Algebraic stability conditions
and contractible stability spaces

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Abstract

Suppose that C is either a locally-finite triangulated category with
finite rank Grothendieck group, or a discrete derived category of finite
global dimension. We prove that any component of the space of stabil-
ity conditions on C is contractible (and that there is only one compo-
nent in the discrete case). More generally, we prove that any ‘finite-type’
component of a stability space is contractible. In particular, the prin-
cipal component of the stability space associated to the Calabi–Yau–N
Ginzburg algebra of an ADE Dynkin quiver is contractible. These results
generalise and unify various known ones for stability spaces of specific
categories, and settle some conjectures about the stability spaces associated
to Dynkin quivers, and to their Calabi–Yau–N Ginzburg algebras.

1 Introduction

Spaces of stability conditions on a triangulated category were introduced in
[10], inspired by the work of Michael Douglas on stability of D-branes in string
theory. The construction associates a space Stab(C) of stability conditions
to each triangulated category C. A stability condition σ ∈ Stab(C) consists of
a slicing — for each ϕ ∈ ℝ an abelian subcategory Pσ(ϕ) of semistable
objects of phase ϕ such that each object of C can be expressed as an iterated
extension of semistable objects — and a central charge Z : KC → C mapping the
Grothendieck group KC linearly to C. The slicing and charge obey a short list
of axioms. The miracle is that the ‘moduli space’ Stab(C) of stability conditions
is a (possibly empty or infinite-dimensional) smooth complex manifold, locally
modelled on a linear subspace of \text{Hom}(KC, C) [10 Theorem 1.2]. Whilst a
number of examples are known it is, in general, difficult to compute Stab(C).
In this paper we use algebraic and combinatorial methods to establish results
about the topology of certain stability spaces. In particular we show that the
components of the stability space of a locally-finite triangulated category with
finite rank Grothendieck group, or of a discrete derived category with finite
global dimension, are contractible. We also show that the principal component
of the stability space associated to the Calabi–Yau–N Ginzburg algebra of an
ADE Dynkin quiver is contractible. These results generalise and unify various
known ones on the topology of stability spaces. The starting point of our analysis
is the relation between stability conditions and t–structures.
Roughly, a slicing can be seen as a real analogue of a $t$–structure, and a stability condition as a complex analogue. Each stability condition $\sigma \in \text{Stab}(C)$ has an associated $t$–structure $D_\sigma$, whose aisle consists of extensions of semistable objects with strictly positive phase. Thus $\text{Stab}(C)$ is a union of (possibly empty) disjoint subsets $S_D$ of stability conditions with fixed associated $t$–structure $D$. Algebraically one moves from one $t$–structure to a neighbouring one by Happel–Reiten–Smalø tilting. The geometry of $\text{Stab}(C)$ reflects this, for example [36, \S 5]:

- If $S_D$ and $S_E$ are in the same component of $\text{Stab}(C)$ then $D$ and $E$ are related by a finite sequence of tilts;
- If $\sigma$ and $\tau$ are close in the natural metric on $\text{Stab}(C)$ then $D_\sigma$ and $D_\tau$ are mutual tilts of some third $t$–structure;
- If $(\sigma_n)$ is a sequence of stability conditions in some fixed $S_D$ with limit $\sigma$ then $D_\sigma$ is a tilt of $D$.

Thus we can think of $\text{Stab}(C)$ as a map of ‘well-behaved’ $t$–structures on $C$, and the tilting relations between them, in which the latter discrete structure has been suitably ‘smoothed out’. Under certain finiteness conditions this discrete structure can be used to build a combinatorial model for the homotopy type of $\text{Stab}(C)$.

The collection of $t$–structures can be made into a poset $T(C)$ with relation $D \subset E$ if there is an inclusion of the respective aisles. The shift in the triangulated category $C$ induces a shift on $T(C)$, such that $D \subset D[-1]$. The relation of tilting is encoded in the poset together with this shift; $E$ is a left tilt of $D$ if and only if $D \subset E \subset D[-1]$. We can define a sub-poset, the tilting poset $\text{Tilt}(C)$, with the same elements, but where now $D \leq E$ if there is a finite sequence of left tilts from $D$ to $E$. The above facts suggest that the topology of $\text{Stab}(C)$ is intimately related to the properties of $\text{Tilt}(C)$. We prove a result in this direction, but where we restrict to the subspace $\text{Stab}_{\text{alg}}(C)$ of ‘algebraic’ stability conditions and the subset of ‘algebraic’ $t$–structures.

We say a $t$–structure is algebraic if its heart is an abelian length category with finitely many simple objects, and that a stability condition is algebraic if its associated $t$–structure is so. (The term ‘finite category’ is often used for an abelian length category with finitely many simple objects, but we prefer to avoid it since the term is overloaded, and this usage potentially ambiguous.) The subspace $\text{Stab}_{\text{alg}}(C)$ of algebraic stability conditions has various nice properties which make it more amenable to analysis. The subset $S_D$ has non-empty interior if and only if $D$ is algebraic (Lemma 3.2). Moreover, it is easy to describe its geometry in this case: $S_D \cong (\mathbb{H} \cup \mathbb{R}_{<0})^n$ where $\mathbb{H}$ is the strict upper half-plane in $\mathbb{C}$ and $n = \text{rk } \mathcal{K}C$. In particular, it has a natural decomposition into contractible submanifolds corresponding to the decomposition into subspaces of the form $\mathbb{H}^i \times \mathbb{R}_{<0}^{n-i}$, and this yields a decomposition of $\text{Stab}_{\text{alg}}(C)$ into submanifolds. This decomposition satisfies the frontier condition, and endows $\text{Stab}_{\text{alg}}(C)$ with the structure of a regular, normal cellular stratified space (Corollary 3.10 and Proposition 3.21). There is a stratum $S_{D,I}$ of codimension $\# I$ for each pair $(D, I)$ of an algebraic $t$–structure $D$ and subset $I$ of simple objects in its heart, and

$$S_{D,I} \subset S_{E,J} \iff D \leq E \leq L_J E \leq L_I D$$
where \( L_I D \) is the left tilt of \( D \) at the torsion theory generated by the simple objects in \( I \), and \( L_J E \) is similarly defined. In particular, the poset of strata \( P\text{Stab}_{\text{alg}}(C) \) — whose elements are the strata ordered by inclusion of their closures — has a combinatorial description (Corollary 3.13) as a certain poset constructed from \( \text{Tilt}(C) \). For any pair of strata \( S \) and \( S' \) the set of strata \( T \) with \( S \subset T \subset S' \) is finite (Lemma 3.26), and if \( \text{Stab}_{\text{alg}}(C) \) is a locally-closed subspace of \( \text{Stab}(C) \) the number and configuration of such strata is determined in Lemma 3.27. In this case \( P\text{Stab}_{\text{alg}}(C) \) is a pure poset of length \( n \), whose closed bounded intervals have a uniform structure — in short the stratification of \( \text{Stab}_{\text{alg}}(C) \) is highly regular.

We prove our main results under certain finiteness conditions. Suppose \( \text{Stab}^0_{\text{alg}}(C) \) is a component of \( \text{Stab}_{\text{alg}}(C) \) and that for each \( \sigma \in \text{Stab}^0_{\text{alg}}(C) \) the \( t \)-structure \( D_\sigma \) has only finitely many algebraic tilts. Then the stratification of \( \text{Stab}^0_{\text{alg}}(C) \) is locally-finite and closure-finite, and has the structure of a regular, totally-normal CW-cellular stratified space. By [19, Theorem 2.50] the classifying space of \( P\text{Stab}^0_{\text{alg}}(C) \) embeds into \( \text{Stab}^0_{\text{alg}}(C) \) as a strong deformation retract, and this component has the homotopy-type of an \( n \)-dimensional CW complex (Corollary 3.22). In particular, the homology of \( \text{Stab}^0_{\text{alg}}(C) \) vanishes above the middle dimension. Under the subtly stronger condition that, for each \( \sigma \in \text{Stab}^0_{\text{alg}}(C) \), the \( t \)-structure \( D_\sigma \) has only finitely many tilts, all of which are algebraic, we prove in Theorem 4.9 that \( \text{Stab}^0_{\text{alg}}(C) \) is actually a contractible component of \( \text{Stab}(C) \); we say such a component is of finite-type. The finiteness condition is crucial for our proof, which proceeds by an induction on the number of strata in certain ‘conical’ subsets. We give various examples of finite-type components: any component of the stability space of a locally-finite triangulated category with finite rank Grothendieck group, or of a discrete derived category with finite global dimension is of finite-type, as is the principal component of the stability space associated to the Calabi–Yau–N Ginzburg algebra of an ADE Dynkin quiver — see respectively Corollaries 5.4, 5.8, and 4.10.

There are three posets which play a pivotal rôle in this paper: the poset \( T(C) \) of \( t \)-structures, the tilting poset \( \text{Tilt}(C) \), and the algebraic tilting poset \( \text{Tilt}_{\text{alg}}(C) \) whose elements are the algebraic \( t \)-structures with \( D \preceq E \) whenever there is a finite sequence of left tilts via algebraic \( t \)-structures from \( D \) to \( E \). Components of the latter are in correspondence with components of \( \text{Stab}_{\text{alg}}(C) \) (Corollary 3.13). There are injective maps of posets

\[
\text{Tilt}_{\text{alg}}(C) \hookrightarrow \text{Tilt}(C) \hookrightarrow T(C).
\]

The topological complexity of \( \text{Stab}(C) \) is governed in large part by the structure of intervals in these posets and the extent to which the above maps fail to be isomorphisms. As evidence for this claim we show that when all three are isomorphic \( \text{Stab}(C) \) is either empty or has a single component consisting of algebraic \( t \)-structures (Lemma 3.10). The first finiteness condition of the previous paragraph is equivalent to the finiteness of the intervals between a \( t \)-structure \( D \) and \( D[-1] \) in the component \( \text{Tilt}^0_{\text{alg}}(C) \) corresponding to \( \text{Stab}^0_{\text{alg}}(C) \). The second is equivalent to this finiteness together with the condition that \( \text{Tilt}^0_{\text{alg}}(C) \) is actually a component of \( \text{Tilt}(C) \). This last has the useful consequence that \( \text{Tilt}^0_{\text{alg}}(C) \) is then a lattice, whose closed bounded intervals are finite (2.4).

We discuss some related work. The idea of ‘exploring’ \( \text{Stab}(C) \) via tilting
is a natural one, and appears in many papers on stability spaces as a technique for constructing stability conditions. The papers [25] and [30] are very similar in spirit to this one. They discuss the case of the derived categories $D(Q)$ of finite-dimensional representations of an acyclic quiver $Q$, and $D(Γ_NQ)$ of the associated Calabi–Yau–N Ginzburg algebras $Γ_NQ$, where $N \geq 2$. Let $\text{Stab}(Q)$ and $\text{Stab}(Γ_NQ)$ be the associated stability spaces. In [25] the oriented exchange graph, whose vertices are $t$–structures and whose edges are simple left tilts between these, is identified in these cases (more precisely they identify the principal component, i.e. the component containing the standard $t$–structure with heart the representations). This carries an action of the Seidel–Thomas braid group, and the quotient is the exchange graph for $(N − 1)$-clusters. In [30] it is shown that for a Dynkin quiver $Q$ the exchange graphs embed into the respective spaces of stability conditions. This is then used to show that $\text{Stab}(Q)$ is simply-connected, and that the same is true for the principal component of $\text{Stab}(Γ_NQ)$ if the Seidel–Thomas braid action on it is faithful.

We can show more. When $Q$ is a Dynkin quiver, $D(Q)$ is both locally-finite and discrete, and the categories $D(Γ_NQ)$ inherit good finiteness properties from it. The (poset generated by the) oriented exchange graph of $C$ embeds into $\text{Tilt}(C)$, and for a finite-type component this is an isomorphism. The embedding of the (barycentric subdivision of the) exchange graph in $\text{Stab}^0(C)$ corresponds to the embedding of the 1-skeleton of $BP\text{Stab}^0(C)$. By considering all tilts, not just simple ones, we obtain a higher-dimensional simplicial complex capturing the entire homotopy-type. In this way we are able to generalise the proof of simply-connectedness in [30], using essentially the same method, to obtain the contractibility of the principal components — see Corollary 4.10 for $\text{Stab}^0(Γ_NQ)$ and Example 5.5 for $\text{Stab}(Q)$. This partially settles Conjectures 5.7 and 5.8 of [30].

The principal component $\text{Stab}^0(Γ_NQ)$ has been identified as a complex space in various cases, and in each of these it is already known to be contractible. When the underlying Dynkin diagram of $Q$ is $A_n$ [22] shows that $\text{Stab}^0(Γ_NQ)$ is the universal cover of the space of degree $n + 1$ polynomials $p_n(z)$ with simple zeros. The central charges are constructed as periods of the quadratic differential $p_n(z)^{N−2}dz^2$ on $\mathbb{P}^1$, using the technique of [14], which treats more general Calabi–Yau-3 categories by considering quadratic differentials on more general Riemann surfaces. The $N = 2$ and $N = 3$ cases were treated previously in [33] and [32] respectively. The $A_2$ case for arbitrary $N$, including $N = ∞$ which corresponds to $\text{Stab}(Q)$, is also proved in [13] using different methods. A similar picture holds for any ADE Dynkin quiver when $N = 2$; the paper [12] identifies $\text{Stab}^0(Γ_2Q)$ as a covering space, using a geometric description related to Kleinian singularities, and [30, Corollary 5.5] shows that it is actually the universal cover.

Slightly more generally than $D(Q)$, one can consider discrete derived categories. The principal component of the stability space for the simplest non-Dynkin case, the bounded derived category of the bound quiver

$$
\begin{align*}
\gamma_1 &
\gamma_2 \\
\gamma_2 &\cdot \gamma_1 = 0,
\end{align*}
$$

was shown to be contractible in [35]. This paper generalises the methods and results of [35] to show that the stability space of any discrete derived category
with finite global dimension is contractible (Corollary 5.8). The same result is obtained independently in [17, Theorem 8.10] using an alternative algebraic interpretation of the poset $P \text{Stab}(C)$ for a discrete derived category $C$ as the poset of silting pairs. In this way, [17] relates the combinatorics of the stability space to silting subcategories, silting mutation, Bongartz completion, and also to co-$t$-structures.

The natural next case is to consider tame representation type quivers. The prototypical example here is that of the Kronecker quiver $K$. The space of stability conditions on $D(K)$ was studied in [29] in geometric guise, using the fact that $D(K)$ is equivalent to the coherent derived category $D(P^1)$ of the projective line. The principal component $\text{Stab}^0(C)$ is identified, and the $t$-structures associated to its elements listed. The situation here is much more complicated: in this case $\text{Stab}^0_{\text{alg}}(C) \neq \text{Stab}^0(C)$, and is neither an open nor a closed subset. The stratification of $\text{Stab}^0_{\text{alg}}(C)$ is neither locally-finite, nor closure-finite. In particular, one cannot apply the machinery of [19], to show that $\text{Stab}^0_{\text{alg}}(C)$ is homotopy equivalent to $BP \text{Stab}^0_{\text{alg}}(C)$ — new ideas will be required to study the tame representation type case from the point of view of this paper. A more positive indication is that the union of orbits $C \cdot \text{Stab}_{\text{alg}}(C)$ of the natural $C$ action is the entirety of $\text{Stab}(C)$ in this case. It seems reasonable to hope that this might always be the case for a tame representation type quiver: $C \cdot \text{Stab}_{\text{alg}}(C)$ is always an open subset (Lemma 3.3) and, given that $S_D$ has empty interior if $D$ is not algebraic, one can hope that $C \cdot \text{Stab}_{\text{alg}}(C)$ is at least dense.

