Generalized Gross-Pitaevskii equation adapted to the $U(5) \supset SO(5) \supset SO(3)$ symmetry for spin-2 condensates

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A generalized Gross-Pitaevskii equation adapted to the $U(5) \supset SO(5) \supset SO(3)$ symmetry has been derived and solved for the spin-2 condensates. The spin-textile and the degeneracy of the ground state (g.s.) together with the factors affecting the stability of the g.s., such as the gap and the level density in the neighborhood of the g.s., have been studied. Based on a rigorous treatment of the spin-degrees of freedom, the spin-textiles can be understood in a $N$-body language. In addition to the ferro-, polar, and cyclic phases, the g.s. might in a mixture of them when $0 < M < 2N$ ($M$ is the total magnetization). The great difference in the stability and degeneracy of the g.s. caused by varying $\varphi$ (which marks the features of the interaction) and $M$ is notable. Since the root mean square radius $R_{\text{rms}}$ is an observable, efforts have been made to derive a set of formulae to relate $R_{\text{rms}}$ and $N, \omega$ (frequency of the trap), and $\varphi$. These formulae provide a way to check the theories with experimental data.

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I. INTRODUCTION

Since the success of trapping cold atoms via optical trap,[1] the study of condensates of atoms with nonzero spin is a hot topic. There are numerous literatures dedicated to the study of the ground state (g.s.) for spin-2 condensates, the g.s. was found to have three phases, namely, the ferro-phase, polar phase and cyclic phase, depending on the features of the interaction.[2–4] On the other hand, it has been revealed that the total Hamiltonian of the spin-2 systems is associated with the $U(5) \supset SO(5) \supset SO(3)$ symmetry.[5, 6] Since there are elegant theory dealing with this symmetry, it is worthy to derive a generalized Gross-Pitaevskii (GP) equation adapted to the $U(5) \supset SO(5) \supset SO(3)$ symmetry. Since the solutions of this equation could provide the details of the low-lying states, we believe that the understanding to the spin-2 condensates could be thereby enriched.

We consider the case that the magnetic field $B = 0$. The symmetry-adapted GP equation is derived firstly. Then, by solving the GP equation, both the spin-textiles and the spatial wave functions of the low-lying states can be known. The emphasis is placed on the g.s. Since the stability of the g.s. is affected by its environment, in addition to the g.s. itself, special effort is made to study the environment, namely, the width of the gap (the energy difference between the first excited state and the g.s.) and the level-density in the neighborhood of the g.s.. This is a topic less studied before. Besides, the degeneracy of the g.s. is an important feature which is also studied in this paper. The total magnetization $M$ is considered as being conserved, which depends on how the system is experimentally prepared. The influence of $M$ on the spin-textiles and the degeneracy of the g.s. is found to be great.

After we have obtained the spatial wave functions, physical quantities related to the spatial degrees of freedom can be known. In particular, we have calculated the root mean square radius $R_{\text{rms}}$. Since $R_{\text{rms}}$ is an observable, theoretical results and experimental data can be compared with each other. This is a notable way to check the theory. The GP equation will be firstly solved analytically under the Thomas-Fermi approximation (TFA). A number of formulae relating the physical quantities have been thereby obtained, and the underlying physics can be understood in an analytical way. Then, strict numerical calculations are performed to check the validity of the TFA.

II. INHERENT SYMMETRY AND THE GROSS-PITAEVSKII EQUATION

A condensate with $N$ spin-2 neutral atoms trapped by an isotropic harmonic potential $\frac{1}{2}m\omega^2r^2$ is considered, in which the spin-orbit coupling is assumed to be weak. It is further assumed that the trap is not so weak (say, $\omega \geq 100 \text{s}^{-1}$) and the spin-dependent interaction is weak so that the spin-modes are much lower than and separated from the spatially excited modes. This is the basic assumption of this paper. In other words, the following results hold only for the systems adapted to this assumption. In this case a group of excited states distinct in their spin-modes would emerge in the neighborhood of the g.s.. In each of these states (including the g.s.) all the particles have a common spatial wave function which

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is most advantageous to binding and adapted to a specific spin-textile. Thus one of these states with a specific spin-mode $\gamma$ can be in general written as

$$\Psi_\gamma = \prod_{j=1}^{N} \phi_\gamma(r_j)\Theta_\gamma,$$

where $\phi_\gamma$ is the common spatial wave function and $\Theta_\gamma$ is a total spin-state which is now unknown. If the particles did not fall into the same state but have some of them getting excited, the total energy would be thereby higher. The group $\{\Psi_\gamma\}$ totally form a band which is called the ground band. The states not in the ground band will contain various spatially excited modes and will not be studied in this paper.

