CONTINUOUS DEPENDENCE AND UNIQUENESS FOR LATERAL CAUCHY PROBLEMS FOR LINEAR INTEGRO-DIFFERENTIAL PARABOLIC EQUATIONS

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Abstract. Via Carleman estimates we prove uniqueness and continuous dependence results for lateral Cauchy problems for linear integro-differential parabolic equations without initial conditions. The additional information supplied prescribes the conormal derivative of the temperature on a relatively open subset of the lateral boundary of the space-time domain.

1. Introduction

In this paper we consider the linear ill-posed integro-differential parabolic problem with no initial condition

\[
\begin{aligned}
D_t u(t, x) - A(x, D)u(t, x) &= Bu(t, x) + f_0(t, x), \\ u(t, x) &= g(t, x), \\ D_{\nu_A} u(t, x) &= D_{\nu_A} g(t, x),
\end{aligned}
\]

for \( (t, x) \in (0, T) \times \Omega \), \( (t, x) \in (0, T) \times \partial \Omega \), and \( (t, x) \in (0, T) \times \Gamma \), respectively. Here \( \Omega \) is a bounded connected open set in \( \mathbb{R}^n \) whose boundary \( \partial \Omega \) is of \( C^2 \)-class, \( \Gamma \subset \partial \Omega \) is a sub-domain of \( \Gamma \), i.e., a relatively open subset of \( \partial \Omega \). Moreover, \( A(x, D) = \sum_{i,j=1}^n D_{x_i} (a_{i,j}(x) D_{x_j}) + \sum_{j=1}^n b_j(x) D_{x_j} + a_0(x) \) is an elliptic operator which generates an analytic semigroup \( \{e^{tA}\}_{t \geq 0} \) in \( L^2(\Omega) \).

The operator \( B \) is defined by

\[
Bu(t, x) = f_1(t, x) u(T_1, x) + f_2(t, x) u(T_2, x) + f_3(t, x) \int_{T_1}^{T_2} \rho_1(\sigma, x) u(\sigma, x) d\sigma
\]

\[ + Bu(t, x) + f_4(t, x) \int_{T_1}^{T_2} \rho_2(\sigma, x) Bu(\sigma, x) d\sigma := \sum_{j=1}^5 B_j u(t, x), \]

where \( 0 < T_1 < T_2 < T \) and

\[
Bu(t, x) = \int_{\Omega} k(t, x, y) u(t, y) dy,
\]

the kernel \( k : (0, T) \times \Omega \times \Omega \to \mathbb{R} \) being a measurable function. The functions \( f_0, f_1, f_2, f_3, f_4, \rho_1, \rho_2, k, g \) are suitably chosen so as to satisfy Hypotheses 2.1 and

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stated in Sections 2 and 3. Finally, \( \nu_A \) denotes the conormal vector related to the operator \( A(x,D) \), i.e., \( (\nu_A(x))_i = \sum_{j=1}^{n} a_{i,j}(x)\nu_j(x) \) for any \( i = 1, \ldots, n \) and \( x \in \partial \Omega \), where \( \nu \) denotes the outward unit normal vector to \( \partial \Omega \) at \( x \).

We consider the inverse problem of determining \( u \) by the knowledge of \( f_0 \) and \( g \).

Our main results are the uniqueness: \( f_0 = 0 \) and \( g = 0 \) imply \( u = 0 \) in \( (0, T) \times \Omega \) and the continuous dependence of \( u \) in terms of \( (f_0, g) \). Continuous dependence means here that \( u \) is estimated in \( C((0, T]; L^2(\Omega)) \cap L^2_{\text{loc}}((0, T]; H^1(\Omega)) \) in terms of the \( H^1(0, T; L^2(\Omega)) \)-norm of \( f_0 \) and the \( L^2(0, T; H^2(\Omega)) \)-norm of \( g \).

When the non-local term \( B \) is not included, that is, when we have to deal with a differential problem, we can apply the Carleman estimate in [4] (see also [3]) and prove the uniqueness and continuous dependence. With the presence of \( B \), to the best knowledge of the authors, results are not available in literature.

Our method is still based on the Carleman estimate in [4], but in order to treat the non-local terms, we need strong conditions on the kernels \( \rho_1, \rho_2 \) and \( k \) in \( B \).

Finally, we stress that, due to the absence of initial conditions, our results can concern both forward and backward parabolic problems.

Carleman estimates are a powerful tool in solving inverse problems. We refer the readers to the pioneering work [2] and also to [6] and the survey [11] related to parabolic inverse problems. Concerning uniqueness and continuous dependence results for Cauchy problems with no initial conditions, we mention the papers [8]-[10]. More specifically, in [8], \( f_1 = f_2 = f_3 = f_4 = 0 \) and only the Dirichlet boundary condition is prescribed on \( \partial \Omega \). Two different additional conditions are assumed. In the first case, \( u \) is assumed to be known in an open subdomain \( \omega \) with \( \omega \subset \Omega \), while in the latter the linear operator \( B \), which transforms spatial arguments, is defined by

\[
Bu(t,x) = k_0(t,x)u(t,\sigma x) + \sum_{j=1}^{n} k_j(t,x)D_{x_j}u(t,px),
\]

for some \( \sigma \in (0, 1) \), where \( \Omega \) is convex with respect to \( x = 0 \). In [2] the case when the elliptic operator \( A(\cdot, D) \) has smooth and unbounded coefficients in a cylinder of \( \mathbb{R}^{m+n} \) and it degenerates on some directions is considered and new Carleman estimates are proved. Finally, in [10] problem [14] is considered with Dirichlet boundary conditions on \( \partial \Omega \) and first order additional conditions on a part of \( \partial \Omega \). Also in this situation, new Carleman type estimates have been the key tool to prove the uniqueness and continuous dependence results.

We conclude this introduction with giving the plan of the paper. In Section 2 we state the problems that we deal with in the paper and introduce the well-known Carleman estimates for linear parabolic operators (e.g., [3]-[5]). In Section 3 we establish the uniqueness result (Theorem 3.1) for our problem and prove it. The proof is based on the Carleman estimate. Finally, Section 4 is devoted to deducing the continuous dependence result in non-weighted \( L^2 \)-spaces (Theorem 4.1).

**Notation.** Throughout the paper we set \( Q_{T_1,T_2} = (T_1, T_2) \times \Omega \) for any \( T_1, T_2 \in \mathbb{R} \) with \( T_1 < T_2 \) and we simply write \( Q_T \) for \( Q_{0,T} \).
2. MAIN ASSUMPTIONS AND PRELIMINARY RESULTS

To begin with, let us introduce our standing assumptions. For this purpose, we introduce the function \( l : [0, T] \to \mathbb{R} \), defined by \( l(t) = t(T-t) \) for any \( t \in [0, T] \), and a function \( \psi \in C^2(\bar{\Omega}) \) which satisfies the following properties:

\[
\psi(x) > 0, \ x \in \Omega, \quad |\nabla \psi(x)| > 0, \ x \in \bar{\Omega},
\]

\[
D_{vA} \psi(x) := \sum_{i,j=1}^n a_{i,j}(x) \nu_j(x) D_i \psi(x) \leq 0, \ x \in \partial \Omega \setminus \Gamma.
\]

For the existence of such a function, we refer the reader to [3].

**Hypotheses 2.1.**

(i) \( \Omega \) is a bounded open set in \( \mathbb{R}^n \), \( \partial \Omega \) being of \( C^2 \)-class;

(ii) \( \Gamma \subset \partial \Omega \) is an arbitrarily fixed sub-domain of \( \Gamma \);

(iii) the coefficients of the operator \( A(x, D) \), defined in (1.2), satisfy the following conditions:

(a) \( a_{i,j} \in C^2(\bar{\Omega}) \), \( a_{j,i} = a_{i,j} \), for any \( i, j = 1, \ldots, n \);

(b) \( a_j \in C^1(\bar{\Omega}) \) for any \( j = 0, \ldots, n \),

(c) \( a_0 \in C(\bar{\Omega}) \);

(d) \( \sum_{i,j=1}^n a_{i,j}(x) \xi_i \xi_j \geq \mu_0 |\xi|^2 \) for any \( x \in \bar{\Omega}, \ \xi \in \mathbb{R}^n \) and some positive constant \( \mu_0 \);

(iv) \( f_0 \in L^2((0, T) \times \Omega) \) and \( g \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega)) \);

(v) \( f_1, f_2, f_3, f_4 \in L^2(0, T; L^\infty(\Omega)) \);

(vi) \( \rho_1, \rho_2 \) belong to \( L^2((0, T); L^\infty(\Omega)) \).

