LONG EXACT SEQUENCES FOR DE RHAM COHOMOLOGY OF DIFFEOLOGICAL SPACES

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Abstract. In this paper we present the notion of de Rham cohomology with compact support for diffeological spaces. Moreover we shall discuss the existence of three long exact sequences. As a concrete example, we show that long exact sequences exist for the de Rham cohomology of diffeological subcartesian spaces.

1. Introduction

Generally, the de Rham cohomology is a cohomology based on differential forms of a topological smooth manifold. In [7], J.-M. Souriau introduced diffeological spaces as generalization of the notions of topological smooth manifolds. Moreover, in [2], P. Iglesias-Zemmour extended the notions of differential forms and de Rham cohomology groups on diffeological spaces.

In Section 2 we discuss about the Hausdorffness, the compactness, the paracompactness, and the normality of $D$-topology of a diffeological space. Since the inclusion is not compatible with $D$-topologies, we need to be cautious in dealing with these notions. But we can show that if a diffeological space is $D$-paracompact and $D$-Hausdorff, then it is $D$-normal. In Section 3 we introduce "diffeological subcartesian spaces" which is a diffeological space locally diffeomorphic to a (not necessarily open) subspace of an Euclidean space. It is shown that every diffeological subcartesian space has a partition of unity subordinate to any $D$-open cover. In Section 4 we discuss de Rham cohomology (with compact support) in respect to diffeological spaces. It is shown that if there is a $D$-open cover $\{A, B\}$ of $X$ such that there exists a partition of unity subordinate to it, then we have a Mayer-Vietoris exact sequence of de Rham cohomology groups (see Theorem 4.3):

$$
\to H^p_{\text{dR}}(X) \xrightarrow{j^1 \oplus j^2} H^p_{\text{dR}}(A) \oplus H^p_{\text{dR}}(B) \xrightarrow{i^1 - i^2} H^p_{\text{dR}}(A \cap B) \xrightarrow{\delta} H^{p+1}_{\text{dR}}(X) \to \cdots .
$$

On the other hand, we shall see in Section 5 that if $X$ is $D$-Hausdorff, then there is a Mayer-Vietoris exact sequence of de Rham cohomology groups with compact support (see Theorem 5.3):

$$
\to H^p_c(A \cap B) \xrightarrow{i_1^* \oplus i_2^*} H^p_c(A) \oplus H^p_c(B) \xrightarrow{j_1^* - j_2^*} H^p_c(X) \xrightarrow{\delta} H^{p+1}_c(A \cap B) \to \cdots .
$$

In particular, if $X$ is a diffeological subcartesian space, then there exist both types of Mayer-Vietoris exact sequences. In Section 6, we introduce a long exact sequence for pair of diffeological spaces. Let $A$ be a $D$-compact set of a diffeological subcartesian
space $X$. If there exists a $D$-open set $M$ of $X$ such that $A$ is a deformation retract of $M$, then we have a long exact sequence (see Theorem 6.2):

\[ \rightarrow H^p_c(X \setminus A) \overset{i^*}{\rightarrow} H^p_c(X) \overset{j^*}{\rightarrow} H^p_c(A) \overset{\delta}{\rightarrow} H^{p+1}_c(X \setminus A) \rightarrow \cdots. \]

I would like to thank Kazuhisa Shimakawa, who suggested me the idea of using differential $p$-forms on a diffeological space with compact support (see Definition 5.1).

2. The $D$-Topology for Diffeological spaces

A diffeological space consists of a set $X$ together with a family $D$ of maps from open subsets of Euclidean spaces into $X$ satisfying the following conditions:

- **Covering:** Any constant parametrization $\mathbb{R}^n \to X$ belongs to $D$.
- **Locality:** A parametrization $P: U \to X$ belongs to $D$ if every point $u$ of $U$ has a neighborhood $W$ such that $P|_W: W \to X$ belongs to $D$.
- **Smooth compatibility:** If $P: U \to X$ belongs to $D$, then so does the composite $P \circ Q: V \to X$ for any smooth map $Q: V \to U$ between open subsets of Euclidean spaces.

We call $D$ a diffeology of $X$, and each member of $D$ a plot of $X$. A map $f: X \to Y$ between diffeological spaces is called smooth if for any plot $P: U \to X$ of $X$, the composite $f \circ P: U \to Y$ is a plot of $Y$. Clearly, the class of diffeological spaces and smooth maps form a category $\textbf{Diff}$.

**Theorem 2.1** ([2, 1.60], [6, 2.1]). The category $\textbf{Diff}$ is complete, cocomplete, and cartesian closed.

Let $X$ be a diffeological space. Let $A$ be a subset of $X$. We say that $A$ is $D$-open in $X$ if for any plot $P: U \to X$ of $X$, $P^{-1}(A)$ is open in $U$. A subset $A$ is called $D$-closed in $X$ if for any plot $P: U \to X$ of $X$, $P^{-1}(A)$ is closed in $U$. We will denote the closure of $A$ by $\overline{A}$. That is to say, $\overline{A}$ is the smallest $D$-closed subset containing $A$.

**Lemma 2.2** ([1, 3.17]). Let $A$ be a $D$-open set of diffeological space $X$. Then a subset $B$ of $A$ is $D$-open if and only if it is $D$-open in $A$.

**Remark.** Let $A$ be a subset of a diffeological space $X$. Then we can give $A$ two topologies:

1. $\tau_1(A)$: the $D$-topology of the sub-diffeology on $A$;
2. $\tau_2(A)$: the sub-topology of the $D$-topology on $X$.

However, these topologies are not always the same. In general, we can only conclude that $\tau_2(A) \subseteq \tau_1(A)$. Therefore we need to be careful when defining separation axioms and compactness.

**Definition 2.3** ($D$-Hausdorff space). A diffeological space $X$ is $D$-Hausdorff if and only if for any elements $x$ and $y$ of $X$, there are $D$-open neighborhoods $U_x$ and $U_y$ of $x$ and $y$, respectively, such that $U_x \cap U_y = \emptyset$.

We have the following by the above remark.

**Lemma 2.4.** Let $A$ be a subspace of $X$. If $X$ is $D$-Hausdorff, then $A$ is also $D$-Hausdorff.
**Definition 2.5** (D-compact space). Let $C$ be a subset of a diffeological space $X$. We say that $C$ is $D$-compact in $X$ if every covers of $C$ consisting of $D$-open sets of $X$ have a finite cover. If $X$ is $D$-compact in $X$, then it is called to be $D$-compact.

Then we have the following by Lemma 2.2

**Proposition 2.6.** Let $A$ be a $D$-open set of $X$. Then a subset $C$ of $A$ is $D$-compact in $X$ if and only if it is $D$-compact in $A$.