When $h\omega$ and $\lambda = \sqrt{\hbar/(m\omega)}$ are used as units for energy and length (where $m$ is the mass and is given at the one of $^{87}$Rb), the total Hamiltonian is

$$H = \sum_i h(i) + \sum_{i<j} V_{ij},$$

where $h(i) = -\frac{1}{2}\nabla_i^2 + \frac{1}{2}r_i^2$, $V_{ij} = \delta(r_i - r_j)\sum_s g_s P_{ij}$. $P_{ij} = \sum_{m_i,m_j}\langle \chi(i)\chi(j)\rangle_{m_i,m_j}\langle \chi(i)\chi(j)\rangle_{m_i,m_j}$. $\chi(i)$ is the spin-state of the $i$-th particle, and the two spins of $i$ and $j$ are coupled to the combined spin $s$ and its $z$-component $m_i$. $s = 0, 2, 4$. Obviously, $P_{ij}$ is the projector onto the $s$-channel. $g_s$ is the strength related to the scattering length of the $s$-channel.

We introduce an operator defined in the total spin-space as $\mathbf{V} \equiv \sum_{i<j} \sum_s g_s P_{ij}$. It has been proved that $\mathbf{V}$ is a linear combination of the set of Casimir operators belonging to a chain of nested algebra as $U(5) \supset SO(5) \supset SO(3)$. Consequently, the eigenstates and the eigenenergies of $\mathbf{V}$ can be exactly known as

$$\mathbf{V}\Theta_\gamma = \tilde{E}_{\gamma}\Theta_\gamma.$$  

Now, $\gamma$ represents a group of quantum numbers $\nu$, $n_{tri}$, $S$, and $M$, where $S$ and $M$ are the total spin and its $z$-component, $\nu$ and $n_{tri}$ will be explained below, and $\Theta_\gamma = \Theta_{\gamma_{n_{tri},SM}}$ is the related eigenstate. For convenience, $M > 0$ is assumed.

When two spins are coupled to zero, they form a singlet pair $\theta_{\text{pair}}(ij) \equiv \langle \chi(i)\chi(j)\rangle_0$. When three spins are coupled to zero, they form a triplet and can be approximately denoted as $\theta_{\text{tri}} \approx \langle \chi\chi\chi\rangle_0$ (this notation is exact when $N \to \infty$). In the triplet every pair of spins is coupled to 2. It turns out that the singlet pair and triplet are basic building blocks. An eigenstate may contain $n_{\text{pair}}$ singlet pairs and $n_{\text{tri}}$ triplets. The number of particles neither in the pairs nor the triplets is $N_{\text{free}} \equiv N - 2n_{\text{pair}} - 3n_{\text{tri}}$, the total spin $S$ is contributed by them. It has been proved that, when $n_{\text{pair}}$, $n_{\text{tri}}$, $S$ and $M$ are given, the total-symmetric eigenstate $\Theta_{\gamma_{n_{tri},SM}}$ is unique. The four quantum numbers are bound by the following conditions: (i) $N_{\text{free}} \geq 0$, (ii) $N_{\text{free}} \leq S \leq 2N_{\text{free}}$, (iii) $S = 2N_{\text{free}} - 1$ is not allowed, and of course (iv) $|M| \leq S$. \[8\]

Let us define $g_{04} = g_0 - g_1, g_{24} = g_2 - g_4$, and $g_{024} = \frac{1}{2}(g_0 + g_2 + g_4)$. Then, the eigenenergy associated with $\Theta_\gamma$ is \[3\]

$$\tilde{E}_{\gamma} = a_1 N + a_2 N(N+1) + g_\nu v(v+3) + g_S S(S+1),$$

where $g_\nu = (-7g_{04} + 10g_{24})/70, g_S = (-g_{24})/14, a_1 = 11\frac{1}{15}g_0 - \frac{1}{15}g_{24} - \frac{2}{70}g_{024}$, and $a_2 = -\frac{1}{70}g_{04} - \frac{1}{15}g_{24} + \frac{1}{21}g_{024}$. Note that $\tilde{E}_{\gamma}$ does not depend on $n_{\text{tri}}$, implying that the levels might be degenerate as shown below.\[8\]

