The Fréchet distance of surfaces is computable

Eike Neumann
Aston University, Birmingham, UK
neumaef1@aston.ac.uk

Abstract

We show that the Fréchet distance of two-dimensional parametrised surfaces in a metric space is computable. This settles a long-standing open question in computational geometry.

1 Introduction

In 1906, Maurice Fréchet introduced a natural pseudometric for parametrised curves [12], which he generalised in 1924 to parametrised surfaces [13]. If \( A : [0,1]^2 \rightarrow X \) and \( B : [0,1]^2 \rightarrow X \) are parametrised surfaces in some metric space, then their Fréchet distance is given by:

\[
\inf_{\varphi, \psi \in \text{Aut}'([0,1]^2)} \max_{x \in [0,1]^2} d(A(\varphi(x)), B(\psi(x))),
\]

where \( \text{Aut}'([0,1]^2) \) denotes the set of orientation-preserving homeomorphisms of \([0,1]^2\).

The problem of computing the Fréchet distance of curves and surfaces has received considerable attention in Computational Geometry (see [1] and references therein). Alt and Godau showed in 1995 [3] that the Fréchet distance between polygonal curves is polytime computable in the usual computational model of computational geometry. Later, Godau showed in his PhD-thesis [14] that computing the Fréchet distance between triangulated surfaces is NP-hard. Alt and Buchin [2] proved in 2010 that the Fréchet distance between triangulated surfaces is upper semicomputable, i.e., there exists an algorithm which takes an input two triangulated surfaces and returns as output a decreasing sequence of rational upper bounds to the Fréchet distance, which converges to the true distance. Note that no assumption is made on the rate of convergence of this sequence, so that in general one has no information on the quality of any such upper bound. It was also shown in [2] that the so-called weak Fréchet distance of surfaces, which is obtained by letting \( \varphi \) and \( \psi \) in \((*)\) range over all surjective (and not necessarily injective) reparametrisations, is polytime computable. The question remained open whether the Fréchet distance of triangulated parametrised surfaces is computable, i.e., whether there exists an algorithm which takes as input two triangulated surfaces and a number \( \varepsilon > 0 \) and produces a rational approximation to the Fréchet distance to error \( \varepsilon \). In this paper we give an affirmative answer to this question. We employ computable analysis, a rigorous mathematical theory of computation with continuous data, (see e.g., [20, 26]) to obtain the following stronger result:

**Theorem 1.** Let \( X \) be a computable metric space. There exists an algorithm which takes as input two parametrised surfaces \( A : [0,1]^2 \rightarrow X \), \( B : [0,1]^2 \rightarrow X \) in \( X \), and returns as output their Fréchet distance.
The main idea behind the proof of Theorem 1 is to compute approximations to the Fréchet distance by replacing Aut'([0,1]^2) in (*) with a suitable computably compact set, using that the minimum of a continuous function over a computably compact set is computable. The proof can be outlined as follows: Section 4 shows that a 2^{-n}-approximation to the Fréchet distance can be obtained by letting the infimum in (*) range over the set of L_n-Lipschitz automorphisms, where L_n is a constant that depends computably on n. This reduces the problem to the problem of computing the closure Aut'([0,1]^2) of the set of reparametrisations as a subset of the space of continuous functions C([0,1]^2,[0,1]^2). While this set is relatively easily seen to be lower semicomputable (which yields upper semicomputability of the Fréchet distance) it is more difficult to establish upper semicomputability. This amounts to showing that there exists an algorithm which takes as input a map \( \phi: [0,1]^2 \to [0,1]^2 \) and halts if and only if the map is not contained in Aut'([0,1]^2). While the set of surjective functions is closed, and it is easy to find an algorithm which halts if and only if a given function is not surjective (which yields computability of the weak Fréchet distance), falsifying injectivity is considerably more difficult, and this is the main part of the proof where some new ideas are needed. The main idea is to “count” for every \( y \in [0,1]^2 \) the solutions to the equation \( \phi(x) = y \) using the Brouwer mapping degree.

In the context of computable analysis, the computability of the Fréchet distance was also recently studied by Park, Park, Park, Seon, and Ziegler [18]. They observed that the Fréchet distance of continuous curves with values in a computable metric space is computable. Regarding the question of computational complexity, the algorithm we obtain from the proof of Theorem 1 makes use of multiple unbounded searches and is therefore not even primitive recursive. We hence do not obtain any nontrivial upper complexity bounds on the problem beyond establishing its computability. The problem of characterising the complexity of the Fréchet distance is therefore far from settled. However, there is some hope that the ideas presented in this paper could be used to design more efficient algorithms which yield better upper complexity bounds.

2 Preliminaries

Let us introduce some notation and terminology and recall some basic definitions from computable analysis. We mainly follow the ideas of Matthias Schröder [21]. For a concise introduction to computable analysis see [19]. See [20] and [26] for classic textbooks on the subject which cover some aspects of the theory we require.

