EQUIVALENCES OF BIPROJECTIVE ALMOST PERFECT NONLINEAR FUNCTIONS

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ABSTRACT. Two important problems on almost perfect nonlinear (APN) functions are the enumeration and equivalence problems. In this paper, we solve these two problems for any biprojective APN function family by introducing a group theoretic method for those functions. Roughly half of the known APN families of functions on even dimensions are biprojective. By our method, we settle the equivalence problem for all known biprojective APN functions. Furthermore, we give a new family of such functions. Using our method, we count the number of inequivalent APN functions in all known biprojective APN families and show that the new family found in this paper gives exponentially many new inequivalent APN functions. Quite recently, the Taniguchi family of APN functions was shown to contain an exponential number of inequivalent APN functions by Kaspers and Zhou (J. Cryptol. 34(1), 2021) which improved their previous count (J. Comb. Th. A 186, 2022) for the Zhou-Pott family. Our group theoretic method substantially simplifies the work required for proving those results and provides a generic natural method for every family in the large super-class of biprojective APN functions that contains these two family along with many others.

1. INTRODUCTION

Almost perfect nonlinear (APN) functions are cryptographically important functions since they give optimal protection against differential attacks when used as a building block of a Substitution Permutation Network. These functions are combinatorially interesting as there are several connections between APN functions and other combinatorial objects like difference sets, distance-regular graphs, symmetric association schemes, uniformly packed codes, and dimensional dual hyperovals (see for instance [21, 8, 7, 9, 23]). Important questions in the study of APN functions are to find new infinite families of (esp. bijective) APN functions, determine equivalences between known functions, enumerating inequivalent APN functions in total or within known families.

There are at least 15 known families of quadratic APN functions (see [22] for the most up-to-date list) and roughly half of them fall into the framework of $(q, r)$-biprojective functions introduced in [11]. In this paper, we will first give a new infinite family of $(q, r)$-biprojective APN functions (Theorem 1). Then we will develop a technique for determining equivalence of two APN functions if one of them is $(q, r)$-biprojective (Theorem 3). Using this technique we are able to count the number of inequivalent functions in all of the known $(q, r)$-biprojective APN function families (Theorem 5). We also show that, apart from one case, all of the known $(q, r)$-biprojective APN function families are pairwise inequivalent (Theorem 3). The standard way of showing equivalence is by using computers to compare invariants in small dimensions. Apart from that, some previous results that show inequivalence between some specific infinite families exist (see for instance [22, 17, 16]), usually relying on long and technical calculations with linear polynomials tailored towards the specific families. In this paper, we will give for the first time a generic method for determining equivalence of APN functions for the highly fertile super-class of $(q, r)$-biprojective functions that contains roughly half of the known families (Theorem 5). The novelty relies on exploiting the existence of a large cyclic subgroup in the automorphism group of the APN function. A similar approach has been successfully employed by the authors for the algebraically similar object of semifields [13, 12].
Moreover, we are going to show that the new family we present contains an exponential number of inequivalent APN functions (Corollary 1). Among the known infinite families, this is only the second family with this property. We also count the number of inequivalent APN functions in all other \((q,r)\)-biprojective families of APN functions, thus completely settling the equivalence question within and between all biprojective families (Table 1). Our group theoretic framework allows us to substantially simplify the problem, without which the treatment of the more complex functions would not be possible. Recently, Kaspers and Zhou [16] showed that the Taniguchi family of APN functions contains an exponential number of inequivalent functions using an intricate analysis of linearized polynomials. With our group theoretic method, which works for any \((q,r)\)-biprojective family, we can, in particular, recover those results in a more natural way and provide similar results for all known biprojective APN families of Table 1 that are out of reach of the method of Kaspers and Zhou. Some of our ideas are inspired from the works of Dempwolff and Yoshiara [8, 25], which also employ group theory, but only cover the equivalence question of monomials, which have a much simpler structure than the biprojective functions we consider here.

In Section 2, we will recall the basic definitions including projective and biprojective polynomials. Then in Section 3, we will prove that a new family of biprojective APN functions exists. In Section 4, we introduce our technique. Then we concentrate on the \((1,q)\)-biprojective APN family introduced by Carlet, and provide equivalence and enumeration results specific to that family in Section 5. Section 6 is devoted to computing a centralizer condition for all the known families that will be used in later sections. In Sections 7 and 8, we prove our all remaining equivalence and enumerations results. The main results here are Theorem 5 and 6. Finally, in Section 9, we compute the Walsh spectrum of the new family.

2. Preliminaries

An \( (n,n) \)-vectorial Boolean function is a map from the \( n \)-dimensional \( \mathbb{F}_2 \)-vector space \( \mathbb{F}_2^n \) to itself. In this paper we are only interested in the case where \( n = 2m \) is even. In that case, we can identify \( \mathbb{F}_2^n \) with \( \mathbb{F}_{2^m} \times \mathbb{F}_{2^m} \), which is the setting that we will be using throughout the paper. An \( (n,n) \)-vectorial Boolean function \( F \) is said to be almost perfect nonlinear (APN) if \( F(x) + F(x+a) = b \) has zero or two solutions for all nonzero \( a \in \mathbb{F}_2^n \) and all \( b \in \mathbb{F}_2^n \). Recall that the absolute trace map on a finite field \( \mathbb{F}_{2^m} \) is defined as

\[
\text{Tr}(x) = \sum_{i=0}^{m-1} x^{2^i}.
\]

Note that these functions are optimal in characteristic two since solutions to \( F(x) + F(x+a) = b \) come in pairs, i.e., \( F(x_0 + a) + F(x_0) = F((x_0 + a) + a) + F(x_0 + a) \).

2.1. Projective and biprojective polynomials. Projective and biprojective APN functions were introduced by the first author in [11]. We recall the definitions in this section, as well as fix some notation we will use throughout the paper.

Let \( \mathbb{M} \) be the finite field with \( 2^m \) elements. Let \( F \) be

\[
F(x,y) = (f(x,y),g(x,y)),
\]
Lemma 1. APN condition was proved in \cite{11}. Taniguchi \cite{20}, Zhou-Pott \cite{26} functions and the two families from \cite{11} by the first component of the Taniguchi functions is often also written (in our notation) as (1)

$$f(x, y) = a_0 x^{q+1} + b_0 x^q y + c_0 x y^q + d_0 y^{q+1},$$

$$g(x, y) = a_1 x^{r+1} + b_1 x^r y + c_1 x y^r + d_1 y^{r+1},$$

where \(a_i, b_i, c_i, d_i \in \mathbb{M}\). We will call \(f(x, y), g(x, y)\) \(q\)-biprojective polynomials and \(F(x, y)\) a \((q, r)\)-biprojective polynomial pair. We are going to use the shorthand notation

$$f(x, y) = (a_0, b_0, c_0, d_0)_q,$$

$$g(x, y) = (a_1, b_1, c_1, d_1)_r.$$

The univariate polynomial \(f(x, 1)\) is called a \(q\)-projective polynomial. The careful study of zeroes of projective polynomials was done by Bluher \cite{6}. Let \(\mathbb{P}^1(\mathbb{M}) = \mathbb{M} \cup \{\infty\}\) denote the projective line over the finite field \(\mathbb{M}\), i.e., the ratios \(v/w\) where \((v, w) \in \mathbb{M} \times \mathbb{M} \setminus \{(0, 0)\}\) where \(v/0\) for all \(v \in \mathbb{M}^*\) is defined to be the symbol \(\infty\).

Define

$$D_f^\infty(x, y) = a_0 x^q + a_0 x + c_0 y^q + b_0 y,$$

$$D_g^\infty(x, y) = a_1 x^r + a_1 x + c_1 y^r + b_1 y,$$

and for \(u \in \mathbb{P}^1(\mathbb{M}) \setminus \{\infty\}\),

$$D_f^u(x, y) = (a_0 u + b_0) x^q + (a_0 u^q + c_0) x + (c_0 u + d_0) y^q + (b_0 u^q + d_0) y,$$

and similarly

$$D_g^u(x, y) = (a_1 u + b_1) x^r + (a_1 u^r + c_1) x + (c_1 u + d_1) y^r + (b_1 u^r + d_1) y.$$

We will view \(F\) as a vectorial Boolean function \(F : \mathbb{M} \times \mathbb{M} \rightarrow \mathbb{M} \times \mathbb{M}\). We will not make any distinction between polynomials and functions and call a \((q, r)\)-biprojective polynomial pair \(F(x, y)\) a \((q, r)\)-biprojective function \(F\). The following straightforward lemma that simplifies checking for the APN condition was proved in \cite{11}.

**Lemma 1.** Let \(F(x, y) = (f(x, y), g(x, y))\) be a \((q, r)\)-biprojective polynomial pair. Then \(F\) is APN on \(\mathbb{M} \times \mathbb{M}\) if and only if \(D_f^u(x, y) = 0 = D_g^u(x, y)\) has exactly two solutions for each \(u \in \mathbb{P}^1(\mathbb{M})\).

Table \(\ref{tab:APN_families}\) lists all known biprojective APN families. We denote the families of Gold \cite{6}, Carlet \cite{6}, Taniguchi \cite{20}, Zhou-Pott \cite{26} functions and the two families from \cite{11} by \(\mathcal{G}, \mathcal{C}, \mathcal{T}, \mathcal{ZP}, \mathcal{F}_1, \mathcal{F}_2\), respectively. The new family we present in Section \(\ref{sec:new_APN}\) is denoted by \(\mathcal{F}_4\) (family \(\mathcal{F}_3\) in \cite{11} is another family of biprojective APN functions that only contains sporadic examples). We want to note that the first component of the Taniguchi functions is often also written (in our notation) as \((1, 0, c, d)_q\). However, it is easy to verify that all these function with \(c \neq 0\) are equivalent to the \(c = 1\) case, and the \(c = 0\) case is a Zhou-Pott function. The \textbf{Count} column refers to the number of (CCZ-)inequivalent functions in each family. For the precise definitions of equivalence, we refer to Section \(\ref{sec:equivalence}\).

3. The new APN family

3.1. Family \(\mathcal{F}_4\). We will now present a new family of \((q, r)\)-biprojective APN functions on \(\mathbb{M} \times \mathbb{M}\) where \(q = 2^k\) with \(\gcd(k, m) = 1\) and \(r = 2^{k+m/2}\). We will use the following notation in this section:
### Table 1. Known infinite families of biprojective APN functions on $\mathbb{F}_q \times \mathbb{F}_q$

| Family | Function | Notes | Count | Proved in |
|--------|----------|-------|-------|-----------|
| $\mathcal{G}$ | $X^{q+1}$ | $q = 2^k$, $\gcd(k, m) = 1$ | $\frac{x(2m)}{2}$ | 10 |
| | $((0,1,1)_q, (1,0,1,1)_q)$ | | | |
| | | $m$ odd. | | |
| | $((0,b,a)_q, (0,1,1,b+1)_q)$ | $m$ even, $\mathcal{T}_{M/2} (a) = 1$, $b = \sum_{i=0}^{k} a^{2^i}$ | $\frac{x(m)}{2}$ | 9 |
| $\mathcal{C}$ | $(xy, (1,b,c,d)_q)$ | | | |
| $\mathcal{T}$ | $((1,0,1,0)_q, (0,0,1,0)_q)$ | $q = 2^k$, $0 < k < m$, $\gcd(k, m) = 1$, $x^{q+1} + bx^q + cx + d \neq 0$ for $x \in \mathbb{F}_q$. | $\frac{x(m)(2m^2 - 2)}{2}$ | 10 |
| $\mathcal{ZP}$ | $((0,0,0)_q, (0,0,1)_q)$ | | | |
| $\mathcal{F}_1$ | $((1,0,1)_q, (1,0,1)_q)$ | $q = 2^k$, $0 < k < m$, $\gcd(3k, m) = 1$. | $\frac{x(m)}{2}$ | 11 |
| $\mathcal{F}_2$ | $((0,1,1)_q, (1,0,1)_q)$ | $q = 2^k$, $0 < k < m$, $\gcd(3k, m) = 1$, $m$ odd. | $\frac{x(m)}{2}$ | 11 |
| $\mathcal{F}_4$ | $((1,0,0)_q, (0,1,\frac{m}{2})_q)$ | $q = 2^k$, $r = 2^{k+m/2}$, $0 < k < m$, $\gcd(k, m) = 1$, $a \in \mathbb{K}^x$, $B \in \mathbb{M}^x \setminus \mathcal{C}_M$, $B^{q+r} \neq a^{q+1}$. | $\frac{x(m)(2m^2 - 2)}{2}$ | 11 |

**Notation 1.**

- Let $F = \mathbb{F}_{2^m}$ with $n = 2m$ even.
- Let $M = \mathbb{F}_{2^m}$ be the finite field with $2^m$ elements where $m$ is even and $m/2$ is odd.
- Let $K = \mathbb{F}_{2^m/2}$ be the finite field with $Q = 2^{m/2}$ elements.
- Let $q = 2^k$ and $r = 2^{k+m/2}$ with $1 \leq k \leq m - 1$.
- Let $\mathcal{C}_M = (\mathbb{M}^x)^3$ be the set of non-zero cubes of $\mathbb{M}$.
- Clearly, $(\mathbb{M}^x)^{Q+1} = K^x \subset \mathcal{C}_M$ (note 3)$Q + 1$ since $m/2$ is odd.
- Let the group of $(Q + 1)^st$ roots of unity be denoted by $(\mathbb{M}^x)^{Q-1}$. It is easy to see that $(\mathbb{M}^x)^{Q-1} \cap K = \{1\}$ and any $x \in \mathbb{M}^x$ can be written uniquely as $x = cg$ where $c \in K^x$ and $g \in (\mathbb{M}^x)^{Q-1}$.

