Optimal Quantum-Enhanced Interferometry

Matthias D. Lang and Carlton M. Caves

1 Center for Quantum Information and Control, University of New Mexico, Albuquerque, New Mexico, 87131-0001, USA
2 Centre for Engineered Quantum Systems, School of Mathematics and Physics, The University of Queensland, St Lucia, QLD 4072, Australia

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We analyze the ultimate bounds on the phase sensitivity of an interferometer, given the constraint that the state input to the interferometer’s initial 50:50 beamsplitter $B$ is a product state of the two input modes. Requiring a product state is a natural restriction: if one were allowed to input an arbitrary, entangled two-mode state $\Xi$ to the beamsplitter, one could generally just as easily input the state $B(\Xi)$ directly into the two modes after the beamsplitter, thus rendering the beamsplitter unnecessary. We find optimal states for a fixed photon number and for a fixed mean photon number.

I. INTERFEROMETRIC SETTING

In this brief sketch, we consider an interferometric setting, depicted in Fig. [1] for determining the differential phase shift imparted to fields in the interferometer’s two arms. Two modes, with annihilation operators $a$ and $b$, are incident on a 50:50 beamsplitter; after the beamsplitter, the two arms experience phase shifts $\varphi_1$ and $\varphi_2$. To make the depicted setup into an interferometer, one would add a second 50:50 beamsplitter, at which the modes in the two arms are recombined. A prime reason for using such an interferometer is that it is insensitive to common-mode noise in the two arms, with each arm acting as a phase reference for the other; the interferometer is sensitive only to the differential phase shift $\phi_d = \varphi_1 - \varphi_2$. In the following, we perform a quantum Fisher analysis to determine the optimal sensitivity for estimating $\phi_d$; this analysis reports the optimal sensitivity without having to consider the second beamsplitter in the interferometer.

We assume that the state input to the beamsplitter is a product state $|\psi_{in}\rangle = |\xi\rangle \otimes |\chi\rangle$, where $|\xi\rangle$ is the state of mode $a$ and $|\chi\rangle$ is the state of mode $b$. The action of the beamsplitter is described by the unitary operator $B = e^{-i(a^{\dagger}b + b^{\dagger}a)\pi/4}$, so the state after the beamsplitter is $B|\psi_{in}\rangle$. The phase shifters are described by the unitary operator $U = e^{i(q(a^{\dagger}a + \varphi_2 b^{\dagger}b) - \varphi_1 a^{\dagger}a)}$. The state after the phase shifters is thus

$$|\psi\rangle = U B |\psi_{in}\rangle$$

This setup is very close to the setting we considered in [1], the physical difference being that in [1], the input state of mode $a$ was required to be a coherent state. We use the same notation here as in [1], except that there modes $a$ and $b$ were called $a_1$ and $a_2$.

Our restriction to product input states is natural—indeed, it is the only sensible assumption—in the case of an interferometric setup. If one allows arbitrary, entangled input states $|\Xi\rangle$ of the two input modes, there is no reason for the initial beamsplitter, since one might just as well input the state $B(\Xi)$ directly into the two arms approaching the phase shifters, thus rendering the beamsplitter unnecessary.

![Fig. 1](image.png)

FIG. 1: Two modes $a$ and $b$ are incident on a 50:50 beamsplitter. After the beamsplitter, phase shifts $\varphi_1$ and $\varphi_2$ are imposed in the two arms. A measurement is then made to detect the differential phase shift $\phi_d = \varphi_1 - \varphi_2$. When the measurement is pushed beyond a second 50:50 beamsplitter, i.e., when the DETECTION box includes a 50:50 beamsplitter before measurement, the result is a Mach-Zehnder interferometer, which is sensitive only to $\phi_d$.

