Speed limits of the trace distance in the interaction picture for open quantum system

Satoshi Nakajima and Yasuhiro Utsumi
Department of Physics Engineering, Faculty of Engineering, Mie University, Tsu, Mie 514-8507, Japan
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We investigate the speed limit of the state transformation in open quantum systems described by the Lindblad type quantum master equation. We obtain universal bounds of the total entropy production described by the trace distance between the initial and final states in the interaction picture. Beyond a previous study by Vu and Saito [Phys. Rev. Lett. 128, 010602 (2022)] applicable only to the completely mixed initial state, our results are applicable to an arbitrary initial state. Our bounds can be tighter than the bound of Vu and Hasegawa [Phys. Rev. Lett. 126, 010601 (2021)] which measures the distance by the modified Wasserstein distance \( d_F \). For a system of which Hilbert space is two-dimensional, the trace distance is greater than or equal to \( d_F \). For this reason, our results can significantly improve Vu-Hasegawa’s bound.

INTRODUCTION

In recent years, studies of time-dependent open systems have been active [1]. These studies relate to quantum pumps [2, 3], excess entropy production [4, 5], the efficiency and power of heat engines [6–10], shortcuts to adiabaticity [11–13], and speed limits [14–18]. Obtaining a fundamental bound on the speed of state transformation is an important issue relevant to broad research fields including quantum control theory [19] and foundations of nonequilibrium statistical mechanics [20]. Speed limits for time-dependent closed quantum systems have been studied more than half century [1]. Since 1945, the Mandelstam-Tamm relation [21] \( \mathcal{L} / \int_0^\tau dt \Delta E \leq 1 \) has been known (In this letter, we set \( h = 1 \)). Here, \( \mathcal{L} \) is the distance between the initial and final states, and \( \Delta E \) is the energy fluctuation. Recently, even in classical systems, it turns out that there exist speed limits expressed in terms of the distance between states [14]. Shiraishi et al. [14] demonstrated that

\[
\sigma \geq \frac{L^2}{\int_0^\tau dt 2A_c(t)}
\] (1)

for a system described by a classical master equation \( \frac{d}{dt}p_n(t) = \sum_{m} W_{nm}p_m(t) \) with the local detailed balance condition [22]. \( W_{nm} \) is the transition matrix and \( p_n(t) \) is the probability of state \( n \) at time \( t \). \( \sigma \) is the total entropy production, \( L := \sum_n |p_n(\tau) - p_n(0)| \) is the \( L^1 \) norm, and \( A_c(t) := \sum_{n \neq m} W_{nm}p_m(t) \) is the activity.

Recently, the speed limits for the open quantum systems described by the Lindblad type quantum master equation,

\[
\frac{d}{dt}\rho(t) = -i[H(t), \rho(t)] + \mathcal{D}(\rho(t)),
\] (2)

have been researched actively. Here, \( \rho(t) \) is the density operator (the state) of the system, \( H(t) := H_S(t) + H_L(t) \), \( H_S \) is the system Hamiltonian, \( H_L \) is the Lamb shift Hamiltonian. \( \mathcal{D}(\rho) \) represents dissipation and is given by \( \mathcal{D}(\rho) = \sum_k \gamma_k D[L_k](\rho) \) with \( D[X](Y) := (XYX^\dagger - \frac{1}{2}X^\dagger XY - \frac{1}{2}YX^\dagger X) \). In this letter, \( X \) and \( Y \) denote linear operators of the system. \( \gamma_k \) are non-negative real numbers which describe the strength of the dissipation. For the Lindblad type quantum master equation, \( [H_L(t), H_S(t)] = 0 \). The label \( k \) is a tuple \( (b, a, \omega) \) where \( b \) is the label of the bath. The jump operators \( L_{b,a,\omega} \) satisfy

\[
[L_{b,a,\omega}, H_S] = \omega L_{b,a,\omega}, \quad L_{b,a,-\omega} = L_{b,a,\omega}^\dagger.
\] (3)

We assume the local detailed balance condition

\[
\gamma_{b,a,-\omega} = e^{-\beta_b \omega} \gamma_{b,a,\omega},
\] (4)

where \( \beta_b \) is the inverse temperature of the bath \( b \). We notice that \( L_k, \omega, \gamma_k \) and \( \beta_b \) can depend on time. The total entropy production rate is given by

\[
\dot{\sigma} := -\text{Tr}[\frac{d\rho}{dt} \ln \rho] - \sum_b \beta_b \text{Tr}[\mathcal{D}_b(\rho)H_S].
\] (5)

Here, \( \mathcal{D}_b(\rho) \) denotes the contribution from the bath \( b \) of \( \mathcal{D}(\rho) \). The total entropy production is given by \( \sigma := \int_0^\tau dt \dot{\sigma} \).

For the system described by (2), there are two approaches to speed limits. The first approach is Funo et al.’s approach [12], which treats the first and second terms of the right-hand side of (2) equally. Funo et al. [15] demonstrated that

\[
\|\rho(\tau) - \rho(0)\|_1 \leq c_1, \quad c_3 \leq \sqrt{2\sigma \int_0^\tau dt \mathcal{A}(t)}.
\] (6)

Here, \( \|\rho(\tau) - \rho(0)\|_1 \) is the trace distance and \( \|X\|_1 := \text{Tr}\sqrt{X^\dagger X} \) is the trace norm. \( c_1 \) corresponds to the contribution from the first term of the right-hand side of (2), \( c_2 \) and \( c_3 \) correspond to the contribution from the second term of the right-hand side of (2) [23]. \( \mathcal{A}(t) \) is defined by

\[
\mathcal{A}(t) := \sum_{n \neq m} W_{nm}p_n(t)
\] (7)
with $W_{mn} := \sum_k \gamma_k |\langle n(t)|L_k|n(t)\rangle|^2$. Here, we used the spectral decomposition of $\rho(t)$:

$$
\rho(t) = \sum_n p_n(t)|n(t)\rangle\langle n(t)|.
$$

(8)

If the quantum master equation reduces to the classical master equation \cite{24, 10} reduces to \cite{11} because $c_1 = c_2 = 0$. For no dissipation limit $\gamma_k = 0$, \cite{6} becomes a Mandelstam-Tamm type \cite{21} relation because of $c_2 = c_3 = 0$.

