Representations of $N = 2$ superconformal vertex algebra

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Introduction

In the last few years $N = 2$ superconformal algebra has attracted very much interests. The most important step in the representation theory of $N = 2$ superconformal algebra was made in the series of papers [FST], [ST], [FSS1]. It was proved that certain categories of modules of $N = 2$ superconformal algebra and affine Lie algebra $A^{(1)}_1$ are equivalent [FST]. The structure of the singular vectors in highest weight representations and embedding diagrams for Verma modules is also very much understood (cf. [D], [ST]).

In this paper we will investigate the representation theory of $N = 2$ superconformal algebra from the vertex algebra point of view. On the irreducible highest weight module $L_c$ exists a natural structure of vertex operator superalgebra (SVOA) (cf. Section 1). We will classify all irreducible $L_c$–modules. The problem of the classification of irreducible $L_c$–modules was initiated by Eholzer and Gaberdiel in [EG]. They proved that if $L_c$ is rational SVOA, then $L_c$ has to be unitary representations, i.e. $c = \frac{3m}{m+2}$ for $m \in \mathbb{N}$. They used the theory of Zhu’s algebra $A(L_c)$, and calculated it in some special cases. Unfortunately, the complicated structure of singular vectors makes the explicit determination of Zhu’s algebra $A(L_c)$ extremely difficult, and they didn’t get the complete classification result.

Instead of explicit calculation of Zhu’s algebra, for problem of the classification of irreducible $L_c$–modules we use the equivalence of categories from [FST]. We interpret the Kazama–Suzuki and anti Kazama–Suzuki mapping from [FST] in the language of vertex algebras and get embedding results between certain simple vertex operator (super)algebras (cf. Sections 4, 5). In this way we can use the representation theory of VOA $L(m, 0)$ associated to
admissible $A_1^{(1)}$–representation from [AM]. As a result we get the complete classification of irreducible $L_c$–modules.

Let us explain the classification result in more details. For $m = \frac{t}{u}$ admissible set $c_m = \frac{3m}{m+2}$, $N = 2u + t - 2$ and

$$S^m = \{n - k(m + 2) \mid k, n \in \mathbb{Z_+}, n \leq N, k \leq u - 1\}.$$

Let

$$W^c_m = \left\{ \left( \frac{jk - \frac{1}{2}}{m + 2}, \frac{j - k}{m + 2} \right) \mid j, k \in \mathbb{N}_{\frac{1}{2}}, 0 < j, k, j + k \leq N + 1 \right\},$$

and if $m \notin \mathbb{N}$ let

$$D^c_m = \{(h, q) \in \mathbb{C}^2 \mid q^2 + \frac{4h}{m+2} = \frac{r(r+2)}{(m+2)^2}, r \in S^m \setminus \mathbb{Z}\}.$$

Note that the set $W^c_m$ is finite and the set $D^c_m$ is union of finitely many rational curves.

Let $L_{h,q,c}$ denotes the irreducible highest weight $N = 2$–module with the highest weight $(h, q, c)$ (see Section 1.).

**Theorem 0.1** Let $m \in \mathbb{N}$. Then the set

$$\{L_{h,q,c_m} \mid (h, q) \in W^c_m\}$$

provides all irreducible $L_{c_m}$–modules.

If $m \in \mathbb{N}$ then $L_{h,q,c_m}, (h, q) \in W^c_m$ are all unitary representations with the central charge $c_m$. So, in that case the irreducible $L_{c_m}$ modules coincide with the unitary representations for $N = 2$ superconformal algebra.

**Theorem 0.2** Let $m \in \mathbb{Q}$ is admissible and $m \notin \mathbb{N}$. Then the set

$$\{L_{h,q,c_m} \mid (h, q) \in W^c_m \cup D^c_m\}$$

provides all irreducible $L_{c_m}$–modules.

SVOA $L_{c_m}, m \notin \mathbb{N}$, has uncountably many non-isomorphic irreducible representations which have a nice description as a union of finite set $W^c_m$ and the infinite set $D^c_m$ which is described with finitely many rational curves.
1 \( N = 2 \) superconformal vertex algebra

In this section we will show that on the vacuum representations of \( N = 2 \) superconformal algebra exists a natural structure of vertex operator super-algebra. This result follows from the results on local generating fields of SVOAs (cf. [K], [L], [P]). This structure was already studied in [EG].

\( N = 2 \) superconformal algebra \( A \) is the infinite-dimensional Lie superalgebra with basis \( L_n, T_n, G^\pm_r, C, n \in \mathbb{Z}, \ r \in \frac{1}{2} + \mathbb{Z} \) and (anti)commutation relations given by

\[
\begin{align*}
[L_m, L_n] &= (m - n)L_{m+n} + \frac{C}{12}(m^3 - m)\delta_{m+n,0} \\
[L_m, G^\pm_r] &= \left( \frac{1}{2}m - r \right)G^\pm_{m+r} \\
[L_m, T_n] &= -nT_{n+m} \\
[T_m, T_n] &= \frac{C}{3}m\delta_{m+n,0} \\
[T_m, G^\pm_r] &= \pm G^\pm_{m+r} \\
\{G^+_r, G^-_s\} &= 2L_{r+s} + (r - s)T_{r+s} + \frac{C}{3}(r^2 - \frac{1}{4})\delta_{r+s,0} \\
[L_m, C] &= [T_n, C] = [G^\pm_r, C] = 0 \\
\{G^+_r, G^+_s\} = \{G^-_r, G^-_s\} &= 0
\end{align*}
\]

for all \( m, n \in \mathbb{Z}, \ r, s \in \frac{1}{2} + \mathbb{Z} \).

We denote the Verma module generated from a highest weight vector \( |h, q, c\rangle \) with \( L_0 \) eigenvalue \( h \), \( T_0 \) eigenvalue \( q \) and central charge \( c \) by \( M_{h,q,c} \). An element \( v \in M_{h,q,c} \) is called singular vector if

\[
L_n v = T_n v = G^\pm_r v = 0, \quad n, r + \frac{1}{2} \in \mathbb{N},
\]

and \( v \) is an eigenvector of \( L_0 \) and \( T_0 \). Let \( J_{h,q,c} \) be the maximal \( U(A) \)-submodule in \( M_{h,q,c} \). Then

\[
L_{h,q,c} = \frac{M_{h,q,c}}{J_{h,q,c}}
\]

is the irreducible highest weight module.