Inserting Eq.14 into the Schrödinger equation, we have

$$\sum_{j=2}^{N} \prod_{j=1}^{N} \phi_\gamma(r_j)\Theta_\gamma |H - E_\gamma| \prod_{j=1}^{N} \phi_\gamma(r_j)\Theta_\gamma = 0,$$

where the integration covers all the degrees of freedom except $r_1$. From this equation and making use of Eq.(3), it is straightforward to obtain the symmetry-adapted GP equation for the normalized $\phi_\gamma(r_1)$ as

$$[h + 2\frac{\tilde{E}_{\gamma}}{N}\phi_\gamma^*\phi_\gamma - \epsilon_\gamma]\phi_\gamma = 0,$$

where $\epsilon_\gamma$ is the chemical potential. After solving Eq.(6), we can obtain the total energy $E_\gamma = N\epsilon_\gamma - V_{\text{tot},\gamma}$, where $V_{\text{tot},\gamma} = \int |\phi_\gamma|^2 d\mathbf{r} \tilde{E}_{\gamma}$ is the total interaction energy. The whole spectrum of the ground band can be obtained from the set $\{E_\gamma\}$. Obviously, this equation is a generalization of the one for spin-1 condensates given in ref.[9] It is also a more accurate version for the one given as Eq.(74) in ref.[2]. In which the spin-dependent interaction has been neglected.

### III. Thomas-Fermi Approximation

Since $N$ is usually large, Eq.(6) can be solved by using the TFA. Neglecting the kinetic term in Eq.(6), when $\hbar\omega$ and $\lambda$ are used as units, the normalized TFA solution is $\phi_\gamma = \sqrt{\frac{15}{8\pi r_0}}\sqrt{1 - (r_0/r)^2}$ and $\epsilon_\gamma = r_0^2/2$, where

$$r_0 = \left(\frac{15}{2\pi N}\tilde{E}_{\gamma}\right)^{1/5},$$

is the TF-radius. From this solution we can obtain the root mean square radius $R_{\text{rms}} \equiv \langle r^2\rangle^{1/2} = \sqrt{3/\epsilon_\gamma}$, and the total energy

$$E_\gamma = \frac{1}{4} \left(\frac{32\gamma^7}{27\pi^2}\gamma^{1/5} N^{3/5} \tilde{E}_{\gamma}^{2/5}\right)^{3/5} = \frac{5}{6} N R_{\text{rms}}^2.$$

Thus, the total energy is directly related to the size (measured by $R_{\text{rms}}$).

When $N$ is large, we know from Eq.(4) that $\tilde{E}_{\gamma}S$ is proportional to $N^2$. Accordingly, $R_{\text{rms}} \propto N^{1/3}$ and $E_\gamma \propto N^{2/5}$.
FIG. 1: (color on line) The ground state energy \( E_{gs} \) (in \( \hbar \omega \)) versus \( \varphi \). \( N = 12000 \) and \( \omega = 300\)\(^{-1} \) are assumed (the same in the following figures except specified). The total magnetization is given at five cases as \( M = 0, 1000, N, 2N \). \( N = 1000 \), and \( 2N \), respectively, for the curves "1" to "5". There are three regions of \( \varphi \) separated by \( \varphi_a = \tan^{-1}(\frac{7(N+M/2+3)}{10(N+M/2+2)}) \) and \( \varphi_b = \tan^{-1}(\frac{-7}{10}) \) marked by the two vertical dotted lines. In each region the g.s. has its specific \( (v, S, d) \) marked in the region, where \( d \) is the degeneracy. In region I, \( E_{gs} \) is \( M \)-independent. In I and II, the g.s. is not degenerate (\( d = 1 \)). In III the degeneracy depends on \( M \) seriously. For the curves "1" to "5", \( d = 1, 167, 2001, 167, 1 \), respectively.

\[ N^{7/5} \text{. Thus the size increases with } N \text{ very slowly, but } E_\gamma \text{ increases with } N \text{ faster than linearly.} \]

