Tropical complete intersection curves

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Abstract

A tropical complete intersection curve \( C \subseteq \mathbb{R}^{n+1} \) is a transversal intersection of \( n \) smooth tropical hypersurfaces. We give a formula for the number of vertices of \( C \) given by the degrees of the tropical hypersurfaces. We also compute the genus of \( C \) (defined as the number of independent cycles of \( C \)) when \( C \) is smooth and connected.

1 Notation and definitions

We work over the tropical semifield \( \mathbb{R}_{tr} = (\mathbb{R}, \oplus, \circ) = (\mathbb{R}, \max, +) \). A tropical (Laurent) polynomial in variables \( x_1, \ldots, x_m \) is an expression of the form

\[
   f = \bigoplus_{a = (a_1, \ldots, a_m) \in A} \lambda_a x_1^{a_1} \cdots x_m^{a_m} = \max_{a \in A} \{ \lambda_a + a_1 x_1 + \cdots + a_m x_m \},
\]

where the support set \( A \) is a finite subset of \( \mathbb{Z}^m \), and the coefficients \( \lambda_a \) are real numbers. (In the middle expression of (1), all products and powers are tropical.) The convex hull of \( A \) in \( \mathbb{R}^m \) is called the Newton polytope of \( f \), denoted \( \Delta_f \).

Any tropical polynomial \( f \) induces a regular lattice subdivision of \( \Delta_f \) in the following way: With \( f \) as in (1), let the lifted Newton polytope \( \tilde{\Delta}_f \) be the polyhedron defined as

\[
   \tilde{\Delta}_f := \text{conv}(\{(a, t) \mid a \in A, t \leq \lambda_a\}) \subseteq \Delta_f \times \mathbb{R} \subseteq \mathbb{R}^m \times \mathbb{R}
\]

Furthermore, we define the top complex \( T_f \) to be the complex whose maximal cells are the bounded facets of \( \tilde{\Delta}_f \). Projecting the cells of \( T_f \) to \( \mathbb{R}^m \) by deleting the last coordinate gives a collection of lattice polytopes contained in \( \Delta_f \), forming a regular subdivision of \( \Delta_f \). We denote this subdivision by \( \text{Subdiv}(f) \).

The standard volume form on \( \mathbb{R}^m \) is denoted by \( \text{vol}_m(\cdot) \), or simply \( \text{vol}(\cdot) \) if the space is clear from the context.

1.1 Tropical hypersurfaces

Note that any tropical polynomial \( f(x_1, \ldots, x_m) \) is a convex, piecewise linear function \( f : \mathbb{R}^m \to \mathbb{R} \).
Definition 1.1. Let \( f : \mathbb{R}^m \to \mathbb{R} \) be a tropical polynomial. The tropical hypersurface \( V_{tr}(f) \) associated to \( f \) is the non-linear locus of \( f \).

It is well known that for any tropical polynomial \( f \), \( V_{tr}(f) \) is a finite connected polyhedral cell complex in \( \mathbb{R}^m \) of pure dimension \( m - 1 \), some of whose cells are unbounded. Furthermore, \( V_{tr}(f) \) is in a certain sense dual to \( \text{Subdiv}(f) \): There is a one-one correspondence between the \( k \)-cells of \( V_{tr}(f) \) and the \( (m - k) \)-cells of \( \text{Subdiv}(f) \). A cell \( C \) of \( V_{tr}(f) \) is unbounded if and only if its dual \( \overline{C} \in \text{Subdiv}(f) \) is contained in the boundary of \( \Delta_f \). (For proofs consult [5] and [6].)

Let \( m \in \mathbb{N} \), and let \( e_1, \ldots, e_m \) denote the standard basis of \( \mathbb{R}^m \). For any \( d \in \mathbb{N}_0 \), we define the simplex \( \Gamma_d^m := \text{conv}\{0, de_1, \ldots, de_m\} \subseteq \mathbb{R}^m \), where \( 0 \) denotes the origin of \( \mathbb{R}^m \). For example, \( \Gamma_3^2 \) is the triangle in \( \mathbb{R}^2 \) with vertices \((0, 0), (3, 0) \) and \((0, 3) \). Note that \( \text{vol}(\Gamma_d^m) = \frac{1}{m!}d^m \).

Definition 1.2. A tropical hypersurface \( X = V_{tr}(f) \subseteq \mathbb{R}^m \) is smooth if every maximal cell of \( \text{Subdiv}(f) \) is a simplex of volume \( \frac{1}{m!} \). If in addition we have \( \Delta_f = \Gamma_d^m \) for some \( d \in \mathbb{N} \), we say that \( X \) is smooth of degree \( d \).

