Exit problems for oscillating compound Poisson process

Tetyana Kadankova *

Key words: oscillating process, scale function, exit from interval
Running head: Exit problems for oscillating compound Poisson process
2000 Mathematics subject classification: 60G40; 60K20

Abstract

In this article we determine the Laplace transforms of the main boundary functionals of the oscillating compound Poisson process. These are the first passage time of the level, the joint distribution of the first exit time from the interval and the value of the overshoot through the boundary. In case when $E \xi_i(1) = 0$, $\sigma_i^2 = E \xi_i(1)^2 < \infty$, $i = 1, 2$, we prove the limit results for the mentioned functionals.

Introduction

Oscillating random walks with two switching levels were considered in [9],[5],[6]. The authors derived the Laplace-Stieltjes transforms of the distributions of the random walks in transient and stationary regimes. In addition, the asymptotic analysis of the stationary distribution was performed.

This article studies the so-called one- and two-sided exit problems for an oscillating compound Poisson process. More specifically, we determine the Laplace transforms of the following boundary characteristics, these are the first passage time of a boundary and the joint distribution of the first exit time from an interval and the value of the overshoot at this instant. The obtained results are given in closed form, namely in terms of the functions involving the scale functions of the auxiliary processes $\xi_i(t)$, $i = 1, 2$ (see below for a definition). The motivation of this study stems from the fact that these processes are used as governing processes for certain oscillating queueing systems. Examples of such systems are queueing models in which service speed or customer arrival rate change depending on the workload level, and dam models in which the release rate depends on the buffer content (see [11] and references therein). To solve the two-sided exit problem, we used a probabilistic approach borrowed from [2].

The rest of the article is structured as follows. In Section 1 we introduce the process and determine boundary characteristics of the auxiliary processes. Section 2 deals with the one-boundary characteristics of the oscillating process. In Section 3 we determine the joint distribution of the first exit time from the interval and the value of the overshoot. The asymptotic results under the conditions that $E \xi_i(1) = 0$, $\sigma_i^2 = E \xi_i(1)^2 < \infty$, $i = 1, 2$ are given in Section 4.

*Vrije Universiteit Brussel, Department of Mathematics, Building G, Brussels, Belgium, e-mail: tetyana.kadankova@vub.ac.be
1 Preliminaries

In this section we introduce the process of interest and the auxiliary processes. Further, we will determine the Laplace transforms of the first passage time and the first exit time for the auxiliary processes. These results will be used to solve a two-sided problem for the oscillating compound Poisson process.

Let \( \{\xi(t); t \geq 0\} \), \( i = 1, 2 \) be real-valued semi-continuous from below compound Poisson processes:

\[
\xi_i(t) = \sum_{k=0}^{N(t)} \xi_i^k - a_i t, \quad t \geq 0, \quad i = 1, 2
\]

where \( \xi_0^i = 0 \), \( \xi_k^i \sim \xi^i > 0 \) are independent identically distributed variables with distribution function \( F_i(x) \); \( \{N_i(t); t \geq 0\} \), \( N_i(0) = 0 \) is an ordinary Poisson process with parameter \( \lambda_i \) independent from \( \{\xi_k^i; k \geq 0\} \), and \( a_i > 0 \) is a drift coefficient. Their Laplace transforms are then of the following form \( \mathbb{E} e^{-z\xi_i(t)} = e^{tk_i(z)} \), where

\[
k_i(z) = a_i z + \lambda_i \int_0^\infty (e^{-xz} - 1) dF_i(x), \quad \Re(z) = 0.
\]

We now introduce the one-boundary characteristics of the processes. Denote by

\[
\tau_i^-(x) = \inf\{t : \xi_i(t) \leq -x\}, \quad x \geq 0
\]

the first passage time of the lower level \(-x\), and by

\[
\tau_i^+(x) = \inf\{t : \xi_i(t) > x\}, \quad T_i^+(x) = \xi_i(\tau_i^+(x)) - x
\]

the first crossing time of the level \(x\) and the value of the overshoot through this level. We set per definition \( \inf\{\emptyset\} = \infty \). Note, that due to the fact that the process \( \xi_i(t) \) has only positive jumps, the negative level \(-x\) is reached continuously. Hence, the value of the overshoot is equal to zero. For a fixed \( b > 0 \) and all \( x \in \mathbb{R} \), \( t \geq 0 \) introduce the process \( \xi(x,t) \in \mathbb{R} \), \( \xi(x,0) = x \) by means of the following recurrence relations:

\[
\xi(x,t) = \begin{cases} 
 x + \xi_2(t), & 0 \leq t < \tau_2^-(x-b), \\
 \xi(b, t - \tau_2^-(x-b)), & t \geq \tau_2^-(x-b), 
\end{cases} \quad x > b, \quad (1)
\]

\[
\xi(x,t) = \begin{cases} 
 x + \xi_1(t), & 0 \leq t < \tau_1^+(b-x), \\
 \xi(b + T_1^+(b-x), t - \tau_1^+(b-x)), & t \geq \tau_1^+(b-x), 
\end{cases} \quad x \leq b.
\]

Let us explain how the process evolves. Observe, that \( b \) is a switching point of the process \( \xi(x,t), \ t \geq 0 \). If \( \xi(x,t_0) > b \), then the increments of the process coincide with the increments of the process \( \xi_2(t-t_0) \) up to the first passage of \( b \). If \( \xi(x,t_0) \leq b \), then the increments of the process coincide with the increments of the process \( \xi_1(t-t_0) \) up to the first passage of \( b \).

To derive the Laplace transforms of the one-boundary characteristics of the processes \( \xi_i(t) \), we will need notion of a resolvent of a compound Poisson process.
Introduce the resolvents $R_i^x(x)$, $x \geq 0$ [12] of the processes $\xi_i(t)$, $t \geq 0$, by means of their Laplace transforms:

$$\int_0^\infty e^{-xz} R_i^x(x) \, dx = (k_i(z) - s)^{-1}, \quad \Re(z) > c_i(s), \quad R_i^x(x) = 0, \ x < 0,$$

where $c_i(s) > 0$, $s > 0$ is the unique root of the equation $k_i(z) - s = 0$, $i = 1, 2$ in the semi-plane $\Re(z) > 0$. Note, that $R_i^x(0) = a_i^{-1} > 0$.

