Uncovering conformal symmetry in the 3D Ising transition: State-operator correspondence from a fuzzy sphere regularization

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The 3D Ising transition, the most celebrated and unsolved critical phenomenon in nature, has long been conjectured to have emergent conformal symmetry, similar to the case of the 2D Ising transition. Yet, the emergence of conformal invariance in the 3D Ising transition has rarely been explored directly, mainly due to unavoidable mathematical or conceptual obstructions. Here, we design an innovative way to study the quantum version of the 3D Ising phase transition on spherical geometry, using the “fuzzy (non-commutative) sphere” regularization. We accurately calculate and analyze the energy spectra at the transition, and explicitly demonstrate the state-operator correspondence (i.e. radial quantization), a fingerprint of conformal field theory. In particular, we have identified 13 parity-even primary operators within a high accuracy and 2 parity-odd operators that were not known before. Our result directly elucidates the emergent conformal symmetry of the 3D Ising transition, a conjecture made by Polyakov half a century ago. More importantly, our approach opens a new avenue for studying 3D CFTs by making use of the state-operator correspondence and spherical geometry.

I. INTRODUCTION

Symmetry is one of the most important organizing principles in physics. As is well known, symmetries present microscopically (e.g. condensed matter systems, ultraviolet (UV) Lagrangians) can be spontaneously broken at low energies, giving rise to various distinct phases of matter such as crystals and magnets. Conversely and rather unexpectedly, symmetries absent microscopically can emerge at low energies, and such a phenomenon is called emergent symmetry. One prominent example is the order-disorder phase transition of 2D Ising model, for which Polyakov discovered emergent conformal symmetry in 1970 [1], 26 years after Onsager’s exact solution [2]. Polyakov’s remarkable discovery of emergent conformal symmetry in the 2D Ising transition gave birth to conformal field theory (CFT) [3], a class of quantum field theories with profound applications in various fields of physics including statistical mechanics, quantum condensed matter, string theory and quantum gravity. In statistical physics, it is a common belief that many universality classes of (classical and quantum) phase transitions are captured by CFTs, however this has not been proven for 3D transitions. The emergence of conformal symmetry at phase transitions is not only aesthetically beautiful, but also useful in understanding the properties of these transitions, such as computing experimentally measurable critical exponents. In 2D the (local) conformal symmetry has an infinite-dimensional algebra, and it makes many 2D CFTs exactly solvable [3, 6]. In $d \geq 2$ dimensions, there is only a finite-dimensional (global) conformal symmetry, i.e. $SO(d + 1, 1)$, with which one is not able to analytically solve CFTs as in 2D. Therefore, CFT beyond 2D are rather poorly understood, with their solutions remaining outstanding for decades despite their broad appeal to physics and mathematics.

Historically, the study of lattice models for 2D classical phase transitions and their quantum cousins $(1 + 1D)$ quantum phase transitions) played a key role in the discovery and understanding of 2D CFTs [1, 2, 7]. Similar progress in the study of conformal symmetry for $d \geq 3$ dimensional theories, however, has stalled due to the natural limitation of the lattice formulation. There are a plenty of papers studying 3D phase transitions on the lattice, e.g. computing critical exponents. However, the perspective of conformal symmetry has rarely been explored [8–13]. The conformal symmetry of a $d$–dimensional CFT is most transparent in geometries such as $\mathbb{R}^d$, $S^d$ as well as $S^{d-1} \times \mathbb{R}$. In particular, CFTs on $S^{d-1} \times \mathbb{R}$ obey a property called state-operator correspondence (i.e. radial quantization), which is a direct consequence of conformal symmetry [7]. Specifically, for a quantum Hamiltonian defined on sphere $S^{d-1}$, its eigenstates are in one-to-one correspondence with the scaling operators (including primary and descendant operators) of the infrared (IR) CFT. Moreover, the energy gaps of these eigenstates are proportional to the scaling dimensions of their corresponding scaling operators [14]. This nice feature can be used to explore various properties of CFTs, including scaling dimensions of operators, operator product expansion coefficients, and even operator algebras [7]. For 2D CFTs, $S^1 \times \mathbb{R}$ is very natural as one just needs to study a $1 + 1D$ quantum lattice model defined on a 1D periodic chain (i.e. $S^1$) [15–18]. However, simulating lattice models of $d \geq 3$ dimensional CFTs on

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$S^{d-1} \times \mathbb{R}$ will be problematic, because a regular lattice cannot be put on a sphere $S^{d-1} \times \mathbb{R}^2$ due to its nontrivial curvature. While efforts have nevertheless been made to discretize the sphere, no signature of state-operator correspondence has been found so far [19, 20].

To overcome this geometric obstacle, in this paper we are pursuing a different direction, namely we fuzzify a sphere [21]. Specifically, we study a $2 + 1D$ quantum Ising transition defined on a fuzzy (non-commutative) sphere in light of Landau level regularization [22]. As a result of this innovative discretization, we have observed almost perfect state-operator correspondence in surprisingly small system sizes. We use exact diagonalization to calculate properties of the $2+1D$ Ising transition for up to 16 effective spins, and we have found its low lying eigenstates (up to 70 lowest states) split into representations of the $3D$ conformal symmetry (i.e. conformal multiplets), hence directly demonstrating the emergence of conformal symmetry. Among these low energy states, we have found 15 conformal primary states, most of which have not been discovered in any previous model studies of the $3D$ Ising transition. Specifically, we have found 13 parity-even primaries, whose scaling dimensions agree well with $3D$ Ising transition, but it is unclear if the proof is correct (see the comment in Appendix B in the first arXiv version of Ref.[13]).

Our observations directly verify conformal symmetry for the $3D$ Ising transition, which was conjectured by Polyakov 50 years ago [1]. Before our results, the most compelling evidence for the $3D$ Ising transition being conformal was from numerical conformal bootstrap [23–27], which assumes conformal symmetry and found critical exponents close to the values obtained by various methods such as Monte Carlo simulation [28, 29] and measured by experiments [30]. In addition, there was an effort [13] to justify the conformal invariance of the $3D$ Ising by showing that the virial current operator does not exist. Our obtained operator spectrum from the state-operator correspondence indeed convincingly shows that the $3D$ Ising transition does not have the virial current, which is a structural explanation of the $3D$ Ising being conformal [32]. A major surprise of our results is that an incredibly small system size ($8 \sim 16$ total spins) is already enough to yield accurate conformal data of the $3D$ Ising CFT. So we expect this approach to open a new avenue for studying higher dimensional phase transitions and CFTs. Firstly, there is a zoo of universalities that can be studied using our approach, which is amenable to various numerical techniques such as exact diagonalization (ED), density-matrix renormalization group (DMRG) and determinantal Monte Carlo. This offers an opportunity to tackle many open questions regarding phase transitions, critical phases and CFTs. Secondly, a number of new universal quantities can be computed once the $3D$ CFT is simulated on a sphere, such as operator product expansion coefficients, $F$ (of $F$-theorem) [33–36], and the spherical binder ratio [37], just to name a few.

The paper is organized as follows. In Sec. II A we will review background knowledge including the radial quantization of CFTs and the state-operator correspondence. The spherical Landau level quantization and related fuzzy sphere are discussed in Sec. II B. Readers familiar with these topics can skip some of these subsections. In Sec. III, we formulate spherical Landau levels to regularize the $3D$ Ising transition on a fuzzy sphere. A global quantum phase diagram is presented. In Sec. IV, we present the low-lying energy spectra at the phase transition point, and analyze their one-to-one correspondence with the scaling operators as predicted by the Ising CFT. This is the main result of this paper. At last, we present a discussion and outlook in Sec. V.

II. REVIEW OF BACKGROUND

A. Radial quantization of CFTs: state-operator correspondence

In this subsection we review some basics of radial quantization, and for an elaborated discussion we refer the readers to CFT lecture notes such as those in [6, 38]. The conformal group in $d$ dimensions $SO(d+1, 1)$ is generated by $d$-dimensional translations $P_\mu = i\partial_\mu$, $d$-dimensional Lorentz rotations $M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$, dilatations $D = ix^\mu \partial_\mu$, and special conformal transformations $K_\mu = i(2x_\mu(x^\nu \partial_\nu) - x^2 \partial_\mu)$. From the operator point of view, a CFT can be thought of as a theory whose operators form an infinite-dimensional representation of the conformal group. Specifically, one can write CFT operators $\{\hat{O}_\alpha\}$ as eigen-operators (i.e. irreducible repre-

![FIG. 1. Through a Weyl transformation, Euclidean flat spacetime $\mathbb{R}^d$ is mapped to the manifold of cylinder $S^{d-1} \times \mathbb{R}$. As a result, a CFT on $\mathbb{R}^d$ quantized on equal radius slices can be described equivalently in terms of a CFT on $S^{d-1} \times \mathbb{R}$ quantized on equal time slices. The states defined on the $S^{d-1} \times \mathbb{R}$ have well-defined quantum numbers of $SO(d)$ Lorentz rotation and dilatation, and thus they are in one-to-one correspondence with operators of the CFT, dubbed as state-operator correspondence.](image-url)
sentations) of the dilatation and Lorentz rotation $SO(d)$.
In particular, the eigenvalue $\Delta$ of dilatation is called scaling dimension of the operator, and it corresponds to the exponent in the power law correlation function of the operator, e.g. $(O(x)O(0)) \sim 1/|x|^{2\Delta}$. One can further categorize operators into primary operators and descendant operators: 1) primary operators are operators that are annihilated by the special conformal transformation $K_\mu$; 2) descendant operators are not annihilated by $K_\mu$, and all of them can be obtained by applying translations $P_\mu$ (multiple times) to the primary operators. Therefore, one can organize CFT operators as primary operators and their descendants, and each primary and its descendants form a set of operators called a conformal multiplet.\(^4\) A CFT has an infinite number of primary operators, which makes it hard to tackle theoretically. A major task of solving a CFT is thus to obtain its low lying (if not full) spectrum of primary operators.

