SUPERSYMMETRIC, INTEGRABLE TODA
FIELD THEORIES: THE B(1,1) MODEL

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ABSTRACT
We study the two-dimensional supersymmetric Toda theory based on the Lie superalgebra $B(1, 1) \equiv Osp(3|2)$ and construct its quantum W-currents. We also investigate the fermionic affinization of this model: we show that despite the non-unitary form of the Lagrangian the $B^{(1)}(1, 1)$ theory has a real particle mass spectrum which is not renormalized at one-loop. We construct the first higher–spin conserved current, prove its conservation to all-loop order, compute one-loop corrections to the corresponding charge and check consistency between charge and mass renormalization.
Affine Toda theories are massive systems obtained as perturbation of the corresponding conformal Toda theories. The perturbation is chosen in such a way that these models possess an infinite number of conserved currents \([1, 2]\) and therefore retain the integrability properties they have at their fixed, conformal points. The existence of higher–spin conserved charges implies elasticity and factorization of the S-matrix which can be determined exactly using unitarity and a bootstrap principle \([3]\). Therefore in these cases, by studying perturbed conformal field theories one can find all the on-shell informations of the massive theories which are encoded in their S-matrix. This program has been completed for all *unitary* affine Toda theories based on simply-laced \([4]\) as well as nonsimply-laced Lie algebras \([5]\): the quantum integrability has been established and the exact S-matrices have been constructed.

Unitary fermionic extensions of these models \([6]\), i.e. the affine Toda theories based on the Lie superalgebras \(A^{(2)}(0, 2n - 1), C^{(2)}(n + 1), B^{(1)}(0, n), A^{(4)}(0, 2n)\), have been considered also, and the construction of their exact S-matrices has been successfully carried out \([6, 8]\). Despite the presence of fermions in the spectrum these theories are not supersymmetric.

If one insists on supersymmetry, one is led to consider affine Toda theories which are *not* manifestly unitary \([6, 9]\). In the untwisted cases these are the affine Toda theories associated to the purely fermionic admissible root systems of the \(A^{(1)}(n, n), B^{(1)}(n, n), D^{(1)}(n, n - 1), \) and \(D^{(1)}(2, 1; \alpha)\) superalgebras \([10, 11]\). The general understanding of such theories is incomplete, however there is evidence to suggest that for special values of the coupling constant they might admit a unitary restriction and describe the perturbation of ”minimal” conformal field theories \([12]\). For some specific bosonic systems these conjectures are supported by the fact that even if the Lagrangian is not manifestly unitary the theory has a particle and soliton spectrum which is real and stable under perturbation \([13]\).

In this paper we consider the supersymmetric theories: we study as an explicit example the supersymmetric affine Toda theory based on the Lie superalgebra \(B^{(1)}(1, 1)\) and address the issue of its quantum integrability. A more general analysis will be presented in Ref. \([14]\).

We work in Minkowski space with light-cone coordinates

\[
\begin{align*}
    z &\equiv x^+ = \frac{1}{\sqrt{2}}(x^0 + x^1) \\
    \bar{z} &\equiv x^- = \frac{1}{\sqrt{2}}(x^0 - x^1) \\
    \partial &\equiv \partial_+ = \frac{1}{\sqrt{2}}(\partial_0 + \partial_1) \\
    \bar{\partial} &\equiv \partial_- = \frac{1}{\sqrt{2}}(\partial_0 - \partial_1) \\
    \Box &\equiv 2\partial\bar{\partial}
\end{align*}
\]

Since the theory we will study is supersymmetric, it is convenient to perform all the calculations in N=1 superspace with coordinates \(Z = (z, \bar{z}, \theta, \bar{\theta})\) and supercovariant spinor
derivatives
\[ D = \partial_\theta + i \theta \bar{\partial} \]
\[ \bar{D} = \partial_{\bar{\theta}} - i \theta \partial \bar{\theta} \]
satisfying the anticommutation relations \( \{ D, \bar{D} \} = 0 \) and \( D^2 = i \partial, \bar{D}^2 = -i \bar{\partial} \). Bosonic and fermionic fields are then components of superfields, \( \Phi \equiv \phi + \frac{1}{\sqrt{2}} \theta \psi + \frac{1}{\sqrt{2}} \bar{\theta} \bar{\psi} + \theta \bar{\theta} F, \psi \) and \( \bar{\psi} \) being Majorana–Weyl fermions.

Two-dimensional Toda theories associated to a Lie superalgebra with \( b \) bosonic and \( f \) fermionic simple roots are represented by actions of the form
\[ S = \frac{1}{\beta^2} \int d^2 z d^2 \theta \left[ D \bar{\Phi} \cdot D \Phi + \theta \bar{\theta} \sum_{i=1}^{b} q_i e^{\bar{\alpha}_i \cdot \Phi} + \sum_{i=1}^{f} q_i e^{\bar{\alpha}_i \cdot \bar{\Phi}} \right] \] (3)
where \( \bar{\alpha}_i \) are the simple roots of the superalgebra, \( q_i \) are the Kač labels, \( \beta \) is the coupling constant and \( \bar{\Phi} = (\Phi_1, \cdots, \Phi_r) \) is a set of \( r = b + f \) superfields.

