A note on the number of partitions of $n$ into $k$ parts
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Abstract

We prove new formulas and congruences for $p(n, k) :=$ the number of partitions of $n$ into $k$ parts and $q(n, k) :=$ the number of partitions of $n$ into $k$ distinct parts. Also, we give lower and upper bounds for the density of the set $\{n \in \mathbb{N} : p(n, k) \equiv i (\mod m)\}$, where $m \geq 2$ and $0 \leq i \leq m – 1$.

Keywords: Restricted integer partitions; Restricted partition function.

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1 Introduction

A partition of a positive integer $n$ is a non-increasing sequence of positive integers whose sum equals $n$. We define $p(n)$ as the number of partitions of $n$ and for convenience, we define $p(0) = 1$. Let $p(n, k)$ be the number of partitions of $n$ with exactly $k$ summands. Let $q(n, k)$ be the number of partitions of $n$ with $k$ distinct parts and let $q(n)$ total number of partitions of $n$ with distinct parts. For instance, there are 5 partitions of 8 with three summands $1 + 1 + 6$, $1 + 2 + 5$, $1 + 3 + 4$, $2 + 2 + 4$, $2 + 3 + 3$, hence $p(8, 3) = 5$ and $q(8, 3) = 2$. Obviously, $p(n, k) = 0$ if and only if $n < k$. Also, $q(n, k) = 0$ if and only if $n < k + \binom{k}{2}$. Moreover, $p(n) = \sum_{k=1}^{n} p(n, k)$ and $q(n) = \sum_{k=1}^{n} q(n, k)$.

The function $p(n, k)$ was studied extensively in the literature; see for instance [8]. However, there is no known closed form for $p(n, k)$.

Let $\mathbf{a} = (a_1, \ldots, a_k)$ be a sequence of positive integers and let $p_\mathbf{a}(n)$ be the restricted partition function associated to $\mathbf{a}$; see Section 2. In Theorem 3.2 we prove new formulas for $p(n, k)$ and $q(n, k)$, using their intrinsic connection with the restricted partition function associated to the sequence $\mathbf{k} := (1, 2, \ldots, k)$. In Proposition 3.3 we give another formulas for $p(n, 3)$ and $q(n, 3)$. In [3] we proved that if a certain determinant is nonzero, then the restricted partition function $p_\mathbf{a}(n)$ can be computed by solving a system of linear equations with coefficients which are values of Bernoulli polynomials and Bernoulli Barnes numbers. Using a similar method, we prove that if a certain determinant $\Delta(k)$, which depends only on $k$, is nonzero, then $p(n, k)$ and $q(n, k)$ can be expressed in terms of values of Bernoulli polynomials and Bernoulli Barnes numbers; see Theorem 3.4.

In Theorem 4.2 respectively in Corollary 4.3 we provide formulas for $P(n, k) =$ the polynomial part of $p(n, k)$, respectively for $Q(n, k) =$ the polynomial part of $q(n, k)$. In Proposition 5.2 we prove formulas for the ”waves” of $p(n, k)$ and $q(n, k)$, defined analogously as the Sylvester ”waves” (see [11],[12]) of the restricted partition function $p_\mathbf{k}(n)$.

In Proposition 6.1 we give new formulas for $p(n, k)$ and $q(n, k)$ in terms of coefficients of a reciprocal polynomial and, as a consequence, in Corollary 6.2 we prove some congruence relations for $p(n, k)$ and $q(n, k)$. In a recent preprint [9], K. Grajdzica found lower and upper bounds for the density of the set $\{n \in \mathbb{N} : p_\mathbf{a}(n) \equiv i (\mod m)\}$ for a fixed integer $0 \leq i \leq m – 1$. Using this, we prove lower and upper bounds for the density of the set $\{n \in \mathbb{N} : p(n, k) \equiv i (\mod m)\}$; see Theorem 6.3.
2 Preliminaries

