THE MOTIVIC SEGAL-BECKER THEOREM FOR ALGEBRAIC K-THEORY

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ABSTRACT. The present paper is a continuation of earlier work by Gunnar Carlsson and the first author on a motivic variant of the classical Becker-Gottlieb transfer and an additivity theorem for such a transfer by the present authors. Here, we establish a motivic variant of the classical Segal-Becker theorem relating the classifying space of a 1-dimensional torus with the spectrum defining algebraic K-theory.

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1. Introduction

A classical result due to Segal from the early 1970s (see [Seg73]) is a theorem that shows the classifying space of the infinite unitary group, namely BU, is a split summand of \( \lim_{n \to \infty} \Omega_+^n ((S^1)^n \wedge \mathbb{CP}^\infty) \). A year later, Becker (see [Beck74]) proved a similar result for the infinite orthogonal group in the place of the infinite unitary group U and BO(2) in the place of \( \mathbb{CP}^\infty \).

The purpose of this paper is to consider similar problems in the motivic world and for Algebraic K-Theory, making use of a theory of the Motivic Becker-Gottlieb transfer worked out by Gunnar Carlsson and the first author in [CJ20] and the Additivity Theorem for such a transfer worked out in [JP20] by the authors. We will adopt the terminology and conventions from [CJ20] as well as other terminology that has now become standard. As such, the base scheme will be a perfect field \( k \) and we will restrict to the category of smooth schemes of finite type over \( k \). This category will be denoted \( \text{Sm}(k) \) and will be provided with the Nisnevich topology. \( \text{PSh}_*(k) \) will denote the category of pointed simplicial presheaves on this site. This category will be made motivic, by inverting the affine line \( k^1 \) as in [MV99]: the pointed simplicial presheaves in this category will be referred to as motivic spaces.

Then, the first observation (see [VV98, 6.2]) is that Algebraic K-theory is represented by the motivic-spectrum with \( \mathbb{Z} \times \text{BGL}_{\infty} \) as the motivic space in each degree, with the structure map given by the Bott-periodicity:

\[
\mathbb{Z} \times \text{BGL}_{\infty} \simeq \Omega_{\mathbf{T}}(\mathbb{Z} \times \text{BGL}_{\infty}).
\]

We will denote this motivic spectrum by \( K \). Therefore,

\[
K(\mathbf{x}) \simeq [\Sigma^F(X_+), X, \mathbb{Z} \times \text{BGL}_{\infty}]
\]

where the first \([\ , \ ]\) (the second \([\ , \ ]\)) denotes the hom in the stable motivic homotopy category (the corresponding unstable pointed motivic homotopy category, respectively.)

We observe in Proposition 2.1 that there is an \( \Omega_{\mathbf{T}} \)-motivic spectrum whose 0-th term is given by the motivic space \( \text{BGL}_{\infty} \). Assuming this, the first main result of this paper is the following theorem, which we call the motivic Segal-Becker Theorem in view of the fact that such a result was proven for topological complex K-theory, making use of complex unitary groups, by Segal (see [Seg73]) and for real K-theory, making use of orthogonal groups, by Becker (see [Beck74]). In fact, Becker’s proof, making use of the transfer, also applies to topological complex K-theory.) For a motivic space \( P \), we will let \( Q(P) = \lim_{n \to \infty} \Omega_{\mathbf{T}}^n(P) \). Of key importance for us is the following map:

\[
(1.0.1) \quad \lambda : Q(\text{BGL}_m) \to Q(\lim_{n \to \infty} \text{BGL}_n) = Q(\text{BGL}_{\infty}) \wedge_\mathbb{Z} \text{BGL}_{\infty},
\]

where the map \( q \) is the obvious one induced by the fact that \( \text{BGL}_{\infty} \) is the 0-th space of an \( \Omega_{\mathbf{T}} \)-spectrum: see Proposition 2.1. The map \( Q(\text{BGL}_m) \to Q(\lim_{n \to \infty} \text{BGL}_n) = Q(\text{BGL}_{\infty}) \) is induced by the inclusion, \( \mathbb{G}_m \to \text{GL}_n \to \text{GL}_{\infty} \), where the first map is the diagonal imbedding.

**Theorem 1.1.** (The motivic Segal-Becker theorem for Algebraic K-Theory) (i) Assume that the base scheme is a field \( k \) of characteristic 0. Then the map in (1.0.1) induces a surjection for every pointed motivic space \( X \) that is a compact object in the unstable pointed motivic homotopy category:

\[
[X, Q(\text{BGL}_m)] \to [X, \text{BGL}_{\infty}].
\]

(Recall that a motivic space \( X \) is a compact object in the unstable pointed motivic homotopy category, if \( \text{Map}(X, \ ) \) commutes with all small colimits in the second argument, and where \( \text{Map}(\ , \ ) \) denotes the simplicial mapping space.)

(ii) Assume that the base scheme is a perfect field \( k \) of positive characteristic \( p > 0 \). Then, after inverting \( p \), the map in (1.0.1) induces a surjection for every pointed motivic space \( X \) that is a compact object in the corresponding unstable pointed motivic homotopy category:

\[
[X, Q(\text{BGL}_m)] \to [X, \text{BGL}_{\infty}].
\]

**Remarks 1.2.** 1. Localizing at the prime \( p \) in the unstable pointed motivic homotopy category, as used in statement (ii) and elsewhere in this paper is discussed in detail in [AFH]. One may also observe that, though \([\ , \ ]\) as used in statement (ii) denotes Hom in the unstable pointed motivic homotopy theory, since the target space is an infinite

\[The assumption that \( k \) be perfect may be dropped in view of recent results such as in [EK] and [BH Theorem 10.12].]
The Motivic Segal-Becker Theorem

1. Use of the adjunction between taking $\Omega_T$-loops and $T$-suspension.

2. One should view the above results as a rather weak form of the Segal-Becker theorem, in the sense that we are able to prove only the surjectivity (and not split surjectivity) of the above maps, and also only for objects $X$ that are compact objects in the corresponding unstable pointed motivic homotopy category. We hope to consider questions on split surjectivity in a sequel to this paper, as it seems to involve considerable additional work and certain techniques used in establishing such splittings classically do not seem to extend readily to the motivic framework.

3. It is possible there is an analogue of the above theorem for Hermitian K-theory (see: [Ho05], [Sch16]) which is represented by the classifying space of the infinite orthogonal groups. In fact, much of the proof for the case of algebraic K-theory seems to carry over to the Hermitian case, the main difficulty being to prove an analogue of Theorem 3.5. We hope to return to this question elsewhere.

Our approach to all of the above is via a theory of motivic transfers. Such a theory of motivic and étale variants of the classical Becker-Gottlieb transfer were developed by Gunnar Carlsson and the first author in [CJ20] and the additivity of transfers was established in a general framework by the present authors in [JP20], though special cases such as Snith-splitting for the suspension spectrum of $BGL_n$ appears in [K18]. Theorem 1.1 is proven by making intrinsic use of this transfer, just as was done by Becker and Gottlieb, making use of the classical Becker-Gottlieb transfer. See [Beck74], (and also, [BG75] and [BG76]).

In fact we summarize the main ideas of the proof of Theorem 1.1 (as well as an overview of the paper) as follows: The splitting provided by the motivic Becker-Gottlieb transfer as in Proposition 2.2 enables us to prove Proposition 2.4. This shows the map

$$\bar{q} = q \circ Q(p): Q(BNGL_\infty(T)) \to Q(\lim_n BNGL_n(T_n)) \to Q(\lim_n BGL_n) = Q(BGL_\infty)^{-1}BGL_\infty$$

induces a surjection

$$[X, Q(BNGL_\infty(T))] \overline{\to} [X, BGL_\infty],$$

for every compact object $X$ in the unstable pointed motivic homotopy category. (Here $NGL_n(T_n)$ denotes the normalizer of the maximal torus of diagonal matrices in $GL_n$.) Then we show in Propositions 2.5 and 2.6 that the map $\bar{q}$, above factors through $\lambda_*$, where $\lambda$ is the map in (1.0.1), thereby proving the theorem.

It may be worth pointing out this involves a second, somewhat different use of the transfer, this time as defined in (3.2.3), and with a key property proven in Corollary 3.7. These occupy most of section 2 of the paper. While Proposition 2.4 is rather straightforward given the properties of the motivic Becker-Gottlieb transfer, Proposition 2.6 is a bit involved: here one needs to know the relationship between maps defined by the transfer as in (3.2.3) and Gysin maps, for at least finite étale maps in orientable motivic cohomology theories. This is discussed in section 3 of the paper.

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1.1. Basic assumptions and terminology. We will assume throughout that the base scheme is a perfect field. (The assumption that $k$ be perfect can be dropped, if one prefers, in view of recent results such as in [EK] and [BH Theorem 10.12].) Then $\text{Spt} = \text{Spt}(k)$ will denote the category of motivic spectra on the big Nisnevich site of $k$, with $\mathcal{SH} = \mathcal{SH}(k)$ denoting the corresponding motivic stable homotopy category. If $k$ is of characteristic 0, no further assumptions are needed.

However, if $\text{char}(k) = p > 0$, then we will only consider $\mathcal{SH}[p^{-1}]$, which is the motivic stable homotopy category on $k$, with the prime $p$ inverted. (The main reason for this restriction is that a theory of Spanier-Whitehead duality holds only after inverting $p$ in this case.) Given a motivic spectrum $E$, and a motivic space $X$, the generalized motivic cohomology represented by $E$ is given by the bi-graded theory

$$h^{p,q}(X, E) = \{\Sigma^\infty X, (S^1)^{p-q} \wedge G^\wedge qE\}$$

with $[\ , \ ]$ denoting the Hom in the motivic stable homotopy category.

In both the above cases, we do not require the existence of a symmetric monoidal structure on the category of spectra itself, that is, it is sufficient to assume the smash product of spectra is homotopy associative and homotopy commutative.
1.2. Geometric classifying spaces. We begin by recalling briefly the construction of the geometric classifying space of a linear algebraic group; see for example, [Tot99 section 1], [MV99 section 4]. Let $G$ denote a linear algebraic group over $S = \text{Spec } k$, that is, a closed subgroup-scheme in $\text{GL}_n$ over $S$ for some $n$. For a (closed) imbedding $i : G \to \text{GL}_n$ as a closed subgroup-scheme, the geometric classifying space $B\text{G}(G;i)$ of $G$ with respect to $i$ is defined as follows. For $m \geq 1$, let $E\text{G}_{gm} = U_m(G) = U(\mathbb{A}^m)$ be the open sub-scheme of $\mathbb{A}^m$ where the diagonal action of $G$ determined by $i$ is free. By choosing $m$ large enough, one can always ensure that $U(\mathbb{A}^m)$ is non-empty and the quotient $U(\mathbb{A}^m)/G$ is a quasi-projective scheme. We will further choose such a family $(U(\mathbb{A}^m)/m)$ so that it satisfies the hypotheses in [MV99 Definition 2.1, section 4.2] defining an admissible gadget.