In general the system of simple roots (or equivalently the Dynkin diagram) associated to a Lie superalgebra is not unique. Unequivalent sets of simple roots can be obtained from a given one by acting on it with a generalized Weyl transformation associated to fermionic roots [11]. Different sets of simple roots have different content of fermionic roots. Therefore different Toda actions can be obtained from a given Lie superalgebra. For superalgebras which admit a set of purely fermionic simple roots a manifestly supersymmetric Toda action can be constructed, since in this case the term \( \theta \bar{\theta} \) in eq.(3) is absent.

We consider here the supersymmetric Toda theory associated to the Lie superalgebra \( B(1, 1) \equiv \text{Osp}(3|2) \). The fermionic Dynkin diagram of this rank–2 superalgebra is shown in Fig.1. An explicit realization of the two fermionic roots can be given in terms of complex vectors \( \bar{\alpha}_1 = (1, -i), \bar{\alpha}_2 = (0, i) \) with scalar products conventionally defined as \( \bar{\alpha}_i \cdot \bar{\alpha}_j = \sum_k \bar{\alpha}_i^k \bar{\alpha}_j^k \) (no complex conjugation). The corresponding supersymmetric Toda action is
\[ S = \frac{1}{\beta^2} \int d^2 z d^2 \theta \left[ D \bar{\Phi}_1 D \Phi_1 + D \Phi_2 D \Phi_2 + e^{\Phi_1 - i \Phi_2} + 2 e^{i \Phi_2} \right] \] (4)
The theory is classically integrable. The first \( d \) classical conserved currents can be constructed by using the general procedure based on the Miura transformation \([15, 9]\), where \( d \) are the dimensions of the vector representation of the algebra. For \( B(1, 1) \) the \( W^{(s)} \) currents are given by
\[ (D + D \bar{\Phi} \cdot \bar{\lambda}_3)(D + D \bar{\Phi} \cdot \bar{\lambda}_4) \cdots (D + D \bar{\Phi} \cdot \bar{\lambda}_1) = \sum_{i=0}^{5} W^{(2-i/2)}(z, \theta) D^i \] (5)
where \( \bar{\lambda}_j, j = 1, \ldots, 5 \) are the weights of the vector representation, \( \bar{\lambda}_1 = -\bar{\lambda}_5 = \bar{\alpha}_1 + \bar{\alpha}_2, \bar{\lambda}_2 = -\bar{\lambda}_4 = \bar{\alpha}_2 \) and \( \bar{\lambda}_3 = 0 \). Explicitly one obtains
\[ (D - D \Phi_1)(D - i D \Phi_2) D(D + i D \Phi_2)(D + D \Phi_1) = W^{(2)} + W^{(1)} D + W^{(1)} D^2 + D^5 \] (6)
with

\[
W^{(1)} = -iD\Phi_1 \partial \Phi_1 - iD\Phi_2 \partial \Phi_2 + D \partial \Phi_2 + 2iD\partial \Phi_1
\]

\[
W^{(2)} = -i \partial^2 \Phi_2 + i D\Phi_1 D\partial \Phi_1 - (\partial \Phi_2)^2 - iD\Phi_2 D\partial \Phi_2
\]

\[
+ 2D\Phi_1 D\partial \Phi_2 - 2i D\Phi_1 D\Phi_2 \partial \Phi_2
\]

\[
W^{(3)} = -iD^2 \Phi_1 + 2D\Phi_1 \partial^2 \Phi_1 - iD\Phi_1 \partial^2 \Phi_2 + iD\partial \Phi_2 \partial \Phi_1 - (\partial \Phi_2)^2 D\Phi_1
\]

\[
+ \partial \Phi_2 D\Phi_2 \partial \Phi_1 - iD\Phi_2 D\partial \Phi_2 D\Phi_1
\]

\[
= \frac{1}{2} D(W^{(2)} + DW^{(1)})
\]  

(7)

It is straightforward to check using the equations of motion from the action in eq.(4), that \(D\) commutes with the differential operator in eq.(5), i.e. the currents \(W^{(s)}\) are superholomorphic

\[
\bar{D}W^{(s)} = 0
\]  

(8)

thus ensuring the classical integrability of the \(B(1,1)\) system. In particular, being \(W^{(1)}\) proportional to the stress-energy tensor, the action in eq.(4) describes a massless supersymmetric model which is conformally invariant. These properties hold at the quantum level too: the holomorphic currents in eq.(7) maintain their form albeit the coefficients of the various terms acquire a coupling constant dependence as we now show.