To quantify the sensitivity of a particular product input, we use quantum Fisher information. Since we are interested in the differential phase shift, the relevant element of the Fisher matrix, called $F_{dd}$ in [1], is

$$F = \langle \psi_{in}|B^\dagger N_d B|\psi_{in}\rangle - \langle \psi_{in}|B^\dagger N_d B|\psi_{in}\rangle^2$$

$$= -\langle \langle a^{\dagger}b - b^{\dagger}a \rangle^2 \rangle + \langle a^{\dagger}b - b^{\dagger}a \rangle^2$$

$$= -\langle \langle a^{\dagger}a^{\dagger}b b \rangle - \langle a^{\dagger}b b\rangle + \langle b^{\dagger} b^{\dagger} a^{\dagger} a \rangle + \langle a^{\dagger} b^{\dagger} b \rangle + \langle b^{\dagger} b^{\dagger} a a \rangle \rangle \rangle$$

$$+ \langle \langle a^{\dagger} b \rangle^2 + \langle b^{\dagger} a \rangle^2 \rangle - 2 \langle \langle a^{\dagger} b \rangle \langle b^{\dagger} a \rangle \rangle \rangle \rangle$$

Here we introduce a notation that we use throughout the following: $\langle O \rangle = \langle \psi_{in}|O|\psi_{in}\rangle$ denotes an expectation value with respect to the input state $|\psi_{in}\rangle$. Notice that the Fisher information is the variance of $N_d$ in the state $B|\psi_{in}\rangle$ after the 50:50 beamsplitter.
To find the optimal performance, we maximize $F$ over all product input states, subject to whatever additional constraints we impose on the input; i.e., find whichever input state maximizes the variance of $N_d$ after the first beamsplitter. The expression (2) is valid for arbitrary inputs; specializing to product inputs gives

$$F = 2N_a N_b + N_a + N_b - \langle a^\dagger a^\dagger b b \rangle - \langle a a \rangle \langle b^\dagger b^\dagger \rangle$$

$$- 2 |\langle a |^2 |\langle b |^2 + |\langle a \rangle|^2 |\langle b |^2 + |\langle a \rangle|^2 |\langle b \rangle|^2 ,$$

(3)

where $N_a = \langle a |^2$ and $N_b = \langle b |^2$ are the mean photon numbers in the two input modes.

In the remainder of the paper, we first find the optimal input state for a fixed photon number and then find the optimal state for a constraint on mean photon number.

II. FIXED PHOTON NUMBER

If we fix the total photon number $N = N_a + N_b$, all product states have the form $|n\rangle \otimes |N - n\rangle$. Under these circumstances, only the first three terms in Eq. (3) contribute to the Fisher information. Finding the maximum reduces to finding the $n$ that maximizes $2n(N - n)$; the maximum is achieved at $n = N/2$ for $N$ even and at $n = (N ± 1)/2$ for $N$ odd, the two signs corresponding to an exchange of the input modes. The maximal Fisher information is

$$F_{\text{max}} = \begin{cases} \frac{N(N + 2)}{2}, & N \text{ even}, \\ \frac{N(N + 2) - 1}{2}, & N \text{ odd}. \end{cases}$$

(4)

The optimal state,

$$|\psi_{\text{in}}\rangle_N = \begin{cases} \langle N/2 | \otimes |N/2 \rangle, & N \text{ even}, \\ \langle (N ± 1)/2 | \otimes |(N ± 1)/2 \rangle, & N \text{ odd}, \end{cases}$$

(5)

is the twin-Fock state for $N$ even [3] and its closest equivalent for $N$ odd. The optimal state gives rise to a Quantum Cramér-Rao Bound (QCRB) for the variance of the phase estimate given by

$$(\Delta \phi_d^{\text{est}})^2 \geq \frac{1}{F_{\text{max}}} = \begin{cases} \frac{2}{N(N + 2)}, & N \text{ even}, \\ \frac{2}{N(N + 2) - 1}, & N \text{ odd}, \end{cases}$$

(6)

which shows an asymptotic Heisenberg scaling. An input twin-Fock state leads to entanglement between the two arms after the beamsplitter [2].

Holland and Burnett [3] introduced the twin-Fock state (for $N$ even) and considered the Heisenberg scaling of its phase sensitivity. Measurements that achieved the Heisenberg scaling were demonstrated in [4, 5]. Robustness of the twin-Fock state against various errors was investigated in [4, 7], and sub-shot-noise precision for interferometry with twin-Fock states was demonstrated experimentally in [8].

If we were to remove the restriction of having a product input to the interferometer, the optimal input would be the state that maximizes the variance of $N_d$ after the first beamsplitter, i.e. the entangled input state that becomes a $N00N$ state $B|\psi_{\text{in}}\rangle = (|N, 0 \rangle + |0, N\rangle)/\sqrt{2}$ after the beamsplitter [9, 11]. The $N00N$ state has a Fisher information $F = N^2$. For $N = 1$ and $N = 2$, the twin-Fock input produces a $N00N$ state on the other side of the beamsplitter and thus has the same Fisher information as the $N00N$ state. For $N \geq 2$, the $N00N$-state Fisher exceeds that of the twin-Fock state and is a factor of 2 larger asymptotically in $N$.

