PERSISTENT CURRENTS and MAGNETIZATION

in TWO-DIMENSIONAL MAGNETIC QUANTUM SYSTEMS

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Abstract

Persistent currents and magnetization are considered for a two-dimensional electron (or gas of electrons) coupled to various magnetic fields. Thermodynamic formulae for the magnetization and the persistent current are established and the "classical" relationship between current and magnetization is shown to hold for systems invariant both by translation and rotation. Applications are given, including the point vortex superposed to an homogeneous magnetic field, the quantum Hall geometry (an electric field and an homogeneous magnetic field) and the random magnetic impurity problem (a random distribution of point vortices).

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1 Introduction

Since the pioneering work of Bloch [1], several questions concerning persistent currents have been answered. The conducting ring case has been largely discussed in the literature [2], as well as the persistent current due to a point-like vortex [3].

However, many questions remain open. For example, we might ask about standard relationships such as

\[ I = -\frac{dE}{d\Phi} \]  \tag{1}  
\[ M = IV \]  \tag{2}

where \( I, M, E, V, \Phi \) are, respectively, the persistent current, magnetization, energy, area (denoted by \( V \) as volume in two dimensions) and magnetic flux through the system.

These relations are clearly understood in the case of a ring. Do they apply to more general, for instance infinite, systems? More precisely, are they still correct, and if yes, in which case?

The aim of this paper is to clarify these points in the case of two-dimensional systems. We will show that (2) holds true as soon as the system is invariant by translation and rotation. We will also insist on the role of the spin coupling, which happens to be crucial in the case of point vortices. We will exemplify these considerations for an electron (or gas of electrons) coupled to

- a point vortex superposed to an homogeneous magnetic field,
- an uniform electric field and an homogeneous magnetic field (quantum Hall geometry),
- a distribution of point-like vortices randomly dropped onto the plane according to Poisson’s law, modeling disordered magnetic systems (analytical and numerical).

The case of electrons on a ring threaded by a flux, and electrons in a plane coupled to a point vortex will be revisited in Appendix A. Some technical details on harmonic regularization for computing partition functions and the so-called “Landau counting rule” will be given in Appendix B.

## 2 Basic Definitions and Formalism

Consider the two-dimensional quantum mechanical problem of an electron with Hamiltonian (we set the electron mass and the Planck constant $m_e = \hbar = 1$)

$$H = \frac{1}{2} \left( \vec{p} - e\vec{A} \right)^2 + V - \frac{e}{2}\sigma_z B$$  \hspace{1cm} (3)

$V$ and $\vec{A}$ are, respectively, the scalar and vector potential ($\vec{B} = \nabla \times \vec{A}$) and $\sigma_z/2$ is the electron spin. The local current is ($\vec{v} = \vec{p} - e\vec{A}$)\n
$$\vec{j}(\vec{r}) = \frac{e}{2} \left( \vec{v} | \vec{r} \rangle \langle \vec{r} | + | \vec{r} \rangle \langle \vec{r} | \vec{v} \right)$$  \hspace{1cm} (4)

The total magnetization is by definition

$$M = \frac{1}{2} \int d\vec{r} (\vec{r} \times \langle \vec{j}(\vec{r}) \rangle) \cdot \vec{k} + \frac{e}{2} \langle \sigma_z \rangle$$  \hspace{1cm} (5)

where $\vec{k}$ is the unit vector perpendicular to the plane and $\langle \rangle$ means average over Boltzmann or Fermi-Dirac distributions. In the Boltzmann case, one obtains the thermal magnetization ($Z_\beta = \text{Tr} \left\{ e^{-\beta H} \right\}$)

$$M_\beta = \frac{e}{2Z_\beta} \text{Tr} \left\{ e^{-\beta H} \left( (\vec{r} \times \vec{v}) \cdot \vec{k} + \sigma_z \right) \right\}$$  \hspace{1cm} (6)
We want to relate the thermal magnetization to a thermodynamical quantity. To this aim, we add a fictitious uniform magnetic field $\vec{B}'$ perpendicular to the plane. In the symmetric gauge, its vector potential is $e\vec{A}' = (e/2)\vec{B}' \times \vec{r}$. $H$ becomes $H(B')$ with the partition function $Z_\beta(B') \equiv \text{Tr} \left\{ e^{-\beta H(B')} \right\}$. First order perturbation theory in $B'$ gives

$$Z_\beta(B') = Z_\beta + \frac{eB'}{2} \text{Tr} \left\{ e^{-\beta H(B')} \left( (\vec{r} \times \vec{v}) \cdot \vec{k} + \sigma_z \right) \right\} + \ldots$$

and thus, the magnetization,

$$M_\beta = \frac{1}{\beta} \lim_{B' \to 0} \frac{\partial \ln Z_\beta(B')}{\partial B'}$$

In the sequel, we will consider the spin-up and spin-down as two distinct physical situations. It follows that in (8), the spin induced part of the magnetization is $\frac{e}{2}\sigma_z$, thus the orbital part of the magnetization is

$$M_{\beta}^{orb} = \frac{1}{\beta} \lim_{B' \to 0} \frac{\partial \ln Z_\beta(B')}{\partial B'} - \frac{e}{2}\sigma_z$$

Note that in the particular case of $H$ containing a homogeneous magnetic field $\vec{B}$ perpendicular to the plane, (8) narrows down to the usual formula

$$M_\beta = \frac{1}{\beta} \frac{\partial \ln Z_\beta}{\partial B}$$

and (9) to

$$M_{\beta}^{orb} = \frac{1}{\beta} \frac{\partial \ln Z_\beta}{\partial B} - \frac{e}{2}\sigma_z$$

Note also that one can compute directly $M_{\beta}^{orb}$ using (8) by simply dropping the coupling of the fictitious $\vec{B}'$ field to the spin in $Z_\beta(B')$, and, accordingly, using (10) in the case of $H$ containing a homogeneous $\vec{B}$ field, the coupling of $\vec{B}$ to the spin in $Z_\beta$.

