HOPF STRUCTURES ON MINIMAL HOPF QUIVERS

HUA-LIN HUANG, YU YE, AND QING ZHAO

Abstract. In this paper we investigate pointed Hopf algebras via quiver methods. We classify all possible Hopf structures arising from minimal Hopf quivers, namely basic cycles and the linear chain. This provides full local structure information for general pointed Hopf algebras.

Keywords Hopf algebra, Hopf quiver

2000 MR Subject Classification 16W30, 16W35, 16S80

1. Introduction

Quivers are oriented diagrams consisting of vertices and arrows. Due to a well-known theorem of Gabriel [11], elementary associative algebras over fields can be presented by path algebras of quivers modulo admissible ideals in some unique manner and their representations given by representations of the corresponding bound quivers. This makes the abstract algebras and their representation theory visible and plays a central role in the modern representation theory of associative algebras.

After Gabriel, quiver theory has been established for some other algebraic structures, in particular for coalgebras and Hopf algebras. A coalgebra over a field is said to be pointed if its simple subcoalgebras are one-dimensional, or equivalently, its simple comodules are one-dimensional. In [6], Chin and Montgomery gave a Gabriel type theorem for pointed coalgebras. Cibils and Rosso introduced the notion of Hopf quivers [9], which are determined by groups with ramification data, and observed that the path coalgebra of a quiver admits a graded Hopf structure if and only if the quiver is a Hopf quiver. A Hopf algebra is called pointed if its underlying coalgebra is so. Van Oystaeyen and Zhang established a Gabriel type theorem for graded pointed Hopf algebras [25].

The aforementioned results motivate our project on the study of general pointed Hopf algebras by taking advantage of quiver methods. The quiver setting gives a visible frame to classification problem and representation theory and other respects. The project of quiver approaches to pointed Hopf algebras consists mainly of the following aspects:
(1) Classify graded Hopf structures on Hopf quivers. This amounts to a complete classification of graded pointed Hopf algebras.

(2) Carry out a proper deformation process for graded Hopf structures to get general pointed Hopf algebras.

(3) Investigate the (co)representation theory as well as other aspects of the obtained Hopf algebras with a help of their quiver presentation.

This paper is conceived as the first step of the project. We classify Hopf algebra structures on minimal Hopf quivers, namely basic cycles and the linear train. Intuitively, it is easy to observe that a general Hopf quiver are compatible combination of these minimal Hopf quivers. Moreover, on a given Hopf quiver there is a clear relation between the sub-Hopf algebras and sub-Hopf quivers, as can be seen from Cibils and Rosso’s description of graded Hopf structures on Hopf quivers \cite{9}. Therefore, our result provides complete basic structure ingredients for general pointed Hopf algebras.

Most of the Hopf structures arising from minimal Hopf quivers are by no means novel. They appeared sporadically in various references of Hopf algebras (see \cite{24, 1, 21} etc.) and quantum groups (see \cite{17, 10} etc). Quiver methods provide these Hopf algebras in a unified setting. Aside from quiver techniques, our arguments also rely on Bergman’s Diamond Lemma \cite{4} which helps to present the Hopf algebras by generators with relations. This is useful in carrying out the preferred deformation procedure in the sense of Gerstenhaber and Schack \cite{13}.

In principle general pointed Hopf algebras can be obtained by compatible gluing of the minimal ones we obtain. By consulting the classical Lie theory, viewing the minimal Hopf structures as quantum version of the Borel part of \(\mathfrak{sl}_2\), it is natural to expect that the gluing problem should turn out to be a rich theory and (quantum versions of) Cartan matrices, Weyl groups and root systems should get involved. We leave this for future work. In the situation of finite-dimensional pointed Hopf algebras with abelian group-likes, Andruskiewitsch and Schneider have made substantial progress in classification problem \cite{2} by different method initiated in \cite{1}.

The paper is organized as follows. In Section 2 we review some necessary facts about Hopf quivers and pointed Hopf algebras. In Sections 3 and 4, the explicit classifications of Hopf structures on minimal Hopf quivers are given. Section 5 is devoted to some applications of the classification results.

Throughout the paper, we work over an algebraically closed field of characteristic zero. The readers are referred to \cite{12, 3} for general knowledge of quivers and representations, and to \cite{23, 18} for that of coalgebras and Hopf algebras.
2. Quiver Approaches to Pointed Hopf Algebras

For the convenience of the reader, in this section we recall some basic notions and facts in [9, 25]. Note that there is a dual approach to elementary Hopf algebras via quivers, see [7, 14, 8, 15] for related works.

2.1. A quiver is a quadruple \( Q = (Q_0, Q_1, s, t) \), where \( Q_0 \) is the set of vertices, \( Q_1 \) is the set of arrows, and \( s, t : Q_1 \to Q_0 \) are two maps assigning respectively the source and the target for each arrow. A path of length \( l \geq 1 \) in the quiver \( Q \) is a finitely ordered sequence of \( l \) arrows \( a_l \cdots a_1 \) such that \( s(a_{i+1}) = t(a_i) \) for \( 1 \leq i \leq l - 1 \). By convention a vertex is said to be a trivial path of length 0.

The path coalgebra \( kQ \) is the \( k \)-space spanned by the paths of \( Q \) with counit and comultiplication maps defined by \( \varepsilon(g) = 1 \), \( \Delta(g) = g \otimes g \) for each \( g \in Q_0 \), and for each nontrivial path \( p = a_n \cdots a_1 \), \( \varepsilon(p) = 0 \),

\[
\Delta(a_n \cdots a_1) = p \otimes s(a_1) + \sum_{i=1}^{n-1} a_n \cdots a_{i+1} \otimes a_i \cdots a_1 + t(a_n) \otimes p.
\]

The length of paths gives a natural gradation to the path coalgebra. Let \( Q_n \) denote the set of paths of length \( n \) in \( Q \), then \( kQ = \bigoplus_{n \geq 0} kQ_n \) and \( \Delta(kQ_n) \subseteq \bigoplus_{n=i+j} kQ_i \otimes kQ_j \). Clearly \( kQ \) is pointed with the set of grouplikes \( G(kQ) = Q_0 \), and has the following coradical filtration

\[
kQ_0 \subseteq kQ_0 \oplus kQ_1 \subseteq kQ_0 \oplus kQ_1 \oplus kQ_2 \subseteq \cdots.
\]

Hence \( kQ \) is coradically graded.

According to [9], a quiver \( Q \) is said to be a Hopf quiver if the corresponding path coalgebra \( kQ \) admits a graded Hopf algebra structure. Hopf quivers can be determined by ramification data of groups. Let \( G \) be a group, \( C \) the set of conjugacy classes. A ramification datum \( R \) of the group \( G \) is a formal sum \( \sum_{C \in C} R_CC \) of conjugacy classes with coefficients in \( \mathbb{N} = \{0, 1, 2, \cdots \} \).

The corresponding Hopf quiver \( Q = Q(G, R) \) is defined as follows: the set of vertices \( Q_0 \) is \( G \), and for each \( x \in G \) and \( c \in C \), there are \( R_c \) arrows going from \( x \) to \( cx \).