The tilting technique was used in [9] to construct an open subset of algebraic stability conditions on a certain triangulated category related to the canonical bundle on $P^2$ (specifically, on the full subcategory of the coherent derived category of the total space on those objects with cohomology supported on the zero section). Each heart of a stability condition which appears in this region is isomorphic to the category of nilpotent representations of a cyclic quiver with three vertices, where the numbers $a$, $b$ and $c$ of arrows between these are positive integral solutions of the Markov equation $a^2 + b^2 + c^2 = abc$. Each of these hearts has an excellent collection of simple objects, and it is shown in [8] that the set of such hearts is closed under simple tilts. Each component of the exchange graph is isomorphic to the Cayley graph for the standard generators of the affine braid group. In this case the existence of symmetries is used to control the tilting process, in place of the finiteness assumptions which we employ. It would be interesting to study the stratification of this open subset in more detail, in particular to see whether it is locally-finite and/or closure-finite. It would also be interesting to know whether the open subset is a component of the algebraic stability conditions. The combinatorics of tilting is known in many similar cases — we can replace $P^2$ by any smooth Fano variety $Z$ with a full exceptional collection, and such that $\dim K(Z) \otimes \mathbb{C} = \dim Z + 1$, see [8] — and so there are potentially many examples of this kind.

Here is a brief summary of the contents. Section 2 primarily contains background material, included for the sake of completeness, and to fix notation. The majority relates to $t$-structures, their abelian analogues torsion structures, and tilting. The only original material is in §2.4 where some basic results about algebraic $t$-structures and tilting between them are proved. Section 3 is a general discussion of the space $\text{Stab}_{\text{alg}}(C)$ of algebraic stability conditions. We
note some elementary properties, and then in §3.1 discuss the stratification in detail, relating it to the tilting poset $\text{Tilt}(C)$, and using the theory of [19] on cellular stratifications with non-compact cells to show that, when locally-finite, $\text{Stab}_\text{alg}(C)$ has the homotopy-type of an $n$-dimensional CW-complex. This is the technical heart of the paper. In §3.2 we note some further good properties of the poset of strata $\text{Pos}_{\text{Stab}_\text{alg}}(C)$; these are not used elsewhere in the paper.

Section 4 is dedicated to proving the main result, Theorem 4.9, that finite-type components are contractible. In §4.1 we show that the principal component $\text{Stab}_0(\Gamma_NQ)$ is of finite-type, hence contractible. Finally, in §5 we deduce that the stability space of a discrete derived category with finite global dimension, and any component of the stability space of a locally-finite triangulated category with finite rank Grothendieck group, is of finite-type, and therefore is contractible.

A warning about terminology is in order: ‘locally-finite’ is used in three different senses in this paper: as a property of stability conditions, as a property of stratifications, and as a property of triangulated categories. All three uses are standard in the literature. All stability conditions will be locally-finite, so no confusion is likely here. The other two uses are compatible; the local-finiteness of a triangulated category directly implies the local-finiteness of the stratification of its stability space.

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2 Preliminaries

2.1 Posets

Let $P$ be a poset. We denote the closed intervals by $[p, q] = \{ r \in P \mid p \leq r \leq q \}$,

$(-\infty, p] = \{ r \in P \mid r \leq p \}$, and $[p, \infty) = \{ r \in P \mid r \geq p \}$.

A chain in a poset $P$ of length $k$ is a sequence $p_0 < \cdots < p_k$ of elements. One says $q$ covers $p$ if $p < q$ and there does not exist $r \in P$ with $p < r < q$. A chain $p_0 < \cdots < p_k$ is said to be unrefinable if $p_i$ covers $p_{i-1}$ for each $i = 2, \ldots, k$. A maximal chain is an unrefinable chain in which $p_i$ is a minimal element and $p_k$ a maximal one. A poset is pure if all maximal chains have the same length; the common length is then called the length of the poset.

A poset determines a simplicial set whose $k$-simplices are the non-strict chains $p_0 \leq \cdots \leq p_k$ in $P$. The classifying space $B(P)$ of $P$ is the geometric realisation of this simplicial set. If we view $P$ as a category with objects the
elements and a (unique) morphism \( p \to q \) whenever \( p \leq q \), the above simplicial set is the *nerve*, and the \( B(P) \) is the classifying space of the category in the usual sense.

Elements \( p \) and \( q \) are said to be in the same *component* of \( P \) if there is a sequence of elements \( p = p_0, p_1, \ldots, p_k = q \) such that either \( p_i \leq p_{i+1} \) or \( p_i \geq p_{i+1} \) for each \( i = 0, \ldots, k-1 \); equivalently if the 0-simplices corresponding to \( p \) and \( q \) are in the same component of the classifying space \( B(P) \).

The classifying space is a rather crude invariant of \( P \). For example, there is a homeomorphism \( B(P) \cong B(P^\text{op}) \), and if each finite set of elements has an upper bound (or a lower bound) then the classifying space \( B(P) \) is contractible (since it is a CW-complex with vanishing higher homology, and hence homotopy, groups).

### 2.2 \( t \)-structures

We fix some notation. Let \( C \) be an additive category. We write \( c \in C \) to mean \( c \) is an object of \( C \). We will use the term *subcategory* to mean strict, full subcategory. When \( S \) is a subcategory we write \( S^\perp \) for the subcategory on the objects

\[ \{ c \in C : \text{Hom}_C(s, c) = 0 \ \forall s \in S \} \]

and similarly \( ^\perp S \) for \( \{ c \in C : \text{Hom}_C(c, s) = 0 \ \forall s \in S \} \). When \( A \) and \( B \) are subcategories of \( C \) we write \( A \cap B \) for the subcategory on objects which lie in both \( A \) and \( B \).

Suppose \( C \) is triangulated with shift functor \([1]\). Exact triangles in \( C \) will be denoted either by \( a \to b \to c \to a[1] \) or by a diagram

\[
\begin{array}{ccc}
\text{a} & \to & \text{b} \\
\downarrow & & \downarrow \\
\text{c} & \to & \text{a}[1]
\end{array}
\]

where the dotted arrow denotes a map \( c \to a[1] \). We will always assume that \( C \) is essentially small so that isomorphism classes of objects form a set. Given sets \( S_i \) of objects for \( i \in I \) let \( \langle S_i | i \in I \rangle \) denote the ext-closed subcategory generated by objects isomorphic to an element in some \( S_i \). We will use the same notation when the \( S_i \) are subcategories of \( C \).

**Definition 2.1.** A *\( t \)-structure* on a triangulated category \( C \) is an ordered pair \( D = (D_{\leq 0}, D_{\geq 1}) \) of subcategories, satisfying:

1. \( D_{\leq 0}[1] \subset D_{\leq 0} \) and \( D_{\geq 1}[-1] \subset D_{\geq 1} \);
2. \( \text{Hom}_C(d, d') = 0 \) whenever \( d \in D_{\leq 0} \) and \( d' \in D_{\geq 1} \);
3. for any \( c \in C \) there is an exact triangle \( d \to c \to d' \to d[1] \) with \( d \in D_{\leq 0} \) and \( d' \in D_{\geq 1} \).

We write \( D^n \) to denote the shift \( D_{\leq 0}[-n] \), and so on. The subcategory \( D_{\leq 0} \) is called the *aisle* and \( D_{\geq 0} \) the *co-aisle* of the \( t \)-structure. The intersection \( D^0 = D_{\geq 0} \cap D_{\leq 0} \) of the aisle and co-aisle is an abelian category \([\text{Théorème 1.3.6}]\) known as the *heart* of the \( t \)-structure.\(^1\)

\(^1\)The metaphors are mixed: perhaps ‘nave’ would be a more consistent term than heart.
The exact triangle \( d \to c \to d' \to d[1] \) is unique up to isomorphism. The first term determines a right adjoint to the inclusion \( D^{\leq 0} \hookrightarrow C \) and the last term a left adjoint to the inclusion \( D^{\geq 1} \hookrightarrow C \).

A \( t \)-structure \( D \) is bounded if any object of \( C \) lies in \( D^{\geq -n} \cap D^{\leq n} \) for some \( n \in \mathbb{N} \). In the sequel we will always assume that \( t \)-structures are bounded. This has two important consequences. Firstly, a bounded \( t \)-structure is completely determined by its heart; the \( t \)-structure is recovered as \( (D^0, D^0[1], D^0[2], \ldots) \).

Secondly, the inclusion \( D^0 \hookrightarrow C \) induces an isomorphism \( K(D^0) \cong K(C) \) of Grothendieck groups. Closely related to this is the easy but important fact that if \( D^0 \subset E^0 \) are hearts of bounded \( t \)-structures then \( D = E \).

**Definition 2.2.** Let \( T(C) \) be the poset of bounded \( t \)-structures on \( C \), ordered by inclusion of the aisles. Abusing notation we write \( D \subset E \) to mean \( D^{\leq 0} \subset E^{\leq 0} \).

(We need to assume \( C \) is essentially small for \( T(C) \) to be a poset— it is known \([31]\) that there is a proper class of \( t \)-structures even on the derived category \( D(\mathbb{Z}) \).)

There is a natural action of \( \mathbb{Z} \) on \( T(C) \) given by shifting: we write \( D[n] \) for the \( t \)-structure \( (D^{\leq -n}, D^{\geq -n+1}) \). Note that \( D[1] \subset D \), and not vice versa.

### 2.3 Torsion structures and tilting

The notion of torsion structure, also known as a torsion–torsion-free pair, is an abelian analogue of that of \( t \)-structure; the notions are related by the process of tilting.

**Definition 2.3.** A torsion structure on an abelian category \( A \) is an ordered pair \( T = (T^{\leq 0}, T^{\geq 1}) \) of subcategories satisfying

1. \( \text{Hom}_A(t, t') = 0 \) whenever \( t \in T^{\leq 0} \) and \( t' \in T^{\geq 1} \);
2. for any \( a \in A \) there is a short exact sequence \( 0 \to t \to a \to t' \to 0 \) with \( t \in T^{\leq 0} \) and \( t' \in T^{\geq 1} \).

The subcategory \( T^{\leq 0} \) is the torsion theory of \( T \), and \( T^{\geq 1} \) the torsion-free theory; the motivating example is the subcategories of torsion and torsion-free abelian groups.

The short exact sequence \( 0 \to t \to a \to t' \to 0 \) is unique up to isomorphism. The first term determines a right adjoint to the inclusion \( T^{\leq 0} \hookrightarrow A \) and the last term a left adjoint to the inclusion \( T^{\geq 1} \hookrightarrow A \). It follows that \( T^{\leq 0} \) is closed under factors, extensions and coproducts and that \( T^{\geq 1} \) is closed under subobjects, extensions and products. Torsion structures in \( A \) ordered by inclusion of their torsion theories form a poset. Note that it is a bounded poset with minimal element \((0, A)\) and maximal element \((A, 0)\).

**Proposition 2.4** ([21, Proposition 2.1], [3 Theorem 3.1]). Let \( D \) be a \( t \)-structure on a triangulated category \( C \). Then there is a canonical isomorphism between the poset of torsion structures in the heart \( D^0 \) and the interval \([D, D[-1]]_\subset \) in \( T(C) \) consisting of \( t \)-structures \( E \) with \( D \subset E \subset D[-1] \).
Let $D$ be a $t$–structure on a triangulated category $C$. It follows from Proposition 2.4 that a torsion structure $T$ in the heart $D^0$ determines a new $t$–structure $L_T D = ((D^{\leq 0}, T^{\leq 1}), (T^{\geq 2}, D^{\geq 2}))$ called the left tilt of $D$ at $T$. The heart of the left tilt is $L_T D^0 = (T^{\leq 1}, T^{\geq 2})$ and $D \subset L_T D \subset D[-1]$. The shifted $t$–structure $R_T D = L_T D[1]$ is called the right tilt of $D$ at $T$. It has heart $R_T D^0 = (T^{\leq 0}, T^{\geq 0})$ and $D[1] \subset R_T D \subset D$. The terminology is explained by the fact that left and right tilting are in verse to one another: $(T^{\geq 1}, T^{\leq 1})$ is a torsion structure on $L_T D^0$, and right tilting with respect to this we recover the original $t$-structure. Similarly, $(T^{\geq 0}, T^{\leq 0})$ is a torsion structure on $R_T D^0$, and left tilting with respect to this we return to $D$. Since there is a correspondence between bounded $t$–structures and their hearts we will, where convenient, speak of the left or right tilt of a heart.

Definition 2.5. Let the tilting poset $\text{Tilt}(C)$ be the poset of $t$–structures with $D \leq E$ if and only if there is a finite sequence of left tilts from $D$ to $E$.

Remark 2.6. If $D \leq E$ then $D \subset E \subset D[-k]$ for some $k \in \mathbb{N}$ — the first inequality is obvious, the second follows from the fact (proved by an easy induction) that $E$ is obtained from $D$ by $k$ left tilts then $E \subset D[-k]$.

It follows that the identity on elements is a map of posets $\text{Tilt}(C) \to T(C)$. We saw above that if $D \subset E \subset D[-1]$ then $D \leq E \iff D \subset E$, so that the map induces an isomorphism $[D, D[-1]] \leq \cong [D, D[-1]]$. 

Lemma 2.7. Suppose $D$ and $E$ are in the same component of $\text{Tilt}(C)$. Then $F \leq D, E \leq G$ for some $F, G$ in that component. (We do not claim that $F$ and $G$ are the infimum and supremum, simply that lower and upper bounds exist.)

Proof. Note that if $D$ and $E$ are left tilts of some $t$–structure $H$ then they are right tilts of $H[-1]$, and vice versa. It follows that we can replace an arbitrary sequence of left and right tilts connecting $D$ with $E$ by a sequence of left tilts followed by a sequence of right tilts, or vice versa. The result follows.

2.4 Algebraic $t$-structures

We say an abelian category is algebraic if it is a length category with finitely many (isomorphism classes of) simple objects. To spell this out, this means it is both artinian and noetherian so that every object has a finite composition series. By the Jordan-Hölder theorem, the graded object associated to such a composition series is unique up to isomorphism. The simple objects form a basis for the Grothendieck group, which is isomorphic to $\mathbb{Z}^n$, where $n$ is the number of simple objects. A $t$–structure $D$ is algebraic if its heart $D^0$ is. If $C$ admits an algebraic $t$–structure then the heart of any other $t$-structure on $C$ which is a length category must also have exactly $n$ simple objects, since the two hearts have isomorphic Grothendieck groups.

Let the algebraic tilting poset $\text{Tilt}_{\text{alg}}(C)$ be the poset consisting of the algebraic $t$–structures, with $D \preceq E$ when $E$ is obtained from $D$ by a finite sequence of left tilts, via algebraic $t$–structures. Clearly

$$D \preceq E \Rightarrow D \leq E \Rightarrow D \subset E,$$

and there is an injective map of posets $\text{Tilt}_{\text{alg}}(C) \to \text{Tilt}(C)$. 

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Remark 2.8. There is an alternative algebraic description of \(\text{Tilt}_{\text{alg}}(C)\) when \(C = D(A)\) is the bounded derived category of a finite-dimensional algebra \(A\), of finite global dimension, over an algebraically-closed field. By [17] Lemma 4.1 the poset \(\mathcal{P}_t(C)\) of silting subcategories in \(C\) is the sub-poset of \(T(C)^{op}\) consisting of the algebraic \(t\)-structures, and under this identification silting mutation in \(\mathcal{P}_t(C)\) corresponds to (admissible) tilting in \(T(C)^{op}\). Moreover, it follows from [11] §2.6 that the partial order in \(\mathcal{P}_t(C)\) is generated by silting mutation, so that \(D \in E \iff D \ll E\) for algebraic \(D\) and \(E\). Hence \(\text{Tilt}_{\text{alg}}(C) \cong \mathcal{P}_t(C)^{op}\).

