Synchronization in fiber lasers arrays

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Abstract

We consider an array of fiber lasers coupled through the nearest neighbors. The model is a generalized nonlinear Schroedinger equation where the usual Laplacian is replaced by the graph Laplacian. For a graph with no symmetries, we show that there is no resonant transfer of energy between the different eigenmodes. We illustrate this and confirm our result on a simple graph. This shows that arrays of fiber ring lasers can be made temporally coherent.

1 Introduction

The dynamics of coupled oscillators continues to be an active area of research due in large measure to its ubiquitous presence in a wide range of disciplines such as social, life and neuro-sciences. It also applies in the traditional areas of classical and quantum mechanics, in particular in nonlinear optical systems describing phenomena such as light localization in waveguide arrays.

This work concentrates on a simple model to understand how synchronization can be achieved when light is propagating in ideally identical optical fibers coupled by a suitable scheme. Two well known models encapsule most of the behavior considered here: (i) The Kuramoto (KM) model

$$\frac{d\phi_i}{dt} = \hat{\omega}_i + K \sum_j sin(\phi_i - \phi_j)$$

(1)
The discrete nonlinear Schrödinger (DNLS) model

\[
\frac{du_i}{dt} = c(u_{i+1} + u_{i-1}) + |u_i|^2u_i
\]  

where the latter is the most relevant to our study. The collective (synchronous) behavior of these coupled optical fibers would produce a maximum power electromagnetic output. Historically, research in optical fiber technology has been driven by the increased demand for better communication systems and a natural by-product has been the development of fiber lasers [3]. These are interesting for applications like for example material cutting where they are starting to replace solid state devices. To increase their power output, several groups are considering arrays of fiber lasers. Here light is amplified in individual, decoupled fiber amplifiers which are then coupled within a single cavity or in a second cavity. The grand challenge is to achieve a coherent power output that scales with the square of number of elements in the array. In this context, temporal coherence is the equivalent of synchronization of oscillators in an array. In practice, this synchronization has proven to be very difficult. Typically the efficiency dramatically diminishes as the number of fiber amplifiers increases. Amongst the various combining schemes a most interesting case is that of Fridman et.al [4]. There the authors combine passively as many as 25 fiber lasers in a two dimensional array. They analyze the power output in a short time interval over many round trips and observe two main effects. First, even though the efficiency of the phase locking is around 20 to 30%, there are rare events where it exceeds 90%. The second observation is that the efficiency depends on the coupling architecture, the best performance is when connectivity is increased.

These fiber arrays at high intensities are well described by the DNLS equation (2). A number of studies of this model suggest that nonlinearity can enhance coherence. Following this, here we propose a more general coupling scheme where the usual discrete Laplacian operator in the DNLS equation is replaced by a graph Laplacian. Our main result is that when choosing a specific form of the coupling we are able to ”separate” the modes so the nonlinearity will only couple them weakly. Then the system can be considered as temporally coherent. This holds for any graph such that the eigenvalues of the Laplacian are distinct, i.e. a graph with no symmetries. We illustrate this by analyzing a simple graph and confirm this result numerically. We calculate the coherence factor of the array for different initial mode distribution and show that fiber ring lasers in an array can be synchronized.

The article is organized as follows. We introduce the model in section 2 and show that the nonlinearity does not couple the linear eigenmodes on average so that their average amplitude is constant. In section 3, we illustrate this general result on a particular graph. This system is analyzed numerically in section 4. There we confirm our predictions and consider different mode distributions. Conclusions are presented in section 5.
Figure 1: The top panels presents two physical arrangements of a four fiber array. On the left the fibers 2 and 3 are closer and thus coupled stronger than the other pairs of fibers. On the top right panel the couple 3 and 4 are closer. The bottom panel shows the graph used to model these devices. The arrows on the branches are oriented arbitrarily and the coupling parameters $\alpha, \beta$ are adjusted to represent each of the physical configurations.

2 The model

We propose a design of a fiber array where the coupling is purposely made irregular. We expect that in a network the coupling will be different due to the arrangement of the single fibers in relation to others. Fig. 1 presents the cross-section of the distribution of fibers in the array and the associated network or graph used to model it. A consequence of this is that we need to generalize the classical 2D nonlinear Schroedinger equation

$$i u_t = \Delta u + |u|^2 u,$$

to the graph NLS

$$i u_t = G u + |u|^2 u,$$  \hspace{1cm} (3)

where the standard 2D Laplacian is replaced by the graph Laplacian $G$. This operator is a natural extension of the continuous Laplacian see [3] for examples of its application. The specific form of (3) written for the graph shown in Fig. 1 gives