We start with some simple lemmas.

**Lemma 2.** Let $\gcd(k, m) = 1$ and $r = qQ$. Then

- $\gcd(q + 1, 2^m - 1) = \gcd(r - 1, 2^m - 1) = \gcd(q^2 - 1, 2^m - 1) = 3$,
- $\gcd(q - 1, 2^m - 1) = \gcd(r + 1, 2^m - 1) = \gcd(q + 1, Q - 1) = 1$.

**Proof.** We only prove $\gcd(r - 1, 2^m - 1) = 3$ and $\gcd(r + 1, 2^m - 1) = 1$, the other statements are obvious/well-known.

Since $\gcd(k, m) = 1$ and $m$ is even, we know that $k$ is odd. Then $\gcd(k + m/2, m) = 2$. Indeed, assume $d$ is an odd divisor of $m$ and $k + m/2$, then it is also a divisor of $m/2$ and thus also of $k$, so $d = 1$. Then $\gcd(r - 1, 2^m - 1) = 2^{\gcd(k+m/2,m)} - 1 = 3$ and $\gcd(r + 1, 2^m - 1) = 1$ since $m/\gcd(k + m/2, m)$ is odd. \hfill $\Box$

We are now ready to prove the APN property of the new family.

**Theorem 1.** Let $B \in \mathbb{M}^x \setminus \mathcal{C}_M$ and $a \in \mathbb{K}^x$ be such that $B^{q+r} \neq a^{q+1}$, and let $F : \mathbb{M} \times \mathbb{M} \to \mathbb{M} \times \mathbb{M}$ be defined as

$$F(x, y) = ((1, 0, 0, B)_q, (0, 1, a/B, 0)_r),$$

where $q = 2^k$ with $m \equiv 2 \pmod{4}$, $\gcd(k, m) = 1$ and $r = qQ$. Then $F$ is APN.
Proof. We are going to use Lemma 1. First
\[ D_f^y(x, y) = B(y^q + y) = 0, \quad \text{and,} \]
\[ D_f^y(x, y) = x^r + \frac{ax}{B} = 0, \]
implies \( y \in \mathbb{F}_2 \) and \( x = 0 \) are the only common solutions since \( \gcd(r - 1, 2^m - 1) = 3 \) and \( a/B \not\in C_M \). Similarly,
\[ D_f^\infty(x, y) = y^q + y = 0, \quad \text{and,} \]
\[ D_f^\infty(x, y) = x + \frac{ax}{B} = 0, \]
have the same common solutions. Now for \( u \in \mathbb{M}^x \),
\[ D_f^u(x, y) = ux^q + u^qx + B(y^q + y) = 0, \quad \text{and,} \]
\[ D_f^u(x, y) = x^r + \frac{ax}{B} uy^r + uy^r = 0. \]
Now,
\[ D_f^u(ux, y) = u^{q+1}(x^q + x) + B(y^q + y) = 0, \quad \text{and,} \]
\[ D_f^u(ux, y) = u^r(x^r + y) + \frac{a}{B} u(x + y^r) = 0. \]
When \( x, y \in \mathbb{F}_2 \), the only solutions are \((x, y) \in \{(0, 0), (1, 1)\}\) since \( u^r + au/B = 0 \) implies \( a/B \) is a cube. We will proceed to show that these are the only solutions for \( x, y \in \mathbb{M} \). Now we can assume \( x, y \in \mathbb{M} \setminus \mathbb{F}_2 \), since \( x \in \mathbb{F}_2 \) implies \( y \in \mathbb{F}_2 \) and vice versa for \( D_f^u(ux, y) = 0 \). Furthermore, \( x = y^r \) implies \( y^r + y = 0 \) and thus \( x = y \in \mathbb{F}_4 \setminus \mathbb{F}_2 \) by Lemma 2. Then \( x^q + x = y^q + y = 1 \), and in turn \( u^{q+1} = B \), which is impossible since \( \gcd(q + 2^m - 1) = 3 \) and \( B \not\in C_M \). The same argument shows \( y \neq x^r \) and we can concentrate on
\[ u^{q+1} = \frac{B(y^q + y)}{x^q + x}, \quad \text{and,} \]
\[ u^{r-1} = \frac{a(x + y^r)}{B(x^r + y^r)} \]
for \( x, y \in \mathbb{M} \setminus \mathbb{F}_2 \) with \( x^r \neq y \) and \( x \neq y^r \). Now assume (1) and (2) holds for such \( x, y \in \mathbb{M} \setminus \mathbb{F}_2 \), and let
\[ \phi_q(x, y) = \frac{y^q + y}{x^q + x} = \frac{1}{cg}, \quad \text{and,} \]
\[ \phi_r(x, y) = \frac{x + y^r}{x^r + y} = dh, \]
for some \( c, d \in \mathbb{K}^x \) and \( g, h \in (\mathbb{M}^x)^{Q-1} \). Multiplying (1) and (2) we get
\[ \phi_q(x, y) \phi_r(x, y) = \frac{u^{q+r}}{a} \in \mathbb{K}, \]
and therefore \( g = h \). Noting that \( q^2 \equiv q^2Q^2 \equiv r^2 \) (mod \( 2^m - 1 \)), we get
\[ 1 = \frac{u^{(q+1)(q-1)}}{u^{(r-1)(r+1)}} = \frac{B^{q+r} \phi_q(x, y)^{q-1}}{a^{q+1} \phi_r(x, y)^{r+1}}, \]
which implies
\[ (cg)^{q-1}(dg)^{r+1} = c^{q-1}d^{q+1}g^{q+r} = c^{q-1}d^{q+1} = \frac{B^{q+r}}{a^{q+1}} \neq 1 \]
since by our assumption $B^{q+r} \neq a^{q+1}$. Now let $z = y + x^Q$ and consider

$$cg(z^q + z + (x^q + x)^Q) = x^q + x, \quad \text{and,}$$

$$(x^q + x) + z^r = dg((x + x^q)^Q + z),$$

which is a rewriting of (3) and (4). Or,

$$cg(x^q + x)^Q + (x^q + x) = cg(z^q + z), \quad \text{and,}$$

$$dg(x^q + x)^Q + (x^q + x) = z^r + dgz.$$

Adding the two equations and also adding $d$ times the former and $c$ times the latter equation, we get the following two equations:

$$\begin{align*}
(6) \quad (c + d)g(x^q + x)^Q &= z^r + cgz^q + (c + d)gz, \quad \text{and,} \\
(7) \quad (c + d)(x^q + x) &= c(z^r + dgz^q).
\end{align*}$$

Now if $c + d = 0$ then either $z = y + x^Q = 0$ and $c = d = 1$ since $(x^q + x)^{Q-1} \in (\mathbb{M})^{Q-1}$ by (3) and (4), which contradicts (4); or $z^r = cgz^q$ by the above equations and consequently $z^{r-q} = z(q(Q-1)) = cg$, and again we reach the contradiction $c = d = 1$ by (4). If $c + d \neq 0$ then comparing $g$ times (7) to the power $Q$ with (6) yields

$$\begin{align*}
(c(z^r + dgz^q))^Qg &= z^r + cgz^q + (c + d)gz, \\
\quad cgz^q + cdz^r &= z^r + cgz^q + (c + d)gz, \\
\quad (cd + 1)z^r &= (c + d)gz,
\end{align*}$$

that is to say $z^{r-1} = \frac{(c + d)g}{cd + 1}$, thus $g \in \mathbb{C}_M$. Note that $cd = 1$ implies $c = d$ contrary to our assumption. But now by (1),

$$u^{q+1} = \frac{B}{cg},$$

a contradiction since the left hand side is a cube and the right hand side is not. \[\square\]

4. A TECHNIQUE TO DETERMINE EQUIVALENCE

We will now develop a technique that allows us to determine when $(q, r)$-biprojective functions are equivalent or not. Denoting by $\mathbb{F} = \mathbb{F}_2^n$, we will first recall the different types of equivalences. Denote by $\Gamma_F = \{(x, F(x)) : x \in \mathbb{F}\}$ the graph of $F$.

**Definition 1.** Two functions $F, G : \mathbb{F} \to \mathbb{F}$ are called

(i) CCZ-equivalent, if there exists an affine permutation

$${\mathcal{A}} : (x, y) \mapsto \left( \begin{array}{cc} M & K \\ N & L \end{array} \right) \left( \begin{array}{c} x \\ y \end{array} \right) + \left( \begin{array}{c} u \\ v \end{array} \right)$$

of $\mathbb{F} \times \mathbb{F}$ such that $\mathcal{A}(\Gamma_F) = \Gamma_G$;

(ii) extended affine equivalent (EA-equivalent),

if $F$ and $G$ are CCZ-equivalent with $K = 0$;

(iii) extended linear equivalent (EL-equivalent),

if $F$ and $G$ are EA-equivalent with $(u, v) = (0, 0)$;

(iv) affine equivalent,

if $F$ and $G$ are EA-equivalent with $K = N = 0$;

(v) linear equivalent,

if $F$ and $G$ are affine equivalent with $(u, v) = (0, 0)$.
EA-equivalence between $F, G$ can be written equivalently as $N(x) + L(F(x)) = G(M(x))$ which is readily checked from the definition.

A major result by Yoshiara [24, Theorem 1] states that two quadratic APN functions $F$ and $G$ are CCZ-equivalent if and only if they are EA-equivalent. It is then straightforward that under the additional condition $F(0) = G(0) = 0$ it even suffices to consider EL-equivalence (see for instance [16 Proposition 2.2]). We summarize these observations in the following theorem.

**Theorem 2.** Two quadratic APN functions $F, G : F \to F$ with $F(0) = G(0) = 0$ are CCZ-equivalent if and only if they are EL-equivalent.

We define the group of EL-mappings (i.e., the set of mappings that correspond to extended linear transformations on graphs) as

$$E LM = \left\{ \begin{pmatrix} M & 0 \\ N & L \end{pmatrix} \in \text{GL}(F \times F) \right\}.$$  

Note that when we refer to linear mappings in this work, we always refer to linearity over the prime field $F_2$, for instance $\text{GL}(F) \cong \text{GL}(n, F_2)$ and $\text{GL}(F \times F) \cong \text{GL}(2n, F_2)$. Further denote by

$$\text{Aut}_{EL}(F) = \{ A \in ELM : A(\Gamma_F) = \Gamma_F \}$$

the group of EL-automorphisms of a function $F$. Clearly, if $F$ and $G$ are EL-equivalent, the corresponding EL-automorphism groups are conjugate in $E LM$. We include the simple proof (essentially just a special case of the orbit-stabilizer theorem) for the convenience of the reader.

**Proposition 1.** Assume $F, G : F \to F$ are EL-equivalent via the EL-mapping $\gamma \in ELM$, i.e., $\gamma(\Gamma_F) = \Gamma_G$. Then $\text{Aut}_{EL}(F) = \gamma^{-1} \text{Aut}_{EL}(G) \gamma$.