If \(A\) does not have finite global dimension, then a similar result holds but we must replace the poset of silting subcategories in \(C\), with the analogous poset in the bounded homotopy category of finitely-generated projective modules.

We collect some elementary facts about algebraic \(t\)-structures.

Lemma 2.9. Suppose \(D\) and \(E\) are \(t\)-structures and that \(E\) is algebraic. Then \(E \subset D[-d]\) for some \(d \in \mathbb{N}\).

Proof. Since \(D\) is bounded each simple object \(s\) of the heart \(E^0\) is in \(D^{\leq k_s}\) for some \(k_s \in \mathbb{Z}\). Then \(E^0 \subset D^{\leq d}\) for \(d = \max_s \{k_s\}\) — the minimum exists since there are finitely many simple objects in \(E^0\) — and this implies \(D \subset D[-d]\).

Remark 2.10. It follows that \(BT(C)\) is contractible, in particular is connected, whenever \(C\) admits an algebraic \(t\)-structure.

Lemma 2.11. Suppose \(D\) and \(E\) are in the same component of \(\text{Tilt}_{\text{alg}}(C)\). Then \(F \ll D, E \ll G\) for some \(F, G\) in that component.

Proof. This proved in exactly the same way as Lemma 2.7 we need only note that all the \(t\)-structures encountered in the construction are algebraic.

It is known that the poset \(T(C)\) of \(t\)-structures is not a lattice in general — see [7] for a detailed discussion and counterexample, and also [15] — and we do not claim that the lower and upper bounds of the previous lemma are infima or suprema. We do however have the following weaker result.

Lemma 2.12. Suppose \(D\) is algebraic (in fact it suffices for it to be a length category). Then for each \(D \subset E, F \subset D[-1]\) there is a supremum \(E \lor F\) and an infimum \(E \land F\) in \(T(C)\).

Proof. We construct only the supremum \(E \lor F\), the infimum is constructed similarly. We claim that \((E^{\leq 0}, F^{\leq 0})\) is the aisle of a bounded \(t\)-structure; it is clear that this \(t\)-structure must then be the supremum in \(T(C)\).

Since \(D \subset E, F \subset D[-1]\) we may work with the corresponding torsion structures \(T_E\) and \(T_F\) on \(D^0\), and show that \(T^{\leq 0} = (T^{\leq 0}_E, T^{\leq 0}_F)\) is a torsion theory, with associated torsion-free theory \(T^{\geq 1} = T^{>0}_E \cap T^{>0}_F\). Certainly \(\text{Hom}_C(t, t') = 0\) whenever \(t \in T^{\leq 0}\) and \(t' \in T^{\geq 1}\), so it suffices to show that any \(d \in D^0\) sits in a short exact sequence \(0 \to t \to d \to t' \to 0\) with \(t \in T^{\leq 0}\) and \(t' \in T^{\geq 1}\). We do this in stages, beginning with the short exact sequence

\[
0 \to e_0 \to d \to e'_0 \to 0
\]

with \(e_0 \in T^{0}_E\) and \(e'_0 \in T^{>1}_E\). Combining this with the short exact sequence

\[
0 \to f_0 \to e'_0 \to f'_0 \to 0
\]

with \(f_0 \in T^{0}_F\) and \(f'_0 \in T^{>1}_F\) we obtain a second short exact sequence

\[
0 \to t \to d \to f'_0 \to 0
\]
where \( t \) is an extension of \( e_0 \) and \( f_0 \), and hence is in \( T^{\leq 0} \). Repeat this process, at each stage using the expression of the third term as an extension via alternately the torsion structure \( T_E \) and \( T_F \). This yields successive short exact sequences, each with middle term \( d \) and first term in \( T^{\leq 0} \), and such that the third term is a quotient of the third term of the previous sequence. Since \( D^0 \) is a length category this process must stabilise. It does so when the third term has no subobject in either \( T^{\leq 0}_E \) or \( T^{\leq 0}_F \), i.e. when the third term is in \( T^{2\geq 1}_E \cap T^{2\geq 1}_F = T^{2\geq 1} \). This exhibits the required short exact sequence and completes the proof.

In general, this cannot be used inductively to show that the components of \( \text{Tilt}_{\text{alg}}(C) \) are lattices, since we do not know that \( E \wedge F \) and \( E \vee F \) are algebraic. For the remainder of this section we impose an assumption that guarantees that they are: let \( \text{Tilt}^0(C) = \text{Tilt}^0_{\text{alg}}(C) \) be a component of the tilting poset consisting entirely of algebraic \( t \)-structures, equivalently a component of \( \text{Tilt}_{\text{alg}}(C) \) closed under all tilts.

**Lemma 2.13.** The component \( \text{Tilt}^0(C) \) is a lattice. Infima and suprema in \( \text{Tilt}^0(C) \) are also infima and suprema in \( T(C) \).

**Proof.** Suppose \( E, F \in \text{Tilt}^0(C) \). As in Lemma 2.7 we can replace an arbitrary sequence of left and right tilts connecting \( E \) with \( F \) by one consisting of a sequence of left tilts followed by a sequence of right tilts, or vice versa, but now using the infima and suprema of Lemma 2.12 at each stage of the process. We can do this since \( \text{Tilt}^0(C) \) consists entirely of algebraic \( t \)-structures, and therefore these infima and suprema are algebraic. Thus \( E \) and \( F \) have upper and lower bounds in \( \text{Tilt}^0(C) \).

We now construct the infimum and supremum. First, convert the sequence of tilts from \( E \) to \( F \) into one of right followed by left tilts by the above process. Then if \( E, F \subset G \) the same is true for each \( t \)-structure along the new sequence. Now convert this new sequence to one of left tilts followed by right tilts, again by the above process. Inductively applying Lemma 2.12 shows that each \( t \)-structure in the resulting sequence is still bounded above in \( T(C) \) by \( G \). In particular the \( t \)-structure \( H \) reached after the final left tilt, and before the first right tilt, satisfies \( E, F \preceq H \subset G \). It follows that \( H \in \text{Tilt}^0(C) \) is the supremum \( E \vee F \) of \( E \) and \( F \) in \( T(C) \).

To complete the proof we need to show that \( E \vee F \preceq G \) whenever \( G \in \text{Tilt}^0(C) \) and \( E, F \preceq G \). This follows since \( E \vee F \preceq (E \vee F) \vee G = G \).

The argument for the infimum is similar.

**Lemma 2.14.** The following are equivalent:

1. Intervals of the form \( [D, D[-1]]_{\preceq} \) in \( \text{Tilt}^0(C) \) are finite;

2. All closed bounded intervals in \( \text{Tilt}^0(C) \) are finite.

**Proof.** Assume that intervals of the form \( [D, D[-1]]_{\preceq} \) in \( \text{Tilt}_{\text{alg}}(C) \) are finite. Given \( D \preceq E \) in \( \text{Tilt}^0(C) \) note that \( E \subset D[-d] \) for some \( d \in \mathbb{N} \) by Lemma 2.8 so that

\[
D \preceq E \preceq E \vee D[-d] = D[-d].
\]

Hence it suffices to show that intervals of the form \( [D, D[-d]]_{\preceq} \) are finite. We prove this by induction on \( d \). The case \( d = 1 \) is true by assumption. Suppose it
is true for $d < k$. In diagrams it will be convenient to use the notation $E \to F$ to mean $F$ is a left tilt of $E$.

By definition of $\text{Tilt}_{\text{alg}}(C)$ any element of the interval $[D, D[-k]]_{\simeq}$ sits in a chain of tilts $D = D_0 \to D_1 \to \cdots \to D_r = D[-k]$ via algebraic $t$–structures. This can be extended to a diagram

$$
D = D_0 \to D_1 \to D_2 \to \cdots \to D_{r-1} \to D_r = D[-k]
$$

of algebraic $t$–structures and tilts, where $D'_1 = D[-1]$, so that $D_1 \to D'_1$ as shown, and $D'_i = D_i \lor D'_{i-1}$ is constructed inductively. The only point that requires elaboration is the existence of the tilt $D'_{r-1} \to D_r$. First note that $D'_1, D_2 \preceq D_r$ so that $D'_2 = D_2 \lor D'_1 \preceq D_r$ too. By induction $D'_{r-1} \preceq D_r$. Since $D_r[1] \preceq D_{r-1} \preceq D'_{r-1} \preceq D_r$

$D_r$ is a left tilt of $D'_{r-1}$ by Proposition 2.4.

The existence of the above diagram shows that each element of the interval $[D, D[-k]]_{\simeq}$ is a right tilt of some element of the interval $[D[-1], D[-k]]_{\simeq}$. By induction the latter has only finitely many elements, and by assumption each of these has only finitely many right tilts. This establishes the first implication. The converse is obvious.

2.5 Simple tilts

Suppose $D$ is an algebraic $t$–structure. Then each simple object $s \in D^0$ determines two torsion structures on the heart, namely $(\langle s \rangle, \langle s \rangle^{\bot})$ and $(\langle s \rangle^{\bot}, \langle s \rangle)$. These are respectively minimal and maximal non-trivial torsion structures in $D^0$. We say the left tilt at the former, and the right tilt at the latter, are simple. We use the abbreviated notation $L_s D$ and $R_s D$ respectively for these tilts.

More generally we have the following notions. A torsion structure $T$ is hereditary if $t \in T^{\leq 0}$ implies all subobjects of $t$ are in $T^{\leq 0}$. It is co-hereditary if $t \in T^{\geq 1}$ implies all quotients of $t$ are in $T^{\geq 1}$. When $T$ is a torsion structure on an algebraic abelian category then the hereditary torsion structures are those of the form $(S, S^{\bot})$ where the torsion theory $S = \langle s_1, \ldots, s_k \rangle$ is generated by a subset of the simple objects. Dually, the co-hereditary torsion structures are those of the form $(\langle s \rangle^{\bot}, S)$. We use the abbreviated notation $L_S D$ and $R_S D$ respectively for the left tilt at $(S, S^{\bot})$ and $R_S D$ for the right tilt at $(\langle s \rangle^{\bot}, S)$. Note that, in the notation of the previous section, $L_S D \wedge L_S D = L_{S^{\bot} S} D$ and $L_S D \vee L_S D = L_{S \cup S^{\bot}} D$.

In general a tilt, even a simple tilt, of an algebraic $t$–structure need not be algebraic. However, if the heart is rigid, i.e. the simple objects have no self-extensions, or has only finitely many isomorphism classes of indecomposable objects then [26, Proposition 5.2] and Lemma 4.2 respectively show that the tilted $t$–structure is also algebraic.
2.6 Stability conditions

Let $C$ be a triangulated category and $K(C)$ be its Grothendieck group. A stability condition $(Z, P)$ on $C$ [10, Definition 1.1] consists of a group homomorphism $Z : K(C) \to \mathbb{C}$ and full additive subcategories $P(\varphi)$ of $C$ for each $\varphi \in \mathbb{R}$ satisfying

1. if $c \in P(\varphi)$ then $Z(c) = m(c) \exp(i\pi\varphi)$ where $m(c) \in \mathbb{R}_{>0}$;
2. $P(\varphi + 1) = P(\varphi)[1]$ for each $\varphi \in \mathbb{R}$;
3. if $c \in P(\varphi)$ and $c' \in P(\varphi')$ with $\varphi > \varphi'$ then $\text{Hom}(c, c') = 0$;
4. for each nonzero object $c \in C$ there is a finite collection of triangles $0 = c_0 \to c_1 \to \cdots \to c_n = c$ with $b_j \in P(\varphi_j)$ where $\varphi_1 > \cdots > \varphi_n$.

The homomorphism $Z$ is known as the central charge and the objects of $P(\varphi)$ are said to be semi-stable of phase $\varphi$. The objects $b_j$ are known as the semi-stable factors of $c$.

We define $\varphi^+(c) = \varphi_1$ and $\varphi^-(c) = \varphi_n$. The mass of $c$ is defined to be $m(c) = \sum_{i=1}^n m(b_i)$.

For an interval $(a, b) \subset \mathbb{R}$ we set $P(a, b) = \{ c \in C \mid \varphi(c) \in (a, b) \}$, and similarly for half-open or closed intervals. Each stability condition $\sigma$ has an associated bounded $t$-structure $D_\sigma = (P(0, \infty), P(-\infty, 0])$ with heart $D_\sigma^0 = P(0, 1]$. Conversely, if we are given a bounded $t$-structure on $C$ together with a stability function on the heart with the Harder–Narasimhan property — the abelian analogue of property 4 above — then this determines a stability condition on $C$ [10, Proposition 5.3].

A stability condition is locally-finite if we can find $\varepsilon > 0$ such that the quasi-abelian category $P(\varphi - \varepsilon, \varphi + \varepsilon)$, generated by semi-stable objects with phases in $(\varphi - \varepsilon, \varphi + \varepsilon)$, has finite length (see [10, Definition 5.7]). The set of locally-finite stability conditions can be topologised so that it is a, possibly infinite-dimensional, complex manifold, which we denote $\text{Stab}(C)$ [10, Theorem 1.2]. The topology arises from the (generalised) metric

$$d(\sigma, \tau) = \sup_{0 \neq c \in C} \max \left( |\varphi^+_{\sigma}(c) - \varphi^+_{\tau}(c)|, |\varphi^-_{\sigma}(c) - \varphi^-_{\tau}(c)|, \left| \log \frac{m_{\sigma}(c)}{m_{\tau}(c)} \right| \right)$$

which takes values in $[0, \infty]$. It follows that for fixed $0 \neq c \in C$ the mass $m_{\sigma}(c)$, and lower and upper phases $\varphi^-_{\sigma}(c)$ and $\varphi^+_{\sigma}(c)$ are continuous functions $\text{Stab}(C) \to \mathbb{R}$. The projection

$$\pi : \text{Stab}(C) \to \text{Hom}(K(C, C) : (Z, P) \mapsto Z$$

is a local homeomorphism.

The group $\text{Aut}(C)$ of automorphisms acts continuously on the space $\text{Stab}(C)$ of stability conditions with an automorphism $\alpha$ acting by

$$(Z, P) \mapsto (Z \circ \alpha^{-1}, \alpha(P)).$$
There is also a smooth right action of the universal cover $G$ of $GL_2^+ \mathbb{R}$. An element $g \in G$ corresponds to a pair $(T_g, \theta_g)$ where $T_g$ is the projection of $g$ to $GL_2^+ \mathbb{R}$ under the covering map and $\theta_g : \mathbb{R} \to \mathbb{R}$ is an increasing map with $\theta_g(t + 1) = \theta_g(t) + 1$ which induces the same map as $T_g$ on the circle $\mathbb{R}/2\mathbb{Z} = \mathbb{R}^2 - \{0\}/\mathbb{R} > 0$. In these terms the action is given by

$$(Z, P) \mapsto (T_g^{-1} \circ Z, P \circ \theta_g).$$

(Here we think of the central charge as valued in $\mathbb{R}^2$.) This action preserves the semistable objects, and also preserves the Harder–Narasimhan filtrations of all objects. The subgroup consisting of pairs for which $T$ is conformal is isomorphic to $\mathbb{C}$ with $\lambda \in \mathbb{C}$ acting via

$$(Z, P) \mapsto (\exp(-i\pi \lambda)Z, P(\varphi + \text{Re} \lambda))$$

i.e. by rotating the phases and rescaling the masses of semistable objects. This action is free and preserves the metric. The action of $1 \in \mathbb{C}$ corresponds to the action of the shift automorphism $[1]$.

