Abstract. We introduce the notion of twisted jets. For a Deligne-Mumford stack $X$ of finite type over $C$, a twisted 1-jet on $X$ is a representable morphism $D \to X$ such that $D$ is a smooth Deligne-Mumford stack with the coarse moduli space $\text{Spec} C[\! [t] \! ]$. We study the motivic measure on the space of the twisted 1-jets on a smooth Deligne-Mumford stack.

As an application, we prove that two birational minimal models with Gorenstein quotient singularities have the same orbifold cohomology with Hodge structure.

1. Introduction

In 1995, Kontsevich produced the theory named motivic integration [Kon95]. Since then, this remarkable idea has become a powerful method for examining both the local and global structures of varieties.

Let $X$ be a variety over $C$. For $n \geq 0$, an $n$-jet on $X$ is a $C[\! [t] \! ]/(t^{n+1})$-point of $X$, where we have followed the convention $(t^1) = (0)$. The $n$-jets of $X$ naturally constitute a variety (or pro-variety if $n = 1$), denoted $L_n X$. For $m \geq n$, the natural surjection $C[\! [t] \! ]/(t^{n+1}) \to C[\! [t] \! ]/(t^{m+1})$ induces the truncation morphism $L_m X \to L_n X$.

Consider the case where $X$ is smooth and of dimension $d$. Then $L_n X$ is a locally trivial affine space bundle over $X$. Whenever $X$ is singular, it fails. For example, for $n = 1$, $L_1 X$ is the tangent space of $X$ and hence not a locally trivial bundle over $X$.) The idea of Kontsevich is to give $L_1 X$ a measure which takes values in the Grothendieck ring $\mathcal{M}$ of $k$-varieties which is localized by the class $L$ of the affine line. For each $n \geq 0$, the family of constructible subsets of $L_n X$ is stable under finite union or finite intersection. In other words, this family is a Boolean algebra. The map

$$f \mapsto \text{constructible subsets of } L_n X \subset \mathcal{M}$$

is a finite additive measure. For each $m > n \geq 0$, because the truncation morphism $L_m X \to L_n X$ is a locally trivial affine space bundle of

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relative dimension \( m \) and \( n \), the pull-back

\[(\binom{m}{n})^{-1} : \text{fcone}	ext{structurable subsets of } L_n X \to \text{fcone}	ext{structurable subsets of } L_m X \text{ and } g;\]

is considered to be an extension of the measure into a bigger Boolean algebra. The motivic measure on \( L_1 X \) is defined to be the limit of these extensions. Demailly and Loeser generalized the motivic measure to the case where \( X \) is singular \[DL99\].

The integral of a function with respect to the motivic measure produces a new invariant. In particular, when \( X \) is smooth and the function is constant equal to 1, then the integral, which is the full volume of \( L_1 X \), is the class of \( X \) in \( M \). It reduces to the Hodge structure of the cohomology of \( X \) if \( X \) is complete. By the transformation rule of Kontsevich, for any resolution \( Z \) of \( X \), it equals the integral of a function on \( L_1 Z \) determined by the relative canonical divisor \( K_{Z-X} \). This implies the following theorem of Kontsevich, which we will generalize:

**Theorem 1.1.** Let \( X \) and \( X^0 \) be smooth complete varieties. Suppose that there are proper birational morphisms \( Z \to X \) and \( Z^0 \to X^0 \) such that \( K_{Z-X} = K_{Z^0-X^0} \). Then the rational cohomologies of \( X \) and \( X^0 \) have the same Hodge structures.

It is key that by the valuative criterion for properness, almost every \( 1 \)-jet on \( X \) lifts to a unique \( 1 \)-jet on \( Z \), and hence the map \( L_1 Z \to L_1 X \) is bijective outside of the zero subset. Note that Batyrev first proved the equality of Betti numbers in the case where \( X \) and \( X^0 \) are Calabi-Yau varieties, with p-adic integration and the Weil conjecture \[Bat99a\].

Let \( X \) be a variety with Gorenstein canonical singularities. Demailly and Loeser gave \( L_1 X \) another measure, called the motivic Gorenstein measure, denoted \( G_{x}^{\text{or}}[DL02] \). As in the case with \( X \) smooth, \( G_{x}^{\text{or}}(L_1 X) \) is calculated by the relative canonical divisor \( K_{Z-X} \) for a resolution \( Z \to X \). This implies:

**Proposition 1.2.** Let \( X \) and \( X^0 \) be varieties with Gorenstein canonical singularities. Suppose that there are proper birational morphisms \( Z \to X \) and \( Z^0 \to X^0 \) such that \( K_{Z-X} = K_{Z^0-X^0} \). Then \( G_{x}^{\text{or}}(L_1 X) = G_{x}^{\text{or}}(L_1 X^0) \).

Quotient singularities form one of the mildest classes of singularities\(^1\). If \( X \) is a variety with quotient singularities, then we can give \( X \) an orbifold structure. In the algebro-geometric context, there is a smooth Deligne-Mumford stack \( X \) such that \( X \) is the coarse moduli space of \( X \) and the automorphism group of general points of \( X \) is trivial. Although the natural morphism \( X \to X \) is proper and birational, not almost every \( 1 \)-jet on \( X \) lifts to a \( C \)-[jet]; point of \( X \) from lack of the strict valuative criterion for properness. However by twisting the source SpecC \( [\eta] \), we can lift almost every \( 1 \)-jet on \( X \) to \( X \). More precisely, a twisted \( 1 \)-jet on \( X \) is a representable morphism

\(^1\)Here the words 'quotient singularities' mean 'quotient singularities with respect to the étale topology', see Definition \[22\].
D ! X such that D is a smooth Deligne-Mumford stack with the coarse moduli space $\text{SpecC} [t]$ and D contains $\text{SpecC} (\{t\})$ as open substack. (A paper of Abramovich and Vistoli [AV02] was the inspiration for this notion. They introduced the notion of twisted stable map.) For almost every $\text{-jet } : \text{SpecC} [t] ! X$, there is a unique twisted $\text{-jet } D ! X$ such that the induced morphism $\text{SpecC} [t] ! X$ of the coarse moduli spaces is. If $\text{L}_1 X$ is the coarse moduli space of the twisted $\text{-jets}$ on $X$, then we define the orbifold measure $\text{Gor}_X$ on $\text{L}_1 X$ in a similar fashion as on $\text{L}_1 X$, though it takes values in the Grothendieck ring of Hodge structures. We show the following close relation between $\text{Gor}_X$ and $\text{L}_1 X$:

\begin{equation}
\text{Gor}_X = \text{L}_1 X
\end{equation}

For the precise meaning, see Theorem 3.15.

Chen and Ruan defined the orbifold cohomology for arbitrary orbifold [CR00]. It originates from string theory on orbifolds [DHVW]. Let $X$ be a variety with Gorenstein quotient singularities and $X$ as above. The inertia stack of $X$, denoted $\text{I}(X)$, is an object in the algebrao-geometric realm that corresponds to the twisted sector. We define the $i$-th orbifold cohomology group of $X$ by:

$$
H^{i}_{\text{orb}} (X ; \mathbb{Q}) = \bigoplus_{Y \in \text{I}(X)} H^{2s(Y)} (\overline{Y} ; \mathbb{Q}) \otimes \mathbb{Q} (s(Y)),
$$

where $Y$ runs over the connected components of $\text{I}(X)$, $\overline{Y}$ is the coarse moduli space of $Y$ and $s(Y)$ is an integer which is representation-theoretically determined.

Remark 1.4. (1) The author guesses, though is not sure, that our orbifold cohomology is equal to one by Chen and Ruan in [CR00]. The point he wonders is whether the cohomology groups of $\overline{Y}$ are isomorphic to ones of the analytic orbifold (V-manifold) associated to $Y$.

(2) The orbifold Hodge numbers (that is, the Hodge numbers associated to $H^{i}_{\text{orb}} (X ; \mathbb{Q})$) are equal to Batyrev’s stringy Hodge numbers [Bat99]. It follows from Lemma 2.13 and Theorem 1.3.

Theorem 1.3 implies the fact that when $X$ is complete, the invariant $\text{Gor}_X (\text{L}_1 X)$ reduces to the alternating sum of the orbifold cohomology groups of $X$. Hence we obtain the following, conjectured by Ruan [Rau00]:

Theorem 1.5 (= Corollary 3.16). Let $X$ and $X^0$ be complete varieties with Gorenstein quotient singularities. Suppose that there are proper birational morphisms $Z ! X$ and $Z ! X^0$ such that $K_{Z-X} = K_{Z-Z^0}$. Then the orbifold cohomologies of $X$ and $X^0$ have the same Hodge structures.
If $X$ and $X^0$ are birational minimal models, that is, $K_X$ and $K_{X^0}$ are nef, then for a common resolution $Z$ of $X$ and $X^0$, we have the equality $K_{Z,X} = K_{Z,X^0}$ (see [KM98, Prop. 3.51]). Hence $X$ and $X^0$ have the same orbifold cohomology with Hodge structure. Note that in the case where $X$ and $X^0$ are global quotients, Theorem 1.1 is due to Batyrev [Bat99b] and Deneef-Loeser [DL02]. After writing out the first version of this paper, I learned by an e-mail message from Ernesto Lupercio that Mainak Poddar and he independently proved Theorem 1.5.

Contents. The paper is organized as follows. In Section 1, we review motivic measures. Section 2 is the central part of the paper. Here we introduce the notion of a twisted jet and examine the space of them. Then we prove the main result. In Section 3, we review Deligne-Mumford stacks and prove some general results on Deligne-Mumford stacks which we need in the preceding section.

Conventions and Notations.

In Section 1 and 2, we work over $\mathbb{C}$.

For a Deligne-Mumford stack $X$, we denote by $X$ the coarse moduli space of $X$.

We denote by $(\text{Sch} = S)$ (resp. $(\text{Sch} = \mathbb{C})$) the category of schemes over a scheme $S$ (resp. over $\mathbb{C}$).

For a $\mathbb{C}$-scheme $X$ (ormore generally a stack over $\mathbb{C}$) and a $\mathbb{C}$-algebra $R$, we denote by $X \times R$ the product $X \times \text{Spec} R$. Then we denote by $X \circlearrowleft[[t]]$ (resp. $X \circlearrowleft[[t^n]]$) the scheme $X \times \mathbb{C}[[t]]$ (resp. $X \times \mathbb{C}[[t^n]]$).

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2. Motivic measure; review

In this section, we would like to review the theory of motivic measures, developed by Kontsevich [Kon95], Batyrev [Bat99b], Deneef and Loeser [DL99], [DL02]. I mention [Cra99] for a nice introduction and [Loo02], [DL01] for surveys.

2.1. Completing Grothendieck rings. Let us first construct the ring in which motivic measures take values.

A variety means a reduced scheme of finite type over $\mathbb{C}$.

Definition 2.1. We denote the Grothendieck ring of varieties, denoted $K_0(\text{Var})$, to be the abelian group generated by the isomorphism classes $fX$ of varieties with the relations $fX = fX$, and $XY + fY$ if $Y$ is a closed subvariety of $X$. The ring structure is defined by $fX gY = fX Y g$. 

In the same fashion, we can define the Grothendieck ring of separated algebraic spaces of finite type. Actually, it is the same thing as \( K_0(\text{Var}) \), since every noetherian algebraic space decomposes into the disjoint union of schemes [Knudsen, Prop. 6.6].

Suppose that \( A \) is a constructible subset of a variety \( X \), that is, \( A \) is a disjoint union of locally closed subvarieties \( A_i \). Then we put \( f \text{A}_g = \sum_i f \text{A}_i g \in K_0(\text{Var}) \), which is independent of the choice of stratifications. We denote the class of \( A^i \) by \( L \) and the localization \( K_0(\text{Var})[L^{-1}] \) by \( M \). For \( m \in \mathbb{Z} \), let \( F_m \) be the subgroup of \( M \) generated by the elements \( fX g_L \) with \( \dim X \geq m \). The collection \( \{ F_m \}_{m \in \mathbb{Z}} \) is a descending filtration of \( M \) with

\[
F_m M \supseteq F_{m+1} M.
\]

Definition 2.2. We define the complete Grothendieck ring of varieties by

\[
\tilde{M} = \lim M = F_m M.
\]

By condition (2.1), it has a natural ring structure.

