LATTICE \( W \) ALGEBRAS AND QUANTUM GROUPS.\(^+\)

Ya.P. Pugay
Landau Institute for Theoretical Physics
142432 Chernogolovka, Russia

ABSTRACT
We represent Feigin’s construction [22] of lattice \( W \) algebras and give some simple results: lattice Virasoro and \( W_3 \) algebras. For simplest case \( g = sl(2) \) we introduce whole \( U_q(sl(2)) \) quantum group on this lattice. We find simplest two-dimensional module as well as exchange relations and define lattice Virasoro algebra as algebra of invariants of \( U_q(sl(2)) \). Another generalization is connected with lattice integrals of motion as the invariants of quantum affine group \( U_q(\hat{g}) \). We show that Volkov’s scheme leads to the system of difference equations for the function from non-commutative variables.

0. Introduction.
In this talk I would like to give a brief introduction to the Feigin [22] construction of lattice \( W \) algebras and represent some simple results. More complete consideration can be find in the forthcoming work [36].

In 1985 Alexander Zamolodchikov [1] investigated the possibility of existence of new additional infinite symmetries in the context of two-dimensional Conformal Field Theory [4], or, equivalently, the existence of a primary field with conformal dimension \((s,0)\) or \((0,s)\). (Hereafter we consider only holomorphic part.) By direct use of bootstrap principle he proved that there might exist primary field \( W_3 \) with conformal dimensions \((3,0)\). Due to the equation
\[
\partial z W_3(z) = 0,
\]
it is the conserved current which generates additional infinite symmetry, while algebra \((W_3,T)\) (where \( T \) is the stress-energy tensor), is a quadratic one. Namely, operator product expansion of two \( W_3 \) currents includes quadratic term on \( T \). In the past few years considerable progress has been made in an understanding of the deep structures underlying these algebras (see for example refs.[2,5-15,20,21]) as well as its classical limits [16-19]. It was shown in the works [2,6,11,12,14] that \( W \) algebras can be considered as the result of quantum Drinfel’d-Sokolov reduction and the fact that generators of \( W \) algebras commute with screenings operators can be taken as the definition of \( W \) algebras. Namely, such as screening operators constitute the nilpotent part of the quantum group [24], mathematically \( W \) algebra is the algebra of invariants of this group. Lukyanov and Fateev [5-8] found

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that such invariants are given by quantum generalization of Miura transformation. We give the lattice version of this picture. We hope that lattice construction can clear up the intrinsic features of $W$ algebras. Our consideration is somewhat different from a number of another works appeared in connection with lattice current algebras [23-31].

Plan of this talk is following:

1. Feigin’s construction of lattice $W$ algebras.
2. Examples: Virasoro and $W_3$ on the lattice.
3. Generalization: $U_q(sl(2))$.
4. 2D-modules of $U_q(sl(2))$. Exchange algebra. Virasoro algebra.
5. Generalizations: affine case and integrals of motion.

1. Feigin’s construction of lattice $W$ algebras.

1.1 W algebra in continuous theory.

Consider bosonic representation of conformal field theory in terms of free scalar fields $\phi^b$ ($b = 1, \ldots, r$ with expansion

$$T(\phi^a(\zeta)\phi^b(z)) = \frac{\phi^{ab}}{k}\log(\zeta - z) + O(\zeta - z),$$

(1.1.1)

Let $g$ be a simple Lie algebra with simple roots $\alpha_i$ ($i=1,\ldots,r=\text{rank}(g)$) and Cartan decomposition $g = n_- \oplus h \oplus n_+$. Following to Feigin and Frenkel [12] one can give the following definition:

**Definition.** Vacuum representation of $W$ algebra associated with simple Lie algebra $g$ has a realization as the intersection of the kernels of screening operators

$$S_{\alpha_i} = \oint \exp(-\alpha_i\phi) :dz , \quad 1 < i < r.$$  (1.1.2)

We have by this means in the continuous theory

$$Wg \sim \text{Ker}(S_{\alpha_1}) \cap \text{Ker}(S_{\alpha_2}) \cap \text{Ker}(S_{\alpha_3}) \ldots .$$

Lukyanov and Fateev found that bosonic realization of $W$ is given by quantum Miura transformation which in the case of $W sl(n)$ has the following form [5-8]:

$$\partial^n + \sum_{i=1}^{n} W_i(x)\partial^{n-i} =: \prod_{i=1}^{n} (\partial - \tilde{h}_i\partial^2(x)),$$

(1.1.3)

and $\tilde{h}_i = \bar{\omega}_1 - \bar{\alpha}_1 - \ldots - \bar{\alpha}_{i-1}$ are fundamental weights of the $n$-dimensional vector representation of $SL(n)$.