Let $B\text{G}_{gm} = V_m(G) = U_m(G)/G$ denote the quotient $S$-scheme (which will be a quasi-projective variety) for the action of $G$ on $U_m(G)$ induced by this (diagonal) action of $G$ on $\mathbb{A}^m$; the projection $U_m(G) \to V_m(G)$ defines $V_m(G)$ as the quotient of $U_m(G)$ by the free action of $G$ and $V_m(G)$ is thus smooth. We have closed imbeddings $U_m(G) \to U_{m+1}(G)$ and $V_m(G) \to V_{m+1}(G)$ corresponding to the imbeddings $Id \times \{0\} : \mathbb{A}^m \to \mathbb{A}^{m+1}$. We set $E\text{G}_m = \{U_m(G)/m\} = \{E\text{G}_{gm,m}/m\}$ and $B\text{G}_m = \{V_m(G)/m\}$ which are ind-objects in the category of schemes. (If one prefers, one may view each $E\text{G}_{gm}$ ($B\text{G}_{gm}$) as a sheaf on the big Nisnevich (étale) site of smooth schemes over $k$ and then view $E\text{G}$ ($B\text{G}$) as the corresponding colimit taken in the category of sheaves on $(\text{Sm}/k)_{Nis}$ or on $(\text{Sm}/k)_{et}$.)

**Definition 1.3.** We will denote $E\text{G}_m$ by $E$ and $B\text{G}_m$ by $B\text{G}$ throughout the paper.

Given a scheme $X$ of finite type over $S$ with a $G$-action, we let $U_m(G) \times_X X$ denote the balanced product, where $(u, x)$ and $(ug^{-1}, gx)$ are identified for all $(u, x) \in U_m \times_X X$ and $g \in G$. Since the $G$-action on $U_m(G)$ is free, $U_m(G) \times_X X$ exists as a geometric quotient which is also a quasi-projective scheme in this setting, in case $X$ is assumed to be quasi-projective: see [MFK Proposition 7.1]. (In case $X$ is an algebraic space of finite type over $S$, the above quotient also exists, but as an algebraic space of finite type over $S$.)

Next we recall a particularly nice way to construct geometric classifying spaces for closed subgroups of $\text{GL}_n$ making use of the Stiefel varieties.

**Definition 1.4.** (Stiefel varieties and Grassmannians). Let $n$ denote a fixed positive integer and let $i \geq 0$ denote an integer. We let $\text{St}_{n+i,n}$ denote the set of all $(n+i) \times n$-matrices of rank $n$, or equivalently the set of all injective linear transformations $\mathbb{A}^n \to \mathbb{A}^{n+i}$. We view this as an open subscheme of the affine space $\mathbb{A}^{(n+i) \times n}$. The group $\text{GL}_n$ acts on $\text{St}_{n+i,n}$ through its action on $\mathbb{A}^n$; we view this as a right action on the set of all $(n+i) \times n$-matrices. This is a free action and the quotient is the Grassmann variety of $n$-planes in $\mathbb{A}^{n+i}$, and denoted $\text{Grass}_{n+i,n}$.

As observed on [MV99 p. 138], for each fixed positive integer $n$, the family $\{\text{St}_{n+i,n}/|i| \geq 0\}$ satisfies the conditions in [MV99 Definition 2.1, p. 133], so that it defines what is there called an admissible gadget. Thus $\{\text{St}_{n+i,n}/|i| \geq 0\}$ forms finite dimensional approximations to the classifying space for any closed subgroup $H$ of $\text{GL}_n$. Therefore, we will make the following definitions.

**Definition 1.5.** (i) $B\text{G}_{gm,1} = \text{St}_{n+i,n}/H$, $B\text{H} = \lim_{i \to \infty} B\text{G}_{gm,i}$, and

(ii) $B\text{G}_{gm,1} = \lim_{i \to \infty} \lim_{n \to \infty} \text{St}_{n+i,n}/\text{GL}_n = \lim_{n \to \infty} \text{St}_{2n,n}/\text{GL}_n$.

For any linear algebraic group $G$, we will let $B\text{G}$ denote the geometric classifying space defined above (as in Definition 1.3 or equivalently in Definition 1.5) as an ind-scheme. (We may view this as a motivic space.)

2. The motivic Segal-Becker Theorem: proof of Theorem 1.1

We begin with the following observation due to Voevodsky.

**Proposition 2.1.** There exists a motivic $\Omega_{\mathbb{T}}$-spectrum, $\mathbb{K}$, whose $0$-th space is given by $B\text{G}_{gm,1}$.

**Proof.** The required spectrum is just $f_1(\mathbb{K})$, where $\mathbb{K}$ denotes the $\Omega_{\mathbb{T}}$-spectrum representing algebraic $K$-theory: see [VV00 Theorem 2.2]. That this is the case follows from [VV01 Lemma 2.2] (which holds unconditionally over any field by [Lev08 Theorem 7.5.1]) and [VV01 Theorem 4.1, Lemma 4.6 and its proof].

2.1. Changing base points. Recall motivic spaces are assumed to be pointed simplicial presheaves. However, it is often necessary for us to consider a motivic space $Y$ viewed as an unpointed simplicial presheaf and then provide it with an extra base point $+$. A typical example, we run into in this paper, is when $Y$ is the geometric classifying space of a linear algebraic group (denoted $B\text{G}$: see Definition 1.3) or a finite degree approximation of it (denoted $B\text{G}_{gm,n}$), both of which are pointed. However, while considering a motivic Becker-Gottlieb transfer
involving $BG (BG_{gm,m})$, one needs to consider $\Sigma_T^\infty BG_+ (\Sigma_T^\infty BG_{gm,m}^\infty)$, which is the $T$-suspension spectrum of $BG (BG_{gm,m})$ provided with an extra base point $+$.  

Observe that there is a natural map $r : BG_+ \to BG$ sending $+$ to the base point of $BG$. Let $a : \Sigma_T^\infty BG \to \Sigma_T^\infty BG_+$ denote a map, so that $\Sigma_T^\infty r \circ a = \text{id}_{\Sigma_T^\infty BG}$. (Since the definition of such a map $a$ is straightforward, we skip the details.)

**Proposition 2.2.** Let $h^\bullet$ denote a generalized motivic cohomology theory defined with respect to a motivic spectrum (with $p$ inverted, if $\text{char}(k) = p > 0$.) Let $G$ denote a linear algebraic group, which is also special in the sense of Grothendieck (see: [Ch]). Then, with $N(T)$ denoting the normalizer of a split maximal torus in $G$, one obtains the commutative square

\[
\begin{array}{cccc}
h^\bullet(\Sigma_T^\infty BN(T)) & \xrightarrow{\pi^\ast} & h^\bullet(\Sigma_T^\infty BN(T)_+) \\
p^\ast & & p^\ast \\
\downarrow & & \downarrow \\
h^\bullet(\Sigma_T^\infty BG) & \xrightarrow{\pi^\ast} & h^\bullet(\Sigma_T^\infty BG_+),
\end{array}
\]

where the map $p : BN(T) \to BG$ is the map induced by the inclusion $N(T) \to G$. The right vertical map and the horizontal maps are all split monomorphisms. Therefore, the left vertical map is also a monomorphism.

**Proof.** That the right vertical map is a split monomorphism is a consequence of the motivic Becker-Gottlieb transfer as proved in [CJ20] 9.2, as well as [JP20] Theorem 1.6] and [An] Theorem 5.1. Moreover, all of these depend on the key identification of the Grothendieck-Witt group with the motivic $\pi_0$ of the motivic sphere spectrum due to Morel: see [Mo4], [Mo12]. The restriction that the characteristic of the base field $k$ be different from 2 is removed in [BH] Theorem 10.12.

Let the motivic Euler characteristic of $G/N(T)$ be denoted $\chi^{A^1}(G/N(T))$ henceforth. Now one may recall from [JP20] Theorem 1.6 that we showed $\chi^{A^1}(G/N(T))$ is 1 in the Grothendieck-Witt group $GW(Spec k)$ ($GW(Spec k)[p^{-1}]$ if $\text{char}(k) = p > 0$), provided $k$ has a square root of $-1$. Hence this conclusion holds whenever the base field $k$ is algebraically or quadratically closed. In positive characteristics $p$, one may see that this already shows that $\chi^{A^1}(G/N(T))$ is a unit in the group $GW(Spec k)[p^{-1}]$, without the assumption on the existence of a square root of $-1$ in $k$. For this, one may first observe the commutative diagram, where $\bar{k}$ is an algebraic closure of $k$:

\[
\begin{array}{cccc}
GW(Spec \bar{k})[p^{-1}] & \xrightarrow{rk} & \mathbb{Z}[p^{-1}] \\
\downarrow & & \downarrow \text{id} \\
GW(Spec k)[p^{-1}] & \xrightarrow{rk} & \mathbb{Z}[p^{-1}].
\end{array}
\]

Here the left vertical map is induced by the change of base fields from $k$ to $\bar{k}$, and $rk$ denotes the rank map. Since the motivic Euler-characteristic of $G/N(T)$ over $Spec k$ maps to the motivic Euler-characteristic of the corresponding $G/N(T)$ over $Spec \bar{k}$, it follows that the rank of $\chi^{A^1}(G/N(T))$ over $Spec k$ is in fact 1. By [An] Lemma 2.9(2), this shows that the $\chi^{A^1}(G/N(T))$ over $Spec k$ is in fact a unit in $GW(Spec k)[p^{-1}]$, that is, when $k$ has positive characteristic. (For the convenience of the reader, we will summarize a few key facts discussed in [An] Proof of Lemma 2.9(2)). It is observed there that when the base field $k$ is not formally real, then $I(k) = \text{kernel}(GW(k) \to \mathbb{Z})$ is the nil radical of $GW(k)$; see [Bae78] Theorem V.8.9, Lemma V.7.7 and Theorem V.7.8. Therefore, if $\text{char}(k) = p > 0$, and the rank of $\chi^{A^1}(G/N(T))$ is 1 in $\mathbb{Z}[p^{-1}]$, then $\chi^{A^1}(G/N(T))$ is $1 + q$ for some nilpotent element $q$ in $I(k)[p^{-1}]$ and the conclusion follows.)