Note that it is not known whether the natural map \( M \to \tilde{M} \) is injective or not.

Recall that a Hodge structure is a finite dimensional \( \mathbb{Q} \)-vector space \( H \) with a bigrading \( H = \bigoplus_{p+q} H^{p,q} \) such that \( H^{p,q} \) is the complex conjugate of \( H^{q,p} \) and each weight sum \( \dim H \) and \( \dim H^{q,p} \) is defined over \( \mathbb{Q} \). The category \( \mathcal{H}S \) of Hodge structures is an abelian category with tensor product.

Definition 2.3. We define the Grothendieck ring of Hodge structures, denoted \( K_0(\mathcal{H}S) \), to be the abelian group which consists of formal differences \( fH \sim gH \), where \( fH \) and \( gH \) are isomorphism classes of Hodge structures. The addition and the multiplication come from \( \mathcal{H}S \) and \( \mathbb{Q} \) respectively.

A mixed Hodge structure is a finite dimensional \( \mathbb{Q} \)-vector space \( H \) with increasing filtration \( W^i H \), called the weight filtration, such that the associated graded \( G^i H \) underlies a Hodge structure having \( G^m H \) as weight \( m \) summand. For a mixed Hodge structure \( H \), we denote by \( fH g \) the element \( fG^{p,q} H \sim gK_0(\mathcal{H}S) \).

The cohomology groups \( H^c(X; \mathbb{Q}) \) with compact supports of a variety \( X \) has a natural mixed Hodge structure.

Definition 2.4. We define the Hodge characteristic \( h(X) \) of \( X \) by

\[
h(X) = (1)^{\dim h} H^c(X; \mathbb{Q}) g 2 K_0(\mathcal{H}S).
\]

Consider the following map:

\( (\text{Varieites}) \to K_0(\mathcal{H}S); X \mapsto h(X) \):

It factors through the map

\( (\text{Varieites}) \to M; X \mapsto fX g; \)
because the following hold:
\[ h(\mathcal{X}, \mathcal{Y}) = h(\mathcal{X}) \circ h(\mathcal{Y}), \]
\[ h(\mathcal{X}) = h(\mathcal{X} \cap \mathcal{Y}) + h(\mathcal{Y}) \text{ if } \mathcal{Y} \text{ is closed}, \]
the Hodge characteristic of the ample line, \( h(\mathcal{A}^1) = fH^2(\mathcal{A}^1; \mathcal{Q}); g \), is invertible.

We also denote by \( \mathcal{H} \) the induced homomorphism \( M ! K_0(\mathcal{H}^S) \).

For \( m \in \mathbb{Z} \), let \( F_m K_0(\mathcal{H}^S) \) be the subgroup generated by the elements \( fH^2 g \) such that the maximum \( m \) weight of \( \mathcal{H} \) is less than or equal to \( m \).

**Definition 2.5.** We denote the complete Grothendieck ring of Hodge structures, denoted \( K_0(\mathcal{H}^S) \), as follows:

\[ K_0(\mathcal{H}^S) := \text{lim } K_0(\mathcal{H}^S); F_m K_0(\mathcal{H}^S); \]

We can see that the natural map \( K_0(\mathcal{H}^S) ! K_0(\mathcal{H}^S) \) is injective. Because the maximal weight of \( H^1(\mathcal{X}; \mathcal{Q}) \) does not exceed \( \dim \mathcal{X} \), \( \mathcal{H} \) extends to \( \mathcal{H} ! K_0(\mathcal{H}^S) \).

2. Jets on schemes. For convenience's sake, we denote by \( (t^1) \) the ideal \( (0) \) of the power series ring \( \mathbb{C}[[t]] \). For \( n \in \mathbb{Z} \), we denote by \( D_n \) the affine scheme \( \text{Spec} \mathbb{C}[[t]]/(t^{n+1}) \).

**Definition 2.6.** Let \( \mathcal{X} \) be a scheme. For \( n \in \mathbb{Z} \), we denote the scheme of \( n \)-jets\(^2\) of \( \mathcal{X} \), denoted \( L_n \mathcal{X} \), to be the scheme representing the functor

\[ (\text{Sch}=\mathbb{C}) ! (\text{Sets}) : \quad \mathcal{U} \mapsto \text{Hom}(\text{Sch}=\mathbb{C}) (\mathcal{U}, D_n \mathcal{X}); \]

Greenberg [Gre61] proved the representability of the functor for \( n < 1 \). Form \( n \in \mathbb{Z} \) with \( n < n \), a canonical closed immersion \( D_n ! D_m \) induces a canonical projection \( L_n \mathcal{X} ! L_m \mathcal{X} \). Since all these projections are a ne morphisms, the projective limit \( \lim L_n \mathcal{X} = \lim L_n \mathcal{X} \) exists in the category of schemes.

If \( \mathcal{X} \) is of finite type, then for \( n < 1 \), so is \( L_n \mathcal{X} \). If \( \mathcal{X} \) is smooth and of pure dimension \( d \), then, for each \( n \geq 0 \), the natural projection \( L_{n+1} \mathcal{X} ! L_n \mathcal{X} \) is a Zariski locally trivial \( \mathbb{A}^d \)-bundle. If \( f : Y! X \) is a morphism of schemes, then for each \( n \), there is a canonical morphism \( f_n : L_n Y! L_n \mathcal{X} \).

2.3. Motivic measure. Let \( \mathcal{X} \) be a scheme of pure dimension \( d \). By abuse of notation, we denote the set of points of \( L_1 \mathcal{X} \) also by \( L_1 \mathcal{X} \). Let \( \pi_n : L_1 \mathcal{X} ! L_n \mathcal{X} \) be the canonical projection.

**Definition 2.7.** A subset \( \mathcal{A} \) of \( L_1 \mathcal{X} \) is stable at level \( n \) if we have:

(1) \( \pi_n(\mathcal{A}) \) is a constructible subset in \( L_n \mathcal{X} \),

\(^2\)In [DL99], an \( 1 \)-jet is called an arc. As it is more convenient, I prefer my terminology.
(2) $A = \binom{1}{n} (A)$,
(3) for any $m \leq n$, the projection $m+1(A) \rightarrow m(A)$ is a piecewise trivial $A^d$-bundle.

(A morphism $f : Y \to X$ of schemes is called a piecewise trivial $A^d$-bundle if there is a stratification $X = \coprod_{i} X_i$ such that $f|_{X_i} : f^{-1}(X_i) \to X_i$ is isomorphic to $X_i \times A^d$ for each $i$.) A subset $A$ of $L_1 X$ is stable if it is stable at level $n$ for some $n \geq 0$.

The stable subsets of $L_1 X$ constitute a Boolean algebra. If $A \subset L_1 X$ is a stable subset, then $f_m(A) \cap L_m 2 M$ is constant for $m \geq 0$. We denote $\mathcal{M}$ by $X(A)$. The map

$$X : \text{stable subsets of } L_1 X \to \mathcal{M}$$

is a finite additive measure. Let us extend this measure to a bigger family of subsets of $L_1 X$.

**Definition 2.8.** A subset $A \subset L_1 X$ is called measurable if, for every $m \geq 2$, there are stable subsets $A_m \subset L_1 X$ and $C_i \subset L_1 X$, $i \geq 2$, such that the symmetric difference $(A \setminus A_m) \cap (A \setminus A_m)$ is contained in $\cup C_i$ and we have $X(C_i) \leq 2 M$ for all $i$, and $\lim_{i \to 1} X(C_i) = 0$ in $\mathcal{M}$.

The measurable subsets of $L_1 X$ also constitute a Boolean algebra. Suppose that $A \subset L_1 X$ is measurable and $A_m \subset L_1 X$, $m \geq 2$ are stable subsets as in the definition above. We put $X(A) = \lim_{m \to 1} X(A_m)$. It is independent of the choice of $A_m$, see [Loo02, Prop. 2.2], [DL02, Th. A.6]. The map

$$X : \text{measurable subsets of } L_1 X \to \mathcal{M}$$

is a finite additive measure.

**Definition 2.9.** We call this the motivic measure on $L_1 X$.

**Definition 2.10.** Let $A \subset L_1 X$ be a measurable subset and $f : A \to Z$ a function. We say that $f$ is a measurable function if the fibers are measurable and $X(\{f=1\}) = 0$. For a measurable function, we formally define the motivic integration of $L$ by

$$L^d \int_A X = \lim_{n \to \infty} \left[ X(\{f=1\})L^n \right]_{n \geq 2}$$

We say that $L^d$ is integrable if the series above converges in $\mathcal{M}$.

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3 This differs from the definition in [CM99], [Bat99b] and [DL99] by a factor $L^d$. 
Example 2.11. Let $I$ be an ideal sheaf on $X$. A point $2 L_1 X$ corresponds to a morphism $\text{Spec } ! L_1 X$ for the residue field of $X$. The function
\[ \text{ord } I : L_1 X \to \mathbb{Z}_0 \{ f \} \]
\[ \text{for } (0) \setminus I = (t^n) \]
is a measurable function by the following lemma.

Lemma 2.12. [Loc02, Prop. 3.1], [DL99, Lem. 4.4]. For a subvariety $Y$ of $X$ of positive codimension, the subset $L_1 Y \subseteq L_1 X$ is of measure zero.

Example 2.13. [Bat99b, Th. 3.6], [Cra99, Th. 2.15]. Let $X$ be a smooth variety of dimension $d$ and $E = \sum_{i=1}^d E_i$ an effective divisor on $X$ with simple normal crossing support. For a subset $J = \{f_1, \ldots, f_r\}$, we define
\[ E_J = \bigcap_{\alpha \in J} E_{f_\alpha} \cap \bigcap_{\beta \in \beta} E_{f_\beta} \cap \bigcap_{\gamma \in \gamma} E_{f_\gamma} \]
If $E_i$ is the ideal sheaf associated to $E$, then we have the following formula:
\[ Z \otimes_{L_1 Y} \mathbb{Z}_0 = X \otimes_{f_\alpha} Y \otimes_{L_{d+1}} \mathbb{Z}_0 \]

2.4. The transformation rule. Let $X$ and $Y$ be varieties of pure dimension $d$ and $f : Y \to X$ be a morphism over $D_1$. We denote the morphisms $f_n : L_n Y \to L_n X$ as follows. For a scheme $U$, a $U$-point of $L_n Y$ is a morphism $0 : D_n U \to D_1$. If $U = P_1$, then we define $f_n(\cdot)$ to be the $U$-point of $L_n X$ corresponding to the composition $D_n U \to Y \to X$. Assume that $Y$ is smooth. We denote the Jacobian ideal sheaf $J_f$ of $f$ as follows. The Jacobian ideal sheaf $J_f$ of $f$ is defined as a subsheaf of $J_f$ and $d f$ is the torsion. The following, called the transformation rule, is the most basic theorem in the theory.

Theorem 2.14. [DL02, Th. 1.16], [Loc02, Th. 3.2]. Let $A$ be a measurable set in $L_1 Y$ and $f : A \to Y \otimes_{L_1 Y} \mathbb{Z}_0$. Suppose that $f_\alpha$ is injective. Then we have the following equality:
\[ Z \otimes_{L_1 Y} \mathbb{Z}_0 = \bigcap_{\alpha} \text{ord } J_f \otimes_{L_1 Y} \mathbb{Z}_0 \]
We will generalize this later (Theorem 3.18).
25. The motivic Gorenstein measure. Let $X$ be a variety with 1-Gorenstein and canonical singularities, that is, the canonical sheaf $\mathcal{I}_X$ is invertible and all discrepancies are 0, (see [K.M.M87, x0-2]). Then there exists a natural morphism $\frac{d}{X} ! X$. The kernel of this morphism is the torsion. We define an ideal sheaf $\mathcal{I}_X$ on $X$ by the equation:

$$\mathcal{I}_X ! X = \text{Im} ( \frac{d}{X} ! X );$$

Then $L^{\text{ord}} \mathcal{I}_X$ is integrable by Example 2.13 and Lemma 2.16.