The resolvent defined in [12] is called a scale function in modern literature (see [8] for more details). The importance of scale functions as a class with which one may express a whole range of fluctuation identities for spectrally one-sided Lévy processes. Scale functions are also an important working tool in risk insurance, more specifically, in optimal barrier strategies. In the rest of the article we will use the term resolvent.

Denote by

$$\mathbf{m}_i^x(s) = \mathbb{E} \left[ e^{-s\tau_i^+(x)} \right], \quad \mathbf{m}_i^{-x}(z, s) = \mathbb{E} \left[ e^{-s\tau_i^+(x) - zT_i^+(x)} \right], \quad \Re(z) \geq 0$$

the Laplace transforms of the first passage time of the negative level $-x$ and the joint distribution of the first crossing time of the level $x$ and the value of the overshoot. one-boundary characteristics of the The lemma below contains the expressions for these Laplace transforms. Observe that these results are valid for Lévy processes whose Laplace exponent is given by (28).

**Lemma 1.** For $s \geq 0$, $i = 1, 2$ the following equalities are valid:

$$\mathbf{m}_i^x(s) = e^{-sc_i(s)}, \quad (2)$$

$$\mathbf{m}_i^{-x}(z, s) = e^{sz} - R_i^x(x) \frac{k_i(z) - s}{z - c_i(s)} - (k_i(z) - s)e^{sz} \int_0^x e^{-uz} R_i^s(u) \, du.$$  

Note that the first equality of (2) is well known (see for instance [15]). Proof of the second relation is given in appendix. We now consider the two-sided exit problem for the auxiliary processes. For $d_i > 0$, $x \in [0, d_i]$ denote by

$$\chi_{i,x}^{d_i} = \inf\{t : x + \xi_i(t) \notin [0, d_i]\}, \quad i = 1, 2$$

the first exit time from the interval $[0, d_i]$ by the process $x + \xi_i(t)$. Introduce the events: $A_i = \{x + \xi_i(x_{i,x}^{d_i}) > d_i\}$ the exit from the interval occurs through the upper boundary; $A_i^c = \{x + \xi_i(x_{i,x}^{d_i}) \leq 0\}$ the exit from the interval occurs through the lower boundary. By $T_i(x) = (x + \xi_i(x_{i,x}^{d_i}) - d_i)\mathbf{I}_{A_i} + 0 \cdot \mathbf{I}_{A_i^c}$ we denote the value of the overshoot at the instant of the first exit. Here $\mathbf{I}_A$ is the indicator of the event $A$. Introduce the Laplace transforms

$$\mathbf{V}_i^{d_i}(x)(s) = \mathbb{E} \left[ e^{-s\chi_{i,x}^{d_i}; A_i} \right], \quad \mathbf{V}_i^{d_i}(z, s) = \mathbb{E} \left[ e^{-s\chi_{i,x}^{d_i} - zT_i(x); A_i^c} \right], \quad \Re(z) \geq 0.$$  

**Lemma 2.** For $s \geq 0$, $i = 1, 2$ the following equalities hold:

$$\mathbf{V}_i^{d_i}(s) = \frac{R_i^x(d_i - x)}{R_i^x(d_i)},$$

$$\mathbf{V}_i^{d_i}(z, s) = \mathbf{m}_i^{d_i - x}(z, s) - \frac{R_i^x(d_i - x)}{R_i^x(d_i)} \mathbf{m}_i^{d_i}(z, s). \quad (3)$$
Note, that the first relation of the lemma was derived in [7] for a compound Poisson process, and in [13] for a spectrally one-sided Lévy process (28). To verify the second relation, we make use of the following equation:

\[
E \left[ e^{-s\tau_i^+(d_i-x)}; T_i^+(d_i - x) \in du \right] = E \left[ e^{-s\chi_i^1}; T_i(x) \in du, A_i \right] + \\
E \left[ e^{-s\chi_i^0}; A_i \right] E \left[ e^{-s\tau_i^+(d_i)}; T_i^+(d_i) \in du \right], \quad x \in [0, d_i].
\]

The latter was derived for spectrally one-sided Lévy processes (28) in [3], [4], and for general Lévy processes in [2]. Now plugging in the expression for \( E \left[ e^{-s\chi_i^1}; A_i \right] \) (the first equality of the lemma), we obtain the second statement of the lemma.

## 2 One-boundary characteristics of the process \( \xi(x, t) \).

In this section we derive the Laplace transforms of the one-boundary characteristics of the process and study their asymptotic behavior. Let us formally define the one-boundary functionals of the process \( \xi(x, t), \ t \geq 0 \). For \( r \leq \min\{x, b\} \) denote by

\[
\tau_x^r(b) = \inf\{t : \xi(x, t) \leq r\}, \quad \bar{\tau}_x^r(s) = E \left[ e^{-s\tau_x^r(b)}; \tau_x^r(b) < \infty \right],
\]

the first passage time of the level \( r \) by the process \( \xi(x, t) \) and its Laplace transform. For \( k \geq \max\{x, b\} \) denote by

\[
\tau_x^k(b) = \inf\{t : \xi(x, t) > k\}, \quad \bar{\tau}_x^k(s) = \xi(x, \tau_x^k(b)) - k
\]

the first crossing time of the level \( k \) and the value of the overshoot by the process \( \xi(x, t) \). The variables \( \tau_x^r(b), \tau_x^k(b), \bar{\tau}_x^k(s) \) are called the one-boundary characteristics of the process. Introduce

\[
\bar{\tau}_x^k(s) = E \left[ e^{-s\tau_x^k(b)}; \tau_x^k(b) < \infty \right], \quad \bar{\tau}_x^k(z, s) = E \left[ e^{-s\tau_x^k(b)-z\bar{\tau}_x^k(b)}; \tau_x^k(b) < \infty \right].
\]

For \( s \geq 0 \) define the function \( K_x^s(u), \ x \in \mathbb{R}, \ u \geq 0 \), by means of its Laplace transform \( \mathbb{K}_x^s(z) \):

\[
\mathbb{K}_x^s(z) = \int_0^\infty e^{-uz}K_x^s(u) \, du = \frac{k_1(z) - s}{k_2(z) - s} \int_0^\infty e^{-uz}R_1^s(x + u) \, du,
\]

where \( \Re(z) > \max\{c_1(s), c_2(s)\} \). Note, that it follows from the definition (4) that for \( x \leq 0 \) \( \mathbb{K}_x^s(z) = e^{xz}(k_2(z) - s)^{-1} \) and \( K_x^s(u) = R_2^s(x + u) \).

