Bosonization Based on Bethe Ansatz Equations and
Spin-Charge Separation in the Hubbard Model with Finite $U$

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Abstract

We develop a bosonization approach for one-dimensional models based on Bethe ansatz equations. The operator formalism of the exact soluble models in the low energy limit provides a systematic method to calculate the asymptotic correlation functions. As examples with and without internal degrees of freedom, the Calogero-Sutherland (C-S) model and the repulsive Hubbard model are considered respectively. We verify that the low energy behavior of the C-S model is controlled by two classes of $c = 1$ conformal field theories, depending on whether the C-S interactions are among bosons or among fermions. For the Hubbard model, we show the explicit charge-spin separation at low energy for arbitrary $U > 0$. The low energy behavior of the system is described by the (semi-) direct product of two independent Virasoro algebras with $c = 1$. 

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I. INTRODUCTION

Recently it has attracted much attention to understand strongly correlated fermion systems in low dimensions \(d \leq 2\). Thanks to the special behaviors of one-dimensional (1-d) systems, we have better knowledge for them than those in higher dimensions, with the help from some non-perturbative methods. There is a number of 1-d systems called exactly soluble, whose spectra are exactly given by the Bethe ansatz equations. However, it is often difficult to calculate the correlation functions by using the Bethe ansatz. On the other hand, the bosonization approach provides a useful tool to compute the asymptotic correlation functions \(I\) in certain cases. Therefore, it is important to understand the interrelationship between exact solutions and bosonization. According to Haldane \(A\), the Bethe ansatz soluble models can be characterized by 1-d Luttinger liquids. Moreover, after renormalization, only forward scatterings are important in the Luttinger liquids. These imply that there is a harmonic fluid description \(\mathcal{B}\) to exact soluble models in the low energy limit, i.e., it should be possible to bosonize these models, at least in the low energy regime. However, how to do bosonization, especially for the case of the systems with internal degrees of freedom, has never been shown explicitly before in the literature. In this paper we will show how to bosonize the exact soluble models at low energy by starting from the Bethe ansatz. The bonus of doing so is a short cut for deriving the low temperature thermodynamics and static correlation functions. We will take two nontrivial models, the Calogero-Sutherland (C-S) model and the Hubbard model as examples.

The C-S model \(\mathcal{C}\) not only presents a beautiful example for soluble 1-d many-body model with long range interactions, but is also closely related to physical phenomena such as the edge excitations in the fractional quantum Hall effect \(\mathcal{D}\). It also provides a realization of some basic physical concept such as exclusion statistics \(\mathcal{E}\). The correlation functions of the C-S model can be exactly calculated in a wide range of the coupling constant \(\mathcal{F}\). Also, Ha \(\mathcal{G}\) proposed a bosonized anyon effective theory at low energy, and found that the asymptotic correlation functions of the effective theory agree with those of exact results.
This raises a question of whether there is an explicit way to derive the bosonization and show this agreement. In our recent work [12], a bosonization approach was developed for 1-d gas with exclusion statistics, which can be modified to deal with this problem. We will apply this approach to the bosonization of the C-S model, as a warm-up for treating models with internal degrees of freedom such as the Hubbard model. A by-product of this exercise is the proof that at low energy the bosonic and fermionic C-S models give rise to two different \( c = 1 \) conformal field theories (CFT’s), because of different selection rules for the quantum numbers. To our knowledge, this result is new in the literature. We also present the correlation functions respectively in each case.

The Hubbard model is known to have many remarkable properties, and has attracted a lot of attentions since the two dimensional Hubbard model was believed to be the right model for the high \( T_c \)-superconductivity about ten years ago. Although the 1-d Hubbard model we studied here may not directly relate to two-dimensional physics, there are still good reasons to investigate it for gaining insights into high \( T_c \)-superconductivity. After Lieb-Wu’s Bethe ansatz solution of the 1-d Hubbard model [13], a lot of efforts have been spent in this model. We list here some relevant developments. As is well-known, there may be string solutions with complex rapidities in some Bethe ansatz soluble models, which may affect the low energy physics of the system [14]. Fortunately, for the Hubbard model, it has been shown that the complex rapidities offer only the states in the upper Hubbard band and, therefore, do not contribute to the low energy physics [15]. So we are allowed to consider only the real rapidities in our treatment. Spin-charge separation is a fascinating property of the Hubbard model. It has been shown that in the strong-coupling limit, the ground-state wave function given by Lieb-Wu’s solution is indeed of the spin-charge separation form [16]. It was also checked from the exclusion statistics point of view that the low-lying states are also of the spin-charge separation form for large \( U \) [14]. Ren and Anderson [17] assumed the separation for arbitrary coupling in the low-energy limit and proposed a bosonized effective theory of the model. The critical exponents of the effective theory agreed with the numerical calculations [16,18] and the phase shift argument [17]. The correlation functions
of the Hubbard model were also discussed by finite-size scaling considerations \cite{19}. It has been proven that there are two independent $c = 1$ Virasoro algebras which describe the critical theory of the Hubbard model. It was also asserted that the low energy behaviors of the system at less than half filling could be characterized by a (semi-)direct product of two Virasoro algebras \cite{19}.

In this paper, we consider the repulsive Hubbard model at less than half filling. The low energy effective theory of the Hubbard model is not Lorentz invariant in $1 + 1$ dimensions, since there are two different Fermi vectors. This generally leads to the gap-less charge- and spin-density waves having different velocities. In this sense, the spin-charge separation is expected to appear not only at large $U$ but also at low energy at arbitrary coupling, as has been assumed in \cite{17}. Starting form the Bethe ansatz solutions, we will explicitly demonstrate this property in the low temperature thermodynamic limit. We show that the free energy of the model at the low temperature is of the form $F(T)/L = F(0)/L - \frac{\pi T^2}{6v_c} - \frac{\pi T^2}{6v_s}$, which implies that the critical behavior of the system can be characterized by the direct product of two independent $c = 1$ Virasoro algebras and coincides with the results of finite-size scaling at $T = 0$ \cite{19}. Meanwhile, by generalizing the bosonization approach (without internal degrees of freedom) to the Hubbard model which incorporates spin degrees of freedom, we arrive at an effective theory by bosonizing the thermodynamic limit of the original theory, resulting in the bosonized effective theory proposed by Ren and Anderson. Then the calculations of the asymptotic correlation functions become systematic.

This paper is organized as follows: In the next section, we present bosonization of the C-S model. In Sec. III, some relevant results about the Hubbard model are reviewed. Then in Sec. IV, the thermodynamic potential of the Hubbard model (and then free energy) at low temperature is derived from the Bethe Ansatz equations and the spin-charge separation at low energy is demonstrated. In Sec. V the bosonization of the Hubbard model is given, and the single-particle correlation functions are systematically calculated for various excitations. The last section is devoted to the conclusions.
II. CALOGERO-SUTHERLAND MODEL

The C-S model we consider here is described by the $N$-body Hamiltonian

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i<j} U(x_i - x_j),$$  \hspace{1cm} (2.1)

with

$$U(x) = g \sum_{n=-\infty}^{\infty} (x + nL)^{-2} = g \frac{\pi^2}{L^2} \sin^{-2} \left( \frac{\pi x}{L} \right),$$  \hspace{1cm} (2.2)

where $L$ is the size of 1-d ring. As $L \to \infty$, the interaction potential \( U(x) \to g/x^2 \).

The ground state is given by

$$\psi_{B,\lambda} = \prod_{i<j} |\sin \pi (x_i - x_j)/L|^{\lambda},$$  \hspace{1cm} (2.3)

or

$$\psi_{F,\lambda} = \prod_{i<j} (z_i - z_j)^{\lambda} \prod_{k} z_k^{-\lambda(n-1)/2},$$  \hspace{1cm} (2.4)

depending on the statistics of the particles, bosons or fermions. Here $z = \exp(i2\pi x/L)$ and \( \lambda = [(1 + 2g)^{1/2} + 1]/2 \). It is obvious that for $\lambda = 1$($g = 0$) the former is the ground state of hard-core bosons and the latter that of free fermions.

Moreover, the model can be exactly solved by using the asymptotic Bethe ansatz or the Jack Polynomials. The periodic boundary conditions give rise to the spectrum determined by the following equations for pseudomomenta $k_i$:

$$Lk_i = 2\pi I_i + \pi(1 - \lambda) \sum_{j<i} \text{sgn}(k_j - k_i),$$  \hspace{1cm} (2.5)

where

$$I_i = \left\{ \begin{array}{ll} (N + 1)/2 \ \text{mod}(1), & \text{for fermion case,} \\ \text{integer}, & \text{for boson case} \end{array} \right\}$$  \hspace{1cm} (2.6)

These selection rules have been discussed by Kawakami and Yang, and for bosons and fermions the behavior of the momentum distribution is quite different. We will see that
it is the difference in the selection rules that gives rise to two different classes of $c = 1$ CFT for low energy effective field theory. The total energy $E$ and the total momentum $P$ are, respectively,

$$E = \sum_i k_i^2, \quad P = \sum_i k_i = \sum_i p_i(k).$$

(2.7)

where $p_i(k) = 2\pi I_i/L$

The thermodynamics of the theory has been fully discussed by Sutherland [5]. Here we would like to examine its relations to CFT, Luttinger liquid and bosonization. Some relevant topics have been partially considered by several authors before [20,10,22,23]. The thermodynamics of the C-S model can also be formulated in terms of an ideal exclusion gas (IEG) [8]. The thermodynamic potential is known to be given by

$$\frac{\Omega}{L} = -\frac{T}{2\pi} \int_{-\infty}^{\infty} dk \ln(1 + w(k, T)^{-1}),$$

(2.8)

with the function $w(k, T) \equiv \rho_a(k)/\rho(k)$, where $\rho_a(k)$ and $\rho(k)$ are the hole and particle densities, satisfying

$$w(k, T)^\lambda [1 + w(k, T)]^{1-\lambda} = e^{(k^2-\mu)/T}.$$  

(2.9)

The particle density is determined by

$$\rho(k, T)(1 + w(k, T)) = \frac{1}{2\pi} - (\lambda - 1)\rho(k, T).$$  

(2.10)

In the ground state, there is a (pseudo-)Fermi surface $k_F$, such that $\rho(k) = 1/2\pi\lambda$ for $|k| < k_F$ and $\rho(k) = 0$ for $|k| > k_F$. Then the Fermi momentum is given by $k_F = \pi \lambda \tilde{d}_0$, and the ground state energy and momentum by $E_0/L = \pi^2 \lambda^2 \tilde{d}_0^2/3, \quad P_0 = 0$, where $\tilde{d}_0 = N_0/L$.

