Constraining $f(R)$ theories with the energy conditions

S. E. Perez Bergliaffa

ICRANET, Piazzale della Repubblica, 10 - 65100 Pescara, Italia
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A new method to constrain gravitational theories depending on the Ricci scalar is presented. It is based on the weak energy condition and yields limits on the parameters of a given theory through the current values of the derivatives of the scale factor of the Friedmann-Robertson-Walker geometry. A further constraint depending on the current value of the snap is also given. Actual constraints (and the corresponding error propagation analysis) are calculated for two examples, which show that the method is useful in limiting the possible $f(R)$ theories.

Introduction

It follows from several observations that the universe is currently expanding with positive acceleration. The many models that have been advanced to explain this situation can be classified in two classes. The first class contains those models that incorporate modifications to the matter side of Einstein’s equations. This matter (known as “dark energy”), can be described by an ideal fluid or by a scalar field with a convenient potential, and it must violate the strong energy condition in order to accelerate the universe in General Relativity (GR). The models in the second class have normal matter as a source but assume that gravitation is not described by GR at low curvatures. As examples of this latter class we can mention theories depending on the Ricci scalar (the so-called $f(R)$ theories), and gravity modified from contributions of extra dimensions. Although $f(R)$ theories offer a chance to explain the acceleration of the universe, they are not free of problems. For instance, the application of their metric formulation to the Friedmann-Robertson-Walker (FRW) geometry yields a fourth order nonlinear differential equation for the scale factor $a(t)$ which in general cannot be analytically solved even for simple $f(R)$.

Since many $f(R)$ give rise to accelerated expansion, another issue is how to reduce the theory space using observations. Constrains have been obtained from cosmological and astrophysical data, solar system tests, fifth force/BBN data, and by requiring that a given theory describes the correct sequence of decelerated-accelerated phases in the evolution of the universe. Most of the cosmological tests involve either some transformation of the theory under scrutiny to an equivalent form with one auxiliary scalar field and/or considerations in different frames (see for instance), or some assumptions regarding the dependence of the Hubble “constant” $H$ with the redshift $\mathcal{R}$. Here instead a new criterion based on model-independent data shall be given, that helps in deciding, without solving the EOM or making frame transformations or assumptions about $H$, whether a given $f(R)$ theory is appropriate to describe the universe. The basic premise will be that the acceleration is due solely to a modified gravitational theory with normal matter as a source. The criterion will then be obtained by imposing the energy conditions on matter, yielding conditions on $f(R)$ and its derivatives w.r.t $R$ in terms of the current value of the derivatives of the scale factor. These conditions are to be satisfied if the theory given by $f(R)$ is to describe the current state of the universe, and they bring forth limits on the parameters that enter the theory under consideration.

Energy Conditions

The energy conditions (EC) are inequalities satisfied by “normal” matter (see for instance). When specialized to a FRW universe, the (local) null, weak, and dominant EC are given by

\[
\text{NEC} \iff \rho + p \geq 0,
\]
\[
\text{WEC} \iff \rho \geq 0 \text{ and } (\rho + p \geq 0),
\]
\[
\text{SEC} \iff (\rho + 3p \geq 0) \text{ and } (\rho + p \geq 0),
\]
\[
\text{DEC} \iff \rho \geq 0 \text{ and } (\rho \pm p \geq 0).
\]

The EC have proved to be useful in the context of cosmological singularities and bounces. Other applications of the energy conditions to cosmology can be found in.

Let us remind the reader that for a theory given by $f(R)$, the EOM are

\[
f’R - 2f + 3f’’ \left(\frac{\ddot{R}}{a} + 3\dot{a}\dot{R}/a\right) + 3f'''\dot{R}^2 + T = 0, \quad (1)
\]
\[
f’R_{tt} + \frac{1}{2}f’’ - 3f’’’\frac{\ddot{R}}{a} + T_{tt} = 0, \quad (2)
\]
This inequality gives a relation between the derivatives of the theory.