In characteristic 0, the commutative diagram

\[
\begin{array}{cccc}
GW(Spec \bar{k}) & \xrightarrow{rk} & \mathbb{Z} \\
\downarrow \text{id} & & \downarrow \text{id} \\
GW(Spec k) & \xrightarrow{rk} & \mathbb{Z}.
\end{array}
\]

shows that once again the rank of $\chi^{A^1}(G/N(T))$ is 1. Therefore, to show that the class $\chi^{A^1}(G/N(T))$ is a unit in $GW(Spec k)$, it suffices to show its signature is 1: this is proven in [An] Theorem 5.1(1). (Again, for the convenience of the reader, we summarize some details from the proof of [An] Theorem 5.1(1).) When the field $k$ is not formally real, the discussion in the last paragraph applies, so that by [An] Lemma 2.12 one reduces to considering only the case when $k$ is a real closed field. In this case, one lets $\mathbb{R}^{alg}$ denote the real closure of $\mathbb{Q}$.
Remark 2.3. There is an extension of the above theorem for linear algebraic groups that are not special, as discussed in [JP20, Theorem 1.5] and [CJ20, Theorem 1.5(1)]. But then we require the field be infinite to prevent certain GL_q being invertible, if p > 0 is the characteristic of the base field k. Then it is shown in [An, Proof of Theorem 5.1(1)] that, in this case, knowing the rank and signature of the motivic Euler characteristic χ_k(G/N(T)) are 1 suffices to prove it is a unit in the Grothendieck-Witt group.

These complete the proof that the right vertical map in (2.1.1) is a split monomorphism.

The horizontal maps in (2.1.1) are split by the map α^*. Since the diagram commutes, it follows that the left vertical map in (2.1.1) is also a monomorphism. This proves the Proposition.

2.2. For the rest of the discussion, we will restrict to the family of groups \{GL_n|n\}. Then the main result of this section is the proof of Theorem 1.1. We will break the proof into several propositions. Given two motivic spaces X, Y, we will let [X, Y] denote the Hom in the unstable pointed motivic homotopy category, with p inverted, if p > 0 is the characteristic of the base field k. For a motivic space P, we will let Q(P) = lim_{n→∞} Ω^n_p(T^\wedge n(P)).

Let (2.2.1) p : BN_{GL∞}(T) = lim_{n→∞} BN_{GL_n}(T_n) → lim_{n→∞} BGL_n = BGL∞ denote the map induced by the inclusion of N_{GL_n}(T_n) in GL_n.

Proposition 2.4. (i) Assume the base field k is of characteristic 0. Then the map

q = q ◦ Q(p) : Q(BN_{GL∞}(T)) = Q(lim_{n→∞} BN_{GL_n}(T_n)) → Q(lim_{n→∞} BGL_n) = Q(BGL∞)\\^{h} BGL∞ induces a surjection for every pointed motivic space X which is a compact object in the unstable pointed motivic homotopy category:

[X, lim_{n→∞} Q(BN_{GL_n}(T_n))] → [X, BGL∞].

(ii) Assume the base field k is perfect and of positive characteristic p. Then the same conclusion holds after inverting the prime p.

Proof. We will follow [Beck74, §4] in this proof. The proof of the second statement follows along the same lines as the proof of the first statement. Therefore, we will discuss a proof of only the first statement. Clearly the map q provides a map of the corresponding spectra:

(2.2.2) \Sigma^∞_T(BN_{GL∞}(T)) = lim_{n→∞} BN_{GL_n}(T_n) → \hat{K},

where \hat{K} is the motivic Ω_T-spectrum whose 0-th space is BGL∞. Let φ denote the above map in (2.2.2).

Let h denote the motivic cohomology theory defined by the mapping cone of the above map φ. Then, for every motivic space X, we obtain a long-exact sequence:

(2.2.3) ... → [X, Q(BN_{GL∞}(T))]\\^{h} [X, BGL∞]\\^{h} h^{0,0}(X) → ... For each n ≥ 0, let

(2.2.4) u_n ∈ [BGL_n, BGL∞]

be the class of the map induced by the imbedding GL_n → GL∞. Then it suffices to show that each such u_n is in the image of the induced map q*, which is equivalent to showing that c(u_n) = 0. (To see this let v_n : BGL_n → Q(BN_{GL_n}(T)) be such that q*(v_n) = [q ◦ v_n] = u_n. Here, if α is a map, [α] denotes the stable homotopy class of α. Since X is assumed to be compact, [X, BGL∞] = lim_{n→∞} [X, BGL_n]. Therefore, giving an α ∈ [X, BGL∞] is equivalent to giving an α_n : X → BGL_n (for some n), so that α = u_n ◦ α_n. Now let β_n = v_n ◦ α_n. Then q*(β_n) = [q ◦ v_n ◦ α_n] = [u_n ◦ α_n] = [α].)
Now we observe the commutative diagram, where \( p_n : BN_{GL_n}(T_n) \to BGL_n \) is the obvious map induced by the imbedding \( N_{GL_n}(T_n) \to GL_n \):

\[
\begin{array}{ccc}
[BGL_n, Q(BN_{GL_n}(T))] & \stackrel{q_*}{\longrightarrow} & R^{0,0}(BGL_n) \\
\downarrow p_n & & \downarrow c \\
[BN_{GL_n}(T_n), Q(BN_{GL_n}(T))] & \stackrel{q_*}{\longrightarrow} & R^{0,0}(BN_{GL_n}(T_n))
\end{array}
\]

(2.2.5)

Recall that \( p_n^* \) is a monomorphism, by Proposition 2.2. Therefore, now it suffices to prove \( p_n^*(c(u_n)) = 0 \), for each \( n \). But the commutativity of the above diagram, shows that this is equivalent to showing that \( c(p_n^*(u_n)) = 0 \).

At this point, we observe the commutative diagram, where \( i_n \) and \( j \) are the obvious maps:

\[
\begin{array}{ccc}
BN_{GL_n}(T_n) & \stackrel{p_n}{\longrightarrow} & BGL_n \\
\downarrow i_n & & \downarrow u_n \\
BN_{GL_n}(T) & \stackrel{j}{\longrightarrow} & BGL \\
\downarrow q & & \downarrow id \\
Q(BN_{GL_n}(T)) & \stackrel{Q(p)}{\longrightarrow} & Q(BG_m) \longrightarrow BGL_n.
\end{array}
\]

The map \( BGL_{\infty} \to Q(BGL_{\infty}) \) is provided by taking the colimit of the maps \( BGL_{\infty} \to \Omega^n_{T} \Sigma^n_{T}(BGL_{\infty}) \), while the map \( Q(BGL_{\infty}) \to BGL_{\infty} \) is provided by the fact \( BGL_{\infty} \) is an infinite \( T \)-loop space. The fact that the composite map \( BGL_{\infty} \to Q(BGL_{\infty}) \to BGL_{\infty} \) is homotopic to the identity map follows from the adjunction between \( \wedge T \) and \( \Omega_T \), along with the \( \Omega_T \)-infinite loop space structure on \( BGL_{\infty} \). In view of the commutative diagram above, \( p_n^*(u_n) = u_n \circ p_n = q \circ Q(p) \circ j \circ i_n = q_*(j \circ i_n) \). Since the two rows in the diagram (2.2.5) are exact, it follows that indeed \( c(q_*(j \circ i_n)) = 0 \), thereby completing the proof that \( c(p_n^*(u_n)) = 0 \).

Let \( \lambda : Q(BG_m) \to Q(BGL_{\infty}) \to BGL_{\infty} \) denote the map considered in (1.0.1).

**Proposition 2.5.** (i) Assume the base field \( k \) is of characteristic 0. Let \( X \) denote any pointed motivic space. Then, there exists a map \( \zeta : Q(BN_{GL_n}(T)) \to Q(BG_m) = Q(\mathbb{P}^\infty) \), so that the triangle

\[
\begin{array}{ccc}
[X, Q(BN_{GL_n}(T))] & \stackrel{c_*}{\longrightarrow} & [X, Q(BG_m)] \\
\downarrow q_* & & \downarrow \lambda_* \\
[X, BGL_{\infty}]
\end{array}
\]

(2.2.6)

commutes.

(ii) Assume the base field \( k \) is perfect and of positive characteristic \( p \). Then the same conclusion holds after inverting the prime \( p \).

**Proof of Theorem 1.1** Before we prove the above proposition, we proceed to show how to complete the proof of Theorem 1.1 given the above Proposition. We simply observe that, for a compact object \( X \), the composition of the maps in

\[
[X, Q(BN_{GL_n}(T))] \stackrel{c_*}{\longrightarrow} [X, Q(BG_m)] \stackrel{\lambda_*}{\longrightarrow} [X, BGL_{\infty}]
\]

is a surjection, thereby proving that the map \( \lambda_* \) is also surjection, which completes the proof of the theorem.

2.3. Therefore, it suffices to prove Proposition 2.5 which we proceed to do presently. Moreover, we will only consider the characteristic 0 case explicitly, as the positive characteristic case follows exactly along the same lines, once the characteristic is inverted. Let \( St_{2n,n} \) denote the Stiefel variety of \( n \)-frames (that is, \( n \)-linearly independent vectors) \((v_1, \cdots, v_n)\) in \( \mathbb{A}^{2n} \). The group \( GL_n \) acts on this variety, by acting on such frames by sending \((v_1, \cdots, v_n)\) to \((v_1, \cdots, v_n) \cdot g \in GL_n \) to \((v_1, \cdots, v_n) \cdot g \in GL_n \). (We view this as a right action because the Stiefel variety \( St_{2n,n} \) identifies with the variety of all \( 2n \times n \)-matrices of rank \( n \), with each vector \( v_i \) in an \( n \)-frame \((v_1, \cdots, v_n) \) written as the \( i \)-th column.) This is a free-action and the quotient is the Grassmannian, \( Grass_{2n,n} \). The Stiefel variety \( St_{2n,n} \) is an open sub-variety of the affine space \( \mathbb{A}^{2n^2} \) and the complement has codimension \( n + 1 \) in \( \mathbb{A}^{2n^2} \).

Next we consider the ind-scheme:

\[
\mathbb{A}^2 \to \mathbb{A}^4 \to \mathbb{A}^6 \cdots \to \mathbb{A}^{2n} \to \mathbb{A}^{2n^2} \to \cdots
\]

(2.3.1)
where the closed immersion $\mathbb{A}^{2n} \to \mathbb{A}^{2n+2}$ sends $(x_1, \cdots, x_{2n})$ to $(x_1, \cdots, x_{2n}, 0, 0)$. Let $e_i, i = 1, \cdots, 2n, 2n + 1, 2n + 2$ denote the standard basis vectors in $\mathbb{A}^{2n+2}$. Then we obtain a closed immersion

\begin{equation}
(2.3.3)
i_n : St_{2n,n} \to St_{2n+2,n+1},
\end{equation}

by sending an $n$-frame $(v_1, \cdots, v_n)$ in $\mathbb{A}^{2n}$ to the $n + 1$-frame $(v_1, \cdots, v_n, \bar{e}_{n+1} = e_{2n+1} + e_{2n+2})$. This induces a closed immersion of the Grassmannians, Grass$_{2n,n} \to$ Grass$_{2n+2,n+1}$. Therefore, the discussion in section 1.2 shows that the above Stiefel varieties may be used as finite dimensional approximations to EGL$_n$ and the Grassmannian, Grass$_{2n,n}$, could be used as a finite dimensional approximation to BGL$_n$.

