NON-COMMUTATIVE CHERN CHARACTERS
OF THE C*-ALGEBRAS OF SPHERES
AND QUANTUM SPHERES

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ABSTRACT. We propose in this paper the construction of non-commutative Chern characters of the C*-algebras of spheres and quantum spheres. The final computation gives us a clear relation with the ordinary \Z/(2)-graded Chern characters of tori or their normalizers.

INTRODUCTION.

For compact Lie groups the Chern character $ch : K^*(G) \otimes \mathbb{Q} \to H^*_{DR}(G; \mathbb{Q})$ were constructed. In [4] - [5] we computed the non-commutative Chern characters of compact Lie group $C^*$-algebras and of compact quantum groups, which are also homomorphisms from quantum $K$-groups into entire current periodic cyclic homology of group $C^*$-algebras (resp., of $C^*$-algebra quantum groups), $ch_{C^*} : K_*(C^*(G)) \to HE_*(C^*(G))$, (resp., $ch_{C^*} : K_*(C^*_\varepsilon(G)) \to HE_*(C^*_\varepsilon(G))$). We obtained also the corresponding algebraic version $ch_{alg} : K_*(C^*(G)) \to HP_*(C^*(G))$, which coincides with the Fedosov-Cuntz-Quillen formula for Chern characters [5]. When $A = C^*_\varepsilon(G)$ we first computed the $K$-groups of $C^*_\varepsilon(G)$ and the $HE_*(C^*_\varepsilon(G))$. Thereafter we computed the Chern charactor $ch_{C^*} : K_*(C^*_\varepsilon(G)) \to HE_*(C^*_\varepsilon(G)$ as an isomorphism modulo torsions.

Using the results from [4] - [5], in this paper we compute the non-commutative Chern characters $ch_{C^*} : K_*(A) \to HE_*(A)$, for two cases $A = C^*(S^n)$, the $C^*$-algebra of spheres and $A = C^*_\varepsilon(S^n)$, the $C^*$-algebras of quantum spheres. For compact groups $G = O(n+1)$, the Chern character $ch : K_*(S^n) \otimes \mathbb{Q} \to H^*_{DR}(S^n; \mathbb{Q})$ of the sphere $S^n = O(n+1)/O(n)$ is an isomorphism (see, [15]). In the paper, we describe two Chern character homomorphisms

$$ch_{C^*} : K_*(C^*(S^n)) \to HE_*(C^*(S^n))$$

and

$$ch_{C^*} : K_*(C^*_\varepsilon(S^n)) \to HE_*(C^*_\varepsilon(S^n)).$$
Finally, we show that there is a commutative diagram

\[
\begin{array}{ccc}
K_\ast(C^\ast(S^n)) & \xrightarrow{ch_{C^\ast}} & HE_\ast(C^\ast(S^n)) \\
\downarrow \cong & & \downarrow \cong \\
K_\ast(C(N_{T_n})) & \xrightarrow{ch_{CQ}} & HE_\ast(C(N_{T_n})) \\
\downarrow \cong & & \downarrow \cong \\
K^\ast(N_{T_n}) & \xrightarrow{ch} & H^\ast_{DR}(N_{T_n})
\end{array}
\]

(Similarly, for \( A = C^\ast_\varepsilon(S^n) \), we have an analogous commutative diagram with \( W \times S^1 \) of place of \( W \times S^n \)), from which we deduce that \( ch_{C^\ast} \) is an isomorphism modulo torsions.

We now briefly review the structure of the paper. In section 1, we compute the Chern character of the \( C^\ast \)-algebras of spheres. The computation of Chern character of \( C^\ast(S^n) \) is based in two crucial points:

i) Because the sphere \( S^n = O(n + 1)/O(n) \) is a homogeneous space and \( C^\ast \)-algebra of \( S^n \) is the transformation group \( C^\ast \)-algebra, following J. Parker [10], we have, \( C^\ast(S^n) \cong C^\ast(O(n)) \otimes K(L^2(S^n)) \).

ii) Using the stability property theorem \( K_\ast \) and \( HE_\ast \) in [5], we reduce it to the computation of \( C^\ast \)-algebras of subgroup \( O(n) \) in \( O(n+1) \) group.

In section 2, we compute the Chern character of \( C^\ast \)-algebras of quantum spheres. For quantum sphere \( S^n \), we define the compact quantum \( C^\ast \)-algebra \( C^\ast_\varepsilon(S^n) \), where \( \varepsilon \) is a positive real number. Thereafter, we prove that

\[
C^\ast_\varepsilon(S^n) \cong C(S^1) \oplus \bigoplus_{\varepsilon \neq \omega \in W} \int_{S^1} \mathcal{K}(H_{w,t})dt,
\]

where \( \mathcal{K}(H_{w,t}) \) is the elementary algebra of compact operators in a separable infinite dimensional Hilbert space \( H_{w,t} \) and \( W \) is the Weyl of a maximal torus \( T_n \) in \( SO(n) \).

