We explore the tan-concavity of the Lagrangian phase operator for the study of the deformed Hermitian Yang-Mills (dHYM) metrics. This new property compensates for the lack of concavity of the Lagrangian phase operator as long as the metric is almost calibrated. As an application, we introduce the tangent Lagrangian phase flow (TLPF) on the space of almost calibrated $(1,1)$-forms that fits into the GIT framework for dHYM metrics recently discovered by Collins-Yau. The TLPF has some special properties that are not seen for the line bundle mean curvature flow (i.e. the mirror of the Lagrangian mean curvature flow for graphs). We show that the TLPF starting from any initial data exists for all positive time. Moreover, we show that the TLPF converges smoothly to a dHYM metric assuming the existence of a $C$-subsolution, which gives a new proof for the existence of dHYM metrics in the highest branch.

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Date: March 10, 2020.
2010 Mathematics Subject Classification. Primary 53C55; Secondary 53C44.
Key words and phrases. deformed Hermitian Yang-Mills metric, parabolic PDE, mirror symmetry.
1. Introduction

Let $X$ be a compact $n$-dimensional complex manifold with a fixed Kähler form $\alpha$ and a closed real $(1, 1)$-form $\hat{\chi}$. For $\phi \in C^\infty(X; \mathbb{R})$, we set $\chi := \hat{\chi} + \sqrt{-1}\partial \bar{\partial} \phi$ (where the forms $\hat{\chi}, \chi$ are not necessary Kähler). We say that $\phi \in C^\infty(X, \mathbb{R})$ is a deformed Hermitian-Yang Mills (dHYM) metric if it satisfies

$$\text{Im}(e^{-\sqrt{-1}\hat{\Theta}}(\alpha + \sqrt{-1}\chi)^n) = 0.$$  \hspace{1cm} (1.1)

We define a topological invariant

$$Z := \int_X (\alpha + \sqrt{-1}\chi)^n$$

and assume $Z \neq 0$. Then the constant angle $\hat{\Theta}$ is uniquely determined (mod. $2\pi$) by the property that

$$Ze^{-\sqrt{-1}\hat{\Theta}} \in \mathbb{R}_{>0}.$$ 

The equation (1.1) first appeared in the physics literature [MMMS00], and [LYZ01] from mathematical side as the mirror object to a special Lagrangian in the setting of semi-flat mirror symmetry. Recently, it has been studied actively (e.g. [CCL20, Che19, CJY15, CY18, HJ20, HY19, JY17, Pin19, SS19]). We define

$$\theta(\lambda) := \sum_{i=1}^n \arctan \lambda_i,$$

where $\arctan$ takes values in $(-\frac{\pi}{2}, \frac{\pi}{2})$ so the image of $\theta$ lies in $(-n\frac{\pi}{2}, n\frac{\pi}{2})$. For an $n \times n$ Hermitian matrix $A$ with eigenvalues $\lambda[A]$, we set

$$\Theta(A) := \theta(\lambda[A]).$$

The function $\Theta$ is smooth by the symmetry of $\theta$. For $\phi \in C^\infty(X; \mathbb{R})$, we set $A[\phi] := \chi_{ij}\alpha^{kj}$. This is a Hermitian endomorphism on $T^{1,0}X$ with respect to $\alpha$. We denote the eigenvalues of $A[\phi]$ by $\lambda[\phi]$. Then according to the argument [JY17], the condition (1.1) is equivalent to

$$\Theta(A[\phi]) = \hat{\Theta} \pmod{2\pi},$$  \hspace{1cm} (1.2)

where we regard $A[\phi]$ as an $n \times n$ Hermitian matrix at each point by taking normal coordinates. We say that $\phi \in C^\infty(X; \mathbb{R})$ is supercritical (resp. hypercritical) if it satisfies $\Theta(A[\phi]) > (n-2)\frac{\pi}{2}$ (resp. $> (n-1)\frac{\pi}{2}$). In particular, since $\arctan(\cdot)$ takes values in $(-\frac{\pi}{2}, \frac{\pi}{2})$, the condition $\Theta(A[\phi]) > (n-1)\frac{\pi}{2}$ yields that all coefficients of $\lambda[\phi]$ are positive, and hence $\chi[\phi]$ is Kähler. Although the equation (1.2) is elliptic, it has several problems on analysis. Most seriously, the operator $\Theta$ fails to be concave in general. Indeed, a straightforward computation shows that $\Theta(A[\phi])$ is concave if and only if $\phi$ is hypercritical. The concavity of the operator is essential to apply the Evans-Krylov theory [Kry82, Wan12] for $C^{2, \beta}$ estimate. To deal with this problem, an important observation shown by Yuan [Yua05] is that the level set $\{\lambda \in \mathbb{R}^n | \theta(\lambda) = \sigma\}$ is still convex as long as $\sigma > (n-2)\frac{\pi}{2}$. This result indicates that one may get a concave function $\zeta \circ \theta$ by composing $\theta$ with a sufficiently concave function $\zeta: \mathbb{R} \to \mathbb{R}$. Indeed, Collins-Picard-Wu [CPW17] showed that the function $-e^{-A\theta(\lambda)}$ is concave for a large enough constant $A = A(\delta)$ as long as $\lambda$ satisfies $\theta(\lambda) \geq (n-2)\frac{\pi}{2} + \delta$ for some $\delta > 0$. 

In this paper, we consider yet another choice of \( \varsigma \). Let us consider the function
\[
f(\lambda) := \tan(\theta(\lambda) - \hat{\Theta}).
\]
The function \( f \) is not globally defined, but it is well-defined restricted to a subset of \( \mathbb{R}^n \);
\[
S := \left\{ \lambda \in \mathbb{R}^n \mid |\theta(\lambda) - \hat{\Theta}| < \frac{\pi}{2} \right\}.
\]
Compared to the result \([CPW17]\), one might find it strange that our choice of \( \varsigma(x) = \tan(x - \hat{\Theta}) \) is not concave, so it is not sure that the composition \( f = \varsigma \circ \theta \) is also a concave function. Our first main theorem is the following;

**Theorem 1.1.** Assume that \( \hat{\Theta} \in ((n-1)\frac{\pi}{2}, n\frac{\pi}{2}) \), then the function \( \theta - \hat{\Theta} \) is tan-concave, i.e. the composition \( f = \tan(\theta - \hat{\Theta}) \) is concave on \( S \).

After the author had posted the preprint on arXiv, he was informed by F. R. Harvey and H. B. Lawson, Jr. that Theorem 1.1 has a relation to their recent work \([HL20]\), in which they proved a “tameness” condition for the composition \( \tan(S\text{Lag}/n) \) to solve the inhomogeneous Dirichlet problem \( S\text{Lag}(D^2u) := \text{Tr}(\arctan(D^2u)) = \psi \) for a function \( u: \Omega \to \mathbb{R} \) where \( \Omega \subset \mathbb{R}^n \) is a bounded domain with smooth strictly convex boundary \( \partial \Omega \), and \( \psi \) is a continuous function on \( \Omega \) satisfying \( \psi(\Omega) \subset ((n-2)\frac{\pi}{2}, n\frac{\pi}{2}) \) (see \([HL20\), Section 5] for more details).

We expect that the tan-concavity property is useful for the study of dHYM metrics or minimal Lagrangian graphs. As a demonstration, we provide a new geometric flow approach to construct dHYM metrics. Associated to the set \( S \subset \mathbb{R}^n \), we define the space of almost calibrated potential functions (cf. \([CCL20]\));
\[
\mathcal{H} := \left\{ \phi \in C^\infty(X; \mathbb{R}) \mid |\Theta(A[\phi]) - \hat{\Theta}| < \frac{\pi}{2} \right\}.
\]
We remark that the set \( \mathcal{H} \) is strictly contained in the space of potentials with super-critical phase when \( \hat{\Theta} > (n-1)\frac{\pi}{2} \). In later arguments, we always assume that \( \mathcal{H} \) is not empty and \( 0 \in \mathcal{H} \) by replacing a reference form \( \hat{\chi} \). For any \( \phi_0 \in \mathcal{H} \), we define
\[
\frac{d}{dt} \phi_t = \tan(\Theta(A[\phi_t]) - \hat{\Theta}). \tag{1.3}
\]
We have \( f_i = \frac{1+\lambda^2}{1+\lambda^2} > 0 \) for all \( i \), which guarantees the ellipticity of the operator in the RHS. So the short time existence follows from general theory. We would like to call \( \text{1.3} \), the tangent Lagrangian phase flow (TLPF). For simplicity, we set \( F(A) := f(\lambda[A]) \) so that the flow equation is given by \( \frac{d}{dt} \phi = F(A[\phi]) \). On the other hand, Jacob-Yau \([JY17]\) introduced the line bundle mean curvature flow (LBMCF)
\[
\frac{d}{dt} \phi_t = \Theta(A[\phi_t]) - \hat{\Theta} \tag{1.4}
\]
as the mirror of the Lagrangian mean curvature flow (LMCF) for graphs. In a formal level, the TLPF is similar to the LBMCF whenever the Lagrangian phase \( \Theta(A[\phi_t]) \) is very close to \( \hat{\Theta} \). However it is expected that the limiting behavior of these two flows are
quite different. A similar observation can be found in the comparison of the Kähler-Ricci flow and the inverse Monge-Ampère flow on Fano manifolds [CHT17]. For the moment, we assume \( \hat{\Theta} \in ((n-1)\frac{\pi}{2}, n\frac{\pi}{2}) \). Then the virtue of the TLPF is the following:

- The TLPF has some special properties that are not seen for the LBMCF (cf. Remark 4.4 and Remark 5.5).
- Our choice of \( \zeta(x) = \tan(x - \hat{\Theta}) \) has a natural geometric meaning; the TLPF perfectly fits into the GIT framework recently discovered by Collins-Yau [CY18] (as the mirror of Solomon’s [Sol13] and Thomas’s [Tho01]) in the sense that it defines the gradient flow of the Kempf-Ness functional \( J \), which is globally convex on \( \mathcal{H} \). This GIT framework gives supporting evidence for the equation (1.3) working well (cf. Remark 4.2).
- From a PDE point of view, the TLPF is more in line with the parabolic equation proposed by Krylov [Kry76] by Theorem 1.1.

