Phase transitions for non-singular Bernoulli actions

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Abstract. Inspired by the phase transition results for non-singular Gaussian actions introduced in [AIM19], we prove several phase transition results for non-singular Bernoulli actions. For generalized Bernoulli actions arising from groups acting on trees, we are able to give a very precise description of their ergodic-theoretical properties in terms of the Poincaré exponent of the group.

Key words: non-singular Bernoulli action, phase transition, strong ergodicity, Krieger type 2020 Mathematics Subject Classification: 37A40 (Primary); 20E08 (Secondary)

1. Introduction

When $G$ is a countable infinite group and $(X_0, \mu_0)$ is a non-trivial standard probability space, the probability measure-preserving (pmp) action

$$G \curvearrowright (X_0, \mu_0)^G : (g \cdot x)_h = x_{g^{-1}h}$$

is called a **Bernoulli action**. Probability measure-preserving Bernoulli actions are among the best-studied objects in ergodic theory and they play an important role in operator algebras [Ioa10, Pop03, Pop06]. When we consider a family of probability measures $(\mu_g)_{g \in G}$ on the base space $X_0$ that need not all be equal, the Bernoulli action

$$G \curvearrowright (X, \mu) = \prod_{g \in G} (X_0, \mu_g)$$

(1.1)

is in general no longer measure-preserving. Instead, we are interested in the case where $G \curvearrowright (X, \mu)$ is **non-singular**, that is, the group $G$ preserves the **measure class** of $\mu$. By Kakutani’s criterion for equivalence of infinite product measures the Bernoulli action (1.1) is non-singular if and only if $\mu_h \sim \mu_g$ for every $h, g \in G$ and

$$\sum_{h \in G} H^2(\mu_h, \mu_{gh}) < +\infty \quad \text{for every } g \in G. \quad (1.2)$$

Here $H^2(\mu_h, \mu_{gh})$ denotes the **Hellinger distance** between $\mu_h$ and $\mu_{gh}$ (see (2.2)).
It is well known that a pmp Bernoulli action $G \curvearrowright (X_0, \mu_0)^G$ is mixing. In particular, it is ergodic and conservative. However, for non-singular Bernoulli actions, determining conservativeness and ergodicity is much more difficult (see, for instance, [BKV19, Dan18, Kos18, VW17]).

Besides non-singular Bernoulli actions, another interesting class of non-singular group actions comes from the Gaussian construction, as introduced in [AIM19]. If $\pi: G \to O(H)$ is an orthogonal representation of a locally compact second countable (lcsc) group on a real Hilbert space $H$, and if $c: G \to H$ is a 1-cocycle for the representation $\pi$, then the assignment

$$\alpha_g(\xi) = \pi_g(\xi) + c(g) \quad (1.3)$$

defines an affine isometric action $\alpha: G \curvearrowright H$. To any affine isometric action $\alpha: G \curvearrowright H$, Arano, Isono and Marrakchi associated a non-singular group action $\hat{\alpha}: G \curvearrowright \hat{H}$, where $\hat{H}$ is the Gaussian probability space associated to $H$. When $\alpha: G \curvearrowright H$ is actually an orthogonal representation, this construction is well established and the resulting Gaussian action is pmp. As explained below [BV20, Theorem D], if $G$ is a countable infinite group and $\pi: G \to \ell^2(G)$ is the left regular representation, the affine isometric representation $(1.3)$ gives rise to a non-singular action that is conjugate with the Bernoulli action $G \curvearrowright \prod_{g \in G} (\mathbb{R}, \nu_{F(g)})$, where $F: G \to \mathbb{R}$ is such that $c_g(h) = F(g^{-1}h) - F(h)$, and $\nu_{F(g)}$ denotes the Gaussian probability measure with mean $F(g)$ and variance 1.

By scaling the 1-cocycle $c: G \to H$ with a parameter $t \in [0, +\infty)$ we get a one-parameter family of non-singular actions $\hat{\alpha}_t: G \curvearrowright \hat{H}_t$ associated to the affine isometric actions $\alpha_t: G \curvearrowright H_t$, given by $\alpha_t(\xi) = \pi_t(\xi) + tc(g)$. Arano, Isono and Marrakchi showed that there exists a $t_{\text{diss}} \in [0, +\infty)$ such that $\hat{\alpha}_t$ is dissipative up to compact stabilizers for every $t > t_{\text{diss}}$ and infinitely recurrent for every $t < t_{\text{diss}}$ (see §2 for terminology).

Inspired by the results obtained in [AIM19], we study a similar phase transition framework, but in the setting of non-singular Bernoulli actions. Such a phase transition framework for non-singular Bernoulli actions was already considered by Kosloff and Soo in [KS20]. They showed the following phase transition result for the family of non-singular Bernoulli actions of $G = \mathbb{Z}$ with base space $X_0 = \{0, 1\}$ that was introduced in [VW17, Corollary 6.3]. For every $t \in [0, +\infty)$ consider the family of measures $(\mu^t_n)_{n \in \mathbb{Z}}$ given by

$$\mu^t_n(0) = \begin{cases} 
1/2 & \text{if } n \leq 4t^2, \\
1/2 + t/\sqrt{n} & \text{if } n > 4t^2.
\end{cases}$$

Then $\mathbb{Z} \curvearrowright (X, \mu_t) = \prod_{n \in \mathbb{Z}}([0, 1], \mu^t_n)$ is non-singular for every $t \in [0, +\infty)$. Kosloff and Soo showed that there exists a $t_1 \in (1/6, +\infty)$ such that $\mathbb{Z} \curvearrowright (X, \mu_t)$ is conservative for every $t < t_1$ and dissipative for every $t > t_1$ [KS20, Theorem 3]. In [DKR20, Example D] the authors describe a family of non-singular Poisson suspensions for which a similar phase transition occurs. These examples arise from dissipative essentially free actions of $\mathbb{Z}$, and thus they are non-singular Bernoulli actions. We generalize the phase transition result from [KS20] to arbitrary non-singular Bernoulli actions as follows.
Suppose that $G$ is a countable infinite group and let $(\mu_g)_{g \in G}$ be a family of equivalent probability measure on a standard Borel space $X_0$. Let $\nu$ also be a probability measure on $X_0$. For every $t \in [0, 1]$ we consider the family of equivalent probability measures $(\mu^t_g)_{g \in G}$ that are defined by
\[
\mu^t_g = (1-t)\nu + t\mu_g.
\]
(1.4)

Our first main result is that in this setting there is a phase transition phenomenon.

**THEOREM A.** Let $G$ be a countable infinite group and assume that the Bernoulli action $G \curvearrowright (X, \mu_1) = \prod_{g \in G}(X_0, \mu_g)$ is non-singular. Let $\nu \sim \mu_e$ be a probability measure on $X_0$ and for every $t \in [0, 1]$ consider the family $(\mu^t_g)_{g \in G}$ of equivalent probability measures given by (1.4). Then the Bernoulli action $G \curvearrowright (X, \mu_t) = \prod_{g \in G}(X_0, \mu^t_g)$ is non-singular for every $t \in [0, 1]$ and there exists a $t_1 \in [0, 1]$ such that $G \curvearrowright (X, \mu_t)$ is weakly mixing for every $t < t_1$ and dissipative for every $t > t_1$.

Suppose that $G$ is a non-amenable countable infinite group. Recall that for any standard probability space $(X_0, \mu_0)$, the pmp Bernoulli action $G \curvearrowright (X_0, \mu_0)^G$ is strongly ergodic. Consider again the family of probability measures $(\mu^t_g)_{g \in G}$ given by (1.4). In Theorem B below we prove that for $t$ close enough to 0, the resulting non-singular Bernoulli action is strongly ergodic. This is inspired by [AIM19, Theorem 7.20] and [MV20, Theorem 5.1], which state similar results for non-singular Gaussian actions.

**THEOREM B.** Let $G$ be a countable infinite non-amenable group and suppose that the Bernoulli action $G \curvearrowright (X, \mu_1) = \prod_{g \in G}(X_0, \mu_g)$ is non-singular. Let $\nu \sim \mu_e$ be a probability measure on $X_0$ and for every $t \in [0, 1]$ consider the family $(\mu^t_g)_{g \in G}$ of equivalent probability measures given by (1.4). Then there exists a $t_0 \in (0, 1]$ such that $G \curvearrowright (X, \mu_t) = \prod_{g \in G}(X_0, \mu^t_g)$ is strongly ergodic for every $t < t_0$.

Although we can prove a phase transition result in large generality, it remains very challenging to compute the critical value $t_1$. However, when $G \subset \text{Aut}(T)$, for some locally finite tree $T$, following [AIM19, §10], we can construct generalized Bernoulli actions of which we can determine the conservativeness behaviour very precisely. To put this result into perspective, let us first explain briefly the construction from [AIM19, §10].

For a locally finite tree $T$, let $\Omega(T)$ denote the set of orientations on $T$. Let $p \in (0, 1)$ and fix a root $\rho \in T$. Define a probability measure $\mu_p$ on $\Omega(T)$ by orienting an edge towards $\rho$ with probability $p$ and away from $\rho$ with probability $1-p$. If $G \subset \text{Aut}(T)$ is a subgroup, then we naturally obtain a non-singular action $G \curvearrowright (\Omega(T), \mu_p)$. Up to equivalence of measures, the measure $\mu_p$ does not depend on the choice of root $\rho \in T$. The Poincaré exponent of $G \subset \text{Aut}(T)$ is defined as
\[
\delta(G \curvearrowright T) = \inf \left\{ s > 0 \text{ for which } \sum_{w \in G \cdot v} \exp(-sd(v, w)) < +\infty \right\}, \quad (1.5)
\]
where $v \in V(T)$ is any vertex of $T$. In [AIM19, Theorem 10.4] Arano, Isono and Marrakchi showed that if $G \subset \text{Aut}(T)$ is a closed non-elementary subgroup, the action $G \acts (\Omega(T), \mu_p)$ is dissipative up to compact stabilizers if $2\sqrt{p(1-p)} < \exp(-\delta)$ and weakly mixing if $2\sqrt{p(1-p)} > \exp(-\delta)$. This motivates the following similar construction.

Let $E(T) \subset V(T) \times V(T)$ denote the set of oriented edges, so that vertices $v$ and $w$ are adjacent if and only if $(v, w), (w, v) \in E(T)$. Suppose that $X_0$ is a standard Borel space and that $\mu_0, \mu_1$ are equivalent probability measures on $X_0$. Fix a root $\rho \in T$ and define a family of probability measures $(\mu_e)_{e \in E(T)}$ by

$$
\mu_e = \begin{cases} 
\mu_0 & \text{if } e \text{ is oriented towards } \rho, \\
\mu_1 & \text{if } e \text{ is oriented away from } \rho.
\end{cases}
$$

Suppose that $G \subset \text{Aut}(T)$ is a subgroup. Then the generalized Bernoulli action

$$
G \acts \prod_{e \in E(T)} (X_0, \mu_e) : \ (g \cdot x)_e = x_{g \cdot e^{-1}}
$$

is non-singular and up to conjugacy it does not depend on the choice of root $\rho \in T$. In our next main result we generalize [AIM19, Theorem 10.4] to non-singular actions of the form (1.7).

**Theorem C.** Let $T$ be a locally finite tree with root $\rho \in T$ and let $G \subset \text{Aut}(T)$ be a non-elementary closed subgroup with Poincaré exponent $\delta = \delta(G \acts T)$. Let $\mu_0$ and $\mu_1$ be equivalent probability measures on a standard Borel space $X_0$ and define a family of equivalent probability measures $(\mu_e)_{e \in E(T)}$ by (1.6). Then the generalized Bernoulli action (1.7) is dissipative up to compact stabilizers if $1 - H^2(\mu_0, \mu_1) < \exp(-\delta/2)$ and weakly mixing if $1 - H^2(\mu_0, \mu_1) > \exp(-\delta/2)$.

2. Preliminaries

2.1. Non-singular group actions. Let $(X, \mu), (Y, \nu)$ be standard measure spaces. A Borel map $\varphi : X \to Y$ is called non-singular if the pushforward measure $\varphi_*\mu$ is equivalent to $\nu$. If in addition there exist conull Borel sets $X_0 \subset X$ and $Y_0 \subset Y$ such that $\varphi : X_0 \to Y_0$ is a bijection we say that $\varphi$ is a non-singular isomorphism. We write $\text{Aut}(X, \mu)$ for the group of all non-singular automorphisms $\varphi : X \to X$, where we identify two elements if they agree almost everywhere. The group $\text{Aut}(X, \mu)$ carries a canonical Polish topology.

A non-singular group action $G \acts (X, \mu)$ of an lcsc group $G$ on a standard measure space $(X, \mu)$ is a continuous group homomorphism $G \to \text{Aut}(X, \mu)$. A non-singular group action $G \acts (X, \mu)$ is called essentially free if the stabilizer subgroup $G_x = \{ g \in G : g \cdot x = x \}$ is trivial for almost every (a.e.) $x \in X$. When $G$ is countable this is the same as the condition that $\mu((x \in X : g \cdot x = x)) = 0$ for every $g \in G \setminus \{e\}$. We say that $G \acts (X, \mu)$ is ergodic if every $G$-invariant Borel set $A \subset X$ satisfies $\mu(A) = 0$ or $\mu(X \setminus A) = 0$. A non-singular action $G \acts (X, \mu)$ is called weakly mixing if for any ergodic pmp action $G \acts (Y, \nu)$ the diagonal product action $G \acts X \times Y$ is ergodic. If $G$ is not compact and $G \acts (X, \mu)$ is pmp, we say that $G \acts X$ is mixing if

$$
\lim_{g \to \infty} \mu(g \cdot A \cap B) = \mu(A)\mu(B) \quad \text{for every pair of Borel subsets } A, B \subset X.
$$
Suppose that $G \curvearrowright (X, \mu)$ is a non-singular action and that $\mu$ is a probability measure. A sequence of Borel subsets $A_n \subset X$ is called almost invariant if

$$\sup_{g \in K} \mu(g \cdot A_n \Delta A_n) \to 0$$

for every compact subset $K \subset G$.

The action $G \curvearrowright (X, \mu)$ is called strongly ergodic if every almost invariant sequence $A_n \subset X$ is trivial, that is, $\mu(A_n)(1 - \mu(A_n)) \to 0$. The strong ergodicity of $G \curvearrowright (X, \mu)$ only depends on the measure class of $\mu$. When $(Y, \nu)$ is a standard measure space and $\nu$ is infinite, a non-singular action $G \curvearrowright (Y, \nu)$ is called strongly ergodic if $G \curvearrowright (Y, \nu')$ is strongly ergodic, where $\nu'$ is a probability measure that is equivalent to $\nu$.

