ON THE THEORY OF POLARIZATION OF GENERALIZED NIJENHUIS TORSIONS

PIERGIULIO TEMPESTA AND GIORGIO TONDO

Abstract. The theory of generalized Nijenhuis torsions, recently introduced, offers new powerful tools to detect the Frobenius integrability of operator fields on a differentiable manifold.

In this work, we introduce the notion of polarization of generalized Nijenhuis torsion and establish several algebraic identities. We also prove that this new vector-valued 2-form is relevant in the characterization of $C^\infty$ Haantjes moduli of operators.

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1. INTRODUCTION

In the last years, the theory of Haantjes operators, namely (1,1)-tensor fields having a vanishing Haantjes torsion [12], has experienced a resurgence of interest. New interesting applications have been found, for instance in the theory of infinite-dimensional integrable systems [17]-[19] and of hydrodynamic-type systems [10].

Recently, Haantjes operators have been related with Hamiltonian integrable and superintegrable models and with the classical problem of separation of variables in Hamiltonian mechanics [26], [24].

In [27] the notion of Haantjes algebra has been introduced. Its relevance is due to the fact that, in the semisimple, Abelian case, one can construct a coordinate chart in which all of the operators of the algebra can diagonalize simultaneously.

Our theory of generalized Nijenhuis torsions of level $m$ has been introduced in [27] and further studied in [28]. In particular, it was proved that the vanishing of a generalized Nijenhuis torsion of a given operator field on a manifold is sufficient to guarantee the mutual integrability of the eigen-distributions of the operator. This
means that each eigendistribution as well as each direct sum of eigendistributions is Frobenius integrable. This result widely generalizes the standard integrability theorems by Nijenhuis [22] and Haantjes [12] to a much wider class of operators. In addition, we proved that coordinate charts exist allowing us to represent the operator in a block-diagonal form.

In [25], generalized Haantjes algebras of level \( m \) have been introduced, namely algebras of operators whose generalized Nijenhuis torsion of a given level \( m \) vanishes. One can ensure, under some technical conditions, the simultaneous block-diagonalization (in suitable coordinate charts) of the operators forming a generalized Haantjes algebra.

The aim of this work is to introduce the notion of polarization of a Nijenhuis torsion of level \( m \). In the spirit of the theory of polarization of homogeneous forms, our construction allows us to define a new vector-valued 2-form, depending on a set of operators \( \{ A_1, \ldots, A_{2m} \} \), \( m \geq 1 \). For \( m = 1 \), it coincides with the standard Frölicher-Nijenhuis bracket. Also, our polarization form can be specialized to recover several algebraic identities.

A direct algebraic application of the polarization of Nijenhuis torsions concerns the theory of Haantjes moduli, namely \( \mathcal{C}^\infty \)-moduli of operators having vanishing Haantjes torsion. Precisely, our main result is Theorem 3.12, stating that the vanishing of the polarization of a generalized Nijenhuis torsion of level \( m \), evaluated on suitable arguments, is necessary and sufficient for the existence of an Abelian generalized Haantjes modulus of the same level.

More important, the theory of polarization forms that we establish in this work offers a natural and unified language allowing us to interpret the theory of generalized Nijenhuis torsions from a new perspective. In turn, this general framework enables us to obtain new results interesting per se from an algebraic viewpoint.

2. On the theory of generalized Nijenhuis torsions

In this section, we shall revise a recent construction of generalized Nijenhuis torsions [27] and Haantjes brackets [28] proposed by us.

Let \( M \) be a differentiable manifold. We shall denote by \( \mathfrak{X}(M) \) the Lie algebra of all smooth vector fields on \( M \) and by \( \mathcal{T}_1^1(M) \) the space of smooth \((1,1)\) tensor fields on \( M \) (operator fields). For the sake of simplicity, from now on the expressions “tensor fields” and “operator fields” will be abbreviated to tensors and operators. In the following, all tensors will be considered to be smooth.

2.1. Generalized Nijenhuis and Haantjes geometries.

**Definition 2.1.** Let \( A : \mathfrak{X}(M) \to \mathfrak{X}(M) \) be an operator \( \in \mathcal{T}_1^1(M) \). The Nijenhuis torsion of \( A \) is the vector-valued 2-form defined by

\[
\tau_A(X,Y) := A^2[X,Y] + [AX,AY] - A\left([X,AY] + [AX,Y]\right),
\]

where \( X, Y \in \mathfrak{X}(M) \) and [ , ] denotes the Lie bracket of two vector fields.

We also recall the definition of generalized torsions, according to the formulation of [27], since it will be crucial in the subsequent discussion.