A Hopf quiver \( Q = Q(G, R) \) is connected if and only if the union of the conjugacy classes with non-zero coefficients in \( R \) generates \( G \). We denote the unit element of \( G \) by \( e \). If \( R_{\{e\}} \neq 0 \), then there are \( R_{\{e\}} \)-loops attached to each vertex; if the order of elements in a conjugacy class \( C \neq \{e\} \) is \( n \) and \( R_C \neq 0 \), then corresponding to these data in \( Q \) there is a sub-quiver \( (n, R_C) \)-cycle (called basic \( n \)-cycle if \( R_C = 1 \)), i.e., the quiver having \( n \) vertices, indexed by the set of integers modulo \( n \), and \( R_C \) arrows going from \( i \) to \( i + 1 \) for each \( i \); if the order of elements in a conjugacy class \( C \) is \( \infty \),
then in $Q$ there is a sub-quiver $R_C$-chain (called linear chain if $R_C = 1$), i.e., a quiver having set of vertices indexed by the set of integral numbers, and $R_C$ arrows going from $j$ to $j + 1$ for each $j$. Therefore, basic cycles (including loop as 1-cycle) and the linear train are basic building brocks of general Hopf quivers.

Due to cibils and Rosso [9], for a given Hopf quiver $Q$, the set of graded Hopf structures on $kQ$ is in one-to-one correspondence with the set of $kQ_0$-Hopf bimodule structures on $kQ_1$. The graded Hopf structures are obtained from Hopf bimodules via quantum shuffle product [22]. The graded Hopf structures can be restricted to sub-Hopf quivers, hence for the very local sub-Hopf structures it suffices to consider those arising from minimal Hopf quivers.

### 2.2. Let $H$ be a pointed Hopf algebra. Denote its coradical filtration by $\{H_n\}_{n=0}^{\infty}$. Define

$$\text{gr}(H) = H_0 \oplus H_1/H_0 \oplus H_2/H_1 \oplus \cdots$$

as the corresponding (coradically) graded coalgebra. Then $\text{gr}(H)$ inherits from $H$ a coradically graded Hopf algebra structure (see e.g. [18]). Note that any generating set of $\text{gr}(H)$ (as an algebra) can be lifted to one for $H$. This useful fact can be verified easily by induction.

**Lemma 2.1.** Assume that $\mathcal{G} \subset \text{gr}(H)$ is a generating set and $\tilde{\mathcal{G}} \subset H$ an arbitrary set of its representatives. Then $\tilde{\mathcal{G}}$ generates $H$.

The procedure from $H$ to $\text{gr}H$ is called degeneration. The converse procedure is called deformation. According to Gerstenhaber and Schack [13], a coalgebra-preserving deformation is called preferred. If we want to classify all the Hopf structures on the whole path coalgebra of a Hopf quiver, or the bialgebras of type one [20], then we only need to carry out preferred deformation procedure.

According to Van Oystaeyen and Zhang [25], if $H$ is a coradically graded pointed Hopf algebra, then there exists a unique Hopf quiver $Q(H)$ such that $H$ can be realized as a large sub-Hopf algebra of a graded Hopf structure on the path coalgebra $kQ(H)$. Here by “large” we mean $H$ contains the subspace $kQ(H)_0 \oplus kQ(H)_1$. This Gabriel type theorem allows us to classify pointed Hopf algebras exhaustively in the quiver setting. The combinatorial structure of Hopf quivers implies clearly a Cartier-Gabriel decomposition theorem (see e.g. [23] [18]) for general pointed Hopf algebras as given by Montgomery [19]. It suffices to study only Hopf structures on connected Hopf quivers.
3. Hopf Structures on Basic Cycles

3.1. Let $G = \langle g \mid g^n = 1 \rangle$ be a cyclic group of order $n$ and let $Z$ denote the Hopf quiver $Q(G, g)$. The quiver $Z$ is a basic $n$-cycle and this is the only possible way it is realized as a Hopf quiver. If $n = 1$, then $Z$ is the one-loop quiver, that is, consisting of one vertex and one loop. It is easy to see that such a quiver provides only the familiar divided power Hopf algebra in one variable, which is isomorphic to the polynomial algebra in one variable.

From now on we assume $n > 1$ and fix a basic $n$-cycle $Z$. For each integer $i$ modulo $n$, let $a_i$ denote the arrow $g^i \to g^{i+1}$. Let $p^l_i$ denote the path $a_{i+l-1} \cdots a_{i+1}a_i$ of length $l$. Then $\{p^l_i \mid 0 \leq i \leq n-1, \ l \geq 0\}$ is a basis of $kZ$.

Before moving on, we fix some notations of Gaussian binomials. For any $q \in k$, integers $l, m \geq 0$, let

$$l_q = 1 + q + \cdots + q^{l-1}, \quad l_q! = 1 \cdot q \cdot \cdots \cdot l_q \cdot q \cdot \cdots \cdot q, \quad \binom{l+m}{l}_q = \frac{(l+m)!_q}{l!_q m!_q}.$$

When $\neq q \in k$ is an $n$-th root of unity of order $d$,

$$\binom{l+m}{l}_q = 0 \text{ if and only if } \left[\frac{l+m}{d}\right] - \left[\frac{m}{d}\right] - \left[\frac{l}{d}\right] > 0,$$

where $[x]$ means the integer part of $x$.

We need the following fact about automorphisms of the path coalgebra $kZ$ in our later argument.

**Lemma 3.1.** Let $d > 1$ be an integer and $kZ[d]$ the subcoalgebra $\bigoplus_{i=0}^{d-1} kZ_i$. For any $\lambda \in k$ the following linear map

$$f^d_{\lambda}(0) : kZ[d] \to kZ[d]$$

$$p^l_i \mapsto p^l_i, \quad p^d_0 \mapsto p^d_0 + \lambda(1 - g^d),$$

$$p^d_i \mapsto p^d_i, \quad 1 \leq i \leq n-1.$$ defines a coalgebra automorphism of $kZ[d]$. There exists a coalgebra automorphism $F^d_{\lambda}(0) : kZ \to kZ$ such that its restriction to $kZ[d]$ is $f^d_{\lambda}(0)$. Similar map $f^d_{\lambda}(j)$ can be defined and be extended to $F^d_{\lambda}(j)$ for any $j$. Moreover, any automorphism of the subcoalgebra $kZ[d]$ with restriction to $kZ[d-1]$ being identity is a finite composition of some $f^d_{\lambda}(j)$’s. Therefore all such automorphisms are extendable to automorphisms of the path coalgebra $kZ$. 
Proof. The claim that \( f_\lambda^d(0) \) is a coalgebra automorphism is obvious. For the second claim, define the map \( F_\lambda^d(0) \) as follows:

\[
F_\lambda^d(0) : \quad k\mathcal{Z} \rightarrow k\mathcal{Z}
\]

\[
p_i^d \mapsto p_i^d, \quad \forall \ i, \ 0 \leq l \leq d - 1,
\]

\[
p_0^d \mapsto p_0^d + \lambda(1 - g^d),
\]

\[
p_i^d \mapsto p_i^d, \quad 1 \leq i \leq n - 1, \text{ and for } l > d,
\]

\[
p_i^d \mapsto \begin{cases} 
  p_0^d - \lambda p_d^{l-d}, & i = 0, \ l \neq d \ (\text{mod } n); \\
  p_0^d + \lambda p_0^{d-l} - \lambda p_d^{l-d}, & i = 0, \ l = d \ (\text{mod } n); \\
  p_i^d + \lambda p_i^{l-d}, & 1 \leq i \leq n - 1, \ i + l \neq d \ (\text{mod } n); \\
  p_i^d, & 1 \leq i \leq n - 1, \ i + l \neq d \ (\text{mod } n).
\end{cases}
\]

