Refined Enumeration of Permutations Sorted with Two Stacks and a $D_8$-Symmetry

Mathilde Bouvel and Olivier Guibert
LaBRI, CNRS and Université de Bordeaux, 351, cours de la Libération, 33405 Talence cedex, France
{bouvel, guibert}@labri.fr
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Abstract. We study permutations that are sorted by operators of the form $S \circ \alpha \circ S$, where $S$ is the usual stack sorting operator introduced by Knuth and $\alpha$ is any $D_8$-symmetry obtained by combining the classical reverse, complement, and inverse operations. Such permutations can be characterized by excluded (generalized) patterns. Some conjectures about the enumeration of these permutations, refined with numerous classical statistics, have been proposed by Claesson, Dukes, and Steingrímsson. We prove these conjectures, and enrich one of them with a few more statistics. The proofs mostly rely on generating trees techniques, and on a recent bijection of Giraudo between Baxter and twisted Baxter permutations.

Keywords: permutations, generalized patterns, stack sorting, symmetries of the square, enumeration, permutation statistics, Baxter permutations, generating trees

1. Introduction

Since the very beginning, the study of permutations with excluded patterns has been intimately linked with sorting devices, and especially with the stack sorting operator. Indeed, permutation patterns have been first introduced by Knuth [24], who characterized permutations sorted by one stack as those avoiding the classical pattern 231. Since the fundamental work of Knuth, many articles have subsequently studied permutations that are sorted by some sorting operators or their composition (see, for instance, [7, 9], or more recently [2, 14, 15, 27]). A large part of the results obtained in this field are (or are related to) characterizations of such families of permutations by excluded patterns.

Pattern avoiding permutations have been intensively studied over the past decades, especially from an enumerative point of view, and many enumeration results have been obtained. This is true in particular for permutation classes that avoid classical patterns, but also for families of permutations defined by excluded generalized patterns of various types (dashed patterns [3], barred patterns [30], vincular or bi-vincular patterns [10], or more recently mesh patterns [11]). One of the important
aspects of these results is the specific attention that is given to enumeration refined according to some statistics. An example of such a result in the context of stack sorting is provided by [8]. In the general framework of pattern-avoiding permutations, enumeration problems refined with the distribution of some statistics are being more and more investigated in the community, under the name of st-Wilf equivalence [16]. These are not only interesting from an enumerative point of view, but also provide more insight in the study of a pattern-avoiding permutation family.

This article deals with these two aspects of the study of permutations sorted by $S \circ \alpha \circ S$, where $S$ is the usual stack sorting operator and $\alpha$ is any $D_8$-symmetry. We first characterize such permutations by excluded patterns. These characterizations are then used to enumerate such permutations, and our enumerative results are refined according to some statistics. More precisely, we provided two statistic-preserving bijections, the first one between permutations sorted by $S \circ S$ and those sorted by $S \circ r \circ S$, $r$ denoting the reverse (see Theorem 3.2), and the second one between permutations sorted by $S \circ i \circ S$ and Baxter permutations, $i$ denoting the inverse (see Theorem 3.4).

2. Definitions and Notations

In this section, we recall the classical definitions about permutations, patterns, $D_8$-symmetries and stack sorting, that are necessary to state our results.

2.1. Permutations, Diagrams, and $D_8$-Symmetries

A permutation of $\mathfrak{S}_n$ is a bijective map from $[1..n] = \{1, 2, \ldots, n\}$ to itself, $n$ being called the size of the permutation. In our context, we will view permutations in two different ways. A permutation $\sigma$ of $\mathfrak{S}_n$ can be seen as a word $\sigma_1 \sigma_2 \cdots \sigma_n$ where $\sigma_i = \sigma(i)$ for all $i \in \{1, \ldots, n\}$, containing exactly once each letter from 1 to $n$. It can also be seen as what we call its diagram: An $n \times n$ grid with exactly one dot per row and per column, the dots being placed in cells of coordinates $(i, \sigma(i))$. For every element of a permutation $\sigma$, corresponding to the dot at coordinates $(i, \sigma(i))$ in its diagram, we call $i$ its index and $\sigma(i)$ its value.

Recall that for $\sigma \in \mathfrak{S}_n$, its reverse (complement, inverse, respectively) is the permutation $r(\sigma)$ ($c(\sigma)$, $i(\sigma)$, respectively) defined by $r(\sigma)(i) = \sigma(n+1-i)$ ($c(\sigma)(i) = n+1-\sigma(i)$, $i(\sigma)(i) = j$ such that $\sigma(j) = i$, respectively). These operations correspond respectively to symmetries with respect to a vertical axis, a horizontal axis and the south-west to north-east diagonal on the diagrams of the permutations (see Figure 1). Hence, these three operations generate the eight element group $D_8$ of the symmetries of the square.

2.2. Permutation Patterns and Some Generalizations

From the word representation of permutations, we inherit basic concepts like word concatenation, or subwords*. A subword (with $k$ letters) of a permutation is however not a permutation in general, as its letters may not consist of all integers

*By subword of a word $w$, we mean a subsequence of the letters of $w$ whose order is preserved. These letters are not necessarily consecutive in $w$. Subwords formed by subsequences of consecutive letters are called factors.
An occurrence of a mesh pattern to a normalized dashes we will make use of three of them here. When introducing represented by diagrams, where elements of \( \pi \) is allowed in the shaded regions of the mesh pattern. For a formal definition of \( \sigma \) is a (classical) occurrence of \( \pi \) in \( \sigma \) and \( \pi \) and \( \sigma \) are order-isomorphic to \( \pi \), i.e., such that \( \sigma_{i_1} \cdots \sigma_{i_k} \) is order-isomorphic to \( \pi \) and \( \pi_{\ell+1} = i_{\ell + 1} \) whenever there is no dash between \( \pi_{\ell} \) and \( \pi_{\ell + 1} \) in \( \pi \). Notice that classical patterns can be viewed as dashed patterns with dashes between \( \pi_{\ell} \) and \( \pi_{\ell + 1} \) for any \( \ell \in \{1, \ldots, k - 1\} \), and that a dashed pattern with no dash corresponds to a normalized factor of the permutation. However, in this article, we take the convention that a pattern written with no dash is always a classical pattern. Pattern avoidance of dashed patterns is defined as in the classical case.

In the same way dashed patterns introduce adjacency constraint between elements of a pattern, we can impose a dual condition on the elements of the permutation that are used to form a given pattern, by imposing that they use consecutive values \( \sigma_i, \sigma_i + 1, \ldots \). Combining these two types of constraints (adjacencies in indices and in values), one obtains the so-called bivincular patterns (see [10]). We consider here a generalization of the bivincular patterns called mesh patterns introduced in [11]. Mesh patterns will be used in Sections 6 and 7. A mesh pattern is a pair \((\pi, R)\), with \( \pi \) a permutation of size \( k \) and \( R \subseteq [0..(k + 1)] \times [0..(k + 1)] \). Mesh patterns can be represented by diagrams, where elements of \( \pi \) correspond to points at coordinates \((i, \pi(i))\) and every pair \((i, j)\) of \( R \) corresponds to a shaded unit square with bottom left corner \((i, j)\). For example, the mesh pattern \( \mu = (21, \{(0, 1), (1, 0), (1, 1)\}) \) is represented as \( \mu = \square \). An occurrence of a mesh pattern \((\pi, R)\) in a permutation \( \sigma \) is a (classical) occurrence of \( \pi \) in \( \sigma \) that additionally satisfies that no element of \( \sigma \) is allowed in the shaded regions of the mesh pattern. For a formal definition of pattern containment of mesh pattern, we refer the reader to [11].
The third generalization of pattern avoidance we consider is the one of patterns with one barred element [30] (or barred patterns† for short). Consider a permutation $\pi$ with one barred element, and denote by $\pi^\sim$ the normalization of the subword of $\pi$ obtained when deleting the barred element. We say that a permutation $\sigma$ contains the barred pattern $\pi$ if there exists a (classical) occurrence of $\pi^\sim$ in $\sigma$ that cannot be extended into a (classical) occurrence of $\pi$ in $\sigma$. Consequently, a permutation $\sigma$ avoids the barred pattern $\pi$ if every occurrence of $\pi^\sim$ in $\sigma$ can be extended into an occurrence of $\pi$ in $\sigma$.

We denote by $\text{Av}(\pi', \pi'', \ldots, \pi''')$ the set of permutations that simultaneously avoid $\pi'$, $\pi''$, ..., and $\pi'''$.

Example 2.1. Permutation $\sigma = 316452$ avoids the classical pattern 2413 but contains the classical pattern $\pi = 2431$, and the subwords 3642 and 3652 are its two occurrences in $\sigma$. Moreover, $\sigma$ avoids 24-3-1 as a dashed pattern, as the elements corresponding to 2 and 4 are never at consecutive indices in an occurrence of $\pi$ in $\sigma$. Furthermore, $\sigma$ avoids the barred pattern $\tau = 3\bar{1}542$ as all the occurrences of $\tau^\sim = \pi$ can be extended with a smallest element to account for $\bar{1}$. Finally, $\sigma$ contains two occurrences of the mesh pattern $\mu$ defined above, namely, 31 and 64.

2.3. Some Sorting Operators on Permutations

The study of the stack sorting operator for permutations has been at the origin of the definition of classical permutation patterns by Knuth [24]. Some of the above mentioned generalizations of permutation patterns also arise from the study of stack sorting. In particular, barred patterns have been introduced by West in [31] for characterizing the permutations that can be sorted by two passes of the stack sorting operator $S$ (see Theorem 3.1 below). More recently, mesh patterns and their generalizations (marked mesh patterns [28], decorated patterns [27]) were used to characterize permutations sorted by three passes of the stack sorting operator $S$ in [15, 27].

We say that a permutation $\sigma \in S_n$ is sorted by an operator $\text{Sort}$ when $\text{Sort}(\sigma) = 12\cdots n$.

The operation of sorting a permutation $\sigma = \sigma_1 \cdots \sigma_n$ through a stack is defined as follows. Consider a stack that satisfies the Hanoi condition, i.e., such that the elements in the stack are in increasing order from the top to the bottom of the stack. Starting from a permutation $\sigma = \sigma_1 \cdots \sigma_n$ and an empty stack, consider the elements $\sigma_i$ of $\sigma$, for increasing $i$ starting at 1. For each $\sigma_i$, all the elements of the stack that are smaller than $\sigma_i$ are popped out, and $\sigma_i$ is pushed on the top of the stack. After $\sigma_n$ has been processed, all the elements that remain in the stack are popped out. This outputs a permutation, denoted $S(\sigma)$, which is the result of the stack sorting of $\sigma$.

More formally, the stack sorting operator is also classically characterized recursively by $S(\varepsilon) = \varepsilon$ and $S(LnR) = S(L)S(R)n$ where $n$ is the maximum of the word $LnR$ of distinct integers (which is not necessarily a permutation) and $\varepsilon$ denotes the empty word. Other sorting operators on permutations have been studied in the literature, in connection with permutation patterns. This is the case in [13] for the tack sorting

† As noticed in [11], this corresponds to a special case of mesh pattern, where $R$ contains only one pair $(i, j)$. 
operator $T$ defined by $T(e) = e$ and $T(LnR) = T(R)T(L)n$ or in [2] for the bubble sort operator $B$ defined by $B(e) = e$ and $B(LnR) = B(L)Rn$.

The tree sorting operator can be easily characterized by the identity $T(\sigma) = S(r(\sigma))$ for every permutation $\sigma$. Therefore, the compositions of these two sorting operators can be interpreted as $S \circ T = S \circ S \circ r$ and $T \circ S = S \circ r \circ S$. Similarly, $T \circ T = S \circ r \circ S \circ r$. Hence, following the line of [31] and looking for a characterization of the permutations that are sorted by these compositions of sorting operators, we are led to the analysis of the permutations that are sorted by $S \circ r \circ S$. We actually address a rather more general question here: We characterize and enumerate permutations that are sorted by $S \circ \alpha \circ S$ for any $\alpha$ in the group $D_8$.

