Asymptotic Behaviour of Ergodic Integrals of ‘Renormalizable’ Parabolic Flows

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Abstract

Ten years ago A. Zorich discovered, by computer experiments on interval exchange transformations, some striking new power laws for the ergodic integrals of generic non-exact Hamiltonian flows on higher genus surfaces. In Zorich’s later work and in a joint paper authored by M. Kontsevich, Zorich and Kontsevich were able to explain conjecturally most of Zorich’s discoveries by relating them to the ergodic theory of Teichmüller flows on moduli spaces of Abelian differentials.

In this article, we outline a generalization of the Kontsevich-Zorich framework to a class of ‘renormalizable’ flows on ‘pseudo-homogeneous’ spaces. We construct for such flows a ‘renormalization dynamics’ on an appropriate ‘moduli space’, which generalizes the Teichmüller flow. If a flow is renormalizable and the space of smooth functions is ‘stable’, in the sense that the Lie derivative operator on smooth functions has closed range, the behaviour of ergodic integrals can be analyzed, at least in principle, in terms of an Oseledec’s decomposition for a ‘renormalization cocycle’ over the bundle of ‘basic currents’ for the orbit foliation of the flow.

This approach was suggested by the author’s proof of the Kontsevich-Zorich conjectures and it has since been applied, in collaboration with L. Flaminio, to prove that the Zorich phenomenon generalizes to several classical examples of volume preserving, uniquely ergodic, parabolic flows such as horocycle flows and nilpotent flows on homogeneous 3-manifolds.

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1. Introduction

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A fundamental problem in smooth ergodic theory is to establish quantitative estimates on the asymptotics of ergodic integrals of smooth functions. For several examples of hyperbolic flows, such as geodesic flows on compact manifolds of negative curvature, the asymptotic behaviour of ergodic integrals is described by the Central Limit Theorem (Y. Sinai, M. Ratner). In these cases, the dynamical system can be described as an approximation of a ‘random’ stochastic process, like the outcomes of flipping a coin. Non-hyperbolic systems are less well understood, with the important exception of toral flows. For generic non-singular area-preserving flows on the 2-torus logarithmic bounds on ergodic integrals of zero-average functions of bounded variation can be derived by the Denjoy-Koksma inequality and the theory of continued fractions. For a general ergodic flow, ergodic integrals are bounded for all times for a special class of functions: coboundaries with bounded ‘transfer’ functions (Gottschalk-Hedlund). In the hyperbolic examples and in the case of generic toral flows, a smooth function is a coboundary if and only if it has zero average with respect to all invariant measures.

In this article, we are interested in flows with parabolic behaviour. Following A. Katok, a dynamical system is called parabolic if the rate of divergence of nearby orbits is at most polynomial in time, while hyperbolic systems are characterized by exponential divergence. Toral flows are a rather special parabolic example, called elliptic, since there is no divergence of orbits. It has been known for many years that typical examples of parabolic flows, such as horocycle flows or generic nilpotent flows are uniquely ergodic, but until recently not much was known about the asymptotic behaviour of ergodic averages, with the exception of some polynomial bounds on the speed of convergence in the horocycle case (M. Ratner, M. Burger), related to the polynomial rate of mixing. We have been able to prove, in collaboration with L. Flamini, that for many examples of parabolic dynamics the behaviour of ergodic averages is typically described as follows.

A smooth flow $\Phi^X$ on a finite dimensional manifold $M$ has deviation spectrum $\{\lambda_1 > \ldots > \lambda_i > \ldots > 0\}$ with multiplicities $m_1, \ldots, m_i, \ldots \in \mathbb{Z}^+$ if there exists a system $\{D_{ij} \mid i \in \mathbb{Z}^+, 1 \leq j \leq m_i\}$ of linearly independent $X$-invariant distributions such that, for almost all $p \in M$, the ergodic integrals of any smooth function $f \in C^\infty_0(M)$ have an asymptotic expansion

$$\int_0^T f(\Phi^X(t,p)) dt = \sum_{i \in \mathbb{N}} \sum_{j=1}^{m_i} c_{ij}(p,T) D_{ij}(f) T^{\lambda_i} + R(p,T)(f),$$

where the real coefficients $c_{ij}(p,T)$ and the distributional remainder $R(p,T)$ have, for almost all $p \in M$, a sub-polynomial behaviour, in the sense that

$$\limsup_{T \to +\infty} \frac{\log \sum_{j=1}^{m_i} |c_{ij}(p,T)|^2}{\log T} = \limsup_{T \to +\infty} \frac{\log \|R(p,T)\|}{\log T} = 0.$$  

(1.2)