Next we make the following identifications.

\begin{equation}
(2.3.5)\bar{v}^{\cdot 2n}_n \text{ denote the standard basis vectors in } \mathbb{A}^{2n}, \text{ could be used as a finite dimensional approximation to BGL}_{2n},
\end{equation}

\begin{equation}
(2.3.6)\text{The fact that the terms appearing on the right-hand-sides in (2.3.3) are indeed approximations to the classifying spaces of the corresponding linear algebraic groups, follows from the discussion in section 1.2}
\end{equation}

Next consider the map $St_{2n,n} \to St_{2n,1}$ sending an $n$-frame $(v_1, v_2, \cdots, v_n)$ to $v_1$. Clearly this factors through the quotient of $St_{2n,n}/1 \times GL_{n-1}$, where $GL_{n-1}$ acts only on the last $n-1$-vectors in the $n$-frame $(v_1, \cdots, v_{n-1}, v_n)$. Therefore, we obtain the map

\begin{equation}
(2.3.4)\phi_n : St_{2n,n}/(1 \times GL_{n-1}) \to St_{2n,1}.
\end{equation}

Moreover, the above map $\phi_n$ is compatible with the obvious action of $GL_n$ on $St_{2n,n}/GL_{n-1}$ where it acts on the vector $v_1$ in an $n$-frame, $(v_1, \cdots, v_{n-1}, v_n)$, and it acts on the 1-frame $v$ in $St_{2n,1}$. Taking quotients, this defines the map

\begin{equation}
(2.3.5)\tilde{\phi}_n : St_{2n,n}/(G_m \times GL_{n-1}) \to St_{2n,1}/G_m.
\end{equation}

One may then observe the commutative square:

\begin{equation}\begin{array}{ccc}
St_{2n,n}/(G_m \times GL_{n-1}) & \xrightarrow{\tilde{\phi}_n} & St_{2n,1}/G_m \\
\downarrow & & \downarrow \\
St_{2n+2,n+1}/(G_m \times GL_{n+1}) & \xrightarrow{\tilde{\phi}_{n+1}} & St_{2n+2,1}/G_m,
\end{array}\end{equation}

where the left vertical map is the closed immersion defined by $i_n : St_{2n,n} \to St_{2n+2,n+1}$ and the right vertical map is induced by the closed immersion $St_{2n,1} \to St_{2n+2,1}$.

One may also observe that clearly $St_{2n,1}/G_m$ is an approximation to the classifying space of $G_m$, so that we will let

\begin{equation}
(2.3.7)BG_m^{gm,n} = St_{2n,1}/G_m.
\end{equation}

We also let

\begin{equation}
(2.3.8)\bar{u}_n : BN_{GL_n}(T_n)^{gm,n} = St_{2n,n}/(G_m \times N_{GL_{n-1}}(T_{n-1})) \to St_{2n,n}/(G_m \times GL_{n-1}), \text{ and}
\end{equation}

\begin{equation}
\bar{u}_n = \bar{\phi}_n \circ \bar{u}_n : BN_{GL_n}(T_n)^{gm,n} = St_{2n,n}/(G_m \times N_{GL_{n-1}}(T_{n-1})) \to St_{2n,1}/G_m = BG_m^{gm,n}.
\end{equation}

Then we also obtain the commutative diagram:

\begin{equation}\begin{array}{ccc}
St_{2n,n}/(G_m \times N_{GL_{n-1}}(T_{n-1})) & \xrightarrow{\bar{u}_n} & St_{2n,n}/(G_m \times GL_{n-1}) \\
\downarrow & & \downarrow \\
St_{2n+2,n+1}/(G_m \times N_{GL_{n+1}}(T_{n})) & \xrightarrow{\bar{u}_{n+1}} & St_{2n+2,1}/G_m,
\end{array}\end{equation}

\begin{equation}\begin{array}{ccc}
St_{2n,n}/(G_m \times N_{GL_{n-1}}(T_{n-1})) & \xrightarrow{\tilde{\phi}_n} & St_{2n,1}/G_m \\
\downarrow & & \downarrow \\
St_{2n+2,n+1}/(G_m \times N_{GL_{n+1}}(T_{n})) & \xrightarrow{\bar{\phi}_{n+1}} & St_{2n+2,1}/G_m.
\end{array}\end{equation}
As pointed out in the introduction, apart from the transfer for passage from $\text{BGL}_{n}^{gm,n}$ to $\text{BN}_{GL_n}(T_n)^{gm,n}$ for the proof of Theorem 1.1, one also needs to invoke a transfer map for the finite étale map

$$r_n : \text{BN}_{GL_n}(T_n)^{gm,n} = \text{St}_{2n,n}/(\mathbb{G}_m \times \text{N}_{GL_{n-1}}(T_{n-1})) \rightarrow \text{St}_{2n,n}/\text{N}_{GL_n}(T_n) = \text{BN}_{GL_n}(T_n)^{gm,n}.$$  

We also need to know that such a transfer map has reasonable properties, like compatibility with base-change, and agreement with Gysin maps defined on orientable generalized motivic cohomology theories. The purpose of the last short section of the paper is to set-up such a transfer and establish these basic properties for it: see (3.2.3) for the definition of such a transfer. Let

$$(2.3.10) \quad \tau_n : \Sigma^{-\infty}_T \text{BN}_{GL_n}(T_n)^{gm,n}_+ \rightarrow \Sigma^{-\infty}_T (\text{BN}_{GL_n}(T_n)^{gm,n})_+$$

denote the corresponding transfer defined as in (3.2.3), and let

$$(2.3.11) \quad \zeta_n : \Sigma^{-\infty}_T \text{BN}_{GL_n}(T_n)^{gm,n}_+ \rightarrow \Sigma^{-\infty}_T \text{BN}_{GL_n}(T_n)^{gm,n}_+ \rightarrow \Sigma^{-\infty}_T \text{BG}_{m,n} \rightarrow \Sigma^{-\infty}_T \text{BG}_{m}$$

denote the composition, where the map $\pi_n$ is the composition of the map $\Sigma^{-\infty}_T \bar{u}_n$ followed by the map that sends the base point $+$ to the base point of $\text{BG}_{m,n}$ as in section 2.1. The last map, denoted $\iota_n$, is the obvious one sending a finite dimensional approximation of $\text{BG}_{m,n}$ to the direct limit of such approximations. Let

$$(2.3.12) \quad \bar{q}_n : Q(\text{BN}_{GL_n}(T_n)^{gm,n}_+) \rightarrow \text{BGL}_\infty$$

denote the composition

$$Q(\text{BN}_{GL_n}(T_n)^{gm,n}_+) \rightarrow Q(\text{BN}_{GL_n}(T)) \xrightarrow{Q(n)} Q(\text{BGL}_\infty) \xrightarrow{\lambda} \text{BGL}_\infty.$$  

Then a key result is the following Proposition, which we show also proves Proposition 2.3.

**Proposition 2.6.** Assume the above situation. Then the following diagrams commute:

$$(2.3.13) \quad \Sigma^{-\infty}_T \text{BN}_{GL_n}(T_n)^{gm,n}_+ \rightarrow \Sigma^{-\infty}_T \text{BN}_{GL_{n+1}}(T_{n+1})^{gm,n+1}_+ \xrightarrow{\zeta_{n+1}} \Sigma^{-\infty}_T \text{BG}_{m,n+1}$$

$$(2.3.14) \quad [X, Q(\text{BN}_{GL_n}(T_n)^{gm,n}_+)] \xrightarrow{\iota_n} [X, Q(\text{BGL}_n)] \xrightarrow{\lambda} [\Sigma^{-\infty}_T X, \Sigma^{-\infty}_T \text{BG}_m] \xrightarrow{\lambda_n} [\Sigma^{-\infty}_T X, \Sigma^{-\infty}_T \text{BG}_{m+1}].$$

where $\lambda : \Sigma^{-\infty}_T \text{BG}_m \rightarrow f_1(K) = \bar{K}$ is the map of spectra corresponding to the infinite loop-space map $\lambda : Q(\text{BGL}_m) \rightarrow \text{BGL}_\infty$. The left vertical map in (2.3.13) is the obvious map induced by the closed immersion $i_n$ in (2.3.2). Moreover, $\left[ \quad , \quad \right]$ in the middle row of (2.3.14) denotes Hom in the motivic stable homotopy category, while $[\quad , \quad]$ in the top row and the bottom row denotes Hom in the unstable pointed motivic homotopy category.

**Proof.** We will first prove the commutativity of the triangle in (2.3.13). For this, one begins with the cartesian square (which also defines $P_n$ and the map $r_n'$):

$$(2.3.15) \quad P_n \xrightarrow{i} \text{BN}_{GL_{n+1}}(T_{n+1})^{gm,n+1} = \text{St}_{2n+2,n+1}/(\mathbb{G}_m \times \text{N}_{GL_{n+1}}(T_{n+1})) \xrightarrow{\iota_n} \text{BN}_{GL_{n+1}}(T_{n+1})^{gm,n+1} = \text{St}_{2n+2,n+1}/\text{N}_{GL_{n+1}}(T_{n+1}).$$

Observe that the right vertical map, and therefore, the left vertical map also is a finite étale map of degree $n+1$. By Proposition 3.2 (which shows the naturality of the transfer with respect to base-change), we observe that the
square below homotopy commutes:

\[ (2.3.16) \]

Then a straightforward calculation, as discussed below, shows that \( P_n = BN_{GL_n(T_n)}(\gamma_{m,n}) \cup BN_{GL_n(T_n)}(\gamma_{m,n}) \) the main observation here is that under the identifications in \((2.3.3)\), the map \( BN_{GL_n(T_n)}(\gamma_{m,n}) \to BN_{GL_{n+1}(T_{n+1})}(\gamma_{m,n+1}) \) lifts to \( BN_{GL_{n+1}(T_{n+1})}(\gamma_{m,n+1}) \), which provides the required splitting. In fact, this splitting may be described in more detail as follows. Observe first that the imbedding \( i : St_{2n,n} \to St_{2n+2,n+1} \) is defined by sending an \( n \)-frame, \((v_1, \ldots, v_n)\) in \( St_{2n,n} \) to the \( n+1 \)-frame, \((v_1, \ldots, v_n, e_{n+1})\) in \( St_{2n+2,n+1} \), where \( e_{n+1} \) is the nonzero vector chosen as in \((2.3.2)\) that lies in the ambient affine space \( \mathbb{A}^{2n+2} \) and is outside of \( \mathbb{A}^{n+2} \). The induced map \( St_{2n,n}/NGL_n(T_n) \to St_{2n+2,n+1}/NGL_{n+1} \) is the map \( i : BN_{GL_n(T_n)}(\gamma_{m,n}) \to BN_{GL_{n+1}(T_{n+1})}(\gamma_{m,n+1}) \) appearing in \((2.3.15)\).