Now we will consider the Verma module \( M_{0,0,c} \). One easily sees that for every \( c \in \mathbb{C} \)

\[
G^\pm_{\frac{1}{2}} |0, 0, c\rangle
\]
are the singular vectors in $M_{0,0,c}$. Set

$$V_c = \frac{M_{0,0,c}}{U(A)G_{-\frac{1}{2}}^+|0,0,c\rangle + U(A)G_{-\frac{1}{2}}^-|0,0,c\rangle}.$$ 

Then $V_c$ is a highest weight $\mathcal{A}$–module. Let $1$ denote the highest weight vector. Let $L_c = L_{0,0,c}$ be the corresponding simple module. Define the following four vectors in $V_c$:

$$\tau^\pm = G_{\pm\frac{1}{2}}^\pm 1, \quad j = T_{-1}1, \quad \nu = L_{-2}1,$$

and set

$$G^+(z) = Y(\tau^+, z) = \sum_{n\in\mathbb{Z}} G_{n+\frac{1}{2}}^+ z^{-n-2},$$

$$G^-(z) = Y(\tau^-, z) = \sum_{n\in\mathbb{Z}} G_{n+\frac{1}{2}}^- z^{-n-2},$$

$$L(z) = Y(\nu, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2},$$

$$T(z) = Y(j, z) = \sum_{n\in\mathbb{Z}} T_n z^{-n-1}. \quad (1.1)$$

It is easy to see that the fields $G^+(z), G^-(z), L(z), T(z)$ are mutually local and the theory of local fields (cf. [K], [Li], [P]) implies the following result.

**Proposition 1.1** There is a unique extension of the fields (1.1) such that $V_c$ becomes vertex operator superalgebra (SVOA). Moreover, $L_c$ is a simple SVOA.

**Definition 1.1** Let $V$ be SVOA. We will say that $V$ is $N = 2$ SVOA if there exist vectors $\tau^\pm, \nu, j \in V$ such that components of the fields $Y(\tau^\pm, z), Y(\nu, z), Y(j, z)$ span $N = 2$ superconformal algebra.

Previous definition has the following obvious but important consequence.

**Corrolary 1.1** Assume that $V$ is $N = 2$ SVOA. Then we have the following:
(1) $V$ is a $U(A)$–module.

(2) $V$ is a module for the SVOA $V_c$.

(3) $U(A).1$ is a subalgebra of $V$, isomorphic to $V_c$ or to certain quotient of $V_c$.

In what follows will present one construction of $N = 2$ SVOA.

**Remark 1.1** In [Z], Zhu constructed an associative algebra $A(V)$ for an arbitrary VOA $V$ and established a one to one correspondence between irreducible representations of $V$ and irreducible representation of $A(V)$. V. Kac and W. Wang extended in [KWn] the definition of Zhu’s algebra to the case of SVOAs. In our case, one can show that the Zhu’s algebra $A(V_c)$ is isomorphic to the polynomial algebra $\mathbb{C}[x,y]$, and Zhu’s algebra $A(L_c)$ is a certain quotient of $\mathbb{C}[x,y]$. This will imply that every irreducible $L_c$–module has to be the irreducible highest weight $U(A)$–module. Eholzer and Gaberdiel in [EG] starting from physical motivated definition of Zhu’s algebra showed that for non-generic $c$ $A(L_c) = \mathbb{C}[x,y]/I$, where $I$ is an ideal in $\mathbb{C}[x,y]$ generated by two polinomials $p_1(x,y)$, $p_2(x,y)$. They also calculated $p_1$, $p_2$ for some special cases (see Table 3.1 in [EG]).

## 2 Vertex operator algebras associated to affine Lie algebra $\hat{sl}_2$

In this section we recall the classification of the irreducible modules for VOAs associated to affine Lie algebra $A^{(1)}_1$ obtained by the author and A. Milas in [AM].

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$. The affine Lie algebra $\hat{\mathfrak{g}}$ associated with $\mathfrak{g}$ is defined as

$$\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$$
with the usual commutation relations. Let \( g = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+ \) and \( \hat{g} = \hat{\mathfrak{n}}_- + \hat{\mathfrak{h}} + \hat{\mathfrak{n}}_+ \) be the usual triangular decompositions for \( g \) and \( \hat{g} \). Let 
\[ P = \mathbb{C}[t] \otimes g \oplus \mathbb{C}c \]
be upper parabolic subalgebra. Let \( U \) be any \( g \)–module. Considering \( U \) as a \( P \)–module, we defined the induced module ( so called generalized Verma module) \( M(m, U) = U(\hat{g}) \otimes U(P) U \), where the central element \( c \) acts as multiplication with \( m \in \mathbb{C} \). Clearly \( \hat{g} \)–module \( M(m, U) \) contains a maximal submodule which intersect \( U \) trivially. Let \( L(m, U) \) be the corresponding quotient. If \( U \) is an irreducible \( g \)–module, then \( L(m, U) \) is an irreducible \( \hat{g} \)–module at the level \( m \). For \( \lambda \in \mathfrak{h}^* \) with \( V(\lambda) \) we denote the irreducible highest weight \( g \)–module. Set \( M(m, \lambda) = M(m, V(\lambda)) \). Let \( L(m, \lambda) \) denotes its irreducible quotient.

**Theorem 2.1** ([FZ]) Every \( M(m, 0) \) \( m \neq -g \) (where \( g \) denotes dual Coxeter number) has the structure of VOA. Let \( U \) be any \( g \)–module. Then every \( M(m, U) \) is a module for \( M(m, 0) \). In particular \( M(m, \lambda) \) is \( M(m, 0) \)–module. On the irreducible highest weight \( \hat{g} \)–module \( L(m, 0) \) exists the structure of simple VOA.

Let now \( g = sl_2 \), with generators \( e, f, h \) and relations \([e, f] = 2h, [h, e] = 2e, [h, f] = -2f\). Let \( \Lambda_0, \Lambda_1 \) denote the fundamental weights for \( g \), and let \( \omega_1 \) the fundamental weight for \( g \). Let \( r \in \mathbb{C} \). Then \( L(m, V(r\omega_1)) = L(m, V(\lambda)) \) is the irreducible highest weight \( \hat{g} \)–module with the highest weight \( (m - r)\Lambda_0 + r\Lambda_1 \).

