An analytically solvable time dependent Jaynes Cummings Model

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Using the underlying su(2) algebra of the Jaynes-Cummings Model (JCM), we construct a time dependent interaction term that allows analytical solution for even off-resonance conditions. Exact solutions for the time evolution of any state has been found. The effect of detuning on the Rabi oscillations and the collapse and revival of inversion is indicated. It is also shown that at resonance, the time dependent JCM is analytically solvable for an arbitrary interaction term.

I. INTRODUCTION

The Jaynes Cummings Model (JCM) [1,2] is one of the major paradigms in quantum optics today. It was introduced as a highly simplified model to explain, qualitatively, the salient features of the interaction of matter with quantised radiation field in a cavity. Despite the fact that it considers matter in the highly simplified form of a two level atom and radiation in the form of a single mode field it demonstrates quite a few uniquely quantum mechanical features. It’s importance has been enhanced recently by experimental realisation in Rydberg atom masers [3,4]. Various generalisations of the basic JCM have been considered in the literature - multiphoton interactions, nonlinear media, deformed radiation fields to name but a few. Recently, a few authors have treated the time dependent version of the JCM [5–10]. In this paper we treat a particular class of time dependent JCM’s, where the time dependence of the interaction term is such that it allows for an analytical solution.

This article is organised as follows. In section II, we discuss the generalisation of the basic JCM due to Yu et al. [11], which encompasses most of the generalized JCMs that have been treated in the literature. In section III we discuss, following [11], the underlying su(2) algebra that allows a reduction of the generalized JCM into the problem of the dynamics of a spin-three half particle in a magnetic field. In this article, we are concerned with time dependent interaction, which will lead to a generalisation over [11] in that it leads to a time dependent magnetic field. Here we point out why the case of zero detuning, which has been treated by most authors in the context of the time dependent model, is analytically solvable for all choices of the time dependence. In section IV, we describe our choice for the time dependence of the interaction term, for which an analytic solution can be obtained for arbitrary detuning. We also outline a possible physical situation where such an interaction could arise. Section V contains details of the analytic solution, while section VI discusses the time evolution of atomic inversion in the one-photon linear medium case, as an example of the results that can be obtained. Finally, section VII has a few concluding remarks, and a few avenues for further investigations are indicated.

II. THE GENERALISED JCM

The basic JCM is defined by the Hamiltonian

$$H = \omega a^\dagger a + \frac{\omega_0}{2} \sigma_3 + (\lambda a^\dagger \sigma_- + \lambda^* a \sigma_+)$$

where $a$ is the annihilation operator for the single mode radiation field, while $\sigma_3, \sigma_- , \sigma_+$ are pseudospin operators describing the two level atom. Various authors have considered different generalisations to this basic Hamiltonian. In what follows we are going to consider the case of real coupling constant $\lambda(t) = \lambda(t)^*$. The wide variety of generalized JCM Hamiltonians that can be exactly solved, at least for time independent couplings, have led Yu et al (11) into a search for an underlying algebraic structure unifying all of them. In their notation a very general class of JCMs can be represented by the generic formula

$$H = r (A_0) + s (A_0) \sigma_3 + \lambda(t) (A_+ \sigma_- + A_- \sigma_+)$$

where $A_0$ is some generalised number operator for the photon field, and $A_+, (A_-)$ represent generalised raising (lowering) operators for it. We have changed the original notation slightly in order to exhibit the time dependence in the interaction term explicitly. Here the first term in the Hamiltonian (2) obviously stands for the energy of a free photon field, whereas the second term stands for the two level atom, the level splitting depending on the strength of the photon field. The last two terms stand for the field-atom interaction in the rotating wave approximation. Here,
the generalisation over the standard JCM is that the operators $A_+, (A_-)$ raise (lower) the eigenvalue of $A_0$ by a number $m$ which can be integer or fractional. Obviously a rather large collection of generalized JCMs, including those mentioned above, can be treated as special cases of this Hamiltonian.

As noted in [11] the algebraic structure of the system can be summarised by

\[ [A_0, A_\pm] = \pm m A_\pm \]
\[ A_+ A_- = \chi(A_0) \]  

By following the same elementary steps as for determination of the angular momenta eigenvalues from their algebra, here we can see that the states for which $\chi(A_0)$ vanishes, while increasing, which we will collectively refer to as "low" states act as lower cut off for ladders of states for which the $A_0$ eigenvalues change in steps of $m$. Each state of the ladder is generated from the previous one by the action of $A_+$. Depending on the function $\chi(n)$, the ladders may or may not have a upper cut off. We will refer to the upper cut off states, if they exist as "high" states.

