SAMPLING AND INTERPOLATION IN RADIAL WEIGHTED SPACES OF ANALYTIC FUNCTIONS

A. BORICHEV, R. DHUEZ, K. KELLAY

Abstract. We obtain sampling and interpolation theorems in radial weighted spaces of analytic functions for weights of arbitrary (more rapid than polynomial) growth. We give an application to invariant subspaces of arbitrary index in large weighted Bergman spaces.

1. Introduction

Let \( h : [0, 1) \rightarrow [0, +\infty) \) be an increasing function such that \( h(0) = 0 \), and \( \lim_{r \to 1} h(r) = +\infty \). We extend \( h \) by \( h(z) = h(|z|), \ z \in \mathbb{D} \), and call such \( h \) a weight function. Denote by \( A_h(\mathbb{D}) \) the Banach space of holomorphic functions on the unit disk \( \mathbb{D} \) with the norm
\[
\|f\|_h = \sup_{z \in \mathbb{D}} |f(z)|e^{-h(z)} < +\infty.
\]

A subset \( \Gamma \) of \( \mathbb{D} \) is called a sampling set for \( A_h(\mathbb{D}) \) if there exists \( \delta > 0 \) such that for every \( f \in A_h(\mathbb{D}) \) we have
\[
\delta \|f\|_h \leq \|f\|_{h, \Gamma} = \sup_{z \in \Gamma} |f(z)|e^{-h(z)}.
\]

A subset \( \Gamma \) of \( \mathbb{D} \) is called an interpolation set for \( A_h(\mathbb{D}) \) if for every function \( a \) defined on \( \Gamma \) such that \( \|a\|_{h, \Gamma} < \infty \) there exists \( f \in A_h(\mathbb{D}) \) such that
\[
a = f \mid \Gamma.
\]
In this case there exists \( \delta = \delta(h, \Gamma) > 0 \) such that for every \( a \) with \( \|a\|_{h, \Gamma} < \infty \) we can find such \( f \in A_h(\mathbb{D}) \) with
\[
\delta \|f\|_h \leq \|a\|_{h, \Gamma}.
\]

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Next we assume that $h \in C^2(D)$, and

$$
\Delta h(z) = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) h(x + iy) \geq 1, \quad z \in D.
$$

We consider the weighted Bergman spaces

$$
A^p_h(D) = \{ f \in \text{Hol} (D) : \| f \|^p_{p,h} = \int_D |f(z)|^p e^{-ph(z)} dm(z) < \infty \},
$$

where $dm$ is area measure, $1 \leq p < \infty$.

A subset $\Gamma$ of $D$ is a sampling set for $A^p_h(D)$ if

$$
\| f \|^p_{p,h,\Gamma} \asymp \| f \|^p_{p,h}, \quad \Gamma = \sum_{z \in \Gamma} e^{-ph(z)} |f(z)|^p \Delta h(z).
$$

A subset $\Gamma$ of $D$ is an interpolation set for $A^p_h(D)$ if there exists $\delta > 0$ such that for every function $a$ defined on $\Gamma$ such that $\| a \|^p_{p,h,\Gamma} < \infty$ there exists $f \in A^p_h(D)$ such that

$$
a = f \big|_{\Gamma}, \quad \delta \| f \|_{p,h} \leq \| a \|_{p,h,\Gamma}.
$$

For a motivation of these definitions let us consider the case $p = 2$. Then $A^2_h(D)$ is a Hilbert space of analytic functions in the unit disc, and we define the reproducing kernel $k_\lambda \in A^2_h(D)$, $\lambda \in D$, by

$$
\langle k_\lambda, f \rangle = f(\lambda), \quad f \in A^2_h(D).
$$

For regular $h$ considered in our paper (for precise conditions on $h$ see the next section) we have (see, for example, Lemmas 3.3 and 4.1 (ii) below)

$$
\| k_\lambda \|^2 \asymp e^{2h(\lambda)} \Delta h(\lambda).
$$

Therefore, a family of normalized reproducing kernels $\{ k_\lambda / \| k_\lambda \| \}_{\lambda \in \Lambda}$ is a frame in $A^2_h(D)$, that is

$$
\sum_{\lambda \in \Lambda} \left| \langle f, \frac{k_\lambda}{\| k_\lambda \|} \rangle \right|^2 \asymp \| f \|^2, \quad f \in A^2_h(D),
$$

if and only if $\Lambda$ is a sampling set for $A^2_h(D)$. In a similar way, a family $\{ k_\lambda / \| k_\lambda \| \}_{\lambda \in \Lambda}$ is a Riesz basic sequence in its closed linear span in $A^2_h(D)$, that is

$$
\| \sum_{\lambda \in \Lambda} a_\lambda \frac{k_\lambda}{\| k_\lambda \|} \|^2 \asymp \sum_{\lambda \in \Lambda} |a_\lambda|^2
$$

for any sequence $\{a_\lambda \}_{\lambda \in \Lambda}$ of complex numbers, if and only if $\Lambda$ is an interpolation set for $A^2_h(D)$.

The famous Feichtinger conjecture (see, for example, [9, 10]) claims that any frame in a Hilbert space is a finite union of Riesz basic sequences. For families of normalized reproducing kernels in $A^2_h(D)$ this
conjecture translates into the question on whether any sampling set for $A^2_h(D)$ is a finite union of interpolation sets for $A^2(D)$. The answer is positive for $h$ we consider in this paper (as follows from Theorems 2.2 and 2.4).

In the plane case, if $h : [0, \infty) \to [0, +\infty)$ is an increasing function such that $h(0) = 0$, $\lim_{r \to \infty} h(r) = +\infty$, we extend $h$ by $h(z) = h(|z|)$, $z \in \mathbb{C}$, and consider the Banach space $A^h(\mathbb{C})$ of entire functions with the norm

$$
\|f\|_h = \sup_{z \in \mathbb{C}} |f(z)| e^{-h(z)} < +\infty,
$$

and the weighted Fock spaces

$$
A^p_h(\mathbb{C}) = \left\{ f \in \text{Hol}(\mathbb{C}) : \|f\|_{p,h}^p = \int_{\mathbb{C}} |f(z)|^p e^{-ph(z)} dm_2(z) < \infty \right\}.
$$

and define the sampling and the interpolation subsets for the spaces $A^h(\mathbb{C})$, $A^p_h(\mathbb{C})$, $1 \leq p < \infty$, like above, in the disc case.

K. Seip and R. Wallstén [20, 24] described sampling and interpolation sets for the Fock spaces $A^h(\mathbb{C})$, $A^2_h(\mathbb{C})$, with $h(z) = c |z|^2$, in terms of Beurling type densities. Later on, K. Seip [21] obtained such a description for the Bergman type spaces $A_h(D)$, $h(z) = \alpha \log \frac{1}{1-|z|}$, $\alpha > 0$, and for $A^2_h(D)$ ($= A^2_h(D)$ with $h = 0$). For motivation and some applications of these results, for example to Gabor wavelets, see a survey [8] by J. Bruna.

The results of K. Seip were extended to the Fock spaces $A^p_h(\mathbb{C})$ (with $h$ not necessarily radial) such that $\Delta h \asymp 1$ in [3] and [19], and to Bergman spaces $A^p_h(D)$, $\Delta h(z) \asymp (1 - |z|^2)^{-2}$, in [22]. Yu. Lyubarskii and K. Seip [16] obtained such results for the spaces $A^h(\mathbb{C})$, $A^2_h(\mathbb{C})$, with $h(z) = m(\arg z)|z|^2$, $m$ being a $2\pi$ periodic 2-trigonometrically convex function. For more results and references see the books [13] and [23].

Recently, N. Marco, X. Massaneda and J. Ortega-Cerdà [18] described sampling and interpolation sets for the Fock spaces $A^p_h(\mathbb{C})$ for a wide class of $h$ such that $\Delta h$ is a doubling measure. These results rely mainly upon the method used by A. Beurling [4] in his work on band-limited functions and on Hörmander-type weighted estimates for the $\bar{\partial}$ equation. Therefore, it is not clear whether they can be extended to weight functions $h$ having more than polynomial growth at infinity.

The aim of our work is to extend previous results to the case of radial $h$ of arbitrary (more than polynomial) growth. For this, we use the method proposed by Yu. Lyubarskii and K. Seip in [16]. First we produce peak functions with precise asymptotics. For example, for
every \( z \in \mathbb{D} \) we find \( f_z \in \mathcal{A}_h(\mathbb{D}) \) such that

\[
|f_z(w)| \asymp e^{h(w) - |w - z|^2 \Delta h(z)/4}
\]

in a special neighborhood of \( z \). (For a different type of peak functions in \( \mathcal{A}_h(\mathbb{D}) \) see [11].) These peak functions permit us then to reduce our problems to those in the standard Fock spaces \( \mathcal{A}_h^p(\mathbb{C}) \), \( h(z) = |z|^2 \).

The construction of peak functions in [16] is based on sharp approximation of \( h \) by \( \log |f|, f \in \text{Hol}(\mathbb{D}) \), obtained in the work of Yu. Lyubarskii and M. Sodin [17]. Here we need a similar construction for radial \( h \) of arbitrary (more than polynomial) growth. This is done in a standard way: we atomise the measure \( \Delta h(z) dm(z) \) and obtain a discrete measure \( \sum \delta_{z_n} \). Since our \( h \) are radial, we try to get sufficiently symmetric sequence \( \{z_n\} \):

\[
\{z_n\} = \bigcup_k \{s_k e^{2\pi im/N_k}\}, \quad s_k \to 1, \quad N_k \to \infty.
\]

For approximation of general \( h \) see the paper [15] by Yu. Lyubarskii and E. Malinnikova and the references there.

Our paper is organized as follows. The main results are formulated in Section 2. We construct peak functions in Section 3. Technical lemmas on sampling and interpolation sets are contained in Section 4. In Section 5 we obtain auxiliary results on asymptotic densities. The theorems on sampling sets are proved in Section 6 and the theorems on interpolation sets are proved in Section 7. In Sections 6–7 we deal with the disc case. Some changes necessary to treat the plane case are discussed in Section 8. Finally, in Section 9 we give an application of our results to subspaces of \( \mathcal{A}_h^p(\mathbb{D}) \) invariant under multiplication by the independent variable.

We do not discuss here the following interesting fact: the families of interpolation (sampling) sets are not monotonic with respect to the weight function \( h \). Also, we leave open other questions related to our results, including whether our interpolation sets are just sets of free interpolation, that is (say, for the spaces \( \mathcal{A}_h^p(\mathbb{C}) \)) the sets \( \Lambda \subset \mathbb{C} \) such that

\[
\ell^\infty(\Lambda) \cdot \mathcal{A}_h^p(\mathbb{C})|\Lambda = \mathcal{A}_h^p(\mathbb{C})|\Lambda.
\]

We hope to return to these questions later on.

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2. Main results

From now on in the disc case we assume that the function

\[
\rho(r) = \left( (\Delta h)(r) \right)^{-1/2}, \quad 0 \leq r < 1,
\]
decreases to 0 near the point 1, and
\[ \rho'(r) \rightarrow 0, \quad r \rightarrow 1. \] (2.1)
Then for any \( K > 0 \), for \( r \in (0, 1) \) sufficiently close to 1, we have \([r - K\rho(r), r + K\rho(r)] \subset (0, 1)\), and
\[ \rho(r + x) = (1 + o(1))\rho(r), \quad |x| \leq K\rho(r), \quad r \rightarrow 1. \] (2.2)
Furthermore, we assume that either (I) the function \( r \mapsto \rho(r)(1 - r)^{-C} \) increases for some \( C < \infty \) and for \( r \) close to 1 or (II) \( \rho'(r) \log 1/\rho(r) \rightarrow 0, \quad r \rightarrow 1. \)
Typical examples for (I) are
\[ h(r) = \log \log \frac{1}{1 - r} \cdot \log \frac{1}{1 - r}, \quad h(r) = \frac{1}{1 - r}, \quad r \rightarrow 1; \]
a typical example for (II) is
\[ h(r) = \exp \frac{1}{1 - r}, \quad r \rightarrow 1. \]
Denote by \( D(z, r) \) the disc of radius \( r \) centered at \( z \), \( D(r) = D(0, r) \). Given \( z, w \in \mathbb{D} \), we define
\[ d_\rho(z, w) = \frac{|z - w|}{\min(\rho(z), \rho(w))}. \]
We say that a subset \( \Gamma \) of \( \mathbb{D} \) is \( d_\rho \)-separated (with constant \( c \)) if
\[ \inf \{d_\rho(z, w) : z, w \in \Gamma, \ z \neq w \} \geq c > 0. \]
Given \( \Gamma \subset \mathbb{D} \), we define its lower \( d_\rho \)-density
\[ D^-_\rho(\Gamma, \mathbb{D}) = \lim_{R \rightarrow \infty} \liminf_{|z| \rightarrow 1, \ z \in \mathbb{D}} \frac{\text{Card}(\Gamma \cap D(z, R\rho(z)))}{R^2}, \]
and its upper \( d_\rho \)-density
\[ D^+\rho(\Gamma, \mathbb{D}) = \limsup_{R \rightarrow \infty} \limsup_{|z| \rightarrow 1, \ z \in \mathbb{D}} \frac{\text{Card}(\Gamma \cap D(z, R\rho(z)))}{R^2}. \]
We remark here that by (2.2), for fixed \( R, 0 < R < \infty \), we have
\[ \lim_{|z| \rightarrow 1, \ z \in \mathbb{D}} \frac{1}{\pi R^2} \int_{D(z, R\rho(z))} \Delta h(z) \ dm_2(z) = 1. \]
We could compare these densities \( D^\pm \) to those defined in the case \( \rho(z) \approx 1 - |z| \) by K. Seip in [21]:

\[
D^-(\Gamma, \mathbb{D}) = \liminf_{r \to 1^-} \inf_{\Phi \in \text{Aut}(\mathbb{D})} \frac{\sum_{z \in \Phi(\Gamma) \cap \mathbb{D}(r) \setminus \mathbb{D}(1/2)} \log \frac{1}{1-|z|}}{\log \frac{1}{1-r}}, \quad (2.3)
\]

\[
D^+(\Gamma, \mathbb{D}) = \limsup_{r \to 1^-} \sup_{\Phi \in \text{Aut}(\mathbb{D})} \frac{\sum_{z \in \Phi(\Gamma) \cap \mathbb{D}(r) \setminus \mathbb{D}(1/2)} \log \frac{1}{1-|z|}}{\log \frac{1}{1-r}}, \quad (2.4)
\]

where \( \text{Aut}(\mathbb{D}) \) is the group of the Möbius automorphisms of the unit disc. In contrast to \( D^\pm \), the densities \( D^\pm \rho \) are rather "local", and correspondingly, it is not difficult to compute \( D^\pm(\Gamma, \mathbb{D}) \) for many concrete \( \Gamma \).

**Theorem 2.1.** A set \( \Gamma \subset \mathbb{D} \) is a sampling set for \( A_h(\mathbb{D}) \) if and only if it contains a \( d_\rho \)-separated subset \( \Gamma^* \) such that \( D^-_\rho(\Gamma^*, \mathbb{D}) > \frac{1}{2} \).

**Theorem 2.2.** A set \( \Gamma \subset \mathbb{D} \) is a sampling set for \( A_h^p(\mathbb{D}) \), \( 1 \leq p < \infty \), if and only if (i) \( \Gamma \) is a finite union of \( d_\rho \)-separated subsets, and (ii) \( \Gamma \) contains a \( d_\rho \)-separated subset \( \Gamma^* \) such that \( D^-_\rho(\Gamma^*, \mathbb{D}) > \frac{1}{2} \).

**Theorem 2.3.** A set \( \Gamma \) is an interpolation set for \( A_h(\mathbb{D}) \) if and only if it is \( d_\rho \)-separated and \( D^+_\rho(\Gamma, \mathbb{D}) < \frac{1}{2} \).