**Definition 2.1** A rational number \( m = t/u \) is called admissible if \( u \in \mathbb{N}, t \in \mathbb{Z}, (t, u) = 1 \) and \( 2u + t - 2 \geq 0 \).

Let \( m = t/u \in \mathbb{Q} \) be admissible. We define the following set of weights at the level \( m \):

\[
P^m = \{ \lambda_{m,k,n} = (m - n + k(m + 2))\Lambda_0 + (n - k(m + 2))\Lambda_1, \]
\[
k, n \in \mathbb{Z}_+, n \leq 2u + t - 2, k \leq u - 1 \}.
\]

Set
\[
S^m = \{ n - k(m + 2)|k, n \in \mathbb{Z}_+, n \leq 2u + t - 2, k \leq u - 1 \}.
\]

**Remark 2.1** The weights \( \lambda \in P^m \) was introduced in [KZ]. The corresponding modules are called admissible.
The classification of irreducible $L(m,0)$–modules was given in [AM]. Recall the following result.

**Theorem 2.2** [AM, Theorem 3.5.3] The set

$$\{L(m,r\omega_1) \mid r \in S^m\}$$

provides all irreducible $L(m,0)$–modules from the category $O$.

**Remark 2.2** Previous theorem was also proved in [DLM2].

Theorem 2.2 shows that admissible representations of level $m$ for $\hat{g}$ can be identified with the irreducible $L(m,0)$–modules in the category $O$. In the case when $m \in \mathbb{N}$ representations defined with (2.1) are all irreducible modules for VOA $L(m,0)$. In the case $m \notin \mathbb{N}$ and $m$ admissible, VOA $L(m,0)$ has uncountably many irreducible representations outside the category $O$. The classification of irreducible $L(m,0)$–modules in the category of weight modules was given in [AM]. We will now recall the classification result.

For every $r, s \in \mathbb{C}$ define $E_{r,s} = t^s C[t,t^{-1}]$. Set $E_i = t^{s+i}$. We define $U(\hat{g})$ action on $E_{r,s}$:

$$e.E_i = -(s+i)E_{i-1}, \quad h.E_i = (-2s-2i+r)E_i, \quad f.E_i = (s+i-r)E_{i+1}. \quad (2.2)$$

Set

$$T^m = \{ (r, s) : r \in S^m \setminus \mathbb{Z}, \ s \notin \mathbb{Z}, \ r - s \notin \mathbb{Z} \}.$$ 

Then $E_{r,s}$ is an irreducible $U(\hat{g})$–module if $(r, s) \in T^m$.

Define $\Omega = ef + fe + \frac{1}{2}h^2$ the Casimir element element of $U(\hat{g})$. The proof of the following lemma is standard.

**Lemma 2.1** Let $w \in E_{r,s}$, $w \in V(r\omega_1)$ or $w \in V(r\omega_1)^*$. Then $\Omega w = \frac{r(r+2)}{2}w$.

We recall the following results from [AM, Section 4].

**Theorem 2.3** [AM] Assume that $m$ is admissible, $m \notin \mathbb{N}$. Let $r \in S^m \setminus \mathbb{Z}$.

Then $\hat{g}$–module $L(m, E_{r,s})$ is an module for the VOA $L(m,0)$. Moreover, $L(m, E_{r,s})$ is an irreducible $L(m,0)$–module if and only if $(r, s) \in T^m$.

We need the following consequence of the Theorem 2.3.
Corollary 2.1 Assume that $m$ is admissible, $m \notin \mathbb{N}$. Let $r \in S^m \setminus \mathbb{Z}$. Then for every $\beta \in \mathbb{C}$, there exists an $L(m,0)$-module $M$ and the weight vector $w \in M_0$ such that

$$\Omega|_{M_0} = \frac{r(r+2)}{2} \text{Id}, \quad h(0)w = \beta w.$$ 

Proof. Let $s = \frac{r-\beta}{2}$ and $M = L(m,E_{r,s})$. Then $M_0 \cong E_{r,s}$. Set $w = E_0$. Then Lemma 2.1 and relation (2.2) imply that

$$\Omega|_{M_0} = \frac{r(r+2)}{2} \text{Id}, \quad h(0)w = \beta w. \quad \square$$

Theorem 2.4 [AM] Let $M = \bigoplus_{n=0}^{\infty} M_n$ be an irreducible $L(m,0)$-module such that $M_0$ is a weight $U(\mathfrak{g})$-module. The $M$ is one of the following modules:

- $L(m,V(r\omega_1))$, $r \in S^m$,
- $L(m,V(r\omega_1)^*)$, $r \in S^m$,
- $L(m,E_{r,s})$, $(r,s) \in T^m$.

Remark 2.3 Modules $L(m,E_{r,s})$ are not in the category $\mathcal{O}$. In particular, these modules are not irreducible quotient of any Verma modules over $\mathfrak{g}$.

In [FST] B. L. Feigin, A. M. Semikhatov, and I. Yu. Tipunin introduced the notion of relaxed Verma module. Our modules $L(m,E_{r,s})$ are irreducible quotients of relaxed Verma modules.

3 Fermionic and lattice construction of vertex superalgebras

In this section we will consider vertex superalgebra $F$ constructed from two charged fermions and vertex superalgebra $F^{-1}$ constructed from rank one
lattice $L = Z\alpha$ such that $\langle \alpha, \alpha \rangle = -1$. In the following sections we will use vertex superalgebras $F$ and $F_{-1}$ for finding the connections between the representation theory of VOA $L(m,0)$ and SVOA $L_c$.

3.1 Fermionic SVOA $F$

Recall the construction of fermionic SVOA $F$. The charged ree fermionic fields are

$$\Psi^\pm(z) = \sum_{i \in \frac{1}{2} + Z} \psi^\pm_i z^{-i - \frac{1}{2}},$$

with the following commutation relations

$$\{\psi^+_i, \psi^-_j\} = \delta_{i,j}, \quad \{\psi^+_i, \psi^+_j\} = 0$$

Let $F$ be the Fock space defined by $\psi^+_i > 0 \psi^-_i = 0$. Then $F$ is a SVOA with the central charge $c = 1$ (see [KWn] and [K] for details).

3.2 Lattice construction of vertex superalgebras

Let $L$ be a lattice. Set $h = C \otimes Z L$ and extend the $Z$-form $\langle \cdot, \cdot \rangle$ on $L$ to $h$. Let $\hat{h} = C[t, t^{-1}] \otimes h \oplus C c$ be the affinization of $h$ (see Section 2). We also use the notation $h(n) = t^n \otimes h$ for $h \in h, n \in Z$.

