The graph isomorphism problem is polynomial

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Abstract

It is known that a graph isomorphism testing algorithm is polynomially equivalent to
a detecting of a graph non-trivial automorphism algorithm. The polynomiality of the lat-
er algorithm, is obtained by consideration of symmetry properties of regular $k$-partitions
that, on one hand, generalize automorphic $k$-partitions (=systems of $k$-orbits of permutation
groups), and, on other hand, schemes of relations (strongly regular 2-partitions or regular
3-partitions), that are a subject of the algebraic combinatorics.

It is shown that the stabilization of a graph by quadrangles detects the triviality of
the graph automorphism group. The result is obtained by linearization of the algebraic
combinatorics.

Keywords: $k$-partitions, symmetry, algebraic combinatorics

1 Introduction

It is known that the graph isomorphism problem is equivalent by complexity to the problem of
exposure of orbits of a graph automorphism group, and these two problems are equivalent to
the problem of detecting of graph non-trivial automorphism. This equivalence was considered
by R. Marthon [5] (s. also [4]) and independently by author [2]. In given text we show that the
latter problem is polynomial.

Attempts to find the complexity of graph isomorphism problem were gone in two directions:
group theory and computational theory, and till today no way brought a result. The literature
to the first way one can find in [3] and to the second in [4]

We will go the first way and, more exactly, study symmetry properties of combinatorial ob-
jects that follows from symmetry properties of $k$-orbits. Earlier this direction led to a sequence
of graph stabilization algorithms (Weisfeiler-Lehman algorithm and its generalizations), to no-
tions of strongly regular graph and distance regular graph, and also to developing of algebraic
combinatorics (that has different origins [1]).

What was of principal in studying of graph isomorphism it was a simplification of a model.
If to consider a graph as a partition $L_2$ of a Cartesian square $V^2$ of a finite set $V$ (i.e. a color
digraph with colored vertices), then it is evident that the less is $|L_2|$ (the coarse is $L_2$) the graph
is simpler; so all specialists were concentrated on the consideration of simple graphs, where
$|L_2| = 3$, classes are symmetrical, i.e. $\langle v_1, v_2 \rangle L_2 \langle v_2, v_1 \rangle$ for any $v_1, v_2 \in V$, and one class is a
diagonal ($\{\langle v_i, v_j \rangle\}$, non-colored vertices). But in this case classes of actual examples are very
large and not observable. About 1986 author discovered for him that there exists another way of
the problem simplification, it is the case where $|L_2|$ is as large as possible and therefore classes
of $L_2$ are small. This way led to construction that gives a very simple local representation of
diculties of the problem and shows the way of combinatorial problem solution.

The main achievement in combinatorics for last 30 years is a developing of algebraic combina-
torics that studies associative schemes of relations: partitions of $V^2$ possessing certain symmetry
properties. By studying of schemes of relations it were obtained some important examples of
strongly regular graphs satisfying to (so-called) k-condition for k > 3. But nevertheless it
was not developed a conceptional theory of symmetries of k-partitions that could open a new
view on graph isomorphism. From here appears an other idea (1983): to consider the symmetry
properties of k-partitions for any k ≥ 2.

And the third algebraic idea (1983) came from consideration of stabilization algorithm for a
3-partition L3. Pure intuitively it is clear that a stabilization of L3 is an equation on n3 (n = |V|)
variables. But if L3 is obtained from an initial partition L2, then we have the same equation on
n2 variables, so it have to exist some overdetermination (in a linear equation system the number
of equations is greater than number of variables). The attempts to find a corresponding system
of algebraic equations were without success many years and it was decided to search for a pure
combinatorial solution, because the applying of this direction was always successful.

So this text was planned as realization of a combinatorial solution. But on the way of
working on the text and thank to the text [3] and the book [1] suddenly it was found an
algebraic approach that was simpler as combinatorial. In that connection in this paper is given

2 k-partitions

Let V be a n-element set, V^k be Cartesian power of V and V^{(k)} be the non-diagonal part of
V^k, i.e. any k-tuple from V^{(k)} consists of k different coordinates. Under k-partition L_k below
we understand a partition of V^{(k)}.

