A simple proof of a strong comparison principle for
semicontinuous viscosity solutions of the prescribed mean
curvature equation *

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Abstract

A strong comparison principle for semicontinuous viscosity solutions of the pre-
scribed mean curvature equation is considered. The difficulties of the problem come
from the fact that this nonlinear equation is non-uniformly elliptic, does not depend
on the value of unknown functions, depends on spatial variables and solutions are
semicontinuous. Our simple proof of the strong comparison principle consists only of
three ingredients, the definition of viscosity solutions, the inf and sup convolutions of
functions, and the theory of classical solutions of quasilinear elliptic equations. Once
we have the strong comparison principle, we can prove a weak comparison principle
for semicontinuous viscosity solutions of the prescribed mean curvature equation in a
bounded domain.

Key words. Prescribed mean curvature equation, strong comparison principle, semicontinuous vis-
cosity solution.

AMS subject classifications. Primary 35J93; Secondary 35D40, 35B50, 35B51

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1 Introduction

We consider the prescribed mean curvature equation of the form

\[
\text{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) = NH \quad \text{in } \Omega,
\]

(1.1)

where \( \Omega \) is a domain in \( \mathbb{R}^N \) and \( N \geq 2 \). The function \( u : \Omega \to \mathbb{R} \) is unknown, \( Du \) denotes the gradient of \( u \) in spatial variables \( x \) and \( H : \Omega \to \mathbb{R} \) is a locally Lipschitz continuous function in \( \Omega \). When the solution \( u \) is Lipschitz continuous, equation (1.1) is regarded as uniformly elliptic. However, when \( u \) is only semicontinuous, equation (1.1) can be non-uniformly elliptic.

Our goal is to prove a strong comparison principle for semicontinuous viscosity solutions of the prescribed mean curvature equation. Here, our strong comparison principle is stated as follows: if a lower semicontinuous viscosity supersolution \( u \) and an upper semicontinuous viscosity subsolution \( v \) satisfy that \( u \geq v \) in \( \Omega \) and \( u(x_0) = v(x_0) \) at some point \( x_0 \in \Omega \), then \( u \equiv v \) in \( \Omega \).

It is well known that for linear elliptic equations the strong comparison principle is equivalent to the strong maximum principle since the difference of two solutions is still a solution. Here, the strong maximum principle is the following: if a subsolution \( u \) satisfies that \( u \leq m \) with some constant \( m \) and \( u(x_0) = m \) at some point \( x_0 \in \Omega \), then \( u \equiv m \) in \( \Omega \). Evidently the strong comparison principle implies the strong maximum principle provided that the constant \( m \) is a supersolution. The strong maximum principle for classical solutions of linear and nonlinear elliptic equations has been well studied (cf. [GT], [PW]). In a book [PS] Theorem 2.1.3 (Tangency Principle), p. 16] we can find the strong comparison principle for classical solutions of nonlinear elliptic equations.

There are some results on the strong maximum principle for weak solutions in the viscosity sense. For notations of viscosity solutions we refer to the literature [CIL] and [Ko]. The strong maximum principle for semicontinuous viscosity solutions has been proved by [KaKu], [BD], [BB], [KoKo], and [BGI]. There are a few papers on the strong comparison principle. Trudinger [T] proved the strong comparison principle for Lipschitz continuous viscosity solutions of uniformly elliptic equations. Ishii and Yoshimura [IY] proved the strong comparison principle for semicontinuous viscosity solutions of uniformly elliptic equations. At the same time Giga and the first author [GO] dealt with the strong comparison principle for semicontinuous viscosity solutions of nonlinear elliptic equations. We recently noticed that the argument in [GO] Proof of Theorem 3.1, pp. 177–179] works
for uniformly elliptic equations of the form $F(D^2 u) = 0$, but it does not work for non-uniformly elliptic equations of the form $F(Du, D^2 u) = f(x)$ such as (1.1).

