On Krull-Gabriel dimension of cluster repetitive
categories and cluster-tilted algebras

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with an Appendix by Grzegorz Bobiński*

Abstract

Assume that $K$ is an algebraically closed field and denote by $\text{KG}(R)$ the
Krull-Gabriel dimension of $R$, where $R$ is a locally bounded $K$-category (or a
bound quiver $K$-algebra). Assume that $C$ is a tilted $K$-algebra and $\tilde{C}, \hat{C}, \breve{C}$ are
the associated repetitive category, cluster repetitive category and cluster-tilted
algebra, respectively. Our first result states that $\text{KG}(\tilde{C}) = \text{KG}(\hat{C}) \leq \text{KG}(\breve{C})$.
Since the Krull-Gabriel dimensions of tame locally support-finite repetitive
categories are known, we further conclude that $\text{KG}(\tilde{C}) = \text{KG}(\hat{C}) = \text{KG}(\breve{C}) \in \{0, 2, \infty\}$. Finally, in the Appendix Grzegorz Bobiński presents a different
way of determining the Krull-Gabriel dimension of the cluster-tilted algebras,
by applying results of Geigle.

1 Introduction

Assume that $K$ is an algebraically closed field and $R$ is a locally bounded $K$-category.
Recall that $R$ is isomorphic to a bound quiver $K$-category associated with some
locally finite bound quiver [12, 18]. Hence we can identify finite locally bounded
$K$-categories with bound quiver $K$-algebras, see [6, I-III]. We denote by $\text{mod}(K)$
the category of all finite dimensional $K$-vector spaces, by $\text{mod}(R)$ the category of
all finitely generated right $R$-modules and by $\text{ind}(R)$ the category of all finitely
generated indecomposable right $R$-modules. Let $\mathcal{F}(R)$ be the category of all finitely
presented contravariant $K$-linear functors from $\text{mod}(R)$ to $\text{mod}(K)$, see [34] for
details on functor categories. A natural approach to study this abelian category is
via the associated Krull-Gabriel filtration [33]

$$
\mathcal{F}(R)_{-1} \subseteq \mathcal{F}(R)_0 \subseteq \mathcal{F}(R)_1 \subseteq \ldots \subseteq \mathcal{F}(R)_\alpha \subseteq \mathcal{F}(R)_{\alpha+1} \subseteq \ldots
$$

of $\mathcal{F}(R)$ by Serre subcategories, defined recursively as follows:

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(1) $\mathcal{F}(R)_{-1} = 0$ and $\mathcal{F}(R)_{\alpha+1}$ is the Serre subcategory of $\mathcal{F}(R)$ formed by all functors having finite length in the quotient category $\mathcal{F}(R)/\mathcal{F}(R)_\alpha$, where $\alpha$ is an ordinal number or $\alpha = -1$.

(2) $\mathcal{F}(R)_\beta = \bigcup_{\alpha < \beta} \mathcal{F}(R)_\alpha$, for any limit ordinal $\beta$.

The Krull-Gabriel dimension $\text{KG}(R)$ of $R$ is the smallest ordinal number $\alpha$ such that $\mathcal{F}(R)_\alpha = \mathcal{F}(R)$, if such a number exists, and $\text{KG}(R) = \infty$ otherwise. The Krull-Gabriel dimension of $R$ is finite if $\text{KG}(R) = n$, for some $n \in \mathbb{N}$. The Krull-Gabriel dimension of $R$ is undefined if $\text{KG}(R) = \infty$. We note that the Krull-Gabriel dimension is defined for any small abelian category similarly as above.

There are several motivations to study the Krull-Gabriel dimension of a locally bounded $K$-category or a bound quiver $K$-algebra, see [35] for details. Our motivation comes from the conjecture of Prest [34] stating that a finite dimensional algebra $A$ is of domestic representation type if and only if the Krull-Gabriel dimension $\text{KG}(A)$ of $A$ is finite. We refer to [36, XIX] for the definitions of finite, tame and wild representation type as well as stratification of tame representation type into domestic, polynomial and non-polynomial growth (introduced in [37]).

The known results support the conjecture of Prest, see the introduction of [30] for a comprehensive and up-to-date list of these results. Let us recall that the first and fundamental fact in this direction, due to Auslander [8 Corollary 3.14], states that an algebra $A$ is of finite representation type if and only if $\text{KG}(A) = 0$. The present paper is in part devoted to determination of the Krull-Gabriel dimension of the cluster-tilted algebras. In particular, we confirm the conjecture of Prest for this class of algebras. Recall that the cluster-tilted algebras play a prominent role in the theory of cluster algebras.

The cluster algebras were introduced by Fomin and Zelevinsky in the seminal paper [17] in order to create a combinatorial framework for the study of canonical bases in quantum groups and for the study of total positivity for algebraic groups. Since then the cluster algebras have become a separate field of study with many connections to other subjects, in particular to the representation theory of finite dimensional algebras. We refer the reader to [27] for a comprehensive survey on the theory of cluster algebras and related topics.

Cluster algebras are linked to the representation theory via tilting theory in cluster categories which was introduced by Buan, Marsh, Reineke, Reiten and Todorov in another seminal paper [13]. If $H$ is a hereditary algebra and $\mathcal{D}^b(H)$ is the derived category of bounded complexes over $\text{mod}(H)$, then the cluster category $\mathcal{C}_H$ is an orbit category of $\mathcal{D}^b(H)$ under the action of the functor $\tau^{-1}[1]$. Here $\tau$ denotes the Auslander-Reiten translation in $\mathcal{D}^b(H)$ and $[1]$ the shift functor. Recall that Keller shows in [26] a deep result that the cluster category $\mathcal{C}_H$ is triangulated.
An object $T$ in $\mathcal{C}_H$ is a cluster-tilting object provided $T$ has no self-extensions in $\mathcal{C}_H$ and the number of isomorphism classes of indecomposable direct summands of $T$ equals the number of simple modules in $\text{mod}(H)$. To each hereditary algebra $H$ one can associate a cluster algebra in such a way that the cluster variables correspond to the indecomposable direct summands of cluster-tilting objects in $\mathcal{C}_H$ and the clusters to the cluster-tilting objects themselves, see [13] for details. In this way cluster categories categorify cluster algebras.

Cluster-tilted algebras were introduced in [14] as (the opposite algebras of) the endomorphisms algebras of cluster-tilting objects. This is analogous to the classical definition of tilted algebras by Happel and Ringel from [21]. Cluster-tilted algebras attracted much attention and are still an active area of research in cluster theory, see [1] for a convenient survey on the topic. Interestingly, it turns out that these algebras can be defined without any reference to cluster categories. Indeed, it is proved in [2] that any cluster-tilted algebra is of the form of the trivial extension $\tilde{C} := C \ltimes \text{Ext}^2(DC, C)$, where $C$ is a tilted algebra and $D$ denotes the standard $K$-duality. This fact allows to show in [3] that a cluster-tilted algebra $\tilde{C}$ is an orbit algebra of a cluster repetitive category $\tilde{\mathcal{C}}$. More specifically, there is a Galois covering $\tilde{\mathcal{C}} \rightarrow \tilde{C}$ with a covering group $\mathbb{Z}$, see Section 3 for details. Note that this point of view is similar to the description of standard self-injective algebras as orbit algebras of repetitive categories, see [38].

Recently the second author showed in [30] a general result that if $R$ is a locally support-finite [15] locally bounded $K$-category and $G$ a torsion-free admissible group of $K$-linear automorphisms of $R$, then $\text{KG}(R) = \text{KG}(R/G)$ where $R/G$ denotes the orbit category [12]. In other words, the induced Galois covering $R \rightarrow R/G$ preserves Krull-Gabriel dimension. This theorem is further applied in [30] Theorem 7.3] and [30] Theorem 8.1] to determine Krull-Gabriel dimensions of tame locally-support finite repetitive categories and standard self-injective algebras, respectively.

