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Dynamical large deviations for homogeneous systems with long range interactions and the Balescu–Guernsey–Lenard equation

Ouassim Feliachi and Freddy Bouchet

Abstract We establish a large deviation principle for time dependent trajectories (paths) of the empirical density of $N$ particles with long range interactions, for homogeneous systems. This result extends the classical kinetic theory that leads to the Balescu–Guernsey–Lenard kinetic equation, by the explicit computation of the probability of typical and large fluctuations. The large deviation principle for the paths of the empirical density is obtained through explicit computations of a large deviation Hamiltonian. This Hamiltonian encodes all the cumulants for the fluctuations of the empirical density, after time averaging of the fast fluctuations. It satisfies a time reversal symmetry, related to the detailed balance for the stochastic process of the empirical density. This explains in a very simple way the increase of the macrostate entropy for the most probable states, while the stochastic process is time reversible, and describes the complete stochastic process at the level of large deviations.

Keywords Plasma · Balescu–Guernsey–Lenard equation · Large deviation theory · Macroscopic fluctuation theory · Widom theorem

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1 Introduction

We consider the Hamiltonian dynamics of particles that interact through a mean-field potential. The dynamics reads

\[
\begin{align*}
\frac{dr_n}{dt} &= v_n \\
\frac{dv_n}{dt} &= -\frac{1}{N} \sum_{m \neq n} \frac{d}{dr_n} W(r_n - r_m)
\end{align*}
\]

where \(\{r_n\}_{1 \leq n \leq N}\) are the positions and \(\{v_n\}_{1 \leq n \leq N}\) the velocities. This set-up is relevant for plasmas in the weak coupling regime \([41]\), self-gravitating systems \([42, 18]\), and many particle systems with long range interactions \([10]\). It also shares many theoretical analogies with two-dimensional and geostrophic turbulence, through the point vortex model \([17, 28]\), or stochastically forced models of two-dimensional and geostrophic turbulence \([11]\). The kinetic theory of systems with mean-field potentials (or long range interactions) is a classical piece of theoretical physics. The relaxation to equilibrium of the empirical density

\[
g_N(r, v, t) = \frac{1}{N} \sum_{n=1}^{N} \delta(r - r_n(t)) \delta(v - v_n(t)),
\]

is described by the Balescu–Guernsey–Lenard kinetic equation in the limit of a large number of particles. This result has been formally derived by Balescu, Guernsey and Lenard \([1, 2]\). In the context of plasma physics where we consider \(N\) charged particles submitted to Coulomb interactions, we refer to Nicholson \([41]\) for a derivation using the BBGKY hierarchy, or to Lifshitz and Pitaevskii \([34]\) who follow the Klimontovich approach.

In this paper we extend this classical kinetic theory by describing the statistics of the large deviations for time dependent trajectories (paths) of the empirical density. For simplicity, we restrict our analysis to paths of the empirical density which remain close to homogenous distributions. We consider the projection of the empirical density on homogeneous distributions:

\[
f_N(v, t) = \frac{1}{N} \sum_{n=1}^{N} \delta(v - v_n(t)),
\]

where \(t\) is the volume of the system. The natural evolution of \(f_N\) occurs on time scales of order \(N\) (except in dimension \(d = 1\) \([48]\)). After time rescaling \(\tau = t/N\), we study the probability of \(f_N(v, \tau) = f_N(v, N\tau)\) (by abuse of notation and for convenience, we still denote \(f_N = f_N\)). We justify that the probability that a path \(\{f_N(\tau)\}_{0 \leq \tau \leq T}\) remains in the neighborhood of a prescribed path \(\{f(\tau)\}_{0 \leq \tau \leq T}\) satisfies the large deviation principle

\[
P\left(\{f_N(\tau)\}_{0 \leq \tau \leq T} = \{f(\tau)\}_{0 \leq \tau \leq T}\right) \approx \frac{1}{N} \int_{-\infty}^{\infty} e^{-N \int_{0}^{T} dt \sum_{p=0}^{p<0} \left[\int dv f_p - H[f, p]\right] e^{-N I_0[f^p]}},
\]

where \(f\) is the time derivative of \(f\), \(p\) is a function over the velocity space and is called the conjugated momentum of \(f\), the Hamiltonian \(H\) is a functional of \(f\) and \(p\) that characterizes the dynamical fluctuations, \(I_0\) is a large deviation rate function for the initial conditions of \(f_N\), and where the symbol \(\approx\) means a large deviation principle.
Dynamical large deviations for particles with long range interactions

(roughly speaking, the log of the left hand side is equivalent to the log of the right hand side, see classical textbooks [19] for a more precise definition). We note that \( H \) is not the Hamiltonian of the microscopic dynamics but it rather defines a statistical field theory that quantifies the probabilities of paths of the empirical density.

The main result of this paper is the first computation of an explicit expression for \( H \) and the study of its symmetry properties. The explicit expression for \( H \) is

\[
H[f,p] = -\frac{1}{4\pi L^3} \sum_k \int d\omega \log \left[ 1 - \mathcal{J}[f,p](k,\omega) \right],
\]

with

\[
\mathcal{J}[f,p](k,\omega) = 4\pi \int dv_1 dv_2 \frac{\partial p}{\partial v_1} \cdot A[f](k,\omega,v_1,v_2) \left\{ \frac{\partial f}{\partial v_2} f(v_1) - f(v_2) \frac{\partial f}{\partial v_1} \right\} + 4\pi \int dv_1 dv_2 \left\{ \frac{\partial p}{\partial v_1} \frac{\partial p}{\partial v_2} - \frac{\partial p}{\partial v_1} \frac{\partial p}{\partial v_2} \right\} : A[f](k,\omega,v_1,v_2) f(v_1) f(v_2),
\]

where the \( A(k,\omega,v_1,v_2) \) is a symmetric tensor which can be expressed through the Fourier transform of the interaction potential and the dielectric function.

Equations (3-4) clearly show that the Hamiltonian is not quadratic in the conjugated momentum \( p \). This shows that the fluctuations that lead to large deviations are not locally Gaussian, by contrast with many other cases, for instance when diffusive limits are involved as in the case of macroscopic fluctuation theory [4], or for plasma fluctuations at scales much smaller than the Debye length [23]. It is striking that it is possible to get explicit formulas (3-4) for the large deviation Hamiltonian, for which cumulants of all order are relevant and non trivial. The four key theoretical ideas and technical tools we use are: making the connection with large deviation theory for slow-fast systems and identifying the statistics of the fast motion, expressing the Hamiltonian as a functional determinant on a space of functions that depend both on time and velocity, using the Szegö–Widom theorem to reduce this functional determinant to a simpler one on a space of functions that depend on velocity only, and finally computing explicitly those determinants on the space of functions that depend on velocity only.

This article is the last of a series of three aimed at establishing dynamical large deviation principles related to the main classical kinetic theories, starting from Hamiltonian dynamics. The first one [8] dealt with the large deviations for dilute gases (associated with the Boltzmann equation), the second one dealt with large deviations for plasma fluctuations at scales much smaller then the Debye length [23] (associated with the Landau equation), and this one deals with the case of systems with mean-field interactions (associated with the Balescu–Guernsey–Lenard equation). The key point of these three works is to establish large deviation principles for particle systems with Hamiltonian dynamics. At first sight it might seem surprising to obtain a stochastic process for an effective kinetic description, starting from a deterministic dynamics. However it is well known that, after taking the limit with an infinite time scale separation between the slow and fast degrees of freedom, the effective dynamics of a slow-fast dynamical system, with chaotic fast degrees of freedom, is stochastic. At the level of large deviations, for deterministic dynamical systems,
mathematicians have proven theorems that establish large deviation principles for the effective stochastic process of the slow variable, from natural hypotheses [29,30]. This behavior can also be illustrated numerically, for instance coupling a slow dynamics with a fast chaotic Lorenz model dynamics [35]. As far as we know, our three works ([8,23] and this one) and [5] establish the first large deviation principles, in kinetic theory that do not start from stochastic dynamics, like for instance in macroscopic fluctuation theory [4]. While in [5] the result is proven for dilute gases in the Boltzmann–Grad limit for times of order of one collision times, our derivations are not mathematical proofs. All the steps of our derivation are however exact computations, once natural hypothesis are made, in the spirit of the most precise classical works by theoretical physicists in kinetic theory, and the result is expected to be valid for times much large that the kinetic times.

The large deviation results of the present paper ([3]) have some direct relations with the large deviations for plasma fluctuations at a scale much smaller than the Debye length, considered in [23]. While the result in the present paper is much more general, we explain in section 5.3 that for plasma fluctuations for scales much smaller than the Debye length, the locally Gaussian large deviation Hamiltonian of [23] can be recovered. We also prove in this paper a conjecture made in [23] about the structure of the cumulant expansion. We stress that the results of [23] are not just a sub-case of the results in the present paper. There, we were also considering questions of a different nature: the relation with the large deviation Hamiltonian for dilute gases, on one hand, and the relation with large deviations for effective mean-field diffusions, on the other hand.

We stress that our large deviation principle for paths immediately implies a gradient flow structure for the Balescu–Guernsey–Lenard operator, adapting to this specific case the general connection between path large deviation and gradient flows first discussed in [35] and simply explained in section 5 of [8]. As far as we know, no gradient flow structure was known before for the Balescu–Guernsey–Lenard operator. As the large deviations are not locally Gaussian, this gradient structure is not a standard one, but a generalized one [36] (please see section 5 of [8] for a simple definition standard of non standard gradient flow structure).

In parallel to our results, many mathematical results have recently been obtained for the kinetic theory of plasma and systems with long range interactions. The derivation of the Vlasov equation from the $N$ particle dynamics had been first proved by Neunzert [40], Braun and Hepp [15] and Dobrushin [20], for interactions through a smooth potential. This question is still under study for interaction potentials with singularities, for instance with the Coulomb interaction (see for instance [23]). Kiessling’s review [27] provides a recent report on the mathematical justification of the Vlasov equation from the microscopic dynamics of interacting particles. The stability of stationary states of the Vlasov equation, for describing the dynamics of the empirical density over time scales that diverge with $N$, but which are much smaller than the kinetic time, has been proven in [16]. The description of Gaussian fluctuations of the potential, for dynamics close to the Vlasov equilibrium, has been established by Braun and Hepp for smooth interaction potentials, or in the book [45]. More recent works [31,82,33,43,46,21] discuss the Gaussian process of the fluctuations of the
potential close to a Vlasov solution. A recent proof has been proposed for the validity of the Balescu–Guernsey–Lenard equation up to time scales of order $N^r$ with $r < 1$ \cite{22}.

We also stress that the mathematics community is also very active in studying large deviation principles related to kinetic theories. Rezakhanlou pioneered large deviation principle results, related to the Boltzmann equations \cite{44}, in the case of stochastic toy models of collisions. The first mathematical result which deals with Hamiltonian dynamics, more specifically hard sphere dynamics, has been obtained by Bodineau, Gallagher, Saint-Raymond, and Simonella \cite{5,6}. They basically prove the validity of the large deviation Hamiltonian for times of order one collision time, in the spirit of Lanford’s proof for the Boltzmann equation. Large deviation principles have also been obtained for Kac’s models \cite{26,3}. As far as we know, no mathematical results exists for the large deviations of plasma or systems with mean-field interactions.