In the present paper we consider lower semicontinuous viscosity supersolutions and upper semicontinuous viscosity subsolutions of (1.1). Therefore, we have to deal with non-uniformly elliptic equations. Our proof is different from usual one. After being reduced to the case where both the supersolution $u$ and the subsolution $v$ are bounded, by virtue of Jensen, Lions and Souganidis [JLS], we introduce the inf and sup convolutions of $u$ and $v$ respectively, where those convolutions are continuous functions and moreover they are monotone with respect to the parameter. Then we consider the Dirichlet problems for (1.1) in every sufficiently small ball centered at a point $x_0$, where $u$ touches $v$. We choose the continuous boundary data as the inf and sup convolutions of $u$ and $v$, respectively. Since $H$ is locally Lipschitz continuous, by the theory of quasilinear elliptic equations (see [GT]), the gradient estimates of classical solutions are available and these problems have unique classical solutions provided that the ball is sufficiently small. Here the strong comparison principle is applicable to these two classical solutions and also a weak comparison principle is applicable to compare $u$ and $v$ with these two classical solutions, respectively. Eventually, these comparisons yield that $u$ and $v$ coincide with each other on the boundary of each small ball centered at a point $x_0$, and hence $u$ and $v$ coincide with each other in a small ball centered at a point $x_0$. Then the conclusion follows from the connectedness of the domain.

The present paper is organized as follows. In section 2 we state our main theorem and prove it. In section 3 we give a weak comparison principle as a corollary of our strong comparison principle. In Appendix we prove a weak comparison principle for (1.1) which compares a lower semicontinuous viscosity supersolution with a classical solution, or an upper semicontinuous viscosity subsolution with a classical solution.

2 Main theorem

Let $\Omega$ be a domain in $\mathbb{R}^N$, $N \geq 2$ and let $u : \Omega \to \mathbb{R}$. For functions $u$ we set

$$M(u) := \text{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right).$$

Here $Du$ denotes the gradient of $u$ in spatial variables $x$. Let $H : \Omega \to \mathbb{R}$ be a locally Lipschitz continuous function in $\Omega$. Then equation (1.1) is written as

$$M(u) = NH \quad \text{in} \quad \Omega.$$  \hspace{1cm} (2.1)
Our main theorem concerns an extension of the strong comparison theorem to semi-
continuous viscosity supersolutions and subsolutions of (2.1). We will use the following
notations:

$$\text{USC}(\Omega) = \{\text{upper semicontinuous functions } u : \Omega \to \mathbb{R}\},$$
$$\text{LSC}(\Omega) = \{\text{lower semicontinuous functions } u : \Omega \to \mathbb{R}\}.$$ 

Also, $\text{USC}(\bar{\Omega}), \text{LSC}(\bar{\Omega})$ are defined similarly.

**Theorem 2.1** Let $u \in \text{LSC}(\Omega)$ be a viscosity supersolution of (2.1), that is,
$$M(u) \leq NH \quad \text{in } \Omega$$
in the viscosity sense, and let $v \in \text{USC}(\Omega)$ be a viscosity subsolution of (2.1), that is,
$$M(v) \geq NH \quad \text{in } \Omega$$
in the viscosity sense. Assume that $u \geq v$ in $\Omega$ and that $u(x_0) = v(x_0)$ at some point $x_0 \in \Omega$. Then $u \equiv v$ in $\Omega$.

**Remark 2.2** A continuous viscosity solution $u$ of (2.1) means that $u \in C(\Omega)$ is both a viscosity supersolution and subsolution of (2.1). Combining the results of [B] and [T] yields the strong comparison principle for continuous viscosity solutions of (2.1). Indeed, it is shown in [B] that continuous viscosity solutions of (2.1) are Lipschitz continuous. Then, equation (2.1) is regarded as uniformly elliptic, and hence thanks to Trudinger’s results in [T] we see that the strong comparison principle for continuous viscosity solutions of (2.1) holds.

The following weak comparison principle, which is proved in more general form in [KaKu, Theorem 3, p. 475], plays a key role in the present paper. Therefore we give a simple proof directly by using the implicit function theorem and the definition of viscosity solutions in the Appendix.