In the present paper we use the above facts to determine the Krull-Gabriel dimension of cluster repetitive categories and cluster-tilted algebras, see Theorem 3.6 for a precise statement of the main result. In particular, we show that if $C$ is a tilted $K$-algebra and $\tilde{\mathcal{C}}, \mathcal{C}$, $\tilde{C}$ are the associated repetitive category, cluster repetitive category and cluster-tilted algebra, respectively, then we have

$$\text{KG}(\tilde{C}) = \text{KG}(\tilde{\mathcal{C}}) = \text{KG}(\mathcal{C}) \in \{0, 2, \infty\}.$$ 

We also confirm the conjecture of Prest for the class of cluster-tilted algebras.

The paper is organized as follows. In the remaining part of Section 1 we fix the notation and terminology used in the paper.

Section 2 is devoted to the introduction of admissible functors between module categories and their examples. A crucial property of an admissible functor $\varphi : \text{mod}(A) \rightarrow \text{mod}(B)$ is that its existence implies that $\text{KG}(B) \leq \text{KG}(A)$, see
Proposition 2.1. In Theorem 2.2 we recall some examples of admissible functors from \([32]\), related with certain Galois coverings. These admissible functors are used in proofs of the main results of \([30,32]\) (see Theorem 2.3) which are directly applied in the present paper. Then we prove Theorem 2.4 which yields a new class of admissible functors playing a prominent role in the proofs our main results, given in Section 3. Later on we introduce \textit{hs-finite} classes of modules and show in Lemma 2.5 their significance in the context of Theorem 2.4.

The most import technical fact of Section 3 is Proposition 3.3. In this proposition we show that some particular class of modules over the repetitive category of a tilted algebra, studied in \([3]\), is hs-finite. This property enables us to make use of facts from Section 2 in the proofs of the main results of the paper which are Theorem 3.4, Theorem 3.6 and Corollary 3.7. The crucial part of Theorem 3.4 states that \(\text{KG}(\tilde{C}) = \text{KG}(\hat{C}) \leq \text{KG}(\check{C})\) where \(C\) is a tilted algebra and \(\tilde{C}, \hat{C}, \check{C}\) are the associated repetitive category, cluster repetitive category and cluster-tilted algebra, respectively. Theorem 3.6 strengthens Theorem 3.4, showing in particular that \(\text{KG}(\tilde{C}) = \text{KG}(\hat{C}) = \text{KG}(\check{C}) \in \{0,2,\infty\}\). In Corollary 3.7 we conclude that if the base field is countable, then \(\tilde{C}\) possesses a super-decomposable pure-injective module (see \([25,34]\)) if and only if \(\tilde{C}\) is of non-domestic type, confirming (another) conjecture of Prest for the class of cluster-tilted algebras.

The final section of the paper is the Appendix written by Grzegorz Bobiński. The author gives an alternative proof of Theorem 3.6, which is based on a result of Geigle given in \([20, \text{Corollary 2.9]}\).

Throughout the paper, we use the following notation and terminology. Fix an algebraically closed field \(K\) and assume that \(R\) is a locally bounded \(K\)-category. Recall that in this case \(R\) is isomorphic with a bound quiver \(K\)-category of a locally finite quiver, see \([12]\) or \([30, \text{Section 2]}\) for more straightforward presentation.

Assume that \(C\) is a full subcategory of \(R\). Then \(C\) is convex if and only if for any \(n \geq 1\) and objects \(x, z_1, \ldots, z_n, y\) of \(R\) the following condition is satisfied: if \(x, y\) are objects of \(C\) and the \(K\)-vector spaces of morphisms \(R(x, z_1), R(z_1, z_2), \ldots, R(z_{n-1}, z_n), R(z_n, y)\) are nonzero, then \(z_1, \ldots, z_n\) are objects of \(C\).

Assume that \(R\) is a bound quiver \(K\)-category of a bound quiver \((Q, I)\) and \(C\) is a full subcategory of \(R\). In this setting, \(C\) is convex if and only if \(C\) is a bound quiver \(K\)-category of a bound quiver \((Q', I')\) such that \(Q'\) is some convex subquiver of \(Q\). Recall that a full subquiver \(Q'\) of \(Q\) is convex if and only if any path in \(Q\) whose source and target belong to \(Q'\) is entirely contained in \(Q'\). We refer to Section 2 of \([30]\) for more details.

Assume that \(C\) is a full subcategory of \(R\). A \textit{convex hull} of \(C\) is a full subcategory of \(R\) whose set of objects is an intersection of sets of objects of all convex subcategories of \(R\) containing \(C\). Thus a convex hull of \(C\) is the smallest convex
subcategory of \( R \) containing \( C \). We say that \( R \) is \textit{intervally-finite} if and only if a convex hull of any finite full subcategory of \( R \) is again finite. This notion is only implicitly contained in \cite[2.1]{12}. The terminology was introduced later.

Assume that \( R \) is a locally bounded \( K \)-category. A \textit{right \( R \)-module} is a \( K \)-linear contravariant functor of the form \( M : R \to \text{Mod}(K) \) where \( \text{Mod}(K) \) denotes the category of all \( K \)-vector spaces. The category of all such modules is denoted by \( \text{Mod}(R) \). A module \( M \in \text{Mod}(R) \) is \textit{finite dimensional} if and only if \( \dim M = \sum_{x \in \text{ob}(R)} \dim_K M(x) < \infty \). We denote by \( \text{mod}(R) \) the full subcategory of \( \text{Mod}(R) \) formed by finite dimensional modules. Moreover, \( \text{ind}(R) \) is the full subcategory of \( \text{mod}(R) \) whose objects are indecomposable modules.

If \( M \in \text{mod}(R) \), then the set \( \text{supp}(M) = \{ x \in R \mid M(x) \neq 0 \} \) is the \textit{support of the module} \( M \). If \( \mathcal{C} \) is a full subcategory of \( \text{mod}(R) \), then the union of all supports of modules belonging to \( \mathcal{C} \) is called the \textit{support of} \( \mathcal{C} \) and denoted by \( \text{supp}(\mathcal{C}) \).

The category \( R \) is \textit{locally support-finite} \cite{15} if and only if for any \( x \in R \) the union of the sets \( \text{supp}(M) \), where \( M \in \text{ind}(R) \) and \( M(x) \neq 0 \), is finite.

Assume that \( R, A \) are locally bounded \( K \)-categories, \( F : R \to A \) is a \( K \)-linear functor and \( G \) a group of \( K \)-linear automorphisms of \( R \) acting freely on the objects of \( R \) (i.e. \( gx = x \) if and only if \( g = 1 \), for any \( g \in G \) and \( x \in \text{ob}(R) \)). Then \( F : R \to A \) is a \textit{Galois covering} \cite{12} if and only if

1. the functor \( F : R \to A \) induces isomorphisms
   \[
   \bigoplus_{g \in G} R(gx, y) \cong A(F(x), F(y)) \cong \bigoplus_{g \in G} R(x, gy)
   \]
   of vector spaces, for any \( x, y \in \text{ob}(R) \),

2. the functor \( F : R \to A \) is surjective on objects,

3. \( Fg = F \), for any \( g \in G \),

4. for any \( x, y \in \text{ob}(R) \) such that \( F(x) = F(y) \) there is \( g \in G \) such that \( gx = y \).

It is well known that a functor \( F : R \to A \) satisfies the above conditions if and only if \( F \) induces an isomorphism \( A \cong R/G \) where \( R/G \) is the \textit{orbit category}, see \cite{12}. We refer to \cite{15} for a general definition of a covering functor.

Assume that \( F : R \to A \cong R/G \) is a Galois covering. The \textit{pull-up} functor \( F_* : \text{Mod}(A) \to \text{Mod}(R) \) associated with \( F \) is the functor \((\cdot) \circ F^{\text{op}} \). The pull-up functor has the left adjoint \( F_{\lambda} : \text{Mod}(R) \to \text{Mod}(A) \) and the right adjoint \( F_{\rho} : \text{Mod}(R) \to \text{Mod}(A) \) which are called the \textit{push-down functors}. We refer to Section 2 of \cite{30} for concrete description of these functors. Here we only mention that push-down functors restrict to the categories of finite dimensional modules and these restrictions coincide, that is, \( F_{\lambda}(\text{mod}(R)) \subseteq \text{mod}(A) \), \( F_{\rho}(\text{mod}(R)) \subseteq \text{mod}(A) \).
and $F_\lambda|_{\text{mod}(R)} = F_\mu|_{\text{mod}(R)}$. The restriction $F_\lambda|_{\text{mod}(R)}$ is also denoted as $F_\lambda$. In the paper we consider only the push-down $F_\lambda : \text{mod}(R) \to \text{mod}(A)$.