The second approach is Vu’s approach \cite{17, 18}, which focuses on the second term of the right-hand side of \cite{24}. Vu and Hasegawa \cite{17} demonstrated that

$$
\sigma \geq \sigma_{\text{VH}} := \frac{d_T(\rho(\tau), \rho(0))^2}{\int_0^\tau dt \ 2B(t)}.
$$

(9)

Here,

$$
B(t) := \text{Tr}[\rho(t) \sum_k \gamma_k L_k^\dagger L_k]
$$

$$
= A(t) + \sum_n \sum_k p_n(t) \gamma_k |\langle n(t)|L_k|n(t)\rangle|^2
$$

(10)

coincides with the activity \cite{22}. The modified Wasserstein distance $d_T$ is defined by $d_T(\rho(\tau), \rho(0)) := \sum_n |b_n - a_n|$ where $\{a_n\}$ and $\{b_n\}$ are increasing eigenvalues of $\rho(0)$ and $\rho(\tau)$. For no dissipation limit, \cite{9} is consistent because $d_T(\rho(\tau), \rho(0)) = 0$ holds with $B(t) = 0$ and $\sigma = 0$.

For a system of which Hilbert space is $d$-dimensional, Vu and Saito \cite{18} demonstrated that

$$
\sigma \geq \frac{||\rho(\tau) - \rho_0||_1^2}{\int_0^\tau dt \ 2B(t)}.
$$

(11)

under the initial condition that the initial state is completely mixed as $\rho(0) = \rho_0 := 1/d$. For no dissipation limit, \cite{14} is also consistent because of $||\rho(\tau) - \rho_0||_1 = 0$.

We consider \cite{9} and \cite{11} possess the following shortcomings, which we would like to improve in the present letter. (i) $d_T$ can be zero between different states: When there is an unitary operator $U$, such that $\rho(\tau) = U\rho(0)U^\dagger$, $d_T$ becomes zero and thus cannot distinguish between the two states. (ii) Even in the classical master equation limit \cite{24, 10} does not lead to \cite{11}: $d_T(\rho(\tau), \rho(0))$ does not become $L$ \cite{23} and $B(t) > A_c(t)$ in general. (iii) In \cite{14}, we cannot replace $\rho_0$ by an any initial state $\rho(0)$. In fact, in the weak dissipation limit $\gamma_k \to 0$, although $\sigma$ and $B(t)$ vanish, $||\rho(\tau) - \rho_0||_1$ remains.



**MAIN RESULTS**

The main results of this letter are

$$
\sigma \geq \sigma_0 \geq \sigma_1 \geq \sigma_2,
$$

(12)

$$
\sigma_0 := \frac{||\tilde{\rho}(\tau) - \rho(0)||_1^2}{\int_0^\tau dt \ 2A_c(\tau)},
$$

(13)

$$
\sigma_1 := \frac{||\tilde{\rho}(\tau) - \rho(0)||_1^2}{\int_0^\tau dt \ [B(t) + B'(t)]},
$$

(14)

$$
\sigma_2 := \frac{||\tilde{\rho}(\tau) - \rho(0)||_1^2}{\int_0^\tau dt \ [B(t) + B_{\infty}(t)]},
$$

(15)

where $\tilde{\rho}(t) := U^\dagger(t)\rho(t)U(t)$ denotes the interaction picture. Here, $U(t)$ is defined by $\frac{d}{dt}U(t) = -iH(t)U(t)$ and $U(0) = 1$. $A_c(t)$ is given by

$$
A_c(t) := \text{Tr} \left( \tilde{\rho}(t) \frac{1}{4} \sum_k \gamma_k |\varphi, \tilde{L}_k\rangle |\varphi, \tilde{L}_k\rangle \right).
$$

(16)

$\varphi(t)$ is defined by

$$
\varphi(t) := \Phi(\tilde{\rho}(t) - \rho(0)).
$$

(17)

Here, $\Phi$ maps a self-adjoint operator $X$ to a self-adjoint operator as $\Phi(X) := \sum_n \text{sign}(x_n) |n\rangle \langle n|$, where the spectral decomposition of $X$ is $X = \sum_n x_n |n\rangle \langle n|$. $\text{sign}(x)$ is the sign of $x$. $B'(t)$ and $B_{\infty}(t)$ are defined by

$$
B'(t) := \text{Tr}(\varphi \tilde{\rho} \bar{\varphi} \sum_k \gamma_k |\varphi, \tilde{L}_k\rangle |\varphi, \tilde{L}_k\rangle),
$$

(18)

$$
B_{\infty}(t) := \sum_k \gamma_k ||\tilde{L}_k||^2 \geq B'(t),
$$

(19)

where $||\tilde{L}_k||^2_{\infty}$ equals to the maximum eigenvalues of $\tilde{L}_k^\dagger \tilde{L}_k$. $||Y||_{\infty}$ is called the spectral norm.

