Neumann and Poincare problems for Poisson’s equations with measurable data

Vladimir Ryazanov

Abstract

First, it is proved the existence theorem on solutions of the Riemann–
Hilbert boundary value problem with arbitrary measurable data for gene-
ralized analytic functions in the unit disc. Then the theorem is extended
to arbitrary Jordan domains with rectifiable boundaries in terms of the
natural parameter and nontangential limits, moreover, to arbitrary Jordan
domains in terms of harmonic measure and principal asymptotic values.
On this basis, it is established the corresponding existence theorems for
the Neumann and Poincare problems with arbitrary measurable data for
the Poisson equations.

2010 Mathematics Subject Classification. AMS: Primary 30C62,
31A05, 31A20, 31A25, 31B25, 35J61 Secondary 30E25, 31C05, 34M50, 35F45,
35Q15

Keywords: Poisson equations, Riemann–Hilbert, Neumann and Poincare
problems, generalized analytic and harmonic functions, logarithmic potential

 Dedicated to the 100th anniversary of the birth of G.D. Suvorov

1 Introduction

The present paper is a natural continuation of the articles [16–20] devoted to
the Riemann–Hilbert and Poincare (in particular, Neumann) boundary values
problems for analytic and harmonic functions, respectively. Here we extend
the corresponding results to generalized analytic and harmonic functions with
arbitrary measurable data, see relevant history notes in the mentioned articles
and necessary comments on previous results below.
Recall that a path in the unit disk $D := \{ z \in \mathbb{C} : |z| < 1 \}$ terminating at $\zeta \in \partial D$ is called **nontangential** if its part in a neighborhood of $\zeta$ lies inside of an angle in $D$ with the vertex at $\zeta$. Hence the limit along all nontangential paths at $\zeta \in \partial D$ also named **angular** at the point. The latter is a traditional tool of the geometric function theory, see e.g. monographs [2], [6], [9], [13] and [14].

It is well–known the uniqueness theorem to the Dirichlet problem in terms of the angular limits e.g. for bounded harmonic functions $u$, see also Corollary IX.1.1 and Theorem IX.2.3. However, in general there is no uniqueness theorem in the Dirichlet problem for the Laplace equation even under zero boundary data.

The following deep (non–trivial) result of Luzin was one of the main theorems of his dissertation, see e.g. his paper [7], dissertation [8], p. 35, and its reprint [9], p. 78, where one may assume that $\Phi(0) = \Phi(1) = 0$.

**Theorem A.** For any measurable function $\varphi : [0,1] \to \mathbb{R}$, there is a continuous function $\Phi : [0,1] \to \mathbb{R}$ such that $\Phi' = \varphi$ a.e.

Just on the basis of Theorem A, Luzin has proved the next significant result of his dissertation, see e.g. [9], p. 80.

**Theorem B.** Let $\varphi(\vartheta)$ be real, measurable, almost everywhere finite and have the period $2\pi$. Then there exists a harmonic function $U$ in the unit disk $D$ such that $U(z) \to \varphi(\vartheta)$ for a.e. $\vartheta$ as $z \to e^{i\vartheta}$ along any nontangential path.

Such a solution of the Dirichlet problem for harmonic functions was given by Luzin in the explicit form through the function $\Phi$ from Theorem A:

$$U(re^{i\vartheta}) = -\frac{r}{\pi} \int_0^{2\pi} \frac{(1 - r^2) \sin(\vartheta - t)}{(1 - 2r \cos(\vartheta - t) + r^2)^2} \Phi(t) \, dt . \quad (1.1)$$

Later on, it was shown in [19] that the construction of Luzin can be described as the Poisson–Stieltjes integral, where $\Phi$ is not in general of bounded variation,

$$U_{\Phi}(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\vartheta - t) \, d\Phi(t) \quad \forall \ z = re^{i\vartheta}, \ r \in (0,1) , \ \vartheta \in [-\pi, \pi] . \quad (1.2)$$
Note that the Luzin dissertation was published in Russian as the book [9] with comments of his pupils Bari and Men’shov only after his death. A part of its results was also printed in Italian [10]. However, Theorem A was published with a complete proof in English in the book [21] as Theorem VII(2.3). Hence Frederick Gehring in [3] has rediscovered Theorem B and his proof on the basis of Theorem A that in fact coincided with the original proof of Luzin.