In section 2 we define the Hamiltonian dynamics, as well as the classical kinetic equation that describes the relaxation to equilibrium. In section 4 we establish a large deviation principle for the empirical density using the slow-fast decomposition of the quasilinear dynamics. In section 4 we provide an explicit computation of the large deviation Hamiltonian. In section 5 we check that this Hamiltonian is fully compatible with the conservation laws of the system, as well as its time-reversal symmetry, and that it is consistent with statistics in the microcanonical ensemble. We discuss perspectives in section 6.

2 Dynamics of particles with long range interactions

In this section we set up the definitions, and present classical results about the kinetic theory of the dynamics of $N$ particles with long range interactions, in the limit of large $N$. In section 2.1 we define the Hamiltonian dynamics. In section 2.2 we introduce the Vlasov equation that describes the evolution of the empirical density on timescales of order one. In section 2.3 we introduce the Balescu–Guernsey–Lenard equation that describes the long time relaxation of the empirical density, from Vlasov stationary solutions to the Maxwell-Boltzmann equilibrium distribution, and some of its important physical properties.

2.1 Hamiltonian dynamics of $N$ particles with long range interactions

We consider $N$ particles with positions $\{r_n\}_{1 \leq n \leq N}$ and velocities $\{v_n\}_{1 \leq n \leq N}$ governed by a Hamiltonian dynamics

\[
\begin{align*}
\frac{dr_n}{dr} &= v_n \\
\frac{dv_n}{dr} &= -\frac{1}{N} \sum_{m \neq n} \frac{d}{dr} W(r_n - r_m)
\end{align*}
\]
where the interaction potential \( W(r) \) is an even function of \( r \). In the following, we consider that \( r_n \) belongs to a 3-dimensional torus of size \( L^3 \), and \( v_n \in \mathbb{R}^3 \). We stress that our results are actually valid for any space dimension \( d > 1 \). We assume that the potential \( W \) is a long range potential: the decay of \( W \) is slow enough, so that the interaction is dominated by the collective effects of the \( N \) particle rather than by local effects. In an infinite space this condition would be met if the potential would be non-integrable, for instance in dimension \( d \) it would decay asymptotically like a power law \( 1/|r|^d \) or more slowly. This condition is met in many physical systems, for instance self-gravitating systems or weak interacting plasma (with a large plasma parameter). For any finite \( L \), the condition that the potential decays more slowly than \( 1/|r|^d \) is a sufficient condition for the potential to be long range.

We call \( \mu \)-space the \((r,v)\) space. The \( \mu \)-space is of dimension 6. Let us define \( g_N \) the \( \mu \)-space empirical density for the positions and velocities of the \( N \) particles

\[
g_N(r,v,t) = \frac{1}{N} \sum_{n=1}^{N} \delta(r - r_n(t)) \delta(v - v_n(t)).
\]

In the following, we will study the stochastic process of the asymptotic dynamics of \( g_N \), as the number of particles \( N \) goes to infinity.

2.2 The Vlasov equation

From equation (5), one immediately obtains the Klimontovich equation

\[
\frac{\partial g_N}{\partial t} + v \cdot \frac{\partial g_N}{\partial r} - \frac{\partial V[g_N]}{\partial r} \cdot \frac{\partial g_N}{\partial v} = 0,
\]

where \( V[g_N](r,v) = \int d\nu \int d\nu' W(r - r') g_N(r',v',t) \). This is an exact equation for the evolution of \( g_N \), if \( W \) is regular enough. For the Coulomb interaction, the formal equation (6) has to be interpreted carefully. In the following, we do not discuss the divergences that might occur related to small scale interactions. At a mathematic level, this would be equivalent to considering a potential which is regularized at small scales, and smooth. The Klimontovich equation (6) contains all the information about the trajectories of the \( N \) particles. We would like to build a kinetic theory, that describes the stochastic process for \( g_N \) at a mesoscopic level.

An important first result is that the sequence \( \{g_N\} \) obeys a law of large numbers when \( N \to +\infty \). More precisely, if we assume that there is a set of initial conditions \( \{g^0_N\} \) such that \( \lim_{N \to +\infty} g_N^0(r,v) = g^0(r,v) \), then over a finite time interval \( t \in [0,T] \), \( \lim_{N \to +\infty} g_N(r,v,t) = g(r,v,t) \) where \( g \) solves the Vlasov equation

\[
\frac{\partial g}{\partial t} + v \cdot \frac{\partial g}{\partial r} - \frac{\partial V[g]}{\partial r} \cdot \frac{\partial g}{\partial v} = 0 \text{ with } g(r,v,t = 0) = g^0(r,v).
\]

While the solution of the Klimontovich equation is a distribution that carries the whole information about the positions and velocities of all the particles, the Vlasov equation describes the evolution of a continuous mesoscopic density for the same evolution. As the Klimontovich and the Vlasov equations are formally the same, this
law of large numbers is actually a stability result for the Vlasov equation in a space of distributions. Such a result has first been proven for smooth potentials by Braun and Hepp [15] and Neunzert [40].

The Vlasov equation has infinitely many Casimir conserved quantities. As a consequence, it has an infinite number of stable stationary states [48]. Any homogeneous distribution $g(r, v) = f(v)$ is a stationary solution of the Vlasov equation. In the following, we will consider dynamics close to any homogeneous $f$ which is a linearly stable stationary solution of the Vlasov equation. This linear stability can be assessed by studying the dielectric susceptibility $\varepsilon[f](k, \omega)$ [41,34], defined by

$$
\varepsilon[f](k, \omega) = 1 - \hat{W}(k) \int dv \frac{k \cdot \frac{\partial f}{\partial v} - \omega - i\eta}{k \cdot v - \omega - i\eta},
$$

(8)

where $\hat{W}(k)$ is the $k$-th Fourier component of the interaction potential: $\hat{W}(k) = \int dr \exp(-i\mathbf{k} \cdot \mathbf{r}) W(r)$. Equation (8) and every other equations involving $\pm i\eta$ have to be understood as the limit as $\eta$ goes to zero with $\eta$ positive. The dielectric susceptibility function $\varepsilon$ plays the role of a dispersion relation in the linearized dynamics, and a solution $f$ is stable if $\varepsilon[f]$ has no zeros except for $\omega$ on the real line. We note that $\varepsilon[f](-k, -\omega) = \varepsilon^*[f](k, \omega)$. Another important property of the dielectric susceptibility is $\varepsilon[I[f]](k, -\omega) = \varepsilon^*[f](k, \omega)$, where $I[f](v) = f(-v)$. This last property, associated to the time-reversal symmetry of the Hamiltonian dynamics, will be used in section 5.2. In this section we have discussed the linear stability of stationary solutions of the Vlasov equation while [48] defines different other notions of stability.

From the point of view of dynamical systems, those homogeneous solutions might be attractors of the Vlasov equation, with some sort of asymptotic stability. At a linear level, this convergence for some of the observables, for instance the potential, is called Landau damping [41,34]. Such a stability might also be true for the full dynamics. Indeed some non-linear Landau damping results have recently been proven [37].

In the following we will study the dynamics of $g_N$, when its initial condition is close to a homogeneous stable state $f(v)$. On time scales of order one, the distribution is stable and remains close to $f$ according to the Vlasov equation. However a slow evolution occurs on a timescale $\tau$ of order $N$, in spaces of dimension $d > 1$. For this reason, such $f$ are called quasi-stationary states [48]. In the following section, we explain that this slow evolution is described by the Balescu–Guernsey–Lenard equation for most initial conditions, or as a law of large numbers.

As a conclusion, the Balescu–Guernsey–Lenard equation appears as a mesoscopic description of the solution of the Klimontovich equation, for homogeneous solutions, which is valid up to time scales of order $N$, while the Vlasov equation is valid only up to time scales of order one. The Balescu–Guernsey–Lenard equation is a crucial correction to the Vlasov equation close to homogeneous solutions. Indeed homogeneous solution have no evolution through the Vlasov equation as they are stationary, while they have an evolution of order one over times scales of order $N$ through the Balescu–Guernsey–Lenard equation.
2.3 The Balescu–Guerney–Lenard equation

With the rescaling of time $\tau = t/N$, we expect a law of large numbers in the sense that “for almost all initial conditions” the empirical density $g_N$ converges to $f$, with $f$ that evolves according to the Balescu–Guerney–Lenard equation

$$\frac{\partial f}{\partial \tau} = \frac{\partial}{\partial v} \int dv_2 B[f](v, v_2) \left( -\frac{\partial f}{\partial v_2} f(v) + f(v_2) \frac{\partial f}{\partial v} \right),$$

(9)

with

$$B[f](v_1, v_2) = \frac{\pi}{L^3} \int^{+\infty}_{-\infty} d\omega \sum_{k \in (2\pi/L)Z^3} \frac{\tilde{W}(k)^2 kk}{|e[f](k, \omega)|^2} \delta(\omega - k \cdot v_1) \delta(\omega - k \cdot v_2),$$

(10)

where $kk$ denotes the tensor product $k \otimes k$. The tensor $B$ is called the collision kernel of the Balescu–Guerney–Lenard equation (by analogy with the Boltzmann equation).

A recent proof has been proposed for the the validity of the Balescu–Guerney–Lenard equation up to time scales $t$ of order $N^r$ with $r < 1$ \cite{22}. We know no mathematical proof of such a result for time scales $t$ of order $N$ ($\tau$ of order one). In the theoretical physics literature, this equation is derived as an exact consequence of the dynamics once natural hypotheses are made. Two classes of derivations are known, either the BBGKY hierarchy detailed in \cite{41} or the Klimontovich approach presented for instance in \cite{34}. The Klimontovich derivation is the more straightforward from a technical point of view. We now recall the main steps of the Klimontovich derivation, that will be useful later.

In the following we will consider statistical averages over measures of initial conditions for the $N$ particle initial conditions $\{r_n^0, v_n^0\}$. We denote $E_\delta$ the average with respect to this measure of initial conditions. As an example the measure of initial conditions could be the product measure $\prod_{n=1}^N g^0 (r_n^0, v_n^0) dr_n dv_n$. But we might consider other measures of initial conditions. We assume that for the statistical ensemble of initial conditions, the law of large numbers $\lim_{N \to \infty} g^0 (r, v) = g^0 (r, v)$ is valid at the initial time. This is true for instance for the product measure. In the following, for simplicity, we restrict the discussion to cases when the initial conditions are statistically homogenous: $g^0 (r, v) = P^0 (v)$. In the following, we define $f$ as the statistical average of $g_N$ over the initial conditions $f(v, t) = E_\delta (g_N(r, v, t))$.