It is not difficult to prove the following.

**Proposition 2.7.** If $X$ is $D$-compact, then every $D$-closed subset of $X$ is also $D$-compact in $X$.

**Proposition 2.8.** If $X$ is $D$-Hausdorff, then every $D$-compact subset of $X$ is $D$-closed in $X$.

We turn to $D$-paracompactness. Let $X$ be a diffeological space. A collection $\{W_\lambda\}_{\lambda \in \Lambda}$ of subsets of $X$ is called locally finite if each $x \in X$ has a $D$-open neighborhood whose intersection with $W_\lambda$ is non-empty only for finitely many $\lambda$.

**Lemma 2.9.** Let $\{W_\lambda\}_{\lambda \in \Lambda}$ be a collection of subsets of $X$. If $\{W_\lambda\}_{\lambda \in \Lambda}$ is locally finite, then $\cup_{\lambda \in \Lambda} \overline{W_\lambda} = \cup_{\lambda \in \Lambda} W_\lambda$ holds.

**Proof.** It is clear that $\cup_{\lambda \in \Lambda} \overline{W_\lambda} \subset \cup_{\lambda \in \Lambda} W_\lambda$. Conversely, let $x \notin \cup_{\lambda \in \Lambda} \overline{W_\lambda}$. Then for any $\lambda \in \Lambda$, there exists $D$-open neighborhood $U_\lambda(x)$ of $x$ such that $U_\lambda(x) \cap W_\lambda = \emptyset$. Since $\{W_\lambda\}_{\lambda \in \Lambda}$ is locally finite, there exist a $D$-open neighborhood $V$ of $x$ and finitely many $\lambda_i \in \Lambda$ ($1 \leq i \leq m$) such that $V \cap W_{\lambda_i} \neq \emptyset$. Let $U = (\cap_{1 \leq i \leq m} U_{\lambda_i}(x)) \cap V$. Then $U$ is $D$-open neighborhood of $x$. Since for any $\lambda \in \Lambda$, $W_\lambda \cap U = \emptyset$, we have $(U_{\lambda_i} \cap W_{\lambda_i}) \cap U = \emptyset$. Therefore $x \notin \cup_{\lambda \in \Lambda} \overline{W_\lambda}$.

Let $\{V_\alpha\}_{\alpha \in I}$ and $\{U_\beta\}_{\beta \in J}$ be two covers of $X$. We say that $\{V_\alpha\}_{\alpha \in I}$ is a refinement of $\{U_\beta\}_{\beta \in J}$ if for any $\alpha \in I$, there exists $\beta \in J$ such that $V_\alpha \subset U_\beta$.

**Definition 2.10** (D-paracompact space). We say that a subset $A$ of a diffeological space $X$ is $D$-paracompact in $X$ if every cover of $A$ consisting of $D$-open sets of $X$ has a locally finite refinement consisting of $D$-open sets of $X$. If $X$ is $D$-paracompact in $X$, then we call it $D$-paracompact.

**Proposition 2.11.** Let $A$ be a $D$-closed subset of $X$. If $X$ is $D$-paracompact, then $A$ is $D$-paracompact in $X$.

**Proof.** Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be a cover of $A$ consisting of $D$-open sets of $X$. Then $U = \{U_\lambda\}_{\lambda \in \Lambda} \cup \{X \setminus A\}$ is $D$-open cover of $X$. Since $X$ is $D$-paracompact, there exists a locally finite refinement $\{V_\alpha\}_{\alpha \in I}$ of $U$. Let $I' = \{\alpha \in I | V_\alpha \cap A \neq \emptyset\}$. Then $\{V_\alpha\}_{\alpha \in I'}$ is a locally finite refinement of $\{U_\lambda\}_{\lambda \in \Lambda}$.

**Definition 2.12** (D-normal space). We say that a diffeological space $X$ is $D$-normal if for any $D$-closed sets $A$ and $B$ of $X$ such that $A \cap B = \emptyset$, there exist $D$-open sets $U_A$ and $U_B$ of $X$ such that $A \subset U_A$, $B \subset U_B$ and $U_A \cap U_B = \emptyset$.

From the definition, it is clear that we have the following.

**Proposition 2.13.** A diffeological space $X$ is $D$-normal if and only if for any $D$-closed set $A$ and $D$-open set $B$ of $X$ such that $A \subset B$, there exists a $D$-open set $U_A$ of $X$ such that $A \subset U_A \subset \overline{U_A} \subset B$. 

Theorem 2.14. If $X$ is $D$-Hausdorff and $D$-paracompact, then it is $D$-normal.

Proof. Let $x$ be an element of $X$. Let $F$ be a $D$-closed set of $X$ such that $x \notin F$ and let $y$ be an element of $F$. Since $X$ is $D$-Hausdorff, there exists a $D$-open neighborhood $U_x$ and $U_y$ of $x$ and $y$, respectively, such that $U_x \cap U_y = \emptyset$. Then $U = \{U_y | y \in F\} \cup \{X \setminus F\}$ is a $D$-open cover of $X$. Since $X$ is $D$-paracompact, there exists a locally finite refinement $\{W_\lambda\}_{\lambda \in \Lambda}$ of $U$. Thus there are a $D$-open neighborhood $V$ of $x$ and finitely many $\lambda_i$ ($1 \leq i \leq m$) such that $W_{\lambda_i} \cap V \neq \emptyset$. Let $I = \{\lambda_i | x \notin W_{\lambda_i}, 1 \leq i \leq m\}$ and $V_0 = V \setminus \bigcup_{\lambda_i \in I} W_{\lambda_i}$. Since $\bigcup_{\lambda_i \in I} W_{\lambda_i} = \bigcup_{\lambda_i \in I} W_{\lambda_i}$ holds, $V_0$ is a $D$-open neighborhood of $x$. Let $J = \{\lambda \in \Lambda | W_\lambda \cap F \neq \emptyset\}$ and $W = \bigcup_{\lambda \in J} W_\lambda$. Then $W$ is a $D$-open set of $X$ such that $F \subset W$. Then we have $V_0 \cap W = \emptyset$.