Definition 2.15. We define the motivic Gorenstein measure $\mathcal{G}_{X}$ on $L_1 X$ as follows:

$$\mathcal{G}_{X} : \text{fm measurable subsets of } L_1 X \ni M$$

$$A \mapsto L^{\text{ord}} \mathcal{I}_X d X :$$

Lemma 2.16. Let $X$ and $X^0$ be complete varieties with 1-Gorenstein canonical singularities.

1. Let $A$ be a measurable subset of $L_1 X$ and $f : Z ! X$ be a resolution. Then

$$L^{\text{ord}} \mathcal{I}_X d X = L^{\text{ord}} K d Z;$$

where $K$ is the ideal sheaf associated to $K_Z X$.

2. Suppose that there exist proper birational morphisms $f : Z ! X$ and $g : Z ! X^0$ with $K_Z X = K_Z X^0$. Then we have $\mathcal{G}_{X} (L_1 X) = \mathcal{G}_{X^0} (L_1 X^0)$.

Proof. (1). By Theorem 2.14, we have

$$L^{\text{ord}} \mathcal{I}_X d X = L^{\text{ord}} K d Z;$$

We have to show

(2.2) $\text{ord} \mathcal{I}_X f \text{ ord} J = \text{ord} K ;$

Pulling back $\mathcal{I}_X ! X = \frac{d}{X} = \text{tors}$, we have $(f^{-1} \mathcal{I}_X) (f ! X) = J_f \frac{d}{Z}$. On the other hand, we have $f ! X = K ! Z = K \frac{d}{Z}$. Hence $(f^{-1} \mathcal{I}_X) K = J_f$. This shows (2.2).

(2) is a direct consequence of (1).

Remark 2.17. The invariant $\mathcal{G}_{X} (L_1 X)$ has been already introduced in [Koen95] by using a resolution of singularities. Denef and Loeser constructed the invariant more directly with the motivic Gorenstein measure [DL02].

Suppose $X$ is complete. The question is whether $\mathcal{G}_{X} (L_1 X)$ is the alternating sum of a kind of cohomology groups, as the case where $X$ is smooth. It is known that when $X$ is a global quotient, the answer is Yes [Bat99b], [DL02]. Our result, Theorem 3.15, says that when $X$ has only quotient singularities, the answer is Yes.
3. Twisted jets

In this section, we need to manage the theory of Deligne-Mumford stacks. See the next section for the generalities about Deligne-Mumford stacks.

3.1. Non-twisted jets on stacks. The following is a direct generalization of the notion of jet on schemes:

Definition 3.1. Let $X$ be a Deligne-Mumford stack. For $n \in \mathbb{Z}_0$ [f1 g], we define the stack of non-twisted $n$-jets of $X$, denoted $L_n X$, as follows. An object of $L_n X$ over $U \times D_n$ is an object of $X$ over $U$. For a morphism $\phi : V \to U$ in $(\text{Sch} = \mathbb{C})$, a morphism in $L_n X$ over $\phi$ is a morphism in $X$ over $\phi$.

Lemma 3.2. For every $n \in \mathbb{Z}_0$ [f1 g], $L_n X$ is a stack.

Proof. It is clear that they satisfy the axioms of category. If $(U_1 \to U)$ is an etale covering in $(\text{Sch} = \mathbb{C})$, then so is $(U_1 \times D_n \to U_1 \times D_n)$.

Let $f : Y \to X$ be a morphism of schemes and $Z \to Y$ a closed subscheme with an ideal sheaf $A$. We say that $f$ is $\mathbb{Z}$-etale if for any ring $A$ and any nilpotent ideal $J \subseteq A$, and for any commutative diagram of solid arrows

$$
\begin{array}{ccc}
\text{Spec} A & \xrightarrow{J} & Y \\
\downarrow & & \downarrow \\
\text{Spec} A & \xrightarrow{X} &
\end{array}
$$

such that $J$ is nilpotent, there is a unique broken arrow which makes the whole diagram commutative. $\mathbb{Z}$-etale is defined for a representable morphism of stacks also in the evident fashion.

Lemma 3.3. (1) Let $M$ be a scheme and $N$ a closed subscheme. We denote by $(L_n M)_N$ the subscheme of $L_n M$ parametrizing the jets with the base point in $N$. Let $p : M \to X$ be an $\mathbb{N}$-etale morphism.

Then, for every $n \in \mathbb{Z}_0$ [f1 g], we have a natural isomorphism:

$$(L_n M)_N = L_n X \times N$$

(2) For every $n \in \mathbb{Z}_0$ [f1 g], $L_n X$ is a Deligne-Mumford stack.

Proof. (1). Let us first show $L_n X \times N$ is an algebraic space. Let $p : L_n X \to X$ be the canonical projection. An object of $L_n X \times N$ is a triple $(x; f; \phi)$ where $x : U \to D_n \times X$, $f : U \to N$, and $(\phi)$ is a morphism in $X$ over $U$. By definition, an automorphism of $(x; f; \phi)$ is an automorphism of $x$ such that $p(x) = \phi$. Hence $p(x)$ must be the identity. We have thus proved that the automorphism of every object of $L_n X \times N$ is trivial and hence that $L_n X \times N$ is an algebraic space (see [LMB00, Cor. 8.1.1]).
The diagram of solid arrows

\[\begin{array}{c}
U \\
\downarrow f \\
N \\
\downarrow p \\
P \\
\downarrow D_n \\
\downarrow X
\end{array}\]

is commutative. We can see that there is a unique broken arrow in the diagram. If \(n < 1\), since \(p\) is \(-\)etale, this is trivial from the definition. If \(n = 1\), since \(D_1\) is the direct limit of \(D_n, 0 \leq n < 1\), this follows from the case \(n < 1\). Sending \((;f;\) to define a morphism \(L_nX \times M \to L_nM\).

The inverse of the morphism is given by \((p_n; M) : (L_nM)_N \to L_nX \times N\) where \(p_n : L_nM \to L_nX\) and \(M : (L_nM)_N \to (L_0M)_N\) are the natural morphisms. We have thus proved (1).

(2). Now suppose that \(p\) is etale and surjective. Consider the following cartesian diagram:

\[\begin{array}{c}
L_nM \\
\downarrow \pi \\
M \\
\downarrow p \\
X
\end{array}\]

Because \(M\) is representable and \(p\) is etale and surjective, \(\pi\) is representable (see [LM B00, Lem. 4.3.3]). This completes the proof (see [LM B00, Prop. 4.5]).

3.2. Twisted jets. For a positive integer \(l\), we put \(1_l = \exp(2\pi i l! = l)\). Let \(1_l = \text{Aut}(1)\) be the group of the \(l\)-th roots of 1. \(1_l\) acts on \(D_n\) by \(1_l : t \mapsto t^l\). We denote by \(D_n^1\) the quotient stack \(D_n/1_l\) with \(m = nl\). The stack \(D_n^1\) has the canonical atlas \(D_n^1 \to D_n\) and the closed point \(\text{Spec} C \to D_n\). We have a morphism \(D_n^1 \to D_n\) such that \(D_n\) is the coarse moduli space of \(D_n^1\) for this morphism, and such that the composition \(D_n^1 \to D_n\)

\[\begin{array}{c}
D_n^1 \\
\downarrow \\
D_n
\end{array}\]

is given by the ring homomorphism \(C[t] = (t^{n+1})! \to C[t] = (t^{n+1}), t \mapsto t^l\).

Definition 3.4. Let \(X\) be a Deligne-Mumford stack. A twisted \(n\)-jet of order 1 on \(X\) is a representable morphism \(D_n^1 \to X\) for an algebraically closed field \(C\).

For a Deligne-Mumford stack \(X\), the inertia stack of \(X\), denoted \(I(X)\), is the stack parametrizing the pairs \((;X;\) such that \(2 \text{Ob} X \to \text{Aut}(;X;)\). For details on inertia stack, see Subsection 4.3. There is a natural forgetting morphism \(I(X) \to X\). For \(12 Z > 0\), let \(I^2(X) = I(X)\) denote the substack parametrizing the pairs \((;X;\) with \(\text{ord}(;X;) = l\).

Let \(D_n^1 \to X\) be a twisted \(n\)-jet of order 1 on \(X\). The canonical morphism

\[\sim : D_m \to D_n^1 \to X\]
is considered to be an -point of \( L^n_m X \) and the canonical morphism

\[
\text{Spec } ! : D^1_m \rightrightarrows D^n_1 \rightrightarrows X
\]
to be an -point of \( X \). Since the automorphism group of the closed point of \( D^n_1 \) is identified with \( \Gamma \), induces an injection \( 1 : \text{Aut}(\Gamma) \). If \( b \in \text{Aut}(\Gamma) \) is the image of \( \Gamma \), then the pair \( (\Gamma; b) \) is regarded as an -point of \( L^1_1(X) \) and the triple \( (\sim; \Gamma; b; \emptyset) \) to be an -point of \( L^1_1(X) \). We define a map by

\[
\text{ftwisted n-jets of order 1 on } X g \colon L^n_m X \times L^1_1(X) \to \Gamma \sim (\Gamma; b; \emptyset):
\]

Lemma 3.5. The subset \( \text{Im } ( ) \subseteq L^n_m X \times L^1_1(X) \) is closed for the Zariski topology.

Proof. Fix an atlas \( p : M \rightrightarrows X \) with \( M \) separated.

We will first characterize the points in \( \text{Im } ( ) \). On account of the arguments on groupoid spaces in Subsection 4.1, we can see that the following are equivalent:

1. to give a commutative diagram

\[
\begin{array}{ccc}
D^1_m & \to & M \\
\downarrow & & \downarrow p \\
D^n_1 & \to & X
\end{array}
\]

such that \( \pi \) is a twisted n-jet of order 1,

2. to give a morphism of groupoid spaces

\[
\begin{array}{ccc}
\text{Spec } ! : (D^1_m) & ! : M \\
\downarrow & \downarrow \text{pr}_1 \\
D^1_m & \to & M
\end{array}
\]

such that the composition

\[
\text{Spec } ! : (D^1_m) ! : M \rightrightarrows M \times M
\]
corresponds to an automorphism of order 1 of the following -point of \( X \):

\[
\text{Spec } ! : D^1_m \rightrightarrows M \rightrightarrows X
\]

3. to give a morphism \( : D^1_m \rightrightarrows M \times M \) such that \( \text{pr}_1 = \text{pr}_2 \) and the composition

\[
\text{Spec } ! : D^1_m \rightrightarrows M \times M
\]
corresponds to an automorphism of order 1 of the following -point of \( X \):

\[
\text{Spec } ! : D^1_m \rightrightarrows M \rightrightarrows M \times M
\]
Any point of \( f_m X \times I^1(X) \) is represented by the triple \( (\, \begin{array}{l} j \cr b \cr \id \end{array}, \, \begin{array}{l} j \cr b \cr \id \end{array} ; b ; \id ) \) such that \( j \) is an -point of \( f_m X \) with an algebraically closed field, and \( j \) is an -point of \( X \) corresponding to the composition

\[
\Spec f_m ! D_m ! X
\]

and \( b \) is an automorphism of \( f_m \). Then the equivalence above implies:

\[
| (\, \begin{array}{l} j \cr b \cr \id \end{array}, \, \begin{array}{l} j \cr b \cr \id \end{array} ; b \in \Im (\, \begin{array}{l} j \cr b \cr \id \end{array}) | \text{ for a lift } : D_m ! M \text{ of } \nu, \text{ where there exists a morphism } : D_m ! M \times M \text{ such that } \nu \text{ and } \nu \text{ correspond to } b.
\]

