Abstract

This manuscript presents an attempt to introduce Lagrangian formalism for mechanical systems using para-quaternionic Kähler manifolds, which represent an interesting multidisciplinary field of research. In addition to, the geometrical-physical results related to para-quaternionic Kähler mechanical systems are also given.

Keywords: Para-quaternionic Kähler geometry, Lagrangian mechanical system.

MSC: 53C15, 70H03, 70H05.
1 Introduction

Today, many branches of science are into our lives. One the branches is mathematics that has multiple applications. In particular, differential geometry and mathematical physics have a lots of different applications. These applications are used in many areas. One of them are on geodesics. Geodesics are known the shortest route between two points. Time-dependent equations of geodesics can be easily found with the help of the Euler-Lagrange equations. We can say that differential geometry provides a good working area for studying Lagrangians of classical mechanics and field theory. The dynamic equation for moving bodies is obtained for Lagrangian mechanic. These dynamic equation is illustrated as follows:

Lagrange Dynamics Equation [1][2][3]: Let $M$ be an $n$-dimensional manifold and $TM$ its tangent bundle with canonical projection $\tau_M : TM \rightarrow M$. $TM$ is called the phase space of velocities of the base manifold $M$. Let $L : TM \rightarrow R$ be a differentiable function on $TM$ called the Lagrangian function. We consider the closed 2-form on $TM$ given by $\Phi_L = -dd_JL$ (if $J^2 = -I$, $J$ is a complex structure and if $J^2 = I$, $J$ is a paracomplex structure, $Tr(J) = 0$). Consider the equation

$$i_X \Phi_L = dE_L. \tag{1}$$

Then $X$ is a vector field, we shall see that (1) under a certain condition on $X$ is the intrinsical expression of the Euler-Lagrange equations of motion. This equation is named as Lagrange dynamical equation. We shall see that for motion in a potential, $E_L = V(L) - L$ is an energy function and $V = J(X)$ a Liouville vector field. Here $dE_L$ denotes the differential of $E$. The triple $(TM, \Phi_L, X)$ is known as Lagrangian system on the tangent bundle $TM$. If it is continued the operations on (1) for any coordinate system $(q^i(t), p_i(t))$, infinite dimension Lagrange’s equation is obtained the form below:

$$\frac{dq^i}{dt} = \dot{q}^i, \quad \frac{d}{dt} \left( \frac{\partial L}{\partial q^i} \right) = \frac{\partial L}{\partial \dot{q}^i}, \quad i = 1, ..., n. \tag{2}$$

There are many studies about Lagrangian dynamics, mechanics, formalisms, systems and equations (see detail [4]). There are real, complex, paracomplex and other analogues. It is well-known that Lagrangian analogues are very important tools. They have a simple method to describe the model for mechanical systems. The models about mechanical systems are given as follows.

Some examples of the Lagrangian is applied to model the problems include harmonic oscillator, charge $Q$
in electromagnetic fields, Kepler problem of the earth in orbit around the sun, pendulum, molecular and fluid dynamics, LC networks, Atwood’s machine, symmetric top etc. Let’s remember some work done. Vries shown that the Lagrangian motion equations have a very simple interpretation in relativistic quantum mechanics [5]. Paracomplex analogue of the Euler-Lagrange equations was obtained in the framework of para-Kählerian manifold and the geometric results on a paracomplex mechanical systems were found by Tekkoyn [6]. Electronic origins, molecular dynamics simulations, computational nanomechanics, multiscale modelling of materials fields were contributed by Liu [7]. Bi-paracomplex analogue of Lagrangian systems was shown on Lagrangian distributions by Tekkoyn and Sari [8]. Tekkoyn and Yayli presented generalized-quaternionic Kählerian analogue of Lagrangian and Hamiltonian mechanical systems. Eventually, the geometric-physical results related to generalized-quaternionic Kählerian mechanical systems are provided [9].

Nowadays, there are many studies about Euler-Lagrangian dynamics, mechanics, formalisms, systems and equations [2, 4, 10, 11, 12] and there in. There are real, complex, paracomplex and other analogues. As known it is possible to produce different analogous in different spaces. Quaternions were invented by Sir William Rowan Hamiltonian as an extension to the complex numbers. Hamiltonian’s defining relation is most succinctly written as:

\[ i^2 = j^2 = k^2 = -1, \quad ijk = -1. \]

Split quaternions are given by

\[ i^2 = -1, \quad j^2 = 1 = k^2, \quad ijk = 1. \]

Generalized quaternions are defined as

\[ i^2 = -a, \quad j^2 = -b, \quad k^2 = -ab, \quad ijk = -ab. \]

If it is compared to the calculus of vectors, quaternions have slipped into the realm of obscurity. They do however still find use in the computation of rotations. Lots of physical laws in classical, relativistic, and quantum mechanics can be written pleasantly by means of quaternions. Some physicists hope they will find deeper understanding of the universe by restating basic principles in terms of quaternion algebra [13, 14, 15, 16, 17, 18, 19].