The notion of a deviation spectrum first arose in the work of A. Zorich and in his joint work with M. Kontsevich on non-exact Hamiltonian flows with isolated saddle-like singularities on compact higher genus surfaces. Zorich discovered
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in numerical experiments on interval exchange transformations an unexpected new phenomenon \[8\]. He found that, although a generic flow on a surface of genus \(g \geq 2\) is uniquely ergodic (H. Masur, W. Veech), for large times the homology classes of return orbits exhibit unbounded polynomial deviations with exponents \(\lambda_1 > \lambda_2 > \ldots > \lambda_g > 0\) from the line spanned in the homology group by the Schwartzmann’s asymptotic cycle. In his later work \[9\], \[10\] and in joint work with M. Kontsevich \[6\], Zorich was able to explain this phenomenon in terms of conjectures on the Lyapunov exponents of the Teichmüller flow on moduli spaces of holomorphic differentials on Riemann surfaces. Kontsevich and Zorich also conjectured that Zorich’s phenomenon is not merely topological, but it extends to ergodic integrals of smooth functions. “There is, presumably, an equivalent way of describing the numbers \(\lambda_i\). Namely, let \(f\) be a smooth function \[6\]. Then for a generic trajectory \(p(t)\), we expect that the number \(\int_0^T f(p(t)) \, dt\) for large \(T\) with high probability has size \(T^{\lambda_i + o(T)}\) for some \(i \in \{1, \ldots, g\}\). The exponent \(\lambda_1\) appears for all functions with non-zero average value. The next exponent, \(\lambda_2\), should work for functions in a codimension 1 subspace of \(C^\infty(S)\), etc.” \[6\]

Around the same time, we proved that for a generic non-exact Hamiltonian flow \(\Phi_X\) on a higher genus surface not all smooth zero average functions are smooth coboundaries \[3\]. In fact, we found that, in contrast with the hyperbolic case and the elliptic case of toral flows, there are \(X\)-invariant distributional obstructions, which are not signed measures, to the existence of smooth solutions of the cohomological equation \(Xu = f\). This result suggested that Zorich’s phenomenon should be related to the presence of invariant distributions other than the (unique) invariant probability measure. In fact, in \[4\] we were able prove the Kontsevich-Zorich conjectures that the deviation exponents are non-zero and that generic non-exact Hamiltonian flows on higher genus surfaces have a deviation spectrum. Recently, in collaboration with L. Flaminio, we have proved that other classical parabolic examples, such as horocycle flows on compact surfaces of constant negative curvature \[1\] and generic nilpotent flows on compact 3-dimensional nilmanifolds \[2\], do have a deviation spectrum, but of countable multiplicity, in contrast with the case of flows on surfaces which have spectrum of finite multiplicity equal to the genus.

We will outline below a general framework, derived mostly from \[4\] and successfully carried out in \[1\], \[2\], for proving that a flow on a pseudo-homogeneous space has a deviation spectrum. Our framework is based on the construction of an appropriate renormalization dynamics on a moduli space of pseudo-homogeneous structures, which generalizes the Teichmüller flow. A renormalizable flow for which the space of smooth functions is stable (in the sense of A. Katok), has a deviation spectrum determined by the Lyapunov exponents of a renormalization cocycle over a bundle of basic currents. Pseudo-homogeneous spaces are a generalization of homogeneous spaces. The motivating non-homogeneous example is given by any punctured Riemann surface carrying a holomorphic differential vanishing only at the punctures. It turns out that renormalizable flows are necessarily parabolic. In fact, the class of renormalizable flows encompasses all parabolic flows which are reasonably well-understood, while not much is known for most non-renormalizable parabolic flows, such as generic geodesic flows on flat surfaces with conical singularities. Our ap-
proach unifies and generalizes several classical quantitative equidistribution results such as the Zagier-Sarnak results for periodic horocycles on non-compact hyperbolic surfaces of finite volume \[\text{II}\] or number theoretical results on the asymptotic behaviour of theta sums \[\text{II}\].

2. Renormalizable flows

Let \(\mathfrak{g}\) be a finite dimensional real Lie algebra. A \(\mathfrak{g}\)-structure on a manifold \(M\) is defined to be a homomorphism \(\tau\) from \(\mathfrak{g}\) into the Lie algebra \(\mathfrak{V}(M)\) of all smooth vector fields on \(M\). This notion is well-known in the theory of transformation groups (originated in the work of S. Lie) under the name of ‘infinitesimal \(G\)-transformation group’ (for a Lie group \(G\) with \(\mathfrak{g}\) as Lie algebra). The second fundamental theorem of Lie states that any infinitesimal \(G\)-transformation group \(\tau\) on \(M\) can be ‘integrated’ to yield an essentially unique local \(G\)-transformation group. A \(\mathfrak{g}\)-structure \(\tau\) will be called faithful if \(\tau\) induces a linear isomorphism from \(\mathfrak{g}\) onto \(T_x M\), for all \(x \in M\).

Let \(\tau\) be a \(\mathfrak{g}\)-structure. For each element \(X \in \mathfrak{g}\), the vector field \(X_{\tau} := \tau(X)\) generates a (partially defined) flow \(\Phi^X_\tau\) on \(M\). Let \(E_0(X_{\tau}) \subset M\) be the closure of the complement of the domain of definition of the map \(\Phi^X_\tau(t, \cdot)\) at time \(t \in \mathbb{R}\). A faithful \(\mathfrak{g}\)-structure will be called pseudo-homogeneous if for every \(X \in \mathfrak{g}\) there exists \(t > 0\) such that \(E_0(X_{\tau}) \cup E_{-t}(X_{\tau})\) has zero (Lebesgue) measure. A manifold \(M\) endowed with a pseudo-homogeneous \(\mathfrak{g}\)-structure will be called a pseudo-homogeneous \(\mathfrak{g}\)-space. All homogeneous spaces are pseudo-homogeneous.