On the other hand, it is noted that, in the units of \( \hbar \omega \) and \( \lambda \), all the strengths \( g_s \propto \sqrt{\omega} \), where \( \omega \) is the magnitude of \( \omega \), i.e., \( \omega = \omega_s^{-1} \). Therefore \( E_{vs} \propto \sqrt{\omega} \). When \( \omega \)-independent units are used, the size is described by \( R_{\text{rms}} \lambda \propto (\omega_s^{-2})^{-5/2} \), i.e., a stronger trap leads to a smaller size. Similarly, one can prove that the chemical potential \( \epsilon_s \hbar \omega \propto (\omega_s^{-6})^{1/5} \), and the total energy \( E_\gamma \hbar \omega \propto (\omega_s^{-6})^{1/5} \). Thus the total energy will increase with \( \omega \) a little faster than linearly.

IV. GROUND STATE AND ITS PHASE

It is obvious from Eq.(4) that which pair of \( v \) and \( S \) is more advantageous to binding depends on the competition between \( g_s \) and \( g_{24} \), which depend on \( g_{04} \) and \( g_{24} \), and they are considered as variable. Therefore, we introduce a variable parameter \( \varphi \) to manifest the competition. In the units of \( \hbar \omega \) and \( \lambda \), the strengths are assumed as \( g_{04} = u(\omega)\)\(^{1/2} \)\( \cos \varphi \), \( g_{24} = u(\omega)\)\(^{1/2} \)\( \sin \varphi \), \( g_{024} = 2.5u(\omega)\)\(^{1/2} \), and \( u = 10^{-3} \) (For a comparison, the realistic strengths for \( ^{87}\text{Rb} \) are \( g_{24} = -0.28u(\omega)\)\(^{1/2} \), \( g_{04} = -0.41u(\omega)\)\(^{1/2} \), and \( g_{024} = 2.39u(\omega)\)\(^{1/2} \)). In fact, it has been shown previously that the phase of the g.s. depends on \( \varphi \).[2][3]

Under the TFA, the ground state energy \( E_{gs} \) is just the lowest \( E_\gamma \) given by Eqs.(8) and (4). Although \( E_\gamma \) does not directly depend on \( M \), the condition \( S \geq |M| \) restricts the possible choice of \( S \), and \( E_{gs} \) is thereby affected. \( E_{gs} \) versus \( \varphi \) with \( M \) given at five values are shown in Fig.1. There are two critical points \( \varphi_a = \tan^{-1}(\frac{7(N+M/2+3)}{10(N+M/2+2)}) \) and \( \varphi_b = \tan^{-1}(\frac{-7}{10}) \). When \( \varphi \) goes through either \( \varphi_a \) or \( \varphi_b \) a phase transition will occur. This phenomenon was found as early as in 2000 by using the mean field theory (in which \( \varphi_a = \tan^{-1}(\frac{-7}{10}) \) and the same \( \varphi_b \) were found.[2] The present more accurate result will tend to the old one when \( N \) is large).

When \( \varphi \) is in \((0, \varphi_a)\), i.e., in region-I, the g.s. does not depend on \( M \) and will have \((v, S, d) = (N, 2N, 1)\), where \( d \) is the degeneracy of the state. Since all the spins align along a common direction in this g.s., it is in the ferro-phase.[3][4]

When \( \varphi \) is in \((\varphi_a, \varphi_b)\), i.e., in region-II, the g.s. depends on \( M \) and will have \((v, S, d) = (M/2, M, 1)\). Since \( v = M/2 \) and \( S = M \), all the \( M/2 \) unpaired spins must align along a common direction so that the total spin can be maximized (i.e., \( S \) can be equal \( M \)). Thus each g.s. in region-II is a mixture of a group of aligned spins together with the \((N - M/2)/2\) singlet pairs, namely, a mixture of ferro-phase and polar phase. When \( M = 0 \), \((v, S, d) = (0, 0, 1)\), thus it is in pure polar phase (we have assumed that \( N \) is even, otherwise \( S = 2 \)). When \( M \) is larger, more spins are in the ferro-phase, and accordingly \( E_{gs} \) gets higher as shown in the figure. When \( M = 2N \), \((v, S, d) = (N, 2N, 1)\), and the g.s. is in pure ferro-phase.