It is straightforward (but a bit tedious) to verify that \( F_\lambda^d(0) \) is the desired coalgebra automorphism of \( k\mathcal{Z} \). The rest claims are easy. \( \square \)

3.2. First we recall the graded Hopf structures on \( k\mathcal{Z} \). By [9], they are in one-to-one correspondence with the \( kG \)-module structures on \( ka_0 \), and in turn with the set of \( n \)-th root of unity. For each \( q \in k \) with \( q^n = 1 \), let \( g.a_0 = qa_0 \) define a \( kG \)-module. The corresponding \( kG \)-Hopf bimodule is \( kG \otimes kG a_0 \otimes kG = ka_0 \otimes kG \). We identify \( a_i = a_0 \otimes g^i \). This is how we view \( k\mathcal{Z}_1 \) as a \( kG \)-Hopf bimodule. The following path multiplication formula

\[
p_i^l \cdot p_j^m = q^{im} \binom{l + m}{l} q_j^{l+m}.
\]

was given in [9] by induction. In particular,

\[
g \cdot p_i^l = q^l p_i^{l+1}, \quad p_i^l \cdot g = p_i^{l+1}, \quad a_0^d = q_p^d p_0^l.
\]

For each \( q \), the corresponding graded Hopf algebra is denoted by \( k\mathcal{Z}(q) \).

We consider in the following lemma the algebraic side of \( k\mathcal{Z}(q) \). The facts are our starting point of the preferred deformation process.

**Lemma 3.2.** As an algebra, \( k\mathcal{Z}(q) \) can be presented by generators with relations as follows:

1. when \( q = 1 \), generators: \( g, \ a_0 \). relations: \( g^n = 1, \ ga_0 = a_0g \).
2. when \( \text{ord}(q) = d > 1 \), generators: \( g, a_0, p_0^d \). relations: \( g^n = 1, \ ga_0 = qa_0g, \ a_0^d = 0, \ a_0 p_0^d = p_0^d a_0, \ gp_0^d = p_0^d g \).

**Proof.** The claim about the generators and the relations they satisfy is direct consequence of (3.1) and (3.2). In particular, for the case \( \text{ord}(q) = d > 1 \), we have

\[
(p_0^d)^l = p_0^{dl}, \quad p_0^{dl} a_0^j = j^l q p_0^{j + dl}.
\]
It suffices to prove conversely the relations are enough to define \( k\mathcal{Z}(q) \).

Let \( H(q) \) denote the algebra defined in the lemma. To avoid confusion, we use new notations for the generators: change \( g \) to \( h \), \( a_0 \) to \( a \), and \( p_0^d \) to \( p \). The relations are obtained by substituting the old notations by the new ones.

For the case \( q = 1 \), by the well-known diamond lemma \[ \{a^i h^i \mid 0 \leq i \leq n - 1, \ l \geq 0 \} \] is a basis of \( H(1) \). Now define a linear map \( f : H(1) \rightarrow k\mathcal{Z}(1), \ a^i h^i \mapsto l! p_i^j \). Apparently this is a linear isomorphism. It remains to check that it respects the multiplication. This is again a direct consequence of (3.1) and (3.2):

\[
 f((a^i h^i)(a^m h^j)) = (l + m)! p_{i+j}^{l+m} = (l! p_i^j)(m! p_j^m) = f(a^i h^i) \cdot f(a^m h^j) .
\]

For the case \( \text{ord}(q) = d > 1 \), the set \( \{p^i a^j h^i \mid 0 \leq i \leq n - 1, \ 0 \leq j \leq d - 1, \ l \geq 0 \} \) is a basis of \( H(q) \), again by the diamond lemma. Define a linear map

\[
 h : H(q) \rightarrow k\mathcal{Z}(q), \ p^i a^j h^i \mapsto j! q_i p_i^{j+dl} .
\]

Similarly one can verify by direct computation with a help of (3.1) and (3.3) that this is an algebra isomorphism:

\[
 f((p^j a^i h^i)(p^j a^i h^i)) = q^{j'}(j + j')! a^{j'+j'+dl} h^{l+l'} = (j! q_i p_i^{j+dl})(j'! q_i p_i^{j'+dl'}) \]

\[
 = f(p^j a^i h^i) \cdot f(p^j a^i h^i) .
\]

\( \square \)

3.3. Now we are ready to state the main result of this section. We classify all the (non-graded) Hopf structures on the path coalgebra \( k\mathcal{Z} \).

**Theorem 3.3.** Let \( H \) be a Hopf structure on \( k\mathcal{Z} \) with \( \text{gr} \ H \cong k\mathcal{Z}(q) \). Then as algebra, it can be presented by generators and relations as follows:

1. if \( q = 1 \), generators: \( g, \ a_0 \). relations: \( g^n = 1, \ ga_0 = a_0g \). In particular, the Hopf algebra \( H \) is isomorphic to \( k\mathcal{Z}(1) \).
2. if \( \text{ord}(q) = n \), generators: \( g, \ a_0, \ p_0^d \). relations: \( g^n = 1, \ a_0^0 = 0, \ ga_0 = qa_0g, \ gp_0^n = p_0^ng, \ a_0 p_0^n = p_0^n a_0 = \lambda a_0 \) with some \( \lambda \in k \).
3. if \( 1 < \text{ord}(q) = d < n \), \( n \neq 2d \), generators: \( g, \ a_0, \ p_0^d \). relations: \( g^n = 1, \ a_0^0 = 0, \ ga_0 = qa_0g, \ gp_0^d = p_0^dg, \ a_0 p_0^d = p_0^da_0 = 0 \). In other words, the Hopf algebra \( H \) is isomorphic to \( k\mathcal{Z}(q) \).
4. if \( n = 2d \) is even and \( \text{ord}(q) = d \), generators: \( g, \ a_0, \ p_0^n \). relations: \( g^n = 1, \ a_0^0 = \mu(1 - g^d), \ ga_0 = qa_0g, \ gp_0^d = p_0^dg, \ a_0 p_0^d = p_0^da_0 = \frac{\mu(1-q)}{(d-1)_q} a_0(1 + g^d) \) with some \( \mu \in k \).
The proof will be separated into several steps. The main idea is to determine all the possible preferred deformations from the graded ones, with a help of the quiver.

3.4. If \( \text{gr} \, H \cong k\mathbb{Z}(1) \), then \( H \) is generated by \( g \) and \( a_0 \), by Lemma 3.2 (1). We only need to give all the possible relations involved by them. This suffices to consider all the possible preferred deformations of the graded generating relations. In this situation, we need to determine the
\[
\text{lower terms} = ga_0g^{-1} - a_0.
\]
By \( \Delta (g \cdot a_0 \cdot g^{-1}) = \Delta (g) \Delta (a_0) \Delta (g^{-1}) = g \cdot a_0 \cdot g^{-1} \otimes 1 + g \otimes g \cdot a_0 \cdot g^{-1} \), we can conclude that \( g \cdot a_0 \cdot g^{-1} \in g(k\mathbb{Z})^1 \), hence
\[
g \cdot a_0 \cdot g^{-1} - a_0 = \lambda(1 - g)
\]
for some \( \lambda \in k \). The relation \( g^n = 1 \) is stable under deformation. Note that
\[
a_0 = g^n \cdot a_0 \cdot g^{-n} = a_0 + n\lambda(1 - g),
\]
This forces \( \lambda = 0 \). Therefore there are no non-trivial preferred deformations for \( k\mathbb{Z}(1) \).

