Wormhole Geometries In $f(R, T)$ Gravity

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Abstract We study wormhole solutions in the framework of $f(R, T)$ gravity where $R$ is the scalar curvature, and $T$ is the trace of the stress-energy tensor of the matter. We have obtained the shape function of the wormhole by specifying an equation of state for the matter field and imposing the flaring out condition at the throat. We show that in this modified gravity scenario, the matter threading the wormhole may satisfy the energy conditions, so it is the effective stress-energy that is responsible for violation of the null energy condition.

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1 Introduction

One of the great discoveries in modern cosmology has been the accelerating expansion of the universe [1,2,3]. Among the various interesting possibilities invoked in order to explain the cosmic speed up, $f(R)$ modified gravity models ($R$ is the scalar curvature) have attracted a lot of attention (see [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15] and references therein). The Einstein field equations of General Relativity were firstly derived from an action principle by Hilbert, by adopting a linear function of the scalar curvature, $R$, in the gravitational Lagrangian density. An extension of the Einstein-Hilbert Lagrangian to the $f(R)$ gravity scenario can be performed naturally as there is no a priori reason why the gravitational action should be linear in the Ricci scalar $R$. Furthermore, higher order terms can naturally appear in low energy effective Lagrangian of quantum gravity and string theory.

In Ref. [16], a generalization of $f(R)$ modified theories of gravity was proposed by including in the theory a coupling of an arbitrary function of the Ricci scalar...
with the trace of the stress-energy tensor $T$, i.e. $f(R, T)$ gravity. They have investigated some astrophysical and cosmological aspects of the scenario by choosing several functional forms for $f$ (see also [17,18]).

In this paper, we explore the possibility whether static and spherically symmetric traversable wormhole geometries are supported by $f(R, T)$ gravity. We show that, in this modified theory, static wormhole throats respecting the null energy condition (NEC) can exist. Note that as is widely known, traversable wormholes as solutions to the Einstein equations can only exist with exotic matter which violates the null energy condition [19,20,21,22,23,24]. The null energy condition holds if $T_{\mu\nu}n^\mu n^\nu \geq 0$ for any null vector field $n^\mu$. The search of realistic physical models providing the wormhole existence represents an important direction in wormhole physics. Various models of such kind include scalar fields [25,26]; wormholes geometries induced by quantum effects [27,28], wormhole solutions in semi-classical gravity [29,30]; solutions in modified gravity [31,32,33,34]; wormholes on the brane [35,36]; wormholes supported by generalized Chaplygin gas [37]; wormhole solutions in Einstein-Gauss-Bonnet theory [38,39]; modified teleparallel gravity [40], etc (for instance see [41] and references therein).

This paper is organized as follows. In section II, we present a brief review of the fundamental concepts of $f(R, T)$ gravity, the action of the scenario and equations of motion. We explore the wormhole geometries in $f(R, T)$ gravity in section III. Firstly, we introduce the space-time metric and the necessary conditions to have a traversable wormhole solution. In the second stage, we choose a special functional form for $f$ and investigate the solutions of the gravitational field equations. By specifying an equation of state for the matter field, we obtain the shape function of the wormhole. We impose that the matter threading the wormhole satisfies the energy conditions. So, it is the effective stress-energy tensor that is responsible for the violation of the null energy condition. Finally, our summery and conclusions are presented in section IV.

