RECI PROCAL TRANSFORMATIONS
FOR STÅCKEL-RELATED LIOUVILLE INTEGRABLE SYSTEMS

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ABSTRACT. We consider the Stäckel transform, also known as the coupling-constant metamorphosis, which under certain conditions turns a Hamiltonian dynamical system into another such system and preserves the Liouville integrability. We show that the corresponding transformation for the equations of motion is nothing but the reciprocal transformation of a special form and we investigate the properties of this transformation. This result is further applied for the study of the $k$-hole deformations of the Benenti systems or more general seed systems.

INTRODUCTION

The Stäckel transform [9], also known as the coupling-constant metamorphosis [14] (cf. e.g. also [25]), is a powerful tool for producing new Liouville integrable systems from the known ones. This is essentially a transformation that maps an $n$-tuple of functions in involution on a $2n$-dimensional Poisson manifold into another $n$-tuple of functions on the same manifold, and these $n$ new functions are again in involution. In the present paper we show that the corresponding transformations for equations of motion are nothing but reciprocal transformations. We also study the properties and present some applications of the latter.

The significance of reciprocal transformations in the theory of integrable nonlinear partial differential equations is well recognized. These transformations were intensively used in the theory of dispersionless systems as well as the theory of soliton systems (see e.g. [20, 22] and references therein). However, the role of the reciprocal transformations in the theory of finite-dimensional dynamical systems is far from being fully explored, and the goal of the present paper is to contribute to such an exploration by developing the theory of reciprocal transformations for Liouville integrable Hamiltonian systems. To the best of our knowledge, such transformations first appeared in the paper [14] by Hietarinta et al., where the concept of the coupling-constant metamorphosis, or the Stäckel transform [9] (cf. also [26] for even more general transformations in the action-angle variables and [15, 16, 27] for more recent developments), was introduced. The reciprocal transformation appeared in this context as a transformation expressing the time (evolution parameter) for the target system through that of the source system [14], but the question of whether it sends the (solutions of) equations of motion for the source system into those of the target system was not addressed in [14].

In fact, as we show below, this transformation, when applied to the equations of motion of the source system, in general does not yield the equations of motion for the target system, unless we restrict the equations of motion onto the level surfaces of the corresponding Hamiltonians, see Propositions 3 and 5 below for details.

Even more broadly, we show that two Liouville integrable systems related by an appropriate Stäckel transform for the constants of motion are related by the reciprocal transformation for the equations of motion restricted to appropriate Lagrangian submanifolds (see e.g. Ch.3 of [10] and references therein for more details on the latter).

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Moreover, we present a multitime extension of the original reciprocal transformation from Hietarinta et al. \cite{14}, and study the applications of this extended transformation to the integration of equations of motion in the Hamilton–Jacobi formalism using the separation of variables (cf. \cite{9}).

As a byproduct, we present reciprocal transformations for a large class of dispersionless, weakly nonlinear hydrodynamic-type systems, the so-called Killing systems \cite{8} that are intimately related to the Stäckel-separable systems \cite{12,13,5}.

In the second part of the paper we consider the relations among classical Liouville integrable Stäckel systems on $2n$-dimensional phase space. In \cite{7} infinitely many classes of the Stäckel systems related to the so-called seed class, namely, the $k$-hole deformations of the latter, were constructed. Here we show that any $k$-hole deformation consists of a sequence of elementary deformations (one-hole deformations). These elementary deformations are nothing but particular cases of the Stäckel transforms considered in the first section of the present paper. Hence the equations of motion for infinitely many classes of Stäckel systems are related, upon restriction onto appropriate Lagrangian submanifolds, to the equations of motion for the systems from the seed class (which is a natural generalization of the Benenti class for the classical Stäckel systems) by a sequence of reciprocal transformations.

The significance of this result stems from the fact that the overwhelming majority of known today Liouville integrable natural dynamical systems that admit orthogonal separation of variables belong to the seed class. The $k$-hole deformations of such systems again are Liouville integrable natural dynamical systems and admit orthogonal separation of variables, but the corresponding separation curves no longer are of the seed type. Our results make it possible to understand how the corresponding dynamics is different from that of the systems from the seed class, and thereby reveal new properties of the deformed systems, which will be discussed in more detail elsewhere.

1. Main results

We start with the following simple results that slightly generalize Proposition 2 from \cite{25} and the results of \cite{14,15}. The proof is by straightforward computation.

**Proposition 1.** Let $(M,P)$ be a Poisson manifold with the Poisson bracket $\{f,g\} = (df,Pdg)$. Consider $k$ functions on $M$ of the form

$$H_i = H_i^{(0)} + \alpha H_i^{(1)}, \quad i = 1, \ldots, k,$$

where $\alpha$ is a parameter, and $H_i^{(0)}$ and $H_i^{(1)}$ are smooth functions on $M$. Assume that $H_i$ are functionally independent for all values of $\alpha$.

Suppose that there exists an $s \in \{1, \ldots, k\}$ such that $H_s^{(1)} \neq 0$ and

$$\{H_s, H_j\} = 0 \quad (2)$$

for all $j = 1, \ldots, k$ and for all values of $\alpha$, and let

$$\tilde{H}_i = \tilde{H}_i^{(0)} + \tilde{\alpha}\tilde{H}_i^{(1)} = H_i^{(0)} - H_i^{(1)}(H_s^{(0)} - \tilde{\alpha})/H_s^{(1)}, \quad i = 1, \ldots, s - 1, s + 1, \ldots, k,$$

$$\tilde{H}_s = \tilde{H}_s^{(0)} + \tilde{\alpha}\tilde{H}_s^{(1)} = -(H_s^{(0)} - \tilde{\alpha})/H_s^{(1)},$$

where $\tilde{\alpha}$ is another parameter.

Then we have

$$\{\tilde{H}_s, \tilde{H}_j\} = 0 \quad (4)$$

for all $j = 1, \ldots, k$ and for all values of $\tilde{\alpha}$.

**Corollary 1.** Under the assumptions of Proposition 1 suppose that we have $\{H_i, H_j\} = 0$ for some (fixed) $i$ and $j$ and for all values of $\alpha$. Then

$$\{\tilde{H}_i, \tilde{H}_j\} = 0$$
for all values of $\tilde{\alpha}$.

The transformation from $H_i$ to $\tilde{H}_i$ is known as a coupling-constant metamorphosis \cite{14} or as a (generalized) Stäckel transform \cite{9,15}.

