MULTIFRACTAL ANALYSIS OF RANDOM WEAK GIBBS MEASURES

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Abstract. We describe the multifractal nature of random weak Gibbs measures on some classes of attractors associated with $C^1$ random dynamics semi-conjugate to a random subshift of finite type. This includes the validity of the multifractal formalism, the calculation of Hausdorff and packing dimensions of the so-called level sets of divergent points, and a $0$-$\infty$ law for the Hausdorff and packing measures of the level sets of the local dimension.

1. Introduction. Weak Gibbs measures are conformal probability measures obtained as eigenvectors of Ruelle-Perron-Frobenius operators associated with continuous potentials on topological dynamical systems. When the system $(X, f)$ has nice enough geometric properties, for instance in the case of a conformal repeller, these measures provide natural, and now standard examples of measures obeying the multifractal formalism: their Hausdorff spectrums and $L^q$-spectrums form Legendre pairs.

Specifically, for such a measure $\mu$ on $(X, f)$, the (lower) $L^q$-spectrum $\tau_\mu : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is defined by

$$\tau_\mu(q) = \lim_{r \to 0} \inf \frac{\log \sup \{ \sum_{i} (\mu(B_i))^q \}}{\log(r)},$$

where the supremum is taken over all families of disjoint closed balls $B_i$ of radius $r$ with centers in $\text{supp}(\mu)$; the Hausdorff spectrum of $\mu$ is defined by

$$d \in \mathbb{R} \mapsto \dim_H E(\mu, d),$$

where $\dim_H$ stands for the Hausdorff dimension,

$$E(\mu, d) = \{ x \in \text{supp}(\mu) : \dim_{\text{loc}}(\mu, x) = d \},$$

with

$$\dim_{\text{loc}}(\mu, x) = \lim_{r \to 0^+} \frac{\log(\mu(B(x, r)))}{\log(r)},$$

and we have the duality relation

$$\dim_H E(\mu, d) = \tau^*(\mu)(d) := \inf_{q \in \mathbb{R}} \{ dq - \tau_\mu(q) \}, \quad \forall d \in \mathbb{R},$$

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a negative dimension meaning that the set is empty. In fact, due to the super and submultiplicativity properties associated with \( \mu \), the same equality holds if we replace the \lim by a \( \liminf \) or a \( \limsup \) in the definition of the local dimension.

The rigorous study of these measures started with the Gibbs measures case, which corresponds to H"older continuous potentials, or continuous potentials possessing the so-called bounded distortions property, and in particular on the so-called “cookie-cutter” Cantor sets associated with a \( C^{1+\alpha} \) expanding map \( f \) on the line \([5, 43]\) (see [41] for an extended discussion of dimension theory and multifractal analysis for hyperbolic conformal dynamical systems). This followed seminal works by physicists of turbulence and statistical mechanics pointing the accuracy of multifractals to statistically and geometrically describe the local behavior of functions and measures \([17, 19]\). In the case of Gibbs measures, the \( L^q \)-spectrum of the Gibbs measure is differentiable, and analytic if the potential \( \phi \) is H"older continuous; it is the unique solution \( t \) of the equation \( P(q\phi + t \log \|Df\|) = 0 \), where \( P(\cdot) \) stands for the topological pressure. The general case of continuous potentials was solved later in \([36, 10, 20, 13]\), with the same formula for the \( L^q \)-spectrum. These progress then led to the multifractal analysis of Bernoulli convolutions associated with Pisot numbers \([14, 12]\). Thermodynamic formalism and large deviations are central tool in these studies.

In the context of random dynamical systems, the multifractal analysis of random Gibbs measures (to be defined below) associated with random H"older continuous potentials on attractors of random \( C^{1+\alpha} \) expanding (or expanding in the mean) random conformal dynamics encoded by random subshifts of finite type has been studied in \([23, 15, 35]\). These works, as well as the dimension theory of attractors of random dynamics \([4, 23, 24, 35]\), are based on the thermodynamic formalism for random transforms \([22, 3, 4, 18, 21, 6, 25, 27, 7, 35, 8]\). The multifractal analysis of random weak Gibbs measures is also implicitly considered in \([15]\) (which deals with the multifractal analysis of Birkhoff averages), but the fibers are deterministic, and the techniques developed there seems difficult to adapt in a simple way in the case of random subshifts.

In this paper we consider, on a base probability space \((\Omega, \mathcal{F}, \mathbb{P}, \sigma)\), random weak Gibbs measures on some class of attractors included in \([0, 1]\) and associated with \( C^1 \) random dynamics conjugate (up to countably many points), or semi-conjugate, to a random subshift of finite type. We provide a study of the multifractal nature of these measures, including the validity of the multifractal formalism, the calculation of Hausdorff and packing dimensions of the so-called level sets of divergent points, and a 0-\( \infty \) law for the Hausdorff and packing measures of the level sets of the local dimension. Compared to the above mentioned works, apart the source of new difficulties coming from the relaxation of the regularity properties of the potentials, our assumptions provide a more general process of construction of the random Cantor set in terms of the distribution of the random family of intervals used to refine the construction at a given step: it can contain contiguous intervals (i.e. without gap in between, and even no gap) with positive probability; thus, for instance, it covers the natural families of Cantor sets one can obtain by picking at random a fiber in a Bedford-McMullen carpet. Extensions of our results to the higher dimensional case will be discussed in remark \([1]\). We focus on the one dimensional case because our model will be used in a companion paper to study the multifractal nature of discrete measures obtained as “inverse” of the random weak Gibbs measures considered here.
Section 2 develops background about random dynamical systems and thermodynamic formalism, and presents our main results, namely theorems 2.6 and 2.7 as well as concrete examples of random attractors. Section 3 provides the basic properties that will used in the proof of this theorem in Section 4.

2. Setting and main result. We first need to expose basic facts from random dynamical systems and thermodynamic formalism.

2.1. Random subshift, relativized entropy, topological pressure and random weak Gibbs measures. Random subshift. Denote by Σ the symbolic space $(Z^+)^N$, and endow it with the standard ultrametric distance: for any $u = u_0u_1\ldots$ and $v = v_0v_1\ldots$ in Σ, $d(u, v) = e^{-\inf\{n\in N: u_n \neq v_n\}}$, with the convention $\inf\emptyset = +\infty$. Let $(\Omega, F, P)$ be a complete probability space and $\sigma$ a $P$-preserving ergodic map. The product space $\Omega \times \Sigma$ is endowed with the $\sigma$-field $F \otimes B(\Sigma)$, where $B(\Sigma)$ stands for the Borel $\sigma$-field of $\Sigma$. Let $\Pi_\Omega : \Omega \times \Sigma \rightarrow \Omega$ be defined by $\Pi_\Omega(\omega, \gamma) = \omega$.

Let $l$ be a $Z^+$ valued random variable (r.v.) such that $\int \log(l) \,dP < \infty$ and $P(\{\omega \in \Omega : l(\omega) \geq 2\}) > 0$. Let $A = \{A(\omega) = (A_{r,s}(\omega)) : \omega \in \Omega\}$ be a random transition matrix such that $A(\omega)$ is a $(l(\omega) \times l(\sigma^k(\omega)))$-matrix with entries 0 or 1. We suppose that the map $\omega \mapsto A_{r,s}(\omega)$ is measurable for all $(r,s) \in Z^+ \times Z^+$ and each $A(\omega)$ has at least one non-zero entry in each row and each column. Let $\Sigma_\omega = \{v = v_0v_1\ldots : 1 \leq v_k \leq l(\sigma^k(\omega))$ and $A_{v_k, v_{k+1}}(\sigma^k(\omega)) = 1$ for all $k \in N\}$, and $F_\omega : \Sigma_\omega \rightarrow \Sigma_{\sigma \omega}$ be the left shift ($F_\omega(v)_i = v_{i+1}$) for any $v = v_0v_1\ldots \in \Sigma_\omega$. Each $\Sigma_\omega$ is endowed with the metric induced by $d$ which makes each $F_\omega$ a continuous map. Define $\Omega_\omega = \{\omega, v : \omega \in \Omega, v \in \Sigma_\omega\}$. It is a measurable subset of $\Omega \times \Sigma$. Define the map $F : \Omega_\omega \rightarrow \Omega_\omega$ as $F((\omega, v)) = (\sigma \omega, F_\omega v)$.

For each $n \in N$, define $\Sigma_{\omega,n}$ as the set of words $v = v_0v_1\ldots v_{n-1}$ of length $n$, i.e. such that $1 \leq v_i \leq \min(l(\sigma^k(\omega)))$ for all $0 \leq k \leq n - 1$ and $A_{v_i, v_{i+1}}(\sigma^k(\omega)) = 1$ for all $0 \leq k \leq n - 2$. Define $\Sigma_{\omega,*} = \cup_n \Sigma_{\omega,n}$. For $v = v_0v_1\ldots v_{n-1} \in \Sigma_{\omega,n}$, we write $|v|$ for the length $n$ of $v$, and we define the cylinder $[v]_\omega = \{w \in \Sigma_\omega : w_i = v_i$ for $i = 0, \ldots, n - 1\}$. For $v = v_0v_1\ldots v_{n-1} \in \Sigma_\omega$ and $v = v_0v_1\ldots v_{m-1} \in \Sigma_\omega$, define $[v]_\omega = [v_0v_1\ldots v_{n-1}]$ and $[v]_\omega = [v_0v_1\ldots v_{m-1}]$ for $m, n \in N$ with $m \geq n$. For $v = v_0v_1\ldots v_{n-1} \in \Sigma_{\omega,n}$ and $w = w_0w_1\ldots w_{m-1} \in \Sigma_{\omega,m}$, define $v \wedge w$ to be the longest common prefix of them, that is $v \wedge w = : v_0v_1\ldots v_{j-1}$ if $v_i = w_i$ for $0 \leq i \leq j - 1$ with $v_j \neq w_j$ or $j = \min\{m, n\}$.

For any $s \in \Sigma_{\omega,1}$, $p \geq M(\omega)$ and $s' \in \Sigma_{\sigma^{p+1}\omega,1}$, there is at least one word $s(s', s')' \in \Sigma_{\sigma^p\omega,p-1}$ such that $sv(s', s')' \in \Sigma_{\omega,p+1}$, since $A(\omega)A(\sigma \omega)\ldots A(\sigma^{p-1}\omega)$ is positive. We fix such a $s(s', s')'$ and denote the word $sv(s', s')'$ by $s*s'$. Similarly, for any $w = w_0w_1\ldots w_{m-1} \in \Sigma_{\omega,m}$ and $w' = w'_0w'_1\ldots w'_{m-1} \in \Sigma_{\sigma^n\omega,m}$ with $n, p, m \in N$ and $p \geq M(\sigma^n(\omega))$, we fix $w_{n-1}, w'_0 \in \Sigma_{\sigma^{n}\omega,p}$ (a word depending on $w_{n-1}$ and $w'_0$ only) so that $w*w' := w_0w_1\ldots w_{n-1}w(w_{n-1}, w'_0)w'_0w'_1\ldots w'_{m-1} \in \Sigma_{\omega,n+m+1}$.

Relativized entropy. The space $\Omega_\omega$ is endowed with the $\sigma$-field obtained as the trace of $F \otimes B(\Sigma)$. Here we denote the restriction of $\Pi_\Omega$ to $\Omega_\omega$ by $\Pi_\omega$ as well.
Using the same notations as in [22, 25, 27], let

\[ P_\rho(\Sigma_\Omega) = \{ \rho, \text{ probability measure on } \Sigma_\Omega : \Pi_{\Omega^*} \rho = \rho \circ \Pi_{\Omega}^{-1} = \mathbb{P} \}, \]

and

\[ T_\rho(\Sigma_\Omega) = \{ \rho \in P_\rho(\Sigma_\Omega) : \rho \text{ is } F\text{-invariant} \}. \]

Any \( \rho \in P_\rho(\Sigma_\Omega) \) extends naturally to \((\Omega \times \Sigma, \mathcal{F} \otimes B(\Sigma))\) and as such disintegrates in \( d\rho(\omega, \nu) = d\rho_\omega(\nu) d\mathbb{P}(\omega) \), where the measures \( \rho_\omega, \omega \in \Omega \), are regular conditional probabilities supported on \( \Sigma_\omega \) almost surely. In particular, for any measurable set \( R \subset \Sigma_\Omega \), for \( \mathbb{P}\)-almost every \( \omega \), \( \rho_\omega(R(\omega)) = \rho(R|\Pi_{\Omega}^{-1}(F)) \), where \( R(\omega) = \{ \nu \in \Sigma_\omega : (\omega, \nu) \in R \} \).

Let \( R = \{ R_i \} \) be a finite or countable partition of \( \Sigma_\Omega \) into measurable sets. Then for all \( \omega \in \Omega \), \( R(\omega) = \{ R_i(\omega) \} \) is a partition of \( \Sigma_\omega \).

Given \( \rho \in P_\rho(\Sigma_\Omega) \), the conditional entropy of \( R \) given \( \Pi_{\Omega}^{-1}(F) \) is defined by

\[
H_\rho(R|\Pi_{\Omega}^{-1}(F)) = -\int \sum_i \rho(R_i|\Pi_{\Omega}^{-1}(F)) \log(\rho(R_i|\Pi_{\Omega}^{-1}(F))) d\mathbb{P}
\]

\[
= \int H_{\rho_\omega}(R(\omega)) d\mathbb{P}(\omega)
\]

where \( H_{\rho_\omega}(A) \) denotes the usual entropy of a partition \( A \).

Now, given a finite or countable partition \( Q \) of \( \Sigma_\Omega \), define the fiber entropy of \( F \), also called the relative entropy of \( F \) with respect to \( Q \), as

\[
h_\rho(F, Q) = \lim_{n \to \infty} \frac{1}{n} H_\rho(\bigvee_{i=0}^{n-1} F^{-1} Q|\Pi_{\Omega}^{-1}(F))
\]

(here \( \vee \) denotes the join of partitions).

Then define

\[
h_\rho(F) = \sup_Q h_\rho(F, Q),
\]

where the supremum is taken over all finite or countable measurable partitions \( Q = \{ Q_i \} \) of \( \Sigma_\Omega \) with finite conditional entropy, that is, \( h_\rho(F, Q) < +\infty \). In our setting, we have \( h_\rho(F) \leq \int \log(l) d\mathbb{P} \). The number \( h_\rho(F) \), also denoted \( h(\rho|\mathbb{P}) \) in the literature, is the relativized entropy of \( F \) given \( \rho \). It is also called the fiber entropy of the bundle random dynamics \( F \).

**Topological pressure and random weak Gibbs measures.** We say that a measurable function \( \Phi \) on \( \Sigma_\Omega \) is in \( L^1_{\Sigma_\Omega}(\Omega, C(\Sigma)) \) if

1. \[
C_\Phi =: \int_\Omega \| \Phi(\omega) \|_\infty d\mathbb{P}(\omega) < \infty,
\]

where \( \| \Phi(\omega) \|_\infty =: \sup_{\nu \in \Sigma_\omega} |\Phi(\omega, \nu)| \).

2. For \( \mathbb{P}\)-a.e. \( \omega \), \( \text{var}_n \Phi(\omega) \to 0 \) as \( n \to \infty \), where \( \text{var}_n \Phi(\omega) = \sup \{ |\Phi(\omega, \nu) - \Phi(\omega, \bar{\nu})| : v_i = w_i, \forall i < n \} \).

Now, if \( \Phi \in L^1_{\Sigma_\Omega}(\Omega, C(\Sigma)) \), due to Kingsman’s subadditive ergodic theorem,

\[
\lim_{n \to \infty} \frac{1}{n} \log \sum_{\nu \in \Sigma_\omega} \sup \exp(S_n \Phi(\omega, \nu))
\]

exists for \( \mathbb{P}\)-a.e. \( \omega \) and does not depend on \( \omega \), where \( S_n \Phi(\omega, \nu) = \sum_{i=0}^{n-1} \Phi(F^i(\omega, \nu)) \).

It is called **topological pressure of** \( \Phi \) and is denoted \( P(\Phi) \).
Also, with $\Phi$ is associated the Ruelle-Perron-Frobenius operator $\mathcal{L}_\Phi^\omega : C^0(\Sigma_\omega) \to C^0(\Sigma_{\sigma \omega})$ defined as
\[
\mathcal{L}_\Phi^\omega h(\omega) = \sum_{f^w_\omega = \omega} \exp(\Phi(\omega, w)) h(w), \quad \forall \omega \in \Sigma_{\sigma \omega}.
\]

**Proposition 1.** Removing from $\Omega$ a set of $\mathbb{P}$-probability 0 if necessary, for all $\omega \in \Omega$ there exists $\lambda(\omega) > 0$ and a probability measure $\tilde{\mu}_\omega$ on $\Sigma_\omega$ such that
\[
(\mathcal{L}_\Phi^\omega)^* \tilde{\mu}_\omega = \lambda(\omega) \tilde{\mu}_\omega.
\]

We call the family $\{\tilde{\mu}_\omega : \omega \in \Omega\}$ a random weak Gibbs measure on $\{\Sigma_\omega : \omega \in \Omega\}$ associated with $\Phi$.

### 2.2. A model of random dynamical attractor.

For any $\omega \in \Omega$, let $U^1_\omega = [a_{\omega,1}, b_{\omega,1}], U^2_\omega = [a_{\omega,2}, b_{\omega,2}], \ldots, U^n_\omega = [a_{\omega,s}, b_{\omega,s}] \ldots$ be closed non trivial intervals with disjoint interiors and $b_{\omega,s} \leq a_{\omega,s+1}$. We assume that for each $s \geq 1$, $\omega \mapsto (a_{\omega,s}, b_{\omega,s})$ is measurable, as well as $a_{\omega,1} \geq 0$ and $b_{\omega,1}(\omega) \leq 1$. Let $f^s_\omega(x) = \frac{x - a_{\omega,s}}{b_{\omega,s} - a_{\omega,s}}$ and consider a measurable mapping $\omega \mapsto T^s_\omega$ from $(\Omega, \mathcal{F})$ to the space of $C^1$ diffeomorphisms of $[0, 1]$ endowed with its Borel $\sigma$-field. We consider the measurable $C^1$ diffeomorphism $T^s_\omega : U^n_\omega \to [0, 1]$ by $T^s_\omega = T^s_\omega \circ f^n_\omega$. We denote the inverse of $T^s_\omega$ by $g^n_\omega$. We also define
\[
U^v_\omega = g^{v^n_\omega} \circ g^{v^{n-1}_\omega} \circ \cdots \circ g^{v_1}_\omega([0, 1]), \quad \forall \omega = v_0 \cdots v_{n-1} \in \Sigma_{\omega,n},
\]
\[
X_\omega = \bigcap_{v \geq 1} \bigcup_{v \in \Sigma_{\omega,n}} U^v_\omega,
\]
\[
X_\Omega = \{(\omega, x) : \omega \in \Omega, x \in X_\omega\},
\]
and for all $\omega \in \Omega$, $s \geq 1$ and $x \in U^s_\omega$,
\[
\psi(\omega, s, x) = -\log |(T^s_\omega)'(x)|.
\]

We say that a measurable function $\tilde{\psi}$ defined on $\tilde{U}_\Omega = \{(\omega, s, x) : \omega \in \Omega, 1 \leq s \leq l(\omega), x \in U^n_\omega\}$ is in $L^1_{X_\Omega}(\Omega, \tilde{C}([0, 1]))$ if
1. $\int_{\Omega} \|\tilde{\psi}(\omega)\|_{L^1(\tilde{C}([0, 1]))} d\mathbb{P}(\omega) < \infty$, where $\|\tilde{\psi}(\omega)\|_{L^1(\tilde{C}([0, 1]))} := \sup_{1 \leq s \leq l(\omega)} \sup_{x \in U^n_\omega} |\tilde{\psi}(\omega, s, x)|$,
2. for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\var(\tilde{\psi}(\omega, \varepsilon)) \to 0$ as $\varepsilon \to 0$, where
\[
\var(\tilde{\psi}(\omega, \varepsilon)) = \sup_{1 \leq s \leq l(\omega)} \sup_{x, y \in U^n_\omega, |x - y| \leq \varepsilon} \left| \tilde{\psi}(\omega, s, x) - \tilde{\psi}(\omega, s, y) \right|.
\]

We will make the following assumption:

The function $\psi$ defined above belongs to $L^1_{X_\Omega}(\Omega, \tilde{C}([0, 1]))$ and $\psi$ satisfies the contraction property in the mean
\[
c_\psi := -\int_{\Omega} \sup_{1 \leq s \leq l(\omega)} \sup_{x \in U^n_\omega} \psi(\omega, s, x) d\mathbb{P}(\omega) > 0.
\]

Under this assumption, there is $\mathbb{P}$-almost surely a natural projection $\pi_\omega : \Sigma_\omega \to X_\omega$ defined as
\[
\pi_\omega(\omega) = \lim_{n \to \infty} g^{v^n_\omega} \circ g^{v^{n-1}_\omega} \circ \cdots \circ g^{v_1}_\omega(0).
\]

This mapping may not be injective, but any $x \in X_\omega$ has at most two preimages in $\Sigma_\omega$. We can define $\Psi(\omega, y) = \psi(\omega, v_0, \pi(\omega))$ for $y = v_0v_1 \cdots \in \Sigma_\omega$, then $\Psi \in L^1_{\Sigma_\omega}(\Omega, C(\Sigma))$. By a standard way, we can easily get that for $\mathbb{P}$-almost every $\omega \in \Omega$, the Bowen-Ruelle formula holds, i.e. $\dim_H X_\omega = t_0$ where $t_0$ is the unique root of the equation $P(t \Phi) = 0$.  

\[\]
\[\]
\[\]
2.3. Multifractal analysis of the random weak Gibbs measures. Our results require some additional definitions related to multifractal formalism.