1.2 Minkowski sums and mixed subdivisions

The set \( \mathcal{K}^m \) of all convex sets in \( \mathbb{R}^m \) has a natural structure of a semiring, as follows: If \( K_1 \) and \( K_2 \) are convex sets, we define binary operators \( \oplus \) and \( \odot \) by

\[
\begin{align*}
K_1 \oplus K_2 &:= \text{conv}(K_1 \cup K_2) \\
K_1 \odot K_2 &:= K_1 + K_2.
\end{align*}
\]

The operator + in \( \text{(3)} \) is the Minkowski sum, defined for any two subsets \( A, B \subseteq \mathbb{R}^m \) by \( A + B := \{a + b \mid a \in A, b \in B\} \). The Minkowski sum of two convex sets are again convex, so \( \text{(3)} \) is well defined. Furthermore, it is easy to see that \( \odot \) distributes over \( \oplus \), and it follows that \( \mathcal{K}^m \) is indeed a semiring.

Lemma 1.3. Let \( \mathbb{R}_{tr}[x_1, \ldots, x_m] \) be the semiring of tropical polynomials in \( n \) variables. The map \( \mathbb{R}_{tr}[x_1, \ldots, x_m] \to \mathcal{K}^{m+1} \) defined by \( f \mapsto \Delta_f \), is a homomorphism of semirings.

Proof. This is a straightforward exercise. The key ingredients are the identities

\[
\text{conv}(A \cup B) = \text{conv}(\text{conv}(A) \cup \text{conv}(B)) \quad \text{and} \quad \text{conv}(A + B) = \text{conv}(A) + \text{conv}(B),
\]

which hold for any (not necessarily convex) subsets \( A, B \subseteq \mathbb{R}^m \).

Let \( f_1, \ldots, f_n \) be tropical polynomials, and set \( f := f_1 \odot \cdots \odot f_n \). As a consequence of Lemma 1.3, we find that \( \text{Subdiv}(f) \) is the subdivision of \( \Delta_f = \Delta_{f_1} + \cdots + \Delta_{f_n} \subseteq \mathbb{R}^m \times \mathbb{R} \) to \( \mathbb{R}^m \) by deleting the last coordinate.
For any cell $\Lambda \in \text{Subdiv}(f)$, the lifted cell $\tilde{\Lambda} \in \mathcal{T}_f$ can be written uniquely as a Minkowski sum $\tilde{\Lambda} = \tilde{\Lambda}_1 + \cdots + \tilde{\Lambda}_n$, where $\tilde{\Lambda}_i \in \mathcal{T}_{f_i}$ for each $i$. Projecting each term to $\mathbb{R}^m$ gives a representation of $\Lambda$ as a Minkowski sum $\Lambda = \Lambda_1 + \cdots + \Lambda_n$. The subdivision $\text{Subdiv}(f)$, together with the associated Minkowski sum representation of each cell, is called the regular mixed subdivision of $\Delta_f$ induced by $f_1, \ldots, f_n$.

Remark 1.4. Note that the representation of $\Lambda$ as a Minkowski sum of cells of the $\text{Subdiv}(f_i)$’s is not unique in general. Following [1], we call the representation obtained from the lifted Newton polytopes as described above, the privileged representation of $\Lambda$.

Definition 1.5. The mixed cells of the mixed subdivision are the cells with privileged representation $\Lambda = \Lambda_1 + \cdots + \Lambda_n$, where $\dim \Lambda_i \geq 1$ for all $i = 1, \ldots, n$.

2 Intersections of tropical hypersurfaces

In this section we go through some basic properties and definitions regarding unions and intersections of tropical hypersurfaces. Most of the material here also appear in the recent article [1].

We begin by observing that any union of tropical hypersurfaces is itself a tropical hypersurface. This follows by inductive use of the following lemma:

Lemma 2.1. If $X$ and $Y$ are tropical hypersurfaces in $\mathbb{R}^m$, and $f, g$ are tropical polynomials such that $X = V_{tr}(f)$ and $Y = V_{tr}(g)$, then $X \cup Y = V_{tr}(f \circ g)$.

Proof. By definition, $V_{tr}(f \circ g)$ is the non-linear locus of the function $f \circ g = f + g$. Since $f$ and $g$ are both convex and piecewise linear, this is exactly the union of the non-linear loci of $f$ and $g$ respectively. \qed

Remark 2.2. Let $U = X_1 \cup \cdots \cup X_n$, where $X_i = V_{tr}(g_i) \subseteq \mathbb{R}^m$ is a tropical hypersurface for each $i$. We denote by $\text{Subdiv}_U$ the mixed subdivision of $\Delta_{g_1} + \cdots + \Delta_{g_n}$ induced by $g_1, \ldots, g_n$. It follows from Lemma 2.1 and the discussion in Section 1.2 that $\text{Subdiv}_U$ is dual to $U$ in the sense explained in Section 1.1.

Moving on to intersections, we will only consider smooth hypersurfaces. Let $I$ be the intersection of smooth tropical hypersurfaces $X_1, \ldots, X_n \subseteq \mathbb{R}^m$, where $n \leq m$. As a first observation, notice that $I$ is a polyhedral complex, since the $X_i$’s are. The intersection is proper if $\dim(I) = m - n$.