Similar to section 1, we first compute the \( K_\ast(C^\ast_\varepsilon(S^n)) \) and \( HE_\ast(C^\ast_\varepsilon(S^n)) \), and we prove that \( ch_{C^\ast} : K_\ast(C^\ast_\varepsilon(S^n)) \rightarrow HE_\ast(C^\ast_\varepsilon(S^n)) \) is an isomorphism modulo torsions.

**Notes on Notation:** For any compact space \( X \), we write \( K^\ast(X) \) for the \( \mathbb{Z}/(2) \)-graded topological \( K \)-theory of \( X \). We use Swan’s theorem to identify \( K^\ast(X) \) with \( \mathbb{Z}/(2) \)-graded \( K_\ast(C(X)) \). For any involutive Banach algebra \( A \), \( K_\ast(A) \), \( HE_\ast(A) \), \( HP_\ast(A) \) are \( \mathbb{Z}/(2) \)-graded algebraic or topological \( K \)-groups of \( A \), entire cyclic homology, and periodic cyclic homology of \( A \), respectively. If \( T \) is a maximal torus of a compact group \( G \), with the corresponding Weyl group \( W \), write \( C(T) \) for the algebra of complex valued functions on \( T \). We use the standard notations from the root theory such as \( P, P^+ \) for the positive highest weights, etc... We denote by \( N_T \) the normalizer of \( T \) in \( G \), by \( N \) the set of natural numbers, \( \mathbb{R} \) the field of real numbers and \( \mathbb{C} \) the field of complex numbers, \( \ell^2_A(N) \) the standard \( \ell^2 \) space of square integrable sequences of elements from \( A \), and finally by \( C^\ast_\varepsilon(G) \) we denote the compact quantum algebras, \( C^\ast(G) \) the \( C^\ast \)-algebra of \( G \).
§1. Non-commutative Chern characters of $C^*$-algebras of spheres.

In this section, we compute non-commutative Chern characters of $C^*$-algebras of spheres. Let $A$ be an involutive Banach algebra. We construct the non-commutative Chern characters $ch_{C^*}: K_*(A) \to HE_*(A)$, and show in [4] that for $C^*$-algebra $C^*(G)$ of compact Lie groups $G$, the Chern character $ch_{C^*}$ is an isomorphism.

**Proposition 1.1** ([5], Theorem 2.6). Let $H$ be a separable Hilbert space and $B$ an arbitrary Banach space. We have

1) $K_*(K(H)) \cong K_*(C)$
2) $K_*(B \otimes K(H)) \cong K_*(B)$
3) $HE_*(K(H)) \cong HE_*(C)$
4) $HE_*(B \otimes K(H)) \cong HE_*(B)$,

where $K(H)$ is the elementary algebra of compact operators in a separable infinite-dimensional Hilbert space $H$.

**Proposition 1.2.** ([5], Theorem 3.1). Let $A$ be an involutive Banach algebra with unity. There is a Chern character homomorphism $ch_{C^*}: K_*(A) \to HE_*(A)$.

**Proposition 1.3.** ([5], Theorem 3.2). Let $G$ be an compact group and $T$ a fixed maximal torus of $G$ with Weyl group $W := N_T/T$. Then the Chern character $ch_{C^*}: K_*(C^*(G)) \to HE_*(C^*(G))$ is an isomorphism modulo torsions, i.e.

$$ch_{C^*}: K_*(C^*(G)) \otimes C \xrightarrow{\cong} HE_*(C^*(G)),$$

which can be identified with the classical Chern character

$$ch : K_*(C(N_T)) \to HE_*(C(N_T)),$$

that is also an isomorphism modulo torsions, i.e.

$$ch : K^*(N_T) \otimes C \xrightarrow{\cong} H^*_D(N_T).$$

Now, for $S^n = O(n+1)/O(n)$, where $O(n)$, $O(n+1)$ are the orthogonal matrix groups. We denote by $T_n$ a fixed maximal torus of $O(n)$ and $N_{T_n}$ the normalizer of $T_n$ in $O(n)$. Following Proposition 1.2, there a natural Chern character $ch_{C^*}: K_*(C^*(S^n)) \to HE_*(C^*(S^n))$. Now, we compute first $K_*(C^*(S^n))$ and then $HE_*(C^*(S^n))$ of $C^*$-algebra of the sphere $S^n$.