**Definition 2.2.** Let \( A : \mathfrak{X}(M) \to \mathfrak{X}(M) \) be an operator. The generalized Nijenhuis torsion of \( A \) of level \( m \), for each integer \( m \geq 1 \), is the vector-valued 2-form defined
We shall denote by 
\[ T_{\alpha, X, Y}^{(m)}(X, Y) = A^2\tau_{\alpha, X, Y}^{(m-1)}(X, Y) + \tau_{\alpha, X, Y}^{(m-1)}(AX, AY) \] 
(2.2) 
- \[ A\left(\tau_{\alpha, X, Y}^{(m-1)}(X, AY) + \tau_{\alpha, X, Y}^{(m-1)}(AX, Y)\right), \quad X, Y \in \mathcal{X}(M). \]

Here the notation \( \tau_{\alpha, X, Y}^{(0)}(X, Y) := [X, Y], \tau_{\alpha, X, Y}^{(1)}(X, Y) := \tau_{\alpha, X, Y}(X, Y) \) is used.

We also remind a useful formula, proved in [14] (Section 4.6), by means of a suitable polynomial representation of (1,2)-tensors:

\[ \tau_{\alpha, X, Y}^{(m)}(X, Y) = \sum_{p=0}^{m} \sum_{q=0}^{m} (-1)^{2m-p-q} \binom{m}{p} \binom{m}{q} A^{p+q} \left[ A^{m-p} X, A^{m-q} Y \right]. \]

Alternatively, this formula can also be proved by induction on \( m \).

Example 2.3. For \( m = 2 \) one finds that \( \tau_{\alpha, X, Y}^{(m)}(X, Y) \) coincides with the Haantjes torsion of \( A \), that is, the vector-valued 2-form defined by [12]

\[ \mathcal{H}_A(X, Y) := A^2\tau_{\alpha, X, Y}(X, Y) + \tau_{\alpha, X, Y}(AX, AY) - A\left(\tau_{\alpha, X, Y}(X, AY) + \tau_{\alpha, X, Y}(AX, Y)\right). \]

Definition 2.4. [28] A generalized Nijenhuis operator of level \( m \) is a (1,1)-tensor whose generalized Nijenhuis torsion of the same level vanishes identically.

In particular, for \( m = 1 \) and \( m = 2 \), we get the standard Nijenhuis and Haantjes operators. The relevance of Haantjes operators in the theory of integrable Hamiltonian classical systems and for the construction of separating variables has been recently discussed in [26], [24], [30].

Another useful geometric object is the vector-valued 3-tensor \( T(A, B) : \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \rightarrow \mathcal{X}(M) \) defined by

\[ T(A, B)(\alpha, X, Y) := (I \otimes AB - A \otimes B)(\alpha, X, Y). \]

We shall denote by \( T^T(\alpha, X, Y) := T(\alpha, Y, X) \) the transposed of \( T \) w.r.t. the last two arguments. We also recall that for each operator \( A, B \), and for all \( \alpha, X, Y \in \mathcal{X}(M), \)

\[ (A \otimes B)(\alpha, X, Y) = \langle \alpha, AX \rangle BY. \]

Thus, one can prove the following

Proposition 2.5. [28] Let \( A : \mathcal{X}(M) \rightarrow \mathcal{X}(M) \) be an operator. Then, for all \( f, g \in C^\infty(M) \) the relations

\[ \tau_{f A}(X, Y) = f^2\tau_{A}(X, Y) - f\left(T(A, A) - T^T(A, A)\right)(df, X, Y) \]

(2.6) 
\[ \tau_{f A^m g I}(X, Y) = f^{2m}\tau_{A^m}(X, Y), \quad m \in \mathbb{N}\setminus\{0,1\}. \]

hold.

According to eq. (2.7), we deduce that \( \tau_{A^m}(X, Y) \) is a homogeneous 2-form of degree \( 2m \) in \( A \).

3. Polarization of generalized Nijenhuis torsions

In this section, we shall introduce the polarization form of a generalized Nijenhuis torsion. To this aim, we shall start with the simplest representative of our construction.
3.1. The Frölicher-Nijenhuis bracket.

**Definition 3.1.** [11] Let \( A, B \in T^1(M) \). The Frölicher–Nijenhuis bracket of \( A \) and \( B \) is the vector-valued 2-form given by

\[
[A, B](X, Y) := \left( AB + BA \right)[X, Y] + [AX, BY] + [BX, AY]
\]

\[
- A\left( [X, BY] + [BX, Y] \right) - B\left( [X, AY] + [AX, Y] \right), \quad X, Y \in \mathfrak{X}(M) .
\]

We observe that the Frölicher–Nijenhuis bracket can be regarded as the polarization of the Nijenhuis torsion. Indeed, as the Nijenhuis torsion (2.1) is a quadratic form involving the entries of the operator \( A \), it can be naturally polarized. As a result, one obtains the associated bilinear form that coincides just with the Frolicher–Nijenhuis bracket:

\[
[A, B](X, Y) = \tau_{A+B}(X, Y) - (\tau_A(X, Y) + \tau_B(X, Y)) .
\]