Corollary 5.1 in [16] has strengthened Theorem B, see also [17], as the next:

**Theorem C.** For each (Lebesgue) measurable function \( \varphi : \partial \mathbb{D} \to \mathbb{R} \), the space of all harmonic functions \( u : \mathbb{D} \to \mathbb{R} \) with the angular limits \( \varphi(\zeta) \) for a.e. \( \zeta \in \partial \mathbb{D} \) has the infinite dimension.

Theorem B of Luzin (as well as Theorem C) was key to establish the corresponding result on the Riemann–Hilbert boundary value problem with arbitrary measurable data for analytic functions in [16], Theorems 2.1 and 5.2:

**Theorem D.** Let \( \lambda : \partial \mathbb{D} \to \mathbb{C}, |\lambda(\zeta)| \equiv 1 \), and \( \varphi : \partial \mathbb{D} \to \mathbb{R} \) be measurable functions. Then there exist analytic functions \( f : \mathbb{D} \to \mathbb{C} \) that has angular limits

\[
\lim_{z \to \zeta} \Re \left\{ \lambda(\zeta) \cdot f(z) \right\} = \varphi(\zeta) \quad \text{for a.e. } \zeta \in \partial \mathbb{D}.
\]

The space of such analytic functions has the infinite dimension.

The proof of Theorem D was reduced to the corresponding two Dirichlet problems with measurable data for harmonic functions in the unit disk. Then Theorem D was extended to arbitrary Jordan domains with rectifiable boundaries in terms of the natural parameter and angular limits, see Theorem 3.1 in [16], and also to arbitrary Jordan domains in terms of harmonic measure and the so-called unique principal asymptotic values, see Theorem 3.1 in [16].

In turn, the results obtained in [16] have been applied in the paper [18] to the study of the Poincare problem on directional derivatives and, in particular, of the Neumann problem with arbitrary measurable data for the harmonic functions. Namely, it was shown that the latter problems can be reduced to the Riemann-Hilbert problem through a suitable choice of the functions \( \lambda \) and \( \varphi \) in [16].
The well-known monograph [22] was devoted to the theory of the generalized analytic functions, i.e., continuous complex valued functions $h(z)$ of the complex variable $z = x + iy$ with generalized first partial derivatives by Sobolev satisfying equations of the form

$$\partial \bar{z} h + ah + bh = c, \quad \partial \bar{z} := \frac{1}{2} \left( \frac{\partial}{\partial x} + i \cdot \frac{\partial}{\partial y} \right),$$

(1.4)

where it was assumed that the complex valued functions $a, b$ and $c$ belong to the class $L^p$ with some $p > 2$ in the corresponding domain $D \subseteq \mathbb{C}$.

In the present paper, to study Neumann and Poincare problems for the Poisson equations with arbitrary measurable boundary data it is first developed the theory of the Riemann–Hilbert problem with arbitrary measurable data for generalized analytic functions satisfying equations of the form

$$\partial \bar{z} h(z) = g(z)$$

(1.5)

with the real valued function $g$ in the class $L^p, p > 2$. We call such functions $h$ generalized analytic functions with sources $g$.

2 Riemann–Hilbert problem with measurable data

In this section, it is studied the Riemann–Hilbert problem with arbitrary measurable data for generalized analytic functions with sources.

**Theorem 1.** Let $\lambda : \partial \mathbb{D} \to \mathbb{C}, \ |\lambda(\zeta)| \equiv 1$, and $\varphi : \partial \mathbb{D} \to \mathbb{R}$ be measurable functions and let $g : \mathbb{D} \to \mathbb{R}$ belong to the class $L^p(\mathbb{D})$ for some $p > 2$.

