Energy minimizing brittle crack propagation II

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15.11.1997

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Note to the reader

This paper is an alternative version of the published paper [Bu3], containing more results obtained in [Bu1] and no numerical results. The paper is dated by the last modification of the file.

1 Introduction

This paper is devoted to the study of quasi-static brittle crack evolution. We work under the following assumptions: a linear elastic body, with or without initial cracks inside, evolves in a quasi-static manner under an imposed path of boundary displacements. During its evolution cracks with unprescribed geometry may appear and/or grow.

The difficulty of brittle crack propagation problems consists in the nature of the main unknown: the crack itself, at various moments in time. The research in this field concerns mainly the constitutive behaviour of a brittle material, like the basic paper of Griffith [G]. The essential was stated in papers like: Eshelby [Es], Irwin [I], Gurtin [Gu1], [Gu2].

In almost all the studies the geometry of the crack is prescribed. There are few exceptions, as the papers of Ohtsuka [Oht1—3] or Stumpf & Le [Stle]. The geometry of the crack can be prescribed in a strong form, like in the case of a plane rectangular or elliptic crack which is supposed to remain plane rectangular or elliptic during its growth. In a weak form, the geometry of the crack can be prescribed by the assumption that the configuration of the body is 2 dimensional and the crack is supposed to have only an edge, which is a point. In this case the evolution of the crack is conveniently reduced to the evolution of a point. Under these assumptions the geometrical nature of the main unknown is obscured.

A new direction of research in brittle fracture mechanics begins with the article of Mumford & Shah [MS] regarding the problem of image segmentation. This problem, which consists in finding the set of edges of a picture and constructing a smoothed version of that picture, it turns to be intimately related to the problem of brittle crack evolution. In the before mentioned article Mumford and Shah propose the following variational approach to the problem of image segmentation: let \( g : \Omega \subset \mathbb{R}^2 \rightarrow [0,1] \) be the original picture, given as a distribution of grey levels (1 is white and 0 is black), let \( u : \Omega \rightarrow \mathbb{R} \) be the smoothed picture and \( K \) be the set of edges. \( K \) represents the set where \( u \) has jumps, i.e. \( u \in C^1(\Omega \setminus K, \mathbb{R}) \). The pair formed by the smoothed picture \( u \) and the set of edges \( K \) minimizes then the functional:

\[
I(u, K) = \int_{\Omega} \alpha \ | \nabla u \|^2 \ dx + \int_{\Omega} \beta \ | u - g \|^2 \ dx + \gamma H^1(K) .
\]

The parameter \( \alpha \) controls the smoothness of the new picture \( u \), \( \beta \) controls the \( L^2 \) distance between the smoothed picture and the original one and \( \gamma \) controls the total length of the edges given by this variational method. The authors remark that
for $\beta = 0$ the functional $I$ might be useful for an energetic treatment of fracture mechanics.

An energetic approach to fracture mechanics is naturally suited to explain brittle crack appearance under imposed boundary displacements. The idea is presented in the followings.

The state of a brittle body is described by a pair displacement-crack. $(u, K)$ is such a pair if $K$ is a crack — seen as a surface — which appears in the body and $u$ is a displacement of the broken body under the imposed boundary displacement, i.e. $u$ is continuous in the exterior of the surface $K$ and $u$ equals the imposed displacement $u_0$ on the exterior boundary of the body.

Let us suppose that the total energy of the body is a Mumford-Shah functional of the form:

$$E(u, K) = \int_{\Omega} w(\nabla u) \, dx + F(u_0, K).$$

The first term of the functional $E$ represents the elastic energy of the body with the displacement $u$. The second term represents the energy consumed to produce the crack $K$ in the body, with the boundary displacement $u_0$ as parameter. Then the crack that appears is supposed to be the second term of the pair $(u, K)$ which minimizes the total energy $E$.

We shall use this idea in the study of quasi-static brittle crack evolution. For doing this, we proceed to a time discretization which transforms the problem of crack evolution into a sequence of energy minimization problems. Francfort & Marigo [Fma] proceed in the same way in the case of brittle brutal damage evolution. However, it is only a belief that when the time step goes to zero, the discretized evolution converges to an almost continuous (with respect to time) evolution. We have found in the frame of generalized minimizing movements, introduced by De Giorgi [DG], stronger mathematical reasons to support this belief. That is why we introduce the notion of energy minimizing movement as a particular case of a generalized minimizing movement.

In section 2, the notion of energy minimizing movement is introduced in a form useful in the sequel. After preliminaries concerning the statics of a brittle body, the Griffith criterion of brittle crack propagation is presented in subsection 3.3., as a selection criterion amongst all possible crack evolutions. At the end of this section we formulate the problem of quasi-static brittle crack evolution in the form (3.3.8).

In section 4, we give an energy minimizing movement formulation to this problem by using a Mumford-Shah energy functional (definition 4.1.). We are assured about the existence of the discretized (or incremental) solution to that problem by theorem 4.1. (in a weak form presented in the section of proofs). In subsection 4.2. we investigate the features of this first model. In this model we have only one material constant connected to fracture, namely the constant of Griffith $G$. We find exact solutions and useful estimations in the case of anti-plane displacements (theorem 4.2.), which tell us that crack appearance is allowed in this model. A part of this results can be found in [Bu2], in connection with fiber-matrix debonding in
composites. We prove also a bad feature of the model: the critical stress which lead to fracture in an uni-dimensional traction experiment is not a constant of material.

In section 5. is presented an improved model, based on a preliminary study of smooth brittle crack propagation (see also [Bu1], [Bu3]). The Griffith criterion is reformulated by using proposition 5.1. and the $K2$ functional (definition 5.3.). The $K2$ functional is a generalized version of the J integral of Rice [R]. After we modify the Griffith criterion by the extension of the functional $K2$, we obtain the differential global criterion of brittle crack appearance (DA). A stronger version of (DA) is the local crack appearance criterion (LA). In subsection 5.4. the improved model is presented. In this model we have two constants of material connected to fracture: $G$ and a quantity with dimension of stress named $\Sigma$. The critical stress which lead to fracture is deduced from the the elastic constants and $\Sigma$, hence this time it is a constant of material.

Section 6. is devoted to a brief introduction to special functions with bounded variation or deformation. Weak versions of theorems 4.1. and 5.2., concerning the existence of the discretized (or incremental) solutions of the models presented here, are given as consequences of more general results due to De Giorgi & Ambrosio [DGA], Ambrosio [A1—3], Belletini, Coscia & Dal Maso, [BCDM].

In section 7. are given the conclusions regarding the features of the two models presented in the paper. We prove a general existence result of the energy minimizing movement described in the first model under the assumption of uniformly bounded power communicated by the rest of the universe to the body. A comparation is made with the model of Ambrosio & Braides [AB], based also on generalized minimizing movements, where the crack appearance is forbidden.

2 General energy minimizing movements

An energy minimizing movement is a particular case of a generalized minimizing movement. The latter notion has been introduced by De Giorgi in [DG], inspired by the paper [ATW] of Almgren, Taylor & Wang. The definition of a generalized minimizing movement is (according to Ambrosio [Amb]) the following:

**Definition 2.1.** Let $S$ be a topological space and

$$F : (1, +\infty) \times N \times S \times S \to \mathbb{R} \cup \{+\infty\}$$

be a function. For any $u_0 \in S$, a function $u : [0, +\infty) \to S$ is a generalized minimizing movement associated to $F$ with initial data $u_0$, if there exists a diverging sequence $(s_i)_{i \in N}$, $s_i > 1$, and there are functions $u_i : N \to S$ such that:

i) $u_i(0) = u_0$;

ii) for any $k \in N$ and any $i$, $u_i(k + 1)$ minimizes the functional

$$v \mapsto F(s_i, k, v, u_i(k))$$

over $S$;
iii) for any $t \geq 0$, $u_i([s,t]) \to u(t)$ in $S$ as $i \to +\infty$.

The canonical example of (generalized) minimizing movement is given by the choice: $S = \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$ Lipschitz continuous and $C^2$ and

$$F(s, k, u, v) = f(u) + \frac{s}{2} | u - v |^2.$$ 

In this case, for any $u_0 \in \mathbb{R}^n$ there is only one minimizing movement, namely the unique solution of the Cauchy problem

$$u'(t) = -\nabla f(u(t)) , u(0) = u_0 .$$

An energy minimizing movement is a generalized minimizing movement associated to a particular function $F$. It is designed to be a "weak stable" solution of an evolution problem of the following type:

$$\begin{cases}
A (u(t), \alpha(t), t) = 0 & \forall \ t \geq 0 \\
\frac{d}{dt} \alpha(t) \leq L (\alpha(t), u(t)) & \forall \ t \geq 0 \\
u(0) = u_0 , \alpha(0) = \alpha_0.
\end{cases} \quad (2.0.1)$$

There are two unknowns in this problem: $u$ and $\alpha$. The evolution of these unknowns is quasi-static with respect to $u$. Suppose that we are not in position to give a proper law of evolution of $\alpha$, or that the law of evolution that we have gives too many solutions. Assume further that we have instead at our disposal the expression of the total energy of the system described by the pair $(u, \alpha)$, name it $f(u, \alpha)$, and a set of constraints, not in a differential form, upon the evolution of $\alpha$. We can make then a time discretization with time step $\delta$ and recursively find $(u_\delta^k, \alpha_\delta^k)$ from $(u_\delta^k, \alpha_\delta^k)$, by a minimization process of the total energy $f$ under some constraints. A weak stable solution of the previous problem will be a limit of sequences $(u_\delta^k, \alpha_\delta^k)_k$ when the time step $\delta$ converges to 0.

In the further definition $S$ may be seen as the space of all pairs $x = (u, \alpha)$, endowed with a topology.

**Definition 2.2.** Let $S$ be a topological space and

$$F : (1, +\infty) \times N \times S \times S \to \mathbb{R} \cup \{+\infty\} ,$$

$$F(s, k, x, y) = f(s, x, y) + \psi(k/s, y)$$

be a function, with $f : N \times S \times S \to \mathbb{R}$ and $\psi : [0, \infty) \times S \to [0, +\infty)$. For any $x_0 \in S$, a generalized minimizing movement $x : [0, +\infty) \to S$ associated to $F$ with initial datum $x_0$ is an energy minimizing movement associated to the energy $f$ with the constraints $\psi$ and initial datum $x_0$.

Let us denote by $S(\lambda)$ the following set:
\[ S(\lambda) = \{ y \in S : \psi(\lambda, y) = 0 \} . \]

From definition 2.2. we see that \( x : [0, +\infty) \to S \) is an energy minimizing evolution associated to \( f \), with the constraints \( \psi \) and initial data \( x_0 \) if there exists a diverging sequence \( (s_i)_{i \in \mathbb{N}}, s_i > 1 \), and there are functions \( x_i : \mathbb{N} \to S \) such that:

i) \( x_i(0) = x_0 \);

ii) for any \( k \in \mathbb{N} \) and any \( i \in \mathbb{N} \), \( x_i(k + 1) \) minimizes the functional \( f \) over the set \( S(k/s_i) \) (in particular \( x_i(k + 1) \) belongs to \( S(k/s_i) \));

iii) for any \( t > 0 \), \( x_i([s_it]) \to x(t) \) in \( S \) as \( i \to +\infty \).

3 Notations and preliminaries

3.1 Notations and constitutive assumptions

The open bounded set \( \Omega \subset \mathbb{R}^3 \) represents the reference configuration of an elastic body and \( u : \Omega \to \mathbb{R}^3 \) is the displacement field of the body with respect to this configuration. We shall always suppose, without mentioning further, that the open set \( \Omega \) and its closure have the same topological boundary.

The expression of the elastic (or free) energy of the body is:

\[
\int_{\Omega} w(\nabla u) \, dx .
\]

The first Piola-Kirchoff stress tensor \( S \) is

\[
S(u) = \frac{dW}{d\nabla}(\nabla u)
\]

and the equilibrium equation of the body in the absence of volumic forces is

\[
div S(u) = 0 \text{ in } \Omega .
\]

In this paper we consider that the body is linear elastic and homogeneous, i.e. the function \( w(u) \) has the form:

\[
w(u) = \frac{1}{2} C \nabla u : \nabla u ,
\]

with the elasticity 4-tensor \( C \) having the symmetries:

\[
C_{ijkl} = C_{jikl} = C_{klij} .
\]

Under these assumptions the stress tensor \( S \) becomes the Cauchy stress tensor:

\[
\sigma = \sigma(u) = C \nabla u = C \epsilon(u) ,
\]
where \( \epsilon(u) \) is the symmetric part of \( \nabla u \), i.e.

\[
\epsilon(u) = \frac{1}{2} \left( \nabla u + (\nabla u)^T \right).
\]

We shall suppose moreover that \( w \) satisfies the growth conditions:

\[
\forall F \in \mathbb{R}^3, \quad F = F^T, \quad c |F|^2 \leq w(F) \leq C |F|^2,
\]

where \( c \) and \( C \) belong to \((0, +\infty)\).

In the case of plane displacement the domain \( \Omega \subset \mathbb{R}^2 \) represents a section in the cylindrical reference configuration of the body \( \Omega \times R \) and \( u : \Omega \to \mathbb{R}^2 \) is a plane displacement. The displacement with respect to the 3 dimensional configuration of the body has the following expression:

\[
(x_1, x_2, x_3) \in \Omega \times \mathbb{R} \mapsto (u_1(x_1, x_2), u_2(x_1, x_2), 0) \in \mathbb{R}^3.
\]

In this case we suppose that the body is linear elastic, homogeneous and isotropic.