3.5. If \( \text{ord}(q) = n \) and \( \text{gr} \, H \cong k\mathbb{Z}(q) \), then \( H \) is generated by \( g, a_0, p^n_0 \) by Lemma 3.2 (2). As before, we need to determine all the possible preferred deformations of the graded generating relations in Lemma 3.2 (2).

Firstly, the relation \( ga_0 = qa_0g \) might be deformed to
\[
 ga_0g^{-1} = qa_0 + \alpha(1-g)
\]
for some \( \alpha \in k \). Let \( \tilde{a}_0 = a_0 - \frac{\alpha}{1-q}(1-g) \), then we have \( g\tilde{a}_0g^{-1} = q\tilde{a}_0 \). Set \( \lambda = \frac{\alpha}{1-q} \) and \( f^1_{\lambda} \) as in Lemma 3.1. It follows that the map \( f^1_{\lambda} \) can be extended to a coalgebra automorphism \( F^1_{\lambda} \) of \( k\mathbb{Z} \). Now the original Hopf structure can be transported through \( F^1_{\lambda} \) to one with \( ga_0 = qa_0g \). By iterative application of the lemma, we can have through coalgebra automorphism (or base change)
\[
a_0g^i = a_i, \quad a_0g^i = l_q!p_l^i, \quad i = 0, 1, \ldots, n-1, \quad l = 1, \ldots, n-1.
\]
Note that under such coalgebra automorphisms, the “new” elements \( g, a_0, p^n_0 \) preserve to be generators of \( H \) according to Lemma 3.2.

Secondly, consider the relation \( a_0^n = 0 \). In order to see to what it may be deformed, we should look at \( \Delta (a_0^n) \). By the Gaussian binomial formula (see (3.4)
e.g. [16], Prop. IV.2.2), we have

\[
\Delta(a^n_0) = (\Delta(a_0))^n = (a_0 \otimes 1 + g \otimes a_0)^n
\]

\[
= \sum_{i=0}^{n} \binom{n}{i} a_0^i g^{n-i} \otimes a_0^{n-i}
\]

\[
= a^n_0 \otimes 1 + 1 \otimes a^n_0 .
\]

This follows that

\[(3.5) \quad a^n_0 = 0,\]

since in \(kZ\) there is no loop attached to 1.

Finally we consider the relations involved \(p^n_0\). Similarly, by

\[
\Delta(gp^n_0 - p^n_0g) = (gp^n_0 - p^n_0g) \otimes g + g \otimes (gp^n_0 - p^n_0g)
\]

we have \(gp^n_0 - p^n_0g = 0\). By (3.4) and (3.5) we have the following equations:

\[
\Delta(a_0p^n_0) = \Delta(a_0)\Delta(p^n_0)
\]

\[
= (a_0 \otimes 1 + g \otimes a_0)(\sum_{l=0}^{n} p^{n-l}_l \otimes p^l_0)
\]

\[
= \sum_{l=0}^{n} a_0 p^{n-l}_l \otimes p^l_0 + \sum_{l=0}^{n} g p^{n-l}_l \otimes a_0 p^l_0
\]

\[
= \sum_{l=0}^{n} p^{n+1-l}_l \otimes p^l_0 + a_0 p^n_0 \otimes 1 + g^{n+1} \otimes a_0 p^n_0
\]

\[
\Delta(p^n_0a_0) = \Delta(p^n_0)\Delta(a_0)
\]

\[
= (\sum_{l=0}^{n} p^{n-l}_l \otimes p^l_0)(a_0 \otimes 1 + g \otimes a_0)
\]

\[
= \sum_{l=0}^{n} p^{n-l}_l a_0 \otimes p^l_0 + \sum_{l=0}^{n} p^{n-l}_l g \otimes p^l_0 a_0
\]

\[
= \sum_{l=0}^{n} p^{n+1-l}_l \otimes p^l_0 + p^n_0 a_0 \otimes 1 + g^{n+1} \otimes p^n_0 a_0
\]

Let \([a_0, p^n_0] = a_0 p^n_0 - p^n_0 a_0\). Then the previous equations give rise to

\[
\Delta([a_0, p^n_0]) = [a_0, p^n_0] \otimes 1 + g \otimes [a_0, p^n_0].
\]

Now from the structure of the space of \((1, g)\)-primitive elements of \(kZ\), we have

\[
[a_0, p^n_0] = \lambda a_0 + \mu (1 - g)
\]

for some \(\lambda, \mu \in k\). By induction, one gets

\[
a^n_0 p^n_0 = p^n_0 a^n_0 + n\lambda a^n_0 + n\mu a^{n-1}_0.
\]
With (3.4) and (3.5), this forces \( \mu = 0 \).

3.6. Now assume \( 1 < \text{ord}(q) = d < n \) and \( \text{gr} \, H \cong k\mathcal{Z}(q) \), then \( H \) is generated by \( g, \, a_0, \, p_0^d \). Repeat the argument in Subsection 3.5, we can assume without loss of generality for \( i = 0, 1, \cdots, n - 1, \, l = 1, \cdots, d - 1 \) that

\[
(3.6) \quad g a_0 = q a_0 g, \, a_0 g^i = a_i, \, a_0^l g^i = l g^i.
\]

Consider \( \Delta(a_0^d) \). By Gaussian binomial formula again, we have

\[
\Delta(a_0^d) = (\Delta(a_0))^d = (a_0 \otimes 1 + g \otimes a_0)^d = \sum_{i=0}^{d} \binom{d}{i}_q a_0^i g^{d-i} \otimes a_0^{d-i} = a_0^d \otimes 1 + g^d \otimes a_0^d.
\]

Since in \( k\mathcal{Z} \) there is no arrow going from 1 to \( g^d \), hence it follows that

\[
(3.7) \quad a_0^d = \mu(1 - g^d),
\]

with some \( \mu \in k \).