Let \(T_\mathfrak{g}(M)\) be the space of all pseudo-homogeneous \(\mathfrak{g}\)-structures on \(M\). The automorphism group \(\text{Aut}(\mathfrak{g})\) acts on \(T_\mathfrak{g}(M)\) by composition on the right. The group \(\text{Diff}(M)\) acts on \(T_\mathfrak{g}(M)\) by composition on the left. The spaces

\[
T_\mathfrak{g}(M) := T_\mathfrak{g}(M)/\text{Diff}_0(M), \quad \mathcal{M}_\mathfrak{g}(M) := T_\mathfrak{g}(M)/\Gamma(M),
\]

where \(\Gamma(M) := \text{Diff}^+(M)/\text{Diff}_0(M)\) is the mapping class group, will be called respectively the Teichmüller space and the moduli space of pseudo-homogeneous \(\mathfrak{g}\)-structures on \(M\). The group \(\text{Aut}(\mathfrak{g})\) acts on the Teichmüller space \(T_\mathfrak{g}(M)\) and on the moduli space \(\mathcal{M}_\mathfrak{g}(M)\), since in both cases the action of \(\text{Aut}(\mathfrak{g})\) on \(T_\mathfrak{g}(M)\) passes to the quotient.

Let \(\text{Aut}^{(1)}(\mathfrak{g})\) be the subgroup of automorphisms with determinant one. An element \(X \in \mathfrak{g}\) will be called a priori renormalizable if there exists a partially hyperbolic one-parameter subgroup \(\{G^X_t\} \subset \text{Aut}^{(1)}(\mathfrak{g}), t \in \mathbb{R} (t \in \mathbb{Z})\), in general non-unique, with a single (simple) Lyapunov exponent \(\mu_X > 0\) such that

\[
G^X_t(X) = e^{t\mu_X} X.
\]

It follows from the definition that the subset of a priori renormalizable elements of a Lie algebra \(\mathfrak{g}\) is saturated with respect to the action of \(\text{Aut}(\mathfrak{g})\). The subgroup \(\{G^X_t\}\) acts on the Teichmüller space and on the moduli space of pseudo-homogeneous \(\mathfrak{g}\)-structures as a ‘renormalization dynamics’ for the family of flows \(\Phi^X_\tau\) generated by the vector fields \(\{X_{\tau} \mid \tau \in T_\mathfrak{g}(M)\}\) on \(M\). It will be called a generalized Teichmüller flow (map). A flow \(\Phi^X_\tau\) will be called renormalizable if \(\tau \in \mathcal{M}_\mathfrak{g}(M)\) is a recurrent
point for some generalized Teichmüller flow (map) $G^X_t$. If $\mu$ is a probability $G^X_t$-invariant measure on the moduli space, then by Poincaré recurrence the flow $\Phi^X_\tau$ is renormalizable for $\mu$-almost all $\tau \in \mathcal{M}_g(M)$.

Let $R$ be an inner product on $\mathfrak{g}$. Every faithful $\mathfrak{g}$ structure $\tau$ induces a Riemannian metric $R_\tau$ of constant curvature on $M$. Let $\omega_\tau$ be the volume form of $R_\tau$. The total volume function $A : T_\mathfrak{g}(M) \to \mathbb{R}^+ \cup \{+\infty\}$ is $\text{Diff}^+(M)$-invariant and $\text{Aut}^{(1)}(\mathfrak{g})$-invariant. Hence $A$ is well-defined as an $\text{Aut}^{(1)}(\mathfrak{g})$-invariant function on the Teichmüller space and on the moduli space. It follows that the subspace of finite-volume $\mathfrak{g}$-structures has an $\text{Aut}^{(1)}(\mathfrak{g})$-invariant stratification by the level hypersurfaces of the total volume function. Since different hypersurfaces are isomorphic up to a dilation, when studying finite-volume spaces it is sufficient to consider the hypersurface of volume-one $\mathfrak{g}$-structures:

$$T^{(1)}_\mathfrak{g}(M) := T_\mathfrak{g}(M) \cap A^{-1}(1), \quad \mathcal{M}^{(1)}_\mathfrak{g}(M) := \mathcal{M}_\mathfrak{g}(M) \cap A^{-1}(1).$$