Furthermore, $\mathcal{G}(R)$ denotes the category of all contravariant additive $K$-linear functors $\text{mod}(R) \to \text{mod}(K)$. If $M \in \text{mod}(R)$, then $R(\cdot, M) = \text{Hom}_R(\cdot, M)$ denotes the contravariant hom-functor. Recall that any homomorphism of modules $f \in R(M, N)$ induces a homomorphism of functors $R(\cdot, f) : R(\cdot, M) \to R(\cdot, N)$ such that the map $R(X, f) : R(X, M) \to R(X, N)$ is defined by $R(X, f)(g) = fg$, for any $g \in R(X, M)$. The Yoneda lemma implies that the function $f \mapsto R(\cdot, f)$ defines an isomorphism $R(M, N) \to \mathcal{G}(R)(R(\cdot, M), R(\cdot, N))$ of vector spaces and this yields $M \cong N$ if and only if $R(-, M) \cong R(-, N)$.

Assume that $F \in \mathcal{G}(R)$. The functor $F$ is finitely presented if and only if there exists an exact sequence of functors $R(-, M) \xrightarrow{\rho} R(-, N) \to F \to 0$, for some $M, N \in \text{mod}(R)$ and $R$-module homomorphism $\rho : M \to N$. This means that $F \cong \text{Coker}(\rho)$ which yields $F(X)$ is isomorphic to the cokernel of the map $R(X, \rho) : R(X, M) \to R(X, N)$. We denote by $\mathcal{F}(R)$ the full subcategory of $\mathcal{G}(R)$ formed by finitely presented functors. Obviously $R(-, M) \in \mathcal{F}(R)$ for any $M \in \text{mod}(R)$. Moreover, the functor $R(-, M)$ is a projective object of the category $\mathcal{F}(R)$ and any projective object of $\mathcal{F}(R)$ is a hom-functor, see [30 IV.6]. If $F \in \mathcal{G}(R)$, then $\text{supp}(F) = \{X \in \text{mod}(R) \mid F(X) \neq 0\}$ is the support of $F$.

We refer the reader to [30] for the background of the representation theory of finite dimensional algebras over algebraically closed fields.

## 2 Admissible functors and Krull-Gabriel dimension

In this section we introduce admissible functors and relate them with Krull-Gabriel dimension. We give examples of such functors in Theorems 2.2 and 2.4. These results are applied in Section 3.

Assume that $\varphi : \text{mod}(A) \to \text{mod}(B)$ is a $K$-linear additive covariant functor. We define $\Lambda_\varphi : \mathcal{F}(B) \to \mathcal{G}(A)$ as the composition $(\cdot) \circ \varphi$. Observe that if $U \in \mathcal{F}(B)$ and $\beta(-, X) \xrightarrow{\eta(-, f)} \beta(-, Y) \to U \to 0$ is exact, then we get the exact sequence $\beta(\varphi(-), X) \xrightarrow{\eta(\varphi(-), f)} \beta(\varphi(-), Y) \to U\varphi \to 0$.

We say that $\varphi : \text{mod}(A) \to \text{mod}(B)$ is admissible if and only if $\varphi$ is dense\footnote{This means that for any module $X \in \text{mod}(B)$ there exists a module $M \in \text{mod}(A)$ such that $\varphi(M) \cong X$. Although the property is often referred as essential surjectivity, we stick to above terminology since it is consistent with our previous work, especially with [30].} and $\text{Im}(\Lambda_\varphi) \subseteq \mathcal{F}(A)$, that is, $U\varphi$ is finitely presented, for any $U \in \mathcal{F}(A)$.

The following fact shows that admissible functors are useful in the study of Krull-Gabriel dimension.
Proposition 2.1. Assume that $\varphi : \text{mod}(A) \to \text{mod}(B)$ is an admissible functor. Then we have $\text{KG}(B) \leq \text{KG}(A)$.

Proof. The functor $\Lambda_\varphi : \mathcal{F}(B) \to \mathcal{F}(A)$ is exact being a composition with a $K$-linear additive functor. We show that $\Lambda_\varphi$ is also faithful. Indeed, let $f = (f_N)_N : U \to V$ be a natural transformation of functors $U, V \in \mathcal{F}(A)$ and assume that $\Lambda_\varphi(f) = 0$. If $N \in \text{mod}(A)$, then $N \cong \varphi(X)$, for some $X \in \text{mod}(B)$ and thus $f_N \cong f_{\varphi(X)} = \Lambda_\varphi(f)_X = 0$. This yields that $f = 0$ and hence the functor $\Lambda_\varphi$ is faithful. Then we conclude from [29, Appendix B] that $\text{KG}(B) \leq \text{KG}(A)$. \hfill $\square$

The following two theorems (Theorem 2.2 and 2.3) are proved in [30] and [32]. The assertion (3) of Theorem 2.2 gives an interesting example of an admissible functor studied in [32].

Theorem 2.2. Assume that $R$ is a locally bounded $K$-category, $G$ is an admissible group of $K$-linear automorphisms of $R$ and $F : R \to A \cong R/G$ the associated Galois covering. Assume that $B$ is a finite convex subcategory of the category $R$. We denote by $\mathcal{E}_B : \text{mod}(B) \to \text{mod}(R)$ the functor of extension by zeros. We call $B$ a fundamental domain of the category $R$ if and only if for any $M \in \text{ind}(R)$ there exists $g \in G$ such that $\text{supp}(gM) \subseteq B$, see [32] (by $gM$ we denote the induced action $gM = M \circ g^{-1}$ of $G$ on $\text{mod}(R)$). We recall the following theorem proved in [32].

Theorem 2.2. Assume that $R$ is a locally bounded $K$-category and $G$ an admissible torsion-free group of $K$-linear automorphisms of $R$. The following assertions hold.

(1) If there exists a fundamental domain $B$ of $R$, then $R$ is locally support-finite.

(2) If $R$ is locally support-finite and intervalually-finite, then there exists a fundamental domain $B$ of $R$.

(3) If $B$ is a fundamental domain of $R$, then the push-down functor $F_\lambda : \text{mod}(R) \to \text{mod}(A)$ is dense and the functor $F_\lambda \mathcal{E}_B : \text{mod}(B) \to \text{mod}(A)$ is admissible. In particular, $\text{KG}(A) \leq \text{KG}(B)$ and thus $\text{KG}(A) \leq \text{KG}(R)$.

Proof. All assertions are proved in [32] and [30]. For convenience we present a short proof of (1). Assume, to the contrary, that $R$ is not locally support-finite. Then there are indecomposable finite dimensional $R$-modules with arbitrarily large supports, because $R$ is a bound quiver $K$-category of a locally finite quiver. In particular, if $B$ is any finite subcategory of $R$, then there is $M \in \text{ind}(R)$ such that $|\text{supp}(M)| > |B|$. Hence there is no $g \in G$ with $\text{supp}(gM) \subseteq B$ and so $B$ is not a fundamental domain of $R$. \hfill $\square$

The following theorem is the main result of the papers [30] and [32], see in particular [32, Theorem 1.5] (Theorem 2.2 is an important ingredient of its proof). We apply this theorem in the next section.
Theorem 2.3. Assume $R$ is a locally bounded $K$-category, $G$ an admissible torsion-free group of $K$-linear automorphisms of $R$ and $F: R \to A \cong R/G$ the Galois covering. If $B$ is a fundamental domain of $R$, then $\text{KG}(R) = \text{KG}(B) = \text{KG}(A)$. □

In the sequel we assume that $A, B$ are arbitrary locally bounded $K$-categories. We aim to present other examples of admissible functors. For that purpose, we introduce the following definition.