For a fixed \( s \geq 0 \) define the function \( F_x(u), \ u \geq 0 \) by means of its Laplace transform \( \mathbb{F}_x(z) \):

\[
\mathbb{F}_x(z) = \int_0^\infty e^{-uz}F_x(u) \, du = \frac{1}{z - c_1(s)} \frac{k_1(z) - s}{k_2(z) - s}, \quad \Re(z) > c_2(s). \quad (5)
\]
Theorem 1. The Laplace transforms of $\mathcal{L}(f)$, $\mathcal{L}(k)$ and of the joint distribution of $\{\mathcal{L}(f), \mathcal{L}(k)\}$ are such that for $s \geq 0$:

$$\mathcal{L}(f)(s) = \frac{C_1 b^{-x}}{C_1 - r(s)}(c_2(s), s), \quad r \leq \min \{x, b\}, \quad (6)$$

$$\mathcal{L}(k)(s) = 1 + s \int_{0}^{b-x} R^k(u) du + s \int_{0}^{d_2} K^s_{b-x}(u) du - \frac{K^s_{b-x}(d_2)}{F_s(d_2)} \left( \frac{s}{c_1(s)} + s \int_{0}^{d_2} F_s(u) du \right), \quad k \geq \max \{x, b\}, \quad (7)$$

$$\mathcal{L}(x, s) = e^{z_{d_2}(k_2(z) - s)} \left( [x_{z_{d_2}} - \int_{0}^{d_2} e^{-u z} K^s_{b-x}(u) du] - \frac{K^s_{d_2}(d_2)}{F_s(d_2)} e^{z_{d_2}(k_2(z) - s)} \left( [x_{z_{d_2}} - \int_{0}^{d_2} e^{-u z} F_s(u) du] \right), \quad k \geq \max \{x, b\}, \quad (8)$$

where $d_2 = k - b$, $C^s_i(x, s) = e^{z_{x}}$, $x < 0$, $C^s_i(x, s) = e^{z_{x}} \left( 1 - (k_i(z) - s) \int_{0}^{x} e^{-u z} R^s_i(u) du \right), \quad x \geq 0$.

Corollary 1. Let $k_1(z) = k_2(z) = k(z)$. Then

$$\mathcal{L}(f)(s) = e^{-z_{d_2}(c_{i})(s)}, \quad r \leq x,$$

$$\mathcal{L}(k)(s) = 1 + s \int_{0}^{b-x} R^k(u) du - \frac{s}{c(s)} R^k(k - x), \quad k \geq x, \quad (9)$$

$$\mathcal{L}(x, s) = e^{(k-x)z} \left( 1 - (k(z) - s) \int_{0}^{k-x} e^{-u z} R^s(u) du - R^s(k - x) \frac{k(z) - s}{z - c(s)} \right),$$

where $R^s(x)$, $x \geq 0$ is the resolvent of the process $\xi(t) = \xi_i(t)$; $c(s) > 0$, $s > 0$ is the unique root of the equation $k(z) = s$ in the semi-plane $\Re(z) > 0$.

Corollary 2. Assume that the conditions $(A): E\xi_i(1) = 0$, $\sigma^2_i = E\xi_i(1)^2 < \infty$ are satisfied. Then the following limiting equalities are valid:

$$\lim_{B \to \infty} E \left[ e^{-s (x_{B})_{i}}/B^2 \right] = \sigma_1 e^{-b_{-}x_{s_1}}/\sigma_{1} c_{1}(k - b s_2) + \frac{\sigma_{2} e^{-b_{-}x_{s_2}}}{\sigma_{2} c_{2}(k - b s_2)} s_{i}, x \leq b,$$

$$\lim_{B \to \infty} E \left[ e^{-s (x_{B})_{i}}/B^2 \right] = \sigma_1 \sigma_{1} c_{1}(x - b s_2) + \frac{\sigma_{2} e^{-b_{-}x_{s_2}}}{\sigma_{2} c_{2}(k - b s_2)} s_{i}, x \in [b, k];$$

$$\lim_{B \to \infty} E \left[ e^{-s (x_{B})_{i}}/B^2 \right] = \sigma_1 \sigma_{1} c_{1}(b - r s_1) + \frac{\sigma_{2} e^{-b_{-}x_{s_2}}}{\sigma_{2} c_{2}(b - r s_2)} s_{i}, x \geq b,$$

$$\lim_{B \to \infty} E \left[ e^{-s (x_{B})_{i}}/B^2 \right] = \sigma_1 \sigma_{1} c_{1}(b - x s_1) + \frac{\sigma_{2} e^{-b_{-}x_{s_2}}}{\sigma_{2} c_{2}(b - x s_2)} s_{i}, x \in [r, b],$$

where $s_i = \sqrt{2s}/\sigma_i$, $i = 1, 2$, $k \geq \max \{x, b\}$, $r \leq \min \{x, b\}$. 