From Proposition 1 and Corollary 1 it is immediate that the transformation (3) preserves (super)integrability: if the dynamical system associated with $H_s$ is Liouville integrable (so $k \geq n = \frac{1}{2}\text{rank} P$ and $H_s$ belongs to a family of $n$ commuting Hamiltonians $H_i$ such that $PdH_i \neq 0$ for all $i$) or superintegrable (i.e., $H_s$ is Liouville integrable and $k > n$), then so is the dynamical system associated with $\tilde{H}_s$.

Note that the relations (3) can be inverted:

\begin{equation}
H^{(0)}_i = \tilde{H}_i - H^{(1)}_s \tilde{H}_s, \quad i = 1, \ldots, s - 1, s + 1, \ldots, k,
\end{equation}

\begin{equation}
H^{(0)}_s = -H^{(1)}_s \tilde{H}_s + \tilde{\alpha}.
\end{equation}

Recall that the equations of motion associated with a Hamiltonian $H$ and a Poisson structure $P$ on $M$ read (see e.g. \cite{3})

\begin{equation}
dx^b / dt = (X_H)^b, \quad b = 1, \ldots, \dim M,
\end{equation}

where $x^b$ are local coordinates on $M$, $X_H = PdH$ is the Hamiltonian vector field associated with $H$, and $t_H$ is the corresponding evolution parameter.

Consider the equations of motion (6) for $H = H_s$ and $t_H = t$ and for $H = \tilde{H}_s$ and $t_H = \tilde{t}$:

\begin{equation}
dx^b / dt = (X_{H_s})^b, \quad b = 1, \ldots, \dim M,
\end{equation}

\begin{equation}
dx^b / d\tilde{t} = (X_{\tilde{H}_s})^b, \quad b = 1, \ldots, \dim M.
\end{equation}

According to \cite{14} we have a reciprocal transformation (see e.g. \cite{20,22,23} for more details on such transformations) relating the times $t$ and $\tilde{t}$:

\begin{equation}
d\tilde{t} = -H^{(1)}_s dt.
\end{equation}

When does (9) turn (7) into (8)? From (9) we find that

\begin{equation}
d/ dt = -H^{(1)}_s d/ d\tilde{t},
\end{equation}

and taking into account (7) and (8) we see that our question boils down to the following: when does the equality

\begin{equation}
X_{H_s} = -H^{(1)}_s X_{\tilde{H}_s}
\end{equation}

hold?

We have $X_{H_s} = PdH_s = PdH^{(0)}_s + \alpha PdH^{(1)}_s$ and

\begin{equation}
X_{\tilde{H}_s} = Pd\tilde{H}_s \overset{3}{=} -\frac{1}{H^{(1)}_s} PdH^{(0)}_s + \frac{H^{(0)}_s - \tilde{\alpha}}{\left(H^{(1)}_s\right)^2} PdH^{(1)}_s.
\end{equation}

Plugging this into (10) and multiplying the resulting equation by $H^{(1)}_s$, which is nonzero by assumption, we obtain the following equation:

\begin{equation}
(H^{(0)}_s + \alpha H^{(1)}_s - \tilde{\alpha}) PdH^{(1)}_s = 0.
\end{equation}

Clearly, (11) holds if and only if either $PdH^{(1)}_s = 0$ or $(H^{(0)}_s + \alpha H^{(1)}_s - \tilde{\alpha}) = 0$. The first possibility immediately yields the following result:

**Proposition 2.** Under the assumptions of Proposition 1, suppose that $H^{(1)}_s$ is a Casimir function for $P$, i.e., $PdH^{(1)}_s = 0$. Then the transformation (7) sends the equations of motion (6) for $H_s$ into the equations of motion (8) for $\tilde{H}_s$. 
The second possibility is slightly more involved and will be of greater interest to us in the sequel:

**Proposition 3.** Under the assumptions of Proposition 1, the transformation (9) sends the equations of motion (7) for $H_s$ restricted onto the level surface $H_s = \tilde{\alpha}$ of $H_s$ into the equations of motion (8) for $\tilde{H}_s$ restricted onto the level surface $\tilde{H}_s = \alpha$ of $\tilde{H}_s$.

*Proof.* Indeed, if $H_s = \tilde{\alpha}$ then $H_s^{(0)} + \alpha X_s^{(1)} - \tilde{\alpha} = 0$ by (11). On the other hand, the condition $\tilde{H}_s = \alpha$ is equivalent to $H_s = \tilde{\alpha}$ by (3). Thus, if $\tilde{H}_s = \alpha$ or $H_s = \tilde{\alpha}$ then (11) holds, and the result follows. □

Note that as the parameters $\alpha$ and $\tilde{\alpha}$ are arbitrary, (9) will transform the equations of motion (7) restricted onto any given level surface of $H_s$ into the equations of motion (8) restricted onto any given level surface of $\tilde{H}_s$. Thus we have a remarkable duality among the deformation parameters and the energy (eigen)values that can be readily transferred to the quantum case.

2. Multitime extension

Now suppose that all $H_i$ are in involution:

$$\{H_i, H_j\} = 0, \quad i, j = 1, \ldots, k. \tag{12}$$

Then by Corollary 1 so are $\tilde{H}_i$:

$$\{\tilde{H}_i, \tilde{H}_j\} = 0, \quad i, j = 1, \ldots, k.$$

Consider the simultaneous evolutions

$$dx^b/dt_i = (X_{H_i})^b, \quad b = 1, \ldots, \dim M, \quad i = 1, \ldots, k, \tag{13}$$

$$dx^b/d\tilde{t}_i = (X_{\tilde{H}_i})^b, \quad b = 1, \ldots, \dim M, \quad i = 1, \ldots, k, \tag{14}$$

and the following extension of (9):

$$d\tilde{t}_s = -\sum_{i=1}^k H_i^{(1)} dt_i, \quad \tilde{t}_i = t_i, \quad i = 1, 2, \ldots, k. \tag{15}$$

It is straightforward to verify that by virtue of (11), (12), and (13) we have

$$\{H_i^{(0)}, H_j^{(0)}\} = 0, \quad \{H_i^{(1)}, H_j^{(1)}\} = 0, \quad (H_i^{(1)})_{ij} = (H_j^{(1)})_{ij}, \quad i, j = 1, \ldots, k,$$

so the transformation (15) for $\tilde{t}_s$ is well-defined. This is where we need the commutativity of $H_i$.