Before proceeding to build with these equations the inequalities that define the energy conditions, let us remark that observations show that the current matter content of the universe (assumed here to be normal matter, as opposed to dark energy) is pressureless. In this case the EC reduce to the inequality \( p \geq -\rho \), which can be taken as the \( n = 1 \) low-curvature limit of

\[
f(R) = R + \frac{\alpha}{R},
\]

a model studied in [3]. Substituting Eqn.(7) in Eqn.(5) for \( \alpha > 0 \) and \( n \) even we get

\[
-3q_0H_0^2nR_0 - \frac{1}{2}R_0^2 - 18H_0^4n(n+1)(q_0 - 2) \geq 0. \tag{8}
\]

Replacing in this equation the numerical values of the parameters and using \( R_0 = 6H_0^2(q_0 - 1) \) we get an inequality that must be satisfied by \( n \):

\[
\phi \equiv -17.64n^2 - 44.50n - 59.62 \geq 0. \tag{9}
\]

Since this equation cannot satisfied by any real \( n \), we conclude that \( n \) cannot be even for \( \alpha > 0 \). The same analysis with odd \( n \) reverses the sign of the inequality [4], so only odd values of \( n \) are allowed for \( \alpha > 0 \). This result generalizes that obtained in [3] for \( n = 1 \) [21]. In the same way, we obtain that only even \( n \) are allowed for \( \alpha < 0 \). These conclusions are valid even when the error coming from the kinematical parameters is taken into account (see Fig[1]). If we knew the value of \( s_0 \), we could get a further constraint for the possible values of \( n \) using Eqn. [9]. This equation will be taken instead as giving the current value of the snap as a function of \( n \) (see fig.[2] [22]. We have also plotted in the figure the error associated with \( s_0 \) (dashed line), which grows as \( n \) for large \( n \). The plot shows that with the current error of the kinematical parameters the method outlined here is

![FIG. 1: Plot of \( \phi \) (solid curve, see Eqn. [9]) and the associated error \( \delta \phi \) (dashed curve) in terms of \( n \).](image-url)
FIG. 2: Plot of the snap for \( f(R) \) given in Eqn. 7 (solid curve) and the associated error (dashed curve) in terms of \( n \).

helpful in determining \( s_0 \) for the theory given by Eqn. 7 only for \( n = 1 \).

Let us next analyze an example of a theory involving two dimension-full parameters, given by [18]

\[
f(R) = R + \alpha \ln \frac{R}{\mu},
\]

where \( \mu < 0 \). In this case it follows from Eqn. 5 and \( \alpha < 0 \) that

\[
0 < \frac{\mu}{R_0} < e^{-g(\beta)}
\]

where \( \beta = \alpha/R_0 \) and

\[
g(\beta) = \frac{1}{2} \left[-6q_0(1 + \beta)\frac{H_0^2}{R_0} + 1 - 36\frac{H_0^4}{R_0^2}\beta(j_0 - q_0 - 2)\right].
\]

Figure 3 shows the permitted values for \( \mu/R_0 \) in the case \( \alpha < 0 \), which are between the horizontal axis and the solid curve, as well as the associated error [23]. The plot shows that \( \mu/R_0 \) tends to a constant value (\( \approx 2.1 \)) for large \( \beta \). Hence, the possible values of \( \mu \) are restricted to \( |\mu| \lesssim 1.2 \times 10^{-41} \text{ m}^{-2} \).

In the case of the theory given in Eqn. 10, the snap would be a function of \( \alpha, \mu \), and of the remaining kinematical parameters.

Discussion

A new method to restrict gravitational theories described by functions of the Ricci scalar has been introduced. It is based essentially in the assumption that normal matter composes the universe, the acceleration being caused by new gravitational dynamics in the low curvature regime, described by \( f(R) \). By imposing that the matter satisfy the weak energy condition, we obtain an inequality that constrain the parameters in the theory. We have shown by way of two examples how the method can be used, and how it conduces to restrictive limits on the parameters, having taken the error into account. We also obtained an equation that depends on the snap, the fourth derivative of the scale factor. Had we any measurements of \( s_0 \), this equation would furnish yet another condition on the parameters of the theory. Since the current value of the snap has not been determined yet, we take this equation as forecasting, for a given \( f(R) \), the current value of the snap. The method presented here could be combined with other approaches (such as avoidance of super-luminal propagation speed [19], compatibility with the PPN limit [20], or those mentioned in the introduction) to restrict the \( f(R) \) theories that are being presented as candidates to model the acceleration of the universe.

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[22] Note that Eqn.(6) is independent of the sign of $\alpha$.
[23] For $\alpha > 0$, the values below the solid curve are the excluded ones.