In the case of anti-plane displacements the domain \( \Omega \subset \mathbb{R}^2 \) represents a section in the cylindrical reference configuration of the body \( \Omega \times R \) too. The anti-plane displacement is a function \( u : \Omega \to R \). The 3 dimensional displacement has the following form:

\[
(x_1, x_2, x_3) \in \Omega \times \mathbb{R} \mapsto (0, 0, u(x_1, x_2)) \in \mathbb{R}^3.
\]

In this case we make the same assumption of isotropic body, therefore the elastic energy takes the form:

\[
\int_{\Omega} \mu | \nabla u |^2 \, dx,
\]

where \( \mu \) is one of the two Lamé’s constants.

### 3.2 Statics of a fractured elastic body

For any measurable set \( B \subset \mathbb{R}^n \), \( |B| = \mathcal{L}^n(B) \) denotes the Lebesgue measure of \( B \) and \( \mathcal{H}^k(B) \) denotes the \( k \) dimensional Hausdorff measure of \( B \).

By a crack set in the body \( \Omega \) we mean (according with Ball [Ba]) a topologically closed countably rectifiable set, generically denoted by \( K \).

Given the function \( f \), a point \( x \in \Omega \) and an unitary vector (or direction) \( n \in \mathbb{R}^n \), the approximate limit of \( f \) in \( x \) with respect to the direction \( n \) is denoted by \( \tilde{f}(x, n) \) and it is defined by the following expression:

\[
\lim_{\rho \to 0^+} \frac{\int_{B_\rho(x) \cap \{ y : (y-x) \cdot n \geq 0 \}} | f(y) - \tilde{f}(x, n) | \, dy}{|B_\rho(x) \cap \{ y : (y-x) \cdot n \geq 0 \}|} = 0.
\]

Whenever a field of normals at \( K \) is chosen, the lateral limits \( f^+ \) and \( f^- \) of any function \( f : \Omega \setminus K \to \mathbb{R}^n \) are \( f^+ : K \to R \) and \( f^- : K \to R \), defined by

\[
f^+(x) = \tilde{f}(x, n(x)), \quad f^-(x) = \tilde{f}(x, -n(x)).
\]
This means that $f^+$ and $f^-$ satisfy the equalities:

$$
\forall x \in K, \quad \lim_{\rho \to 0^+} \frac{\int_{B_{\rho}(x) \cap \{y : (y-x) \cdot n \geq 0\}} |f(y) - f^+(x)| \, dy}{|B_{\rho}(x) \cap \{y : (y-x) \cdot n \geq 0\}|} = 0 ,
$$

$$
\forall x \in K, \quad \lim_{\rho \to 0^+} \frac{\int_{B_{\rho}(x) \cap \{y : (y-x) \cdot n \leq 0\}} |f(y) - f^-(x)| \, dy}{|B_{\rho}(x) \cap \{y : (y-x) \cdot n \leq 0\}|} = 0 .
$$

Remark that for any $x \in K$ the object $(f^+(x), f^-(x), n(x))$ it is unique to a change of signs, i.e.

$$(f^+(x), f^-(x), n(x)) \sim (f^-(x), f^+(x), -n(x)) .$$

For a given crack set $K$ in $\Omega$, by an admissible displacement with respect to $K$ we mean a function $u : \Omega \setminus K \to \mathbb{R}^k$ (where $k$ might be 1,2 or 3) which is $C^1$ and posses continuous lateral limits on $K$. In this section we shall consider the space $W^{1,2}(\Omega \setminus K) \cap L^\infty(\Omega)$ as the set of weak admissible displacements with respect to the crack set $K$.

Let $n$ be the dimension of the reference configuration $\Omega$. For a given $u_0 \in H^2(\partial \Omega, R^n) \cap L^\infty(\partial \Omega, R^n)$ and for a given rectifiable crack set $K$, such that $\mathcal{H}^{n-1}(\partial \Omega \setminus K) > 0$, the following problem has a solution $u = u(u_0, K)$, unique to rigid displacements of $\Omega \setminus K$ equals to 0 on $\partial \Omega$:

\[
\begin{cases}
    \text{div } \sigma(u) = 0 & \text{in } \Omega \setminus K \\
    \sigma^+(u)n = \sigma^-(u)n = 0 & \text{on } K \\
    u = u_0 & \text{on } \partial \Omega \setminus K.
\end{cases}
\]  

We use the same notation — $u = u(u_0, K)$ — in the anti-plane case, when $n = 2$, $k = 1$ and the problem becomes

\[
\begin{cases}
    \mu \text{ div } \nabla u = 0 & \text{in } \Omega \setminus K \\
    (\nabla u)^+n = (\nabla u^-n = 0 & \text{on } K \\
    u = u_0 & \text{on } \partial \Omega \setminus K.
\end{cases}
\]  

The solution $u(u_0, K)$ of the problem minimizes the functional

$$E(v) = \int_\Omega w(\nabla v) \, dx$$

over the following set of weak admissible displacements with respect to the crack set $K$:

$\{ v \in W^{1,2}(\Omega \setminus K, R^n) \cap L^\infty(\Omega, R^n) : v = u_0 \text{ on } \partial \Omega \setminus K \}$ .

By standard arguments it follows that the functional

$$v \in W^{1,2}(\Omega, R^n) \mapsto \int_\Omega \sigma(u_0, K) : \nabla v \, dx$$

is lower semicontinuous and coercive.

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depends only on the trace of \( v \) on \( \partial \Omega \), hence it give raise to the linear continuous function:

\[
T(K) : H^\frac{1}{2}(\partial \Omega, \mathbb{R}^n) \cap L^\infty(\partial \Omega, \mathbb{R}^n) \to H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^n),
\]

\[
\langle T(K)u_0, v \rangle = \int _\Omega \sigma(u(u_0, K)) : \nabla v' \, dx \quad \text{for any } v' = v \text{ on } \partial \Omega.
\] (3.2.3)

In the latter definition \( \langle \cdot, \cdot \rangle \) is the duality product of the pair of spaces \( H^\frac{1}{2}(\partial \Omega, \mathbb{R}^n) \) and \( H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^n) \). The function \( T(K) \) is called the Dirichlet-to-Neumann map of the elastic body \( \Omega \) with the crack set \( K \).

Under the assumptions concerning the elastic energy density \( w \), more precise because of the symmetries of the elasticity tensor \( C \), the function \( T(K) \) is also self-adjoint, i.e. for any \( u, v \) we have

\[
\langle T(K)u, v \rangle = \langle T(K)v, u \rangle.
\]

In the same way can be defined the Dirichlet-to Neumann map associated to the problem (3.2.2).

Remark finally that, under the assumptions considered for \( w \), the elastic energy of the body can be expressed using the Dirichlet-to-Neumann map. Indeed, we have:

\[
\int _\Omega w(\nabla u(u_0, K)) \, dx = \frac{1}{2} \langle T(K)u_0, u_0 \rangle.
\]

3.3 The Griffith criterion of brittle crack propagation

Let us consider in the elastic body \( \Omega \) an initial crack set \( K_0 \) which evolves and becomes at the moment \( t \) the crack set \( K_t \). We assume that the crack set always increase in time, i.e.

\[
\forall 0 < t < t', \quad K_t \subset K_{t'}.
\] (3.3.4)

We suppose that the evolution of the body is quasi-static. At the moment \( t \) the state of the body is characterized by the pair \( (u(t), K_t) \), where \( u(t) \) is the displacement of the body, admissible with respect to \( K_t \). Let us denote by \( u_0(t) \) the trace of \( u(t) \) on \( \partial \Omega \). We have then the equality \( u(t) = u(u_0(t), K_t) \).

The power given to the body by the rest of the universe at the moment \( t \) has the following expression:

\[
P(t) = \int _{\partial \Omega} S(u(t)) u \cdot \dot{u}_0(t) \, dx = \langle T(K_t)u_0(t), \dot{u}_0(t) \rangle.
\]

Let us consider a given curve \( t \mapsto (u(t), K_t) \), such that for any \( t \) we have \( u(t) = u(u_0(t), K_t) \). For a given \( t \) we introduce the following curve of displacements:

\[
\forall \tau \geq 0, \quad w(\tau) = u(u_0(t + \tau), K_t).
\]
\( w(\tau) \) represents the displacement the body at the moment \( t + \tau \) in the presence of the crack \( K_t \). An easy calculation lead us to the equality:

\[
\frac{d}{d\tau} \int_{\Omega} w(\nabla w(\tau)) \, dx |_{\tau=0} = P(t) .
\]  

Therefore \( P(t) \) represents the power consumed at the moment \( t \) by the body in order to modify its displacement, constrained to follow the path of imposed boundary displacements \( t \mapsto u_0(t) \), without any modification of the actual crack set \( K_t \).

The Griffith criterion of brittle crack propagation asserts that during the propagation of the crack \( K_t \) the following inequality is true at any moment \( t \):

\[
\frac{d}{dt} \left\{ \int_{\Omega} w(\nabla u(t)) \, dx + G\mathcal{H}^{n-1}(K_t) \right\} \leq P(t) .
\]  

Here \( G \) is the constant of Griffith, supposed to be a material constant.

The relation (3.3.6) can be written in a different form using the Dirichlet-to-Neumann map \( T(K_t) \). Let us assume that the crack evolution is smooth in the sense that the function \( t \mapsto T(K_t) \) is differentiable, i.e. the Dirichlet-to-Neumann map varies smoothly in time. The Griffith criterion takes the following form:

\[
\frac{1}{2} \frac{d}{dt} [T(K_t)] u_0(t), u_0(t) + \frac{1}{2} \langle T(K_t) \dot{u}_0(t), u_0(t) \rangle + G \frac{d}{dt} \{ \mathcal{H}^{n-1}(K_t) \} \leq \langle T(K_t) u_0(t), \dot{u}_0(t) \rangle .
\]  

The function \( T(K_t) \) is self-adjoint, therefore we obtain the following expression of the Griffith criterion:

\[
\frac{1}{2} \frac{d}{dt} [T(K_t)] u_0(t), u_0(t) + G \frac{d}{dt} \{ \mathcal{H}^{n-1}(K_t) \} \leq 0 .
\]  

We can see that we have the following equality:

\[
P(t) - \frac{d}{dt} \int_{\Omega} w(\nabla u(t)) \, dx = -\frac{1}{2} \frac{d}{dt} [T(K_t)] u_0(t), u_0(t) .
\]

The quantity from the left of the previous equality is usually called the energy release rate due only to crack propagation.

It is obvious that \( u_0(t) \) plays the role of a time-dependent parameter, since in the last inequality \( \dot{u}_0(t) \) does not appear.

The problem of quasi-static brittle propagation of an initial crack in an elastic body under a time-dependent imposed displacement \( u_0(t) \) can be formally put in the form (2.0.1). If we put apart the constraint (3.3.4), we have the following formulation:

\[
\left\{ \begin{array}{ll}
\mathbf{u}(t) - \mathbf{u}(u_0(t), K_t) &= 0 \\
\frac{1}{2} [\frac{d}{dt} [T(K_t)]] \mathbf{u}_0(t), \mathbf{u}_0(t) + G \frac{d}{dt} \{ \mathcal{H}^{n-1}(K_t) \} &= \leq 0 \\
\mathbf{u}(0) &= \mathbf{u}_0 , \quad K_0 = K .
\end{array} \right. \]  

\[ (3.3.8) \]
4 The first model

In the left term of the Griffith criterion (3.3.6) appears the time-derivative of an energetic functional. Let us consider the following set of admissible pairs displacement-crack:

\[
M = \{(u, K) : K \text{ is a crack set and } u \in C^1(\Omega \setminus K, \mathbb{R}^n) \text{ such that } (u^+, u^-, n) \text{ exists on } K\}.
\]

The Mumford-Shah energy functional over \(M\) has the following expression:

\[
I : M \rightarrow \mathbb{R} \cup \{+\infty\}, \quad I(u, K) = \int_\Omega w(\nabla u) \, dx + G\mathcal{H}^{n-1}(K) \quad (4.0.1)
\]

4.1 Introduction of the first model

It is natural to try to give an energy minimizing formulation to the problem (3.3.8) by using the functional defined at (4.0.1). According to definition 2.2. and the constraint (3.3.4), we give the following formulation:

**Definition 4.1.** Let us consider the space \(M\) endowed with the topology given by the convergence:

\[
(u_h, K_h) \rightarrow (u, K) \text{ if } \begin{cases}
u h L^2 \rightarrow u \\
\mathcal{H}^{n-1}(K_h \Delta K) \rightarrow 0
\end{cases}.
\]

We define the functions

\[
J : M \times M \rightarrow \mathbb{R}, \quad J((u, K), (v, L)) = \int_\Omega w(\nabla v) \, dx + G\mathcal{H}^{n-1}(L \setminus K),
\]

\[
\Psi : [0, \infty) \times M \rightarrow \{0, +\infty\}, \quad \Psi(\lambda, (v, K)) = \begin{cases} 0 & \text{if } v = u_0(\lambda) \text{ on } \partial \Omega \setminus K \\ +\infty & \text{otherwise}
\end{cases}.
\]

We consider the initial data \((u_0, K) \in M\) such that \(u_0 = u(u_0(0), K)\).