The second inequality of \cite{12} leads to \cite{11} for $\rho(0) = \rho_0$ because $B'(t) = B(t)$ and $||\tilde{\rho}(\tau) - \rho_0||_1 = ||\rho(\tau) - \rho_0||_1$ hold. In the classical master equation limit, the first inequality of \cite{12} leads to \cite{11}: In this limit, $||\tilde{\rho}(\tau) - \rho(0)||_1 = 0$ and $A_c(t) = A_c$ hold (Appendix D). Even for no dissipation limit, \cite{12} is consistent because $||\tilde{\rho}(\tau) - \rho(0)||_1 = 0$ holds with $\sigma = 0$ and $A_c(t) = B(t) = B'(t) = B_{\infty}(t) = 0$.

**DISTANCES IN TWO-DIMENSIONAL SYSTEM**

We discuss a system of which Hilbert space is two-dimensional. In this case, the state of the system can be written as $\tilde{\rho}(t) = \frac{1}{2}(1 + r(t) \cdot \tau)$. Here, $\tau = (\tau_x, \tau_y, \tau_z)$, $\tau_i$ is the Pauli matrix, and $r(t)$ is the Bloch vector. Using the Bloch vector, the trace distance and $d_T$ are expressed as

$$
||\tilde{\rho}(\tau) - \rho(0)||_1 = |r(\tau) - r(0)|,
$$

(20)

$$
d_T(\tilde{\rho}(\tau), \rho(0)) = ||r(\tau) - |r(0)||,
$$

(21)

with $|x| := \sqrt{x \cdot x}$. Then, we obtain

$$
d_T(\tilde{\rho}(\tau), \rho(0)) \leq ||\tilde{\rho}(\tau) - \rho(0)||_1.
$$

(22)
We notice that $d_T(\hat{\rho}(\tau), \hat{\rho}(0)) = d_T(\rho(\tau), \rho(0))$. Thus, our bounds $\sigma_k (k = 0, 1, 2)$ can be better than $[27]$. A specific instance is shown below.

**APPLICATION TO A QUANTUM DOT**

We apply our inequality [12, 20] to a spinless quantum dot coupled to a single lead [2, 6]. The quantum master equation is given by

$$\frac{d\rho}{dt} = -i[H_S, \rho] + \gamma(1 - f(\varepsilon))\hat{D}[a](\rho) + \gamma f(\varepsilon)\hat{D}[a^\dagger](\rho)$$

(23)

with $H_S = \varepsilon a^\dagger a$. Here, $a$ is the annihilation operator of the electron of the system, $\varepsilon$ is the energy level of the system, $f(\varepsilon) = \frac{1}{e^{\beta\varepsilon} + 1}$ is the Fermi distribution, $\beta$ is the inverse temperature of the lead, and $\gamma$ is the coupling strength. The equation of the motion of the Bloch vector $\mathbf{r} = (x, y, z)$ is given by

$$\frac{d}{dt}x = -\frac{1}{2} \gamma x, \quad \frac{d}{dt}y = -\frac{1}{2} \gamma y,$$

$$\frac{d}{dt}z = -\gamma(z - [1 - 2f(\varepsilon)]).$$

(24)

We calculate the trace distance and $d_T$ by [20] and [21]. The activity $B(t)$ and its upper limit $B_\infty(t)$ are given by $B(t) = \gamma(1 + [2f(\varepsilon) - 1]z(t))/2$ and $B_\infty(t) = \gamma$. $B'(t)$ is calculated as $B(t)$ by replacing $z(t)$ with $z'(t)|z'(t)\rangle\langle z'(t)|$. We can demonstrate

$$r' = -r + \frac{2\langle r - r(0)\rangle}{|r - r(0)|^2}(r - r(0))$$

(25)

using $\varphi(t) = \frac{1}{|r - r(0)|}\{r - r(0)\} \cdot \tau$.

Figure [b] shows that the direction of the Bloch vector changes from the $y$-direction to the $z$-direction. In this process, the norm of the Bloch vector doesn’t change much. Then, $d_T(\hat{\rho}(t), \hat{\rho}(0))$ is small. At $\gamma t = 0.547 \cdots$, although $\rho(t) \neq \hat{\rho}(0)$, the distance $d_T(\hat{\rho}(t), \hat{\rho}(0))$ becomes zero. On the other hand, $||\hat{\rho}(t) - \hat{\rho}(0)||_1^2$ is much larger than $d_T(\hat{\rho}(t), \hat{\rho}(0))^2$. Figure [b] shows our bounds $\sigma_1$ and $\sigma_2$ are far superior to Vu-Hasegawa’s bound $\sigma_{VH}$. The Vu-Saito relation [11] is not applicable in this case.

**DERIVATION OF THE MAIN RESULT**

We derive our main results. Our key idea is the use of the trace distance in the interaction picture within Vu’s framework [17, 18]. In this framework, the semi-inner product is defined by

$$\langle X, Y \rangle_{\hat{\rho}}^{(b)} := \text{Tr}[X^\dagger \hat{O}_{\hat{\rho}}^{(b)}(Y)],$$

$$\hat{O}_{\hat{\rho}}^{(b)}(Y) := \frac{1}{2} \sum_{a, \omega} \frac{1}{\gamma a, a, \omega} \hat{L}_{b, a, \omega}[\hat{\rho}] \beta_{\omega}[\hat{L}_{b, a, \omega}^\dagger, Y])$$

(27)

with $\{\rho\}_{c}(X) := e^{-c/2} \int_0^1 ds e^{sc} \rho^s X \rho^{1-s} [27]$. The semi-inner product satisfies $\langle \langle X, Y \rangle \rangle_{\hat{\rho}}^{(b)} = \langle \langle Y, X \rangle \rangle_{\hat{\rho}}^{(b)}$ and $\langle \langle X, X \rangle \rangle_{\hat{\rho}}^{(b)} \geq 0$.