$$\begin{align*}
 iu_1 &= (u_2 - u_1) + |u_1|^2 u_1, \\
 iu_2 &= - (u_2 - u_1) + \alpha (u_3 - u_2) + (u_4 - u_2) + |u_2|^2 u_2, \\
 iu_3 &= -\alpha (u_3 - u_2) + \beta (u_4 - u_3) + |u_3|^2 u_3, \\
 iu_4 &= (u_2 - u_4) - \beta (u_3 - u_4) + |u_4|^2 u_4,
\end{align*}$$
where \( u_i, \ i = 1 - 4 \) are the values of the field amplitude, where \( \beta \) is the coupling for the branch between nodes 1 and 2, where \( \alpha \) is the coupling for the branch between nodes 2 and 3 and where 1 is the coupling for the branch between nodes 2 and 4. The system of equations above can be written symbolically as

\[
iU_t = GU + N(U).
\]

In the above equation the vector \( U \), the graph Laplacian \( G \) and the nonlinearity are respectively

\[
U = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}, \quad G = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 - \alpha & \alpha & 1 \\ 0 & \alpha & -\alpha - \beta & \beta \\ 0 & 1 & \beta & -1 - \beta \end{pmatrix}, \quad N(U) = \begin{pmatrix} |u_1|^2 & u_1 \\ u_2^2 & u_2 \\ u_3^2 & u_3 \\ u_4^2 & u_4 \end{pmatrix}.
\]

This system preserves the "mass"

\[
M = \sum_i |u_i|^2,
\]

and the energy

\[
H = \sum_i (\nabla u)^2 + \sum_i |u_i|^2 - |u_i|^4,
\]

where \( \nabla \) is the discrete gradient associated to the graph \([5]\).

Since the matrix \( G \) is symmetric it is natural, following \([5]\), to use as a basis for \( U \) the eigenvectors of \( G \), such that \( Gz_i = -\omega_i^2 z_i \). These are the columns of the orthogonal \( Z \) matrix such that

\[
GZ = ZD,
\]

where \( D \) is the diagonal matrix of diagonal \(-\omega_1^2, -\omega_2^2 \ldots \). We then introduce the vector \( \Gamma \) of components \( \gamma_k, k = 1, \ldots n \) such that

\[
U = Z\Gamma.
\]

In terms of these new coordinates \( \gamma_k \), equation (4) reduces to

\[
i\dot{\gamma}_k = -\omega_k^2 \gamma_k + \sum_{j=1}^4 z_{jk}|u_j|^2u_j,
\]

where we have used the orthogonality of the eigenvectors of \( G \), \( Z^{-1} = Z^T \). The term \( |u_j|^2u_j \) can be written as

\[
|u_j|^2u_j = \sum_{lmn} z_{jl} z_{jm} z_{jn} \gamma_l \gamma_m \gamma_n^*.
\]

We then get the final equation

\[
i\dot{\gamma}_k = -\omega_k^2 \gamma_k + \sum_{jlmn} z_{jl} z_{jm} z_{jn} \gamma_l \gamma_m \gamma_n^*.
\]
This is the equation that we will analyze throughout the article.

Let us assume that all the eigenvalues $\omega_j$ are simple. This is the generic case for a graph without symmetries. Then we can simplify the equation (7) by eliminating the first term on the right hand side. For that we introduce

$$\gamma_k = e^{i\omega_k^2 t} \beta_k,$$

to obtain

$$i \dot{\beta}_k = \sum_{jlmn} z_{jl} z_{jm} z_{jn} \beta_l \beta_m \beta_n^* e^{i(\omega_l^2 + \omega_m^2 - \omega_n^2 - \omega_k^2) t}. \quad (8)$$

Only the resonant terms such that

$$\omega_l^2 + \omega_m^2 - \omega_n^2 - \omega_k^2 = 0, \quad (9)$$

will contribute on the long term to $\beta_k$. Then, because the frequencies are all different, the resonant condition is satisfied if $l = n$ and $m = k$. We then obtain the resonant evolution of $\beta_k$ as

$$i \dot{\beta}_k = \beta_k \sum_{jl} z_{jl}^2 z_{jk} |\beta_l|^2. \quad (10)$$

This equation is such that the intensity in each mode is constant

$$I_k = |\beta_k|^2, \quad \dot{I}_k = 0.$$ 

The solution of equation (10) is then

$$\beta_k(t) = \beta_k(0) \exp(-i t \Omega_k), \quad (11)$$

where the nonlinear correction to the frequency is

$$\Omega_k = \sum_{jl} z_{jl}^2 z_{jk} |\beta_l|^2. \quad (12)$$

Returning back to the original variables $\gamma_k$ we get our main result

$$\gamma_k(t) = \gamma_k(0) \exp \left[ i t (\omega_k^2 - \Omega_k) \right]. \quad (13)$$

There is no long term energy transfer between the modes. For a given initial condition, the modal distribution is fixed and the system will keep this for all times.