(vii) \( k \) is a measurable function in \( (0, T) \times \Omega \times \Omega \). Moreover, the functions \( (t, y) \mapsto (l(t))^{\gamma-3} \|k(t, \cdot, y)\|^2_{L^1(\Omega)} \) and \( (t, x) \mapsto (l(t))^{\gamma-3} \|k(t, x, \cdot)\|^2_{L^1(\Omega)} \) are bounded in \( QT \).

**Remark 2.2.** The conditions on \( \rho_1, \rho_2 \) and \( k \) will be refined in Section [3].

In this paper, our first main problem is:

(IP1): estimate the solution \( u \) in \( C((0, T); L^2(\Omega)) \cap L^2_{\loc}((0, T); H^1(\Omega)) \) to the problem

\[
\begin{align*}
&\begin{cases}
  u \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega)), \\
  D_t u(t, x) - A(x, D) u(t, x) = B u(t, x) + f_0(t, x), & (t, x) \in (0, T) \times \Omega, \\
  u(t, x) = g(t, x), & (t, x) \in (0, T) \times \partial \Omega, \\
  D_{vA} u(t, x) = D_{vA} g(t, x), & (t, x) \in (0, T) \times \Gamma,
\end{cases}
\end{align*}
\]

where the linear operator \( B \) is defined by [1.3] and [1.4].

We can consider another problem:

(IP1'): estimate in \( C((0, T); L^2(\Omega)) \cap L^2_{\loc}((0, T); H^1(\Omega)) \) the solution \( u \) to the problem

\[
\begin{align*}
&\begin{cases}
  u \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega)), \\
  D_t u(t, x) + A(x, D) u(t, x) = B u(t, x) + f_0(t, x), & (t, x) \in (0, T) \times \Omega, \\
  u(t, x) = g(t, x), & (t, x) \in (0, T) \times \partial \Omega, \\
  D_{vA} u(t, x) = D_{vA} g(t, x), & (t, x) \in (0, T) \times \Gamma.
\end{cases}
\end{align*}
\]

By the change of the unknown function \( w(t, x) = u(T - t, x) \) for \( (t, x) \in (0, T) \times \Omega \), the problem (IP1') changes to problem (IP1) with \( (B, f_0, g) \) being replaced by
\((\mathcal{B}, \hat{f}_0, -\hat{g})\), where \(\hat{h}(t, x) = -h(T - t, x)\) for a given function \(h\) and the linear operator \(\mathcal{B}\) is defined by

\[
\mathcal{B}w(t, x) = \hat{f}_1(t, x)w(\hat{T}_2, x) + \hat{f}_2(t, x)w(\hat{T}_1, x) - \hat{f}_3(t, x) \int_{\hat{T}_1}^{\hat{T}_2} \hat{\rho}_1(\sigma, x)w(\sigma, x) \, d\sigma
\]

\[+ \mathcal{B}w(t, x) + \hat{f}_4(t, x) \int_{\hat{T}_1}^{\hat{T}_2} \hat{\rho}_2(\sigma, x)\mathcal{B}w(\sigma, x) \, d\sigma =: \sum_{j=1}^{5} \hat{B}_j u(t, x),
\]

where \(\hat{T}_1 = T - T_2, \hat{T}_2 = T - T_1\) and

\[\hat{B}w(t, x) = \int_{\Omega} \tilde{k}(t, x, y)w(t, y) \, dy.\]

Thus (IP1') is led back to the problem (IP1), which is a forward problem in time.

**Remark 2.3.** It is a simply task to check that, if Hypotheses \(\text{H2.1}\) hold true for \((f_0, f_1, f_2, f_3, f_4, \rho_1, \rho_2, k)\), then they hold true also for \((\hat{f}_0, \hat{f}_1, \hat{f}_2, \hat{f}_3, \hat{f}_4, \hat{\rho}_0, \hat{\rho}_1, \hat{k})\).

We stress that, if the triplet \((\rho_1, \rho_2, k)\) satisfies the forthcoming Hypotheses \(\text{H3.1}\) then the triplet \((\hat{\rho}_1, \hat{\rho}_2, \hat{k})\) satisfies the same conditions with the same constants, since \((T - t) = l(t)\) for any \(t \in [0, T]\).

Coming back to problem (IP1) and introducing the function \(v = u - g\), where \(u\) is the solution to problem (IP1), we can reduce (IP1) to the problem with homogeneous boundary condition:

\[
(IP2) \begin{cases}
v \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega)), \\
D_\lambda v(t, x) - A(x, D)v(t, x) = \mathcal{B}v(t, x) + \hat{f}(t, x), & (t, x) \in (0, T) \times \Omega, \\
v(t, x) = 0, & (t, x) \in (0, T) \times \partial \Omega, \\
D_\nu v(t, x) = 0, & (t, x) \in (0, T) \times \Gamma,
\end{cases}
\]

where

\[
\hat{f} = f_0 - D_t g + A(\cdot, D)g + \mathcal{B}g.
\] (2.1)

Therefore, we mainly consider problem (IP2).

Now we state a key Carleman estimate. For this purpose, we introduce the functions \(\varphi_\lambda : \overline{\Omega} \to \mathbb{R}\) and \(\alpha_\lambda : [0, T] \times \overline{\Omega} \to \mathbb{R}\) with \(\lambda \in [1, +\infty)\), defined by

\[
\varphi_\lambda(x) = e^{\lambda \psi(x)}, \quad \alpha_\lambda(t, x) = \frac{e^{2\lambda \psi(x)} - e^{2\lambda \psi(x)}}{l(t)}, \quad t \in (0, T), x \in \overline{\Omega}. \tag{2.2}
\]

By \(\text{[5]}\) Lemma 2.4 (see also \(\text{[5]}\) 4) and since \(\varphi_\lambda(x) \geq 1\) for all \(x \in \overline{\Omega}\), there exists \(\tilde{\lambda}\) such that for any \(\lambda \geq \tilde{\lambda}\) we can choose \(\tilde{s}_0 = \tilde{s}_0(\lambda) > 0\) and \(C_1 = C_1(\lambda) > 0\) such that the following Carleman estimate

\[
s^3 \int_{Q_T} (l(t))^{-3} |v(t, x)|^2 \exp[2s\alpha_\lambda(t, x)] \, dt \, dx
\]

\[+ s \int_{Q_T} (l(t))^{-1} |\nabla_x v(t, x)|^2 \exp[2s\alpha_\lambda(t, x)] \, dt \, dx
\]

\[+ s^{-1} e^{-\lambda \psi(x)} \int_{Q_T} l(t) \left[ |D_x v(t, x)|^2 + \sum_{i,j=1}^{n} |D_{x_i} D_{x_j} v(t, x)|^2 \right] \exp[2s\alpha_\lambda(t, x)] \, dt \, dx
\]
we conclude that \( \nabla \) \( \exp \) exponentially decay to 0 at We stress that the condition (3.1) implies that the kernel \( \text{Remark 3.2.} \) \((v, v)\) Indeed, since \( \text{derivatives of } v\) vanish almost everywhere on \((0, T) \times \Gamma\). 