By virtue of (13) we have

$$\frac{d}{dt_s} = -H_i^{(1)} \frac{d}{dt_s}, \quad \frac{d}{dt_i} = \frac{d}{dt_i} - H_i^{(1)} \frac{d}{dt_s}, \quad i = 1, 2, \ldots, s - 1, s + 1, \ldots, k.$$

In view of (13) and (14) we search for conditions when

$$X_{H_i} = -H_i^{(1)} X_{\tilde{H}_i}, \quad X_{\tilde{H}_i} = X_{\tilde{H}_i} - H_i^{(1)} X_{\tilde{H}_s}, \quad i = 1, 2, \ldots, s - 1, s + 1, \ldots, k. \tag{16}$$

Plugging (11) and (3) into (16) yields, after some simplifications,

$$(H_i^{(0)} + \alpha H_i^{(1)} - \tilde{\alpha}) PdH_i^{(1)} = 0, \quad i = 1, \ldots, k.$$

Hence we have the following generalizations of Propositions 2 and 3:

**Proposition 4.** Under the assumptions of Proposition 1, suppose that

$$\{H_i, H_j\} = 0, \quad i, j = 1, \ldots, k,$$

and that $H_i^{(1)}, i = 1, \ldots, k,$ are Casimir functions for $P$, i.e., $PdH_i^{(1)} = 0, i = 1, \ldots, k$. Then (13) transforms (15) into (14).
Proposition 5. Under the assumptions of Proposition 4 suppose that
\[ \{H_i, H_j\} = 0, \quad i, j = 1, \ldots, k. \]
Then the reciprocal transformation (15) sends the equation of motion (13) restricted onto the level surface \( H_s = \bar{\alpha} \) into the equations of motion (17) restricted onto the level surface \( \bar{H}_s = \alpha \).

3. Canonical Poisson structure

Let \( P \) be a canonical Poisson structure on \( M = \mathbb{R}^{2n} \). Then the Hamilton–Jacobi equations for \( H_1 \) and \( \bar{H}_1 \) have a common solution, cf. [9]. Namely, we have the following extension of the results of [9] to the Hamiltonians that are not necessarily quadratic in the momenta:

Proposition 6. Under the assumptions of Proposition 4 let \( M = \mathbb{R}^{2n} \), \( P \) be a canonical Poisson structure on \( M \), and \( \lambda_i, \mu_i, \ i = 1, \ldots, n \), be the Darboux coordinates for \( P \), i.e., \( \{\lambda_i, \mu_j\} = \delta_{ij} \). Let \( \lambda = (\lambda_1, \ldots, \lambda_n) \) and \( \mu = (\mu_1, \ldots, \mu_n) \).

Let \( S = S(\lambda, \alpha, E_s, a_1, \ldots, a_{n-1}) \), where \( a_i \) are arbitrary constants, be a complete integral of the stationary Hamilton–Jacobi equation for the Hamiltonian \( H_s = H_s(\lambda, \mu) \),

\[ \bar{H}_s(\alpha, \lambda, \partial S/\partial \lambda) = E_s. \]

If we set \( E_s = \bar{\alpha} \) and \( \alpha = \bar{E}_s \) then \( S = S(\lambda, \alpha, E_s, a_1, \ldots, a_{n-1}) \) is a complete integral of the stationary Hamilton–Jacobi equation for the Hamiltonian \( \bar{H}_s = \bar{H}_s(\alpha, \lambda, \mu) \),

\[ \bar{H}_s(\alpha, \lambda, \partial S/\partial \lambda) = \bar{E}_s. \]

Moreover, let \( \{H_i, H_j\} = 0, \ i, j = 1, \ldots, k \), and let

\[ S = S(\lambda, \alpha, E_1, \ldots, E_k, a_1, \ldots, a_{n-k}) \]

where \( a_i \) are arbitrary constants, be a complete integral for the system of stationary Hamilton–Jacobi equations

\[ H_i(\alpha, \lambda, \partial S/\partial \lambda) = E_i, \quad i = 1, \ldots, k. \]

If we set

\[ \alpha = \bar{E}_s, \quad E_s = \bar{\alpha}, \quad E_i = \bar{E}_i, \quad i = 1, 2, \ldots, s - 1, s + 1, \ldots, k, \]

then \( S \) (17) is also a complete integral for the system

\[ \bar{H}_i(\alpha, \lambda, \partial S/\partial \lambda) = \bar{E}_i, \quad i = 1, \ldots, k. \]

As for the equations of motion, we have, in addition to general Propositions 4 and 5, a somewhat more explicit result:

Corollary 2. Under the assumptions of Proposition 4 let \( M = \mathbb{R}^{2n} \), \( P \) be a canonical Poisson structure on \( M \), and \( \lambda_i, \mu_i, \ i = 1, \ldots, n \) be the Darboux coordinates for \( P \), i.e., \( \{\lambda_i, \mu_j\} = \delta_{ij} \). Let \( \lambda = (\lambda_1, \ldots, \lambda_n) \) and \( \mu = (\mu_1, \ldots, \mu_n) \).

Suppose that \( k = n, \partial H^{(1)}_i/\partial \mu = 0 \) for all \( i = 1, \ldots, n \) and that \( \lambda_i \) can be chosen as local coordinates on the Lagrangian submanifold \( N_E \) = \{ \( (\lambda, \mu) \in M \mid H_i(\alpha, \lambda, \mu) = E_i, \ i = 1, \ldots, n \} \) (in other words, the system \( H_i(\alpha, \lambda, \mu) = E_i, \ i = 1, \ldots, n \) can be solved for \( \mu \)), and that we have

\[ \bar{E}_s = \alpha, \quad E_s = \bar{\alpha}, \quad \bar{E}_i = E_i, \quad i = 1, 2, \ldots, s - 1, s + 1, \ldots, n. \]

Then the reciprocal transformation (15) turns the system

\[ d\lambda/dt_i = (\partial H_i/\partial \mu)|_{N_E}, \quad i = 1, \ldots, n, \]

(19)

into

\[ d\lambda/d\bar{t}_i = (\partial \bar{H}_i/\partial \mu)|_{\bar{N}_E}, \quad i = 1, \ldots, n, \]

(20)

where \( \bar{N}_E \) = \{ \( (\lambda, \mu) \in M \mid \bar{H}_i(\alpha, \lambda, \mu) = \bar{E}_i, \ i = 1, \ldots, n \} \).
For instance, if we have
\[ H_i = \frac{1}{2} (\mu, G_i(\lambda) \mu) + V_i(\lambda) + \alpha W_i(\lambda), \quad i = 1, \ldots, n, \]  
(21)
where \((\cdot, \cdot)\) stands for the scalar product in \(\mathbb{R}^n\) and \(G_i(\lambda)\) are \(n \times n\) matrices, then the system (19) reads
\[ \frac{d\lambda}{dt_i} = G_i(\lambda) M, \]
(22)
where \(M = M(\lambda, \alpha, E_1, \ldots, E_n)\) is a general solution of the system \(H_i(\alpha, \lambda, \mu) = E_i, \ i = 1, \ldots, n\).