Now let us examine possible excitations in the model. First there are density fluctuations due to particle-hole excitations, i.e., the sound waves with velocity $v_s = v_F \equiv 2k_F$ (see below). Moreover, by adding extra $M$ particles to the ground state one can create particle excitations, and by Galileo boost a persistent current. It is easy to verify that the velocities of these three classes of elementary excitations in the model satisfy a fundamental relation,
\( v_s = \sqrt{v_Nv_J} \), that Haldane years ago used to characterize the Luttinger liquid \(^{25}\). Indeed, shifting \( N_0 \) to \( N = N_0 + M \), the change in the ground state energy is \( \delta_1 E_0 = \pi(\lambda k_F)M^2 \), while a persistent current, created by the boost of the Fermi sea \( k \rightarrow k + \pi J/N_0 \), leads to the energy shift \( \delta_2 E_0 = \pi(k_F/\lambda)J^2 \). Therefore the total changes in energy and in momentum, due to charge and current excitations, are

\[
\begin{align*}
\delta E_0 &= (\pi/2L)(v_NM^2 + v_JJ^2), \\
\delta P_0 &= \pi(\bar{d}_0 + M/L)J,
\end{align*}
\]

with

\[
v_N = v_s\lambda, \quad v_J = v_s/\lambda, \quad v_s = \sqrt{v_Nv_J}.
\]

These coincide with the well-known relations \(^{25}\) in the Luttinger liquid theory, if we identify \( \lambda \) with Haldane’s controlling parameter \( \exp(-2\varphi) \). It is easy to check that the selection rule \(^{2.6}\) can be rewritten as \(^{20}\)

\[
J = \begin{cases} 
M \text{ mod}(2), & \text{for fermion case,} \\
\text{even} & \text{for boson case}
\end{cases}
\]

It has been claimed from finite-size scaling \(^{20}\) that the low energy behavior of the C-S model is controlled by \( c = 1 \) CFT. A review of this point will help to understand the behavior of the Hubbard model. The central charge \( c \) of a conformal invariant system is related to thermodynamic observable by finite-size scaling \(^{26}\). Generally, the free energy of the system is given by

\[
\frac{F(T)}{L} - \frac{F(0)}{L} = -\frac{\pi T^2 c}{6v_s}.
\]

For the C-S model, we consider the low energy expansion of the thermodynamic potential \(^{2.8}\). Following Yang and Yang \(^{27,5}\), we introduce the dressed energy \( \epsilon(k, T) \) by writing

\[
w(k, T) = e^{\epsilon(k, T)/T}.
\]

And \(^{2.9}\) reads
\[ \epsilon(k, T) = k^2 - \mu - T(1 - \lambda) \ln(1 + e^{-\epsilon(k,T)/T}). \] (2.16)

Because there is no singularity in \( \epsilon(k, T) \) at \( T = 0 \), the zero temperature dressed energy is given by

\[ \epsilon(k) = \begin{cases} \frac{(k^2 - k_F^2)}{\lambda}, & |k| < k_F, \\
 k^2 - k_F^2, & |k| > k_F. \end{cases} \] (2.17)

At low energy, one can consider only the excitation around the Fermi surface and the thermodynamic potential is given by \[28\]

\[ \frac{\Omega(T)}{L} = \frac{1}{2\pi} \int_{-k_F}^{k_F} dk \epsilon(k) - \frac{T}{\pi} \int_{-\delta}^{\delta} d\delta p_F \ln(1 + e^{-|\epsilon(k(p), T)|/T}), \] (2.18)

where \( \delta \) is a cut-off and the first term on the right hand side of the last equality is recognized as \( \Omega(0)/L \). \( \delta p_F \) is the deformation of the Fermi surface, i.e., \( \delta p_F/2\pi = \rho(k_F)\delta k_F \). It is known that \( \epsilon(k) = v_s \delta p_F \) near the Fermi point \( p_F \). After a little calculation, (2.18) reads

\[ \frac{\Omega(T)}{L} - \frac{\Omega(0)}{L} = -\frac{\pi T^2}{6v_s}, \] (2.19)

which implies that the theory is actually cut-off independent at low temperature. Notice that \( F = \Omega - \mu N \). Only the particle-hole excitations near the Fermi surface contribute to thermal excitations, leading to \( N(T) - N(0) = 0 \); this can be checked by an explicit calculation in terms of the definition (2.10) of \( \rho(k, T) \). Thus, we have

\[ \frac{F(T)}{L} - \frac{F(0)}{L} = \frac{\Omega(T)}{L} - \frac{\Omega(0)}{L} = -\frac{\pi T^2}{6v_s}. \] (2.20)

Comparing this to (2.14), we see \( c = 1 \) for the C-S model.

The central charge of a conformal invariant system can also be given by the finite-size scaling in the spatial direction \[23\]. In Appendix A, we confirm \( c = 1 \) for the C-S model from such a calculation, with some subtleties not noticed before in the literature. Now let us work out explicitly the effective theory at low energy with \( c = 1 \) conformal invariance. For this purpose, we will develop a bosonization approach. Our starting point is the observation that the grand partition function \( Z_G \), corresponding to the thermodynamic potential (2.8),
is of the form of that for a system of ideal fermions with a complicated, \( T \)-dependent energy dispersion given by the dressed energy:

\[
Z_G = \prod_k (1 + e^{-\epsilon(k,T)/T}). \tag{2.21}
\]

This fermion representation is not very useful, because of the implicit \( T \)-dependence in the dressed energy. Nevertheless, as we have seen in ref. [12,28], in the low-\( T \) limit the dressed energy is effectively \( T \)-independent: Namely we have \( \epsilon(k,T) = \epsilon(k) + O(Te^{-|\epsilon(k)|/T}) \), as a result of (2.19). Hence, the grand partition function can be obtained from an effective Hamiltonian, given by

\[
H_{\text{eff}} = \sum_k \epsilon(k) \, c_k^\dagger c_k, \tag{2.22}
\]

where \( c_k^\dagger \) are fermion creation operators.

Another simplification in the low-\( T \) limit is that we need to consider only low-energy excitations near the right and left Fermi points \( |k| = \pm k_F \), which are completely decoupled and have a linearized energy dispersion

\[
\epsilon_{\pm}(k) = \begin{cases} 
\pm v_F (k \mp k_F), & |k| > k_F, \\
\pm v_F (k \mp k_F)/\lambda, & |k| < k_F.
\end{cases} \tag{2.23}
\]

We note the ‘refraction’ at \( k = \pm k_F \). In spite of this peculiarity, we have succeeded in bosonizing the effective Hamiltonian as follows [12]: The density fluctuation operator at \( k \sim k_F \) is constructed as follows

\[
\rho_q^{(+)} = \sum_{k > k_F} : c_{k+q}^\dagger c_k : + \sum_{k < k_F - \lambda q} : c_{k+\lambda q}^\dagger c_k : \\
+ \sum_{k_F - \lambda q < k < k_F} : c_{k-k_F + k_F + q}^\dagger c_k : \tag{2.24}
\]

for \( q > 0 \). A similar density operator \( \rho_q^{(-)} \) can also be defined at \( k \sim -k_F \),

\[
\rho_q^{(-)} = \sum_{k < -k_F} : c_{k-q}^\dagger c_k : + \sum_{k > -k_F + \lambda q} : c_{k-\lambda q}^\dagger c_k : \\
+ \sum_{-k_F + \lambda q > k > -k_F} : c_{k+k_F - k_F - q}^\dagger c_k : \tag{2.25}
\]
Within the Tomonaga approximation \([23]\), in which commutators are taken to be their ground-state expectation value, we obtain

\[
\left[ \rho_q^{(\pm)} : \rho_{q'}^{(\pm)\dagger} \right] = \frac{L}{2\pi} q \delta_{q,q'}, \quad \left[ H_{\pm}, \rho_q^{(\pm)} \right] = \pm v_F q \rho_q^{(\pm)},
\]

which describe 1-d free phonons with the sound velocity \(v_s = v_F\). Introducing normalized boson annihilation operators \(b_q = \sqrt{2\pi/qL} \rho_q^{(+)}\) and \(\tilde{b}_q = \sqrt{2\pi/qL} \rho_q^{(-)\dagger}\), the bosonized Hamiltonian satisfying (2.26) is given by

\[
H_B = v_s \left\{ \sum_{q>0} q (b_q^\dagger b_q + \tilde{b}_q^\dagger \tilde{b}_q) + \frac{\pi}{2L} \left[ \lambda M^2 + \frac{1}{\lambda} J^2 \right] \right\}.
\]

The bosonized total momentum operator, corresponding to the fermionized \(P = \sum_k p(k) c_k^\dagger c_k\), is

\[
P = \sum_{q>0} q (b_q^\dagger b_q - \tilde{b}_q^\dagger \tilde{b}_q) + \pi (\tilde{d}_0 + M/L) J.
\]

In the coordinate-space formulation, the normalized density field \(\rho(x)\) is given by \(\rho(x) = \rho_R(x) + \rho_L(x)\):

\[
\rho_R(x) = M_R \frac{L}{L} + \sum_{q>0} \sqrt{q/2\pi L} \lambda (e^{iqx} b_q + e^{-iqx} b_q^\dagger),
\]

and \(\rho_L(x)\) is similarly constructed from \(\tilde{b}_q\) and \(\tilde{b}_q^\dagger\):

\[
\rho_L(x) = M_L \frac{L}{L} + \sum_{q>0} \sqrt{q/2\pi L} \lambda (e^{-iqx} \tilde{b}_q + e^{iqx} \tilde{b}_q^\dagger),
\]

where \(M_R, L\) are given by \(M = M_R + M_L\). The boson field \(\phi(x)\), which is conjugated to \(\rho(x)\) and satisfies \([\phi(x), \rho(x')] = i\delta(x - x')\), is \(\phi(x) = \phi_R(x) + \phi_L(x)\) with

\[
\phi_R(x) = \frac{\phi_0}{2} + \frac{\pi J_R X}{L} + i \sum_{q>0} \sqrt{\pi/2qL} \lambda (e^{iqx} b_q - e^{-iqx} b_q^\dagger),
\]

\[
\phi_L(x) = \frac{\phi_0}{2} + \frac{\pi J_L X}{L} + i \sum_{q>0} \sqrt{\pi/2qL} \lambda (e^{-iqx} \tilde{b}_q - e^{iqx} \tilde{b}_q^\dagger),
\]

with \(J = J_R + J_L\). Here \(M\) and \(J\) are operators with integer eigenvalues obeying the selection rule (2.13), and \(\phi_0\) is an angular variable conjugated to \(M\): \([\phi_0, M] = i\). The Hamiltonian (2.27) becomes
\[ H_B = \frac{v_s}{2\pi} \int_0^L dx \left[ \Pi(x)^2 + (\partial_x X(x))^2 \right], \]  
\[ (2.32) \]

where \( \Pi(x) = \pi \lambda^{1/2} \rho(x) \) and \( X(x) = \lambda^{-1/2} \phi(x) \). With \( X(x,t) = e^{iHt}X(x)e^{-iHt} \), the Lagrangian density reads

\[ \mathcal{L} = \frac{v_s}{2\pi} \partial_\alpha X(x,t) \partial^\alpha X(x,t). \]  
\[ (2.33) \]

We recognize in no ambiguity that \( \mathcal{L} \) is the Lagrangian of a \( c = 1 \) CFT \[30\]. Since \( \phi_0 \) is an angular variable, there is a hidden invariance in the theory under \( \phi \to \phi + 2\pi \). The field \( X \) is thus said to be “compactified” on a circle, with a radius that is determined by the coupling constant:

\[ X \sim X + 2\pi R, \quad R^2 = 1/\lambda. \]  
\[ (2.34) \]

(Incidentally, \( \lambda \) can be identified with the exclusion statistics parameter \[12\] of the quasiparticle excitations in the C-S model.) States \( V[X]|0\rangle \) or operators \( V[X] \) are allowed only if they respect this invariance, so the quantum numbers of quasiparticles are strongly constrained. For the boson case in \( (2.13) \), the zero-mode term in \( (2.27) \) can be written as

\[ (\pi/2L)[M^2/R^2 + 4R^2D^2], \]  

where \( D = J/2 = M \mod(1) \) are integers. This means that the Hamiltonian \( (2.27) \) in this case has duality

\[ R \leftrightarrow 1/2R, \quad M \leftrightarrow D, \]  
\[ (2.35) \]

which is just the famous duality in the usual \( c = 1 \) CFT \[30\]. For the fermion case in \( (2.13) \), the duality of \( (2.27) \) reads

\[ R \leftrightarrow 1/R, \quad M \leftrightarrow J, \]  
\[ (2.36) \]

which corresponds to the so called exclusion particle-hole duality mentioned in \[31\]

\[ \lambda \leftrightarrow 1/\lambda. \]  
\[ (2.37) \]
Moreover, the partition function in the fermion case (in the low-$T$ limit) can be rewritten as $Z = Tr_H[q^{-1} \bar{q}^{L} L^R]$, where $q = e^{ivs/T}$. The zeroth generators of the Virasoro algebra are

\[
L_0^{R,L} = v_s^{-1} H_{R,L} + \frac{\pi}{4L} [JR \mp M/R]^2,
\]
\[
H_R = v_s \sum_{q>0} q b_q^\dagger b_q, \quad H_L = v_s \sum_{q>0} \bar{q} \bar{b}_q^\dagger \bar{b}_q.
\] (2.38)

The constraint $J = M \mod (2)$ makes the spectrum and duality relation different from those in the $c = 1$ CFT compactified on a circle. Indeed, the fermion case corresponds to the $c = 1$ CFT compactified on an interval (or an orbifold) $S^1/Z_2$; see [30] for more details.

