Reducing the Computational Complexity of Pseudoinverse for the Incremental Broad Learning System on Added Inputs

Hufei Zhu Chenghao Wei
Big Data Institute, College of Computer Science and Software Engineering, Shenzhen University, Shenzhen 518060, Guangdong, China
{zhuhufei, chenghao.wei}@szu.edu.cn

ABSTRACT
In this brief, we improve the Broad Learning System (BLS) [7] by reducing the computational complexity of the incremental learning for added inputs. We utilize the inverse of a sum of matrices in [5] to improve a step in the pseudoinverse of a row-partitioned matrix. Accordingly we propose two fast algorithms for the cases of $q > k$ and $q < k$, respectively, where $q$ and $k$ denote the number of additional training samples and the total number of nodes, respectively. Specifically, when $q > k$, the proposed algorithm computes only a $k \times k$ matrix inverse, instead of a $q \times q$ matrix inverse in the existing algorithm. Accordingly it can reduce the complexity dramatically. Our simulations, which follow those for Table V in [7], show that the proposed algorithm and the existing algorithm achieve the same testing accuracy, while the speedups in BLS training time of the proposed algorithm dramatically. Accordingly it can reduce the computational complexity of incremental learning for added inputs, by utilizing the inverse of a row-partitioned matrix. We develop two fast algorithms for the cases of $q > k$ and $q < k$, respectively, where $q$ and $k$ denote the number of additional training samples and the total number of nodes, respectively.

In Section II, the existing incremental BLS on added inputs are introduced. Then, in Section III, we propose the improved pseudoinverse to reduce the computational complexity of incremental learning for added inputs. Section IV compares the performances of the existing algorithm and the proposed algorithm, in Modified National Institute of Standards and Technology database (MNIST) classification. Finally we conclude this brief in Section V.

1. INTRODUCTION
Single layer feedforward neural networks (SLFN) with universal approximation capability have been widely applied in classification and regression [1][3]. For SLFN, traditional train methods, i.e., Gradient-descent-based learning algorithms [4][5], converge slowly. They may halt at a local minimum, and their generalization performance is sensitive to the training parameters (e.g., learning rate). A different train method, the random vector functional-link neural network (RVFLNN) [2], has fast learning speed, and offers the generalization capability in function approximation [3].

For a new added node or input, a dynamic step-wise updating algorithm was proposed in [6] to update the output weights of the RVFLNN easily, by only computing the pseudoinverse of the corresponding added node or input. Then in [7], Broad Learning System (BLS) was proposed to improve the previous scheme in three aspects. Firstly, BLS generates the mapped features from the input data to form the feature nodes, which are enhanced as the enhancement nodes with random weights, and finally it uses the feature nodes and the enhancement nodes as its input. Secondly, BLS updates the output weights easily for any number of new added nodes or inputs, by utilizing only one iteration to compute the pseudoinverse of the corresponding added nodes or inputs. Lastly, the ridge regression approximation of the pseudoinverse is applied to compute the output weights for BLS, to achieve a better generalization performance.

This brief focuses on the incremental BLS on added inputs. We reduce the computational complexity of the incremental learning for added inputs, by utilizing the inverse of a sum of matrices in [5] to improve a step in the pseudoinverse of a row-partitioned matrix. We develop two fast algorithms for the cases of $q > k$ and $q < k$, respectively, where $q$ and $k$ denote the number of additional training samples and the total number of nodes, respectively.

In Section II, the existing incremental BLS on added inputs are introduced. Then, in Section III, we propose the improved pseudoinverse to reduce the computational complexity of incremental learning for added inputs. Section IV compares the performances of the existing algorithm and the proposed algorithm, in Modified National Institute of Standards and Technology database (MNIST) classification. Finally we conclude this brief in Section V.

2. EXISTING INCREMENTAL BROAD LEARNING SYSTEM ON ADDED INPUTS
In the BLS, the original input data $X$ is transferred into the mapped features in the feature nodes. Then the feature nodes are enhanced as the enhancement nodes. The expanded input matrix, which consists of all the $n$ groups of feature nodes and the $m$ groups of enhancement nodes, is denoted as $A^m$.

The BLS includes the incremental learning for the additional input training samples. When encountering new input samples with the corresponding output labels, the modeled BLS can be remodeled in an incremental way without a complete retraining process. It updates the output weights incrementally, without retraining the whole network from the beginning.