To facilitate later analysis of our numerical results, we will elaborate a bit more about the operator contents of the global conformal primaries are called quasi-primaries. We also remark that conserved operators (i.e. global operators have two quantum numbers ($\Delta$, $\ell$) with quantum number ($\Delta + 2n$, $\ell$) ≥ 0. Here and hereafter the repeated operators and their descendants, and each primary and its descendants form a set of operators called a conformal multiplet.\(^4\) A CFT has an infinite number of primary operators, which makes it hard to tackle theoretically. A major task of solving a CFT is thus to obtain its low lying (if not full) spectrum of primary operators.

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\[ \partial_{\nu_1} \cdots \partial_{\nu_j} \square^n O, \quad n, j \geq 0, \quad (1) \]

with quantum number ($\Delta + 2n + j$, $\ell$) for $\ell \geq i > 0$, $n, j \geq 0$. Here and hereafter the repeated indices shall be contracted. The descendents of a spin-$\ell$ primary operator $O_{\mu_1 \cdots \mu_n}$ are a bit more complicated as there are two different types. The first type can be written as,

\[ \partial_{\nu_1} \cdots \partial_{\nu_j} \partial_{\mu_1} \cdots \partial_{\mu_l} \square^n O_{\mu_1 \cdots \mu_l}, \quad (2) \]

with quantum number ($\Delta + 2n + j + \ell$, $\ell + j - i$) for $\ell \geq i \geq 0$, $n, j \geq 0$. Here and hereafter the repeated indices shall be contracted. The other type will involve the $\varepsilon$ tensor of $SO(3)$, and can be written as,

\[ \varepsilon_{\mu_1 \nu_1} \partial_{\nu_1} \cdots \partial_{\nu_j} \partial_{\mu_1} \cdots \partial_{\mu_l} \square^n O_{\mu_1 \cdots \mu_l}, \quad (3) \]

with quantum number ($\Delta + 2n + j + i + 1$, $\ell + j - i$) for $\ell - 1 \geq i \geq 0$, $n, j \geq 0$. We note that the $\varepsilon$ tensor alters spacetime parity symmetry of $O_{\mu_1 \cdots \mu_l}$.

We also remark that conserved operators (i.e. global symmetry current $J_\mu$ and energy momentum tensor $T_{\mu \nu}$) should be treated a bit differently, because they satisfy the conservation equations $\partial_\mu J_\mu = 0$ and $\partial_\mu T_{\mu \nu} = 0$. Therefore, their descendents in Eq. (2) and (3) should have $i = 0$.\(^5\)

Now we turn to the state perspective of CFTs. To define states of a CFT, we first need to quantize it, or in other words find a Hilbert space construction of it. A quantum phase transition, namely a quantum Hamiltonian realization of a $d$-dimensional CFT in $d-1$ space dimensions, can be viewed as a way to quantize the CFT. The states of the CFT are nothing but the quantum Hamiltonian's eigenstates. Formally, the quantization of CFTs can be more general than quantum phase transitions. Specifically, one can foliate $d$-dimensional spacetime into $d-1$-dimensional surfaces, and each leaf of the foliation is endowed with its own Hilbert space. One convenient quantization is radial quantization, which has the $d$-dimensional Euclidean space $\mathbb{R}^d$ foliated to $S^{d-1} \times \mathbb{R}$, as shown in the left hand side of Fig. 1. In the radial quantization, the $SO(d)$ Lorentz rotation acts on the $S^{d-1}$ sphere, while the dilatation acts as the scaling of sphere radius. Therefore, the states defined on the foliation $S^{d-1}$ have well-defined quantum numbers of $SO(d)$ rotation and dilatation, and they are indeed in one-to-one correspondence with operators of the CFT, dubbed as state-operator correspondence.

For a quantum Hamiltonian realization, the radial quantization described above is not natural, and instead one may want a quantization scheme that has an identical Hilbert space on each leaf of foliation. A quantum Hamiltonian is usually defined on the $M^{d-1} \times \mathbb{R}$ manifold: $\mathbb{R}$ is the time direction, while $M^{d-1}$ is a $d-1$-dimensional space manifold (e.g. sphere, torus, etc.), the leaf of foliation, on which the Hilbert space (and the quantum state) lives. In order to discuss state-operator correspondence in such a quantization scheme, one needs to map $\mathbb{R}^d$ to the cylinder $S^{d-1} \times \mathbb{R}$ using a Weyl transformation [7, 14], as shown in Fig. 1. Under the Weyl transformation the dilation $\tau \rightarrow e^{\Delta} \tau$ of $\mathbb{R}^d$ becomes the translation along the time direction $\tau \rightarrow \tau + \lambda$ of $S^{d-1} \times \mathbb{R}$. If the theory has conformal symmetry, we can simply relate correlators and states on $\mathbb{R}^d$ to those on $S^{d-1} \times \mathbb{R}$. Moreover, we still have the state-operator correspondence on the cylinder $S^{d-1} \times \mathbb{R}$. In particular, the state-operator correspondence on the cylinder has a nice physical interpretation, namely the eigenstates $|\psi_n\rangle$ of the CFT quantum Hamiltonian on $S^{d-1}$ are in one-to-one correspondence with the CFT operators, and the energy gaps $\delta E_n$ of these states are proportional to the scaling dimensions $\Delta_n$ of CFT operators [7, 14],

\[ \delta E_n = E_n - E_0 = \frac{v}{R} \Delta_n, \quad (4) \]

where $R$ is the radius of sphere $S^{d-1}$ and $v$ is the velocity of light that is model dependent. Also the $SO(d)$ rotation

\(^4\) Here we are talking about primary operators under the global conformal symmetry $SO(d+1,1)$. For 2D CFTs one usually talks about primary operators under Virasoro symmetries, and the global conformal primaries are called quasi-primaries.

\(^5\) The conformal multiplet of a conserved operator is called a short multiplet.
symmetry of $S^{d-1}$ is identified with the $SO(d)$ Lorentz rotation of the conformal group, so the $SO(d)$ quantum numbers of $|\psi_n\rangle$ are identical to those of CFT operators.

We emphasize that in contrast to radial quantization on $\mathbb{R}^d$, conformal symmetry is indispensable for the state-operator correspondence of radial quantization on the cylinder $S^{d-1} \times \mathbb{R}$. Therefore, observing the state-operator correspondence on the cylinder $S^{d-1} \times \mathbb{R}$ will be direct evidence for the conformal symmetry of the theory or phase transition. For $d = 2$, the cylinder $S^1 \times \mathbb{R}$ corresponds to nothing but a quantum Hamiltonian defined on a periodic chain, and there are very nice results studying the resulting state-operator correspondence [15–18]. In higher dimensions, one needs to study a quantum Hamiltonian defined on $S^{d-1}$, however, it is highly nontrivial for a discrete lattice model as $S^{d-1} \times \mathbb{Z}$ has a curvature.

B. Spherical Landau levels, fuzzy two-sphere and lowest Landau level projection

As originally shown by Landau, electrons moving in 2D space under a magnetic field will form completely flat bands called Landau levels, which is the key to the quantum Hall effect. Landau level quantization can be considered on any orientable manifold, and Haldane [39] first introduced Landau levels on spherical geometry to study the fractional quantum Hall physics.

For electrons moving on the surface of a radius-$r$ sphere with a $4\pi s$ monopole ($2s \in \mathbb{Z}$) placed at the origin (Fig. 2), the Hamiltonian is

$$H_0 = \frac{1}{2M_e r^2} \Lambda_n^2,$$

where $M_e$ is the electron’s mass and $\Lambda_n = \partial_\theta + iA_\theta$ is the covariant angular momentum, $A_\theta$ is the gauge field of the monopole. As usual we take $\hbar = e = c = 1$. The eigenstates will be quantized into spherical Landau levels, whose energies are $E_n = [n(n+1) + (2n+1)s]/(2M_e r^2)$, with $n = 0, 1, 2, \cdots$ the Landau level index. The $(n+1)$th Landau level is $(2s + 2n + 1)$-fold degenerate, and the single particle states in each Landau level are called Landau orbitals. Assuming all interactions are much smaller than the energy gap between Landau levels, we can just consider the lowest Landau level (LLL) $n = 0$, which is $2s+1$-fold degenerate. The wave-functions for each Landau orbital on LLL are called monopole harmonics [40]

$$\Phi_m(\theta, \varphi) = N_m e^{im\theta} \cos^{s+m} \left( \frac{\theta}{2} \right) \sin^{s-m} \left( \frac{\theta}{2} \right),$$

with $m = -s, -s+1, \cdots, s$ and $N_m = \sqrt{\frac{(2s+1)!}{4\pi(s+m)!(s-m)!}}$. Here $(\theta, \varphi)$ is the spherical coordinate.