If we eliminate \(M\) from (22) then we obtain the dispersionless Killing systems (cf. [8])
\[ \lambda_{t_i} = G_i(G_s)^{-1} \lambda_{t_s}, \quad i = 1, 2, \ldots, s - 1, s + 1, \ldots, n, \]
(23)
and the reciprocal transformation (15), which in our case reads
\[ \frac{d\tilde{t}_s}{d\lambda_i} = -\sum_{i=1}^{n} W_i(\lambda)(dt_i), \quad \tilde{t}_i = t_i, \quad i \neq s, \]
(24)
turns (22) into
\[ \lambda_{t_i} = \tilde{G}_i(\tilde{G}_s)^{-1} \lambda_{\tilde{t}_s}, \quad i = 1, 2, \ldots, s - 1, s + 1, \ldots, n, \]
(25)
where \(\tilde{G}_s = -G_s/W_s\) and \(\tilde{G}_i = G_i - W_i G_s/W_s, \ i = 1, 2, \ldots, s - 1, s + 1, \ldots, n,\) are related to the Hamiltonians
\[ \tilde{H}_i = \frac{1}{2} (\mu, \tilde{G}_i(\lambda) \mu) + \tilde{V}_i(\lambda) + \tilde{\alpha} W_i(\lambda), \quad i = 1, \ldots, n. \]
(26)

4. Solving the Reduced Equations of Motion

Under the assumptions of Corollary 3 we can apply Proposition 6 in order to obtain the solutions of equations of motion (19) and (20) as follows:

**Corollary 3.** Under the assumptions of Corollary 2, suppose that
\[ S = S(\lambda, \alpha, E_1, \ldots, E_n) \]
(26)
is a complete integral for the system of stationary Hamilton–Jacobi equations
\[ H_i(\alpha, \lambda, \partial S/\partial \lambda) = E_i, \quad i = 1, \ldots, n. \]
Then a general solution of (19) for \(i = r\) can be written in implicit form as
\[ \partial S/\partial E_j = \delta_{jr} t_r + b_j, \quad j = 1, \ldots, n, \]
(27)
where \(b_j\) are arbitrary constants, and by virtue of (18) a general solution of (20) for \(i = r\) can be written in implicit form as
\[ \partial S/\partial \tilde{E}_j = \delta_{jr} \tilde{t}_r + b_j, \quad j = 1, \ldots, n. \]
(28)

Comparing (27) and (28) and using (18) we readily see that, in perfect agreement with (15), \(t_i = \tilde{t}_i\) for \(i \neq s\), but \(t_s = \partial S/\partial E_s - b_s = \partial S/\partial \tilde{\alpha} - b_s\) while \(\tilde{t}_s = \partial S/\partial \tilde{E}_s - b_s = \partial S/\partial \alpha - b_s\), so this approach does not yield an explicit formula expressing \(\tilde{t}_s\) as a function of \(\lambda, \mu,\) and \(t_i\).

In order to find a complete integral (26) we can use separation of variables as follows (see e.g. [7] and references therein). Under the assumptions of Corollary 3 suppose that \(\lambda_i, \mu_i, \ i = 1, \ldots, n\) are separation coordinates for the Hamiltonians \(H_i, \ i = 1, \ldots, n,\) that is, the system of equations \(H_i(\alpha, \lambda, \mu) = E_i, \ i = 1, \ldots, n,\) is equivalent to the following one:
\[ \varphi_i(\lambda_i, \mu_i, \alpha, E_1, \ldots, E_n) = 0, \quad i = 1, \ldots, n, \]
(29)
that is, the separation relations on the Lagrangian submanifold \(N_E.\)

Consider the system of stationary Hamilton–Jacobi equations for \(H_i\)
\[ H_i(\alpha, \lambda, \partial S/\partial \lambda) = E_i, \quad i = 1, \ldots, n. \]
(30)
By the above, (30) is equivalent to the system
\[ \varphi_i(\lambda_i, \partial S/\partial \lambda_i, \alpha, E_1, \ldots, E_n) = 0, \quad i = 1, \ldots, n. \] (31)

Suppose that (29) can be solved for \( \mu_i, \ i = 1, \ldots, n: \)
\[ \mu_i = M_i(\lambda_i, \alpha, E_1, \ldots, E_n), \quad i = 1, \ldots, n. \]

Then there exists a complete integral of (31), and hence of (30), of the form (cf. e.g. [7] and references therein)
\[ S = \sum_{i=1}^{n} \int M_i(\lambda_i, \alpha, E_1, \ldots, E_n)d\lambda_i, \] (32)
and general solutions for (19) and (20) can be found using the method of Corollary 3.

The formulas (27) take the form
\[ \sum_{i=1}^{n} \int \left( \frac{\partial M_i(\lambda_i, \alpha, E_1, \ldots, E_n)}{\partial E_j} \right) d\lambda_i = \delta_{jr} t_r + b_j, \quad j = 1, \ldots, n. \]

For \( r = s \) we have
\[ \tilde{t}_s + b_s = \partial S/\partial \tilde{E}_s = \partial S/\partial \alpha = \sum_{i=1}^{n} \int \left( \frac{\partial M_i(\lambda_i, \alpha, E_1, \ldots, E_n)}{\partial \alpha} \right) d\lambda_i. \]

5. Deformations of seed systems

Under the assumptions of Corollary 2, suppose that \( \lambda_i, \mu_i, i = 1, \ldots, n \) are separation coordinates for the Hamiltonians \( H_i, i = 1, \ldots, n \), then the Lagrangian submanifold \( N_E \) is defined by \( n \) separation relations (29). Further assume that all functions \( \varphi_i \) are identical,
\[ \varphi_i = \varphi(\lambda_i, \mu_i, \alpha, E_1, \ldots, E_n), \quad i = 1, \ldots, n. \] (33)

Then the relations (29) mean that the points \( \lambda_i, \mu_i \) belong to the separation curve
\[ \varphi(\lambda, \mu, E_1, \ldots, E_n) = 0 \] (34)
for all \( i = 1, \ldots, n. \)