**Proof.** Assume $\delta \in \text{Aut}_{EL}(G)$. Then $\gamma^{-1} \delta \gamma \in \text{Aut}_{EL}(F)$. Indeed

$$(\gamma^{-1} \delta \gamma)(\Gamma_F) = (\gamma^{-1} \delta)(\Gamma_G) = \gamma^{-1} (\Gamma_G) = \Gamma_F.$$  

We conclude $\text{Aut}_{EL}(F) \subseteq \gamma^{-1} \text{Aut}_{EL}(G) \gamma$. The other inclusion follows by symmetry. $\square$

**4.1. The group theoretic framework.** In this section, we will develop a framework that allows us to prove EL-inequivalence of two functions $F, G$ that are both biprojective polynomial pairs.

Generally, it is very difficult to determine when two (APN) functions are equivalent or not. Usually, it is only possible to check inequivalence in low dimensions via computer using certain invariants. Thus, the usual way to argue that a family of APN functions is “new” is to show that they are not equivalent to the known APN functions in low dimensions. This is of course on several levels unsatisfying: One does not gain any theoretical insight as to why the mappings are inequivalent, and one does not get any information on the behavior in larger dimensions.

The key idea in our approach is inspired by the inequivalence results of Yoshiara [25] and Dempwolff [8] on power functions. In both papers, the authors establish CCZ-inequivalence between power functions by exploiting the existence of a large cyclic subgroup in the automorphism groups of power functions. Similarly, we will identify a large cyclic subgroup in the group $\text{Aut}_{EL}(F)$ if $F$ is a biprojective polynomial pair and use this subgroup and Proposition 1 to prove inequivalences between biprojective polynomial pairs. Note that the general approach of this technique is quite flexible to determine (in)equivalence of combinatorial objects; it was for instance also used by the authors to determine isotopies for semifields in [13].

We start by fixing some further notation that will be used throughout this section:
Notation 2.

- $F = F_{2^n}$, $M = F_{2^m}$ with $n = 2m$.
- $q = 2^k$, $r = 2^l$, $\overline{q} = 2^{m-k}$, $\overline{r} = 2^{m-l}$.
- We denote by $p$ a primitive divisor of $2^m - 1$, i.e., a prime that divides $2^m - 1$ but not $2^i - 1$ for $i < m$. Such a prime exists if $m > 1$, $m \neq 6$ by Zsigmondy’s theorem (cf. [14], Chapter IX., Theorem 8.3.). Note that $p \neq 2$ since $p | 2^m - 1$.
- $P$ is the unique Sylow $p$-subgroup of $M^\times$.
- For $a \in M^\times$ we denote by $m_a \in GL(M)$ the linear map $x \mapsto ax$.
- For $A, B, C, D \in GL(M)$ we write diagonal matrices as $\text{diag}(A, B) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in GL(F)$ and
  \[
  \text{diag}(A, B, C, D) = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & D \end{pmatrix} \in GL(F \times F).
  \]
- $Z^{(q,r)} = \{ \text{diag}(m_a, m_a, m_{q+1}, m_{q+1}) : a \in M^\times \}$ is a cyclic subgroup of $GL(F \times F)$ of order $2^m - 1$.
- $Z^{(q,r)}_p = \{ \text{diag}(m_a, m_a, m_{q+1}, m_{q+1}) : a \in P \}$ is the unique Sylow $p$-subgroup of $Z^{(q,r)}$ of order $|P|$.
- For a $(q, r)$-biprojective polynomial pair $F$ we denote by $C_F$ the centralizer of $Z^{(q,r)}_p$ in $\text{Aut}_{EL}(F)$ (see Lemma 3 for a proof that $Z^{(q,r)}_p \leq \text{Aut}_{EL}(F)$).

The reason to consider $Z^{(q,r)}$ is the simple (but crucial) fact that $Z^{(q,r)} \leq \text{Aut}_{EL}(F)$ for any $(q, r)$-biprojective polynomial pair $F$.

Lemma 3. Let $F = (f_1(x, y), f_2(x, y))$ be a $(q, r)$-biprojective polynomial pair. Then
\[
Z^{(q,r)}_p \leq Z^{(q,r)} \leq \text{Aut}_{EL}(F).
\]

Proof. The first inclusion is obvious. A simple calculation yields
\[
\begin{pmatrix} m_a \\ m_a \\ m_{q+1} \\ m_{q+1} \end{pmatrix} \begin{pmatrix} x \\ y \\ f_1(x, y) \\ f_2(x, y) \end{pmatrix} = \begin{pmatrix} ax \\ ay \\ a^{q+1}f_1(x, y) \\ a^{r+1}f_2(x, y) \end{pmatrix}.
\]
We have $a^{q+1}f_1(x/a, y/a) = f_1(x, y)$ and $a^{r+1}f_2(x/a, y/a) = f_2(x, y)$, so
\[
\begin{pmatrix} ax \\ ay \\ a^{q+1}f_1(x, y) \\ a^{r+1}f_2(x, y) \end{pmatrix} : x, y \in M^\times \}
\]
so $Z^{(q,r)} \leq \text{Aut}_{EL}(F)$ as claimed. \hfill \Box

A key idea of the approach is the following: Instead of working with the entire automorphism group $\text{Aut}_{EL}(F)$ (which is in general very difficult to determine), we focus only on the much simpler subgroup $Z^{(q,r)}$. The following lemma makes this possible by establishing that under certain conditions $Z^{(q,r)}_p$ is not only a Sylow subgroup of $Z^{(q,r)}$, but also of $\text{Aut}_{EL}(F)$. The proof of the Lemma is inspired by a similar result in [25], Corollary 2].
Lemma 4. Let $m > 2$, $m \neq 6$ and $F: \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ be a \((q, r)\)-biprojective polynomial pair such that $C_F$ contains $Z^{(q,r)}$ as a subgroup with index $I$ and $p$ does not divide $I$. Then $Z^{(q,r)}_P$ is a Sylow $p$-subgroup of $\text{Aut}_{EL}(F)$.

Proof. First define $S = \{\text{diag}(m_a, m_b, m_c, m_d) : a, b, c, d \in P\}$. Clearly, $|S| = |P|^4$. We show that $S$ is a Sylow $p$-subgroup of $\text{GL}(\mathbb{F} \times \mathbb{F})$. We have $|\text{GL}(\mathbb{F} \times \mathbb{F})| = |\text{GL}(4m, \mathbb{F}_2)| = 2^{2m(4m-2)} \prod_{i=1}^{m} (2^i - 1)$. Clearly, $2^{km+i} - 1 \equiv 2^i - 1 \pmod{2^m - 1}$ for $k \in \mathbb{N}$. As $p$ is a $2$-primitive divisor of $2^m - 1$, all integers $2^i - 1$ in $[1, 4m]$ are coprime to $p$, except for $j = m, 2m, 3m, 4m$. The $p$-part of $2^{2m - 1} = (2^m - 1)(2^m + 1)$ and $2^{4m - 1} = (2^{2m - 1})(2^m + 1)$ is clearly $|P|$ (recall that $p \neq 2$). With a little more effort, $2^{3m} - 1 = (2^{m} - 1)(2^{2m} + 2m + 1)$ has also $p$-part $|P|$ since $2^{2m} + 2m + 1 = 3(2^{m} - 1) + (2^m - 1)$ is not divisible by $p$ because $p \neq 3 = 2^2 -$ if $m > 2$.

We conclude that the $p$-part of $|\text{GL}(\mathbb{F} \times \mathbb{F})|$ is $|P|^4$ and $S$ is a Sylow $p$-subgroup of $\text{GL}(\mathbb{F} \times \mathbb{F})$ as claimed.

In particular, by the second Sylow theorem, all Sylow $p$-subgroups of $\text{GL}(\mathbb{F} \times \mathbb{F})$ are abelian since $S$ is abelian. Then also the Sylow $p$-subgroups of $\text{Aut}_{EL}(F)$ are abelian. Let $R$ be such an abelian Sylow $p$-subgroup of $\text{Aut}_{EL}(F)$ that contains the $p$-group $Z^{(q,r)}_P$. Since $R$ is abelian, $R$ is a subgroup of the centralizer $C_F$ of $Z^{(q,r)}_P$ in $\text{Aut}(F)$, which contains $Z^{(q,r)}_P$ as an index $I$ subgroup. Since $p$ does not divide $I$ by our assumption, we know that $R = Z^{(q,r)}_P$ and $Z^{(q,r)}_P$ is a Sylow $p$-subgroup of $C_F$. □

The following lemma was proven by the authors in [13].

Lemma 5. Let

$$M_P = \{\text{diag}(m_a, m_a) : a \in P\},$$

and

$$M = \{\text{diag}(m_a, m_a) : a \in \mathbb{M}^X\}.$$

Let $N_{\text{GL}(\mathbb{F})}(M_P), N_{\text{GL}(\mathbb{F})}(M)$ and $C_{\text{GL}(\mathbb{F})}(M_P), C_{\text{GL}(\mathbb{F})}(M)$ be the normalizers and centralizers of $M$ and $M_P$ in $\text{GL}(\mathbb{F})$. Then

(a) $N_{\text{GL}(\mathbb{F})}(M_P) = N_{\text{GL}(\mathbb{F})}(M) = \left\{ \begin{pmatrix} m_{c_1} & m_{c_2} \\ m_{c_1} & m_{c_2} \end{pmatrix} : c_1, c_2, c_3, c_4 \in \mathbb{M}^X, \tau \in \text{Gal}(\mathbb{M}/\mathbb{F}_2) \right\} \cap \text{GL}(\mathbb{F}) \cong \text{GL}(2, \mathbb{M}),$

(b) $C_{\text{GL}(\mathbb{F})}(M_P) = C_{\text{GL}(\mathbb{F})}(M) = \left\{ \begin{pmatrix} m_{c_1} & m_{c_2} \\ m_{c_3} & m_{c_4} \end{pmatrix} : c_1, c_2, c_3, c_4 \in \mathbb{M}^X \right\} \cap \text{GL}(\mathbb{F}) \cong \text{GL}(2, \mathbb{M}).$

4.2. A theorem to determine equivalence of projective polynomial pairs. First, we compute $\lambda^{-1} \text{diag}(m_a, m_a, m_a^{q+1}, m_a^{r+1}) \lambda$ for $\lambda \in \text{ELM}$. We have for $\lambda = \begin{pmatrix} M & 0 \\ N & L \end{pmatrix}$ with $M, L \in \text{GL}(\mathbb{F})$ that

$$\lambda^{-1} = \begin{pmatrix} M^{-1} & 0 \\ L^{-1}NM^{-1} & L^{-1} \end{pmatrix}$$

and

$$(8) \quad \lambda^{-1} \text{diag}(m_a, m_a, m_a^{q+1}, m_a^{r+1}) \lambda = \begin{pmatrix} M^{-1} \text{diag}(m_a, m_a)M \\ 0 \\ L^{-1} \text{diag}(m_a^{q+1}, m_a^{r+1})L \end{pmatrix}.$$

Theorem 3. Assume $m > 2$ and $m \neq 6$. Let $F$ and $G$ be \((q_1, r_1)\)- and \((q_2, r_2)\)-biprojective polynomial pairs respectively, with $q_i = 2^{k_i}$ and $r_i = 2^{l_i}$ where $k_1, l_1 \neq m/2$ and $k_1 \neq \pm l_1 \pmod{m}$. Assume further that

(C) $p$ does not divide $[C_F : Z^{(q_1, r_1)}]$. 