As Bowknekt McCarthy and Pilch have shown, the operators $S_{\alpha_i}$ satisfy to the q-Serre relations and realize the representation of quantum group $U_q(n_+)$ [24]. Therefore, roughly speaking, $W$ algebra is the algebra of invariants of $U_q(n_+)$. 

$$W \sim Inv[U_q(n_+)].$$
It will be our key principle in the lattice construction. Let us note here that finding of integrals of motion in the W-symmetric Conformal Field Theory perturbed by relevant operator [37]

\[ \Phi =: \exp(-\alpha_0 \phi) : \]

where \( \alpha_0 \) is the affine root, reduce in the bosonized picture to the determination of \( \text{Inv}[U_q(\hat{n}_+)] \):

\[ IM \sim \text{Inv}[U_q(\hat{n}_+)] = \text{Ker}(S_{\alpha_0}) \cap \text{Ker}(S_{\alpha_1}) \cap \text{Ker}(S_{\alpha_2}) \cap \text{Ker}(S_{\alpha_3}) \ldots . \]

1.2 Feigin’s construction of lattice \( \mathbf{W} \) algebras. ([22]).

Let us consider an infinite set of points \( x_{-\infty}, \ldots, x_1, x_2, x_3, \ldots, x_\infty \) on the line (fig.1). One can think of \( x_i \) as a vertex operator

\[ V_\alpha =: \exp(\alpha \phi) : \quad (1.2.1) \]

in the point \( i \). Denote by \( \Lambda \) the root lattice of \( g \) endowed by standard scalar product

\[ \langle \alpha_i, \alpha_j \rangle = a_{ij} \quad , \quad (1.2.2) \]

where \( (a_{ij}) \) is a Cartan matrix of \( g \). Define the multigrading on \( x_i \) by the map:

\[ x_i \rightarrow \Lambda \\
\text{deg}(x_i) = \alpha_j, \quad i \in \mathbb{Z}, \quad j = 1, \ldots, r. \quad (1.2.3) \]

Imitating the exchange relations for vertex operators

\[ V_\alpha(z)V_\beta(\zeta) = q^{<\alpha, \beta>}V_\beta(\zeta)V_\alpha(z) \quad , \quad (1.2.4) \]

determine now skew polynomial algebra with basic relations:

\[ x_i x_j = q^{<\text{deg}(i), \text{deg}(j)>} x_j x_i \quad i < j. \quad (1.2.5) \]

Having in our mind to find the analogue of screening operators, put

\[ S_{\alpha_i} = \sum_{\text{deg}(x_j) = \alpha_i} x_j . \quad (1.2.6) \]

One can immediately prove the following lemma:

**Lemma.** Operators \( S_{\alpha_i} \) satisfy to the \( q \)-Serre relations:

\[ (ad^{1-a_{ij}}_{S_{\alpha_i}})_q S_{\alpha_j} = 0 \quad . \quad (1.2.7) \]

These operators \( S_{\alpha_j} \) constitute the \( U_q(n_+) \) algebra, and formulas for comultiplication, antipod and counit are of the form:

\[ \Delta S_{\alpha_j} = S_{\alpha_j} \otimes 1 + q^{h_i} \otimes S_{\alpha_j} \quad , \]
\[ S(S_{\alpha_j}) = -S_{\alpha_j} \quad , \]
\[ \epsilon(S_{\alpha_j}) = 0 \quad , \quad (1.2.8) \]
while the actions of operators $h_s$ will look like

$$h_i(P) = <\deg P, \alpha_i > P,$$
$$h_i h_j = h_j h_i$$
$$S(h_i) = -h_i$$
$$\Delta h_i = h_i \otimes 1 + 1 \otimes h_i$$
$$\epsilon(h_i) = 0.$$  (1.2.9)

Due to the eqs. (1.27)-(1.2.9) and property

$$q^{h_i} S_{\alpha_j} = q^{a_{ij}} S_{\alpha_j} q^{h_i}$$

operators $h_i$ and $S_{\alpha_i}$ constitute the borel part $U_q(b_+)$ of quantum universal enveloping algebra $U_q(g)$.