For any \(s \geq 1\) we recursively define \((u^s, K^s) : N \rightarrow M\) like this:

i) \((u^s, K^s)(0) = (u_0, K)\);

ii) for any \(k \in N\) \(u^s, L^s)(k+1) \in M\) minimizes the functional

\[
(v, L) \in M \mapsto J(((u^s, K^s)(k), (v, L)) + \Psi((k+1)/s, (v, L))
\]

over \(M\). In order to verify the constraint (3.3.4), \(K^s(k+1)\) is defined by the formula:

\[
K^s(k+1) = K^s(k) \cup L^s(k+1).
\]

An energy minimizing movement associated to \(J\) with the constraints (3.3.4), \(\Psi\) and initial data \((u_0, K)\) is any \((u, K) : [0, +\infty) \rightarrow M\) having the property: there is a diverging sequence \((s_i)\) such that for any \(t > 0\)

\[
(u^s, K^s)([s_i, t]) \rightarrow (u, K)(t) \text{ as } i \rightarrow \infty.
\]
In the previous definition \(1/s\) is the step of the discretization of the time variable, hence \((u^s(k), K^s(k))\) represents the approximate pair displacement-crack at the time \(k/s\). We name any function \((u^s, K^s) : N \rightarrow M\) an incremental solution if it verifies i) and ii) from the definition 4.1.

When \(s_i\) converges to \(\infty\) the time step goes to 0 and the incremental solution \((u^s_i, K^s_i)((s_i; t))\) converges to \((u, K)(t)\), for any \(t > 0\).

A necessary condition for the existence of an energy minimizing movement introduced in definition 4.1. is that for any given \(s\) the incremental solution \(k \in N \mapsto (u^s, K^s)(k)\) exists. The following theorem provides an answer to this existence query. In the general case \(n = 3\) this theorem is true, to our knowledge, only in a weak form, presented in section 6. In the anti-plane case, however, due to partial regularity results for the minimizers of Mumford-Shah functional from [DGCL], the theorem is certainly true.

**Theorem 4.1.** Let \(\Omega \subset \mathbb{R}^n\) be a bounded open set with piecewise smooth boundary, let \((u_0, K)\) be a given admissible pair displacement-crack in \(\Omega\) and let

\[ u_0 : N \rightarrow H^{1/2}(\partial \Omega, R^n) \cap L^\infty(\partial \Omega, R^n) \]

be a given sequence of imposed displacements such that \(u_0 = u(u_0(0), K)\) on \(\partial \Omega \setminus K\).

Then there exists the sequence \((u, K) : N \rightarrow M\) such that:

1) \(u(0) = u_0\) and \(K(0) = K\);

2) for any \(k \in N\) there is a crack set \(L(k + 1)\) such that \((u(k + 1), L(k + 1)) \in M\), \(u(k + 1) = u_0(k + 1)\) on \(\partial \Omega \setminus L(k + 1)\) and \((u(k + 1), L(k + 1))\) is a minimizer of the functional

\[ (v, L) \in M, v = u_0(k + 1) \text{ on } \partial \Omega \setminus L \mapsto J((u(k), K(k)), v, L) \]

The set \(K(k + 1)\) is given by the formula

\[ K(k + 1) = K(k) \cup L(k + 1) \]

### 4.2 Features of the first model

We shall investigate further the behaviour of the model proposed in definition 4.1. in the particular case of anti-plane displacement. There are some obvious adjustments to be made. \(\Omega\) is now a bounded domain in \(R^2\) and the displacement is a scalar function \(u\). The functional \(J\) will take the form:

\[ J((u, K), (v, L)) = \int_\Omega \mu \ | \nabla v |^2 \ dx + GH^1(L \setminus K) \]
Let us consider a particular type of imposed displacement on $\partial \Omega$. We split the boundary of the body in three parts:

$$\partial \Omega = \Gamma^1_u \cup \Gamma^2_u \cup \Gamma_f$$

$$\Gamma^i_u \cap \Gamma_f = \emptyset , \Gamma^i_u \cap \Gamma^j_u = \emptyset , \mathcal{H}^1(\Gamma^i_u) \cdot \mathcal{H}^1(\Gamma^2_u) \cdot \mathcal{H}^1(\Gamma_f) > 0 .$$

At any moment $t \geq 0$, $\Gamma_f$ is force free, i.e. the displacement is not prescribed on this part of the boundary. On $\Gamma^1_u$ and $\Gamma^2_u$ the imposed displacement is defined by the formula:

$$u_0(t)(x) = \begin{cases} 0 & \text{on } \Gamma^1_u \\ t\delta & \text{on } \Gamma^2_u , \end{cases}$$

where $\delta$ is a positive constant with dimension of speed. This displacement is homogeneous with respect to the time variable:

$$\forall t > 0 , u_0(t) = tu_0(1) .$$

We suppose further that at the moment $t = 0$ there are no cracks in the body. This assumption takes the form $K = \emptyset$. At $t = 0$ we have $u_0(0) = 0$, hence the initial data are $(u_0 = 0, K = \emptyset)$.

Let us consider a time discretization given by the parameter $1/s$ and the incremental solution $k \in N \mapsto (u^k, K^k)(k)$ introduced in definition 4.1. for the initial data and the imposed boundary described above. In order to shorten the notations we shall omit for the moment the superscript $s$.

The incremental solution $(u, K) : N \rightarrow M$ is recursively defined by the following two rules:

i) $u(0) = 0$ and $K(0) = \emptyset$;

ii) for any $k \in N$ we seek for the crack set $L(k+1)$ and for the displacement $u(k+1)$ such that $(u(k+1), L(k+1)) \in M$, $u(k+1) = (k+1)/s \ u_0(1)$ on $(\Gamma^1_u \cup \Gamma^2_u) \setminus L(k+1)$ and $(u(k+1), L(k+1))$ is a minimizer of the functional

$$(v, L) \in M , v = (k+1)/s \ u_0(1) \text{ on } (\Gamma^1_u \cup \Gamma^2_u) \setminus L \mapsto J((u(k), K(k)), (v, L)) .$$

The set $K(k+1)$ is given by the formula

$$K(k+1) = K(k) \cup L(k+1) .$$

Let us denote by $u_0$ the displacement of the body $\Omega$, without cracks, under the prescribed displacement on the boundary $u_0(1)$. With the use of a notation made before, $u_0$ is defined by $u_0 = u(0(1), \emptyset)$. For any $k \in N$ we have $(k/s \ u_0, \emptyset) \in M$ and $k/s \ u_0 = k/s \ u_0(1)$ on $\Gamma^1_u \cup \Gamma^2_u$. Therefore, for any $k \in N$ we have

$$J((u(k), K(k)), (u(k+1), L(k+1))) \leq J((u(k), K(k)), ((k+1)/s \ u_0, \emptyset)) .$$

The last inequality reads:

$$\int_{\Omega} \mu \ | \nabla u(k+1) |^2 \ dx + G\mathcal{H}^1(L(k+1) \setminus K(k)) \leq \left( \frac{k}{s} \right)^2 \int_{\Omega} \mu \ | \nabla u_0 |^2 \ dx .$$

(4.2.2)
We can always find a curve in $\overline{\Omega}$ which separates $\Gamma_1^u$ from $\Gamma_2^u$. Moreover, we can find such a curve which is a length minimizer in the family of all curves in $\overline{\Omega}$ separating $\Gamma_1^u$ from $\Gamma_2^u$. Let us denote this curve by $S$ (which exists but it might not be unique). The domain $\overline{\Omega}$ has the following decomposition with respect to $S$:

$$\overline{\Omega} = \Omega^1 \cup \Omega^2 , \Gamma_1^u \subset \Omega^1 , \Gamma_2^u \subset \Omega^2 , \Omega^1 \cap \Omega^2 = \emptyset ,$$

Let us define the following displacement:

$$u_S(x) = \begin{cases} 0 & x \in \Omega^1 \\ \delta & x \in \Omega^2 \end{cases}$$

It is easy to see that for any $k \in \mathbb{N}$ we have $(k/s u_S, S) \in M$ and $k/s u_S = k/s u_0$ on $(\Gamma_1^u \cup \Gamma_2^u) \setminus S$. Therefore we obtain the following inequality:

$$\int_{\Omega} \mu | \nabla u(k+1) |^2 \, dx + GH^1(L(k+1) \setminus K(k)) \leq GH^1(S \setminus K(k)) \quad (4.2.3)$$

From (4.2.3) we derive the following conclusion: for large time $k/s$ the crack set $K(k)$ is not void. Indeed, suppose that the function $k \in \mathbb{N} \mapsto (k/s u_0, \emptyset)$ is an incremental solution constructed by the rules i) and ii) above. Then for any $k \in \mathbb{N}$ the inequality (4.2.2) becomes an equality and the inequality (4.2.3) takes the following form:

$$\left( \frac{k}{s} \right)^2 \int_{\Omega} \mu | \nabla u_0 |^2 \, dx \leq GH^1(S \setminus K(k)) \ , \quad (4.2.4)$$

which lead to contradiction. Therefore this model can predict crack appearance. The critical step $k$, after which a crack appears in the body (as the incremental solution predict), is the greatest natural with the property (4.2.4).

The following theorem contains stronger informations regarding the minimizers of the Mumford-Shah functional in our particular case.

**Theorem 4.2.** Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with piecewise smooth boundary $\partial \Omega$ and let $n$ be the field of outward normals over the boundary. Let us suppose that the boundary of $\Omega$ has the following decomposition:

$$\partial \Omega = \overline{\Gamma_u} \cup \overline{\Gamma_u} \cup \overline{\Gamma_f}$$

$$\Gamma_u^1 \cap \Gamma_f = \emptyset , \overline{\Gamma_u^1} \cap \overline{\Gamma_u^2} = \emptyset , \mathcal{H}^1(\Gamma_u^1) \cdot \mathcal{H}^1(\Gamma_u^2) \cdot \mathcal{H}^1(\Gamma_f) > 0 \ .$$

Let us consider the functional

$$I(v, K) = \frac{1}{2} \int_{\Omega} | \nabla v | \, dx + GH^1(K)$$

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defined over the set
\[ \{(v, K) : v \in C^1(\Omega \setminus K, \mathbb{R})\} . \]

Let, for any \( D \in R, u(D) : \Omega \to R \) be the solution of the problem:
\[
\begin{aligned}
\text{div} \nabla v &= 0 & \text{in} & \Omega \\
\nabla v \mathbf{n} &= 0 & \text{on} & \Gamma_f \\
v &= 0 & \text{on} & \Gamma^1_u \\
v &= D & \text{on} & \Gamma^2_u.
\end{aligned}
\]

We suppose that exist strictly positive numbers \( c \) and \( C \) such that for any \( x \in \Gamma^1_u \cup \Gamma^2_u \)
\[ C \geq |\nabla u(1)\mathbf{n}|(x) \geq c . \]

There exist then two numbers \( m \leq M \), which depends only on \( \Omega, \Gamma^1_u, \Gamma^2_u \) and \( \Gamma_f \), such that:

i) if \( D^2 < m \) then \((u(D), \emptyset)\) is the only minimizer of the functional \( I \) over the set
\[ M(D) = \{(v, K) : v \in C^1(\Omega \setminus K, \mathbb{R}) \, , \, v = u(D) \text{ on } \Gamma^1_u \cup \Gamma^2_u\} . \]

ii) if \( D^2 > M \) then any minimizer of the functional \( I \) over the set \( M(D) \) has the form \((u_K, K)\), with \(|\nabla u_K| = 0 \) almost everywhere in \( \Omega \) and \( K \) geodesic in \( \Omega \) (i.e. length minimizer) separating \( \Gamma^1_u \) from \( \Gamma^2_u \).

Moreover, if \( c = C \) then \( M = m \), hence if \( \nabla u(1)\mathbf{n} \) is piecewise constant on \( \Gamma^1_u \cup \Gamma^2_u \) then we have only two kinds of minimizing crack sets.

The theorem assures us that for small time \( k/s \) the body remains sane and for large time \( k/s \) a crack with a particular shape appears in the body. Precisely, for small \( k/s \) we have \((u(k), K(k)) = (k/s u_\emptyset, \emptyset)\) and for large \( k/s \) we have \((u(k), K(k)) = (k/s u_S, S)\). The theorem help us to find particular cases when the passage from the first type of minimizer to the second one is brutal. Indeed, consider that \( \Omega \) is a rectangle \((0,a) \times (0,L)\),
\[ \Gamma^1_u = (0,a) \times \{0\} , \Gamma^1_u = (0,a) \times \{L\} \]
and \( \Gamma_f \) is the remaining part of the boundary. Let us consider, for simplicity, that \( \delta = 1 \). With the notations from the theorem 4.2. we have \( c = C \) therefore we have only two kinds of pairs displacement-crack which compete. We use \[ \text{(4.2.4)} \] in order to find the critical time \( k/s \) when the incremental solution \((u, K) : N \to M \) switches from \((k/s u_\emptyset, \emptyset)\) to \((k/s u_S, S)\), where \( S \) is, for example, \((0,a) \times \{L/2\}\). We find that the critical \( k/s \) is determined by the double inequality:
\[ \left( \frac{k}{s} \right)^2 \leq \frac{GL}{\mu} \leq \left( \frac{k+1}{s} \right)^2 . \]

We are lead to the definition of the critical moment \( t_c \), given by the formula
\[ t_c^2 = \frac{GL}{\mu} . \]
$t_c$ is proportional with the square root of $L$. The anti-plane stress existing in the sane body at the moment $t_c$ has the following expression:

$$
\mu \nabla (t_c u_0) = \left( 0, \frac{t_c}{L} \right).
$$

We can see that this stress depends on $L$, hence on the geometry of the body. Because $G$ is supposed to be a material constant we obtain the following conclusion:

*the model described above is not compatible with any model of crack appearance based on a critical stress as material constant.*

## 5 The improved model

We have seen that the first model allows crack appearance but it is not compatible with any critical fracture stress based model. Our purpose is to improve the first model in order to allow the existence of a critical stress which damages a structure. We shall find a way to make this improvement by studying first how smooth brittle propagation of cracks can be described with the Mumford-Shah energetic functional.