**Lemma 2.15.** For any $g \in G$ the $t$–structures $D_{g \cdot \sigma}$ and $D_{\sigma}$ are related by a finite sequence of tilts.

**Proof.** Since $G$ is connected $\sigma$ and $g \cdot \sigma$ are in the same component of $\text{Stab}(\mathbb{C})$. Hence by [36 Corollary 5.2] the $t$–structures $D_{\sigma}$ and $D_{\tau}$ are related by a finite sequence of tilts.

2.7 Cellular stratified spaces

A CW-cellular stratified space, in the sense of [19], is a generalisation of a CW-complex in which non-compact cells are permitted. In §3 we will show that (parts of) stability spaces have this structure, and use it to show their contractibility. Here, we recall the definitions and result we will require.

A *k-cell structure* on a subspace $e$ of a topological space $X$ is a continuous map $\alpha : D \to X$ where $\text{int}(D^k) \subset D \subset D^k$ is a subset of the $k$-dimensional disk containing the interior, such that $\alpha(D) = \overline{e}$, the restriction of $\alpha$ to $\text{int}(D^k)$ is a homeomorphism onto $e$, and $\alpha$ does not extend to a map with these properties defined on any larger subset of $D^k$. We refer to $e$ as a *cell* and to $\alpha$ as a *characteristic map* for $e$.

**Definition 2.16.** A *cellular stratification* of a topological space $X$ consists of a filtration

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_k \subset \cdots$$

by subspaces, with $X = \bigcup_{k \in \mathbb{N}} X_k$, such that $X_k - X_{k-1} = \bigcup_{\lambda \in \Lambda_k} e_\lambda$ is a disjoint union of $k$-cells for each $k \in \mathbb{N}$. A *CW-cellular stratification* is a cellular stratification satisfying the further conditions that

1. the stratification is closure-finite: the boundary $\partial e = \overline{e} - e$ of any $k$-cell is contained in a union of finitely many lower-dimensional cells;

2. $X$ has the weak topology determined by the closures $\overline{e}$ of the cells in the stratification: a subset $A$ of $X$ is closed if, and only if, its intersection with each $\overline{e}$ is closed.
When the domain of each characteristic map is the entire disk then a CW-cellular stratification is nothing but a CW-complex structure on \( X \). Although the collection of cells and characteristic maps is part of the data of a cellular stratified space we will suppress it from our notation for ease-of-reading — since we never consider more than one stratification of any given topological space there is no possibility for confusion.

A cellular stratification is said to be **regular** if each characteristic map is a homeomorphism, and **normal** if the boundary of each cell is a union of lower-dimensional cells. Note that a regular, normal cellular stratification induces cellular stratifications on the domain of the characteristic map of each of its cells. Finally, we say a CW-cellular stratification is **regular and totally-normal** if it is regular, normal, and in addition for each cell \( e_\lambda \) with characteristic map \( \alpha_\lambda : D_\lambda \to X \) the induced cellular stratification of \( \partial D_\lambda = D_\lambda - \text{int}(D^k) \) extends to a regular CW-complex structure on \( \partial D^k \). (The definition of totally-normal CW-cellular stratification in [19] is more subtle, as it handles the non-regular case too, but it reduces to the above for regular stratifications. A regular CW-complex is totally-normal, but regularity alone does not even entail normality for a CW-cellular stratified space.) Note that any union of strata in a regular, totally-normal CW-cellular stratified space is itself a regular, totally-normal CW-cellular stratified space.

A normal cellular stratified space \( X \) has a poset of strata (or face poset) \( P(X) \) whose underlying set is the set of cells, and where \( e_\lambda \leq e_\mu \iff e_\lambda \subset e_\mu \). When \( X \) is a regular CW-complex there is a homeomorphism from the classifying space \( BP(X) \) to \( X \). More generally,

**Theorem 2.17** ([19, Theorem 2.50]). Suppose \( X \) is a regular, totally-normal CW-cellular stratified space. Then \( BP(X) \) embeds in \( X \) as a strong deformation retract, in particular there is a homotopy equivalence \( X \simeq BP(X) \).

### 3 Algebraic stability conditions

We say a stability condition \( \sigma \) is **algebraic** if the corresponding \( t \)-structure \( D_\sigma \) is algebraic. Let \( \text{Stab}_{\text{alg}}(C) \subset \text{Stab}(C) \) be the subspace of algebraic stability conditions.

Write \( S_D = \{ \sigma \in \text{Stab}(C) : D_\sigma = D \} \) for the set of stability conditions with associated \( t \)-structure \( D \). Recall from [12, Lemma 5.2] that when \( D \) is algebraic a stability condition in \( S_D \) is uniquely determined by a choice for each simple object in the heart of a central charge in

\[
\{ r \exp(i \pi \theta) \in \mathbb{C} : r > 0 \text{ and } \theta \in (0, 1] \} = \mathbb{H} \cup \mathbb{R}_{<0} \tag{1}
\]

where \( \mathbb{H} \) is the strict upper half-plane. Hence, in this case, an ordering of the simple objects determines an isomorphism \( S_D \cong (\mathbb{H} \cup \mathbb{R}_{<0})^n \). In particular, if \( C \) has an algebraic \( t \)-structure then \( \text{Stab}_{\text{alg}}(C) \neq \emptyset \).

The action of \( \text{Aut}(C) \) restricts to an action on the subspace \( \text{Stab}_{\text{alg}}(C) \). In contrast \( \text{Stab}_{\text{alg}}(C) \) need not be preserved by the action of \( C \) on \( \text{Stab}(C) \). The action of \( i \mathbb{R} \subset \mathbb{C} \) uniformly rescales the masses of semistable objects; this does not change the associated \( t \)-structure and so preserves \( \text{Stab}_{\text{alg}}(C) \). However, \( \mathbb{R} \subset \mathbb{C} \) acts by rotating the phases of semistables. This alters the \( t \)-structure by a finite sequence of tilts, and can result in a non-algebraic \( t \)-structure. In
fact, the union of orbits $\mathbb{C} \cdot \text{Stab}_{\text{alg}}(\mathbb{C})$ consists of those stability conditions $\sigma$ for which $(P_\sigma(\theta, \infty), P_\sigma(-\infty, \theta])$ is an algebraic $t$-structure for some $\theta \in \mathbb{R}$. The choice of $\theta = 0$ for the associated $t$-structure is purely conventional. If we define

$$\text{Stab}_\theta^{\text{alg}}(\mathbb{C}) = \{\sigma \in \text{Stab}(\mathbb{C}) \mid (P_\sigma(\theta, \infty), P_\sigma(-\infty, \theta]) \text{ is algebraic}\}$$

then there is a commutative diagram

$$\begin{array}{ccc}
\text{Stab}_{\text{alg}}(\mathbb{C}) & \longrightarrow & \text{Stab}(\mathbb{C}) \\
\downarrow & & \downarrow_{\sigma \mapsto \theta \cdot \sigma} \\
\text{Stab}_\theta^{\text{alg}}(\mathbb{C}) & \longrightarrow & \text{Stab}(\mathbb{C})
\end{array}$$

in which the vertical maps are homeomorphisms. So $\text{Stab}_\theta^{\text{alg}}(\mathbb{C})$ is independent up to homeomorphism of the choice of $\theta \in \mathbb{R}$, but the way in which it is embedded in $\text{Stab}(\mathbb{C})$ is not.

**Lemma 3.1.** The space of algebraic stability conditions is contained in the union of full components of $\text{Stab}(\mathbb{C})$, i.e. those components locally homeomorphic to $\text{Hom}(\mathbb{K}C, \mathbb{C})$. A stability condition $\sigma$ in a full component of $\text{Stab}(\mathbb{C})$ is algebraic if and only if $P_\sigma(0, \epsilon) = \emptyset$ for some $\epsilon > 0$.

**Proof.** The assumption that $\text{Stab}_{\text{alg}}(\mathbb{C}) \neq \emptyset$ implies that $\mathbb{K}C \cong \mathbb{Z}^n$ for some $n \in \mathbb{N}$, since it is isomorphic to the Grothendieck group of an algebraic abelian category. It follows from the description above that any component containing an algebraic stability condition is full.

Suppose $D$ is algebraic. Then for any $\sigma \in S_D$ the simple objects are semistable. Since there are finitely many simple objects there is one, $s$ say, with minimal phase $\varphi^-_\sigma(s) = \epsilon > 0$. It follows that $P_\sigma(0, \epsilon) = \emptyset$.

Conversely, suppose $P_\sigma(0, \epsilon) = \emptyset$ for some stability condition $\sigma$ in a full component. Then the heart $P_\sigma(0, 1) = P_\sigma(\epsilon, 1]$. Since $1 - \epsilon < 1$ we can apply [11, Lemma 4.5] to deduce that the heart of $\sigma$ is an abelian length category. It follows that the heart has $n$ simple objects (forming a basis of $\mathbb{K}C$), and hence is algebraic.

**Lemma 3.2.** The interior of $S_D$ is non-empty precisely when $D$ is algebraic.

**Proof.** The explicit description of $S_D$ for algebraic $D$ above shows that the interior is non-empty in this case. Conversely, suppose $D$ is not algebraic and $\sigma \in S_D$. Then by Lemma 3.1 there are $\sigma$-semistable objects of arbitrarily small phase $\varphi > 0$. It follows that the $\mathbb{C}$ orbit through $\sigma$ contains a sequence of stability conditions not in $S_D$ with limit $\sigma$. Hence $\sigma$ is not in the interior of $S_D$. Since $\sigma$ was arbitrary the latter must be empty.

**Corollary 3.3.** The subset $\mathbb{C} \cdot \text{Stab}_{\text{alg}}(\mathbb{C}) \subset \text{Stab}(\mathbb{C})$ is open, and consists of those stability conditions in full components of $\text{Stab}(\mathbb{C})$ for which the phases of semistable objects are not dense in $\mathbb{R}$.

**Proof.** A stability condition $\sigma \in \mathbb{C} \cdot \text{Stab}_{\text{alg}}(\mathbb{C})$ clearly lies in a component of $\text{Stab}(\mathbb{C})$ meeting $\text{Stab}_{\text{alg}}(\mathbb{C})$, and hence in a full component. By Lemma 3.1 if $\sigma$ is in a full component then $\sigma \in \mathbb{C} \cdot \text{Stab}_{\text{alg}}(\mathbb{C})$ if and only if $P_\sigma(t, t + \epsilon) = \emptyset$ for some $t \in \mathbb{R}$ and $\epsilon > 0$, equivalently if and only if the phases of semistable objects are not dense in $\mathbb{R}$.
To see that $\mathbb{C} \cdot \text{Stab}_{\text{alg}}(C)$ is open note that if $\sigma \in \mathbb{C} \cdot \text{Stab}_{\text{alg}}(C)$ and $d(\sigma, \tau) < \epsilon/4$ then $P_\sigma(t + \epsilon/4, t + 3\epsilon/4) = \emptyset$ and so $\tau \in \mathbb{C} \cdot \text{Stab}_{\text{alg}}(C)$ too. \hfill \Box

**Example 3.4.** Let $X$ be a smooth complex projective algebraic curve with genus $g(X) > 0$. Then the space $\text{Stab}(X)$ of stability conditions on the bounded derived category of coherent sheaves on $X$ is a single orbit of the $G$ action (see [14]), through the stability condition with associated heart the coherent sheaves, and central charge $Z(\mathcal{E}) = -\deg \mathcal{E} + i \text{rk} \mathcal{E}$, see [10] Theorem 9.1 for $g(X) = 1$ and [28] Theorem 2.7 for $g(X) > 1$. It follows from the fact that there are semistable sheaves of any rational slope when $g(X) > 0$ that the phases of semistable objects are dense for every stability condition in $\text{Stab}(X)$. Hence $\text{Stab}_{\text{alg}}(D(X)) = \emptyset$.

By [18], [3.5] we know that for some higher dimensional varieties, e.g. for $X = \mathbb{P}^1 \times \mathbb{P}^1$, or $\mathbb{P}^n$ blown up in finitely many points where $n \geq 2$, there exist stability conditions for which the phases of semistable objects are dense in some non-empty open interval of $\mathbb{R}$. In these cases $\text{Stab}_{\text{alg}}(X) \neq \text{Stab}(X)$; we conjecture that this is always the case for smooth projective varieties.

**Example 3.5.** Let $Q$ be a finite connected quiver, and $\text{Stab}(Q)$ the space of stability conditions on the bounded derived category of its finite-dimensional representations over an algebraically-closed field. When $Q$ has underlying graph of ADE Dynkin type the phases of semistable objects form a discrete set. [13] Lemma 3.13]; when it has extended ADE Dynkin type the phases either form a discrete set or have accumulation points $t + \mathbb{Z}$ for some $t \in \mathbb{R}$ (all cases occur) [18] Corollary 3.15]; for any other acyclic $Q$ there exists a family of stability conditions for which the phases are dense in some non-empty open interval [18] Proposition 3.32]; and for $Q$ with oriented loops there exist stability conditions for which the phases of semistable objects are dense in $\mathbb{R}$ by [18] Remark 3.33. It follows that $\text{Stab}_{\text{alg}}(Q) = \text{Stab}(Q)$ only in the Dynkin case; that $\mathbb{C} \cdot \text{Stab}_{\text{alg}}(Q) = \text{Stab}(Q)$ in the Dynkin or extended Dynkin cases; and that $\mathbb{C} \cdot \text{Stab}_{\text{alg}}(Q) \neq \text{Stab}(Q)$ when $Q$ has oriented loops. For a general acyclic quiver, we do not know whether $\mathbb{C} \cdot \text{Stab}_{\text{alg}}(Q) = \text{Stab}(Q)$ or not. It does in the particular case of the Kronecker quiver — see Example 3.8 below.

**Remark 3.6.** The density of the phases of semistable objects for a stability condition is an important consideration in other contexts too. Proposition 4.1 of [38] states that if phases for $\sigma$ are dense in $\mathbb{R}$ then the orbit of the universal cover $G$ of $GL_2^+ \mathbb{R}$ through $\sigma$ is free, and the induced metric on the quotient $G \cdot \sigma/\mathbb{C} \cong G/\mathbb{C} \cong H$ of the orbit is half the standard hyperbolic metric.

**Lemma 3.7.** Suppose there exists a uniform lower bound on the maximal phase gap of algebraic stability conditions, i.e. there exists $\delta > 0$ such that for each $\sigma \in \text{Stab}_{\text{alg}}(C)$ there exists $\varphi \in \mathbb{R}$ with $P_\sigma(\varphi - \delta, \varphi + \delta) = \emptyset$. Then $\mathbb{C} \cdot \text{Stab}_{\text{alg}}(C)$ is closed, and hence is a union of components of $\text{Stab}(C)$.

*Proof.* Suppose $\sigma \in \mathbb{C} \cdot \text{Stab}_{\text{alg}}(C) \setminus \mathbb{C} \cdot \text{Stab}_{\text{alg}}(C)$. Let $\sigma_n \to \sigma$ be a sequence in $\mathbb{C} \cdot \text{Stab}_{\text{alg}}(C)$ with limit $\sigma$. Write $\varphi_n^+ \equiv \varphi_n^-$, and so on.

Fix $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that $d(\sigma_n, \sigma) < \epsilon$ for $n \geq N$. By Corollary 3.3 the phases of semistable objects for $\sigma$ are dense in $\mathbb{R}$. Thus, given $\varphi \in \mathbb{R}$, we can find $\theta$ with $|\theta - \varphi| < \epsilon$ such that $P_\sigma(\theta) \neq \emptyset$. So there exists $0 \neq c \in \mathbb{C}$ such that $\varphi^+_n(c) \to \theta$. Hence $c \in P_N(\theta - \epsilon, \theta + \epsilon) \subseteq P_N(\varphi - 2\epsilon, \varphi + 2\epsilon)$. In
particular the latter is non-empty. Since $\varphi$ is arbitrary we obtain a contradiction by choosing $\epsilon < \delta/2$. Hence $C \cdot \text{Stab}_{\text{alg}}(C)$ is closed.