Consider now the algebra of formal Loran’s series $C[x_i, x_i^{-1}]$ with

$$\deg(x_i^{-1}) = -\deg(x_i)$$  (1.2.10)

In according to the general rule the adjoint action of quantum group $U_q(n_+)$ is determined by q-commutation with screening operator

$$S_{\alpha_i}(P) = S_{\alpha_i} P - q^{<\deg P, \alpha_i>} P S_{\alpha_i}$$  (1.2.11)

We can give the following definition of lattice $W$ algebra:

**Definition.** Generators of lattice $W$ algebra associated with simple Lie algebra $g$ constitute the functional basis of space

$$Inv_{U_q(b_+)}(C[x_i, x_i^{-1}]) .$$

Here we added new requirements:

$$h_i(W) = 0$$  (1.2.12)

to have an scaling invariance $x_i \rightarrow \lambda x_i$ and to satisfy the requirement of finiteness. Another essential argument is that we are looking for local expression for generators: namely, like local fields $W, T$ lattice generators must to commute if they far enough from each other.

So the problem is to find solution of the system of difference equations from infinite number non-commutative variables. It is significant that commutation relations (1.2.5) depend on the sign of the difference $(i - j)$ only. We should try to find all solutions of the system:

$$[S_{\alpha_i}, w_1]_q = 0 \quad i = 1, ..., r,$$
$$h_i(w_1) = <\deg(w_1), \alpha_i > w_1 = 0$$  (1.2.13)

where $w_1 = w_1(x_1, x_2, x_3, ..., x_{k-1}, x_k)$. Then we will obtain the whole set of generators by shift:

$$w_2 = w_1[x_1 \rightarrow x_2, x_2 \rightarrow x_3, x_3 \rightarrow x_4, ...] ,$$
$$w_3 = w_1[x_1 \rightarrow x_3, x_2 \rightarrow x_4, x_3 \rightarrow x_5, ...] ,$$  (1.2.14)

e\text{tc.}
2. Examples: Virasoro and $W_3$ on the lattice.

In this section we turn our attention to the explicit construction of lattice algebras. Let us assume for simplicity that deformation parameter $q$ is in generic position. In this case one can apply to the classical limit $q \to 1$ and reduce rather complicated system (1.2.13) of the difference equations to the ordinary one. Screening operators in this limit turn to be differential operators of first order acting on the manifold with the coordinates $x_i$. One can easily obtain nonstandard realization of universal enveloping algebras $U(b_+)$ and solve the system of ordinary differential equations. Classical solution help us to "guess" the right answer for the quantum case. Moreover, if we consider deformation in the generic position, then it is possible to determine the necessary number of variables to find non-trivial invariant. Indeed, in the classical case we have $dim(b_+)$ constraints and trivial solution $w = const$. So one can expect that nontrivial invariant would depend on $[dim(b_+)+1]$ variables at least. (It is not true for special values of $q$ and non-regular maps (1.2.3))

2.1 Faddeev-Takhtajan-Volkov algebra.

Virasoro algebra is connected with $sl(2)$ algebra. In this case we have the following basic relations:

$$x_i x_j = qx_j x_i , \quad i < j$$

and system of equations is:

$$deg(\sigma) = 0 ,$$

$$\sum x_i, \sigma = 0 .$$

(2.1.2)

As we noted before, in fact, one need to solve the following equation:

$$(x_1 + x_2 + x_3)\sigma(x_1, x_2, x_3) = \sigma(x_1, x_2, x_3)(x_1 + x_2 + x_3)$$

(2.1.3)

and the solution must have zero grading. We have two obvious solutions of this equation:

$$x_1 + x_2 + x_3 ,$$

$$x_1 x_2^{-1} x_3 ,$$

(2.1.4)

and zero-grading invariant is

$$\sigma_1 = (x_1 + x_2 + x_3)x_1^{-1} x_2 x_3^{-1} .$$

(2.1.5)