Using the semi-inner product, we can obtain

$$\frac{d}{dt}||\hat{\rho}(t) - \hat{\rho}(0)||_1 = \text{Tr}[\varphi(t) \frac{d\hat{\rho}}{dt}]$$

$$= \sum_b \langle \langle \varphi(t), -\ln \hat{\rho} - \beta_b \hat{H}_{S} \rangle \rangle_{\hat{\rho}}^{(b)}.$$  

(28)

We used $\frac{d}{dt}\text{Tr}[f(X(t))] = \text{Tr}[f'(X(t)) \frac{dX(t)}{dt}]$ for a self-adjoint operator $X(t)$ and a differentiable function $f(x)$ in the first line. In the second line, we used the quantum master equation in the interaction picture (Appendix [B]).
leads to

\[ ||\hat{\rho}(\tau) - \hat{\rho}(0)||_1 = \sum_b \int_0^\tau dt \langle \varphi(t), -\ln \hat{\rho} - \beta_b \hat{H}_S \rangle^{(b)}_{\hat{\rho}} \]

\[ \leq \sum_b \left[ \int_0^\tau dt \langle \varphi(t), \varphi(t) \rangle^{(b)}_{\hat{\rho}} \cdot \int_0^\tau dt \dot{\sigma}^{(b)} \right]^{\frac{1}{2}} \]

\[ \leq \left[ \int_0^\tau dt \sum_b \langle \varphi(t), \varphi(t) \rangle^{(b)}_{\hat{\rho}} \right]^{\frac{1}{2}} \sqrt{\sigma}. \tag{30} \]

Here, we used the Cauchy-Schwarz inequality

\[ \left| \int_0^\tau dt \langle \langle X(t), Y(t) \rangle^{(b)}_{\hat{\rho}} \right| \leq \sqrt{\int_0^\tau dt \langle \langle X(t), X(t) \rangle^{(b)}_{\hat{\rho}} \int_0^\tau dt \langle \langle Y(t), Y(t) \rangle^{(b)}_{\hat{\rho}}. \tag{31} \]

The entropy production rate can be written by using the semi-inner product [17]:

\[ \dot{\sigma}^{(b)} := \text{Tr} \left[ D_b(\rho)(-\ln \rho - \beta_b \hat{H}_S) \right] = \langle -\ln \hat{\rho} - \beta_b \hat{H}_S, -\ln \hat{\rho} - \beta_b \hat{H}_S \rangle^{(b)}_{\rho}. \tag{32} \]

\( \sigma \) is given by \( \sigma = \int_0^\tau dt \sum_b \dot{\sigma}^{(b)} \). In [30], we can demonstrate (Appendix C)

\[ \sum_b \langle \varphi(t), \varphi(t) \rangle^{(b)}_{\hat{\rho}} \leq 2A_\varphi(t), \tag{33} \]

which leads to the tightest inequality of (12). We can demonstrate

\[ 2A_\varphi(t) \leq B(t) + B'(t) \leq B(t) + B_\infty(t) \tag{34} \]

using \( \text{Tr}(\hat{\rho}(t)\{\varphi, \hat{L}_k\}) \geq 0, \varphi(t)^2 = 1, \) and \( B_\infty(t) \geq B'(t) \). Here, \( \{X,Y\} = XY + YX \). Then, we obtain (12). We notice that a Mandelstam-Tamm type relation can not be obtained for no dissipation limit in this framework.

**DISCUSSION**

We compare the derivations of (9) and (12). [9] can be derived as follows [17, 18]. For the spectral decomposition \( \hat{\rho}(t) = \sum_n p_n(t)|\tilde{n}(t)\rangle \langle \tilde{n}(t)| \), we put

\[ \tilde{\phi}(t) := \sum_n c_n(t)|\tilde{n}(t)\rangle \langle \tilde{n}(t)|, \quad c_n(t)^2 = 1. \tag{35} \]

Then, \( \langle \langle \tilde{\rho}(t), \tilde{\phi}(t) \rangle = 0 \) and \( \tilde{\phi}(t)^2 = 1 \) hold. For any \( c_n(t) \), \( \tilde{\phi}(t) \) can not describe \( \varphi \) in general [21]. For \( c_n(t) = \text{sign}(p_n(t) - p_n(0)) \) or \( c_n(t) = \text{sign}(p_n(t) - p_n(0)) \),

\[ dV(\tilde{\rho}(t), \tilde{\rho}(0)) := \sum_n |p_n(t) - p_n(0)| \]

holds. By repeating similar calculations from (30) and by exploiting \( dV(\tilde{\rho}(t), \tilde{\rho}(0)) \geq dV(\hat{\rho}(t), \hat{\rho}(0)) \) (Appendix E and (32)) \( dV(\hat{\rho}(t), \hat{\rho}(0)) = dV(\hat{\rho}(t), \hat{\rho}(0)) \), we can derive (Appendices C and D).

In open quantum systems described by the Lindblad type quantum master equation, we obtained universal bounds of the total entropy production described by the trace distance between the initial and final states. We considered the trace distance in the interaction picture instead of the modified Wasserstein distance [17] and trace distance in the Schrödinger picture [18]. Our bounds can be tighter than the bound of Vu and Hasegawa [17]. Our results are applicable to an arbitrary initial state, beyond Vu-Saito’s bound [18] applicable only to the completely mixed initial state. In the classical master equation limit, our tightest inequality leads to the inequality by Shiraishi et al. [14].

