PATA TYPE CONTRACTIONS INVOLVING RATIONAL
EXPRESSIONS WITH AN APPLICATION TO
INTEGRAL EQUATIONS

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Abstract. In this paper, we introduce the notion of rational Pata type con-
traction in the complete metric space. After discussing the existence and
uniqueness of a fixed point for such contraction, we consider a solution for
integral equations.

1. Introduction and preliminaries. Fixed point theory firstly appeared in the
paper of Liouville [14] in 1837. In this paper, Liouville used implicitly the fixed
point theorem, namely, method of successive approximations for a solution of cer-
tain differential equations. This approaches were abstracted by Banach [5] in 1922
under the name of contraction mapping principle. Since then fixed point theory is
developed independently. One of the recent results in this direction was given by
Pata [17].

Theorem 1.1. [17] Let \((X, d)\) be a complete metric space and let \(\Lambda \geq 0, \alpha \geq 1,
\beta \in [0, \alpha]\) be fixed constants. The mapping \(T : X \to X\) has a fixed point in \(X\) if the
inequality
\[
d(Tu, Tv) \leq (1 - \varepsilon) d(u, v) + \Lambda (\varepsilon) \alpha d(v,v) [1 + \|u\| + \|v\|]^{\beta},
\]
is satisfied for every \(\varepsilon \in [0, 1]\) and \(\psi \in \Psi\), where \(\|u\| = d(u,w)\) for an arbitrary but
fixed \(w \in X\).

Remark 1. We considered here the set
\[
\Psi = \{\psi : [0, 1] \to [0, \infty) \mid \psi \text{ is continuous at zero with } \psi(0) = 0\}.
\]

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Definition 1.2. (See [13]) A self-mapping $T$ on a metric space $(X,d)$ is said to satisfy $C$-condition if
\[ \frac{1}{2}d(x,Tx) \leq d(x,y) \implies d(Tx, Ty) \leq d(x, y), \forall x, y \in X. \]

The aim of Suzuki [20] is to extend the well-known Edelstein’s Theorem by using the notion of $C$-condition.

Let $(\mathcal{X}, d)$ be a metric space and $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a function. A mapping $T : \mathcal{X} \to \mathcal{X}$ is called $\alpha$-orbital admissible if the following condition holds:
\[ \alpha(u, Tu) \geq 1 \implies \alpha(Tu, T^2u) \geq 1, \quad (1) \]
for all $u \in \mathcal{X}$.

Let us consider the following set:
\[ \Phi = \left\{ \phi : [0, \infty) \to [0, \infty) \mid \phi \text{ is nondecreasing and } \sum_{n \geq 1} \phi^n(x) < \infty \text{ for each } x > 0 \right\}, \]
where we denote by $\phi^n$ the $n$-th iterate of $\phi$. This class is called a set of $c$-comparison function. It can be proved that every function $\phi \in \Phi$ has the following properties:
\begin{enumerate}
  \item $\phi(x) < x$, for any $x > 0$;
  \item $\phi(0) = 0$.
\end{enumerate}

2. Main results. In this section we shall introduce two contractions that involve different kind of rational expression. The following statement express the first notion that we discuss in the sequel.

Definition 2.1. Let $(\mathcal{X}, d)$ be a metric space and a function $\phi \in \Phi$. Let $\Lambda \geq 0$, $\lambda \geq 1$ and $\beta \in [0, \lambda]$ be fixed constants. An $\alpha$-orbital admissible mapping $T : \mathcal{X} \to \mathcal{X}$ is called Pata-Jaggi contraction if for every $\varepsilon \in [0, 1]$ the following condition is satisfied
\[ \frac{1}{2}d(u, Tu) \leq d(u, v) \implies \alpha(u, Tu)\alpha(v, Tv)d(Tv, Tv) \leq \phi(\mathcal{R}(u, v)) \quad (2) \]
for all $u, v \in X$, where
\[ \mathcal{R}(u, v) = (1 - \varepsilon) \max \left\{ d(u, v), \frac{\ell(u, Tu)\ell(v, Tv)}{d(u, v)} \right\} + \lambda \varepsilon^{\lambda} \psi(\varepsilon) \left[ 1 + \|u\| + \|v\| + \|Tu\| + \|Tv\| \right]^2. \quad (3) \]

Theorem 2.2. If a continuous self-mapping $T$ on a complete metric spaces $(\mathcal{X}, d)$ fulfills the following three assumptions,
\begin{enumerate}
  \item $(i)$ $T$ on $\mathcal{X}$ is Pata-Jaggi contraction;
  \item $(ii)$ there exists $u_0 \in \mathcal{X}$ such that $\alpha(u_0, Tu_0) \geq 1$;
\end{enumerate}
then $T$ has a fixed point. In addition, we assume the following condition,
\item $(iii)$ $\alpha(u^*, Tu^*) \geq 1$ for all $u^* \in \text{Fix}(T)$,
we conclude that $T$ possesses a unique fixed point.