Let \( \mu \) be a compactly supported positive and finite Borel measure on \( \mathbb{R}^n \).

**Definition 2.1.** The (lower) \( L^q \) spectrum \( \tau_m : \mathbb{R} \to \mathbb{R} \cup \{-\infty\} \) and the upper-\( L^q \) spectrum \( \tau_M : \mathbb{R} \to \mathbb{R} \cup \{-\infty\} \) are respectively defined by

\[
\tau_m(q) = \liminf_{r \to 0} \frac{\log \sup \{ \sum_i (\mu(B_i))^q \}}{\log(r)},
\]

\[
\tau_M(q) = \limsup_{r \to 0} \frac{\log \sup \{ \sum_i (\mu(B_i))^q \}}{\log(r)},
\]

where the supremum is taken over all families of disjoint closed balls \( B_i \) of radius \( r \) with centers in \( \text{supp}(\mu) \).

By construction, the function \( \tau_m \) is non decreasing and concave over its domain, which equals \( \mathbb{R} \) or \( \mathbb{R}^+ \) (see [29, 2]).

**Definition 2.2.** The lower and upper large deviations spectra \( LD \) and \( LD \) are given by

\[
LD_m(d) = \lim_{\varepsilon \to 0} \liminf_{r \to 0} \frac{\log \# \{ i : r^{d+\varepsilon} \leq \mu(B(x_i, r)) \leq r^{d-\varepsilon} \}}{\log(r)},
\]

\[
LD_M(d) = \lim_{\varepsilon \to 0} \limsup_{r \to 0} \frac{\log \# \{ i : r^{d+\varepsilon} \leq \mu(B(x_i, r)) \leq r^{d-\varepsilon} \}}{\log(r)},
\]

where the supremum is taken over all families of disjoint closed balls \( B_i = B(x_i, r) \) of radius \( r \) with centers \( x_i \) in \( \text{supp}(\mu) \).

**Definition 2.3.** For all \( x \in \text{supp}(\mu) \), define

\[
\dim_{\text{loc}}(\mu, x) = \liminf_{r \to 0^+} \frac{\log \mu(B(x, r))}{\log r} \quad \text{and} \quad \overline{\dim}_{\text{loc}}(\mu, x) = \limsup_{r \to 0^+} \frac{\log \mu(B(x, r))}{\log r}.
\]

Then, for \( d \leq d' \in \mathbb{R} \), define

\[
E(\mu, d) = \{ x \in \text{supp}(\mu) : \dim_{\text{loc}}(\mu, x) = d \},
\]

\[
\overline{E}(\mu, d) = \{ x \in \text{supp}(\mu) : \overline{\dim}_{\text{loc}}(\mu, x) = d \},
\]

\[
E(\mu, d, d') = E(\mu, d) \cap \overline{E}(\mu, d'),
\]

\[
E(\mu, d, d') = \{ x \in \text{supp}(\mu) : \dim_{\text{loc}}(\mu, x) = d, \overline{\dim}_{\text{loc}}(\mu, x) = d' \}.
\]

It is clear that since \( \mu \) is bounded, \( E(\mu, d, d') = \emptyset \) if \( d' < 0 \).

Finally, define

\[
\dim_H(\mu) = \sup \{ s : \text{ for } \mu\text{-almost every } x \in \text{supp}(\mu), \dim_{\text{loc}}(\mu, x) \geq s \}
\]

and

\[
\dim_p(\mu) = \sup \{ s : \text{ for } \mu\text{-almost every } x \in \text{supp}(\mu), \overline{\dim}_{\text{loc}}(\mu, x) \geq s \}
\]

Equivalent definitions are (see [9]):

\[
\dim_H(\mu) = \inf \{ \dim_H E : E \text{ Borel set, } \mu(E) > 0 \}
\]

and

\[
\dim_p(\mu) = \inf \{ \dim_p E : E \text{ Borel set, } \mu(E) > 0 \}.
\]
Definition 2.4. (Legendre Transform) For any function \( f : \mathbb{R} \to \mathbb{R} \cup \{-\infty\} \) with non-empty domain, its Legendre transform \( f^* \) is defined on \( \mathbb{R} \) by

\[
f^*(d) = \inf_{q \in \mathbb{R}} \{dq - f(q)\} \in \mathbb{R} \cup \{-\infty\}.
\]

One always has (see [27, 29])

\[
\dim_H E(\mu, d) \leq \min(\dim_H E(\mu, d), \dim_H \overline{E}(\mu, d), \dim_P E(\mu, d))
\]

\[
\leq \max(\dim_H E(\mu, d), \dim_H \overline{E}(\mu, d), \dim_P E(\mu, d)) \leq \tau^*_\mu(d).
\]

and (see [2])

\[
\dim_H E(\mu, d) \leq LD_\mu(d) \leq \tau^*_\mu(d) \quad (3)
\]

\[
\dim_P E(\mu, d) \leq \overline{LD}_\mu(d) \leq \tau^*_\mu(d). \quad (4)
\]

Definition 2.5. (Multifractal formalism) We say that \( \mu \) obeys the multifractal formalism at \( d \in \mathbb{R} \cup \{\infty\} \) if \( \dim_H E(\mu, d) = \tau^*_\mu(d) \), and that the multifractal formalism holds (globally) for \( \mu \) if it holds at any \( d \in \mathbb{R} \cup \{\infty\} \) (here a negative dimension means that the set is empty).

The reader should have in mind that if the domain of \( \tau^*_\mu \) is the whole interval \( \mathbb{R} \), then \( \tau^*_\mu(d) \geq 0 \) if and only if \( \tau'_\mu(d) > -\infty \), i.e. \( d \in [\tau'_\mu(+\infty), \tau'_\mu(-\infty)] \). Also, if the multifractal formalism holds at \( d \), then

\[
\dim_H E(\mu, d) = \dim_P E(\mu, d) = \overline{LD}_\mu(d) = \tau^*_\mu(d).
\]

Let \( \phi \in L^1_{\Sigma}(\Omega, \mathcal{H}([0, 1])) \) and consider the function

\[
\Phi(\omega, y) = \phi(\omega, \nu_0, \pi(y)) \quad (y = \nu_0 \nu_1 \cdots \in \Sigma_\omega).
\]

We have \( \Phi \in L^1_{\Sigma}(\Omega, C(\Sigma)) \). Let \( \mu \) be the random weak Gibbs measure on \( \{X_\omega : \omega \in \Omega\} \) obtained as \( \mu = \pi_\omega \mu_{\omega} := \pi_\omega \circ \pi_{\omega}^{-1} \), where \( \bar{\mu} \) is obtained from proposition [1]. Without changing the random measures \( \bar{\mu}_{\omega} \) and \( \mu_{\omega} \), we can assume \( P(\Phi) = 0 \). Then, since equation (2), for any \( q \in \mathbb{R} \), there exists a unique \( T(q) \in \mathbb{R} \) such that \( P(q\Phi - T(q)\Psi) = 0 \), and the mapping \( T \) is concave and non decreasing.

Theorem 2.6. For \( \mathbb{P} \)-a.e. \( \omega \in \Omega \),

1. For all \( q \in \mathbb{R} \), \( \tau_{\mu_{\omega}}(q) = \tau_{\mu_{\omega}}(q) = T(q) = \min_{\rho \in \mathcal{E}(\Sigma_\omega)} \left\{ \frac{h_\rho(F) + q \int \Phi d\rho}{\int \Psi d\rho} \right\} \).

2. The multifractal formalism holds globally for \( \mu_{\omega} \). Furthermore, for all \( d \in [T'(+\infty), T'(-\infty)] \), one has

\[
\dim_H E(\mu_{\omega}, d) = T^*(d) = \max_{\rho \in \mathcal{E}(\Sigma_\omega)} \left\{ \frac{h_\rho(F)}{\int \Psi d\rho} : \int \Psi d\rho = d \right\}.
\]

3. For all \( d \leq d' \in [T'(+\infty), T'(-\infty)] \),

\[
\dim_H E(\mu_{\omega}, d, d') = \inf\{T^*(d), T^*(d')\}
\]

and

\[
\dim_P E(\mu_{\omega}, d, d') = \sup\{T^*(\beta) : \beta \in [d, d']\}.
\]

4. For all \( d \in [T'(+\infty), T'(-\infty)] \),

\[
\dim_H E(\mu_{\omega}, d) = T^*(d), \quad \dim_P E(\mu_{\omega}, d) = \sup\{T^*(d') : d' \geq d\},
\]

\[
\dim_H \overline{E}(\mu_{\omega}, d) = T^*(d) \text{ and } \dim_P \overline{E}(\mu_{\omega}, d) = \sup\{T^*(d') : d' \leq d\}.
\]
We refer to [34] for the notions of generalized Hausdorff and packing measures $\mathcal{H}^g$ and $\mathcal{P}^g$ associated with a gauge function $g$, i.e. a function $g : [0, \infty) \to [0, \infty)$ which is non decreasing and satisfies $g(0) = 0$.

**Theorem 2.7.** For all $d \in [T'(+\infty), T'(-\infty)]$ such that $T^*(d) < \max(T^*)$, for all gauge function $g$, the following zero-infinity laws hold:

$$\mathcal{H}^g(E(\mu, d)) = \begin{cases} 0 & \text{if } \limsup_{r \to 0} \frac{\log g(r)}{\log r} > T^*(d), \\ +\infty & \text{otherwise} \end{cases}$$

and

$$\mathcal{P}^g(E(\mu, d)) = \begin{cases} 0 & \text{if } \liminf_{r \to 0} \frac{\log g(r)}{\log r} > T^*(d), \\ +\infty & \text{otherwise} \end{cases}.$$

Let us put our result in perspective with respect to the existing literature.

The study achieved in [41, 42] leads to the multifractal nature of Gibbs measures projected on some random Cantor sets whose construction assumes a strong separation condition for the pieces of the construction. About the same time, the multifractal analysis of random Gibbs measures and Birkhoff averages on random Cantor sets and the whole torus were obtained in [23, 24]; when the support of the measure is a Cantor set, a strong separation condition is assumed as well. More recently, in [15], the multifractal analysis for disintegrations of Gibbs measures on $\{1, \ldots, m\}^\mathbb{N} \times \{1, \ldots, m\}^\mathbb{N}$ was achieved as a consequence of the multifractal analysis of conditional Birkhoff averages of random continuous potentials (not $C^\alpha$). The approach developed there could, with some effort, be adapted to derive our results on weak Gibbs measures if we worked with random fullshifts only. However, as we already said it in the beginning of the introduction, the method cannot be extended easily to the random subshift, and our view point will be different. In [15], the authors start by establishing large deviations results, and then use them to construct by concatenation Moran sets of arbitrary large dimension in the level sets $E(\mu_\omega, d)$; we will concatenate information provided by random Gibbs measures associated with Hölder potentials which approximate the continuous potentials associated with the random weak Gibbs measure and the random maps generating the attractor $X_\omega$. This will provide us with a very flexible tool from which, for instance, we will deduce the result about the sets $E(\mu_\omega, d, d')$. In this sense, our results also complete a part of those obtained in [35] which, in particular, achieves the multifractal analysis of random Gibbs measures on random Cantor sets obtained as the repeller of random conformal maps.

The multifractal analysis of Birkhoff averages on random conformal repellers of $C^1$ expanding maps is studied in [46], where the random dynamics is in fact coded by a non random subshift of finite type, and the random potentials that are considered satisfy an equicontinuity property stronger than the one we require.

The sets $E(\mu, d, d')$ were studied for deterministic Gibbs measures on conformal repellers and for self-similar measures in [16, 38, 11, 39].

Finally, in [31], zero-infinity laws are established for Besicovitch subsets of self-similar sets of the line. This inspired theorem 2.7 of which the results in [31] turn out to be a special case. Also, in [32], a zero-infinity law is established for the Hausdorff and packing measure of sets of generic points of invariant measures on a conformal repeller.

**Remark 1.** The approach used in this paper can be extended to the higher dimension if the random attractor can be represented as follows: For any $\omega \in \Omega$, let
$U^1_\omega, U^2_\omega, \cdots, U^{d(\omega)}_\omega$ be closed sets which are the closures of their interiors supposed to be pairwise disjoint. We assume that

1. $U_s^\omega \subset [0,1]^d$ for any $1 \leq s \leq l(\omega)$.
2. There exists a measurable $C^1$ diffeomorphism $T_s^\omega : U_s^\omega \to [0,1]^d$. We denote the inverse of $T_s^\omega$ by $g_s^\omega$. Furthermore, $T : \tilde{U}_\Omega = \{ (\omega, s, x) : \omega \in \Omega, 1 \leq s \leq l(\omega), x \in U_s^\omega \} \to [0,1]^d$ which is defined as $(\omega, s, x) \mapsto T_s^\omega(x)$ is measurable.
3. There exists a function $\psi \in L^1_{\tilde{X}_\Omega}(\Omega, \tilde{C}([0,1]^d))$ such that

$$\exp(-\var(\psi, \omega, d(x,y))) \leq \frac{d(T_s^\omega(x), T_s^\omega(y))}{\exp(-\psi(\omega, s, x))d(x,y)} \leq \exp(\var(\psi, \omega, d(x,y)))$$

for any $x, y \in U_s^\omega$ with $1 \leq s \leq l(\omega)$.

Here, we say that a measurable function $\bar{\psi}$ defined from $\tilde{U}_\Omega$ to $\mathbb{R}$ is in $L^1_{\tilde{X}_\Omega}(\Omega, \tilde{C}([0,1]^d))$ if

- $\int_{\Omega} \| \bar{\psi}(\omega) \|_\infty dP(\omega) < \infty$, where $\| \bar{\psi}(\omega) \|_\infty := \sup_{1 \leq s \leq l(\omega)} \sup_{x \in U_s^\omega} |\bar{\psi}(\omega, s, x)|$,
- for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\var(\bar{\psi}, \omega, \varepsilon) \to 0$ as $\varepsilon \to 0$, where

$$\var(\bar{\psi}, \omega, \varepsilon) = \sup_{1 \leq s \leq l(\omega)} \sup_{x,y \in U_s^\omega} \sup_{d(x,y) \leq \varepsilon} |\bar{\psi}(\omega, s, x) - \bar{\psi}(\omega, s, y)|.$$

4. Equation (2) holds.

Now, we can define

$$U^\omega_\nu = g_0^\nu \circ g_1^\nu \circ \cdots \circ g_{n-1}^\nu([0,1]^d), \quad \forall \nu = v_0 \cdots v_{n-1} \in \Sigma_{\omega,n}$$

$$X_\omega = \bigcap_{n \geq 1} \bigcup_{v \in \Sigma_{\omega,n}} U^v_\omega$$

$$X_\Omega = \{ (\omega, x) : \omega \in \Omega, x \in X_\omega \}.$$

Nevertheless there is a difference in the estimation of the local dimensions of measures. As we will see in this paper, a building block in our proofs is the comparison of the mass assigned by a random Gibbs measure to neighboring basic intervals of the form $U^\nu_\omega$, in order to control the mass assigned to centered intervals by a random weak Gibbs measure, as well as some auxiliary measures obtained by concatenation of pieces of random Gibbs measures (this point of view is fruitful in the study of the discrete inverses of random weak Gibbs measures in the companion paper mentioned at the beginning of this introduction). In higher dimension the situation is different in general. If a strong separation condition is satisfied by the basic sets $U^\nu_\omega$, there is no much difference with the 1 dimensional case. Otherwise, one can adapt the method used in [10] for self-conformal measures and under the open set condition, which, for any Gibbs measure $\nu$, consists in controlling the asymptotic behavior of the distance of $\nu$-almost every point $x$ to the boundary of the basic set of the $n$-th generation containing $x$. We omit the details.

### 2.4. Examples of random attractor

We end this section with examples illustrating our assumptions on the random attractors considered in this paper. As a first example, one has the fibers of McMullen-Bedford self-affine carpets, and more generally the Gatzouras-Lalley self-affine carpets [28], which naturally illustrate the idea that at a given step of the construction two consecutive intervals $U^s_\omega$ and $U^{s+1}_\omega$ may touch each other. In [30], Luzia considers a class of expanding maps of the 2-torus of the form $f(x,y) = (a(x,y), b(y))$ that are $C^\infty$-perturbations of Gatzouras-Lalley carpets, whose fibers illustrate our purpose with nonlinear maps. These
examples are associated with random fullshifts. Let us give a first more explicit example associated with a random subshift and a piecewise linear random maps.

Let \( \Omega = \Gamma := \mathbb{Z}^+ \times \mathbb{Z}^+ \times \cdots, \mathcal{F} \) be the \( \sigma \)-algebra generated by the cylinders \([n_1 n_2 \cdots n_k] \ (k \in \mathbb{N}, n_i \in \mathbb{Z}^+ \text{ for } 1 \leq i \leq k)\), and \( \mathbb{P} \) the probability measure on \((\Omega, \mathcal{F})\) defined by

\[
\mathbb{P}([n_1 n_2 \cdots n_k]) = \frac{1}{n_1(n_1 + 1)} \cdot \frac{1}{n_2(n_2 + 1)} \cdots \frac{1}{n_k(n_k + 1)}.
\]

Also, let \( \sigma \) be the shift map on \( \Omega \). Such a system is ergodic. It satisfies the conditions we need.

For \( \underline{n} = n_1 n_2 \cdots n_k \cdots \in \Gamma \), define \( l(\underline{n}) = n_1 \) and \( A(\underline{n}) \) the \( n_1 \times n_2 \)-matrix with all entries equal to 1 if \( n_2 \neq n_2 - 1 \) or \( n_1 = 2 \), and the \( n_1 \times n_2 \)-matrix whose \( n_1 - 1 \) first rows have entries equal to 1 and the entries of the \( n_1 \)-th row equal 0 except that \( A_{n_1-1,1}(\underline{n}) = 1 \). It is easy to check that both \( l \) and \( A \) are measurable, that \( \int \log l \, d\mathbb{P} < +\infty \), and \( l \) and \( A \) define a random subshift, which is not a fullshifts. Also, the integer \( M = \inf\{m \in \mathbb{N} : A(\omega)A(\sigma \omega) \cdots A(\sigma^{m-1} \omega) \text{ is positive} \} \) is measurable, since for any \( k \in \mathbb{N} \),

\[
\{ \underline{n} \in \Omega : M(\underline{n}) = k \} = [(k + 1)k(k - 1) \cdots 2].
\]

Notice that both \( l \) and \( M \) are unbounded.

Then we set \( T^{x}_{\underline{n}}(x) = n_1 x \mod 1 \) for \( x \in [\frac{1}{n_1}, \frac{1}{n_1}] \) and for \( i = 1, 2, \cdots, n_1 \).

In fact, the measure \( \mathbb{P} \) defined above is a special example of a Gibbs measure on \((\Omega, \mathcal{F}, \sigma)\) (see [44, 45]). So we can enrich the previous construction by considering any such measure \( \mathbb{P} \) for which \( \int \log l \, d\mathbb{P} < +\infty \). For the mappings maps \( T_{\omega}^{x} \), here is a way to provide a non trivial example, which seems to be not covered by the existing literature.

Start with a family \( \{ \varphi_{\omega, s} \}_{s \in \mathbb{N}} \) of random \( C^1 \) diffeomorphisms of \([0, 1]\) such that at least one \( \varphi'_{s, \omega} \) is nowhere \( C^r \) with positive probability. Assume that there exists a random variable \( a_0 \) taking values in \((0, 1]\) and such that

\[
\inf_{1 \leq s \leq l(\omega), x \in [0, 1]} |\varphi'_{s, \omega}(x)| \geq a_0(\omega).
\]

Let \( T_{\omega}^{x} = \varphi_{s, \omega} \circ f_{\omega}^{s} \), where \( f_{\omega}^{s} \) is the linear map from \( U_{\omega}^{s} \) onto \([0, 1]\). Then, the constant \( c_{\omega} \) of equation (2) satisfies

\[
c_{\omega} \geq \int_{\Omega} \left[ \log(a_0(\omega)) - \sup_{1 \leq s \leq l(\omega)} \log(|U_{\omega}^{s}|) \right] \, d\mathbb{P}(\omega).
\]

Thus, we require that

\[
\int_{\Omega} \left[ \log(a_0(\omega)) - \sup_{1 \leq s \leq l(\omega)} \log(|U_{\omega}^{s}|) \right] \, d\mathbb{P}(\omega) > 0.
\]

This allows some \( T_{\omega}^{x} \) be not uniformly expanding, but ensures expansiveness in the mean. It is easily seen that the Lebesgue measure of \( X_{\omega} \) is almost surely bounded by \( \prod_{i=0}^{n-1} \left( \sum_{1 \leq s \leq l(\omega)} |U_{\sigma^{i} \omega}^{s}| / a_0(\sigma^{i} \omega) \right) \) for all \( n \geq 1 \). Thus, if we strengthen our requirement by assuming that

\[
\int_{\Omega} \left[ \log(a_0(\omega)) - \log \left( \sum_{1 \leq s \leq l(\omega)} |U_{\omega}^{s}| \right) \right] \, d\mathbb{P}(\omega) > 0,
\]

then the Lebesgue measure of \( X_{\omega} \) is 0 almost surely.
Now let us provide a completely explicit illustration of the last idea (we will work with a random fullshifts for simplicity of the exposition).