These LLL Landau orbitals indeed form a $SO(3)$ spin-s irreducible representation. This can be understood by constructing the $SO(3)$ angular momentum operator [41],

$$L_\mu = \Lambda_\mu + sX_\mu,$$

which satisfies the $SO(3)$ algebra $[L_\mu, L_\nu] = i\epsilon_{\mu\nu\rho}L_\rho$. Projecting the system into the LLL, the kinetic energy of the covariant angular momentum will be quenched, so effectively we have $L_\mu \sim s^2x_\mu/r$. ($\tilde{x}_\mu$ denotes the coordinates in the projected LLL.) As a result, the coordinates $\tilde{x}_\mu$ of electrons will not actually commute, instead we have

$$[\tilde{x}_\mu, \tilde{x}_\nu] = i\epsilon_{\mu\nu\rho} \tilde{x}_\rho.$$

This defines a fuzzy two-sphere [21]. Moreover, Landau orbitals (6) are in one-to-one correspondence with states on the fuzzy two-sphere. Formally, a system defined on the LLL can be equivalently viewed as a system defined on a fuzzy two-sphere. We will not delve into details along that direction, and refer the reader to [42] for more discussions.

As is usually done in the literature, we will consider the limit where the interaction strength is much smaller than the Landau level gap, so we can project the system into the LLL. Technically, this can be done by rewriting the annihilation operator $\psi(\theta, \varphi)$ on the LLL as

$$\hat{\psi}(\theta, \varphi) = \frac{1}{\sqrt{2s+1}} \sum_{m=-s}^s \Phi^*_m \hat{c}_m.$$  

$\hat{c}_m$ stands for the annihilation operator of Landau orbital $m$, and is independent of coordinates $(\theta, \varphi)$. The density-density interaction $n(\theta, \varphi) = \psi^\dagger \psi$ can be written as,

$$n(\theta, \varphi) = \frac{1}{2s+1} \sum_{m_1, m_2} \Phi_{m_1} \Phi^*_{m_2} c_{m_1}^\dagger c_{m_2}.$$  

Any interaction can be straightforwardly (though perhaps tediously) written in the second quantized form using Landau orbital operators $c_{m_i}^\dagger c_m$. For example, the density-density interaction

$$H_I = \int d^2r_a d^2r_b U(r_a - r_b)n(r_a)n(r_b)$$

can be written as,

$$H_I = (2s + 1)^2 \int d\Omega_a d\Omega_b U(\theta_a, \varphi_a; \theta_b, \varphi_b)n(\theta_a, \varphi_a)n(\theta_b, \varphi_b)$$

$$= \sum_{m_1, m_2, m_3, m_4} V_{m_1, m_2, m_3, m_4} c_{m_1}^\dagger c_{m_2} c_{m_3} c_{m_4},$$

where $V_{m_1, m_2, m_3, m_4}$ can be further expanded using the so-called Haldane pseudopotential $V$ [39], corresponding to the two-fermion scattering in the spin-$2s - l$ channel (see Appendix Sec. A).

In summary, the model we are working with is a fermionic Hamiltonian enclosing $2s+1$-Landau orbitals with long-range $SO(3)$ invariant interactions. Interestingly, all the orbitals form an $SO(3)$ spin-$s$ irrep. Furthermore, the length scale of the system is $\sqrt{2s+1}$ instead of $2s+1$ since the spatial dimension is $d = 2$, and the thermodynamic limit corresponds to taking $s$ to infinity.

III. MODEL ON A FUZZY-TWO SPHERE

A. Hamiltonian
Here we explicitly define the model, which is spinful electrons in the LLL. \(^6\) In spatial space, the Hamiltonian takes the form

\[
H = \int (2s + 1)^2 d\Omega_a d\Omega_b U(\Omega_{ab}) \left[n^0(\theta_a, \varphi_a)n^0(\theta_b, \varphi_b) - n^z(\theta_a, \varphi_a)n^z(\theta_b, \varphi_b)\right] - \hbar \int (2s + 1)d\Omega n^z(\theta, \varphi),
\]

where \(n^\alpha(\theta, \varphi)\) is a local density operator given by

\[
n^\alpha(\theta, \varphi) = (\bar{\psi}_\uparrow(\theta, \varphi), \psi_\downarrow(\theta, \varphi)) \sigma^\alpha \begin{pmatrix} \bar{\psi}_\uparrow(\theta, \varphi) \\ \psi_\downarrow(\theta, \varphi) \end{pmatrix},
\]

with \(\sigma^{z,y,z}\) being Pauli matrices, \(\sigma^0 = I_{2 \times 2}\), and \(U(\Omega_{ab})\) the local density-density interactions (defined below). The first term behaves like an Ising ferromagnetic interaction, while the second term is the transverse field. By projecting the Hamiltonian into the LLL, we obtain

\[
H = H_{00} + H_{zz} + H_t,
\]

\[
H_{00} = \frac{1}{2} \sum_{m_{1,2,3,4},l = \pm s} V_{m_1,m_2,m_3,m_4}(c_{m_1}^\dagger c_{m_4})(c_{m_2}^\dagger c_{m_3}) \delta_{m_1+m_2,m_3+m_4},
\]

\[
H_{zz} = -\frac{1}{2} \sum_{m_{1,2,3,4},l = \pm s} V_{m_1,m_2,m_3,m_4}(c_{m_1}^\dagger \sigma^z c_{m_4})(c_{m_2}^\dagger \sigma^z c_{m_3}) \delta_{m_1+m_2,m_3+m_4},
\]

\[
H_t = -\hbar \sum_{m = \pm s} c_m^\dagger \sigma^x c_m,
\]

where \(c_m^\dagger = (c_{m_1}^\dagger, c_{m_2}^\dagger, c_{m_3}^\dagger)\) is the fermion creation operator on the \(m_{th}\) Landau orbital. The parameter \(V_{m_1,m_2,m_3,m_4}\) is connected to the Haldane pseudopotential \(V_1\) by

\[
V_{m_1,m_2,m_3,m_4} = \sum_l V_l (4s - 2l + 1) \begin{pmatrix} s & s & 2s-l & -2s-l \\ m_1 & m_2 & m_3 & m_4 \end{pmatrix},
\]

where \(\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\) is the Wigner 3j-symbol. In this paper we will only consider ultra-local density-density interactions in real space, i.e. \(U(\Omega_{ab}) = g_0 \frac{1}{\Omega} \delta(\Omega_{ab}) + g_1 \frac{1}{\Omega^2} \nabla^2 \delta(\Omega_{ab})\), and the associated Haldane pseudopotentials involve \(V_0, V_1\) (see Appendix Sec. A). Next we will set \(V_1 = 1\) as energy unit and vary \(V_0, \hbar\) to study the phase diagram.

We consider the half-filling case with the LLL filled by \(N = 2s + 1\) electrons. When \(h = 0\) and \(V_0, V_1 > 0\), the ground state is an Ising ferromagnet that spontaneously breaks \(Z_2\) symmetry. In quantum Hall literature this phase is called quantum Hall ferromagnetism \([43, 44]\). The two-fold degenerate ground states are \(|\Psi^+\rangle = \prod_{m = -s}^s c_{m_1}^\dagger |0\rangle\) and \(|\Psi^-\rangle = \prod_{m = -s}^s c_{m_4}^\dagger |0\rangle\). When \(h > V_0, V_1\), the ground state is a trivial paramagnet that preserves Ising symmetry, \(|\Psi_\psi\rangle = \prod_{m = -s}^s (c_{m_1}^\dagger + c_{m_4}^\dagger) |0\rangle\). Therefore, we expect a \(2 + 1D\) Ising transition as increasing \(h\). The global phase diagram of the model is as shown in Fig. 2(b).

### B. Symmetries and order parameter

The Hamiltonian (14) has three symmetries,

1. Ising \(Z_2\) symmetry: \(c_m \rightarrow \sigma^z c_m\).
2. \(SO(3)\) sphere rotation symmetry: \(c_{m = -s, \ldots, s}\) form the spin-s representation of \(SO(3)\).
3. Particle-hole symmetry: \(c_m \rightarrow i\sigma^y c_m^\dagger, i \rightarrow -i\).

\( ^6\) The spin degree of freedom should be thought as a pseudospin as it does not couple to the Zeeman field of the magnetic monopole.