We remind that the local expression of the components of the Frölicher–Nijenhuis bracket reads

\[
[A, B]^i_{jk} = \sum_{l=1}^n \left( A^i_l \partial_l B^k_j - A^i_l \partial_l B^j_k + B^i_l \partial_l A^k_j - B^i_l \partial_l A^j_k \right) .
\]

This bracket has relevant geometric applications [20], in particular in the theory of almost-complex structures and in the detection of obstructions to integrability [13]. The bracket is symmetric and \( R \)-linear (but not \( C^\infty(M) \)-linear) in \( A \) and \( B \). In fact, it satisfies the identity

\[
[fA, B](X, Y) = f[A, B](X, Y) - (T(B, A) - T^T(B, A)(df, X, Y))
\]

Choosing \( B = A \) in Eq. (3.1), one gets twice the Nijenhuis torsion:

\[
[A, A](X, Y) = 2 \tau_A(X, Y) .
\]

In order to generalize equation (3.2) to the case of Nijenhuis torsions of level \( m \), we will follow the approach proposed in [8]. Precisely, we first introduce the notion of defect of a torsion of level \( m \), and consequently the polarization 2-form associated with it.

**Definition 3.2.** Let \( A_1, \ldots, A_n \in T^1(M) \). The defect of index \( k \) of the generalized Nijenhuis torsion of level \( m \) is the vector-valued 2–form defined by

\[
\Delta^{(m)}(A_1, A_2, \ldots, A_k)(X, Y) := \sum_{I \subseteq \{1, 2, \ldots, k\}} (-1)^{k-|I|} \tau^{(m)}_{\sum_{i \in I} A_i}(X, Y) .
\]

Here \( k, m \in \mathbb{N} \setminus \{0\} \), \( I \) denotes any non-empty subset extracted from \( \{1, \ldots, k\} \) and \( |I| \) is the cardinality of \( I \).

The \( k \)-th defect is by construction symmetric in its arguments \( (A_1, \ldots, A_k) \). It easy to prove from Eq. (3.5) that the defect fulfills the recurrence relations

\[
\Delta^{(m)}_{k+1}(A_1, A_2, \ldots, A_k, A_{k+1})(X, Y) = \Delta^{(m)}(A_1, A_2, \ldots, A_k, A_{k+1})(X, Y)
\]

\[
- \Delta^{(m)}_{k}(A_1, A_3, \ldots, A_k, A_{k+1})(X, Y) - \Delta^{(m)}_k(A_2, A_3, \ldots, A_k, A_{k+1})(X, Y)
\]

\[1\] For sake of clarity, in this article we have renounced to the usual unified notation \([\cdot, \cdot]\) which, depending on the context, should stand for both the standard Lie bracket of vector fields and the Frölicher–Nijenhuis bracket of operators. Instead, we have preferred to maintain the symbol \([\cdot, \cdot]\) for the Frölicher–Nijenhuis bracket and to introduce the notation \([\cdot, \cdot]\) for the Lie bracket of two vector fields as well as the commutator of two operators.
involving the first argument. Analogous relations hold for each of the remaining arguments.

As a direct consequence of the recurrence relations (3.6) and their analogous ones, we can state the following

**Proposition 3.3.** The defect of index \( k \) is additive in each of the entries \( A_i \) if and only if satisfies the relation

\[
\Delta^{(m)}_{k+1} (A_1, A_2, \ldots, A_{k+1}) (X, Y) = 0.
\]

According to the polarization identity stated in [16] (pag. 42), by specializing the \( k \)-th defect to the case \( k = 2m \), we are able to define the new vector-valued 2-form representing the polarization form associated with our generalized Nijenhuis torsions.

**Definition 3.4.** The polarization of the Nijenhuis torsion of level \( m \) is the vector-valued 2-form

\[
P^{(m)}(A_1, A_2, \ldots, A_{2m})(X, Y) := \Delta^{(m)}_{2m} (A_1, A_2, \ldots, A_{2m})(X, Y)
\]

where \( m \in \mathbb{N} \setminus \{0\} \) and \( A_1, \ldots, A_{2m} \in \mathcal{T}_1(M) \).