Let us now turn to the persistent current and consider in the plane a semi infinite line $\mathcal{D}$ starting at $\vec{r}_0$ and making an angle $\theta_o$ with the horizontal $x$-axis. The orbital persistent
current $I^{orb}(\vec{r}_0, \theta_0)$ through the line is

$$I^{orb}(\vec{r}_0, \theta_0) \equiv \int_D d|\vec{r} - \vec{r}_0| \frac{(\vec{r} - \vec{r}_0) \times \langle \vec{j}(\vec{r}) \rangle}{|\vec{r} - \vec{r}_0|} \cdot \vec{k}$$

(12)

where $\frac{(\vec{r} - \vec{r}_0) \times \langle \vec{j}(\vec{r}) \rangle}{|\vec{r} - \vec{r}_0|} \cdot \vec{k}$ is the orthonormal component of $\vec{j}(\vec{r})$ to the line. It obviously depends on $\vec{r}_0$ and $\theta_0$. Consider now systems rotationally invariant around $\vec{r}_0$. $I^{orb}(\vec{r}_0, \theta_0)$ no longer depends on $\theta_0$ and, without loss of generality, can be averaged over $\theta_0$. So

$$I^{orb}(\vec{r}_0) = \frac{1}{2\pi} \int d\vec{r} \frac{(\vec{r} - \vec{r}_0) \times \langle j_\theta(\vec{r}) \rangle}{|\vec{r} - \vec{r}_0|^2} \cdot \vec{k}$$

(13)

and for the Boltzmann distribution

$$I^{orb}_\beta(\vec{r}_0) = \frac{e}{2\pi Z_\beta} \text{Tr} \left\{ e^{-\beta H(\vec{r} - \vec{r}_0)} \frac{(\vec{r} - \vec{r}_0) \times \vec{v}}{|\vec{r} - \vec{r}_0|^2} \cdot \vec{k} \right\}$$

(14)

Again, we want to establish a link between (14) and a thermodynamical quantity. We add to the system a fictitious vortex $\phi'$ located at $\vec{r}_0$ the origin of the line, with potential vector and magnetic field

$$e\vec{A}_v = \alpha' \frac{\vec{k} \times (\vec{r} - \vec{r}_0)}{|\vec{r} - \vec{r}_0|^2}$$

(15)

$$e\vec{B}_v = 2\pi \alpha' \delta(\vec{r} - \vec{r}_0) \vec{k}$$

(16)

where $\alpha' = \phi'/\phi_o - \phi_o$ is the flux quantum and can always be chosen in the interval $[0, 1]$ by periodicity. The resulting Hamiltonian is

$$H(\alpha') = \frac{1}{2} \left( \vec{p} - e\vec{A} - e\vec{A}_v \right)^2 + V - \frac{e}{2} \sigma_z (B + B_v)$$

(17)

with partition function $Z_\beta(\alpha')$.

Though being a well-established fact [4], we recall here that the spin coupling term $\sigma_z (B + B_v)$ in (17) is necessary to define in a non ambiguous way point vortices quantum systems. The behavior of the wavefunctions at the location $\vec{r}_0$ of the fictitious point
vortex (or of a physical point vortex contained in the original Hamiltonian (3)) has to be specified properly. This is precisely achieved in a scale invariant way with the spin coupling term, which materializes either a repulsive hard-core (spin-down) $\pi \alpha' \delta(\vec{r} - \vec{r}_0)$ or attractive (spin-up) $-\pi \alpha' \delta(\vec{r} - \vec{r}_0)$ point contact interaction at $\vec{r}_0$. In the standard Aharonov-Bohm problem, the wavefunctions are chosen to vanish at $\vec{r}_0$, i.e. the spin-down repulsive prescription. However, we might as well take the attractive spin-up prescription. Still, the persistent current should clearly be insensitive to the nature of the fictitious vortex introduced to define it, and in particular on the way it is regularized at short distance.

First order perturbation theory in $\alpha'$ gives

$$Z_{\beta}(\alpha') = Z_{\beta} + \alpha' \beta \left( \text{Tr} \left\{ e^{-\beta H} \frac{(\vec{r} - \vec{r}_0) \times \vec{v}}{|\vec{r} - \vec{r}_0|^2} \cdot \vec{k} \right\} + \pi \sigma_z G_{\beta}(\vec{r}_0, \vec{r}_0) \right) + \ldots$$

(18)

where $G_{\beta}(\vec{r}, \vec{r}') \equiv \langle \vec{r} | e^{-\beta H} | \vec{r}' \rangle$ is the thermal propagator for the Hamiltonian (3). We define the total current around $\vec{r}_0$ as

$$I_{\beta}(\vec{r}_0) \equiv \frac{e}{2\pi \beta} \lim_{\alpha' \to 0} \frac{\partial \ln Z_{\beta}(\alpha')}{\partial \alpha'}$$

(19)

and, from (14),

$$I_{\beta}^{\text{orb}}(\vec{r}_0) = \frac{e}{2\pi \beta} \lim_{\alpha' \to 0} \frac{\partial \ln Z_{\beta}(\alpha')}{\partial \alpha'} - \frac{e}{2 \sigma_z} G_{\beta}(\vec{r}_0, \vec{r}_0) Z_{\beta}$$

(20)

Note that in the particular case of $H$ containing a vortex $\phi$ at $\vec{r}_0$, or consisting of a ring of center $\vec{r}_0$ threaded by a magnetic flux $\phi = \alpha \phi_0$, the orbital persistent current generated in the plane around $\vec{r}_0$, or in the ring\(^2\) (21) narrows down to

$$I_{\beta}^{\text{orb}}(\vec{r}_0) = \frac{e}{2\pi \beta} \frac{\partial \ln Z_{\beta}}{\partial \alpha} - \frac{e}{2 \sigma_z} G_{\beta}(\vec{r}_0, \vec{r}_0) Z_{\beta}$$

(21)

If the first term in (20) is usually discussed in the literature under the weaker form appearing in (21), valid for the point vortex and the ring [1, 3], the second term in (20,21)\(^2\)In the case of a ring, the propagator at the center of the ring is by definition nul.
triggered by the spin coupling to the propagator has been so far ignored. However, both terms are crucial, and if the spin term is absent, some contradictions do arise in the computation of orbital persistent currents. Indeed, orbital persistent currents—as well as orbital magnetizations—should vanish when $H$ becomes free, and this is not the case if the spin term is ignored, as we will see later for the point vortex, or for the point vortex superposed to an homogeneous field. We will also encounter in the attractive point vortex case a situation where the orbital current by itself is not defined (in fact infinite), and where only the total (orbital + spin) current has a sensible physical meaning.

To summarize, orbital magnetizations and currents are respectively given by (9) and (20). They should both vanish when the original Hamiltonian $H$ becomes free. Total magnetizations and currents are given by (8) and (19). They obviously depend, as well as their orbital counterparts, on the spin degree of freedom defined in the original Hamiltonian (3).

