All Sampling Methods Produce Outliers
Samuel Epstein

Abstract—Given a computable probability measure \( P \) over natural numbers or infinite binary sequences, there is no computable, randomized method that can produce an arbitrarily large sample such that none of its members are outliers of \( P \). In addition, given a binary predicate \( \gamma \), the length of the smallest program that computes a complete extension of \( \gamma \) is less than the size of the domain of \( \gamma \) plus the amount of information that \( \gamma \) has with the halting sequence.

Index Terms—Kolmogorov Complexity, Statistics.

I. INTRODUCTION

An outlier is a data point that varies noticeably from other data points in a sample or collection. There is no exact mathematical definition of what constitutes an outlier. Though there are known partial indicators, the determination of an outlier remains a subjective endeavor.

Outliers can have many causes, such as due to variability in system performance, human mistakes, instrument malfunctions, contamination from elements outside the population or by inherent standard deviations in populations.

In algorithmic information theory, outliers are precisely defined algorithmically with respect to computable probability measures over either natural numbers or infinite sequences. The probability measure represents the model, and natural numbers and infinite sequences are assumed to be data points with respect to these models. The level or score to which a data point is an outlier to a model (probability measure) is given by the deficiency of randomness function. It is defined by \( d(x|P) = -\log P(x) - K(x|P) \), where \( x \) is the data point and \( P \) is the probability measure. The term \( K \) is the Kolmogorov complexity of a string, formally defined in Section III. \( d(x|P) \) is the difference between length a string’s P-code and its optimal description. If \( x \) is not in the support of \( P \), then \( d(x|P) = \infty \). The function \( d \) is optimal, in the following manner.

Given a computable probability measure \( P \) over \( \mathbb{N} \), an expectation bounded test is a function \( d : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0} \) that is lower semi-computable and

\[
\sum_{x \in \mathbb{N}} P(x) 2^{d(x)} \leq 1.
\]

Lower semi-computability is formally defined in Section III. Typical numbers \( x \) of \( P \) will have a low test score. An expectation bounded test \( d \) is universal if for every expectation bounded test \( d' \), there is a \( c_{d'} \in \mathbb{N} \), such that for all \( x \in \mathbb{N} \),

\[
d(x) + c_{d'} > d'(x).
\]

It can be shown that the deficiency of randomness, \( d \), is a universal expectation test, in that there is a constant \( c \in \mathbb{N} \), where for any expectation bounded test \( d \), for any \( x \in \mathbb{N} \),

\[
d(x|P) + K(d|P) > d(x) - c.
\]

In this paper we show that all sampling methods produce outliers and provide a lower bound on the rate in which they occur. A sampling algorithm \( A \) to a semi-measure \( P \) is a computable function that takes as input a parameter \( n \in \mathbb{N} \) and a random source of bits, and outputs, with probability one, an encoding of \( 2^n \) unique natural numbers.

Note that if \( P \) is a computable probability measure on \( \mathbb{N} \), then for each \( n \in \mathbb{N} \), there is only a finite number of \( x \in \mathbb{N} \) where \( d(x|P) \leq n \). This is because there is an algorithm that on input \( k \in \mathbb{N} \), can enumerate a list \( M \) of numbers \( x \) by order of \( P(x) \) convergence time and stop when total \( P \) mass of \( M \) is \( > 1 - 2^{-k} \). \( M \) is a finite set. Each \( y \) in the support of \( P \) and not in \( M \) can be identified by a Shannon-Fano code of size \( K(y|P) < + \log P(y) - k + K(k) \) and thus has a deficiency of randomness \( > k - K(k) \). Thus any sampling method \( A \) to a computable probability measure will, with increasing \( n \) as input, produce samples containing members with increasing outlier scores,

\[
\omega(1) < \max_{a \in A(n)} d(a|P).
\]

For semi-measures in general, this bound is not necessarily guaranteed. For example the universal semi-computable semi-measure \( m \), defined in Section III, has no outliers by definition. In this paper we improve the bounds of the above equation to a logarithmic scale, and prove the property holds for computable semi-measures.

Corollary. For computable semi-measure \( P \) over \( \mathbb{N} \), for sampling method \( A \), there is a constant \( c_{P,A} \in \mathbb{N} \), such that for all \( n, k \in \mathbb{N} \),

\[
\Pr(n - \max_{a \in A(n)} d(a|P) > k) < 2^{-k + O(K(k,n)) + c_{P,A}}.
\]

To achieve this result, we show that all sufficiently large sets will either have an outlier or high mutual information with the halting sequence.

Theorem. Relativized to computable semi-measure \( P \) over \( \mathbb{N} \), for any finite set \( D \subset \mathbb{N} \),

\[
s = [\log \sum_{a \in D} m(a)/P(a)] < \log \max_{a \in D} m(a)/P(a) + I(D : \mathcal{H}) + O(K(s) + K(I(D : \mathcal{H})))
\]

The term \( I(D : \mathcal{H}) = K(D) - K(D|\mathcal{H}) \) is the mutual information that \( D \) has with the halting sequence. There is no computable method to produce sets \( D \) that have arbitrary high mutual information with the halting sequence. We use this property to derive impossibility results of sampling methods.

We also prove the same bounds with sampling methods over infinite sequences. The deficiency of randomness of an infinite sequence \( \alpha \in \{0,1\}^{\infty} \) with respect to a computable probability measure \( P \) over infinite sequences

\[
\omega(1) < \max_{a \in A(n)} d(a|P).
\]
is \( D(\alpha | P) = \max_{\alpha} - \log P(\alpha | 1, n) - K(\alpha | 1, n) | P) \). If \( \alpha \) is not in the support of \( P \), then \( D(\alpha | P) = \infty \). \( D \) is universal over integral tests (see Section IX). A continuous sampling method \( A \) to a probability measure \( P \) takes in a parameter \( n \) and an infinite source of random bits and outputs \( 2^n \) unique infinite sequences, encoded in the form \( \alpha_1[1] \alpha_2[1] \ldots \alpha_{2^n}[1] \alpha_1[2] \alpha_2[2] \ldots \alpha_{2^n}[2] \ldots \). We get the following sampling corollary which is analogous to the discrete case.

**Corollary.** For computable measure \( P \) over \( \{0,1\}^\infty \), for continuous sampling method \( A \), there is a constant \( c_{P,A} \in \mathbb{N} \), such that for all \( n,k \in \mathbb{N} \),
\[
\Pr(n-\max_{\alpha \in A(n)} D(\alpha | P) > k) < 2^{-k + O(\log k + K(n)) + c_{P,A}},
\]

This theorem was derived similarly to the discrete case, by first showing that large sets of infinite sequences with low \( D \) scores have high information with the halting sequence. The information term \( I \) over infinite sequences used in this paper was introduced in [Lev74]. The continuous sampling no-go corollary is derived from the following theorem, similarly to the discrete case. The term \( \mathcal{Z} \) is defined in Section III.

**Theorem.** Relativized to computable probability measure \( P \) over \( \{0,1\}^\infty \), for any \( Z \subseteq \{0,1\}^\infty \), if \( N \supseteq s < \log \sum_{n \in Z} 2^{D(\alpha | P)} \), then \( s < \sup_{\alpha \in \mathcal{Z}} D(\alpha | P) + I((Z) : \mathcal{H}) + O(K(s) + \log I((Z) : \mathcal{H})). \)

### A. Binary Predicates

In this paper, we also prove upper bounds on the size of the smallest program that computes a complete extension of a given binary predicate \( \gamma \). We prove that for non-exotic predicates, this size is not more than the number of elements of \( \gamma \). Exotic predicates have high mutual information with the halting sequence, and thus no algorithm can generate such predicates.

More formally, a binary predicate is defined to be a function of the form \( f : D \to \{0,1\} \), where \( D \subseteq \mathbb{N} \). We say that binary predicate \( \lambda \) is an extension of \( \gamma \), if for all \( i \in \text{Dom}(\gamma) \), \( \gamma(i) = \lambda(i) \). If a binary predicate has a domain of \( \mathbb{N} \) and is an extension of binary predicate \( \gamma \), then we say it is a complete extension of \( \gamma \). In this paper we prove the following result.

**Theorem.** For binary predicate \( \gamma \) and the set \( \Gamma \) of complete extensions of \( \gamma \),
\[
\min_{\gamma \in \Gamma} K(\gamma) < \log |\text{Dom}(\gamma)| + I(\gamma : \mathcal{H}).
\]

### II. Related Work

The study of Kolmogorov complexity originated from the work of [Kol65]. The canonical self-delimiting form of Kolmogorov complexity was introduced in [ZL70] and treated later in [Cha75]. The universal probability \( m \) was introduced in [Sol64]. More information about the history of the concepts used in this paper can be found the textbook [LV08].

Information conservation laws were introduced and studied in [Lev74], [Lev84]. Information asymmetry and the complexity of complexity were studied in [G75]. A history of the origin of the mutual information of a string with the halting sequence can be found in [VV04b].