Next, let $A$ and $B$ be $D$-closed sets of $X$ such that $A \cap B = \emptyset$. By the above condition, for any $a \in A$, there exist a $D$-open neighborhood $U_a$ of $a$ and $D$-open set $W(a)$ of $X$ such that $B \subset W(a)$ and $U_a \cap W(a) = \emptyset$. Let $U' = \{U_a | a \in A\} \cup \{X \setminus A\}$. Then $U'$ is a $D$-open cover of $X$. Since $X$ is $D$-paracompact, there exists a locally finite refinement $\{L_\lambda\}_{\lambda \in \Lambda}$ of $U'$. For any $b \in B$, there exist a $D$-open neighborhood $M_b$ of $b$ and finitely many $\lambda_i$ ($1 \leq i \leq m$) such that $M_b \cap L_{\lambda_i} \neq \emptyset$. Let $I = \{\lambda_i | a \notin L_{\lambda_i}, 1 \leq i \leq m\}$ and let $M = \bigcup_{\lambda \in \Lambda} M_b$, where $M_b = M_b \setminus \bigcup_{\lambda_i \in I} L_{\lambda_i}$ is a $D$-open neighborhood of $b$. Let $J = \{\lambda \in \Lambda | L_\lambda \cap A \neq \emptyset\}$ and let $L = \bigcup_{\lambda \in J} L_\lambda \supset A$. Then we have $L \cap M = \emptyset$. □

3. DIFFEOREGICAL SUBCARTESAN SPACES

In this section we present the notion of diffeological subcartesian spaces. Moreover we prove that every diffeological subcartesian space has a partition of unity subordinate to arbitrary $D$-open cover.

Definition 3.1 (diffeological subcartesian space). We say that a diffeological space $X$ is a diffeological subcartesian space if the following conditions are satisfied.

(1) $X$ is $D$-Hausdorff and $D$-paracompact.

(2) For each $x$ in $X$, there exists a diffeomorphism $\varphi_x : U \to U'$, called a chart at $x$, from a $D$-open neighborhood $U$ of $x$ to a subspace $U'$ of an Euclidean space $\mathbb{R}^{n_x}$, where $n_x \geq 0$ and $U'$ need not be open in $\mathbb{R}^{n_x}$.

It is clear that a diffeological subcartesian space is $D$-normal by Theorem 2.14.

Lemma 3.2 (Shrink Lemma). If a diffeological space $X$ is $D$-Hausdorff and $D$-paracompact, then for every $D$-open cover $\{U_\lambda\}_{\lambda \in \Lambda}$ of $X$, there exists a locally finite $D$-open cover $\{W_\lambda\}_{\lambda \in \Lambda}$ such that $\overline{W_\lambda} \subset U_\lambda$ holds for any $\lambda \in \Lambda$.

Proof. Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be a $D$-open cover of $X$. Since $\bigcup_{\lambda \in \Lambda} U_\lambda = X$ holds, we have

$$\bigcap_{\lambda \in \Lambda} (X \setminus U_\lambda) = \bigcap_{\lambda \neq \lambda' \in \Lambda} (X \setminus U_{\lambda'}) \cap (X \setminus U_\lambda) = \emptyset.$$ 

Since $X$ is $D$-normal by Theorem 2.14 there exist two $D$-open sets $S_\lambda$ and $W$ of $X$ such that

$$\bigcap_{\lambda \neq \lambda' \in \Lambda} (X \setminus U_{\lambda'}) \subset S_\lambda, \ X \setminus U_\lambda \subset W \text{ and } S_\lambda \cap W = \emptyset.$$ 

Thus we have $\overline{S_\lambda} \subset U_\lambda \text{ and } S_\lambda \cup (\bigcup_{\lambda \neq \lambda' \in \Lambda} X \setminus U_{\lambda'}) = X$ since $S_\lambda \subset X \setminus W \subset U_\lambda$ holds. Let $U = \{S_\lambda | \lambda \in \Lambda\}$. Then $U$ is a $D$-open cover of $X$. Since $X$ is $D$-paracompact, there exists a locally finite refinement $\{W_j\}_{j \in J}$ of $U$. For any $\lambda \in \Lambda$, 4

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Let \( J_\lambda = \{ j_\lambda \in J | W_{j_\lambda} \subset S_\lambda \} \) and let \( V_\lambda = \bigcup_{j_\lambda \in J_\lambda} W_{j_\lambda} \). Then we have
\[
\nabla_\lambda = \bigcup_{j_\lambda \in J_\lambda} W_{j_\lambda} \subset S_\lambda \subset U_\lambda.
\]

Then \( \{ V_\lambda | \lambda \in \Lambda \} \) is a locally finite \( D \)-open cover of \( X \). \( \square \)

Let \( X \) be a diffeological space. If \( f: X \to \mathbb{R} \) is a real-valued smooth map on \( X \), the support of \( f \), denoted by \( \text{supp} f \), is the closure of the set of points where \( f \) is nonzero:
\[
\text{supp} f = \{ p \in X; f(p) \neq 0 \}.
\]

Let \( G = \{ A_\lambda \}_{\lambda \in \Lambda} \) be an arbitrary \( D \)-open cover of \( X \). We say that a collection \( \{ \phi_\lambda: X \to \mathbb{R} \}_{\lambda \in \Lambda} \) is a partition of unity subordinate to \( G \) if the following conditions are satisfied:
\begin{enumerate}
  \item \( 0 \leq \phi_\lambda(x) \leq 1 \) for all \( \lambda \in \Lambda \) and all \( x \in X \),
  \item \( \text{supp} \phi_\lambda \subset A_\lambda \),
  \item the set \( \{ \text{supp} \phi_\lambda | \lambda \in \Lambda \} \) of supports is locally finite, and
  \item \( \sum_{\lambda \in \Lambda} \phi_\lambda(x) = 1 \) for all \( x \in X \).
\end{enumerate}

**Theorem 3.3.** Let \( X \) be a diffeological subcartesian space. Then for any \( D \)-open cover \( U \) of \( X \), there exists a partition of unity subordinate to \( U \).

**Proof.** Let \( U = \{ U_\lambda \}_{\lambda \in \Lambda} \) be a \( D \)-open cover of \( X \). Without loss of generality we may assume that the elements of \( U \) are chart domains. By Shrink Lemma, there exist locally finite \( D \)-open covers \( \{ V_\lambda \} \) and \( \{ W_\lambda \} \) of \( X \) such that
\[
\nabla_\lambda \subset W_\lambda \subset U_\lambda.
\]

Let \( \varphi_\lambda: U_\lambda \to U_\lambda' \) be a chart for each \( \lambda \in \Lambda \) and \( U_\lambda' \) is the subset of \( \mathbb{R}^{n_\lambda} \). Then there are a closed subset \( \tilde{V}_\lambda \) and an open subset \( \tilde{W}_\lambda \) in \( \mathbb{R}^{n_\lambda} \) such that \( \varphi_\lambda^{-1}(\tilde{V}_\lambda) = \nabla_\lambda \) and \( \varphi_\lambda^{-1}(\tilde{W}_\lambda) = W_\lambda \), respectively. By [12, 2.19], there exists a smooth function \( g_\lambda: \mathbb{R}^{n_\lambda} \to \mathbb{R} \) such that \( \text{supp} g_\lambda \subset \tilde{W}_\lambda \), \( g_\lambda|_{\tilde{V}_\lambda} = 1 \) and \( g_\lambda|_{\mathbb{R}^{n_\lambda} \setminus \tilde{W}_\lambda} \equiv 0 \). We define \( f_\lambda: X \to \mathbb{R} \) by
\[
f_\lambda(x) = \begin{cases} 
g_\lambda \circ \varphi_\lambda(x) & x \in U_\lambda \\
0 & x \in X \setminus U_\lambda. \end{cases}
\]