**Proposition 2.3** Let $\Omega$ be a bounded domain in $\mathbb{R}^N$. Let $u \in \text{LSC}(\Omega)$ be a viscosity supersolution of (2.1) and let $v \in C^2(\Omega) \cap C(\bar{\Omega})$ be a classical solution of (2.1). Assume that $u \geq v$ on $\partial \Omega$, then $u \geq v$ in $\Omega$. Similarly, it holds for a classical solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and a viscosity subsolution $v \in \text{USC}(\Omega)$ of (2.1).

Now we are in a position to prove Theorem 2.1.
Proof of Theorem 2.1

1st step: Reduction to the case where both u and v are bounded: Let $E$ be a bounded domain with $E \subset \Omega$ and $x_0 \in E$. Since $u$ is lower semicontinuous and $v$ is upper semicontinuous, there exists $K > 0$ such that

$$u > -K \quad \text{and} \quad v < K \quad \text{in} \quad E.$$

We use a notation $B(x, r)$ as an open ball in $\mathbb{R}^N$ of radius $r > 0$ centered at $x \in \mathbb{R}^N$. For simplicity we write in particular $B_r := B(x_0, r)$ for every $r > 0$. Choose a positive number $R > 0$ satisfying

$$-\frac{1}{R} \leq H \leq \frac{1}{R} \quad \text{in} \quad E$$

and $B_R \subset E$.

For $x \in \overline{B}_R$ we set

$$u_R(x) := \min\{u(x), K + \sqrt{R^2 - |x - x_0|^2}\},$$

$$v_R(x) := \max\{v(x), -K - \sqrt{R^2 - |x - x_0|^2}\}.$$

Then $u_R \in \text{LSC}(\overline{B}_R)$ and $v_R \in \text{USC}(\overline{B}_R)$ are bounded in $\overline{B}_R$. Moreover, $u_R$ is a viscosity supersolution of (2.1) in $\overline{B}_R$ and $v_R$ is a viscosity subsolution of (2.1) in $\overline{B}_R$.

Since $v \leq u \in \Omega$ and $v(x_0) = u(x_0)$, we have $v \leq v_R \leq u_R \leq u$ in $\overline{B}_R$ and $v_R(x_0) = u_R(x_0)$. By the definition of $u_R$ and $v_R$ we see that if $u_R \equiv v_R$ in $\overline{B}_R$, then $u \equiv v$ in $\overline{B}_R$.

Thus we may assume that $u$ and $v$ are bounded in $\overline{B}_R$.

2nd step: Introducing the inf and sup convolutions of the super and subsolutions: We introduce the inf and sup convolutions of $u$ and $v$, respectively, as in [JLS]. For each $\epsilon > 0$, we set

$$u_\epsilon(x) := \inf\limits_{y \in \overline{B}_R}\left\{u(y) + \frac{|x-y|^2}{2\epsilon}\right\} \quad \text{for} \ x \in \overline{B}_R,$$

$$v_\epsilon(x) := \sup\limits_{y \in \overline{B}_R}\left\{v(y) - \frac{|x-y|^2}{2\epsilon}\right\} \quad \text{for} \ x \in \overline{B}_R.$$

Notice that $u_\epsilon, \ v_\epsilon \in C(\overline{B}_R)$, and at each point $x \in \overline{B}_R$ the inf convolution $u_\epsilon(x)$ increases to $u(x)$ and the sup convolution $v_\epsilon(x)$ decreases to $v(x)$ as $\epsilon$ decreases to 0.