Following [AIM19, Definition A.16], we say that a non-singular action $G \curvearrowright (X, \mu)$ is dissipative up to compact stabilizers if each ergodic component is of the form $G \curvearrowright G/K$, for a compact subgroup $K \subset G$. By [AIM19, Theorem A.29] a non-singular action $G \curvearrowright (X, \mu)$, with $\mu(X) = 1$, is dissipative up to compact stabilizers if and only if

$$\int_G \frac{d g \mu}{d \mu}(x) d\lambda(g) < +\infty \quad \text{for a.e. } x \in X,$$

where $\lambda$ denotes the left invariant Haar measure on $G$. We say that $G \curvearrowright (X, \mu)$ is infinitely recurrent if for every non-negligible subset $A \subset X$ and every compact subset $K \subset G$ there exists $g \in G \setminus K$ such that $\mu(g \cdot A \cap A) > 0$. By [AIM19, Proposition A.28] and Lemma 2.1 below, a non-singular action $G \curvearrowright (X, \mu)$, with $\mu(X) = 1$, is infinitely recurrent if and only if

$$\int_G \frac{d g \mu}{d \mu}(x) d\lambda(g) = +\infty \quad \text{for a.e. } x \in X.$$

A non-singular action $G \curvearrowright (X, \mu)$ is called dissipative if it is essentially free and dissipative up to compact stabilizers. In that case there exists a standard measure space $(X_0, \mu_0)$ such that $G \curvearrowright X$ is conjugate with the action $G \curvearrowright G \times X_0 : g \cdot (h, x) = (gh, x)$. A non-singular action $G \curvearrowright (X, \mu)$ decomposes, uniquely up to a null set, as $G \curvearrowright D \sqcup C$, where $G \curvearrowright D$ is dissipative up to compact stabilizers and $G \curvearrowright C$ is infinitely recurrent. When $G$ is a countable group and $G \curvearrowright (X, \mu)$ is essentially free, we say that $G \curvearrowright X$ is conservative if it is infinitely recurrent.

**Lemma 2.1.** Suppose that $G$ is an lcsc group with left invariant Haar measure $\lambda$ and that $(X, \mu)$ is a standard probability space. Assume that $G \curvearrowright (X, \mu)$ is a non-singular action that is infinitely recurrent. Then we have that

$$\int_G \frac{d g \mu}{d \mu}(x) d\lambda(g) = +\infty \quad \text{for a.e. } x \in X.$$

**Proof.** Note that the set

$$D = \left\{ x \in X : \int_G \frac{d g \mu}{d \mu}(x) d\lambda(g) < +\infty \right\}$$

is $G$-invariant. Therefore, it suffices to show that $G \curvearrowright X$ is not infinitely recurrent under the assumption that $D$ has full measure.
Let \( \pi : (X, \mu) \to (Y, \nu) \) be the projection onto the space of ergodic components of \( G \bowtie X \). Then there exist a conull Borel subset \( Y_0 \subset Y \) and a Borel map \( \theta : Y_0 \to X \) such that \( (\pi \circ \theta)(y) = y \) for every \( y \in Y_0 \).

Write \( X_y = \pi^{-1}(y) \). By [AIM19, Theorem A.29], for a.e. \( y \in Y \) there exists a compact subgroup \( K_y \subset G \) such that \( G \bowtie X_y \) is conjugate with \( G \bowtie G/K_y \). Let \( G_n \subset G \) be an increasing sequence of compact subsets of \( G \) such that \( \bigcup_{n \geq 1} G_n = G \). For every \( x \in X \), write \( G_x = \{ g \in G : g \cdot x = x \} \) for the stabilizer subgroup of \( x \). Using an argument as in [MRV11, Lemma 10], one shows that for each \( n \geq 1 \) the set \( \{ x \in X : G_x \subset G_n \} \) is Borel. Thus, for every \( n \geq 1 \) the set

\[
U_n = \{ y \in Y_0 : K_y \subset G_n \} = \{ y \in Y_0 : G_{\theta(y)} \subset G_n \}
\]

is a Borel subset of \( Y \) and we have that \( \nu(\bigcup_{n \geq 1} U_n) = 1 \). Therefore, the sets

\[
A_n = \{ g \cdot \theta(y) : g \in G_n, y \in U_n \}
\]

are analytic and exhaust \( X \) up to a set of measure zero. So there exist an \( n_0 \in \mathbb{N} \) and a non-negligible Borel set \( B \subset A_{n_0} \). Suppose that \( h \in G \) is such that \( h \cdot B \cap B \neq \emptyset \). Then there exist \( y \in U_{n_0} \) and \( g_1, g_2 \in G_{n_0} \) such that \( h g_1 \cdot \theta(y) = g_2 \cdot \theta(y) \), and we get that \( h \in G_{n_0} K_y G_{n_0}^{-1} \subset G_{n_0} G_{n_0} G_{n_0}^{-1} \). In other words, for \( h \in G \) outside the compact set \( G_{n_0} G_{n_0} G_{n_0}^{-1} \) we have that \( \mu(h \cdot B \cap B) = 0 \), so that \( G \bowtie X \) is not infinitely recurrent. \( \square \)

We will frequently use the following result of Schmidt and Walters. Suppose that \( G \bowtie (X, \mu) \) is a non-singular action that is infinitely recurrent and suppose that \( G \bowtie (Y, \nu) \) is pmp and mixing. Then by [SW81, Theorem 2.3] we have that

\[
L^\infty(X \times Y)^G = L^\infty(X)^G \overline{\otimes} 1,
\]

where \( G \bowtie X \times Y \) acts diagonally. Although [SW81, Theorem 2.3] demands proper ergodicity of the action \( G \bowtie (X, \mu) \), the infinite recurrence assumption is sufficient as remarked in [AIM19, Remark 7.4].

2.2. The Maharam extension and crossed products. Let \( (X, \mu) \) be a standard measure space. For any non-singular automorphism \( \varphi \in \text{Aut}(X, \mu) \), we define its Maharam extension by

\[
\varphi : X \times \mathbb{R} \to X \times \mathbb{R} : \quad \varphi(x, t) = (\varphi(x), t + \log(d \varphi^{-1}_t \mu/d \mu)(x)).
\]

Then \( \varphi \) preserves the infinite measure \( \mu \times \exp(-t)dt \). The assignment \( \varphi \mapsto \varphi \) is a continuous group homomorphism from \( \text{Aut}(X) \) to \( \text{Aut}(X \times \mathbb{R}) \). Thus, for each non-singular group action \( G \bowtie (X, \mu) \), by composing with this map, we obtain a non-singular group action \( G \bowtie X \times \mathbb{R} \), which we call the Maharam extension of \( G \bowtie X \). If \( G \bowtie X \) is a non-singular group action, the translation action \( \mathbb{R} \bowtie X \times \mathbb{R} \) in the second component commutes with the Maharam extension \( G \bowtie X \times \mathbb{R} \). Therefore, we get a well-defined action \( \mathbb{R} \bowtie L^\infty(X \times \mathbb{R})^G \), which is the Krieger flow associated to the action \( G \bowtie X \). The Krieger flow is given by \( \mathbb{R} \bowtie \mathbb{R} \) if and only if there exists a \( G \)-invariant \( \sigma \)-finite measure \( \nu \) on \( X \) that is equivalent to \( \mu \).
Suppose that $M \subset B(H)$ is a von Neumann algebra represented on the Hilbert space $H$ and that $\alpha: G \curvearrowright M$ is a continuous action on $M$ of an lcsc group $G$. Then the crossed product von Neumann algebra $M \rtimes_\alpha G \subset B(L^2(G, H))$ is the von Neumann algebra generated by the operators \{\pi(x)\}_{x \in M}$ and \{uh\}_{h \in G}$ acting on $\xi \in L^2(G, H)$ as
\[(\pi(x)\xi)(g) = \alpha_g^{-1}(x)\xi(g), \quad (uh\xi)(g) = \xi(h^{-1}g).
\]
In particular, if $G \curvearrowright (X, \mu)$ is a non-singular group action, the crossed product $L^\infty(X) \rtimes G \subset B(L^2(G \times X))$ is the von Neumann algebra generated by the operators
\[(\pi(H)\xi)(g, x) = H(g \cdot x)\xi(g, x), \quad (uh\xi)(g, x) = \xi(h^{-1}g, x),
\]
for $H \in L^\infty(X)$ and $h \in G$. If $G \curvearrowright X$ is non-singular essentially free and ergodic, then $L^\infty(X) \rtimes G$ is a factor. Moreover, when $G$ is a unimodular group, the Krieger flow of $G \curvearrowright X$ equals the flow of weights of the crossed product von Neumann algebra $L^\infty(X) \rtimes G$. For non-unimodular groups this is not necessarily true, motivating the following definition.

**Definition 2.2.** Let $G$ be an lcsc group with modular function $\Delta: G \rightarrow \mathbb{R}_{>0}$. Let $\lambda$ denote the Lebesgue measure on $\mathbb{R}$. Suppose that $\alpha: G \curvearrowright (X, \mu)$ is a non-singular action. We define the modular Maharam extension of $G \curvearrowright X$ as the non-singular action

$$\beta: G \curvearrowright (X \times \mathbb{R}, \mu \times \lambda): \quad g \cdot (x, t) = (g \cdot x, t + \log(\Delta(g)) + \log(dg^{-1} \mu/d\mu)(x)).$$

Let $L^\infty(X \times \mathbb{R})^\beta$ denote the subalgebra of $\beta$-invariant elements. We define the flow of weights associated to $G \curvearrowright X$ as the translation action $\mathbb{R} \curvearrowright L^\infty(X \times \mathbb{R})^\beta: (t \cdot H)(x, s) = H(x, s - t)$.

As we explain below, the flow of weights associated to an essentially free ergodic non-singular action $G \curvearrowright X$ equals the flow of weights of the crossed product factor $L^\infty(X) \rtimes G$, justifying the terminology. See also [Sa74, Proposition 4.1].

Let $\alpha: G \curvearrowright X$ be an essentially free ergodic non-singular group action with modular Maharam extension $\beta: G \curvearrowright X \times \mathbb{R}$. By [Sa74, Proposition 1.1] there is a canonical normal semifinite faithful weight $\varphi$ on $L^\infty(X) \rtimes_\alpha G$ such that the modular automorphism group $\sigma^\varphi$ is given by

$$\sigma^\varphi_t(\pi(H)) = \pi(H), \quad \sigma^\varphi_t(u_h) = \Delta(g)^{it}u_h\pi((dg^{-1} \mu/d\mu)^{it}),$$

where $\Delta: G \rightarrow \mathbb{R}_{>0}$ denotes the modular function of $G$.

For an element $\xi \in L^2(\mathbb{R}, L^2(G \times X))$ and $(g, x) \in G \times X$, write $\xi_{g, x}$ for the map given by $\xi_{g, x}(s) = \xi(s, g, x)$. Then by Fubini’s theorem $\xi_{g, x} \in L^2(\mathbb{R})$ for a.e. $(g, x) \in G \times X$. Let $U: L^2(\mathbb{R}, L^2(G \times X)) \rightarrow L^2(G, L^2(X \times \mathbb{R}))$ be the unitary given on $\xi \in L^2(\mathbb{R}, L^2(G \times X))$ by

$$(U\xi)(g, x, t) = \mathcal{F}^{-1}(\xi_{g, x})(t + \log(\Delta(g)) + \log(dg^{-1} \mu/d\mu)(x)),$$

where $\mathcal{F}^{-1}: L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ denotes the inverse Fourier transform. One can check that conjugation by $U$ induces an isomorphism

$$\Psi: (L^\infty(X) \rtimes_\alpha G) \rtimes_{\sigma^\varphi} \mathbb{R} \rightarrow L^\infty(X \times \mathbb{R}) \rtimes_\beta G.$$
Let \( \kappa : L^\infty(X \times \mathbb{R}) \to L^\infty(X \times \mathbb{R}) \rtimes_\beta G \) be the inclusion map and let \( \gamma : \mathbb{R} \acts L^\infty(X \times \mathbb{R}) \rtimes_\beta G \) be the action given by

\[ \gamma_t(\kappa(H))(x, s) = \kappa(H)(x, s - t), \quad \gamma_t(ug) = ug. \]

Then one can verify that \( \Psi \) conjugates the dual action \( \hat{\sigma}_\varphi : \mathbb{R} \acts (L^\infty(X) \rtimes_\alpha G) \rtimes_\sigma \mathbb{R} \) and \( \gamma \). Therefore, we can identify the flow of weights \( \mathbb{R} \acts Z((L^\infty(X) \rtimes_\alpha G) \rtimes_\sigma \mathbb{R}) \) with \( \mathbb{R} \acts Z((L^\infty(X \times \mathbb{R}) \rtimes_\beta G) \cong L^\infty(X \times \mathbb{R}) \beta : \) the flow of weights associated to \( G \acts X \).

**Remark 2.3.** It will be useful to speak about the Krieger type of a non-singular ergodic action \( G \acts X \). In light of the discussion above, we will only use this terminology for countable groups \( G \), so that no confusion arises with the type of the crossed product von Neumann algebra \( L^\infty(X) \rtimes G \). So assume that \( G \) is countable and that \( G \acts (X, \mu) \) is a non-singular ergodic action. Then the Krieger flow is ergodic and we distinguish several cases. If \( \nu \) is atomic, we say that \( G \acts X \) is of type I. If \( \nu \) is non-atomic and finite, we say that \( G \acts X \) is of type II_1. If \( \nu \) is non-atomic and infinite, we say that \( G \acts X \) is of type II\( \infty \). If the Krieger flow is properly ergodic (that is, every orbit has measure zero), we say that \( G \acts X \) is of type III_0.