3. Main Results

For any sorting operator $\text{Sort}$, let us denote by $\text{ld}_n(\text{Sort})$ the set of permutations of size $n$ that are sorted by $\text{Sort}$, i.e., $\text{ld}_n(\text{Sort}) = \{ \sigma \in \mathfrak{S}_n : \text{Sort}(\sigma) = 12 \cdots n \}$. Let us also write $\text{ld}(\text{Sort}) = \cup_n \text{ld}_n(\text{Sort})$. It has been known since Knuth [24] that $\text{ld}(S) = \text{Av}(231)$, and West [31] has proved that $\text{ld}(S \circ S) = \text{Av}(2341, 3\overline{5}241)$. Theorem 3.1 below has been proved by Albert et al. [1]. However, this result being unpublished, we give its proof here. We can also notice that the methodology introduced by Claesson and Úlfarsson in [15,27] applies here, and could also provide a proof of Theorem 3.1.

**Theorem 3.1.** The sets of permutations that are sorted by $S \circ \alpha \circ S$, for any $\alpha$ in $D_8$ are characterized by

(i) $\text{ld}(S \circ S) = \text{ld}(S \circ i \circ c \circ r \circ S) = \text{Av}(2341, 3\overline{5}241)$;
(ii) $\text{ld}(S \circ c \circ S) = \text{ld}(S \circ i \circ r \circ S) = \text{Av}(231)$;
(iii) $\text{ld}(S \circ r \circ S) = \text{ld}(S \circ i \circ c \circ S) = \text{Av}(1342, 31-4-2) = \text{Av}(1342, 3\overline{5}142)$;
(iv) $\text{ld}(S \circ i \circ S) = \text{ld}(S \circ r \circ c \circ S) = \text{Av}(3412, 3-4-21)$.

**Proof.** Let us recall that for any permutation $\sigma$ and any pattern $\pi$ (classical, dashed, or barred), we have $\sigma \in \text{Av}(\pi) \iff r(\sigma) \in \text{Av}(r(\pi)) \iff c(\sigma) \in \text{Av}(c(\pi)) \iff i(\sigma) \in \text{Av}(i(\pi))$. Let us also notice that the following equivalence holds, for any $\alpha$ in $D_8$: $\sigma \in \text{ld}(S \circ \alpha \circ S) \iff \alpha(S(\sigma)) \in \text{ld}(S) = \text{Av}(231)$. It will be used in the proof of every point of Theorem 3.1.

(i) With the result of [31], it is enough to prove that $\text{ld}(S \circ S) = \text{ld}(S \circ i \circ c \circ r \circ S)$. And indeed, for any permutation $\sigma$, we have $\sigma \in \text{ld}(S \circ S) \iff S(\sigma) \in \text{Av}(231) \iff r \circ S(\sigma) \in \text{Av}(132) \iff c \circ r \circ S(\sigma) \in \text{Av}(312) \iff i \circ c \circ r \circ S(\sigma) \in \text{Av}(231) \iff \sigma \in \text{ld}(S \circ i \circ c \circ r \circ S)$.

(ii) Similarly, we have $\sigma \in \text{ld}(S \circ c \circ S) \iff c \circ S(\sigma) \in \text{Av}(231) \iff S(\sigma) \in \text{Av}(213) \iff r \circ S(\sigma) \in \text{Av}(312) \iff i \circ r \circ S(\sigma) \in \text{Av}(231) \iff \sigma \in \text{ld}(S \circ i \circ r \circ S)$.

Furthermore, we claim that $S(\sigma) \in \text{Av}(213) \iff \sigma \in \text{Av}(231)$. Indeed, if $\sigma \in \text{Av}(231)$, then $\sigma$ is sorted by $S$, hence $S(\sigma) = 12\cdots n$ (with $n$ denoting the size of $\sigma$) thus avoids $213$. Conversely, if $\sigma \notin \text{Av}(231)$, then consider an occurrence $bca$ of $231$ in $\sigma$ such that $c$ is maximal: $S(\sigma)$ contains the subsequence $bac$ which is an occurrence of $213$. 


(iii) We prove first that \( \text{ld}(S \circ r \circ S) = \text{ld}(S \circ i \circ c \circ S) \). This follows from the following equivalences: \( \sigma \in \text{ld}(S \circ r \circ S) \iff r \circ S(\sigma) \in \text{Av}(231) \iff S(\sigma) \in \text{Av}(132) \iff c \circ S(\sigma) \in \text{Av}(312) \iff i \circ c \circ S(\sigma) \in \text{Av}(231) \iff \sigma \in \text{ld}(S \circ i \circ c \circ S) \). We next prove that \( S(\sigma) \in \text{Av}(132) \iff \sigma \in \text{Av}(1342, 31-4-2) \).

If \( \sigma \) contains an occurrence \( \sigma_i \sigma_j \sigma_k \sigma_\ell \) of 1342 (31-4-2, respectively) then necessarily \( S(\sigma) \) contains the subsequence \( \sigma_i \sigma_j \sigma_\ell \) (\( \sigma_j \sigma_i \sigma_\ell \), respectively) which is an occurrence of 132. Next, we prove by induction that \( \sigma \) contains 1342 or 35142 as long as \( S(\sigma) \) contains 132. This is clear when \( \sigma = e \), and consider a non-empty permutation \( \sigma \). Recall that writing \( \sigma = LnR \), with \( n \) the maximum of \( \sigma \), we have \( S(\sigma) = S(L)S(R)n \). Therefore, any occurrence of 132 in \( S(\sigma) \) is actually a subsequence \( acb \) of \( S(L)S(R) \). If it is contained in \( S(L) \) (\( S(R) \), respectively) then we conclude by induction that \( L \) (\( R \), respectively), and consequently \( \sigma \), contains an occurrence of 1342 or 35142. Assume now that \( a \) is in \( S(L) \) and \( cb \) is in \( S(R) \). Since, for any sequence \( \tau \), decreasing subsequences \( yx \) in \( S(\tau) \) necessarily come from occurrences \( yzx \) of the pattern 231, we obtain that there is an element \( d \) larger than \( c \) such that \( cd b \) is a subsequence of \( R \). An occurrence of 1342 in \( \sigma \) is then given by \( acdb \). The last possibility is that \( ac \) is in \( S(L) \) and \( b \) is in \( S(R) \). Depending on the order in which \( a \) and \( c \) appear in \( L \), we have that either \( acnb \) or \( canb \) is a subsequence of \( \sigma \), which are occurrences of 1342 and 35142, respectively.

Finally, proving the identity \( \text{Av}(1342, 31-4-2) = \text{Av}(1342, 35142) \) is easily done observing that \( \sigma \) avoids 31-4-2 if and only if \( \sigma \) avoids 35142.

(iv) As before, we have \( \sigma \in \text{ld}(S \circ i \circ S) \iff i \circ S(\sigma) \in \text{Av}(231) \iff S(\sigma) \in \text{Av}(312) \iff c \circ S(\sigma) \in \text{Av}(132) \iff r \circ c \circ S(\sigma) \in \text{Av}(231) \iff \sigma \in \text{ld}(S \circ r \circ c \circ S) \), so that \( \text{ld}(S \circ i \circ S) = \text{ld}(S \circ r \circ c \circ S) \).

Let us now prove that \( S(\sigma) \in \text{Av}(312) \iff \sigma \in \text{Av}(3412, 3-4-21) \). If \( \sigma \) contains an occurrence \( \sigma_i \sigma_j \sigma_\ell \sigma_k \) of 3412 (3-4-21, respectively) then necessarily \( S(\sigma) \) contains the subsequence \( \sigma_i \sigma_\ell \sigma_k \) (\( \sigma_j \sigma_\ell \sigma_k \), respectively) which is an occurrence of 312. Conversely, if \( S(\sigma) \) contains an occurrence of 312, we follow the same ideas as in point (iii) to exhibit an occurrence of 3412 or 3-4-21 in \( \sigma \).

A natural question is then to look at the enumeration sequences \( (c_n) \) of the sets \( \mathcal{C} \) of pattern-avoiding permutations that appear in Theorem 3.1. Of course, the set \( \text{Av}(231) \) (which corresponds to one-stack sortable permutations) is enumerated by the Catalan numbers \( \text{Cat}_n = \frac{1}{n+1} \binom{2n}{n} \) (see [24]); and it has been proved that the set \( \text{Av}(2341, 35241) \) of two-stack sortable permutations is enumerated by \( \frac{2(3n)!}{(n+1)!(2n+1)!} \) as conjectured by West [31]. This formula was first proved analytically by Zeilberger in [34], and two combinatorial proofs have been given a few years after by West and Goulden in [22], and by Dulucq et al. in [17, 18]. For the two other sets, conjectures on their enumeration have been proposed by [13], and refined with the distribution of some statistics. These conjectures are enriched and stated as Theorems 3.2 and 3.4, that we prove in this article.

Theorem 3.2. The two sets \( \text{ld}(S \circ S) \) and \( \text{ld}(S \circ r \circ S) \) are equienumerated, i.e., for any integer \( n \), we have \( \#\text{ld}_n(S \circ S) = \#\text{ld}_n(S \circ r \circ S) \). Moreover, the tuple of statistics \( (\text{udword}, \text{rmax}, \text{imax}, \text{zeil}, \text{indmax}, \text{slmax}, \text{slmax} \circ r) \) has the same distribution on both sets.
Definition 3.3 and Table 3 below give the definitions of the permutation statistics that appear in Theorem 3.2, together with other classical statistics on permutations.

The equidistribution of the statistic udword implies that many classical permutation statistics — that depend only on the up-down word — are also equidistributed in $\text{Id}(S \circ \pi)$ and $\text{Id}(S \circ r \circ S)$, for instance, des, asc, maj, among many others. Therefore, in Theorem 3.2, the tuple of statistics whose distribution is jointly preserved can be extended with all these statistics.

**Definition 3.3.** (Some classical permutation statistics)
Let $\pi$ be a permutation of size $n$.

A descent of $\pi$ is an index $i \in [1..(n-1)]$ such that $\pi_i > \pi_{i+1}$.

An ascent of $\pi$ is an index $i \in [1..(n-1)]$ such that $\pi_i < \pi_{i+1}$.

The major index of $\pi$ is the sum of indices $i$ that are descents in $\pi$.

The number of components of $\pi$ is the largest $k$ such that $\pi$ can be written as the concatenation $\pi = \alpha_1 \cdots \alpha_k$ with for all $i < j$, for all $a_i \in \alpha_i$ and $a_j \in \alpha_j$, $a_i < a_j$.

The segments $\alpha_i$ are called the components of $\pi$.

A right-to-left maximum (left-to-right maximum, respectively) of $\pi$ is an index $i$ such that $\pi_i > \pi_j$ for all $j > i$ ($j < i$, respectively).

A right-to-left minimum (left-to-right minimum, respectively) of $\pi$ is an index $i$ such that $\pi_i < \pi_j$ for all $j > i$ ($j < i$, respectively).

A valley of $\pi$ is an index $i \in [2..(n-1)]$ such that $\pi_{i-1} > \pi_i < \pi_{i+1}$.

A peak of $\pi$ is an index $i \in [2..(n-1)]$ such that $\pi_{i-1} < \pi_i > \pi_{i+1}$.

A double descent (double ascent, respectively) of $\pi$ is an index $i \in [2..(n-1)]$ such that $\pi_{i-1} > \pi_i > \pi_{i+1}$ ($\pi_{i-1} < \pi_i < \pi_{i+1}$, respectively).

The length of the rightmost (leftmost, respectively) increasing run of $\pi$ is the largest $k$ such that $\pi_{n-k+1} < \cdots < \pi_{n-1} < \pi_n$ ($\pi_1 < \pi_2 < \cdots < \pi_k$, respectively).