We take \((\Omega,\mathcal{F},\mathbb{P},\sigma)\) as the fullshifts \(((0,1,2)^\mathbb{N},\mathcal{F},\mathbb{P},\sigma)\). For any \(n\)-th cylinder \([\omega_0\omega_1\cdots\omega_{n-1}]\subset\Omega\) we set \(\mathbb{P}([\omega_0\omega_1\cdots\omega_{n-1}]) = \frac{1}{3^n}\). It is the unique ergodic measure of maximal entropy for the shift map.

Let \(l\) be a random variable depending on \(\omega_0\) only, which is given by
\[
l(\omega) = \begin{cases} 
4 & \omega_0 = 0 \\
1 & \omega_0 = 1 \\
3 & \omega_0 = 2 
\end{cases}
\]
The entries of the random transition matrix are always 1 (we consider the random fullshifts). We assume that the map \(T(\omega,x)\) just depends on \(\omega_0\) and \(x\).

If \(\omega_0 = 0\), let \(\varphi_{s,\omega}(x) = x\) for \(s = 1,2,3,4\) and \(U^1_\omega = [0,1/4]\), \(U^2_\omega = [1/4,1/2]\), \(U^3_\omega = [1/2,3/4]\) and \(U^4_\omega = [3/4,1]\). In this case, we know that \(a_0(\omega) = 1\); notice that the intervals \(U^j_\omega\), \(1 \leq s \leq 4\) cover the interval \([0,1]\).

If \(\omega_0 = 1\), let \(h(x) = 6 + \sum_{j=1}^{\infty} j^{-2}\sin(2\pi jx)\). Define
\[
\varphi_{1,\omega}(x) = \int_0^x h(t)\,dt,
\]
and \(U^1_\omega = [0,1]\). In this case we can choose \(a_0(\omega) = 1/2\). It is easy to check that \(T^1_\omega\) is not expanding on some interval; furthermore it is just of class \(C^1\) since \(h\) is \(\epsilon\)-Hölder for any \(\epsilon \in (0,1)\).

If \(\omega_0 = 2\), let \(\varphi_{s,\omega}(x) = x\) for \(s = 1,3\) and \(\varphi_{2,\omega}(x) = \frac{7x}{8} + \frac{x^2}{8}\), and \(U^1_\omega = [0,1/9]\), \(U^2_\omega = [1/9,2/9]\), \(U^3_\omega = [2/3,7/9]\). It is easily checked that the left derivative of \(T^1_\omega\) and the right derivative of \(T^2_\omega\) do not coincide, so the dynamics is not the restriction of a random conformal map. In this case we can choose \(a_0(\omega) = 7/8\).

Also,
\[
\int_\Omega \left[ \log(a_0(\omega)) - \log\left( \sum_{1 \leq s \leq l(\omega)} |U^s_\omega| \right) \right] \,d\mathbb{P}(\omega) = \frac{\log 21 - \log 16}{3} > 0,
\]
so that all the conditions hold.

3. Basic properties of random weak Gibbs and random Gibbs measures. Approximation of \((\Phi,\Psi)\) by random Hölder potentials. This section prepares the proofs of our main results. Sections 3.1 and 3.2 present basics properties of random weak Gibbs and random Gibbs measures. Section 3.3.1 provides an approximation of \((\Phi,\Psi)\) by a family \(\{(\Phi_i,\Psi_i)\}_{i \geq 1}\) of random Hölder potentials. Section 3.3.2 provides the approximation of \((T,T^*)\) by \((T_i,T^*_i)\). Section 3.3.3 derives some related properties of the associated pressure functions, which yield the variational formulas appearing in theorem \ref{3.6}. Section 3.3.4 presents properties related to the random Gibbs measures associated with the couple \((\Phi_i,\Psi_i)\), which will be used as building blocks in the concatenation of measures used in the proof of the main parts of theorem \ref{2.6} and of theorem \ref{2.7} (section 4).

3.1. Properties of weak Gibbs measures. Fix a potential \(\Phi \in \mathbb{L}_1^{1,\infty}(\Omega,C(\Sigma))\) (here \(P(\Phi)\) may not be 0). Since \(\text{var}_n \Phi(\omega) \to 0\) as \(n \to 0\) and \((\text{var}_n \Phi)_{n \geq 1}\) is bounded in \(L^1\) norm, using Maker’s ergodic theorem \ref{33}, we can get
\[
\lim_{n \to \infty} \frac{1}{n} V_n \Phi(\omega) = 0, \quad \mathbb{P}\text{-almost surely,}
\]
where
\[ V_n \Phi(\omega) := \sum_{i=0}^{n-1} \text{var}_{\sigma^i}(\Phi(\omega)) = o(n). \]

Due to [1] and Birkhoff ergodic theorem, setting \( S_n \| \Phi(\omega) \|_\infty = \sum_{i=0}^{n-1} \| \Phi(\sigma^i(\omega)) \|_\infty \), for any sequence \((a_n)_{n \geq 0}\) of positive numbers such that \( a_n = o(n) \), \( \mathbb{P} \)-almost surely we have
\[ |S_n \| \Phi(\omega) \|_\infty - S_n - a_n \| \Phi(\omega) \|_\infty| = nC_\Phi - (n - a_n)C_\Phi + o(n) = o(n). \]

**Definition 3.1.** A family \( u = \{u_{n,\omega} : \Sigma_{n,\omega} \to \Sigma_{\omega}\}_{n \in \mathbb{N}} \) of measurable maps satisfying \((u_{n,\omega}(v)))_n = v\) for all \((n, \omega) \in \mathbb{N} \times \Omega\) and \(v \in \Sigma_{n,\omega}\) is called an extension. We say that it is measurable if the mapping \((\omega, \varphi) \mapsto u_{n,\omega}(\varphi)\) is measurable for all \(n \in \mathbb{N}\).

Let \( u = \{u_{n,\omega}\} \) be an extension and \( \Phi \in L^1_{\Sigma_1}(\Omega, C(\Sigma)) \). Then for \((n, \omega) \in \mathbb{N} \times \Omega\)
\[ Z_{u,n}(\Phi(\omega)) := \sum_{v \in \Sigma_{n,\omega}} \exp \left( S_n(\Phi(\omega), u_{n,\omega}(v)) \right) \]
is called \(n\)-th partition function of \(\Phi\) in \(\omega\) with respect to \(u\).

Due to the assumption \(\log(l) \in L^1(\Omega, \mathbb{P})\), using the same method as in [3, 18, 21, 25, 27, 7, 8], it is easy to prove the following lemma.

**Lemma 3.2.** Let \( u \) be any extension and \( \Phi \in L^1_{\Sigma_1}(\Omega, C(\Sigma)) \).

Then \( \lim_{n \to \infty} \frac{1}{n} \log Z_{u,n}(\Phi(\omega)) = P(\Phi) \) for \(\mathbb{P}\)-almost every \(\omega \in \Omega\). This limit is independent of \(u\).

Furthermore, the following variational principle holds:
\[ P(\Phi) = \sup_{\rho \in \mathcal{F}(\Sigma_1)} \left\{ h_\rho(F) + \int \Phi d\rho \right\}. \]

Now let
\[ \lambda(\omega, n) = \lambda(\omega) \cdot \lambda(\sigma \omega) \cdot \cdots \cdot \lambda(\sigma^{n-1} \omega), \]
where \(\lambda(\omega)\) is defined as in proposition [1]. The following lemma is direct when the potential \(\Phi\) possesses bounded distorsions so that the Ruelle-Perron-Frobenius theorem holds for the operator \(\mathcal{L}_\Phi^\omega\). For general potentials in \(L^1_{\Sigma_1}(\Omega, C(\Sigma))\) we need a proof.

**Lemma 3.3.** One has \( \lim_{n \to \infty} \frac{\log \lambda(\omega, n)}{n} = P(\Phi) \) for \(\mathbb{P}\)-almost every \(\omega \in \Omega\).

**Proof.** For any \(M_1 \in \mathbb{N}\), let \(\Omega^{M_1} = \{\omega \in \Omega : M(\omega) \leq M_1\}\). Fix \(M_1\) large enough so that \(\mathbb{P}(\Omega^{M_1}) > 0\). For each \(\omega \in \Omega\), let \(b_k(\omega)\) be the \(k\)-th return time of \(\omega\) to the set \(\Omega^{M_1}\), i.e. \(b_1 = \inf\{n \in \mathbb{N} : \sigma^n \omega \in \Omega^{M_1}\}\) and for \(k > 1\), \(b_k = \inf\{n \in \mathbb{N} : \sigma^n \omega \in \Omega^{M_1}, n > b_{k-1}\}\). By the Birkhoff ergodic theorem we get \(\lim_{k \to \infty} \frac{b_k}{k} = \frac{1}{\mathbb{P}(\Omega^{M_1})}\) for \(\mathbb{P}\)-almost every \(\omega \in \Omega\). Then \(\lim_{k \to \infty} \frac{b_{k+1} - b_k}{b_k} = 0\) for \(\mathbb{P}\)-almost every \(\omega \in \Omega\), which implies that \(M(\sigma^n \omega) = o(n)\).

Now we claim that for any \(\psi \in \Sigma_{\sigma^n \omega}\) we have
\[ Z_{u,n-M(\sigma^n \omega)}(\Phi(\omega) \exp(-o(n))) \leq \mathcal{L}_\Phi^\omega \psi \leq Z_{u,n}(\Phi(\omega) \exp(o(n))). \]
The second inequality uses the fact that we work with a subshift as well as \( (5). \) We just prove the first inequality: for \( n \) large enough so that \( M(\sigma^n_\omega) \leq n, \)

\[
L_\Phi^{\omega,n}(v) = \sum_{w \in \Sigma_{\omega,n}, wv \in \Sigma_\omega} \exp(S_n \Phi(\omega, w)) \geq \sum_{w' \in \Sigma_{\omega,n-M(\sigma^n_\omega)} \Phi'} \exp \left( S_n - M(\sigma^n_\omega)(\omega) \right) - o(n) \]

\[
= Z_{\omega,n-M(\sigma^n_\omega)}(\Phi, \omega) \exp(-o(n)).
\]

By using the topological mixing property and preserving for each \( w' \in \Sigma_{\omega,n-M(\sigma^n_\omega)} \) only one path of length \( M(\sigma^n_\omega) \) from \( w' \) to \( v \), the inequality follows from \( (5), \) \( M(\sigma^n_\omega) = o(n) \) and \( (6). \)

Now, since \( \lambda(\omega, n) = \int L_\Phi^{\omega,n}(v) d\tilde{\mu}_{\sigma^n_\omega}(v) \), we can easily get the result from lemma 3.3 and the fact that \( M(\sigma^n_\omega) = o(n). \)

For each \( \omega \in \Omega \), let

\[
D(\omega) = \frac{1}{\lambda(\omega, M(\omega))} \exp(-S_M(\omega)\|\Phi(\omega)\|_\infty).
\]

Then \( D(\omega) > 0 \) for \( \mathbb{P} \)-almost every \( \omega \in \Omega \). From \( M(\sigma^n_\omega) = o(n) \), \( (6) \) and lemma 3.3 we can get that \( \log D(\sigma^n_\omega) = o(n) \) \( \mathbb{P} \)-almost surely.

Recall that by proposition 1 for \( \mathbb{P} \)-a.e. \( \omega \in \Omega \), the measures \( \tilde{\mu}_\omega \) satisfy

\[
(L_\Phi^s)^{s} \tilde{\mu}_{\sigma^n_\omega} = \lambda(\omega)\tilde{\mu}_\omega.
\]

**Proposition 2.** For \( \mathbb{P} \)-a.e. \( \omega \in \Omega \), for any \( n \in \mathbb{N} \), for all \( v = v_0 v_1 \ldots v_{n-1} \in \Sigma_{\omega,n} \), one has

\[
\frac{D(\sigma^n_\omega)}{\lambda(\omega, n)} \exp \left( \inf_{v \in [v]_\omega} S_n \Phi(\omega, v) \right) \leq \tilde{\mu}_\omega([v]_\omega) \leq \frac{1}{\lambda(\omega, n)} \exp \left( \sup_{v \in [v]_\omega} S_n \Phi(\omega, v) \right),
\]

so that

\[
\exp(-\epsilon_n n) \leq \frac{\tilde{\mu}_\omega([v]_\omega)}{\exp(S_n \Phi(\omega, v) - \log(\lambda(\omega, n)))} \leq \exp(\epsilon_n n)
\]

for any \( v \in [v]_\omega \), where \( \epsilon_n \) does not depend on \( v \) and tends to 0 as \( n \to \infty \).

**Proof.** Let us deal first with the case \( n = 1 \).

Fix \( 1 \leq i \leq l(\omega) \). For any \( 1 \leq j \leq l(\sigma^M(\omega)_\omega) \), there exists \( w \in \Sigma_{\sigma^n_\omega, M(\omega)_1} \) such that \( iwj \in \Sigma_{\omega,1+M(\omega)} \). Due to proposition 1 we have

\[
\tilde{\mu}_\omega([iwj]_\omega) = \frac{1}{\lambda(\omega, 1)} \int \lambda(\omega, M(\omega)) \Phi(\omega, w) d\tilde{\mu}_{\sigma^{M(\omega)}_\omega},
\]

where \( L_\Phi^{\omega,n} = L_\Phi^{\omega,n-1} \circ \cdots \circ L_\Phi^0 \). This implies

\[
\tilde{\mu}_\omega([iwj]) \geq \inf_{w \in [iwj]_\omega} \exp(S_{M(\omega)} \Phi(\omega, w)) = \lambda(\omega, 1) \tilde{\mu}_{\sigma^M(\omega)_\omega}([j]_{\sigma^M(\omega)_\omega}).
\]

Then \( \tilde{\mu}_\omega([i]) \geq D(\omega) \) follows after summing over \( 1 \leq j \leq l(\sigma^M(\omega)_\omega) \). The upper bound \( \tilde{\mu}_\omega([i]) \leq 1 \) is obvious.

The general case is achieved similarly:

If \( v \in \Sigma_{\omega,n} \), for each \( 1 \leq j \leq l(\sigma^{n+M(\sigma^n_\omega)-1}_\omega) \), there exists \( w \in \Sigma_{\sigma^n_\omega, M(\sigma^n_\omega)-1} \) such that \( vwj \in \Sigma_{\omega,n+M(\sigma^n_\omega)} \). One has

\[
\tilde{\mu}_\omega([vwj]_\omega) = \frac{1}{\lambda(\omega, n)} \lambda(\sigma^n_\omega, M(\sigma^n_\omega)) \int L_\Phi^{\omega,n+M(\sigma^n_\omega)} 1_{[vwj]_\omega} d\tilde{\mu}_{\sigma^n+M(\sigma^n_\omega)}.
\]
from which we get
\[ \bar{\mu}_\omega([v\omega]) \geq \frac{1}{\lambda(\omega, n)} \cdot D(\sigma^n \omega) \exp(\inf_{\omega \in [v\omega]} S_n \Phi(\omega, v)) \bar{\mu}_{\sigma^n M(\sigma^n \omega) \omega}(\sigma^n \omega) \cdot \bar{\mu}_{\sigma^n M(\sigma^n \omega) \omega}(\sigma^n \omega) \exp(\inf_{\omega \in [v\omega]} S_n \Phi(\omega, v)). \]

Then, taking the sum over \( 1 \leq j \leq l(\sigma^n + M(\sigma^n) - 1) \omega \) we get
\[ \bar{\mu}_\omega([v\omega]) \geq \frac{1}{\lambda(\omega, n)} \cdot D(\sigma^n \omega) \exp(\inf_{\omega \in [v\omega]} S_n \Phi(\omega, v)). \]

The inequality \( \bar{\mu}_\omega([v\omega]) \leq \frac{1}{\lambda(\omega, n)} \cdot D(\sigma^n \omega) \exp(\sup_{\omega \in [v\omega]} S_n \Phi(\omega, v)) \) is direct from the equality
\[ \bar{\mu}_\omega([v\omega]) = \frac{1}{\lambda(\omega, n)} \int \mathcal{L}^\omega \mu_{[v\omega]} \, d\bar{\mu}_{\sigma^n \omega}. \]

Finally we conclude with \( \square \) and log \( D(\sigma^n \omega) = o(n) \).

For any \( \gamma \in L^1_{\mathcal{X}}(\Omega, \bar{C}([0,1])) \) and any \( z \in U^v_\omega \), let
\[ S_n \gamma(\omega, z) = \sum_{i=0}^{n-1} \gamma(\sigma^i \omega, v_i, T_{\sigma^i \omega}^{-1} \cdots T_{\omega}^{-1} z). \]

From the Lagrange’s finite-increment theorem, distortions and proposition 2 using standard estimates we can get the following proposition.

**Proposition 3.** For \( \mathbb{P} \)-almost every \( \omega \in \Omega \), there are sequences \((e(\psi, \omega, n))_{n \geq 0}\) and \((e(\phi, \omega, n))_{n \geq 0}\) of positive numbers, that we also denote as \((e(\Psi, \omega, n))_{n \geq 0}\) and \((e(\Phi, \omega, n))_{n \geq 0}\), decreasing to 0 as \( n \to +\infty \), such that for all \( n \in \mathbb{N} \), for all \( v = v_0 v_1 \cdots v_n \in \Sigma_{\omega,n} \), we have:

1. For all \( z \in U^v_\omega \),
   \[ \exp(S_n \psi(\omega, z) - n e(\psi, \omega, n)) \leq |U^v_\omega| \leq \exp(S_n \psi(\omega, z) + n e(\psi, \omega, n)), \]
   hence for all \( \nu \in [v\omega] \),
   \[ \exp(S_n \Psi(\omega, \nu) - n e(\Phi, \omega, n)) \leq |U^\nu_\omega| \leq \exp(S_n \Psi(\omega, \nu) + n e(\Phi, \omega, n)). \]
   Consequently, for all \( \nu \in \Sigma_{\omega,n} \) define \( X^\nu_\omega := X_\omega \cap U^\nu_\omega : \)
   \[ |X^\nu_\omega| \leq |U^\nu_\omega| \leq \exp(S_n \Psi(\omega, \nu) + n e(\Psi, \omega, n)). \]
2. If we assume that \( P(\Phi) = 0 \), then for all \( \nu \in [v\omega] \),
   \[ \exp(S_n \Phi(\omega, \nu) - n e(\Phi, \omega, n)) \leq \bar{\mu}_\omega([v\omega]) \leq \exp(S_n \Phi(\omega, \nu) + n e(\Phi, \omega, n)), \]
   hence for all \( z \in U^v_\omega \),
   \[ \exp(S_n \phi(\omega, z) - n e(\phi, \omega, n)) \leq \bar{\mu}_\omega(X^\nu_\omega) \leq \mu_\omega(X^\nu_\omega) = \mu_\omega(U^v_\omega), \]
   as well as \( \bar{\mu}_\omega(U^v_\omega) \leq \exp(S_n \phi(\omega, z) + n e(\phi, \omega, n)) \) if \( \bar{\mu}_\omega \) is atomless.

### 3.2. Properties of random Gibbs measures.

Random Gibbs measures are associated with random Hölder continuous potentials.

We say that a function \( \Phi : \Sigma_\Omega \to \mathbb{R} \) is a random Hölder continuous potential if

1. \( \Phi \in L^1_{\mathcal{X}}(\Omega, C(\Sigma)); \)
2. \( \exists \kappa \in (0,1] \text{ such that } \varphi_n(\omega) \leq K(\omega) e^{-\kappa n}, \text{ with } \int \log K(\omega) d\mathbb{P}(\omega) < \infty. \)
Theorem 3.4 ([26 [27]). Assume that $F$ is a countably generated $\sigma$-algebra, $F$ is a topological mixing subshift of finite type and $\Phi$ is a random Hölder potential.

For $P$-almost every $\omega \in \Omega$, there exists some random variables $C = C^\Phi(\omega) > 0$, $\lambda = \lambda^\Phi(\omega) > 0$, a function $h = h(\omega) = h(\omega, v) > 0$ and a measure $\bar{\mu} \in P_\Theta(\Sigma_\Omega)$ with disintegrations $\mu_\omega$ satisfying

$$\int |\log C^\Phi| \, dP < +\infty, \quad \int |\log \lambda^\Phi| \, dP < +\infty \text{ and } \log h \text{ is a Hölder potential,}$$

and such that

$$L^\Phi_0 h(\omega) = \lambda(\omega) h(\sigma \omega), \quad (L^\Phi_0)^+ \mu_\sigma \omega = \lambda(\omega) \bar{\mu}_\omega, \quad \int h(\omega) \, d\bar{\mu}_\omega = 1. \quad (8)$$

Let $m_\omega = m^\Phi_\omega$ be given by $dm_\omega = h(\omega) d\bar{\mu}_\omega$ and set $dm(\omega, v) = d\bar{\mu}_\omega(v) dP(\omega)$. Then $m \in I^1(\Sigma_\Omega)$, and for $P$-almost every $\omega \in \Omega$, for all $v = v_0 v_1 \ldots v_{n-1} \in \Sigma_{\omega, n}$, and for all $v \in [v]_\omega$

$$\frac{1}{C^\Phi} \leq \exp(\sum_{i=0}^{n-1} \Phi(F^i(\omega, v))) - \log \lambda^\Phi(\sigma^{n-1} \omega) \cdot \ldots \cdot \lambda^\Phi(\omega) \leq C^\Phi.$$

The family of measures $(m_\omega)_{\omega \in \Omega}$ is called a random (or relative) Gibbs measure for the potential $\Phi$. Moreover, $m$ is the unique maximizing $F$-invariant probability measure in the variational principle, i.e. such that

$$P(\Phi) = h_m(F) + \int F \, dm, \quad \text{and one has } P(\Phi) = \int \log \lambda(\omega) \, dP.$$

Each time we need to refer to the function $\Phi$, we denote the measures $m$ and $m_\omega$ as $m^\Phi$ and $m^\Phi_\omega$, and denote $\lambda$ as $\lambda^\Phi$.

