Einstein–Cartan gravity with Holst term and fermions

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We investigate the consequences of the ambiguity of minimal coupling procedure for Einstein–Cartan gravity with Holst term and fermions. A new insight is provided into the nature and physical relevance of coupling procedures considered hitherto in the context of Ashtekar–Barbero–Immirzi formalism with fermions. The issue of physical effects of the Immirzi parameter in semi–classical theory is reinvestigated. We argue that the conclusive answer to the question of its measurability will not be possible until the more fundamental problem of nonuniqueness of gravity–induced fermion interaction in Einstein–Cartan theory is solved.

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I. INTRODUCTION

In the last few years one can observe an increase of interest in coupling fermionic matter to Einstein–Cartan (EC) gravity modified by the presence of a new term in the gravitational action, the so called Holst term $\beta$. Its inclusion simplifies the structure of constraints of the theory and enables nonperturbative canonical quantisation of gravitational field in the lines of Loop Quantum Gravity (LQG) program. The history of this term can be traced back to the introduction by Ashtekar of new variables for general relativity (GR) and reducing constraints of the theory to the polynomial form. In order to avoid difficulties concerning reality conditions, necessary in Ashtekar complex approach, Barbero proposed a real alternative. The relation between these two approaches was then clarified by Immirzi and Holst. It appeared that the addition of a new term to the gravitational action allows for a unified treatment of both approaches. The Real parameter $\beta$ whose inverse precedes the new term is called an Immirzi parameter. In the covariant Holst approach, the Ashtekar and Barbero formulation can be recovered for $\beta = \pm i$ and $\beta = \pm 1$, respectively. However, it appeared that one can proceed in the program of canonical quantisation for $\beta$ assuming a generic real value. The parameter proved to play an important role in the quantum theory of geometry by entering the spectra of area and volume operators.

The Holst modification of the standard Palatini action does not change classical field equations, so long as matter with nonvanishing spin distribution is not included. However, the first order formalism employed by LQG leads in a natural way to EC gravity, rather than standard GR. Hence, the space–time manifold acquires a non–vanishing torsion when the spin of matter is not negligible. This makes the Holst term modify field equations. In [1], fermionic matter was considered, modeled by the standard Dirac Lagrangian. It was concluded that the Immirzi parameter can be observable in principle, even if quantum aspects of gravitational field are not taken into account. The coupling constant in front of the well known term describing point fermion interaction of EC gravity appeared to involve Immirzi parameter. The possibility of establishing experimental bounds on the parameter without invoking the complete theory of quantum geometry would be of great importance, since it seems there is a long way for LQG to develop before definite experimental predictions can be extracted. One of the successes of LQG has been the derivation of the formula for black hole entropy, which appeared to involve the Immirzi parameter. This allowed for the establishing of theoretical bounds on possible values of the Immirzi parameter by black hole entropy calculations and comparison with the Bekenstein–Hawking formula. A precise value of the parameter was given shortly after [18]. If we were also able to find experimental limits on this value, as it was hoped in [1], a comparison of these two results would provide the first test of physical relevance of LQG approach.

The minimal coupling of fermions to gravity was employed in [1]. The necessity of that coupling scheme and hence all physical predictions resulting from [1] were questioned shortly after. Two different one–parameter families of possible non–minimal couplings were suggested in [2] and [4]. The family considered in [2], which was claimed general, led authors to the conclusion that the minimally coupled theory is parity invariant, whereas the non–minimally coupled is not. This conclusion was based on the analysis of transformation properties of the effective action, obtained by establishing the connection between torsion and matter and inserting the result back to the initial action so that torsion was effectively eliminated from the theory. In fact, the minimally coupled theory is also not parity invariant,
although its effective action is. To see this, one should take into account the transformation properties of equations connecting torsion with matter fields. Parity breaking in the case of minimal coupling was first observed in [4], but it was misinterpreted as an indication of internal inconsistency of a theory. After criticizing the minimal procedure, a one–parameter family of non–minimal couplings was proposed in [4], apparently different from the one considered in [2] (in contradiction with the statements of [2] concerning generality). Parity violation was avoided by an appropriate choice of the parameter of the non–minimal coupling, suitably adopted to the value of the Immirzi parameter. After this choice was made, the theory reduced to the one which is often viewed as usual EC gravity with fermions (as explained in [15], the hitherto accounts on EC gravity with fermions were fairly incomplete). Finally, in [6] there was considered the four–parameter family of non–minimal couplings generalising the two families of [2] and [4]. It has been concluded that the Immirzi parameter is not measurable, as it can be “hidden” in the coupling parameters. It has not been taken into account that the possibility of measuring the Holst term induced physical effects might be regained, if we were able to decide on theoretical grounds which coupling procedure is physically appropriate.

The leading idea of this paper is that the minimal coupling scheme is the one that has been historically successful in constructing models, which could withstand the rigor of experimental testing whenever such tests were feasible. The experimental successes of the standard model of particle physics and general theory of relativity seem to support the minimal approach. Indeed, the Yang–Mills theories, which constitute the formal basis for the standard model, employ minimal coupling on the fundamental level. The necessity of using non–minimal couplings when describing effectively composed objects does not hold much relevance as long as we aim to incorporate elementary point–like fermions (quarks and leptons) into the theory of gravity. As is well known, the EC gravity can be formulated as a gauge theory of Yang–Mills type for the Poincaré group [20, 21, 22, 23]. Hence, one could hope that the application of minimal coupling would lead to the physically relevant model in this case as well. According to this viewpoint, special importance should be attached to the original analysis of [1], rather than to the later analyses employing non–minimal couplings. It is also important to stress that the conjecture of [4] that the Holst term should not produce any physical effects in the classical theory is not justified. We cannot know that before a thorough theoretical analysis is completed.