The length of the rightmost (leftmost, respectively) decreasing run of $\pi$ is the largest $k$ such that $\pi_{n-k+1} > \cdots > \pi_{n-1} > \pi_n$ ($\pi_1 > \pi_2 > \cdots > \pi_k$, respectively).

The index of the maximal element of $\pi$ is the index $i$ such that $\pi_i = n$.

The Zeilberger’s statistic of $\pi$ is the largest $k$ such that $n(n-1) \cdots (n-k+1)$ is a subword of $\pi$.

The number of letters to the left of the second left-to-right maximum of $\pi \cdot (n+1)$ is the largest $k$ such that $\pi_1 \geq \pi_i$ for all $i \in [1..k]$.

The up-down word of $\pi$ is the word reporting all the ascents (by $u$) and descents (by $d$) of $\pi$ (see [29] for instance): $w \in \{u, d\}^{n-1}$, with $w_i = u$ ($d$, respectively) if $\pi_i < \pi_{i+1}$ ($\pi_i > \pi_{i+1}$, respectively). Notice that the up-down word of $\pi$ is another but equivalent way of describing the descent set of $\pi$.

**Theorem 3.4.** The three sets $\text{Id}(S \circ i \circ S)$, $\text{Bax}$, and $\text{TBax}$ are equienumerated, i.e., for any integer $n$, we have $\#\text{Id}_n(S \circ i \circ S) = \#\text{Bax}_n = \#\text{TBax}_n$. Moreover, the triple of statistics (lmax, des, comp) has the same distribution on $\text{Id}(S \circ i \circ S)$ and $\text{Bax}$. Furthermore, it also has the same distribution as the triple of statistics (lmax, occ$\mu$, comp) on the set $\text{TBax}$, where occ$\mu$ denotes the number of occurrences of the mesh pattern $\mu = \n\n\n$.

The set $\text{Bax}$ of Baxter permutations was first defined in [5] and can be characterized by excluded dashed patterns as $\text{Bax} = \text{Av}(2-41-3, 3-14-2)$ (see [25], for
example). In this article, we take this as the definition of Baxter permutations. The Baxter numbers \(Bax_n\) enumerate the set of Baxter permutations, and we have \(Bax_n = \frac{2^n}{n(n+1)} \sum_{k=1}^{n} \binom{n+1}{k} \binom{n+1}{k+1}\) (see [12]). It has been proved, for example, by Guibert [23], West [33], or Giraudo [21], that Baxter numbers also count the permutations in \(TBax = Av(2\text{-}4\text{-}1\text{-}3, 3\text{-}4\text{-}1\text{-}2)\), and these are called the \textit{twisted Baxter permutations}.

| Notation | Definition (given for a permutation \(\pi \in S_n\)) |
|----------|--------------------------------------------------|
| des      | Number of descents                               |
| asc      | Number of ascents                                |
| maj      | Major index                                      |
| comp     | Number of components                             |
| rmax     | Number of right-to-left maxima                   |
| lmax     | Number of left-to-right maxima                   |
| rmin     | Number of right-to-left minima                   |
| lmin     | Number of left-to-right minima                   |
| valley   | Number of valleys                                |
| peak     | Number of peaks                                  |
| ddes     | Number of double descents                        |
| dasc     | Number of double ascents                         |
| rir      | Length of the rightmost increasing run           |
| rdr      | Length of the rightmost decreasing run           |
| lir      | Length of the leftmost increasing run            |
| ldr      | Length of the leftmost decreasing run            |
| indmax   | Index of the maximal element                     |
| zeil     | Zeilberger’s statistic                           |
| slmax    | Number of letters to the left of the second left-to-right maxima of \(\pi \cdot (n+1)\) |
| udword   | Up-down word                                     |

Table 1: Notations for some statistics on permutations.

In the next section, we start by reviewing the method to establish the existence of statistic-preserving bijections through generating trees and rewriting systems. Then Section 5 uses this method to prove Theorem 3.2. Theorem 3.4 is proved in two steps: Section 6 establishes a bijection between \(ld(S \circ i \circ S)\) and \(TBax\), using again a generating tree; and Section 7 refines the bijection of [21] between \(TBax\) and \(Bax\), according to the statistics of Theorem 3.4.

4. Rewriting Systems and Generating Trees

\textit{Generating trees} have been first introduced by West in [32] in the context of pattern-avoiding permutations. Barcucci et al. [4] have extended it to other combinatorial objects, as well as described it in a formal way, by means of the \textit{ECO-method}. Here,
we only recall some basics on generating trees, before we apply this method in the proofs of Theorems 3.2 and 3.4.

4.1. Generating Trees

A generating tree is an infinite rooted tree associated to a set $C$ of permutations, whose vertices are permutations. The root is the permutation $1 \in C$, and permutations at distance $n - 1$ from the root are exactly the permutations of size $n$ in $C$, whose set is denote $C_n$.

There are several possible definitions of the edges of a generating tree. The four usual ones are as follows: The parent of a permutation $\pi \in C_n$ is the permutation of $C_{n-1}$ obtained from $\pi$ by removing the largest (smallest, rightmost, leftmost, respectively) element of the permutation, and normalizing. This obviously defines uniquely the parent of a permutation $\pi \neq 1$. Of course, depending on $C$ and on the operation of removing and normalizing that is chosen, it may result in a permutation that does not belong to $C$. This is never the case when $C$ is a pattern class, i.e., when the excluded patterns are classical patterns, but it may happen when the excluded patterns are generalized patterns. In the case of barred patterns, the situations in which it does not happen have been characterized in [17].

The above describes the way a parent is obtained from any of its children in the generating tree. In order to build the tree efficiently from its root, it is certainly more convenient to describe instead how the children of a permutation $\pi$ are built from $\pi$.

There are of course four possible insertion rules corresponding to the four possibilities of removing an element that have been described earlier. For each insertion rule, we define the sites of a permutation $\pi \in C$ to be the places in which a new element may be inserted, resulting in a permutation $\pi'$. A site is said active when $\pi' \in C$.

**Insertion rule Largest.** The sites of a permutation $\pi \in C_n$ are located at the beginning, at the end, and between any two consecutive indices $i$ and $i + 1$ of $\pi$, and are above $\pi$. The children of $\pi$ are the permutations of $C_{n+1}$ obtained by inserting a largest element in an active site of $\pi$.

**Insertion rule Smallest.** The sites of a permutation $\pi \in C_n$ are located at the beginning, at the end, and between any two consecutive indices $i$ and $i + 1$ of $\pi$, and are below $\pi$. The children of $\pi$ are the permutations of $C_{n+1}$ obtained by inserting a smallest element in an active site of $\pi$.

**Insertion rule Rightmost.** The sites of a permutation $\pi \in C_n$ are located below, above, and between any two consecutive values $i$ and $i + 1$ of $\pi$, and are to the right of $\pi$. The children of $\pi$ are the permutations of $C_{n+1}$ obtained by inserting a rightmost element in an active site of $\pi$.

**Insertion rule Leftmost.** The sites of a permutation $\pi \in C_n$ are located below, above, and between any two consecutive values $i$ and $i + 1$ of $\pi$, and are to the left of $\pi$. The children of $\pi$ are the permutations of $C_{n+1}$ obtained by inserting a leftmost element in an active site of $\pi$.

Notice that each of the four insertion rules corresponds to adding a new element on one of the four sides of the square around the diagram that represent the permutation (see Figure 2).
4.2. Rewriting Systems

The shape of a generating tree associated to a set $C$ of permutations contains information, even without the permutations labeling the vertices, in particular for enumeration. Indeed, even when considering a generating tree where the permutations labeling the vertices have been removed, we still have a bijection between the vertices of this infinite tree and the permutations of $C$, which maps the size of a permutation to the level a vertex in the tree (the level denoting the distance to the root $+1$ here).

Rewriting systems are a way to describe the shape of a generating tree without the need of labeling each vertex by a permutation. Instead, we can label the vertices of the tree by tuples, that are called labels, in such a way that the label of each vertex contains enough information on the corresponding permutation $\pi$ to build the labels of all the children of $\pi$. In general, these tuples indicate the values of some statistics on $\pi$, such as the number of active sites, the size, .... The shape of the tree is then completely described by a rewriting system on the labels, that encapsulate the parent-child relation on the permutations.

Such a rewriting system consists of a starting point $\ell_0$ (the label of permutation $1$) and a set of rewriting rules of the form $\ell \leadsto L$ with $L = \{\ell_a, \ldots, \ell_b\}$, that describe the labels $\ell_a, \ldots, \ell_b$ of the children of a permutation whose label is $\ell$. There-
there is a bijection between permutations of size $n$ in $C$ and sequences of labels $(\ell_0, \ell_1, \ell_2, \ldots, \ell_{n-1})$ such that for any $i$, there is a rewriting rule $\ell_i \rightsquigarrow L_i$ in the system such that $\ell_{i+1} \in L_i$. Indeed, such sequences of labels correspond to paths from the root to some node at level $n$ in the tree.

Consequently, when the same rewriting system is obtained for two sets $C$ and $D$ of permutations, this implies that there is a bijection between $C$ and $D$, that preserves the size of the permutation. Namely, $\pi \in C$ and $\sigma \in D$ are in correspondence through this bijection if and only if they correspond to the same vertex in the generating tree, or equivalently to the same sequence $(\ell_0, \ell_1, \ell_2, \ldots, \ell_{n-1})$ of labels as above.

Notice that it is also possible to enrich the labels of the vertices to take into account the value of one or more statistics on the corresponding permutations, and follow the evolution of these statistics in the generating tree. In the case there is a common rewriting system for $C$ and $D$ where the labels take into account statistics $(s_1, \ldots, s_k)$ in $C$ and $(r_1, \ldots, r_k)$ in $D$, then these statistics are equidistributed. More precisely, because the same rewriting system for $C$ and $D$ is refined, these statistics are jointly equidistributed. Sections 5 and 6 provide examples of this use of generating trees.

5. A Statistic-Preserving Bijection Between $\text{Id}(S \circ S)$ and $\text{Id}(S \circ r \circ S)$

We first prove that $\text{Id}(S \circ r \circ S)$ is enumerated by the same sequence as $\text{Id}(S \circ S)$. This is achieved by the description of a generating tree that is common for $\text{Id}(S \circ S)$ and $\text{Id}(S \circ r \circ S)$. We shall also see that the underlying bijection $\Phi$ additionally preserves every statistic listed in Theorem 3.2, concluding the proof of this theorem.

5.1. A Simple Rewriting System

We describe a common generating tree for $\text{Id}(S \circ S)$ and $\text{Id}(S \circ r \circ S)$. This tree is in fact identical to the common generating tree for $\text{Av}(3214, \bar{2}4135)$ and $\text{Av}(3241, \bar{2}4153)$ given in [17]. Indeed, we have the following correspondence:

Remark 5.1. The $c \circ i$ operation provides a bijection between $\text{Av}(3214, \bar{2}4135)$ ($\text{Av}(3241, \bar{2}4153)$, respectively) and $\text{Id}(S \circ S)$ ($\text{Id}(S \circ r \circ S)$, respectively).

Proof. The proof is straightforward from the characterizations $\text{Id}(S \circ S) = \text{Av}(2341, 35241)$ and $\text{Id}(S \circ r \circ S) = \text{Av}(1342, 35142)$ of Theorem 3.1. Indeed, a permutation $\sigma$ avoids $3214$ ($24135$, $3241$, $24153$, respectively) if and only if $c \circ i(\sigma)$ avoids $2341$ ($35241$, $1342$, $35142$, respectively).