If we consider systems that are both invariant by translation and rotation, we can proceed further by averaging (14) over $\vec{r}_0$. Taking advantage of the identity (22)

$$\int d\vec{r}_0 \frac{(\vec{r} - \vec{r}_0) \times \vec{v}}{|\vec{r} - \vec{r}_0|^2} = \pi \vec{r} \times \vec{v}$$

we can write, using (6,14),

$$I_{orb} = \frac{1}{V} \int d\vec{r}_0 I(\vec{r}_0) = \frac{1}{V} M_{orb}$$

which generalizes the conducting ring situation to translation and rotation invariant two-dimensional systems. Furthermore, since for a translation invariant system $G_\beta(\vec{r}_0, \vec{r}_0) = Z_\beta/V$, (23) holds also for the total current and magnetization

$$I = \frac{1}{V} M$$

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Finally, \((23,24)\) still hold for non rotational invariant systems if the persistent current \(I_{\text{orb}}\) is understood as averaged over all directions \(\theta_0\) of the semi infinite straight line.

To have access to the orbital magnetization and persistent currents of a gas of non-interacting electrons, \(Z_\beta(B')\) or \(Z_\beta(\alpha')\) have to be replaced in \((3,20)\), and \(Z_\beta\) in \((11,21)\), by the corresponding grand-partition functions. For instance, \((11)\) and \((21)\) respectively become

\[
M_{\text{orb}} = \frac{1}{\beta} \ln \Xi_\beta(\mu) - N \frac{e}{2} \sigma_z \tag{25}
\]

\[
I_{\text{orb}} = e \frac{\partial}{2\pi} \ln \Xi_\beta(\mu) - \frac{e}{2} \sigma_z \text{Tr} \left\{ \delta(\vec{r} - \vec{r}_0) f(H) \right\} \tag{26}
\]

where \(\Xi_\beta(\mu)\) is the grand-partition function \(-\ln \Xi_\beta(\mu) = \text{Tr} \left\{ \ln(1 + e^{-\beta(H - \mu)}) \right\}\) for chemical potential \(\mu\) and \(f(E)\) is the Fermi-Dirac distribution. At zero temperature, the second quantized magnetization and persistent current at Fermi energy \(E_F\) are by definition, using the integral representation of the step function \(\theta(E_F - H)\),

\[
M_{E_F} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dt \frac{e^{iE_F t'}}{t' - i\eta'} Z_\beta M_\beta |_{\beta \rightarrow it' + \epsilon'} \tag{27}
\]

\[
I_{E_F} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dt \frac{e^{iE_F t'}}{t' - i\eta'} Z_\beta I_\beta |_{\beta \rightarrow it' + \epsilon'} \tag{28}
\]

where \(\eta', \epsilon' \rightarrow 0^+\). One should bear in mind that the number of electrons \(N\) may be fixed, implicitly determining \(E_F\) by

\[
N = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dt' \frac{e^{iE_F t'}}{t' - i\eta'} Z_\beta |_{\beta \rightarrow it' + \epsilon} \tag{29}
\]

We insist here that we regard the first quantized thermalized magnetizations and currents \((3,20)\), or \((8,19)\), as the basic objects from which their second quantized versions can be computed, for instance via \((27)\) and \((28)\) at zero temperature. In particular if \(M_\beta\) or \(I_\beta\) happens not to depend on the temperature, that is to say if the magnetization or current for a quantum state do not depend on the energy of the quantum state, then the second
quantized magnetization and current are simply given by \( M = N M_{\beta} \) or \( I = N I_{\beta} \) - see for instance in Appendix A the current for the point vortex case.

## 3 Applications

### 3.1 The homogeneous magnetic field case

As a warm-up exercise, consider an homogeneous magnetic field perpendicular to the plane. The spin-up \((\sigma_z = +1)\) or down \((\sigma_z = -1)\) partition function

\[
Z_{\beta}^{\sigma_z} = \frac{V}{2\pi \beta} \frac{b}{\sinh b} \exp(\sigma_z b) \tag{30}
\]

where \( b \equiv \beta \omega_c \) and \( \omega_c = eB/2 \) (\( eB > 0 \) is assumed without loss of generality) leads to the orbital magnetization

\[
M_{\beta}^{\text{orb}} = \frac{1}{\beta} \frac{\partial}{\partial B} \ln Z_{\beta}^{\sigma_z} - \frac{e}{2} \left( \frac{1}{b} - \coth b \right) \tag{31}
\]

which in this particular case does not depend on \( \sigma_z \).

Since the system is invariant by translation and rotation, we know a priori that the orbital persistent current is given by \( I_{\beta}^{\text{orb}} = M_{\beta}^{\text{orb}}/V \). Let us however illustrate the considerations above by computing directly the orbital persistent current around a point \( \vec{r}_0 \). Adding a vortex \( \alpha' \) located at this point with a repulsive (spin-down) or attractive (spin-up) prescription, we get \( \mathbb{F} \) for \( 0 \leq \alpha' \leq 1/2 \) the partition function (see Appendix B)

\[
Z_{\beta}^{\sigma_z}(\alpha') = Z_{\beta}^{\sigma_z} + \frac{e^{\sigma_z b}}{2\sinh b} \left( \alpha' - e^{-(\sigma_z + \alpha') b} \frac{\sinh \alpha' b}{\sinh b} \right) \tag{32}
\]

One has also

\[
G_{\beta}^{\sigma_z}(\vec{r}, \vec{r}_0) = \frac{1}{V} Z_{\beta}^{\sigma_z} \tag{33}
\]
Thus, (20) gives the orbital persistent current

\[ I_{\beta}^{\text{orb}} = \frac{e}{2\pi^2} \lim_{\alpha' \to 0} \frac{\partial \ln Z_{\beta}^{\sigma_z}(\alpha')}{\partial \alpha'} - \sigma_z \frac{e}{2V} = \frac{e}{2V} \left( \frac{1}{b - \coth b} \right) \]  

which does not depend on \( \vec{r}_0 \), and indeed satisfies \( M_{\beta}^{\text{orb}} = I_{\beta}^{\text{orb}} V \). Note that it does not depend on \( \sigma_z \), i.e. in particular on the short distance behavior of the fictitious vortex.