\[ \text{Remark 2.4.} \text{Note that the Carleman estimate in } [11, \text{Lemma 2.4}] \text{ actually contains \( \text{the } L^2\)-norms of } e^{\alpha_1 v}, e^{\alpha_2 D_1 v} \text{ and } e^{\alpha_3 D_x v} \text{ for } j = 1, \ldots, n \text{ on } (0, T) \times \Gamma \text{ on its right-hand side. In our situation all these terms identically vanish on } (0, T) \times \partial \Omega. \text{ Indeed, since } v = 0 \text{ almost everywhere on } (0, T) \times \Omega, D_i v \text{ and the tangential spatial derivatives of } v \text{ vanish almost everywhere on } (0, T) \times \partial \Omega \text{ as well. On the other hand, since the conormal derivative of } v \text{ vanishes on } (0, T) \times \Gamma \text{ and for any } x \in \Gamma \text{ we can split an arbitrary vector of } \mathbb{R}^n \text{ along } \nu_{\Gamma}(x) \text{ and the tangential directions, we conclude that } \nabla v \text{ vanishes almost everywhere on } (0, T) \times \Gamma. \]

3. Uniqueness result

In this section \( \lambda \geq \hat{\lambda} \) is fixed and for notational convenience we set

\[ c_{1, \lambda}(\psi) := e^{2\lambda \|\psi\|_\infty} - e^{\lambda \psi_m} \]

where \( \psi_m \) denotes the minimum of the function \( \psi \).

We also assume the following additional set of assumptions.

**Hypotheses 3.1.** There exist five positive constants \( K_j \) \((j = 1, \ldots, 5)\) and \(0 < T_1 < T_2 < T\) such that

\[ |\rho_1(t, x)| \leq K_1 \exp[s_0 \alpha_1(t, x)], \quad (t, x) \in Q_T, \quad (3.1) \]

\[ |\rho_2(t, x)| \leq K_2 \exp[s_0 \alpha_2(t, x)], \quad (t, x) \in Q_{T_1, T_2}, \quad (3.2) \]

Moreover,

\[ K_3 := \text{ess sup}_{(t, x) \in Q_T} (l(t))^\gamma \int_\Omega |k(t, x, y)| \, dy < +\infty, \quad (3.3) \]

for some \( \gamma \in [0, 3] \) and

\[ \int \{x \in \Omega : \psi(x) > \psi(y)\} \frac{|k(t, x, y)| \, dx}{1 + |k(t, x, y)|} \leq K_4 (l(t))^{\gamma-3} \exp[-2s_0 c_{1, \lambda}(\psi)(l(t))^{-1}], \quad (3.4) \]

\[ \int \{x \in \Omega : \psi(x) \leq \psi(y)\} \frac{|k(t, x, y)| \, dx}{1 + |k(t, x, y)|} \leq K_5 (l(t))^{\gamma-3}, \quad (3.5) \]

for any \((t, y) \in Q_T\).

**Remark 3.2.** We stress that the condition (3.1) implies that the kernel \( \rho_1 \) should exponentially decay to 0 at \( t = 0 \) and \( t = T \).
Next, we choose $s_0 \geq \hat{s}_0$ so as to satisfy the inequalities

$$H_0(s_0) := 6C_1 \left\{ \left. \begin{array}{l}
2^{-6}T^6[(T_2 - T_1)^{-1} + s_0^{1+\delta}] + 2^{-2}T^3s_0c_1,\lambda(\psi) \right. \\
+ 2^{-6}T^6(T_2 - T_1)K_2^2\|f_3\|_{L^2(0,T;L^\infty(\Omega))} \\
+ (T_2 - T_1)K_2^2K_3(K_4 + K_5)\|f_4\|_{L^2(0,T;L^\infty(\Omega))} \right\} \leq \frac{1}{2}s_0^3, \quad (3.6)
$$

$$H_1(s_0) := 6C_1M_{T_1,T_2}^{-1}s_0^{-(1+\delta)} \sum_{j=0}^1 \|f_j\|_{L^\infty(\Omega;L^2(0,T))} \leq \frac{1}{2}s_0^{-1}e^{-\lambda\|\psi\|_\infty}, \quad (3.7)
$$

$C_1$ being the positive constant in estimate (2.3), $K_3, K_4, K_5$ being given in (3.3)- (3.5) and $M_{T_1,T_2} = [\min\{T_1(T - T_1), T_2(T - T_2)\}]^{-1}$. Observe then that, for all $(t, x) \in Q_T$, we have

$$\exp[-2s_0c_1,\lambda(\psi)(l(t))^{-1}] \leq \exp[2s_0\alpha_\lambda(t, x)] \leq 1, \quad (3.8)
$$

Then we show our first main result.

**Theorem 3.3.** Let Hypotheses 2.1-3.1 and conditions (3.6), (3.7) be satisfied. Further, let $u$ be a strong solution to problem (IP1). Then, the following weighted estimate

$$\frac{1}{2}s_0^3 \int_{Q_T} (l(t))^{-3}|v(t, x)|^2 \exp[2s_0\alpha_\lambda(t, x)] \, dtdx + \cdots \leq 6C_1 \int_{Q_T} |\tilde{f}(t, x)|^2 \exp[2s_0\alpha_\lambda(t, x)] \, dtdx, \quad (3.9)
$$

holds true with $v = u - g$ and $s \geq \hat{s}_0$. In particular, problem (IP1) admits at most one solution.

The rest of this section is devoted to the proof of Theorem 3.1.

### 3.1. Estimating $B_1$ and $B_2$

First we need some weighted trace results.

**Lemma 3.4.** The following estimate holds true for all $w \in H^1(T_1, T_2; L^2(\Omega))$, $r_0 \geq 0$, $\varepsilon > 0$ and $j = 1, 2$:

$$\int_{\Omega} |w(T_j, x)|^2 \exp[2r_0\alpha_\lambda(T_j, x)] \, dx \leq \varepsilon^2 \int_{Q_{T_1,T_2}} |D_tw(t, x)|^2 \exp[2r_0\alpha_\lambda(t, x)] \, dtdx + \cdots \quad (3.10)$$
Proof. By a density argument, we can assume that $w$ is smooth enough. We arbitrary fix $x \in \Omega$. From the identity
\[
|w(t,x)|^2 \exp[2r_0\alpha(t,x)] - |w(T_j,x)|^2 \exp[2r_0\alpha(T_j,x)]
\]
\[
= \int_{T_j}^t D_s \{ |w(s,x)|^2 \exp[2r_0\alpha(s,x)] \} \, ds
\]
\[
= 2 \int_{T_j}^t w(s,x) D_s w(s,x) \exp[2r_0\alpha(s,x)] \, ds
\]
\[
+ 2r_0 \int_{T_j}^t |w(s,x)|^2 (e^{2\lambda\|\psi\|_\infty} - e^{\lambda\psi(x)})l'(s)(l(s))^{-2} \exp[2r_0\alpha(t,x)] \, ds,
\]
which holds true for $j = 1, 2$, and Young inequality we easily deduce that the following inequality holds for all $t \in (T_1, T_2)$, $\varepsilon \in \mathbb{R}_+$ and for $j = 1, 2$:
\[
|w(t,x)|^2 \exp[2r_0\alpha(T_j,x)]
\]
\[
\leq |w(t,x)|^2 \exp[2r_0\alpha(T_j,x)] + \varepsilon^2 \int_{T_j}^t |D_s w(s,x)|^2 \exp[2r_0\alpha(s,x)] \, ds
\]
\[
+ \varepsilon^{-2} \int_{T_j}^t |w(s,x)|^2 \exp[2r_0\alpha(s,x)] \, ds
\]
\[
+ 2r_0 \int_{T_j}^t |w(s,x)|^2 c_1,\lambda(\psi)|l'(s)|(l(s))^{-2} \exp[2r_0\alpha(t,x)] \, ds
\]
\[
\leq |w(t,x)|^2 \exp[2r_0\alpha(T_j,x)] + \varepsilon^2 \int_{T_1}^{T_2} |D_s w(s,x)|^2 \exp[2r_0\alpha(s,x)] \, ds
\]
\[
+ \int_{T_1}^{T_2} |w(t,x)|^2 [\varepsilon^{-2} + 2r_0 c_1,\lambda(\psi)|l'(s)|(l(s))^{-2}] \exp[2r_0\alpha(t,x)] \, ds.
\]
Integrating over $(T_1, T_2)$ the first and last side of the previous chain of inequalities yields
\[
|w(T_j,x)|^2 \exp[2r_0\alpha(T_j,x)]
\]
\[
\leq \varepsilon^2 \int_{T_1}^{T_2} |D_t w(t,x)|^2 \exp[2r_0\alpha(t,x)] \, dt
\]
\[
+ \int_{T_1}^{T_2} |w(t,x)|^2 \{(T_2 - T_1)^{-1} + \varepsilon^{-2} + 2r_0 c_1,\lambda(\psi)|l'(t)|(l(t))^{-2}\} \exp[2r_0\alpha(t,x)] \, dt.
\]
Finally, an integration over $\Omega$ leads to the assertion. \qed