We can also define a random Gibbs measure on the random attractor $X_\omega$ by setting $\mu_\omega = m_\omega \circ \pi_v^{-1}$.

Given a random Hölder potential $\Phi$, from (8) we can define the normalized potential $\Phi'(\omega, v) = \Phi(\omega, v) + \log h(\omega, v) - \log h(F(\omega, v)) - \log \lambda(\omega)$, which satisfies $L^\Phi_0 1 = 1$ for $P$-almost every $\omega \in \Omega$. This implies that $\Phi' \leq 0$ for $P$-almost every $\omega \in \Omega$. Also, we have the following fact:

**Proposition 4.** Suppose that $\Phi$ is a random Hölder potential. If $P(\Phi) = 0$, there exists some $\varpi > 0$ such that for $P$-almost every $\omega \in \Omega$, there exists $N(\omega)$ such that for any $n \geq N(\omega)$ and any $v \in \Sigma_{\omega, n}$, one has

$$\sup_{v \in [v]_\omega} S_n \Phi(\omega, v) \leq -n \varpi.$$

As a consequence, $\mu_\omega$ is atomless.

If we need to refer explicitly to $\Phi$, we will use the notations $N_\Phi(\omega)$ and $\varpi_\Phi$ instead of $N(\omega)$ and $\varpi$.

The main idea of the proof is from [15].

**Proof.** Since $P(\Phi) = 0$, we have $\sup\{ \int \Phi \, d\rho : \rho \in I_\sigma(\Sigma_\Omega) \} = 0$. We claim that:

$$\sup\{ \int \Phi \, d\rho : \rho \in I_\sigma(\Sigma_\Omega) \} < 0.$$

Choose $M_2 \in \mathbb{N}$ large enough such that $P(\{ \omega : M(\omega) < M_2, l(\omega) \geq 2 \}) > 0$. For any $\omega \in \Omega$ such that $M(\omega) < M_2$ and $l(\omega) \geq 2$ we have $L^\Phi_0 1 = 1$, hence
\[ S_{M_2} \Phi'(\omega, v) < 0 \text{ for any } v \in \Sigma_\omega \text{ and } \int S_{M_2} \Phi'(\omega, v) d\rho_\omega < 0 \text{ for any probability measure } \rho_\omega \text{ on } \Sigma_\omega. \] Since, moreover, we have \[ S_{M_2} \Phi' \leq 0, \] we conclude that \sup\{\Phi' d\rho, \rho \in \mathcal{I}_\Omega(\Sigma_\Omega)\} < 0, hence \sup\{\Phi' d\rho : \rho \in \mathcal{I}_\Omega(\Sigma_\Omega)\} < 0.

Now, let \(-2\varpi := \sup\{\Phi' d\rho : \rho \in \mathcal{I}_\Omega(\Sigma_\Omega)\}. \] If the conclusion of the proposition does not hold, there exists a subsequence \((n_k)_{k \geq 1}\) such that \#\{v : |v| = n_k, \sup_{\omega \in \Sigma_\omega} \Phi(\omega, v) > -\varpi\} \geq 1, hence for any \(q \in \mathbb{N}^+\),

\[
P(q\Phi) = \lim_{k \to \infty} \frac{\log \sum_{v \in \Sigma_{\omega,n_k}} \exp \left( qS_{n_k} \Phi(\omega, u_{n_k,\omega}(v)) \right)}{n_k} \geq \lim_{k \to \infty} \frac{-q\varpi n_k}{n_k} = -q\varpi.
\]

However, due to [7], we have

\[
P(q\Phi) = \sup_{\rho \in \mathcal{I}_\Omega(\Sigma_\Omega)} \left\{ h_\rho(F) + \int q\Phi d\rho \right\} \leq \sup_{\rho \in \mathcal{I}_\Omega(\Sigma_\Omega)} \left\{ q\Phi d\rho \right\} + \sup_{\rho \in \mathcal{I}_\Omega(\Sigma_\Omega)} \left\{ h_\rho(F) \right\} \leq -2q\varpi + \sup_{\rho \in \mathcal{I}_\Omega(\Sigma_\Omega,F)} \left\{ h_\rho(F) \right\}.
\]

Since \(\sup_{\rho \in \mathcal{I}_\Omega(\Sigma_\Omega,F)} \left\{ h_\rho(F) \right\} = \int \log \lambda d\mathcal{P} < \infty\), letting \(q\) tend to \(\infty\) we get a contradiction. \(\square\)

3.3. Approximation of \((\Phi, \Psi)\) by random H"older potentials, and related properties. We mainly introduce objects and related properties which will be used in the following sections. Also, we explain the variational formulas appearing in the statement of theorem 2.6.

3.3.1. Approximation of \((\Phi, \Psi)\) by random H"older potentials. Now we approximate the potentials \(\Phi\) and \(\Psi\) associated with \(\{\mu_\omega\}_{\omega \in \Omega}\) and \(\{X_\omega\}_{\omega \in \Omega}\) by more regular potentials: for any \(i \geq 1\), for any \(\omega \in \Omega\) for any \(v = v_0 v_1 \cdots v_i \cdots \in [v]_{\omega} \subseteq \Sigma_\omega\) with \(v = v_0 v_1 \cdots v_{i-1} \in \Sigma_{\omega,i-1}\) define

\[
\Phi_i(\omega, v) = \frac{\max\{\Phi(\omega, w), w \in [v]_{\omega}\} + \min\{\Phi(\omega, w), w \in [v]_{\omega}\}}{2},
\]

\[
\Psi_i(\omega, v) = \frac{\max\{\Psi(\omega, w), w \in [v]_{\omega}\} + \min\{\Psi(\omega, w), w \in [v]_{\omega}\}}{2}.
\]

These functions \(\Phi_i\) and \(\Psi_i\) are piecewise constant with respect to the second variable. They are random H"older continuous potentials. Indeed, if we take

\[ K_{\Phi_i}(\omega) = (2 \sup_{v \in \Sigma_\omega} |\Phi(\omega, v)| + 1)e^i \text{ and } \kappa = 1, \]

then

\[
\text{var}_n \Phi_i(\omega) \begin{cases} 
\leq 2 \sup_{v \in \Sigma_\omega} |\Phi(\omega, v)| \leq K_{\Phi_i}(\omega) \exp(-\kappa n) & \text{if } n \leq i, \\
= 0 & \text{if } n > i.
\end{cases}
\]

Furthermore,

\[
\log((2 \sup_{v \in \Sigma_\omega} |\Phi(\omega, v)| + 1)e^i) \leq i + 2 \sup_{v \in \Sigma_\omega} |\Phi(\omega, v)|,
\]
and the right hand side is integrable since $\Phi \in L^1_{2\alpha}(\Omega, C(\Sigma))$. Also, since for $\mathbb{P}$-almost every $\omega$ we have $\var_r\Phi(\omega) \to 0$ as $n \to +\infty$, and

$$\|\Phi(\omega) - \Phi_i(\omega)\|_\infty \leq \var_r\Phi(\omega), \quad (9)$$

we have $\Phi_i \to \Phi$ uniformly as $i \to \infty$ for $\mathbb{P}$-almost every $\omega$. The same property holds for $\Psi_i$ and $\Psi$. Consequently, without loss of generality we can also assume that $P(\Phi_i) = 0$ since $P(\Phi_i)$ converges to $P(\Phi)$ as $i$ tends to $+\infty$. Also,

3.3.2. *Approximation of $(T, T^*)$ by $(T_i, T_i^*)$. Due to our assumptions on $(\Phi, \Psi)$ and the definition of $(\Phi_i, \Psi_i)_{i \in \mathbb{N}}$, we have $c_{q_i} > 0$, hence for the same reason as for $(\Phi, \Psi)$, for any $q \in \mathbb{R}$, for any $i \in \mathbb{N}$, there exists a unique $T_i(q)$ such that $P(q\Phi_i - T_i(q)\Psi_i) = 0$ and the function $T_i$ is concave and non-decreasing. Also, the function $T_i$ is differentiable since for Hölder potentials the associated random Gibbs measure is the unique invariant measure that maximizes the variation principle (see [18, 25, 26, 35].)

**Lemma 3.5.** For any $q \in \mathbb{R}$, one has that $T_i(q) \to T(q)$ as $i \to \infty$.

**Proof.** At first, we recall that for any $\Phi \in L^1_{2\alpha}(\Omega, C(\Sigma))$ one has

$$P(\Phi) = \sup_{\rho \in \mathcal{I}_\rho(\Sigma_\alpha)} \{ h_\rho(F) + \int \Phi \, d\rho \}.$$ 

Also, for any $q \in \mathbb{R}$, we have $P(q\Phi - T(q)\Psi) = P(q\Phi_i - T_i(q)\Psi_i) = 0$. Thus

$$\inf_{\rho \in \mathcal{I}_\rho(\Sigma_\alpha)} \left( \int [q(\Phi - \Phi_i) - T(q)(\Psi - \Psi_i) - (T(q) - T_i(q))\Psi_i] \, d\rho \right) \leq 0, \quad (10)$$

and

$$\sup_{\rho \in \mathcal{I}_\rho(\Sigma_\alpha)} \left( \int [q(\Phi - \Phi_i) - T(q)(\Psi - \Psi_i) - (T(q) - T_i(q))\Psi_i] \, d\rho \right) \geq 0. \quad (11)$$

The inequality (10) implies that for any $\varepsilon > 0$, there exists a measure $\rho \in \mathcal{I}_\rho(\Sigma_\alpha)$ such that

$$\int [q(\Phi - \Phi_i) - T(q)(\Psi - \Psi_i) - (T(q) - T_i(q))\Psi_i] \, d\rho \leq \varepsilon.$$ 

Then

$$\int (T(q) - T_i(q))\Psi_i \, d\rho \geq \int q(\Phi - \Phi_i) - T(q)(\Psi - \Psi_i) \, d\rho - \varepsilon,$$

and

$$T(q) - T_i(q) \leq \frac{\int [q(\Phi - \Phi_i) - T(q)(\Psi - \Psi_i)] \, d\rho - \varepsilon}{\int (\Psi_i) \, d\rho} \leq \frac{\int [q(\var_r\Phi) - |T(q)|\var_r\Psi] \, d\mathbb{P} - \varepsilon}{\int (\Psi_i) \, d\rho} \leq \frac{\int [q(\var_r\Phi) - |T(q)|\var_r\Psi] \, d\mathbb{P} - \varepsilon}{-c_{q_i}/2}$$

since $\int (\Psi_i) \, d\rho \leq \frac{-c_{q_i}}{2} < 0$ for $i$ large enough. Letting $i \to \infty$, from the arbitrariness of $\varepsilon$ we get

$$\liminf_{i \to \infty} T_i(q) \geq T(q).$$

Using (11) similarly we can get $\limsup_{i \to \infty} T_i(q) \leq T(q)$. Finally $\lim_{i \to \infty} T_i(q) = T(q)$. \qed
Lemma 3.6. Let $\tilde{T} : \mathbb{R} \to \mathbb{R}$ be a concave function. Suppose that $(\tilde{T}_i)_{i \geq 1}$ is a sequence of differentiable concave functions from $\mathbb{R}$ to $\mathbb{R}$ which converges pointwise to $\tilde{T}$. Then $(\tilde{T}_i)_{i \geq 1}$ converges pointwise to $\tilde{T}^*$ over the interior of the domain of $\tilde{T}^*$.

Proof. Let $\alpha$ be an interior point of $\text{dom}(\tilde{T}^*)$. Let $q_\alpha \in \mathbb{R}$ be the unique point such that $\alpha \in [\tilde{T}^*(q_\alpha), \tilde{T}^*(q_\alpha^-)]$, and $\tilde{T}^*(\alpha) = \alpha q_\alpha - \tilde{T}(q_\alpha)$.

By (11) proposition 2.5(i)], there exists a sequence $(q_i)_{i \geq 1}$ such that for $i$ large enough one has $\tilde{T}_i(q_i) = \alpha$. Without loss of generality we can assume that this sequence converges to $q' \in \mathbb{R}$ or diverges to $-\infty$ or $\infty$.

Suppose first that it converges to $q' \in \mathbb{R}$. If $q'_0 = q_\alpha$ then we are done since $(\tilde{T}_i)_{i \geq 1}$ converges uniformly on compact sets. Suppose that $q'_0 \neq q_\alpha$ and $q'_0 > q_\alpha$.

Using the uniform convergence of $(\tilde{T}_i)_{i \geq 1}$ in a compact neighborhood of $[q_\alpha, q'_0]$ and the inequality $\tilde{T}_i(q) \leq \tilde{T}_i(q_i) + \tilde{T}_i(q_i)(q - q_i)$ ($\tilde{T}_i$ is concave), we can get $\tilde{T}(q_\alpha) \leq \tilde{T}(q'_0) + \alpha(q_\alpha - q'_0)$. On the other hand, since $T$ is concave, we have $\tilde{T}(q_\alpha) + \tilde{T}'(q_\alpha) + (q_\alpha - q'_0) \geq \tilde{T}(q'_0)$ and $\tilde{T}'(q_\alpha) \leq \alpha$. This implies that $\alpha = \tilde{T}'(q_\alpha) + \tilde{T}^*(q'_0) = \lim_{i \to \infty} (\alpha q_i - \tilde{T}_i(q_i) = \tilde{T}_i^*(\alpha))$.

The case $q'_0 \neq q_\alpha$ and $q'_0 < q_\alpha$ is similar. Now suppose that $(q_i)_{i \geq 1}$ diverges to $\infty$ (the case where it diverges to $-\infty$ is similar). If $\tilde{T}$ is affine over $[q_\alpha, \infty)$ with slope $\alpha$, $\alpha$ is not an interior point of $\text{dom}(\tilde{T}^*)$. Consequently, there exists $q'_0$ and $\epsilon > 0$ such that $\tilde{T}'(q'_0) < \alpha - \epsilon$, and $\tilde{T}(q) \leq \tilde{T}(q'_0) + (\alpha - \epsilon)(q - q'_0)$ for all $q \geq q'_0$.

On the other hand, since $\tilde{T}_i'$ is non increasing for all $i$, for $i$ large enough we have $\tilde{T}_i(q) \geq \tilde{T}_i(q_i) + (q - q_i^0)$ for all $q \in [q'_0, q_i]$. Since $(q_i)_{i \geq 1}$ diverges to $\infty$, this contradicts the convergence of $(\tilde{T}_i)_{i \geq 1}$ to $\tilde{T}$.

$\blacksquare$

3.3.3. Explanation of some variational formulas in theorem 2.6.

Lemma 3.7. For any $q \in \mathbb{R}$ we have

$$T(q) = \min_{\rho \in \mathcal{I}_\Psi(\Sigma_\alpha)} \left\{ \frac{h_\rho(F) + q \int \Phi d\rho}{\int \Psi d\rho} \right\}.$$  (12)

Furthermore, for any $d \in [T'(0), T'(+\infty)]$ we have

$$T^*(d) = \max_{\rho \in \mathcal{I}_\Psi(\Sigma_\alpha)} \left\{ \frac{-h_\rho(F)}{\int \Psi d\rho} : \frac{\int \Phi d\rho}{\int \Psi d\rho} = d \right\}. $$  (13)

Proof. Property (12) is a direct consequence of the variation principle (7) and the entropy map is upper semi-continuous.

Regarding property (13), on the one hand, for any $d \in \mathbb{R}, q \in \mathbb{R},$

$$qd - T(q) = \inf_{\rho \in \mathcal{I}_\Psi(\Sigma_\alpha)} \left\{ \frac{h_\rho(F) + q \int \Phi d\rho}{\int \Psi d\rho} \right\}$$

$$= \sup_{\rho \in \mathcal{I}_\Psi(\Sigma_\alpha)} \left\{ \frac{-h_\rho(F)}{\int \Psi d\rho} + q \left( d - \frac{\int \Phi d\rho}{\int \Psi d\rho} \right) \right\}$$

$$\geq \sup_{\rho \in \mathcal{I}_\Psi(\Sigma_\alpha)} \left\{ \frac{-h_\rho(F)}{\int \Psi d\rho} : \frac{\int \Phi d\rho}{\int \Psi d\rho} = d \right\}.$$ 

Taking infimum for $q \in \mathbb{R}$, we get

$$T^*(d) \geq \inf_{q \in \mathbb{R}} \{ qd - T(q) \} \sup_{\rho \in \mathcal{I}_\Psi(\Sigma_\alpha)} \left\{ \frac{-h_\rho(F)}{\int \Psi d\rho} : \frac{\int \Phi d\rho}{\int \Psi d\rho} = d \right\}.$$
On the other hand, for any \( d \in (T'(\infty), T'(\infty)) \), by the proof of lemma \( 3.6 \) there exists \( i \) large enough and \( q_i \in \mathbb{R} \) such that \( T_i'(q_i) = d \) and \( P(q_i, \Phi_i - T_i(q_i) \Psi_i) = 0 \). Then (see [27, theorem 4.4.1]) there exists a unique \( \rho_i \in \mathcal{I}_\varnothing(\Sigma_\Omega) \) such that \( h_{\rho_i}(F) + \int (q_i, \Phi_i - T_i(q_i) \Psi_i) d\rho_i = 0 \) (see ), and adapting standard arguments from the deterministic subshift of finite type context (see [14] for instance) yields \( T_i'(q_i) = d = \int \frac{\Phi_i d\rho_i}{\Psi_i d\rho_i} \), and then \( h_{\rho_i}(F) = T_i'(d) \). Now, Lemma 3.6 tells us that \( T_i'(d) \to T^*(d) \) as \( i \to \infty \) and we know that \( \mathcal{I}_\varnothing(\Sigma_\Omega) \) is compact for the weak* topology (see [25, 27]). Thus, there exists a limit point \( \rho' \) of \( (\rho_i) \in \mathcal{I}_\varnothing(\Sigma_\Omega) \) such that \( \int \frac{\Phi d\rho'}{\Psi d\rho'} = T^*(d) \) and \( d = \int \frac{\Phi d\rho'}{\Psi d\rho'} \), since the entropy map is upper semi-continuous and \((\Phi_i, \Psi_i)\) converges uniformly to \((\Phi, \Psi)\). Finally, we get

\[
T^*(d) = \max_{\rho \in \mathcal{I}_\varnothing(\Sigma_\Omega)} \left\{ -\frac{h_{\rho}(F)}{\int \frac{\Phi d\rho}{\Psi d\rho}} : \frac{\int \Phi d\rho}{\int \Psi d\rho} = d \right\}.
\]

The case \( d \in \{ T'(\infty), T'(\infty) \} \) now follows by approximating \( d \) by a sequence \((d_k)_{k \geq 0}\) of elements of \((T'(\infty), T'(\infty))\) and for each \( k \) picking \( \rho_k \) which realizes

\[
\max_{\rho \in \mathcal{I}_\varnothing(\Sigma_\Omega)} \left\{ -\frac{h_{\rho}(F)}{\int \frac{\Phi d\rho}{\Psi d\rho}} : \frac{\int \Phi d\rho}{\int \Psi d\rho} = d_k \right\}.
\]

Then, \( T^* \) being continuous at \( d \) (it is lower semi-continuous as a concave function and upper semi-continuous as a Legendre transform), for any limit point \( \rho \) of the sequence \((\rho_k)_{k \geq 0}\) we have \( -\frac{h_{\rho}(F)}{\int \frac{\Phi d\rho}{\Psi d\rho}} = T^*(d) \) and \( \int \frac{\Phi d\rho}{\Psi d\rho} = d \).

\[ \square \]

3.3.4. Preparation of the lower bound estimate of the Hausdorff spectrum; simultaneous control for random Gibbs measures associated with \((\Phi_i, \Psi_i)\). In this quite technical subsection, we prepare the “concatenation of random Gibbs measures” approach that will be used in the next sections to construct auxiliary measures with nice properties.

First series of basic properties, and an almost doubling property for random Gibbs measures on \( \mathbb{X}_\omega \). Let \( D \) be a dense and countable subset of \((T'(\infty), T'(\infty))\). Fix a sequence \( \{ \varepsilon_i \}_{i \in \mathbb{N}} \) of real numbers decreasing to 0 as \( i \to \infty \). Let \( \{ D_i \}_{i \in \mathbb{N}} \) be an increasing sequence of finite sets such that \( \cup_{i \in \mathbb{N}} D_i = D \). Due to lemma \( 3.5 \) and the proof of lemma \( 3.6 \) for any \( i \in \mathbb{N} \), since \( D_i \) is finite there exists \( j_i \in \mathbb{N} \) such that for any \( d \in D_i \), there exists \( q_i = q_i(d) \in \mathbb{R} \) with the following properties:

1. \( T_{j_i}'(q_i) = d \), \( |T_{j_i}'(d) - T^*(d)| \leq \varepsilon_i \).
2. \( \int \var_\Omega \Phi d\mathbb{P} \leq \varepsilon_i^3 \) and \( \int \var_\Omega \Psi d\mathbb{P} \leq \varepsilon_i^3 \).

