ON THE EXISTENCE AND UNIQUENESS OF SOLUTION TO A STOCHASTIC SIMPLIFIED LIQUID CRYSTAL MODEL

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Abstract. We study in this article a stochastic version of a 2D simplified Ericksen-Leslie systems, which model the dynamic of nematic liquid crystals under the influence of stochastic external forces. We prove the existence and uniqueness of strong solution. The proof relies on a new formulation of the model proposed in [19] as well as a Galerkin approximation.

1. Introduction. Stochastic partial differential equations (SPDE) are used to model physical systems subjected to influence of internal, external or environmental noises or to describe systems that are too complex to be described deterministically, e.g. a flow of a chemical substance in a river subjected by wind and rain, an airflow around an airplane wing perturbed by the random state of the atmosphere and weather, a laser beam subjected to turbulent movement of the atmosphere, spread of an epidemic in some regions and the spatial spread of infectious diseases. SPDEs are also used in the physical sciences (e.g. in plasmas turbulence, physics of growth phenomena such as molecular beam epitaxy and fluid flow in porous media with applications to the production of semiconductors and to the oil industry) and biology (e.g. bacteria growth and DNA structure). Models related to the so called passive scalar equations have potential applications to the understanding of waste (e.g. nuclear) convection under the earths surface, [5, 3, 39, 40].

The presence of noise can lead to new and important phenomena. For example, the 2-dimensional Navier-Stokes equations with sufficiently degenerate noise have a unique invariant measure and hence exhibit ergodic behavior in the sense that the time average of a solution is equal to the average over all possible initial data. Despite continuous efforts in the last thirty years, such a property has so far not been found for the deterministic counterpart of these equations. This property could lead to profound understanding of the nature of turbulence. The aforementioned Navier-Stokes Equations (NSE) are now a widely accepted model of fluid motion, see for instance the well known monograph [52, 51]. The theory of NSE is reasonable well understood. For instance, in the case of 2-dimensional domains, it is known since the pioneering works of Lions and Prodi in the 1960s (see for instance [37]) that the solutions exist for all times and are unique. In the 3-dimensional case it is known

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that the weak solutions exist for all times, see celebrated work of Leray [26], and that the strong solutions are unique. However, despite many efforts in the recent years the questions whether the weak solutions are unique or strong solutions exist for all times, remain unresolved, see for instance [53]. To our best knowledge, the first work on the stochastic NSE (SNSE) written from the mathematical point of view is a paper [1]. Later the motivation for the large deviations paper of Faris and Jona-Lasinio [15] was clearly the stochastic fluid dynamics as they wrote, roughly speaking, the motion of a viscous incompressible fluid is described by the Navier-Stokes equations. However, these equations are only approximate. In particular, they take into account only the macroscopic nature of the fluid motion.

Liquid crystal is often viewed as the fourth state of matter besides gas, liquid and solid, or as an intermediate state between liquids and solids. It possesses either no or partial positional order but displays an orientational order at the same time. The nematic phase is the simplest among all liquid crystal phases and is close to the liquid phase. The molecules float around as in a liquid phase, but have the tendency of aligning along a preferred direction due to their orientation. The hydrodynamic theory of liquid crystals due to Ericken and Leslie was developed around the 1960s, [12, 13, 27, 28]. Their theory, now referred to as the Ericksen-Leslie (EL) dynamic theory, is one of the most successful theories used to model many dynamic phenomena in nematic liquid crystals, [56, 19].

As recalled in [20], the mathematical studies on the dynamical liquid crystal systems started with the work of [33, 35, 32, 34], where the authors established the global existence of weak solutions, in both 2D and 3D, to the Ginzburg-Landau approximation of the liquid crystal system, see [8, 50] for some generalizations to the general liquid crystal systems. Global existence of weak solutions to the original liquid crystal systems in 2D, without the Ginzburg-Landau approximation, was established in [36, 21, 23, 25, 54, 20]. In particular, it was shown that global weak solutions to liquid crystal system in 2D have at most finite many singular times, while the uniqueness of weak solutions to liquid crystal system in 2D was proved in [30, 29, 55, 59]; global existence (but without uniqueness) of weak solutions to the liquid crystal system in 3D was recently established in [31], under the assumption that the initial director filed takes value from the upper half unit sphere. If the initial data are suitably smooth, then the liquid crystal system has a unique local strong solution, see [22, 54, 56, 57, 25]. Moreover, if the initial data is suitably small, or the initial director filed satisfies some geometrical condition in 2D, then the local strong solution to the liquid crystal system can be extended to be a global one, see [30, 19]. It is worth to mention that some mathematical analysis concerning the global existence of weak solutions and local or global well-posedness of strong solutions of the non-isothermal liquid crystal systems were addressed in [17, 16].

We borrow from [6] the following motivations for the problem studied this article. Most of the physical systems confront dynamical instabilities. The instability befalls at some critical value of the control parameter (which is in our case some random external noise) of the system. In our predicament the dynamics are quite intricate because the evolution of the director field $\mathbf{d}$ is coupled to the velocity field $\mathbf{v}$. In [42], the author has studied the stationary orientational correlations of the director field of a nematic liquid crystal near the Fréedericksz transition. In this transition the molecules tend to reorient due to some random external perturbations. It has been studied by [49] that the decay time, required for the system is shortened by the field fluctuations to leave an unstable state, which is built by switching on
the field to a value beyond instability point. See also [24] and references there in, for more details. A nematic drifts very much like a typical organic liquid with molecules of indistinguishable size. Since, the transitional motions are coupled to inner, orientational motions of the molecules, in most cases the flow muddles the alignment. Conversely, by implementation of an external field, a change in the alignment will generate a flow in the nematic. This is an important motivation for studying the flows of nematic liquid crystals, effected by altering external forces.

The aim of this article is to study a class of stochastic EL models. Our model include an abstract and general form of random external forces depending eventually on the velocity $v$ of the fluid. We prove the existence of a strong solution in a two-dimensional bounded domain. The proof of the existence of solution is based on a Galerkin scheme similar to that of [2, 7] in the case of the 2D Navier-Stokes and the 3D Lagrange averaged Navier-Stokes equations. In [4], the authors proved several results (including the existence and uniqueness of weak solutions) of a simplified stochastic EL model with Ginzburg-Landau approximation. The main difference of our work and that of [4] is that the model considered in [4] is a simplified EL model with a Ginzburg-Landau type approximation in which the term $|\nabla d|^2 d$ is replaced by a polynomial $f(d)$ that satisfies some reasonable growth assumptions. There is a large amount of literature on the existence and uniqueness solutions for stochastic partial differential equations. We refer the reader to [47, 11, 60, 9, 10, 3, 58, 61, 43, 45, 44]. However, the existing results in the literature do not cover the situation considered in this paper.

The article is organized as follows. In the next section we present the simplified EL model, the reformulation and its mathematical setting. The main result appear in Section 3.

2. A stochastic simplified Ericksen-Leslie model and its mathematical setting.

2.1. Governing equations. In this article, we consider a stochastic version of a simplified Ericksen-Leslie model in a two-dimensional domain. More precisely, we assume that the domain $\mathcal{M}$ of the fluid is a bounded domain in $\mathbb{R}^2$. Then, we consider the system

$$
\begin{align*}
\frac{\partial v}{\partial t} - \nu_1 \Delta v + (v \cdot \nabla)v + \nabla p &= -\lambda \nabla \cdot [\nabla d \otimes \nabla d] + g_1(t, v) + g_2(t, v) \dot{W}_t, \\
\text{div } v &= 0, \\
\frac{\partial d}{\partial t} + v \cdot \nabla d &= \nu_2 (\Delta d + |\nabla d|^2 d), \quad |d| = 1,
\end{align*}
$$

in $\mathcal{M} \times [0, T]$. In (2.1), the unknown functions are the velocity $v = (v_1, v_2)^T$ of the fluid, $d = (d^1, d^2)^T$ is the director field, which stands for the averaged macroscopic/continuum molecular orientation in $\mathbb{R}^2$. Finally, $p$ denotes the pressure of the fluid. The positive constants $\nu_1, \lambda, \nu_2$ are the viscosity of the fluid, the competition between the kinetic and the potential energy and the microscopic elastic relaxation time, respectively. Note that $\Lambda = \frac{1}{2}([\nabla v + (\nabla v)^T])$ denotes the rate of strain tensor. The symbol $\otimes$ is the usual Kronecker product, e.g. $(a \otimes b)_{ij} = a^i b^j$ for $a, b \in \mathbb{R}^2$. The notation $\nabla d \otimes \nabla d$ denotes the $2 \times 2$ matrix, whose $(i, j)^{th}$ entry is given by $\partial_i d \cdot \partial_j d$. The external volume force $g_1(t, v)$ is given. The term $g_2(t, v) \dot{W}_t$ represents random external forces depending eventually on $v$, where $\dot{W}_t$ denotes the time derivative of a cylindrical
Wiener process. Equation (2.1)\textsubscript{1} and (2.1)\textsubscript{3} represent the conservation of the linear momentum and angular momentum respectively.

We endow (2.1) with the boundary condition
\begin{equation}
\begin{aligned}
    v = 0, \quad d = d^0 \text{ on } \partial M, \quad |d^0| = 1,
\end{aligned}
\end{equation}
where \( \partial M \) is the boundary of \( M \) and \( \eta \) is its outward normal.

The initial condition is given by
\begin{equation}
    (v, d)(0) = (v^0, d^0) \text{ in } M.
\end{equation}

Let us recall that the two main difficulties associated with the mathematical analysis of the EL model (2.1) are related to the presence of the quadratically increasing term \( |\nabla d|^2 d \) and the nonlinear term \( d \otimes (\Delta d + |\nabla d|^2 d) \), which bring extra technical difficulties to proving the existence of solutions, compared to the model considered in [4].