The quantization and renormalization of the conservation laws in eq.(8) could be studied using for example light-cone quantization procedures as described in Ref. [16]; however since we are mainly interested in the affine extension of the \(B(1,1)\) model, we follow here the approach of Ref. [17] which applies to conformal as well as massive systems. We extend to superspace the techniques introduced in Ref.[17] for the corresponding analysis of higher–spin bosonic currents. The quantum Lagrangian is defined by normal ordering the exponentials so that the theory is free of any ultraviolet divergences. We use superspace propagators

\[
\langle \Phi_i(Z, \bar{Z})\Phi_j(0,0) \rangle = -\delta_{ij} \frac{\beta^2}{4\pi} \bar{D}D[log(2z\bar{z})\delta^{(2)}(\theta)]
\]  

(9)

and look for potential anomalies in

\[
\bar{D}_Z \langle W^{(s)}(Z, \bar{Z}) \rangle \equiv \bar{D}_Z \left( W^{(s)}(Z, \bar{Z}) \exp \left( \frac{i}{\beta^2} \int d^2w d^2\theta' L_{\text{int}} \right) \right)_0
\]  

(10)

We compute local contributions to the above expression and see if we can cancel them by adding to the classical currents coupling constant dependent corrections. The Wick contractions in eq.(10) produce a number of \(D\) and \(\bar{D}\) from the various terms in the currents and from the superspace propagators: one first reduces them using their commutation relations and then the D-algebra is performed loop by loop using \(\bar{D}D\delta^{(2)}(\theta - \theta')_{|\theta = \theta'} = 1\).
As for the bosonic calculation [17], local contributions can arise only by expanding the exponential in eq.(10) to first order in $L^\text{int};$ once the D-algebra has been performed one obtains terms of the form

$$
\bar{D} Z \int d^2 w A(z, \bar{z}, \theta, \bar{\theta}) \bar{D} Z \frac{1}{(z-w)^{n}} B(w, \bar{w}, \theta, \bar{\theta}) \tag{11}
$$

where $A, B$ are products of superfields and their $D$-derivatives. Finally a local contribution is produced using

$$
\bar{D} Z \bar{D} Z \frac{1}{(z-w)^{n}} = -i \bar{\partial}_z \frac{1}{(z-w)^{n}} = \frac{2\pi}{(n-1)!} \partial_w^{n-1} \delta^{(2)}(z-w) \tag{12}
$$

For the $W^{(1)}$ current one obtains one-loop contributions from Wick contracting the $D\Phi_1 \partial \Phi_1$ and $D\Phi_2 \partial \Phi_2$ terms in the current with the exponentials in the interaction Lagrangian. The $O(\beta^2)$ corrections are

$$
D \left< W^{(1)} \left( \frac{i}{\beta^2} \int d^2 w d^2 \theta' e^{\Phi_1-i\Phi_2} \right) \right> \sim 0
$$

$$
\bar{D} \left< W^{(1)} \left( \frac{i}{\beta^2} \int d^2 w d^2 \theta' e^{i\Phi_2} \right) \right> \sim \frac{\beta^2}{8\pi} \partial \Phi_2 e^{i\Phi_2} \tag{13}
$$

which can be cancelled renormalizing the terms $D\partial \Phi_1$ and $D\partial \Phi_2$ in the classical current, thus obtaining the quantum holomorphic stress-energy tensor

$$
W^{(1)} = -i D\Phi_1 \partial \Phi_1 - i D\Phi_2 \partial \Phi_2 + (1 - \frac{\beta^2}{4\pi}) D\partial \Phi_2 + 2i(1 - \frac{\beta^2}{8\pi}) D\partial \Phi_1 \tag{14}
$$

In the same way one can compute the quantum corrections to the $W^{(2)}$ current. In this case since the current contains up to three factors of fields the Wick contractions lead to contributions up to two loops, which actually vanish because of the D-algebra. The calculation is not too difficult and one obtains

$$
W^{(2)} = -i \left(1 - \frac{\beta^2}{4\pi}\right) \partial^2 \Phi_2 + i \left(1 - \frac{\beta^2}{2\pi}\right) D\Phi_1 \partial \Phi_1 - (\partial \Phi_2)^2 - i D\Phi_2 D\partial \Phi_2 + 2(1 - \frac{\beta^2}{4\pi}) D\Phi_1 D\partial \Phi_2 - 2i D\Phi_1 D\Phi_2 \partial \Phi_2 \tag{15}
$$

The corresponding calculation for spin $s = 2$ needs not be performed since the $W^{(2)}$ current is linearly dependent on $W^{(1)}$ and $W^{(2)}$ (see eq.(7)) and therefore its renormalized expression is easily obtained using eqs.(14) and (15).