III. FIXED MEAN PHOTON NUMBER

We now move on to a constraint on the mean photon number $N_a + N_b$; in this section $N$ denotes this total mean photon number. The proof for the optimal input state consists of two steps. The first step finds the optimal states $|\xi \rangle$ and $|\chi \rangle$ under the assumption that both $N_a$ and $N_b$ have fixed values. It turns out that these optimal states are independent of how the total mean photon number $N$ is divided up between $N_a$ and $N_b$. In the second step, we show that the optimal split of resources is an equal division, $N_a = N_b = N/2$.

We begin the first step by noticing that the quantity on the second line of Eq. (3) is either negative or zero. We ignore this quantity for the moment. As we see shortly, the product input state that maximizes the top line has $\langle a | = \langle b | = 0$ and, therefore, also maximizes the quantum Fisher information $F$. Furthermore, in this first step, $N_a$ and $N_b$ are assumed to be fixed, so maximizing the top line reduces to maximizing

$$- \langle a^\dagger a^\dagger b b \rangle - \langle a a \rangle \langle b^\dagger b^\dagger \rangle .$$

(7)

We can always choose the phase of mode $a$, i.e., multiply $a$ by a phase factor, to make $\langle a a \rangle$ real and positive, i.e., $\langle a a \rangle = \langle a^\dagger a^\dagger \rangle \geq 0$. With this choice we need to maximize

$$- \langle a a \rangle \left( \langle b b \rangle + \langle b^\dagger b^\dagger \rangle \right) = \langle a a \rangle \left( \langle p^2 \rangle - \langle x^2 \rangle \right) ,$$

(8)

where in the second form we introduce the quadrature components $x$ and $p$ for mode $b$, i.e., $b = (x + ip)/\sqrt{2}$.

The proof continues along the lines of [11]:

$$\left( \langle p^2 \rangle - \langle x^2 \rangle \right)^2 = \left( \langle p^2 \rangle + \langle x^2 \rangle \right)^2 - 4\langle x^2 \rangle \langle p^2 \rangle$$

$$= (2N_b + 1)^2 - 4\langle x^2 \rangle \langle p^2 \rangle$$

$$\leq (2N_b + 1)^2 - 4(\Delta x)^2 (\Delta p)^2$$

$$\leq (2N_b + 1)^2 - 1$$

$$= 4N_b(N_b + 1) .$$

The first inequality is saturated if and only if $\langle x \rangle = \langle p \rangle = 0$; equality is achieved in the second inequality if and only
if the input state $|\chi\rangle$ of mode $b$ is a minimum-uncertainty state. This situation is identical to that in [1]: the optimal choice for $|\chi\rangle$ is squeezed vacuum with $x$ the squeezed quadrature and $p$ the anti-squeezed quadrature.

What is left now is to maximize

$$2\sqrt{N_b(N_b + 1)} |aa\rangle = \sqrt{N_b(N_b + 1)}(|aa\rangle + \langle a^\dagger a^\dagger\rangle)$$

(10)

over the input states $|\xi\rangle$ of mode $a$ for which $|aa\rangle$ is real and positive. This is the same maximization we just performed for mode $b$, except for a sign change, whose effect is to exchange the squeezed and anti-squeezed quadratures. The optimal state $|\xi\rangle$ is squeezed vacuum with $p$ as the squeezed quadrature and $x$ as the anti-squeezed quadrature.

Summarizing, the optimal input state is $S_a(\gamma r) |0\rangle \otimes S_b(r') |0\rangle$, where $r$ and $r'$ are real and positive and $S_c(\gamma) = \exp[\frac{i}{2} (\gamma a^2 - \gamma^* a^2)]$ is the squeeze operator for a field mode $c$. The values of the squeeze parameters are determined by $N_a = \sinh^2 r$ and $N_b = \sinh^2 r'$. Notice that, as promised, the optimal state has $\langle a \rangle = \langle b \rangle = 0$ and thus maximizes the Fisher information [3]: the maximum value is

$$F = 2N_a N_b + N_a + N_b + 2\sqrt{N_a(N_a + 1)N_b(N_b + 1)}.$$ 

(11)