The notion of the deficiency of randomness with respect to a measure follows from the work of [She83], and also studied in [KU87], [VY87], [She99]. At a Tallinn conference in 1973, Kolmogorov formulated the notion of a two part code and introduced the structure function (see [VV04b] for more details). Related aspects involving stochastic objects were studied in [She83], [She99], [VY87], [VY99].

The combination of complexity with distortion balls can be seen in [FLV06]. The work of Kolmogorov and the modeling of individual strings using a two-part code was expanded upon in [VV04b], [GTV01]. These works introduced the notion of using the prefix of a “border” sequence to define a universal algorithmic sufficient statistic of strings. The generalization and synthesis of the work and the development of algorithmic rate distortion theory can be seen in the works of [VV04a], [VV10]. More information on algorithmic statistics can be found in [VS17], [SV15].

The outlier theorem is an extension to the “Sets Have Simple Members” theorem, first appearing in [EL11]. This theorem was derived from the work in [EB11], which introduced a variant of Theorem 6 in [VV04a]. The first game theoretic proof to the “Sets Have Simple Members” theorem can be found in [She12].

The formulas in this paper involving information with the halting sequence are compatible with the Independence Postulate, detailed in [Lev84], [Lev13]. The Independence Postulate is a generalization of the Church-Turing thesis.

### III. Conventions

We use \( \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \Sigma, \Sigma^* \), and \( \Sigma^\infty \) to represent natural numbers, integers, rational numbers, reals, bits, finite strings, and infinite strings. Let \( X_{\geq 0} \) and \( X_{> 0} \) be the sets of non-negative and of positive elements of \( X \). The length of a string \( x \in \Sigma^n \) is denoted by \( |x| = n \). The removal of the last bit of a string is denoted by \( (p0^-)^n = (p1^-)^n \), for \( p \in \Sigma^* \). For the empty string \( \emptyset \), \( (\emptyset^-) \) is undefined. We use \( \Sigma^\infty \) to denote \( \Sigma^* \cup \Sigma^\infty \), the set of finite and infinite strings. For \( x \in \Sigma^\infty \), \( y \in \Sigma^\infty \), we say \( x \sqsubseteq y \) if \( x = y \) or \( x \in \Sigma^* \) and \( y = xx \) for some \( z \in \Sigma^* \). Also \( x \sqsubset y \) if \( x \sqsubseteq y \) and \( x \neq y \). The \( i \)th bit of a string \( x \in \Sigma^\infty \) is denoted by \( x[i] \). The first \( n \) bits of a string \( x \) in \( \Sigma^\infty \) is denoted by \( x[0..n] \). The indicator function of a mathematical statement \( A \) is denoted by \( [A] \), where if \( A \) is true then \( [A] = 1 \), otherwise \( [A] = 0 \). The size of a finite set \( S \) is denoted to be \( |S| \). We use \( \langle x \rangle \) to represent a self delimiting code for \( x \in \Sigma^* \), such as \( 1||x||0x \). The self delimiting code for a finite set of strings \( \{a_1, \ldots, a_n\} \) is \( \langle \{a_1, \ldots, a_n\} \rangle = \langle n \rangle |\langle a_1 \rangle | \langle a_2 \rangle | \ldots | \langle a_n \rangle \rangle \). For two infinite strings \( \alpha \) and \( \beta \), \( \langle \alpha, \beta \rangle = \alpha_1 \beta_1 \alpha_2 \beta_2 \ldots \). For sets \( Z \) of infinite strings, \( Z_{\leq n} = \{ a[0..n] : a \in Z \} \) and \( Z = \langle Z_{\leq 1} \rangle | \langle Z_{\leq 2} \rangle | \langle Z_{\leq 3} \rangle | \ldots \).

As is typical of the field of algorithmic information theory, the theorems in this paper are relative to a fixed universal machine, and therefore their statements are only relative up to additive and logarithmic precision. For positive real functions \( f \) the terms \( <+f, >+f, =+f \) represent \( <f + O(1), >f - O(1), \)
and \( f = O(1) \), respectively. In addition \( <f, g> \) denote \( <f/O(1), g/O(1)> \). The terms \( <f, g> \) denote \( <f, g> \) for nonnegative real function \( f \), the terms \( \log f \), \( \log f \) represent the terms \( <f+O(\log(f+1)), f-O(\log(f+1)) \), and \( =O(\log(f+1)) \), respectively. A discrete measure is a nonnegative function \( Q : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0} \) over natural numbers. The support of a measure \( Q \) is the set of all elements whose \( Q \) value is positive, with \( \text{Supp}(Q) = \{a : Q(a) > 0\} \). A measure is elementary if its support is finite and its range is a subset of \( \mathbb{Q} \). We say \( Q \) is a semi-measure if \( \sum a Q(a) \leq 1 \). We say that \( Q \) is probability measure if \( \sum a Q(a) = 1 \).

\( T_y(x) \) is the output of algorithm \( T \) (or \( \bot \) if it does not halt) on input \( x \in \Sigma^* \) and auxiliary input \( y \in \Sigma^{\leq \infty} \). \( T \) is prefix-free if for all \( x, s \in \Sigma^* \) with \( s \neq \emptyset \), and \( y \in \Sigma^{\leq \infty} \), either \( T_y(x) = \bot \) or \( T_y(xs) = \bot \). The complexity of \( x \in \Sigma^* \) with respect to \( T_y \) is \( K_T(x\|y) = \min\{\|p\| : T_y(p) = x\} \).

There exists optimal for \( K \) prefix-free algorithm \( U \), meaning that for all prefix-free algorithms \( T \), there exists \( c_T \in \mathbb{N} \), where \( K_T(x\|y) \leq K_U(x\|y) + c_T \) for all \( x \in \Sigma^* \) and \( y \in \Sigma^{\leq \infty} \). For example, one can take a universal prefix-free algorithm \( U \), where for each prefix-free algorithm \( T \), there exists \( t \in \Sigma^* \), with \( U_y(tx) = T_y(x) \) for all \( x \in \Sigma^* \) and \( y \in \Sigma^{\leq \infty} \). The function \( K(x\|y) \), defined to be \( K_U(x\|y) \), is the Kolmogorov complexity of \( x \in \Sigma^* \) relative to \( y \in \Sigma^{\leq \infty} \). When we say that a universal Turing machine is relativized to an object, this means that an encoding of the object is provided to the universal Turing machine on an auxiliary tape.

A function \( f : \mathbb{N} \rightarrow \mathbb{R} \) is computable if there is a total recursive function \( g(x, n) \) over all \( x \in \mathbb{N} \) and \( n \in \mathbb{N} \) where \( |f(x) - g(x, n)| < 1/n \). The complexity of such a computable function \( f \), is \( K(f) \), the minimal length of a \( U \)-program to compute \( f \). A function \( f : \mathbb{N} \rightarrow \mathbb{R} \) is lower semi-computable if the set \( S = \{(x, r) : x \in \mathbb{N}, r \in \mathbb{Q}, r < f(x)\} \) is recursively enumerable. If \( f \) is not computable but lower semi-computable, then its complexity \( K(f) \) is equal to the size of smallest \( U \)-program that on input \( x \), enumerates \( \{r : f(x) > r\} \).

The chain rule for Kolmogorov complexity is \( K(x, y) = K(x) + K(y\|x, K(x)) \). The mutual information in finite strings \( x \) and \( y \) relative to \( z \in \Sigma^* \) is \( I(x : y | z) = K(x|z) + K(y|z) - K((x, y)|z) = K(x|z) - K(x|y, K(y|z), z) \).

The universal probability of a number \( a \in \mathbb{N} \) is \( m(a) = \sum_z [U_y(z) = a] 2^{-\|z\|} \). The coding theorem states \(-\log m(a) = K(a)\). The halting sequence \( H \in \Sigma^\infty \) is the infinite string where \( H[i] = [U(i) \neq \bot] \) for all \( i \in \mathbb{N} \). As mentioned in the introduction, the amount of information that \( a \in \mathbb{N} \) has with \( H \) is denoted by \( I(a : H) = K(a) - K(a|H) \).

IV. ALGORITHMIC STATISTICS

Algorithmic Statistics is the study of the separation of information, i.e. a string \( x \in \Sigma^* \), into two parts. The first part is the model containing the “denoised” information of \( x \). The second part is the data-to-model code representing the remaining randomness in \( x \). The algorithmic statistics that we use in this paper are computable semi-measures \( P \) which have \( x \) in their support. Other models studied in the literature are finite sets of numbers and total recursive functions. For semi-measures, the model is an encoding or Turing number of an algorithm that computes \( P \). The data-to-model code is the Shannon Fano encoding of length \( + \log P(x) \) of \( x \) with respect to \( P \). If \( x \) is typical of a model then it has a low deficiency of randomness \( d(x|P) = -\log P(x) - K(x|P) \).