### III. THE UNDERLYING ALGEBRA

One next notes that the Hamiltonian (2) admits of a conserved quantity

\[ \Delta = A_0 + \frac{(1 + \sigma_3)}{2} m \]

which not only commutes with the Hamiltonian, does so with every one of it’s four terms. Thus the dynamics splits into a number of two dimensional subspaces spanned by $|n, \uparrow\rangle$ and $|n + m, \downarrow\rangle$, the first label being the $A_0$ eigenvalue. On the other hand, the states $|l, \downarrow\rangle$ and $|h, \uparrow\rangle$ where the $A_0$ eigenstates $|l\rangle$ and $|h\rangle$ belong to the "low" and "high" states, respectively span one dimensional invariant subspaces under the dynamics. Time evolution of these latter subspaces is trivial, since the interaction term vanishes there.

Leaving aside the one-dimensional subspaces, where the operator $\chi(\Delta)$ vanishes, we can define three spin operators

\[ J_1 = \frac{1}{2\sqrt{\chi(\Delta)}} (A_+ \sigma_- + A_- \sigma_+) \]
\[ J_2 = \frac{1}{2\sqrt{\chi(\Delta)}} (A_+ \sigma_- - A_- \sigma_+) \]
\[ J_3 = \frac{1}{2} \sigma_3 \]

These operators obey the pauli matrix algebra

\[ J_i J_j = i \epsilon_{ijk} J_k \frac{1}{2} + \delta_{ij} I. \]

Now, one can rewrite (2) in terms of $\Delta$:

\[ H = \Omega(\Delta) + \delta(\Delta) \sigma_3 + \lambda(t) (A_+ \sigma_- + A_- \sigma_+) \]

Here the terms in this Hamiltonian is related to those in the original one by

\[ \Omega(\Delta) = \frac{r(\Delta-m)+r(\Delta)}{2} + \frac{s(\Delta-m)-s(\Delta)}{2} \]
\[ \delta(\Delta) = \frac{r(\Delta-m)-r(\Delta)}{2} + \frac{s(\Delta-m)+s(\Delta)}{2} \]

Before proceeding further, we give a specific example for the sake of concreteness. We consider the case of the $m$-photon JCM in a Kerr medium. This is described by (2) with

\[ A_0 = a^\dagger a \]
\[ A_+ = (a^\dagger)^m \]
\[ A_- = a^m \]
\[ \chi(n) = \frac{n!}{(n-m)!} \]
\[ r(n) = \omega n + \kappa(n^2 - n) \]
\[ s(n) = \frac{\omega}{2} \]
This leads to

\[ \Omega(\Delta) = \left( \Delta - \frac{m^2}{2} \right) \omega + \kappa \left( \Delta^2 - (m+1)\Delta + \frac{m(m+1)}{2} \right) \]

\[ \delta(\Delta) = \delta + \kappa \left( \frac{m(m+1)}{2} - m\Delta \right) \]

where the detuning parameter \( \delta \) is defined as \( \omega_0 - m\omega \). In this example the photon number states \(|0\rangle, |1\rangle, \ldots, |m-1\rangle \) comprise the low states \(|l\rangle \), and there are no high states. Putting \( \kappa = 0 \) gives us the \( m \)-photon JCM.

Resuming our analysis, the time evolution operator can be written as

\[ U(t) = e^{-i\Omega(\Delta)(t-t_0)} U_i(t) \]  

(13)

with \( U_i \) satisfying the interaction picture Schrödinger equation

\[ i \frac{\partial U_i}{\partial t} = H_i(t) U_i(t) \]  

(14)

where, in terms of the pseudospin operators the interaction Hamiltonian becomes that of a negatively charged spin-\( \frac{1}{2} \) in a time varying magnetic field:

\[ H_i = J \cdot B, \]  

(15)

where the magnetic field \( B \) is given by:

\[ B = 2 \left( \lambda(t) \sqrt{\chi(\Delta)} \right) \]  

(16)

In the time independent JCMs, this reduces to the motion of a spin-\( \frac{1}{2} \) particle in a constant magnetic field the case that was treated in [11]. Our interest lies in the case where the coupling is time dependent. It is easy to see that for JCMs in linear media the case of zero detuning \( \delta = 0 \) is special in that there the magnetic field is unidirectional. Thus the interaction Hamiltonian at different times commute, leading to a exactly solvable time evolution operator.