Then $F, G$ are EL-equivalent if and only if they are EL-equivalent via a mapping $\gamma = \begin{pmatrix} M & 0 \\ N & L \end{pmatrix} \in \text{ELM}$ where $M, L \in \text{GL}(\mathbb{F})$ and, writing $M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$, $L = \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix}$, the subfunctions $M_1, M_2, M_3, M_4, L_1, L_2, L_3, L_4 : M \to M$ are monomials (written as linearized polynomials) of the same degree, say $2^t$, or zero. Furthermore, we have either

$\bullet \ L_2 = L_3 = 0$,
$\bullet \ k_1 \equiv \pm k_2 \pmod{m}$, $l_1 \equiv \pm l_2 \pmod{m}$,
$\bullet \ If \ k_1 \equiv k_2 \pmod{m} \ (\text{resp.} \ l_1 \equiv l_2 \pmod{m})$, then $L_1$ (resp. $L_4$) has degree $2^t$,
$\bullet \ If \ k_1 \equiv -k_2 \pmod{m} \ (\text{resp.} \ l_1 \equiv -l_2 \pmod{m})$, then $L_1$ (resp. $L_4$) has degree $2^{m-t}$,

or

$\bullet \ L_1 = L_4 = 0$,
$\bullet \ k_1 \equiv \pm l_2 \pmod{m}$, $l_1 \equiv \pm k_2 \pmod{m}$,
$\bullet \ If \ k_1 \equiv k_2 \pmod{m} \ (\text{resp.} \ k_1 \equiv l_2 \pmod{m})$, then $L_2$ (resp. $L_3$) has degree $2^t$,
$\bullet \ If \ k_1 \equiv -k_2 \pmod{m} \ (\text{resp.} \ k_1 \equiv -l_2 \pmod{m})$, then $L_2$ (resp. $L_3$) has degree $2^{m-t}$.

If additionally $k_1, l_1 \not\equiv 0 \pmod{m}$ then $N = 0$, that is $F$ and $G$ are even linear equivalent.

**Proof.** Assume $F, G$ are EL-equivalent by the EL-mapping $\alpha \in \text{ELM}$, i.e., $\alpha(\Gamma_F) = \Gamma_G$. For $\alpha^{-1}Z_p^{(q_2, r_2)}\alpha \leq \text{Aut}_{\text{EL}}(F)$ we have $|\alpha^{-1}Z_p^{(q_2, r_2)}\alpha| = |P| = |Z_p^{(q_1, r_1)}|$, so $\alpha^{-1}Z_p^{(q_2, r_2)}\alpha$ is a Sylow $p$-subgroup of $\text{Aut}_{\text{EL}}(F)$ by Lemma [4] as long as condition [C] holds. In particular, it is conjugate to $Z_p^{(q_1, r_1)}$ in $\text{Aut}_{\text{EL}}(F)$ by Sylow’s theorem, i.e., there exists a $\lambda \in \text{Aut}_{\text{EL}}(F)$ such that

$$\lambda^{-1}\alpha^{-1}Z_p^{(q_2, r_2)}\alpha = (\alpha\lambda)^{-1}(\alpha\lambda)^{-1}Z_p^{(q_2, r_2)}(\alpha\lambda) = Z_p^{(q_1, r_1)}.$$

Note that $F, G$ are also EL-equivalent by the mapping $(\alpha\lambda) \in \text{ELM}$ since $(\alpha\lambda)(\Gamma_F) = \alpha(\Gamma_F) = \Gamma_G$.

Writing

$$(\alpha\lambda) = \begin{pmatrix} M & 0 \\ N & L \end{pmatrix}$$

with $M, L \in \text{GL}(\mathbb{F})$, this means $G \circ M = L \circ F + N$. We immediately get using Eqs. [9] and [10] that $\text{diag}(m_a, m_a)M = M \text{diag}(m_b, m_b)$ for all $a \in P$ and some $b = \pi(a)$, where $\pi : P \to P$ is a bijection, i.e., $M$ is in the normalizer of $\{\text{diag}(m_a, m_a) : a \in P\}$. From Lemma [5] we deduce that $M_1, M_2, M_3, M_4$ are zero or monomials of the same degree, say $2^t$.

We now compute $G \circ M = L \circ F + N$.

Writing $F, G$ as biprojective polynomial pairs, we have $F(x, y) = (f_1(x, y), f_2(x, y))$ and $G(x, y) = (g_1(x, y), g_2(x, y))$ with

$$g_1(x, y) = \alpha_1 x^{2^r+1} + \beta_1 x^{2^{r'}} y + \gamma_1 xy^{2^r} + \delta_1 y^{2^r+1} \quad \text{and} \quad g_2(x, y) = \alpha_2 x^{2^r+1} + \beta_2 x^{2^{r'}} y + \gamma_2 xy^{2^r} + \delta_2 y^{2^r+1}$$

and

$$(G \circ M)(x, y) = (g_1(M_1(x) + M_2(y), M_3(x) + M_4(y)), g_2(M_1(x) + M_2(y), M_3(x) + M_4(y))).$$
Denoting by $g'_1$ and $g'_2$ the two components of $G \circ M$, we get

$$g'_1(x, y) = \alpha_1((c_1x^{2^i} + c_2y^{2^i})^{q_2+1} + \beta_1(c_1x^{2^i} + c_2y^{2^i})^q_2(c_3x^{2^i} + c_4y^{2^i})$$

$$+ \gamma_1(c_1x^{2^i} + c_2y^{2^i})(c_3x^{2^i} + c_4y^{2^i})^{q_2} + \delta_1(c_3x^{2^i} + c_4y^{2^i})^{q_2+1}$$

$$g'_2(x, y) = \alpha_2((c_1x^{2^i} + c_2y^{2^i})^{r_2+1}) + \beta_2(c_1x^{2^i} + c_2y^{2^i})^{r_2}(c_3x^{2^i} + c_4y^{2^i})$$

$$+ \gamma_2(c_1x^{2^i} + c_2y^{2^i})(c_3x^{2^i} + c_4y^{2^i})^{r_2} + \delta_2(c_3x^{2^i} + c_4y^{2^i})^{r_2+1}.$$ 

We have $L \circ F = [L_1(f_1(x, y)) + L_2(f_2(x, y)), L_3(f_1(x, y)) + L_4(f_2(x, y))]$. Observe that the only terms that appear in $g'_1$ are of the form $x^{q_2+1}$ and $y^{q_2+1}$, and the same for $g'_2$ with $g_2$ substituted by $r_2$. Just by comparing these degrees to the possible degrees of $L \circ F$, we conclude that, since $k_1 \neq \pm l_1$ (mod $m$), either $L_2 = L_3 = 0$ and

$$(10) \quad k_1 \equiv \pm k_2 \pmod{m}, l_1 \equiv \pm l_2 \pmod{m}$$

or $L_1 = L_4 = 0$ and

$$(11) \quad k_1 \equiv \pm l_2 \pmod{m}, k_2 \equiv \pm l_1 \pmod{m}.$$ 

Furthermore, the non-zero $L_i$ for $i \in \{1, 2, 3, 4\}$ are monomials of degree $2^i$ or $2^{m-i}$ depending on the signs in Eqs. (10) and (11) since $k_1, l_1 \neq m/2$ (mod $m$).

We also deduce that if $k_1, l_1 \neq 0$ (mod $m$), then also $k_2, l_2 \neq 0$ (mod $m$) by our previous considerations, and both $G \circ M$ and $L \circ F$ have no linear terms, so that $N$ is the only linear part in the entire equation $G \circ M = L \circ F + N$. This implies $N = 0$. \hfill \square

Theorem 3 essentially answers when two biprojective polynomial pairs are EL-equivalent as long as Condition (C) on the centralizer stated in the theorem is satisfied. We will prove this condition later for most of the known APN biprojective polynomial pairs, which results in a comprehensive inequivalence result of the known APN functions that are constructed via projective polynomials.

Remark 1. It is possible to derive a slightly more general version of Theorem 3 where the conditions $k_1, l_1 \neq m/2$ and $k_1 \neq \pm l_1$ (mod $m$) are not needed, however the statement gets more involved. We decided to exclude these cases for the sake of readability since, in this paper, we will only need the cases that are covered by Theorem 3 as it stands.

5. THE CARLET FAMILY $\mathcal{C}$

The case of $(1, r)$-biprojective polynomial pairs requires some extra considerations since in this case the question of equivalence cannot be reduced to linear equivalence (see Theorem 3). The APN functions of this type are precisely the ones in the Carlet family $\mathcal{C}$ (see Table I). We will consider the special properties of this family in this section.

The following lemma can be easily deduced from a classical result by Bluher [4], as is done for example in [2] Lemma 4.

Lemma 6 ([2] Lemma 4]). Let $q = 2^k$ with $\gcd(k, m) = 1$. The number of polynomials $f(x, y) = (p_1, p_2, p_3, p_4)q$ with $p_1, p_2, p_3, p_4 \in M$ such that $p_1 \neq 0$ and $f(x, 1)$ has no roots in $M$ is precisely $\frac{(2^m+1)(2^m)^2(2^m-1)^2}{3}$.

Note that for $f(x, y) = (p_1, p_2, p_3, p_4)q$ with $p_1 \neq 0$ we have that $f(x, 1)$ has no roots if and only if $f(x, y) = 0$ holds only for $x = y = 0$. Indeed, we have $f(x, y) = y^{q+1}f(x, y, 1)$ for $y \neq 0$ and $f(x, 0) = p_1x^{q+1} = 0$ if and only if $x = 0$. 
Similarly to [2], we define a group action by $G = \mathbb{M}^\times \times \text{GL}(2, \mathbb{M})$ on the set of biprojective polynomial pairs $f = (p_1, p_2, p_3, p_4)_q$ with $p_1 \neq 0$ as follows: $M = \begin{pmatrix} c_1 & c_3 \\ c_2 & c_4 \end{pmatrix} \in \text{GL}(2, \mathbb{M})$ acts on a biprojective polynomial pair $f(x, y) = (p_1, p_2, p_3, p_4)_q$ via

$$Mf = \begin{pmatrix} x & y \end{pmatrix} M \begin{pmatrix} p_1 & p_3 \\ p_2 & p_4 \end{pmatrix} (M^q)^t \begin{pmatrix} x^q \\ y^q \end{pmatrix}$$

where $M^t$ denotes as usual the transpose of $M$ and $M^q$ the matrix where every entry is taken to the $q$-th power. A straightforward calculation shows $Mf(x, y) = f(c_1x + c_2y, c_3x + c_4y)$. $\mathbb{M}^\times$ acts on $f$ by regular multiplication, i.e., for $a \in \mathbb{M}^\times$ we have $af = (ap_1, ap_2, ap_3, ap_4)_q$. Note that $|G| = (2^m - 1) \cdot |\text{GL}(2, \mathbb{M})| = (2^m - 1)^3(2^m + 1)2^m$. Further, if $f(x, 1)$ has no roots in $\mathbb{M}$ then $((a, M)f)(x, 1)$ also has no roots in $\mathbb{M}$. This is obvious in the case of the scalar multiplication, and for the matrix action we have $Mf(x, 1) = f(c_1x + c_2, c_3x + c_4) = 0$ if and only if $c_2 = c_1x$ and $c_4 = c_3x$, which leads to $\det(M) = c_1c_4 + c_2c_3 = xc_1c_3 + xc_1c_3 = 0$. Moreover, $(a, M)f(x, y) = (p_1', p_2', p_3', p_4')_q$ for some values $p_1', p_2', p_3', p_4'$ with $p_1' \neq 0$. This is again obvious for the scalar multiplication, and for the matrix action, the coefficient of the $x^{q+1}$-term is (by simply expanding $f(c_1x + c_2y, c_3x + c_4y)$):

$$p_1c_1^{q+1} + p_2c_2^{q+3} + p_3c_1c_3^{q+1} + p_4c_3^{q+1} = f(c_1, c_3) \neq 0$$

since $(c_1, c_3) \neq (0, 0)$. Thus we can view the action of $G$ purely on the set of projective polynomials $f = (p_1, p_2, p_3, p_4)_q$ with $p_1 \neq 0$ and $f(x, 1) \neq 0$ for all $x \in \mathbb{M}$.

We have the following central result about this action:

**Lemma 7.** Let $q = 2^k$ be fixed with $\gcd(k, m) = 1$. Then $G$ acts transitively on the set of polynomials $f = (p_1, p_2, p_3, p_4)_q$ with $p_1 \neq 0$ such that $f(x, 1)$ has no roots in $\mathbb{M}$, i.e., all such polynomials are in the same orbit under $G$. The size of the stabilizer of any polynomial in this set is $3(2^m - 1)$.

