Criteria for Poisson process convergence with applications to inhomogeneous Poisson-Voronoi tessellations

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Abstract

This article employs the relation between probabilities of two consecutive values of a Poisson random variable to derive conditions for the weak convergence of point processes to a Poisson process. As applications, we consider the starting points of \(k\)-runs in a sequence of Bernoulli random variables and point processes constructed using inradii and circumscribed radii of inhomogeneous Poisson-Voronoi tessellations.

1 Introduction and main results

Let \(X\) be a random variable taking values in \(\mathbb{N}_0 = \mathbb{N} \cup \{0\}\) and let \(\lambda > 0\). It is well-known that
\[
    k \Pr(X = k) = \lambda \Pr(X = k - 1), \quad k \in \mathbb{N},
\]
if and only if \(X\) follows a Poisson distribution with parameter \(\lambda\). We use this observation to establish weak convergence to a Poisson process. Indeed, we will prove that a tight sequence of point processes \(\xi_n, n \in \mathbb{N}\), satisfies
\[
    \lim_{n \to \infty} k \Pr(\xi_n(B) = k) - \lambda(B) \Pr(\xi_n(B) = k - 1) = 0, \quad k \in \mathbb{N},
\]
for any \(B\) in a certain family of sets and some locally finite measure \(\lambda\), if and only if \(\xi_n\) converges in distribution to a Poisson process with intensity measure \(\lambda\). Many different methods to investigate Poisson process convergence are available in the literature; we refer to surveys and classical results \([19, 23, 24]\). Using Stein’s method, one can even derive quantitative bounds for the Poisson process approximation; see e.g. \([1, 3, 4, 5, 9, 10, 13]\).

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and the references therein. In contrast to these results, our findings are purely qualitative and do not provide rates of convergence, but they have the advantage that the underlying conditions are easy to verify. This is demonstrated in Sections 3.2 and 3.3 where weak convergence of point processes constructed using inradii and circumscribed radii of inhomogeneous Poisson-Voronoi tessellations is established.

The proof of our abstract criterion for Poisson process convergence relies on characterizations of point process convergence from [18, 19] and the characterizing equation (1.1) for the Poisson distribution.

Let us now give a precise formulation of our results. Let \( S \) be a locally compact second countable Hausdorff space (lcscH space) with Borel \( \sigma \)-field \( S \). A non-empty class \( U \) of subsets of \( S \) is called a ring if it is closed under finite unions and intersections, as well as under proper differences. Let \( \hat{S} \) denote the class of relatively compact sets of \( S \). We say that a measure \( \lambda \) on \( S \) is non-atomic if \( \lambda(\{x\}) = 0 \) for all \( x \in S \), and we define
\[
\hat{S}_\lambda = \{ B \in \hat{S} : \lambda(\partial B) = 0 \},
\]
where \( \partial B \) indicates the boundary of \( B \).

Let \( M(S) \) be the space of all locally finite measures on \( S \), endowed with the vague topology induced by the mappings \( \pi_f : \mu \mapsto \mu(f) = \int f \, d\mu \), \( f \in C^K_\infty(S) \), where \( C^K_\infty(S) \) denotes the set of non-negative and continuous functions with compact support. Note that \( M(S) \) is a Polish space (see e.g. [18, Theorem A2.3]). Let \( \mathcal{N}(S) \subset M(S) \) denote the set of all locally finite counting measures. A random measure \( \xi \) on \( S \) is a random element in \( M(S) \) measurable with respect to the \( \sigma \)-field generated by the vague topology, and it is a point process if it takes values in \( \mathcal{N}(S) \).

Our first main result provides a characterization of weak convergence to a Poisson process.

**Theorem 1.1.** Let \( \xi_n, n \in \mathbb{N}, \) be a sequence of point processes, and let \( \lambda \) be a non-atomic locally finite measure on \( S \). Let \( U \subseteq \hat{S}_\lambda \) be a ring containing a countable topological basis of \( S \). Then the following statements are equivalent:

(i) For all open sets \( B \in U \) and \( k \in \mathbb{N}, \) \( \xi_n(B), n \in \mathbb{N}, \) is tight and
\[
\lim_{n \to \infty} k \mathbb{P}(\xi_n(B) = k) - \lambda(B) \mathbb{P}(\xi_n(B) = k - 1) = 0.
\]

(ii) \( \xi_n, n \in \mathbb{N}, \) converges in distribution to a Poisson process with intensity measure \( \lambda. \)

**Remark 1.2.** Note that the sequence \( \xi_n(B), n \in \mathbb{N}, \) in Theorem 1.1 is tight by the Markov inequality if \( \mathbb{E}[\xi_n(B)] \to \lambda(B). \)

**Remark 1.3.** For a point process \( \varrho \), the function \( f : S \times \mathcal{N}(S) \to [0, \infty) \) defined as
\[
f(x, \mu) = 1_B(x) 1\{\mu(B) = k\}
\]
with \( k \in \mathbb{N} \) and \( B \in U \) satisfies
\[
\mathbb{E} \sum_{x \in \varrho} f(x, \varrho) - \int_S \mathbb{E}[f(x, \varrho + \delta_x)] d\lambda(x) = k \mathbb{P}(\varrho(B) = k) - \lambda(B) \mathbb{P}(\varrho(B) = k - 1),
\]
where \( \delta_x \) is the Dirac delta function at \( x \).
where $\delta_x$ denotes the Dirac measure centered at $x \in S$. By the Mecke formula, the left-hand side of (1.3) equals zero for all integrable functions $f : S \times \mathcal{N}(S) \to [0, \infty)$ if and only if $\varrho$ is a Poisson process with intensity measure $\lambda$ (see e.g. [24, Theorem 4.1]). Theorem 1.4 shows that, if we replace $\varrho$ by $\xi_n, n \in \mathbb{N}$, satisfying a tightness assumption, then the left-hand side of (1.4) vanishes as $n \to \infty$ for all $f$ of the form (1.3) if and only if $\xi_n, n \in \mathbb{N}$, converges weakly to a Poisson process with intensity measure $\lambda$.

Next we apply Theorem 1.4 to investigate point processes on $S$ that are constructed from an underlying Poisson or binomial point process on a measurable space $(Y, \mathcal{Y})$. By $\mathcal{N}_\sigma(Y)$ we denote the set of all $\sigma$-finite counting measures on $Y$, which is equipped with the $\sigma$-field generated by the sets

$$\{\mu \in \mathcal{N}_\sigma(Y) : \mu(B) = k\}, \quad k \in \mathbb{N}_0, B \in \mathcal{Y}.$$ 

For $t \geq 1$ let $\eta_t$ be a Poisson process on $Y$ with a $\sigma$-finite intensity measure $P_t$ (i.e. $\eta_t$ is a random element in $\mathcal{N}_\sigma(Y)$), while $\beta_n$ is a binomial point process of $n \in \mathbb{N}$ independent points in $Y$ which are distributed according to a probability measure $Q_n$. For a family of measurable functions $h_t : V_t \times \mathcal{N}_\sigma(Y) \to S$ with $V_t \in \mathcal{Y}, t \geq 1$, we are interested in the point processes

$$\sum_{x \in \eta_t \cap V_t} \delta_{h_t(x, \eta_t)}, \quad t \geq 1, \quad \text{and} \quad \sum_{x \in \beta_t \cap V_t} \delta_{h_n(x, \beta_t)}, \quad n \in \mathbb{N}. $$

In order to deal with both situations simultaneously, we introduce a joint notation. In the sequel, we study the point processes

$$\xi_t = \sum_{x \in \zeta_t \cap U_t} \delta_{g_t(x, \xi_t)}, \quad t \geq 1, \quad \text{(1.5)}$$

where $\zeta_t = \eta_t, g_t = h_t$ and $U_t = V_t$ in the Poisson case, while $\zeta_t = \beta_{[t]}, g_t = h_{[t]}$ and $U_t = V_{[t]}$ in the binomial case. We assume

$$\mathbb{P}(\xi_t(B) < \infty) = 1 \quad \text{for all } B \in \mathcal{S}$$

so that $\xi_t$ is locally finite. Let $M_t$ be the intensity measure of $\xi_t$. By $K_t$ we denote the intensity measure of $\zeta_t$, i.e. $K_t = P_t$ if $\zeta_t = \eta_t$ and $K_t = [t] Q_{[t]}$ if $\zeta_t = \beta_{[t]}$. Moreover, we define $\zeta_t = \eta_t$ in the Poisson case and $\zeta_t = \beta_{[t]} - 1$ in the binomial case. From Theorem 1.1 we derive the following criterion for convergence of $\xi_t, t \geq 1$, to a Poisson process.

**Theorem 1.4.** Let $\xi_t, t \geq 1$, be a family of point processes on $S$ given by (1.5) and let $M$ be a non-atomic locally finite measure on $S$. Fix any ring $U \subset \mathcal{S}_M$ containing a countable topological basis, and assume that

$$\lim_{t \to \infty} M_t(B) = M(B) \quad \text{(1.6)}$$

for all open sets $B \in U$. Then,

$$\lim_{t \to \infty} \int_{U_t} \mathbb{E} \left[ 1\{g_t(x, \zeta_t + \delta_x) \in B\} \left\{ \sum_{y \in \zeta_t \cap U_t} \delta_{g(y, \zeta_t + \delta_x)}(B) = m \right\} \right] dK_t(x)$$

$$- M(B) \mathbb{P}(\xi_t(B) = m) = 0 \quad \text{(1.7)}$$

for all open sets $B \in U$ and $m \in \mathbb{N}_0$, if and only if $\xi_t, t \geq 1$, converges weakly to a Poisson process with intensity measure $M$. 

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Remark 1.5. One is often interested in Poisson process convergence for \( S = \mathbb{R}^d, d \geq 1 \), and for the situation that the intensity measure of the Poisson process is absolutely continuous (with respect to the Lebesgue measure). In this case, we can apply Theorem 1.1 and Theorem 1.4 in the following way. The family \( \mathcal{R}^d \) of sets in \( \mathbb{R}^d \) that are finite unions of Cartesian products of bounded intervals is a ring contained in the relatively compact sets of \( \mathbb{R}^d \). For any absolutely continuous measure the boundaries of sets from \( \mathcal{R}^d \) have zero measure. By \( \mathcal{I}^d \) we denote the subset of open sets of \( \mathcal{R}^d \), which contains a countable topological basis of \( \mathbb{R}^d \). Note that the sets of \( \mathcal{I}^d \) are finite unions of Cartesian products of bounded open intervals. Thus, we prove weak convergence for sequences of point processes of the form (1.5), one has to deal with the dependence between topological basis of \( \mathcal{I}^d \). That is, for any fixed \( x \) we say that a statistic is locally dependent if its value at a given point depends only on a local and deterministic neighborhood. By (1.9), the last expression can be approximated by

\[
\frac{1}{k} \sum_{y \in \delta \cap U_t} \delta_{g_t(y, \eta)}(B) = m
\]

for \( x \in Y \). Under the assumption (1.8), the integral in (1.7) coincides with

\[
\mathbb{E} \left[ \frac{1}{k} \sum_{y \in \eta \cap U_t} \delta_{g_t(y, \eta)}(B) = m \right] dK_t(x).
\]

By (1.9), the last expression can be approximated by

\[
\mathbb{E} \left[ \frac{1}{k} \sum_{y \in \eta \cap A_{t,x}} \delta_{g_t(y, \eta)}(B) = m \right] dK_t(x).
\]

Due to the independence of \( \eta_t | A_{t,x} \) and \( \eta_t | A_{t,x}^c \), this can be rewritten as

\[
\mathbb{E} \left[ \frac{1}{k} \sum_{y \in \eta \cap A_{t,x}} \delta_{g_t(y, \eta)}(B) = m \right] dK_t(x).
\]
Using once more (1.8) and (1.9), the previous term can be approximated by

\[
\mathbb{P}(\xi_t(B) = m) \int_{U_t} \mathbb{E}\left[1\{g_t(x, \eta_t + \delta_x) \in B\}\right] dK_t(x) = \mathbb{P}(\xi_t(B) = m) M_t(B),
\]

where the last equality follows from the Mecke formula. Consequently, the expression on the left-hand side of (1.7) becomes small if the approximation in (1.9) is good.

In Section 3, we provide examples for applying our abstract main results Theorem 1.1 and Theorem 1.4. Our first example in Subsection 3.1 are \(k\)-runs, i.e. at least \(k\) successes in a row in a sequence of Bernoulli random variables. For the situation that the success probabilities converge to zero, we show that the rescaled starting points of the \(k\)-runs behave like a Poisson process if some independence assumptions on the underlying Bernoulli random variables are satisfied.

