1. Introduction

Fix a number field \( K \) with ring of integers \( \mathcal{O}_K \), as well as a finite set \( S \) of places of \( K \) that contains the archimedean places. Denote by \( \mathcal{O}_{K,S} \) the corresponding ring of \( S \)-integers. The purpose of this short note is to introduce a qualitative conjecture, in the spirit of Campana, to the effect that certain subsets of rational points on a variety over \( K \) or a Deligne–Mumford stack over \( \mathcal{O}_{K,S} \) cannot be Zariski dense; see Conjecture 1.2. This conjecture interpolates, in a way that we make precise, between Lang’s conjecture for rational points on varieties over \( K \) of general type, and the conjecture of Lang and Vojta that asserts that \( \mathcal{O}_{K,S} \)-points on a variety of logarithmic general type are not Zariski-dense. One might thus expect our conjecture to follow from Vojta’s quantitative conjecture on integral points; we show this is the case. As an application we show, assuming Conjecture 1.2, that for a fixed positive integer \( g \), there is an integer \( m_0 \) such that, for any \( m > m_0 \), no principally polarized abelian variety \( A/K \) of dimension \( g \) with semistable reduction outside of \( S \) has full level-\( m \) structure.
Remark 1.3. CAMPANA stated the case of Conjecture 1.2 for curves in [4, Conjecture 4.5 and Remark 4.6]. ABRAMOVICH gave a higher dimensional statement of it in [3, Conjecture 2.4.19]. Conjecture 1.2 is a streamlined version of this generalization.

Remark 1.4. Setting $\epsilon_i = 1$ for all $i$ in Conjecture 1.2, the condition $n_q(\mathcal{D}, x) \geq 1$ is automatically satisfied for all rational points and all $q$. Also $K_X + \sum (1 - \epsilon_i)D_i = K_X$, hence we recover LANG’s conjecture for rational points on varieties of general type. At the other end of the spectrum, setting all the $\epsilon_i = 0$, the condition $n_q(\mathcal{D}, x) \geq 1$ can only be satisfied if $n_q(\mathcal{D}, x) = 0$ at all $q \subset \mathcal{O}_{K,S}$, so $x$ is $S$-integral on $X \setminus \mathcal{D}$. Hence we get the LANG–VOJTA conjecture: $S$-integral points on a variety of logarithmic general type are not Zariski-dense. We show in §3 that Conjecture 1.2 follows from VOJTA’s conjecture.

1.1. Application. We give an application of Conjecture 1.2, in the spirit of our recent work [2, 1]. Recall that, for a positive integer $m$, a full level-$m$ structure on an abelian variety $A/K$ of dimension $g$ is an isomorphism of group schemes on the $m$-torsion subgroup

\begin{equation}
\phi: A[m] \xrightarrow{\sim} (\mathbb{Z}/m\mathbb{Z})^g \times (\mu_m)^g.
\end{equation}

Theorem 1.5. Let $K$ be a number field, $S$ a finite set of places, and let $g$ be a positive integer. Assume Conjecture 1.2. Then there is an integer $m_0$ such that, for any $m > m_0$, no principally polarized abelian variety $A/K$ of dimension $g$ with semistable reduction outside $S$ has full level-$m$ structure.

Remark 1.6. In [1], we prove a version of Theorem 1.5 without a semistability assumption on the abelian variety $A$, at the cost of assuming VOJTA’s conjecture.

The idea behind Theorem 1.5 is the following. Let $(\tilde{A}_g)_K$ denote the moduli stack parametrizing principally polarized abelian varieties of dimension $g$ over $K$. We show in Proposition 2.2 that for $\epsilon > 0$ and $\vec{\epsilon} = (\epsilon, \epsilon, \ldots)$, if $X \subseteq (\tilde{A}_g)_K$ is closed, then the set $X(K, S)_{\geq m_0}$ of $K$-rational points of $X$ corresponding to abelian varieties $A/K$ admitting full level-$m$ structure for some $m \geq m_0$ and having semistable reduction outside of $S$ lies inside an $\vec{\epsilon}$-Campana set, for $m_0 \gg 0$. We then use a result on logarithmic hyperbolicity [2, Theorem 1.6] to verify the hypothesis of Conjecture 1.2 in this case, and a Noetherian induction argument to conclude the proof of Theorem 1.5.