### 5.1 Smooth brittle crack propagation

There are two steps in order to define the notion of smooth brittle crack propagation. The first step consists in smoothness demands on the initial crack set $K$. We shall suppose that $K$ is endowed with the structure of manifold with boundary. The boundary of $K$, denoted by $\partial K$, represents the edge of the crack. The second step consists in smoothness demands on the evolution $t \mapsto K_t$ of the crack. We shall restrict our attention only to evolutions of the initial crack $K$ obtained by smooth deformations of $K$. The initial crack may be as complex as we wish, because the structure of manifold with boundary allows that, but this complexity remains the same during the propagation of the crack.

We shall work with deformations of the initial crack set $K$ by endomorphisms of $\Omega$. Let us consider the following set of diffeomorphisms:

$$
\mathcal{D}^s = \{ \phi \in C^\infty(\Omega, \Omega) \cap W^{s,2}(\Omega, R^n) : \phi^{-1} \in C^\infty(\Omega, \Omega) \text{ and } supp (\phi - 1_\Omega) \subset \Omega \}.
$$

We have denoted by $n$ the dimension of the space where $\Omega$ lies. The introduction of the Sobolev space $W^{s,2}(\Omega, R^n)$ has been made for mathematical reasons (to be found for example in Ebin & Marsden [EbM]) and the number $s$ is chosen to be greater than $\frac{n}{2} + 2$. In the paper [Bu3] a rigorous mathematical description of smooth brittle crack propagation can be found. We mention that the number $s$ controls the variation of the smoothness of the deformed crack set $\phi(k)$ from the smoothness of the initial crack set $K$.

The condition $supp (\phi - 1_\Omega) \subset \Omega$ means that near the boundary of $\Omega$ $\phi$ equals the identity map.
**Definition 5.1.** A smooth fracture curve is a function
\[ t \in [0, T] \mapsto \phi_t \in \mathcal{D}^s \]
which has the following properties:
\begin{enumerate}
  
  \item \[ \phi_0 = 1_{\Omega} \]
  
  \item The map \( t \in [0, T] \mapsto \phi_t \in \mathcal{D}^s \) is continuous with respect to the topology induced by the norm \( \max (\| \cdot \|_{L^\infty}, \| \cdot \|_{W^{s,2}}) \); for every \( t \in [0, T] \) \( \dot{\phi}_t \) exists and \[ \eta_t = \dot{\phi}_t \phi_t^{-1} \in W^{s,2}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n) \]
  
  \item For any \( t < t' \) we have \( \phi_t(K) \subset \phi_{t'}(K) \).
\end{enumerate}

We have used the notation \( f \circ g \) for the composition of the function \( f \) with \( g \).

A crack evolution curve is associated to the smooth fracture curve \( t \mapsto \phi_t \) and initial crack \( K \) by the formula:
\[ K_t = \phi_t(K) \]

There are infinitely many smooth crack propagation curves \( t \mapsto \phi_t \) with the same associated crack evolution curve \( t \mapsto K_t \).

Under smoothness assumptions on the initial crack set \( K \), for a smooth crack propagation curve \( t \mapsto \phi_t \) the condition that the crack grows implies that for any \( t \geq 0 \) we have:
\begin{align*}
  \int_{\phi_t(K)} \text{div}_s \eta_t \, d\mathcal{H}^{n-1} &\geq 0 \\
  \{ \eta_t \cdot n = 0 \text{ on } \phi_t(K) \}
\end{align*}

The integral from the last inequality equals the variation of the area of the crack set (see Allard [All]). The operator \( \text{div}_s \) is the tangential derivative with respect to the surface \( \phi_t(K) \) and it has the following form:
\[ \text{div}_s \eta = \text{div} \eta - n \cdot (\nabla \eta) n \]

where \( n \) is the normal to the surface \( \phi_t(K) \).

**5.2 K2 functional and the Griffith criterion**

We want now to reformulate the Griffith criterion of brittle crack propagation (3.3.6) in terms of smooth crack propagation curves. Our assumptions on the evolution of the body are the following:

\begin{enumerate}
  
  \item The evolution of the linear elastic body \( \Omega \) is quasi-static,
  
  \item A smooth curve of imposed displacements \( t \mapsto u_0(t) \) is given on the boundary \( \partial \Omega \).
\end{enumerate}

The assumption A2) can be modified by the replacement of \( \partial \Omega \) with a fixed part of the boundary \( \Gamma_u \); on the remaining part \( \Gamma_f \) we suppose that the body is force free.
At any moment $t$ the state of the body is described by the admissible pair displacement-crack $(u(t), K_t)$. The assumption A1) implies that the displacement $u(t)$ is determined by the knowledge of $K_t$ and boundary condition $u_0(t)$. With a notation used several times before, we have $u(t) = u(u_0(t), \phi_t(K))$. We shall change for our purposes this notation by writing:

$$u(t) = u(u_0(t), \phi_t) .$$

**Definition 5.2.** Let $t \in [0, T] \mapsto u^0(t) \in C(\partial \Omega, R^n)$ be a $C^1$ curve of imposed displacements on the exterior boundary of the body. A balanced fracture curve is any $C^1$ function

$$t \in [0, T] \mapsto (u^*, \phi_t) \in W^{1,2}(\Omega \setminus K, R^n) \times D^n ,$$

satisfying the following items:

i) for any $t \in [0, T]$ we have

$$u^*, \phi_t^{-1} = u(t) = u(u_0(t), \phi_t) ,$$

ii) $t \mapsto \phi_t$ is a smooth crack propagation curve.

For given curve of boundary displacement and initial crack set $K$, for any smooth crack propagation curve there is only one associated balanced fracture curve. The Griffith criterion of brittle crack propagation will act as a selection criterion amongst all smooth crack propagation curves.

We have seen that an equivalent form of the Griffith criterion is (3.3.7). With the change of notation $T(\phi_t) = T(\phi_t(K))$, we say that a smooth crack propagation curve is compatible with the Griffith criterion if for any $t$ we have

$$\frac{1}{2} \frac{d}{dt} [T(\phi_t) u_0(t), u_0(t)] + G \frac{d}{dt} \{H^{n-1}(\phi_t(K))\} \leq 0 . \quad (5.2.3)$$

The first term from (5.2.3) represents the variation of the elastic energy of the body calculated for the following variation of the displacement:

$$\tau \mapsto u(t + \tau) = u(u_0(t), \phi_{t+\tau}) .$$

The dependence of $u(t + \tau)$ with respect to $\phi(t + \tau)$ is implicit. An explicit variation of the displacement would be preferable, like this one:

$$v(\tau) = u(t).\phi_t.\phi_t^{-1} .$$

We are lead, by a change of variables, to the following equality:

$$2 \frac{d}{d\tau} \left( \int_{\Omega} w(\nabla v(\tau)) \, dx \right) \bigg|_{\tau=0} = \int_{\Omega} \{[C \nabla u(u_0(t), \phi_t) : \nabla u(u_0(t), \phi_t)] \, \text{div} \, \eta_t - \}

\quad (5.2.4)$$
Proposition 5.1. Let $t \mapsto \phi_t$ be a smooth crack propagation curve, $\eta_t = \dot{\phi}_t \phi_t^{-1}$ and $u_0 \in H^\frac{1}{2}(\partial \Omega) \cap L^\infty(\partial \Omega)$. Let us define, for fixed $t$, $v(\tau) = u(t, \phi_t) \phi_t^{-1} + \tau$. Then the following inequality is true:

\[
\left( \frac{d}{dt} [T(\phi_t)] u_0, u_0 \right) \leq 2 \frac{d}{d\tau} \int_\Omega w(\nabla v(\tau)) \, dx \bigg|_{\tau=0}.
\]

(5.2.5)

This proposition, together with the equality (5.2.4), allows us to introduce a generalization of the J integral. We use the notation $\eta \in W^{s,2}_0(\Omega, R^n)$ for $\eta \in W^{s,2}_0(\Omega, R^n)$ with null trace on $\partial \Omega$.

Definition 5.3. The generalized J integral is the following functional

\[
K^2 : D^s \times \left\{ W^{s,2}_0(\Omega, R^n) \cap L^\infty(\partial \Omega, R^n) \right\} \times \left\{ H^\frac{1}{2}(\partial \Omega) \cap L^\infty(\partial \Omega, R^n) \right\} \rightarrow \mathbb{R},
\]

\[
K^2(\phi, \eta, u_0) = \int_\Omega \left\{ -\frac{1}{2} [C \nabla u(0, \phi) : \nabla u(0, \phi)] \, \text{div} \, \eta + [C \nabla u(0, \phi)]_{ij} [\nabla u(0, \phi)]_{ik} [\nabla \eta]_{kj} \right\} \, dx.
\]

(5.2.6)

In order to explain why the functional $K^2$ is the generalization J integral, we begin by a temporary introduction of easier notations:

\[
\sigma = C \nabla u(0, \phi), \quad u = u(0, \phi), \quad w = \frac{1}{2} C \nabla u(0, \phi) : \nabla u(0, \phi).
\]

We define the tubular neighbourhood, of radius $r$, of the edge $\partial (K)$ of the crack set $K$:

\[
B_r = B_r(\partial \phi(K)) = \cup_{x \in \partial \phi(K)} B(x, r).
\]

The field of normals over $\partial B_r(\partial \phi(K))$ will be denoted by $\nu$, without specifying the parameter $r$.

If $u$ belongs to $C^2$ then we have

\[
w \eta_i,i - \sigma_{ij} u_{i,k} \eta_{k,j} = \left[ w \eta_i - \sigma_{ij} u_{i,k} \eta_k \right]_i - \sigma_{km} u_{k,mi} \eta_i + \sigma_{lj} u_{l,ki} \eta_k + \sigma_{li} u_{l,k} \eta_k.
\]

(5.2.7)

According to the assumption $A1)$, the divergence of the stress field $\sigma$ equals 0. We integrate the equality (5.2.7) over $\Omega \setminus B_r$ and we obtain:

\[
K^2(\phi, \eta, u_0) = - \lim_{r \to 0} \int_{\Omega \setminus B_r} \left[ w \eta_i - \sigma_{ij} u_{i,k} \eta_k \right]_i \, dx.
\]
By a flux-divergence formula, we are lead to the following expression of $K_2$:

$$K_2(\phi, \eta, u_0) = \lim_{r \to 0} \left\{ \int_{\partial B_r(\partial \phi(K))} \left\{ -w \eta \cdot \nu + \sigma_{il,k} \eta_k \nu_i \right\} \right\} + \int_{\phi(K)} [w] \eta \cdot n \ d\mathcal{H}^{n-1}.$$  

The functional $K_2$ is interesting in the case when $\eta \cdot n = 0$ on $\phi(K)$, as (5.1.2) suggests. In this case we have:

$$K_2(\phi, \eta, u_0) = \lim_{r \to 0} \left\{ \int_{\partial B_r(\partial \phi(K))} \left\{ w \eta \cdot \nu - \sigma_{il,k} \eta_k \nu_i \right\} \ d\mathcal{H}^{n-1} \right\}.$$  

Let us consider that we are in the case of plane displacements (hence $n = 2$) and that the crack set $\phi(K)$ lies on the $Ox_1$ axis. If we take $\eta$ equal to $(1, 0)$ in a neighbourhood of the edge of the crack then we have:

$$K_2(\phi, \eta, u_0) = \lim_{r \to 0} \left\{ \int_{\partial B_r(\partial \phi(K))} \left\{ w \nu_1 - \sigma_{k,i} u_{k,1} \nu_i \right\} \ d\mathcal{H}^{n-1} \right\}.$$  

We recognize in the right term of the equality above the expression of the classical J integral.

We propose the following selection criterion for smooth crack propagation curves:

A smooth crack propagation curve $t \mapsto \phi_t$ satisfies the generalized Griffith criterion if at any moment $t \geq 0$ we have $\eta_t \cdot n = 0$ on $\phi_t(K)$ and

$$K_2(\phi_t, \eta_t, u_0(t)) \geq G \frac{d}{dt} \mathcal{H}^{n-1}(\phi_t(K)).$$  

Ohtsuka [Oht1—4] proves that under stronger smoothness assumptions on $K$ and on the curve $t \mapsto K_t$ always exists a smooth crack propagation curve $t \mapsto \phi_t$ such that for any $t$ and with our notations we have:

$$\left\langle \frac{d}{dt} [T(\phi_t)] u_0(t), u_0(t) \right\rangle + K_2(\phi_t, \eta_t, u_0(t)) = 0.$$  

For this reason we consider that (5.2.9) is not too strong with respect to the classical Griffith criterion.

5.3 Extension of $K_2$ and admissible cracks

We want to extend the Griffith criterion of brittle fracture propagation in order to allow crack appearance. The leading idea is to consider crack evolution curves $t \mapsto K_t$ which are limits of crack evolution curves of the form $t \mapsto \phi_t(K)$.