**Proposition 2.4** For each $\epsilon > 0$, $v_\epsilon \geq u_\epsilon \ in \ \overline{B}_{\frac{R}{2}}$.

Proof. By setting $\rho = \frac{R}{2}$, we observe that for every $0 < r \leq \rho$

$$\frac{1}{r} \geq \frac{N}{N-1}|H| \quad \text{in} \ \partial B_r \quad \text{and} \quad \int_{B_r} |H|^N dx < \omega_N, \quad (2.2)$$

where $\omega_N$ denotes the Lebesgue measure of the unit ball in $\mathbb{R}^N$. 5
Fix $\varepsilon_0 > 0$ arbitrarily. Let us show that

$$v^{\varepsilon_0} \geq u_{\varepsilon_0} \text{ in } \overline{B}_\rho.$$ 

For each $\varepsilon \in (0, \varepsilon_0]$, we set

$$\delta_\varepsilon := \min_{y \in \overline{B}_\rho} (u_\varepsilon(y) - v^\varepsilon(y)).$$

Since $u_\varepsilon - v^\varepsilon$ is continuous in $\overline{B}_R$, $\delta_\varepsilon$ is well defined. By observing that

$$\min_{y \in \overline{B}_\rho} (u_\varepsilon(y) - v^\varepsilon(y)) \leq u_\varepsilon(x_0) - v^\varepsilon(x_0) \leq u(x_0) - v(x_0) = 0,$$

we know $\delta_\varepsilon \leq 0$. Since $u_\varepsilon(x)$ increases to $u(x)$ and $v^\varepsilon(x)$ decreases to $v(x)$ as $\varepsilon$ decreases to 0 at each $x \in \overline{B}_R$, $\delta_\varepsilon$ is monotone increasing as $\varepsilon$ decreases to 0. Let us show a lemma.

**Lemma 2.5**

$$\lim_{\varepsilon \to 0} \delta_\varepsilon = 0.$$

**Proof.** We may set

$$\lim_{\varepsilon \to 0} \delta_\varepsilon = -\lambda$$

for some number $\lambda \geq 0$. For each $\varepsilon > 0$ there exists a point $y_\varepsilon \in \overline{B}_\rho$ such that $\delta_\varepsilon = u_\varepsilon(y_\varepsilon) - v^\varepsilon(y_\varepsilon)$ and moreover there exist points $y_{1,\varepsilon}, y_{2,\varepsilon} \in \overline{B}_R$ such that

$$u_\varepsilon(y_\varepsilon) := u(y_{1,\varepsilon}) + \frac{|y_\varepsilon - y_{1,\varepsilon}|^2}{2\varepsilon},$$

$$v^\varepsilon(y_\varepsilon) := v(y_{2,\varepsilon}) - \frac{|y_\varepsilon - y_{2,\varepsilon}|^2}{2\varepsilon}.$$ 

Since $u_\varepsilon(y_\varepsilon), v^\varepsilon(y_\varepsilon), u(y_{1,\varepsilon})$ and $v(y_{2,\varepsilon})$ are bounded, we must have

$$y_\varepsilon - y_{1,\varepsilon} \to 0 \quad \text{and} \quad y_\varepsilon - y_{2,\varepsilon} \to 0 \quad \text{as} \quad \varepsilon \to 0. \quad (2.3)$$

On the other hand, the Bolzano-Weierstrass theorem yields that there exist a sequence $\{\varepsilon_j\}$ which decreases to 0 as $j \to \infty$ and $x_* \in \overline{B}_\rho$ satisfying

$$y_{\varepsilon_j} \to x_* \quad \text{as} \quad j \to \infty. \quad (2.4)$$

Then it follows from (2.3) and (2.4) that

$$y_{1,\varepsilon_j}, y_{2,\varepsilon_j} \to x_* \quad \text{as} \quad j \to \infty. \quad (2.5)$$