We shall consider a k-partition L_k with a set of coloring functions F = {f : L_k → Cr},
where Cr is a set of “colors” which can have different identity: numbers, vectors, tensors and
other. Under an automorphism group Aut(L_k) we understand the maximal permutation group
G(V) that maintains each class of L_k and so each function f ∈ F.

Let P, Q be partitions of a set M, then P ∩ Q and P ∪ Q denote the union and intersection
of P and Q. If P is a subpartition of Q, then we write P ⊂ Q.

The action of Aut(L_k) on V^{(k)} forms a partition Orb_k(Aut(L_k)) ⊂ L_k that consists of orbits
of this action or (as one say) of k-orbits of Aut(L_k). If L_k = Orb_k(Aut(L_k)), then we say that
L_k is an automorphic partition (or system of k-orbits of a permutation group G = Aut(L_k)).

It is convenient to represent a k-set (k-relation, k-class) U_k ⊂ V^{(k)} as a matrix M(U_k),
whose line is a k-tuple of U_k and i-th column consists of values of i-th coordinate of k-tuples. So
the matrix of a k-set is defined accurate to line order. If L_k is an automorphic partition, then
its class is an automorphic k-set (k-orbit). An automorphic k-partition and its classes possess
evident symmetry properties. Consider those properties.

3 Regular and pq-stable k-partitions

We say that a k-partition L_k is s-symmetrical, if for any class U_k of L_k any k-relation U'^k_k
that differs from U_k by order of coordinates (order of columns in matrix M(U_k)) belongs to L_k.

Let α_k = ⟨v_1, ..., v_k⟩ be a k-tuple, l < k and α_l = ⟨v_1, ..., v_l⟩ be a l-tuple that is a projection
(l-projection) of α_k on a subspace W = {i_j : j ∈ [1, l], i_j ∈ [1, k]}, then, using projecting operator ̂p(W), we write α_l = ̂p(W)α_k. The set of all k (k − 1)-projections of α_k we
write as ̂pα_k. Here the projections are considered in natural order of coordinates determinate
by α_k. The reverse to ̂p operator ̂q assembles k (k − 1)-projections ̂qα_k in k-tuple α_k = ̂q̂pα_k.
From this definition follows the action of projecting and assembling operators on k-relations and
k-partitions.

For a k-relation U_k is U_k ⊂ ̂q̂pU_k. So we call U_k l-full if ̂q^{k-l} ̂p^{k-l}U_k = U_k (here ̂p^{k-l}U_k is a
set of \binom{k}{l} projections of U_k on l-subspaces). A k-partition L_k is l-full, if all its classes are l-full:
\( \hat{q}^{k-l}\hat{p}^{k-l}L_k = L_k \). If \( L_k \) is \( l \)-full, then it is evidently \( (l + 1) \)-full.

We say that a \( k \)-partition \( L_k \) is \( p \)-symmetrical, if \( \hat{p}L_k \) is a \((k - 1)\)-partition, i.e. any two \((k - 1)\)-
projections of any two classes of \( L_k \) are either equal or disjoint. It is clear that \( \hat{p}\hat{q}L_k \sqsubset L_k \). So we say that a \( k \)-partition \( L_k \) is \( pq \)-stable if \( L_k = \hat{p}\hat{q}L_k \).

**Proposition 1** Let a \( k \)-partition \( L_k \) be \( p \)-symmetrical, then a \((k - 1)\)-partition \( L_{k-1} = \hat{p}L_k \) is \( p \)-symmetrical too.

**Proof:** It is sufficient to consider the case \( k = 3 \) (for \( k > 3 \) the proof is similar). Let \( A = Cl((1, 2, 3)) \) be a class of \( L_3 \) containing 3-tuple \( (1, 2, 3) \), \( B = Cl((1, 4, 5)) \in L_3 \) and \( C = Cl((1, 2, 4)) \in L_3 \), so that the intersection of projections of \( A \) and \( B \) on a subspace \( W = (1) \) is not trivial: \( \hat{p}((1))L_k ≠ \hat{p}((1))B \). But the inequality is not valid because of \( \hat{p}((1))A = \hat{p}((1))\hat{p}((1, 2))A = \hat{p}((1))\hat{p}((1, 2))C = \hat{p}((1))\hat{p}((1, 4))C = \hat{p}((1))\hat{p}((1, 4))B = \hat{p}((1))B \).