**Proposition 1.4.**

$$HE_*(C^*(S^n)) \cong H^W_D(T_n)$$

**Proof:** We have

$$HE_*(C^*(S^n)) = HE_*(C^*(O(n+1)/O(n)))$$

$$\cong HE_*(C^*(O(n)) \otimes K(L^2(O(n+1)/O(n))))$$
Moreover, by a result of Khalkhali [8] - [9], we have

$$H \cong HE_\ast(C^\ast(O(n))) \quad \text{(by Proposition 1.1)}$$

$$H \cong HE_\ast(CN_{T_n}) \quad \text{(see [5]).}$$

Thus, we have $HE_\ast(C^\ast(S^n)) \cong HE_\ast(CN_{T_n})$.

Apart from that, because $CN_{T_n}$ is the commutative $C^\ast$-algebra, by a result Cuntz-Quillen’s [1], we have an isomorphism

$$HP_\ast(CN_{T_n}) \cong H_{DR}^\ast(N_{T_n}).$$

Moreover, by a result of Khalkhali [8] - [9], we have

$$HP_\ast(CN_{T_n}) \cong HE_\ast(CN_{T_n}).$$

We have, hence

$$HE_\ast(C^\ast(S^n)) \cong HE_\ast(CN_{T_n}) \cong HP_\ast(CN_{T_n})$$

$$\cong H_{DR}^\ast(N_{T_n}) \cong H_{DR}^W(T_n) \quad \text{(by [15]).}$$

**Remark 1.** Because $H_{DR}^W(T_n)$ is the de Rham cohomology of $T_n$, invariant under the action of the Weyl group $W$, following Watanabe [15], we have a canonical isomorphism $H_{DR}^W(T_n) \cong H^\ast(SO(n)) = \Lambda_C(x_3, x_7, \ldots, x_{2i+3})$, where $x_{2i+3} = \sigma(p_i) \in H^{2n+3}(SO(n))$ and $\sigma : H^\ast(BSO(n), R) \to H^\ast(SO(n), R)$ for a commutative ring $R$ with a unit $1 \in R$, and $p_i = \sigma_i(t_1^2, t_2^2, \ldots, t_i^2) \in H^\ast(BT_n, Z)$ the Pontryagin classes.

Thus, we have

$$HE_\ast(C^\ast(S^n)) \cong \Lambda_C(x_3, x_7, \ldots, x_{2i+3}).$$

**Proposition 1.5.**

$$K_\ast(C^\ast(S^n)) \cong K_\ast(N_{T_n}).$$

**Proof.** We have

$$K_\ast(C^\ast(S^n)) = K_\ast(C^\ast(O(n + 1)/O(n)))$$

$$\cong K_\ast(C^\ast(O(n)) \otimes K(L^2(O(n + 1)/O(n)))) \quad \text{(see [10])}$$

$$\cong K_\ast(C^\ast(O(n))) \quad \text{(by Proposition 1.1)}$$

$$\cong K_\ast(CN_{T_n})$$

$$\cong K^\ast(N_{T_n}) \quad \text{(by Lemma 3.3, from [5]).}$$

Thus,

$$K_\ast(C^\ast(S^n)) \cong K^\ast(N_{T_n}).$$

**Remark 2.** Following Lemma 4.2 from [5], we have

$$K^\ast(N_{T_n}) \cong K^\ast(SO(n + 1)) / \text{torsion}$$

$$= \Lambda_Z(\beta(\lambda_1), \ldots, \beta(\lambda_{n-3}), \varepsilon_{n+1}),$$
where $\beta : R(SO(n)) \to \tilde{K}^{-1}(SO(n))$ be the homomorphism of Abelian groups assigning to each representation $\rho : SO(n) \to U(n+1)$ the homotopy class $\beta(\rho) = [i_n \rho] \in [SO(n), U] = \tilde{K}^{-1}(SO(n))$, where $i_n : U(n+1) \to U$ is the canonical one, $U(n+1)$ and $U$ be the $n$–th and infinite unitary groups respectively and $\varepsilon_{n+1} \in K^{-1}(SO(n+1))$. We have, finally

$$K_*(C^*(S^n)) \cong \Lambda_2(\beta(\lambda_1), ..., \beta(2n-3), \varepsilon_{n+1}).$$

Moreover, the Chern character of $SU(n+1)$ was computed in [14], for all $n \geq 1$. Let us recall the result. Define a function $\phi : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{Z},$ given by

$$\phi(n, k, q) = \sum_{i=1}^{k} (-1)^{i-1} \binom{n}{k-1} i^q - 1.$$

**Theorem 1.6.** Let $T_n$ be a fixed maximal torus of $O(n)$ and $T$ the fixed maximal torus of $SO(n)$, with Weyl groups $W : = \mathcal{N}_T / T$, the Chern character of $C^*(S^n)$

$$ch_{C^*} : K_*(C^*(S^n)) \to HE_*(C^*(S^n))$$

is an isomorphism, given by

$$ch_{C^*} (\beta(\lambda_k)) = \sum_{i=1}^{n} ((-1)^{i-1} 2/(2i - 1)! ) \phi(2n + 1, k, 2i) x_{2i+3}, \ (k = 1, 2, ..., n-1)$$

$$ch_{C^*} (\varepsilon_{n+1}) = \sum_{i=1}^{n} ((-1)^{i-1} 2/(2i - 1)! ) ((1/2^n) \sum_{k=1}^{n} \phi(2n + 1, k, 2i)) x_{2i+3}.$$