**Proof.** Consider first the case that $m$ is even. Set $f(x, y) = x^{q+1} + uy^{q+1} = (1, 0, 0, u)_q$ for a non-cube $u \in \mathbb{M}^\times$, and observe that $f(x, 1)$ has no roots in $\mathbb{M}$. By [2, Lemma 6], the stabilizer of $f$ under $G$ has order $3(2^m - 1)$. By the orbit-stabilizer theorem, the size of the orbit is then

$$|G|/(3 \cdot (2^m - 1)) = 2^m(2^m - 1)^2(2^m + 1)/3,$$

so the entire set of polynomials with no roots by Lemma [6].

Now let $m$ be odd and consider $f(x, y) = (1, 0, 1, u)_q$ with $u \in \mathbb{M}^\times$ such that $f(x, 1)$ has no roots in $\mathbb{M}$. An element $(a, M) \in G$ acts on $f$ by definition as follows

$$(a, M)f = \begin{pmatrix} x & y \end{pmatrix} aM \begin{pmatrix} 1 & 1 \\ 0 & u \end{pmatrix} (M^q)^t \begin{pmatrix} x^q \\ y^q \end{pmatrix}$$

Now assume that $(a, M)$ is in the stabilizer of $f$ and set $M = \begin{pmatrix} c_1 & c_3 \\ c_2 & c_4 \end{pmatrix}$. Then

$$aM \begin{pmatrix} 1 & 1 \\ 0 & u \end{pmatrix} (M^q)^t = \begin{pmatrix} 1 & 1 \\ 0 & u \end{pmatrix}.$$ 

This implies that $a^2 \det(M)^q = 1$. Since $m$ is odd, the mapping $x \mapsto x^{q+1}$ permutes $\mathbb{M}$ and we can find a $\gamma \in \mathbb{M}$ such that $\gamma^{q+1} = a$. It is enough to consider the case $a = \det(M) = 1$, all other cases for $a$ are covered by scaling $M$ by $\gamma$. So let $a = \det(M) = 1$. In this case $((M^q)^t)^{-1} = \begin{pmatrix} c_4 & c_2 \\ c_3 & c_1 \end{pmatrix}$. The
condition
\[
\begin{pmatrix}
c_1 & c_3 \\
c_2 & c_4
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
0 & u
\end{pmatrix}
= \begin{pmatrix}
1 & 1 \\
0 & u
\end{pmatrix}
\begin{pmatrix}
c_4^q & c_2^q \\
c_3^q & c_1^q
\end{pmatrix}
\]
yields one equation per entry in the matrix:
\[
\begin{align*}
(12) & \quad c_1 = c_4^q + c_3^q \\
& \quad c_1 + uc_3 = c_2^q + c_1^q \\
(13) & \quad c_2 = uc_3^q \\
& \quad c_2 + uc_4 = uc_1^q.
\end{align*}
\]
Observe that all possible solutions of these equations yield an invertible matrix $M$ as long as $(c_3, c_4) \neq (0, 0)$: Indeed, we have using Eqs. (12) and (13) \( \det(M) = c_1c_4 + c_2c_3 = c_4^{q+1} + c_3^q c_4 + uc_3^{q+1} \) and \( \det(M) \neq 0 \) no matter the choice of $(c_3, c_4) \neq (0, 0)$ since \( \det(M) \neq 0 \) for $c_3 = 0$, $c_4 \neq 0$ and if $c_3 \neq 0$, we substitute $c_4 \mapsto c_3c_4$ which gives $c_3^{q+1} + c_3^q c_4 + uc_3^{q+1} = c_3^{q+1}f(c_4, 1)$ which is never 0 since $f(x, 1)$ has no root in $\mathbb{M}$. After eliminating $c_1, c_2$ using Eqs. (12), (13), we get (after slightly reordering the equations)
\[
\begin{align*}
(14) & \quad c_4^q + c_3^q = uqc_3^q + c_3^q + c_1^q + uc_3 \\
(15) & \quad uc_4 + uc_3^q = uc_3^q + uc_3^{q+1}.
\end{align*}
\]
Dividing the second equation by $u$ and then adding the two equations yields
\[
c_4 + c_3^q = uc_3 + uqc_3^q.
\]
Taking this equation to the power $q$ gives
\[
c_4^q + c_3^{q^2} = uqc_3^q + uqc_3^{q+1}.
\]
A comparison with Eq. (14) yields
\[
uqc_3^q + c_3^q + uc_3 = uqc_3^q + uqc_3^{q+1}
\]
and, after slightly reordering, \( uqc_3^q + c_3^q + uc_3 = (uqc_3^q + c_3^q + uc_3)q \), which implies \( uqc_3^q + c_3^q + uc_3 \in \mathbb{F}_2 \).
Observe that \( g(x) := uq^{x^q} + x^q + ux = xq^q(x^q-1) \) where \( g'(x) = uq^{x^q+1} + x + u \). We show that $g(x)$ permutes $\mathbb{M}$, i.e., \( \ker(g) = \{0\} \). This is clearly the case if $g'$ has no root in $\mathbb{M}$. Assume to the contrary \( g'(x) = uq^{x^q+1} + x + u = 0 \). Substituting $x \mapsto x/u$ and then multiplying the entire equation by $u$ yields \( x^{q+1} + x + u^2 = 0 \). But $f(x, 1) = x^{q+1} + x + u$ has no root in $\mathbb{M}$, so \( x^{q+1} + x + u^2 = 0 \) has also no solution in $\mathbb{M}$. We conclude that $g'$ has no roots in $\mathbb{M}$. Thus $g$ has trivial kernel, and permutes $\mathbb{M}$. In particular, \( g(c_3) = 0 \) if and only if $c_3 = 0$ and \( g(c_3) = 1 \) only for one value, say $c_3^q$. For $c_3 = 0$, Eq. (14) yields $c_4 = 1$ (recall that $x \mapsto x + x^q$ is a 2-to-1 mapping since \( \gcd(k, m) = 1 \) and $c_4 = 0$ is not possible since $M$ is invertible), and Eqs. (12), (13) then give $c_1 = 1$, $c_2 = 0$. This solution thus yields precisely the “trivial” case where $M$ is the identity matrix. For the non-zero solution $c_3^q$, we know that there are either 2 or 0 possible values for $c_4$ by Eqs. (15) and (14), again since $x \mapsto c_4 + c_3^q$ is a 2-to-1 mapping. If we find admissible values for $c_4$, the coefficients $c_1, c_2$ are uniquely determined by Eqs. (12), (13). We thus have 1 or 3 solutions for $a = \det(M) = 1$. As mentioned above, other values of $a$ are covered by simply scaling $M$ by $\gamma$ where $\gamma^{q+1} = a$. In total, the stabilizer of $f$ under $G$ has thus size $K(2^m-1)$ where $K \in \{1, 3\}$. By the orbit-stabilizer theorem, the orbit of $f$ has then size $(2^m - 1)^2(2^m+1)2^m/K$. Since the orbit can only contain functions $f$ such that $f(x, 1)$ has no roots in $\mathbb{M}$, Lemma 6 implies that we must have $K = 3$ and $G$ acts transitively on the set of all biprojective polynomials $f$ such that $f(x, 1)$ has no roots in $\mathbb{M}$.
\[\square\]
With this result we can show that, if \( m \) is even, all Carlet functions are EA-equivalent to Zhou-Pott functions, i.e., a function in the Family \( ZP \) in particular it can be written as

\[
\text{Let } C = (f(x, y), g(x, y)) = (xy, (1, \beta, \gamma, \delta)_q) \text{ be a Carlet function. We have } C(c_1 x + c_2 y, c_3 x + c_4 y) = (f(c_1 x + c_2 y, c_3 x + c_4 y), g(c_1 x + c_2 y, c_3 x + c_4 y)). \text{ Note that }
\]

\[
f(c_1 x + c_2 y, c_3 x + c_4 y) = c_1 c_3 x^2 + (c_1 c_4 + c_2 c_3) xy + c_4 y^2,
\]

in particular it can be written as \( N_1(x) + N_2(y) + d_1 xy \) where \( N_1, N_2 \) are linear functions and \( d_1 \in \mathbb{M}_x \) no matter the choice of \( c_1, c_2, c_3, c_4 \). By Lemma 7 we can choose \( c_1, c_2, c_3, c_4 \) with \( c_1 c_4 + c_2 c_3 \neq 0 \) in a way such that \( g(c_1 x + c_2 y, c_3 x + c_4 y) = d_2((1, 0, 0, u)_q) \) for an arbitrary non-cube \( u \) and some \( d_2 \in \mathbb{M}_x \). We conclude that \( L \circ C \circ M + N \) is a Zhou-Pott function, when we set \( L = \text{diag}(m_1/d_1, m_1/d_2) \),

\[
M = \begin{pmatrix} m_{c_1} & m_{c_2} \\
    m_{c_3} & m_{c_4} \end{pmatrix} \text{ and } N = \begin{pmatrix} N_1 \\
    0 \end{pmatrix}. \qedhere
\]

For the \( m \) odd case, we will employ Theorem 3. We will do this later in Section 7, since we need to check the centralizer condition (C) first.

**Remark 2.** After proving Theorem 4, we found out that the result was recently published as part of the PhD thesis of Christian Kaspers [15]. The proof in [15] is different from ours in the sense that it relies on a result proved in [15, Theorem 2.1.] that characterizes (using our language) the orbits of the polynomials \( x^{q+1} - u \) under \( G \), where \( u \) is a non-cube. In particular, it does not translate to the \( m \) odd case, which is left as an open problem in [15, Theorem 2.1.] In that sense, Lemma 7 can be seen as a generalization of [15, Theorem 2.1.] to the \( m \) odd case, solving this open problem. As we will see in later in Section 7, Lemma 7 will allow us to completely settle the equivalence question for the Carlet functions also in the \( m \) odd case, again closing a gap left in [15, Theorem 2.1.]

### 6. The Centralizer Condition

We now determine the centralizers to check condition (C) in Theorem 3. This is not very difficult, but quite technical, since we will compute the centralizers for many different families of functions.

**Lemma 8.** Let \( m > 2, m \neq 6 \) and let \( F \) be a \((q = 2^k, r = 2^l)\)-biprojective polynomial pair with \( l \notin \{0, m/2\}, k \neq \pm l \) (mod \( m \)) and \( C_F \) the centralizer of \( Z_p^{(q,r)} \) in \( \text{Aut}_{EL}(F) \). Then

- If \( F \) is a Zhou-Pott function with \( j \neq 0 \) or contained in the family \( \mathcal{F}_4 \), then \( C_F = \bigcup_{\omega \in \mathbb{F}_q^\times} Z_\omega^{(q,r)} \),

where we define

\[
Z_\omega^{(q,r)} = \{ \text{diag}(m_a, m_{a \omega}, m_{a r+1}, m_{a r+1 \omega}); a \in \mathbb{M}_x \}
\]

for \( \omega \in \mathbb{F}_q^\times \). Observe that \( Z_1^{(q,r)} = Z^{(q,r)} \).

- If \( F \) is a Taniguchi function, then \( C_F = Z^{(q,r)} \).

- If \( F \) is a function in the families \( \mathcal{F}_1 \) or \( \mathcal{F}_2 \), then \( C_F = Z^{(q,r)} \cup A \cup B \), where

\[
A = \begin{pmatrix} m_a & m_a & 0 & 0 \\
    m_a & 0 & 0 & 0 \\
    0 & 0 & m_{a r+1} & 0 \\
    0 & 0 & 0 & m_{a r+1} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & m_a & 0 & 0 \\
    m_a & m_a & 0 & 0 \\
    0 & 0 & m_{a r+1} & 0 \\
    0 & 0 & 0 & m_{a r+1} \end{pmatrix} ; a \in \mathbb{M}_x \}
\]

- If \( F \) is a Carlet function for odd \( m \) then \( C_F \) contains \( Z^{(q,r)} \) as an index 3 subgroup.
In all these cases, $C_F = \mathbb{Z}^{(q,r)}$ or $\mathbb{Z}^{(q,r)}$ is an index 3 subgroup of $C_F$. In particular, $p$ does not divide the index and condition (C) is satisfied for all families.

**Proof.** Assume $\lambda = \begin{pmatrix} M & 0 \\ N & L \end{pmatrix} \in C_F$. Then by Eq. (8), $M$ is in the centralizer of $\{\text{diag}(m_a, m_a) : a \in P\}$ in $\text{GL}(F)$.