Assume that $R \subseteq \text{mod}(A)$ is a class of $A$-modules and $N \in \text{mod}(A)$. A homomorphism $\alpha_N: M_N \to N$, where $M_N \in R$, is a right $R$-approximation of $N$ if and only if for any $L \in R$ and $a: L \to N$ there is $b: L \to M_N$ such that $\alpha_Nb = a$, that is, the following diagram

$$
\begin{array}{ccc}
N & \xrightarrow{\alpha_N} & L \\
\downarrow{a} & & \downarrow{b} \\
M_N & \xrightarrow{\quad} & 
\end{array}
$$

is commutative. Further, we say that $R$ is contravariantly finite if and only if any module $N \in \text{mod}(A)$ has a right $R$-approximation.

Observe that if $R \subseteq \text{mod}(A)$ is a contravariantly finite class of $A$-modules and $S$ is the smallest full subcategory of $\text{mod}(A)$ closed under isomorphisms and direct summands such that $R \subseteq \text{ob}(S)$, then $S$ is a contravariantly finite subcategory of $\text{mod}(A)$ in the classical sense of [9].

Assume that $\varphi: \text{mod}(A) \to \text{mod}(B)$ is a $K$-linear additive covariant functor and $R \subseteq \text{mod}(A)$ is a class of $A$-modules. We denote by $\text{Ker}(\varphi)$ the kernel of $\varphi$, that is, the class of all homomorphisms $f$ in $\text{mod}(A)$ such that $\varphi(f) = 0$. We say that a homomorphism $f: X \to Y$ in $\text{mod}(A)$ factorizes through $R$ if and only if there is a module $M \in R$ and homomorphisms $g: X \to M$ and $h: M \to Y$ in $\text{mod}(A)$ such that $f = hg$.

The following theorem shows important examples of admissible functors.

**Theorem 2.4.** Assume that $\varphi: \text{mod}(A) \to \text{mod}(B)$ is a $K$-linear additive covariant functor which is full and dense. Moreover, assume that there is a contravariantly finite class of modules $R_{\varphi} \subseteq \text{mod}(A)$ such that $\text{Ker}(\varphi)$ equals the class of all homomorphisms in $\text{mod}(A)$ which factorize through $R_{\varphi}$. Then $\varphi: \text{mod}(A) \to \text{mod}(B)$ is admissible and thus $\text{KG}(B) \leq \text{KG}(A)$.

**Proof.** It is enough to show that a functor $B(\varphi(-), Z): \text{mod}(A) \to \text{mod}(K)$ belongs to $\mathcal{F}(A)$, for any $Z \in \text{mod}(B)$. Indeed, this follows from the fact that for any $U \in \mathcal{F}(B)$ the functor $U\varphi \in \mathcal{G}(A)$ is a cokernel of a morphism between such functors (see the beginning of this section) and the category $\mathcal{F}(A)$ is abelian. Since $\varphi: \text{mod}(A) \to \text{mod}(B)$ is dense, it is sufficient to show that $B(\varphi(-), \varphi(N)) \in \mathcal{F}(A)$, for any $N \in \text{mod}(A)$. We fix a module $N \in \text{mod}(A)$.\n
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Observe that \( \tilde{\varphi} = (\tilde{\varphi}_X)_{X \in \text{mod}(A)} \) where \( \tilde{\varphi}_X : A(X, N) \to B(\varphi(X), \varphi(N)) \) is given by the formula \( \tilde{\varphi}_X(f) = \varphi(f) \), for any \( X \in \text{mod}(A) \) and \( f \in A(X, N) \), is a natural transformation of functors \( A(-, N) \to B(\varphi(-), \varphi(N)) \). Namely, the fact that the functor \( \varphi \) preserves the composition (as any covariant functor) implies that the following diagram
\[
\begin{array}{ccc}
A(X, N) & \xrightarrow{\tilde{\varphi}_X} & B(\varphi(X), \varphi(N)) \\
\downarrow{(-)g} & & \downarrow{(-)\varphi(g)} \\
A(Y, N) & \xrightarrow{\tilde{\varphi}_Y} & B(\varphi(Y), \varphi(N))
\end{array}
\]
commutes, for any homomorphism \( g : Y \to X \in \text{mod}(A) \). Since \( \varphi \) is full, we get that \( \tilde{\varphi} : A(-, N) \to B(\varphi(-), \varphi(N)) \) is an epimorphism of functors.

Assume that \( \alpha_N : M_N \to N \) is a right \( \mathcal{R}_\varphi \) approximation of the module \( N \), for some \( M_N \in \mathcal{R}_\varphi \). We show that \( \text{Im}(A(-, \alpha_N)) = \text{Ker}(\tilde{\varphi}) \), equivalently, the sequence
\[
A(-, M_N) \xrightarrow{A(-, \alpha_N)} A(-, N) \xrightarrow{\tilde{\varphi}} B(\varphi(-), \varphi(N)) \to 0
\]
is exact. We show that \( \text{Im}(A(X, \alpha_N)) = \text{Ker}(\tilde{\varphi}_X) \), for any \( X \in \text{mod}(A) \). Indeed, if \( a \in \text{Im}(X, \alpha_N) \), then \( a = \alpha_N b \), for some \( b \in A(X, M_N) \) and thus \( a \) factorizes through \( M_N \in \mathcal{R}_\varphi \). This yields \( a \in \text{Ker}(\tilde{\varphi}_X) \) and hence \( \text{Im}(A(X, \alpha_N)) \subseteq \text{Ker}(\tilde{\varphi}_X) \). To show the converse inclusion, assume that \( f \in \text{Ker}(\tilde{\varphi}_X) \). Then \( f \) factorizes through some \( L \in \mathcal{R}_\varphi \), so \( f = hg \), for some homomorphisms \( g : X \to L \) and \( h : L \to N \). Since \( \mathcal{R}_\varphi \) is contravariantly finite, we get that \( h = \alpha_N j \), for some \( j : L \to M_N \). This implies \( f = hg = \alpha_N jg \in \text{Im}(A(X, \alpha_N)) \) which shows the second inclusion and proves that the above exact sequence gives a projective presentation of the functor \( B(\varphi(-), \varphi(N)) \).

Summing up, we get \( B(\varphi(-), \varphi(N)) \in \mathcal{F}(A) \), for any module \( N \in \text{mod}(A) \), which yields \( U\varphi \in \mathcal{F}(A) \), for any \( U \in \mathcal{F}(B) \) and so the functor \( \varphi : \text{mod}(A) \to \text{mod}(B) \) is admissible. Consequently, we get \( \text{KG}(B) \leq \text{KG}(A) \) by Proposition 2.1. \( \square \)

Assume that \( \mathcal{R} \subseteq \text{mod}(A) \) is some class of \( A \)-modules. If \( T \in \mathcal{F}(A) \), then the class \( \text{supp}_\mathcal{R}(T) = \{ X \in \mathcal{R} \mid T(X) \neq 0 \} \) is called the \( \mathcal{R} \)-support of \( T \). We shall call the class \( \mathcal{R} \) hom-support finite (in short, hs-finite) if and only if the \( \mathcal{R} \)-support of a hom-functor \( A(-, N) \) is finite, for any \( N \in \text{mod}(A) \).

The following lemma is a generalized version of a well-known fact for finite subcategories of \( \text{mod}(A) \), see for example [10, Proposition 4.2]. We apply the lemma in the next section. Let us denote by \( \text{add}(\mathcal{R}) \) the class of all finite direct sums of modules from the class \( \mathcal{R} \).

