Weyl collineations that are not curvature collineations

Ibrar Hussain\textsuperscript{a}, Asghar Qadir\textsuperscript{a} and K. Saifullah\textsuperscript{b}

\textsuperscript{a}Centre for Advanced Mathematics and Physics, National University of Sciences and Technology, Rawalpindi, Pakistan

\textsuperscript{b}Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan

Electronic address: ibrar\_msw@yahoo.com, aqadirs@comsats.net.pk, saifullah@qau.edu.pk

Abstract: Though the Weyl tensor is a linear combination of the curvature tensor, Ricci tensor and Ricci scalar, it does not have all and only the Lie symmetries of these tensors since it is possible, in principle, that “asymmetries cancel”. Here we investigate if, when and how the symmetries can be different. It is found that we can obtain a metric with a finite dimensional Lie algebra of Weyl symmetries that properly contains the Lie algebra of curvature symmetries. There is no example found for the converse requirement. It is speculated that there may be a fundamental reason for this lack of “duality”.
1. Introduction

In general relativity the Ricci tensor and Ricci scalar combine to give the matter content of the spacetime and the Weyl tensor gives the gravitational field with the matter content removed [1]. As such the Weyl tensor plays a fundamental role in understanding the purely gravitational field for a given metric. Since it is conformally invariant [2], i.e. remains unchanged under infinitesimal re-scalings, its local symmetries are of particular interest. Local symmetries of the metric tensor, called isometries or Killing vectors (KVs), are given by

\[ \mathcal{L}_X g = 0, \]  

(1.1)

where \( \mathcal{L}_X \) is the Lie derivative along the vector field \( X \) and \( g \) is the metric tensor. Replacing \( g \) by any tensor field gives the local symmetries of that tensor, called collineations [3]. Putting \( \lambda g \) on the right side of Eq. (1.1), gives the conformal Killing vectors (CKVs). If \( \lambda \) reduces from being any (differentiable) function to a constant number, \( X \) is called a homothetic vector (HV) or a homothety. The proper solution of the non-homogeneous equation is called a proper HV, while the complementary function gives linear combination of KVs. The complete general solution gives the set of HVs, \( \{ \text{HVs} \} \), which contains \( \{ \text{KVs} \} \) properly if there exists a proper HV and otherwise \( \{ \text{HVs} \} \equiv \{ \text{KVs} \} \). Special significance attaches to \( \{ \text{HVs} \} \) as they are the Noether symmetries [4] of the Einstein-Hilbert Lagrangian \( \sqrt{|g|} R \), where \( R \) is the Ricci scalar [5]. Since the metric tensor is everywhere non-singular, \( \{ \text{KVs} \} \) form a finite dimensional Lie algebra of dimension \( \leq n(n+1)/2 \) for a manifold of dimension \( n \).

The curvature tensor or Ricci tensor can be “degenerate” in the sense that their “determinant” is zero. (The 4th rank curvature tensor of 4-dimensions can be represented by a 6 dimensional matrix, on account of its algebraic symmetries, whose rank gives the rank of the tensor.) In this case the system of equations can become under-determined and the resulting Lie algebra can become infinite dimensional. By Noether’s theorem [6] the homotheties give the conservation laws for the spacetime given by \( g \). Clearly, special interest attaches to the case where there are finitely many conserved quantities and hence the Lie algebra of curvature collineations \( \{ \text{CCs} \} \) is finite dimensional. Clearly \( \{ \text{CCs} \} \supseteq \{ \text{HVs} \} \).
The Weyl tensor, $C$, can be written in components form as

$$C_{cd}^{ab} = R_{cd}^{ab} - 2\delta^a_c \delta^b_d R + \frac{1}{3} \delta^a_c \delta^b_d R,$$

where $R_{abcd}$ is the curvature tensor $R_{ab}$ is the Ricci tensor. It is trace-free. Replacing $g$ by $C$, in component form Eq. (1.1) becomes

$$C_{abcd,f} X^f + C_{fcd}^a X^f_a + C_{bdf}^a X^f_a + C_{bcf} X^f_a - C_{bcd} X^f_a = 0,$$