The field of algorithmic statistics studies properties of algorithmic sufficient statistics, i.e. statistics whose sum of the model complexity and data-to-model code length is equal (up to a small error term) to \( K(x) \). For probability distributions, these are such \( P \) where \( K(P) - \log P(x) \approx K(x) \). A minimal sufficient statistic is an algorithmic sufficient statistic with the smallest model complexity, i.e. one that minimizes \( K(P) \). According to Occam’s razor, out of all the algorithmic sufficient statistics, the minimal ones summarize the relevant information of \( x \) in the most concise manner.

This paper is connected to algorithmic statistics in two ways. First, the main theorem is a result about deficiencies of randomness, \( d \). The deficiency function \( d \) and its relation to models are one of the central areas of study in algorithmic statistics. Second, Lemma 2 is a statement about the stochasticity measure of a finite set of strings. The stochasticity term is related to those used in algorithmic statistics in that it measures whether a string is typical of a simple probability measure. The extended deficiency of randomness \( d(x) \) with respect to elementary measure \( Q \) and \( v \in \mathbb{N} \) is \( d(x|Q, v) = [-\log Q(x) - K(x|Q, v)] \). The stochasticity of \( a \in \mathbb{N} \), conditional to \( b \in \mathbb{N} \), is measured by

**Definition 1 (Stochasticity):**

\( \Lambda(a|b) = \min\{K(Q|b) + 3\log \max\{d(a|Q, b), 1\} : Q \text{ is an elementary probability measure}\} \)

We have \( \Lambda(a) = \Lambda(a|0) \), with \( \Lambda(a|b) < + \Lambda(a) + K(b) \). Thus if \( a \) has low \( \Lambda(a) \), then it is typical for a simple probability measure. Stochasticity is an important area of research because the stochasticity measure of an elementary object lower bounds the amount of information that the object has with the halting sequence, as shown in Section VII. Objects with high mutual information with the halting sequence are exotic in that there is no (randomized) method to produce them, due to information nongrowth laws. Thus the study of stochasticity yields insight into the properties of objects that can and cannot be produced by algorithms.

V. GAMES

In this section we introduce a generalization to the so-called “Epstein-Levin” game, introduced in [She12]. This new generalized game consists of a finite bipartite graph \( E \subseteq L \times R \), with \( L \subseteq \mathbb{N} \) and \( R \subseteq \mathbb{N} \). There is a computable probability distribution \( P \) over the right vertices. The game is between Alice and Bob and is defined by four additional parameters.

1) An integer \( k \).
2) A positive rational \( l \).
3) A positive rational number \( \delta \).
4) A computable function \( W : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0} \).
The rules of the game are as follows. Alice assigns increasing rational nonnegative weights to vertices on L, which are all initially 0. The sum $\sum_{a \in L} W(a) \cdot \text{weight}(a)$ cannot exceed 1. After each turn by Alice, Bob can mark vertices on L and R. Once a vertex is marked, it will stay marked. There are restrictions on how Bob can mark the left and right vertices. The sum $W(a)$ over all marked left vertices cannot exceed $l$. Furthermore, the total $P$-probability of marked vertices on the right is at most $\delta$.

Bob wins if every R vertex whose combined weight of its L-neighbors is equal to or greater than $2^{-k}$ either has a marked neighbor or is marked itself. Note that this is a generalization of the “Epstein-Levin” game in [She12], whose instantiation is equivalent to setting $W(a) = 1$ for all $a \in N$.

**Lemma 1:** For $l = O(2^k \log(1/\delta))$, Bob has a computable winning strategy.

Note that the game can be made finite by making the weights restricted to the form $2^{-m}$ for $m \in \mathbb{Z}$. Since this new game changes the weights by a factor of at most 2, Bob can compensate by changing $k$ by 1. In addition, the minimal weight is changed to be an $m \in \mathbb{Z}$ where $2^{-m} \max_{a \in L} W(a)/|L| < 1$ so the sum $\sum_{a \in L} W(a) \cdot \text{weight}(a) \leq 2$, which is a constant factor. Thus this game is a finite game with full information so either Alice or Bob has a winning strategy. We prove that Bob has a probabilistic strategy that has a non-zero chance of winning. Thus Alice can’t have a winning strategy, otherwise Bob’s strategy would succeed with probability 0. Bob’s simple probabilistic strategy is unchanged from that in [She12]:

- If Alice increases the weight on a vertex $a \in L$, by some value $\epsilon \in (0, 1)$, then Bob marks that vertex with probability $c2^k \epsilon$, where $c > 1$ is a constant to be chosen later. If $c2^k \epsilon > 1$, then Bob marks the vertex.
- If a vertex on R has neighbors in L with total weight not less than $2^{-k}$ but no marked neighbors, then Bob immediately marks this vertex.

To prove that Bob has a non-zero chance of succeeding, we prove the following two events each have probability less than 1/2.

1) The total $P$-measure of marked R-vertices exceeds $\delta$.

2) The sum of $W(a)$ over all marked left vertices $a \in L$ is more than $l$.

For (1), for each $y \in R$, with left neighbors with weights increasing $\epsilon_1, \ldots, \epsilon_m$, with $\sum \epsilon_i \geq 2^{-k}$, the probability that all its neighbors are unmarked is not more than

$$(1 - c2^k \epsilon_1) \cdots (1 - c2^k \epsilon_m) \leq e^{-c2^k(\epsilon_1 + \cdots + \epsilon_m)} \leq e^{-c}.$$

For every $P$-measure, the expected $P$-measure of marked vertices in R does not exceed $e^{-c}$. For (1) to be less than 1/2, it suffices for $c = \ln(1/\delta) + O(1)$.

For (2), the requirement that $\sum_{a \in L} W(a) \cdot \text{weight}(a) \leq 1$ guarantees the following bound on the expectation \[ E\left[ \sum_{a \in L} \{W(a) : a \text{ is a left marked vertex}\} \right] \leq \sum_{a \in L} W(a) \cdot \text{weight}(a)c2^k \leq c2^k. \]

Thus (2) is satisfied for $l = c2^{k+2} = O(2^k \log(1/\delta))$, thus proving the lemma.

### A. Stochasticity

The above game can be applied to the following statement about the stochasticity of finite sets of natural numbers.

**Lemma 2:** Let $\eta : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ be a lower semi-computable function, $W : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ be a computable function with $\sum_{a \in \mathbb{N}} W(a)\eta(a) \leq 1$. Then for every finite $D \subset \mathbb{N}$ with $\log \sum_{a \in D} \eta(a) \geq s \in \mathbb{Z}$ there is $a \in D$ with $K(a) <^{\text{cd}} - \log W(a) - s + \Lambda(D) + 2K(s)$. Note the above is true relative to any oracle $\alpha$.

**Proof.** Let $Q$ be any elementary probability measure witnessing $\Lambda(\langle D \rangle|s)$. The randomness deficiency of $\langle D \rangle$ with respect to $Q$, conditional to $s$, is $d = \max\{d(\langle D \rangle|Q,s),1\}$. From $Q$ we create the following generalized Epstein-Levin game. The bipartite graph $E \subseteq L \times R$ is created by having R be the encoded sets $\langle G \rangle$ in the support of $Q$. The combined members of encoded sets in $R$ are set to $L$ and there is a connection between a vertex $a \in L$ and an encoded set $\langle G \rangle \in R$, if and only if $a \in G$. Alice approximates the weights $\eta$ from below. At each round, Alice increases the weight of a vertex in $L$ by the amount specified in the corresponding round of the lower enumeration of $\eta$. We set the parameters $k = -s$ and $\delta = 2^{-cd}$, for a constant $c \in \mathbb{N}$ solely dependent on the universal Turing machine to be determined later. The elementary probability is $P = Q$. By Lemma 1, Bob has a winning strategy where the sum of all $W(a)$ over left vertices marked by Bob is at most

$$l = O(2^k \log(1/\delta)) = O(cd2^{-s}).$$

The right vertex $\langle D \rangle$ is not marked. Otherwise, since the $Q$ measure of vertices that are marked is not more than $2^{-cd}$, and right vertices are marked during the course of the game, the function $Q' = \langle D \cdot 2^{cd} \rangle$ restricted to marked right vertices is a lower semi-computable semi-measure. This semi-measure can be lower computed using $Q$, $d$, $s$, and $c$. Hence the $Q'$ code of $D$ would have the size $=^{+} \log(Q(\langle D \rangle)2^{cd})$. Thus the following contradiction occurs for large enough $c \in \mathbb{N}$ dependent solely on the universal Turing machine $U$,

$$K(\langle D \rangle|\langle Q \rangle, d, s, c) <^{+} - \log Q(\langle D \rangle) - cd$$

$$cd <^{+} - \log Q(\langle D \rangle) - K(\langle D \rangle)|\langle Q \rangle, s) + K(c, d)$$

$$cd <^{+} d + K(c, d).$$

Therefore, since $\langle D \rangle$ is not marked, and since $\sum_{a \in D} \eta(a) \geq 2^s = 2^{-k}$, by the rules of the game, $D$ has a marked $a \in L$. The semi-measure $p(a) = W(a)/l$ for Bob’s marked $L$
vertices is lower semi-computable relative to $Q$, $s$, and $d$, so
\[
K(a|Q, s, d)
\]
\[
<^+ - \log P(a)
\]
\[
<^+ - \log W(a) + \log l
\]
\[
<^+ - \log W(a) - s + \log d
\]
\[
K(a) <^+ - \log W(a) - s + K(d) + \log d + K(Q|s) + K(s)
\]
\[
K(a) <^+ - \log W(a) - s + \Lambda(D)|s) + K(s)
\]
\[
K(a) <^+ - \log W(a) - s + \Lambda(D)) + 2K(s).
\]
\end{proof}

\section{B. Stochastic Sets}

The above lemma can be applied to the following result showing that large sets of numbers with low randomness deficiencies are exotic.