Because of the $c \circ i$ transform, the insertion rule $\text{Largest}$ that was used in the generating tree of [17] is naturally transformed into the insertion rule $\text{Rightmost}$ in the generating tree for $\text{Id}(S \circ S)$ and $\text{Id}(S \circ r \circ S)$. This incremental construction of permutations where active sites are on the right is also known in the literature under the name of $\text{staff representation}$ of permutations (see [6], for example). Furthermore, for the same reason, the sites on the right for $\text{Id}(S \circ S)$ and $\text{Id}(S \circ r \circ S)$ are numbered from top to bottom, so that it mimics the numbering from left to right for $\text{Av}(3214, \bar{2}4135)$ and $\text{Av}(3241, \bar{2}4153)$. 
Theorem 5.2. The generating trees for both $\text{ld}(S \circ S)$ and $\text{ld}(S \circ r \circ S)$ with the insertion rule Rightmost are characterized by the following rewriting system

$$\mathcal{R}_\Phi \begin{cases} (2, 1, (1)) \\ (x, k, (p_1, \ldots, p_k)) \leadsto (2 + p_j, j, (p_1, \ldots, p_{j-1}, i)) \\
\text{for } 1 \leq j \leq k \text{ and } p_{j-1} < i \leq p_j \\
(x + 1, k + 1, (p_1, \ldots, p_k, i)) \\
\text{for } p_k < i \leq x 
\end{cases}$$

where in the label $(x, k, (p_1, \ldots, p_k))$ of a permutation $\pi$,

- $x$ denotes the number of active sites of $\pi$,
- $k$ is the number of right-to-left maxima in $\pi$, and
- $p_\ell$ denotes the number of active sites above the $\ell$-th right-to-left maximum in $\pi$ (for the decreasing order of their values).

In Theorem 5.2 and its proof, we use the convention that $p_0 = 0$.

An immediate consequence of Theorem 5.2 is that the rewriting system $\mathcal{R}_\Phi$ provides a bijection between $\text{ld}(S \circ S)$ and $\text{ld}(S \circ r \circ S)$, denoted $\Phi$, that preserves the size and the number of right-to-left maxima.

Proof of Theorem 5.2. It is enough to use the bijective correspondence via the $c \circ i$ transformation between $\text{ld}(S \circ S)$ and $\text{Av}(3214, 24135)$, and $\text{ld}(S \circ r \circ S)$ and $\text{Av}(3241, 24153)$, respectively. Indeed, Theorem 5.2 is a direct translation of [17, Proposition 13] in this context. We however give the construction used in the proof, since it will have to be further analyzed in the next subsection. We omit the proof that this construction is correct, and refer the reader to [17] for details.

Permutation 1 belongs to $\text{ld}(S \circ S)$ and $\text{ld}(S \circ r \circ S)$, and has two active sites, one right-to-left maximum, and one active site above this right-to-left maximum. Let us now examine the permutations that are obtained when inserting a rightmost element into the active site of a permutation $\pi$ of $\text{ld}(S \circ S)$ ($\text{ld}(S \circ r \circ S)$, respectively) labeled $(x, k, (p_1, \ldots, p_k))$.

Consider $\pi \in \text{ld}(S \circ S)$ ($\text{ld}(S \circ r \circ S)$, respectively), and one of its active site $s$, which is the $i$-th active site in the numbering from top to bottom. Denote by $\pi'$ the permutation obtained from $\pi$ by the insertion of a rightmost element in site $s$.

Suppose first that site $s$ is above the rightmost element of $\pi$. Then define $j$ such that the largest right-to-left maximum of $\pi$ that is below $s$ is the $j$-th one, and $s_j$ to be the site that immediately below (above, respectively) $j$. Notice that in the case of $\text{ld}(S \circ S)$, the site $s_j$ is always active in $\pi$. Then any site above $s_j$ (included) is active in $\pi'$ if and only if it was active in $\pi$, and the two sites that have been created around the inserted element are both active. As for the sites below $s_j$, they all become inactive (all but the bottommost site, which is always active, respectively).

If on the contrary site $s$ is below the rightmost element of $\pi$, then any site is active in $\pi'$ if and only if it was active in $\pi$, and the two sites that have been created around the inserted element are both active.

Figure 3 gives a graphical view of these two cases of insertion.
Permutations Sorted with Two Stacks and a Symmetry

Figure 3: The two cases of insertion of a rightmost element into a permutation of \( \text{ld}(S \circ S) \) and \( \text{ld}(S \circ r \circ S) \).

In the first case, we have that \( 1 \leq j \), and \( p_{j-1} < i \leq p_j \), and the label of \( \pi' \) is \((2 + p_j, j, (p_1, \ldots, p_{j-1}, i))\). In the second case, we have \( p_k < i \leq x \) and the label of \( \pi' \) is \((x + 1, k + 1, (p_1, \ldots, p_k, i))\).

Figures 4 and 5 show respectively two permutations in bijective correspondence by \( \Phi \), and the first levels of the generating tree corresponding to the rewriting system \( R_\Phi \).

The rewriting system given by [17] for the sets \( \text{Av}(3241, 24153) \) and \( \text{Av}(3214, 24135) \) is actually more precise. It takes into account the number of left-to-right maxima of a permutation \( \pi \) but also the number of ascents in \( \pi^{-1} \). After the \( c \circ i \) transformation, these statistics correspond to the number of right-to-left maxima and the number of descents for permutations in \( \text{ld}(S \circ S) \) and \( \text{ld}(S \circ r \circ S) \). Hence, these two statistics are preserved by the bijection \( \Phi \) between \( \text{ld}(S \circ S) \) and \( \text{ld}(S \circ r \circ S) \). Actually many more statistics are preserved by \( \Phi \). For each of them, we provide in the following a refinement of the rewriting system \( R_\Phi \). This will complete the proof of Theorem 3.2.

5.2. Refinements of the Rewriting System

The rewriting system \( R_\Phi \) can be refined to take into account many statistics, that are equidistributed in \( \text{ld}(S \circ r \circ S) \) and \( \text{ld}(S \circ S) \) (in \( \text{Av}(3241, 24153) \) and
for an active site as the largest $qs$ and $kq$ for an inactive site $(j \succeq 6)) \in 14 + ks + sp < s = k(p1, \ldots, pk)((1, 1))(3.2, (1, 2))(3.1, (1)) \ (4.2, (1, 3)) \ (5.2, (1, 3)) \ (5.2, (1, 3)) \ (5.2, (1, 3)) \ (5.2, (1, 3)) \ (5.2, (1, 3)) \ (5.2, (1, 3))$.

The insertion into an active site does not change the number of left-to-right maxima, in the part of the label denoted by $\pi$. The rewriting system $R_\Phi$ can be refined as follows to account for the number of left-to-right maxima, in the part of the label denoted by $q$:

$$R_{\Phi}^{\max} = \begin{cases} (2, 1, (1), 1) \\
(x, k, (p_1, \ldots, p_k), q) \mapsto (2 + p_1, 1, (1), q + 1) \\
(2 + p_j, j, (p_1, \ldots, p_{j-1}, i), q) \\
\text{for } 1 \leq j \leq k \text{ and } p_{j-1} < i \leq p_j, i \neq 1 \\
(x + 1, k + 1, (p_1, \ldots, p_k, i), q) \\
\text{for } p_k < i \leq x \end{cases}$$

**Proof.** The insertion into an active site does not change the number of left-to-right maxima of $\pi$, unless the insertion takes place in the topmost site of $\pi$. This site is always active, and in this case (corresponding to $i = j = 1$), one left-to-right maximum is created.

### 5.2.2. Zeilberger’s Statistic

Recall that Zeilberger’s statistic is defined for any permutation $\pi$ as the largest $k$ such that $n(n - 1) \cdots (n - k + 1)$ is a subword of $\pi$. The rewriting system $R_\Phi$ can be refined as follows to account for Zeilberger’s statistic, in the part of the label denoted by $q$:

$$R_{\Phi}^{\text{Zeilberger}} = \begin{cases} (2, 1, (1), 1) \\
(x, k, (p_1, \ldots, p_k), q) \mapsto (2 + p_1, 1, (1), q + 1) \\
(2 + p_j, j, (p_1, \ldots, p_{j-1}, i), q) \\
\text{for } 1 \leq j \leq k \text{ and } p_{j-1} < i \leq p_j, i \neq 1 \\
(x + 1, k + 1, (p_1, \ldots, p_k, i), q) \\
\text{for } p_k < i \leq x \end{cases}$$

**Proof.** The insertion into an active site does not change the number of left-to-right maxima, in the part of the label denoted by $q$. The rewriting system $R_\Phi$ can be refined as follows to account for Zeilberger’s statistic, in the part of the label denoted by $q$.
Figure 5: The first levels of the generating tree corresponding to the rewriting system $R_\Phi$. 
by $q$:

$$
\mathcal{R}^\text{zeil}_{\Phi} = \begin{cases} 
(2, 1, (1), 1) \\
(x, k, (p_1, \ldots, p_k), q) & \rightsquigarrow (2 + p_j, j, (p_1, \ldots, p_{j-1}), r) \\
& \text{for } 1 \leq j \leq k \text{ and } p_{j-1} < i \leq p_j \\
(x+1, k+1, (p_1, \ldots, p_k), r) & \text{for } p_k < i \leq x 
\end{cases}
$$

where $r$ is the number $s$ of the site in which the insertion is performed if $s \leq q+1$, $q$ otherwise.

It should be noticed that the numbering of the sites that is referred to is the numbering of all sites, not only the active sites. Furthermore, recall that sites are numbered from top to bottom.

**Proof.** Consider a permutation $\pi$ of size $n$ whose Zeilberger’s statistic has value $q$. This means that $n(n-1) \cdots (n-q+1)$ is a subword of $\pi$ but that $n(n-1) \cdots (n-q)$ is not. If an element is inserted in the $s$-th site (assuming it is active) for $s \leq q$ then $(n+1)n \cdots (n+1-s+1)$ is a subword of the resulting permutation but $(n+1)n \cdots (n+1-s)$ is not, so that Zeilberger’s statistic has value $r = s$ for the resulting permutation. For $s = q+1$ ($s > q+1$, respectively), Zeilberger’s statistic has value $r = s = q+1$ ($r = q$, respectively) for the permutation obtained after the insertion, since $(n+1)n \cdots (n-q+1) \ (n+1)n \cdots (n+1-q+1)$, respectively) is a subword of the resulting permutation but not $(n+1)n \cdots (n-q) \ (n+1)n \cdots (n+1-q)$, respectively).

5.2.3. Index of the Maximal Element

The rewriting system $\mathcal{R}_\Phi$ can be refined as follows to account for the index of the maximal element, in the part of the label denoted by $q$:

$$
\mathcal{R}^\text{indmax}_{\Phi} = \begin{cases} 
(2, 1, (1), 1) \\
(x, k, (p_1, \ldots, p_k), q, n) & \rightsquigarrow (2 + p_1, 1, (1), n+1, n+1) \\
& (2 + p_j, j, (p_1, \ldots, p_{j-1}, i), q, n+1) \\
& \text{for } 1 \leq j \leq k \text{ and } p_{j-1} < i \leq p_j, i \neq 1 \\
(x+1, k+1, (p_1, \ldots, p_k, i), q, n+1) & \text{for } p_k < i \leq x 
\end{cases}
$$

It is necessary in this case to also account for the size of the permutations, denoted by $n$.

**Proof.** The insertion into an active site does not change the index of the maximal element of $\pi$, unless the insertion takes place in the topmost site of $\pi$. This site is always active, and in this case (corresponding to $i = j = 1$), the maximal element is the rightmost one, whose index equals the size of the permutation resulting from the insertion.
5.2.4. The slmax Statistic

Recall that the slmax statistic is defined, for any permutation \( \pi \), as the largest \( k \) such that \( \pi_{1} \geq \pi_{i} \) for all \( i \in [1..k] \). The rewriting system \( \mathcal{R}_\Phi \) can be refined as follows to account for the slmax statistic in the part of the label denoted by \( q \):

\[
\mathcal{R}_{\text{slmax}}^{\Phi} = \begin{cases}
(2, 1, (1), 1, 1) \\
(x, k, (p_1, \ldots, p_k), q, n) & \mapsto (2 + p_1, 1, (1), q, n + 1) \\
& \quad (2 + p_j, j, (p_1, \ldots, p_{j-1}, i), q + \delta_{q,n}, n + 1) \\
& \quad \text{for } 1 \leq j \leq k \text{ and } p_{j-1} < i \leq p_j, i \neq 1 \\
& \quad (x + 1, k + 1, (p_1, \ldots, p_k, i), q + \delta_{q,n}, n + 1) \\
& \quad \text{for } p_k < i \leq x 
\end{cases}
\]

Here, we also introduce the size of the permutations, denoted by \( n \).