We define the fluctuations $\delta g_N$ by $g_N(r, v, t) = f(v) + \delta g_N / \sqrt{N}$. The scaling $1/\sqrt{N}$ is natural when we see the Vlasov equation \cite{7} as a law of large numbers for the empirical density. For the potential we obtain $V[g_N] = V[\delta g_N] / \sqrt{N}$, as $f$ is homogeneous. If we introduce this decomposition in the Klimontovich equation \cite{6}, we obtain

$$\frac{\partial f}{\partial \tau} + v \frac{\partial f}{\partial r} = \frac{1}{N} E_\delta \left( \frac{\partial V[\delta g_N]}{\partial \tau} - \frac{\partial V[\delta g_N]}{\partial \tau} \frac{\partial \delta g_N}{\partial \tau} \right),$$

(11)

$$\frac{\partial \delta g_N}{\partial \tau} + v \frac{\partial \delta g_N}{\partial r} = \frac{1}{\sqrt{N}} \left[ \frac{\partial V[\delta g_N]}{\partial \tau} + \frac{\partial \delta g_N}{\partial \tau} \right] - E_\delta \left( \frac{\partial V[\delta g_N]}{\partial \tau} - \frac{\partial \delta g_N}{\partial \tau} \right),$$

(12)
In the first equation, the right hand side of the equation \( \frac{1}{N} \mathbb{E}_S \left( \frac{\partial V}{\partial r} | \frac{\partial g_N}{\partial v} \right) \) is called the averaged non-linear term and is responsible for the long term evolution of the distribution \( f \). The right hand side of the second equation \( \frac{1}{\sqrt{N}} \mathbb{E}_S \left( \frac{\partial V}{\partial r} | \frac{\partial g_N}{\partial v} \right) \) describes the fluctuations of the non-linear term. For stable distributions \( f \), and on timescales much smaller than \( \sqrt{N} \), we can neglect this term, following Klimontovich and classical textbooks [34]. Please see [16] for a mathematical proof of a sufficient condition of stability on time scales of order \( N^\alpha \), for some \( \alpha < 1 \).

Neglecting the terms much smaller than \( \sqrt{N} \) closes the hierarchy of the correlation functions. The Bogoliubov approximation then amounts to using the time scale separation between the evolution of \( f \) and \( \delta g_N \).

One computes the correlation function \( \mathbb{E}_S \left( \frac{\partial V}{\partial r} | \frac{\partial g_N}{\partial v} \right) \) resulting from (12) with fixed \( f \), and argues that this two point correlation function converges to a stationary quantity on time scales much smaller than \( \sqrt{N} \). Using this quasi-stationary correlation function \( \mathbb{E}_S \left( \frac{\partial V}{\partial r} | \frac{\partial g_N}{\partial v} \right) \), one can compute the right hand side of (11) as a function of \( f \).

After time rescaling \( \tau = t/N \), we define \( g_N^s(\mathbf{r}, \mathbf{v}, \tau) = g_N(\mathbf{r}, \mathbf{v}, N\tau) \). By abuse of notation and for convenience, we still denote \( g_N^s(\tau) = g_N(\tau) \). The closed equation for \( g_N^s(\tau) \) resulting from (11) is the Balescu–Guernsey–Lenard equation (9).

We do not reproduce these lengthy and classical computations that can be found in plasma physics textbooks, for instance in Chapter 51 of [34]. A natural conjecture is that we have a law of large numbers \( \lim_{N \to \infty} g_N(\mathbf{r}, \mathbf{v}, \tau) = f(\mathbf{v}, \tau) \), where \( f \) solves the Balescu–Guernsey–Lenard equation (9), and valid for any finite time \( \tau \).

Symmetries and conservation properties. The Balescu–Guernsey–Lenard equation (9) has several important physical properties:

1. It conserves the mass \( M[f] \), momentum \( P[f] \) and total kinetic energy \( E[f] \) defined by

\[
M[f] = \int d\mathbf{v} \, f(\mathbf{v}) \, , \quad P[f] = \int d\mathbf{v} \, \mathbf{v} \, f(\mathbf{v}) \, , \quad E[f] = \int d\mathbf{v} \, \frac{\mathbf{v}^2}{2} \, f(\mathbf{v}) . \tag{13}
\]

2. It increases monotonically the entropy \( S[f] \) defined by

\[
S[f] = -k_B \int d\mathbf{v} \, f(\mathbf{v}) \log f(\mathbf{v}) ,
\]

where \( k_B \) is the Boltzmann constant.

3. It converges towards the Boltzmann distribution for the corresponding energy

\[
f_B(\mathbf{v}) = \frac{\beta^{3/2}}{(2\pi)^{3/2}} \exp \left( -\frac{\beta \mathbf{v}^2}{2} \right) .
\]
Derivation of the large deviation principle from the quasi-linear dynamics

In this section, we derive a large deviation principle for the empirical density of \( N \) particles with long range interactions, directly from the dynamics (5).

In section 3.1, we introduce the quasi-linear dynamics of the empirical density of \( N \) long range interacting particles, for which the law of large numbers is the Balescu–Guernsey–Lenard kinetic theory. In section 3.1, we explain that this quasi-linear dynamics for the empirical density can be seen as a slow-fast system, for which we can define the path large deviation functional for the slow variable. In section 3.2, we characterize the stochastic process for the quasi-linear dynamics of the fluctuations of the empirical density as a stationary Gaussian process.

3.1 The Klimontovich approach, quasilinear and slow-fast dynamics

We begin by equations which are similar to (11-12), but by contrast to the discussion of the previous section, we will not compute just the average for the effect of fluctuations on the evolution of \( f_N \), but all the cumulants after time averaging.

We consider the empirical density

\[
g_N (r,v,t) = \frac{1}{N} \sum_{n=1}^{N} \delta (v - v_n(t)) \delta (r - r_n(t)),
\]

of \( N \) particles which interact through a long range pair potential according to the dynamics (5). From these equations of motion, we can deduce the Klimontovich equation

\[
\frac{\partial g_N}{\partial t} + v \cdot \frac{\partial g_N}{\partial r} - \nabla V[g_N] \cdot \frac{\partial g_N}{\partial v} = 0.
\] (14)

We consider the decomposition

\[
g_N (r,v,t) = f_N (v) + \frac{1}{\sqrt{N}} \delta g_N (r,v,t),
\]

where \( f_N (v,t) = \frac{1}{L^3} \int dr g_N (r,v,t) \) is the projection of \( g_N \) on homogeneous distributions (distributions that depend on velocity only) and \( \delta g_N \) describes the inhomogeneous fluctuations of the empirical density \( g_N \). Alternately, we can understand \( f_N \) as the empirical density of the \( N \) particles in the velocity space: \( f_N (r,v,t) = N^{-1} L^{-3} \sum_{n=1}^{N} \delta (v - v_n(t)) \). From the Klimontovich equation (14), we straightforwardly write

\[
\frac{\partial f_N}{\partial t} = \frac{1}{NL^3} \int dr \left( \frac{\partial V[\delta g_N]}{\partial r} \cdot \frac{\partial \delta g_N}{\partial v} \right),
\] (15)

\[
\frac{\partial \delta g_N}{\partial t} = -v \cdot \frac{\partial \delta g_N}{\partial r} + \frac{\partial V[\delta g_N]}{\partial r} \cdot \frac{\partial f_N}{\partial v} + \frac{1}{\sqrt{N}} \left( \frac{\partial V[\delta g_N]}{\partial r} \cdot \frac{\partial \delta g_N}{\partial v} - \frac{1}{L^3} \int dr \left( \frac{\partial V[\delta g_N]}{\partial r} \cdot \frac{\partial \delta g_N}{\partial v} \right) \right). (17)
\]
Just like in the previous section, we will consider statistical averages over a probability measure for the initial conditions \( \{ \mathbf{r}_n^0, \mathbf{v}_n^0 \} \) of the \( N \) particles. As the microscopic dynamics is deterministic, the only source of randomness is the ensemble of initial conditions. We assume that this ensemble of initial conditions is sampled from a spatially homogeneous measure and that the set of corresponding \( g_N \) is concentrated close to homogeneous distributions in the \((\mathbf{r}, \mathbf{v})\) space. Moreover we assume that the large deviation principle

\[
P \left( f_N (\tau = 0) = f^0 \right) \sim \frac{1}{N} e^{-N h_0 [f^0]},
\]

holds, where \( I_0 \) is a large deviation rate function for \( f_N (\tau = 0) \) the initial conditions of \( f_N \). As an example, the measure of initial conditions \( \{ \mathbf{r}_n^0, \mathbf{v}_n^0 \} \) could be the homogeneous product measure \( \prod_{n=1}^N P^0 (\mathbf{v}_n^0) \, \mathrm{d}\mathbf{v}_n / L^3 \), with \( \int \mathrm{d}\mathbf{v} P^0 (\mathbf{v}) = 1 \). Then \( I_0 \) would then be the Kullback–Leibler divergence of \( f^0 \) with respect to \( P^0 \). But we might consider other ensembles of initial conditions.

We now assume the validity of the quasi-linear approximation, which amounts to neglecting terms of order \( N^{-1/2} \) in the evolution equation for \( \delta g_N \). We also change the timescale \( \tau = t/N \) and obtain the quasilinear dynamics

\[
\frac{\partial f_N}{\partial \tau} = \frac{1}{L^3} \int \mathrm{d}\mathbf{r} \left( \frac{\partial V}{\partial \delta g_N} \cdot \frac{\partial \delta g_N}{\partial \mathbf{r}} \right),
\]

\[
\frac{\partial \delta g_N}{\partial \tau} = N \left\{ -\mathbf{v} \cdot \frac{\partial \delta g_N}{\partial \mathbf{r}} + \frac{\partial V}{\partial \delta g_N} \frac{\partial f_N}{\partial \mathbf{r}} \right\}.
\]

When \( N \) goes to infinity, we observe that the equation for \( \delta g_N \) is a fast process, with timescales for \( \tau \) of order \( 1/N \), while the equation for \( f_N \) is a slow one with timescales for \( \tau \) of order \( 1 \). For such slow-fast dynamics, it is natural to consider \( f_N \) fixed (frozen) in equation (20) on time scales for \( \tau \) of order \( 1/N \). For fixed \( f_N \) the dynamics for \( \delta g_N \) is linear and can be solved. Computing then the average of the term \( \int \mathrm{d}\mathbf{r} \frac{\partial V}{\partial \delta g_N} \cdot \frac{\partial \delta g_N}{\partial \mathbf{r}} \), for the asymptotic process for \( \delta g_N \) for fixed \( f_N \) leads to the Guernsey–Lenard–Balescu equation, as explained in section 3. Those computation can be found in classical textbooks [34].

In the following we want to go beyond these classical computations, by estimating not just the average of the left hand side in (18), \( \int \mathrm{d}\mathbf{r} \frac{\partial V}{\partial \delta g_N} \cdot \frac{\partial \delta g_N}{\partial \mathbf{r}} \), but all the cumulants of the time averages \( \int_0^\tau \frac{\partial V}{\partial \delta g_N} \cdot \frac{\partial \delta g_N}{\partial \mathbf{r}} \) in order to describe the large deviations for the process \( f_N \). For slow-fast dynamics, the theory for the large deviations of the effective evolution of the slow variable is a classical one both in theoretical physics (see for instance [9]) and mathematics. In the mathematics literature, it is for instance treated for diffusions [24,47], or chaotic deterministic systems [29,30]. The result for the path large deviations for the slow dynamics is explained in section 2.2.2 of [25] (see equations (20-21)). After rescaling time \( \tau = t/N \), we then have

\[
P \left( \{ f_N (\mathbf{v}, \tau) \}_{0 \leq \tau \leq T} = \{ f (\mathbf{v}, \tau) \}_{0 \leq \tau \leq T} \right) \sim \frac{1}{N} e^{-NL^3S_{\delta \mathbf{p}, \mathbf{v}}^T \mathbf{t} \int \mathrm{d} \mathbf{v} f \mathcal{P} (f, \mathbf{p})} e^{-Nh_0 [f_0]},
\]

(21)
where $I_0$ is a large deviation rate function for the initial conditions of $f_N$, see equation (18), and with

$$H[f,p] = \lim_{T \to \infty} \frac{1}{TL^3} \log \mathbb{E}_f \left[ \exp \left( \int_0^T dt \int dv \, p(v) \int dv' \frac{\partial V[\delta g_N]}{\partial r'} \frac{\partial \delta g_N}{\partial v} \right) \right]$$

(22)

and where $\mathbb{E}_f$ denotes the expectation on the process for $\delta g_N$, evolving according to

$$\frac{\partial \delta g_N}{\partial t} = -v \cdot \frac{\partial \delta g_N}{\partial r} + \frac{\partial V[\delta g_N]}{\partial r} \frac{\partial f}{\partial v}.$$  

(23)

In this equation, $f_N = f$ is fixed and time independent. We note that the classical mathematical results to justify (22) would require to prove mixing properties for the fast process, and stability of the invariant measure, that nobody has proven yet for (23).