5
Proof. Let us verify (3). Set \( x = b \). In view of the definition of the process \( \xi(x, t) \) (1), spatial homogeneity of the processes \( \xi_i(t) \) and Markov property of \( \chi^d_{1,x} \) we can write the following equation:

\[
\int_0^{b}(s) = \int_1^{d}1 \times (s) + \int_0^{\infty} \bar{V}_{1,1,1}(du, s)e^{-uc_2(s)} \int_0^{b}(s), \quad d_1 = b - r,
\]

where \( V_{2,1}(du, s) = \mathbb{E} \left[ e^{-\xi(x, t)}; T_1(x) \in du, A_1 \right] \), \( x \in [0, d_1] \). It follows from (2), (3) that

\[
\bar{V}_{1,x}(z, s) = C_{1}^{d_1 - x} (z, s) - \frac{R^*_1(d_1) - x}{R^*_1(d_1)} C_{1}^{d_1}(z, s).
\]

Taking into account the latter equality and (11), we derive \( \int_x^{b}(s) \). Let \( x > b \). Then a

\[
\int_x^{b}(s) = e^{-(x-b)c_2(s)} \int_0^{b}(s) = \frac{e^{-(x-b)c_2(s)}}{C_{1}^{d_1} (c_2(s), s)} = \frac{C_{1}^{d_1} (c_2(s), s)}{C_{1}^{b-x} (c_2(s), s)}.
\]

If \( x \in [r, b] \), then we have from \( \int_x^{b}(s) = \int_0^{b}(s) \int_x^{b}(s) \) that

\[
\int_x^{b}(s) = \frac{\int_x^{b}(s)}{\int_x^{b}(s)} = \frac{C_{1}^{d_1} (c_2(s), s)}{C_{1}^{b-x} (c_2(s), s)}, \quad x \in [r, b].
\]

Hence, we showed that (3) is valid for all \( x \geq r \).

We now verify (4). Set first \( x = b \). Then taking into account the defining formula (11) of the process \( \xi(x, t) \), spatial homogeneity of the processes \( \xi_i(t) \) and Markov property of \( \tau^+_1(x) \), we can write

\[
\bar{F}_{b}(z, s) = e^{zd_2} \int_0^{d_2} m_1^0(du, s)e^{-uz} + \int_0^{d_2} m_1^0(du, s) \bar{V}_{2,1}(y, z, s)
\]

\[
+ \int_0^{d_2} m_1^0(du, s) \frac{R^*_2(d_2 - u)}{R^*_2(d_2)} \bar{F}_{b}(z, s), \quad d_2 = k - b,
\]

where \( m_1^0(du, s) = \mathbb{E} \left[ e^{-s\tau^+_1(x)}; T_1^+(x) \in du \right] \). By means of this equation we can determine the function \( \bar{F}_{b}(z, s) \). Making use of the expression for the function \( F_1(u) \), \( u \geq 0 \) (5), equalities (2), (3), after performing some calculations, we find

\[
\bar{F}_{b}(z, s) = C_{2}^{d_2} (z, s) - \frac{R^*_2(d_2)}{F_1(d_2)} e^{zd_2}(k_2(z) - s) \left( \bar{F}_1(z) - \int_0^{d_2} e^{-uz} F_1(u)du \right).
\]

Let \( x \in (b, k] \). Then the function \( \bar{F}_x(z, s) \) can be found from the following equation:

\[
\bar{F}_x(z, s) = \bar{V}_{2,x-b}(z, s) + \frac{R^*_2(k-x)}{R^*_2(k)} \bar{F}_b(z, s), \quad x \in (b, k].
\]
In view of (12) we derive
\[ \mathcal{F}^k_x(z, s) = C^k_x(z, s) - \frac{R^k_0(k - x)}{F_s(d_2)} \mathcal{F}^s_{d_2}(z), \quad x \in [b, k], \] (13)
where \( \mathcal{F}^s_{d_2}(z) = e^{zd_2}(k_2(z) - s) \left( \mathbb{F}_s(z) - \int_0^{d_2} e^{-uz} F_s(u) du \right). \) Let \( x < b. \) Then we can determine the function \( \mathcal{F}^k_x(z, s) \) from the following relation:
\[ \mathcal{F}^k_x(z, s) = e^{zd_2} \int_{d_2}^{\infty} m^{b-x}(du, s)e^{-uz} + \int_0^{d_2} m^{b-x}(du, s) \mathcal{F}^k_{u+b}(z, s). \]
Employing (2), (3), the definition of the function \( K^s_x(u) \) (1) and the formula (13), we obtain
\[ \mathcal{F}^k_x(z, s) = \mathcal{K}^{d_2}_{b-x}(z, s) - \frac{K^s_{b-x}(d_2)}{F_s(d_2)} \mathcal{F}^s_{d_2}(z), \quad x < b, \] (14)
where
\[ \mathcal{K}^{d_2}_{b-x}(z, s) = e^{zd_2}(k_2(z) - s) \left( \mathbb{K}^s_{b-x}(z) - \int_0^{d_2} e^{-uz} K^s_{b-x}(u) du \right). \]
Note that for \( x \in [b, k] \) it follows from the definition of the function \( K^s_x(u) \) (1) that
\[ \mathcal{K}^{d_2}_{b-x}(z, s) = C^k_x(z, s), \quad K^s_{b-x}(d_2) = R^s_0(k - x). \]
Hence, the formula (14) is valid for all \( x \leq k. \) Since \( \mathcal{F}^k_x(s) = \mathcal{F}^k_x(0, s) \), then (14) follows from (8) when \( z = 0. \) We now verify statements of Corollary (11). In case when \( k_1(z) = k_2(z) = k(z) \) we have
\[ C^k_x(c_2(s), s) = e^{xc(s)}, \quad \mathbb{F}_s(z) = (z - c(s))^{-1}, \quad F_s(u) = e^{uc(s)}, \]
\[ \mathbb{K}^s_x(z) = e^{xz} \int_x^{\infty} e^{-uz} R^s(u) du, \quad K^s_x(u) = R^s(x + u). \]
These equalities and (6) imply the formulae (14). The limiting equalities are derived in Section 4. \( \square \)

