Collective Ion Drag Force

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Abstract—The forces acting on the ensemble of dust grains in a plasma flow are analyzed. It is shown that the nonreciprocal character of the forces results in the appearance of an ion drag force, which depends on the intergrain distance. Results of calculations for two grains and an unbounded hexagonal lattice are presented. Estimates show that the collective ion drag force under typical dusty plasma conditions can be comparable with the weight of an individual grain.

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1. INTRODUCTION

Under typical ground-based laboratory conditions, an ensemble of negatively charged dust grains placed in gas-discharge plasma levitates at a certain height over the horizontal electrode. The equilibrium levitation height is determined by the balance between the gravitational force and the electric repulsion from the electrode. The near-electrode region is characterized by the presence of an electrode-directed ion flow, whose speed is comparable with the ion-acoustic velocity. These and many other aspects of dusty plasma physics were discussed in more detail, e.g., in recent monographs [1–3].

At the same time, under zero-gravity conditions, dust grains are arranged in the plasma volume, where, in addition to the electric forces, a significant role is played by the ion drag force or the friction force, whereas the gravity force is absent. The ion drag force arises due to two small-scale processes: ion scattering in the electric field of a grain and absorption of ions upon their direct collisions with the grain. This results in the momentum exchange between the dust grain and the ion flow. However, numerical simulations (see, e.g., [4]) have demonstrated that the ion drag force acting on an individual dust grain in the near-electrode region is much smaller than the forces of gravity and electric repulsion from the wall and can thus be neglected in the general force balance.

Figure 1 schematically shows the forces acting on a dust grain in the near-wall sheath. The equilibrium levitation height $z_0$ of an individual dust grain is determined by the balance of forces, $Mg + QE(z_0) = 0$, where $M$ and $Q$ are the mass and charge of the dust grain, respectively; $g$ is the gravitational acceleration; and $E(z)$ is the electric field in the sheath. The presence of an ion flow (shown by the vector $u$ in Fig. 1) leads to a strong asymmetry of the electric field created by a point charge. Due to the Cherenkov effect, wakefield ion-acoustic oscillations are excited and the electric field of the dust grains within the Mach cones (shaded areas in Fig. 1) is an oscillating function of coordinates. Beyond the Mach cones, the electric field usually decreases monotonically with distance from the dust grain.

The complicated distribution of the field around a point charge in a plasma flow was studied both analytically and numerically by using different plasma models (see [1–3]). In the general case, the electric potential $\phi(r)$ of the field produced by a point charge placed

![Fig. 1. Scheme of forces acting on grains in the near-wall sheath.](image-url)
2. INTERACTION BETWEEN DUST GRAINS

In a uniform plasma flow, the potential of an immobile charged particle located in the coordinate origin is given by

\[
\phi(r) = \frac{4\pi Q}{(2\pi)^3} \int d^3 k \frac{\exp(i k \cdot r)}{k^2 \varepsilon(-k \cdot u, k)},
\]

where \(Q\) is the particle charge and \(u\) is the flow velocity. Here, it is assumed that the plasma flow is directed toward \(z \rightarrow -\infty\) (Fig. 1). In what follows, all spatial variables are normalized to the characteristic length \(\lambda = u/\omega_p\), where \(\omega_p\) is the ion plasma frequency, and the interparticle forces are normalized to \(Q^2/\lambda^2\). In dimensionless variables, the force acting on another particle with the same charge \(Q\) located at a point \(r\) is \(F(r) = -\nabla U(r)\). The potential function \(U(r) = U(\rho, z)\) is obtained from expression (1) by integration over the polar angle,

\[
U(\rho, z) = \frac{1}{4\pi \rho^3} \int_0^\infty \int_0^\infty \frac{k J_1(k \rho) \exp(i k z)}{(k^2 + k^2)(k^2 + k^2)} \ dv dk.
\]

It is assumed that the plasma consists of a cold ion flow and Maxwellian electrons with a temperature \(T_e\). In this case, the plasma permittivity in dimensionless variables has the form

\[
\epsilon(\omega, k) = 1 + \frac{M^2}{k^2} - \frac{1}{\alpha(\omega + i\nu)}.
\]