We note that to obtain equation (22) from equation (21) of [23], we have considered $f_N$ as a function of the $\mu$-space. Then the conjugated momentum $p(r,v)$ should also be a function of the $\mu$-space and the scalar product be the one of the $\mu$-space. However, recognizing that for homogeneous $f$, $p$ should also be homogeneous ($p(r,v) = p(v)$), and performing trivial integration over $r$ leads to (22). The $L^3$ factor in the large deviation principle (21) also comes from a trivial integration over $r$ of $fp$.

In the definition of $H$, in (22), we have divided the scaled cumulant generating function by $L^3$ for convenience, such that the action in (21) appears as a natural action for homogeneous distributions.

The goal of the following sections and the contribution of this work is to obtain an explicit expression for (22).

### 3.2 The quasi-stationary Gaussian process for $\delta g_N$

In order to compute (22), we need to estimate averages over the stochastic process which corresponds to generic sets of initial condition for $\delta g_N$, and where $\delta g_N$ satisfies equation (23). We first note that for fixed $f$, equation (23) is linear. If the set of initial conditions for $\delta g_N(t=0)$ is a Gaussian random variable, then the stochastic process $\{\delta g_N(t)\}_{t \geq 0}$ will be a Gaussian process. Several recent mathematical works [31, 32, 33, 43, 46, 21] discuss some properties of the Gaussian process of the fluctuations close to a Vlasov solution. For instance [33] proves that, when starting from sets of Gaussian initial conditions of the form of relevant central limit theorems, at long times, the stochastic process converges to a statistically stationary Gaussian process. The fact that for generic sets of initial conditions, the stochastic process of the fluctuations $\delta g_N$ converges to a stochastically stationary process, which is independent of the initial conditions, has long been understood by physicists. This is for instance explained in §51 of [34], where the asymptotic stationary process is precisely characterized. This striking convergence result is related to the Landau damping and the fact that we deal with particle systems. The work [4] derives another characterization of this stationary process, based on an integral equation, and illustrates numerically the convergence. In the following we will thus consider averages in equation (22) as
averages over this stationary Gaussian process. Such stationary averages are denoted \( \mathbb{E}_S \).

We do not reproduce the classical and lengthy computations of the correlation functions of this stationary process, but just report the formulas which can be found for instance in §51 of [34]. The potential autocorrelation function are homogeneous because of the space translation symmetry. Then

\[
\mathbb{E}_S \left( V [\delta g_N] (r_1, t_1) V [\delta g_N] (r_2, t_2) \right) = \mathcal{G}_{VV} (r_1 - r_2, t_1 - t_2),
\]

We define \( \phi \) the space-time Fourier transform of a function \( \varphi \) as

\[
\hat{\phi}(k, \omega) = \int_{[0, t]} \int_{-\infty}^{\infty} dr e^{-i(k \cdot r - \omega t)} \varphi(r, t),
\]

following the same convention as in [34]. According to equation (51.20), §51 of [34], with the identification \( V = e\phi \) and \( \hat{W}(k) = 4\pi e^2/k^2 \), the space-time Fourier transform of the autocorrelation function of the potential then reads

\[
\mathcal{G}_{VV}(k, \omega) = 2\pi \int dv \int \left( \delta(v - v') \frac{\delta(\omega - k \cdot v)}{[\epsilon(f)(k, \omega)]^2} \right) \frac{\hat{W}(k)^2}{[\epsilon(f)(k, \omega)]^2}.
\]

Similarly the time stationary correlation function between the potential and distribution fluctuations is space-time homogeneous

\[
\mathbb{E}_S (V [\delta g_N] (r_1, t_1) [\delta g_N] (r_2, v, t_2)) = \mathcal{G}_{VG} (r_1 - r_2, t_1 - t_2, v).
\]

According to equation (51.21) of [34], its space-time Fourier transform reads

\[
\mathcal{G}_{VG}(k, \omega, v) = -\frac{k}{\omega - k \cdot v + i\eta} \frac{\partial f}{\partial v}(v) \mathcal{G}_{VV}(k, \omega) + 2\pi \frac{\hat{W}(k)}{[\epsilon(f)(k, \omega)]} f(v) \delta(\omega - k \cdot v).
\]

We also define the autocorrelation function of the distribution fluctuations

\[
\mathbb{E}_S (\delta g_N (r_1, v_1, t_1) \delta g_N (r_2, v_2, t_2)) = \mathcal{G}_{GG} (r_1 - r_2, t_1 - t_2, v_1, v_2).
\]

According to equation (51.23) of [34], its space-time Fourier transform reads

\[
\mathcal{G}_{GG}(k, \omega, v_1, v_2) = 2\pi \delta(v_1 - v_2)f(v_1) \delta(\omega - k \cdot v_1)
\]

\[
+ \frac{\hat{\mathcal{G}}_{VV}(k, \omega)}{(\omega - k \cdot v_1 + i\eta)(\omega - k \cdot v_2 - i\eta)} \frac{\partial f}{\partial v}(v_1)k \frac{\partial f}{\partial v}(v_2)
\]

\[
- 2\pi \hat{W}(k)k \frac{\partial f}{\partial v}(v_1) \frac{f(v_1)\delta(\omega - k \cdot v_2)}{\epsilon(f)(k, \omega)(\omega - k \cdot v_1 + i\eta)}
\]

\[
- 2\pi \hat{W}(k)k \frac{\partial f}{\partial v}(v_2) \frac{f(v_1)\delta(\omega - k \cdot v_1)}{\epsilon^*(f)(k, \omega)(\omega - k \cdot v_2 - i\eta)}.
\]

We note that the order in the correlation functions for \( V \) and \( g_N \) matters. We have

\[
\mathbb{E}_S (\delta g_N (r_1, v, t_1) V [\delta g_N] (r_2, t_2)) = \mathcal{G}_{GV} (r_1 - r_2, t_1 - t_2, v),
\]
with
\[ \widehat{C}_{VG}(\mathbf{k}, \omega, \mathbf{v}) = \widehat{C}_{GV}(-\mathbf{k}, -\omega, \mathbf{v}) = \widehat{C}_{GV}^*(\mathbf{k}, \omega, \mathbf{v}). \]

We also note the symmetry property for \( \widehat{C}_{GG} \):
\[ \widehat{C}_{GG}(\mathbf{k}, \omega, \mathbf{v}_1, \mathbf{v}_2) = \widehat{C}_{GG}(\mathbf{k}, -\omega, \mathbf{v}_2, \mathbf{v}_1). \]

It is a consequence of the symmetry \( C_{GG} \):
\[ C_{GG}(\mathbf{r}, \mathbf{t}, \mathbf{v}_1, \mathbf{v}_2) = C_{GG}(\mathbf{r}, -\mathbf{t}, \mathbf{v}_2, \mathbf{v}_1). \]
Moreover, since \( C_{GG} \) is real, we have
\[ \widehat{C}_{GG}(\mathbf{k}, -\omega, \mathbf{v}_2, \mathbf{v}_1) = \widehat{C}_{GG}^*(\mathbf{k}, \omega, \mathbf{v}_2, \mathbf{v}_1). \]

We thus have the symmetry
\[ \widehat{C}_{GG}(\mathbf{k}, \omega, \mathbf{v}_1, \mathbf{v}_2) = \widehat{C}_{GG}^*(\mathbf{k}, \omega, \mathbf{v}_2, \mathbf{v}_1). \] (28)

We note that as a mere consequence of the definition of \( V[\delta g_N] \), we have the following relations between the two-point correlation functions
\[ \widehat{C}_{VG}(\mathbf{k}, \omega, \mathbf{v}_1) = \hat{W}(\mathbf{k}) \int d\mathbf{v}_2 \widehat{C}_{GG}(\mathbf{k}, \omega, \mathbf{v}_1, \mathbf{v}_2), \] (29)
\[ \widehat{C}_{VV}(\mathbf{k}, \omega) = (\hat{W}(\mathbf{k}))^2 \int d\mathbf{v}_1 d\mathbf{v}_2 \widehat{C}_{GG}(\mathbf{k}, \omega, \mathbf{v}_1, \mathbf{v}_2). \] (30)

4 Computation of the large deviation Hamiltonian

In this section, we obtain an explicit formula for the large deviation functional of the empirical density of \( N \) particles with long range interactions, starting from equation (22). We noticed in section 3.2 that the fluctuations of the homogeneous part empirical density are described by the average of a quadratic form over a Gaussian stationary process. In section 4.1, we explain how this makes the computation of the Hamiltonian (22) equivalent to the computation of a functional determinant. In section 4.2, this functional determinant is explicitly computed, using the Szegő–Widom theorem and an explicit computation of determinants in the space of observable over velocity distributions.

4.1 The large deviation Hamiltonian as a functional Gaussian integral

Within the quasi-linear approximation, the fluctuations of the empirical density \( \delta g_N \) describe a stationary Gaussian process over functions of the \( \mu \)-space. The goal of this subsection is to show that the computation of the large deviation Hamiltonian is equivalent to the computation of a Gaussian functional integral of the fluctuations of the empirical density \( \delta g_N \).

We consider \( \mathcal{H}_v \), the Hilbert space of complex functions over the velocity space, with \( \langle \cdot, \cdot \rangle \), the Hermitian product: \( \langle a, b \rangle = \int d\mathbf{v} a^*(\mathbf{v}) b(\mathbf{v}) \). We can conveniently express the argument of the exponential in the formula (22) for the large deviation Hamiltonian using a spatial Fourier decomposition of the fluctuations of the empirical density \( \hat{\delta} g(\mathbf{k}, \mathbf{v}, t) = \int_{[0,L]} dr e^{-i\mathbf{k} \cdot \mathbf{r}} \delta g_N(\mathbf{r}, \mathbf{v}, t). \) Using this Fourier decomposition, the definition of the potential \( V[\delta g_N] \) and partial integration with respect to the velocity integral, we obtain
\[
\int \text{d}v \langle \exp \{\hat{H}(v)\} \rangle = \frac{1}{L^3} \int \frac{\partial V[\delta g_N]}{\partial \delta g_N} \frac{\partial \delta g_N}{\partial v} = \frac{1}{2} \sum_{k \in (2\pi/L)^3} \langle \delta \hat{g}_N(k, ., t) , M(k) | \delta \hat{g}_N(k, ., t) \rangle ,
\]
where we define the Hermitian operator \( M(k) \) acting on \( \varphi \in \mathcal{H} \) as
\[
M(k) | \varphi \rangle = \int \text{d}v_2 M(k; v_1, v_2) \varphi(v_2) ,
\]
with the kernel \( M \) defined by
\[
M(k; v_1, v_2) = \frac{i}{L^3} \hat{W}(k) k \left\{ -\frac{\partial p}{\partial v}(v_1) + \frac{\partial p}{\partial v}(v_2) \right\} .
\]
There is a factor 1/2 on the r.h.s. of (31) because we chose to symmetrize the expression, such that \( M(k; v_1, v_2) = M(k; v_2, v_1)^* \). \( M(k) \) is then an Hermitian operator.