Let $\tau$ be a faithful $\mathfrak{g}$-structure and let $X \in \mathfrak{g}$. If the linear map $\text{ad}_X$ on $\mathfrak{g}$ has zero trace, the flow $\Phi^X_\tau$ preserves the volume form $\omega_\tau$ and $X_\tau$ defines a symmetric operator on $L^2(M, \omega_\tau)$ with domain $C^\infty_0(M)$. If $\tau$ is pseudo-homogeneous, by E. Nelson’s criterion $X_\tau$ is essentially skew-adjoint. It turns out that any a priori renormalizable element $X \in \mathfrak{g}$ is nilpotent, in the sense that all eigenvalues of the linear map $\text{ad}_X$ are equal to zero, hence the flow $\Phi^X_\tau$ is volume preserving and parabolic. In all the examples we have considered, the Lie algebra $\mathfrak{g}$ is traceless, in the sense that for every element $X \in \mathfrak{g}$, the linear map $\text{ad}_X$ has vanishing trace. In this case, any pseudo-homogeneous $\mathfrak{g}$-structure induces a representation of the Lie algebra $\mathfrak{g}$ by essentially skew-adjoint operators on the Hilbert space $L^2(M, \omega_\tau)$ with common invariant domain $C^\infty_0(M)$.

3. Examples

Homogeneous spaces provide a wide class of examples. Let $G$ be a finite dimensional (non-compact) Lie group with Lie algebra $\mathfrak{g}$ and let $M = G/\Gamma$ be a (compact) homogeneous space. The Teichmüller space $T_G(M) \subset T_\mathfrak{g}(M)$ and the moduli space $\mathcal{M}_G(M) \subset \mathcal{M}_\mathfrak{g}(M)$ of all homogeneous $G$-space structures on $M$ are respectively isomorphic to the Lie group $\text{Aut}(G)$ and to the homogeneous space $\text{Aut}(G)/\text{Aut}(G, \Gamma)$, where $\text{Aut}(G, \Gamma) < \text{Aut}(G)$ is the subgroup of automorphisms which stabilize the lattice $\Gamma$. The Teichmüller and moduli spaces $T^{(1)}_G(M)$ and $\mathcal{M}^{(1)}_G(M)$ of homogeneous volume-one $G$-space structures are respectively isomorphic to the subgroup $\text{Aut}^{(1)}(G)$ of orientation preserving, volume preserving automorphisms and to the homogeneous space $\text{Aut}^{(1)}(G)/\text{Aut}^{(1)}(G, \Gamma)$.

In the Abelian case $\mathfrak{g} = \mathbb{R}^n$, any $X \in \mathbb{R}^n$ is a priori renormalizable. In fact, the group $\text{Aut}^{(1)}(\mathbb{R}^n) = \text{SL}(n, \mathbb{R})$ acts transitively on $\mathbb{R}^n$ and $X_1 = (1, 0, ..., 0)$ is renormalized by the one-parameter group $G_t := \text{diag}(e^{s}, e^{-t/n}, ..., e^{-t/n}) \subset \text{SL}(n, \mathbb{R})$. Finite volume Abelian homogeneous spaces are diffeomorphic to $n$-dimensional tori $\mathbb{T}^n$. The generalized Teichmüller flow $G_t$ on the moduli space of all volume-one Abelian homogeneous structures on $\mathbb{T}^n$ is a volume preserving Anosov flow on the
finite-volume non-compact manifold $\text{SL}(n, \mathbb{R})/\text{SL}(n, \mathbb{Z})$. Hence, in this case, almost all homogeneous flows are renormalizable, by Poincaré recurrence theorem. The dynamics of the flow $G_t$ has been investigated in depth by D. Kleinbock and G. Margulis in connection with the theory of Diophantine approximations.

In the semi-simple case, let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ be the unique 3-dimensional simple Lie algebra. There is a basis $\{H, H^\perp, X\}$ with commutation relations $[X, H] = H$, $[X, H^\perp] = -H^\perp$, and $[H, H^\perp] = 2X$. The elements $H$, $H^\perp$ are renormalized by the one-parameter group $G_t := \text{diag}(e^t, e^{-t}, 1) \subset \text{Aut}^{(1)}(\mathfrak{g})$, while $X$ is not a priori renormalizable. The unit tangent bundle of any hyperbolic surface $S$ can be identified to a homogeneous space $M := \text{PSL}(2, \mathbb{R})/\Gamma$. The vector fields $H$, $H^\perp \in \mathfrak{g}$ generate the horocycle flows and the vector field $X$ generates the geodesic flow on $S$. Since $G_t$ is a group of inner automorphisms, it is in fact generated by the geodesic vector field $X$, every point of the moduli space is fixed under $G_t$. Hence horocycle flows are renormalizable on every homogeneous space $\text{PSL}(2, \mathbb{R})/\Gamma$.