3 Exit from the interval by the process \( \xi(x, t). \)

For \( B > 0, \ x, b \in [0, B], \) introduce the following random variable:
\[ \chi_x(b) = \inf\{t : \xi(x, t) \notin [0, B]\}, \quad i = 1, 2 \]
i.e. the first exit time from the interval \([0, B]\) by the process \( \xi(x, t). \) Introduce the events: \( \mathcal{A} = \{\xi(x, \chi_x(b)) > B\} \) the process exits the interval through the upper boundary; \( \mathcal{A} = \{\xi(x, \chi_x(b)) \leq 0\} \) the process exits the interval through the lower boundary. Denote by \( T(x) = (\xi(x, \chi_x(b)) - B)\mathbb{I}_\mathcal{A} + 0 \cdot \mathbb{I}_{\bar{\mathcal{A}}} \) the value of the overshoot at the instant of the first exit. Define
\[ \mathcal{V}_x(s) = \mathbb{E}\left[e^{-s\chi_x(b)}, \mathcal{A}\right], \quad \mathcal{V}_x(z, s) = \mathbb{E}\left[e^{-s\chi_x(b) - zT(x), \bar{\mathcal{A}}}\right], \quad \Re(z) \geq 0. \]
Then the following expansions hold as
\[ \text{Corollary 4.} \]
are valid:
\[ \text{Theorem 2.} \]

where \( \mathbf{V}_x(s) = \mathbf{E} \left[ e^{-sx_x(b)} \mathbf{A} \right] \),
\[ \mathbf{R}_x^B(s) = 1 + s \int_0^x R_1^s(u) du + s \int_0^u K_2^s(v) dv, \quad x \in \mathbb{R}, \ u \geq 0; \]

and
\[ \mathbf{R}_x^B(z, s) = e^{uz} \left( C_1^x(z, s) - (k_2(z) - s) \int_0^u e^{-uz} K_2^s(v) dv \right), \quad x \in \mathbb{R}, \ u \geq 0. \]

**Corollary 3.** Assume that \( k_1(z) = k_2(z) = k(z) \). Then the following equalities are valid:
\[ \mathbf{V}_x(s) = \frac{R^s(B - x)}{R^s(B)}, \quad \mathbf{V}_x(s) = C_{B-x}^s(s) - \frac{R^s(B - x)}{R^s(B)} C_s^B(s), \]
\[ \mathbf{V}_x(z, s) = C_{B-x}^s(z, s) - \frac{R^s(B - x)}{R^s(B)} C_s^B(z, s), \]

where \( C_s^x = 1 + s \int_0^x R^s(u) du, \)
\[ C_s^x(z, s) = e^{xz} \left( 1 - (k(z) - s) \int_0^x e^{-uz} R^s(u) du \right), \]

\( R^s(x), \ x \geq 0 \) are the resolvents of the processes \( \xi(t) = \xi_i(t); \ c(s) > 0, \ s > 0 \) is the unique root of the equation \( k(z) = s \) in the semi-plane \( \mathbb{R}(z) > 0 \).

**Corollary 4.** Assume that \( \mathbf{E}\xi_i(1) = 0, \ \sigma_i^2 = \mathbf{E}\xi_i(1)^2 < \infty, \ x, b \in (0, 1) \).

Then the following expansions hold as \( B \to \infty \)
\[ \mathbf{E} \left[ e^{-sx_x(b)B^2/B^2} \mathbf{A} \right] \]
\[ \mathbf{E} \left[ e^{-sx_x(b)B^2/B^2} \mathbf{A} \right] \]
\[ \mathbf{E} \left[ e^{-sx_x(b)B^2/B^2} \mathbf{A} \right] \]
\[ \mathbf{E} \left[ e^{-sx_x(b)B^2/B^2} \mathbf{A} \right] \]

where \( s_i = \sqrt{2s}/\sigma_i, \ i = 1, 2, \ b = 1 - b. \)

\[ \sigma_{1\text{sh}}((b - x)s_1) \text{ch}(bs_2) + \sigma_{2\text{sh}}(bs_2) \text{ch}((b - x)s_1), \]
\[ \sigma_{1\text{sh}}(bs_1) \text{ch}(bs_2) + \sigma_{2\text{sh}}(bs_2) \text{ch}(bs_1), \]
\[ \sigma_{1\text{sh}}(bs_1) \text{ch}(bs_2) + \sigma_{2\text{sh}}(bs_2) \text{ch}(bs_1), \]
\[ \sigma_{2\text{sh}}(1 - x)s_2 \]
\[ \sigma_{1\text{sh}}(bs_1) \text{ch}(bs_2) + \sigma_{2\text{sh}}(bs_2) \text{ch}(bs_1), \]
\[ \sigma_{2\text{sh}}(1 - x)s_2 \]
\[ \sigma_{1\text{sh}}(bs_1) \text{ch}(bs_2) + \sigma_{2\text{sh}}(bs_2) \text{ch}(bs_1), \]
\[ \sigma_{2\text{sh}}(1 - x)s_2 \]
\[ \sigma_{1\text{sh}}(bs_1) \text{ch}(bs_2) + \sigma_{2\text{sh}}(bs_2) \text{ch}(bs_1), \]
\[ \sigma_{2\text{sh}}(1 - x)s_2 \]
Formula (8) implies that integral equations with respect to the Laplace transforms
\[
\mathbb{E} \left[ e^{-s\sum_{i}^{(b)}(b)} \right] = \mathbb{V}_{x}(s), \quad V_{x}(du, s) = \mathbb{E} \left[ e^{-s\sum_{i}^{(b)}(b)} ; T(x) \in du, \mathcal{A} \right],
\]
where
\[
\tau = \sum_{i}^{(b)}(b),
\]
\[
\mathbb{E} \left[ e^{-s\tau} \right] = \mathbb{V}_{x}(s) + \int_{0}^{\infty} V_{x}(du, s) \mathbb{E} \left[ e^{-s\tau^{(b)}(b)} \right], \quad x, b \in [0, B],
\]
\[
\mathbb{E} \left[ e^{-s\tau^{B}(b)} ; T^{B} \in du \right] = V_{x}(du, s) + V_{x}(s) \mathbb{E} \left[ e^{-s\tau^{B}(b)} ; T^{B} \in du \right].
\] (19)