One may see that from \( i \) one obtains an induced map

\[ (2.3.17) \]

since the action of \( \mathbb{G}_m \times NGL_{n-1}(T_{n-1}) \) on \( St_{2n,n} \) and the action of \( \mathbb{G}_m \times NGL_n(T_n) \) on \( St_{2n+2,n+1} \) are compatible. (For this one identifies \( St_{2n,n} \) as imbedded in \( St_{2n+2,n+1} \) using the imbedding \( i_n \) considered in \((2.3.2)\), and identifies the group \( \mathbb{G}_m \times NGL_{n-1}(T_{n-1}) \) with the subgroup \( \mathbb{G}_m \times NGL_{n-1}(T_{n-1}) \times 1 \) of \( \mathbb{G}_m \times NGL_n(T_n) \).) Moreover, one may see that this map is a closed immersion and that one obtains a commutative triangle:

\[ (2.3.18) \]

The left inclined map is a finite \( \acute{e} \)tale map of degree \( n \), while the right inclined map is a finite \( \acute{e} \)tale map of degree \( n+1 \). Since the top horizontal map is also \( \acute{e} \)tale (see \[Mi\] Chapter I, Corollary 3.6)) and a closed immersion, it is the open (and closed) imbedding of a connected component in \( P_n \). Let the complement in \( P_n \) of \( St_{2n,n}/(\mathbb{G}_m \times NGL_{n-1}(T_{n-1})) \) be denoted \( C_n \). Then the induced map \( C_n \to St_{2n,n}/NGL_n(T_n) \) is a bijective and finite \( \acute{e} \)tale map, so is an isomorphism, showing that \( St_{2n,n}/NGL_n(T_n) = BN_{GL_n(T_n)}(\gamma_{m,n}) \) is a split summand of \( St_{2n+2,n+1}/(\mathbb{G}_m \times NGL_n(T_n)) = BN_{GL_{n+1}(T_{n+1})}(\gamma_{m,n+1}) \). (One may also obtain an explicit description of the above splitting as given by sending the \( n \)-frame \((v_1, \ldots, v_n)\) in \( St_{2n,n} \) to the \( n+1 \)-frame, \((e_{n+1}, v_1, \ldots, v_n)\) in \( St_{2n+2,n+1} \), \( \mathbb{G}_m \) acts on \( e_{n+1} \) by multiplication by scalars. Let \( s \) denote this imbedding of \( St_{2n,n}/NGL_n(T_n) \) in \( St_{2n+2,n+1}/(\mathbb{G}_m \times NGL_n(T_n)) \).

Moreover, one observes from the above description of the splitting \( P_n = BN_{GL_n(T_n)}(\gamma_{m,n}) \cup BN_{GL_n(T_n)}(\gamma_{m,n}) \), that under the composite map

\[ (2.3.19) \]

the copy of \( BN_{GL_n(T_n)}(\gamma_{m,n}) = St_{2n,n}/NGL_n(T_n) \) in \( P_n \) (under the above splitting of \( P_n \)) is sent to the base point. (Observe that the \( n \)-frames \((v_1, \ldots, v_n)\) coming from \( St_{2n,n} \), under the above imbedding \( s \), get sent to the last \( n \)-frames.) Since the diagram

\[ \text{(which is the same as the diagram (2.3.9)) also commutes, combining these diagrams and composing with the inclusions into \( BG_m \) proves the commutativity of the triangle (2.3.13).} \]
The commutativity of the part of the above diagram on and above the second row has already been proven. The bottom square on the right commutes by (2.3.19). Proposition 3.2(i) proves that $\tau^n = \tau_n + \text{id}$, where $\text{id}$ denotes the identity map of $\Sigma^n_T \text{BN}_{GL}(T_n)^{gm,n}$. Finally the observation in (2.3.19) completes the proof.

Next we consider the commutativity of the diagram (2.3.14). Since the top square there evidently commutes, it suffices to consider the commutativity of the bottom triangle there. The key observation is that, in order to prove the commutativity of the bottom triangle in (2.3.14), it suffices to take $X = \text{BN}_{GL}(T_n)^{gm,n}$ and show that the triangle commutes for the class $u \in [X, Q(\text{BN}_{GL}(T_n)^{gm,n})]$ denoting the class corresponding to the identity map $\Sigma^n T X_+ \to \Sigma^n_X$.

Let $u$ denote the class considered in the last line. Then $\tilde{q}_n(u) = (q \circ Q(p))_*(u)$ denotes the class of the vector bundle of rank $n$ associated to the principal $N_{GL}(T_n)$-bundle over $\text{BN}_{GL}(T_n)^{gm,n}$. One may see this readily as follows: First the natural map $p : \text{BN}_{GL}(T_n)^{gm,n} \to BGL_\infty$ corresponds to the rank $n$ vector bundle over $\text{BN}_{GL}(T_n)^{gm,n}$ associated to the principal $N_{GL}(T_n)$-bundle over $\text{BN}_{GL}(T_n)^{gm,n}$. Then the homotopy commutative diagram:

$$
\begin{array}{ccc}
\text{BN}_{GL}(T_n)^{gm,n} & \overset{p}{\to} & BGL_\infty \\
\downarrow & & \downarrow \text{id} \\
Q(\text{BN}_{GL}(T_n)^{gm,n}) & \overset{q}{\to} & Q(BGL_\infty) \to BGL_\infty
\end{array}
$$

completes the proof. We will denote this vector bundle over

(2.3.19) \[ \text{BN}_{GL}(T_n)^{gm,n} = \text{St}_{2n,n}/N_{GL}(T_n) \] by $\alpha$.

Let $\beta$ denote the line bundle associated to the principal $\mathbb{G}_m$-bundle

(2.3.19) \[ \text{St}_{2n,n}/(1 \times N_{GL}(T_n)) \to \text{St}_{2n,n}/(\mathbb{G}_m \times N_{GL}(T_n)) = \text{BN}_{GL}(T_n)^{gm,n}. \]

Then $\lambda'_*(\zeta_n(u))$ is the image of $\beta$ under the transfer map $\tau_n^* : \tilde{K}^0,0(\text{BN}_{GL}(T_n)^{gm,n}) \to \tilde{K}^0,0(\text{BN}_{GL}(T_n)^{gm,n})$, where $\zeta_n$ is the map in (2.3.14). This results from the following observations:

(i) the map $\lambda' \circ j_n : \Sigma^n_T B\mathbb{G}_m^{gm,n} \to \Sigma^n_T B\mathbb{G}_m \to f_1(K) = \tilde{K}$ corresponds to a map $\tilde{\lambda}_n' : \text{St}_{2n,1}/\mathbb{G}_m = B\mathbb{G}_m^{gm,n} \to BGL_\infty$

(ii) the map $\tilde{\lambda}_n' : \text{St}_{2n,1}/\mathbb{G}_m = B\mathbb{G}_m^{gm,n} \to BGL_\infty$ corresponds to the line bundle on $\text{St}_{2n,1}/\mathbb{G}_m$ corresponding to the $\mathbb{G}_m$-bundle $\text{St}_{2n,1} \to \text{St}_{2n,1}/\mathbb{G}_m$.

(iii) The above line bundle on $\text{St}_{2n,1}/\mathbb{G}_m$ pulls back under the map $\tilde{\phi}_n$ (see (2.3.5)) to the line bundle on $\text{St}_{2n,n}/(\mathbb{G}_m \times \text{GL}_{n-1})$ corresponding to the $\mathbb{G}_m$-bundle $\text{St}_{2n,1}/(1 \times \text{GL}_{n-1}) \to \text{St}_{2n,1}/(\mathbb{G}_m \times \text{GL}_{n-1})$.

(iv) The above line bundle on $\text{St}_{2n,n}/(\mathbb{G}_m \times \text{GL}_{n-1})$ pulls back to $\beta$ by the map $u_n : \text{BN}_{GL}(T_n)^{gm,n} = \text{St}_{2n,n}/(\mathbb{G}_m \times N_{GL}(T_n)) \to \text{St}_{2n,n}/(\mathbb{G}_m \times \text{GL}_{n-1})$.

(v) Recall that $\pi_n = \Sigma^n_T \tilde{u}_n$. The above observations now show that the composite map

$$
\lambda' \circ j_n \circ \pi_n = \lambda' \circ j_n \circ \Sigma^n_T \tilde{u}_n + \lambda' \circ j_n \circ \Sigma^n_T \tilde{u}_n + \Sigma^n_T \phi_n + \Sigma^n_T \tilde{\phi}_n \to \Sigma^n_T B\mathbb{G}_m^{gm,n} \to \Sigma^n_T B\mathbb{G}_m^{gm,n} \to f_1(K) = \tilde{K}
$$

corresponds to the bundle $\beta$.

(vi) Now $\lambda' \circ \zeta_n = \lambda' \circ j_n \circ \pi_n + \tau_n = \lambda' \circ j_n \circ \Sigma^n_T \tilde{u}_n + \tau_n = \lambda' \circ j_n \circ \Sigma^n_T \tilde{u}_n + \Sigma^n_T \phi_n + \tau_n$: see (2.3.11).
Therefore, it follows that \( \lambda^* \) corresponds to the composite map:

\[
\begin{align*}
\sum_{T}^\infty \text{BN}_{\text{GL}_n}(T_n)^{\text{gm},n} & \xrightarrow{\text{id} + \sum_{T}^\infty \text{BN}_{\text{GL}_n}(T_n)^{\text{gm},n} \tau_n \circ \sum_{T}^\infty \text{BN}_{\text{GL}_n}(T_n)^{\text{gm},n} g_m} \sum_{T}^\infty \text{BN}_{\text{GL}_n}(T_n)^{\text{gm},n} \circ \sum_{T}^\infty \text{BN}_{\text{GL}_n}(T_n)^{\text{gm},n} g_m \\
\xrightarrow{j_n \circ \tau_n \circ \sum_{T}^\infty \text{BN}_{\text{GL}_n}(T_n)^{\text{gm},n} g_m} \text{BG}_m & \xrightarrow{f_1} K = K.
\end{align*}
\]

Therefore, at this point, in order to prove the commutativity of the bottom triangle in (2.3.14), it suffices to prove that

\[
(3.2.1) \quad \tau_n^* (\beta) = \alpha,
\]

where \( \tau_n^* \) denotes the transfer map induced by the transfer \( \tau_n \) on the Grothendieck groups. This is a straightforward computation making use of the direct-images of coherent sheaves under finite étale maps as discussed in the following paragraphs, as well as Corollary 3.6. (See [Beck74, p. 142] for very similar arguments in the topological case.)