In the nilpotent, non-Abelian case, let $\mathfrak{n}$ be the Heisenberg Lie algebra, spanned by elements $\{X, X^\perp, Z\}$ such that $[X, X^\perp] = Z$ and $Z$ is a generator of the one-dimensional center $\mathbb{Z}_n$. The element $X$ is renormalized by the one-parameter subgroup $G_t := \text{diag}(e^t, e^{-t}, 1) \subset \text{Aut}^{(1)}(\mathfrak{n})$. Since the group $\text{Aut}(\mathfrak{n})$ acts transitively on $\mathfrak{n} \setminus \mathbb{Z}_n$, every $Y \in \mathfrak{n} \setminus \mathbb{Z}_n$ is a priori renormalizable, while the elements of the center are not. A compact nilmanifold modeled over the Heisenberg group $N$ is a homogeneous space $M = N/\Gamma$, where $\Gamma$ is a co-compact lattice. These spaces are topologically circle bundles over $\mathbb{T}^2$ classified by their Euler characteristic. The moduli space $\mathcal{M}_N^{(1)}(M)$ of volume-one homogeneous $\mathfrak{n}$-structures on $M$ is a 5-dimensional finite-volume non-compact orbifold which fibers over the modular surface $\text{SL}(2, \mathbb{R})/\text{SL}(2, \mathbb{Z})$ with fiber $\mathbb{T}^2$. The generalized Teichmüller flow is an Anosov flow on $\mathcal{M}_N^{(1)}(M)$ [2].

The motivation for our definition of a pseudo-homogeneous space comes from the theory of Riemann surfaces of higher genus. Any holomorphic (Abelian) differential $h$ on a Riemann surface $S$ of genus $g \geq 2$, vanishing at $Z_h \subset S$, induces a (non-unique) pseudo-homogeneous $\mathbb{R}^2$-structure on the open manifold $M_h := S \setminus Z_h$. In fact, the frame $\{X, X^\perp\}$ of $TS|_{M_h}$ uniquely determined by the conditions
\begin{equation}
\frac{\sqrt{-1}}{2} i_X (h \wedge \tilde{h}) = \Im(h), \quad \frac{\sqrt{-1}}{2} i_{X^\perp} (h \wedge \tilde{h}) = -\Re(h)
\end{equation}
satisfies the Abelian commutation relation $[X, X^\perp] = 0$ and the homomorphism $\tau_h : \mathbb{R}^2 \to \mathcal{V}(M_h)$ such that $\tau_h(1, 0) = X$, $\tau_h(0, 1) = X^\perp$ is a pseudo-homogeneous $\mathbb{R}^2$-structure on $M_h$. Let $Z \subset S$ be a given subset of cardinality $\sigma \in \mathbb{N}$ and let $\kappa = (k_1, ..., k_\sigma) \in (\mathbb{Z}^+)^\sigma$ with $\sum k_i = 2g - 2$. Let $\mathcal{H}_\kappa(S, Z)$ be the space of Abelian differentials $h$ with $Z_h = Z$ and zeroes of multiplicities $(k_1, ..., k_\sigma)$. The projection of the set $\{\tau_h \in \mathcal{T}_{\mathbb{R}^2}(M) \mid h \in \mathcal{H}_\kappa(S, Z)\}$ into the moduli space $\mathcal{M}_{\mathbb{R}^2}(M)$ of pseudo-homogeneous $\mathbb{R}^2$-structures on $M := S \setminus Z$ is isomorphic to a stratum $\mathcal{H}(\kappa)$ of the moduli space of Abelian differentials on $S$. The flow induced on $\mathcal{H}(\kappa)$ by the one-parameter group of automorphism $G_t := \text{diag}(e^t, e^{-t}) \subset \text{SL}(2, \mathbb{R})$ coincides with the Teichmüller flow on the stratum $\mathcal{H}(\kappa)$.
4. Cohomological equations

Let $\mathfrak{g}$ be a finite dimensional Lie algebra endowed with an inner product. Any pseudo-homogeneous $\mathfrak{g}$-structure $\tau$ on a manifold $M$ induces a Sobolev filtration $\{W^s_\tau(M)\}_{s \geq 0}$ on the space $W^0_\tau(M) := L^2(M, \omega)$ of square-integrable functions. Let $\Delta$ be the non-negative Laplace-Beltrami operator of the Riemannian metric $R_\tau$ on $M$. The Laplacian is densely defined and symmetric on the Hilbert space $W^0_\tau(M)$ with domain $C^\infty_0(M)$, but it is not in general essentially self-adjoint. In fact, if $\mathfrak{g}$ is traceless, by a theorem of E. Nelson, $\Delta$ is essentially self-adjoint if and only if the representation $\tau$ of the Lie algebra $\mathfrak{g}$ on $W^0_\tau(M)$ by essentially skew-adjoint operators induces a unitary representation of a Lie group. Let then $\tilde{\Delta}$ be the Friederichs extension of $\Delta$. The Sobolev space $W^s_\tau(M)$, $s > 0$, is defined as the maximal domain of the operator $(I + \tilde{\Delta})^{s/2}$ endowed with the norm

$$\|f\|_{s, \tau} := \|(I + \tilde{\Delta})^{s/2}f\|_{0, \tau}. \tag{4.1}$$

The Sobolev spaces $W^{-s}_\tau(M)$ are defined as the duals of the Hilbert spaces $W^s_\tau(M)$, for all $s > 0$. Let $C^B_\tau(M)$ be the space of continuous bounded functions on $M$. The pseudo-homogeneous space $(M, \tau)$ will be called of bounded type if there is a continuous (Sobolev) embedding $W^s_\tau(M) \subset C^B_\tau(M)$ for all $s > \dim(M)/2$. The bounded-type condition is essentially a geometric property of the pseudo-homogeneous structure.

