A Generalized Nyquist Stability Criteria for Heterogeneous Networks

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Abstract—A decentralized stability criteria is derived for a network of heterogeneous agents. We define a closed convex hull, spanned by the heterogeneous agents, wherein the characteristic loci must be contained. The method preserves information about phase and gain. This allows us to guarantee stability using the generalized Nyquist criteria, with a minimum amount of conservatism. The decentralized stability criteria does not require any information about the network topology or of other agents. But, with knowledge about the eigenvalues of the network Laplacian, the proposed method allows us to be less conservative than passivity based methods. This allows us to include agents or subsystems with nonminimum phase actuators.

I. INTRODUCTION

Typically when considering frequency control in a power system, we only care about the average frequency (or center of inertia) mode. That is, we ignore the stability of higher order dynamics, such as poorly damped inter-area modes. With an increasing share of inverter based renewable generation the inertia is decreasing. Consequently, faster frequency control is required. Because of this, the assumption that inter-area modes can be ignored is no longer a good assumption. Especially if we are providing frequency control with nonminimum phase (NMP) power sources, such as hydropower. With limited knowledge of the network, conservative assumptions are necessary in order to guarantee stability. This work presents a novel method, readily allowing for various degrees of conservatism.

First we will present a stability guarantee for a network of homogeneous plants. In a homogeneous network, each plant must stabilize all the modes of the system. In general, this is fairly conservative as we actually can achieve stability even if a plant has a destabilizing effect on one of the modes. This can happen if we have heterogeneous plants, and if the destabilizing plant is small, or if it has poor observability/controllability of the mode compared to other plants.

If we do not have any information about the network, we need to make some conservative assumptions in order to guarantee stability. Assuming identical or proportional subsystems is one such conservative assumption. Other methods, allowing for truly heterogeneous plants, guarantee stability using passivity assumptions or by using the small gain theorem [1]. Here, we will present a less conservative stability criterion using the generalized Nyquist criteria [2]. Appendix A. The results are in line with [3] but our approach differs. The novelty in this work is mainly in the transparency in which the stability criteria is formulated. The presented method preserves information about gain and phase, and allows us to conveniently track the NMP zeros of each plant. Since stability is guaranteed by directly applying the Nyquist criteria, we allow for NMP and time-delayed plants. In general, we also allow for unstable plants. But unstable plants cannot be allowed in a truly decentralized system where we do not have information about all the other plants. However, typically the plants can (and should) be stabilized locally, prior to connecting to the network.

II. STABILITY IN A HETEROGENEOUS NETWORK

Consider the linear time-invariant system

$$s \begin{bmatrix} \delta \\ \mathcal{N} \omega \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\mathcal{L} & -\mathcal{F}(s) \end{bmatrix} \begin{bmatrix} \delta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} d.$$ \hspace{1cm} (1)

For instance describing a power system of $n$ synchronous machines connected over a loss-less undirected network characterized by the Laplacian matrix $\mathcal{L} \in \mathbb{R}^{n \times n}$, where $\delta = [\delta_1, \ldots, \delta_n]^T$ are the rotor phase angles and $\omega = [\omega_1, \ldots, \omega_n]^T$ are the rotor frequency deviations from the linearization point. The input $d = [d_1, \ldots, d_n]^T$ is the difference between local active power loads and generation set-points. The diagonal matrices $\mathcal{M} = \text{diag}(M_1, \ldots, M_n)$ represent the inertia constants of each machine, $\mathcal{F}(s) = \text{diag} [F_1(s), \ldots, F_n(s)]$ represents frequency dependent loads, governors etc., while $0$ and $I$ are appropriately sized zero and identity matrices, respectively. Let $0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_n$ denote the eigenvalues of $\mathcal{L}$. An eigenvalue decomposition of the Laplacian $\mathcal{L}$ is given by

$$\Lambda = \text{diag}(0, \lambda_2, \ldots, \lambda_n) = \nu^T \mathcal{L} \nu \hspace{1cm} (2)$$

where $\nu \in \mathbb{R}^{n \times 1}$ is a unitary matrix of eigenvectors $\nu = [v_1, \ldots, v_n]$ so that $\nu^T \nu = I$. The zero eigenvalue $\lambda_1 = 0$ corresponds to the average frequency mode of the system. The eigenvector $v_1 = 1/\sqrt{n}$, where $1 \in \mathbb{R}^n$ is a vector of ones.

