

**Theorem 3.1**

Let \( G \) be a finite group and \( S = \{ s_1, s_2, s_3, s_4, s_5, s_6 \} \) be a subset of \( G \) which \( S^{+} = S \) and \( s_3, s_6 \) are involutions. Suppose that \( S \) contains at least three involutions and there exists an involution \( h \) in \( G \backslash S \) such that \( s_2 = hs_1, s_3 = hs_2, s_4 = hs_3, s_5 = hs_4, s_6 = hs_5, s_7 = hs_6 \).

Then the Cayley graph \( \text{Cay}(G, S) \) is not normal.

**Proof**

By the equations (*) and because \( h \not\in S \), we have \( 1 \not\in S \). Consider \( \sigma = (s_3 s_4)(s_5 s_6) \). Clearly \( s_5 s_6 \neq s_4 \) and \( s_6 s_3 \neq s_3 \), because if \( s_5 s_6 \neq s_4 \) then by the last equation of (*) we have \( s_6 h s_5 = s_4 \) so \( s_6 s_3 = s_4 \) and it implies that \( s_6 = 1 \), a contradiction. Thus \( s_5 s_6 \neq s_4 \). Similarly we can see \( s_6 s_3 \neq s_4 \). It shows that \( \sigma \) is a permutation on \( G \).

Let \( X = \text{Cay}(G, S) \) and \( A = \text{Aut}(X) \). Denote by \( A_1 \) the stabilizer of 1 in \( A \). To prove that \( X = \text{Cay}(G, S) \) is not normal, by Proposition 2.3, it suffices to show that \( \sigma \in A_1 \) and \( \sigma \not\in \text{Aut}(G, S) \).

By the equations (*), \( s_4 = s_4 h s_1 = s_4 h s_5 = s_4 h \) and \( s_3 = s_1 h h s_5 h = h = h h s_5 h = h s_5 h = s_5 h \).

Since \( s_5 s_6 \) are involutions and by the assumption, \( S \) contains at least three involutions, implying that either \( s_4 \) and \( s_1 \) or \( s_3 \) and \( s_5 \) are involutions. If \( s_4 \) and \( s_1 \) are involutions, then \( s_5 s_6 \) must be involutions. It means that \( s_3 \) and \( s_4 \) have different orders. If \( s_3 \) and \( s_4 \) are involutions, then \( s_3^{-1} = (hs_i)^{-1} = s_4^{-1} h^{-1} = h = s_3 h \), which means that \( s_3 \) and \( s_4 \) are not involutions and \( s_5 \) and \( s_6 \) have different orders. Thus, \( \sigma \in \text{Aut}(G, S) \) because \( \sigma \) permutes \( s_3 \) to \( s_5 \). Further, \( s_3 s_4 \neq 1 \) and \( s_5 s_6 \neq 1 \) because \( s_5 \) is an involution. Hence, \( \sigma \) fixes \( \sigma \). So we need only to show that \( \sigma \in A = \text{Aut}(X) \) and for this, it is enough to show that \( \sigma \) keeps adjacency of edges.

Let \( T = \{ w \}, h \omega, s_5 \omega, h s_4 \omega \} \). For any \( \omega \in T \), we have \( T = \{ \omega \}, h \omega, s_5 \omega, h s_4 \omega \} \). For example if \( \omega = s_5 \omega \), then

\[
\omega = h s_3 = h s_3 h = s_4
\]

\[
s_6 \omega = s_5 s_6 \omega = s_6
\]

Also if \( \omega = s_4 \omega \), then

\[
\omega = h s_5 \omega = s_5 h s_5 \omega = s_6 s_4 \omega
\]

Similarly, if \( \omega = s_4 \) or \( \omega = s_5 s_4 \) the same result is obtained. Thus it is assumed that for any \( \omega \in T \), \( T = \{ \omega \}, h \omega, s_5 \omega, h s_4 \omega \} \). Clearly, \( \sigma \) fixes every element in \( G \).