As the second and third example, we consider statistics related to inradii and circumscribed radii of inhomogeneous Poisson-Voronoi tessellations. We study the Voronoi tessellation generated by a Poisson process \(\eta_t, t > 0, \) on \(\mathbb{R}^d\) with intensity measure \(t\mu,\) where \(\mu\) is a locally finite and absolutely continuous measure with density \(f.\) In Section 3.2 for any cell with the nucleus in a compact set, we take the \(\mu\)-measure of the ball centered at the nucleus and with twice the inradius as the radius. We prove that the point process formed by these statistics converges in distribution after a transformation depending on \(t\) to a Poisson process as \(t \to \infty\) under some minor assumptions on the density \(f.\) Our transformation allows us to describe the behavior of the balls with large \(\mu\)-measures. In Section 3.3, we consider for each cell with the nucleus in a compact convex set the \(\mu\)-measure of the ball around the nucleus with the circumscribed radius as radius and establish, after rescaling with a power of \(t,\) convergence in distribution to a Poisson process for \(t \to \infty.\) This result requires continuity of \(f,\) but under weaker assumptions on \(f,\) we provide lower and upper bounds for the tail distribution of the minimal \(\mu\)-measure of these balls having the circumscribed radii as radii.

In [8], the limiting distributions of the maximal inradius and the minimal circumscribed radius of a stationary Poisson-Voronoi tessellation were derived. In our work, we extend these results in two directions. First, our findings imply Poisson process convergence of the transformed inradii and circumscribed radii for the stationary case. This implies the mentioned results from [8] and allows to deal with the \(m\)-th largest (or smallest) value or combinations of several order statistics. Second, we deal with inhomogeneous Poisson-Voronoi tessellations. In [11] some general results for the extremes of stationary tessellations were deduced, but they cannot be applied to inhomogeneous Poisson-Voronoi tessellations. For stationary Poisson-Voronoi tessellations the convergence of the nuclei of extreme cells to a compound Poisson process was studied in [12].

As our Theorem 1.4 deals with underlying Poisson and binomial point processes, we expect that one can extend our results on inradii and circumscribed radii of Poisson-Voronoi tessellations to Voronoi tessellations constructed from an underlying binomial point process.

Before we discuss our applications in Section 3, we prove our main results in the next section.
2 Proofs of the main results

Recall that $S$ is a locally compact second countable Hausdorff space, which is abbreviated as lcscH space. A topological space is second countable if its topology has a countable basis, and it is locally compact if every point has an open neighborhood whose topological closure is compact. A family of sets $C \subset \hat{S}$ is called dissecting if

(i) every open set $G \subset S$ can be written as a countable union of sets in $C$,

(ii) every relatively compact set $B \in \hat{S}$ is covered by finitely many sets in $C$.

Lemma 2.1. A countable topological basis $T$ of $S$ is dissecting.

Proof. By the definition of a countable topological basis $T$ has property (i) of a dissecting family of sets. Since, for any $B \in \hat{S}$, $\bigcup_{T \in T} T = S \supset B$, the compactness of $B$ implies that (ii) is satisfied. 

Let us now state a consequence of [19, Theorem 4.15] and [18, Theorem 16.16] or [19, Theorem 4.11]. This result will be used in the proof of Theorem 1.1. We write $\xrightarrow{d}$ to denote convergence in distribution.

Lemma 2.2. Let $\xi_n, n \in \mathbb{N}$, be a sequence of point processes on $S$, and let $\gamma$ be a Poisson process on $S$ with a non-atomic locally finite intensity measure $\lambda$. Let $U \subset \hat{S}_\lambda$ be a ring containing a countable topological basis. Then the following statements are equivalent:

(i) $\xi_n \xrightarrow{d} \gamma$.

(ii) $\xi_n(B) \xrightarrow{d} \gamma(B)$ for all open sets $B \in U$.

Proof. Observe that [21, Theorem 3.6] ensures the existence of a Poisson process $\gamma$ with intensity measure $\lambda$. Since $\lambda$ has no atoms, from [21, Proposition 6.9] it follows that $\gamma$ is a simple point process (i.e. $\mathbb{P}(\gamma(\{x\}) \leq 1$ for all $x \in S) = 1$). Elementary arguments also yield $\hat{S}_\lambda = \{B \in \hat{S} : \gamma(\partial B) = 0$ a.s. $\} =:\hat{S}_\gamma$.

It follows from Lemma 2.1 that $U$ is dissecting. By [18, Theorem 16.16 (ii)] or [19, Theorem 4.11] with $U$ as dissecting semi-ring, we obtain that (i) implies (ii).

Conversely, if $\xi_n(U) \xrightarrow{d} \xi(U)$ for all $U \in U$, the desired result follows from [19, Theorem 4.15], whose conditions are satisfied with $U$ as dissecting ring and semi-ring. Thus, it is enough to show that (ii) implies $\xi_n(U) \xrightarrow{d} \xi(U)$ for all $U \in U$.

For any $U \in U$ there exists a sequence of open sets $A_j, j \in \mathbb{N}$, such that

$U \subset A_j, \quad A_{j+1} \subset A_j \quad \text{and} \quad \overline{U} = \cap_{j \in \mathbb{N}} A_j.$

Since $U$ contains a countable topological basis, for any $A_j$ one can find a countable family of open sets $B^{(j)}_\ell, \ell \in \mathbb{N}$, in $U$ such that $\cup_{\ell \in \mathbb{N}} B^{(j)}_\ell = A_j$. In particular, they cover the compact set $\overline{U}$. So there exists a finite subcover of elements from $B^{(j)}_\ell, \ell \in \mathbb{N}$, that covers $\overline{U}$. Since $U$ is a ring, the union of the elements belonging to this subcover of $\overline{U}$ is in $U$ for each $j \in \mathbb{N}$. Because $U$ is closed under finite intersections, we can make this family of sets from $U$ that contain $\overline{U}$ monotonously decreasing in $j$. Thus, without loss of generality, we may assume $A_j \in U$ for all $j \in \mathbb{N}$. 

Since $\mathcal{U}$ is a ring and contains a countable topological basis, for the interior $\text{int}(U)$ of $U$ there exists a sequence of open sets $B_j \in \mathcal{U}$, $j \in \mathbb{N}$, such that

$$B_j \subset U, \quad B_j \subset B_{j+1} \quad \text{and} \quad \text{int}(U) = \bigcup_{j \in \mathbb{N}} B_j.$$  

For a fixed $m \in \mathbb{N}$, we have that

$$\mathbb{P}(\xi_n(B_j) \geq m) \leq \mathbb{P}(\xi_n(U) \geq m) \leq \mathbb{P}(\xi_n(A_j) \geq m)$$

for all $n \in \mathbb{N}$. By $\xi_n(U') \xrightarrow{d} \gamma(U')$ for all open sets $U' \in \mathcal{U}$, we obtain

$$\mathbb{P}(\gamma(B_j) \geq m) \leq \liminf_{n \to \infty} \mathbb{P}(\xi_n(U) \geq m) \leq \limsup_{n \to \infty} \mathbb{P}(\xi_n(U) \geq m) \leq \mathbb{P}(\gamma(A_j) \geq m). \quad (2.1)$$

Moreover, from $U \in \mathbf{S}_\lambda$, whence $\lambda(\partial U) = 0$, it follows that $\lambda(B_j) \to \lambda(\text{int}(U)) = \lambda(U)$ and $\lambda(A_j) \to \lambda(\overline{U}) = \lambda(U)$ as $j \to \infty$. Thus, letting $j \to \infty$ in (2.1) and using that $\gamma$ is a Poisson process lead to

$$\lim_{n \to \infty} \mathbb{P}(\xi_n(U) \geq m) = \mathbb{P}(\gamma(U) \geq m).$$

This establishes $\xi_n(U) \xrightarrow{d} \xi(U)$ and concludes the proof.

We are now in the position to prove the first main result of this paper.

**Proof of Theorem 1.1.** Let us show (i) implies (ii). By Lemma 2.2 it is enough to prove that $\xi_n(B) \xrightarrow{d} \gamma(B)$ for all open sets $B \in \mathcal{U}$. Since $\mathbb{P}(\xi_n(B) = 0), n \in \mathbb{N}$, is a bounded sequence in $[0, 1]$, there exists a subsequence such that $\lim_{j \to \infty} \mathbb{P}(\xi_{n_j}(B) = 0)$ exists; then repeated applications of (1.2) yield for $k \in \mathbb{N}$ that

$$\lim_{j \to \infty} \mathbb{P}(\xi_{n_j}(B) = k) = \frac{\lambda(B)^k}{k!} \lim_{j \to \infty} \mathbb{P}(\xi_{n_j}(B) = 0). \quad (2.2)$$

Consequently we have for any $N \in \mathbb{N}$,

$$\sum_{k=0}^{N} \lim_{j \to \infty} \mathbb{P}(\xi_{n_j}(B) = k) = \lim_{j \to \infty} \mathbb{P}(\xi_{n_j}(B) \in \{0, \ldots, N\})$$

$$= 1 - \lim_{j \to \infty} \mathbb{P}(\xi_{n_j}(B) \in \{N + 1, N + 2, \ldots \}).$$

By tightness of $\xi_{n_j}(B)$, $j \in \mathbb{N}$, the right-hand side of the equation converges to 1 as $N \to \infty$ so that

$$\sum_{k \in \mathbb{N}_0} \lim_{j \to \infty} \mathbb{P}(\xi_{n_j}(B) = k) = 1.$$

Thus, from (2.2) we deduce $\lim_{j \to \infty} \mathbb{P}(\xi_{n_j}(B) = 0) = e^{-\lambda(B)}$. Together with (2.2), this proves that

$$\lim_{j \to \infty} \mathbb{P}(\xi_{n_j}(B) = k) = \frac{\lambda(B)^k}{k!} e^{-\lambda(B)}$$
for all \( k \in \mathbb{N}_0 \). In conclusion, since for any subsequence \((n_i)_{i \in \mathbb{N}}\) there exists a further subsequence \((n_{i_i})_{i_i \in \mathbb{N}}\) such that \( \mathbb{P}(\xi_{n_i}(B) = 0) \), \( i \in \mathbb{N} \), converges to \( e^{-\lambda(B)} \), we obtain

\[
\lim_{n \to \infty} \mathbb{P}(\xi_n(B) = k) = \frac{\lambda(B)}{k!} e^{-\lambda(B)}
\]

for all \( k \in \mathbb{N}_0 \). The result follows by applying Lemma 2.2.

Conversely, let us assume \( \xi_n \xrightarrow{d} \gamma \) for some Poisson process \( \gamma \) with intensity measure \( \lambda \). It follows from Lemma 2.2 that, for any open set \( B \in \mathcal{U} \), \( \xi_n(B) \xrightarrow{d} \gamma(B) \) so that \( \xi_n(B), n \in \mathbb{N} \), is tight and

\[
0 = k \mathbb{P}(\gamma(B) = k) - \lambda(B) \mathbb{P}(\gamma(B) = k - 1)
= \lim_{n \to \infty} k \mathbb{P}(\xi_n(B) = k) - \lambda(B) \mathbb{P}(\xi_n(B) = k - 1)
\]

for \( k \in \mathbb{N} \), which shows (i).

Finally, we derive Theorem 1.4 from Theorem 1.1.

Proof of Theorem 1.4. By (1.6) and the Markov inequality we deduce that \( \xi_t(B), t \geq 1 \), is tight for all open \( B \in \mathcal{U} \). Let \( f : S \times \mathcal{N}(S) \to [0, \infty) \) be the function given by

\[
f(x, \mu) = 1_B(x) 1\{\mu(B) = k\}
\]

for \( k \in \mathbb{N} \) and \( B \in \mathcal{U} \). Then, by applying the Mecke equation (if \( \zeta_t = \eta_t \)) and the identity

\[
\mathbb{E} \sum_{x \in \beta_n} u(x, \beta_n) = n \int_Y \mathbb{E}[u(x, \beta_{n-1} + \delta_x)] d\mathbb{Q}_n(x)
\]

for any measurable function \( u : Y \times \mathcal{N}(S) \to [0, \infty) \) (if \( \zeta_t = \beta_{[t]} \)), we obtain

\[
k \mathbb{P}(\xi_t(B) = k) = \mathbb{E} \sum_{z \in \xi_t} f(z, \xi_t) = \mathbb{E} \sum_{x \in \zeta_t \cap U_t} f(g_t(x, \zeta_t), \xi_t(\zeta_t))
= \int_{U_t} \mathbb{E}\left[1\{g_t(x, \hat{\zeta}_t + \delta_x) \in B\} 1\left\{ \sum_{y \in \hat{\zeta}_t \cap U_t} \delta_{g_t(y, \hat{\zeta}_t + \delta_x)}(B) = k - 1 \right\} \right] dK_t(x).
\]

Thus, Theorem 1.4 yields the equivalence between (1.7) and the convergence in distribution of \( \xi_t, t \geq 1 \), to a Poisson process with intensity measure \( M \).