Let \( j \) be a point of \( f_m X \times I^1(X) \). Suppose that \( j \in \Im (\, \begin{array}{l} j \cr b \cr \id \end{array}) \) and \( j \) is a specialization of \( b \). It suffices to show that \( j \in \Im (\, \begin{array}{l} j \cr b \cr \id \end{array}) \). By [LM B00, Prop. 7.2.1], there is a complete discrete valuation ring \( \mathcal{R} \) with algebraically closed residue field and quotient field \( \mathcal{K} \) such that there is a commutative diagram as follows:

\[
\begin{array}{c}
\Spec \mathcal{K} \\
\downarrow \\
\Spec \mathcal{R} \\
\downarrow \\
\Spec : \\
\end{array}
\]

\[
\begin{array}{c}
\Spec \mathcal{R} \\

\begin{array}{c}
\Spec f_m X \times I^1(X) \\
\downarrow \\
\Spec : \\
\end{array}
\end{array}
\]

(Here by abuse of notation, the arrows and in the diagram are representatives of the points and respectively.) If corresponds to a triple \( (\, \begin{array}{l} j \cr b \cr \id \end{array} ; b ; \id ) \), then, the pullbacks \( (\, \begin{array}{l} j \cr b \cr \id \end{array} ; b \in \mathcal{K} \times \mathcal{K} ; \id ) \) and \( (\, \begin{array}{l} j \cr b \cr \id \end{array} ) \) correspond to and respectively. By extending \( \mathcal{R} \), we can assume that \( \mathcal{R} : \Spec \mathcal{R} ! \mathcal{X} \) lifts to \( \mathcal{R} : \Spec \mathcal{R} ! \mathcal{M} \). Since is etale, \( \mathcal{R} : D_m ! \mathcal{X} \) uniquely lifts to \( \mathcal{R} : D_m ! \mathcal{R} ! \mathcal{M} \) such that the diagram

\[
\begin{array}{c}
\Spec \mathcal{R} \\
\downarrow \\
\Spec : \\
\downarrow \\
D_m \\
\mathcal{R} \\
\mathcal{X}
\end{array}
\]

is commutative. Let \( \overline{\mathcal{K}} \) be the algebraic closure of \( \mathcal{K} \), let \( \overline{\nu} \) be the composition \( D_m \overline{\mathcal{K}} ! D_m \overline{\mathcal{R}} ! \mathcal{X} \) and let \( \overline{\nu} : \Spec \overline{\mathcal{K}} ! \mathcal{M} \times \mathcal{M} \) be the lift of \( \overline{\nu} \) which corresponds to \( \overline{b} \). From \( | \) and the assumption, there is a morphism \( : D_m \overline{\mathcal{K}} ! M \times M \) such that \( \nu \) and \( \nu \) correspond to and respectively, and the composition

\[
\Spec \overline{\mathcal{K}} ! D_m \overline{\mathcal{K}} ! M \times M
\]
equals the com position

$$\text{Spec} K \overset{b_0}{\longrightarrow} \text{Spec} R \quad \text{for } M \times M :$$

We can replace $K$ with a finite extension $K^0$ of $K$. Moreover, replacing $R$ with its normalization in $K^0$, we can assume that $K^0 = K$ and that $b_0$ and induce the same morphism $\text{Spec} K \overset{b_0}{\longrightarrow} \text{Spec} R \quad \text{for } M \times M$.

Consider the unique morphism $\cdot : D_m \rightarrow R \quad M \times M$ such that $p_{1, *} = \sim$. Then the two morphism $p_{1,*}$ and $p_{2,*}$ is the same morphism because of the separatedness of $M$. Then the composition $D_m \overset{1}{\longrightarrow} D_m \overset{R}{\longrightarrow} R \quad M \times M$ satisfies the condition in $\sim$. Hence 2 $\text{Im} (\cdot )$. The proof is now complete.

Definition 3.6. We define the stack of twisted n-jets of order 1 on $X$, denoted $L^1_nX$, to be the reduced closed substack of $L^m_X \times I^m(\mathcal{X})$ with support $\text{Im} (\cdot )$. We denote the stack of twisted n-jets on $X$, denoted $L^0_nX$, to be the disjoint sum $\bigcup \limits_{0} L^1_nX$. In particular, $L^0_nX$ is the inertia stack $I(\mathcal{X})$.

If we set

$$l_0 = \max \{ j \mid \text{ord } j \} \text{ for some } 2 \text{ ob } X \text{ and for some } 2 \text{ Aut}(\cdot )g;$$

then for any $l > l_0, L^1_nX = \cdot$. So the disjoint sum above is indeed a finite sum.

3.3. The formal neighborhood of $I(\mathcal{X})$ and its canonical automorphism. Let $X$ be a smooth Deligne-Mumford stack, $x : \text{Spec} \overset{1}{\longrightarrow} X$ its closed point. Then the tangent space $T_xX$ is defined to be $T_vM$ for an atlas $M \overset{1}{\longrightarrow} X$ and a lift $v : \text{Spec} \overset{1}{\longrightarrow} X$ of $x$, uniquely determined up to unique isomorphism. Then $\text{Aut}(x)$ naturally acts on $T_xX$. We now globalize it.

Let $Y$ be a connected component of the inertia stack $I(\mathcal{X}), F : Y \overset{1}{\longrightarrow} X$ the forgetting map. Let $X_0$ be the image of $Y$ by $F$. The completion $\mathcal{O}^\cdot_X$ of $O_X$ along $X_0$ is considered to be an $O^\cdot_{X_0}$-algebra. Then we define a coherent sheaf $A$ on $Y$ to be the pullback of $O^\cdot_X$ by $F : Y \overset{1}{\longrightarrow} X_0$.

Definition 3.7. We define $N = \text{Spec} A$ and call it the formal neighborhood of $Y$.

If we set $\hat{X} = \text{Spec} \mathcal{O}^\cdot_X$ where we consider $\mathcal{O}^\cdot_X$ to be an $O^\cdot_X$-algebra, there is a natural morphism $\hat{X} \overset{1}{\longrightarrow} X$, which is $X_0$-etale. Since $Y \overset{1}{\longrightarrow} X_0$ is unramified and at $X_0$, it is etale. Hence the natural morphism $N \overset{1}{\longrightarrow} \hat{X}$ is also etale and the composition $N \overset{1}{\longrightarrow} X \overset{1}{\longrightarrow} Y$-etale.
Let $U, V$ be varieties. Let $\phi : U \to X$ be an etale morphism, $\psi : V \to U$ a morphism and an automorphism of $V$. Suppose that $\phi \circ \psi : V \to Y$ is etale.

Then we obtain a commutative diagram

\[
\begin{array}{c}
V \\
\downarrow \phi \\
U \\
\downarrow \psi \\
Y \\
\downarrow \psi \\
X
\end{array}
\]

where $\phi \circ \psi$ denotes a 2-morphism. Let $\sim : V \to U \times U$ be the corresponding morphism.

If $\hat{O}_U$ is the completion of $O_U$ along $(V)$, then $A_V = \hat{O}_U$. We have a canonical automorphism of $A_V$

\[ A_V \xrightarrow{p_2} \sim \hat{O}_U \xrightarrow{(p_1)} V \]

and hence a canonical automorphism of $A$ and $N$. Now this automorphism of $N$ is considered to be a globalization of the action on $T_xX$ mentioned above.

3.4. Shift number and orbifold cohomology. Suppose that $Y$ is contained in $T^1(X)$ for an integer $l \geq 1$. Let $(\kappa, \lambda)$ be a closed point of $Y$ where $\kappa$ is a closed point of $X$ and $2 \text{Aut}(\kappa)$. Then acts on the tangent space $T_xX$. For a suitable basis, this automorphism is given by a diagonal matrix

\[ \text{diag}(a_1^{\lambda};\ldots; a_d^{\lambda}) \]

with $l \leq a_j \leq 1$ and $d = \dim X$.

Definition 3.8. We define the shift number of $Y$ by

\[ s(Y) = \dim X \cdot \sum_{j=1}^l \frac{1}{a_j} \cdot \left( \frac{1}{x_j^{\lambda}} \right) \]

This is determined by the rank of the eigenbundles of $N$ for the canonical action. Hence it depends only on $Y$.

Suppose that the coarse moduli space $X = X$ is a variety with Gorenstein quotient singularities and $X$ has no re-echs. Then the matrix $\text{diag}(a_1^{\lambda};\ldots; a_d^{\lambda})$ is in $\text{SL}_d(\mathbb{C})$ (see [W atz74]). Hence $s(Y)$ is an integer.
Now, let us denote the orbifold cohomology.

Definition 3.9. Assume $X$ is complete. Then we define the $i$-th orbifold cohomology group along with Hodge structure as follows:

$$H^i_{orb}(X; \mathbb{Q}) = H^{2s(Y)}(\overline{Y}; \mathbb{Q}) \otimes (s(Y)),$$

where $Y$ runs over the connected components of $\mathfrak{I}(X)$ and $Q(s(Y))$ is a Tate twist $Q(1)^{s(Y)}$.

Since the natural morphism $\overline{Y} \to X$ is quasi-finite, $\overline{Y}$ is a scheme (see LMBO04 Th. A.2). Because of this and Corollary 4.23, $\overline{Y}$ is a complete variety with quotient singularities. Therefore the rational cohomology groups of $Y$ have pure Hodge structures. For the projective case, see Dan78 Cor. 14.4. For the general case, it follows from the following two facts: one is that the intersection cohomology of every complete variety has pure Hodge structures [Sa90], the other is that the rational cohomology of a variety with quotient singularities equals the intersection cohomology.

3.5. The motivic measure on twisted 1-jets. Let $X$ be a smooth Deligne-Mumford stack of pure dimension $d$. By abuse of notation, we also denote by $L_1 X$ the set of points $\mathcal{L}_1 X_j = \mathcal{J}_{L_1 X}$, we denote by $\mathcal{L}_1 X$ the natural morphism $L_1 X \to \mathcal{L}_1 X$.

Definition 3.10. A subset $A$ of $L_1 X$ is stable at level $n$ if we have:

1. $n(A) = \mathcal{L}_n X$ is a constructible subset in $\mathcal{L}_n X$,
2. $A = \mathcal{L}_n (\mathcal{L}_n X)$,

A subset $A$ of $L_1 X$ is stable if it is stable at level $n$ for some $n \geq 0$.

We denote the notion of the measurable subset similarly. Then we define the motivic measure $\mu X$ on $L_1 X$ by

$$\mu X(A) = L^{nd} \text{Hom}(\mathfrak{d} \mathfrak{g}(\mathfrak{d} (\mathfrak{g}(\mathfrak{d}))) \otimes K_0(H), n = 0;$$

where $L = P_0(1)g = h(A)$. It is well-defined by the following:

Proposition 3.11. Let $n \geq 0$. Let $B$ be a constructible subset and let $C$ be the inverse image of $B$ by the natural morphism $L_{n+1} X \to L_n X$. Then we have the equality $\text{Hom}(C) = L^{nd} \text{Hom}(B)$.

Proof. Let $Y$ be a connected component of $I_1(X) = L_1 X$ and put $(L_n X)_Y = L_n X \cap Y$. Let $\Lambda = (\mathfrak{d} \mathfrak{g}(\mathfrak{d} (\mathfrak{g}(\mathfrak{d}))) \otimes K_0(H), L_n X \times Y$ be an $X$-point. Then we have the following commutative diagram of solid arrows:

$$\begin{array}{ccc}
D_0 & \xrightarrow{(P_b)} & Y \\
\downarrow & & \downarrow \\
D_m & \xrightarrow{(P)} & X
\end{array}$$
Since \( N \) is \( Y \)-etale, there is a unique broken arrow  fitting into the diagram. Sending to determine a closed immersion

\[
(L_n X)_Y \setminus L_m N
\]

Let \( g \) be the canonical automorphism of \( N \). In view of the definition of \( N \) and \( Y \), in the proof of Lemma 3.12, we see that for \( 2 L_m N \), \( 2 \) \( N \) and

\[
\text{since} \quad \text{Lemma 4.26.}
\]

Then we have that

\[
\text{the representation of the jet maps the only point of the finite group action.}
\]

\[
\text{analytically trivialization of an affine space by a linear group action.}
\]

\[
\text{Let } H \text{ be a subgroup of } G \text{ and } V^0 \text{ a connected component of the locus of the points with stabilizer } H. \text{ Let } w \in W \text{ be a close point. As is well known, there is a representation } H \rightarrow G \text{ which describes the } H \text{-action on an analytic neighborhood of } w. \text{ Let } (L_n V)_w, L_n V \text{ be the subset of the jets which maps the only point of } D_n \text{ to } w. \text{ Then the induced } H \text{-action on } (L_n V)_w = \Lambda^m \dim Y \text{ is given by } \Lambda^m. \text{ Therefore } (L_{n+1} V)_w = H ! (L_n V)_w = H \text{ is an analytically locally trivialization of } A^m \dim Y = H. \text{ Let } G^0 \text{ be the subgroup of the elements keeping } W \text{ stable. Then } H \text{ is a normal subgroup of } G^0. \text{ It is easy to see that the image of } (L_n V)_w = H \text{ in } (L_n V)_w = G \text{ is naturally isomorphic to } (L_n V)_w = H = (G^0 \setminus H). \text{ Since } G^0 \text{ freely acts on } (L_n V)_w, \text{ the assertion follows.}
\]

Lemma 3.12. Let \( G \) be a finite group, \( V \) a smooth \( G \)-variety and \( W \) a smooth closed subvariety consisting of \( G \)-invariant points.