**Theorem 2.4.** A set \( \Gamma \) is an interpolation set for \( A_h^p(\mathbb{D}) \), \( 1 \leq p < \infty \), if and only if it is \( d_\rho \)-separated and \( D^+_\rho(\Gamma, \mathbb{D}) < \frac{1}{2} \).

In the plane case we assume that \( h \in C^2(\mathbb{C}) \), \( \Delta h(z) \geq 1 \), \( z \in \mathbb{C} \), that the function \( \rho(r) = [(\Delta h)(r)]^{-1/2} \), \( 0 \leq r < \infty \), decreases to 0 at infinity, and that \( \rho'(r) \to 0 \), \( r \to \infty \).

Then for any \( K > 0 \), for sufficiently large \( r \), we have \( r > K \rho(r) \), and

\[
\rho(r + x) = (1 + o(1))\rho(r), \quad |x| \leq K \rho(r), \quad r \to \infty.
\]

Furthermore, we assume that either (I\( _C \)) the function \( r \mapsto \rho(r)r^C \) increases for some \( C < \infty \) and for large \( r \) or (II\( _C \)) \( \rho'(r) \log 1/\rho(r) \to 0 \) as \( r \to \infty \).

Typical examples for (I\( _C \)) are

\[
h(r) = r^2 \log \log r, \quad h(r) = r^p, \quad p > 2, \quad r \to \infty;
\]

a typical example for (II\( _C \)) is

\[
h(r) = \exp r, \quad r \to \infty.
\]
Next, we introduce \( d_\rho \), and the notion of \( d_\rho \)-separated subsets of \( \mathbb{C} \) as above. Given \( \Gamma \subset \mathbb{C} \), we define its lower \( d_\rho \)-density

\[
D^-_\rho(\Gamma, \mathbb{C}) = \liminf_{R \to \infty} \frac{\text{Card}(\Gamma \cap D(z, R\rho(z)))}{R^2},
\]

and its upper \( d_\rho \)-density

\[
D^+_\rho(\Gamma, \mathbb{C}) = \limsup_{R \to \infty} \frac{\text{Card}(\Gamma \cap D(z, R\rho(z)))}{R^2}.
\]

**Theorem 2.5.** A set \( \Gamma \subset \mathbb{C} \) is a sampling set for \( A_h(\mathbb{C}) \) if and only if it contains a \( d_\rho \)-separated subset \( \Gamma^* \) such that

\[
D^-_\rho(\Gamma^*, \mathbb{C}) > \frac{1}{2}.
\]

**Theorem 2.6.** A set \( \Gamma \subset \mathbb{C} \) is a sampling set for \( A^p_h(\mathbb{C}), 1 \leq p < \infty \), if and only if (i) \( \Gamma \) is a finite union of \( d_\rho \)-separated subsets and (ii) \( \Gamma \) contains a \( d_\rho \)-separated subset \( \Gamma^* \) such that

\[
D^-_\rho(\Gamma^*, \mathbb{C}) > \frac{1}{2}.
\]

**Theorem 2.7.** A set \( \Gamma \) is an interpolation set for \( A_h(\mathbb{C}) \) if and only if it is \( d_\rho \)-separated and

\[
D^+_\rho(\Gamma, \mathbb{C}) < \frac{1}{2}.
\]

**Theorem 2.8.** A set \( \Gamma \) is an interpolation set for \( A^p_h(\mathbb{C}), 1 \leq p < \infty \), if and only if it is \( d_\rho \)-separated and

\[
D^+_\rho(\Gamma, \mathbb{C}) < \frac{1}{2}.
\]

### 3. Peak functions

In this section we first approximate \( h \) by \( \log |f| \), for a special infinite product \( f \). Then, using this construction, and an estimate on the partial products for the Weierstrass \( \sigma \)-function, we approximate the function \( w \mapsto h(w) - |w - z|^2 \Delta h(z)/4 \) in a fixed \( d_\rho \)-neighborhood of \( z \in \mathbb{D} \).

**Proposition 3.1.** There exist sequences \( \{r_k\}, \{s_k\}, 0 = r_0 < s_0 < r_1 < \ldots r_k < s_k < r_{k+1} < \ldots < 1 \), and a sequence \( N_k, k \geq 0 \), of natural numbers, such that \( N_{k+1} \geq N_k \) for large \( k \), and

(i) \[
\lim_{k \to \infty} \frac{r_{k+1} - r_k}{\rho(r_k)} = \sqrt{2\pi}, \quad \lim_{k \to \infty} N_k(r_{k+1} - r_k) = 2\pi,
\]

(ii) \[
\lim_{k \to \infty} \frac{r_{k+1} - r_k}{r_k - r_{k-1}} = 1, \quad \lim_{k \to \infty} \frac{r_{k+1} - s_k}{r_{k+1} - r_k} = \frac{1}{2},
\]

\[
\text{if } \Lambda = \{ s_k e^{2\pi i m / N_k} \}_{j \geq 0, 0 \leq m < N_k}, \text{ and if}
\]

\[
f(z) = \lim_{r \to 1^-} \prod_{\lambda \in \Lambda \cap \mathbb{D}} \left( \frac{1 - z/\lambda}{1 - z} \right),
\]

then the products in the right hand side converge uniformly on compact subsets of the unit disc, and

\[
|f(z)| \asymp e^{h(z) \text{dist}(z, \Lambda)} \rho(z), \quad z \in \mathbb{D}.
\]
Given $s \in (0, 1)$ sufficiently close to the point 1, we can define $\{r_k\}$, $\{N_k\}$, $\{s_k\}$ and $\Lambda$ as above in such a way that $s \in \Lambda$ and (3.1) holds (uniformly in $s$).

**Proof.** (i) We choose the sequence $\{r_k\}$ in the following way: $r_0 = 0$; given $r_k$, $k \geq 0$, the number $r^*_k$ is defined by

$$
(r^*_k - r_k) \int_{r_k \leq |z| < r^*_k} \Delta h(w) \frac{dm_2(w)}{2\pi} = 2\pi,
$$

and $r_{k+1}$ is the smallest number in the interval $[r^*_k, 1)$ such that

$$
N_k = \int_{r_k \leq |z| < r_{k+1}} \Delta h(w) \frac{dm_2(w)}{2\pi} \in \mathbb{N}.
$$

Here we use that by (2.1),

$$
\Delta h(r)^{-1/2} = \rho(r) = o(1 - r), \quad r \to 1. \quad (3.2)
$$

Furthermore, since $\Delta h(r)$ increases for $r$ close to 1, we obtain that $N_k$ do not decrease for large $k$, and

$$
\lim_{k \to \infty} N_k(r_{k+1} - r_k) = 2\pi.
$$

By (2.2) we obtain

$$
\lim_{k \to \infty} \frac{(r_{k+1} - r_k)^2}{\rho(r_k)^2} = 2\pi,
$$

$$
\lim_{k \to \infty} \frac{r_{k+1} - r_k}{r_k - r_{k-1}} = 1.
$$

Next we define $s_k$ by the relations

$$
\log \frac{1}{s_k} = \frac{1}{N_k} \int_{r_k \leq |z| < r_{k+1}} \Delta h(w) \log \frac{1}{|w|} \frac{dm_2(w)}{2\pi}, \quad k \geq 0. \quad (3.3)
$$

Clearly,

$$
\log \frac{1}{r_{k+1}} < \log \frac{1}{s_k} < \log \frac{1}{r_k}, \quad k \geq 0.
$$

By (2.2) and (3.2) we have

$$
\lim_{k \to \infty} \frac{r_{k+1} - s_k}{r_{k+1} - r_k} = \frac{1}{2}.
$$

(ii) First of all we note that

$$
f(z) = \lim_{L \to \infty} \prod_{0 \leq m \leq L} \frac{1 - z^{N_m} s_k^{N_m}}{1 - z^N s_k^{N_m}}.
$$
Set

\[ W_m = \log \left| \frac{1 - z^{N_m}s_{m-N_m}}{1 - z^{N_m}s_{m}} \right|. \]

If \( m > k \), then for some constant \( c > 0 \),

\[ s_m - r \geq c \frac{m - k}{N_m}. \]

Since

\[ \frac{x}{y} \leq e^{x-y}, \quad 0 \leq x \leq y \leq 1, \quad (3.4) \]

we get

\[ r^{N_m}s_{m-N_m} \leq e^{-c(m-k)}. \]

Next we use that if \( |\zeta| \leq c < 1, |\zeta'| \leq 1 \), then

\[ \left| \log \left| \frac{1 - \zeta}{1 - \zeta'} \right| \right| \leq c_1 |\zeta|, \quad (3.5) \]

with \( c_1 \) depending only on \( c \).

Therefore,

\[ |W_m| \leq c_1 e^{-c(m-k)}, \quad m > k. \]

Summing up, we obtain

\[ \sum_{m>k} |W_m| \leq c_1 \sum_{m>k} e^{-c(m-k)} \leq c_2, \quad (3.6) \]

for some positive constants \( c_1, c_2 \).

Thus,

\[ f(z) = \prod_{m \geq 0} \frac{1 - z^{N_m}s_{m-N_m}}{1 - z^{N_m}s_{m}}. \]

Suppose that \( z \in \mathbb{D} \setminus \Lambda, r_k \leq r < r_{k+1} \), where \( r = |z| \), and for some \( d, 0 \leq d < N_k \),

\[ |\arg z - \frac{2\pi d}{N_k}| \leq \frac{\pi}{N_k}. \]

Now we set

\[ A(z) = \log |f(z)| - h(z) - \log \frac{\text{dist}(z, \Lambda)}{\rho(z)}. \]

By Green’s formula,

\[ h(r) = \int_{D(r)} \Delta h(w) \log \frac{r}{|w|} \frac{dm_2(w)}{2\pi}. \quad (3.7) \]
Therefore,
\[ A(z) + \log \frac{\text{dist}(z, \Lambda)}{\rho(z)} = \]
\[ \sum_{0 \leq m \leq k-1} \left[ \log \left| \frac{1-z^{N_m s_m^{-N_m}}}{1-z^{N_m s_m}} \right| - \int_{r_m \leq |w| < r_{m+1}} \Delta h(w) \log \frac{r}{|w|} \frac{dm_2(w)}{2\pi} \right] \]
\[ - \int_{r_k \leq |w| < r} \Delta h(w) \log \frac{r}{|w|} \frac{dm_2(w)}{2\pi} + \sum_{m \geq k} \log \left| \frac{1-z^{N_m s_m^{-N_m}}}{1-z^{N_m s_m}} \right| \]
\[ = \sum_{0 \leq m \leq k-1} U_m - V + \sum_{m \geq k} W_m. \]

First,
\[ |V| \leq \left( \sup_{r_k \leq |w| < r} \log \frac{r}{|w|} \right) \int_{r_k \leq |w| < r} \Delta h(w) \frac{dm_2(w)}{2\pi} \]
\[ \leq c N_k (r_{k+1} - r_k) \leq c, \quad (3.8) \]
for some constants \(c, c_1\).

Next we are to verify that
\[ \left| W_k - \log \frac{\text{dist}(z, \Lambda)}{\rho(z)} \right| \leq c, \quad (3.9) \]
for some constant \(c\). Indeed, \(z^{N_k s_k^{-N_k}} \leq c_1\), for some constant \(c_1 < 1\), and it remains to estimate \(W^*(z)\), where
\[ W^*(\zeta) = \log \left| \frac{\rho(r_k)(1 - \zeta^{N_k s_k^{-N_k}})}{\zeta - s_k e^{2\pi i d/N_k}} \right|, \quad \zeta \in \mathbb{D}. \]

Consider the set
\[ \Omega = \left\{ r e^{i\theta} : r_k \leq r \leq r_{k+1}, \ |\theta - \frac{2\pi d}{N_k}| \leq \frac{\pi}{N_k} \right\}. \]

For \(\zeta \in \partial \Omega\) we have \(\rho(r_k) \asymp |\zeta - s_k e^{2\pi i d/N_k}|, \ |\zeta^{N_k s_k^{-N_k}}| \asymp 1\), and either \(|\arg(\zeta^{N_k s_k^{-N_k}})| \geq c_1\) or \(|1 - |\zeta^{N_k s_k^{-N_k}}|| \geq c_1\), for some positive constant \(c_1\). Therefore, \(|W^*(\zeta)| \leq c_2\), \(\zeta \in \partial \Omega\), for some constant \(c_2\). Since \(W^*\) is harmonic on \(\Omega\), we obtain, by the maximum principle, that \(|W^*(z)| \leq c_2\), and \((3.9)\) follows.

It remains to verify that
\[ \sum_{0 \leq m < k} |U_m| \leq c, \quad (3.10) \]
with $c$ independent of $r$. This together with (3.8), (3.9) and (3.6) implies that $A$ is bounded uniformly in $z \in \mathbb{D} \setminus \Lambda$, and (3.1) follows.

Since
\[
\int_{r_m \leq |w| < r_{m+1}} \Delta h(w) \log \frac{r}{|w|} \frac{dm_2(w)}{2\pi} = N_m \log \frac{r}{s_m},
\]
we have
\[
U_m = \log \left| \frac{1 - s_m^N w^N}{1 - w^N s_m^N} \right|, \quad 0 \leq m < k.
\]

Next we consider two cases. If $\rho$ satisfies the property (I$D$), then we define
\[
U^*_m(w) = \log \left| \frac{1 - s_m^N w^N}{1 - w^N s_m^N} \right|, \quad 0 \leq m < k, \quad s_m < |w| \leq 1,
\]
and divide $m$, $0 \leq m < k$, into the groups
\[
S_t = \{ m : 1 - s_m \in [2^t(1-r), 2^{t+1}(1-r)] \}, \quad t \in \mathbb{Z}_+.
\]
Put $\rho_t = \rho(1 - 2^t(1-r))$, $t \geq 0$. Then by (I$D$) we have
\[
\rho(s_m) \asymp \rho_t, \quad m \in S_t, \quad t \in \mathbb{Z}_+,
\]
and hence,
\[
s_{m+1} - s_m \asymp \rho_t, \quad m \in S_t, \quad t \in \mathbb{Z}_+.
\]
By (3.2), $\rho_t = o(2^t(1-r))$ as $2^t(1-r) \to 0$. Therefore, for $t \geq 1$, $m \in S_t$, and for some $c < 1$ independent of $m,r$, we have by (3.4) that
\[
s_m^N (1 - 2^{t-1}(1-r))^{-N_m} \leq c. \tag{3.11}
\]
Now, (3.4) implies that for $\zeta \in (1 - 2^{t-1}(1-r))\mathbb{T}$,
\[
|U^*_m(\zeta)| \leq c \left( \frac{s_m}{1 - 2^{t-1}(1-r)} \right)^{N_m}
\leq c_1 \exp \left[ - \frac{c_2((1 - s_m) - 2^{t-1}(1-r))}{\rho_t} \right] \leq c_1 \exp \left[ - \frac{c_2 2^{t-1}(1-r)}{\rho_t} \right],
\]
and
\[
|\sum_{m \in S_t} U^*_m(\zeta)| \leq c_3 \frac{2^t(1-r)}{\rho_t} \exp \left[ - \frac{c_2 2^{t-1}(1-r)}{\rho_t} \right] \leq c_4, \quad t \geq 1,
\tag{3.12}
\]
with $c, c_1, c_2, c_3, c_4$ independent of $r, t$.

Furthermore,
\[
|\sum_{m \in S_0} U^*_m(r)| \leq c \sum_{m \in S_0} \left( \frac{s_m}{r} \right)^{N_m} \leq c \sum_{m \in S_0} e^{-c_1(k-m)} \leq c_2,
\]
with positive $c, c_1, c_2$ independent of $r$. 

Since \( U^*_m(w) = 0 \), \( w \in \mathbb{T} \), and for \( t \geq 1 \), \( m \in S_t \), \( U^*_m \) are harmonic in the annulus \( \{ w : 1 - 2^{t-1}(1-r) \leq w \leq 1 \} \), we deduce from (3.12) that

\[
\left| \sum_{m \in S_t} U^*_m(r) \right| \leq c2^{-t}, \quad t \geq 1,
\]

with \( c \) independent of \( r, t \), and (3.10) is proved.