The needed estimates for $B_1$ and $B_2$ follow from \eqref{5.10}, if we choose $\varepsilon = s_0^{-\alpha/(1+\delta)/2}$, with $\delta \in (0, 1)$, and observe that $l(t) \geq M_{T_1,T_2} := \min\{T_1(T - T_1), T_2(T - T_2)\}$ for any $t \in [T_1, T_2]$ and $l(t) \leq 2^{-2}T^2$, $|l'(t)| \leq T$ for any $t \in [0, T]$. Indeed,
\[
\int_{Q_T} |B_j v(t,x)|^2 \exp[2s_0\alpha(T_j,x)] \, dt \, dx
\]
\[
= \int_{\Omega} |v(T_j,x)|^2 \exp[2s_0\alpha(T_j,x)] \, dx \int_0^T |f_j(t,x)|^2 \, dt
\]
\[
\leq \|f_j\|_{L^2(0,T),L^2(\Omega)}^2 \int_{\Omega} |v(T_j,x)|^2 \exp[2s_0\alpha(T_j,x)] \, dx
\]
\[
\begin{align*}
&\leq s_0^{-\frac{1}{2}} \|f_j\|_{L^2(0,T;L^\infty(\Omega))}^2 \int_{Q_{T_1, T_2}} (l(t))^{-\frac{1}{2}} l(t) \left| D_t v(t,x) \right|^2 \exp \left[ 2s_0\alpha_\lambda(t,x) \right] dtdx \\
&+ \|f_j\|_{L^2(0,T;L^\infty(\Omega))}^2 \int_{Q_{T_1, T_2}} \left\{ \left[ (T_2 - T_1)^{-1} + s_0^{1+\delta} \right] (l(t))^3 + 2s_0\alpha_\lambda(t) \right\} \|l(t)\| dtdx \\
&\leq s_0^{-\frac{1}{2}} \|f_j\|_{L^2(0,T;L^\infty(\Omega))}^2 M_{T_1, T_2}^{-1} \int_{Q_{T_1, T_2}} (l(t))^{-\frac{1}{2}} l(t) \left| D_t v(t,x) \right|^2 \exp \left[ 2s_0\alpha_\lambda(t,x) \right] dtdx \\
&+ \|f_j\|_{L^2(0,T;L^\infty(\Omega))}^2 \left( 2^{-6} T^6 \left[ (T_2 - T_1)^{-1} + s_0^{1+\delta} \right] + 2^{-2} T^3 s_0\alpha_\lambda(t) \right) \\
&\times \int_{Q_{T_1, T_2}} (l(t))^{-\frac{3}{2}} \left| v(t,x) \right|^2 \exp \left[ 2s_0\alpha_\lambda(t,x) \right] dtdx, \quad (3.11)
\end{align*}
\]

for \( j = 1, 2. \)

### 3.2. Estimating \( B_3 \)

Using (3.1), (3.8) and again the condition \( \|l\|_\infty \leq 2^{-2}T^2 \), we easily obtain the following chain of inequalities:

\[
\begin{align*}
&\int_{Q_T} |B_3v(t,x)| \exp \left[ 2s_0\alpha_\lambda(t,x) \right] dtdx \\
&= \int_{Q_T} \exp \left[ 2s_0\alpha_\lambda(t,x) \right] |f_3(t,x)|^2 \left| \int_{T_1}^{T_2} \rho_1(\sigma,x) v(\sigma,x) d\sigma \right|^2 dtdx \\
&\leq (T_2 - T_1) \int_{Q_T} \exp \left[ 2s_0\alpha_\lambda(t,x) \right] |f_3(t,x)|^2 dtdx \int_{T_1}^{T_2} \rho_1(\sigma,x)^2 |v(\sigma,x)|^2 d\sigma \\
&= (T_2 - T_1) \int_{Q_{T_1, T_2}} \rho_1(\sigma,x)^2 |v(\sigma,x)|^2 d\sigma dx \int_{T_1}^{T} \exp \left[ 2s_0\alpha_\lambda(t,x) \right] |f_3(t,x)|^2 dt \\
&\leq (T_2 - T_1) \int_{Q_{T_1, T_2}} \rho_1(\sigma,x)^2 |v(\sigma,x)|^2 dx d\sigma \int_{T_1}^{T} \left| f_3(t,\cdot) \right|^2_{L^\infty(\Omega)} dt \\
&\leq (T_2 - T_1) \|f_3\|_{L^2(0,T;L^\infty(\Omega))}^2 K_2^2 \int_{Q_{T_1, T_2}} \exp \left[ 2s_0\alpha_\lambda(\sigma,x) \right] |v(\sigma,x)|^2 d\sigma dx \\
&\leq (T_2 - T_1) \|f_3\|_{L^2(0,T;L^\infty(\Omega))}^2 K_2^2 \int_{Q_T} \left( l(\sigma) \right)^3 (l(\sigma))^{-3} \exp \left[ 2s_0\alpha_\lambda(\sigma,x) \right] |v(\sigma,x)|^2 d\sigma dx \\
&\leq (T_2 - T_1) \|f_3\|_{L^2(0,T;L^\infty(\Omega))}^2 2^{-6} T^6 K_2^2 \\
&\times \int_{Q_T} \left( l(\sigma) \right)^{-3} \exp \left[ 2s_0\alpha_\lambda(\sigma,x) \right] |v(\sigma,x)|^2 d\sigma dx. \quad (3.12)
\end{align*}
\]

### 3.3. Estimating \( B_4 = B \)

Via Hölder’s inequality, we obtain

\[
\begin{align*}
&\int_{Q_T} |B_4v(t,x)| \exp \left[ 2s_0\alpha_\lambda(t,x) \right] dtdx \\
&= \int_{Q_T} \left| \int_{\Omega} k(t,x,y) v(t,y) dy \right|^2 \exp \left[ 2s_0\alpha_\lambda(t,x) \right] dtdx \\
&\leq K_2 \int_0^T \left( l(t) \right)^{-\gamma} dt \int_{\Omega} \exp \left[ 2s_0\alpha_\lambda(t,x) \right] dx \int_{\Omega} \left| k(t,x,y) \right| |v(t,y)|^2 dy \\
&\leq K_2 \int_{Q_T} \left( l(t) \right)^{-3} |v(t,y)|^2 dtdy \int_{\Omega} \left( l(t) \right)^{3-\gamma} \exp \left[ 2s_0\alpha_\lambda(t,x) \right] |k(t,x,y)| dx, \quad (3.13)
\end{align*}
\]
$K_2$ being defined by $[3.3]$. Setting $h_{s_0,\lambda}(t,x,y) = (l(t))^{3-\gamma} \exp \{2s_0[\alpha_\lambda(t,x) - \alpha_\lambda(t,y)]\}$, we easily deduce the estimates

$$h_{s_0,\lambda}(t,x,y) \leq (l(t))^{3-\gamma} \exp \{2s_0[\exp (\lambda \|\psi\|_\infty) - \exp (\lambda \psi_m)](l(t))^{-1}\}$$