Define new function \( \phi_\lambda: X \to \mathbb{R} \) by
\[
\phi_\lambda(x) = \frac{f_\lambda(x)}{\sum_{\lambda' \in \Lambda} f_{\lambda'}(x)}.
\]

Then \( \{ \phi_\lambda: X \to \mathbb{R} | \lambda \in \Lambda \} \) is a partition of unity subordinate to \( U \). \( \square \)

**Corollary 3.4.** Let \( A \) be a \( D \)-closed subset of a diffeological subcartesian space \( X \). Let \( U_\lambda A \) be a \( D \)-open subset of \( X \) containing \( A \). Then there exists a function \( \varphi: X \to \mathbb{R} \) such that \( \text{supp} \varphi \subset U_\lambda A \) and \( \varphi \equiv 1 \) on \( A \).

**Proof.** Since \( \{ U_\lambda, X \setminus A \} \) is a \( D \)-open cover of \( X \), there exists a partition of unity subordinate \( \{ \varphi_{U_\lambda}, \varphi_{X \setminus A} \} \) to \( \{ U_\lambda, X \setminus A \} \) by Theorem [3.3]. Since \( \varphi_{X \setminus A} \equiv 0 \) on \( A \), the function \( \varphi_{U_\lambda} \) has the required properties. \( \square \)
4. de Rham cohomology of diffeological spaces

In this section we shall show that there exists a Mayer-Vietoris exact sequence with respect to de Rham cohomology of diffeological spaces.

We first recall from [2] the notion of differential forms on a diffeological space. A covariant antisymmetric \( p \)-tensor of \( \mathbb{R}^n \) [2, 6.11] is called a linear \( p \)-form of \( \mathbb{R}^n \). The vector space of linear \( p \)-forms of \( \mathbb{R}^n \) is denoted by \( \Lambda^p(\mathbb{R}^n) \). Let \( U \) be an open set of \( \mathbb{R}^n \). Let \( d \) be a linear map from \( C^\infty(U, \Lambda^p(\mathbb{R}^n)) \) to \( C^\infty(U, \Lambda^{p+1}(\mathbb{R}^n)) \) defined by [2, 6.24]. Then we have \( d \circ d = 0 \).

**Definition 4.1** ([2, 6.28]). Let \( X \) be a diffeological space. We say that \( \alpha \) is a differential \( p \)-form on \( X \) if the following two conditions are fulfilled

1. For all integers \( n \), for all \( n \)-plots \( P : U \to X \), we have
   \[
   \alpha(P) \in C^\infty(U, \Lambda^k(\mathbb{R}^n)).
   \]
2. For all open sets \( V \) of \( \mathbb{R}^m \), \( m \geq 0 \), for all smooth parametrizations \( F : V \to U \), we have
   \[
   \alpha(P \circ F) = F^*(\alpha(P)),
   \]
   where \( F^*(\alpha(P)) \) (cf. [2, 6.22]) is defined by
   \[
   F^*(\alpha(P))(\nu)(x_1) \cdots (x_p) = \alpha(P)(F(\nu))(D(F)(\nu)(x_1)) \cdots (D(F)(\nu)(x_p))
   \]
   for all \( \nu \in V \) and for all \( k \)-vectors \( x_1, \cdots, x_p \in \mathbb{R}^m \).

The condition \( \alpha(P \circ F) = F^*(\alpha(P)) \) is called the smooth compatibility condition. The set of differential \( p \)-forms on \( X \) is clearly a real vector space, and will be denoted by \( \Omega^p(X) \). We define a linear map \( d \) from \( \Omega^p(X) \) to \( \Omega^{p+1}(X) \) by

\[
(d\alpha)(P) = d(\alpha(P))
\]

for any \( \alpha \in \Omega^p(X) \) and any plot \( P : U \to X \) of \( X \). Clearly, \( d \circ d = 0 \) holds. Therefore we have a cochain complex \( \{\Omega^p(X), d\} \), called the de Rham complex. We define

\[
Z^p(X) = \text{Ker}[d : \Omega^p(X) \to \Omega^{p+1}(X)] \text{ and } B^p(X) = \text{Im}[d : \Omega^{p-1}(X) \to \Omega^p(X)].
\]

Since \( B^p(X) \) is a linear subspace of \( Z^p(X) \), we can define the \( p \)-th de Rham cohomology group of \( X \) to be the quotient vector space

\[
H^p_{\text{dR}}(X) = Z^p(X)/B^p(X).
\]

Let \( f : X \to Y \) be a smooth map between diffeological spaces. We define

\[
f^* : \Omega^p(Y) \to \Omega^p(X)
\]

by \( f^*(\alpha)(P) = \alpha(f \circ P) \) for any \( \alpha \in \Omega^p(Y) \) and any plot \( P \) of \( X \). Then we have \( f^*(d\alpha) = d(f^*\alpha) \) [2, 6.25]. Thus \( f \) induces a homomorphism \( f^* : H^p_{\text{dR}}(Y) \to H^p_{\text{dR}}(X) \). Let \( A \) be a \( D \)-open set of \( X \). For any \( \alpha \in \Omega^p(X) \), we define \( \alpha|_A = i_A^*(\alpha) \in \Omega^p(A) \), where \( i_A : A \to X \) is the inclusion map.