\textbf{Theorem 1:} Relativized to computable semi-measure $P$ over $\mathbb{N}$, for any finite set $D \subseteq \mathbb{N}$, if $\exists s < \log \sum_{a \in D} m(a)/P(a)$, then $s <^+ \log \max_{a \in D} m(a)/P(a) + \Lambda(D) + 2K(s)$.

\textbf{Proof.} We invoke Lemma 2. $W(a)$ is set to $P(a)$. $\eta(a)$ is set to $m(a)/P(a)$ and is thus lower semi-computable. In addition $\sum_{a \in D} W(a)\eta(a) \leq 1$ and $\sum_{a \in D} \eta(a) \geq 2^n$. The lemma produces an $a \in \mathbb{N}$ such that $K(a) <^+ - \log P(a) - s + \Lambda(D) + 2K(s)$. Some reworking proves the theorem.

\textbf{Corollary 1:} Relativized to computable semi-measure $P$ over $\mathbb{N}$, for any finite set $D \subseteq \mathbb{N}$, if $\exists s < \log |D|$, then $s <^+ \log \max_{a \in D} m(a)/P(a) + \Lambda(D) + 2K(s)$.

\section{VI. Helper Lemmas}

The following elementary lemmas are used, in a helper capacity, throughout the paper. The terminology $O(f)$ for some function $f: \mathbb{N} \rightarrow \mathbb{N}$ signifies Big Oh notation of $f$ with the parameters solely dependent on the choice of the universal Turing machine $U$. This holds also for the $f <^+ g$ inequality, which is equal to $f < g + O(1)$, for functions $f, g$ between $\mathbb{N}$.

\textbf{Lemma 3:} For every $c, n \in \mathbb{N}$ there exists $c' \in \mathbb{N}$ where if $x < y + c$ for some $x, y \in \mathbb{N}$ then $x + nK(x) < y + nK(y) + c'$.

\textbf{Proof.} $K(x) <^+ K(y) + K(y - x)$ as $x$ can be computed from $y$ and $(y - x)$. So $nK(x) - nK(y) < nK(y - x) + O(n)$. Assume not, then there exists $x, y, c \in \mathbb{N}$ where $x < y + c$ and $y - x < nK(x) - nK(y) < nK(y - x) + O(n)$, which is a contradiction for $c' = O(n) + 2c + \max_{a \in \mathbb{N}}[2n \log a - a]$.

\textbf{Lemma 4:} For $d, d', n, m \in \mathbb{N}$ there exists $d'' \in \mathbb{N}$ where for any $f, g, h \in \mathbb{N}$, if $f < g + nK(g) + d$ and $g < h + mK(h) + d'$, then $f < h + (m + 2n)K(h) + d''$.

\textbf{Proof.} If $g < h + d'$, then due to Lemma 3 applied to $x = g$, $y = h, n$, and $c = d'$, there exists $c'$ dependent on $d'$ and $n$ where $g + nK(g) < h + nK(h) + c'$ and thus $f < h + nK(h) + d + c'$, proving the lemma. Thus $h + d' \leq g$ and $g - h < mK(h) + d'$, which implies $K(g - h) <^+ 2 \log mK(h) + 2 \log d'$. Therefore $K(g) <^+ K(h) + K(g - h) <^+ K(h) + 2 \log m + 2 \log K(h) + 2 \log d'$. So,
\[
f < g + nK(g) + d
\]
\[
<^+ h + mK(h) + nK(g) + d + d'
\]
\[
<^+ h + mK(h) + nK(h) + K(g - h) + O(1) + d + d'
\]
\[
<^+ h + mK(h) + nK(h) + 2 \log m
\]
\[
+ 2 \log K(h) + 2 \log d' + O(1) + d + d'
\]
\[
< h + (m + 2n)K(h) + O(n \log m) + 2n \log d' + d + d'
\]
\[
< h + (m + 2n)K(h) + d''
\]
\[
\text{where } d'' = O(n \log m) + 2n \log d' + d + d'.
\]

\textbf{Lemma 5:} For every $c, n \in \mathbb{N}$, there exists $c' \in \mathbb{N}$ where for all $x, y \in \mathbb{N}$, if $x < y + n \log x + c$ then $x < y + 2n \log y + c'$.

\textbf{Proof.}
\[
\log x < \log y + \log \log x + \log cn
\]
\[
2 \log x - 2 \log \log x < 2 \log y + 2 \log cn
\]
\[
2 \log x < 2 \log y + 2 \log cn.
\]

Combining with the original inequality
\[
x < y + n \log x + c
\]
\[
x < y + n(2 \log y + 2 \log cn) + c
\]
\[
= y + 2n \log y + c',
\]
\[
\text{where } c' = 2n \log cn + c.
\]

\textbf{Lemma 6:} For every $d \in \mathbb{N}$ there exists $d' \in \mathbb{N}$ where if $x < y + K(x) + d$ then $x < y + 2K(y) + d'$.

\textbf{Proof.} It must be that $y + d < x$, otherwise the lemma is trivially solved. Thus $x - y < K(x) + d$, so $K(x) <^+ 2 \log K(x) + 2 \log d$. So $K(x) <^+ K(y) + K(x - y) <^+ K(y) + 2 \log K(x) + 2 \log d$. By Lemma 5, when $n = 2$ and $c = 2 \log d + O(1)$, there is a $c' \in \mathbb{N}$, where $K(x) < K(y) + 4 \log K(y) + c' < 2K(y) + c' + O(1)$. So
\[
x < y + K(x) + d
\]
\[
< y + 2K(y) + c' + d + O(1)
\]
\[
= y + 2K(y) + d',
\]
\[
\text{where } d' = c' + d + O(1).
\]

\textbf{Lemma 7:} For all $d, m \in \mathbb{N}$ there is a $d' \in \mathbb{N}$ where if $x + mK(x) + d > y$ then $x + d' > y - 2mK(y)$.

\textbf{Proof.} If $x + d > y$, then the lemma is satisfied, so $x + d' \leq y$. Thus $y - x < mK(x) + d$ implies $K(y - x) <^+ 2 \log mK(x) + 2 \log dm$. Thus $K(x) <^+ K(y) + K(y - x) <^+ K(y) + 2 \log K(x) + 2 \log dm$. Applying Lemma 5 where $c = 2 \log dm + O(1)$ and $n = 2$, we get a $c'$ dependent on $c$ and $n$
where \( \mathbf{K}(x) < \mathbf{K}(y) + 4 \log \mathbf{K}(y) + c' < 2\mathbf{K}(y) + c' + O(1) \).
So
\[
\begin{align*}
  x + m\mathbf{K}(x) + d &> y \\
  x + m(2\mathbf{K}(y) + c' + O(1)) + d &> y \\
  x + d' &> y - 2m\mathbf{K}(y),
\end{align*}
\]
where \( d' = m(c' + O(1)) + d \).
\[\square\]

\section{VII. Left-Total Machines}

We say \( x \in \Sigma^* \) is total with respect to a machine if the machine halts on all sufficiently long extensions of \( x \). More formally, \( x \) is total with respect to \( T_y \) for some \( y \in \Sigma^{\infty} \) if there exists a finite prefix free set of strings \( Z \subset \Sigma^* \) where \( \sum_{z \in Z} 2^{-\|z\|} = 1 \) and \( T_y(xz) \neq \bot \) for all \( z \in Z \). We say \( \alpha \in \Sigma^{\infty} \) is to the “left” of \( \beta \in \Sigma^{\infty} \), and use the notation \( \alpha \triangleleft \beta \), if there exists \( x \in \Sigma^* \) such that \( x0 \leq \alpha \) and \( x1 \leq \beta \). A machine \( T \) is left-total if for all auxiliary strings \( \alpha \in \Sigma^{\infty} \) and for all \( x, y \in \Sigma^* \) with \( x < y \), one has that \( T_\alpha(y) \neq \bot \) implies that \( x \) is total with respect to \( T_\alpha \). An example left-total machine can be seen in Figure 1.