Let $X \in \mathfrak{g}$. Following A. Katok, the space $W^s_\tau(M)$ is called $W^s_\tau$-stable with respect to the flow $\Phi^X_\tau$ if the subspace

$$R^s(X_\tau) := \{f \in W^s_\tau(M) \mid f = X_\tau u, \ u \in W^s_\tau(M)\} \tag{4.2}$$

is closed in $W^s_\tau(M)$. The flow $\Phi^X_\tau$ will be called tame (of degree $\ell > 0$) if $W^s_\tau(M)$ is $W^{-s-\ell}_\tau$-stable with respect to $\Phi^X_\tau$ for all $s > \ell$. In all the examples of §3, generic renormalizable flows are tame. In particular, it is well known that generic toral flows are tame, horocycle flows and generic nilpotent flows on 3-dimensional compact nilmanifolds were proved tame of any degree $\ell > 1$ in [1], [2], generic non-exact Hamiltonian flows on higher genus surfaces were proved tame in [3]. These results are based on the appropriate harmonic analysis: in the homogeneous cases, the theory of unitary representations for the Lie group $SL(2, \mathbb{R})$ [1] and the Heisenberg group [2]; in the more difficult non-homogeneous case of higher genus surfaces, the theory of boundary behaviour of holomorphic functions on the unit disk plays a crucial role [4].

If the Sobolev space $W^s_\tau(M)$ is stable with respect to the flow $\Phi^X_\tau$, the closed range $R^s(X_\tau)$ of the operator $X_\tau$ coincides with the distributional kernel $\mathcal{I}^s(X_\tau) \subset W^{-s}_\tau(M)$ of $X_\tau$, which is a space of $X_\tau$-invariant distributions. Let $X$ be any smooth vector field on a manifold $M$. A distribution $\mathcal{D} \in \mathcal{D}'(M)$ is called $X$-invariant if $X\mathcal{D} = 0$ in $\mathcal{D}'(M)$. Invariant distributions are in bijective correspondence with (homogeneous) one-dimensional basic currents for the orbit foliation $\mathcal{F}(X)$ of the flow $\Phi^X$. A one-dimensional basic current $C$ for a foliation $\mathcal{F}$ on $M$ is a continuous linear functional on the space $\Omega^1_0(M)$ of smooth 1-forms with compact support such
that, for all vector fields $Y$ tangent to $\mathcal{F}$,  
\[ i_Y C = L_Y C = 0 \quad (\iff i_Y C = dC = 0). \]  

(4.3)

It follows from the definitions that the one-dimensional current $C := i_X D$ is basic for $\mathcal{F}(X)$ if and only if the distribution $D$ is $X$-invariant. Let $\mathcal{I}(X)$ be the space of all $X$-invariant distributions and $\mathcal{B}(X)$ be the space of all one-dimensional basic currents for the orbit foliation $\mathcal{F}(X)$. The linear map $i_X : \mathcal{I}(X) \to \mathcal{B}(X)$ is bijective.

Let $(M, \tau)$ be a pseudo-homogeneous space. There is a well-defined Hodge (star) operator and a space $C^0_\tau(M)$ of square-integrable 1-forms on $M$ associated with the metric $R_\tau$. Since the Laplace operator $\triangle_\tau$ extends to $C^2_\tau(M)$ with domain $\Omega^1_\tau(M)$, it is possible to define, as in the case of functions, a Sobolev filtration $\{C^s_\tau(M)\}_{s \geq 0}$, on the space $C^0_\tau(M)$. The Sobolev spaces $C^s_{\tau,+}(M)$ are defined as the duals of the Sobolev spaces $C^s(M)$, for all $s > 0$. Let $\mathcal{B}^s(X_\tau) := \mathcal{B}(X_\tau) \cap C^s_{\tau,+}(M)$ be the subspaces of basic currents of Sobolev order $\leq s$ for the orbit foliation $\mathcal{F}(X_\tau)$. The space $\mathcal{B}^s(X_\tau)$ is the image of $\mathcal{I}^s(X_\tau) := \mathcal{I}(X_\tau) \cap W^{-s}_{\tau}(M)$ under the bijective map $i_X : \mathcal{I}(X) \to \mathcal{B}(X)$. In the case of minimal toral flows the space $\mathcal{B}^s(X_\tau)$ is one-dimensional for all $s > 0$ (as all invariant distributions are scalar multiples of the unique invariant probability measure). In the parabolic examples we have studied, $\mathcal{B}^s(X_\tau)$ has countable dimension, as soon as $s > 1/2$, for horocycle flows or generic nilpotent flows, while for generic non-exact Hamiltonian flows on higher genus surfaces the dimension is finite for all $s > 0$ and grows linearly with respect to $s > 0$. This finiteness property seems to be an exceptional low dimensional feature.