1. Assume that \( V \) and \( W \) are a ne, say \( V = \text{Spec} R \) and \( W = \text{Spec} R = p. \) Moreover assume that \( p \) is generated by \( c = \text{codim } (W; V) \) elements. Then the completion of \( V \) along \( W \) is isomorphic to \( \text{Spec}( \mathbb{R} = p)[x_1, \ldots, x_c] \), \( G \rightarrow \text{GL}_e(\mathbb{R} = p). \)

2. Assume that \( G \) is a finite cyclic group. Then there is an a ne open covering \( \{ V_i \} \text{ of } V \) such that for every \( i \), if we write \( V_i = \text{Spec} R \) and \( W \setminus V_i = \text{Spec} R = p, \) the completion of \( V_i \) along \( W \setminus V_i \) is isomorphic as \( G \)-schemes to \( \text{Spec}( \mathbb{R} = p)[x_1, \ldots, x_c] \), \( G \rightarrow \text{GL}_e(\mathbb{C}). \)
Proof. (1). We denote by \( \hat{R} \) the completion of \( R \) with respect to an ideal \( p \) and by \( \hat{R}_p \) the completion of the localized ring \( R_p \) with respect to the maximal ideal. Let \( K \) be the quotient field of \( R = p \) and \( f_1 \) generators of \( p \). It is well known that there is an isomorphism \( \hat{R}_p ! K [k_1; \ldots; x_c] \) sending \( f_i \) to \( x_i \). We have a natural injection \( \hat{R} ! \hat{R}_p = K [k_1; \ldots; x_c] \). Clearly the image contains the subring \( (R = p)[k_1; \ldots; x_c] \). Consider the injection \( (R = p)[k_1; \ldots; x_c] ! \hat{R} \). Since the induced map \( R = p ! \hat{R} = \hat{p} \) is the identity and the images \( f_i \) of \( x_i \) generate \( \hat{p} \), is a surjection and hence an isomorphism (see [Sai 89], we can regard the isomorphisms above of cohomology groups as one of mixed Hodge structures. This implies the assertion.

Remark 3.13. The author guesses that even in the case of a general finite group, the action on \( N_W \) is etale locally realizable in \( C \). From facts on splitting fields of finite groups (see [CR 88]), this is true at least over the generic point of \( W \).

Lemma 3.14. Let \( T \) and \( S \) be varieties and \( f : T ! S \) an analytically locally trivial fibration of \( A^d = G \) for a finite group \( G = GL_d (C) \). Then \( h(T) = h (f) L^d \).

Proof. Since the fiber is a quotient of an affine space, the higher direct images of \( Q_T \) vanishes;

\[
R^i f Q_T = \begin{cases} Q_S & (i = 0) \\ 0 & (i > 0). \end{cases}
\]

Hence the spectral sequence is degenerate and it follows that \( H^i (T; \mathbb{Q}) = H^i (S; \mathbb{Q}) \) for every \( i \).

Taking a strata tion of \( S \), we may assume that \( S \) is smooth. Since \( S \) and \( T \) have at most quotient singularities (in the analytic sense), by Poincaré duality, we conclude that \( H^d_c (T; \mathbb{Q}) = H^d_c (S; \mathbb{Q}) \).

Regarding the sheaves as mixed Hodge modules, studied by Saito [Sai 89] (see also [Sa 89]), we can regard the isomorphisms above of cohomology groups as one of mixed Hodge structures. This implies the assertion.
Lemma 3.17. Since the coarse moduli space \( \mathcal{L} \) is a twisted n-jet on \( X \) of order \( l \), then it induces a morphism \( \mathcal{L}_{l}^{0} \) to \( X \) of the coarse moduli spaces. We denote the map \( \mathcal{L}_{l}^{0} : \mathcal{L}_{l} X ! \mathcal{L}_{l} X \) by \( \mathcal{T}^{l} \).

The following is our main result.

Theorem 3.15. Let \( B = \mathcal{L}_{l} X \) be a measurable subset and put \( A = A_{l}(B) \). Then we have the following equation in \( K_{0}(\mathcal{H}) \):

\[
\begin{align*}
\mathcal{L}_{X}^{0}(A) & = \mathcal{L}_{X}^{0}(1) \mathcal{L}_{X}^{0}(Y) \\
& = \left( 1 \right) \mathcal{H}_{l} \mathcal{L}_{X}^{0} (X ; \mathcal{Q})g
\end{align*}
\]

where \( Y \) runs over the connected components of \( I(X) \).

The proof is postponed until the end of the section.

Corollary 3.16. Let \( X \) and \( X^{0} \) be complete varieties with Gorenstein quotient singularities. Suppose that there are proper birational morphisms \( Z ! X \) and \( Z ! X^{0} \) such that \( K_{Z = X} = K_{Z = X^{0}} \). Then the orbifold cohomology groups of \( X \) and \( X^{0} \) have the same Hodge structure.

Proof. By Theorem 3.15 and Proposition 3.11, we have

\[
\begin{align*}
\mathcal{L}_{X}^{0}(A) & = \mathcal{L}_{X}^{0}(1) \mathcal{L}_{X}^{0}(Y) \\
& = \left( 1 \right) \mathcal{H}_{l} \mathcal{L}_{X}^{0} (X ; \mathcal{Q})g
\end{align*}
\]

From Lemma 2.16, we have

\[
\mathcal{P}(A_{l}) (1) \mathcal{H}_{l} \mathcal{L}_{X}^{0} (X ; \mathcal{Q})g = \mathcal{P}(1) \mathcal{H}_{l} \mathcal{L}_{X}^{0} (X ; \mathcal{Q})g.
\]

Since \( \mathcal{H}_{l} \mathcal{L}_{X}^{0} (X ; \mathcal{Q}) \) and \( \mathcal{H}_{l} \mathcal{L}_{X}^{0} (X ; \mathcal{Q}) \) have a pure Hodge structure of weight \( 1 \), \( \mathcal{H}_{l} \mathcal{L}_{X}^{0} (X ; \mathcal{Q})g = \mathcal{H}_{l} \mathcal{L}_{X}^{0} (X ; \mathcal{Q})g \) for every \( i \).

Lemma 3.17. Let \( X_{\text{sing}} \) denote the singular locus of \( X \) with reduced subscheme structure.

1. The subset \( A_{l}(\mathcal{L}_{1} X)_{\text{sing}} \) of \( \mathcal{L}_{1} X \) is of measure zero.
2. The map \( A_{l}(\mathcal{L}_{1} X)_{\text{sing}} \) is bijective over \( \mathcal{L}_{1} X \times L_{1} (X_{\text{sing}}) \).

Proof. (1). It suffices to show that for every \( n \), \( \mathcal{L}_{1} X n \mathcal{L}_{1} (X_{\text{sing}}) = \mathcal{L}_{1} X \). But this is clear by the local description of \( \mathcal{L}_{1} X \) in the proof of Proposition 3.11.

(2). Surjectivity. Let \( \mathcal{L}_{1} X \times X \) be an \( \mathcal{L}_{1} = \mathcal{L}_{1} X \) with an algebraically closed field. We denote \( D \) to be the normalization of the fiber product \( X \times X \). Then the Deligne-Mumford stack \( D \) contains the scheme \( \mathcal{L}_{1}(\mathcal{L}_{1}(X_{\text{sing}})) \) as open substack. Therefore the coarse moduli space \( D \) of \( D \) contains \( \mathcal{L}_{1}(\mathcal{L}_{1}(X_{\text{sing}})) \) as open subscheme. The scheme \( D \) must be the spectrum of a local ring \( \mathcal{S} \). From the universality of coarse moduli space, there is a natural
morphism $D^! \to \text{Spec} \{t\}$. So we have $\{t\} \to S \to (t)$. Because $\{t\}$ and $(t)$ are the only intermediate rings between $\{t\}$ and $(t)$, the ring $S$ must be $\{t\}$. Suppose that $E = \text{Spec} S$ is an atlas of $D$ and $S$ is a regular local ring. Since $S = \{t\}$ is henselian, the natural morphism $E \to D$ is finite (EGA IV, Th. 18.5.11). Hence $S$ is complete, see EGA IV, Cor. 7.6). So $S = \{t\}$. Consider the groupoid space $E \to D \to E$. The scheme $E \to D \to E$ must be the disjoint sum of spectra of complete regular local rings. Since the first projection $pr_1 : E \to D \to E$ is etale, there is an isomorphism

$$a \quad a \quad E = E \to D \to E$$

such that the composition

$$a \quad a \quad E = E \to D \to E^{pr_1}$$

is isomorphism on each component. If $l$ denote the number of the components in $E \to D \to E$, then the second projection $pr_2 : E \to D \to E$ determines the action of some group $G$ on $E$ with $|G| = l$. Since this action is effective, the group $G$ is isomorphic to $\mathbb{Z}$ for some $l$. For a suitable isomorphism $\mathbb{Z} = G$, the action is given by $\tau \cdot t$. Hence the stack $D$ is isomorphic to $D^!$ and the morphism $D^! \to D = X$ is a twisted $1$-jet on $X$. The image of this twisted $1$-jet by $\tau$ is.

In injectivity, let $\tau_1, \tau_2 : D^! \to X$ be two twisted $1$-jets on $X$ of order $1$. We suppose that $\tau_1(1) = \tau_2(1)$ and $2 \mathfrak{L}_1 \cap \mathfrak{L}_1 (X_{\text{sing}})$. Construct $D$ from $\tau_1$ as above. Then for each $i \geq 1$, there is a unique morphism $h_i : D^! \to D$ such that the following diagram is commutative:

$$\begin{tikzcd}
\text{Spec} \{t\} \ar{r}{h_1} & D^! \ar{d}
\ar{r}[swap]{\tau_1} & \ar{d}[swap]{\tau_2}
\ar{d}{\tau_2} & X
\end{tikzcd}$$

Let $E$ be an atlas of $D$ as above. Then the natural morphism $E \to D \ni h_1$ ($D^!$) is a birational morphism of smooth 1-dimensional schemes. Therefore it is an isomorphism, and so is $h_1$ (LM B00, Prop. 3.8.1). Then we can easily see that $\tau_1$ and $\tau_2$ have the same image in $\mathfrak{L}_1 X_{\text{sing}}$. 

To prove Theorem 3.14, we now need to generalize the transformation rule. Let $V$ be a Deligne-Mumford stack over $D$ of pure relative dimension $d$. For each $n \geq 2$, we denote $V_n$ to be the moduli stack of the $D_n$-morphism $\mathbb{S} D_n ! V$. Then for $n \geq 1$, there is a natural projection $V_m \to V_n$. So we can denote the motivic measure $V$ over $V_1$ which takes values in $K^0(\mathbb{S} X)$, in a similar fashion as before. We should replace condition (3) in
Definition 3.18. Let \( h : A \to \mathbb{Z} \) be injective. Let \( \mathcal{J} \) be a measurable function on \( h(A) \). Then we define \( h^n \mathcal{J} \) to see that it is an \( h^n \mathcal{J} \) is injective, that \( \text{ord} \mathcal{J}_h \) is constant equal to \( e < 1 \) and that \( \text{ord} \mathcal{J}(V=D_1) \) and \( \text{ord} \mathcal{J}(W=D_1) \) are bounded from above on \( h(A) \) and \( A \) respectively. Then for \( n>0 \), \( h^n : h^n A \to h^n A \) is a piecewise trivial \( A^n \)-bundle.

