Conjugacy stability of parabolic subgroups of Artin-Tits groups of spherical type

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July 20, 2018

Abstract

We study conjugacy stability of standard parabolic subgroups of Artin-Tits groups of spherical type. Up to some exceptions, we show that irreducible subgroups are conjugacy stable whereas reducible ones are not. This answers a question asked by Ivan Marin and generalizes a theorem obtained by González-Meneses in the specific case of Artin braid groups.

1 Introduction

Let $S$ be a finite set. A Coxeter matrix over $S$ is a symmetric square matrix $M = (m_{s,t})_{s,t \in S}$ indexed by the elements of $S$, such that $m_{s,s} = 1$, and $m_{s,t} \in \{2, 3, 4, \ldots, \infty\}$ for all $s, t \in S$, $s \neq t$. Such a Coxeter matrix is usually represented by its Coxeter graph, denoted by $\Gamma = \Gamma_S = \Gamma(M)$. This is a labeled graph whose set of vertices is $S$, in which two distinct vertices $s$ and $t$ are connected by an edge if $m_{s,t} \geq 3$; if in addition $m_{s,t} \geq 4$, the corresponding edge wears the label $m_{s,t}$.

The Coxeter system of $\Gamma$ is the pair $(W, S)$, where $W = W_{\Gamma}$ is the group

$$W_{\Gamma} = \left\langle S \mid s^2 = 1 \quad \Pi(s, t; m_{s,t}) = \Pi(t, s; m_{s,t}) \text{ for all } s, t \in S, s \neq t, m_{s,t} \neq \infty \right\rangle.$$

For $m \geq 2$,

$$\Pi(a, b; m) = \begin{cases} (ab)^k & \text{if } m = 2k, \\ (ab)^{k+a} & \text{if } m = 2k + 1. \end{cases}$$
The group $W_\Gamma$ is called the Coxeter group of $\Gamma$; sometimes we shall also use the notation $W_S$ for this group.

Let $\Sigma_S := \{ \sigma_s, s \in S \}$ be a set in one-to-one correspondence with $S$. The Artin-Tits system of $\Gamma$ is the pair $(A, \Sigma)$, where $A = A_\Gamma$ is the group

$$A_\Gamma = \left\langle \Sigma \mid \Pi(\sigma_s, \sigma_t; m_{s,t}) = \Pi(\sigma_t, \sigma_s; ms, t) \quad \text{for all } s, t \in S, s \neq t, m_{s,t} \neq \infty \right\rangle.$$

The group $A_\Gamma$ is called the Artin-Tits groups of $\Gamma$; sometimes we shall also use the notation $A_S$ for this group.

There is a canonical epimorphism $\theta : A_\Gamma \to W_\Gamma$, defined by $\theta(s) = \sigma_s$. Its kernel is called the Colored Artin-Tits groups (or pure Artin-Tits group) of $\Gamma$ and denoted by $CA_\Gamma$.

A set-theoretic section $\tau : W_\Gamma \to A_\Gamma$ for $\theta$ can be defined in the following way. For $w \in W_\Gamma$, choose any reduced expression $s_i \ldots s_i$ for $w$ (i.e. a word-representative on $S$ of the shortest possible length) and set $\tau(w) = \sigma_i \ldots \sigma_i$; this assignment is well-defined thanks to Matsumoto’s Lemma (see for instance [8, Theorem 1.2.2]).

This information can be synthesized in the following short exact sequence:

$$1 \to CA_\Gamma \to A_\Gamma \overset{\theta}{\to} W_\Gamma \to 1 \quad (1)$$

The groups $A_\Gamma$ and $W_\Gamma$ are called irreducible if the graph $\Gamma$ is connected. The group $A_\Gamma$ is said to be of spherical type if $W_\Gamma$ is a finite group. By extension, we say that $\Gamma$ is of spherical type if $W_\Gamma$ is finite. We recall Coxeter’s classification of irreducible finite Coxeter groups, hence of irreducible Artin-Tits groups of spherical type: this result is synthesized in Figure 1.

Figure 1: Connected Coxeter diagrams of spherical type with a specific enumeration of the standard generators of the associated Artin-Tits groups.

Let $X$ be a subset of $S$ and set $\Sigma_X = \{ \sigma_x, x \in X \}$. The standard parabolic subgroup associated to $X$ is the subgroup of $W_\Gamma$ ($A_\Gamma$, respectively) generated by $X$ (by $\Sigma_X$, respectively); denote it by $W_X$ ($A_X$, respectively). Consider
the subgraph $\Gamma_X$ of $\Gamma = \Gamma_S$ generated by $X$ (the set of vertices is $X$ and the edges are exactly the edges of $\Gamma_S$ which connect two vertices in $X$). It is known [2, 12] that $(W_X, X)$ ($(A_X, \Sigma_X)$, respectively) is the Coxeter system (Artin-Tits system, respectively) of $\Gamma_X$. A parabolic subgroup is a subgroup conjugate to a standard parabolic subgroup.

The flagship example of an Artin-Tits group is the braid group on $n$ strands $B_n$ ($n \geq 2$). It is associated to the Coxeter graph $A_{n-1}$ depicted in Figure 1; the corresponding Coxeter group is the symmetric group $S_n$. Let $m$ and $n$ be two positive integers such that $2 \leq m \leq n$. Considering only the $m-1$ first vertices of the graph $A_{n-1}$ furnishes a fundamental example of a standard (irreducible) parabolic subgroup: the braid group $B_m$ embedded in $B_n$ by adding $n-m$ straight strands to any $m$-strand braid.