When \( \varphi \) is in \((\varphi_b, 2\pi)\), i.e., in region-III, the g.s. depends also on \( M \) and has \((v, S, d) = (N, M, d)\). Note that these g.s. have \( v = N \), therefore they do not contain any singlet pairs, but the spins in triplets are allowed. It is recalled that, in regions I and II, the g.s. has \( S = 2v \). It implies that all the unpaired spins must be aligned, and therefore there is no room for the triplets. Whereas in III, the number of triplets \( n_{tri} \) can have different choices in a specific domain under the constraints of symmetry given in points (i) to (iv) in the paragraph above Eq.(4) (while those spins not in the triplets are coupled to \( S = M \)). This leads to the degeneracy of the g.s.[5] When \( M = 0 \) the g.s. has \( S = 0 \) and therefore \( N_{free} = 0 \) (point (ii)). Thus, all the particles are in the triplets (i.e., in a pure cyclic phase), and henceforth \( d = 1 \) (we have assumed that \( N \) is a multiple of 3 to simplify the discussion). When \( M = 2N \), the g.s. has \( S = 2N \) and therefore \( N_{free} = N \). Thus, neither the pairs nor the triplets emerge (i.e., in a pure ferro-phase), and henceforth \( d = 1 \) also. When \( M \) is neither 0 nor \( 2N \), the g.s. is in general degenerate. In particular, when \( M = N \), \( N_{free} \) can be ranged from \( M/2 \) to \( M \) (each step is 3). In this case the degeneracy \( d \) is maximized (the curve "3") in III has \( d = 2001 \). The appearance of highly degenerate g.s. in the region III is a notable feature.

It is reminded that the polar phase is composed of \( s = 0 \) pairs. Therefore, when \( g_0 \) is less positive than \( g_2 \) and \( g_4 \), the g.s. will prefer this phase. The cyclic phase is composed of \( s = 0 \) triplets in which every pair of spins must be coupled to \( s = 2 \). Therefore, when \( g_2 \) is less
positive, the g.s. will prefer this phase. While the ferro-phase is composed of aligned spins, in which every two must be coupled to $s = 4$. Therefore, when $g_4$ is less positive, the g.s. will prefer this phase. Thus the phase transitions manifested above reflects the competition of the interactions of the three $s$-channels. The competition depends on $M$ because the least number of spins that must be aligned is determined by $M$.

When $M = 2N$ is given, the ferro-phase is the only choice disregarding how the interaction is as shown by the uppermost curve in Fig.1. When $N$ is sufficiently large the terms $\sim 1/N$ can be neglected, then we have

$$\tilde{E}_{\text{gs}} \approx (a_2 + g_v + 4g_s)N^2$$

$$= N^2 [-\frac{1}{6}(\cos \varphi + \sin \varphi) + 1.25]u(\bar{\omega})^{1/2}. \quad (9)$$

From this formula we know that $\tilde{E}_{\text{gs}}$ has a minimum at $\varphi = \pi/4$ and a maximum at $\varphi = 5\pi/4$ where $\tilde{E}_{\text{gs}} = (\frac{15\sqrt{2}}{12})N^2u(\bar{\omega})^{1/2}$. Since the total energy is proportional to $N^{3/2}(\tilde{E}_{\text{gs}})^{1/2}$ as shown in Eq.(8), with $N = 12000$ and our parameters, the minimum at $\varphi = \pi/4$ has $E_{\text{gs}} = 51612\hbar\omega$, and the maximum at $\varphi = 5\pi/4$ has $E_{\text{gs}} = 60125\hbar\omega$. This explains the origin of the dip and the highest peak in Fig.1.

Since the root mean square radius is an observable, it is desirable to study $R_{\text{rms}}$ so that the theory can be checked via experimental measurement. According to Eq.(7), $R_{\text{rms}}$ is proportional to $(E_{\text{gs}})^{1/2}$, thus the variation of $R_{\text{rms}}$ versus $\varphi$ with the minimum and maximum is predicted. In particular, when $\varphi = \pi/4$ ($5\pi/4$) and the unit $\lambda$ is replaced by $\mu m$, the minimum (maximum) in Fig.1 is associated with $R_{\text{rms}} = 3.53$ ($3.81$) $\mu m$. In region II and III, it is predicted that the increase of $M$ would lead to the increase of the size. This is also a point to be checked.