where “,” denotes the partial derivative. Though $C$ and the curvature tensor have similar forms, the local symmetries of the curvature tensor (i.e. CCs) and Weyl collineations (WCs) are different. There has been very little work done which even mentions WCs [3, 7, 8, 9, 10], some of which has errors as mentioned in the conclusion. Indeed if the Ricci tensor, $R$, is zero, i.e. for vacuum with zero cosmological term, \{WCs\} \equiv \{CCs\}, as the Weyl tensor reduces to the curvature tensor [3]. The tensor $R$ is degenerate if its matrix in any coordinate basis is of rank 3 or less. Clearly, it is possible to arrange that one be degenerate without the other being degenerate. For example, if the spacetime is of Petrov type O the Weyl tensor is zero (except for Minkowski space) while the curvature tensor is not. We can choose a metric of type O with non-degenerate curvature tensor. Then the Lie algebra of \{CCs\} will be of dimension less then or equal to 6 and of \{WCs\} infinite dimensional such that every vector field is a WC. An example is the De-Sitter (or anti De-Sitter) spacetime.

The question arises whether the case of a finite dimensional Lie algebra of \{WCs\} and an infinite dimensional Lie algebra of \{CCs\}, can be found. It is not a priori obvious that it will, since the curvature tensor has up to 10 independent components while the Weyl tensor has only 6 (due to the trace-free condition). In this paper we have investigated the relation between the \{WCs\} and \{CCs\} in specific cases, with a view to finding more general statements about the relation. Where examples are found the existence of such metrics is obviously proved but when they are not found it does not prove that they do not exist. Better methods would be needed to obtain the final answer in that case. It would be of interest to obtain answers to these questions for at least some classes of metrics.

2. Examples of unequal \{WCs\} and \{CCs\}

The simplest attempt to find the desired examples, is to consider a non-vacuum
spacetime. The examples that spring to mind are the Schwarzschild interior and Reissner-Nordstrom metrics [11]. In the former the Ricci scalar is non-zero while in the latter the Ricci scalar is zero but the Ricci tensor is non-zero. The Schwarzschild interior solution is Petrov type O [12] and thus every vector field is a WC while KVs, HVs and CCs are four. The Reissner-Nordstrom spacetime is of Petrov type D and there are the same four WCs, KVs, HVs and CCs. But when we take pressure as constant the Schwarzschild interior has the following non-zero components of curvature and Weyl tensor

\[ R_{212}^1 = \frac{8\pi G p}{c^4} r^2, \quad R_{313}^3 = R_{212}^1 \sin^2 \theta \]

\[ C_{101}^0 = \frac{k(3r - 1)}{r(1 - kr^2)}, \quad C_{202}^0 = -\frac{1}{6} kr(3r + 5), \quad C_{212}^1 = \frac{1}{2} kr(3r - \frac{5}{3}), \]

\[ C_{303}^0 = C_{202}^0 \sin^2 \theta, \quad C_{313}^1 = C_{212}^1 \sin^2 \theta, \quad C_{323}^2 = -\frac{4k}{3} r \sin^2 \theta, \]

where \( k = 8\pi G p/c^4 \). In this case the CCs are arbitrary and \( \{\text{WCs}\} = \{\text{KVs}\} = \{\text{HV}s\} = 4 \), with generators given by

\[ X_0 = \frac{\partial}{\partial t}, \quad X_1 = -\sin \phi \frac{\partial}{\partial \theta} - \cot \theta \cos \phi \frac{\partial}{\partial \phi}, \]

\[ X_2 = \cos \phi \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{\partial}{\partial \phi}, \quad X_3 = \frac{\partial}{\partial \phi}. \]

Thus \( \{\text{WCs}\} \) is properly contained in \( \{\text{CCs}\} \). In the case of Reissner-Nordstrom metric the Ricci scalar is zero. However, \( \{\text{KVs}\} = \{\text{HV}s\} = \{\text{CCs}\} = \{\text{WCs}\} \) given by the Lie group \( G_4 = SO(3) \otimes \mathbb{R} \) (where \( \otimes \) denotes direct product) with the generators given above.

