The behavior of $f(R)$ gravity in the solar system, galaxies and clusters

Pengjie Zhang$^{1,2,\ast}$

$^1$Shanghai Astronomical Observatory, Chinese Academy of Science, 80 Nandan Road, Shanghai, China, 200030
$^2$Joint Institute for Galaxy and Cosmology (JOINGC) of SHAO and USTC

For cosmologically interesting $f(R)$ gravity models, we derive the complete set of the linearized field equations in the Newtonian gauge, under environments of the solar system, galaxies and clusters respectively. Based on these equations, we confirmed previous $\gamma = 1/2$ solution in the solar system. However, $f(R)$ gravity models can be strongly environment-dependent and the high density (comparing to the cosmological mean) solar system environment can excite a viable $\gamma = 1$ solution for some $f(R)$ gravity models. Although for $f(R) \propto -1/R$, it is not the case; for $f(R) \propto -\exp(-R/\lambda^2 A^2 H^2)$, such $\gamma = 1$ solution does exist. This solution is virtually indistinguishable from that in general relativity (GR) and the value of the associated curvature approaches the GR limit, which is much higher than value in the $\gamma = 1/2$ solution. We show that for some forms of $f(R)$ gravity, this solution is physically stable in the solar system and can smoothly connect to the surface of the Sun.

The derived field equations can be applied directly to gravitational lensing of galaxies and clusters. We find that, despite significant difference in the environments of galaxies and clusters comparing to that of the solar system, gravitational lensing of galaxies and clusters can be virtually identical to that in GR, for some forms of $f(R)$ gravity. Fortunately, galaxy rotation curve and intra-cluster gas pressure profile may contain valuable information to distinguish these $f(R)$ gravity models from GR.

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INTRODUCTION

The standard theory of gravity (the general relativity (GR)) combined with the standard model of particle physics failed to explain a wide range of independent observations, from the expansion of the Universe, the cosmic microwave background, the large scale structure of the universe to galaxy and cluster dynamics. To reconcile observations, dark matter and dark energy, as modifications to particle physics, were proposed and work surprisingly well [1]. However, equally reasonable in logic, one can modify gravity instead to reconcile observations. It has been shown that the modified Newtonian dynamics (MOND) and its relativistic version Tensor-Vector-Scalar theory [2] can replace dark matter at galaxy scales to reproduce galaxy rotation curves, and 5-D DGP gravity [3] and $f(R)$ gravity [4, 5] can replace dark energy to reproduce the accelerated expansion of the universe.

Like dark matter and dark energy, viable modifications in gravity must pass all sorts of tests from the large scale structure of the universe [6, 7, 8, 9, 10, 11] to galaxy and cluster dynamics to the solar system tests (SST). Unlike GR, which involves metric derivatives no higher than second order, $f(R)$ gravity involves also third and fourth order derivatives, which caused complications in the calculation [12].

An outstanding question is whether $f(R)$ gravity is consistent with SST, which have put stringent constraints on the PPN parameter $\gamma = 1 \pm O(10^{-4})$ [13]. In this parameterization, the Schwarzschild metric takes the form of $ds^2 = (1-2GM/r+\cdots)dt^2 -(1+2\gamma GM/r+\cdots)dr^2 + r^2d\Omega$. For GR, $\gamma = 1$. Various authors have discussed the conditions for $f(R)$ gravity or its extensions to pass SST [14]. Utilizing the equivalence between $f(R)$ gravity and a special class of scalar-tensor theory, Chiba concluded that the solution of $f(R) = -\mu^2/R$ gravity has $\gamma = 1/2$ and is thus ruled out by SST [15]. The equivalence between $f(R)$ gravity and special scalar-tensor theory evoked some controversies (e.g. [16, 17]). Controversies also exist in approaches without resort to scalar-tensor theory. There are different approaches to vary the $f(R)$ action to obtain the field equations. One is the metric formalism, in which the only independent variable is the metric. Another is the Palatini form, in which the connection is also an independent variable. Based on the Palatini form, $\llbracket$18$\rrbracket$ concluded that $f(R)$ gravity can in general be perfectly consistent with SST. Same conclusion is reached in the metric formalism by several groups (e.g. [19, 20, 21]). However, based also on the metric formalism, $\llbracket$22$\rrbracket$ confirmed the $\gamma = 1/2$ solution for the $f(R) = -\mu^2/R$ gravity. Furthermore, they pointed out that the $\gamma = 1$ vacuum solution with constant curvature can not connect smoothly to the surface of the Sun. This work has been extended to a wide range of $f(R)$ gravity models [24].