Then there exist generalized analytic functions $h : \mathbb{D} \to \mathbb{C}$ with the source $g$ that have the angular limits

$$\lim_{z \to \zeta} \text{Re} \left\{ \frac{\lambda(\zeta) \cdot h(z)}{\overline{\lambda(\zeta)}} \right\} = \varphi(\zeta) \quad \text{for a.e.} \quad \zeta \in \partial \mathbb{D}. \quad (2.1)$$

Furthermore, the space of such functions $h$ has the infinite dimension.
Proof. First of all, let us extend the function \( g \) by zero outside of \( \mathbb{D} \) and consider the logarithmic (Newtonian) potential \( N_G \) of the source \( G = 2g \),

\[
N_G(z) := \frac{1}{2\pi} \int_{\mathbb{C}} \ln|z - w| G(w) \, dm(w) .
\]  

(2.2)

It is known that \( \triangle N_G = G \) in the generalized sense, see Theorem 3.7.4 in [15]:

\[
\int_{\mathbb{C}} N_G(z) \triangle \psi(z) \, dm(z) = \int_{\mathbb{C}} \psi(z) G(z) \, dm(z) \quad \forall \ \psi \in C^\infty_0(\mathbb{C}) .
\]  

(2.3)

Note that \( N_G \) is the convolution \( \Psi * G \), where \( \Psi(\zeta) = \ln|\zeta| \), hence \( \partial \Psi * G / \partial z = \partial \Psi / \partial z * G \), see e.g. (4.2.5) in [5], and by elementary calculations

\[
\frac{\partial}{\partial z} \ln|z - w| = \frac{1}{2} \cdot \frac{1}{z - w}, \quad \frac{\partial}{\partial \bar{z}} \ln|z - w| = \frac{1}{2} \cdot \frac{1}{\bar{z} - w} .
\]

Consequently,

\[
\frac{\partial N_G(z)}{\partial z} = \frac{1}{4} \cdot T_G(z) , \quad \frac{\partial N_G(z)}{\partial \bar{z}} = \frac{1}{4} \cdot \overline{T_G(z)} ,
\]

where \( T_G \) is the known integral operator

\[
T_G(z) := \frac{1}{\pi} \int_{\mathbb{C}} G(w) \frac{dm(w)}{z - w} .
\]

Thus, \( N_G \in W^{2,p}_{\text{loc}}(\mathbb{C}) \) by Theorems 1.36–1.37 in [22] if \( p > 1 \). Moreover, \( N_G \in C^{1,\alpha}_{\text{loc}}(\mathbb{C}) \) with \( \alpha = (p - 2)/p \) by Theorem 1.19 in [22] if \( p > 2 \).

Now, by the above arguments \( \triangle U = G \) for \( U = N_G \) and, setting \( u = U_x \) and \( v = -U_y \), we have that \( u_x - v_y = G \) and \( u_x + v_y = 0 \). Thus, it is clear by elementary calculations that \( H := u + iv \) is just a generalized analytic function with the source \( g \). Moreover, the function

\[
\varphi_g(\zeta) := \lim_{z \to \zeta} \Re \left\{ \overline{\lambda(\zeta)} \cdot H(z) \right\} = \Re \left\{ \overline{\lambda(\zeta)} \cdot H(\zeta) \right\} , \quad \forall \zeta \in \partial \mathbb{D} ,
\]  

(2.4)

is measurable because the function \( H \) is continuous in the whole plane \( \mathbb{C} \).

Next, by Theorem 2.1 in [16] there exist analytic functions \( A \) in \( \mathbb{D} \) such that along any non-tangential path

\[
\lim_{z \to \zeta} \Re \left\{ \overline{\lambda(\zeta)} \cdot A(z) \right\} = \Phi(\zeta) \quad \text{for a.e. } \zeta \in \partial \mathbb{D} \quad (2.5)
\]
for the measurable function $\Phi(\zeta) := \varphi(\zeta) - \varphi_g(\zeta)$, $\zeta \in \partial \mathbb{D}$. Finally, the functions $h := A + H$ are desired generalized analytic functions with the source $g$. The space of such functions has the infinite dimension, see Theorem 5.2 and Remark 5.2 in [16], as well as Remark 3.1 in [17].