Let $t \mapsto \phi_t$ be a smooth crack propagation curve. At any moment $t$ the vector field $\eta_t = \dot{\phi}_t.\dot{\phi}_t^{-1}$ represents the propagation speed of the edge $\partial \phi_t(K)$ of the crack
Precisely the restriction of $\eta_t$ to $\partial \phi_t(K)$ represents the distribution of speed of propagation of the points belonging to this $n-2$ surface. The appearance of a new crack at the moment $t$ is seen as a limit of processes of smooth crack propagation, when the distribution of speed $\eta_t$ develops jumps.

We shall consider therefore a sequence $\eta_h(t)$ in the space $W^{s,2}(\Omega, R^n) \cap L^\infty(\Omega, R^n)$. For each $h$ we define the following flow $\tau \mapsto \phi^h_\tau$:

$$\phi^h_0 = 1_\Omega , \quad \phi^h_\tau = 1_\Omega + \tau \eta_h .$$

For small times $\tau$ we have $\phi^h_\tau \in D^s$ and for any $h$ we see that

$$\phi^h_0 (\phi^h_0)^{-1} = \eta_h .$$

Let us suppose that $\eta_h$ converges almost everywhere to $\eta$. Then for any $\tau$ $\phi^h_\tau$ converges almost everywhere to $\phi^\eta_\tau = 1_\Omega + \tau \eta$. We make the following supplementary assumptions:

S1) for small $\tau$ $\phi^\eta_\tau$ is almost everywhere injective,

S2) $\Omega \setminus \phi^\eta_\tau(\Omega)$ has finite Hausdorff $n-1$ measure,

S2) let us denote by $S_\eta$ the set where $\eta$ has no approximate limit (the complementary of the Lebesgue set of $\eta$, see for this the section of proofs) and by $\| Df \|$ the total variation measure associated to the distributional derivative of the function $f$; then we have

$$\| D^s \eta \| (\Omega \setminus S_\eta) = 0 .$$

The assumption S1) assures us that $\phi^\eta_\tau$ almost everywhere maps different points from $\Omega$ in different places in $\phi^\eta_\tau(\Omega)$; S2) prevents the case where in the limit appear holes in the configuration $\phi^\eta_\tau(\Omega)$ and S3) is a more sophisticated condition which says that no strange Cantor sets appear in $\phi^\eta_\tau(\Omega)$.

Under these assumptions it is easy to prove that on $S_\eta$ the jump of $\eta$ satisfies the relation:

$$[\eta] \cdot n = 0 .$$

Indeed, suppose that in a neighbourhood of $x \in S_\eta$ we have $[\eta] \cdot n < 0$. Then the assumption S2) is contradicted because a solid neighbourhood of $x$ is transformed by $\phi^\eta_\tau$ in a neighbourhood with a hole, when $\tau > 0$; if we have $[\eta] \cdot n > 0$ then S1) is contradicted because even if local injectivity is respected, the global injectivity in the form S1) is not.

We shall consider therefore pairs $(\eta, N)$ where $N$ is a topologically closed countably rectifiable set, $\eta$ has tangential jumps on the surface $N$ and satisfies the smoothness assumption $\eta \in W^{s,2}_0(\Omega \setminus N, R^n) \cap L^\infty(\Omega, R^n)$. The subscript 0 in the notation $W^{s,2}_0(\Omega \setminus N, R^n)$ means that $\eta = 0$ on $\partial \Omega$ in the sense of traces.

Let us suppose that there is no initial crack in the body: $K = \emptyset$. We perform the same calculation for $K^2(1_\Omega, \eta, u_0)$ as we did after definition 5.3. and we obtain from (5.2.7) the expression:

$$K^2(1_\Omega, \eta, u_0) = - \int_\Omega [w \eta_i - \sigma_{ij} u_{i,k} \eta_k]_i .$$
Let us suppose that \( N \) is a surface with boundary and let \( \partial N \) be a tubular neighbourhood of \( \partial N \), of radius \( r \). Because of the assumption \([\eta] \cdot n = 0\) on \( N \), we obtain:

\[
K^2(1, \eta, \mathbf{u}_0) = \int_N \sigma_{ii}(\mathbf{u}, \eta) \, d\mathcal{H}^{n-1}.
\]

It is natural to try to modify the Griffith criterion \((5.2.9)\) in order to have a control on the integral from above. We propose the following differential criterion of crack appearance \((DA)\), which makes a selection amongst all crack sets which can appear in the body. The constant \( \Sigma \), with the dimension of a stress, which appears in this criterion, is postulated to be a constant of material.

\[(DA). \text{ Let us consider the elastic body } \Omega \text{ and the imposed boundary displacement } \mathbf{u}_0. \text{ A crack set } N \text{ can appear in the body if there exists a vector field } \eta \in W^{s,2}_0(\Omega \setminus N, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n), \text{ such that } \eta \cdot n = 0 \text{ on } N \text{ and}
\]

\[
\int_N \sigma_{ii}(\mathbf{u}_0, \mathbf{0}) \, n_i u_{i,k}(\mathbf{u}_0, \mathbf{0})[\eta_k] \, d\mathcal{H}^{n-1} \geq ||\eta||_{L^\infty} \Sigma \mathcal{H}^{n-1}(N). \quad (5.3.10)
\]

We give also a criterion of local crack appearance \((LA)\). This criterion tells us if in the point \( x \in \Omega \) a crack with normal \( n \) can appear.

\[(LA). \text{ Let us consider the elastic body } \Omega \text{ and the imposed boundary displacement } \mathbf{u}_0. \text{ In the point } x \in \Omega \text{ a crack with normal } n \text{ can appear if}
\]

\[
\sup \{ \sigma_{ii}(\mathbf{u}_0, \mathbf{0}) n_i u_{i,k}(\mathbf{u}_0, \mathbf{0}) \nu_k : \nu \in \mathbb{R}^n, |\nu| = 1, n \cdot \nu = 0 \} \geq \Sigma. \quad (5.3.11)
\]

From \((5.3.10)\) and \((5.3.11)\) we see that if \( N \) is smooth enough and if for any \( x \in N \), the criterion \((LA)\) is satisfied for the pair \((x, n(x))\), where \( n(x) \) is the normal to \( N \) at \( x \), then \( N \) satisfies the global criterion \((DA)\).

Let us suppose that the body \( \Omega \) is a cylinder \( \omega \times [0, L] \) and \( \mathbf{u}_0 \) imposed on the top and bottom of this cylinder such that

\[
\mathbf{u}(\mathbf{u}_0, \mathbf{0})(x_1, x_2, x_3) = (0, 0, ax_3), \quad a > 0.
\]

The stress \( \sigma(\mathbf{u}_0, \mathbf{0}) \) has the form:

\[
\sigma(\mathbf{u}_0, \mathbf{0}) = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \sigma
\end{pmatrix},
\]

where \( \sigma = Ea. \) If we denote by \( \alpha \) the angle between \( n \) and the \( Ox_3 \) axis, we have:

\[
\sup \{ \sigma_{ii}(\mathbf{u}_0, \mathbf{0}) n_i u_{i,k}(\mathbf{u}_0, \mathbf{0}) \nu_i : \nu \in \mathbb{R}^n, |\nu| = 1 \} = \frac{1}{2E} \sigma^2 \sin(2\alpha).
\]
The maximum value of this expression is attained for \( \alpha = \pi/4 \). Therefore in the experience of uniaxial traction the (LA) criterion affirms that a crack can appear if

\[
\frac{1}{2E} \sigma^2 \geq \Sigma ,
\]

and if we have equality in the relation above then the normal of the crack predicted by (LA) makes the angle \( \pi/4 \) with the axis of the cylinder. The relation (5.3.12) gives us the value of the critical stress for uniaxial traction (which is a constant of material this time):

\[
\sigma_{cr} = \sqrt{2E\Sigma} .
\]

### 5.4 The improved model

In this section we propose an improved energy minimizing movement formulation to the problem of brittle fracture evolution (3.3.8). The model is based on two constants of material connected to fracture, namely \( G \) and \( \Sigma \) previously introduced. In this formulation the critical stress which lead to fracture in a traction experiment, defined by (5.3.13), is a constant of material.

We denote by \( S_n \) the set of all \( \nu \in \mathbb{R}^n \) with \( |\nu| = 1 \). Let us define the following function:

\[
f_\infty : \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^n \times S_n \to \mathbb{R} \cup \{ +\infty \},
\]

\[
f_\infty(\sigma, F, n) = \begin{cases} G & \text{if } \sup \{\sigma_{l,k} n_l n_k : \nu \in \mathbb{R}^n, |\nu| = 1, n \cdot \nu = 0\} \geq \Sigma \\ +\infty & \text{otherwise} \end{cases}.
\]

The physical dimension of \( f_\infty \) is the same as the one of \( G \).

With the use of the function \( f_\infty \) the criterion of brittle crack appearance (LA), takes the form: \emph{given the imposed boundary displacement \( u_0, x \in \Omega \) and \( n \in S_n \), a crack of normal \( n \) can pass by \( x \) if:}

\[
f_\infty(\sigma(u_0, \emptyset), \nabla u(u_0, \emptyset), n) < +\infty .
\]

The idea of the improved model is to consider only pairs displacement-crack \((v, L) \in M\) admissible with respect to (LA).

**Definition 5.4.** Let us consider the space \( M \) of admissible pairs displacement-crack endowed with the topology given by the convergence:

\[
(u_h, K_h) \to (u, K) \text{ if } \begin{cases} u_h L^2 \to u \\ H^{-1}(K_h \Delta K) \to 0 \end{cases}.
\]

We define the functions

\[
J_\infty : M \times M \to \mathbb{R} ,
\]
\[ J_\infty ((u, K), (v, L)) = \int_\Omega w(\nabla v) \, dx + \int_{L\setminus K} f_\infty(\sigma(u), \nabla u, n) \, d\mathcal{H}^{n-1}(x) \]

\[
\Psi : [0, \infty) \times M \to \{0, +\infty\} \,, \quad \Psi(\lambda, (u, K)) = \begin{cases} 0 & \text{if } v = u_0(\lambda) \text{ on } \partial\Omega \setminus K \\ +\infty & \text{otherwise} \end{cases} \,.
\]

We consider the initial data \((u_0, K) \in M\) such that \(u_0 = u(u_0(0), K)\).

For any \(s \geq 1\) we recursively define \((u^s, K^s) : \mathbb{N} \to \mathbb{M}\) like this:

i) \((u^s, K^s)(0) = (u_0, K)\);  
ii) for any \(k \in \mathbb{N}\) \((u^s, L^s)(k+1) \in \mathbb{M}\) minimizes the functional \((v, L) \in \mathbb{M} \mapsto J_\infty(((u^s, K^s)(k), (v, L)) + \Psi((k+1)/s, (v, L)))\) over \(\mathbb{M}\). \(K^s(k+1)\) is defined by the formula:

\[ K^s(k+1) = K^s(k) \cup L^s(k+1) \,.
\]

An energy minimizing movement associated to \(J_\infty\) with the constraints \((3.3.4)\), \(\Psi\) and initial data \((u_0, K)\) is any \((u, K) : [0, +\infty) \to \mathbb{M}\) having the property: there is a diverging sequence \((s_i)\) such that for any \(t > 0\)

\[ (u^{s_i}, K^{s_i})([s_it]) \to (u, K)(t) \text{ as } i \to \infty \,.
\]

We are interested if for fixed \(s\) an incremental solution exists. There is no result to our knowledge that assures the existence of a minimizer of the functional

\[ (v, L) \in \mathbb{M} \mapsto \int_\Omega w(\nabla v) \, dx + \int_{L\setminus K} f_\infty(\sigma(u), \nabla u, n) \, d\mathcal{H}^{n-1}(x) \]

in our case. That is why we prefer to modify the function \(f_\infty\). This function imposes a cost equal to \(+\infty\) to the pairs displacement-crack which are not compatible with the (LA) criterion. We shall demand a finite but great cost instead, hoping that non admissible pairs will not enter in competition with admissible ones.

Let us consider a number \(C > G\) (with the same physical dimension as \(G\)) and a function

\[ f_C : \mathbb{M} \to \mathbb{R} \]

with the following properties:

i) \(f_C\) is positively 1-homogeneous with respect to the third variable, 
ii) for any \(n \in \mathbb{R}^n\) such that \(|n| = 1\), if

\[ \sup \{\sigma_l n_i F_{l,k} \nu_k : \nu \in \mathbb{R}^n, \ |\nu| = 1, \ n \cdot \nu = 0\} \geq \Sigma \]

then \(f_C(\sigma, F, n) = G\), 
iii) for any \(n \in \mathbb{R}^n\) such that \(|n| = 1\), if

\[ \sup \{\sigma_l n_i F_{l,k} \nu_k : \nu \in \mathbb{R}^n, \ |\nu| = 1, \ n \cdot \nu = 0\} < \Sigma \]

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then $f_C(\sigma, F, n) > G$.

iv) for any $(\sigma, F, n) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times S_n$ we have $f_C(\sigma, F, n) \leq C$.

As for theorem 4.1., the following statement has been proven to be true only in a weak sense, described in the next section.