Set

$$\hat{h}^+ = C[t] \otimes h; \quad \hat{h}^- = t^{-1} C[t^{-1}] \otimes h.$$ 

Then $\hat{h}^+$ and $\hat{h}^-$ are abelian subalgebras of $\hat{h}$. Let $U(\hat{h}^-) = S(\hat{h}^-)$ be the universal enveloping algebra of $\hat{h}^-$. Consider the induced $\hat{h}$-module

$$M(1) = U(\hat{h}) \otimes_{U(C[t] \otimes h \oplus C c)} C \simeq S(\hat{h}^-) \ (\text{linearly}),$$

where $C[t] \otimes h$ acts trivially on $C$ and $c$ acts on $C$ as multiplication by 1.

Let $\hat{L}$ be the canonical central extension of $L$ by the cyclic group $\langle \pm 1 \rangle$. Form the induced $\hat{L}$-module

$$C\{L\} = C[\hat{L}] \otimes_{\langle \pm 1 \rangle} C \simeq C[L] \ (\text{linearly}),$$

with the following commutation relations

$$\{\psi^+_i, \psi^-_j\} = \delta_{i,j}, \quad \{\psi^+_i, \psi^+_j\} = 0$$

Let $F$ be the Fock space defined by $\psi^+_i > 0 \psi^-_i = 0$. Then $F$ is a SVOA with the central charge $c = 1$ (see [KWn] and [K] for details).
where $C[\cdot]$ denotes the group algebra and $-1$ acts on $C$ as multiplication by $-1$. For $a \in \hat{L}$, write $\iota(a)$ for $a \otimes 1$ in $C\{L\}$. Then the action of $\hat{L}$ on $C\{L\}$ is given by: $a \cdot \iota(b) = \iota(ab)$ and $(-1) \cdot \iota(b) = -\iota(b)$ for $a, b \in \hat{L}$.

Furthermore we define an action of $h$ on $C\{L\}$ by: $h \cdot \iota(a) = \langle h, \bar{a} \rangle \iota(a)$ for $h \in h, a \in \hat{L}$. Define $z^h \cdot \iota(a) = z^{(h, \bar{a})} \iota(a)$.

The untwisted space associated with $L$ is defined to be

$$V_L = C\{L\} \otimes_C M(1) \cong C\{L\} \otimes S(h^-) \text{ (linearly)}.$$ 

Then $\hat{L}, h, z^h (h \in h)$ act naturally on $V_L$ by acting on either $C\{L\}$ or $M(1)$ as indicated above.

For $h \in h$ set $h(z) = \sum_{n \in Z} h(n) z^{-n-1}$. We use a normal ordering procedure, indicated by open colons, which signify that in the enclosed expression, all creation operators $h(n)$ ($n < 0), a \in \hat{L}$ are to be placed to the left of all annihilation operators $h(n), z^h (h \in h, n \geq 0)$. For $a \in \hat{L}$, set

$$Y(\iota(a), z) = \circ e^f(a(z)-\bar{a}(0)z^{-1}) a z^a \circ.$$ 

Let $a \in \hat{L}; h_1, \cdots, h_k \in h; n_1, \cdots, n_k \in Z (n_i > 0)$. Set

$$v = \iota(a) \otimes h_1(-n_1) \cdots h_k(-n_k) \in V_L.$$ 

Define vertex operator

$$Y(v, z) = \circ \left( \frac{1}{(n_1-1)!} \left( \frac{d}{dz} \right)^{n_1-1} h_1(z) \right) \cdots \left( \frac{1}{(n_k-1)!} \left( \frac{d}{dz} \right)^{n_k-1} h_k(z) \right) Y(\iota(a), z) \circ.$$ 

This gives us a well-defined linear map

$$Y(\cdot, z) : V_L \to (\text{End}V_L)[[z, z^{-1}]]$$

$$v \mapsto Y(v, z) = \sum_{n \in Z} v_n z^{-n-1}, \ (v_n \in \text{End}V_L).$$

Let $\{ \alpha_i | i = 1, \cdots, d \}$ be an orthonormal basis of $h$ and set

$$\omega = \frac{1}{2} \sum_{i=1}^d \alpha_i(-1) \alpha_i(-1) \in V_L.$$ 

Then $Y(\omega, z) = \sum_{n \in Z} L_n z^{-n-2}$ gives rise to a representation of the Virasoro algebra on $V_L$ and

$$L_0 (\iota(a) \otimes h_1(-n_1) \cdots h_n(-n_k))$$

$$= \left( \frac{1}{2} (\bar{a}, \bar{a}) + n_1 + \cdots + n_k \right) (\iota(a) \otimes h_1(-n_1) \cdots h_k(-n_k)). \ (3.1)$$
Now we will assume that $L = \mathbb{Z}\alpha$ is a rank one lattice. The following theorem is a special case of the results of Kac [K, Proposition 5.5], and Dong and Lepowsky [DL, Remark 6.17, Remark 9.21].

**Theorem 3.1** Assume that $L = \mathbb{Z}\alpha$ is a rank one lattice, $\langle \alpha, \alpha \rangle = n$ and $n$ is an integer. Then $V_L$ is vertex superalgebra. Moreover, if $L$ is a positive definite (i.e. $n > 0$) then $V_L$ is vertex operator superalgebra.

**Remark 3.1** If $n < 0$, then relation (3.1) gives that $V_L$ is a $1/2\mathbb{Z}$-graded with the respect to $L_0$ and the weight subspaces are not bounded below. This implies that $V_L$ is vertex superalgebra which is not vertex operator superalgebra (we follow the definitions from [DL] and [Li]).

**Remark 3.2** If $\langle \alpha, \alpha \rangle = 1$, then $V_L$ is SVOA which is isomorphic to the SVOA $F$ constructed from two charged fermions. This fact is in conformal field theory known as boson-fermion correspondence. If $\langle \alpha, \alpha \rangle = 3$, then $V_L$ is isomorphic to $N = 2$ SVOA $L_2$ with $c = 1$ (cf. [K]).

The following discussion is similar as in [DLM]. In [DLM] the authors considered the case of positive definite even lattice $L$ when $V_L$ is VOA. We are interested in the case when $L$ is negative definite lattice of rank one.