Proof. It is a direct consequence of the following lemma.

Lemma 3.19. Let \( \mathcal{J} \) be a measurable set. Suppose that \( h_j, j \in A \) ! \( V_j \) is injective. Let \( \mathcal{J} \) be a measurable function on \( h(A) \).

Theorem 3.18. Let \( W_1 \) be a measurable set. Suppose that \( h_1, j_1 \in A ! V_1 \) is injective. Let \( \mathcal{J} \) be a measurable function on \( h(A) \).

Proof. Looijenga's proof \cite{Looi02, Lem. 9.2} works also in this setting.

Take a non-twisted 1-jet \( : \text{Spec}([t]) \to W \) in \( A \), and put \( m = (t) \) \([t]\). Let \( q \) be the image of the closed point by \( e \). Take another 2 \( A \) such that \( n e(\cdot) = n e(\cdot) \). Then the morphism

\[ e^n m \to D_1 ! m^n e^{n+1} m^{n+1} \]

is a \( C([t]) \)-derivation. So it defines an \( [t]\)-module homomorphism

\[ @ : W_{-D_1} ! m^n e^{n+1} m^{n+1} \]

The length of the torsion of \( W_{-D_1} \) equals \( \text{ord} \mathcal{J}(W_{-D_1}) \), hence it is bounded. So, since \( n>0 \), the composition map

\[ @ : W_{-D_1} ! m^n e^{n+1} m^{n+1} \]

annihilates the torsion. Conversely, every \( [t]\)-module homomorphism \( W_{-D_1} ! m^n e^{n+1} m^{n+1} \) which annihilates the torsion is \( @ \) for some \( n \).

After some work, we can see that if \( 2A \) is such that \( h_2 n(\cdot) = h_n n(\cdot) \), then \( n e(\cdot) = n e(\cdot) \) (see \cite{Looi02, Lem. 9.2}). So \( @ \) is defined. It is easy to see that \( n(1) = n(2) i @ 1 = @ 1 \) and that \( h_2 n(1) = h_2 n(2) i @ 1 \) and \( @ 2 \) have the same image in \( H_{\text{cm}}(\mathbb{Z}) \).
Hence $h_n^{-1} h_n \ (\ )$ is isom orphic to an a ne space,

$$\text{Hom} [\{h\}] ( W \rightarrow; m^n e + 1 = m^{n+1} )$$

$$= \ker \text{Hom} [\{h\}] ( W \rightarrow; m^n (\text{tors}); m^n e + 1 = m^{n+1} )$$

$$\neq \text{Hom} [\{h\}] ( h; m^n e + 1 = m^{n+1} );$$

The length of $W = V$ equals $e = \text{ord} J_h (\ )$. So $h_n^{-1} h_n \ (\ )$ is isom orphic to an a ne space of dimension $e$.

The rest is easy.

Proof of Theorem 3.15. Let $Y$ be a connected component of $I^1(X)$ and $N$ its formal neighborhood. We may assume that $B$ is contained in $\overline{1^1(Y)}$. Let $N$ be the quotient of $N$ by the canonical automorphism $g$, that is, $\text{Spec} A^g$ and $A$ is the subsheaf of the $g$-invariant sections. Then the natural morphism $N ! X$ factors as

$$N ! N ! X :$$

In the proof of Proposition 3.11, we saw that for each $n; m$ with $m = n!$, there is a closed immersion $\overline{1(Y)X} \rightarrow; \overline{1(N)}$. Let $|l| : D_m ! D_n$ be the morphism associated to the ring homomorphism defined by $t^l t^m$. If $D_m ! X$ is a twisted $n$-jet in $n (B)$, then $(\ )$ fits into the diagram

$$\begin{array}{c}
D_m \xrightarrow{|l|} N \\
\uparrow \\
D_n \longrightarrow \overline{N}:
\end{array}$$

Then $(\ ) = f_1 (\ )$. We define the subset $B' \overline{\text{I}_X N}$ to be the image of $B$ by the map $T$. Then $A = f_1 (\overline{B})$ and $f_1 \overline{B}$ is bijective outside of subsets of measure zero. Let $I_X$ (resp. $I_N$) be the ideal sheaf on $X$ defined by

$$I_X \! X = \text{Im} ( d_X ! X )$$

$$\text{(resp. } I_N \! N = \text{Im} ( d_X ! N ));$$

and define $G_{or}$ and $G_{or}$ to be $\text{I}^{\text{ord}}_X X$ and $\text{I}^{\text{ord}}_N N$, respectively. Since the morphism $f$ has no ramification divisor, by a similar argument as the proof of Lemma 2.13, we see $f \! X = J_f \overline{I}$, where $J_f$ is the jacobian ideal sheaf. So, by Theorem 3.18, we obtain

$$G_{or} (\overline{B}) = G_{or} (A):$$

We have thus reduced the problem to the case of a cyclic quotient; it suffices to show the following lemma.

Lem m a 3.20. Let the notation as above. We have $\text{L}^s (Y) X (B) = h N \overline{G_{or} (B)}$. 


Proof. The proof is essentially by a trick used in [DL02]. We first consider an easy case where \( X \) is a quotient stack \( [\mathbb{A}^n_\mathbb{R} = \mathbb{G}] \) of an affine space over a ring \( R \) whose spectrum is a smooth variety of dimension \( d \) over \( \mathbb{C} \), and \( G = SL_2(\mathbb{C}) \), a finite cyclic group of order \( 1 \) generated by \( g = \text{diag}(a_1; \cdots; a_1) \), \( a_1 < 1 \). Suppose that \( Y \) is the commutant associated to \( g \). Then \( N = \mathbb{A}^c_\mathbb{R} \) (Spec \([x_1; \cdots; x_c]]) \), its canonical automorphism is \( g = \text{diag}(a_1; \cdots; a_c) \), and \( N = \mathbb{A}^c_\mathbb{R} = \mathbb{G} \). Since the natural morphisms \( \mathbb{A}^c_\mathbb{R} \to \mathbb{A}^c_\mathbb{R} \) and \( \mathbb{A}^c_\mathbb{R} \to \mathbb{G} \) are (Spec \( R \))-etale, and since we consider only jets which send the only closed point into Spec \( R \), it makes no matter to replace \( \mathbb{A}^c_\mathbb{R} \), \( \mathbb{A}^c_\mathbb{R} = \mathbb{G} \) with \( \mathbb{A}^c_\mathbb{R} \).

Consider three \( R \)-algebra homomorphisms: (i) \( u : R[[t]][x] \to R[[t]][x] \), \( x_1 \mapsto t^a x_1 \), \( x_c \mapsto x_c \), (ii) : \( R[[t]][x] \to R[[t]][x] \), \( x_1 \mapsto x_1 \), \( t \mapsto t^a \), (iii) \( : R[[t]][x] \to R[[t]][x] \), the composition of and the inclusion \( R[[t]][x] \to R[[t]][x] \). Since \( R[[t]][x] \) is generated by the monomials \( x_1^{a_1+1} \cdots x_c^{a_c} \) with \( a_1 \cdots a_c \equiv 0 \) (mod \( l \)), there is a \( R[[t]] \)-homomorphism \( v : R[[t]][x] \to R[[t]][x] \) with \( u = v \).

Here the horizontal arrows are \( R[[t]] \)-algebra homomorphisms and the vertical ones send \( t \mapsto t^a \). Write the diagram of the associated schemes as follows:

\[
\begin{array}{ccc}
R[[t]][x] & \xrightarrow{u} & R[[t]][x] \\
\downarrow & & \downarrow \\
R[[t]][x] & \xrightarrow{v} & R[[t]][x]
\end{array}
\]

Here \( E_1 \) are copies of \( \mathbb{A}^c_\mathbb{R} \).

Let \( B_0 \) be the image of \( B \) by \( \sim : (L \times X)_\gamma \to L_1 N \). Then for \( 0 \leq B_0 \leq 1 \), \( \sim(x_1) \) is of the form

\[
(3.2) \quad \sim(x_1) = r_0 t^{a_1} + r_1 t^{a_1+1} + r_2 t^{a_1+2} + \cdots
\]

If we put \( \tau = u_1^{-1}(0) \), then we have

\[
(x_1) = r_0 + r_1 t^1 + r_2 t^2 + \cdots
\]

Therefore if we define \( 2 L_1 E_1 \) by

\[
(3.3) \quad (x_1) = r_0 + r_1 t + r_2 t^2 + \cdots
\]
then we have the following commutative diagram,

\[
\begin{array}{ccc}
E_2 [t] & \longrightarrow & D_2 \\
\downarrow & & \downarrow \\
E_1 [t] & \longrightarrow & D_1
\end{array}
\]

Here \([l]\) is the morphism denoted by \(t^1 \cdot t \). Let \(B_1 \to L_1 \{E_1\}\) be the image of \(B_0\) by the map \(0 \to t\). It is easy to see that if \(B\) is stable at level \(n\), then so is \(B_1\), and that \(f_{n} (\mathcal{G})g = f_{n} (\mathcal{B}_0)g = f_{n} (\mathcal{B}_1)g \mathcal{L}^{c}\), where \(c\) are truncation morphisms of \(L_1 X\), \(L_1 N\), and \(L_1 E_1\) respectively. Therefore

\( \text{(3.4)} \)

\[ h_{E_1} (\mathcal{B}_1) = \chi (\mathcal{B}) \mathcal{L}^{c} \]

Put \( = v_1 (\cdot)\). The chain of the correspondences, \(T^0 \to T^1 \to T^2\), defines a map \((L_1 X)_Y \mapsto L_1 N_Y\) which is the same as one in the proof of the theorem (see diagram \( \text{3.4} \)) and compare it with the last two ones.

Shrinking \(\text{Spec} \mathcal{R}\) to an open subset, suppose that the canonical sheaf \(\mathcal{O}_{\text{Spec} \mathcal{R}}\) of \(\text{Spec} \mathcal{R}\) is generated by a section \(e^0\). Consider a d-form \(e = dx_1 \wedge \cdots \wedge dx_d\) on \(N\). This is stable under the \(G\)-action. If \(r\) denotes the natural morphism \(N \to N\), the canonical sheaf \(\mathcal{O}_{\text{Spec} \mathcal{R}}\) of \(N\) is generated by a d-form \(e^0\) with \(r e = e\). Direct computation gives \(e^1 = t \cdot a_1^{-1}(dx_1 \wedge \cdots \wedge dx_d)\). Hence we have the following equations of subsheaves of \(\mathcal{O}_{E_1} [\mathcal{B}] \to \mathcal{D} \).

\[ (t \cdot a_1^{-1})_N J_N \quad \forall \mathcal{N} = \mathcal{D} ; \quad (t \cdot a_1^{-1})_N J_N = (t \cdot a_1^{-1})_N J_N = (t \cdot a_1^{-1})_N J_N = (t \cdot a_1^{-1})_N J_N \quad \forall \mathcal{N} = \mathcal{D} ; \quad (t \cdot a_1^{-1})_N J_N = (t \cdot a_1^{-1})_N J_N = (t \cdot a_1^{-1})_N J_N = (t \cdot a_1^{-1})_N J_N \]

This means that \(\mathcal{O}_{J_N} = \mathcal{O}_{J_N} \quad \forall \mathcal{N} = \mathcal{D} \). From the transformation rule, we obtain that \(h_{\text{Spec} \mathcal{R}} (\mathcal{A}) = \mathcal{L} \cdot a_1^{-1} (\mathcal{E}_1)\), and granting \(\text{3.4}\), that \(h_{\text{Spec} \mathcal{R}} (\mathcal{A}) = \mathcal{L} \cdot a_1^{-1} (\mathcal{E}_1)\). We have proved the assertion in this case.

