Minimum strongly biconnected spanning directed subgraph problem

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Abstract

Let \( G = (V, E) \) be a strongly biconnected directed graph. In this paper we consider the problem of computing an edge subset \( H \subseteq E \) of minimum size such that the directed subgraph \( (V, H) \) is strongly biconnected.

Keywords: Directed graphs, Connectivity, Approximation algorithms, Graph algorithms, strongly connected graphs

1. Introduction

In 2010, Wu and Grumbach [44] introduced the concept of strongly biconnected directed graphs. A directed graph \( G = (V, E) \) is strongly biconnected if \( G \) is strongly connected and the underlying undirected graph of \( G \) has no articulation point (Note that the underlying graph of \( G \) is connected if \( G \) is strongly connected). Let \( G = (V, E) \) be a strongly biconnected directed graph. In this paper we consider the problem of computing an edge subset \( H \subseteq E \) of minimum size such that the directed subgraph \( (V, H) \) is strongly biconnected. Observe that optimal solutions for minimum strongly connected spanning subgraph problem are not necessarily strongly biconnected, as shown in Figure 1.

The minimum strongly connected spanning subgraph problem is NP-complete [15]. Note that a strongly biconnected graph has a strongly biconnected spanning subgraph with \( n \) edges if and only if it has a hamiltonian cycle. Therefore, the minimum strongly biconnected spanning subgraph problem is also NP-complete. Khuller et al. [30], Zhao et al. [45] and Vetta [43] provided approximation algorithms for the minimum strongly connected spanning subgraph problem. Wu and Grumbach [44] introduced the concept of strongly biconnected directed graph and strongly biconnected components. Strongly biconnected directed graphs and twinless strongly connected directed graphs have been received a lot of attention in [44, 15, 4, 18, 22, 23, 24, 26, 27, 28, 29]. Articulation points and blocks of an undirected graph can be calculated in \( O(n + m) \) time using Tarjan’s algorithm [38, 40].
Figure 1: (a) A strongly biconnected directed graph. (b) An optimal solution for the minimum strongly connected spanning subgraph problem. (c) An optimal solution for the minimum strongly biconnected spanning subgraph problem. Note that this subgraph has strong articulation points but its underlying graph has no articulation points.
Tarjan [38] gave the first linear time algorithm for calculating strongly connected components. Pearce [37] and Nuutila et al. [36] provided improved versions of Tarjan’s algorithm. Efficient linear time algorithms for finding strongly connected components were given in [41, 19, 7, 32, 8]. Strong articulation points and strong bridges can be computed in linear time in directed graphs using the algorithms of Italiano et al. [20, 21, 11]. The algorithms of Italiano et al. [20, 21] are based on a strong connection between strong articulation points, strong bridges and dominators in flowgraphs. Dominators can be found efficiently in flowgraphs [1, 5, 17, 16, 31]. In the following section we consider the minimum strongly biconnected spanning subgraph problem.

2. Approximation algorithms for the minimum strongly biconnected spanning subgraph problem

In this section we study the minimum strongly biconnected spanning subgraph problem. The following lemma shows an obvious connection between the size of an optimal solution for the minimum strongly biconnected spanning subgraph problem and the size of an optimal solution for the minimum strongly connected spanning subgraph problem.

Lemma 2.1. Let $G = (V, E)$ be a strongly biconnected directed graph. Then the size of an optimal solution for the minimum strongly connected spanning subgraph problem is less than or equal to the size of an optimal solution for the minimum strongly biconnected spanning subgraph problem.

Proof. Let $t$ be the size of an optimal solution for the minimum strongly connected spanning subgraph problem. By definition, every strongly biconnected spanning subgraph $G_1 = (V, E_1)$ of $G$ is strongly connected. Therefore, we have $|E_1| \geq t$. □

A strongly connected spanning subgraph with size $2(n - 1)$ of a strongly connected graph $G = (V, E)$ can be constructed by computing the union of outgoing branching rooted at $v \in V$ and incoming branching rooted at $v$ ([12, 30]). But this subgraph is not necessarily strongly biconnected.

Algorithm 2.2 can compute a feasible solution for the minimum strongly biconnected spanning subgraph problem.

Lemma 2.3. The output of Algorithm 2.2 is strongly biconnected.

Proof. Lines 1–9 computes a strongly connected spanning subgraph $G_v = (V, E_v)$ of $G$ since there is a path from $v$ to $w$ and a path from $w$ to $v$ for all $w \in V$. The while loop of lines 10–14 removes all articulation points of the underlying graph of $G_v = (V, E_v)$. □
Algorithm 2.2.
Input: A strongly biconnected graph $G = (V, E)$.
Output: a strongly biconnected subgraph $G_v = (V, E_v)$ in $G$

1. select a vertex $v \in V$
2. $E_v \leftarrow \emptyset$
3. build a spanning tree $T$ rooted at $v$ in $G$
4. for each edge $(u, w)$ in $T$ do
   5. $E_v \leftarrow E_v \cup \{(u, w)\}$
6. build $G_r = (V, E_r)$, where $E_r = \{(w, u) \mid (w, u) \in E\}$
7. construct a spanning tree $T_v$ rooted at $v$ in $G_r$
8. for each edge $(u, w)$ in $T_v$ do
   9. $E_v \leftarrow E_v \cup \{(w, u)\}$
10. while the underlying graph of $G_v = (V, E_v)$ is not biconnected do
11.   compute the strongly biconnected components of $G_v$
12.   find an edge $(u, w) \in E \setminus E_v$ such that $u, w$ do not
13.      belong to the same strongly connected component of $G_v$
14.   $E_v \leftarrow E_v \cup \{(u, w)\}$.