To show the difference between two CFT’s, we check the primary fields in each theory. The primary fields of both CFT’s are given by

\[
\phi_{M,J}(x) \sim e^{i(M\lambda_1^1/2 + J/\lambda_1^1)X_R(x)} e^{i(M\lambda_1^1/2 - J/\lambda_1^1)X_L(x)} ;
\] (2.39)

The charge-1 operators correspond to $M = 1$. Therefore, charge-1 operators are given by $M = 1$ and $J = 2m$ with integer $m$ for the boson case, while $J = 2m + 1$ for the fermion case. Hence, the charge-1 primary fields in the boson case are bosons, and those in the fermion case are fermions:

\[
\Psi_{B}^\dagger (x,t) = \rho(x)^{1/2} \sum_{m=-\infty}^{\infty} e^{iO_m} : e^{i(\lambda_1^1/2 + 2m/\lambda_1^1)X_R(x_-)} : \]
\[
: e^{i(\lambda_1^1/2 - 2m/\lambda_1^1)X_L(x_+)} : ,
\] (2.40)

and

\[
\Psi_{F}^\dagger (x,t) = \rho(x)^{1/2} \sum_{m=-\infty}^{\infty} e^{iO_m} : e^{i(\lambda_1^1/2 + (2m+1)/\lambda_1^1)X_R(x_-)} : \]
\[
: e^{i(\lambda_1^1/2 - (2m+1)/\lambda_1^1)X_L(x_+)} : ,
\] (2.41)

where the hermitian, constant-valued operators $O_m$ satisfy $[O_m, O_m'] = i\pi(m - m')$, which give rise to the Klein factor necessary for the correct commutation relations for $\Psi_{B(F)}$ and $\Psi_{B(F)}^\dagger$.

It is clear that the correlation functions of the bosons are different from those of the fermions, e.g., for $m = 0$, one has respectively
\[ \langle \Psi_B(x)\Psi_B^\dagger(0) \rangle \sim x^{-\lambda/2}, \] (2.42)

and

\[ \langle \Psi_F(x)\Psi_F^\dagger(0) \rangle \sim x^{-(\lambda+1/\lambda)/2} \cos k_Fx. \] (2.43)

In summary, we have bosonized the C-S model and shown that the low energy behavior is controlled by \( c = 1 \) CFT. Depending on whether the C-S interactions refer to those between boson or between fermions, we have two different classes of \( c = 1 \) CFT, governed by different selection rules for quasi-particle quantum numbers. To our knowledge, the appearance of two different classes of \( c = 1 \) CFT’s in the low energy limit of the C-S model was not notified before in the literature.

In addition to the above charge-1 operators, there are more allowed operators, the so-called anyons or exclusons, in the theory constructed as follows:

\[ \Psi_{B,\lambda}^\dagger(x) =: \Psi_B^\dagger(x)e^{i\lambda^{1/2}(X_R(x) - X_L(x))}, \] (2.44)

for boson case and

\[ \Psi_{F,\lambda}^\dagger(x) =: \Psi_F^\dagger(x)e^{i(\lambda^{1/2}-1/\lambda^{1/2})(X_R(x) - X_L(x))}, \] (2.45)

for fermion case. Those operators satisfy

\[ \Psi_{\lambda}^\dagger(x)\Psi_{\lambda}^\dagger(x') - e^{i\pi\lambda\text{sgn}(x-x')}\Psi_{\lambda}^\dagger(x')\Psi_{\lambda}^\dagger(x) = 0 \]

for \( x \neq x' \). The multi-sector density operator for exclusons is

\[ \hat{\rho}(x) = \Psi_{\lambda}^\dagger(x)\Psi_{\lambda}(x) = \Psi_{B(F)}^\dagger(x)\Psi_{B(F)}(x) \]

\[ = \rho(x) \sum_m : \exp\{i2m[X_R(x) - X_L(x)]/\lambda^{1/2}\} : \] (2.46)

The correlation functions for density fluctuations and for anyons (or exclusons) are the same in the above-mentioned two classes of \( c = 1 \) CFT’s:

\[ \langle \hat{\rho}(x,t)\hat{\rho}(0,0) \rangle \approx d_0^2 \left[ 1 + \frac{1}{(2\pi d_0)^2 \lambda} \left( \frac{1}{x_+^2} + \frac{1}{x_-^2} \right) \right] \]

\[ + \sum_{m=1}^{\infty} A_m \frac{1}{[x_+ x_-]^{m^2/\lambda}} \cos(2\pi d_0 mx), \] (2.47)
\[ G(x, t; \lambda) \equiv \langle \Psi^\dagger_\lambda(x, t) \Psi_\lambda(0, 0) \rangle \]
\[ \approx \bar{d}_0 \sum_{m=-\infty}^{\infty} B_m \frac{1}{x_-^{(m+\lambda)^2/\lambda}} \frac{1}{x_+^{m^2/\lambda}} e^{i(2\pi(m+\lambda/2)x+\mu t)}, \]

with \( A_m \) and \( B_m \) regularization-dependent constants. We notice that the correlation functions (2.48) coincide with the asymptotic ones calculated in [10] for the C-S model.

III. THE LIEB-WU SOLUTION AND RELEVANT DEVELOPMENTS FOR HUBBARD MODEL

In the previous section, we achieved bosonization of the C-S model which has no internal degree of freedom. The bosonization of 1-d models with internal degrees of freedom such as spin is more complicated, because there are backward scatterings and umklapp scatterings. We will take the Hubbard model as an example to investigate. In this section, we first review Lieb-Wu’s Bethe ansatz solution and other developments which are relevant in the coming sections.

The general form of the Hubbard Hamiltonian reads

\[ H = -t \sum_{\langle ij \rangle, \sigma} c^\dagger_i c_j \sigma + U \sum_i n_i^\uparrow n_i^\downarrow + \mu \sum_i n_i - \frac{\hbar}{2} \sum_i (n_i^\uparrow - n_i^\downarrow), \]

where the summation \( \langle ij \rangle \) runs over nearest-neighbors. We have set the electrons on a 1-d ring with size \( L \). This model has been exactly solved by Lieb and Wu [32], and the spectrum of the model is shown to be determined by the following Bethe ansatz equations,

\[ L k_i = 2\pi I_i + \sum_{\alpha} 2 \tan^{-1} \frac{4(\sin k_i - \lambda_\alpha)}{u}, \]
\[ \sum_{i=1}^{N} 2 \tan^{-1} \frac{4(\lambda_\alpha - \sin k_i)}{u} = 2\pi J_\alpha + \sum_{\beta=1}^{N} 2 \tan^{-1} \frac{2(\lambda_\alpha - \lambda_\beta)}{u}, \]

where \( N \) (\( N_\downarrow \)) is the number of electrons (with down-spin) and \( u = U/t \). The total energy and momentum are related to the pseudomomenta \( k_i \) by

\[ E = -2t \sum_{i=1}^{N} \cos k_i + \mu N + h(N_\downarrow - \frac{N}{2}), \]
\[ P = \sum_{i=1}^{N} k_i = \frac{2\pi}{L} \left( \sum_i I_i + \sum_\alpha J_\alpha \right). \]
Although the variation of rapidity $\lambda_\alpha$ seems to be related to only the density variation of spin-down electrons in the $\lambda$-space, it reflects the spinon density fluctuation in the real space, because of the antiferromagnetic ground state. Hence, the spin wave rapidity $\lambda_\alpha$ characterizes the spinon dynamics of the system. The quantum numbers $I_i$ and $J_\alpha$ obey the selection rules

$$I_i = \frac{N_\downarrow}{2} \mod(1), \quad J_\alpha = \frac{N - N_\downarrow + 1}{2} \mod(1). \quad (3.4)$$

We would like to discuss the low energy behavior of the model with repulsive interaction ($U > 0$) at less than half filling. We have two kinds of gap-less excitations. Fortunately, it has been proven that the excitations with complex $k$ and $\lambda$ have gaps of the order $u = U/t$. We can only consider the real $k$ and $\lambda$ in the low energy limit. It has been shown that the finite-size corrections to the thermodynamic potential at $T = 0$ are

$$\Omega_L(0) - \Omega_\infty(0) = -\frac{\pi}{6L}(v_c + v_s) + O(1/L^2), \quad (3.5)$$

which implies that the low energy effective theory of the Hubbard model is not Lorentz invariant. There are two Fermi velocities $v_c$ and $v_s$, which correspond to the charge- and spin-wave velocities respectively, and hence four Fermi points $\pm k_F$ and $\pm \lambda_F$. The finite-size scaling (3.5) also shows that the low energy spectrum can be characterized by two $c = 1$ Virasoro algebras. To derive (3.5), it is important to realize that not only the thermodynamic potential (or the total energy) is scaled but also the distributions of the charge and spin-wave. In the next section, we will give (3.5) another consistent check from finite-size scaling in the temperature direction.

Again, we proceed to consider the thermodynamic limit. The zero temperature behavior has been well-understood recently. Around the Fermi points, there are three kinds of excitations. The first kind is charge- and spin-wave fluctuations in real space, which corresponds to non-zero modes in $k$-$\lambda$ space. We shall leave those to the coming sections. Other two kinds are corresponding to zero modes in $k$-$\lambda$ space. One of them is created by adding extra number of particles to the ground state. The other is persistent currents. Both
of excitations shift the ground state energy. It is known that the ground state energy per site is given by

$$
\varepsilon_0 = \frac{1}{2\pi} \int_{-k_F}^{k_F} dk \varepsilon_c(k),
$$

(3.6)

where $\varepsilon_c$ is the zero-temperature dressed energy obeying

$$
\varepsilon_c(k) = \varepsilon_c^{(0)} + \frac{1}{2\pi} \int_{-\lambda_F}^{\lambda_F} d\lambda K_1(\sin k - \lambda)\varepsilon_s(\lambda),
$$

$$
\varepsilon_s(\lambda) = \varepsilon_s^{(0)} + \frac{1}{2\pi} \int_{-k_F}^{k_F} dk \cos kK_1(\lambda - \sin k)\varepsilon_c(k)
- \frac{1}{2\pi} \int_{-\lambda_F}^{\lambda_F} d\lambda K_2(\lambda - \lambda')\varepsilon_s(\lambda').
$$

(3.7)

The bare energies $\varepsilon_c^{(0)} = \mu - h/2 - 2t \cos k$ and $\varepsilon_s^{(0)} = h$. And the functions $K_1$ and $K_2$ are defined by

$$
K_1(x) = \frac{8u}{u^2 + 16x^2}, \quad K_2(x) = \frac{4u}{u^2 + 4x^2}.
$$

(3.8)

The solutions of the integral equations (3.7) define the energy bands. So, we can determine the $k_F$ and $\lambda_F$ by

$$
\varepsilon_c(\pm k_F) = 0, \quad \varepsilon_s(\pm \lambda_F) = 0.
$$

(3.9)

Alternatively, we can fix the values of $k_F$ and $\lambda_F$ through the particle numbers:

$$
n_c = \frac{N}{L} = \int_{-k_F}^{k_F} dk \rho_c(k),
$$

$$
n_\downarrow = \frac{N_\downarrow}{L} = \int_{-\lambda_F}^{\lambda_F} d\lambda \rho_\downarrow(\lambda),
$$

(3.10)

where the distributions of charge- and spin-wave, $\rho(k)$ and $\rho_\downarrow(\lambda)$, are given by a set of integral equations:

$$
\rho_c(k) = \frac{1}{2\pi} + \frac{\cos k}{2\pi} \int_{-\lambda_F}^{\lambda_F} d\lambda K_1(\sin k - \lambda)\rho_\downarrow(\lambda),
$$

$$
\rho_\downarrow(\lambda) = \frac{1}{2\pi} \int_{-k_F}^{k_F} dk K_1(\lambda - \sin k)\rho_c(k)
- \frac{1}{2\pi} \int_{-\lambda_F}^{\lambda_F} d\lambda' K_2(\lambda - \lambda')\rho_\downarrow(\lambda').
$$

(3.11)
Generally speaking, the zero mode excitations can be thought as the fluctuations of the Fermi surface. The corresponding shift to the ground state energy is given by
\[
\delta E_0 = \frac{\partial^2 E_0}{\partial k_F^2}(\delta k_F)^2 + \frac{\partial^2 E_0}{\partial \lambda_F^2}(\delta \lambda_F)^2,
\]
(3.12)
where the fact (3.3) insures there are no cross derivatives. It is easy to see \( \partial^2 E_0/\partial k_F^2 \sim v_c \) and \( \partial^2 E_0/\partial \lambda_F^2 \sim v_s \), where
\[
v_c = \frac{\varepsilon'_c(k_F)}{2\pi \rho_c(k_F)}, \quad v_s = \frac{\varepsilon'_s(\lambda_F)}{2\pi \rho_s(\lambda_F)},
\]
are the charge- and spin-wave velocities.