\[ U_i = \exp \left[ 2i \sqrt{\chi(\Delta)} \left( \int_{t_0}^{t} dt' \lambda(t') \right) J_1 \right]. \]  

(17)

Most of the authors who have treated time dependent versions of the JCM analytically have dealt with the case of zero detuning \( \delta = 0 \). It should be noted, however, that in the case of presence of Kerr or other nonlinearities, the function \( \delta(\Delta) \) would not reduce to a constant, and the magnetic field will not be unidirectional for all the invariant subspaces even for zero detuning.

Our interest in the current article lies in looking for a time dependent interaction term, which will allow analytical solution even when the detuning is non-zero. In this we benefit from the fact that the spin - field interaction has been widely studied.

**IV. THE TIME DEPENDENCE**

In our version of the JCM, we take the coupling constant \( \lambda \) to be time dependent of the form :

\[ \lambda(t) = \lambda_0 \text{sech}(t/(2\tau)). \]  

(18)

In this form the coupling increases from a very small value at large negative times to a peak at time \( t = 0 \), to decrease exponentially at large times. Thus, depending on the value of \( \tau \) and the initial time \( t_0 \), various limits such as adiabatically or rapidly increasing (for \( t_0 < t < 0 \)) or decrasing (for \( 0 < t < t_0 \)) coupling can be conveniently studied. This allows us to investigate, analytically, the effect of transients in various different limits of the effect of switching the interaction on and off in the cavity atom-maser system. The choice of the interaction was guided by the fact that the JCM has su(2) as an underlying algebra [11], and the above interaction is known to give rise to exactly solvable dynamics for a spinning particle in a time dependent magnetic field [12,13].

It should be noted that the time dependence specified in (18) is only one of a class of generalised interactions that afford analytical solutions [14]. One reason why we confine ourselves to this particular interaction is that it is the simplest one among this class. Also, the form of the potential (18) is not unfamiliar in the subject of two level atoms interacting with radiation. The famous area and shape stable solution for the loss less Maxwell equation coupled with the optical Bloch equation that is encountered in the study of self induced transparency has the same time dependence [15,16]. Phase modulation effects in such a medium is another place where this shows up [17,18]. One possible use of this potential, then, could be the investigation of the effects of quantizing the radiation field in these problems.
V. THE ANALYTICAL SOLUTION

Following the Wei-Norman formalism \[12,17\] we write the evolution operator in the form (where we have suppressed the \(\Delta\) dependence for convenience)

\[
U_i = e^{-i\delta(t-t_0)\sigma_3} \tilde{U}_i, \tag{19}
\]

where \(\tilde{U}_i\) is

\[
\tilde{U}_i = e^{h(t)\sigma_3} e^{g(t)J_+} e^{-f(t)J_-}. \tag{20}
\]

We introduce the new functions

\[
\mathcal{H} = e^{-h}, \quad \mathcal{F} = f e^{-h}, \quad \mathcal{G} = g e^h. \tag{21}
\]

It can be shown that \(\mathcal{H}, \mathcal{G}^*\) and \(\mathcal{F}\) obeys the second order differential equation

\[
\frac{d^2X}{dt^2} + \left[ -\frac{d}{dt} \ln \lambda(t) + i2\delta(n+m) \right] \frac{dX}{dt} + \chi(n+m)\lambda(t)^2 X = 0, \tag{22}
\]

with the initial conditions

\[
\mathcal{H}(t_0) = 1, \quad \dot{\mathcal{H}}(t_0) = 0, \quad \mathcal{F}(t_0) = 0, \quad \dot{\mathcal{F}}(t_0) = i\sqrt{\chi(n+m)}\lambda(t_0).
\]

The initial conditions for \(\mathcal{F}\) and \(\mathcal{G}^*\) are identical, implying

\[
\mathcal{G} = \mathcal{F}^*. \tag{23}
\]

In terms of these new functions the operator \(\tilde{U}_i\) is

\[
\tilde{U}_i = \begin{pmatrix} \mathcal{H}^* & \mathcal{F}^* \\ -\mathcal{F} & \mathcal{H} \end{pmatrix}. \tag{24}
\]