To clarify this crucial issue, we derive the complete set of linearized field equations of the two Newtonian potentials $\phi$ and $\psi$, for cosmologically interesting $f(R)$ gravity models, under the metric formalism. We draw the attention that results in the Palatini form are in general different. The field equations turn out to take simple forms under the environments of the solar system, galax-
ies or clusters. Based on these equations, we confirmed
the $\gamma = 1/2$ solution. However, we also find that these
equations accept the $\gamma = 1$ solution for some forms of
$f(R)$ gravity, due to the presence of non-negligible inter-
planetary dust and planets. This large matter density in
the solar system, when compared to the cosmological mean,
can significantly suppress the corrections induced
to GR by $f(R)$ gravity and reduces the field equations to
the GR limit. We further checked on the stability of the
$\gamma = 1$ solution and discussed ways to make it stable.

Galaxies and clusters are excellent laboratories to test
$f(R)$ gravity through gravitational lensing, cluster X-ray
and SZ flux and galaxy rotation curve. We perturb the
FRW background and derive the equations applicable to
galaxies and clusters. The derived equations can be ap-
plied directly to address the above observational prop-
ties. For some $f(R)$ gravity models, the gravitational
lensing effect is virtually identical to that in GR. How-
ever, galaxy rotation curve, cluster pressure profile, the
relation between the cluster mass, X-ray temperature,
X-ray luminosity and the SZ flux, are modified, even for
these $f(R)$ gravity models. These observations then have
discriminating power for a wide range of $f(R)$ gravity
models.

**LINEARIZED FIELD EQUATIONS OF $f(R)$ GRAVITY**

The $f(R)$ gravity takes the action

$$L = \int (R + f(R))\sqrt{-g}dt^4 x ,$$

and the field equation

$$R_{\mu \nu} - \frac{1}{2}g_{\mu \nu}(R + f(R)) + f(R)R_{\mu \nu} + g_{\mu \nu}\Box R - f_{R\mu \nu} = 8\pi G T_{\mu \nu}. \quad (2)$$

Throughout the paper, we assume that $T_{\mu \nu}$ takes the
form of ideal fluid with negligible pressure. For $f(R)$
gravity models of cosmological interest, the $f(R)$ term
must vanish at high redshifts and $R$ should approach its
GR limit, in order not to conflict with early time physics
such as BBN and CMB. At these high redshift the cos-
omological density is comparable to the solar system local
density where SST were carried out. So we would expect
that the correction induced by $f(R)$ can be significantly
suppressed. However, this is only true if the GR limit,
namely $R \rightarrow 8\pi G \rho$, is reached. In this paper, we will
explicitly investigate on the feasibility of such solution.

In the solar system, galaxies and clusters, we expect
that the gravitational field is weak and the time varia-
tion of the field is negligible. Due to different environ-
ments and different boundary conditions, the linearized
field equations in the solar system differ slightly from
that in galaxies and clusters. Thus we will treat the two
cases separately.

**Field equations applicable to the solar system**

In the solar system, we choose a static metric with the
proper time

$$ds^2 = -g_{\mu \nu} dx^\mu dx^\nu = (1 + 2\psi(x))dt^2 - (1 + 2\phi(x)) \sum_i dx_i^2. \quad (3)$$