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[21] L. Mandelstam and I. Tamm, “The uncertainty relation between energy and time in nonrelativistic quantum mechanics”, J. Phys. (Moscow) 9, 249 (1945).

[22] $W_{nm} = \sum_b W_{nm}^{(b)}$ where $W_{nm}^{(b)}$ is the contribution from the bath $b$. The local detailed balance condition is $W_{nm}^{(b)} e^{-\beta_b E_n} = W_{nm}^{(b)} e^{-\beta_b E_m}$ where $E_n$ is the energy of the state $n$ and $\beta_b$ is the inverse temperature of the bath $b$.

[23] $c_1 := \int_0^t dt \| i[H, \rho] \|$, $c_2 := \int_0^t dt \| D_{out}(\rho) \|$, and $c_3 := \int_0^t dt \| D_{in}(\rho) \|$, where $D_{in}(\rho) := \sum_n |n\rangle \langle n| D(\rho) |n\rangle \langle n|$ and $D_{out}(\rho) := \sum_{n \neq m} |n\rangle \langle n| D(\rho) |m\rangle \langle m|$. Here, $\{ |n\rangle \}$ is defined by $\{ 5 \}$.

[24] If (i) the eigenstate $|E_n\rangle$ of $H_S$ are time-independent, (ii) $H_S$ is non-degenerate, (iii) the system state can be represented as $|\rho(t)\rangle = \sum_n p_n(t) |E_n\rangle \langle E_n|$, the quantum master equation reduces to the classical master equation. In this case, $W_{nm}$ corresponds to $W_{nm}$.

[25] Precisely, in Ref. [17], $\sigma_{VH}$ is $d_T(\rho(\tau), \rho(0))^2 / \int_0^t dt \ 2B_{\infty}(t)$ where $B_{\infty}(t) = \sum_{k} \gamma_k ||L_k||^2 \geq B(\tau)$. However, $B_{\infty}(t)$ can be replaced by $B(\tau)$ as is done in Ref. [18].
Appendix A: Notation

In this section, we look back at the notations in the main text. We denote the Hilbert space of the system by $\mathcal{H}$ and the set of the linear operators of $\mathcal{H}$ by $\mathcal{L}(\mathcal{H})$.

We consider the Lindblad type quantum master equation
\[
\frac{d}{dt}\rho(t) = -i[H(t), \rho(t)] + \mathcal{D}(\rho(t)), \quad \mathcal{D}(\rho) = \sum_k \gamma_k \tilde{D}_k[\rho](\rho),
\] (A1)
where
\[
\tilde{D}[X](Y) := \left( XYX^\dagger - \frac{1}{2} X^\dagger XY - \frac{1}{2} YX^\dagger X \right) \quad (X, Y \in \mathcal{L}(\mathcal{H})).
\] (A2)

Here, $H(t) = H_S(t) + H_L(t)$ where $H_S$ is the system Hamiltonian and $H_L$ is the Lamb shift Hamiltonian. $[H_L(t), H_S(t)] = 0$ holds. The label $k$ is a tuple $(b, a, \omega)$ where $b$ is the label of the bath. The jump operators $\tilde{L}_{b,a,\omega}$ satisfy
\[
[L_{b,a,\omega}, H_S] = \omega L_{b,a,\omega}, \quad L_{b,a,\omega} = L_{b,a,\omega}^\dagger.
\] (A3)

We assume the local detailed balance condition
\[
\gamma_{b,a,-\omega} = e^{-\beta_b} \gamma_{b,a,\omega},
\] (A4)
where $\beta_b$ is the inverse temperature of the bath $b$. The total entropy production rate is given by
\[
\dot{\sigma} = \sum_b \dot{\sigma}^{(b)} = \left\{ \text{Tr}[\mathcal{D}_b(\rho)(-\ln \rho - \beta_b H_S)] \right\}.
\] (A5)

Here, $\mathcal{D}_b(\rho)$ denotes the contribution from the bath $b$ of $\mathcal{D}(\rho)$. The total entropy production is given by
\[
\sigma = \int_0^\tau dt \dot{\sigma}.
\] (A6)

For each time-dependent operator $X(t)$, the corresponding operator in the interaction picture is defined by $\tilde{X}(t) = U(t)^\dagger X(t) U(t)$. Here, $U(t)$ is defined by
\[
\frac{d}{dt} U(t) = -i H(t) U(t)
\] (A7)
and $U(0) = 1$. The quantum master equation in the interaction picture is given by
\[
\frac{d}{dt} \tilde{\rho}(t) = \sum_b \tilde{D}_b(\tilde{\rho}), \quad \tilde{D}_b(Y) := \sum_{a,\omega} \gamma_{b,a,\omega} \tilde{D}[\tilde{L}_{b,a,\omega}](Y).
\] (A8)

On the other hand,
\[
\frac{d}{dt} \|\tilde{\rho}(t) - \tilde{\rho}(0)\|_1 = \text{Tr} \left[ \varphi(t) \frac{d\tilde{\rho}}{dt} \right], \quad \varphi(t) = \Phi(\tilde{\rho}(t) - \tilde{\rho}(0))
\] (A9)
holds. $\Phi$ maps a self-adjoint operator $X$ to a self-adjoint operator as
\[
\Phi(X) := \sum_n \text{sign}(x_n) |n\rangle \langle n|,
\] (A10)
where the spectral decomposition of $X$ is $X = \sum_n x_n |n\rangle \langle n|$. $\text{sign}(x)$ is the sign of $x$. $\varphi(t)$ satisfies
\[
\varphi(t)^2 = 1.
\] (A11)