In the present paper, we present equations related to Lagrangian mechanical systems on a generalized-quaternionic Kähler manifold.
2 Preliminaries

In this study, all the manifolds and geometric objects are $C^\infty$ and the Einstein summation convention is in use. Also, $A$, $F(TM)$, $\chi(TM)$ and $\Lambda^1(TM)$ denote the set of paracomplex numbers, the set of (para)-complex functions on $TM$, the set of (para)-complex vector fields on $TM$ and the set of (para)-complex 1-forms on $TM$, respectively. The definitions and geometric structures on the differential manifold $M$ given in [20] may be extended to $TM$ as follows:

3 Conformal Geometry

In mathematics, a conformal map is a function which preserves angles. In the most common case the function is between domains in the complex plane. Conformal maps can be defined between domains in higher dimensional Euclidean spaces, and more generally on a Riemann or semi-Riemann manifold. Conformal geometry is the study of the set of angle-preserving (conformal) transformations on a space. In two real dimensions, conformal geometry is precisely the geometry of Riemann surfaces. In more than two dimensions, conformal geometry may refer either to the study of conformal transformations of "flat" spaces (such as Euclidean spaces or spheres), or, more commonly, to the study of conformal manifolds which are Riemann or pseudo-Riemann manifolds with a class of metrics defined up to scale. A conformal manifold is a differentiable manifold equipped with an equivalence class of (pseudo) Riemann metric tensors, in which two metrics $g'$ and $g$ are equivalent if and only if

$$g' = \lambda^2 g$$

where $\lambda > 0$ is a smooth positive function. An equivalence class of such metrics is known as a conformal metric or conformal class [21].

4 Conformal Structure

The linear distance between two points can be found easily by Riemann metric, which is very useful and is defined inner product. Many scientists have used the Riemann metric. Einstein was one of the first studies in this field. Einstein discovered which the Riemannian geometry and successfully used it to describe General Relativity
in the 1910 that is actually a classical theory for gravitation. However, the universe is really completely not like Riemannian geometry. Each path between two points is not always linear. Also, orbits of move objects may change during movement. So, each two points in space may not be linear geodesic and need not to be. Therefore, new metric is needed for non-linear distances like spherical surface. Then, a method is required for converting nonlinear distance to linear distance. Weyl introduced a metric with a conformal transformation in 1918.

**Definition 1.** Let $M$ an $n$-dimensional smooth manifold. A *conformal structure* on $M$ is an equivalence class $G$ of Riemann metrics on $M$. A manifold with a conformal structure is called a *conformal manifold*.

(i) Two Riemann metrics $g$ and $g'$ on $M$ are said to be equivalent if and only if

$$g' = e^\lambda g$$

where $\lambda$ is a smooth function on $M$. The equation given by (4) is called a *conformal structure*

(ii) A *Weyl structure* on $M$ is a map $F : G \to \wedge^1 M$ satisfying

$$F(e^\lambda g) = F(g) - d\lambda$$

where $G$ is a conformal structure. Note that a Riemann metric $g$ and a one-form $\varphi$ determine a Weyl structure, namely $F : G \to \wedge^1 M$ where $G$ is the equivalence class of $g$ and $F(e^\lambda g) = \varphi - d\lambda$.

**Theorem 1.** A connection on the metric bundle $\varphi$ of a conformal manifold $M$ naturally induces a map $F : G \to \wedge^1 M$ and (5), and conversely. Parallel translation of points in $\varphi$ by the connection is the same as their translation by $F$.

5 Generalized-Quaternionic Kähler Manifolds

A generalized almost quaternion structure on the manifold $M$ is a subbundle of the bundle of endomorphisms of the tangent bundle of $M$, whose standard fibre is the algebra of generalized quaternions. A generalized almost quaternion structure on a pseudo-Riemannian manifold is called a generalized quaternion-Hermitian if the following conditions hold:

i) The endomorphisms $F, G$ and $H$ of $T_x M$ satisfy

$$F^2 = -aI, \ G^2 = -bI, \ H^2 = -abI, \ FG = H, GH = bF, HF = aG,$$

(6)
ii) The compatibility equations are given by, for $X, Y \in T_x M$,

$$g(FX, FY) = a g(X, Y), \quad g(GX, GY) = bg(X, Y), \quad g(HX, HY) = abg(X, Y),$$  \hspace{1cm} (7)