3 Applications

All our examples throughout this section concern point processes on \( \mathbb{R} \). By Remark 1.5 it is sufficient for the convergence of such point processes to a Poisson process on \( \mathbb{R} \) with absolutely continuous locally finite intensity measure to show (1.2) or (1.6) and (1.7) for all sets from \( \mathcal{I} \), i.e. for all finite unions of open and bounded intervals.
3.1 Long head runs

Consider a sequence of Bernoulli random variables. A \( k \)-head run is defined as an uninterrupted sequence of \( k \) successes, where \( k \) is a positive integer. For example, for \( k = 1 \), one simply studies the successes, while for \( k = 2 \) one considers the occurrence of two consecutive successes in a row. Several authors have investigated the number of \( k \)-head runs in a sequence of Bernoulli random variables; for an overview on this topic, we refer to [2]. Let the starting point of a \( k \)-head run be the index of its first success. Our goal is to find explicit conditions under which the point process of rescaled starting points of the \( k \)-head runs converges weakly to a Poisson process. Our investigation relies on two assumptions: the probability of having a \( k \)-head run is the same for all \( k \) consecutive elements of the sequence, and the Bernoulli random variables are independent if far away.

We will see that if these conditions are satisfied and if the probability of having a \( k \)-head run goes to 0 slower than the probability of having a \( k \)-head run with at least another \( k \)-head run nearby, then the aforementioned point process converges in distribution to a Poisson process.

Let us now give a precise formulation of our result. Let \( X_i^{(n)}, i, n \in \mathbb{N} \), be an array of Bernoulli distributed random variables and let \( k \in \mathbb{N} \). Assume that there exists a function \( f : \mathbb{N} \to \mathbb{N} \) such that for all \( q, n \in \mathbb{N} \) the random variable \( X_q^{(n)} \) is independent of \( \{X_{\ell}^{(n)} : |q - \ell| \geq f(n), \ell \in \mathbb{N}\} \) and that

\[
y_n := \mathbb{P}(X_q^{(n)} = 1, \ldots, X_{q+k-1}^{(n)} = 1) > 0
\]

does not depend on \( q \). If \( X_i^{(n)}, i \in \mathbb{N} \), are i.i.d. for \( n \in \mathbb{N} \), then \( y_n = p_n^K \) with \( p_n := \mathbb{P}(X_1^{(n)} = 1) \). Define

\[
I_i^{(n)} = \mathbb{1}\{X_i^{(n)} = 1, \ldots, X_{i+k-1}^{(n)} = 1\}, \quad i \in \mathbb{N}.
\]

Let \( \xi_n \) be the point process of the \( k \)-head runs for \( X_i^{(n)}, i \in \mathbb{N} \), that is

\[
\xi_n = \sum_{i=1}^{\infty} I_i^{(n)} \delta_{iy_n}.
\]

For any \( i_0 \in \mathbb{N} \), let

\[
W_{i_0}^{(n)} = \sum_{j \in \mathbb{N} : 1 \leq |j-i_0| \leq f(n)+k-2} I_j^{(n)}.
\]

We denote by \( \lambda_1 \) the restriction of the Lebesgue measure to \([0, \infty)\).

**Theorem 3.1.** Let \( \xi_n, n \in \mathbb{N} \), be the sequence of point processes given by (3.1). Assume that \( f(n)y_n \to 0 \) and that

\[
\lim_{n \to \infty} \sup_{i \in \mathbb{N}} y_n^{-1} \mathbb{E}[I_i \mathbb{1}\{W_i^{(n)} > 0\}] = 0.
\]

Then \( \xi_n \) converges weakly to a Poisson process with intensity measure \( \lambda_1 \).

For underlying independent Bernoulli random variables, the Poisson approximation of the random variable \( \xi_n((0, u)), u > 0 \), is considered in e.g. [1, 5, 14, 20] and the Poisson
process convergence follows from the results of [1]. Quantitative bounds for the Poisson
process approximation of 2-runs in the i.i.d. case were derived in [29, Proposition 3.C] and
[30, Theorem 6.3]; see also [9, Subsection 3.5], where the Poisson process approximation
for the more general problem of counting rare words is considered.

As a consequence of Theorem 3.1, we can study the limiting distribution of
$$T_n = \min\{i \in \mathbb{N} : I_i^{(n)} = 1\},$$
which gives the first arrival time of a $k$-head run for a sequence of Bernoulli random
variables.

**Corollary 3.2.** If the assumptions of Theorem 3.1 are satisfied, then $y_n T_n$ converges in
distribution to an exponentially distributed random variable with parameter $1$.

Clearly, in the case when the Bernoulli random variables $(X_i^{(n)})_{i \in \mathbb{N}}$ are i.i.d. with
parameter $p_n > 0$, if $p_n$ converges to $0$, the assumptions of Theorem 3.1 are fulfilled with
$f(n) \equiv 1$, and so $\xi_n$ converges in distribution to a Poisson process. Other conditions for
weak convergence are given in the following corollary.

**Corollary 3.3.** Let $\xi_n$, $n \in \mathbb{N}$, be the sequence of point processes given by (3.1). Let us
assume that $f(n) y_n \rightarrow 0$ and
$$\limsup_{n \rightarrow \infty} y_n^{-1} \sum_{j \in \mathbb{N} : 1 \leq |i-j| \leq f(n) + k - 2} \mathbb{E}[I_i^{(n)} I_j^{(n)}] = 0.$$ 
Then $\xi_n$ converges weakly to a Poisson process with intensity measure $\lambda_1$.

Let us now prove the main result of this section, Theorem 3.1.

**Proof of Theorem 3.1.** For any bounded interval $A \subset [0, \infty)$, the assumptions on $X_i^{(n)}$, $i \in \mathbb{N}$, imply that
$$\mathbb{E}[\xi_n(A)] = y_n \sum_{i=1}^{\infty} \delta_{y_n i}(A) = (\sup(A)y_n^{-1} + b_n)y_n - (\inf(A)y_n^{-1} + a_n)y_n$$
for some $a_n, b_n \in [-1, 1]$. By $y_n \rightarrow 0$, we have $\mathbb{E}[\xi_n(A)] \rightarrow \lambda_1(A)$ and, consequently,$$\mathbb{E}[\xi_n(B)] \rightarrow \lambda_1(B)$$ for all $B \in \mathcal{I}$. Moreover, $\xi_n(B)$, $n \in \mathbb{N}$, is tight (see Remark 1.2).

Then, we can write $\xi_n(B)$ as
$$\xi_n(B) = \sum_{i \in \mathcal{A}_n} I_i^{(n)}$$
with $\mathcal{A}_n = \{i \in \mathbb{N} : y_n i \in B\}$.

For $i_0 \in \mathcal{A}_n$, we have for any $m \in \mathbb{N}$ that
$$\left| \mathbb{E}\left[I_{i_0}^{(n)} \{\xi_n(B) - I_i^{(n)} = m - 1\} \right] - \mathbb{E}\left[I_{i_0}^{(n)} \{\xi_n(B) - W_{i_0}^{(n)} - I_i^{(n)} = m - 1\} \right]\right|$$
$$\leq \mathbb{E}\left[I_{i_0}^{(n)} \{W_{i_0}^{(n)} > 0\}\right].$$
Together with $\mathbb{E}[\xi_n(B)] = |\mathcal{A}_n| y_n$, this yields
$$H_n := \left| \sum_{i \in \mathcal{A}_n} \mathbb{E}\left[I_i^{(n)} \{\xi_n(B) - I_i^{(n)} = m - 1\} \right] - \sum_{i \in \mathcal{A}_n} \mathbb{E}\left[I_i^{(n)} \{\xi_n(B) - W_i^{(n)} - I_i^{(n)} = m - 1\} \right]\right|$$
$$\leq \sum_{i \in \mathcal{A}_n} \mathbb{E}\left[I_i^{(n)} \{W_i^{(n)} > 0\}\right] \leq \left( \sup_{i \in \mathbb{N}} y_n^{-1} \mathbb{E}\left[I_i \{W_i^{(n)} > 0\}\right] \right) \mathbb{E}[\xi_n(B)].$$
Therefore from (3.2), we obtain $H_n \to 0$. From the independence of $I_{i_0}^{(n)}$ and $\xi_n(B) - W_{i_0}^{(n)} - I_{i_0}^{(n)}$ for $i_0 \in \mathcal{A}_n$, it follows that
\[
\mathbb{E}\left[I_{i_0}^{(n)} 1\{\xi_n(B) - W_{i_0}^{(n)} - I_{i_0}^{(n)} = m - 1\}\right] = \mathbb{E}[I_{i_0}^{(n)}]\mathbb{P}(\xi_n(B) - W_{i_0}^{(n)} - I_{i_0}^{(n)} = m - 1).
\]
Combining the previous arguments implies for $m \in \mathbb{N}$ that
\[
\limsup_{n \to \infty} \left|m\mathbb{P}(\xi_n(B) = m) - \lambda_1(B)\mathbb{P}(\xi_n(B) = m - 1)\right|
= \limsup_{n \to \infty} \left|\sum_{i \in \mathcal{A}_n} \mathbb{E}[I_{i}^{(n)} 1\{\xi_n(B) - I_{i}^{(n)} = m - 1\}] - \lambda_1(B)\mathbb{P}(\xi_n(B) = m - 1)\right|
= \limsup_{n \to \infty} \left|\sum_{i \in \mathcal{A}_n} \mathbb{E}[I_{i}^{(n)}]\mathbb{P}(\xi_n(B) - W_{i}^{(n)} - I_{i}^{(n)} = m - 1) - \sum_{i \in \mathcal{A}_n} \mathbb{E}[I_{i}^{(n)}]\mathbb{P}(\xi_n(B) = m - 1)\right|
\leq \limsup_{n \to \infty} \sum_{i \in \mathcal{A}_n} \mathbb{E}[I_{i}^{(n)}]\mathbb{P}(W_{i}^{(n)} + I_{i}^{(n)} > 0) \leq \lambda_1(B) \limsup_{n \to \infty} \sup_{i \in \mathbb{N}} \mathbb{P}(W_{i}^{(n)} + I_{i}^{(n)} > 0).
\]
Finally, the inequality
\[
\mathbb{P}(W_{i}^{(n)} + I_{i}^{(n)} > 0) \leq (2k + 2f(n) - 3)y_n, \quad i \in \mathbb{N},
\]
and the assumption $f(n)y_n \to 0$ lead to
\[
\lim_{n \to \infty} \left|m\mathbb{P}(\xi_n(B) = m) - \lambda_1(B)\mathbb{P}(\xi_n(B) = m - 1)\right| = 0.
\]
The result follows by applying Theorem 1.1.

**Proof of Corollary 3.3** This follows directly from Theorem 3.1 and
\[
\mathbb{E}[I_i 1\{W_i^{(n)} > 0\}] \leq \mathbb{E}[I_i^{(n)}W_i^{(n)}] = \sum_{j \in \mathbb{N} : 1 \leq |i-j| \leq f(n)+k-2} \mathbb{E}[I_i^{(n)}I_j^{(n)}]
\]
for any $i \in \mathbb{N}$.

### 3.2 Inradii of an inhomogeneous Poisson Voronoi tessellation

In this section, we consider the inradii of an inhomogeneous Voronoi tessellation generated by a Poisson process with a certain intensity measure $t\mu$, $t > 0$; recall that the inradius of a cell is the largest radius for which the ball centered at the nucleus is contained in the cell. We study the point process on $\mathbb{R}$ constructed by taking for any cell with the nucleus in a compact set, a transform of the $\mu$-measure of the ball centered at the nucleus and with twice the inradius as the radius. The aim is to continue the work started in [S] by extending the result on the largest inradius to inhomogeneous Poisson-Voronoi tessellations and proving weak convergence of the aforementioned point process to a Poisson process.
For any locally finite counting measure \( \nu \) on \( \mathbb{R}^d \), we denote by \( N(x, \nu) \) the Voronoi cell with nucleus \( x \in \mathbb{R}^d \) generated by \( \nu + \delta_x \), that is

\[
N(x, \nu) = \{ y \in \mathbb{R}^d : \| x - y \| \leq \| y - x' \|, x \neq x' \in \nu \},
\]

where \( \| \cdot \| \) denotes the Euclidean norm. For \( x \in \nu \) we have \( N(x, \nu) = N(x, \nu - \delta_x) \).