Looking through the complete classification of spherically symmetric static metrics by KVs, CCs and RCs [13, 14, 15] did not yield any interesting case. We, therefore, looked at the corresponding classification of cylindrically symmetric static spacetimes [16, 17, 18] and plane symmetric static spacetimes [19, 20, 21] for this purpose.

The general cylindrically symmetric static metric is

\[ ds^2 = e^{\nu(r)} dt^2 - dr^2 - a^2 e^{\lambda(r)} d\theta^2 - e^{\mu(r)} dz^2. \]
This metric has 3 KVs in general, which generate the Lie group $SO(2) \otimes \mathbb{R} \otimes \mathbb{R}$,

\[
X_0 = \frac{\partial}{\partial t}, \quad X_1 = \frac{1}{a} \frac{\partial}{\partial \theta}, \quad X_2 = \frac{\partial}{\partial z},
\]

where, in Eq. (2.1) $a$ is a constant with dimensions of length and $\nu$, $\lambda$ and $\mu$ are arbitrary functions [12]. In the case $\lambda = \text{constant}$, we get a cylindrical analogue of the Bertotti-Robinson metrics and we can choose $\lambda = 0$. If this is not the case we naturally choose the function so that $a$ gets replaced by $r$ and we are left with some other general function of $r$. The plane symmetric general metric can be written as [12]

\[
ds^2 = e^{\nu(x)} dt^2 - dx^2 - e^{\mu(x)} (dy^2 + dz^2).
\]

This metric has 4 KVs in general, which generate the Lie group $[SO(2) \otimes_s \mathbb{R}^2] \otimes \mathbb{R}$, (where $\otimes_s$ denotes semi-direct product) given by

\[
X_0 = \frac{\partial}{\partial t}, \quad X_1 = \frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial z}, \quad X_3 = - z \frac{\partial}{\partial y} + y \frac{\partial}{\partial z}.
\]

The first case of interest is when $\nu = 0$ in Eq. (2.1) and $e^\lambda = e^\mu = (r/a)^2$. In this case the stress-energy tensor is given by

\[
T_{00} = - \frac{1}{kr^2} = -T_{11}, \quad T_{22} = 0 = T_{33},
\]

so that it is not a realistic spacetime. The non-zero component of the curvature tensor is

\[
R_{323}^2 = - \frac{1}{a^2}
\]

and those of the Weyl tensor are

\[
C_{101}^0 = - \frac{1}{3r^2}, \quad C_{202}^0 = \frac{1}{6} = C_{212}^1, \quad C_{303}^0 = \frac{1}{6a^2} = C_{313}^1, \quad C_{323}^2 = - \frac{1}{3a^2}.
\]

Here the Ricci tensor is of rank 2 and is degenerate.

This case has one extra KV

\[
X_3 = - z \frac{\partial}{a \partial \theta} + a \theta \frac{\partial}{\partial z},
\]
one proper HV

\[ X_4 = \frac{t}{\partial t} + \frac{r}{\partial r}, \]

and one additional WC

\[ X_5 = \frac{1}{2} (t^2 + r^2) \frac{\partial}{\partial t} + tr \frac{\partial}{\partial r}. \]

There are infinitely many CCs and RCs. Clearly, here \( \{ \text{KVs} \} \subset \{ \text{HVs} \} \subset \{ \text{WCs} \} \subset \{ \text{CCs} \} \).

A case with the reverse inclusion appears for the plane symmetric metric, with \( e^\nu = e^\mu = (x/a)^b \) \((a, b \neq 0 \in \mathbb{R})\) in Eq. (2.2). Here

\[ T_{00} = -\frac{bx^b}{\kappa 4a^2x^2} (4 - 3b) = -T_{22} = -T_{33}, T_{11} = \frac{3b^2}{\kappa 4a^2}, \]

which represents a realistic spacetime for \( 0 < b < \frac{1}{4} \) and satisfies the positive energy condition \( (T > 0) \) for \( 0 < b < \frac{1}{4} \). It is clearly an anisotropic spacetime. This is a Petrov type O metric with non-zero curvature tensor components

\[ R_{101}^{0} = \frac{b}{4} (\frac{2b - b^2}{x^2}), R_{212}^{1} = R_{313}^{1} = (\frac{x^b}{a})(\frac{(2b - b^2)}{4a^2}), \]

\[ R_{202}^{0} = -\frac{b^2 x^b}{4a^2 x^2} = R_{303}^{0} = R_{323}^{2}. \]