Before giving the second main theorem, we recall briefly some existence results of dHYM metrics. In the analysis of the equation (1.2), it is crucial to give a proper notion of \( C \)-subsolutions whose existence implies a priori estimates to all orders. In this direction, Collins-Jacob-Yau [CJY15] showed the following result by using the method of continuity;

**Theorem 1.2** ([CJY15]). Let \( X \) be a compact complex manifold with a Kähler form \( \alpha \), and \( \hat{\chi} \) a closed real \((1,1)\)-form. Assume that \( \hat{\Theta} \in ((n-2)\frac{\pi}{2}, n\frac{\pi}{2}) \) and there is a \( C \)-subsolution \( \phi \) satisfying \( \Theta(A[\phi]) > (n-2)\frac{\pi}{2} \). Then there exists a deformed Hermitian Yang-Mills metric (1.2).

We remark that for any \( C \)-subsolution \( \phi \), the supercritical phase condition \( \Theta(A[\phi]) > (n-2)\frac{\pi}{2} \) is automatically satisfied if \( \hat{\Theta} \geq ((n-2)\frac{\pi}{2} + \frac{\pi}{n}) \frac{\pi}{2} \), and hence is not vacuous. However it is expected that the condition \( \Theta(A[\phi]) > (n-2)\frac{\pi}{2} \) can be improved when \( \hat{\Theta} \in ((n-2)\frac{\pi}{2}, ((n-2)\frac{\pi}{2} + \frac{\pi}{2}) \). Pingali [Pin19] showed that this is actually true when \( n = 3 \) using a new continuity path obtained by rewriting (1.2) as a generalized complex Monge-Ampère equation. In [CJY15], they also claimed that one can show a weaker existence result by using the method of continuity;

**Theorem 1.3** ([CJY15], Remark 7.4). Let \( X \) be a compact complex manifold with a Kähler form \( \alpha \), and \( \hat{\chi} \) a closed real \((1,1)\)-form. Assume that \( \hat{\Theta} \in ((n-1)\frac{\pi}{2}, n\frac{\pi}{2}) \) and there is a \( C \)-subsolution \( \phi \) satisfying \( \Theta(A[\phi]) > (n-1)\frac{\pi}{2} \). Then the line bundle mean curvature flow (1.4) with \( \phi_0 := \phi \) exists for all time, and converges to the deformed Hermitian Yang-Mills metric in the \( C^\infty \)-topology.

When proving Theorem 1.3, the point is that the hypercritical phase condition \( \Theta(A[\phi_0]) > (n-1)\frac{\pi}{2} \) is preserved under the LBMCF so that the operator \( \Theta(A[\phi_t]) \) remains to be concave. The author does not know whether the hypercritical phase condition \( \Theta(A[\phi_0]) > (n-1)\frac{\pi}{2} \) in Theorem 1.3 can be replaced by \( \phi_0 \in \mathcal{H} \). One reason is that the LBMCF defines the gradient flow of the volume functional \( \mathcal{V} \) (cf. Section 2.1), but we do not know whether \( \mathcal{V} \) is globally convex. In the original LMCF case,
it is conjectured by Thomas-Yau [TY02, Section 7] that we have to assume an additional condition, so called the “flow-stability” for the initial Lagrangian to obtain the convergence of the flow (see also [Nev13] for expositions).

Now we give the second main theorem. We show that the TLPF potentially has more global existence and convergence properties. With the aid of Theorem 1.1 and already known methods, it is standard to show the following:

**Theorem 1.4.** Let $X$ be a compact complex manifold with a Kähler form $\alpha$, and $\tilde{\chi}$ a closed real $(1,1)$-form. Assume that $\tilde{\Theta} \in ((n - 1)\frac{\pi}{2}, n\frac{\pi}{2})$. For any $\phi_0 \in \mathcal{H}$, let $\phi_t$ be the tangent Lagrangian phase flow (1.3) starting from $\phi_0$. Then

1. The flow $\phi_t$ exists for all positive time.
2. Moreover, if there is a $C$-subsolution, then the flow $\phi_t$ converges to the deformed Hermitian Yang-Mills metric $\phi_\infty \in \mathcal{H}$ in the $C^\infty$-topology.

Our notion of $C$-subsolutions coincides with that defined in [CJY15]. Indeed, the analysis of the TLPF proceeds very closely to general theory of fully non-linear parabolic equations whose RHS is concave [PT17] (based on the elliptic case [Sze18]). We make several comments on Theorem 1.4;

- The initial condition $\phi_0 \in \mathcal{H}$ is sharp (otherwise, the TLPF (1.3) is not well-defined). Meanwhile, Neves [Nev13] shows that without the almost calibrated assumption a finite-time singularity occurs under the LMCF. We expect that the same is true of the LBMCF.
- Theorem 1.4 removes the hypercritical phase condition $\Theta(A[\phi_0]) > (n - 1)\frac{\pi}{2}$ imposed in Theorem 1.3 and gives a new proof for the existence of dHYM metrics (we note that if $\tilde{\Theta} > (n - 1)\frac{\pi}{2}$, then any $C$-subsolution $\phi$ is almost calibrated by Remark 5.2, and hence we can take this $\phi$ as the initial data for instance).
- The assumption of Theorem 1.4 is stronger than that of Theorem 1.2. However, we emphasize that Theorem 1.4 has a more natural meaning from geometric/variational point of view, and the estimates involved are much simpler than those in [CJY15].

Even if there are no dHYM metrics, we expect that the TLPF is useful to construct an optimally destabilizing one-parameter subgroups in GIT in analogy with the Harder-Narasimhan filtrations of unstable vector bundles (for instance, see related results [DS16, HIs19, Xia19] studied in Kähler settings).

This paper is organized as follows. In Section 2, we fix some notations by following [CXY17, JY17], and study the basic properties of the space $\mathcal{H}$ and functionals on it. In Section 3, we give a proof of Theorem 1.1, that is the core of this paper. In Section 4, we prove some monotonicity formulas along the TLPF. Then we prove the long time existence of the flow which is the first part of Theorem 1.4. In Section 5, we recall the notion of $C$-subsolutions defined in [CJY15]. Then in accordance with [PT17], we establish the $C^k$ estimates and the $C^\infty$-convergence of the TLPF, which shows the second part of Theorem 1.4.
Acknowledgment. The author expresses his gratitude to Prof. F. R. Harvey and H. B. Lawson, Jr. for pointing out a relation between Theorem 1.1 and their work [HL20].

2. Foundations

2.1. Notations and formulas. Let $X$ be an $n$-dimensional compact complex manifold with a Kähler form $\alpha$ and $\hat{\chi}$ a closed real $(1,1)$-form. First we fix some notations. For $\phi \in C^\infty(X; \mathbb{R})$, we define a Hermitian metric $\eta$ on $T^{1,0}X$ by

$$\eta_{ij} := \alpha_{ij} + \chi_{i\bar{k}} \alpha^{k\bar{j}} \chi_{k\bar{j}}.$$ 

Let

$$v := \left| \frac{(\alpha + \sqrt{-1} \chi \phi)^n}{\alpha^n} \right| = \sqrt{\prod_{i=1}^n (1 + \lambda_i^2)},$$

where $\lambda_i$ denotes the eigenvalues of $A[\phi]$. In particular, this implies $v \geq 1$. In terms of $V$ and $\Theta(A[\phi])$, the $\mathbb{C}$-valued function $(\alpha + \sqrt{-1} \chi \phi)^n / \alpha^n$ is expressed as

$$\frac{(\alpha + \sqrt{-1} \chi \phi)^n}{\alpha^n} = ve^{-\sqrt{-1} \Theta(A[\phi])}.$$ 

Integrating with respect to $\alpha^n$ we get

$$e^{-\sqrt{-1} \hat{\Theta}} Z = \int_X ve^{-\sqrt{-1} (\Theta(A[\phi]) - \hat{\Theta})} \alpha^n = \int_X v \cos(\Theta(A[\phi]) - \hat{\Theta}) \alpha^n + \sqrt{-1} \int_X v \sin(\Theta(A[\phi]) - \hat{\Theta}) \alpha^n.$$ 

So by the definition of $\hat{\Theta}$ we know that $e^{-\sqrt{-1} \hat{\Theta}} Z$ is real so that

$$|Z| = \int_X v \cos(\Theta(A[\phi]) - \hat{\Theta}) \alpha^n, \quad \int_X v \sin(\Theta(A[\phi]) - \hat{\Theta}) \alpha^n = 0.$$ 

According to [CY18 Section 2], we define functionals on the space of almost calibrated potentials $\mathcal{H}$. We define the Calabi-Yau functional $CY_\mathbb{C}$ by

$$CY_\mathbb{C}(\phi) := \frac{1}{n+1} \sum_{j=0}^n \int_X \phi(\alpha + \sqrt{-1} \chi \phi)^j \wedge (\alpha + \sqrt{-1} \chi)^{n-j}, \quad \phi \in \mathcal{H}.$$ 

This is a $\mathbb{C}$-valued functional. Then the variational formula of $CY_\mathbb{C}$ is given by

$$\delta CY_\mathbb{C}(\delta \phi) = \int_X \delta \phi (\alpha + \sqrt{-1} \chi \phi)^n.$$ 

Also we set

$$C(\phi) := \text{Re}(e^{-\sqrt{-1} \hat{\Theta}} CY_\mathbb{C}(\phi)),$$ 

$$J(\phi) := -\text{Im}(e^{-\sqrt{-1} \hat{\Theta}} CY_\mathbb{C}(\phi)).$$

Then the variational formula of $CY_\mathbb{C}$ yields that

$$\delta C(\delta \phi) = \int_X \delta \phi \text{Re}(e^{-\sqrt{-1} \hat{\Theta}} (\alpha + \sqrt{-1} \chi \phi)^n),$$ 

$$\delta J(\delta \phi) = -\int_X \delta \phi \text{Im}(e^{-\sqrt{-1} \hat{\Theta}} (\alpha + \sqrt{-1} \chi \phi)^n).$$
Also we define the volume functional by

\[ \mathcal{V}(\phi) := \int_X v_\phi \alpha^n. \]

The function \( \mathcal{V} \) is non-negative. More precisely, we have \( \mathcal{V}(\phi) \geq |Z| \) for all \( \phi \in \mathcal{H} \) (cf. [JY17, Proposition 3.2]). The variational formula of \( \mathcal{V} \) (cf. [JY17, Proposition 3.4]) is given by

\[ \delta \mathcal{V}(\delta \phi) = \int_X \langle d\Theta(A[\phi]), d\delta \phi \rangle \eta v_\phi \alpha^n. \]

These functionals have the following properties;

**Proposition 2.1.** For any \( \phi \in \mathcal{H} \) and \( c \in \mathbb{R} \) we have

1. \( CY_C(\phi + c) = CY_C(\phi) + cZ \).
2. \( \mathcal{C}(\phi + c) = \mathcal{C}(\phi) + c|Z| \).
3. \( J(\phi + c) = J(\phi) \).