2. Proof of Theorem 1.5

2.1. Moduli spaces and toroidal compactifications. We follow the notation of [1, §4], working over $\text{Spec} \mathbb{Z}$:

- $\tilde{\mathcal{A}}_g \subset \mathcal{A}_g$ a toroidal compactification of the moduli stack of principally polarized abelian varieties of dimension $g$
- $\mathcal{A}_g \subset \tilde{\mathcal{A}}_g$ the resulting compactification of the moduli space of principally polarized abelian varieties of dimension $g$
\[ \tilde{A}_g^{[m]} \subset A_g^{[m]} \] a compatible toroidal compactification of the moduli stack of principally polarized abelian varieties of dimension \( g \) with full level-\( m \) structure

\[ A_g^{[m]} \subset \mathcal{A}_g^{[m]} \] the resulting compactification of the moduli space of principally polarized abelian varieties of dimension \( g \) with full level-\( m \) structure

As noted in [1], we may use a construction by Faltings and Chai [5] of the stack \( \tilde{A}_g^{[m]} \), which is a priori smooth over \( \text{Spec} \mathbb{Z}[1/m, \zeta_m] \), where \( \zeta_m \) is a primitive \( m \)-th root of unity, to obtain a stack we denote by \( (\tilde{A}_g^{[m]})_{\mathbb{Z}[1/m]} \) smooth over \( \mathbb{Z}[1/m] \). This stack is extended over all of \( \text{Spec} \mathbb{Z} \) by defining \( \tilde{A}_g^{[m]} \) as the normalization of \( \tilde{A}_g^{[m]} \) in \( (\tilde{A}_g^{[m]})_{\mathbb{Z}[1/m]} \). Unfortunately, even the interior of this stack over primes dividing \( m \) does not have a modular interpretation. See, however, the results of Madapusi Pera in [1, Appendix A].

2.2. Semistability and integrality. We require the following well-known statement essentially contained in [5].

**Proposition 2.1.** Let \( K \) be a number field, \( S \) a finite set of places. Let \( A/K \) be a principally polarized abelian variety with full level-\( m \) structure, and with semistable reduction outside of \( S \). Then the point \( x_m: \text{Spec} K \to \tilde{A}_g^{[m]} \) associated to \( A \) extends to an integral point \( \xi_m: \text{Spec} \mathcal{O}_{K,S} \to \tilde{A}_g^{[m]} \).

**Proof.** First consider the case \( m = 1 \). By [5, Theorem IV.5.7(5)] the extension exists if and only if, for every prime \( q \notin S \) and for any strictly henselization \( V \) of \( \mathcal{O}_{K,q} \) with valuation \( v \), the bimultiplicative form \( v \circ b \) corresponds to a point of a cone only depending on \( q \). (Here \( b \) is the symmetric bimultiplicative form associated to the degeneration of \( A \) at \( q \) by the theory of degenerations, as indicated in [5, Proposition IV.5.1].) This condition is automatic for a Dedekind domain such as \( \mathcal{O}_{K,q} \), see [5, Remark IV.5.3], hence our proposition holds in case \( m = 1 \).

To prove the statement in general, consider the point \( x: \text{Spec} K \to \tilde{A}_g \) obtained by composing \( x_m \) with \( \tilde{A}_g^{[m]} \to \tilde{A}_g \). Since the proposition holds for \( m = 1 \), the point \( x \) extends to \( \xi: \text{Spec} \mathcal{O}_{K,S} \to \tilde{A}_g \). Since \( \tilde{A}_g^{[m]} \to \tilde{A}_g \) is representable and finite, the stack \( Z := \text{Spec} \mathcal{O}_{K,S} \times_{\tilde{A}_g} \tilde{A}_g^{[m]} \), where the projection on the left is \( \xi \), is in fact a scheme finite over \( \text{Spec} \mathcal{O}_{K,S} \). The point \( x_m \) defines a point \( \text{Spec} K \to Z \), which extends to a point \( \text{Spec} \mathcal{O}_{K,S} \to Z \) by the valuative criterion for properness. Composing with the projection \( Z \to \tilde{A}_g^{[m]} \) gives the desired point \( \xi_m \).