**Proposition 4.2.** Let \( X \) be a diffeological space. Let \( \{X_\lambda\} \) be a collection of subspaces of \( X \) such that \( X = \bigsqcup_{\lambda \in A} X_\lambda \). Then \( H^p_{\text{dR}}(X) \) and \( \prod_{\lambda \in A} H^p_{\text{dR}}(X_\lambda) \) are isomorphic for each \( p \).
Proof: It is clear that for each $\lambda \in \Lambda$, $X_\lambda$ is $D$-open in $X$ by the definition of coproduct spaces. We define a homomorphism

$$\prod i^*_\lambda : \Omega^p(X) \to \prod_{\lambda \in \Lambda} \Omega^p(X_\lambda)$$

by $\prod i^*_\lambda(\omega) = (i^*_\lambda(\omega))_{\lambda \in \Lambda}$ for any $\omega \in \Omega^p(X)$, where $i^*_\lambda : \Omega^p(X) \to \Omega^p(X_\lambda)$ is the cochain map induced by the inclusion $i^*_\lambda : X_\lambda \to X$. Let $\omega$ be an element of $\text{Ker}(\prod i^*_\lambda)$. Since $(\prod i^*_\lambda)(\omega) = ((i^*_\lambda(\omega))_{\lambda \in \Lambda} = (\omega|_{X_\lambda})_{\lambda \in \Lambda} = 0$, we have $\omega = 0$. Let $(\tau_\lambda)_{\lambda \in \Lambda}$ be an element of $\prod_{\lambda \in \Lambda} \Omega^p(X_\lambda)$. We define $\tau \in \Omega^p(X)$ by for any $\lambda \in \Lambda$, $\tau|_{X_\lambda} = \tau_\lambda$. Then we have $(\prod i^*_\lambda)(\tau) = (\tau_\lambda)_{\lambda \in \Lambda}$ since $X_\lambda \cap X_{\lambda'}$ is empty for each $\lambda \neq \lambda'$.

Let $A$ and $B$ be two $D$-open sets of $X$ such that $X = A \cup B$ holds. Then we have a diagram:

$$
\begin{array}{ccc}
A \cap B & \xrightarrow{i_1} & A \\
| && | \\
\downarrow{i_2} & & \downarrow{j_1} \\
B & \xrightarrow{j_2} & X = A \cup B
\end{array}
$$

consisting of inclusions. Now, consider the following sequence:

$$
0 \to \Omega^p(X) \xrightarrow{j^*_1 \oplus j^*_2} \Omega^p(A) \oplus \Omega^p(B) \xrightarrow{i^*_1 - i^*_2} \Omega^p(A \cap B) \to 0,
$$

where

$$(j^*_1 \oplus j^*_2)(\omega) = (j^*_1(\omega), j^*_2(\omega)), \quad (i^*_1 - i^*_2)(\omega) = i^*_1(\omega) - i^*_2(\omega).$$

Then we have the following.

**Theorem 4.3** (Mayer-Vietoris exact sequence). Let $X$ be a diffeological space. Let $\{A, B\}$ be a $D$-open cover of $X$. If there exists a partition of unity $\phi_i : X \to \mathbb{R}$ ($i = A, B$) subordinate to $\{A, B\}$, then we have a long exact sequence:

$$
\cdots \to H^p_{dR}(X) \xrightarrow{j^*_1 \oplus j^*_2} H^p_{dR}(A) \oplus H^p_{dR}(B) \xrightarrow{i^*_1 - i^*_2} H^p_{dR}(A \cap B) \xrightarrow{\delta} H^{p+1}_{dR}(X) \to \cdots.
$$

**Proof.** To see existence of the Mayer-Vietoris exact sequence, it suffices to show that the sequence (2) is exact for each $p$. We shall show that $j^*_1 \oplus j^*_2$ is injective. Let $\alpha$ be an element of $\text{Ker}(j^*_1 \oplus j^*_2)$. Since $\alpha|_A = 0 = \alpha|_B$, we have $\alpha = 0$. Let $(j^*_1 \oplus j^*_2)(\omega)$ be an element of $\text{Im}(j^*_1 \oplus j^*_2)$. Since $i^*_1 j^*_1(\alpha) = \alpha|_{A \cap B} = i^*_2 j^*_2(\alpha)$, we have

$$(i^*_1 - i^*_2) \circ (j^*_1 \oplus j^*_2)(\omega) = i^*_1 j^*_1(\omega) - i^*_2 j^*_2(\omega) = 0.$$ 

Thus $\text{Im}(j^*_1 \oplus j^*_2) \subset \text{Ker}(i^*_1 - i^*_2)$. Let $\alpha, \beta$ be an element of $\text{Ker}(i^*_1 - i^*_2)$. We define $\omega \in \Omega^p(X)$ by

$$\omega = \begin{cases} 
\alpha & \text{on } A \\
\beta & \text{on } B,
\end{cases}$$

that is, for any plot $P : U \to X$ of $X$, $\omega(P) = \alpha(P|_{P^{-1}(A)})$ on $P^{-1}(A)$ and $\omega(P) = \beta(P|_{P^{-1}(B)})$ on $P^{-1}(B)$. Then $\omega$ is well-defined since $\alpha|_{A \cap B} = \beta|_{A \cap B}$ holds. Clearly, we have $(j^*_1 \oplus j^*_2)(\omega) = (\alpha, \beta)$. Thus $\text{Ker}(i^*_1 - i^*_2) \subset \text{Im}(j^*_1 \oplus j^*_2)$. We
shall show that \((i_1^* - i_2^*)\) is surjective. Let \(\sigma\) be an element of \(\Omega^p(A \cap B)\). Since \(\text{supp} \varphi_B \cap A \subset A \cap B\), we can define \(\eta_A \in \Omega^p(A)\) by

\[
\eta_A = \begin{cases} 
\varphi_B \times \sigma & \text{on } A \cap B \\
0 & \text{on } A \setminus \text{supp} \varphi_B,
\end{cases}
\]

that is, for any plot \(P: V \to A\) of \(A\), \(\eta_A(P) = (\varphi_B \circ P|_{P^{-1}(A \cap B)}) \times \sigma(P|_{P^{-1}(A \cap B)})\) on \(P^{-1}(A \cap B)\) and \(\eta_A(P) = 0\) on \(P^{-1}(A \setminus \text{supp} \varphi_B)\). Then \(\eta_A\) satisfies the smooth compatibility condition \(F^*(\eta_A(P)) = \eta_A(P \circ F)\) for every smooth map \(F\) from an open set of Euclidean spaces to the domain of \(P\). Similarly, we define \(\eta_B \in \Omega^p(B)\) by

\[
\eta_B = \begin{cases} 
-\varphi_A \times \sigma & \text{on } A \cap B \\
0 & \text{on } A \setminus \text{supp} \varphi_A.
\end{cases}
\]

Then we have \((i_1^* - i_2^*)(\eta_A, \eta_B) = \varphi_B \times \sigma + \varphi_A \times \sigma = (\varphi_B + \varphi_A) \times \sigma = \sigma\). Therefore \((i_1^* - i_2^*)\) is surjective. \(\square\)

5. de Rham cohomology with compact support

In this section we define the de Rham cohomology of diffeological spaces with compact support. We shall show that there exists a Mayer-Vietoris exact sequence for de Rham cohomology with compact support.