**Remark 3.5.** We omit the factor \( \frac{1}{(2m)!} \) usually adopted in polarization formulae (see e.g. [16]) in order to recover exactly the Frölicher-Nijenhuis bracket (3.1) for \( m = 1 \):

\[
P^{(1)}(A_1, A_2)(X, Y) = \llbracket A_1, A_2 \rrbracket (X, Y).
\]

By way of an example, let us consider the new case \( m = 2 \). The polarization of the Haantjes torsion reads explicitly:

\[
P^{(2)}(A_1, A_2, A_3, A_4)(X, Y) = \mathcal{H}(A_1 + A_2 + A_3 + A_4)(X, Y)
- \mathcal{H}(A_1 + A_2 + A_3) + \mathcal{H}(A_1 + A_2 + A_4) + \mathcal{H}(A_1 + A_3 + A_4) + \mathcal{H}(A_2 + A_3 + A_4)
+ \mathcal{H}(A_1 + A_2) + \mathcal{H}(A_1 + A_3) + \mathcal{H}(A_1 + A_4) + \mathcal{H}(A_2 + A_3) + \mathcal{H}(A_2 + A_4) + \mathcal{H}(A_3 + A_4)
- \mathcal{H}(A_1) + \mathcal{H}(A_2) + \mathcal{H}(A_3) + \mathcal{H}(A_4)(X, Y),
\]

where the Haantjes torsion \( \mathcal{H}(A)(X, Y) := \mathcal{H}_A(X, Y) \) is defined by Eq. (2.3).

Following the approach of [23], we deduce an equivalent, useful definition of polarization:

\[
P^{(m)}(A_1, A_2, \ldots, A_{2m})(X, Y) = \frac{\partial^{2m}}{\partial \lambda_1 \ldots \partial \lambda_{2m}} \tau^{(m)}_{\sum_{i=1}^{2m} \lambda_i A_i}(X, Y),
\]

the \( \lambda_i \) being distinct arbitrary real parameters.

**Lemma 3.6.** The following relation holds

\[
\Delta^{(m)}_1 (A)(X, Y) = \tau^{(m)}_{A}(X, Y) = \frac{1}{(2m)!} P^{(m)}(\underbrace{A, \ldots, A}_{2m-times}).
\]
Both the two statements can be proved by induction on $n$.

Let us prove a result crucial for the sequel of our analysis.

**Theorem 3.7.** The polarization of the Nijenhuis torsion of level $m$ fulfills the following recurrence relation:

\begin{equation}
(3.13)
\mathcal{P}^{(m)}(A_{1}, A_{2}, \ldots, A_{2m})(X, Y) = \sum_{j,k,i_{1},i_{2} \neq t_{1},t_{2} \in T_{n}}^{2m} (A_{j}A_{k} + A_{k}A_{j})\mathcal{P}^{(m-1)}(A_{i_{1}}, \ldots, A_{i_{2m-2}})(X, Y)
\end{equation}

where $m \in \mathbb{N} \setminus \{0,1\}$, and is $\mathbb{R}$-multilinear, that is, it is $\mathbb{R}$-linear in each of the $A_{i} \in T_{1}^{1}(M)$ for $m \in \mathbb{N} \setminus \{0\}$.

**Proof.** Both the two statements can be proved by induction on $m$.

For $m = 1$, we obtain the relation $\mathcal{P}^{(1)}(A_{1}, A_{2}) = [A_{1}, A_{2}]$ which is $\mathbb{R}$-bilinear (see Definition 3.1).

For $m = 2$, after a lengthy calculation, we get

\begin{equation}
(3.14)
\mathcal{P}^{(2)}(A_{1}, A_{2}, A_{3}, A_{4})(X, Y) =
\end{equation}
which is a special case of Eq. (3.13) in view of Eq. (3.9). Indeed, Eq. (3.14) implies that \( \mathcal{P}^{(2)}(A_1, A_2, A_3, A_4) \) is \( \mathbb{R} \)-linear in \( A_1, A_2, A_3, A_4 \) thanks to the bilinearity of the standard Frölicher-Nijenhuis bracket (3.1).

Also, let us assume as induction hypotheses that the recursion relation (3.13) holds for the polarization of the Nijenhuis torsion of level \( m - 1 \) and that this polarization is linear in \( A_1, \ldots, A_{2m-2} \). Let us show that Eq. (3.13) holds true for the polarized Nijenhuis torsion of level \( m \).

\[
(3.15) \quad \mathcal{P}^{(m)}(A_1, A_2, \ldots, A_{2m})(X, Y) = \frac{\partial^{2m}}{\partial \lambda_1 \ldots \partial \lambda_{2m}} \gamma^{(m)}_{\sum \lambda_i A_i}(X, Y) \\
= \frac{\partial^{2m}}{\partial \lambda_1 \ldots \partial \lambda_{2m}} \left( (\sum_{i=1}^{2m} \lambda_i A_i)^2 \tau^{(m-1)}_{\sum \lambda_i A_i}(X, Y) + \tau^{(m-1)}_{m \sum \lambda_i A_i} \left( \sum_{i=1}^{2m} \lambda_i A_i X, \sum_{i=1}^{2m} \lambda_i A_i Y \right) \\
- \left( \sum_{i=1}^{2m} \lambda_i A_i \right) \left( \sum_{i=1}^{m-1} \lambda_i A_i X, \sum_{i=1}^{m-1} \lambda_i A_i Y \right) + \tau^{2m}_{m \sum \lambda_i A_i} \left( X, \sum_{i=1}^{2m} \lambda_i A_i Y \right) \right)
\]