Let $a := (a_1, a_2, \ldots, a_k)$ be a sequence of positive integers, $k \geq 1$. The restricted partition function associated to $a$ is $p_a : \mathbb{N} \rightarrow \mathbb{N}$, $p_a(n) :=$ the number of integer solutions $(x_1, \ldots, x_k)$ of $\sum_{i=1}^k a_i x_i = n$ with $x_i \geq 0$. Note that the generating function of $p_a(n)$ is

$$\sum_{n=0}^\infty p_a(n)z^n = \frac{1}{(1 - z^{a_1}) \cdots (1 - z^{a_k})}. \tag{2.1}$$

Let $D$ be a common multiple of $a_1, a_2, \ldots, a_k$. Bell [3] has proved that $p_a(n)$ is a quasi-polynomial of degree $k - 1$, with the period $D$, i.e.

$$p_a(n) = d_{a,k-1}(n)n^{k-1} + \cdots + d_{a,1}(n)n + d_{a,0}(n), \tag{2.2}$$

where $d_{a,m}(n + D) = d_{a,m}(n)$ for $0 \leq m \leq k - 1$ and $n \geq 0$, and $d_{a,k-1}(n)$ is not identically zero. Sylvester [11], [12] decomposed the restricted partition in a sum of “waves”:

$$p_a(n) = \sum_{j \geq 1} W_j(n, a), \tag{2.3}$$

where the sum is taken over all distinct divisors $j$ of the components of $a$ and showed that for each such $j$, $W_j(n, a)$ is the coefficient of $t^{-1}$ in

$$\sum_{0 \leq \nu < j, \gcd(\nu, j) = 1} \rho_j^{-\nu m} e^{\nu t} \left(1 - \rho_j^{m \nu} e^{-\nu t}\right),$$

where $\rho_j = e^{2\pi i/j}$ and $\gcd(0, 0) = 1$ by convention. Note that $W_j(n, a)$’s are quasi-polynomials of period $j$. Also, $W_1(n, a)$ is called the polynomial part of $p_a(n)$ and it is denoted by $P_a(n)$.

It is well known that $p(n, k)$, the number of partitions of $n$ with exactly $k$ summands, equals to the number of partitions of $n$ whose largest part is $k$. It follows that

$$p(n, k) = \begin{cases} p(1, 2, \ldots, k)(n - k), & n \geq k \\ 0, & n < k \end{cases}. \tag{2.4}$$

There is a 1-to-1 correspondence between the partitions of $n$ with $k$ distinct parts and the partitions of $n - \binom{k}{2}$ with $k$ parts, given by

$$a_1 < a_2 < \ldots < a_k \mapsto a_1 \leq a_2 - 1 \leq \cdots \leq a_k - (k - 1).$$

Hence

$$q(n, k) = \begin{cases} p(n - \binom{k}{2}, k), & n \geq k + \binom{k}{2} \\ 0, & n < k + \binom{k}{2} \end{cases}. \tag{2.5}$$

From (2.1), (2.3) and (2.5) it follows that

$$\sum_{n=0}^\infty p(n, k)z^n = \frac{z^k}{(1 - z)(1 - z^2) \cdots (1 - z^k)}, \quad \sum_{n=0}^\infty q(n, k)z^n = \frac{z^{k+\binom{k}{2}}}{(1 - z)(1 - z^2) \cdots (1 - z^k)},$$

are the generating functions for $p(n, k)$ and $q(n, k)$ respectively.
3 Main results

Let $D_k$ be the least common multiple of $1, 2, \ldots, k$.

**Proposition 3.1.** We have that

$$p(n, k) = f_{k,k-1}(n)n^{k-1} + \cdots + f_{k,1}(n)n + f_{k,0}(n) \text{ for all } n \geq k,$$

where $f_{k,m}(n) = d_{k,m}(n-k)$, and $k = (1, 2, \ldots, k)$.

