q-Bosonization of the quantum group $GL_q(2)$ based on the Gauss decomposition

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Abstract.

The new method of q-bosonization for quantum groups based on the Gauss decomposition of a transfer matrix of generators is suggested. The simplest example of the quantum group $GL_q(2)$ is considered in some details.

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1. The method of construction of Lie group and algebra representations using the realization their generators in terms of boson oscillator creation and annihilation operators is well known. The example of such construction gives the famous Schwinger representation (bosonization) for angular momentum operators. For investigations of representations of quantum groups and algebras [1,2] the similar construction is also useful. However in this case it is natural to replace the usual boson oscillators on deformed ones [3-6]. This procedure is called $q$-bosonization.

For the simple quantum algebras of all classical series from Cartan list $q$-bosonization procedure is described with sufficient completeness in [7] (see also [6] for the noncompact $su_q(1,1)$ and [8-10] for super case). However, the situation for quantum groups is quite different. It seems first attempts to this end have been done for the quantum group $GL_q(n)$ in the special case $q^n = 1$ in [11,12]. In the general case, $q \in C\backslash\{0\}$, the examples of $q$-bosonization were given in [13] and [14] for the $GL_q(2)$ and $GL_q(3)$ respectively. The attempt to generalize these results to the $GL_q(n)$ was undertaken in somewhat complicated fashion in Ref. [15]. The receipt given in [15] based on the special construction of the representation in the $q$-analog of the highest weight module. The application of this receipt to the $n > 3$ case needs very tedious computations. The deficiency of this approach [13,15] consists in the use of specific features of the Fock representation for the $q$-oscillators. As a result, a wide class of non Fock representations (listed, for instance, in [5,16-17]) are excluded from considerations ab initio. We remark also that the direct ways to extend this receipt to quantum groups of other series are absent. Let us also mention the interesting works [18,19] concerned with the similar problem for the matrix pseudogroups $S\mu U(n)$ and used somewhat different form of $q$-oscillators. Thus we see that the search of new ways of realization of $q$-bosonization procedure seems very desirable.

In this paper we want to show that the Gauss decomposition [20] for the $GL_q(2)$ suggests $q$-bosonization in a very simple and pure algebraic way. Let us note that similar methods applied to the general case of the $GL_q(n)$ quantum group [21] as well as to quantum groups of $B_n$, $C_n$ and $D_n$ series [22]

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2. Let us recall [1] that quantum group $GL_q(2)$ is defined as associative unital algebra freely generated by four generators $a, b, c, d$ (usually written as entries of the matrix $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$) factorized by 2-ideal generated by commutation relations of the form

\[
ab = qba, \quad ac = qca, \quad bd = qdb, \quad cd = qdc \\
bc = cb, \quad ad - da = \lambda bc,
\]

where $\lambda \equiv q - q^{-1}$. From this commutation relations it follows that the quantum ($q$-
determinant
\[ D \equiv D_q(T) \equiv \det_q(t) := ad - qbc = da - q^{-1}bc \]
commutes with all generators and thus belongs to the center of quantum group \( GL_q(2) \). Let us suppose that \( D_q(T) \equiv \det_q(t) \neq 0 \). The additional condition \( D_q(T) = 1 \) pick out the quantum group \( SL_q(2) \). Let us recall that the quantum groups \( GL_q(2) \) and its subgroup \( SL_q(2) \) are not groups in usual sense. In particular, the entries of the matrix \( T^n \) subject to the modified commutation relations (1) with \( q \) replaced by \( q^n \).