2.3. **Non-singular Bernoulli actions.** Suppose that \( G \) is a countable infinite group and that \( (\mu_g)_{g \in G} \) is a family of equivalent probability measures on a standard Borel space \( X_0 \). The action

\[ G \acts (X, \mu) = \prod_{h \in G} (X_0, \mu_h) : (g \cdot x)_h = x_{g^{-1}h} \]  

is called the Bernoulli action. For two probability measures \( \nu, \eta \) on a standard Borel space \( Y \), the Hellinger distance \( H^2(\nu, \eta) \) is defined by

\[ H^2(\nu, \eta) = \frac{1}{2} \int_Y \left( \sqrt{d\nu/d\zeta} - \sqrt{d\eta/d\zeta} \right)^2 d\zeta, \]

where \( \zeta \) is any probability measure on \( Y \) such that \( \nu, \eta \prec \zeta \). By Kakutani’s criterion for equivalence of infinite product measures \([\text{Kak}48]\) the Bernoulli action (2.1) is non-singular if and only if

\[ \sum_{h \in G} H^2(\mu_h, \mu_{gh}) < +\infty \quad \text{for every } g \in G. \]

If \( (X, \mu) \) is non-atomic and the Bernoulli action (2.1) is non-singular, then it is essentially free by \([\text{BKV}19, \text{Lemma 2.2}]\).

Suppose that \( I \) is a countable infinite set and that \( (\mu_i)_{i \in I} \) is a family of equivalent probability measures on a standard Borel space \( X_0 \). If \( G \) is an lcsc group that acts on \( I \), the action

\[ G \acts (X, \mu) = \prod_{i \in I} (X_0, \mu_i) : (g \cdot x)_i = x_{g^{-1}i} \]  

is called the Bernoulli action. For two probability measures \( \nu, \eta \) on a standard Borel space \( Y \), the Hellinger distance \( H^2(\nu, \eta) \) is defined by

\[ H^2(\nu, \eta) = \frac{1}{2} \int_Y \left( \sqrt{d\nu/d\zeta} - \sqrt{d\eta/d\zeta} \right)^2 d\zeta, \]

where \( \zeta \) is any probability measure on \( Y \) such that \( \nu, \eta \prec \zeta \). By Kakutani’s criterion for equivalence of infinite product measures \([\text{Kak}48]\) the Bernoulli action (2.1) is non-singular if and only if

\[ \sum_{h \in G} H^2(\mu_h, \mu_{gh}) < +\infty \quad \text{for every } g \in G. \]

If \( (X, \mu) \) is non-atomic and the Bernoulli action (2.1) is non-singular, then it is essentially free by \([\text{BKV}19, \text{Lemma 2.2}]\).
is called the generalized Bernoulli action and it is non-singular if and only if 
\[ \sum_{i \in I} H^2(\mu_i, \mu_{g_i}) < +\infty \] for every \( g \in G \). When \( v \) is a probability measure on \( X_0 \) such that \( \mu_i = v \) for every \( i \in I \), the generalized Bernoulli action (2.3) is pmp and it is mixing if and only if the stabilizer subgroup \( G_i = \{g \in G : g \cdot i = i\} \) is compact for every \( i \in I \). In particular, if \( G \) is countable infinite, the pmp Bernoulli action \( G \acts (X_0, \mu_0)^G \) is mixing.

2.4. Groups acting on trees. Let \( T = (V(T), E(T)) \) be a locally finite tree, so that the edge set \( E(T) \) is a symmetric subset of \( V(T) \times V(T) \) with the property that vertices \( v, w \in V(T) \) are adjacent if and only if \( (v, w), (w, v) \in E(T) \). When \( T \) is clear from the context, we will write \( E \) instead of \( E(T) \). Also we will often write \( T \) instead of \( V(T) \) for the vertex set. For any two vertices \( v, w \in T \) let \([v, w]\) denote the smallest subtree of \( T \) that contains \( v \) and \( w \). The distance between vertices \( v, w \in T \) is defined as \( d(v, w) = |V([v, w])| - 1 \). Fixing a root \( \rho \in T \), we define the boundary \( \partial T \) of \( T \) as the collection of all infinite line segments starting at \( \rho \). We equip \( \partial T \) with a metric \( d_\rho \) as follows. If \( \omega, \omega' \in \partial T \), let \( v \in T \) be the unique vertex such that \( d(\rho, v) = \sup_{\omega' \in \omega' \cap \partial T} d(\rho, v) \) and define \[ d_\rho(\omega, \omega') = \exp(-d(\rho, v)). \]

Then, up to homeomorphism, the space \((\partial T, d_\rho)\) does not depend on the chosen root \( \rho \in T \). Furthermore, the Hausdorff dimension \( \dim_H \partial T \) of \((\partial T, d_\rho)\) is also independent of the choice of \( \rho \in T \).

Let \( \text{Aut}(T) \) denote the group of automorphisms of \( T \). By [Tit70, Proposition 3.2], if \( g \in \text{Aut}(T) \), then either:

- \( g \) fixes a vertex or interchanges a pair of vertices (in this case we say that \( g \) is elliptic);
- or there exists a bi-infinite line segment \( L \subset T \), called the axis of \( g \), such that \( g \) acts on \( L \) by non-trivial translation (in this case we say that \( g \) is hyperbolic).

We equip \( \text{Aut}(T) \) with the topology of pointwise convergence. A subgroup \( G \subset \text{Aut}(T) \) is closed with respect to this topology if and only if for every \( v \in T \) the stabilizer subgroup \( G_v = \{g \in G : g \cdot v = v\} \) is compact. An action of an lcsc group \( G \) on \( T \) is a continuous homomorphism \( G \to \text{Aut}(T) \). We say that the action \( G \acts T \) is cocompact if there is a finite set \( F \subset E(T) \) such that \( G \cdot F = E(T) \). A subgroup \( G \subset \text{Aut}(T) \) is called non-elementary if it does not fix any point in \( T \cup \partial T \) and does not interchange any pair of points in \( T \cup \partial T \). Equivalently, \( G \subset \text{Aut}(T) \) is non-elementary if there exist hyperbolic elements \( h, g \in G \) with axes \( L_h \) and \( L_g \) such that \( L_h \cap L_g \) is finite. If \( G \subset \text{Aut}(T) \) is a non-elementary closed subgroup, there exists a unique minimal \( G \)-invariant subtree \( S \subset T \) and \( G \) is compactly generated if and only if \( G \acts S \) is cocompact (see [CM11, §2]).

Recall from (1.5) the definition of the Poincaré exponent \( \delta(G \acts T) \) of a subgroup \( G \subset \text{Aut}(T) \). If \( G \subset \text{Aut}(T) \) is a closed subgroup such that \( G \acts T \) is cocompact, then we have that \( \delta(G \acts T) = \dim_H \partial T \).

3. Phase transitions for non-singular Bernoulli actions: proof of Theorems A and B

Let \( G \) be a countable infinite group and let \( (\mu_g)_{g \in G} \) be a family of equivalent probability measures on a standard Borel space \( X_0 \). Let \( v \) also be a probability measure on \( X_0 \). For \( t \in [0, 1] \) we define the family of probability measures
\[ \mu_t^g = (1-t)\nu + t\mu_g, \quad g \in G. \] (3.1)

We write \( \mu_t \) for the infinite product measure \( \mu_t = \prod_{g \in G} \mu_t^g \) on \( X = \prod_{g \in G} X_0 \). We prove Theorem 3.1 below, which is slightly more general than Theorem A.

**Theorem 3.1.** Let \( G \) be a countable infinite group and let \( (\mu_g)_{g \in G} \) be a family of equivalent probability measures on a standard probability space \( X_0 \), which is not supported on a single atom. Assume that the Bernoulli action \( G \curvearrowright \prod_{g \in G}(X_0, \mu_g) \) is non-singular. Let \( \nu \) also be a probability measure on \( X_0 \). Then for every \( t \in [0, 1] \) the Bernoulli action

\[ G \curvearrowright (X, \mu_t) = \prod_{g \in G}(X_0, (1-t)\nu + t\mu_g) \] (3.2)

is non-singular. Assume, in addition, that one of the following conditions holds.

1. \( \nu \sim \mu_e \).
2. \( \nu \prec \mu_e \) and \( \sup_{g \in G} | \log d\mu_g/d\mu_e(x) | < +\infty \) for a.e. \( x \in X_0 \).

Then there exists a \( t_1 \in [0, 1] \) such that \( G \curvearrowright (X, \mu_t) \) is dissipative for every \( t > t_1 \) and weakly mixing for every \( t < t_1 \).

**Remark 3.2.** One might hope to prove a completely general phase transition result that only requires \( \nu \prec \mu_e \), and not the additional assumption that \( \sup_{g \in G} | \log d\mu_g/d\mu_e(x) | < +\infty \) for a.e. \( x \in X_0 \). However, the following example shows that this is not possible.

Let \( G \) be any countable infinite group and let \( G \curvearrowright \prod_{g \in G}(C_0, \eta_g) \) be a conservative non-singular Bernoulli action. Note that Theorem 3.1 implies that \( G \curvearrowright \prod_{g \in G}(C_0, (1-t)\eta_e + t\eta_g) \) is conservative for every \( t < 1 \). Let \( X_0 = C_0 \cup C_1 \) such that \( 0 < \sum_{g \in G} \mu_g(C_1) < +\infty \) and such that \( \mu_g|\{0\} = \mu_g(C_0)\eta_g \). Then the Bernoulli action \( G \curvearrowright (X, \mu) = \prod_{g \in G}(X_0, \mu_g) \) is non-singular with non-negligible conservative part \( C_0^G \subset G \) and dissipative part \( X \setminus C_0^G \). Taking \( \nu = \eta_e \prec \mu_e \), for each \( t < 1 \) the Bernoulli action \( G \curvearrowright (X, \mu_t) = \prod_{g \in G}(X_0, (1-t)\eta_e + t\mu_g) \) is constructed in the same way, by starting with the conservative Bernoulli action \( G \curvearrowright \prod_{g \in G}(C_0, (1-t)\eta_e + t\eta_g) \). So for every \( t \in (0, 1) \) the Bernoulli action \( G \curvearrowright (X, \mu_t) \) has non-negligible conservative part and non-negligible dissipative part.

We can also prove a version of Theorem B in the more general setting of Theorem 3.1.

**Theorem 3.3.** Let \( G \) be a countable infinite non-amenable group. Make the same assumptions as in Theorem 3.1 and consider the non-singular Bernoulli actions \( G \curvearrowright (X, \mu_t) \) given by (3.2). Assume, moreover, that:

1. \( \nu \sim \mu_e \), or
2. \( \nu \prec \mu_e \) and \( \sup_{g \in G} | \log d\mu_g/d\mu_e(x) | < +\infty \) for a.e. \( x \in X_0 \).

Then there exists a \( t_0 > 0 \) such that \( G \curvearrowright (X, \mu_t) \) is strongly ergodic for every \( t < t_0 \).
Proof of Theorem 3.1. Assume that $G \bowtie (X, \mu_1) = \prod_{g \in G} (X_0, \mu_g)$ is non-singular. For every $t \in [0, 1]$ we have that
\[
\sum_{h \in G} H^2(\mu^g_h, \mu^l_{gh}) \leq t \sum_{h \in G} H^2(\mu_h, \mu_{gh}) \quad \text{for every } g \in G,
\]
so that $G \bowtie (X, \mu_t)$ is non-singular for every $t \in [0, 1]$. The rest of the proof we divide into two steps.

Claim 1. If $G \bowtie (X, \mu_t)$ is conservative, then $G \bowtie (X, \mu_s)$ is weakly mixing for every $s < t$.

Proof of Claim 1. Note that for every $g \in G$ we have that
\[
(\mu^g_s)^r = (1 - r)v + r \mu^g_s = (1 - r)v + r(1 - s)v + rs \mu_g = \mu^g_{sr},
\]
so that $(\mu_s s) r = \mu_{sr}$. Therefore, it suffices to prove that $G \bowtie (X, \mu_s)$ is weakly mixing for every $s < 1$, assuming that $G \bowtie (X, \mu_1)$ is conservative.

The claim is trivially true for $s = 0$. So assume that $G \bowtie (X, \mu_1)$ is conservative and fix $s \in (0, 1)$. Let $G \bowtie (Y, \eta)$ be an ergodic pmp action. Define $Y_0 = X_0 \times X_0 \times [0, 1]$ and define the probability measures $\lambda$ on $[0, 1]$ by $\lambda(0) = s$. Define the map $\theta : Y_0 \to X_0$ by
\[
\theta(x, x', j) = \begin{cases} x & \text{if } j = 0, \\ x' & \text{if } j = 1. \end{cases}
\]
Then for every $g \in G$ we have that $\theta_* (\mu^g_s \times \nu \times \lambda) = \mu^g_s$. Write $Z = [0, 1]^G$ and equip $Z$ with the probability measure $\lambda^G$. We identify the Bernoulli action $G \bowtie (Y_0^G)$ with the diagonal action $G \bowtie X \times X \times Z$. By applying $\theta$ in each coordinate we obtain a $G$-equivariant factor map
\[
\Psi : X \times X \times Z \to X : \quad \Psi(x, x', z)_h = \theta(x_h, x'_h, z_h).
\]
Then the map $\text{id}_Y \times \Psi : Y \times X \times X \times Z \to Y \times X$ is $G$-equivariant and we have that $(\text{id}_Y \times \Psi)_* (\eta \times \mu_1 \times \mu_0 \times \lambda^G) = \eta \times \mu_s$. The construction above is similar to [KS20, §4].

Take $F \in L^\infty(Y \times X, \eta \times \mu_s)^G$. Note that the diagonal action $G \bowtie (Y \times X, \eta \times \mu_1)$ is conservative, since $G \bowtie (Y, \eta)$ is pmp. The action $G \bowtie (X \times Z, \mu_0 \times \lambda^G)$ can be identified with a pmp Bernoulli action with base space $(X_0 \times [0, 1], \nu \times \lambda)$, so that it is mixing. By [SW81, Theorem 2.3] we have that
\[
L^\infty(Y \times X \times X \times Z, \eta \times \mu_1 \times \mu_0 \times \lambda^G)^G = L^\infty(Y \times X, \eta \times \mu_1)^G \otimes 1 \otimes 1,
\]
which implies that the assignment $(y, x, x', z) \mapsto F(y, \Psi(x, x', z))$ is essentially independent of $x'$ and $z$. Choosing a finite set of coordinates $F \subset G$ and changing, for $g \in F$, the value $z_g$ between 0 and 1, we see that $F$ is essentially independent of the $x_g$-coordinates for $g \in F$. As this is true for any finite set $F \subset G$, we have that $F \in L^\infty(Y)^G \otimes 1$. The action $G \bowtie (Y, \eta)$ is ergodic and therefore $F$ is essentially constant. We conclude that $G \bowtie (X, \mu_s)$ is weakly mixing. \qed
CLAIM 2. If \( v \sim \mu_e \) and if \( G \curvearrowright (X, \mu_1) \) is not dissipative, then \( G \curvearrowright (X, \mu_s) \) is conservative for every \( s < 1 \).