2.2. The deterministic case and its reformulation. In this part, we recall from [19, 14] a reformulation of the deterministic simplified EL model, in which the constraints \( |d| = 1 \) is automatically satisfied. We start with the following simplified deterministic EL model
\begin{equation}
\begin{aligned}
    v_t - \nu_1 \Delta v + (v \cdot \nabla)v + \nabla p &= -\lambda \nabla \cdot [\nabla \theta \otimes \nabla \theta], \\
    \text{div } v &= 0, \\
    d_t + v \cdot \nabla d &= \nu_2 (\Delta d + |\nabla d|^2 d), \quad |d| = 1.
\end{aligned}
\end{equation}
Let us set \( d = (\cos \theta, \sin \theta)^T \). It follows that (see [19] for details)
\begin{equation}
\begin{aligned}
\Delta d + |\nabla d|^2 d &= \Delta \theta \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}, \\
\nabla d \otimes \nabla d &= \nabla \theta \otimes \nabla \theta.
\end{aligned}
\end{equation}
Thus we derive the following reformulation to (2.4) (see [19, 14] for details)
\begin{equation}
\begin{aligned}
    v_t - \nu_1 \Delta v + (v \cdot \nabla)v + \nabla p &= -\lambda \nabla \cdot [\nabla \theta \otimes \nabla \theta], \\
    \text{div } v &= 0, \\
    \theta_t + v \cdot \nabla \theta - \nu_2 \Delta \theta &= 0.
\end{aligned}
\end{equation}
Note that the constraints \( |d| = 1 \) is automatically satisfied since \( d = (\cos \theta, \sin \theta)^T \).

The equivalence between the systems (2.4) and (2.6) is discussed in [14], where the authors studied the large-time behavior of (2.4) under some regularity conditions (referred to as a trigonometric condition) on the initial direction field. To simplify the notations, hereafter we set \( \nu_1 = \lambda = \alpha = 1 \).

Using similar idea as in (2.4), we rewrite (2.1) in the following form:
\begin{equation}
\begin{aligned}
    v_t - \nu_1 \Delta v + (v \cdot \nabla)v + \nabla p &= -\lambda \nabla \cdot [\nabla \theta \otimes \nabla \theta] + g_1(t, v) + g_2(t, v)\dot{W}_t, \\
    \text{div } v &= 0, \\
    \theta_t + v \cdot \nabla \theta - \nu_2 \Delta \theta &= 0.
\end{aligned}
\end{equation}
For simplicity, we associate to (2.7) the following initial and boundary conditions
\begin{equation}
    (v, \theta) = (0, 0) \text{ on } \partial M, \quad (v, \theta)(0) = (v^0, \theta^0) \text{ in } M.
\end{equation}

Remark 1. The boundary condition
\begin{equation}
    \frac{\partial d}{\partial \eta} = 0 \text{ on } \partial M,
\end{equation}

where \( \eta \) is the outward normal to \( \partial \mathcal{M} \), is often used for \( \mathbf{d} \) (see [4]). If we set \( \mathbf{d} = (\cos \theta, \sin \theta)^T \), it follows from (2.9) that

\[
(- \sin \theta, \cos \theta)^T \frac{\partial \theta}{\partial \eta} = 0 \text{ on } \partial \mathcal{M}. \tag{2.10}
\]

Taking the scalar product in \( \mathbb{R}^2 \) of (2.10) with \((- \sin \theta, \cos \theta)^T \) gives

\[
\frac{\partial \theta}{\partial \eta} = 0 \text{ on } \partial \mathcal{M}. \tag{2.11}
\]

The results presented in this paper are also valid if we used (2.11) for the boundary condition for \( \theta \) instead.

2.3. Mathematical setting. We first introduce a weak formulation of (2.7)-(2.8). Hereafter, we assume that the domain \( \mathcal{M} \) is bounded with a smooth boundary \( \partial \mathcal{M} \) (e.g., of class \( C^2 \)).

Hereafter, if \( X \) is a real Hilbert space with inner product \( \langle \cdot, \cdot \rangle_X \), we will denote the induced norm by \( |\cdot|_X \), while \( X^* \) will indicate its dual. If \( X \) is a Banach space, we will denote by \( X^* \) the dual space of \( X \). To simplify the notations, the duality paring between \( X \) and \( X^* \) will be denoted \( \langle \cdot, \cdot \rangle \) and the norm in \( X^* \) will be denoted \( \|\cdot\|_s \).

We set

\[
V_1 = \{ u \in C_c^\infty(\mathcal{M}) : \text{div } u = 0 \text{ in } \mathcal{M} \}.
\]

We denote by \( H_1 \) and \( V_1 \) the closure of \( V_1 \) in \((L^2(\mathcal{M}))^2 \) and \((H^1_0(\mathcal{M}))^2 \) respectively. The scalar product in \( H_1 \) is denoted by \( \langle \cdot, \cdot \rangle_{L^2} \) and the associated norm by \( |\cdot|_{L^2} \).

Moreover, the space \( V_1 \) is endowed with the scalar product

\[
\langle (u, v) \rangle = \sum_{i=1}^{2} (\partial_{x_i} u, \partial_{x_i} v)_{L^2}, \quad \|u\| = ((u, u))^{1/2}.
\]

We now define the operator \( A_0 \) by

\[
A_0 u = -\mathcal{P} \Delta u, \quad \forall u \in D(A_0) = (H^2(\mathcal{M}))^2 \cap V_1,
\]

where \( \mathcal{P} \) is the Leray-Helmholtz projector in \( L^2(\mathcal{M}) \) onto \( H_1 \). Then, \( A_0 \) is a self-adjoint positive unbounded operator in \( H_1 \) which is associated with the scalar product defined above. Furthermore, \( A_0^{-1} \) is a compact linear operator on \( H_1 \) and \( |A_0 \cdot|_{L^2} \) is a norm on \( D(A_0) \) that is equivalent to the \( H^2 \)-norm.

We denote by \( A_1 \) the Dirichlet Laplacian on \( \mathbb{R}^2 \), that is

\[
D(A_1) = \{ \theta \in H^2(\mathcal{M}), \quad \theta = 0 \text{ on } \partial \mathcal{M} \}, \quad A_1 \theta = -\Delta \theta, \quad \forall \theta \in D(A_1).
\]

Recall that

\[
\langle A_1 \theta_1, \theta_2 \rangle = \int_{\mathcal{M}} \nabla \theta_1 \nabla \theta_2 dx, \quad \forall \theta_1, \theta_2 \in H^1_0(\mathcal{M}).
\]

Now we define the Hilbert spaces \( \mathcal{H} \) and \( \mathcal{U} \) by

\[
\mathcal{H} = H_1 \times H^1_0(\mathcal{M}), \quad \mathcal{U} = V_1 \times D(A_1), \tag{2.12}
\]

endowed with the scalar products whose associated norms are respectively

\[
|\langle v, \theta \rangle|^2_\mathcal{H} = |v|^2_{L^2} + |\nabla \theta|^2_{L^2} \quad \text{and} \quad |\langle v, \theta \rangle|^2_\mathcal{U} = \|v\|^2 + |A_1 \theta|^2_{L^2}. \tag{2.13}
\]

Hereafter, we set

\[
H_2 = L^2(\mathcal{M}), \quad V_2 = H^1_0(\mathcal{M}). \tag{2.14}
\]
We introduce the bilinear operators $B_0$ and $B_1$ (and their associated trilinear forms $b_0, b_1$) defined from $V_1 \times V_1$ into $V_1^*$ and $V_1 \times D(A_1)$ into $L^2(M)$ respectively by:

\[
(B_0(u,v),w) = \int_M [(u \cdot \nabla)v] \cdot w \, dx = b_0(u,v,w), \quad \forall u, v, w \in V_1,
\]

\[
(B_1(u,\theta,\rho) = \int_M [(u \cdot \nabla)\theta] \rho \, dx = b_1(u,\theta,\rho), \quad \forall u \in V_1, \quad \theta, \rho \in D(A_1).
\]

(2.15)

We recall that $B_0$ and $B_1$ satisfy the following estimates

\[
\begin{align*}
|B_0(u,v)|_{V_1^*} &\leq c\|u\|_{L^2}^{1/2}\|u\|_{L^2}^{1/2}\|v\|_{L^2}^{1/2}, \quad \forall u, v \in V_1, \\
|B_1(u,\theta)|_{V_1^2} &\leq c\|u\|_{L^2}^{1/2}\|\theta\|_{L^2}^{1/2}\|\theta\|_{L^2}^{1/2}, \quad \forall u \in V_1, \theta \in V_2.
\end{align*}
\]

(2.16)

We introduce the trilinear form $r_0$ defined by:

\[
r_0(\theta,\psi,v) = -\sum_{i,j}^2 \int_M \partial\theta \partial\psi \partial v_i \partial x_j \partial x_i \, dx, \quad \forall \theta, \psi \in W_0^{1,4}, \quad v \in V_1.
\]

(2.17)

**Proposition 1.** There exists a constant $c > 0$ such that

\[
|r_0(\theta,\psi,v)| \leq c\|\theta\|_{L^4}\|\nabla\psi\|_{L^4}\|\nabla v\|_{L^4} \leq c\|\theta\|_{L^4}\|\nabla\psi\|_{L^4}\|\nabla v\|_{L^4}, \quad \forall \theta, \psi, v \in D(A_1), \quad v \in V_1. \tag{2.18}
\]