Electric charges of fermions are gapped in the entire phase diagram (see Appendix Sec. C1), while the Ising spins of fermions are the degrees of freedom that go through the phase transitions. Therefore, all the gapless degrees of freedom at the phase transition are charge-neutral. In particular, the order parameter of the transition is a particle-hole excitation of fermions,

\[
M = \sum_{m = -s}^s c_m^\dagger \sigma^z c_m.
\]

We emphasize an important point for the Landau level regularization of the Ising transition: the electrons are sitting on a fuzzy sphere due to the monopole, but the
Ising spins are sitting on a normal sphere (for any finite \( N = 2s + 1 \)) since they are charge neutral. This is the key difference between our Landau level regularization and the non-commutative field theory [45], namely the latter always has quantum fields defined on a fuzzy manifold as long as the physical volume is finite.

To further analyze the Ising transition in our system, we will relate the UV symmetries of our Landau level model to the IR symmetries of the 3D Ising CFT. It is obvious we can identify the Ising \( Z_2 \) and \( SO(3) \) sphere rotation between UV and IR. A slightly non-trivial symmetry is the particle-hole symmetry, which turns out to be the spacetime parity symmetry of 3D Ising CFT. To understand this relation, we can write an \( SO(3) \) vector,

\[
 n_{m=0, \pm 1}^x = \sum_{m_1 = -s}^s (-1)^{m_1} \left( \begin{array}{rr}
 s & 1 \\
 m_1 & -m_1 \\
 \end{array} \right) c_{m_1}^\dagger \sigma^x c_{m_1-m},
\]

and find it transforms as

\[
 \begin{pmatrix}
  n_{m=0}^x \\
  n_{m=1}^x \\
  n_{m=-1}^x
\end{pmatrix} \rightarrow
 \begin{pmatrix}
  0 & 0 & 1 \\
  0 & 1 & 0 \\
-1 & 0 & 0
\end{pmatrix}
 \begin{pmatrix}
  n_{m=0}^x \\
  n_{m=1}^x \\
  n_{m=-1}^x
\end{pmatrix},
\]

under particle-hole transformation. The particle-hole acts as an improper \( Z_2 \) of \( O(3) \), so it can be identified as the spacetime parity of the 3D Ising CFT.

C. Finite size scaling

The phase diagram in Fig. 2(b) is obtained by the conventional finite size scaling of the \( Z_2 \) order parameter \( M \) in Eq. (16). We have simulated \( N = 2s + 1 = 8, 10, \cdots, 24 \) using ED for smaller sizes (\( N \leq 16 \)) and DMRG for larger sizes \( N > 16 \) (the length scale in this 2 + 1D system is \( L_x = \sqrt{N} \)). At the phase transition point, the \( Z_2 \) order parameter should scale as \( \langle M^2 \rangle / N^{2-\Delta} \approx L_x^{-2\Delta} = N^{2-\Delta} \) [28], where \( \Delta \approx 0.5181489 \) is the scaling dimension of Ising order parameter [23, 24]. Fig. 3 (a) depicts \( \langle M^2 \rangle / N^{2-\Delta} \) with respect to the transverse field strength \( h \) of different \( N \) for \( V_0 = 4.75 \). All the curves nicely cross at \( h_c \approx 3.16 \), which we identify as the transition point. Similarly for other \( V_0 \) we have identified the critical \( h_c \) and obtained the phase diagram as shown in Fig. 2(b).

We have also computed the binder cumulant

\[
 U_4 = \frac{3}{2} \left( 1 - \frac{1}{3} \left( \frac{\langle M^4 \rangle}{\langle M^2 \rangle^2} \right) \right),
\]

where \( U_4 \) is a RG-invariant quantity, and \( U_4 = 1, 0 \) at the thermodynamic limit corresponds to the ordered phase and disordered phase, respectively. At the phase transition \( U_4 \) will be a universal quantity related to the four point correlator of the order parameter field \( \sigma \) of CFT [37]. Fig. 3 (b) shows \( U_4 \) with respect to the transverse field strength \( h \) for different \( N \) for \( V_0 = 4.75 \). Clearly, at small \( h \) the model is in the Ising ferromagnetic phase, while at large \( h \) the model is in the disordered phase. To estimate the value of binder ratio at the critical point \( U_4^c \), we perform a detailed crossing-point analysis (Appendix Sec. B). With the data on hand, the best estimate we can give is 0.28 \( \leq U_4^c \leq 0.40 \). It will be interesting to evaluate \( U_4 \) from conformal bootstrap and compare with...
our estimate. In practice, for small \( N \) (as we simulated numerically), finite-size effects are inevitable. One common source is from the couplings of irrelevant operators, which are typically present in microscopic models. Tuning along the critical line in the 2-dimensional parameter space \((V_0, h)\) shown in Fig. 2(b) generically modifies the coupling strength of irrelevant operators and therefore the magnitude of finite-size effects (while the relevant operators flow to the same fixed point). In the following section, we will present the data of the state-operator correspondence at a particular point \( V_0 = 4.75, h_c = 3.16 \), where we find the finite size effects are smallest (Appendix Sec. D).

### IV. STATE-OPERATOR CORRESPONDENCE

We now turn to the central results of our paper: the state-operator correspondence of the 3D Ising transition. As explained in Sec. II A, on \( S^3 \times \mathbb{R} \) the eigenstates of the quantum Hamiltonian are in one-to-one correspondence with the scaling operators of its corresponding CFT. In particular, the energy gaps of each state will be proportional to the scaling dimensions of the scaling operators [14]. Therefore, we explore energy spectra at the critical point by utilizing exact diagonalization and compare it with CFT predictions.

To match the Ising transition’s energy spectra with the 3D Ising CFT’s operator spectrum, we first need to rescale the energy spectrum with a non-universal (i.e. model- and size-dependent) numerical factor. The natural calibrator is the energy momentum tensor \( T_{\mu_1\mu_2} \), a conserved operator that any local CFT possesses. For any \( 3D \) CFT, \( T_{\mu_1\mu_2} \) will be a global symmetry singlet, Lorentz spin \( \ell = 2 \) operator with scaling dimension \( \Delta_T = 3 \). Our model has exact \( SO(3) \) Lorentz rotation, Ising \( Z_2 \), and spacetime parity symmetries, so every eigenstate has well-defined quantum numbers \((Z_2, P, \ell)\) of these three symmetries. The energy-momentum tensor will be the lowest state in the \((Z_2 = 1, P = 1, \ell = 2)\) sector. We rescale the full spectrum by setting the energy momentum tensor to exactly \( \Delta_T = 3 \), and then examine if the low-lying states form representations of 3D conformal symmetry up to a finite size correction.

We analyze the low-lying spectra according to the following steps,

1. For each \( Z_2 = \pm 1 \) sector, we find the lowest-lying energy state (regardless of \( \ell \) and \( P \)), and identify it as a primary state.

2. Based on the representation theory of the 3D conformal group as summarized in Eq. (1), (2), (3), we enumerate the descendant states of the identified primary state and examine if all of descendant states (up to \( \Delta = 7 \)) exist in our energy spectrum.

3. We remove the identified conformal multiplet (i.e. primary and its descendants) from the energy spectrum, and for the remaining states we repeat the step 1,2.

Remarkably, we found that the lowest-lying 70 eigenstates form representations of the 3D conformal symmetry up to a small finite size correction, with no extra or missing state. This is a direct and unambiguous demonstration of the emergent conformal symmetry of the 3D Ising transition.

After verifying the emergent conformal symmetry, we further compare our scaling dimensions of the identified primary operators with the numerical conformal bootstrap data [23, 24], and we find a good agreement for all of them. Table I lists all the primary operators we have identified with \( N = 16 \) ED data. We have found 12 parity-even primary operators besides the energy-momentum tensor, and all of them have less than a 1.6% discrepancy from the bootstrap data [23, 24]. In Appendix Sec. D we list concrete values of each conformal multiplet, as one can see the numerical accuracy is unexpectedly high, particularly given that it is from a small system size (\( N = 16 \) total spins): around 10 operators have relative numerical error around 3% ~ 5.5%, and the rest of them have relative numerical error smaller than 3%. Fig. 4 plots conformal multiplets of a few representative primary operators, which clearly illustrate the emergent conformal symmetry and agree well with numerical conformal bootstrap results.