**Example 3.8.** Let $\text{Stab}(\mathbb{P}^1)$ be the space of stability conditions on the bounded derived category $D(\mathbb{P}^1)$ of coherent sheaves on $\mathbb{P}^1$. Theorem 1.1 of [29] identifies $\text{Stab}(\mathbb{P}^1) \cong \mathbb{C}^2$. In particular there is a unique component, and it is full. The category $D(\mathbb{P}^1)$ is equivalent to the bounded derived category $D(K)$ of finite-dimensional representations of the Kronecker quiver $K$. In particular, $\text{Stab}_{\text{alg}}(\mathbb{P}^1)$ is non-empty. The Kronecker quiver has extended ADE Dynkin type so by Example 3.5 the phases of semistable objects for any $\sigma \in \text{Stab}(\mathbb{P}^1)$ are either discrete or accumulate at the points $t + \mathbb{Z}$ for some $t \in \mathbb{R}$. The subspace $\text{Stab}(\mathbb{P}^1) \cap \text{Stab}_{\text{alg}}(\mathbb{P}^1)$ consists of those with phases accumulating at $\mathbb{Z} \subset \mathbb{R}$. Therefore $C \cdot \text{Stab}_{\text{alg}}(\mathbb{P}^1) = \text{Stab}(\mathbb{P}^1)$ and $\text{Stab}_{\text{alg}}(\mathbb{P}^1)$ is not closed. Neither is it open [35, p20]: there are stability conditions for which each semistable object has phase in $\mathbb{Z}$ which are the limit of stability conditions with phases accumulating at $\mathbb{Z}$.

An explicit analysis of the semistable objects for each stability condition, as in [29], reveals that there is no lower bound on the maximum phase gap of algebraic stability conditions, so that whilst this condition is sufficient to ensure $C \cdot \text{Stab}_{\text{alg}}(C) = \text{Stab}(C)$ it is not necessary.

### 3.1 The stratification of algebraic stability conditions

In this section we define and study a natural stratification of $\text{Stab}_{\text{alg}}(C)$ with contractible strata. Suppose $D$ is an algebraic $t$-structure on $C$, so that $S_D = (\mathbb{H} \cup \mathbb{R}_{<0})^n$ where $n = \text{rk}(KC)$. For a subset $I$ of the simple objects in the heart of $D$ we define a subset of $\text{Stab}(C)$

$$S_{D,I} = \{ \sigma \mid D = D_\sigma \text{ and } \varphi_\sigma(s) = 1 \iff s \in I \}$$

$$= \{ \sigma \mid D = D_\sigma \text{ and } P_\sigma(1) = \langle I \rangle \}$$

$$= \{ \sigma \mid D = (P_\sigma(0, \infty), P_\sigma(-\infty, 0)) \text{ and } L_I D = (P_\sigma[0, \infty), P_\sigma(-\infty, 0)) \}.$$  

Clearly $S_D = \bigcup_I S_{D,I}$ and there is a decomposition

$$\text{Stab}_{\text{alg}}(C) = \bigcup_{D \in \text{alg}} S_D = \bigcup_{D \in \text{alg}} \left( \bigcup_I S_{D,I} \right).$$  

(2)

into strata of the form $S_{D,I}$. A choice of ordering of the simple objects of $D^b$ determines a homeomorphism $S_D \cong (\mathbb{H} \cup \mathbb{R}_{<0})^n$ under which the decomposition into strata corresponds to the the apparent decomposition of $(\mathbb{H} \cup \mathbb{R}_{<0})^n$ with $S_{D,I} \cong \mathbb{H}^{n-\#I} \times \mathbb{R}^{\#I}$. In particular each stratum $S_{D,I}$ is contractible.

Consider the closure $\overline{S_{D,I}}$ of a stratum. For $I \subset K \subset \{ s_1, \ldots, s_n \}$ let

$$\partial_K S_{D,I} = \{ \sigma \in S_{D,I} \mid \text{Im } Z_\sigma(s) = 0 \iff s \in K \},$$

so that $\overline{S_{D,I}} = \bigcup_K \partial_K S_{D,I}$ (as a set). For example $\partial S_{D,I} = S_{D,I}$.

**Lemma 3.9.** For any $t$-structure $E$, not necessarily algebraic, the intersection $S_E \cap \partial_K S_{D,I}$ is a union of components of $\partial_K S_{D,I}$. Each component in $\text{Stab}_{\text{alg}}(C)$ is a stratum $S_{E,J}$ for some $E$ and subset $J$ of the simple objects in $E$, with $\# J = \# K$.  

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Proof. Suppose $\sigma_n \to \sigma$ in Stab $(C)$. Then $P_\sigma(0) = \{0 \neq c \in C \mid \varphi_n^+(c) \to 0\}$. If $\sigma_n \in S_D$ for all $n$ then

$$P_\sigma(0) = \left\{ 0 \neq d \in D^0 \mid \varphi_n^+(d) \to 0 \right\}, \left\{ 0 \neq d \in D^0 \mid \varphi_n^-(d) \to 1 \right\} \right. \left[ -1 \right].$$

Furthermore, $D_\sigma$ is the right tilt of $D$ at the torsion theory

$$\left\{ 0 \neq d \in D^0 \mid \varphi_n^-(d) \not\to 0 \right\} = -\left\{ 0 \neq d \in D^0 \mid \varphi_n^+(d) \to 0 \right\}. \tag{3}$$

Now suppose $\sigma \in \partial_K S_{D,I}$ and $(\sigma_n)$ is a sequence in $S_{D,I}$ with limit $\sigma$. If $\varphi_n^+(d) \to 0$ for some $0 \neq d \in D^0$ then $Z_n(d) \to Z_\sigma(d) \in \mathbb{R}_{>0}$. Hence $d \in \langle K \rangle$. For $d \in \langle K \rangle$ there are three possibilities:

1. $\varphi_n^+(d) \to 0$ and $d \in P_\sigma(0)$;
2. $\varphi_n^+(d) \to 1$ and $d \in P_\sigma(1)$;
3. $\varphi_n^-(d) \to 0$, $\varphi_n^+(d) \to 1$, and $d$ is not $\sigma$-semistable.

Since the upper and lower phases of $d$ are continuous in Stab $(C)$, and the possibilities are distinguished by discrete conditions on the limiting phases, we deduce that the torsion theory [3] is constant for $\sigma$ in a component of $\partial_K S_{D,I}$. Hence the component is contained in $S_E$ for some $t$-structure $E$, and $S_E \cap \partial_K S_{D,I}$ is a union of components of $\partial_K S_{D,I}$ as claimed.

Now suppose that $\sigma \in S_{E,J} \cap \partial_K S_{D,I}$ for some algebraic $E$. It follows from the definitions of $S_{E,J}$ and of $\partial_K S_{D,I}$ that

$$\langle [t] \mid t \in J \rangle = \mathbb{Z}_\sigma^{-1}(\mathbb{R}) = \langle [s] \mid s \in K \rangle$$

as (free) subgroups of $KC$. Hence their ranks are equal, i.e. $\#J = \#K$.

By a similar argument to that used for the first part of this proof

$$\left\{ 0 \neq d \in D^0 \mid \varphi_n^-(d) \to 1 \right\}$$

is constant for $\sigma$ in a component of $\partial_K S_{D,I}$. It follows that $P_\sigma(0)$ is constant in a component. By the first part $E$ is fixed by the choice of component. As $\langle J \rangle = P_\sigma(1) = P_\sigma(0)[1]$ the subset $J$ of $E$ is also fixed. So each component of $\text{Stab}_{alg}(C) \cap \partial_K S_{D,I}$ is contained in some stratum $S_{E,J}$.

Finally we show that $S_{E,J} \subset \partial_K S_{D,I}$. Consider a perturbation $\tau$ of $\sigma$ which does not lie in $\partial_K S_{D,I}$. There must be an $s \in K$ such that $Z_\tau(s) \notin \mathbb{R}$. Hence neither $s$ nor $s[1]$ are in $P_\tau(1)$. Therefore $P_\tau(1) \neq \langle J \rangle$ and so $\tau \notin S_{E,J}$. It follows that $S_{E,J} \subset \partial_K S_{D,I}$. Since $S_{E,J}$ is connected, it must be a component of $\partial_K S_{D,I}$.

Corollary 3.10. The decomposition [3] of $\text{Stab}_{alg}(C)$ satisfies the frontier condition, i.e. if $S_{E,J} \cap \overline{S_{D,I}} \neq \emptyset$ then $S_{E,J} \subset \overline{S_{D,I}}$. In particular, the closure of each stratum is a union of lower-dimensional strata. Moreover,

$$S_{E,J} \subset \overline{S_{D,I}} \Rightarrow E \leq D \leq L_I D \leq L_J E.$$
Proof. The frontier condition follows immediately from Lemma 3.9 Suppose that $S_{E,I} \subset S_{D,I}$ and choose $\sigma$ in the intersection. Let $\sigma_n \to \sigma$ where $\sigma_n \in S_{D,I}$. Then $D^{\leq 0} = P_n(0,\infty)$, $D^{\neq 0} = P_\sigma(0,\infty)$, and $E^{\leq 0} = P_\sigma(0,\infty)$. Since $P_n(0,\infty)$ and $P_\sigma(0,\infty)$ do not vary with $n$, and the lower phase $\varphi^{-}(c)$ of any $0 \neq c \in \mathbb{C}$ is continuous in $\tau$,

$$P_\sigma(0,\infty) \subset P_n(0,\infty) \subset P_\sigma(0,\infty) \subset P_\sigma[0,\infty],$$

i.e. that $E \subset D \subset L_{I}D \subset L_{J}E$. Since all these $t$-structures are in the interval between $E$ and $E[-\infty]$, Remark 2.6 implies that $E \leq D \leq L_{I}D \leq L_{J}E$. 

Lemma 3.11. Suppose $D, E$ are algebraic $t$-structures, and that $I$ and $J$ are subsets of simple objects in the respective hearts. If $E \leq D \leq L_{I}D \leq L_{J}E$ then $S_{E,I} \subset S_{D,J}$.

Proof. Fix $\sigma \in S_{E,J}$. Since $E \leq D \leq L_{I}E$ we know that $D = L_{T}E$ for some torsion structure $T$ on $E^0$, and moreover that $T^{\leq 0} \subset (J) = P_\sigma(1)$. Any simple object of $D^0$ lies either in $T^{\leq 1}$ or in $T^{\geq 1}$. Hence any simple object $s$ of $D^0$ lies in $P_\sigma[0,1]$, and $s \in P_\sigma(0)$ if and only if $s \in T^{\leq 1}$. Moreover, if $s \in I$ then $s[1] \in L^{1}_{I}D^{\leq 0} \subseteq L_{J}E^{\leq 0} = P_\sigma[0,\infty)$. Thus $s \in I \Rightarrow s \in P_\sigma(1)$.

Since the simple objects of $D^{0}$ form a basis of $K/C$ we can perturb $\sigma$ by perturbing their charges. Given $\delta > 0$ we can always make such a perturbation to obtain a stability condition $\tau$ with $d(\sigma, \tau) < \delta$ for which $Z_\tau(s) \in \mathbb{H} \cup \mathbb{R}_{>0}$ for all simple $s$ in $D$, and $Z_\tau(s) \in \mathbb{Z}$ for all simple $s$ in $D$. We will prove that $\omega \in S_{D}$. Since the perturbation and rotation can be chosen arbitrarily small it will follow that $\sigma \in S_{D}$. And since $s \in P_\sigma(1)$ whenever $s \in I$ we can refine this statement to $\sigma \in S_{D,I}$ as claimed.

It remains to prove $\omega \in S_{D}$. For this it suffices to show that each simple $s$ in $D^{0}$ is $\tau$-semistable. For then $s$ is $\omega$-semistable too, and the choice of $Z_\omega$ implies that $s \in P_\sigma[0,1]$. The hearts of distinct (bounded) $t$-structures cannot be nested, so this implies $D = D_{\omega}$, or equivalently $\omega \in S_{D}$ as required.

Since $E$ is algebraic Lemma 3.1 guarantees that there is some $\delta > 0$ such that $P_\sigma(0,2\delta) = \emptyset$. Provided $d(\sigma, \tau) < \delta$ we have

$$P_\sigma(0,1] = P_\sigma(2\delta,1] \subset P_\sigma[\delta, 1 + \delta] \subset P_\sigma[0,1 + 2\delta] = P_\sigma(0,1].$$

It follows that the Harder–Narasimhan $\tau$-filtration of any $e \in E^0 = P_\sigma(0,1]$ is a filtration by subobjects of $e$ in the abelian category $P_\sigma(0,1]$. Consider a simple $s'$ in $D^0$ with $s'[1] \in T^{\leq 0}$. Since $T^{\leq 0}$ is a torsion theory any quotient of $s'[1]$ is also in $T^{\leq 0}$, in particular the final factor in the Harder–Narasimhan $\tau$-filtration, $t$, say, is in $T^{\leq 0}$. Hence $t[-1] \in D^0$ and $[t] = -\sum m_s[s \in KC]$ where the sum is over the simple $s$ in $D^0$ and the $m_s \in \mathbb{N}$. Since $\text{Im} Z_\tau(s) \geq 0$ for all simple $s$ it follows that $\text{Im} Z_\tau(t) = -\sum m_s \text{Im} Z_\tau(s) \leq 0$. Combined with the fact that $t$ is $\tau$-semistable with phase in $(\delta, 1 + \delta)$ we have $\varphi^{-}(s'[1]) = \varphi^{-}(t) \geq 1$. Hence $s' \in P_\tau[1, 1 + \delta]$. Since $Z_\tau(s'[1]) \in \mathbb{R}_{<0}$ we see that $s'[1] \in P_\tau(1)$, and in particular is $\tau$-semistable.

Now suppose $s' \in T^{\geq 1}$. Since $T^{\geq 1}$ is a torsion-free theory in $P_\sigma(0,1]$ any subobject of $s'$ is also in $T^{\geq 1}$. In contrast, $s'$ cannot have any proper quotients in $T^{\geq 1}$: if it did we would obtain a short exact sequence

$$0 \to f \to s \to f' \to 0$$

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in $\mathbb{P}_s[0,1]$ with $f, f' \in T^{\geq 1}$. This would also be short exact in $D^0$, contradicting the fact that $s'$ is simple. It follows that any proper quotient of $s'$ is in $T^{\leq 0}$. The argument of the previous paragraph then shows that either $s'$ is $\tau$-semistable (with no proper semistable quotient), or $s' \in \mathbb{P}_s[1,1+\delta]$. But $\text{Im} Z_{\tau}(s') > 0$ so the latter is impossible, and $s'$ must be $\tau$-semistable. This completes the proof.

**Definition 3.12.** Let $\text{Int}(C)$ be the poset whose elements are intervals in the poset $\text{Tilt}(C)$ of $t$-structures of the form $[D, L_I D]$ where $D$ is algebraic and $I$ is a subset of the simple objects in the heart of $D$. We order these intervals by inclusion. We do not assume that $L_I D$ is algebraic.

**Corollary 3.13.** There is an isomorphism $\text{Int}(C)^{op} \rightarrow P\text{Stab}_{alg}(C)$ of posets given by the correspondence $[D, L_I D] \leftrightarrow S_{D,I}$. The components of $\text{Stab}_{alg}(C)$ correspond to components of $\text{Tilt}_{alg}(C)$.