**Proof.** The proof is given in [4, 6]. For the reader convenience, we repeat it. From (2.17), we easily derive that for any $\theta, \psi \in D(A_1), \quad v \in V_1$, we have

\[
|r_0(\theta,\psi,v)| \leq c\|\nabla\theta\|_{L^4}\|\nabla\psi\|_{L^4}\|\nabla v\|_{L^4} \leq c\|\theta\|_{L^4}\|\nabla\psi\|_{L^4}\|\nabla v\|_{L^4}, \tag{2.19}
\]

and (2.18) is proved. \qed

**Proposition 2.** There exists a bilinear operator $R_0$ defined on $D(A_1)$ with values in $V_1^*$ such that

\[
\langle R_0(\theta,\psi), v \rangle = r_0(\theta,\psi,v), \quad \forall \theta, \psi \in D(A_1), \quad v \in V_1.
\]

Moreover, there exists a constant $c > 0$ such that

\[
\|R_0(\theta,\psi)\|_{V_1^*} \leq c\|\theta\|_{L^4}\|\nabla\psi\|_{L^4}\|\nabla v\|_{L^4}, \quad \forall \theta, \psi \in D(A_1).
\]

(2.21)

**Proof.** The first part of the proposition follows from the fact that for any $v \in V_1$, the mapping $r_0(\cdot,\cdot,v)$ defined on $D(A_1)$ with values in $\mathbb{R}$ is continuous. The estimate (2.21) also follows directly from (2.18). \qed

**Proposition 3.** For any $v \in V_1$ and $\theta \in D(A_1)$, we have

\[
\langle B_1(v,\theta), A_1 \theta \rangle = -\langle R_0(\theta,\theta), v \rangle. \tag{2.22}
\]

**Proof.** To simplify the notations, we assume the summation over repeated indexes. Taking into account the fact that $\text{div}v = 0$ as well as the boundary conditions, as in [4] we derive that

\[
\begin{align*}
-\langle B_1(v,\theta), A_1 \theta \rangle &= \int_M v_i \partial\theta \partial x_i \partial^2 \theta \partial x_i \partial x_j \partial x_j \, dx \\
&= -\int_M v_i \partial x_i \partial\theta \partial x_i \partial x_j \partial x_j \, dx - \int_M v_i \partial\theta \partial x_i \partial x_i \partial x_j \partial x_j \\
&= -\int_M v_i \partial x_i \partial\theta \partial x_i \partial x_j \partial x_j \, dx - \frac{1}{2} \int_M v_i \partial\nabla\theta^2 \partial x_i \partial x_i \\
&= -\int_M \partial x_i \partial x_i \partial x_i \partial x_j \partial x_j \partial x_i \partial\theta \partial x_i \partial x_j \partial x_j \\
&= r_0(\theta,\theta,v) = \langle R_0(\theta,\theta), v \rangle.
\end{align*}
\]
Remark 2. We recall from [18] that
\[
\mathrm{div} (\nabla \theta \otimes \nabla \theta) = \nabla \left( \frac{1}{2} |\nabla \theta|^2 \right) - A_1 \theta (\nabla \theta)^T. \tag{2.24}
\]
It is clear that \(\mathrm{div}(\nabla \theta \otimes \nabla \psi) \in L^2(M)\) for \(\theta, \psi \in D(A_1)\), therefore
\[
R_0(\theta, \psi) = \mathcal{P}_1 [\mathrm{div} (\nabla \theta \otimes \nabla \psi)], \quad \forall \theta, \psi \in D(A_1). \tag{2.25}
\]
It follows from (2.24)-(2.25) that
\[
R_0(\theta, \theta) = \mathcal{P}_1 [\mathrm{div} (\nabla \theta \otimes \nabla \theta)] = \mathcal{P}_1 \left[ \nabla \left( \frac{1}{2} |\nabla \theta|^2 \right) - A_1 \theta (\nabla \theta)^T \right]. \tag{2.26}
\]
Using the notations above, we rewrite (2.7)-(2.8) as
\[
\begin{align*}
\left\{\begin{array}{l}
v_0 + A_0 v + B_0 (v, v) - R_0(\theta, \theta) = g_1(t, v) + g_2(t, v) \dot{W}_t, \\
\theta_t + A_1 \theta + B_1 (v, \theta) = 0,
\end{array}\right. \\
(v, \theta)(0) = (v^0, \theta^0) \in \mathcal{H},
\end{align*}
\tag{2.27}
\]
or equivalently
\[
\begin{align*}
v(t) + \int_0^t (A_0 v(s) + B_0 (v(s), v(s))) ds = v^0 + \int_0^t R_0(\theta(s), \theta(s)) ds \\
+ \int_0^t g_1(s, v(s)) ds + \int_0^t g_2(s, v(s)) dW_s,
\end{align*}
\tag{2.28}
\]
\[
\theta(t) + \int_0^t (A_1 \theta(s) + B_1 (v(s), \theta(s))) ds = \theta^0,
\]
\(\mathbb{P}\)-a.s., and for all \(t \in [0, T]\).

We also recall from [19] some basic energy laws for (2.4) and (2.6). By multiply (2.4) by \(v\), (2.4) by \(\Delta d + |\nabla d|^2 d\) and adding the resulting equalities gives
\[
\frac{1}{2} \frac{d}{dt} (|\nabla v|^2_{L_2} + |\nabla d|^2_{L_2}) + \|v\|^2 + |A_1 d + |\nabla d|^2 d|^2_{L_2} = 0, \tag{2.29}
\]
which is a basic energy law for (2.4).

Substituting \(d = (\cos \theta, \sin \theta)^T\) into (2.29), we derive
\[
\frac{1}{2} \frac{d}{dt} (|v|^2_{L_2} + |\nabla \theta|^2_{L_2}) + \|v\|^2 + |A_1 \theta|^2_{L_2} = 0, \tag{2.30}
\]
which is a basic energy law for (2.6).

Note that (2.30) can also be derived by multiply (2.6) by \(v\), (2.6) by \(A_1 \theta\) and adding the resulting equalities.

Hereafter, as in [7] we assume that \((\Omega, \mathcal{F}, \mathbb{P})\) is a complete probability space, and let \(\{\mathcal{F}_t\}_{t \in [0, T]}\) be an increasing and right continuous family of sub \(\sigma\)-algebras of \(\mathcal{F}\), such that \(\mathcal{F}_0\) contains all the \(\mathbb{P}\)-null sets of \(\mathcal{F}\). Let \(\{\beta_t^j, \ t \geq 0, \ j = 1, 2, \ldots\}\) be a given sequence of mutually independent standard real \(\mathcal{F}\)–Wiener processes defined on this space, and suppose given \(\mathcal{R}\), a separable Hilbert space, and \(\{e_j, \ j = 1, 2, \ldots\}\), an orthonormal basis of \(\mathcal{R}\). We denote by \(\{W_t, \ t \geq 0\}\), the cylindrical Wiener process with values in \(\mathcal{R}\) defined by
\[
W_t = \sum_{j=1}^{\infty} \beta_t^j e_j.
\]
For a separable Banach space \( X \) and \( p \in [1, \infty] \), we denote by \( M^p_{\mathcal{F}_t}(0, T; X) \) the space of all processes \( \rho \in L^p(\Omega \times (0, T), d\mathbb{P} \times dt; X) \) that are \( \mathcal{F}_t \)-progressively measurable. The space \( M^2_{\mathcal{F}_t}(0, T; X) \) is a Banach subspace of \( L^p(\Omega \times (0, T), d\mathbb{P} \times dt; X) \). We will also denote by \( L^p(\Omega; C([0, T]; X)) \), for \( 1 \leq p < \infty \), the space of all continuous and \( \mathcal{F}_t \)-progressively measurable \( X \)-valued processes \( \{\rho_t; 0 \leq t \leq T\} \), satisfying

\[
E \left( \sup_{t \in [0, T]} ||\rho_t||_X^p \right) < \infty.
\]

For any Hilbert space \( Z \) with scalar product \( (\cdot, \cdot)_Z \), we will denote by \( \mathcal{L}^2(\mathbb{R}; Z) \) the separable Hilbert space of Hilbert-Schmidt operators from \( \mathbb{R} \) into \( Z \), and by \( ((\cdot, \cdot))_{\mathcal{C}(\mathbb{R}, Z)} \) and \( ||\cdot||_{\mathcal{C}(\mathbb{R}, Z)} \) the scalar product and its associated norm in \( \mathcal{L}^2(\mathbb{R}; Z) \).

For any process \( \rho \in M^2_{\mathcal{F}_t}(0, T; \mathcal{L}^2(\mathbb{R}; Z)) \), the stochastic integral of \( \rho \) with respect to the Wiener process \( W_t \) is denoted \( \int_0^t \rho(s)dW_s, \ 0 \leq t \leq T \) and is defined as the unique continuous \( Z \)-valued \( \mathcal{F}_t \)-martingale, such that for all \( z \in Z \), we have

\[
\left( \left( \int_0^t \rho(s)dW_s, z \right) \right)_Z = \sum_{j=1}^{\infty} \int_0^t (\rho(s)e_j, z)_Z d\beta_t^j, \ 0 \leq t \leq T,
\]

where the integral with respect to \( d\beta_t^j \) is understood in the sense of Itô, and the series converges in \( L^2(\Omega; C([0, T])) \), [7]. For some properties of this stochastic integral, the reader is referred to classical textbook such as [48].