All other basic generators of lattice Virasoro algebra are obtained by simple shift. This algebra was found from another point of view by Volkov and its classical version was appeared in the work Talhtadjan and Faddeev [25]. At the classical level lattice Virasoro has the following Poisson brackets:

$$\{S_1, S_2\} = S_1 S_2 (1 - S_1 - S_2) ,$$

$$\{S_1, S_3\} = S_1 S_2 S_3 ,$$

$$\{S_1, S_i\} = 0 , \quad |i - 1| > 2 ,$$

and

$$\{S_2, S_3\} = S_1 S_2 S_3.$$
where
\[ S_i = \frac{1}{\sigma_i + 1} \]  
and Poisson brackets of any \( S_i \) are obtained by shift \([1 \to i], [2 \to i + 1], [3 \to i + 2]\) etc. Faddeev and Takhtajan found this Poisson structure by studying of Volterra system:
\[ \dot{S}_i = \{H, S_i\} = S_i(S_{i+1} - S_{i-1}) \]  
where Hamiltonian \( H \) has the form \( H = \sum \ln(S_i) \).

### 2.2 Lattice \( W_3 \) algebra.

Let us consider following example of lattice algebra associated to Lie algebra \( sl(3) \). There are several ways to define the grading. We put regular coloring of the points, exactly as in the fig.2:
\[ \text{deg}(x_{2n}) = \alpha_1 \quad \text{deg}(x_{2n+1}) = \alpha_2 \]  
and the commutation relations:
\[ x_n x_{2n+k} = q x_{2n+k} x_n \quad k > 0 \]  
\[ x_n x_{2n+k+1} = q^{-\frac{1}{2}} x_{2n+k+1} x_n \]  
Applying to the classical limit [35] one can prove that invariants of \( U_q(sl(3))_+ \) has the form
\[ \tau_1 = (x_4 x_5 + x_2 x_5 + x_2 x_3) x_6 x_1 (x_2 x_1 + x_4 x_1 + x_4 x_2)^{-1} \]  
\[ (x_4 x_3 + x_6 x_5 + x_6 x_3)^{-1} \]  
\[ \tau_2 = \tau_1 (x_1 \to x_2, x_2 \to x_3, x_3 \to x_4, \text{ etc.}) \]  
\[ \text{etc.} \]
These functions from non-commutative variables determine the functional basis in the invariant space of \( U_q(sl(3))_+ \).

It is rather cumbersome matter to find an algebra of these invariants but in the classical limit
\[ \{\tau_i, \tau_j\} = \lim_{q \to 1} \frac{1}{1 - q} |\tau_i, \tau_j| \]  
the calculations gives us the following result:
\[ \{\tau_1, \tau_2\} = -\tau_1 \tau_3 \tau_4 \tau_6 \]  
\[ \{\tau_1, \tau_3\} = \tau_1 \tau_5 [\tau_2 \tau_3 + \tau_3 \tau_4 - \tau_3] \]  
\[ \{\tau_1, \tau_4\} = -\tau_1 \tau_4 [\tau_1 \tau_2 + \tau_2 \tau_3 + \tau_2 \tau_4 - \tau_2 - \tau_3] \]  
\[ \{\tau_1, \tau_5\} = \tau_1 \tau_5 [\tau_1 \tau_2 + \tau_2 \tau_3 - \tau_1 - \tau_2 - \tau_3 + 1] \]  
\[ \{\tau_1, \tau_6\} = -\tau_1 \tau_2 [\tau_1 \tau_2 - \tau_1 - \tau_2 + 1] \]  
\[ \{\tau_1, \tau_i\} = 0 \quad |i - 1| > 5 \]  
The remaining Poisson brackets are obtained through these Poisson brackets by shifts. Nevertheless this Poisson structure seems to be unwieldy there exist some diagramm which show some kind of symmetry:
\[ \{\tau_1, \tau_i\} = \tau_1 \tau_i \times \Gamma_i \]  
\[ \text{(2.2.6)} \]
where $\Gamma$ is given by fig.(3-4). (May be it is possible to write down the Poisson brackets for $\mathcal{W}sl(n)$ without explicit knowledge of lattice bosonization?)