We introduce
\[
A_\varphi(t) = \text{Tr} \left( \tilde{\rho}(t) \frac{1}{4} \sum_k \gamma_k [\varphi, \tilde{L}_k]^\dagger [\varphi, \tilde{L}_k] \right),
\] (A12)
\[
B(t) = \text{Tr} \left[ \rho(t) \sum_k \gamma_k L_k^\dagger L_k \right],
\] (A13)
\[
B'(t) = \text{Tr} \left( \varphi \tilde{\rho} \varphi \sum_k \gamma_k \tilde{L}_k^\dagger \tilde{L}_k \right),
\] (A14)
\[
B_\infty(t) = \sum_k \gamma_k ||L_k||_\infty^2.
\] (A15)
and
\[
\langle X, Y \rangle^{(b)}_\rho = \text{Tr}[X^\dagger \hat{O}^{(b)}_\rho (Y)] \quad (X, Y \in \mathcal{L}(\mathcal{H})),
\]
\[
\hat{O}^{(b)}_\rho (Y) = \frac{1}{2} \sum_{a, \omega} e^{-\beta_\omega s/2} \gamma_{b, a, \omega}[\hat{L}_{b, a, \omega}, \{ \hat{\rho} \} \beta_\omega ((\hat{L}^\dagger_{b, a, \omega}, Y)] \quad (Y \in \mathcal{L}(\mathcal{H})),
\]
\[
\{ \hat{\rho} \} \rho_c(X) = e^{-\epsilon s/2} \int_0^1 ds \ e^{s \rho s} X \rho^{1-s} \quad (X \in \mathcal{L}(\mathcal{H})).
\]

The semi-inner product $\langle \langle X, Y \rangle \rangle^{(b)}_\rho$ satisfies
\[
(\langle \langle X, Y \rangle \rangle^{(b)}_\rho)^* = \langle \langle Y, X \rangle \rangle^{(b)}_\rho, \quad \langle \langle X, X \rangle \rangle^{(b)}_\rho \geq 0.
\]

The Cauchy-Schwarz inequality \[12\]
\[
\left| \int_0^T dt \ \langle \langle X(t), Y(t) \rangle \rangle^{(b)}_\rho \right| \leq \left[ \int_0^T dt \ \langle \langle X(t), X(t) \rangle \rangle^{(b)}_\rho \ \int_0^T dt \ \langle \langle Y(t), Y(t) \rangle \rangle^{(b)}_\rho \right]^{1/2}
\]
holds $(X(t), Y(t) \in \mathcal{L}(\mathcal{H}))$.

**Appendix B: Total entropy production rate**

In this section, we demonstrate that $\hat{\sigma}^{(b)} = \langle \langle - \ln \hat{\rho} - \beta_b \tilde{H}_S, - \ln \hat{\rho} - \beta_b \tilde{H}_S \rangle \rangle^{(b)}_\rho$.

First, we demonstrate that \[1\]
\[
\{ \hat{\rho} \} \rho_c([X, \ln \rho] - cX) = e^{-\epsilon s/2} X \rho - e^{\epsilon s/2} \rho X \quad (X \in \mathcal{L}(\mathcal{H})).
\]

In fact,
\[
\{ \hat{\rho} \} \rho_c([X, \ln \rho] - cX) = e^{-\epsilon s/2} \int_0^1 ds \ e^{s \rho s} (X \ln \rho - \ln \rho X - cX) \rho^{1-s}
\]
\[
= -e^{-\epsilon s/2} \int_0^1 ds \ e^{s \rho s} \ln \rho + cX e^{(1-s) \ln \rho} + e^{s \rho s} \ln \rho X (\ln \rho) e^{(1-s) \ln \rho}
\]
\[
= -e^{-\epsilon s/2} \int_0^1 ds \frac{d}{ds} [e^{s \rho s} \ln \rho X e^{(1-s) \ln \rho}]
\]
\[
= -e^{\epsilon s/2} \rho X + e^{\epsilon s/2} X \rho.
\]

Then, we obtain
\[
e^{-\beta_\omega s/2} \hat{L}^\dagger_{b, a, \omega} \hat{\rho} - e^{-\beta_\omega s/2} \rho \hat{L}^\dagger_{b, a, \omega} = \{ \hat{\rho} \} \beta_\omega (\{ \hat{L}^\dagger_{b, a, \omega}, \ln \hat{\rho} \} - \beta_b \omega \hat{L}^\dagger_{b, a, \omega})
\]
\[
= \{ \hat{\rho} \} \beta_\omega (\{ \hat{L}^\dagger_{b, a, \omega}, \ln \hat{\rho} + \beta_b \tilde{H}_S \}).
\]