**Proof.** The insertion in the topmost site of \( \pi \) (which is always active) does not change the value of the slmax statistic. The insertion in any other site may change the value of this statistic, increasing it by 1. This happens exactly when \( \pi \) starts with its maximum, and this situation is characterized by the equality \( q = n \). \( \blacksquare \)

5.2.5. The slmax or Statistic

By definition, the slmax or statistic is defined, for any permutation \( \pi \), as the largest \( k \) such that \( \pi_{n} \geq \pi_{i} \) for all \( i \in [(n-k+1)..n] \). The rewriting system \( \mathcal{R}_\Phi \) can be refined as follows to account for the slmax or statistic in the part of the label denoted by \( q \):

\[
\mathcal{R}_{\text{slmax or}}^{\Phi} = \begin{cases}
(2, 1, (1), 1) \\
(x, k, (p_1, \ldots, p_k), q) & \mapsto (2 + p_j, j, (p_1, \ldots, p_{j-1}, i), q + 1) \\
& \quad \text{for } 1 \leq j \leq k \text{ and } p_{j-1} < i \leq p_j \\
& \quad (x + 1, k + 1, (p_1, \ldots, p_k, i), 1) \\
& \quad \text{for } p_k < i \leq x 
\end{cases}
\]

**Proof.** The insertion in any site of \( \pi \) that is above the rightmost element of \( \pi \) increases the value of the slmax or statistic by 1. This corresponds to the first case of the rewriting rule. When the insertion is performed in a site that is below the rightmost element of \( \pi \), the slmax or statistic takes value 1 after the insertion. This corresponds to the second case of the rewriting rule. \( \blacksquare \)

5.2.6. Up-Down Word

The rewriting system \( \mathcal{R}_\Phi \) can be refined as follows to account for the udword statistic in the part of the label denoted by \( w \):

\[
\mathcal{R}_{\text{udword}}^{\Phi} = \begin{cases}
(2, 1, (1), \varepsilon) \\
(x, k, (p_1, \ldots, p_k), w) & \mapsto (2 + p_j, j, (p_1, \ldots, p_{j-1}, i), w \cdot u) \\
& \quad \text{for } 1 \leq j \leq k \text{ and } p_{j-1} < i \leq p_j \\
& \quad (x + 1, k + 1, (p_1, \ldots, p_k, i), w \cdot d) \\
& \quad \text{for } p_k < i \leq x 
\end{cases}
\]
Notice that, unlike every statistic previously examined, the part of the label recorded by $w$ is not an integer, but a word of $\{u, d\}^{n-1}$, where $n$ denotes the size of the permutation. Starting from the empty word $\varepsilon$, the words $w$ are obtained by concatenation of one letter at the end of the word at every iteration of the rewriting rule.

**Proof.** Consider a permutation $\pi$ of size $n$, and denote by $\pi'$ the permutation obtained after the insertion in an active site $s$ of $\pi$. If $s$ is above $\pi_n$ (which corresponds to the first case of the rewriting rule), then $\pi'_n < \pi'_{n+1}$ and $u$ should be appended to the up-down word of $\pi$ to obtain the one of $\pi'$. If $s$ is below $\pi_n$ (which corresponds to the second case of the rewriting rule), then $\pi'_n > \pi'_{n+1}$ and $d$ should be appended to the up-down word of $\pi$ to obtain the one of $\pi'$.

6. A Statistic-Mapping Bijection Between $\text{Id}(S \circ i \circ S)$ and $\text{TBax}$

This section starts the proof of Theorem 3.4, which will be concluded in Section 7. We describe a bijection $\Psi$ between $\text{Id}(S \circ i \circ S)$ and $\text{TBax}$ by means of a generating tree, thus proving that $\text{Id}(S \circ i \circ S)$ is enumerated by the Baxter numbers. As far as we know, this generating tree for Baxter numbers has not been described previously in the literature. We next show that the rewriting system we provide can be refined according to some statistics, mapping the triple $(l_{\text{max}}, \text{des}, \text{comp})$ on $\text{Id}(S \circ i \circ S)$ onto the triple $(l_{\text{max}}, \text{occ}_\mu, \text{comp})$ on $\text{TBax}$. Recall that $\mu$ is the mesh pattern $\mu = \begin{array}{c|c|c|c} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}$ and that $\text{occ}_\mu$ denotes the number of occurrences of $\mu$.

6.1. A Simple Rewriting System

We describe a common generating tree for $\text{Id}(S \circ i \circ S)$ and $\text{TBax}$.

**Theorem 6.1.** The generating trees for $\text{Id}(S \circ i \circ S)$ and $\text{TBax}$ are characterized by the following rewriting system:

$$
\mathcal{R}_\Psi = \begin{cases} (2, 0) \\
(q, r) \leadsto (i+1, q+r-i) \text{ for } 1 \leq i \leq q \\
(q, r-j) \text{ for } 1 \leq j \leq r
\end{cases}
$$

It is obtained with the insertion rule Smallest for $\text{Id}(S \circ i \circ S)$ and with the insertion rule Rightmost for $\text{TBax}$. The numbering of the active sites is from left to right for $\text{Id}(S \circ i \circ S)$ and from top to bottom for $\text{TBax}$. The label $(q, r)$ of a permutation $\pi$ is interpreted as follows.

* For permutations in $\text{Id}(S \circ i \circ S)$:
  * $q$ is the index of the second element in the first ascent of $\pi$ (or $n + 1$ if $\pi = n(n-1) \cdots 21$), i.e., $q = \text{ldr}(\pi) + 1$,
  * $r$ is the number of active sites to the right of the first ascent of $\pi$ (or $0$ if $\pi = n(n-1) \cdots 21$).

* For permutations in $\text{TBax}$:
  * $q$ is the number of active sites above $\pi_n$ plus 1 (the site immediately below $\pi_n$).
• r is the number of active sites below \( \pi_n \) minus 1 (the site immediately below \( \pi_n \)).

Proof. Let us first study \( \text{ld}(S \circ i \circ S) = \text{Av}(3412, 3-4-21) \) with the insertion rule Small-
est. It is readily checked that if \( \pi \) avoids 3412 and 3-4-21, then so does the permutation obtained from \( \pi \) by deleting its smallest element and normalizing. This justifies that we have a generating tree, whose root 1 is labeled \((2, 0)\) according to the interpretation of the labels given in Theorem 6.1. It can further be noticed that a site which is inactive in \( \pi \) cannot be active in any child of \( \pi \).

It is easily proved that a site \( s \) of \( \pi \) is inactive if and only if there are elements \( a < b < c \) such that \( bca \) is a subsequence of \( \pi \) and \( s \) is either between \( c \) and \( a \) or follows \( a \) immediately.

In particular, every site that precedes the first ascent of \( \pi \) and the site inside this first ascent are active. The number of such active sites corresponds to \( q \) in the label of \( \pi \). The insertion of 1 in such a site of \( \pi \) (say, the \( i \)-th such site) does not deactivate any site that was active in \( \pi \), and the sites that were inactive in \( \pi \) remain inactive. Hence, \( q \) children of \( \pi \) are obtained by insertions of this first case, whose labels are \((i + 1, q + r - i)\) for \( 1 \leq i \leq q \).

The other children \( \pi' \) of \( \pi \) are obtained by insertion of 1 in active sites that are to the right of the first ascent of \( \pi \), that we denote by \( bc \). The insertion of 1 in such a site (say the \( j \)-th such site) creates a subsequence \( bc1 \) in \( \pi' \) so that the sites between \( c \) and 1 and just after 1 are inactive in \( \pi' \). So \( j \) active sites to the right of the first ascent become inactive. For the other sites, as before they are active in \( \pi' \) if and only if they were active in \( \pi \). Hence, \( r \) children of \( \pi \) are obtained by insertions of this second case, whose labels are \((q, r - j)\) for \( 1 \leq j \leq r \).

Figure 6 gives a graphical view of the different cases of insertion.

We now turn to the study of \( \text{TBax} = \text{Av}(2-41-3, 3-41-2) \) with the insertion rule Rightmost. As above, if \( \pi \) avoids 2-41-3 and 3-41-2, then so does the permutation obtained from \( \pi \) by deleting its rightmost element, so that we have a generating tree. Its root 1 is labeled \((2, 0)\) according to the interpretation of the labels. Furthermore, denoting \( n \) the size of \( \pi \), the followings can be readily proved:

- the site immediately below \( \pi_n \) is active;
- when inserting \( \pi'_{n+1} \) into an active site of \( \pi \), the sites below \( \pi'_{n+1} \) are active in \( \pi' \) if and only if they were active in \( \pi \);
- the sites above \( \pi_n \) are active if and only if they are located between two adjacent right-to-left maxima of \( \pi \) (including the topmost site of \( \pi \)).

With these three facts, a careful examination allows to prove that the insertion of \( \pi'_{n+1} \) in the \( i \)-th active site above \( \pi_n \), for \( 1 \leq i \leq q - 1 \) (in the site immediately below \( \pi_n \), in the \((j + 1)\)-th active site below \( \pi_n \), for \( 1 \leq j \leq r \), respectively), produces a child of \( \pi \) labeled by \((i + 1, q + r - i)\), \((q + 1, r)\), \((q, r - j)\), respectively).

The various cases of insertion are shown in Figure 7.

Figures 8 and 9 show respectively two permutations in bijective correspondence by \( \Psi \), and the first levels of the generating tree corresponding to the rewriting system.

\[ \text{By two adjacent right-to-left maxima, we mean two right-to-left maxima whose values are consecutive integers.} \]
Figure 6: Insertion of a smallest element into a permutation \( \pi \) of \( \text{ld}(S \circ i \circ S) \), for \( 1 \leq i < q, i = q \) and \( 1 \leq j \leq r \).

\( R_{\Psi} \). In Figure 8, in addition to the labels of the permutations, we have also reported the statistics preserved by the rewriting system, that are studied in the following paragraphs.

6.2. Refinements of the Rewriting System

The rewriting system \( R_{\Psi} \) can be refined to take into account three statistics, which are identically distributed in \( \text{ld}(S \circ i \circ S) \) and in \( \text{TBax} \). This could of course be described in a single rewriting system, but for the sake of clarity, we provide one rewriting system for each of the three statistics.

6.2.1. Number of Left-to-Right Maxima

The rewriting system \( R_{\Psi} \) can be refined as follows to account for the number of left-to-right maxima, in the part of the label denoted by \( m \):

\[
R_{\Psi}^{\text{lmax}} \begin{cases} 
(2, 0, 1) \\
(q, r, m) \quad \rightsquigarrow \quad (2, q + r - 1, m + 1) \\
(i + 1, q + r - i, m) \text{ for } 2 \leq i \leq q \\
(q, r - j, m) \text{ for } 1 \leq j \leq r 
\end{cases}
\]

\textbf{Proof.} In the case of \( \text{ld}(S \circ i \circ S) \) (\( \text{TBax} \), respectively), the insertion of a smallest (rightmost, respectively) element into an active site does not change the number of
Figure 7: Insertion of a rightmost element into a permutation $\pi$ of $\text{TBax}$, in a site above $\pi_n$, the site immediately below $\pi_n$, and a site strictly below $\pi_n$.

left-to-right maxima of $\pi$, unless the insertion takes place in the leftmost (topmost, respectively) site of $\pi$. This site is always active, and in this case (corresponding to $i = 1$ in the first part of the rewriting rule), one left-to-right maximum is created.