**Proof.** It follows from (2.2) and (2.4).

**Theorem 3.2.** (1) For $n \geq k$ we have that:

$$p(n, k) = \frac{1}{(k-1)!} \sum_{0 \leq j_1 \leq \frac{D_k}{k-1}, \ldots, 0 \leq j_k \leq \frac{D_k}{k-1}} \prod_{\ell=1}^{k-1} \left( \frac{n-k-j_1-2j_2-\ldots-kj_k}{D_k} + \ell \right).$$

(2) For $n \geq k + \left(\frac{k}{2}\right)$ we have that:

$$q(n, k) = \frac{1}{(k-1)!} \sum_{0 \leq j_1 \leq \frac{D_k}{k-1}, \ldots, 0 \leq j_k \leq \frac{D_k}{k-1}} \prod_{\ell=1}^{k-1} \left( \frac{n-k-\left(\frac{k}{2}\right)-j_1-2j_2-\ldots-kj_k}{D_k} + \ell \right).$$

**Proof.** (1) The result follows from [4, Corollary 2.10] and (2.4).

(2) It follows from (1) and (2.5).

The **unsigned Stirling numbers** $[n \choose k]$ are defined by the identity

$$(x)^n := x(x+1)\cdots(x+n-1) = \sum_{k=0}^{n} [n \choose k] x^k.$$

The **Bernoulli numbers** $B_\ell$’s are defined by the identity

$$\frac{t}{e^t-1} = \sum_{\ell=0}^{\infty} \frac{t^\ell}{\ell!} B_\ell.$$

$B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$ and $B_n = 0$ is $n$ is odd and greater than 1.

**Proposition 3.3.** (1) For $n \geq 3$, we have that:

$$p(n, 3) = \sum_{m=1}^{3} \frac{(-1)^{m-1}}{6(m-1)!} \sum_{i_1+i_2+i_3=2-m} \frac{B_{i_1}B_{i_2}B_{i_3}2^{i_2}3^{i_3}(n-3)^{m-1}}{i_1!i_2!i_3!} + \sum_{j=2}^{1} \sum_{j=1}^{j} \rho_j \sum_{k=0}^{2} \frac{1}{6^k} \left[ \begin{array}{c} 3 \\ k + 1 \end{array} \right] \sum_{0 \leq j_1 \leq 5, 0 \leq j_2 \leq 2, 0 \leq j_3 \leq 1, j_1+j_2+3j_3 \equiv \ell \pmod{3} \} \right) (j_1 + 2j_2 + 3j_3)^k,$$

where $\rho_j = e^{2\pi i j}$. 

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(2) For \( n \geq 6 \), we have that:

\[
q(n, 3) = \sum_{m=1}^{3} (-1)^{m-1} \frac{6(m-1)!}{12} \sum_{i_1+i_2+i_3=2-m} B_{i_1} B_{i_2} B_{i_3} 2^{i_2} 3^{i_3} (n-6)^{m-1} + \\
+ \frac{1}{12} \sum_{j=2}^{3} \sum_{\ell=1}^{j} \rho_j^\ell \sum_{k=0}^{2} \frac{1}{6^k} \sum_{0 \leq j_1 \leq 5, 0 \leq j_2 \leq 2, 0 \leq j_3 \leq 1} (j_1 + 2j_2 + 3j_3)^k,
\]

Proof. (1) Since 1, 2, 3 are coprime, the conclusion follows from \([5\text{, Proposition 4.3}]\) and the fact that \( p(n, 3) = p(1,2,3)(n-3) \) for \( n \geq 3 \).

(2) Follows from (1) and the fact that \( q(n, 3) = p(n-3, 3) \) for \( n \geq 6 \).

The Bernoulli polynomials are defined by

\[
B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k.
\]

For \( a = (a_1, \ldots, a_k) \), the Bernoulli-Barnes numbers (see \([11]\)) are

\[
B_j(a) = \sum_{i_1+\cdots+i_k=j} \binom{j}{i_1, \ldots, i_k} B_{i_1} \cdots B_{i_k} a_1^{i_1} \cdots a_k^{i_k}.
\]

We consider the determinant:

\[
\Delta(k) := \begin{vmatrix}
\frac{B_1(x)}{1!} & \cdots & \frac{B_1(1)}{1!} & \cdots & \frac{B_k(x)}{k!} & \cdots & \frac{B_k(1)}{k!} \\
\frac{B_2(x)}{2!} & \cdots & \frac{B_2(1)}{2!} & \cdots & \frac{B_{k+1}(x)}{k+1!} & \cdots & \frac{B_{k+1}(1)}{k+1!} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{B_k x}{k!} & \cdots & \frac{B_k(1)}{k!} & \cdots & \frac{B_{k+k-1}(x)}{(k+k-1)!} & \cdots & \frac{B_{k+k-1}(1)}{(k+k-1)!}
\end{vmatrix}.
\]