We denote Sierpiński space by \( S \). If \( X \) and \( Y \) are represented spaces, we write \( C(X,Y) \) for their exponential in the category of represented spaces. We write \( \Theta(X) \) for the open subsets of \( X \) identified with \( C(X,S) \), \( A(X) \) for the closed subsets of \( X \) identified with their complement as an element of \( \Theta(X) \), \( K(X) \) for the (saturated) compacts of \( X \) identified in the usual manner with a subspace of \( \Theta(\Theta(X)) \), and \( V(X) \) for the (closed) overts of \( X \) identified in the usual manner with a subspace of \( \Theta(\Theta(X)) \). A subset of \( X \) is called semi-decidable if it is a computable point of the space \( \Theta(X) \). A closed subset of \( X \) is called lower semicomputable if it is a computable point of the space \( V(X) \). It is called upper semicomputable if it is a computable point of the space \( A(X) \). A saturated compact subset of \( X \) is called lower semicomputable if it is a computable point of the space \( V(X) \). It is called upper semicomputable if it is a computable point of the space \( A(X) \). A closed or compact set is called
computable if it is both lower and upper semicomputable. We denote by \( \mathbb{R}_< \)
the space of lower reals, where a real number \( x \) is encoded by a sequence \( (l_n)_n \)
of rational numbers which converges from below to \( x \). We denote by \( \mathbb{R}_> \)
the space of upper reals, where a real number \( x \) is encoded by a sequence \( (r_n)_n \)
of rational numbers which converges from above to \( x \). A real-valued function \( F : X \to \mathbb{R} \)
is called lower semicomputable if it is computable as a function to the lower reals \( \mathbb{R}_< \)
and upper semicomputable if it is computable as a function to the upper reals \( \mathbb{R}_> \). If \( f : X \to Y \)
is a partial function, we will often say that \( f(x) \) is uniformly computable in \( x \) to express that \( f \) is a computable function. Throughout this paper we endow \( \mathbb{R}^n \) with the maximum norm
\[
|\mathbf{x}| = \max \{|x_i| \mid i = 1, \ldots, n\}.
\]
It will be convenient to write \( D^n \) for the unit cube \([0, 1]^n\) and \( S^{n-1} \) for its boundary \( \partial [0, 1]^n \). If \( A \) is a subset of a metric space \( M \), let
\[
d(x, A) = \inf \{d(x, y) \mid y \in A\}
\]
denote the distance function of \( A \). If \( \varepsilon > 0 \) is a real number, we call the set
\[
A^\varepsilon = \{x \in M \mid d(x, A) \leq \varepsilon\}
\]
the closed \( \varepsilon \)-thickening of \( A \). Analogously, the set
\[
A^\varepsilon = \{x \in M \mid d(x, A) < \varepsilon\}
\]
is called the open \( \varepsilon \)-thickening of \( A \).

Our proof of Theorem 1 is based on the following simple observation:

**Proposition 2.**

1. The infimum of an upper semicomputable function over a lower semicomputable closed set is uniformly upper semicomputable. More formally, for every represented space \( X \), the function
\[
\inf : \forall (X) \times \mathcal{C}(X, \mathbb{R}_>) \to \mathbb{R}_>, (A, f) \mapsto \inf \{f(x) \mid x \in A\}
\]
is computable.
2. The minimum of a lower semicomputable function over an upper semicomputable compact set is uniformly lower semicomputable. More formally, for every represented space \( X \), the function
\[
\min : \forall (X) \times \mathcal{C}(X, \mathbb{R}_<) \to \mathbb{R}_<, (K, f) \mapsto \min \{f(x) \mid x \in K\}
\]
is computable.

3 The Brouwer mapping degree

The main topological tool for proving Theorem 1 will be the Brouwer mapping degree. We will summarise here the main facts we need in the sequel. Very readable constructions of the degree are given in [17] and [23].

**Theorem 3.** There exists a unique function
\[
\deg : \forall C(\mathbb{R}^n, \mathbb{R}^n) \times \mathcal{O}(\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{Z}
\]
with domain
\[
\text{dom}(\deg) = \{(f, U, y) \mid U \text{ bounded, } y \not\in f(\partial U)\}
\]
satisfying the following properties:
1. **Translation invariance:** \( \deg(f, U, y) = \deg(f - y, U, 0) \).

2. **Normalisation:** \( \deg(id, U, y) = 1 \) for all \( y \in U \).

3. **Additivity:** If \( U_1 \) and \( U_2 \) are open disjoint subsets of \( U \) with \( y \notin f(U_1 \cup U_2) \) then \( \deg(f, U, y) = \deg(f, U_1, y) + \deg(f, U_2, y) \).

4. **Homotopy invariance:** If \( H(t, x) \) is a homotopy from \( f \) to \( g \) with \( y \notin H(t, \partial U) \) for all \( t \in [0, 1] \) then \( \deg(f, U, y) = \deg(g, U, y) \).

**Proposition 4.** If \( \deg(f, U, y) \) is well-defined and non-zero, then the equation \( f(x) = y \) has a solution in \( U \).

It can be shown that the degree is computable when the open sets are correctly topologised. Let \( \mathcal{U}(\mathbb{R}^n) \) denote the space of open subsets of \( \mathbb{R}^n \) which is obtained by identifying an open set \( U \) with its two-sided distance function:

\[
d_{\text{two-sided}}(\cdot, U) : \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto \begin{cases} d(x, \partial U) & \text{if } x \notin U, \\ -d(x, \partial U) & \text{if } x \in U. \end{cases}
\]

Note that the underlying representation is much stronger than the standard representation of open sets.

**Theorem 5.** The partial map

\[
\deg : \mathcal{C}(\mathbb{R}^n) \times \mathcal{U}(\mathbb{R}^n) \times \mathbb{R}^n, (f, U, y) \mapsto \deg(f, U, y)
\]

is computable with semi-decidable domain.