We turn now to the study of the supersymmetric massive perturbation of the system in eq.(4). It is obtained by a fermionic affinization of the $B(1,1)$ superalgebra [10, 11] which corresponds to the addition of the lowest fermionic root $\bar{\alpha}_0 = -(\bar{\alpha}_1 + 2\bar{\alpha}_2) = (-1, -i)$. The affine $B^{(1)}(1,1)$ Toda action is given by

$$
S = \frac{1}{\beta^2} \int d^2 x d^2 \theta \left[ D\bar{\Phi} \cdot \bar{D}\Phi + e^{\Phi_1-i\Phi_2} + 2e^{i\Phi_2} + e^{-\Phi_1-i\Phi_2} \right] \tag{16}
$$
where the Kač labels have been chosen so that the one-point functions vanish.

Expanding the exponentials up to third order, we obtain the classical mass spectrum and the 3-point coupling of the theory:

\[ \mathcal{L}^{(2)} \equiv -M_1 \Phi_1^2 - M_2 \Phi_2^2 = \Phi_1^2 - 2\Phi_2^2, \quad \mathcal{L}^{(3)} = -i\Phi_1^2 \Phi_2 \]  

(17)

It follows that the bosonic masses are \((m_j^2 = 2M_j^2)\)

\[ m_1^2 = 2, \quad m_2^2 = 8 \]  

(18)

Despite the manifest non-unitarity of the theory the classical mass spectrum is real. We note that since \(m_2 = 2m_1\) the two particles are at threshold and divergent contributions, similar to the ones discussed in Ref. [18], could be produced in the on-shell amplitudes. However one can show that wave-function renormalization is sufficient to render the theory free of any threshold singularities.

We want to establish now the existence of quantum higher–spin conserved currents for the \(B^{(1)}(1,1)\) model. Since the theory contains a mass scale, the stress-energy tensor acquires a non-vanishing trace. It satisfies a conservation law of the form

\[ \bar{D}J^{(s)} + DJ^{(s)} = 0 \]  

(19)

with \(s = 1\). The most efficient way to analyze the situation for \(s > 1\) is to use the massless perturbation techniques introduced above, with superfield propagators as in eq.(9) and treating the whole exponentials in eq.(16) as interaction terms [17]. We compute

\[ \bar{D}_Z \langle J^{(s)}(Z, \bar{Z}) \rangle \equiv \bar{D}_Z \left( J^{(s)}(Z, \bar{Z}) \exp \left( \frac{i}{\beta^2} \int d^2 w d^2 \theta' \mathcal{L}_{\text{int}} \right) \right) \]  

(20)

An anomaly would spoil the conservation law if the computation of the r.h.s. of eq.(20) would produce local terms which were not expressible as \(D\)-derivatives of some appropriate \(\bar{J}\). Since we are not interested in the actual form of \(\bar{J}\) we discard total \(D\)-derivatives, freely integrating by parts on \(z, \theta\). Moreover any current of the form \(J^{(s)} = D\mathcal{J}^{(s - \frac{1}{2})}\) is not relevant since it trivially satisfies eq.(19). In this manner we find that the \(B^{(1)}(1,1)\) perturbed system does not have conserved currents of spin \(s = \frac{3}{2}\) or \(s = 2\). In particular the \(W^{(\frac{3}{2})}\) current is not conserved in the affine case because it does not respect the symmetry \(\Phi_1 \rightarrow -\Phi_1\) of the action in eq.(16), while \(W^{(2)}\) is trivial being a total \(D\)-derivative.

The first nontrivial higher–spin current appears at \(s = 3\), in accordance with the general statement proven for purely bosonic affine Toda theories, that the spins of the conserved charges should be given by the exponents of the algebra modulo the Coxeter number [2, 19]. Indeed in this case one has \(s = 1, 3 \mod 4\).
In order to study the renormalization of the spin-3 current we proceed as follows: we consider up to total $D$-derivative the most general expression, even in $\Phi_1$ on the basis of the $\Phi_1 \rightarrow -\Phi_1$ symmetry,

\[ J^{(3)} = a D\partial\Phi_1 \partial^2\Phi_1 + b D\partial\Phi_2 \partial^2\Phi_2 + c D\Phi_1 \partial^2\Phi_2 + d(\partial\Phi_1)^2 D\partial\Phi_2 + e(\partial\Phi_1)^3 D\Phi_1 + f(\partial\Phi_2)^3 D\Phi_2 + g D\Phi_1 \partial\Phi_1(\partial\Phi_2)^2 + h D\Phi_1 D\partial\Phi_1 D\partial\Phi_2 + k(\partial\Phi_1)^2 \partial\Phi_2 D\Phi_2 \]