This is just the widely adopted Newtonian gauge in cos-
ology when dropping the time dependence, where $\phi$
and $\psi$ are two Newtonian potentials. Since $|\phi|, |\psi| \ll 1$, non-
vanishing Ricci tensor components are $R_{00} \simeq \nabla^2\psi$ and
$R_{ij} \simeq -\nabla^2\phi_{ij} - (\phi + \psi)_{ij}$. The curvature scalar
$R \simeq -2\nabla^2\psi - 4\nabla^2\phi$. In Eq. (2) it is safe to neglect
terms $(R + f)\phi$ and $(R + f)\psi$ with respect to $R + f$ and
neglect terms $\phi \Box R, \psi \Box R$ with respect to $\Box R$, since
$\phi, \psi$ are small. Also, one can approximate the covariant
derivative $f_{R\mu \nu}$ as the ordinary derivative $f_{R\mu \nu}$, since
$|\psi| \ll 1$. We then obtain

$$R_{00}(1 + f_R) + \frac{1}{2}(R + f) - \Box R = 8\pi G \rho , \quad (4)$$

$$R_{i\mu}(1 + f_R) - \frac{1}{2}(R + f) - \Box R - (f_R)_{,ii} = 0 , \quad (5)$$

$$R_{ij}(1 + f_R) - (f_R)_{,ij} = 0$ when $i \neq j. \quad (6)$$

Before proceeding to the final results, we point out a
generic constraint exerted by Eq. (6) for constant curva-
ture solutions. For this kind of solutions, $f_{R\mu \nu} = f_{R,\mu \nu} = 0$ and $(\phi + \psi)_{ij} = 0$. Thus the coefficient of the $r^{-1}$ term
in $\phi$ and $\psi$ must be equal (with opposite sign). In another
word, the constant curvature solution must have $\gamma = 1$.

Since $|\phi + \psi| \ll 1$, Eq. (6) can be integrated to give

$$(\phi + \psi)(1 + f_R) = -f_R + C_0 r^2 + \text{const.} . \quad (7)$$

This result is straightforward to check. From Eq. (2) we
get $(\phi + \psi)(1 + f_R) + (\phi + \psi)_{ij} f_R + (\phi + \psi),i f_{R,j} + (\phi + \psi)_{j} f_{R,i} = -f_{R,ij}$. Since $|\phi + \psi| \ll 1$, the last three terms
in the left hand side is negligible comparing to the right
hand side, we then obtain $(\phi + \psi)_{,ij}(1 + f_R) \simeq -f_{R,ij}$. This is the Eq. (6) to begin with. The integral would
produce a term $ax_i + bx_j$ in the right hand side of Eq. (7).
However there is no special direction in the Universe, so
it vanishes. The term $C_0 r^2$ is necessary. It reflects the
fact that, the flat Minkowski space-time is no longer the
true background in $f(R)$ gravity.

Combining Eq. (6) and (7) we obtain

$$\nabla^2(\phi - \psi) = -\frac{8\pi G \rho + 2C_0}{1 + f_R} . \quad (8)$$

Eq. (6) and (7) completely determine the gravitational
field of $f(R)$ gravity, up to a constant $C_0$. $C_0$ can
be determined by either Eq. 4 or the trace of Eq. 2 \((f_R - 1)R - 2f + 3\Box f_R = -8\pi G\rho\). For example, when \(f = \Lambda\) is the cosmological constant, one can show \(C_0 = f/8\).

These equations do not require the condition of spherical symmetry and can be applied to various environments including star forming regions and star or black hole accretion disk. However, to clarify the issue whether \(f(R)\) gravity is consistent with SST, we apply them to an idealized case, where a point source with mass \(M\) (the Sun) is embedded in a background with density \(\rho_{\text{back}} (\rho = M\delta_d(r) + \rho_{\text{back}})\) to sufficiently large radii.

The \(\gamma = 1/2\) solution found in the literature \cite{23, 24} corresponds to a small perturbation induced by the central star (Sun) on top of the homogeneous and isotropic vacuum background with curvature scalar \(R_0\). For this solution, the perturbation in the curvature scalar, is \(\Delta R \approx -2GM/3f_{RR}(R_0)r\), where \(R_0\) is the curvature scalar of the vacuum background. The overall curvature scalar \(R \equiv R_0 + R_1 \ll 8\pi G\rho_{\text{back}}\). Following the notation of \cite{30}, we call it the low curvature solution. This \(\gamma = 1/2\) solution is not only viable for the vacuum \((\rho_{\text{back}} = 0, \gamma_0 = 0, \gamma_2 \neq 0)\) but also hold for the realistic configuration of \(\gamma_{\text{back}} \neq 0\) around the Sun, as explicitly shown by \cite{25}. One can also check that the \(\gamma = 1/2\) solution is indeed a solution of Eq. \(\gamma = 7\) and \(\gamma = 8\).