### 3.2 The point vortex + a homogeneous magnetic field case

We consider a point vortex carrying a flux \( \phi/\phi_0 = \alpha \) at the origin of the plane superposed to a homogeneous \( \vec{B} \) field, and concentrate on the repulsive spin-down case, bearing in mind that the analysis in the attractive spin-up case would follow the same lines as in the point vortex case (see Appendix A). Using the partition function (32) with \( \alpha' \to \alpha \), \( \sigma_z = -1 \), (11) gives the orbital magnetization in the spin-down case

\[ M_{\beta}^{\text{orb}} = \frac{e}{2} \left( \frac{1}{b} - \coth b \right) - \frac{e\alpha}{2V} \frac{1}{b^2} \left( \alpha - e^{-\alpha b} \frac{\sinh \alpha b}{\sinh b} \right) + \frac{b}{\sinh b} e^{-2(\alpha-1)b} (\alpha - e^{(\alpha-1)b} \frac{\sinh \alpha b}{\sinh b}) + \ldots \]  

where \( \ldots \) denote corrections which are subleading in volume. The vortex does not affect the leading volume term, which is therefore identical to the pure homogeneous magnetic field magnetization [31].

Let us now turn to the persistent current around the vortex location at the origin of the plane.

Since, when \( \alpha > 0 \),

\[ G_{\beta}(\vec{0}, \vec{0}) = 0 \]  

\[ I_{\beta}(\vec{0}) = I_{\beta}^{\text{orb}}(\vec{0}) \]. From (21), the persistent current at leading order in volume reads

\[ I_{\beta}^{\text{orb}}(\vec{0}) = \frac{e}{2V} \left( \frac{1}{b} - \frac{2e^{-2\alpha b}}{1 - e^{-2b}} \right) \]
with indeed $M^{orb}_\beta \neq I^{orb}_\beta V$.

When $\alpha = 0$ on the other hand, the orbital persistent current should narrow down to the homogeneous magnetic field result \([34]\). However, this does not happen for the current in \([37]\). But, if one pays attention to the fact that, when $\alpha = 0$,

$$G_{\beta}(\vec{0}, \vec{0}) = \frac{Z_{\beta}}{V}$$  \((38)\)

where $Z_{\beta}$ stands for the partition function of the homogeneous $\vec{B}$ field, then the spin term is such that in \([21]\) the orbital persistent current correctly narrows down to the orbital current for the homogeneous magnetic field, i.e.

$$I^{orb}_{\beta} = \frac{e}{2V} \left( \frac{1}{b} - \frac{2}{1 - e^{-2b}} \right) + \frac{e}{2V} = \frac{e}{2V} \left( \frac{1}{b} - \coth b \right)$$  \((39)\)

### 3.3 The case of the plane with an electric field

Consider now a homogeneous magnetic field and an electric field $\vec{E}$ in the horizontal $x$-direction (Hall geometry) and ask how the magnetization and the persistent current \((31,34)\) are affected by the electric field. The spectrum is unbounded from below, making the use of thermodynamical quantities as the partition function hazardous. We propose to circumvent this difficulty by confining the electron in a harmonic well (for more details on harmonic regularization see Appendix B). The Hamiltonian in the symmetric gauge $\vec{A} = B\vec{k} \times \vec{r}/2$ reads

$$H = \frac{1}{2}(\vec{p} - e\vec{A})^2 - eE x + \frac{1}{2} \omega^2 \vec{r}^2$$  \((40)\)

with indeed a bounded spectrum. Its propagator is

$$G_{\beta}(\vec{r}, \vec{r}') = G^{(B,\omega)}_{\beta} (\vec{r} - e \frac{\vec{E}}{\omega^2}, \vec{r}' - e \frac{\vec{E}}{\omega^2}) e^{-i\omega \cdot \vec{k} \times (\vec{r} - \vec{r}')} e^{\frac{\mu}{2} (\frac{\vec{k} \cdot \vec{r}}{\omega^2})^2}$$  \((41)\)
where
\[
G_\beta^{(B,\omega)}(\vec{R}, \vec{R}') = \frac{\omega_t}{2\pi \sinh \beta \omega_t} e^{-\frac{\omega_t t}{2 \sinh \beta \omega_t} \left( \cosh \beta \omega_c (\vec{R} - \vec{R}')^2 + 2i \sinh \beta \omega_c (\vec{R} \times \vec{R}') \cdot \vec{k} + (\cosh \beta \omega_t - \cosh \beta \omega_c) (\vec{R}^2 + \vec{R}'^2) \right) }
\]
(42)
is the propagator for the homogeneous $\vec{B}$ field in a harmonic well and $\omega_t = \sqrt{\omega^2 + \omega_c^2}$.

Note that in the thermodynamic limit $\omega \to 0$,
\[
G_\beta(\vec{r}, \vec{r}') = \frac{\omega_c}{2\pi \sinh \beta \omega_c} \exp \left\{ - \frac{\omega_c}{2 \sinh \beta \omega_c} \left[ \cosh \beta \omega_c (\vec{r} - \vec{r}')^2 + 2i \sinh \beta \omega_c (\vec{r} \times \vec{r}') \cdot \vec{k} \right. \right.
\]
\[
\left. - \sinh \beta \omega_c (x + x') \frac{\beta e E}{\omega_c} + \frac{\sinh \beta \omega_c}{2} \left( \frac{1}{\beta \omega_c} - \coth \beta \omega_c \right) \frac{\beta e E}{\omega_c} \left( \frac{\beta e E}{\omega_c} + 2i (y - y') \right) \right]\}
\]
(43)
still depends on $\vec{r}$ at coinciding points $\vec{r} = \vec{r}'$, a direct manifestation of the breaking of translation invariance. The partition function reads (without the spin coupling to the $\vec{B}$ field since we are interested in the orbital magnetization only)
\[
Z_\beta = \frac{1}{2 \cosh \beta \omega_t} \frac{1}{\cosh \beta \omega_c} e^{\frac{\beta E}{\omega_c} \left( \frac{\beta E}{\omega_c} \right)^2}
\]
(44)
If $E \neq 0$, the partition function blows up in the thermodynamic limit $\omega \to 0$. On the contrary, when $E = 0$, it correctly yields back the partition function for the homogeneous magnetic field if the proper thermodynamic limit prescription $\lim_{\omega \to 0} \frac{2\pi \beta}{(\beta \omega)^2} \to V$ is taken.

The orbital magnetization reads
\[
M_{\beta}^{\text{orb}} = \frac{e \sinh \beta \omega_c - \frac{\omega_c}{\omega} \sinh \beta \omega_t}{2 \cosh \beta \omega_t - \cosh \beta \omega_c}
\]
(45)
which happens to coincide with (31) in the thermodynamic limit. The electric field has no effect on the magnetization of the system, a result that can be understood at the semiclassical level by the fact that the electric field only induces a translation of cyclotron orbits.