We can also assume that \( j_{i+1} > j_i \) for each \( i \in \mathbb{N} \). We set

\[ Q_i = \{ q_i : d_i \in D_i \}. \]

For any \( q \in Q_i \), we define \( \Lambda_{i, q} = q \Phi_{j_i} - T_{j_i}(q) \Psi_{j_i} \). To simplify the writing, from now on we denote \((\Phi_{j_i}, \Psi_{j_i}, T_{j_i})\) by \((\Phi_i, \Psi_i, T_i)\), so that

\[ \Lambda_{i, q} = q \Phi_i - T_i(q) \Psi_i. \]

We also define

\[ \var_\mathbb{W}_i \Upsilon(\omega) := \var_{j_i} \Upsilon(\omega), \quad \Upsilon \in \{ \Phi, \Psi \}. \]
This quantity will play an important role due to the approximation of \((\Phi, \Psi)\) by \((\Phi_{j_n}, \Psi_{j_n})\) (see [9]).

We now start a series of estimates that will be useful later. At first, using Birkhoff ergodic theorem and Egorov’s theorem yields \(C > 0\) and a measurable subspace \(\tilde{\Omega}\) of \(\Omega\) such that \(P(\tilde{\Omega}) > 7/8\) and for all \(\omega \in \tilde{\Omega}\), \(n \in \mathbb{N}\) and for \(\Upsilon \in \{\Phi, \Psi\}\), one has

\[
\max \left(\frac{1}{n} S_n \| \Upsilon(\omega) \|_\infty, \frac{1}{n} S_n \| \Upsilon(\sigma^{-n+1} \omega) \|_\infty, \frac{1}{n} S_n (\log l(\omega)) \right) \leq C. \tag{14}
\]

Also, for each \(i \geq 1\), there exist three positive integers \(L(i), M(i), N(i)\), a positive number \(\varpi_i > 0\), a measurable set \(\Omega(i) \subset \tilde{\Omega}\), and a sequence \(\{c_{i,n}\}_{n \geq 1}\) of real numbers decreasing to 0 as \(n \to \infty\) (without loss of generality we also ask that \(\{nc_{i,n}\}_{n \geq 1}\) is nondecreasing and \(\sum_{n=1}^{\infty} \exp(-nc_{i,n}) < +\infty\)) (by remark 2 below) such that: \(P(\Omega(i)) > 3/4\), and for any \(\omega \in \Omega(i), n \in \mathbb{N}\) and \(\Upsilon \in \{\Phi, \Psi\}\) one has:

- \(M(\omega) < M(i), L(\omega) \leq L(i)\);
- \(V_n \Upsilon(\sigma^{M(i)} \omega) \leq nc_{i,n}, \quad \epsilon(\Upsilon, \sigma^{M(i)} \omega, n) \leq c_{i,n}\), and
- \(\left| S_n \varphi_n \Upsilon(\omega) - n \int_\Omega \varphi_n \Upsilon(\omega) \, dP \right| \leq nc_{i,n}; \tag{15}\)
- if \(n \geq N(i)\)
  \[ \sup_{\varphi \in [\nu], \omega} S_n \Psi(\omega, \varphi) \leq (-nc_{\Psi}/2), \quad \forall \varphi \in \Sigma_{\omega}, \tag{16} \]
  and
  \[ \sup_{\varphi \in [\nu], \omega} S_n \Psi_i(\omega, \varphi) \leq (-nc_{\Psi}/2), \quad \forall \varphi \in \Sigma_{\omega}, \tag{17} \]
  where \(c_{\Psi} = c_{\psi}\) is defined in [2], and (17) comes from the definition of \(\Psi_i\);
- for any \(q \in Q_i\), the Gibbs measure \(\mu_{\sigma^{M(i)} \omega}^{\Lambda_i,q}\) is well defined, and
  \[ V_n \Lambda_i,q(\sigma^{M(i)} \omega) \leq nc_{i,n}, \tag{18} \]
  \[ \epsilon(\Lambda_i,q, \sigma^{M(i)} \omega, n) \leq c_{i,n}; \tag{19} \]
  also, if \(n \geq N(i)\), one has
  \[ \sup_{\varphi \in [\nu], \omega} S_n \Lambda_{i,q}(\sigma^{M(i)} \omega, \varphi) \leq (-n \varpi_i). \tag{20} \]

**Remark 2.** In the estimates \([15]\) to \([20]\), Birkhoff ergodic theorem and Egorov’s theorem have been used, combined with [5], proposition [3] and proposition [4].

The previous list of properties being established, for \(s \in \mathbb{N}\), we let \(\theta_1(i, \omega, s)\) be the \(s\)-th return time of the point \(\omega\) to the set \(\Omega(i)\) under the map \(\sigma\), that is

\[ \theta_1(i, \omega, 1) = \inf\{ n \in \mathbb{N} \cup \{0\} : \sigma^n \omega \in \Omega(i) \}, \]

and for \(s \geq 2\),

\[ \theta_1(i, \omega, s) = \inf\{ n \in \mathbb{N} : n > \theta_1(i, \omega, s-1), \sigma^n \omega \in \Omega(i) \}. \]

Then for any \(i \in \mathbb{N}\), due to the Birkhoff ergodic theorem, for \(P\)-almost every \(\omega\)

\[ \lim_{s \to \infty} \frac{\theta_1(i, \omega, s)}{s} = \frac{1}{P(\Omega(i))}. \]
Consequently,
\[
\lim_{s \to \infty} \frac{\theta_1(i, \omega, s) - \theta_1(i, \omega, s - 1)}{\theta_1(i, \omega, s)} = 0. \tag{21}
\]
Since \(\mathbb{N}\) is countable, there exists \(\tilde{\Omega}' \subset \Omega\) of full probability such that the previous properties of \(\theta_1(i, \omega, \cdot)\) hold for all \(\omega \in \tilde{\Omega}'\) and \(i \in \mathbb{N}\).

Now we prove an almost everywhere almost doubling property for the Gibbs measures \(\mu^t_{\sigma^{M(i)}_\omega}\) (lemmas 3.8 and 3.9 and remark 3 below). This requires some preparation.

At first, we introduce for each \(i \in \mathbb{N}\) a new sequence which is the translation by \(M(i)\) of a subsequence of the sequence \(\theta_1(i, \omega, \cdot)\). The resulting sequence possesses a property similar to (21).

Given \(\omega \in \tilde{\Omega}'\) and \(i \in \mathbb{N}\), let \(s_1 = \min\{s \in \mathbb{N} : \theta_1(i, \omega, s) \geq M(i)\}\) and define
\[
n_1^i(\omega) = \theta_1(i, \omega, s_1) - M(i).
\]
Then, by induction, for \(k \geq 2\), let \(s_k\) be the smallest \(s \in \mathbb{N}\) such that
\[
\theta_1(i, \omega, s) - n_{k-1}^i(\omega) \geq \max \left( M(i), N(i), n_{k-1}^i(\omega)(c_i n_{k-1}^i(\omega))^{\frac{1}{2}} + \sqrt{\theta_1(i, \omega, s)} \right), \tag{22}
\]
and define
\[
n_k^i(\omega) = \theta_1(i, \omega, s_k) - M(i).
\]
The relatively involved choice for the sequence \((n_k^i(\omega))_{k \in \mathbb{N}}\) will be fully justified later when we will establish the lower bound for the Hausdorff spectrum. For the moment, properties (21) and (22) clearly imply that
\[
\lim_{k \to \infty} \frac{n_k^i(\omega) - n_{k-1}^i(\omega)}{n_{k-1}^i(\omega)} = 0. \tag{23}
\]

For \(v \in \Sigma_{\sigma^{M(i)}_\omega, \mathbb{N}}\) such that \(U^v_{\sigma^{M(i)}_\omega}\) is neither the leftmost nor the rightmost interval in the collection \(\{U^v_{\sigma^{M(i)}_\omega}\}_{w \in \Sigma_{\sigma^{M(i)}_\omega, \mathbb{N}}}\), we denote by \(v^+\) and \(v^-\) the elements of \(\Sigma_{\sigma^{M(i)}_\omega, \mathbb{N}}\) such that \(U^v_{\sigma^{M(i)}_\omega}\) and \(U^{v^+}_{\sigma^{M(i)}_\omega}\) are the closest intervals of the collection that are neighboring \(U^v_{\sigma^{M(i)}_\omega}\), and with the convention that \(U^{v^-}_{\sigma^{M(i)}_\omega}\) is on the left of \(U^v_{\sigma^{M(i)}_\omega}\). If \(U^v_{\sigma^{M(i)}_\omega}\) is the leftmost (resp. rightmost) interval we only define \(v^+\) (resp. \(v^-\)).

For any integer \(k \geq 2\), let
\[
\mathcal{U}(i, \sigma^{M(i)}_\omega, k) = \bigcup_{v \in \Sigma_{\sigma^{M(i)}_\omega, \mathbb{N}} : \text{for all } v', \text{either } |v \vee v'| \leq n_{k-1}^i(\omega) \text{ or } |v \wedge v'| \leq n_{k-1}^i(\omega)} U^v_{\sigma^{M(i)}_\omega},
\]
that is \(\mathcal{U}(i, \sigma^{M(i)}_\omega, k)\) is the union of those intervals \(U^v_{\sigma^{M(i)}_\omega}\) of the \(n_k^i(\omega)\)-th generation of the construction of \(X_\omega\) such that the length of the longest common prefix \(v \vee v'\) or \(v \wedge v'\) and \(v\) or \(v^+\) or \(v^-\) is less than or equal to \(n_{k-1}^i(\omega)\).

**Lemma 3.8.** For all \(i \in \mathbb{N}\), for all \(\omega \in \tilde{\Omega}' \cap \Omega(i)\), and all \(q \in Q_1\), we have
\[
\mu_{\sigma^{M(i)}_\omega} \left( \bigcap_{N=1}^{\infty} \bigcup_{k \geq N} \mathcal{U}(i, \sigma^{M(i)}_\omega, k) \right) = 0.
\]

**Remark 3.** Lemma 3.8 and property \(\lim_{k \to \infty} \frac{n_k^i(\omega) - n_{k-1}^i(\omega)}{n_{k-1}^i(\omega)} = 0\) imply that for \(\mu_{\sigma^{M(i)}_\omega}\) a.e. \(x \in X_{\sigma^{M(i)}_\omega}\) with \(x \in U^v_{\sigma^{M(i)}_\omega}\), we have \(|v \vee v'| \sim |v|\) for \(v' \in \{v^+, v^-\}\) as \(|v| \to \infty\).
Proof. Let $i \in \mathbb{N}$, $\omega \in \bar{\Omega} \cap \Omega(i)$ and $q \in Q_i$, and simply denote $(n^i_j(\omega))_{j \in \mathbb{N}}$ by $(n^i_j)_{j \in \mathbb{N}}$. For any $v$ and $v'$ such that $|v| = n^i_{k-1}, |v'| = n^i_k - n^i_{k-1}$ and $vv' \in \Sigma_{\sigma(M(\omega))}$, by definition of $(n^i_j(\omega))_{j \in \mathbb{N}}$ and $\mu^\Lambda_{i,q}$, and by proposition 3 and properties 15, 19 and 20, one has

$$\frac{\mu^\Lambda_{i,q}(U_{\sigma(M(\omega))}(\omega)))}{\mu^\Lambda_{\sigma(M(\omega))}(U_{\sigma(M(\omega))}(\omega)))} \leq \exp(-(n^i_k - n^i_{k-1})\omega_i + 4n^i_k c_{i,n^i_k}).$$

We will use this fact to estimate the $\mu^\Lambda_{\sigma(M(\omega))}$-measure of $U(i, \sigma(M(\omega)), k)$. Notice that for any $v \in \Sigma_{\sigma(M(\omega))}$, there are at most 2 words $v'$ such that $vv' \in \Sigma_{\sigma(M(\omega))}$ and $|vv'| \leq n^i_{k-1}$ or $|vv'| - \lambda(n^i_{k-1}) \leq n^i_{k-1}$. Indeed, $U^e_{\sigma(M(\omega))}$ is either the leftmost or the rightmost interval of generation $n^i_k$ inside $U^e_{\sigma(M(\omega))}$. Consequently,

$$\mu^\Lambda_{\sigma(M(\omega))}(U(i, \sigma(M(\omega)), k)) \leq \sum_{v \in \Sigma_{\sigma(M(\omega))}, n^i_{k-1} \leq |v| \leq n^i_k} 2 \exp(-(n^i_k - n^i_{k-1})\omega_i + 4n^i_k c_{i,n^i_k}) \mu^\Lambda_{\sigma(M(\omega))}(U^e_{\sigma(M(\omega))}(\omega)) = 2 \exp(-(n^i_k - n^i_{k-1})\omega_i + 4n^i_k c_{i,n^i_k}).$$

Since $\omega_i > 0$, (22) and $\sum_{n=1}^{\infty} \exp(-nc_{i,n}) < +\infty$, we get

$$\sum_{k=1}^{\infty} \mu^\Lambda_{\sigma(M(\omega))}(U(i, \sigma(M(\omega)), k)) < +\infty,$$

and the Borel-Cantelli’s lemma yields $\mu^\Lambda_{\sigma(M(\omega))}(\bigcap_{N=1}^{\infty} \bigcup_{k \geq N} U(i, \sigma(M(\omega)), k)) = 0$. \hfill $\Box$

For any $x \in X_\omega$, for any $n \in \mathbb{N}$, define $x|_n = v$, where $v \in \Sigma_{\omega,n}$ and $x \in U^e_{\sigma(M(\omega))}$. For any $\varepsilon > 0$, $\beta \geq 0$ and $k, p \geq 1$ and $\omega \in \bar{\Omega} \cap \Omega(i)$ we now define the following sets:

$$F_{i,\beta,k}(\sigma(M(\omega)), \varepsilon) = \begin{cases} x \in X_{\sigma(M(\omega))} : & \forall v \in \Sigma_{\sigma(M(\omega))}, n^i_k \text{ satisfying } |v \wedge x|_n^i \geq n^i_{k-1}, \\ \text{for any } v \in [v]_{\sigma(M(\omega))}, & \left| S_{n^i_k} \Phi_i(\sigma(M(\omega)), v) - \beta \right| \leq \varepsilon \end{cases}$$

and

$$E_{i,\beta}(\sigma(M(\omega)), \varepsilon) = \bigcup_{p \geq k \geq k} F_{i,\beta,k}(\sigma(M(\omega)), \varepsilon).$$

Remark 4. For $x \in F_{i,\beta,k}(\sigma(M(\omega)), \varepsilon)$, we can get the control

$$\frac{|S_{n^i_k} \Phi_i(\sigma(M(\omega)), v)|}{S_{n^i_k} \Phi_i(\sigma(M(\omega)), v)} - \beta \leq \varepsilon$$

not only for $[x]_{\sigma(M(\omega))}$ but also for the cylinders $[v]_{\sigma(M(\omega))}$ if they have the same prefix with length at least $n^i_{k-1}$.

Lemma 3.9. For all $i \in \mathbb{N}$, for any $\varepsilon > 0$, for all $\omega \in \bar{\Omega} \cap \Omega(i)$ and for all $q \in Q_i$, set $E_{i,\beta}(\sigma(M(\omega)), \varepsilon)$ has full $\mu^\Lambda_{\sigma(M(\omega))}$-measure.
Proof. Fix \( i \in \mathbb{N} \). Let \( \omega \in \Omega' \cap \Omega(i) \). Fix \( \varepsilon > 0 \). We will show that the \( \mu^{\Lambda_{i,q}}_{\sigma^{M(i)}(\omega)} \) measure for the complementary set of \( E_{i,T'(q)}(\sigma^{M(i)}(\omega), \varepsilon) \) vanishes. Let

\[
S_{i,q,k} = \mu^{\Lambda_{i,q}}_{\sigma^{M(i)}(\omega)} \left( X_{\sigma^{M(i)}(\omega)} \setminus F_{i,T'(q),k}(\sigma^{M(i)}(\omega), \varepsilon) \right).
\]

We have, using the second equality in the definition of \( F_{i,\beta,k}(\sigma^{M(i)}(\omega), \varepsilon) \),

\[
S_{i,q,k} \leq \sum_{\gamma \in \{-1,1\} \in \Sigma_{\sigma^{M(i)}(\omega), n_k}} \sum_{\nu \in \Sigma_{\sigma^{M(i)}(\omega), n_k}^*} \sum_{\nu_1, \ldots, \nu_{n_k} \in \nu \wedge \nu' \geq n_{k-1}} \mu^{\Lambda_{i,q}}_{\sigma^{M(i)}(\omega)}(U_{\sigma^{M(i)}(\omega)}^\nu)
\]

\[
\cdot \exp \left( - \gamma \eta (T'(q) - \gamma \varepsilon) S_{n_k} \Psi_i(\sigma^{M(i)}(\omega), \nu') - S_{n_k} \Phi_i(\sigma^{M(i)}(\omega), \nu') \right)
\]

\[
= \sum_{\gamma \in \{-1,1\} \in \Sigma_{\sigma^{M(i)}(\omega), n_k}} \sum_{\nu \in \Sigma_{\sigma^{M(i)}(\omega), n_k}^*} \sum_{\nu_1, \ldots, \nu_{n_k} \in \nu \wedge \nu' \geq n_{k-1}} \exp((q + \gamma \eta) S_{n_k} \Phi_i(\sigma^{M(i)}(\omega), \nu'))
\]

\[
\cdot \exp((-T_i(q) + \gamma \eta T'(q) - \varepsilon \eta) S_{n_k} \Psi_i(\sigma^{M(i)}(\omega), \nu'))
\]

\[
\cdot \exp(-\gamma \eta ((T'(q) - \varepsilon \eta) S_{n_k} \Psi_i(\sigma^{M(i)}(\omega), \nu') - S_{n_k} \Phi_i(\sigma^{M(i)}(\omega), \nu')))
\]

\[
\cdot \exp(\gamma \eta (S_{n_k} \Phi_i(\sigma^{M(i)}(\omega), \nu') - S_{n_k} \Phi_i(\sigma^{M(i)}(\omega), \nu')))
\]

\[
\cdot \exp(n_k \epsilon (\Lambda_{i,q}, \sigma^{M(i)}(\omega), n_k))
\]

Since \( T_i \) is in fact not only differentiable, but analytic using the same method as in [14] [17], we have

\[
T_i(q + \gamma \eta) = T_i(q) + T'_i(q) \gamma \eta + O(\eta^2),
\]

uniformly in \( q \in Q_i \). Thus, there exists \( b > 0 \) such that for \( \eta \) small enough, for all \( q \in Q_i \), we have

\[
|T_i(q + \gamma \eta) - T_i(q) - T'_i(q) \gamma \eta| \leq b \eta^2.
\]

Consider such an \( \eta \) in \( (0, \frac{\varepsilon}{2b}] \). We have

\[
S_{i,q,k} \leq \sum_{\gamma \in \{-1,1\} \in \Sigma_{\sigma^{M(i)}(\omega), n_k}} \sum_{\nu \in \Sigma_{\sigma^{M(i)}(\omega), n_k}^*} (l(\sigma^{M(i)}(\omega) + n_k) - l(\sigma^{M(i)}(\omega) + n_k - 1))
\]

\[
\cdot \exp(S_{n_k} ((q + \gamma \eta) \Phi_i - T_i(q + \gamma \eta) \Psi_i) (\sigma^{M(i)}(\omega), \nu'))
\]

\[
\cdot \exp((\varepsilon \eta - b \eta^2) S_{n_k} \Psi_i (\sigma^{M(i)}(\omega), \nu') + n_k \epsilon (\Lambda_{i,q}, \sigma^{M(i)}(\omega), n_k))
\]

\[
\leq \sum_{\gamma \in \{-1,1\} \in \Sigma_{\sigma^{M(i)}(\omega), n_k}} \exp((-\varepsilon \eta - b \eta^2) n_k \epsilon / 4)
\]

\[
\leq \sum_{\gamma \in \{-1,1\} \in \Sigma_{\sigma^{M(i)}(\omega), n_k}} \exp((-\varepsilon \eta - b \eta^2) n_k \epsilon / 4) \quad \text{see [14], [17] and [19]}
\]

\[
= 2 \exp \left( -\frac{\varepsilon^2}{4b} n_k \epsilon / 4 \right).
\]

Consequently, \( \sum_{k=1}^{+\infty} S_{i,q,k} < \infty \), which by the Borel-Cantelli lemma yields the desired conclusion.

\( \square \)

Remark 5. Combining proposition [3] with lemmas [3.8] and [3.9] shows that for \( \mu^{\Lambda_{i,q}}_{\sigma^{M(i)}(\omega)} \)-almost every \( x \), for \( n \) large enough, the mass assigned by \( \mu^{\Lambda_{i,q}}_{\sigma^{M(i)}(\omega)} \) to \( U_{\sigma^{M(i)}(\omega)}^x \) and its neighbours in the family \( \{U_{\sigma^{M(i)}(\omega)}^x : \nu \in \Sigma_{\sigma^{M(i)}(\omega), n}\} \) have asymptotically the same scaling exponent. This is this property that we qualify as almost doubling.
Collecting additional facts from the previous properties. Now we can collect the following facts.