**Lemma 2.5.** Assume that \( \mathcal{R} \subseteq \text{mod}(A) \) is a hs-finite class of \( A \)-modules. Then the class \( \text{add}(\mathcal{R}) \) is contravariantly finite.
Proof. Assume that $N \in \text{mod}(A)$. We set

$$M_N = \bigoplus_{X \in \mathcal{R}} (A(X, N) \otimes_K X)$$

and define $\alpha_N : M_N \to N$ as $\alpha_N(f \otimes x) = f(x)$, for any $f \in A(X, N)$ and $x \in X$. Observe that $M_N$ is a finite dimensional $A$-module, because $\mathcal{R} \subseteq \text{mod}(A)$ is hs-finite and so there is only a finite number of modules $X \in \mathcal{R}$ such that $A(X, N) \neq 0$. We show that $\alpha_N : M_N \to N$ is a right $\text{add}(\mathcal{R})$-approximation of $N$. Indeed, for $X \in \mathcal{R}$ and a homomorphism $a : X \to N$ define $b : X \to M_N$ as $b(x) = a \otimes x$, for any $x \in X$. Then we have $(\alpha_N b)(x) = \alpha_N(b(x)) = \alpha_N(a \otimes x) = a(x)$, hence $\alpha_N b = a$. This implies that any homomorphism $a : Y \to N$ such that $Y \in \text{add}(\mathcal{R})$ factorizes through $\alpha_N$, and so the assertion follows.

Remark. It is convenient to note that if supp$_{\mathcal{R}}(A(-, N)) = \{X_1, \ldots, X_m\}$ and $A(X_i, N)$ is generated, as a $K$-vector space, by the homomorphisms $f_{i1}, \ldots, f_{im}$, for any $i = 1, \ldots, m$, then $M_N \cong \bigoplus_{i=1}^m X_i^n$ and $\alpha_N \cong [f_{11} \ldots f_{1m} \ f_{21} \ldots f_{2m} \ldots f_{m1} \ldots f_{mn}]$. Although such setting may seem straightforward, the approach presented in the proof above makes argumentation more concise.

3 The main results

We start with recalling some basic facts on trivial extensions of $K$-algebras and their Galois coverings by some special locally finite dimensional $K$-algebras (or locally bounded $K$-categories, equivalently). From our point of view, the most important cases of these locally finite dimensional algebras are repetitive algebras [22] and cluster repetitive algebras [3].

Assume that $C$ is an algebra and $E$ is a non-zero $C$-$C$-bimodule. Consider a locally finite dimensional $K$-algebra $C_E$ of the form

$$C_E = \begin{bmatrix}
\vdots & & 0 \\
\vdots & & C_{-1}
E_0 & C_0 & \text{E}
C_1 & \ddots & \iddots
0 & \text{E} & \ddots & \iddots
\end{bmatrix}$$

where $C_i = C$ and $E_i = E$, for any $i \in \mathbb{Z}$, and there are only finitely many non-zero entries. The multiplication is naturally induced from that of $C$ and the $C$-$C$-bimodule structure of $E$. Further, the identity maps $C_i \to C_{i-1}$ and $E_i \to E_{i-1}$ induce an automorphism $\nu := \nu_{C_E}$ such that the orbit algebra $C_E / \langle \nu \rangle$ is isomorphic to the trivial extension $C \ltimes E$ of $C$ by $E$.

Observe that $C_E$ may be viewed as a locally bounded $K$-category as follows. Assume that $\{e_1, \ldots, e_n\}$ is a complete set of primitive orthogonal idempotents of
Consider $C$. Then the objects of $C_E$ are of the form $e_{m,i}$, for $m \in \{1, \ldots, n\}$, $i \in \mathbb{Z}$, and the morphism spaces are defined in the following way

$$C_E(e_{m,j}, e_{l,i}) = \begin{cases} 
  e_{l}Ce_{m}, & i = j, \\
  e_{l}Ee_{m}, & i = j + 1, \\
  0, & \text{otherwise}.
\end{cases}$$

Moreover, the projection functor $C_E \to C_E/\langle \nu \rangle \cong C \rtimes E$ is a Galois covering with an admissible torsion-free covering group $\langle \nu \rangle \cong \mathbb{Z}$.

In the case $E = D(C)$, the algebra $C_E$ is called the repetitive algebra of $C$, denoted by $\hat{C}$, and it is a self-injective algebra [22]. The automorphism $\nu_{\hat{C}}$ is the Nakayama automorphism of $\hat{C}$ and $\hat{C}/\langle \nu_{\hat{C}} \rangle$ is isomorphic to the trivial extension algebra $T(C) = C \rtimes D(C)$. Let us mention that the repetitive algebras of tilted algebras play an important role in the classification problems of self-injective algebras, see e.g. [16], [38].

If $C$ is a tilted algebra and $E = \text{Ext}^2_C(DC, C)$, then the above matrix algebra $C_E$ is called the cluster repetitive algebra of $C$ and denoted by $\check{C}$ [3]. In this case, the trivial extension $C \rtimes E$ of $C$ by $E = \text{Ext}^2_C(DC, C)$ is called the relation extension algebra and denoted by $\tilde{C} [2]$. It also follows from [2] that $\tilde{C}$ is a cluster-tilted algebra in the sense of [14] (i.e. the endomorphism algebra of a cluster-tilting object in a cluster category) and every cluster-tilted algebra occurs in that way.

Let $G: \check{C} \to \check{C}/\langle \nu_{\check{C}} \rangle \cong \tilde{C}$ be the Galois covering of the cluster-tilted algebra $\tilde{C}$ by the cluster repetitive category $\check{C}$. We denote by $G_{\lambda}: \text{mod}(\check{C}) \to \text{mod}(\tilde{C})$ the associated push-down functor.

**Theorem 3.1.** Assume that $C$ is a tilted algebra, $\check{C}$ the associated cluster repetitive $K$-category and $\tilde{C}$ the cluster-tilted algebra. There exists a fundamental domain $B$ of $\check{C}$ and hence $K_{\check{G}}(\check{C}) = K_{\tilde{G}}(\tilde{C})$.

**Proof.** Let $C$ be a tilted algebra. Consider the matrix algebra $\overline{C} = \begin{bmatrix} C_0 & 0 \\ E & C_1 \end{bmatrix}$, where $C_0 = C_1 = C$ and $E = \text{Ext}^2_C(DC, C)$, endowed with the ordinary matrix addition and the multiplication induced from that of $C$ and from the $C$-$C$-bimodule structure of $E = \text{Ext}^2_C(DC, C)$ (called the cluster duplicated algebra of $C$ in [3]). Lemma 5 of [3] describes the quiver $Q_\check{C}$ of the cluster repetitive category $\check{C}$ and we easily conclude that $\overline{C}$ is a finite convex subcategory of $\check{C}$. Moreover, in the proof of [3, Theorem 22] there is defined a full subcategory $\hat{\Omega}$ of $\text{ind}(\check{C})$ such that the restriction

$$G_{\lambda}|_{\hat{\Omega}}: \hat{\Omega} \to \text{ind}(\tilde{C})$$

of the push-down functor $G_{\lambda}: \text{mod}(\check{C}) \to \text{mod}(\tilde{C})$ is bijective on objects, faithful, preserves irreducible morphisms and almost split sequences. Let us add that the

\[\text{---}\]

We note that $\hat{\Omega}$, also denoted by $\Omega$ in [3], is called by the authors a fundamental domain as well. This definition differs only slightly from ours in Section 2.
objects of $\tilde{\Omega}$ are the successors of $\Sigma_0$ and also proper predecessors of $\Sigma_1$ in $\text{ind}(\tilde{C})$, where $\Sigma_i$ denotes the image in $\text{mod}(C_i)$ of a complete slice $\Sigma$ from $\text{mod}(C)$ under the isomorphisms $C_i \cong C$, $i \in \mathbb{Z}$. Further, it is shown that $\text{ob}(\tilde{C}) = \text{supp}(\tilde{\Omega})$.

Let now $X \in \text{ind}(\tilde{C})$. Then $G_\lambda(X) \in \text{ind}(\tilde{\Omega})$. Since $G_\lambda|_{\tilde{\Omega}}$ is bijective on objects, there exists a module $Y \in \tilde{\Omega}$ such that $G_\lambda(X) \sim G_\lambda(Y)$ and hence $\gamma X \cong Y$, for some $g \in G$ (see e.g. [13, 2.5]). We conclude that for any $X \in \text{ind}(\tilde{C})$ there is $g \in G$, such that $gX \in \tilde{\Omega}$. This implies that $\text{supp}(gX) \subseteq \text{supp}(\tilde{\Omega})$, which means that $\tilde{C}$ is a fundamental domain of $\tilde{\Omega}$. Applying Theorem 2.3 to the Galois covering $G : \tilde{C} \to \tilde{C}/\langle \nu \rangle \cong \tilde{\Omega}$ we get that $\text{KG}(\tilde{C}) = \text{KG}(\tilde{\Omega})$.