5. The renormalization cocycle

The Sobolev spaces $C^s_\tau(M)$ of one-dimensional currents form a smooth infinite dimensional vector bundle over $T_0(M)$. Such bundles can be endowed with a flat connection with parallel transport given locally by the identity maps $C^s_\tau(M) \to C^s_\tau(M)$, for any $\tau \approx \tau' \in T_0(M)$. Since the diffeomorphism group $\text{Diff}(M)$ acts on $C^s_\tau(M)$ by push-forward, we can define (orbifold) vector bundles $C^s_\tau(M)$ over the Teichmüller space $T_0(M)$ or the moduli space $\mathcal{M}_g(M)$ of pseudo-homogeneous structures on $M$. If $X \in \mathfrak{g}$ is a priori renormalizable, a generalized Teichmüller flow (map) $G^X_t$ can be lifted by parallel transport to a ‘renormalization cocycle’ $R^X_t$ on the bundles of currents $C^s_\tau(M)$ over the Teichmüller space or the moduli space. It follows from the definitions that the sub-bundles $\mathcal{B}^s_\tau(X) \subset C^s_\tau(M)$ with fibers the subspaces of basic currents $\mathcal{B}^s(X_\tau) \subset C^s_\tau(M)$ are $R^X_t$-invariant. It can be proved that, for any $G_\ell$-ergodic probability measure $\mu$ on the moduli space, if the flows $\Phi^X_\tau$ are tame of degree $\ell > 0$ for $\mu$-almost all $\tau \in \mathcal{M}_g(M)$, then the sub-bundles $\mathcal{B}^s_\tau(X)$ are $\mu$-almost everywhere defined with closed (Hilbert) fibers of constant rank, for all $s > \ell$.

In the examples considered, with the exception of flows on higher genus surfaces, the Hilbert bundles of basic currents $\mathcal{B}^s_\tau(X)$ are infinite dimensional, and to the author’s best knowledge, available Oseledec-type theorems for Hilbert bundles
do not apply to the renormalization cocycle. However, the cocycle has a well defined Lyapunov spectrum and an Oseledec decomposition. We are therefore led to formulate the following hypothesis:

\( H_1(s) \). The renormalization cocycle \( R^X_s \) on the bundle \( \mathcal{B}^s(X) \) over the dynamical system \( (G^X_s, \mu) \) has a Lyapunov spectrum \( \{ \nu_1 > ... > \nu_k > ... > 0 > ... \} \) and an Oseledec’s decomposition

\[
\mathcal{B}^s(X) = E^s_0(\nu_1) \oplus ... \oplus E^s_0(\nu_k) \oplus ... \oplus N^s_0,
\]

in which the components \( E^s_0(\nu_k) \) correspond to the Lyapunov exponents \( \nu_k > 0 \), while the component \( N^s_0 \) has a non-positive top Lyapunov exponent. Our result on the existence of a deviation spectrum requires an additional technical hypothesis, verified in our examples.

\( H_2(s) \). Let \( \gamma^1_s(p) \) be the one-dimensional current defined by the time \( T = 1 \) orbit-segment of the flow \( \Phi^X_s \) with initial point \( p \in M \). (a) The essential supremum of the norm \( \| \gamma^1_s(p) \|_{r,s} \) over \( p \in M \) is locally bounded for \( r \in \text{supp}(\mu) \subset \mathcal{M}_g(M) \); (b) The orthogonal projections of \( \gamma^1_s(p) \) on all subspaces \( E^s_0(\nu_k) \subset C^0_s(M) \) are non-zero for \( \mu \)-almost all \( r \in \mathcal{M}_g(M) \) and almost all \( p \in M \).

Let \( X \in \mathfrak{g} \) be a priori renormalizable and let \( \mu \) be a \( G^X_s \)-invariant Borel probability measure on \( \mathcal{M}_g(M) \), supported on a stratum of bounded-type \( \mathfrak{g} \)-structures. If the flow \( \Phi^X_s \) is tame of degree \( \ell > 0 \) and the hypotheses \( H_1(s), H_2(s) \) are verified for \( s > \ell + \dim(M)/2 \), for \( \mu \)-almost \( r \in \mathcal{M}_g \), the flow \( \Phi^X_r \) has a deviation spectrum with deviation exponents

\[
\nu_1/\mu_X > ... > \nu_k/\mu_X > ... > 0
\]

and multiplicities given by the decomposition (5.1) of the renormalization cocycle.

In the homogeneous examples, the Lyapunov spectrum of the renormalization cocycle is computed explicitly in every irreducible unitary representation of the structural Lie group. In the horocycle case, the existence of an Oseledec’s decomposition (5.1) is equivalent to the statement that the space of horocycle-invariant distributions is spanned by (generalized) eigenvectors of the geodesic flow, well-known in the representation theory of semi-simple Lie groups as conical distributions. In the non-homogeneous case of higher genus surfaces, the Oseledec’s theorem applies since the bundles \( \mathcal{B}^s(X) \) are finite dimensional. We have found in all examples a surprising heuristic relation between the Lyapunov exponents of the renormalization cocycle and the Sobolev regularity of basic currents (or equivalently of invariant distributions): the subspaces \( E^s_0(\nu_k) \) are generated by basic currents of Sobolev order \( 1 - \nu_k/\mu_X \geq 0 \). The Sobolev order of a one-dimensional current \( C \) is defined as the infimum of all \( s > 0 \) such that \( C \in C^0_s(M) \).