Denote the additional input training samples as $X_a$. The incremental feature nodes and enhancement nodes corresponding to $X_a$ can be represented as the matrix $A_a$. Then
the expanded input matrix $A^m_n$ should be updated into
\[ xA^m_n = \begin{bmatrix} A^m_n \\ A^m_x \end{bmatrix}, \tag{1} \]
which can be regarded as a row-partitioned matrix. Accordingly the output weights
\[ W^m_n = (A^m_n)^+ Y \tag{2} \]
should be updated into
\[ xW^m_n = (xA^m_n)^+ \begin{bmatrix} Y \\ Y_a \end{bmatrix}, \tag{3} \]
where $Y$ and $Y_a$ are the output labels corresponding to the input $X$ and the additional input $X_a$, respectively.

In [7], the pseudoinverse of $A^m_n$ is computed by
\[ \left( xA^m_n \right)^+ = \left( \left( A^m_n \right)^+ - BD^T \right) B \tag{4} \]
where
\[ B = \begin{cases} C^+ & \text{if } C \neq 0 \\ \left( A^m_n \right)^+ D (I + D^T D)^{-1} & \text{if } C = 0, \tag{5a} \end{cases} \]
and
\[ D^T = A_x (A^m_n)^+. \tag{6} \]

The corresponding output weights are computed by
\[ xW^m_n = W^m_n + B (Y_a - A_x W^m_n). \tag{8} \]

### 3. IMPROVED PSEUDOINVERSE TO REDUCE THE COMPUTATIONAL COMPLEXITY OF INCREMENTAL LEARNING FOR ADDED INPUTS

We can assume that $A^m_n$ and $A_x$ in (11) are $l \times k$ and $q \times k$, respectively, where $l$, $q$, and $k$ denote the number of training samples, the number of additional training samples and the total number of nodes, respectively. Usually the number of training samples is greater than or equal to the total number of nodes $[6, 7]$, i.e., $l \geq k$ for the $l \times k$ matrix $A^m_n$. Moreover, in Tables V and VI of [7] where the simulations include the increment of input pattern, $l \geq 2k$ is satisfied in most cases. Thus usually we can assume that $A^m_n$ is full column rank. Correspondingly as in [6], we can show $C = 0$, of which the details are in the next paragraph.

Since $A^m_n$ is full column rank, the pseudoinverse must be the left inverse
\[ \left( A^m_n \right)^+ = \left( \left( A^m_n \right)^T A^m_n \right)^{-1} A^m_n^T \tag{9} \]
Substitute (7) into (6) to obtain
\[ C = A_x^T - \left( A^m_n \right)^T A_x \left( A^m_n \right)^+ A^m_n, \tag{10} \]
into which substitute (9) to obtain
\[ C = \left( A_x - A_x \left( \left( A^m_n \right)^T A^m_n \right)^{-1} \left( A^m_n \right)^T A^m_n \right)^T, \tag{11} \]
\[ C = (A_x - A_x I)^T = 0. \tag{12} \]

Now we can focus on the case of $C = 0$, i.e., (11). Substitute (7) into (5b) to obtain
\[ B = \left( A^m_n \right)^+ D (I + A_x (A^m_n)^+ D)^{-1}, \tag{13} \]
which can be written as
\[ B = \bar{D} (I + A_x \bar{D})^{-1}, \tag{14} \]
where
\[ \bar{D} = \left( A^m_n \right)^+ D. \tag{15} \]

For the $q \times k$ $A_x$ and $k \times q$ $\bar{D}$, obviously (15) with a $k \times k$ matrix inverse is more efficient when $q > k$, and (13) or (5b) with a $q \times q$ matrix inverse is more efficient when $q < k$. Since $l \geq k$ is always satisfied, the computational complexity of $A_x \bar{D}$ in (15) is lower than that of $D^T D$ in (5b), and then we should use (15) instead of (5b). Finally we summarize (12) and (15), to change (5) into
\[ B = \begin{cases} C^+ & \text{if } C \neq 0 \\ \bar{D} (I + A_x \bar{D})^{-1} & \text{if } C = 0 \& q \leq k \tag{16a} \\ (I + \bar{D} A_x)^{-1} \bar{D} & \text{if } C = 0 \& q \geq k \tag{16b} \end{cases} \]
where $\bar{D}$ is computed by (14).

### 4. EXPERIMENT AND DISCUSSION

The proposed fast algorithms (i.e., (16b) and (16c)) and the existing algorithm (i.e., (5b)) are compared on MATLAB software platform under a Microsoft-Windows Server with 128 GB of RAM. We strictly follow the simulations for Table V in [7], to give the experimental results for the incremental BLS on added inputs. The tested BLS includes 10×10 feature nodes and 5000 enhancement nodes, and then the total number of nodes is $k = 5100$. The initial network is trained by the first $I = 10000$ training samples in the MNIST data set, and then the incremental learning algorithm adds $q = 10000$ training samples each time until all the 60000 training samples are fed. From these simulation parameters, it can be easily seen that actually the proposed (16c) is utilized. The snapshot results of each update are in Table I, where $T_{\text{proposed}}$ denotes the ratio between the training time of the BLS using the existing (5b) and that of the BLS using the proposed (16c). From Table I, it can be seen that the proposed algorithm and the existing algorithm achieve the same testing accuracy, while the speedups in BLS training time of the proposed algorithm over the existing algorithm are $1.24 \sim 1.30$.