where $I$ denotes the identity tensor of type $(1,1)$ in $M$. In particular, 2-form $Q$ defined by $Q(X, Y) = (X, FY) = (X, GY) = (X, HY)$ on $M$ is called the Kähler form of the endomorphisms $F, G$ and $H$. If the Kähler form $Q$ on $M$ is closed, i.e. $dQ = 0$, the manifold $M$ is called a generalized-quaternionic Kähler manifold \cite{23}. If $a = b = 1$, $M$ is quaternion manifold. If $a = 1, b = -1$, $M$ is para-quaternion manifold. The bundle $V$ is a set that locally admits a basis $\{F, G, H\}$ satisfying \cite{11} and \cite{7} in any coordinate neighborhood $U \subset M$ such that $M = \bigcup U \ [14]$. Then $V$ is called a generalized-quaternionic structure in $M$. The pair $(M, V)$ denotes a generalized-quaternionic manifold with $V$. The structure $V$ with such a Riemannian metric $g$ is called a generalized-quaternionic metric structure. The triple $(M, g, V)$ denotes a generalized-quaternionic metric manifold. Let $\{x_i, x_{n+i}, x_{2n+i}, x_{3n+i}\}$, $i = \overline{1, n}$ be a real coordinate system on a neighborhood $U$ of $M$, and let $\{\partial / \partial x_i, \partial / \partial x_{n+i}, \partial / \partial x_{2n+i}, \partial / \partial x_{3n+i}\}$ and $\{dx_i, dx_{n+i}, dx_{2n+i}, dx_{3n+i}\}$ be natural bases over $R$ of the tangent space $T(M)$ and the cotangent space $T^*(M)$ of $M$, respectively. Taking into consideration \cite{11}, then we can obtain the expressions as follows:

$$F(\partial / \partial x_i) = a \partial / \partial x_{n+i}, \quad F(\partial / \partial x_{n+i}) = -a \partial / \partial x_i, \quad F(\partial / \partial x_{2n+i}) = a \partial / \partial x_{3n+i}, \quad F(\partial / \partial x_{3n+i}) = -a \partial / \partial x_{2n+i},$$

$$G(\partial / \partial x_i) = -b \partial / \partial x_{n+i}, \quad G(\partial / \partial x_{n+i}) = b \partial / \partial x_i, \quad G(\partial / \partial x_{2n+i}) = -b \partial / \partial x_{3n+i}, \quad G(\partial / \partial x_{3n+i}) = b \partial / \partial x_{2n+i},$$

$$H(\partial / \partial x_i) = -ab \partial / \partial x_{n+i}, \quad H(\partial / \partial x_{n+i}) = -ab \partial / \partial x_i, \quad H(\partial / \partial x_{2n+i}) = -ab \partial / \partial x_{3n+i}, \quad H(\partial / \partial x_{3n+i}) = -ab \partial / \partial x_{2n+i}.$$  \hspace{1cm} (8)

6 Generalized-Quaternionic Conformal Kähler Manifolds

**Definition 2.** Let $(M, g, \nabla, J_\pm)$ be an almost para/pseudo-Hermitian Weyl manifold. If $\nabla(J_\pm) = 0$, then one says that this is a (para)-Kähler Weyl manifold. Note that necessarily $J_\pm$ is integrable in this setting.

**Theorem 2.** If $(M, g, \nabla, J_\pm)$ is a (para)-Kähler Weyl manifold with dimension $n \geq 6$ and with $H^1(M; R) = 0$, then the underlying Weyl structure on $M$ is trivial.

**Theorem 3.** If $(M, g, \nabla, J_\pm)$ is a curvature (para)-Kähler Weyl manifold with dimension $n \geq 6$ and with $H^1(M; R) = 0$, then the underlying Weyl structure on $M$ is trivial.

**Theorem 4.** Let $n \geq 6$. If $(M, g, J_\pm, \nabla)$ is a Kähler–Weyl structure, then the associated Weyl structure is trivial, i.e. there is a conformally equivalent metric $\tilde{g} = e^{2f}g$ so that $(M, \tilde{g}, J_\pm)$ is Kähler and so that $\nabla = \nabla^{\tilde{g}}$. \cite{21} \cite{25} \cite{29}. 

After this part $W$ will be used instead of $J$. A manifold with a Weyl structure is known as a Weyl manifold. The second structure was chosen the minus sign. Because the condition of the structure required to provide.

$$W_{±}^2 = ±Id$$

If we rewrite (8) equation with conformal structure, we obtain the following equations:

$$W_F(\frac{∂}{∂x_i}) = ae^λ \frac{∂}{∂x_{n+i}}, W_F(\frac{∂}{∂x_{n+i}}) = -ae^{-λ} \frac{∂}{∂x_i}, W_F(\frac{∂}{∂x_{2n+i}}) = ae^λ \frac{∂}{∂x_{3n+i}},$$

$$W_F(\frac{∂}{∂x_{2n+i}}) = -ae^{-λ} \frac{∂}{∂x_i}, W_F(\frac{∂}{∂x_{3n+i}}) = be^λ \frac{∂}{∂x_{n+i}}, W_F(\frac{∂}{∂x_{n+i}}) = be^{-λ} \frac{∂}{∂x_{2n+i}},$$

$$W_H(\frac{∂}{∂x_{2n+i}}) = -abe^{-λ} \frac{∂}{∂x_{2n+i}}, W_H(\frac{∂}{∂x_{3n+i}}) = -abe^{-λ} \frac{∂}{∂x_{n+i}}, W_H(\frac{∂}{∂x_{n+i}}) = -abe^{-λ} \frac{∂}{∂x_{2n+i}}.$$  