Fig. 1: The above diagram represents the domain of a left total machine \( T \) with the 0 bits branching to the left and the 1 bits branching to the right. For \( i \in \{1,\ldots,5\} \), \( x_i \triangleleft x_{i+1} \) and \( x_i < y \). Assuming \( T(y) \) halts, each \( x_i \) is total. This also implies each \( x_i^- \) is total as well.

For the remaining part of this paper, we can and will change the universal self delimiting machine \( U \) into an optimal left-total machine \( U' \) by the following definition. The algorithm \( U' \) enumerates all strings \( p \in \Sigma^* \) in order of their convergence time of \( U(p) \) and successively assigns them consecutive intervals \( i_p \subset [0,1] \) of width \( 2^{-\|p\|} \). Then \( U' \) outputs \( U(p) \) on input \( p' \) if the open interval corresponding to \( p' \) and not that of \( (p')^- \) is strictly contained in \( i_p \). The open interval in \([0,1]\) corresponding with \( p' \) is \( (p')^-2^{-\|p'\|},(p')^+2^{-\|p'\|} \) where \( (p) \) is the value of \( p \) in binary. For example, the value of both strings 011 and 0011 is 3. The value of 0100 is 4. The same definition applies for the machines \( U'_{\alpha} \) and \( U_{\alpha} \), over all \( \alpha \in \Sigma^{\infty} \). We now set \( U \) to equal \( U' \).

Without loss of generality, the complexity terms of this paper are defined with respect to the optimal left total machine \( U \). The infinite border sequence \( \mathcal{B} \in \Sigma^{\infty} \) represents the unique infinite sequence such that all its finite prefixes have total and non total extensions. The term “border” is used because for any string \( x \in \Sigma^* \), \( x \triangleleft \mathcal{B} \) implies that \( x \) total with respect to \( U \) and \( \mathcal{B} \triangleleft x \) implies that \( U \) will never halt when given \( x \) as an initial input. Figure 2 shows the domain of \( U \) with respect to \( \mathcal{B} \).

\subsection{A. Properties of Total Strings}

This section uses the notion of a Martin Löf random infinite sequence. An infinite sequence \( \alpha \in \Sigma^{\infty} \) is Martin Löf random if there is a constant \( c \in \mathbb{N} \) such that for all \( n \in \mathbb{N} \), \( \mathbf{K}(\alpha[0..n]) > n - c \). Let \( \Omega = \sum_{x} m(x) \) be Chaitin’s Omega, the probability that \( U \) will halt. It is well known that the binary expansion of \( \Omega \) is Martin Löf random.

\begin{proposition}
If \( b \in \Sigma^* \) is total and \( b^- \) is not, then \( b^- \triangleleft \mathcal{B} \).
\end{proposition}

\begin{proof}
The border sequence is the binary expansion of Chaitin’s Omega for machine \( U \), because the probability that a random infinite sequence contains a prefix that is a halting program is precisely the probability that the random sequence is at the left of the border sequence. If \( b \in \Sigma^* \) is total and \( b^- \) is not, then \( b^- \) has a total extension \( b^-0 \) and a non total extension \( b^-1 \), thus by the definition of the border sequence, \( b^- \triangleleft \mathcal{B} \).
\end{proof}

\begin{lemma}
If \( b \in \Sigma^* \) is total and \( b^- \) is not, and \( x \in \Sigma^* \), then \( \mathbf{K}(b) + \mathcal{I}(x;\mathcal{H}|b) < \log \mathbf{I}(x;\mathcal{H}) + \mathbf{K}(b|x,\|b\|) \).
\end{lemma}

\begin{proof}
By Proposition 1, \( b^- \triangleleft \mathcal{B} \) is a prefix of the border sequence and thus \( \|b\| < \mathbf{K}(b). \) Since \( \mathcal{B} \) is computable from
the halting sequence $\mathcal{H}$, we have that $b$ is computable from $\|b\|$ and $\mathcal{H}$, with $K(b|\mathcal{H}) \bowtie K(\|b\|)$.

The chain rule gives the equality $K(b) + K(x|b, K(b)) = K(x) + K(b|x, K(x))$. Combined with the inequalities $K(x|b) \bowtie K(x|b, K(b)) + K(b|x, K(x))$, we get

$$K(b) + K(x|b) \bowtie K(x) + K(b|x) + K(K(b)).$$

Subtracting $K(x|b, \mathcal{H})$ from both sides results in

$$K(b) + K(x|b) - K(x|b, \mathcal{H}) \bowtie K(x) + K(b|x) + K(K(b)).$$

Proof. Due to Proposition 1, by the definition of $b$, $b$ is total and $b^*$ is not, so $b^* \sqsubseteq B$ is a prefix of border, and is thus a random string, with $\|b\| \bowtie K(b)$. Due to Lemma 8, with the second term removed,

$$\|b\| \bowtie K(b) \bowtie K(x : \mathcal{H}) + K(b|\|b\|)$$

$$\|b\| \bowtie \log I(x : \mathcal{H}).$$

$\square$

B. Stochasticity and the Halting Sequence

Left-total machines can be used to prove properties of stochasticity. As mentioned earlier, the stochasticity of a string lower bounds the amount of mutual information it has with the halting sequence. The following lemma was first introduced in [EL11].

Lemma 10: For $x \in \mathbb{N}$, $\Lambda(x) \bowtie I(x : \mathcal{H}) + 6K(I(x : \mathcal{H}))$.

Proof. Using the optimal left-total Turing machine, let $U(x^*) = x$, $\|x^*\| = K(x)$, and $v$ be the shortest total prefix of $x^*$. We define the elementary probability measure $Q$ such that $Q(a) = \sum_w 2^{-\|w\|} U(wv) = a$. A graphical depiction of these definitions can be seen in Figure 3. Thus $Q$ is computable relative to $v$. In addition, since $v \sqsubseteq x^*$, one has the lower bound $Q(x) \geq 2^{-\|x^*\|+\|v\|} = 2^{\kappa(x)+\|v\|}$. Therefore

$$\text{d}(x|Q, v) = -\log Q(x) - K(x|Q, v)$$

$$= \kappa(x) - \log Q(x) = K(x|v)$$

$$\bowtie \kappa(x) - \|v\| - K(x|v)$$

$$\bowtie (K(v) + K(x|v)) - \|v\| - K(x|v)$$

$$\bowtie (\|v\| + K(\|v\|) + K(x|v)) - \|v\| - K(x|v)$$

$$\bowtie K(\|v\|).$$

Figure 3: A graphical depiction of the terms used in Lemma 10. The shortest program for $x \in \mathbb{N}$ is $x^* = 0110010$, with $U(x^*) = x$. The shortest total prefix of $x^*$ is $v = 01100$, with $v^- = 0110$ being a prefix of border $B$. Assuming $x^*$ is the only extension of $v$ that is a program for $x$, then $Q(x) = 2^{-\|x^*\|+\|v\|} = 2^{-2}$. Since $v$ is total and $v^*$ is not total, by Proposition 1, $v^*$ is a prefix of the border sequence $B$. In addition, $Q$ is computable from $v$. Therefore

$$K(x|\mathcal{H}) \bowtie K(x|Q) + K(Q|\mathcal{H})$$

$$\bowtie K(x|Q) + K(v|\mathcal{H})$$

$$\bowtie K(x) - \|v\| + K(\|v\|)$$

$$\bowtie K(x) - K(x|\mathcal{H}) + K(\|v\|)$$

$$\|v\| \bowtie I(x : \mathcal{H}) + 2K(I(x : \mathcal{H})).$$

Equation (3) is due to $B$ being computable from $\mathcal{H}$, therefore $v^- \sqsubseteq B$ is simple relative to $\mathcal{H}$ and $\|v\|$. Equation (4) is from Lemma 6. Since $Q$ is computable from $v$, one gets

$$\Lambda(x) \bowtie K(v) + 3 \log \max\{d(x|Q, v), 1\}$$

$$\bowtie \|v\| + K(\|v\|) + 3 \log \max\{d(x|Q, v), 1\}$$

$$\bowtie \|v\| + K(\|v\|) + 3 \log K(\|v\|)$$

$$\bowtie \|v\| + 2K(\|v\|).$$

Applying Lemma 4 to $f = \Lambda(x)$, $g \equiv \|v\|$, and $h \equiv I(x : \mathcal{H})$, with $n = 2$ and $m = 2$, gets $\Lambda(x) \bowtie I(x : \mathcal{H}) + 6K(I(x : \mathcal{H})).$ $\square$

VIII. DISCRETE SAMPLING

Theorem 1 has applications to sampling no-go theorems. In this section, we use this theorem to show that any sampling method will eventually produce outliers. The greater the sample size the greater the outlier score of an element in the sample. We first rework Theorem 1 to be in terms of mutual information with the halting sequence and not stochasticity.