**Proof.** By Proposition 1.5, we have

$$K_*(C^*(S^n)) \cong K_*(\mathcal{C}(N_{T_n})) \cong K^*(N_{T_n})$$

and

$$HE_*(C^*(S^n)) \cong HE_*(\mathcal{C}(N_{T_n})) \cong H_{DR}^*(N_{T_n}) \quad \text{ (by Proposition 1.4)}.$$
Moreover, by the results of Watanabe [15], the Chern character \( ch : K^*(\mathcal{N}_{
abla, n}) \otimes \mathbb{C} \rightarrow H^*_DR(\mathcal{N}_{\nabla, n}) \) is an isomorphism.

Thus, \( ch_{C^*} : K_*(C^*(S^n)) \rightarrow HE_*(C^*(S^n)) \) is an isomorphism (by Proposition 1.4 and 1.5), given by

\[
ch_{C^*}(\lambda_k) = \sum_{i=1}^{n} ((-1)^{i-1} 2/(2i-1)! \phi(2n+1, k, 2i)) x_{2i+3}, \quad (k = 1, 2, ..., n-1),
\]

\[
ch_{C^*}(\varepsilon_{n+1}) = \sum_{i=1}^{n} ((-1)^{i-1} 2/(2i-1)! \left(1/2^n \right) \phi(2n+1, k, 2i)) x_{2i+3},
\]

where:

\[
K_*(C^*(S^n)) \cong \Lambda^\mathbb{Z}(\lambda_1, ..., \lambda_{n-3}, \varepsilon_{n+1}),
\]

\[
HE_*(C^*(S^n)) \cong \Lambda^\mathbb{C}(x_3, x_7, ..., x_{2n+3}).
\]

\[\square\]

§2. Non-Commutative Chern character of \( C^* \)-algebra of quantum spheres.

In this section, we at first recall definitions and main properties of compact quantum spheres and their representations. More precisely, for \( S^n \), we define \( C^*_\varepsilon(S^n) \), the \( C^* \)-algebras of compact quantum spheres as the \( C^* \)-completion of the *-algebra \( \mathcal{F}_\varepsilon(S^n) \) with respect to the \( C^* \)-norm, where \( \mathcal{F}_\varepsilon(S^n) \) is the quantized Hopf subalgebra of the Hopf algebra, dual to the quantized universal enveloping algebra \( U(\mathcal{G}) \), generated by matrix elements of the \( U(\mathcal{G}) \) modules of type \( 1 \) (see [3]). We prove that

\[
C^*_\varepsilon(S^n) \cong C(S^1) \oplus \bigoplus_{e \neq \omega \in W} \int_{S^1} \mathcal{K}(H_{w,t}) dt,
\]

where \( \mathcal{K}(H_{w,t}) \) is the elementary algebra of compact operators in a separable infinite-dimensional Hilbert space \( H_{w,t} \) and \( W \) is the Weyl group of \( S^n \) with respect to a maximal torus \( \mathcal{T} \).

After that, we first compute the K-groups \( K_*(C^*_\varepsilon(S^n)) \) and the \( HE_*(C^*_\varepsilon(S^n)) \), respectively. Thereafter we define the Chern character of \( C^* \)-algebras quantum spheres, as a homomorphism from \( K_*(C^*_\varepsilon(S^n)) \) to \( HE_*(C^*_\varepsilon(S^n)) \), and we prove that \( ch_{C^*} : K_*(C^*_\varepsilon(S^n)) \rightarrow HE_*(C^*_\varepsilon(S^n)) \) is an isomorphism modulo torsions.

Let \( G \) be a complex algebraic group with Lie algebra \( \mathcal{G} = \text{Lie} \ G \) and \( \varepsilon \) is real number, \( \varepsilon \neq -1 \).

**Definition 2.1.** ([3], Definition 13.1). The quantized function algebra \( \mathcal{F}_\varepsilon(G) \) is the subalgebra of the Hopf algebra dual to \( U_\varepsilon(G) \), generated by the matrix elements of the finite-dimensional \( U_\varepsilon(G) \)-modules of type \( 1 \).