Thus, by Lemma 5, all possible mappings that are contained in $C_F$ are EL-mappings

$$\lambda = \begin{pmatrix} M & 0 \\ N & L \end{pmatrix}$$

where $M = \begin{pmatrix} m_{c_1} & m_{c_2} \\ m_{c_3} & m_{c_4} \end{pmatrix}$, i.e., we can write $M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$ where the mappings $M_1, \ldots, M_4$ are monomials $M_i = c_i x$. Since $\lambda \in \text{Aut}_{\text{EL}}(F)$, we can proceed identically to the proof of Theorem 3 with $t = 0$, and we infer $N = 0$ and $L_1 = d_1 x$, $L_2 = L_3 = 0$, $L_4 = d_4 x$, except when $k \equiv 0 \pmod{m}$, in which case $N$ may be nonzero. We check which of these mappings are contained in $\text{Aut}_{\text{EL}}(F)$.

Let $F = [(\alpha_1, \beta_1, \gamma_1, \delta_1), (\alpha_2, \beta_2, \gamma_2, \delta_2), r]$ be a biprojective polynomial pair. Spelling out the equation $F \circ M = L \circ F + N$ yields

$$\begin{aligned}
\alpha_1((c_1 x + c_2 y)^{q+1}) + \beta_1(c_1 x + c_2 y)^{q}(c_3 x + c_4 y) + \gamma_1(c_1 x + c_2 y)(c_3 x + c_4 y)^{q} + \delta_1(c_3 x + c_4 y)^{q+1} \\
= d_1(\alpha_1 x^{q+1} + \beta_1 x^{q} y + \gamma_1 x y^{q} + \delta_1 y^{q+1})
\end{aligned}
$$

(16)

$$\begin{aligned}
\alpha_2((c_1 x + c_2 y)^{r+1}) + \beta_2(c_1 x + c_2 y)^{r}(c_3 x + c_4 y) + \gamma_2(c_1 x + c_2 y)(c_3 x + c_4 y)^{r} + \delta_2(c_3 x + c_4 y)^{r+1} \\
= d_4(\alpha_2 x^{r+1} + \beta_2 x^{r} y + \gamma_2 x y^{r} + \delta_2 y^{r+1}),
\end{aligned}
$$

(17)

with added linear function $N_1(x) + N_2(y)$ in Eq. (16) if $k \equiv 0 \pmod{m}$. We now consider Eqs. (16) and (17) for various cases of $F$.

**Zhou-Pott functions:** Let $F$ be a Zhou-Pott function. Eqs. (16) and (17) yield

$$\begin{aligned}
(c_1 x + c_2 y)^{q+1} + d(c_3 x + c_4 y)^{q+1} &= d_1(x^{q+1} + dy^{q+1}) \\
(c_1 x + c_2 y)(c_3 x + c_4 y)^{r} &= d_4 xy^{r}.
\end{aligned}
$$

(18)

(19)

Comparing the coefficients of these polynomials leads to several defining equations. We start with Eq. (19)

$$\begin{aligned}
c_1 e_3^r &= 0 \\
c_2 e_3^r &= 0 \\
c_1 e_4^r &= d_4 \\
c_2 e_4^r &= 0.
\end{aligned}
$$

$d_4$ is nonzero, so these equations imply $c_2 = c_3 = 0$ since $c_1$ and $c_4$ are necessarily nonzero. Substituting this in Eq. (18) and comparing the coefficients yields

$$\begin{aligned}
c_1^{q+1} &= d_1 \\
d c_4^{q+1} &= d \cdot d_1
\end{aligned}
$$

In total, we thus have $c_4^{q+1} = c_4^{q+1} = d_1$ and $c_1 c_4^r = d_4$. Since $\gcd(k, m) = 1$ and $m$ is even, the mapping $x \mapsto x^{q+1}$ is 3-to-1 on $\mathbb{F}_4^\times$, so we conclude that $c_4 = \omega c_1$ with $\omega \in \mathbb{F}_4^\times$. Further $c_1 c_4^r = c_4^{q+1} \omega^r = d_4$. 

Taniguchi functions: Outside of the special case \((k, m) = (1, 4)\) (where we have \(2k = m/2\)), Eqs. (16) and (17) yield

\[
(21) \quad (c_1x + c_2y)^{q+1} + (c_1x + c_2y)(c_3x + c_4y)^q + d(c_3x + c_4y)^{q+1} = d_1(x^{q+1} + xy^q + dy^{q+1})
\]

Comparing the coefficients again yields for Eq. (21): 

\[
\begin{align*}
&c_1^{q+1}c_3 = 0 \\
c_1^q c_4 &= d_1 \\
c_2^q c_3 = 0 \\
c_2^q c_4 &= 0.
\end{align*}
\]

This again leads to \(c_2 = c_3 = 0\). Substituting this in Eq. (20) and comparing the coefficients now gives

\[
\begin{align*}
c_1^{q+1} &= d_1 \\
dc_4^{q+1} &= d \cdot d_1 \\
c_1 c_4^q &= d_1.
\end{align*}
\]

Comparing the first and the third equation directly yields \(c_1 = c_4\), and \(d_1 = d_1^{q+1}, d_4 = c_1^{q+1}\) immediately follow.

Family \(F_1\): Eqs. (16) and (17) yield

\[
(22) \quad (c_1x + c_2y)^{q+1} + (c_1x + c_2y)(c_3x + c_4y)^q + (c_3x + c_4y)^{q+1} = d_1(x^{q+1} + xy^q + dy^{q+1})
\]

This leads again to further conditions

\[
\begin{align*}
&c_1^{q+1} + c_1 c_3^q + c_3^{q+1} = d_1, & c_1^{q+1} + c_1^q c_3 + c_3^{q+1} &= d_4, \\
&c_1^q c_2 + c_3^q c_2 + c_2^{q+1} c_4 = 0, & c_1^q c_2 + c_1^q c_4 + c_3^q c_4 &= d_4, \\
&c_1 c_2^q + c_1 c_4^q + c_3 c_4^q = d_1, & c_1 c_2^q + c_2^q c_3 + c_3 c_4^q &= 0, \\
&c_2^{q+1} + c_2 c_4^q + c_4^{q+1} = d_4, & c_2^{q+1} + c_2^q c_4 + c_4^{q+1} &= d_4.
\end{align*}
\]

Case \(c_4 = c_3\):
Note that because of Eq. (22), we have \(d_1 = c_1^{q+1}\) and in particular \(c_1 = c_3 \neq 0\). Eq. (23) then shows \(c_4 = 0\). Eq. (26) gives \(d_4 = c_1^{q+1}\) and then Eq. (27) yields \(c_2 = c_1\). It is then easy to verify that all other equations are satisfied, so this case yields the set \(A\) defined in the theorem.

Case \(c_4 = 0\):
From Eq. (22) and Eq. (24) we immediately deduce \(d_1 = c_1^{q+1}\) (in particular \(c_3 \neq 0\)) and \(c_3 = c_4\); and from Eq. (20) we get \(d_4 = c_3^{q+1}\). Eq. (28) leads to \(c_3^q (c_2 + c_4) = 0\), which implies \(c_2 = c_3 = c_4\). With these relations, it is again easy to check that all equations are satisfied, and this case gives the set \(B\) defined in the theorem.
Case $c_3 = 0$:
Eq. (22) and Eq. (23) give $d_1 = c_1^{q+1}$, $c_1 \neq 0$, and $c_2 = 0$. Eqs. (24) and (25) then also imply $c_4 = c_1$ and Eq. (26) yields $d_4 = c_1^{q+1}$, so we have $c_1 = c_4$, $c_2 = c_3 = 0$, $d_1 = c_1^{q+1}$, $d_4 = c_1^{q+1}$. All equations are satisfied, and this case gives precisely $Z(q,r)$.

Case $c_1, c_3 \neq 0$, $c_1 \neq c_3$:
From Eq. (23) and Eq. (28), we get
\[
c_2^{q^2} = \frac{c_3 c_4^{q^2}}{(c_1 + c_3)^{q^2}} = \frac{c_3 c_4^{q^2}}{(c_1 + c_3)},
\]
which directly gives $c_3^3 (c_1 + c_3) = c_3 (c_1 + c_3)^{q^3}$ and (simplifying further) $c_3^3 c_1 = c_1^3 c_3$ and finally $c_3^{q^3} = c_3^{q^3-1}$. Since $\gcd(3k, m) = 1$, we have $\gcd(q^3 - 1, 2^m - 1) = 1$, so the mapping $x \mapsto x^{q^3-1}$ is a bijection on $M$. We conclude $c_1 = c_3$, which contradicts our assumption.

Family $\mathcal{F}_2$: Eqs. (16) and (17) yield in this case
\[
(c_1 x + c_2 y)^{q+1} + (c_1 x + c_2 y)(c_3 x + c_4 y)^{q+1} + (c_3 x + c_4 y)^{q+1} = d_1 (x^{q+1} + xy^q + dy^{q+1})
\]
\[
(c_1 x + c_2 y)^{q^3} (c_3 x + c_4 y) + (c_1 x + c_2 y)(c_3 x + c_4 y)^{q^3} = d_4 (x^{q^3} y + xy^{q^3}).
\]
This leads to the conditions
\[
\begin{align*}
(30) & \quad c_1^{q+1} + c_1 c_3^{q+1} = d_1, & \quad (34) & \quad c_1^3 c_3 + c_1 c_3^{q+1} = 0, \\
(31) & \quad c_1^2 c_2 + c_3^3 c_2 + c_3^3 c_4 = 0, & \quad (35) & \quad c_1^3 c_4 + c_2 c_3^{q+3} = d_4, \\
(32) & \quad c_1 c_3^{q+1} + c_3^{q+1} = d_1, & \quad (36) & \quad c_2^3 c_3 + c_1 c_4^{q+1} = d_4, \\
(33) & \quad c_2^{q+1} + c_2 c_4^{q+1} = d_1, & \quad (37) & \quad c_2^3 c_4 + c_2 c_4^{q+3} = 0.
\end{align*}
\]
From Eq. (34), we get $c_1 = 0$, $c_3 = 0$, or $c_1^{q-1} = c_3^{q-1}$. Since $\gcd(3k, m) = 1$, this last condition is equivalent to $c_1 = c_3$. We will distinguish these three cases.

Case $c_1 = c_3$:
Eq. (30) implies $d_1 = c_1^{q+1}$, in particular $c_1 = c_3 \neq 0$. By Eq. (31), we then get $c_4 = 0$ and by Eq. (32) $c_2 = c_1 = c_3$. Clearly, $d_4 = c_1^{q+1}$. Then all equations are satisfied; this case gives the set $A$ specified in the theorem.

Case $c_1 = 0$:
Eq. (30) implies $d_1 = c_1^{q+1}$, in particular $c_3 \neq 0$. Eq. (32) gives $c_4 = c_3$ and Eq. (31) yields $c_2 = c_3 = c_4$. Clearly, with $d_4 = c_3^{q+1}$, all equations are then satisfied, and we get the set $B$.

Case $c_3 = 0$:
Eq. (30) implies $d_1 = c_1^{q+1}$, in particular $c_1 \neq 0$. Eq. (31) yields $c_2 = 0$ and Eq. (32) gives $c_4 = c_1$. Clearly, with $d_4 = c_1^{q+1}$, all equations are again satisfied, and we get the set $Z(q,r)$.

The family $\mathcal{F}_4$: Let $F$ be a function in the family $\mathcal{F}_4$. Recall that by Lemma 2 the $(qQ - 1)$-st powers in $\mathbb{F}_{2^m}$ are precisely the cubes.
Eqs. (16) and (17) yield
\begin{align}
(c_1 x + c_2 y)^{q+1} + B(c_3 x + c_4 y)^{q+1} &= d_1 (x^{q+1} + By^{q+1}) \\
(c_1 x + c_2 y)^{qQ}(c_3 x + c_4 y) + a/B(c_1 x + c_2 y)(c_3 x + c_4 y)^{qQ} &= d_4 (x^{qQ} y + a/Bxy^{qQ}).
\end{align}

We compare the coefficients.
\begin{align*}
c_1^{q+1} + Bc_3^{q+1} &= d_1 \\
c_1^q c_2 + Bc_3^q c_4 &= 0 \\
c_1 c_2^q + Bc_3 c_4^q &= 0 \\
c_2^{q+1} + Bc_4^{q+1} &= d_1 B.
\end{align*}

Let us consider the case $c_1 c_2 c_3 c_4 \neq 0$ first, and set $c_1 = z_1 c_1$ and $c_2 = z_4 c_2$ with $z_1, z_2 \in \mathbb{M}^\times$. The second and third equations on the left then yield $B = z_1^q z_2 = z_1 z_2^q$, which implies $z_1^{q-1} = z_2^{q-1}$ and since $\gcd(q - 1, 2m - 1) = 1$, we conclude $z_1 = z_2$. Then $B = z_1^{q+1}$, in particular $B$ is a cube since $\gcd(q + 1, 2m - 1) = 3$. So we arrive at a contradiction and $c_1 c_2 c_3 c_4 = 0$. By the second and third equation on the left and the bijectivity of $M$, we infer that there are only two cases to check, $c_1 = c_4 = 0$ and $c_2 = c_3 = 0$.