V. STABILITY OF THE GROUND STATE

It is believed that the stability of the g.s. is assured by the gap $E_{\text{gap}}$, namely, the energy difference between the first excited state and the g.s.. The former has its $(v, S)$ slightly different from that of the g.s., $(v_g, S_g)$, and can be easily obtained from Eq.(4). An example of $E_{\text{gap}}$ is shown in Fig.2. Note that when $M = 2N$, $(v, S) = (N, 2N)$ is the only choice, no choice other than $(N, 2N)$ is allowed. Thus, under $M$-conservation, there are no excited spin-modes. Hence, in order to show the gap for a very large $M$, the curve "5" in Fig.2 is not given as $M = 2N$ but $M = 2N - 4$. Fig.2 has the following features:

(i) Since Eq.(4) manifests that the energy is nearly $\propto v^2$ and $S^2$, the gap would be in general large if both $v_g$ and $S_g$ are large. In this case, a small deviation from $(v, S)$ will cause a great change in energy. Whereas if one of them is zero, the gap would be much lower. In region I the g.s. is in the ferro-phase with $(v_g, S_g) = (N, 2N)$. Therefore, the gap is very high and accordingly

![FIG. 2: (color on line) The energy gap $E_{\text{gap}}$ (in $\hbar\omega$) versus $\varphi$. The parameters are the same as in Fig.1, except that the curve "5" has $M = 2N - 4$ instead of $M = 2N$.](image)

the g.s. is very stable in the ferro-phase. In II and III, the upmost curve in Fig.2 has $M = 2N - 4$. Accordingly, $(v_g, S_g) = (N - 2, 2N - 4)$ and $(N, 2N - 4)$, respectively. Since they are also large, the gap is also high. When $M = 2N - 4$, the g.s. is very close to be in ferro-phase and therefore is stable. However, when $M$ decreases, the stability reduces. In particular, when $M = 0, (v_g, S_g) = (0, 0)$ and $(N, 0)$ in II and III, respectively. In this case $E_{\text{gap}} \leq 0.00012$ was found in both II and III as shown by the lowest curve in Fig.2. Thus the g.s. in polar and cyclic phases with a small $M$ are highly unstable.

(ii) $E_{\text{gap}}$ in every region appears as a peak, namely, in the middle part it is higher but very low when $\varphi$ is close to the borders. For Fig.2 the g.s. in I has $(v, S) = (N, 2N)$, while the first excited state has $(v, S) = (N, 2N - 2)$ if $\varphi \leq 0.4722\pi$. Thus the excitation from $(N, 2N)$ to $(N, 2N - 2)$ is caused by a change in $S$. However, when $\varphi \to 0$, accordingly $g_s \to 0$. Therefore the gap is zero. On the other hand, the first excited state has $(v, S) = (N - 2, 2N - 4)$ when $(0.4722\pi < \varphi \leq \varphi_a)$. From Eq.(4), we know the gap

$$E_{N,2N} - E_{N-2,2N-4}$$

$$= \frac{4}{70} [7(N + \frac{1}{2})g_{04} + 10(N - 2)g_{24}], \quad (10)$$

One can prove that the gap is zero when $\varphi = \varphi_{a'} \equiv \tan^{-1} \frac{1}{2(N + 1/2)}$. Since $\varphi_{a'}$ is extremely close to $\varphi_a$, this explains the decline of the gap when $\varphi \to \varphi_a$. Based on Eq.(4), the decline of the peak in II and III can be similarly explained. Thus, in the neighborhoods of the borders, the g.s. is highly unstable.

In addition to the gap, another factor that could affect the stability of the g.s. is the level density in the neighborhood of the lowest level. This density can be calculated based on Eq.(8). As an example, the number of levels with excitation energy $\leq 0.1\hbar\omega$ is given in Table III in which seven choices of $\varphi$ and four choices of
$M$ are chosen. The other parameters are the same as in Fig.1. If the degeneracy of a level is $d$, then the number contributed by this level is $d$.

