Origin and Propagation of Fluctuations of Turbulent Magnetic Fields

K. Petrovay

Instituto de Astrofísica de Canarias, La Laguna (Tenerife), E-38200 Spain

Abstract. The degree of linear polarization has recently been found to show wide random variations over the solar disk. These variations are presumably at least partly due to fluctuations in the flux density of turbulent photospheric magnetic fields and associated variations in the degree of Hanle depolarization. In order to understand the origin of such large scale fluctuations of the turbulent magnetic flux density we develop a phenomenological model to calculate the spatial Fourier spectrum of the fluctuations of turbulent magnetic fields in the solar photosphere and convective zone. It is found that if the model parameters are fitted to turbulence closure models and numerical experiments the characteristic scale of the fluctuations is by about an order of magnitude larger than the turbulence scale (the scale of the granulation), owing to the more effective quenching of small-scale fluctuations by turbulent diffusion.

1. Introduction

In a highly conductive and strongly turbulent plasma like the solar photosphere random small-scale deviations of the flow field from mirror symmetry are expected to incessantly generate small magnetic flux loops of random orientation. This is a miniature analogue of the large-scale dynamo process occurring in a globally non-mirror-symmetric flow, for which reason it is known as small-scale dynamo action (Leorat, Pouquet, & Frisch 1981, Kida, Yanase, & Mizushima 1991, Petrovay & Szakály 1993). Owing to the random orientation of the loops, no net large-scale field will arise, but a non-zero mean magnetic energy density $B^2$ and mean unsigned flux density $|B|$ results. Most of this turbulent flux resides in magnetic structures with scales much smaller than the characteristic scale of the turbulent velocity field (which is the granular scale of $\sim 1000$ km in the solar case). Traditional Zeeman magnetography is “blind” to these fields as the net circular polarization of the mixed polarity small-scale field cancels out in a resolution element.

It has long been recognized that the Hanle effect (magnetic depolarization of linearly polarized radiation) may offer a way to detect the turbulent fields...
The observational study of turbulent fields has however been hampered by the shortage of observational data, by the lack of a reliable radiative transfer theory for polarized radiation in magnetic fields and by the fact that beside turbulent fields, the Hanle effect is also due to resolved magnetic elements (network, ephemeral active regions) and to the overlying canopy fields (especially for lines formed higher in the atmosphere). Nevertheless, in recent years important advances have been made in all these areas (Faurobert-Scholl et al. 1993, Faurobert-Scholl et al. 1994, Bianda, Solanki, & Stenflo 1998, Bianda, Stenflo, & Solanki 1998, Landi Degl’Innocenti 1996, Trujillo Bueno & Landi Degl’Innocenti 1997, Landi Degl’Innocenti 1998).

One of the most intriguing recent discoveries is the finding of Stenflo, Keller, & Gandorfer (1998) that the degree of linear polarization $Q/I$ shows large amplitude random variations over the solar disk. While, as mentioned above, canopy fields, network elements and ephemeral active regions may also contribute to the observed Hanle depolarization, it is still likely that at least part of this observed spatial variation is due to the presence of similar fluctuations in the flux density $|B|$ of the turbulent photospheric magnetic field.

The presence of fluctuations should not come as a surprise from a theoretical point of view. After all, in the turbulent solar photosphere any physical quantity should show fluctuations, and even variations of an amplitude comparable to the mean value are commonplace. What is more surprising is the spatial scale of the fluctuations. While the observations have a resolution of about $1''$ along the slit, the dominant variations seem to occur on the much larger scale of $\sim 10^4$ km. Variations on the granular scale are of much smaller amplitude. (See e.g. Fig. 3 in Stenflo, Keller, & Gandorfer 1998.) The real theoretical challenge is therefore to understand how the dominant scale of fluctuations of turbulent flux density can be so much larger than the turbulence scale?

One popular explanation for the existence of large-scale photospheric structures (such as supergranulation) is that they are the “imprints” of processes going on deeper down in the convective zone where the characteristic scales (determined by the pressure scale height $H$) are larger. Alternatively, it is of course also possible that the large scales are due to some local photospheric process like an inverse cascade. In order to resolve this problem, we need a model for the generation and transport of turbulent magnetic flux that takes into account both the generation and saturation processes constituting the small-scale dynamo and the turbulent transport of $|B|$ throughout the underlying convective zone. In the following such a model will be presented.