Denoting the total space of the vector bundle \( \alpha \) by \( E(\alpha) \), observe that \( E(\alpha) = \text{St}_{2n,n} \times_{\text{N}_{\text{GL}_n}(T_n)} W \), where \( W \) corresponds to the \( n \)-dimensional representation of \( \text{GL}_n \) forming the fibers of the vector bundle \( \alpha \). We will let \( W' \) denote the representation \( \text{W} \), but viewed as a representation of \( \text{N}_{\text{GL}_n}(T_n) \). Let \( T_{n-1} \) denote the \( n-1 \)-dimensional split torus forming the last \( n-1 \)-factors in the split maximal torus \( T_n \). Observe that on further restricting to the action of the subgroup \( H = \mathbb{G}_m \times \text{N}_{\text{GL}_n}(T_n) \), \( W' \) is the representation of \( \text{N}_{\text{GL}_n}(T_n) \) that is induced from a \( 1 \)-dimensional representation \( V \) of the subgroup \( H \), that is, if \( \{ \sigma_i | H \} = 1, \cdots n \) is the complete set of left cosets of \( H \) in \( \text{N}_{\text{GL}_n}(T_n) \), then \( W' \cong \oplus_{1 \leq i \leq V} V_i \) with each \( V_i = V \) and where \( \text{N}_{\text{GL}_n}(T_n) \) acts on \( W' \) as follows. For \( g \in \text{N}_{\text{GL}_n}(T_n) \), if \( g \cdot \sigma = \sigma_k h, h \in H \), then \( g \cdot v_i = h_i \), with \( v_i = v_k \in V \). (This may be seen by observing that the normalizer of the maximal torus \( \text{N}_{\text{GL}_n}(T_n) \) is the semi-direct product of the symmetric group \( \Sigma_n \) and the maximal torus \( T_n \).)

Next observe that \( p : \text{BN}_{\text{GL}_n}(T_n)^{\text{gm},n} = \text{St}_{2n,n}/H \to \text{St}_{2n,n}/\text{N}_{\text{GL}_n}(T_n) = \text{BN}_{\text{GL}_n}(T_n)^{\text{gm},n} \) is a finite étale map of degree \( n \). Then \( \beta \) identifies with the line bundle, with structure group \( \mathbb{G}_m \), defined by \( \text{St}_{2n,n} \times \text{V} \) on \( \text{BN}_{\text{GL}_n}(T_n)^{\text{gm},n} = \text{St}_{2n,n}/H \). Clearly \( p_\alpha(\beta) = \alpha \). Therefore, it suffices to show that the transfer \( \tau_n^* = p_* \) in this case. That is, it suffices to prove that the transfer \( \tau_n^* \) on Grothendieck groups identifies with the push-forward in this case, which is proven in more generality in the next section of this paper: see Corollary 3.6. This, therefore, completes the proof of the proposition.

**Proof of Proposition 2.5** One observes from Proposition 2.6 that

\[
\tilde{q}_n = \lambda_* \circ \tilde{\zeta}_n,
\]

and that the direct system of maps \( \{ \tilde{\zeta}_n : n \} \) are compatible. Therefore, it follows that the maps \( \{ \tilde{q}_n : n \} \) are also compatible, and taking the direct limit, we obtain \( \tilde{q}_n = \lim_{n \to \infty} \tilde{q}_n = \lambda_* \circ \lim_{n \to \infty} \tilde{\zeta}_n = \lambda_* \circ \zeta \), which proves Proposition 2.5.

**3. The motivic transfer and the motivic Gysin maps associated to projective smooth morphisms**

3.1. One may observe from the discussion in (2.3.10) and (2.3.21) that we need to define a transfer for all finite étale maps between smooth schemes with reasonable properties, such as compatibility with base-change, and then show that such a transfer induces the pushforward map at the level of algebraic K-theory. The definition of such a transfer map for finite étale maps is relatively straightforward. In fact there are existing constructions in the literature, as the referee has pointed out, which either provide such transfers directly or can be used to provide such transfers with a little bit of effort: we will in fact discuss some of these below in Remarks 3.3.

However, proving that such transfers coincide with the pushforward map on Algebraic K-theory seems a bit involved: in the approach we take, one needs to first show that these transfers coincide with Gysin maps for all orientable generalized motivic cohomology theories, and then observe that such Gysin maps on Algebraic K-theory agree with pushforward maps. A careful examination of the proof of the first statement will show that it takes more or less the same effort to define a transfer map for all projective smooth maps and show that it agrees with a Gysin map up to multiplication by a certain Euler class, which will trivialize when the maps are finite étale. Therefore, we adopt this approach: observe that, as a consequence we are able to derive the precise relationship between the transfer and Gysin maps associated to projective smooth maps between smooth quasi-projective schemes on all orientable generalized motivic cohomology theories: we believe this result is of independent interest, though not used in the rest of the paper in this generality.

Therefore, the general context in which we work in this section will be the following. Let

\[
(3.1.1) \quad p : E \to B
\]

denote a projective smooth map of quasi-projective smooth schemes over the base field.
3.2. The definition of a transfer for projective smooth maps. In order to motivate this construction, we will quickly review the corresponding Thom-Pontrjagin construction in the context of classical algebraic topology. Here \( p : E \to B \) will denote a smooth fiber bundle between compact manifolds \( E \) and \( B \). Then one may obtain a closed imbedding of \( E \) in \( B \times \mathbb{R}^N \) for \( N \) sufficiently large. We will denote this imbedding by \( i \). Therefore, one obtains the Thom-Pontrjagin collapse map

\[
\text{TP} : B_+ \wedge S^N \to \text{Th}(\nu)
\]

where \( \nu \) denotes the normal bundle associated to the closed imbedding \( i \). (One may recall that this is the starting point of the classical Atiyah duality (see [At61], [SpWh55], [DPS4]) as well as its étale variant as in [J86] and [J87] in the context of étale homotopy theory as in [AM69].)

We proceed to define a corresponding construction in the motivic context, making use of the Voevodsky collapse in the place of the Thom-Pontrjagin collapse. In the situation in (3.1.1), as the schemes \( E \) and \( B \) are assumed to be quasi-projective, one obtains a closed immersion \( i : E \to B \times \mathbb{R}^N \) for a large enough \( N \). Therefore, the discussion in [Voev03] Proposition 2.7, Lemma 2.10 and Theorem 2.11 (see also [CJ20, 10.4, Definition 10.8]) provides the Voevodsky collapse map

\[
\tau \colon V : B^+_+ \wedge T^n \to \text{Th}(\nu)
\]

for a suitably large \( n \), and where \( \nu \) denotes the vector bundle on \( E \) we call the virtual normal bundle: see [CJ20 §10.8]. (See also [Ho] 5.3 for a discussion on the collapse, which in this framework is originally due to Voevodsky.)

Let \( \tau = \tau_{E/B} \) denote the relative tangent bundle associated to \( p : E \to B \). Assume the relative dimension of \( p \) is \( d \). Then it follows from [Voev03] Proposition 2.7 through Theorem 2.11 (see also [CJ20 10.4, Definition 10.8]) that \( \nu \oplus \tau \) is a trivial bundle on pull-back to \( \tilde{E} \), where \( \tilde{E} \) is a (functorial) affine replacement of \( E \) provided by the technique of Jouanolou (see [Joun73]).

**Definition 3.1.** Therefore, we may define the Becker-Gottlieb transfer in the situation of (3.1.1) as follows:

\[
\text{tr} : B_+^+_+ \wedge T^n \xrightarrow{\nu} \text{Th}(\nu^+) \xrightarrow{\tau^+} \text{Th}(\nu \oplus \tau) \simeq E_+^+_+ \wedge T^n
\]

where \( i_\nu \) is the map induced by the obvious inclusion \( \nu \to \nu \oplus \tau \). (See, for example, the proof of [BG75 Theorem 4.3].)

**Proposition 3.2. (Some basic properties of the transfer.)**

(i) Assume that in (3.1.1), \( E = E_0 \sqcup E_1 \). Denoting the corresponding transfers \( t_{\nu} : B_+^+_+ \wedge T^n \to E_0^+_+ \wedge T^n \), and \( t_{\nu} : B_+^+_+ \wedge T^n \to E_0^+_+ \wedge T^n \), \( t_{\nu} = t_{\nu}^0 + t_{\nu}^1 \) in any generalized motivic cohomology theory.

(ii) In case \( E = B \) in (3.1.1), and the map \( p \) is the identity map on \( B \), then \( t_{\nu} = \text{id} \) on any orientable generalized motivic cohomology theory.

(iii) Assume that the square

\[
\begin{array}{ccc}
E' & \xrightarrow{p'} & E \\
\downarrow p & & \downarrow p \\
B' & \xrightarrow{p} & B
\end{array}
\]

is cartesian. Then we obtain the following homotopy commutative diagram of transfer maps:

\[
\begin{array}{ccc}
B_+^+_+ \wedge T^n & \xrightarrow{V'} & \text{Th}(\nu') \\
\downarrow & & \downarrow \nu'_i \\
B_+^+_+ \wedge T^n & \xrightarrow{\nu'_i} & \text{Th}(\nu') \times \text{Th}(\nu \oplus \tau) \simeq E_+^+_+ \wedge T^n
\end{array}
\]

where \( V' \) and \( i'_\nu \) are the maps corresponding to \( V \) and \( i_\nu \) when \( B \) and \( E \) are replaced by \( B' \) and \( E' \).

**Proof.** The proofs of the first and last statements are straightforward from the construction of the transfer. The second statement follows Corollary 3.6 by taking the map \( p \) to be the identity. \( \square \)

**Remarks 3.3.** Here we briefly discuss other possible constructions of the transfer associated to finite étale maps \( p : E \to B \), where \( E \) and \( B \) are quasi-projective smooth schemes over the base field \( k \). One may find one such construction in [RO 2.3], as pointed out by the referee. It is verified in op.cit that this transfer is compatible with base-change. (Making use of the 6-functor formalism in motivic homotopy theory as in [AV], the second author has also sketched a construction of a transfer for finite étale maps. As this is not all that different from the one...
in [RO 2.3], we do not discuss this any further here.) Therefore, what one needs to show is that this transfer on Algebraic K-theory agrees with the pushforward. As this is not discussed in [RO], all one can say is that a proof of this fact will likely follow the same steps as outlined above and discussed below in detail, except that the relative tangent bundle to the map $p$ will be trivial in this case.

The referee has also pointed out that the discussions in [EHK+] and [HJN+], making use of framed correspondences, provide a transfer map for finite étale maps that identify with pushforwards (of vector bundles) on Algebraic K-theory.