Some years ago, Ivan Marin asked whether standard parabolic subgroups of Artin-Tits groups of spherical type are conjugacy stable. A proper subgroup $H$ of a group $G$ is said to be conjugacy stable if any two elements of $H$ which are conjugated in $G$ must be conjugated through an element of $H$; this is equivalent to saying that the conjugacy classes of $H$ do not fuse (or merge) in $G$.

A first partial answer was given by González-Meneses who showed that standard irreducible parabolic subgroups of braid groups are conjugacy stable [10].

In this paper, we answer completely Marin’s question by proving the following result (we use notation from Figure 1):

**Theorem 1.** Let $A_\Gamma = A_S$ be an irreducible Artin-Tits group of spherical type. Let $X \subset S$.

1) If $A_X$ is irreducible, $A_X$ is conjugacy stable in $A_S$ except in the following cases:

- $A_X$ is of type $D_5$ and $A_S$ is of type $E_6, E_7$ or $E_8$,
- $A_X$ is of type $D_7$ and $A_S$ is of type $E_8$,
- $A_X$ is of type $E_7$ and $A_S$ is of type $E_8$,
- $A_X$ is of type $D_{2k}$ ($k \geq 2$),
- $A_X$ is of type $H_3$ and $A_S$ is of type $H_4$.

2) If $A_X$ is reducible, $A_X$ is not conjugacy stable in $A_S$ except in the following cases:

- $A_S$ is of type $B_n$ ($n \geq 3$) and $A_X = A_{s_1} \times A_X'$, with $X' \subset \{s_3, \ldots, s_n\}$ and $A_X'$ irreducible.
- $A_S$ is of type $F_4$ and $A_X = A_{X_1} \times A_{X_2}$ has two irreducible components, satisfying $X_1 \subset \{s_1, s_2\}$ and $X_2 \subset \{s_3, s_4\}$.

If $A_S$ is reducible and is expressed as $A_S = A_{S_1} \times \ldots \times A_{S_r}$, where $r \geq 1$ and each $A_{S_i}$ is non-trivial and irreducible, for a subset $X \subset S$ we can consider $X_i = X \cap S_i$ ($i = 1, \ldots, r$) and decompose $A_X$ as a direct product of parabolic
subgroups $A_X = A_{X_1} \times \ldots \times A_{X_r}$—notice that $A_{X_i}$ might be trivial (when $X_i$ is empty) or reducible. Since elements in distinct components of $A_S$ commute pairwise, $A_X$ is conjugacy stable in $A_S$ if and only if $A_{X_i}$ is conjugacy stable in $A_{S_i}$ for all $i$. Therefore Theorem 1 above answers completely Marin’s question, for any Artin-Tits group of spherical type.

González-Meneses’ proof in the specific case of braids relies heavily on the identification between braids and mapping classes of punctured disks: Birman-Lubotzky-McCarthy’s *Canonical Reduction Systems* of mapping classes play a fundamental role. Although more combinatorial in spirit, our approach shares a common point with González-Meneses’; instead of the Canonical Reduction System, we use the parabolic closure of elements of Artin-Tits groups of spherical type introduced recently in [5]; see Theorem 2.

The main step in our proof consists in reducing conjugacy stability of $A_X$ in $A_S$ to a problem in the respective Coxeter groups:

**Definition 1.** We say that $(W_X, W_S)$ satisfies Property $\star_W$ if for all $Y_1, Y_2 \subset X$, for all $w \in W_S$ such that $Y_1 w = w Y_2$, there exists $v \in W_X$ such that $v^{-1} y v = w^{-1} y w$ for all $y \in Y_1$.

Our main technical result is the following:

**Proposition 1.** If $(W_X, W_S)$ satisfies Property $\star_W$, then $A_X$ is conjugacy stable in $A_S$.

Proving Property $\star_W$ for finite Coxeter groups is a matter of case-checking which is mostly contained in the study of the so-called Coxeter classes in [5, Section 2.3]. We can deduce from the results there that for irreducible $W_X$, Property $\star_W$ holds in all the cases which are not listed in Theorem 1 (1) (Section 3). Property $\star_W$ in special reducible cases (Theorem 1 2) also follows. Section 4 finishes the proof of Theorem 1; counterexamples are given when $A_X$ is not conjugacy stable in $A_S$.

**Acknowledgements.** We thank to Ivan Marin for suggesting the problem under study, to Luis Paris for his interest and comments on the problem, and to Jesús Juyumaya for showing Geck-Pfeiffer’s book to the first author. The first author was supported by Fondecyt Regular 1180335, the third author was supported by the Spanish projects MTM2016-76453-C2-1-P and FEDER.

## 2 The Property $\star_W$ and conjugacy stability

### 2.1 The Properties $\star_W$ and $\star_A$

Fix a Coxeter system $(W_S, S)$ with Coxeter graph $\Gamma = \Gamma_S$. Consider also the corresponding Artin-Tits system $(A_S, \Sigma_S)$. In this section, we do not make any assumption on $W_S$: in particular, $W_S$ need not be finite.

The following basic facts about the length and reduced expressions in $W_S$ can be found for instance in [11, Section 1.4]. For $w \in W_S$, the length $\ell(w)$ of $w$ is
defined as the length of the shortest possible word on $S$ which represents $w$; such a word is said to be a reduced expression for $w$. For $s \in S$, $w \in W_S$, we have

$$|\ell(ws) - \ell(w)| = |\ell(sw) - \ell(w)| = 1.$$  

Moreover, the following two statements are equivalent:

- $\ell(ws) = \ell(w) + 1$ ($\ell(sw) = \ell(w) + 1$, respectively)
- for every reduced expression $s_1 \ldots s_{\ell(w)}$ of $w$, the word $s_1 \ldots s_{\ell(w)}s$ (the word $ss_1 \ldots s_{\ell(w)}$, respectively) is a reduced expression as well.