Now let \( \{ u, v \} \in E(X) \). We aim to prove that \( \{ u, v \} \in E(X) \). Consider two cases:

**Case 1.** If \( \{ u, v \} \cap T = \emptyset \), then \( \{ u, v \} \notin T \) and \( \{ u, v \} \notin T \). Consider two cases:

**Case 2.** If \( \{ u, v \} \cap T \neq \emptyset \), without loss of generality we can assume \( u \in T \), then \( T = \{ u, h u, s_4 u, h s_4 u \} \) and \( \sigma = (u h u)(s_4 u s_4 h u) \). Thus \( u^{\sigma} = h u, h u, s_5 u, h s_5 u \). Thus \( u^{\sigma} = h u, s_5 u, h s_5 u \) and \( \{ u, v \} \in E(X) \), we have \( v = s_i u \) for some \( i \), where \( 1 \leq i \leq 6 \).
If \(v = s_j u\), then \(v = s_j h u\) and \(\{u, v\}^c = \{h u, s_j u\} \subseteq E(X)\). Similarly if \(v = s_j u\), then \(\{u, v\}^c = \{h u, s_j h u\} \subseteq E(X)\). Now, suppose that \(v = s_j u\) for some \(j\), \(1 \leq j \leq 4\). It is clear that \(v = s_j u \neq u\) and \(v \neq h u\). Because if \(v = h u\), then \(s_j u = h u\) for some \(j\), \(1 \leq j \leq 4\). So \(s_j = h\) and it is a contradiction. Similarly, \(v = s_j u \neq s_j u\) or \(s_j h u\) for some \(j\), \(1 \leq j \leq 4\). Therefore \(v \not\in T\) and \(v^c = v\). Now If \(j = 1\) then \(v = s_1 u = s h u\) and \(\{u, v\}^c = \{h u, s_1 h u\} \subseteq E(X)\). If \(j = 2\), then \(v = s_2 u = h s u\) and \(\{u, v\}^c = \{h u, h s u\} \subseteq E(X)\). Similarly, for \(j = 3, 4\), we have \(\{u, v\}^c \subseteq E(X)\).

Therefore, both Cases 1, 2 implies that \(\sigma \not\in A\). Thus \(\sigma \in A^c\) and \(\sigma \in Aut(G, S)\), by Lemma 2.3, \(Cay(G, S)\) is not normal.

**Theorem 3.2**

Let \(G\) be a finite group and \(S = \{s_1, s_2, s_3, s_4, s_5, s_6\}\) be a subset of \(G\) such that \(1 \not\in S\), \(G = \langle S \rangle\) and \(S^c = S\). Suppose that \(s_j\) is an involution, \(N = \{1, s_1, s_2, s_3\}\) be a subgroup of \(G\) and \(H = \langle s_1, s_2, s_3 \rangle\) such that \(s_j, s_j \in H\). If \(|G:H| \geq 4\) and \(\{s_1 s_2, s_1 s_3, s_2 s_3\} = \{s_2 s_3, s_1 s_3, s_1 s_2\}\), then the Cayley graph \(Cay(G, S)\) is not normal.

**Proof**

Since \(|G:H| \geq 4\) and \(s_1, s_2, s_3, s_4, s_5, s_6 \in H\), there is a coset \(Hg\) such that \(s_j \not\in Hg\) for each \(i\), \(1 \leq i \leq 6\). It implies that \(Hg \neq H\). Let \(X = Cay(G, S)\) and \(A = Aut(X)\). Now, we define a permutation \(s\) on \(G\). If \(v \in Hg\), then \(v^c = s_j v\) and if \(v \in G \setminus Hg\), then \(v^c = v\). Clearly for each \(i\), \(1 \leq i \leq 6\), \(s_i^\sigma = s_i^\sigma\). Further, \(1 \not\in Hg\). If \(\sigma \in Aut(G)\), then \(\sigma\) fixes each element of \(G\), because \(G = \langle S \rangle\) and \(s_i^\sigma = s_i^\sigma\), and it means \(\sigma = 1\), a contradiction. Thus, \(\sigma \not\in Aut(G, S)\), and it is enough to show that \(\sigma \not\in Aut(X)\).

Let \(\{u, v\} \subseteq E(X)\). We claim that \(\{u, v\}^c \subseteq E(X)\). For this we consider two cases.

**Case 1.** If \(\{u, v\} \cap Hg = \emptyset\), then \(u, v \not\in Hg\) and \(\{u, v\}^c = \{u, v\} \) so \(\sigma \in Aut(X)\).