Define the Schur polynomials $p_r(x_1, x_2, \cdots) (r \in \mathbb{Z}_+)$ in variables $x_1, x_2, \cdots$ by the following equation:

$$\exp\left(\sum_{n=1}^{\infty} \frac{x_n y^n}{n}\right) = \sum_{r=0}^{\infty} p_r(x_1, x_2, \cdots)y^r. \quad (3.2)$$

For any monomial $x_1^{n_1}x_2^{n_2}\cdots x_r^{n_r}$ we have an element $h(-1)^{n_1}h(-2)^{n_2}\cdots h(-r)^{n_r} \mathbf{1}$ in $V_L$ for $h \in \mathbf{h}$. Then for any polynomial $f(x_1, x_2, \cdots), f(h(-1), h(-2), \cdots) \mathbf{1}$ is a well-defined element in $V_L$. In particular, $p_r(h(-1), h(-2), \cdots) \mathbf{1}$ for $r \in \mathbb{Z}_+$ are elements of $V_L$.

Suppose $a, b \in \hat{L}$ such that $\bar{a} = \alpha, \bar{b} = \beta$. Then

$$Y(\iota(a), z)\iota(b) = z^{(\alpha, \beta)} \exp\left(\sum_{n=1}^{\infty} \frac{\alpha(-n) z^n}{n}\right) \iota(ab)$$

$$= \sum_{r=0}^{\infty} p_r(\alpha(-1), \alpha(-2), \cdots) \iota(ab) z^{r+(\alpha, \beta)}. \quad (3.3)$$
Thus
\[ \iota(a)_i \iota(b) = 0 \quad \text{for } i \geq -\langle \alpha, \beta \rangle. \quad (3.4) \]

Especially, if \( \langle \alpha, \beta \rangle \geq 0 \), we have \( \iota(a)_i \iota(b) = 0 \) for all \( i \in \mathbb{Z}_+ \), and if \( \langle \alpha, \beta \rangle = -n < 0 \), we get
\[ \iota(a)_{i-1} \iota(b) = p_{n-i}(\alpha(-1), \alpha(-2), \cdots) \iota(ab) \quad \text{for } i \in \mathbb{Z}_+. \quad (3.5) \]

From now on we will assume that \( L = \mathbb{Z} \alpha \) and \( \langle \alpha, \alpha \rangle = -1 \).

Set \( F_{-1} = V_L \). Let \( a \in \hat{L} \) such that \( \bar{a} = \alpha \). Set \( e = \iota(a), f = \iota(a^{-1}), k = \alpha(-1)1 \). Set \( e^n = \iota(a^n), f^n = \iota(a^{-n}) \).

The relations \((3.4)\) and \((3.5)\) in the case of vertex superalgebra \( F_{-1} \) give the following proposition.

**Proposition 3.1** The following relations are hold in the vertex superalgebra \( F_{-1} \).

\begin{enumerate}
  \item[(a)] \( e_i f = 0 \) for \( i \geq -1 \), \( e_{-2}f = 1 \), \( e_{-3}f = k \);
  \item[(b)] \( f_i e = 0 \) for \( i \geq -1 \), \( f_{-2}e = 1 \), \( f_{-3}e = -k \);
  \item[(c)] \( e_i e = 0 \) for \( i \geq 1 \), \( e_0 e = e^2 \);
  \item[(d)] \( f_i f = 0 \) for \( i \geq 1 \), \( f_0 f = f^2 \);
  \item[(e)] \( k_i e = 0 \) for \( i \geq 1 \), \( k_0 e = -e \);
  \item[(f)] \( k_i f = 0 \) for \( i \geq 1 \), \( k_0 f = f \).
\end{enumerate}

4 Embedding of \( N = 2 \) SVOA \( L_c \) into SVOA \( F \otimes L(m, 0) \)

The tensor product of the SVOA \( F \) and the VOA \( L(m,0) \) is SVOA \( F \otimes L(m,0) \). We will show that the SVOA \( L_{c_m} \) can be realized as a subalgebra of the SVOA \( F \otimes L(m,0) \), where \( c_m = \frac{3m}{m+2} \).
Define the following vectors in $F \otimes L(m, 0)$:

\[
\tau^+ = \psi^{-\frac{1}{2}} \mathbf{1} \otimes e(-1) \mathbf{1}, \\
\tau^- = \frac{2}{m + 2} \psi^{-\frac{1}{2}} \mathbf{1} \otimes f(-1) \mathbf{1}, \\
j = \frac{m}{m + 2} \psi^+ \psi^{-\frac{1}{2}} \mathbf{1} \otimes \mathbf{1} - \frac{1}{m + 2} \mathbf{1} \otimes h(-1) \mathbf{1}, \\
\nu = \frac{1}{m + 2} \mathbf{1} \otimes e(-1) f(-1) \mathbf{1} - \frac{m}{m + 2} \psi^+ \psi^{-\frac{1}{2}} \mathbf{1} \otimes \mathbf{1} \\
- \frac{1}{m + 2} \psi^+ \psi^{-\frac{1}{2}} \mathbf{1} \otimes h(-1) \mathbf{1}.
\]

**Remark 4.1** Our relations (4.1)-(4.4) are similar to relations (3.1)-(3.3) from [FST].

The following lemma can be proved by direct calculations.

**Lemma 4.1** $F \otimes L(m, 0)$ is $N = 2$ SVOA, i.e. the component of the fields $Y(\tau^+, z)$, $Y(\tau^-, z)$, $Y(j, z)$ and $Y(\omega, z)$ span $N = 2$ superconformal algebra with the central charge $c = c_m$.

Now, Corollary 1.1 and Lemma 4.1 imply that $F \otimes L(m, 0)$ is an $U(\mathcal{A})$–module and $U(\mathcal{A}).(\mathbf{1} \otimes \mathbf{1})$ is an subalgebra of the SVOA $F \otimes L(m, 0)$ isomorphic to certain quotient of the SVOA $V_{c_m}$.

The mapping $U(\mathcal{A}) \rightarrow F \otimes L(m, 0)$ is a special case of Kazama-Suzuki mapping considered by Feigin, Semikhatov and Tipunin in [FST]. They constructed functor between certain categories of $\mathfrak{sl}_2$–modules and $N = 2$–modules. They showed that this functor is an equivalence of categories.

Applying this functor in our case we see that $U(\mathcal{A}).(\mathbf{1} \otimes \mathbf{1})$ is an irreducible $U(\mathcal{A})$–module, and it is isomorphic to $L_{c_m}$. We get the following theorem.