Theorem 3.4. If \( \Delta(k) \neq 0 \), then \( p(n, k) \) can be computed in terms of \( B_j \left( \frac{x}{k} \right) \), \( 1 \leq v \leq k \), \( 1 \leq j \leq kD_k \) and \( B_j(k) \), \( 0 \leq j \leq kD_k \), where \( k = (1, 2, \ldots, k) \).

Proof. According to \([5\text{, Formula (1.8)]}\), we have that:

\[
\sum_{m=0}^{k-1} \sum_{v=1}^{D_k} d_{k,m}(v) D_{n+m} \frac{B_{n+m+1}(\frac{x}{k})}{n+m+1} = \frac{(-1)^{n-1}k!}{(n+k)!} B_{n+k}(k) - \delta_0 n, \ (\forall) n \geq 0,
\]

where \( \delta_0 n \) is the Kronecker’s symbol. Giving values \( 0 \leq n \leq kD_k - 1 \) in \([3.1]\), and seeing \( d_{k,m}(v) \)'s as variables, we obtain a system of \( kD_k \) linear equations, with the determinant equal to \( \pm D_k^N \Delta(k) \) for some integer \( N \geq 1 \). By hypothesis, \( \Delta(k) \neq 0 \), hence we can determine \( d_{k,m}(v) \) by solving the system. From (2.2), one has

\[
p_k(n) = d_{k,k-1}(n)n^{k-1} + \cdots + d_{k,1}(n)n + d_{0,0}(n).
\]

Hence, the conclusion follows from (2.4) and (2.5).
4 The polynomial part of $p(n, k)$ and $q(n, k)$

We recall the following basic facts on quasi-polynomials [10, Proposition 4.4.1]:

**Proposition 4.1.** The following conditions on a function $f : \mathbb{N} \to \mathbb{C}$ and integer $D > 0$ are equivalent.

(i) $f(n)$ is a quasi-polynomial of period $D$.

(ii) $\sum_{n=0}^{\infty} f(n)z^n = \frac{L(z)}{M(z)}$, where $L(z), M(z) \in \mathbb{C}[z]$, every zero $\lambda$ of $M(z)$ satisfies $\lambda^D = 1$ (provided $\frac{L(z)}{M(z)}$ has been reduced to lowest terms), and $\deg L(z) < \deg M(z)$.

(iii) For all $n \geq 0$, $f(n) = \sum_{\lambda=1}^{D} F_\lambda(n)\lambda^{-n}$, where each $F_\lambda(n)$ is a polynomial function. Moreover, $\deg F_\lambda(n) \leq m(\lambda) - 1$, where $m(\lambda) =$ multiplicity of $\lambda$ as a root of $M(z)$.

We define the polynomial part of $f(n)$ to be the polynomial function $F(n) = F_1(n)$, with the notation of Proposition 4.1. The polynomial part $F(n)$ of a quasi-polynomial $f(n)$ gives a rough approximation of $f(n)$, which is useful for studying the asymptotic behaviour of $f(n)$, when $n \gg 0$. If $a = (a_1, \ldots, a_k)$ is a sequence of positive integers and $p_a(n)$ is the restricted partition function associated to $a$, we denote $P_a(n)$, the polynomial part of $p_a(n)$. Several formulas of $P_a(n)$ were proved in [2], [7] and [3].

We consider the following functions:

$$P(n, k) = P_{1,2,\ldots,k}(n-k), \ n \geq k,$$

$$Q(n, k) = P_{1,2,\ldots,k}(n-k - \binom{k}{2}), \ n \geq k + \binom{k}{2},$$

and we called them, the polynomial part of $p(n, k)$ and $q(n, k)$, respectively.