**Proof.** The existence of the isomorphism is direct from Corollary 3.10 and Lemma 3.11. In particular, components of these posets are in 1-to-1 correspondence. The second statement follows because components of $\text{Stab}_{alg}(C)$ correspond to components of $P(\text{Stab}_{alg}(C))$, and components of $\text{Int}(C)$ correspond to components of $\text{Tilt}_{alg}(C)$.

**Remark 3.14.** Following Remark 2.8 we note an alternative description of $\text{Int}(C)$ when $C = D(A)$ is the bounded derived category of a finite-dimensional algebra $A$ over an algebraically-closed field, and has finite global dimension. By Lemma 4.1 $\text{Int}(C)^{op} \cup \{0\} \cong \mathbb{P}_2(C)$ is the poset of silting pairs defined in §3, where $0$ is a formally adjoined minimal element. Hence, by the above corollary, $P\text{Stab}_{alg}(C) \cup \{0\} \cong \mathbb{P}_2(C)$.

**Remark 3.15.** If $D$ and $E$ are not both algebraic then $D \leq E \leq D[-1]$ need not imply $S_D \cap S_E \neq \emptyset$, see [35, p20] for an example. Thus components of $\text{Stab}_{alg}(C)$ may not correspond to components of $\text{Tilt}(C)$. In general we have maps

\[
\begin{array}{ccc}
\pi_0 \text{Stab}_{alg}(C) & \longrightarrow & \pi_0 \text{Stab}(C) \\
\downarrow & & \downarrow \\
\pi_0 \text{Tilt}_{alg}(C) & \longrightarrow & \pi_0 \text{Tilt}(C) \longrightarrow \pi_0 T(C).
\end{array}
\]

The maps in the bottom row are induced from the maps of posets $\text{Tilt}_{alg}(C) \rightarrow \text{Tilt}(C) \rightarrow T(C)$, the vertical equality holds by the above corollary, and the vertical map exists because $S_D$ and $S_E$ in the same component of $\text{Stab}(C)$ implies that $D$ and $E$ are related by a finite sequence of tilts [36, Corollary 5.2].

**Lemma 3.16.** Suppose that $\text{Tilt}_{alg}(C) = \text{Tilt}(C) = T(C)$ are non-empty. Then $\text{Stab}_{alg}(C) = \text{Stab}(C)$ has a single component.

**Proof.** It is clear that $\text{Stab}(C) = \text{Stab}_{alg}(C) \neq \emptyset$. Suppose that $\sigma, \tau \in \text{Stab}(C)$. Since $\text{Tilt}_{alg}(C) = \text{Tilt}(C)$ the associated $t$-structures $D_{\sigma}$ and $D_{\tau}$ are algebraic, so that $D_{\sigma} \subset D_{\tau}[-d]$ for some $d \in \mathbb{N}$ by Lemma 2.9. Since $\text{Tilt}_{alg}(C) = T(C)$ this implies $D_{\sigma} \not\simeq D_{\tau}[-d]$, and thus $D_{\sigma}$ and $D_{\tau}$ are in the same component of $\text{Tilt}_{alg}(C)$. Hence by Corollary 3.13 $\sigma$ and $\tau$ are in the same component of $\text{Stab}_{alg}(C) = \text{Stab}(C)$.
Lemma 3.17. Suppose $C = D(A)$ for a finite-dimensional algebra $A$ over an algebraically closed field, with finite global dimension. Then $\text{Stab}_{\text{alg}}(C)$ is connected. Moreover, any component of $\text{Stab}(C)$ other than that containing $\text{Stab}_{\text{alg}}(C)$ consists entirely of stability conditions for which the phases of semistable objects are dense in $\mathbb{R}$.

Proof. By Remark 2.8 $\text{Tilt}_{\text{alg}}(C)$ is the sub-poset of $T(C)$ consisting of the algebraic $t$-structures. The proof that $\text{Stab}_{\text{alg}}(C)$ is connected is then the same as that of the previous result. For the last part note that if $\sigma$ is a stability condition for which the phases of semistable objects are not dense then acting on $\sigma$ by some element of $C$ we obtain an algebraic stability condition. Hence $\sigma$ must be in the unique component of $\text{Stab}(C)$ containing $\text{Stab}_{\text{alg}}(C)$.

Remark 3.18. To show that $\text{Stab}(C)$ is connected when $C = D(A)$ as in the previous result it suffices to show that there are no stability conditions for which the phases of semistable objects are dense. For example, from Example 3.5 and the fact that the path algebra of an acyclic quiver is a finite-dimensional algebra of global dimension 1, we conclude that $\text{Stab}(Q)$ is connected whenever $Q$ is of ADE Dynkin, or extended Dynkin, type. (Later we show that $\text{Stab}(Q)$ is contractible in the Dynkin case; it was already known to be simply-connected by [30].)

By Remark 3.6 $G$ acts freely on a component consisting of stability conditions for which the phases are dense. In contrast, it does not act freely on a component containing algebraic stability conditions since any such contains stability conditions for which the central charge is real, and these have non-trivial stabiliser. Hence, the $G$ action also distinguishes the component containing $\text{Stab}_{\text{alg}}(C)$ from the others, and if there is no component on which $G$ acts freely $\text{Stab}(C)$ must be connected.

Suppose $\text{Stab}_{\text{alg}}(C) \neq \emptyset$. Let $\text{Bases}(KC)$ be the groupoid whose objects are pairs consisting of an ordered basis of the free abelian group $KC$ and a subset of this basis, and whose morphisms are automorphisms relating these bases (so there is precisely one morphism in each direction between any two objects; we do not ask that it preserve the subsets). Fix an ordering of the simple objects in the heart of each algebraic $t$-structure. This fixes isomorphisms

$$S_{D,I} \cong \mathbb{H}^{n-\#I} \times \mathbb{R}^{\#I}_{>0}.$$  

Regard the poset $\text{Int}(C)$ as a category, and let $F_C : \text{Int}(C) \to \text{Bases}(KC)$ be the functor taking $[D,L_I D]$ to the pair consisting of the ordered basis of classes of simple objects in $D$ and the subset of classes of $I$. This uniquely specifies $F_C$ on morphisms.

Proposition 3.19. The functor $F_C$ determines $\pi : \text{Stab}_{\text{alg}}(C) \to \text{Hom}(KC, \mathbb{C})$ up to homeomorphism as a space over $\text{Hom}(KC, \mathbb{C})$.

Proof. As sets there is a commutative diagram

$$
\begin{array}{ccc}
\text{Stab}_{\text{alg}}(C) & \xrightarrow{\beta} & \sum_{D,I} \mathbb{H}^{n-\#I} \times \mathbb{R}^{\#I}_{>0} \\
\downarrow \pi & & \downarrow \sum \pi_{D,I} \\
\text{Hom}(KC, \mathbb{C}) & & 
\end{array}
$$
where the map $\pi_{D,I}$ is determined from the pair \( F_C \left( \left[ D, L_I D \right], \right) \) of basis and subset, and $\beta$ is defined using the bijections $S_{D,I} \cong H^{\alpha - #I} \times \mathbb{R}^{#I}_{<0}$. The subsets 

\[
U_{E,J} = \bigcup_{E \leq D \leq L_I D \leq L_J E} \pi_{D,I}^{-1} U,
\]

where $U$ is open in $\text{Hom} \left( (KC, \mathbb{C}) \right)$, form a base for a topology. With this topology, $\beta$ is a homeomorphism. To see this note that

\[
\beta^{-1} U_{E,J} = \left( \bigcup_{E \leq D \leq L_I D \leq L_J E} S_{D,I} \right) \cap \pi^{-1} U
\]

is the intersection of an open subset with an upward-closed union of strata, hence open. So $\beta$ is continuous. Moreover, all sufficiently small open neighbourhoods of a point of $\text{Stab}_{\text{alg}}(C)$ have this form, so the bijection $\beta$ is an open map, hence a homeomorphism. \qed

A more practical approach is to study the homotopy-type of $\text{Stab}_{\text{alg}}(C)$. In good cases this is encoded in the poset $P_{\text{Stab}_{\text{alg}}}(C) \cong \text{Int}(C)^{op}$. A stratification is \textit{locally-finite} if any stratum is contained in the closure of only finitely many other strata, and \textit{closure-finite} if the closure of each stratum is a union of finitely many strata.

**Lemma 3.20.** The following are equivalent:

1. The stratification of $\text{Stab}_{\text{alg}}(C)$ is locally-finite;
2. The stratification of $\text{Stab}_{\text{alg}}(C)$ is closure-finite;
3. Each interval $[D,D[-1]]_<$ in $\text{Tilt}_{\text{alg}}(C)$ is finite.

\textbf{Proof.} This follows easily from Corollary 3.13 which states that $S_{E,J} \subset S_{D,I}$ if $D \leq E \leq L_I D \leq L_J E$. Thus the size of the interval $[D,D[-1]]_<$ is precisely

\[
\# \{ E \in \text{Tilt}_{\text{alg}}(C) \mid S_E \cap S_D \neq \emptyset \} = \# \{ E \in \text{Tilt}_{\text{alg}}(C) \mid S_E \cap S_{E[1]} \neq \emptyset \}.
\]

The result follows because each $S_D$ is a finite union of strata, and each stratum is in some $S_D$. \qed

**Proposition 3.21.** The space $\text{Stab}_{\text{alg}}(C)$ of algebraic stability conditions, with the decomposition into the strata $S_{D,I}$, can be given the structure of a regular, normal cellular stratified space. It is a regular, totally-normal CW-cellular stratified space precisely when $\text{Stab}_{\text{alg}}(C)$ is locally-finite.

\textbf{Proof.} First we define a cell structure on $S_{D,I}$. Let $\pi \colon \text{Stab}(C) \to \text{Hom} \left( (KC, \mathbb{C}) \right)$ be the projection onto the central charge. Consider $\text{Hom} \left( (KC, \mathbb{C}) \right)$ as a real vector space. Note that

\[
\overline{S_{D,I} \cap \text{Stab}_{alg}(C)} \cong \pi \left( \overline{S_{D,I}} \cap \text{Stab}_{alg}(C) \right) \subset \pi \left( S_{D,I} \right)
\]

and that $\overline{S_{D,I}}$ is the real convex closed polyhedral cone

\[
C = \{ \text{charge} \mid \text{Im} \ Z(s) \geq 0 \text{ for } s \notin I \text{ and } \text{Im} \ Z(s) = 0, \text{ Re} \ Z(s) \leq 0 \text{ for } s \in I \}
\]

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in \( \text{Hom}(K\mathbb{C}, \mathbb{C}) \). The projection \( \pi \) identifies the stratum \( S_{D,I} \) with the (relative) interior of \( C \). By Corollary 3.10, \( S_{D,I} \cap \text{Stab}_{\text{alg}}(C) \) is a union of strata. Moreover, the projection of each boundary stratum

\[ S_{E,J} \subset S_{D,I} \cap \text{Stab}_{\text{alg}}(C) \]

is cut out by a finite set of (real) linear equalities and inequalities. Therefore we can subdivide \( C \) into a union of real convex polyhedral sub-cones in such a way that each stratum is identified with the (relative) interior of one of these sub-cones.

Let \( A(1,2) \) be the open annulus in \( \text{Hom}(K\mathbb{C}, \mathbb{C}) \) consisting of points of distance in the range \((1,2)\) from the origin, and \( A[1,2] \) its closure. Then we have a continuous map

\[ S_{D,I} \cap \text{Stab}_{\text{alg}}(C) \xrightarrow{\pi} C \cong C \cap A(1,2) \hookrightarrow C \cap A[1,2] \]

where \( C \) is identified with \( C \cap A(1,2) \) via a radial contraction. The subdivision of \( C \) into cones induces the structure of a compact curvilinear polyhedron on the intersection \( C \cap A[1,2] \). A choice of homeomorphism from \( C \cap A[1,2] \) to a closed cell yields a map from \( S_{D,I} \cap \text{Stab}_{\text{alg}}(C) \) to a closed cell which is a homeomorphism onto its image. The inverse from this image is a characteristic map for the stratum \( S_{D,I} \), and the collection of these gives \( \text{Stab}_{\text{alg}}(C) \) the structure of a regular, normal cellular stratified space.

When the stratification of \( \text{Stab}_{\text{alg}}(C) \) is locally-finite the cellular stratification is closure-finite by Lemma 3.20, and any point is contained in the interior of a closed union of finitely many cells. This guarantees that \( \text{Stab}_{\text{alg}}(C) \) has the weak topology arising from the cellular stratification, which is therefore a CW-cellular stratification. We can also choose the above subdivision of \( C \) to have finitely many sub-cones. In this case the curvilinear polyhedron \( C \cap A[1,2] \) has finitely many faces, and therefore has a CW-structure for which the strata of \( S_{D,I} \cap \text{Stab}_{\text{alg}}(C) \) are identified with certain open cells. It follows that the cellular stratification is totally-normal. Conversely, if the stratification is CW-cellular then it is closure-finite, and hence by Lemma 3.20 it is locally-finite.

**Corollary 3.22.** Suppose the stratification of \( \text{Stab}_{\text{alg}}(C) \) is locally-finite. Then there is a homotopy equivalence \( \text{Stab}_{\text{alg}}(C) \simeq BP(\text{Stab}_{\text{alg}}(C)) \).

**Proof.** This is direct from Proposition 3.21 and Theorem 2.17.

**Corollary 3.23.** Suppose the stratification of \( \text{Stab}_{\text{alg}}(C) \) is locally-finite. Then the integral homology groups \( H_i(\text{Stab}_{\text{alg}}(C)) = 0 \) for \( i > n = \text{rk}(K\mathbb{C}) \).

**Proof.** By Corollary 3.22, \( \text{Stab}_{\text{alg}}(C) \simeq BP(\text{Stab}_{\text{alg}}(C)) \). A chain in the poset \( P(\text{Stab}_{\text{alg}}(C)) \) consists of a sequence of strata of \( \text{Stab}_{\text{alg}}(C) \) of decreasing codimension, each in the closure of the next. Since the maximum codimension of any stratum is \( n \), the length of any chain is less than or equal to \( n \). Hence \( BP(\text{Stab}_{\text{alg}}(C)) \) is a CW-complex of dimension \( \leq n \), and the result follows.

**Remark 3.24.** If \( \text{Stab}_{\text{alg}}(C) \) is locally-finite then any union \( U \) of strata of \( \text{Stab}_{\text{alg}}(C) \) is a regular, totally-normal CW-cellular stratified space. Hence there is a homotopy equivalence \( U \simeq BP(U) \) and \( H_i(U) = 0 \) for \( i > n = \text{rk}(K\mathbb{C}) \).
Example 3.25. We continue Example 3.8. The ‘Kronecker heart’ \( \langle O, O(-1)[1] \rangle \) of \( D(\mathbb{P}^1) \) is algebraic. There are infinitely many torsion structures on this heart such that the tilt is a \( t \)-structure with heart isomorphic to the Kronecker heart \( [35, §3.2] \). It quickly follows from Corollary 3.13 that the stratification of \( \operatorname{Stab}_{\text{alg}}(\mathbb{P}^1) \) is neither closure-finite nor locally-finite — see [35, Figure 5] for a diagram of the codimension 2 strata in the closure of the stratum corresponding to the Kronecker heart.

3.2 More on the poset of strata

Corollary 3.22 shows that when \( \operatorname{Stab}_{\text{alg}}(C) \) is closure-finite and locally-finite its homotopy-theoretic properties are encoded in the poset \( P(\operatorname{Stab}_{\text{alg}}(C)) \). In the remainder of this section we elucidate some of the latter’s good properties.