Adding extra particles \( M \) and \( M_\downarrow \) to the ground state leads to a deformation of the Fermi surface given by
\[
\begin{pmatrix}
\delta M_{k_F} \\
\delta M_{\lambda_F}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial k_F}{\partial N} & \frac{\partial k_F}{\partial N_\downarrow} \\
\frac{\partial \lambda_F}{\partial N} & \frac{\partial \lambda_F}{\partial N_\downarrow}
\end{pmatrix} \begin{pmatrix}
M \\
M_\downarrow
\end{pmatrix}.
\]
(3.14)

For the persistent currents, which correspond to the quantum number shifts \( I \rightarrow I + D_I \) and \( J \rightarrow J + D_J \), we have the Fermi surface deformation as follows
\[
\begin{pmatrix}
\delta J_{k_F} \\
\delta J_{\lambda_F}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial k_F}{\partial D_I} & \frac{\partial k_F}{\partial D_J} \\
\frac{\partial \lambda_F}{\partial D_I} & \frac{\partial \lambda_F}{\partial D_J}
\end{pmatrix} \begin{pmatrix}
D_I \\
D_J
\end{pmatrix}.
\]
(3.15)

In general, the matrices in (3.14) and (3.15) are related to the dressed charge matrix of the theory [19]. For the case of zero magnetic field, \( h = 0 \), which we are interested in, we have
\[
\begin{align*}
\delta M_{k_F} &= \frac{1}{2\xi} \frac{M_c}{L}, & \delta M_{\lambda_F} &= \frac{1}{2\sqrt{2}} \frac{M_s}{L}, \\
\delta J_{k_F} &= \frac{\xi}{2} \frac{J_c}{L}, & \delta J_{\lambda_F} &= \frac{1}{\sqrt{2}} \frac{J_s}{L},
\end{align*}
\]
(3.16)
where \( M_c = M, M_s = M_\uparrow - M_\downarrow, J_c = 2D_I + D_J \) and \( J_s = -D_J \). And \( \xi = \xi(\sin k_F) \) with the function \( \xi(x) \) determined by integral equation
\[
\xi(x) = 1 + \frac{1}{2\pi} \int_{-k_F}^{k_F} K(x - x') \xi(x'),
\]

\[
K(x) = \int_{-\infty}^{\infty} e^{-|\omega|u/4+i\omega x} \frac{d\omega}{2 \cosh \omega u/4}.
\]  

(3.17)

In particular, we have

\[
\rho_c(k) = \frac{1}{2\pi} + \frac{\cos k}{2\pi} \int_{-k_F}^{k_F} dk' K(\sin k - \sin k') \rho_c(k'),
\]  

(3.18)

\[
\varepsilon_c(k) = \varepsilon^{(0)}(k) + \frac{1}{2\pi} \int_{-k_F}^{k_F} dk' \cos k' K(\sin k - \sin k') \varepsilon_c(k'),
\]  

(3.19)

Hence, the zero mode part of the excitation spectrum at \( h = 0 \), according to our above discussion, is

\[
\delta E_0 = (\pi/2L)(v_{c,N}M_c^2 + v_{c,J_c}J_c^2) + (\pi/2L)(v_{s,N}M_s^2 + v_{s,J_s}J_s^2),
\]  

(3.20)

where the velocity relations are given by

\[
v_{c(s),N} = v_{c(s)} \lambda_{c(s)}, \quad v_{c(s),J} = v_{c(s)}/\lambda_{c(s)},
\]  

(3.21)

with \( \lambda_c = \xi^{-2} \) and \( \lambda_s = 1/2 \). The energy shift (3.20) and the velocity relations (3.21), for each component (spin or charge), resemble those in the C-S model, (2.11) and (2.12). Also, it is easy to show the momentum shift caused by zero modes is given by

\[
\delta P_0 = \pi (n_0 + M_c/L)J_c + \pi M_s J_s/L,
\]  

(3.22)

which can be thought of as the two-component generalization of the momentum shift in (2.11).

The selection rules (3.4) imply that the excitation quantum numbers obey the following selection rules [19]

\[
D_I = \frac{M_c - M_s}{2} \mod(1), \quad D_J = \frac{M_c}{2} \mod(1).
\]  

(3.23)

For \( J_c \) and \( J_s \), the fact \( M_c = M_s \mod(1) \) and (3.23) imply that

\[
J_c = \frac{M_c}{2} \mod(1), \quad J_s = \frac{M_s}{2} \mod(1).
\]  

(3.24)
IV. LOW TEMPERATURE BEHAVIOR AND SPIN-CHARGE SEPARATION

In this section, we would like to present the thermodynamics of the Hubbard model at low temperature and show spin-charge separation for arbitrary $U > 0$. Also, we will confirm the finite-size scaling result (3.3).

A. Statistics and thermodynamics

Recently an exclusion statistics description has been developed for the Bethe ansatz soluble models [9,33,14]. For the Hubbard model, the statistics matrix is given by

$$
\begin{align*}
  g^{cc}(k, k') &= \delta(k - k'), \\
  g^{c\downarrow}(k, \lambda) &= -\frac{4}{\pi u} \frac{\cos k}{1 + 16(\sin k - \lambda)^2 / u^2}, \\
  g^{\downarrow\downarrow}(\lambda, \lambda') &= \delta(\lambda - \lambda') + \frac{1}{\pi} \frac{1}{1 + 4(\lambda - \lambda')^2 / u^2}, \\
  g^{\downarrow\uparrow}(\lambda, k) &= -\frac{1}{\pi} \frac{2}{1 + 16(\lambda - \sin k)^2 / u^2}.
\end{align*}
$$

The equation for $g^{\downarrow\downarrow}$ shows the mutual statistics between different spin-wave rapidities, while $g^{c\downarrow}$ and $g^{\downarrow\uparrow}$ give the mutual statistics between $k$ and $\lambda$. However, the constraint (3.9) means that the low energy excitations near the Fermi surface have no mutual statistics between states described by $k$ and $\lambda$.

We showed before that for Bethe ansatz soluble models with no internal degree of freedom, one can use an effective statistics parameter to characterizes the statistics of low energy excitations, which can be read off from the Luttinger-liquid velocity relations [12]. For the present case, the general velocity relation (3.21) seems to suggest that there is no effective mutual statistics and there are two effective statistics parameters, $\lambda_c$ and $\lambda_s$, characterizing the statistics of low energy excitations. Below, we will see that indeed there are two types of anyonic (or exclusonic) excitations in the theory.
If only the real $k$ and $\lambda$ are taken into account, the thermodynamic potential can be written down in terms of the principle of exclusion statistics \[9\] and the statistics matrix (4.1). It has been given in \[14\]:

$$
\frac{\Omega}{L} = -\frac{T}{2\pi} \int_{-\pi}^{\pi} dk \ln[1 + e^{-\varepsilon_c(k,T)/T}],
$$

(4.2)

where the dressed energy $\varepsilon_c(k,T)$, which is the finite temperature generalization of (3.7), is given by

$$
\varepsilon_c(k,T) = \varepsilon_c^{(0)} - \frac{T}{2\pi} \int_{-\infty}^{\infty} d\lambda K_1(\sin k - \lambda) \times \ln[1 + e^{-\varepsilon_s(\lambda,T)/T}],
$$

$$
\varepsilon_s(\lambda,T) = \varepsilon_s^{(0)} - \frac{T}{2\pi} \int_{-\pi}^{\pi} dk \cos k K_1(\lambda - \sin k) \times \ln[1 + e^{-\varepsilon_c(k,T)/T}]
$$

$$
+ \frac{T}{2\pi} \int_{-\infty}^{\infty} d\lambda' K_2(\lambda - \lambda') \ln[1 + e^{-\varepsilon_s(\lambda',T)/T}].
$$

(4.3)

The dressed energy $\varepsilon_s(\lambda,T)$ does not explicitly appear in (4.2) since the number of bare single particle states for spinon is zero.

Corresponding to the zero temperature dressed energy (3.19) at $h = 0$, its finite temperature counterpart is

$$
\varepsilon_c(k,T) = \varepsilon_c^{(0)} - \frac{T}{2\pi} \int_{-\pi}^{\pi} dk' \cos k' K(\sin k - \sin k') \ln[1 + e^{-\varepsilon_c(k,T)}].
$$

(4.4)

Similarly, the charge- and spin-wave densities at finite temperature are also given by a set of integral equations

$$
\rho_c(k,T) = [1 + e^{\varepsilon_c(k,T)/T}]^{-1} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda K_1(\sin k - \lambda) \rho_\downarrow(\lambda,T) \right\},
$$

$$
\rho_\downarrow(\lambda,T) = [1 + e^{\varepsilon_s(\lambda,T)/T}]^{-1} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} dk K_1(\lambda - \sin k) \rho_c(k) 
$$

$$
- \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda' K_2(\lambda - \lambda') \rho_\downarrow(\lambda') \right\}. \quad (4.5)
$$
B. Finite-size scaling and spin-charge separation at low temperature

In this subsection, we would like to do finite-size scaling at finite temperature, to show two new results. First, we want to confirm the finite-size scaling (3.5) at finite temperature, as promised above. On the other hand, the spin-charge separation in the Hubbard model has been previously shown only in the strong coupling limit for a given temperature [16,14]. Here we want to show the spin-charge separation at low temperature for arbitrary \( U > 0 \), which was assumed in the Luttinger liquid interpretation of the Hubbard model [17].