We assume that \( g_1 \) and \( g_2 \) are measurable Lipschitz and sublinear mappings from \( \Omega \times (0, T) \times H_1 \) into \( \mathcal{V}_*^1 \) and from \( \Omega \times (0, T) \times H_1 \) into \( \mathcal{L}^2(\mathbb{R}; H_1) \) respectively. More precisely, for all \( v_1, v_2 \in \mathcal{V}_1, g_1(\cdot, v_1) \) and \( g_2(\cdot, v_1) \) are \( \mathcal{F}_t \)-progressively measurable, and \( d\mathbb{P} \times dt \)-a.e. in \( \Omega \times (0, T) \)

\[
\begin{align*}
||g_1(t, v_1) - g_1(t, v_2)||_{\mathcal{V}_1^*} & \leq L_1|v_1 - v_2|_{L^2}, \\
g_1(t, 0) & \in M^2_{\mathcal{F}_t}(0, T; \mathcal{V}_1^*), \\
||g_2(t, v_1) - g_2(t, v_2)||_{\mathcal{L}^2(\mathbb{R}; H_1)} & \leq L_2|v_1 - v_2|_{L^2}, \\
g_2(t, 0) & \in M^2_{\mathcal{F}_t}(0, T; \mathcal{L}^2(\mathbb{R}; H_1)).
\end{align*}
\]

Finally, we assume that

\[
(v^0, \theta^0) \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathcal{H}).
\]

Hereafter, for any \((w, \psi) \in \mathcal{H})

\[
E(w, \psi) = ||w||_{L^2}^2 + ||\nabla \psi||_{L^2}^2.
\]

**Definition 2.1.** A variational solution to (2.27) or (2.28) is a process \((v, \theta) \in M^2_{\mathcal{G}_t}(0, T; \mathcal{U}) \cap L^2(\Omega, L^\infty(0, T; \mathcal{H}))\), weakly continuous with values in \( \mathcal{H} \) such that (2.28) is satisfied in \( \mathcal{U}^*, \mathbb{P} \)-a.s, for all \( t \in [0, T] \).

3. **Existence and uniqueness of solutions.** In this section, we prove the existence and uniqueness of variational solution to (2.27) or (2.28).

**Proposition 4.** If \((v, \theta) \) is a solution to (2.27) or (2.28), then \((v, \theta) \in L^2(\Omega; C([0, T]; \mathcal{H}))\), and

\[
E(v, \theta)(t) + 2 \int_0^t ||v(s)||^2 + ||A_1 \theta(s)||_{L^2}^2 ds = E(v^0, \theta^0) + 2 \int_0^t \langle g_1(s, v(s)), v(s) \rangle ds + 2 \int_0^t \langle g_2(s, v(s)), v(s) dW_s \rangle + \int_0^t ||g_2(s, v(s))||_{H^2_{\mathcal{V}_1}}^2 ds,
\]
There exists at most one variational solution to (2.27) or (2.28).

Proposition 5.

Let

we derive that

Now we take the duality of (3.2) holds. We also note that (3.2) holds. We also note that (3.1) holds true.

Note that if \((v, \theta)\) is a solution to (2.27) or (2.28), then from (2.16), (2.21) and (2.31), we have

Integrating (3.3) from 0 to \(t\), adding the resulting equation to (3.2) and using (2.22), we derive (3.1).

Proposition 5. There exists at most one variational solution to (2.27) or (2.28).

Proof. Let \((v_1, \theta_1), (v_2, \theta_2)\) be variational solutions to (2.27). Let \((w, \psi) = (v_1, \theta_1) - (v_2, \theta_2)\). Then \((w, \psi)\) satisfies

\[
\begin{align*}
\frac{dw}{dt} &+ [A_0w + B_0(v_2, w) + B_0(w, v_1)]dt = [R_0(\theta_2, \psi) + R_0(\psi, \theta_1)]dt \\
+ & [g_1(t, v_1) - g_1(t, v_2)]dt + [g_2(t, v_1) - g_2(t, v_2)]dW, \\
\psi_1 + A_1\psi &+ B_1(v_2, \psi) + B_1(w, \theta_1) = 0. \\
(w(0, \psi)(0)) & = (0, 0) 
\end{align*}
\]

Let \(L_1 > 0, L_2 > 0\) be constants such that

\[
\begin{align*}
\|g_1(t, v_1) - g_1(t, v_2)\|_{H_1^0} & \leq L_1|v_1 - v_2|_{L^2}, \\
\|g_2(t, v_1) - g_2(t, v_2)\|_{H_1^0} & \leq L_2|v_1 - v_2|_{L^2}.
\end{align*}
\]

Using Itô’s formula, we derive that

\[
\begin{align*}
\frac{dw}{dt} &+ 2\int_0^t (\|w\|^2 + b_0(w, v_1, w))ds = 2\int_0^t \langle R_0(\theta_2, \psi) + R_0(\psi, \theta_1), w \rangle ds \\
+ & 2\int_0^t \langle g_1(t, v_1) - g_1(t, v_2), w \rangle ds + \int_0^t \|g_2(s, v_1) - g_2(s, v_2)\|^2_{H_1^0}ds \\
+ & 2\int_0^t (\langle R_0(\theta_2, \psi), w \rangle + \langle R_0(\psi, \theta_1), w \rangle - b_1(w, \theta_1, A_1\psi) - b_1(v_2, \psi, A_1\psi))ds \\
+ & 2\int_0^t \langle g_1(t, v_1) - g_1(t, v_2), w \rangle ds + \int_0^t \|g_2(s, v_1) - g_2(s, v_2)\|^2_{H_1^0}ds.
\end{align*}
\]

Now we take the duality of (3.4) with \(A_1\psi\). Adding the resulting equality to (3.6), we derive that

\[
\begin{align*}
\frac{dw}{dt} &+ |\nabla \psi(t)|_{L^2}^2 + \int_0^t (|w|^2 + |A_1\psi|^2_{L^2})ds = -2\int_0^t b_0(w, v_1, w)ds \\
+ & 2\int_0^t (\langle R_0(\theta_2, \psi), w \rangle + \langle R_0(\psi, \theta_1), w \rangle - b_1(w, \theta_1, A_1\psi) - b_1(v_2, \psi, A_1\psi))ds \\
+ & 2\int_0^t \langle g_1(t, v_1) - g_1(t, v_2), w \rangle ds + \int_0^t \|g_2(s, v_1) - g_2(s, v_2)\|^2_{H_1^0}ds.
\end{align*}
\]
where $\|g_1(t_1,0)\|_{L^2} (\|v,\theta\|_{L^2} \leq 1)$.

Let

\[ \mathcal{Y}_2(t) = |w(t)|^2_{L^2} + |\nabla\psi(t)|^2_{L^2}, \]

and

\[ K_1(t) = c(\|v_1\|^2 + \|\theta_2\|^2|A_1\theta_2|^2 + \|\theta_1\|^2|A_1\theta_1|^2 + |v_2|^2_{L^2} |v_2|^2_{L^2}). \]

Applying Itô's formula to the process $\kappa(t)\mathcal{Y}_2(t)$ and using (3.7)-(3.14), we derive that

\[ \kappa(t)\mathcal{Y}_2(t) + \int_0^t \kappa(s)(\|w\|^2 + |A_1\psi|^2_{L^2})ds \leq \int_0^t \kappa(s)\mathcal{Y}_2(s)ds + 2 \int_0^t \kappa(s)\langle g_2(s,v_1) - g_2(s,v_2), wdW_s \rangle. \]

Since $0 < \kappa(t) \leq 1$, the expectation of the stochastic integral in (3.15) vanishes and we derive that

\[ \mathbb{E}(\kappa(t)\mathcal{Y}_2(t)) \leq \mathbb{E} \int_0^t \kappa(s)\mathcal{Y}_2(s)ds, \quad 0 \leq t \leq T. \]

It follows from the Gronwall lemma that $\mathcal{Y}_2(t) = 0$ $\mathbb{P}$-a.s., for all $t \in [0,T]$, which gives $(v_1,\theta_1) = (v_2,\theta_2)$, $\mathbb{P}$-a.s., for all $t \in [0,T]$. 

**Proposition 6.** We assume all the above hypotheses. Moreover, we suppose that $g_1(\cdot,0) \in L^4(\Omega, L^2(0,T; L^2(\Omega, \mathbb{F}, \mathbb{P}; H)))$, $g_2(\cdot,0) \in L^4(\Omega, L^2(0,T; \mathcal{G}^2(\mathbb{F}; L^2))),$ and $(v^0, \theta^0) \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathcal{X})$ satisfies $\mathbb{E} \left[ (v^0, \theta^0)^2 \right] < \infty$. Then, there exists a unique solution $(v, \theta) \in L^4(\Omega, C(0,T; \mathcal{X})) \cap L^4(\Omega, L^2(0,T; \mathcal{U}))$.

Furthermore, the following estimate holds:

\[ \mathbb{E} \sup_{t \in [0,T]} \| (v, \theta)(t) \|^4_{L^2(H)} + \mathbb{E} \int_0^T \| (v(s)) \|^2 + |A_1\theta(s)|^2_{L^2}ds \leq C, \]

\[ \mathbb{E} \sup_{t \in [0,T]} \| (v, \theta)(t) \|^4_{L^2(H)} \leq C, \quad \mathbb{E} \left( \int_0^T \| (v, \theta)(s) \|^2 ds \right)^2 \leq C, \]

where $C$ is a constants depending only on the data.
**Proof.** The proof follows similar steps as in [7, 2]. We note that because of the stronger nonlinearity present in the model, some of the a priori estimates are more involved.

**Step 1: Galerkin approximation.** Let \( \{(w_i, \psi_i), i = 1, 2, 3, \ldots \} \subset U \) be an orthonormal basis of \( H \), where \( \{w_i, i = 1, 2, \ldots \} \) and \( \{\psi_i, i = 1, 2, \ldots \} \) are eigenvectors of the Stokes operator \( A_0 \) and the Dirichlet Laplacian operator \( A_1 \) respectively. We set \( U_m = H_m = \text{span} \{(w_1, \psi_1), \ldots (w_m, \psi_m)\} \).