This spacetime has two extra KVs

\[ X_4 = \frac{z}{\partial t} + \frac{t}{\partial z}, X_5 = \frac{y}{\partial t} + \frac{t}{\partial y}, \]

and one proper HV which is also a CC and RC

\[ X_6 = (t \frac{\partial}{\partial t} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}), (b \neq 2). \]

In case \( b = 2 \), the curvature tensor becomes degenerate and the Lie algebra of \( \{ \text{CCs} \} \) infinite dimensional. However, not every vector field will be a CC, while every vector field is a WC. Clearly \( \{ \text{CCs} \} \subset \{ \text{WCs} \} \) for this metric (for all \( b \)).

A more interesting case is \( e^\nu = (r/a)^4 \), \( e^\lambda = e^\mu = (r/a)^2 \) in Eq. (2.1). Here

\[ T_{00} = -\frac{r^2}{\kappa a^4}, T_{11} = \frac{5}{\kappa r^2}, T_{22} = \frac{4}{\kappa}, T_{33} = \frac{4}{\kappa a^2}, \]

\[ \text{(continued on the next page)} \]
which is also a non-realistic spacetime. The non-zero components of the curvature tensor are

\[ R_{101}^0 = -\frac{2}{r^2}, \quad R_{202}^0 = 2, \quad R_{303}^0 = -\frac{2}{a^2}, \quad R_{323}^2 = -\frac{1}{a^2}, \]

and those of the Weyl tensor are

\[ C_{101}^0 = \frac{1}{3r^2}, \quad C_{202}^0 = \frac{1}{6} = C_{212}^1, \]
\[ C_{303}^0 = \frac{1}{6a^2} = C_{313}^1 = -C_{323}^2. \]

This case has one extra KV

\[ X_3 = -\frac{z}{a} \frac{\partial}{\partial \theta} + a\theta \frac{\partial}{\partial z}, \]

one proper HV which is also a CC

\[ X_4 = -t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r}, \]

and one additional WC which is also a conformal vector field

\[ X_5 = -\frac{1}{2} \left( \frac{b^4}{r^2} + t^2 \right) \frac{\partial}{\partial t} + tr \frac{\partial}{\partial r}. \]

Here \( X_3 \) gives a local spatial rotation between the axial and rotational symmetry directions and \( X_4 \) and \( X_5 \) are scaling symmetries. Clearly \( \langle X_1, X_2, X_3 \rangle \) gives the plane symmetry group and \([X_0, X_4] = -X_0, [X_0, X_5] = -X_4, [X_4, X_5] = -X_5\), while \( \{\text{CCs}\} \subset \{\text{WCs}\} \equiv \{\text{RCs}\}\).  

3. Conclusion

We found only two papers that address Weyl collineations properly, both of which have errors and address only limited aspects of the problem [9, 10]. For example in [9] it is claimed that there are 10 WCs for De-Sitter and anti De-Sitter metrics. This is obviously wrong as these spaces are of Petrov type O, for which all vector fields are WCs. Again in Ref. 10 only infinite dimensional Lie algebras for Weyl collineations of pp-waves are found. In that paper \( R_{11} \) is taken to be the only non-zero component...
of the Ricci tensor, while for pp-waves $R_{00} \neq 0$. Correcting the error does not provide any interesting example.