The notation for the subcategory $\tilde{\Omega}$ (= $\Omega$) in the above theorem comes from [3]. This notation is slightly confusing because of its similarity to the usual symbol for the syzygy functor, nevertheless we use it to be consistent with [3]. From now on $\Omega$ is reserved for the syzygy functor.

Our aim now is to show that $\text{KG}(\tilde{C}) \leq \text{KG}(\hat{C})$, for any tilted algebra $C$. We denote by $K_C$ the set

$$\{ \tilde{P}_x, \tau^{1-i}\Omega^{-1}(C) \mid x \in (\hat{C})_0, i \in \mathbb{Z} \}$$

of modules from $\text{mod}(\hat{C})$ where $\tilde{P}_x$ is an indecomposable projective $\hat{C}$-module at the vertex $x \in (\hat{C})_0$ and $\tau = \tau_{\hat{C}}$ is the Auslander-Reiten translation in $\text{mod}(\hat{C})$. We have the following fact, see [3, Lemma 8, Theorem 9].

**Proposition 3.2.** Assume that $C$ is a tilted algebra. There exists an additive $K$-linear functor $\phi : \text{mod}(\hat{C}) \to \text{mod}(\tilde{C})$ which is full and dense such that $\text{Ker}(\phi)$ equals the class of all homomorphisms in $\text{mod}(\hat{C})$ which factorize through $\text{add}(K_C)$.

The following property of $K_C$ is crucial in applying the results from Section 2.

**Proposition 3.3.** Assume that $C$ is a tilted algebra. The class $K_C$ is a hs-finite class of $\text{mod}(\hat{C})$.

**Proof.** We view a tilted algebra $C$ as a $\hat{C}$-module whose support is the set $e_{1,0}, \ldots, e_{n,0}$ of objects of $\hat{C}$. The proof is divided into three cases.

**Case 1.** Let $C$ be a tilted algebra and $N \in \text{mod}(\hat{C})$. Then $\tilde{C}(\tilde{P}_x, N) \cong N(x)$, for any $x \in \text{ob}(\hat{C})$. Hence the number of indecomposable projective modules in $\text{mod}(\tilde{C})$ belonging to the support of the functor $\tilde{C}(\cdot, N)$ is finite and equals the cardinality of the set $\text{supp}(N)$. Note that this argument works for an arbitrary finite dimensional algebra $C$.

**Case 2.** Let $C$ be a tilted algebra of a Dynkin type $\Delta$. Recall that the stable part $\Gamma^*_{\tilde{C}}$ of the Auslander-Reiten quiver $\Gamma_{\tilde{C}}$ of $\hat{C}$ is of the form $\mathbb{Z}\Delta$. Assume that $P$ is an indecomposable direct summand of $C$. Since $P$ is not a projective-injective $\hat{C}$-module, it lies on a stable section $\Sigma$ in $\mathbb{Z}\Delta$ [6, VIII.1]. The repetitive category $\tilde{C}$
is locally representation finite by \[ \text{[4]} \] (see also \[ \text{[23, Theorem 5.1]} \]) and so we conclude that the support of the functor \( \hat{C}(-, N) \) is finite. This yields the existence of some integers \( k, m \) such that \( k > m \) and the support of \( \hat{C}(-, N) \) is contained in the full subquiver \( \mathcal{D} \) of \( \Gamma_{\hat{C}} \) consisting of all indecomposable modules lying between \( \tau^k \Sigma \) and \( \tau^m \Sigma \) (more formally, all modules being both successors of \( \tau^k \Sigma \) and predecessors of \( \tau^m \Sigma \)).

In what follows we assume that the module \( P \) is a successor of the module \( N \) and hence \( k > m \geq 0 \). For other cases the arguments are similar.

Assume now that \( X \) is an arbitrary indecomposable non-projective (and non-injective) \( \hat{C} \)-module. Then \( \Omega(X) \) is a proper predecessor of \( X \) and \( \Omega^{-1}(X) \) is a proper successor of \( X \). Indeed, this follows from the fact that there are non-zero homomorphisms \( \Omega(X) \to P(X), P(X) \to X, X \to I(X) \) and \( I(X) \to \Omega^{-1}(X) \) where \( P(X) \) and \( I(X) \) denote the projective cover of \( X \) and the injective envelope of \( X \) in \( \text{mod}(\hat{C}) \), respectively (recall that there are no oriented cycles in \( \Gamma_{\hat{C}} \)). This implies that for \( i > 0 \) we have \( \tau^{-i} \Omega^{-i}(P) \in \tau^l \Sigma \) for some \( l < 0 \) and thus \( \tau^{-i} \Omega^{-i}(P) \notin \mathcal{D} \). Moreover, if \( 1 - i > k \) we get \( \tau^{-i} \Omega^{-i}(P) \in \tau^s \Sigma \) for some \( s > k \) and hence \( \tau^{-i} \Omega^{-i}(P) \notin \mathcal{D} \) also in this case. Therefore we conclude that if \( \tau^{-i} \Omega^{-i}(P) \in \mathcal{D} \), then \( i \in \{-k + 1, -k + 2, \ldots, -1, 0\} \). Note that any direct summand of a module \( \tau^{-i} \Omega^{-i}(C) \) is of the form \( \tau^{-i} \Omega^{-i}(P) \), for some direct summand \( P \) of \( C \). Together with Case 1, these arguments yield the class \( K_C \) is hs-finite in the case \( C \) is tilted of Dynkin type.

Case 3. Let \( C \) be a tilted algebra of Euclidean or wild type \( \Delta \). Then the Auslander-Reiten quiver \( \Gamma_{\hat{C}} \) of \( \hat{C} \) has a decomposition

\[
\Gamma_{\hat{C}} = \bigvee_{q \in \mathbb{Z}} (X_q \vee C_q)
\]

such that for each \( q \in \mathbb{Z} \), \( X_q \) is a component whose stable part \( X_q^s \) is of the form \( \mathbb{Z} \Delta \) and \( C_q \) is an infinite family of components whose stable part \( C_q^s \) is a union either of stable tubes (if \( \Delta \) is of Euclidean type \[ \text{[5]}, \text{[38]} \]) or of components of the form \( \mathbb{Z} A_{\infty} \) (if \( \Delta \) is of wild type \[ \text{[16, 3.5]} \]). Moreover, the following statements hold:

(a) for each pair \( p, q \in \mathbb{Z} \) with \( q > p \), we have \( \text{Hom}_{\hat{C}}(X_q, X_p \vee C_p) = 0 \) and \( \text{Hom}_{\hat{C}}(C_q, C_p \vee X_{p+1}) = 0 \);

(b) for each \( q \in \mathbb{Z} \), we have \( \nu_{\hat{C}}(X_q) = X_{q+2} \) and \( \nu_{\hat{C}}(C_q) = C_{q+2} \);

(c) for each \( q \in \mathbb{Z} \), we have \( \text{Hom}_{\hat{C}}(X_q, X_p \vee C_p) = 0, \text{Hom}_{\hat{C}}(C_q, C_p \vee X_{p+1}) = 0 \) for \( p > q + 2 \);

(d) for each \( q \in \mathbb{Z} \), we have \( \Omega_{\hat{C}}(C_{q+1}) = C_q^s \) and \( \Omega_{\hat{C}}(X_{q+1}) = X_q^s \).
Let $N \in X_t \lor C_t$ for some $t \in \mathbb{Z}$. By (a) we have that the support of $\hat{\mathcal{C}}(-, N)$ consists of modules from $\bigvee_{p \leq t}(X_p \lor C_p)$. Hence by (c) we obtain that the support of $\hat{\mathcal{C}}(-, N)$ is contained in the family

$$D = (X_{t-3} \lor C_{t-3}) \lor (X_{t-2} \lor C_{t-2}) \lor (X_{t-1} \lor C_{t-1}) \lor (X_t \lor C_t).$$

Assume that $P$ is an indecomposable direct summand of $C$. Since $P$ is not a projective-injective $\hat{\mathcal{C}}$-module, we obtain that $P \in X^s_i \lor C^s_i \lor X^s_{i+1} \lor C^s_{i+1}$. Therefore we conclude that if $\tau^{1-i}\Omega^{-i}(P) \in D$, then at least one of the following conditions holds: $X^s_i \subseteq D$, $C^s_i \subseteq D$, $X^s_{i+1} \subseteq D$ or $C^s_{i+1} \subseteq D$. Equivalently, if $\tau^{1-i}\Omega^{-i}(P) \in D$, then $i \in \{t-4, t-3, \ldots, t\}$. As before, any direct summand of a module $\tau^{1-i}\Omega^{-i}(C)$ is of the form $\tau^{1-i}\Omega^{-i}(P)$, for some direct summand $P$ of $C$. Hence, together with Case 1, these arguments yield the class $\mathcal{K}_C$ is hs-finite in the case $C$ is tilted of Euclidean or wild type.