The assumptions that \( \operatorname{Stab}_{\text{alg}}(C) \) is locally-finite and closure-finite are respectively equivalent to the statements that the unbounded closed intervals \([S, \infty)\) and \((-\infty, S]\) are finite for each \( S \in P(\operatorname{Stab}_{\text{alg}}(C)) \). It follows of course that closed bounded intervals are also finite, but in fact the latter holds without these assumptions.

Lemma 3.26. Suppose \( S_{E,J} \subset \overline{S_{D,I}} \). Then the closed interval \([S_{E,J}, S_{D,I}]\) in \( P(\operatorname{Stab}_{\text{alg}}(C)) \) is isomorphic to a sub-poset of \([I, K]^\text{op}\). Here the subset \( K \) is uniquely determined by the requirement that \( S_{E,J} \subset \partial K S_{D,I} \), and subsets of the simple objects in \( D^0 \) are ordered by inclusion.

Proof. Fix \( \sigma \in S_{E,J} \cap \partial K S_{D,I} \). Using the fact that \( \operatorname{Stab}(C) \) is locally isomorphic to \( \operatorname{Hom}(KC, C) \) we can choose an open neighbourhood \( U \) of \( \sigma \) in \( \operatorname{Stab}(C) \) so that \( U \cap \partial L S_{D,I} \) is non-empty and connected for any subset \( I \subset L \subset K \), and empty when \( L \not\subset K \). It follows that \( U \) meets a unique component of \( \partial L S_{D,I} \) for each \( I \subset L \subset K \). The strata in \([S_{E,J}, S_{D,I}]\) correspond to those components for which the heart is algebraic. Since \( \partial L S_{D,I} \subset \overline{\partial L S_{D,I}} \iff L' \subset L \) the result follows.

We have seen that \( \operatorname{Stab}_{\text{alg}}(C) \) need be neither open nor closed as a subset of \( \operatorname{Stab}(C) \). The next two results shows that whether or not it is locally-closed is closely related to the structure of the bounded closed intervals in \( P(\operatorname{Stab}_{\text{alg}}(C)) \).

Lemma 3.27. The first of the statements below implies the second and third, which are equivalent. When \( \operatorname{Stab}_{\text{alg}}(C) \) is locally-finite all three are equivalent.

1. The inclusion \( \operatorname{Stab}_{\text{alg}}(C) \) is locally-closed as a subspace of \( \operatorname{Stab}(C) \).
2. The inclusion \( \operatorname{Stab}_{\text{alg}}(C) \cap \overline{S_D} \hookrightarrow \overline{S_D} \) is open for each algebraic \( D \).
3. For each pair of strata \( S_{E,J} \subset \overline{S_{D,I}} \) there is an isomorphism \([S_{E,J}, S_{D,I}] \cong [I, K]^\text{op} \), where \( K \) is uniquely determined by the requirement that \( S_{E,J} \subset \partial K S_{D,I} \).

Proof. Suppose \( \operatorname{Stab}_{\text{alg}}(C) \) is locally-closed. Let \( \sigma \in \operatorname{Stab}_{\text{alg}}(C) \cap \overline{S_D} \) where \( D \) is algebraic. Then there is a neighbourhood \( U \) of \( \sigma \) in \( \operatorname{Stab}(C) \) such that \( U \cap \operatorname{Stab}_{\text{alg}}(C) \) is closed in \( U \). Then \( U \cap S_D \subset U \cap \operatorname{Stab}_{\text{alg}}(C) \) so

\[
U \cap \overline{S_D} \subset U \cap \operatorname{Stab}_{\text{alg}}(C)
\]
and Stab_{alg}(C) \cap \overline{S_D} is open in \overline{S_D}.

Now suppose Stab_{alg}(C) \cap \overline{S_D} is open in \overline{S_D}. Then we can choose a neighbourhood \( U \) of \( \sigma \) so that \( U \cap \partial_L S_{D,I} \) is non-empty and connected for each \( I \subset L \subset K \) and, moreover, \( U \cap \overline{S_D} \subset \text{Stab}_{alg}(C) \). It follows, as in the proof of Lemma 3.26, that \([S_E,J,S_{D,I}] \cong [I,K]^{op}\).

Conversely, if \([S_E,J,S_{D,I}] \cong [I,K]^{op}\) then given a neighbourhood \( U \) with \( U \cap \partial_L S_{D,I} \) non-empty and connected for each \( I \subset L \subset K \) we see that it meets only components of the \( \partial_L S_{D,I} \) which are in Stab_{alg}(C). Hence Stab_{alg}(C) \cap \overline{S_D} is open in \overline{S_D}.

Finally, assume the stratification of Stab_{alg}(C) is locally-finite and that Stab_{alg}(C) \cap \overline{S_D} is open for each algebraic \( D \). Fix \( \sigma \in \text{Stab}_{alg}(C) \). There are finitely many algebraic \( D \) with \( \sigma \in \overline{S_D} \). There is an open neighbourhood \( U \) of \( \sigma \) in Stab(C) such that

\[ U \cap \overline{S_D} \subset \overline{S_D} \cap \text{Stab}_{alg}(C) \]

for any algebraic \( D \) (the left-hand side is empty for all but finitely many such). Hence

\[ U \cap \text{Stab}_{alg}(C) = U \cap \bigcup_{D \text{ alg}} S_D \subset U \cap \bigcup_{D \text{ alg}} \overline{S_D} = \bigcup_{D \text{ alg}} U \cap \overline{S_D} \subset U \cap \text{Stab}_{alg}(C) \]

and so \( U \cap \text{Stab}_{alg}(C) = \bigcup_{D \text{ alg}} U \cap \overline{S_D} \). The latter is a finite union of closed subsets of \( U \), hence closed in \( U \). Therefore each \( \sigma \in \text{Stab}_{alg}(C) \) has an open neighbourhood \( U \ni \sigma \) such that \( U \cap \text{Stab}_{alg}(C) \) is closed in \( U \). It follows that \( \text{Stab}_{alg}(C) \) is locally-closed.

**Corollary 3.28.** Suppose \( \text{Stab}_{alg}(C) \) is locally-closed. Then \( P\text{Stab}_{alg}(C) \) is pure of length \( n = \text{rk}(KC) \).

*Proof.* The stratum \( S_{D,I} \) contains \( S_{D_1,\{s_1,\ldots,s_n\}} \) in its closure, and is in the closure of \( S_{D,\emptyset} \). It follows that any maximal chain in \( P(\text{Stab}_{alg}(C)) \) is in a closed interval of the form \([S_{D_1,\{s_1,\ldots,s_n\}}, S_{E,J}]\). As Stab(C) is locally-closed this is isomorphic to the poset of subsets of an \( n \)-element set by Lemma 3.27. This implies \( P\text{Stab}_{alg}(C) \) is pure of length \( n \).

**Example 3.29.** We continue Examples 3.8 and 3.25. The subspace Stab_{alg}(\mathbb{P}^1) is not locally-closed: if it were then Stab(\mathbb{P}^1) − Stab_{alg}(\mathbb{P}^1) = A \cup U for some closed \( A \) and open \( U \). This subset consists of those stability conditions for which the phases of semistable objects accumulate at \( Z \subset \mathbb{R} \), and this has empty interior. Hence the only possibility is that \( U = \emptyset \), in which case Stab_{alg}(\mathbb{P}^1) would be open. This is not the case, so Stab_{alg}(\mathbb{P}^1) cannot be locally-closed. Nevertheless, from the explicit description of stability conditions in \[95\] one can see that the poset of strata is pure (of rank 2), and that the second two conditions of Lemma 3.27 are satisfied.

## 4 Finite-type components

We say a component \( \text{Tilt}^0(C) \) has **finite-type** if each \( t \)-structure in it is algebraic and has **finite tilting type**, i.e. has only finitely many torsion-structures in its heart. A \( t \)-structure has finite tilting type if and only if the interval \([D,D[-1]] \leq \)
in Tilt(C) is finite. It follows from Lemmas 2.13 and 2.14 that a finite-type component $\text{Tilt}^0(C)$ is a lattice, and that closed bounded intervals in it are finite.

**Lemma 4.1.** Suppose that the set $S$ of $t$-structures obtained from some $D$ by finite sequences of simple tilts consists entirely of algebraic $t$-structures of finite tilting type. Then $S$ is (the underlying set of) a finite-type component of Tilt(C). Moreover, every finite-type component arises in this way.

**Proof.** If $D$ has finite tilting type then any tilt of $D$ can be decomposed into a finite sequence of simple tilts. It follows that $S$ is a component of Tilt(C) as claimed. It is clearly of finite-type. Conversely if $\text{Tilt}^0(C)$ is a finite-type component, and $D \in \text{Tilt}^0(C)$, then every $t$-structure obtained from $D$ by a finite sequence of simple tilts is algebraic, of finite tilting type. Hence $D$ contains the set $S$, and by the first part $S = \text{Tilt}^0(C)$.

If the heart of a $t$-structure contains only finitely many isomorphism classes of indecomposable objects, then it is of finite tilting type (because a torsion theory is determined by the indecomposable objects it contains). Therefore, whilst we do not use it in this paper, the following result may be useful in detecting finite-type components, particularly if up to automorphism there are only finitely many $t$-structures which can be reached from $D$ by finite sequences of simple tilts. In very good cases — for instance when tilting at a 2-spherical simple object $s$ with the property that $\text{Ext}_C^i(s, s') = 0$ for $i \neq 1$ for any other simple object $s'$ — the tilted $t$-structure itself is obtained by applying an automorphism of $C$ and hence inherits the property of being algebraic of finite tilting type. A similar situation arises if $D$ is an algebraic $t$-structure in which all simple objects are rigid, i.e. have no self extensions. In this case [25, Proposition 5.2] states that all simple tilts of $D$ are also algebraic.

**Lemma 4.2.** Suppose that $D$ is an algebraic $t$-structure on a triangulated category $C$ whose heart has only finitely many isomorphism classes of indecomposable objects. Then any simple tilt of $D$ is algebraic.

**Proof.** It suffices to prove that the claim holds for any simple right, since the simple left tilts are shifts of these. Since there are only finitely many indecomposable objects in $D^0$ there are in particular only finitely many simple objects. Let these be $s_1, \ldots, s_n$ and consider the right tilt at $s_1$. Let $\sigma \in S_D$ be the unique stability condition with $Z_{\sigma}(s_1) = 1$ and $Z_{\sigma}(s_j) = -1$ for $j = 2, \ldots, n$. Let $\tau$ be obtained by acting on $\sigma$ by $-1/2 \in C$. Then $D_\tau$ is the right tilt of $D_\sigma$ at $s_1$. As there are only finitely many indecomposable objects in $D^0$ the set of $\varphi \in \mathbb{R}$ such that $P_{\sigma}(\varphi) \neq 0$ is discrete. The same is therefore true for $\tau$. It follows that $P_{\tau}(0, \epsilon) = 0$ for some $\epsilon > 0$. The component of Stab(C) containing $\sigma$ and $\tau$ is full since $\sigma$ is algebraic. Hence by Lemma 3.1 the stability condition $\tau$ is algebraic too.

**Lemma 4.3.** Let $\text{Tilt}^0(C)$ be a finite-type component of Tilt(C). Then

$$\text{Stab}^0(C) = \bigcup_{D \in \text{Tilt}^0(C)} S_D$$

(4)

is a component of Stab(C).
Proof. Clearly $\text{Tilt}^0(C)$ is also a component of $\text{Tilt}_{\text{alg}}(C)$. By Corollary 3.13 there is a corresponding component $\text{Stab}^0_{\text{alg}}(C)$ of $\text{Stab}_{\text{alg}}(C)$ given by the RHS of (1). Let $\text{Stab}^0(C)$ be the unique component of $\text{Stab}(C)$ containing $\text{Stab}^0_{\text{alg}}(C)$. Recall from [26, Corollary 5.2] that the $t$–structures associated to stability conditions in a component of $\text{Stab}(C)$ are related by finite sequences of tilts. Thus, each stability condition in $\text{Stab}^0(C)$ has associated $t$–structure in $\text{Tilt}^0(C)$. In particular, the $t$–structure is algebraic and $\text{Stab}^0_{\text{alg}}(C) = \text{Stab}^0(C)$ is actually a component of $\text{Stab}(C)$.

A finite-type component $\text{Stab}^0(C)$ of $\text{Stab}(C)$ is one which arises in this way from a finite-type component $\text{Tilt}^0(C)$ of $\text{Tilt}(C)$.

**Lemma 4.4.** Suppose $\text{Stab}^0(C)$ is a finite-type component. The stratification of $\text{Stab}^0(C)$ is locally-finite and closure-finite.

**Proof.** This is immediate from Lemma 3.20 and the obvious fact that the interval $[D_\sigma, D_\sigma[-1]]$ of algebraic tilts is finite when the interval $[D_\sigma, D_\sigma[-1]]$ of all tilts is finite.

**Corollary 4.5.** Suppose $\text{Stab}^0(C)$ is a finite-type component. There is a homotopy equivalence $\text{Stab}^0(C) \simeq BP(\text{Stab}^0(C))$, in particular $\text{Stab}^0(C)$ has the homotopy-type of a CW-complex.

**Proof.** This is immediate from Lemma 4.4 and Corollary 3.22.

We now prove that finite-type components are contractible. Our approach is modelled on the proof of the simply-connectedness of the stability spaces of representations of Dynkin quivers [30, Theorem 4.6] (although the details are a little different). The key is to show that certain ‘conical unions of strata’ are contractible.

The open star $S^*_{D,I}$ of a stratum $S_{D,I}$ is the union of all strata containing $S_{D,I}$ in their closure. An open star is contractible: $S^*_{D,I} \simeq BP(S^*_{D,I})$ by Remark 3.24 and, since $P(S_{D,I})$ is a poset with lower bound $S_{D,I}$, its classifying space is conical, hence contractible.

**Definition 4.6.** For a finite set $F$ of $t$–structures in $\text{Tilt}^0(C)$ let the cone $C(F) = \{(E, J) \mid F \preceq E \preceq L_J E \preceq \sup F \text{ for some } F \in F\}$.

Let $V(F) = \bigcup_{(E, J) \in C(F)} S_{E,J}$ be the union of the corresponding strata; we call such a subspace conical. For example, $V(\{F\}) = S_{F,\emptyset}$.

**Remark 4.7.** If $(E, J) \in C(F)$ then $F \preceq E \preceq \sup F$. Since $[\inf F, \sup F]_<$ is finite, and there are only finitely many possible $J$ for each $E$, it follows that $C(F)$ is a finite set. Let $c(F) = \#C(F)$ be the number of elements, which is also the number of strata in $V(F)$.

Note that $V(F)$ is an open subset of $\text{Stab}^0(C)$ since $S_{D,I} \subset V(F)$ and $S_{D,I} \subset S_{E,J}$ implies $F \preceq D \preceq E \preceq L_J E \preceq L_D \preceq \sup F$ for some $F \in F$ so that $S_{E,J} \subset V(F)$ too. It is also non-empty since it contains $S_{\sup F,\emptyset}$.
Proposition 4.8. The conical subspace $V(F)$ is contractible for any finite set $F \subset \text{Tilt}^0(C)$.

Proof. Let $C = C(F)$, $c = c(F)$, and $V = V(F)$. We prove this result by induction on the number of strata $c$. When $c = 1$ we have $C = \{\sup F, 0\}$ so that $V = \sup F, \emptyset$ is contractible as claimed. Suppose the result holds for all conical subspaces of with strictly fewer than $c$ strata.

Recall from Remark 3.24 that $V \simeq BP(V)$ so that $V$ has the homotopy-type of a CW-complex. Hence it suffices, by the Hurewicz and Whitehead Theorems, to show that the integral homology groups $H_i(V) = 0$ for $i > 0$. Choose $(D, I) \in C$ such that

1. $\not\exists (E, J) \in C$ with $E \not\supset D$;
2. $(D, I') \in C \iff I' \subseteq I$.