Hereafter, we denote by \( P_m = (P_m^1, P_m^2) \) the orthogonal projection from \( H \) onto \( U_m = H_m \equiv \text{span} \{(w_1, \psi_1), (w_2, \psi_2), \ldots (w_m, \psi_m)\} \). We look for \( (v_m, \theta_m) \in H_m \) solution to Itô stochastic ordinary differential equations

\[
\begin{align*}
\langle v_m(t), w_k \rangle + \int_0^t \langle A_0 v_m + B_0 (v_m, v_m) - R_0(\theta_m, \theta_m), w_k \rangle ds \\
= \langle v_0^m, w_k \rangle + \int_0^t \langle g_1(s, v_m), w_k \rangle ds + \sum_{j=1}^m \int_0^t \langle g_2(s, v_m) e_j, w_k \rangle d\beta^j_s, \tag{3.18}
\end{align*}
\]

where \( (v_0^m, \theta_0) = P_m (v^0, \theta^0) \) in \( H \) as \( m \to \infty \).

As in the proof of Theorem 1.2.1 of [38], we can obtain the existence and uniqueness of a solution \( (v_m, \theta_m) \in M^2_{\theta_k} (0, T; U_m) \) of (3.18) with continuous trajectories.

**Step 2: Some estimates for the approximating sequence.**

Now using Itô’s formula, we derive that

\[
\begin{align*}
|v_m(t)|^2_{L^2} + 2 \int_0^t |v_m|^2 ds &= \sum_{k=1}^m \langle \langle v_0^m, w_k \rangle \rangle^2 + 2 \int_0^t \langle R_0(\theta_m, \theta_m), v_m \rangle ds \\
+ 2 \int_0^t \langle g_1(s, v_m), v_m \rangle ds + 2 \sum_{j=1}^m \int_0^t \langle g_2(s, v_m(s)) e_j, v_m(s) \rangle d\beta^j_s \\
+ \sum_{j,k=1}^m \int_0^t \|g_2(s, v_m) e_j, w_k\|^2 ds. \tag{3.19}
\end{align*}
\]

Following the proof of Proposition 1, we can check that \( (v_m, \theta_m) \) satisfies

\[
\begin{align*}
\mathcal{E}(v_m, \theta_m)(t) + 2 \int_0^t (|v_m(s)|^2 + |A_1 \theta_m(s)|^2_{L^2}) ds \\
= \mathcal{E}(v_m, \theta_m)(0) + 2 \int_0^t \langle g_1(t, v_m), v_m \rangle ds \\
+ 2 \sum_{j=1}^m \int_0^t \langle g_2(s, v_m(s)) e_j, v_m(s) \rangle d\beta^j_s + \sum_{j,k=1}^m \int_0^t \|g_2(s, v_m) e_j, w_k\|^2 ds. \tag{3.20}
\end{align*}
\]

In (3.20), we use the fact that

\[
\langle R_0(\theta_m, \theta_m), v_m \rangle = \langle B_1(v_m, \theta_m), A_1 \theta_m \rangle.
\]

Note that

\[
\sum_{j,k=1}^m \|g_2(s, v_m) e_j, w_k\|^2 ds \leq \sum_{j,k=1}^m \|g_2(s, v_m)\|^2_{L^2(\mathbb{R}; H_1)} \leq \|v_m\|^2_{L^2} + c \|g_2(s, 0)\|^2_{L^2(\mathbb{R}; H_1)},
\]

\[
|2 \langle g_1(s, v_m), v_m \rangle | \leq 2L_1 |v_m|_{L^2} \|v_m\| + 2 |g_1(s, 0)| v^*_1 \|v_m\|. \tag{3.21}
\]
It follows from (3.20)–(3.22) that
\[
E(v_m, \theta_m)(t) + \int_0^t \left( \|v_m(s)\|^2 + \|\nabla \theta_m(s)\|_{L^2}^2 \right) ds \\
\leq E(v^0, \theta^0) + (cL_1^2 + L_2^2) \int_0^t E(v_m, \theta_m)(s) ds \\
+ c \int_0^t \left( \|g_1(s, 0)\|_{V_1}^2 + \|g_2(s, 0)\|_{L^2}^2 \right) ds + 2 \sum_{j=1}^m \int_0^t \langle g_2(s, v_m(s))e_j, v_m(s) \rangle d\beta^j_s.
\]

For each \( n \geq 1 \), we consider the \( \mathcal{F}_t \)-stopping time \( \tau_n \) defined by:
\[
\tau_n = \min \left( T, \inf \left\{ t \in [0, T]; E(v_m, \theta_m)(t) + \int_0^t \left( \|v_m(s)\|^2 + \|\nabla \theta_m(s)\|_{L^2}^2 \right) ds \geq n^2 \right\} \right).
\]

For fixed \( m \), the sequence \( \{\tau_n; n \geq 1\} \) is increasing to \( T \). From (3.23), we derive that
\[
\sup_{s \in [0, t \wedge \tau_n]} [E(v_m, \theta_m)(s)]^2 + c \left[ \int_0^{t \wedge \tau_n} \left( \|v_m(s)\|^2 + \|\nabla \theta_m(s)\|_{L^2}^2 \right) ds \right] \leq [E(v^0, \theta^0)]^2 \\
+ c \int_0^t \sup_{r \in [0, s \wedge \tau_n]} [E(v_m, \theta_m)(r)]^2 ds + c \left[ \int_0^T \left[ \|g_1(s, 0)\|_{V_1}^2 + \|g_2(s, 0)\|_{L^2}^2 \right] ds \right] \sum_{j=1}^m \int_0^s \langle g_2(r, v_m(s))e_j, v_m(r) \rangle d\beta^j_s.
\]
Taking the expectation and using Doob’s inequality, we derive that
\[
E \sup_{s \in [0, t \wedge \tau_n]} \left[ \sum_{j=1}^m \int_0^s \langle g_2(r, v_m(s))e_j, v_m(r) \rangle d\beta^j_s \right]^2 \\
\leq cE \sum_{j=1}^m \int_0^{t \wedge \tau_n} \langle g_2(r, v_m(s))e_j, v_m(s) \rangle^2 ds \\
\leq cE \left( \sup_{s \in [0, t \wedge \tau_n]} \|v_m(s)\|_{L^2}^2 \int_0^{t \wedge \tau_n} \|g_2(s, v_m(s))\|_{L^2}^2 ds \right) \\
\leq \frac{1}{2} E \sup_{s \in [0, t \wedge \tau_n]} [E(v_m, \theta_m)(s)]^2 + cE \int_0^{t \wedge \tau_n} \left( \sup_{s \in [0, t \wedge \tau_n]} [E(v_m, \theta_m)(s)]^2 + cE \left[ \int_0^{t \wedge \tau_n} \|g_2(s, 0)\|_{L^2}^2 ds \right] \right)^2.
\]
It follows from the Gronwall lemma and the fact that \( \tau_n \uparrow T \) as \( n \) goes to \( \infty \) that
\[
E \left( \sup_{s \in [0, T]} [E(v_m, \theta_m)(s)]^2 \right) + E \left[ \int_0^T \left( \|v_m(s)\|^2 + \|\nabla \theta_m(s)\|_{L^2}^2 \right) ds \right] \\
\leq cE [E(v^0, \theta^0)]^2 + cE \left[ \int_0^T \|g_1(s, 0)\|_{V_1}^2 ds \right] + cE \left[ \int_0^T \|g_2(s, 0)\|_{L^2}^2 ds \right] \geq T.
\]
We also note that
\[
|B_0(v_m, v_m)|_{V_t^2} \leq c |v_m|_{L^2} \|v_m\|,
|R_0(\theta_m, \theta_m)|_{V_t^2} \leq c \|\theta_m\|_{L^2} \|A_1 \theta_m\|_{L^2},
|B_1(v_m, \theta_m)|_{L^2} \leq c |v_m|_{L^2}^{1/2} \|v_m\|^{1/2} \|\theta_m\|^{1/2} \|A_1 \theta_m\|_{L^2}^{1/2}.
\] (3.27)

It follows from (3.26)-(3.27) that
\[
\mathbb{E} \sup_{[0,T]} |(v_m, \theta_m)|_{\mathcal{H}}^2 \leq C, \quad \mathbb{E} \int_0^T \|\theta_m(s)\|^2 ds \leq C, \quad \mathbb{E} \int_0^T \left[ |B_0(v_m, v_m)|_{V_t^2} + |R_0(\theta_m, \theta_m)|_{V_t^2} + |B_1(v_m, \theta_m)|_{L^2}^2 \right] ds \leq C, \quad (3.28)
\]
\[
\mathbb{E} \sup_{[0,T]} |(v_m, \theta_m)|_{\mathcal{H}}^4 \leq C, \quad \mathbb{E} \left[ \int_0^T \|\theta_m(s)\|^2 ds \right]^2 \leq C.
\]

**Step 3: Taking the limit.**

From (3.28), we can find a subsequence still denoted \((v_m, \theta_m)\) such that
\[
(v_m, \theta_m) \to (v, \theta) \text{ in } M_{\mathcal{H}}^2(0, T; \mathcal{U}),
(v_m(0), \theta_m(0)) \to (\zeta_1, \zeta_2) \text{ in } L^4(\Omega; \mathcal{F}_0; \mathbb{P}; \mathcal{H}),
B_0(v_m, v_m) \to \beta_0 \text{ in } M_{\mathcal{H}}^2(0, T; V_t^2),
R_0(A_1 \theta_m, \theta_m) \to \nu_0 \text{ in } M_{\mathcal{H}}^2(0, T; V_t^2),
B_1(v_m, \theta_m) \to \nu_1 \text{ in } M_{\mathcal{H}}^2(0, T; H_2),
g_1(t, v_m) \to g_1^0 \text{ in } M_{\mathcal{H}}^2(0, T; V_1^2),
g_2(t, v_m) \to g_2^0 \text{ in } M_{\mathcal{H}}^2(0, T; \mathbf{L}^2(\mathbb{R}; H_1)).
\] (3.29)

As in [7], we can check that \((v, \theta) \in L^2(\Omega; C(0, T; \mathcal{H}))\) and satisfies for all \(0 \leq t \leq T\)
\[
v(t) + \int_0^t A_0 v ds + \int_0^t \beta_0(s) ds = v^0 + \int_0^t (v_0^0(s) + g_1^0(s)) ds + \int_0^t g_2^0(s) dW_s,
\theta(t) + \int_0^t A_1 \theta ds + \int_0^t \beta_1(s) ds = \theta^0.
\] (3.30)

Let us prove that
\[
\beta_0 = B_0(v, v), \quad \nu_0 = R_0(A_1 \theta, \theta), \quad \nu_1 = B_1(v, \theta).
\]

Let us set
\[
(\tilde{v}_m, \tilde{\theta}_m) = \mathcal{P}_m(v, \theta),
\]
where \(\mathcal{P}_m\) is the orthogonal projection of \(\mathcal{H}\) onto \(\mathcal{U}^m\).