Having such a Hamiltonian structure one can define differential-difference chain of nonlinear equations:

$$H = \sum \ln(\tau_i)$$

$$\dot{\tau}_j = \{\tau_j, H\} = \tau_j \times \sum_i \Gamma_i$$  

(2.2.7)

where $\sum \Gamma_i$ denote that we should summarize all terms in the diagramm fig.4. For example we have:

$$\dot{\tau}_1 = \tau_1 \{[-\tau_1 \tau_2 + \tau_2 \tau_3 - \tau_3 \tau_4 + \tau_2 + \tau_3]
- [-\tau_1 \tau_0 + \tau_0 \tau_{-1} - \tau_{-1} \tau_{-2} + \tau_0 + \tau_{-1}]\}$$

(2.2.8)

etc.

Probably it would be interesting to investigate this non-linear chain which seems to be integrable.

3. Generalization: $U_q(sl(2))$.  

In this section we turn our attention to the question what is the role of second part of quantum group. To begin with let us consider some local lattice field $F$ which has in the neighborhood of point 1 the form (fig.5):

$$F(1)_0 = F(x_1, x_2, x_3, ..., x_k) = \sum C_{\{\beta\}} x_{\beta_1} x_{\beta_2} ... x_{\beta_k}.$$  

(3.1)

The worth important observation is that screening operator $S_{\alpha_i}$ acts on local field $F(1)_0$ only by half of infinite sum:

$$S_{\alpha_i} = U_- + \sum_{i=1}^{k} [x_i] + U_+,$$

$$U_- = \sum_{i=-\infty}^{0} x_i,$$

$$U_+ = \sum_{i=k+1}^{\infty} x_i.$$  

(3.2)

Namely,

$$S_{\alpha_i}(F_1) = [x_1 + x_2 + x_3 + ... + x_k, F_1]_q + [U_+, F_1]_q =$$

$$= [x_1 + x_2 + x_3 + ... + x_k, F_1]_q + [1 - q^{2degF_1}] U_+ F_1.$$  

(3.3)

The expressions like $U_+ F_1$ may be considered as the lattice analogues of the fields dressed by screening operator (fig.6-7):

$$\left(\phi : [exp(-\alpha_i \phi) : F dz) \rightarrow (U_+ F_1)\right)$$

(3.4)

To investigate lattice representation of quantum group $sl_2$ if we should extend our space $C[x_i, x_i^{-1}]$ by non-local expressions which are given by the action of screening operators:

$$C[x_i, x_i^{-1}] \otimes U_q(sl_2)_+.$$
To introduce action of $U_q(sl_2)$ on this space, let us define a q-derivation of the "screening variable" $U_+$:

$$D_q \equiv \frac{\partial}{\partial U_+} ,$$

$$D_q(U_+) = 1 ,$$

$$D_q(x_i) = 0 \quad (i \text{ finite}) ,$$

$$D_q(fg) = D_q(f)g + q^{\deg f}fD_q(g) .$$

One can immediately show that operators $(D_q, h, S)$ constitute twisted $sl(2)_q$ algebra:

$$D_q S - qS D_q = 1 - q^{2h} ,$$

$$q^h D_q q^{-h} = q^{-1} D_q ,$$

$$q^h S q^{-h} = qS .$$

(3.5)

One can immediately show that operators $(D_q, h, S)$ constitute twisted $sl(2)_q$ algebra:

After the change

$$D_q \rightarrow \frac{q^{-h} D_q}{1 - q}$$

we will obtain an ordinary quantum group $sl(2)_q$.

4. 2D-module of $U_q(sl(2))$. Exchange algebra.

In previous section we received the realization of quantum group $U_q(sl(2))$ on the lattice. One can try to investigate the representations of quantum group $U_q(sl(2))$ on this lattice.