2.3. Intersection multiplicities. Let \((\mathcal{X}, \mathcal{D})\) be a normal crossings model, and let \( I_{\mathcal{D}_i} \) denote the ideal of \( \mathcal{D}_i \). Given a maximal ideal \( q \) of \( \mathcal{O}_{K,S} \) with localization \( \mathcal{O}_{K,q} \), and a point
Let $x \in \mathcal{X}(\mathcal{O}_{K,q})$, define $n_q(\mathcal{D}_i, x)$ through the equality of ideals in $\mathcal{O}_{K,q}$

$$I_{\mathcal{D}_i} = q^{n_q(\mathcal{D}_i, x)}.$$ 

We call $n_q(\mathcal{D}_i, x)$ the intersection multiplicity of $x$ and $\mathcal{D}_i$.

### 2.4. Notation for substacks

Let $X \subseteq (\overline{A}_g)_K$ be a closed substack, let $X' \to X$ be a resolution of singularities, $X' \subseteq \overline{X}'$ a smooth compactification with $D = \overline{X}' \setminus X'$ a normal crossings divisor. Assume that the rational map $f: \overline{X}' \to \overline{A}_g$ is a morphism. Let $X_m' = X' \times_{\overline{A}_g} \overline{A}_g^{[m]}$, and let $\overline{X}_m' \to \overline{X}' \times_{\overline{A}_g} \overline{A}_g^{[m]}$ be a resolution of singularities with projections $\pi_m: \overline{X}_m' \to \overline{X}'$ and $f_m: \overline{X}_m' \to \overline{A}_g^{[m]}$.

We now spread these objects over $\mathcal{O}_{K,S}$ for a suitable finite set of places $S$ containing the archimedean places. Let $(\mathcal{X}, \mathcal{D})$ be a normal crossings model of $(\overline{X}', D)$ over Spec $\mathcal{O}_{K,S}$. As above, write $\mathcal{D} = \sum_i \mathcal{D}_i$. Such a model exists, even for Deligne–Mumford stacks, by [7, Proposition 2.2]. To avoid clutter, in the special case when $\mathcal{D} = (\epsilon, \epsilon, \ldots)$, an $\epsilon$-Campana point of $\mathcal{X}'$ shall be called an $\epsilon$-Campana point, and we write $\mathcal{X}'(\mathcal{O}_{K,S})_{\epsilon, \mathcal{D}}$ for the set of $\epsilon$-Campana points of $(\mathcal{X}', \mathcal{D})$.

### 2.5. Levels and Campana points

Let $X(K, S)_{[m]}$ be the set of $K$-rational points of $X$ corresponding to principally polarized abelian varieties $A/K$ with semistable reduction outside $S$, admitting full level-$m$ structure. Define

$$X(K, S)_{\geq m_0} := \bigcup_{m \geq m_0} X(K, S)_{[m]}.$$

#### Proposition 2.2

Fix $\epsilon > 0$. Then there exists $m_0$ such that $X(K, S)_{\geq m_0}$ is contained in the set $\mathcal{X}'(\mathcal{O}_{K,S})_{\epsilon, \mathcal{D}}$ of $\epsilon$-Campana points of $(\mathcal{X}', \mathcal{D})$.

**Proof.** Let $x_m \in X'_m(K)$, and write $\pi_m(x_m) =: x$ for its image in $\overline{X}'(K)$. Let $q \notin S$ be a finite place of $K$, and let $\mathcal{O}_{K,q}$ be the corresponding local ring. Let $\xi: \text{Spec} \mathcal{O}_{K,q} \to \overline{X}'$ and $\xi_m: \text{Spec} \mathcal{O}_{K,q} \to \overline{X}_m'$ be the extensions of $x$ and $x_m$ to Spec $\mathcal{O}_{K,q}$, which exist by Proposition 2.1.