The goal of the following of this subsection is to express the sum on the r.h.s. of (31) as a sum of independent terms to make the computation of (22) easier. Since for every \( k \in (2\pi/L)^3 \), \( M(k) \) is an Hermitian operator, \( M(k)^* = M(-k) \), and
\[
\delta \hat{g}_N(k, ., t) = \delta \hat{g}_N(-k, ., t)
\]
we have the following relation
\[
\langle \delta \hat{g}_N(k, ., t) , M(k) | \delta \hat{g}_N(k, ., t) \rangle = \langle \delta \hat{g}_N(-k, ., t) , M(-k) | \delta \hat{g}_N(-k, ., t) \rangle .
\]
This implies that on the r.h.s. of (31), the contribution of an index \( k \in (2\pi/L)^3 \) will be equal to the contribution of its negative \(-k\).

Because the stochastic process \( \delta g_N(r, ., t) \), for the fluctuations of the distribution function is spatially homogeneous, the stochastic process \( \delta \hat{g}_N(k, ., t) \) is statistically mutually independent with every other \( \delta \hat{g}_N(k', ., t) \) as long as \( k' \neq -k \). Because \( \delta \hat{g}_N(k, ., t) \) is not statistically independent from \( \delta \hat{g}_N(-k, ., t) \), it is useful to treat them together. We define \( \mathbb{Z}_+^3 = \mathbb{Z}^3 / \mathbb{Z}_2 \) the quotient of group \( \mathbb{Z}^3 \) with \( \mathbb{Z}_2 \), the cyclic group of order 2. In other words, \( \mathbb{Z}_+^3 \) is the set of triplets of integers where we identify a triplet \((a, b, c) \in \mathbb{Z}_+^3 \) with its negative \((-a, -b, -c) \). Then, using (34), the sum over \( k \in (2\pi/L)^3 \) can be rewritten as
\[
\frac{1}{L^3} \sum_{k \in (2\pi/L)^3} \langle \delta \hat{g}_N(k, ., t) , M(k) | \delta \hat{g}_N(k, ., t) \rangle = \sum_{k \in (2\pi/L)^3_+} \langle \delta \hat{g}_N(k, ., t) , M(k) | \delta \hat{g}_N(k, ., t) \rangle .
\]
As a consequence, the r.h.s of (35) is a sum of statistically independent terms. We can then use the fact that the expected value of a product of independent random variables is the product of their expected values, as well as equations (31) and (35) to obtain
\[
\mathbb{E} \left[ \exp \left( \int_0^T \text{d}r \int \text{d}v_p(v) \int \text{d}v_r \frac{\partial V[\delta g_N]}{\partial \delta g_N} \frac{\partial \delta g_N}{\partial v_r} \right) \right] = \prod_{k \in (2\pi/L)^3_+} \mathbb{E} \left[ \exp \left( \int_0^T \text{d}r \langle \delta \hat{g}_N(k, ., t) , M(k) | \delta \hat{g}_N(k, ., t) \rangle \right) \right] .
\]
We can then go back to (22) using (34) and (36) to express the large deviation Hamiltonian as a sum over the wavevectors
\[
H[f, p] = \sum_{k \in (2\pi/L)^3} \hat{H}[f, p](k) ,
\]
where
\[
\hat{H}[f, p](k) = \lim_{T \to \infty} \frac{1}{2TL^3} \log \mathbb{E} \left[ \exp \left( \int_0^T dt \langle \delta \delta_N(k, \cdot, t), M(k) [\delta \delta_N(k, \cdot, t)] \rangle \right) \right].
\] (38)

### 4.2 Application of the Szegő–Widom theorem

The computation of (37-38) requires to estimate large time large deviations of a quadratic functional of a Gaussian stochastic process. More precisely, the Gaussian process involved in (38) is the stochastic process of the \( k \)-th Fourier mode of the fluctuations of the empirical density \( \delta \delta_N(k, \cdot, t) \), and the quadratic functional is defined by the Hermitian operator \( M(k) \). Since \( \delta \delta_N(k, \cdot, t) \) is a Gaussian process, it is possible to compute (38) via functional determinants. Thanks to the Szegő–Widom theorem, it is possible to evaluate the asymptotics of this Fredholm determinant in terms of much simpler determinants of an operator on \( H_v \). This program was first implemented in [14], with a nice application to a model inspired by 2D and geophysical turbulence. In appendix A we explain the details of this program, easily adapting [14] for Gaussian processes with complex variables. The result (58) of the appendix A adapted to the case where the Hilbert space is \( H_v \), reads
\[
\log \mathbb{E} \left[ \exp \left( \int_0^T dt \langle \delta \delta_N(k, \cdot, t), M(k) [\delta \delta_N(k, \cdot, t)] \rangle \right) \right] \sim -\frac{T}{2\pi} \int d\omega \log \det \left\{ u_{\omega, k} \right\},
\] (39)
where, for any \( k \) and \( \omega \), and \( \varphi \in H_v \), \( u_{\omega, k} \varphi \) is defined by
\[
u_{\omega, k} \varphi(v_1) = \varphi(v_1) + \int dv_2 dv_3 M(k; v_1, v_2) \overline{\delta \delta_N(k, \omega, v_2, v_3)} \varphi(v_3).
\]

\( u_{\omega, k} \) is a linear operator of \( H_v \). The subscript \( H_v \) in (39) indicates that the determinant is a determinant of an operator over \( H_v \). Then, combining equations (38) and (39) yields
\[
\hat{H}[f, p](k) = -\frac{1}{4\pi L^3} \int d\omega \log \det \left\{ u_{\omega, k} \right\}.
\] (40)

Our next task is to obtain an explicit formula for \( \hat{H}[f, p](k) \) and thus for the full large deviation Hamiltonian \( H \). The determinant can be easily computed once we realize that the range of \( u_{\omega, k} - \text{Id} \) is two-dimensional. The explicit computation is performed in appendix B. The result reads
\[
\det \left\{ u_{\omega, k} \right\} = 1 - f[f, p](k, \omega),
\] (41)
with
\[ \mathcal{J} [f, p] (\kappa, \omega) = -2 \int \mathrm{d}v_1 \kappa \frac{\partial p}{\partial v_1} \Im \left\{ \overline{\mathcal{G}}_{VG} (\kappa, \omega, v_1) \right\} \]
\[ - \int \mathrm{d}v_1 \mathrm{d}v_2 \kappa \frac{\partial p}{\partial v_1} \kappa \frac{\partial p}{\partial v_2} \left\{ \overline{\mathcal{G}}_{VG} (\kappa, \omega, v_1) \overline{\mathcal{G}}_{VG} (\kappa, \omega, v_2)^* \right\} \]
\[ - \overline{\mathcal{G}}_{VV} (\kappa, \omega) \overline{\mathcal{G}}_{GG} (\kappa, \omega, v_1, v_2) \right\} \right\}. \quad (42) \]

Using the expressions of the two-point correlation functions \[ (25-27) \], we obtain that
\[ \mathcal{J} [f, p] = \mathcal{L} [f, p] + Q [f, p, p], \]
where \( \mathcal{L} \) depends linearly on \( p \) and \( Q \) depends on \( p \) as a quadratic form. We have
\[ \mathcal{L} [f, p] (\kappa, \omega) = 4\pi \int \mathrm{d}v_1 \mathrm{d}v_2 A [f] (\kappa, \omega, v_1, v_2) : \frac{\partial p}{\partial v_1} \left\{ \frac{\partial f}{\partial v_2} f (v_1) - f (v_2) \frac{\partial f}{\partial v_1} \right\} \]
and
\[ Q [f, p, q] (\kappa, \omega) = 2\pi \int \mathrm{d}v_1 \mathrm{d}v_2 A [f] (\kappa, \omega, v_1, v_2) : \left\{ \frac{\partial p}{\partial v_2} \frac{\partial q}{\partial v_1} - \frac{\partial p}{\partial v_1} \frac{\partial q}{\partial v_2} \right\} f (v_1) f (v_2), \]
\[ \quad (44) \]
with
\[ A [f] (\kappa, \omega, v_1, v_2) = \frac{\pi \kappa \kappa \mathcal{W}(\kappa)^2}{\left[ \epsilon [f] (\kappa, \omega) \right]^2} \delta (\omega - \kappa v_1) \delta (\omega - \kappa v_2). \quad (46) \]

We note that the tensor \( A \) is related to the tensor \( B \) of the Balescu–Guernsey–Lenard equation \[ (29) \]:
\[ B [f] (v_1, v_2) = \frac{1}{L^2} \sum_{\kappa} \int \mathrm{d}\omega A [f] (\kappa, \omega, v_1, v_2), \]
and that it shares all of its properties: it is symmetric as a tensor, it is symmetric in its velocities argument
\[ A (\kappa, \omega, v_1, v_2) = A (\kappa, \omega, v_1, v_2) \]
(momentum conservation), and we have
\[ A (\kappa, \omega, v_1, v_2), (v_1 - v_2) = 0 \]
(energy conservation). These properties are related to the conservation laws of the physical system, as we will see in section \[ 5.1 \]. Using \( \epsilon [f] (-\kappa, -\omega) = \epsilon^* [f] (\kappa, \omega) \), we also have
\[ A [f] (\kappa, \omega, v_1, v_2) = A [f] (-\kappa, -\omega, v_1, v_2). \]
\( A \) also has a symmetry property related to the time reversal symmetry. Recalling that
\( I [f] (v) = f (-v) \) is the velocity inversion involution, we recall that \( \epsilon [I [f]] (\kappa, -\omega) = \epsilon^* [f] (\kappa, \omega) \) and as a consequence
\[ A [I [f]] (\kappa, -\omega, -v_1, -v_2) = A [f] (\kappa, \omega, v_1, v_2). \]
We will discuss more deeply this property in section 5.2.

Using equations (37), (40) and (41) we obtain an explicit formula for the large deviation Hamiltonian

\[ H[f,p] = -\frac{1}{4\pi L^3} \sum_k \int d\omega \log \{ 1 - \mathcal{J}[f,p](k,\omega) \}, \tag{47} \]

where \( \mathcal{J}[f,p](k,\omega) \) is defined in equations (43)-(45).

As a conclusion, in this section, we have established the path large deviation principle

\[ P(\{ f_N(\tau) \}_{0 \leq \tau \leq T} = \{ f(\tau) \}_{0 \leq \tau \leq T} \uparrow \lim_{N \to \infty} e^{-NL^3} \int_0^T d\tau \sup_p \{ \int d\nu \mathcal{J}[f,p] \} e^{-NL_0[f(\tau=0)]}, \]

where \( H \) is given by (47) and where \( \tau = t/N \).