**Proof Sketch.** The degree \( \deg(f, U, y) \) is defined so long as \( y \notin f(\partial U) \), and this is uniformly semi-decidable for continuous \( f \) and \( U \in \mathcal{U} \). To compute \( \deg(f, U, y) \), compute a sufficiently good twice differentiable approximation \( \tilde{f} \) to \( f \) and a sufficiently good approximation \( \tilde{y} \) to \( y \), which is a regular value of \( \tilde{f} \). It suffices to choose \( \tilde{y} \) with \( |y - \tilde{y}| < d(y, f(\partial U)) \) and \( \tilde{f} \) with \( |f - \tilde{f}| < d(y, f(\partial U)) \). The fact that \( \tilde{y} \) can be chosen to be a regular value follows from Sard’s theorem. Then \( \deg(f, U, y) \) can be computed using the determinant formula:

\[
\deg(f, U, y) = \deg(\tilde{f}, U, \tilde{y}) = \sum_{x \in \tilde{f}^{-1}(\tilde{y})} \sgn(\det(D\tilde{f}(x))).
\]

For more details refer to the construction of the mapping degree in [23, Chapter 16].

A similar result was proved by Miller [16] for the fixed point index on rational cubical complexes (see also [7, 5, 6]). His proof uses computational homology rather than the determinant formula. An analogous result based on computational homology is stated in Collins [10].

## 4 Reduction to a compact search problem

For a map \( f : X \rightarrow Y \), let \( \Gamma_f \) denote its graph. Define a new distance function \( d_\Gamma \) on \( C(D^2, D^2) \) by

\[
d_\Gamma(f, g) = d_H(\Gamma_f, \Gamma_g)
\]

where \( d_H \) is the Hausdorff distance on the metric space \( D^2 \times D^2 \) with the product metric

\[
d((x_0, y_0), (x_1, y_1)) = \max\{d(x_0, x_1), d(y_0, y_1)\}.
\]
We will call this the graph distance on \( C(D^2, D^2) \). With respect to \( d_T \), the metric space \( C(D^2, D^2) \) is totally bounded but incomplete. The total boundedness is what will allow us to reduce the problem of computing the Fréchet distance to the problem of computing a minimum over a compact set. A similar idea is used in [13] to compute the Fréchet distance of curves.

Recall that if \( f : X \to Y \) is a uniformly continuous map between metric spaces, then a modulus of (uniform) continuity for \( f \) is a function \( \omega : \mathbb{N} \to \mathbb{N} \) such that for all \( x, y \in X \) we have the implication:

\[
d(x, y) \leq 2^{-\omega(n)} \implies d(f(x), f(y)) < 2^{-n}.
\]

The graph distance is useful, as good approximations of the reparametrisations with respect to the graph distance yield good approximations of the Fréchet distance.

**Lemma 6.** For a function \( \varphi : D^2 \to D^2 \), let

\[
F_{A,B}(\varphi) = \max_{x \in D^2} d(A(x), B(\varphi(x)))
\]

Let \( \mu_A \) and \( \mu_B \) be moduli of continuity of \( A \) and \( B \) respectively. Then we have the implication

\[
d_T(\varphi, \psi) \leq 2^{-\mu_A(n+1)-\mu_B(n+1)} \implies d(F_{A,B}(\varphi), F_{A,B}(\psi)) \leq 2^{-n}.
\]

**Lemma 7.** Let \( f : \partial[0,1]^n \to \mathbb{R}^n \) be a map. Let \( \tilde{f} : [0,1]^n \to \mathbb{R}^n \) denote its radial extension

\[
\tilde{f}(x) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{x}{|x|} & \text{otherwise.} \end{cases}
\]

If \( L \) is a Lipschitz constant for \( f \), then \( L + |f| \) is a Lipschitz constant for \( \tilde{f} \).

**Proof.** Let \( x, y \in [0,1]^n \) with \( |y| \leq |x| \). Let \( r = |x| \) and \( s = |y| \). If \( s = 0 \) then

\[
|\tilde{f}(x) - \tilde{f}(y)| = r \cdot |f(\frac{x}{r})| \leq |x - y| \cdot |f|.
\]

If \( s > 0 \), we calculate:

\[
|\tilde{f}(x) - \tilde{f}(y)| = |r \cdot f(\frac{x}{r}) - s \cdot f(\frac{y}{s})| \\
\leq |s \cdot f(\frac{x}{s}) - s \cdot f(\frac{y}{s})| + |r \cdot f(\frac{x}{r}) - s \cdot f(\frac{y}{s})| \\
\leq s \cdot L \cdot |\frac{x}{s} - \frac{y}{s}| + |f| \cdot |r - s| \\
\leq L \cdot |\frac{x}{s} - \frac{y}{s}| + |f| \cdot |x - y|. \\
\leq L \cdot |x - y| + |f| \cdot |x - y|.
\]

For the last line, note that the point \( \frac{x}{s} \) is the projection of \( x \) onto the set \( \{ z \in \mathbb{R}^n \mid |z| = s \} \). \( \square \)

The following Lemma is the main result of this section. It shows that the space of automorphisms of \( D^2 \) is totally bounded with respect to the graph distance.

**Lemma 8.** Let \( f : D^2 \to D^2 \) be an automorphism. For all \( n \in \mathbb{N} \) there exists an automorphism \( \tilde{f} : D^2 \to D^2 \) with \( d_T(f, \tilde{f}) < 2^{-n} \) such that \( \tilde{f} \) has Lipschitz constant

\[
4^n \times 4^n \times (3 \times 4^n + 3) + 1.
\]
Proof. Subdivide $D^2$ into a uniform square grid of mesh width $2^{-n}$. We will replace $f$ on the edges of this grid by a map with small Lipschitz constant and extend this map radially to obtain the desired approximation. An edge of the grid is simply an edge of one of the squares of the subdivision. We view these edges as subsets of $D^2$. Hence if two squares meet at an edge, this edge is counted as one edge and not as two. An edge is called an interior edge if it is not completely contained in the boundary of $D^2$. Otherwise it is called a boundary edge.