Changing variable to

\[
z(t) = \frac{e^{t/\tau}}{1 + e^{t/\tau}}, \tag{25}
\]

the equation (22) becomes

\[
z(1-z) \frac{d^2X}{dz^2} + (\gamma - z) \frac{dX}{dz} + \alpha^2 X = 0, \tag{26}
\]

where \(\gamma = 1/2 + i2\delta(n+m)\tau\) and \(\alpha = 2\lambda_0\tau \sqrt{\chi(n+m)}\). This is the hypergeometric equation with \(\beta = -\alpha\) and thus

\[
\mathcal{H} = A_h \ _2F_1(\alpha, -\alpha; \gamma; z) + B_h \ z^{1-\gamma} \ _2F_1(\alpha - \gamma + 1, -\alpha - \gamma + 1; 2 - \gamma; z) \tag{27}
\]

\[
\mathcal{F} = A_f \ _2F_1(\alpha, -\alpha; \gamma; z) + B_f \ z^{1-\gamma} \ _2F_1(\alpha - \gamma + 1, -\alpha - \gamma + 1; 2 - \gamma; z) \tag{28}
\]

The initial conditions lead to :
This is given, in the one-photon case, by

\[ A_h = (1 - z_0)^{-\gamma} 2 F_1(\alpha - \gamma + 1, -\alpha - \gamma + 1; 1 - \gamma; z_0), \]

\[ B_h = \frac{z^2 - z_0}{2(1 - z_0)^{-\gamma}} \left( \frac{z}{1 - z_0} \right)^{\gamma-1} 2 F_1(\alpha + 1, -\alpha + 1; \gamma + 1; z_0), \]

\[ A_f = -i \alpha \left( \frac{z}{1 - z_0} \right)^{1 - \gamma} 2 F_1(\alpha - \gamma + 1, -\alpha - \gamma + 1; 2 - \gamma; z_0), \]

\[ B_f = -i \alpha \left( \frac{z}{1 - z_0} \right)^{1 - \gamma} 2 F_1(\alpha - \alpha; \gamma; z_0) \] (29)

where \( z_0 = z(t_0) \).

If we now write the state of the system at time \( t \), by

\[ |\psi(t)\rangle = \sum_{n=0}^{\infty} [u_n(t)|n, \uparrow\rangle + v_n(t)|n, \downarrow\rangle], \] (30)

the time dependent functions \( u_n(t) \) and \( v_n(t) \) can be seen to be

\[ u_n(t) = \exp\left[-i(\Omega(n + m) + \delta(n + m))(t - t_0)\right] \left[H_{n+m}^+ u_n(t_0) + \mathcal{F}_{n+m}^0 v_{n+m}(t_0)\right], \]

\[ v_{n+m}(t) = \exp\left[-i(\Omega(n + m) - \delta(n + m))(t - t_0)\right] \left[-\mathcal{F}_{n+m}^0 u_n(t_0) + H_{n+m}^+ v_{n+m}(t_0)\right]. \] (31)

Here the subscripts on \( H \) and \( \mathcal{F} \) denote that they correspond to the subspace \( \Delta = n + m, i.e. \) the subspace spanned by \( |n, \uparrow\rangle \) and \( |n + m, \downarrow\rangle \). The evolution equation (21) is valid for all nonnegative integer \( n \). The one dimensional subspaces, spanned by the "low" states, \( |l, \downarrow\rangle \) evolve according to

\[ v_l(t) = e^{-i(\Omega(t) - \delta(t))(t - t_0)} v_l(t_0). \] (32)

while the evolution of the "high" states \( |h, \uparrow\rangle \) is given by

\[ u_h(t) = e^{-i(\Omega(h + m) + \delta(h + m))(t - t_0)} u_h(t_0). \] (33)

The equations (31), (32) and (33) describe the complete time evolution of the state of the Jaynes-Cummings system.

**VI. TIME EVOLUTION OF ATOMIC INVERSION**

Although all the results presented so far are equally applicable to all the generalized JCM’s that can be described by (3) in what follows we will deal only with the standard one-photon JCM with time dependent coupling given by (3), mainly for the sake of brevity. A quantity of physical interest is the time evolution of the atomic inversion \( \sigma_3 \). This is given, in the one-photon case, by,

\[ \langle \sigma_3(t) \rangle = -|v_0(t_0)|^2 + \sum_{n=0}^{\infty} \left[ (1 - 2|\mathcal{F}_{n+1}|^2) \left(|u_n(t_0)|^2 - |v_{n+1}(t_0)|^2\right)^2 + 4\text{Re} \left( H_{n+1} \mathcal{F}_{n+1}^* u_n(t_0)^* v_{n+1}(t_0)\right) \right], \] (34)