Density-current formulation of the large deviation principle. We define the current as

\[ j_N(v,t) = -\frac{1}{NL^3} \int dr \left( \frac{\partial V[\delta g_N]}{\partial r} \delta g_N \right). \]

In appendix D, we prove that the large deviation principle (4.2) is equivalent to an empirical density-current formulation:

\[ P(\{ f_N(\tau), j_N(\tau) \}_{0 \leq \tau \leq T} = \{ f(\tau), j(\tau) \}_{0 \leq \tau \leq T} \uparrow \lim_{N \to \infty} e^{-NL^3} \int_0^T d\tau \sup_E \{ \int d\nu j \cdot E - \tilde{H}[f,E] \} e^{-NL_0[f(\tau=0)]}, \]

where \( j_N(\tau) \) should be interpreted as a time-averaged current after time rescaling, with

\[ \mathcal{J}[f,j] = \begin{cases} L^3 \int_0^T d\tau \tilde{L}[f,j] & \text{if } f + \frac{\partial}{\partial v} j = 0, \\ +\infty & \text{otherwise.} \end{cases} \]

and where \( \tilde{L}[f,j] = \sup_E \{ \int d\nu j \cdot E - \tilde{H}[f,E] \} \), and \( \tilde{H} \) is defined by \( H[f,p] = \tilde{H}[f,\partial p/\partial v] \).

5 Properties of the large deviation Hamiltonian

In this section we check that the large deviation Hamiltonian (47) satisfies all the expected symmetry properties. In section 5.1 we check that the Hamiltonian (47) is consistent with the mass, momentum and energy conservation laws. In section 5.2 we show that the Hamiltonian (47) has a time-reversal symmetry, and has the negative of the entropy, with conservation law constraints and up to constants, as a quasipotential.
5.1 Conservation laws

It is a classical exercise to prove that any conservation law is equivalent to a symmetry property of the large deviation Hamiltonian, see for instance section 7.3.2 of [8]. From section 7.3.2 of [8], we know that a functional \( C[f] \) is a conserved quantity of the large deviation principle (21) if and only if for any \( f \) and \( p \)

\[
\int dv \, \frac{\delta H}{\delta p(v)} \frac{\delta C}{\delta f(v)} = 0, \tag{48}
\]

or equivalently, if for any \( f, p \) and \( \alpha \in \mathbb{R} \):

\[
H[f, p] = H[f, p + \alpha \frac{\delta C}{\delta f}], \tag{49}
\]

We will need the expression of the functional derivative of the Hamiltonian \( H \) with respect to its conjugate momentum \( p \) throughout this section. It reads

\[
\frac{\delta H}{\delta p(v)}[f, p] = \frac{1}{4\pi L^3} \sum_k \int d\omega \frac{\delta \mathcal{J}[f, p]}{\delta p(v)} \frac{A(k, \omega, v)}{1 - \mathcal{J}[f, p](k, \omega)}, \tag{50}
\]

with

\[
\frac{\delta \mathcal{J}}{\delta p(v)}[f, p](k, \omega) = -4\pi \int dv_2 \frac{\partial}{\partial v_1} \left\{ A(k, \omega, v_2, v_2) \left[ \frac{\partial f}{\partial v_2}(v_1) - \frac{\partial f}{\partial v_1}(v_2) + 2f(v_2) \left( \frac{\partial p}{\partial v_1} - \frac{\partial p}{\partial v_2} \right) \right] \right\}. \tag{51}
\]

**Mass conservation.** The conservation of the total mass \( M[f] = \int dv \, f \) is immediately visible from equation (49) as \( H \) only depends on the derivative of the conjugate momentum \( p \).

**Momentum conservation.** We define the total momentum \( \mathbf{P}[f] = \int dv \, \mathbf{v} \cdot f \). It follows that \( \frac{\delta \mathbf{P}}{\delta f(v)} = \mathbf{v} \). Using equation (51) and partial integration, the relation

\[
\int dv_1 \frac{\delta \mathcal{J}}{\delta p(v_1)}[f, p](k, \omega) \frac{\delta \mathbf{P}}{\delta f(v_1)} = 0
\]

is a direct consequence of the symmetry \( A(k, \omega, v_1, v_2) = A(k, \omega, v_2, v_1) \). Then, using the relation (50) between the functional derivatives of \( \mathcal{J} \) and \( H \), we obtain

\[
\int dv_1 \frac{\delta H}{\delta p(v_1)}[f, p](k, \omega) \frac{\delta \mathbf{P}}{\delta f(v_1)} = 0.
\]

We have thus checked that the large deviation principle conserves momentum.

The conservation of mass and momentum should have been expected as momentum and mass conservations were already granted from the expression of the Hamiltonian (22), as a direct consequence of mass and momentum conservations for \( f_N \) that can be deduced from either equation (13) or equation (19).
Energy conservation. We define the total kinetic energy $E[f] = \int d\mathbf{v} \frac{\mathbf{v}^2}{2} f$. It follows that $\frac{\delta E}{\delta f} = \mathbf{v}^2 / 2$. Using equation (51) and partial integration, one can check that the relation

$$\int d\mathbf{v} \frac{\delta f}{\delta p} (\mathbf{v}_1) \frac{\delta E}{\delta f} (\mathbf{v}_1) = 0$$

is a direct consequence of the following symmetries of the tensor:

$$A(k, \omega, \mathbf{v}_1, \mathbf{v}_2) = A(k, \omega, \mathbf{v}_2, \mathbf{v}_1).$$

Then, using the relation (50) between the functional derivatives of $\mathcal{J}$ and $H$, we obtain

$$\int d\mathbf{v}_1 \frac{\delta H}{\delta p} (\mathbf{v}_1) \frac{\delta E}{\delta f} (\mathbf{v}_1) = 0.$$ 

From the result (48) we deduce that the large deviation principle conserves the kinetic energy.

The conservation of the kinetic energy is not a trivial consequence of equation (15) or equation (19). Indeed, from equation (15) or equation (19), at any time some energy can be exchanged between the kinetic part $\int d\mathbf{v} \frac{\mathbf{v}^2}{2} f_N$ and the potential part related to $\delta g_N$. However $\int_0^T \int d\mathbf{v} \frac{\mathbf{v}^2}{2} \partial_t f_N$ is equal to the negative of the variations of the potential energy. Then, over any time $T$, these variations should remain bounded, for the system to stay close to the set of homogenous solutions. As a consequence, in accordance with our hypothesis of spatial homogeneity, $\lim_{T \to \infty} \frac{1}{T} \int_0^T \int d\mathbf{v} \frac{\mathbf{v}^2}{2} \partial_t f_N = 0$. This is the reason why we should have expected the conservation of kinetic energy by the large deviation principle. The conservation of kinetic energy by the large deviation principle, which is a conservation for the slow effective dynamics for the empirical density, should thus be interpreted as a conservation for time averages for the fast process. If the system became inhomogeneous, this conservation could be broken.

5.2 Time-reversal symmetry, quasipotential, and entropy

For the Hamiltonian dynamics (5), we consider the microcanonical measure with fixed energy $E$ and momentum fixed and equal to zero, and denote $\mathcal{E}_m$ averages with respect to the microcanonical measure. We expect the stationary probability to observe $f_N = f$, to satisfy a large deviation principle

$$\mathcal{E}_m [\delta (f_N - f)] \geq \exp \{ -NU[f] \},$$

where this large deviation principle defines the quasipotential $U$.

From classical equilibrium statistical mechanics considerations, for this system with long range interactions, it is easy to justify that the quasipotential is

$$U[f] = \begin{cases} \frac{-S[f]}{k_B} + \frac{S_m(E)}{k_B} & \text{if } \int d\mathbf{v} f = 1, \int d\mathbf{v} \mathbf{v} f = 0, \text{ and } \int d\mathbf{v} \frac{\mathbf{v}^2}{2} = E; \\ +\infty & \text{otherwise}, \end{cases}$$

(53)
where
\[ S[f] = -k_B \int df \log f \]
is the entropy of the macrostate \( f \) and
\[ S_m(E) = -k_B \inf_f \left\{ \int df \log f \left| \begin{array}{c}
\int df = 1, \\
\int df f = 0, \\
\int df f^2 = E
\end{array} \right. \right\}. \]
is the equilibrium entropy. We have \( S_m(E) = k_B \left[ 3 \log(E)/2 + 3 \log (4\pi)/2 + 3/2 \right] \).

It is also classically known that the Hamiltonian dynamics (55) is time-reversible: the dynamics is symmetric by the change of variable \((t, r_n, v_n) \to (-t, r_n, -v_n)\). This is equivalent to say that if \( \{r_n(t), v_n(t)\}_{t \in [0,T]} \) is a solution of the Hamiltonian dynamics, then \( \{r_n(T-t), -v_n(T-t)\}_{t \in [0,T]} \) is also a solution. In order to take into account the change of sign for the velocity, we define the linear operator on the set of function of the velocity \( I[f](v) = f(-v) \). We note that \( I \) is an involution: \( I^2 = \text{Id} \).

From the relation (53) between the quasipotential \( U \) and the entropy \( S \), using the conservation law symmetries of the large deviation Hamiltonian (49) we can conclude that the stochastic process for the empirical density \( f_N \) should verify a generalized detailed balance symmetry. This symmetry writes
\[ \mathbf{P}_T (f_0(T) = f_2 | f_0(0) = f_1) \mathbf{P}_m (f_N = f_1) = \mathbf{P}_T (f_0(T) = I[f_2] | f_0(0) = I[f_2]) \mathbf{P}_m (f_N = I[f_2]), \] (54)
where \( \mathbf{P}_m \) is the stationary measure with respect to the microcanonical measure, \( \mathbf{P}_T \) are the transition probabilities for the microcanonical measure. The term “generalized” means that the symmetry holds using the involution \( I \). It is a classical exercise, see for instance section 7.3.2 of [8], to prove that the detailed balance condition (54) implies a detailed balance symmetry at the level of the Hamiltonian: for any \( f \) and \( p \),
\[ H[I[f], -I[p]] = H \left[ f, p + \frac{\delta U}{\delta f} \right]. \] (55)

From the relation (53) between the quasipotential \( U \) and the entropy \( S \), using the conservation law symmetries of the large deviation Hamiltonian (49) we can conclude that the generalized detailed balance symmetry (55) is equivalent to the symmetry: for any \( f \) and \( p \),
\[ H[I[f], -I[p]] = H \left[ f, p - \frac{1}{k_B} \frac{\delta S}{\delta f} \right]. \] (56)

One may directly check this symmetry, from (47), using the time reversal symmetry for \( A \):
\[ A[I[f]](k, -\omega, -v_1, -v_2) = A[f](k, \omega, v_1, v_2). \]
It is however simpler to first note that for spatially homogeneous systems, which is the case in this paper, one has the further symmetry:
\[ H[I[f], I[p]] = H[f, p]. \]
This symmetry can be checked starting from (47) and (43), using
\[ A[I[f]](k, -\omega, -v_1, -v_2) = A[f](k, \omega, v_1, v_2), \]
to conclude that
\[ \mathcal{J} [I[f], I[p]](k, \omega) = \mathcal{J} [f, p](k, -\omega). \]

With this remark, we can conclude that the generalized detailed balance condition is equivalent to: for any \( f \) and \( p \),
\[
H[f, -p] = H \left[ f, p - \frac{1}{k_B} \frac{\delta S}{\delta f} \right].
\]

This last condition is a detailed balance condition at the level of large deviations (see for instance section 7.3.2 of [8]). In order to check directly (57), one can start from (47) and (43), and see that this follows from
\[
\mathcal{J} \left[ f, p - k_B^{-1} \delta S / \delta f \right] = \mathcal{J} \left[ f, -p \right] = 0.
\]

One can see that this last equality is equivalent to the relation: for any \( f \) and \( p \),
\[
L[f, p] = Q \left[ f, p, k_B^{-1} \frac{\delta S}{\delta f} \right],
\]
using (43) and that \( L \) is linear and \( Q \) quadratic with respect to \( p \). Using (44) and (45) and \( \partial / \partial v (\delta S / \delta f) / k_B = 1 / f \partial f / \partial v \), this is easily verified using \( A[f](k, \omega, v_1, v_2) = A[f](k, \omega, v_2, v_1) \).