Turning now to the persistent current around a point $\vec{r}_0$, a fictitious vortex $\alpha'$ located at $\vec{r}_0$ is added to the system, that we take to be repulsive. All what is needed is the
partition function of an homogeneous magnetic field in presence of an electric field and a harmonic well, and a repulsive vortex at $\vec{r}_0$, at leading order in $\alpha'$. It can be obtained in perturbation theory along the lines developed in [7]. The harmonic well has not only the virtue, as we have seen, to make tractable the computation of an otherwise diverging partition function with an $\vec{E}$ field, but it is also needed [7] to give an unambiguous meaning to the perturbative analysis in $\alpha'$, which is otherwise ill-defined.

The first order perturbative correction to the partition function $Z_{\beta}(\alpha') = Z_\beta + \alpha' Z^1_{\beta} + \ldots$

$$Z^1_{\beta} = 2\beta \int dz \; d\bar{z} \frac{1}{z - \bar{z}_0} (\partial_z - \frac{1}{2} \omega_c \bar{z}) G_\beta(\vec{r}, \vec{r}') |_{\vec{r}' = \vec{r}}$$

where $z = x + iy$ is the complex coordinate in the plane, gives an orbital persistent current

$$I^{orb\beta}(\vec{r}_0) = \frac{e}{2\pi \beta} \frac{Z^1_{\beta}}{Z_\beta} + \frac{e}{2Z_\beta} G_\beta(\vec{r}_0, \vec{r}_0)$$

equal to

$$I^{orb\beta}(\vec{r}_0) = \frac{e}{2\pi} e^{-\frac{\omega t}{\sinh \beta \omega t}} (\cosh \beta \omega t - \cosh \beta \omega_c) |\vec{r}_0 - e \frac{\vec{E}}{\omega^2}|^2 \frac{\omega_t}{\sinh \beta \omega t} \left( \frac{\sinh \beta \omega_c - \omega_c}{\omega_t} \sinh \beta \omega_t \right)$$

The current is exponentially damped when the point $\vec{r}_0$ around which it is estimated differs from the equilibrium position in the harmonic well $\vec{r}_0 - e \frac{\vec{E}}{\omega^2} = 0$. When $\vec{E} = 0$, (48) coincides, in the thermodynamic limit, with the orbital current for the homogeneous magnetic field (34), as it should. When $\vec{E} \neq 0$, this is still the case if $\vec{r}_0 = e\vec{E}/\omega^2$, which means that the effect of the electric field it to locate the current at the edge of the sample. However, as soon as $\vec{r}_0 \neq e\vec{E}/\omega^2$, the orbital current $I^{orb\beta}(\vec{r}_0)$ vanishes in the thermodynamic limit.
4 The Magnetic Impurity Problem

Consider a random Poissonian distribution \( \{ \vec{r}_i, i = 1, 2, \ldots, N \} \) of \( N \) hard-core vortices at position \( \vec{r}_i \), carrying a flux \( \phi = \alpha \phi_o \), where \( \alpha \) can always be taken in the interval \([0, 1/2]\) without loss of generality. After averaging over disorder, the system is invariant by translation and rotation, so it is sufficient to compute \( M^{\text{orb}} \) to get the orbital persistent current, since they are related by (B). The Hamiltonian is

\[
H = \frac{1}{2} \left( \vec{p} - e \vec{A}(\vec{r}) \right)^2 + \frac{e}{2} B(\vec{r})
\]  

(49)

with

\[
e \vec{A}(\vec{r}) = \alpha \sum_{i=1}^{N} \vec{k} \times \frac{(\vec{r} - \vec{r}_i)}{|\vec{r} - \vec{r}_i|^2}
\]

(50)

\[
e B(\vec{r}) = 2\pi \alpha \sum_{i=1}^{N} \delta(\vec{r} - \vec{r}_i)
\]

(51)

\( B(\vec{r}) \) might be replaced by its mean value

\[
\langle B \rangle = \frac{2\pi \rho \alpha}{e}
\]

(52)

a mean field approximation that is valid in the limit \( \alpha \to 0 \), and \( \rho \equiv N/V \), the vortex density, fixed.

In a Brownian motion approach, for a given configuration of vortices, the partition function is

\[
Z_\beta = Z_\beta^0 \left\langle \exp \left( i \sum_{i=1}^{N} 2\pi n_i \alpha \right) \right\rangle_{\{C\}}
\]

(53)

where \( \left\langle \right\rangle_{\{C\}} \) means averaging over all closed Brownian curves of length \( \beta \), and \( n_i \) is the winding number around the vortex at location \( \vec{r}_i \). \( Z_\beta^0 = V/(2\pi \beta) \) is the free partition function.
Averaging over all vortices configurations leads to

\[ Z_\beta = Z_\beta^0 \left\langle \exp \left( \rho \sum_n S_n (e^{2i\pi\alpha n} - 1) \right) \right\rangle_{\{C\}} \tag{54} \]

where \( S_n \) stands for the arithmetic area of the \( n \)-winding sector of the given Brownian curve chosen in \( \{C\} \). Due to scaling properties of Brownian curves \([5, 8]\), the random variable \( S_n \) scales like \( \beta \), thus the rescaled variables

\[ S = \frac{2}{\beta} \sum_n S_n \sin^2(\pi \alpha n) \quad A = \frac{1}{\beta} \sum_n S_n \sin(2\pi \alpha n) \tag{55} \]

The probability distributions of \( S \) and \( A \) are actually independent of \( \beta \) with \( \langle S \rangle_{\{C\}} = \pi \alpha (1 - \alpha) \) and \( \langle A \rangle_{\{C\}} = 0 \).

To get the magnetization, we add a fictitious magnetic field \( \vec{B}' \), with partition function (without the spin coupling to the fictitious \( \vec{B}' \) field since we are interested in the orbital magnetization only)

\[ Z_\beta(B') = Z_\beta^0 \left\langle e^{-\beta \rho (S - iA) + i\beta B' A} \right\rangle_{\{C\}} \tag{56} \]

where \( A \equiv (1/\beta) \sum_n nS_n \) is the rescaled algebraic area enclosed by the Brownian curve in \( \{C\} \). From (8), and since \( S \to S, A \to -A \), and \( A \to -A \) when one spans a Brownian curve in the opposite direction, the orbital magnetization is

\[ M_{\beta}^{\text{orb}} = -\frac{\langle e^{-\beta \rho S} \sin(\beta \rho A) A \rangle_{\{C\}}}{\langle e^{-\beta \rho S} \cos(\beta \rho A) \rangle_{\{C\}}} \tag{57} \]