We continue to consider the relations involved \( p_0^d \). By

\[
\Delta(g p_0^d - p_0^d g) = (g p_0^d - p_0^d g) \otimes g + g^{d+1} \otimes (g p_0^d - p_0^d g)
\]

it follows \( g p_0^d - p_0^d g = \nu (g - g^{d+1}) \) for some \( \nu \in k \). By induction we have \( g^n p_0^d - p_0^d g^n = n \nu (1 - g^d) \). Since \( g^n = 1 \), we conclude that \( \nu = 0 \) and

\[
(3.8) \quad g p_0^d = p_0^d g.
\]
Finally we consider $\Delta([a_0, p_0^d])$. By (3.6), (3.7) and (3.8) we have the following:

$$
\Delta(a_0 p_0^d) = \Delta(a_0) \Delta(p_0^d) = (a_0 \otimes 1 + g \otimes a_0) \left( \sum_{l=0}^{d} p_l^{d-l} \otimes p_0^l \right)
$$

$$
= \sum_{l=0}^{d} a_0 p_l^{d-l} \otimes p_0^l + \sum_{l=0}^{d} g p_l^{d-l} \otimes a_0 p_0^l
$$

$$
= \sum_{l=1}^{d} p_l^{d+1-l} \otimes p_0^l + a_0 p_0^d \otimes 1 + g^{d+1} \otimes a_0 p_0^d
$$

$$
= \sum_{l=1}^{d} p_l^{d+1-l} \otimes p_0^l + a_0 p_0^d \otimes 1 + g^{d+1} \otimes a_0 p_0^d
$$

$$
+ a_0 p_1^{d-1} \otimes a_0 + g p_1^{d-1} \otimes a_0 p_0^d
$$

$$
= \Delta(p_0^d a_0) = \Delta(p_0^d) \Delta(a_0) = \left( \sum_{l=0}^{d} p_l^{d-l} \otimes p_0^l \right)(a_0 \otimes 1 + g \otimes a_0)
$$

$$
= \sum_{l=0}^{d} p_l^{d-l} a_0 \otimes p_0^l + \sum_{l=0}^{d} p_l^{d-l} g \otimes p_0^l a_0
$$

$$
= \sum_{l=1}^{d} p_l^{d+1-l} \otimes p_0^l + p_0^d a_0 \otimes 1 + g^{d+1} \otimes p_0^d a_0
$$

$$
+ p_1^{d-1} a_0 \otimes a_0 + p_1^{d-1} g \otimes p_0^{d-1} a_0
$$

$$
= \sum_{l=1}^{d} p_l^{d+1-l} \otimes p_0^l + a_0 p_0^d \otimes 1 + g^{d+1} \otimes a_0 p_0^d
$$

$$
+ q \frac{\mu}{(d-1)_q} (g - g^{d+1}) \otimes a_0 + \frac{\mu}{(d-1)_q} p_1^d \otimes (g - g^{d+1})
$$

These equations lead to

$$
\Delta([a_0, p_0^d]) = \frac{\mu (1 - q)}{(d-1)_q} (a_0 + p_1^d)
$$

$$
= \{ [a_0, p_0^d] - \frac{\mu (1 - q)}{(d-1)_q} (a_0 + p_1^d) \} \otimes 1 + g^{d+1} \otimes \{ [a_0, p_0^d] - \frac{\mu (1 - q)}{(d-1)_q} (a_0 + p_1^d) \}.
$$

It follows as before that

$$
[a_0, p_0^d] = \frac{\mu (1 - q)}{(d-1)_q} a_0 (1 + g^d) + \lambda (1 - g^{d+1})
$$
for some $\lambda \in k$. Again by induction we have
\[ a_0^d p_0^d = p_0^d a_0^d + \frac{\mu^2(1 - q)}{(d - 1)q!}(1 - g^{2d}) + d\lambda a_0^{d-1}. \]
So $\mu = \lambda = 0$ if $n \neq 2d$, or $\lambda = 0$ otherwise. Now we can conclude that

\[
[a_0, p_0^d] = \begin{cases} 
\mu(1 - q)(d - 1)q! a_0(1 + g^d), & \text{if } n = 2d; \\
0, & \text{otherwise.}
\end{cases}
\]

3.7. So far we have proved that the Hopf structures on $k\mathbb{Z}$ must be generated by $g, a_0, p_0^d$ (where $d = \text{ord}(q)$) and satisfy the relations presented in Theorem 3.3. To complete the proof it suffices to verify that these relations are enough to define Hopf structures on $k\mathbb{Z}$. We only need to prove the cases of $\text{ord}(q) = n$ and $\text{ord}(q) = n/2$, since otherwise the Hopf structures are graded and was done in Lemma 3.2.

The verification is sort of routine, similar to that for graded case. We only prove the case of $\text{ord}(q) = n$. The case of $\text{ord}(q) = n/2$ can be done in a similar manner. Assume an algebra $C(q, \lambda)$ is defined by generators $h, a, p$ with relations
\[ h^n = 1, \ a^n = 0, \ ha = qah, \ hp = ph, \ ap - pa = \lambda a. \]
By the diamond lemma, the algebra has
\[ \{p^k a^j h^i \ | \ 0 \leq i, j \leq n - 1, \ k \geq 0\} \]
as a basis. Since the Hopf algebra $H$ has a basis of similar form
\[ \{(p_0^k a_0^j g^i \ | \ 0 \leq i, j \leq n - 1, \ k \geq 0\}, \]
we can define a linear isomorphism $F : C(q, \lambda) \rightarrow H$ by sending $p^k a^j h^i$ to $(p_0^k a_0^j g^i)$. It is an algebra map by direct calculation. This completes the proof.

3.8. We summarize in the following all the Hopf algebra structures living on $k\mathbb{Z}$. We denote by $k\mathbb{Z}(n, q, \lambda)$ the Hopf algebra defined by (2) of Theorem 3.3, and by $k\mathbb{Z}(\frac{n}{2}, q, \mu)$ the Hopf algebra defined by (4) of Theorem 3.3. The verification of the statements about Hopf algebra isomorphism is routine, so is omitted.

**Theorem 3.4.** Let $\mathbb{Z}$ be a basic $n$-cycle and $k\mathbb{Z}$ the associated path coalgebra.

(1) If $n$ is odd, then the Hopf structures on $k\mathbb{Z}$ are given by $k\mathbb{Z}(q)$ (graded) and $k\mathbb{Z}(n, q, \lambda)$ (non-graded). We have Hopf algebra isomorphism $k\mathbb{Z}(q) \cong k\mathbb{Z}(q')$ if and only if $q = q'$; $k\mathbb{Z}(n, q, \lambda) \cong$
$k\mathbb{Z}(n, q', \lambda')$ if and only if $q = q'$ and there exists some $0 \neq \zeta \in k$ such that $\lambda = \zeta n \lambda'$.

(2) If $n$ is even, then the Hopf structures on $k\mathbb{Z}$ are given by $k\mathbb{Z}(q)$ (graded), $k\mathbb{Z}(n, q, \lambda)$ and $k\mathbb{Z}(\frac{n}{2}, q, \mu)$ (non-graded). We have Hopf algebra isomorphism $k\mathbb{Z}(\frac{n}{2}, q, \mu) \cong k\mathbb{Z}(\frac{n}{2}, q', \mu')$ if and only if $q = q'$ and there exists some $0 \neq \zeta \in k$ such that $\mu = \zeta \frac{n}{2} \mu'$.

4. Hopf Structures on the Linear Chain

4.1. Let $G = \langle g \rangle$ be an infinite cyclic group and let $\mathcal{A}$ denote the Hopf quiver $Q(G, g)$. Then $\mathcal{A}$ is the linear chain. We remark that this is the only possible way to view it as a Hopf quiver. Let $e_i$ denote the arrow $g^i \rightarrow g^{i+1}$ and $p^i_l$ the path $e_{i+l-1} \cdots e_i$ of length $l \geq 1$, for each $i \in \mathbb{Z}$. The notation $p^d_0$ is understood as $e$. Similar to the case of basic cycle, we need the following lemma to make appropriate base change in later argument. The proof is almost identical to Lemma 3.1, so we omit the detail.