(21)

and compute all the local contributions which arise from Wick contracting with the interaction Lagrangian. Dropping total $D$-derivatives and using various identities valid up to integration by parts, after a fairly amount of algebra we obtain ($\alpha \equiv \frac{\beta^2}{2\pi}$)

\[
\tilde{D} \left\langle J^{(3)} \left( \frac{i}{\beta^2} \int d^2w d^2\theta' e^{2i\Phi_2} \right) \right\rangle \\
\sim \left[ b + (1 - \frac{9\alpha}{4} + \frac{\alpha^2}{4}) D\partial^2\Phi_2 \partial\Phi_2 + \left[ -\frac{1}{2} c + i(1 - \frac{\alpha}{4})g + (\frac{1}{2} - \frac{\alpha}{4})h \right] D\Phi_1 D\partial^2\Phi_1 \\
+ \left[ i c + id + (2 - \frac{\alpha}{2})g + (1 - \frac{\alpha}{2})k \right] \partial\Phi_1 \partial^2\Phi_1 + g D\Phi_1 \partial\Phi_1 D\partial\Phi_2 \right) e^{2i\Phi_2}
\]

(22)

and

\[
\tilde{D} \left\langle J^{(3)} \left( \frac{i}{\beta^2} \int d^2w d^2\theta' e^{\Phi_1 - i\Phi_2} \right) \right\rangle \\
\sim \left[ -a - \left( \frac{i}{2} + \frac{i\alpha}{4} \right)c - \left( i + \frac{i\alpha}{4} \right)d + \left( 1 + \frac{9\alpha}{4} + \frac{\alpha^2}{4} \right)e \\
-(\frac{\alpha}{4} + \frac{\alpha^2}{8})g - \frac{i\alpha}{4} h - (\frac{\alpha}{2} + \frac{\alpha^2}{8})k \right] \partial\Phi_1 \partial^2\Phi_1 \\
+ \left[ ia + \left( \frac{1}{2} - \frac{\alpha}{4} \right)c - \frac{i\alpha}{4} d + (2i + \frac{3i\alpha}{4} - \frac{i\alpha^2}{4})e - \left( \frac{5i\alpha}{4} - \frac{i\alpha^2}{8} \right)g \\
+(\frac{1}{2} - \frac{\alpha}{4})h - (i + i\alpha - \frac{i\alpha^2}{8})k \right] \partial\Phi_2 \partial^2\Phi_1 \\
+ \left[ ib - \left( 1 + \frac{\alpha}{4} \right)c - \left( 1 + \frac{\alpha}{4} \right)d + 2ie + \left( \frac{3i\alpha}{4} + \frac{i\alpha^2}{4} \right)f \\
-(i + \alpha + \frac{i\alpha^2}{8})g + \frac{1}{2} h - (2i + \frac{5i\alpha}{4} + \frac{i\alpha^2}{8})k \right] \partial\Phi_1 \partial^2\Phi_2 \\
+ \left[ -\frac{i}{2} c - \frac{3\alpha}{4} f + (1 + \frac{\alpha}{2})g + \left( \frac{i}{2} - \frac{i\alpha}{8} \right)h + (1 + \frac{\alpha}{4})k \right] D\Phi_1 D\partial^2\Phi_2
\]
\[
\begin{align*}
&+ \left[ \frac{i}{2}c - (3 + \frac{3\alpha}{4})e + \frac{\alpha}{4}g + \frac{i\alpha}{8}h + (1 + \frac{\alpha}{2})k \right] D\Phi_2 D\partial^2 \Phi_1 \\
&+ \left[ id + 3e + \frac{\alpha}{2}g + (\frac{i}{2} - \frac{i\alpha}{2})h - (1 + \frac{\alpha}{2})k \right] D\partial \Phi_1 D\partial \Phi_2 \\
&+ \left[ -3ie + ig - \frac{1}{2}h + 2ik \right] \partial \Phi_2 D\partial \Phi_1 D\Phi_2 + [2e + 2f - 2g - 2k](\partial \Phi_2)^2 \partial \Phi_1 \\
&+ \left[ -3if + 2ig + \frac{1}{2}h + ik \right] \partial \Phi_2 D\partial \Phi_2 D\Phi_1 \right) e^{\Phi_1 - i\Phi_2} \\
\end{align*}
\]

(23)

The third exponential needs not be considered because the theory is symmetric under \( \Phi_1 \rightarrow -\Phi_1 \).