Summing up, the above three cases show that the class $\mathcal{K}_C$ is a hs-finite class of mod($\hat{\mathcal{C}}$) for any tilted algebra $C$.

We are now in a position to prove the first main result of the paper.

**Theorem 3.4.** Assume that $C$ is a tilted algebra and $\hat{\mathcal{C}}, \tilde{\mathcal{C}}, \check{\mathcal{C}}$ are the associated repetitive category, cluster repetitive category and cluster-tilted algebra, respectively. The following assertions hold.

1. The functor $\phi$: mod($\hat{\mathcal{C}}$) $\rightarrow$ mod($\check{\mathcal{C}}$) is admissible.

2. We have $\text{KG}(\tilde{\mathcal{C}}) = \text{KG}(\check{\mathcal{C}}) \leq \text{KG}(\hat{\mathcal{C}})$.

**Proof.** (1) Proposition 3.3 yields the class $\mathcal{K}_C$ is hs-finite, so the functor $\phi$: mod($\hat{\mathcal{C}}$) $\rightarrow$ mod($\check{\mathcal{C}}$) is admissible by Proposition 3.2, Theorem 2.4 and Lemma 2.5. The assertion of (2) follows from (1), Theorem 2.4 and Theorem 3.1.

We shall apply the following classification of Krull-Gabriel dimensions of locally support-finite repetitive $K$-categories over an algebraically closed field $K$.

**Theorem 3.5.** Assume that $K$ is an algebraically closed field and $A$ is a finite dimensional basic and connected $K$-algebra such that $\hat{A}$ is locally support-finite. Then $\text{KG}(\hat{A}) \in \{0, 2, \infty\}$ and the following assertions hold.

1. $\text{KG}(\hat{A}) = 0$ if and only if $\hat{A} \cong \check{B}$ where $B$ is some tilted algebra of Dynkin type.

2. $\text{KG}(\hat{A}) = 2$ if and only if $\hat{A} \cong \tilde{B}$ where $B$ is some representation-infinite tilted algebra of Euclidean type.
(3) $\text{KG}(\widehat{A}) = \infty$ if and only if $\widehat{A}$ is wild or $\widehat{A} \cong \widehat{B}$ where $B$ is a tubular algebra.

The following theorem is the second main result of the paper. In this theorem we refine the assertion (2) of Theorem 3.4 and determine the Krull-Gabriel dimension of cluster-tilted algebras. We also conclude that the class of cluster-tilted algebras supports the conjecture of Prest.

**Theorem 3.6.** Assume that $K$ is an algebraically closed field, $C$ is a tilted $K$-algebra and $\tilde{C}, \check{C}, \hat{C}$ are the associated repetitive category, cluster repetitive category and cluster-tilted algebra, respectively. Then $\text{KG}(\tilde{C}) = \text{KG}(\check{C}) = \text{KG}(\hat{C}) \in \{0, 2, \infty\}$ and the following assertions hold.

1. $C$ is tilted of Dynkin type if and only if $\text{KG}(\tilde{C}) = 0$.
2. $C$ is tilted of Euclidean type if and only if $\text{KG}(\tilde{C}) = 2$.
3. $C$ is tilted of wild type if and only if $\text{KG}(\tilde{C}) = \infty$.

In particular, a cluster-tilted algebra $\tilde{C}$ has finite Krull-Gabriel dimension if and only if $\tilde{C}$ is of domestic representation type.

**Proof.** We apply freely Theorem 3.5 and the assertion (2) of Theorem 3.4 stating that $\text{KG}(\tilde{C}) = \text{KG}(\check{C}) \leq \text{KG}(\hat{C})$. We prove all assertions simultaneously.

If $C$ is of Dynkin type, then $\text{KG}(\tilde{C}) = \text{KG}(\check{C}) \leq \text{KG}(\hat{C}) = 0$, so $\text{KG}(\tilde{C}) = \text{KG}(\check{C}) = \text{KG}(\hat{C}) = 0$.

If $C$ is of Euclidean type, then $\text{KG}(\tilde{C}) = \text{KG}(\check{C}) \leq \text{KG}(\hat{C}) = 2$, but $\text{KG}(\check{C}) \neq 0$ since $\tilde{C}$ is of infinite representation type (see [14]) and $\text{KG}(\tilde{C}) \neq 1$ by [28]. This yields $\text{KG}(\tilde{C}) = \text{KG}(\check{C}) = \text{KG}(\hat{C}) = 2$.

If $C$ is of wild type, then $\tilde{C}$ is also of wild type by [14] and hence we obtain $\text{KG}(\tilde{C}) = \text{KG}(\check{C}) = \text{KG}(\hat{C}) = \infty$.

Since the algebra $C$ is a tilted algebra either of Dynkin, or of Euclidean or of wild type, we conclude that the above implications can be replaced by equivalences. This also yields the fact that $\text{KG}(\tilde{C}) = \text{KG}(\check{C}) = \text{KG}(\hat{C}) \in \{0, 2, \infty\}$. Moreover, if $C$ is tilted of Dynkin or Euclidean type, then $\tilde{C}$ is of domestic representation type (assuming that finite type is contained in domestic type). This implies that the class of cluster-tilted algebras supports the conjecture of Prest on Krull-Gabriel dimension and domestic algebras.

We finish the section with natural application of the above theorem to the theory of super-decomposable pure-injective modules. Assume that $R$ is a ring with a unit. An $R$-module $M \neq 0$ is super-decomposable if and only if $M$ does not have an

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Observe that this argument gives another proof of the fact that cluster-tilted algebras of Dynkin type are representation finite.
indecomposable direct summand. For the concept of pure-injectivity we refer to [24]. The problem of the existence of super-decomposable pure-injective \( R \)-modules is studied for the first time in [39]. The case when \( R \) is a finite dimensional algebra over a field is studied in many papers, see [25] for an up-to-date list of results concerning this case and [30, Theorem 8.3] for the most recent one about self-injective algebras. It is conjectured by Prest that if \( R \) is a finite dimensional algebra over an algebraically closed field, then \( R \) is of domestic representation type if and only if there is no super-decomposable pure-injective \( R \)-module, see for example [34]. This conjecture is sometimes restricted only to countable fields, see [25] for details. The following theorem supports the conjecture.

**Corollary 3.7.** Assume that \( C \) is a tilted algebra over an algebraically closed field \( K \) and \( \tilde{C} \) is the corresponding cluster-tilted algebra. If \( \tilde{C} \) is of domestic type, then it has no super-decomposable pure-injective module. The converse implication holds if the field \( K \) is countable.

**Proof.** If \( \tilde{C} \) is domestic, then by assertions (1), (2) of Theorem 3.6 we have that \( KG(\tilde{C}) \) is finite and thus super-decomposable pure-injective \( \tilde{C} \)-module does not exist, see for example [34]. Conversely, if \( \tilde{C} \) is non-domestic, then \( \tilde{C} \) is wild, so it possesses a super-decomposable pure-injective module by [31, Theorem 3.2].