In the opposite case, we have the following two equivalent statements:

- $\ell(ws) = \ell(w) - 1$ ($\ell(sw) = \ell(w) - 1$, respectively)
- $w$ admits a reduced expression finishing with the letter $s$ (starting with the letter $s$, respectively).

In order to prove Proposition \[\text{II}\] we first establish an intermediate property, which is the analogue of Property $\star_W$ at the level of Artin-Tits groups. For this purpose, we will first need to prove a technical lemma. Roughly speaking, the next lemma says that the section $\tau : W_S \rightarrow A_S$ (see the Introduction) allows to transform conjugators in $W_S$ between subsets of $S$ into conjugators in $A_S$ between subsets of $\Sigma_S$.

**Lemma 1.** Let $Y_1, Y_2 \subset S$. Suppose that $w \in W_S$ is such that $Y_1w = wY_2$ and write, for $y \in Y_1$, $\pi(y) = w^{-1}yw \in Y_2$. Then $\tau(w)$ satisfies $\tau(w)^{-1}\sigma_y\tau(w) = \sigma_{\pi(y)}$, for all $y \in Y_1$.

**Proof.** Fix $y \in Y_1$. We know that $\ell(yw) = \ell(w\pi(y)) = \ell(w) \pm 1$. Suppose first that $\ell(yw) = \ell(w) + 1$. If $s_1 \ldots s_{\ell(w)}$ is a reduced expression for $w$, then both $ys_1 \ldots s_{\ell(w)}y$ and $s_1 \ldots s_{\ell(w)}\pi(y)$ are reduced expressions for $yw = w\pi(y)$. Then $\tau(yw)$ can be written as $\sigma_y\sigma_{s_1} \ldots \sigma_{s_{\ell(w)}}$, but also as $\sigma_{s_1} \ldots \sigma_{s_{\ell(w)}}\sigma_{\pi(y)}$. In other words, in $A_S$ we have $\sigma_y\tau(w) = \tau(w)\sigma_{\pi(y)}$.

Now, suppose that $\ell(yw) = \ell(w\pi(y)) = \ell(w) - 1$. This means that $w$ admits a reduced expression starting with the letter $y$, but also a reduced expression finishing with the letter $\pi(y)$. We then can write $w = yw' = w''\pi(y)$, with $\ell(w') = \ell(w'') = \ell(w) - 1$. But we observe that $w'' = w\pi(y) = yw = w'$. Hence $yw' = w'\pi(y)$, with $\ell(yw') = \ell(w') + 1$. According to the discussion in the above paragraph, in $A_S$ we have $\sigma_y\tau(w') = \tau(w')\sigma_{\pi(y)}$. It finally follows from $\tau(w) = \sigma_y\tau(w')$ that $\tau(w)^{-1}\sigma_y\tau(w) = \tau(w')\sigma_y\tau(w') = \tau(w')^{-1}\sigma_y\tau(w') = \sigma_{\pi(y)}$. \[\square\]

**Definition 2.** Let $A_X$ a standard parabolic subgroup of $A_S$. We say that $(A_X, A_S)$ satisfies Property $\star_A$ if for all $Y_1, Y_2 \subset X$, for all $g \in A_S$ such that $\Sigma_{Y_1}g = g\Sigma_{Y_2}$, there exists $h \in A_X$ such that $h^{-1}\sigma_yh = g^{-1}\sigma_yg$ for all $y \in Y_1$. 

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Proposition 2. \((W_X, W_S)\) satisfies Property \(\star_W\) if and only if \((A_X, A_S)\) satisfies Property \(\star_A\).

Proof. Suppose first that \((W_X, W_S)\) satisfies Property \(\star_W\). Let \(Y_1, Y_2 \subseteq X\) and consider \(g \in A_S\) such that \(\Sigma Y_1g = g \Sigma Y_2\). Let \(\pi : Y_1 \rightarrow Y_2\) be the bijection given for \(y \in Y_1\) by \(g^{-1} \sigma_y g = \sigma_{\pi(y)} \in \Sigma Y_2\). Using the canonical surjection \(\theta : A_S \rightarrow W_S\), we have \(\theta(g)^{-1}y\theta(g) = \pi(y)\) in \(W_S\). Now, Property \(\star_W\) provides an element \(h \in W_X\) satisfying \(h^{-1} y h = \theta(g)^{-1} y \theta(g)\) for all \(y \in Y_1\). Applying Lemma 1, we see that \(\tau(h)\) satisfies \(\tau(h)^{-1} \sigma_y \tau(h) = \sigma_{\pi(y)} = g^{-1} \sigma_y g\), for all \(y \in Y_1\). This shows Property \(\star_A\) for \((A_X, A_S)\), as \(\tau(h) \in A_X\).

Conversely, assume that \((A_X, A_S)\) has Property \(\star_A\). Let \(Y_1, Y_2 \subseteq S\) and consider \(w \in W_S\) such that \(Y_1w = wY_2\). Let \(\pi : Y_1 \rightarrow Y_2\) be the bijection given for \(y \in Y_1\) by \(w^{-1} y w = \pi(y) \in Y_2\). By Lemma 1, we have \(\tau(w)^{-1} \sigma_y \tau(w) = \sigma_{\pi(y)}\), for all \(y \in Y_1\). Thanks to Property \(\star_A\), \(\tau(w)\) may be replaced by \(\sigma_{\pi(y)}\) in all \((A_X, A_S)\), satisfying \(h^{-1} \sigma_y h = \sigma_{\pi(y)}\). Apply \(\theta\) to this equality, to obtain \(\theta(h)^{-1} \sigma_y \theta(h) = \sigma_{\pi(y)}\) and observe that \(\theta(h) \in W_X\). This shows Property \(\star_W\) for \((W_X, W_S)\).