It follows that
\[
|(|\tilde{v}_m, \tilde{\theta}_m)|_{\mathcal{H}} \leq \|(v, \theta)\|_{\mathcal{H}}, \quad \|(|\tilde{v}_m, \tilde{\theta}_m)|_{\mathcal{U}} \leq \|(v, \theta)\|_{\mathcal{U}},
(\tilde{v}_m, \tilde{\theta}_m) \to (v, \theta) \text{ in } M_{\mathcal{H}}^2(0, T; \mathcal{U}).
\] (3.31)

From (3.18) and (3.30), we derive that for \(1 \leq k \leq m\),
\[
\langle \tilde{v}_m(t) - v_m(t), w_k \rangle + \int_0^t \langle A_0(\tilde{v}_m - v_m), w_k \rangle ds + \int_0^t \langle \beta_0^k - B_0(v_m, v_m), w_k \rangle ds
= \int_0^t \langle \nu_0^k - R_0(\theta_m, \theta_m), w_k \rangle ds + \int_0^t \langle g_1^k - g_1(s, v_m), w_k \rangle ds
+ \sum_{j=1}^m \int_0^t \langle g_2^j c_j - g_2(s, v_m) c_j, w_k \rangle d\beta_2^j + \sum_{j=m+1}^\infty \int_0^t \langle g_2^j(s) c_j, w_k \rangle d\beta_2^j,
\]
\[\tilde{\theta}_m(t) - \theta_m(t) + \int_0^t A_1(\tilde{\theta}_m - \theta_m)ds + \int_0^t (\beta_1^m - B_1(v_m, \theta_m))ds = 0. \quad (3.32)\]

Note that

\[
\beta_0^m - B_0(v_m, v_m) = \beta_0^m - B_0(\tilde{v}_m, \tilde{v}_m) + B_0(\tilde{v}_m - v_m, \tilde{v}_m) + B_0(v_m, \tilde{v}_m - v_m), \\
\beta_0^m - R_0(\theta_m, \theta_m) = R_0(\tilde{\theta}_m, \tilde{\theta}_m) + R_0(A_1(\tilde{\theta}_m - \theta_m), \tilde{\theta}_m) + R_0(\theta_m, \tilde{\theta}_m) - \tilde{\theta}_m - \theta_m, \quad (3.33)
\]

\[\beta_1^m - B_1(v_m, \theta_m) = \beta_1^m - B_1(\tilde{v}_m, \tilde{\theta}_m) + B_1(\tilde{v}_m - v_m, \tilde{\theta}_m) + B_1(v_m, \tilde{\theta}_m - \theta_m).\]

Let us set \( v_m = \tilde{v}_m - v_m, \rho_m = \tilde{\theta}_m - \theta_m. \)

From the Itô's formula, we have

\[
d(\vartheta_m, w_k)^2 = 2d(\vartheta_m, w_k)d(\vartheta_m, w_k) + \sum_{j=1}^{m} \left[ (g_1^e e_j - g_1(t, v_m)e_j, w_k) \right] dt \\
+ \sum_{j=m+1}^{\infty} \left[ (g_1^e e_j, w_k) \right] dt.
\]

It follows that

\[
|\vartheta_m(t)|^2 + 2 \int_0^t (|\vartheta_m|^2 + \beta_0^m - B_0(v_m, v_m), \vartheta_m)ds \\
= 2 \int_0^t (\vartheta_m - B_0(v_m, v_m))ds + 2 \int_0^t (g_1^e - g_1(s, v_m), \vartheta_m)ds + 2 \sum_{j=1}^{m} \int_0^t (g_2e_j - g_1(s, v_m), \vartheta_m)ds \\
- g_1(s, v_m)e_j, \vartheta_m)\\n+ 2 \sum_{j=1}^{m} \int_0^t (g_1^e e_j, \vartheta_m)ds + \sum_{j=m+1}^{\infty} \int_0^t |\vartheta_m|^2 + 2 \int_0^t (\vartheta_m, \rho_m)ds.
\]

From (3.32) we have

\[
\frac{d}{dt} |\rho_m|^2 + 2|A_1 \rho_m|^2 + 2(\beta_0^m - B_1(\tilde{v}_m, \tilde{\theta}_m), A_1 \rho_m) \\
+ 2(B_1(\vartheta_m, \theta_m), A_1 \rho_m) + 2(B_1(v_m, \rho_m), A_1 \rho_m).
\]

Let us set \( v_m = \tilde{v}_m - v_m, \rho_m = \tilde{\theta}_m - \theta_m. \)

Note that

\[
\langle \beta_0^m - B_0(v_m, v_m), \theta_m \rangle = \langle \beta_0^m - B_0(\tilde{v}_m, \tilde{v}_m), \vartheta_m \rangle + \langle B_0(\vartheta_m, v_m), \vartheta_m \rangle \\
\leq |\beta_0^m - B_0(\tilde{v}_m, \tilde{v}_m), \vartheta_m| + \frac{1}{4} |\vartheta_m|^2 + c |v_m|^2|\vartheta_m|^2, \quad (3.36)
\]

\[\beta_1^m - B_1(v_m, \theta_m), A_1 \rho_m \]

\[\leq |\beta_1^m - B_1(\tilde{v}_m, \tilde{\theta}_m), A_1 \rho_m| + \langle B_1(\vartheta_m, \tilde{\theta}_m), A_1 \rho_m \rangle + \langle B_1(v_m, \rho_m), A_1 \rho_m \rangle \\
\leq |\beta_1^m - B_1(\tilde{v}_m, \tilde{\theta}_m), A_1 \rho_m| + \frac{1}{4} (|\vartheta_m|^2 + |A_1 \rho_m|^2) \\
+ c |\vartheta_m|^2|A_1 \theta_m|^2|\vartheta_m|^2 + c |v_m|^2|\vartheta_m|^2|A_1 \rho_m|^2, \quad (3.37)
\]

\[
(\vartheta_0^m - R_0(\theta_m, \theta_m), \vartheta_m) \\
= (\vartheta_0^m - R_0(\tilde{\theta}_m, \tilde{\theta}_m), \vartheta_m) + (R_0(\rho_m, \tilde{\theta}_m), \vartheta_m) + (R_0(\theta_m, \rho_m), \vartheta_m) \\
\leq (\vartheta_0^m - R_0(\tilde{\theta}_m, \tilde{\theta}_m), \vartheta_m) + \frac{1}{4} (|\vartheta_m|^2 + |A_1 \rho_m|^2).
\]
\[ \sum_{j=1}^{m} |\Pi_m(g_2^b e_j - g_2(s, v_m)e_j)^2_{L^2} \leq \|g_2^b - g_2(s, v_m)\|^2_{L^2(\mathbb{R}; H_1)} \]

\[ \leq \|g_2(s, v) - g_2(s, v_m)\|^2_{L^2(\mathbb{R}; H_1)} + 2\langle g_2^b - g_2(s, v_m), g_2^b - g_2(s, v_m) \rangle \]

\[ - \|g_2(s, v) - g_2(s, v_m)\|^2_{L^2(\mathbb{R}; H_1)} \leq 2L_2 |v - \tilde{v}_m|_{L^2}^2 + 2L_2 |v_m - \tilde{v}_m|_{L^2}^2 + 2\langle g_2^b - g_2(s, v_m), g_2^b - g_2(s, v_m) \rangle \]

\[ - \|g_2(s, v) - g_2(s, v_m)\|^2_{L^2(\mathbb{R}; H_1)} \]

\[ s.t. \quad \langle g_1(s, \tilde{v}_m) - g_1(s, v_m), \vartheta_m \rangle \leq L_1 \|\vartheta_m\| L^2\|\vartheta_m\|^2 \leq \frac{1}{4} \|\vartheta_m\|^2 + cL_1^2 \|\vartheta_m\|^2_{L^2}. \] (3.40)

Let

\[ Z(t) = \pi \|\vartheta_m(t)\|^2 + \|\rho_m(t)\|^2 = \|((\tilde{v}_m - v_m)(t))_{L^2}^2 + \|((\tilde{\theta}_m - \theta_m)(t))_{L^2}^2, \]

\[ Y_1(t) = e^{c\|v_m\|^2 + c\|\theta_m\|^2 + c\|\vartheta_m\|^2} A_1 \|\vartheta_m\|_{L^2}^2 + cL_2^2 cL_1. \]

\[ K_2(t) = \|\vartheta_m\|^2 + |A_1 \rho_m|^2_{L^2}. \]