Since $\frac{|y_{\varepsilon_j} - y_{1,\varepsilon_j}|^2}{2\varepsilon_j}$ and $\frac{|y_{\varepsilon_j} - y_{2,\varepsilon_j}|^2}{2\varepsilon_j}$ are bounded, by taking a subsequence if necessary, we may suppose that

$$\frac{|y_{\varepsilon_j} - y_{1,\varepsilon_j}|^2}{2\varepsilon_j} \to \beta_1 (\geq 0) \quad \text{and} \quad \frac{|y_{\varepsilon_j} - y_{2,\varepsilon_j}|^2}{2\varepsilon_j} \to \beta_2 (\geq 0) \quad \text{as} \quad j \to \infty.$$
for some numbers $\beta_1, \beta_2$. The lower semicontinuity of $u$ and $-v$ at $x_*$ yields that, for every $\eta > 0$, there exists $\gamma > 0$ such that if $|x - x_*| < \gamma$ then

$$u(x) > u(x_*) - \eta \quad \text{and} \quad -v(x) > -v(x_*) - \eta.$$ 

By (2.5) there exists $n_0 \in \mathbb{N}$ such that if $j \geq n_0$ then

$$u(y_{1,\varepsilon_j}) > u(x_*) - \eta \quad \text{and} \quad -v(y_{2,\varepsilon_j}) > -v(x_*) - \eta.$$ 

Hence, for $j \geq n_0$

$$\delta_{\varepsilon_j} = u_{\varepsilon_j}(y_{\varepsilon_j}) - v^{\varepsilon_j}(y_{\varepsilon_j}) = u(y_{1,\varepsilon_j}) + \frac{|y_{1,\varepsilon_j} - y_{2,\varepsilon_j}|^2}{2\varepsilon_j} - v(y_{2,\varepsilon_j}) + \frac{|y_{1,\varepsilon_j} - y_{2,\varepsilon_j}|^2}{2\varepsilon_j} > u(x_*) - v(x_*) - 2\eta + \frac{|y_{1,\varepsilon_j} - y_{2,\varepsilon_j}|^2}{2\varepsilon_j}.$$ 

Letting $j \to \infty$ yields that for every $\eta > 0$

$$0 \geq -\lambda \geq u(x_*) - v(x_*) - 2\eta + \beta_1 + \beta_2.$$ 

Since $u(x_*) \geq v(x_*)$, $\beta_1 \geq 0$ and $\beta_2 \geq 0$, we see that $0 \leq \lambda \leq 2\eta$ and $0 \leq \beta_1 + \beta_2 \leq 2\eta$ for every $\eta > 0$. Thus we conclude that $\lambda = \beta_1 = \beta_2 = 0$ and the proof of Lemma 2.5 is finished.

We return to the proof of Proposition 2.4. In order to prove that

$$v^{\varepsilon_0} \geq u_{\varepsilon_0} \quad \text{in} \quad \overline{B_r},$$

let us show that for each $r$ with $0 < r \leq \rho$,

$$v^{\varepsilon_0}(x) \geq u_{\varepsilon_0}(x) \quad \text{for all} \quad x \in \partial B_r.$$ 

Suppose that there exist $r$ with $0 < r \leq \rho$ and a point $x_1 \in \partial B_r$ so that $u_{\varepsilon_0}(x_1) - v^{\varepsilon_0}(x_1) > 0$. Since $u_{\varepsilon_0}(x) - v^{\varepsilon_0}(x)$ is continuous, there exist $r_1$ with $0 < r_1 < \frac{r}{2}$ and $\beta > 0$ such that

$$u_{\varepsilon_0}(x) - v^{\varepsilon_0}(x) \geq \beta \quad \text{in} \quad \overline{B(x_1, r_1)}.$$ 

(2.6)

We divide $\partial B_r$ into two pieces:

$$\Gamma_+ := \partial B_r \cap \overline{B(x_1, r_1)} \quad \text{and} \quad \Gamma_0 := \partial B_r \setminus \Gamma_+.$$ 

Clearly $\partial B_r$ is a disjoint union of $\Gamma_+$ and $\Gamma_0$.