\(\square\)

2.2 Proof of Proposition 1

Let us suppose again that \((W_S, S)\) is a Coxeter system with Coxeter graph \(\Gamma = \Gamma_S\) and \((A_S, \Sigma_S)\) is the corresponding Artin-Tits system. We now assume in addition that \(A_S\) is of spherical type. Let \(X \subseteq S\). We need to show that if \((W_X, W_S)\) satisfies Property \(\star_W\), then \(A_X\) is conjugacy stable in \(A_S\). Thanks to Proposition 2, Proposition 1 will follow from:

Proposition 3. Assume that \(A_S\) is an Artin-Tits group of spherical type and that \(A_X\) is a standard parabolic subgroup of \(A_S\). If \((A_X, A_S)\) has Property \(\star_A\), then \(A_X\) is conjugacy stable in \(A_S\).

Before proceeding to the proof, we recall the important and recently defined notion of parabolic closure of elements of Artin-Tits groups of spherical type:

Theorem 2. \([6, \text{Section 7, Lemma 8.1}]\) For each \(a \in A_S\), there is a unique minimal (with respect to inclusion) parabolic subgroup \(P_a\) of \(A_S\) which contains \(a\) and the parabolic closure of \(a\). For \(g \in A_S\), we have \(g^{-1}P_aga = P_g^{-1}ag\).

Proof (of Proposition 3). Let \(a, b \in A_X\) and \(c \in A_S\) satisfying \(c^{-1}a c = b\). According to Theorem 4, the respective parabolic closures \(P_a\) and \(P_b\) are conjugated by \(c: c^{-1}P_ac = P_b\).

Choose elements \(s_a, s_b\) of \(A_X\) and subsets \(Y_a, Y_b\) of \(X\) such that \(s_a^{-1}P_as_a = A_{Y_a}\) and \(s_b^{-1}P_bs_b = A_{Y_b}\); this is possible thanks to the much stronger result \([6, \text{Theorem 4}]\). Note that \(s_a^{-1}cs_b\) conjugates \(A_{Y_a}\) into \(A_{Y_b}\); according to \([6, \text{Proposition 2.1 (3)\text{]}}, we can find \(u \in A_S\) with \(Y_a = uY_b\) and \(v \in A_{Y_b}\) such that \(s_a^{-1}cs_b = uv\).

By Property \(\star_A\), we can find \(u' \in A_X\) such that \(u'^{-1} \sigma_y u' = u^{-1} \sigma_y u\) for every \(y \in Y_a\). It follows that \(s_a u' u s_b^{-1}\), which is an element of \(A_X\), conjugates \(a\) into \(b\).
3 Checking Property $\star_W$

This section is devoted to checking Property $\star_W$ for an irreducible proper standard parabolic subgroup $W_X$ of a finite irreducible Coxeter group $W_S$, when $(W_X, W_S)$ is not one of the special cases described in Theorem 1. Throughout, the reader is implicitly referred to Figure whenever considering a Coxeter (Artin-Tits) group of some specific (spherical) type and some letters in the corresponding generating sets.

The proof goes through Coxeter’s classification, according to the type of $W_X$. Observe that the assumption that $W_X$ is a proper subgroup implies that $W_X$ cannot be of type $E_6, F_4, H_4$ or $I_{2m}, m > 5$. Subgroups $W_X$ of type $H_3, E_7$ or $D_{2n}$ are excluded as special cases in Theorem 1 as well as subgroups $W_X$ of type $D_5$ or $D_7$ inside a group of type $E_{6,7,8}$. If $W_X$ is of type $A_n, E_6$, or $I_{2,5}$, we shall prove in Lemma 2 a stronger statement than Property $\star_W$; Lemma 3 deals with the situation where $W_X$ is of type $B$ or $D$.

Recall that every finite Coxeter group contains a unique longest element, usually denoted by $w_0$; here, following the context, we will use this notation or make it more precise by adding the specification on the Coxeter group under consideration: for instance, $w_0(F_4)$ will denote the longest element of the Coxeter group of type $F_4$. Relevant facts concerning the longest element of a finite Coxeter group can be found in Sections 1.5 and 1.6. It is known that $w_0(W_S)$ is an involution and that the inner automorphism of $W_S$ associated to $w_0(W_S)$ fixes the standard generating set $S$. Moreover, $w_0(W_S)$ is central if $W_S$ is of type $B_n$ ($n \geq 2)$, $D_n$ ($n \geq 4$, even), $I_{2m}$ (m even), $H_3, H_4, F_4, E_7$ or $E_8$. In the other irreducible cases, the conjugation action of $w_0(W_S)$ on $S$ is given by $s_i \mapsto s_{n-i+1}$ in type $A_n$, $s_1 \mapsto s_2, s_2 \mapsto s_1, s_i \mapsto s_i(i \geq 3)$ in type $D_n$ (n odd) and by the permutation $[s_1, s_6][s_3, s_5]$ in type $E_6$.