**Theorem 4.1** Let $m \neq -2$.

1. SVOA $L_{c_m} \cong U(\mathcal{A}).(\mathbf{1} \otimes \mathbf{1})$ is a subalgebra of SVOA $F \otimes L(m, 0)$.

2. Assume that $M$ is $L(m, 0)$–module. Then $F \otimes M$ is a module for the SVOA $L_{c_m}$.
Remark 4.2 The irreducibility of the submodule $U(A)(1 \otimes 1)$ was also proved in [EG, Appendix A] using the calculations of vacuum characters (see also [FSST]).

5 Embedding of $\hat{sl}_2$ VOA $L(m,0)$ into vertex superalgebra $L_c \otimes F_{-1}$.

In this section we will consider tensor product $N = 2$ SVOA $L_c$ with the lattice vertex superalgebra $F_{-1}$. We will show that the simple VOA $L(m,0)$ is a subalgebra of $L_c \otimes F_{-1}$. Our costruction is the vertex operator algebra interpretation of the ‘anti’-Kazama-Suzuki mapping which are considered in [FST].

Let $m \in \mathbb{C}, m \neq -2$. Recall the definition of $e, f \in F_{-1}$ from Section 3.

Set

$x = G^+_{-\frac{1}{2}}1 \otimes f,$ \hspace{1cm} (5.1)

$y = \frac{m + 2}{2}G^-_{-\frac{1}{2}}1 \otimes e,$ \hspace{1cm} (5.2)

$h = -m1 \otimes \alpha(-1)1 + (m + 2)T_{-1}1 \otimes 1.$ \hspace{1cm} (5.3)

Remark 5.1 The relations (5.1)-(5.3) are similar to relation (3.13) from [FST].

Set $Y(x, z) = \sum_{n \in \mathbb{Z}} x(n) z^{-n-1}, Y(y, z) = \sum_{n \in \mathbb{Z}} y(n) z^{-n-1}, Y(h, z) = \sum_{n \in \mathbb{Z}} h(n) z^{-n-1}$.

Then

$x(n) = \sum_{i \in \mathbb{Z}} G^+_{i + \frac{1}{2}} \otimes f_{n-i-2},$ \hspace{1cm} (5.4)

$y(n) = \frac{m + 2}{2} \sum_{i \in \mathbb{Z}} G^-_{i + \frac{1}{2}} \otimes e_{n-i-2},$ \hspace{1cm} (5.5)

$h(n) = -m \text{Id} \otimes \alpha(n) + (m + 2)T_n \otimes \text{Id}.$ \hspace{1cm} (5.6)

Relations (5.4)-(5.6) and Proposition 3.1 imply the following lemma.
Lemma 5.1

(a) \( x(n)x = 0, \forall n \in \mathbb{Z}_+ \) \( y(n)y = 0 \) \( \forall n \in \mathbb{Z}_+ \),

(b) \( x(n)y = 0 \) for \( n \geq 2 \), \( x(1)y = m1 \), \( x(0)y = h \),

(c) \( h(n)x = 0 \) for \( n \geq 1 \), \( h(0)x = 2x \),

(d) \( h(n)y = 0 \) for \( n \geq 1 \), \( h(0)y = -2y \),

(e) \( h(n)h = 0 \) for \( n \geq 2 \), \( h(1)h = 2m1 \), \( h(0)y = 0 \).

Lemma 5.1 implies that \( L_{cm} \otimes F_{-1} \) is a module for affine Lie algebra \( \hat{\mathfrak{sl}}_2 \) at the level \( m \). The mapping \( U(\hat{\mathfrak{sl}}_2) \rightarrow L_{cm} \otimes F_{-1} \) is a special case of 'anti'-Kazama-Suzuki mapping considered in [FST]. This mapping gives a functor from the category of \( N = 2 \)-modules to the category of \( \hat{\mathfrak{sl}}_2 \)-modules. By using properties of this functor we get that \( U(\hat{\mathfrak{sl}}_2)(1 \otimes 1) \) is an irreducible \( U(\hat{\mathfrak{sl}}_2) \)-module isomorphic to \( L(m,0) \).

We have obtained the following theorem.

Theorem 5.1 Let \( m \in \mathbb{C} \), \( m \neq -2 \), and \( c = cm \).

(1) VOA \( L(m,0) \cong U(\hat{\mathfrak{sl}}_2)(1 \otimes 1) \) is a subalgebra of vertex superalgebra \( L_c \otimes F_{-1} \).

(2) Assume that \( M \) is a module for SVOA \( L_{cm} \). Then \( M \otimes F_{-1} \) is a module for VOA \( L(m,0) \).

6 Modules for SVOA \( L_c \)

The representation theory of the SVOA \( L_c \) with the central charge \( c = cm \) is interesting only in the case when \( m \) is an admissible rational number. Otherwise, \( L_c = V_c \) and every highest weight \( A \)-module is an \( L_c \)-module. In this section we will present a construction of certain set of modules for the SVOA \( L_c \) with the central charge \( c = cm \). This construction is based on the realization of the SVOA \( L_c \) as a subalgebra of the tensor product SVOA \( F \otimes L(m,0) \) and the representation theory of the simple VOA \( L(m,0) \). When \( m \in \mathbb{N} \) then VOA \( L(m,0) \) has finitely many irreducible representations, and we will obtain only finitely many irreducible representations for the SVOA \( L_{cm} \). When \( m \) is admissible rational number and \( m \notin \mathbb{Q} \) then VOA \( L(m,0) \)
has uncountably many irreducible representations (see Section 2), and we will get uncountably many irreducible $L_c$-representations.

For an irreducible $L(m,0)$–module $M$, the top level $M_0$ is an irreducible $U(sl_2)$–module.

Now the relations (4.1)-(4.4) imply the following lemma (see also [EG, Section 4]).

**Lemma 6.1** Let $M$ be any $L(m,0)$–module and $w \in M_0$ such that $h(0)w = \beta w$, and $\Omega w = \gamma w$, for every $w \in M_0$. Then $U(A)(1 \otimes w)$ is a highest weight $U(A)$–module with the highest weight $h, q, c_m$, where

$$h = \frac{\gamma}{2(m+2)} - \frac{\beta^2}{4(m+2)}, \quad q = -\frac{\beta}{m+2}. \quad (6.1)$$

Moreover, $L_{h,q,c_m}$ is an irreducible $L_{c_m}$–module.