**Definition 5.1.** Let \(X\) be a diffeological space. Let \(\alpha\) be a differential \(p\)-form on \(X\). An element \(x\) in \(X\) is a support element of \(\alpha\) if and only if for any plot \(P: U \to X\) of \(X\) and any \(r \in U\) such that \(P(r) = x\), \(\alpha(P)(r)\) is nonzero. We call the closure of the set of support elements the support of \(\alpha\), and it will be denoted by \(\text{supp} \alpha\):

\[
\text{supp} \alpha = \{x \in X | \forall P: U \to X, \forall r \in U \text{ s.t. } P(r) = x, \alpha(P)(r) \neq 0\}.
\]

We say that \(\alpha\) is a compactly supported \(p\)-form on \(X\) if the support of \(\alpha\) is \(D\)-compact in \(X\). The set of compactly supported \(p\)-forms of \(X\) is denoted by \(\Omega^p_c(X)\).

It is clear that \(\{\Omega^p_c(X), d\}\) is the subcomplex of the de Rham complex \(\{\Omega^p(X), d\}\). We define the \(p\)-th de Rham cohomology group of \(X\) with compact support to be the quotient space

\[
H^p_c(X) = Z^p_c(X)/B^p_c(X),
\]

where,

\[
Z^p_c(X) = \text{Ker}[d: \Omega^p_c(X) \to \Omega^{p+1}_c(X)] \quad \text{and} \quad B^p_c(X) = \text{Im}[d: \Omega^{p-1}_c(X) \to \Omega^p_c(X)].
\]

We say that a smooth map \(f: X \to Y\) is a proper map if for any \(D\)-compact set \(C\) in \(Y\), the inverse image \(f^{-1}(C)\) is \(D\)-compact in \(X\). Now for any element \(\alpha \in \Omega^p_c(Y)\), \(f^*(\alpha)\) is an element of \(\Omega^p_c(X)\) since \(f^{-1}(\text{supp} \alpha)\) is \(D\)-compact in \(X\). Therefore \(f\) induces a homomorphism \(f^*: H^p_c(Y) \to H^p_c(X)\). If \(X\) is \(D\)-compact, then we have \(H^p_c(X) = H^p_{\text{dr}}(X)\) since \(\Omega^p_c(X) = \Omega^p(X)\) holds.

Let \(X\) be a \(D\)-Hausdorff space. Let \(A\) be a \(D\)-open set of \(X\). If a subset \(C\) of \(A\) is \(D\)-compact in \(A\), then it is \(D\)-compact in \(X\) by Proposition 2.9. Hence the inclusion \(i: A \to X\) induces a map

\[
i_*: \Omega^p_c(A) \to \Omega^p_c(X)
\]
Proposition 5.2. Let \( X \) be a \( D \)-Hausdorff space. Let \( \{X_\lambda\} \) be a collection of subspaces of \( X \) such that \( X \) can be written as a coproduct \( X = \coprod_{\lambda \in \Lambda} X_\lambda \). Then \( \oplus_{\lambda \in \Lambda} H^p_c(X_\lambda) \) and \( H^p_c(X) \) are isomorphic for each \( p \).

**Proof.** Let \( \sum i_{\lambda*}: \oplus_{\lambda \in \Lambda} \Omega^p_c(X_\lambda) \to \Omega^p_c(X) \) be the map defined by

\[
\left( \sum i_{\lambda*} \right) (\omega_{\lambda})_{\lambda \in \Lambda} = \sum_{\lambda \in \Lambda} i_{\lambda*}(\omega_{\lambda}),
\]

where \( i_{\lambda*}: \Omega^p_c(X_\lambda) \to \Omega^p_c(X) \) is the chain map induced by the inclusion \( i_{\lambda} \). Let \((\omega_{\lambda})_{\lambda \in \Lambda}\) be an element of \( \text{Ker}(\sum i_{\lambda*}) \). For each \( \lambda \in \Lambda \) and any plot \( P: U \to X_\lambda \) of \( X_\lambda \), it is also a plot of \( X \). Then we have

\[
\left( \sum i_{\lambda*} \right) (\omega_{\lambda})_{\lambda \in \Lambda}(P) = \sum_{\lambda \in \Lambda} i_{\lambda*}(\omega_{\lambda})(P) = \omega_{\lambda}(P) = 0.
\]

Thus \( \sum i_{\lambda*} \) is injective since \( \omega_{\lambda} = 0 \) for each \( \lambda \in \Lambda \). Next we shall show that \( \sum i_{\lambda*} \) is surjective. Let \( \tau \) be an element of \( \Omega^p_c(X) \). For each \( \lambda \in \Lambda \), \( X_\lambda \) is \( D \)-open in \( X \) by the properties of coproduct diffeology. Since \( \text{supp} \tau \) is \( D \)-compact in \( X \) and \( \{X_\lambda\}_{\lambda \in \Lambda} \) is a cover of \( \text{supp} \tau \), there exists a finite cover \( \{X_{\lambda_i}\}_{1 \leq i \leq m} \) of \( \text{supp} \tau \). We define \((\tau_j)_{j \in \Lambda} \in \oplus_{\lambda \in \Lambda} \Omega^p_c(X_j) \) by

\[
(\tau_j)_{j \in \Lambda} = \left\{ \begin{array}{cl}
\tau|_{X_{\lambda_i}} & j = \lambda_i \\
0 & j \neq \lambda_i.
\end{array} \right.
\]

Then \((\tau_j)_{j \in \Lambda}\) is well-defined since \( X_\lambda \cap X_{\lambda'} \) is empty for each \( \lambda \) and \( \lambda' \) in \( \Lambda \) such that \( \lambda' \neq \lambda \). Clearly, we have \((\sum i_{\lambda*})(\tau_j)_{j \in \Lambda} = \tau. \) \( \square \)

**Theorem 5.3** (Mayer-Vietoris exact sequence). Let \( X \) be a \( D \)-Hausdorff space. Let \( \{A, \lambda \} \) be a \( D \)-open cover of \( X \). If there exists a partition of unity \( \varphi_i: X \to R \) \( i = A, B \) subordinate to \( \{A, B\} \), then we have a long exact sequence:

\[
\to H^p_c(A \cap B) \xrightarrow{i_{1*} \oplus i_{2*}} H^p_c(A) \oplus H^p_c(B) \xrightarrow{j_{1*} - j_{2*}} H^p_c(X) \xrightarrow{\delta} H^{p+1}_c(A \cap B) \to \cdots.
\]