Case $c_2 = c_3 = 0$: The second and third equation on the right yield $d_4 = c_1^q c_4 = c_1 c_4^q$, so $c_1^{q-1} = c_4^{q-1}$ and $c_4 = \omega c_1$ with $\omega \in \mathbb{F}_4^\times$ since $\gcd(qQ - 1, 2m - 1) = 3$ by Lemma 2. Similarly, the first and fourth equations on the left yield $d_1 = c_1^{q+1} = c_1^q$ (observe that $\omega^{q+1} = 1$). It is then easy to verify that for this choice of coefficients, all equations hold. So this case gives precisely the sets $Z_{\omega}^{(q,r)}$ for $\omega \in \mathbb{F}_4^\times$.

Case $c_1 = c_4 = 0$: The first and fourth equations on the left give $d_1 = Bc_3^{q+1} = c_2^{q+1}/B$, so $c_2^{q+1} = B^2 c_3^{q+1}$. In particular, since $\gcd(q + 1, 2m - 1) = 3$, we know that both $c_3^{q+1}$ and $c_2^{q+1}$ are cubes, but $B^2$ is not a cube since $B$ is a non-cube. So this case cannot occur.

**The Carlet family:** Let $F = (x, y, (1, \beta, \gamma, \delta)_r)$ be a Carlet function. Eqs. (16) and (17) then yield
\begin{align}
(c_1 x + c_2 y)(c_3 x + c_4 y) &= d_4 xy + N_1(x) + N_2(y), \\
(c_1 x + c_2 y)^{r+1} + \beta(c_1 x + c_2 y)^r(c_3 x + c_4 y) + \gamma(c_1 x + c_2 y)(c_3 x + c_4 y)^r + \delta(c_3 x + c_4 y)^{r+1} &= d_1 (x^{r+1} + \beta xy^r + \gamma xy^{r+1} + \delta y^{r+1})
\end{align}

Observe that Eq. (40) can always be satisfied for given $c_1, c_2, c_3, c_4$ by the (unique) choice $d_4 = c_1 c_4 + c_2 c_3 = \det(M) \neq 0$, $N_1(x) = c_1 c_3 x^2$, $N_2(y) = c_2 c_4 y^2$. We thus only have to care about Eq. (41). This equation is satisfied if and only if $(d_1, M)$ is in the stabilizer of $f(x, y) = (1, \beta, \gamma, \delta)_r$ under the action of $G$ (see Section 5). The size of this stabilizer is precisely $3(2^m - 1)$ by Lemma 7.

7. Inequivalences within Biprojective families

We now combine Lemma 8 with Theorems 2 and 3 to obtain CCZ-inequivalence results inside the APN families that can be written as biprojective polynomial pairs.

We start by investigating the equivalences inside different families. In the case of the Zhou-Pott and Taniguchi functions, this was already done in [16, 17]. Note that the approach in these papers is very technical, for instance the proof of the inequivalence of the Taniguchi functions in [16] relies on an intricate investigation of certain linear polynomials that takes more than two dozen pages. Theorem 8
allows us to minimize the effort since we only have to consider linear equivalences of a very special type. This also allows us to deal with the more complicated functions in the other families, which seem to be out of reach using the approach in [16, 17]. We skip an alternative proof of the Zhou-Pott and Taniguchi functions (which can easily be constructed based on the approach we outline for the other families), and instead only deal with the remaining families for which the (in)equivalence question has not been settled yet. In particular, note that Theorem 5 together with Theorem 4 completely settles the equivalence problem for the family of Carlet functions, thus solving the problem left open in [17].

Theorem 5. Let \( q = 2^k \) and \( \overline{q} = 2^{m-k} \). We consider the following functions defined on \( M \times M \) with \( m > 2, m \neq 6 \).

(a) Let \( F_q, F_{q'} \) be two functions in the families \( F_1 \) with parameters \( q \) and \( q' \).

(i) \( F_q \) is CCZ-equivalent to \( F_{q'} \) if and only if \( q' = q \) or \( q' = \overline{q} \).

(ii) There are \( \varphi(m)/2 \) CCZ-inequivalent functions in both families \( F_1 \), and \( F_2 \).

(b) Let \( F_q, F_{q'} \) be two functions in the families \( F_2 \) with parameters \( q \) and \( q' \).

(i) \( F_q \) is CCZ-equivalent to \( F_{q'} \) if and only if \( q' = q \) or \( q' = \overline{q} \).

(ii) There are \( \varphi(m)/2 \) CCZ-inequivalent functions in both families \( F_1 \), and \( F_2 \).

(c) Let \( F_{q,B,a} \) be a function in the family \( F_4 \) with parameters \( q, B, a \).

(i) \( F_{q,B,a} \) is linear equivalent to \( F_{\overline{q},B,a'} \).

(ii) For each choice of \( B' \) and \( a' \) there exists an \( a \) such that \( F_{q,B,a} \) is linear equivalent to \( F_{q,B',a'} \).

(iii) \( F_{q,B,a} \) is linear equivalent to \( F_{q,B',a'} \) for at most \( m \) choices of \( a' \).

(iv) There are in total at least \( \varphi(m)(2^{m/2} - 2)/(2m) \) CCZ-inequivalent functions in Family \( F_4 \).

(d) Let \( C(q; 1, \beta, \gamma, \delta) := (xy, (1, \beta, \gamma, \delta)y) \) be a Carlet function with \( m \) odd.

(i) \( C(q; 1, \beta, \gamma, \delta) \) is CCZ-equivalent to \( C(q'; 1, \beta_2, \gamma_2, \delta_2) \) if and only if \( q' = q \) or \( q' = \overline{q} \) (no matter the choice of the other coefficients).

(ii) There are in total \( \varphi(m)/2 \) CCZ-inequivalent Carlet functions.

Here \( \varphi \) denotes Euler’s totient function.

Proof. By Theorem 3, we just have to test for linear equivalence, i.e., for two functions \( F, F' \) we have to check \( F \circ M = L \circ F' \), where the subfunctions \( L_i, M_i \) are zero or monomials, except for the Carlet family, where we do have to test for EL-equivalence, i.e., \( F \circ M = L \circ F' + N \).

For functions \( F_q, F_{q'} \) in the families \( F_1, F_2 \), the inequivalences follow directly from Theorem 3 except in the case where \( q' = \overline{q} \). In this case, the equivalence can be stated directly: For both families, the equivalence \( F_q \circ M = L \circ F_{\overline{q}} \) can be achieved by setting \( M_1 = M_4 = L_2 = L_3 = 0, M_2 = M_3 = x^{\overline{q}}, L_1 = x, \) and \( L_4 = x^{\overline{q}} \) for the Family \( F_1 \) and \( L_4 = x^q \) for Family \( F_2 \).

Family \( F_4 \): Consider \( F_{q,B,a}, F_{q',B',a'} \). The two functions are CCZ-inequivalent if \( k \neq \pm k' \pmod{m} \) by Theorem 3 and the fact that \( k' \neq \pm (k + m/2) \pmod{m} \) since \( \gcd(k + m/2, m) = 2 \) (see the proof of Lemma 2). If \( q' = \overline{q} \), we have \( F_{q,B,a} \circ M = L \circ F_{\overline{q},B,BQ^{-1}/a} \) by setting \( M_1 = M_4 = x^{\overline{q}}, L_1 = x, L_4 = (BQ/a)x^Q \) and \( M_2 = M_3 = L_2 = L_3 = 0 \). Observe that \( (BQ^{-1}/a) \in F_Q \), so the conditions of the family are not violated.

It thus only remains to consider the case \( q' = q \). Consider now \( M_4 = c_q^2 x^{2^q} \). By Theorem 3, this immediately gives \( L_1 = d_1 x^{2^q}, L_4 = d_4 x^{2^q} \) and \( L_2 = L_3 = 0 \). \( F_{q,B,a} \) and \( F_{q,B',a'} \) are then linear.
From the two equations on the left, we deduce (setting $C = M$):

\begin{align}
(42) \quad (c_1 x + c_2 y)^{q^{2^1+2^j}} + B(c_3 x + c_4 y)^{q^{2^1+2^j}} &= d_1 (x^{q^{2^1+2^j}} + B^2 y^{q^{2^1+2^j}}) \\
(43) \quad (c_1 x + c_2 y)^{qQ^2} (c_3 x + c_4 y) + a/B(c_1 x + c_2 y)(c_3 x + c_4 y)^{qQ^2} &= d_4 (x^{qQ^2} y + (a'/B')^{2^1} x y^{qQ^2}).
\end{align}

Just like in the proof of Theorem 3, we again compare the coefficients which leads to the equations

\begin{align*}
c_1^{q+1} + B c_3^{q+1} &= d_1 \\
c_1^q c_2 + B c_3 c_4 &= 0 \\
c_1 c_2^q + B c_3 c_4 &= 0 \\
c_2^{q+1} + B c_4^{q+1} &= d_1 B^{2^j}.
\end{align*}

From the second and third equation on the left, like in the corresponding case of Theorem 3, we again infer that $c_2 = c_3 = 0$ or $c_1 = c_4 = 0$.

Case $c_2 = c_3 = 0$: The first and last equations on the left imply $d_1 = c_1^{q+1}$ and $B/(B^{2^j}) c_4^{q+1} = c_1^{q+1}$. In particular, $B/(B^{2^j}) = r^{q+1}$ must be a cube and $c_1/c_4 = r \omega$ for $\omega \in \mathbb{F}_q^\times$. Note that, no matter the choice of $B, B'$ we can always find a $t$ such that $B/(B^{2^j})$ is a cube. Indeed, $1/B' \in B\mathcal{M}$ or $1/B' \in \mathcal{B}\mathcal{M}$.

In the second case, $B/B' \in \mathcal{C}_\mathcal{M}$, so we can choose $t = 0$, in the second case $B/(B^{2^j}) \in \mathcal{C}_\mathcal{M}$, so we can choose $t = 1$.

The second and third equations of the right can be satisfied if and only if

\[ \left( \frac{c_1}{c_4} \right)^{qQ^2} = \frac{a}{a'^{2^j}} B^{2^j} \left( \frac{c_1}{c_4} \right)^{q+1}, \]

or, equivalently, $(c_1/c_4)^{q(Q+1)} = a/(a'^{2^j})$. Note that $(c_1/c_4)^{q(Q+1)} \in \mathbb{F}_Q$, in particular there is for a fixed $a$ always precisely one $a'$ that satisfies the equation. We have thus shown that for arbitrary $a, B, B'$ there exists an $a'$ such that $F_{q,u',a'}$ is linear equivalent to $F_{q,u,a}$. We can thus assume from now on that $B' = B$.

Then $r^{q+1} = 1/B^{2^j-1}$, so $t$ is necessarily even. Then $(c_1/c_4)^{q(Q+1)} = r^{q(Q+1)} = a/(a'^{2^j})$, so for a fixed choice of $a, t$, precisely one $a'$ solves the equation.

Case $c_1 = c_4 = 0$: From the two equations on the left, we deduce (setting $B = B'$): $(c_2/c_3)^{q+1} = B^{2^j+1}$, which implies that $t$ is odd. Similarly, from the two equations on the right, we infer $(c_2/c_3)^{qQ^2} = B^{2^j+1}/(a'^{2^j} a)$ and, combining this with the previous condition, $(c_3/c_2)^{q(Q+1)} = 1/(a'^{2^j} a)$. Just like in the last case, a fixed choice for $t, a$ yields a unique $a'$ that satisfies the condition.