There is however an issue of fundamental importance that makes things more involved. If torsion of space–time does not vanish, the standard minimal coupling procedure (MCP) is not unique. Equivalent flat space Lagrangians (generating the same flat space field equations) give rise to generically non–equivalent curved theories. The problem has been discussed since the very beginning of gauge formulation of gravity [20]. Apart from the search for the criteria of choice of the most physical flat space Lagrangian, a more radical solution was proposed by Saa in [24, 25]. The introduction of a connection–compatible volume element, instead of a metric–compatible one, together with the requirement for torsion trace to be derivable from a potential, eliminated the ambiguity. The new procedure leads to interesting effects, such as a propagating torsion or coupling gauge fields to torsion without breaking gauge symmetry. Although Saa’s idea provides a very interesting solution to the problem, it results in significant departures from standard GR, which are not certain to withstand the confrontation with observable data [26, 27] without some assumptions of rather artificial nature, such as demanding a priori that part of the torsion tensor vanish [28].

The consequences of the above–mentioned ambiguity for EC theory with fermions were investigated in [19]. It appears that what was considered in [1, 3] as standard EC theory with fermions is only one possibility from the two–parameter family of theories, this family being related to the freedom of divergence addition to the flat space Lagrangian density (see Section III). After this result is at hand, one can no longer acknowledge the analysis of [1] as satisfying, as it corresponds to a particular flat fermionic Lagrangian randomly selected from the infinity of possibilities. In this paper we aim to exhaust the possibilities left by MCP. Contrary to the approach presented in [1], we aim to limit the multiplicity of “equivalent” flat space Lagrangians by imposing reasonable restrictions on them, rather than choosing a particular one without any justification.

An interesting result of the following paper is the observation that all the non–equivalent theories obtained by different non–minimal coupling procedures of [2, 4, 6] can be interpreted as resulting from the application of MCP to the suitably chosen fermionic flat space Lagrangians (see Section III). Hence, even if MCP is given the priority, we cannot claim those procedures to be worse than the one of [1]. However, from this viewpoint, the problem of choosing among them may be reduced to the choice of the most physically reasonable flat space Lagrangian for fermions. Some suggestions concerning this choice are given at the end of Section III.

The paper is organized as follows: In Section IV we explain the origins of ambiguity of MCP and impose some obvious restrictions on flat space fermionic Lagrangians, which leaves us with a two–parameter family. In Section IV we show that all the coupling procedures discussed so far in the context of fermions in Ashtekar–Barbero–Immirzi formalism can be realised as minimal couplings for appropriate flat space Lagrangians for fermions. We comment on their physical relevance. We also discuss a truly non–minimal procedure, which cannot be reinterpreted in this manner. In Section V we briefly recall the formalism of EC theory with Holst term and rederive the effective action taking into account the freedom of addition of a divergence to the flat space matter Lagrangian. We also comment on the possibility of detecting the physical effects produced by the Immirzi parameter. In Section V we draw conclusions.
II. NONUNIQUENESS OF MINIMAL COUPLING PROCEDURE

A classical field theory in flat Minkowski space is defined by the action functional

\[ S = \int \mathcal{L}, \]

where \( \mathcal{L} \) is a Lagrangian density and \( \mathcal{L} = \mathcal{L} \, d^4x = \mathcal{L} \, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \) a Lagrangian four–form. It is well known that the addition of a divergence of a vector field \( V \) to \( \mathcal{L} \) changes \( \mathcal{L} \) by a differential

\[ \partial_{\mu}V^\mu \, d^4x = L_V \, d^4x = d(V \, d^4x), \quad (\text{II.1}) \]

where \( L \) denotes Lie derivative and \( \partial \) the internal product. Thus, such a transformation does not change field equations generated by \( S \). In order to proceed from Minkowski space to the general Riemann–Cartan (RC) manifold with metric \( g_{\mu\nu} \) and the metric compatible connection \( \nabla \) (not–necessarily torsion free)\(^1\), we can apply MCP

\[ \int \mathcal{L}(\phi, \partial_{\mu}\phi, \ldots) \, d^4x \longrightarrow \int \mathcal{L}(\phi, \nabla_{\mu}\phi, \ldots) \epsilon, \quad (\text{II.2}) \]

where \( \epsilon = \sqrt{-g} \, d^4x \) is a volume–form, \( g \) being the determinant of a matrix of components \( g_{\mu\nu} \) of the metric tensor in the basis \( \partial_{\mu} \), and \( \phi \) represents fields of the theory. Dots in (II.2) correspond to the possibility of different Lagrangian four–form on the RC manifold, the difference being

\[ \nabla_{\mu}V^\mu \epsilon = \nabla_{\mu}V^\mu \epsilon - T_{\mu}V^\mu \epsilon, \quad (\text{II.3}) \]

where \( \nabla \) is the torsion free Levi–Civita connection and \( T_{\mu} = T^{\nu}_{\mu\nu} \) the torsion trace vector. The first term in (II.3) is a differential, \( \nabla_{\mu}V^\mu \epsilon = d(V \, \epsilon) \), whereas the second is not. Hence, the equivalent flat Lagrangians yield non–equivalent theories on RC space.

One could hope differential forms formalism would fix the problem and argue that the last expression of (II.1), rather than the first, should be adopted to curved space. Then, \( d(V \, d^4x) \) would transform into \( d(V \, \epsilon) \), which is again a differential. However, this is not a good solution, since decomposition of a given Lagrangian four–form \( \mathcal{L}_1 \, d^4x \) to the sum of another Lagrangian four–form \( \mathcal{L}_2 \, d^4x \) and the term \( d(V \, d^4x) \) is by no means unique. We should rather use the identity \( d(V \, d^4x) = - (\ast d\phi) \wedge \cdots \wedge d\phi \), where \( \ast \) is a hodge star (see Section VI), and minimally couple gravity by the passage \( dV^\mu \longrightarrow DV^\mu = dV^\mu + \omega_{\mu\nu}V^\nu \) (where \( \omega_{\mu\nu} \) are connection one–forms) and by the change of a hodge star of flat Minkowski metric to the one of curved metric on the final manifold, but this would give the result identical to (II.3).