If we have local field $F_0 = (..., F(1)_0, F(2)_0, ...)$ and modules created from $F$ by the action of screenings then their exchange algebra determined by universal $R$-matrix (Drinfeld) for $U_q(sl(2))$:

$$F(1)_i = [S, [S, ...[S, F(1)_0]] \quad (i - times) ,$$

$$F(2)_j = [S, [S, ...[S, F(2)_0]] \quad (j - times) ,$$

$$F(1)_i F(2)_j = (R)^{kl}_{ij} F(2)_l F(1)_k .$$

(4.1)

Simplest possibility of such a module is two dimensional module. Unfortunately, naive "highest weight" $x^{-\frac{1}{2}}$ isn’t true because it creates reducible module. One can prove, however, that proper expression for “highest weight” in this case is given by the expression

$$F(1)_{-\frac{1}{2}} = x_1^\frac{1}{4} x_2^{-\frac{1}{4}} (x_1 + x_2)^{-\frac{1}{2}} ,$$

(4.2)

while second vector in this module is given by the formula:

$$F(1)_{\frac{1}{2}} = [S, A_{-\frac{1}{2}}]_q = (1 - q)U_+ x_1^\frac{1}{4} x_2^{-\frac{1}{4}} (x_1 + x_2)^{-\frac{1}{2}} ,$$

(4.3)

where

$$U_+ = \sum_3^\infty x_i .$$

(4.4)

By shift $x_1, x_2 \rightarrow x_3, x_4$ we find another module

$$F(2)_{-\frac{1}{2}} = x_3^\frac{1}{4} x_4^{-\frac{1}{4}} (x_3 + x_4)^{-\frac{1}{2}} ,$$

$$F(2)_{\frac{1}{2}} = [S, B_{-\frac{1}{2}}] = (1 - q)[U_+ - x_3 - x_4] x_3^\frac{1}{4} x_4^{-\frac{1}{4}} (x_3 + x_4)^{-\frac{1}{2}} .$$

(4.5)
and $R$ matrix in the representation $\frac{1}{2}, \frac{1}{2}$ has the well-known form. In this picture one can find alternative expression for lattice Virasoro algebra. Indeed, the expressions like

$$\Delta_{13} = F(1)_{-\frac{1}{2}}F(2)_{\frac{1}{2}} - q^{\frac{1}{2}}F(1)_{\frac{1}{2}}F(2)_{-\frac{1}{2}}$$

(4.7)

belong to the invariant space of $U_q(sl(2))$ and therefore have to be invariants. Really, the condition

$$D_q(\Delta_{13}) = 0$$

denotes that this expression is local, while invariance under the action of $S$ and $h$ is the definition of Virasoro algebra (sec.1.2).

Even without knowledge of exchange algebra one can determine the invariants like

$$\Delta_{12} = F(1)_{-\frac{1}{2}}F(1/2)_{\frac{1}{2}} - q^{\frac{1}{2}}F(1)_{\frac{1}{2}}F(1/2)_{-\frac{1}{2}}$$

(4.8)

where (fig.8)

$$F(1/2)_{-\frac{1}{2}} = x_{x_2}^{x_3} (x_2 + x_3)^{-\frac{1}{2}}$$

$$F(1/2)_{\frac{1}{2}} = [S, F(1/2)_{-\frac{1}{2}}] = (1 - q)[U_+ - x_3]x_{x_2}^{x_3} (x_2 + x_3)^{-\frac{1}{2}}$$

(4.9)

It is interesting that 3-point invariant

$$\Sigma = \Delta_{13}^{-1}\Delta_{12} = (x_3 + x_4)^{-\frac{1}{2}} x_{x_4}^{x_3} (x_2 + x_3)^{-\frac{1}{2}}$$

(4.10)

on the classical level coincides with Tahtadjan-Faddeev Virasoro algebra:

$$\Sigma^{-2} = \sigma + 1$$

(4.11)

On the quantum level I have not proved that $\Sigma$ can be expressed through $\sigma$ and vice versa. But I think it is right.

Let us note, that 3-dimensional module is generated by the $S$-operator action on $x^{-1}$. The exchange relation in this case are determined through the quantum $R^{1,1}$ matrix for $U_q(sl_2)$.

Hence we can start from exchange algebra [10] without explicit knowledge of vectors in module and define generators of $W$ algebra as an invariant of $U_q(g)$. The algebra of invariants gives us $W$ lattice algebra.