For compact quantum groups the unitary representation of \( \mathcal{F}_\varepsilon(G) \) are parameterized by pairs \((w, t)\), where \( t \) is an element of a fixed maximal torus of the compact real form of \( G \) and \( w \) is an element of the Weyl group \( W \) of \( \mathcal{T} \) in \( G \).

Let \( \lambda \in P^+, V_\varepsilon(\lambda) \) be the irreducible \( U_\varepsilon(G) \)-module of type \( 1 \) with the highest weight \( \lambda \). Then \( V_\varepsilon(\lambda) \) admits a positive definite hermitian form \((.,.)\), such that
Proposition 2.3. (\[3\], 13.1.9). Let $\{v_\mu^\nu\}$ be an orthogonal basis for weight space $V_\mu^\nu(\lambda)$, $\mu \in P^+$. Then $\bigcup \{v_\mu^\nu\}$ is an orthogonal basis for $V_\mu^\nu(\lambda)$. Let $C^\lambda_{\nu,s;\mu,r}(x) = (xv_\mu^\nu, v_\mu^s)$ be the associated matrix elements of $V_\mu^\nu(\lambda)$. Then the matrix elements $C^\lambda_{\mu,s;\mu,r}$ (where $\lambda$ runs through $P^+$, while $(\mu,r)$ and $(\nu,s)$ runs independently through the index set of a basis of $V_\mu^\nu(\lambda)$) form a basis of $\mathcal{F}_\mu^\nu(G)$ (see \[3\]).

Now very irreducible *-representation of $\mathcal{F}_\mu^\nu(SL_2(\mathbb{C}))$ is equivalent to a representation belonging to one of the following two families, each of which is parameterized by $S^1 = \{t \in \mathbb{C} | |t| = 1\}$,

i) the family of one-dimensional representation $\pi_t$

ii) the family $\pi_t$ of representation in $\ell^2(\mathbb{N})$ (see \[3\])

Moreover, there exists a surjective homomorphism $\mathcal{F}_\mu^\nu(G) \rightarrow \mathcal{F}_\mu^\nu(SL_2(\mathbb{C}))$ induced by the natural inclusion $SL_2(\mathbb{C}) \hookrightarrow G$ and by composing the representation $\pi_{-1}$ of $\mathcal{F}(SL_2(\mathbb{C}))$ with this homomorphism, we obtain a representation of $\mathcal{F}_\mu^\nu(G)$ in $\ell^2(\mathbb{N})$ denoted by $\pi_{s_i}$, where $s_i$ appears in the reduced decomposition $w = s_{i_1}s_{i_2}\ldots s_{i_k}$. More precisely, $\pi_{s_i} : \mathcal{F}_\mu^\nu(G) \rightarrow \mathcal{L}(\ell^2(\mathbb{N}))$ is of class CCR (see \[11\]), i.e its image is dense in the ideal of compact operators in $\mathcal{L}(\ell^2(\mathbb{N}))$.

The representation $\pi_t$ is one-dimensional and is of the form

$$
\pi_t(C^\lambda_{\nu,s;\mu,r}(x)) = \delta_{r,s}\delta_{\mu,\nu} \exp(2\pi\sqrt{-1}\mu(x)),
$$

if $t = \exp(2\pi\sqrt{-1}x) \in T$, for $x \in \text{Lie } T$, (see \[3\]).

Proposition 2.2 (\[3\], 13.1.7). Every irreducible unitary representation of $\mathcal{F}_\mu^\nu(G)$ on a separable Hilbert space is the completion of a unitarizable highest weight representation. Moreover, two such representations are equivalent if and only if they have the same highest weight. \hfill \Box

Proposition 2.3. (\[3\], 13.1.9). Let $\omega = s_{i_1}s_{i_2}\ldots s_{i_k}$ be a resuced decomposition of an element $w$ of the Weyl group $W$ of $G$. Then

i) the Hilbert space tensor product $\rho_{w,t} = \pi_{s_{i_1}} \otimes \pi_{s_{i_2}} \otimes \ldots \otimes \pi_{s_{i_k}} \otimes \pi_t$ is an irreducible *-representation of $\mathcal{F}_\mu^\nu(G)$ which is associated to the Schubert cell $S_w$;

ii) up to equivalence, the representation $\rho_{w,t}$ does not depend on the choice of the reduced decomposition of $w$;

iii) every irreducible *-representation of $\mathcal{F}_\mu^\nu(G)$ is equivalent to some $\rho_{w,t}$. \hfill \Box