If we consider the first term and the induction hypothesis about the linearity of \( \mathcal{P}^{(m-1)} \) we get

\[
(3.16) \quad \frac{\partial^{2m}}{\partial \lambda_1 \ldots \partial \lambda_{2m}} \left( (\sum_{i=1}^{2m} \lambda_i A_i)^2 \tau^{(m-1)}_{\sum \lambda_i A_i}(X, Y) \right) \\
= \frac{1}{(2m - 2)!} \frac{\partial^{2m}}{\partial \lambda_1 \ldots \partial \lambda_{2m}} \left( \sum_{i=1}^{2m} \lambda_i A_i \right)^2 \mathcal{P}^{(m-1)}(A_1, \ldots, \sum_{i=1}^{2m} \lambda_i A_i)(X, Y) \\
= \frac{1}{(2m - 2)!} \frac{\partial^{2m}}{\partial \lambda_1 \ldots \partial \lambda_{2m}} \left( \sum_{i_1, \ldots, i_{2m-2}} \lambda_{i_1} \ldots \lambda_{i_{2m-2}} \mathcal{P}^{(m-1)}(A_{i_1}, \ldots, A_{i_{2m-2}})(X, Y) \right) \\
= \frac{1}{(2m - 2)!} \frac{\partial^{2m}}{\partial \lambda_1 \ldots \partial \lambda_{2m}} \left( \sum_{j, \lambda_1, \ldots, \lambda_{2m}} \lambda_1 \ldots \lambda_{2m} \sum_{j, \lambda_1, \ldots, \lambda_{2m} \neq \lambda_{i_1, \ldots, i_{2m-2}}} (A_j A_k + A_k A_j) \mathcal{P}^{(m-1)}(A_{i_1}, \ldots, A_{i_{2m-2}})(X, Y) \right),
\]

therefore we obtain the first summation in the right term of Eq. (3.13).
Thus, we have obtained the second summation in the right term of Eq. (3.15) we get

\[ \left( \frac{\partial^{2m}}{\partial \lambda_1 \ldots \partial \lambda_{2m}} \right)^{\tau_{(m-1)}^{(m-1)}} \left( \sum_{k=1}^{2m} \lambda_k A_k \right) \left( \lambda_1 - \lambda_j \lambda_j \lambda_j \sum_{i=1}^{2m} \lambda_i A_i X \right) \]

\[ = \left( \frac{\partial^{2m}}{\partial \lambda_1 \ldots \partial \lambda_{2m}} \right)^{\tau_{(m-1)}^{(m-1)}} \left( \lambda_1 \lambda_j \sum_{i=1}^{2m} \lambda_i A_i X \right) \]

\[ = \frac{1}{(2m-2)!} \left( \frac{\partial^{2m}}{\partial \lambda_1 \ldots \partial \lambda_{2m}} \right)^{\tau_{(m-1)}^{(m-1)}} \left( \sum_{i,j=1}^{2m} \lambda_i \lambda_j \sum_{i=1}^{2m} \lambda_k A_k \right) \sum_{i=1}^{2m} \lambda_i A_i X \]

\[ = \frac{1}{(2m-2)!} \left( \frac{\partial^{2m}}{\partial \lambda_1 \ldots \partial \lambda_{2m}} \right)^{\tau_{(m-1)}^{(m-1)}} \left( \lambda_1 \lambda_j \sum_{i=1}^{2m} \lambda_i A_i X \right) \]

\[ = \frac{1}{(2m-2)!} \left( \frac{\partial^{2m}}{\partial \lambda_1 \ldots \partial \lambda_{2m}} \right)^{\tau_{(m-1)}^{(m-1)}} \left( \sum_{i,j,k=1}^{2m} \lambda_i \lambda_j \lambda_k A_k \right) \sum_{i=1}^{2m} \lambda_i A_i X \]

Thus, we have obtained the second summation in the right term of Eq. (3.15). In the same manner, from the third term of Eq. (3.15) we get

\[ \left( \frac{\partial^{2m}}{\partial \lambda_1 \ldots \partial \lambda_{2m}} \right)^{\tau_{(m-1)}^{(m-1)}} \left( \sum_{i=1}^{2m} \lambda_i A_i X \right) \]

\[ = \frac{1}{(2m-2)!} \left( \frac{\partial^{2m}}{\partial \lambda_1 \ldots \partial \lambda_{2m}} \right)^{\tau_{(m-1)}^{(m-1)}} \left( \sum_{i,j,k=1}^{2m} \lambda_i \lambda_j \lambda_k A_k \right) \sum_{i=1}^{2m} \lambda_i A_i X \]

\[ = \frac{1}{(2m-2)!} \left( \frac{\partial^{2m}}{\partial \lambda_1 \ldots \partial \lambda_{2m}} \right)^{\tau_{(m-1)}^{(m-1)}} \left( \sum_{i,j,k=1}^{2m} \lambda_i \lambda_j \lambda_k A_k \right) \sum_{i=1}^{2m} \lambda_i A_i X \]
The polarization of the Nijenhuis torsion is

\[ \Delta_k(m) = \sum_{i_1 < \ldots < i_{2m-2}} \mathcal{P}^{(m-1)}(A_{i_1}, \ldots, A_{i_{2m-2}})(A_k X, Y) \]