Note that (2.6) gives in particular

$$u_{\varepsilon_0}(x) \geq v^{\varepsilon_0}(x) + \beta \quad \text{on} \quad \Gamma_+.$$ 

(2.7)
Then for every \( 0 < \varepsilon \leq \varepsilon_0 \), by the monotonicity of \( u_\varepsilon \) and \( v^\varepsilon \)

\[
u_\varepsilon(x) \geq u_{\varepsilon_0}(x) \geq v^{\varepsilon_0}(x) + \beta \geq v^\varepsilon(x) + \beta \quad \text{on } \Gamma_+,
\]

and by the definition of \( \delta_\varepsilon \) we see that

\[
u_\varepsilon(x) \geq v^\varepsilon(x) + \delta_\varepsilon \quad \text{on } \partial B_r.
\]

(2.8)

Since \( H \) is Lipschitz continuous in \( \overline{B}_r \), we see that the interior estimates of [GT, Corollary 16.7, p. 407] are available. Therefore it follows from (2.2) and the theory of the prescribed mean curvature equation ([GT, Theorem 16.10, p. 408]) that there exist \( \hat{u}_\varepsilon, \hat{v}^\varepsilon \in C^2(B_r) \cap C(\overline{B}_r) \) satisfying

\[
M(\hat{u}_\varepsilon) = M(\hat{v}^\varepsilon) = NH \quad \text{in } B_r,
\]

\[
\hat{u}_\varepsilon = u_\varepsilon \quad \text{and} \quad \hat{v}^\varepsilon = v^\varepsilon \quad \text{on } \partial B_r.
\]

Notice that \( M(\hat{v}^\varepsilon + \delta_\varepsilon) = NH \) in \( B_r \). By Proposition 2.3 and (2.8) we observe that

\[
v + \delta_\varepsilon \leq \hat{v}^\varepsilon + \delta_\varepsilon \leq \hat{u}_\varepsilon \leq u \quad \text{in } \overline{B}_r.
\]

(2.9)

Also, \( \hat{u}_\varepsilon \) increases and \( \hat{v}^\varepsilon \) decreases as \( \varepsilon \) decreases to 0 by the monotonicity of \( u_\varepsilon \) and \( v^\varepsilon \).

The boundedness of \( \{\hat{u}_\varepsilon\} \) and \( \{\hat{v}^\varepsilon\} \) together with the interior estimates of [GT, Corollary 16.7, p. 407] yields that there exist \( \hat{u}_0, \hat{v}^0 \in C^2(B_r) \) such that

\[
\hat{u}_\varepsilon \to \hat{u}_0 \quad \text{and} \quad \hat{v}^\varepsilon \to \hat{v}^0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{uniformly on compact sets in } B_r,
\]

\[
M(\hat{u}_0) = M(\hat{v}^0) = NH \quad \text{in } B_r.
\]

Therefore, since \( \lim_{\varepsilon \to 0} \delta_\varepsilon = 0 \), we observe from (2.9) that

\[
v \leq \hat{v}^0 \leq \hat{u}_0 \leq u \quad \text{in } B_r.
\]

Since \( v(x_0) = u(x_0) \), we have that \( \hat{v}^0(x_0) = \hat{u}_0(x_0) \). By the strong comparison principle for classical solutions we see that

\[
\hat{v}^0(x) \equiv \hat{u}_0(x) \quad \text{in } B_r.
\]

(2.10)

However, by (2.7)

\[
\hat{u}_{\varepsilon_0}(x) \geq \hat{v}^{\varepsilon_0}(x) + \beta \quad \text{on } \Gamma_+.
\]

Hence it follows from the continuity of \( \hat{u}_{\varepsilon_0} \) and \( \hat{v}^{\varepsilon_0} \) that there exist \( \beta_3 \) with \( 0 < \beta_3 \leq \beta \) and \( r_2 \) with \( 0 < r_2 \leq r_1 \) satisfying

\[
\hat{u}_{\varepsilon_0}(x) \geq \hat{v}^{\varepsilon_0}(x) + \beta_3 \quad \text{in } B_r \cap \overline{B(x_1,r_2)}.
\]
By the monotonicity of \(u_\varepsilon\) and \(v^\varepsilon\) for \(0 < \varepsilon \leq \varepsilon_0\) we observe that
\[
\hat{u}_\varepsilon(x) \geq \hat{u}_\varepsilon(0) \geq \hat{v}^\varepsilon_0(x) + \beta_3 \geq \hat{v}^\varepsilon(x) + \beta_3 \quad \text{in} \ B_r \cap \overline{B(x_1, r_2)}.
\]
Therefore letting \(\varepsilon \to 0\) yields that
\[
\hat{u}_0(x) \geq \hat{v}^0(x) + \beta_3 \quad \text{in} \ B_r \cap \overline{B(x_1, r_2)}.
\]
This contradicts (2.10) .