Suppose now that \( \rho \) satisfies the property (II_D). For some constant \( c > 0 \),

\[
r - s_m \geq \frac{c}{N_m}, \quad m < k,
\]

and we obtain that

\[
(s_m/r)^{N_m} \leq e^{-c},
\]

and again by (3.5),

\[
|U_m| \leq c_1 (s_m/r)^{N_m}, \quad m < k,
\]

for some positive constant \( c_1 \).

By (2.2) and (3.4) we obtain that

\[
\sum_{m<k} |U_m| \leq c \sum_{m<k} e^{N_m(s_m-r)}
\]

\[
\leq c_1 \sum_{m<k} \int_{r_m}^{r_{m+1}} e^{-c_2(r-x)/\rho(x)} \frac{dx}{\rho(x)} \leq c_1 \int_0^r e^{-c_2(r-x)/\rho(x)} \frac{dx}{\rho(x)}, \quad (3.13)
\]

with positive \( c, c_1, c_2 \) independent of \( k, r \).

Choose \( y \) such that \( \rho(y) = 2\rho(r) \). Then for \( x < y \) close to 1,

\[
\rho(x) \log \frac{1}{\rho(x)} \leq \rho(r) \log \frac{1}{\rho(r)} + (r-x) \sup_{[x,r]} \left| \rho \log \frac{1}{\rho} \right|,
\]

\[
\rho(x) \log \frac{1}{\rho(x)} \leq c_2(r-x).
\]

Hence,

\[
\int_0^y e^{-c_2(r-x)/\rho(x)} \frac{dx}{\rho(x)} \leq c_3 + \int_0^y e^{-\log(1/\rho(x))} \frac{dx}{\rho(x)} \leq c_3 + 1, \quad (3.14)
\]

with \( c_3 \) independent of \( r \).

Finally, for \( r \) close to 1

\[
\int_y^r e^{-c_2(r-x)/\rho(x)} \frac{dx}{\rho(x)} \leq \int_y^r e^{-c_2(r-x)/(2\rho(r))} \frac{dx}{\rho(r)} \leq c_3,
\]

with \( c_3 \) independent of \( r \). These inequalities together with (3.13) prove (3.10), and hence, (3.1).
(iii) Given $s_k \leq s < s_{k+1}$, $k \geq 0$, we may find $0 < r_k' < s < r_{k+1}' < 1$ such that

$$N_k = N_k' = \int_{r_k' \leq |z| < r_{k+1}'} \Delta h(w) \frac{dm_2(w)}{2\pi},$$

and

$$\log \frac{1}{s} = \frac{1}{N_k} \int_{r_k' \leq |z| < r_{k+1}'} \Delta h(w) \log \frac{1}{|w|} \frac{dm_2(w)}{2\pi}.$$ 

After that, we define $r_n^*, r_n', N_{n-1}', n > k + 1$, as in part (i).

Furthermore, we define by induction, on the step $t \geq 1$, the number $r_{k-t+1}' \in (0, r_{k-t+1}')$ by the equality

$$(r_{k-t+1}' - r_{k-t+1}^*) \int_{r_{k-t+1}' \leq |z| < r_{k-t+1}'} \Delta h(w) \frac{dm_2(w)}{2\pi} = 2\pi,$$

and the number $r_{k-t}'$ as the largest number in the interval $(0, r_{k-t+1}']$ such that

$$N_{k-t}' = \int_{r_{k-t}' \leq |z| < r_{k-t+1}'} \Delta h(w) \frac{dm_2(w)}{2\pi} \in \mathbb{N}.$$ 

We continue this induction process until either

$$A_p = r_p' \int_{|z| < r_p'} \Delta h(w) \frac{dm_2(w)}{2\pi} < 2\pi$$

or $A_p \geq 2\pi$ and

$$\int_{r < |z| < r_p'} \Delta h(w) \frac{dm_2(w)}{2\pi} \notin \mathbb{N}, \quad r \in [0, r_p^*].$$

It is clear that in both cases $r_p' \leq c(h) < 1$. Next, we modify $h$ on $r_{p+1}'D$ in such a way that the modified function $h^*$ is smooth, radial, subharmonic, $|h^*(0)| \leq c_1(h)$, and

$$N = \int_{|z| < r_{p+1}'} \Delta h^*(w) \frac{dm_2(w)}{2\pi} \in \mathbb{N}.$$ 

Finally, we set $r_0 = 0$, $N_0 = N$, $r_m = r_{m+p}'$, $N_m = N_{m+p}'$, $m \geq 1$, and define $s_m$, $m \geq 0$, by \[3.3\]. We apply the above argument to $h = h^* - h^*(0)$ to obtain all the estimates from (i)–(ii) uniformly in $s$ together with the property $s \in \Lambda$. \[3.15\]

Given $0 < r \leq 1$, we define

$$h_r(w) = \begin{cases} h(w), & |w| < r, \\ h(r) + \log \frac{|w|}{r} \int_{D(r)} \Delta h(z) \frac{dm_2(z)}{2\pi}, & |w| \geq r. \end{cases}$$
Note that
\[ h_r(w) = \int_{D(r)} \Delta h(z) \left( \log \frac{|w|}{|z|} \right) \frac{dm_2(z)}{2\pi}, \quad |w| \geq r. \]

The proof of Proposition 3.1 gives us immediately

\textbf{Lemma 3.2.} In the notations of Proposition 3.1 if \( s_{k-1} \leq r < s_k, \) and
\[ f(\zeta) = \prod_{\lambda \in \Lambda \cap D(r)} \left( \frac{1 - \zeta/\lambda}{1 - \zeta^{\lambda}} \right), \]
then
\[ |f(\zeta)| \asymp e^{h_{r_k}(\zeta)} \min \left( 1, \frac{\text{dist}(\zeta, \Lambda \cap rD)}{\rho(\zeta)} \right), \quad \zeta \in D. \]

If \( |z| = r, \) then we can divide this \( f \) by three factors \( \zeta - \lambda_j, \lambda_j \in \Lambda \cap rD \cap D(z, 5\rho(z)), j = 1, 2, 3, \) and multiply it by \( \rho(z)^3, \) to obtain

\textbf{Lemma 3.3.} Given \( z \in D \) such that \( r = |z| \) is sufficiently close to 1, there exists a function \( g_z \) analytic and bounded in \( D \) and such that uniformly in \( z, \)
\[ |g_z(w)|e^{-h(w)} \asymp 1, \quad |w - z| < \rho(z), \] \[ |g_z(w)|e^{-h(w)} \leq c(h) \min \left( 1, \frac{\min[\rho(z), \rho(w)]}{|z - w|} \right)^3, \quad w \in D. \] (3.17)

We need only to verify that for some \( c, \)
\[ e^{h_r(w) - h(w)} \rho(z)^3 \leq c \rho(w)^3, \quad 0 \leq r = |z| \leq |w| < 1. \]
This follows from the inequality \( h(t) \geq h_r(t) \) and the estimate
\[ \frac{d}{dt}[h(t) - h_r(t)] = \frac{1}{t} \int_{D(t) \setminus D(r)} \Delta h(w) \frac{dm_2(w)}{2\pi} \geq \frac{3}{\rho(t)}, \quad t \geq r + B \rho(r), \]
for some \( B > 0 \) independent of \( r. \)

Next, we obtain an asymptotic estimate for partial products of the Weierstrass \( \sigma \)-function.

\textbf{Lemma 3.4.} Given \( R \geq 10, \) we define \( \Sigma = \Sigma_R = (\mathbb{Z} + i\mathbb{Z}) \cap D(R^2), \)
\[ P_R(z) = z \prod_{\lambda \in \Sigma \setminus \{0\}} \left( 1 - \frac{z}{\lambda} \right) \]
Then uniformly in $R$
\[
|P_R(z)| \asymp \text{dist}(z, \Sigma) e^{(\pi/2)|z|^2}, \quad |z| \leq R, \quad (3.18)
\]
\[
|P_R(z)| \geq c \text{dist}(z, \Sigma) \left( \frac{e|z|^2}{R^2} \right)^{\pi/2}, \quad |z| > R. \quad (3.19)
\]

**Proof.** For every $\lambda \in \Sigma$ denote
\[ Q_\lambda = \{ w \in \mathbb{C} : |\text{Re}(w - \lambda)| < 1/2, |\text{Im}(w - \lambda)| < 1/2 \}. \]
Set
\[ Q = \bigsqcup_{\lambda \in \Sigma} Q_\lambda. \]
If $z \in Q$, then we denote by $\lambda_0$ the element of $\Sigma$ such that $z \in Q_{\lambda_0}$. For $\lambda \in \Sigma \setminus \{0\}$ we define
\[
B_\lambda = \int_{Q_\lambda} \log |1 - \frac{z}{w}| \, dm_2(w) - \log |1 - \frac{z}{\lambda}|
= \int_{Q_0} \left[ \log \left| \frac{\lambda - z + w}{\lambda - z} \right| - \log \left| \frac{\lambda + w}{\lambda} \right| \right] \, dm_2(w).
\]
We use that
\[
|\log |1 + a| - \text{Re}(a - \frac{a^2}{2})| = O(|a|^3), \quad a \to 0.
\]
Since
\[
\int_{Q_0} w \, dm_2(w) = 0, \quad \int_{Q_0} w^2 \, dm_2(w) = 0,
\]
we conclude that
\[
|B_\lambda| \leq c \left[ \frac{1}{|\lambda|^3} + \frac{1}{|z - \lambda|^3} \right], \quad \lambda \in \Sigma \setminus \{0, \lambda_0\}.
\]
Furthermore, we define
\[
B_0 = \int_{Q_0} \log |1 - \frac{z}{w}| \, dm_2(w) - \log |z|
= \int_{Q_0} \left[ \log \left| \frac{1}{w} \right| + \log \left| 1 - \frac{w}{z} \right| \right] \, dm_2(w).
\]
If $\lambda_0 \neq 0$, then $|B_0| \leq c$ for an absolute constant $c$. Similarly, in this case,
\[
B_{\lambda_0} + \log |z - \lambda_0| = \int_{Q_0} \left[ \log |w + (\lambda_0 - z)| - \log \left| 1 + \frac{w}{\lambda_0} \right| \right] \, dm_2(w),
\]
and, hence, $|B_{\lambda_0} + \log |z - \lambda_0| | \leq c$ for an absolute constant $c$. In the same way, if $\lambda_0 = 0$, then $|B_0 + \log |z| | \leq c$ for an absolute constant $c$.  


Therefore,
\[
\left| \int_Q \log \left| 1 - \frac{z}{w} \right| dm_2(w) - \log \frac{|P_R(z)|}{\text{dist}(z, \Sigma)} \right| = O(1), \ |z| < R^2 + 1,
\]
\[
\left| \int_Q \log \left| 1 - \frac{z}{w} \right| dm_2(w) - \log |P_R(z)| \right| = O(1), \ |z| \geq R^2 + 1.
\]

Next we use the identity
\[
\frac{2}{\pi} \int_{D(R^2+1)} \log \left| 1 - \frac{z}{w} \right| dm_2(w) = 4 \int_0^{\min(|z|, R^2+1)} \log \frac{|z|}{s} s \, ds
\]
\[
= \begin{cases} 
(R^2 + 1)^2 + 2(R^2 + 1)^2 \log \frac{z}{R^2 + 1}, & |z| \geq R^2 + 1, \\
|z|^2, & |z| < R^2 + 1,
\end{cases}
\]
and the estimates
\[
\left| \int_{D(R^2+1) \setminus Q} \log \left| 1 - \frac{z}{w} \right| dm_2(w) \right|
\leq \left| \int_{D(R^2+1) \setminus Q} \text{Re} \frac{z}{w} \, dm_2(w) \right| + c \int_{D(R^2+1) \setminus Q} \left| \frac{z}{w} \right|^2 \, dm_2(w)
\leq \frac{c|z|^2}{R^2}, \ |z| \leq R^{3/2}, \quad (3.22)
\]
and
\[
\left| \int_{D(R^2+1) \setminus Q} \log \left| 1 - \frac{z}{w} \right| dm_2(w) \right|
\leq m_2(D(R^2 + 1) \setminus Q) \cdot \log \left( (R^2 + 1)(|z| + (R^2 + 1)) \right), \ z \in \mathbb{C}, \quad (3.23)
\]
for an absolute constant $c$. Now, (3.18) follows from (3.20) - (3.22); (3.19) follows from (3.20) - (3.23). 

**Proposition 3.5.** Given $R \geq 100$, there exists $\eta(R) > 0$ such that for every $z \in \mathbb{D}$ with $|z| \geq 1 - \eta(R)$, there exists a function $g = g_{z, R}$ analytic in $\mathbb{D}$ such that uniformly in $z, R$ we have
\[
|g(w)| e^{-h(w)} \asymp e^{-|z-w|^2/[4\rho(z)^2]}, \quad w \in \mathbb{D} \cap D(z, R\rho(z)),
\]
\[
|g(w)| e^{-h(w)} \leq c(h) \left[ \frac{R^2 \rho(z)^2}{e|z-w|^2} \right]^{R^2/4}, \quad w \in \mathbb{D} \setminus D(z, R\rho(z)). \quad (3.24)
\]

**Proof.** Without loss of generality we may assume that $z \in (0, 1)$. By Proposition 3.1 we find \{r_k\}, \{N_k\}, \{s_k\}, $\Lambda = \{s_k e^{2\pi i m/N_k}\}$, and $f \in$
such that $z = s_k$ for some $k$, $Z(f) = \Lambda$, and

$$|f(w)| \asymp e^{h(w)} \frac{\text{dist}(w, \Lambda)}{\rho(w)}, \quad w \in \mathbb{D}. \quad (3.26)$$

We define $\Sigma = \Sigma_{R/\sqrt{2\pi}}$ (see Lemma 3.4), and denote

$$\lambda_{a,b} = s_k + e^{2\pi i b/N_k} + a, \quad a + bi \in \Sigma.$$

Then

$$\max_{a + bi \in \Sigma} \left| a + bi - \frac{\lambda_{a,b} - \lambda_{0,0}}{\sqrt{2\pi} \rho(z)} \right| \leq \varepsilon(k),$$

with $\varepsilon(k) \to 0$ as $k \to \infty$. Denote

$$Q(w) = \frac{w - z}{\rho(z)} \prod_{a + bi \in \Sigma \setminus \{0\}} \left( \frac{w - \lambda_{a,b}}{z - \lambda_{a,b}} \right),$$

and put $g = f/Q$. Now, estimates (3.24) and (3.25) follow for fixed $R$ when $k$ is sufficiently large, and correspondingly, $\varepsilon(k)$ is sufficiently small.