**Proof.** To see existence of the Mayer-Vietoris exact sequence, it suffices to show that the sequence

\[
0 \to \Omega^p_c(A \cap B) \xrightarrow{i_{1*} \oplus i_{2*}} \Omega^p_c(A) \oplus \Omega^p_c(B) \xrightarrow{j_{1*} - j_{2*}} \Omega^p_c(X) \to 0
\]

is exact for each \( p \). It is not difficult to prove that \( i_{1*} \oplus i_{2*} \) is injective and that \( \text{Im}(i_{1*} \oplus i_{2*}) \) is a linear subspace of \( \text{Ker}(j_{1*} - j_{2*}) \). Let \( (\alpha, \beta) \) be an element of \( \text{Ker}(j_{1*} - j_{2*}) \). Since \( j_{1*}(\alpha) = j_{2*}(\beta) \) holds, \( \alpha|_{A \cap B} = \beta|_{A \cap B} \) and \( \alpha = \beta = 0 \) on \( X \setminus (\text{supp} \alpha \cap \text{supp} \beta) \). Therefore we have \((i_{1*} \oplus i_{2*})(\alpha|_{A \cap B}) = (\alpha, \beta) \). We shall show that \( (j_{1*} - j_{2*}) \) is surjective. Let \( \omega \) be an element of \( \Omega^p_c(X) \). Then
Proposition 6.1. homotopic. γ: X are homotopic. Since supp(φA × ω) ⊂ A and supp(φB × ω) ⊂ B, we have

\[ (j_1 - j_2) (j_1 (φA × ω), j_2 (φB × ω)) = j_1 (φA × ω) - j_2 (φB × ω) = (φA + φB) × ω = ω. \]

6. LONG EXACT SEQUENCE FOR PAIR OF DIFFEOREOLOGICAL SPACES

In this section we shall prove Theorem 6.2. Let \( f_0, f_1: X \to Y \) be two smooth maps between diffeological spaces. We say that \( f_0 \) and \( f_1 \) are homotopic if there exists a homotopy \( F: X \times [0,1] \to Y \) satisfying \( F(x,0) = f_0(x) \) and \( F(x,1) = f_1(x) \). If \( f_0 \) and \( f_1 \) are homotopic then \( f_0^*: H^p_{\text{dR}}(Y) \to H^p_{\text{dR}}(X) \) by [2, 6.88].

Let \( A \) be a subspace of a diffeological space \( X \). If there exists a retraction \( \gamma: X \to A \) such that \( |A| = 1 \), then \( A \) is called a retract of \( X \). Moreover we say that \( A \) is a deformation retract of \( X \) if \( \gamma \) and the identity map \( 1_X: X \to X \) are homotopic.

Proposition 6.1. Let \( A \) and \( C \) be subsets of a diffeological space \( X \) such that \( C \subset A \). If there exists a \( D \)-open set \( V \) of \( X \) such that \( A \) is a retract of \( V \), then \( C \) is \( D \)-compact in \( A \) if and only if \( C \) is \( D \)-compact in \( X \).

Proof. If \( C \) is \( D \)-compact in \( A \), then it is \( D \)-compact in \( X \) since all \( D \)-open sets of \( X \) are \( D \)-open in \( A \). Conversely, let \( C \) be a \( D \)-compact set of \( X \). Let \( \{U_\lambda \mid \lambda \in \Lambda\} \) be a cover of \( C \), where \( U_\lambda \) is \( D \)-open in \( A \) for each \( \lambda \in \Lambda \). Then we have

\[ C \subset \bigcup_{\lambda \in \Lambda} U_\lambda = \bigcup_{\lambda \in \Lambda} (\gamma^{-1}(U_\lambda) \cap A) \subset \bigcup_{\lambda \in \Lambda} \gamma^{-1}(U_\lambda), \]

where \( \gamma: V \to A \) is a retraction. Since \( \gamma^{-1}(U_\lambda) \) is \( D \)-open in \( V \), it is \( D \)-open in \( X \) by Lemma 6.2. Since \( C \) is \( D \)-compact in \( X \), \( C \subset \bigcup_{1 \leq i \leq m} \gamma^{-1}(U_\lambda) \) holds. Then we have

\[ C \subset \bigcup_{1 \leq i \leq m} (\gamma^{-1}(U_\lambda) \cap A) = \bigcup_{1 \leq i \leq m} U_\lambda. \]

Therefore \( C \) is \( D \)-compact in \( A \). □

We will prove the following theorem.

Theorem 6.2. Let \( A \) be a \( D \)-compact set in a diffeological subcartesian space \( X \). If there exists a \( D \)-open set \( M \) of \( X \) such that \( A \) is a deformation retract of \( M \), then we have a long exact sequence:

\[ \to H^p_c(X \setminus A) \xrightarrow{i^*} H^p_c(X) \xrightarrow{j^*} H^p_c(A) \xrightarrow{\delta} H^{p+1}_c(X \setminus A) \xrightarrow{i_*} \cdots, \]

where \( i: X \setminus A \to X \) and \( j: A \to X \) are inclusions.

We prove this by using the argument similar to that of [3, Proposition 13.11]. The map \( j \) coincides with the composite of the inclusions:

\[ A \xrightarrow{j_1} M \xrightarrow{j_2} X. \]

Since \( A \) is \( D \)-compact by Proposition 6.1 and it is a deformation retract of \( M \), a map \( \gamma^*: H^p_c(A) = H^p_{\text{dR}}(A) \to H^p_{\text{dR}}(M) \) is an isomorphism, where \( \gamma: M \to A \) is a retraction. Moreover, there exists a \( D \)-open set \( U_A \) of \( X \) such that

\[ A \subset U_A \subset \overline{U_A} \subset M. \]
since $X$ is $D$-normal. Therefore there exists a function $\varphi: X \to \mathbb{R}$ such that $\text{supp}\varphi \subset M$ and $\varphi \equiv 1$ on $\overline{U}_A$ by Corollary 3.4. Then we have the following lemma.

**Lemma 6.3.**

1. $j^*: \Omega^p_c(X) \to \Omega^p_c(A)$ is surjective.
2. For any $\omega$ in $Z^p(A)$, there exists $\tau$ in $\Omega^p(A)$ such that $j^*(\tau) = \omega$ and $d\tau|_{U_A}$ is zero.
3. For any $\omega$ in $\Omega^p_c(X)$ such that $\text{supp}(d\tau) \cap A$ is empty and $j^*(\omega)$ is zero, there exists $\sigma$ in $\Omega^{p-1}_c(X)$ such that $(\omega - d\sigma)|_{U_A}$ is zero.