In the present paper we have found non-trivial examples of WCs that are not simply CCs. Of course, the Petrov type O example is trivial in another sense, namely that all vector fields are Weyl collineations, thus $\{\text{CCs}\} \subsetneq \{\text{WCs}\}$. The first example of cylindrical symmetry discussed in this paper was trivial in yet another sense, namely that the Lie algebra of curvature collineations is infinite dimensional. Thus $\{\text{WCs}\} \subsetneq \{\text{CCs}\}$. However the last case is entirely non-trivial as it has $\{\text{CCs}\} \subsetneq \{\text{WCs}\}$ and both have finite dimensional Lie algebras. The question arises whether there are cases in which $\{\text{WCs}\} \subsetneq \{\text{CCs}\}$ and both have finite dimensional Lie algebras.

It is possible that the rank of the $6 \times 6$ Weyl matrix is greater or less than the rank of the corresponding curvature matrix. If the rank of the curvature matrix is $\geq 4$ then the Lie algebra of CCs is finite dimensional [17]. From here it seems possible that there do exist cases with $\{\text{WCs}\} \subsetneq \{\text{CCs}\}$ that remain finite dimensional. However, the process of calculation seemed to indicate that such cases may not be possible. It would be worth while to either find such a case or to provide a definite proof that it does not exist. If it does not, the “duality” between curvature and Weyl collineations that may have been expected would be shown to be violated.

**Acknowledgments**

Useful discussions with Ugur Camci are acknowledged. IH would like to thank the Higher Education Commission of Pakistan and Quaid-i-Azam University, Islamabad for financial support provided during this work.

**References**

[1] Penrose, R., Rindler, W., *Spinors and Spacetime* (Cambridge University Press) 1986.

[2] Hawking, S. W., and Ellis, G. F. R., *The Large Scale Structure of Spacetime* (Cambridge University Press) 1973.

[3] Katzin, G. H., Levine, J., and Davis, W. R., *J. Math. Phys.* 10 (1969) 617.
[4] Stephani, H., *Applications of Lie Groups to Differential Equations* (Springer-Verlag) 1993.

[5] Landau, L. D., and Lifshitz, E. M., *The Classical Theory of Fields* (Pergamon Press) 1962.

[6] Bluman, G., and Kumei, S., *Symmetries and Differential Equations* (Springer-Verlag) 1989.

[7] Hall, G. S., *Gravitation & Cosmology* 2 (1996) 270.

[8] Hall, G. S., *Gen. Rel. Grav.* 32 (2000) 933.

[9] Bokhari, A. H., Ahmad, S., and Pervez, A., *Int. J. Theor. Phys.* 35 (1996) 1013.

[10] Shabbir, G., *J. Pure & App. Sci. Islamia Univ. Bahawalpur* 21 (2002) 1.

[11] Misner, C.W., Thorne, K. S., and Wheeler, J. A., *Gravitation* (W.H. Freeman and Company) 1973.

[12] Kramer, D., Stephani, H., MacCullum, M. A. H., and Herlt, E., *Exact Solutions of Einstein Field Equations* (Cambridge University Press) 1980.

[13] Qadir, A., and Ziad, M., *Nuovo Cimento* B110 (1995) 317.

[14] Bokhari, A. H., Kashif, A. R., Qadir, A., and Shaikh, A. G., *Nuovo Cimento* B115 (2000) 383.

[15] Ziad, M., *Gen. Rel. Grav.* 35 (2003) 915.

[16] Qadir, A., and Ziad, M., *Nuovo Cimento* B110 (1995) 277.

[17] Bokhari, A. H., Kashif, A. R., and Qadir, A., *Gen. Rel. Grav.* 35 (2003) 1059; Kashif, A. R., Ph.D. Thesis, Quaid-i-Azam University, Islamabad (2003).

[18] Qadir, A., Saifullah, K., and Ziad, M., *Gen. Rel. Grav.* 35 (2003) 1927; Saifullah, K., Ph.D. Thesis, Quaid-i-Azam University, Islamabad (2003).

[19] Feroze, T., Qadir, A., and Ziad, M., *J. Math. Phys.* 42 (2001) 4947.

[20] Bokhari, A. H., Kashif, A. R., and Qadir, A., *J. Math. Phys.* 41 (2000) 2167.

[21] Farid, T. B., Qadir, A., and Ziad, M., *J. Math. Phys.* 36 (1995) 5812.