Eventually, we conclude that \(v^\varepsilon_0 \geq u^\varepsilon_0\) on \(\partial B_r\). Since this holds for every \(0 < r \leq \rho\), we have
\[
v^\varepsilon_0 \geq u^\varepsilon_0 \quad \text{in} \ \overline{B_\rho}.
\]

3rd step: Completion of the proof of Theorem 2.1 By Proposition 2.4 letting \(\varepsilon \to 0\) yields that
\[
v \geq u \quad \text{in} \ \overline{B_\rho},
\]
which shows that \(u \equiv v\) in \(\overline{B_\rho}\). Since \(\Omega\) is connected, we conclude that \(u \equiv v\) in \(\Omega\).

3 A weak comparison principle for semicontinuous viscosity solutions of the prescribed mean curvature equation

The strong comparison principle proved in section 2 yields a weak comparison principle for semicontinuous viscosity solutions of the prescribed mean curvature equation in a bounded domain.

**Theorem 3.1** Let \(\Omega\) be a bounded domain in \(\mathbb{R}^n\). Let \(u \in \text{LSC}(\overline{\Omega})\) and \(v \in \text{USC}(\overline{\Omega})\) be viscosity super and subsolutions of (2.1), respectively. Assume that \(u \geq v\) on \(\partial \Omega\). Then \(u \geq v\) in \(\Omega\), and hence either \(u \equiv v\) in \(\Omega\) or \(u > v\) in \(\Omega\).

**Proof.** Suppose that there exists a point \(x_1 \in \overline{\Omega}\) satisfying
\[
\theta := \min_{x \in \overline{\Omega}} (u - v)(x) = (u - v)(x_1) < 0.
\]
Hence \(x_1 \in \Omega\), since \(u \geq v\) on \(\partial \Omega\). Then we observe that
\[
u \geq u + \theta \quad \text{in} \ \Omega \quad \text{and} \quad u(x_1) = v(x_1) + \theta.
\]
Note that \(v + \theta\) is also a viscosity subsolution of (2.1). By Theorem 2.1 we have that \(u \equiv v + \theta\) in \(\Omega\), which contradicts the assumption that \(u \geq v\) on \(\partial \Omega\). Therefore we see that \(u \geq v\) in \(\Omega\).

Moreover, if there exists a point \(x_2 \in \Omega\) so that \(u(x_2) = v(x_2)\), then \(u \equiv v\) in \(\Omega\) by Theorem 2.1 which concludes that either \(u \equiv v\) in \(\Omega\) or \(u > v\) in \(\Omega\).
Appendix

Although Proposition 2.3 was already proved in [KaKu, Theorem 3, p. 457], for convenience we will give a simple proof directly by using the implicit function theorem and the definition of viscosity solutions.

*Proof of Proposition 2.3.* By the argument in [MS, Theorem A.1, p. 253], which applies Sard’s theorem to a smooth function being comparable to the distance function to the closed set $\mathbb{R}^N \setminus \Omega$ due to Calderón and Zygmund [Z, Lemma 3.6.1, p. 136] (see also [CZ, Lemma 3.2, p. 185]), we observe that for each small $\varepsilon > 0$ there exists a smooth open set $\Omega_{\varepsilon} \subset \subset \Omega$ with $\Omega_{\varepsilon'} \subset \subset \Omega_{\varepsilon}$ if $\varepsilon < \varepsilon'$ and $\Omega_{\varepsilon} \to \Omega$ as $\varepsilon \to 0$. Since $\Omega$ is bounded, we notice that $\Omega_{\varepsilon}$ is a union of a finite number of smooth domains.