Indeed, by (3.26), for $u = ge^{-h}$ we have

$$|u(w)| \asymp \frac{\text{dist}(w, \Lambda)}{\rho(w)} \cdot \frac{\rho(z)}{|w - z|} \cdot \prod_{a + bi \in \Sigma \setminus \{0\}} \left| \frac{z - \lambda_{a,b}}{w - \lambda_{a,b}} \right|.$$  

If $w = z + \sqrt{2\pi} \rho(z)w'$, then

$$|u(w)| \asymp \frac{\text{dist}(w, \Lambda)}{\rho(w)} \cdot \frac{1}{|w'|} \cdot \prod_{a + bi \in \Sigma \setminus \{0\}} \left| \frac{(z - \lambda_{a,b})/(\sqrt{2\pi} \rho(z))}{w' + (z - \lambda_{a,b})/(\sqrt{2\pi} \rho(z))} \right|.$$  

For small $\varepsilon(k)$ and for $\text{dist}(w', \Sigma) > 1/10$ we have

$$|u(w)| \asymp \frac{\text{dist}(w, \Lambda)}{\rho(w)} \cdot \frac{1}{|w'|} \cdot \prod_{a + bi \in \Sigma \setminus \{0\}} \left| \frac{a + bi}{w' - (a + bi)} \right| \rho(w)|P_{R/\sqrt{2\pi}}(w')|.$$  

Now, for $z$ sufficiently close to 1, by Lemma 3.4 and by the maximum principle, we have

$$|u(w)| \asymp \frac{\text{dist}(w', \Sigma)}{|P_{R/\sqrt{2\pi}}(w')|} \asymp \exp \left[ -\frac{\pi}{2} |w'|^2 \right]$$

$$= \exp \left[ -\frac{|z - w|^2}{4\rho(z)^2} \right], \quad |w'| \leq \frac{R}{\sqrt{2\pi}}.$$
and
\[ |u(w)| \leq c \frac{\text{dist}(w', \Sigma)}{|P_{R/\sqrt{2\pi}}(w')|} \leq c_1 \left[ \frac{R^2}{2\pi e|w'|^2} \right]^{(\pi/2)-(R^2/(2\pi))} \]
\[ = c_1 \left[ \frac{R^2 \rho(z)^2}{e|z-w|^2} \right]^{R^2/4}, \quad |w'| > \frac{R}{\sqrt{2\pi}}. \]

**Proposition 3.6.** Given \( R \geq 100 \), there exists \( \eta(R) > 0 \) such that for every \( z \in \mathbb{D} \) with \( |z| \geq 1 - \eta(R) \), there exists a function \( g = g_{z,R} \) analytic in \( \mathbb{D} \) such that uniformly in \( z, R \) we have
\[ |g(w)| e^{-h(z)} \leq c(h) \left[ \frac{R^2 \min[\rho(z), \rho(w)]^2}{e|z-w|^2} \right]^{R^2/4}, \quad w \in \mathbb{D} \cap D(z, R\rho(z)). \]

**Proof.** We use the argument from the above proof, and just replace Proposition 3.1 by Lemma 3.2. Furthermore, we use the argument from the proof of Lemma 3.3. \( \square \)

### 4. \( d_\rho \)-separated sets

Here we establish several elementary properties of \( d_\rho \)-separated sets, sets of sampling, and sets of interpolation.

**Lemma 4.1.** Let \( 0 < R < \infty \), let \( z \) be sufficiently close to the unit circle, \( \eta^*(R) < |z| < 1 \), and let \( f \) be bounded and analytic in \( D = D(z, R\rho(z)) \). Then
\[ |f(z_1)e^{-h(z_1)} - f(z_2)e^{-h(z_2)}| \leq c(R, h) d_\rho(z_1, z_2) \max_D |f e^{-h}|, \quad z_1, z_2 \in D(z, R\rho(z)/2), \]
\[ |f(z)| e^{-h(z)} \leq c(R, h) \rho(z)^2 \int_D |f(w)| e^{-h(w)} dm_2(w). \]

**Proof.** We may assume that \( \rho(\zeta) \asymp \rho(z) \), \( \zeta \in D \). We suppose that \( \max_D |f e^{-h}| = 1 \) and define
\[ H(w) = h(z + w\rho(z)), \quad |w| \leq 1. \]
Then
\[ \Delta H(w) = \frac{R^2 \rho(z)^2}{\rho(z + w\rho(z))^2} \asymp R^2, \quad |w| \leq 1. \]
Set
\[ G(w) = \int_{\mathbb{D}} \log \left| \frac{z - w}{1 - \bar{z}w} \right| \Delta H(z) \, dm(z), \quad |w| \leq 1. \]

Then \(|G(w)| + |\nabla G(w)| \leq c, w \leq 1,\) for some \(c\) depending only on \(h\) and \(R,\) and \(H_1 = H - G\) is real and harmonic in \(\mathbb{D}.\) Denote by \(\tilde{H}_1\) the harmonic conjugate of \(H_1,\) and consider
\[ F(w) = f(z + wR\rho(z))e^{-H_1(w) - i\tilde{H}_1(w)}. \]

Then \(F\) is analytic and bounded in \(\mathbb{D},\) and hence,
\[ |F(w_1) - F(w_2)| \leq c|w_1 - w_2|, \quad w_1, w_2 \in \mathcal{D}(1/2). \]

Since
\[ |F(w)| = |f(z + wR\rho(z))|e^{-h(z + wR\rho(z))}e^{G(w)}, \]
we obtain assertion (i). Assertion (ii) follows by the mean value property for \(F.\)

**Corollary 4.2.** Every set of sampling for \(A_h(\mathbb{D})\) contains a \(d_{\rho}\)-separated set of sampling for \(A_h(\mathbb{D}).\)

**Corollary 4.3.** Every set of interpolation for \(A_h(\mathbb{D})\) is \(d_{\rho}\)-separated.

**Corollary 4.4.** Every set of interpolation for \(A_h^p(\mathbb{D}), 1 \leq p < \infty,\) is \(d_{\rho}\)-separated.

**Lemma 4.5.** For every \(\varepsilon > 0, 1 \leq p < \infty,\) we have \(A_h^p(\mathbb{D}) \subset A_{(1+\varepsilon)h}(\mathbb{D}).\)

**Proof.** By (2.1) and (3.7),
\[ \frac{|p'(r)|}{\rho(r)} = o\left( \frac{1}{\rho(r)} \right) = o\left( h'(r) \right), \quad r \to 1, \]
and hence,
\[ e^{\varepsilon h(z)} \rho(z)^2 \to \infty, \quad |z| \to 1. \quad (4.1) \]

Applying Hölder’s inequality and Lemma 4.1 (ii) with \(R = 1,\) we obtain our assertion:
\[ |f(z)|e^{(-1+\varepsilon)h(z)} \leq \frac{c \cdot e^{-\varepsilon h(z)}}{\rho(z)^2} \int_{\mathcal{D}(z, \rho(z))} |f(z)|e^{-h(z)} \, dm_2(z) \]
\[ \leq \frac{c \cdot e^{-\varepsilon h(z)}}{\rho(z)^{2/p}} \left( \int_{\mathcal{D}(z, \rho(z))} |f(z)|^p e^{-ph(z)} \, dm_2(z) \right)^{1/p} \leq c, \quad z \in \mathbb{D}. \]
Lemma 4.6. Let $\Gamma$ be a $d_\rho$-separated (with constant $\gamma$) subset of $\mathbb{D}$. If $R > 0$, $\Omega(R, \Gamma) = \{w : \min_{z \in \Gamma} d_\rho(w, z) \leq R\}$, and if $f$ is analytic in $\Omega(R, \Gamma)$, then
\[
\|f\|^p_{p,h,\Gamma} \leq c(\gamma, R, h, p) \int_{\Omega(R,\Gamma)} |f(w)|^p e^{-ph(w)} dm_2(w).
\]

Proof. The assertion follows from Lemma 4.1 (ii).

Lemma 4.7. Let $\Gamma \subset \mathbb{D}$. Then
\[
\|f\|^p_{p,h,\Gamma} \leq c(\Gamma)\|f\|^p_{p,h}, \quad f \in \mathcal{A}^p_h(\mathbb{D}), \quad (4.2)
\]
if and only if $\Gamma$ is a finite union of $d_\rho$-separated subsets.

Proof. For every $z \in \mathbb{D}$ with $|z|$ close to 1, we apply Lemma 3.3 to obtain the function $f = g_z$ such that
\[
|f(w)e^{-h(w)}| > 1, \quad |w - z| < \rho(z),
\]
\[
|f(w)e^{-h(w)}| \leq c(h) \frac{\rho(z)^3}{|z - w|^3}, \quad w \in \mathbb{D}. \quad (4.3)
\]
By (4.3),
\[
\|f\|^p_{p,h,\Gamma} \geq \sum_{w \in \Gamma \cap \mathcal{D}(z, \rho(z))} |f(w)|^p e^{-ph(w)} \rho(w)^2 
\]
\[
\geq c \text{Card}(\Gamma \cap \mathcal{D}(z, \rho(z))) \rho(z)^2. \quad (4.5)
\]
Furthermore, by (4.3)–(4.4),
\[
\int_{|w - z| < \rho(z)} |f(w)|^p e^{-ph(w)} dm_2(w) \asymp \rho(z)^2,
\]
\[
\int_{|w - z| \geq \rho(z)} |f(w)|^p e^{-ph(w)} dm_2(w) \leq c(h) \rho(z)^2,
\]
and hence, $f \in \mathcal{A}^p_h(\mathbb{D})$ and
\[
\|f\|^p_{p,h} \asymp \rho^2(z). \quad (4.6)
\]
Now, (4.2), (4.5), and (4.6) imply that
\[
\sup_{z \in \mathbb{D}} \text{Card}(\Gamma \cap \mathcal{D}(z, \rho(z))) < \infty,
\]
and hence, $\Gamma$ is a finite union of $d_\rho$-separated subsets.

In the opposite direction, if $\Gamma$ is $d_\rho$-separated, then (4.2) follows from Lemma 4.6.

Lemma 4.8. Every set of sampling for $\mathcal{A}^p_h(\mathbb{D})$ contains a $d_\rho$-separated set of sampling for $\mathcal{A}^p_h(\mathbb{D})$. 

Proof. Let $\Gamma$ be a set of sampling for $A^p_h(D)$. For every $\varepsilon > 0$ we can find a $d_\rho$-separated subset $\Gamma^*$ of $\Gamma$ such that
\[
\sup_{w \in \Gamma} \min_{z \in \Gamma^*} d_\rho(z, w) \leq \varepsilon.
\]
Suppose that there exists $f \in A^p_h(D)$ such that
\[
\|f\|_{2,h} \asymp 1, \quad \|f\|_{p,h,\Gamma} \asymp 1, \quad \|f\|_{p,h,\Gamma^*} \leq \varepsilon.
\]
By Lemma 4.7, for some $N, K$ independent of $\varepsilon$, both $\Gamma$ and $\Gamma^*$ are unions of $N$ subsets $d_\rho$-separated with constant $K$. Without loss of regularity we can assume that $|z| + \rho(z) < 1$, $z \in \Gamma$. For every $z_k \in \Gamma^*$ we choose $w_k \in D(z_k, \rho(z_k))$ such that
\[
2|f(w_k)|^p e^{-ph(w_k)} \geq u_k^p = \sup_{D(z_k, \rho(z_k))} |f|^p e^{-ph}.
\]
Then the sequence $\{w_k\}$ is the union of $c(N, K)$ subsets $d_\rho$-separated with constant $c_1(N, K)$. By Lemma 4.6
\[
\sum_{z_k \in \Gamma^*} u_k^p \rho(z_k)^2 \leq C\|f\|_{p,h}^p,
\]
with $C$ independent of $\varepsilon$. Furthermore, by Lemma 4.1 (i), for every $k$ and for every $w \in D(z_k, \varepsilon \rho(z_k))$,
\[
||f(w)|e^{-h(w)} - |f(z_k)|e^{-h(z_k)}| \leq C\varepsilon u_k.
\]
Therefore,
\[
\|f\|_{p,h,\Gamma}^p = \sum_{w \in \Gamma} |f(w)|^p e^{-ph(w)} \rho(w)^2
\]
\[
\leq C \sum_{z_k \in \Gamma^*} \left( |f(z_k)|^p e^{-ph(z_k)} + \varepsilon^p u_k^p \right) \rho(z_k)^2
\]
\[
\leq C\|f\|_{p,h,\Gamma^*}^p + \varepsilon^p \|f\|_{p,h}^p \leq C\varepsilon^p,
\]
with $C$ independent of $\varepsilon$. This contradiction implies our assertion. \qed

5. Asymptotic densities

Given a set $\Gamma \subset \mathbb{D}$ such that $D^-_\rho(\Gamma, \mathbb{D}) < \infty$, we denote
\[
q_-(R) = \liminf_{|z| \to 1, z \in \mathbb{D}} \frac{\text{Card} \Gamma(z, R)}{R^2}, \quad 0 < R < \infty,
\]
where
\[
\Gamma(z, R) = \Gamma \cap D(z, R\rho(z)).
\]
In this section we study the behavior of the function $q_-$ and obtain a Beurling type result (Lemma 5.3).
We use the following
Lemma 5.1. \( (i) \) If \( R > 0 \), and \( 0 < \varepsilon < \varepsilon(R) \), then for \( R'' \geq R'(R, \varepsilon) \), and for \( z \in \mathbb{D} \) such that \( |z| \geq \eta_1(R'') \) we have
\[
\frac{\text{Card} \Gamma(z, R'')}{R''^2} \geq q_-(R) - \varepsilon;
\]

\( (ii) \) if \( \delta > 0, R > 0, R'' \geq R'(R, \delta), |z| \geq \eta_2(R'') \),
\[
E = \{ w \in \mathcal{D}(z, R'' \rho(z)/2) : \frac{\text{Card} \Gamma(w, R)}{R^2} \geq q_-(R) + \delta \},
\]
and
\[
m_2 E \geq \delta m_2 \mathcal{D}(z, R'' \rho(z)/2),
\]
then
\[
\frac{\text{Card} \Gamma(z, R'')}{R''^2} \geq q_-(R) + \frac{\delta^2}{5}.
\]

Proof. By (2.2),
\[
\max_{w \in \mathcal{D}(z, R'' \rho(z))} \left| \frac{\log \frac{\rho(z)}{\rho(w)}}{R'} \right| = o(1), \quad |z| \to 1,
\]
and hence, for small \( \varepsilon \), for fixed \( R'' \), and for \( |z| \) close to 1 we have
\[
\text{Card} \left( \Gamma \cap \mathcal{D}(w, (R + \varepsilon^3) \rho(z)) \right) \geq \text{Card} \Gamma(w, R) \geq (q_-(R) - \varepsilon^3) R^2, \quad w \in \mathcal{D}(z, R'' \rho(z)). \quad (5.1)
\]

In the same way,
\[
E \subset E' = \{ w \in \mathcal{D}(z, R'' \rho(z)/2) : \text{Card} \left( \Gamma \cap \mathcal{D}(w, (R + \varepsilon^3) \rho(z)) \right) \geq (q_-(R) + \delta) R^2 \}. \quad (5.2)
\]
We use that by the Fubini theorem, for \( 0 < r_1 < r_2 \) and for \( F \subset \mathcal{D}(r_2 - r_1), \)
\[
\text{Card} F = \frac{1}{\pi r_1^2} \int_{\mathcal{D}(r_2)} \text{Card}(F \cap \mathcal{D}(w, r_1)) \, dm_2(w). \quad (5.3)
\]
(i) By (5.3),
\[
\frac{\text{Card} \Gamma(z, R'')}{R''^2} \geq \int_{\mathcal{D}(z, (R'' - R - \varepsilon^3) \rho(z))} \frac{\text{Card} \left( \Gamma \cap \mathcal{D}(w, (R + \varepsilon^3) \rho(z)) \right)}{\pi (R + \varepsilon^3)^2 \rho(z)^2 R''^2} \, dm_2(w). \quad (5.4)
\]
Therefore, by (5.1), for $|z|$ close to 1,
\[
\frac{\text{Card } \Gamma(z, R'')}{R''^2} \geq (q_-(R) - \varepsilon^3) \left(\frac{R'' - R - \varepsilon^3}{R''}\right)^2 \left(\frac{R}{R + \varepsilon^3}\right)^2 \geq q_-(R) - \varepsilon,
\]
for $\varepsilon < \varepsilon(R)$, $R'' \geq R'(R, \varepsilon)$.