A. Synchronization in a Homogeneous Network

Consider a set of classical machines

$$\delta_i = \frac{1}{s} \frac{1}{sM_i + F_i(s)} d_i, \quad i \in \{1, \ldots, n\}$$

connected through the output, $y_i = \delta_i$, over a loss-less network described by the Laplacian matrix $\mathcal{L}$ as in (1). With constant frequency dependent load/production, $F_i(s) = D_i > 0$, asymptotic synchronization on the average frequency mode is achieved [4].

If we consider identical plants this can be generalized:
Theorem 1: Consider \( n \) identical linear SISO plants
\[
\begin{bmatrix}
y_1 \\
\vdots \\
y_n
\end{bmatrix} = \begin{bmatrix}
g(s) & & & \\
& \ddots & & \\
& & g(s) & \\
& & & g(s)
\end{bmatrix} \begin{bmatrix}
u_1 \\
\vdots \\
u_n
\end{bmatrix} \iff y = g(s)u
\]
coupled through the output over a network described by a Laplacian matrix \( L \in \mathbb{R}^{n \times n} \), i.e., let \( u = -Ly \). Let \( 0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \) denote the eigenvalues of \( L \). Then the system asymptotically synchronizes on \( y_{\lambda i} = v_1^T y = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} y_i \) if and only if
\[
g(s) = \frac{1}{1 + \lambda_j g(s)} \tag{3}
\]
is stable for all \( j \in \{2, \ldots, n\} \). The system is exponentially stable if and only if \( g(s) \) is stable.

The proof is given in [4, Theorem 8.4] on state-space form. Here, we will present it on transfer function form. This allows us to extend the analysis to heterogeneous subsystems coupled over a loss-less network, and to conveniently track limitations imposed by NMP zeros.

Proof: Diagonalize the system using the eigenvalue decomposition (2). Stability of the network is then equivalent to closed-loop stability of the diagonal system in Fig. 1. ■

B. Characteristic Loci of a Heterogeneous Network

The closed-loop system is stable if and only if the open-loop fulfills the generalized Nyquist criteria. For a homogeneous network, this simplifies to applying the Nyquist criteria to \( \lambda_j g(s) \), \( j \in \{2, \ldots, n\} \). The problem in the heterogeneous case is that there is no convenient way to calculate the characteristic loci without complete knowledge of the network. However, we will show that it is possible to define a convex hull wherein the characteristic loci of the heterogeneous network must be contained. This will allow us to formulate a conservative stability criteria based on generalized Nyquist criteria.

Notation: Consider a set of \( n \) heterogeneous subsystems \( G(s) = \text{diag} [g_1(s), \ldots, g_n(s)] \in \mathbb{C}^{n \times n} \) coupled through the output over a network described by the Laplacian matrix \( L \) as shown in Fig. 2. Let eigenvalues \( \Lambda \) and the matrix of eigenvectors \( V \) be defined as in (2) so that \( \Lambda = V^T L V \).

Since \( G(s) \) is diagonal, the characteristic loci at \( s = j\omega \),
\[
\lambda_i (G(j\omega)) = g_i(j\omega), \quad i \in \{1, \ldots, n\}.
\tag{4}
\]

Remark 1: Note that we have defined \( \lambda_i \) as both an operator and as the eigenvalues of the Laplacian \( L \). That is, \( \lambda_i (\mathcal{L}) = \lambda_i \).