Voronoi tessellations, i.e. tessellations consisting of Voronoi cells \( N(x, \nu), x \in \nu \), arise in different fields such as biology [26], astrophysics [27] and communication networks [6]. For more details on Poisson-Voronoi tessellations, i.e. Voronoi tessellations generated by different fields such as biology [26], astrophysics [27] and communication networks [6].

For any locally finite counting measure \( \nu \) on \( \mathbb{R}^d \), we have

\[
\{ y \in \mathbb{R}^d : \| x - y \| \leq \| y - x' \|, x \neq x' \in \nu \}.
\]

where \( \| \cdot \| \) denotes the Euclidean norm. For \( x \in \nu \) we have \( N(x, \nu) = N(x, \nu - \delta_x) \). Voronoi tessellations, i.e. tessellations consisting of Voronoi cells \( N(x, \nu), x \in \nu \), arise in different fields such as biology [26], astrophysics [27] and communication networks [6]. For more details on Poisson-Voronoi tessellations, i.e. Voronoi tessellations generated by an underlying Poisson process, we refer the reader to e.g. [7, 22, 28]. The inradius of the Voronoi cell \( N(x, \nu) \) is given by

\[
c(x, \nu) = \sup\{ R \geq 0 : B(x, R) \subset N(x, \nu) \},
\]

where \( B(x, r) \) denotes the open ball centered at \( x \in \mathbb{R}^d \) with radius \( r > 0 \).

Let \( \eta_t, t > 0 \), be a Poisson process on \( \mathbb{R}^d \) with intensity measure \( t\mu \), where \( \mu \) is a locally finite measure on \( \mathbb{R}^d \) with density \( f : \mathbb{R}^d \rightarrow [0, \infty) \). Consider a compact set \( W \subset \mathbb{R}^d \) with \( \mu(W) = 1 \), and assume that there exists a bounded open set \( A \subset \mathbb{R}^d \) with \( W \subset A \) such that \( f_{\min} := \inf_{x \in A} f(x) > 0 \) and \( f_{\max} := \sup_{x \in A} f(x) < \infty \). For any Voronoi cell \( N(x, \eta_t) \) with \( x \in \eta_t \), we take the \( \mu \)-measure of the ball around \( x \) with twice the inradius as radius, and we define the point process \( \xi_t \) on \( \mathbb{R} \) as

\[
\xi_t = \xi_t(\eta_t) = \sum_{x \in \eta_t \cap W} \delta_{\mu(B(x, 2c(x, \eta_t))) - \log(t)}.
\]

(3.3)

Let \( M \) be the measure on \( \mathbb{R} \) given by \( M([u, \infty)) = e^{-u} \) for \( u \in \mathbb{R} \).

**Theorem 3.4.** Let \( \xi_t, t > 0 \), be the family of point processes on \( \mathbb{R} \) given by (3.3). Then \( \xi_t \) converges in distribution to a Poisson process with intensity measure \( M \).

The next theorem shows that if the density function \( f \) is Hölder continuous, it is possible to take out the factor 2 from \( \mu(B(x, 2c(x, \eta_t))) \) and to consider \( 2^d \mu(B(x, c(x, \eta_t))) \). Recall that a function \( h : \mathbb{R}^d \rightarrow \mathbb{R} \) is Hölder continuous with exponent \( b > 0 \) if there exists a constant \( C > 0 \) such that

\[
|h(x) - h(y)| \leq C\|x - y\|^b
\]

for all \( x, y \in \mathbb{R}^d \). We define the point process \( \hat{\xi}_t \) as

\[
\hat{\xi}_t = \hat{\xi}_t(\eta_t) = \sum_{x \in \eta_t \cap W} \delta_{2^d \mu(B(x, c(x, \eta_t))) - \log(t)}.
\]

**Theorem 3.5.** Let \( f \) be Hölder continuous. Then \( \hat{\xi}_t, t > 0 \), converges in distribution to a Poisson process with intensity measure \( M \).

As a corollary of the previous theorems, we have the following generalization to the inhomogeneous case of the result obtained in [8, Theorem 1, Equation (2a)] for the stationary case; see also [11, Section 5] for the maximal inradius of a stationary Poisson-Voronoi tessellation and of a stationary Gauss-Poisson-Voronoi tessellation.
Corollary 3.6. For $u \in \mathbb{R}$,

$$\lim_{t \to \infty} \mathbb{P}\left( \max_{x \in \eta \cap W} t \mu(B(x, 2c(x, \eta_t))) - \log(t) \leq u \right) = e^{-e^{-u}}. \tag{3.4}$$

Moreover, if $f$ is Hölder continuous,

$$\lim_{t \to \infty} \mathbb{P}\left( \max_{x \in \eta \cap W} 2^d t \mu(B(x, c(x, \eta_t))) - \log(t) \leq u \right) = e^{-e^{-u}}. \tag{3.5}$$

For an underlying binomial point process, (3.4) was shown under similar assumptions in [15]. The related problem of maximal weighted $r$-th nearest neighbor distances for the points of a binomial point process was studied in [16]; see also [17].

For the proofs of Theorem 3.4 and Theorem 3.5, we will use the quantities $v_t(x, u)$ and $q_t(x, u)$, which are introduced in the next lemma.

Lemma 3.7. For any $u \in \mathbb{R}$ there exists a $t_0 > 0$ such that for all $x \in W$ and $t > t_0$ the equations

$$t \mu(B(x, 2v_t(x, u))) = u + \log(t) \quad \text{and} \quad 2^d t \mu(B(x, q_t(x, u))) = u + \log(t) \tag{3.6}$$

have unique solutions $v_t(x, u)$ and $q_t(x, u)$, respectively, which satisfy

$$\max\{v_t(x, u), q_t(x, u)\} \leq \left( \frac{u + \log(t)}{2^d f_{\min} k_d} \right)^{1/d}, \tag{3.7}$$

where $k_d$ is the volume of the $d$-dimensional unit ball.

Proof. Let $u \in \mathbb{R}$ be fixed and set $m = \inf\{||x - y|| : x \in \partial W, y \in \partial A\} \in (0, \infty)$. Note that $B(x, m) \subset A$ for all $x \in W$. Choose $t_0 > 0$ such that

$$\frac{u + \log(t)}{t} < f_{\min} k_d m^d$$

for $t > t_0$. For $x \in W$ and $t > t_0$ this implies that

$$2^d t \mu(B(x, m)) \geq t \mu(B(x, m)) \geq t f_{\min} k_d m^d > u + \log(t)$$

and, obviously, $t \mu(B(x, 0)) = 0$. Since the function $[0, m] \ni a \to \mu(B(x, a))$ is continuous and strictly increasing because of $f_{\min} > 0$, by the intermediate value theorem, the equations in (3.6) have unique solutions $v_t(x, u)$ and $q_t(x, u)$. Since $\max\{2v_t(x, u), q_t(x, u)\} < m$, we obtain

$$\frac{u + \log(t)}{t} = \mu(B(x, 2v_t(x, u))) \geq 2^d f_{\min} k_d v_t(x, u)^d$$

and

$$\frac{u + \log(t)}{t} = 2^d \mu(B(x, q_t(x, u))) \geq 2^d f_{\min} k_d q_t(x, u)^d,$$

which prove (3.7).
Let $M_t$ be the intensity measure of $\xi_t$. Then from the Mecke formula and Lemma 3.7 it follows that for any $u \in \mathbb{R}$ there exists a $t_0 > 0$ such that for $t > t_0$,

$$M_t([u, \infty)) = t \int_W \mathbb{P}(t \mu(B(x, 2c(x, \eta_t + \delta_x))) \geq u + \log(t)) f(x) dx$$

$$= t \int_W \mathbb{P}(c(x, \eta_t + \delta_x) \geq v_t(x, u)) f(x) dx$$

$$= t \int_W \mathbb{P}(\eta_t(B(x, 2v_t(x, u))) = 0) f(x) dx$$

$$= t \int_W e^{-t \mu(B(x, 2v_t(x, u)))} f(x) dx = te^{-u-\log(t)} \mu(W) = e^{-u} = M([u, \infty)), \quad (3.8)$$

where we used (3.6) and $\mu(W) = 1$ in the last steps. For any $y \in \mathbb{R}$ and point configuration $\nu$ on $\mathbb{R}^d$ with $y \in \nu$, we denote by $h_t(y, \nu)$ the quantity

$$h_t(y, \nu) = t \mu(B(y, 2c(y, \nu))) - \log(t), \quad (3.9)$$

where $c(y, \nu)$ is the inradius of the Voronoi cell with nucleus $y$ generated by $\nu$. So we can rewrite $\xi_t$ as

$$\xi_t = \sum_{x \in \eta_t \cap W} \delta_{h_t(x, \eta_t)}.$$

**Proof of Theorem 3.4.** From Theorem 1.4 and (3.8) it follows that it is enough to show that

$$\lim_{t \to \infty} t \int_W \mathbb{E} \left[ \mathbf{1}\{h_t(x, \eta_t + \delta_x) \in B\} \mathbf{1}\{ \sum_{y \in \eta \cap W} \delta_{h_t(y, \eta_t + \delta_x)}(B) = m\} \right] f(x)dx$$

$$- M(B)\mathbb{P}(\xi_t(B) = m) = 0$$

for any $m \in \mathbb{N}_0$ and $B \in \mathcal{I}$. Let $B = \bigcup_{j=1}^{\ell}(u_{2j-1}, u_{2j})$ with $u_1 < u_2 < \ldots < u_{2\ell}$ and $\ell \in \mathbb{N}$. By Lemma 3.7 there is a $t_0 > 0$ such that $v_t(x, u_k)$ exists for all $k = 1, \ldots, 2\ell$, $x \in W$ and $t > t_0$. Assume $t > t_0$ in the following. Elementary arguments imply that

$$t \int_W \mathbb{E} \left[ \mathbf{1}\{h_t(x, \eta_t + \delta_x) \in B\} \mathbf{1}\{ \sum_{y \in \eta \cap W} \delta_{h_t(y, \eta_t + \delta_x)}(B) = m\} \right] f(x)dx$$

$$= \sum_{j=1}^{\ell} t \int_W \mathbb{E} \left[ \mathbf{1}\{h_t(x, \eta_t + \delta_x) \in (u_{2j-1}, u_{2j})\} \mathbf{1}\{ \sum_{y \in \eta \cap W} \delta_{h_t(y, \eta_t + \delta_x)}(B) = m\} \right] f(x)dx. \quad (3.11)$$

For each $k = 1, \ldots, 2\ell$, set $w_{k,t,x} = 2v_t(x, u_k)$ Since $h_t(x, \eta_t + \delta_x) \in (u_{2j-1}, u_{2j})$ if and only if $c(x, \eta_t + \delta_x) \in (v_t(x, u_{2j-1}), v_t(x, u_{2j}))$, or equivalently, $\eta_t(B(x, u_{2j-1}, t, x)) = 0$ and $\eta_t(B(x, u_{2j-1}, t, x)) > 0$, we obtain that

$$S_j : = t \int_W \mathbb{E} \left[ \mathbf{1}\{h_t(x, \eta_t + \delta_x) \in (u_{2j-1}, u_{2j})\} \mathbf{1}\{ \sum_{y \in \eta \cap W} \delta_{h_t(y, \eta_t + \delta_x)}(B) = m\} \right] f(x)dx$$

$$= t \int_W \mathbb{E} \left[ \mathbf{1}\{\eta_t(B(x, w_{2j-1}, t, x)) = 0\} \times \mathbf{1}\{\eta_t(B(x, w_{2j}, t, x)) > 0\} \sum_{y \in \eta \cap W} \delta_{h_t(y, \eta_t + \delta_x)}(B) = m\} \right] f(x)dx.$$
For any point configuration \( \nu \) on \( \mathbb{R}^d \) and \( x \in W \), let \( \xi_{t,x}(\nu) \) be the counting measure given by \( \xi_{t,x}(\nu) = \sum_{y \in \nu \cap W} \delta_{h_t(y,\nu+\delta_x)} \) so that

\[
S_j = t \int_W \mathbb{E} \left[ 1 \{ \eta_t(B(x, w_{2j-1}, t, x)) = 0 \} \right. \\
\times \left. 1 \{ \eta_t(B(x, w_{2j}, t, x)) > 0, \xi_{t,x}(\eta_t|B(x,w_{2j-1},t,x)^c)(B) = m \} \right] f(x)dx
= t \int_W \mathbb{P} \left( \eta_t(B(x, w_{2j-1}, t, x)) = 0 \right) \\
\times \mathbb{P} \left( \eta_t(B(x, w_{2j}, t, x) \setminus B(x, w_{2j-1}, t, x)) > 0, \xi_{t,x}(\eta_t|B(x,w_{2j-1},t,x)^c)(B) = m \right) f(x)dx.
\]