A crucial step to obtain the \(\gamma = 1/2\) solution starts with the trace equation \((f_R - 1)R - 2f + 3\Box f_R = -8\pi G\rho\). It can be rewritten in the form of \(\Box f_R = -8\pi G\rho/3 + (2f - (f_R - 1)R)/3\). In the limit that \(8\pi G\rho \gg |2f - (f_R - 1)R|/3\), one has \(\Box f_R \approx 0\). This key equation to reach the \(\gamma = 1/2\) solution. We call the condition \(8\pi G\rho \gg |2f - (f_R - 1)R|\) as the low curvature condition. As explicitly shown in \cite{23, 24, 25}, the low curvature condition is satisfied for the \(\gamma = 1/2\) solution.

However, the trace equation can have another branch of solution. It can be rewritten in the form of \(R = 8\pi G\rho + (f_R R - 2f + 3\Box f_R)/3\). In the limit that \(f_R R - 2f + 3\Box f_R \ll 8\pi G\rho\), \(R \approx 8\pi G\rho\). We call it the high curvature solution and call the condition \(|f_R R - 2f + 3\Box f_R| \ll 8\pi G\rho\) as the high curvature condition. We will show that, given the fact that the local density \(\rho_{\text{back}}\) is much higher than \(\rho_c\) \((\rho_{\text{back}}/\rho_c \geq 10^6 - 10^8\), \cite{11}\), the high curvature condition can be satisfied and a viable \(\gamma = 1\) solution is excited, for some forms of \(f(R)\) gravity. The low curvature condition differ significantly from the high curvature condition \cite{31}.

Hence the existence of the \(\gamma = 1/2\) solution does not invalidate the existence of the \(\gamma = 1\) solution. Contrast to the \(\gamma = 1\) vacuum solution, the new \(\gamma = 1\) solution can connect smoothly to the surface of the Sun and can pass all SST. In the solar environment, it virtually reduces to GR, with \(R \approx 8\pi G\rho\). For this solution, \(f \rightarrow 0\) and \(f_R \ll 1\).

The solution of Eq. \(\gamma = 8\) under the spherical symmetry is

\[
\phi - \psi = \frac{1}{1 + f_R(r = 0)} \left[ \frac{2GM}{r} \int dr \left( r^{-2} \int \left( \frac{8\pi G\rho_{\text{back}} + 2C_0}{1 + f_R} r^2 dr \right) \right) \right].
\]

Since now \(|f_R| \ll 1\) we obtain

\[
\phi - \psi \approx 2GM \int \frac{dr}{r^2} \left( \int (8\pi G\rho_{\text{back}} + 2C_0) r^2 dr \right).
\]

Combining with Eq. \(\gamma = 7\) we obtain

\[
\phi = \frac{GM}{r} - \int \frac{dr}{r^2} \left( 4\pi G\rho_{\text{back}} r^2 dr + \frac{C_0 r^2}{3} + \text{const.} \right),
\]

\[
\psi = -\frac{GM}{r} + \int \frac{dr}{r^2} \left( 4\pi G\rho_{\text{back}} r^2 dr + \frac{2C_0 r^2}{3} + \text{const.} \right),
\]

where the constant \(C_0\) is given by the trace of Eq. \(\gamma = 2\). By variable transform \(r \rightarrow \tilde{r} = r(1 + \phi)\), one can express the solution (Eq. \(\gamma = 10\)) in the form of the Schwarzschild metric and then verifies the \(\gamma = 1\) result.

Is this solution consistent with all approximations we made? To be specific, we discuss two forms of \(f(R)\) gravity discussed in the literature, \(f(R) = -\mu^4/R\) \cite{4} and \(f(R) = -\lambda_1H_0^2 \exp(-\lambda_2H_0^2)\) \cite{11}. For \(f(R) = -\mu^4/R\) to drive the late time acceleration, \(\mu \sim H_0\) where \(H_0\) is the present day Hubble constant. For \(f(R) = -\lambda_1H_0^2 \exp(-\lambda_2H_0^2)\) proposed in \cite{11}, \(\lambda_1 \sim 1\) and \(2 \sim 10^3\) can produce virtually degenerate expansion rate to that of \(\Lambda\)CDM cosmology.