Arguing similarly to the proof of Theorem 1 and applying Theorem 3.1 instead of Theorem 2.1 in [16], we obtain the next result.

**Theorem 2.** Let $D$ be a Jordan domain in $\mathbb{C}$ with a rectifiable boundary, $\lambda : \partial D \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, and $\varphi : \partial D \to \mathbb{R}$ be measurable functions with respect to the natural parameter on $\partial D$ and let $g : D \to \mathbb{R}$ be in $L^p(D)$ for some $p > 2$.

Then there exist generalized analytic functions $h : D \to \mathbb{C}$ with the source $g$ that have the angular limits

$$\lim_{z \to \zeta} \text{Re} \left\{ \lambda(\zeta) \cdot h(z) \right\} = \varphi(\zeta) \quad \text{for a.e. } \zeta \in \partial D \tag{2.6}$$

Furthermore, the space of such functions $h$ has the infinite dimension.

The conceptions of a harmonic measure introduced by R. Nevanlinna in [11] and a principal asymptotic value based on one nice result of F. Bagemihl [1] make also possible with a great simplicity and generality to formulate the existence theorems the Riemann-Hilbert problem in arbitrary Jordan domains.

First of all, given a measurable set $E \subseteq \partial \mathbb{D}$ and a point $z \in \mathbb{D}$, a **harmonic measure** of $E$ at $z$ relative to $\mathbb{D}$ is the value at $z$ of a bounded harmonic function $u$ in $\mathbb{D}$ with the boundary values 1 a.e. on $E$ and 0 a.e on $\partial \mathbb{D} \setminus E$ in the sense of the angular limits. Such a function can be calculated through the Poisson integral, see e.g. Corollary IX.1.1 in [4].

Since the harmonic measure zero is invariant under conformal mappings between Jordan domains, given a Jordan domain $D$, a set $E \subseteq \partial D$ will be called measurable with respect to harmonic measures in $D$ if $E = \omega(E)$ is measurable with respect to the linear measure on $\partial \mathbb{D}$ where $\omega$ is a conformal mapping of $D$ onto the unit disk $\mathbb{D}$, see e.g. arguments in the proof of Theorem 4.1.
Next, a Jordan curve generally speaking has no tangents. Hence we need a replacement for the notion of a nontangential limit. In this connection, recall Theorem 2 in [1], see also Theorem III.1.8 in [12], stating that, for any function \( \Omega : \mathbb{D} \rightarrow \mathbb{C} \), for all pairs of arcs \( \gamma_1 \) and \( \gamma_2 \) in \( \mathbb{D} \) terminating at \( \zeta \in \partial \mathbb{D} \), except a countable set of \( \zeta \in \partial \mathbb{D} \),

\[
C(\Omega, \gamma_1) \cap C(\Omega, \gamma_2) \neq \emptyset
\]

(2.7)

where \( C(\Omega, \gamma) \) denotes the **cluster set of** \( \Omega \) **at** \( \zeta \) **along** \( \gamma \), i.e.,

\[
C(\Omega, \gamma) = \{ w \in \mathbb{C} : \Omega(z_n) \to w, \ z_n \to \zeta, \ z_n \in \gamma \}.
\]

Immediately by the theorems of Riemann and Caratheodory, this result is extended to an arbitrary Jordan domain \( D \) in \( \mathbb{C} \). Given a function \( \Omega : D \rightarrow \mathbb{C} \) and \( \zeta \in \partial D \), denote by \( P(\Omega, \zeta) \) the intersection of all cluster sets \( C(\Omega, \gamma) \) for arcs \( \gamma \) in \( D \) terminating at \( \zeta \). Later on, we call the points of the set \( P(\Omega, \zeta) \) **principal asymptotic values** of \( \Omega \) at \( \zeta \). Note that, if \( \Omega \) has a limit along at least one arc in \( D \) terminating at a point \( \zeta \in \partial D \) with the prproperty (2.7), then the principal asymptotic value is unique.