For every $r \in \mathbb{Z}_+$, $m = \frac{t}{u}$ admissible and $i \in \{0,1,\ldots,r\}$ we define

$$h_{i,r} = \frac{r(r+2)}{4(m+2)} - \frac{(r-2i)^2}{4(m+2)},$$

$$q_{i,r} = -\frac{(r-2i)}{m+2}.$$

Set $N = 2u + t - 2$. Note that if $m \in \mathbb{N}$, then $N = m$. Define the following finite set

$$W_{c_m} = \{(h_{i,r}, q_{i,r}) \mid 0 \leq r \leq N, \ 0 \leq i \leq r\}.$$

Assume now that $M$ is an irreducible $L(m,0)$–module such that $M_0$ is finite-dimensional. Then $M_0 \cong V(r\omega_1)$ for certain $r \in \{0, \ldots, N\}$. Then for every weight vector $w \in M_0$ such that $h(0)w = \beta w$ we have that $\beta = r - 2i$ for certain $i \in \{0, \ldots, r\}$. Now Lemma 6.1 implies the following theorem.

**Theorem 6.1** Assume that $m \in \mathbb{Q}$ is admissible. Then for every $(h, q) \in W_{c_m}$, the $A$–module $L_{h,q,c_m}$ is the irreducible $L_{c_m}$–module.

**Remark 6.1** Theorem 6.1 shows that starting from $L(m,0)$–modules $M$ such that $M_0$ is finite-dimensional we can construct only finitely many irreducible modules for SVOA $L_{c_m}$. 
We shall now give another parametrisation of the set $W^c_m$.
Let $N_{\frac{1}{2}} = \{\frac{1}{2}, \frac{3}{2}, \ldots\}$. Set
\[
j = i + \frac{1}{2} \quad k = r + \frac{1}{2} - i.
\]
Then we have that
\[
j, k \in N_{\frac{1}{2}}, \quad 0 < j, k, j + k \leq N + 1,
\]
and obtain
\[
h_{i,r} = \frac{jk - \frac{1}{4}}{m + 2}, \quad q_{i,r} = \frac{j - k}{m + 2}.
\]
We get
\[
W^c_m = \left\{ \left( \frac{jk - \frac{1}{4}}{m + 2}, \frac{j - k}{m + 2} \right) \mid j, k \in N_{\frac{1}{2}}, 0 < j, k, j + k \leq N + 1 \right\},
\]
In the case $m \in \mathbb{N}$ this is exactly the parametrisation of the unitary discrete series of $N = 2$ minimal models (see [4]). So, Theorem 6.1 shows that every unitary minimal model is a module for SVOA $L^c_m$.

Now we assume that $m = \frac{t}{u}$ is admissible and $m \notin \mathbb{N}$. Then the irreducible $L(m,0)$–modules are given in Theorem 2.4. Starting from $L(m,0)$–modules $M$ such that $M_0$ is finite-dimensional we constructed finitely many irreducible $L^c_m$–modules $L_{h,q,c_m}^c$, $(h, q) \in W^c_m$. Now we consider the case when $M_0$ is infinite-dimensional.

**Proposition 6.1** Assume that $r \in S^m \setminus \mathbb{Z}$, and that $(h, q)$ satisfies the equation
\[
q^2 + \frac{4h}{m + 2} = \frac{r(r + 2)}{(m + 2)^2}.
\]
Then $L_{h,q,c_m}^c$ is an irreducible $L^c_m$–module.

**Proof.** Assume that $(h, q)$ satisfies the equation (6.2). Set $\beta = -q(m + 2)$. By using Corollary 2.1 we get that there is a $L(m,0)$–module $M$ and the weight vector $w \in M_0$ such that
\[
\Omega | M_0 = \frac{r(r + 2)}{2} \text{Id}, \quad h(0)w = \beta w.
\]
Since \((h, q)\) satisfies \((6.2)\), we get

\[
q = -\frac{\beta}{m + 2}, \quad h = \frac{r(r+2)}{2(m+2)} - \frac{\beta^2}{4(m+2)}.
\]

Now Lemma \(6.1\) implies that \(L_{h,q,c,m}\) is a \(L_{c,m}\)-module. Irreducibility is clear. \(\square\)

Define the following set

\[
D^{c_m} = \{ (h, q) \in C^2 \mid q^2 + \frac{4h}{m+2} = \frac{r(r+2)}{(m+2)^2}, \quad r \in S^m \setminus Z \}.
\]

Theorem \(6.1\) and Proposition \(6.1\) give the following theorem.

**Theorem 6.2** Assume that \(m \in Q\) is admissible and \((h, q) \in W^{c_m} \cup D^{c_m}\). Then \(L_{h,q,c,m}\) is an irreducible \(L_{c,m}\)-module.

**Remark 6.2** Theorem \(6.2\) implies that for every admissible nonintegral \(m\), the SVOA \(L_{c,m}\) has uncountably many non-isomorphic irreducible modules. These modules can be parametrized as the union of finitely many rational curves in \(C^2\). So, we have constructed uncountably many irreducible modules over \(N = 2\) superconformal algebra which are annihilated with the fields \(Y(v, z)\), where \(v\) is a vector from the maximal submodule of \(V_{c,m}\). This gives an difference with the case of Virasoro (cf. \([W]\)) and Neveu-Schwarz algebra (cf. \([A]\)) where there exists only finitely many such irreducible modules.

### 7 Classification of irreducible \(L_{c,m}\)-modules

In this section we will give the complete classification of the irreducible \(L_{c,m}\)-modules. We will prove that \(L_{c,m}\)-modules constructed in Section 6 gives all irreducible modules for SVOA \(L_{c,m}\). The proof of the classification result will use the fact that VOA \(L(m,0)\) is a subalgebra of SVOA \(L_{c,m} \otimes F_{-1}\)-modules and the classification of all irreducible \(L(m,0)\)-modules obtained in \([AM]\).

Let \(g = sl_2\) and \(\hat{g} = \hat{sl}_2\) as before.
Lemma 7.1 Let $c = c_m$ for $m$ admissible. Assume that $L_{h, q, c_m}$ is a $L_{c_m}$-module. Then we have that
\[
\frac{4h}{m+2} + q^2 = \frac{r(r+2)}{(m+2)^2}
\]
for certain $r \in S^m$.