**Proof.** We shall show the condition (2). Let $\omega$ be an element of $Z^p(A)$. Since $\varphi \times \gamma^*(\omega) \in \Omega^p_c(M)$, $\tau = k_2\ast(\varphi \times \gamma^*(\omega))$ is an element of $\Omega^p_c(X)$. Then for any plot $P: U \to A$ of $A$, we have

$$j^*(\tau)(P) = j^*k_2\ast(\varphi \times \gamma^*(\omega))(P) = (\varphi \times \gamma^*(\omega))(P) = \varphi(P) \times \omega(\gamma \circ P).$$

But $\varphi \equiv 1$ and $\gamma \circ P = P$ on $A$, we have $j^*(\tau)(P) = \omega(P)$. Moreover for any plot $Q: V \to X$ of $X$, we have

$$d\tau|_{U_A}(Q) = d\tau(Q|_{Q^{-1}(U_A)}) = d\omega(\gamma \circ Q|_{Q^{-1}(U_A)}) = 0.$$

Similarly, we can prove the condition (1) in the same argument. We shall show the condition (3). Let $\omega$ be an element of $\Omega^p_c(X)$ such that $\text{supp}(d\tau) \cap A = \emptyset$ and $j^*(\omega) = 0$. Then $k_2\ast(\omega) \in \Omega^p(M)$. We have

$$k_1\ast[k_2\ast(\omega)] = [k_1\ast k_2\ast(\omega)] = [(k_2 \circ k_1)\ast(\omega)] = [j^*(\omega)] = 0.$$

Since $k_1: H^p_{\text{dR}}(M) \to H^p_{\text{dR}}(A) = H^p_c(A)$ is an isomorphism, $[k_2\ast(\omega)] = 0$ holds. Thus there exists $\sigma_0$ in $\Omega^{p-1}_c(M)$ such that $d\sigma_0 = k_2\ast(\omega) = \omega|_{U_A}$. Then $\varphi \times \sigma_0$ is in $\Omega^{p-1}_c(M)$ and $d(\varphi \times \sigma_0)|_{U_A} = \omega|_{U_A}$ since $\text{supp}\varphi \subset M$ and $\varphi \equiv 1$ on $U_A$. Let $\gamma = k_2\ast(\varphi \times \sigma_0) \in \Omega^p_c(X)$. Clearly, $(\varphi - \omega)|_{U_A} = 0$ holds. 

Let $\Omega^p_c(X, A)$ be the kernel of the chain map $j^*: \Omega^p_c(X) \to \Omega^p_c(A)$. Let us denote the cohomology of the subcomplex $\{\Omega^p_c(X, A), d\}$ by $H^p_c(X, A)$. By the property (1) of Lemma 6.3, we have a short exact sequence:

$$0 \to \Omega^p_c(X, A) \xrightarrow{i_*} \Omega^p_c(X) \xrightarrow{j^*} \Omega^p_c(A) \to 0,$$

where $i_*$ is the inclusion map. Thus we have the following long exact sequence:

$$\begin{array}{c}
0 \to \Omega^p_c(X, A) \xrightarrow{i_*} \Omega^p_c(X) \xrightarrow{j^*} \Omega^p_c(A) \xrightarrow{\delta} H^{p+1}_c(X, A) \to \cdots.
\end{array}$$

Now, we get a cochain map $i_*: \Omega^p_c(X \setminus A) \to \Omega^p_c(X, A)$ since for any $\tau$ in $\Omega^p_c(X \setminus A), j^*i_*(\tau) = 0$ holds. Then we have the following proposition.

**Proposition 6.4.** $i_*: H^p_c(X \setminus A) \to H^p_c(X, A)$ is an isomorphism.

**Proof.** We shall show that $i_*$ is injective. Let $[\omega]$ be an element of $\text{Ker} i_*$. Since $i_*[\omega] = 0$ holds, there exists $\tau$ in $\Omega^{p-1}_c(X, A)$ such that $d\tau = i_*[\omega]$. Then for any plot $P$ of $A$, we get $d\tau(P) = i_*(\omega)(P) = 0$. Thus $\text{supp}(d\tau) \cap A$ is empty and $j^*(\tau)$ is zero. Then there exists $\sigma$ in $\Omega^{p-2}(X)$ such that $(\tau - d\sigma)|_{U_A} = 0$. 

by the property (3) of Lemma 6.3. Hence \( \check{\tau} = (\tau - d\sigma)|_{X \setminus A} \) is an element of \( \Omega^{p-1}_c(X \setminus A) \). Hence we have
\[
d\check{\tau} = (d\tau - dd\sigma)|_{X \setminus A} = d\tau|_{X \setminus A} = i_* (\omega)|_{X \setminus A} = \omega.
\]
Therefore \( i_* \) is injective since \( [\omega] = 0 \) holds. Next, we shall show that \( i_* \) is surjective. Let \( [\omega] \) be an element of \( H^p_c(X, A) \). Since \( \omega \) in \( Z^p_c(X, A) \), there exists \( \sigma \) in \( \Omega^{p-1}_c(X) \) such that \( (\omega - d\sigma)|_{U \setminus A} = 0 \) by the property (3) of Lemma 6.3. Since we have
\[
d(j^*(\sigma)) = j^*(d\sigma) = j^*(\omega) = 0,
\]
there exists \( \tau \) in \( \Omega^{p-1}_c(X) \) such that \( j^*(\sigma) = j^*(\tau) \) and \( d\tau|_{U \setminus A} = 0 \) by the property (2) of Lemma 6.3. Then \( \sigma - \tau \) is an element of \( \Omega^{p-1}_c(X, A) \) since \( j^*(\sigma - \tau) = 0 \) holds. Let \( \check{\omega} = (\omega - d(\sigma - \tau))|_{X \setminus A} = (\omega - d\sigma)|_{X \setminus A} + d\tau|_{X \setminus A} \). Then \( \check{\omega} \) in \( \Omega^{p-1}_c(X \setminus A) \) and we get
\[
i_*[\check{\omega}] = i_*[(\omega - d\sigma)|_{X \setminus A}] = [i_* i^*(\omega - d\sigma)] = [\omega - d\sigma] = [\omega].
\]
Therefore \( i_* \) is surjective.
\[\square\]

Therefore we have Theorem 6.2 by the exact sequence (3) and Proposition 6.4.

**Corollary 6.5.** Let \( X \) be a \( D \)-compact diffeological subcartesian space. Let \( A \) be a \( D \)-closed subset of \( X \). If there exists a \( D \)-open subset \( M \) of \( X \) such that \( A \) is a deformation retract of \( M \), then we have a long exact sequence:
\[
\rightarrow H^p_c(X \setminus A) \xrightarrow{i_*} H^p_{dR}(X) \xrightarrow{j^*} H^p_{dR}(A) \xrightarrow{\delta} H^{p+1}_c(X \setminus A) \rightarrow \cdots.
\]

**Proof.** Clearly, \( H^p_c(X) = H^p_{dR}(X) \) and \( H^p_c(A) = H^p_{dR}(A) \) since \( A \) is \( D \)-compact by Proposition 2.7 and Proposition 6.1. Therefore the exactness of the sequence above follows from Theorem 6.2. \[\square\]

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