It is possible to choose such a $D$ since $C$ is finite; note that $D$ is necessarily in $F$. It is then possible to choose such an $I$ because if $S_{D, I'}, S_{D, I''} \subseteq V$ then $L_{I'} D, L_{I''} D \equiv \sup F$ which implies $L_{I' \cup I''} D = L_I D \cup L_{I''} D \equiv \sup F$. Consider the relative long exact sequence

$$\cdots \rightarrow H_i(V) \rightarrow H_i(V - S_D) \rightarrow H_i(V, V - S_D) \rightarrow \cdots.$$  

By choice of $D$ the subspace $V - S_D = V(F')$ is also conical, with $F' = F \cup \{L_s D \mid s \in D^0 \text{ simple}, L_s D \equiv \sup F\} - \{D\}$. Note that $\sup F' = \sup F$. Moreover, $V(F')$ has fewer strata than $V$ so by induction it is contractible. Hence $H_i(V - S_D) = 0$ for $i > 0$. The choice of $D$ also ensures that $V \cap S_D$ is closed in $V$. The choice of $I$ ensures that $V \cap S_D \subseteq V \cap S_{D, I}^*$. Hence $V - S_{D, I}^*$ is a closed subset of $V - S_D$, which is open. Excising $V - S_{D, I}^*$ is contractible. By induction $S_{D, I}^* - S_D$ is also contractible: if it is the conical subspace

$$\bigcup_{D \lesssim E \lesssim L, E \lesssim I, D} S_{E, J} = V \{L_s D \mid s \in I\},$$

and this has fewer strata than $V$. Hence $H_i(V) \cong H_i(V - S_D)$ for all $i$, and the result follows.

Theorem 4.9. The component $\text{Stab}^0(C)$ of $\text{Stab}(C)$ is contractible.

Proof. By Lemma 3.2 $\text{Stab}^0(C)$ is a locally-finite stratified space. Thus a singular integral $i$-cycle in $\text{Stab}^0(C)$ has support meeting only finitely many strata, say the support is contained in $\{F \mid F \in F\}$. Therefore the cycle has support in $V(F)$, and so is null-homologous whenever $i > 0$ by Proposition 4.3. This shows that $H_i(\text{Stab}^0(C)) = 0$ for $i > 0$. Since $\text{Stab}^0(C)$ has the homotopy type of a CW-complex it follows from the Hurewicz and Whitehead Theorems that $\text{Stab}^0(C)$ is contractible.
4.1 Calabi–Yau–N Ginzburg algebras

Let $Q$ be a quiver whose underlying unoriented graph is an ADE Dynkin diagram. Let $\Gamma_N Q$ be the associated Calabi–Yau–N Ginzburg algebra, $D(\Gamma_N Q)$ the bounded derived category of finite-dimensional representations of $\Gamma_N Q$ over an algebraically-closed field $k$, and $\text{Stab}(\Gamma_N Q)$ the space of stability conditions on $D(\Gamma_N Q)$. See [23, §7] for the details of the construction of the differential-graded algebra $\Gamma_N Q$ and its derived category.

**Corollary 4.10.** The principal component $\text{Stab}^0(\Gamma_N Q)$ of the stability space, containing the stability conditions with heart the representations of $\Gamma_N Q$, is of finite-type, and hence is contractible.

**Proof.** By Corollary 8.4 of [25] each $t$–structure obtained from the standard one, whose heart is the representations of $\Gamma_N Q$, by a finite sequence of simple tilts is algebraic. Lemma 5.1 and Proposition 5.2 of [30] show that each of these $t$–structures is of finite tilting type. Hence by Lemma 1.1 the component $\text{Tilt}^0(C)$ containing the standard $t$–structure has finite-type, and by Theorem 4.9 the corresponding component $\text{Stab}^0(\Gamma_N Q)$ is contractible. □

This affirms the second part of Conjecture 5.8 of [30]. It is in accord with the known computations of $\text{Stab}^0(\Gamma_N Q)$ — by [22, Theorem 1.1] in the $A_n$ case for any $N$ (see also [13, Theorem 1.1] for $A_2$), or by [12, Theorem 1.1] and [30, Corollary 5.5] for arbitrary $Q$ and $N = 2$, the principal component $\text{Stab}^0(\Gamma_N Q)$ is the universal cover of $h^\text{reg}/W$ where $h^\text{reg}$ is the complement of the root hyperplanes in the Cartan subalgebra of the Lie algebra associated to the underlying Dynkin diagram, and $W$ the associated Weyl group.

Theorem 5.4 of [30] states that there is a surjective homomorphism

$$\text{Br}(Q) \longrightarrow \pi_1(\text{Stab}^0(\Gamma_N Q)/\text{Br}(\Gamma_N Q))$$

where $\text{Br}(Q)$ is the braid group of the Dynkin diagram underlying $Q$, and $\text{Br}(\Gamma_N Q)$ the Seidel–Thomas braid group, i.e. the subgroup of automorphisms of $D(\Gamma_N Q)$ generated by the spherical twist functors associated to the simple objects of the standard heart. Combining this with Corollary 4.10 we obtain a surjective homomorphism $\text{Br}(Q) \rightarrow \text{Br}(\Gamma_N Q)$. In the $A_n$ case, or generally for $N = 2$, this surjection is an isomorphism. Conjecture 5.7 of [30] is that there should be such an isomorphism for any $N$, and for any acyclic $Q$.

5 Two classes of examples

We discuss two classes of examples of triangulated categories in which each component of the stability space is of finite-type, and hence is contractible. Each class contains the bounded derived category of finite-dimensional representations of ADE Dynkin quivers, so these can be seen as two ways to generalise from these.

5.1 Locally-finite triangulated categories

We recall the definition of locally-finite triangulated category from [26]. Let $C$ be a triangulated category. The *abelianisation* $\text{Ab}(C)$ is the full subcategory...
of functors $F: \mathcal{C}^{\text{op}} \to \text{Ab}$ to the category of abelian groups on those additive functors fitting into an exact sequence

$$\text{Hom}_C(-, c) \to \text{Hom}_C(-, c') \to F \to 0$$

for some $c, c' \in \mathcal{C}$. The fully faithful Yoneda embedding $\mathcal{C} \to \text{Ab}(\mathcal{C})$ is the universal cohomological functor on $\mathcal{C}$, in the sense that any cohomological functor to an abelian category factors, essentially uniquely, as the Yoneda embedding followed by an exact functor. A triangulated category $\mathcal{C}$ is locally-finite if idempotents split and its abelianisation $\text{Ab}(\mathcal{C})$ is a length category. The following ‘internal’ characterisation is due to Auslander $[3, \text{Theorem 2.12}].$

**Proposition 5.1.** A triangulated category $\mathcal{C}$ with split idempotents is locally-finite if and only if for each $c \in \mathcal{C}$

1. there are only finitely many isomorphism classes of indecomposable objects $c' \in \mathcal{C}$ with $\text{Hom}_\mathcal{C}(c', c) \neq 0$;
2. for each indecomposable $c' \in \mathcal{C}$, the $\text{End}_\mathcal{C}(c')$-module $\text{Hom}_\mathcal{C}(c', c)$ has finite length.

The category $\mathcal{C}$ is locally-finite if and only if $\mathcal{C}^{\text{op}}$ is locally-finite so that the above properties are equivalent to the dual ones.

Locally-finite triangulated categories have many good properties: they have a Serre functor, they have Auslander–Reiten triangles, the inclusion of any thick subcategory has both left and right adjoints, any thick subcategory, or quotient thereof, is also locally-finite. See $[26, 2, 37]$ for further details.

**Lemma 5.2** (cf. $[16, \text{Proposition 6.1}]).$ Suppose that $\mathcal{C}$ is a locally-finite triangulated category $\mathcal{C}$ with $\text{rk} \mathcal{K}_\mathcal{C} < \infty$. Then any $t$–structure on $\mathcal{C}$ is algebraic, with only finitely many isomorphism classes of indecomposable objects in its heart.

**Proof.** Let $d$ be an object in the heart of a $t$–structure, and suppose it has infinitely many pairwise non-isomorphic subobjects. Write each of these as a direct sum of the indecomposable objects with non-zero morphisms to $d$. Since there are only finitely many isomorphism classes of such indecomposable objects, there must be one of them, $c$ say, such that $c^{\oplus k}$ appears in these decompositions for each $k = 1, 2, \ldots$. Hence $c^{\oplus k} \hookrightarrow d$ for each $k$, which contradicts the fact that $\text{Hom}_\mathcal{C}(c, d)$ has finite length as an $\text{End}_\mathcal{C}(c)$-module. We conclude that any object in the heart has only finitely many pairwise non-isomorphic subobjects. It follows that the heart is a length category. Since $\text{rk} \mathcal{K}_\mathcal{C} < \infty$ it has finitely many simple objects, and so is algebraic.

To see that there are only finitely many indecomposable objects (up to isomorphism) note that any indecomposable object in the heart has a simple quotient. There are only finitely many such simple objects, and each of these admits non-zero morphisms from only finitely many isomorphism classes of indecomposable objects.

**Remark 5.3.** Since a torsion theory is determined by its indecomposable objects it follows that a $t$–structure on $\mathcal{C}$ as above has only finitely many torsion structures on its heart, i.e. it has finite tilting type.

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Our default assumption that all categories are essentially small is necessary here.
Corollary 5.4. Suppose $C$ is a locally-finite triangulated category and that $\text{rk}KC < \infty$. Then the stability space is a (possibly empty) disjoint union of finite-type components, each of which is contractible.

Proof. Combining Lemma 5.2 with Lemma 4.1 shows that each component of the tilting poset is of finite-type. The result follows from Theorem 4.9.

Example 5.5. Let $Q$ be a quiver whose underlying graph is an ADE Dynkin diagram. Then the bounded derived category $D(Q)$ of finite-dimensional representations of $Q$ over an algebraically-closed field is a locally-finite triangulated category [23, §2]. The space $\text{Stab}(Q)$ of stability conditions is non-empty and connected (by Remark 5.18 or the results of [24]), and hence by Corollary 5.4 is contractible. This affirms the first part of Conjecture 5.8 in [30]. Previously $\text{Stab}(Q)$ was known to be simply-connected [30, Theorem 4.6].

Example 5.6. For $m \geq 1$ the cluster category $C_m(Q) = D(Q)/\Sigma_m$ is the quotient of $D(Q)$ by the automorphism $\Sigma_m = \tau^{-m}[m-1]$, where $\tau$ is the Auslander–Reiten translation. Each $C_m(Q)$ is locally-finite [26, §2], but $\text{Stab}(C_m(Q)) = \emptyset$ because there are no $t$–structures on $C_m(Q)$.

Remark 5.6 of [30] proposes that $\text{Stab}(\Gamma_N Q)/\text{Br}(\Gamma_N Q)$ should be considered as an appropriate substitute for the stability space of $C_{N-1}(Q)$. Our results show that the former is homotopy equivalent to the classifying space of the braid group $Br(\Gamma_N Q)$, which might be considered as further support for this point of view.

5.2 Discrete derived categories

This class of triangulated categories was introduced and classified by Vossieck [34]; we use the more explicit classification in [6]. The contractibility of the stability space, Corollary 5.8 below, follows from the results of this paper combined with the detailed analysis of $t$–structures on these categories in [16]. Theorem 7.1 of [17] provides an independent proof of the contractibility of $\text{BInt}(C)$ for a discrete derived category $C$, using the interpretation of $\text{Int}(C)$ in terms of the poset $P_2(C)$ of silting pairs (Remark 3.14). Combining this with Corollary 3.22 one obtains an alternative proof [17, Theorem 8.10] of the contractibility of the stability space.

Let $A$ be a finite-dimensional associative algebra over an algebraically-closed field. Let $D(A)$ be the bounded derived category of finite-dimensional right $A$-modules.

Definition 5.7. The derived category $D(A)$ is discrete if for each map (of sets) $\mu: \mathbb{Z} \to K(D(A))$ there are only finitely many isomorphism classes of objects $d \in D(A)$ with $[H^i d] = \mu(i)$ for all $i \in \mathbb{Z}$.

The derived category $D(Q)$ of a quiver whose underlying graph is an ADE Dynkin diagram is discrete. Theorem A of [6] states that if $D(A)$ is discrete but not of this type then it is equivalent as a triangulated category to $D(\Lambda(r, n, m))$ for some $n \geq r \geq 1$ and $m \geq 0$ where $\Lambda(r, n, m)$ is the path algebra of the bound quiver in Figure 4. Indeed, $D(A)$ is discrete if and only if $A$ is tilting-cotilting equivalent either to the path algebra of an ADE Dynkin quiver or to one of the $\Lambda(r, n, m)$. 32
Figure 1: The algebra $\Lambda(r,n,m)$ is the path algebra of the quiver $Q(r,n,m)$ above with relations $\gamma_{n-r+1}\gamma_{n-r+2} = \cdots = \gamma_n\gamma_1 = 0$.

Discrete derived categories form an interesting class of examples as they are intermediate between the locally-finite case considered in the previous section and derived categories of tame representation type algebras. More precisely, the distinctions are captured by the Krull–Gabriel dimension of the abelianisation, which measures how far the latter is from being a length category. In particular, $\text{KGdim } \text{Ab}(C) \leq 0$ if and only if $C$ is locally-finite [27]. Krause conjectures [27, Conjecture 4.8] that $\text{KGdim } \text{Ab}(D(A)) = 0$ or $1$ if and only if $D(A)$ is discrete. As evidence he shows that $\text{KGdim } \text{Ab}(D_b(\text{proj } k[\epsilon])) = 1$ where $\text{proj } k[\epsilon]$ is the full subcategory of finitely generated projective modules over the algebra $k[\epsilon]$ of dual numbers. The category $D_b(\text{proj } k[\epsilon])$ is discrete — there are infinitely many indecomposable objects, even up to shift, but no continuous families — but not locally-finite. Finally, by [20, Theorem 4.3] $\text{KGdim } (D(A)) = 2$ when $A$ is a tame hereditary Artin algebra, for example the path algebra of the Kronecker quiver $K$.

Since the Dynkin case was covered in the previous section we restrict to the categories $D(\Lambda(r,n,m))$. These have finite global dimension if and only if $r < n$, and we further restrict to this situation.

**Corollary 5.8** (cf. [17, Theorem 8.10]). Suppose $C = D(\Lambda(r,n,m))$, where $n > r \geq 1$ and $m \geq 0$. Then the stability space $\text{Stab}(C)$ is contractible.

*Proof.* By [10] Proposition 6.1 any $t$–structure on $C$ is algebraic with only finitely many isomorphism classes of indecomposable objects in its heart. Lemma 4.1 then shows that each component of the tilting poset has finite-type. By Theorem 4.9 $\text{Stab}(C) = \text{Stab}_{\text{alg}}(C)$, and is a union of contractible components. By Lemma 3.17 $\text{Stab}_{\text{alg}}(C)$ is connected. Hence $\text{Stab}(C)$ is contractible. □

**Example 5.9.** The space of stability conditions in the simplest case, $(n,r,m) = (2,1,0)$, was computed in [35] and shown to be $\mathbb{C}^2$. (The category was described geometrically in [35], as the constructible derived category of $\mathbb{P}^1$ stratified by a point and its complement, but it is known that in this case the constructible derived category is equivalent to the derived category of the perverse sheaves, and these have a nearby / vanishing-cycle description as representations of the quiver $Q(2,1,0)$ with relation $\gamma_2\gamma_1 = 0$.)

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