Consider an interior edge $e$ of this grid. The map $f$ sends $e$ to a simple curve $C$. We can isometrically identify $e$ with the interval $[0, 2^{-n}]$ and think of the curve $C$ as being parametrised over this interval by a continuous function $\gamma : [0, 2^{-n}] \to D^2$.

Up to slightly perturbing $f$ we can assume that all curves $C$ of this form intersect the mesh in a non-degenerate manner in the sense that the intersection of $C$ with the edges of the mesh is zero-dimensional, i.e., $\gamma$ never maps an interval into an edge, and that $C$ does not intersect any vertices of the mesh.

By the assumption that $C$ intersects the mesh in a non-degenerate manner there exists a unique square $S_0$ such that the initial segment $\gamma(0, b)$ of the curve is completely contained in $S_0$ for some $\delta > 0$.

For a square $S$ we say that $\gamma$ is staying at $S$ in the interval $[a, b]$ if $\gamma(t) \in S$ for $t < a$, $\gamma(a + \delta) \in S$, for all sufficiently small $\delta > 0$, $\gamma(b) \in S$ and $\gamma(t) \in S$ for all $t > b$.

Note that the intervals $[a, b]$ in which $\gamma$ is staying at some square $S$ form a forest $F$ (i.e., a finite union of trees) with respect to the usual inclusion order. The interval $[0, 2^{-n}]$ decomposes into the maximal intervals of this order.

If $\gamma$ is staying at $S$ in the interval $[a, b]$, we call a restriction $\gamma|_{[c, d]}$ with $[c, d] \subset [a, b]$ an arm of $\gamma$ in $[a, b]$ if $\gamma(c) \in S$ and $\gamma(d) \in S$, but $\gamma(t) \notin S$ for all $c < t < d$. We call the interval $[c, d]$ the domain of the arm $A$. The arms of $\gamma$ in $[a, b]$ can be linearly ordered by comparing the left endpoints of their domains with respect to the usual order on $\mathbb{R}$. If $S'$ is another square and $A$ is an arm, we say that $A$ reaches $S'$ if it intersects $S'$. We say that an arm $A$ of $\gamma$ in $[a, b]$ reaches new squares if there exists a square $S'$ such that $A$ reaches $S'$ but no arm $A'$ of $\gamma$ in $[a, b]$ with $A' \leq A$ reaches $S'$.

We replace $\gamma$ with a simpler curve that only has arms which reach new squares. Let $I$ be a maximal interval in the forest $F$ and let $S$ be the square at which $\gamma$ is staying in $I$. Let $A$ be an arm of $\gamma$ in $I$ which does not reach any new squares. Let $[c, d]$ be the domain of $A$. Then there exist $c', d' \in I$ with $c' < c < d < d'$ such that $A$ is the only arm of $\gamma$ in the interval $[c', d']$ and $\gamma(c')$ and $\gamma(d')$ are interior points of $S$. Hence, the arm $A$ can be pruned away by replacing $\gamma$ on $[c', d']$ with the linear interpolation in $\gamma(c')$ and $\gamma(d')$. Use this method to prune away all arms in $I$ which do not reach new squares.

Note that we can do this in such a way that the new curve does not intersect itself. Do the same for all maximal intervals in $F$, and apply this procedure recursively to all intervals on the next level, so that we eventually obtain a new curve $\zeta$ which does not have any arms that do not reach new squares.

Replace $\zeta$ with a suitable linear approximation $\eta$ which is constructed as follows: consider the intersection of $\zeta$ with the edges of our grid. Since $\zeta$ intersects the grid in a nondegenerate manner this is a finite set of points. Order them according to the order in which they are visited by $\zeta$. If the line segment between two consecutive points $\zeta(t_0)$ and $\zeta(t_1)$ is contained in an edge of the grid, add an additional point $\zeta(s)$ with $s \in (t_0, t_1)$ to the set. This point is necessarily contained in the interior of some square. Consider the polygonal
chain \( P \) which interpolates these points in the given order. This chain could have some self-intersections. Note however that if we have two segments in the chain whose endpoints lie on the edges of the grid, then the two segments intersect if and only if the original curve \( \zeta \) has a self-intersection. Hence, the only self-intersections of \( P \) can happen between segments where at least one endpoint is an interior point of a square. In this case, we can move this endpoint closer to the boundary of the square to resolve the intersection. Doing this finitely many times yields a simple polygonal chain which intersects the same squares as \( \zeta \). Let \( \eta : [0, 2^{-n}] \to D^2 \) be the parametrisation of this chain where each segment is traversed at the same speed.