Thus, by the Bagemihl theorem, arguing similarly to the proof Theorem 1 and applying Theorem 4.1 instead of Theorem 2.1 in [16], we also obtain the following result.

**Theorem 3.** Let \( D \) be a Jordan domain in \( \mathbb{C} \), \( \lambda : \partial D \to \mathbb{C} \), \( |\lambda(\zeta)| \equiv 1 \), and \( \varphi : \partial D \to \mathbb{R} \) be measurable functions with respect to harmonic measures in \( D \) and let \( g : D \to \mathbb{R} \) be in \( L^p(D) \) for some \( p > 2 \).

Then there exist generalized analytic functions \( h : \mathbb{D} \to \mathbb{C} \) with the source \( g \) that have the limits in the sense of the unique principal asymptotic value

\[
\lim_{z \to \zeta} \Re \left\{ \overline{\lambda(\zeta)} \cdot h(z) \right\} = \varphi(\zeta) \quad \text{for a.e. } \zeta \in \partial D
\]

(2.8)

with respect to the harmonic measure in \( D \). Furthermore, the space of such functions \( h \) has the infinite dimension.
Remark 1. As it follows from the proofs of Theorems 1–3, the generalized analytic functions \( h \) with a source \( g \in L^p \), \( p > 2 \), satisfying the Hilbert boundary conditions can be represented in the form of the sums \( A + H \) with analytic functions \( A \) satisfying the corresponding Hilbert boundary conditions and a generalized analytic function \( H = u + iv \) with the same source \( g \), \( u = P_x \) and \( v = -P_y \), where \( P \) is the logarithmic (Newtonian) potential \( N_G \), \( G = 2g \), in the class \( W^{2,p}_{\text{loc}}(\mathbb{C}) \cap C^{1,\alpha}_{\text{loc}}(\mathbb{C}) \), \( \alpha = (p - 2)/p \), that satisfies the equation \( \triangle P = G \).

In particular, under \( \lambda \equiv 1 \) we obtain the following consequences on the Dirichlet problem for the generalized analytic functions.

**Corollary 1.** Let \( D \) be a Jordan domain in \( \mathbb{C} \) and let \( g : D \to \mathbb{R} \) belong to the class \( L^p(D) \) for some \( p > 2 \).

If \( \varphi : \partial D \to \mathbb{R} \) is a measurable function with respect to harmonic measure, then there exist generalized analytic functions \( h : D \to \mathbb{C} \) with the source \( g \) and

\[
\lim_{z \to \zeta} \text{Re} \ h(z) = \varphi(\zeta)
\]

for a.e. \( \zeta \in \partial D \) with respect to the harmonic measure in the sense of the unique principal asymptotic value.

If the domain \( D \) has a rectifiable boundary and \( \varphi : \partial D \to \mathbb{R} \) is measurable with respect to the natural parameter on \( \partial D \), then (2.9) holds for a.e. \( \zeta \in \partial D \) with respect to the natural parameter on \( \partial D \), too.

Moreover, the space of such functions \( h \) has the infinite dimension in the both cases.

3 The Poincare problem for Poisson’s equations

In this section, we consider the Poincare problem on the directional derivatives for the Poisson equations

\[
\triangle U(z) = G(z)
\]
with real valued functions $G$ of a class $L^p(D)$, $p > 2$, in the corresponding domain $D \subseteq \mathbb{C}$. For short, we call solutions $U$ of (3.1) in the class $W^{2,p}_{\text{loc}} \cap C^{1,\alpha}_{\text{loc}}$, $\alpha = (p-2)/p$, \textit{generalized harmonic functions with the sources $G$}.

Note that the directional derivative
\[
\frac{\partial U}{\partial \nu} = \text{Re} \, \nu \cdot \nabla U = \text{Re} \, \nu \cdot \nabla U = (\nu, \nabla U) \tag{3.2}
\]
is the scalar product of $\nu$ and the gradient $\nabla U$ interpreted as vectors in $\mathbb{R}^2$. Hence the following theorem is a direct consequence of Theorem 1 and Remark 1 on the Riemann-Hilbert problem in the unit disk $D$ with $\lambda(\zeta) = \nu(\zeta)$, $\zeta \in \partial D$.