Similar arguments as used to compute \( M_t([u, \infty)) \) for \( u \in \mathbb{R} \) imply for \( x \in W \) that

\[
t \mathbb{P} \left( \eta_t(B(x, w_{2j-1}, t, x)) = 0 \right) = e^{-u_{2j-1}},
\]

and so we deduce that

\[
S_j = e^{-u_{2j-1}} \int_W \mathbb{P} \left( \xi_{t,x}(\eta_t|B(x,w_{2j-1},t,x)^c)(B) = m \right) f(x)dx \\
- e^{-u_{2j-1}} \int_W \mathbb{P} \left( \eta_t(B(x, w_{2j}, t, x) \setminus B(x, w_{2j-1}, t, x)) = 0, \right. \\
\left. \xi_{t,x}(\eta_t|B(x,w_{2j-1},t,x)^c)(B) = m \right) f(x)dx.
\]

Furthermore, we can rewrite the second integral as

\[
\int_W \mathbb{P} \left( \eta_t(B(x, w_{2j}, t, x) \setminus B(x, w_{2j-1}, t, x)) = 0 \right) \mathbb{P} \left( \xi_{t,x}(\eta_t|B(x,w_{2j},t,x)^c)(B) = m \right) f(x)dx \\
= \int_W e^{-u_{2j} + u_{2j}} \mathbb{P} \left( \xi_{t,x}(\eta_t|B(x,w_{2j},t,x)^c)(B) = m \right) f(x)dx \\
= e^{-u_{2j} + u_{2j}} \int_W \mathbb{P} \left( \xi_{t,x}(\eta_t|B(x,w_{2j},t,x)^c)(B) = m \right) f(x)dx.
\]

Combining this and (3.12) yields

\[
S_j = e^{-u_{2j-1}} \int_W \mathbb{P} \left( \xi_{t,x}(\eta_t|B(x,w_{2j-1},t,x)^c)(B) = m \right) f(x)dx \\
- e^{-u_{2j}} \int_W \mathbb{P} \left( \xi_{t,x}(\eta_t|B(x,w_{2j},t,x)^c)(B) = m \right) f(x)dx.
\]

Substituting this into (3.11) implies that to prove (3.10) and to complete the proof, it is enough to show for all \( x \in W \) and \( k = 1, \ldots, 2\ell \) that

\[
\lim_{t \to \infty} \mathbb{P} \left( \xi_{t,x}(\eta_t|B(x,w_{2k},t,x)^c)(B) = m \right) - \mathbb{P}(\xi_t(B) = m) = 0. \quad (3.13)
\]

Let \( x \in W, k \in \{1, \ldots, 2\ell\} \) and \( \varepsilon > 0 \) be fixed. Set

\[
a_t = 2 \left( \frac{u_{2\ell} + \log(t)}{2^d f_{\min,k,d}^2} \right)^{1/d}.
\]
By Lemma 3.7 there exists a $t' > 0$ such that $w_{k,t,y} \leq w_{2t,t,y} \leq a_t$ for all $y \in W$ and $t > t'$. Without loss of generality we may assume that $2a_t < \min \{ \parallel z_1 - z_2 \parallel : z_1 \in \partial W, z_2 \in \partial A \}$. Therefore the observation

$$h(y, \nu) \in B \text{ if and only if } h(y, \nu|_{B(y,w_{2t,t,y})}) \in B$$

for any point configuration $\nu$ on $\mathbb{R}^d$ with $y \in \nu$ leads to

$$\left| \mathbb{P} \left( \xi_{t,x}(\eta|_{B(x,w_{k,t,x})^c})(B) = m \right) - \mathbb{P} \left( \xi_t(B) = m \right) \right| \leq \mathbb{E} \left[ |\xi_{t,x}(\eta|_{B(x,w_{k,t,x})^c})(B) - \xi_t(B) | \right]$$

$$\leq \mathbb{E} \sum_{y \in \eta \cap B(x,2a_t) \cap B(x,w_{k,t,x})^c} 1\{ h_t(y, \eta|_{B(x,w_{k,t,x})^c} + \delta_x \in B \} + \mathbb{E} \sum_{y \in \eta \cap B(x,2a_t)} 1\{ h_t(y, \eta|_B) \in B \}$$

$$\leq \mathbb{E} \sum_{y \in \eta \cap B(x,2a_t) \cap B(x,w_{k,t,x})^c} 1\{ h_t(y, \eta|_{B(x,w_{k,t,x})^c} + \delta_x > u_1 \}$$

$$+ \mathbb{E} \sum_{y \in \eta \cap B(x,2a_t)} 1\{ h_t(y, \eta) > u_1 \}.$$

Then, the Mecke formula and (3.9) imply that

$$\left| \mathbb{P} \left( \xi_{t,x}(\eta|_{B(x,w_{k,t,x})^c})(B) = m \right) - \mathbb{P} \left( \xi_t(B) = m \right) \right|$$

$$\leq t \int_{B(x,2a_t) \cap B(x,w_{k,t,x})^c} \mathbb{P}(h_t(y, \eta|_{B(x,w_{k,t,x})^c} + \delta_x + \delta_y) > u_1) f(y) dy$$

$$+ t \int_{B(x,2a_t)} \mathbb{P}(h_t(y, \eta + \delta_y) > u_1) f(y) dy$$

$$= t \int_{B(x,2a_t) \cap B(x,w_{k,t,x})^c} \mathbb{P}(t \mu(B(y,2c(y, \eta|_{B(x,w_{k,t,x})^c} + \delta_x + \delta_y)) > u_1 + \log(t)) f(y) dy$$

$$+ t \int_{B(x,2a_t)} \mathbb{P}(t \mu(B(y,2c(y, \eta + \delta_y)) > u_1 + \log(t)) f(y) dy.$$ 

Since $c(y, \nu + \delta_y + \delta_x) > v_t(y, u_1)$ only if $c(y, \nu + \delta_y) > v_t(y, u_1)$ for any point configuration $\nu$ on $\mathbb{R}^d$ and $x, y \in \mathbb{R}^d$, it follows for $x \in W$ and $y \in B(x,2a_t) \cap B(x,w_{k,t,x})^c$ that

$$\mathbb{P}(t \mu(B(y,2c(y, \eta|_{B(x,w_{k,t,x})^c} + \delta_x + \delta_y)) > u_1 + \log(t))$$

$$= \mathbb{P}(c(y, \eta|_{B(x,w_{k,t,x})^c} + \delta_x + \delta_y) > v_t(y, u_1)) \leq \mathbb{P}(c(y, \eta|_{B(x,w_{k,t,x})^c} + \delta_y) > v_t(y, u_1))$$

$$\leq \mathbb{P}(\eta|_{B(x,w_{k,t,x})^c}(B(y,2v_t(y, u_1))) = 0) = \exp(-t \mu(B(y,2v_t(y, u_1)) \cap B(x,w_{k,t,x})^c)).$$

Let $\lambda_d$ denote the Lebesgue measure on $\mathbb{R}^d$. For $A_1, A_2 \in \mathcal{B}(\mathbb{R}^d)$ with $A_1, A_2 \subset A$ and $\lambda_d(A_2) > 0$ we obtain

$$\frac{\mu(A_1)}{\mu(A_2)} \geq \frac{f_{\min} \lambda_d(A_1)}{f_{\max} \lambda_d(A_2)}.$$

With $\tau := f_{\min} / f_{\max} \in (0, 1]$,

$$A_1 = B(y,2v_t(y, u_1)) \cap B(x,w_{k,t,x})^c \text{ and } A_2 = B(y,2v_t(y, u_1)),$$

this implies for $x \in W$ and $y \in B(x,2a_t) \cap B(x,w_{k,t,x})^c$ that

$$t \mu(B(y,2v_t(y, u_1)) \cap B(x,w_{k,t,x})^c) \geq \frac{\tau}{2} t \mu(B(y,2v_t(y, u_1))) = \frac{\tau}{2} (u_1 + \log(t)).$$
Moreover, we have that
\[ \mathbb{P}(t \mu(B(y, 2c(y, \eta + \delta_y)))) > u_1 + \log(t)) = \mathbb{P}(\eta(B(y, 2v_1(y, u_1))) = 0) = e^{-u_1-\log(t)}. \]

In conclusion, combining the previous bounds leads to
\[ |\mathbb{P}(\xi_t, \eta(B(x, w_{h,t,x})) - \mathbb{P}(\xi_t(B)) = m)| \leq t^{1/2} e^{-u_1/2} \mu(B(x, 2a_t)) + e^{-u_1} \mu(B(x, 2a_t)) \leq (2a_t)^{d} f_{\max} [t^{1/2} e^{-u_1/2} + e^{-u_1}], \]

where in the last step we used the fact that \( f \) is bounded by \( f_{\max} \) in \( A \) and, by the choice of \( a_t, B(x, 2a_t) \subset A \). Again, from the definition of \( a_t \) it follows that the right-hand side converges to 0 as \( t \to \infty \). This shows (3.13) and concludes the proof. \( \square \)

Next, we derive Theorem 3.5 from Theorem 3.4.

**Proof of Theorem 3.5.** Assume that \( f \) is Hölder continuous with exponent \( b > 0 \). From Lemma 2.2, Theorem 3.4 and Remark 1.5 we obtain that it is enough to show that \( \mathbb{E}(|\xi_t(B) - \xi_t(B)|) \to 0 \) as \( t \to \infty \) for all \( B \in \mathcal{I} \). By Lemma 3.7 for any \( u \in \mathbb{R} \) there exists a \( t_0 > 0 \) such that
\[
\mu(B(x, 2v_t(x, u))) = 2^d \mu(B(x, q_t(x, u)))
\]
\[
= 2^d k_d f(x)q_t(x, u) + 2^d \int_{B(x, q_t(x, u))} f(y) - f(x) dy
\]
\[
= \mu(B(x, 2q_t(x, u))) - \int_{B(x, 2q_t(x, u))} f(y) - f(x) dy + 2^d \int_{B(x, q_t(x, u))} f(y) - f(x) dy
\]
for all \( x \in W, t > t_0 \), where
\[
\max\{v_t(x, u), q_t(x, u)\} \leq \left(\frac{u + \log(t)}{2^d f_{\min} k_d t}\right)^{1/d}.
\]

Thus, the Hölder continuity of \( f \) and elementary arguments establish that
\[
|\mu(B(x, 2v_t(x, u))) - \mu(B(x, 2q_t(x, u)))| \leq C \left(\frac{u + \log(t)}{t}\right)^{1+b/d}, \quad x \in W, t > t_0, \quad (3.14)
\]
for some \( C > 0 \). In particular, from the definition of \( v_t(x, u) \) it follows that
\[
\mu(B(x, 2v_t(x, u))) = \frac{u + \log(t)}{t},
\]
\[
\mu(B(x, 2q_t(x, u))) \geq \frac{u + \log(t)}{t} - C \left(\frac{u + \log(t)}{t}\right)^{1+b/d}, \quad (3.15)
\]
for \( t > t_0 \). Next, we write \( B = \bigcup_{j=1}^{\ell} (u_{2j-1}, u_{2j}) \) for some \( \ell \in \mathbb{N} \) and \( u_1 < \cdots < u_{2\ell} \). The triangle inequality yields
\[
\mathbb{E}[|\xi_t(B) - \hat{\xi}_t(B)|] \leq \sum_{j=1}^{\ell} \mathbb{E}[|\xi_t((u_{2j-1}, u_{2j})) - \hat{\xi}_t((u_{2j-1}, u_{2j}))|]
\]
\[
\leq \sum_{j=1}^{\ell} \mathbb{E}[|\xi_t((u_{2j-1}, \infty)) - \hat{\xi}_t((u_{2j-1}, \infty))|] + \mathbb{E}[|\xi_t([u_{2j}, \infty)) - \hat{\xi}_t([u_{2j}, \infty))|]. \quad (3.16)
\]
Moreover, the Mecke formula establishes for \( u \in \mathbb{R} \) that
\[
\mathbb{E}[|\xi_t((u, \infty)) - \tilde{\xi}_t((u, \infty))|] \\
\leq \mathbb{E} \sum_{x \in \eta_t \cap W} |1\{c(x, \eta_t + \delta_x) > v_t(x, u)\} - 1\{c(x, \eta_t + \delta_x) > q_t(x, u)\}| \\
= t \int_W \mathbb{P}(\eta_t(B(x, 2v_t(x, u))) = 0, \eta_t(B(x, 2q_t(x, u))) > 0) f(x) dx \\
+ t \int_W \mathbb{P}(\eta_t(B(x, 2v_t(x, u))) > 0, \eta_t(B(x, 2q_t(x, u))) = 0) f(x) dx \\
\leq f_{\max} t \int_W \left[ \exp \left( - t\mu(B(x, 2v_t(x, u))) \right) + \exp \left( - t\mu(B(x, 2q_t(x, u))) \right) \right] \\
\times \left[ 1 - \exp \left( - t|\mu(B(x, 2q_t(x, u))) - \mu(B(x, 2v_t(x, u)))| \right) \right] dx.
\]