We collect the properties of \( f \) that we will need;

**Proposition 2.2.** Suppose we have real numbers \( \lambda_1 \geq \ldots \geq \lambda_n \) which satisfy \( \theta(\lambda) = \sigma \) for \( \sigma \in ((n-2)\frac{\pi}{2}, n\frac{\pi}{2}) \). Then \( \lambda = (\lambda_1, \ldots, \lambda_n) \) have the following properties;

1. \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1} > 0 \) and \( \lambda_{n-1} \geq |\lambda_n| \). In particular, \( \lambda \) lies in the set \( \{ \lambda \in \mathbb{R}^n | \sum_{i=1}^n \lambda_i > 0 \} \).
2. The set \( \{ \lambda \in \mathbb{R}^n | \theta(\lambda) \geq \sigma \} \) is convex with boundary a smooth, convex hypersurface.

Furthermore, if \( \sigma \geq (n-1)\frac{\pi}{2} \), then

3. \( \lambda_n > 0 \).

In addition, if \( \sigma \geq (n-2)\frac{\pi}{2} + \beta \), then there exist constants \( \varepsilon(\beta) > 0 \) and \( C(\beta) > 0 \) such that

4. if \( \lambda_n \leq 0 \), then \( \lambda_{n-1} \geq \varepsilon(\beta) \).
5. \( |\lambda_n| < C(\beta) \).

*Proof.* The property (3) is trivial. See [CJY15, Lemma 3.1] for other statements. \( \square \)

We end up this subsection with showing some properties of \( F \). Let us write \( F^{ij} \) for the derivative of \( F \) with respect to the \( ij \)-entry of \( A \). Then at a diagonal matrix \( A \) we have

\[ F^{ij} = \delta_{ij} f_i, \quad (2.1) \]

\[ F^{ij,rs} = f_{ir} \delta_{ij} \delta_{rs} + \frac{f_i - f_j}{\lambda_i - \lambda_j} (1 - \delta_{ij}) \delta_{is} \delta_{jr}. \quad (2.2) \]

The first formula shows that the operator \( F(A[\phi]) \) is elliptic. In the second formula, we note that \( \frac{f_i - f_j}{\lambda_i - \lambda_j} \leq 0 \). In particular, we have \( f_i \leq f_j \) if \( \lambda_i \geq \lambda_j \) (see [Szé18, Section 4] for more details).
2.2. Fundamental estimates on $\mathcal{H}$. First, by Proposition 2.2 (1), we have the following;

**Lemma 2.3** (Green function estimate). For any $\phi \in \mathcal{H}$, we have a uniform bound $\Delta_\alpha \phi \geq -C$ for some uniform constant $C > 0$ depending only on $\alpha$ and $\hat{\chi}$. In particular, we have

$$\sup_X \phi \leq \int_X \phi \alpha^n + C'$$

for some uniform constant $C' > 0$ depending only on $\alpha$ and $\hat{\chi}$.

**Proposition 2.4** (Harnack type inequality). Assume $\hat{\Theta} \in ((n-1)\frac{\pi}{2}, n\frac{\pi}{2})$. Then for any $\phi \in \mathcal{H}$, there exists a constant $C$ and $C'$ depending only on $\alpha$, $\hat{\chi}$, $\hat{\Theta}$, $\inf_X \Theta(A[\phi])$ and $\mathcal{C}(\phi)$ such that

$$\sup_X \phi \leq -C \inf_X \phi + C'.$$

*Proof.* For any $\phi \in \mathcal{H}$, we set $\tilde{\phi} := \phi - |Z|^{-1}\mathcal{C}(\phi)$ so that $\mathcal{C}(\tilde{\phi}) = 0$. We may assume that $0 \in \mathcal{H}$. We connect $\tilde{\phi}$ with the base point $0$ by a segment $s\tilde{\phi}$ ($s \in [0,1]$), so we have $\chi_{s\tilde{\phi}} = s\chi_{\tilde{\phi}} + (1-s)\hat{\chi}$. Since $\hat{\Theta} > (n-1)\frac{\pi}{2}$, we have a trivial upper bound

$$\Theta(A[s\tilde{\phi}]) - \hat{\Theta} < n\frac{\pi}{2} - \hat{\Theta} < \frac{\pi}{2}.$$

Moreover, by [CY18, Lemma 3.1 (7)], for all $s \in [0,1]$ we have

$$\Theta(A[s\tilde{\phi}]) - \hat{\Theta} \geq \min\{\Theta(A[0]), \Theta(A[\tilde{\phi}])\} - \hat{\Theta} \geq \min\{\inf_X \Theta(A[0]), \inf_X \Theta(A[\tilde{\phi}])\} - \hat{\Theta} > -\frac{\pi}{2}.$$

(Indeed, the set $\mathcal{H}$ is convex when $\hat{\Theta} > (n-1)\frac{\pi}{2}$ as pointed out in [CY18, Section 2]). Combining with $\nu_{s\tilde{\phi}} \geq 1$ we have

$$\text{Re}(e^{-\sqrt{-1}\Theta}(\alpha + \sqrt{-1}X_{s\tilde{\phi}})^n) = v_{s\tilde{\phi}} \cos(\Theta(A[s\tilde{\phi}]) - \hat{\Theta}) \alpha^n \geq c\alpha^n$$

for some constant $c > 0$ depending only on $\hat{\chi}$, $\hat{\Theta}$ and $\inf_X \Theta(A[\phi])$. Thus $\nu_{\tilde{\phi}} := \int_0^1 \text{Re}(e^{-\sqrt{-1}\Theta}(\alpha + \sqrt{-1}X_{s\tilde{\phi}})^n)ds$ defines a positive measure with volume $|Z|$ and a uniform lower bound $c\alpha^n$. On the other hand, the variational formula of $\mathcal{C}$ implies that

$$0 = \mathcal{C}(\tilde{\phi}) = \int_0^1 \int_X \tilde{\phi} \text{Re}(e^{-\sqrt{-1}\Theta}(\alpha + \sqrt{-1}X_{s\tilde{\phi}})^n)ds = \int_X \tilde{\phi} \nu_{\tilde{\phi}}.$$

In particular, this shows that $\sup_X \tilde{\phi} \geq 0$ and $\inf_X \tilde{\phi} \leq 0$. Keeping this in mind, we compute

$$\int_X \tilde{\phi} \alpha^n = -\frac{1}{c} \int_X \tilde{\phi}(\nu_{\tilde{\phi}} - c\alpha^n) \leq -\frac{1}{c} \left[ \int_X (\nu_{\tilde{\phi}} - c\alpha^n) \right] \inf_X \tilde{\phi} = -C_1 \inf_X \tilde{\phi},$$
where we put $C_1 := c^{-1}|Z| - [\alpha]^n$. Combining with the Green function estimate (cf.Lemma 2.3) we have

$$\sup_X \tilde{\phi} \leq \int_X \tilde{\phi} \alpha^n + C_2 \leq C_1 \inf_X \tilde{\phi} + C_2.$$ 

Putting $\tilde{\phi} = \phi - |Z|^{-1}C(\phi)$ into the above, we obtain the desired formula.

\[ \Box \]

3. Tan-concavity

Now we give a proof of Theorem 1.1.