Proof of Claim 2. Again it suffices to assume that \( G \curvearrowright (X, \mu_1) \) is not dissipative and to show that \( G \curvearrowright (X, \mu_s) \) is conservative for every \( s < 1 \).

When \( s = 0 \), the statement is trivial, so assume that \( G \curvearrowright (X, \mu_1) \) is not dissipative and fix \( s \in (0, 1) \). Let \( C \subseteq X \) denote the non-negligible conservative part of \( G \curvearrowright (X, \mu_1) \).

As in the proof of Claim 1, write \( Z = [0, 1]^G \) and let \( \lambda \) be the probability measure on \([0, 1]\) given by \( \lambda(0) = s \). Writing \( \Psi : X \times X \times Z \to X \) for the \( G \)-equivariant map (3.4).

We claim that \( \Psi_*((\mu_1 \times \mu_0 \times \lambda^G)|_{C \times X \times Z}) \sim \mu_s \), so that \( G \curvearrowright (X, \mu_s) \) is a factor of a conservative non-singular action, and therefore must be conservative itself.

As \( \Psi_*((\mu_1 \times \mu_0 \times \lambda^G)|_{C \times X \times Z}) \sim \mu_s \), we have that \( \Psi_*((\mu_1 \times \mu_0 \times \lambda^G)|_{C \times X \times Z}) \sim \mu_s \). Let \( U \subseteq X \) be the Borel set, uniquely determined up to a set of measure zero, such that \( \Psi_*((\mu_1 \times \mu_0 \times \lambda^G)|_{C \times X \times Z}) \sim \mu_s \). We have to show that \( \mu_s(X \setminus U) = 0 \). Fix a finite subset \( F \subseteq G \). For every \( t \in [0, 1] \) define

\[
(X_1, \gamma_1^t) = \prod_{g \in F} (X_0, (1-t)v + t\mu_g),
\]

\[
(X_2, \gamma_2^t) = \prod_{g \in G \setminus F} (X_0, (1-t)v + t\mu_g).
\]

We shall write \( \gamma_1 = \gamma_1^1, \gamma_2 = \gamma_2^1 \). Also define

\[
(Y_1, \xi_1) = \prod_{g \in F} (X_0 \times X_0 \times [0, 1], \mu_g \times v \times \lambda),
\]

\[
(Y_2, \xi_2) = \prod_{g \in G \setminus F} (X_0 \times X_0 \times [0, 1], \mu_g \times v \times \lambda).
\]

By applying the map (3.3) in every coordinate, we get factor maps \( \Psi_j : Y_j \to X_j \) that satisfy \( (\Psi_j)_\ast(\xi_j) = \gamma_j \) for \( j = 1, 2 \). Identify \( X_1 \times X_2 \cong X \times (X_0 \times \{0, 1\})^G \) and define the subset \( C' \subseteq X_1 \times X_2 \) by \( C' = C \times (X_0 \times \{0, 1\})^G \). Let \( U' \subseteq X \) be Borel such that

\[
(id_{X_1} \times \Psi_2)_\ast((\gamma_1 \times \xi_2)|_{C'}) \sim (\gamma_1 \times \gamma_2)|_{U'}.
\]

Identify \( Y_1 \times Y_2 \cong X \times (X_0 \times \{0, 1\})^F \) and define \( V \subseteq Y_1 \times X_2 \) by \( V = U' \times (X_0 \times \{0, 1\})^F \). Then we have that

\[
(\Psi_1 \times id_{X_2})_\ast((\xi_1 \times \gamma_2)|_{V}) \sim (\Psi_1 \times id_{X_2})_\ast(id_{Y_1} \times \Psi_2)_\ast((\gamma_1 \times \xi_1)|_{C'} \times v^F \times \lambda^F) \]

\[
= \Psi_*((\xi_1 \times \xi_2)|_{C \times X \times Z}) \sim \mu_s|_{U'}.\]

Let \( \pi : X_1 \times X_2 \to X_2 \) and \( \pi' : Y_1 \times X_2 \to X_2 \) denote the coordinate projections. Note that by construction we have that

\[
\pi'_\ast((\xi_1 \times \gamma_2)|_{V}) \sim \pi'_\ast((\gamma_1 \times \gamma_2)|_{U'}) \sim \pi_\ast(\mu_s|_{U'}).
\]

(3.5) Let \( W \subseteq X_2 \) be Borel such that \( \pi_\ast(\mu_s|_{U'}) \sim \gamma_2|_{W} \). For every \( y \in X_2 \) define the Borel sets

\[
U_y = \{ x \in X_1 : (x, y) \in U \} \quad \text{and} \quad U'_y = \{ x \in X_1 : (x, y) \in U' \}.
\]
As $\pi_*((\gamma_1 \times \gamma_2^s)|_{\mathcal{U}}) \sim \gamma_2^s|_W$, we have that

$$\gamma_1(\mathcal{U}_y^s) > 0 \quad \text{for } \gamma_2^s\text{-a.e. } y \in W.$$

The disintegration of $(\gamma_1 \times \gamma_2^s)|_{\mathcal{U}}$ along $\pi$ is given by $(\gamma_1|_{\mathcal{U}_y^s})_{y \in W}$. Therefore, the disintegration of $((\xi_1 \times \gamma_2^s)|_{\mathcal{V}}$ along $\pi'$ is given by $(\gamma_1|_{\mathcal{U}_y') \times \nu^F \times \lambda^F)_{y \in W}$. We conclude that the disintegration of $(\Psi_1 \times \text{id}_{X_2})_*((\xi_1 \times \gamma_2^s)|_{\mathcal{V}})$ along $\pi$ is given by $((\Psi_1)_*(\gamma_1|_{\mathcal{U}_y^s} \times \nu^F \times \lambda^F))_{y \in W}$. The disintegration of $\mu_s|_{\mathcal{U}}$ along $\pi$ is given by $(\gamma_2^s|_{\mathcal{U}_y^s})_{y \in W}$. Since $\mu_s|_{\mathcal{U}} = (\Psi_1 \times \text{id}_{X_2})_*((\xi_1 \times \gamma_2^s)|_{\mathcal{V}})$, we conclude that

$$(\Psi_1)_*(\gamma_1|_{\mathcal{U}_y^s} \times \nu^F \times \lambda^F) \sim \gamma_1^s|_{\mathcal{U}_y^s} \quad \text{for } \gamma_2^s\text{-a.e. } y \in W.$$ 

As $\gamma_1(\mathcal{U}_y^s) > 0$ for $\gamma_2^s\text{-a.e. } y \in W$, and using that $\nu \sim \mu_e$, we see that

$$\gamma_1^s \sim \nu^F \sim (\Psi_1)_*(\gamma_1 \times \nu^F \times \lambda^F)|_{\mathcal{U}_y^s \times X_0^F \times \{1\}^F} \sim (\Psi_1)_*(\gamma_1|_{\mathcal{U}_y^s} \times \nu^F \times \lambda^F).$$

for $\gamma_2^s\text{-a.e. } y \in W$. It is clear that also $(\Psi_1)_*(\gamma_1|_{\mathcal{U}_y^s} \times \nu^F \times \lambda^F) < \gamma_1^s$, so that $\gamma_1^s|_{\mathcal{U}_y^s} \sim \gamma_1^s$ for $\gamma_2^s\text{-a.e. } y \in W$. Therefore, we have that $\gamma_1^s(\mathcal{X}_1 \setminus \mathcal{U}_y^s) = 0$ for $\gamma_2^s\text{-a.e. } y \in W$, so that

$$\mu_s(\mathcal{U}\Delta(\mathcal{X}_0^F \times W)) = 0.$$

Since this is true for every finite subset $\mathcal{F} \subset G$, we conclude that $\mu_s(\mathcal{X} \setminus \mathcal{U}) = 0$. \hfill \Box

The conclusion of the proof now follows by combining both claims. Assume that $G \actson (X, \mu_t)$ is not dissipative and fix $s < t$. Choose $r$ such that $s < r < t$.

$v \sim \mu_e$. By Claim 2 we have that $G \actson (X, \mu_r)$ is conservative. Then by Claim 1 we see that $G \actson (X, \mu_s)$ is weakly mixing.

$v \prec \mu_e$. As $v \prec \mu_e$, the measures $\mu_e^l$ and $\mu_e^s$ are equivalent. We have that

$$\frac{d\mu_e^l}{d\mu_e^s} = \left(1 - t\right)\frac{d\nu}{d\mu_e} + t\frac{d\mu_g}{d\mu_e} \frac{d\mu_e}{d\mu_e^l}.$$

So if $\sup_{g \in G} |\log d\mu_g/d\mu_e(x)| < +\infty$ for a.e. $x \in X_0$, we also have that

$$\sup_{g \in G} |\log d\mu_g/d\mu_e(x)| < +\infty \quad \text{for a.e. } x \in X_0.$$

It follows from [BV20, Proposition 4.3] that $G \actson (X, \mu_t)$ is conservative. Then by Claim 1 we have that $G \actson (X, \mu_s)$ is weakly mixing. \hfill \Box

Remark 3.4. Let $I$ be a countably infinite set and suppose that we are given a family of equivalent probability measures $(\mu_i)_{i \in I}$ on a standard Borel space $X_0$. Let $v$ be a probability measure on $X_0$ that is equivalent to all the $\mu_i$. If $G$ is an lcsc group that acts
on $\mathcal{I}$ such that for each $i \in \mathcal{I}$ the stabilizer subgroup $G_i = \{g \in G : g \cdot i = i\}$ is compact, then the pmp generalized Bernoulli action

$$G \curvearrowright \prod_{i \in \mathcal{I}} (X_0, \nu), \quad (g \cdot x)_i = x_{g^{-1} \cdot i}$$

is mixing. For $t \in [0, 1]$ write

$$(X, \mu_t) = \prod_{i \in \mathcal{I}} (X_0, (1-t)\nu + t\mu_i)$$

and assume that the generalized Bernoulli action $G \curvearrowright (X, \mu_1)$ is non-singular.

Since [SW81, Theorem 2.3] still applies to infinitely recurrent actions of lcsc groups (see [AIM19, Remark 7.4]), it is straightforward to adapt the proof of Claim 1 in the proof of Theorem 3.1 to prove that if $G \curvearrowright (X, \mu_t)$ is infinitely recurrent, then $G \curvearrowright (X, \mu_s)$ is weakly mixing for every $s < t$. Similarly, we can adapt the proof of Claim 2, using that a factor of an infinitely recurrent action is again infinitely recurrent. Together, this leads to the following phase transition result in the lcsc setting.

Assume that $G_i = \{g \in G : g \cdot i = i\}$ is compact for every $i \in \mathcal{I}$ and that $\nu \sim \mu_e$. Then there exists a $t_1 \in [0, 1]$ such that $G \curvearrowright (X, \mu_t)$ is dissipative up to compact stabilizers for every $t > t_1$ and weakly mixing for every $t < t_1$.

Recall the following definition from [BKV19, Definition 4.2]. When $G$ is a countable infinite group and $G \curvearrowright (X, \mu)$ is a non-singular action on a standard probability space, a sequence $(\eta_n)$ of probability measures on $G$ is called strongly recurrent for the action $G \curvearrowright (X, \mu)$ if

$$\sum_{h \in G} \eta_n^2(h) \int_X \frac{d\mu(x)}{\sum_{k \in G} \eta_n(hk^{-1})dk^{-1}\mu/d\mu(x)} \quad n \to +\infty \quad 0.$$

We say that $G \curvearrowright (X, \mu)$ is strongly conservative if there exists a sequence $(\eta_n)$ of probability measures on $G$ that is strongly recurrent for $G \curvearrowright (X, \mu)$.

**Lemma 3.5.** Let $G \curvearrowright (X, \mu)$ and $G \curvearrowright (Y, \nu)$ be non-singular actions of a countable infinite group $G$ on standard probability spaces $(X, \mu)$ and $(Y, \nu)$. Suppose that $\psi : (X, \mu) \to (Y, \nu)$ is a measure-preserving $G$-equivariant factor map and that $\eta_n$ is a sequence of probability measures on $G$ that is strongly recurrent for the action $G \curvearrowright (X, \mu)$. Then $\eta_n$ is strongly recurrent for the action $G \curvearrowright (Y, \nu)$.

**Proof.** Let $E : L^0(X, [0, +\infty)) \to L^0(Y, [0, +\infty))$ denote the conditional expectation map that is uniquely determined by

$$\int_Y E(F)H \ d\nu = \int_X F(H \circ \psi) \ d\mu$$

for all positive measurable functions $F : X \to [0, +\infty)$ and $H : Y \to [0, +\infty)$. Since

$$\frac{dk^{-1}\nu}{dv} \quad \frac{dk^{-1}\mu}{d\psi_{*}\mu} = E\left(\frac{dk^{-1}\mu}{d\mu}\right)$$
for every $k \in G$, we have that
\[
\sum_{k \in G} \eta_n(hk^{-1}k^{-1}) \frac{d\nu}{dv}(y) = E\left( \sum_{k \in G} \eta_n(hk^{-1}k^{-1}) \frac{d\mu}{d\nu}(y) \right) \quad \text{for a.e. } y \in Y. \tag{3.6}
\]
By Jensen’s inequality for conditional expectations, applied to the convex function $t \mapsto 1/t$, we also have that
\[
\frac{1}{E(\sum_{k \in G} \eta_n(hk^{-1}k^{-1}) \frac{d\mu}{d\nu}(y))} \leq E\left( \frac{1}{\sum_{k \in G} \eta_n(hk^{-1}k^{-1}) \frac{d\mu}{d\nu}(y)} \right) \quad \text{for a.e. } y \in Y. \tag{3.7}
\]
Combining (3.6) and (3.7), we see that
\[
\sum_{h \in G} \eta_n^2(h) \int_Y \frac{dv(y)}{\sum_{k \in G} \eta_n(hk^{-1}k^{-1}) \frac{d\nu}{d\nu}(y)}
\]
\[
\leq \sum_{h \in G} \eta_n^2(h) \int_Y E\left( \frac{1}{\sum_{k \in G} \eta_n(hk^{-1}k^{-1}) \frac{d\mu}{d\nu}(y)} \right) dv(y)
\]
\[
= \sum_{h \in G} \eta_n^2(h) \int_X \frac{d\mu(x)}{\sum_{k \in G} \eta_n(hk^{-1}k^{-1}) \frac{d\mu}{d\mu}(x)},
\]
which converges to 0 as $\eta_n$ is strongly recurrent for $G \curvearrowright (X, \mu)$. \hfill \Box

We say that a non-singular group action $G \curvearrowright (X, \mu)$ has an invariant mean if there exists a $G$-invariant linear functional $\varphi \in L^\infty(X)^*$. We say that $G \curvearrowright (X, \mu)$ is amenable (in the sense of Zimmer) if there exists a $G$-equivariant conditional expectation $E: L^\infty(G \times X) \to L^\infty(X)$, where the action $G \curvearrowright G \times X$ is given by $g \cdot (h, x) = (gh, g \cdot x)$.