Let

\[ \sigma(t) = \exp \left( - \int_0^t Y_1(s) ds \right), \]

Using (3.34)-(3.40), it follows from Ito’s formula that

\[ \sigma(t) Z(t) + \int_0^t \sigma(s) K_2(s) ds + \int_0^t \sigma(s) \|g_2^b - g_2(s, v)\|^2_{L^2(\mathbb{R}; H_1)} \]

\[ \leq c \int_0^t \sigma(s) ( - \beta_0^b + B_0(\tilde{v}_m, \tilde{\theta}_m), \vartheta_m) ds + c \int_0^t \sigma(s) ( - \beta_1^b + B_1(\tilde{v}_m, \tilde{\theta}_m), A_1 \rho_m) ds \]

\[ + c \int_0^t \sigma(s) ( - R_0 A_1 \tilde{\theta}_m, \vartheta_m) ds + c \int_0^t \sigma(s) (g_1 - g_1(s, \tilde{v}_m), \vartheta_m) ds \]

\[ + 2L_2 \int_0^t \sigma(s) |v - \tilde{v}_m|_{L^2}^2 ds \]

\[ + \sum_{j=m+1}^{\infty} \int_0^t \sigma(s) \|\Pi_m g_2^b e_j\|^2_{L^2} ds \]

\[ + 2 \int_0^t \sigma(s) (g_2^b - g_2(s, v_m), g_2^b - g_2(s, v)) ds \]

\[ + 2 \sum_{j=1}^m \int_0^t \sigma(s) (g_2^b e_j - g_2(s, v_m)e_j, \vartheta_m) ds \beta_s^b + \sum_{j=m+1}^{\infty} \int_0^t \sigma(s) (g_2^b e_j, \vartheta_m) ds \beta_s^b. \] (3.41)

For each \( n \geq 1 \), we consider the \( \tilde{\mathbb{F}_t} \)-stopping time \( \tau_n \) defined by:

\[ \tau_n = \min \left\{ t \in [0, T] ; \| (v, \theta) (t) \|^2_{H} + \int_0^t \| (v, \theta) \|^2_{H} ds \geq n^2 \right\}. \] (3.42)

We derive from (3.41) that

\[ \mathbb{E} \sigma(\tau_n) Z(\tau_n) + c \mathbb{E} \int_0^{\tau_n} \sigma(s) K_2(s) ds + \mathbb{E} \int_0^{\tau_n} \sigma(s) \|g_2^b - g_2(s, v)\|^2_{L^2(\mathbb{R}; H_1)} ds \]
and

\[
\text{Claim 1. The right side of (3.43) goes to 0 as } m \text{ goes to } +\infty.
\]

(i). We first note that

\[
\lim_{m \to \infty} \left( \sum_{j=m+1}^{\infty} E \int_0^T \sigma(s) |g_2 e_j|_{L^2}^2 ds + 2E \int_0^T \sigma(s) |v(s) - \tilde{v}_m|_{L^2}^2 ds \right) = 0. \tag{3.44}
\]

Moreover, since

\[
g_2(t, v_m) \to g_2^*(t) \text{ in } M_{\delta_1}^2(0, T; L^2(\mathbb{R}; H_1)),
\]

and

\[
1_{[0, \tau_m]} \sigma(t) (g_2^*(t) - g_2(t, v)) \in M_{\delta_1}^2(0, T; L^2(\mathbb{R}; H_1)),
\]

we derive that

\[
\lim_{m \to \infty} E \int_0^{\tau_m} \sigma(s) \langle g_2^* - g_2(s, \tilde{v}_m), \tilde{\theta}_m \rangle ds = 0. \tag{3.45}
\]

(ii). Let us now prove that

\[
\lim_{m \to \infty} E \int_0^{\tau_m} \sigma(s) \langle - \beta_0^* + B_0(\tilde{v}_m, \tilde{\theta}_m), \tilde{\theta}_m \rangle ds = 0. \tag{3.46}
\]

We recall that

\[
(v_m, \theta_m) \to (v, \theta), \quad (\tilde{v}_m, \tilde{\theta}_m) \to (v, \theta),
\]

\[
(\tilde{v}_m, \tilde{\theta}_m) - (v_m, \theta_m) \to (0, 0) \text{ in } M_{\delta_1}^2(0, T; \mathcal{U}). \tag{3.47}
\]

We also have

\[
\lim_{m \to \infty} E \int_0^{\tau_m} \sigma(s) \langle - \beta_0^* + B_0(\tilde{v}_m, \tilde{\theta}_m), \tilde{\theta}_m \rangle ds
\]

\[
= \lim_{m \to \infty} E \int_0^{\tau_m} \sigma(s) \langle - \beta_0^* + B_0(v, v), \theta_m \rangle ds \tag{3.48}
\]

\[
+ \lim_{m \to \infty} E \int_0^{\tau_m} \sigma(s) \langle - B_0(v, v) + B_0(\tilde{v}_m, \tilde{v}_m), \tilde{\theta}_m \rangle ds.
\]

From (3.47) and the fact that \(1_{[0, \tau_m]} \sigma(t)(- \beta_0^* + B_0(v, v)) \in M_{\delta_1}^2(0, T; V_1^*)\), it follows that

\[
\lim_{m \to \infty} E \int_0^{\tau_m} \sigma(s) \langle - \beta_0^* + B_0(v, v), \theta_m \rangle ds = 0. \tag{3.49}
\]

We also note that

\[
\| - B_0(v, v) + B_0(\tilde{v}_m, \tilde{v}_m) \|_{V_1^*} \leq c \| \tilde{v}_m - v \|_{L^2}^{1/2} \| \tilde{v}_m - v \|_{L^2}^{1/2} (\| \tilde{v}_m \|_{L^2}^{1/2} + \| v \|_{L^2}^{1/2} + \| v \|_{L^2}^{1/2}),
\]

which implies that

\[
\|
1_{[0, \tau_m]}(- B_0(v, v) + B_0(\tilde{v}_m, \tilde{v}_m))\|_{V_1^*} \to 0 \quad \text{as } m \to \infty, \quad dt \times d\mathbb{P} - \text{a.e.,}
\]

\[
\|
1_{[0, \tau_m]}(- B_0(v, v) + B_0(\tilde{v}_m, \tilde{v}_m))\|_{V_1^*} \leq c \| v \| \in M_{\delta_1}^2(0, T; \mathbb{R}).
\]
We conclude from (3.49) and (3.50) that

\[
\lim_{m \to \infty} E \int_0^{\tau_n} \sigma(s) \langle -B_0(v, v) + B_0(\tilde{v}_m, \tilde{\theta}_m), \vartheta_m \rangle ds = 0. \tag{3.50}
\]

We conclude from (3.49) and (3.50) that

\[
\lim_{m \to \infty} E \int_0^{\tau_n} \sigma(s) \langle -\beta_0^2 + B_0(\tilde{v}_m, \tilde{\theta}_m), \vartheta_m \rangle ds = 0, \tag{3.51}
\]

which proves (3.46).

(iii). Next we will prove that

\[
\lim_{m \to \infty} E \int_0^{\tau_n} \sigma(s) \langle \nu_0^b - R_0(\tilde{\theta}_m, \tilde{\theta}_m), \vartheta_m \rangle ds = 0. \tag{3.52}
\]

From (3.47) and the fact that \(1_{[0, \tau_n]} \sigma(t)(r_0^b - R_0(\theta, \theta)) \in M_2^2(0, T; V_0^*)\), we also have

\[
\lim_{m \to \infty} E \int_0^{\tau_n} \sigma(s) \langle r_0^b - R_0(\theta, \theta), \vartheta_m \rangle ds = 0. \tag{3.53}
\]

We also note that

\[
||R_0(\tilde{\theta}_m, \tilde{\theta}_m) - R_0(\theta, \theta)||_{V_0^*} \leq c|A_1(\tilde{\theta}_m - \theta)|_{L^2}^{1/2} ||\tilde{\theta}_m - \theta||_{L^2}^{1/2} ||\theta||_{L^2}^{1/2} |A_1\theta|_{L^2}^{1/2} + c||\tilde{\theta}_m - \theta||_{L^2}^{1/2} |A_1(\tilde{\theta}_m - \theta)|_{L^2}^{1/2} |\tilde{\theta}_m|_{L^2}^{1/2},
\]

which implies that

\[
\|1_{[0, \tau_n]}(R_0(\tilde{\theta}_m, \tilde{\theta}_m) - R_0(\theta, \theta))\|_{V_0^*} \to 0 \quad \text{as} \quad m \to \infty, \quad dt \times d\mathbb{P} - \text{a.e.},
\]

\[
\|1_{[0, \tau_n]}(R_0(\tilde{\theta}_m, \tilde{\theta}_m) - R_0(\theta, \theta))\|_{V_0^*} \leq cm|A_1\theta|_{L^2} \in M_2^2(0, T; \mathbb{R}).
\]

It follows that

\[
\lim_{m \to \infty} E \int_0^{\tau_n} \sigma(s) \langle R_0(A_1 \tilde{\theta}_m, \tilde{\theta}_m) - R_0(A_1 \theta, \theta), \vartheta_m \rangle ds = 0. \tag{3.54}
\]

We conclude from (3.53) and (3.54) that

\[
\lim_{m \to \infty} E \int_0^{\tau_n} \sigma(s) \langle \nu_0^b - R_0(\tilde{\theta}_m, \tilde{\theta}_m), \vartheta_m \rangle ds = 0, \tag{3.55}
\]

which proves (3.52).