The following lemma shows that the approximation factor of Algorithm 2.2 is 3

Lemma 2.4. The number of edges in the output of Algorithm 2.2 is at most $3(n - 1)$.

Proof. Lines 1–9 computes a strongly connected spanning subgraph $G_v = (V, E_v)$ of size $2n - 2$ since each spanning tree has only $n - 1$ edges. The while loop of lines 10–14 removes all articulation points of the underlying graph of $G_v = (V, E_v)$ by adding at most $n - 1$ edges to the subgraph $G_v = (V, E_v)$ because the number of strongly biconnected components of any directed graph is at most $n$.

Theorem 2.5. Algorithm 2.2 runs in $O(nm)$ time.

Proof. A spanning tree of a strongly biconnected graph can be constructed in $O(n + m)$ time using depth first search or breadth first search. Furthermore, lines 10–14 take $O(nm)$ time since the number of strongly biconnected components of any directed graph is at most $n$.

Lemma 2.6. Let $G = (V, E)$ be a strongly biconnected directed graph. Let $G_1 = (V, E_1)$ be the output of a $t$-approximation algorithm for the minimum strongly connected spanning subgraph problem for the input $G$. A strongly
A strongly biconnected subgraph can be obtained from \( G_1 = (V, E_1) \) with size at most \((1 + t)h\) in polynomial time, where \( h \) is the size of an optimal solution for the minimum strongly biconnected spanning subgraph problem.

Proof. A strongly biconnected subgraph can be obtained from \( G_1 = (V, E_1) \) by running the while loop of lines 10–14 of Algorithm 2.2 on \( G_1 = (V, E_1) \). By lemma 2.1, the size of an optimal solution for the minimum strongly connected spanning subgraph problem is less than or equal to the size of an optimal solution for the minimum strongly biconnected spanning subgraph problem. This while loop takes \( O(nm) \) time. The while loop of lines 10–14 adds at most \( n-1 \) edges to \( G_1 \). Clearly, each strongly biconnected spanning subgraph of \( G \) has at least \( n \) edges. \( \square \)

The following lemma shows an obvious connection between the size of an optimal solution for the minimum strongly biconnected spanning subgraph problem and the size of an optimal solution for the minimum 2-vertex connected spanning undirected subgraph problem.

**Lemma 2.7.** Let \( G = (V, E) \) be a strongly biconnected directed graph. Then the size of an optimal solution for minimum 2-vertex connected spanning undirected subgraph problem is less than or equal to the size of an optimal solution for the minimum strongly biconnected spanning subgraph problem.

Proof. Let \( t \) be the size of an optimal solution for the minimum strongly connected spanning subgraph problem. By definition, the underlying graph of every strongly biconnected spanning subgraph \( G_1 = (V, E_1) \) is biconnected. Therefore, we have \( |E_1| \geq t \). \( \square \)

**Lemma 2.8.** There is a \( 17/6 \) approximation algorithm for the minimum strongly biconnected spanning subgraph problem.

Proof. Let \( G = (V, E) \) be a strongly biconnected directed graph. A strongly biconnected subgraph can be obtained from \( G \) by calculating a strongly connected spanning subgraph of \( G \) and a biconnected spanning subgraph of the underlying graph of \( G \). Moreover, let \( h \) be the size of an optimal solution for the minimum strongly biconnected spanning subgraph problem. Let \( i \) be the size of an optimal solution for the minimum strongly connected spanning subgraph problem and let \( s \) be the size of an optimal solution for the minimum 2-vertex connected spanning undirected subgraph problem. By lemma 2.1 we have \( h \geq i \). Moreover, by lemma 2.7 we have \( h \geq s \). A feasible solution of size at most \((17/6)h\) for the minimum strongly biconnected spanning subgraph problem can be obtained by running Vetta’s algorithm [43] on \( G \) and the algorithm of Vempala and Vetta [42] on the underlying graph of \( G \). \( \square \)
3. Open Problems

Results of Mader \[34, 35\] imply that the number of edges in each minimal $k$-vertex-connected undirected graph is less than or equal to $kn$ \[6\]. Results of Edmonds \[10\] and Mader \[33\] imply that the number of edges in each minimal $k$-vertex-connected directed graph is at most $2kn$ \[6\]. Jaberi \[24\] proved that each minimal 2-vertex-strongly biconnected directed graph has at most $7n$ edges. The proof in is based on results of Mader \[34, 35, 33\] and Edmonds \[10\].

We leave as open problem whether the number of edges in each minimal strongly biconnected directed graph is at most $2n$ edges. An important question is whether there are better algorithms for the problems in \[28, 24, 9\].

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