**Proposition 3.6.** Let $G$ be a countable infinite group and let $(\mu_t)_{t \in G}$ be a family of equivalent probability measures on a standard Borel space $X_0$ that is not supported on a single atom. Let $\nu$ be a probability measure on $X_0$ and for each $t \in [0, 1]$ consider the Bernoulli action (3.2). Assume that $G \curvearrowright (X, \mu_t)$ is non-singular.

1. If $G \curvearrowright (X, \mu_t)$ has an invariant mean, then $G \curvearrowright (X, \mu_s)$ has an invariant mean for every $s < t$.
2. If $G \curvearrowright (X, \mu_t)$ is amenable, then $G \curvearrowright (X, \mu_s)$ is amenable for every $s > t$.
3. If $G \curvearrowright (X, \mu_t)$ is strongly conservative, then $G \curvearrowright (X, \mu_s)$ is strongly conservative for every $s < t$.

**Proof.** (1) We may assume that $t = 1$. So suppose that $G \curvearrowright (X, \mu_1)$ has an invariant mean and fix $s < 1$. Let $\lambda$ be the probability measure on $[0, 1]$ that is given by $\lambda(0) = s$. Then by [AIM19, Proposition A.9] the diagonal action $G \curvearrowright (X \times X \times [0, 1]^G, \mu_1 \times \mu_0 \times \lambda^G)$ has an invariant mean. Since $G \curvearrowright (X, \mu_s)$ is a factor of this diagonal action, it admits a $G$-invariant mean as well.

(2) It suffices to show that $G \curvearrowright (X, \mu_t)$ is amenable whenever there exists a $t \in (0, 1)$ such that $G \curvearrowright (X, \mu_t)$ is amenable. Write $\lambda$ for the probability measure on $[0, 1]$ given by $\lambda(0) = t$. Then $G \curvearrowright (X, \mu_t)$ is a factor of the diagonal action $G \curvearrowright (X \times X \times$
(0, 1)^G, \mu_1 \times \mu_0 \times \lambda^G), so by [Zim78, Theorem 2.4] also the latter action is amenable. Since \( G \acts (X \times \{0, 1\}^G, \mu_0 \times \lambda^G) \) is pmp, we have that \( G \acts (X, \mu_1) \) is amenable.

(3) We may again assume that \( t = 1 \). Suppose that \( (\eta_n) \) is a strongly recurrent sequence of probability measures on \( G \) for the action \( G \acts (X, \mu_1) \). Fix \( s < 1 \) and let \( \lambda \) be the probability measure on \( \{0, 1\} \) defined by \( \lambda(0) = s \). As the diagonal action \( G \acts (X \times X \times \{0, 1\}^G, \mu_1 \times \mu_0 \times \lambda^G) \) is pmp, the sequence \( \eta_n \) is also strongly recurrent for the diagonal action \( G \acts (X \times X \times \{0, 1\}^G, \mu_1 \times \mu_0 \times \lambda^G) \). Since \( G \acts (X, \mu_t) \) is a factor of \( G \acts (X \times X \times \{0, 1\}^G, \mu_1 \times \mu_0 \times \lambda^G) \), it follows from Lemma 3.5 that the sequence \( \eta_n \) is strongly recurrent for \( G \acts (X, \mu_t) \).

We finally prove Theorem 3.3. The proof relies heavily upon the techniques developed in [MV20, §5].

**Proof of Theorem 3.3.** For every \( t \in (0, 1] \) write \( \rho^t \) for the Koopman representation \( \rho^t : G \acts L^2(X, \mu_t) : (\rho^t_g(\xi))(x) = \left( \frac{dg \mu_t}{d \mu_t} (x) \right)^{1/2} \xi(g^{-1} \cdot x) \).

Fix \( s \in (0, 1) \) and let \( C > 0 \) be such that \( \log(1 - x) \geq -Cx \) for every \( x \in [0, s) \). Then for every \( t < s \) and every \( g \in G \) we have that
\[
\log(\langle \rho^t_g(1), 1 \rangle) = \sum_{h \in G} \log(1 - H^2(\mu^t_{gh}, \mu^t_h)) \\
\geq \sum_{h \in G} \log(1 - tH^2(\mu^t_{gh}, \mu^t_h)) \\
\geq -Ct \sum_{h \in G} H^2(\mu^t_{gh}, \mu^t_h).
\]

Because \( G \acts (X, \mu_1) \) is non-singular we get that
\[
\langle \rho^t_g(1), 1 \rangle \to 1 \quad \text{as } t \to 0, \quad \text{for every } g \in G. \tag{3.8}
\]

We claim that there exists a \( t' > 0 \) such that \( G \acts (X, \mu_t) \) is non-amenable for every \( t < t' \). Suppose, to the contrary, that \( t_n \) is a sequence that converges to zero such that \( G \acts (X, \mu_{t_n}) \) is amenable for every \( n \in \mathbb{N} \). Then it follows from [Nev03, Theorem 3.7] that \( \rho^{t_n} \) is weakly contained in the left regular representation \( \lambda_G \) for every \( n \in \mathbb{N} \). Write \( 1_G \) for the trivial representation of \( G \). It follows from (3.8) that \( \bigoplus_{n \in \mathbb{N}} \rho^{t_n} \) has almost invariant vectors, so that
\[
1_G \prec \bigoplus_{n \in \mathbb{N}} \rho^{t_n} \prec \infty \lambda_G < \lambda_G,
\]
which is in contradiction to the non-amenability of \( G \). By Theorem 3.1 there exists a \( t_1 \in [0, 1] \) such that \( G \acts (X, \mu_t) \) is weakly mixing for every \( t < t_1 \). Since every dissipative action is amenable (see, for example, [AIM19, Theorem A.29]) it follows that \( t_1 \geq t' > 0 \).
Write $Z_0 = [0, 1)$ and let $\lambda$ denote the Lebesgue probability measure on $Z_0$. Let $\rho^0$ denote the reduced Koopman representation
\[
\rho^0 : G \curvearrowright L^2(X \times Z_0^G, \mu_0 \times \lambda^G) \oplus C1 : \quad (\rho^0_g(\xi))(x) = \xi(g^{-1} \cdot x).
\]
As $G$ is non-amenable, $\rho^0$ has stable spectral gap. Suppose that for every $s > 0$ we can find $0 < s' < s$ such that $\rho^s$ is weakly contained in $\rho^{s'} \otimes \rho^0$. Then there exists a sequence $s_n$ that converges to zero, such that $\rho^{s_n}$ is weakly contained in $\rho^{s_n} \otimes \rho^0$ for every $n \in \mathbb{N}$. This implies that $\bigoplus_{n\in\mathbb{N}} \rho^{s_n}$ is weakly contained in $\bigoplus_{n\in\mathbb{N}} \rho^{s_n} \otimes \rho^0$. But by (3.8), the representation $\bigoplus_{n\in\mathbb{N}} \rho^{s_n}$ has almost invariant vectors, so that $\bigoplus_{n\in\mathbb{N}} \rho^{s_n}$ weakly contains the trivial representation. This is in contradiction to $\rho^0$ having stable spectral gap.

We conclude that there exists an $s > 0$ such that $\rho^s$ is not weakly contained in $\rho^0 \otimes \rho^0$ for every $t < s$.

We prove that $G \curvearrowright (X, \mu_t)$ is strongly ergodic for every $t < \min\{t', s\}$, in which case we can apply [MV20, Lemma 5.2] to the non-singular action $G \curvearrowright (X, \mu_t)$ and the pmp action $G \curvearrowright (X \times Z_0^G, \mu_0 \times \lambda^G)$ by our choice of $t'$ and $s$. After rescaling, we may assume that $G \curvearrowright (X, \mu_1)$ is ergodic and that $\rho^t$ is not weakly contained in $\rho^t \otimes \rho^0$ for every $t \in (0, 1)$.

Let $t \in (0, 1)$ be arbitrary and define the map
\[
\Psi : X \times X \times Z_0^G \rightarrow X : \quad \Psi(x, y, z)_h = \begin{cases} x_h & \text{if } z_h \leq t, \\ y_h & \text{if } z_h > t. \end{cases}
\]
Then $\Psi$ is $G$-equivariant and we have that $\Psi(\mu_1 \times \mu_0 \times \lambda^G) = \mu_t$. Suppose that $G \curvearrowright (X, \mu_t)$ is not strongly ergodic. Then we can find a bounded almost invariant sequence $f_n \in L^\infty(X, \mu_t)$ such that $\|f_n\|_2 = 1$ and $\mu_t(f_n) = 0$ for every $n \in \mathbb{N}$. Therefore, $\Psi^*_n(f_n)$ is a bounded almost invariant sequence for $G \curvearrowright (X \times X \times Z_0^G, \mu_1 \times \mu_0 \times \lambda^G)$. Let $E : L^\infty(X \times X \times Z_0^G) \rightarrow L^\infty(X)$ be the conditional expectation that is uniquely determined by $\mu_1 \circ E = \mu_1 \times \mu_0 \times \lambda^G$. By [MV20, Lemma 5.2] we have that $\lim_{n \rightarrow \infty} \|(E \circ \Psi^*_n)(f_n) - \Psi^*_n(f_n)\|_2 = 0$. As $\Psi$ is measure-preserving we get, in particular, that
\[
\lim_{n \rightarrow \infty} \|(E \circ \Psi^*_n)(f_n)\|_2 = 1. \quad (3.9)
\]
Note that if $\mu_t(f) = 0$ for some $f \in L^2(X, \mu_t)$, we have that $\mu_1((E \circ \Psi^*_n)(f)) = 0$. So we can view $E \circ \Psi^*_n$ as a bounded operator
\[
E \circ \Psi^*_n : L^2(X, \mu_t) \oplus C1 \rightarrow L^2(X, \mu_1) \oplus C1.
\]
**Claim.** The bounded operator $E \circ \Psi^*_n : L^2(X, \mu_t) \oplus C1 \rightarrow L^2(X, \mu_1) \oplus C1$ has norm strictly less than $1$.

The claim is in direct contradiction to (3.9), so we conclude that $G \curvearrowright (X, \mu_t)$ is strongly ergodic.

**Proof of claim.** For every $g \in G$, let $\varphi_g$ be the map
\[
\varphi_g : L^2(X_0, \mu'_g) \rightarrow L^2(X_0, \mu_g) : \quad \varphi_g(F) = tF + (1 - t)\nu(F) \cdot 1.
\]
Then \( E \circ \Psi_e : L^2(X_0, \mu_t) \to L^2(X, \mu_1) \) is given by the infinite product \( \bigotimes_{g \in G} \varphi_g \). For every \( g \in G \) we have that
\[
\|F\|_{2, \mu_g} = \|(d\mu_g^t/d\mu_g)^{-1/2}F\|_{2, \mu_g^t} \leq t^{-1/2}\|F\|_{2, \mu_g},
\]
so that the inclusion map \( \iota_g : L^2(X_0, \mu_t^g) \to L^2(X_0, \mu_g) \) is given by the infinite product \( \bigotimes_{g \in G} \phi_g \).

For every \( g \in G \) we have that
\[
\|\iota_g F\|_{2, \mu_t^g} = \|(d\mu_t^g/d\mu_g)^{-1/2}F\|_{2, \mu_t^g} \leq t^{-1/2}\|F\|_{2, \mu_t^g},
\]
so that the inclusion map \( \iota_g : L^2(X_0, \mu_t^g) \to L^2(X_0, \mu_g) \) satisfies \( \|\iota_g\| \leq t^{-1/2} \) for every \( g \in G \).

We have that \( \varphi_g(F) = t(F - \mu_g(F) \cdot 1) + \mu_t(F) \cdot 1 \) for every \( F \in L^2(X_0, \mu_t^g) \).

So if we write \( P_t^g \) for the projection map onto \( L^2(X_0, \mu_t^g) \odot C1 \), and \( P_g \) for the projection map onto \( L^2(X_0, \mu_g) \odot C1 \), we have that
\[
\varphi_g \circ P_t^g = t(P_g \circ \iota_g) \quad \text{for every} \quad g \in G. \tag{3.10}
\]

For a non-empty finite subset \( F \subset G \) let \( V(F) \) be the linear subspace of \( L^2(X, \mu_t) \odot C1 \) spanned by
\[
\bigotimes_{g \in F} L^2(X_0, \mu_t^g) \odot C1 \otimes \bigotimes_{g \in G \setminus F} 1.
\]

Then, using (3.10), we see that
\[
\|(E \circ \Psi_e)(f)\|_2 \leq t|F|/2 \|f\|_2 \quad \text{for every} \quad f \in V(F).
\]

Since \( \bigoplus_{F \neq \emptyset} V(F) \) is dense inside \( L^2(X, \mu_t) \odot C1 \), we have that
\[
\|(E \circ \Psi_e)|_{L^2(X, \mu_t) \odot C1}\| \leq t^{1/2} < 1. \tag*{\Box}
\]

This also concludes the proof of Theorem 3.3. \( \tag*{\Box} \)

4. Non-singular Bernoulli actions arising from groups acting on trees: proof of Theorem C

Let \( T \) be a locally finite tree and choose a root \( \rho \in T \). Let \( \mu_0 \) and \( \mu_1 \) be equivalent probability measures on a standard Borel space \( X_0 \). Following [AIM19, §10], we define a family of equivalent probability measures \( (\mu_e)^e \in E \) by
\[
\mu_e = \begin{cases} 
\mu_0 & \text{if} \ e \text{ is oriented towards} \ \rho, \\
\mu_1 & \text{if} \ e \text{ is oriented away from} \ \rho.
\end{cases} \tag{4.1}
\]

Let \( G \subset \text{Aut}(T) \) be a subgroup. When \( g \in G \) and \( e \in E \), the edges \( e \) and \( g \cdot e \) are simultaneously oriented towards, or away from, \( \rho \), unless \( e \in E([\rho, g \cdot \rho]) \). As \( E([\rho, g \cdot \rho]) \) is finite for every \( g \in G \), the generalized Bernoulli action
\[
G \curvearrowright (X, \mu) = \prod_{e \in E} (X_0, \mu_e) : \quad (g \cdot x)_e = x_{g^{-1} \cdot e} \tag{4.2}
\]
is non-singular. If we start with a different root \( \rho' \in T \), let \( (\mu'_e)^e \in E \) denote the corresponding family of probability measures on \( X_0 \). Then we have that \( \mu_e = \mu'_e \) for all but finitely many \( e \in E \), so that the measures \( \prod_{e \in E} \mu_e \) and \( \prod_{e \in E} \mu'_e \) are equivalent. Therefore, up to conjugacy, the action (4.2) is independent of the choice of root \( \rho \in T \).
Lemma 4.1. Let $T$ be a locally finite tree such that each vertex $v \in V(T)$ has degree at least 2. Suppose that $G \subset \text{Aut}(T)$ is a countable subgroup. Let $\mu_0$ and $\mu_1$ be equivalent probability measures on a standard Borel space $X_0$ and fix a root $\rho \in T$. Then the action $\alpha : G \curvearrowright (X, \mu)$ given by (4.2) is essentially free.