**Case 2.** If \(\{u, v\} \cap Hg \neq \emptyset\). We may assume that \(u \in Hg\), thus \(u^c = s_j u\) and since \(\{u, v\} \subseteq E(X)\) it is easy to see \(v = s_j u\) for some \(k\), \(1 \leq k \leq 6\). If \(k = 1\), then \(v = s_1 u \in Hg\) because \(s_1 \in H\) and \(u \in Hg\), so we have \(\{u, v\}^c = \{s_j u, u\} \subseteq E(X)\). If \(k = 2\), then \(v = s_2 u \not\in Hg\) because \(s_2 \not\in H\). Since \(N = \{1, s_1, s_2, s_3\}\) be a group of order 4 and \(s_3\) is an involution, we have \(s_2 = s_3^2\). Therefore, \(\{u, v\}^c = \{s_j u, s_j u\} = \{s_j u, s_j s_3^2 u\} \subseteq E(X)\). Similarly if \(k = 3\), then \(v = s_3 u \in Hg\) and \(\{u, v\}^c = \{s_j u, s_j u\} = \{s_j u, s_j s_3 u\} \subseteq E(X)\). If \(k = 4\), then \(v = s_4 u \not\in Hg\) because \(s_4 \not\in H\) and \(u \not\in Hg\). So \(v^c = s_4 u\). By the assumption, we know that \(s_4 \in \{s_1 s_4, s_2 s_4, s_3 s_4\}\). Thus there is an \(s_j\) such that \(s_j s_4 = s_j s_4\), where \(l = 4\) or 5 or 6. So \(\{s_j u, s_j s_4 u\} = \{s_j u, s_j s_4 u\} \subseteq E(X)\). Similarly, if \(k = 5\), then \(v = s_5 u \in Hg\) and \(\{u, v\}^c = \{s_j u, s_j u\} = \{s_j u, s_j u\} \subseteq E(X)\), where \(r = 4\) or 5 or 6.

It implies that in each case, \(\{u, v\}^c \subseteq E(X)\) and so \(\sigma \in Aut(X)\). Therefore, \(\sigma \in A\), but \(\sigma \not\in Aut(G, S)\) and by Lemma 2.3, \(Cay(G, S)\) is not normal.

## 4. Conclusion

Now we construct an infinite family of non-normal Cayley graphs of valency 6 by using Theorem 3.1 in the following example.

### Example 4.1

Let \(n(>2)\) be an even integer and \(m > 1\). If \(G = \langle a, b, c \mid a^n = b^x = c^w = 1, b^{-1} a b = a^{-1}, b^{-1} c b = c^{-1}\rangle\), then, the Cayley graph

\[
\text{Cay}\left(G, \left\{a^2, a^2 b, c, c^{-1}, bc, cb\right\}\right)
\]

is a non-normal Cayley graph of valency 6.

**Proof**

It is clear that, \(a^2, c, c^{-1} \neq 1\). Further \(bc \neq 1\), because if \(bc = 1\), then \(c = b\) a contradiction. Similarly, \(cb \neq 1\). Also \(a^2 b \neq 1\), because if \(a^2 b = 1\), then \(a^2 \left(a^2 a^2\right) = b = a^2 b = b \in S\) a contradiction. Thus, \(1 \not\in S\).

Now, let \(h = b \in G \setminus S\) and consider \(s_1 = cb, s_2 = c^{-1}, s_3 = s_1, s_4 = s_3, s_5 = s_2^{a^2}, s_6 = a^2 b\). It is easy to see that \(h\) is an involution, \(S\) has at least three involutions and \(s_5 s_6 = s_5 = s_1, s_4 = s_3, s_5 = s_2^{a^2}, s_6 = a^2 b\). Thus the conditions of Theorem 3.1 are hold and \(Cay(G, S)\) is not normal.

In following examples some non-normal Cayley graphs of valency 6 on \(A_6\) and \(A_7\) are determined.

### Example 4.2

Let \(W_1 = \{(1, 2), (4, 5), (5, 6, 4), (4, 6, 5), (6, 4, 1, 2), (5, 6, 3, 4), (1, 2)(3, 4)\}\). Then the \(Cay(A_6, W_1)\) is not normal.

**Proof**

Consider \(h = (1, 2)(5, 6)\). It is clear that \(h \in A_6 \setminus W_1\) and \(h\) is an involution. Now suppose that \(s_1 = (1, 2)(4, 5), s_2 = (5, 6, 4), s_3 = (4, 6, 5), s_4 = (6, 4)(1, 2), s_5 = (5, 6)(3, 4)\) and \(s_6 = (1, 2)(3, 4)\). It is easy to see the conditions of Theorem 3.1 are hold and the Cayley graph \(Cay(A_6, W_1)\) is not normal.
Example 4.3
Let \( W_5 = \{(1 \ 3) \ (2 \ 6), \ (5 \ 2 \ 6), \ (2 \ 5 \ 6), \ (3 \ 2) \ (1 \ 3), \ (5 \ 6) (2 \ 4), \ (1 \ 3) \ (2 \ 4)\} \). Then the \( \text{Cay}(A_5, W_5) \) is not normal.