The thermodynamic potential (4.2) looks like that of a single-component system (2.8), because the dressed energy \( \varepsilon_s(\lambda, T) \) does not explicitly appear. However, it is necessary to emphasize that \( \varepsilon_c(k, T) \) is coupled to \( \varepsilon_s(\lambda, T) \) through (4.3). Even at \( T = 0 \), the dressed energies are still coupled each other through (3.7). To see the finite-size scaling in \( T \), we fix \( N = N_0 \) and \( N_\downarrow = N_0 / 2 \). As shown in Appendix B, \( \varepsilon_{c(s)}(k(\lambda), T) = \varepsilon_{c(s)}(k(\lambda), 0) + O(T^2) \), i.e., one has

\[
\varepsilon_c(k, T) = \varepsilon_c(k) + \tilde{\varepsilon}_c(k, T) + O(T^3/\nu^2),
\]
\[
\varepsilon_s(\lambda, T) = \varepsilon_s(\lambda) + \tilde{\varepsilon}_s(\lambda, T) + O(T^3/\nu^2).
\] (4.6)

According to (4.3), \( \tilde{\varepsilon}_c(k, T) \) and \( \tilde{\varepsilon}_s(\lambda, T) \) can be determined by the following integral equations

\[
\tilde{\varepsilon}_c(k, T) = -\frac{\pi T^2}{6\varepsilon_c'(k_F)}K_1(\sin k - \lambda_F) + \int_{\lambda_F}^{k_F} \frac{d\lambda}{2\pi} \tilde{\varepsilon}_s(\lambda, T),
\]
\[
\tilde{\varepsilon}_s(\lambda, T) = -\frac{\pi T^2}{6\varepsilon_s'(\lambda_F)}K_1(\sin k - \lambda) \cos k_F
\]
\[
+ \frac{\pi T^2}{6\varepsilon_c'(k_F)}K_1(\sin k_F - \lambda) \cos k_F
\]
\[
- \int_{-k_F}^{k_F} \frac{d\lambda}{2\pi} \cos k K_1(\sin k - \lambda) \tilde{\varepsilon}_c(k, T)
\]
\[
- \int_{\lambda_F}^{\lambda_F'} \frac{d\lambda'}{2\pi} K_2(\lambda - \lambda') \tilde{\varepsilon}_s(\lambda, T).
\] (4.7)

Solving the integral equations, one finds that
\[
\int_{-k_F}^{k_F} \frac{d k}{2 \pi} \tilde{e}_c(k, T) = -\frac{\pi T^2}{6 \tilde{\varepsilon}_c(k_F)} \tilde{f}_c - \frac{\pi T^2}{6 \tilde{\varepsilon}_s(\lambda_F)} \tilde{f}_s,
\]

where

\[
\tilde{f}_c = 2\pi \rho_c(k_F) - 1, \quad \tilde{f}_s = 2\pi \rho_s(\lambda_F).
\]

So, we have

\[
\frac{1}{2\pi} \int_{-k_F}^{k_F} d k \tilde{e}_c(k, T) = -\frac{\pi T^2}{6 \tilde{\varepsilon}_c(k_F)} \left(1 - \frac{1}{2\pi \rho(k_F)}\right) - \frac{\pi T^2}{6 \tilde{\varepsilon}_s(\lambda_F)}.
\]

Because we have fixed the particle numbers, the free energy can be expanded as

\[
\frac{F(T)}{L} = \frac{\Omega(T)}{L} - \mu N_0 - \frac{h}{2}(N_\uparrow - N_\downarrow)
\]

\[
\approx \frac{1}{2\pi} \int_{-\pi}^{\pi} d k \ln[1 + e^{-\varepsilon_c(k)/T}] + \frac{T}{2\pi} \int_{-\pi}^{\pi} \frac{d k}{1 + e^{\varepsilon_c(k)/T}} \tilde{e}_c(k, T)
\]

\[-\mu N_0 - \frac{h}{2}(N_\uparrow - N_\downarrow)
\]

\[
\approx \frac{F(0)}{L} - \frac{\pi T^2}{6 \tilde{\varepsilon}_c(k_F)} + \frac{1}{2\pi} \int_{-k_F}^{k_F} d k \tilde{e}_c(k, T),
\]

where \(F(0)/L \equiv \varepsilon_0\) (3.6). Using (4.10), we have

\[
\frac{F(T)}{L} - \frac{F(0)}{L} = -\frac{\pi T^2}{6 \tilde{\varepsilon}_c(k_F)} - \frac{\pi T^2}{6 \tilde{\varepsilon}_s(\lambda_F)}.
\]

In fact, this result is valid for arbitrary \(N = N_0 + M\) and \(N_\downarrow = N_0/2 + M_\downarrow\) if \(M \ll N_0\) and \(M_\downarrow \ll N_0/2\).

Finally, we get the free energy in the low temperature limit, consistent with (3.3) and resulting in two \(c = 1\) Virasoro algebras. Combining (4.12) and (3.20), we see that at \(h = 0\), the contributions of excitations, including both zero modes and non-zero modes, to the thermodynamic potential are separated into spin and charge parts. In this sense, we can say that there is a spin-charge separation in the Hubbard model at low temperature.

In the low-\(T\) limit, the thermodynamic potential can be rewritten as

\[
\frac{\Omega(T)}{L} - \frac{\Omega(0)}{L} = -\frac{2T}{\pi} I_s(\lambda_F, T) 2\pi \rho_s(\lambda_F)
\]

\[-\frac{2T}{\pi} I_c(k_F, T) 2\pi \rho_c(k_F),
\]

(4.13)
where

\[ I_s(\lambda_F, T) = \int_{\lambda_F}^{\lambda_F + \delta} d\lambda \ln(1 + e^{-\varepsilon_s(\lambda)/T}), \]
\[ I_c(k_F, T) = \int_{k_F}^{k_F + \delta} dk \ln(1 + e^{-\varepsilon_c(k)/T}). \]

(4.14)

The physical momenta corresponding to the charge and spin excitations are defined by

\[ dp = 2\pi \rho_c(k) dk, \quad d\bar{p} = 2\pi \rho_c(\lambda) d\lambda. \]

(4.15)

Finally, the thermodynamic potential in the low-\( T \) limit can be expressed as

\[ \frac{\Omega(T)}{L} = -\frac{T}{2\pi} \int_{-\pi}^{\pi} dk \ln[1 + e^{-\varepsilon_c(k, \pm)/T}] - \frac{T}{\pi} \int_{-\delta}^{\delta} d\delta p_s \ln(1 + e^{-|\varepsilon_s(\delta p_s)|/T}), \]

(4.16)

where \( \delta p_s = p_s - p_s(\lambda_F). \) Eq. (4.16) is useful to bosonize the Hubbard model in the next section.

V. BOSONIZATION OF THE HUBBARD MODEL

Bosonization of the Hubbard model can generally be achieved by Tomonaga-Luttinger’s bosonization techniques \[34\]. However, the existence of spin degree of freedom brings scattering processes other than the forward scattering, say the backward and umklapp scatterings. They can not be exactly diagonalized. Fortunately in the Hubbard model case, Luther and Emery showed that the Hamiltonian still can be diagonalized at particular values of coupling constants \[35\] and the backward and umklapp scatterings can develop gaps for spin- and charge-waves respectively. Then reasoning with renormalization group analysis \[36\], one expects that the backward and umklapp scatterings are irrelevant. In this section, we will confirm these ideas in the bosonization approach based on Bethe ansatz equations, which we have developed in Sec. II to bosonize the C-S model.
A. Bosonized Hamiltonian and Lagrangian

Because of the similarity between (2.8) and (4.2), the grand partition function of the Hubbard model can be put into a form like (2.21). And again this grand partition function corresponds to an ideal fermion system with a complicated, T-dependent dressed energy $\varepsilon_c(k, T)$. To exploit this fermion representation, in the case of the C-S model we noticed that $\varepsilon(k, T) = \varepsilon(k, 0) + O(e^{-|\varepsilon|/T})$, so that we could use $\varepsilon(k, 0)$ to define an effective Hamiltonian to derive the low-$T$ grand partition function. However, this trick cannot be repeated for the Hubbard model, since $\varepsilon_c(k, T) = \varepsilon_c(k) + O(T^2/\nu)$. So we cannot simply ignore the $T$-dependent part of $\varepsilon_c(k, T)$ in discussing the low-$T$ thermodynamics of the system.

Instead of the original thermodynamic potential (4.2), we begin with the low-$T$ thermodynamic potential (4.16). Now, both $\varepsilon_c(k)$ and $\varepsilon_s(\delta p_s)$ are $T$-independent. In principle, one may try to introduce two kinds of fermions with dispersions $\varepsilon_c(k)$ and $\varepsilon_s(\delta p_s)$ respectively. However, the fermions with $|p_s| > p_{sf} + \delta$ and $|p_s| < p_{sf} - \delta$ are not defined. (One may take these states to have zero energy, but it leads to degeneracy deep inside the Fermi sea.) Rather, noting the cut-off $\delta$-independence of the low-$T$ thermodynamic potential, we first keep a finite $\delta$, and take the limit $\delta \to 0$ after all calculations. Thus we rewrite (4.16) as

$$\frac{\Omega(T)}{L} = \lim_{\delta \to 0} \frac{\Omega(T, \delta)}{L},$$

$$\frac{\Omega(T, \delta)}{L} = \frac{\Omega_c(T)}{L} + \frac{\Omega_s(T, \delta)}{L},$$

with

$$\frac{\Omega_c(T)}{L} = -\frac{T}{2\pi} \int_{-\pi}^{\pi} dk \ln(1 + e^{-\varepsilon_c(k, \pm)/T}),$$

$$\frac{\Omega_s(T, \delta)}{L} = -\frac{T}{2\pi} \int_{-n_s \pi}^{n_s \pi} dp_s \ln(1 + e^{-\varepsilon_s(p_s, \delta)/T}).$$

Here $\varepsilon_s(p_s, \delta)$ is defined by

$$\varepsilon_s(p_s, \delta) = \begin{cases} 
\pm v_s(p_{sf} \mp k_F), & p_{sf} - \delta < |p_s| < p_{sf} + \delta, \\
\pm v_s(k \mp k_F)e^{-|p_{sf} \mp k_F|/\delta}, & 0 \leq |p_s| < p_{sf} - \delta \\
or |p_s| > p_{sf} + \delta.
\end{cases}$$

(5.3)
If we keep $\delta$ until all calculations are finished, the degeneracy of the states far from the Fermi surface will be removed. Now both $\varepsilon_c(k)$ and $\varepsilon_s(p_s, \delta)$ become $T$-independent. Therefore, the low-$T$ grand partition function can be obtained by an effective Hamiltonian that incorporates two kinds of fermions with dispersions $\varepsilon_c(k)$ and $\varepsilon_s(p_s, \delta)$:

$$Z_G = \prod_k (1 + e^{-\varepsilon_c(k)/T}) \prod_{p_s} (1 + e^{-\varepsilon_s(p_s, \delta)/T}) = \text{Tr} e^{-\beta H_{\text{eff}}}.$$

(5.4) The effective Hamiltonian $H_{\text{eff}}$ is given by

$$H_{\text{eff}} = \sum_k \varepsilon_c(k)c_k^\dagger c_k + \sum_{p_s} \varepsilon_s(p_s, \delta)s_{p_s}^\dagger s_{p_s} + \delta E_0,$$

(5.5) where $c_k^\dagger$ and $s_{p_s}^\dagger$ are spinless fermion creation operators, which can be identified as the charge- and spin-excitations. Possible zero-mode excitations are included in the last term, $\delta E_0$, of the effective Hamiltonian.