If the relations
\[ \varphi(\lambda_i, \mu_i, \alpha, H_1, \ldots, H_n) = 0, \quad i = 1, \ldots, n, \]
uniquely determine the Hamiltonians \( H_i \) for \( i = 1, \ldots, n \), then for the sake of brevity we shall say that \( H_i \) for \( i = 1, \ldots, n \) have the separation curve
\[ \varphi(\lambda, \mu, \alpha, H_1, \ldots, H_n) = 0. \]

Fixing values of all Hamiltonians \( H_i = E_i, i = 1, \ldots, n \), picks a particular Lagrangian submanifold from the Lagrangian foliation. Setting \( \alpha = 0 \) in the above formulas we see that \( \lambda_i, \mu_i, i = 1, \ldots, n \) are separation coordinates for the Hamiltonians \( H_i^{(0)}, i = 1, \ldots, n \), as well, so the system of equations \( H_i^{(0)}(\lambda, \mu) = E_i, i = 1, \ldots, n, \) is equivalent to
\[ \varphi_0(\lambda_i, \mu_i, E_1, \ldots, E_n) = 0, \quad i = 1, \ldots, n, \] (35)
where \( \varphi_0(\lambda_i, \mu_i, E_1, \ldots, E_n) = \varphi(\lambda_i, \mu_i, \alpha, E_1, \ldots, E_n)|_{\alpha=0}. \)

In what follows we shall restrict ourselves to the separable systems whose separation curves \( \varphi_0 = 0 \) read
\[ H_i^{(0)} \lambda^{\beta_1} + H_2^{(0)} \lambda^{\beta_2} + \cdots + H_n^{(0)} \lambda^{\beta_n} = \psi(\lambda, \mu) \] (36)
where \( n + k - 1 = \beta_1 > \beta_2 > \ldots > \beta_n = 0, k \in \mathbb{N} \) and \( \psi(\lambda, \mu) \) is a smooth function. Each class of systems (36) is labeled by a sequence \( (\beta_1, \ldots, \beta_n) \) while a particular system from a class is given by a
particular choice of \( \psi(\lambda, \mu) \). In particular, the choice \( \psi(\lambda, \mu) = \frac{1}{2} f(\lambda)\mu^2 + \gamma(\lambda) \) yields the well-known classical Stäckel systems.

For \( k = 0 \) there is only one class given by
\[
H^{(0)}_1 \lambda^{n-1} + H^{(0)}_2 \lambda^{n-2} + \cdots + H^{(0)}_n = \psi(\lambda, \mu)
\]
(37)
which is precisely the Benenti class of Stäckel systems \([1, 2]\) if \( \psi(\lambda, \mu) = \frac{1}{2} f(\lambda)\mu^2 + \gamma(\lambda) \). All these systems separate in the same set of coordinates \((\lambda_i, \mu_i)\) by construction. We shall refer below to the systems with the separation curve \((37)\) as to the systems from the seed class.

In \([7]\) it was shown that an arbitrary class of the systems with the separation curve \((36)\) is obtained via the so-called \( k \)-hole deformation from the seed class \((37)\).

Below we demonstrate that an arbitrary \( k \)-hole deformation is nothing but a sequence of \( k \) Stäckel transforms \((3)\), and hence all separable classes \((36)\) are Stäckel-equivalent to the seed class in the sense of \([9]\). In order to do this we first introduce an alternative notation for different classes \((36)\), which is more convenient for further considerations as well as for the bi-Hamiltonian extension.

We shall call a polynomial of the form \( \sum_{j=1}^{n+k} a_{m+j}\lambda^{n-k-(m+j)} \) a sub-chain of length \( s \) if \( a_k \neq 0 \) for \( k = m + 1, \ldots, m + s \), and a string of holes of length \( s \) if \( a_k = 0 \) for \( k = m + 1, \ldots, m + s \).

Now consider a polynomial \( \sum_{i=1}^{n+k} a_i \lambda^{n+k-i} \), where precisely \( k \) of the coefficients \( a_i \) equal zero, but \( a_1 \neq 0 \) and \( a_{n+k} \neq 0 \). Denote \( n \) non-vanishing coefficients \( a_i \) by \( (H^{(0)}_1, \ldots, H^{(0)}_n) \), where \( 0 \neq a_1 = H^{(0)}_1 \) and \( 0 \neq a_{n+k} = H^{(0)}_n \) by assumption. In what follows we also assume that if \( 0 \neq a_k = H^{(0)}_{k'} \) and \( 0 \neq a_l = H^{(0)}_{l'} \) and \( k > l \) then \( k' > l' \).

It is immediate that any class of separation curves \((36)\) is uniquely determined by a sequence of sub-chains and strings of holes \((n_1, m_1, n_2, m_2, \ldots, m_{l-1}, n_l)\), \( \sum_{i=1}^{l} n_i = n, \sum_{i=1}^{l} m_i = k \), where \( n_i \) is the length of \( i \)-th sub-chain and \( m_i \) is the length of \( i \)-th string of holes. The separation curve corresponding to a sequence \((n_1, m_1, n_2, m_2, \ldots, m_{l-1}, n_l)\), \( \sum_{i=1}^{l} n_i = n, \sum_{i=1}^{l} m_i = k \), reads
\[
\sum_{j=1}^{n_1} H^{(0)}_j \lambda^{n_k-j} + \sum_{r=1}^{l-2} \sum_{j=1}^{n_{l-r}} H^{(0)}_j \lambda^{n_k-j} + \sum_{r=1}^{n_k} \sum_{j=1}^{n_r} H^{(0)}_j \lambda^{n_k-j} + \sum_{j=1}^{n_1} H^{(0)}_{n_{l+1}+j} \lambda^{n_k-j} = \psi(\lambda, \mu).
\]
(38)

For \( k = 0 \) we have only one chain (the seed class). For \( k = 1 \) there are \((n-1)\) different classes consisting of two sub-chains \((n_1, 1, n_2)\), \( n_1 + n_2 = n \), separated by one hole. For \( k = 2 \) we have \( \frac{1}{2}(n-1)(n-2) \) different classes, where \((n-1)\) of these classes consist of two sub-chains separated by a two-hole string \((n_1, 2, n_2)\), \( n_1 + n_2 = n \), while the remaining cases consist of three sub-chains \((n_1, 1, n_2, 1, n_3)\), \( n_1 + n_2 + n_3 = n \), separated by single holes, and so on.