Proof. By the assumption $(ii)$ of the theorem, there exists a point $u_0 \in \mathcal{X}$ so that $\alpha(u_0, Tu_0) \geq 1$. On account of the fact that $T$ is an $\alpha$-orbital admissible mapping, we derive that
\[ \alpha(u_0, Tu_0) \geq 1 \implies \alpha(Tu_0, T^2u_0) \geq 1, \]
and iteratively, we find
\[ \alpha(T^nu_0, T^{n+1}u_0) \geq 1 \text{ for every } n \in \mathbb{N}. \quad (4) \]
We denote \( d(u, u_0) = \| u \|, \forall u \in X \). Starting to the point \( u_0 \) we build an iterative sequence \( \{ u_n \} \) where \( u_n = T u_{n-1} = T^n u_0 \) for \( n = 1, 2, 3, \cdots \), and we presume that any two consequent terms of this sequence are distinct. Indeed, if we can find \( l_0 \in \mathbb{N} \) such that

\[ u_{l_0} = u_{l_0+1} = T u_{l_0}, \]

then, \( u_{l_0} \) is a fixed point. To avoid this, we will assume in the following that for all \( n \in \mathbb{N} \)

\[ u_n \neq u_{n+1} \] which is equivalent to \( d(u_n, u_{n+1}) > 0 \).

Since \( \frac{1}{2} d(u_{n-1}, T u_{n-1}) = \frac{1}{2} d(u_{n-1}, u_n) \leq d(u_{n-1}, u_n) \), by recalling that \( T \) is a Pata-Jaggi contraction, we deduce

\[
d(u_n, u_{n+1}) = d(T u_{n-1}, T u_n) \\
\leq \alpha(u_{n-1}, T u_{n-1}) d(u_n, T u_n) d(T u_{n-1}, T u_n) \\
\leq \phi(R(u_n, u_{n+1})) ,
\]

where

\[
R(u_n, u_{n+1}) = (1 - \varepsilon) \max \left\{ d(u_{n-1}, u_n), \frac{d(u_{n-1}, T u_{n-1}) d(u_n, T u_n)}{d(u_{n-1}, u_n)} \right\} + \lambda \varepsilon^{\lambda} \psi(\varepsilon) [1 + \| u_{n-1} \| + \| u_n \| + \| T u_{n-1} \| + \| T u_n \|]^{\beta} \\
= (1 - \varepsilon) \max \left\{ d(u_{n-1}, u_n), \frac{d(u_{n-1}, u_n) d(u_n, u_{n+1})}{d(u_{n-1}, u_n)} \right\} + \lambda \varepsilon^{\lambda} \psi(\varepsilon) [1 + \| u_{n-1} \| + \| u_n \| + \| T u_{n-1} \| + \| T u_n \|]^{\beta} \\
= (1 - \varepsilon) \max \left\{ d(u_{n-1}, u_n), d(u_n, u_{n+1}) \right\} + \lambda \varepsilon^{\lambda} \psi(\varepsilon) [1 + \| u_{n-1} \| + \| u_n \| + \| u_{n+1} \|]^{\beta} .
\]

Taking into account the properties of function \( \phi \) and together with \( (6) \), the expression \( (5) \) yields that

\[
d(u_n, u_{n+1}) \leq \phi(R(u_n, u_{n+1})) < R(u_n, u_{n+1}) \\
= (1 - \varepsilon) \max \left\{ d(u_{n-1}, u_n), d(u_n, u_{n+1}) \right\} + \lambda \varepsilon^{\lambda} \psi(\varepsilon) [1 + \| u_{n-1} \| + 2 \| u_n \| + \| u_{n+1} \|]^{\beta} .
\]

After the observations in \((7)\), we are ready to claim that the sequence \( \{ d(u_{n-1}, u_n) \} \) is non-increasing. Indeed, if on the contrary, we have

\[
\max \left\{ d(u_{n-1}, u_n), d(u_n, u_{n+1}) \right\} = d(u_{n-1}, u_n) ,
\]

then we obtain

\[
d(u_n, u_{n+1}) < (1 - \varepsilon) d(u_{n-1}, u_n) + \lambda \varepsilon^{\lambda} \psi(\varepsilon) [1 + \| u_{n-1} \| + 2 \| u_n \| + \| u_{n+1} \|]^{\beta} .
\]

Since it is true for all \( \varepsilon \in [0, 1] \), we derive

\[
d(u_n, u_{n+1}) < d(u_n, u_{n+1}) ,
\]

for \( \varepsilon = 0 \). Clearly, it is a contradiction. Thus, the sequence \( \{ d(u_{n-1}, u_n) \} \) is non-increasing and there is some non-negative real number \( \delta \) such that

\[
\lim_{n \to \infty} d(u_{n-1}, u_n) = \delta .
\]

Throughout the proof, we presume the following:

\[
\frac{1}{2} d(u_{n-1}, u_n) \leq d(u_{n-1}, u_0) \text{ or } \frac{1}{2} d(u_n, u_{n+1}) \leq d(u_n, u_0) ,
\]
Note the the opposite cases does not bring anything to discuss. Indeed, if we suppose, on the contrary, that
\[ \frac{1}{2} d(u_{n-1}, u_n) > d(u_{n-1}, u_0) \text{ and } \frac{1}{2} d(u_n, u_{n+1}) > d(u_n, u_0), \]
then, by the triangle inequality, we derive
\[ d(u_{n-1}, u_n) \leq d(u_{n-1}, u_0) + d(u_0, u_n) < \frac{1}{2} d(u_{n-1}, u_0) + \frac{1}{2} d(u_n, u_{n+1}) < d(u_{n-1}, u_n), \]
which is a contradiction.

On the other hand, to prove that \( \delta = 0 \) we need to show that \( \{\mu_n\} \) is bounded, where \( \mu_n = \|u_n\| = d(u_n, u_0) \).