Let $C$ be a non-empty cell of $I$. Then $C$ can be written uniquely as $C = C_1 \cap \cdots \cap C_n$, where for each $i$, $C_i$ is a cell of $X_i$ containing $C$ in its relative interior. (The relative interior of a point must here be taken to be the point itself.)

Regarding $C$ as a cell of the union $U = X_1 \cup \cdots \cup X_n$, we consider the dual cell $C^\vee \in \text{Subdiv}_U$ (cf. Remark 2.2). From Section 1.2 we know that $C^\vee$ has a privileged representation as a Minkowski sum of cells of the subdivisions dual to the $X_i$’s. It is not hard to see that this representation is precisely $C^\vee = C_1^\vee + \cdots + C_n^\vee$. In particular, since $\dim C_i \leq m - 1$, and therefore $\dim C_i^\vee \geq 1$, for each $i$, $C^\vee$ is a mixed cell of $\text{Subdiv}_U$. 
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Figure 1: Tropical planes intersecting in a tropical line.

Figure 2: A proper intersection which is not transversal.

**Definition 2.3.** With the notation as above, the intersection $X_1 \cap \cdots \cap X_n$ is *transversal along $C$* if

\[(4) \quad \dim C^\vee = \dim C_1^\vee + \cdots + \dim C_n^\vee.\]

More generally, the intersection $X_1 \cap \cdots \cap X_n$ is said to be *transversal* if for any subset $J \subseteq \{1, \ldots, n\}$ (of size at least two), the intersection $\bigcap_{i \in J} X_i$ is proper and transversal along each cell.

**Remark 2.4.** Definition 2.3 implies that if smooth tropical hypersurfaces $X_1, \ldots, X_n$ intersect transversely, then $\text{Subdiv}_U$ is a *tight coherent mixed subdivision* (see e.g. [7]).

Recall from standard theory that the $k$-*skeleton* $X^{(k)}$ of a polyhedral complex $X$, is the subcomplex consisting of all cells of dimension less or equal to $k$. It is not hard to see from Definition 2.3 that if $X$ and $Y$ are tropical hypersurfaces intersecting transversely in $\mathbb{R}^n$, then

\[(5) \quad X^{(j)} \cap Y^{(k)} = \emptyset\]

for all nonnegative integers $j, k$ such that $j + k < n$. More generally, we find that:

**Lemma 2.5.** Suppose $X_1, \ldots, X_n$ intersect transversally, and let $I_J = \bigcap_{i \in J} X_i$, where $J$ is a subset of $\{1, 2, \ldots, n\}$. For each $s \notin J$ we have

$$I_J^{(j)} \cap X_s^{(k)} = \emptyset,$$

for all $j, k$ such that $j + k < n$.

**Example 2.6.** Figure 1 shows a *tropical line* in $\mathbb{R}^3$ as the transversal intersection of two tropical planes (i.e., tropical hypersurfaces of degree 1).

**Example 2.7.** Figure 2 shows an intersection in $\mathbb{R}^3$ which is proper, but not transversal. The surfaces are $X = V_{tr}(0x \oplus 0y \oplus 0)$ and $Y = V_{tr}(0xy \oplus 0z \oplus 0xyz)$. (Since the “spines” meet in a point, the intersection is not transversal.)
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2.1 Intersection multiplicities

Let $X_1, \ldots, X_n \subseteq \mathbb{R}^m$ be smooth tropical hypersurfaces such that the intersection $I = X_1 \cap \cdots \cap X_n$ is transversal. Let $U = X_1 \cup \cdots \cup X_n$ and denote by $\text{Subdiv}_U$ the mixed subdivision associated to $U$. In [1, Definition 4.3], a general formula is given for the intersection multiplicity at each cell of $I$. For our purposes, two special cases suffice. If $P \in I^{(0)}$, let $P^\vee$ be the associated dual cell in $\text{Subdiv}_U$.

**Definition 2.8.** Suppose $n = m$, so $I$ consists of finitely many points. The intersection multiplicity at a point $P \in I$ is defined by $m_P = \text{vol}(P^\vee)$.

**Remark 2.9.** This generalizes the standard definition of intersection multiplicities of tropical plane curves.

**Definition 2.10.** Suppose $n = m - 1$, so $I$ is one-dimensional. The intersection multiplicity at a vertex $P \in I$ is defined by $m_P = 2 \text{vol}(P^\vee)$.

**Remark 2.11.** It follows from the definition of transversality that $P^\vee$ has a privileged representation of the form $P^\vee = \Lambda_1 + \cdots + \Lambda_{n-1} + \Delta$, where each $\Lambda_i$ is a primitive lattice interval, and $\Delta$ is a primitive lattice triangle. It follows from this that $\text{vol}(P^\vee)$ is always a positive multiple of $\frac{1}{2}$.