2 $f(R, T)$ gravity

The action of $f(R, T)$ gravity is of the following form [16]

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} f(R, T) + \int d^4x \sqrt{-g} L_m$$

(1)

Here $f(R, T)$ is an arbitrary function of the scalar curvature, $R = R^\mu_\mu$, and the trace $T = T^\mu_\mu$ of the stress-energy tensor of the matter, $T_{\mu\nu}$. $L_m$ is the Lagrangian density of the matter and is related to the stress-energy tensor as follows

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta (\sqrt{-g} L_m)}{\delta g^{\mu\nu}}.$$  

(2)

Assuming that the Lagrangian density of matter $L_m$ depends only on the metric $g_{\mu\nu}$, we deduce that

$$T_{\mu\nu} = g_{\mu\nu} L_m - \frac{\partial L_m}{\partial g^{\mu\nu}}$$

(3)
Varying the action (1) with respect to the metric provides the field equations of \( f(R, T) \) gravity \[16\]

\[
\begin{align*}
 f_R(R, T) \left( R_{\mu\nu} - \frac{1}{3} R g_{\mu\nu} \right) &+ \frac{1}{6} f(R, T) g_{\mu\nu} = \\
8\pi G &\left( T_{\mu\nu} - \frac{1}{3} T g_{\mu\nu} \right) - f_T(R, T) \left( T_{\mu\nu} - \frac{1}{3} T g_{\mu\nu} \right) \\
&- f_T(R, T) \left( \Theta_{\mu\nu} - \frac{1}{3} \Theta g_{\mu\nu} \right) + \nabla_\mu \nabla_\nu f_R(R, T) .
\end{align*}
\] (4)

where we have denoted \( f_R(R, T) = \partial f(R, T) / \partial R \) and \( f_T(R, T) = \partial f(R, T) / \partial T \), respectively and

\[
\Theta_{\mu\nu} \equiv g^{\alpha\beta} \delta T_{\alpha\beta} / \delta g^{\mu\nu} .
\] (5)

In this paper, we assume that the matter Lagrangian is given by \( \mathcal{L}_m = -\rho \), where \( \rho \) is the energy density (see \[42, 43, 33\]). As a result, Equ. (5) takes the following form

\[
\Theta_{\mu\nu} = -2T_{\mu\nu} - \rho g_{\mu\nu} .
\] (6)

3 Whormhole geometries in \( f(R, T) \) gravity

3.1 Spacetime metric and the Gravitational Field Equations

For the wormhole metric, we consider the following line element \[22\]

\[
ds^2 = -e^{2\Phi(r)} dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) ,
\] (7)

where \( \Phi(r) \) and \( b(r) \) are two arbitrary functions of \( r \) known as the redshift and shape functions respectively. The radial coordinate \( r \) is non-monotonic such that it decreases from infinity to a minimum value \( r_0 \), representing the location of the throat of the wormhole, where \( b(r_0) = r_0 \), and then it increases from \( r_0 \) towards infinity. To have a traversable wormhole solution, it is necessary to impose the flaring out condition, given by \( (b - b')/b^2 > 0 \), \[22, 31\], and at the throat with \( b(r_0) = r = r_0 \), the conditions \( b'(r_0) < 1 \) and \( 1 - b(r)/r > 0 \) are imposed. For the wormhole to be traversable, one must demand that there are no horizons present, which are identified as the surfaces with \( e^{2\Phi} \to 0 \), so that \( \Phi(r) \) must be finite everywhere. In the following analysis, for simplicity, we consider that the redshift function is constant so, \( \Phi' = 0 \).

In the rest of this paper, we assume that \( f(R, T) = R + 2f(T) \), where \( f(T) \) is an arbitrary function of the trace of the stress-energy tensor. The gravitational field equations (4), by the definition (6) take the following form

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} + 2f(T)_{\mu\nu} + (2f + 2f') g_{\mu\nu} .
\] (8)

where \( f = f(T) \) and \( F = \frac{df}{dT} \). Supposing \( 8\pi G \equiv 1 \), this equation can be recast in the form

\[
G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T^{(\text{eff})}_{\mu\nu} ,
\] (9)
where the effective stress-energy tensor is defined by $T^{(\text{eff})}_{\mu\nu} = T^{(m)}_{\mu\nu} + \tilde{T}^{(m)}_{\mu\nu}$. The latter term is given by

\[ \tilde{T}^{(m)}_{\mu\nu} = 2F T_{\mu\nu} + (2\rho F + f) g_{\mu\nu}. \]