Let \( \mathcal{G}_\lambda (j\omega) \) denote the closed convex hull
\[
\mathcal{G}_\lambda (j\omega) = \left\{ v^T g(j\omega) v \mid v^T v = 1 \right\} \tag{5}
\]
spanned by all the subsystems \( g_i(j\omega) \), \( i \in \{1, \ldots, n\} \). Since \( \lambda_1 = 0 \), we have that \( \mathcal{L} = \mathcal{V} \Lambda \mathcal{V}^T = \mathcal{V}_{2:n} \Lambda_2:n \mathcal{V}_{2:n}^T \), where
\[
\Lambda_2:n = \text{diag}(\lambda_2, \ldots, \lambda_n) \quad \text{and} \quad \mathcal{V}_{2:n} = [v_2:n, \ldots, v_n]. \tag{6}
\]

Eigenvalues are invariant under the similarity transform. Therefore, the non-zero eigenvalues
\[
\lambda_j (L \mathcal{G}(j\omega)) = \lambda_j \left( \Lambda_2:n \mathcal{V}_{2:n}^T \mathcal{G}(j\omega) \mathcal{V}_{2:n} \right). \tag{7}
\]
This means that synchronization properties of the closed-loop system in Fig. 2 can be assessed by applying the generalized Nyquist criteria to the characteristic loci (7). To simplify the notation, let
\[
\mathcal{X}(j\omega) = \mathcal{V}_{2:n}^T \mathcal{G}(j\omega) \mathcal{V}_{2:n}.
\tag{8}
\]

Lemma 1: The diagonal element in \( \mathcal{X}(j\omega) \),
\[
v_j^T \mathcal{G}(j\omega) v_j \in \mathcal{G}_\lambda (j\omega). \tag{8}
\]

Proof: Consider \( \mathcal{G}(s) = \text{diag} [g_1(s), g_2(s)] \) then
\[
v_j^T \mathcal{G}(j\omega) v_j = g_1(j\omega) \left[ 1 - v_{2:j}^2 \right] + g_2(j\omega) v_{2:j}^2 \]
where \( v_{2:j} \) is the second element in \( v_j \), as shown in Fig. 3a. Now add a third subsystem \( g_3(s) \) to \( \mathcal{G}(s) \). As shown in Fig. 3b, the resulting non-zero eigenvalues must lie in the closed convex hull (5). ■

Lemma 2: The characteristic loci
\[
\lambda_j (\mathcal{X}(j\omega)) \in \mathcal{G}_\lambda (j\omega), \forall j \in \{1, \ldots, n-1\}. \tag{9}
\]

Proof: From (4) and (5), and the fact that the similarity transform does not change eigenvalues, it follows that
\[
\lambda_j (V^T \mathcal{G}(j\omega) V) = \lambda_j (\mathcal{G}(j\omega)) \in \mathcal{G}_\lambda (j\omega)
\]
Omitting $v_1$ to form $V_{2:n}$ means that we remove one of the directions, $v_1$, in which $G(j\omega)$ is projected. Formally,

$$\lambda_j\left(\mathcal{X}(j\omega)\right) \in \left\{ v^T G(j\omega) v \mid v^T v = 0, v^T v = 1 \right\} \subseteq G_L(j\omega).$$

**Remark 2:** In general, the eigenvalues of the MIMO frequency response (9) are not the same as its diagonal elements (8). However, they all lie in the closed convex hull (5).

**Theorem 2:** The weighted characteristic loci

$$\lambda_j\left(\Lambda_{2:n}\mathcal{X}(j\omega)\right) \in G_L(j\omega), \forall j \in \{1, \ldots, n-1\}$$

where $G_L(j\omega) = \{ \alpha G_L(j\omega) \mid \alpha \in [\lambda_2, \lambda_n] \}$.

**Proof:** Applying the similarity transform,

$$\lambda_j\left(\Lambda_{2:n}\mathcal{X}(j\omega)\right) = \lambda_j\left(\Lambda_{2:n}^{1/2}\mathcal{X}(j\omega)\Lambda_{2:n}^{1/2}\right).$$

We see that, left multiplication with $\Lambda_{2:n}$ amplifies the directions in which $G_L(j\omega)$ is projected. The operation modifies the convex hull wherein the eigenvalues can be found. Formally,

$$G_L(j\omega) = \left\{ v^T G(j\omega) v \mid v^T v = 0, v^T v = \alpha, \alpha \in [\lambda_2, \lambda_n] \right\}.$$