Now we are at a position to bosonize the theory, as we have done for the C-S model, except that there are two kinds of fermions. Because of the continuity of $\varepsilon'_c(k)$ and $\varepsilon'_s(p_s, \delta)$ at Fermi surface, the bosonization is standard. At low-$T$, only low energy excitations near the Fermi surface are relevant. So, instead of the full expressions of the dispersions, the linear dispersions are employed

$$\varepsilon_c^\pm(k) = \pm \varepsilon'_c(k_F)(k \mp k_F) = \pm v_c(p \mp p_F),$$

$$\varepsilon_s^\pm(p_s) = \pm v_s(p_s \mp p_sF),$$

(5.6) and the effective Hamiltonian is separated into the right- and left- moving parts

$$H_{\text{eff}} = H_{c,+} + H_{c,-} + H_{s,+} + H_{s,-} + \delta E_0,$$

$$H_{c,\pm} = \sum_p \varepsilon_c^\pm(k)c_p^\dagger c_p, \quad H_{s,\pm} = \sum_{p_s} \varepsilon_s^\pm(p_s)s_{p_s}^\dagger s_{p_s}.$$  

(5.7)

By using the well-known bosonization technique, we have

$$\rho_{c,q}^{(+)} = \sum_{p \sim p_F} :c_{p+q}^\dagger c_p :,$$

$$\rho_{c,q}^{(-)} = \sum_{p \sim -p_F} :c_{p-q}^\dagger c_p :,$$

$$\rho_{s,q}^{(+)} = \sum_{p_s \sim p_sF} :s_{p_s+q}^\dagger s_{p_s} :,$$

$$\rho_{s,q}^{(-)} = \sum_{p_s \sim -p_sF} :s_{p_s-q}^\dagger s_{p_s} :.$$  

(5.8)
They satisfy the following commutation relations:

\[
\begin{align*}
\left[ \rho_{c(s),q}^{(\pm)}, \rho_{c(s),q'}^{(\mp)\dagger} \right] &= \frac{L}{2\pi} q \delta_{q,q'}, \\
\left[ H_{c(s),\pm}, \rho_{c(s),q}^{(\pm)} \right] &= \pm v_c \rho_{c(s),q}^{(\pm)},
\end{align*}
\] (5.9)

which describe two 1-d free phonons with the sound velocities \( v_c \) and \( v_s \). At \( \hbar = 0 \), the bosonized Hamiltonian satisfying (5.9) is given by

\[
H_B = v_c \sum_{q>0} (\hat{a}_q^{\dagger} a_q + \hat{\tilde{a}}_q^{\dagger} \tilde{a}_q) + \frac{1}{2L} (v_{c,N} M_c^2 + v_{c,J} J_c^2) \\
+ v_s \sum_{q>0} (\hat{b}_q^{\dagger} b_q + \hat{\tilde{b}}_q^{\dagger} \tilde{b}_q) + \frac{1}{2L} (v_{s,N} M_s^2 + v_{s,J} J_s^2),
\] (5.10)

where

\[
\begin{align*}
a_q &= \sqrt{2\pi/qL} \rho_{c,q}^{(+)} , & \tilde{a}_q &= \sqrt{2\pi/qL} \rho_{c,q}^{(-)\dagger} , \\
b_q &= \sqrt{2\pi/qL} \rho_{s,q}^{(+)} , & \tilde{b}_q &= \sqrt{2\pi/qL} \rho_{s,q}^{(-)\dagger}
\end{align*}
\] (5.11)

which are normalized boson annihilation operators. The cut-off \( \delta \)-dependence only contributes a constant to (5.10), which vanishes as \( \delta \to 0 \). So, we can take \( \delta \to 0 \) at this stage, which does not affect any results that we will obtain below.

The bosonized total momentum operator may also be obtained by bosonizing the fermionized \( \rho_{c(s)} = \sum_p \rho_{c(s),p}^{\dagger} c_p + \sum_p \rho_{s,p}^{\dagger} s_p \). It reads

\[
P = \sum_{q>0} q (a_q^{\dagger} a_q - \tilde{a}_q^{\dagger} \tilde{a}_q) + \pi J_c (n_{c0} + M_c / L) \\
+ \sum_{q>0} q (b_q^{\dagger} b_q - \tilde{b}_q^{\dagger} \tilde{b}_q) + \pi J_s M_s / L.
\] (5.12)

In the coordinate-space formulation, the charge-density field and spin-density field are given by

\[
\rho_{c(s)}(x) = \rho_{c(s),R}(x) + \rho_{c(s),L}(x),
\] (5.13)
where \( \Pi \) with obeying the selection rule (3.24), and

\[
M = \begin{bmatrix} \phi(t) \\ \phi(x) \end{bmatrix},
\]

Here \( M_{c(s)} = M_{c(s),R} + M_{c(s),L} \). The conjugate field \( \phi_{c(s)}(x) \) of \( \rho_{c(s)}(x) \), which obeys

\[
[\phi_{c(s)}(x),\rho_{c(s)}(x')] = i\delta(x-x'), \quad \text{is} \quad \phi_{c(s)}(x) = \phi_{c(s),R}(x) + \phi_{c(s),L}(x) \text{ with}
\]

\[
\phi_{c,R}(x) = \phi_{c,0,R} + \frac{\pi J_{c,R} x}{L} + i \sum_{q>0} \sqrt{\frac{\pi \lambda_c}{2qL}} (e^{iqx} a_q - e^{-iqx} a_q^\dagger),
\]

\[
\phi_{c,L}(x) = \phi_{c,0,L} + \frac{\pi J_{c,L} x}{L} + i \sum_{q>0} \sqrt{\frac{\pi \lambda_c}{2qL}} (e^{-iqx} \tilde{a}_q - e^{iqx} \tilde{a}_q^\dagger),
\]

\[
\phi_{s,R}(x) = \phi_{s,0,R} + \frac{\pi J_{s,R} x}{L} + i \sum_{q>0} \sqrt{\frac{\pi \lambda_s}{2qL}} (e^{iqx} b_q - e^{-iqx} b_q^\dagger),
\]

\[
\phi_{s,L}(x) = \phi_{s,0,L} + \frac{\pi J_{s,L} x}{L} + i \sum_{q>0} \sqrt{\frac{\pi \lambda_s}{2qL}} (e^{-iqx} \tilde{b}_q - e^{iqx} \tilde{b}_q^\dagger),
\]

(5.14)

where \( J_{c(s)} = J_{c(s),R} + J_{c(s),L} \). Here \( M_{c(s)} \) and \( J_{c(s)} \) are operators with integer eigenvalues obeying the selection rule (3.24), and \( \phi_{c,0} = \phi_{c,0,R} + \phi_{c,0,L} \) and \( \phi_{s,0} = \phi_{s,0,R} + \phi_{s,0,L} \) are angular variables conjugated to \( M_c \) and \( M_s \): \( [\phi_{c(s)}(0), M_{c(s)}] = i \).

In the coordinate-space formulation, the bosonized Hamiltonian (5.10) reads

\[
H_B = H_{B,c} + H_{B,s},
\]

\[
H_{B,c} = \frac{v_c}{2\pi} \int_0^L dx \left[ \Pi_c(x)^2 + (\partial_x X_c(x))^2 \right],
\]

\[
H_{B,s} = \frac{v_s}{2\pi} \int_0^L dx \left[ \Pi_s(x)^2 + (\partial_x X_s(x))^2 \right],
\]

(5.16)

where \( \Pi_{c(s)}(x) = \pi^{1/2} \rho_{c(s)}(x) \) and \( X_{c(s)} = \lambda_{c(s)}^{-1/2} \phi_{c(s)}(x) \). With the Heisenberg operators \( A(x,t) = e^{iH_B t} A(x) e^{-iH_B t} \), the Lagrangian density is given by

\[
\begin{align*}
\end{align*}
\]

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\[ \mathcal{L} = \mathcal{L}_c + \mathcal{L}_s, \quad (5.17) \]
\[ \mathcal{L}_c = \frac{v_c}{2\pi} \partial_\alpha X_c(x,t) \partial^\alpha X_c(x,t), \quad (5.18) \]
\[ \mathcal{L}_s = \frac{v_s}{2\pi} \partial_\alpha X_s(x,t) \partial^\alpha X_s(x,t). \quad (5.19) \]

The Lagrangians (5.18) and (5.19) give rise to two \( c = 1 \) Virasoro algebras. We see that there is no mixture between spin and charge sectors in the total Lagrangian (5.17). Therefore, in charge-spin basis, the low energy behavior of the system is characterized by the direct product of two independent Virasoro algebras both with central charges \( c = 1 \) [24].

Note the selection rule (3.24); then one can take \( D_{c(s)} = 2J_{c(s)} = M_{c(s)} \text{mod}(1) \). The zero-mode part in \( H_{B,c(s)} \) becomes

\[ \frac{v_{c(s)}\pi}{2L} \left( \lambda_{c(s)} M_{c(s)}^2 + \frac{1}{4\lambda_{c(s)}} D_{c(s)}^2 \right). \quad (5.20) \]

So each sector has the duality relation as in the usual \( c = 1 \) CFT.

Changing the right-and-left-moving representation to the \( \theta-\phi \) representation with \( \theta = \phi_R - \phi_L \), which Haldane has used in discussing the Luttinger liquid theory [25], we see that our bosonization theory precisely agrees with the Luttinger liquid theory for the Hubbard model proposed by Ren and Anderson [17]. The selection rules chosen in [17] were

\[ J_{\uparrow,\downarrow} = M_{\uparrow,\downarrow} \text{mod}(2), \quad (5.21) \]

where

\[ J_c = \frac{J_{\uparrow} + J_{\downarrow}}{2}, \quad J_s = \frac{J_{\uparrow} - J_{\downarrow}}{2}. \quad (5.22) \]

The selection rules (5.21) of \( J_{\uparrow} \) and \( J_{\downarrow} \) are a restriction on the quantum numbers more constrained than that given by (3.24). For example, the choice \( M_c = M_s = 1, J_c = 1/2 \) and \( J_s = -1/2 \), which equals to \( M_{\uparrow} = 1, M_{\downarrow} = 0 \) and \( J_{\uparrow} = 0 \) and \( J_{\downarrow} = 1 \), is forbidden by (5.21) but is allowed by (3.24). This choice corresponds to a charge-1, spin-up boson excitation near \( k_F \) as we will see below. We believe that (3.24) gives the correct selection rules for the quantum numbers, since it is derived on the basis of the Bethe ansatz equations.
B. Correlation functions of single quasiparticles

In order to calculate the correlation functions in the low energy limit, we first need to determine the allowed operator in the theory. Since there are two angular variables, \( \phi_{c0} \) and \( \phi_{s0} \), there are hidden symmetries in the theory under

\[
\phi_{c(s)} \rightarrow \phi_{c(s)} + 2\pi. \tag{5.23}
\]

Then \( X_{c(s)} \) is compactified on a circle with radius \( R_{c(s)} = \lambda_{c(s)}^{-1/2} \),

\[
X_{c(s)} \sim X_{c(s)} + 2\pi R_{c(s)} \tag{5.24}
\]

Operators \( V[X] \) are allowed if only if they are invariant under (5.23) (or (5.24)). Because the theory is described by the direct product of two Virasoro algebras, the primary fields of the theory have a similar structure. The primary fields satisfy the periodic boundary conditions are

\[
\phi_{[M_c,J_c;M_s,J_s]}(x) \sim \ e^{i(M_c\lambda_{c}/2 + J_c/\lambda_{c}/2)X_{R,c}(x)} \\
\times e^{i(M_s\lambda_{s}/2 - J_s/\lambda_{s}/2)X_{L,c}(x)} \\
\times e^{i(M_s\lambda_{s}/2 + J_s/\lambda_{s}/2)X_{R,s}(x)} \\
\times e^{i(M_s\lambda_{s}/2 - J_s/\lambda_{s}/2)X_{L,s}(x)} \ . \tag{5.25}
\]

The fermion operator, say of charge-1 and spin-up, is given by \( M_c = M_s = 1 \):

\[
\Psi_{F}^{1}(x) \sim \sum_{J_c + J_s = \text{odd}} e^{iO_J} \cdot e^{i(\lambda_{c}/2 + J/\lambda_{c}/2)X_{R,c}(x)} \\
\times e^{i(\lambda_{s}^{1/2} - J_s/\lambda_s^{1/2})X_{L,c}(x)} \\
\times e^{i(\lambda_{s}^{1/2} + J_s/\lambda_s^{1/2})X_{R,s}(x)} \\
\times e^{i(\lambda_{s}^{1/2} - J_s/\lambda_s^{1/2})X_{L,s}(x)} \ , \tag{5.26}
\]

where \( O_J \) are constant value operators satisfying \([O_J, O_{J'}] = i[(J_c + J_s)/2 - (J'_c + J'_s)/2]\).