For the matter content of the wormhole, we consider an anisotropic fluid source whose stress-energy tensor satisfies the energy conditions and is given by [22]

\[ T_{\mu\nu} = (\rho + p_t) u_\mu u_\nu + p_t g_{\mu\nu} + (p_r - p_t) \chi_\mu \chi_\nu. \]  

(10)

Here $\rho$, $p_t$ and $p_r$ are the energy density, the perpendicular (to the inhomogeneous direction) pressure, and the parallel pressure respectively as measured in the fluid elements rest frame. The vector $u_\mu$ is the fluid four-velocity and $\chi_\mu$ is a space-like vector orthogonal to $u_\mu$. With these considerations, the stress-energy tensor takes a diagonal form, i.e., $T'_{\mu\nu} = \text{diag}[-\rho(r), p_r(r), p_t(r), p_t(r)]$. Thus, the gravitational field equations (9) are given as follows

\[ \frac{b'}{r^2} = \rho - f, \]

(11)

\[ -\frac{b}{r^3} = p_r (1 + 2F) + 2\rho F + f, \]

(12)

\[ \frac{b - b' r}{2r^3} = p_t (1 + 2F) + 2\rho F + f. \]

(13)

Now we assume that $f(T) = \lambda T$, where $\lambda$ is a constant [16] and with Eq. (11), $T = -\rho + p_r + 2p_t$. Thus the field equations (11)-(13) yield the following results

\[ \rho = \frac{b'}{r^2(1 + 2\lambda)}, \]

(14)

\[ p_r = -\frac{b}{r^3(1 + 2\lambda)}, \]

(15)

\[ p_t = \frac{(b - b' r)}{2r^3(1 + 2\lambda)}. \]

(16)

These equations describe the matter threading the wormhole, as a function of the shape function and the coupling parameter $\lambda$. Note that in the case $\lambda = 0$, the general relativistic limit can be recovered. The system of equations (14)-(16) are three equations with four unknown functions $\rho(r)$, $p_r(r)$, $p_t(r)$ and $b(r)$. There are different strategies to solve the field equations. For example, by specifying an equation of state for the matter field, one can obtain the shape function and the stress-energy components.

3.2 Energy Conditions

As has been mentioned in the introduction, the existence of the wormhole solution in general relativity relies on the violation of the null energy condition. The null energy condition holds if

\[ T_{\mu\nu} n^\mu n^\nu \geq 0 \]

for any null vector field $n^\mu$. However, if the theory of gravity is chosen to be more complicated than Einstein gravity, one may circumvent this issue and possess a
throat region which respects energy conditions. Thus, considering a radial null
vector, violation of the Null energy condition, i.e., $T^\mu_{\nu}n^\mu n^\nu < 0$ is given by
\[ \rho^{(\text{eff})} + p_r^{(\text{eff})} = (1 + 2\lambda)(\rho + p_r) < 0. \] (17)
On the other hand, with the field equations (14)-(16) we deduce that
\[ \rho^{(\text{eff})} + p_r^{(\text{eff})} = \frac{b' r - b}{r^3}. \] (18)
Using the flaring out condition i.e., $(b' r - b)/b^2 < 0$, this term is negative. If we
suppose the matter threading the wormhole to satisfy the energy conditions,
imposing the weak energy condition (WEC) given by $\rho \geq 0$ and $\rho + p_r \geq 0$, we see
that the coupling parameter $\lambda$ is limited to $\lambda \leq -\frac{1}{2}$.