6.2.2. Number of Descents in $\text{ld}(S \circ i \circ S)$ and Number of Occurrences of $\mu$ in $\text{TBax}$

The rewriting system $\mathcal{R}_\Psi^\text{des}$ can be refined as follows to account for the number of descents in $\text{ld}(S \circ i \circ S)$, and for the number of occurrences of the mesh pattern $\mu$ in $\text{TBax}$, in the part of the label denoted by $d$:

$$\mathcal{R}_\Psi^\text{des} = \begin{cases} (2, 0, 0, ()) \\ (q, r, d, (x_1, \ldots, x_r)) \leadsto (i + 1, q + r - i, d, (0, \ldots, 0, 1, x_1, \ldots, x_r)) \\ \text{for } 1 \leq i \leq q - 1 \\ (q + 1, r, d + 1, (x_1, \ldots, x_r)) \\ (q, r - j, d + x_j, (x_{j+1}, x_{j+2}, \ldots, x_r)) \\ \text{for } 1 \leq j \leq r \end{cases}$$

The additional labels $x_j \in \{0, 1\}$ are interpreted as follows. For permutations in $\text{ld}(S \circ i \circ S)$, $x_j$ equals 1 if and only if the $j$-th active site to the right of the first ascent is an ascent (or is the site at the end of the permutation). For permutations in $\text{TBax}$,
Figure 8: Permutations 251364798 ∈ \text{ld}(S \circ i \circ S)$ and 213786495 ∈ TBax are in bijective correspondence by $\Psi$.

$x_j$ equals 1 if and only if there exists an element $\pi_a$ in $\pi$ such that when inserting a rightmost element $\pi'_{j+1}$ in the $j$-th active site below $\pi_n$ (without counting the site immediately below $\pi_n$), $(\pi_a, \pi'_{j+1})$ is an occurrence of the mesh pattern $\mu$.

Figure 10 shows a graphical interpretation of the labels $x_j$ in the case of TBax. It is readily proved, but very useful in the proof that the above rewriting system meets our requirements.

**Proof.** Let us first refine the rewriting rule for $\text{ld}(S \circ i \circ S)$. Denote by $\pi$ a permutation of $\text{ld}(S \circ i \circ S)$ labeled by $(q, r, d, (x_1, \ldots, x_r))$, and denote by $\pi'$ the permutation obtained after the insertion of a smallest element in an active site $s$ of $\pi$. Obviously, if $s$ is in a descent (an ascent, respectively) of $\pi$, this descent (ascent, respectively) is transformed into a valley (i.e., a descent followed by an ascent) in $\pi'$. This simple fact will be used several times in the following proof.

Consider first the case where $s$ is in the left of the first ascent of $\pi$. Necessarily, $s$ is in a descent of $\pi$. Consequently, the number of descents is unchanged in $\pi'$.

Remember from the proof of Theorem 6.1 that all sites to the left of the first ascent of $\pi$ are active. It follows that $s$ is the $i$-th active site of $\pi$, for some $i < q$. The insertion creates an ascent $\pi'_{j+1}$, so that the active sites of $\pi$ numbered from $i + 1$ to $q$ are now located after the first ascent of $\pi'$. As we have seen in the proof of Theorem 6.1, these $q - i$ sites remain active in $\pi'$. All of them correspond to descents of $\pi'$ except for the last one, that is, the one in the first ascent of $\pi$ (now, the second ascent of $\pi'$). The label of $\pi'$ is therefore the one given in the first case of the rewriting rule of $\mathcal{R}_\Psi^\text{des}$.

If $s$ is the site in the first ascent $\pi_{q-1}$ of $\pi$ (remember that this site is always active), then the insertion creates one descent $\pi'_{q-1} = (\pi_{q-1} + 1)1$ and the first ascent of $\pi'$ is $\pi'_{q-1} = 1(\pi_{q-1} + 1)$. As in the proof of Theorem 6.1, the sites to the right of the first ascent are unaffected by this insertion, so that the label of $\pi'$ is the one given in the second case of the rewriting rule of $\mathcal{R}_\Psi^\text{des}$.

If $s$ is the $j$-th active site to the right of the first ascent of $\pi$, for some $j \in [1..r]$, then the number of descents is increased by 1 (remains constant, respectively) if $s$ is in an ascent (in a descent, respectively). Therefore, the number of descents of $\pi'$
is $d + x_j$. The proof of Theorem 6.1 ensures that the active sites between the first ascent and $s$ ($s$ included) become inactive whereas the active sites to the right of $s$ are unaffected by the insertion. This completes the proof that the label of $\pi'$ is the one given in the third case of the rewriting rule of $\mathcal{R}_\Psi$.

We now turn to the refinement of the rewriting rule for TBax. Denote by $\pi$ a permutation of TBax, of size $n$, and labeled by $(q, r, d, (x_1, \ldots, x_r))$. Denote by $\pi'$ the permutation obtained after the insertion of a rightmost element in an active site $s$ of $\pi$.

Let us first notice that all occurrences of $\mu$ in $\pi$ are also present in $\pi'$, since the inserted element is a rightmost one. Notice also that the only occurrences of $\mu$ that may be created in $\pi'$ involve the inserted element $\pi'_{n+1}$ as the rightmost element of $\mu$. Finally, notice that an element $y$ may be involved in at most one occurrence $(x, y)$ of $\mu$. Indeed, if it were involved in two occurrences $(x_1, y)$ and $(x_2, y)$ with $x_1 > x_2$, then $x_2$ would lie in a region that is supposed to be empty.
Figure 10: Situation where $x_j = 1$, for any $j$-th active site below $\pi_n$ (without counting the site immediately below $\pi_n$) in a permutation $\pi$ of size $n$ in TBax.

Let us now examine the label of the child $\pi'$ of $\pi$, obtained after the insertion in site $s$.

Consider first the case where $s$ is an active site above $\pi_n$. We can write that $s$ is the $i$-th active site of $\pi$ for some $i < q$. The number of occurrences of $\mu$ is unchanged, as $\pi_{n+1}'$ cannot be involved in an occurrence of $\mu$. Indeed, if there were an occurrence of $\mu$ with $\pi_{n+1}'$ as rightmost element, then $\pi_n' = \pi_n$ would lie in a region that is supposed to be empty, providing a contradiction. Any active site of $\pi$ that was below $\pi_n$ (except for the one immediately below) remains active in $\pi'$; and Figure 10 ensures that the corresponding values $x_j$ are unchanged. The site immediately below $\pi_n$ also remains active in $\pi'$, and is now located strictly below $\pi_{n+1}'$. From Figure 10, we deduce that the value $x_j$ associated to this site is also 1. Finally, the $q - i - 1$ active sites of $\pi$ below $s$ (except the one immediately below) and above $\pi_n$ remain active in $\pi'$ and are now located strictly below $\pi_{n+1}'$. The corresponding $x_j$ all have value 0, since otherwise $\pi_n' = \pi_n$ would lie in a region that is supposed to be empty. We conclude that in this case, $\pi'$ has label $(i + 1, q + r - i, d, (0, \ldots, 0, 1, x_1, \ldots, x_r))$.

Next, if $s$ is the site immediately below $\pi_n$ (remember that this site is always active), then $(\pi_n', \pi_{n+1}')$ is a new occurrence of $\mu$ in $\pi'$. By the remarks at the beginning of our proof, no other occurrence of $\mu$ is created. Furthermore, the active sites that are below $\pi_{n+1}'$ (except the one immediately below) are exactly the active sites that were below $\pi_n$ in $\pi$ (except the one immediately below). From Figure 10, we get that the corresponding values $x_j$ are unchanged, so that in this case $\pi'$ has label $(q + 1, r, d + 1, (x_1, \ldots, x_r))$.

Finally, if $s$ is the $j$-th site below $\pi_n$ (without counting the one immediately below) for some $j \in [1..r]$, then as before at most one occurrence of $\mu$ may be created, and this happens exactly when $x_j = 1$. Furthermore, the active sites that are below $\pi_{n+1}'$ (except the one immediately below) are exactly the active sites that were below $s$ in $\pi$. From Figure 10, we get that the corresponding values $x_j$ are unchanged, so that in this case $\pi'$ has label $(q, r - j, d + x_j, (x_{j+1}, \ldots, x_r))$. □
6.2.3. Number of Components

The rewriting system $\mathcal{R}_{\Psi}$ can be refined as follows to account for the number of components, in the part of the label denoted by $c$:

$$
\mathcal{R}_{\Psi}^{\text{comp}} = \begin{cases} 
(2, 0, 1, (2)) \\
(q, r, c, (z_1, \ldots, z_c)) 
\leadsto (2, q + r - 1, c + 1, (2, z_1 - 1, z_2, \ldots, z_c)) \\
(i + 1, q + r - i, c, (z_1 + 1, z_2, \ldots, z_c)) 
& \text{for } 2 \leq i \leq q \\
(q, r - j, c - k + 1, (\sum_{i=1}^{k} z_i - j, z_k + 1, \ldots, z_c)) 
& \text{for } 1 \leq j \leq r
\end{cases}
$$

where $k = \min\{h \mid \sum_{i=1}^{h} z_i \geq q + j\}$

The additional labels $z_i$ are interpreted as follows. For permutations in $\text{Id}(S \circ i \circ S)$, $z_i$ is the number of active sites in the $i$-th component from left to right (including the site immediately to the right of the component, if active). Notice that $z_1$ also counts the leftmost site of the permutation, which is always active. For permutations in $\text{TBax}$, $z_i$ is the number of active sites in the $i$-th component from right to left, or equivalently from top to bottom (including the site immediately below of the component, if active). Notice that $z_1$ also counts the topmost site of the permutation, which is always active.

**Proof.** Let us first refine the rewriting rule for $\text{Id}(S \circ i \circ S)$. Denote by $\pi$ a permutation of $\text{Id}(S \circ i \circ S)$ labeled by $(q, r, c, (z_1, \ldots, z_c))$, and denote by $\pi'$ the permutation obtained after the insertion of a smallest element in an active site $s$ of $\pi$.

Assume first that $s$ is located to the left of or inside the first ascent of $\pi$. Recall that all such sites of $\pi$ are active, so that $s$ is both the $i$-th site and the $i$-th active site of $\pi$, for some $1 \leq i \leq q$.

If $i = 1$, $s$ is the leftmost site of $\pi$. The insertion of a smallest element $\pi'_1$ in $s$ creates a new component, consisting only of $\pi'_1$ at the beginning of $\pi'$. It also creates an active site before $\pi'_1$, and all the active sites of $\pi$ remain active. The leftmost site of $\pi$ that contributed to $z_1$ is now counted in the first component of $\pi'$, so that $\pi'$ has label $(2, q + r - 1, c + 1, (2, z_1 - 1, z_2, \ldots, z_c))$.

If $1 < i < q$, then the site $s$ is somewhere in the decreasing run of $\pi$ starting at the beginning of $\pi$. The smallest element of $\pi$ (that necessarily belongs to the first component) is therefore located to the right of $s$, so that $s$ is a site inside the first component of $\pi$. Similarly, if $i = q$, then $s$ is the site inside the first ascent of $\pi$. The smallest element of $\pi$ is therefore either to the right of $s$ or immediately to the left of $s$, so that $s$ is a site inside the first component of $\pi$, or the site on its right boundary. Hence in these two cases the element inserted in $s$ belongs to the first component. We conclude that $\pi'$ has label $(i + 1, q + r - i, c, (z_1 + 1, z_2, \ldots, z_c))$.

Otherwise, $s$ is located to the right of the first ascent of $\pi$, and we can write that it is the $j$-th such site, for some $j \in [1..r]$. Let us denote by $k$ the index of the component such that $s$ is inside (or on the right boundary of) the $k$-th component of $\pi$. Equivalently, this index $k$ can be characterized as the minimal $h$ satisfying $\sum_{i=1}^{h} z_i \geq q + j$. The insertion of a smallest element in $s$ causes the merge of the first
components of $\pi$ into a single component of $\pi'$ (which is of course the first one), and leaves the other components unaffected. Furthermore, the $j$ active sites of $\pi$ that become inactive in $\pi'$ are located to the left of $s$ (including $s$), so that they belong to the first component of $\pi'$. We deduce that $\pi'$ has label $\left(q, r - j, c - k + 1, \left(\sum_{i=1}^{k} z_i - j, z_{k+1}, \ldots, z_r\right)\right)$ in this case.