\textbf{Theorem 4.} Let $\nu : \partial D \to \mathbb{C}$, $|\nu(\zeta)| \equiv 1$, and $\varphi : \partial D \to \mathbb{R}$ be measurable functions and let $G : \mathbb{D} \to \mathbb{R}$ be $L^p(\mathbb{D})$, $p > 2$. Then there exist generalized harmonic functions $U : \mathbb{D} \to \mathbb{R}$ with the source $G$ that have the angular limits
\[
\lim_{z \to \zeta} \frac{\partial U}{\partial \nu}(z) = \varphi(\zeta) \quad \text{for a.e. } \zeta \in \partial \mathbb{D}. \tag{3.3}
\]
Furthermore, the space of such functions $U$ has the infinite dimension.

\textbf{Remark 2.} We are able to say more in the case of $\text{Re} \, n(\zeta)\nu(\zeta) > 0$, where $n(\zeta)$ is the inner normal to $\partial \mathbb{D}$ at the point $\zeta$. Indeed, the latter magnitude is a scalar product of $n = n(\zeta)$ and $\nu = \nu(\zeta)$ interpreted as vectors in $\mathbb{R}^2$ and it has the geometric sense of projection of the vector $\nu$ onto $n$. In view of (3.3), since the limit $\varphi(\zeta)$ is finite, there is a finite limit $U(\zeta)$ of $U(z)$ as $z \to \zeta$ in $\mathbb{D}$ along the straight line passing through the point $\zeta$ and being parallel to the vector $\nu$ because along this line
\[
U(z) = U(z_0) - \int_0^1 \frac{\partial U}{\partial \nu} (z_0 + \tau(z - z_0)) \, d\tau. \tag{3.4}
\]
Thus, at each point with condition (3.3), there is the directional derivative
\[
\frac{\partial U}{\partial \nu}(\zeta) := \lim_{t \to 0} \frac{U(\zeta + t \cdot \nu) - U(\zeta)}{t} = \varphi(\zeta). \tag{3.5}
\]

In particular, in the case of the Neumann problem, $\text{Re} \, n(\zeta)\nu(\zeta) \equiv 1 > 0$, where $n = n(\zeta)$ denotes the unit interior normal to $\partial \mathbb{D}$ at the point $\zeta$, and we have by Theorem 4 and Remark 2 the following significant result.
Corollary 2. For each measurable function $\varphi : \partial \mathbb{D} \to \mathbb{R}$, one can find generalized harmonic functions $U : \mathbb{D} \to \mathbb{R}$ with the source $G$ such that, at a.e. point $\zeta \in \partial \mathbb{D}$, there exist:

1) the finite radial limit

$$U(\zeta) := \lim_{r \to 1} U(r \zeta) \ ,$$

(3.6)

2) the normal derivative

$$\frac{\partial U}{\partial n} (\zeta) := \lim_{t \to 0} \frac{U(\zeta + t \cdot n) - U(\zeta)}{t} = \varphi(\zeta) \ ,$$

(3.7)

3) the angular limit

$$\lim_{z \to \zeta} \frac{\partial U}{\partial n} (z) = \frac{\partial U}{\partial n} (\zeta) \ .$$

(3.8)

Furthermore, the space of such functions $U$ has the infinite dimension.

Similarly, the next result in domains bounded by rectifiable Jordan curves is a direct consequence of the relation (3.2), Theorem 2 and Remark 1.

Theorem 5. Let $D$ be a Jordan domain in $\mathbb{C}$ with a rectifiable boundary, $\nu : \partial D \to \mathbb{C}$, $|\nu(\zeta)| = 1$, and $\varphi : \partial D \to \mathbb{R}$ be measurable functions with respect to the natural parameter and let $G : D \to \mathbb{R}$ be in the class $L^p$, $p > 2$.