if $t \in [0,T]$ and $\psi(x) > \psi(y)$, and

$$h_{s_0,\lambda}(t,x,y) \leq (l(t))^{3-\gamma},$$

if $t \in [0,T]$ and $\psi(x) \leq \psi(y)$. Then from $[3.4]$, $[3.5]$, $[3.14]$ and $[3.15]$, we obtain

$$\int_{\Omega} h_{s_0,\lambda}(t,x,y)|k(t,x,y)| \, dx = \int_{\{x \in \Omega : \psi(x) > \psi(y)\}} h_{s_0,\lambda}(t,x,y)|k(t,x,y)| \, dx$$

$$+ \int_{\{x \in \Omega : \psi(x) \leq \psi(y)\}} h_{s_0,\lambda}(t,x,y)|k(t,x,y)| \, dx$$

$$\leq (l(t))^{3-\gamma} \exp \{2s_0c_1,\lambda(\psi)(l(t))^{-1}\}$$

$$\times \int_{\{x \in \Omega : \psi(x) > \psi(y)\}} |k(t,x,y)| \, dx$$

$$+ (l(t))^{3-\gamma} \int_{\{x \in \Omega : \psi(x) \leq \psi(y)\}} |k(t,x,y)| \, dx$$

$$\leq K_4 + K_5,$$  \hspace{1cm} (3.16)

for any $(t,y) \in Q_T$. Hence, from $[3.13]$ and $[3.10]$ we easily deduce the estimate

$$\int_{Q_T} \exp \{2s_0\alpha_\lambda(t,x)\}|B_4v(t,x)|^2 \, dt \, dx$$

$$\leq K_3(K_4 + K_5) \int_{Q_T} (l(t))^{-3}|v(t,x)|^2 \exp \{2s_0\alpha_\lambda(t,x)\} \, dt \, dx.$$  \hspace{1cm} (3.17)

### 3.4. Estimating $B_5$

By the definition of $B_5$ in $[1.3]$, estimates $[3.2]$, $[3.8]$ and $[3.14]$ we obtain

$$\int_{Q_T} |B_5v(t,x)|^2 \exp \{2s_0\alpha_\lambda(t,x)\} \, dt \, dx$$

$$= \int_{Q_T} \exp \{2s_0\alpha_\lambda(t,x)\}|f_4(t,x)|^2 \left| \int_{T_1}^{T_2} \rho_2(\sigma,x)Bv(\sigma,x) \, d\sigma \right|^2 \, dt \, dx$$

$$\leq (T_2 - T_1) \int_{Q_T} \exp \{2s_0\alpha_\lambda(t,x)\}|f_4(t,x)|^2 \, dt \, dx \int_{T_1}^{T_2} |\rho_2(\sigma,x)|^2 |Bv(\sigma,x)|^2 \, d\sigma$$

$$\leq (T_2 - T_1) \int_{Q_T} |f_4(t,x)|^2 \, dt \, dx \int_{T_1}^{T_2} |\rho_2(\sigma,x)|^2 |Bv(\sigma,x)|^2 \, d\sigma$$

$$\leq (T_2 - T_1) \int_{0}^{T} \|f_4(t,\cdot)\|_{L^2(\Omega)}^2 \, dt \int_{Q_{T_1,T_2}} |\rho_2(\sigma,x)|^2 |Bv(\sigma,x)|^2 \, d\sigma \, dx$$

$$\leq (T_2 - T_1)K_1^2 \|f_4\|_{L^2(0,T;L^\infty(\Omega))}^2 \int_{Q_{T_1,T_2}} \exp \{2s_0\alpha_\lambda(\sigma,x)\} |Bv(\sigma,x)|^2 \, d\sigma \, dx$$

$$\leq (T_2 - T_1)K_1^2 K_3(K_4 + K_5) \|f_4\|^2_{L^2(0,T;L^\infty(\Omega))}.$$
Theorem 4.1. Under Hypotheses 3.3. Then from (2.3), with 
\[ s > \varepsilon \]
for any \( 1 \) holds true for the solution to problem (IP1). Moreover, the continuous dependence results estimate the solution in \( H^1(\Omega) \). We can conclude that \( u = 0 \) in \( C([0, T]; L^2(\Omega)) \), i.e., uniqueness holds true for the solution to problem (IP1). Moreover, the continuous dependence results estimate the solution in \( C((0, T]; L^2(\Omega)) \cap L^2_{\text{loc}}((0, T]; H^1(\Omega)) \) and the data in \( L^2(Q_T) \times [H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))] \).

Remark 4.2. If \( f_0 = g = 0 \), then \( v = 0 \) in \( [2 \varepsilon T, T] \times \Omega \) for all \( \varepsilon \in (0, 1/2) \). This implies \( u = g = 0 \) in \( (0, T] \times \Omega \). In particular, since \( u \in H^1(0, T; L^2(\Omega)) \Rightarrow C([0, T]; L^2(\Omega)) \), we can conclude that \( u = 0 \) in \( C([0, T]; L^2(\Omega)) \), i.e., uniqueness holds true for the solution to problem (IP1). Moreover, the continuous dependence results estimate the solution in \( C((0, T]; L^2(\Omega)) \cap L^2_{\text{loc}}((0, T]; H^1(\Omega)) \) and the data in \( L^2(Q_T) \times [H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))] \).

Remark 4.3. From estimates 3.3 and 3.5 it follows that
\[
\sup_{(t, y) \in Q_T} \int_\Omega |k(t, x, y)| dx \leq K_0 = \max \{ K_4, K_5 \}.
\] (4.1)

We will use this estimate in the proof of Theorem 4.1.

In the proof of Theorem 4.1 we need the following lemma from [11], which we state here as a lemma.

Lemma 4.4 (Theorem 4.9 of [11]). Let \( z \in C([0, T]) \) and \( b, k \in L^1(0, T) \) be nonnegative functions which satisfy the integral inequality
\[
z(\tau) \leq a + \int_0^\tau b(s)z(s)\,ds + \int_0^\tau k(s)(z(s))^{1/2}\,ds, \quad \tau \in [0, T],
\]
where $p \in (0,1)$ and $a \geq 0$ are given constants. Then, the following estimate
\[
|z(\tau)| \leq \exp \left( \int_0^\tau b(s) \, ds \right) \left[ \sqrt{a} + \frac{1}{2} \int_0^\tau k(s) \exp \left( -\frac{1}{2} \int_0^s b(\sigma) \, d\sigma \right) \, ds \right]^2
\]
holds true for any $\tau \in [0,T]$.

Proof of Theorem 4.1 Let us introduce a family of functions $\sigma \in W^{1,\infty}(0,T)$ ($\varepsilon \in (0,T_1/(2T))$) such that

\[
0 \leq \sigma \leq 1, \quad \sigma(t) = 0, \; t \in [0,\varepsilon T], \quad \sigma(t) = 1, \; t \in [2\varepsilon T, T].
\]