Let us now consider two Lagrangian densities differing by a divergence of a vector field \( V^\mu(\phi) \) (we wish \( V^\mu \) not to depend on derivatives of \( \phi \) in order for both Lagrangians to depend on first derivatives only)

\[ \mathcal{L} - \mathcal{L}' = \partial_{\mu}V^\mu = \frac{\partial V^\mu}{\partial \phi} \partial_{\mu}\phi. \quad (\text{II.4}) \]

Here, \( V \) is required to transform as a vector under proper Lorentz transformations: if \( \phi \rightarrow \phi' \) represents the action of a relevant representation of a proper Lorentz group in the space of fields, we have \( V^\mu(\phi) \rightarrow V^\mu(\phi') = \Lambda^\mu_{\nu}V^\nu(\phi) \). Hence, \( \partial_{\mu}V^\mu \) is a Lorentz scalar and \( \mathcal{L}' \) is a Lorentz scalar (if \( \mathcal{L} \) is). All Lagrangian densities considered by us are also required to be real, which implies the reality of \( V \). Let us then focus our attention on the Dirac field \( \psi \). The requirement for Lagrangians to be real and quadratic in the fields suggests the following form of \( V \)

\[ V^\mu = \bar{\psi}B^\mu\psi, \quad (\text{II.5}) \]

where the matrices \( B^\mu \) obey the reality condition \( B^{\mu\dagger} = \gamma^0 B^\mu \gamma^0 \). Together with the requirement of vector transformation properties under the action of proper Lorentz group, it leads to \( B^\mu = a\gamma^\mu + b\gamma^\mu \gamma^5 \) for some real numbers \( a \) and \( b \). Here \( \gamma^\mu \) are the Dirac matrices obeying \( \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \), \( \gamma^5 := -i\gamma^0 \gamma^1 \gamma^2 \gamma^3 \) and \( \bar{\psi} := \psi\gamma^5 \), where \( \gamma^5 \) is a Hermitian conjugation of a column matrix. Hence,

\[ V^\mu = aJ^\mu_{(V)} + bJ^\mu_{(A)}, \quad (\text{II.6}) \]

where \( J^\mu_{(V)} = \bar{\psi}\gamma^\mu\psi, \ J^\mu_{(A)} = \bar{\psi}\gamma^\mu\gamma^5\psi \) denote Dirac vector and axial current.

\(^1\) For more general considerations concerning not–necessarily metric connections see [28].
III. NON–MINIMAL COUPLINGS FROM A DIFFERENT PERSPECTIVE

A. Apparent non–minimal couplings

According to [6], the general non–minimally coupled fermion action quadratic in fermionic field is an integral of a Lagrangian four–form

\[ L_{F} = i 2 \left( \bar{\psi} \gamma^{\alpha}(\zeta - i \xi \gamma^{5})D_{\alpha} \psi - D_{\alpha} \bar{\psi}(\zeta - i \xi \gamma^{5}) \gamma^{\alpha} \psi \right) \epsilon, \]  

(III.1)

where bar means complex conjugation while acting on numbers and Dirac conjugation while acting on spinors (see Section VI for the definition of \( D_{\alpha} \)). In fact, this action can be obtained via MCP from the flat space Lagrangian

\[ L_{F} = i 2 \left( \bar{\psi} \gamma^{\mu}(\zeta - i \xi \gamma^{5}) \partial_{\mu} \psi - \partial_{\mu} \bar{\psi}(\zeta - i \xi \gamma^{5}) \gamma^{\mu} \psi \right). \]  

(III.2)

Following Alexandrov, we shall denote

\[ \zeta = \eta + i \theta, \quad \xi = \rho + i \tau, \quad \eta, \theta, \rho, \tau \in \mathbb{R}. \]  

(III.3)

Then we get

\[ L_{F} = \eta i 2 \left( \bar{\psi} \gamma^{\mu} \partial_{\mu} \psi - \partial_{\mu} \bar{\psi} \gamma^{\mu} \psi \right) - \theta \frac{\partial_{\mu} \bar{\psi} \gamma^{\mu} \psi}{2} + \rho \frac{\partial_{\mu} \bar{\psi} \gamma^{5} \gamma^{\mu} \psi}{2} + \tau i \frac{\partial_{\mu} \bar{\psi} \gamma^{\mu} \gamma^{5} \psi}{2}. \]  

(III.4)

As we can now see, the first component is just the mass–free part of the standard Dirac Lagrangian density

\[ L_{F0} = i 2 \left( \bar{\psi} \gamma^{\mu} \partial_{\mu} \psi - \partial_{\mu} \bar{\psi} \gamma^{\mu} \psi \right) - m \bar{\psi} \psi, \]  

(III.5)

multiplied by a scaling constant \( \eta \), which could be set to one. The next two terms are divergences of vector and axial currents and they correspond to the families of ‘non–minimal’ couplings considered in [2] and [4], respectively. What is the most interesting is the last term. It seems to have been overlooked in [2] and [4] and brought to life in [6]. The Lagrangian obtained by setting \( \theta = \rho = 0 \), which can be conveniently rewritten in the form

\[ L_{1} = i 2 \left( \bar{\psi} (\eta - \tau \gamma^{5}) \gamma^{\mu} \partial_{\mu} \psi - \partial_{\mu} \bar{\psi} (\eta - \tau \gamma^{5}) \gamma^{\mu} \psi \right), \]  

(III.6)

generates the equation

\[ (\eta - \tau \gamma^{5}) i \gamma^{\mu} \partial_{\mu} \psi = 0, \]  

(III.7)

which is equivalent to the mass–free Dirac equation if \( \eta^{2} \neq \tau^{2} \) (under this condition the matrix \( \eta - \tau \gamma^{5} \) is invertible). At first, one could suppose that (III.6) might be generalised by

\[ L_{A} = i 2 \left[ \bar{\psi} A \gamma^{\mu} \partial_{\mu} \psi - \partial_{\mu} \bar{\psi} A \gamma^{\mu} \psi \right], \]  

(III.8)

where \( A \) is an invertible \( 4 \times 4 \) matrix, but the reality condition for (III.8)

\[ \gamma^{\mu} \gamma^{0} A^{\dagger} = A \gamma^{\mu} \gamma^{0} \]  

(III.9)

implies that \( A = \eta - \tau \gamma^{5} \), where \( \eta, \tau \) are real numbers, leaving us with (III.6). In the following, we will use the letter \( A \) just to denote the matrix \( \eta - \tau \gamma^{5} \), where the invertibility condition \( \eta^{2} \neq \tau^{2} \) should be understood to hold.