**Theorem 5.2.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with piecewise smooth boundary, let $(u_0, K)$ be a given admissible pair displacement-crack in $\Omega$ and let

$$u_0 : N \to H^{\frac{1}{2}}(\partial\Omega, R^n) \cap L^\infty(\partial\Omega, R^n)$$

be a given sequence of imposed displacements such that $u_0 = u(u_0(0), K)$ on $\partial\Omega \setminus K$. Let us consider the functional

$$J_C : M \times M \to \mathbb{R},$$

$$J_C((u, K), (v, L)) = \int_\Omega w(\nabla v) \, dx + \int_{L \setminus K} f_C(\sigma(u), \nabla u, n) \, d\mathcal{H}^{n-1}(x),$$

where $f_C$ is chosen to satisfy the assumptions i)—iv) from above. Then there exists the sequence $(u, K) : N \to M$ such that:

i) $u(0) = u_0$ and $K(0) = K$;

ii) for any $k \in N$ there is a crack set $L(k + 1)$ such that $(u(k + 1), L(k + 1)) \in M$, $u(k + 1) = u_0(k + 1)$ on $\partial\Omega \setminus L(k + 1)$ and $(u(k + 1), L(k + 1))$ is a minimizer of the functional

$$(v, L) \in M, v = u_0(k + 1) \text{ on } \partial\Omega \setminus L \implies J_C((u(k), K(k)), v, L)).$$

The set $K(k + 1)$ is given by the formula

$$K(k + 1) = K(k) \cup L(k + 1).$$

### 6 Proofs

#### 6.1 Weak versions of theorems 4.1. and 5.2.

This section is dedicated to a brief voyage through the spaces $\text{SBV}$ and $\text{SBD}$. The weak forms of theorems 4.1. and 4.2. are direct applications of results listed below.

The space $\text{SBV}(\Omega, \mathbb{R}^n)$ of special functions with bounded variation was introduced by De Giorgi and Ambrosio in the study of a class of free discontinuity problems ([DGA], [A1], [A2]). For any function $u \in L^1(\Omega, \mathbb{R}^n)$ let us denote by $Du$ the distributional derivative of $u$ seen as a vector measure. The variation of $Du$
is a scalar measure defined like this: for any Borel measurable subset $B$ of $\Omega$ the variation of $Du$ over $B$ is

$$\left| Du \right| (B) = \sup \left\{ \sum_{i=1}^{\infty} | Du(A_i) | : \cup_{i=1}^{\infty} A_i \subset B , A_i \cap A_j = \emptyset \forall i \neq j \right\} .$$

A function $u$ has bounded variation if the total variation of $Du$ is finite. We send the reader to the book of Evans & Gariepy [EG] for basic properties of such functions.

The space $SBV(\Omega, R^n)$ is defined as follows:

$$SBV(\Omega, R^n) = \{ u \in L^1(\Omega, R^n) : | Du | (\Omega) < +\infty , | D^s u | (\Omega \setminus Su) = 0 \} .$$

The Lebesgue set of $u$ is the set of points where $u$ has approximate limit. The complementary set is a $L^1$ negligible set denoted by $Su$. If $u$ is a special function with bounded variation then $Su$ is also $\sigma$ (i.e. countably) rectifiable.

From the Calderon & Zygmund [CZ] decomposition theorem we obtain the following expression of $Du$, the distributional derivative of $u \in SBV(\Omega, R^n)$, seen as a measure:

$$Du = \nabla u(x) \, dx + \left[ u \right] \otimes n \, dH^{n-1}_{|K} .$$

We shall use further the notation $\mu \ll \lambda$ if the measure $\mu$ is absolutely continuous with respect to the measure $\lambda$.

Let us define the following Sobolev space associated to the crack set $K$ (see [ABF]):

$$W^{1,2}_K = \left\{ u \in SBV(\Omega, R^n) : \int_{\Omega} |\nabla u|^2 \, dx + \int_{K} |u|^2 \, dH^{n-1} < +\infty , | D^s u | \ll H^{n-1}_{|K} \right\} .$$

It has been proved in [DGCL] the following equality:

$$W^{1,2}(\Omega \setminus K, R^n) \cap L^{\infty}(\Omega, R^n) = W^{1,2}_K(\Omega, R^n) \cap L^{\infty}(\Omega, R^n) . \quad (6.1.1)$$

Therefore if $u = u(u_0, K)$ and $u_0 \in L^{\infty}(\partial \Omega, R^n)$ then $u$ is a special function with bounded variation.

A similar description can be made for the space of special functions with bounded deformation $SBD(\Omega)$ can be found in Ambrosio, Coscia & Dal Maso [ACDM]. For any function $u \in L^1(\Omega, R^n)$ we denote by $Eu$ the symmetric part of the distributional derivative of $u$, seen as a vector measure. We denote also by $Ju$ the subset of $\Omega$ where $u$ has different approximate limits with respect to a point-dependent direction. The difference between $Su$ and $Ju$ is subtle. Let us quote only the fact that for a function $u \in SBV(\Omega, R^n)$ the difference of these sets is $H^{n-1}$-negligible.

The definition of $SBD(\Omega)$ is the following:

$$SBD(\Omega, R^n) = \{ u \in L^1(\Omega, R^n) : | Eu | (\Omega) < +\infty , | E^s u | (\Omega \setminus Ju) = 0 \} .$$

If $u$ is a special function with bounded deformation then $Ju$ is countably rectifiable. We have a decomposition theorem for $SBD$ functions, similar to Calderon &
Zygmund result applied for SBV functions. The decomposition theorem is due to Belletini, Coscia & Dal Maso [BCDM] and asserts that

\[ \mathbf{E}u = \epsilon(u)(x) \, dx + [u] \otimes n \, d\mathcal{H}^{n-1}_{\mathbf{J}u} . \]

Here \( \otimes \) means the symmetric part of tensor product and \( \epsilon(u) \) is the approximate symmetric gradient, hence the approximate limit of the symmetric part of the gradient of \( u \).

In order to give weak versions of theorems 4.1. and 5.2. let us weaken first the space \( M \) of pairs displacement-crack. We introduce the new set of weak pairs displacement-crack \( \mathcal{M} \):

\[ \mathcal{M} = \{(u, K) : K \text{ is } \sigma \text{-rectifiable, } u \in \text{SBD}(\Omega) \text{ and } |E^s u| (\Omega \setminus K) = 0\} . \] (6.1.2)

Given \((u, K) \in \mathcal{M}\), the set \( K \) is countably rectifiable but it is not necessarily closed; we impose also weaker conditions on the regularity of the displacement \( u \). A direct consequence of (6.1.1) is that any pair displacement-crack \((u, K)\) such that \( u \in L^\infty(\Omega, \mathbb{R}^n) \) belongs to the set \( \mathcal{M} \).

Let us define the functional \( J \), the weak version of the functional \( J \) introduced at definition 4.1.:

\[ J : \mathcal{M} \times \mathcal{M} \to \mathbb{R} , \quad J((u, K), (v, L)) = \int_{\Omega} w(\epsilon(v)) \, dx + G\mathcal{H}^{n-1}(L \setminus K) . \]

Before we introduce the correspondent of the function \( \Psi \) from the same definition, let us explain what we mean by \( u = u_0 \) on the boundary of \( \Omega \). We consider, for simplicity, that \( u_0 : \partial \Omega \to \mathbb{R}^n \) is a continuous and therefore bounded function. Then, for any \( u \in \text{SBD}(\Omega) \), \( u = u_0 \) if the approximate limit of \( u \) equals \( u_0 \) in any point of \( \partial \Omega \) where the first exists, i.e.:

\[ \forall x \in \partial \Omega, \text{ if } \exists v(x) \text{ such that } \lim_{\rho \to 0^+} \frac{\int_{B_\rho(x) \cap \Omega} |u(y) - v(x)| \, dy}{|B_\rho(x) \cap \Omega|} = 0 \text{ then } v(x) = u_0(x) . \]

Let us consider a curve of imposed displacements \( \lambda \mapsto u_0(\lambda) \in C(\partial \Omega, \mathbb{R}^n) \). The function \( \ominus \), introduced instead of \( \Psi \), is defined as follows:

\[ \ominus : [0, +\infty) \times \mathcal{M} \to \{0, +\infty\} , \]

\[ \ominus(\lambda, (u, K)) = \begin{cases} 0 & \text{if } u = u_0 \text{ and } \mathcal{H}^{n-1}(K \setminus \mathbf{J}u) = 0 \\ +\infty & \text{otherwise} . \end{cases} \]

**Definition 6.1.** (weak version of definition 4.1.) Let us consider the space \( \mathcal{M} \) endowed with the topology given by the convergence:

\[ (u_h, K_h) \to (u, K) \text{ if } \begin{cases} u_h \to u \text{ in } L^2 , \\ \mathcal{H}^{n-1}(K_h \Delta K) \to 0 . \end{cases} \]
Let us consider also the function $J$, the curve of imposed displacements $t \mapsto u_0(t)$ with the associated function $\ominus$ and the initial data $(u_0, K) \in M$ such that $u_0 = u(u_0(0), K)$.

For any $s \geq 1$ we recursively define $(u^s, K^s) : N \to M$ like this:

i) $(u^s, K^s)(0) = (u_0, K)$;

ii) for any $k \in N$ $(u^s, L^s)(k + 1) \in M$ minimizes the functional

$$(v, L) \in M \mapsto J(((u^s, K^s)(k), (v, L)) + \ominus((k + 1)/s, (v, L))$$

over $M$. In order to verify the constraint (3.3.4), $K^s(k+1)$ is defined by the formula:

$$K^s(k+1) = K^s(k) \cup J_{u^s(k+1)}.$$  \hfill (6.1.3)

An energy minimizing movement associated to $J$ with the constraints (3.3.4), $\ominus$ and initial data $(u_0, K)$ is any $(u, K) : [0, +\infty) \to M$ having the property: there is a diverging sequence $(s_i)$ such that for any $t > 0$

$$(u^s_i, K^s_i)_{[s_i, t]} \to (u, K)(t) \text{ as } i \to \infty.$$ 

Let us remark that the disappearance of the set $L^s(k+1)$ from the crack-growth condition (6.1.3) is only apparent, because if $(u^s, L^s)(k+1)$ minimizes the functional

$$(v, L) \in M \mapsto J(((u^s, K^s)(k), (v, L)) + \ominus((k + 1)/s, (v, L))$$

then $\ominus((k + 1)/s, (u^s, L^s)(k+1)) = 0$, hence

$$\mathcal{H}^{n-1}(K \setminus J_{u^s}) = 0.$$ 

In [ACDM] has been proven that functionals like $J$ are $L^1$ inferior semi-continuous and coercive, hence on closed subspaces $V$ of $SBD(\Omega)$ the functional

$$v \in V \mapsto J(((u^s, K^s)(k), (v, J_v))$$

has a minimizer. Such a closed subspace of $SBD(\Omega)$ is the space of all weak displacements $v$ with $v = u_0$, where $u_0$ is a given boundary displacement. Therefore the following theorem is true by a trivial induction:

**Theorem 4.1.** (weak version) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with piecewise smooth boundary, let $(u_0, K)$ be a given admissible pair displacement-crack in $\Omega$ and let

$$u_0 : N \to C(\partial\Omega, \mathbb{R}^n)$$

be a given sequence of imposed displacements such that $u_0 = u(u_0(0), K)$ on $\partial\Omega \setminus K$.

Then there exists the sequence $(u, K) : N \to M$ such that:

i) $u(0) = u_0$ and $K(0) = K$.
ii) for any \( k \in \mathbb{N} \) there is a countably rectifiable set \( L(k+1) \) such that \( (u(k+1), L(k+1)) \in M, u(k+1) = u_0(k+1) \) on \( \partial \Omega \), and \( (u(k+1), L(k+1)) \) is a minimizer of the functional 
\[
(v, L) \in M, v = u_0(k+1) \text{ on } \partial \Omega \quad \Rightarrow \quad J(((u(k), K(k)), v, L)) .
\]
The set \( K(k+1) \) is given by the formula 
\[
K(k+1) = K(k) \cup Ju_{(k+1)} .
\]

The weak version of theorem 5.2. is obtained in the same way. We start by relaxing the functional \( J_C \) to the functional \( J_{C'} \):
\[
J_C : M \times M \rightarrow \mathbb{R} ,
\]
\[
J_C(((u, K), (v, L))) = \int_\Omega w(\epsilon(v)) \, dx + \int_{L\backslash K} f_C(\sigma(u), \nabla u, n) \, dH^{n-1}(x) .
\]
Here \( \sigma(u) = C\epsilon(u) \) and \( \epsilon(u) \) is the approximate symmetric gradient of \( u \).

We have the following definition of an energy minimizing movement associated to \( J_C \) with the usual constraints:

**Definition 6.2.** (weak version of definition 5.4. adapted for \( J_C \)) Let us consider the space \( M \) endowed with the topology given by the convergence:
\[
(u_h, K_h) \rightarrow (u, K) \quad \text{if} \quad \begin{cases} u_h \rightarrow u \quad \text{in } L^2, \\ H^{n-1}(K_h \Delta K) \rightarrow 0 .
\end{cases}
\]

Let us consider also the function \( J_C \), the curve of imposed displacements \( t \mapsto u_0(t) \) with the associated function \( \ominus \) and the initial data \( (u_0, K) \in M \) such that \( u_0 = u(u_0(0), K) \).