2.2 Tropical versions of Bernstein’s Theorem and Bezout’s Theorem

Given polytopes $\Delta_1, \ldots, \Delta_m$ in $\mathbb{R}^m$, we consider the map $\gamma : (\mathbb{R}_{\geq 0})^m \to \mathbb{R}$ defined by $(\lambda_1, \ldots, \lambda_m) \mapsto \text{vol}(\lambda_1\Delta_1 + \cdots + \lambda_m\Delta_m)$. One can show that $\gamma$ is given by a homogeneous polynomial in $\lambda_1, \ldots, \lambda_m$ of degree $m$. We define the *mixed volume* of $\Delta_1, \ldots, \Delta_m$ to be the coefficient of $\lambda_1\lambda_2 \cdots \lambda_m$ in the polynomial expression for $\gamma$. The following tropical version of Bernstein’s Theorem is proved in [1, Corollary 4.7]:

**Theorem 2.12.** Suppose tropical hypersurfaces $X_1, \ldots, X_m \subseteq \mathbb{R}^m$ with Newton polytopes $\Delta_1, \ldots, \Delta_m$ intersect in finitely many points. Then the total number of intersection points counted with multiplicities is equal to the mixed volume of $\Delta_1, \ldots, \Delta_m$.

As a special case of this we get a tropical version of Bezout’s Theorem:

**Corollary 2.13.** Suppose the tropical hypersurfaces $X_1, \ldots, X_m \subseteq \mathbb{R}^m$ have degrees $d_1, \ldots, d_m$, and intersect in finitely many points. Then the number of intersection points counting multiplicities is $d_1 \cdots d_m$.

**Proof.** By Theorem 2.12, the number of intersection points, counting multiplicities, is the coefficient of $\lambda_1\lambda_2 \cdots \lambda_m$ in

$$\text{vol}(\lambda_1\Gamma_{d_1}^m + \cdots + \lambda_m\Gamma_{d_m}^m) = \text{vol}(\Gamma_{\lambda_1 d_1 + \cdots + \lambda_m d_m}^m) = \frac{1}{m!}(\lambda_1 d_1 + \cdots + \lambda_m d_m)^m.$$ 

By the multinomial theorem, the wanted coefficient is $d_1 \cdots d_m$, as claimed. \qed
3 Tropical complete intersection curves

A tropical complete intersection curve $C$ is a transversal intersection of $n$ smooth tropical hypersurfaces $X_1, \ldots, X_n \subseteq \mathbb{R}^{n+1}$, for some $n \geq 2$. It is a one-dimensional polyhedral complex, some of whose edges are unbounded. We say that $C$ is smooth if the intersection multiplicity is 1 at each vertex (cf. Definition 2.10).

Recall that any cell $C$ of $C$ is also a cell of the tropical hypersurface $U = X_1 \cup \cdots \cup X_n$. In particular, the notation $C^\vee$ always refers to the cell of $\text{Subdiv}_U$ dual to $C \subseteq U$.

Lemma 3.1. Each vertex of $C$ has valence 3.

Proof. If $P$ is a vertex of $C$, then by Remark 2.11, $P^\vee$ has a privileged representation $P^\vee = \Lambda_1 + \cdots + \Lambda_{n-1} + \Delta$, where each $\Lambda_i$ is a primitive interval, and $\Delta$ is a primitive lattice triangle. If $E$ is any edge of $C$ adjacent to $P$, then $E^\vee$ must be a mixed cell of $\text{Subdiv}_U$ which is also a facet of $P^\vee$. This means that $E^\vee = \Lambda_1 + \cdots + \Lambda_{n-1} + \Delta'$, where $\Delta'$ is a side of $\Delta$. Hence there are exactly 3 such adjacent edges - one for each side of $\Delta$. \hfill \qed

Our first goal is to calculate the number of vertices of $C$. Before stating the general formula, let us discuss the easiest case as a warm up example:

3.1 Example: Complete intersections in $\mathbb{R}^3$

Let $C = X \cap Y \subseteq \mathbb{R}^3$ be a tropical complete intersection curve, where $X = V_{\text{tr}}(f)$ and $Y = V_{\text{tr}}(g)$ are smooth tropical surfaces of degrees $d$ and $e$ respectively.

Theorem 3.2. The number of vertices of $C$, counting multiplicities, is $de(d+e)$.

Proof. The idea is to look at all the vertices of the union $X \cup Y$, and their dual polytopes in the subdivision corresponding to $X \cup Y$. Since the intersection of $X$ and $Y$ is transversal, we can write the set of vertices of $X \cup Y$ as a disjoint union,

\[ (X \cup Y)^{(0)} = X^{(0)} \sqcup Y^{(0)} \sqcup (X \cap Y)^{(0)}. \]

Now, any element $P \in (X \cup Y)^{(0)}$ corresponds to a maximal cell $P^\vee$ in $\text{Subdiv}(f \circ g)$. The privileged representation of $P^\vee$ is of one of the following forms:

- $P^\vee = (3\text{-cell of Subdiv}(f)) + (0\text{-cell of Subdiv}(g)) \implies P \in X^{(0)}.$
- $P^\vee = (0\text{-cell of Subdiv}(f)) + (3\text{-cell of Subdiv}(g)) \implies P \in Y^{(0)}.$
- $P^\vee = (2\text{-cell of Subdiv}(f)) + (1\text{-cell of Subdiv}(g))$ or $P^\vee = (1\text{-cell of Subdiv}(f)) + (2\text{-cell of Subdiv}(g)) \implies P \in (X \cap Y)^{(0)}.$

Hence, dualizing (6) and taking volumes, we get the relation

\[ \sum_{P \in (X \cup Y)^{(0)}} \text{vol}(P^\vee) = \sum_{P \in X^{(0)}} \text{vol}(P^\vee) + \sum_{P \in Y^{(0)}} \text{vol}(P^\vee) + \sum_{P \in (X \cap Y)^{(0)}} \text{vol}(P^\vee). \]
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Now, if \( P \in (X \cap Y)^{(0)} \), the volume of \( P^\vee \) is \( \frac{1}{2}m_P \) (by definition of intersection multiplicity). Hence, (7) gives

\[
\text{vol}(\Delta_{f \odot g}) = \text{vol}(\Delta_f) + \text{vol}(\Delta_g) + \sum_{P \in (X \cap Y)^{(0)}} \frac{1}{2}m_P.
\]

Since \( \Delta_f = \Gamma^3_{d_1}, \Delta_g = \Gamma^3_{e_r} \), and \( \Delta_{f \odot g} = \Gamma^3_{d_1} + \Gamma^3_{e_r} = \Gamma^3_{d+e} \), we find that

\[
\sum_{P \in (X \cap Y)^{(0)}} m_P = 2\left[\frac{(d+e)^3}{6} - \frac{d^3}{6} - \frac{e^3}{6}\right] = de(d + e).
\]

\[\square\]

3.2 The number of vertices in the general case

In this section we prove the following generalization of Theorem 3.2:

**Theorem 3.3.** Let \( C = X_1 \cap \cdots \cap X_n \) be a tropical complete intersection curve in \( \mathbb{R}^{n+1} \), where \( X_1, \ldots, X_n \) are smooth of degrees \( d_1, \ldots, d_n \). The number of vertices of \( C \), counting multiplicities, is

\[
\sum_{P \in C^{(0)}} m_P = d_1d_2\cdots d_n(d_1 + d_2 + \cdots + d_n).
\]

To prove Theorem 3.3 we will use the same setup as in the previous section. Note that in the proof of the case \( n = 3 \), the relation (8) is the key giving us control over \( (X \cap Y)^{(0)} \). So as an auxiliary lemma, we first state and prove a generalization of this.

To simplify the writing, we introduce the following notation: Let \([n] = \{1, 2, \ldots, n\}\). For any nonempty subset \( J = \{j_1, \ldots, j_k\} \subseteq [n] \), we put

\[
U_J := X_{j_1} \cup \cdots \cup X_{j_k},
\]

\[
I_J := X_{j_1} \cap \cdots \cap X_{j_k}.
\]

In the special case \( J = [n] \), we simply write \( U \) and \( I \), i.e. \( U := U_{[n]} \) and \( I = C = I_{[n]} \).

By the assumption of transversality, we have \( I_{j}^{(0)} \cap I_{k}^{(0)} = \emptyset \) whenever \( J, K \subseteq [n] \) are distinct nonempty subsets. Thus we can split the 0-cells of \( U = X_1 \cup \cdots \cup X_n \) into a disjoint union:

\[
U^{(0)} = \bigsqcup_{J \subseteq [n]} I_J^{(0)}.
\]

Similarly, for any nonempty subset \( J \subseteq [n] \), we get

\[
U_J^{(0)} = \bigsqcup_{J' \subseteq J} I_{J'}^{(0)}.
\]
Lemma 3.4. For a transversal intersection of tropical hypersurfaces $X_1, \ldots, X_n$, we have:

\[ I^{(0)} \sqcup \bigcup_{|J|=n-1} U_J^{(0)} \sqcup \bigcup_{|J|=n-3} U_J^{(0)} \sqcup \cdots = U^{(0)} \sqcup \bigcup_{|J|=n-2} U_J^{(0)} \sqcup \bigcup_{|J|=n-4} U_J^{(0)} \sqcup \cdots. \tag{10} \]

Proof. By applying (9) to every set $U_J^{(0)}$ in (10), we see that the following expression is equivalent to (10):

\[ I^{(0)} \sqcup \bigcup_{|J|=n-1} I_{J'}^{(0)} \sqcup \bigcup_{|J|=n-3} I_{J'}^{(0)} \sqcup \cdots = \bigcup_{|J'\subseteq J} I_{J'}^{(0)} \sqcup \bigcup_{|J'|\subseteq J} I_{J'}^{(0)} \sqcup \bigcup_{|J'|\subseteq J} I_{J'}^{(0)} \sqcup \cdots. \tag{11} \]