We first check the high curvature condition \(|f_R R - 2f + 3\Box f_R| \ll 8\pi G\rho\). Since now \(R \simeq 8\pi G\rho\), \(|f_R| \ll 1\), the condition reduces to \(|\Box f_R| \ll R \simeq 8\pi G\rho\). The high curvature condition requires \(|d^2f/dR^2 \ll 1\) and \(|d^2f/dR^2|/R^2 \ll 1\) in general. Here, we have adopted the approximations \(R \sim R/r \) and \(R^2 \sim R^2/r^2\) where \(R \neq 0\).

For \(f(R) = -\mu^4/R\), \(|d^2f/dR^2| \sim |d^3f/dR^3[R/r^2]| \sim \mu^4/R^3 \sim (\rho_c/\rho)^3(H_0^2v^2/c^2) \sim 10^{12}(AU/r)^2 \gg 1\). Clearly, \(f(R) = -\mu^4/R\) model does no accept the \(\gamma = 1\) solution and contradicts with SST.

For \(f(R) = -\lambda_1H_0^2 \exp(-\lambda_2H_0^2)\) and \(\lambda_2 \sim 10^3\), \(\exp(-\lambda_2H_0^2) \sim 10^{-400}\). So \(|d^2f/dR^2| \sim \exp(-\lambda_2H_0^2)/(H_0^2v^2/c^2) \sim 10^{-370}(AU/r)^2 \ll 1\). We can further check \(|d^3f/dR^3[R/r^2]| \ll 1\). Since \(f(R) = -\lambda_1H_0^2 \exp(-\lambda_2H_0^2)\), the high curvature condition is satisfied and \(\gamma = 1\) high curvature solution is accepted. Furthermore, the weak field condition is also satisfied since \(|\phi|, |\psi| \sim 10^{-8}(AU/r) \ll 1\). The condition \(|f_R| \ll 1\) is perfectly satisfied too. Hence-after, we only discuss this form of \(f(R)\) gravity, unless explicitly specified.

Is it consistent with SST? Yes. Since \(C_0 \ll G\rho\), Eq. \(\gamma = 10\) is virtually identical to that in GR and causes no conflict with SST. Furthermore, in the solar system,
\(|\phi|, |\psi| \sim 10^{-8}(M/M_{\text{Sun}})(\text{AU}/r) \gg |f_R|\) is generally satisfied. \(|C_\nu^2/\phi| \ll 1\) is also satisfied. We then have \(\phi + \psi \simeq 0\) and \(\nabla^2(\phi - \psi) \simeq -8\pi G \rho\). Thus the field equations are virtually identical to that in GR and we should sense no difference between \(f(R)\) gravity and GR.

This solution relies on the condition that \(\phi_{\text{back}} \gg \rho_c\). Clearly, at sufficiently large distance, this condition is violated. However, from Eq. 9 and the definition of \(\gamma\), it is clear that the value of \(\gamma\) is determined by the behavior of \(f_R\) at \(r \to 0\) instead of \(r \to \infty\). So the \(\gamma = 1\) conclusion is not affected.

Is it stable against small perturbations? The stability of solutions in \(f(R)\) gravity models has been widely discussed \([9, 10, 27, 28, 29, 30]\). For example, \([30]\) found that, the high curvature cosmological solutions may not satisfy the success of early universe physics such as BBN and \(H - z\) relation at \(z \lesssim 2\) and SST, while being physically stable in the solar system \([32]\).