In the rest of this section, we do the following:

(i) Making use of the same Voevodsky-collapse used in the construction of the transfer, we proceed to define a Gysin map associated to projective and smooth maps between smooth quasi-projective schemes in all orientable generalized motivic cohomology theories.

(ii) Then we show that this Gysin map agrees with the Gysin maps defined by more traditional means, typically by factoring the given map $p : E \to B$ as the composition of a closed immersion of $E$ into a relative projective space $B \times \mathbb{P}^n$ followed by the projection $\pi : B \times \mathbb{P}^n \to B$.

(iii) At this point, standard comparison results (see [P09, 2.9.1] which invokes [T-T, 3.16, 3.17 and 3.18]) show that the above Gysin maps identify with the push-forward maps on the Algebraic K-theory of smooth quasi-projective schemes.

(iv) Finally, we show that on orientable generalized motivic cohomology theories, the map induced by the transfer and the Gysin maps constructed below differ only by multiplication by the Euler class of the relative tangent bundle to the map $p$. As a result, when $p$ is a finite étale map, the relative tangent bundle to the map $p$ trivializes and the map induced by the transfer agrees with the push-forward on Algebraic K-theory.

3.3. Gysin maps associated to projective smooth maps on orientable generalized motivic cohomology theories. We begin by quickly reviewing the corresponding situation in Algebraic Topology. For any generalized cohomology theory $h^*$, the Thom-Pontrjagin collapse in (3.2.1) induces the map

$$\text{TP}^* : h^*(\text{Th}(\nu)) \to h^*(B_+ \wedge S^N).$$

We will further assume that $h^*$ is an orientable cohomology theory in the sense that it has a Thom-class $T(\nu) \in h^*(\text{Th}(\nu))$ (where $c$ is the codimension of $E$ in $B \times S^N$), so that the cup product with this class defines the Thom-isomorphism: $h^*(E) \to h^{*+c}(\text{Th}(\nu))$. Moreover, in this case one also observes the suspension isomorphism: $h^*(B_+ \wedge S^N) \cong h^{*-N}(B)$. Thus the composition

$$(3.3.1) \quad p_* : h^*(E) \xrightarrow{\cup T(\nu)} h^{*+c}(\text{Th}(\nu)) \xrightarrow{\text{TP}^*} h^{*+c}(B_+ \wedge S^N) \cong h^{*+c-N}(B)$$

defines a Gysin map. One may observe that if the relative dimension of $E$ over $B$ is $d$, then $c = N - d$, so that $h^{*+c-N}(B) = h^{*-d}(B)$ as required of a Gysin map.

We proceed to define a corresponding Gysin map in the motivic context, for orientable generalized motivic cohomology theories in the sense of [PY02 §2] (see also [P09]), making use of the Voevodsky collapse in the place of the Thom-Pontrjagin collapse. In the situation in (3.1.1), as the schemes $E$ and $B$ are assumed to be quasi-projective, one obtains a closed immersion $i : E \to B \times \mathbb{P}^n$ for a large enough $N$. In this context, we recall the Voevodsky collapse

$$(3.3.2) \quad V : B_+ \wedge T^n \to \text{Th}(\nu)$$
as discussed above in (3.2.2). It should be clear that, with this collapse map replacing the Thom-Pontrjagin and generalized motivic cohomology theories that are orientable (and bi-graded), one obtains a Gysin map

$$(3.3.3) \quad p_* : h^{*\bullet}(E) \xrightarrow{\cup T(\nu)} h^{*+2c\bullet+c}(\text{Th}(\nu)) \xrightarrow{V^*} h^{*+2c\bullet+c}(B_+ \wedge T^n) \cong h^{*+2c-2n\bullet+c-n}(B) = h^{*-2d\bullet-d}(B),$$

if $d$ is the relative dimension of $E$ over $B$, $T(\nu)$ is the Thom-class of the bundle $\nu$, and $c$ is the rank of the vector bundle $\nu$.

Next we proceed to show that the Gysin map defined above indeed agrees with Gysin maps that are defined by other more traditional means, such as in [P09] or [PY02] §4 and §5. (See also [Deg].) For this, we need to first recall the framework for defining the Voevodsky collapse. One may observe from Voev03 pp. 69-70 that one needs to consider the sequence of closed immersions

$$(3.3.4) \quad E \xrightarrow{i} B \times \mathbb{P}^{d+2d} \to B \times \mathbb{P}^d \times \mathbb{P}^{d+2d} \to B \times \mathbb{P}^{d+2d},$$
where Segre denotes the Segre imbedding. We will let \( m = d^2 + 2d \) henceforth. Let \( \nu \) denote the normal bundle to the above composite closed immersion and let \( c \) denote the codimension of this closed immersion. Then one obtains the following sequence of maps:

\[
\begin{align*}
(3.3.5) \quad h^\ast \cdot (E) \xrightarrow{(\text{Th}(\nu))} h^{s+2c} \cdot \ast c \rightarrow h^{s+2c} \cdot \ast c (B \times \mathbb{P}^m/(B \times \mathbb{P}^m - E)) \xrightarrow{\text{Gysin}_1} h^{s+2c} \cdot \ast c (B \times \mathbb{P}^n) \\
\xrightarrow{\text{Gysin}_2} h^{s+2c-2m} \cdot \ast c (B).
\end{align*}
\]

Here the map denoted Gysin\(_1\) precomposed with the cup product with the Thom class \( \text{Th}(\nu) \) is the usual Gysin map associated to the composite closed immersion \( E \rightarrow B \times \mathbb{P}^m \) (see [PY02 \S4]) and the map denoted Gysin\(_2\) is the usual Gysin map associated to the projection \( B \times \mathbb{P}^m \rightarrow B \); see [PY02 Definition 5.1].

3.3.6. Deformation to the normal cone. In order to relate the Gysin maps in (3.3.3) and (3.3.5), we first invoke the technique of deformation to the normal cone from [PY02 1.2.1, Theorem 1.2]. (See also [PY02 2.2.8 Theorem].)

Let \( i: Y \rightarrow X \) denote a closed immersion of smooth schemes of finite type over \( k \) with normal bundle \( \mathcal{N} \). Then there exists a smooth scheme \( \tilde{X} \) together with a smooth map \( p: \tilde{X} \rightarrow A^1 \) and a closed immersion \( i: Y \times A^1 \rightarrow \tilde{X} \), so that the composition \( p \circ i: Y \times A^1 \rightarrow A^1 \) coincides with the projection \( Y \times A^1 \rightarrow A^1 \). Moreover the following additional properties hold:

1. The fiber of \( p \) over \( 1 \in A^1 \) is isomorphic to \( X \) and the base change of \( Y \) to \( A^1 \) corresponds to the given imbedding \( Y \rightarrow X \).
2. The fiber of \( p \) over \( 0 \in A^1 \) is isomorphic to \( \mathcal{N} \) and the base change of \( i \) by the imbedding \( 0 \in A^1 \) corresponds to the 0-section imbedding \( Y \rightarrow \mathcal{N} \).
3. If \( Z \rightarrow Y \) is a closed immersion of a smooth subscheme of \( Y \), then one obtains the following diagram

\[
\begin{array}{ccc}
(\mathcal{N}, \mathcal{N} - Z) \rightarrow (\tilde{X}, \tilde{X} - Z \times A^1) & \xrightarrow{i} & (X, X - Z),
\end{array}
\]

and hence the following diagram for any orientable generalized motivic cohomology theory \( h^\ast \cdot \ast \) with both the horizontal maps being isomorphisms:

\[
(3.3.7) \quad h^\ast \cdot (\mathcal{N}, \mathcal{N} - Z) \xrightarrow{i} h^\ast \cdot (\tilde{X}, \tilde{X} - Z \times A^1) \xrightarrow{i} h^\ast \cdot (X, X - Z).
\]

4. Moreover, in the above situation the normal bundle to the composite closed immersion \( Z \rightarrow Y \rightarrow \mathcal{N} \) is isomorphic to the sum \( \mathcal{N}_{Z,Y} \oplus \mathcal{N}_Z \), where \( \mathcal{N}_{Z,Y} \) denotes the normal bundle associated to the closed immersion \( Z \rightarrow Y \).

**Proposition 3.4.** Assume the above situation. Then the Gysin map defined in (3.3.3) agrees with the Gysin map defined in (3.3.5).

**Proof.** We will follow the constructions in [Voov93 pp. 69-70]. Accordingly the second \( \mathbb{P}^d \) is the projective space and \( \mathcal{H} \) will denote the incidence hyperplane in \( \mathbb{P}^d \times \mathbb{P}^d \). Then it is observed there that, \( \mathbb{P}^d = \mathbb{P}^d \times \mathbb{P}^d - \mathcal{H} \), considered as a scheme over \( \mathbb{P}^d \) by the projection to the first factor (\( p_1 \)) is an affine space bundle; in fact, this is an instance of what is known as Jouanolou’s trick.

Let \( N \) denote the normal bundle to the Segre imbedding of \( B \times \mathbb{P}^d \times \mathbb{P}^d \) in \( B \times \mathbb{P}^n \). If \( j: B \times \mathbb{P}^d \rightarrow B \times \mathbb{P}^d \times \mathbb{P}^d \) denotes the open immersion, we let \( j^\ast (N) \) be the pull-back of \( N \) to \( B \times \mathbb{P}^d \). Since \( B \times \mathbb{P}^d \rightarrow B \times \mathbb{P}^d \) is the diagonal imbedding.