We claim that for each fixed subset $J' \subseteq [n]$, the set $I_{J'}^{(0)}$ appears equally many times on each side of (11). By inspection, this is true for $J' = [n]$. Assume now $|J'| = k < n$. Then for any integer $s$ with $k \leq s \leq n$, there are exactly \( \binom{n-k}{s-k} \) sets $J \subseteq [n]$ containing $J'$ such that $|J| = s$. Hence, the number of times $I_{J'}^{(0)}$ appears on the left side of (11) is \( \left( \binom{n-k}{s-k} \right) + \left( \binom{n-k}{s-3-k} \right) + \cdots = \binom{n-k}{1} + \binom{n-k}{2} + \cdots = 2^{n-k-1} \), while the number of appearances on the right side is \( \binom{n-k}{0} + \binom{n-k}{2} + \cdots = 2^{n-k-1} \). This proves the claim, and the lemma follows. \( \square \)

Proof of Theorem 2.3. Suppose $C$ and $X_1, \ldots, X_n$ are as in the statement of the theorem. We assume that for each $i$, $X_i$ has degree $d_i$, so the associated Newton polytope is the simplex $\Gamma_+^{n+1}$. Let $U$ denote the union $X_1 \cup \cdots \cup X_n$, and Subdiv$_U$ the associated subdivision of $\Gamma_+^{n+1}$.

For each nonempty $J = \{j_1, \ldots, j_k\} \subseteq [n]$, let $U_J$ and $I_J$ be as in (8). In particular, $U_J$ is a tropical hypersurface (set-theoretically contained in $U$) with an associated subdivision Subdiv$_{U_J}$ of the simplex $\Delta_J := \Gamma_+^{d_{j_1} + \cdots + d_{j_k}}$.

Each vertex of $U_J$ is also a vertex of $U$, and therefore corresponds to a maximal cell of Subdiv$_U$. Let $S_J$ be the set of maximal cells of Subdiv$_U$ corresponding to the vertices of $U_J$. By transversality, the elements of $S_J$ are simply translations of the maximal cells of Subdiv$_{U_J}$. Hence the total volume of the cells of $S_J$, denoted $\text{vol}(S_J)$, is

\[ \text{vol}(S_J) = \sum_{P \in U_J^{(0)}} \text{vol}(P^\vee) = \text{vol}(\Delta_J) = \frac{1}{(n+1)!} (d_{j_1} + \cdots + d_{j_k})^{n+1}. \]

Now we turn to Lemma 3.3. Dualizing (10), we find that

\[ \sum_{P \in I^{(0)}} \text{vol}(P^\vee) + \sum_{|J|=n-1} \text{vol}(S_J) + \cdots = \text{vol}(S) + \sum_{|J|=n-2} \text{vol}(S_J) + \cdots. \tag{12} \]

By the definition of intersection multiplicity, the dual $P^\vee \in \text{Subdiv}_U$ of a vertex $P \in I^{(0)}$ has volume $\frac{1}{2} m_P$. It follows that

\[ \sum_{P \in I^{(0)}} \frac{1}{2} m_P = \frac{1}{(n+1)!} \sum_{\{j_1, \ldots, j_k\} \subseteq [n]} (-1)^{n-k} (d_{j_1} + \cdots + d_{j_k})^{n+1}, \]
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which after some elementary manipulation reduces to
\[ \sum_{P \in I(0)} m_P = d_1 d_2 \cdots d_n (d_1 + d_2 + \cdots + d_n). \]

3.3 The genus of tropical complete intersection curves

Definition 3.5. The \textit{genus} \( g = g(C) \) of a tropical complete intersection curve \( C \) is the first Betti number of \( C \), i.e., the number of independent cycles of \( C \).

Lemma 3.6. For a connected tropical complete intersection curve \( C \), we have
\[ 2g(C) - 2 = v - x, \]
where \( v \) is the number of vertices, and \( x \) the numbers unbounded edges of \( C \).

For the proof, recall that a graph is called \textit{3-valent} if every vertex has 3 adjacent edges. Furthermore, we apply the following terminology: A one-dimensional polyhedral complex in \( \mathbb{R}^m \) with unbounded edges is regarded as a graph, where the 1-valent vertices have been removed. For example, a tropical line in \( \mathbb{R}^3 \) is considered a 3-valent graph with 2 vertices and 5 edges.