Corollary 2: Relativized to computable semi-measure $P$ over $\mathbb{N}$, for any finite set $D \subset \mathbb{N}$, if $\mathbb{N} \ni s <
log ∑a∈A(n) m(a)/P(a), then s ≺ ∗ log max_a∈A(n) m(a)/P(a) + I(D : H) + 6K(I(D : H)) + 2K(n).

Proof. This follows from the application of Lemma 10 to Theorem 1.

Corollary 3: Relativized to computable semi-measure P over N, for any finite set D ⊆ N, if n < log |D|, n < ∗ log max_a∈D m(a)/P(a) + I(D : H) + 6K(I(D : H)) + 2K(n).

We recall that for a semi-measure P over N, a sampling method A is a total computable function that takes in a parameter n and a random source of bits and outputs, with probability one, an encoding of 2^n unique natural numbers.

Corollary 4: For computable semi-measure P over N, for sampling method A, there is a constant c_P, A ∈ N, where for all n, k ∈ N, Pr(n − log max_a∈A(n) m(a)/P(a) > k) < 2^{−k + O(K(n,k))} + c_P, A.

Proof. Given a fixed n and A, let X = \{x_i\} be the (possibly infinite) prefix free set of finite sequences representing the random seeds that cause A to halt, with for each i, A(n, x_i) = y_i, where y_i ∈ Σ^*. An encoding of 2^n natural numbers.

Let X ⊆ X be the subset of X such that for all x ∈ X, A(n, x) = y, and I(y : H) ≥ c. Let c_A ∈ N be the size of a program that takes in x_i and a program for n and uses A to output y_i. Thus c_A is a constant solely dependent on A and the universal Turing machine U. Let d ∈ R^≥0 be defined by 2^d = sup_x m(x)2^K(x). Then over all c, \sum x \in X_c 2^{−\Vert x\Vert} ≤ 2^{−c + K(n) + c_A + d}. Otherwise there is a c where,

2^d < \sum x \in X_{c_1} 2^{−\Vert x\Vert − K(n) − c_A} 2^c ≤ \sum x \in X_{c_2} 2^{−\Vert x\Vert − K(n) − c_A} 2I(y_i : H) ≤ \sum y_i A(n, x_i) = y, x_i \in X_c, 2^{−\Vert y_i\Vert − K(n) − c_A} 2I(y_i : H) ≤ \sum y_i 2^{d_2 − K(y_i | H)} ≤ 2^{d_2},

causing a contradiction. So for all n,

Pr(n − log max_a∈A(n) m(a)/P(a) > k) < Pr(c_P + I(A(n) : H) + 6K(I(A(n) : H)) + 2K(n) > k)

(5)

< Pr(c_P + I(A(n) : H) > k − 2K(n) − 12K(k − 2K(n)))

< Pr(c_P + I(A(n) : H) > k − 2K(n) − 12K(k − 2K(n)))

< Pr(c_P + I(A(n) : H) > k − 2K(n) − 12K(k − 2K(n)))

Equation 5 comes from the application of Corollary 3. Equation 6 comes from Lemma 7. The term c_P ∈ N is a constant solely dependent on P and the universal Turing machine U. □

IX. INFINITE SEQUENCES

In Section I, the deficiency of randomness, d, of natural numbers was defined. In this section, we define the deficiency of randomness D of infinite sequence. This notion will be used in the no-go sampling theorems over infinite sequences. Before introducing D, we review some standard notions of measures and integration.

A set of subsets of a set X is called an algebra if it is closed under finite intersections and unions and under complements. It is called a σ-algebra if it is closed under countable intersections and unions and under complements.

A nonnegative function µ defined over some subsets of X is monotonic if A ⊆ B implies µ(A) ≤ µ(B). Such a function is additive if whenever µ is defined on disjoint E_1, . . . , E_n, then µ is defined on E = \bigcup_{i=1}^n E_i, and µ(E) = \sum_{i=1}^n µ(E_i).

Before introducing D, we review some standard notions of measures and integration.

A set of subsets of a set X is called an algebra if it is closed under finite intersections and unions and under complements. It is called a σ-algebra if it is closed under countable intersections and unions and under complements.

A nonnegative function µ defined over some subsets of X is monotonic if A ⊆ B implies µ(A) ≤ µ(B). Such a function is additive if whenever µ is defined on disjoint E_1, . . . , E_n, then µ is defined on E = \bigcup_{i=1}^n E_i, and µ(E) = \sum_{i=1}^n µ(E_i).

A pair (X, A) consisting of a set X and a σ-algebra A over X is a measurable space. A measure µ is nonnegative σ-additive function over A. It is a probability measure if µ(X) = 1. The triplet (X, A, µ) is called a measure space.

For this paper, we focus our attention on, Σ^∞, the set of infinite sequences. For a string x ∈ Σ^*, let the set of all infinite strings that start with x, denoted xΣ^∞, is called a cylinder set. For infinite strings, measures can be derived by functions on strings, µ : Σ^∗ → R≥0, where µ(x) = µ(x[0]) + µ(x[1]). Such functions are also referred to as measures. This is because µ can be defined on cylinder sets in the standard way, and then by the Carathéodory’s extension theorem, to all Borel sets B of infinite sequences, which is the smallest σ-algebra containing the cylinder sets. Thus (Σ^∞, B, µ) defines a measure space. Such measures µ are called probability measures if µ(Σ^∞) = 1. A measure µ : Σ^∗ → R≥0 is computable if it computable as defined in Section III.

Another example of a measurable space is (C, k), where C are the Borel sets of C, i.e. the smallest σ-algebra containing the open intervals (a, b) ⊆ C. We say a function f : Σ^∞ → R is measurable if and only if f^{−1}(C) ∈ B whenever C ∈ C. We say f is continuous if for every x ∈ Σ^∞, for every ε > 0, there is a cylinder set Z ⊇ x, such that |f(x) − f(z)| < ε for every z ∈ Z. A function f : Σ^∞ → R is lower semi-continuous if for every r ∈ R, the set \{z ∈ Σ^∞ : f(z) > r\} is open. All lower semi-computable functions are by definition, lower semi-continuous.

A measurable function g is simple if its range is finite: \{a_1, . . . , a_k\}. The (Lebesgue) integral of such g is \int f dµ = \sum_{i=1}^k a_i µ(g^{−1}(a_i)). The integral of a measurable function f, is \int f dµ = sup\{\int f dµ : g ≤ f, g is simple\}.

A function D : Σ^∞ → R≥0 ∪ ∞ is an integrable test with respect to computable probability measure P if it is lower semi-computable and \int_{Σ^∞} 2^{D(α)} P(dα) ≤ 1.

Theorem. (G21) For computable probability measure P over Σ^∞, there exists a universal integrable test D : Σ^∞ → R≥0 ∪ ∞, where for all other integrable tests D,

D(α) < ∗ D(α|P) + K(D|P).

As shown in the following theorem, any such universal integrable test D is equal, up to an additive constant, to a supremum of a term that uses the finite prefix of an infinite
Theorem. ([G21]) For universal integrable test $D$ for computable probability measure $P$ over $\Sigma^\infty$,

$$D(\alpha|P) = \sup_{n \in \mathbb{N}} - \log P(\alpha[0..n]) - K(\alpha[0..n]|P),$$

where the constant depends on $P$.

This justifies the following definition.

Definition 2 (Deficiency of Randomness of an Infinite Sequence): $D(\alpha|P) = \sup_{n \in \mathbb{N}} - \log P(\alpha[0..n]) - K(\alpha[0..n]|P)$. As we look at sampling with respect to infinite sequences, we will need an information function between infinite sequences, and more specifically the amount of information that a specific sequence $\alpha$ has with the halting sequence $H$. We use the symmetric function $I : \Sigma^\infty \times \Sigma^\infty \to \mathbb{R}$, where

$$I(\alpha : \beta|c) = \log \sum_{x,y \in \Sigma^\ast} m(x|c,\alpha)m(y|c,\beta)2^{I(x:y|c)}.$$ 

This function was introduced in [Lev74]. The following theorem was stated in [Lev74], and a proof of it can be found in [Ver21].

Theorem 2: Assume that a family $P_\rho$, $\rho \in \Omega$, of probability distributions on $\Omega$ is fixed. Assume that there is a Turing machine $T$ that for all $\rho$ computes $P_\rho$ having oracle access to $\rho$. Then for all $\alpha, \rho \in \Omega$, there is a probability bounded (and even expectation) $P_\rho$-test $t_{\alpha, \rho, T}$ such that

$$I(\langle \rho, \omega \rangle : \alpha) \leq I(\rho : \alpha) + t_{\alpha, \rho, T}(\omega) + c_T,$$

for all $\omega \in \Omega$, where $c_T$ does not depend on $\rho$, $\alpha$, $\omega$.