Let us now estimate the Lipschitz constant of the curve \( \eta \). As there are \( 4^n \) squares in total, the forest \( \mathcal{F}_{\eta} \) contains at most \( 4^n \) trees. By the same argument each tree has height at most \( 4^n \). Since \( \eta \) intersects the mesh in a non-degenerate manner, each node in the tree has at most four children. Hence each tree has at most \( 4^4 \times 4^n \) elements, and thus the forest has at most \( 4^n \times 4^4 \times 4^n \) elements in total. By construction, the number of vertices used in the linear interpolation of \( \zeta \) in each element of the forest is bounded by

\[
3 \times (\text{the number of arms}) + 3.
\]

As every arm has to reach at least one new square, there are at most \( 4^n \) arms, so that every element of the forest contributes at most \( 3 \times 4^n + 3 \) vertices. In total there are at most

\[
N = 4^n \times 4^4 \times (3 \times 4^n + 3)
\]

vertices, and just as many line segments. Thus, the interval \([0, 2^{-n}]\) is divided into \( N \) segments of length \( 2^{-n} / N \). Each line segment has diameter at most \( 2^{-n} \). Hence, the Lipschitz constant of \( \eta \) is bounded by \( N \).

If \( e \) is a boundary edge, then \( f(e) \) is contained in the boundary of \([0, 1]\). Hence, we can construct a piecewise linear function with the same image, which uses at most 5 pieces. Its Lipschitz constant is therefore bounded by \( 5 \times 2^n < N \).

We have constructed a piecewise linear curve \( \eta \) for every edge \( e \) of the mesh. Similarly as in the construction of each individual curve, we can make sure that these curves do not intersect by potentially moving certain vertices in the interiors of the squares of the grid closer to the boundary (which does not change the estimate of the Lipschitz constant). Hence, these curves define a bijective map \( f \) on the 1-skeleton of the mesh. By Lemma 7 this map extends to a bijection with Lipschitz constant \( N + 1 \) via radial extension to the interiors of the squares. By construction, the curve \( \eta \) intersects a square if and only if \( \gamma \) does so. Hence, the image of every square \( S \) under \( f \) is \( 2^{-n} \)-close in the Hausdorff distance to the image of \( \gamma \). Since every square has diameter at most \( 2^{-n} \), the graphs of \( f \) and \( \gamma \) are \( 2^{-n} \)-close in the Hausdorff distance. Hence, \( f \) has all the desired properties. \( \square \)

Finally, we observe that the compact space of \( L \)-Lipschitz functions is uniformly upper semicomputably compact in \( L \) as a subset of \( C(D^2, D^2) \). This is a special case of the constructive version of the Arzelà-Ascoli theorem, which was proved in [4].

**Theorem 9.** The map

\[
\text{mod} : \mathbb{N}^\mathbb{N} \to \mathcal{K} \left( C(D^2, D^2) \right),
\]

\[
\omega \mapsto \{ f : D^2 \to D^2 \mid \omega \text{ is a modulus of continuity for } f \}
\]

7
is computable. In particular, the map
\[
\text{lip} : \mathbb{R} \rightarrow x \left[ C \left( (D^2, D^2) \right) \right],
\]
\[
L \mapsto \left\{ f : D^2 \rightarrow D^2 \mid \forall x, y \in D^2, | f(x) - f(y) | \leq L \cdot | x - y | \right\}
\]
is computable.

5 Computability of automorphisms

The goal of this section is to establish the following result:

**Theorem 10.** The closure \( \text{Aut}'(D^2) \) of the set of orientation-preserving automorphisms of the unit square \( D^2 \) is computable as a closed subset of the space \( C(D^2, D^2) \) of continuous self-maps of \( D^2 \).

**Proof.** This follows from Theorems 12 and 13 below.

Let \( f : S^1 \rightarrow S^1 \) be a self-map of the unit circle. Then \( f \) is called an orientation-preserving pseudo-automorphism (or just pseudo-automorphism for short) if \( f \) lifts to a surjective monotonically increasing map \( \tilde{f} : [0, 1] \rightarrow [0, 1] \) with respect to suitable orientation-preserving bijective parametrisations of its domain and codomain. Note that being a pseudo-automorphism is computably falsifiable, i.e., the pseudo-automorphisms are an upper semi-computable closed subset of \( C(S^1, S^1) \). It is easy to see that they are also lower semicomputable.

**Definition 11.** For \( f : D^2 \rightarrow D^2 \), let \( \tilde{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) denote the radial extension \( \tilde{f}(r \cdot x) = r \cdot f(x) \), where \( x \in \partial D^2 \), \( r \in [1, +\infty) \).

Call \( f : D^2 \rightarrow D^2 \) an orientation preserving pseudo-automorphism (or just pseudo-automorphism) if it satisfies the following conditions:

1. \( f(\partial D^2) \subseteq \partial D^2 \).
2. \( f|_{\partial D^2} : S^1 \rightarrow S^1 \) is a pseudo-automorphism of the unit circle.
3. \( f \) is surjective.
4. If \( U_1, \ldots, U_n \) are disjoint open subsets of \( \mathbb{R}^2 \) and \( y \in \tilde{f}(\bigcup U_i) \setminus \tilde{f}(\partial \bigcup U_i) \) then \( \sum_{i=1}^n \text{deg}(\tilde{f}, U_i, y) = 1 \).

Let \( \text{PseudAut}^2 \) denote the set of all (orientation preserving) pseudo-automorphisms of \( D^2 \).

**Theorem 12.** The set \( \text{PseudAut}^2 \) is computable as a closed subset of the space \( C(D^2, D^2) \).

**Proof.** This essentially follows from the definition. On the one hand, all the conditions in Definition 11 define upper semicomputable closed sets. Thus their intersection is again upper semicomputable. On the other hand, piecewise linear automorphisms which are given by matrices with rational entries on a rational polygonal subdivision of \( D^2 \) are dense in the set of all automorphisms, which are in turn dense in the set of all pseudo-automorphisms by Theorem 13 below. As we can semi-decide for a given piecewise linear map which is specified by rational data if it is bijective, the set \( \text{PseudAut}^2 \) admits a computably enumerable dense sequence and therefore is lower semicomputable.