**Lemma 2.** Suppose that $W_X$ is a Coxeter group of type $A_n, E_6$ or $I_{2,5}$. For $Y_1, Y_2 \subset X$, if $\phi$ is a bijection $\phi : Y_1 \longrightarrow Y_2$ compatible with the defining relations of $W_X$, then we can find $v \in A_X$ such that $\phi(y) = v^{-1}yv$, for all $y \in Y_1$. In particular, if $W_X$ is a standard parabolic subgroup of a bigger Coxeter group $W_S$, then $(W_X, W_S)$ satisfies Property $\star_W$.

**Proof.** For Property $\star_W$, observe that a conjugation in $W_S$ between subsets $Y_1$ and $Y_2$ of $X$ induce in particular a bijection $Y_1 \longrightarrow Y_2$ compatible with the defining relations of $W_X$.

Under the assumption, the standard parabolic subgroups $W_Y$ and $W_Y$ are isomorphic; according to Proposition 2.3.8 (A_n case), Table A.1 (E_6 case), and because conjugation by the longest element of $I_{2,5}$ swaps $s_1$ and $s_2$, it follows that in all cases under consideration, the sets $Y_1$ and $Y_2$ are conjugated by an element of $W_X$. We need to show in addition that there is such a conjugation which realizes $\phi$ pointwise on $Y_1$. To this end, according to the above paragraph, it is sufficient to check, for some representative $Y$ of each conjugacy class of subsets of $S$, that...
every bijection \( \psi : Y \rightarrow Y \) compatible with the defining relations of \( W_X \) can be realized pointwise on \( Y \) as a conjugation by an element of \( W_X \).

If \( W_X \) is of type \( A_n \) (i.e. \( W_X \) is the symmetric group \( S_{n+1} \)), then \( W_Y \) must be a direct product of some symmetric groups of smaller ranks:

\[
W_Y \cong S_n^{m_1} \times S_{n+1}^{m_2} \times \ldots \times S_{n+1}^{m_n},
\]

for some \( m_1, \ldots, m_n \geq 0 \). A bijection \( \psi : Y \rightarrow Y \) compatible with the defining relations of \( W_X \) consists in some permutation of the irreducible components of same rank combined possibly with a conjugation by an arbitrary product of the respective longest elements of the irreducible components. This can be realized by a conjugation inside \( W_X \).

Suppose now that \( W_X \) has type \( E_6 \). If \( Y \subset \{ s_1, s_3, s_4, s_5, s_6 \} \), the latter paragraph shows the desired claim, as the parabolic subgroup generated by \( s_1, s_3, s_4, s_5, s_6 \) is of type \( A_5 \). The other conjugacy classes of standard parabolic subgroups of the Coxeter group of type \( E_6 \) (see [8, Table A1]) are listed in the table below; the first column indicates the isomorphism class of the subgroup, the second column gives a special representative \( Y \) for the class, the third column indicates all possible bijective \( \psi : Y \rightarrow Y \) compatible with the group relations and for each one, we give in the fourth column an element \( v \) of \( W_X \) such that \( v^{-1}yv = \psi(y) \), for \( y \in Y \).

| Isomorphism class | \( Y \) | \( \psi \) | \( v \) |
|-------------------|---------|---------|-------|
| \( S_3 \times Z_2 \) | \( \{ s_1, s_2, s_3, s_6 \} \) | \( s_2, s_4 \) | \( s_2s_4s_2 \) |
| \( D_4 \) | \( \{ s_2, s_3, s_4, s_5 \} \) | \( s_3, s_5 \) \( s_2, s_5 \) \( s_2, s_3 \) | \( w_0(E_6) \) \( w_0(W_{\{s_1, s_2, s_3, s_4, s_5\}}) \) \( w_0(W_{\{s_2, s_3, s_4, s_5, s_6\}}) \) |
| \( S_3^2 \times Z_2 \) | \( \{ s_1, s_2, s_3, s_5, s_6 \} \) | \( s_1, s_3 \) \( s_5, s_6 \) \( s_2, s_3 \) \( s_1, s_5 \) \( s_3, s_5 \) | \( s_1s_3s_1 \) \( s_5s_6s_5 \) \( s_1s_3s_1s_5s_6s_5 \) \( w_0(E_6) \) |
| \( S_5 \times Z_2 \) | \( \{ s_1, s_2, s_4, s_5, s_6 \} \) | \( s_2, s_6 \) \( s_4, s_6 \) \( s_1, s_6 \) | \( s_3, s_5 \) | \( w_0(W_{\{s_2, s_3, s_5, s_6\}}) \) \( w_0(E_6) \) |

Finally, if \( W_X \) has type \( I_{2,5} \), \( Y \) may have one or two elements; in any case, the claim holds as \( s_1 \) and \( s_2 \) are conjugated in \( I_{2,5} \).

Notice that using [8, Table A.2] and similar arguments as in the above proof, the statement of Lemma 2 can be shown as well for \( W_X \) of type \( E_8 \) (and \( I_{2m} \) \( m \) odd); however we shall not need this: these cases are excluded by the assumption that \( W_X \) is a proper subgroup of the finite Coxeter group \( W_S \).

We now consider the cases where \( W_X \) is of type \( B \) and \( D \). By contrast with Lemma 2 in order to obtain the desired conjugating element in \( W_X \), we need the fact that the bijection under consideration \( Y_1 \rightarrow Y_2 \) comes from a conjugation by an element of a Coxeter group \( W_S \) containing \( W_X \) as a standard parabolic subgroup.
Lemma 3. Let \( W_X \) be a proper standard irreducible parabolic subgroup of an irreducible finite Coxeter group \( W_S \). Suppose that \( W_X \) is of type \( B_n \) (\( n \geq 2 \)) or that \( W_X \) is of type \( D_n \) (\( n \) odd) and \( W_S \) is of type \( D_m \) (\( m > n \)). Then \((W_X, W_S)\) satisfies Property \(*_W\).