**F1.** Due to lemmas 3.8 and 3.9 as well as remark 3, proposition 3, and the relation $\Lambda_{i,q} = q\Phi_i - T_i(q)\Psi_i$: for all $i \in \mathbb{N}$, for any $\varepsilon_i > 0$ (pay attention to the fact that $\varepsilon_i$ differs from $\varepsilon$), for all $\omega \in \Omega' \cap \Omega(i)$, there exists an integer $\mathcal{K}_i = \mathcal{K}_i(\sigma^M(\omega))$ such that for any $q \in Q_i$, there exists $E_{i,q} = E_{i,q}(\sigma^M(\omega)) \subset X_{\sigma^M(\omega)}$ such that

1. $\mu_{\sigma^M(\omega)}^\Lambda_{i,q}(E_{i,q}) > 1 - \varepsilon_i$
2. $M(i) \leq n_{k_{i}}^i \varepsilon_i^3$, \hspace{1cm} (24)
3. $\varepsilon_i n_{k_{i}}^i \leq \varepsilon_i^3$, \hspace{1cm} (25)
4. $\forall k \geq \mathcal{K}_i$, $n_{k_{i}}^i - n_{k-1_{i}}^i \leq n_{k-1_{i}}^i \varepsilon_i^3$ \hspace{1cm} (26)
5. for any $v \in E_{i,q}$, for any $v \in \Sigma_{\sigma^M(\omega)}$, with $k \geq \mathcal{K}_i$ such that $v \in [v]_{\sigma^M(\omega)}$, one has $|v \vee v| \geq n_{k-1_{i}}^i$ and $|v \wedge v| \geq n_{k-1_{i}}^i$. Furthermore, for any $w \in \{v, v+, v-\}$, there exists (in fact for all) $w \in [w]_{\sigma^M(\omega)}$ such that

$$\frac{|S_{n_k}^i \Phi_i(\sigma^M(\omega), w)|}{S_{n_k}^i \Psi_i(\sigma^M(\omega), w)} - T_i(q) \leq \varepsilon_i,$$ \hspace{1cm} (27)

$$\frac{\log \mu_{\sigma^M(\omega)}^\Lambda_{i,q}(U_{\sigma^M(\omega), w})}{S_{n_k}^i \Psi_i(\sigma^M(\omega), w)} - T_i(q) \leq \varepsilon_i,$$ \hspace{1cm} (28)

and

$$\frac{|S_{[v]} \Phi_i(\sigma^M(\omega), w)|}{S_{[v]} \Psi_i(\sigma^M(\omega), w)} - T_i(q) \leq \varepsilon_i,$$ \hspace{1cm} (29)

In fact with a suitable change of $\varepsilon_i$ (taking $2\varepsilon_i$ is enough), we can get the following additional properties from (27), (28), and (29) above:

$$\frac{|S_{n_k}^i \Phi_i(\sigma^M(\omega), w)|}{S_{n_k}^i \Psi_i(\sigma^M(\omega), w)} - d_i \leq \varepsilon_i,$$ \hspace{1cm} (30)

$$\frac{\log \mu_{\sigma^M(\omega)}^\Lambda_{i,q}(U_{\sigma^M(\omega), w})}{S_{[v]} \Psi_i(\sigma^M(\omega), w)} - T_i^*(d_i) \leq \varepsilon_i,$$ \hspace{1cm} (31)

and

$$\frac{|S_{[v]} \Phi_i(\sigma^M(\omega), w)|}{S_{[v]} \Psi_i(\sigma^M(\omega), w)} - T_i^*(d_i) \leq \varepsilon_i.$$ \hspace{1cm} (32)

**F2.** We can change $\Omega(i)$ to $\Omega_i \subset \Omega' \cap \Omega(i)$ a bit smaller such that $\mathbb{P}(\Omega_i) \geq 1/2$ and there exist deterministic integers $\kappa_i$ and $n(i)$ such that for any $\omega \in \Omega_i$, $\mathcal{K}_i(\sigma^M(\omega)) \leq \kappa_i$ and $n_{k_{i}}^i(\omega) \leq n(i)$ and the properties listed in **F1** hold.

We let $\theta(i, \omega, s)$ denote the $s$-th return time to the set $\Omega_i$ for the point $\omega$. Again, since $\mathbb{N}$ is countable, there exists $\Omega'' \subset \Omega'$ of full probability such that for all $\omega \in \Omega''$, we have for any $\Psi \in \{\Phi, \Psi\}$

$$\lim_{n \to \infty} \frac{V_n \mathcal{Y}(\omega)}{n} = 0.$$ \hspace{1cm} (33)
and for any \( i \in \mathbb{N} \)
\[
\lim_{s \to \infty} \frac{\theta(i, \omega, s) - \theta(i, \omega, s - 1)}{\theta(i, \omega, s - 1)} = 0. \tag{34}
\]

4. Multifractal analysis of random weak Gibbs measures: Proof of Theorems 2.6 and 2.7. This section consists of three subsections. In the first one we obtain the sharp lower bound for the lower \( L^1 \)-spectrum of \( \mu_\omega \). Next, in the second subsection, we prove the validity of the multifractal formalism (see (2) in theorem 2.6). Due to (3), (4) and lemma 3.7 we just need to prove \( \dim_H E(\mu_\omega, d) = \tau^*_\mu(\omega, d) \). There, our approach to construct suitable auxiliary measures already prepares the material used to establish in the third subsection the refinements gathered in (3)(4) of theorem 2.6 and theorem 2.7.

4.1. Lower bound for \( \tau_{\mu_\omega} \) and upper bound for \( \tau^*_{\mu_\omega} \). Fix a countable and dense subset \( D \) of \( \mathbb{R} \). Let \( \Omega \) be a set of full \( \mathbb{P} \)-probability, such that:

1. for all \( q \in D \) the weak Gibbs measure \( \{\mu^T(\omega, q)\}_\omega \in \Omega \) are defined;
2. for all \( \omega \in \Omega \) the conclusions of proposition 3 hold all the potentials \( q\Phi - T(\omega, \Psi, q) \in D \);
3. for all \( n \) large enough, for all \( \nu \in \Sigma_\omega \):

\[
\exp(-2nC_\Psi) \leq \exp(S_n(\Psi, \nu, \nu)) \leq \exp(-nc_\Psi/2),
\]

which follows from (14) and (16).

We will establish the lower bound \( \tau_{\mu_\omega}(q) \geq T(\omega, q) \) for all \( \omega \in \Omega \) and \( q \in D \). Since \( D \) is dense and both \( \tau_{\mu_\omega} \) and \( T \) are continuous, this will yield \( \tau_{\mu_\omega} \geq T \) for all \( \omega \in \Omega \). By using the multifractal formalism, this immediately yields the desired upper bound \( T^* \) for \( \tau^*_{\mu_\omega} \) and the various spectra we consider for \( \mu_\omega \). The equality \( \tau_{\mu_\omega} = \tau^*_{\mu_\omega} \) then follows from standard considerations in large deviations theory.

Fix \( \omega \in \Omega \). Let \( r > 0 \) and consider \( B = \{B_i\} \), a packing of \( X_\omega \) by disjoint balls \( B_i \) with center \( x_i \) and radius \( r \). For each ball \( B_i \), choose \( n = n_i \) and \( v(x_i) \in \Sigma_{\omega,n} \) such that \( x_i \in U^v_{\omega}(x_i) \) and \( |U^v_{\omega}(x_i)| \leq r \), but \( |U^v_{\omega}(x_i)|^{-1} \geq r \). By removing a set of probability 0 from \( \hat{\Omega} \) if necessary, for any \( \mu \in [v(x_i)]_\omega \), we have

\[
r \geq |U^v_{\omega}(x_i)| \geq \exp(S_n(\Psi, v, v) - n\varepsilon(\Psi, v, n)) \geq \exp(-2nC_\Psi - n\varepsilon(\Psi, v, n)).
\]

Thus \( n \geq -\frac{\log r}{3C_\Psi} \) for \( r \) small enough. On the other hand, for \( r \) small enough, for any \( \mu \in [v(x_i)]_\omega \), we have

\[
r \leq |U^v_{\omega}(x_i)|^{-1} \leq \exp(S_{n-1}(\Psi, v) + (n-1)\varepsilon(\Psi, v, n-1)) \leq \exp(-2(n-1)c_\Psi + (n-1)(\Psi, v, n-1)),
\]

so \( n \geq -\frac{3\log r}{3C_\Psi} \). Thus, for \( r \) small enough, independently on the choice of \( B \), if \( v(x_i) \in \Sigma_{\omega,n} \) and \( \mu \in [v(x_i)]_\omega \), we have

\[
-\frac{\log r}{3C_\Psi} \leq n \leq -\frac{3\log r}{c_\Psi}. \tag{35}
\]

Case \( q \in D \cap (-\infty, 0) \).

For each \( B_i \in B \), one has \( X_{\omega}^v(x_i) \supset B_i \), so for any \( \mu \in [v(x_i)]_\omega \)

\[
(\mu_\omega(B_i))^q \leq (\mu_\omega(X_{\omega}^v(x_i)))^q
\]
for any sequence \( \{\omega\} \) is a partition of \([0, \infty)\). Letting \( r \to 0 \) yields \( \tau_{\mu_{\omega}}(q) \geq T(q) \).

**Case** \( q \in D \cap [0, +\infty) \). Define

\[
V(\omega, n, r) = \{ v \in \Sigma, n : |U_{\omega}^n| \geq 2r, \exists s \text{ such that } v_s \in \Sigma, n+1, |U_{\omega}^n| < 2r \},
\]

\[
V'(\omega, n, r) = \{ v \in V(\omega, n, r), \text{ there is no } k < n \text{ such that } |v|_k \in V(\omega, k, r) \},
\]

and

\[
V(\omega, r) = \bigcup_{n \geq 1} V'(\omega, n, r).
\]

Then \( \{U_{\omega}^n : v \in V(\omega, r)\} \) is a partition of \([0, 1]\). Define \( n(\omega, r) = \max\{|v| : v \in V(\omega, r)\} \) and \( n'(\omega, r) = \min\{|v| : v \in V(\omega, r)\} \). Then, from Proposition 3, we know that for some positive constants \( B_1 \) and \( B_2 \), for \( r \) small enough, we have \( -B_1 \log(r) \leq n'(\omega, r) \leq B_2 \log(r) \).

For any \( v \in V(\omega, r) \), \( U_{\omega}^n \) meets at most \( O(-\log r) \) many balls of \( B_1 \) and for any \( B_1, B_2 \), \( U_{\omega}^n \) meets at most two intervals of \( U_{\omega}^n \). Consequently, since \( (\mu_{\omega}(B_1))^q \leq 2^q((\mu_{\omega}(U_{\omega}^n))^q + (\mu_{\omega}(U_{\omega}^n))^q) \), we have

\[
\sum_{B_i \in B} (\mu_{\omega}(B_i))^q \leq \exp(o(-\log r)) 2^q \sum_{n'(\omega, r) \leq n \leq n(\omega, r)} \sum_{v \in \Sigma, n \cap \Sigma' \omega(n(\omega, r))}
\]

Using the same argument as for \( q < 0 \), we can prove that

\[
(\mu_{\omega}(U_{\omega}^n))^q \leq (\mu_{\omega}(U_{\omega}^n))^{q - T(q)}(U_{\omega}^n)^q \exp(o(-\log r)),
\]

so that

\[
\sum_{B_i \in B} (\mu_{\omega}(B_i))^q \leq r^{T(q)} \exp(o(-\log r)) \sum_{n'(\omega, r) \leq n \leq n(\omega, r)} \sum_{v \in \Sigma, n \cap \Sigma'} \mu_{\omega}(U_{\omega}^n)^q
\]

\[
= r^{T(q)} \exp(o(-\log r)),
\]

independently on the choice of \( \{B_i\} \), where we used the fact that \( \{U_{\omega}^n : v \in V(\omega, r)\} \) is a partition of \([0, 1]\). Letting \( r \to 0 \) yields \( \tau_{\mu_{\omega}}(q) \geq T(q) \).

**4.2. Lower bound for the Hausdorff spectrum.** Recall facts F1 and F2 derived at the end of section 3.3.4. For any \( \omega \in \Omega \), for any \( d_i \in [T'(+\infty), T'(-\infty)] \), for any sequence \( \{d_i\} \) with \( d_i \in D_1 \), such that \( \lim_{i \to \infty} d_i = d \), and consequently \( \lim_{i \to \infty} T^*(d_i) = T^*(d) \) by continuity of \( T^* \), we will construct a probability measure \( \eta_{d} \) supported on a set \( K_1 \) of \( \{d_i\} \) such that

- \( K_1 \) is a partition of \([0, \infty)\).
- For any \( x \in K_1 \), \( \lim_{r \to 0} \frac{\log(\eta_{d}(B(x, r)))}{\log r} \geq T^*(d) \).

This will imply that \( \dim_{H} \eta_{d} \geq T^*(d) \), and then

\[
\dim_{H}(E(\mu_{\omega}, d)) \geq \dim_{H}(K_1) \geq T^*(d).
\]
Fix a sequence \( \{\epsilon_i\}_{i\in\mathbb{N}} \) small enough such that \( \Pi_{i\geq 1}(1-\epsilon_i) \geq \frac{1}{2} \). For each \( i \in \mathbb{N} \), \( F1 \) will be applied with this \( \epsilon_i \).

From now on we only deal with points \( \omega \) in the set \( \bar{\Omega}'' \) of \( P \)-probability 1 defined at the end of section 3.3.4, recall that such points satisfies equations (33) and (34).

We will build a family of Moran structures indexed by the elements of \( \prod_{i\geq 1} D_i \). The construction consists of four steps.

**Step 1.** For any \( \omega \in \bar{\Omega}'' \), recall that \( \theta(1,\omega,1) \) is the smallest \( n \in \mathbb{N} \) such that \( \sigma^n \omega \in \Omega_1 \subset \bar{\Omega}'' \cap \Omega(1) \). Define \( m_1 := \theta(1,\omega,1) + M(1) \). Fact F2 (applied to \( \sigma^{\theta(1,\omega,1)} \omega \) instead of \( \omega \)) tell us that for any \( d_1 \in D_1 \), there exists \( q_1 \in Q_1 \) such that \( T^*_1(q_1) = d_1 \) and a set \( E_1,q_1(\sigma^{m_1} \omega) \subset X_{\sigma^{m_1} \omega} \) such that the results in F1 hold. For any \( k \geq 1 \), we denote \( n^1_k(\sigma^{\theta(1,\omega,1)} \omega) \) by \( n^1_k \).

Choose \( \kappa'_1 > \kappa_1 \) large enough so that:

- \( n^1_{\kappa'_1} \geq \frac{1}{\epsilon_2^3} \max\{m_1, M(2), n(2)\} \),
- for all \( p \geq m_1 + n^1_{\kappa'_1} \), \( \epsilon(\Psi,\omega,p) \leq \epsilon_2^3 \), \( \epsilon(\Phi,\omega,p) \leq \epsilon_3^3 \) and \( V^*(\Psi(\omega)) \leq p\epsilon_2^3 \)
- for any \( s \in \mathbb{N} \) such that the return time \( \theta(2,\omega,s) \) satisfies \( \theta(2,\omega,s) \geq m_1 + n^1_{\kappa'_1} \) (see also (34)), one has
\[
\frac{\theta(2,\omega,s) - \theta(2,\omega,s-1)}{\theta(2,\omega,s-1)} \leq \epsilon_2^3.
\]

Let \( s_2 := \min\{s \in \mathbb{N} : \theta(2,\omega,s) \geq m_1 + n^1_{\kappa'_1} \} \), and then \( \pi_1 := \max\{k \in \mathbb{N} : m_1 + n^1_k \leq \theta(2,\omega,s_2)\} \). By construction one has \( \pi_1 \geq \kappa'_1 \), and due to (26) one has
\[
\theta(2,\omega,s_2) - m_1 - n^1_{\kappa_1} \leq n^1_{\kappa_1 + 1} - n^1_{\kappa_1} \leq \epsilon_1 n^1_{\kappa_1}.
\]

Figure 1 illustrates the beginning of the construction.

**Figure 1.** Choice of \( \pi_1 \).

For each \( q \in Q_1 \) and \( k \geq 1 \), let
\[
\mathcal{V}(\sigma^{m_1} \omega, 1, q, k) = \left\{ v \in \Sigma_{\sigma^{m_1} \omega,n_k^1} : E_1,q(\sigma^{m_1} \omega) \cap X_{\sigma^{m_1} \omega}^v \neq \emptyset \right\}.
\]

Also, set
\[
\mathcal{V}(\sigma^{m_1} \omega, 1, q) = \mathcal{V}(\sigma^{m_1} \omega, 1, q, \pi_1).
\]

Fix \( w_0 \in \Sigma_{\omega,\theta(1,\omega,1)} \), for any \( d_1 \in D_1 \), we define:
\[
R_1(d_1) = \{ w_0 * v : v \in \mathcal{V}(\sigma^{m_1} \omega, 1, q_1) \},
\]
and
\[
R_1 = \bigcup_{d_1 \in D_1} R_1(d_1)
\]
(recall that the meaning of \( s * s' \) is specified at the beginning of section 2.1).
Step 2. Suppose that \( \theta(i + 1, \omega, s_{i+1}), \bar{\pi}_i, R_i \) have been chosen. Define
\[
m_{i+1} := \theta(i + 1, \omega, s_{i+1}) + M(i + 1)
\]
and
\[
n^+_k = n^+_k(\sigma^{\theta(i+1,\omega,s_{i+1})}(\omega)).
\]
Like in the case \( i = 0 \), for any \( d_{i+1} \in D_{i+1} \), there exists \( q_{i+1} \in Q_{i+1} \) with
\[
T'_{i+1}(q_{i+1}) = d_{i+1}
\]
as well as a set \( E_{i+1,q_{i+1}}(\sigma^{m_{i+1}\omega}) \subset X_{m_{i+1}\omega} \) such that the results in F1 hold.

Then choose \( \kappa'_{i+1} \) large enough so that:
\[
n^+_{\kappa'_{i+1}} \geq \frac{1}{\varepsilon^3_{i+2}} \max\{m_{i+1}, M(i + 2), n(i + 2)\}
\] (36)

- for all \( p \geq m_{i+1} + n^+_{\kappa'_{i+1}} \),
\[
\max\{\epsilon(\Phi, \omega, p), \epsilon(\Phi, \omega, p), \frac{V_p \Psi(\omega)}{\rho} \} \leq \varepsilon^3_{i+2},
\] (37)

- for any \( s \in \mathbb{N} \) such that the return time \( \theta(i + 2, \omega, s) \) satisfies \( \theta(i + 2, \omega, s) \geq m_{i+1} + n^+_{\kappa'_{i+1}} \) (see also (34)), one has
\[
\frac{\theta(i + 2, \omega, s) - \theta(i + 2, \omega, s - 1)}{\theta(i + 2, \omega, s - 1)} \leq \varepsilon^3_{i+2}.
\]

Let \( s_{i+2} =: \min\{s \in \mathbb{N} : \theta(i + 2, \omega, s) \geq m_{i+1} + n^+_{\kappa'_{i+1}} \} \), and then \( \bar{\pi}_{i+1} =: \{k \in \mathbb{N} : m_{i+1} + n^+_{\kappa'_{i+1}} \leq \theta(i + 2, \omega, s_{i+2})\} \). By construction one has \( \bar{\pi}_{i+1} \geq \kappa'_{i+1} \), and due to (26) one has
\[
\theta(i + 2, \omega, s_{i+2}) - m_{i+1} - n^+_{\kappa'_{i+1}} \leq n^+_{\kappa'_{i+1}} - n^+_{\kappa'_{i+1}} \leq \varepsilon^3_{i+1} n^+_{\kappa'_{i+1}}. 
\] (38)

Remark 6. By construction, we can take \( n^+_{\kappa'_{i+1}} \) as big as we want (in the construction \( m_{i+1} \leq \varepsilon^3_{i+2} n^+_{\kappa'_{i+1}} \)). This implies that we can get the sequence \( \{m_i\}_{i \in \mathbb{N}} \) increasing as fast as we want. We can also impose that \( m_{i+2} - m_{i+1} \geq n^+_{\kappa'_{i+1}} \) and \( m_{i+1} = o(n^+_{\kappa'_{i+1}}) \). Thus, the speed we fix for the growth of \( \{n_{\kappa'_{i+1}}\}_{i \in \mathbb{N}} \) directly impacts the growth speed of \( \{m_i\}_{i \in \mathbb{N}} \).

Here again we illustrate this construction (see figure [2]).