**Theorem 13.** We have \( \text{PseudAut}^2 = \text{Aut}'(D^2) \).
Proof. Clearly every automorphism is a pseudo-automorphism. The proof of the converse takes up the rest of this section. It follows from Lemma 15 below.

We now prove the remaining direction of Theorem 13: Given a pseudo-automorphism \( f : D^2 \to D^2 \) and a number \( \varepsilon > 0 \) we construct an automorphism \( f_\varepsilon \) which is \( \varepsilon \)-close to it. The key observation is the following:

Lemma 14. Let \( f : D^2 \to D^2 \) be a pseudo-automorphism. Then the preimage under \( f \) of every connected set is connected and the preimage under \( f \) of every simply connected set is simply connected.

Proof. Let \( C \) be connected. Let \( A_1 \cap A_2 \supseteq C \) be a partition of \( f^{-1}(C) \) into non-empty closed subsets \( A_1 \) and \( A_2 \). Then \( f(A_1) \cup f(A_2) \) is a partition of \( C \) into non-empty closed subsets. Since \( C \) is connected, there exists \( y \in f(A_1) \cap f(A_2) \). If \( A_1 \) and \( A_2 \) are disjoint, they can be separated by open neighbourhoods \( U_1 \supset A_1 \) and \( U_2 \supset A_2 \) in \( \mathbb{R}^2 \). Recall that \( f \) denotes the radial extension of \( f \) to \( \mathbb{R}^2 \). Since \( f \) is a pseudo-automorphism we have \( \deg(f, U_1, y) = 1 \) for \( i = 1, 2 \) and \( \deg(f, U_1, y) + \deg(f, U_2, y) = 1 \). Contradiction! It follows that \( A_1 \cap A_2 \neq \emptyset \), and so \( f^{-1}(C) \) is connected.

Now, assume that \( C \) is simply connected. Let \( L \) be a loop in \( f^{-1}(C) \). If \( L \) cannot be deformed into a point, then there exists \( x \) in the region \( U \) which is bounded by \( L \) with \( f(x) \notin C \). In particular the degree \( \deg(f, U, f(x)) \) is well defined. As \( f \circ L \) is a closed curve in \( C \) and \( C \) is simply connected it follows from \( f(x) \notin C \) that \( f(x) \) is not contained in any region that is bounded by \( f \circ L \). This implies \( \deg(f, U, f(x)) = 0 \). On the other hand, since \( f \) is a pseudo-automorphism we have \( \deg(f, U, f(x)) = 1 \). Contradiction! It follows that \( f^{-1}(C) \) is simply connected.

We now construct \( f_\varepsilon \) based on Lemma 14. We divide the codomain \( D^2 \) into a uniform square mesh \( \mathcal{M} \) of mesh width \( h < \varepsilon/2 \). Introducing some notation from the proof of Lemma 8 call a vertex of the mesh an interior vertex if it is contained in the interior of \( D^2 \). Call it a boundary vertex otherwise. Call an edge an interior edge, if at least one of its endpoints is an interior vertex. Call it a boundary edge otherwise. Assign to each interior vertex \( v \) an arbitrarily chosen point \( x(v) \) in \( f^{-1}(v) \). Assign to each boundary vertex \( v \) an arbitrarily chosen point \( x(v) \) in \( f^{-1}(v) \). Such a point exists since \( f \mid \partial D^2 \) is a surjective map onto the boundary. For each boundary edge \( e \) with endpoints \( v_0 \) and \( v_1 \), let \( \alpha(e) \) denote the arc in the boundary that joins the two points \( x(v_0) \) and \( x(v_1) \). As \( f \mid \partial D^2 \) is pseudo-automorphism of circles, different boundary edges \( e_0 \) and \( e_1 \) are associated with arcs \( \alpha(e_0) \) and \( \alpha(e_1) \) that do not intersect except in their endpoints. Now consider an interior edge \( e \) with endpoints \( v_0 \) and \( v_1 \). Let \( U = e^{h/4} \) be the open \( h/4 \)-thickening of \( e \). By Lemma 14 the set \( f^{-1}(U) \) is connected. In particular there exists an arc \( \alpha(e) \) in \( f^{-1}(U) \) which connects \( x(v_0) \) and \( x(v_1) \). We can choose these arcs such that arcs associated with different edges do not intersect except in their endpoints. Then for each square \( S \in \mathcal{M} \) of the mesh we have four arcs, corresponding to the four edges of the square, which form a simple closed curve \( C(S) \subseteq D^2 \). This curve bounds an open region \( \Omega(S) \subseteq D^2 \). Now, choose for each edge \( e \) with endpoints \( v_0 \) and \( v_1 \) a bijective map \( \phi_e : \alpha(e) \to e \) which sends \( x(v_0) \) to \( v_0 \) and \( x(v_1) \) to \( v_1 \). For each square \( S \in \mathcal{M} \) this yields a bijective map \( \phi_S : \Omega(S) \to S \) by radial extension. Now assign to each \( x \in D^2 \) which is contained in some region \( \Omega(S) \) the value \( \phi_S(x) \). This defines a partial map

\[
f_\varepsilon : \subseteq D^2 \to D^2.
\]
Our next goal is to show that this map is well-defined, bijective, and ε-close to $f$.