Therefore

\[ \mathcal{P}^{(m-1)}(A_{i_1}, \ldots, A_{i_{2m-2}})(X, A_k Y) \]

In this way, we have deduced the third summation in the right term of Eq. (3.13). Concerning \( \mathbb{R} \)-multilinearity, it is easily deduced by assuming that the property holds for each polarization of level \( m - 1 \) in Eq. (3.13) (all of them involving \( 2m - 2 \) arguments), and taking also into account that in each addend the explicit dependence on the remaining arguments \( A_j \) and \( A_k \) \( (j \neq k) \) is linear.

**Proposition 3.8.** If the operators \( A_i \in T_1^1(M) \) pairwise commute, then for \( m \geq 2 \) the polarization of the Nijenhuis torsion is \( C^\infty(M) \)-multilinear.

**Proof.** Once again, this statement can be proved by induction on \( m \).

In view of Eq. (3.4) we obtain

\[ \mathcal{P}^{(2)}(f A_1, A_2, A_4)(X, Y) = f \mathcal{P}^{(2)}(A_1, A_2, A_4)(X, Y) + \]

\[ - \left( A_2[A_3, T(A_4, A_1)] - T^T(A_4, A_1) \right) + \left( A_2[A_3, T(A_5, A_1)] - T^T(A_5, A_1) \right) \]

\[ - \left( A_2[A_4, T(A_5, A_1)] - T^T(A_5, A_1) \right) + \left( A_2[A_4, T(A_5, A_1)] - T^T(A_5, A_1) \right) \]

\[ + \left( A_2[A_5, T(A_5, A_1)] - T^T(A_5, A_1) \right) + \left( A_2[A_5, T(A_5, A_1)] - T^T(A_5, A_1) \right) \]

where \( T \) is defined by Eq. (2.5). As the operators \( A_i \) by assumption commute pairwise, each of the commutators of the form

\[ [A_i, T(A_j, A_k)] = 0 \quad i, j, k \neq \]

vanishes. Therefore \( \mathcal{P}^{(2)}(A_1, A_2, A_4) \) is \( C^\infty(M) \)-linear. Again by induction on \( m \) and taking into account Eq. (3.13), we deduce the \( C^\infty(M) \)-linearity of the polarization of the Nijenhuis torsion for any level \( m \geq 2 \).

**Proposition 3.9.** For \( k > 2m \), we have

\[ \Delta_k(m)(A_1, A_2, \ldots, A_k)(X, Y) = 0 \]

**Proof.** It is a simple consequence of Proposition 3.3 and Theorem 3.7 and the recursion relations (3.6).

Observe that the defect of index 2 for any \( m \) has the simple form

\[ \Delta_2(m)(A, B)(X, Y) = r_{A+B}^m(X, Y) - r_A^m(X, Y) - r_B^m(X, Y) \]

Thus, we can state the following result.
Lemma 3.10. The relation
\( \Delta_2^{(m)}(A, B)(X, Y) = \frac{1}{(2m)!} \left( \sum_{k=1}^{2m-1} \binom{2m}{k} \mathcal{P}^{(m)}(A, \ldots, A, B, \ldots, B)(X, Y) \right) \)
holds.

Proof. Using eq. (3.20) we have
\( \tau_{A+B}^{(m)} - \tau_A^{(m)} - \tau_B^{(m)})(X, Y) = \)
\[ \frac{1}{(2m)!} \left( \mathcal{P}^{(m)}(A + B, \ldots, A + B) - \mathcal{P}^{(m)}(A, \ldots, A) - \mathcal{P}^{(m)}(A, \ldots, A) \right)(X, Y) \]
In view of the additivity property of \( \mathcal{P}^{(m)} \), by induction on \( m \) it can be proved that
\[ \mathcal{P}^{(m)}(A + B, \ldots, A + B)(X, Y) = \sum_{k=0}^{2m} \binom{2m}{k} \mathcal{P}^{(m)}(A, \ldots, A, B, \ldots, B)(X, Y) , \]
which completes the proof. \( \square \)

3.2. Generalized Haantjes modules and vector spaces. In the following analysis, we shall further illustrate the relevance of the polarization of the generalized Nijenhuis torsion (3.15). Indeed they play a crucial role in the study of the \( C^\infty(M) \)-modules of generalized Nijenhuis operators. In [27], we have introduced the notion of Haantjes modules, namely modules of Haantjes operators. This notion has been further generalized in the recent work [25], where the generalized Haantjes algebras of level \( m \) have been introduced. Here we are interested in the case of generalized modules of operators, according to the following