**Proposition 2** Let a \( k \)-partition \( L_k \) be \( p \)-symmetrical and \( k > 2 \), then it is \( s \)-symmetrical.

**Proof:** It is sufficient to consider the case \( k = 3 \) and assume that \( L_3 \) is a \( 2 \)-full \( 3 \)-partition. Further we use the induction on \( n \). For \( n = 3 \) the statement is easy verified. Let \( n = 4 \), and \( L_3 \) be \( p \)-symmetrical \( 2 \)-full \( 3 \)-partition. Let \( L_1 = \hat{p}^2L_3 ≠ \{V\} \), then this case is reduced to cases \( n < 4 \) and therefore the statement is correct. Let \( L_1 = \{V\} = \{v_1, v_2, v_3, v_4\} \) and let \( L_3' \) be a \( 3 \)-partition obtained from \( L_3 \) by removal from it all 3-tuples containing value of a coordinate \( v_4 \), then \( L_3' \) is also \( p \)-symmetrical \( 2 \)-full \( 3 \)-partition for that statement is correct. From here it follows that the statement is correct for \( L_3 \). The generalization on any \( n \) is evident.

**Proposition 3** Let \( L_k \) be a \( pq \)-stable \((k - 1)\)-full partition, then \( \hat{p}\hat{q}L_k = \hat{q}\hat{p}L_k = L_k \).

Let \( U_k \) be a \( k \)-relation and \( mU_l \) be a multiprojection of \( U_k \) on a \( l \)-dimensional subspace \( W \), that we write as \( mU_l = \hat{\hat{m}}(W)U_k \). It means that a matrix \( M(mU_l) \) is obtained from the matrix \( M(U_k) \) by removal of columns that do not belong to \( W \). We call a \( k \)-relation \( U_k \) \( mp \)-symmetrical, if \( \hat{\hat{m}}(W)U_k \) is homogenous (i.e each line of \( M(mU_l) \) has the same multiplicity) for any possible subspace \( W \). A \( k \)-partition \( L_k \) is \( mp \)-symmetrical, if every its class is \( mp \)-symmetrical.

We have described three necessary properties of an automorphic partition: \( s \)-, \( p \)- and \( mp \)-symmetry, at that \( p \)-symmetry involves \( s \)-symmetry (proposition 2). A \( k \)-partition that possesses these three symmetries we call regular \( k \)-partition. A \( k \)-partition that is a projection of a regular \( k \)-partition we call strongly-regular. One can see that regular and strongly-regular graphs satisfy corresponding conditions. It is clear that strongly-regular partition is \( pq \)-stable. Reverse statement is not correct, a counterexample is a 8-point cubic graph obtained from a cube in which two parallel edges \( \{1, 2\} \) and \( \{3, 4\} \), belonging to one cube face, are changed with edges \( \{1, 3\} \) and \( \{2, 4\} \). This graph is point-transitive, its \( 2 \)-partition \( L_2 \) (on edges and not edges) is assembling in \( 3 \)-partition \( L_3 \), but \( L_3 \) is not \( mp \)-symmetrical. We will prove below the next

**Theorem 4** Let \( k ≥ 3 \) and \( L_k \) be a regular, \( pq \)-stable \( k \)-partition, then \( L_k \) is strongly-regular.