**Proof of Theorem 1.1.** From a direct computation, we have

$$df = (1 + f^2)d\theta, \quad d\theta = \sum_i \frac{d\lambda_i}{1 + \lambda_i^2},$$

$$\nabla^2 \theta = -\sum_i \frac{2\lambda_i}{(1 + \lambda_i)^2} d\lambda_i \otimes d\lambda_i,$$

$$\nabla^2 f = 2f(df \otimes d\theta + (1 + f^2)\nabla^2 \theta) = 2f(1 + f^2)d\theta \otimes d\theta + (1 + f^2)\nabla^2 \theta = (1 + f^2)(2f d\theta \otimes d\theta + \nabla^2 \theta).$$

If we set $T(\lambda) := \tan((n - 1)\frac{\pi}{2} - \theta(\lambda))$, we observe that $f < -T$ since

$$-\frac{\pi}{2} < (n - 1)\frac{\pi}{2} - \hat{\Theta} - \theta < \frac{\pi}{2}$$

from the assumption. Thus

$$\nabla^2 f \leq (1 + f^2)(-2T d\theta \otimes d\theta + \nabla^2 \theta)$$

since $d\theta \otimes d\theta$ is semipositive. In the standard coordinates, the form $-\frac{1}{2}(-2T d\theta \otimes d\theta + \nabla^2 \theta)$ has the following matrix representation

$$M := \left( \frac{T + \delta_{ij}\lambda_i}{(1 + \lambda_i^2)(1 + \lambda_j^2)} \right).$$

Then it suffices to show that $M \geq 0$, i.e. $M$ is positive semidefinite. For $k = 1, \ldots, n$ and $I \subset \{1, \ldots, n\}$, let $M_I$ be the principal submatrix of $M$ associated to $I$, i.e.

$$M_I = \left( \frac{T + \delta_{ij}\lambda_i}{(1 + \lambda_i^2)(1 + \lambda_j^2)} \right)_{i,j \in I}.$$ 

Since $M$ is symmetric, $M \geq 0$ holds if and only if $\det M_I \geq 0$ for all $I \subset \{1, \ldots, n\}$. Moreover, since the term $1/(1 + \lambda_i^2)$ appears precisely in the $(i, k)$-entries or $(k, i)$-entries of $M$ for $k = 1, \ldots, n$, we see that

$$\prod_{i \in I} (1 + \lambda_i^2)^2 \cdot \det M_I = \det \tilde{M}_I,$$
where \( \tilde{M}_I \) denotes the principal submatrix of the \( n \times n \) symmetric matrix \( \tilde{M} := (T + \delta_{ij} \lambda_i) \). By the symmetry of \( f \), we may only consider the point \( \lambda \in \mathcal{S} \) with \( \lambda_1 \geq \ldots \geq \lambda_n \). A standard induction argument shows that

\[
\det \tilde{M}_I = \prod_{i \in I} \lambda_i \cdot \left( 1 + \sum_{i \in I} \frac{T}{\lambda_i} \right)
\]
as long as \( \lambda_n \neq 0 \). Set \( x_i := \arctan \lambda_i, y_i := \frac{\pi}{2} - x_i \) \( (i = 1, \ldots, n) \). The argument is divided into two cases;

**Case 1** \((\theta(\lambda) \geq (n - 1)\frac{\pi}{2})\); In this case, we know that \( T \leq 0 \) and \( \lambda_i > 0 \) for all \( i = 1, \ldots, n \). So the condition \( \det \tilde{M}_I \geq 0 \) is equivalent to

\[
1 + \sum_{i \in I} \frac{T}{\lambda_i} \geq 0.
\]

This is clearly true for all \( I \) when \( T = 0 \), so we assume \( T < 0 \), or equivalently \( \theta(\lambda) > (n - 1)\frac{\pi}{2} \). Since \( \sum_{i \in I} \frac{T}{\lambda_i} \geq \sum_{i = 1}^{n} \frac{T}{\lambda_i} \), we may only consider the case \( I = \{1, \ldots, n\} \). From the assumption, we observe that

\[
0 < \sum_{i = 1}^{n} y_i < \frac{\pi}{2}, \quad 0 < y_1 \leq \ldots \leq y_n < \frac{\pi}{2}.
\]

So combining with the formula

\[
\tan \left( \sum_{i = 1}^{n} y_i \right) = \frac{\tan(\sum_{i = 1}^{n-1} y_i) + \tan y_n}{1 - \tan(\sum_{i = 1}^{n-1} y_i) \tan y_n},
\]

we know that \( 0 < 1 - \tan \left( \sum_{i = 1}^{n-1} y_i \right) \tan y_n < 1 \), and hence

\[
\tan \left( \sum_{i = 1}^{n} y_i \right) \geq \tan \left( \sum_{i = 1}^{n-1} y_i \right) + \tan y_n.
\]

Repeating this, we obtain

\[
-\frac{1}{T} = \tan \left( \sum_{i = 1}^{n} y_i \right) \geq \sum_{i = 1}^{n} \tan y_i = \sum_{i = 1}^{n} \frac{1}{\lambda_i}.
\]

**Case 2** \(( (n - 2)\frac{\pi}{2} < \theta(\lambda) < (n - 1)\frac{\pi}{2} )\); In this case, we have \( T > 0 \). If \( \lambda_n > 0 \), we know that \( \tilde{M} \geq 0 \) since it is decomposed as

\[
\tilde{M} = T \cdot E_n + \text{diag}(\lambda_1, \ldots, \lambda_n),
\]

where \( E_n \) denotes the \( n \times n \) matrix whose all entries are equal to one. Now we assume that \( \lambda_n < 0 \). We note that \( \lambda_i > 0 \) for \( i = 1, \ldots, n - 1 \). If \( n \notin I = \{i_1, \ldots, i_\ell\} \), we have a decomposition \( \tilde{M}_I = T \cdot E_\ell + \text{diag}(\lambda_{i_1}, \ldots, \lambda_{i_\ell}) \) with \( \lambda_{i_1}, \ldots, \lambda_{i_\ell} > 0 \), so \( \det \tilde{M}_I \geq 0 \) is
clear. If \( n \in I \), we have \( \sum_{i \in I} \frac{T}{\lambda_i} \leq \sum_{i=1}^n \frac{T}{\xi_i} \). Eventually we may assume \( I = \{1, \ldots, n\} \). Then the condition \( \det \tilde{M}_I \geq 0 \) is equivalent to
\[
1 + \sum_{i \in I} \frac{T}{\lambda_i} \leq 0.
\]
From the assumption, we have
\[
0 < \sum_{i=1}^{n-1} y_i < \frac{\pi}{2} < \sum_{i=1}^n y_i < \pi, \quad 0 < y_1 \leq \ldots \leq y_{n-1} < \frac{\pi}{2} < y_n < \pi.
\]
So from the formula
\[
\tan \left( \sum_{i=1}^n y_i \right) = \frac{\tan(\sum_{i=1}^{n-1} y_i) + \tan y_n}{1 - \tan(\sum_{i=1}^{n-1} y_i) \tan y_n},
\]
we know that \( 1 < 1 - \tan(\sum_{i=1}^{n-1} y_i) \tan y_n, \tan(\sum_{i=1}^{n-1} y_i) + \tan y_n < 0 \) and hence we obtain
\[
\tan \left( \sum_{i=1}^n y_i \right) \geq \tan \left( \sum_{i=1}^{n-1} y_i \right) + \tan y_n.
\]
Applying the same argument as in Case 1 to the first term, we obtain \( \tan(\sum_{i=1}^{n-1} y_i) \geq \sum_{i=1}^{n-1} \tan y_i \). Thus
\[
\frac{-1}{T} = \tan \left( \sum_{i=1}^n y_i \right) \geq \sum_{i=1}^n \tan y_i = \sum_{i=1}^{n-1} \frac{1}{\lambda_i}.
\]
This completes the proof. \( \square \)

Remark 3.1. When \( \hat{\Theta} < (n-1)\frac{\pi}{2} \) the function \( f \) is no longer concave since its level set is no longer convex (cf. [Yua05]).

4. The tangent Lagrangian phase flow

4.1. Monotonicity formulas. We start with some monotonicity properties of functionals along the TLPF defined in Section 2.

Proposition 4.1. Along the TLPF \( \phi_t \) with \( \phi_0 \in \mathcal{H} \) we have

(1) \( \mathcal{C}(\phi_t) \) is constant.
(2) \( \mathcal{J}(\phi_t) \) is monotonically decreasing.
(3) \( \mathcal{V}(\phi_t) \) is monotonically decreasing.

Proof. From the variational formula of \( \mathcal{C} \) and \( \mathcal{J} \), we compute
\[
\frac{d}{dt} \mathcal{C}(\phi_t) = \int_X \frac{d}{dt} \dot{\phi} \cdot \text{Re}(e^{-\sqrt{-1}\hat{\Theta}} (\alpha + \sqrt{-1}X_\phi)^n)
= \int_X \tan(\Theta(A[\phi]) - \hat{\Theta}) \text{Re}(e^{-\sqrt{-1}\hat{\Theta}} (\alpha + \sqrt{-1}X_\phi)^n)
= \int_X v \sin(\Theta(A[\phi]) - \hat{\Theta}) \alpha^n
= 0,
\]
\[
\frac{d}{dt} J(\phi_t) = -\int_X \frac{d}{dt} \phi \cdot \text{Im}(e^{-\sqrt{-1} \hat{\Theta}}(\alpha + \sqrt{-1} X_\phi)^n)
\]
\[
= -\int_X \tan(\Theta(A[\phi]) - \hat{\Theta}) \text{Im}(e^{-\sqrt{-1} \hat{\Theta}}(\alpha + \sqrt{-1} X_\phi)^n)
\]
\[
= -\int_X \tan^2(\Theta(A[\phi]) - \hat{\Theta}) \text{Re}(e^{-\sqrt{-1} \hat{\Theta}}(\alpha + \sqrt{-1} X_\phi)^n)
\]
\[
\leq 0
\]
since \(\phi_t\) stays in the set \(\mathcal{H}\) as long as it exists (cf. Lemma 4.5) and hence the form \(\text{Re}(e^{-\sqrt{-1} \hat{\Theta}}(\alpha + \sqrt{-1} X_\phi)^n)\) defines a positive measure on \(X\). Also we have
\[
\frac{d}{dt} V(\phi_t) = -\int_X \langle d\Theta(A[\phi]), \frac{d}{dt} \phi \rangle \eta_{\phi} = -\int_X (1 + F^2) |d\Theta(A[\phi])|^2 \eta_{\phi} \leq 0.
\]

**Remark 4.2.** In [CY18, Section 2], they discovered a GIT/momentum map interpretation for dHYM metrics in which the \(J\)-functional plays a role of the Kempf-Ness functional, and dHYM metrics are characterized as critical points of \(J\). Also the space \(\mathcal{H}\) has a natural Riemannian structure defined by
\[
\|\delta \phi\|^2_\phi := \int_X (\delta \phi)^2 \text{Re}(e^{-\sqrt{-1} \hat{\Theta}}(\alpha + \sqrt{-1} X_\phi)^n), \quad \delta \phi \in T_{\phi} \mathcal{H}.
\]
Moreover, the Riemannian manifold \(\mathcal{H}\) is equipped with the Levi-Civita connection, and the sectional curvature is non-positive as shown in the recent work [CCL20]. The \(J\)-functional is convex along geodesics with respect to this Riemannian structure. From the proof of Proposition 4.1, we know that the TLPF defines the gradient flow of the \(J\)-functional.