As for the general case, the proof follows along almost the same lines: We take the fiber product \(N \times V\) for an atlas \(V\) to linearize the canonical automorphism. Define \(B_Y, B_{VY}\), and \(B_Y\) in the evident fashion. By the argument for the preceding case and a comment on argument, after replacing \(B\), we obtain that \(B_{VY}\) and \(B_Y\) are stable at level \(n\) and a morphism \(v_n : B_{VY} \to B_Y\) is a trivial line space bundle of the expected relative dimension. Here we have used Lemmas \text{3.2.3} and \text{3.4.3} instead of the transformation rule itself. The natural morphism \(n \cdot B_Y \to B_Y\) is an affine space bundle of relative dimension \(c\) which is trivial Zariski locally on \(V\). (Recall \text{3.2} and \text{3.3}.) This bundle results from the truncation \(L_n \cdot (\text{Spec} \mathcal{R})\). Let \(L_n (\text{Spec} \mathcal{R})\) and the identity of \((\mathcal{E}_1)_{0 \cdot \mathcal{N}} \). Hence \(n \cdot B_Y \to B_Y\) is also a Zariski locally trivial affine space bundle of the expected relative dimension. By the same argument as the proof of Lemma \text{3.1.1}, we can conclude that
is an analytically locally trivial branization of a quotient of an affine space. The lemma follows from Lemma 3.14.

4. General results on Deligne-Mumford stacks

In this section, we give some general results on Deligne-Mumford stacks which we need in the preceding section. There are good references for stacks today (for example [DM 69], [V is 89], [G om 01] and [LM B 00]).

We fix a base scheme $S$.

4.1. Deligne-Mumford stacks. A stack is a category bered in groupoids over $(\text{Sch} = S)$ such that every Isom functor is a sheaf and every descent datum is effective. A morphism of stacks $X ! Y$ is representable\(^4\) if for any $U \to S$ and any morphism $U ! X$, the fiber product $U \times_X Y$ is represented by a scheme.

Definition 4.1. Let $P$ be a property of morphism $f : Y ! X$ of $S$-schemes, stable under base change and local in the etale topology on $X$ (for example: surjective, proper etc). We say that a representable morphism $f : Y ! X$ of stacks has property $P$ if for every $S$-scheme $U$ and every morphism $U ! X$, the projection $U \times_X Y ! U$ has property $P$.

Definition 4.2. A (separated) Deligne-Mumford stack is a stack $X$ which satisfies the following:

1. the diagonal $\Delta : X ! X \times X$ is representable and finite,
2. there exists a scheme $M$ and a morphism $M ! X$ (necessarily representable after (1)), which is etale and surjective.

A scheme $M$ in (2) is called an atlas of $X$. A Deligne-Mumford stack $X$ is of finite type if there is an atlas of finite type.

Definition 4.3. Let $P$ be a property of morphism $f : Y ! X$ of $S$-schemes, stable under etale base change and local in the etale topology on $X$ (for example: birational, being an open immersion with dense image etc). We say that a representable morphism $f : Y ! X$ of stacks has property $P$ if for every $S$-scheme $U$ and every etale morphism $U ! X$, the projection $U \times_X Y ! U$ has property $P$.

Definition 4.4. Let $Q$ be a property of schemes local in the etale topology (for example: reduced, smooth, normal, locally integral etc). Let $X$ be a Deligne-Mumford stack. We say that $X$ has property $Q$ if an atlas of $X$ has property $Q$.

Definition 4.5. A (not necessarily representable) morphism $f : Y ! X$ of Deligne-Mumford stacks of finite type is proper if there is a $S$-scheme $Z$ and a proper surjective morphism $g : Z ! Y$ such that $f = g$ is (necessarily representable and) proper.

\(^4\)In [LM B 00], this is called schem atique.
A Deligne-Mumford stack $X$ of finite type is complete if it is proper over $S$.

Although our condition appears weaker than one of $[\text{DM} 69, \text{Def. 4.11}]$, they are actually equivalent by Chow’s lemma (see $[\text{DM} 69, \text{Def. 4.12}], [\text{LM B} 00, \text{Th. 16.6}], [\text{LM B} 03, \text{Prop. 2.6}]$).

Example 4.6. Let $Z$ be a $S$-scheme and $G$ a finite group acting on $X$. The quotient stack $[Z=G]$ is defined as follows: an object over $U_2$ (Sch=$S$) is a $G$-torsor $P/U$ with a $G$-equivariant morphism $P/Z$, and a morphism over $U^0$. $U$ is a cartesian diagram

$$
\begin{array}{ccc}
p^0 & \to & p \\
\downarrow & & \downarrow \\
U^0 & \to & U
\end{array}
$$

which is compatible with the $G$-equivariant morphism $sP^0!Z$ and $P!Z$. It is a Deligne-Mumford stack with a canonical atlas $Z=[Z=G]$.

Here we denote points of a Deligne-Mumford stack. For details, see $[\text{LM B} 00, \text{Ch. 5}]$.

Definition 4.7. Let $X$ be a Deligne-Mumford stack. A point of $X$ is a $S$-morphism $\text{Spec} K ! X$ for a field $K$ with a morphism $\text{Spec} K ! S$.

Let $x_i : \text{Spec} K_i ! X$ ($i=1,2$) be points of $X$. We say that $x_1$ and $x_2$ are equivalent if there is a field $K_3$ such that $K_3/K_1; K_2$ and the diagram

$$
\begin{array}{ccc}
\text{Spec} K_3 & \to & \text{Spec} K_2 \\
\downarrow & & \downarrow \\
\text{Spec} K_1 & \to & X
\end{array}
$$

commutes.

Definition 4.8. We denote the set of points of $X$, denoted $X^0$ to be the set of the equivalence classes of points of $X$.

The Zariski topology on $X^0$ is defined as follows: an open subset is $\{x \mid j \in X^0 \}$ for an open substack $U \subset X$. There is a 1-1 correspondence between the closed subsets of $X$ and the reduced closed substacks of $X$.

We now introduce the notion of (etale) groupoid space which is equivalent to Deligne-Mumford stack. Some references are $[\text{Vie} 89, \text{p. 668}], [\text{LM B} 00, (2.4.3), (3.4.3), \text{Prop. 3.8}, (4.3)]$ and $\{\text{com 01}, \text{Subsec. 2.4}]$.

Definition 4.9. An (etale) groupoid space consists of the following data:

1. two $S$-schemes $X_0$ and $X_1$,
2. four morphisms: source and target $q_i : X_1 ! X_0$ ($i=1,2$), origin $!: X_0 ! X_1$, inverse $: X_1 ! X_0$ and composition $m : X_1 q_1 \times_{X_0} X_2 X_1 ! X_1$ which satisfies the following:
Definition 4.10. Given a groupoid space $X_1 \times X_0$, we denote the category formed in groupoids $[X_1 \times X_0]$ as follows: an object over $U$ (Sch=S) is a morphism $U \to X_0$ of $S$-schemes and a morphism of such $U \to V$ such that $q_1 = h = b$. Then $[X_1 \times X_0]$ is a prestack (see [LM B00, 3.1]).

The stack $X = [X_1 \times X_0]$ is a Deligne-Mumford stack with a canonical atlas $X_0 ! X$. We can identity the fiber product $X_0 \times X_0$ with $X_1$. Conversely, given a Deligne-Mumford stack $X$ and an atlas $X_0 ! X$, then the scheme $X_0$ and $X_1 = M \times M$ underlies a natural groupoid space structure with $q_1 = pr_1$ and $q_0 = M \times X$. The associated stack $[M \times M]$ is canonically isomorphic to $X$. In summary, giving a groupoid space $X_1 \times X_0$ is equivalent to giving a Deligne-Mumford stack $X$ and an atlas $X_0!X$.

Let $f : U ! X$ be a morphism, which is considered as an object of $X$. If $f$ lifts to $0 : U \to X_0$, then the automorphism group of $f$ is identified with the set of morphisms $U ! X_1$ with $q_1 = q_0 = 0$.

Definition 4.11. A morphism $f : (Y_1 \times Y_0) ! (X_1 \times X_0)$ of groupoid spaces is a pair of morphisms $f_i : Y_i ! X_i$ $(i = 0, 1)$ which respects the groupoid space structures.

Given a morphism $f : (Y_1 \times Y_0) ! (X_1 \times X_0)$, then we have a natural morphism of prestacks $[f] : [Y_1 \times Y_0] ! [X_1 \times X_0]$ and hence a natural morphism of stacks $[f] : [Y_1 \times Y_0] ! [X_1 \times X_0]$ from [LM B00, Lem. 3.2].

Conversely, consider a commutative diagram

\[
\begin{array}{ccc}
Y_0 & \xrightarrow{f_0} & X_0 \\
\downarrow & & \downarrow \\
Y & \xrightarrow{g} & X
\end{array}
\]

such that $X, Y$ are Deligne-Mumford stacks and the vertical arrows are atlases. If we denote $Y_1 = Y_0 \times Y_0$ and $X_1 = X_0 \times X_0$, and if $f_1 : Y_1 ! X_1$ is the natural morphism, then the pair of $(f_0 ; f_1)$ determines a morphism $f : (Y_1 \times Y_0) ! (X_1 \times X_0)$ of groupoid spaces. Evidently, $[f] = g$.

Example 4.12. Let $X = [Z = G]$ be a quotient stack with $G$ finite. There is a canonical atlas $Z ! X$. Then the groupoid space $Z \times Z$ is isomorphic to the groupoid space

\[
\begin{array}{ccc}
Z & \xrightarrow{G \text{ action}} & Z \\
\downarrow & \downarrow \text{pr}_1 & \\
Z & & Z
\end{array}
\]
whose origin, inverse and composition are induced by the group structure of $G$.

Let $X$ be a locally integral Deligne-Mumford stack, associated to a groupoid space $X_1 \to X_0$. Let $X_1^{\text{nor}}$ be the normalization of $X_1$ respectively. Then the lifts of the structure morphisms of $X_1 \to X_0$ induce a groupoid space $X_1^{\text{nor}} \to X_0^{\text{nor}}$.

Definition 4.13. [V.89, Def. 1.18] We define the normalization $X^{\text{nor}}$ of $X$ to be the stack associated to $X_1^{\text{nor}} \to X_0^{\text{nor}}$.

It is easy to show the uniqueness and the universality of the normalization.

4.2. Quasi-coherent sheaves.

Definition 4.14. A quasi-coherent sheaf $F$ on a Deligne-Mumford stack $X$ consists of the following data:

1. For each etale morphism $U \to X$ with a scheme $U$, a quasi-coherent sheaf $F_U$ on $U$,
2. For each diagram of etale morphisms

\[
\begin{array}{ccc}
V & \rightarrow & U \\
\downarrow & & \downarrow \\
X & \rightarrow & \ \\
\end{array}
\]

with $V, U$ schemes, an isomorphism $F_V \rightarrow F_U$ which satisfies the cocycle condition.

Example 4.15. (1) The structure sheaf $O_X$ on $X$ is defined by $(O_X)_U = O_U$.

(2) The sheaf of differentials $\omega_X$ is defined by $(\omega_X)_U = \omega_{U/S}$ and by the canonical isomorphism.

(3) Let $f : Y \to X$ be a morphism of Deligne-Mumford stacks. We define the sheaf of relative differentials $\omega_{Y/X}$ of $Y$ over $X$ to be the unique sheaf such that for each commutative diagram

\[
\begin{array}{ccc}
V & \rightarrow & Y \\
\downarrow & & \downarrow \\
U & \rightarrow & X \\
\end{array}
\]

$(\omega_{Y/X})_V = \omega_{U/V}$ and we have the following exact sequence

$f_\ast \omega_X \rightarrow \omega_Y \rightarrow \omega_X$.

Definition 4.16. In Definition 4.14, if every $F_U$ is an $O_U$-algebra and every $F_V$ is a homomorphism of $O_V$-algebras, then we say that $F$ is an $O_X$-algebra.

As the case of schemes, to an $O_X$-algebra $F$, we can associate a representable and a morphism $\text{Spec} F \to X$. For details, see [LM B00, (14.2)].
43. Inertia stacks. In this subsection, we study the inertia stack. It is an algebraic-geometric object corresponding to the twisted sector which was introduced by Kawasaki [Kaw78] and used by Chen and Ruan to define the orbifold cohomology [CR00].