## 4 Partition stabilization algorithm

It is clear that any \( k \)-partition \( L_k \) can be stabilized by \( pq \)-stabilization to a \( pq \)-stable partition \( R_k = (\hat{p}\hat{q})^\nu L_k \), where \( \nu \) is a number of iterations. From theorem 4 it follows that for \( k ≥ 3 \) \( R_k \) is strongly-regular, if \( L_k \) is regular. One can see that the algorithm of the regularization of a \( k \)-partition follows immediately from its definition and is polynomial. Concern of graph isomorphism it is of interest whether exists a number \( k \) for that \( pq \)-stabilization (of a regular \( k \)-partition) leads to an automorphic \( k \)-partition or at least to a strongly regular \( k \)-partition with non-trivial automorphism group. If such number \( k \) exists, then the graph isomorphism problem is polynomial (because of complexity equivalence, considered above). We show below that corresponding \( k \) exists and is equal to 3.
5 Automorphic $k$-partitions

Let $L_k$ be an automorphic $k$-partition, i.e. $L_k = Orb_k(Aut(L_k))$, then we have the next

**Theorem 5** Let $L_{(k+i)} = \hat{q}^i L_k$, $i \in [1, n-k]$, then $L_{(k+i)}$ is automorphic.

**Proof:** Since $L_k$ is automorphic, $\hat{p}\hat{q}^i L_k = L_k$ for any $i \in [1, n-k]$. So $L_{(k+i)}$ is $k$-full and $pq$-stable (or $L_{(k+i)} = \hat{p}\hat{q} L_{(k+i)} = \hat{q}\hat{p} L_{(k+i)}$) for any $i$. It follows that $\hat{p}^{n-k-i} \hat{q}^{n-k-i} L_{(k+i)} = L_{(k+i)}$. $\square$

A permutation group $G(V)$ is called $k$-closed, if $Aut(Orb_k(G)) = G$.

**Corollary 6** Let $G$ be a permutation group, then it is $k$-closed group iff its $n$-orbit is $k$-full.

Let $G$ be a 1-closed group, then it is a cartesian product of symmetric groups acting on a partition of $V$.

6 Algebraic combinatorics of strongly regular $k$-partitions

The purpose of this section is a proof of theorem and a proof of the polynomial complexity of the algorithm, detecting graph non-trivial automorphism (and therefore a proof of polynomiality of the graph isomorphism problem).

As we wrote above the contemporary algebraic combinatorics is a theory of strongly regular $2$-partitions (or one can say strongly regular color digraphs) that have historically many other names. With certain restriction with an additional condition one obtains distance regular graphs. But this theory cannot tell many about possible symmetries on $k$-partitions, so in order to obtain such information one has to consider $k$-partitions for $k > 2$. The main difficulty of such undertaking is that by $k > 2$ one cannot apply especially good developed matrix theory. So in order to find an approach to investigation of $k$-partitions we put a question: what is the most important in representation of $k$-partition? And an answer could be: of course, it is its coloring function. Now we begin a search for an appropriate coloring function.

6.1 Level invariant transformation

Let $A = \{a_1, \ldots, a_d\}$ and $B = \{b_1, \ldots, b_d\}$ be sets of colors, $L_k$ be a $k$-partition with $d$ classes and $f : L_k \rightarrow A, g : L_k \rightarrow B$ be bijections. Let $T$ be a transformation of $A$ to $B$: $B = TA$, so that $T$ is also a transformation of $f$ to $g$: $g = Tf$. Such transformations maintain $L_k$ or level surfaces of $f$. We call $T$ a level invariant transformation for function $f$. Such transformations, applied to a coloring function of $k$ variables $f(x_1, \ldots, x_k)$, give different possibility for algebraic approach to investigation of $k$-partitions. Here it will be of interest for us two level invariant transformations. One of them is a polynomial $T(a) = P_{d-1}(a)$ of degree $d-1$ that is defined by the next system of linear equations:

\[ b_i = x_0 + x_1 a_i + \ldots + x_{d-1} a_i^{d-1}, i \in [1, d]. \]  \hspace{1cm} (1)

And the second is a matrix $T$ that transforms the vector $\vec{a} = (a_1, \ldots, a_d)$ to the vector $\vec{b} = (b_1, \ldots, b_d)$:

\[ \vec{b} = T \vec{a} \]  \hspace{1cm} (2)

We say that two functions $f$ and $g$ are equivalent $f \sim g$, if they have the same level surfaces: $L_k(f) = L_k(g)$. 