Proof. Take $g \in G \setminus \{e\}$. It suffices to show that $\mu(\{x \in X : g \cdot x = x\}) = 0$. If $g$ is elliptic, there exist disjoint infinite subtrees $T_1, T_2 \subset T$ such that $g \cdot T_1 = T_2$. Note that

$$(X_1, \mu_1) = \prod_{e \in E(T_1)} (X_0, \mu_e) \quad \text{and} \quad (X_2, \mu_2) = \prod_{e \in E(T_2)} (X_0, \mu_e)$$

are non-atomic and that $g$ induces a non-singular isomorphism $\varphi : (X_1, \mu_1) \to (X_2, \mu_2) : \varphi(x)_e = x_g^{-1}$. We get that

$$\mu_1 \times \mu_2(\{(x, \varphi(x)) : x \in X_1\}) = 0.$$

A fortiori $\mu(\{x \in X : g \cdot x = x\}) = 0$. If $g$ is hyperbolic, let $L_g \subset T$ denote its axis on which it acts by non-trivial translation. Then $\prod_{e \in E(L_g)} (X_0, \mu_e)$ is non-atomic and by [BKV19, Lemma 2.2] the action $g^\mathbb{Z} \rtimes \prod_{e \in E(L_g)} (X_0, \mu_e)$ is essentially free. This implies that also $\mu(\{x \in X : g \cdot x = x\}) = 0$. \qed

We prove Theorem 4.2 below, which implies Theorem C and also describes the stable type when the action is weakly mixing.

Theorem 4.2. Let $T$ be a locally finite tree with root $\rho \in T$. Let $G \subset \text{Aut}(T)$ be a closed non-elementary subgroup with Poincaré exponent $\delta = \delta(G \curvearrowright T)$ given by (1.5). Let $\mu_0$ and $\mu_1$ be non-trivial equivalent probability measures on a standard Borel space $X_0$. Consider the generalized non-singular Bernoulli action $\alpha : G \curvearrowright (X, \mu)$ given by (4.2). Then $\alpha$ is:

- weakly mixing if $1 - H^2(\mu_0, \mu_1) > \exp(-\delta/2)$;
- dissipative up to compact stabilizers if $1 - H^2(\mu_0, \mu_1) < \exp(-\delta/2)$.

Let $G \curvearrowright (Y, \nu)$ be an ergodic pmp action and let $\Lambda \subset \mathbb{R}$ be the smallest closed subgroup that contains the essential range of the map

$$X_0 \times X_0 \to \mathbb{R} : (x, x') \mapsto \log(d\mu_0/d\mu_1)(x) - \log(d\mu_0/d\mu_1)(x').$$

Let $\Delta : G \to \mathbb{R}_{>0}$ denote the modular function and let $\Sigma$ be the smallest subgroup generated by $\Lambda$ and $\log(\Delta(G))$.

Suppose that $1 - H^2(\mu_0, \mu_1) > \exp(-\delta/2)$. Then the Krieger flow and the flow of weights of $\beta : G \curvearrowright X \times Y$ are determined by $\Lambda$ and $\Sigma$ as follows.

1. If $\Lambda$ (respectively, $\Sigma$) is trivial, then the Krieger flow (respectively, flow of weights) is given by $\mathbb{R} \curvearrowright \mathbb{R}$.
2. If $\Lambda$ (respectively, $\Sigma$) is dense, then the Krieger flow (respectively, flow of weights) is trivial.
3. If $\Lambda$ (respectively, $\Sigma$) equals $a\mathbb{Z}$, with $a > 0$, then the Krieger flow (respectively, flow of weights) is given by $\mathbb{R} \curvearrowright \mathbb{R}/a\mathbb{Z}$. 

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In general, we do not know the behaviour of the action (4.2) in the critical situation $1 - H^2(\mu_0, \mu_1) = \exp(-\delta/2)$. However, if $T$ is a regular tree and $G \acts T$ has full Poincaré exponent, we prove in Proposition 4.3 below that the action is dissipative up to compact stabilizers. This is similar to [AIM19, Theorems 8.4 and 9.10].

**Proposition 4.3.** Let $T$ be a $q$-regular tree with root $\rho \in T$ and let $G \subset \text{Aut}(T)$ be a closed subgroup with Poincaré exponent $\delta = \delta(G \acts T) = \log(q - 1)$. Let $\mu_0$ and $\mu_1$ be equivalent probability measures on a standard Borel space $X_0$.

If $1 - H^2(\mu_0, \mu_1) = (q - 1)^{-1/2}$, then the action (4.2) is dissipative up to compact stabilizers.

Interesting examples of actions of the form (4.2) arise when $G \subset \text{Aut}(T)$ is the free group on a finite set of generators acting on its Cayley tree. In that case, following [AIM19, §6] and [MV20, Remark 5.3], we can also give a sufficient criterion for strong ergodicity.

**Proposition 4.4.** Let the free group $F_d$ on $d \geq 2$ generators act on its Cayley tree $T$. Let $\mu_0$ and $\mu_1$ be equivalent probability measures on a standard Borel space $X_0$.

Then the action (4.2) is dissipative if $1 - H^2(\mu_0, \mu_1) \leq (2d - 1)^{-1/2}$ and weakly mixing and non-amenable if $1 - H^2(\mu_0, \mu_1) > (2d - 1)^{-1/2}$. Furthermore, the action (4.2) is strongly ergodic when $1 - H^2(\mu_0, \mu_1) > (2d - 1)^{-1/4}$.

The proof of Theorem 4.2 below is similar to that of [LP92, Theorem 4] and [AIM19, Theorems 10.3 and 10.4].

**Proof of Theorem 4.2.** Define a family $(X_e)_{e \in E}$ of independent random variables on $(X, \mu) = \prod_{e \in E}(X_0, \mu_e)$ by

$$X_e(x) = \begin{cases} \log(d\mu_1/d\mu_0)(x_e) & \text{if } e \text{ is oriented towards } \rho, \\ \log(d\mu_0/d\mu_1)(x_e) & \text{if } e \text{ is oriented away from } \rho. \end{cases}$$

(4.3)

For $v \in T$ we write

$$S_v = \sum_{e \in E([\rho, v])} X_e.$$ 

Then we have that

$$\frac{dg\mu}{d\mu} = \exp(S_{g\rho}) \quad \text{for every } g \in G.$$

Since $G \subset \text{Aut}(T)$ is a closed subgroup, for each $v \in T$ the stabilizer subgroup $G_v = \{g \in G : g \cdot v = v\}$ is a compact open subgroup of $G$.

Suppose that $1 - H^2(\mu_0, \mu_1) < \exp(-\delta/2)$. Then we have that

$$\int_X \sum_{v \in G_{\rho}} \exp(S_v(x)/2) \, d\mu(x) = \sum_{v \in G_{\rho}} (1 - H^2(\mu_0, \mu_1))^{2d(\rho, v)} < +\infty,$$

by definition of the Poincaré exponent. Therefore, we have that $\sum_{v \in G_{\rho}} \exp(S_v(x)/2) < +\infty$ for a.e. $x \in X$. Let $\lambda$ denote the left invariant Haar measure on $G$ and define $L = \lambda(G_{\rho})$, where $G_{\rho} = \{g \in G : g \cdot \rho = \rho\}$. Then we have that
\[
\int_G \frac{d\mu(x)}{d\mu}(g) \, d\lambda(g) = L \sum_{v \in G-p} \exp(S_v(x)) < +\infty \quad \text{for a.e. } x \in X.
\]

We conclude that \( G \cong (X, \mu) \) is dissipative up to compact stabilizers.

Now assume that \( 1 - H^2(\mu_0, \mu_1) > \exp(-\delta/2) \). We start by proving that \( G \cong (X, \mu) \) is infinitely recurrent. By [AIM19, Theorem 8.17] we can find a non-elementary closed compactly generated subgroup \( G' \subset G \) such that \( 1 - H^2(\mu_0, \mu_1) > \exp(-\delta(G')/2) \). Let \( T' \subset T \) be the unique minimal \( G' \)-invariant subtree. Then \( G' \) acts cocompactly on \( T' \) and we have that \( \delta(G') = \dim_H \partial T' \). Let \( X \) and \( Y \) be independent random variables with distributions \( (\log d\mu_1/d\mu_0)_\ast \mu_0 \) and \( (\log d\mu_0/d\mu_1)_\ast \mu_1 \), respectively. Set \( Z = X + Y \) and write
\[
\phi(t) = \mathbb{E}(\exp(tZ)).
\]

The assignment \( t \mapsto \phi(t) \) is convex, \( \phi(t) = \phi(1-t) \) for every \( t \) and \( \phi(1/2) = (1 - H^2(\mu_0, \mu_1))^2 \). We conclude that
\[
\inf_{t \geq 0} \phi(t) = (1 - H^2(\mu_0, \mu_1))^2.
\]

Write \( R_k \) for the sum of \( k \) independent copies of \( Z \). By the Chernoff–Cramér theorem, as stated in [LP92], there exists an \( M \in \mathbb{N} \) such that
\[
\mathbb{P}(R_M \geq 0) > \exp(-M\delta(G')). \tag{4.4}
\]

Below we define a new unoriented tree \( S \). This means that the edge set of \( S \) consists of subsets \( \{v, w\} \subset V(S) \). Fix a vertex \( \rho' \in T' \) and define the unoriented tree \( S \) as follows.

- \( S \) has vertices \( v \in T' \) so that \( d_{T'}(\rho', v) \) is divisible by \( M \).
- There is an edge \( \{v, w\} \in E(S) \) between two vertices \( v, w \in S \) if \( d_{T'}(v, w) = M \) and \( [\rho', v]_{T'} \subset [\rho', w]_{T'} \).

Here the notation \( [\rho', v]_{T'} \) means that we consider the line segment \( [\rho', v] \) as a subgraph of \( T' \). We have that \( \dim_H \partial S = M \dim_H \partial T' = M\delta(G') \). Form a random subgraph \( S(x) \) of \( S \) by deleting those edges \( \{v, w\} \in E(S) \) where
\[
\sum_{e \in E([v, w]_{T'})} X_e(x_e) < 0.
\]

This is an edge percolation on \( S \), where each edge remains with probability \( p = \mathbb{P}(R_M \geq 0) \). So by (4.4) we have that \( p \exp(\dim_H S) > 1 \). Furthermore, if \( \{v, w\} \) and \( \{v', w'\} \) are edges of \( S \) so that \( E([v, w]_{T'}) \cap E([v', w']_{T'}) = \emptyset \), their presence in \( S(x) \) constitutes independent events. So the percolation process is a quasi-Bernoulli percolation as introduced in [Lyo89]. Taking \( w \in (1, p \exp(\dim_H S)) \) and setting \( w_n = w^{-n} \), it follows from [Lyo89, Theorem 3.1] that percolation occurs almost surely, that is, \( S(x) \) contains an infinite connected component for a.e. \( x \in X \). Writing
\[
S'(v)(x) = \sum_{e \in E([v', v]_{T'})} X_e(x_e),
\]
this means that for a.e. $x \in (X, \mu)$ we can find a constant $a_x > -\infty$ such that $S'_v(x) > a_x$ for infinitely many $v \in T'$. As $T'/G'$ is finite, there exists a vertex $w \in T'$ such that

$$\sum_{v \in G' \cdot w} \exp(S'_v(x)) = +\infty$$

with positive probability. \hfill (4.5)

Therefore, by Kolmogorov’s zero–one law, we have that $\sum_{v \in G' \cdot w} \exp(S'_v(x)) = +\infty$ almost surely. Since a change of root results in a conjugate action, we may assume that $\rho = w$. Then (4.5) implies that $\sum_{v \in G \cdot \rho} \exp(S_v(x)) = +\infty$ for a.e. $x \in X$. Writing again $L$ for the Haar measure of the stabilizer subgroup $G_\rho = \{ g \in G : g \cdot \rho = \rho \}$, we see that

$$\int_G \frac{dg\mu}{d\mu} \, d\lambda(g) = L \sum_{v \in G \cdot \rho} \exp(S_v) = +\infty$$

almost surely. We conclude that $G \curvearrowright (X, \mu)$ is infinitely recurrent. We prove that $G \curvearrowright (X, \mu)$ is weakly mixing using a phase transition result from the previous section. Define the measurable map

$$\psi : X_0 \to (0, 1] : \psi(x) = \min\{d\mu_1/d\mu_0(x), 1\}.$$

Let $\nu$ be the probability measure on $X_0$ determined by

$$\frac{d\nu}{d\mu_0}(x) = \rho^{-1} \psi(x) \quad \text{where} \quad \rho = \int_{X_0} \psi(x) \, d\mu_0(x).$$

Then we have that $\nu \sim \mu_0$ and for every $s > 1 - \rho$ the probability measures

$$\eta_0^s = s^{-1}(\mu_0 - (1 - s)\nu), \quad \eta_1^s = s^{-1}(\mu_1 - (1 - s)\nu)$$

are well defined. We consider the non-singular actions $G \curvearrowright (X, \eta_0^s) = \prod_{e \in E(X_0, \eta_0^s)}$, where

$$\eta_0^s = \begin{cases} \eta_0^s & \text{if } e \text{ is oriented towards } \rho, \\ \eta_1^s & \text{if } e \text{ is oriented away from } \rho. \end{cases}$$

By the dominated convergence theorem we have that $H^2(\eta_0^s, \eta_1^s) \to H^2(\mu_0, \mu_1)$ as $s \to 1$. So we can choose $s$ close enough to 1, but not equal to 1, such that $1 - H^2(\eta_0^s, \eta_1^s) > \exp(-\delta/2)$. By the first part of the proof we have that $G \curvearrowright (X, \eta_0^s)$ is infinitely recurrent. Note that

$$\mu_j = (1 - s)\nu + s\eta_j^s \quad \text{for } j = 0, 1.$$

Since we assumed that $G \subseteq \text{Aut}(T)$ is closed, all the stabilizer subgroups $G_v = \{ g \in G : g \cdot v = v \}$ are compact. By Remark 3.4 we conclude that $G \curvearrowright (X, \mu)$ is weakly mixing.