3. **Quantum deformed oscillator** (or simply **q-oscillator**) is [3-5] an associative algebra \( A_q \) generated by three elements \( a(= a_-), \ a^\dagger(= a_+) \) and \( N \), which fulfill the following commutation relations

\[ [a, a^\dagger]_q \equiv aa^\dagger - qa^\dagger a = q^{-N}, \quad [N, a] = -a, \quad [N, a^\dagger] = a^\dagger. \]

From this relations it follows that \([N, a^\dagger a] = 0 = [N, aa^\dagger]\). The element \( \zeta = q^{-N}([N] - a^\dagger a) \), where \([N] = \frac{q^N - q^{-N}}{q - q^{-1}}\), commutes with every generator and thus belong to the center of the q-oscillator algebra \( A_q \). It is worth-while to stress that the deformed boson oscillator algebra has nontrivial center unlike the undeformed case. This nontriviality of the center is one of the reasons for existence of the non-Fock representations of the q-oscillator algebra \( A_q \) [5,16,17]. The q-analog of the Fock representation can be constructed easily from the usual Fock representation of boson oscillator. Let \( \{|n >: n = 0, 1, 2, \ldots\} \) be the standard basis in the Fock space \( H \) with vacuum \(|0 >\). Then the operators \( a, a^\dagger \) and \( N \), defined by the following action on this basis

\[ N|n > = n|n >, \quad a|n > = \sqrt{|n||n - 1 >}, \quad a^\dagger|n > = \sqrt{|n + 1||n + 1 >}, \]

fulfill the commutation relations (3) and define the Fock representation for q-oscillator algebra \( A_q \). As in the usual case the basic vector \(|n >\) can be obtained from the vacuum state by repeated action of the q-creation operator \( a^\dagger \), namely, \(|n > = ([n]!)^{-1/2} (a^\dagger)^n |0 >\). On the Fock space q-oscillator operators \( a, a^\dagger \) are connected with usual boson oscillator operators \( b, b^\dagger \) by the relations

\[ a = b([N]/N)^{1/2}, \quad a^\dagger = ([N]/N)^{1/2} b^\dagger, \quad N = b^\dagger b. \]

From these relations it follows that Fock basis is not deformed (but the q-analogs of coherent states was subjects to deformation [5]). Now, let us note that on the Fock space \( H \) the central element \( \zeta \) vanishes (\( \zeta = 0 \)). From this fact it follows that on the Fock space \( H \) special relations

\[ [N] = a^\dagger a, \quad [N + 1] = aa^\dagger \]

(3)
are valid, as well as the additional commutation relation

\[ aa^\dagger - q^{-1}a^\dagger a = q^N, \]  

which realize the specific \( q \leftrightarrow q^{-1} \) symmetry inherited to the Fock representation only. We would like to stress that this additional relations (4)-(5) are not valid in other representations in which the central element \( \zeta \) takes the values different from zero. This means that including of the relation (5) (or, equivalent, relations (4)) into the list of defining commutation relations (3) drastically restricts possible representations to the Fock representation only. It can be shown (see [26],for example) that such extension of the commutation relations (3) generates the algebra \( A_{q,1/q} \) isomorphic to the quantum algebra \( U_q sl(2) \). Let us remark that there is one more important difference between \( q \)-oscillator algebra \( A_q \) and its restricted form \( A_{q,1/q} \). Namely, the \( A_{q,1/q} \) may be endowed with a Hopf algebra structure inherited from quantum algebra \( U_q sl(2) \) one, but such structure for \( A_q \) is still unknown and probably doesn’t exist at all.

4. The first example of \( q \)-bosonization of quantum group \( GL_q(2) \) was suggested in [13] (the early results [10-12] concern with the specific particular case ”\( q \) is a root of unity”). The example given in [13] i is based on the isomorphism between Fock space for \( q \)-oscillator and \( q \)-analogue f a Verma module (constructed in [13]) in which representation of the quantum group \( GL_q(2) \) acts. As a result the authors received two following variants of \( q \)-bosonization of the quantum group \( GL_q(2) \)

\[
T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda \mu \nu q^N a_- & \mu q^N \\ \nu q^N & \alpha_+ \end{pmatrix},
\]

\[
T_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -\lambda \mu a_+ & \mu q^{-N} \\ \nu q^{-N} & q^{-N} \alpha_- \end{pmatrix},
\]

where, as usual, \( \lambda = q - q^{-1} \) and \( \mu, \nu \) are parameters of the realization.