(ii) By (5.1)–(5.2), for small $\varepsilon > 0$, fixed $R''$ and $|z|$ close to 1 we have
\[
\int_{D(z, (R'' - R - \varepsilon^3)\rho(z))} \frac{\text{Card}(\Gamma \cap D(w, (R + \varepsilon^3)\rho(z)))}{\pi(R + \varepsilon^3)^2 \rho(z)^2 R''^2} \, dm_2(w) = \int_{E'} \ldots + \int_{D(z, (R'' - R - \varepsilon^3)\rho(z)) \setminus E'} \frac{R^2 m_2 E'}{\pi(R + \varepsilon^3)^2 \rho(z)^2 R''^2} (q_-(R) + \delta) + \frac{R^2 (\pi R'' - R - \varepsilon^3)^3 \rho(z)^2 m_2 E'}{\pi(R + \varepsilon^3)^2 \rho(z)^2 R''^2} (q_-(R) - \varepsilon^3).
\]
If $m_2 E' \geq \delta m_2 D(z, R''\rho(z)/2)$, then by (5.4) we obtain
\[
\frac{\text{Card } \Gamma(z, R'')}{R''^2} \geq \int_{D(z, (R'' - R - \varepsilon^3)\rho(z))} \frac{\text{Card}(\Gamma \cap D(w, (R + \varepsilon^3)\rho(z)))}{\pi(R + \varepsilon^3)^2 \rho(z)^2 R''^2} \, dm_2(w) \geq \left(\frac{R}{R + \varepsilon^3}\right)^2 \left[\left(\frac{R'' - R - \varepsilon^3}{R''}\right)^2 (q_-(R) - \varepsilon^3) + \frac{\delta}{4}(\delta + \varepsilon^3)\right] \geq q_-(R) + \frac{\delta^2}{5}
\]
for $\varepsilon = \varepsilon(\delta)$, $R'' \geq R'(R, \delta)$, $|z| \geq \eta_2(R'')$. 

By Lemma 5.1(i), for every $R_0$ and $\varepsilon$ such that $0 < \varepsilon < \varepsilon(R_0)$, we have
\[
D_\rho^-(\Gamma, \mathbb{D}) = \liminf_{R \to \infty} q_-(R) \geq q_-(R_0) - \varepsilon.
\]
Therefore, we obtain

**Corollary 5.2.**
\[
\lim_{R \to \infty} q_-(R) = D_\rho^-(\Gamma, \mathbb{D}), \quad (5.5)
\]
and
\[
q_-(R) \leq D_\rho^-(\Gamma, \mathbb{D}), \quad R > 0. \quad (5.6)
\]
Given closed subsets $A$ and $B$ of $C$, the Fréchet distance $[A, B]$ is the smallest $t > 0$ such that $A \subset B + tD$, $B \subset A + tD$. A sequence $\{A_n\}, A_n \subset C$, converges weakly to $A \subset C$ if for every $R > 0$,

$$[(A_n \cap D(R)) \cup RT, (A \cap D(R)) \cup RT] \to 0, \quad n \to \infty.$$ 

In this case we use the notation $A_n \rightharpoonup A$. Given any sequence $\{A_n\}, A_n \subset C$, we can choose a weakly convergent subsequence $\{A_{n_k}\}$.

**Lemma 5.3.** If $\Gamma \subset D$, and $D^-(\Gamma, D) \leq \frac{1}{2}$, then there exists a sequence of points $z_j \in D, |z_j| \to 1$, a sequence $R_j \to \infty, j \to \infty$, and a subset $\Gamma_0$ of $C$ such that

$$\Gamma^#(z_j, R_j) \rightharpoonup \Gamma_0, \quad j \to \infty, \quad (5.7)$$

$$\liminf_{R \to \infty} \frac{\text{Card}(\Gamma_0 \cap D(R))}{R^2} \leq \frac{1}{2}, \quad (5.8)$$

where

$$\Gamma^#(z, R) = \{w \in C : z + w\rho(z) \in \Gamma(z, R)\}$$

$$= \{w \in D(R) : z + w\rho(z) \in \Gamma\}.$$

**Proof.** Choose a sequence of positive numbers $\delta_k, \sum_{k \geq 1} \delta_k \leq 1$, set $r_k = 2^k, k \geq 1$, and apply Lemma 5.1(ii) to find $\varepsilon_1 > 0, 0 < \eta_1 < 1, R_1$ such that for $\eta_1 \leq |w| < 1$, if

$$\frac{\text{Card} \Gamma(w, R_1)}{R_1^2} \leq q_-(R_1) + \varepsilon_1,$$

then there exists $z = z_1(w) \in D(w, R_1\rho(w)/2)$ such that

$$\frac{\text{Card} \Gamma(z, r_1)}{r_1^2} \leq q_-(r_1) + \delta_1.$$

Applying Lemma 5.1(ii) repeatedly, we find $\varepsilon_m \to 0, \eta_m \to 1, R_m \to \infty, m \to \infty$, such that for $m \geq 1, \eta_m \leq |w| < 1$, if

$$\frac{\text{Card} \Gamma(w, R_m)}{R_m^2} \leq q_-(R_m) + \varepsilon_m,$$

then there exists $z = z_m(w) \in D(w, R_m\rho(w)/2)$, such that

$$\frac{\text{Card} \Gamma(z, r_k)}{r_k^2} \leq q_-(r_k) + \delta_k, \quad 1 \leq k \leq m.$$

Next, by the definition of $q_-(R_m)$, we can find $w_m \in D, \eta_m \leq |w_m| < 1, \eta_m \leq |w_m| < 1$, such that

$$\frac{\text{Card} \Gamma(w_m, R_m)}{R_m^2} \leq q_-(R_m) + \varepsilon_m,$$
and define $z_m = z_m(w_m)$. We obtain

$$\limsup_{m \to \infty} \frac{\text{Card}(\Gamma(z_m, r_k))}{r_k^2} \leq D_\rho^-(\Gamma, \mathbb{D}), \quad k \geq 1. \quad (5.9)$$

Finally, we choose a sequence $\{m_k\}$ and a set $\Gamma_0 \in \mathbb{C}$ such that

$$\Gamma^\#(z_{m_k}, r_{m_k}) \rightharpoonup \Gamma_0, \quad m_k \to \infty. \quad (5.8)$$

The property (5.8) follows from (5.9). □

Analogously, we have

**Lemma 5.4.** If $\Gamma \subset \mathbb{D}$, and $D_\rho^+(\Gamma, \mathbb{D}) \geq \frac{1}{2}$, then there exists a sequence of points $z_j \in \mathbb{D}$, $|z_j| \to 1$, a sequence $R_j \to \infty$, $j \to \infty$, and a subset $\Gamma_0$ of $\mathbb{C}$ such that

$$\Gamma^\#(z_j, R_j) \rightharpoonup \Gamma_0, \quad j \to \infty, \quad (5.10)$$

$$\limsup_{R \to \infty} \frac{\text{Card}(\Gamma_0 \cap \mathcal{D}(R))}{R^2} \geq \frac{1}{2}. \quad (5.11)$$

6. **Sampling theorems**

We set $\beta(z) = |z|^2/4$.

**Proof of Theorem 2.1.** By Corollary 4.2, every sampling set for $\mathcal{A}_h(\mathbb{D})$ contains a $d_\rho$-separated subset which is also a sampling set for $\mathcal{A}_h(\mathbb{D})$.

(A) Suppose that $\Gamma$ is $d_\rho$-separated, and $D_\rho^-(\Gamma, \mathbb{D}) \leq \frac{1}{2}$. We follow the scheme proposed in [16]. We apply Lemma 3.3 to obtain $z_j, R_j$, and $\Gamma_0$ satisfying (5.7)–(5.8). Fix $\varepsilon > 0$. By the theorem of Seip on sampling in Fock type spaces [20, Theorem 2.3], there exists $f \in \mathcal{A}_\beta(\mathbb{C})$ such that

$$\|f\|_{\beta} = 1, \quad \|f\|_{\beta, \Gamma_0} \leq \varepsilon.$$

For $K > 1$ we set $f_K(z) = f((1 - K^{-3/2})z)$. Then

$$|f_K(z)|e^{-\beta(z)} \leq |f_K(z)|e^{-(1-K^{-3/2})^2\beta(z)}$$

$$\leq |f(z)|e^{-\beta(z)} + \|f(z)|e^{-\beta(z)} - |f((1 - K^{-3/2})z)|e^{-(1-K^{-3/2})^2\beta(z)}|$$

$$= |f(z)|e^{-\beta(z)} + o(1), \quad |z| \leq K, \quad K \to \infty,$$

where in the last relation we use [20] Lemma 3.1 (for a similar estimate see Lemma 14 (i)). Furthermore,

$$|f_K(z)|e^{-\beta(z)} = o(1), \quad |z| > K, \quad K \to \infty.$$

Therefore, for sufficiently large $K$ we get

$$\|f_K\|_{\beta} \approx 1, \quad \|f_K\|_{\beta, \Gamma_0} \leq 2\varepsilon.$$
We fix such \( K \) and for \( N \geq 0 \) set
\[
T_N f_K(z) = \sum_{0 \leq n \leq N} c_n z^n,
\]
where
\[
f_K(z) = \sum_{n \geq 0} c_n z^n.
\]
As in [10, page 169], by the Cauchy formula,
\[
|c_n| \leq c \cdot \inf_r \exp\left[(1 - \frac{1}{2})^2 r^2 / 4\right],
\]
and hence,
\[
\sum_{n \geq 0} |c_n z^n| e^{-\beta(z)} \leq c(1 + |z|)^4 e^{[(1-\frac{1}{2})^2-1] |z|^2 / 4}, \quad z \in \mathbb{C}. \tag{6.1}
\]
Therefore, for sufficiently large \( N \) we have
\[
\|T_N f_K\|_{\beta, \Gamma} \leq 3 \varepsilon.
\]
We fix such \( N \), set \( P = T_N f_K \), and choose \( a \in \mathbb{C} \) such that
\[
|P(a)| e^{-\beta(a)} \approx 1.
\]
By (5.7), we can find large \( R > |a| \) and \( z \) close to the unit circle such that
\[
|P(w)| \leq \varepsilon |w|^R, \quad |w| \geq R,
\]
\[
\|P\|_{\beta, \Gamma} \leq 4 \varepsilon.
\]
We set \( z^* = z + a \rho(z) \), apply Proposition 3.5 to get \( g = g_{z,R} \), and define
\[
f(w) = g(w) P\left(\frac{w - z}{\rho(z)}\right).
\]
Then \( f \in A_h(\mathbb{D}) \),
\[
|f(z^*)| e^{-\beta(z^*)} \approx |g(z^*)| e^{-\beta(z^*)} \cdot |P(a)| \approx e^{-|a|^2 / 4} e^{\beta(a)} = 1,
\]
\[
|f(w)| e^{-\beta(w)} \leq e^{-|w-z|^2/(4\rho(z)^2)} \cdot 4 \varepsilon e^{w-z|^2/(4\rho(z)^2)} = 4 \varepsilon, \quad w \in \Gamma(z, R),
\]
\[
|f(w)| e^{-\beta(w)} \leq c \left[ \frac{R^2 \rho(z)^2}{e|z-w|^2} \right]^{R^2 / 4} \cdot \varepsilon \left[ \frac{|z-w|}{\rho(z)} \right]^R
\leq c \varepsilon, \quad w \in \mathbb{D} \setminus D(z, R \rho(z)),
\]
with \( c \) independent of \( \varepsilon, R \). Since \( \varepsilon \) can be chosen arbitrarily small, this shows that \( \Gamma \) is not a sampling set for \( A_h(\mathbb{D}) \).

(B) Now we assume that \( \Gamma \) is a \( d_{\rho} \)-separated subset of \( \mathbb{D} \), \( D_{\rho} \Gamma, \mathbb{D} \) \( > \frac{1}{2} \), and \( \Gamma \) is not a sampling set for \( A_h(\mathbb{D}) \). Then there exist functions \( f_n \in A_h(\mathbb{D}) \) such that \( \|f_n\|_h = 1 \) and \( \|f_n\|_{h, \Gamma} \to 0, n \to \infty \). By
the normal function argument, either (B1) \( f_n \) tend to 0 uniformly on compact subsets of the unit disc or (B2) there exists a subsequence \( f_{n_k} \) converging uniformly on compact subsets of the unit disc to \( f \in \mathcal{A}_h(\mathbb{D}) \), \( f \neq 0 \), with \( f \big| \Gamma = 0 \).

In case (B1), using Proposition 3.5 we can find \( z_n \in \mathbb{D} \), \( R_n \to \infty \), \( n \to \infty \), such that the functions

\[
F_n(w) = \frac{f(z_n + w\rho(z_n))}{g_{z_n,R_n}(z_n + w\rho(z_n))}
\]
satisfy the conditions:

\[
\begin{align*}
|F_n(0)| &\leq 1, \\
|F_n(w)| &\leq e^{\frac{|w|^2}{4}}, |w| \leq R_n, \\
\sup_{\Gamma^\#(z_n,R_n)} |F_n| &\to 0, \quad n \to \infty.
\end{align*}
\]

By Corollary 5.2 we can find \( q, \frac{1}{2} < q < D_{\rho}^{-}(\Gamma, \mathbb{D}) \), and \( 0 < C < R'_n < R_n \), \( R'_n \to \infty \) as \( n \to \infty \), such that

\[
\text{Card } \Gamma^\#(z_n, r) \geq qr^2, \quad C \leq r \leq R'_n.
\]

Again by the normal function argument, we can choose a sequence \( n_k \to \infty \), \( k \to \infty \), such that \( F_{n_k} \) converge uniformly on compact subsets of \( \mathbb{C} \) to \( F \in \mathcal{A}_\beta(\mathbb{C}) \), and \( \Gamma(z_{n_k}, R_{n_k}) \to \Gamma^* \) such that

\[
\begin{align*}
F(0) &\neq 0, \\
F \big| \Gamma^* &\neq 0, \\
\text{Card}(\Gamma^* \cap \text{clos } \mathcal{D}(r)) &\geq qr^2, \quad r \geq C.
\end{align*}
\]

To get the last inequality we use that \( \Gamma \) is \( d_{\rho} \)-separated.

However, by Jensen’s inequality,

\[
-\infty < \log |F(0)| \leq \frac{1}{2\pi} \int_0^{2\pi} \log |F(re^{i\theta})|d\theta - \sum_{w_k \in \Gamma^* \cap \mathcal{D}(r)} \log \frac{r}{|w_k|}
\]

\[
\leq \frac{r^2}{4} - \int_0^r \log \frac{r}{s} n(s) ds = \frac{r^2}{4} - \int_0^r \frac{n(s)}{s} ds
\]

\[
\leq \frac{r^2}{4} - \frac{qr^2}{2} + O(1) \to -\infty, \quad r \to \infty,
\]

where \( n(r) = \text{Card}(\Gamma^* \cap \text{clos } \mathcal{D}(r)) \). This contradiction implies our assertion in case (B1).
In case (B2), without loss of generality we can assume that $0 \not\in \Gamma$, $f(0) \neq 0$. By Jensen’s inequality,

$$-\infty < \log |f(0)| \leq \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})|d\theta - \sum_{w_k \in \Gamma \cap D(r)} \log \frac{r}{|w_k|}.$$  \hspace{1cm} (6.2)

Furthermore, we choose $\varepsilon > 0$ and large $R > R(\varepsilon)$. Then

$$\sum_{w_k \in \Gamma \cap D(r)} \log \frac{r}{|w_k|} = \sum_{w_k \in \Gamma \cap D(r)} \log \frac{r}{|w_k|} \cdot \frac{1}{\pi R^2 \rho(w_k)^2} \int_{D(w_k,R\rho(w_k))} dm_2(w)$$

$$\geq O(1) + (1 - \varepsilon) \int_{D(r-R^2\rho)} \frac{1}{\pi R^2 \rho(w)^2} \log \frac{r}{|w|} dm_2(w)$$

$$\times \text{Card}\{w_k \in \Gamma : w \in D(w_k,R\rho(w_k))\} = O(1) + (1 - \varepsilon)^2 q_-(R - \varepsilon) \int_{D(r-R^2\rho)} \frac{1}{\rho(w)^2} \log \frac{r}{|w|} dm_2(w)$$

$$\geq O(1) + \frac{(1 - \varepsilon)^2}{\pi} (R - \varepsilon) \int_{D(r-R^2\rho)} \frac{1}{\rho(w)^2} \log \frac{r}{|w|} dm_2(w)$$

$$= O(1) + \frac{(1 - \varepsilon)^2}{\pi} (R - \varepsilon) 2\pi h(r), \quad r \to 1,$$

that contradicts to (6.2) for small $\varepsilon > 0$ and $R > R(\varepsilon)$. This proves our assertion. $\square$

**Proof of Theorem 2.2.** By Lemmas 4.7 and 4.8, every sampling set for $\mathcal{A}_p^\rho(D)$ is a finite union of $d_\rho$-separated subsets and contains a $d_\rho$-separated subset which is also a sampling set for $\mathcal{A}_p^\rho(D)$.