Write $E$ for the boundary divisors of $(\overline{A}_g)_K$; on $\overline{X}'$ we have an equality of divisors

$$f^*E = \sum a_i D_i,$$

where each $a_i > 0$; see [2, Equation (4.3)]. It follows from [1, Proposition 4.3] that there exists an integer $M$ depending only on $g$ such that

$$n_q(\mathcal{D}, x) \geq \frac{m}{M \cdot \max\{a_i\}}.$$ 

We note that in our case we can take $M = 1$ because $x$ and $x_m$ could be extended to $\mathcal{O}_{K,q}$-points, as the proof of [1, Proposition 4.3] shows. Thus, if $m \geq \max\{a_i\}/\epsilon$, the point $x \in \overline{X}'(K)$ is an $\epsilon$-Campana point. ✠
Proof of Theorem 1.5. We proceed by Noetherian induction. For each integer $i \geq 1$, let

$$W_i = \overline{A_g(K)}_{\geq i}.$$ 

Note that $W_i$ is a closed subset of $A_g$, and that $W_i \supseteq W_{i+1}$ for every $i$. The chain of $W_i$ must stabilize by the Noetherian property of the Zariski topology of $A_g$. Say $W_n = W_{n+1} = \cdots$.

We claim that $W_n$ has dimension $\leq 0$. Suppose not, and let $X \subseteq W_n$ be an irreducible component of positive dimension. Fix $\epsilon > 0$ so that $K_X + (1-\epsilon)D$ is big: such an $\epsilon$ exists by [2, Corollary 1.7]. Hence the hypothesis of Conjecture 1.2 holds (with all $\epsilon_i$ equal to $\epsilon$.) By Conjecture 1.2, the set $\mathcal{X}(O_{K,S})_{\epsilon \mathcal{D}}$ of $\epsilon$-Campana points is not Zariski dense in $X$. On the other hand, Proposition 2.2 shows there is an integer $m_0$ such that $X(K,S)_{\geq m_0} \subseteq \mathcal{X}(O_{K,S})_{\epsilon \mathcal{D}}$, from which is follows that $X(K,S)_{\geq m_0}$ is not Zariski-dense in $X$. Thus $W_{m_0}$, which equals $W_n$, does not contain $X$, and thus $X$ is not an irreducible component after all. This proves that $\dim W_n \leq 0$.

Finally, if $W_n$ is a finite set of points, then we can apply the Mordell–Weil theorem to conclude that $W_n(K)[m] = \emptyset$ for all $m \gg 0$. ✷

3. Vojta’s conjecture and Campana points

3.1. Counting functions for integral points. Let $(\mathcal{X}, \mathcal{D})$ be a normal crossings model. For $q \in O_{K,S}$, we denote by $\kappa(q)$ the residue field of the associated local ring. Following Vojta [8, p. 1106], for $x \in \mathcal{X}(O_{K,S})$, define the counting function

$$N(D, x) = \sum_{q \in \text{Spec} \ O_{K,S}} n_q(D, x) \log |\kappa(q)|,$$

as well as the truncated counting function

$$N^{(1)}(D, x) = \sum_{q \in \text{Spec} \ O_{K,S}, n_q(D, x) > 0} \log |\kappa(q)|.$$

The quantities on the right hand sides of (3.1) and (3.2) depend on the model $(\mathcal{X}, \mathcal{D})$ and the finite set $S$ only up to functions bounded on $X(O_{K,S})$. Since we are interested in these quantities only up to such functions, the notation $N(D, x)$ and $N^{(1)}(D, x)$ does not reflect the model $(\mathcal{X}, \mathcal{D})$ or the finite set $S$.

3.2. Vojta’s conjecture for integral points. For a smooth proper Deligne–Mumford stack $\mathcal{X} \to \text{Spec} \ O_{K,S}$ over the ring of $S$-integers $O_{K,S}$ of a number field $K$, we write $X = \mathcal{X}_K$ for the generic fiber, which we assume is irreducible, and $\underline{X}$ for the coarse moduli space of $X$. Similarly, for a normal crossings divisor $\mathcal{D}$ of $\mathcal{X}$, we write $D$ for its generic fiber.

For a divisor $H$ on $\underline{X}$, we denote by $h_H(x)$ the Weil height of $x$ with respect to $H$, which is well-defined up to a bounded function on $\underline{X}(\overline{K})$. If $H$ is only a divisor on $X$, then some positive integer multiple $rH$ descends to $\underline{X}$. Given a point $x \in X(\overline{K})$ we define $h_H(x) = \frac{1}{r}h_H(\underline{x})$, where $\underline{x}$ is the image of $x$ in $\underline{X}(\overline{K})$. 
The following is a version of Vojta’s conjecture for stacks, applied to integral points:

**Conjecture 3.1.** Let \( \mathcal{X} \to \text{Spec} \mathcal{O}_{K,S} \), \( X \), \( X_\mathcal{X} \), and \( D \) be as above. Suppose that \( X_\mathcal{X} \) is projective, and let \( H \) be a big line bundle on it. Fix \( \delta > 0 \). Then there is a proper Zariski-closed subset \( Z \subset X \) containing \( D \) such that

\[
N^{(1)}(D, x) \geq h_{KX(D)}(x) - \delta h_{H}(x) - O(1)
\]

for all \( x \in \mathcal{X}(\mathcal{O}_{K,S}) \setminus Z(K) \).