Let us investigate the possibility of adding a mass term to the Lagrangian (III.6). An obvious choice would be to add

\[ -m \bar{\psi} A \psi \]  

(III.10)

to (III.6). Then the variation with respect to \( \bar{\psi} \) would yield the massive Dirac equation. However, the Lagrangian would no longer be real, as the reality condition (III.9) does not guarantee the reality of (III.10). The appropriate reality condition for (III.10) is \( \gamma^{0} A^{\dagger} \gamma^{0} = A \), which will be fulfilled if and only if \( \tau = 0 \). Note that for a non–real
Lagrangian it is not guaranteed that the variation with respect to \( \psi \) will give the equation which is equivalent to the one obtained from variation with respect to \( \overline{\psi} \). Indeed, straightforward calculation of this variation gives the equation

\[
-i \partial_\mu \overline{\psi} \gamma^\mu \psi - m \overline{\psi} A = 0. \tag{III.11}
\]

After the Dirac conjugation is performed and the reality condition \( (III.9) \) is applied, this equation can be rewritten as

\[
i A \gamma^\mu \partial_\mu \psi - m \gamma^0 A^\gamma A^0 \psi = 0. \tag{III.12}
\]

It is now clear that it would not be equivalent to the massive Dirac equation unless \( \gamma^0 A^\gamma A^0 = A \) and hence \( \tau = 0 \). One could try the addition of a more general mass term \(-m \overline{\psi} B \psi\), where \( B \) is a general \( 4 \times 4 \) matrix obeying the reality condition \( \gamma^0 B^\gamma A^0 = B \). Then the variation with respect to \( \overline{\psi} \) and \( \psi \) would yield the same equation. But this equation would not be equivalent to the Dirac one, unless \( B = A \), which leads again to \( \tau = 0 \).

We have argued that it seems impossible to find an appropriate mass term for the Lagrangian \( (III.9) \) for \( \tau \neq 0 \). All fermions that have been detected in nature are massive. Although their masses are not inserted \emph{a priori} into the Lagrangian, but rather arise as a result of spontaneous symmetry breaking via Higgs mechanism, it is still important that the initial Dirac Lagrangian allow for the addition of mass term in a consistent way. Otherwise the Higgs mechanism could not yield an expected result. This is why we will not take into account the subfamily \( (III.2) \) corresponding to \( \tau \neq 0 \) in further considerations. We are therefore left with the two–parameter family of Lagrangians (we set \( \eta = 1 \) to exclude also scaling) that can be constructed from \( (III.5) \) by the addition of a divergence of a linear combination of axial and vector Dirac currents. As shown in the previous section, this is all the freedom we can gain by adding a divergence to the Lagrangian density, under the requirements for it to be real, second order in field powers, first order in derivatives and invariant under the proper Lorentz transformations. The consequences of this freedom will be exploited in Subsection \( V.B \). In the following, we will use the parameters \( a = -\theta/2, b = \rho/2 \) introduced in \( (II.6) \). It seems very difficult to reduce the remaining freedom on theoretical grounds. In the case of EC theory without Holst modification, one could demand the Lagrangian density to be parity invariant, which would correspond to \( b = 0 \). However, if we allow for the addition of Holst term to the standard Palatini term of gravitational action, which clearly behaves differently under parity transformation, there is no reason for excluding pseudo–scalar term corresponding to \( b \neq 0 \) from the flat Lagrangian density.

### B. Genuine non–minimal couplings

All the “non–minimal” couplings of gravity to fermions considered so far could be reinterpreted as minimal couplings for suitable fermionic Lagrangians. Are there any truly non–minimal couplings which cannot be viewed as minimal ones for any choice of the flat space Lagrangian? To answer this question, let us first note that the Dirac Lagrangian four–form obtained from \( (III.5) \) via MCP

\[
\mathcal{L}_{F0} = -\frac{i}{2} (\ast e_a) \wedge (\overline{\psi} \gamma^a D \psi - \overline{D \psi} \gamma^a \psi) - m \overline{\psi} \psi \epsilon
\]

decomposes as

\[
\mathcal{L}_{F0} = \mathcal{L}_{F0}^0 - \frac{1}{8} S_\alpha J^\alpha (A) \epsilon, \tag{III.13}
\]

where \( \mathcal{L}_{F0}^0 \) is a part determined by the Levi–Civita connection. Here \( S_\alpha = \epsilon_{abcd} T^{bc} \), where \( T^{ab} \) are components of the torsion tensor in an unhonolomic tetrad basis (in a holonomic frame \( \partial_\mu \) the components are given by \( T^\rho_{\mu \nu} = -\Gamma^\rho_{\mu \nu} + \Gamma^\rho_{\nu \mu} \), where the connection coefficients are defined by \( \nabla_\mu \partial_\nu = \Gamma^\rho_{\nu \mu} \partial_\rho \)). This form of decomposition suggests the non–minimal coupling

\[
\mathcal{L}_{F,nm} = \mathcal{L}_{F0}^0 + \eta_1 S_\alpha J^\alpha (A) \epsilon, \tag{III.14}
\]