Field equations applicable to galaxies and clusters

Galaxies and clusters are virialized objects embedded in the FRW background. Approximately they are static in the physical coordinate. So the time scale of the field variation is the Hubble time and it is safe to neglect all the time derivatives of \(\phi\) and \(\psi\). For a galaxy or cluster at \(z = 1/a - 1\), we choose a metric

\[
d^2 s = (1 + 2\psi)dt^2 - a^2(1 + 2\phi) \sum_i dx_i^2. \tag{11}
\]

Eq. (7) is then replaced by \([11]\)

\[
\phi + \psi = -f_R(R_{\text{FRW}} + \delta R) + f_R(R_{\text{FRW}}) \tag{12}
\]

Here, \(R = R_{\text{FRW}} + \delta R = 6(\dot{a}^2/a^2 + \ddot{a}/a) - 2\nabla^2 \psi - 4 \nabla^2 \phi\). Throughout this section, the derivative is with respect to the physical coordinate. The term \(C_\nu^2\) presented in the solar system solution vanishes, because now the FRW background is the right background. One can simply verify \(C_\nu = 0\) by the boundary condition that when \(r \to \infty\), \(\phi \to 0\), \(\psi \to 0\) and \(R \to R_{\text{FRW}}\).

The Poisson equation (Eq. 5) is replaced by

\[
\nabla^2 (\phi - \psi) = -8\pi G (\rho_m - \bar{\rho}_b). \tag{13}
\]

Here, \(\rho_m\) is the matter density of galaxies or clusters and \(\bar{\rho}_b\) is the background matter density. These equations have been derived under the quasi-static approximation at sub-horizon scales \([11]\). Equations applicable to galaxies and clusters are very similar. A good thing is that now the quasi-static approximation is well satisfied at all relevant scales in galaxy and cluster environments.

The gravitational lensing is governed by the combination \(\phi - \psi\). So, given the same matter distribution, the gravitational lensing effect of a galaxy or a cluster in \(f(R)\) gravity is identical to that in GR, except a change in the Newton’s constant from \(G\) to \(G/(1 + f_R)\). For \(f(R)\) to drive late time acceleration, \(f_R > 0\) in general, thus the lensing signal will be smaller by a factor \(f_R/(1 + f_R)\). However, for some \(f(R)\) gravity models, such as \(f(R) = -\lambda_1 H_0^2 \exp(-R/\lambda_2 H_0^2) + \alpha R^2/H_0^2\) proposed above, \(|f_R| \ll 1\) in galaxy and cluster environments, so the difference may not be observable.

However, galaxy rotation curve and intra-cluster gas pressure profile can be significantly different to that in GR. The acceleration of a test particle is \(\ddot{\mathbf{r}} = -\nabla \psi\). The gas pressure \(p\) is determined by \(\nabla p = -\rho \nabla \psi\). The matter density decreases from \(\sim 10^4 \rho_c\) close to the center to \(\sim 40 \rho_c\) at virial radius. The corresponding variation in \(f_R\) is then comparable to variations in \(\phi, \psi\) and thus \(\phi + \psi = 0\).
no longer holds. So in galaxy and cluster environments, \( \psi \) in \( f(R) \) gravity does not follow the Poisson equation, as that in GR does. Whether the difference caused is observable is currently under investigation.

**SUMMARY**

To investigate \( f(R) \) gravity in the solar system, galaxies and clusters, we derive the complete sets of the field equations which determine the two Newtonian potentials \( \phi \) and \( \psi \), under corresponding environments. In the solar system, we found that some \( f(R) \) gravity models of cosmological interest do contain physically stable solutions which are virtually indistinguishable from GR.

We predict that gravitational lensing effect of quasi-static celestial objects such as galaxy and clusters in \( f(R) \) gravity is virtually the same as in GR, up to a change in the Newton’s constant. However, galaxy rotation curve and cluster gas pressure profile differ intrinsically from that in GR. Thus observations of galaxies and cluster dynamics are promising to put useful constraints on \( f(R) \) gravity.

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[32] As seen from later discussion, the high curvature condition is equivalent to \( |\phi| \ll 8\pi G \rho \), while the low curvature condition is equivalent to \( |\phi| \approx 8\pi G \rho \).

[33] The cosmological solution of \( f(R) \) is still unstable. Whether one can find suitable correction to \( f(R) = -\lambda_1 H_0^2 \exp(-R/\lambda_2 H_0^2) + \alpha R^2/H_0^2 \) with \( \alpha \ll 10^{-30} \) is still unstable. Whether one can find suitable correction to \( f(R) = -\lambda_1 H_0^2 \exp(-R/\lambda_2 H_0^2) \) to make both
the high curvature solar system solution and the cosmological solution stable requires further investigation.