Then there exist generalized harmonic functions $U : D \to \mathbb{R}$ with the source $G$ that have the angular limits

$$\lim_{z \to \zeta} \frac{\partial U}{\partial \nu} (z) = \varphi(\zeta) \quad \text{for a.e.} \quad \zeta \in \partial D \quad (3.9)$$

with respect to the natural parameter. Furthermore, the space of such functions $U$ has the infinite dimension.

Arguing similarly to Remark 2, we have by Theorem 5 the following.

Corollary 3. Let $D$ be a domain in $\mathbb{C}$ bounded by a rectifiable Jordan curve and $\varphi : \partial D \to \mathbb{R}$ be a measurable function with respect to the natural parameter and let $G : D \to \mathbb{R}$ be in the class $L^p$, $p > 2$. 

Then one can find generalized harmonic functions $U : D \rightarrow \mathbb{R}$ with the source $G$ such that, at a.e. point $\zeta \in \partial D$ with respect to the natural parameter, there exist:

1) the finite normal limit

$$U(\zeta) := \lim_{z \to \zeta} U(z) ,$$

(3.10)

2) the normal derivative

$$\frac{\partial U}{\partial n}(\zeta) := \lim_{t \to 0} \frac{U(\zeta + t \cdot n) - U(\zeta)}{t} = \varphi(\zeta) ,$$

(3.11)

3) the angular limit

$$\lim_{z \to \zeta} \frac{\partial U}{\partial n}(z) = \frac{\partial U}{\partial n}(\zeta) .$$

(3.12)

Furthermore, the space of such functions $U$ has the infinite dimension.

Here we have also applied the well-known fact that any rectifiable curve has a tangent a.e. with respect to the natural parameter. Moreover, note that the tangent function $\tau(s)$ to $\partial D$ is measurable with respect to the natural parameter $s$ as the derivative $d\zeta(s)/ds$ and, thus, the inner normal $n(s)$ to $\partial D$ is measurable with respect to the natural parameter, too.

Finally, the following result in arbitrary Jordan domains is a direct consequence of the relation (3.10), Theorem 3 and Remark 1.

**Theorem 6.** Let $D$ be a Jordan domain in $\mathbb{C}$, $\nu : \partial D \rightarrow \mathbb{C}$, $|\nu(\zeta)| = 1$, and $\varphi : \partial D \rightarrow \mathbb{R}$ be measurable functions with respect to harmonic measures in $D$ and let $G : D \rightarrow \mathbb{R}$ be in $L^p(D)$ for some $p > 2$.

Then there exist generalized harmonic functions $U : \mathbb{D} \rightarrow \mathbb{C}$ with the source $G$ that have the limits in the sense of the unique principal asymptotic value

$$\lim_{z \to \zeta} \frac{\partial U}{\partial \nu}(z) = \varphi(\zeta) \quad \text{for a.e.} \quad \zeta \in \partial D$$

(3.13)

with respect to the harmonic measure in $D$. Furthermore, the space of such functions $U$ has the infinite dimension.
**Remark 3.** As it follows from the proofs of Theorems 4–6, the generalized harmonic functions $U$ with a source $G \in L^p$, $p > 2$, satisfying the Poincare (Neumann) boundary conditions can be represented in the form of the sums $N_G + H$ of the logarithmic (Newtonian) potential $N_G$ that is a generalized harmonic function with the source $G$ and harmonic functions $H$ satisfying the corresponding Poincare (Neumann) boundary conditions.

The corresponding results on the boundary value problems for generalized analytic and harmonic functions with arbitrary measurable data can be also proved for the systems of arcs by Bagemihl–Seidel, see the paper [20] devoted to similar problems for analytic functions.

**ACKNOWLEDGMENTS.** This work was partially supported by grants of Ministry of Education and Science of Ukraine, project number is 0119U100421.

**References**

[1] F. Bagemihl, *Curvilinear cluster sets of arbitrary functions*, Proc. Nat. Acad. Sci. U.S.A., 41 (1955), 379–382.