Indeed, from the triangle inequality and since the sequence \( \{d(u_n, u_{n+1})\} \) is non-increasing, we find that
\[ \mu_n = d(u_n, u_0) \leq d(u_n, u_{n+1}) + d(u_{n+1}, u_1) + d(u_1, u_0) = d(u_n, u_{n+1}) + d(Tu_n, Tu_0) + \mu_1 \leq d(Tu_n, Tu_0) + 2\mu_1. \]

On the other hand, on account of (4), we have
\[ \alpha(u_n, Tu_n)\alpha(u_0, Tu_0) \geq 1. \]

Taking (9) into account and regarding that \( T \) is Pata-Jaggi contraction, and the property \( \phi(t) < t \) for any \( t > 0 \) of the function \( \phi \), we have
\[ d(Tu_n, Tu_0) \leq \alpha(u_n, Tu_n)\alpha(u_0, Tu_0)d(Tu_n, Tu_0) \leq \phi(R(u_n, u_0)) < R(u_n, u_0) \]
\[ = (1 - \varepsilon) \max \left\{ d(u_n, u_0), \frac{d(u_n, Tu_n) + d(u_0, Tu_0)}{\alpha(u_n, Tu_0)} \right\} + \Lambda \varepsilon^\lambda \psi(\varepsilon) \left[ 1 + \|u_n\| + \|u_0\| + \|Tu_n\| + \|Tu_0\| \right]^\beta \]
\[ = (1 - \varepsilon) \max \left\{ d(u_n, u_0), \frac{d(u_n, u_{n+1}) + d(u_0, u_1)}{\alpha(u_n, u_0)} \right\} + \Lambda \varepsilon^\lambda \psi(\varepsilon) \left[ 1 + \|u_n\| + \|u_0\| + \|u_{n+1}\| + \|u_1\| \right]^\beta \]
\[ < (1 - \varepsilon) \max \left\{ \mu_n, \frac{\mu_1^2}{\mu_n} \right\} + \Lambda \varepsilon^\lambda \psi(\varepsilon) \left[ 1 + 2\mu_n + 2\mu_1 \right]^\beta. \]

Consequently, from (10) and the previous inequality we derive that
\[ \mu_n < (1 - \varepsilon) \max \left\{ \mu_n, \frac{\mu_1^2}{\mu_n} \right\} + \Lambda \varepsilon^\lambda \psi(\varepsilon) \left[ 1 + 2\mu_n + 2\mu_1 \right]^\beta + 2\mu_1. \]

It is easy to see that if for any \( n \in \mathbb{N} \), \( \max \left\{ \mu_n, \frac{\mu_1^2}{\mu_n} \right\} \) then \( \mu_n \leq \mu_1 \), so that the sequence \( \{\mu_n\} \) is bounded. On the contrary, if \( \mu_n > \mu_1 \), then \( \max \left\{ \mu_n, \frac{\mu_1^2}{\mu_n} \right\} = \mu_n \) and since \( \beta \leq \lambda \) we get
\[ \varepsilon \mu_n < \Lambda \varepsilon^\lambda \psi(\varepsilon) \left[ 1 + 2\mu_n + 2\mu_1 \right]^\beta + 2\mu_1 \leq \Lambda \varepsilon^\lambda \psi(\varepsilon) \left[ 1 + 2\mu_n + 2\mu_1 \right] \lambda + 2\mu_1 \]
\[ = \Lambda \varepsilon^\lambda \psi(\varepsilon)(1 + 2\mu_1)^\lambda \left[ 1 + \frac{2\mu_1}{1 + 2\mu_1} \right] \lambda + 2\mu_1 \]
\[ \leq \Lambda \varepsilon^\lambda \psi(\varepsilon)2^\lambda \mu_1^\lambda \left( 1 + \frac{1}{2\mu_n} \right)^\lambda (1 + 2\mu_1)^\lambda + 2\mu_1. \]
If we suppose that the sequence \( \{\mu_n\} \) is unbounded we can find a subsequence \( \mu_{n_k} \) of \( \{\mu_n\} \) such that \( \mu_{n_k} \to \infty \) as \( k \to \infty \) and then, choosing \( \varepsilon = \varepsilon_k = \frac{1+2\mu_k}{\mu_{n_k}} \) in the above inequality we obtain
\[
1 \leq 2\lambda \left(1 + 2\mu_1\right)^{2\lambda} \left(1 + \frac{1}{2\mu_{n_k}}\right)^{\lambda\psi(\varepsilon_k)} \to 0 \text{ as } k \to \infty.
\]

Since this is a contradiction, we conclude that our assumption is false. Consequently, the sequence \( \{\mu_n\} \) is bounded. Therefore, there is \( m > 0 \) such that \( \mu_n \leq m \) for \( n \in \mathbb{N} \).

By using the observation that the sequence \( \{\mu_n\} \) is bounded, we shall indicate that \( \delta = 0 \), where \( \delta = \lim_{n \to \infty} d(u_n, u_{n+1}) \). Suppose, on the contrary, that \( \delta > 0 \).