It is a simple task to show that the function $v_\varepsilon = \sigma \varepsilon v$, where $v$ is the solution to problem (IP2), solves the following initial and boundary-value problem:

\[
(DP1) \begin{cases}
 v_\varepsilon \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)), \\
 D_tv_\varepsilon(t,x) - A(x,D)v_\varepsilon(t,x) \\
 = Bv_\varepsilon(t,x) + \sigma'\varepsilon(t)v(t,x) + \sigma\varepsilon(t) \sum_{j \neq 4} B_j v(t,x) \\
 + \tilde{f}_\varepsilon(t,x), \quad (t,x) \in Q_T, \\
 v_\varepsilon(0,x) = 0, \quad x \in \Omega, \\
 v_\varepsilon(t,x) = 0, \quad (t,x) \in (0,T) \times \partial\Omega,
\end{cases}
\]

where $\tilde{f}_\varepsilon = \sigma \varepsilon f$ and the operator $B_j$ ($j = 1,2,3,5$) are defined in (1.3). Recall now that $-A(\cdot,D)$ satisfies the following estimate for all $w \in H^2(\Omega) \cap H^1_0(\Omega)$:

\[
- \int_\Omega wA(\cdot,D)wdx = - \int_\Omega \sum_{i,j=1}^n D_{x_i}(a_{i,j}D_{x_j}w)w \, dx \\
+ \int_\Omega \sum_{j=1}^n a_j w D_{x_j}w \, dx + \int_\Omega a_0 w^2 \, dx \\
= \int_\Omega \sum_{i,j=1}^n a_{i,j} D_{x_i}w D_{x_j}w \, dx + \int_\Omega \sum_{j=1}^n a_j w D_{x_j}w \, dx + \int_\Omega a_0 w^2 \, dx \\
\geq \mu_0 \|\nabla_w w\|_{L^2(\Omega)}^2 - \left( \sum_{j=1}^n \|a_j\|_{L^\infty}^2 \right)^{1/2} \int_\Omega |\nabla_w w| \|w\| \, dx \\
- \mu_0 \|\nabla_w w\|_{L^2(\Omega)}^2 - \|a_0\|_{L^\infty} \|w\|_{L^2(\Omega)}^2 \\
- \left( \sum_{j=1}^n \|a_j\|_{L^\infty}^2 \right)^{1/2} \left( \varepsilon \|\nabla_w w\|_{L^2(\Omega)}^2 + \varepsilon^{-1} \|w\|_{L^2(\Omega)}^2 \right),
\]

where $\mu_0$ is the ellipticity constant in Hypothesis (2.1)(iii). Hence, choosing $\varepsilon$ properly, we conclude that

\[
- \int_\Omega wA(\cdot,D)wdx \geq \mu_0 \|\nabla_w w\|_{L^2(\Omega)}^2 - \mu_1 \|w\|_{L^2(\Omega)}^2,
\]

for some positive constant $\mu_1$. Fix $\tau \in [\varepsilon T,T)$. Multiplying both sides of the differential equation in $(DP1)$ by $v_\varepsilon$, integrating in $[\varepsilon T,\tau] \times \Omega$ and taking the
previous estimate into account, we get
\[
\|v_\varepsilon(t, \cdot)\|_{L^2(\Omega)}^2 + \mu_0 \int_{\varepsilon T}^T \| \nabla_x v_\varepsilon(t, \cdot) \|_{L^2(\Omega)}^2 dt - 2\mu_1 \int_{\varepsilon T}^T \| v_\varepsilon(t, \cdot) \|_{L^2(\Omega)}^2 dt
\]
\[
\leq 2 \int_{\varepsilon T}^T (Bv_\varepsilon(t, \cdot), v_\varepsilon(t, \cdot))_{L^2(\Omega)} dt + 2 \int_{\varepsilon T}^T \sigma'_\varepsilon(t) \| v_\varepsilon(t, \cdot) \|_{L^2(\Omega)} \| v(t, \cdot) \|_{L^2(\Omega)} dt
\]
\[
+ 2 \sum_{j \neq 4} \int_{\varepsilon T}^T \sigma_\varepsilon(t) \| v_\varepsilon(t, \cdot) \|_{L^2(\Omega)} \| B_j v(t, \cdot) \|_{L^2(\Omega)} dt
\]
\[
+ 2 \int_{\varepsilon T}^T \| v_\varepsilon(t, \cdot) \|_{L^2(\Omega)} \| \tilde{f}_\varepsilon(t, \cdot) \|_{L^2(\Omega)} dt. \tag{4.2}
\]

Let us estimate the terms in the right-hand side of (4.2). The last one is straightforward to estimate using Hölder inequality. Hence, we focus our attention on the other terms.

According to [8, 9, 11], we can estimate
\[
\| Bv_\varepsilon(t, \cdot) \|_{L^2(\Omega)} \leq \sqrt{K_3 K_6} (l(t))^{\gamma - 3} \| v_\varepsilon(t, \cdot) \|_{L^2(\Omega)}, \quad t \in (0, T), \tag{4.3}
\]
and, consequently,
\[
\int_{\varepsilon T}^T (Bv_\varepsilon(t, \cdot), v_\varepsilon(t, \cdot))_{L^2(\Omega)} dt \leq \sqrt{K_3 K_6} \int_0^T \chi_{(\varepsilon T, T)}(t) (l(t))^{\gamma - 3} \| v_\varepsilon(t, \cdot) \|_{L^2(\Omega)}^2 dt. \tag{4.4}
\]

Further, using the inclusion \( \text{supp } \sigma'_\varepsilon \subset [\varepsilon T, 2\varepsilon T] \), we obtain the inequality
\[
\int_{\varepsilon T}^T |\sigma'_\varepsilon(t)|^2 \| v(t, \cdot) \|_{L^2(\Omega)} \| v_\varepsilon(t, \cdot) \|_{L^2(\Omega)} dt \leq \int_{\varepsilon T}^T |\sigma'_\varepsilon(t)|^2 \| v(t, \cdot) \|_{L^2(\Omega)}^2 dt
\]
\[
\leq \| \sigma'_\varepsilon \|_{L^2(0, T)}^2 \int_{\varepsilon T}^{2\varepsilon T} \| v(t, \cdot) \|_{L^2(\Omega)}^2 dt. \tag{4.5}
\]

Now, we estimate terms in (4.2) containing the operators \( \tilde{B}_j v, j = 1, 2, 3, 5 \). Using the inclusion \( \text{supp } \sigma_\varepsilon \subset [\varepsilon T, T] \), we have the inequalities
\[
2 \int_0^T \| v_\varepsilon(t, \cdot) \|_{L^2(\Omega)}^2 \| B_j v(t, \cdot) \|_{L^2(\Omega)} dt
\]
\[
\leq \int_0^T \| v_\varepsilon(t, \cdot) \|_{L^2(\Omega)}^2 dt + \int_0^T |\sigma_\varepsilon(t)|^2 \| B_j v(t, \cdot) \|_{L^2(\Omega)}^2 dt
\]
\[
\leq \int_0^T \| v_\varepsilon(t, \cdot) \|_{L^2(\Omega)}^2 dt + \int_{\varepsilon T}^T \| B_j v(t, \cdot) \|_{L^2(\Omega)}^2 dt. \tag{4.6}
\]

From the definition of \( B_j, j = 1, 2, \) Lemma 3.4 with \( s_0 = 0 \) and \( \varepsilon = 1 \), we deduce
\[
\int_{\varepsilon T}^T \| B_j v(t, \cdot) \|_{L^2(\Omega)}^2 dt = \int_{\varepsilon T}^T dt \int_{\Omega} |f_j(t, x)|^2 \| v(T_j, x) \|^2 dx
\]
\[
\leq \| f_j \|_{L^2(0, T; L^\infty(\Omega))} \int_{\Omega} |v(T_j, x)|^2 dx
\]
\[
\leq \| f_j \|_{L^2(0, T; L^\infty(\Omega))} \int_{Q_{T_1, T_2}} |D_t v(t, x)|^2 dt dx.
\]
+ \| f_j \|_{L^2(0,T;L^\infty(\Omega))}^2 \int_{Q_{T_1,T_2}} \left[ (T_2 - T_1)^{-1} + 1 \right] |v(t,x)|^2 \, dt \, dx.