Here, \(v\) is the ion–neutral collision frequency, which is assumed to be infinitely small, and \(M = (n_e/n_i) u_i / T_e\), where \(n_e\) and \(n_i\) are the equilibrium electron and ion densities and \(m_i\) is the mass of an ion. In the general case, plasma in the near-wall sheath is nonneutral, i.e., \(n_e \neq n_i\); however, the plasma nonuniformity is disregarded. Of interest is the case of a purely ion flow, when the effect of electrons can be neglected and \(M = 0\). For quasineutral plasma with \(n_e = n_i\), the parameter \(M\) coincides with the Mach number of the flow, i.e., with the ratio of the speed \(u\) to the ion-acoustic velocity.

The integrand in expression (2) as a function of \(k_e\) has four poles. In the limit \(\nu \rightarrow 0\), two of them are complex conjugated, \(k_e = \pm i q_e(k)\), whereas two other lie near the real axis, \(k_e = \pm q_e(k) - i 0\). In an explicit form, they can be written as \(q_e(k) = (\pm 2\alpha(k) + \beta(\omega)) / 2\), where \(\alpha(k) = \kappa^2 + M^2 - 1\) and \(\beta(\omega) = \sqrt{\alpha(\omega)} + 4k^2\). The integral with respect to \(k_e\) in expression (2) can easily be calculated and the resulting potential is divided into the sum of two terms:
$U(\rho, z) = V(\rho, z) + \theta(-z)W(\rho, z)$, where $\theta(z)$ is the Heaviside step function. The first term,

$$V(\rho, z) = \int_0^\infty dk \frac{kq_i(k)}{\beta(k)} e^{-\rho k|z|} J_0(k\rho), \quad (4)$$

is caused by the complex conjugated poles $k_\pm = \pm q_i(k)$, whereas the second one,

$$W(\rho, z) = 2\int_0^\infty dk \frac{kq_i(k)}{\beta(k)} \sin(q_i(k)z)J_0(k\rho), \quad (5)$$

appears due to the real poles $k_e = \pm q_i(k)$. The even function of $z$ defined by integral (4) is usually regarded as a screened Coulomb potential, while integral (5) describes wakefield ion-acoustic oscillations (or purely ion oscillations at $M = 0$) formed downstream of the point charge.

Integrals (4) and (5) are calculated numerically; different limiting cases were considered in detail in the literature. In the general case, expression (2) is a smooth function at $\rho \neq 0$ and $z \neq 0$. Near the coordinate origin ($z \to 0$, $\rho \to 0$), the total potential tends to the Coulomb potential $U(\rho, z) = 1/\sqrt{\rho^2 + z^2}$. Note that integral (5) acquires an additional singularity in the cold ion approximation. Since $q_i(k) \to 1$ at $k \to \infty$, expression (5) logarithmically diverges at $\rho \to 0$. This divergence is smoothed when the finite ion temperature is taken into account. Below, we will be mainly interested in the behavior of the potential at $z = 0$ and $\rho > 0$; hence, expressions (4) and (5) are sufficient for the purposes of this study.

Although total potential (2) is a smooth function of coordinates at $z = 0$, the derivatives of integrals (4) and (5) are discontinuous at $z = 0$. A typical example is presented in Fig. 2, where the solid line shows $U(\rho, z)$ as a function of $z$ and the dashed line shows even function (4). At $z > 0$, both curves coincide, while the difference at $z < 0$ is related to wakefield ion-acoustic waves (5). Due to the excitation of wakefield oscillations, the maximum of $U(\rho, z)$ at a fixed value of $\rho$ shifts from the plane $z = 0$ and the potential derivative is nonzero, $\partial U(\rho, 0) \neq 0$.

Let us assume that two particles levitate at the same height $z = z_i$ and the distance between them is somehow kept constant. For example, this can be achieved by superposing an additional field preventing the grains from moving in the horizontal plane. Digressing from dusty plasma, one can imagine an elongated spacecraft in which opposite ends carry charges of the same sign in the solar wind flow.\(^1\)

As shown in Fig. 1, the first grain pushes the second one upward (or downward), while the second grain does the same with the first one in the same direction. The vertical forces acting on each grain $f_z(\rho) = -U_z(\rho, 0)$ are equal and depend on the distance between the grains. Since this effect disappears with increasing intergrain distance, we call them collective ion drag forces. In the absence of an external confining potential, such forces should lead to the acceleration of a pair of grains, i.e., such a complex behaves as a molecular motor. Note that, depending on the plasma parameters, the direction of the collective ion drag force can be either positive or negative and the pair of grains can shift either downstream or upstream.