Since, trivially we have \( \frac{1}{2}d(u_{n-1}, Tu_{n-1}) \leq d(u_{n-1}, u_n) \) and the \( T \) is Pata-Jaggi contraction, we find
\[
d(u_n, u_{n+1}) \leq \phi(R(u_{n-1}, u_n)),
\]
where
\[
R(u_{n-1}, u_n) < (1 - \varepsilon)d(u_{n-1}, u_n) + \Lambda\varepsilon^{\lambda}(\varepsilon + |u_{n-1}| + 2|u_n| + |u_{n+1}|)^{\beta}
\]
\[
= (1 - \varepsilon)d(u_{n-1}, u_n) + \Lambda\varepsilon^{\lambda}(\varepsilon + \mu_{n-1} + 2\mu_n + \mu_{n+1})^{\beta}
\]
\[
\leq (1 - \varepsilon)d(u_{n-1}, u_n) + \Lambda\varepsilon^{\lambda}(1 + 4m)^{\beta}(\varepsilon)
\]
Taking into account that \( \phi \) is increasing and (7), we derive from (11) that
\[
d(u_n, u_{n+1}) \leq \phi(R(u_{n-1}, u_n)) \leq \phi((1 - \varepsilon)d(u_{n-1}, u_n) + \Lambda\varepsilon^{\lambda}(1 + 4m)^{\beta}(\varepsilon)).
\]

Letting \( n \to \infty \) in the above inequality, we get
\[
\delta \leq (1 - \varepsilon)\delta + \Lambda\varepsilon^{\lambda}(1 + 4m)^{\beta}(\varepsilon).
\]

or equivalent
\[
\delta \leq \Lambda\varepsilon^{\lambda-1}(1 + 4m)^{\beta}(\varepsilon).
\]

Since the function \( \psi \) is continuous at zero, and \( \psi(0) = 0 \) we get that \( \delta = 0 \) which is a contradiction.

In the next step, we shall show that \( \{u_n\} \) is a Cauchy sequence. Returning in (12) we have
\[
d(u_n, u_{n+1}) \leq \phi((1 - \varepsilon)d(u_{n-1}, u_n) + \Lambda\varepsilon^{\lambda}(1 + 4m)^{\beta}(\varepsilon))
\]
for any \( \varepsilon \in [0, 1] \). Particularly, for \( \varepsilon = 0 \), taking into account the properties of function \( \phi \) and since the sequence \( \{d(u_{n-1}, u_n)\} \) is non-increasing we get that
\[
d(u_n, u_{n+1}) \leq \phi(d(u_{n-1}, u_n)) \leq \phi^n(d(u_0, u_1)) = \phi^n(\mu_1).
\]

Of course, using the triangle inequality we have
\[
d(u_n, u_{n+1}) \leq d(u_n, u_{n+1}) + d(u_{n+1}, u_{n+2}) + ... + d(u_{n+i-1}, u_{n+i})
\]
\[
\leq \sum_{j=n}^{n+i-1} \phi^j(\mu_1).
\]

But, since \( \phi \in \Phi \), there is \( n(\zeta) \in \mathbb{N} \) such that \( \sum_{j \geq n(\zeta)} \phi^j(\mu_1) < \zeta \) so that, from
\[
d(u_n, u_{n+1}) \leq \sum_{j=n}^{n+i-1} \phi^j(\mu_1) \leq \sum_{j \geq n(\zeta)} \phi^j(\mu_1) < \zeta
\]
we can conclude that the sequence \( \{(u_n)\} \) is Cauchy in a complete metric space. Therefore, there is \( u^* \in \mathcal{X} \), such that \( u_n \to u^* \), as \( n \to \infty \). Furthermore, since \( \mathcal{X} \) is a continuous mapping, we get that \( Tu^* = u^* \), that is \( u^* \) is a fixed point of \( T \).
We will prove now the uniqueness of the fixed point. For this purpose let \( v^* \) be another fixed points of \( T \), such that \( d(u^*, v^*) > 0 \). Certainly, from the presumption \((iii)\), we have \( \alpha(u^*, Tu^*) \geq 1 \) and \( \alpha(v^*, Tv^*) \geq 1 \). Thus, we have \( \frac{1}{2}d(u^*, Tu^*) = 0 \leq d(u^*, v^*) \) and since \( T \) is Pata-Jaggi contraction we get

\[
d(u^*, v^*) \leq \alpha(u^*, T u^*) \alpha(v^*, T v^*) d(T u^*, T v^*) \leq \phi(\mathcal{R}(u^*, v^*)) < \mathcal{R}(u^*, v^*)
\]

\[
= (1 - \varepsilon) \max \left\{ d(u^*, v^*), \frac{d(u^*, T u^*) d(T u^*, T v^*)}{\phi(u, v)} \right\} + \\
+ \Lambda \varepsilon^\lambda \psi(\varepsilon) \left[ 1 + \|u^*\| + \|v^*\| + \|Tv^*\| \right]^{\beta} \\
= (1 - \varepsilon)d(u^*, v^*) + \Lambda \varepsilon^\lambda \psi(\varepsilon) \left[ 1 + 2 \|u^*\| + 2 \|v^*\| \right]^{\beta}.
\]

Letting \( \varepsilon \to 0 \) in the above inequality, we have \( d(u^*, v^*) < d(u^*, v^*) \) which is a contradiction. Hence, \( d(u^*, v^*) = 0 \), that is \( u^* = v^* \). \( \square \)