Up to D-derivatives the terms in the r.h.s. of eqs.(22) and (23) are all independent and they are not total derivatives. Therefore the \( J^{(3)} \) current will satisfy the conservation equation (19) if the various coefficients separately vanish. This leads to a set of equations for \( a, b, \ldots, k \) which can be solved nontrivially. Thus we obtain, up to an overall normalization factor, the quantum spin-3 conserved current

\[
J^{(3)} = \frac{i}{2} \left( -4 + \frac{13}{8\pi} \beta^2 + \frac{7}{32\pi^2} \beta^4 - \frac{\beta^6}{64\pi^3} \right) D\partial \Phi_1 \partial^2 \Phi_1 \\
- \frac{i}{2} \left( 1 - \frac{7}{8\pi} \beta^2 - \frac{7}{32\pi^2} \beta^4 + \frac{\beta^6}{64\pi^3} \right) D\partial \Phi_2 \partial^2 \Phi_2 \\
+ \left( -6 + \frac{3}{2\pi} \beta^2 \right) D\Phi_1 \partial \Phi_1 \partial^2 \Phi_2 + \left( 6 - \frac{9}{4\pi} \beta^2 + \frac{3}{16\pi^2} \beta^4 \right) (\partial \Phi_1)^2 D\partial \Phi_2 \\
- \frac{i}{2} \left( 1 - \frac{\beta^2}{2\pi} \right) (\partial \Phi_1)^3 D\Phi_1 + i \left( 1 + \frac{\beta^2}{4\pi} \right) (\partial \Phi_2)^3 D\Phi_2 \\
- 6D\Phi_1 D\partial \Phi_1 D\Phi_2 \partial \Phi_2 + \frac{3i}{4\pi} \beta^2 (\partial \Phi_1)^2 \partial \Phi_2 D\Phi_2 \\
\] (24)

The classical current is obtained by setting \( \beta^2 = 0 \). It is worth noticing that at the classical level \( J^{(3)} \) is equivalent, up to total \( D \)-derivative terms, to the classical current \( W^{(1)}DW^{(1)} + 2W^{(1)}W^{(\frac{3}{2})} \). While in the conformal theory the two terms are separately holomorphic currents, in the affine case only the linear combination above satisfies the classical conservation law. At the quantum level we could not use the renormalized expressions in eqs.(14) and (15) since \( J^{(3)} \) is a composite operator and the explicit calculation in eqs.(22), (23) was necessary in order to determine its complete renormalized form.

The corresponding spin-3 charge

\[
Q^{(3)} = \int dzd\theta J^{(3)}(z, \theta) \\
\] (25)

satisfies \( \bar{D}Q^{(3)} = 0 \) and it commutes with the Hamiltonian of the system. Single particle states are eigenstates of the charge operator with eigenvalues proportional to the
particle charge $\omega_j$, defined as $Q_j^{(3)} = \beta^2 \omega_j p_{+j}^3 |p_j\rangle$. (We follow here the notations of Ref. [17].) For an on-shell 3-point correlation function $\langle \Phi_a \Phi_b \Phi_c \rangle$ momentum and charge conservations lead to

$$ p_{+a} + p_{+b} + p_{+c} = 0 \quad , \quad \omega_a p_{+a}^3 + \omega_b p_{+b}^3 + \omega_c p_{+c}^3 = 0 \quad (26) $$

Specializing these relations to the vertex function $\langle \Phi_1 \Phi_1 \Phi_2 \rangle$ of the $B^{(1)}(1,1)$ theory and writing momenta in terms of rapidities ($p_{+j} = \frac{m_j}{\sqrt{2}} e^{\theta_j}$), in a frame of reference where one of the particle has rapidity zero and the other two $\pm i \Theta$, we obtain

$$ 2 m_1 \cos \Theta = m_2 \quad , \quad 2 \omega_1 m_1^3 \cos 3 \Theta = \omega_2 m_2^3 \quad (27) $$

Therefore the charges and masses of the theory must satisfy

$$ \frac{\omega_2}{\omega_1} = 1 - 3 \frac{m_1^2}{m_2^2} \quad (28) $$

We verify up to one-loop level that this relation is indeed valid.

The charges $\omega_j$, $j = 1, 2$ can be computed in terms of the on-shell matrix elements $\langle p_j | J^{(3)}(0) | p_j \rangle$ (see Ref. [17] for details on the general procedure). At the classical level they are obtained from the quadratic terms in $J^{(3)}$. One finds $\omega_1^{(0)} = 4$, $\omega_2^{(0)} = 1$ so that using the classical mass ratio $m_1^2/m_2^2 = 1/4$, eq.(28) is satisfied. We compute now first order corrections to the masses and the charges. One-loop corrections to the mass spectrum are given by on-shell self-energy supergraphs with massive propagators

$$ \langle \Phi_i(Z, \bar{Z}) \Phi_j(0, 0) \rangle = -i \delta_{ij} \beta^2 (D D + M_i) \delta^{(2)}(\theta) \quad (29) $$

We obtain the following contributions to the effective action ($e^{iS} \to e^{i\Gamma}$)

$$ (A) : \quad -2 \int d^2 \theta \Phi_1 (D D + M_1 + M_2) \Phi_1 \Sigma(p^2; m_1^2, m_2^2) \quad (30) $$

$$ (B) : \quad - \int d^2 \theta \Phi_2 (D D + 2 M_1) \Phi_2 \Sigma(p^2; m_1^2, m_2^2) $$

where

$$ \Sigma(p^2; m_1^2, m_2^2) = \frac{1}{(2\pi)^2} \int \frac{d^2 k}{(k^2 - m_1^2)(k^2 - m_2^2)} \quad (31) $$

We note that when evaluated on-shell with $p_2^2 = 8 = (2m_1)^2$, $\Sigma(p_2^2; m_1^2, m_2^2)$ is divergent since the two particles are at threshold. In any event using the on-shell conditions $D D \Phi_j = M_j \Phi_j$ the kinematic factors in (A) and (B) vanish so that at one-loop the classical masses are not renormalized.