5. Generalizations: affine case and integrals of motion.

5.1 Lattice integrals of motion [22].

Let now turn our attention to the integrals of motion in perturbed conformal field theories with $W$ symmetry [37]. The integrable perturbation of such a theories is given by the relevant field

$$V_{\alpha_0} = :exp(\alpha_0 \phi) :$$

(5.1.1)
where $\alpha_0$ is the additional (affine) root of the affinization of Lie algebra $g$. As Zamolodchikov proved [33], the determination of integrals of motion in perturbed conformal field theories is reduced to the finding of the kernel of operator

$$S_{\alpha_0} = \oint : \left[ \exp(-\alpha_0 \phi) \right] : dz$$

in the vacuum representation of $W$ algebra. Dealing with bozonized picture we can rewrite this problem [23] to the finding of screening operators kernels intersection

$$S_{\alpha_i} = \oint : \left[ \exp(-\alpha_i \phi) \right] : dz \quad 0 < i < r.$$

which constitute the nilpotent part of quantum affine group $U_q(\hat{g})$. In according to the lattice ideology, we have to define additional screening operator through our non-commutative variables $x_i$. Consider, for example, $\hat{sl}(n)$ case with Cartan matrix $a_{ij}$.

Define the multigrading in this case by the regular map (fig.7):

$$\deg(x_i [\text{mod}(n-1)]) = \alpha_i.$$ (5.1.5)

We have $(n-1)$ ordinary screening operators

$$S_i = \sum_{k=-\infty}^{\infty} x_{i+k(n-1)} \quad i = 1, ..., n - 1.$$ (5.1.6)

Our idea is to construct affine generator as following sum:

$$S_0 = \sum_{k=-\infty}^{\infty} (x_{1+k(n-1)} x_{2+k(n-1)} \cdots x_{n-1+k(n-1)})^{-1}$$ (5.1.7)

which has proper grading and corresponds in the continuous limit to the operator $V_{\alpha_0}$. It is rather simple matter to prove the following lemma:

**Lemma.** Operators $S_i$, $(i = 0, ..., n - 1)$ satisfy to the $q$-Serre relations for the quantum affine group $U_q(\hat{sl}(n))$.

$$(ad_{S_{\alpha_i}}^{1-a_{ij}})_q S_{\alpha_j} = 0,$$ (5.1.8)

where $i \neq j; i, j = 0, ..., n - 1$.

Now operators $S_i$ gives us the representation of the nilpotent part of quantum affine Lie algebra $U_q(\hat{sl}(n)_+)$. Therefore, mathematically, integrals of motion problem is defined by similar way:

**Definition.** Integrals of motion on the lattice constitute the invariant space of nilpotent part of quantum affine Lie algebra:

$$IM \sim \text{Inv}_{U_q(\hat{sl}(n)_+)}(C[x_i, x_i^{-1}]).$$

Such as we have infinitely many generators of quantum affine group $U_q(sl(n)_+)$ (i.e. the number of constraints is infinite) then there is no hope to find local invariants. But we
can try to find the "local density" of "integrals". Namely, such functions, commutator of which with screening operator is given by "total derivation":

\[ [S_i, I(x)] = D_l(P) = P(x, x_{i+1}, x_{i+2}, ..., x_{i+k}) - P(x_{i+l}, x_{i+l+1}, x_{i+l+2}, ..., x_{i+l+k}) \quad l \in Z \]  

(5.1.8)

5.2 Volkov’s scheme.

In this section we will represent Volkov’s method possessing to determine the invariant space of quantum affine group through the solution of some system of difference equation.

For simplicity consider first example of integrals of motion in the conformal field theory perturbed by field \( \Phi_{13} \). Such as such a field is represented by vertex operator

\[ \Phi_{13} = : \exp(-\phi) : \]  

(5.2.1)

then its resonable lattice analogue has the form \( x^{-1} \). One can immediatly prove that

\[ (ad_{S_{\alpha_i}})_q S_{\alpha_j} = 0 \quad i \neq j \quad i, j = 0, 1 \]  

(5.2.2)

and these generators constitute \( U_q(\hat{sl}(2)_+ ) \) algebra. Hence, we have two screenings in this case:

\[ S_0 = \sum x_i^{-1} \]  
\[ S_1 = \sum x_i \]  

(5.2.3)