As a final remark, we note that the quasipotential and the entropy are solutions to the stationary Hamilton-Jacobi equation
\[
H \left[ f, \frac{\delta U}{\delta f} \right] = H \left[ f, -\frac{1}{k_B} \frac{\delta S}{\delta f} \right] = 0.
\]

Those are direct consequences of any of the detailed balance symmetries: (57), (55) or (56).

In this section we have explained that \( U \) (53) is the quasipotential. We have argued that the large deviation Hamiltonian satisfies the generalized detailed balance symmetry (56) as a consequence of the microscopic time reversibility, and checked directly this relation from the explicit Hamiltonian equations. We have moreover justified that that the large deviation Hamiltonian satisfies the detailed balance symmetry (57). This proves that \( U \) satisfies the stationary Hamilton-Jacobi equation.

5.3 A remark on Gaussian approximations of the large deviation principle and cumulant expansions

In a recent paper [23] dedicated to the path large deviations for homogeneous plasma, for fluctuations on scales \( k \lambda_D \gg 1 \) where \( \lambda_D \) is the Debye length and \( k \) a wavenumber, we obtained a large deviation Hamiltonian which is quadratic in \( p \) featuring locally Gaussian distribution of the large deviations. We have obtained this quadratic in \( p \) Hamiltonian, either from the large deviation Hamiltonian associated with the Boltzmann equation, or from an expansion using \( k \lambda_D \gg 1 \) of the expression (22). The technical approach in [23] is different. In [23] we used a tedious cumulant expansion. This paper rather uses the Szegő–Widom theorem, a very efficient approach.

The present paper considers any interaction, and not just the Coulomb interaction case, but it is clear that the result in [23] should be recovered from the results of the present paper. In appendix C we show that the large deviation Hamiltonian of this paper (47), is fully consistent with our previous perturbative result for \( k \lambda_D \gg 1 \). We
also justify an assumption we used in [23] about the cumulant series expansion of (22) from the formula (47) for the large deviation Hamiltonian associated with the Balescu–Guernsey–Lenard equation.

6 Perspectives

The main result of this paper is the derivation of a large deviation principle (2), for the velocity empirical density, for the Hamiltonian dynamics of \( N \) particles which interact through mean-field interactions. We have obtained an explicit formula for the large deviation Hamiltonian (3-4) and we have checked all its symmetry properties. This result opens many mathematical and theoretical questions, as well as interesting applications.

This large deviation result relies on natural assumptions. Some of these assumptions are also required to establish the Balescu–Guernsey–Lenard kinetic equation, but the hypotheses made to obtain the large deviation principle seem stronger. The first assumption is the validity of the quasilinear approximation: we neglected non linear terms of order \( 1/\sqrt{N} \) in the equation for the fluctuations of the empirical density. This amounts to neglecting possible effects of large deviations of the fluctuations, and describing the fluctuation process at a Gaussian level only. The second assumption is the convergence of the process of fluctuations to a stationary Gaussian process and more specifically the convergence of the large time asymptotics for the large deviation estimates over this process. A proof would also require the study of the mixing properties for this Gaussian processes. The mixing properties are critical to justify the Markov behavior described by the slow-fast large deviation principle. While the proofs of these assumptions are beyond the scope of this paper, they open very interesting questions for both theoretical physicists and mathematicians.

Systems with long range interactions are important for many phenomena. However, more elaborate models than the one we used in this paper could be more appropriate to describe physical situations where rare event are important for applications. Of special interest, would be the derivation of large deviation principles for inhomogeneous systems with long range interactions, for instance self-gravitating systems. This is an exciting application, that would open the way to the study of the rare destabilization of globular clusters or galaxies, or the formation of inhomogeneous structures of smaller scales in self gravitating systems. Another very interesting generalization would be for the dynamics of \( N \) point-vortices for two-dimensional hydrodynamics. Another generalization should also consider dynamics of particles driven by stochastic forces, which generically lead to irreversible stochastic processes. For those systems, explicit results for the large deviation theory would be extremely useful for explaining non-equilibrium phase transitions in two dimensional [13] and geostrophic turbulence [12], or in systems with long range interactions [39,38].
A Long time large deviations for quadratic observables of Gaussian processes, functional determinants and the Szegö–Widom theorem for Fredholm determinants

In this appendix, we explain how we can use the Szegö–Widom theorem in order to evaluate the large time asymptotics of Fredholm determinants that appears when computing the cumulant generating function of a quadratic observable of a Gaussian process. We follow the ideas in [13], adapting the discussion for the case of Gaussian processes with complex variables.

Let $Y_t$ be a stationary $\mathbb{C}^n$-valued Gaussian process with correlation matrix $C(t) = \mathbb{E}(Y_t \otimes Y_t^\ast)$ and with a zero relation matrix $R(t) = \mathbb{E}(Y_t \otimes Y_t^\ast) = 0$, let $M \in \mathcal{M}_n(\mathbb{C})$ be a $n \times n$ Hermitian matrix. The aim of this appendix is to prove that

$$\log \mathbb{E} \exp \left( \int_0^T dt Y_t^\ast M Y_t \right) \sim \frac{T}{2\pi i} \int d\omega \log \det \left( I_n - M \tilde{C}(\omega) \right),$$

where $\tilde{C}(\omega) = \int_\mathbb{R} e^{i\omega t} C(t) dt$ is the Fourier transform of the correlation matrix $C(t)$ and $I_n$ is the $n \times n$ identity matrix. We note that the determinant of the r.h.s. of (58) is a real number. Indeed, as $Y_t$ is a stationary process, $\tilde{C}(\omega)$ and $M$ are Hermitian matrices, then the determinant is the determinant of a Hermitian operator and is a real number.

For pedagogical reasons, in this appendix the result [55] is stated for a process $Y_t$ that takes values in a finite-dimensional space. However with adapted hypotheses, this result can be generalized when $Y_t$ is a stationary $\mathcal{H}$-valued Gaussian process, where $\mathcal{H}$ is a Hilbert space, and where $M$ is a Hermitian operator on $\mathcal{H}$.

In section A.1, we state the Szegö–Widom theorem. In section A.2, we explain that the left hand side of (58) is the log of the determinant of a Gaussian integral, that this quantity can be expressed as a functional determinant for linear operators on $L^2([0,T], \mathbb{C}^n)$, and that Szegö–Widom theorem reduces it to the computation of frequency integrals of determinants of operators on the space $\mathbb{C}^n$, as expressed by [55].

A.1 The Szegö–Widom theorem

We first define integral operators on $L^2([0,T], \mathbb{C}^n)$. We considers maps $\varphi : [0,T] \rightarrow \mathbb{C}^n$ and $K : \mathbb{R} \rightarrow \mathcal{M}_n(\mathbb{C})$, where $\mathcal{M}_n(\mathbb{C})$ is the set of $n \times n$ complex matrices. We define the integral operator $K_T$ by

$$K_T \varphi(t) = \int_0^T K(t-s) \varphi(s) ds,$$

$K_T$ is a linear operator of $L^2([0,T], \mathbb{C}^n)$. $K$ is called the kernel of the operator $K_T$.

The Szegö–Widom theorem allows to compute large $T$ asymptotics of the logarithm of the Fredholm determinant of the integral operator $\text{Id} + K_T$. The result is

$$\log \det \left( \text{Id} + K_T \right) \sim \frac{T}{2\pi i} \int d\omega \log \det \left( I_n + \int_\mathbb{R} e^{i\omega t} K(t) dt \right),$$

where $I_n$ is the $n \times n$ identity matrix. Whereas the determinant on the l.h.s. of this expression, denoted by the subscript $[0,T]$ is a Fredholm determinant, the determinant on the r.h.s. is a matrix determinant which can be more easily computed. Further details about this theorem and its possible applications can be found in [13].

A.2 Expectation of functionals of Gaussian processes

Let $Y_t$ be a $\mathbb{C}^n$-valued stationary Gaussian process with correlation matrix $C(t) = \mathbb{E}(Y_t \otimes Y_t^\ast)$,
In this appendix, we compute the determinant of the operator

\[ B \]

by

\[ u \]

which is the result (58). In these expressions, the determinant to be computed on the r.h.s. is the determinant

\[ (\text{59}). \]

We assume that for all times \( t \), \( M(t) \) is a \( n \times n \) Hermitian matrix. As \( Y_t \) is a Gaussian process we can compute the expectation as a Gaussian integral. It is straightforward to check that

\[ \text{det} (\text{Id} - (MC)_T)^{-1}, \]

where \((MC)_T\) is the integral operator whose kernel is \((M \ast C)(t)\) the convolution product on \([0, T]\) of the kernels \(M(t)\) and \(C(t)\).

Then, we can deduce the following expression for \( u \)

\[ \mathcal{U}(T) = -\log \text{det} (\text{Id} - (MC)_T), \]

where the determinant is the Fredholm determinant of the integral operator \(\text{Id} - (MC)_T\). Generally, it is not obvious how to compute this kind of Fredholm determinant. Fortunately, we can use the Szegő–Widom theorem to obtain an expression for large \( T \) asymptotics as a finite-dimensional determinant. Using the result [60] from section [A.1] we get

\[ \mathcal{U}(T) \sim -\frac{T}{2\pi} \int \omega \log \text{det} \left( I_n - \frac{1}{2\pi} \int_0^T \text{e}^{-i\omega t} (M \ast C)(t) \, dt \right). \]

In the special case when \( M_T \) is a diagonal integral operator, i.e. when its kernel is \( M(t) = M\delta(t) \), we can write

\[ \mathcal{U}(T) \sim -\frac{T}{2\pi} \int \omega \log \text{det} \left( I_n - M \frac{1}{2\pi} \int_0^T \text{e}^{-i\omega t} C(t) \, dt \right), \]

which is the result [55]. In these expressions, the determinant to be computed on the r.h.s. is the determinant of a \( n \times n \) matrix.

B Computation of the determinant of the operator \( u_{k,\omega} \)

In this appendix, we compute the determinant of the operator \( u_{k,\omega} \), encountered in section [4.2] and defined by

\[ u_{k,\omega} \]

for any \( \varphi \in \mathcal{H}_c, \mathcal{H}_c \) being the Hilbert space of complex functions over the velocity space. Using equation [40] we can simplify this expression

\[ u_{k,\omega} \]

We note that the operator \( u_{k,\omega} \) has the form

\[ u_{k,\omega} : \varphi \mapsto \varphi - (w, Q\varphi) v - (v, Q\varphi) w, \]

where \( Q \) is a Hermitian operator over \( \mathcal{H}_c \), \( w \) and \( v \) are complex functions over the velocity space, and \( (\cdot, \cdot) \) denotes the Hermitian product: \( (a, b) = \int dv a^* (v) b(v) \). The connection is made between formulas [61] and [62] by setting \( v(v) = -i k, w(v) = \tilde{W}(k) \) and \( Q\varphi(v) = \int dv \tilde{G}_{GG}(k, \omega, v_2, v_3) \varphi(v_3) \).