Proof. Suppose that \( W_X \) is of type \( B_n \); observe that \( W_S \) must be of type \( F_4 \) (if \( n \leq 3 \)) or \( B_m \) (\( m > n \)). Assume first that \( W_S \) is of type \( B_m \) (\( m > n \)). For a subset \( Y \subset X \subset S \), denote by \( Y' \) the biggest subset of \( Y \) containing \( s_1 \) and such that \( W_{Y'} \) is irreducible (set \( Y' = \emptyset \) if \( s_1 \notin Y \)) and write \( Y'' = Y \setminus Y' \). According to [8] Proposition 2.3.10], two subsets \( Y_1, Y_2 \) of \( X \subset S \) are conjugated inside \( W_S \) if and only if \( Y'_1 = Y'_2 \) and the groups \( W_{Y'_1} \) and \( W_{Y'_2} \) are isomorphic. Moreover, such a conjugation must fix \( Y'_1 \) pointwise; conjugation of \( Y'_1 \) into \( Y''_1 \) by an element of \( W_S \) can be replaced by a conjugation by an element of \( W_{\{s_2, \ldots, s_n\}} \subset W_X \), according to Lemma 2 as \( W_{\{s_2, \ldots, s_n\}} \) is of type \( A_{n-1} \). This shows that two subsets \( Y_1, Y_2 \) of \( X \) are conjugated in \( W_S \) whenever they are conjugated in \( W_X \). If on another hand, \( W_S \) is of type \( F_4 \), it is readily checked that any conjugation in \( W_S \) between subsets of \( X \) is already realized in \( W_X \).

Suppose now that \( W_X \) is of type \( D_n \), \( n \) odd and that \( W_S \) is of type \( D_m \), \( m > n \). Recall that conjugation by the longest element \( w_0(W_X) \) leaves invariant \( s_1, \ldots, s_n \) and swaps \( s_1 \) and \( s_2 \). Let \( Y_1, Y_2 \) be two subsets of \( X \) and \( w \in W_S \) such that \( w^{-1}Y_1w = Y_2 \). If \( Y_1 \) and \( Y_2 \) each contain at most one of \( s_1, s_2 \), up to replacing one of \( Y_1 \) (\( i = 1, 2 \)) by \((w_0(W_X))^{-1}Y_i(w_0(W_X))\), we may assume that both \( Y_1, Y_2 \) are subsets of \( \{s_1, s_3, \ldots, s_n\} \); the latter set defines a Coxeter group of type \( A_{n-1} \) and by Lemma 2 we can find \( v \in W_{\{s_1, s_3, \ldots, s_n\}} \subset W_X \) such that \( v^{-1}yw = w^{-1}yw \) for all \( y \in Y_1 \). If on the contrary \( Y_1 \) contains both \( s_1 \) and \( s_2 \), then \( Y_2 \) also contains both \( s_1 \) and \( s_2 \) (\( s_1 \) and \( s_2 \) can be simultaneously conjugated to letters in \( W_S \) only if they are fixed or swapped). For \( i = 1, 2 \), consider the (product of) irreducible component(s) \( W_{Y''_i} \) of \( W_{Y_i} \) which contains \( s_1 \) and \( s_2 \). Note that \( Y'_i = Y''_i \). Let \( Y''''_i = Y''_i \setminus Y''_i \). Then \( w^{-1}Y''''_iw = Y''''_2 \); as \( Y''''_i \) \( \subset \{s_4, \ldots, s_n\} \) and \( W_{\{s_4, \ldots, s_n\}} \) is of type \( A_{n-3} \), Lemma 2 gives an element \( v \) of \( W_{\{s_4, \ldots, s_n\}} \subset W_X \) such that \( v^{-1}yw = w^{-1}yw \) for all \( y \in Y''''_1 \). If \( w^{-1}s_1w = s_1 \), then \( v^{-1}yw = w^{-1}yw \) for all \( y \in Y_1 \); otherwise, \( (w_0(W_X))^{-1}y(vw_0(W_X)) = w^{-1}yw \) for all \( y \in Y_1 \). \(\square\)

4 Proof of Theorem 1

4.1 Reducible \( A_X \)

We first consider the case of a reducible subgroup \( A_X \). We need to show that \( A_X \) is not conjugacy stable except in the two special cases of Theorem 1. The following observation is the key of the proof –note that it is valid for any Artin-Tits group:

Lemma 4. Two letters \( \sigma_s, \sigma_t \in \Sigma_S \) are conjugated in \( A_S \) if and only if the vertices \( s \) and \( t \) of the Coxeter graph \( \Gamma_S \) can be connected in \( \Gamma_S \) through a path.
following only edges with odd labels (or no label).

Proof. This follows from Lemma 1 and the corresponding fact for Coxeter groups (see for example [2, Chap. IV, §1, no. 3, Proposition 3]).

Assume first that $A_X$ has at least three irreducible components. Observe that a connected Coxeter graph of spherical type as at most one even-labeled edge. It follows from Lemma 4 that we can find $\sigma_s, \sigma_t \in \Sigma_X$ belonging to distinct components of $A_X$ such that $\sigma_s$ and $\sigma_t$ are conjugated in $A_S$; this contradicts conjugacy stability.

The same argument applies if $A_X$ has only two components but $A_S$ is not of type $B_n$ nor $F_4$ since in this case, all the edges in the Coxeter graph $\Gamma_S$ are odd-labeled.