For \( q_{i+1} \in Q_{i+1} \) and \( k \geq 1 \), define
\[
\mathcal{V}(\sigma^{m_{i+1}\omega}, i + 1, q_{i+1}, k) = \left\{ v \in \Sigma^{\sigma_{m_{i+1}\omega} q_{i+1}}_{\sigma_{m_{i+1}\omega} n_{k+1}^+} : E_{i+1,q}(\sigma^{m_{i+1}\omega}) \cap X_{\sigma^{m_{i+1}\omega} q_{i+1}}^v \neq \emptyset \right\},
\]
and
\[
\mathcal{V}(\sigma^{m_{i+1}\omega}, i + 1, q_{i+1}) = \mathcal{V}(\sigma^{m_{i+1}\omega}, i + 1, q_{i+1}, \bar{\pi}_{i+1}).
\]
As in the case $i = 0$, for any $d_{i+1} \in D_{i+1}$, we can define
\[
R_{i+1}(d_1, d_2, \ldots, d_i, d_{i+1}) = \left\{ w * v \mid w \in R_i(d_1, d_2 \cdots, d_i), \quad v \in \mathcal{V}(\sigma^{m_{i+1}+1} \omega, i+1, q_{i+1}) \right\}
\]
and
\[
R_{i+1} = \left\{ w * v \mid w \in R_i, \quad v \in \mathcal{V}(\sigma^{m_{i+1}+1} \omega, i+1, q_{i+1}) \right\}.
\]

**Step 3.** For any $d \in [T'(+\infty), T'(-\infty)]$, there exists $\{d_i\}_{i \in \mathbb{N}} \in \prod_{i \in \mathbb{N}} D_i$, such that $\lim_{i \to \infty} d_i = d$ and $\lim_{i \to \infty} T^*(d_i) = T^*(d)$.

Moreover, if $d \in (T'(+\infty), T'(-\infty))$, $T^*_i(d_i)$ converges to $T^*(d)$ directly from lemma 3.6. If $d \in \{T'(+\infty), T'(-\infty)\}$, again due to lemma 3.6, we can choose $(d_i)_{i \geq 1}$ to be piecewise constant to make sure that $T^*_i(d_i) - T^*(d_i)$ tends to 0 as $i \to \infty$, so that $T^*_i(d_i)$ converges to $T^*(d)$ as well.

We fix such a sequence and suppose that the correspondence $q$ are $\{q_i\}_{i \in \mathbb{N}}$.

Define
\[
K(\omega, \{d_i\}_{i \geq 1}) = \bigcap_{i \geq 1} \bigcup_{v \in R_i(d_1, d_2, \ldots, d_i)} U^v.
\]

We are going to prove that for any $x \in K(\omega, \{d_i\}_{i \geq 1})$, one has $\dim_{\text{loc}}(\mu_x, x) = d$, hence $K(\omega, \{d_i\}_{i \geq 1}) \subset E(\mu_x, d)$. To do so, we first establish two general estimates:

**First estimate.** This estimate will tell us that the Birkhoff sum $S_{m_{i+1}+n_{i+1}} \mathcal{Y}(\cdot)$ for $\mathcal{Y} \in \{\Phi, \Psi\}$ can be well approximated by
\[
\sum_{p=1}^{i} S_{n_p}^p \mathcal{Y}_p(F^{n_p} \cdot) + S_{n_{i+1}}^{i+1} \mathcal{Y}_{i+1}(F^{m_{i+1}} \cdot).
\]

For any $w \in R_i$ and $v \in \mathcal{V}(\sigma^{m_{i+1}} \omega, i+1, q_{i+1}, k)$, for any $k \leq \kappa_{i+1}$, for any $v \in \{w * v\}_\omega$, (remembering the notations introduced at the beginning of section 3.3.4 and the fact that by construction $|\mathcal{Y}(\omega, \mathcal{V}) - \mathcal{Y}_p(\omega, \mathcal{V})| \leq \bar{\var}_p \mathcal{Y}(\omega)$.) we have:
\[
\left| S_{m_{i+1}+n_{i+1}} \mathcal{Y}(\omega, \mathcal{V}) - \sum_{p=1}^{i} S_{n_p}^p \mathcal{Y}_p(F^{n_p} \cdot) + S_{n_{i+1}}^{i+1} \mathcal{Y}_{i+1}(F^{m_{i+1}} \cdot) \right|
\leq \sum_{p=1}^{i} \sum_{t=0}^{n_p} \left| (F^t \mathcal{Y}_p(F^{m_t} \cdot)) - S_{m_t}^t \mathcal{Y}(F^t \cdot) \right|
\leq \sum_{p=1}^{i} \sum_{t=0}^{n_p} \left| (F^t \mathcal{Y}_p(F^{m_t} \cdot)) - S_{(m_{i+1})^t}^{i+1} \mathcal{Y}_{i+1}(F^{m_{i+1}+t} \cdot) \right|
\leq \sum_{p=1}^{i} \sum_{t=0}^{n_p} \left( \bar{\var}_p \mathcal{Y}(\sigma^{m_t} \cdot) + (m_{i+1} \kappa_{i+1}) \mathcal{Y}(\sigma^{m_{i+1}+t} \cdot) \right)
\leq \sum_{p=1}^{i} (m_{p+1} - m_p - n_p^0) C.
\]

Also, since $\int_{\Omega} \bar{\var}_p \mathcal{Y}(\omega) \, d\omega \leq \varepsilon_p^3 \sigma^{m_p} \omega \in \Omega(p)$, and (15) and (25) hold, we have
\[
\sum_{i=0}^{n_{p_{m_{p}}} - 1} (\bar{\text{var}}_{p}(\sigma^{m_{p}+i} \omega)) = S_{n_{p_{m}}} \bar{\text{var}}_{p}(\sigma^{m_{p}} \omega) \\
\leq |S_{n_{p_{m}}} \bar{\text{var}}_{p}(\sigma^{m_{p}} \omega) - n_{p_{m}}^{p} \int_{\Omega} \bar{\text{var}}_{p}(\omega) \, d\mu| + n_{p_{m}}^{p} \int_{\Omega} \bar{\text{var}}_{p}(\omega) \, d\mu \\
\leq 2n_{p_{m}}^{p} \varepsilon_{p}^{3}.
\]

One estimates the term invoking \(\bar{\text{var}}_{i+1} \bar{Y}\) similarly. Then, recalling that \(m_{p+1} - m_{p} = M(p+1) + \theta(s+1, \omega, s_{p+1}) - m_{p} - n_{p_{m}}^{p}\), and using \([24]\) and \([38]\), one gets

\[
|S_{m_{i}+n_{i}^{k}} \bar{\text{var}}_{p}(\omega, \nu) - \sum_{i=1}^{n_{p}} \bar{\text{var}}_{p}(F^{m_{p}}(\omega, \nu)) - S_{n_{i}} \bar{\text{var}}_{p}(F^{m_{i+1}}(\omega, \nu))| \\
\leq \sum_{i=1}^{n_{p}} n_{p_{m}}^{i} \varepsilon_{p}^{3} + n_{i}^{k} \varepsilon_{i}^{3} + m_{1} C + C \sum_{i=1}^{n_{p}} n_{p_{m}}^{i} \varepsilon_{p}^{3} + \varepsilon_{i+1}^{3},
\]

\[
\leq \sum_{i=1}^{n_{p}} \varepsilon_{p_{m}}^{i} ((1+2 C) \varepsilon_{p}^{3}) + n_{i}^{k} \varepsilon_{i+1}^{3} + m_{1} C
\]

\[
\leq m_{i} (1+2 C) + m_{i+1} (1+2 C) \varepsilon_{i}^{3} + n_{i}^{k} \varepsilon_{i+1}^{3} + m_{1} C
\]

\[
\leq (m_{i} + n_{i}^{k}) (\varepsilon_{i}^{3})
\]

for \(i, k\) large enough, where in the last inequality we have used the fact that \(m_{i} \leq m_{i+1} \varepsilon_{i+1}^{3} + 1\) and \(\varepsilon_{i} \geq \varepsilon_{i+1} > 0\).

**Second estimate.** This estimate will tell us that if two intervals of the same generation are encoded by words with a very long common prefix, then their lengths are almost the same. For \(i \in \mathbb{N}\) large enough, for any \(k\) with \(\kappa_{i+1} > k \leq \kappa_{i+1}\) for any \(v, v' \in \Sigma_{\omega, m_{i}+n_{i}^{k}}\) satisfying \(|v \land v'| \geq m_{i} + n_{i}^{k}\), we have

\[
\frac{|U_{\omega}^{v}|}{|U_{\omega}^{v'}|} \leq \exp((m_{i} + n_{i}^{k}) \varepsilon_{i}^{2}).
\]

Indeed,

\[
\left| \log |U_{\omega}^{v}| - \log |U_{\omega}^{v'}| \right| \\
\leq V_{m_{i}+n_{i}^{k}} \Psi(\omega) + 2(m_{i} + n_{i}^{k}) \varepsilon(\Psi, \omega, m_{i} + n_{i}^{k}) \\
+ 2(n_{i}^{k+1} - n_{i}^{k}) C \text{ (by proposition 3 and the definition of } V_{n}) \\
\leq (4(m_{i} + n_{i}^{k}) + 2C) n_{i}^{k+1} \varepsilon_{i}^{3} + m_{1} C \\
\text{ (by } [37], [26] \text{ and } [14]).
\]

At last we get \(\left| \log |U_{\omega}^{v}| - \log |U_{\omega}^{v'}| \right| \leq (m_{i} + n_{i}^{k}) \varepsilon_{i}^{2}\) for \(i\) large enough, and \([40]\) follows.

We can now estimate the local dimension of \(\mu_{\omega}\). Fix \(x \in K(\omega, \{d_{i}\}_{i \geq 1})\). If \(r\) is small enough, we can choose the largest \(i\), and then the largest \(k = k_{i+1}\).
such that $\kappa_{i+1} < k \leq \pi_{i+1}$, such that the following property holds: there exists $w \in R_i(d_1, d_2, \cdots, d_i)$ and $v \in \mathcal{V}(\sigma^{m_{i+1}} \omega, i + 1, q_{i+1}, k)$ satisfying $x \in U_{\omega}^{u,v}$ and

$$|U_{\omega}^{u,v}| \geq 2 r \exp((m_{i+1} + n_{k}^{i+1}) \varepsilon_i^2).$$

By construction, if $U_{\omega}^{u,v^+}$ and $U_{\omega}^{u,v^-}$ are the neighboring intervals of $U_{\omega}^{u,v}$, then $|v \wedge v^+| , |v \wedge v^-|$ are larger than $n_{k}^{i+1}$. Then by (40) we have $|U_{\omega}^{u,v^+} \geq 2 r$ and $|U_{\omega}^{u,v^-} \geq 2 r$. So there exists $v' \in \{v^-, v^+\}$ such that $(B(x, r) \cap X_\omega) \subset U_{\omega}^{u,v} \cup U_{\omega}^{u,v'}$ (see figure 3).

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure3}
\caption{A cover for $B(x, r) \cap X_\omega$.}
\end{figure}

Now, using estimates similar to those leading to (40) with $\Psi$ replaced by $\Phi$ we get that for any $w \ast v \in [w \ast v]_\omega$,

$$\mu_\omega(B(x, r)) \leq \mu_\omega(U_{\omega}^{u,v}) + \mu_\omega(U_{\omega}^{u,v'})$$

$$\leq 2 \exp(S_{m_{i+1} + n_{k}^{i+1}} \Phi(\omega, w \ast v) + (m_{i+1} + n_{k}^{i+1}) \varepsilon_i^2).$$

Consequently, using (39), we get

$$\log \mu_\omega(B(x, r))$$

$$\leq \log 2 + \sum_{p=1}^{i} S_n \Phi_p(F^{m_p}(\omega, v)) + S_{n_{k}^{i+1}} \Phi_{i+1}(F^{m_{i+1}}(\omega, v))$$

$$+ 2(m_{i+1} + n_{k}^{i+1}) \varepsilon_i^2$$

$$\leq \sum_{p=1}^{i} S_n \Phi_p(F^{m_p}(\omega, v)) + S_{n_{k}^{i+1}} \Phi_{i+1}(F^{m_{i+1}}(\omega, v)) + 3(m_{i+1} + n_{k}^{i+1}) \varepsilon_i^2.$$

Let $T_p^\Phi = S_n \Phi_p(F^{m_p}(\omega, v))$ and $T_{i+1,k}^\Phi = S_{n_{k}^{i+1}} \Phi_{i+1}(F^{m_{i+1}}(\omega, v))$. Then

$$\log \mu_\omega(B(x, r)) \leq \left( \sum_{p=1}^{i} T_p^\Phi \right) + T_{i+1,k}^\Phi + 3(m_{i+1} + n_{k}^{i+1}) \varepsilon_i^2. \quad (41)$$

Now let us estimate $\log r$ from below:

\textbf{If} $k < \pi_{i+1}$, \textbf{from} our choice of $n_{k}^{i}$, \textbf{there} exists $\tilde{v}$ such that $|w \ast \tilde{v}| = m_{i+1} + n_{k}^{i+1}$, $x \in U_{\omega}^{u,v}$ and

$$|U_{\omega}^{u,v}| \leq 2 r \exp((m_{i+1} + n_{k}^{i+1}) \varepsilon_i^2). \quad (42)$$
Then, due to (26) and (14), one has \( \mathcal{I}^\Psi_{i+1,k+1} - \mathcal{I}^\Psi_{i+1,k} \geq -Cn_{i+1}^k \varepsilon_{i+1}^3 \) (where \( \mathcal{I}^\Psi_p \) is defined similarly as \( I^\Psi_p \)), so from (42) and (39) we can get

\[
\log r \geq \left( \sum_{p=1}^i \mathcal{I}^\Psi_p \right) + \mathcal{I}^\Psi_{i+1,k+1} - 2(m_{i+1} + n_{i+1}^i)\varepsilon_i^2 - \log 2
\]

\[
\geq \left( \sum_{p=1}^i \mathcal{I}^\Psi_p \right) + \mathcal{I}^\Psi_{i+1,k} - Cn_{i+1}^k \varepsilon_{i+1}^3 - 2(m_{i+1} + n_{i+1}^i)\varepsilon_i^2 - \log 2
\]

\[
\geq \left( \sum_{p=1}^i \mathcal{I}^\Psi_p \right) + \mathcal{I}^\Psi_{i+1,k} - 3(m_{i+1} + n_{i+1}^i)\varepsilon_i^2.
\]

Consequently,

\[
\log r \geq \left( \sum_{p=1}^i \mathcal{I}^\Psi_p \right) + \mathcal{I}^\Psi_{i+1,k} - 3(m_{i+1} + n_{i+1}^i)\varepsilon_i^2. \tag{43}
\]

**If** \( k = \pi_{i+1} \), there exists \( \bar{v} \) such that \( |w * \bar{v}| = m_{i+2} + n_{\pi_{i+2}+1}^i \), \( x \in U^w_{\bar{v}} \) and

\[ |U^w_{\bar{v}}| \leq 2\exp((m_{i+2} + n_{\pi_{i+2}+1}^i)\varepsilon_{i+1}^2). \]

Then, using proposition 3 to estimate \( |U^w_{\bar{v}}| \) yields

\[
\log r \geq \left( \sum_{p=1}^{i+1} \mathcal{I}^\Psi_p \right) + \mathcal{I}^\Psi_{i+2,\pi_{i+2}+1} - 2(m_{i+2} + n_{\pi_{i+2}+1}^i)\varepsilon_i^2 - \log 2
\]

(see proposition 3 and (37))

\[
\geq \left( \sum_{p=1}^i \mathcal{I}^\Psi_p \right) + \mathcal{I}^\Psi_{i+1,\pi_{i+1}} - 3(m_{i+1} + n_{\pi_{i+1}}^i)\varepsilon_i^2
\]

(see fact F2 and [36, 14]).

This implies that (43) holds as well.

Finally, for any \( \frac{1}{2} \in (w * \bar{v})_w \), (41) and (43) imply

\[
\frac{\log \mu_\omega(B(x,r))}{\log r} \geq \frac{\left( \sum_{p=1}^i \mathcal{I}^\Psi_p \right) + \mathcal{I}^\Psi_{i+1,k}}{\left( \sum_{p=1}^i \mathcal{I}^\Psi_p \right) + \mathcal{I}^\Psi_{i+1,k} - 3(m_{i+1} + n_{i+1}^i)\varepsilon_i^2}. \tag{44}
\]

Due to (30), we have \( |\mathcal{I}^\Psi_p - d_p| \leq \varepsilon_p \) and \( |\mathcal{I}^\Psi_{i+1,k} - d_{i+1}| \leq \varepsilon_{i+1} \) for \( k \geq \pi_{i+1} \). It follows from Stolz-Cesàro theorem that

\[
\liminf_{r \to 0} \frac{\log(\mu_\omega(B(x,r)))}{\log r} \geq d. \tag{45}
\]

It remains to prove that \( \limsup_{r \to 0} \frac{\log \mu_\omega(B(x,r))}{\log r} \leq d \). Choose the smallest \( i \) and then the smallest \( k = k_{i+1} \) such that \( \pi_{i+1} \leq k \leq \pi_{i+1} \), and there exist \( w \in R_i(d_1, d_2, \cdots, d_i) \) and \( v \in \mathcal{V}(\sigma^{m_{i+1}+1} \omega, i+1, q_{i+1}, k) \) for which \( x \in U^w_{\bar{v}} \) and \( |U^w_{\bar{v}}| \leq r \) (see figure 4).

Then using proposition 3 we have

\[
\mu_\omega(B(x,r)) \geq \mu_\omega(U^w_{\bar{v}}) \geq \exp(\left( \sum_{p=1}^i \mathcal{I}^\Psi_p \right) + \mathcal{I}^\Psi_{i+1,k} - 2(m_{i+1} + n_{i+1}^i)\varepsilon_i^2).
\]
If \( k > \kappa_{i+1} \), then \( \tilde{v} \), the father of \( v \), belongs to \( \mathcal{V}(\sigma^{m_{i+1}}\omega, i + 1, d_{i+1}, \kappa_{i+1} - 1) \) and \( x \in U^{w*\tilde{v}}_\omega \). We then have \( |U^{w*\tilde{v}}_\omega| \geq r \), so
\[
\log r \leq \log |U^{w*\tilde{v}}_\omega| \leq (\sum_{p=1}^{i} \mathcal{I}^\Psi_p) + \mathcal{I}^\Psi_{i+1,k-1} + 2(m_{i+1} + n_{i+1}^{i+1})\varepsilon_i^2.
\]

If \( k = \kappa_{i+1} \), then there exists \( w' \in R_{i-1}(d_1, d_2, \ldots, d_{i-1}) \), \( v' \in \mathcal{V}(\sigma^{m_i}\omega, i, d_i, \pi_i) \) with \( x \in U^{w'v'}_\omega \) and \( |U^{w'v'}_\omega| \geq r \), so
\[
\log r \leq \log |U^{w'v'}_\omega| \leq (\sum_{p=1}^{i} \mathcal{I}^\Psi_p) + \mathcal{I}^\Psi_{i+1,k} + 3(m_{i+1} + n_{i+1}^{i+1})\varepsilon_i^2.
\]

In both cases we have
\[
\log r \leq (\sum_{p=1}^{i} \mathcal{I}^\Psi_p) + \mathcal{I}^\Psi_{i+1,k} + 3(m_{i+1} + n_{i+1}^{i+1})\varepsilon_i^2.
\]

Finally we have
\[
\log(\mu_\omega(B(x, r))) \leq (\sum_{p=1}^{i} \mathcal{I}^\Psi_p) + \mathcal{I}^\Psi_{i+1,k} - 2(m_{i+1} + n_{i+1}^{i+1})\varepsilon_i^2,
\]

which yields
\[
\limsup_{r \to 0} \frac{\log(\mu_\omega(B(x, r)))}{\log r} \leq d.
\]

The inclusion \( K(\omega, \{d_i\}_{i \geq 1}) \subset E(\mu_\omega, d) \) is established.

**Step 4.** Now we turn to the estimate \( \dim_H K(\omega, \{d_i\}_{i \geq 1}) \) from below. Recall that \( w_0 \) has been defined in step 1. For any \( v = w_0 * v' \in R_1(d_1) \) (i.e. with \( v' \in \mathcal{V}(\sigma^{m_1}\omega, 1, q_1) \)), define
\[
\eta_\omega(U^{w_0v'}_\omega) := \frac{\mu_{\sigma^{m_1}\omega}(U^{v'}_{\sigma^{m_1}\omega})}{\sum_{v'' \in \mathcal{V}(\sigma^{m_1}\omega, 1, q_1)} \mu_{\sigma^{m_1}\omega}(U^{v''}_{\sigma^{m_1}\omega})}.
\]
Then inductively, for any \( w \in R_i(d_1, d_2, \ldots, d_i) \) and \( v \in \mathcal{V}(\sigma^{m_i+\omega}, i + 1, q_{i+1}) \), so that \( w * v \in R_{i+1}(d_1, d_2, \ldots, d_i, d_{i+1}) \), define:

\[
\eta_{\omega}(U_w^{w*v}) := \frac{\mu_{\sigma^{m_i+\omega}))(U_w^{w*v})}{\sum_{\omega'} \mu_{\sigma^{m_i+\omega}(U_{\omega'}^{w*v})}},
\]

We can extend \( \eta_{\omega} \) in a unique way to a probability measure on the \( \sigma \)-algebra generated by \( \bigcup_{i \geq 1}(U_w^{v} : v \in R_i(d_1, d_2, \ldots, d_i)) \). This measure is supported on \( K(\omega, \{d_i\}_{i \geq 1}) \).

Since for each \( i \geq 1 \) we have \( \sum_{v \in \mathcal{V}_i} \mu_{\sigma^{m_i+\omega}}(U_{\omega'}^{w*v}) \geq \mu_{\sigma^{m_i+\omega}}(E_{i,q_i}) \geq 1 - \epsilon_i \) and \( \prod_{i=1}^{\infty}(1 - \epsilon_i) \geq \frac{1}{2} \), using the same approach as in the proof of (45), by (31) (or see also (28), (29) and (32)) we can get that for any \( x \in K(\omega, \{d_i\}_{i \geq 1}) \),

\[
\liminf_{r \to 0} \frac{\log(\eta_{\omega}(B(x,r)))}{\log r} \geq \liminf_{i \to \infty} T^*(d_i) = T^*(d).
\]

This yields \( \dim_H(E(\mu_\omega,d)) \geq T^*(d) \).