**Lemma 4.1.** Let $k\mathcal{A}[d]$ be the subcoalgebra $\bigoplus_{i=0}^{d} k\mathcal{A}_i$. For any $\lambda \in k$ the following linear map

$$f^d_{\lambda} : k\mathcal{A}[d] \rightarrow k\mathcal{A}[d]$$

$$p^i_l \mapsto p^i_l, \quad \forall i, 0 \leq l \leq d - 1,$$

$$p^d_0 \mapsto p^d_0 + \lambda(1 - g^d),$$

$$p^d_i \mapsto p^d_i, \quad i \neq 0.$$

defines a coalgebra automorphism of $k\mathcal{A}[d]$. There exists a coalgebra automorphism $F^d_{\lambda} : k\mathcal{A} \rightarrow k\mathcal{A}$ such that its restriction to $k\mathcal{A}[d]$ is $f^d_{\lambda}$.

4.2. We collect in this subsection some useful results of graded Hopf structures on $k\mathcal{A}$. The graded Hopf structures are in one-one correspondence to the left $kG$-module structures on $ke_0$, in turn to non-zero elements of $k$. Assume $g.e_0 = q e_0$ for some $0 \neq q \in k$. The corresponding $kG$-Hopf bimodule is $ke_0 \otimes kG$. We identify $e_i$ and $e_0 \otimes g^i$, and in this way we have a $kG$-Hopf bimodule structure on $k\mathcal{A}_1$. We denote the corresponding graded Hopf algebra by $k\mathcal{A}(q)$. The following lemma give the presentation of $k\mathcal{A}(q)$ by generators with relations. The proof is routine as Lemma 3.2, so is omitted.

**Lemma 4.2.** The algebra $k\mathcal{A}(q)$ can be presented via generators with relations as follows:

1. when $q = 1$, generators: $g, g^{-1}, e_0$. relations: $gg^{-1} = 1 = g^{-1}g$, $ge_0 = e_0 g$. 

(2) when \( q \neq 1 \) is not a root of unity, generators: \( g, g^{-1}, e_0 \). relations: 
\[ gg^{-1} = 1 = g^{-1}g, \quad ge_0 = qe_0g. \]

(3) when \( q \neq 1 \) is a root of unity of order \( d \), generators: \( g, g^{-1}, e_0, p_0^d \).
relations: 
\[ gg^{-1} = 1 = g^{-1}g, \quad e_0^d = 0, \quad ge_0 = qe_0g, \quad gp_0^d = p_0^d g, \quad e_0p_0^d = p_0^d e_0. \]

4.3. With the algebraic characterization of \( kA(q) \), we can proceed to the possible preferred deformation. The classification of Hopf structures on \( kA \) is given in the following.

**Theorem 4.3.** Let \( H \) be a Hopf structure on \( kA \) with \( \text{gr} \ H \cong kA(q) \). Then as algebra, it can be presented by generators and relations as follows:

1. if \( q = 1 \), generators: \( g, g^{-1}, e_0 \). relations: 
\[ gg^{-1} = 1 = g^{-1}g, \quad ge_0g^{-1} = e_0 + \lambda(1 - g) \text{ with } \lambda \in \{0, 1\}. \]

2. if \( q \neq 1 \) and is not a root of unity, generators: \( g, g^{-1}, e_0 \). relations: 
\[ gg^{-1} = 1 = g^{-1}g, \quad ge_0 = qe_0g. \text{ In particular, } H \text{ is isomorphic to } kA(q). \]

3. if \( q \neq 1 \) is a root of unity of order \( d \), generators: \( g, g^{-1}, e_0, p_0^d \).
relations: 
\[ gg^{-1} = 1 = g^{-1}g, \quad e_0^d = 0, \quad ge_0 = qe_0g, \quad e_0p_0^d = p_0^d e_0, \quad gp_0^d - p_0^d g = \lambda(g - g^{d+1}) \text{ with } \lambda \in k. \]

The idea of the proof is the same as that of the basic cycle case.

4.4. Firstly we consider the case of \( q = 1 \). Assume that \( H \) is a Hopf algebra on \( kA \) with \( \text{gr} \ H \cong kA(1) \). Then as algebra, it is generated by \( g, g^{-1}, e_0 \) according to Lemmas 2.1 and 4.2. So in order to get the defining relations, all we need to do is determine the deformations of \( ge_0g^{-1} = e_0 \). By

\[ \Delta(ge_0g^{-1}) = (ge_0g^{-1}) \otimes 1 + g \otimes (ge_0g^{-1}), \]
we have \( ge_0g^{-1} = e_0 + \lambda(1 - g) \) for some \( \lambda \in k \). If \( \lambda \neq 0 \), then let \( E := \frac{g}{\lambda} \).
we have \( gEg^{-1} = E + (1 - g) \). Note that \( H \) is generated by \( g, E \), therefore through the coalgebra automorphism

\[ F : \quad kA \longrightarrow kA \]
\[ g^i \mapsto g^i, \quad \forall \ i \in \mathbb{Z}, \]
\[ e_0 \mapsto e_0 \lambda, \quad e_i \mapsto e_i, \quad \forall \ i \neq 0, \]
\[ p_0^l \mapsto \lambda^l p_0^l, \quad \text{if } e_0 \text{ appears } t \text{ times in } p_0^l, \quad \forall \ i \in \mathbb{Z}, \quad \forall \ l \geq 2 \]
we can always reduce the relation in \( H \) to the equation \( ge_0g^{-1} = e_0 + (1 - g) \) when \( \lambda \neq 0 \).
On the contrary, similar to the argument in Subsection 3.7, it is not difficult to verify that the relations in Theorem 4.3 (1) are actually enough to define the algebra structure of \( H \).

4.5. Next we consider the case when \( q \neq 1 \) is not a root of unity. Assume that \( H \) is a Hopf algebra on \( kA \) with \( \text{gr} \, H \cong kA(q) \). Again, as algebra, it is generated by \( g, \, g^{-1}, \, e_0 \) according to Lemmas 2.1 and 4.2. So we need to determine the deformation of \( ge_0g^{-1} = qe_0 \) to get defining relations for \( H \).

Similar to the previous argument, we have
\[
ge e_0g^{-1} = qe_0 + \lambda(1 - g)
\]
for some \( \lambda \in k \). Now let \( \tilde{e}_0 = e_0 + \frac{\lambda}{1-q} \), then we have
\[
\tilde{g} \tilde{e}_0 \tilde{g}^{-1} = q \tilde{e}_0.
\]
By Lemma 4.1, we can have a coalgebra isomorphism for the coalgebra \( kA \) which sends \( e_0 \) to \( \tilde{e}_0 \). Now under the isomorphism, the original Hopf structure can be transported to a new one with relation
\[
ge e_0g^{-1} = qe_0.
\]
So in this case, the Hopf structures are graded, and we are done with (2) of the theorem.