Let $G \curvearrowright (Y, v)$ be an ergodic pmp action. To determine the Krieger flow and the flow of weights of $\beta : G \curvearrowright X \times Y$ we use a similar approach to [AIM19, Theorem 10.4] and [VW17, Proposition 7.3]. First we determine the Krieger flow and then we deal with the flow of weights.
As before, let $G' \subset G$ be a non-elementary compactly generated subgroup such that $1 - H^2(\mu_0, \mu_1) > \exp(-\delta(G')/2)$. By [AIM19, Theorem 8.7] we may assume that $G/G'$ is not compact. Let $T' \subset T$ be the minimal $G'$-invariant subtree. Let $v \in T'$ be as in Lemma 4.5 below so that

$$\bigcap_{g \in G} \left(E(gT') \cup E([v, g^{-1} \cdot v])\right) = \emptyset. \quad (4.6)$$

Since changing the root yields a conjugate action, we may assume that $\rho = v$. Let $(Z_0, \zeta_0)$ be a standard probability space such that there exist measurable maps $\theta_0, \theta_1 : Z_0 \to X_0$ that satisfy $(\theta_0)_* \zeta_0 = \mu_0$ and $(\theta_1)_* \zeta_0 = \mu_1$. Write

$$(Z, \zeta) = \prod_{e \in E(T) \setminus E(T')} (Z_0, \zeta_0),$$

$$(X_1, \rho_1) = \prod_{e \in E(T) \setminus E(T')} (X_0, \mu_e),$$

$$(X_2, \rho_2) = \prod_{e \in E(T')} (X_0, \mu_e).$$

By the first part of the proof we have that $G' \curvearrowright (X_2, \rho_2)$ is infinitely recurrent. Define the pmp map

$$\Psi : (Z, \zeta) \to (X_1, \rho_1) : \quad (\Psi(z))_e = \begin{cases} \theta_0(ze) & \text{if } e \text{ is oriented towards } \rho, \\ \theta_1(ze) & \text{if } e \text{ is oriented away from } \rho. \end{cases}$$

Consider

$$U = \{e \in E(T) : e \text{ is oriented towards } \rho\}.$$

Since $gU \Delta U = E(T)([\rho, g \cdot \rho]) \subset E(T')$ for any $g \in G'$, the set $(E(T) \setminus E(T')) \cap U$ is $G'$-invariant. Therefore, $\Psi$ is a $G'$-equivariant factor map. Consider the Maharam extensions

$$G' \curvearrowright Z \times X_2 \times Y \times \mathbb{R} \quad \text{and} \quad G \curvearrowright X \times Y \times \mathbb{R}$$

of the diagonal actions $G' \curvearrowright Z \times X_2 \times Y$ and $G' \curvearrowright X \times Y \times \mathbb{R}$, respectively. Identifying $(X, \mu) = (X_1, \rho_1) \times (X_2, \rho_2)$, we obtain a $G'$-equivariant factor map

$$\Phi : Z \times X_2 \times Y \times \mathbb{R} \to X_1 \times X_2 \times Y \times \mathbb{R} : \quad \Phi(z, x, y, t) = (\Psi(z), x, y, t).$$

Take $F \in L^\infty(X \times Y \times \mathbb{R})^G$. By [AIM19, Proposition A.33] the Maharam extension $G' \curvearrowright Z_2 \times Y \times \mathbb{R}$ is infinitely recurrent. Since $G' \curvearrowright Z$ is a mixing pmp generalized Bernoulli action we have that $F \circ \Phi \in L^\infty(Z \times X_2 \times Y \times \mathbb{R})^G \subset 1 \overline{\otimes} L^\infty(X_2 \times Y \times \mathbb{R})^G$ by [SW81, Theorem 2.3]. Therefore, $F$ is essentially independent of the $E(T) \setminus E(T')$-coordinates. Thus, for any $g \in G$ the assignment

$$(x, y, t) \mapsto F(g \cdot x, y, t) = F(x, y, t - \log(dg^{-1}\mu/d\mu)(x))$$

is essentially independent of the $E(T) \setminus E(gT')$-coordinates. Since $\log(dg^{-1}\mu/d\mu)$ only depends on the $E([\rho, g^{-1} \cdot \rho])$-coordinates, we deduce that $F$ is essentially independent of
the $E(T) \setminus (E(gT') \cup E([\rho, g^{-1} \cdot \rho]))$-coordinates, for every $g \in G$. Therefore, by (4.6), we have that $F \in 1 \mathcal{L} L^\infty(Y \times \mathbb{R})$.

So we have proven that any $G$-invariant function $F \in L^\infty(X \times Y \times \mathbb{R})$ is of the form $F(x, y, t) = H(y, t)$, for some $H \in L^\infty(Y \times \mathbb{R})$ that satisfies

$$H(y, t) = H(g \cdot y, t + \log(dg^{-1}_/\mu)/\mu)(x)) \quad \text{for a.e. } (x, y, t) \in X \times Y \times \mathbb{R}.$$ 

Since $0$ is in the essential range of the maps $\log(dg_\mu/\mu)$, for every $g \in G$, we see that $H(g \cdot y, t) = H(y, t)$ for a.e. $(y, t) \in Y \times \mathbb{R}$. By ergodicity of $G \curvearrowright Y$, we conclude that $H$ is of the form $H(y, t) = P(t)$, for some $P \in L^\infty(\mathbb{R})$ that satisfies

$$P(t) = P(t + \log(dg^{-1}_/\mu)/\mu)(x)) \quad \text{for a.e. } (x, t) \in X \times \mathbb{R}, \text{ for every } g \in G. \quad (4.7)$$

Let $\Gamma \subset \mathbb{R}$ be the subgroup generated by the essential ranges of the maps $\log(dg_\mu/\mu)$, for $g \in G$. If $\Gamma = \{0\}$ we can identify $L^\infty(X \times Y \times \mathbb{R})^G \cong L^\infty(\mathbb{R})$. If $\Gamma \subset \mathbb{R}$ is dense, then it follows that $P$ is essentially constant so that the Maharam extension $G \curvearrowright X \times Y \times \mathbb{R}$ is ergodic, that is, the Krieger flow of $G \curvearrowright X \times Y$ is trivial. If $\Gamma = a\mathbb{Z}$, with $a > 0$, we conclude by (4.7) that we can identify $L^\infty(X \times Y \times \mathbb{R})^G \cong L^\infty(\mathbb{R}/a\mathbb{Z})$, so that the Krieger flow of $G \curvearrowright X \times Y$ is given by $\mathbb{R}/a\mathbb{Z}$. Finally, note that the closure of $\Gamma$ equals the closure of the subgroup generated by the essential range of the map

$$X_0 \times X_0 \to \mathbb{R}: \quad (x, x') \mapsto \log(d\mu_0/d\mu_1)(x) - \log(d\mu_0/d\mu_1)(x').$$

So we have calculated the Krieger flow in every case, concluding the proof of the theorem in the case where $G$ is unimodular.

When $G$ is not unimodular, let $G_0 = \ker \Delta$ be the kernel of the modular function. Let $G \curvearrowright X \times Y \times \mathbb{R}$ be the modular Maharam extension and let $\alpha: G_0 \curvearrowright X \times Y \times \mathbb{R}$ be its restriction to the subgroup $G_0$. Then we have that

$$L^\infty(X \times Y \times \mathbb{R})^G \subset L^\infty(X \times Y \times \mathbb{R})^\alpha.$$ 

By [AIM19, Theorem 8.16] we have that $\delta(G_0) = \delta$, and we can apply the argument above to conclude that $L^\infty(X \times Y \times \mathbb{R})^\alpha \subset 1 \mathcal{L} 1 \mathcal{L} L^\infty(\mathbb{R})$. So for every $F \in L^\infty(X \times Y \times \mathbb{R})^G$ there exists a $P \in L^\infty(\mathbb{R})$ such that

$$P(t) = P(t + \log(dg^{-1}_/\mu)/\mu)(x) + \log(\Delta(g))) \quad \text{for a.e. } (x, t) \in X \times \mathbb{R}, \text{ for every } g \in G. \quad (4.8)$$

Let $\Pi$ be the subgroup of $\mathbb{R}$ generated by the essential range of the maps

$$x \mapsto \log(dg^{-1}_/\mu)/\mu)(x) + \log(\Delta(g)) \quad \text{with } g \in G.$$ 

As $0$ is contained in the essential range of $\log(dg^{-1}_/\mu)/\mu)$, for every $g \in G$, we get that $\log(\Delta(G)) \subset \Pi$. Therefore, $\Pi$ also contains the subgroup $\Gamma \subset \mathbb{R}$ defined above. Thus, the closure of $\Pi$ equals the closure of $\Sigma$, where $\Sigma \subset \mathbb{R}$ is the subgroup as in the statement of the theorem. From (4.8) we conclude that we may identify $L^\infty(X \times Y \times \mathbb{R})^G \cong L^\infty(\mathbb{R})^\Sigma$, so that the flow of weights of $G \curvearrowright X \times Y$ is as stated in the theorem. \qed
LEMMA 4.5. Let $T$ be a locally finite tree and let $G \subset \text{Aut}(T)$ be a closed subgroup. Suppose that $H \subset G$ is a closed compactly generated subgroup that contains a hyperbolic element and assume that $G/H$ is not compact. Let $S \subset T$ be the unique minimal $H$-invariant subtree. Then there exists a vertex $v \in S$ such that

$$\bigcap_{g \in G} (gS \cup [v, g^{-1} \cdot v]) = \{v\}.$$  \hfill (4.9)

Proof. Let $k \in H$ be a hyperbolic element and let $L \subset T$ be its axis, on which $k$ acts by a non-trivial translation. Then $L \subset S$, as one can show for instance as in the proof of [CM11, Proposition 3.8]. Pick any vertex $v \in L$. We claim that this vertex will satisfy (4.9). Take any $w \in V(T) \setminus \{v\}$. As $G/H$ is not compact, one can show as in [AIM19, Theorem 9.7] that there exists a $g \in G$ such that $g \cdot w \not\in S$. Since $k$ acts by translation on $L$, there exists an $n \in \mathbb{N}$ large enough such that

$$[v, k \cdot v] \subset [v, k^n g \cdot v] \quad \text{and} \quad [v, k^{-1} \cdot v] \subset [v, k^{-n} g \cdot v],$$

so that in particular we have that $w \not\in [v, k^n g \cdot v] \cap [v, k^{-n} g \cdot v] = \{v\}$. Since $S$ is $H$-invariant, we also have that $k^n g \cdot w \not\in S$ and $k^{-n} g \cdot w \not\in S$ and we conclude that

$$w \not\in ((k^n g)^{-1}S \cup [v, k^n g \cdot v]) \cap ((k^{-n} g)^{-1}S \cup [v, k^{-n} g \cdot v]).$$

Proof of Proposition 4.3. Define the family $(X_e)_{e \in E}$ of independent random variables on $(X, \mu)$ by (4.3) and write

$$S_v = \sum_{e \in E(v, \rho)} X_e.$$

Claim. There exists a $\delta > 0$ such that

$$\mu(\{x \in X : S_v(x) \leq -\delta \text{ for every } v \in T \setminus \{\rho\}\}) > 0.$$

Proof of claim. Note that $\mathbb{E}(\exp(X_e/2)) = 1 - H^2(\mu_0, \mu_1)$ for every $e \in E$. Define a family of random variables $(W_n)_{n \geq 0}$ on $(X, \mu)$ by

$$W_n = \sum_{v \in T, d(v, \rho) = n} \exp(S_v/2).$$

Using that $1 - H^2(\mu_0, \mu_1) = (q - 1)^{-1/2}$, one computes that

$$\mathbb{E}(W_{n+1} \mid S_v, d(v, \rho) \leq n) = W_n \quad \text{for every } n \geq 1.$$

So the sequence $(W_n)_{n \geq 0}$ is a martingale, and since it is positive it converges almost surely to a finite limit when $n \to +\infty$. Write $\Sigma_n = \{v \in T : d(v, \rho) = n\}$. As $W_n \geq \max_{v \in \Sigma_n} \exp(S_v/2)$ we conclude that there exists a positive constant $C < +\infty$ such that

$$\mathbb{P}(S_v \leq C \text{ for every } v \in T) > 0.$$

For any vertex $w \in T$, write $T_w = \{v \in T : [\rho, w] \subset [\rho, v]\}$: the set of children of $w$, including $w$ itself. Using the symmetry of the tree and changing the root from $\rho$ to $w \in T$, we also have that
\[ \mathbb{P}(S_v - S_w \leq C \text{ for every } v \in T_w) > 0 \text{ for every } w \in T. \quad (4.10) \]

Set \( v_0 = (\log d\mu_1/d\mu_0)_*\mu_0 \) and \( v_1 = (\log d\mu_0/d\mu_1)_*\mu_1 \). Because \( 1 - H^2(\mu_0, \mu_1) \neq 0 \) we have that \( \mu_0 \neq \mu_1 \), so that there exists a \( \delta > 0 \) such that
\[ v_0 * v_1((-\infty, -\delta)) > 0. \]