Likewise we can estimate
\[
\int_0^T \| B_3 v(t,\cdot) \|_{L^2(\Omega)}^2 \, dt = \int_0^T \int_\Omega |f_3(t,x)|^2 \left| \int_{T_1}^{T_2} \rho_1(\sigma,x) v(\sigma,x) \, d\sigma \right|^2 \, dx
\]
\[
\leq \| f_3 \|_{L^2(0,T;L^\infty(\Omega))}^2 \times \int_\Omega \left( \int_{T_1}^{T_2} |\rho_1(\sigma,x)|^2 \, d\sigma \right) \left( \int_{T_1}^{T_2} |v(\sigma,x)|^2 \, d\sigma \right) \, dx
\]
\[
\leq \| f_3 \|_{L^2(0,T;L^\infty(\Omega))}^2 \| \rho_1 \|_{L^2(0,T;L^\infty(\Omega))} \int_{T_1}^{T_2} \| v(\sigma,\cdot) \|_{L^2(\Omega)}^2 \, d\sigma.
\]
(4.8)

and, taking 4.3 (with \( v \) replacing \( v_\gamma \) into account,
\[
\int_0^T \| B_3 v(t,\cdot) \|_{L^2(\Omega)}^2 \, dt = \int_0^T \int_\Omega |f_4(t,x)|^2 \left| \int_{T_1}^{T_2} \rho_2(\sigma,x) Bv(\sigma,x) \, d\sigma \right|^2 \, dx
\]
\[
\leq \| \rho_2 \|_{L^2(T_1,T_2;L^\infty(\Omega))}^2 \| f_4 \|_{L^2(0,T;L^\infty(\Omega))} \int_{T_1}^{T_2} \| Bv(\sigma,\cdot) \|_{L^2(\Omega)}^2 \, d\sigma
\]
\[
\leq \| \rho_2 \|_{L^2(T_1,T_2;L^\infty(\Omega))}^2 \| f_4 \|_{L^2(0,T;L^\infty(\Omega))}^2
\]
\[
\times K_3 K_6 \int_{T_1}^{T_2} (|l(\sigma)|)^{\gamma - 3} (|l(\sigma)|)^{-3} \| v(\sigma,\cdot) \|_{L^2(\Omega)}^2 \, d\sigma
\]
\[
\leq K_3 K_6 \max \left\{ M_3^{\gamma - 3}, 2^{\delta + 3\gamma - 7}\right\} \| \rho_2 \|_{L^2(T_1,T_2;L^\infty(\Omega))}^2
\]
\[
\times \| f_4 \|_{L^2(0,T;L^\infty(\Omega))} \int_{T_1}^{T_2} (|l(\sigma)|)^{-3} \| v(\sigma,\cdot) \|_{L^2(\Omega)}^2 \, d\sigma,
\]
(4.9)

where we also used the estimate \( \| l \|_{L^\infty} \leq T^2/4 \). Therefore, from 4.2 and 4.4-4.9 we get the integral inequality:
\[
\| v_\gamma(t,\cdot) \|_{L^2(\Omega)}^2 + \mu_0 \int_0^T \| \nabla_x v_\gamma(t,\cdot) \|_{L^2(\Omega)}^2 \, dt
\]
\[
\leq \int_0^T b_\gamma(t) \| v_\gamma(t,\cdot) \|_{L^2(\Omega)}^2 \, dt + \int_0^T \| \tilde{f}_\gamma(t,\cdot) \|_{L^2(\Omega)} \| v_\gamma(t,\cdot) \|_{L^2(\Omega)} \, dt
\]
\[
+ J_1(f_1, f_2) \int_{T_1}^{T_2} \| D_1 v(t,\cdot) \|_{L^2(\Omega)}^2 \, dt + J_2(f_1, f_2, f_3, \rho_1) \int_{T_1}^{T_2} \| v(t,\cdot) \|_{L^2(\Omega)}^2 \, dt
\]
\[
+ J_3(f_4, \rho_2) \int_{T_1}^{T_2} (|l(t)|)^{-3} \| v(t,\cdot) \|_{L^2(\Omega)}^2 \, dt + 2 \| \sigma_\gamma \|_{L^\infty(0,T)} \int_{T_1}^{T_2} \| v(t,\cdot) \|_{L^2(\Omega)}^2 \, dt,
\]
(4.10)

where we have set
\[
b_\gamma(t) = 2 \mu_0 + 1 + \sqrt{K_3 K_6 \chi_{(\varepsilon,T)}(t)(|l(t)|)^{3 - \gamma}}, \quad t \in (0,T),
\]
\[
J_1(f_1, f_2) = \sum_{j=1}^{2} \| f_j \|_{L^2(0,T;L^\infty(\Omega))}^2,
\]
\[ J_2(f_1, f_2, f_3, \rho_1) = \left( (T_2 - T_1)^{-1} + 1 \right) \sum_{j=1}^{2} \| f_j \|_{L^2(0,T; L^\infty(\Omega))}^2 \]
\[ + \| \rho_1 \|_{L^2(0,T; L^\infty(\Omega))}^2. \]
\[ J_3(f_4, \rho_2) = K_3 K_6 \max \{ M_{T_1, T_2, 2}^{2\gamma - 3}, 2^{6 - 4\gamma} T^{4\gamma - 6} \} \| \rho_2 \|_{L^2(0, T_2; L^\infty(\Omega))}^2 \| f_4 \|_{L^2(0,T; L^\infty(\Omega))}^2. \]

Since \( \varepsilon \in (0, T_1/(2T)) \), it follows that \( 2\varepsilon T < T_1 \) and (4.10) implies the integral inequality
\[ \| v(\tau, \cdot) \|_{L^2(\Omega)}^2 + \mu_0 \int_0^\tau \| \nabla_x v(t, \cdot) \|_{L^2(\Omega)}^2 \, dt \]
\[ \leq \int_0^\tau b_\varepsilon(t) \| v(t, \cdot) \|_{L^2(\Omega)}^2 \, dt + \int_0^\tau \| f_\varepsilon(t, \cdot) \|_{L^2(\Omega)} \| v(t, \cdot) \|_{L^2(\Omega)} \, dt \]
\[ + J_1(f_1, f_2) \int_T^{T_2} \| D_t v(t, \cdot) \|_{L^2(\Omega)}^2 \, dt + J_3(f_4, \rho_2) \int_T^{T_2} (l(t))^{-3} \| v(t, \cdot) \|_{L^2(\Omega)}^2 \, dt \]
\[ + J_4(\varepsilon, f_1, f_2, f_3, \rho_1) \int_T^{T_2} \| v(t, \cdot) \|_{L^2(\Omega)}^2 \, dt, \]
where
\[ J_4(\varepsilon, f_1, f_2, f_3, \rho_1) = J_2(f_1, f_2, f_3, \rho_1) + 2\| \sigma_\varepsilon \|_{L^2(0,T)}^2. \]

Now, from (2.2) we deduce the inequalities
\[ (l(t))^j \exp[2s_0 \alpha_\lambda(t, x)] \geq \left( \min_{t \in [\varepsilon T, T_2]} l(t) \right)^j \exp \left\{ -2s_0 c_{1,\lambda}(\psi) \left( \min_{t \in [\varepsilon T, T_2]} l(t) \right)^{-1} \right\} \]
\[ =: C_{2+j}(\varepsilon, T_2, T), \]
\[ (l(t))^{-3} \exp[2s_0 \alpha_\lambda(t, x)] \geq 2^{6-4\gamma} T^{-6} \exp \left\{ -2s_0 c_{1,\lambda}(\psi) \left( \min_{t \in [\varepsilon T, T_2]} l(t) \right)^{-1} \right\} \]
\[ =: C_{4}(\varepsilon, T_2, T), \]
for all \( t \in [\varepsilon T, 2\varepsilon T] \) and \( j = 0, 1 \). Hence, from the Carleman type estimate (3.9), we obtain
\[ \int_T^{T_2} \| D_t^j v(t, \cdot) \|_{L^2(\Omega)}^2 \, dt \]
\[ \leq (C_{4-j}(\varepsilon, T_2, T))^{-1} \int_T^{T_2} \| l(t) \|^{-3+4j} \exp[2s_0 \alpha_\lambda(t, x)] \| D_t^j v(t, \cdot) \|_{L^2(\Omega)}^2 \, dt \]
\[ \leq 12 C_1 (C_{4-j}(\varepsilon, T_2, T))^{-1} s_0^{4j-3} e^{\lambda \| \psi \|} \| \int_T^{T_2} |f(t, x)|^2 \exp[2s_0 \alpha_\lambda(t, x)] \, dt dx \]
\[ \leq 12 C_1 (C_{4-j}(\varepsilon, T_2, T))^{-1} s_0^{4j-3} e^{\lambda \| \psi \|} \| \int_{Q_T} |f|^2 \, dx \, dt, \]
for \( j = 0, 1 \) and
\[ \int_T^{T_2} \| l(t) \|^{-3} \| v(t, \cdot) \|_{L^2(\Omega)}^2 \, dt \]
\[ \leq (C_2(\varepsilon, T_2, T))^{-1} \int_T^{T_2} \| l(t)^{-3} \exp[2s_0 \alpha_\lambda(t, x)] \| v(t, \cdot) \|_{L^2(\Omega)}^2 \, dt \]
\[ \leq 12 C_1 (C_2(\varepsilon, T_2, T))^{-1} s_0^{-3} \| \int_{Q_T} |f|^2 \, dx \, dt. \]
Finally, from (4.9), (4.11) and (4.12) we deduce the fundamental integro-differential inequality