Proof. By Lemma 3.1, \( C \) is 3-valent. Since \( C \) is connected, it has a spanning tree \( T \), such that \( C \setminus T \) consists of \( g \) edges. While \( T \) is not 3-valent, we can construct a 3-valent tree \( T' \) from \( T \) by adding unbounded edges wherever necessary. Clearly, we must add exactly \( 2g \) such edges. Thus if \( C \) has \( v \) vertices and \( e \) edges, \( T' \) has \( v \) vertices and \( e + g \) edges. Since \( T' \) is 3-valent, it is easy to see (for example by induction) that the number of edges is one more that twice the number of vertices, i.e.,
\[ e + g - 1 = 2v. \]

On the other hand, since \( C \) is 3-valent, we must have \( e = \frac{1}{2}(3v + x) \). Combining this with (13) gives the wanted result. \( \square \)

Lemma 3.7. Let \( C \) be the transversal intersection of \( X_1, \ldots, X_n \subseteq \mathbb{R}^{n+1} \), where each \( X_i = V_{tr}(f_i) \) is a smooth tropical hypersurface of degree \( d_i \). If \( C \) is smooth, the number of unbounded edges of \( C \) is \( x = (n + 2)d_1 \cdots d_n \).

Proof. Let \( U = X_1 \cup \cdots \cup X_n \), and let \text{Subdiv}_{U} \) be the associated subdivision of the simplex \( \Gamma := \Delta^{n+1}_{d_1 + \cdots + d_n} \). The unbounded edges of \( C \) are then in one-one correspondence with the mixed \( n \)-cells of \text{Subdiv}_{U} contained in the boundary of \( \Gamma \). To prove the lemma, it therefore suffices to show that there are exactly \( d_1 \cdots d_n \) mixed \( n \)-cells in each of the \( n + 2 \) facets of \( \Gamma \). By symmetry it is enough to consider the facet \( \Gamma' \) with \( e_1 = (1, 0, \ldots, 0) \) as an inner normal vector. In the following we identify \( \mathbb{R}^n \) with the hyperplane in \( \mathbb{R}^{n+1} \) orthogonal to \( e_1 \).
For each $i = 1, \ldots, n$ let $S_i$ be the subdivision induced by $\text{Subdiv}(f_i)$ on the facet of $\Gamma_i^{n+1}$ with $e_i$ as an inner normal vector. We can then regard $S_i$ as the subdivision associated to the tropical hypersurface $X_i' := V_{tr}(f_i) \subseteq \mathbb{R}^n$, where $f_i'$ is the tropical polynomial obtained from $f_i$ by removing all terms containing $x_1$. Furthermore, $X_i'$ is homeomorphic to the intersection $X_i \cap H$, where $H$ is any (classical) hyperplane with equation $x_1 = k$ and $k << 0$. Note that $\deg X_i' = \deg X_i = d_i$.

Let $S$ be the subdivision of $\Gamma'$ induced by $\text{Subdiv}_U$. As above, we regard $S$ as the subdivision associated to the union $X_1' \cup \cdots \cup X_n' \subseteq \mathbb{R}^n$. Thus, the (finitely many) points in the intersection $I := X_1' \cap \cdots \cap X_n'$ are precisely the duals of the mixed $n$-cells of $S$. We know from Theorem 2.13 that the number of points in $I$ is $d_1 \cdots d_n$ when counting with intersection multiplicities; in other words (by Definition 2.8) we have $\sum_{Q \in I} vol_n(Q^\vee) = d_1 \cdots d_n$.

All that remains is to show that if $Q \in I$, then $vol_n(Q^\vee) = 1$. This is where smoothness of $C$ comes in: Regarding $Q^\vee$ as an $n$-cell in $\text{Subdiv}_U$, let $P$ be the vertex of $C$ such that $Q^\vee$ is a facet of $P^\vee \in \text{Subdiv}_U$. Writing (as in Remark 2.11) $P^\vee = \Lambda_1 + \cdots + \Lambda_{n-1} + \Delta$, where the $\Lambda_i$’s are primitive intervals and $\Delta$ a primitive triangle, we must have $Q^\vee = \Lambda_1 + \cdots + \Lambda_{n-1} + \Delta'$, where $\Delta'$ is a side in $\Delta$. Since $vol_{n+1}(P^\vee) = \frac{1}{2}$ (by smoothness), it follows from this that $vol_n(Q^\vee) = 1$. 

**Theorem 3.8.** Let $C$ be the transversal intersection of $n$ smooth tropical hypersurfaces in $\mathbb{R}^{n+1}$ of degrees $d_1, \ldots, d_n$. If $C$ is smooth and connected, the genus $g$ of $C$ is given by

$$2g - 2 = d_1 \cdots d_n(d_1 + \cdots + d_n - (n + 2)).$$

**Proof.** Since $C$ is smooth, it has exactly $v = d_1 d_2 \cdots d_n(d_1 + d_2 + \cdots + d_n)$ vertices (by Theorem 3.3) and $x = (n + 2)d_1 \cdots d_n$ unbounded edges (by Lemma 3.7). Combined with Lemma 3.6 this proves the theorem. 

**Remark 3.9.** In complex projective space it is well known that any complete intersection curve is connected. This follows from standard cohomological arguments (see also [1, Section 3.4.6] for a direct geometric argument due to Serre). In the tropical setting, it is known that any transversal intersection of tropical hyperplanes is a tropical variety, i.e., the tropicalization of an algebraic variety defined over the field of Puiseux series ([2, Section 3, and Lemma 1.2 for the relation to Puiseux series]). Furthermore, if a tropical variety is the tropicalization of an irreducible variety, then it is connected ([3, Theorem 2.2.7]). This suggests that - at least in the general case - a tropical complete intersection curve is connected. However, to the author’s knowledge, this has not been proved.