The second step of the proof is now trivial. For a constraint on the total mean photon number $N = N_a + N_b$, it is straightforward to see that Eq. (11) is maximized by splitting the photons equally between the two modes, i.e., $N_a = N_b = N/2$. The resulting optimal input state has $r = r'$,

$$|\psi_{in, opt}\rangle = S_a(\gamma r) |0\rangle \otimes S_b(r') |0\rangle,$$

(12)

and the maximal Fisher information and corresponding QCRB are

$$F = N(N + 2), \quad \langle \Delta \phi_{opt}^2 \rangle \geq \frac{1}{F_{max}} = \frac{1}{N(N + 2)}.$$ 

(13)

This again exhibits Heisenberg scaling and, without the factor of 2 that appears in the fixed-photon-number result [4], achieves the $1/N$ Heisenberg limit.

A question that naturally arises is that of the optimal measurement. Here again we can refer to [1], which showed in the Supplemental Material that the classical Fisher information of photon counting after a second 50:50 beam splitter is the same as the quantum Fisher information, provided the coefficients of the expansion of the input state in the number basis are real. This requirement is met by the optimal state (12).

Unlike the situation where there is a strong mean field, however, the interferometer with dual squeezed-vacuum inputs runs on modulated noise, so the mean of the differenced photocount after a second 50:50 beamsplitter gives no information about the phase. One strategy for extracting the phase information is to look directly at the fluctuations by squaring the differenced photocount and thus effectively measuring $N^2$ [14,12]. One can show [14] that sensitivity at the optimal operating point achieves the QCRB [13].

Notice now that since $BaB^\dagger = (a+ib)/\sqrt{2}$ and $BbB^\dagger = (b+ia)/\sqrt{2}$, we have $B(a^2 - b^2)B^\dagger = a^2 - b^2$. Thus the beamsplitter leaves unchanged the product of squeeze operators in the optimal input state (12).

$$BS_a(\gamma r) S_b(r) B^\dagger = S_a(-r) S_b(r),$$

(14)

and this in turn means that the optimal input state is an eigenstate of the beamsplitter,

$$B|\psi_{in, opt}\rangle = BS_a(\gamma r) S_b(r) |0, 0\rangle = |\psi_{in, opt}\rangle.$$ 

(15)

Thus the state after the 50:50 beamsplitter is the same product of squeezed vacua as before the beamsplitter. The Heisenberg limit is thus achieved without any entanglement between the arms of the interferometer. In fact, Jiang, Lang, and Caves [2] showed that the state $|\psi_{in, opt}\rangle$ is the only nonclassical product state, i.e., not a coherent state, that produces no modal entanglement after a beamsplitter. These results indicate that, as in [14], modal entanglement is not a crucial resource for quantum-enhanced interferometry.

Caves pointed out that using squeezed states in an interferometer allows one to achieve sensitivities below the shot-noise limit [15]; this original scheme, often simply dubbed “squeezed-state interferometry,” involves injecting squeezed vacuum into the secondary input port of an interferometer. That squeezing the light into the primary input port, in addition to inputting squeezed light into the secondary port, is advantageous was first shown by Bondurant and Shapiro [16] and further investigated by Kim and Sanders [17]. All these papers, however, included a mean field in at least one of the input modes. Paris argued [18] that if one considers arbitrary squeezed-coherent states as interferometer inputs, putting all the available power into the squeezing, instead of into a mean field, yields better fringe visibility. Under a Gaussian constraint, Refs. [19,20] showed that a state that maximizes the Fisher information for a detecting a differential phase shift after a beamsplitter is $S_a(r) S_b(r) |0\rangle$. Relative to these last results, our contribution in this paper is to remove the assumption of Gaussianity, replacing it with a restriction to product inputs.

A problem with using Fisher information to find optimal states under a mean-number constraint is that one can come up with states that seemingly violate the Heisenberg limit. This was noted for single-mode schemes by Shapiro [21] and later by Rivas [22]. For the former case, Braunstein and co-authors showed that under a precise asymptotic analysis, no violation of the Heisenberg limit occurs [23,25]. For the latter case, it was shown that the Fisher information does not provide a tight bound, which makes a Fisher analysis uninformative [26,27].
same problem \cite{28}. Requiring product inputs removes this pathology of the Fisher information, therefore providing additional motivation for our product constraint.

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