It is also possible, of course, to consider another variants of \( q \)-bosonization for \( GL_q(2) \). For example, realization, using only creation operator \( a_+ \), has the form

\[
T_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \mu q^N & a_+ \\ a_+ & \mu^{-1} q^{-N} (\mathcal{D}_q + qa_+^2) \end{pmatrix}.
\]

Another example using two independent (mutually commuting) \( q \)-oscillators \( \{a_\pm^{(i)}, N_i; i = 1, 2\} \) has the form

\[
T_3 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda \mu a_+^{(1)} a_-^{(1)} q^{N_2-1} & \mu \nu \sigma^{-1} X_2 \\ \sigma q^{N_2} & \nu W_1^{-1} \end{pmatrix},
\]
where \( X_2 = \lambda a_+^{(2)} a_-^{(2)} + q^{-N_2} \), \( W_1 = q a_+^{(1)} a_-^{(1)} + q^{-N_1} \) and \( W_1^{-1} \) is understand in the sense of the formal power series. In this case \( D_q = -\mu \nu q^{-1} \).

5. We’ll show now, that the Gauss decomposition of a quantum matrix \( T \) allows us to give the rather simple procedure for construction of \( q \)-bosonization for \( GL_q(2) \), which has more or less natural generalization not only to the case of quantum groups \( GL_q(n) \) with \( n > 2 \), but also for quantum groups of the series \( B_n, C_n \) and \( D_n \) \cite{21,22}. Let us return to the simplest case of the quantum group \( GL_q(2) \) and consider the Gauss decomposition of it’s quantum matrix \( T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \):

\[
T = T_L T_D T_R = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} = \begin{pmatrix} A + uBz & uB \\ Bz & B \end{pmatrix}.
\]

Such Gauss decomposition was firstly considered in the work \cite{20} (remark, that such decomposition is used also in \cite{23-25}) where it was noticed that such decomposition gives the new set of generators for \( GL_q(2) \), which have more simple commutation rules than relations (1) for original generators. Indeed, this relations have the following (\( q \)-Weyl) form

\[
AB = BA, \ Au = quA, \ Az = qzA, \ uB = qBu, \ zB = qBz, \ uz = zu.
\]

If we suppose the invertibility of the initial generator \( d \) in \( GL_q(2) \) (or adding \( d^{-1} \) to the list of generators) we have

\[
B = d, \ z = d^{-1}c, \ u = bd^{-1}, \ A = a - bd^{-1}c,
\]

from which it follows that

\[
T = \begin{pmatrix} A + uBz & uB \\ Bz & B \end{pmatrix} = \begin{pmatrix} 1 & bd^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a - bd^{-1}c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ d^{-1}c & 1 \end{pmatrix}.
\]

The element \( d^{-1} \) has the following commutation relations

\[
d^{-1}c = qcd^{-1}, \ d^{-1}b = qbd^{-1}, \ dd^{-1} = d^{-1}d,
\]

\[
d^{-1}a - q^2ad^{-1} = (1 - q^2)D_q(d^{-1})^2, \ D_qd^{-1} = d^{-1}D_q.
\]

Note, that quantum determinant has the same value as before and is equal to the usual determinant of central diagonal matrix of the Gauss decomposition (6)

\[
D_q = ad - qbc = (a - bd^{-1}c)d = AB = \det T_D.
\]

Let us remark also, that if we take another variant of the Gauss decomposition

\[
T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & Az \\ uA & uAz + B \end{pmatrix},
\]

5
instead of (6), then we receive the same commutation rules (7) for modified generators as before.