For any \( s \geq 1 \) we recursively define \( (u^s, K^s) : N \rightarrow M \) like this:

i) \( (u^s, K^s)(0) = (u_0, K) \);

ii) for any \( k \in \mathbb{N} \) \( (u^s, L^s)(k+1) \in M \) minimizes the functional 
\[
(v, L) \in M \quad \Rightarrow \quad J_C(((u^s, K^s)(k), (v, L)) + \ominus((k+1)/s, (v, L))
\]
over \( M \). In order to verify the constraint [3.3.4], \( K^s(k+1) \) is defined by the formula:
\[
K^s(k+1) = K^s(k) \cup Ju_{s(k+1)} .
\]

An energy minimizing movement associated to \( J_C \) with the constraints [3.3.4], \( \ominus \) and initial data \( (u_0, K) \) is any \( (u, K) : [0, +\infty) \rightarrow M \) having the property: there is a diverging sequence \( (s_i) \) such that for any \( t > 0 \)
\[
(u^{s_i}, K^{s_i})([s_it]) \rightarrow (u, K)(t) \quad \text{as} \quad i \rightarrow \infty .
\]
The assumptions i)—iv) on $f_C$ from the previous section allow us to apply the main existence result from [ACDM] to the functional $J_C$. We have therefore the following weak version of the theorem 5.2.:

**Theorem 5.2.** (weak version) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with piecewise smooth boundary, let $(u_0, K)$ be a given admissible pair displacement-crack in $\Omega$ and let

$$u_0 : N \rightarrow C(\partial \Omega, \mathbb{R}^n)$$

be a given sequence of imposed displacements such that $u_0 = u(u_0(0), K)$ on $\partial \Omega \setminus K$.

Then there exists the sequence $(u, K) : N \rightarrow M$ such that:

i) $u(0) = u_0$ and $K(0) = K$;

ii) for any $k \in N$ there is a countably rectifiable set $L(k+1)$ such that $(u(k+1), L(k+1)) \in M$, $u(k+1) = u_0(k+1)$ on $\partial \Omega$ and $(u(k+1), L(k+1))$ is a minimizer of the functional

$$(v, L) \in M, \quad v = u_0(k+1) \text{ on } \partial \Omega \Rightarrow J_C((u(k), K(k)), v, L).$$

The set $K(k+1)$ is given by the formula

$$K(k+1) = K(k) \cup J_u(k+1).$$

In the anti-plane case we have to consider the space $SBV(\Omega, R)$ instead of $SBD(\Omega, \mathbb{R}^n)$. In this case the partial regularity results of De Giorgi, Carriero & Leaci [DGCL] and Ambrosio [A...] tell us that the classical Mumford-Shah functional has minimizers in the set of pairs displacement-crack $M$. Both theorems are therefore true in the strong form, in the anti-plane case.

### 6.2 Theorem 4.2.

For any $D > 0$ let $u_0(D)$ be the following boundary displacement:

$$u_0(D)(x) = \begin{cases} 0 & \text{on } \Gamma^1_u \\ D & \text{on } \Gamma^2_u. \end{cases}$$

For any crack set $K$ we denote by $u_K$ the displacement $u_K = u(u_0(1), K)$, i.e. the solution (or one of the solutions) of the problem:

$$\begin{align*}
\text{div} \nabla u &= 0 \quad \text{in } \Omega \setminus K \\
\nabla u \cdot n &= 0 \quad \text{on } \Gamma_f \cup K \\
u &= 0 \quad \text{on } \Gamma^1_u \\
u &= 1 \quad \text{on } \Gamma^2_u.
\end{align*}$$
More general, we shall use the notation $u_K(D) = u(u_0(D), K)$. It is obvious that $u_K(D) = D u_K$. In the body of Theorem 4.2, we have introduced the Mumford-Shah functional:

$$I(v, K) = \frac{1}{2} \int_{\Omega} |\nabla v| \, dx + G\mathcal{H}^1(K).$$

The displacement $u_K$ has the minimum property

$$I(u_K, K) \leq I(v, K), \quad \forall (v, K) \in M, \quad v = u_0(1) \text{ on } \Gamma_1^u \cup \Gamma_2^u.$$

That is why it is reasonable to redefine the functional $I$ as a functional depending only on the crack set $K$:

$$\tilde{I}(K) = I(u_K, K).$$

With this notation we have, for any $D > 0$, the inequality:

$$D^2 \frac{1}{2} \int_{\Omega} |\nabla u_K| \, dx + G\mathcal{H}^1(K) \leq I(v, K), \quad \forall (v, K) \in M, \quad v = u_0(D) \text{ on } \Gamma_1^u \cup \Gamma_2^u.$$

We make the notation:

$$\tilde{I}(K, D) = D^2 \frac{1}{2} \int_{\Omega} |\nabla u_K| \, dx + G\mathcal{H}^1(K).$$

We shall need further the function $\overline{u}$, which is defined modulo an additive constant by the relations:

$$\frac{\partial \overline{u}}{\partial x_1} = \frac{\partial u_\emptyset}{\partial x_2}, \quad \frac{\partial \overline{u}}{\partial x_2} = -\frac{\partial u_\emptyset}{\partial x_1}.$$

The level sets of $\overline{u}$ form a congruence of curves in $\overline{\Omega}$. The part of the boundary $\Gamma_f$ belongs to this congruence. We define the following system of open neighbourhoods named $V(\overline{u})$, with the aid of this congruence:

$$\forall A \in V(\overline{u}), \quad \partial A \setminus \Gamma_u \text{ it is locally a level set of } \overline{u}. $$

For any $A \in V(\overline{u})$ we denote by $\partial_u A$ the part of the boundary of $A$ belonging to $\Gamma_1^u$ or $\Gamma_2^u$, i.e.

$$\partial_u A = \partial A \cap (\Gamma_1^u \cup \Gamma_2^u).$$

The remaining part of $\partial A$ is denoted by $\partial_f A$.

Let $K$ be a rectifiable curve and $\Omega' \in V(\overline{u})$ such that

$$K \setminus \Gamma_f \subset \Omega' \cup \partial_u \Omega'.$$

For the couple $(K, \Omega')$ we introduce the following stress field:

$$\sigma = \begin{cases} \nabla u_\emptyset & \text{in } \Omega \setminus \Omega' \\ 0 & \text{in } \Omega'. \end{cases}$$
This stress field is statically admissible with respect to the body with reference configuration $\Omega \setminus K$ and boundary displacement $u_0(1)$. Therefore we have the following inequality:

$$\frac{1}{2} \int_{\Omega} |\nabla u_K|^2 \, dx \geq \int_{\Gamma_u} (\sigma n) \cdot u_0(1) \, d\mathcal{H} - \frac{1}{2} \int_{\Omega} |\sigma|^2 \, dx .$$

The latter inequality can be put in terms of Mumford-Shah functional $I$ like this:

$$\tilde{I}(\emptyset) - \tilde{I}(K) \leq \frac{1}{2} \int_{\Omega'} |\nabla u_\emptyset|^2 \, dx - G\mathcal{H}^1(K \setminus \Gamma_f) .$$

The reason for which we have put $H^1(K \setminus \Gamma_f)$ instead of $H^1(K)$ is that $u_K = u_K \setminus \Gamma_f$ but $\tilde{I}(K) \geq \tilde{I}(K \setminus \Gamma_f)$. As a consequence, sets $K$ with a part on $\Gamma_f$ are disqualified to be minimizers of $\tilde{I}$.

Let us denote by $\tau$ the tangent vector field in direct sense to $\partial \Omega'$. After few calculations we obtain from the previous inequality the estimation:

$$\tilde{I}(\emptyset) - \tilde{I}(K) \leq \frac{1}{2} \int_{\Omega'} u_\emptyset \cdot (\nabla u_\emptyset \tau) - G\mathcal{H}^1(K \setminus \Gamma_f) . \quad (6.2.4)$$

We deduce that for any $D > 0$ we have:

$$\tilde{I}(\emptyset, D) - \tilde{I}(K, D) \leq \frac{D^2}{2} \int_{\partial\Omega'} u_\emptyset \cdot (\nabla u_\emptyset \tau) - G\mathcal{H}^1(K \setminus \Gamma_f) . \quad (6.2.5)$$

Let us return to the congruence of curves defined by $\overline{\pi}$ and consider the projection function on $\Gamma^2_u$ with respect to the congruence. For any set $B \subset \Gamma^2_u$ we denote by $V\overline{\pi}(B)$ the variation of $\overline{\pi}$ on $B$. We see that:

$$\frac{D^2}{2} \int_{\partial\Omega'} (\nabla u_\emptyset n) \cdot u_\emptyset \, d\mathcal{H} \geq \frac{1}{2} D^2 V\overline{\pi}(P(K)) , \quad (6.2.6)$$

because of the equality:

$$\inf \left\{ \frac{D^2}{2} \int_{\partial\Omega'} (\nabla u_\emptyset n) \cdot u_\emptyset \, d\mathcal{H} \mid \Omega' \in V(\overline{\pi}), K \setminus \Gamma_f \subset \Omega' \right\} = \frac{1}{2} D^2 V\overline{\pi}(P(K)) .$$

From the inequalities (6.2.5) and (6.2.6) we obtain the improved estimation:

$$\tilde{I}(\emptyset, D) - \tilde{I}(K, D) \leq \frac{1}{2} D^2 V\overline{\pi}(P(K)) - G\mathcal{H}^1(K \setminus \Gamma_f) . \quad (6.2.7)$$

Let us remark that

$$V\overline{\pi}(K) \geq V\overline{\pi}(P(K))$$

therefore we have:

$$\tilde{I}(\emptyset, D) - \tilde{I}(K, D) \leq \frac{1}{2} D^2 V\overline{\pi}(K) - G\mathcal{H}^1(K \setminus \Gamma_f) .$$
Due to the assumption (recall the notation \( u(D) = u(u_0(D), \emptyset) \))

\[
C \geq |\nabla u(1)n|^2 \geq c > 0
\]

we have \( \overline{\pi}(K) \leq C \mathcal{H}^1(K) \) hence (if we suppose that \( K \cap \Gamma_f \) is \( \mathcal{H}^1 \) negligible):

\[
\tilde{I}(\emptyset, D) - \tilde{I}(K, D) \leq \left[ \frac{1}{2} D^2 C - G \right] \mathcal{H}^1(K).
\]

(6.2.9)

Therefore, if

\[
\frac{1}{2} D^2 C < G,
\]

from (6.2.9) we see that \( \emptyset \) minimizes \( \tilde{I}(\cdot, D) \), which proves the point i) of the theorem.

Let us go back to (6.2.7) and introduce \( \Gamma(K) \) as the curve with the properties:

- p1) for any \( \Omega' \in V(\overline{\pi}) \), if \( K \setminus \Gamma_f \subset \Omega' \) then \( \Gamma(K) \subset \Omega' \),
- p2) \( \Gamma(K) \) is a length minimizer in the class of curves that fulfills p1).

We remark that \( \Gamma(K) \) might not be unique, but it always exists.

It is straightforward that \( \Gamma(K) = \Gamma(P(K)) \) and \( V\overline{\pi}(P(K)) = V\overline{\pi}(\Gamma(K)) \). We have then:

\[
\tilde{I}(\emptyset, D) - \tilde{I}(K, D) \leq \frac{1}{2} D^2 V\overline{\pi}(\Gamma(K)) - G \mathcal{H}^1(\Gamma(K)).
\]

(6.2.10)

From the assumption (6.2.8) we see that

\[
\frac{1}{2} D^2 V\overline{\pi}(P(K)) - G \mathcal{H}^1(\Gamma(K)) \geq \left[ \frac{D^2}{2} c - G \right] \mathcal{H}^1(\Gamma(K))显然, (6.2.11)

Therefore, if

\[
\frac{D^2}{2} c > G
\]

then the right member of (6.2.11) is positive and it attains the maximum when \( \mathcal{H}^1(\Gamma(K)) \) is maximal. This happens when \( \Gamma(K) \) separates \( \Gamma_u^1 \) from \( \Gamma_u^2 \). In this case is easy to see that we have equality in the relation (6.2.10), which proves the point ii) of the theorem.

The proof of iii) it is now straightforward. If \( C = c \) then

\[
\frac{1}{2} D^2 C = \frac{D^2}{2} c.
\]

### 7 Conclusions and perspectives

The first model contains only a constant connected to fracture, namely the constant of Griffith \( G \). The main qualities of this model are:

- i) crack appearance is allowed, together with crack propagation,
ii) there is no restriction concerning the pattern of the crack during its evolution.

We have seen that in the first model the critical stress which lead to fracture (or crack appearance) is not a constant of material.

The second model contains two constants of material connected to fracture: \( G \) and a constant with the dimension of a stress named \( \Sigma \). In this model the critical stress which lead to crack appearance is a constant of material, related to \( \Sigma \). This model has the same qualities as the first.

These two models are fully macroscopical, in the sense that no fracture mechanism based on micro-cracks or other micro-defects was supposed.

The main open theoretical problem is the general existence of an energy minimizing movement according to our definitions. Below is described an existence result based on a sound physical assumption \((7.0.1)\). Nevertheless, we do not know if \((7.0.1)\) can be proved from the basic assumptions of the model.