By Lemma 4.5, Proposition 4.1 (1) and Proposition 2.4, we obtain the following;

**Corollary 4.3.** Assume \(\hat{\Theta} \in ((n-1) \frac{\pi}{2}, n \frac{\pi}{2})\). Then along the TLPF \(\phi_t\), the Harnack type inequality
\[
\sup_X \phi_t \leq -C \inf_X \phi_t + C'
\]
holds for some uniform constant \(C, C' > 0\) depending only on \(\alpha, \hat{\chi}, \hat{\Theta}\) and the initial data \(\phi_0 \in \mathcal{H}\).

**Remark 4.4.** In the proof of Proposition 2.4 we see that the normalization \(\bar{\phi} = \phi - |Z|^{-1} C(\phi)\) is a significant issue. As for the LBMCF (1.4), the \(C(\phi_t)\) is not constant or even monotone. Instead, one can easily show that if \(\xi < \inf_X \hat{\Theta}(A[\phi_0]) \leq \sup_X \Theta(A[\phi_0]) < \xi + \frac{\pi}{2}\) for some \(\xi \in \mathbb{R}\), the \(\xi\)-twisted \(C\)-functional
\[
C_\xi(\phi) := \text{Re}(e^{-\sqrt{-1} \xi \hat{\Theta} C(\phi)})
\]
is decreasing along the flow by applying Jensen’s inequality to \(\tan(x - \xi)\) for \(x \in (\xi, \xi + \frac{\pi}{2})\) (see [Tak19, Proposition 2.1]). However this argument does not apply when \(\text{osc}_X \Theta(A[\phi_0]) \geq \frac{\pi}{2}\).
4.2. Long time existence. Now let us consider the TLPF $\phi_t$ with $\phi_0 \in H$ for $t \in [0, T)$ (where $T > 0$ is not necessarily the maximal existence time).

**Lemma 4.5** (see [PT17], Lemma 6). Along the TLPF $\phi_t$ with $\phi_0 \in H$, we have a uniform control

$$\inf_X \Theta(A[\phi_0]) \leq \Theta(A[\phi_t]) \leq \sup_X \Theta(A[\phi_0]).$$

So $\| \frac{d}{dt} \phi \|_{C^0} \leq C$ for some constant $C > 0$ depending only on $\hat{\Theta}$ and $\phi_0$. In particular, the flow stays in $H$ as long as it exists.

**Proof.** A straightforward computation shows that

$$\frac{d}{dt} F(A[\phi_t]) = F^{ij} \partial_i \partial_j \frac{d}{dt} \phi = F^{ij} \partial_i \partial_j (F(A[\phi_t])).$$

So by a maximum principle we have $\inf_X F(A[\phi_0]) \leq F(A[\phi_t]) \leq \sup_X F(A[\phi_0])$, which shows that $\phi_t \in H$ as long as it exists. So by the monotonicity of $\tan: (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$, we have $\inf_X \Theta(A[\phi_0]) \leq \Theta(A[\phi_t]) \leq \sup_X \Theta(A[\phi_0])$ as desired. $\square$

Integrating $\frac{d}{dt} \phi$ on $[0, T)$ we obtain the following;

**Corollary 4.6.** Along the TLPF $\phi_t$ with $\phi_0 \in H$, we have $\| \phi_t \|_{C^0} \leq C_T$ for some constant $C_T > 0$ depending only on $\hat{\Theta}$, $\phi_0$ and $T$.

Also, combining with Proposition 2.2 we have;

**Corollary 4.7.** Along the TLPF $\phi_t$ with $\phi_0 \in H$, there is a uniform constant $C > 0$ depending only on $\phi_0$ such that $|\lambda_n| < C$.

Set

$$\mathcal{F}(\lambda[\phi]) := \sum_{i=1}^n f_i(\lambda[\phi]).$$

Then Lemma 4.5 further implies;

**Corollary 4.8.** For the TLPF $\phi_t$ with $\phi_0 \in H$, there is a constant $C > 0$ depending only on $\hat{\Theta}$ and $\phi_0$ such that

$$\frac{1}{C} < \mathcal{F}(\lambda[\phi_t]) < C. \quad (4.1)$$

**Proof.** We compute

$$\mathcal{F}(\lambda[\phi_t]) = (1 + f^2) \sum_{i=1}^n \frac{1}{1 + \lambda_i^2}.$$

The upper bound of $f$ follows from the assumption $\hat{\Theta} > (n - 1) \frac{\pi}{2}$. The lower bound of $f$ is uniformly controlled by $\inf_X \Theta(A[\phi_0])$ by Lemma 4.5. Combining with the uniform control of $|\lambda_n|$ we obtain the desired estimate. $\square$

Set

$$\mathcal{L} := \frac{d}{dt} - F^{ik} \nabla_k \nabla_i.$$
Lemma 4.9. Assume $\hat{\Theta} \in ((n-1)\frac{\pi}{2}, n\frac{\pi}{2})$ and let $\phi_t$ be the TLPF with $\phi_0 \in \mathcal{H}$. Then we have

$$|\sqrt{-1} \partial \bar{\partial} \phi_t|_{\alpha} \leq C_T,$$

where the constant $C_T > 0$ depends only on $\alpha$, $\hat{\Theta}$, $\phi_0$ and $T$.

Proof. Take $T' < T$ and let $\nabla$ be the Chern connection with respect to $\alpha$. The strategy is applying the maximum principle to the function $G := \log \lambda_1 - Dt$

on $X \times [0, T']$. The constant $D > 0$ is determined in later argument. Assume that the function $G$ attains its maximum on $X \times [0, T']$ at some $(x_0, t_0)$. We want to apply the operator $\mathcal{L}$ to $G$. More precisely, since $\lambda_1$ may not be differentiable ($\lambda_1$ is only continuous on $X$), we use the perturbation technique as in [Szé18, Section 4]. We take a normal coordinates with respect to $\alpha$ centered at $x_0$ to identify $A[\phi(x_0, t_0)]$ as a matrix valued function on it which is diagonal at the origin with eigenvalues $\lambda_1, \ldots, \lambda_n$, and then adjust $A$ by subtracting a small constant diagonal matrix $B = \text{diag}(B^{ii})$ with $0 = B^{11} < B^{22} < \ldots < B^{nn}$. At the origin the matrix $\tilde{A} := A - B$ has eigenvalues

$$\tilde{\lambda}_1 = \lambda_1, \quad \tilde{\lambda}_i = \lambda_i - B^{ii} < \tilde{\lambda}_1 \quad (i > 1).$$

These are distinct, and define smooth functions near the origin. Set

$$\tilde{\Theta} := \log \tilde{\lambda}_1 - Dt.$$

Then we have $\tilde{G}(x, t) \leq G(x, t)$ and $\tilde{G}$ achieves its maximum $\tilde{G}(x_0, t_0) = G(x_0, t_0)$ at $(x_0, t_0)$. It suffices to show that $\tilde{\lambda}_1$ is bounded from above. We may assume $\tilde{\lambda}_1 \geq 1$ at the origin. We compute

$$\mathcal{L} \log \tilde{\lambda}_1 = \frac{1}{\tilde{\lambda}_1} \mathcal{L} \tilde{\lambda}_1 + F^{kk} |\nabla_k \tilde{\lambda}_1|^2 \lambda_1^2,$$

$$\nabla_k \tilde{\lambda}_1 = \nabla_k \chi_{11} - \nabla_k B^{11},$$

$$\nabla_k \nabla_k \tilde{\lambda}_1 = \nabla_k \nabla_k \chi_{11} + \sum_{p>1} \left| \nabla_k \chi_{1p} \right|^2 + \left| \nabla_k \chi_{p1} \right|^2 \frac{1}{\lambda_1 - \lambda_p} \nabla_k \chi_{11} + \sum_{p>1} \left| \nabla_k \chi_{1p} \right|^2 + \left| \nabla_k \chi_{p1} \right|^2 \frac{1}{\lambda_1 - \lambda_p} \nabla_k \chi_{11} + \sum_{p>1} \left| \nabla_k B^{1p} \right|^2 + \left| \nabla_k B^{p1} \right|^2 \frac{1}{\lambda_1 - \lambda_p} \nabla_k B^{11} - 2 \text{Re} \sum_{p>1} \nabla_k \chi_{1p} \nabla_k B^{1p} + \nabla_k \chi_{1p} \nabla_k B^{p1} + \tilde{\lambda}_1 \nabla_k B^{pq} \nabla_k B^{rs}$$

(for instance, see [Szé18 equation (70)]). Evaluating this expression at the origin, and using that $B$ is constant we have

$$\nabla_k \tilde{\lambda}_1 = \nabla_k \chi_{11},$$

$$\nabla_k \nabla_k \tilde{\lambda}_1 = \nabla_k \nabla_k \chi_{11} + \sum_{p>1} \left| \nabla_k \chi_{1p} \right|^2 + \left| \nabla_k \chi_{p1} \right|^2 \frac{1}{\lambda_1 - \lambda_p} \nabla_k \chi_{11} + \sum_{p>1} \left| \nabla_k B^{1p} \right|^2 + \left| \nabla_k B^{p1} \right|^2 \frac{1}{\lambda_1 - \lambda_p} \nabla_k \chi_{11} + \sum_{p>1} \left| \nabla_k B^{pq} \right|^2 + \left| \nabla_k B^{rs} \right|^2 \frac{1}{\lambda_1 - \lambda_p} \nabla_k \chi_{11}.$$