We now turn to the refinement of the rewriting rule for $TBax$. Recall that in this case the components are numbered from top to bottom. Denote by $\pi$ a permutation of $TBax$, of size $n$, and labeled by $(q, r, c, \langle z_1, \ldots, z_r \rangle)$. Denote by $\pi'$ the permutation obtained after the insertion of a rightmost element in an active site $s$ of $\pi$.

Assume first that $s$ is located above $\pi_n$, or immediately below $\pi_n$, so that $s$ is the $i$-th active site of $\pi$ (from top to bottom), for some $1 \leq i \leq q$.

If $i = 1$, $s$ is the topmost site of $\pi$ (which is always active). The insertion of an element in $s$ creates a new maximal element at the end of $\pi$. Therefore, it creates a new component consisting only of $\pi'_{n+1} = n + 1$. It also creates an active site above $\pi'_{n+1}$, and all the active sites of $\pi$ remain active. The topmost site of $\pi$ that contributed to $z_1$ is now counted in the first component of $\pi'$, so that $\pi'$ has label $(2, q + r - 1, c + 1, (2, z_1 - 1, z_2, \ldots, z_r))$.

If $1 < i < q$ ($i = q$, respectively), then the site $s$ is above (immediately below, respectively) $\pi_n$ and hence is inside (on the lower boundary of, respectively) the first component of $\pi$. Hence, $\pi'_{n+1}$ (the rightmost element inserted in site $s$) also belongs to this component. Therefore, we conclude that $\pi'$ has label $(i + 1, q + r - i, c, (z_1 + 1, z_2, \ldots, z_r))$.

Otherwise, $s$ is located below $\pi_n$, and not immediately below. We can write that it is the $j$-th such site, for some $j \in [1..r]$. Let us denote by $k$ the index of the component such that $s$ is inside (or on the lower boundary of) the $k$-th component of $\pi$, from top to bottom. This index $k$ is also characterized by $k = \min \{ h \mid \sum_{i=1}^{h} z_i \geq q + j \}$. The insertion of a rightmost element in $s$ causes the merge of the first $k$ components of $\pi$ into a single component of $\pi'$ (which is of course the first one), and leaves the other components unaffected (see Figure 11 for an illustration of this statement). Furthermore, the $j$ active sites of $\pi$ that become inactive in $\pi'$ are located above $s$ (including $s$), so that they belong to the first component of $\pi'$. We deduce that $\pi'$ has label $(q, r - j, c - k + 1, \left(\sum_{i=1}^{k} z_i - j, z_{k+1}, \ldots, z_r\right))$ in this case.

7. A Statistic-Mapping Bijection Between $Bax$ and $TBax$

It is known (see for instance, [23, 33]) that the sets of permutations $Bax$ and $TBax$ are both enumerated by the Baxter numbers. To conclude the proof of Theorem 3.4, we prove in addition that the triples of statistics $(l_{\text{max}}, \text{des}, \text{comp})$ on $Bax$ and $(l_{\text{max}}, \text{occ}_{\mu}, \text{comp})$ on $TBax$ are equidistributed. Contrary to the previous sections, this result is not obtained by generating trees, but by refining a recent bijection between $Bax$ and $TBax$ given by Giraudo in [21].

7.1. Giraudo’s Bijection Between $Bax$ and $TBax$

Giraudo describes in [21] a bijective correspondence between Baxter permutations of size $n$, pairs of twin binary trees with $n$ internal nodes, and twisted Baxter
permutations of size $n$. We first define pairs of twin binary trees (introduced in [19]) and then present some aspects of this bijection. We do not describe it in full details, but focus on some of its properties that are sufficient for our purpose.

To any (unlabeled) binary tree $\tilde{T}$ with $n$ internal nodes (and hence $n + 1$ leaves), one can associate a word $w$ of $\{\ell, r\}^{n-1}$ as follows: $w_i = \ell$ ($r$, respectively) if the $(i + 1)$-th leaf of $T$ is oriented to the left (to the right, respectively). Here, leaves are numbered according to the depth-first search traversal that always visits a left child before its right sibling. This word is called the canopy of $T$. Notice that the leftmost and rightmost leaves of $T$ (which are always oriented to the left and to the right respectively) are left aside from the construction of the canopy of $T$. Two (unlabeled) binary trees $T$ and $T'$ both with $n$ internal nodes form a pair of twin binary trees when their canopies $w$ and $w'$ are complementary, i.e., when they are such that $w_i = \ell$ if and only if $w'_i = r$ for any $i \in [1..(n-1)]$.

The bijection of Giraudo is made efficient by several algorithms described in [21]. To present them, let us first recall the classical process of building an (unlabeled) binary tree from a permutation by insertion of leaves. Consider a permutation $\sigma$ of size $n$, and write $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$. Starting from an empty tree $T$, a labeled binary tree can be obtained by insertions of $\sigma_1, \sigma_2, \ldots, \sigma_n$ in $T$ as follows. For any $i$ from 1 to $n$, if $T$ is the empty tree, then replace $T$ by a tree having a root labeled by $\sigma_i$, with two pending leaves; otherwise, compare $\sigma_i$ with the label $x$ of the root of $T$. If $\sigma_i < x$ ($\sigma_i > x$, respectively), then recursively insert $\sigma_i$ in the left (right, respectively) subtree of $T$. This produces a labeled binary tree, but forgetting the labels on internal nodes (that actually carry redundant information) gives the desired construction.

From either Bax or TBax to pairs of twin binary trees, the bijection of [21] associates to a permutation $\sigma$ of size $n$ the pair $(T_L, T_R)$ where $T_L$ ($T_R$, respectively) is the (unlabeled) binary tree obtained from $\sigma$ ($r(\sigma) = \sigma_n \cdots \sigma_2 \sigma_1$, respectively) by insertion of leaves. For the proof that $(T_L, T_R)$ is a pair of twin binary trees, we refer the reader to [21]. Figure 12 shows an example of this construction.

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§ Here, we consider only full binary trees, i.e., trees where every internal node has exactly two children.
Figure 12: The pair of (labeled) twin binary trees \((T^{lab}_L, T^{lab}_R)\) associated to \(\sigma = 52143786 \in \text{Bax}\). The pair \((T_L, T_R)\) is obtained by removing the labels of internal nodes.

From pairs of twin binary trees to \(\text{Bax}\) (\(\text{TBax}\), respectively), the correspondence is provided by the algorithms \(\text{EXTRACTBAXTER}\) (\(\text{EXTRACTMIN}\), respectively) of [21]. We do not need the details of these correspondences here, but only the fact that with the above construction they provide bijections between the set of pairs of twin binary trees and \(\text{Bax}\) and \(\text{TBax}\) respectively. This induces a bijection between \(\text{Bax}\) and \(\text{TBax}\), that we further analyze in the sequel.

7.2. Statistics Transported by the Bijection

We prove that the statistics \(l_{\text{max}}\) and \(\text{comp}\) are preserved by the above bijection from \(\text{Bax}\) to \(\text{TBax}\) and that it maps the statistic \(\text{des}\) to the statistic \(\text{occ}_\mu\).

7.2.1. Number of Left-to-Right Maxima

**Lemma 7.1.** The number of left-to-right maxima is preserved by the bijection of S. Giraudo. More precisely, the number of left-to-right maxima of \(\sigma \in \text{Bax}\) (\(\text{TBax}\), respectively) is mapped to the length of the rightmost branch from the root of \(T_L\), where \((T_L, T_R)\) is the pair of twin binary trees associated to \(\sigma\) by the bijection of Section 7.1.

**Proof.** Lemma 7.1 is actually a direct consequence of the following remark: Let \(\sigma\) be any permutation of size \(n\), and let \(T\) be the labeled binary tree obtained from the empty tree by leaf insertion of \(\sigma_1, \sigma_2, \ldots, \sigma_n\); then the elements on the rightmost branch from the root of \(T\) are the left-to-right maxima of \(\sigma\). Indeed, when inserting a leaf \(\sigma_i\), this element is appended to the rightmost branch from the root of \(T\) if and only if it is larger than any element \(\sigma_j\) for \(j < i\), which matches the definition of left-to-right maximum. \[\square\]

7.2.2. Number of Descents in \(\text{Bax}\) and Number of Occurrences of \(\mu\) in \(\text{TBax}\)

We prove that the number of descents (occurrences of \(\mu\), respectively) in \(\sigma \in \text{Bax}\) (\(\text{TBax}\), respectively) is mapped to the number of left edges in \(T_L\), where \((T_L, T_R)\) is the pair of twin binary trees associated to \(\sigma\) by the bijection of Section 7.1.

**Lemma 7.2.** Let \(\sigma\) be a permutation of \(\text{Bax}\), and let \(T = T_L\) be the labeled binary tree associated to \(\sigma\). A value \(i\) is the second element of a descent of \(\sigma\) if and only if there exists a left edge whose lower label is \(i\) in \(T\).
Proof. Let us first consider an element \( i \) such that \( i \) is the second element of a descent of \( \sigma \). Then we can write \( \sigma = \sigma_\Lambda ji \sigma_c \) with \( j > i \). If no element \( x \) of \( \sigma_\Lambda \) is such that \( i < x < j \), then the insertion of \( i \) creates a left edge from \( j \) to \( i \) in \( T \). Otherwise, there is an element \( x \) in \( \sigma_\Lambda \) such that \( i < x < j \). Choosing the smallest such element \( x \) allows to write \( \sigma = \sigma_\Lambda x \sigma_b ji \sigma_c \), with either \( y < i \) or \( x < y \) for every \( y \) in \( \sigma_\Lambda \) or in \( \sigma_b \). We now claim that \( x < y \) holds for every \( y \) in \( \sigma_b \). This claim is proved in the next paragraph. This will conclude the proof of Lemma 7.2, since in this situation the insertion of \( i \) creates a left edge from \( x \) to \( i \) in \( T \).

If \( \sigma_b \) is empty, the above claim is clear. Otherwise, consider the rightmost element \( e \) in \( \sigma_b \). Then either \( e < i \) or \( x < e \). In the first case, the subsequence \( xeji \) of \( \sigma \) would form an occurrence of the pattern 3-14-2 in \( \sigma \), contradicting that \( \sigma \in Bax \). Consequently, \( x < e \). Assume now that there is an element \( f \) in \( \sigma_b \) such that \( f < i \). We can take \( f \) to be the rightmost such element. Because \( x < e \), the element \( g \) that immediately follows \( f \) in \( \sigma \) is an element of \( \sigma_b \) such that \( x < g \). But then \( xfgi \) is an occurrence of 3-14-2 in \( \sigma \), contradicting that \( \sigma \in Bax \), and hence proving the claim.

Conversely, assume now that there is a left edge from \( x \) to \( i \) in \( T \). This means that \( x > i \) and that \( \sigma = \sigma_\Lambda x \sigma_b i \sigma_c \). If \( \sigma_b \) is empty, then \( xi \) is a descent of \( \sigma \). Otherwise, the edge from \( x \) to \( i \) in \( T \) proves that no element \( y \) of \( \sigma_\Lambda \) nor of \( \sigma_b \) is such that \( i < y < x \). Let us denote by \( j \) the rightmost element of \( \sigma_b \), and let us prove by contradiction that \( j > i \). So assume that \( j < i \). Then we have \( j < i < x \). Because there is a left edge from \( x \) to \( i \) (and not from \( x \) to \( j \)) in \( T \), this means that there is an element \( e \) in \( \sigma_\Lambda \) or in \( \sigma_b \) such that \( j < e < i \). Taking \( e \) to be the largest such element, we can furthermore ensure that \( e \in \sigma_\Lambda \) (or we would have an edge from \( x \) to \( e \) in \( T \)). Then the subsequence \( exji \) of \( \sigma \) is an occurrence of the classical pattern 2413. It may not be an occurrence of 2-41-3. But \( \sigma_b \) contains no element of value in \( [e..x] \), and in \( x\sigma_b \) there is an element of value at least \( x \) (namely, \( x \)) and an element of value smaller than \( e \) (namely, \( j \)) to its right. Hence there exists a descent \( ba \) in \( x\sigma_b \) such that \( b \geq x \) and \( a < e \). The subsequence \( eba \) is then an occurrence of 2-41-3 in \( \sigma \), contradicting that \( \sigma \in Bax \). We conclude that \( j > i \), so that \( ji \) is a descent of \( \sigma \).