\[
    z_\varepsilon(\tau) := \|v_\varepsilon(\tau, \cdot)\|^2_{L^2(\Omega)} + \mu_0 \int_0^\tau \|\nabla_x v_\varepsilon(t, \cdot)\|^2_{L^2(\Omega)} \, dt \\
    \leq \int_0^\tau b_\varepsilon(t) z_\varepsilon(t) \, dt + \int_0^\tau \|\bar{f}(t, \cdot)\|_{L^2(\Omega)} \chi_{(\varepsilon T, T)}(t) (z_\varepsilon(t))^{1/2} \, dt \\
    + J_5(\varepsilon, f_1, f_2, f_3, f_4, \rho_1, \rho_2, \lambda, \psi) \|\bar{f}\|^2_{L^2(\Omega_{\varepsilon T})},
\]

for any \( \tau \in (\varepsilon T, T) \) (and, hence, for any \( \tau \in [0, T] \) since \( v_\varepsilon(t, \cdot) = 0 \) for any \( t \in [0, \varepsilon T] \)), where

\[
    J_5(\varepsilon, f_1, f_2, f_3, f_4, \rho_1, \rho_2, \lambda, \psi) = 12C_1(C_3(\varepsilon, T_2, T))^{-1}J_1(f_1, f_2)s_0e^{\lambda\|\psi\|_\infty} \\
    + 12C_1(C_2(\varepsilon, T_2, T))^{-1}J_3(f_4, \rho_2)s_0^{-3} \\
    + 12C_1(C_4(\varepsilon, T_2, T))^{-1}J_4(\varepsilon, f_1, f_2, f_3, \rho_1)s_0^{-3}.
\]

From Lemma 4.13 we deduce the fundamental estimate

\[
    \|v_\varepsilon(\tau, \cdot)\|^2_{L^2(\Omega)} + \mu_0 \int_0^\tau \|\nabla_x v_\varepsilon(t, \cdot)\|^2_{L^2(\Omega)} \, dt \\
    \leq \left[J_5(\varepsilon, f_1, f_2, f_3, f_4, \rho_1, \rho_2, \lambda, \psi)^{1/2} \|\bar{f}\|_{L^2(\Omega_{\varepsilon T})} \exp\left(\frac{1}{2} \int_0^\tau b_\varepsilon(r) \, dr\right) \\
    + \int_0^\tau \exp\left(\frac{1}{2} \int_0^\tau b_\varepsilon(r) \, dr\right) \chi_{(\varepsilon T, T)}(t) \|\bar{f}(t, \cdot)\|_{L^2(\Omega)} \, dt\right]^2,
\]

for any \( \tau \in [0, T] \). In particular, for all \( \tau \in [2\varepsilon T, T] \), we easily find the desired estimate for \( u = v + g \):

\[
    \|u(\tau, \cdot)\|^2_{L^2(\Omega)} + \mu_0 \int_{2\varepsilon T}^\tau \|\nabla_x u(t, \cdot)\|^2_{L^2(\Omega)} \, dt \\
    \leq 2\|g(\tau, \cdot)\|^2_{L^2(\Omega)} + 2\mu_0 \int_0^\tau \|\nabla_x g(t, \cdot)\|^2_{L^2(\Omega)} \, dt \\
    + 2 \left[J_5(\varepsilon, f_1, f_2, f_3, f_4, \rho_1, \rho_2, \lambda, \psi)^{1/2} \exp\left(\frac{1}{2} \|b_\varepsilon\|_{L^1(0, T)}\right) \|\bar{f}\|_{L^2(\Omega_{\varepsilon T})} \\
    + \exp\left(\frac{1}{2} \|b_\varepsilon\|_{L^1(0, T)}\right) \int_0^\tau \chi_{(\varepsilon T, T)}(t) \|\bar{f}(t, \cdot)\|_{L^2(\Omega)} \, dt\right]^2, \\
    (4.13)
\]

for any \( \varepsilon \in (0, T_1/(2T)) \). Finally, observe that from (2.1) and (4.3) we easily deduce the estimate

\[
    \|\bar{f}(t, \cdot)\|_{L^2(\Omega)} \leq \|f_0(t, \cdot)\|_{L^2(\Omega)} + \|D_1 g(t, \cdot)\|_{L^2(\Omega)} + \|A(\cdot, D) g(t, \cdot)\|_{L^2(\Omega)} \\
    + \|B g(t, \cdot)\|_{L^2(\Omega)} \\
    \leq \|f_0(t, \cdot)\|_{L^2(\Omega)} + \|D_1 g(t, \cdot)\|_{L^2(\Omega)} + \|A(\cdot, D) g(t, \cdot)\|_{L^2(\Omega)} \\
    + \sum_{j=1}^2 \|f_j(t, \cdot)\|_{L^\infty(\Omega)} \|g(T_j, \cdot)\|_{L^2(\Omega)} \\
    + \|f_3(t, \cdot)\|_{L^\infty(\Omega)} \|\rho_1\|_{L^2(T_1, T_2, L^\infty(\Omega))} \|g\|_{L^2(Q_{T_1, T_2})} \\
    + \sqrt{K_3 K_6} \|l(t)\|^{-3} \|g(t, \cdot)\|_{L^2(\Omega)} \\
    + \sqrt{K_3 K_6 M_{T_1, T_2}^{-3}} \|f_4(t, \cdot)\|_{L^\infty(\Omega)} \|\rho_2\|_{L^2(T_1, T_2, L^\infty(\Omega))} \|g\|_{L^2(Q_{T_1, T_2})},
\]
where, as usual, $M_{T_1,T_2} = \inf_{t \in [T_1,T_2]} l(t)$. Hence,
\[
\int_0^T \chi_{(\epsilon T,T)}(t) \| \tilde{f}(t,\cdot) \|_{L^2(\Omega)} dt \\
\leq \sqrt{T} \left( \| f_0 \|_{L^2(Q_T)} + \| D_1 g \|_{L^2(Q_T)} + \| A(\cdot,D)g \|_{L^2(Q_T)} + \sum_{j=1}^2 \| f_j \|_{L^2(Q_T)} \| g(T_j,\cdot) \|_{L^2(\Omega)} \\
+ \| f_3 \|_{L^2(0,T;L^\infty(\Omega))} \| p_1 \|_{L^2(T_1,T_2;L^\infty(\Omega))} \| g \|_{L^2(Q_{T_1,T_2})} \\
+ \sqrt{K_3 K_6 M_{T_1,T_2}^{-3}} \| f_4 \|_{L^2(0,T;L^\infty(\Omega))} \| p_2 \|_{L^2(T_1,T_2;L^\infty(\Omega))} \| g \|_{L^2(Q_T)} \right) \\
+ \sqrt{K_3 K_6} \left( \int_{\epsilon T}^T (l(t))^{2\gamma-6} dt \right)^{1/2} \| g \|_{L^2(Q_T)}.
\]
Replacing this estimate in (4.13), the assertion follows at once. \(\surd\)

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