where \( \eta_1 \in \mathbb{R} \) is a coupling parameter. This family cannot be attained by MCP from any flat Lagrangian. In fact, there does not exist a flat space Lagrangian which would produce via MCP the Lagrangian four–form differing from \( (III.14) \) by a differential. To see it, imagine that such a flat space Lagrangian four–form, say \( \mathcal{L}_{\eta_1} \), exists. Let \( \mathcal{L}_{\eta_1} \) denote the result of application of MCP to \( \mathcal{L}_{\eta} \). If \( \mathcal{L}_{\eta_1} \) differs form \( (III.14) \) by a differential, then the flat space limits of these two four–forms (obtained for the flat Minkowski metric and vanishing torsion) ought to differ by a differential as well. But for \( (III.14) \) this limit would be just \( \mathcal{L}_{F0} \). Hence, we would have

\[
\mathcal{L}_{\eta_1} = \mathcal{L}_{F0} + \partial_\mu V^\mu \tag{III.15}
\]
for some vector field $V$. But then application of MCP would yield the relation
\[
\mathcal{L}_{\eta_1} = \mathcal{L}_{F_0} + \frac{\partial}{\partial \mu} V^\mu \epsilon - T_\mu V^\mu = \mathcal{L}_{F, nm} + \frac{\partial}{\partial \mu} V^\mu = \left[\frac{1}{8} + \eta_1\right] S_a J^a_\theta + T_a V^a \epsilon .
\] (III.16)

Although the second component in the final expression is a differential, the third one is not (for generic torsion) which contradicts our assumption.

Hence, we see that (III.14) represents a truly non–minimal coupling. It is a part of a family of couplings
\[
\mathcal{L}_{\eta, non-min} = \mathcal{L}_{F_0} + \eta_1 S_a J^a_\theta + \eta_2 T_a J^a_\theta \epsilon ,
\] (III.17)
discussed in [24]. The second part corresponding to $\eta_2$ is not really non–minimal, as it arises as a result of the application of MCP to the flat space Lagrangian constructed from (III.5) via (II.4) with $\omega$ cotetrad,

\[
S_V
\]
for some vector field $\nabla$ the metric connection $L$ and $
abla \phi$

where

\[
\Gamma^a_{bc} = \Gamma^a_{bc} e^c
\]
are connection one–forms (spin connection) obeying antisymmetry condition $\omega_{ab} = -\omega_{ba}$ and $\Omega^a_{bc} := d\omega^a_{bc} + \omega^a_c \wedge \omega^c_b = \frac{1}{2} R^a_{bcde} e^c \wedge e^d$ are the curvature two–forms. The connection coefficients $\Gamma^a_{bc}$ are related to the metric connection $\nabla$ on the RC manifold by $\nabla e_v e_b = \Gamma^a_{bc} e_a$, where $e_a = e_a^\mu \partial_\mu$ is an orthonormal tetrad (a basis of vector fields which is dual to one–form field basis $e^\mu$). Variation is given by

\[
\delta \mathcal{L} = \delta e^a \wedge \left( \frac{\delta \mathcal{L}_G}{\delta e^a} + \frac{\delta \mathcal{L}_{hol}}{\delta e^a} + \frac{\delta \mathcal{L}_m}{\delta e^a} \right) + \delta \omega^{ab} \wedge \left( \frac{\delta \mathcal{L}_G}{\delta \omega^{ab}} + \frac{\delta \mathcal{L}_{hol}}{\delta \omega^{ab}} + \frac{\delta \mathcal{L}_m}{\delta \omega^{ab}} \right) + \delta \phi^A \wedge \frac{\delta \mathcal{L}_m}{\delta \phi^A} ,
\]

\[
\phi^A
\]
representing matter fields (we used the independence of $\mathcal{L}_G$ and $\mathcal{L}_{hol}$ on $\phi^A$). Explicitly,

\[
\frac{\delta \mathcal{L}_G}{\delta e^a} = -\frac{1}{4k} \epsilon_{abcd} e^a \wedge e^b \wedge \Omega^{cd} , \quad \frac{\delta \mathcal{L}_G}{\delta \omega^{ab}} = -\frac{1}{2k} \epsilon_{cd} e^a \wedge e^b \wedge \Omega^{cd} ,
\]

\[
\frac{\delta \mathcal{L}_{hol}}{\delta e^a} = \frac{1}{k} e^b \wedge \omega^{ab} + \frac{1}{k} DQ_a , \quad \frac{\delta \mathcal{L}_{hol}}{\delta \omega^{ab}} = -\frac{1}{2k} D \left( e^a \wedge e^b \right) = \frac{1}{k} Q[a \wedge e^b] ,
\] (IV.3)

where $Q[a] := D e^a = \frac{1}{2} T_a^{bc} e^b \wedge e^c$ is a torsion two–form whose components in a tetrad basis we are denoting by $T^a_{bc}$.

The resulting field equations are

\[
\frac{\delta \mathcal{L}_G}{\delta e^a} + \frac{\delta \mathcal{L}_{hol}}{\delta e^a} + \frac{\delta \mathcal{L}_m}{\delta e^a} = 0 \iff \ G^a_{b} := R^a_{b} - \frac{1}{2} R e^a_{b} = k t^a_{b} + \frac{1}{2k} \epsilon^{cdef} R_{bced}
\]

\[
\frac{\delta \mathcal{L}_G}{\delta \omega^{ab}} + \frac{\delta \mathcal{L}_{hol}}{\delta \omega^{ab}} + \frac{\delta \mathcal{L}_m}{\delta \omega^{ab}} = 0 \iff \ T^{cab} - T^a \eta^{bc} + T^{b} \eta^{ac} = \frac{k}{1 + \beta^2} \left( S^{abc} - \frac{1}{2 \beta} e^a_{de} S^{dec} \right)
\] (IV.4)

\[
\frac{\delta \mathcal{L}_m}{\delta \phi^A} = 0
\]
where \( R^a_{\ b} := \eta^{ac} R^d_{\ cdb} \), \( R := R^a_{\ b} \), \( T^a := T^{ba}_{\ b} \), and the dynamical definitions of energy–momentum and spin density tensors on Riemann–Cartan space are

\[
t_a^\ b := \frac{\delta \mathcal{L}_m}{\delta e^a} \wedge e^b, \quad S^{abc} (\ast e_c) := \frac{\delta \mathcal{L}_m}{\delta \omega_{ab}}.
\]