The first equation of this system means that the process \( \xi(x, t) \) can reach the lower boundary 0 either on the sample paths which do not cross the upper boundary \( B \), or on the sample paths which do cross the upper boundary and then pass the lower boundary. The second equation is written analogously. Observe, that the mathematical expectations which enter the equations of the system are determined by (3)-(8). Taking into account the formulae (6), (19), we derive
\[
\frac{C_{1}^{h-x}(c_{2}(s), s)}{C_{1}^{h}(c_{2}(s), s)} = \mathbb{V}_{x}(s) + \frac{e^{-c_{2}(s)(B-b)}}{C_{1}^{h}(c_{2}(s), s)} \mathbb{V}_{x}(c_{2}(s), s),
\]
\[
\mathbb{V}_{x}(c_{2}(s), s) = \mathbb{V}_{x}(c_{2}(s), s) + \mathbb{V}_{x}(s) \mathcal{F}^{B}(c_{2}(s), s).
\] (20)

Formula (8) implies that
\[
\mathbb{F}^{B}(c_{2}(s), s) = C_{1}^{h-x}(c_{2}(s), s)e^{c_{2}(s)(B-b)} - \frac{K_{b-x}^{s}(B-b)}{F_{b}(B-b)} \mathbb{F}(s), \quad x \in [0, B],
\]
where \( \mathbb{F}(s) = \left( k_{1}(c_{2}(s) - s)(c_{2}(s) - c_{1}(s))^{-1}e^{c_{2}(s)(B-b)} \right) \). Solving system (20) with respect to two unknown functions \( \mathbb{V}_{x}(s), \mathbb{V}_{x}(c_{2}(s), s) \), we find for all \( x \in [0, B] \) that
\[
\mathbb{V}_{x}(s) = \frac{K_{b-x}^{s}(d_{2})}{K_{h}^{d}(d_{2})},
\]
\[
\mathbb{V}_{x}(c_{2}(s), s) = e^{c_{2}(s)d_{2}} \left( C_{1}^{h-x}(c_{2}(s), s) - \frac{K_{b-x}^{s}(d_{2})}{K_{h}^{d}(d_{2})} C_{1}^{h}(c_{2}(s), s) \right),
\]
where \( d_{2} = B - b \). It follows from the second equation from the system (19) and from (8) that
\[
\mathbb{V}_{x}(z, s) = \mathbb{V}_{x}(z, s) - \frac{K_{b-x}^{s}(d_{2})}{K_{h}^{d}(d_{2})} \mathbb{F}_{0}(z, s) = \mathcal{R}_{0}^{d_{2}}(z, s) - \frac{K_{b-x}^{s}(d_{2})}{K_{h}^{d}(d_{2})} \mathcal{R}_{0}^{d_{2}}(z, s),
\]
where
\[
\mathcal{R}_{x}^{d_{2}}(z, s) = e^{zd_{2}} \left( C_{1}^{x}(z, s) - (k_{2}(z) - s) \int_{0}^{d_{2}} e^{-u(z)K_{x}(u) du} \right), \quad x \in \mathbb{R}.
\]
The second equality (15) can be derived from (16) for \( z = 0 \). If \( k_{1}(z) = k_{2}(z) = k(z) \), then
\[
\mathcal{R}_{x}^{d_{2}}(z, s) = C_{x}^{d_{2}}(z, s), \quad \mathcal{R}_{x}^{d_{2}}(s) = 1 + s \int_{0}^{d_{2}} R^{s}(u) du, \quad x \in \mathbb{R}.
\]
The formulae (15), (16) of Theorem 2 imply the statements of Corollary 3. \( \square \)
4 Asymptotic behavior

In this section we assume that the following conditions are fulfilled (A): $E\xi_i(1) = 0$, $\sigma_i^2 = E\xi_i(1)^2 < \infty$, $i = 1, 2$. It is a well-known fact (see for instance [1], [7], [14]) that

$$\lim_{B \to \infty} \frac{1}{B} R_i^{s/B^2}(xB) = \frac{2}{\sigma_i \sqrt{2s}} \text{sh}(x^+ s_i), \quad \lim_{B \to \infty} Bc_i(s/B^2) = s_i,$$

where $s_i = \sqrt{2s}/\sigma_i$, $i = 1, 2$, $x^+ = \max\{0, x\}$. We now verify the limiting relations for the functions which appear in Theorems 1, 2. Observe, that under the condition (A) the following expansion is valid as $B \to \infty$, $z > 0$

$$k_i(z/B) = \frac{1}{\sigma_i^2 z^2 / B^2} + o(B^{-2}).$$

Then in view of the definition of the function $K^s(z)$ (4) we can write

$$\tilde{k}_x^s(z) = \lim_{B \to \infty} \frac{1}{B^2} K_1^{s/B}(z/B) = e^{xz}, \quad x \leq 0,$$

$$\tilde{k}_x^s(z) = \frac{1}{\frac{1}{2} \sigma_1^2 z^2 - s} \left( \frac{z \sigma_1}{\sqrt{2s}} \text{sh}(s_1 x) + \text{ch}(s_1 x) \right), \quad x \geq 0. \quad (22)$$

For $\Re(z) > \sqrt{2s}/\sigma_2$ the right-hand sides of these equalities are the Laplace transforms:

$$\tilde{k}_x^s(z) = \int_0^{\infty} e^{-zu} k_x^s(u) \, du, \quad \Re(z) > \sqrt{2s}/\sigma_2.$$