(9)

We continue our studies thinking of the $(M, g, ∇, W_{±})$ instead of the almost para/pseudo-Kähler Weyl manifolds $(M, g, ∇, J_{±})$.

### 7 Conformal Euler-Lagragian Mechanical Systems

Here, we obtain Euler-Lagrange equations for quantum and classical mechanics by means of a canonical local basis $\{F, G, H\}$ of $V$ that they defined on a generalized-quaternionic Kähler manifold $(M, g, V)$. Firstly, let $F$ take a local basis element on the generalized-quaternionic Kähler manifold $(M, g, V)$, and $\{x_i, x_{n+i}, x_{2n+i}, x_{3n+i}\}$ be its coordinate functions. Let semispray be the vector field $X$ determined by

$$X = X^i \frac{∂}{∂x_i} + X^{n+i} \frac{∂}{∂x_{n+i}} + X^{2n+i} \frac{∂}{∂x_{2n+i}} + X^{3n+i} \frac{∂}{∂x_{3n+i}},$$

(10)

where $X^i = \dot{x}_i, X^{n+i} = \dot{x}_{n+i}, X^{2n+i} = \dot{x}_{2n+i}, X^{3n+i} = \dot{x}_{3n+i}$ and the dot indicates the derivative with respect to time $t$. The vector field defined by

$$V_F(L) = F(X) = aX^i e^λ \frac{∂L}{∂x_n+i} - aX^{n+i} e^{-λ} \frac{∂L}{∂x_i} + aX^{2n+i} e^λ \frac{∂L}{∂x_{3n+i}} - aX^{3n+i} e^{-λ} \frac{∂L}{∂x_{2n+i}}$$

is named a conformal *Liouville vector field* on the generalized-quaternionic Kähler manifold $(M, g, V)$. For $F$, the closed generalized-quaternionic Kähler form is the closed 2-form given by $Φ^F_L = -dd_xL$ such that

$$d_xL = ae^λ \frac{∂L}{∂x_{n+i}} dx_i - ae^{-λ} \frac{∂L}{∂x_i} dx_{n+i} + ae^λ \frac{∂L}{∂x_{3n+i}} dx_{2n+i} - ae^{-λ} \frac{∂L}{∂x_{2n+i}} dx_{3n+i} : F(M) → \wedge^3 M.$$  

(11)
Then we have

\[ \Phi_L^F = ae^\lambda \frac{\partial L}{\partial x_j} \frac{\partial L}{\partial x_{n+i}} \frac{\partial L}{\partial x_{n+j}} \frac{\partial L}{\partial x_{n+i+j}} \frac{\partial L}{\partial x_{n+i}} \frac{\partial L}{\partial x_{n+j}} \frac{\partial L}{\partial x_{n+i+j}} \frac{\partial L}{\partial x_{n+i}} \frac{\partial L}{\partial x_{n+j}} \]

Then we calculate

\[ i_X \Phi_L^F = aX^2 e^{\lambda} \frac{\partial L}{\partial x_{n+i}} \frac{\partial L}{\partial x_{n+j}} \frac{\partial L}{\partial x_{n+i+j}} \frac{\partial L}{\partial x_{n+i}} \frac{\partial L}{\partial x_{n+j}} \frac{\partial L}{\partial x_{n+i+j}} \frac{\partial L}{\partial x_{n+i}} \frac{\partial L}{\partial x_{n+j}} \]

(12)
\[-aX^{2n+i}e^\lambda \frac{\partial L}{\partial x_{n+i}} \frac{\partial L}{\partial x_{n+i}} dx_i - aX^{2n+i}e^\lambda \frac{\partial L}{\partial x_{n+i}} \frac{\partial^2 L}{\partial x_{n+i}^2} dx_i - aX^{2n+i}e^\lambda \frac{\partial L}{\partial x_{n+i}} \frac{\partial L}{\partial x_{n+i}} dx_{n+i}\]

\[+ aX^{2n+i}e^{-\lambda} \frac{\partial^2 L}{\partial x_{n+i}^2} dx_{n+i} + aX^{2n+i}e^{-\lambda} \frac{\partial L}{\partial x_{n+i}} \frac{\partial L}{\partial x_{n+i}} dx_{n+i} - aX^{2n+i}e^{-\lambda} \frac{\partial L}{\partial x_{n+i}} \frac{\partial L}{\partial x_{n+i}} dx_{2n+i}\]