**Definition 2.3.** Let \((\mathcal{X}, d)\) be a metric space and a function \( \phi \in \Phi \). Let \( \Lambda \geq 0, \lambda \geq 1 \) and \( \beta \in [0, \lambda] \) be fixed constants. An \( \alpha \)-orbital admissible mapping \( T : \mathcal{X} \to \mathcal{X} \) is called rational type Pata contraction if for every \( \varepsilon \in [0, 1] \) the following condition is satisfied

\[
\frac{1}{2}d(u, Tu) \leq d(u, v) \text{ implies } \alpha(u, Tu) \alpha(v, Tv) d(Tu, Tv) \leq \phi(Q(u, v))
\]

for all \( u, v \in \mathcal{X} \), where

\[
Q(u, v) = (1 - \varepsilon) \max \left\{ d(u, v), \frac{d(v, Tu)(1 + d(u, Tu))}{1 + d(u, v)} \right\} + \\
+ \Lambda \varepsilon^\lambda \psi(\varepsilon) \left[ 1 + \|u\| + \|v\| + \|Tu\| + \|Tv\| \right]^{\beta}.
\]

**Theorem 2.4.** If a continuous self-mapping \( T \) on a complete metric spaces \((\mathcal{X}, d)\) fulfills the following three assumptions,

(i) \( T \) on \( \mathcal{X} \) is rational type Pata contraction;
(ii) there exists \( u_0 \in \mathcal{X} \) such that \( \alpha(u_0, Tu_0) \geq 1 \);

then \( T \) has a fixed point. In addition, we assume the following condition,

(iii) \( \alpha(u^*, Tu^*) \geq 1 \) for all \( u^* \in Fix(T) \),

we conclude that \( T \) possesses a unique fixed point.

**Proof.** By following the initial lines from the proof of Theorem 2.2, we conclude that there is a sequence \( \{u_n\} \), starting from \( u_0 \in \mathcal{X} \) such that

\[
d(u_n, u_{n+1}) > 0 \text{ and } \alpha(u_n, u_{n+1}) \geq 1 \text{ for any } n \in \mathbb{N}.
\]

Since

\[
\frac{1}{2}d(u_{n-1}, Tu_{n-1}) = d(u_{n-1}, u_n) < d(u_{n-1}, u_n),
\]

and since \( T \) is rational type Pata contraction, for \( u = u_{n-1} \) and \( v = u_n \) in (14) we have

\[
d(u_n, u_{n+1}) \leq \alpha(u_{n-1}, Tu_{n-1}) \alpha(u_n, Tu_n) d(Tu_{n-1}, Tu_n) \leq \phi(Q(u_{n-1}, u_n)),
\]
where
\[
d(u_n, u_{n+1}) \leq \phi(Q(u_{n-1}, u_n)) < Q(u_{n-1}, u_n)
\]
\[
= (1 - \varepsilon) \max \left\{ d(u_{n-1}, u_n), \frac{d(u_{n}, Tu_n) (1 + d(u_{n-1}, Tu_{n-1}))}{1 + d(u_{n-1}, u_n)} \right\} + \\
+ \Lambda \varepsilon^\lambda \psi(\varepsilon) \left[ 1 + \|u_{n-1} + \|u_n + \|Tu_{n-1} + \|Tu_n \| \right]^\beta
\]
\[
= (1 - \varepsilon) \max \left\{ d(u_{n-1}, u_n), \frac{d(u_{n}, u_{n+1}) (1 + d(u_{n-1}, u_n))}{1 + d(u_{n-1}, u_n)} \right\} + \\
+ \Lambda \varepsilon^\lambda \psi(\varepsilon) \left[ 1 + \|u_{n-1} + \|u_n + \|u_{n+1} \| \right]^\beta
\]
\[
= (1 - \varepsilon) \max \left\{ d(u_{n-1}, u_n), d(u_n, u_{n+1}) \right\} + \\
+ \Lambda \varepsilon^\lambda \psi(\varepsilon) \left[ 1 + \|u_{n-1} + \|u_n + \|u_{n+1} \| \right]^\beta.
\]

where the property \(\phi(t) < t, t > 0\) is used above.

Using the observed inequality above, we shall show that the sequence \(\{d(u_n, u_{n+1})\}\) is non-increasing. Suppose, on the contrary, that there exists \(n_0 \in \mathbb{N}\) such that \(d(u_{n_0-1}, u_{n_0}) \leq d(u_{n_0}, u_{n_0+1})\) for \(n \geq n_0\), then
\[
d(u_{n_0}, u_{n_0+1}) < (1-\varepsilon)d(u_{n_0}, u_{n_0+1}) + \Lambda \varepsilon^\lambda \psi(\varepsilon) \left[ 1 + \|u_{n_0-1} + \|u_{n_0} + \|u_{n_0+1} \| \right]^\beta,
\]
or, equivalent
\[
d(u_{n_0}, u_{n_0+1}) < \Lambda \varepsilon^\lambda \psi(\varepsilon) \left[ 1 + \|u_{n_0-1} + \|u_{n_0} + \|u_{n_0+1} \| \right]^\beta.
\]

Since the inequality above holds for all \(\varepsilon \in [0, 1]\), we get \(d(u_n, u_{n+1}) < 0\), for \(\varepsilon = 0\), a contradiction. Consequently, we derive that
\[
d(u_{n-1}, u_n) > d(u_n, u_{n+1})\]
for every \(n \in \mathbb{N}\),

that is, the sequence \(\{d(u_n, u_{n+1})\}\) is non-increasing. Consequently, we find that
\[
d(u_n, u_{n+1}) \leq d(u_0, u_1) = \mu_1.
\]

Moreover, there exists \(\delta \geq 0\) such that \(\lim_{n \to \infty} d(u_n, u_{n+1}) = \delta\).