**Lemma 15.**

1. If $S_0$ and $S_1$ are adjacent squares which intersect only in an edge $e$, then the corresponding regions $\overline{\Omega(S_0)}$ and $\overline{\Omega(S_1)}$ intersect only in the arc $A(e)$. Furthermore, the arc $A(e)$ without its endpoints is contained in the interior of the union of $\overline{\Omega(S_0)}$ and $\overline{\Omega(S_1)}$.

2. If $S_0$ and $S_1$ are adjacent squares which intersect only in a vertex $v$, then the corresponding regions $\overline{\Omega(S_0)}$ and $\overline{\Omega(S_1)}$ intersect only in the point $x(v)$.

3. If $S_0$ and $S_1$ are non-adjacent squares, then the corresponding regions $\overline{\Omega(S_0)}$ and $\overline{\Omega(S_1)}$ are disjoint.

**Proof.**

1. The curves $C(S_0)$ and $C(S_1)$ intersect only in $A(e)$ by construction. Hence, if $\overline{\Omega(S_0)}$ and $\overline{\Omega(S_1)}$ intersect in further points, one set must contain the other. Let $c_0$ denote the centre of $S_0$. Then $\overline{\Omega(S_0)}$ contains $f^{-1}(c_0)$ but $\overline{\Omega(S_1)}$ is disjoint from $f^{-1}(c_0)$. Hence, $\overline{\Omega(S_1)}$ cannot contain $\overline{\Omega(S_0)}$. By symmetry, $\overline{\Omega(S_0)}$ cannot contain $\overline{\Omega(S_1)}$. It follows that $\overline{\Omega(S_0)}$ and $\overline{\Omega(S_1)}$ intersect each other only in $A(e)$. The arc $A(e)$ is then contained in the bounded region of the curve whose trace is the union of the remaining arcs of $C(S_0)$ and $C(S_1)$. It follows that $A(e)$ without its endpoints is contained in the interior of the union of $\overline{\Omega(S_0)}$ and $\overline{\Omega(S_1)}$.

2. Follows from an analogous argument.

3. By construction, $\overline{\Omega(S_0)}$ and $\overline{\Omega(S_1)}$ are contained in the preimages of the $h/4$-thickenings of $S_0$ and $S_1$ respectively. If $S_0$ and $S_1$ are non-adjacent then these sets are disjoint, and then so are $\overline{\Omega(S_0)}$ and $\overline{\Omega(S_1)}$.

**Lemma 16.** The map $f_\varepsilon$ is well-defined, total, bijective, and ε-close to $f$.

**Proof.** By Lemma 15 the map is well-defined, as every point in its domain is assigned at most one value. The domain of $f_\varepsilon$ is a finite union of closed sets, and hence closed. If $x$ is a point in the domain, it is either contained in an open region $\Omega(S)$ or on an arc $A(e)$. In the latter case it is either contained in the boundary $\partial D^2$ or it is contained in a common arc of two regions $\overline{\Omega(S_0)}$ and $\overline{\Omega(S_1)}$. If it is contained in an arc, it is either an endpoint of the arc or an “interior point”, i.e., not an endpoint. If it is an “interior point”, by Lemma 15 it is contained in the interior of the union of the two regions, and hence in particular in the interior of the domain of $f_\varepsilon$. If it is an endpoint, then by a similar argument as in Lemma 15 it is contained in the interior of the union of the four regions which meet at this point. Hence, every point of the domain of $f_\varepsilon$ is an interior point. It follows that the domain of $f_\varepsilon$ is open and closed. Hence $f_\varepsilon$ is total. By construction $f_\varepsilon$ is bijective. Finally, if $x \in D^2$ is mapped by $f$ to a square $S$, then $f_\varepsilon(x)$ is contained in an adjacent square. Hence, $f_\varepsilon$ is ε-close to $f$.

### 6 Computability of the Fréchet distance

Putting it all together we can prove Theorem 1. Let us restate the result more formally:
**Theorem 17.** Let $X$ be a computable metric space. Then the function
\[ d_{\text{Fréchet}} : C(D^2, X) \times C(D^2, X) \to \mathbb{R}, \]
\[ (A, B) \mapsto \inf_{\varphi, \psi \in \text{Aut}(D^2)} \max_{x \in D^2} d(A(\varphi(x)), B(\psi(x))) \]
is computable.

**Proof.** It follows from Theorem 10 and Proposition 2.1 that the Fréchet distance is uniformly upper semicomputable in $A$ and $B$. It remains to show that the Fréchet distance is uniformly lower semicomputable in $A$ and $B$. For a number $L \in \mathbb{R}$, let
\[ \text{Aut}_L(D^2) = \text{Aut}(D^2) \cap \{ f : D^2 \to D^2 \mid \forall x, y \in D^2. \, |f(x) - f(y)| \leq L \cdot |x - y| \} \]
denote the closure of the set of $L$-Lipschitz orientation-preserving automorphisms of $D^2$. Let $n \in \mathbb{N}$. By Lemma 8 and Lemma 6, the number
\[ d_n = \inf_{\varphi, \psi \in \text{Aut}_L(D^2)} \max_{x \in D^2} d(A(\varphi(x)), B(\psi(x))), \]
where
\[ L_n = 4^{a(n)} \times 4^{4^{4^{a(n)}}} \times (3 \times 4^{a(n)} + 3) + 1 \]
and
\[ a(n) = \mu_A(n + 1) + \mu_B(n + 1) \]
is $2^{-n}$-close to the Fréchet distance $d_{\text{Fréchet}}(A, B)$. By Theorem 10 and Theorem 9, the set $\text{Aut}_L(D^2)$ is uniformly computably compact in $n$. Hence, by Proposition 2.2 the numbers $d_n - 2^{-n}$ are uniformly lower semicomputable in $A$, $B$, and $n$. As these numbers converge from below to the Fréchet distance, the Fréchet distance itself is uniformly lower semicomputable in $A$ and $B$. \qed

It is easy to modify the proof to show computability of the Fréchet distance of surfaces which are parametrised over the sphere $S^2$ rather than the square $[0,1]^2$. The proof of Theorem 10 can be used to show that the set of orientation-reversing automorphisms and the set of all automorphisms are computable as well. This allows us to compute further variations of the Fréchet distance.