**Theorem 4.2.** For $n \geq k$, we have that:

1. $P(n, k) = \frac{1}{D_k(k-1)!} \sum_{0 \leq j_1 \leq \frac{D_k}{D_k-1}, \ldots, 0 \leq j_k \leq \frac{D_k}{D_k-1}} \prod_{\ell=1}^{k-1} \left( \frac{n-k-j_1-2j_2-\ldots-kj_k+\ell}{D_k} \right).$

2. $P(n, k) = \frac{1}{k!} \sum_{u=0}^{k-1} \frac{(-1)^u}{(k-1-u)!} \sum_{i_1+\ldots+i_k=u} B_{i_1} \cdots B_{i_k} \frac{1}{i_1! \cdots i_k!} \left( k^i (n-k)^{k-1-u}. \right.$

**Proof.** (1) It follows from [4 Corollary 3.6] and (2.24). See also [7, Theorem 1.1].

(2) It follows from [4 Corollary 3.11] and (2.4). See also [2, p.2].

**Corollary 4.3.** For $n \geq k + \binom{k}{2}$, we have that:

1. $Q(n, k) = \frac{1}{D_k(k-1)!} \sum_{0 \leq j_1 \leq \frac{D_k}{D_k-1}, \ldots, 0 \leq j_k \leq \frac{D_k}{D_k-1}} \prod_{\ell=1}^{k-1} \left( \frac{n-k-\binom{k}{2}-j_1-2j_2-\ldots-kj_k+\ell}{D_k} \right).$

2. $Q(n, k) = \frac{1}{k!} \sum_{u=0}^{k-1} \frac{(-1)^u}{(k-1-u)!} \sum_{i_1+\ldots+i_k=u} B_{i_1} \cdots B_{i_k} \frac{1}{i_1! \cdots i_k!} \left( n-k-\binom{k}{2} \right)^{k-1-u}.\$

**Proof.** It follows from Theorem 4.2 and (2.5).
5 The Sylvester waves of $p(n, k)$ and $q(n, k)$

Let $k := (1, 2, \ldots, k)$. According to [23], the restricted partition function $p_k(n)$ can be written as a sum of "waves", $p_k(n) = \sum_{j=1}^{k} W_j(n, k)$. We define the functions

$$W_j(n, k) = W_j(n - k, k), \quad n \geq k,$$

$$\tilde{W}_j(n, k) := W_j(n - k - \left(\begin{array}{c} k \\ 2 \end{array}\right), k), \quad n \geq k + \left(\begin{array}{c} k \\ 2 \end{array}\right),$$

and we call them the "waves" of $p(n, k)$ and $q(n, k)$, respectively.

**Remark 5.1.** Note that $P(n, k) = W_1(n, k)$ and $Q(n, k) = \tilde{W}_1(n, k)$ are the polynomial parts of $p(n, k)$ and $p(n, k)$.

**Proposition 5.2.** (1) For any positive integers $1 \leq j \leq k \leq n$, we have that:

$$W_j(n, k) = \frac{1}{D_k(k-1)!} \sum_{m=1}^{k} \sum_{t=1}^{j} \rho_j \sum_{t=m-1}^{k-1} \left[\begin{array}{c} k \\ t \end{array}\right] (-1)^t m+1 \left(\begin{array}{c} t \\ m-1 \end{array}\right).$$

$$\cdot \sum_{0 \leq j_1 \leq D_k-1, \ldots, 0 \leq j_k \leq D_k-1, j_1 + \cdots + j_k \equiv \ell (\text{mod} j)} D_k^{-t}(j_1 + \cdots + k j_k)^{t-m+1}(n-k)^{m-1}.$$

(2) For any positive integers $1 \leq j \leq k$ and $n \geq k + \left(\begin{array}{c} k \\ 2 \end{array}\right)$, we have that:

$$\tilde{W}_j(n, k) = \frac{1}{D_k(k-1)!} \sum_{m=1}^{k} \sum_{t=1}^{j} \rho_j \sum_{t=m-1}^{k-1} \left[\begin{array}{c} k \\ t \end{array}\right] (-1)^t m+1 \left(\begin{array}{c} t \\ m-1 \end{array}\right).$$

$$\cdot \sum_{0 \leq j_1 \leq D_k-1, \ldots, 0 \leq j_k \leq D_k-1, j_1 + \cdots + j_k \equiv \ell (\text{mod} j)} D_k^{-t}(j_1 + \cdots + k j_k)^{t-m+1}(n-k - \left(\begin{array}{c} k \\ 2 \end{array}\right))^{m-1}.$$