Let \( E \) denote the pull-back of \( p_1^\ast (\tau_{B \times \mathbb{P}^d/B}) \), (where \( \tau_{B \times \mathbb{P}^d/B} \) denotes the relative tangent bundle to \( B \times \mathbb{P}^d \) over \( B \)), and let \( \nu_1 \) denote the normal bundle to the closed immersion \( i: E \rightarrow B \times \mathbb{P}^d \). Let \( p: \tilde{E} \rightarrow E \) denote the map induced by \( p_1 : B \times \mathbb{P}^d \rightarrow B \times \mathbb{P}^d \) when \( \tilde{E} \) is defined by the cartesian square:

\[
\begin{array}{ccc}
\tilde{E} & \xrightarrow{p} & B \times \mathbb{P}^d \\
\downarrow & & \downarrow p_1 \\
E & \xrightarrow{\nu_1} & B \times \mathbb{P}^d.
\end{array}
\]

Let \( \nu_1 \) denote the normal bundle to the induced closed immersion \( \tilde{E} \rightarrow B \times \mathbb{P}^d \). Then denoting the Thom-class of the bundle \( E_1 = (E \oplus j^\ast (N)) \) by \( \nu_1 \) by \( T(E_1) \), we obtain the sequence of maps:

\[
(3.3.8) \quad h^\ast \cdot (\tilde{E}) \xrightarrow{(Th(E_1))} h^{s+2c} \cdot \ast c (Th(E_1)) \rightarrow h^{s+2c} \cdot \ast c (Th(E \oplus j^\ast (N))).
\]

\[2\]This may be viewed as a cohomology-variant of the purity Theorem: see [MV99 Theorem 2.23, p. 115].
The last map is obtained from the observation that the normal bundle to the composite closed immersion \( \tilde{E} \to B \times \mathbb{P}d^0 \) is \( \xi_1 \); see (3.3.6(4)) above. At this point we make use of the identification \( j^*(N) = p_1^*(\Delta^*(N)) \), so that \( \xi_1 = p_1^*(\tau_{B \times \mathbb{P}d/B} \oplus \Delta^*(N)) \), and \( E \oplus N = p_1^*(\tau_{B \times \mathbb{P}d/B} \oplus \Delta^*(N)) \), which then readily provides the commutativity of the following diagram with \( \xi_0 = t^*(\tau_{B \times \mathbb{P}d/B} \oplus \Delta^*(N)) \oplus \nu_1 \):

\[
(3.3.9) \quad h^\bullet(\tilde{E}) \xrightarrow{\cup T(\xi_1)} h^{*+2c,\bullet+c}(Th(\xi_1)) \xrightarrow{\approx} h^{*+2c,\bullet+c}(Th(\xi_1')) \xrightarrow{\approx} h^{*+2c,\bullet+c}(B \times \mathbb{P}m/(B \times \mathbb{P}m - E)) \xrightarrow{\approx} h^{*+2c,\bullet+c}(B \times \mathbb{P}m/(B \times \mathbb{P}m - (B \times \Delta \mathbb{P}d))).
\]

The left-most vertical map above is an isomorphism as \( p : \tilde{E} \to E \) is an affine space bundle. To see that the next vertical map is an isomorphism, one needs to observe that

\[
Th(\xi_1) = Th(p^*(\tau_{B \times \mathbb{P}d/B} \oplus \Delta^*(N)) \oplus \nu_1)) \simeq Th(p^*(\tau_{B \times \mathbb{P}d/B} \oplus \Delta^*(N)) \oplus \nu_1).\]

At this point we make use of the isomorphisms in (3.3.7) and (3.3.6(4)) to obtain the isomorphism:

\[
h^{*\bullet}(Th(\tau_{B \times \mathbb{P}d/B} \oplus \Delta^*(N)) \oplus \nu_1)) \cong h^{*\bullet}(B \times \mathbb{P}m)/(B \times \mathbb{P}m - E)).
\]

to see the last vertical map in (3.3.9) is an isomorphism, one first observes that

\[
Th(\xi_1') = Th(p^*(\tau_{B \times \mathbb{P}d/B} \oplus \Delta^*(N))) \simeq Th(p^*(\tau_{B \times \mathbb{P}d/B} \oplus \Delta^*(N))),
\]

and then adopts a similar argument to obtain the isomorphism:

\[
h^{*\bullet}(\tau_{B \times \mathbb{P}d/B} \oplus \Delta^*(N)) \cong h^{*\bullet}(B \times \mathbb{P}m)/(B \times \mathbb{P}m - (B \times \Delta \mathbb{P}d)).
\]

We also obtain the following commutative diagram:

\[
(3.3.10) \quad h^{*+2c,\bullet+c}(B \times \mathbb{P}m/(B \times \mathbb{P}m - B \times (\mathbb{P}d \times \mathbb{P}d))) \xrightarrow{\simeq} h^{*+2c,\bullet+c}(B \times \mathbb{P}m/(B \times \mathbb{H}_\infty)) \xrightarrow{\Ss} h^{*+2c,\bullet+c}(B \times \mathbb{P}m/(B \times \mathbb{P}m - (B \times \mathbb{P}d \times \mathbb{P}d))) \xrightarrow{\Ss} h^{*+2c,\bullet+c}(B \times \mathbb{P}m/(B \times \mathbb{P}m - (B \times \Delta \mathbb{P}d))) \xrightarrow{\Ss} h^{*+2c,\bullet+c}(B \times \mathbb{P}m/(B \times \mathbb{P}m - (B \times \Delta \mathbb{P}d))).
\]

where \( \mathbb{H}_\infty \) is a hyperplane in \( B \times \mathbb{P}m \) which pulls back to the incidence hyperplane in \( B \times \mathbb{P}d \times \mathbb{P}d \) under the Segre imbedding. Next we recall the following identification (see [Voev03] proof of Lemma 2.10):

\[
Th(\xi_1') \simeq B \times \mathbb{P}m/(B \times \mathbb{P}m - B \times (\mathbb{P}d \times \mathbb{P}d)) \cup B \times \mathbb{H}_\infty).
\]

This shows that in this case, one obtains a composite collapse map

\[
(3.3.10) \quad V : B_+ \times T^m \to Th(\xi_1') \to Th(\xi_1),
\]

and hence that one may compose the maps forming the top row of the diagram (3.3.9) followed by the maps forming the top row of the diagram (3.3.10). In view of the fact that \( p : \tilde{E} \to E \) is an affine replacement, \( Th(\xi_1) \cong Th(\xi_0) \), so that the collapse map in (3.3.10) defines a collapse \( V : B_+ \times T^m \to Th(\xi_0) \), which differs from the collapse map in (3.2.2) only by the addition of a trivial bundle, and hence a \( T \)-suspension of some finite degree on both the source and the target: see [Voev03] proof of Proposition 2.7 and Theorem 2.11]. Therefore, one may now observe that the composition of the maps in the top rows of the two diagrams followed by the suspension isomorphism forming the right most vertical map in the second diagram identifies with the map \( p_\ast \) in (3.3.3).

Observe that there is a natural map:

\[
h^{*+2c,\bullet+c}(B \times \mathbb{P}m/(B \times \mathbb{P}m - (B \times \Delta \mathbb{P}d))) \to h^{*+2c,\bullet+c}(B \times \mathbb{P}m/(B \times \mathbb{P}m - (B \times \Delta \mathbb{P}d))).
\]

Therefore one may compose the maps forming the bottom rows of the two diagrams (3.3.9) and (3.3.10). The composition of the maps forming the bottom rows of the two diagrams defines the Gysin map in (3.3.5). The commutativity of the two diagrams proves these two maps are the same.

**Theorem 3.5.** Let \( h^\bullet \) denote a generalized motivic cohomology which is orientable in the above sense. Let \( tr \) denote the transfer as in (3.2.3). Then if \( ev_t(\tau) \) denotes the Euler class of the bundle \( \tau \), we obtain the relation:

\[
(3.3.11) \quad tr^\ast(\alpha) = p_\ast(\alpha \cup ev_t(\tau)), \quad \alpha \in h^\bullet(E)
\]

where \( p_\ast \) denotes the Gysin map defined above in (3.3.3).
Proof. As shown in [BG75, Theorem 4.3], and adopting the terminology as in (3.2.2) and (3.3.3), it suffices to prove the commutativity of the following diagram:

\[
\begin{array}{ccc}
\h^* \cdot (E) & \xrightarrow{\cup eu(\tau)} & \h^{*+2d} \cdot +d(E) \\
\downarrow \cong & & \downarrow p_* \\
h^{*+2n} \cdot +n(E_+ \wedge T^n) & \cong & h^{*+2n} \cdot +n(Th(\nu \oplus \tau)) \\
\end{array}
\]

Here \(d\) is the relative dimension of \(E\) over \(B\) and \(i\) denotes the map of Thom-spaces induced by the inclusion \(\nu \to \nu \oplus \tau\). The definition of the Gysin map as in (3.3.3) readily proves the commutativity of the right square, so that it suffices to prove the commutativity of the left square. This results from the commutativity of the diagram:

\[
\begin{array}{ccc}
\h^* \cdot (E) & \xrightarrow{\cup eu(\tau)} & \h^{*+2d} \cdot +d(E) \\
\downarrow \cup T(\tau) & & \downarrow \cup T(\nu) \\
h^{*+2d} \cdot +d(Th(\tau)) & \cong & h^{*+2d} \cdot +d(Th(\nu)) \\
\downarrow \cup T(\nu) & & \downarrow \cup T(\pi^*_1(\nu)) \\
h^{*+2n} \cdot +n(Th(\tau \oplus \nu)) & \cong & h^{*+2n} \cdot +n(Th(\pi^*_1(\nu))). \\
\end{array}
\]

Here, if \(\alpha\) denotes a vector bundle, \(Th(\alpha) (T(\alpha))\) denotes the Thom space (The Thom-class, respectively) of \(\alpha\). Observe that the composition of the top row and the right vertical map in the left square of (3.3.12) equals the composition of the maps in the top row of (3.3.13). The composition of the map in the left column and the first bottom map in (3.3.12) clearly equals the composition of the two vertical maps in the left most column of (3.3.13). Since \(E(\tau)\) denotes the total space of the vector bundle \(\tau\), we obtain the isomorphism \(h^{*+2d} \cdot +d(E) \cong h^{*+2d} \cdot +d(E(\tau))\) and also the isomorphism \(h^{*+2n} \cdot +n(Th(\nu)) \cong h^{*+2n} \cdot +n(Th(\pi^*_1(\nu)))\), where \(\pi_1 : E(\tau) \to E\) denotes the projection. Moreover, under the above isomorphisms, the map denoted \(i^*\) in (3.3.12) identifies with the bottom most map in (3.3.13). These observations prove the commutativity of the diagram (3.3.13) and hence the commutativity of the diagram (3.3.12) as well.

Corollary 3.6. Let \(p : E \to B\) denote a finite étale map between smooth quasi-projective schemes. If \(h^{*+*}\) is an orientable generalized motivic cohomology theory defined by a motivic spectrum, then one has the equality:

\[ tr^* = p_* \]

where \(tr^*\) denotes the map induced by the motivic Becker-Gottlieb transfer \(tr\) (see (3.2.3)) in the abouve cohomology theory and \(p_*\) denotes the Gysin map. Moreover, for Algebraic K-Theory, the Gysin map \(p_*\) agrees with the finite pushforward defined for coherent sheaves.

Proof. The first statement is an immediate consequence of Theorem 3.5 once one observes that the Euler class \(eu(\tau)\) is trivial, which follows from the fact that \(p\) is finite étale and \(\tau\) denotes the relative tangent bundle of the map \(p\). The second statement on the Gysin map \(p_*\) for Algebraic K-Theory follows from [P09, 2.9.1], invoking [L-T] 3.16, 3.17 and 3.18. Observe that pushforward by finite étale maps send vector bundles to vector bundles, and for smooth quasi-projective schemes over \(k\), the K-theory of coherent sheaves identifies with the K-theory of vector bundles.

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