**TABLE I: The number of levels lower than $0.1\hbar\omega$ versus $M$ and $\varphi$. $\varphi_a$ and $\varphi_b$ denote the borders, $\delta = 0.01\pi$. The first column gives the region of $\varphi$.**

| Region | $\varphi$ | Number of levels |
|--------|-----------|-----------------|
|        | $M = 0$   | $M = N$         | $M = 2N - 4$ |
| I      | $\pi/4$   | 1               | 1            | 1            |
| I      | $\varphi_a - \delta$ | 4               | 4            | 2            |
| II     | $\varphi_a + \delta$ | 23059          | 7            | 2            |
| II     | $\pi$     | 20596           | 1            | 1            |
| II     | $\varphi_b - \delta$ | 167171         | 19           | 2            |
| III    | $\varphi_b + \delta$ | 3372           | 7999         | 2            |
| III    | $3\pi/2$  | 885             | 2001         | 1            |

From the table we know that

(i) When $\varphi$ is in region I or $M$ is close to $2N$ (i.e., in or close to the ferro-phase), the density of low-lying states is rather diffuse. Together with the big gap, both factors assure the stability of the g.s. (When the number $= 1$ as shown in the second row, there is no excited states lower than $0.1\hbar\omega$).

(ii) When $\varphi$ is in region II and $M$ is very small (i.e., in or close to the polar phase), the density is extremely dense. Together with the very small gap, both factors lead to a highly unstable g.s. This situation can be greatly improved when $M$ gets larger.

(iii) When $\varphi$ is in region III and $M$ is not very large (say, $M \leq N$), the low-lying spectrum is also dense but not as dense as in the polar phase. Note that, when $M$ is not close to 0 or $2N$ (say, $0.05 \times 2N \leq M \leq 0.95 \times 2N$), it has been mentioned that the states in cyclic phase could be highly degenerate. The great degeneracy contributes to the level-density substantially (say, in the last row of Table I, the number 2001 arises completely from the degeneracy of the g.s.).

**VI. NUMERICAL SOLUTIONS OF THE SYMMETRY ADAPTED GP-EQUATION**

By making use of the TFA we have obtained analytical solutions, thereby the related physics can be understood in an analytical way. In order to evaluate the accuracy of the TFA, the equation with the kinetic energy included is solved numerically and the results are given below for a comparison. Firstly, we found that the curves for $E_{gs}$ from the numerical solutions as a whole is an upward shift of those plotted in Fig.1. It implies that the amount of total kinetic energies contained in various states with very different spin-textiles are similar, and the $E_{gs}$ given under the TFA are correct when $N = 12000$ (except the omission of the kinetic energy). The shift is about $3.5 \times 10^3\hbar\omega$ (thus the kinetic energy is about 6% of the total energy in our cases). Secondly, we found that the curves for $E_{gap}$ from the numerical solutions overlap the curves from TFA plotted in Fig.2. It implies that the amount of kinetic energies contained in the g.s. and in the first excited state is similar. Thus the spin-excitation does not remarkably affect the spatial motion.

**TABLE II: Root Means Square Radius $R_{rms}$ in $\mu$m from numerical calculation versus $N$, $\varphi$, and $M$ with $\omega = 300s^{-1}$.**

The values from TFA are given inside the parentheses.

| $N$ | $\varphi$ | $M$ | $R_{rms}$ | $N$ | $\varphi$ | $M$ | $R_{rms}$ | $N$ | $\varphi$ | $M$ | $R_{rms}$ |
|-----|-----------|-----|----------|-----|-----------|-----|----------|-----|-----------|-----|----------|
| 1200| $\pi/4$   | 0, $N$, $2N$ | 2.48(2.23) | 6000| $\pi/4$   | 0, $N$, $2N$ | 3.19(3.07) | 12000| $\pi/4$   | 0, $N$, $2N$ | 3.62(3.53) |
|     | $\pi$     | 0   | 2.57(2.35) |     | $\pi$     | 0   | 3.34(3.24) |     | $\pi$     | 0   | 3.80(3.72) |
|     |           | $N$ | 2.58(2.36) |     |           | $N$ | 3.36(3.25) |     |           | $N$ | 3.81(3.73) |
|     |           | 2$N$| 2.60(2.38) |     |           | 2$N$| 3.39(3.29) |     |           | 2$N$| 3.85(3.77) |
|     | $5\pi/4$  | 0   | 2.56(2.33) |     | $5\pi/4$  | 0   | 3.33(3.22) |     | $5\pi/4$  | 0   | 3.78(3.70) |
|     |           | $N$ | 2.58(2.35) |     |           | $N$ | 3.35(3.25) |     |           | $N$ | 3.80(3.73) |
|     |           | 2$N$| 2.62(2.40) |     |           | 2$N$| 3.42(3.32) |     |           | 2$N$| 3.88(3.81) |