### 5. CONCLUSION

In this brief, we utilize the inverse of a sum of matrices to improve a step in the pseudoinverse of a row-partitioned matrix. Accordingly we can reduce the computational complexity of the incremental BLS on added inputs.

Firstly we show that usually the condition of $C = 0$ is satisfied, to focus on (5b) in the existing BLS algorithm. To reduce the computational complexity of (5b), we deduce (16b) from (5b), and then utilize the inverse of a sum of matrices to deduce (16c) from (16b). The proposed (16b) and (16c) are two fast algorithms for the cases of $q < k$.
### Table 1: Snapshot Results Using Incremental Learning: Increment of Input Patterns

| Number of Input Patterns | Structure | Testing Accuracy (%) | Speedups in Each Additional Training Time | Speedups in Accumulative Training Time | Speedups in Each Additional Testing Time | Speedups in Accumulative Testing Time |
|--------------------------|-----------|----------------------|------------------------------------------|---------------------------------------|-----------------------------------------|---------------------------------------|
|                          | Existing  | Proposed             |                                           |                                       |                                         |                                       |
| 10000                    | 10000     | 97.16                | 1.0000                                   | 1.0000                                | 1.0000                                  | 1.0000                                |
| 20000                    | 20000     | 97.92                | 1.3020                                   | 1.2427                                | 0.9957                                  | 0.9998                                |
| 30000                    | 30000     | 98.08                | 1.2567                                   | 1.2508                                | 1.0029                                  | 0.9999                                |
| 40000                    | 40000     | 98.12                | 1.2820                                   | 1.2618                                | 0.9855                                  | 0.9989                                |
| 50000                    | 50000     | 98.15                | 1.2554                                   | 1.2600                                | 0.9825                                  | 0.9979                                |

and \( q > k \), respectively, where \( q \) and \( k \) denote the number of additional training samples and the total number of nodes, respectively. When \( q > k \), the proposed (16c) computes only a \( k \times k \) matrix inverse, instead of a \( q \times q \) matrix inverse in the existing (5b), to reduce the complexity dramatically. The simulation results in Table I show that when the proposed (16c) and the existing (5b) are applied in the incremental BLS on added inputs, they achieve the same testing accuracy, while the speedups in BLS training time of the proposed algorithm over the existing algorithm are \( 1.24 \sim 1.30 \).

### 6. REFERENCES

[1] M. Leshno, V. Y. Lin, A. Pinkus, and S. Schocken, “Multilayer feedforward networks with a nonpolynomial activation function can approximate any function,” *Neural Netw.*, vol. 6, no. 6, pp. 861-867.

[2] Y.-H. Pao and Y. Takefuji, “Functional-link net computing: Theory, system architecture, and functionalities,” *Computer*, vol. 25, no. 5, pp. 76-79, May 1992.

[3] Y.-H. Pao, G.-H. Park, and D. J. Sobajic, “Learning and generalization characteristics of the random vector functional-link net,” *Eurocomputing*, vol. 6, no. 2, pp. 163-180, 1994.

[4] Y. LeCun et al., “Handwritten digit recognition with a back-propagation network,” *Proc. Neural Inf. Process. Syst. (NIPS)*, 1990, pp. 396-404.

[5] J. S. Denker et al., “Neural network recognizer for handwritten zip code digits,” *Advances in Neural Information Processing Systems*, D. S. Touretzky, Ed. San Francisco, CA, USA: Morgan Kaufmann, 1989, pp. 323-331.

[6] C. L. Philip Chen and J. Z. Wan, “A rapid learning and dynamic stepwise updating algorithm for flat neural networks and the application to timeseries prediction”, *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 29, no. 1, pp. 62-72, Feb. 1999.

[7] C. L. Philip Chen and Z. Liu, “Broad Learning System: An Effective and Efficient Incremental Learning System Without the Need for Deep Architecture”, *IEEE Transactions on Neural Networks and Learning Systems*, vol. 29, no. 1, Jan. 2018.

[8] H. V. Henderson and S. R. Searle, “On Deriving the Inverse of a Sum of Matrices”, *SIAM Review*, vol. 23, no. 1, January 1981.