6. Now let us turn to the problem of $q$-bosonization. Because of the commutation rules for the new generators are simpler than fundamental relations (1) it is natural to begin with them. We start with the case of two independent (mutually commuting) $q$-oscillators $\{a_{\pm}^{(i)}, N_i; i = 1, 2\}$. Comparing the commutation rules (3) and (7) we suppose that

$$u = \Psi_u(a_{\pm}^{(1)}, N_1)a_{\pm}^{(1)}; \quad z = \Psi_z(a_{\pm}^{(2)}, N_2)a_{\pm}^{(2)};$$

$$A = \Psi_A(a_{\pm}^{(i)}, N_i); \quad B = \Psi_B(a_{\pm}^{(i)}, N_i);$$

Here functions of the type $\Psi_u$, etc. are understanding as formal series or polynomials in its arguments and a priori are rather arbitrary. Under such assumptions the commutation relations $[u, z] = 0 = [A, B]$ are fulfilled automatically, but other $q$-commutators give some additional restrictions. For example, for $u \sim a_{\pm}^{(i)}$ commutator $Au = quA$ reads

$$\Psi_A(a_{\pm}^{(i)}, N_i)a_{\pm}^{(1)} = qa_{\pm}^{(1)}\Psi_A(a_{\pm}^{(i)}, N_i). \quad (7)$$

From this we received (after expanding $\Psi_A$ into the series and moving $a_{\pm}^{(1)}$ to the right side)

$$a_{\pm}^{(1)}\Psi_A(a_{\pm}^{(i)}, a_{\mp}^{(i)}, N_i) = \Psi_A(q^{-1}a_{\pm}^{(1)}a_{\pm}^{(1)} - q^{-N_1}, a_{\pm}^{(i)}, a_{\pm}^{(1)}, N_1 - 1, N_2)a_{\pm}^{(1)}$$

If we insert the last expression into the former one we arrived to the relation

$$\Psi_A(a_{\pm}^{(1)}, a_{\mp}^{(1)}, N_1) = q\Psi_A(q^{-1}a_{\pm}^{(1)}a_{\pm}^{(1)} - q^{-N_1}, N_1 - 1)a_{\pm}^{(1)}$$ \quad (8)

where we omit variables related to the second oscillator which are not changing in such processes. After repeating this procedure with all other commutators from (7) we received

$$\Psi_A(a_{\pm}^{(i)}, a_{\mp}^{(i)}, N_i) = q\Psi_A(q^{-1}a_{\pm}^{(i)}a_{\mp}^{(i)} + q^{-N_i}, N_i \mp 1, N_2) \quad (9)$$

$$\Psi_B(a_{\pm}^{(i)}, a_{\mp}^{(i)}, N_i) = q^{-1}\Psi_B(q^{+1}a_{\pm}^{(i)}a_{\mp}^{(i)} + q^{-N_i}, N_i \mp 1, N_2) \quad (10)$$

Let us note that $q$-commutators does not limit choice of the functions $\Psi_u, \Psi_z$. Although we can not give general solution of the equations (8)-(11) the concrete particular solutions can be finded without difficulty. For example the choice

$$\Psi_u = \alpha, \quad \Psi_z = \beta, \quad \Psi_A = \gamma q^{N_1 - N_2}, \quad \Psi_B = \delta q^{N_2 - N_1}$$

gives

$$u = \alpha a_{\pm}^{(1)}, \quad z = \beta a_{\pm}^{(2)}, \quad A = \gamma q^{N_1 - N_2}, \quad B = \delta q^{N_2 - N_1}; \quad (11)$$
and defines the following realization for the matrix $T$ of $GL_q(2)$-generators

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \gamma q^{N_1-N_2} + \alpha\beta\delta q^{N_2-N_1}a_+^{(1)}a_-^{(2)} \\ \beta\delta q^{N_2-N_1}a_-^{(2)} \end{pmatrix}.$$

$$D_q = \det_q T = AB = \gamma\delta.$$

Slightly more general choice

$$u = \alpha a_+^{(1)}, \ z = \beta a_-^{(2)}, \ A = \gamma X_1 Y_2, \ B = \delta Y_1 X_2,$$

where

$$X_i = \lambda a_+^{(i)} a_-^{(i)} + q^{-N_i}, \ Y_i = \lambda a_+^{(i)} a_-^{(i)} - q^{-N_i+1}, \ \lambda = q - q^{-1},$$