The sphere $S^n$, can be realized as the orbit under the action of the compact group $SU(n+1)$ of the highest weight vector $v_0$ in its natural $(n+1)$-dimensional representation $V^h$ of $SU(n+1)$. If $t_{rs}, 0 \leq r, s, \leq n$, are the matrix entries of $V^h$, the algebra of functions on the orbit is generated by the entries in the “first column” $t_{s0}$ and their complex conjugates. In fact,

$$
\mathcal{F}(S^n) := \mathbb{C}[t_{00}, \ldots , t_{n0}, \bar{t}_{00}, \ldots , \bar{t}_{n0}] / \sim,
$$

where $\sim$ is the following equivalence relation

$$
t_{s0}\bar{t}_{s0} \iff \sum_{s=0}^n t_{s0}\bar{t}_{s0} = 1.
$$
Proposition 2.4. ([3], 13.2.6). The *-structure on Hopf algebra $\mathcal{F}_\varepsilon(SL_{n+1}(\mathbb{C}))$, is given by
\[
t_{rs}^* = (-\varepsilon)^{r-s}q \det(\hat{T}_{rs}),
\]
where $\hat{T}_{rs}$ is the matrix obtained by removing the $r^{th}$ row and the $s^{th}$ column from $T$.

Definition 2.5. ([3], 13.2.7). The *-subalgebra of $\mathcal{F}_\varepsilon(SL_{n+1}(\mathbb{C}))$ generated by the elements $t_{s0}$ and $t_{s0}^*$, for $s = 0, \ldots, n$, is called the quantized algebra of functions on the sphere $S^n$, and is denoted by $\mathcal{F}_\varepsilon(S^n)$. It is a quantum $SL_{n+1}(\mathbb{C})$-space.

We set $z_s = t_{s0}$ from now on. Using Proposition 2.4, it is easy to see that the following relations hold in $\mathcal{F}_\varepsilon(S^n)$:
\[
\begin{cases}
z_r z_s = \varepsilon^{-1} z_s z_r & \text{if } r < s \\
z_r z_s^* = -\varepsilon^{-1} z_s^* z_r & \text{if } r \neq s \\
z_r z_s^* - z_r^* z_r + (\varepsilon^{-2} + 1) \sum_{s > r} z_s z_s^* = 0, & (*) \\
\sum_{s=0}^{n} z_s z_s^* = 0.
\end{cases}
\]

Hence, $\mathcal{F}_\varepsilon(S^n)$ has (*) as its defining relations. The construction of irreducible *-representations of $\mathcal{F}_\varepsilon(S^n)$, is given by

Theorem 2.6. ([3], 13.2.9). Every irreducible *-representation of $\mathcal{F}_\varepsilon(S^n)$ is equivalent exactly to one of the following:

i) the one-dimensional representation $\rho_{0,t}$, $t \in S^1$, given by $\rho_{0,t}(z_0^*) = t^{-1}, \rho_{0,t}(z_r^*) = 0$ if $r > 0$.

ii) the representation $\rho_{r,t}$, $1 \leq r \leq n$, $t \in S^1$, on the Hilbert space tensor product $\ell^2(\mathbb{N})^\otimes r$, given by
\[
\rho_{r,t}(z_s^*)(e_{k_1} \otimes \cdots \otimes e_{k_r}) =
\begin{cases}
\varepsilon^{-(k_1 + \cdots + k_s + s)(1 - \varepsilon^{-2(k_s+1)+1})/2} e_{k_1} \otimes \cdots \otimes e_{k_s} \otimes e_{k_{s+1}} + 1 \otimes e_{k_{s+2}} & \text{if } s < r \\
0 & \text{if } s > r \\
t^{-1} \varepsilon^{-(k_1 + \cdots + k_r + r)} e_{k_1} \otimes \cdots \otimes e_{k_r} & \text{if } r = s
\end{cases}
\]

The representation $\rho_{0,t}$ is equivalent to the restriction of the representation $\mathcal{T}_t$ of $\mathcal{F}_\varepsilon(SL_{n+1}(\mathbb{C}))$ (cf. 2.3); and for $r > 0, \rho_{r,t}$ is equivalent to the restriction of $\pi_{s_1} \otimes \cdots \otimes \pi_{s_r} \otimes \mathcal{T}_t$. $\square$

From Theorem 2.6, we have
\[
\bigcap_{(w,t) \in W \times T} \ker \rho_{w,t} = \{e\},
\]
i.e. the representation $\bigoplus_{w \in W} \int_T \rho_{w,t} dt$ is faithful and
\[
\dim \rho_{w,t} =
\begin{cases}
1 & \text{if } w = e \\
0 & \text{if } w \neq e.
\end{cases}
\]
We recall now the definition of compact quantum of spheres $C^*$-algebra.

**Definition 2.7.** The $C^*$-algebraic compact quantum sphere $C^*_\varepsilon(S^n)$ is the $C^*$

completion of the *-algebra $\mathcal{F}_\varepsilon(S^n)$ with respect to the $C^*$-norm

$$
\|f\| = \sup_\rho \|\rho(f)\| \quad (f \in \mathcal{F}_\varepsilon(S^n)),
$$

where $\rho$ runs through the *-representations of $\mathcal{F}_\varepsilon(S^n)$ (cf., Theorem 2.6) and the norm on the right-hand side is the operator norm.