\[+ aX^{2n+i}e^{-\lambda} \frac{\partial^2 L}{\partial x_{n+i}^2} dx_{2n+i} - aX^{2n+i}e^{-\lambda} \frac{\partial L}{\partial x_{n+i}} \frac{\partial L}{\partial x_{n+i}} dx_{3n+i}\]

\[+ aX^{2n+i}e^{-\lambda} \frac{\partial^2 L}{\partial x_{n+i}^2} dx_{3n+i} + aX^{2n+i}e^{-\lambda} \frac{\partial L}{\partial x_{n+i}} \frac{\partial L}{\partial x_{n+i}} dx_{3n+i} + aX^{2n+i}e^{-\lambda} \frac{\partial^2 L}{\partial x_{n+i}^2} dx_{3n+i}\]

\[+ aX^{2n+i}e^{-\lambda} \frac{\partial L}{\partial x_{n+i}} \frac{\partial L}{\partial x_{n+i}} dx_{3n+i} + aX^{2n+i}e^{-\lambda} \frac{\partial L}{\partial x_{n+i}} \frac{\partial L}{\partial x_{n+i}} dx_{3n+i} + aX^{2n+i}e^{-\lambda} \frac{\partial^2 L}{\partial x_{n+i}^2} dx_{3n+i}\]

\[+ aX^{3n+i}e^{-\lambda} \frac{\partial^2 L}{\partial x_{n+i}^2} dx_{3n+i} + aX^{3n+i}e^{-\lambda} \frac{\partial L}{\partial x_{n+i}} \frac{\partial L}{\partial x_{n+i}} dx_{3n+i}\]

\[+ aX^{3n+i}e^{-\lambda} \frac{\partial L}{\partial x_{n+i}} \frac{\partial L}{\partial x_{n+i}} dx_{3n+i} - aX^{3n+i}e^{-\lambda} \frac{\partial L}{\partial x_{n+i}} \frac{\partial L}{\partial x_{n+i}} dx_{n+i}\]  

Energy function is

\[E^F = V^F(L) - L = aX^ie^\lambda \frac{\partial L}{\partial x_{n+i}} - aX^{n+i}e^{-\lambda} \frac{\partial L}{\partial x_{n+i}} - aX^{2n+i}e^\lambda \frac{\partial L}{\partial x_{3n+i}} - aX^{3n+i}e^{-\lambda} \frac{\partial L}{\partial x_{2n+i}} - L\]  

and hence

\[dE^F = aX^i e^\lambda \frac{\partial L}{\partial x_{n+i}} dx_j + aX^i e^{-\lambda} \frac{\partial^2 L}{\partial x_{n+i}^2} dx_j - aX^{n+i}e^{-\lambda} \frac{\partial^2 L}{\partial x_{n+i}^2} dx_j + aX^{n+i}e^{-\lambda} \frac{\partial L}{\partial x_{n+i}} \frac{\partial L}{\partial x_{n+i}} dx_j\]

\[+ aX^{n+i}e^{-\lambda} \frac{\partial^2 L}{\partial x_{n+i}^2} dx_j - aX^{n+i}e^{-\lambda} \frac{\partial L}{\partial x_{n+i}} \frac{\partial L}{\partial x_{n+i}} dx_j + aX^{2n+i}e^\lambda \frac{\partial L}{\partial x_{3n+i}} \frac{\partial L}{\partial x_{n+i}} dx_j\]

\[+ aX^{3n+i}e^{-\lambda} \frac{\partial^2 L}{\partial x_{n+i}^2} dx_j - aX^{3n+i}e^{-\lambda} \frac{\partial L}{\partial x_{n+i}} \frac{\partial L}{\partial x_{n+i}} dx_j + aX^{3n+i}e^{-\lambda} \frac{\partial^2 L}{\partial x_{n+i}^2} dx_{n+i}\]

\[+ aX^{3n+i}e^{-\lambda} \frac{\partial L}{\partial x_{n+i}} \frac{\partial L}{\partial x_{n+i}} dx_{n+i} + aX^{3n+i}e^{-\lambda} \frac{\partial L}{\partial x_{n+i}} \frac{\partial^2 L}{\partial x_{n+i}^2} dx_{n+i}\]

\[+ aX^{3n+i}e^{-\lambda} \frac{\partial L}{\partial x_{n+i}} \frac{\partial L}{\partial x_{n+i}} dx_{n+i} - aX^{3n+i}e^{-\lambda} \frac{\partial L}{\partial x_{n+i}} \frac{\partial L}{\partial x_{n+i}} dx_{n+i}\]