**Theorem 7.1.** Let us consider for a given \( s \) an incremental solution \( k \mapsto (u^s(k), K^s(k)) \in M \), according to definition 4.1. For any \( k \in \mathbb{N} \) we introduce the displacement

\[
v^s(k+1) = u((k+1)/s, K^s(k)) \ .
\]

Let us suppose that the power communicated by the rest of the universe to the body is bounded at any moment \( t \). The incremental form of this assumption consists in the existence of a constant \( P \) such that for any \( k \) and \( s \) we have

\[
\langle T(K^s(k)) \frac{1}{2} (u_0((k+1)/s) + u_0(k/s)) , u_0((k+1)/s) - u_0(k/s) \rangle \leq P/s \ . \quad (7.0.1)
\]

Then for any \( t > 0 \) there exist diverging sequences \((s_i)_i\) and \((k_i)_i\) such that \( k_i/s_i \) converges to \( t \) and \((u^s_i, K^s_i)(k_i)\) converges to an element of \( M(\mathbf{u}, K)(t) \).

**Proof:** From the minimality assumption on the incremental solution we have for any \( k \in \mathbb{N} \) the inequality:

\[
J((u^s(k), K^s(k)), (v^s(k+1), K^s(k))) \geq J((u^s(k), K^s(k)), (u^s(k+1), K^s(k+1))) \ .
\]

This inequality means that:

\[
\int_{\Omega} w(\nabla v^s(k+1)) \, dx \geq \int_{\Omega} w(\nabla u^s(k+1)) \, dx + G\mathcal{H}^{n-1}(K^s(k+1) \setminus K^s(k)) \ .
\]

The crack growth condition \( K^s(k) \subset K^s(k+1) \) implies that the latter relation can be put in the following form:

\[
\left( \int_{\Omega} w(\nabla v^s(k+1)) \, dx - \int_{\Omega} w(\nabla u^s(k)) \, dx \right) + G\mathcal{H}^{n-1}(K^s(k+1)) \geq \int_{\Omega} w(\nabla u^s(k+1)) \, dx + G\mathcal{H}^{n-1}(K^s(k+1)) \ . \quad (7.0.2)
\]
This is the incremental form of the Griffith criterion of crack propagation (3.3.6).
Indeed, we have the chain of equalities:

\[
\int_\Omega w(\nabla v^s(k + 1)) \, dx - \int_\Omega w(\nabla u^s(k)) \, dx = \frac{1}{2} \langle T(K^s(k)) u_0((k + 1)/s), u_0((k + 1)/s) \rangle - \frac{1}{2} \langle T(K^s(k)) u_0(k/s), u_0(k/s) \rangle =
\]

\[
= \langle T(K^s(k)) u_0((k + 1)/s) + u_0(k/s), u_0((k + 1)/s) - u_0(k/s) \rangle .
\]

\(v^s(k + 1)\) represents the displacement of the body with the boundary displacement \(u_0(k/s + 1)\) in the presence of the crack \(K^s(k)\). \(u^s(k)\) represents the displacement of the body with the boundary displacement \(u_0(k/s)\) in the presence of the same crack \(K^s(k)\). According to (3.3.5), the quantity

\[
\left( \int_\Omega w(\nabla v^s(k + 1)) \, dx - \int_\Omega w(\nabla u^s(k)) \, dx \right) / \left( \frac{1}{s} \right)
\]

is the discretized expression of the power communicated by the rest of the universe to the body at the moment \(k/s\), when a time discretization with step \(1/s\) is considered.

We deduce from the inequality (7.0.2) that

\[
P/s + \int_\Omega w(\nabla u^s(k)) \, dx + GH^{n-1}(K^s(k)) \geq \int_\Omega w(\nabla u^s(k + 1)) \, dx + GH^{n-1}(K^s(k + 1)) .
\]

We have therefore:

\[
P k/s \geq \int_\Omega w(\nabla u^s(k + 1)) \, dx + GH^{n-1}(K^s(k + 1)) .
\]

From the compactness theorem for SBD space and the latter inequality we deduce that for any \(t > 0\) there exist diverging sequences \((s_i)_i\) and \((k_i)_i\) such that \(k_i/s_i\) converges to \(t\) and \((u^{s_i}, K^{s_i})(k_i)\) converges to an element of \(M(u, K)(t)\).

In the paper [AB] Ambrosio & Braides introduce a generalized minimizing movement based model for the propagation of a crack in the presence of viscous forces in the body. They give as initial datum at \(t = 0\) the anti-plane displacement \(u_0 \in SBV(\Omega, R) \cap L^\infty(\Omega, R)\). For a given \(s\) they recursively define a sequence \((u^s_k)_k\) in \(SBV(\Omega, R)\) and an increasing sequence of closed rectifiable sets \((K^s_k)_k\) as follows: \(u_0^s = u_0, K_0^s = \emptyset\) and \(u_{k+1}^s = w, K_{k+1}^s = S_w \cup K_k^s\), where \(w\) is a minimizer of the functional

\[
v \mapsto \int_\Omega |\nabla v|^2 \, dx + \mathcal{H}^{n-1}(S_v \setminus K^s_k) + s \int_\Omega |v - u^s_k|^2 \, dx \quad (7.0.3)
\]

over the set of all \(v\) such that:

\[
v \in SBV(\Omega, R) , \quad \|v\|_\infty \leq \|u_0\|_\infty .
\]
The generalized minimizing movements obtained as limits of such incremental solutions, when \( s \) diverges, correspond to the following situation: a body evolves from the initial state \( u_0 \), with the initial crack \( S_{u_0} \), under a constant imposed boundary displacement. The equation of evolution for the displacement is:

\[
\text{div} \nabla u(t) + \dot{u}(t) = 0 .
\]

The authors obtain an existence result for the generalized minimizing movement introduced by them. After the introduction of the piecewise constant function:

\[
u^s(t) = u^s_{[st]},
\]

they find the following estimation:

\[
\|u^s(t') - u^s(t)\|_{L^2} \leq M \sqrt{t' - t} + \frac{1}{s} \text{ if } t' \geq t .
\]

Therefore there exists a diverging sequence \((s_i)\), such that \( u^{s_i} \) converges to \( u \) uniformly in \( L^\infty([0,T], L^2(\Omega, R)) \), for all \( T > 0 \). A consequence of this result is that the crack appearance is forbidden in this model.

This result is obtained under the assumption of constant imposed boundary displacement, equal to the trace on the boundary of the initial datum \( u_0 \).

It is natural to introduce the Lamé constant \( \mu \) and the viscosity \( \lambda \) in the expression of the functional (7.0.3) and modify it like this:

\[
v \mapsto \int_\Omega \mu |\nabla v|^2 \, dx + \mathcal{H}^{n-1}(S_v \setminus K^s_k) + \lambda s \int_\Omega |v - u^s_k|^2 \, dx .
\]

We obtain the more physical case of an anti-plane displacement satisfying at any moment \( t \) the equation:

\[
\text{div} \mu \nabla u(t) + \lambda \dot{u}(t) = 0 .
\]

The estimation (7.0.4) becomes

\[
\|u^s(t') - u^s(t)\|_{L^2} \leq M \sqrt{t' - t} + \frac{1}{\lambda s} \text{ if } t' \geq t .
\]

We expect to obtain our first model, in the case of anti-plane displacements, when the viscosity \( \lambda \) converges to 0. It is easy to see that if \( \lambda \) converges to 0 then the uniform estimation from above is lost, hence there is no contradiction between the fact that in our model crack appearance is allowed and the fact that in the model of Ambrosio & Braides crack appearance is forbidden.

As a conclusion, an open direction of research consists in the use of more general minimizing movements in order to study the propagation of a crack in the presence of viscous effects (as is the paper [AB]) or in the case of an elasto-plastic body.
The models presented in the paper are of applicative interest. In order to use them we have to know how to minimize a Mumford-Shah functional. This can be done by approximating, in the sense of variational convergence, the original functional by a less strange one. The idea is to replace the pair displacement-crack \((u, K)\) with the pair \((u, f)\), where \(f\) is a smoothed version of the characteristic function of the crack set \(K\), taking values in the interval \([0, 1]\). The original functional may be replaced by an Ambrosio-Tortorelli approximation, introduced in \([AT1]\), \([AT2]\). This opens the path to future interesting numerical results.
References

A1 L. Ambrosio, Variational problems in SBV and image segmentation, Acta Appl. Mathematicæ17, 1989, 1-40

A2 L. Ambrosio, Existence Theory for a New Class of Variational Problems, Arch. Rational Mech. Anal., vol. 111, 1990, 291-322

A3 L. Ambrosio, The space $SBV(\Omega)$ and free discontinuity problems, in Variational and Free Boundary Problems, editori A. Friedman, J. Spruck, IMA Vol. in Math. and Its Appl., vol. 53, Springer-Verlag, 1994, 1-24

AB L. Ambrosio, A. Braides, Energies in SBV and Variational Models in Fracture Mechanics, Proceedings of the EurHomogenization congress, Nizza, Gakuto Int. Series, Math. Sci. and Appl., 9, 1–22, 1997.

ABF L. Ambrosio, G. Buttazzo, I. Fonseca, Lower semicontinuity problems in Sobolev spaces with respect to a measure, J. Math. Pures Appl. 75, 1996, 211-224

ACDM L. Ambrosio, A. Coscia, G. Dal Maso, Fine Properties of Functions with Bounded Deformation, Preprint SISSA 8/96/M, 1996

All W.K. Allard, On the first variation of a varifold, Ann. of Math., vol. 95, no. 3, 1972, 417-491

AT1 L. Ambrosio, V.M. Tortorelli, Approximation of Functionals Depending on Jumps by Elliptic Functionals via $\Gamma$-convergence, Comm. Pure Appl. Math., vol. 43, 1990, 999-1036

AT2 L. Ambrosio, V.M. Tortorelli, On the approximation of free discontinuity problems, Boll. U.M.I., 6-B, 1992, 105-123

BCDM G. Bellettini, A. Coscia, G. Dal Maso, Compactness and lower semicontinuity properties in $SBD(\Omega)$, Preprint S.I.S.S.A. 86/96/M, 1996

Ba J. Ball, Some recent developments in nonlinear elasticity and its applications to material sciences, to appear in Proc. EPSRC Spring School 1995, Cambridge Univ. Press

Bu1 M. Buliga, Variational Formulations in Brittle Fracture Mechanics, PhD Thesis, Institute of Mathematics of the Romanian Academy, 1997

Bu2 M. Buliga, Modelisation de la décohésion d’interface fibres-matrice dans les matériaux composites, mémoire de D.E.A., Ecole Polytechnique, 1995

Bu3 M. Buliga, Energy concentration and brittle crack propagation, J. of Elasticity, 52, 3, 201-238, 1999
CZ A.P. Calderon, A. Zygmund, On the differentiability of functions which are of bounded variation in Tonelli’s sense, Rev. Un. Mat. Argentina, 20, 1960, 102-121

DGA E. De Giorgi, L. Ambrosio, Un nuovo funzionale del calcolo delle variazioni, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., 82, 1988, 199-210

DGCL E. De Giorgi, G. Carriero, A. Leaci, Existence theory for a minimum problem with free discontinuity set, Arch. Rational Mech. Anal., vol. 108, 1989, 195-218

DP G. Del Piero, Recent developments in the mechanics of materials which do not support tension, in Free Boundary Problems: Theory and Applications, vol I, Eds. Hoffmann K. H., Sprekels J., Pitman res. notes in math. series, Longman Scientific & Technical, 1990

EbM D. G. Ebin, J. Marsden, Groups of diffeomorphisms and the motion of an incompressible fluid, Ann. Math., vol. 92, no.1, 1970, 102-163

Es J. D. Eshelby, Energy relations and the energy-momentum tensor in continuum mechanics, Inelastic Behavior of Solids, ed. M.F. Kanninen et al., New York: McGraw-Hill, 1970, 77-115

EG L. C. Evans, R. F. Gariepy, Measure Theory and Fine Properties of Functions, CRC Press, 1992

FMa G. Francfort, J.-J. Marigo, Stable damage evolution in a brittle continuous medium, Eur. J. Mech., A/Solids, 12, no. 2, 1993, 149-189

G A.A. Griffith, The phenomenon of rupture and flow in solids, Phil. Trans. Royal Soc. London, A 221, 1920, 163-198

Gu1 M. E. Gurtin, On the Energy Release Rate in Quasistatic Elastic Crack Propagation, J. of Elasticity, vol 9, no. 2, 1979, 187-195

Gu2 M. E. Gurtin, Thermodynamics and the Griffith Criterion for Brittle Fracture, Int. J. Solids Structures, vol. 15, 1979, 553-560

I G.R. Irwin, Structural Mechanics, Pergamon Press, London, England, 1960

MS D. Mumford, J. Shah, Optimal approximation by piecewise smooth functions and associated variational problems, Comm. on Pure and Appl. Math., vol. XLII, no. 5, 1989, 577-685

Oht1 K. Ohtsuka, Generalized J-integral and Its Applications I. Basic Theory, Japan J. Appl. Math., 2, 1985, 21-52

Oht2 K. Ohtsuka, Generalized J-integral and three-dimensional fracture mechanics I, Hiroshima Math. J.,11, 1981, 329-350
Oht3 K. Ohtsuka, Generalized J-integral and three-dimensional fracture mechanics II, Hiroshima Math. J., 16, 1986, 327-352

Oht4 K Ohtsuka, Generalized J-integral and its applications, RIMS Kokyuroku A62, 149-165

R J.R. Rice, Mathematical analysis in the mechanics of fracture, in Fracture: an Advanced Treatise, vol. 2, ed. H. Liebowitz, Academic Press, 1969, 191-311

StLe H. Stumpf, K. Ch. Le, Variational principles of nonlinear fracture mechanics, Acta Mechanica 83, 1990, 25-37