\( M_{\beta}^{\text{orb}} \) is actually only a function of \( \beta \rho \) and \( \alpha \), odd in \( \alpha \). Thus, for \( \alpha \in [0, 1/2] \), necessarily

\[ M_{\beta}^{\text{orb}} = (1 - 2\alpha) F(\beta \rho, \alpha(1 - \alpha)) = (1 - 2\alpha) \sum_{n=1}^{\infty} (\beta \rho)^n \sum_{m \geq n} a_{mn}(\alpha(1 - \alpha))^m \tag{58} \]

which can in principle be obtained in perturbation theory \([3, 4]\).
The $\rho^n\alpha^m$ term describes an electron interacting $m$ times with $n$ vortices, so $m \geq n$. The mean magnetic field (52) scales like $\rho\alpha$, therefore the mean field terms are those with $m = n$. They are in fact easy to obtain, since $\alpha \to 0$ corresponds to the mean field regime. Thus, from (31,52,58),

$$\langle M^{orb}\rangle_{\beta \gamma} = \frac{e}{2} \left( \frac{1}{\langle b \rangle} - \coth \langle b \rangle \right)$$

(59)

where $\langle b \rangle = \beta \langle \omega_c \rangle$, with $\langle \omega_c \rangle = e\langle B \rangle / 2 = \pi \rho \alpha$. From (58), one infers that $M^{orb}_{\beta \gamma}$ necessarily contains

$$M^{orb}_{\beta \gamma}|_{mean} = (1 - 2\alpha)e \left( \frac{1}{\langle b \rangle'} - \coth \langle b \rangle' \right)$$

(60)

where $\langle b \rangle' = \beta \pi \rho \alpha (1 - \alpha)$.

It is even possible to go a little bit further, by observing that the $\rho^n\alpha^{n+1}$ terms come from the homogeneous $(B)$ field + one vortex case [6]. The first correction to the mean field approximation (60) is

$$M^{orb}_{\beta \gamma} = M^{orb}_{\beta \gamma}|_{mean} + e\alpha (1-\alpha)(1-2\alpha) \left( \frac{1}{2\langle b \rangle'} + \frac{\langle b \rangle' - 1 - e^{-2\langle b \rangle'}}{1 - e^{-2\langle b \rangle'}} + \frac{2\langle b \rangle'(1-\langle b \rangle')e^{-2\langle b \rangle'}}{(1 - e^{-2\langle b \rangle'})^2} \right) + \cdots$$

(61)

In Fig. 1 ($\beta \rho = 1$), the agreement is rather good between (61) - full curve - and the numerical simulations - points - based on (57). We generated 2000 random walks of 100000 steps each one. However, the situation becomes less transparent for higher $\beta \rho$ values. Clearly, the perturbative analytical approach need more and more corrections coming from $(B)$ + two vortices, .... The numerical simulations indicate that, for $0 < \alpha < 1/2$, $M^{orb}_{\beta \gamma}$ is negative whatever $\beta \rho$ is. Even if it is physically clear why it should be so since the vortices enforce a clockwise rotation of the electrons when $0 < \alpha < 1/2$, an analytical proof is missing.
A Appendix A: Persistent current for the ring and the point vortex

A.1 The case of the ring revisited

Consider a ring of radius $R$ threaded by a homogeneous magnetic field $B$ perpendicular to the ring, with flux $\phi$ through the ring. Let us denote as usual $\alpha = e\phi / 2\pi = \phi / \phi_0$ the value of $\phi$ in unit of the quantum of flux. The system is periodic in $\alpha$ with period 1. Reversing the sign of $B$ does not change the physics either -the partition function is symmetric around $\alpha = 1/2$, whereas the magnetization and current are antisymmetric-, so one can always restrict $0 \leq \alpha \leq 1/2$. Clearly, the orbital persistent current changes its sign with $\phi$. But $\phi = \pm 1/2$ are identical, thus necessarily $I_\beta^{orb}(\alpha = 1/2) = 0$.

To summarize, both first and second quantized orbital currents should be defined in the interval $\alpha \in [0, 1/2]$, and due to obvious symmetry considerations, they should vanish when $\alpha = 0, 1/2$.

In unit $2R^2 = 1$ the partition function is

$$Z_\beta(\alpha) = \sum_{m=-\infty}^{m=\infty} e^{-\beta (m-\alpha)^2}$$  \hspace{1cm} (62)

The thermal current is formally defined as in (20). However, it is equivalent to thread the ring with an homogeneous $\vec{B}$ field or at its center with a vortex line of flux $\phi$. It follows that (20) narrows down to (21). Clearly no distinction has to be made either between the orbital and total magnetization (or current) since by geometry the propagator at the center of the ring is zero. Thus

$$I_\beta = I_\beta^{orb} = \frac{e}{2\pi \beta} \frac{\partial \ln Z_\beta}{\partial \alpha}$$  \hspace{1cm} (63)

and

$$M_\beta = M_\beta^{orb} = \frac{1}{\beta} \frac{\partial \ln Z_\beta}{\partial B}$$  \hspace{1cm} (64)
Since $\phi = \alpha 2\pi / e = B\pi R^2$, one necessarily has $M_\beta = I_\beta \pi R^2$.

One obtains

$$ I_\beta = -\frac{e}{2\pi} \left( 2\alpha + 2 \sum_{k=1}^{\infty} \frac{\sinh(2k\beta\alpha)}{\sinh(k\beta)}(-1)^k \right) $$

or equivalently, using the reciprocity formula,

$$ Z_\beta(\alpha) = \sqrt{\frac{\pi}{\beta}} e^{-\alpha^2 \beta} \frac{i\alpha\beta}{\pi} $$

$$ I_\beta = \frac{e}{\beta} \sum_{k=1}^{\infty} \frac{\sin(2\pi k\alpha)}{\sinh(k\pi^2/\beta)}(-1)^k $$

Clearly, from (65, 67)

$$ I_\beta(\alpha = 0) = 0 = I_\beta(\alpha = 1/2) $$

as it should. Note that (65, 67) are, to our knowledge, the simplest expressions obtained so far for the thermalized persistent current on a ring. (67) is the sine Fourier series of a periodic and odd function, making the symmetries of $I_\beta$ explicit.