4
6.2 Non-linear number coloring function

Let \( L_k \) be a \( pq \)-stable \( k \)-partition and \( \sigma_k \equiv \| \sigma_{1...i_k} \| \) be an associated coloring tensor on \( L_k \). Let \( L_{k+1} = qL_k \) and \( \sigma_{k+1} \equiv \| \sigma_{1...i_{k+1}} \| \) be a tensor associated with \( L_{k+1} \). We can represent the tensor \( \sigma_{k+1} \) through the tensor \( \sigma_k \) as:

\[
\sigma_{i_1...i_{k+1}} = \sigma_{i_1...i_k} \prod_{(j_1...j_{k-1}) \in \hat{p}(i_1...i_k)} \sigma_{j_1...j_{k-1}i_{k+1}},
\]

where \( \hat{p} \) is the projection of \( \sigma_{i_1...i_{k+1}} \) to \( L_k \) and \( \hat{p}(i_1...i_k) \) is the projection of \( i_1...i_k \) to \( L_k \).

Since \( L_k \) is a projection of \( L_{k+1} \), we can represent the tensor on \( L_k \) through the tensor on \( L_{k+1} \) as:

\[
\sigma_{i_1...i_k} = \sum_{\alpha \in C_{k+1}} \sigma_{i_1...i_k*}(\alpha),
\]

where \( C_{k+1} = \{ \sigma_{i_1...i_{k+1}} \} \) is a set of colors on \( L_{k+1} \) and \( \langle i_1...i_k* \rangle(\alpha) \) is a representative \( (k+1) \)-tuple of a class \( \alpha \). If there exists no such representative of a class \( \alpha \), then \( \sigma_{i_1...i_k*}(\alpha) = 0 \). The factor \( \sigma_{i_1...i_k} \) in (3) does not change the structure of the product and can be omitted. Then using described above the level invariant transformation (1) we find an equation on a tensor of \( pq \)-stable \( k \)-partition in form:

\[
\sum_{\alpha \in C_{k+1}} \prod_{(j_1...j_{k-1}) \in \hat{p}(i_1...i_k)} \sigma_{j_1...j_{k-1}*}(\alpha) = P_{d-1}(\sigma_{i_1...i_k}).
\]

For a strongly regular \( k \)-partition \( L_k \) (that is \( mp \)-symmetrical) we can rewrite equality (4) as:

\[
\sigma_{i_1...i_k} = \sum_{\alpha \in C_{k+1}} \sigma_{i_1...i_k*}(\alpha) r_{i_1...i_k*}(\alpha) = \sum_{l=1,n} \sigma_{i_1...i_k,l},
\]

where \( r_{i_1...i_k*}(\alpha) \) is a multiplicity of a \( k \)-tuple \( (i_1...i_k) \) in a multiprojection of the class of color \( \alpha \) of \( L_{k+1} \) by removing the latter column in a matrix of the class \( \alpha \).

Thus for a strongly regular \( k \)-partition \( L_k \) the equation (5) takes a form:

\[
\sum_{l} \prod_{(j_1...j_{k-1}) \in \hat{p}(i_1...i_k)} \sigma_{j_1...j_{k-1}l} = P_{d-1}(\sigma_{i_1...i_k}),
\]

where we assume that elements with equal indices are zero and \( l \neq i_1, \ldots, i_k \) (this condition is implied in corresponding sums below).

6.3 Linear number coloring function

We can represent the tensor \( \sigma_{k+1} \) through the tensor \( \sigma_k \) also as:

\[
\sigma_{i_1...i_{k+1}} = \sigma_{i_1...i_k} x_0 + \sum_{(j_1...j_{k-1}) \in \hat{p}(i_1...i_k)} \sigma_{j_1...j_{k-1}i_{k+1}} x_{\nu(j_1...j_{k-1})},
\]

where \( \nu(j_1...j_{k-1}) = m, m \in [1,k] \) is the index of coordinate in \( \{i_1, \ldots, i_k\} \setminus \{j_1, \ldots, j_{k-1}\} \) and \( x_0, x_1, \ldots, x_k \) are free parameters.