(A) Suppose that $\Gamma$ is a $d_\rho$-separated subset of the unit disc and $D^-_\rho(\Gamma, \mathcal{D}) \leq \frac{1}{2}$. As in part (A) of the proof of Theorem 2.1, we apply Lemma 5.3 to obtain $z_j, R_j,$ and $\Gamma_0$ satisfying (5.7)–(5.8). Furthermore, $\Gamma_0$ is uniformly separated, that is

$$\inf\{|z_1 - z_2| : z_1, z_2 \in \Gamma_0, z_1 \neq z_2\} > 0.$$

Then, by a version of a result of Seip [20, Lemma 7.1] (see also Lemma 4.7),

$$\sum_{z \in \Gamma_0} e^{-p\beta(z)} |g(z)|^p \leq c \|g\|_{p,\beta}^p, \quad g \in \mathcal{A}_p^\rho(\mathbb{C}). \hspace{1cm} (6.3)$$

Fix $\varepsilon > 0$. By the theorem of Seip on sampling in Fock spaces [20, Theorem 2.1], there exists $f \in \mathcal{A}_p^\rho(\mathbb{C})$ such that

$$\|f\|_{p,\beta} = 1, \quad \|f\|_{p,\beta, \Gamma_0} \leq \varepsilon.$$
We approximate $f$ by a polynomial $P$ in the norm of $A^p_\beta(\mathbb{C})$ and obtain, using (6.3), that
\[
1 - \varepsilon \leq \|P\|_{p,\beta} \leq 1 + \varepsilon, \quad \|P\|_{p,\beta,\Gamma_0} \leq 2\varepsilon.
\]
For some $M > 0$ we have
\[
\int_{D(M)} |P(z)|^p e^{-p\beta(z)} dm_2(z) \geq \frac{1}{2}.
\] (6.4)
By (5.7), we can find large $R > M$ and $z$ close to the unit circle such that
\[
|P(w)| \leq \varepsilon |w|^R, \quad |w| \geq R.
\] (6.5)
We apply Proposition 3.5 to get $g = g_{z,R}$, and define
\[
f(w) = \frac{g(w)}{\rho(z)^{2/p}} P\left(\frac{w-z}{\rho(z)}\right).
\]
Then, by (3.24) and (6.4),
\[
\|f\|^p_{p,h} \geq \int_{|w-z| \leq R\rho(z)} |f(w)|^p e^{-ph(w)} dm_2(w)
\]
\[
\times \int_{|w-z| \leq R\rho(z)} \frac{1}{\rho(z)^2} \left|P\left(\frac{w-z}{\rho(z)}\right)\right|^p e^{-p|w-z|^2/(4\rho(z)^2)} dm_2(w)
\]
\[
= \int_{D(R)} |P(w)|^p e^{-p|w|^2/4} dm_2(w) \geq \frac{1}{2}.
\] (6.7)
On the other hand,
\[
\|f\|^p_{p,h,\Gamma} = \sum_{w \in \Gamma} |f(w)|^p e^{-ph(w)} \rho(w)^2 = \sum_{w \in \Gamma(z,R)} \ldots + \sum_{w \in \Gamma \setminus \Gamma(z,R)} \ldots
\]
By (3.24) and (6.6),
\[
\sum_{w \in \Gamma(z,R)} \frac{\rho(w)^2}{\rho(z)^2} |g(w)|^p e^{-ph(w)} \left|P\left(\frac{w-z}{\rho(z)}\right)\right|^p
\]
\[
\leq c \sum_{w \in \Gamma^\#(z,R)} e^{-p|w|^2/4} |P(w)|^p \leq C\varepsilon^p.
\]
Since $\Gamma$ is $d_\rho$-separated, by Lemma 4.6 we have
\[
\sum_{w \in \Gamma \setminus \Gamma(z,R)} |f(w)|^p e^{-ph(w)} \rho(w)^2 \leq c \int_{D \setminus D(z,(R-1)\rho(z))} |f(w)|^p e^{-ph(w)} dm_2(w).
\]
Furthermore, by \((3.24)\), \((3.25)\), and \((6.5)\),
\[
\int_{D(\mathbb{D}) \setminus D(\mathbb{D}, (R-1)\rho(z))} \frac{1}{\rho(z)^2} |g(w)|^p e^{-ph(w)} \left| P\left(\frac{w - z}{\rho(z)}\right)\right|^p dm_2(w)
\leq c\varepsilon^p \int_{D(\mathbb{D}) \setminus D(\mathbb{D}, R\rho(z))} \frac{1}{\rho(z)^2} \left[\frac{|w - z|}{\rho(z)}\right]^{pR} \left[\frac{R^2 \rho(z)^2}{|w - z|^2}\right]^{pR^2/4} dm_2(w)
+ c\varepsilon^p \int_{D(\mathbb{D}, R\rho(z)) \setminus D(\mathbb{D}, (R-1)\rho(z))} \frac{1}{\rho(z)^2} R^{pR} e^{-p|w - z|^2/[4\rho(z)^2]} dm_2(w)
= c\varepsilon^p \int_{|w| > R} |w|^p R^2 \left[\frac{2}{|w|^2}\right]^{pR^2/4} dm_2(w)
+ c\varepsilon^p \int_{R-1 < |w| < R} R^{pR} e^{-p|w|^2/4} dm_2(w) \leq c\varepsilon^p,
\]
with \(c\) independent of \(R\). This together with \((6.7)\) shows that \(\Gamma\) is not a sampling set for \(A_2^p(\mathbb{D})\).

(B) Now we assume that \(\Gamma\) is a \(d_{\rho}\)-separated subset of \(\mathbb{D}\), \(D_{\rho}(\Gamma, \mathbb{D}) > \frac{1}{2}\). Then, by Theorem 2.1, we can fix small \(\varepsilon > 0\) such that \(\Gamma\) is a sampling set for \(A_{(1+\varepsilon)h}(\mathbb{D})\). Following the method of \([7, Section 6]\), we are going to prove that \(\Gamma\) is a sampling set for \(A_h^p(\mathbb{D})\).

We set
\[
A_{(1+\varepsilon)h,0}(\mathbb{D}) = \{ F \in Hol(\mathbb{D}) : \lim_{|z| \to 1} F(z)e^{-(1+\varepsilon)h(z)} = 0 \},
\]
\[
R_\Gamma : F \in A_{(1+\varepsilon)h,0}(\mathbb{D}) \mapsto \{ F(z_k)e^{-(1+\varepsilon)h(z_k)} \}_{z_k \in \Gamma} \in c_0.
\]
Since \(\Gamma\) is a sampling set for \(A_{(1+\varepsilon)h}(\mathbb{D})\), \(R_\Gamma\) is an invertible linear operator onto a closed subspace \(V\) of the space \(c_0\). Therefore, linear functionals \(E_z : v \in V \mapsto (R_\Gamma^{-1}v)(z)e^{-(1+\varepsilon)h(z)}\), \(\|E_z\| > 1\), \(z \in \mathbb{D}\), (see also Lemma 3.2) extend to linear functionals on \(c_0\) bounded uniformly in \(z\). Thus, for every \(z \in \mathbb{D}\), there exist \(b_k(z)\), \(k \geq 1\), such that
\[
\sum_{k \geq 1} |b_k(z)| \leq C, \quad (6.8)
\]
with \(C\) independent of \(z\), and
\[
F(z)e^{-(1+\varepsilon)h(z)} = \sum_{k \geq 1} b_k(z)F(z_k)e^{-(1+\varepsilon)h(z_k)}. \quad (6.9)
\]
Let \(f \in A_h^p(\mathbb{D})\). By Lemma 4.3, \(f \in A_{(1+\varepsilon)h,0}(\mathbb{D})\). For every \(z \in \mathbb{D}\), we use Lemma 3.3 (with \(h\) replaced by \(\varepsilon h\)) to get \(g_z\) satisfying \((3.16)\)–\((3.17)\) (with \(\rho\) replaced by \(\varepsilon^{-1/2}\rho\)). Now, we apply \((6.9)\) to \(F = fg_z\).
By (3.16) we obtain
\[ |f(z)|e^{-h(z)} \leq C \sum_{k \geq 1} |b_k(z)||f(z_k)|e^{-h(z_k)}|g_z(z_k)|e^{-\varepsilon h(z_k)}. \]

By (6.8),
\[ |f(z)|p e^{-ph(z)} \leq C \sum_{k \geq 1} |f(z_k)|^p e^{-ph(z_k)}|g_z(z_k)|^p e^{-p \varepsilon h(z_k)} \]
and hence,
\[ \|f\|_{p,h} \leq C \sum_{k \geq 1} |f(z_k)|^p e^{-ph(z_k)} \int_{D} |g_z(z_k)|^p e^{-p \varepsilon h(z_k)} dm_2(z). \]

It remains to note that by (3.17),
\[
\int_{D} |g_z(z_k)|^p e^{-p \varepsilon h(z_k)} dm_2(z) \leq c(\varepsilon) \int_{D} \min \left[ 1, \frac{\rho(z_k)^{3p}}{|z - z_k|^{3p}} \right] dm_2(z) \leq C(\varepsilon) \rho(z_k)^2.
\]
Then \( \|f\|_{p,h} \leq C\|f\|_{p,h,\Gamma} \), and using Lemma 4.7 we conclude that \( \Gamma \) is a sampling set for \( A_{\rho}^p(D) \).

7. Interpolation theorems

Proof of Theorem 2.3. Corollary 4.3 claims that every interpolation set for \( A_h(D) \) is \( d_\rho \)-separated.

(A) Let \( \Gamma \) be a \( d_\rho \)-separated subset of \( D \), and let \( D^+(\Gamma, D) \geq 1/2 \). Suppose that \( \Gamma \) is an interpolation set for \( A_h(D) \). We apply Lemma 5.4 to obtain \( z_j, R_j \), and \( \Gamma_0 = \{z_k^0\}_{k \geq 1} \) satisfying (5.10), (5.11). Suppose that \( a_k \in \mathbb{C} \) satisfy the estimate
\[ |a_k|e^{-\beta(z_k^0)} \leq 1. \]
By (5.10), we can choose a subsequence \( \{z'_j\} \) of \( \{z_j\} \) and \( R'_j \to \infty \) satisfying the following properties: \( \Gamma(z'_j, R'_j) \) are disjoint,
\[ B_j = \text{Card} \Gamma(z'_j, R'_j) = \text{Card} [\Gamma_0 \cap D(R'_j)], \]
and we can enumerate \( \Gamma(z'_j, R'_j) = \{w_{jk}\}_{1 \leq k \leq B_j} \) in such a way that
\[ \max_{1 \leq k \leq B_j} \left| z_k^0 - \frac{w_{jk} - z'_j}{\rho(z'_j)} \right| \to 0, \quad j \to \infty. \]

Without loss of generality, we can assume that \( |z'_j| > 1 - \eta(R'_j) \). Therefore, by Proposition 3.5 there exist \( g_j = g_{z'_j, R'_j} \) satisfying (3.22) and (3.20).
For \( j \geq 1 \) we consider the following interpolation problem:

\[
f_j(w) = \begin{cases} 
  a_k g_j(w_{jk}), & w = w_{jk} \in \Gamma(z_j', R_j'), \\
  0, & w \in \Gamma \setminus \Gamma(z_j', R_j').
\end{cases}
\]  

(7.2)

By our assumption on \( \Gamma \), we can find \( f_j \in \mathcal{A}_h(\mathbb{D}) \) satisfying (7.2). Then the functions

\[
F_j(w) = \frac{f_j(z_j' + w\rho(z_j'))}{g_j(z_j' + w\rho(z_j'))}
\]

satisfy the properties

\[
F_j\left(\frac{w_{jk} - z_j'}{\rho(z_j')}\right) = a_k, \quad 1 \leq k \leq B_j,
\]

\[
|F_j(w)| \leq c e^{\beta(w)}, \quad |w| \leq R_j'.
\]

By a normal families argument and by (7.1), we conclude that there exists an entire function \( F \in \mathcal{A}_{\beta}(\mathbb{C}) \) such that

\[
F(z_k^0) = a_k, \quad k \geq 1.
\]

Thus, \( \Gamma_0 \) is a set of interpolation for \( \mathcal{A}_{\beta}(\mathbb{C}) \). However, by the theorem of Seip on interpolation in the Fock type spaces [20, Theorem 2.4], this is impossible for \( \Gamma_0 \) satisfying (5.11). This contradiction proves our assertion.

(B) Now we assume that \( \Gamma \) is a \( d_\rho \)-separated subset of \( \mathbb{D} \), \( D_\rho^+(\Gamma, \mathbb{D}) < \frac{1}{2} \). First of all, if \( \Gamma \) is an interpolation set for \( \mathcal{A}_h(\mathbb{D}) \), and \( \lambda \in \mathbb{D} \setminus \Gamma \), then \( \Gamma \cup \{\lambda\} \) is also an interpolation set for \( \mathcal{A}_h(\mathbb{D}) \). (Later on, to deal with the plane case, we add to \( \Gamma \) an infinite sequence in such a way that the modified \( \Gamma \) satisfies the same conditions, and then use that if \( \Gamma \) is an interpolation set for \( \mathcal{A}_h(\mathbb{C}) \), and \( \lambda \in \mathbb{C} \setminus \Gamma \), \( \mu \in \Gamma \), then \( \Gamma \cup \{\lambda\} \setminus \{\mu\} \) is also an interpolation set for \( \mathcal{A}_h(\mathbb{C}) \).)

For every sufficiently large \( R, R \geq R_0 \), we can find \( \eta_1(R) < 1 \) such that the family of sets \( \Gamma^\#(z, R), z \in \Gamma \setminus \mathcal{D}(\eta_1(R)), R \geq R_0 \), satisfies the uniform estimates

\[
\sup_{w \in \mathbb{C}, r \geq R_0} \frac{\text{Card}(\mathcal{D}(w, r) \cap \Gamma^\#(z, R))}{r^2} < \frac{1}{2},
\]

\[
\inf_{w_1 \neq w_2, w_1, w_2 \in \Gamma^\#(z, R)} |w_1 - w_2| > 0.
\]

(7.3)

Therefore, by (a variant of) the theorem of Seip-Wallstén on interpolation in the Fock type spaces [20, Theorem 2.4], [21, Theorem 1.2], for some \( c < \infty, \varepsilon > 0 \), and for every \( R \geq R_0 \), \( z \in \Gamma \setminus \mathcal{D}(\eta_1(R)) \), there
exists $F_{z,R} \in A_{(1-\varepsilon)\beta}(\mathbb{C})$, such that

$$
\begin{align*}
F_{z,R}(0) &= 1, \\
F_{z,R} \big| \Gamma^{\#}(z, R) \setminus \{0\} &= 0, \\
\|F_{z,R}\|_{(1-\varepsilon)\beta} &\leq c.
\end{align*}
\tag{7.4}
$$

To continue, we need a simple estimate similar to (6.1).