Considering now a second quantized gas of electrons on the ring, starting from $I_\beta$, the second quantized persistent current at zero temperature and Fermi energy $E_F$ is given by (28). However, since individual persistent currents for each quantum states are simple to obtain, one may sum individual currents up to the Fermi energy $E_F$. The current carried by the state $|m> = \cos(\theta)$

It follows that if $N$ is even,

$$ I_{E_F} = \frac{e}{2\pi} N(1 - 2\alpha) $$

and if $N$ is odd,

$$ I_{E_F} = -\frac{e}{\pi} \alpha N $$

3From now on we revisit the approach followed in [2].
These results seem misleading since the currents in (69), (70) do not vanish either at $\alpha = 0$ or $\alpha = 1/2$. However, their derivation in terms of individual quantum states is valid for a spectrum with no degenerate states, i.e. if and only if $0 < \alpha < 1/2$. Indeed, when $\alpha = 0$, the states $|m>$ and $|-m>$ are degenerate, whereas when $\alpha = 1/2$, the states $|1-m>$ and $|m>$ are degenerate. It follows that (69) is uncorrect when $\alpha = 0$, and (70) is uncorrect when $\alpha = 1/2$. If the degeneracy is properly taken into account, meaning that a given degenerate doublet of states contributes to the current as $I_m + I_{-m}$ when $\alpha = 0$, or $I_{1-m} + I_m$ when $\alpha = 1/2$, the total current ends up to vanish as it should when $\alpha = 0, 1/2$.

We conclude that the persistent current for a gas of electrons at zero temperature on a ring threaded by a flux $\alpha \phi_0$ is respectively given by (69) or (70) when $N$ is even or odd if $0 < \alpha < 1/2$, and 0 if $\alpha = 0, 1/2$.

A.2 The case of point vortex revisited

Consider finally a point vortex of flux $\phi = \alpha \phi_0$ at the origin of the plane, which can be either hard-core (spin-down repulsive prescription), or attractive (spin-up attractive prescription), clearly two different physical situations.

As in the magnetic impurity and ring cases, one can always restrict the interval of definition of $\alpha \in [0, 1/2]$. However, to illustrate the non trivial role of the spin, we will explicit the results in the whole interval $\alpha \in [0, 1]$.

The spin-down and (up) partition function respectively read (see Appendix B)

$$Z^- - Z^0 = \frac{1}{2} \alpha (\alpha - 1) \quad 0 \leq \alpha < 1 \quad (71)$$

$$Z^+ - Z^0 = \frac{1}{2} \alpha (\alpha + 1) \quad 0 \leq \alpha \leq \frac{1}{2}$$

$$Z^+ + Z^- = \frac{1}{2} (\alpha - 1)(\alpha - 2) \quad \frac{1}{2} \leq \alpha < 1 \quad (72)$$
leading to the total current around the origin

\[ I_\beta^- = \frac{e}{V} \left( \alpha - \frac{1}{2} \right) \quad 0 \leq \alpha < 1 \]  
\[ (73) \]

\[ I_\beta^+ = \frac{e}{V} \left( \alpha + \frac{1}{2} \right) \quad 0 \leq \alpha < \frac{1}{2} \]
\[ I_\beta^+ = \frac{e}{V} \left( \alpha - \frac{3}{2} \right) \quad \frac{1}{2} < \alpha < 1 \]  
\[ (74) \]

which is discontinuous at \( \alpha = 1/2 \) in the spin-up case.

We would like to identify the orbital part \( I_\beta^{orb} \) of the currents \((73, 74)\).

If \( \alpha = 0 \), i.e. the free case, by definition

\[ G_\beta^+ (\vec{0}, \vec{0}) = G_\beta^- (\vec{0}, \vec{0}) = \frac{Z_\beta^0}{V} \]  
\[ (75) \]

One rightly deduces

\[ I_\beta^{orb} = \lim_{\alpha \to 0} \frac{e}{V} (\alpha + \sigma_z \frac{1}{2}) - \sigma_z \frac{e}{2V} = 0 \]  
\[ (76) \]

as it should.

If \( 0 < \alpha < 1 \), in the repulsive spin-down case,

\[ G_\beta^- (\vec{0}, \vec{0}) = 0 \]  
\[ (77) \]

We conclude that the orbital persistent current generated by a repulsive vortex at the origin is

\[ I_\beta^{orb-} = 0 \quad \alpha = 0 \]  
\[ (78) \]

\[ I_\beta^{orb-} = \frac{e}{V} (\alpha - \frac{1}{2}) \quad 0 < \alpha < 1 \]  
\[ (79) \]

\(^4\)Computing the current around a point \( \vec{r}_0 \) would require the partition function of a vortex \( \alpha \) at the origin and another fictitious vortex \( \alpha' \) at \( \vec{r}_0 \) at leading order in \( \alpha' \).
On the other hand, in the attractive spin-up case, the propagator at the location of the vortex is infinite as soon as $\alpha \neq 0$, thus $I^\text{orb+}_\beta$ diverges. We conclude that the orbital persistent current generated by an attractive vortex is not defined (in fact infinite), only the total current (orbital + spin) is defined and given by \([74]\).

The thermalized persistent currents \([73, 74]\) do not depend on the temperature. It follows trivially that for a gas of $N$ electrons, in the interval $0 < \alpha < 1/2$ for instance, the persistent current rewrites as

$$I^{\sigma_z} = e \rho_e (\alpha + \sigma_z^2)$$

where $\rho_e = N/V$ is the electron density.

The magnetization induced by a vortex in the plane \([42]\) is deduced from the partition function of a vortex $\alpha + a$ fictitious $\vec{B}'$ field, i.e. \([32]\), with $\alpha' \to \alpha$ and $\vec{B} \to \vec{B}'$. One gets

$$M^{\text{orb}}_{\beta} - \beta = -\frac{1}{V} e^{\pi \beta \frac{3}{\alpha}} (1 - \alpha)(1 - \alpha) 0 < \alpha < 1$$

and

$$M^{\text{orb}}_{\beta} + \beta = -\frac{1}{V} e^{\pi \beta \frac{3}{\alpha}} (1 + \alpha)(1 + \alpha) 0 < \alpha < 1$$

$$M^{\text{orb}}_{\beta} = -\frac{1}{V} e^{\pi \beta \frac{3}{\alpha}} (1 - \alpha)(2 - \alpha)(3 - \alpha) 1/2 < \alpha < 1$$

which are subleading in volume compared to the homogeneous field case. The magnetization for a gas of electrons is, for $0 \leq \alpha < 1/2$,

$$M^{\text{orb}, \sigma_z} = -\frac{e}{6} \alpha (\sigma_z + \alpha) (\sigma_z^2 + \alpha) \frac{1}{e^{-\beta \mu} + 1}$$

with the additional Fermi factor at zero energy. If it is to be compared to the current $I^{\sigma_z}$ in \([80]\), one sees that both the current and the magnetization induced by a point vortex in the plane for a gas of electrons have the same leading volume behavior.
Appendix B: Harmonic regularization and Naive Landau counting rule

B.1 Harmonic regularization

Partition functions for systems with, in the thermodynamic limit, a continuous spectrum (point vortex) or a discrete but infinitely degenerated spectrum (homogeneous magnetic field) can be obtained in a non ambiguous way by regularizing the system at long distance by adding an harmonic term $\omega^2 \vec{r}^2/2$ to the Hamiltonian, computing the partition function in the presence of the regulator, and then taking the thermodynamic limit $\omega \to 0$ with the prescription $1/(\beta \omega)^2 \to V/\lambda^2$, where $\lambda = \sqrt{2\pi\beta}$ is the thermal wavelength.