In [Gei12], it is shown that the above theorem implies the following.

Theorem 3: Let $(P_\alpha)_{\alpha \in \Sigma^\infty}$ be a family of uniformly $\alpha$-computable continuous probability measures. Then for all $\alpha, \beta \in \Sigma^\infty$ we have

$$P_\alpha(\{ \gamma \in \Sigma^\infty : I(\alpha : \gamma) \geq I(\alpha : \beta) > m \}) \leq 2^{-m+c_{\alpha, \beta}},$$

where $c_{\alpha, \beta}$ is a positive constant dependent solely on $\alpha$ and $\beta$.

In addition [Gei12] contains a short proof for the following theorem.

Theorem 4: For partial recursive $f : \Sigma^\infty \to \Sigma^\infty$, $\alpha, \beta \in \Sigma^\infty$, $I(f(\alpha) : \beta) <^+ I(\alpha : \beta) + K(f)$.

X. CONTINUOUS SAMPLING

This section proves sampling no-go theorems for infinite sequences. Theorem 5 uses the following definitions. We recall that $x \preceq y$ for $x, y \in \Sigma^\ast$ implies that $x$ is a prefix of $y$ or equal to $y$. For a string $x \in \Sigma^\ast$, let $D(x|P) = \max_{y \preceq x}(\log(m(y|P)/P(y)))$. Let $bb(b) = \max\{U(p) : p \ll b, \text{ or } p \supseteq b\}$ be the largest number produced by a program that extends $b$ or is to the left of $b$.

Theorem 5: Relativized to computable probability measure $P$ over $\Sigma^\infty$, for $Z \subseteq \Sigma^\infty$, if $\mathbb{N} \ni s < \log \sum_{x \in Z} 2^{D(\alpha|x|P)}$, then $s < \sup_{\alpha \in Z} D(\alpha|P) + I(Z : H) + O(K(s) + \log I(Z : H)).$

Informal Proof: The proof starts off by determining an $N \in \mathbb{N}$, such that $\sum_{x \in Z \subseteq \Sigma^\infty} 2^{D(\alpha|x|P)} > 2^s$. This is equal to $bb(b)$ for some total string $b$. Then Lemma 2 is invoked with $W(x) = P(x)$, $\eta(x) = [x \in \Sigma^\ast]2^{D(\alpha|x|P)}$, $D = Z \subseteq \mathbb{N}$, relativized to $b$. This produces $x \in D$ where $K(x|b) <^{+} - \log P(x) + \Lambda(D(b)) + O(K(s))$. Using Lemma 10, the $\Lambda(D(b))$ term is replaced with $I(D : H|b)$. The conditioning on $b$ is removed using Lemma 8. Finally the $I(D : H)$ term is replaced with $I(Z : H)$ to achieve the theorem.

Proof.

1. Determination of $N$. For a total $b \in \Sigma^\ast$, let $m_b(x|y) = \{2^{-\|y\|} : U_y(z) = x, U_y(z) \text{ halts in } bb(b) \text{ time}\}$ be the algorithmic weight of $x$ using solely programs that are running in $bb(b)$ time. For $x \in \Sigma^\ast$, let $D_b(x|P) = \max_{y \preceq x}(\log(m_b(y|P)/P(y)))$, with $D_b \leq D$. We set $b$ to be the shortest total string with

1) $N = bb(b)$.
2) $\sum_{z \in Z} 2^{D_b(z|x|P)} > 2^s$.

2. Invocation of Lemma 2. We let $W(x) = P(x)$, $\eta(x) = 2^{D(\alpha|x|P)}[x \in \Sigma^\ast]$, and $D = Z \subseteq \mathbb{N}$. Since the universal Turing machine is relativized to $P$, it must be that

$$\sum_{x \in Z} W(x)\eta(x) = \sum_{x \in \Sigma^\ast} P(x)2^{D(\alpha|x|P)} = \int P(x)2^{D(\alpha|x|P)}dP(\alpha) \leq \int 2^{D(\alpha|x|P)}dP(\alpha) = 1.$$ 

Lemma 2, relativized to $b$, gives $x \in D$ with

$$K(x|b) < - \log P(x) - s + \Lambda(D(b)) + O(K(s)).$$

3. Replace $\Lambda(D(b))$ with $I(D : H|b)$.

Due to Lemma 10,

$$K(x|b) < - \log P(x) - s + I(D : H|b) + O(K(s) + \log I(D : H|b)) + \Lambda(D(b)) + O(K(s) + \log I(D : H|b) + K(b))).$$

4. Remove conditioning of $b$.

By Lemma 8,

$$K(b) + I(D : H|b) <^{+} I(D : H) + K(b|D, \|b\|).$$

Therefore

$$s \leq \log(m_b(x|P(x))) + I(D : H) + K(b|D, \|b\|) + O(K(s) + \log I(D : H) + K(b|D, \|b\|)).$$
Since $D \subseteq \Sigma^{bb(b)}$, $K(b|D, ||b||) = O(1)$, as a program can output the leftmost total string $y$ of length $||b||$ such that $bb(y)$ is the length of the strings in $D$. Hence

$$s \leq \log(m(x)/P(x)) + I(D : \mathcal{H}) + O(K(s) + \log I(D : \mathcal{H})).$$

5. Replace $I(D : \mathcal{H})$ with $I(\{Z : \mathcal{H}\}$.

We have that $K(D|\{Z\}) <^+ K(||b||) + K(s)$, as $D$ is computable from $||b||$, $s$, and $\{Z\}$; thus, so is $D = Z_{\leq bb(b)}$. By Definition 3 of mutual information between infinite sequences,

$$I(D : \mathcal{H}) <^+ I(\{Z : \mathcal{H}\} + K(D|\{Z\})$$

$$<^+ I(\{Z : \mathcal{H}\} + K(||b||) + K(s)$$

$$<^+ I(\{Z : \mathcal{H}\} + 2\log I(D : \mathcal{H}) + K(s)$$

$$<^+ I(\{Z : \mathcal{H}\} + K(s).$$

Where Equation 7 is due to the application of Lemma 9, noting $K(b|D, ||b||) = O(1)$. Equation 8 is due to Lemma 5. So

$$s \leq \log(m(x)/P(x)) + I(D : \mathcal{H}) + O(K(s) + \log I(D : \mathcal{H}))$$

$$\leq \sup_{\alpha \in Z} D(\alpha|P) + I(\{Z : \mathcal{H}\} + O(K(s) + \log I(\{Z : \mathcal{H}\})).$$

Proof. We use $\gamma \sim \mathcal{U}$ to represent infinite sequences distributed according to the uniform distribution.

$$\Pr_{\gamma \sim \mathcal{U}} (n - \max_{\alpha \in A(n, \gamma)} D(\alpha|P) > k)$$

$$< \Pr_{\gamma \sim \mathcal{U}} (c_P + I(A(n, \gamma) : \mathcal{H}) + O(\log I(A(n, \gamma) : \mathcal{H})) < k - O(K(n))$$

$$< \Pr_{\gamma \sim \mathcal{U}} (c_P + I(A(n, \gamma) : \mathcal{H}) > k - O(K(n) + log k))$$

$$< \Pr_{\gamma \sim \mathcal{U}} (I(\gamma : \mathcal{H}) > k - O(K(n) + log k) - c_P - c_A)$$

$$< 2^{-k+O(log k + K(n))} + c_P + c_A.$$ 

Equation 9 comes from Corollary 5, where $c_P \in \mathbb{N}$ is a constant solely dependent on $P$ and the universal Turing machine $U$. Equation 11 comes from the fact that $a < i + O(log i)$ implies that either $a < i$ or then $O(log a) > O(log i)$ and then $a - O(log a) < i$. Equation 12 comes from Theorem 4, where $f = A(n, \gamma)$, with $K(f) = O(K(n))$. Thus $c_A \in \mathbb{N}$ is a constant solely dependent on $A$ and the universal Turing machine $U$. Equation 13 comes from the application of Theorem 3, where $\alpha = 0^\infty$, $\beta = \mathcal{H}$, and $P_\alpha = \mathcal{U}$. □

XI. COMPLETING BINARY PREDICATES

A binary predicate is defined to be a function of the form $f : D \to \Sigma$, where $D \subseteq \mathbb{N}$. We say that binary predicate (or finite string) $\lambda$ is an extension of $\gamma$, if for all $i \in Dom(\gamma)$, $\gamma(i) = \lambda(i)$. If a binary predicate has a domain of $\mathbb{N}$ and is an extension of binary predicate $\gamma$, then we say it is a complete extension of $\gamma$. The self-delimiting code for a binary predicate $\gamma$ with a finite domain is $\langle \{x_1, \lambda(x_1), \ldots, x_n, \lambda(x_n)\} \rangle$. The Kolmogorov complexity of a binary predicate $\lambda$ with an infinite sized domain is $K(\lambda) = K(f)$, where $f : \mathbb{N} \to \mathbb{N}$ is a partial computable function where $f(i) = \lambda(i)$ if $i \in Dom(\lambda)$ and $f(i)$ is undefined otherwise. If there is no such partial computable function, then $K(\lambda) = \infty$.