Proof. Assume that $L_{h, q, c}$ is $L_c$-module. Let $v_{h, q, c}$ be the highest weight vector in $L_{h, q, c}$. Since $L(m, 0)$ is a subalgebra of $L_c \otimes F_1$ (Theorem 5.1), we have that $L_{h, q, c} \otimes F_1$ is a module for VOA $L(m, 0)$-module. In particular $M = U(\hat{\mathfrak{g}})(v_{h, q, c} \otimes 1)$ is an $L(m, 0)$-module. Set $M_0 = U(\mathfrak{g})(v_{h, q, c} \otimes 1)$. Since
\[(g \otimes t^m)M_0 = 0 \quad \text{for} \quad n \geq 1,
\]
we conclude that the top level of $L(m, 0)$-module $M$ is $M_0 = U(\mathfrak{g})(v_{h, q, c} \otimes 1)$. Then the representation theory of VOA $L(m, 0)$ (see Theorem 2.4) easily implies that $M_0$ is an irreducible $U(\mathfrak{g})$-module which is isomorphic to one of the following modules:

$V(r\omega_1), V(r\omega_1^*), E_{r,s}$ for $r \in S^m$, $(r, s) \in T^m$.

Let $\Omega = x(0)y(0) + y(0)x(0) + \frac{1}{2}h(0)^2$ be the Casimir. Then Lemma 2.4 imply that
\[
\Omega|_{M_0} = \frac{r(r+2)}{2} \text{Id} \quad \text{for certain} \quad r \in S^m. \quad (7.1)
\]

Let $w \in M_0$. Then there is $f \in U(\mathfrak{g})$ such that $w = f(v_{h, q, c} \otimes 1)$. Since the action of $\Omega$ commutes with $U(\mathfrak{g})$, then from relations (5.4)-(5.6) we get
\[
\Omega w = f(\Omega(v_{h, q, c} \otimes 1))
\]
\[
= f(x(0)y(0) + y(0)x(0) + \frac{1}{2}h(0)^2)(v_{h, q, c} \otimes 1)
\]
\[
= (2(m+2)h + \frac{1}{2}(m+2)^2q^2)w. \quad (7.2)
\]

So, we have proved
\[
\Omega|_{M_0} = (2(m+2)h + \frac{1}{2}(m+2)^2q^2) \text{Id}. \quad (7.3)
\]

Now from (7.1) and (7.3) follow that
\[
\frac{4h}{m+2} + q^2 = \frac{r(r+2)}{(m+2)^2}
\]
for certain $r \in S^m$. □
Lemma 7.2 Assume that $L_{h,q,c_m}$ is $L_{cm}$–module and
\[ \frac{4h}{m+2} + q^2 = \frac{r(r+2)}{(m+2)^2} \tag{7.4} \]
for $r \in S^m \cap \mathbb{Z}_+$. Then $(h, q) \in W_{cm}$.

Proof. Let $M = U(\hat{g})(v_{h,q,c} \otimes 1)$ be a $L(m,0)$–module as in the proof of Lemma 7.1. Since in the relation (7.4) we have $r \in \mathbb{Z}_+$, we conclude that $M_0$ is an irreducible finite-dimensional $U(\mathfrak{g})$–module isomorphic to $V(r\omega_1)$. This implies that the vector $v_{h,q,c} \otimes 1$ is a weight vector of $V(r\omega_1)$, i.e.
\[ h(0)(v_{h,q,c} \otimes 1) = (r - 2i)(v_{h,q,c} \otimes 1) \]
for certain $i \in \{0, \ldots, r\}$. Applying the formulae (5.6) for the action of $h(0)$ on $L_{h,q,c} \otimes F_{-1}$ we get $q = \frac{r - 2i}{m+2}$. Now the relation (7.4) implies that $h = \frac{r(r+2)}{4(m+2)} - \frac{(r-2i)^2}{4(m+2)}$, and we conclude that $(h, q) \in W_{cm}$. \[ \square \]

Theorem 7.1 Assume that $m \in \mathbb{N}$ and $c = c_m$. Then the set
\[ \{L_{h,q,c} \mid (h, q) \in W_{cm}\} \]
provides all $L_c$ irreducible modules for the SVOA $L_c$. So, irreducible $L_c$–modules are exactly all unitary modules for $N = 2$ superconformal algebra with the central charge $c$.

Proof. We proved in Theorem 6.1 that for every $(h, q) \in W_{cm}$, $L_{h,q,c}$ is a $L_c$–module. It remains to prove that if $L_{h,q,c}$ is a $L_c$–module, then $(h, q) \in W_{cm}$.

Assume now that $L_{h,q,c}$ is a $L_c$–module. Then Lemma 7.1 implies that
\[ \frac{4h}{m+2} + q^2 = \frac{r(r+2)}{(m+2)^2} \]
for certain $r \in S^m$. Since $S^m \subset \mathbb{Z}_+$ for $m \in \mathbb{N}$, we have that $r \in \mathbb{Z}_+$. Now Lemma 7.2 implies that $(h, q) \in W_{cm}$. \[ \square \]

Remark 7.1 Theorem 7.1 shows that SVOA $L_{cm}$ for $m \in \mathbb{N}$ has exactly $\frac{(m+2)(m+1)}{2}$ non-isomorphic irreducible modules.
Theorem 7.2 Assume that $m \in \mathbb{Q}$ is admissible such that $m \notin \mathbb{N}$. Let $c = c_m$. Then the set
\[
\{ L_{h,q,c} \mid (h,q) \in W^c_m \cup D^c_m \}
\]
provides all $L_c$ irreducible modules for the SVOA $L_c$.

Proof. Theorem 6.2 gives that $L_{h,q,c}$ is a $L_c$–module for every $(h,q) \in W^c_m \cup D^c_m$. In order to prove theorem we have to prove that if $L_{h,q,c}$ is a $L_c$–module, then $(h,q) \in W^c_m \cup D^c_m$.

Assume now that $L_{h,q,c}$ is a $L_c$–module. Then Lemma 7.1 implies that
\[
\frac{4h}{m+2} + q^2 = \frac{r(r+2)}{(m+2)^2}
\]
for certain $r \in S^m$. If $r \in S^m \setminus \mathbb{Z}$, then $(h,q) \in D^c_m$. If $r \in \mathbb{Z}_+$, then from Lemma 7.2 follows that $(h,q) \in W^c_m$. So, we get $(h,q) \in W^c_m \cup D^c_m$. \qed

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