On the other hand, the evolution equation of the TLPF implies that

$$\frac{d}{dt} \tilde{\lambda}_1 = \frac{d}{dt} \nabla_1 \nabla_1 \phi = F^{kk, sf} \nabla_1 \chi_{1k} \nabla_1 \chi_{ss} + F^{kk} \nabla_1 \nabla_1 \chi_{kk}.$$
Thus we compute $\mathcal{L}\tilde{\lambda}_1$ as
\[
\mathcal{L}\tilde{\lambda}_1 = F^{kk}\left(\nabla_1\nabla_k\chi_{kk} - \nabla_k\nabla_k\chi_{111} - \sum_{p>1} \left|\nabla_k\chi_{1p}\right|^2 + \left|\nabla_k\chi_{p1}\right|^2\right) \\
+ F^{t\ell,s\ell}\nabla_1\chi_{t\ell}\nabla_1\chi_{s\ell}.
\]
The first two terms are estimated as
\[
\nabla_1\nabla_k\chi_{kk} - \nabla_k\nabla_k\chi_{111} = \nabla_1\nabla_k\tilde{\chi}_{kk} - \nabla_k\nabla_k\tilde{\chi}_{111} + \nabla_1\nabla_1\phi_{kk} - \nabla_k\nabla_k\phi_{11} \\
\leq C_1 + \text{Rm} \ast \nabla\nabla\phi \\
\leq C_2(\lambda_1 + 1)
\]
since $\nabla\nabla\phi$ is controlled by $\lambda_1$, where $\text{Rm}$ denotes the Riemannian curvature tensor of $\alpha$, and the constants $C_1$, $C_2$ only depends on $\alpha$ and $\tilde{\chi}$. Thus we obtain
\[
\mathcal{L}\log\tilde{\lambda}_1 \leq C_2(1 + \lambda_1^{-1})F + \frac{1}{\lambda_1}F^{t\ell,s\ell}\nabla_1\chi_{t\ell}\nabla_1\chi_{s\ell} + F^{kk}\left|\nabla_k\chi_{11}\right|^2 + \frac{\lambda}{\lambda_1^2}.
\] (4.2)

For the first term of (4.2), we have $C_2(1 + \lambda_1^{-1})F < C_3$ at the origin by $\lambda_1 \geq 1$ and Corollary 4.8. From the concavity of $f$, the second term is non-positive. The third term is zero at $(x_0, t_0)$ by $\nabla\tilde{G} = 0$. Thus applying the maximum principle to the function $\tilde{G}$ with $D := C_3 + 1$, we conclude that $t_0 = 0$. This gives the desired bound. □

With the $C^2$-estimate in hand, we obtain a uniform control of the eigenvalues along the flow. Moreover if we assume $\hat{\Theta} \in ((n - 1)\frac{\pi}{2}, n\frac{\pi}{2})$, then the operator $F(A[\phi_1])$ in the RHS of (1.3) is uniformly elliptic and concave. So we apply the Evans-Krylov theory [Kry82, Wan12] to obtain;

**Lemma 4.10.** Let $\phi_t$ be the TLPF with $\phi_0 \in \mathcal{H}$. Assume $\hat{\Theta} \in ((n - 1)\frac{\pi}{2}, n\frac{\pi}{2})$ and $\|\sqrt{-1}\partial\tilde{\phi}\|_{C^0} \leq C_0$. Then there exist constants $C > 0$ and $\beta \in (0, 1)$ depending only on $\alpha$, $\tilde{\chi}$ and $C_0$ such that
\[
\|\sqrt{-1}\partial\tilde{\phi}\|_{C^\beta(X \times [0, T])} \leq C.
\]

The higher order regularity of the flow follows from the Schauder estimates and a standard bootstrapping argument. We omit the detailed proofs. Finally, a standard argument using Ascoli-Arzelà theorem shows that;

**Theorem 4.11.** Let $\phi_t$ be the tangent Lagrangian phase flow with $\phi_0 \in \mathcal{H}$. Assume that $\|\sqrt{-1}\partial\tilde{\phi}\|_{C^{\beta}(X \times [0, T])}$ is uniformly controlled for some $\beta \in (0, 1)$ (where the constant $\beta$ may depend on $T$). Then the flow $\phi_t$ extends beyond $T$. In particular, if $\hat{\Theta} \in ((n - 1)\frac{\pi}{2}, n\frac{\pi}{2})$, the above assumption is automatically satisfied, and hence the flow $\phi_t$ exists for all positive time.

5. **Convergence of the TLPF under the existence of a $C$-subsolution**

5.1. **$C$-subsolution.** Let $\Gamma_n$ be the positive orthant of $\mathbb{R}^n$. Collins-Jacob-Yau [CJY15] introduced the notion of $C$-subsolution;
Definition 5.1. A function $\phi \in C^\infty(X; \mathbb{R})$ is called a $C$-subsolution if for any $x \in X$, the set
\[
\{ \mu \in \Gamma_n | \theta(\lambda[\phi(x)] + \mu) = \hat{\Theta} \}
\]
is bounded.

Remark 5.2. From [CJY15, Lemma 3.3], any $C$-subsolution $\phi$ must satisfy $\Theta(A[\phi]) > \frac{\hat{\Theta}}{n-1}(\hat{\Theta} - \frac{n}{2})$. In particular, any $C$-subsolution $\phi$ is almost calibrated when $\hat{\Theta} > (n-1)\frac{n}{2}$.

In particular, a genuine solution to (1.2) is clearly a $C$-subsolution. In [CJY15], the notion of $C$-subsolutions is used to study the elliptic equation (1.2). In the next subsection, we will see that the same notion is also useful to study the limiting behavior of the TLPF. Set
\[
g(\lambda, \tau) := f(\lambda) + \tau, \quad (\lambda, \tau) \in S \times \mathbb{R}.
\]
The condition $f_i > 0$ shows that at each point $(x, t)$, the ray $\{ (\lambda[\phi(x, t)] + s\mu, s\tau) | s \geq 0 \}$ generated by any non-zero element $(\mu, \tau) \in \Gamma_n \times \mathbb{R}_{\geq 0}$ intersects transversely with the level set $\{ g = 0 \}$ just once. So by the compactness, there is a $\delta > 0$ and $K > 0$ such that at each $(x, t) \in X \times [0, T)$, any element in the set
\[
\{ (\mu, \tau) \in \Gamma_n \times \mathbb{R}_{\geq 0} | f(\lambda[\phi(x, t)] - \delta I + \mu) + \tau - \delta = 0 \}
\]
satisfies $|\mu| + |\tau| < K$, where $I$ denotes the vector $(1, \ldots, 1)$ of eigenvalues of the identity matrix. In later arguments, we fix this $\delta$ and $K$.

5.2. Up to $C^k$-estimates. In the remaining of the paper, we prove the second part of Theorem 1.4. Again we note that the proof is almost a word-by-word copy of general theory of fully non-linear parabolic equations [PT17] once we establish Theorem 1.1 and define a proper notion of $C$-subsolutions. However our function $f$ does not have the structural properties imposed in [PT17, Szé18]. On the contrary, the function $f$ can not be extended to a symmetric cone $\Gamma \subset \mathbb{R}^n$ containing $S$ since $f(\lambda) \to -\infty$ as $\lambda$ reaches the boundary $\partial S$. For this reason, we need to check carefully to see if every argument in [PT17] carries over to our case. We will explain there is no substantial differences from [PT17] except the gradient estimate (cf. Lemma 5.7).

In what follows, we assume that $\hat{\Theta} \in ((n-1)\frac{n}{2}, n\frac{n}{2})$ and there is a $C$-subsolution $\phi$. Set $\hat{\chi} = \chi_\phi$ and let $\phi_t$ be the TLPF with $\phi_0 \in \mathcal{H}$. One can prove the following two lemmas exactly as in [PT17];

Lemma 5.3 (see [PT17], Lemma 1). There exists a uniform constant $C > 0$ depending only on $\alpha, \hat{\chi}, \hat{\Theta}$ and $\phi_0$ such that $\|\phi_t\|_{C^0} \leq C$.

Proof. This lemma is based on the parabolic version of the Alexandrov-Bakelman-Pucci estimates due to [Iso85]. It is straightforward to check that the proof requires only the lower bound $\Delta_\alpha \phi_t \geq -C$ (cf. Lemma 2.3), the ellipticity of the operator and the boundedness of the set (5.1). Unlike the elliptic case, the argument for parabolic case is more subtle, which just provides a uniform lower bound for $\phi_t$ as mentioned in [PT17]. So we apply the Harnack type equality (cf. Corollary 4.3) to get the full estimate of $\phi_t$. \qed
Lemma 5.4 (see [PT17], Lemma 3). There exists a constant \( \rho = \rho(\delta, K) > 0 \) (where the constants \( \delta, K \) are defined in (5.1)) so that if \( |\lambda[\phi_t] - \lambda[\phi]| > K \), then either
\[
\mathcal{L}\phi_t > \rho \mathcal{F}(\lambda[\phi_t])
\]
or we have for any \( i = 1, \ldots, n \),
\[
F_i^i(A[\phi_t]) > \rho \mathcal{F}(\lambda[\phi_t]).
\]

Proof. The proof requires only the ellipticity and convexity of the level set of \( g \), that are available in our case. \( \square \)

Remark 5.5. As for the elliptic operator \( \Theta \), Collins-Jacob-Yau [CJY15, Proposition 3.5] proved a similar inequality based on [Szé18, Proposition 6] only by using the convexity of the level set of \( \theta \). Indeed as pointed out in [CJY15], the proof of [Szé18, Proposition 6] only requires the ellipticity and the convexity of the level set of \( \theta \). However this argument can not be extended directly to the parabolic case, i.e. the LBMCF case (1.4). Indeed, if we set \( h(\lambda, \tau) := \theta(\lambda) + \tau \), then the each level set \( h(\lambda, \tau) = c \) defines the graph of the function \( \tau = c - \theta(\lambda) \), which is convex if and only if the function \( \theta \) itself is concave. This fails as soon as \( \theta(\lambda) < (n - 1)\frac{\pi}{2} \).