[2] P.L. Duren, *Theory of $H^p$ spaces*, Pure and Applied Mathematics, 38, Academic Press, New York-London, 1970.

[3] F.W. Gehring, *On the Dirichlet problem*, Michigan Math. J., 3 (1955–1956), 201.

[4] G.M. Goluzin, *Geometric theory of functions of a complex variable*, Transl. of Math. Monographs, 26, American Mathematical Society, Providence, R.I. 1969.

[5] L. Hörmander, *The analysis of linear partial differential operators. V. I. Distribution theory and Fourier analysis*, Grundlehren der Mathematischen Wissenschaften 256, Springer-Verlag, Berlin, 1983.

[6] P. Koosis, *Introduction to $H^p$ spaces*, Cambridge Tracts in Mathematics 115, Cambridge Univ. Press, Cambridge, 1998.

[7] N.N. Luzin, *K osnovnoi theoreme integral’nogo ischisleniya [On the main theorem of integral calculus]*, Mat. Sb. 28 (1912), 266–294 (in Russian).

[8] N.N. Luzin, *Integral i trigonometriceskii ryady [Integral and trigonometric series]*, Dissertation, Moskwa, 1915 (in Russian).

[9] N.N. Luzin, *Integral i trigonometriceskii ryady [Integral and trigonometric series]*, Editing and commentary by N.K. Bari and D.E. Men’shov. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951 (in Russian).
[10] N. Luzin, *Sur la notion de l'integrale*, Annali Mat. Pura e Appl. 26, no. 3 (1917), 77-129.

[11] R. Nevanlinna, *Eindeutige analytische Funktionen*, Ann Arbor, Michigan, 1944.

[12] K. Noshiro *Cluster sets*, Springer-Verlag, Berlin etc., 1960.

[13] Ch. Pommerenke, *Boundary behaviour of conformal maps*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 299, Springer-Verlag, Berlin, 1992.

[14] I.I. Priwalow, *Randeigenschaften analytischer Funktionen*, Hochschulbücher für Mathematik 25, Deutscher Verlag der Wissenschaften, Berlin, 1956.

[15] T. Ransford, *Potential theory in the complex plane*, London Mathematical Society Student Texts 28, Cambridge University Press, Cambridge, 1995.

[16] V. Ryazanov, *On the Riemann-Hilbert problem without index*, Ann. Univ. Buchar. Math. Ser. 5(LXIII), no. 1 (2014), 169–178.

[17] V. Ryazanov, *Infinite dimension of solutions of the Dirichlet problem*, Open Math. (the former Central European J. Math.) 13, no. 1 (2015), 348–350.

[18] V. Ryazanov, *On Neumann and Poincare problems for Laplace equation*, Anal. Math. Phys. 7, no. 3 (2017), 285–289.

[19] V. Ryazanov, *The Stieltjes integrals in the theory of harmonic functions*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 467 (2018), Issledovaniya po Lineinym Operatoram i Teorii Funktsii. 46, 151–168; transl. in J. Math. Sci.

[20] V. Ryazanov, *On Hilbert and Riemann problems. An alternative approach*, Ann. Univ. Buchar. Math. Ser. 6(LXIV) (2015), no. 2, 237–244.

[21] S. Saks, *Theory of the integral*, Warsaw, 1937; Dover Publications Inc., New York, 1964.

[22] I.N. Vekua, *Generalized analytic functions*, Pergamon Press, London-Paris-Frankfurt; Addison-Wesley Publishing Co., Inc., Reading, Mass. 1962.

Institute of Applied Mathematics and Mechanics of National Academy of Sciences of Ukraine, 84100, Ukraine, Slavyansk, 1st Dobrovolskogo Str.,
Email: Ryazanov@nas.gov.ua

Bogdan Khmelnitsky National University of Cherkasy,
Physics Department, Laboratory of Mathematical Physics,
18001, Ukraine, Cherkasy, 81 Blvd. Shevchenko
Email: vl.ryazanov1@gmail.com