Let us consider two points \( x_1 \) and \( x_2 \). The main idea is to add "spectral parameter" \( \beta \) to the two-point screening operators and define some analogue of "R" matrix:

\[ \begin{align*} 
(\beta x_1 + x_2)R(x_1, x_2) &= R(x_1, x_2)(x_1 + \beta x_2) \\
(\beta x_1^{-1} + x_2^{-1})R(x_1, x_2) &= R(x_1, x_2)(x_1^{-1} + \beta x_2^{-1})
\end{align*} \]  

(5.2.4)

If we could solve these equations then we construct \( R_{i,i+1} \) by simple shift of variables and the product

\[ R = \prod_{-\infty}^{\infty} R_{i,i+1} \]  

(5.2.5)

(or more explicitly, logarithm of this product). gives us the generating function for integrals of motion. Let now \( R_{1,2} = R_{1,2}(x_1, x_2^{-1}; \beta) = R_{1,2}(u; \beta) \). Then both equations (5.2.4) are reduced to the following linear difference equation:

\[ (\beta u + 1)R_{1,2}(q^{-1}u; \beta) = (u + \beta)R_{1,2}(u; \beta) \]  

(5.2.6)

which was appeared in the work [27]. For \( q \) in generic position one of the the solutions of this equation has the form:

\[ R(u, \beta) = \prod_{i=0}^{\infty} \frac{1 + \beta u q^{-i}}{\beta + q^{-i}u} \]  

(5.2.7)
This expression is rather interesting: S.Kryukov notice that it can be formally represented as the two-points correlation function of two q-deformed bosonic fields. Moreover, in the limit $q \to 1$ it leads to the expressions like dilogorithms.

5.3 Integrals of motion. Example $\hat{sl}(3)$.

Let us describe direct generalization of this scheme for the case of $\hat{sl}(3)$ algebra.

Additional screening now has the form

$$S_{α_0} = \sum_{k \in \mathbb{Z}} [(x[2k]x[2k+1])^{-1}]_+. 
$$

It is easy to check by induction, that

$$S_i^2 S_j - [q^{\frac{1}{2}} + q^{-\frac{1}{2}}]S_i S_j S_i + S_j S_i^2 = 0 \quad i \neq j, \quad i, j = 0, 1, 2. \quad (5.3.1)$$

Correspondent difference equations system now has the form:

$$\begin{align*}
[βx_1 + x_3]R &= R[x_1 + βx_3] , \\
[βx_2 + x_4]R &= R[x_2 + βx_4] , \\
[β(x_1x_3)^{-1} + (x_2x_4)^{-1}]R &= R[(x_1x_3)^{-1} + β(x_2x_4)^{-1}] 
\end{align*}$$

Assuming, that

$$R = R(u_1, u_2)$$
$$u_1 = x_1/x_3 , \quad u_2 = x_2/x_4 , \quad u_1u_2 = q^{-1}u_2u_1 .
$$

we obtain the following system:

$$\begin{align*}
(qβu_1 + 1)R(u_1, u_2) &= qu_1R(qu_1, u_2) + βR(qu_1, q^{-1}u_2) \\
(qβu_2 + 1)R(u_1, u_2) &= qu_2R(q^{-1}u_1, qu_2) + βR(u_1, qu_2) \quad (5.3.2)
\end{align*}$$

For the moment I don’t know the proper way to solve this system of difference equations from non-commutative entries, but trivial power expansion gives us the following solution:

$$R_{1,2} = F(-qβu_1, q^{-\frac{β}{2}}u_2) \quad (5.3.3)$$

where $F$ is:

$$F(x, y) = \sum \frac{q^{ax} (\frac{1}{2})_n}{(β)_n(ββ)^{n-m}} x^n y^m , \quad (5.3.4)$$

One can write down similar difference equations and solutions for the any case $\hat{sl}(n)$ in the form of generalized q-hypergeometric Gorn’s series from non-commutative variables.

5.Conclusion.

There are many directions in this approach to be developed:
1. Consider similar picture for the lattice $W$-algebras associated to other simple (affine) algebras.

2. Investigate similar models for more complicated case when $q^p = 1$, $p \in \mathbb{Z}$.

3. Consider non-regular coloring of points.

4. Construct the realization of quantum affine Lie algebras and investigate it representations.

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