Lemma 5.6. We have the estimate
\[
|\sqrt{-1} \partial \bar{\partial} \phi_t|_\alpha \leq C(1 + \sup_{X \times [0,T']} |\nabla \phi_t|_\alpha^2),
\]
where the constant \( C > 0 \) depends only on \( \alpha, \tilde{\chi}, \tilde{\Theta} \) and \( \phi_0 \).

The proof of this lemma also proceeds along the same line as in [PT17, Lemma 2]. However, using the uniform control of \( |\lambda_n| \) and the vanishing of the torsion tensor of \( \alpha \), we can simplify the argument. We give a proof for the sake of completeness.

Proof of Lemma 5.6. Take \( T' < T \) and consider the function
\[
G := \log \lambda_1 + \Phi(|\nabla \phi|^2) + \Psi(\phi)
\]
on \( X \times [0, T'] \), where the functions \( \Phi, \Psi \) are specified by
\[
\Phi(s) := -\frac{1}{2} \log \left( 1 - \frac{s}{2P} \right), \quad s \in [0, P],
\]
\[
\Psi(s) := De^{-s}, \quad s \in \left[ \inf_{X \times [0,T']} \phi, \sup_{X \times [0,T']} \phi \right],
\]
where \( P := \sup_{X \times [0,T']} (|\nabla \phi|^2 + 1) \), and the large constant \( D > 0 \) is chosen in the course of the proof. Then we note that
\[
\frac{1}{4P} < \Phi' < \frac{1}{2P}, \quad \Phi'' = 2(\Phi')^2 > 0.
\]
Assume that \( G \) attains its maximum on \( X \times [0, T'] \) at some \((x_0, t_0)\). Now we use a perturbation technique similar to the one used in Lemma 4.9. We will apply the maximum principle to the function
\[
\tilde{G} := \log \tilde{\lambda}_1 + \Phi(|\nabla \phi|^2) + \Psi(\phi),
\]
(5.3)
where we adopt the same notations as in the proof of Lemma 4.9. Since $|\lambda_n|$ is uniformly controlled along the flow, we may assume that at the origin, $\lambda$ satisfies:

- $\lambda_1 \geq 1$.
- $|\lambda[\phi(x_0, t_0)] - \lambda[\phi(x_0)]| > K$
- $\kappa \lambda_1 \geq -\lambda_n$.
- $\frac{1}{1 + \lambda^2} \leq \frac{\rho}{1 + \lambda^2}$

where the constant $\rho = \rho(\delta, K) > 0$ is determined in Lemma 5.4 and $\kappa = \kappa(D, \|\phi\|_{C^0}) > 0$ is determined in later arguments (if $\lambda$ does not satisfy any of the above four conditions, then the desired estimate already holds). We may assume that $t_0 > 0$. As for the term of $\mathcal{LG}$ we already observe in the proof of Lemma 4.9 that

$$\mathcal{L} \log \tilde{\lambda}_1 \leq C_1 F + \frac{1}{\lambda_1} F^{ik, st} \nabla_1 \chi_{ik} \nabla_1 \chi_{st} + F^{kk}[\nabla_k \tilde{\lambda}_1]^2 \lambda_1^2,$$

where we used the lower bound $\lambda_1 \geq 1$ to obtain the first term. For the second term of $\mathcal{LG}$ we compute

$$\mathcal{L}(\Phi' (|\nabla \phi|^2)) = \Phi' \cdot \mathcal{L}(|\nabla \phi|^2) - \Phi'' \cdot F^{qq} |\nabla_q (|\nabla \phi|^2)|^2$$

$$= \Phi' \cdot (\nabla^j \phi \cdot \mathcal{L}(\nabla_j \phi) + \nabla^j \phi \cdot \mathcal{L}(\nabla_j \phi) - F^{qq} (|\nabla_q \nabla \phi|^2 + |\nabla_q \nabla \phi|^2))$$

$$- \Phi'' \cdot F^{qq} |\nabla_q (|\nabla \phi|^2)|^2,$$

$$\nabla_j \frac{d}{dt} \phi = F^{kk} \nabla_j \chi_{kk}.$$

It follows that

$$\mathcal{L}(\nabla_3 \phi) = F^{kk} (\nabla_j \chi_{kk} - \nabla_k \nabla_k \nabla_3 \phi)$$

$$= F^{kk} (\nabla_j \tilde{\chi}_{kk} - R_{kk} m_k \nabla_m \phi).$$

Using $(4P)^{-1} < \Phi' < (2P)^{-1}$ we get

$$\Phi' \cdot \nabla^j \phi \cdot \mathcal{L}(\nabla_j \phi) \leq C_2 F$$

for some constant $C_2 > 0$ depending only on $\alpha$ and $\tilde{\chi}$. The similar estimate also holds for $\Phi' \cdot \nabla^j \phi \cdot \mathcal{L}(\nabla_j \phi)$. Thus

$$\mathcal{L} \Phi(|\nabla \phi|^2) \leq C_3 F - F^{qq} (|\nabla_q \nabla \phi|^2 + |\nabla_q \nabla \phi|^2) - \Phi'' \cdot F^{qq} |\nabla_q (|\nabla \phi|^2)|^2.$$

The estimate for the last term of $\mathcal{LG}$ is straightforward;

$$\mathcal{L}(\Psi(\phi)) = \Psi' \mathcal{L} \phi - \Psi'' F^{kk} |\nabla_k \phi|^2.$$

Summarizing the above estimates we get

$$0 \leq \mathcal{LG} \leq F^{kk} \left(\frac{|\nabla_k \tilde{\lambda}_1|^2}{\lambda_1^2} + \frac{1}{\lambda_1} F^{ik, st} \nabla_1 \chi_{ik} \nabla_1 \chi_{st} + C_4 F$$

$$- F^{qq} (|\nabla_q \nabla \phi|^2 + |\nabla_q \nabla \phi|^2) - \Phi'' \cdot F^{qq} |\nabla_q (|\nabla \phi|^2)|^2$$

$$+ \Psi' \mathcal{L} \phi - \Psi'' F^{kk} |\nabla_k \phi|^2 \right).$$ (5.4)

To deal with the first bad term, we will use the second, fifth and last good terms. Set

$$I := \{i|F^{ii} > \kappa^{-1} F^{11}\}.$$
We note that $1 \notin I$ since $\kappa < 1$. Then at the maximum point we have $\nabla \tilde{G} = 0$, which yields that
\[
\sum_{k \notin I} F^{kk} \frac{|\nabla_k \tilde{\lambda}_1|^2}{\lambda_1^2} = \sum_{k \notin I} F^{kk} |\Phi' \nabla_k (|\nabla \phi|^2) + \Psi' \nabla_k \phi|^2
\]
\[
\leq 2(\Phi')^2 \sum_{k \notin I} F^{kk} |\nabla_k (|\nabla \phi|^2)|^2 + 2(\Psi')^2 \sum_{k \notin I} F^{kk} |\nabla_k \phi|^2
\]
\[
\leq \Phi'' \sum_{k \notin I} F^{kk} |\nabla_k (|\nabla \phi|^2)|^2 + 2(\Psi')^2 \kappa^{-1} F^{11} P.
\]
On the other hand,
\[
2\kappa \sum_{k \in I} F^{kk} \frac{|\nabla_k \tilde{\lambda}_1|^2}{\lambda_1^2} \leq 2\kappa \Phi'' \sum_{k \in I} F^{kk} |\nabla_k (|\nabla \phi|^2)|^2 + 4\kappa (\Psi')^2 \sum_{k \in I} F^{kk} |\nabla_k \phi|^2.
\]
Choose $\kappa = \kappa(D, \|\phi\|_{C^0})$ sufficiently small so that $4\kappa (\Psi')^2 \leq \frac{1}{2} \Phi''$. Then
\[
\begin{align*}
0 & \leq \frac{1}{\lambda_1} F^{\ell k, sf} \nabla_1 \chi_{\ell k} \nabla_1 \chi_{sf} + (1 - 2\kappa) \sum_{k \in I} F^{kk} \frac{|\nabla_k \tilde{\lambda}_1|^2}{\lambda_1^2} - F^{q q} (|\nabla_q \nabla \phi|^2 + |\nabla_q \nabla \phi|^2) \\
& \quad + \Psi' \mathcal{L} \phi + 2(\Psi')^2 \kappa^{-1} F^{11} P + C_4 F.
\end{align*}
\]
We note that $\nabla_1 \chi_{\ell k} = \nabla_k \tilde{\lambda}_1 = \nabla_k \tilde{\lambda}_1$ since $d\chi = 0$ and we are working at a point in normal coordinates. By the concavity and symmetry of $f$ we have
\[
F^{\ell k, sf} \nabla_1 \chi_{\ell k} \nabla_1 \chi_{sf} \leq \sum_{k \in I} \frac{F^{11} - F^{kk}}{\lambda_1 - \lambda_k} |\nabla_1 \chi_{11}|^2 = \sum_{k \in I} \frac{F^{11} - F^{kk}}{\lambda_1 - \lambda_k} |\nabla_k \tilde{\lambda}_1|^2
\]
since $\frac{F^{11} - F^{kk}}{\lambda_1 - \lambda_k} \leq 0$ (cf. [Sze 18, equation (67)]). We know that
\[
1 - \frac{\kappa}{\lambda_1 - \lambda_k} \geq \frac{1 - 2\kappa}{\lambda_1}.
\]
Indeed,
\[
(1 - \kappa)\lambda_1 - (1 - 2\kappa)(\lambda_1 - \lambda_k) = \kappa \lambda_1 + (1 - 2\kappa)\lambda_k.
\]
This expression is clearly positive when $\lambda_k \geq 0$. Otherwise, we have $k = n$, and by using the assumption $\kappa \lambda_1 \geq -\lambda_n$, we get
\[
\kappa \lambda_1 + (1 - 2\kappa)\lambda_n \geq -2\kappa \lambda_n > 0.
\]
Thus we have
\[
\sum_{k \in I} \frac{F^{11} - F^{kk}}{\lambda_1 - \lambda_k} |\nabla_k \tilde{\lambda}_1|^2 \leq - \sum_{k \in I} \frac{(1 - \kappa) F^{kk}}{\lambda_1 - \lambda_k} |\nabla_k \tilde{\lambda}_1|^2 \leq - \frac{1 - 2\kappa}{\lambda_1} \sum_{k \in I} F^{kk} |\nabla_k \tilde{\lambda}_1|^2.
\]
So the first and second terms of (5.5) are estimated as
\[
\frac{1}{\lambda_1} F^{\ell k, sf} \nabla_1 \chi_{\ell k} \nabla_1 \chi_{sf} + (1 - 2\kappa) \sum_{k \in I} F^{kk} \frac{|\nabla_k \tilde{\lambda}_1|^2}{\lambda_1^2} \leq 0.
\]
since $\lambda_1 \geq 1$. As for the forth good term of (5.5), we use the following estimate
\[ F^{q,i}(|\nabla_q \phi|^2 + |\nabla_q \overline{\nabla} \phi|^2) \geq F^{1i}|\lambda_1 - \overline{\chi}_i|^2 \geq F^{1i} \frac{\lambda_i^2}{2} - C_5 F \]
for some constant $C_5$ depending only on $\alpha$ and $\overline{\chi}$. Putting all things together we obtain
\[ 0 \leq F^{11} \left( 2(\Psi')^2 \kappa^{-1} P - \frac{\lambda_i^2}{2} \right) + \Psi' \mathcal{L} \phi + C_6 F \]
with $C_6 := C_4 + C_5$. Now we invoke Lemma 5.4. From the assumption $\frac{1}{1+\lambda_i^2} \leq \frac{\rho}{1+\lambda_i^2}$, we observe that $\frac{1}{1+\lambda_i^2} < \rho \sum_{i=1}^{n} \frac{1}{1+\lambda_i^2}$, or equivalently $F^{11} < \rho F$. So we have $\mathcal{L} \phi \geq \rho F$. Since $\Psi' < 0$, the above inequality yields that
\[ 0 \leq F^{11} \left( 2(\Psi')^2 \kappa^{-1} P - \frac{\lambda_i^2}{2} \right) + (\rho \Psi' + C_6) F. \]
We take $D > 0$ sufficiently large so that $\rho \Psi' + C_6 < 0$ (this is possible since the constant $C_6$ does not depend on $\kappa$). Then we have
\[ \frac{\lambda_i^2}{2} \leq 2(\Psi')^2 \kappa^{-1} P. \]
This yields the desired bound. \hfill \Box