In the near-wall sheath, the additional force confining dust grains can be approximately written as

$$F_{z0}(z) = -\omega_k^2(z - z_0),$$

where $z_0$ is the equilibrium levitation height and $\omega_k$ is the oscillation frequency of an individual dust grain. The vertical intergrain forces lead to a change in the equilibrium levitation height $z_i$ of a pair of grains. The height can be found from the relationship $f_z(\rho) = -\omega_k^2(z_i - z_0) = 0$.

Since the potential $U(\rho, z)$ is continuous, its derivative can be calculated using integral (4) in the limit $z \to +0$. For numerical calculations, it is convenient to subtract the Coulomb potential from expression (4) and represent the result in the form $V(\rho, z) = 1/\sqrt{\rho^2 + z^2} + \delta V(\rho, z)$, where

$$\delta V(\rho, z) = \int_0^\infty dk \left[ \frac{kq_i(k)}{\beta(k)} e^{-\rho k|z|} - e^{-k|z|} \right] J_0(k\rho).$$

Then, the force is determined by a well-converging integral

$$f_z(\rho) = \int_0^\infty kdk \chi(k) J_0(k\rho), \quad (6)$$

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\(^1\) The feasibility of such a project is not discussed here.
where \( \chi(k) = \frac{q_2(k)}{\beta(k)} \). In a particular case of a purely ion flux \( M = 0 \) considered in [7], we have \( \chi(k) = 1/(1 + k^2) \) and function (6) can be written explicitly as \( f_\rho(p) = - K_0(p) \), where \( K_0(p) \) is the modified Bessel function of the second kind. For \( M > 0 \), integral (6) can easily be calculated by standard numerical methods.

Figure 3 presents several examples of the dependence of the vertical component of the force on the intergrain distance for different Mach numbers: \( M = (1) 2, (2) 0.8, \) and \( (3) 0. \)

The collective ion drag force increases appreciably with increasing number of grains. Let us consider a hexagonal lattice formed of grains levitating at the same height \( z = z_0 \), which is determined by the total balance of vertical forces. The coordinates of grains in the \((x, y)\) plane are \( \rho_{n_1, n_2} = a(n_1 g_1 + n_2 g_2) \), where \( a \) is the lattice constant and \( n_1, n_2 \) are integers. The unit primitive vectors of the lattice can be chosen in the form \( g_1 = (1, 0) \) and \( g_2 = (1, \sqrt{3})/2 \). By virtue of the lattice symmetry, the horizontal balance of forces is ensured for an arbitrary axisymmetric interaction.

The vertical forces acting on each grain are equal. For a dust grain with \( n_1 = n_2 = 0 \), the force is caused by the effect of all other grains of the lattice and can be written as the sum

\[
F_z(a) = \sum_{n_1 \neq 0, n_2 \neq 0} f_\rho(\rho_{n_1, n_2}) \tag{7}
\]

where \( f_\rho(\rho) \) is determined by integral (6).

At smaller values of the lattice constant, it is necessary to take into account a larger number of terms in sum (7). For example, it was found experimentally that the relative accuracy on the order of \( 10^{-2} \) at \( a = 0.5 \) is reached if all lattice points with \( \rho_{n_1, n_2} \leq 20a \) are taken into account. This leads to a rather complicated numerical procedure, which requires at least 120 calculations of integral (6) for different values of \( \rho \).

The calculations are significantly accelerated using the following trick resembling Ewald's method for calculating lattice sums (see, e.g., [11]).