Definition 3.11. A generalized Haantjes module of level \( m \) is a pair \( (M, \mathcal{H}_M^{(m)}) \) which satisfies the following conditions:

- \( M \) is a differentiable manifold of dimension \( n \);
- \( \mathcal{H}_M^{(m)} \) is a set of Haantjes operators \( \mathcal{K} : \mathcal{X}(M) \to \mathcal{X}(M) \) such that
\[ \tau_{fK_1 + gK_2}^{(m)}(X, Y) = 0 , \quad \forall X, Y \in \mathcal{X}(M) , \quad \forall f, g \in C^\infty(M) , \quad \forall K_1, K_2 \in \mathcal{H}_M^{(m)} . \]

Thus, a generalized Haantjes module is a free module of generalized Nijenhuis operators of level \( m \) over the ring of smooth functions on \( M \). If property (3.23) is satisfied when \( f, g \) are simply real constants, we shall use the denomination of level \( m \)-generalized Haantjes vector space.

Our main goal is to determine the tensorial compatibility conditions ensuring the existence of the generalized Haantjes modulus of level \( m \) generated by two operators \( A, B : \mathcal{X}(M) \to \mathcal{X}(M) \). We construct these conditions in the most general case of non-semisimple operators, under the unique hypothesis that they commute.

3.2.1. The general case. We shall start our analysis from the following identity, valid for all \( X, Y \in \mathcal{X}(M) : \)
\[ \tau_{\lambda A + \mu B}^{(m)}(X, Y) = \lambda^{2m} \tau_A^{(m)}(X, Y) + \mu^{2m} \tau_B^{(m)}(X, Y) + \Delta_2^{(m)}(\lambda A, \mu B)(X, Y) . \]
For $m \geq 1$, this identity holds $\forall \lambda, \mu \in \mathbb{R}$ as a consequence of Eqs. (3.20) and (2.6). For $m \geq 2$, it holds more generally $\forall \lambda, \mu \in C^\infty(M)$ due to Eq. (2.7).

**Theorem 3.12.** Let $M$ be a differentiable manifold and let $A, B : \mathfrak{X}(M) \to \mathfrak{X}(M)$ be two Nijenhuis operators of level $m$. Then for $m \geq 1$, $A$ and $B$ generate a Haantjes vector space of level $m$ if and only if the $(2m-1)$ differential conditions

$$
\mathcal{P}^{(m)}(A, \ldots, A, B, \ldots, B)(X, Y) = 0, \quad k = 1, \ldots, 2m - 1
$$

are satisfied. In addition, for $m \geq 2$, if $A$ and $B$ commute, they generate a $C^\infty(M)$ Haantjes module of the same level if and only if the same conditions (3.25) are fulfilled.

**Proof.** From Eqs. (3.24) it follows that $A$ and $B$ generate a generalized Haantjes vector space (resp. a generalized Haantjes module) of level $m$ if and only if the defect $\Delta_2^{(m)}(\lambda A, \mu B)$ vanishes for all $\lambda$ and $\mu \in \mathbb{R}$ (resp. $\in C^\infty(M)$). From Eq. (3.21) and by the multilinearity of the polarization of the Nijenhuis torsion of level $m$, such a defect takes the form

$$
\Delta_2^{(m)}(\lambda A, \mu B)(X, Y) = \frac{1}{(2m)!} \left( \sum_{k=1}^{2m-1} \binom{2m}{k} \lambda^{(2m-k)} \mu^k \mathcal{P}^{(m)}(A, \ldots, A, B, \ldots, B)(X, Y) \right),
$$

for $m \geq 1$ and $\lambda, \mu \in \mathbb{R}$, and for $m \geq 2$ and $\lambda, \mu \in C^\infty(M)$ when $A$ and $B$ commute. \hfill \Box

3.3. **Higher Haantjes brackets.** Let $M$ be a differentiable manifold and $A, B : \mathfrak{X}(M) \to \mathfrak{X}(M)$ two operators.

**Definition 3.13.** [28] Let $M$ be a differentiable manifold of dimension $n$ and let $A, B : \mathfrak{X}(M) \to \mathfrak{X}(M)$ be two $(1, 1)$ tensors. The **Haantjes bracket of level $m$** in $\mathfrak{X}(M)$ is defined, for any $X, Y \in \mathfrak{X}(M)$, by the relations

$$
\mathcal{H}^{(1)}_{A,B}(X, Y) := [A, B](X, Y)
$$

and

$$
\mathcal{H}^{(m)}_{A,B}(X, Y) := \left( AB + BA \right) \mathcal{H}^{(m-1)}_{A,B}(X, Y) + \mathcal{H}^{(m-1)}_{A,B}(AX, BY) + \mathcal{H}^{(m-1)}_{A,B}(BX, AY) - A \left( \mathcal{H}^{(m-1)}_{A,B}(X, BY) + \mathcal{H}^{(m-1)}_{A,B}(BX, Y) \right) - B \left( \mathcal{H}^{(m-1)}_{A,B}(X, AY) + \mathcal{H}^{(m-1)}_{A,B}(AX, Y) \right), \quad m \geq 2.
$$

(3.26)

Each of these brackets is $\mathbb{R}$-homogeneous of degree $m$ in each of its arguments $A$ and $B$, as we shall state in Proposition 3.14; moreover, it is symmetric in the interchange of $A$ and $B$.