2. The spectrum of fluctuations: a linear model

The evolution equation for the unsigned flux density $|B|$ of the turbulent field should read something like

\[
\partial_t |B| = \nabla (\beta \nabla |B|) + \frac{|B|}{\tau_+} - \frac{(|B|/B_0)^q |B|/\tau_-}{q > 0}
\]  

(1)
As the transport of the highly intermittent magnetic field in a turbulent plasma is due to advection of flux tubes irrespective of their polarity, the transport terms are expected to be identical to those for the signed field $B$. For simplicity, here we consider an isotropic turbulent diffusion with diffusivity $\beta = l v / 3$ (first term on the r.h.s.), $l$ and $v$ being the correlation length and r.m.s. amplitude of the turbulent velocity field. The linear generation term corresponding to a small scale dynamo would lead to an exponential growth of a homogeneous $|B|$ field to infinity, were it not for a higher order term leading to the saturation of $|B|$ at a finite value $B_0$, induced by the curvature force (last term on r.h.s.). Numerical simulations and closure calculations (Kida, Yamasaki, & Mizushima 1991, Nordlund et al. 1992, Durney, De Young, & Roxburgh 1993, De Young 1980) show that $B_0$ is about an order of magnitude lower than the equipartition flux density: $B_0 \sim B_{eq}/10$.

As the coefficients $1/\tau_+$ and $1/\tau_-$ are functionals of the turbulent velocity field they are expected to show considerable fluctuations around their mean values. Hence, the flux density $|B|$ determined by the equilibrium of generation and nonlinear saturation processes will also fluctuate around its mean value $|B| \sim B_0$. It is plausible to assume

$$\frac{1}{\tau_-} = \frac{1}{\tau_+} = \frac{1}{\tau} v/l.$$  \hspace{1cm} (2)

Furthermore, introducing the notation $|B|/B_0 = 1 + b$, it greatly simplifies the treatment if one assumes $b \ll 1$. (In Section 3 we will consider the problem to what extent this simplification affects the results.) This allows us to linearize equation (1). In the case when $b$ varies much faster with depth than $B_0$ we obtain

$$\partial_t b = \nabla(\beta \nabla b) - q b/\tau + (1/\tau_+ - 1/\tau_-)'$$  \hspace{1cm} (3)

Now the second term on the r.h.s. describes the net mean restoring effect of generation and saturation terms, tending to reduce the fluctuation $b$, while the last term is the fluctuation generation term or forcing term arising owing to the turbulent fluctuations in the coefficients.

We further introduce the (physically plausible) assumption that the Fourier spectrum of this forcing term is dominated by those isotropic modes whose vertical phase is such that at the depth where the pressure scale height equals the inverse of their horizontal wavenumber $k$ they are maximal:

$$(1/\tau_+ - 1/\tau_-)' = \frac{1}{\tau} \sum_k \hat{b}_f(k, z) \exp[i(kx + ky + \omega t)]$$  \hspace{1cm} (4)

(Note that throughout this paper we take the logarithmic pressure $\ln P$ as independent variable in the vertical direction, the depth $z$ being just a shorthand notation for a function $z(\ln P)$, determined by a convection zone model. $x$ and $y$ are the horizontal coordinates.) For the solution of equation (3) we take a similar Ansatz:

$$b = \sum_k \hat{b}(k, z) \exp[i(kx + ky + \omega t)]$$  \hspace{1cm} (5)

Let us consider one mode only. (Note that in this case $\hat{b}$ can always be considered real as by virtue of our assumption about the vertical phase of modes
\( \hat{b}(k, z) \) may be written as \( \hat{b}_1(k) \cos[\ln P - \ln P_0(k)] \), and the initial phase in \( \hat{b}_1 \) can be chosen freely by a displacement of the time axis.) We substitute (4) and (5) into (3), simplify, and take the real part. To simplify the notation, from this point onwards we omit the hats. With this we arrive at

\[
-d_z(\beta d_z b) + (2k^2 \beta + q/\tau) b = b_f/\tau
\]

This equation determines the Fourier amplitude \( b(z, k) \) of each mode of horizontal wavenumber \( k \) in the spectrum of turbulent magnetic field fluctuations. Vertical diffusion is now separated in the first term; horizontal diffusion and the restoring force constitute the second and third terms on the l.h.s.