Therefore, from (3.14) and (3.15), it follows that
\[
\lim_{t \to \infty} \mathbb{E}[|\xi_t((u, \infty)) - \tilde{\xi}_t((u, \infty))|] = 0. \quad (3.17)
\]

Together with (3.16) and a similar computation for the half-closed intervals on the right-hand side of (3.16), this concludes the proof. \( \square \)

**Proof of Corollary 3.6** Let \( u \in \mathbb{R} \) be fixed. By Markov’s inequality we have for \( u_0 > u \) that
\[
\mathbb{P}(\xi_t((u, u_0)) > 0) \leq \mathbb{P}(\xi_t((u, \infty)) > 0) = \mathbb{P}(\max_{x \in \eta_t \cap W} t\mu(B(x, 2c(x, \eta_t))) - \log(t) > u) \\
\leq \mathbb{P}(\xi_t((u, u_0)) > 0) + \mathbb{E}[\xi_t([u_0, \infty))].
\]

Thus, Theorem 3.4 and 3.8 yield
\[
\lim_{t \to \infty} \sup \mathbb{P}(\max_{x \in \eta_t \cap W} t\mu(B(x, 2c(x, \eta_t))) - \log(t) > u) - 1 + e^{-\mathbb{M}(u,u_0)} \leq e^{-u_0}.
\]

Then, letting \( u_0 \to \infty \) leads to (3.4). Since, for \( u > 0 \),
\[
|\mathbb{P}(\xi_t((u, \infty)) > 0) - \mathbb{P}(\tilde{\xi}_t((u, \infty)) > 0)| \leq \mathbb{E}[|\xi_t((u, \infty)) - \tilde{\xi}_t((u, \infty))|],
\]
(3.4) and (3.17) imply (3.5). \( \square \)

### 3.3 Circumscribed radii of an inhomogeneous Poisson Voronoi tessellation

In this last section, we consider the circumscribed radii of an inhomogeneous Voronoi tessellation generated by a Poisson process with a certain intensity measure \( t\mu, t > 0 \); recall that the circumscribed radius of a cell is the smallest radius for which the ball centered at the nucleus contains the cell. We study the point process on the non-negative real line constructed by taking for any cell with the nucleus in a compact convex set, a transform of the \( \mu \)-measure of the ball centered at the nucleus and with the circumscribed radius as the radius. The aim is to continue the work started in [8] by extending the result
on the smallest circumscribed radius to inhomogeneous Poisson-Voronoi tessellations and by proving weak convergence of the aforementioned point process to a Poisson process.

More precisely, let $\mu$ be an absolutely continuous measure on $\mathbb{R}^d$ with continuous density $f : \mathbb{R}^d \to [0, \infty)$. Consider a Poisson process $\eta$ with intensity measure $t\mu$, $t > 0$. The circumscribed radius of the Voronoi cell $N(x, \eta)$ with $x \in \eta$ is given by

$$C(x, \eta) = \inf \{ R \geq 0 : B(x, R) \supset N(x, \eta) \},$$

with the convention $\inf \emptyset = \infty$; see Section 3.2 for more details on Voronoi tessellations.

Let $W \subset \mathbb{R}^d$ be a compact convex set with $f > 0$ on $W$. We consider the point process

$$\xi_t = \sum_{x \in \eta_t \cap W} \delta_{s_2 t^{(d+2)/(d+1)} \mu(B(x, C(x, \eta_t)))},$$

(3.18)

Here the positive constant $\alpha_2$ is given by

$$\alpha_2 = \left( \frac{2d(d+1)}{(d+1)!} p_{d+1} \right)^{1/(d+1)},$$

with

$$p_{d+1} := \mathbb{P}\left( N(0, \sum_{j=1}^{d+1} \delta_{Y_j}) \subseteq B(0, 1) \right),$$

where $Y_1, \ldots, Y_{d+1}$ are independent and uniformly distributed random points in $B(0, 2)$.

We write $M$ for the measure on $[0, \infty)$ satisfying $M([0, u]) = \mu(W) u^{d+1}$ for $u \geq 0$.

**Theorem 3.8.** Let $\xi_t$, $t > 0$, be the family of point processes on $[0, \infty)$ given by (3.18). Then $\xi_t$ converges in distribution to a Poisson process with intensity measure $M$.

An immediate consequence of this theorem is that a transform of the minimal $\mu$-measure of the balls, having circumscribed radii and nuclei of the Voronoi cells as radii and centers respectively, converges to a Weibull distributed random variable. This generalizes [8, Theorem 1, Equation (2d)]. For the situation that, in contrast to Theorem 3.8, the density of the intensity measure of the underlying Poisson process is not continuous, we can still derive some upper and lower bounds.

**Theorem 3.9.** Let $\zeta_t$ be a Poisson process with intensity measure $t \vartheta$, where $t > 0$ and $\vartheta$ is an absolutely continuous measure on $\mathbb{R}^d$ with density $\phi$. Let $f_1, f_2 : \mathbb{R}^d \to [0, \infty)$ be continuous and $f_1, f_2 > 0$ on $W$.

(i) If there exists $s \in (0, 1]$ such that $s \phi \leq f_1 \leq \phi$, then

$$\limsup_{t \to \infty} \mathbb{P}\left( s \alpha_2 t^{(d+2)/(d+1)} \min_{x \in \zeta_t \cap W} \vartheta(B(x, C(x, \zeta_t))) > u \right) \leq \exp \left( -s \vartheta(W) u^{d+1} \right)$$

for $u \geq 0$.

(ii) If there exists $r \geq 1$ such that $\phi \leq f_2 \leq r \phi$, then

$$\liminf_{t \to \infty} \mathbb{P}\left( r \alpha_2 t^{(d+2)/(d+1)} \min_{x \in \zeta_t \cap W} \vartheta(B(x, C(x, \zeta_t))) > u \right) \geq \exp \left( -r \vartheta(W) u^{d+1} \right)$$

for $u \geq 0$.  

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Let us now prepare the proof of Theorem 3.8. We first have to study the distribution of $C(x, \eta_t + \delta_x)$, which is defined as the circumscribed radius of the Voronoi cell with nucleus $x \in \mathbb{R}^d$ generated by $\eta_t + \delta_x$. To this end, we define $g : W \times T \to [0, \infty)$ by the equation

$$\mu(B(x, g(x, u))) = u$$

for $T := [0, \mu(W)]$. Since $W$ is compact and convex and $f > 0$ on $W$, we have that (3.19) admits a unique solution $g(x, u)$ for all $(x, u) \in W \times T$. As this is the only place where we use the convexity of $W$, we believe that one can omit this assumption. However, we refrained from doing so in order to not further increase the complexity of the proof. Set

$$s_t = \alpha_2 t^{(d+2)/(d+1)}.$$

Thus, we may write

$$P(s_t \mu(B(x, C(x, \eta_t + \delta_x)))) \leq u$$

for all $u \in T$, $g(\cdot, u) : W \to \mathbb{R}$ is continuous and

$$\lim_{u \to 0^+} \sup_{x \in W} |g(x, u)| = 0.$$ 

**Lemma 3.10.** For any $u \in T$, $g(\cdot, u) : W \to \mathbb{R}$ is continuous and

$$\lim_{u \to 0^+} \sup_{x \in W} |g(x, u)| = 0.$$ 

**Proof.** First we show that $g(\cdot, u)$ is continuous for any fixed $u \in T$. For $u = 0$, we obtain $g(x, u) = 0$ for all $x \in W$. Assume $u > 0$ and let $x_0 \in W$ and $\varepsilon > 0$. Then for all $x \in B(x_0, \varepsilon')$ with $\varepsilon' := \min\{g(x_0, u)/2, \varepsilon\}$, we have that

$$B(x_0, g(x_0, u) + \varepsilon') \subset B(x, g(x_0, u)) \quad \text{and} \quad B(x, g(x_0, u) - \varepsilon') \subset B(x_0, g(x_0, u)).$$

Together with (3.19), this leads to

$$\mu(B(x, g(x_0, u) + \varepsilon')) \geq u \quad \text{and} \quad \mu(B(x, g(x_0, u) - \varepsilon')) \leq u.$$ 

Now it follows from the definition of $g$ that

$$g(x, u) \leq g(x_0, u) + \varepsilon' \quad \text{and} \quad g(x, u) \geq g(x_0, u) - \varepsilon'.$$

This yields

$$|g(x, u) - g(x_0, u)| \leq \varepsilon' \leq \varepsilon$$

for all $x \in B(x_0, \varepsilon')$ so that $g$ is continuous at $x_0$. In conclusion since

$$\lim_{u \to 0^+} g(x, u) = 0$$

and $g(x, u_1) < g(x, u_2)$ for all $x \in W$ and $0 \leq u_1 < u_2$, Dini’s theorem implies that $\sup_{x \in W} |g(x, u)| \to 0$ as $u \to 0$. 

We define

$$\beta = \min_{x \in W} f(x) > 0.$$
Lemma 3.11. There exists $u_0 \in T$ such that
\[ g(x,u) \leq \left(\frac{2u}{\beta k_d}\right)^{1/d} \] (3.21)
for all $x \in W$.

Proof. Since $f$ is continuous and $f > 0$ on $W$, it follows that
\[ \min_{x \in W + B(0,\delta)} f(x) > \frac{\beta}{2} \]
for some $\delta > 0$. Furthermore, by Lemma 3.10 we obtain that there exists $u_0 \in T$ such that $g(x,u) \leq \delta$ for all $u \in [0,u_0]$ and $x \in W$. Then, we obtain
\[ u = \mu(B(x,g(x,u))) = \int_{B(x,g(x,u))} f(y)dy \geq \frac{\beta}{2}k dg(x,u)^d \]
for all $x \in W$ and $u \in [0,u_0]$, which shows (3.21).

For $x \in W$ and $u \geq 0$, we consider a sequence of independent and identically distributed random points $(X_i^{(x,u)})_{i \in \mathbb{N}}$ in $\mathbb{R}^d$ with distribution
\[ \mathbb{P}(X_i^{(x,u)} \in E) = \frac{\mu(B(x,2u) \cap E)}{\mu(B(x,2u))}, \quad i \in \mathbb{N}, E \in \mathcal{B}(\mathbb{R}^d). \]
Recall that, for $k \geq d + 1$, $N\left(x, \sum_{j=1}^{k} \delta_{X_j^{(x,u)}}\right)$ denotes the Voronoi cell with nucleus $x$ generated by $X_1^{(x,u)}, \ldots, X_k^{(x,u)}$ and $x$. Then the distribution function of $C(x, \eta_t + \delta_x)$ is equal to
\[ \mathbb{P}(C(x, \eta_t + \delta_x) \leq u) = \sum_{k=d+1}^{\infty} \mathbb{P}(\eta_t(B(x,2u)) = k)p_k(x,u) \] (3.22)
for $u \geq 0$, with $p_k(x,u)$ defined as
\[ p_k(x,u) = \mathbb{P}\left(N\left(x, \sum_{j=1}^{k} \delta_{X_j^{(x,u)}}\right) \subseteq B(x,u)\right). \]
Combining (3.20) and (3.22) establishes for $u/s_t \in T$ that
\[ \mathbb{P}(s_t \mu(B(x,C(x, \eta_t + \delta_x)))) \leq u) \]
\[ = \sum_{k=d+1}^{\infty} \mathbb{P}(\eta_t(B(x,2g(x,u/s_t))) = k)p_k(x,g(x,u/s_t)). \] (3.23)
For $k \in \mathbb{N}$ with $k \geq d + 1$, we define the probability
\[ p_k = \mathbb{P}\left(N\left(0, \sum_{j=1}^{k} \delta_{Y_j}\right) \subseteq B(0,1)\right), \]
where $Y_1, \ldots, Y_k$ are independent and uniformly distributed random points in $B(0,2)$. As discussed in [8, Section 3], one can reinterpret $p_k$ as the probability to cover the unit sphere with $k$ independent spherical caps with random radii. In the next lemma, we prove that $p_k(x,r) \to p_k$ as $r \to 0$ for all $x \in W$ and $k \geq d + 1$, which together with Lemma 3.11 yields $p_k(x,g(u/s_t)) \to p_k$ as $t \to \infty$. 