Lemma 7.3. Let \( \sigma \) be a permutation of \( T Bax \), and let \( T = T_L \) be its labeled binary tree. A subword \( ji \) of \( \sigma \) is an occurrence of \( \mu \) if and only if there is a left edge from \( j \) to \( i \) in \( T \).

Proof. Let us first assume that a subword \( ji \) of \( \sigma \) is an occurrence of \( \mu \), and consider the tree obtained by leaf insertions of the elements of \( \sigma \) from left to right, until \( j \) is inserted. Because \( ji \) is an occurrence of \( \mu \), for every element \( x \) of \( \sigma \) that lies between \( j \) and \( i \), we have \( x > j \). Hence, just before the insertion of \( i \), there is no left edge going out of \( j \) in the tree. Furthermore, \( ji \) being an occurrence of \( \mu \) ensures that \( i < j \) and that no element \( x \) of \( \sigma \) that lies before \( i \) satisfies \( i < x < j \). Consequently, \( i \) is inserted as a left child of \( j \), so that there is a left edge from \( j \) to \( i \) in \( T \).

Conversely, assume that there is a left edge from \( j \) to \( i \) in \( T \). Then \( i < j \) and we can write \( \sigma = \sigma_\Lambda ji \sigma_b i \sigma_c \). Because \( i \) is inserted as a left child of \( j \), no element \( x \) of \( \sigma_\Lambda \) nor of \( \sigma_b \) is such that \( i < x < j \). To conclude the proof that \( ji \) is an occurrence of \( \mu \) in \( \sigma \), it is now enough to prove that no element \( x \) of \( \sigma_b \) is such that \( x < i \). Let us assume that there is an element \( x \in \sigma_b \) such that \( x < i \), and take \( x \) to be the leftmost such element. Because \( x \) is not inserted as a left child of \( j \), there is an element \( y \in \sigma_\Lambda \) such
that \( x < y < j \). This implies that \( y < i \), so that \( yjxi \) is an occurrence of the classical pattern 2413. Since \( x \) does not necessarily immediately follow \( j \) in \( \sigma \), this may not be an occurrence of 2-41-3. However, in this case, the element \( z \) that immediately precedes \( x \) in \( \sigma \) necessarily satisfies \( z > j \). Therefore, \( yzxi \) is an occurrence of 2-41-3, contradicting that \( \sigma \in TBax \).

Remark 7.4. We can see from its proof that Lemma 7.3 does not hold only for permutations \( \sigma \in TBax \) but also for every permutation \( \sigma \in Av(2-41-3) \).

7.2.3. Number of Components

**Lemma 7.5.** The bijection of [21] between \( Bax \) and \( TBax \) preserves the number of components.

Unlike Lemmas 7.1–7.3, to prove Lemma 7.5 we do not find an analogue of this statistic on the corresponding pair of twin binary trees. Instead, we invoke some properties of the bijection of [21] proved by Giraudo in his Ph.D. thesis [20].

To prove Lemma 7.5, we need to review a few more aspects of [21]. To a pair of twin binary trees \((T_L, T_R)\) with \( n \) internal nodes, the construction of [21] actually associates a set of permutations of size \( n \), which contains all the permutations \( \sigma \) such that the binary tree obtained from \( \sigma \) \((r(\sigma), \text{respectively})\) by insertion of leaves is \( T_L \) \((T_R, \text{respectively})\). This set can be described as an equivalence class for the equivalence relation \( R \) that consists in applying rewritings of the form 2-41-3 ⇋ 2-14-3 or 3-41-2 ⇋ 3-14-2 on permutations. This should be understood as: in every occurrence of the dashed pattern 2-41-3 (2-14-3, 3-41-2, 3-14-2, respectively) in \( \sigma \), the permutation obtained by swapping the elements corresponding to the 1 and the 4 is equivalent to \( \sigma \) in the sense of \( R \). As proved in [21], such an equivalence class contains exactly one permutation of \( Bax \) and exactly one permutation of \( TBax \). By definition, the bijection of Section 7.1 puts these two permutations in correspondence.

**Proof of Lemma 7.5.** In [20, Proposition 5.3.8], Giraudo has proved that if \( \sigma \in Bax \) has only one component, then so does every permutation in the equivalence class of \( \sigma \); in particular, the permutation of \( TBax \) that corresponds to \( \sigma \) also has one component. This can be generalized to state that if \( \sigma \in Bax \) has exactly \( k \) components, then the corresponding permutation in \( TBax \) also has exactly \( k \) components. Indeed, consider a permutation \( \sigma \in Bax \) that has \( k \) components, that we denote \( \sigma^{(1)}, \sigma^{(2)}, \ldots, \sigma^{(k)} \) from left to right. Up to normalization, every \( \sigma^{(i)} \) is a permutation. It has exactly one component by definition, and because it corresponds to a factor of \( \sigma \), it also belongs to \( Bax \). Hence, we can apply [20, Proposition 5.3.8] to it, getting that the corresponding permutation \( \tau^{(i)} \in TBax \) has exactly one component. Now we claim that the permutation \( \tau \) whose \( k \) components are \( \tau^{(1)}, \tau^{(2)}, \ldots, \tau^{(k)} \) from left to right is the permutation of \( TBax \) that corresponds to \( \sigma \). Proving this claim will conclude the proof. First, \( \tau \) is indeed equivalent to \( \sigma \) for \( R \) since every \( \tau^{(i)} \) is obtained from \( \sigma^{(i)} \) by rewritings of \( R \). And to prove that \( \tau \in TBax \) it is enough to notice that by definition every \( \tau^{(i)} \) avoids 2-41-3 and 3-41-2, and that no occurrence of 2-41-3 (3-41-2, respectively) in \( \tau \) can span more than one \( \tau^{(i)} \).
An example of two permutations of \( Bax \) and \( TBax \) in correspondence by the bijection of [21] is provided in Figure 13. This figure also shows the mapping of statistics through the bijection. Combined with Figure 8, Figure 13 provides a complete example of our bijection between \( Id(S \circ i \circ S) \), \( TBax \), and \( Bax \).

![Figure 13](image)

*Figure 13:* Two permutations of \( Bax \) and \( TBax \) and the pair of twin binary trees in correspondence by the bijection of [21].

8. Conclusion and Perspectives

This article has solved the problem of enumerating permutations sorted by two passes through a stack with any \( D_8 \)-symmetry in between, i.e., by operators of the form \( S \circ \alpha \circ S \) for \( \alpha \in D_8 \). The enumeration results obtained are refined with the joint equidistribution of a large number of statistics between \( Id(S \circ S) \) and \( Id(S \circ r \circ S) \) on one hand, and between \( Id(S \circ i \circ S) \) and \( Bax \) on the other hand.

A natural continuation for this work is to study the permutations sorted by \( k \) passes through a stack, with \( D_8 \)-symmetries between any two successive stacks, for any \( k \geq 3 \). The case of three stacks (with no symmetry) has been studied in [27], and \( Id(S \circ S \circ S) \) has been characterized by ten excluded patterns (that are classical, mesh and decorated patterns). It would be interesting to examine whether the methodology of [15, 27] could also be used for describing permutations sorted by \( S \circ \alpha \circ S \circ \beta \circ S \) for any \( \alpha, \beta \in D_8 \).

On the enumeration side, the case of \( Id(S \circ S \circ S) \) is still open, as well as the more general problem of enumerating \( Id(S \circ \alpha \circ S \circ \beta \circ S) \) for any \( \alpha, \beta \in D_8 \). However, we have obtained one first result in this direction, for \( \alpha = r \circ c \) and \( \beta = c \). This family of permutation can be described with excluded mesh patterns, and from this description it is possible to establish a bijection between them and colored Motzkin paths where each horizontal step has a color chosen in \( \{h_1, h_2\} \) and each up step has a color chosen in \( \{u_1, u_2\} \). These objects are enumerated by sequence A071356 of [26].

When \( k \) grows, it is very likely that even decorated patterns will not be enough to describe permutations sorted by \( k \) passes through a stack, possibly with \( D_8 \)-symmetries between any two successive stacks. But examining what kind of generalizations of permutation patterns are needed to characterize \( Id(S \circ \alpha_1 \circ S \circ \alpha_2 \circ \cdots \circ S \circ \alpha_k \circ S) \) for \( \alpha_1, \alpha_2, \ldots, \alpha_k \in D_8 \) is a natural way to start building the hierarchy of generalized
permutation patterns that Úlfarsson suggests in the Open problems section of [27].

Even without a description by excluded patterns of \( \text{Id}(S \circ \alpha_1 \circ S \circ \alpha_2 \circ \cdots \circ S \circ \alpha_k \circ S) \) for \( k \geq 2 \) and \( \alpha_1, \alpha_2, \ldots, \alpha_k \in D_8 \), some computer experiments are possible. Running such experiments has driven us to formulate the following conjecture:

**Conjecture 8.1.** Fix any \( k \geq 1 \). For any \((k - 1)\)-tuple \( (\alpha_2, \ldots, \alpha_k) \in \{\text{id}, r\}^{k-1}\), permutations sorted by \( S \circ \text{id} \circ S \circ \alpha_2 \circ \cdots \circ S \circ \alpha_k \circ S \) and by \( S \circ r \circ S \circ \alpha_2 \circ \cdots \circ S \circ \alpha_k \circ S \) are enumerated by the same sequence, even though they are different sets.

Notice that the above statement clearly holds for \( k = 0 \) (only one stack), and that Theorem 3.2 proves it for \( k = 1 \). As in Theorem 3.2, we would like to investigate whether the conjectured underlying bijection between \( \text{Id}(S \circ \text{id} \circ S \circ \alpha_2 \circ \cdots \circ S \circ \alpha_k \circ S) \) and \( \text{Id}(S \circ r \circ S \circ \alpha_2 \circ \cdots \circ S \circ \alpha_k \circ S) \) preserves the distribution of classical permutation statistics. Computer experiments suggest that some might be jointly equidistributed, like the statistics udword, zeil, and the indices of the right-to-left and left-to-right maxima.

From computer experiments, it seems that Conjecture 8.1 may even be strengthened to the following statement:

**Conjecture 8.2.** Fix any \( k \geq 1 \). For any \( (\alpha_1, \alpha_2, \ldots, \alpha_k) \) and \( (\beta_1, \beta_2, \ldots, \beta_k) \) in \( D_k \), we have:

- either \( \text{Id}(S \circ \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_k \circ S) = \text{Id}(S \circ \beta_1 \circ \beta_2 \circ \cdots \circ \beta_k \circ S) \);
- or these sets are not enumerated by the same sequence;
- or they fall into Conjecture 8.1.

Our belief is that proving Conjecture 8.1 will not be possible with the generating tree methodology, used here for \( k = 1 \). However, finding another statistic-preserving bijection (or another way of presenting our bijection) between \( \text{Id}(S \circ S) \) and \( \text{Id}(S \circ r \circ S) \) — a one that would be more likely to be generalized — is a path that we are investigating to try and prove Conjecture 8.1, refined with the joint equidistribution of statistics. We think that proving Conjecture 8.2 is a much harder task.

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