The r.h.s. is the Fourier amplitude of the forcing. \( b_f \) is the fluctuation amplitude produced in time \( \tau \) by the forcing if other terms were not present. As the forcing is due to the action of the fluctuating flow field on the existing magnetic field, it is plausible to assume \( b_f/\tau \propto B_k/B_0 \tau_k \), where \( B_k \) is the corresponding Fourier amplitude in the spectrum of \( |B| \), and \( \tau_k(k, z) \) is the eddy turnover time. This spectrum may be determined from closure calculations, numerical simulations and observations. Its simplest representation is by two power laws joining in a peak at \( k \sim 10/l \). Thus, we represent the r.h.s. by

\[
b_f(z, k)/\tau(z, k) = 10^{2/3-p} (k/k_0)^p, \quad k_0 \sim 1/H(z),
\]

where \( p = p_1 \) for \( k < 10k_0 \) and \( p_2 \) otherwise. (Note that in fact a spectrum with two breaks might be more realistic as the spectrum of \( \tau \) is peaked at a lower

Figure 1. Vertical profiles of some Fourier modes in the spectrum of fluctuations of the turbulent magnetic field. The modes are labelled by their value of \( 1/k \) in Megameter units. Dashed: \( B_0 \)
wavenumber of $k \sim 1/l$; our purpose here, however, is just to present a simple example calculation.) On the basis of the observed properties of photospheric magnetic fields and motion, $p_1 \sim 1.5$ seems to be a realistic choice. The value of $p_2$ will be found to be irrelevant to the solution below.

For the solution of equation (6) the parameters $\beta$ and $\tau$ are interpolated from a convective zone model. $q$ is evaluated by fitting a solution of equation (1), with the diffusive term and the perturbations of the coefficients neglected, to profiles of $|B|^2(t)$ resulting from the closure model of De Young (1980); this yields $q \simeq 0.1$. The boundary conditions may be chosen as closed ($F = \beta d_z b = 0$) or open ($F = v b$); experimenting has shown that this does no exert a very strong influence on the resulting mode profiles. For the numerical solution of equation (6) we take $\ln P$ as independent variable and use a relaxational method.

The vertical profiles of some Fourier modes in the spectrum of fluctuations of the turbulent magnetic field are shown in Figure 1. As expected, the modes with larger horizontal scales have their maxima in deeper layers.

![Figure 1](image1)

**Figure 1.** Vertical profiles of some Fourier modes in the spectrum of fluctuations of the turbulent magnetic field.

Computing a large number of modes and plotting their amplitudes near the surface against $k$ yields the Fourier spectrum of the fluctuations in the turbulent magnetic field (Fig. 2). Experimenting with the parameters in equation (6) we find that the choice of $p_2$ is irrelevant for the spectrum, $B_0/B_{eq}$ determines its amplitude, while $p_1$ and $q$ determine the shape of the spectrum. A striking feature of the spectra in Figure 2 is that their maxima fall to significantly lower wavenumbers than $k_0 = 1/l$, i.e. the characteristic scale of the fluctuations in
turbulent magnetic flux density is much larger than the typical scale of turbulent motions. The physical background of this phenomenon is that the high wavenumber components are more efficiently suppressed by diffusion \(2k^2 \beta b\) term in eq. (6)). The wavenumber of the maximum increases with \(q\) (dashed vs. dash-dotted curves), as a stronger nonlinear saturation reduces the role of the diffusive terms in the equation. On the other hand, the spectral amplitude at even lower wavenumbers depends strongly on the spectral index \(p_1\) of the forcing (dashed vs. solid curves) — clearly, the shallower the forcing spectrum, the more energy is input at the larger scales.

3. The effect of nonlinearity

The results presented in the previous section were computed under the assumption that the fluctuations of \(|B|\) have a small relative amplitude. This assumption is rather dubious in the light of the large Fourier amplitudes found in the model (Fig. 1). In order to have an idea about the extent to which nonlinear effects may modify the linear results, in this section we present an alternative model that calculates the fluctuating field without the assumption of linearity, at the cost of a strong simplification of the geometry: only \(k = 0\) modes are considered. This obviously implies that spatial spectra or correlation lengths cannot be studied; instead, we will compare the temporal autocorrelation of the fluctuations in the linear and nonlinear cases.

\[
\frac{\partial |B|}{\partial t} = d_x (\beta d_x |B|) + |B|/\tau [1 - (|B|/B_0)^q] + B_f/\tau
\]

where the last term corresponds to fluctuation forcing by to the turbulent fluctuations of the velocity field. We model this term as a Gaussian stationary
random process with correlation time $\tau(z)$, vertical correlation “length” 1 (in ln $P$ units), and a mean displacement of $0.3B_0$ over $\tau$.