Let $u \in LSC(\Omega)$ be a viscosity supersolution of (2.1) and let $v \in C^2(\Omega) \cap C(\Omega)$ be a classical solution of (2.1). Assume that $u \geq v$ on $\partial \Omega$. Since $H$ is locally Lipschitz continuous, we see that the interior estimates of [GT, Corollary 16.7, p. 407] are available. Therefore, with the aid of the Schauder interior estimates for elliptic equations, there exists a number $\alpha$ with $0 < \alpha < 1$ depending on $\varepsilon$ such that $v \in C^{2,\alpha}(\Omega_{\varepsilon}).$ We can write

$$\Omega_{\varepsilon} = \bigcup_{j=1}^{n(\varepsilon)} D_{\varepsilon,j}. \quad (3.1)$$

Consider an arbitrary $D_{\varepsilon,j}$. For simplicity we will write $D$ instead of $D_{\varepsilon,j}$. Note that $\partial D$ is close to $\partial \Omega$ if $\varepsilon > 0$ is sufficiently small. Since $u \in LSC(\Omega)$, $v \in C(\Omega)$ and $u \geq v$ on $\partial \Omega$, there exists $\tau(\varepsilon) > 0$ satisfying $\lim_{\varepsilon \to 0} \tau(\varepsilon) = 0$ and $u > v - \tau(\varepsilon)$ on $\partial D$.

Set $w_{\varepsilon} := v - \tau(\varepsilon)$ and we have

$$M(w_{\varepsilon}) = NH \quad \text{in} \quad D.$$

We set

$$X := \{ f \in C^{2,\alpha}(\overline{D}) \mid f = 0 \text{ on } \partial D \},$$

$$F : X \times \mathbb{R} \ni (f, s) \mapsto M(w_{\varepsilon} + f) - N(H + s) \in C^\alpha(\overline{D}).$$

We use the implicit function theorem for $X$ and $F$ (see [AP, Theorem 2.3, p. 38], [D, Theorem 15.1, p. 148] for instance ). For each $0 < \delta << 1$ there exists $\tilde{w}_{\varepsilon,\delta} \in C^{2,\alpha}(\overline{D})$ satisfying

$$M(\tilde{w}_{\varepsilon,\delta}) = N(H + \delta) \quad \text{in} \quad D,$$

$$\tilde{w}_{\varepsilon,\delta} = w_{\varepsilon} \quad \text{on} \quad \partial D.$$

Note that $\tilde{w}_{\varepsilon,0} = w_{\varepsilon}$. Then we have

$$u \geq \tilde{w}_{\varepsilon,\delta} \quad \text{in} \quad \overline{D}. \quad (3.2)$$
Indeed, suppose that there exists a point \( z \in \overline{D} \) satisfying
\[
\min_{x \in \overline{D}} (u - \tilde{w}_{\varepsilon, \delta})(x) = (u - \tilde{w}_{\varepsilon, \delta})(z) < 0. \tag{3.3}
\]
Hence \( z \in D \), since \( u - \tilde{w}_{\varepsilon, \delta} = u - w_\varepsilon > 0 \) on \( \partial D \). Moreover, since \( u \) is a viscosity supersolution of (2.1), we have from (3.3) that \( M(\tilde{w}_{\varepsilon, \delta})(z) \leq NH(z) \). This contradicts the fact that \( M(\tilde{w}_{\varepsilon, \delta})(z) = N(H(z) + \delta) \) with \( \delta > 0 \).

Letting \( \delta \to 0 \) in (3.2) yields that
\[
u \geq w_\varepsilon \quad \text{in} \quad \overline{D}.
\]
Hence it follows from (3.1) that
\[
u \geq v - \tau(\varepsilon) \quad \text{in} \quad \Omega_\varepsilon.
\]
Thus, letting \( \varepsilon \to 0 \) yields that \( u \geq v \) in \( \Omega \), which completes the proof of Proposition 2.3.

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