Theorem 6: For binary predicate $\gamma$ and the set $\Gamma$ of complete extensions of $\gamma$, $\min_{\gamma} K(\gamma) < \log |Dom(\gamma)| + I(\gamma : \mathcal{H}).$

Proof. We recall that $bb(b) = \max \{U(p) : p < b, \ p \not\equiv b\}$ is the largest number produced by a program that extends or is to the left of $b$. The theorem is meaningless if $|Dom(\gamma)| = \infty$, so we can assume $q = |Dom(\gamma)| \leq \infty$. Let $n = \max \{i : i \in Dom(\gamma)\}$. Let $b$ be the shortest total string where $bb(b) \geq n$. Let $b' = bb(b)$. It must be that $K(b(b') | ||b||) = O(1)$ as there is a program that can enumerate, from the left, total strings of length $||b||$. This program returns the first total string $b'$ such that $bb(b') \geq n$. This $b'$ is equal to $b$, otherwise $b' < b$, and thus $bb(b') \geq bb(b) \geq n$, contradicting the definition of $b$. Let $D$ be the set of all strings of length $N$, that extends $\gamma$.

Lemma 2, relative to $b$, with $W(a) = 1$, and $\eta(a) = ||a|| = N|2^{-N}, s = \log \sum_{a \in D} \eta(a) = -q$ results in $a \in D$, with

$$K(a|b) <^+ q + \Lambda(D|b) + 2K(q).$$

Lemma 10 applied to Equation 14, results in

$$K(a|b) < \log q + I(D : \mathcal{H}|b).$$
Since $K(D|b) <^{+} K(\gamma|b)$ and $K(\gamma|b, H) <^{+} K(D|b, H)$,

$$K(a|b) <^{\log} q + I(\gamma : H|b)$$

$$K(a) <^{\log} q + K(b) + I(\gamma : H|b).$$  \hspace{1cm} (15)

Lemma 8, applied to Equation 15, results in

$$K(a) <^{\log} q + I(\gamma : H) + K(b) \langle \gamma \rangle, |b|)$$

$$K(a) <^{\log} |\text{Dom}(\gamma)| + I(\gamma : H).$$  \hspace{1cm} (16)

Thus there exists a complete extension $g' \in \Gamma$, of $\gamma$, that is equal to $a[i]$ for all $i \leq |a||$, and 0 otherwise. This $g'$ can be computed with a program of size $<^{+} K(a)$, thus combined with Equation 16,

$$\min_{g \in \Gamma} K(g) \leq K(g') <^{+} K(a) <^{\log} |\text{Dom}(\gamma)| + I(\gamma : H).$$

\hspace{1cm}\Box

\hspace{1cm} XII. DISCUSSION

One area of progress is to improve the bounds in Corollary 6 to match that of the discrete case. There are several extensions or variants that can be made to the results in this paper. One is to replicate the result on deficiencies of randomness with respect probability measures over general spaces. In [Eps20], a variant to Theorem 1 was used to provide new bounds between different algorithmic quantum entropies, one introduced in [Vit00], and the other in [G01]. By leveraging the work in [Rom03], a conditional complexity alternative to [EL11] can be proven, that shows all natural sets of strings contain members that are simple to all its other members. In general, there are many ways of leveraging stochasticity to reason about combinatorial objects that are created by randomized methods.

\hspace{1cm} REFERENCES

[Cha75] G. J. Chaitin. A Theory of Program Size Formally Identical to Information Theory. Journal of the ACM, 22(3):329–340, 1975.

[EB11] S. Epstein and M. Betke. An Information Theoretic Representation of Agent Dynamics as Set Intersections. In Proceedings of the Fourth Conference on Artificial General Intelligence, volume 6830 of Lecture Notes in Artificial Intelligence, pages 72–81. Springer, 2011.

[EL11] Samuel Epstein and Leonid Levin. On sets of high complexity strings. CoRR, abs/1107.1458, 2011.

[Eps20] Samuel Epstein. An extended coding theorem with application to quantum complexities. Information and Computation, 275, 2020.

[FLV06] L. Fortnow, T. Lee, and N. Vereshchagin. Kolmogorov complexity with error. In Proceedings of the 23rd Annual conference on Theoretical Aspects of Computer Science, pages 137–148, 2006.

[G75] P. Gács. On the Symmetry of Information. Soviet Mathematics Doklady, 15(6):1477–1480, 1975.

[G01] P. Gács. Quantum Algorithmic Entropy. Journal of Physics A Mathematical General, 34(35), 2001.

[G21] Peter Gács. Lecture notes on descriptive complexity and randomness. CoRR, abs/2103.04704, 2021.

[Gei12] Philipp Geiger. Mutual information and Gödel incompleteness. PhD thesis, Heidelberg University, 10 2012.

[GTVO1] P. Gács, J. Tromp, and P. Vitányi. Algorithmic Statistics. IEEE Transactions on Information Theory, 47(6):2443–2463, 2001.

[Kol65] A. N. Kolmogorov. Three approaches to the quantitative definition of information. Problems in Information Transmission, 1:1–7, 1965.

[KU87] A. N. Kolmogorov and V. A. Uspensky. Algorithms and Randomness. SIAM Theory of Probability and Its Applications, 32(3):389–412, 1987.

[Lev74] L. A. Levin. Laws of Information Conservation (Non-growth) and Aspects of the Foundations of Probability Theory. Problemy Peredachi Informatsii, 10(3):206–210, 1974.

[Lev84] L. A. Levin. Randomness conservation inequalities; information and independence in mathematical theories. Information and Control, 61(1):15–37, 1984.

[Lev13] L. A. Levin. Forbidden information. J. ACM, 60(2), 2013.

[LV08] M. Li and P. Vitányi. An Introduction to Kolmogorov Complexity and Its Applications. Springer Publishing Company, Incorporated, 3 edition, 2008.

[Rom03] Andrei E. Romashchenko. Extracting the mutual information for a triple of binary strings. In IEEE Conference on Computational Complexity, pages 221–229. IEEE Computer Society, 2003.

[She83] A. Shen. The concept of (alpha,beta)-stochasticity in the Kolmogorov sense, and its properties. Soviet Mathematics Doklady, 28(1):295–299, 1983.

[She99] A. Shen. Discussion on Kolmogorov Complexity and Statistical Analysis. The Computer Journal, 42(4):340–342, 1999.

[She12] A. Shen. Game Arguments in Computability Theory and Algorithmic Information Theory. In Proceedings of 5th Conference on Computability in Europe , volume 7318 of LNCS, pages 655–666, 2012.

[Sol64] R. J. Solomonoff. A Formal Theory of Inductive Inference, Part II. Information and Control, 7:1–22, 1964.

[SV15] A. Shen and N. Vereshchagin. Algorithmic statistics revisited. In Measures of Complexity. Festschrift for Alexey Chervonenkis, chapter 17, pages 235–252. Springer Verlag, 2015.

[Ver21] N. Vereshchagin. Proofs of conservation inequalities for levins notion of mutual information of 1974. Theoretical Computer Science, 856, 2021.

[Vit00] P Vitányi. Three Approaches to the Quantitative Definition of Information in an Individual Pure Quantum State. In Proceedings of the 15th Annual IEEE Conference on Computational Complexity, COCO ’00, page 263. IEEE Computer Society, 2000.

[VS17] Nikolay K. Vereshchagin and Alexander Shen. Algorithmic statistics: Forty years later. In Computability and Complexity, pages 669–737, 2017.

[VV04a] N. Vereshchagin and P. Vitányi. Algorithmic Rate Distortion Theory, 2004. http://arxiv.org/abs/cs.IT/0411014.

[VV04b] N. Vereshchagin and P. Vitányi. Kolmogorov’s Structure Functions and Model Selection. IEEE Transactions on Information Theory, 50(12):3265 – 3290, 2004.

[VV10] N. Vereshchagin and P. Vitányi. Rate Distortion and Denoising of Individual Data using Kolmogorov Complexity. IEEE Transactions on Information Theory, 56, 2010.

[VY87] V.V. ’Yugin. On Randomness Defect of a Finite Object Relative to Measures with Given Complexity Bounds. SIAM Theory of Probability and Its Applications, 32:558–563, 1987.

[VY99] V.V. ’Yugin. Algorithmic complexity and stochastic properties of finite binary sequences. The Computer Journal, 42:294–317, 1999.

[ZL70] A. K. Zvonkin and L. A. Levin. The complexity of finite objects and the development of the concepts of information and randomness by means of the theory of algorithms. Russian Math. Surveys, page 11, 1970.