On account of (16) we have \(\alpha(u_n, Tu_n) \alpha(u_0, Tu_0) \geq 1\) for any \(n \in \mathbb{N}\). On the other hand, as we discussed in Theorem 2.2 at (9), we have
\[
\frac{1}{2} d(u_{n-1}, u_n) \leq d(u_{n-1}, u_0) \quad \text{or} \quad \frac{1}{2} d(u_n, u_{n+1}) \leq d(u_n, u_0).
\]

Let \(\{\mu_n\}\) where \(\mu_n = \|u_n\| = d(u_n, u_0).\) Since \(T\) is an rational type Pata contraction, we have
\[
d(Tu_n, Tu_0) \leq \alpha(u_n, Tu_n) \alpha(u_0, Tu_0) \leq d(Tu_n, Tu_0) \leq \phi(Q(u_n, u_0)),
\]
where
\[
Q(u_n, u_0) = (1 - \varepsilon) \max \left\{ d(u_n, u_0), \frac{d(u_n, Tu_n) (1 + d(u_{n+1}, Tu_{n+1}))}{1 + d(u_{n+1}, u_n)} \right\} + \\
+ \Lambda \varepsilon^\lambda \psi(\varepsilon) \left[ 1 + \|u_n + \|u_0 + \|Tu_n + \|Tu_0 \| \right]^\beta
\]
\[
= (1 - \varepsilon) \max \left\{ d(u_n, u_0), \frac{d(u_n, u_{n+1}) (1 + d(u_{n+1}, u_n))}{1 + d(u_n, u_0)} \right\} + \\
+ \Lambda \varepsilon^\lambda \psi(\varepsilon) \left[ 1 + \|u_n + \|u_0 + \|u_{n+1} + \|u_1 \| \right]^\beta
\]
\[
\leq (1 - \varepsilon) \max \left\{ \mu_n, \mu_n (1 + \mu_1) \right\} + \\
+ \Lambda \varepsilon^\lambda \psi(\varepsilon) \left[ 1 + \mu_n + \mu_0 + d(u_{n+1}, u_n) + \mu_n + \mu_1 \right]^\beta
\]
\[
\leq (1 - \varepsilon) \max \{ \mu_n, \mu_1 (1 + \mu_1) \} + \Lambda \varepsilon^\lambda \psi(\varepsilon) \left[ 1 + 2\mu_n + 2\mu_1 \right]^\beta.
\]
Returning to (19) and taking into account the properties of function $\phi$ we obtain
\[
d(Tu_n, Tu_0) < q(u_n, u_0) \leq (1 - \varepsilon) \max \{\mu_n, \mu_1(1 + \mu_1)\} + \Lambda \varepsilon^\lambda \psi(\varepsilon) [1 + 2\mu_n + 2\mu_1]^{\beta}.
\]
On the other hand, using the triangle inequality, we have that
\[
\mu_n = d(u_n, u_0) \leq d(u_n, u_{n+1}) + d(u_{n+1}, u_1) + d(u_1, u_0) \leq 2\mu_1 + d(Tu_n, Tu_0).
\]
Combining the two previous relations we get
\[
\mu_n \leq 2\mu_1 + (1 - \varepsilon) \max \{\mu_n, \mu_1(1 + \mu_1)\} + \Lambda \varepsilon^\lambda \psi(\varepsilon) [1 + 2\mu_n + 2\mu_1]^{\beta}.
\]
(20)
We shall prove now, by *reductio ad absurdum* that the sequence \(\{\mu_n\}\) is bounded. Indeed, if we suppose that \(\{\mu_n\}\) is unbounded, we can find a sub-sequence \(\{\mu_{n_l}\}\) such that \(\lim_{l \to \infty} \mu_{n_l} = \infty\). Thus, there is a natural number \(N_0\) such that for every \(l \geq N_0\) we have
\[
\mu_{n_l} \geq \max \{\mu_1(1 + \mu_1), 1 + 2\mu_1\}.
\]
(21)
Therefore,
\[
\mu_{n_l} \leq 2\mu_1 + (1 - \varepsilon)\mu_{n_l} + \Lambda \varepsilon^\lambda \psi(\varepsilon) [1 + 2\mu_{n_l} + 2\mu_1]^{\beta},
\]
that is
\[
\varepsilon \mu_{n_l} \leq 2\mu_1 + \Lambda \varepsilon^\lambda \psi(\varepsilon) [1 + 2\mu_{n_l} + 2\mu_1]^{\beta} \\
\leq 2\mu_1 + \Lambda \varepsilon^\lambda \psi(\varepsilon) 2^{2\beta}\mu_{n_l}^{\beta} \left[1 + \frac{1}{2\mu_{n_l}}\right]^{\beta} \\
\leq 2\mu_1 + \Lambda \varepsilon^\lambda \psi(\varepsilon) 2^{2\beta}\mu_{n_l}^{\beta} [1 + 2\mu_1]^{\beta}.
\]
Choosing \(\varepsilon = \varepsilon_l = \frac{1 + 2\mu_1}{\mu_{n_l}} \in [0, 1]\), for \(l \geq N_0\) and using the assumption \(\beta \in [0, \lambda]\), we obtain
\[
1 + 2\mu_1 \leq 2\mu_1 + \Lambda \frac{1 + 2\mu_1}{\mu_{n_l}}^\lambda \varepsilon_l [1 + 2\mu_1]^{\beta} \\
< 2\mu_1 + \Lambda \frac{1 + 2\mu_1}{\mu_{n_l}}^\lambda \varepsilon_l 2^{2\lambda}\mu_{n_l}^{\lambda} [1 + 2\mu_1]^{\lambda} \\
= 2\mu_1 + \Lambda \varepsilon(\varepsilon_l) 2^{2\lambda}(1 + 2\mu_1)^{2\lambda}.
\]
Inasmuch as \(\psi\) is continuous at 0 and \(\psi(0) = 0\), letting \(l \to \infty\) in the previous inequality we get
\[
1 + 2\mu_1 \leq 2\mu_1,
\]
which is a contradiction. Accordingly, the sequence \(\{\mu_n\}\) is bounded. Therefore, there exists a positive real number \(m\) such that \(\mu_n \leq m\), for any \(n \in \mathbb{N}\).