\[+ aX^{3n+i}e^{-\lambda} \frac{\partial L}{\partial x_{n+i}} \frac{\partial L}{\partial x_{n+i}} dx_{n+i} - aX^{3n+i}e^{-\lambda} \frac{\partial L}{\partial x_{n+i}} \frac{\partial L}{\partial x_{n+i}} dx_{2n+i}\]

\[+ aX^{3n+i}e^{-\lambda} \frac{\partial L}{\partial x_{n+i}} \frac{\partial L}{\partial x_{n+i}} dx_{2n+i} - aX^{3n+i}e^{-\lambda} \frac{\partial L}{\partial x_{n+i}} \frac{\partial L}{\partial x_{n+i}} dx_{2n+i}\]

\[+ aX^{3n+i}e^{-\lambda} \frac{\partial L}{\partial x_{n+i}} \frac{\partial L}{\partial x_{n+i}} dx_{2n+i} - aX^{3n+i}e^{-\lambda} \frac{\partial L}{\partial x_{n+i}} \frac{\partial L}{\partial x_{n+i}} dx_{2n+i}\]  

\[+ aX^{3n+i}e^{-\lambda} \frac{\partial L}{\partial x_{n+i}} \frac{\partial L}{\partial x_{n+i}} dx_{2n+i} + aX^{3n+i}e^{-\lambda} \frac{\partial L}{\partial x_{n+i}} \frac{\partial L}{\partial x_{n+i}} dx_{2n+i}\]  

\[+ aX^{3n+i}e^{-\lambda} \frac{\partial L}{\partial x_{n+i}} \frac{\partial L}{\partial x_{n+i}} dx_{2n+i} + aX^{3n+i}e^{-\lambda} \frac{\partial L}{\partial x_{n+i}} \frac{\partial L}{\partial x_{n+i}} dx_{2n+i}\]  

\[+ aX^{3n+i}e^{-\lambda} \frac{\partial L}{\partial x_{n+i}} \frac{\partial L}{\partial x_{n+i}} dx_{2n+i} + aX^{3n+i}e^{-\lambda} \frac{\partial L}{\partial x_{n+i}} \frac{\partial L}{\partial x_{n+i}} dx_{2n+i}\]  

\[+ aX^{3n+i}e^{-\lambda} \frac{\partial L}{\partial x_{n+i}} \frac{\partial L}{\partial x_{n+i}} dx_{2n+i} + aX^{3n+i}e^{-\lambda} \frac{\partial L}{\partial x_{n+i}} \frac{\partial L}{\partial x_{n+i}} dx_{2n+i}\]
Using (1) and also considering an integral curve of $X$, then we obtain the equation given by

$$\begin{align*}
-aX^i e^\lambda \frac{\partial}{\partial x_{n+i}} dx_i - aX^i e^\lambda \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - aX^{n+i} e^\lambda \frac{\partial}{\partial x_{n+j}} dx_j \\
-aX^{n+i} e^\lambda \frac{\partial}{\partial x_{n+j}} dx_{n+i} - aX^{2n+i} e^\lambda \frac{\partial}{\partial x_{n+j}} dx_{n+i} - aX^{3n+i} e^\lambda \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{n+i} + \frac{\partial L}{\partial x_j} dx_j \\
-aX^i e^{-\lambda} \frac{\partial}{\partial x_j} dx_{n+i} + aX^i e^{-\lambda} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{n+i} - aX^{n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{n+i} + aX^{2n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{n+i} \\
+aX^{n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{n+i} - aX^{2n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{n+i} + aX^{3n+i} e^{-\lambda} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{n+i} + \frac{\partial L}{\partial x_j} dx_{n+i}
\end{align*}$$

Then we have the equations

$$\begin{align*}
\frac{\partial}{\partial t} \left( e^{-\lambda} \frac{\partial L}{\partial x_i} \right) + \frac{\partial L}{\partial x_{n+i}} &= 0, \quad \frac{\partial}{\partial t} \left( e^\lambda \frac{\partial L}{\partial x_{n+i}} \right) - \frac{\partial L}{\partial x_i} = 0, \\
\frac{\partial}{\partial t} \left( e^{-\lambda} \frac{\partial L}{\partial x_{n+i}} \right) + \frac{\partial L}{\partial x_{n+i}} &= 0, \quad \frac{\partial}{\partial t} \left( e^\lambda \frac{\partial L}{\partial x_{n+i}} \right) - \frac{\partial L}{\partial x_{n+i}} = 0,
\end{align*}$$

such that the equations calculated in (17) are named Euler-Lagrange equations constructed on a generalized-quaternionic Kähler manifold $(M, g, V)$ by means of $\Phi^F_L$ and thus the triple $(M, \Phi^F_L, X)$ is called a mechanical system on a generalized-quaternionic Kähler manifold $(M, g, V)$. Considering the above operations and also taking the following vector fields