**Remark 7.** In fact, if in the construction the sequence \( (m_i)_{i \in \mathbb{N}} \) is replaced by another one growing faster (with the effect to modify \( K(\omega, \{d_i\}_{i \in \mathbb{N}}), \eta_{\omega} \)), for example \( m_{i-1} \leq m_i \epsilon_i^3 \); see also remark [4], for all \( x \in K(\omega, \{d_i\}_{i \in \mathbb{N}}) \),

\[
\liminf_{r \to 0} \frac{\log \eta_{\omega}(B(x,r))}{\log r} = \liminf_{i \to \infty} T^*(d_i). \tag{46}
\]

\[
\limsup_{r \to 0} \frac{\log \eta_{\omega}(B(x,r))}{\log r} = \limsup_{i \to \infty} T^*(d_i). \tag{47}
\]

This property will be used in the next subsection.

**Proof of (46) and (47).** Now we recall inequality (32). The basic method of the proof is based on step 3 of the previous proof.

We have

\[
\liminf_{r \to 0} \frac{\log \eta_{\omega}(B(x,r))}{\log r} \geq \liminf_{i \to \infty} \frac{\sum_{p=1}^{i} \mathcal{I}_{p,q} + \mathcal{I}_{i+1,k+1} + 3(m_{i+1} + n_{k+1})\epsilon_i^2}{\sum_{p=1}^{i} \mathcal{I}_{p} + \mathcal{I}_{i+1,k+1} - 3(m_{i+1} + n_{k+1})\epsilon_i^2} \geq \liminf_{i \to \infty} T^*(d_i),
\]

where for the first inequality we have used the same estimates as those used to get (44). Also,

\[
\liminf_{r \to 0} \frac{\log \eta_{\omega}(B(x,r))}{\log r} \leq \liminf_{i \to \infty} \frac{\log \eta_{\omega}(B(x,|U_w^v|))}{\log |U_w^v|} \quad \text{(here } v \in R_i(d_1, \ldots, d_i) \text{ and } x \in U_w^v) \]

\[
\leq \liminf_{i \to \infty} \frac{\sum_{p=1}^{i} \mathcal{I}_{p,q} + 3(m_{i} + n_{r_{i}})\epsilon_i^2}{\sum_{p=1}^{i} \mathcal{I}_{p} - 3(m_{i} + n_{r_{i}})\epsilon_i^2} = \liminf_{i \to \infty} T^*(d_i),
\]

if we assume that \( m_{i-1} \leq m_i \epsilon_i^3 \). Thus equation (46) is established.
To get (47), observe that
\[
\limsup_{r \to 0} \frac{\log \eta_x(B(x, r))}{\log r} = \limsup_{i \to \infty} \frac{\log \eta_x(B(x, \frac{1}{2} |U_v^i| \exp(-(m_i + n_k^i) \varepsilon_{i-1}))}{\log |U_v^i| - (m_i + n_k^i) \varepsilon_{i-1}^2 - \log 2}
\]
(here \(v \in R(d_1, \ldots, d_i)\) and \(x \in U_v^i\))
\[
\geq \limsup_{i \to \infty} \frac{(\sum_{p=1}^{i} T_p^{\Lambda_p} + 2) + 3(m_i + n_k^i) \varepsilon_{i-1}^2}{(\sum_{p=1}^{i} T_p^\psi) + 2} = \limsup_{i \to \infty} T^*(d_i),
\]
if we assume that \(m_i - 1 \leq m_i \varepsilon_{i}^2\). On the other hand,
\[
\leq \limsup_{i \to \infty} \frac{(\sum_{p=1}^{i} T_p^{\Lambda_p} + 2) + 3(m_i + n_k^i) \varepsilon_{i-1}^2}{(\sum_{p=1}^{i} T_p^\psi) + 2} \leq \limsup_{i \to \infty} T^*(d_i).
\]

\[\Box\]

### 4.3. Proofs of (3)(4) in theorem 2.6 and theorem 2.7

We first notice that (4) in theorem 2.6 follows from (3) of it and proposition 1.3(2) in [2].

**Proof of theorem 2.6 (3).** We first deal with the lower bounds for the dimensions.

At first, for the Hausdorff dimension, let us take two sequences \((d_i)_{i \geq 1}\) and \((d'_i)_{i \geq 1}\) in \(\prod_{i \geq 1} D_i\) such that \(\lim_{i \to \infty} d_i = d\) and \(\lim_{i \to \infty} d'_i = d'\), with the properties:

\[
\lim_{i \to \infty} T^*(d_i) = T^*(d), \quad \lim_{i \to \infty} T^*(d'_i) = T^*(d').
\]

Set \(\tilde{d}_i = d_i\) and \(\tilde{d}_{i+1} = d'_i\).

We can use the same construction as in the previous subsection and get a set \(K_1(\omega, \{\tilde{d}_i\}_{i \geq 1}) \subset E(\mu_\omega, d, d')\), as well as a probability measure \(\eta_\omega\) supported on \(K_1(\omega, \{\tilde{d}_i\}_{i \geq 1})\). According to the proof of remark 7, one can also ensure that for all \(x \in K_1(\omega, \{\tilde{d}_i\}_{i \geq 1})\) one has \(\liminf_{r \to 0} \frac{\log \eta_x(B(x, r))}{\log r} = \inf \{T^*(d), T^*(d')\}\). Consequently we get

\[
\dim_H E(\mu_\omega, d, d') \geq \inf \{T^*(d), T^*(d')\}
\]
by definition of \(\dim_H(\eta_\omega)\).

For the packing dimension, we choose three sequences \((d_i)_{i \geq 1}\), \((d'_i)_{i \geq 1}\) and \((d''_i)_{i \geq 1}\) in \(\prod_{i \geq 1} D_i\) in such a way that \(\lim_{i \to \infty} d_i = d\), \(\lim_{i \to \infty} d'_i = d'\), \(\lim_{i \to \infty} d''_i = d''\), \(\lim_{i \to \infty} T^*(d_i) = T^*(d)\), \(\lim_{i \to \infty} T^*(d'_i) = T^*(d')\), and \(\lim_{i \to \infty} T^*(d''_i) = T^*(d'')\).

Here again, we get \(K_2(\omega, \{\tilde{d}_i\}_{i \geq 1}) \subset E(\mu_\omega, d, d')\) and the measure \(\eta_\omega\), which satisfies \(\limsup_{r \to 0} \frac{\log \eta_x(B(x, r))}{\log r} = \sup \{T^*(d), T^*(d'), T^*(d'')\}\).

Consequently we get \(\dim_P E(\mu_\omega, d, d') \geq T^*(d'')\) by definition of \(\dim_P(\eta_\omega)\).

The upper bound of the dimensions directly come from proposition 1.3(1) and (1.4) and (1.5) in [2].

**Proof of theorem 2.7** The properties \(H^q(E(\mu_\omega, d)) = 0\) if \(\limsup_{r \to 0^+} \frac{\log(g(r))}{\log(r)} > T^*(d)\) and \(P^q(E(\mu_\omega, d)) = 0\) if \(\liminf_{r \to 0^+} \frac{\log(g(r))}{\log(r)} > T^*(d)\) follow from standard estimates.
Suppose \( d \in [T'(\infty), T'(-\infty)] \) and \( T^*(d) < \max(T^*) \). The estimates used in the proof of (45) and the proof of remark 7 can be used to get a sequence \((\varepsilon_i')_{i \in \mathbb{N}}\) of positive numbers decreasing to 0 and a constant \( C' > 0 \) such that, independently on \( \{d_i\}_{i \in \mathbb{N}} \in \prod_{i \geq 1} D_i \), for all \( x \in K(\omega, \{d_i\}_{i \in \mathbb{N}}) \), for \( i \) large enough, if \( \exp(-m_{i+1}c_\Psi/2) < r \), then

\[
\eta_\omega(B(x, r)) \leq C' \min\{T^*(d_i) - \varepsilon'_i: 1 \leq j \leq i\}. \tag{48}
\]

Since \( T^*(d) < \max(T^*) \), we can find \( \{d_i\}_{i \in \mathbb{N}} \in \prod_{i \in \mathbb{N}} D_i \) such that \( d_i \to d \) as \( i \to \infty \), \( T^*(d_i) - \varepsilon'_i \geq T^*(d) + \varepsilon'_i \) for \( i \) large enough, \( T^*(d_i) \to T^*(d) \) as \( i \to \infty \), and \( T^*(d_i) - \varepsilon'_i \) is ultimately non increasing.

For any gauge function \( g \) such that \( \limsup_{r \to 0} \frac{\log g(r)}{\log r} \leq T^*(d) \), there exists a sequence \( \{v_r\}_{r > 0} \) of positive numbers such that both \( v_r \) and \( r \) decrease to 0 as \( r \) decreases to 0 and

\[
g(r) \geq r^{T^*(d) + v_r} \quad (r \leq 1).
\]

Due to (48), for \( i \) large enough, for any \( r \) such that \( \exp(-m_{i+1}c_\Psi/2) \leq r \leq \exp(-m_ic_\Psi/2) \), for any \( x \in K(\omega, \{d_i\}_{i \geq 1}) \) one has

\[
\eta_\omega(B(x, r)) \leq C' \min\{T^*(d_i) - \varepsilon'_i: 1 \leq j \leq i\} \leq C' r^{T^*(d_i) - \varepsilon'_i} \leq C' r^{T^*(d) + \varepsilon'_i}.
\]

Notice that \( g(r)r^{v_r} \geq r^{T^*(d) + 2v_r} \). So, if we can impose \( 2v_r \leq \varepsilon'_i \), we will have \( \eta_\omega(B(x, r)) \leq C' g(r)r^{v_r} \), hence \( g(r) \geq C'^{-1} \eta_\omega(B(x, r))r^{-v_r} \). Then, for any positive real number \( \delta > 0 \), this will yield \( \mathcal{H}_0^\delta(K(\omega, \{d_i\}_{i \geq 1})) \geq C'^{-1} \delta^{-v_\omega} \), and letting \( \delta \to 0 \), \( \mathcal{H}^\delta(K(\omega, \{d_i\}_{i \geq 1})) = +\infty \) so \( \mathcal{H}^\delta(E(\mu_\omega, d)) = +\infty \).

Now, if we choose \( m_i \) large enough so that

\[
v_{\exp(-m_i c_\Psi/2)} \leq \varepsilon'_i/2,
\]
then for \( \exp(-m_{i+1}c_\Psi/2) \leq r \leq \exp(-m_ic_\Psi/2) \), we have \( 2v_r \leq \varepsilon'_i \) since \( v_r \leq v_{\exp(-m_i c_\Psi/2)} \).

Finally, suppose that \( g \) is a gauge function such that \( \liminf_{r \to 0} \frac{\log g(r)}{\log r} \leq T^*(d) \). There exist \( \{r_j\}_{j \in \mathbb{N}} \in (0, 1]^\mathbb{N} \) and \( \{r_{j,r}\}_{j \in \mathbb{N}} \in (0, \infty)^\mathbb{N} \) such that \( v_{r_j} \in (0, 1] \) and \( r_{j,r} \) decrease to 0 as \( j \) tends to \( \infty \), and

\[
g(2r_j) \geq r_j^{T^*(d) + v_{r_j}}.
\]

Using the same approach as above, we can choose \( \{d_{i,j}\}_{i \geq 1} \in \prod_{i \geq 1} D_i \) such that \( \lim_{i \to \infty} d_i = d, T^*(d_{i,j}) \) converges slowly to \( T^*(d) \) from above, and in the construction of \( K(\omega, \{d_{i,j}\}_{i \geq 1}, \eta_\omega), m_i \) tends fast enough to \( \infty \) so that, for some \( j_0 \in \mathbb{N} \), for all \( j \geq j_0 \), for any \( x \in K(\omega, \{d_{i,j}\}_{i \geq 1}) \),

\[
\eta_\omega(B(x, r_j)) \leq C'(2r_j)^{T^*(d) + 2v_{r_j}}.
\]

Now, let \( A \subset K(\omega, \{d_{i,j}\}_{i \geq 1}) \) be of positive \( \eta_\omega \)-measure. For any given \( \delta > 0 \), take \( j_0 \) such that \( r_{j_0}^\delta \leq \delta \) consider the following family of closed balls

\[
B_{\delta} = \{B(x, r_j) : x \in A, j \geq j_0\},
\]
which is a covering of \( A \). Due to Besicovitch covering theorem, we can extract an at most countable subfamily of pairwise disjoint balls \( \{B(x_i, \rho_i)\}_{i \in I} \) such that \( \eta_\omega(\bigcup_{i \in I} B_i) \geq \eta_\omega(A)/\Gamma_1 > 0 \) where \( \Gamma_1 \) is the constant with respect to dimension...
1. This family is a $\delta$-packing of $A$, and ($P_{0,\delta}^g$ stands for the prepacking measure associated with $g$)

$$P_{0,\delta}^g(A) \geq \sum_i g(B(x_i, \rho_i)) \geq \sum_i \rho_i^{T(d) + \nu \rho_i} \geq \sum_i \rho_i^{-\nu \rho_i} \eta_\omega(B(x_i, \rho_i)) \geq \rho_{j_0}^{-\nu \rho_i} \eta_\omega(A)/\Gamma_1.$$ 

As $j_0' \to \infty$ when $\delta \to 0$, we can conclude that $P_{0}^g(A) = +\infty$. Then, since any at most countable covering of $K(\omega, \{d_i\}_{i \geq 1})$ must contain a set $A$ of positive $\eta_\omega$-measure, we finally get $P^g(K(\omega, \{d_i\}_{i \geq 1})) = +\infty$, so $P^g(E(\mu_\omega, d)) = +\infty$. \hfill $\square$

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Index of notations

* : concatenation of two words, defined at the beginning of section 2.1
$A(\omega)$ : random transition matrix.
$c_\Psi$, $C_\Phi$ and $C_\Psi$ : constants defined in (2) and (1).
$\epsilon(\cdot, \omega, n)$ : a measure of distortion defined in proposition 3.
$\kappa_i$ : deterministic integer defined in facts $F2$ in section 3.3.4.
$l(\omega)$ : number of random types.
$\Lambda_i, \Psi_i : q\Phi_i - T_i(q)\Psi_i.$
$M(\omega)$ : random integer such that $A(\omega)A(\sigma_\omega) \ldots A(\sigma^{M(\omega) - 1}\omega)$ is positive.
$\mu^\Phi: \omega \in \Omega)$ : random weak Gibbs measures on $\{X_\omega : \omega \in \Omega\}$ associated with the potential $\Phi$.
$(n_k(\omega))_{k \in \mathbb{N}} :$ sequence defined in section 3.3.4.
n$(\cdot)$ : deterministic integer defined in facts $F2$ in section 3.3.4
$P(\cdot)$ : topological pressure.
$(\Psi_i, \Phi_i)$ : the $i$-th approximation of $(\Psi, \Phi)$ by Hölder potentials.
$T(q)$ : root of $P(q\Phi - t\Psi) = 0$.
$T_i(q)$ : root of $P(q\Phi_i - t\Psi_i) = 0$ and also the root of $P(q\Phi_j - t\Psi_j) = 0$.
$(\theta_i(\cdot, \omega, s))_{s \in \mathbb{N}}$ : the $s$-th return time of point $\omega$ to the set $\Omega(i)$ defined in the first series of properties derived in section 3.3.4.
$(\theta(\cdot, \omega, s))_{s \in \mathbb{N}}$ : the $s$-th return time of point $\omega$ to the set $\Omega_i \subset \Omega(i)$ defined in the second half of section 3.3.4.
$\text{var}_n$ : $n$-th variation.

REFERENCES

[1] I. S. Baek, L. Olsen and N. Snigireva, Divergence points of self-similar measures and packing dimension, Adv. Math., 214 (2007), 267–287.
[2] J. Barral, Inverse problems in multifractal analysis, Ann. Sci. Ec. Norm. Sup. (4), 48 (2015), 1457–1510.
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[2] T. Bogenschütz, Entropy, pressure, and a variational principle for random dynamical systems, *Random Comput. Dynam.*, 1 (1992/93), 99–116.

[3] T. Bogenschütz and V. M. Gundlach, Ruelle’s transfer operator for random subshifts of finite type, *Ergod. Th. & Dynam. Sys.*, 15 (1995), 413–447.

[4] P. Collet, J. L. Lebowitz and A. Porzio, The dimension spectrum of some dynamical systems, *J. Statist. Phys.*, 47 (1987), 609–644.

[5] M. Denker and M. Gordin, Gibbs measures for fibred systems, *Adv. Math.*, 148 (1999), 161–192.

[6] M. Denker, Y. Kifer and M. Stadlbauer, Thermodynamic formalism for random countable Markov shifts, *Discrete Contin. Dyn. Syst.*, 22 (2008), 3052–3077.

[7] M. Denker, K.-S. Lau and J. Wu, Ergodic limits on the conformal repellers, *Adv. Math.*, 169 (2002), 58–91.

[8] M. Denker and M. Gordin, Gibbs measures for fibred systems, *Adv. Math.*, 148 (1999), 161–192.

[9] M. Denker, Y. Kifer and M. Stadlbauer, Corrigendum to: Thermodynamic formalism for random countable Markov shifts, *Discrete Contin. Dyn. Syst.*, 29 (2009), 885–918.

[10] Y. Kifer, Equilibrium states for random expanding transformations, *Random Comput. Dynam.*, 1 (1992/93), 1–31.

[11] Y. Kifer, On the topological pressure for random bundle transformations, in *Topology, ergodic theory, real algebraic geometry*, Amer. Math. Soc. Transl. Ser. 2, 202 (2001), 197–214.

[12] Y. Kifer, Thermodynamic formalism for random transformations revisited, *Stoch. Dyn.*, 8 (2008), 77–102.

[13] Y. Kifer and P.-D. Liu, Random dynamics, in *Handbook of dynamical systems*, Elsevier B. V., Amsterdam, 1B (2006), 379–499.

[14] S. P. Lalley and D. Gatzouras, Hausdorff and box dimensions of certain self-affine fractals, *Indiana Univ. Math. J.*, 41 (1992), 533–568.

[15] K.-S. Lau and S.-M. Ngai, Multifractal measures and a weak separation condition, *Adv. Math.*, 141 (1999), 45–96.

[16] N. Luzia, A variational principle for the dimension for a class of non-conformal repellers, *Ergod. Th. & Dynam. Sys.*, 26 (2006), 821–845.
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[31] J.-H. Ma, Z.-Y. Wen and J. Wu, Besicovitch subsets of self-similar sets,  Ann. Inst. Fourier, 52 (2002), 1061–1074.

[32] J.-H. Ma and Z.-Y. Wen, Hausdorff and Packing measure of sets of generic points: A Zero-Infinity Law, J. London Math. Soc., 69 (2004), 383–406.

[33] P. T. Maker, The ergodic theorem for a sequence of functions, Duke Math. J., 6 (1940), 27–30.

[34] P. Mattila, Geometry of Sets and Measures in Euclidean Spaces, Fractals and Rectifiability, Cambridge Studies in Advanced Mathematics, 44. Cambridge University Press, Cambridge, 1995.

[35] V. Mayer, B. Skorulski and M. Urbaniak, Distance Expanding Random Mappings, Thermodynamical Formalism, Gibbs Measures and Fractal Geometry, Lecture Notes in Mathematics, 2036, Springer, Heidelberg, 2011.

[36] E. Olivier, Multifractal analysis in symbolic dynamics and distribution of pointwise dimension for g-measures, Nonlinearity, 12 (1999), 1571–1585.

[37] L. Olsen, A multifractal formalism, Adv. Math., 116 (1995), 82–196.

[38] L. Olsen, Multifractal analysis of divergence points of deformed measure theoretical Birkhoff averages, J. Math. Pures Appl., 82 (2003), 1591–1649.

[39] L. Olsen, Multifractal analysis of divergence points of deformed measure theoretical Birkhoff averages. IV: Divergence points and packing dimension, Bull. Sci. Math., 132 (2008), 650–678.

[40] N. Patzschke, Self-Conformal Multifractal Measures, Adv. Appl. Math., 19 (1997), 486–513.

[41] Y. Pesin and H. Weiss, On the dimension of deterministic and random Cantor-like sets, symbolic dynamics, and the Eckmann-Ruelle conjecture, Comm. Math. Phys., 182 (1996), 105–153.

[42] Y. Pesin and H. Weiss, A multifractal analysis of equilibrium measures for conformal expanding maps and Moran-like geometric constructions, J. Statist. Phys., 86 (1997), 233–275.

[43] D. A. Rand, The singularity spectrum f(α) for cookie-cutters, Ergod. Th. & Dynam. Sys., 9 (1989), 527–541.

[44] O. Sarig, Thermodynamic formalism for countable Markov shifts, Ergod. Th. & Dynam. Sys., 19 (1999), 1565–1593.

[45] O. Sarig, Thermodynamic formalism for countable Markov shifts, Hyperbolic dynamics, fluctuations and large deviations, Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, RI, 89 (2015), 81–117.

[46] L. Shu, The multifractal analysis of Birkhoff averages for conformal repellers under random perturbations, Monatsh. Math., 159 (2010), 81–113.

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