It suffices to show that $\|f\|$ is finite for all $f \in \mathcal{F}_\varepsilon(S^n)$, for it is clear that $\|\cdot\|$ is a $C^*$-norm, i.e. $\|f^*f\| = \|f\|^2$. We now prove the following result about the structure of compact quantum $C^*$-algebra of sphere $S^n$.

**Theorem 2.8.** With notation as above, we have

$$
C^*_\varepsilon \cong \mathcal{C}(S^1) \oplus \bigoplus_{e \neq w \in W} \int_{S^1} \mathcal{K}(H_{w,t}) dt,
$$

where $\mathcal{C}(S^1)$ is the algebra of complex valued continuous functions on $S^1$ and $\mathcal{K}(H)$ the ideal of compact operators in a separable Hilbert space $H$.

**Proof:** Let $w = s_{i_1}s_{i_2}\ldots s_{i_k}$ be a reduced decomposition of the element $w \in W$ into a product of reflections. Then by Proposition 2.6, for $r > 0$, the representation $\rho_{w,t}$ is equivalent to the restriction of $\pi_{s_{i_1}} \otimes \pi_{s_{i_2}} \otimes \ldots \otimes \pi_{s_{i_k}} \otimes T_t$, where $\pi_{s_i}$ is the composition of the homomorphism of $\mathcal{F}_\varepsilon(G)$ onto $\mathcal{F}_\varepsilon(SL_2(\mathbb{C}))$ and the representation $\pi_{-1}$ of $\mathcal{F}_\varepsilon(SL_2(\mathbb{C}))$ in the Hilbert space $\ell^2(\mathbb{N})^\otimes r$; and the family of one-dimensional representations $T_t$, given by

$$
T_t(a) = t, \quad T_t(b) = T_t(c) = 0, \quad T_t(d) = t^{-1},
$$

where $t \in S^1$ and $a, b, c, d$ are given by: Algebra $\mathcal{F}_\varepsilon(SL_2(\mathbb{C}))$ is generated by the matrix elements of type $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Hence, by construction, the representation $\rho_{w,t} = \pi_{s_{i_1}} \otimes \pi_{s_{i_2}} \otimes \ldots \otimes \pi_{s_{i_k}} \otimes T_t$. Thus, we have

$$
\pi_{s_i} : C^*_\varepsilon(S^n) \longrightarrow C^*_\varepsilon(SL_2(\mathbb{C})) \xrightarrow{\pi_{-1}} \mathcal{L}(\ell^2(\mathbb{N})^\otimes r).
$$

Now, $\pi_{s_i}$ is CCR (see, [11]) and so, we have $\pi_{s_i}(C^*_\varepsilon(S^n)) \cong \mathcal{K}(H_{w,t})$. Moreover $T_t(C^*_\varepsilon(S^n)) \cong \mathbb{C}$.

Hence,

$$
\rho_{w,t}(C^*_\varepsilon(S^n)) = \left(\pi_{s_{i_1}} \otimes \ldots \otimes \pi_{s_{i_k}} \otimes T_t\right)(C^*_\varepsilon(S^n))
$$

$$
= \pi_{s_{i_1}}(C^*_\varepsilon(S^n)) \otimes \ldots \otimes \pi_{s_{i_k}}(C^*_\varepsilon(S^n)) \otimes T_t(C^*_\varepsilon(S^n))
$$

$$
\cong \mathcal{K}(H_{s_{i_1}}) \otimes \ldots \otimes \mathcal{K}(H_{s_{i_k}}) \otimes \mathbb{C}
$$

$$
\cong \mathcal{K}(H_{w,t}),
$$

where $H_{w,t} = H_{s_{i_1}} \otimes \ldots \otimes H_{s_{i_k}} \otimes \mathbb{C}$. 


Thus, \( \rho_{w,t}(C_{\mathbb{C}}^*(S^n)) \cong \mathcal{K}(H_{w,t}) \).

Hence, \( \bigoplus_{w \in W} \int_{S^1} \rho_{w,t}(C_{\mathbb{C}}^*(S^n)) \cong \bigoplus_{w \in W} \int_{S^1} \mathcal{K}(H_{w,t}) \, dt \).

Now, recall a result of S. Sakai’s from [11]: Let \( A \) be a commutative \( C^* \)-algebra and \( B \) be a \( C^* \)-algebra. Then, \( C_0(\Omega, B) \cong A \otimes B \), where \( \Omega \) is the spectrum space of \( A \).