Here, we used (A3) in the second line. Using this, we obtain
\[
\hat{O}^{(b)}_\rho ( - \ln \hat{\rho} - \beta_b \tilde{H}_S) = \frac{1}{2} \sum_{a, \omega} e^{-\beta_\omega s/2} \gamma_{b, a, \omega}[\hat{L}_{b, a, \omega}, \{ \hat{\rho} \} \beta_\omega ((\hat{L}^\dagger_{b, a, \omega}, - \ln \hat{\rho} - \beta_b \tilde{H}_S))]
\]
\[
= \frac{1}{2} \sum_{a, \omega} e^{-\beta_\omega s/2} \gamma_{b, a, \omega}[\hat{L}_{b, a, \omega}, \{ \beta_\omega \} (\hat{L}^\dagger_{b, a, \omega}, - \ln \hat{\rho} - \beta_b \tilde{H}_S)]
\]
\[
= \frac{1}{2} \sum_{a, \omega} \gamma_{b, a, \omega} ((\hat{L}_{b, a, \omega}, \hat{L}^\dagger_{b, a, \omega}) \hat{\rho} - \hat{L}^\dagger_{b, a, \omega} \hat{L}_{b, a, \omega}) + e^{-\beta_\omega s/2} \rho \hat{L}^\dagger_{b, a, \omega} - e^{-\beta_\omega s/2} \hat{L}^\dagger_{b, a, \omega} \hat{\rho}
\]
\[
= \frac{1}{2} \sum_{a, \omega} \gamma_{b, a, \omega} ((\hat{L}_{b, a, \omega}, \hat{\rho}) \hat{L}^\dagger_{b, a, \omega} - \hat{L}^\dagger_{b, a, \omega} \hat{\rho} + \hat{L}^\dagger_{b, a, \omega} \hat{\rho} \hat{L}_{b, a, \omega})
\]
\[
= \hat{D}_b(\hat{\rho}).
\]
Here, we used \([A3]\) and \([A4]\) in the fourth line. Then, we obtain

\[
\langle -\ln \tilde{\rho} - \beta_b \tilde{H}_S, -\ln \tilde{\rho} - \beta_b \tilde{H}_S \rangle^{(b)}_{\tilde{\rho}} = \text{Tr}[(-\ln \tilde{\rho} - \beta_b \tilde{H}_S) \tilde{O}^{(b)}_{\tilde{\rho}} (-\ln \tilde{\rho} - \beta_b \tilde{H}_S)] \\
= \text{Tr}\{(-\ln \tilde{\rho} - \beta_b \tilde{H}_S) \tilde{D}_b(\tilde{\rho})\} \\
= \text{Tr}\{(-\ln \rho - \beta_b H_S) D_b(\rho)\} \\
= \delta^{(b)}. \tag{B5}
\]

Appendix C: Activity

In this section, we demonstrate that \(B(t) := \sum_b \langle \varphi(t), \varphi(t) \rangle^{(b)}_{\tilde{\rho}} \leq 2A \varphi(t)\). \(B(t)\) is given by

\[
B(t) = \text{Tr} \left[ \frac{1}{2} \sum_k e^{-\beta_\omega/2} \gamma_k \{\tilde{\rho}, \tilde{\beta}_\omega \{[\tilde{L}_k^\dagger, \varphi]\} \right] \\
= \text{Tr} \left[ \frac{1}{2} \sum_k e^{-\beta_\omega/2} \gamma_k [\varphi, \tilde{L}_k] \{\tilde{\rho}, \tilde{\beta}_\omega \{[\tilde{L}_k^\dagger, \varphi]\} \right] \\
= \frac{1}{2} \sum_k e^{-\beta_\omega/2} \gamma_k \{[\tilde{L}_k^\dagger, \varphi], \{\tilde{\rho}, \tilde{\beta}_\omega \{[\tilde{L}_k^\dagger, \varphi]\} \right] \\
= \frac{1}{2} \sum_k e^{-\beta_\omega/2} \gamma_k M_{\beta_\omega}(\tilde{\rho}, [\tilde{L}_k^\dagger, \varphi]). \tag{C1}
\]

Here, \(\langle X, Y \rangle := \text{Tr}(X^\dagger Y)\) \((X, Y \in \mathcal{L}(\mathcal{H})\) and \(M_c(\rho, X) := \langle X, \{\rho\}_c(X) \rangle\). If we diagonalize \(\rho(t)\) as

\[
\rho(t) = \sum_n p_n(t) |n(t)\rangle \langle n(t)|, \tag{C2}
\]

we obtain

\[
M_c(\rho, X) = e^{-c/2} \int_0^1 ds \text{Tr}[X^\dagger e^{sc} \rho^s X \rho^{1-s}] \\
= e^{-c/2} \sum_{n,m} \langle n|X^\dagger|m\rangle \langle m|X|n\rangle \int_0^1 ds \ e^{sc} p_n p_m^{1-s} \\
= e^{-c/2} \sum_{n,m} \langle n|X^\dagger|m\rangle \langle m|X|n\rangle p_n \int_0^1 ds \ e^{s(c+\ln p_m/p_n)} \\
= e^{-c/2} \sum_{n,m} \langle n|X^\dagger|m\rangle \langle m|X|n\rangle p_n e^{c \ln p_m/p_n - 1} \\
= \sum_{n,m} \langle n|X^\dagger|m\rangle \langle m|X|n\rangle \frac{e^{c/2 p_m - e^{-c/2} p_n}}{c + \ln p_m/p_n} \\
= \sum_{n,m} \langle n|X^\dagger|m\rangle \langle m|X|n\rangle \Psi(e^{c/2 p_m}, e^{-c/2} p_n). \tag{C3}
\]