**Proof**
Consider \( h = (1 \ 3)(5 \ 6) \). It is clear that \( h \in A_5 \setminus W_5 \) and \( h \) is an involution. Let \( s_1 = (1 \ 3)(2 \ 6), \ s_2 = (5 \ 2 \ 6), \ s_3 = (2 \ 5 \ 6), \ s_4 = (5 \ 2)(1 \ 3), \ s_5 = (5 \ 6)(2 \ 4) \) and \( s_6 = (1 \ 3)(2 \ 4) \). It is easy to see the conditions of Theorem 3.1 are hold and the Cayley graph \( \text{Cay}(A_5, W_5) \) is not normal.

Example 4.4
Let \( W_6 = \{(1 \ 4)(2 \ 6), (3 \ 2 \ 6) (2 \ 3 \ 6), (3 \ 2)(1 \ 4), (3 \ 6)(2 \ 5), (1 \ 4)(2 \ 5)\} \). Then the \( \text{Cay}(A_6, W_6) \) is not normal.

**Proof**
Similarly, by consider \( h = (1 \ 4)(3 \ 6) \), the \( \text{Cay}(A_6, W_6) \) is not normal.

Example 4.5
Let \( W_6 = \{(1 \ 5)(3 \ 4), (2 \ 3 \ 4), (2 \ 3)(1 \ 5), (2 \ 4)(3 \ 6), (1 \ 5)(3 \ 6)\} \). Then the \( \text{Cay}(A_6, W_6) \) is not normal.

**Proof**
Similarly, by consider \( h = (1 \ 5)(2 \ 4) \), the \( \text{Cay}(A_6, W_6) \) is not normal.

Example 4.6
Let \( W_6 = \{(1 \ 6)(3 \ 5), (2 \ 5 \ 3), (3 \ 5 \ 2), (2 \ 5)(1 \ 6), (2 \ 3)(4 \ 5), (1 \ 5)(4 \ 3)\} \). Then the \( \text{Cay}(A_6, W_6) \) is not normal.

**Proof**
Similarly, by consider \( h = (1 \ 6)(2 \ 3) \), the \( \text{Cay}(A_6, W_6) \) is not normal.

Example 4.7
Let \( U_4 = \{(1 \ 2)(4 \ 5), (3 \ 5 \ 4), (4 \ 5 \ 3), (3 \ 5)(1 \ 2), (1 \ 4)(2 \ 3), (1 \ 3)(2 \ 4)\} \). Then the \( \text{Cay}(A_4, U_4) \) is not normal.

**Proof**
Consider \( h = (1 \ 2)(3 \ 4) \). It is clear that \( h \in A_4 \setminus U_4 \) and \( h \) is an involution. Let \( s_1 = (1 \ 2)(4 \ 5), s_2 = (3 \ 5 \ 4), s_3 = (4 \ 5 \ 3), s_4 = (3 \ 5)(1 \ 2), s_5 = (1 \ 4)(2 \ 3) \) and \( s_6 = (1 \ 3)(2 \ 4) \). It is easy to see that \( s_2 = h s_1, s_4 = h s_3 \) and \( s_5 s_6 = h \). So by the Theorem 3.1, the \( \text{Cay}(A_4, U_4) \) is not normal.

Example 4.8
Let \( U_6 = \{(1 \ 5)(2 \ 4), (2 \ 3 \ 4), (2 \ 4 \ 3), (3 \ 4)(1 \ 5), (1 \ 3)(2 \ 5), (1 \ 2)(5 \ 3)\} \). Then the \( \text{Cay}(A_6, U_6) \) is not normal.

**Proof**
Consider \( h = (1 \ 5)(2 \ 3) \). It is clear that \( h \in A_6 \setminus U_6 \) and \( h \) is an involution. Similarly we have \( s_2 = h s_1, s_3 = h s_4, s_5 = s_2 h \) and \( s_6 s_5 = h \). So by the Theorem 3.1, the \( \text{Cay}(A_6, U_6) \) is not normal.

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