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Lemma 3.12. For any $k \geq d + 1$ and $x \in W$,

$$\lim_{r \to 0^+} p_k(x, r) = p_k.$$ 

Proof. In this proof, to simplify the notation, for any $x \in W$, $k \geq d + 1$ and $y_1, \ldots, y_k \in \mathbb{R}^d$, we denote by $K_k^{(x)}(y_1, \ldots, y_k)$ the Voronoi cell $N(x, \sum_{j=1}^{k} \delta_y)$ with nucleus $x$ generated by $y_1, \ldots, y_k$ and $x$. Thus, we may write

$$p_k(x, r) = \mathbb{P}(K_k^{(x)}(X_1^{(x,r)}, \ldots, X_k^{(x,r)}) \subseteq B(x, r)),$$

and so from the independence of $X_1^{(x,r)}, \ldots, X_k^{(x,r)}$ it follows that

$$p_k(x, r) = \frac{1}{\mu(B(x, 2r))^k} \int_{B(x, 2r)^k} 1\{K_k^{(x)}(z_1, \ldots, z_k) \subseteq B(x, r)\} \prod_{i=1}^{k} f(z_i) \, dz_1 \ldots dz_k$$

$$= \frac{(2r)^kd}{\mu(B(x, 2r))^k} \int_{B(0,1)^k} 1\{K_k^{(x)}(x + 2rz_1, \ldots, x + 2rz_k) \subseteq B(x, r)\}$$

$$\times \prod_{i=1}^{k} f(x + 2rz_i) \, dz_1 \ldots dz_k.$$

Furthermore, by the definition of $K_k^{(x)}$ we deduce that

$$1\{K_k^{(x)}(x + 2rz_1, \ldots, x + 2rz_k) \subseteq B(x, r)\} = 1\{K_k^{(0)}(2z_1, \ldots, 2z_k) \subseteq B(0, 1)\}$$

for all $z_1, \ldots, z_k \in B(0, 1)$, whence

$$p_k(x, r) = \frac{(2r)^kd}{\mu(B(x, 2r))^k} \int_{B(0,1)^k} 1\{K_k^{(0)}(2z_1, \ldots, 2z_k) \subseteq B(0, 1)\}$$

$$\times \prod_{i=1}^{k} f(x + 2rz_i) \, dz_1 \ldots dz_k.$$

Using the dominated convergence theorem for the integral, the continuity of $f$ and

$$\lim_{t \to \infty} \frac{(2r)^kd}{\mu(B(x, 2r))^k} = \frac{1}{k^d f(x)^k},$$

we obtain

$$\lim_{t \to \infty} p_k(x, r) = \frac{1}{k^d} \int_{B(0,1)^k} 1\{K_k^{(0)}(2z_1, \ldots, 2z_k) \subseteq B(0, 1)\} \, dz_1 \ldots dz_k$$

$$= \frac{1}{(2k^d)^k} \int_{B(0,2)^k} 1\{K_k^{(0)}(z_1, \ldots, z_k) \subseteq B(0, 1)\} \, dz_1 \ldots dz_k = p_k,$$

which concludes the proof. \qed
Let $M_t$ be the intensity measure of $\xi_t$ and let

$$
\widehat{M}_t([0,u]) = t \int_W \mathbb{E} \left[ 1 \{ s_t \mu(B(x,C(x,\eta_t + \delta_x))) \in [0,u] \} \times 1 \{ \eta_t \left( B \left( x, 4 \left( \frac{2u}{\beta s_t k_d} \right)^{1/d} \right) \right) = d + 1 \} \right] f(x) dx
$$

and

$$
\theta_t([0,u]) = t \int_W \mathbb{E} \left[ 1 \{ s_t \mu(B(x,C(x,\eta_t + \delta_x))) \in [0,u] \} \times 1 \{ \eta_t \left( B \left( x, 4 \left( \frac{2u}{\beta s_t k_d} \right)^{1/d} \right) \right) > d + 1 \} \right] f(x) dx
$$

for $u \geq 0$. Observe that

$$
M_t([0,u]) = \widehat{M}_t([0,u]) + \theta_t([0,u]), \quad u \geq 0.
$$

Lemma 3.13. For any $u \geq 0$,

$$
\lim_{t \to \infty} \frac{\widehat{M}_t([0,u])}{M_t([0,u])} = \mu(W) u^{d+1}
$$

and

$$
\lim_{t \to \infty} \frac{\theta_t([0,u])}{M_t([0,u])} \leq \frac{t \int_W \mathbb{P} \left( \eta_t \left( B \left( x, 4 \left( \frac{2u}{\beta s_t k_d} \right)^{1/d} \right) \right) > d + 1 \right) f(x) dx}{\mu(W)} \to 0 \quad \text{as} \quad t \to \infty.
$$

Proof. Let $u \geq 0$ be fixed and $u_t := u/s_t$. Without loss of generality we may assume $u_t \in T$. For $x \in W$ we deduce from (3.19), $g(x,u_t) \to 0$ as $t \to \infty$ and the continuity of $f$ that

$$
\lim_{t \to \infty} \frac{\mu(B(x,2g(x,u_t)))}{u_t} = \lim_{t \to \infty} \frac{\mu(B(x,2g(x,u_t)))}{2^d k_d g(x,u_t)^d} \frac{2^d}{\mu(B(x,g(x,u_t)))} = \frac{2^d f(x)}{f(x)} = 2^d.
$$

Together with $u_t = u/s_t$ and $s_t = \alpha_2 t^{(d+2)/(d+1)}$, this leads to

$$
\lim_{t \to \infty} t^{d+2} \mu(B(x,2g(x,u_t)))^{d+1} = (2^d u/\alpha_2)^{d+1}.
$$

Similarly, we obtain from Lemma 3.11 that for $t$ sufficiently large,

$$
\sup_{x \in W} t^{d+2} \mu(B(x,2g(x,u_t)))^{d+1} \leq t^{d+2} \left( 2^d u/\beta \right)^{d+1} \sup_{x \in W} \sup_{y \in B(x,2g(x,u_t))} f(y)^{d+1} \leq (2^d u/(\alpha_2 \beta))^{d+1} \sup_{y \in W+B(0,1)} f(y)^{d+1}.
$$

Let us now compute the limit of $\widehat{M}_t([0,u])$. By Lemma 3.11 we obtain for $\ell_t := 4 \left( \frac{2u}{\beta k_d} \right)^{1/d}$ that there exists $t_0 > 0$ such that $2g(x,u_t) \leq \ell_t$ for all $t > t_0$ and $x \in W$. From (3.23)
we deduce for $x \in W$ that $s_t \mu(B(x, C(x, \eta_t + \delta_x))) \in [0, u]$ only if there are at least $d + 1$ points of $\eta_t$ in $B(x, 2g(x, u_t))$. Then for $t > t_0$, we have

$$
M_t([0, u]) = t \int_W \mathbb{P}(\eta_t(B(x, 2g(x, u_t))) = d + 1)p_{d+1}(x, g(x, u_t))
\times \mathbb{P}(\eta_t(B(x, \ell_t) \setminus B(x, 2g(x, u_t))) = 0)f(x)dx
= \int_W \frac{t^{d+2} \mu(B(x, 2g(x, u_t)))}{(d + 1)!}e^{-t \mu(B(x, \ell_t))}p_{d+1}(x, g(x, u_t))f(x)dx.
$$

Elementary arguments imply that

$$
\lim_{t \to \infty} t \mu(B(x, \ell_t)) = 0.
$$

Therefore combining (3.25) and Lemma 3.12 yields

$$
\lim_{t \to \infty} \frac{t^{d+2} \mu(B(x, 2g(x, u_t)))}{(d + 1)!}e^{-t \mu(B(x, \ell_t))}p_{d+1}(x, g(x, u_t))f(x)
= \left( \frac{2^d u}{\alpha_2} \right)^{d+1} \frac{p_{d+1}}{(d + 1)!}f(x) = u^{d+1}f(x),
$$

where we used $\alpha_2 = \left( \frac{2^d (d + 1)!}{(d + 1)^{d+1}} \right)^{1/(d+1)}$ in the last step. Thus, by (3.26) and the dominated convergence theorem we obtain

$$
\lim_{t \to \infty} M_t([0, u]) = u^{d+1} \int_W f(x)dx = \mu(W)u^{d+1}.
$$

Finally, let us compute the limit of $\theta_t([0, u])$. For a Poisson distributed random variable $Z$ with parameter $v > 0$ we have

$$
\mathbb{P}(Z \geq d + 2) = \sum_{k=d+2}^{\infty} \frac{v^k}{k!}e^{-v} \leq v^{d+2} \sum_{k=0}^{\infty} \frac{v^k}{k!}e^{-v} = v^{d+2}.
$$

This implies that

$$
\theta_t([0, u]) \leq t \int_W \mathbb{P}\left( \eta_t\left( B\left( x, 4\left( \frac{2u_t}{\beta_k} \right)^{1/d} \right) \right) > d + 1 \right)f(x)dx
\leq t^{d+3} \int_W \mu\left( B\left( x, 4\left( \frac{2u_t}{\beta_k} \right)^{1/d} \right) \right)^{d+2}f(x)dx
\leq \sup_{y \in W + B\left( 0.4\left( \frac{2u_t}{\beta_k} \right)^{1/d} \right)} f(y) \int_W f(x)dx \frac{2^{2d+5d+2}}{\beta^{d+2}}t^{d+3}u_t^{d+2}
= \sup_{y \in W + B\left( 0.4\left( \frac{2u_t}{\beta_k} \right)^{1/d} \right)} f(y) \mu(W) \frac{2^{2d+5d+2}}{\beta^{d+2}} \frac{1}{\alpha_2^{d+2}} t^{-\frac{1}{d+2}} u^{d+2}.
$$

Here, the supremum converges to a constant as $t \to \infty$ so that the second inequality in the assertion is proven. \qed
In the next lemma, we show a technical result, which will be needed in the proof of Theorem 3.8. For \( A \subset \mathbb{R}^d \), let \( \text{conv}(A) \) denote the convex hull of \( A \).

**Lemma 3.14.** Let \( x_0, \ldots, x_{d+1} \in \mathbb{R}^d \) be in general position (i.e., no \( k \)-dimensional affine subspace of \( \mathbb{R}^d \) with \( k \in \{0, \ldots, d-1\} \) contains more than \( k+1 \) of the points) and assume that \( N(x_0, \sum_{j=0}^{d+1} \delta_{x_j}) \) is bounded. Then,

a) \( x_0 \in \text{int}(\text{conv}(\{x_1, \ldots, x_{d+1}\})) \);

b) \( N(x_i, \sum_{j=0}^{d+1} \delta_{x_j}) \) is unbounded for any \( i \in \{1, \ldots, d+1\} \).

**Proof.** Assume that \( x_0 \notin \text{int}(\text{conv}(\{x_1, \ldots, x_{d+1}\})) \). By the hyperplane separation theorem for convex sets there exists a hyperplane through \( x_0 \) with a normal vector \( u \in \mathbb{R}^d \) such that \( \langle u, x_i \rangle \leq \langle u, x_0 \rangle \) for all \( i \in \{1, \ldots, d+1\} \), where \( \langle \cdot, \cdot \rangle \) denotes the standard scalar product on \( \mathbb{R}^d \). Define the set \( R = \{x_0 + ru : r \in [0, \infty)\} \). For any \( y \in R \), \( x_0 \) is the closest point to \( y \) out of \( \{x_0, \ldots, x_{d+1}\} \), whence \( R \subset N(x_0, \sum_{j=0}^{d+1} \delta_{x_j}) \) and \( N(x_0, \sum_{j=0}^{d+1} \delta_{x_j}) \) is unbounded. This gives us a contradiction and, thus, proves part a).

Let \( i \in \{1, \ldots, d+1\} \) and assume that \( N(x_i, \sum_{j=0}^{d+1} \delta_{x_j}) \) is bounded. It follows from part a) that \( x_i \in \text{int}(\text{conv}(\{x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{d+1}\})) \). On the other hand, again by part a), we have that \( x_0 \in \text{int}(\text{conv}(\{x_1, \ldots, x_{d+1}\})) \). This implies that

\[
\text{conv}(\{x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{d+1}\}) = \text{conv}(\{x_0, \ldots, x_{d+1}\}) = \text{conv}(\{x_1, \ldots, x_{d+1}\}),
\]

and, thus, either \( x_i, x_0 \in \text{conv}(\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{d+1}\}) \) or \( x_i = x_0 \). This gives us a contradiction and concludes the proof of part b).