3.3 Special Solution: $p_r = \alpha \rho$

In this section, we adopt a special equation of state for the matter field threading
the wormhole and obtain the solution of the field equations (14)-(16). An inter-
esting equation of state is a linear relation between the radial pressure and the
energy density, i.e., $p_r = \alpha \rho$, where $\alpha$ is a constant. Using this equation of state
provides the following shape function
\[ b(r) = r_0 \left( \frac{\rho_0}{r} \right)^{1/\alpha}. \] (19)
Note that with the condition at the throat $1 - \frac{b(r)}{r} \geq 0$ (which is equivalent
to $r - b(r) \geq 0$), the allowed region for the parameter $\alpha$ is restricted to $\alpha > 0$ and
$\alpha < -1$. In Figs. 1 and 2 the shape function versus $r$ is plotted for $\alpha = 0.6$ and
$\alpha = -1.5$ respectively. As the figures show, the fundamental wormhole condition,
i.e. $b(r) < r$ is fulfilled.

Using the shape function (19) with gravitational field equations (14)-(16), the
stress-energy components are given by
\[ p_r = \alpha \rho = -\frac{C r^{-(3+1/\alpha)}}{(1+2\lambda)}, \] (20)
and
\[ p_t = C(\alpha + 1) r^{-(3+1/\alpha)} \frac{1}{2\alpha(1+2\lambda)}, \] (21)
where $C = r_0^{1+1/\alpha}$. Fig. 3 shows the energy density versus $r$ for $\alpha = 0.6$ and
$\alpha = -1.5$. Obviously, choosing a negative value for $\alpha$ in the allowed region, leads
to a negative energy density for the matter threading the wormhole.

Now, using expression (21), the region satisfying the NEC condition at the
throat is depicted in Fig. 4 for $\alpha = 0.6$ and $\alpha = -1.5$ respectively. As the figures show,
the stress-energy tensor satisfies the null energy condition. As a result, for
a positive value of $\alpha$, the weak energy condition is satisfied. On the other hand,
by choosing a negative value for $\alpha$ in the allowed region, only the null energy
condition is respected.
Fig. 1 The shape function $b(r)$ versus $r$ for $\alpha = 0.6$ and $r_0 = 1$.

Fig. 2 The behavior of $b(r)$ versus $r$ for $\alpha = -1.5$ and $r_0 = 1$. 
Fig. 3 The energy density versus $r$ for $\alpha = 0.6$ (solid line) and $\alpha = -1.5$ (dashed line) with $\lambda = -1$ and $r_0 = 1$.

Fig. 4 The null energy condition is respected for $\alpha = 0.6$ (solid line) and $\alpha = -1.5$ (dashed line) with $\lambda = -1$ and $r_0 = 1$. 
4 Conclusion

The existence of the traversable wormholes as solutions to the Einstein field equations relies on the presence of some form of exotic matter which violate the null energy condition. However, in the framework of a modified theory of gravity, the situation may be completely different. In this paper, we have investigated wormhole solutions in \( f(R, T) \) modified gravity where \( R \) is the curvature scalar and \( T \) is the trace of the stress-energy tensor. We have shown that in the context of \( f(R, T) \) gravity, traversable wormhole solutions can be obtained, without the need to introducing any form of exotic matter. Violation of the energy conditions, which is essential for the existence of the wormhole solutions \([22]\), is realized via the presence of an effective stress-energy tensor generated by the additional curvature and matter terms.

To find the wormhole solutions in \( f(R, T) \) scenario, we have assumed that \( f(R, T) = R + 2f(T) \), where \( f(T) \) is an arbitrary function of the trace of the stress-energy tensor and we have derived the gravitational field equations. Then we have specified an equation of state for the matter threading the wormhole and by imposing the flaring out condition at the throat, we have obtained the shape function. We have shown that the stress-energy tensor of the matter threading the wormhole satisfies the null energy condition in some subspaces of the model parameter space. However, one can explore the wormhole solutions in \( f(R, T) \) modified gravity in more complicated situation than here. For example, in a recent work \([15]\), we investigated the wormhole solutions in the case that the gravity sector is also modified and we studied the energy conditions in this case. Furthermore one can consider an explicit coupling between \( T \) and \( R \) and explore the wormhole geometries.

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