We conclude that, in total, for a function $F_{q,B,a}$ there are at most $m$ choices of $a'$ such that $F_{q,B,a'}$ are linear equivalent (since there are $m$ choices for $t$, with $t$ even and odd appearing in the first and second case, respectively). We have thus in total $\varphi(m)/2$ choices for (inequivalent) $q$, each of which yields at least $(2^{m/2} - 2)/m$ inequivalent functions.

The Carlet family: $C(q;1,\beta,\gamma,\delta)$ is linear equivalent to $C(q;1,\beta,\gamma,\delta)$ by setting $L_1 = x, L_4 = M_1 = M_4 = x, M_2 = M_3 = 0$. Two Carlet functions $C(q;1,\beta_1,\gamma_1,\delta_1)$ and $C(q;1,\beta_2,\gamma_2,\delta_2)$ are then EL-equivalent if and only if $q = q_2$ and there are coefficients $c_1, c_2, c_3, c_4, d_1, d_4$ such that (compare Eq. (41), Eq. (40))

\begin{align}
(44) \quad (c_1 x + c_2 y)(c_3 x + c_4 y) &= d_1 xy + N_1(x) + N_2(y), \\
(c_1 x + c_2 y)^{q+1} + \beta_1 (c_1 x + c_2 y)^{q} (c_3 x + c_4 y) + \gamma_1 (c_1 x + c_2 y)(c_3 x + c_4 y)^{q} + \delta_1 (c_3 x + c_4 y)^{q+1} &= d_4 x^{q+1} + \beta_2 x^q y + \gamma_2 xy^q + \delta_2 y^{q+1}.
\end{align}
Equivalences of Biprojective Almost Perfect Nonlinear Functions

Identically to the corresponding case of the proof of Lemma 8, Eq. (14) is always satisfied by the unique choice \( N_1(x) = c_1c_3x^2, N_2(y) = c_2c_4y^2, d_1 = c_1c_4 + c_2c_3 = \det(M) \neq 0 \). Eq. (15) is satisfied if and only if \((1, \beta_2, \gamma_2, \delta_2)_q\) is in the orbit of \((1, \beta_1, \gamma_1, \delta_1)_q\) under the action of \(G\) defined in Section 6. By Lemma 7, this action is transitive on the set of all biprojective polynomials such that \( f(x, 1) \) has no zeros in \( \mathbb{F} \), so all such functions are in the same orbit. We conclude that two Carlet functions \( C(q; \alpha_2, \beta_2, \gamma_2, \delta_2) \) are always EL-equivalent, no matter the choice of coefficients. \( \square \)

We want to emphasize that this result implies that the new Family \( \mathcal{F}_4 \) we found in this paper is only the second known infinite family that yields exponentially (in \( n \)) many inequivalent APN functions, next to the Taniguchi family. Theorem 5 actually not only gives a lower bound, but also an upper bound on the number of inequivalent functions inside \( \mathcal{F}_4 \), in total yielding a quite tight estimate:

**Corollary 1.** Let \( N(m) \) be the number of inequivalent functions inside Family \( \mathcal{F}_4 \) on \( \mathbb{F} \times \mathbb{F} \) where \( m \equiv 2 \pmod{4} \). Then

\[
\frac{\varphi(m)(2^{m/2} - 2)}{2m} \leq N(m) \leq \frac{\varphi(m)(2^{m/2} - 2)}{2}
\]

8. Equivalence between the families

We now prove inequivalences between the different biprojective APN families. This, again, is only made possible by the simplifications that Theorem 6 allows. Recall that we proved already in Theorem 4 that the Carlet family for \( m \) even is contained in the Zhou-Pott family, so we need not consider that case.

**Theorem 6.** Let \( m > 2, m \neq 6 \) and \( F \) and \( G \) be \((q_1, r_1)-\) and \((q_2, r_2)-\)biprojective polynomial pairs respectively; defined on \( \mathbb{F} \times \mathbb{F} \) in distinct families from the following list (see Table 7):

- The Gold functions \( \mathcal{G} \),
- The Zhou-Pott functions \( \mathcal{ZP} \),
- The Taniguchi functions \( \mathcal{T} \), with \((q, m) \neq (2, 4)\),
- \( \mathcal{F}_1 \) with \((q, m) \neq (2, 4)\),
- \( \mathcal{F}_2 \),
- \( \mathcal{F}_3 \),
- The Carlet functions \( \mathcal{C} \) for \( m \) odd.

Then \( F, G \) are CCZ-inequivalent.

**Proof.** The inequivalence between Gold, Zhou-Pott and Taniguchi functions was shown in [16, 17] (although it is not particularly challenging to give an alternative proof with the help of Theorem 3). The other results follow from Lemma 8 and Theorems 2 and 3. Thu functions \( F \) and \( G \) can only be CCZ-equivalent if \( q_1 \in \{q_2, \overline{q_2}\} \) and \( r_1 \in \{r_2, \overline{r_2}\} \) or \( q_1 \in \{q_2, \overline{q_2}\} \) and \( r_1 \in \{q_2, \overline{q_2}\} \). This already proves most inequivalences. The remaining cases have to be checked by hand for linear equivalence under the conditions given in Theorem 3. It thus only remains to check the inequivalences between the families \( \mathcal{F}_1 \) and Taniguchi functions; and between Zhou-Pott functions and functions in the families \( \mathcal{F}_1, \mathcal{F}_4 \).

So let \( F = (f_1(x, y), f_2(x, y)) \) be a Taniguchi or Zhou-Pott function. In both cases, we have \( f_2 = xy^{r_1} \). Assume that \( F \) is linearly equivalent to a function \( G = (g_1(x, y), g_2(x, y)) = ((\alpha_1, \beta_1, \gamma_1, \delta_1)_{q_2}, (\alpha_2, \beta_2, \gamma_2, \delta_2)_{r_2}) \). Then \( G = L \circ F \circ M \) for some bijective mappings \( M \) and \( L \). Here, by Theorem 3, the subfunctions \( M_1, \ldots, M_4 \) are monomials of the same degree 2 or zero, and either \( L_1 = L_4 = 0 \) or \( L_2 = L_3 = 0 \). Set
\( F \circ M = (h_1(x, y), h_2(x, y)) \). Then

\[
h_2(x, y) = (ax + by)^{2^t}(cx + dy)^{2^t} = (ac^{2^t}x^{2^t} + bd^{2^t}y^{2^t} + ad^{2^t}x^{2^t}y^{2^t} + bc^{2^t}y^{2^t} + bd^{2^t}x^{2^t}y^{2^t} + bd^{2^t}x^{2^t}y^{2^t})
\]

for some \( a, b, c, d \in \mathbb{F} \). Since either \( L_2 = L_3 = 0 \) or \( L_1 = L_4 = 0 \) we then have either \( L_4(h_2(x, y)) = g_2(x, y) \) (in the first case) or \( L_2(h_2(x, y)) = g_1(x, y) \) (in the second case). If \( L_2 \) or \( L_4 \), respectively, is a monomial \( ex^{2^t} \), we can write

\[
L_2(h_2(x, y)) = e(a'c^{2^t}x^{2^t} + b'd^{2^t}y^{2^t} + c'd^{2^t}x^{2^t}y^{2^t} + d'c^{2^t}y^{2^t} + d'a^{2^t}x^{2^t} + b'a^{2^t})
\]

(or equivalently for \( L_4 \)). We now compare the coefficients of \( x^{2^t}y^{2^t}, x^{2^t}y^{2^t}, x^{2^t}y^{2^t}, y^{2^t}y^{2^t} \). We get either \( t' = 0 \) and the necessary conditions

\[
e a'c^{2^t} = \alpha_z \\
e b'd^{2^t} = \beta_z \\
e a'd^{2^t} = \gamma_z \\
e b'a^{2^t} = \delta_z
\]

or \( 2^t = 1 \) and

\[
e a'c^{2^t} = \alpha_z \\
e b'd^{2^t} = \gamma_z \\
e a'd^{2^t} = \beta_z \\
e b'a^{2^t} = \delta_z
\]

for \( z = 1 \) or \( z = 2 \). It is easy to see that these conditions cannot be satisfied if \( \alpha_z = \beta_z = \delta_z = 1 \) and \( \gamma_z = 0 \) or \( \alpha_z = \gamma_z = \delta_z = 1 \) and \( \beta_z = 0 \), which shows inequivalence to the family \( \mathcal{F}_1 \). Similarly, the conditions cannot be satisfied for \( \alpha_z, \delta_z \neq 0 \) and \( \beta_z = \gamma_z = 0 \) or \( \alpha_z = \delta_z = 0 \), \( \beta_z, \gamma_z \neq 0 \), which shows inequivalence to the family \( \mathcal{F}_4 \).

Note that we exclude the cases \( (q, m) = (2, 4) \) for Family \( \mathcal{F}_1 \) and the Taniguchi family since the conditions of Theorem 3 are not satisfied.

9. Walsh spectra of biprojective APN functions

One of the most important properties of APN functions in even dimension is their Walsh spectrum.

**Definition 2.** Let \( F : \mathbb{F} \to \mathbb{F} \) be a mapping. We define

\[
W_F(b, a) = \sum_{x \in \mathbb{F}} (-1)^{Tr(bF(x) + ax)} \in \mathbb{Z}
\]

for all \( a, b \in \mathbb{F} \). We call the multisets

\[
\{ W_F(b, a) : b \in \mathbb{F}^n, a \in \mathbb{F} \} \text{ and } \{ |W_F(b, a)| : b \in \mathbb{F}^n, a \in \mathbb{F} \}
\]

the Walsh spectrum and the extended Walsh spectrum of \( F \), respectively.

The extended Walsh spectrum is invariant under CCZ-equivalence. Most known APN functions in even dimension \( n \) have the so called classical (or Gold-like) extended Walsh spectrum, which contains the values \( 0, 2^{n/2}, 2^{(n+2)/2} \) precisely \( (2^n - 1)2^{n-2} \) times, \( (2^n - 1)2^{n+1}/3 \) times and \( (2^n - 1)2^n/3 \) times, respectively.
We will now show that all functions in the Family $F_4$ have classical Walsh spectrum. This is already known for the Taniguchi, Zhou-Pott, Carlet functions as well as the functions from $F_1$ and $F_2$ [19][1][18], so all known infinite families of biprojective APN functions known so far share the same Walsh spectrum. For our proof, we will employ a criterion from [18]. In fact, we will show that all functions in the Family $F_4$ are 3-to-1 functions. APN 3-to-1 functions are particularly interesting since they have the smallest possible image set for APN functions, which is the reason they were studied in detail in [18]. There, it was also shown that all quadratic (or, more generally, all plateaued) 3-to-1 APN functions have the classical Walsh spectrum, which allowed simple proofs of the Walsh spectra of (among others) the Zhou-Pott functions and the functions from Families $F_1$ and $F_2$. The criterion states:

**Theorem 7** ([18]). Let $n$ be even and $F: \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ be a quadratic APN function such that

- $F(0) = 0$, and
- Every $y \in \text{Im}(F) \setminus \{0\}$ has at least 3 preimages.

Then $F(x) = 0$ if and only if $x = 0$ and every $y \in \text{Im}(F) \setminus \{0\}$ has precisely 3 preimages (i.e., $F$ is 3-to-1). Additionally, $F$ has classical Walsh spectrum.

Using this result, determining the Walsh spectrum of the family is reduced to a simple verification.

**Theorem 8.** All APN functions $F: \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ with $n = 2m$ from the Family $F_4$ are 3-to-1 and have classical Walsh spectrum.

**Proof.** We check the conditions of Theorem 7. The first condition is clearly satisfied. Recall that

$$F(x, y) = (x^{q+1} + By^{q+1}, x^{qQ}y + (a/B)xy^{qQ}),$$

so $F(x, y) = F(\omega x, \omega^2 y)$ for any $\omega \in \mathbb{F}_2^\times$ since $qQ = 2k+m/2$ where $k + m/2$ is even (recall that $k$ and $m/2$ are both odd), so $\omega^{qQ} = \omega$. Thus both conditions of Theorem 7 hold and $F$ is 3-to-1 with classical Walsh spectrum. \qed

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