The above fermion operator satisfies our selection rule (3.24) and, since \( J_c + J_s = \text{odd} \), the selection rule (5.21) of Ren and Anderson as well. However, our rule (3.24) allows
more possible quantum numbers, say charge-1, spin-up bosonic excitations described by the primary fields:

\[
\Psi_B^\dagger(x) \sim \sum_{J_c+J_s=\text{even}} e^{iO_J} \cdot e^{i(\lambda_c^{1/2}+J/\lambda_c^{1/2})X_{R,c}(x)}
+ e^{i(\lambda_c^{1/2}-J_c/\lambda_c^{1/2})X_{L,c}(x)}
+ e^{i(\lambda_s^{1/2}+J_s/\lambda_s^{1/2})X_{R,s}(x)}
+ e^{i(\lambda_s^{1/2}-J_s/\lambda_s^{1/2})X_{L,s}(x)}.
\] (5.27)

The selection rule for bosons reads

\[J_{\uparrow,\downarrow} = (M_{\uparrow,\downarrow} + 1)\text{mod}(2),\] (5.28)

which is different with that for fermions, (5.21).

Besides the fermion and boson excitations, there are two kinds of anyonic excitations which are charge-1 but non-periodic primary fields. The anyon operators read

\[
\Psi_{\lambda_c}^\dagger = \Psi_B^\dagger e^{i\lambda_c^{1/2}(X_{R,c}-X_{L,c})},
\Psi_{\lambda_s}^\dagger = \Psi_B^\dagger e^{i\lambda_s^{1/2}(X_{R,s}-X_{L,s})},
\] (5.29)

which obey

\[
\Psi_{\lambda_c(\uparrow)}^\dagger(x)\Psi_{\lambda_c(\downarrow)}^\dagger(x') - e^{i\lambda_c(\uparrow)\pi}\Psi_{\lambda_c(\uparrow)}^\dagger(x')\Psi_{\lambda_c(\downarrow)}^\dagger(x) = 0.
\] (5.30)

Thus, the statistics of \(\Psi_{\lambda_c}^\dagger\) is interaction-dependent, while \(\Psi_{\lambda_s}^\dagger\) is a semion, in the half way between fermion and boson.

The dynamic correlation functions of fermionic and bosonic single-particle operators, (5.26) and (5.27), can be easily calculated:

\[
G_F(x, t) \equiv \langle \Psi_F(x, t)\Psi_F^\dagger(0, 0) \rangle
\sim \sum_{n=-\infty, m=4n+1}^{\infty} A_m
\frac{\frac{1}{2}(e^{-\nu+me\nu})^2}{x_{R,c}} - \frac{\frac{1}{2}(e^{-\nu-me\nu})^2}{x_{L,c}} + \frac{1}{2}e^{imk_F}
+ \sum_{n=-\infty, m=4n+3}^{\infty} B_m
\frac{\frac{1}{2}(e^{-\nu+me\nu})^2}{x_{R,c}} - \frac{\frac{1}{2}(e^{-\nu-me\nu})^2}{x_{L,c}} + \frac{1}{2}e^{imk_F},
\] (5.31)
\[ G_B(x, t) \equiv \langle \Psi_B(x, t) \Psi_B^\dagger(0, 0) \rangle \]
\[ \sim \sum_{n=-\infty}^{\infty} \sum_{m=4n+1}^{4n+3} C_m \]
\[ \frac{-\frac{1}{2}(e^{-\varphi} + me\varphi)^2}{x_{R,c}} \frac{-\frac{1}{2}(e^{-\varphi} - me\varphi)^2}{x_{L,c}} \frac{-\frac{1}{2}}{x_{L,s}} e^{imk_F} \]
\[ + \sum_{n=-\infty}^{\infty} \sum_{m=4n+4}^{4n+7} D_m \]
\[ \frac{-\frac{1}{2}(e^{-\varphi} + me\varphi)^2}{x_{R,c}} \frac{-\frac{1}{2}(e^{-\varphi} - me\varphi)^2}{x_{L,c}} \frac{-\frac{1}{2}}{x_{R,s}} e^{imk_F}, \] (5.32)

where \( A_m, B_m, C_m \) and \( D_m \) are regularization-dependent constants, and \( x_{R,c(s)} = x - v_{c(s)} t \) and \( x_{L,c(s)} = x + v_{c(s)} t \). \( e^{-2\pi} = 2\lambda_c \). To derive (5.31) and (5.32), we have taken the leading term only for each given \( m \). For the fermion case, \( J_s \) is taken to be \( \frac{1}{2} \) or \( -\frac{1}{2} \) with respect to \( m = 4n + 1 \) or \( 4n + 3 \). For the boson case, \( J_s \) is taken to be \( -\frac{1}{2} \) or \( \frac{1}{2} \) with respect to \( m = 4n + 1 \) or \( 4n + 3 \). We see that the only difference between fermion and boson is that when the fermion has a left(right)-moving spinon-part in a given \( mk_F \)-oscillation, the boson has a right(left)-moving one. They have the same momentum distribution singularities:

\[ n(k) \sim \text{const. - const. sgn}(k - k_F)|k - k_F|^{(e^{-2\varphi} + e^{2\varphi} - 2)/4}, \] (5.33)

for \( k_F \)-oscillations and

\[ n(k) \sim \text{const. - const. sgn}(k - 3k_F)|k - 3k_F|^{(9e^{-2\varphi} + e^{2\varphi} - 2)/4}, \] (5.34)

for \( 3k_F \)-oscillations, etc. They are consistent with those given in [17]. In the strong coupling limit, \( U/t \to \infty \), one has \( (e^{-2\varphi} + e^{2\varphi} - 2)/4 = \frac{1}{8} \) and \( (9e^{-2\varphi} + e^{2\varphi} - 2)/4 = \frac{9}{8} \) as \( e^{-2\varphi} = \frac{1}{2} \).

This implies that there is a similarity of boson and fermion in the two-component Luttinger liquid theory.

The anyon correlation functions read

\[ G_{\lambda_c}(x, t) \equiv \langle \Psi_{\lambda_c}(x, t) \Psi_{\lambda_c}^\dagger(0, 0) \rangle \]
\[ \sim \sum_{n=-\infty}^{\infty} \sum_{m=4n+1}^{4n+3} C_m x_{R,c}^{\frac{1}{2}(2e^{-\varphi} + me\varphi)^2} \]
\[ \frac{-\frac{1}{2}(me\varphi)^2}{x_{L,c}} \frac{-\frac{1}{2}}{x_{L,s}} e^{imk_F} \]
\[ + \sum_{n=-\infty, m=4n+3}^{\infty} B_m x_{R,c} \frac{1}{4}(2e^{2\varphi}+me^{2\varphi})^2 \]
\[ \sum_{n=-\infty, m=4n+3} x_{L,c}^{-\frac{4}{3}}(me^{2\varphi})^2 x_{R,s}^{-\frac{4}{3}} e^{imk_F}, \]

(5.35)

and

\[ G_{\lambda_s}(x, t) \equiv \langle \Psi_{\lambda_s}(x, t)\Psi_{\lambda_s}^\dagger(0, 0) \rangle \]
\[ \sim \sum_{n=-\infty, m=4n+1}^{\infty} A_n x_{R,c}^{-\frac{4}{3}}(e^{-\varphi+me^{2\varphi}})^2 x_{L,c}^{-\frac{4}{3}}(e^{-\varphi-me^{2\varphi}})^2 x_{R,s}^{-\frac{4}{3}}x_{L,s}^{-\frac{4}{3}} e^{imk_F} \]
\[ + \sum_{n=-\infty, m=4n+3}^{\infty} B_m x_{R,c}^{-\frac{4}{3}}(e^{-\varphi+me^{2\varphi}})^2 x_{L,c}^{-\frac{4}{3}}(e^{-\varphi-me^{2\varphi}})^2 x_{R,s}^{-\frac{4}{3}}x_{L,s}^{-\frac{4}{3}} e^{imk_F}. \]

(5.36)

The momentum distributions corresponding to (5.35) vanish faster than those for fermions or bosons near \( mk_F \). So, this kind of excitations may be more difficult to be observed. The distributions corresponding to (5.36) near \((4n+1)k_F\), say \( k_F \), also decay fast but those near \((4n+3)k_F\), say \( 3k_F \), do decay slower:

\[ n(k) \sim \text{const.} - \text{const.} \frac{\text{sgn}(k - 3k_F) |k - 3k_F|^6}{(e^{-e^{2\varphi}}+e^{2\varphi}-3)/4} \]
\[ \rightarrow \text{const.} - \text{const.} \frac{\text{sgn}(k - 3k_F) |k - 3k_F|^6}{(e^{-e^{2\varphi}}+e^{2\varphi}-3)/4}, \quad U/t \rightarrow \infty. \]

(5.37)

So, the semion may possibly be observed.

As shown by (2.46), for the C-S model, it is the same density operator no matter that the single-particle excitations are fermionic, bosonic or anyonic. This remains true for the charge density and spin density in the Hubbard model. And so for the charge-charge and spin-spin density correlation functions. We can define the pairing operators in the theory. One finds that the most important contributions to the spin-singlet and triplet pairings, up to \( 2k_F \)-oscillation, still come from the fermionic (or bosonic) excitations. All of these correlation functions have been calculated in the reference [17], and can be reproduced easily within our approach.
VI. CONCLUSIONS

We have developed a new approach for bosonizing 1-d exactly soluble many-body models at low energy. In this approach, one first derives the low temperature partition function from the Bethe ansatz equations, and then bosonizes the partition function. The asymptotic correlation functions can then be calculated by field theory techniques. Because our bosonization started with the Bethe ansatz equations, we call it the bosonization based on Bethe ansatz equations. We explicitly carried out this program with the C-S model and the Hubbard model as examples. Some new results were obtained during this exercise.

For the C-S model, we showed that the low energy effective theory is described by two different classes of $c = 1$ CFT’s, depending on whether the Hamiltonian describes the interactions between bosons or between fermions. Using the standard terminology in CFT, it is a $c = 1$ CFT compactified on a circle $S^1$ for the bosonic case, compactified on an interval $S^1/Z_2$ for the fermionic case. A piece of evidence for this difference is that the charge-1 primary field in the two theories are different. It is a fermion in the theory of interacting fermions, while it is a boson for the interacting bosons. Their correlation functions have totally different asymptotic behavior. Also our bosonization of the C-S model prepares some techniques for bosonizing multi-component models.

The repulsive Hubbard model at less than half filling is an example of the two-component models. In the above the spin-charge separation in this model has been explicitly verified not only for the strong-coupling limit but also for finite $U$. The low energy theory is shown to be controlled by the direct product of two usual $c = 1$ Virasoro algebras. We confirmed this point by the finite-size scaling in the temperature direction. Furthermore, the explicit low-$T$ grand partition function enabled us to identify the system with a two-component Luttinger liquid theory by using the bosonization technique. We showed that the backforward scattering processes are suppressed at low energy, in agreement with the renormalization group analysis. We pointed out that not only the fermionic excitations are allowed near the Fermi surface, but also the bosonic and anyonic ones. The single-particle correlation
functions of those excitations were systematically calculated in a simple way.

Our bosonization approach can be easily generalized to any 1-d many-body models that is exactly soluble by Bethe ansatz, including thermodynamic Bethe ansatz, and provides a simple way to calculate asymptotic correlation functions using field theory techniques.