where use has been made of \( |H_{n+1}|^2 + |\mathcal{F}_{n+1}^2| = 1 \). In particular, we consider the initial condition where the radiation field is in the number state \( |n\rangle \) and the atom is in a superposition \( c_\uparrow |\uparrow\rangle + c_\downarrow |\downarrow\rangle \); the atomic inversion evolves as

\[ \langle \sigma_3(t) \rangle = p_c \left( 1 - 2|\mathcal{F}_{n+1}|^2 \right) - (1 - p_c) \left( 1 - 2|\mathcal{F}_{n}|^2 \right). \] (35)

A uniquely quantum mechanical feature of the Jaynes-Cummings model is the collapse and revival of atomic inversion when the radiation field is initially in a coherent state. We consider a special case when the atom is initially in the excited state, and derive

\[ \langle \sigma_3(t) \rangle = \sum_{n=0}^{\infty} \left( 1 - 2|\mathcal{F}_{n+1}|^2 \right) \frac{\bar{n}^n}{n!} e^{-\bar{n}}, \] (36)

where \( \bar{n} \) is the mean photon number in the coherent state.

In Fig. 1 we plot the variation of the atomic inversion with time, for initial time \( t_0 = -10\tau \), for two different detunings \( \delta = 0 \) and \( \delta = \tau \), respectively. In this, we have assumed the initial condition that the atom is in the excited
state, \( p = 1 \), and the radiation field is in the number state \( n = 3 \). The effect of the detuning on both the frequency and the amplitude of the Rabi oscillations is quite marked. Note, that as the coupling strength varies, the frequency of Rabi oscillations (the Rabi frequency) changes. The vanishing of the interaction at large positive times leads to the levelling out of the inversion. Fig. 2 shows the inversion for an initial state where the atom is excited and radiation is in a coherent state \( \bar{n} = 10 \) for \( \lambda_2 \tau = 5.0 \) for various detunings. The initial time is \( t_0 = -10\tau \), so the interaction starts at a fairly low value, peaks and then drops off again. The inversion shows a single revival and collapse after the initial collapse, finally levelling out at large times as expected. Finally Fig. 3 shows the same quantities, but with \( t_0 = 0 \), so that the interaction decreases monotonically. It is seen that the interaction drops off too fast to show secondary collapses, after the initial rapid one. It is also seen that though the effect of the detuning is quite marked for \( t_0 = -10\tau \) it is small for \( t_0 = 0 \).

VII. CONCLUSIONS

We have seen one example of a time dependent JCM which can be solved analytically for arbitrary detunings. Of course, a large number of other physical quantities, e.g. photon number distribution, squeezing of the radiation field, atom-field relative phase can be studied for this particular model. Of particular interest may be the effect of introducing a Kerr nonlinearity in the problem.

As pointed out before, (18) is one member of a large subclass of interactions which have been studied in the NMR literature which lead to solvable time dependent problems. In particular, one can consider cases where the time dependence is asymmetric about \( t = 0 \), and investigate the effects of such asymmetry on physical observables.

Apart from the fact that the reduction of the Jaynes Cummings system to the spin - magnetic field system helps us to find interactions for which analytical solutions can be found, the algebraic structure also allows us to develop a convenient approximation scheme for general time dependences. We can treat the case of small detuning as a perturbation on the exactly solvable zero detuning case. Algebraic unitarity preserving perturbation methods, such as the Magnus-Fer method [12,13] could be particularly convenient here. Such extensions will be the subject of a future article.

The routine developed by Perger et al [19] was used to calculate the hypergeometric function.

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• **Figure 1** - Atomic inversion for an initially excited atom, with the radiation field in a number state $n = 3$, at initial time $t_0 = -10\tau$. The peak strength of the interaction is $\lambda_0 = 5\tau$. The solid line describes the system at resonance, $\delta = 0$, while the broken line is for $\delta = \tau$.

• **Figure 2** Atomic inversion for $\lambda_0 = 5\tau$ for an initially excited atom and the radiation field in a coherent state with $\bar{n} = 10$ at the initial time $t_0 = -10\tau$. The solid line is for $\delta = 0$ and the broken line is for $\delta = 0.5\tau$.

• **Figure 3** Same as in Fig. 2, except for $t_0 = 0$. 
Inversion

Time