Using this representation and linear transformation (2), we obtain an equation on coloring tensor of strongly regular \( k \)-partition \( L_k \) in a form:

\[
\sum_{l} \sum_{(j_1...j_{k-1}) \in \hat{p}(i_1...i_k)} \sigma_{j_1...j_{k-1}l} x_{\nu(j_1...j_{k-1})} = T_{i_1...i_k}^{j_1...j_k} \sigma_{j_1...j_k},
\]

where the summand \( \sigma_{i_1...i_k} x_0 \) is omitted and the right part of equation is a conventional sum by \( j \)-indices that represents a coloring of \( L_k \).
We can rewrite (9) in the equivalence form as:

$$\sum_{l} \sum_{(j_1, \ldots, j_k) \in \mathcal{P}(i_1, \ldots, i_k)} \sigma_{j_1, \ldots, j_{k-1} \mid x_{l_{j_1, \ldots, j_{k-1}}}} \sim \sigma_{i_1, \ldots, i_k}. \quad (10)$$

### 6.4 Projective convolution of tensors

Now we introduce an operation on tensors that we applied in left part of equality (7).

Let \( \{\sigma_k^j : j = [1,k]\} \) be a set of \( k \) tensors of a rank \( k \), then we introduce a convolution operation:

$$\sum_{l} \prod_{j=1,k} \sigma_{\alpha(j)}^j \equiv \sigma_k^1 \circ \cdots \circ \sigma_k^k, \quad (11)$$

where \( \alpha(j) = \langle i_1 \ldots i_{j-1} i_{j+1} \ldots i_k \rangle \).

For a tensor of the rank 2 it is the conventional matrix product.

### 6.5 (0,1)-Tensor coloring function

Consider the coloring of \( k \)-partition \( L_k, |L_k| = d \), through \( d \) \((0,1)\)-tensors \( a_k^0 \) with elements \( a_{i_1 \ldots i_k}^0 \in \{0,1\}, \alpha \in [1,d], i_1, \ldots, i_k \in [1,n] \). The value of \( a_{i_1 \ldots i_k}^0 \) is 1 if a \( k \)-tuple \( \langle i_1 \ldots i_k \rangle \) belongs to the class \( U_k(\alpha) \) of \( L_k \), else \( a_{i_1 \ldots i_k}^0 = 0 \). So these \((0,1)\)-tensors are linear independent and any linear combination of them is a coloring of \( L_k \).

Let \( L_k \) be strongly regular, then we obtain an equation:

$$a_k^1 \circ \cdots \circ a_k^k = \sum_{\alpha=1,d} \lambda_{i_1 \ldots i_k}^\alpha a_k^\alpha \quad (12)$$

It is a generalization of equation on associative scheme \( \mathcal{P} \). We try here only to show the possibility of \( k \)-partition representation and do not develop corresponding theory. Now we consider some examples.

### 6.6 Strongly regular 2-partitions

A strongly regular simple graph \( \Gamma(n,m,\lambda,\mu) \) is a strongly regular 2-partition \( L_2 \) that consists of \( d = 2 \) symmetrical classes \( \langle \{v_1 v_2\} \rangle_{L_2} \langle v_2 v_1 \rangle \) for any \( v_1, v_2 \in V \). If \( L_3 = \beta L_2 \), then parameters of \( \Gamma \) are \( n = |V| \), multiplicity \( m \) of a point in a class of \( L_2 \) (that represent edges in the graph) and multiplicities \( \lambda, \mu \) of pairs from two classes of \( L_2 \) in corresponding two classes of \( L_3 \) (that represent triangles with 3 and 1 edges in the graph).

It is known that an adjacency matrix \( A \) of the graph \( \Gamma \) satisfies to equation

$$A^2 = mE + \lambda A + \mu \bar{A}, \quad (13)$$

where \( E \) is the unity matrix, \( \bar{A} = I - A \) and \( I \) is the 1-matrix (\( \{I_{ij}\} = \{1\} \)).