Furthermore, the size of the system is expected to increase by including the kinetic energy. The results from the numerical calculation are shown in Table II where


FIG. 3: (color on line) $\log_{10} R_{\text{rms}}$ versus $\log_{10} N$ with $\omega = 300\text{ s}^{-1}$ where $R_{\text{rms}}$ is in $\mu\text{m}$. $M$ is given at three values. $\varphi = \pi/4$, $\pi$, and $5\pi/4$ for the curves "1" to "3", respectively. The lower panels are for the slopes of the curves.

FIG. 4: (color on line) $\log_{10} R_{\text{rms}}$ versus $\log_{10} \tilde{\varphi}$ with $N = 12000$. Refer to Fig.3. Note that the vertical scale in 4d to 4f is very small. It implies that the curves in 4a to 4c are very close to straight lines.

The $R_{\text{rms}}$ from TFA are given inside the parentheses and are smaller. From the table we know that the TFA leads to a $\sim 8\%$ reduction of the $R_{\text{rms}}$ (if $N = 1200$), or a $\sim 2\%$ reduction of the $R_{\text{rms}}$ (if $N = 12000$). Nonetheless, the two features found previously under the TFA remain unchanged, namely, (i) When the g.s. is not in the ferro-phase, a larger $M$ leads to a larger size. (ii) When the g.s. is in the ferro-phase ($\varphi \leq \varphi_a$ or $M = 2N$), the size is minimized when $\varphi = \pi/4$ and is maximized when $\varphi = 5\pi/4$.

It has been predicted based on the TFA that $R_{\text{rms}} \propto N^{1/5}$. To check this relation $\log_{10} R_{\text{rms}}$ from the numerical solutions are plotted in Fig.3 versus $\log_{10} N$. In 3a to 3c the curves are not exactly straight lines. The slopes depend on $N$ and are plotted in 3d to 3f, respectively. Disregarding $M$ and $\varphi$ the slopes tend to 1/5 when $N$ increases as predicted. It has been predicted based on the TFA that $R_{\text{rms}} \propto (\tilde{\omega})^{-2/5}$. To check this relation $\log_{10} R_{\text{rms}}$ from the numerical solutions are plotted in Fig.4 versus $\log_{10} \tilde{\omega}$. The slopes of the curves tend to $-2/5$ when $\tilde{\omega}$ increases as predicted.

VII. SUMMARY

The generalized GP equation adapted to the $U(5) \supset SO(5) \supset SO(3)$ symmetry has been derived for spin-2 condensates. This equation has been solved analytically under the TFA and by strict numerical calculation. It was found that the TFA is applicable if $N$ is large (say, $N \geq 10^4$). The emphasis is placed on the g.s.. Based on a rigorous treatment of the spin-degrees of freedom, the detailed spin-textiles, i.e., the ferro-, polar, and cyclic phases, and their mixing, are explained in a many-body way and thus the underlying physics can be understood beyond the mean-field-theory. Besides, the variation of the spin-textiles versus $M$ in regions II and III is notable. Note that the factors affecting the stability of the g.s., such as the gap and the neighboring level density, together with the degeneracy of the g.s. itself, are less touched in existing literatures. These factors are studied in detail in this paper. The great difference in the stability and degeneracy caused by varying $\varphi$ (which marks the features of the interaction) and $M$ is notable (this is true even when $\varphi$ varies within a region, i.e., the g.s. remains in the same phase). We believe that the effect of these factors would be serious when the temperature is very low. Since $R_{\text{rms}}$ is an observable, efforts have also been made to clarify the relation between $R_{\text{rms}}$ and $N$, $\omega$, and $\varphi$. This provides a way for checking the theories with experimental data.

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When $N \to \infty$, $\theta_{tr}$ tends to $(\chi \chi \chi)_0$. Otherwise, a revision depending on $N^{-1}$ is necessary. The details are referred to [5, 6].

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