A few remarks are in order. 1) We verify the emergent conformal symmetry of the 3D Ising transition by showing that the low-lying spectra of our model form representations of 3D conformal symmetry. This procedure does not rely on any input of previous knowledge such as numerical bootstrap data. 2) A spinning \((\ell > 0)\) parity-even (parity-odd) primary operator can have parity-odd (parity-even) descendant operators as written in Eq. (3). This nontrivial structure from the CFT’s algebra matches our ED spectrum. 3) The energy momentum tensor \( T_{\mu_1\mu_2} \) is a conserved operator, so it does not have any \( \ell < 2 \) descendant. This structure is clearly shown in our data. 4) All the parity-even primary operators that we found have been reported in the bootstrap study of mixed correlators \( \langle \sigma \sigma \sigma \rangle, \langle \epsilon \epsilon \epsilon \rangle, \langle \sigma \epsilon \epsilon \rangle \). The mixed-correlator bootstrap study is only capable of detecting operators in the \( \sigma \times \sigma, \epsilon \times \epsilon \) and \( \sigma \times \epsilon \) OPE, so it will

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7 For models on the non-conformal manifold such as \( T^2 \times \mathbb{R} \) or \( T^3 \), which Monte Carlo usually simulates, \( U_4 \) cannot be computed using the \( R^3 \) four-point correlator from conformal bootstrap.

8 We have targeted the lowest 100 eigenstates using ED without explicitly imposing the value of \( \ell \), and we only looked at states with \( \ell \leq 4 \) which roughly contains 70 states.

9 To recall, the UV particle-hole symmetry becomes the spacetime parity symmetry of the IR CFT.
TABLE I. Low-lying primary operators identified via state-operator correspondence on a fuzzy sphere with $N = 16$ electrons. The operators in the first and second row are $Z_2$ odd and even operators, respectively. We highlight that two new parity-odd primary operators $\sigma^{P-}$ and $\epsilon^{P-}$ are found. The conformal bootstrap data is from Ref. [24].

|       | $\sigma$ | $\sigma'$ | $\sigma_{\mu_1\mu_2}$ | $\sigma'_{\mu_1\mu_2}$ | $\sigma_{\mu_1\mu_2\mu_3\mu_4}$ | $\sigma^{P-}$ |
|-------|----------|-----------|-------------------------|--------------------------|----------------------------------|--------------|
| Bootstrap | 0.518    | 5.291     | 4.180                   | 6.987                    | 4.638                            | 6.113        |
| Fuzzy sphere | 0.524    | 5.303     | 4.214                   | 7.048                    | 4.609                            | 6.069        |

|       | $\epsilon$ | $\epsilon'$ | $\epsilon''$ | $T_{\mu\nu}$ | $T'_{\mu\nu}$ | $\epsilon_{\mu_1\mu_2\mu_3\mu_4}$ | $\epsilon'_{\mu_1\mu_2\mu_3\mu_4}$ | $\epsilon^{P-}$ |
|-------|------------|------------|--------------|--------------|--------------|----------------------------------|----------------|--------------|
| Bootstrap | 1.413     | 3.830      | 6.896        | 3            | 5.509       | 5.023                            | 6.421          | NA           |
| Fuzzy sphere | 1.414     | 3.838      | 6.908        | 3            | 5.583       | 5.103                            | 6.347          | 10.014       |

FIG. 4. Conformal multiplet of several low lying primary operators: scaling dimension $\Delta$ versus Lorentz spin $\ell$. We plot conformal bootstrap data with lines: lines in red are parity even, non-degenerate operators; lines in green are parity odd, non-degenerate operators; lines in black are parity even, two-fold degenerate operators. Symbols are our numerical data of parity even (red circle) and odd (green square) operators. The discrepancy is typically more significant for the larger $\Delta$.

miss $(Z_2 = 1, P = 1, \text{odd } \ell)$ primary operators (in addition to $P = -1$ primaries). Our approach should be able to detect operators in these quantum number sectors, including the candidate of virial current [4] 10, namely the lowest primary in the $(Z_2 = 1, P = 1, \ell = 1)$ sector. We have not observed any primary operators in the $(Z_2 = 1, P = 1, \text{odd } \ell)$ sector below $\Delta = 7$, and so this gives a lower bound for the virial current candidate, which is higher than the previous estimate [13].

5) We have identified two previously unknown (parity-odd) primary operators in the $(Z_2 = 1, P = -1, \ell = 0)$ and $(Z_2 = -1, P = -1, \ell = 0)$ sectors with $\Delta \approx 10.01$ and $\Delta \approx 11.19$, respectively. To access $P = -1$ primary operators in the bootstrap calculation, one has to bootstrap correlation functions of the spinning operator: for example, the energy momentum tensor. Such study has only been initiated in Ref. [46] but no $P = -1$ primary

10 Strictly speaking, virial current refers to an operator with scaling dimension $\Delta = 2$. If such an operator exists, one may have a theory that is scale-invariant but not conformal-invariant.
A guiding principle for the model design will be kinematical quantity $F$, and their scaling dimensions are related to the critical exponents $\eta$, $\nu$ and $\omega$ [28, 29].

V. SUMMARY AND DISCUSSION

We have designed an innovative scheme to numerically study the 3D Ising transition on the space-time geometry $S^2 \times \mathbb{R}$, and in our calculation we have found almost perfect state-operator correspondence of the 3D CFT, supporting the conjecture that the 3D Ising transition has emergent conformal symmetry. In detail, we considered the 3D Ising transition realized in a fermionic model defined on a fuzzy sphere, which we achieved by projecting spinful electrons into the lowest spherical Landau level where the spin degrees of freedom go through an order-disorder transition. We are able to identify 13 parity-even primary operators and 2 parity-odd primary operators, and around 60 descendant operators, in agreement with the predictions of underlying CFT within a high accuracy.

Our results have now offered a novel solution to the long-standing quest of simulating 3D CFTs on the sphere (more generally on the curved space), and even more remarkably, the finite size effect of our model is much smaller than the conventional approach (i.e. 3D classical Ising model) used to study 3D CFTs. Therefore, our results open a new avenue for studying 3D CFTs in a microscopic way. Thanks to the state-operator correspondence on $S^2 \times \mathbb{R}$, many universal quantities such as operator product expansion coefficients, four-point correlators, and thermal correlators of CFTs are ready to compute directly. These will lead to many insights of CFTs and may reveal a new connection between physics and mathematics.

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11 A CFT at finite temperature on boundary is dual to a black hole of quantum gravity in the bulk.

12 A guiding principle for the model design will be kinematical properties such as symmetries and anomalies. Here we start with spinful fermions, whose maximal spin symmetry is $SU(2)$. We further add interactions that break $SU(2)$ down to the anomaly-free Ising $Z_2$ symmetry. Similarly, for other transitions one can start with $N$-component fermions, and consider various interactions that break the maximal $SU(N)$ symmetry to the desired symmetries.
this work were initiated, and for the Galileo Galilei Institute and the organizers of the GGI workshop “Bootstrapping Nature: Non-perturbative Approaches to Critical Phenomena” for hospitality during the completion of this work.

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Appendix A: Haldane pseudopotential on spherical geometry

This section describes the expression for a general matrix element of a two-body scalar potential \( V(r) \) on spherical geometry by projecting onto the lowest Landau level \( n = 0 \) [39, 41]. The general second-quantization form of the Hamiltonian is,

\[
H = \sum_{\sigma, \sigma'} \sum_{m_1, m_2, m_3, m_4} c_{m_1, \sigma}^\dagger c_{m_2, \sigma}^\dagger c_{m_3, \sigma'} c_{m_4, \sigma'} \delta_{m_1 + m_2, m_3 + m_4} \langle m_1, \sigma; m_2, \sigma' | V | m_3, \sigma' ; m_4, \sigma \rangle
\]  

(A1)

where \( m \) is orbital momentum and \( \sigma = \uparrow, \downarrow \) is pseudospin index. Here we just take the interaction with the same pseudospin \( \sigma = \sigma' \) as an example. The matrix element is given by

\[
\langle m_1; m_2 | V | m_3; m_4 \rangle = \int dr_1 \int dr_2 \Phi^*_{m_1}(r_1) \Phi^*_{m_2}(r_2) V(r_1, r_2) \Phi_{m_3}(r_2) \Phi_{m_4}(r_1)
\]  

(A2)

where \( \Phi_m \) is monopole harmonics functions defined in Eq. 6. If we expand both the initial and final state vectors in the coupled angular momentum basis, we can rewrite the two-body matrix element in the following form:

\[
\langle m_1; m_2 | V | m_3; m_4 \rangle = \sum_{l, l'} \langle m_1; m_2 | l, m_1 + m_2 \rangle \langle l', m_3 + m_4 | m_3; m_4 \rangle \langle l, m_1 + m_2 | V | l', m_3 + m_4 \rangle
\]  

(A3)

where the coefficient \( \langle l, m_1 + m_2 | l', m_3 \rangle \) is the Clebsch-Gordan coefficient. \( \langle l, m | V | l', m' \rangle = V_l \delta_{l, l'} \) and the pair pseudopotential \( V_l \) describes the interaction energy of a pair of electrons as a function of their pair angular momentum \( l \) [39].

To perform calculations in this paper, we used the second-quantization form of Hamiltonian Eq. A1 with the interaction elements shown in Eq. A3. Since we only focus on the short-ranged interactions in real-space (e.g. \( \delta(\Omega_{ab}), \nabla^2 \delta(\Omega_{ab}) \)), the choice of pseudopotential are limited to \( V_0, V_1 \). The explicit relation between the pseudopotentials and the two-body interaction potential in real-space will be also presented below.