Applying this result, for \( B = \mathcal{K}(H_{w,t}) \) and \( A = \mathbb{C}(W \times S^1) \) be a commutative \( C^* \)-algebra. Thus, we have

\[
C_{\mathbb{C}}^*(S^n) \cong \mathbb{C}(S^1) \bigoplus \bigoplus_{e \neq w \in W} \int_{S^1} \mathcal{K}(H_{w,t}) \, dt.
\]

Now, we first compute the \( K^*_*(C_{\mathbb{C}}^*(S^n)) \) and the \( HE^*_*(C_{\mathbb{C}}^*(S^n)) \) of \( C^* \)-algebra of quantum sphere \( S^n \).

**Proposition 2.9.**

\[
HE^*_*(C_{\mathbb{C}}^*(S^n)) \cong H^*_{DR}(W \times S^1).
\]

**Proof.** We have

\[
HE^*_*(C_{\mathbb{C}}^*(S^n)) = HE^*_*(\mathbb{C}(S^1) \bigoplus \bigoplus_{e \neq w \in W} \int_{S^1} \mathcal{K}(H_{w,t}) \, dt)
\]

\[
\cong HE^*_*(\mathbb{C}(S^1)) \bigoplus HE^*_*(\bigoplus_{e \neq w \in W} \int_{S^1} \mathcal{K}(H_{w,t}) \, dt)
\]

\[
\cong HE^*_*(\mathbb{C}(W \times S^1) \otimes \mathcal{K}) \quad \text{(by Proposition 1.1 §1)}
\]

\[
\cong HE^*_*(\mathbb{C}(W \times S^1)).
\]

Since \( \mathbb{C}(W \times S^1) \) is a commutative \( C^* \)-algebra, by Proposition 1.5, §1, we have

\[
HE^*_*(C_{\mathbb{C}}^*(S^n)) \cong HE^*_*(\mathbb{C}(W \times S^1)) \cong H^*_{DR}(W \times S^1)
\]

**Proposition 2.10.**

\[
K^*_*(C_{\mathbb{C}}^*(S^n)) \cong K^*(W \times S^1).
\]

**Proof.** We have

\[
K^*_*(C_{\mathbb{C}}^*(S^n)) = K^*_*(\mathbb{C}(S^1) \bigoplus \bigoplus_{e \neq w \in W} \int_{S^1} \mathcal{K}(H_{w,t}) \, dt)
\]

\[
\cong K^*_*(\mathbb{C}(S^1)) \bigoplus K^*_*(\bigoplus_{e \neq w \in W} \int_{S^1} \mathcal{K}(H_{w,t}) \, dt)
\]

\[
\cong K^*_*(\mathbb{C}(W \times S^1) \otimes \mathcal{K})
\]

\[
\cong K^*_*(\mathbb{C}(W \times S^1)) \quad \text{(by proposition 1.1 §1)}.
\]
In virtue of Proposition 1.5, §1, we have
\[ K_\ast(C(W \times S^1)) \cong K^\ast(W \times S^1) \]

**Theorem 2.11.** With notation above, the Chern character of \( C^\ast \)-algebra of quantum sphere \( C^\ast_\varepsilon(S^n) \)
\[ ch_{C^\ast} : K_\ast(C^\ast_\varepsilon(S^n)) \longrightarrow HE_\ast(C^\ast_\varepsilon(S^n)) \]
is an isomorphism.

**Proof.** By Proposition 2.9 and 2.10, we have:
\[ HE_\ast(C^\ast_\varepsilon(S^n)) \cong HE_\ast(C(W \times S^1)) \cong H_{DR}^\ast(W \times S^1), \]
\[ K_\ast(C^\ast_\varepsilon(S^n)) \cong K_\ast(C(W \times S^1)) \cong K^\ast(W \times S^1). \]

Now, consider the commutative diagram
\[
\begin{array}{ccc}
K_\ast(C^\ast_\varepsilon(S^n)) & \xrightarrow{ch_{C^\ast}} & HE_\ast(C^\ast_\varepsilon(S^n)) \\
\| & & \| \\
K_\ast(C(W \times S^1)) & \xrightarrow{ch_{CQ}} & HE_\ast(C(W \times S^1)) \\
\| & & \| \\
K^\ast(W \times S^1) & \xrightarrow{ch} & H_{DR}^\ast(W \times S^1)
\end{array}
\]

Moreover, following Watanabe [15], the \( ch : K^\ast(W \times S^1) \otimes \mathbb{C} \longrightarrow H_{DR}^\ast(W \times S^1) \)
is an isomorphism.

Thus, \( ch_{C^\ast} : K^\ast(C^\ast_\varepsilon(S^n)) \longrightarrow HE_\ast(C^\ast_\varepsilon(S^n)) \) in an isomorphism.

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