(IV.5)

**B. Effective action**

If the spin density tensor \( S^{abc} \) does not depend on the connection, which is the case for fermions modelled by the Dirac Lagrangian, the second equation of (IV.4) represents an invertible\(^2\) algebraic relation between the components of the torsion tensor and the spin density tensor. We can see from (IV.4) that this important feature of EC theory remains true after the addition of the Holst term (recall that real values of the Immirzi parameter are considered in this paper). This makes the torsion vanish wherever the distribution of matter vanishes (the torsion waves do not exist). Then the connection becomes the Levi–Civita one, determined by the metric, and the first equation of (IV.4) reduces to the usual vacuum Einstein equation (note that the \( \beta \) dependent term of this equation vanishes then on account of Bianchi identity). This is a desirable feature of both the theories (with and without the Holst term), as it renders them compatible with all the experimental tests of GR that are based on vacuum solutions.

Wherever the spin density does not vanish, a nonzero torsion must appear. This may have significance either for the classical theory of self–gravitating matter (star formation, singularity theorems etc.) or for the semi–classical description of quantum fields. In the latter case, the EC theory is believed to differ from GR by the presence of gravity–induced point fermion interaction. The character of this interaction for different Dirac Lagrangians was studied in [19]. Here we will enquire whether this character changes after the addition of Holst term to the action.

In order to investigate the physics emerging from the theory for the space–time metric approaching the flat Minkowski’s one and to compare the predictions of GR, EC theory and Holst–modified EC theory, it is extremely useful to express the torsion through matter fields by means of second equation of (IV.4) and to insert the result back to the initial action. In this way an effective action is obtained, which does not depend on torsion anymore. We shall now derive this action for Holst–modified gravity with fermions. Let us define the contortion one–forms

\[
K^a_{\ b} = K^a_{\ bc} e^c := \omega^a_{\ b} - \omega^a_{\ b}
\]

(objects with \( \circ \) above will always denote torsion–free objects, related to LC connection). The curvature two–form decomposition

\[
\Omega^e_{\ b} = \Omega^e_{\ b} + \overset{\circ}{D} K^a_{\ b} + K^a_{\ c} \wedge K^c_{\ b}
\]

results in

\[
\mathcal{L}_G := -\frac{1}{4k} \epsilon_{abcd} e^a \wedge e^b \wedge \Omega^{cd} = \overset{\circ}{D}_G - \frac{1}{4k} \epsilon_{abcd} e^a \wedge e^b \wedge K^c_{\ e} \wedge K^{ed} - \frac{1}{4k} \overset{\circ}{D} \left( \epsilon_{abcd} e^a \wedge e^b \wedge K^{cd} \right)
\]

where \( \overset{\circ}{D}_{abcd} = 0 \) was used. Here \( k = 8\pi G \), where \( G \) is a gravitational constant. Since all Lorentz indexes in the last term are contracted, \( \overset{\circ}{D} \) acts like a usual differential. Similarly, for Holst term we get

\[
\mathcal{L}_{\text{Hol}} := -\frac{1}{2k\beta} e^a \wedge e^b \wedge \Omega_{ab} = \frac{1}{2k\beta} e^a \wedge e^b \wedge K_{ac} \wedge K^c_{\ b} + \frac{1}{2k\beta} \overset{\circ}{D} \left( e^a \wedge e^b \wedge K_{ab} \right)
\]

(IV.6)

which yields the combined result

\[
\mathcal{L}_G + \mathcal{L}_{\text{Hol}} = \overset{\circ}{D}_G + \frac{1}{2k\beta} e^a \wedge e^b \wedge \left( K_{ac} \wedge K^c_{\ b} - \frac{\beta}{2} \epsilon_{abcd} K^c_{\ e} \wedge K^{ed} \right) + \mathcal{O}(\ldots)
\]

(IV.7)

The last term is a differential and its particular form will not be needed). Using the relation between components of contortion and torsion tensors

\[
K_{abc} = \frac{1}{2} (T_{cab} + T_{bac} - T_{abc})
\]

(IV.8)

\(^2\) Note that this is not a generic feature of EC theory and may not be true if the spin density tensor depended on torsion. For example, in the case of Proca field invertibility breaks down for some values of the field, which gives rise to the notion of torsion singularities [23].
and decomposing torsion into its irreducible parts

\[ T_{abc} = \frac{1}{3}(\eta_{ac} T_b - \eta_{ab} T_c) + \frac{1}{6} \epsilon_{abcd} S^d + q_{abc}, \]

\[ T_a := T^b_{ab}, \quad S_a := \epsilon_{abcd} T^{bcd}, \]

we can finally obtain

\[ \mathcal{L}_G + \mathcal{L}_{Hol} = \mathcal{L}_G + \mathcal{L}_{P0} + \frac{3k\beta}{16(1 + \beta^2)} \left( J_a^{(A)} J_a^{(A)} - 4V_a V_a + \frac{4}{\beta} J_a^{(A)} V_a \right) \epsilon. \]  

Similarly, the Dirac Lagrangian four–form decomposes according to (IV.13). The addition of a divergence of a vector field \( V \) to the flat space Lagrangian results in one more term (IV.13). Ultimately, we have the following four–form on RC space representing gravity with Holst term and fermions

\[ \mathcal{L} = \mathcal{L}_G + \mathcal{L}_{P0} + \frac{3k\beta}{16(1 + \beta^2)} \left( J_a^{(A)} J_a^{(A)} - 4V_a V_a + \frac{4}{\beta} J_a^{(A)} V_a \right) \epsilon. \]  