In the special case of non-exact Hamiltonian flow on higher genus surfaces the Lyapunov exponents of the renormalization cocycle are related to those of the Teichmüller flow. In fact, let \( S \) be compact orientable surface of genus \( g \geq 2 \) and let \( \mathcal{H}(\kappa) \) be a stratum of Abelian differentials vanishing at \( Z \subset S \). Let \( \mathcal{B}_s(X) \subset \mathcal{B}_{\mathcal{H}^2}(X) \) be the measurable bundle of basic currents over \( \mathcal{H}(\kappa) \subset \mathcal{M}_{\mathcal{H}^2}(S \setminus Z) \) and let...
$H^1_\kappa(S \setminus \mathbb{Z}, \mathbb{R})$ be the bundle over $\mathcal{H}(\kappa)$ with fibers isomorphic to the real cohomology $H^1(S \setminus \mathbb{Z}, \mathbb{R})$. Since basic currents are closed, there exists a cohomology map $j_\kappa : B_\kappa(X) \to H^1_\kappa(S \setminus \mathbb{Z}, \mathbb{R})$ such that, as proved in [4], the restrictions $j_\kappa|B^s_\kappa(X)$ are surjective for all $s >> 1$ and, for all $s \geq 1$, there are exact sequences

$$0 \to \mathbb{R} \to B^{s-1}_\kappa(X) \xrightarrow{\delta_\kappa} B^s_\kappa(X) \xrightarrow{j_\kappa} H^1_\kappa(S \setminus \mathbb{Z}, \mathbb{R}).$$

(5.3)

The renormalization cocycle $R^X_t$ on $B^s_\kappa(X)$ projects for all $s >> 1$ onto a cocycle on the cohomology bundle $H^1_\kappa(S \setminus \mathbb{Z}, \mathbb{R})$, introduced by M. Kontsevich and A. Zorich in order to explain the homological asymptotic behaviour of orbits of the flow $\Phi^X_\tau$ for a generic $\tau \in \mathcal{H}(\kappa) \subset \mathcal{M}_{\mathbb{R}^2}(S \setminus \mathbb{Z})$ [6]. The Lyapunov exponents of the Kontsevich-Zorich cocycle on $H^1_\kappa(S \setminus \mathbb{Z}, \mathbb{R})$,

$$\lambda_1 = 1 > \lambda_2 \geq \cdots \geq \lambda_g \geq 0 = \cdots = 0 \geq -\lambda_g \geq \cdots \geq -\lambda_2 > -\lambda_1 = -1,$$

(5.4)

are related the Lyapunov exponents of the Teichmüller flow on $\mathcal{H}(\kappa)$ [6], [4]. Since the bundle map $\delta_\kappa$ shifts Lyapunov exponents by $-1$ and, as conjectured in [6] and proved in [4], the Kontsevich-Zorich exponents $\lambda_1 = 1 > \lambda_2 \geq \cdots \geq \lambda_g$ are non-zero, the strictly positive exponents of the renormalization cocycle coincide with the Kontsevich-Zorich exponents. This reduction explains why in the case of non-exact Hamiltonian flows on surfaces the Lyapunov exponents of the Teichmüller flow are related to the deviation exponents for the ergodic averages of smooth functions.

References

[1] L. Flaminio & G. Forni, Invariant distributions and time averages for horocycle flows, preprint.
[2] L. Flaminio & G. Forni, Equidistribution of nilflows and applications to theta sums, in preparation.
[3] G. Forni, Solutions of the cohomological equation for area preserving flows on higher genus surfaces, *Ann. of Math. (2)* 646 (1997), no. 2, 295–344.
[4] G. Forni, Deviations of ergodic averages for area preserving flows on higher genus surfaces, *Ann. of Math. (2)* 5154 (2001), no. 1, 1–103.
[5] S. Helgason, A duality for symmetric spaces with applications to group representations, *Advances in Math.* 5 (1970), 1–154.
[6] M. Kontsevich, Lyapunov exponents and Hodge theory, in *The mathematical beauty of physics* (Saclay, 1996), 318–332, Adv. Ser. Math. Phys., 24, World Sci. Publishing, River Edge, NJ, 1997.
[7] E. Nelson, Analytic vectors, *Ann. of Math. (2)* 70 1959, 572–615.
[8] A. Zorich, Asymptotic flag of an orientable measured foliation on a surface, in *Geometric study of foliations* (Tokyo, 1993), 479–498, World Sci. Publishing, River Edge, NJ, 1994.
[9] A. Zorich, Finite Gauss measure on the space of interval exchange transformations. Lyapunov exponents, *Ann. Inst. Fourier* 46 (1996), no. 2, 325–370.
[10] A. Zorich, Deviation for interval exchange transformations, *Ergodic Theory Dynam. Systems* 17 (1997), no. 6, 1477–1499.