**Proposition 3.14 ([28]).** Let $M$ be a differentiable manifold and $A, B : \mathfrak{X}(M) \to \mathfrak{X}(M)$ two commuting operators. For any $f, g, h, k \in C^\infty(M)$, $X, Y \in \mathfrak{X}(M)$ and for each integer $m \geq 2$, we have

$$
\mathcal{H}^{(m)}_{fI+gA, hI+kB}(X, Y) = g^m k^m \mathcal{H}^{(m)}_{A,B}(X, Y).
$$

(3.27)
Notice that the notation $\mathcal{H}(A, B)(X, Y) := \mathcal{H}^{(2)}_{A, B}(X, Y)$ will be used.

We shall analyse now the case $m = 2$ to offer a new interpretation of our results in [28] in terms of the theory of polarization forms.

$$\mathcal{H}(A, B)(X, Y) = (AB + BA)[A, B](X, Y) + [A, B](AX, BY) + [A, B](BX, AY)$$

$$-A\left([A, B](X, BY) + [A, B](BX, Y) - B([A, B](X, AY) + [A, B](X, Y)(AX, Y))\right).$$

The following result clarifies the geometric meaning of the Haantjes bracket of level 2.

**Lemma 3.15.** [28] Let $M$ be a differentiable manifold and $A, B : \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ two commuting $(1, 1)$ tensors which can be simultaneously diagonalized in a local chart of $M$. Then for any $X, Y \in \mathfrak{X}(M)$, the Haantjes bracket $\mathcal{H}_{A, B}(X, Y)$ vanishes.

In order to prove Theorem 20 and Corollary 21 of Ref. [28], we have introduced there the auxiliary Haantjes brackets $\mathcal{H}_1(A, B)$, $\mathcal{H}_2(A, B)$ defined by

$$\mathcal{H}_1(A, B)(X, Y) := B^2\tau_A(X, Y) + \tau_A(BX, BY) - B(\tau_A(BX, Y) + \tau_A(X, BY))$$

$$+ A^2\tau_B(X, Y) + \tau_B(AX, AY) - A(\tau_B(AX, Y) + \tau_B(X, AY)),$$

$$\mathcal{H}_2(A, B)(X, Y) := (AB + BA)\tau_A(X, Y) + \tau_A(AX, BY) + \tau_A(BX, AY)$$

$$- A(\tau_A(BX, Y) + \tau_A(X, BY)) - B(\tau_A(AX, Y) + \tau_A(X, AY))$$

$$+ A^2[A, B](X, Y) + [A, B](AX, AY) - A([A, B](AX, Y) + [A, B](X, AY)).$$

We are now able to give an interesting algebraic interpretation to the brackets $\mathcal{H}(A, B)$, $\mathcal{H}_1(A, B)$ and $\mathcal{H}_2(A, B)$ in terms of the polarization multilinear form of level 2.

**Proposition 3.16.** The identities

$$\mathcal{P}^{(2)}(A, A, B, B)(X, Y) = 4(\mathcal{H}(A, B) + \mathcal{H}_1(A, B))(X, Y)$$

$$\mathcal{P}^{(2)}(A, A, A, B)(X, Y) = 6\mathcal{H}_2(A, B)(X, Y)$$

$$\mathcal{P}^{(2)}(A, B, B, B)(X, Y) = 6\mathcal{H}_2(B, A)(X, Y),$$

hold.

**Proof.** These identities are derived directly by Eq. (3.14), setting $A_1 = A, A_2 = B, A_3 = A, A_4 = B$. 

We point out that in the light of the latter result, Corollary 21 of [28] turns out to be a special case of the novel Theorem 3.12.

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Departamento de Física Teórica, Facultad de Ciencias Físicas, Universidad Complutense de Madrid, 28040 – Madrid, Spain, and Instituto de Ciencias Matemáticas, C/ Nicolás Cabrera, No 13–15, 28049 Madrid, Spain

Email address: piergiulio.tempesta@icmat.es, ptempesta@ucm.es

Dipartimento di Matematica e Geoscienze, Università degli Studi di Trieste, piazz.le Europa 1, I-34127 Trieste, Italy.

Email address: tondo@units.it