The formulae (22) imply the following relation

$$\lim_{B \to \infty} \frac{1}{B} K_x^{s/B^2}(uB) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{zu} \tilde{k}_x^s(z) \, dz =$$

$$= \begin{cases} \frac{2}{\sigma_2 \sqrt{2s}} \text{sh}(x+u)^+ s_2, \quad x \leq 0, \\ \frac{2}{\sigma_2 \sqrt{2s}} (\sigma_1 \text{sh}(x s_1) \text{ch}(u s_2) + \sigma_2 \text{sh}(u s_2) \text{ch}(x s_1)), \quad x \geq 0, \end{cases} \quad (23)$$

where $\gamma > \sqrt{2s}/\sigma_2$. Taking into account the latter equality, we can easily obtain the limiting relations for the functions, which enter the statements of Theorems 1, 2

$$\lim_{B \to \infty} R_1^{s/B^2}(s/B^2) = \begin{cases} \text{ch}(x+u)^+ s_2, \quad x \leq 0, \\ \frac{\sigma_1}{\sigma_2} \text{sh}(x s_1) \text{sh}(u s_2) + \text{ch}(u s_2) \text{ch}(x s_1), \quad x \geq 0, \end{cases} \quad (24)$$

$$\lim_{B \to \infty} F_1^{s/B^2}(uB) = \frac{\sigma_1}{\sigma_2} (\sigma_1 \text{ch}(u s_2) + \sigma_2 \text{sh}(u s_2)), \quad u \geq 0, \quad (25)$$
\[
\lim_{B \to \infty} C_1^{x B} (c_2(s/B^2), s/B^2) = \begin{cases} 
  e^{xs_2}, & x \leq 0, \\
  \frac{\sigma_1}{\sigma_2} \text{sh}(xs_1) + \text{ch}(xs_1), & x \geq 0.
\end{cases}
\]

We now verify the limiting equalities of Corollary 2. Let \( k \geq \max\{x, b\}, \ x \leq b. \) Then taking into account (7), (21), (23) and (25), we have as \( B \to \infty \)

\[
\mathbb{E} \left[ e^{-s\chi_B^B(bB)/B^2} \right] = \int_{xB}^B (s/B^2) \to \frac{\sigma_1}{\sigma_2} \text{sh}(b - x)s_1 \text{ch}(d_2 s_2) + \text{ch}(x - b)s_1 \text{ch}(d_2 s_2) \\
- \frac{\sigma_1 \text{sh}(d_2 s_2) + \sigma_2 \text{ch}(d_2 s_2)}{\sigma_1 \text{ch}(d_2 s_2) + \sigma_2 \text{sh}(d_2 s_2)} \left( \frac{\sigma_1}{\sigma_2} \text{sh}((b - x)s_1) \text{ch}(d_2 s_2) + \text{ch}((b - x)s_1) \text{sh}(d_2 s_2) \right) = \\
= \frac{\sigma_1 e^{-(b-x)s_1}}{\sigma_1 \text{ch}((k-b)s_2) + \sigma_2 \text{sh}((k-b)s_2)}, \quad x \leq b,
\]

where \( d_2 = k - r. \) Similarly, we can derive the second formula of Corollary 2

\[
\lim_{B \to \infty} \mathbb{E} \left[ e^{-s\chi_B^B(bB)/B^2} \right] = \frac{\sigma_1 \text{ch}((x-b)s_2) + \sigma_2 \text{sh}((x-b)s_2)}{\sigma_1 \text{ch}((k-b)s_2) + \sigma_2 \text{sh}((k-b)s_2)}, \quad x \in [b, k].
\]

Let \( r \leq \min\{x, b\}. \) Then the following relation follows from (6) and (26) as \( B \to \infty \)

\[
\mathbb{E} \left[ e^{-s\chi_B^B(bB)/B^2} \right] = \int_{xB}^B (s/B^2) \to \frac{C_1^{(b-x)B} (c_2(s/B^2), s/B^2)}{C_1^{(b-r)B} (c_2(s/B^2), s/B^2)} \\
\to \frac{\sigma_2 e^{-(b-x)s_2}}{\sigma_1 \text{sh}((b-r)s_1) + \sigma_2 \text{ch}((b-r)s_1)}, \quad x \geq b,
\]

\[
\mathbb{E} \left[ e^{-s\chi_B^B(bB)/B^2} \right] \to \frac{\sigma_1 \text{sh}((b-x)s_1) + \sigma_2 \text{ch}((b-x)s_1)}{\sigma_1 \text{sh}((b-r)s_1) + \sigma_2 \text{ch}((b-r)s_1)}, \quad x \in [r, b].
\]

We now derive \( \mathcal{L}. \) The following relation follows from the first formula of (15) and from (23) for \( x \in (0, b], \) as \( B \to \infty \)

\[
\mathbb{E} \left[ e^{-s\chi_B^B(bB)/B^2} : \mathcal{L} \right] = \mathbb{V}_B^B (s/B^2) = \frac{K_{(b-x)B}^B (\mathcal{B} B)}{K_{(b-x)B}^{s/B^2} (\mathcal{B} B)} \\
\to \frac{\sigma_1 \text{sh}((b-x)s_1) \text{ch}(b s_2) + \sigma_2 \text{sh}(b s_2) \text{ch}((b-x)s_1)}{\sigma_1 \text{sh}(b s_1) \text{ch}(b s_2) + \sigma_2 \text{sh}(b s_2) \text{ch}(b s_1)}, \quad x \in (0, b].
\]

where \( \mathcal{B} = 1 - b. \) Taking into account the second formula of (15) and (23), (24) we can write for \( x \in (0, b], \) as \( B \to \infty \)

\[
\mathbb{E} \left[ e^{-s\chi_B^B(bB)/B^2} : \mathcal{L} \right] = \mathbb{V}_B^B (s/B^2) = \frac{K_{(b-x)B}^B (\mathcal{B} B)}{K_{(b-x)B}^{s/B^2} (\mathcal{B} B)} - \\
- \frac{K_{(b-x)B}^{s/B^2} (\mathcal{B} B) \mathcal{R}_{(b-x)B}^{s/B^2} (s/B^2)}{K_{(b-x)B}^{s/B^2} (\mathcal{B} B)} \to \frac{\sigma_1 \text{sh}(xs_1)}{\sigma_1 \text{sh}(b s_1) \text{ch}(b s_2) + \sigma_2 \text{sh}(b s_2) \text{ch}(b s_1)}, \quad x \in (0, b].
\]