The total differential has been omitted in the final formula. Variation of the resulting action with respect to \( T_a, S_a \) and \( q_{abc} \) yields the equations

\[ T^a = \frac{3k\beta}{4(1 + \beta^2)} \left( 2\beta V^a - J_a^{(A)} \right), \quad S^a = \frac{-3k\beta}{1 + \beta^2} \left( 2V^a + \beta J_a^{(A)} \right), \quad q_{abc} = 0. \]  

Inserting these results into (IV.11) we finally get the effective Lagrangian four–form

\[ \mathcal{L}_{eff} = \mathcal{L}_G + \mathcal{L}_{P0} + \frac{3k\beta}{16(1 + \beta^2)} \left( J_a^{(A)} J_a^{(A)} - 4V_a V_a + \frac{4}{\beta} J_a^{(A)} V_a \right) \epsilon. \]  

By taking the limit \( \beta \to \infty \) we can recover the effective Lagrangian for the usual EC theory

\[ \mathcal{L}_{EC\ eff} = \mathcal{L}_G + \mathcal{L}_{P0} + \frac{3k}{16} \left( J_a^{(A)} J_a^{(A)} - 4V_a V_a \right) \epsilon, \]  

whose physical implications were discussed in [19]. One of the striking differences between the theory with and without Holst term concerns the consequences of the requirement of parity invariance of the final theory. If Holst term is not present, this condition appears to be equivalent to the requirement of parity invariance of the flat space Lagrangian density for fermions. After it is imposed, we are left with the one–parameter family of theories, parameterized by \( a \) (parameter \( b \) has to be equal to zero). It seems difficult to provide theoretical arguments which would allow for further restriction of the remaining freedom, except for some speculations based on the resulting form of the spin density tensor [19]. Things look different if we adopt Holst–modified Lagrangian as representing gravitational field. Here, the requirement of parity invariance of the final theory fixes things uniquely. To see that this is the case, note that the second equation in (IV.12) is parity invariant if and only if \( V \) is an axial vector. But then, the first equation in (IV.12) is parity invariant only if \( V^a = \frac{1}{2\alpha} J_a^{(A)} \). This choice corresponds to what was done in [4] (the whole family of couplings considered in [4] can be obtained by taking \( V^a = \frac{1}{2\alpha} J_a^{(A)} \)). Hence, it is interesting to note that Holst modification together with the symmetry requirements provides us with uniqueness, which seemed to be impossible on the grounds of the standard EC theory.

Note that this result does not decide the question of measurability of the Immirzi parameter. After appropriate experiments are performed, whether it is possible to decide about the physical relevance of Holst term or not will depend on the particular outcomes. To see it, note that although the EC theory with \( V = 0 \) is indeed equivalent to the Holst–modified theory with \( V = \frac{1}{2\alpha} J_a^{(A)} \), the EC theory with \( V \neq 0 \) is NOT equivalent to the Holst–modified theory for any choice of \( V \) in the latter one, as can be easily seen from equations (IV.12) and their limiting forms for \( \beta \to \infty \). Let us imagine that we can actually measure the torsion itself and after a series of clever experiments we have established that the torsion trace vector \( T^a \) assumes a certain non–zero value. Then we can try to determine \( S^a \). If EC theory without Holst term is valid, we should get \( S^a = -3kJ_a^{(A)} \). If Holst–modified theory is appropriate, we ought to have \( S^a = -3kJ_a^{(A)} - \frac{1}{\beta} T^a \) instead. In the latter case, the result would provide an information about the value of Immirzi parameter.
C. Is it possible to distinguish between GR, EC and Holst–modified EC theory by measuring the strength of interactions?

In the previous section we have argued that it is possible that we could choose between EC and Holst–modified EC theory by measuring torsion. Here we shall consider the more realistic possibility, based on measurements of strength of point interactions between fermions. In the limit of vanishing Riemannian curvature, the effective fermionic Lagrangian density for all theories under consideration assumes the form

$$L_{\text{eff}} = L_{F0} + C_{AA} J^a_{(A)} J^a_{(A)} + C_{AV} J^a_{(A)} J^a_{(V)} + C_{VV} J^a_{(V)} J^a_{(V)}.$$  \text{(IV.15)}

For GR, all coupling constants vanish. For EC theory we have

$$C_{AA} = \frac{3k}{16}(1 - 4b^2), \quad C_{AV} = \frac{-3k}{2}ab, \quad C_{VV} = \frac{-3k}{4}a^2,$$  \text{(IV.16)}

whereas for Holst–modified EC theory

$$C_{AA} = \frac{3k\beta}{16(1 + \beta^2)} \left[4b + \beta(1 - 4b^2)\right], \quad C_{AV} = \frac{3k\beta}{4(1 + \beta^2)}a(1 - 2\beta b), \quad C_{VV} = \frac{-3k\beta^2}{4(1 + \beta^2)}a^2.$$  \text{(IV.17)}

As long as we get experimental values of all coupling constants indistinguishable from zero, we are not able to say which of the three theories of gravitation is correct. Measuring a non–zero value of at least one of them would provide an argument against standard GR. Of course, for any values of the coupling constants, one could adopt from the beginning \text{(IV.15)} itself as representing fermionic field in Minkowski space–time and use the torsionless approach of standard GR to include gravity. The presence of a point fermion interaction does not really contradict GR. However, on the grounds of EC and Holst–modified EC theory, the interaction terms arise naturally as a necessary consequence of the relation between torsion and matter, which was explained in Subsection \text{IV.B}. In this paper, we aim to treat all the theories in the most natural manner, adopting the simplest Dirac theory of fermions in flat space as a starting point in each case. Then, the most traditional method of minimal coupling is used to incorporate gravity and the results are compared.