gives

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \gamma X_1 Y_2 + \alpha\beta\delta q Y_1 X_2 a_+^{(1)}a_-^{(2)} \\ \beta\delta X_1 Y_2 a_-^{(2)} \end{pmatrix}.$$

$$D_q = \det_q T = AB = \gamma\delta X_1 X_2 Y_1 Y_2.$$

There is also a one $q$-boson realization. In particular, the above mentioned variant from [13] can be obtained if we take

$$u = \mu q^N W^{-1}a_-, \ z = \nu q^{-1} W a_-, \ A = \mu\nu(\lambda - q^{N+1}W^{-1})q^N a_-, \ B = a_+,$$

where $W = qa_+ a_- + q^{-N}$.

7. The developed above procedure of construction of $q$-bosonization is pure algebraic one and not based on concrete choice of representation of quantum group as well as $q$-deformed oscillator. At the same time it allows using concrete representations of $q$-oscillator algebras to build the quantum group representations in respected spaces. Let us consider the Fock representation for two independent $q$-oscillators with the basis

$$|n, m > = |n > \otimes |m > = ([n]![m]!)^{-1/2}(a_+^{(1)})^n(a_-^{(2)})^m|0 > \otimes |0 >$$

in the Fock space $\mathbf{H} \otimes \mathbf{H}$. Then for realization (12) we have

$$u|n, m > = \alpha \sqrt{[n+1]}|n+1, m >, \ z|n, m > = \beta \sqrt{[m]}|n, m-1 >, \ $$

$$A|n, m > = \gamma q^{n-m}|n, m >, \ B|n, m > = \delta q^{m-n}|n, m >.$$

These relations define the following representation of the quantum group $GL_q(2)$

$$a|n, m > = (A + uBz)|n, m >.$$
\[ \gamma q^{n-m}|n, m > + \alpha \beta \delta q^{m-n-1}|n, m > \sqrt{[n+1][m]}|n+1, m-1 >, \]
\[ b|n, m >= uB|n, m >= \beta \delta q^{m-n}\sqrt{[n+1][m]}|n+1, m >, \]
\[ c|n, m >= Bz|n, m >= \beta \delta q^{m-n-1}\sqrt{[m]}|n, m-1 >, \]
\[ d|n, m >= B|n, m >= \delta q^{m-n}|n, m > . \]

Let us consider the well-known realization (see f.ex. [6]) of the Fock representation for q-oscillator by q-difference derivative in the space \( P \) of polynomials \( p(w) \) on variable \( w \) with basis \( \{w^n\}_{n=0}^{\infty} \):

\[ p(w)a_+ \equiv M_w : p(w) \mapsto wp(w); w^n \mapsto w^{n+1}, \]
\[ a_- \equiv q D_w : p(w) \mapsto q D_w p(w) = \frac{p(qw)-p(q^{-1}w)}{(q-q^{-1})w}; w^n \mapsto [n]w^{n-1}, \]
\[ q^N \equiv q K_w : p(w) \mapsto q K_w p(w) = p(qw); w^n \mapsto q^n w^n, \]
\[ N \equiv w \frac{d}{dw} : p(w) \mapsto wp'(w) = p(qw); w^n \mapsto nw^n. \]

Then for the same q-bosonization (11) of the quantum group \( GL_q(2) \) we have the following realization of generators on the space \( P \otimes P \) by q-difference derivatives

\[ a = \gamma (q K_w)(q K_v)^{-1} + \alpha \beta \delta M_w (q K_w)^{-1} (q K_v)(\Phi_v), \]
\[ b = \alpha \delta M_w (q K_w)^{-1} (q K_v), \quad c = \beta \delta (q K_w)^{-1} (q K_v)(\Phi_v), \quad d = \delta (q K_w)^{-1} (q K_v). \]

Analogous formulae, of course, are valid and for other q-bosonizations of the quantum group \( GL_q(2) \).

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