**Remark 3.10.** The formula (14) coincides with the genus formula for a smooth complete intersection in $\mathbb{P}^{n+1}_C$ of $n$ hypersurfaces of degrees $d_1, \ldots, d_n$.

4 **Example: Tropical elliptic curves in $\mathbb{R}^3$**

By a tropical quadric surface in $\mathbb{R}^3$, we mean a smooth tropical hypersurface of degree 2. In this section we take a closer look at intersections of tropical quadric surfaces in
**EXAMPLE: TROPICAL ELLIPTIC CURVES IN \( \mathbb{R}^3 \)**

\( \mathbb{R}^3 \), i.e., smooth tropical hypersurfaces in \( \mathbb{R}^3 \) of degree 2. Figure 4 shows a typical tropical quadric surface.

Let \( C \) be a smooth, connected complete intersection curve of two tropical quadric surfaces in \( \mathbb{R}^3 \). We call \( C \) a tropical elliptic curve. The name is justified by Theorem 3.8 which tells us the the genus \( g \) of \( C \) satisfies \( 2g - 2 = 2 \cdot 2 \cdot (2 + 2 - 4) \), that is, \( g = 1 \). In particular, \( C \) contains a unique cycle.

Since \( C \) is smooth, it has exactly \( 2 \cdot 2 \cdot (2 + 2) = 16 \) vertices, by Theorem 3.3. We divide these into two categories: Those on the cycle (called *internal vertices*), and the rest (*external vertices*). Clearly, \( C \) has at least 3 internal vertices. But what is the maximum number of internal vertices? As the following example shows, all 16 vertices can be internal:

**Example 4.1.** Let \( Q_1 = V_{tr}(f) \) and \( Q_2 = V_{tr}(g) \), where

\[
\begin{align*}
f(x, y, z) &= (-6) \oplus 13x \oplus (-3)y \oplus (-4)z \oplus 10x^2 \oplus 2xy \oplus 4xz \oplus (-9)y^2 \oplus 5yz \oplus (-9)z^2, \\
g(x, y, z) &= (-15) \oplus (-10)x \oplus (-4)y \oplus 2z \oplus (-7)x^2 \oplus (-2)xy \\
&\quad \quad \oplus 0xz \oplus 2y^2 \oplus 15yz \oplus (-1)z^2.
\end{align*}
\]

Figure 3: A tropical quadric surface in \( \mathbb{R}^3 \).

Figure 4: The subdivision \( \text{Subdiv}(f) \).

Figure 5: The subdivision \( \text{Subdiv}(g) \).
Figure 6: The intersection \( C = Q_1 \cap Q_2 \).

Figure 7: \( C \) has 16 internal vertices.

Figures 4 and 5 show the subdivisions of \( \Gamma_3^2 \) induced by \( f \) and \( g \) respectively.

The intersection curve \( C = Q_1 \cap Q_2 \) has genus 1 and 16 internal vertices. Figure 6 shows the two quadrics intersecting. In Figure 7 we see the intersection curve alone from a different angle, clearly showing the cycle with all its 16 vertices.

**Remark 4.2.** A computer search shows that for every integer \( m \), with \( 3 \leq m \leq 16 \), there exist two tropical quadric surfaces in \( \mathbb{R}^3 \) intersecting transversally in a tropical elliptic curve with \( m \) internal vertices.

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**References**

[1] B. Bertrand and F. Bihan. Euler characteristic of real nondegenerate tropical complete intersections. Preprint, [http://arxiv:math.AG/0710.1222](http://arxiv:math.AG/0710.1222), 2007.

[2] T. Bogart, A. N. Jensen, D. Speyer, B. Sturmfels, and R. R. Thomas. Computing tropical varieties. *J. Symbolic Comput.*, 42(1-2):54–73, 2007.

[3] M. Einsiedler, M. Kapranov, and D. Lind. Non-Archimedean amoebas and tropical varieties. *J. Reine Angew. Math.*, 601:139–157, 2006.

[4] R. Hartshorne. Complete intersections and connectedness. *Amer. J. Math.*, 84:497–508, 1962.

[5] G. Mikhalkin. Enumerative tropical algebraic geometry in \( \mathbb{R}^2 \). *J. Amer. Math. Soc.*, 18(2):313–377 (electronic), 2005.
REFERENCES

[6] J. Richter-Gebert, B. Sturmfels, and T. Theobald. First steps in tropical geometry. In Idempotent mathematics and mathematical physics, volume 377 of Contemp. Math., pages 289–317. Amer. Math. Soc., Providence, RI, 2005.

[7] B. Sturmfels. Viro’s theorem for complete intersections. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 21(3):377–386, 1994.