In this case, going back to (17) we have
\[
d(u_n, u_{n+1}) < (1 - \varepsilon)d(u_n, u_{n+1}) + \Lambda \varepsilon^\lambda \psi(\varepsilon) [1 + 4m]^{\beta},
\]
and as \(n \to \infty\) in the above inequality,
\[
\delta < (1 - \varepsilon)\delta + \Lambda \varepsilon^\lambda \psi(\varepsilon) [1 + 4m]^{\beta},
\]
that is
\[
\delta < \Lambda \varepsilon^\lambda - 1 \psi(\varepsilon) [1 + 4m]^{\beta},
\]
Taking \(\varepsilon \to 0\) and taking into account that \(\lambda \geq 1\), we obtain \(\delta \leq 0\). That is a contradiction. Thus, \(\lim_{n \to \infty} d(u_n, u_{n+1}) = 0\).

On the other hand, undoubtedly, for \(\varepsilon = 0\) we have
\[
d(u_n, u_{n+1}) \leq \phi(d(u_{n-1}, u_n)) \leq \ldots \leq \phi^n(d(u_0, u_1)) = \phi^n(\mu_1),
\]
because $Q(u_{n-1}, u_n) = d(u_{n-1}, u_n)$ when $\varepsilon = 0$. From this point, a verbatim repetition of the lines of Theorem 2.2 gives us that the sequence $u_n$ is Cauchy in a complete metric space, so that there exists $u^*$ such that $\lim_{n \to \infty} u_n = u^*$.

We claim that this point is actually a fixed point for the mapping $T$. For this scope, we will prove firstly than for each $n \geq 1$, at least one of the following presumptions holds.

$$\frac{1}{2}d(u_{n-1}, u_n) \leq d(u_{n-1}, u^*) \text{ or } \frac{1}{2}d(u_n, u_{n+1}) \leq d(u_n, u^*).$$

By using the method of Reductio ad Absurdum, if

$$\frac{1}{2}d(u_{n-1}, u_n) > d(u_{n-1}, u^*) \text{ and } \frac{1}{2}d(u_n, u_{n+1}) > d(u_n, u^*),$$

for some $n \geq 1$, then, taking into account that $\{d(u_n, u_{n+1})\}$ is a non-increasing, the triangle inequality infer

$$d(u_{n-1}, u_n) \leq d(u_{n-1}, u^*) + d(u^*, u_n) \leq \frac{1}{2}[d(u_{n-1}, u_n) + d(u_n, u_{n+1})] \leq d(u_{n-1}, u_n).$$

That is a contradiction. Hence our the presumptions hold.

Now, returning to our claim, we will suppose on the contrary, that $Tu^* \neq u^*$. Taking the presumption (iii) and (4) into account, we have

$$d(Tu^*, u_{n+1}) \leq \alpha(u^*, Tu^*)\alpha(u_n, Tu_n) d(Tu^*, Tu_n) \leq \phi(Q(d(Tu^*, u_n))) < Q(d(Tu^*, u_n)) = (1 - \varepsilon) \max \left\{d(u^*, u_n), \frac{d(u^*, Tu_n)(1 + d(u_n, Tu_n))}{1 + d(u^*, u_n)} \right\} + \Lambda \varepsilon^\lambda \psi(\varepsilon) \left[1 + \|u^*\| + \|u_n\| + \|Tu^*\| + \|Tu_n\|\right]^\beta.$$

Using the boundedness of $\{u_n\}$ and letting $n \to \infty$ we have

$$d(Tu^*, u^*) < (1 - \varepsilon)d(u^*, Tu^*) + \Lambda \varepsilon^\lambda \psi(\varepsilon) \left[1 + 2m + \|Tu^*\| + \|u^*\|\right]^\beta.$$

Taking limit as $\varepsilon \to 0$ and keeping in mind the properties of function $\psi$, we obtain a contradiction. So that, $Tu^* = u^*$. Let's assume now that the fixed point of $T$ is not unique, that is, there exists $v^* \in Fix(T)$ such that $v^* \neq u^*$. Of course, $\frac{1}{2}d(v^*, Tu^*) = 0 \leq d(v^*, u)$, for any $u \in \mathcal{X}$ and due to the presumption (iv), we have

$$d(v^*, u^*) \leq \alpha(v^*, Tu^*)\alpha(u^*, Tu^*) d(Tv^*, Tv^*) \leq \phi(Q(v^*, u^*)) < Q(v^*, u^*),$$

where

$$Q(v^*, u^*) = (1 - \varepsilon) \max \left\{d(v^*, u^*), \frac{d(v^*, Tu^*)(1 + d(v^*, Tv^*))}{1 + d(v^*, v^*)} \right\} + \Lambda \varepsilon^\lambda \psi(\varepsilon) \left[1 + \|v^*\| + \|u^*\| + \|Tv^*\| + \|Tu^*\|\right]^\beta.$$