A Appendix (by Grzegorz Bobiński)

Throughout this appendix \( K \) is a fixed field. All considered algebras are finite dimensional \( K \)-algebras. For simplicity we also assume that all considered categories are Krull–Schmidt \( K \)-categories with finite dimensional homomorphism spaces.

The aim of this appendix is to present an alternative proof of the following result proved in Theorem 3.6.

**Theorem A.1.** Let \( C \) be a cluster-tilted algebra.

1. If \( C \) is of Dynkin type, then \( KG(C) = 0 \).
2. If \( C \) is of Euclidean type, then \( KG(C) = 2 \).
3. If \( C \) is of wild type, then \( KG(C) = \infty \).

In fact we prove the following equivalent version of the above result.

**Theorem A.2.** If \( H \) is a hereditary algebra and \( C \) is a cluster-tilted algebra of type \( H \), then \( KG(C) = KG(H) \).

The above mentioned equivalence follows from the following well-known description of the Krull–Gabriel dimension of the hereditary algebras.
Proposition A.3. Let $H$ be a hereditary algebra.

(1) If $H$ is of Dynkin type, then $\text{KG}(H) = 0$.

(2) If $H$ is of Euclidean type, then $\text{KG}(H) = 2$.

(3) If $H$ is of wild type, then $\text{KG}(H) = \infty$.

Proof. (1) follows from [8, Corollary 3.14], (2) from [19, Theorem 4.3], and (3) from [11, Theorem 4.3].

We recall first the definition of the Krull–Gabriel dimension of an abelian category. Let $\mathcal{A}$ be an abelian category and $\mathcal{A}_{-1} = 0$. For any $\alpha$ being either an ordinal number or $-1$, let $\mathcal{A}_{\alpha+1}$ be the Serre subcategory of $\mathcal{A}$ consisting of those objects in $\mathcal{A}$ which have finite length after passing to the quotient category $\mathcal{A}/\mathcal{A}_{\alpha}$. Moreover, if $\beta$ is a limit ordinal, then $\mathcal{A}_\beta = \bigcup_{\alpha<\beta} \mathcal{A}_\alpha$. By the Krull–Gabriel dimension $\text{KG-dim}(\mathcal{A})$ of $\mathcal{A}$ we mean the smallest ordinal number $\alpha$ such that $\mathcal{A}_\alpha = \mathcal{A}$. If there is not such number, then we put $\text{KG-dim}(\mathcal{A}) = \infty$.

Abelian categories we are interested in are of special form. Namely, let $\mathcal{C}$ be an additive category and denote by $\mathcal{F}(\mathcal{C})$ the category of all contravariant finitely presented functors from $\mathcal{C}$ to the category mod $K$ of finite dimensional vector spaces. A category $\mathcal{C}$ is called coherent, if the category $\mathcal{F}(\mathcal{C})$ is abelian. There are two important examples of coherent categories: abelian and triangulated ones. If $\mathcal{C}$ is an algebra, then we put $\mathcal{F}(\mathcal{C}) = \mathcal{F}(\text{mod} \mathcal{C})$ and $\text{KG}(\mathcal{C}) = \text{KG-dim}(\mathcal{F}(\mathcal{C}))$, where mod $\mathcal{C}$ is the category of finite dimensional $\mathcal{C}$-modules.

If $\mathcal{B}$ is a full subcategory of a category $\mathcal{C}$, then we denote by $[\mathcal{B}]$ the ideal of morphisms in $\mathcal{C}$, which factor through objects in $\mathcal{B}$. Next, if $X$ is an indecomposable object of $\mathcal{C}$, then we denote by $S_X$ the functor $S_X: \mathcal{C} \to \text{mod} K$ such that $S_X(X) = K$ and $S_X(Y) = 0$, for each indecomposable object $Y$ of $\mathcal{C}$ nonisomorphic to $X$. The following result due to Geigle [20, Corollary 2.9] will play a crucial role in our proof.

Proposition A.4. Let $\mathcal{C}$ be a coherent category and $\mathcal{B}$ be a full subcategory of $\mathcal{C}$ with only finitely many indecomposable objects up to isomorphism. If $S_X \in \mathcal{F}(\mathcal{C})$, for each indecomposable object $X$ of $\mathcal{B}$, then

$$\text{KG-dim}(\mathcal{F}(\mathcal{C})) = \text{KG-dim}(\mathcal{F}(\mathcal{C}/[\mathcal{B}])).$$ 

Observe that $X \in \mathcal{F}(\mathcal{C})$, for an indecomposable object $X$ of $\mathcal{C}$, if and only if there is a source map for $X$, i.e. a map $f: X \to M$ such that $f$ is not a section and every $f': X \to M'$ which is not a section factors through $f$.

Let $H$ be a hereditary algebra. One defines the cluster category $\mathcal{C}_H$ as the quotient of the bounded derived category $D^b(\text{mod} H)$ by the action of the functor $\tau^{-1} \circ \Sigma$, where $\tau$ is the Auslander–Reiten translation and $\Sigma$ is the shift functor.
It is known that $\mathcal{C}_H$ is a triangulated category, whose shift functor is induced by the shift functor $\Sigma$ in $\mathcal{D}^b(\text{mod } H)$ [26, Theorem 1], with almost split triangles [13, Proposition 1.3]. In particular, there is a source map for every indecomposable object of $\mathcal{C}_H$, hence we may apply Proposition A.4 with $C = \mathcal{C}_H$.

An object $T$ of $\mathcal{C}_H$ is called cluster-tilting if $\text{Hom}_{\mathcal{C}_H}(T, \Sigma T) = 0$ and if for each indecomposable object $X$ of $\mathcal{C}_H$ the equality $\text{Hom}_{\mathcal{C}_H}(X, \Sigma T) = 0$ implies that $X$ is a direct summand of $T$. It is known that every cluster-tilting objects has $n$ pairwise nonisomorphic indecomposable direct summands, where $n$ is the number of pairwise nonisomorphic simple $H$-modules [13, Theorem 3.3]. The cluster-tilted algebras of type $H$ are by definition the opposite algebras of the endomorphism algebras of the cluster-tilting objects in $\mathcal{C}_H$. The following fundamental result [14, Theorem A] will be of importance to us.

**Proposition A.5.** Let $H$ be a hereditary algebra, $T$ a cluster-tilting object, and $C = \text{End}_{\mathcal{C}_H}(T)^{\text{op}}$. Then $\text{Hom}_{\mathcal{C}_H}(T, -)$ induces an equivalence

$$\mathcal{C}_H/[\text{add } \Sigma T] \simeq \text{mod } C.$$  

As an immediate consequence of Propositions A.4 and A.5 we obtain the following.

**Corollary A.6.** If $H$ is hereditary algebra and $C$ a cluster-tilted algebra of type $H$, then

$$\text{KG}(C) = \text{KG-dim}(\mathcal{F}(\mathcal{C}_H)).$$

**Proof.** Let $C = \text{End}_{\mathcal{C}_H}(T)^{\text{op}}$, for a cluster-tilting object $T$ in $\mathcal{C}_H$. We know from Proposition A.5 that

$$\text{KG}(C) = \text{KG-dim}(\mathcal{F}(\mathcal{C}_H/[\text{add } \Sigma T])),$$

thus it is sufficient to apply Proposition A.4 with $C = \mathcal{C}_H$ and $B = \text{add } \Sigma T$.  

Now we are ready to prove Theorem A.2.

**Proof of Theorem A.2.** Let $C$ be a cluster-tilted algebra of type $H$. Then

$$\text{KG}(C) = \text{KG-dim}(\mathcal{F}(\mathcal{C}_H)), \tag{A.1}$$

by Corollary A.6. On the other hand, it is well known that $H$ itself is a cluster-tilted algebra of type $H$ ($H$ viewed as an object of $\mathcal{C}_H$ is a cluster-tilting object with $\text{End}_{\mathcal{C}_H}(H)^{\text{op}} \simeq H$), hence using again Corollary A.6 we obtain

$$\text{KG}(H) = \text{KG-dim}(\mathcal{F}(\mathcal{C}_H)). \tag{A.2}$$

Now the claim follows from (A.1) and (A.2).  

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