**Acknowledgements**

This work was supported by EU Horizon 2020 MSCA RISE project 731143. The majority of this work was undertaken while the author was visiting KAIST, Daejeon, Republic of Korea. The author would like to thank Martin Ziegler for bringing this problem to his attention.

**References**

[1] H. Alt. The Computational Geometry of Comparing Shapes. In S. Albers, H. Alt, and S. Näher, editors, *Efficient Algorithms*, volume 5760 of *Lecture Notes in Computer Science*, pages 235–248. Springer, Berlin, Heidelberg, 2009.

[2] H. Alt and M. Buchin. Can we compute the similarity between surfaces? *Discrete & Computational Geometry*, 43(1):78–99, 2010.
[3] H. Alt and M. Godau. Computing the Fréchet distance between two polygonal curves. *Internat. J. Comput. Geom. Appl.*, 5:75–91, 1995.

[4] E. Bishop and D. Bridges. *Constructive Analysis*. Springer-Verlag, 1985.

[5] V. Brattka, S. Le Roux, J. S. Miller, and A. Pauly. The Brouwer Fixed Point Theorem Revisited. In A. Beckmann, L. Bienvenu, and N. Jonoska, editors, *Pursuit of the Universal: 12th Conference on Computability in Europe*, volume 9709 of *Lecture Notes in Computer Science*, pages 58–67. Springer, Cham, 2016.

[6] V. Brattka, S. Le Roux, J. S. Miller, and A. Pauly. Connected Choice and the Brouwer Fixed Point Theorem. *arXiv:1206.4809v2*, 2016.

[7] V. Brattka, S. Le Roux, and A. Pauly. On the Computational Content of the Brouwer Fixed Point Theorem. In S. B. Cooper, A. Dawar, and B. Löwe, editors, *How the World Computes: Turing Centenary Conference and 8th Conference on Computability in Europe*, volume 9712 of *Lecture Notes in Computer Science*, pages 56–67. Springer, Berlin, Heidelberg, 2016.

[8] V. Brattka and G. Presser. Computability on subsets of metric spaces. *Theoretical Computer Science*, 305(1–3):43–76, 2003.

[9] M. Buchin. *On the Computability of the Fréchet Distance Between Triangulated Surfaces*. PhD thesis, Freie Universität Berlin, 2007.

[10] P. Collins. Computability and Representations of the Zero Set. *Electronic Notes in Theoretical Computer Science*, 221:37–43, 2008.

[11] M. Escardó. Synthetic topology of data types and classical spaces. *Electronic Notes in Theoretical Computer Science*, 87:150pp., 2004.

[12] M. Fréchet. Sur quelques points du calcul fonctionnel. *Rendiconti Circ. Mat. Palermo*, 22:1–74, 1906.

[13] M. Fréchet. Sur la distance de deux surfaces. *Ann. Soc. Polonaise Math.*, 3:4–19, 1924.

[14] M. Godau. On the complexity of measuring the similarity between geometric objects in higher dimensions. PhD thesis, Freie Universität Berlin, 1998.

[15] V. Kreinovich. *Categories of space-time models*. PhD thesis, Soviet Academy of Sciences, Novosibirsk, 1979. (Russian).

[16] J. Miller. $\Pi^0_1$-classes in computable analysis and topology. PhD thesis, Cornell University, 2002.

[17] J. Milnor. *Topology from the Differentiable viewpoint*. University Press of Virginia, 1965.

[18] C. Park, J.-W. Park, S. Park, D. Seon, and M. Ziegler. Computable Operations on Compact Subsets of Metric Spaces with Applications to Fréchet Distance and Shape Optimization. *arXiv:1701.08402*, 2017.

[19] A. Pauly. On the topological aspects of the theory of represented spaces. *Computability*, 5(2):159–180, 2016.

[20] M. B. Pour-El and J. I. Richards. *Computability in Analysis and Physics*. Springer, 1989.

[21] M. Schröder. *Admissible Representations for Continuous Computations*. PhD thesis, FernUniversität Hagen, 2002.

[22] M. Schröder. Extended admissibility. *Theoretical Computer Science*, 284:519–538, 2002.
[23] G. Teschl. Topics in real and functional analysis. Available online at https://www.mat.univie.ac.at/gerald/ftp/book-fa/fa.pdf - retrieved 1st September 2017, Version: July 27, 2017.

[24] A. M. Turing. On Computable Numbers, with an Application to the Entscheidungsproblem. *Proceedings of the London Mathematical Society*, 42:230–265, 1936.

[25] A. M. Turing. On Computable Numbers, with an Application to the Entscheidungsproblem: A correction. *Proceedings of the London Mathematical Society*, 2(43):544–546, 1937.

[26] K. Weihrauch. *Computable Analysis*. Springer, 2000.