Now define the transformation from the \( k \)-hole case to the \((k+1)\)-hole one. Without loss of generality we can restrict ourselves to considering the following subcases only:

(i) \((n_1, m_1, n_2, m_2, \ldots, m_{l-1}, n_l) \rightarrow (n_1, m_1, n_2, m_2, \ldots, m_{l-1} + 1, n_l)\), \( n_l = n_l' \),
(ii) \((n_1, m_1, n_2, m_2, \ldots, m_{l-1}, n_l) \rightarrow (n_1, m_1, n_2, m_2, \ldots, m_{l-1}, n_l + 1)\), \( n_l + n_{l+1} = n_l' \).

In the case (i) the number of sub-chains and their lengths are preserved, while the length of the last string of holes is increased by one. In the case (ii) the last sub-chain is split into two sub-chains by inserting an additional hole. Notice that we can reach an arbitrary \( s \)-hole deformation in a unique way from the seed class \((n)\) by applying the above recursion step \( s \) times.

Passing from the \( k \)-hole deformation to the \((k+1)\)-hole one means, according to our recursion, that for the separation curve we have
\[
H^{(0)}_i \lambda^{n_k-i} \rightarrow H^{(0)}_i \lambda^{n_k-i+1}, \quad i = 1, \ldots, n_{l+1}
\]
If $n_l = 0$ and $n_{t+1} = n'_l$ then we have the case (ii) for $n_{t+1} < n'_l$ we have the case (ii).

For the sake of convenience we now formally merge the cases (i) and (ii) into a single transformation

$$(n_1, m_1, n_2, m_2, \ldots, m_{t-1}, n'_l) \to (n_1, m_1, n_2, m_2, \ldots, m_{t-1}, n_l, 1, n_{t+1}), \quad n_l + n_{t+1} = n'_l$$

where $n_l = 0$ for the case (i) and $1 \leq n_l \leq n'_l - 1$ for the case (ii). Here $n_l = 0$ corresponds to a void sub-chain (sub-chain of zero length).

**Proposition 7.** Consider two $n$-tuples of Hamiltonians $\{H_i^{(0)}\}$ and $\{\tilde{H}_i^{(0)}\}$ with the separation curves of the form

$$
\sum_{j=1}^{n'_l} H_j^{(0)} \lambda^{n+k-j} + \sum_{r=1}^{l-2} \sum_{j=1}^{n_l-r} H^{(0)} \lambda^{n+k-n_l'-m_{l-1}-r\sum_{s=2}^{r} (n_{l+1-s}+m_{l-s})} + \sum_{j=1}^{n_l} H_{n_l+n_{t+1}+j} \lambda^{n_l'-j} = \psi(\lambda, \mu)
$$

(39)

and

$$
\sum_{j=1}^{n_{t+1}} \tilde{H}_j^{(0)} \lambda^{n+k+1-j} + \sum_{j=1}^{n'_l} \tilde{H}_{n_{t+1}+j} \lambda^{n+k-n_{t+1}-j} + \sum_{r=2}^{l-1} \sum_{j=1}^{n_{t+1}+r} \tilde{H}^{(0)} \lambda^{n+k+1-r\sum_{s=1}^{r} (n_{l+1-s}+m_{l-s})} + \sum_{j=1}^{n_{t+1}} \tilde{H}_{n_l+n_{t+1}+j} \lambda^{n_l'-j} = \psi(\lambda, \mu),
$$

(40)

corresponding to the $(k+1)$-hole deformation of the seed class $[37]$, respectively. Here $n_l = 0$ and $n_{t+1} = n'_l$ for the case (i) and $1 \leq n_l \leq n'_l - 1$ and $n'_l = n_{t+1} + n_l$ for the case (ii); $n_l = 1$ in both cases.

Then the St"ackel transform from $\{H_i^{(0)}\}$ to $\{\tilde{H}_i^{(0)}\}$ reads

$$
\tilde{H}_1^{(0)} = -\frac{1}{V_{n_{l+1}}^{(n+k)} H_{n_{l+1}}^{(0)}},
$$

$$
\tilde{H}_i^{(0)} = H_{i-1}^{(0)} - \frac{V_{n_{l+1}}^{(n+k)} H_{n_{l+1}}^{(0)}}{V_{n_{l+1}}^{(n+k)}}, \quad i = 2, \ldots, n_{t+1},
$$

(41)

$$
\tilde{H}_i^{(0)} = H_{i}^{(0)} - \frac{V_i^{(n+k)} H_{n_{l+1}}^{(0)}}{V_{n_{l+1}}^{(n+k)}}, \quad i > n_{t+1},
$$

where $V_i^{(n+k)}$ are separable potentials defined by the relation

$$
\lambda^{n+k} + \sum_{j=1}^{n'_l} V_j^{(n+k)} \lambda^{n+k-j} + \sum_{r=1}^{l-2} \sum_{j=1}^{n_l-r} V_j^{(n+k)} \lambda^{n+k-n_l'-m_{l-1}-r\sum_{s=2}^{r} (n_{l+1-s}+m_{l-s})} + \sum_{j=1}^{n_l} V_{n_l+n_{t+1}+j} \lambda^{n_l'-j} = 0
$$

(42)

that must hold for $\lambda = \lambda_i, i = 1, \ldots, n$.

**Proof.** In order to compare the Hamiltonians we should reduce the separation curve (39) for $\tilde{H}_i^{(0)}$ to that for $H_i^{(0)}$. To this end we get rid of the highest monomial $\tilde{H}_1^{(0)} \lambda^{n+k}$ in (40) by expressing $\lambda^{n+k}$ from (12). Then comparing coefficients of the separation curves for $\tilde{H}_i^{(0)}$ and $H_i^{(0)}$ yields

$$
H_i^{(0)} = \tilde{H}_{i+1}^{(0)} - V_{n_{l+1}}^{(n+k)} \tilde{H}_1^{(0)}, \quad i = 1, \ldots, n_{l+1} - 1,
$$

$$
H_{n_{l+1}}^{(0)} = -V_{n_{l+1}}^{(n+k)} \tilde{H}_1^{(0)},
$$

$$
H_i^{(0)} = \tilde{H}_i^{(0)} - V_i^{(n+k)} \tilde{H}_1^{(0)}, \quad i = n_{l+1} + 1, \ldots, n
$$