Before presenting a case study, a few remarks are in place. Our analysis is valid for any graph, of arbitrarily large size. For our approximate analysis to be valid it is important that eigenvalues be different and well separated. Then the phase resonance condition will be well satisfied. We will illustrate this below by comparing our prediction to the numerical solution of the full problem. Also note that when there are multiple eigenvalues, the change of variable from $\gamma$ to $\beta$ is no longer valid and the degeneracy will induce a chaotic temporal dynamics.
\[
\lambda_i = -\omega_i^2 = \frac{1}{2}(\delta - 2\alpha - 3) - \frac{1}{2}(\delta + 2\alpha + 3)
\]

| \(z_i\) | 1/2 | 1/\sqrt{2} | 1/n_3 | 1/n_4 |
|---|---|---|---|---|
| 1/2 | 0 | \(\frac{1}{2n_3}(\delta - 2\alpha - 1)\) | \(-\frac{1}{2n_4}(\delta + 2\alpha + 1)\) | \(1/n_4\) |
| 1/2 | 0 | \(\frac{1}{2n_3}(\delta - 2\alpha + 3)\) | \(\frac{1}{2n_4}(\delta + 2\alpha - 3)\) | \(1/n_4\) |
| 1/2 | -1/\sqrt{2} | 1/n_3 | | |

Table 1: Eigenmodes \((\lambda_i, z_i)\) for the tree.

3 A case study: a graph with a swivel

We now illustrate the above considerations on a simple example derived from the network shown in Fig. 1. This will enable us to put numbers on the frequencies \(\omega_k\) and \(\Omega_k\). In the study [5], we introduced the notion of "soft node" where the eigenvector has a zero coordinate. These are important because if applied to them the damping or forcing of the network is ineffective. We will consider the simplest such network that has a soft node. It corresponds to the tree with \(\beta = 0\).

The eigenvalues and eigenvectors \(z_i\) (columns of the \(z\) matrix) can be computed analytically, they are given in Table 1 in terms of \(\alpha\),

\[
\delta = \sqrt{4\alpha^2 - 4\alpha + 9},
\]

\[
n_3 = \sqrt{2 + (\delta - 2\alpha - 1)^2/4 + (\delta - 2\alpha + 3)^2/4},
\]

and

\[
n_4 = \sqrt{2 + (\delta + 2\alpha + 1)^2/4 + (\delta + 2\alpha - 3)^2/4}.
\]
Figure 2: Plot of the eigenfrequencies $\omega_i$, $i = 1 - 4$ as a function of $\alpha$ for the graph shown in Fig. 1 with $\beta = 1$.

The mode $z_2$ which is independent of $\alpha$ is shown schematically in Fig. 3 where we have plotted the magnitude and sign of the coordinate using vertical arrows. The arrows are opposite and equal for $z_2^1$ and $z_2^2$ because the nodes 1 and 4 play a symmetric role i.e. the graph is invariant by the automorphism transforming node 1 to node 4.

Figure 3: Schematic representation of the constant mode $z_2$, corresponding to $\omega_2 = 1$. 
Let us compute the frequencies of oscillations $\Omega_k$ of the resonant modes. The right hand side of relation (11) can be written in matrix form

$$MI = \Omega$$

where the vector $I = \begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix}$ with $I_j = |\beta_j|^2$, where the vector $\Omega = \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \\ \Omega_4 \end{pmatrix}$ and

where the matrix $M$ has a general term

$$m_{kl} = \sum_j z_{jk} z_{l\beta}^2.$$

Introducing the matrix $Z$ from Table 1 and computing $M$ using Matlab we get the general linear system

$$\begin{pmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 \\ 0 & m_{32} & m_{33} & m_{34} \\ 0 & m_{42} & m_{43} & m_{44} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix} = \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \\ \Omega_4 \end{pmatrix}.$$  

(14)

This matrix enables to compute the nonlinear corrections $\Omega_k$ once the mode distribution $\gamma_k$ is given. The matrix $M$ has rank 3 and its image is

$$\text{Im}(M) = \{\vec{x}, \vec{z}, \vec{t}\}.$$  

Note that $\Omega_2 = 0$. Taking for example $I_1 = 2, I_2 = I_3 = I_4 = 0$ we get $\Omega_1 = 0$ and $\Omega_2 = \Omega_3 = \Omega_4 = 0$.