In order to compute one-loop corrections to the $\omega$-charges we need evaluate the one-loop supergraphs with one insertion of the current $J^{(3)}$ as shown in Fig.2. The relevant
terms in the current are

\[ J^{(3)} \sim i(-4 + \frac{13}{8\pi} \beta^2)D\partial\Phi_1 \partial^2\Phi_1 + i(-1 + \frac{7}{8\pi} \beta^2)D\partial\Phi_2 \partial^2\Phi_2 \]
\[ -6D\Phi_1 \partial\Phi_1 \partial^2\Phi_2 + 6(\partial\Phi_1)^2D\partial\Phi_2 \]  

The diagrams in Fig. 2a, b correspond to one-loop contributions from the classical part of the current, while Fig. 2c corresponds to tree-level contributions from the \( O(\beta^2) \) terms in the current and to wave-function renormalization corrections. As usual, in order to compute the supergraphs one first perform the D-algebra in the loop and then evaluates the momentum integrals. We obtain, for \( \omega_1 \)

\[(a) : -\frac{3}{4\pi} \beta^2 \quad , \quad (b) : -\frac{1}{8\pi} \beta^2 \quad , \quad (c) : (-\frac{13}{4} + 1)\frac{\beta^2}{2\pi} \]  

where in (c) the two terms correspond to the \( O(\beta^2) \) current insertion and to the contribution from wave-function renormalization, respectively. For \( \omega_2 \), the one-loop corrections are

\[(a) : \frac{3}{4\pi} \beta^2 \quad , \quad (b) : (-\frac{3}{4} - \frac{1}{16} I_0)\frac{\beta^2}{2\pi} \quad , \quad (c) : (-\frac{7}{4} + \frac{1}{16} I_0)\frac{\beta^2}{2\pi} \]

where again in (c) the two types of contributions are listed separately and \( I_0 \) denotes the divergent integral

\[ I_0 = \int_0^1 \frac{dx}{(x - \frac{1}{2})^2} \]  

which arises from threshold effects in the computation of \( \Sigma(8; 2, 2) \). Summing the terms in (a), (b) and (c) the divergence cancels and we find that the \( O(\beta^2) \) corrections maintain the classical charge ratio \( \omega_2/\omega_1 = 1/4 \), in agreement with the absence of mass renormalization.

As a further consistency check, we have computed the on-shell vertex function \( \langle \Phi_2 \Phi_2 \Phi_2 \rangle \) at one-loop and obtained a zero result (the corresponding \( \langle \Phi_1 \Phi_1 \Phi_1 \rangle \) corrections are absent because of the coupling). Indeed the vanishing of the \( \Phi_2^3 \) and \( \Phi_2^2 \) couplings is required by eq.(26) (with \( a = b = c \)) and the fact that the charges \( \omega_1 \) and \( \omega_2 \) have a non-vanishing one-loop correction.

We conclude summarizing our results: using massless perturbation in \( N = 1 \) two-dimensional superspace we have constructed the quantum W-supercurrents of the supersymmetric Toda theory based on the Lie superalgebra \( B(1, 1) \). The system possesses two independent holomorphic currents, the stress-energy tensor \( W^{(1)} \) and \( W^{(2)} \) whose first component is fermionic. Then we have considered the supersymmetric affine extension of the model. The addition of the perturbation is such that the first nontrivial conserved current appears now at \( s = 3 \). We have obtained its renormalized expression to all-loop
order and computed the corresponding charge up to one-loop. The $B^{(1)}(1,1)$ theory is not manifestly unitary, nonetheless its particle mass spectrum is real and not renormalized to lowest order in perturbation theory.

Finally we observe that the action in eq.(16) reduces to the supersymmetric sine-Gordon model if we set $\Phi_1 = 0$. Therefore the spectrum of the theory contains soliton solutions which are given by $\Phi_1 = 0$ and $\Phi_2$ assuming the field configurations of the super sine-Gordon solitons [20]. Here again the masses are real. Since the system is integrable one might hope to be able to construct the corresponding S-matrix and determine if and when the non-unitary sector decouples.