**Example 1:** Consider the frequency response of a network with 4 subsystems given in Appendix B. Dropping the $(j\omega)$ notation, the frequency response is

$$G = \text{diag}(g_1, g_2, g_3, g_4).$$

The non-zero eigenvalues of the Laplacian are $0 < \lambda_2 < \lambda_3 < \lambda_4$. The convex hulls $G_1$ and $G_2$ are defined as in Lemma 2 and Theorem 2, respectively. In Fig. 4, we see that the open-loop diagonal elements are confined to $\lambda_2 G_2$, $\lambda_3 G_3$, and $\lambda_4 G_4$, respectively, in accordance with Lemma 1. We also see that the open-loop eigenvalues

$$\lambda_j\left(\Lambda_{2:4} G_{2:4}^T G_{2:4}\right), \quad j \in \{1, 2, 3\}$$

may differ from the open-loop diagonal elements. They are however always confined to $G_1$, in accordance with Theorem 2.

The reader is invited to test Theorem 2 for the randomly generated heterogeneous network presented in Appendix C.

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**C. Synchronization in a Heterogeneous Network**

Without knowledge of the eigenvectors, a conservative guarantee for asymptotic synchronization is given by:

**Theorem 3:** Consider $n$ heterogeneous linear SISO plants

$$\begin{bmatrix}
  y_1 \\
  \vdots \\
  y_n
\end{bmatrix} = \begin{bmatrix}
  g_1(s) \\
  \vdots \\
  g_n(s)
\end{bmatrix} \begin{bmatrix}
  u_1 \\
  \vdots \\
  u_n
\end{bmatrix} \quad \Rightarrow \quad y = g(s)u$$

coupled through the output over a network described by a Laplacian matrix $L \in \mathbb{R}^{n \times n}$, i.e., let $u = -Ly$. Let $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ denote the eigenvalues of $L$. Then the system is guaranteed to achieve asymptotic synchronization on $y_\lambda$, if

$$\frac{g_i(s)}{1 + \alpha g_i(s)}$$

is stable for all $\alpha \in [\lambda_2, \lambda_n]$ and $i \in \{1, \ldots, n\}$. The average mode, $y_\lambda$, is stable if and only if $v_1^T G_L(s)v_1$ is stable.

**Proof:** Follows from Theorems 1 and 2 by applying the generalized Nyquist criterion on the closed convex hull $G_L(j\omega)$ as visualized in Fig. 5.

**Corollary 1:** If we do not have information about all the subsystems $g_i(s)$, then we would have to assume that the origin is included in $G_L(j\omega)$. Consequently, we cannot allow for unstable plants or $\arg(g_j(j\omega)) < -180^\circ$ for frequencies where $|\lambda_2 g_i(j\omega)| > 1$. Theorem 3 is then a combination of having negative imaginary plants and the small gain theorem. In this case we only need to check (10) for $\alpha = \lambda_n$.

**Remark 3:** The open-loop eigenvalues are typically inside, or at least very close to, the corresponding convex hull $G_L(j\omega)$. This can be verified using Gershgorin’s circle theorem. Theorem 3 can then possibly be relaxed by checking the generalized Nyquist criteria for $\lambda_j G_L(j\omega), j \in \{2, \ldots, n\}$ instead of $G_L(j\omega)$.

**Remark 4:** Typically the open-loop eigenvalues are very close to the diagonal elements $\lambda_j v_j^T G(j\omega) v_j$. If we have confident knowledge about the controllability/observability of the subsystems in mode $j$, Theorem 3 can be relaxed.

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**III. Stability in a Heterogeneous Power System**

Consider a power system modeled as (1) with $F_i(s) = D_i + K_i(s)$ where $D_i$ is constant and $K_i(s)$ include possible governors controlled to improve frequency stability, e.g., using local frequency droop. Assume that the heterogeneous ensemble

$$g_i(s) = \frac{1}{s M_i + D_i + K_i(s)}, \quad i \in \{1, \ldots, n\}$$
is coupled through the output over a loss-less network described by a Laplacian matrix \( \mathcal{L} \). Then asymptotic synchronization on \( y_{\lambda_i} \) is guaranteed if \( \alpha_k(s) \) forms a stable closed-loop system (3) for all \( \alpha \in [\lambda_2, \lambda_n] \) and \( i \in \{1, \ldots, n\} \).