1. Connection of the pseudopotential with real-space interactions

If the potential \( V \) only depends on \(|r_1 - r_2|\) on the spherical geometry, it can be expanded in Legendre polynomials,

\[
V(|r_1 - r_2|) = \sum_{k=0}^{\infty} U_k P_k(\cos \theta_{12})
\]  

(A4)

and the real-space interaction can be rewritten in terms of a new set of parameters \( U_k \):

\[
U_k = \frac{1}{2} \int_0^\pi d\theta V(|r_1 - r_2|) P_k(\cos \theta) \sin \theta.
\]  

(A5)

If we insert the potential form Eq. A4 into the matrix element, we have

\[
\langle m_1; m_2 | V | m_3; m_4 \rangle = \int d\Omega_1 \int d\Omega_2 \Phi^*_{m_1}(\Omega_1) \Phi^*_{m_2}(\Omega_2) \sum_k U_k \frac{1}{2k + 1} \sum_{m = -k}^{k} \Phi_{m_3}(\Omega_1) \Phi_{m_4}(\Omega_2) \Phi_{m_3}(\Omega_1)
\]  

\[
= \sum_k U_k \frac{1}{2k + 1} \sum_{m = -k}^{k} \int d\Omega_1 \Phi^*_{m_1}(\Omega_1) Y_{km}(\Omega_1) \int d\Omega_2 \Phi^*_{m_2}(\Omega_2) Y_{km}(\Omega_2) \Phi_{m_3}(\Omega_2)
\]  

\[
= \sum_k U_k (-)^{6s + m_2 + m_3} (2s + 1)^2 \left( \begin{array}{ccc} s & k & s \\ -m_1 & m_1 - m_3 & m_3 \end{array} \right) \left( \begin{array}{ccc} s & k & s \\ -m_2 & m_2 - m_4 & m_4 \end{array} \right) \left( \begin{array}{ccc} s & k & s \\ -s & 0 & s \end{array} \right)^2.
\]  

(A6)

Here, for a general Wigner 3j coefficient, \( \left( \begin{array}{ccc} s_1 & s_2 & s_3 \\ m_1 & m_2 & m_3 \end{array} \right) \), it is non-zero only when \( m_1 + m_2 + m_3 = 0 \) and when \( s_1, s_2, s_3 \) together satisfy the triangle inequality, \( |s_1 - s_2| \leq s_3 \leq s_1 + s_2 \).
Through this matrix element form, one can obtain that the pseudopotential of particles in the lowest Landau level is connected with parameter $U_k$ via

$$V_{2s-l} = \sum_{k=0}^{2s} U_k (-)^{2s+l}(2s+1)^2 \left\{ \begin{array}{ccc} l & s & s \\ k & s & s \end{array} \right\} \left( \begin{array}{ccc} s & k & s \\ -s & 0 & s \end{array} \right)^2,$$

(A7)

where $\left\{ \begin{array}{ccc} l & s & s \\ k & s & s \end{array} \right\}$ is Wigner 6j coefficient.

In this paper, we only consider the short-ranged potentials:

- For short-ranged potential $U(\Omega_{ab}) = \delta(\Omega_{ab})$, by using the expansion

$$\delta(\Omega_a - \Omega_b) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{l,m}^*(\Omega_a)Y_{l,m}(\Omega_b) = \sum_{l=0}^{\infty} (2l+1)P_l(\cos \theta_{ab}),$$

(A8)

we have $U_l = 2l+1$. With the help of Eq. A7, we get the pseudopotentials related to the short-ranged potential $U(\Omega_{ab}) = \delta(\Omega_{ab})$ as

$$V_{2s-l} = \begin{cases} \frac{(2s+1)^2}{14s+1}, & l = 2s \\ 0, & l \neq 2s \end{cases}.$$

(A9)

- For short-ranged potential $U(\Omega_{ab}) = \nabla^2 \delta(\Omega_{ab})$, by using the expansion

$$\nabla^2 \delta(\Omega_a - \Omega_b) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \nabla^2 Y_{l,m}^*(\Omega_a)Y_{l,m}(\Omega_b) = \sum_{l=0}^{\infty} (-l(l+1))(2l+1)P_l(\cos \theta_{ab}),$$

(A10)

we have $U_l = -(l(l+1))(2l+1)$. With the help of Eq. A7, we get the pseudopotentials related to the short-ranged potential $U(\Omega_{ab}) = \nabla^2 \delta(\Omega_{ab})$:

$$V_{2s-l} = \begin{cases} \frac{-s(2s+1)^2}{4s+1}, & l = 2s \\ \frac{s(2s+1)^2}{4s-1}, & l = 2s - 1 \\ 0, & l < 2s \end{cases}.$$

(A11)

In a word, for a general two-body interaction potential on the spherical geometry $U(\Omega_{ab}) = g_0 \delta(\Omega_{ab}) + g_1 \nabla^2 \delta(\Omega_{ab})$, one can use Eq. A9 and A11 to connect with the pseudopotentials $V_l$ as defined in Eq. A3.

### Appendix B: Physical observables across the phase transition

In this section, we provide more detailed analysis on the finite size scaling of physical observable $M^2$ and binder ratio $U_4$. In Fig. 5, order parameter $\langle M^2 \rangle$ is almost unchanged near the critical point $h \approx h_c$, which signals the phase transition point. In comparison, we notice that, as $N$ increases the crossing point of $U_4^c$ is less converged, which is not as perfect as the crossing of order parameter. In the finite-size scaling, the crossing value of the cumulant itself approaches its thermodynamic limit $U_4^c \approx 0.2849 \pm 0.0063$. And we also estimate the upper bound by the lowest value that we get in the DMRG calculation. In a word, with the data up to $N = 36$ the best estimate we can give is $U_4^c \sim (0.28, 0.40)$.

The larger uncertainty of binder ratio is likely due to that the $U_4$ suffers a much larger finite size effect, since at the phase transition $U_4$ is related to the four point correlator of the order parameter field $\sigma$ in CFT. Similar finite size effect has also been observed in Monte Carlo simulations of 3D classical or $2 + 1D$ quantum Ising transitions with much larger system size.

Additionally, we shall note that for the same universality defined or realized on distinct manifolds, $U_4$ will be generically different even in the thermodynamic limit. In principle $U_4$ on the conformal manifold (e.g. $R^3$, $S^2 \times R$) can be computed using the $R^3$ four-point correlator.\footnote{An approximate $R^3$ four-point correlator can be reconstructed using the data from conformal bootstrap.} For $3D$ Ising transition on the non-conformal manifold such as $T^2 \times R$ or $T^3$, which Monte Carlo usually simulates, $U_4$ cannot computed using the $R^3$ four-point correlator.
FIG. 5. A finite-size scaling analysis of the Binder cumulant $U^c_4$ at the phase transition. Each data point is determined by the crossing point on system size pair $(N, 2N)$. The analysis is according to the scaling form $U^c_4(N) = aN^{-\omega/2} + b$. Inset: Example of finite-size crossing point $U^c_4(N)$ with $N = 12$ and $2N = 24$.

FIG. 6. Finite size scaling of charge excitation gap at the phase transition. The charge gap is defined as $\Delta_c = E_0(N_e + 1, N) + E_0(N_e - 1, N) - 2E_0(N_e, N)$, and $E_0(N_e, N)$ is the ground state energy by filling $N_e$ electrons.

Appendix C: Excitation gap

1. Charge gap

In the discussion of quantum magnetism in electron systems, one preliminary question is if or not the charge excitation gap vanishes. Here we define the charge gap as $\Delta_c(N) = E_0(N_e + 1, N) + E_0(N_e - 1, N) - 2E_0(N_e, N)$, where $E_0(N_e, N)$ is the ground state energy on $N$ LLL orbitals by filling $N_e$ electrons. After obtaining the charge gap on each system size, we perform a finite-size scaling to estimate the charge gap in the thermodynamic limit. As shown in Fig. 6, the charge gap at the critical point $h = h_c$ is nonzero on all system sizes, and the value in the thermodynamic limit is also finite. Thus, we conclude that the low-energy excitation is dominated by the spin excitation other than the charge excitation.
took around 30 minutes on a M1 Macbook. The calculations can be done on the laptop, where the IR CFT emerges in incredibly small system sizes. The best illustration is the observation that our model with the largest system size that we can reach using ED. Here, to further elucidate that the numerical findings indeed reflect finite size scaling and off-critical behavior will be interesting for the future work.

Another interesting point is that, we identify almost perfect state-operator correspondence in surprisingly small system sizes. In the main text, we only present the numerical data at a given system size, i.e. \( N = 16 \), which is the largest system size that we can reach using ED. Here, to further elucidate that the numerical findings indeed reflect the physics in the thermodynamic limit, we show the energy spectra on different system sizes. In Fig. 8, we show the energy spectra obtained on different system sizes from the physics in the thermodynamic limit, we show the energy spectra on different system sizes. In Fig. 7, we show three typical plots of excitation gap in Ising ferromagnet phase, paramagnet phase and at the phase transition point. It is clear that, the excitation gaps are finite for ferromagnet and paramagnet phase, but the system becomes gapless at the transition point \( h \approx h_c \). The most interesting thing is, these critical excitations at finite system sizes form a characteristic conformal tower structure as discussed in the main text, which calls for a CFT description of 3D Ising criticality.