Finally, if $A_S$ is of type $B_n$ and $A_X$ has two components which both contain some letter in $\{\sigma_s^2, \ldots, \sigma_s^n\}$, the same argument again applies (because all letters in $\{\sigma_s^2, \ldots, \sigma_s^n\}$ are conjugated in $A_S$) and $A_S$ is not conjugacy stable.

In any of these two cases, we need to show that $A_X$ is conjugacy stable. This will follow from Property $\star W$, according to Proposition 1.

Assume that $W_S$ is of type $B_n$ ($n \geq 3$), $X = \{s_1\} \cup X'$, with $X' \subset \{s_3, \ldots, s_n\}$ and $W_{X'}$ irreducible. Let $Y_1, Y_2 \subset X$ and $w \in W_S$ such that $Y_1w = wY_2$. Notice that $s_1 \in Y_1$ if and only if $s_1 \in Y_2$ because $s_1$ is the unique letter in its conjugacy class in $W_S$ ([2, Chap. IV, §1, no. 3, Proposition 3]). Notice that $W_{X'}$ is of type $A_m$ ($m < n$). It follows from Lemma 2 that there is a $v \in W_{X'} \subset W_X$ so that $v^{-1}yv = w^{-1}yw$ for all $y \in Y_1$; this is Property $\star W$ for $(W_X, W_S)$.

Finally, assume that $W_S$ is of type $F_4$ and $W_X = W_{X_1} \times W_{X_2}$, with $X_1 \subset \{s_1, s_2\}$ and $X_2 \subset \{s_3, s_4\}$. The easy verification of this case is left to the reader.

4.2 Irreducible $A_X$

Suppose that the pair $(A_X, A_S)$ is not in the list of Theorem 1. Section 3 and Proposition 1 show that $A_X$ is conjugacy stable in $A_S$, as desired.

It remains to be proven that $A_X$ is not conjugacy stable in $A_S$ when $(A_X, A_S)$ is one of the special cases mentioned in Theorem 1. 1). Given any Artin-Tits group of spherical type $A_S$, define $\Delta_S$ as the image of the longest element $w_0(W_S)$ under the map $f : W_S \to A_S$. Call an element of $A_S$ positive if it can be expressed as a word on letters in $S$, without using inverses. The support $\text{Supp}(g)$ of a positive element of $A_S$ is the set of letters which appear in any positive word on $S$ representing $g$.

In order to find a counterexample for each case, we will use some objects called ribbons:

Definition 3. Let $A_S$ be an Artin–Tits group of spherical type. Given $X \subseteq S$ and $t \in S$, we define the positive element

$$r(t,X) = \Delta_X^{-1} \Delta_X \cup \{t\}$$
and we call it a ribbon. If moreover \( t \) is adjacent to \( X \) in the Coxeter graph \( \Gamma_S \), we say that \( r(t,X) \) is an adjacent ribbon.

Note that \( r(t,X) \) conjugates \( X \) to some subset \( X' \) of \( X \cup \{ t \} \) [\cite{5} Lemma 4.2]. Using this concept, we have the following:

**Lemma 5.** Let \( g, h \) be positive elements of \( A_S \) such that \( \text{supp}(g) = Y \subseteq S \) and \( \text{supp}(h) = Z \subseteq S \). If \( g \) and \( h \) are conjugate in \( A_S \), then there are subsets \( Y = Y_0, \ldots, Y_n = Z \) of \( S \) and adjacent ribbons \( r(t, Y_{i-1}) \) (\( i = 1, \ldots, n \)) conjugating \( Y_{i-1} \) to \( Y_i \).

**Proof.** Let \( y \in A_S \) such that \( y^{-1}gy = h \); up to multiplying by a power of \( \Delta_S^2 \) (which is central), we may assume that \( y \) is a positive element. Set \( Y_0 = Y \). By [\cite{5} Proposition 6.3], \( y \) can be decomposed as a product \( r_1 \cdots r_m \) such that \( g_i := (r_1 \cdots r_i)^{-1}g(r_1 \cdots r_i) \) is a positive element of \( A_S \), for \( 1 \leq i \leq m \) and, if \( Y_i = \text{Supp}(g_i) \), \( r_i \) is either an element of \( A_{Y_{i-1}} \) or an adjacent ribbon of the form \( r(t, Y_{i-1}) \). Now, by [\cite{5} Corollary 6.5], we know that \( (r_1 \cdots r_i)^{-1}A_{Y_0}(r_1 \cdots r_i) = A_{Y_i} \). So, if \( r_i \) belongs to \( A_{Y_{i-1}} \), then \( r_i \) normalizes \( A_{Y_{i-1}} \), that is, \( A_{Y_{i-1}} = A_{Y_i} \) and we will have that \( r_1 \cdots r_{i-1}r_{i+1} \cdots r_m \) still conjugates \( A_{Y_0} \) to \( A_{Y_m} = Z \). Therefore, we can consider, in the decomposition \( y = r_1 \cdots r_m \), only the factors which are adjacent ribbons. This yields the desired conclusion. \( \square \)

Thanks to the previous lemma, we are able to give the following counterexamples:

- If \( A_X \) is of type \( D_5 \) and \( A_S \) is of type \( E_6, E_7 \) or \( E_8 \), then \( A_X \) is not conjugacy stable in \( A_S \).