Equation (8) is then integrated numerically starting from an arbitrary perturbed initial state. An example solution is presented in Figure 3. Figure 4 shows the autocorrelation of the fluctuating flux density as a function of the time shift for different cases. It is apparent (dotted vs. dashed curves) that the correlation time is primarily determined by the value of the $q$ nonlinearity parameter, lower $q$ values corresponding to longer correlation times. This result is a close analogue to the findings of Section 2 with respect to the spatial correlations. On the other hand, switching off the diffusive term in equation (8) (dash-dotted vs. dashed curves) has no significant effect, showing that diffusive quenching of small spatial scales is not the mechanism responsible for the extended correlations here. Replacing equation (8) with its linear equivalent (solid vs. dashed curves) does not lead to great modifications in the results. This may reassure us to some extent of the reliability of the findings of Section 2 above.

![Figure 4. Autocorrelation of the fluctuating flux density as a function of the time difference at a fixed point in the convective zone in the nonlinear model with $q = 0.1$. Dashed: reference model; dotted: $q = 1$; dash-dotted: reference model with diffusive term switched off; solid: linear equivalent.](image)

4. Conclusion

The results of Section 2 now enable us to answer the question posed at the end of the Introduction. Turbulent transport processes indeed result in a dominant scale for the fluctuations of the turbulent magnetic flux density that is an order
Fluctuations of Turbulent Magnetic Field

of magnitude larger than the turbulence scale. The physical mechanism behind this phenomenon is that small-scale fluctuations are more efficiently damped by diffusion, shifting the spectral peak to lower wavenumbers. The low value of the \( q \) nonlinearity parameter suggested by closure calculations and simulations also favors larger dominant scales.

On the other hand, the possibility that deeper structures are “imprinted” on the photospheric pattern can apparently be discarded. Switching off the first term (vertical diffusion) in equation (6) does not lead to any reduction of the dominant scales (dotted vs. dashed curves in Fig. 2; dash-dotted vs. dashed in Fig. 4).

All this shows that the observations of strongly varying Hanle depolarization over the solar disk may be understood from a theoretical point of view even if all the depolarization is attributed to turbulent fields. Further advances on both the observational and theoretical side may make more detailed comparisons between the models and the observations possible, thereby offering the prospect of a direct observational diagnostics of the properties of the turbulent dynamo.

Acknowledgments. This work was funded by the DGES grant no. 95-0028.

References

Bianda, M., Solanki, S. K., & Stenflo, J. O. 1998, A&A, 331, 760
Bianda, M., Stenflo, J. O., & Solanki, S. K. 1998, A&A in press
De Young, D. S. 1980, ApJ, 241, 81
Durney, B. R., De Young, D. S., & Roxburgh, I. W. 1993, Solar Phys., 145, 207
Faurobert-Scholl, M. 1993, A&A, 268, 765
Faurobert-Scholl, M., Feautrier, N., Machefert, F., Petrovay, K., & Spielfiedel, A. 1994, A&A, 298, 289
Kida, S., Yanase, S., & Mizushima, J. 1991, Phys. Fluids A, 3, 457
Landi Degl’Innocenti, E. 1996, Solar Phys., 164, 21
Landi Degl’Innocenti, E. 1998, Nature, 392, 256
Leorat, J., Pouquet, A., & Frisch, U. 1981, J. Fluid Mech., 104, 419
Nordlund, Å., Brandenburg, A., Jennings, R. L., Rieutord, M., Ruokolainen, J., Stein, R. F., & Tuominen, I. 1992, ApJ, 392, 647
Petrovay, K., & Szakály, G. 1993, A&A, 274, 543
Stenflo, J. O. 1982, Solar Phys., 80, 209
Stenflo, J. O., Keller, C. U., & Gandorfer, A. 1998, A&A, 329, 319
Trujillo Bueno, J., & Landi Degl’Innocenti, E. 1997, ApJ, 482, L183