**Lemma 7.1.** If $F(z) = \sum_{n \geq 0} c_n z^n$, $|F(z)| \leq \exp |z|^2$, $z \in \mathbb{C}$, $N \in \mathbb{Z}_+$, and

$$(T_N F)(z) = \sum_{0 \leq n \leq N} c_n z^n,$$

then

$$
\begin{align*}
|(F - T_N F)(z)| &\leq 2^{-(N-3)/2}, \quad |z| \leq \sqrt{N/(4e)}, \\
|(T_N F)(z)| &\leq (N+1) \exp |z|^2, \quad \sqrt{N/(4e)} < |z| \leq \sqrt{N/2}, \\
|(T_N F)(z)| &\leq (N+1)|z|^{N(2e/N)^{N/2}}, \quad |z| > \sqrt{N/2}.
\end{align*}
$$

**Proof.** By the Cauchy formula, we have

$$|c_n| \leq \inf_{r > 0} [r^{-n} \exp r^2] = \exp \left[ -\frac{n}{2} \log \frac{n}{2e} \right],$$

and hence,

$$
\sum_{n > N} |c_n|r^n \leq \sum_{n > N} \left( \frac{n}{2e} \cdot \frac{4e}{N} \right)^{-n/2} \leq 2^{-(N-3)/2}, \quad r \leq \sqrt{N/(4e)},
$$

and

$$
\sum_{0 \leq n \leq N} |c_n|r^n \leq (N+1) \max_{0 \leq n \leq N} \exp \left[ -\frac{n}{2} \log \frac{n}{2e} + n \log r \right], \quad r \geq 0.
$$

Furthermore,

$$-\frac{n}{2} \log \frac{n}{2e} + n \log r \leq r^2, \quad r \geq 0,$$

and

$$-\frac{n}{2} \log \frac{n}{2e} + n \log r \leq N \log r - \frac{N}{2} \log \frac{N}{2e}, \quad r \geq \sqrt{N/2}, \quad 0 \leq n \leq N.$$
Corollary 7.2. If \( \varepsilon > 0 \), \( F \in A_{(1-\varepsilon)\beta}(\mathbb{C}) \), \( \|F\|_{(1-\varepsilon)\beta} \leq 1 \), \( N \in \mathbb{Z}_+ \), \( T_N F \) is defined as above, and \( R = \sqrt{2N/(1 - \varepsilon)} \) is sufficiently large, \( R > R(\varepsilon) \), then for some \( c = c(\varepsilon) > 0 \) independent of \( z, R \) we have

\[
| (F - T_N F)(z)| e^{-|z|^2/4} \leq e^{-\varepsilon R^2}, \quad |z| \leq R,
\]

\[
| (T_N F)(z)| e^{-|z|^2/4} \leq 2e^{-c|z|^2}, \quad |z| \leq R,
\]

\[
| (T_N F)(z)| \left( \frac{|z|^2}{R^2} \right)^{-R^2/4} \leq \left( \frac{|z|^2}{R^2} \right)^{-\varepsilon R^2/5}, \quad |z| > R.
\]

For large \( N \) we set \( R = \sqrt{2N/(1 - \varepsilon)} \), and for \( z \in \Gamma \) sufficiently close to the unit circle, define, using \( g_{z,R} \) from Proposition 3.6 and \( F_{z,R} \) from (7.4),

\[
U_z(w) = g_{z,R}(w) \cdot (T_N F_{z,R}) \left( \frac{w - z}{\rho(z)} \right).
\]

(7.5)

If \( \Gamma = \{z_n\}_{n \geq 1} \), \( a_n \in \mathbb{C} \), \( n \geq 1 \), and

\[
\sup_{n \geq 1} |a_n| e^{-h(z_n)} \leq 1,
\]

(7.6)

then for \( |z_n| \geq \eta = \max(\eta(R), \eta_1(R)) \) \( (\eta(R)) \) is introduced in Proposition 3.6 we put

\[
V_n = \frac{a_n U_{z_n}}{U_{z_n}(z_n)}.
\]

(7.7)

Then

\[
V_n(z_n) = a_n,
\]

(7.8)

and by the estimates in Proposition 3.6 and in Corollary 7.2 we obtain for \( |z_n| \geq \eta \) that

\[
|V_n(z_k)| e^{-h(z_k)} \leq c_0 e^{-\varepsilon R^2}, \quad |z_k - z_n| \leq R\rho(z_n), \quad k \neq n,
\]

(7.9)

\[
|V_n(z)| e^{-h(z)} \leq c_0 e^{-R|z - z_n|^2/[\rho(z_n)]^2}, \quad |z - z_n| \leq R\rho(z_n),
\]

(7.10)

\[
|V_n(z)| e^{-h(z)} \leq c_0 \left( \frac{R^2 \min[\rho(z_n), \rho(z)]^2}{e|z - z_n|^2} \right)^{\varepsilon R^2/5}, \quad |z - z_n| > R\rho(z_n),
\]

(7.11)

for some \( c_0 \) independent of \( z_n, z, R \).

For \( R > 1 \) we define

\[
A_R(z) = \sum_{z_n \in \Gamma, |z - z_n| > R\rho(z_n)} \left( \frac{R^2 \min[\rho(z_n), \rho(z)]^2}{e|z - z_n|^2} \right)^{\varepsilon R^2/5}.
\]
Suppose that for every $\delta > 0$, we can find arbitrarily large $R$ such that

$$\sup_{D} A_R \leq \delta. \quad (7.12)$$

Then for $0 < \delta < 1/(2c_0)$, for sufficiently large $R$, and for $\eta = \max(\eta(R), \eta_1(R))$ we can define

$$f_1 = \sum_{z_n \in \Gamma \backslash D(\eta)} V_n,$$

and obtain that for some $B$ independent of $\{a_n\}$ satisfying (7.6),

$$\|f_1\|_h \leq B, \quad (7.13)$$

$$\sup_{z_n \in \Gamma \backslash D(\eta)} |f_1(z_n) - a_n| e^{-h(z_n)} \leq \frac{1}{2}, \quad (7.14)$$

Indeed, by (7.10), (7.11), and (7.12), for $z \in D$ we have

$$|f_1(z)| e^{-h(z)} \leq \sum_{z_n \in \Gamma \backslash D(\eta), |z-z_n| \leq R\rho(z_n)} |V_n(z)| e^{-h(z)} + \sum_{z_n \in \Gamma \backslash D(\eta), |z-z_n| > R\rho(z_n)} |V_n(z)| e^{-h(z)} \leq c + c_0 \delta.$$ 

By (7.8), (7.9), (7.11), and (7.12), for $z_k \in \Gamma \backslash D(\eta)$ we have

$$|f_1(z_k) - a_k| e^{-h(z)} \leq \sum_{|z_k-z_n| \leq R\rho(z_n), k \neq n} |V_n(z)| e^{-h(z)} + \sum_{z_n \in \Gamma \backslash D(\eta), |z_k-z_n| > R\rho(z_n)} |V_n(z)| e^{-h(z)} \leq cR^2 e^{-cR^2} + c_0 \delta \leq \frac{1}{2}$$

for sufficiently large $R$. We fix such $\delta, R, \eta$.

Iterating the approximation construction and using (7.13) and (7.14), we obtain $f_2 \in A_h(D)$ such that

$$\|f_2\|_h \leq B/2,$$

$$\sup_{z_n \in \Gamma \backslash D(\eta)} |f_2(z_n) + f_1(z_n) - a_n| e^{-h(z_n)} \leq \frac{1}{4}.$$ 

Continuing this process, we arrive at $f = \sum_{n \geq 1} f_n$ such that

$$\|f\|_h \leq 2B,$$

$$f(z_n) = a_n, \quad z_n \in \Gamma \backslash D(\eta).$$

Thus, $\Gamma \backslash D(\eta)$ is a set of interpolation for $A_h(D)$, and hence, $\Gamma$ is a set of interpolation for $A_h(D)$. 

---

**Note:** The document provides a detailed mathematical proof involving sampling and interpolation in radial weighted spaces, focusing on the conditions and approximations for defining and bounding the functions and their derivatives within specific domains. The proof is structured to ensure clarity and rigor, aligning with the requirements of mathematical analysis in this field.
It remains to estimate $A_R$ for large $R$. Since $\Gamma$ is $d_\rho$-separated, using (2.2) we obtain
\[
\sum_{z_n \in \Gamma, |z-z_n| > R \rho(z_n)} \left( \frac{R^2 \min[\rho(z_n), \rho(z)]^2}{\epsilon |z-z_n|^2} \right)^{\epsilon R^2/5} \leq C(\Gamma, h) \int_{D \setminus D(z, R \rho(z))} \left( \frac{R^2 \min[\rho(w), \rho(z)]^2}{\epsilon |z-w|^2} \right)^{\epsilon R^2/5} dm_2(w) \rho(w)^2
\]
\[
\leq R^2 \int_{|\zeta| > 1} \left( \frac{1}{\epsilon |\zeta|^2} \right)^{\epsilon R^2/5} dm_2(\zeta) = o(1), \quad R \to \infty, \quad (7.15)
\]
because for any $z, w \in D$, $R \geq \sqrt{5/\epsilon}$,
\[
\left( \frac{\min[\rho(w), \rho(z)]}{\rho(z)} \right)^{2\epsilon R^2/5} \left( \frac{\rho(z)}{\rho(w)} \right)^2 \leq 1.
\]
This completes the proof of our assertion. \qed

**Proof of Theorem 2.4.** By Corollary 4.4 every set of interpolation for $A_p^\beta(D)$ is $d_\rho$-separated.

(A) The argument is analogous to that in the part (A) of the proof of Theorem 2.3. We just use [20, Theorem 2.2] instead of [20, Theorem 2.4].

(B) Now we assume that $\Gamma$ is a $d_\rho$-separated subset of $D$, $D_\rho^+(\Gamma, D) < \frac{1}{2}$. As in the part (B) of the proof of Theorem 2.3 we find $c < \infty$, $\epsilon > 0$, and $R_0 > 1$, such that for $R \geq R_0$, $z \in \Gamma \setminus D(\eta_1(R))$, the sets $\Gamma^\#(z, R)$ satisfy (7.3), there exist $F_{z,R} \in A_{p,1-2\epsilon/\beta}^p(C)$, such that
\[
\left\{ \begin{array}{l}
F_{z,R}(0) = 1, \\
F_{z,R} \mid \Gamma^\#(z, R) \setminus \{0\} = 0, \\
\|F_{z,R}\|_{p,1-2\epsilon/\beta} \leq c.
\end{array} \right. \quad (7.16)
\]
Instead of Lemma 7.1 and Corollary 7.2 we use

**Lemma 7.3.** If $F \in A_{p,1-2\epsilon/\beta}(C)$, $\|F\|_{p,1-2\epsilon/\beta} \leq 1$, if $F(z) = \sum_{n \geq 0} c_n z^n$, $T_N F$ is defined as in Lemma 7.1 and if $R = \sqrt{2N/(1-\epsilon)}$ is sufficiently large, $R > R(\epsilon)$, then for some $c = c(\epsilon) > 0$ independent of $z, R$ we have
\[
\begin{align*}
\|(F - T_N F)(z)|e^{-|z|^2/4} &\leq e^{-cR^2}, \quad |z| \leq R, & (7.17) \\
\|(T_N F)(z)| &\leq \left( \frac{e|z|^2}{R^2} \right)^{-R^2/4} \left( \frac{\epsilon |z|^2}{R^2} \right)^{-\epsilon R^2/5}, \quad |z| > R, & (7.18) \\
\int_{D(R)} |(T_N F)(z)|^{p} e^{-p|z|^2/4} dm_2(z) &\leq 1. & (7.19)
\end{align*}
\]
Proof. We just use Lemma 4.5 and Corollary 7.2 to deduce (7.17)–(7.18). Inequality (7.19) is evident. □

Next, we define \( U_z \) as in (7.5) using \( F_z,R \) from (7.16). If \( \Gamma = \{ z_n \} \) \( n \geq 1 \), \( a_n \in \mathbb{C}, n \geq 1 \), and

\[
\sum_{n \geq 1} |a_n|^p e^{-ph(z_n)} \rho(z_n)^2 \leq 1, \tag{7.20}
\]

then for \( |z_n| \geq \eta = \max(\eta(R), \eta_1(R)) \) (\( \eta(R) \) is introduced in Proposition 3.6) we define \( V_n \) by (7.7). Set \( \gamma_n = |a_n| e^{-h(z_n)} \). As above, we obtain

\[
V_n(z) = a_n, \quad |V_n(z_k)| e^{-h(z_k)} \leq c_1 \gamma_n \cdot e^{-cR^2}, \quad |z_k - z_n| \leq R\rho(z_n), \quad k \neq n,
\]

\[
\int_{D(z, R\rho(z_n))} |V_n(z)|^p e^{-ph(z)} dm_2(z) \leq c_1 \gamma_n^p \cdot \rho(z_n)^2,
\]

\[
|V_n(z)| e^{-h(z)} \leq c_1 \gamma_n \cdot \left( \frac{R^2 \min[\rho(z_n), \rho(z)]^2}{e|z - z_n|^2} \right)^{\varepsilon R^2/5}, \quad |z - z_n| > R\rho(z_n),
\]

for some \( c, c_1 \) independent of \( z_n, z, R \).

For \( z, \zeta \in \mathbb{D} \) we set

\[
W(z, \zeta) = \left( \frac{R^2 \min[\rho(\zeta), \rho(z)]^2}{e|z - \zeta|^2} \right)^{\varepsilon R^2/5} (1 - \chi_{D(z, R\rho(z_n))}(z)). \tag{7.21}
\]

Now, to complete the proof as in part (B) of Theorem 2.3, we need only to verify that for every \( \delta > 0 \) there exists arbitrarily large \( R \) such that

\[
\sum_{k \geq 1} \left( \sum_{n \geq 1} \gamma_n W(z_k, z_n) \right)^p \rho(z_k)^2 \leq \delta,
\]

\[
\int_{\mathbb{D}} \left( \sum_{n \geq 1} \gamma_n W(z, z_n) \right)^p dm_2(z) \leq 1.
\]

Furthermore, using (7.21), we can deduce these inequalities from the inequality

\[
\int_{\mathbb{D}} \left( \sum_{n \geq 1} \gamma_n W(z, z_n) \right)^p dm_2(z) \leq \delta_1, \tag{7.21}
\]

with small \( \delta_1 \). By (7.15), for any small \( \delta_2 \) \( > 0 \) we can find large \( R \) such that

\[
\sum_{n \geq 1} W(z, z_n) \leq \delta_2, \quad z \in \mathbb{D}. \tag{7.22}
\]
An estimate analogous to (7.15) gives us for large $R$

$$
\int_{D} W(z, z_n) \, dm_2(z) \leq c \rho(z_n)^2.
$$

Therefore,

$$
\int_{D} \left( \sum_{n \geq 1} \gamma_n W(z, z_n) \right)^p \, dm_2(z)
\leq \int_{D} \left( \sum_{n \geq 1} W(z, z_n) \right)^{p-1} \cdot \left( \sum_{n \geq 1} \gamma_n^p W(z, z_n) \right) \, dm_2(z)
\leq \delta_2^{p-1} \cdot \sum_{n \geq 1} \gamma_n^p \int_{D} W(z, z_n) \, dm_2(z)
\leq c \delta_2^{p-1} \cdot \sum_{n \geq 1} \gamma_n^p \rho(z_n)^2 = c \delta_2^{p-1}.
$$

This completes the proof of our assertion in the case $p > 1$. If $p = 1$, then (7.21) follows from (7.20) and (7.22).

\[\square\]

8. The plane case

The results of Sections 3–7 easily extend to the plane case. First, we can approximate $h$ by $\log |f|$ for a special infinite product $f$.