Let us divide the function \( \chi(k) \) in integral (6) into two parts: \( \chi(k) = \chi_z(k) + 1/(k^2 + \kappa^2) \), where \( \kappa > 0 \) is an auxiliary parameter. Then, the force acting on an individual grain is written in the form \( f_\rho(p) = f_{\rho_z}(p) - K_0(\kappa p) \), where

\[
f_{\rho_z}(p) = - \int_0^\infty dk \chi_z(k) J_0(kp). \tag{8}
\]

Note that the function \( f_{\rho_z}(p) \) is regular at \( p = 0 \). Sum (7) is also divided into two parts: \( F_z(a) = F_{\rho_z}(a) + F_{\rho_z}(a) \), where

\[
F_{\rho_z}(a) = \sum_{n_1 \neq 0, n_2 \neq 0} f_{\rho_z}(\rho_{n_1, n_2}), \tag{9}
\]

\[
F_{\rho_z}(a) = - \sum_{n_1 \neq 0, n_2 \neq 0} K_0(\kappa \rho_{n_1, n_2}). \tag{10}
\]

Sum (10) rapidly converges and can easily be calculated with an arbitrary accuracy. Let us choose an auxiliary parameter \( \kappa \) such that \( f_{\rho_z}(0) = 0 \). Then, since integral (8) at \( p = 0 \) can be calculated in an explicit form, we should set

\[
\kappa^2 = \begin{cases} \exp(M^2), & 0 \leq M < 1, \\ eM^2, & M > 1. \end{cases} \tag{11}
\]

Now, one can add a point with the index \( n_1 = n_2 = 0 \) to sum (9) and, using Poisson's formula
(see, e.g., [11]), pass to summation over the reciprocal lattice,

\[ F_{12}(a) = \frac{4\pi}{\sqrt{3}a^2} \sum_{n,m} \chi_1(n_1 b_1 + n_2 b_2), \]

(12)

where \( b_{1,2} \) are primitive vectors of the reciprocal lattice, which are defined by the relationship \( g_i \cdot b_j = \delta_{ij}2\pi \). Since \( \chi_1(k) \) is an algebraic function and \( \chi_1(k) \sim 1/k^4 \) at \( k \to \infty \), sum (12) also can be calculated with an arbitrary accuracy.

Some examples of the collective ion drag force per one grain as a function of the lattice constant are shown in Fig. 4. For comparison, the figure also shows the force obtained in the nearest neighbors approximation (curve 4). Although there is no direct proof, the collective ion drag force in the case of a subsonic or purely ion flow \( M < 1 \) is apparently always positive (Fig. 4; curves 2, 3), i.e., it is directed downstream. For a supersonic flow \( M > 1 \), the force is directed upstream (Fig. 4, curve 1).

The asymptotic behavior of the total force at small intergrain distances can be estimated using sums (10) and (12). It is evident that, at \( a \to 0 \), sum (10) diverges as \( F_{1z}(a) \sim \log a \), while sum (12) behaves in a more singular way: \( F_{1z}(a) \sim \chi_1(0)/a^2 \). Therefore, we have \( F_{1z}(a) \sim 1/a^2 \) for \( M > 1 \) and \( F_{1z}(a) \sim -1/a^2 \) for \( 0 \leq M < 1 \).

4. CONCLUSIONS

Let us estimate the characteristic value of the ion drag force. In this paper, we used the following force scale: \( F_0 = Q^2 \omega_{pi}^2/u^2 \). The natural scale of forces acting on a grain in the near-wall sheath is determined by weight of an individual grain, \( P = Mg \). Taking, as an example, the characteristic values \( Q = 10^4 e \), \( M = 10^{-10} \), \( n_i = 10^{10} \text{ cm}^{-3} \), and \( T_e = 2 \text{ eV} \), for argon ions and flow velocity equal to the ion-acoustic velocity, we find that \( F_0/P = 2 \). Thus, the collective ion drag force can be comparable with the weight of an individual grain.

It is well known that, as the intergrain distance decreases, a single-layer plasma crystal becomes unstable. Conditions for the onset of this instability depend on the external confining potential, which is usually approximated by a parabolic potential well. There exists a certain minimum intergrain distance for which the static model considered here becomes inapplicable.

Actually, the confining potential has a finite depth and the single-layer plasma crystal can be destroyed by another mechanism. In a subsonic flow, the collective ion drag force can merely carry the grains away from the confining potential well. On the other hand, this force facilitates the existence of a crystal in a supersonic flow.

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