Appendix D: Details of numerical data

In this section, we present the data of energy spectra which are organized by the good quantum numbers and conformal multiplet of various primary fields, e.g. \( \epsilon \) (Tab. II), \( \epsilon' \) (Tab. III), \( T_{\mu_1\mu_2} \) (Tab. IV), \( T'_{\mu_1\mu_2} \) (Tab. V), \( \epsilon_{\mu_1,\mu_2,\mu_3,\mu_4} \) (Tab. VI), \( \sigma \) (Tab. VII), \( \sigma' \) (Tab. VIII), \( \sigma_{\mu_1\mu_2} \) (Tab. IX), and \( \sigma'_{\mu_1\mu_2\mu_3} \) (Tab. X). For comparison, we also list the results from conformal bootstrap (CB) method [23, 24]. These data are used for plotting Fig. 4. For the primary fields, the discrepancies are really small (\(< 1.6\%)\). Generally, fields with higher energies (conformal weights) have larger discrepancies, which is attributed to the finite-size effect. We add that these detailed data also gives a good quantification for the numerical error without inputting other results such as numerical bootstrap. The idea is that, since the conformal symmetry predicts the integer spacings between primaries and their descendants, we can examine how good is this preserved in our spectrum. Based on this we can give a conservative estimate for numerical errors of primaries and low lying descendants, which are 3% relative errors. A rigorous error analysis based on the finite size scaling and off-critical behavior will be interesting for the future work.

Another interesting point is that, we identify almost perfect state-operator correspondence in surprisingly small system sizes. In the main text, we only present the numerical data at a given system size, i.e. \( N = 16 \), which is the largest system size that we can reach using ED. Here, to further elucidate that the numerical findings indeed reflect the physics in the thermodynamic limit, we show the energy spectra on different system sizes. In Fig. 8, we show the energy spectra obtained on different system sizes from \( N = 8 \) to \( N = 16 \). As one can see that, the energies on all system sizes match the prediction of 3D CFT quite well.

Finally, as we discussed in the main text, one of the most surprising aspects of the fuzzy sphere scheme is that the IR CFT emerges in incredibly small system sizes. The best illustration is the observation that our model with only \( N = 4 \) spins (electrons) (Tab. XI) already produces 6 primaries and the approximate conformal invariance. All calculations can be done on the laptop, where the \( N = 16 \) spins requires around 16G memory, and the computation took around 30 minutes on a M1 Macbook.
FIG. 8. The energy spectra corresponding to primary fields for various system sizes $N = 8 - 16$. The dashed color lines denote the numerical values from conformal bootstrap method.

| Operator     | Quantum Number | CB data     | $N = 16$ | Errors     |
|--------------|----------------|-------------|----------|------------|
| $\epsilon$  | $\ell = 0$    | 1.412625(10)| 1.41355766| 0.066%     |
| $\partial_\mu \epsilon$ | $\ell = 1$    | 2.412625(10)| 2.40776449| 0.201%     |
| $\partial_\mu_1 \partial_\mu_2 \epsilon$ | $\ell = 2$    | 3.412625(10)| 3.41455749| 0.057%     |
| $\Box \epsilon$ | $\ell = 0$    | 3.412625(10)| 3.47303235| 1.770%     |
| $\partial_\mu_1 \partial_\mu_2 \partial_\mu_3 \epsilon$ | $\ell = 3$    | 4.412625(10)| 4.38113022| 0.714%     |
| $\Box^2 \epsilon$ | $\ell = 0$    | 4.412625(10)| 4.55437869| 3.212%     |
| $\partial_\mu_1 \partial_\mu_2 \partial_\mu_3 \Box \epsilon$ | $\ell = 3$    | 5.412625(10)| 5.5514904 | 3.227%     |
| $\partial_\mu_1 \Box^2 \epsilon$ | $\ell = 1$    | 5.412625(10)| 6.412625(10)| 6.66423677| 3.924%     |

| Operator     | Quantum Number | CB data     | $N = 16$ | Errors     |
|--------------|----------------|-------------|----------|------------|
| $\epsilon'$ | $\ell = 0$    | 3.829682(23)| 3.83772859| 0.210%     |
| $\partial_\mu \epsilon'$ | $\ell = 1$    | 4.829682(23)| 4.83973617| 0.208%     |
| $\partial_\mu_1 \partial_\mu_2 \epsilon'$ | $\ell = 2$    | 5.829682(23)| 5.82918219| 0.009%     |
| $\Box \epsilon'$ | $\ell = 0$    | 5.829682(23)| 5.9605325 | 2.245%     |
| $\partial_\mu_1 \partial_\mu_2 \partial_\mu_3 \epsilon'$ | $\ell = 3$    | 6.829682(23)| 6.76617638| 0.930%     |
| $\partial_\mu_1 \Box^2 \epsilon'$ | $\ell = 1$    | 6.829682(23)| 7.05458433| 3.293%     |
TABLE IV. Conformal multiplet of $T_{\mu_1 \mu_2}$.

| Operator         | Quantum Number | Exact value | $N = 16$ | Errors   |
|------------------|----------------|-------------|----------|----------|
| $T_{\mu_1 \mu_2}$ | $\ell = 2$    | 3           | 3        | 0.000%   |
| $\partial_{\nu} T_{\mu_1 \mu_2}$ | $\ell = 3$    | 4           | 4.03219819 | 0.805%   |
| $\epsilon_{\mu_2 \nu \rho} \partial_{\rho} T_{\mu_1 \mu_2}$ | $\ell = 2$, $P = -1$ | 4           | 4.07392075 | 1.848%   |
| $\partial_{\nu} \partial_{\mu_2} T_{\mu_1 \mu_2}$ | $\ell = 4$    | 5           | 4.96734107 | 0.653%   |
| $\epsilon_{\mu_2 \nu \rho} \partial_{\rho} \partial_{\nu} T_{\mu_1 \mu_2}$ | $\ell = 3$, $P = -1$ | 5           | 5.14602926 | 2.921%   |
| $\Box T_{\mu_1 \mu_2}$ | $\ell = 2$    | 5           | 5.17292963 | 3.459%   |
| $\partial_{\mu_1} \Box T_{\mu_1 \mu_2}$ | $\ell = 3$    | 6           | 6.04586808 | 0.764%   |
| $\epsilon_{\mu_2 \nu \rho} \partial_{\rho} \partial_{\nu} \partial_{\mu_2} T_{\mu_1 \mu_2}$ | $\ell = 4$, $P = -1$ | 6           | 6.06221026 | 1.037%   |
| $\epsilon_{\mu_2 \nu \rho} \partial_{\rho} \Box T_{\mu_1 \mu_2}$ | $\ell = 2$, $P = -1$ | 6           | 6.29074558 | 4.846%   |

TABLE V. Conformal multiplet of $T'_{\mu_1 \mu_2}$.

| Operator         | Quantum Number | CB data      | $N = 16$ | Errors   |
|------------------|----------------|--------------|----------|----------|
| $T'_{\mu_1 \mu_2}$ | $\ell = 2$    | 5.50915(44)  | 5.5827144 | 1.335%   |
| $\partial_{\nu} T'_{\mu_1 \mu_2}$ | $\ell = 3$    | 6.50915(44)  | 6.57137975 | 0.956%   |
| $\epsilon_{\mu_2 \nu \rho} \partial_{\rho} T'_{\mu_1 \mu_2}$ | $\ell = 2$, $P = -1$, $\sigma$ | 6.50915(44)  | 6.57557892 | 1.020%   |
| $\partial_{\mu_1} T'_{\mu_1 \mu_2}$ | $\ell = 1$    | 6.50915(44)  | 6.74639599 | 3.645%   |

TABLE VI. Conformal multiplet of $\epsilon_{\mu_1 \mu_2 \mu_3 \mu_4}$.

| Operator         | Quantum Number | CB data      | $N = 16$ | Errors   |
|------------------|----------------|--------------|----------|----------|
| $\epsilon_{\mu_1 \mu_2 \mu_3 \mu_4}$ | $\ell = 4$    | 5.022665(28) | 5.1029942 | 1.599%   |
| $\epsilon_{\mu_4 \nu \rho} \partial_{\rho} \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4}$ | $\ell = 4$, $P = -1$, $\sigma$ | 6.022665(28) | 6.17684603 | 2.560%   |
| $\partial_{\mu_1} \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4}$ | $\ell = 3$    | 6.022665(28) | 6.19439341 | 2.851%   |

TABLE VII. Conformal multiplet of $\sigma$.