**Proof.** (1) It follows from [23] Proposition 4.2 and (5.1).

(1) It follows from [23] Proposition 4.2 and (5.2). \qed

6 New formulas and congruences for $p(n, k)$ and $q(n, k)$

We consider the function:

$$f(n, k) := \#\{(j_1, \ldots, j_k) : j_1 + 2 j_2 + \cdots + k j_k = n, 0 \leq j_i \leq \frac{D_k}{k} - 1, 1 \leq i \leq k\},$$

where $D_k$ is the least common multiple of $1, 2, \ldots, k$. Let $d_k := kD_k - \left(\begin{array}{c} k+1 \\ 2 \end{array}\right)$. Note that

$$f(n, k) = f(d - n, k), \quad \text{for} \ 0 \leq n \leq d, \ \text{and} \ f(n, k) = 0 \ \text{for} \ n \geq d + 1.$$

It follows that $F(n, k) = \sum_{i=0}^{d_k} f(n, k) x^i$ is a reciprocal polynomial. With the above notations we have:
Proposition 6.1. (1) For \( n \geq k \) we have that:

\[
p(n, k) = \sum_{j=\left\lceil \frac{n-k}{k} \right\rceil - k}^{\left\lfloor \frac{n-k}{k} \right\rfloor} \binom{k+j-1}{j} f(n-k-jD_k).
\]

(2) For \( n \geq k + \binom{k}{2} \) we have that:

\[
q(n, k) = \sum_{j=\left\lceil \frac{n}{k} \right\rceil - k}^{\left\lceil \frac{n}{k} \right\rceil - \binom{k}{2}} \binom{k+j-1}{j} f(n-k-(\binom{k}{2})-jD_k).
\]

Proof. (1) It follows from [6, Proposition 2.2], [6, Corollary 2.3] and (2.4).

(2) It follows from (1) and (2.5). \( \square \)

Corollary 6.2. (1) For \( n \geq k \) we have that:

\[
(k-1)!p(n, k) \equiv 0 \mod (j + \ell + 1)(j + \ell + 2) \cdots (j + k - 1),
\]

where \( \ell = \left\lfloor \frac{n-k}{k} \right\rfloor - \left\lfloor \frac{n+\binom{k}{2}}{k} \right\rfloor + k \).

(2) For \( n \geq k + \binom{k}{2} \) we have that:

\[
(k-1)!p(n, k) \equiv 0 \mod (j + \ell' + 1)(j + \ell' + 2) \cdots (j + k - 1),
\]

where \( \ell' = \left\lfloor \frac{n-k+\binom{k}{2}}{k} \right\rfloor - \left\lceil \frac{n}{k} \right\rceil + k \).

Theorem 6.3. (1) The inequality

\[
\lim_{N \to \infty} \frac{\#\{n \leq N : p(n, k) \equiv 1(\mod 2)\}}{N} \leq \frac{2}{3},
\]

holds for infinitely many positive integers \( k \). Moreover, if the above inequality is not satisfied for some positive integers \( k \), then it holds for \( k + 1 \).

(2) Let \( m > 1 \) be a positive integer. For each positive integer \( k \) we have

\[
\lim_{N \to \infty} \frac{\#\{n \leq N : p(n, k) \not\equiv 0(\mod m)\}}{N} \geq \frac{1}{\binom{k+1}{2}}.
\]

The above results hold if we replace \( p(n, k) \) with \( q(n, k) \).

Proof. (1) Since \( p(n, k) = p_k(n-k) \) for \( n \geq k \), where \( k = (1, 2, \ldots, k) \), the result follows from [9, Theorem 4.2]. (The asymptotic behaviour is not changed if we replaced \( p_k(n) \) with \( p(n, k) \) or \( q(n, k) \).)

(2) As above, the result follows from [9, Theorem 5.2]. \( \square \)
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