Thus,

$$d(v^*, u^*) < d(v^*, u^*) + \Lambda \varepsilon^\lambda \psi(\varepsilon) \left[1 + 2\|v^*\| + 2\|u^*\|\right]^\beta.$$
Letting \( \varepsilon \to 0 \) and taking into account that \( \psi \in \Psi \) and \( \lambda \geq 1 \), we obtain \( d'(v^*, u^*) < d'(v^*, u^*) \), which is a contradiction. Therefore, \( v^* = u^* \). \( \square \)

3. Application. In this section, as an application of our main results, we consider the problem of existence and uniqueness of the solution for the boundary value problem for second order differential equation. Let \( J = [0, 1] \), \( h : J \times \mathbb{R} \to \mathbb{R} \) a continuous function and the second order differential boundary value problem

\[
\begin{align*}
  u''(t) &= -h(t, u(t)), \quad t \in J, \\
  u(0) &= u(1) = 0,
\end{align*}
\]

which is equivalent to the integral equation

\[ u(t) = \int_{0}^{1} G(t, s)h(s, u(s))ds. \]

Let \( \mathcal{X} = C(J) \) be the space of all continuous functions defined on \( J \) and consider the metric

\[ d(u, v) = \sup_{t \in J} |u(t) - v(t)|. \]

**Theorem 3.1.** The problem (23) has a unique solution \( u \in \mathcal{X} \) provided that there exist a continuous function \( \xi : [0, 1] \to \mathbb{R} \) and \( \varepsilon \in [0, 1] \), such that \( \sup_{t \in J} |\xi(t)| = \varepsilon^2 \) and

\[
\frac{1}{2} \left| u(t) - \int_{0}^{1} G(t, s)h(s, u(s))ds \right| \leq |u(t) - v(t)|
\]

implies

\[
|h(t, u(t)) - h(t, v(t))| \leq |\xi(t)||u(t) - v(t)|,
\]

for all \( t \in J \) and \( u, v \in \mathcal{X} \).

**Proof.** Letting the mapping \( T : \mathcal{X} \to \mathcal{X} \) be defined by

\[ Tu(t) = \int_{0}^{1} G(t, s)h(s, u(s))ds, \]

the equation (24) can be write as \( Tu = u \). In this case, we get also that

\[
\frac{1}{2} |u(t) - Tu(t)| = \frac{1}{2} \left| u(t) - \int_{0}^{1} G(t, s)h(s, u(s))ds \right| \leq |u(t) - v(t)|
\]

implies

\[
|Tu(t) - Tv(t)| = \left| \int_{0}^{1} G(t, s)(h(s, u(s) - h(s, u(s)))ds \right|
\]

\[
\leq \int_{0}^{1} G(t, s) |h(s, u(s) - h(s, u(s))| ds
\]

\[
\leq \int_{0}^{1} G(t, s) |\xi(s)||u(s) - v(s)| ds.
\]

Consequently, since \( \sup_{t \in J} \int_{0}^{1} G(t, s)ds = \frac{1}{8} \) for each \( t \in J \) we have

\[
\begin{align*}
  d(Tu, Tv) &= \sup_{t \in J} |Tu(t) - Tv(t)| \\
  &\leq \varepsilon^2 |u(t) - v(t)| \int_{0}^{1} G(t, s)ds \leq \frac{1}{8} \varepsilon^2 d(u, v) \\
  &\leq \frac{1}{8} (2\varepsilon^2 d(u, v) - \varepsilon^2 d(u, v) + \varepsilon d(u, v)) \\
  &\leq \frac{1}{8} (1 - \varepsilon) \varepsilon d(u, v) + 2\varepsilon^2 d(u, v)
\end{align*}
\]
\[
\begin{align*}
\leq \frac{1}{8}((1-\varepsilon)\delta(u,v)) + 2\varepsilon^2[\delta(u,u_0) + \delta(u_0,v)] \\
\leq \frac{1}{8}((1-\varepsilon)\delta(u,v)) + 2\varepsilon^2[1 + \|u\| + \|v\|] \\
\leq \frac{1}{8}((1-\varepsilon)\delta(u,v)) + 2\varepsilon^2[1 + \|u\| + \|v\| + \|Tu\| + \|Tv\|] \\
= \frac{Q(u,v)}{8} = \phi(Q(u,v)).
\end{align*}
\]

Therefore, choosing \(\alpha(u,v) = 1\) for every \(u,v \in \mathcal{X}\), \(\phi(t) = \frac{t}{8}\), \(\psi(\varepsilon) = \varepsilon\), \(\Lambda = 2\), \(\lambda = 1\) and \(\beta = 1\) we can easily see that \(T\) is an rational type Pata contraction and all the presumption of Theorem 2.4 are satisfied, which guarantees that the problem (23) has a unique solution. \(\Box\)

4. Conclusion. A new methodology to known as Pata type contraction was suggested in this work to provide the existence and uniqueness of more complex integral equations. The concept was used within the framework of complete metric space. To test the capability of the newly introduced method, we applied it the second order differential boundary value problem. The study presented here will open doors to new investigations. It is our expectation that there will be a wide application potential of Pata type of fixed point in the frame of such contractions on integral equations representing some fractional dynamic equations.

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