Take \( g = \sigma_1\sigma_3\sigma_4 \) and \( h = \sigma_2\sigma_4\sigma_5 \), hence \( Y = \{ \sigma_1, \sigma_3, \sigma_4 \} \) and \( Z = \{ \sigma_2, \sigma_4, \sigma_5 \} \). In \( E_6, E_7 \) or \( E_8 \), \( Y \) and \( Z \) (and \( g \) and \( h \)) are conjugate by the following product of ribbons (each arrow indicates the conjugation by its label):

![Diagram](https://via.placeholder.com/150)

However, in \( A_X \) with \( X = \{ \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5 \} \) the possibilities of conjugation by adjacent ribbons are:

![Diagram](https://via.placeholder.com/150)
Hence, by Lemma 5, $g$ and $h$ are not conjugates in $A_X$.

- If $A_X$ is of type $D_7$ and $A_S$ is of type $E_8$, then $A_X$ is not conjugacy stable in $A_S$.

  Take $g = \sigma_2 \sigma_3 \sigma_4$ and $h = \sigma_3 \sigma_4 \sigma_5$, hence $Y = \{\sigma_2, \sigma_3, \sigma_4\}$ and $Z = \{\sigma_3, \sigma_4, \sigma_5\}$. In $E_8$, $Y$ and $Z$ (and $g$ and $h$) are conjugate by the following product of ribbons:

  $$
  \xymatrix{ Y \ar[r]^{r(\sigma_1, Y)} & Y_2 = \{\sigma_1, \sigma_3, \sigma_4\} \ar[r]^{r(\sigma_5, Y_2)} & Z }
  $$

  However, in $A_X$ with $X = \{\sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8\}$ the only possible adjacent ribbon of $Y$ is $r(\sigma_5, Y)$, which normalizes $Y$. Hence, by Lemma 5, $g$ and $h$ are not conjugates in $A_X$.

- If $A_X$ is of type $E_7$ and $A_S$ is of type $E_8$, then $A_X$ is not conjugacy stable in $A_S$.

  Take $g = \sigma_1 \sigma_3 \sigma_4 \sigma_5 \sigma_6$ and $h = \sigma_2 \sigma_4 \sigma_5 \sigma_6 \sigma_7$, hence $Y = \{\sigma_1, \sigma_3, \sigma_4, \sigma_5, \sigma_6\}$ and $Z = \{\sigma_2, \sigma_4, \sigma_5, \sigma_6, \sigma_7\}$. In $E_8$, $Y$ and $Z$ (and $g$ and $h$) are conjugate by the following product of ribbons:

  $$
  \xymatrix{ Y \ar[r]^{r(\sigma_7, Y)} & Y_2 = \{\sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7\} \ar[r]^{r(\sigma_8, Y_2)} & Z \ar[r]^{r(\sigma_2, Y_3)} & Y_3 = \{\sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8\} }
  $$

  However, in $A_X$ with $X = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7\}$ the possibilities of conjugation by adjacent ribbons are:

  $$
  \xymatrix{ Y \ar[r]^{r(\sigma_7, Y)} & Y_2 = \{\sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7\} \ar[r]^{r(\sigma_2, Y_2)} & Y_3 = \{\sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8\} }
  $$

  Hence, by Lemma 5, $g$ and $h$ are not conjugates in $A_X$.

- If $A_X$ is of type $D_{2k}$ ($k \geq 2$), then $A_X$ is not conjugacy stable in any $A_S$. In this case it exists $A_{X'} \subseteq A_S$ of type $D_{2k+1}$ such that $A_X \subseteq A_{X'}$. Follow the notation of Figure 1 for $\Gamma_X$, and let $\Gamma_X$ correspond to the only possible subgraph of type $D_{2k}$. Let $Y = \{\sigma_1, \sigma_3, \ldots, \sigma_{2k}\}$, $Z = \{\sigma_2, \sigma_3, \ldots, \sigma_{2k}\}$ and $Y_1 = \{\sigma_3, \sigma_4, \ldots, \sigma_{2k+1}\}$. In $A_{X'}$, we have that $Y$ and $Z$ (and $\sigma_1 \sigma_3 \cdots \sigma_{2k}$ and $\sigma_2 \sigma_3 \cdots \sigma_{2k}$) are conjugate in the following way.
However, in $A_X$ the only possible adjacent ribbon $r(\sigma_2, Y)$ normalizes $Y$. Therefore, by Lemma 5, the elements $\sigma_1 \sigma_3 \cdots \sigma_{2k}$ and $\sigma_2 \sigma_3 \cdots \sigma_{2k}$ are conjugates in $A'_X$ (hence in $A_S$) but not in $A_X$.

- If $A_X$ is of type $H_3$ and $A_S$ is of type $H_4$, then $A_X$ is not conjugacy stable in $A_S$. We are going to prove that $\sigma_1 \sigma_3 \sigma_3$ and $\sigma_3 \sigma_1 \sigma_1$ are conjugate in $H_4$ but not in $H_3$.

One can easily verify that the conjugation by

$$r(\sigma_4, \{\sigma_1, \sigma_3\}) \cdot r(\sigma_2, \{\sigma_1, \sigma_4\}) \cdot \Delta(\sigma_2, \sigma_3, \sigma_4) \cdot r(\sigma_1, \{\sigma_2, \sigma_4\}) \cdot r(\sigma_3, \{\sigma_1, \sigma_4\})$$

normalizes the subgroup $\{\sigma_1, \sigma_3\}$ and permutes $\sigma_1$ and $\sigma_3$. Now we need to show that there is no element in $H_4$ permuting $\sigma_1$ and $\sigma_3$ under conjugation. This follows from Lemma 6 observing that $H_3$ contains no $\mathbb{Z}^2$ standard parabolic subgroup other than $\langle \sigma_1, \sigma_3 \rangle$.

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