Analogously, the formulae of the corollary can be verified for \( x \in [b, 1). \)
5 Appendix

Let $\xi(t) \in \mathbb{R}$, $\xi(0) = 0$, $\mathbb{E}e^{-p\xi(t)} = e^{tk(p)}$, $\Re(p) = 0$ be a general Lévy process. Denote by

$$\xi_t^+ = \sup_{u \leq t} \xi(u), \quad \xi_t^- = \inf_{u \leq t} \xi(u)$$

the running supremum and infimum of the process. For $x \geq 0$ define

$$\tau_x^+ = \inf\{t > 0 : \xi(t) \geq x\}, \quad T_x^+ = \xi(\tau_x^+) - x$$

the first crossing time of a barrier $x$ and the value of the overshoot. Then the following relation is valid ([10]):

$$\int_0^\infty e^{-pz} \mathbb{E}\left[e^{-s\tau_x^+-zT_x^+}\right] dx = \frac{1}{p-z} \left(1 - \frac{\mathbb{E}e^{-p\xi_{\nu_s}^+}}{\mathbb{E}e^{-z\xi_{\nu_s}^+}}\right), \quad \Re(p), \Re(z) \geq 0, \quad (27)$$

where $\nu_s \sim \exp(s)$, $s > 0$ is an exponential random variable independent from the process $\xi(t)$. For a spectrally positive Lévy process with Laplace exponent

$$k(z) = az + \frac{\sigma^2z^2}{2} + \int_0^\infty \left(e^{-zx} - 1 + zI_{0<x\leq1}\right) \Pi(dx), \quad i = 1, 2 \quad (28)$$

we have

$$\mathbb{E}e^{-z\xi_{\nu_s}^+} = \frac{s}{c(s)} \frac{p-c(s)}{k(p)-s}, \quad \Re(p) \geq 0,$$

where $c(s) > 0$, $s > 0$ is the unique root of the equation $k(z) - s = 0$, in the semi-plane $\Re(z) > 0$. It follows from the latter relation and from (27) that

$$\int_0^\infty e^{-pz} \mathbb{E}\left[e^{-s\tau_x^+-zT_x^+}\right] dx = \frac{1}{p-z} \left(1 - \frac{p-c(s)}{k(p)-s} \frac{k(z)-s}{z-c(s)}\right). \quad (29)$$

Introduce the resolvents $R^s(x)$, $x \geq 0$ [12] of the spectrally one-sided Lévy process $\xi(t)$, $t \geq 0$, by means of their Laplace transforms:

$$\int_0^\infty e^{-xz} R^s(x) dx = (k(z) - s)^{-1}, \quad \Re(z) > c(s), \quad R^s(x) = 0, \quad x < 0,$$

Making use of the definition of the resolvent and inverting the Laplace transform with respect to $p$ ($\Re(p) > c(s)$) in both sides of (29), we find

$$\mathbb{E}\left[e^{-s\tau_x^+-zT_x^+}\right] = e^{xz} - R^s(x) \frac{k(z)-s}{z-c(s)} - (k(z)-s)e^{xz} \int_0^x e^{-uz} R^s(u) du,$$

which is the second equality of Lemma 1.

References

[1] Borovskikh, Yu. V. and Korolyuk, V. S. Analytic problems of asymptotics of probability distributions Naukova Dumka, Kiev, 1981
)[2] Kadankov, V. F., Kadankova, T. On the distribution of the first exit time from an interval and the value of overshoot through the boundaries for processes with independent increments and random walks. Ukr. Math. J., 57(10), 1359–1384 (2005)

[3] V.F. Kadankov, T.V. Kadankova. On the distribution of duration of stay in an interval of the semi-continuous process with independent increments. Random Oper. and Stoch. Equ. (ROSE), 12(4), 365-388 (2004)

[4] Kadankova, T. On the distribution of the number of the intersections of a fixed interval by the semi-continuous process with independent increments. Theory of Stochastic Processes. 1-2, 73–81 (2003)

[5] Kim, D. K.; Lotov, V. I. (2003). Oscillating random walks with two levels of switching. Siberian Adv. Math. 14(1) (2004), 7–46.

[6] Kim, D. K.; Lotov, V. I. (2004). Asymptotics of the stationary distribution of an oscillating random walk. Siberian Math. J. 45(5) (2004), 915–930

[7] Korolyuk, V.S. Boundary problems for compound Poisson processes Kiev, Naukova Dumka 1975

[8] Kyprianou, A. E. Introductory lectures on fluctuations of Lévy processes with applications. Springer-Verlag, Berlin, 2006

[9] Lotov, V. I. (1996). On oscillating random walks. (Russian) Sibirsk. Mat. Zh. 37(4), 869–880; translation in Siberian Math. J. 37(4), 764–774 (1996)

[10] Pecherskii, E.A. and Rogozin, B.A. The combined distributions of the random variables connected with the fluctuations of a process with independent increments. Theor. Prob. and its Appl. 14(3), 431-444 (1969)

[11] Pacheco, A., Ribeiro, H.: Consecutive customer losses in regular and oscillating $M^X|G|1|n$ systems. Queueing Systems 58, 121–136 (2008)

[12] Suprun, V.N. and Shurenkov, V.M. On the resolvent of a process with independent increments terminating at the moment when it hits the negative real semi-axis Studies in the Theory of Stochastic processes, Institute of Mathematics, Academy of Sciences of UKrSSR, Kiev, 170–174 (1975)

[13] Suprun, V.N. Ruin problem and the resolvent of a terminating process with independent increments. Ukr. Math. J., 28, 53–61 (1976)

[14] Shurenkov, V.M. Limit distribution of the first exit time and the position of the process from a wide interval for the processes with independent increments of one sign. Theor. Prob. and its Appl., 23, 419–425 (1978)

[15] Zolotarev, V.M. The first passage time of a level and the behavior at infinity for a class of processes with independent increments. Theor. Prob. and its Appl., 9, 653–664 (1964).