and (41) follows. \qed
Notice that after the renumeration of Hamiltonians $\tilde{H}^{(0)}_i$,
\begin{equation}
\begin{aligned}
\tilde{H}^{(0)}_1 &= \tilde{H}^{(0)}_{n+1}, \\
\tilde{H}^{(0)}_i &= \tilde{H}^{(0)}_{i-1}, \quad i = 2, \ldots, n+1, \\
\tilde{H}^{(0)}_i &= \tilde{H}^{(0)}_i, \quad i > n+1,
\end{aligned}
\end{equation}
we deal with a particular case of Proposition 1 where $H^{(1)}_i = V^{(n+k)}_i$, $\tilde{\alpha} = 0$, $s = n+1$. Thus, the reciprocal transformation
\begin{equation}
d\tilde{t}_{n+1} = -\sum_{j=1}^{n} V^{(n+k)}_j dt_j, \quad \tilde{t}_i = t_i, \quad i \neq n+1,
\end{equation}
transforms the equations of motion for Hamiltonians $H_i = H^{(0)}_i + \alpha H^{(1)}_i$ of the $k$-hole deformation restricted onto the level surface $H_{n+1} = 0$ into the equations of motion for Hamiltonians $\tilde{H}^{(0)}_i$ of the $(k+1)$-hole deformation restricted onto the level surface $\tilde{H}^{(0)}_{n+1} = \tilde{H}^{(0)}_1 = \alpha$.

We can readily extend $\tilde{H}^{(0)}_i$ to $\tilde{H}_i$ using (3) and the results of [7]. Namely, let
\begin{equation}
\begin{aligned}
\tilde{H}^{(1)}_i &= \frac{V^{(n+k)}_i}{V^{(n+k)}_{n+1}} = \tilde{V}^{(n+k-n+1)}_i, \quad i \neq n+1, \\
\tilde{H}^{(1)}_{n+1} &= \frac{1}{V^{(n+k)}_{n+1}} = \tilde{V}^{(n+k-n+1)}_{n+1}.
\end{aligned}
\end{equation}
Then the separation curves for $H_i$ and $\tilde{H}_i$ are of the form
\begin{equation}
\alpha \lambda^{n+k} + H_{n+1} \lambda^{n+k-n+1} + \sum_{i \neq n+1} H_i \lambda^i = \psi(\lambda, \mu),
\end{equation}
\begin{equation}
\tilde{H}_{n+1} \lambda^{n+k} + \tilde{\alpha} \lambda^{n+k-n+1} + \sum_{i \neq n+1} \tilde{H}_i \lambda^i = \psi(\lambda, \mu),
\end{equation}
and the reciprocal transformation (44) transforms the equations of motion for Hamiltonians $H_i$ restricted onto the level surface $H_{n+1} = \tilde{\alpha}$ into the equations of motion for Hamiltonians $\tilde{H}_i$ restricted onto the level surface $\tilde{H}_{n+1} = \alpha$.

As the whole procedure is recursive, it means that the $n$-tuple of integrable Hamiltonian dynamical systems described by the separation curve (36), restricted onto a given Lagrangian submanifold, are related to an $n$-tuple of Hamiltonian dynamical systems from the seed class (also restricted onto an appropriate Lagrangian submanifold) via the sequence of reciprocal transformations.

6. Example

As a simple illustration of the above results, consider the Hénon–Heiles system on a four-dimensional phase space with the coordinates $(p_1, p_2, q_1, q_2)$ and canonical symplectic structure. The corresponding Hamiltonian
\begin{equation}
H_1 = \frac{1}{2} p_1^2 + \frac{1}{2} p_2^2 + q_1^3 + \frac{1}{2} q_1 q_2^2 - \alpha q_1,
\end{equation}
is in involution with
\begin{equation}
H_2 = \frac{1}{2} q_1^2 p_1 p_2 - \frac{1}{2} q_1^3 p_2^2 + \frac{1}{16} q_2^4 + \frac{1}{4} q_1^2 q_2^2 - \frac{1}{4} \alpha q_2^2.
Consider the following one-hole deformation (Stäckel transform) of \( H_1 \) and \( H_2 \):
\[
\mathcal{H}_1 = \frac{1}{2} \frac{1}{q_1} p_1^2 + \frac{1}{2} \frac{1}{q_1} p_2^2 + q_1^2 + \frac{1}{2} q_2^2 - \bar{\alpha} \frac{1}{q_1}.
\]
\[
\mathcal{H}_2 = \frac{1}{2} q_2 p_1 p_2 - \frac{1}{2} q_1 p_2 - \frac{1}{2} \frac{q_2^2}{q_1} p_1^2 - \frac{1}{8} \frac{q_2^2}{q_1} p_2^2 - \frac{1}{16} q_2^4 + \bar{\alpha} \frac{1}{4} q_1^2.
\]

The corresponding reciprocal transformation
\[
dt_1 = q_1 dt_1 + \frac{1}{4} q_2^2 dt_2, \quad t_2 = t_2
\]
defines the map between equations of motion restricted to the respective Lagrangian submanifolds, \( N_E = \{ H_1 = \bar{\alpha}, H_2 = E \} \) and \( \bar{N}_E = \{ \mathcal{H}_1 = \alpha, \mathcal{H}_2 = E \} \).

The separation coordinates \((\lambda, \mu)\) are related to \( p \)'s and \( q \)'s by the formulas
\[
q_1 = \lambda_1 + \lambda_2, \quad q_2 = 2 \sqrt{-\lambda_1 \lambda_2},
\]
\[
p_1 = \frac{\lambda_1 \mu_1}{\lambda_1 - \lambda_2} + \frac{\lambda_2 \mu_2}{\lambda_2 - \lambda_1}, \quad p_2 = \sqrt{-\lambda_1 \lambda_2} \left( \frac{\mu_1}{\lambda_1 - \lambda_2} + \frac{\mu_2}{\lambda_2 - \lambda_1} \right),
\]
are common for \( H_1, H_2 \) and \( \mathcal{H}_1, \mathcal{H}_2 \) and the respective separation curves read
\[
\alpha \lambda^2 + \lambda H_1 + H_2 = \frac{1}{2} \lambda \mu^2 + \lambda^4,
\]
\[
\mathcal{H}_1 \lambda^2 + \tilde{\alpha} \lambda + H_2 = \frac{1}{2} \lambda \mu^2 + \lambda^4.
\]

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References

[1] S. Benenti, Orthogonal separable dynamical systems, Differential geometry and its applications (Opava, 1992), 163–184, Math. Publ., 1, Silesian University in Opava, Opava, 1993; available online at [http://www.emis.de/proceedings/5ICDGA/](http://www.emis.de/proceedings/5ICDGA/)

[2] S. Benenti, Intrinsic characterization of the variable separation in the Hamilton-Jacobi equation, J. Math. Phys. 38 (1997), 6578–6602; preprint version available at [http://www2.dm.unito.it/~benenti/Ricerca/ICVS.pdf](http://www2.dm.unito.it/~benenti/Ricerca/ICVS.pdf)

[3] M. Blaszak, Multi-Hamiltonian theory of dynamical systems, Springer-Verlag, Berlin etc., 1998.