The equation \( \mathcal{L} \) follows also from \( \mathcal{H} \). For \( k = 2 \) and \( d = 2 \) we have a system of equations (for \( i \neq j \))

$$\sum_{l} \sigma_{il} \sigma_{jl} = x + y \sigma_{ij} \quad (14)$$

Let \( \sigma_{ij} \in \{\mu_0, \lambda_0\} \) and the tensor \( \sum_{l} \sigma_{il} \sigma_{jl} \) has values \( \lambda \) and \( \mu \) that color classes \( \lambda_0 \) and \( \mu_0 \) of \( L_2 \) correspondingly, then we obtain the system of equations of coloring transformation:

$$\lambda = x + y \cdot \lambda_0$$

$$\mu = x + y \cdot \mu_0.$$
So the system (14) takes a form:

$$\sum_l \sigma_{il} \sigma_{jl} = \frac{\mu \lambda_0 - \mu_0 \lambda}{\lambda_0 - \mu_0} + \frac{\lambda - \mu}{\lambda_0 - \mu_0} \sigma_{ij}.$$  

For the 0,1-tensor $\sigma_{ij}$ ($\mu_0 = 0$ and $\lambda_0 = 1$) we obtain

$$\sum_l \sigma_{il} \sigma_{jl} = \mu + (\lambda - \mu) \sigma_{ij}. \quad (15)$$

Or using, matrix notation ($\| \sigma_{ij} \| = A$)

$$A^2 - mE = \mu I + (\lambda - \mu) A = \lambda A + \mu A.$$  

### 6.7 pq-Stable Regular 3-partition

Let $L_3$ be a pq-stable regular 3-partition and $\sigma_{ijk}$ be its coloring tensor. Let $L_4 = qL_3$, then there exists a coloring $\sigma_{ijkl}$ of $L_4$ that can be represented accordingly to (3) as:

$$\sigma_{ijkl} = \sigma_{ijk} \sigma_{ijkl} \sigma_{ijkl} \sigma_{ijkl}.$$  

Let $L_2 = pL_3$ and $\sigma_{ij}$ be a tensor on $L_2$, representing $\sigma_{ijk}$ as:

$$\sigma_{ijk} = \sigma_{ij} \sigma_{ik} \sigma_{jk}, \quad (17)$$

then

$$\sigma_{ijkl} = \sigma_{ij} \sigma_{ik} \sigma_{jk} \sigma_{kl} \sigma_{kl} \sigma_{kl}.$$  

We consider here the mp-symmetry of $L_4$. In this case a power and first three factors of equality (18) are not of principal, so we can rewrite that as:

$$\sigma_{ijkl} = \sigma_{ij} \sigma_{jl} \sigma_{kl}. \quad (19)$$

Let $\langle ijk \rangle, \langle i'j'k' \rangle \in U_3 \in L_3$ and $\langle ijk \rangle \in U_4 \in L_4$. We will show that the multiplicity of 3-tuples $\langle ijk \rangle$ and $\langle i'j'k' \rangle$ in 4-relation $U_4$ are equal. Consider sums $S(\langle ijk \rangle) = \sum_l \sigma_{ijkl} \sigma_{kl}$ and $S(\langle i'j'k' \rangle) = \sum_l \sigma_{ijkl} \sigma_{kl}$. Since $\sum_m \sigma_{ml} = \sigma_l$ is a coloring of $L_1 = pL_2$, then we find that $\sum_k S(\langle ijk \rangle) = \sum_k S(\langle i'j'k' \rangle)$, $\sum_j S(\langle ijk \rangle) = \sum_j S(\langle i'j'k' \rangle)$ and $\sum_i S(\langle ijk \rangle) = \sum_i S(\langle i'j'k' \rangle)$. These equalities are valid by different coloring tensor of $L_2$, hence $S(\langle ijk \rangle) = S(\langle i'j'k' \rangle)$ also for different coloring of $L_2$. It proves theorem 4.

### 6.8 Strongly Regular 3-partitions

Here we show that

**Theorem 7** A non-trivial strongly regular 3-partition contains non-trivial automorphism.

and therefore prove the polynomiality of graph isomorphism problem.

Let $L_3$ be a strongly regular 3-partition and $\sigma_{ijk}$ be its coloring tensor, then the equivalence (10) takes a form:

$$\sum_l (\sigma_{ijlz} + \sigma_{ikly} + \sigma_{jklx}) \sim \sigma_{ijk}. \quad (20)$$

Let $L_3$ be 2-full, then we can represent the coloring tensor of $L_3$ through some coloring tensor of $L_2 = pL_3$, so that
\[ \sigma_{ijk} = \sigma_{ikp} + \sigma_{jkq}, \]  

(21)

where \( p, q \) are also free parameters.