According to this standpoint, such a non–zero value of a coupling constant would discredit GR. It could either agree with both remaining theories, which would be the case if the sets of equations \text{(IV.16)} and \text{(IV.17)} had a solution for measured values of the coupling constants, or contradict both of them. It appears that we cannot get a result which would agree with EC theory and disagree with Holst–modified theory, or conversely. For given values of the $C$–constants, either both \text{(IV.16)} and \text{(IV.17)} have a solution with respect to $a$ and $b$, or both are inconsistent. To see this, note that the change of parameters

$$a \rightarrow -\frac{\beta}{\sqrt{1 + \beta^2}}a, \quad b \rightarrow \frac{1 - 2\beta b}{2\sqrt{1 + \beta^2}}$$  \text{(IV.18)}

in \text{(IV.16)} leads directly to \text{(IV.17)}. Hence, as long as we cannot cope with ambiguities resulting from the freedom of addition of a divergence to the flat Lagrangian, it is not possible to discriminate between EC theory and Holst–modified EC theory by measuring the strength of four–fermion interactions, although it is in principle possible to rule them both out, or to strengthen their position and discredit GR.

V. CONCLUSIONS

The ambiguity of the minimal coupling procedure in the presence of torsion allows for the reinterpretation of all the non–minimal coupling procedures considered in the literature concerning Holst–modified gravity with fermions. They can be viewed as minimal couplings for appropriate flat space Lagrangians for fermions. There exist genuine non–minimal couplings which cannot be viewed in this way.

After some reasonable requirements are imposed on the Lagrangian formulation of the theory of Dirac field in flat space, the above–mentioned ambiguity is reduced to the two–parameter freedom in the final theory of Holst–modified EC gravity with fermions. The resulting family of theories is equivalent to the combined families considered in \text{[2]} and \text{[4]}. The richer four–parameter family of couplings introduced in \text{[6]} consists of the above mentioned two–parameter one, scaling and the new one–parameter family. However, the latter is not physically relevant from the viewpoint adopted in this article, as it corresponds to the flat space Lagrangian for mass–less fermions which does not allow for consistent addition of a mass term.
Unlike the standard EC theory, the theory with Holst term becomes unique under the requirement for it to be parity invariant. This property can be traced back to the difference in the behavior of standard gravitational action and Holst term under parity transformation. The resulting unique theory is the one that is usually considered as EC gravity with fermions – in fact, the EC gravity with fermions and without Holst term necessarily contains a one-parameter ambiguity, even if parity invariance requirement is imposed. This surprising feature of the Holst–modified gravity with fermions – in fact, the EC gravity with fermions and without Holst term necessarily contains a one-parameter ambiguity, even if parity invariance requirement is imposed. This surprising feature of the Holst–modified gravity with fermions – in fact, the EC gravity with fermions and without Holst term necessarily contains a one-

As long as we cannot solve the nonuniqueness problem theoretically, the theories with and without Holst term are indistinguishable on the effective level. This means that it is impossible to observe the physical effects of the Immirzi parameter by measuring the strength of gravity–induced interactions between fermions. However, if we were able to perform direct measurements of the space–time torsion, the effects of the Immirzi parameter could be detectable. The final answer to the question of its measurability would depend on the particular outcomes of experiments. We wish to stress that all the conclusions resulting from the analyses of this paper reflect the current state of knowledge and may intrinsically change if a satisfactory solution to the problem of nonuniqueness of EC theory is found.

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VI. APPENDIX: NOTATION AND CONVENTIONS

Throughout the paper we use the units $c = \hbar = 1$. The indexes $a, b, \ldots$ correspond to an orthonormal tetrad, whereas $\mu, \nu, \ldots$ correspond to a holonomic frame. For inertial frame of flat Minkowski space, which is both holonomic and orthonormal, we use $\mu, \nu, \ldots$. The metric components in an orthonormal tetrad basis $\tilde{e}_a$ are $g(\tilde{e}_a, \tilde{e}_b) = (\eta_{ab}) = \text{diag}(1, -1, -1, -1)$. By $e^a$ we shall denote an orthonormal cotetrad – the basis of one-form fields which is dual to the tetrad basis, $e^a(\tilde{e}_b) = \delta^a_b$. Lorenz indexes are shifted by $\eta_{ab}$. $\epsilon = \epsilon^0 \wedge \epsilon^1 \wedge \epsilon^2 \wedge \epsilon^3$ denotes the canonical volume four–form whose components in orthonormal tetrad basis obey $\epsilon_{0123} = -\epsilon^{0123} = 1$. The action of a covariant exterior differential $D$ on any $(r, s)$-tensoral type differential $m$-form

$$T^{a_1 \ldots a_r b_1 \ldots b_s} = \frac{1}{m!} T^{a_1 \ldots a_r b_1 \ldots b_s \mu_1 \ldots \mu_m} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_m}$$

is given by

$$DT^{a_1 \ldots a_r b_1 \ldots b_s} := dT^{a_1 \ldots a_r b_1 \ldots b_s} + \sum_{i=1}^r \omega^{a_i}{}_{c} \wedge T^{a_1 \ldots c \ldots a_r b_1 \ldots b_s} - \sum_{i=1}^s \omega^c{}_{b_i} \wedge T^{a_1 \ldots a_r b_1 \ldots c \ldots b_s}.$$ 

The covariant derivative of a Dirac bispinor field is

$$D\psi = (D_a \psi) \ e^a := d\psi - \frac{i}{2} \omega_{ab} \Sigma^{ab} \psi, \quad D\bar{\psi} = (D\psi)^\dagger \gamma^0, \quad \Sigma^{ab} := \frac{i}{2} [\gamma^a, \gamma^b],$$

where $\gamma^a$ are Dirac matrices, $\omega_{ab} = -\omega_{ba}$ are connection one–forms (spin connection). The hodge star action on external products of orthonormal cotetrad one–forms is given by

$$\star e_a = \frac{1}{3!} \epsilon_{abcd} e_b \wedge e_c \wedge e_d, \quad \star (e_a \wedge e_b) = \frac{1}{2!} \epsilon_{abcd} e_c \wedge e_d, \quad \star (e_a \wedge e_b \wedge e_c) = \epsilon_{abcd} e_d,$$

which by linearity determines the action of $\star$ on any differential form.

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