Let us illustrate this point by computing the partition function for the repulsive point vortex \( \text{eq}[71] \). In the absence of a harmonic well, the spectrum is the continuous free spectrum with for eigenstates Bessel functions with shifted angular momentum $m \to m - \alpha$. It is quite tedious and subtle to compute the partition function directly in the thermodynamic limit \[9\]. Consider rather the same problem in an harmonic well. The spectrum is now discrete with a finite degeneracy

$$E_{n,m} = \omega(2n + |m - \alpha| + 1)$$

with $n \geq 0$. This is nothing but the spectrum of a two-dimensional harmonic oscillator with shifted angular momentum $m \to m - \alpha$. The partition function reads

$$Z^{\omega}_{\beta} = \frac{\cosh \beta \omega(\alpha - 1/2)}{2 \sinh \beta \omega \sinh \beta \omega/2}$$

(85)

Of course it diverges when $\omega \to 0$, but the difference \[H\]

$$Z^{\omega}_{\beta} - Z^{0\omega}_{\beta} = \frac{\sinh \beta \omega \alpha/2 \sinh \beta \omega(\alpha - 1)/2}{\sinh \beta \omega \sinh \beta \omega/2} \to_{\omega \to 0} \frac{\alpha(\alpha - 1)}{2}$$

(86)
remains finite in the thermodynamic limit and yields (74).

The partition function for an homogeneous magnetic field (30) can be obtained along the same lines, with the spectrum

\[ E_{nm} = \omega_t(2n + |m| + 1) - \omega_c m \]  

and in the case repulsive point vortex + homogeneous \( B \) field, with the spectrum

\[ E_{nm} = \omega_t(2n + |m - \alpha| + 1) - \omega_c (m - \alpha) \]  

In the latter case, one computes \( Z_{\omega}(B, \alpha) - Z_{\omega}(B, 0) \) to obtain, in the thermodynamic limit \( \omega \to 0 \), (32).

B.2 Naive Landau counting rule

The so-called “Landau counting rule” in terms of individual states happens to lead to sometimes uncorrect results, although one sums only over a finite number of Landau quantum states. Again, correct results are recovered if the harmonic regularization is used instead.

Consider for example the spectrum (88) where, in the thermodynamic limit \( \omega \to 0 \), i.e. \( \omega_t \to \omega_c \), the vortex induces a modification of the Landau spectrum only for a finite number of states of energy \( \omega_c (2j + 1 + 2\alpha) \), \( j \geq 0 \), namely the states of orbital angular momentum \( -j \leq m \leq 0 \). Computing the partition function (88) directly in the thermodynamic limit with this shifted Landau spectrum would lead to a result different from (32). The harmonic regulator adds a contribution to the partition function from the states \( m > 0 \) in (88), that remains finite in the thermodynamic limit, eventhough it is an infinite sum of vanishing terms, and yields the correct expression (32).

\(^{5}\)The same result can be obtained by Zeta function regularization - see the second reference CMO in [3].
One can illustrate the failure of the “Landau counting rule” on an other example. Consider the thermalized orbital persistent current for an homogeneous magnetic field (34)

\[ I_{\text{orb}}^{\beta} = \frac{1}{Z_\beta} \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} I_{n,m}^{\beta} e^{-\beta E_{n,m}} \] (89)
in terms of the individual persistent current \( I_{n,m} \) for the Landau quantum states \(| n, m \rangle\) of orbital momentum \( m \) and energy \( E_{n,m} = (2n + 1 + |m|)\omega_c - m\omega_c \) (i.e. (87) in the thermodynamic limit). The individual currents write

\[ I_{n,m}^{\text{orb}} \equiv \frac{e}{2\pi} \langle n, m | \frac{v}{r} | n, m \rangle = \frac{e\omega_c}{2\pi} \left( \frac{m}{|m|} - 1 \right) \quad \text{if } m \neq 0 \]
\[ = \frac{e\omega_c}{2\pi} (-1) \quad \text{if } m = 0 \] (90)

Despite the fact that the number of states in a given Landau level is infinite, again only a finite number of states contribute to the persistent current for each Landau level. Thus the sum (89) is convergent. Yet, following this procedure, one gets the uncorrect result

\[ I_{\beta}^{\text{orb}} = -\frac{e}{2\pi} \coth b. \] This is again due to the fact that, in this naive “Landau counting rule”, if the \( m > 0 \) states yield vanishing individual currents, they are still an infinite number of them. To give an unambiguous meaning to this infinite summation of vanishing contributions, a long distance regulator is needed, bearing in mind that the thermodynamic limit should be taken afterwards the summation over the quantum numbers \( n, m \). Indeed, if one adds the harmonic regulator to the Landau Hamiltonian (the spectrum is now given by (87)), one finds

\[ I_{n,m}^{\text{orb}} = \frac{e\omega_t}{2\pi} \left( \frac{m}{|m|} - \frac{\omega_c}{\omega_t} \right) \quad \text{if } m \neq 0 \]
\[ = \frac{e\omega_c}{2\pi} (-1) \quad \text{if } m = 0 \] (91)

where \( \omega_t = \sqrt{\omega^2 + \omega_c^2} \). Now the \( m > 0 \) do have individual non vanishing persistent currents, and their total contribution to the thermalized current does not vanish in the
thermodynamic limit. It is precisely equal to $\frac{e}{\pi}$, a contribution which, when added to those of the $m \leq 0$ states, yield the correct result (34) for $I_{\beta}^{\text{orb}}$.

The same situation happens for the thermalized current induced in the plane by a repulsive point vortex superposed to an homogeneous $\vec{B}$ field with individual currents

$$I_{n,m}^{\text{orb}} = \frac{e\omega t}{2\pi} \left( \frac{m - \alpha}{|m - \alpha|} - \frac{\omega t}{\omega c} \right)$$

(92)

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Figure 1: The orbital magnetization in unit $e = 1$ in the magnetic impurity problem for $\beta \rho = 1$. Comparison between the analytical computation (61), full curve, and numerical simulations (57). The points were obtained by generating 2000 curves of 100000 steps each one.