Finally, we are in position to prove the main result of this section.

**Proof of Theorem 3.8.** From Lemma 3.13 and (3.24) we deduce that \( M_t(B) \to M(B) \) as \( t \to \infty \) for all \( B \in \mathcal{I} \). Then, by Theorem 1.4 it is sufficient to show

\[
\lim_{t \to \infty} t \int_W \mathbb{E} \left[ 1 \{ s_{t\mu}(B(x, C(x, \eta_t + \delta_x))) \in B \} \times 1 \left\{ \sum_{y \in \eta_t \cap W} \delta_{s_{t\mu}(B(y, C(y, \eta_t + \delta_x)))}(B) = m \right\} f(x) dx - M(B) \mathbb{P}(\xi_t(B) = m) = 0
\]

for \( m \in \mathbb{N}_0 \) and \( B \in \mathcal{I} \). Put \( \overline{\mu} = \sup(B) \) and let \( \ell_t = 4 \left( \frac{2\pi}{\beta_n k_d} \right)^{1/d} \). We write

\[
t \int_W \mathbb{E} \left[ 1 \{ s_{t\mu}(B(x, C(x, \eta_t + \delta_x))) \in B \} \left\{ \sum_{y \in \eta_t \cap W} \delta_{s_{t\mu}(B(y, C(y, \eta_t + \delta_x)))}(B) = m \right\} f(x) dx
\]

\[
= t \int_W \mathbb{E} \left[ 1 \{ s_{t\mu}(B(x, C(x, \eta_t + \delta_x))) \in B \} \{ \eta_t(B(x, \ell_t)) = d + 1 \}
\times 1 \left\{ \sum_{y \in \eta_t \cap W} \delta_{s_{t\mu}(B(y, C(y, \eta_t + \delta_x)))}(B) = m \right\} f(x) dx
\]

\[
+ t \int_W \mathbb{E} \left[ 1 \{ s_{t\mu}(B(x, C(x, \eta_t + \delta_x))) \in B \} \{ \eta_t(B(x, \ell_t)) > d + 1 \}
\times 1 \left\{ \sum_{y \in \eta_t \cap W} \delta_{s_{t\mu}(B(y, C(y, \eta_t + \delta_x)))}(B) = m \right\} f(x) dx
\]

=: A_t + R_t.
By Lemma 3.13 we obtain $R_t \to 0$ as $t \to \infty$. Let us study $A_t$. From Lemma 3.11 it follows that there exists $t_0 > 0$ such that $\overline{\eta}/s_t \in T$ and $\ell_t \geq 4g(y, \overline{\eta}/s_t)$ for all $y \in W$ and $t > t_0$. Assume $t > t_0$. In case there are only $d+1$ points of $\eta_t$ in $B(x, \ell_t)$, we deduce that $s_t \mu(B(x, C(x, \eta_t + \delta_x))) \in B$ only if the $d + 1$ points belong to $B(x, 2g(x, \overline{\eta}/s_t))$. Then, by $\ell_t \geq 4g(x, \overline{\eta}/s_t)$ we obtain

$$A_t = t \int_{W} \mathbb{E} \left[ 1 \{ s_t \mu(B(x, C(x, \eta_t + \delta_x))) \in B \} \times 1 \{ \eta_t(B(x, \ell_t) \setminus B(x, \ell_t/2)) = 0, \eta_t(B(x, \ell_t/2)) = d + 1 \} \times 1 \left\{ \sum_{y \in \eta_t \cap W} \delta_{s_t \mu(B(y, C(y, \eta_t + \delta_x)))}(B) = m \right\} f(x) dx. \]$$

(3.27)

Furthermore, since $\ell_t \geq 4g(y, \overline{\eta}/s_t)$ for all $y \in W$, we have that $B(y, 2g(y, \overline{\eta}/s_t)) \cap B(x, \ell_t/2) = \emptyset$, $y \in B(x, \ell_t)^c \cap W$.

Now the observation that $s_t \mu(B(y, C(y, \eta_t + \delta_x))) \in B$ if and only if $s_t \mu(B(y, C(y, (\eta_t + \delta_x)|_{B(y, 2g(y, \overline{\eta}/s_t))}))) \in B$ for $y \in \eta_t$ establishes that

$$A_t = t \int_{W} \mathbb{E} \left[ 1 \{ s_t \mu(B(x, C(x, \eta_t + \delta_x))) \in B \} \times 1 \{ \eta_t(B(x, \ell_t) \setminus B(x, \ell_t/2)) = 0, \eta_t(B(x, \ell_t/2)) = d + 1 \} \times 1 \left\{ \xi_t(\eta_t|_{B(x, \ell_t)^c})(B) + \sum_{y \in \eta_t \cap B(x, \ell_t/2) \cap W} \delta_{s_t \mu(B(y, C(y, \eta_t + \delta_x)))}(B) = m \right\} f(x) dx. \]$$

Suppose that $s_t \mu(B(x, C(x, \eta_t + \delta_x))) \in B$ and that there are exactly $d + 1$ points $y_1, \ldots, y_{d+1}$ of $\eta_t$ in $B(x, \ell_t/2)$ and $\eta_t \cap B(x, \ell_t) \cap B(x, \ell_t/2)^c = \emptyset$. From Lemma 3.14 it follows that $x \in \text{int}(\text{conv}\{\{y_1, \ldots, y_{d+1}\}\})$ and that the Voronoi cells $N(y_i, \eta_t|_{B(x, \ell_t)} + \delta_x), i = 1, \ldots, d + 1$, are unbounded. In particular, we have $C(y_i, \eta_t + \delta_x) > \ell_t/4 > g(y_i, \overline{\eta}/s_t), \quad i = 1, \ldots, d + 1$.

Together with the same arguments used to show (3.27), this implies that

$$A_t = t \int_{W} \mathbb{E} \left[ 1 \{ s_t \mu(B(x, C(x, \eta_t + \delta_x))) \in B \} \times 1 \{ \eta_t(B(x, \ell_t) \setminus B(x, \ell_t/2)) = 0, \eta_t(B(x, \ell_t/2)) = d + 1 \} \times 1 \left\{ \xi_t(\eta_t|_{B(x, \ell_t)^c})(B) = m \right\} f(x) dx \right. \left. - \mathbb{P}(\xi_t(\eta_t|_{B(x, \ell_t)^c})(B) = m) f(x) dx \right].$$

Furthermore, we obtain

$$\mathbb{P}(\xi_t(\eta_t|_{B(x, \ell_t)^c})(B) = m) - \mathbb{P}(\xi_t(B) = m) \leq \mathbb{P}(\eta_t(B(x, \ell_t)) > 0) < t \mu(B(x, \ell_t))$$
for any $x \in W$, where we used the Markov inequality in the last step. Combining the
previous formulas leads to

$$|A_t - M(B)\mathbb{P}(\xi_t(B) = m)|$$

$$\leq |M_t(B) - M(B)|$$

$$+ t \int_W \mathbb{E}\left[1\{s_t \mu(B(x, C(x, \eta_t + \delta_z))) \in B\}1\{\eta_t(B(x, \ell_t)) > d + 1\}\right] f(x)dx$$

$$+ \int_W \mathbb{E}\left[1\{s_t \mu(B(x, C(x, \eta_t + \delta_z))) \in B\}1\{\eta_t(B(x, \ell_t)) = d + 1\}\right]$$

$$\times \left|\mathbb{P}(\xi_t(\eta_t|B(x, \ell_t)) = B) = m) - \mathbb{P}(\xi_t(B) = m)\right| f(x)dx$$

$$\leq |M_t(B) - M(B)| + t \int_W \mathbb{P}(\eta_t(B(x, \ell_t)) > d + 1)f(x)dx + \tilde{M}_t([0, \tau]) \sup_{x \in W} t \mu(B(x, \ell_t))\].$$

It follows from Lemma 3.13 that, as $t \to \infty$, $\tilde{M}_t([0, \tau]) \to M([0, \tau])$, $M_t(B) \to M(B)$ and
the integral on the right-hand side vanishes. Without loss of generality we may assume $\ell_t \leq 1$, and thus the continuity of $f$ on $W + \overline{B(0, 1)}$ implies that

$$t \mu(B(x, \ell_t)) \leq k_d \max_{z \in W + \overline{B(0, 1)}} f(z) t \ell_t^d$$

for all $x \in W$. Now $\ell_t = 4\left(\frac{\sigma_t}{\beta_t \ell_t}\right)^{1/d}$ and $s_t = \alpha_2 t^{(d+2)/(d+1)}$ yield that the right-hand side vanishes as $t \to \infty$. Thus, we obtain

$$\lim_{t \to \infty} A_t - M(B)\mathbb{P}(\xi_t(B) = m) = 0,$$

which together with $R_t \to 0$ as $t \to \infty$ concludes the proof.

Proof of Theorem 6.16. Let $\gamma$ be a Poisson process on $\mathbb{R}^d \times [0, \infty)$ with the restriction of the
Lebesgue measure as intensity measure. Let $\mu_1$ and $\mu_2$ denote the absolutely continuous measures with densities $f_1$ and $f_2$, respectively. Then, [21, Corollary 5.9 and Proposition
6.16] imply that

$$\varrho_t^{(1)} = \sum_{(x, y) \in \gamma} 1\{y \leq tf_1(x)\} \delta_x, \quad \varrho_t^{(2)} = \sum_{(x, y) \in \gamma} 1\{y \leq tf_2(x)\} \delta_x$$

and

$$\varrho_t = \sum_{(x, y) \in \gamma} 1\{y \leq t\phi(x)\} \delta_x$$

are Poisson processes on $\mathbb{R}^d$ with intensity measures $t \mu_1, t \mu_2$ and $t \phi$, respectively. They satisfy

$$\varrho_t^{(1)}(A) \leq \varrho_t(A) \leq \varrho_t^{(2)}(A) \text{ a.s. and } \varrho_t^d = \zeta_t, \quad A \subset \mathbb{R}^d, t > 0.$$  

Therefore for any $v > 0$, we obtain

$$\mathbb{P}\left(\min_{x \in \gamma \cap W} 1\{B(x, C(x, \zeta_t)) > v\}\right) \leq \mathbb{P}\left(\min_{x \in \gamma^{(1)} \cap W} \mu_1(B(x, C(x, \varrho_t^{(1)}))) > v\right)$$

(3.28)
and similarly
\[ \mathbb{P}\left( \min_{x \in \xi \cap W} \mu_2(B(x, C(x, \zeta_t))) > v \right) \geq \mathbb{P}\left( \min_{x \in \xi \cap W} \mu_2(B(x, C(x, \tilde{q}_t^{(2)}))) > v \right). \] (3.29)

From Theorem 3.8, it follows for \( j = 1, 2 \) and \( \nu(t) = u(\alpha t(t+d+2)/(d+1))^{-1} \) with \( u \geq 0 \), that
\[ \lim_{t \to \infty} \mathbb{P}\left( \min_{x \in \xi \cap W} \mu_j(B(x, C(x, \tilde{q}_t^{(j)}))) > \nu(t) \right) = e^{-\mu_j(W)u^{d+1}}. \]

If \( s \phi \leq f_1 \leq \phi \) for some \( s \in (0, 1] \), combining (3.28), the previous limit with \( j = 1 \), and the inequality
\[ \mathbb{P}\left( \min_{x \in \xi \cap W} s \vartheta(B(x, C(x, \zeta_t))) > \nu(t) \right) \leq \mathbb{P}\left( \min_{x \in \xi \cap W} \mu_1(B(x, C(x, \zeta_t))) > \nu(t) \right) \]
implies that
\[ \limsup_{t \to \infty} \mathbb{P}\left( \min_{x \in \xi \cap W} s \vartheta(B(x, C(x, \zeta_t))) > \nu(t) \right) \leq e^{-\mu_1(W)u^{d+1}}. \]

Then, \( s \vartheta(W) \leq \mu_1(W) \) concludes the proof of (i). Analogously, if \( \phi \leq f_2 \leq r \phi \) for some \( r \geq 1 \), combining (3.29), the limit above with \( j = 2 \), the inequality
\[ \mathbb{P}\left( \min_{x \in \xi \cap W} r \vartheta(B(x, C(x, \zeta_t))) > \nu(t) \right) \geq \mathbb{P}\left( \min_{x \in \xi \cap W} \mu_2(B(x, C(x, \zeta_t))) > \nu(t) \right) \]
and \( \mu_2(W) \leq r \vartheta(W) \) for \( u \geq 0 \) shows (ii).

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