4 Numerical results

To validate our approach we now solve numerically the original equations (4) for the tree configuration described above ($\beta = 0$). We chose the initial mode configuration and let it evolve. The resolution was done using the ode45 subroutine of Matlab [7].
Fig. 5 shows the time evolution of the energies $I_k$ for $k = 1 - 4$ for two values of the coupling, $\alpha = 0.5$ (left panel) and $\alpha = 3.5$ (right panel). The initial conditions are the same $\beta_1 = 0, \beta_2 = 1.3, \beta_3 = 0$ and $\beta_4 = 2$. Notice how $I_4$ is approximately constant in agreement with our prediction. On the other hand in both cases the modes 2 and 3 have close frequencies so they couple strongly. For $\alpha = 0.5$ the frequencies are closer so that more interaction occurs. For $\alpha = 3.5$ we have $\omega_4 >> \omega_2$ so the phase term in (9) is larger and the approximation is better. Furthermore, it is clear that for this case a slow dynamics will show a fixed point for mode 4; and homoclinic orbits for modes 1-3. All this is seen in the right panel of Fig. 5.

![Figure 5: Time evolution of the energies in each mode $I_k$ for $k = 1 - 4$ for two values of the coupling, $\alpha = 0.5$ (left panel) and $\alpha = 3.5$ (right panel).](image)

To relate our results to figures of merit in experimental work on coherent beam combining, and in particular paying attention to that of Fridman et al. [4], we introduce the coherence factor $C$

$$C = \frac{\sum_i u_i}{\sum_i |u_i|}.$$  \hfill (15)

For the modes $\omega_i, i = 2 - 4$ this coherence factor will be smaller than one because the eigenvectors $z_i$ have to have positive and negative coordinates, which could be interpreted as being in (same sign) or out (different sign) of phase data. As an example consider that

$$\gamma_1 = \gamma_2 = \gamma_3 = 0, \quad \gamma_4 \neq 0.$$  

Then for $\alpha$ large we have

$$u_2 = z_{24}\gamma_4 \approx -0.8\gamma_4, \quad u_3 = z_{34}\gamma_4 \approx 0.6\gamma_4,$$

and $u_1 = u_4 = 0$. This clearly represents an out of phase combination leading to a low coherence coefficient value,

$$C = \frac{|u_2 + u_4|}{|u_2| + |u_4|} = 0.14.$$
If however one considers instead of the nodes $u_i$ the modes $\gamma$, say by applying the linear transformation given by the matrix $Z$, then the coherence coefficient becomes 1. This could be done using transformational optics.

Another direction would be to use the Goldstone mode $z_1$ corresponding to all node components $u_i$ equal. For this we need to guarantee that the mixing with the other frequencies remains small. This can be done for the tree as we now show. We fix $\gamma_1 = 2$, $\gamma_4 = 0$, and choose two sets of the pair $(\gamma_2, \gamma_3)$, $\gamma_2 = 0.1$, $\gamma_3 = 0$ and $\gamma_2 = 0$, $\gamma_3 = 0.1$. The results are shown in Fig. 6 with $\gamma_2 = 0.1$, $\gamma_3 = 0$ on the left panel and $\gamma_3 = 0.1$, $\gamma_2 = 0$ on the right panel. As expected on the left panel we see a stronger coupling than on the right panel because

$$\omega_1^2 - \omega_2^2 < \omega_1^2 - \omega_3^2,$$

so that the rotating wave approximation leading to (13) is better satisfied. Considering the coherence factor, the parameters $\gamma_1 = 2$, $\gamma_4 = 0$, and $\gamma_3 = 0.1$, $\gamma_2 = 0$ will give $C \approx 1$.

![Figure 6: Time evolution of the energies in each mode $I_k$ for $k = 1-4$ for $\gamma_1 = 2$, $\gamma_4 = 0$. The left panel is for $\gamma_2 = 0.1$, $\gamma_3 = 0$ and the right panel for $\gamma_3 = 0.1$, $\gamma_2 = 0$. The other parameter is $\alpha = 3.5$.](image)

The results of this section show that if the array is "prepared" in a given mode, it will stay on that mode on average. For this preparation, one can force the array at resonance like in [5]. As an example, for our tree configuration we select mode 4 by damping nodes 1 or 4 and forcing only nodes 2 or 3.

5 Conclusions

Motivated by studies in fiber laser arrays and how the different coupling schemes between fibers can affect the coherence property of the output, we introduced an array where the coupling between fibers is irregular. This leads to a discrete nonlinear Schroedinger equation where the usual discrete Laplacian is replaced by a graph Laplacian.

To connect our work with experiments, we computed a coherence coefficient and showed that we can synchronize the array. This study then opens the way
of using a new approach based in graph Laplacians for designing new coupling schemes for arrays of fiber lasers.

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