In [1] we showed that a stronger conjecture, which applies to points \( x \in X(\overline{K}) \) with \([K(x) : K]\) bounded, follows from Vojta’s original conjecture for schemes [8]. Here we have stated only its outcome for integral points.

### 3.3. From Vojta’s conjecture to Conjecture 1.2.

**Lemma 3.2.** If \( x \in \mathcal{X}(\mathcal{O}_{K,S}) \) is an \( \bar{\epsilon} \)-Campana point, then

\[
N^{(1)}(D, x) \leq N(D_{\bar{\epsilon}}, x) \leq h_{D_{\bar{\epsilon}}}(x) + O(1).
\]

**Proof.** If \( x \in \mathcal{X}(\mathcal{O}_{K,S})_{\bar{\epsilon}} \) is an \( \bar{\epsilon} \)-Campana point, and \( n_q(\mathcal{D}, x) > 0 \), then we have

\[
\sum_{i} \epsilon_i \cdot n_q(\mathcal{D}_i, x) = n_q(\mathcal{D}, x) \geq 1.
\]

The definition (3.2) of \( N^{(1)}(D, x) \) gives

\[
N^{(1)}(D, x) = \sum_{n_q(\mathcal{D}, x) > 0} \log |\kappa(q)|
\leq \sum_{n_q(\mathcal{D}, x) > 0} n_q(\mathcal{D}, x) \log |\kappa(q)|
= N(D_{\bar{\epsilon}}, x)
\leq h_{D_{\bar{\epsilon}}}(x) + O(1),
\]

where the last inequality follows as in [8, p. 1113] or [6, Theorem B.8.1(e)].

**Corollary 3.3.** Vojta’s conjecture 3.1 implies the \( \bar{\epsilon} \)-Campana conjecture 1.2.

**Proof.** Since the divisor \( K_X + \sum \epsilon_i(1 - \epsilon_i)D_i \) is big, we may choose an ample \( \mathbb{Q} \)-divisor \( H \) such that \( K_X + \sum \epsilon_i(1 - \epsilon_i)D_i - H \) is effective. Let \( x \in \mathcal{X}(\mathcal{O}_{K,S})_{\bar{\epsilon}} \) be an \( \bar{\epsilon} \)-Campana point. Applying Conjecture 3.1, we obtain a proper Zariski closed subset \( \mathcal{Z} \subset \mathcal{X} \) such that if \( x \notin \mathcal{Z} \), the inequality

\[
N^{(1)}(D, x) \geq h_{KX(D)}(x) - \delta h_{H}(x) - O(1)
\]

holds. By Lemma 3.2, we may replace the left hand side with \( h_{D_{\bar{\epsilon}}}(x) + O(1) \) to get

\[
h_{D_{\bar{\epsilon}}}(x) + O(1) \geq h_{KX(D)}(x) - \delta h_{H}(x) - O(1).
\]
This implies that
\[ O(1) \geq h_{K_X}(\sum_i (1-\epsilon_i)D_i)(x) - \delta h_H(x). \]

By our choice of \( H \), we have
\[ O(1) \geq (1 - \delta)h_{K_X}(\sum_i (1-\epsilon_i)D_i)(x). \]

Since \( K_X + \sum_i (1-\epsilon_i)D_i \) is big, the set of \( x \in \mathcal{X}(O_{K,S})_{\mathcal{F}_x} \) that avoid \( \mathcal{F} \) is finite [6, Theorem B.3.2(g)]. Since \( \mathcal{F} \) is a proper Zariski-closed set, we conclude that the set \( \mathcal{X}(O_{K,S})_{\mathcal{F}_x} \) of \( \mathcal{F} \)-Campana points is not Zariski-dense in \( X \).

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