Definition 4.17. For a Deligne-Mumford stack $X$, its inertia stack, denoted $I(X)$, is the stack defined as follows: an object over $U \times \text{Sch}$ is a pair $(x, \phi)$ where $2 \text{Ob } X$ and $2 \text{Aut}(\phi)$, and a morphism $(x, \phi) \to (x', \phi')$ is a morphism $x \to x'$ in $X$ such that $\phi' = \phi$.

Hence if $X$ is complete, then so is $I(X)$.

The following lemma may be well-known:

Lemma 4.18. Let $Z$ be a scheme with an action of a finite group $G$. Then we have an isomorphism:

$$I(\mathbb{Z}[G]) = \bigoplus_{g \in \text{Conj}(G)} \mathbb{Z}^g \cong C(g);$$

where $\text{Conj}(G)$ is a set of representatives of the conjugacy classes, $\mathbb{Z}^g$ is the locus of fixed points under the $g$-action and $C(g)$ is the centralizer of $g$.

Proof. Let $U \times \text{Sch} \to U$ be a connected scheme. An object of $I(\mathbb{Z}[G])$ over $U$ is a $G$-torsor $P$ over $U$ with a $G$-equivariant morphism $P \to Z$. Its automorphism is an automorphism of a $G$-torsor $P$ over $U$ compatible with $P \to Z$. For some étale surjective $V \to U$, the fiber product $P_V = P \times_U V$ is isomorphic to $G \times V$ as $G$-torsors over $V$. Here $G \times V$ is a $G$-torsor for the right action of $G$. The pullback $V$ is represented by the left action of some $V$. For $g \in G$, $g$ is determined up to conjugacy and we can assume $g \in 2 \text{Conj}(G)$.

Let $V = P_V \to P$ be the natural morphism. Now let us show that if $a \in C(g)$ and $b \in C(g)$, then we have $(a \circ g)(V) \cap (b \circ g)(V) = \emptyset$. Let $x$ (resp. $y$) be a geometric point of $V$ (resp. $b \circ g(V)$) and assume $e(x) = e(y)$. Then we have

$$e(x) = (g \circ x \circ g^{-1}) = (y)g^{-1} = (y)g^{-1} \circ (y);$$

This is a contradiction. So $P$ decomposes into $C(g)$-torsors as $P = P^0 \times J$ where $P^0 = C(g \times V)$ and $J$ is a finite set. Let $f_V$ denote the com position $f$. In the following diagram,
we have $g \circ f = f_Y \circ v$, since $v$ equals the right action of $g$ on $C (g) \overset{V}{\rightarrow}$ and $f_Y$ is $G$-equivariant. We also have $f_Y = f_Y \circ v$ and hence $g \circ f = f_Y$. It implies that $f_Y (C (g) \overset{V}{\rightarrow})$ is in $Z^0$ and hence so is $f \circ \phi$. Thus the $C (g)$-torsor $P^0 \overset{V}{\rightarrow} W$ with $f : P^0 \overset{V}{\rightarrow} Z^0$ is an object of $[Z^0 = C (g)]$. For a non-connected $U \overset{2}{\rightarrow} (S \overset{=}{\rightarrow} S)$ and an object of $I (Z = G)$ over $U$, we can assign it an object of $g_2 \overset{Conj (G)}{\rightarrow} [Z^0 = C (g)]$, in the obvious way. We leave the rest for the reader.

Definition 4.19. A morphism $f : Y \overset{X}{\leftarrow} X$ of stacks is barely faithful if for every object of $Y$, the map $\text{Aut}(f) ! Aut(f(\cdot))$ is bijective.

Clearly, all barely faithful morphisms are faithful functors. From [LM B00, Prop. 23 and Cor. 8.1.2], all barely faithful morphisms of Deligne-Mumford stacks are representable in the sense of [LM B00, Def. 3.9]. Because all separated and quasi-affine morphisms of algebraic spaces are schematic [LM B00, Th. A 2], all barely faithful quasi-affine morphisms of Deligne-Mumford stacks are representable for our definition.

Example 4.20. All immersions are barely faithful. All morphisms of schemes are barely faithful.

Lemma 4.21. Barely faithful morphisms are stable under base change.

Proof. Let $f : Y \overset{X}{\leftarrow} X$ be a barely representable morphism of stacks and $a : X^0 \overset{X}{\leftarrow} X$ any morphism of stacks. An object of the fiber product $Y \times_X X^0$ is a triple $(\cdot; \cdot; \cdot)$, where $Y$ is an object of $Y$, $X$ is an object of $X$ and $(\cdot) : f (\cdot)$ is an object of $X_Y$ for some $U \overset{2}{\rightarrow} (S \overset{=}{\rightarrow} S)$. Its automorphism is a pair of automorphisms $\overset{2}{\rightarrow} \text{Aut}(\cdot)$ and $\overset{2}{\rightarrow} \text{Aut}(\cdot)$ with $f'(\cdot) = a(\cdot)$.

Since the map $f : \text{Aut}(\cdot) ! Aut(f(\cdot))$ is bijective, for each , there is one and only one with $f'(\cdot) = a(\cdot)$. We have thus proved the lemma.

Proposition 4.22. Let $f : Y \overset{X}{\leftarrow} X$ be a barely faithful morphism of Deligne-Mumford stacks. Then the inertia stack $I (Y)$ is naturally isomorphic to the fiber product $Y \times_{I (X)} I (X)$.

Proof. The natural morphism $I (Y) \overset{\cdot}{\rightarrow} Y \times_{I (X)} I (X)$ is defined as follows: for an object $Y$ and its automorphism $\cdot$, the pair $(\cdot; \cdot; \cdot)$, which is an object of $I (Y)$, is mapped to the triple $(\cdot; f(\cdot); f(\cdot); f(\cdot)); x_1)$. $x_1$.

We will first show a fully faithful functor. Let be an object of $Y$. The automorphism group of $(\cdot; f(\cdot); f(\cdot); x_1)$ is a pair of automorphisms $\overset{2}{\rightarrow} \text{Aut}(\cdot)$ and $\overset{2}{\rightarrow} \text{Aut}(f(\cdot); f(\cdot)) = C (f(\cdot))$ such that $f(\cdot) = \cdot$. Hence is barely faithful. Let be another object of $Y$ and an automorphism of $\cdot$. It suffices to show that if $\text{Hom}(I (Y) ; (\cdot); (\cdot)) = \cdot$, then $\text{Hom}(Y \times_{I (X)} (\cdot); (\cdot)) = \cdot$. Suppose that there is an element $(\cdot)$ of $\text{Hom}(Y \times_{I (X)} (\cdot); (\cdot))$, where is a morphism and "
is a morphism \( f(\cdot) \) such that the diagram

\[
\begin{array}{ccc}
  f(\cdot) & \xrightarrow{f} & f(\cdot) \\
  \downarrow & & \downarrow \\
  f(\cdot) & \xrightarrow{f} & f(\cdot)
\end{array}
\]

is commutative and \( f(\cdot) = \cdot \). Since \( f \) is barely faithful, the diagram

\[
\begin{array}{ccc}
  & & \\
  & & \\
  & & \\
\end{array}
\]

is commutative, that is, \( \text{Hom}_X((\cdot); (\cdot); (\cdot)) \in \cdot \).

Now, let us show \( \tau \) is an isomorphism, that is, an equivalence of categories. Let \((\cdot); (\cdot); (\cdot)\) be an object of \(Y \times I(X)\) where \(\tau\) is an isomorphism \(f(\cdot)\). Then there is a natural bijection \(\text{Aut}(\cdot) \times \text{Aut}(\cdot) \to \text{Aut}(\cdot)\). Let \(2 \times \text{Aut}(\cdot)\) be an automorphism corresponding to \(2 \times \text{Aut}(\cdot)\). Then we can see that \((\cdot; \cdot; \cdot)\) is isomorphic to \((\cdot; \cdot; \cdot; \cdot; \cdot)\). We have thus completed the proof.

**Corollary 4.23.** If \(S = \text{Spec} C\) and \(X\) is a smooth Deligne-Mumford stack, then \(\text{I}(X)\) is also smooth.

**Proof.** From Lemma 4.26 and Lemma 4.21, there is an etale, surjective and barely faithful morphism \(\tau_M[i_1\ldots i_n]\) such that each \(M_i\) is smooth and each \(G_i\) is a finite group. Then the assertion follows from Lemma 4.18 and Proposition 4.22.

### 4.4. Coarse moduli space.

**Definition 4.24.** Let \(X\) be a Deligne-Mumford stack. The coarse moduli space of \(X\) is an algebraic space \(X\) with a morphism \(X \to X\) such that:

1. for any algebraically closed field \(K\) with a morphism \(\text{Spec } K \to X\), \(X(\cdot) \to X(\cdot)\) is a bijection,
2. for any algebraic space \(Y\), any morphism \(X \to Y\) uniquely factors as \(X \to X \to Y\).

Keel and Mori proved that the coarse moduli space always exists \([KM 97, Cor. 1.3]\).

**Example 4.25.** Let \(Z\) be an algebraic space and \(G\) a finite group acting on \(Z\). Then the coarse moduli space of the quotient stack \([Z\to G]\) is the quotient algebraic space \(Z/G\).

The following lemma is well-known.

**Lemma 4.26.** e.g. \([AV 02, Lem. 2.2.3]\) Let \(X\) be a Deligne-Mumford stack and \(X\) its coarse moduli space. Then there is an etale covering \((X_1 \to X)_1\)
such that $X$ is isomorphic to a quotient stack $Z_i = G_i$ with a scheme $Z_i$ and a finite group $G_i$. Hence the canonical morphism $X \to X$ is proper.

Now we assume $S = \text{Spec} C$.

**Definition 4.27.** Let $X$ be a variety. We say that $X$ has quotient singularities if there is an etale covering $(U_i \to X)_i$ with a smooth variety $U_i$ and a finite group $G_i$.

Lemma 4.28 shows that for a variety $X$, $X$ has quotient singularities if it is the coarse moduli space of some smooth Deligne-Mumford stack. In fact, only if also holds (Lemma 4.29).

Let $X$ be a smooth Deligne-Mumford stack and $x : \text{Spec} C \to X$ a closed point. Then $\text{Aut}(x)$ acts on the tangent space $T_x X$.

**Definition 4.28.** We say that $\text{Aut}(x)$ is a resection of the subspace of the -fold points $(T_x X)$ of codimension 1.

Lemma 4.29. Let $X$ be a $k$-variety with quotient singularities. Then there is a smooth Deligne-Mumford stack $X$ without resections such that the automorphism group of general geometric points is trivial and $X$ is the coarse moduli space of $X$.

**Proof.** We will give only a sketch. There is a finite set of pairs $(V_i ! X, G_i)_i$ such that:

1. $V_i$ is a smooth variety,
2. $G_i$ is a finite group acting effectively on $V_i$ without resections,
3. $V_i ! X$ is a morphism etale in codimension 1 which factors as $V_i ! X \to G_i ! X$ etale.

Let $V_{ij}$ be the normalization of $V_i \times V_j$. Then the natural morphisms $V_{ij} ! V_i$ and $V_{ij} ! V_j$ are etale in codimension 1. From the purity of branch locus, they are actually etale, and hence $V_{ij}$ is smooth. The diagonal $\Delta : V_i ! V_i \times V_j$ factors through $0 : V_i ! V_{ij}$. Then, with the suitable multiplication morphism, the diagram

$$
\begin{array}{c}
V_{ij} \to \to \to V_i
\end{array}
$$

has the structure of groupoid space. We set $X$ to be the associated stack. Clearly $X$ has no resections. The canonical morphism $X \to X$ makes $X$ the coarse moduli space of $X$ (see [G184, Prop. 9.2]). Any geometric point $x$ of $X$ has a lift $x : \text{Spec} C \to V_i$ with algebraically closed $C$. The automorphism group of $x$ is identified with $\text{Aut}(x) : \text{Spec} C \to V_{ij}$. If $y = x^g$. If $y$ is over the smooth locus of $X$, then this group is trivial.

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