Here, \(\Psi(a, b) := (b - a)/(\ln b/a)\) is the logarithmic mean of \(a\) and \(b\). Using \(\Psi(a, b) \leq \frac{a+b}{2}\), we obtain \(\leq 1\)

\[
M_c(\rho, X) \leq \frac{1}{2} \sum_{n,m} \langle n|X^\dagger|m\rangle \langle m|X|n\rangle (e^{c/2} p_m + e^{-c/2} p_n) \\
= \frac{1}{2} \sum_n ((\langle n|X^\dagger|n\rangle e^{c/2} p_n + \langle n|X^\dagger X|n\rangle e^{-c/2} p_n) \\
= \frac{1}{2} [e^{c/2} \text{Tr}(\rho X^\dagger) + e^{-c/2} \text{Tr}(\rho X^\dagger X)]. \tag{C4}
\]
Then,

\[
B(t) \leq \frac{1}{2} \sum_k \gamma_k \left[ \frac{1}{2} \text{Tr}(\tilde{\rho} X_k X_k^\dagger) + \frac{1}{2} e^{-\beta_k \omega} \text{Tr}(\tilde{\rho} X_k^\dagger X_k) \right]
\]

\[
= \frac{1}{2} \sum_k \gamma_k \text{Tr}(\tilde{\rho} X_k X_k^\dagger)
\]

\[
= 2 A_\varphi(t)
\]

(C5)

holds with \( X_k := [\tilde{L}_k^\dagger, \varphi] \). We used (A3) and (A4) in the second line of (C5). Using

\[
\text{Tr}(\tilde{\rho}(t)\{\varphi, \tilde{L}_k\}^\dagger \{\varphi, \tilde{L}_k\}) \geq 0,
\]

(C6)

\( \varphi(t)^2 = 1 \), and \( B_\infty(t) \geq B'(t) \), we obtain

\[
B(t) \leq 2 A_\varphi(t) \leq B(t) + B'(t) \leq B(t) + B_\infty(t).
\]

(C7)

### Appendix D: Partial activity

Because of (36) in the main text,

\[
\sigma \geq d_V(\rho(\tau), \rho(0))^2
\]

(D1)

can be shown in the same way as (30) in the main text and Appendix C. Here, \( A_\varphi(t) \) is given by (A12) replacing \( \varphi \) with \( \tilde{\varphi} \) and corresponds to the partial activity. \( A_\varphi(t) \) can be rewritten as

\[
A_\varphi(t) = \text{Tr} \left( \rho(t) \frac{1}{4} \sum_k \gamma_k [\varphi, L_k]^\dagger [\varphi, L_k] \right)
\]

(D2)

with

\[
\phi(t) := \sum_n c_n(t) |n(t))\langle n(t)|,
\]

(D3)

and \( c_n(t) = \text{sign}(p_n(t) - p_n(0)) \) or \( c_n(t) = \text{sign}(p_n(\tau) - p_n(0)) \). We obtain

\[
A_\varphi(t) = \sum_{n,m} \frac{1}{4} (c_m(t) - c_n(t))^2 p_n(t) \sum_k \gamma_k |m(t)|L_k|n(t))|^2
\]

\[
= \sum_{c_n(t) \neq c_m(t)} p_n(t) \sum_k \gamma_k |m(t)|L_k|n(t))|^2
\]

\[
\leq A(t).
\]

(D4)

Here, \( A(t) \) is given in (7) in the main text and can be rewritten as

\[
A(t) = \sum_{n \neq m} p_n(t) \sum_k \gamma_k |m(t)|L_k|n(t))|^2.
\]

(D5)

\( A_\varphi(t) \) corresponds to the partial activity [17]. In the classical master equation limit, \( A_\varphi(t) \) of (A12) becomes \( A_\varphi(t) \) with \( c_n(t) = \text{sign}(p_n(t) - p_n(0)) \).

### Appendix E: Distances \( d_T \) and \( d_V \)

The modified Wasserstein distance \( d_T \) is defined by

\[
d_T(\rho(\tau), \rho(0)) := \sum_n |b_n - a_n|
\]

(E1)
where \( \{a_n\} \) and \( \{b_n\} \) are increasing eigenvalues of \( \rho(0) \) and \( \rho(\tau) \). \( d_V \) is defined by

\[
d_V(\rho(\tau), \rho(0)) := \sum_n |p_n(\tau) - p_n(0)|
\]

(E2)

using the spectral decomposition (C2) with differentiable eigenstates \( \{|n(t)\} \). \( d_V(\rho(\tau), \rho(0)) \) can be rewritten as

\[
d_V(\rho(\tau), \rho(0)) = \sum_n |b_{\chi(n)} - a_n|
\]

(E3)

Here, \( \chi \) is a permutation. For any two increasing sequences \( \{x_n\} \) and \( \{y_n\} \), we can demonstrate that

\[
\sum_n |y_{\sigma(n)} - x_n| \geq \sum_n |y_n - x_n|
\]

(E4)

for an arbitrary permutation \( \sigma \). Then, we obtain

\[
d_V(\rho(\tau), \rho(0)) \geq d_T(\rho(\tau), \rho(0)).
\]

(E5)

If the eigenvalues of \( \rho(t) \) don’t intersect as functions of time, \( d_V(\rho(\tau), \rho(0)) = d_T(\rho(\tau), \rho(0)) \) holds.

We prove (E4). For \( i < j \) with \( \sigma(i) > \sigma(j) \), we can prove that

\[
|y_{\sigma(i)} - x_i| + |y_{\sigma(j)} - x_j| \geq |y_{\sigma(j)} - x_i| + |y_{\sigma(i)} - x_j|.
\]

(E6)

The above equation leads to (E4).

[1] T. V. Vu and Y. Hasegawa, “Geometrical Bounds of the Irreversibility in Markovian Systems”, Phys. Rev. Lett. 126, 010601 (2021).

[2] T. V. Vu and K. Saito, “Finite-Time Quantum Landauer Principle and Quantum Coherence”, Phys. Rev. Lett. 128, 010602 (2022).