[4] M. Blaszak, On separability of bi-Hamiltonian chain with degenerated Poisson structures, J. Math. Phys. 39 (1998), 3213–3235.

[5] M. Blaszak and Wen-Xiu Ma, Separable Hamiltonian equations on Riemann manifolds and related integrable hydrodynamic systems, J. Geom. Phys. 47 (2003) 21–42; preprint nlin.SI/0209014 (arXiv.org)

[6] M. Blaszak and A. Sergueyev, Maximal superinegrability of Benenti systems, J. Phys. A: Math. Gen. 38 (2005), L1–L5; preprint nlin.SI/0412018 (arXiv.org)

[7] M. Blaszak, Separable systems with quadratic in momenta first integrals, J. Phys. A: Math. Gen. 38 (2005), 1667–1685; preprint nlin.SI/0312025 (arXiv.org)

[8] M. Blaszak and K. Marciniak, From Stäckel systems to integrable hierarchies of PDE’s: Benenti class of separation relations, J. Math. Phys. 47 (2006) 032904; preprint nlin.SI/0511062 (arXiv.org)

[9] C.P. Boyer, E.G. Kalnins, and W. Miller, Jr., Stäckel-equivalent integrable Hamiltonian systems, SIAM J. Math. Anal., 17 (1986), 778–797; see also [http://www.ima.umn.edu/~miller/stackel.pdf](http://www.ima.umn.edu/~miller/stackel.pdf)
[10] A. Cannas da Silva, *Lectures on symplectic geometry*. Lecture Notes in Mathematics, 1764. Springer-Verlag, Berlin, 2001.

[11] M. Crampin and W. Sarlet, *A class of non-conservative Lagrangian systems on Riemannian manifolds*, J. Math. Phys. **42** (2001), 4313–4326; preprint version available online at [http://kraz.rug.ac.be/pub/tm/2001/2001_w_cofactor_jmp.ps](http://kraz.rug.ac.be/pub/tm/2001/2001_w_cofactor_jmp.ps)

[12] E. V. Ferapontov, *Integration of weakly nonlinear hydrodynamic systems in Riemann invariants*, Phys. Lett. A **158** (1991) 112–118.

[13] E. V. Ferapontov and A. P. Fordy, *Separable Hamiltonians and integrable systems of hydrodynamic type*, J. Geom. Phys. **21** (1997) 169–182.

[14] J. Hietarinta, B. Grammaticos and B. Dorizzi and A. Ramani, *Coupling-Constant Metamorphosis and Duality between Integrable Hamiltonian Systems*, Phys. Rev. Lett. **53**, (1984) 1707–1710.

[15] E.G. Kalnins, J.M. Kress, W. Miller, Jr., P. Winternitz, *Superintegrable systems in Darboux spaces*, J. Math. Phys. **44** (2003), no.12, 5811–5847; preprint version at [http://wwwIMA.umn.edu/preprints/jun2003/1929.pdf](http://wwwIMA.umn.edu/preprints/jun2003/1929.pdf)

[16] E.G. Kalnins, J.M. Kress, W. Miller, Jr. *Second order superintegrable systems in conformally flat spaces. II. The classical two-dimensional Stäckel transform*, J. Math. Phys. **46** (2005), no. 5, 053510; preprint version at [http://wwwIMA.umn.edu/miller/supstructure2.pdf](http://wwwIMA.umn.edu/miller/supstructure2.pdf)

[17] E.G. Kalnins, J.M. Kress, W. Miller, Jr. *Second order superintegrable systems in conformally flat spaces. IV. The classical 3D Stäckel transform and 3D classification theory*, J. Math. Phys. **47** (2006), no. 4, 043514; preprint version at [http://wwwIMA.umn.edu/miller/supstructure3.pdf](http://wwwIMA.umn.edu/miller/supstructure3.pdf)

[18] E. Kalnins, *Separation of variables for Riemannian spaces of constant curvature*, John Wiley & Sons, New York, 1986. Online at [http://wwwIMA.umn.edu/miller/variablesolation.html](http://wwwIMA.umn.edu/miller/variablesolation.html)

[19] W. Miller Jr, *Symmetry and separation of variables*, Addison-Wesley, Reading, MA etc., 1977; available online at [http://wwwIMA.umn.edu/miller/separationofvariables.html](http://wwwIMA.umn.edu/miller/separationofvariables.html)

[20] W. Oevel and C. Rogers, *Gauge transformations and reciprocal links in (2+1) dimensions*, Rev. Math. Phys. **5** (1993) 299–330; preprint version available at [http://math-uni-panderborn.de/walter/publications/Gauges_2+1_13_4_93.ps](http://math-uni-panderborn.de/walter/publications/Gauges_2+1_13_4_93.ps)

[21] S. Rauch-Wojciechowski, K. Marciniak and M. Błaszak, *Two Newton decompositions of stationary flows of KdV and Harry Dym hierarchies*, Physica A **233** (1996), 307–330; see also [http://wwwITN.liu.se/kgzma/publications/](http://wwwITN.liu.se/kgzma/publications/)

[22] C. Rogers and W.F. Shadwick, *Bäcklund Transformations and Their Applications*, Mathematics in Science and Engineering Series, New York, Academic Press, 1982.

[23] B.L. Roždestvenski˘ı, N.N. Janenko, *Systems of quasilinear equations and their applications to gas dynamics*, Translations of Mathematical Monographs, Vol. 55, Providence, RI, AMS, 1983.

[24] E.K. Sklyanin, *Separation of variables—new trends*, Progr. Theoret. Phys. Suppl. **118** (1995), 35–60; preprint [solv-int/9504001](http://arXiv.org)

[25] A.V. Tsiganov, *Canonical transformations of the extended phase space, Toda lattices and the Stäckel family of integrable systems*, J. Phys. A: Math. Gen. **33** (2000) 4169–4182; preprint [solv-int/9909006](http://arXiv.org)

[26] A.P. Veselov, *Time change in integrable systems*, Vestnik Moskov. Univ. Ser. I Mat. Mekh., no. 5 (1987), 25–29 and 104 (in Russian).

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