4.6. Finally we deal with the case when \( q \neq 1 \) is a root of unity of order \( d \). Assume that \( H \) is a Hopf algebra on \( kA \) with \( \text{gr} \, H \cong k \mathbb{Z}(q) \). In this situation, the Hopf algebra \( H \) is generated by \( g, \, g^{-1}, \, e_0, \, p_0^d \). By a similar argument we have in the first place
\[
ge e_0 = qe_0g, \quad e_0^d = \lambda(1 - g^d), \quad gp_0^d g^{-1} = p_0^d + \alpha(1 - g^d),
\]
\[
[e_0, p_0^d] = \frac{\lambda(1 - q)}{(d - 1)q^!} e_0 (1 + g^d) + \mu(1 - g^{d+1}) \]
with some \( \lambda, \alpha, \mu \in k \). By induction we have
\[
e_0^d p_0^d = p_0^d e_0^d + d \frac{\lambda(1 - q)}{(d - 1)q^!} e_0 (1 + g^d) + d \mu e_0^{d-1}.
\]
Combined with the previous equalities, we have
\[
\lambda d \alpha(g^d - g^{2d}) + d \frac{\lambda^2(1 - q)}{(d - 1)q^!} (1 - g^{2d}) + d \mu e_0^{d-1} = 0.
\]
It follows from this equality that \( \lambda = \mu = 0 \). By a similar argument we can prove the relations are enough defining relations for \( H \).
4.7. To conclude this section, we summarize all the Hopf structures arising from the linear chain $A$. Let $kA(1, \lambda)$ denote the Hopf algebra defined by (1) of Theorem 4.3, and $kA(d, q, \lambda)$ the Hopf algebra defined by (3) of Theorem 4.3. The condition of isomorphism is also given.

**Theorem 4.4.** Let $A$ be the linear chain and $kA$ the associated path coalgebra. All the Hopf algebra structures are given by $kA(q)$ with $0 \neq q \in k$ an arbitrary element (graded), $kA(1, \lambda)$ and $kA(d, q, \lambda)$ with $q \neq 1$ a primitive $d$-th root of unity (non-graded). We have Hopf algebra isomorphism $kA(q) \cong kA(q')$ if and only if $q = q'$; $kA(1, \lambda) \cong kA(1, \lambda')$ if and only if $\lambda = \lambda'$; and $kA(d, q, \lambda) \cong kA(d', q', \lambda')$ if and only if $d = d'$, $q = q'$, $\lambda = \lambda'$.

5. Applications

In this section, we give some direct applications of the obtained classification results to bialgebras of type one of Nichols [20], and simple-pointed Hopf algebras of Radford [21].

5.1. Recall that the bialgebras of type one in the sense of Nichols are pointed Hopf algebras that are generated as algebras by group-like and skew-primitive elements. In the quiver terminology, such Hopf algebras live in Hopf quivers and as algebras are generated by vertices and arrows. We are going to investigate all the possible bialgebras of type one living in minimal Hopf quivers. Note that not all pointed Hopf algebras are bialgebras of type one. Later on we can see that in general quiver Hopf algebras are not so.

The case of loop quiver is trivial. The quiver Hopf algebra is generated by the only vertex and arrow, hence is bialgebra of type one. For the cases of basic $n$-cycles ($n \geq 2$) and the linear chain, things turn out to be very different. The idea of classifying bialgebras of type one is similar to those of quiver Hopf algebras. First we classify the graded ones, and then determine all the possible deformations.

5.2. We deal with the basic cycle case first. Keep the notations of Section 3. The graded bialgebras of type one are the graded sub-Hopf algebras on Hopf quivers generated by vertices and arrows, so the classification of such algebras can be obtained as a direct consequence of Lemma 3.2.

**Lemma 5.1.** Let $BZ(q)$ denote the sub-Hopf algebra of $kZ(q)$ generated by vertices and arrows.

1. If $q = 1$, then $BZ(q) \cong kZ(1)$.
2. If $\text{ord}(q) = d > 1$, then $BZ(q)$ can be presented by generators $g, a_0$ with relations $g^n = 1$, $a_0^d = 0$, $ga_0 = qa_0g$. 
We remark that when \( q \) is a non-trivial root of unity, the bialgebra of type one \( BZ(q) \) is a very interesting Hopf algebra. In particular, when \( \text{ord}(q) = n \), it is the well-known Taft algebra \([24]\). It also appears as the Borel subalgebra of Lusztig’s small quantum \( sl_2 \) \([17]\). For general \( \text{ord}(q) = d \), the algebra \( BZ(q) \) is a generalization of the Taft algebra.

It follows directly from Lemmas 3.2 and 5.1 that the only finite dimensional subcoalgebras of \( kZ \) which admit Hopf algebra structures are \( kZ[d] \) with \( d = \text{ord}(q) \), where \( q \) is a non-trivial \( n \)-th root of unity. Hence \( d \) is a factor of \( n \). This is the Theorem 3.1 of \([5]\), which plays an important role in classifying the monomial Hopf algebras. The argument in this paper simplifies the old one.

For the non-graded bialgebras of type one living in the Hopf quiver \( Z \), it suffices to determine the preferred deformations of \( BZ(q) \). This actually was done in \([5]\), though not in terms of deformation. It turns out that these are all the connected monomial Hopf algebras. For completeness, we include the result here.

**Theorem 5.2.** (\([5]\), Theorem 3.6) All the possible preferred deformations of \( BZ(q) \) can be presented by generators \( g, a \) with relations

\[
g^n = 1, \quad a^d = \mu(1 - g^d), \quad ga = qag,
\]

where \( \mu \in \{0, 1\} \).

5.3. Now we consider the case of linear chain. Keep the notations of Section 4. First by Lemma 4.2, we can classify the graded bialgebras of type one.

**Lemma 5.3.** Let \( B\mathcal{A}(q) \) denote the sub-Hopf algebra of \( k\mathcal{A}(q) \) generated by vertices and arrows.

1. If \( q \) is not a non-trivial root of unity, then \( B\mathcal{A}(q) \cong k\mathcal{A}(q) \).
2. If \( q \) is a root of unity with order \( \text{ord}(q) = d > 1 \), then \( B\mathcal{A}(q) \) can be presented by generators \( g, g^{-1}, e_0 \) with relations \( gg^{-1} = 1 = g^{-1}g, \quad e_0^d = 0, \quad \ge_0 = qe_0g \).

Next we consider the possible deformation of \( B\mathcal{A}(q) \). For the case of \( q \) being not a non-trivial root of unity, it is done in Theorem 4.3. When \( q \) is a root of unity with order \( \text{ord}(q) = d > 1 \), it suffices to deform the relations \( e_0^d = 0, \quad \ge_0 = qe_0g \). By the same argument as those in Subsections 3.5 and 4.6, we can always preserve the relation \( \ge_0 = qe_0g \), while deform \( e_0^d = 0 \) to \( e_0^d = \lambda(1 - g^d) \). In this situation, as coalgebra \( B\mathcal{A}(q) \) is identical to the subcoalgebra \( k\mathcal{A}[d - 1] \) of the path coalgebra \( k\mathcal{A} \), namely the subcoalgebra spanned by paths of length strictly less than \( d \). We record the results as follows.
Theorem 5.4. If \( H \) is a bialgebra of type one with \( Q(H) = A \), then \( H \) is one of the following:

1. \( kA(q) \), where \( q \) is not a non-trivial root of unity.
2. \( H \) can be presented by generators \( g, g^{-1}, e \) with relations \( gg^{-1} = 1 = g^{-1}g, \ ge = qeg, \ e^{d} = \mu(1 - g^{d}) \), where \( q \) is a root of unity of order \( d > 1 \) and \( \mu \in \{0, 1\} \).