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APPENDIX A: FINITE SIZE SCALING IN THE L-DIRECTION FOR THE C-S MODEL

This appendix is devoted to resolve a puzzle raised in [20], where it was found that if we take \( k = \frac{2\pi \lambda}{L} n \) in the thermodynamic limit, naively the discrete sum for the total energy of the C-S model differs from the continuous integration by \( -\frac{\pi \lambda v_s}{6L} \). This seems to imply a central charge \( c = \lambda \), in conflict to (2.19). However, as we see below, a careful consideration for the finite size scaling shows that the naive result is wrong. There is an additional term of order \( O(1/L^3) \) when we go to the continuous limit of (2.5) from the discrete version. This will cause a rescaling of the particle density together with that of the total energy. Taking into account this subtlety, we will finally achieve the correct finite size scaling of the ground state energy

\[
E_{0,L} - E_0 = -\frac{\pi v_s}{6L} + O\left(\frac{1}{L^2}\right).
\]  

(A1)

To see this, a useful formula relating the discrete sum to the integration is

\[
\frac{1}{L} \sum_{n=N_1}^{N_2} f\left(\frac{I_n}{L}\right) = \int_{(N_1+1/2)/L}^{(N_2-1/2)/L} dx f(x)
\]

\[
+ \frac{1}{24L^2} [f'((N_1-1)/L) - f'((N_2+1)/L)] + O(1/L^3).
\]  

(A2)

Using (A2), the discrete version of the density \( \rho_L(k) \) can be written as
\[ \rho_L(k) = \frac{1}{2\pi} + (1 - \lambda)\rho_L(k) + \frac{\pi}{12L^2}(1 - \lambda)\lambda \frac{d}{dk_F}[\delta(k - k_F) - \delta(k + k_F)]. \]  

(A3)

The discrete version differs from its continuous counterpart by a term of the order \( O(1/L^2) \).

Denote

\[ \rho_L(k) = \rho(k) + \rho_1(k); \]  

(A4)

one has

\[ \rho_1(k) = -\frac{\pi}{12L^2}(1 - \lambda)\frac{d}{dk_F}[\delta(k + k_F) - \delta(k - k_F)], \]  

(A5)

while \( \rho(k) = 1/2\pi\lambda \). Now let’s examine the finite size scaling of the ground state energy.

Using (A2) again, we have

\[ \frac{E_L}{L} = \int_{-k_F}^{k_F} dk\rho_L(k)\epsilon_0 \]

\[ + \frac{1}{24L^2\rho(k_F)}[\epsilon'_0(k)|_{-k_F} - \epsilon'_0(k)|_{k_F}] \]

\[ = \int_{-k_F}^{k_F} dk\rho(k)\epsilon_0(k) + \int_{-k_F}^{k_F} dk\rho_1(k)\epsilon_0(k) - \frac{\pi \lambda v_F}{6L^2}. \]  

(A6)

The second term of the last equation is easy to calculate and one has

\[ \int_{-k_F}^{k_F} dk\rho_1(k)\epsilon_0(k) = -\frac{\pi v_F}{6L^2}(1 - \lambda). \]  

(A7)

Substituting this into (A6), we have

\[ \frac{E_{0,L}}{L} - \frac{E_0}{L} = -\frac{\pi v_F}{6L^2}. \]  

(A8)

This confirms the result of (2.20).

**APPENDIX B: PROOF OF** \( \varepsilon(T) = \varepsilon(0) + O(T^2/\nu) \)

In this appendix, we derive the low temperature expansions of \( \varepsilon_c(k, T) \) and \( \varepsilon_s(\lambda, T) \), by solving the integral equations (4.3) by iteration.
At the zeroth-order we take
\[ \varepsilon^{(0)}_c(k, T) = \varepsilon^{(0)}_c(k), \quad \varepsilon^{(0)}_s(\lambda, T) = \varepsilon^{(0)}_c(\lambda) = \frac{h}{2}, \]  
(B1)

Then at the first order, i.e., after one iteration, the dressed energies are
\[ \varepsilon^{(1)}_c(k, T) = \varepsilon^{(0)}_c + O(Te^{-h/2T}), \]  
(B2)

\[ \varepsilon^{(1)}_s(\lambda, T) = \varepsilon^{(0)}_s(\lambda) + \frac{1}{2\pi} \int_{-k_F^{(0)}}^{k_F^{(0)}} dk K_1(\lambda - \sin k)\varepsilon^{(0)}_c(k) \]
\[ - \frac{1}{2\pi} \int_{-k_F^{(0)}}^{k_F^{(0)}} d\lambda' K_2(\lambda - \lambda')\varepsilon^{(0)}_s(\lambda') \]
\[ - \frac{T}{\pi} \int_{k_F^{(0)}-\delta}^{k_F^{(0)}} dk K_1(\lambda - \sin k) \ln(1 + e^{-\varepsilon^{(0)}_c}) \]
\[ + O(Te^{-h/2T}), \]  
(B3)

where \( k_F^{(0)} \) is determined by \( \varepsilon^{(0)}_c(k) = 0 \). We denote \( \varepsilon^{(1)}_c(k) \equiv \varepsilon^{(0)}_c(k) \) and the first three terms on the right hand side of (B3) by \( \varepsilon^{(1)}_s(\lambda) \). The fourth term on the right hand side of (B3) is proportional to \( T^2/v_c \), if we consider only the contributions from near the Fermi points.

Thus, we have
\[ \varepsilon^{(1)}_c(k, T) = \varepsilon^{(1)}_c(k) + O(Te^{-h/2T}), \]
\[ \varepsilon^{(1)}_s(\lambda, T) = \varepsilon^{(1)}_c(\lambda) + O(T^2/v_c). \]  
(B4)

To get \( \varepsilon^{(2)}_c(k, T) \) and \( \varepsilon^{(2)}_s(k, T) \), it is enough to replace \( \varepsilon^{(1)}_c(k, T) \) and \( \varepsilon^{(1)}_s(k, T) \) by \( \varepsilon^{(1)}_c(k, 0) \) and \( \varepsilon^{(1)}_s(k, 0) \) on the iteration equations. Then by using similar techniques, we have
\[ \varepsilon^{(2)}_c(k, T) = \varepsilon^{(2)}_c(k) + O(T^2/v_s), \]
\[ \varepsilon^{(2)}_s(\lambda, T) = \varepsilon^{(2)}_c(\lambda) + O(T^2/\nu). \]  
(B5)

where \( \varepsilon^{(2)}_c(k) \) and \( \varepsilon^{(2)}_c(\lambda) \) are given by the iteration of integral equations (3.7) and \( \nu \) could be \( v_s \) or \( v_c \). Repeating the iteration, we finally have
\[ \lim_{n \to \infty} \varepsilon^{(n)}_c(k, T) = \lim_{n \to \infty} \varepsilon^{(n)}_c(k) + O(T^2/\nu), \]
\[ \lim_{n \to \infty} \varepsilon^{(n)}_s(\lambda, k) = \lim_{n \to \infty} \varepsilon^{(n)}_s(\lambda) + O(T^2/\nu). \]  
(B6)
Because of the convergence of the series for \( \varepsilon^{(n)}_{c(s)}(k, T) \), (B6) implies that

\[
\varepsilon_c(k, T) = \varepsilon_c(k) + O(T^2/\nu), \tag{B7}
\]

\[
\varepsilon_s(\lambda, k) = \varepsilon_s(\lambda) + O(T^2/\nu),
\]

which is what we intend to prove.

**APPENDIX C: AN ALTERNATIVE PROOF OF (1.12)**

We give an alternative proof of (1.12). For the simplicity, we consider the \( h = 0 \) case in which, as Lieb and Wu pointed out, there are no particle-like spinon excitations. The total energy is given by

\[
\frac{E(T)}{L} = \int_{-\pi}^{\pi} dk \rho_c(k, T) \varepsilon^{(0)}_c(k, T) + \int_{-\infty}^{\infty} d\lambda \rho_s(\lambda, T) \varepsilon^{(0)}_s(\lambda, T). \tag{C1}
\]

Substituting (1.3) into (C1), one has

\[
\frac{E(T)}{L} = \int_{-\pi}^{\pi} dk \rho_c(k, T) \varepsilon_c(k, T)
\]

\[
+ \int_{-\pi}^{\pi} d\lambda T \ln[1 + e^{-\varepsilon_c(k, T)/T}] \cos k \frac{\cos k}{2\pi}
\]

\[
\times \int_{-\infty}^{\infty} d\lambda \rho_s(\lambda, T) K_1(\lambda - \sin k)
\]

\[
+ \int_{-\infty}^{\infty} d\lambda \rho_s(\lambda, T) \varepsilon_s(\lambda, T)
\]

\[
+ \int_{-\infty}^{\infty} d\lambda T \ln[1 + e^{-\varepsilon_s(\lambda, T)/T}]
\]

\[
\times \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} dk \rho_c(k, T) K_1(\lambda - \sin k)
\right.
\]

\[
- \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda' \rho_s(\lambda', T) K_2(\lambda - \lambda') \right]. \tag{C2}
\]

Then divide the integration range of \( k \) into the intervals \([-\pi, -k_F], [-k_F, -k_F + \delta], [-k_F + \delta, k_F - \delta], [k_F - \delta, k_F] \) and \([k_F, \pi] \) and the interval of \( \lambda \) into \([-\lambda_F, -\lambda_F + \delta], [-\lambda_F + \delta, \lambda_F - \delta] \) and \([\lambda_F - \delta, \lambda_F] \). Use the fact that \( e^{-|\varepsilon(k, T)|/T} \) decay rapidly as \( T \rightarrow 0 \) and (3.11). After some algebras, one gets
\[
\frac{E(T)}{L} = \frac{1}{2\pi} \int_{-k_F}^{k_F} dk \varepsilon_c(k, T) + 2 \int_{k_F-\delta}^{k_F+\delta} dk \rho_c(k_F) \frac{|\varepsilon_c(k)|}{1 + e^{[|\varepsilon_c(k)|]/T}}
+ 2 \int_{k_F-\delta}^{k_F+\delta} dk T \ln(1 + e^{-|\varepsilon_c(k)|/T}) (\rho_c(k_F) - \frac{1}{2\pi})
+ 2 \int_{\lambda_F-\delta}^{\lambda_F} d\lambda \rho_s(\lambda_F) \frac{|\varepsilon_s(\lambda)|}{1 + e^{[|\varepsilon_s(\lambda)|]/T}}
+ 2 \int_{\lambda_F-\delta}^{\lambda_F} d\lambda T \ln(1 + e^{-|\varepsilon_s(\lambda)|/T}) \rho_s(\lambda_F).
\] (C3)

Here \(\delta\) is a cut-off. Using the integral formulas
\[
\int_0^\infty dx \frac{x}{1 + e^x} = \int_0^\infty dx \ln(1 + e^{-x}) = \frac{\pi^2}{12},
\] (C4)

we have, in the limit \(T \to 0\),
\[
\frac{E(T)}{L} = \frac{1}{2\pi} \int_{-k_F}^{k_F} dk \varepsilon_c(k) + \frac{1}{2\pi} \int_{-k_F}^{k_F} dk \varepsilon_c(k, T)
+ \frac{\pi T^2}{6v_c} + \frac{\pi T^2}{6v_s} \left(1 - \frac{1}{2\pi \rho_c(k_F)}\right),
\] (C5)

On the other hand, at \(h = 0\), in terms of (4.4), we have an integral equation for \(\tilde{\varepsilon}_c(k, T)\),
\[
\tilde{\varepsilon}_c(k, T) = -\frac{\pi T^2}{6v_c} \tilde{f}(k),
\] (C6)

\[
\tilde{f}(k) = \cos k_F \bar{K} (\sin k - \sin k_F)
+ \frac{1}{2\pi} \int_{-k_F}^{k_F} dk' \cos k' \bar{K} (\sin k - \sin k') \tilde{f}(k').
\]

Then comparing (3.18) and \(\int_{-k_F}^{k_F} \frac{dk}{2\pi} \tilde{\varepsilon}_c(k, T)\), one has
\[
\int_{-k_F}^{k_F} \frac{dk}{2\pi} \tilde{\varepsilon}_c(k, T) = -\frac{\pi T^2}{6v_c} \left(1 - \frac{1}{2\pi \rho_c(k_F)}\right).
\] (C7)

So
\[
\frac{E(T)}{L} = \frac{1}{2\pi} \int_{-k_F}^{k_F} dk \varepsilon_c(k) + \frac{\pi T^2}{6v_c} + \frac{\pi T^2}{6v_s}
\] (C8)

Finally, using the thermodynamic relation
\[
E(T) = \frac{\partial(\beta \Omega(T))}{\partial \beta},
\] (C9)

we obtain the thermodynamic potential as given by (4.12).
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