(iv). Let us now prove that

\[
\lim_{m \to \infty} E \int_0^{\tau_n} \sigma(s) \langle -\beta^2_1 + B_1(\tilde{v}_m, \tilde{\theta}_m), A_1\vartheta_m \rangle ds = 0.
\]

Following similar steps as in (3.51) and (3.55), can check that

\[
\lim_{m \to \infty} E \int_0^{\tau_n} \sigma(s) \langle -\beta_1^2 + B_1(\nu, \theta), A_1\vartheta_m \rangle ds
\]

\[
= \lim_{m \to \infty} E \int_0^{\tau_n} \sigma(s) \langle -\beta_1^2 + B_1(\nu, \theta), A_1\vartheta_m \rangle ds
\]

\[
+ \lim_{m \to \infty} E \int_0^{\tau_n} \sigma(s) \langle -B_1(\nu, \theta) + B_1(\tilde{v}_m, \tilde{\theta}_m), A_1\vartheta_m \rangle ds = 0.
\]

(v). Let us also prove that

\[
\lim_{m \to \infty} E \int_0^{\tau_n} \sigma(s) \langle \gamma_1^b(s) - g_1(s, \tilde{v}_m), \vartheta_m \rangle ds = 0.
\]
From (3.47) and the fact that
\[ 1[[0, \tau_n] \sigma(t) (g_1(t, v) - g_1(t, \bar{v}_m)) \rightarrow 0 \in M^2_{\delta_1} (0, T; V^*_1) \text{ as } m \rightarrow \infty, \]
we derive that
\[ \lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \sigma(s) \langle g^1_1(s) - g_1(s, v), \vartheta_m \rangle ds = 0, \]
\[ \lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \sigma(s) \langle g^1_1(s) - g_1(s, \bar{v}_m), \vartheta_m \rangle ds = 0. \] (3.56)

Therefore, we derive that
\[ \lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \sigma(s) \langle g^1_1(s) - g_1(s, v), \vartheta_m \rangle ds = 0, \]
\[ \lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \sigma(s) \langle g^1_1(s) - g_1(s, \bar{v}_m), \vartheta_m \rangle ds = 0. \] (3.57)

Finally we conclude from (i)-(v) that the right side of (3.43) goes to 0 as \( m \) goes to \( +\infty. \)

Now using the fact that \( 1[[0, \tau_n] \sigma(t) \leq 1, \) we derive from (3.43) that
\[ \lim_{m \rightarrow \infty} \mathbb{E}((\theta_m, \psi_m)(\tau_n))_{\mathcal{H}}^2 = \lim_{m \rightarrow \infty} \mathbb{E} \left( \left( \bar{g}_m, \bar{\vartheta}_m \right)(\tau_n) - (v_m, \vartheta_m)(\tau_n) \right)_{\mathcal{H}}^2 = 0, \]
\[ \lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} K_2(s) ds = \lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_n} \left( ||\theta_m||^2 + |A_1 \rho_m|^2 \right) ds = 0, \]
\[ \mathbb{E} \int_0^{\tau_n} \| g^2_2 - g_2_2 \|_{L^2(\mathcal{R}; H_1)}^2 ds = 0. \] (3.58)

**Claim 2.** The following hold true.
\[ g_2(t, v) = g^2_2(t) \in M^2_{\delta_1} (0, T; \mathcal{L}^2(\mathcal{R}; H_1)), \]
\[ B_0(v, v) = \beta^0_2 \in M^2_{\delta_1} (0, T; V^*_1), \]
\[ R_0(\theta, \theta) = r^0_2 \in M^2_{\delta_1} (0, T; V^*_1), \]
\[ B_1(v, \theta) = \beta^1_2 \in M^2_{\delta_1} (0, T; H_2), \]
\[ g_1(t, v) = g^1_1(t) \in M^2_{\delta_1} (0, T; V^*_1). \] (3.59)

To prove (3.59)\(_1\), we note that from (3.58)\(_3\) and the fact that the sequence \( \{\tau_n; n \geq 1\} \) is increasing to \( T, \) we derive that
\[ g^2_2(t) = g_2(t, v) \in M^2_{\delta_1} (0, T; \mathcal{L}^2(\mathcal{R}; H_1)). \] (3.60)

This proves (3.59)\(_1\).

To prove (3.59)\(_2, \) we proceed as follows. We note that from (3.58)\(_2\) and (3.31), we also have
\[ (v_m, \theta_m)|_{[0, \tau_n]} \rightarrow (v, \theta)|_{[0, \tau_n]} \in M^2_{\delta_1} (0, T; \mathcal{U}). \]

Therefore, for any \( w \in M^2_{\delta_1} (0, T; V^*_1), \) we have
\[ \mathbb{E} \int_0^{\tau_n} \langle B_0(v, v) - B_0(v_m, v_m), w \rangle ds \]
\[ \leq c \|w\|_{M^2_{\delta_1} (0, T; V^*_1)} \times \mathbb{E} \int_0^{\tau_n} \|v_m - v\|^{1/2} ||v_m - v||_L^2 \|v\| + \|v_m\| ds, \]
which gives
\[ \lim_{m \to \infty} \mathbb{E} \int_0^{\tau_n} \langle B_0(v, v) - B_0(v_m, v_m), w \rangle \, ds = 0. \] (3.61)

From (3.29)3 and (3.61), we derive that
\[ \mathbb{E} \int_0^{\tau_n} \langle B_0(v, v) - \beta_0^0, w \rangle = 0, \forall w \in M_{\tilde{\mathcal{F}}_t}^2(0, T; V_1). \]

Since \( \tau_n \uparrow T \) and \( M_{\tilde{\mathcal{F}}_t}^\infty(0, T; V_1) \) is dense in \( M_{\tilde{\mathcal{F}}_t}^2(0, T; V_1) \), we conclude that
\[ B_0(v, v) = \beta_0^0 \text{ in } M_{\tilde{\mathcal{F}}_t}^2(0, T; V_1^*). \]

This proves (3.59)2.

To prove (3.59)3, we note that
\[ \mathbb{E} \int_0^{\tau_n} (R_0(\theta, \theta) - R_0(\theta_m, \theta_m), w) \, ds \leq c\|w\|_{M_{\tilde{\mathcal{F}}_t}^\infty(0, T; V_1)} \times \]
\[ \mathbb{E} \int_0^{\tau_n} (|A_1(\theta_m - \theta)|_{L^2} \|\theta\|_{L^2} + \|\theta_m - \theta\|_{L^2} |A_1(\theta_m - \theta)|_{L^2} |A_1\theta_m|_{L^2}) \, ds, \]

which gives
\[ \lim_{m \to \infty} \mathbb{E} \int_0^{\tau_n} (R_0(\theta, \theta) - R_0(\theta_m, \theta_m), w) \, ds = 0. \] (3.62)

From (3.29)4 and (3.62), we derive that
\[ \mathbb{E} \int_0^{\tau_n} (R_0(\theta, \theta) - r_0^0, w) = 0, \forall w \in M_{\tilde{\mathcal{F}}_t}^\infty(0, T; V_1), \]

which gives
\[ R_0(\theta, \theta) = r_0^0 \text{ in } M_{\tilde{\mathcal{F}}_t}^2(0, T; V_1^*), \]

and (3.59)3 is proved.

Similarly, we can prove that
\[ B_1(v, \theta) = \beta_1^0 \text{ in } M_{\tilde{\mathcal{F}}_t}^2(0, T; H_2), \ g_1(t, v) = g_1^0(t) \text{ in } M_{\tilde{\mathcal{F}}_t}^2(0, T; V_1^*). \]

We conclude from Claims 1 and 2 that \((v, \theta)\) is the solution to (2.27).

Finally, we note that the estimate (3.17) is derived as in (3.26) using the stopping time given by (3.24).

We recall from [38, 7] the following lemma.

**Lemma 3.1.** Let \( \{Q_m; m \geq 1\} \subset M_{\tilde{\mathcal{F}}_t}^2(0, T; \mathbb{R}) \) be a sequence of continuous real processes, and let \( \{\tau_n; n \geq 1\} \) be a sequence of \( \tilde{\mathcal{F}}_t \)-stopping times such that \( \tau_n \uparrow T \); \( \sup_m \mathbb{E}|Q_m(T)|^2 < \infty \), and \( \lim_{m \to \infty} \mathbb{E}|Q_m(\tau_n)| = 0 \), for \( n \geq 1 \). Then \( \lim_{m \to \infty} \mathbb{E}|Q_m(T)| = 0 \).

Applying Lemma 3.1 to \( Q_m(t) = |(v, \theta) - (v_m, \theta_m)|_{L^2}^2 \) and \( \sigma_n = \tau_n \) and using (3.17), (3.26), (3.58)1 and the uniqueness of \((v, \theta)\), we conclude that the whole sequence given by (3.18) satisfies
\[ \lim_{m \to \infty} \mathbb{E}|(v, \theta) - (v_m, \theta_m)|_{L^2}^2 = 0, \forall t \in [0, T]. \]

Similarly, applying Lemma 3.1 to \( Q_m(t) = \int_0^t \|(v, \theta)(s) - (v_m, \theta_m)(s)\|_{L^2}^2 \, ds \) and using (3.17), (3.58)2, we conclude that the whole sequence \((v_m, \theta_m)\) converges to \((v, \theta)\) strongly in \( M_{\tilde{\mathcal{F}}_t}^2(0, T; \mathcal{U}) \), i.e.,
\[ \lim_{m \to \infty} \mathbb{E} \int_0^t \|(v, \theta)(s) - (v_m, \theta_m)(s)\|_{L^2}^2 \, ds = 0, \forall t \in [0, T]. \]
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