We remark that when \( q \) is not a root of unity, the Hopf algebra \( kA(q) \) is the well-known Borel subalgebra of the quantum group \( U_{\nu}(sl_{2}) \), here \( \nu = \sqrt{q} \); while \( q \) is a non-trivial root of unity, the Hopf algebra in (2), denoted by \( BA(q, \mu) \), is closely related to the (Borel subalgebra of) De Concini-Kac quantum group \( U_{\nu}(sl_{2}) \) at roots of unity \([10]\).

5.4. By comparing the previous classification of bialgebras of type one with the classification of quiver Hopf algebras, it is clear that there is no hope to extend the Gabriel type theorem of Van Oystaeyen and Zhang to non-graded pointed Hopf algebras. For example, when \( q \) is a non-trivial root of unity, the Hopf algebras \( BZ(q) \) and \( BA(q) \) do have non-trivial deformations, while the quiver Hopf algebras \( kZ(q) \) and \( kA(q) \) do not in general. In other words, these Hopf algebras living in a proper subcoalgebra of \( kZ \) and \( kA \) can not be extended to the whole path coalgebras, hence they are not sub-Hopf algebras of any Hopf algebra structures on the path coalgebras of the corresponding Hopf quivers.

5.5. Now we apply the obtained results to simple-pointed Hopf algebras. A Hopf algebra \( H \) is said to be simple-pointed, if it is pointed, not cocommutative, and if \( L \) is a proper sub-Hopf algebra of \( H \), then \( L \subseteq kG(H) \). Very naturally the “simplicity” of such Hopf algebras can be visualized by their corresponding Hopf quivers. We remark that the definition of simple-pointed Hopf algebras adopted here is from \([26]\) which includes infinite-dimensional situation. In there the complete list of classification was obtained by different methods from ours.

Theorem 5.5. A Hopf algebra \( H \) is simple-pointed if and only if its graded version \( gr H \) is simple-pointed, if and only if it is a bialgebra of type one living in either the Hopf quiver \( Z \) or \( A \). Hence it is one of the following: \( kZ(1), BZ(q, \mu) \) (\( q \) is a root of unity of order \( > 1 \), \( \mu \in \{0, 1\} \)), \( kA(q) \) (\( q \) is not a non-trivial root of unity), \( kA(1, 1), BA(q, \mu) \) (\( q \) is a root of unity of order \( > 1 \), \( \mu \in \{0, 1\} \)).

Proof. Assume that \( H \) is simple-pointed, then the Hopf quiver \( Q(H) \) must be connected. It is not hard to deduce from the definition of simple-pointed
Hopf algebras that there is exactly one arrow going from the unit of the group $G(H)$ to some non-unit element. So by the definition of Hopf quiver it follows at once that $G(H)$ must be a cyclic group and $Q(H)$ must be either $\mathbb{Z}$ or $A$. Now the theorem follows directly from Lemma 5.1, Theorem 5.2, Lemma 5.3, and Theorem 5.4.

□

Acknowledgements: The authors were supported partially by the NSF of China (Grant No. 10501041, 10526037, 10601052). Part of the work was done while the first author was visiting the Abdus Salam International Centre for Theoretical Physics (ICTP). He expresses his sincere gratitude to the ICTP for its support.

References

[1] N. Andruskiewitsch, H.-J. Schneider, Lifting of quantum linear spaces and pointed Hopf algebras of order $p^3$, J. Algebra 209 (1998) 658-691.
[2] N. Andruskiewitsch, H.-J. Schneider, On the classification of finite-dimensional pointed Hopf algebras, Ann. Math., to appear. math.QA/0502157.
[3] I. Assem, D. Simson, A. Skowronski, Elements of the Representation Theory of Associative Algebras. Vol. 1. Techniques of Representation Theory. London Mathematical Society Student Texts, 65. Cambridge University Press, Cambridge, 2006.
[4] G. Bergman, The diamond lemma for ring theory, Adv. Math. 29 (1978) 178-218.
[5] X.-W. Chen, H.-L. Huang, Y. Ye, P. Zhang, Monomial Hopf algebras, J. Algebra 275 (2004) 212-232.
[6] W. Chin, S. Montgomery, Basic coalgebras, Modular interfaces (Riverside, CA, 1995), 41-47, AMS/IP Stud. Adv. Math. 4, Amer. Math. Soc., Providence, RI, 1997.
[7] C. Cibils, A quiver quantum group, Comm. Math. Phys. 157 (1993) 459-477.
[8] C. Cibils, M. Rosso, Algèbres des chemins quantiques, Adv. Math. 125 (1997) 171-199.
[9] C. Cibils, M. Rosso, Hopf quivers, J. Algebra 254 (2002) 241-251.
[10] C. De Concini, V.G. Kac, Representations of quantum groups at roots of 1, in “Operator Algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory”, ed. A. Connes etc (2000), Birkhäuser, 471-506.
[11] P. Gabriel, Unzerlegbare Darstellungen I, Manuscripta Math. 6 (1972) 71-103.
[12] P. Gabriel, A.V. Roiter, Representations of finite-dimensional algebras. Translated from the Russian. With a chapter by B. Keller. Reprint of the 1992 English translation. Springer-Verlag, Berlin, 1997.
[13] M. Gerstenhaber, S.D. Schack, Bialgebra cohomology, deformations, and quantum groups, Proc. Nat. Acad. Sci. 87 (1990) 478-481.
[14] E.L. Green, Constructing quantum groups and Hopf algebras from coverings, J. Algebra 176 (1995) 12-33.
[15] E.L. Green, Ø. Solberg, Basic Hopf algebras and quantum groups, Math. Z. 229 (1998) 45-76.
[16] C. Kassel, Quantum groups, Graduate Texts in Mathematics, 155. Springer-Verlag, New York, 1995.
[17] G. Lusztig, Finite dimensional Hopf algebras arising from quantized universal enveloping algebras, Journal of the AMS 3 (1990) 257-296.
[18] S. Montgomery, Hopf Algebras and Their Actions on Rings, CBMS Regional Conf. Series in Math. 82, Amer. Math. Soc., Providence, RI, 1993.
[19] S. Montgomery, Indecomposable coalgebras, simple comodules and pointed Hopf algebras, Proc. of the Amer. Math. Soc. 123 (1995) 2343-2351.
[20] W.D. Nichols, Bialgebras of type one, Communications in Algebra 6(15) (1978) 1521-1552.
[21] D. Radford, Finite-dimensional simple-pointed Hopf algebras, J. Algebra 211 (1999) 686-710.
[22] M. Rosso, Quantum groups and quantum shuffles, Invent. Math. 133 (1998) 399-416.
[23] M. Sweedler, Hopf algebras, W. A. Benjamin, Inc., New York, 1969.
[24] E.J. Taft, The order of the antipode of finite dimensional Hopf algebras, Prc. Nat. Acad. Sci. USA 68 (1971) 2631-2633.
[25] F. Van Oystaeyen, P. Zhang, Quiver Hopf algebras, J. Algebra 280(2) (2004) 577-580.
[26] P. Zhang, Hopf algebras on Schurian quivers, Comm. Algebra 34(11) (2006) 4065-4082.

School of Mathematics, Shandong University, Jinan 250100, China
E-mail address: hualin@sdu.edu.cn

Department of Mathematics, University of Science and Technology of China, Hefei 230026, China
E-mail address: yeyu@ustc.edu.cn

Department of Mathematics, University of Science and Technology of China, Hefei 230026, China
E-mail address: qzhao@mail.ustc.edu.cn