**Lemma 5.7.** There is a constant $C > 0$ depending on $\alpha$, $\overline{\chi}$, $\widehat{\Theta}$ and $\phi_0$ such that
\[ \sup_{X \times [0,T]} |\nabla \phi_i|^2 \alpha \leq C. \quad (5.6) \]

In [PT17 Lemma 4], they give the gradient estimate like Lemma 5.7 by the blowup argument combined with Székelyhidi’s Liouville theorem for $\Gamma$-solutions (cf. [Sze18, Section 5]). This argument does not apply to the TLPF due to the lack of the structural properties for $f$ as mentioned in the beginning of this subsection. However, in our case, since we have a uniform lower bound for the eigenvalues by Corollary 4.7, the argument is rather simple. We follow closely to the argument [CJY15 Proposition 5.1];

**Proof of Lemma 5.7.** Assume that (5.6) does not hold. Then there exists a sequence $(x_k, t_k) \in X \times [0,T)$ with $t_k \to T$ such that
\[ C_k := |\nabla \phi(x_k, t_k)| \alpha = \sup_{X \times [0,t_k]} |\nabla \phi|^2 \alpha \to \infty \]
as $k \to \infty$. By passing to a subsequence, we may further assume that $\{x_k\}$ converges to some point $x \in X$. From the previous arguments, there is a uniform constant $C > 0$ such that
\begin{itemize}
  \item $\overline{\chi} + \sqrt{-1} \partial \overline{\partial} \phi_t \geq -C \alpha$ on $X \times [0,T)$.
  \item $\sup_{X \times [0,T]} |\phi_t| \leq C$.
  \item $|\sqrt{-1} \partial \overline{\partial} \phi(x, t_k)| \alpha \leq C(1 + \sup_{X \times [0,t_k]} |\nabla \phi|^2) \alpha$ for all $x \in X$.
\end{itemize}
For each $k$, we take a local coordinates $\{U_k, (z_1, \ldots, z_n)\}$ centered at $x_k$, identifying with the ball $B_1(0)$ of radius 1, where $\alpha = \text{Id} + O(|z|^2)$. By replacing $C$ by a slightly large number, we may further assume that $\alpha$ is the Euclidean metric. We define $\phi_k(z) := \phi(z/C_k, t_k)$ defined on the ball of radius $C_k$. Then we have the following properties;
ψ satisfies the same heat equation as

\[ |\nabla \phi| \leq 2C \text{ on } B_{C_k}(0). \]

\[ |\nabla \phi_k| \leq C. \]

\[ |\nabla \phi_k|_{\alpha} \leq 1 = |\nabla \phi_k(0)|_{\alpha} \text{ on } B_{C_k}(0). \]

The proof can now be completed exactly as in [CJY13, Proposition 5.1]. So for a fixed \( \beta \in (0, 1) \), we know that by passing to a subsequence, \( \phi_k \) converges to \( \phi_\infty : \mathbb{C}^n \to \mathbb{R} \) in \( C^{1,\beta}_{\text{loc}} \) as \( k \to \infty \). Moreover, the function \( \phi_\infty \) is continuous, uniformly bounded, has \( |\nabla \phi_\infty(0)|_{\alpha} = 1 \) and satisfies \( \sqrt{-1} \partial \bar{\partial} \phi_\infty \geq 0 \) in the sense of distributions. Such a function must be a constant (cf. [Ron74]), which contradicts to \( |\nabla \phi_\infty(0)|_{\alpha} = 1. \)

5.3. Convergence of the flow. Now we will finish the proof of Theorem 1.4 by showing that;

**Theorem 5.8.** Let \( X \) be a compact complex manifold with a Kähler form \( \alpha \), and \( \tilde{\chi} \) a closed real \((1, 1)\)-form. Assume that \( \tilde{\Theta} \in ((n - 1)^\frac{\epsilon}{2}, n\frac{\epsilon}{2}) \) and there is a \( C \)-subsolution. Then the tangent Lagrangian phase flow \( \phi_t \) starting from any potential \( \phi_0 \in \mathcal{H} \) converges to the deformed Hermitian Yang-Mills metric \( \phi_\infty \in \mathcal{H} \) in the \( C^\infty \)-topology.

We can show this by using the argument in [PTT17, Lemma 7].

**Proof of Theorem 5.8.** From the previous subsection, Lemma 4.10 and a standard bootstrapping argument we obtain a uniform \( C_k \) control along the flow \( \phi_t \) for any non-negative integer \( k \). To prove the \( C^\infty \)-convergence, we set \( \psi_t := \frac{d}{dt} \phi_t + A \) for some uniform constant \( A > 0 \) such that \( \psi_t > 0 \) for all \( t \in [0, \infty) \) by using Lemma 4.5. Then \( \psi \) satisfies the same heat equation as \( \phi \);

\[
\frac{d}{dt} \psi = F^{\alpha \beta} \partial_\alpha \partial_\beta \psi.
\]

(5.7)

Since we already know that the RHS of (5.7) is uniformly elliptic by the \( C^2 \)-estimate, we can apply the differential Harnack inequality on compact Hermitian manifolds to (5.7), and obtain

\[
\text{osc}_X \frac{d}{dt} \phi(\cdot, t) = \text{osc}_X \psi(\cdot, t) \leq C_1 e^{-C_2 t}
\]

for some constants \( C_1, C_2 > 0 \) (see [Gil11, Section 6, Section 7] for more details). On the other hand, in the proof of Proposition 4.1 we observe that

\[
\int_X \frac{d}{dt} \phi \cdot \text{Re}(e^{-\sqrt{-1} \tilde{\Theta}}(\alpha + \sqrt{-1} \chi_\phi)^n) = 0,
\]

which in particular shows that there exists a point \( y = y(t) \in X \) such that \( \frac{d}{dt} \phi(y, t) = 0 \) since the measure \( \text{Re}(e^{-\sqrt{-1} \tilde{\Theta}}(\alpha + \sqrt{-1} \chi_\phi)^n) \) is positive along the flow. Therefore for any \( x \in X \) we have

\[
\left| \frac{d}{dt} \phi(x, t) \right| = \left| \frac{d}{dt} \phi(x, t) - \frac{d}{dt} \phi(y, t) \right| \leq \text{osc}_X \frac{d}{dt} \phi(\cdot, t) \leq C_1 e^{-C_2 t},
\]

and hence

\[
\frac{d}{dt} \left( \phi_t + \frac{C_1}{C_2} e^{-C_2 t} \right) = \frac{d}{dt} \phi - C_1 e^{-C_2 t} \leq 0.
\]
So the function $\phi_t + \frac{C_1}{C_2} e^{-C_2 t}$ is decreasing in $t$, and uniformly bounded by the $C^0$ estimate. Thus it converges to a function $\phi_\infty$. By the higher order estimates, we know that this convergence is actually in $C^\infty$. The function $\phi_t$ also converges to the same function $\phi_\infty$ in $C^\infty$. The convergence $\frac{d}{dt}\phi \to 0$ yields that the function $\phi_\infty$ must satisfies the equation $F(A[\phi_\infty]) = 0$, so we have $\Theta(A[\phi_\infty]) = \hat{\Theta}$. This completes the proof. □

Remark 5.9. Using the monotonicity of $\mathcal{C}$ and $\mathcal{V}$ (cf. Proposition 4.1) together with the uniqueness result of dHYM metrics [JY17, Theorem 1.1], one can easily obtain an alternative proof of the $C^\infty$-convergence of the TLPF in the same way as in the proof of [Tak19, Theorem 1.1].
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