Further details and the extension to the other supersymmetric Toda theories will be presented in a separate publication [14].
References

[1] A.V. Mikhailov, M.A. Olshanetsky and A.M. Perelomov, Comm. Math. Phys. 79 (1981) 473;
A.B. Zamolodchikov, “Integrable Field Theory from Conformal Field Theory”, Proceedings of the Taniguchi Symposium, Kyoto (1988);
A. Bilal and J.L. Gervais, Nucl. Phys. B314 (1989) 646.

[2] D. Olive and N. Turok, Nucl. Phys. B257[FS14] (1985) 277.

[3] A.B. Zamolodchikov and Al.B. Zamolodchikov, Ann. Phys. (NY) 120 (1979) 253.

[4] A.E. Arinshtein, V.A. Fateev and A.B. Zamolodchikov, Phys. Lett. 87B (1979) 389;
J.L. Cardy and G. Mussardo, Phys. Lett. 225B (1989) 275;
P.G.O. Freund, T.R. Klassen and E. Melzer, Phys. Lett. 229B (1989) 243;
C. Destri and H.J. de Vega, Phys. Lett. 233B (1989) 336;
P. Christe and G. Mussardo, Nucl. Phys. B330 (1990) 465; Int. J. Mod. Phys. A5 (1990) 4581;
T.R. Klassen and E. Melzer, Nucl. Phys. B338 (1990) 485;
H.W. Braden, E. Corrigan, P.E. Dorey and R. Sasaki, Nucl. Phys. B338 (1990) 689.

[5] G.W. Delius, M.T. Grisaru and D. Zanon, Phys. Lett. 277B (1992) 414; "Exact S-matrices for nonsimply-laced affine Toda theories", CERN-TH.6337/91, IFUM 413/FT (1991), to be published in Nucl. Phys. B.

[6] M. A. Olshanetksy, Comm. Math. Phys. 88 (1983) 63.

[7] M.T. Grisaru, S. Penati and D. Zanon, Phys. Lett. 253B (1991) 357.

[8] G.W. Delius, M.T. Grisaru, S. Penati and D. Zanon, Phys. Lett. 256B (1991) 164;
Nucl. Phys. B359 (1991) 125;
C. Destri, H.J. de Vega and V.A. Fateev, Phys. Lett. 256B (1991) 173.

[9] J. Evans and T. Hollowood, Nucl. Phys. B352 (1991) 723.

[10] V.G. Kač, Adv. Math. 26 (1977) 8; Adv. Math. 30 (1978) 85;
J.F. Cornwell, “Group theory in Physics” vol. III, Academic Press (1989).

[11] D.A. Leites, M.V. Saviliev and V.V. Serganova, in “Group theoretical methods in physics”, VNU Science Press (1986);
L. Frappat, A. Sciarrino and P. Sorba, Comm. Math. Phys. 121 (1989) 457.
[12] A. LeClark, Phys. Lett. B 230 (1989) 103;
    D. Bernard and A. LeClark, Nucl. Phys. B 340 (1990) 721;
    N. Reshetikhin and F.A. Smirnov, “Hidden quantum group symmetry and integrable
    perturbations of conformal theories”, HUTMP89/B246 (1989);
    F.A. Smirnov, Int. J. Mod. Phys. A 4 (1989) 4213; Nucl. Phys. B 337 (1990) 156.

[13] T. Hollowood, “Quantum solitons in affine Toda field theories”, PUPT–1286 (1991).

[14] A. Gualzetti, S. Penati and D. Zanon, “Quantum conserved currents in supersymmetric Toda field theories”, in preparation.

[15] V.A. Fateev and S.I. Lukyanov, “Additional symmetries and exactly soluble models
    in two–dimensional conformal field theory”, I–III, Kiev/Moscow preprints (1988/89);
    L. Palla, Nucl. Phys. B 341 (1990) 714;
    P. Mansfield and B. Spence, Nucl. Phys. B 362 (1991) 294;
    T. Inami and H. Kanno, Comm. Math. Phys. 136 (1991) 519.

[16] P. Mansfield, Nucl. Phys. B 222 (1983) 419;
    T.J. Hollowood and P. Mansfield, Nucl. Phys. B 330 (1990) 720.

[17] G.W. Delius, M.T. Grisaru and D. Zanon, “Quantum conserved currents in affine
    Toda theories”, CERN–TH.6336/91, IFUM 412/FT (1991), to be published in Nucl.
    Phys. B.

[18] M.T. Grisaru, S. Penati and D. Zanon, Nucl. Phys. B 369 (1992) 373.

[19] D. Olive and N. Turok, Nucl. Phys. B 265[FS15] (1986) 469.

[20] P. di Vecchia and S. Ferrara, Nucl. Phys. B 130 (1977) 93;
    C. Imbimbo and S. Mukhi, Nucl. Phys. B 247 (1984) 471.
Figure 1: Fermionic Dynkin diagrams

Figure 2: Diagrams for the calculation of the charge; the wavy line indicates the insertion of the current.