$$\begin{align*}
Y &= Y^i \frac{\partial}{\partial x_i} + Y^{n+i} \frac{\partial}{\partial x_{n+i}} + Y^{2n+i} \frac{\partial}{\partial x_{2n+i}} + Y^{3n+i} \frac{\partial}{\partial x_{3n+i}}, \\
Z &= Z^i \frac{\partial}{\partial x_i} + Z^{n+i} \frac{\partial}{\partial x_{n+i}} + Z^{2n+i} \frac{\partial}{\partial x_{2n+i}} + Z^{3n+i} \frac{\partial}{\partial x_{3n+i}},
\end{align*}$$

we obtain the following Euler-Lagrange equations for quantum and classical mechanics by means of $\Phi^G_L$ and $\Phi^H_L$ on a generalized-quaternionic Kähler manifold $(M, g, V)$, respectively:

$$\begin{align*}
b \frac{\partial}{\partial t} \left( e^{-\lambda} \frac{\partial L}{\partial x_i} \right) + \frac{\partial L}{\partial x_{2n+i}} &= 0, \quad b \frac{\partial}{\partial t} \left( e^\lambda \frac{\partial L}{\partial x_{2n+i}} \right) - \frac{\partial L}{\partial x_i} = 0, \\
b \frac{\partial}{\partial t} \left( e^{-\lambda} \frac{\partial L}{\partial x_{2n+i}} \right) + \frac{\partial L}{\partial x_i} &= 0, \quad b \frac{\partial}{\partial t} \left( e^\lambda \frac{\partial L}{\partial x_{2n+i}} \right) - \frac{\partial L}{\partial x_{2n+i}} = 0.
\end{align*}$$
\[
\begin{align*}
abla \frac{\partial}{\partial t} \left( e^{-\lambda} \frac{\partial L}{\partial x_i} \right) + \frac{\partial L}{\partial x_{3n+i}} &= 0, \quad \nabla \frac{\partial}{\partial t} \left( e^{-\lambda} \frac{\partial L}{\partial x_{3n+i}} \right) + \frac{\partial L}{\partial x_{2n+i}} &= 0, \\
abla \frac{\partial}{\partial t} \left( e^{\lambda} \frac{\partial L}{\partial x_{3n+i}} \right) + \frac{\partial L}{\partial x_{2n+i}} &= 0, \quad \nabla \frac{\partial}{\partial t} \left( e^{\lambda} \frac{\partial L}{\partial x_{3n+i}} \right) + \frac{\partial L}{\partial x_{i}} &= 0.
\end{align*}
\] (21)

Thus the equations introduced by (20) and (21) infer Conformal Euler-Lagrange equations constructed by means of \( \Phi^G_L \) and \( \Phi^H_L \) on a generalized-quaternionic Kähler manifold \((M, g, V)\) and then the triples \((M, \Phi^G_L, X)\) and \((M, \Phi^H_L, X)\) are named mechanical systems on a generalized-quaternionic Kähler manifold \((M, g, V)\). Hence the equations found by (17, 20, 21) easily seen extremely useful in applications from Euler-Lagrange Mechanics, Quantum Physics, Optimal Control, Biology and Fluid Dynamics [27, 28].

8 Conclusion

Given the above equations, Euler-Lagrangeian mechanical systems have intrinsically been defined taking into account a canonical local basis \( \{F, G, H\} \) of \( V \) that they are defined on a generalized-quaternionic Kähler manifold \((M, g, V)\). The paths of semisprays \( X, G, H \) on the generalized-quaternionic Kähler manifold are the solutions Euler-Lagrange equations raised in (17), (20) and (21). These equations are introduced by a canonical local basis \( \{F, G, H\} \) of vector bundle \( V \) on a generalized-quaternionic Kähler manifold \((M, g, V)\). If this equations \( a = 1 \) and \( b = 1 \) are selected, the equations given in (17), (20) and (21) we say to be Euler-Lagrange equations on a quaternionic Kähler manifold. If \( a = 1 \) and \( b = -1 \), the equations given in (17), (20) and (21) we say to be Euler-Lagrange equations on a para-quaternionic Kähler manifold. In these days, Lagrangian models arise to be a very important tool since they present a simple method to describe the model for mechanical systems. One can be proved that the obtained equations are very important to explain the rotational spatial mechanical-physical problems. For this reason, the found equations are only considered to be a first step to realize how a generalized-quaternionic geometry has been used in solving problems in different physical area.

In the literature, the equations, which explains the linear orbits of the objects, were presented. This study explained the non-linear orbits of the objects in the space by the help of revised equations using conformal structure.

Our proposal for future research, the Lagrange mechanical equations derived on a generalized-quaternionic Kähler manifold are suggested to deal with problems in electrical, magnetical and gravitational fields of quantum and classical mechanics of physics.
References

[1] J. Klein, Escapes variationnels et Méc onique, Ann. Inst. Fourier, Grenoble, 12 , pp. 1-124,(1962).