**Proposition 8.1.** There exist sequences $\{r_k\}$, $\{s_k\}$, $0 = r_0 < s_0 < r_1 < \ldots r_k < s_k < r_{k+1} < \ldots < \infty$, and a sequence $N_k, k \geq 0$, of natural numbers, such that $N_{k+1} \geq N_k$ for large $k$, and

\begin{align*}
(i) \quad & \lim_{k \to \infty} \frac{r_{k+1} - r_k}{\rho(r_k)} = \sqrt{2\pi}, \quad \lim_{k \to \infty} \frac{N_k(r_{k+1} - r_k)}{r_k} = 2\pi, \\
& \lim_{k \to \infty} \frac{r_{k+1} - r_k}{r_k - r_{k-1}} = 1, \quad \lim_{k \to \infty} \frac{r_{k+1} - s_k}{r_k - r_{k-1}} = \frac{1}{2}, \\
(ii) \quad & \text{if } \Lambda = \left\{ s_k e^{2\pi i m / N_k} \right\}_{k \geq 0, 0 \leq m < N_k}, \text{ and if} \\
& f(z) = \lim_{r \to \infty} \prod_{\lambda \in \Lambda \cap r \mathbb{D}} \left( 1 - \frac{z}{\lambda} \right),
\end{align*}

then the products in the right hand side converge uniformly on compact subsets of the plane, and

$$
|f(z)| \asymp e^{h(z)} \frac{\text{dist}(z, \Lambda)}{\rho(z)}, \quad z \in \mathbb{C}.
$$
The proof in the case (II_c) is analogous to that of Proposition 3.1 in the case (II_d). We need only to mention that in the estimate (3.14) we use that
\[
\int_0^{y-1} e^{-c_2(r-x)/\rho(x)} \frac{dx}{\rho(x)} \leq c \int_0^{y-1} e^{-c_2(r-x)/\rho(x)^2} \frac{dx}{\rho(x)^2} \leq c_1 \int_0^{y-1} \frac{dx}{(r-x)^2} \leq c_1.
\]

In the case (I_c), to estimate \( \sum U_m \), \( U_m = \log |1 - s_m^{N_m} z^{-N_m}|, \quad 0 \leq m < k, \)
we divide \( m, 0 \leq m < k \), into the groups
\[
S_t = \{ m : 2^t \rho(r) \leq \rho(s_m) < 2^{t+1} \rho(r) \}, \quad t \geq 0.
\]
Then for some \( M < \infty \)
\[
\text{Card } S_t \leq \frac{c r}{2^t / \rho(r)}, \quad r \geq 1,
\]
and
\[
\sum_{m \in S_t} e^{-N_m(r-s_m)} \leq c_1 e^{-c r^2/(2^t \rho(r))} \cdot \frac{r}{2^t / \rho(r)} \leq c_2 2^{-t/M}, \quad t > 0, r \geq 1,
\]
for some \( c, c_1, c_2 \) independent of \( t > 0, r \geq 1 \), and
\[
\sum_{m \in S_0} e^{-N_m(r-s_m)} \leq \sum_{k \geq 0} e^{-ck(r)/\rho(r)} \leq c_1,
\]
for some \( c, c_1 \) independent of \( r \geq 1 \).

Using Proposition 8.1 we arrive at analogs of Propositions 3.5 and 3.6. For example, we have

**Proposition 8.2.** Given \( R \geq 100 \), there exists \( \eta(R) < \infty \) such that for every \( z \in \mathbb{C} \) with \( |z| \geq \eta(R) \), there exists a function \( g = g_{z,R} \) analytic in \( \mathbb{C} \) such that
\[
|g(w)| e^{-h(w)} \asymp e^{-|z-w|^2/[4 \rho(z)^2]}, \quad w \in \mathcal{D}(z, R \rho(z)),
\]
\[
|g(w)| e^{-h(w)} \leq c(h) \left[ \frac{R^2 \min[\rho(z), \rho(w)]^2}{e|z-w|^2} \right]^{R^2/4}, \quad w \in \mathbb{C} \setminus \mathcal{D}(z, R \rho(z)).
\]

After that, the arguments in Sections 4–7 extend to the plane case, and we obtain Theorems 2.5 and 2.8.
9. Subspaces of large index

Let $X$ be a Banach space of analytic functions in the unit disc, such that the operator $M_z$ of multiplication by the independent variable $f \mapsto zf$ acts continuously on $X$. Examples of such spaces are the Hardy spaces $H^p$ and the weighted Bergman spaces $A^{p,h}(D)$, $1 \leq p < \infty$. A closed proper subspace $E$ of $X$ is said to be $z$-invariant if $M_z E \subset E$. The index of a $z$-invariant subspace $E$ is defined as

$$\text{ind} E = \dim E/M_z E.$$

Every $z$-invariant subspace of the Hardy space $H^2$ has index 1. In 1985, C. Apostol, H. Bercovici, C. Foiaş and C. Pearcy [2] proved (in a non-constructive way) that every space $A^{2,h}(D)$ has $z$-invariant subspaces of index equal to 1, 2, \ldots, $+\infty$. Later on, H. Hedenmalm [12] and H. Hedenmalm, S. Richter, K. Seip [14] produced concrete examples of $z$-invariant subspaces of index bigger than 1 in $A^{p,h}(D)$, $A^{p,h}(D)$, with $h_\alpha(z) = \alpha \log \frac{1}{1-|z|}$, $\alpha \geq 0$. These examples are based on Seip’s description of sampling sets in $A^{p,h}(D)$.

In this section we give a construction of $z$-invariant subspaces of large index in $A^{p,h}(D)$, $1 \leq p < \infty$, with $h$ satisfying $(I_D)$ or $(II_D)$ based on our results above. For other constructions suitable for large classes of Banach spaces of analytic functions in the unit disc and for other information on index of $z$-invariant subspaces see [5], [1], [6].

Given a non-empty subset $\Lambda \subset D$, $1 \leq p < \infty$, set

$$I(\Lambda) = \{ f \in A^{p,h}(D) : f(\lambda) = 0, \lambda \in \Lambda \}.$$

If $I(\Lambda) \neq \{0\}$, then $I(\Lambda)$ is a closed $z$-invariant subspace of $A^{p,h}(D)$, and $\text{ind} I(\Lambda) = 1$.

Given $z$-invariant subspaces $E_\alpha$, $\alpha \in A$, denote by $\vee_{\alpha \in A} E_\alpha$ the minimal $z$-invariant subspace containing all $E_\alpha$. It is known that if $\text{ind} E_\alpha = 1$, $\alpha \in A$, then

$$\text{ind} \vee_{\alpha \in A} E_\alpha \leq \text{Card} A.$$

**Theorem 9.1.** If $h$ satisfies either $(I_D)$ or $(II_D)$, and if $1 \leq p < \infty$, then there exist subsets $\Lambda_d \subset D$, $0 \leq d < \infty$, such that

$$\text{ind} \vee_{0 \leq d < u} I(\Lambda_d) = u, \quad 1 \leq u \leq \infty.$$

**Proof.** We restrict ourselves by the (most difficult) case $u = +\infty$, and use the method proposed in [14]. First, by (4.1), we can find $\tilde{h} > h$
satisfying the same conditions as $h$ and such that
\[ \tilde{h}(r) = (1 + o(1))h(r), \quad r \to 1, \]
\[ \tilde{\rho}(r) = (1 + o(1))\rho(r), \quad r \to 1, \]
\[ \log \frac{1}{\rho(r)} = o(\tilde{h}(r) - h(r)), \quad r \to 1, \]
where $\tilde{\rho}(r) = \left[ (\Delta \tilde{h})(r) \right]^{-1/2}$, $0 \leq r < 1$. Then we apply Proposition 3.1 to $\tilde{h}$ to obtain $\Lambda = \{ s_k e^{2\pi im/N_k} \}_{k \geq 0, 0 \leq m < N_k}$, and $f \in \mathcal{A}_{\tilde{h}}(\mathbb{D})$ such that $f(0) = 1$,

\[ |f(z)| \asymp e^{\tilde{h}(z) \text{dist}(z, \Lambda) / \tilde{\rho}(z)}, \quad z \in \mathbb{D}, \]
\[ |f'(\lambda)| \asymp \frac{e^{\tilde{h}(\lambda) / \tilde{\rho}(\lambda)}}{\tilde{\rho}(\lambda)}. \quad \lambda \in \Lambda. \]

An argument similar to that in the proof of Lemma 4.5 shows that for every $g \in \mathcal{A}_{\tilde{h}}^p(\mathbb{D})$ we have

\[ \lim_{|z| \to 1} |g(z)| e^{-\tilde{h}(z)} = 0. \quad (9.1) \]

For $g \in \mathcal{A}_{\tilde{h}}^p(\mathbb{D})$ and for $k \geq 1$ by the residue calculus we have

\[ |g(0)| = \frac{|g(0)|}{|f(0)|} \leq \sum_{\lambda \in \Lambda \cap D(r_k)} \frac{|g(\lambda)|}{|\lambda f'(\lambda)|} + \frac{1}{2\pi} \int_{r_k T} \frac{|g(\zeta)|}{|\zeta f(\zeta)|} |d\zeta|. \]

Passing to the limit $k \to \infty$, and using (9.1) and the fact that $\Lambda$ is $d_{\rho}$-separated and hence,

\[ \sum_{\lambda \in \Lambda} \tilde{\rho}(\lambda)^2 \leq c, \]

we conclude that

\[ |g(0)| \leq \sum_{\lambda \in \Lambda} \frac{|g(\lambda)|}{|\lambda f'(\lambda)|} \]
\[ \leq c \left( \sum_{\lambda \in \Lambda} |g(\lambda)|^p e^{-p\tilde{h}(\lambda)} \tilde{\rho}(\lambda)^2 \right)^{1/p} \left( \sum_{\lambda \in \Lambda} \tilde{\rho}(\lambda)^2 \right)^{(p-1)/p} \]
\[ \leq c \left( \sum_{\lambda \in \Lambda} |g(\lambda)|^p e^{-ph(\lambda)} \rho(\lambda)^2 \right)^{1/p} = c \|g\|_{p,h,\Lambda}. \quad (9.2) \]

We should note here that $\Lambda$ is neither a sampling set for $\mathcal{A}_{\tilde{h}}^p(\mathbb{D})$ nor that for $\mathcal{A}_{\tilde{h}}(\mathbb{D})$. 
Now we set
\[ \Lambda_d^* = \left\{ s_k e^{2\pi i m/N_k}, k = 2^{d+1}(2v+1), v \geq 0, 0 \leq m < N_k \right\}, \]
\[ \Lambda_d = \Lambda \setminus \Lambda_d^*, \quad d \geq 0. \]

It is clear that
\[ D_\rho^+(\Lambda, \mathbb{D}) = D_\rho^-(\Lambda, \mathbb{D}) = \frac{1}{2}, \]
\[ D_\rho^+(\Lambda_d, \mathbb{D}) = D_\rho^-(\Lambda_d, \mathbb{D}) = \frac{1 - 2^{-d-1}}{2}, \quad d \geq 0. \]

Using Theorem 2.3 we obtain that \( I(\Lambda_d) \neq \{0\}, d \geq 0. \) To complete our proof we apply the following criterion from [14, Theorem 2.1].

Suppose that for every \( d \geq 0 \) there exists \( c_d > 0 \) such that
\[ c_d |g(0)| \leq \|g + g_1\|_{p,h}, \quad g \in I(\Lambda_d), g_1 \in \vee_{l \geq 0, l \neq d} I(\Lambda_l). \] (9.3)

Then
\[ \operatorname{ind}_{d \geq 0} I(\Lambda_d) = +\infty. \]

It remains to verify (9.3). Since \( \Lambda \) is \( d_\rho \)-separated and \( \Lambda_d^* \) are pairwise disjoint, by Lemma 4.7 and by (9.2) we obtain
\[ \|g + g_1\|_{p,h} \geq c\|g + g_1\|_{p,h,\Lambda} \geq c\|g\|_{p,h,\Lambda} = c\|g\|_{p,h,\Lambda^*} \geq c_1 |g(0)|. \]

This proves our theorem. \( \square \)

References

[1] E. Abakumov, A. Borichev, Shift invariant subspaces with arbitrary indices in \( \ell^p \) spaces, Journ. of Funct. Anal. 188 (2002), no. 1, 1–26.
[2] C. Apostol, H. Bercovici, C. Foiaș, C. Pearcy, Invariant subspaces, dilation theory, and the structure of the predual of a dual algebra, I, Journ. of Funct. Anal. 63 (1985), 369–404.
[3] B. Berndtsson, J. Ortega-Cerdà, On interpolation and sampling in Hilbert spaces of analytic functions, J. Reine Angew. Math. 464 (1995) 109–128.
[4] A. Beurling, The collected works of Arne Beurling. Vol. 2 Harmonic Analysis, Eds. L. Carleson, P. Malliavin, J. Neuberger and J. Werner, Birkhäuser, Boston, 1989, pp. 341–365.
[5] A. Borichev, Invariant subspaces of given index in Banach spaces of analytic functions, J. Reine Angew. Math. 505 (1998) 23–44.
[6] A. Borichev, H. Hedenmalm, A. Volberg, Large Bergman spaces: invertibility, cyclicity, and subspaces of arbitrary index, Journ. Funct. Anal. 207 (2004) 111–160.
[7] S. Brekke, K. Seip, Density theorems for sampling and interpolation in the Bargmann–Fock space, III, Mathematica Scand. 73 (1993) 112–126.
[8] J. Bruna, Sampling in complex and harmonic analysis, European Congress of Mathematics, Vol. I (Barcelona, 2000), Progr. Math., 201 (2001) 225–246.
[9] P. Casazza, O. Christensen, A. Lindner, R. Vershynin, *Frames and the Feichtinger conjecture*, Proc. Amer. Math. Soc. **133** (2005) no. 4, 1025–1033.

[10] P. Casazza, M. Fickus, J. Tremain, E. Weber, *The Kadison-Singer Problem in Mathematics and Engineering*, Preprint, 2006.

[11] K. Cichon, K. Seip, *Weighted holomorphic spaces with trivial closed range multiplication operators*, Proc. Amer. Math. Soc. **131** (2002) 201–207.

[12] H. Hedenmalm, *An invariant subspace of the Bergman space having the codimension two property*, J. Reine Angew. Math. **443** (1993) 1–9.

[13] H. Hedenmalm, B. Korenblum, K. Zhu, *Theory of Bergman spaces*, Springer–Verlag, New York, 2000.

[14] H. Hedenmalm, S. Richter, K. Seip, *Interpolating sequences and invariant subspaces of given index in the Bergman spaces*, J. Reine Angew. Math. **477** (1996) 13–30.

[15] Yu. Lyubarskii, E. Malinnikova, *On approximation of subharmonic functions*, J. Anal. Math. **83** (2001) 121–149.

[16] Yu. Lyubarskii, K. Seip, *Sampling and interpolation of entire functions and exponential systems in convex domains*, Ark. Mat. **32** (1994) no. 1, 157–193.

[17] Yu. Lyubarskii, M. Sodin, *Analogues of sine type functions for convex domains*, Preprint no.17, Inst. Low Temp. Phys. Eng., Ukrainian Acad. Sci., Kharkov, 1986.

[18] N. Marco, X. Massaneda, J. Ortega-Cerdà, *Interpolating and sampling sequences for entire functions*, Geom. and Funct. Anal. **13** (2003) 862–914.

[19] J. Ortega-Cerdà, K. Seip, *Beurling-type density theorems for weighted $L_p$ spaces of entire functions*, J. d’Anal. Math. **75** (1998) 247–266.

[20] K. Seip, *Density theorems for sampling and interpolation in the Bargmann–Fock space*, I, J. Reine Angew. Math. **429** (1992) 91–106.

[21] K. Seip, *Beurling type density theorems in the unit disk*, Inv. Math. **113** (1993) 21–31.

[22] K. Seip, *Developments from nonharmonic Fourier series*, Documenta Math. J. DMV, Extra Volume ICM (1998) 713–722.

[23] K. Seip, *Interpolating and sampling in spaces of analytic functions*, American Mathematical Society, Providence, 2004.

[24] K. Seip, R. Wallstén, *Density theorems for sampling and interpolation in the Bargmann–Fock space*, II, J. Reine Angew. Math. **429** (1992) 107–113.

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