By substitute the right part of (21) for the tensor of \( L_3 \) in (20) we obtain an equivalence:

\[ \sum_l (\sigma_{il}x + \sigma_{jl}y + \sigma_{kl}z) \sim \sigma_{ikp} + \sigma_{jkq} = \sigma_{ijk} \]  

(22)

(here \( x, y, z \) are new parameters).

Now we consider what for equations follow from equivalence (22). Let tensor \( \|\sigma_{ij}\| \) (and correspondingly tensor \( \|\sigma_{ijk}\| \)) has non-trivial automorphism \( \phi \), then

\[ \sigma_{\phi(i)\phi(j)\phi(k)} = \sigma_{ijk} \]

and

\[ \sigma_{\phi(i)\phi(l)x + \sigma_{\phi(j)\phi(l)y + \sigma_{\phi(k)\phi(l)z = \sigma_{il}x + \sigma_{jl}y + \sigma_{kl}z}. \]

So we have a bijection between summands of sums that represents \( \sigma_{ijk} \) and \( \sigma_{\phi(i)\phi(j)\phi(k)} \) in equivalent coloring. It gives a possibility to reduce in (22) the number of independent variables \( \sigma_{ij} \) and at the same time the number of independent related by equivalence lines (for different 3-tuples \( \langle ijk \rangle \)), by substitution anywhere in (22) \( \sigma_{ij} \) for \( \sigma_{\phi(i)\phi(j)} \) \( (\phi \in Aut(\|\sigma_{ij}\|), i, j \in [1, n]). \)

Let now 2-partition \( L_2 = L(\|\sigma_{ij}\|) \) and 3-partition \( L_3 = qL_2 = L(\|\sigma_{ijk}\|) \) be faithful strongly regular, then there exist 3-tuples \( \langle ijk \rangle \) and \( \langle i'j'k' \rangle \) that belong to the same class of \( L_3 \) and are connected with no automorphism. Then \( \sigma_{ijk} = \sigma_{i'j'k'} \) and

\[ \sum_l (\sigma_{il}x + \sigma_{jl}y + \sigma_{kl}z) = \sum_l (\sigma_{il}x + \sigma_{jl}y + \sigma_{kl}z), \]

(23)

Since \( x, y, z \) are free parameters then from (23) it follows equalities of three subsums:

\[ \sum_{l \neq i,j,k} \sigma_{il} = \sum_{l \neq i',j',k'} \sigma_{i'l}, \]

(24)

\[ \sum_{l \neq i,j,k} \sigma_{jl} = \sum_{l \neq i',j',k'} \sigma_{j'l}, \]

(25)

\[ \sum_{l \neq i,j,k} \sigma_{kl} = \sum_{l \neq i',j',k'} \sigma_{k'l}. \]

(26)

Because of \( s \)-symmetry this three systems of subsums equalities (for different pairs of 3-tuples) are equal. Thus it is sufficient to consider the system \( S_x \) given by expression (24) and choose only such equations in this system that are independent by automorphisms and by transitivity. This system of equations has solution if the number of equations \( |S_x| \) is less than number of variables, because the equalities in \( S_x \) are independent. From here immediately follows that, in the case of strongly regular 3-partition \( L_3 \), the system \( S_x \) can be solved if \( Aut(L_3) \) is enough rich on automorphisms, because, when \( Aut(L_3) \) is trivial, the number of equations in \( S_x \) is \( O(n^3) \) and the numner of variables is \( O(n^2) \). This proves theorem 7.

**Conclusion**

In given solution of graph isomorphism problem were used symmetry properties of \( k \)-orbits. Other texts of author connected with consideration of \( k \)-orbits one can find in ”www.arxiv.org”. Those texts are not mistake free, but they contains new original ideas and a direction of investigation, and therefore could be of interest. Author hopes that investigation of symmetry properties of \( k \)-orbits can bring new ideas for simplifying of simple finite group classification.
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