**Corollary 2:** Stability is guaranteed even for \( \lambda_n \to \infty \) if the actuators are passive, \( D_i + K_i(s) > 0 \), \( \forall i, \omega \). In which case \( G(s) \) is a negative imaginary system.

Assume that \( |M(s)| \gg |D_i + K_i(s)|, \forall i, \omega \). Then \( V^T G(s) V \approx \frac{V^T S V}{\lambda_j v_j} \) and the gain cross-over

\[
\lambda_j v_j^T G(s) v_j = 1, \quad \text{for } \omega_c = \sqrt{\lambda_j v_j^T S V}
\]

which in (1) is the frequency of the \( j \)th undamped inter-area mode. For stability, we need to ensure positive phase margin at \( \omega = \omega_c \). Which is guaranteed by Theorem 3. Thus we see that if \( K_i(s) \) has an NMP zero at \( s = z \). Then we cannot guarantee stability of inter-area modes with undamped frequency \( \omega_c > z \) if the gain is large in \( K_i(s) \). Note that this problem is due to the NMP actuator in \( K_i(s) \). Therefore it cannot be fixed by adding more measurements or by information sharing between plants. Instead, the damping improvement has to come from another actuator, e.g., \( D_{j_k}, \ k \in \{1, \ldots, n\} \). If Theorem 3 is to be used, stability improvement has to come from another power supply acting directly on machine \( i \). The only other option is to reduce the gain of the destabilizing controller \( K_i(s) \), consequently sacrificing the governors contribution to the frequency stability, i.e., the attenuation/damping of the average frequency mode \( y_{\lambda_i} \).

**APPENDIX**

**A. The Generalized Nyquist Criteria [2]**

**Definition 1:** The Nyquist \( D \)-contour includes the entire \( j \omega \)-axis and an infinite semi-circle into the RHP, making small indentations to avoid poles directly on the \( j \omega \)-axis.

**Theorem 4:** Suppose that the observable and controllable MIMO transfer function \( y = G(s)u \) has \( p_N \) unstable poles. The closed-loop system, with \( u = -y \), is then stable if and only if: as \( s \) goes clockwise around the Nyquist \( D \)-contour

1) the characteristic loci of the open-loop transfer function \( G(s) \) make a net sum of \( p_N \) anti-clockwise encirclements of the critical point -1;
2) the characteristic loci do not pass through the point -1.

**B. Matlab Code - Example 1**

```matlab
%% Example system
n = 4; % Number of nodes
V = [0.8207 -0.2564 -0.1035
     -0.3482 -0.4128 0.6771
     -0.4526 -0.1851 -0.7148
     -0.0200 0.8542 0.1412]; % Eigenvectors
e = [2; 6; 12]; % Eigenvalues
g = [ -0.10 - 0.20i
      -0.06 - 0.20i
      -0.07 + 0.07i
      -0.09 - 0.35i]; % Frequency response

%% Plot
X = V * diag(g) * V; eX = diag(e) * X; % Open-loop system
eig_eX = eig(eX);
%% Draw the global convex hull ----
e_g = kron(e,g); % Need e and g to be column vectors
```

REFERENCES

[1] L. Huang, H. Xin, and F. Dörfler, “H∞-Control of grid-connected converters: Design, objectives and decentralized stability certificates,” IEEE Trans. Smart Grid, vol. 11, no. 5, pp. 3805–3816, Sep. 2020.

[2] I. Postlethwaite and A. G. J. MacFarlane, A Complex Variable Approach to the Analysis of Linear Multivariable Feedback Systems. Berlin: Springer, 1979, no. 12.

[3] R. Pates and E. Mallada, “Robust scale-free synthesis for frequency control in power systems,” IEEE Control Syst. Syst., vol. 6, no. 3, pp. 1174–1184, Sep. 2019.

[4] F. Bullo, Lectures on Network Systems. Kindle Direct Publishing, 2020.