| Operator         | Quantum number | CB data      | $N = 16$ | Errors   |
|------------------|----------------|--------------|----------|----------|
| $\sigma$         | $\ell = 0$    | 0.5181489(10) | 0.52428857 | 1.185%   |
| $\partial_{\nu} \sigma$ | $\ell = 1$    | 1.5181489(10) | 1.50941793 | 0.575%   |
| $\Box \sigma$    | $\ell = 0$    | 2.5181489(10) | 2.51722181 | 0.037%   |
| $\partial_{\nu} \partial_{\mu_2} \sigma$ | $\ell = 2$    | 2.5181489(10) | 2.55937503 | 1.637%   |
| $\Box \partial_{\mu_1} \sigma$ | $\ell = 1$    | 3.5181489(10) | 3.50635346 | 0.335%   |
| $\partial_{\mu_1} \partial_{\mu_2} \partial_{\mu_3} \sigma$ | $\ell = 3$    | 3.5181489(10) | 3.6059226 | 2.495%   |
| $\Box \partial_{\mu_1} \partial_{\mu_2} \sigma$ | $\ell = 2$    | 4.5181489(10) | 4.47002281 | 1.065%   |
| $\Box \sigma$    | $\ell = 0$    | 4.5181489(10) | 4.57231367 | 1.199%   |
| $\partial_{\mu_1} \partial_{\mu_2} \partial_{\mu_3} \partial_{\mu_4} \sigma$ | $\ell = 4$    | 4.5181489(10) | 4.52727499 | 0.202%   |
| $\partial_{\mu_1} \partial_{\mu_2} \partial_{\mu_3} \Box \sigma$ | $\ell = 3$    | 5.5181489(10) | 5.36761913 | 2.728%   |
| $\partial_{\mu_1} \partial_{\mu_2} \Box \sigma$ | $\ell = 1$    | 5.5181489(10) | 5.60563429 | 1.585%   |
| $\partial_{\mu_1} \partial_{\mu_2} \partial_{\mu_3} \partial_{\mu_4} \Box \sigma$ | $\ell = 4$    | 6.5181489(10) | 6.24268467 | 4.226%   |
| $\partial_{\mu_1} \partial_{\mu_2} \Box \sigma$ | $\ell = 2$    | 6.5181489(10) | 6.58905267 | 1.088%   |
| $\Box \sigma$    | $\ell = 0$    | 6.5181489(10) | 6.74334514 | 3.455%   |
TABLE VIII. Conformal multiplet of $\sigma'$.

| Operator | Quantum number | CB data | $N = 16$ | Errors |
|----------|----------------|---------|----------|--------|
| $\sigma'$ | $\ell = 0$    | 5.2906(11) | 5.30346641 | 0.243% |
| $\partial_{\mu_1}\sigma'$ | $\ell = 1$    | 6.2906(11) | 6.27713785 | 0.214% |

TABLE IX. Conformal multiplet of $\sigma_{\mu_1\mu_2}$.

| Operator | Quantum number | CB data | $N = 16$ | Errors |
|----------|----------------|---------|----------|--------|
| $\sigma_{\mu_1\mu_2}$ | $\ell = 2$    | 4.180305(18) | 4.21382989 | 0.802% |
| $\partial_{\nu_1}\sigma_{\mu_1\mu_2}$ | $\ell = 3$    | 5.180305(18) | 5.23649044 | 1.085% |
| $\partial_{\mu_1}\sigma_{\mu_1\mu_2}$ | $\ell = 1$    | 5.180305(18) | 5.31578984 | 2.615% |
| $\varepsilon_{\mu_2\nu_1}\partial_{\nu_1}\sigma_{\mu_1\mu_2}$ | $\ell = 2$, $P = -1$ | 5.180305(18) | 5.25315317 | 1.426% |
| $\partial_{\nu_1}\partial_{\nu_2}\sigma_{\mu_1\mu_2}$ | $\ell = 4$    | 6.180305(18) | 6.18724938 | 0.112% |
| $\varepsilon_{\mu_2\nu_1}\partial_{\mu_1}\sigma_{\mu_1\mu_2}$ | $\ell = 3$, $P = -1$ | 6.180305(18) | 6.26160085 | 1.315% |
| $\partial_{\nu_1}\sigma_{\mu_1\mu_2}$ | $\ell = 2$    | 6.180305(18) | 6.29114975 | 1.794% |
| $\Box\sigma_{\mu_1\mu_2}$ | $\ell = 2$    | 6.180305(18) | 6.39595149 | 3.489% |
| $\varepsilon_{\mu_2\nu_1}\partial_{\nu_1}\sigma_{\mu_1\mu_2}$ | $\ell = 1$, $P = -1$ | 6.180305(18) | 6.42999132 | 4.040% |
| $\partial_{\nu_1}\partial_{\nu_2}\sigma_{\mu_1\mu_2}$ | $\ell = 0$    | 6.180305(18) | 6.52321841 | 5.548% |

TABLE X. Conformal multiplet of $\sigma_{\mu_1\mu_2\mu_3}$.

| Operator | Quantum number | CB data | $N = 16$ | Errors |
|----------|----------------|---------|----------|--------|
| $\sigma_{\mu_1\mu_2\mu_3}$ | $\ell = 3$    | 4.63804(88) | 4.60892045 | 0.628% |
| $\partial_{\nu_1}\sigma_{\mu_1\mu_2\mu_3}$ | $\ell = 4$    | 5.63804(88) | 5.56345584 | 1.323% |
| $\varepsilon_{\mu_3\nu_1\nu_2}\partial_{\nu_1}\sigma_{\mu_1\mu_2\mu_3}$ | $\ell = 3$, $P = -1$ | 5.63804(88) | 5.67044594 | 0.575% |
| $\partial_{\nu_1}\sigma_{\mu_1\mu_2\mu_3}$ | $\ell = 2$    | 5.63804(88) | 5.79746571 | 2.828% |
| $\Box\sigma_{\mu_1\mu_2\mu_3}$ | $\ell = 3$    | 6.63804(88) | 6.74065848 | 1.546% |
| $\partial_{\nu_1}\partial_{\nu_1}\sigma_{\mu_1\mu_2\mu_3}$ | $\ell = 3$    | 6.63804(88) | 6.88182226 | 3.672% |
| $\varepsilon_{\mu_3\nu_1\nu_2}\partial_{\nu_1}\sigma_{\mu_1\mu_2\mu_3}$ | $\ell = 4$, $P = -1$ | 6.63804(88) | 6.57417625 | 0.962% |
| $\varepsilon_{\mu_3\nu_1\nu_2}\partial_{\nu_1}\sigma_{\mu_1\mu_2\mu_3}$ | $\ell = 2$, $P = -1$ | 6.63804(88) | 6.93332726 | 4.418% |
| $\partial_{\nu_1}\partial_{\nu_2}\sigma_{\mu_1\mu_2\mu_3}$ | $\ell = 1$    | 6.63804(88) | 6.94900991 | 4.685% |

TABLE XI. The rescaled energy gaps of all states of the fuzzy sphere model with $N = 4$ spins (electrons). All the states seem to be the Ising CFT states with a small finite size corrections for most of them.

| Operator | Quantum number | CB data | $N = 16$ | Errors |
|----------|----------------|---------|----------|--------|
| $\sigma$ | $\partial_{\mu_1}\sigma$ | $\Box\sigma$ | $\partial_{\mu_1}\partial_{\mu_2}\sigma$ | $\partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3}\sigma$ | $\partial_{\mu_1}\Box\sigma$ | $\sigma_{\mu_1\mu_2}$ | $\sigma_{\mu_1\mu_2\mu_3}$ |
| Bootstrap | 0.518 | 1.518 | 2.518 | 2.518 | 3.518 | 3.518 | 4.180 | 4.638 |
| $N = 4$ | 0.530 | 1.522 | 2.427 | 2.428 | 2.847 | 3.291 | 4.241 | 4.618 |
| Errors | 2.3% | 0.3% | 3.6% | 3.6% | 20% | 6.5% | 1.5% | 0.4% |

| Operator | Quantum number | CB data | $N = 16$ | Errors |
|----------|----------------|---------|----------|--------|
| $\epsilon$ | $\partial_{\mu_1}\epsilon$ | $T_{\mu_1\mu_2}$ | $\partial_{\mu_1}\partial_{\mu_2}\epsilon$ | $\Box\epsilon$ | $\epsilon'$ | $\partial_{\mu_1}T_{\mu_1\mu_2}$ | $\epsilon_{\mu_2\nu_1}\partial_{\mu_1}\sigma_{\mu_1\mu_2}$ | $\partial_{\mu_3}\partial_{\mu_4}T_{\mu_1\mu_2}$ |
| Bootstrap | 1.413 | 2.413 | 3 | 3.413 | 3.413 | 3.830 | 4 | 4 | 5 |
| $N = 4$ | 1.382 | 2.337 | 3 | 3.126 | 3.577 | 4.019 | 3.663 | 4.054 | 4.856 |
| Errors | 2.2% | 3.1% | NA | 8.4% | 4.8% | 4.9% | 8.4% | 1.4% | 2.9% |