Using
where $B = \{ L, H \}$.

The second equality defines (28), we see that $Q$ is a Hermitian operator. We note that, whenever $\frac{\partial}{\partial w}$ is not a constant in the velocity space, $v$ and $w$ are linearly independent.

Formula (26) shows that $n_{\alpha, \omega} - Id$ is a rank two linear operator. Then $\det_{\alpha, \omega} n_{\alpha, \omega}$ is the determinant of the operator $n_{\alpha, \omega}$ restricted to span $(v, w)$:

\[
\det_{\alpha, \omega} n_{\alpha, \omega} = \left[ 1 - \langle w, Qv \rangle - \langle w, Qw \rangle \right]
- \langle v, Qv \rangle - \langle v, Qw \rangle.
\]

Then

\[
\det_{\alpha, \omega} n_{\alpha, \omega} = 1 - 2\Re \left( \langle v, Qw \rangle + \langle v, Qw \rangle^* - \langle w, Qv \rangle \langle v, Qv \rangle \right).
\]

where we have used $\langle w, Qv \rangle = \langle Qw, v \rangle = \langle v, Qw \rangle^*$, as $Q$ is an Hermitian operator.

We can explicitly compute the determinant of (61). We have

\[
\langle v, Qv \rangle = \int dv_1 dv_2 k \frac{\partial}{\partial v_1} k \frac{\partial}{\partial v_2} \tilde{\psi}_{\psi V} (k, w, v_1, v_2),
\]

\[
\langle v, Qw \rangle = i \int dv_1 k \frac{\partial}{\partial v_1} \tilde{\psi}_{\psi \psi V} (k, w, v_1)^*,
\]

and

\[
\langle w, Qv \rangle = \tilde{\psi}_{\psi V} (k, w),
\]

where $\tilde{\psi}_{\psi V}, \tilde{\psi}_{\psi V}$ and $\tilde{\psi}_{\psi V}$ are the two-point correlations functions of the quasi-linear problem computed in section 1.2 and we have used (29) 90.

We conclude that

\[
\det_{\alpha, \omega} (n_{\alpha, \omega}) = 1 + 2 \int dv_1 dv_2 k \frac{\partial}{\partial v_1} \left[ \tilde{\psi}_{\psi V} (k, w, v_1) \right]
+ \int dv_1 dv_2 k \frac{\partial}{\partial v_1} k \frac{\partial}{\partial v_2} \left\{ \tilde{\psi}_{\psi \psi V} (k, w, v_1) \tilde{\psi}_{\psi \psi V} (k, w, v_2)^* - \tilde{\psi}_{\psi V} (k, w) \tilde{\psi}_{\psi V} (k, w, v_1, v_2) \right\}.
\]

C Consistency and validation of the cumulant series expansion

In this appendix, we expand $H$ from the formula (47) in powers of $p$. This amounts at a cumulant expansion for the statistics of the fluctuations. We use this expansion to prove a conjecture made in the paper (23), and to recover the Gaussian (order two) truncation we computed in (23).

We expand the logarithm in formula (47) to obtain

\[
H [f, p] = \frac{1}{4dL^3} \sum_k \int d\omega \sum_{n=1}^{\infty} \frac{1}{n} \left( J [f, p] (k, \omega) \right)^n = \sum_{n=1}^{\infty} H^{(n)} [f, p].
\]

The second equality defines $H^{(n)}$ as being the terms homogeneous of order $n$ in $p$ in this expansion. It is the $n$-th cumulant.

We also define $B^{(n)}$ as

\[
B^{(n)} (v_1, \ldots, v_{2n}) \left( \frac{2\pi}{4\pi dL^3} \sum_k \int d\omega \frac{W (k) \omega^{2n}}{|v | \omega | v |} \right) \sum_{k = 1}^{2n} \delta (w - k v).
\]

$B^{(n)}$ is a rank $2n$ tensor. $l^{(k)}$ and $q^{(k)}$ are defined by the relations. We have $J [f, p] = \mathcal{L} [f, p] + Q [f, p, p]$, where $\mathcal{L}$ and $Q$ are defined in equations (44) and (45). We will need to compute (\mathcal{L} [f, p] \delta, which is $\mathcal{L} [f, p]$ to the power $k$. We define $l^{(k)}$ and $q^{(k)}$ by

\[
(l^{(k)} [f, p])^k = \int dv_1 \cdots dv_{2k} \mathcal{L} [f, p] \prod_{j=1}^{k} A (k, w, v_{2j-1}, v_{2j}),
\]
Expansion thus (64) justifies this conjecture. The conservative nature of the dynamics is visible because the large deviation Hamiltonian \( H \) does depend on the conjugate momentum \( p \) only through its gradient \( \partial H / \partial v \). We define \( \bar{H} \) as \( H [f, \partial p / \partial v] = H [f, p] \). We start from the definition of the large deviation Lagrangian

\[
L [f, \bar{f}] = \text{Sup}_p \left\{ \int \text{d}v \left( f - H [f, p] \right) \right\}.
\]

and

\[
(Q [f, p, p])^k = \int \text{d}v_1 \cdots \text{d}v_{2k} \, q^{(k)} [f, p, p] \prod_{j=1}^k A \left( k, \omega, v_{2j-1}, v_{2j} \right).
\]

\( l^{(1)} \) and \( q^{(k)} \) are both tensors of order \( 2k \). \( l^{(1)} \) depends on \( p \) as a homogeneous function of order \( k \), \( q^{(k)} \) depends on \( p \) as a homogeneous function of order \( 2k \).

In the expansion of \( (J [f, p])^n \) using \( J [f, p] = J [f, p] + Q [f, p, p] \), we see that for all \( m \in [n/2, n] \cap \mathbb{N} \), \( 2m - n \, [f, p]^{m-n} [f, p] \) is homogeneous of order \( n \) in \( p \). Using this remark, from equation (62) we obtain

\[
H^{(n)} [f, p] = \sum_{m=\lfloor n/2 \rfloor}^{n} \int \text{d}v_1 \cdots \text{d}v_{2m} \left( \frac{m}{2m-n} \right) \frac{1}{(2\pi)^{2m-n}} B^{(m)} (v_1, \ldots, v_{2m}) ; f^{(2m-n)} [f, p] q^{(m-n)} [f, p],
\]

(64)

where the symbol " : " means a contraction of two tensors of order \( 2m \) with another tensor of order \( 2m \).

This result ensures that as soon as \( n \geq 2 \), \( H^{(n)} \) only includes terms proportional to the tensors \( B^{(m)} \) with \( m \geq n/2 \geq 2 \). In (63), we used a conjecture on \( H^{(n)} \) to justify the Gaussian truncation of the cumulant series expansion of (62). The conjecture was that only the two first cumulants \( H^{(1)} \) and \( H^{(2)} \) do involve the tensor \( B = B^{(1)} \), whereas all the other cumulants \( H^{(n)} \) for \( n > 2 \) only involve tensors \( B^{(m)} \) with \( m \geq 2 \). Expansion thus (64) justifies this conjecture.

In (63), from a cumulant series expansion, we obtained that

\[
H [f, p] = H_{\text{quad}} [f, p] + O (p^3),
\]

where

\[
H_{\text{quad}} [f, p] = \int \text{d}v_1 \text{d}v_2 B (v_1, v_2) : \frac{\partial f}{\partial v_1} \frac{\partial f}{\partial v_2} (v_1 - f (v_2)) \frac{\partial f}{\partial v_1}
\]

\[
+ \int \text{d}v_1 \text{d}v_2 B (v_1, v_2) : \frac{\partial f}{\partial v_1} \frac{\partial f}{\partial v_2} (v_1 - f (v_2)) \left( \frac{\partial f}{\partial v_1} \frac{\partial f}{\partial v_2} \right) f (v_1) f (v_2)
\]

\[
+ \int \text{d}v_1 \text{d}v_2 \text{d}v_3 \text{d}v_4 B^{(2)} (v_1, v_2, v_3, v_4) : \frac{\partial f}{\partial v_1} \frac{\partial f}{\partial v_2} \left( f (v_1) f (v_2) \right) \frac{\partial f}{\partial v_3} \frac{\partial f}{\partial v_4}
\]

\[
- 2 f (v_1) \frac{\partial f}{\partial v_2} f (v_3) \frac{\partial f}{\partial v_4} + \frac{\partial f}{\partial v_3} \frac{\partial f}{\partial v_4} f (v_1) f (v_2),
\]

(65)

and where \( O (p^3) \) designates terms that are of order more than two in the conjugate momentum \( p \), and the symbol " : " means a contraction of two tensors of order 2 and 4, for the two first lines and the third line, respectively. We see that \( H^{(1)} + H^{(2)} = H_{\text{quad}} \). We conclude that (47) is consistent with the quadratic approximation of the large deviation Hamiltonian we obtained in (63).

### D Current formulation of the large deviation principle

Because the particle number is conserved, it is clear that the dynamics of the empirical density has a conservative form \( \frac{\partial \rho}{\partial t} + \frac{1}{N} \sum_{i=1}^{N} \mathbf{f} (\mathbf{r}_i) = 0 \). For the microscopic dynamics (before time averaging), this is a consequence of equations (15) or (19) with

\[
J_V (v, t) = - \frac{1}{NL^3} \int \text{d}x \left( \frac{\partial \rho}{\partial x} \delta V (\delta \rho) \right).
\]

After time averaging, we could have obtained the path large deviations by studying the large deviations of the time averaged current. Alternatively, we can rephrase our large deviation principle as a large deviation principle for the current, through a change of variable. This is the subject of this appendix.

The conservative nature of the dynamics is visible because the large deviation Hamiltonian \( H \) does depend on the conjugate momentum \( p \) only through its gradient \( \partial H / \partial v \). We define \( \bar{H} \) as \( H [f, \partial p / \partial v] = H [f, p] \). We start from the definition of the large deviation Lagrangian

\[
L [f, \bar{f}] = \text{Sup}_p \left\{ \int \text{d}v \left( f - H [f, p] \right) \right\}.
\]
Writing \( f \) as the divergence of a current \( f + \partial / \partial v \cdot \mathbf{j} = 0 \), we have

\[
L[f, j] = \sup_{\{j / H \to 0\}} \sup_p \left\{ - \int dv \, p \frac{\partial}{\partial v} j - H[f, p] \right\}.
\]

Using \( H[f, p] = \tilde{A}[f, \partial p / \partial v] \), and integrating by part, we have

\[
L[f, j] = \sup_{\{j / H \to 0\}} \left\{ \int dv \cdot \mathbf{E} - \tilde{A}[f, \mathbf{E}] \right\}.
\]

where \( \mathbf{E} \) designates the conjugate quantity of the current \( j \).

We thus have the large deviation principle

\[
P\left( \{ f(\tau) \}_{0 \leq \tau \leq T} = \{ f(\tau) \}_{0 \leq \tau \leq T} \right) \approx e^{-\frac{1}{T^2} \sup_{\{j / H \to 0\}} \int_0^T dt L[f, j]} e^{-N_0 f[j(0)]},
\]

with

\[
\mathcal{A}[f, j] = \begin{cases} \int_0^T \frac{\partial}{\partial \tau} L[f, j] & \text{if } j / H \to 0, \\
\frac{1}{\infty} & \text{otherwise.}
\end{cases}
\]

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