[2] M. De Leon, P.R. Rodrigues, Methods of Differential Geometry in Analytical Mechanics, North-Holland Mathematics Studies, Vol.152, (Elsevier, Amsterdam, 1989).

[3] R. Abraham, J. E. Marsden, T. Ratiu, Manifolds, tensor analysis and applications, Springer, pp: 483-542, (2001).

[4] M. De Leon, P.R. Rodrigues, Second-Order Differential Equations and Non-Conservative Lagrangian Mechanics, J. Phys. A: Math. Gen. 20 (1987), 5393-5396.

[5] H. de Vries, Understanding relativistic quantum field theory, The Hamiltonian and Lagrangian densities, Chapter 22, (http://www.physics-quest.org/Book_Chapter_Lagrangian.pdf), (2009).

[6] M. Tekkoyun, On para-Euler Lagrange and para-Hamiltonian equations, Physics Letters A, Vol. 340, 7-11, (2005).

[7] W.K. Liu, S. Jun, Computational nanomechanics of materials, American Scientific Publishers, Stevenson Ranch, CA, (2005).

[8] M. Tekkoyun, M. Sari., Bi-para-mechanical systems on the bi-Lagrangian manifold, Physica B-Condensed Matter, Vol. 405, Issue 10, 2390-2393, (2010).

[9] M. Tekkoyun, Y. Yayli, Mechanical Systems on Generalized-Quaternionic Kähler Manifolds, International Journal of Geometric Methods in Modern Physics, Vol. 8, No. 7, 1-13, (2011).

[10] M. Zambine, Hamiltonian Perspective on Generalized Complex Structure, Communications in Mathematical Physics. 263 (2006), 711-722.

[11] M. Tekkoyun, On Para-Euler-Lagrange and Para-Hamiltonian Equations, Phys. Lett. A, 340 (2005), 7-11.

[12] M. Tekkoyun, Mechanical Systems on Para-Quaternionic Kähler Manifolds, XVI. Geometrical Seminar, 20-25 September, Vršac, Serbia, 2009.
[13] D. Stahlke, Quaternions in Classical Mechanics, Phys. 621. http://www.stahlke.org/dan/phys-papers/quaternion-paper.

[14] A. S. Dancer - H. R. Jørgensen - A. F. Swann, Metric Geometries over the Split Quaternions, arXiv: 0412.215.

[15] L. Kula L. and Y. Yayli, Split Quaternions and Rotations in Semi-Euclidean Space $E^4_2$, J. Korean Math. Soc. 44 (2007), 1313-1327.

[16] L. Kula and Y. Yayli, Dual Split Quaternions and Screw Motion in Minkowski 3-Space, Iranian journal of Sci. & Tech., Transaction A. 30 (2006).

[17] E. Ata and Y. Yayli, Split Quaternions and Semi-Euclidean Projective Spaces, Chaos, Solitons & Fractals. 41 (2009), 1910-1915.

[18] E. Ata and Y. Yayli, A Global Condition for the Triviality of an Almost Split Quaternionic Structure on Split Complex Manifolds, International Journal of Computational and Mathematical Sciences 2:1 (2008).

[19] M. Jafari, Y. Yayli, Generalized Quaternions and their Algebraic Properties, Submitted for publication.

[20] V. Cruceanu, P. A. Gadea, J. M. Masqué, Para-Hermitian and para-Kähler manifolds, http://digital.csic.es/bitstream/10261/15773/1/RockyCFG.PDF).

[21] http://en.wikipedia.org/wiki/Conformal_class.

[22] G. B. Folland, Weyl manifolds, J. Differential Geometry, 4, 145-153, (1970).

[23] O E Arsen’eva, On the Geometry of Quaternion-Hermitian Manifolds, Russian Mathematical Surveys, 48(5):159 (1993).

[24] P. Gilkey, S. Nikčević, Kähler and para-Kähler curvature Weyl manifolds, arXiv:1011.4844v1, (2010).

[25] P. Gilkey, S. Nikcevic, Kähler–Weyl manifolds dimensional 4, http://arxiv.org/abs/1109.4532v1 (2011).

[26] H. Pedersen, Y.S. Poon, A. Swann, The Einstein-Weyl equations in complex and quaternionic geometry. Differential Geom. Appl. 3, no.4, 309321, (1993).

[27] R. Miron, D. Hrimiuc, H. Shimada, S. V. Sabau, The geometry of Hamilton and Lagrange spaces, eBook ISBN: 0-306-47135-3, Kluwer Academic Publishers, New York, (2002).
[28] H. Weyl, Space-Time-Matter, Dover Publ. 1922. Translated from the 4th German edition by H. Brose.

London: Methuen. Reprint New York: Dover (1952).