Here \( v_0 * v_1 \) denotes the convolution product of \( v_0 \) with \( v_1 \). Therefore, there exists \( N \in \mathbb{N} \) large enough such that
\[ \mathbb{P}(S_w \leq -C - \delta \text{ for every } w \in \Sigma_N \text{ and } S_{w'} \leq -\delta \text{ for every } w' \in \Sigma_n \text{ with } n \leq N) > 0. \quad (4.11) \]

Since for any \( w \in \Sigma_N \) and \( w' \in \Sigma_n \) with \( n \leq N \), we have that \( S_v - S_w \) is independent of \( S_{w'} \) for every \( v \in T_w \), and since \( \Sigma_N \) is a finite set, it follows from (4.10) and (4.11) that
\[ \mathbb{P}(S_v \leq -\delta \text{ for every } v \in T \setminus \{\rho\}) > 0. \]

This concludes the proof of the claim. \( \square \)

Let \( \delta > 0 \) be as in the claim and define
\[ \mathcal{U} = \{ x \in X : S_v(x) \leq -\delta \text{ for every } v \in T \setminus \{\rho\} \}, \]
so that \( \mu(\mathcal{U}) > 0 \). Let \( G_\rho \) be the stabilizer subgroup of \( \rho \). Note that for every \( g, h \in G \) we have that \( S_{hg,\rho}(x) = S_{g,\rho}(h^{-1} \cdot x) + S_{h,\rho}(x) \) for a.e. \( x \in X \), so that for \( h \in G \) we have that
\[ h \cdot \mathcal{U} \subset \{ x \in X : S_{h,\rho}(x) \leq -\delta + S_{h,\rho}(x) \text{ for every } g \notin G_\rho \}. \]

It follows that if \( h \notin G_\rho \), we have that
\[ \mathcal{U} \cap h \cdot \mathcal{U} \subset \{ x \in X : S_{h,\rho}(x) \leq -\delta \text{ and } S_{h,\rho}(x) \geq \delta \} = \emptyset. \]

Since \( G \subset \text{Aut}(T) \) is closed, we have that \( G_\rho \) is compact. So the action \( G \curvearrowright (X, \mu) \) is not infinitely recurrent. Let \( \lambda \) denote the left invariant Haar measure on \( G \). By an adaptation of the proof of [BV20, Proposition 4.3], the set
\[ D = \left\{ x \in X : \int_G \frac{dg\mu}{d\mu}(x) \, d\lambda(g) < +\infty \right\} = \left\{ x \in X : \int_G \exp(S_{g,\rho}(x)) \, d\lambda(g) < +\infty \right\} \]
satisfies \( \mu(D) \in (0, 1) \). Since \( G \curvearrowright (X, \mu) \) is not infinitely recurrent, it follows from [AIM19, Proposition A.28] that \( \mu(D) > 0 \), so that we must have that \( \mu(D) = 1 \). By [AIM19, Theorem A.29] the action \( G \curvearrowright (X, \mu) \) is dissipative up to compact stabilizers. \( \square \)

We use a similar approach to [MV20, §6] in the proof of Proposition 4.4.

**Proof of Proposition 4.4.** It follows from Theorem 4.2 and Proposition 4.3 that the action \( G \curvearrowright (X, \mu) \), given by (4.2), is dissipative when \( 1 - H^2(\mu_0, \mu_1) \leq (2d - 1)^{-1/2} \) and weakly mixing when \( 1 - H^2(\mu_0, \mu_1) > (2d - 1)^{-1/2} \). So it remains to show that \( G \curvearrowright (X, \mu) \) is non-amenable when \( 1 - H^2(\mu_0, \mu_1) > (2d - 1)^{-1/2} \) and strongly ergodic when \( 1 - H^2(\mu_0, \mu_1) > (2d - 1)^{-1/4} \).
Phase transitions for non-singular Bernoulli actions

Assume first that $1 - H^2(\mu_0, \mu_1) > (2d - 1)^{-1/2}$. By taking the kernel of a surjective homomorphism $\mathbb{F}_d \to \mathbb{Z}$ we find a normal subgroup $H_1 \subset \mathbb{F}_d$ that is free on infinitely many generators. By [RT13, Théorème 0.1] we have that $\delta(H_1) = (2d - 1)^{-1/2}$. Then, using [Sul79, Corollary 6], we can find a finitely generated free subgroup $H_2 \subset H_1$ such that $H_1 = H_2 * H_3$ for some free subgroup $H_3 \subset H_1$ and such that $1 - H^2(\mu_0, \mu_1) > \exp(-\delta(H_2)/2)$. Let $\psi: H_1 \to H_3$ be the surjective group homomorphism uniquely determined by

$$
\psi(h) = \begin{cases} e & \text{if } h \in H_2, \\
h & \text{if } h \in H_3.
\end{cases}
$$

We set $N = \ker \psi$, so that $H_2 \subset N$ and we get that $1 - H^2(\mu_0, \mu_1) > \exp(-\delta(N)/2)$. Therefore, $N \acts (X, \mu)$ is ergodic by Theorem 4.2. Also we have that $H_1/N \cong H_3$, which is a free group on infinitely many generators. Therefore, $H_1 \acts (X, \mu)$ is non-amenable by [MV20, Lemma 6.4]. A posteriori also $\mathbb{F}_d \acts (X, \mu)$ is non-amenable.

Let $\pi$ be the Koopman representation of the action $\mathbb{F}_d \acts (X, \mu)$:

$$
\pi: G \acts L^2(X, \mu): \quad (\pi_g(\xi))(x) = \left(\frac{d \mu}{d \mu}(x)\right)^{1/2} \xi(g^{-1} \cdot x).
$$

Claim. If $1 - H^2(\mu_0, \mu_1) > (2d - 1)^{-1/4}$, then $\pi$ is not weakly contained in the left regular representation.

Proof of claim. Let $\eta$ denote the canonical symmetric measure on the generator set of $\mathbb{F}_d$ and define

$$
P = \sum_{g \in \mathbb{F}_d} \eta(g)\pi_g.
$$

The $\eta$-spectral radius of $\alpha: \mathbb{F}_d \acts (X, \mu)$, which we denote by $\rho_\eta(\alpha)$, is by definition the norm of $P$, as a bounded operator on $L^2(X, \mu)$. By [AIM19, Proposition A.11] we have that

$$
\rho_\eta(\alpha) = \lim_{n \to \infty} \langle P^n(1), 1 \rangle^{1/n}
= \lim_{n \to \infty} \left( \sum_{g \in \mathbb{F}_d} \eta^n(g)(1 - H^2(\mu_0, \mu_1))^{2|g|} \right)^{1/n},
$$

where $|g|$ denotes the word length of a group element $g \in \mathbb{F}_d$. By [AIM19, Theorem 6.10] we then have that

$$
\rho_\eta(\alpha) = \frac{(1 - H^2(\mu_0, \mu_1))^2}{2d} \left( (2d - 1) + (1 - H^2(\mu_0, \mu_1))^{-4} \right)
$$

if $1 - H^2(\mu_0, \mu_1) > (2d - 1)^{-1/4}$, and

$$
\rho_\eta(\alpha) = \frac{\sqrt{2d - 1}}{d}
$$

if $1 - H^2(\mu_0, \mu_1) \leq (2d - 1)^{-1/4}$. Therefore, if $1 - H^2(\mu_0, \mu_1) > (2d - 1)^{-1/4}$, we have that $\rho_\eta(\alpha) > \rho_\eta(\mathbb{F}_d)$, where $\rho_\eta(\mathbb{F}_d)$ denotes the $\eta$-spectral radius of the left regular
representation. This implies that $\alpha$ is not weakly contained in the left regular representation (see, for instance, \cite{AD03}, §3.2).

Now assume that $1 - H^2(\mu_0, \mu_1) > (2d - 1)^{-1/4}$. As in the proof of Theorem 4.2 there exist probability measures $\nu$, $\eta_0$ and $\eta_1$ on $X_0$ that are equivalent to $\mu_0$ and a number $s \in (0, 1)$ such that

$$\mu_j = (1 - s)\nu + s\eta_j \quad \text{for } j = 0, 1,$$

and such that $1 - H^2(\eta_0, \eta_1) > (2d - 1)^{-1/4}$. Consider the non-singular action

$$\mathbb{F}_d \curvearrowright (X, \eta) = \prod_{e \in E(T)} (X_0, \eta_e) \quad \text{where } \eta_e = \begin{cases} \eta_0 & \text{if } e \text{ is oriented towards } \rho, \\ \eta_1 & \text{if } e \text{ is oriented away from } \rho. \end{cases}$$

By Theorem 4.2 the action $\mathbb{F}_d \curvearrowright (X, \eta)$ is ergodic. Write $\rho$ for the Koopman representation associated to $\mathbb{F}_d \curvearrowright (X, \eta)$. By the claim, $\rho$ is not weakly contained in the left regular representation. Let $\lambda$ be the probability measure on $\{0, 1\}$ given by $\lambda(0) = s$. Let $\rho^0$ be the reduced Koopman representation of the pmp generalized Bernoulli action $\mathbb{F}_d \curvearrowright (X \times \{0, 1\}^{E(T)}, \nu^{E(T)} \times \lambda^{E(T)})$. Then $\rho^0$ is contained in a multiple of the left regular representation. Therefore, as $\rho$ is not weakly contained in the left regular representation, $\rho$ is not weakly contained in $\rho \otimes \rho^0$.

Define the map

$$\Psi: X \times X \times \{0, 1\}^{E(T)} \to X: \Psi(x, y, z)_e = \begin{cases} x_e & \text{if } z_e = 0, \\ y_e & \text{if } z_e = 1. \end{cases}$$

Then $\Psi$ is $\mathbb{F}_d$-equivariant and we have that $\Psi_\ast(\eta \times \nu^{E(T)} \times \lambda^{E(T)}) = \mu$. Suppose that $\mathbb{F}_d \curvearrowright (X, \mu)$ is not strongly ergodic. Then there exists a bounded almost invariant sequence $f_n \in L^\infty(X, \mu)$ such that $\|f_n\|_2 = 1$ and $\mu(f_n) = 0$ for every $n \in \mathbb{N}$. Therefore, $\Psi_\ast(f_n)$ is a bounded almost invariant sequence for the diagonal action $\mathbb{F}_d \curvearrowright (X \times X \times \{0, 1\}^{E(T)}, \eta \times \nu^{E(T)} \times \lambda^{E(T)})$. Let $E: L^\infty(X \times X \times \{0, 1\}^{E(T)}) \to L^\infty(X)$ be the conditional expectation that is uniquely determined by $\mu \circ E = \eta \times \nu^{E(T)} \times \lambda^{E(T)}$. By \cite{MV20}, Lemma 5.2 we have that $\lim_{n \to \infty} \| (E \circ \Psi_\ast)(f_n) - \Psi_\ast(f_n) \|_2 = 0$, and in particular we get that

$$\lim_{n \to \infty} \| (E \circ \Psi_\ast)(f_n) \|_2 = 1. \quad (4.12)$$

But just as in the proof of Theorem 3.3 we have that

$$\left\| (E \circ \Psi_\ast)\big|_{L^2(X, \mu) \ominus C^1} \right\| < 1,$$

which is in contradiction with (4.12). We conclude that $\mathbb{F}_d \curvearrowright (X, \mu)$ is strongly ergodic.

Proposition 4.6 below complements Theorem 4.2 by considering groups $G \subset \text{Aut}(T)$ that are not closed. This is similar to \cite{AIM19}, Theorem 10.5.

**Proposition 4.6.** Let $T$ be a locally finite tree with root $\rho \in T$. Let $G \subset \text{Aut}(T)$ be an lcsc group such that the inclusion map $G \to \text{Aut}(T)$ is continuous and such that
Phase transitions for non-singular Bernoulli actions

$G \subset \text{Aut}(T)$ is not closed. Write $\delta = \delta(G \acts T)$ for the Poincaré exponent given by (1.5). Let $\mu_0$ and $\mu_1$ be non-trivial equivalent probability measures on a standard Borel space $X_0$. Consider the generalized non-singular Bernoulli action $\alpha: G \acts (X, \mu)$ given by (4.2). Let $H \subset \text{Aut}(T)$ be the closure of $G$. Then the following assertions hold.

- If $1 - H^2(\mu_0, \mu_1) > \exp(-\delta/2)$, then $\alpha$ is ergodic and its Krieger flow is determined by the essential range of the map

  $$X_0 \times X_0 \rightarrow \mathbb{R}: (x, x') \mapsto \log(d\mu_0/d\mu_1)(x) - \log(d\mu_0/d\mu_1)(x') \quad (4.13)$$

  as in Theorem 4.2.

- If $1 - H^2(\mu_0, \mu_1) < \exp(-\delta/2)$, then each ergodic component of $\alpha$ is of the form $G \acts H/K$, where $K$ is a compact subgroup of $H$. In particular, there exists a $G$-invariant $\sigma$-finite measure on $X$ that is equivalent to $\mu$.

**Proof.** Let $H \subset \text{Aut}(T)$ be the closure of $G$. Then $\delta(H) = \delta$ and we can apply Theorem 4.2 to the non-singular action $H \acts (X, \mu)$.

If $1 - H^2(\mu_0, \mu_1) > \exp(-\delta/2)$, then $H \acts X$ is ergodic. As $G \subset H$ is dense, we have that

$$L^\infty(X)^G = L^\infty(X)^H = \mathbb{C}1,$$

so that $G \acts X$ is ergodic. Let $H \acts X \times \mathbb{R}$ be the Maharam extension associated to $H \acts X$. Again, as $G \subset H$ is dense, we have that

$$L^\infty(X \times \mathbb{R})^G = L^\infty(X \times \mathbb{R})^H.$$

Note that the subgroup generated by the essential ranges of the maps $\log(dg^{-1}\mu/d\mu)$, with $g \in G$, is the same as the subgroup generated by the essential ranges of the maps $\log(dh^{-1}\mu/d\mu)$, with $h \in H$. Then one determines the Krieger flow of $G \acts X$ as in the proof of Theorem 4.2.

If $1 - H^2(\mu_0, \mu_1) < \exp(-\delta/2)$, the action $H \acts (X, \mu)$ is dissipative up to compact stabilizers. By [AIM19, Theorem A.29] each ergodic component is of the form $H \acts H/K$ for a compact subgroup $K \subset H$. Therefore, each ergodic component of $G \acts (X, \mu)$ is of the form $G \acts H/K$, for some compact subgroup $K \subset H$.

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