ESTIMATING THE OUTPUT ENTROPY OF A TENSOR PRODUCT OF TWO QUANTUM CHANNELS

G. G. Amosov

For a class of bipartite quantum states, we find a nontrivial lower bound on the entropy gain resulting from the action of a tensor product of the identity channel with an arbitrary channel. We use the obtained result to bound the output entropy of the tensor product of a dephasing channel with an arbitrary channel from below. We characterize phase-damping channels that are particular cases of dephasing channels.

Keywords: quantum channel, bipartite quantum system, entanglement, von Neumann entropy

1. Introduction

One of the most important tasks in quantum information theory is to calculate how informational capacities change under an external action on a quantum system. Most of the used capacities are determined based on the von Neumann entropy. We consider the problem of finding a lower bound on the entropy gain for a state of a bipartite system in the case where the perturbation affects only its subsystem. The non-triviality of this problem is connected with a phenomenon of entanglement for bipartite quantum systems; this phenomenon is absent from classical systems.

We consider a state $\rho \in \mathcal{S}(K)$, where $\mathcal{S}(K)$ denotes the set of all positive unit-trace operators in the Hilbert space $K$. Given a quantum channel (a linear completely positive trace-preserving map) $\Omega$ on the set of all bounded operators $B(K)$ on $K$, we can then consider the entropy gain with respect to the action of such a channel,

$$S(\Omega(\rho)) - S(\rho).$$

(1)

Here, $S$ denotes the von Neumann entropy, $S(\rho) = -\text{Tr}(\rho \log \rho)$ with $0 \leq S(\rho) \leq +\infty$. The entropy gain is relevant because it measures the mixing property of the quantum channel [1].

We consider the Kraus decomposition of $\Omega$

$$\Omega(\rho) = \sum_{j=1}^{+\infty} V_j \rho V_j^*.$$  

(2)

Under the condition that the strong limit

$$\underset{N \to +\infty}{s\text{-lim}} \sum_{j=1}^{N} V_j V_j^* = \Omega(I)$$

is achieved.

*Steklov Mathematical Institute, RAS, Moscow, Russia, e-mail: gramos@mi.ras.ru.

1Hilbert spaces are assumed to be infinite dimensional unless explicitly stated otherwise.
exists and with the assumption that \( S(\rho) < +\infty \), the lower bound for entropy gain (1)

\[
S(\Omega(\rho)) - S(\rho) \geq -\text{Tr}(\rho \log \Omega(I))
\]  

was found in [2].

Moving to the context of bipartite systems, it is interesting to evaluate the entropy gain for a state \( \rho \in \mathcal{S}(H \otimes K) \) under the action of the tensor product of the identity channel \( \text{Id}: B(H) \to B(H) \) and the channel \( \Omega \), i.e.,

\[
S((\text{Id} \otimes \Omega)(\rho)) - S(\rho).
\]  

We consider the convex closure \( S_\Omega \) of the output entropy for the state \( \rho \in \mathcal{S}(K) \) defined in [3] as

\[
S_\Omega(\rho) = \inf_{\rho = \sum \pi_j \rho_j} \sum_j \pi_j S(\Omega(\rho_j)).
\]  

We conjecture the following relation between quantities (4) and (5).

**Conjecture 1.** We have the inequality

\[
S((\text{Id} \otimes \Omega)(\rho)) - S(\rho) \geq S_\Omega(\text{Tr}_H(\rho)).
\]  

In this paper, we find a sufficiently broad class of states \( \rho \) for which (6) holds.

It is known that if inequality (6) holds for all states of the form

\[
\rho = (\Phi \otimes \text{Id})(\sigma), \quad \sigma \in \mathcal{S}(H \otimes K),
\]

then the minimal output entropy is additive under tensor product of \( \Phi \). Proving this property was one of the motivations for also addressing Conjecture 1 (see [4]). We note that states of the form \( (\Phi \otimes \text{Id})(|e\rangle\langle e|) \) were studied in [5] in a different context.

The quantum channel \( \Phi \) is said to be **dephasing** [6], [7] if there exists an orthonormal basis \( \{e_n\} \) in \( H \) such that

\[
\Phi(|e_n\rangle\langle e_m|) = \lambda_{nm}|e_n\rangle\langle e_m|,
\]

where \( \lambda_{nm} \) is a positive definite matrix with \( \lambda_{nn} = 1 \). Quantum dephasing channels are known to be complementary to entanglement-breaking channels [7].

We show that (6) is related to the inequality

\[
S((\Phi \otimes \Omega)(|e\rangle\langle e|)) \geq S((\Phi \otimes \text{Id})(|e\rangle\langle e|)) + \sum_n \pi_n S(\Omega(|h_n\rangle\langle h_n|)),
\]

where

\[
|e\rangle\langle e| = \sum_{n,m} \lambda_n \bar{\lambda}_m |e_n\rangle\langle e_m| \otimes |h_n\rangle\langle h_m|
\]

with the orthonormal system \( \{e_n\} \) in \( H \), the unit vectors \( h_n \in K \), a dephasing channel \( \Phi \), and an arbitrary channel \( \Omega \). Inequality (8) is closely related to the property of strong superadditivity for the channel \( \Phi \) introduced in [3] and widely discussed in [8].

If \( H = L^2(\mathbb{R}) \), then the notion of a dephasing channel can be extended to the channel \( \Phi \) for which the output state \( \Phi(|\psi\rangle\langle \psi|) \) is an integral operator of the form [9]

\[
(\Phi(|\psi\rangle\langle \psi|)\psi)(x) = \int_{\mathbb{R}} \lambda(x,y)\psi(x)\bar{\psi}(y)\phi(y) \, dy,
\]
where \( \phi \in L^2(\mathbb{R}) \) and \( \lambda(x,y) \) is a positive-definite kernel. We call (9) a generalized dephasing channel.

Using the generalized eigenvectors of the position operator, we can represent the action of (9) in the form

\[
\Phi(|x\rangle\langle y|) = \lambda(x,y)|x\rangle\langle y|.
\] (10)

Equation (10) should be understood in the sense of (9).

We consider a pure state

\[
|e\rangle\langle e| = \int_{\mathbb{R}^2} \lambda(x)\overline{\lambda}(y)|x\rangle\langle y| \otimes |h_x\rangle\langle h_y| dx dy,
\] (11)

where \( \int_{\mathbb{R}} |\lambda(x)|^2 dx = 1 \), \( x \to h_x \) is a measurable function, and \( h_x \) are unit vectors in \( K \). More precisely, (11) means that for a scalar product of the bipartite system, we have

\[
\langle \phi_1 \otimes \psi_1 |e\rangle\langle e|\phi_2 \otimes \psi_2 \rangle = \int_{\mathbb{R}^2} \lambda(x)\overline{\lambda}(y)\overline{\phi_1}(x)\phi_2(y)|h_x\rangle\langle h_y|\psi_1\rangle\langle \psi_2| dx dy,
\]

where \( \phi_1, \phi_2 \in H \) and \( \psi_1, \psi_2 \in K \). Similarly to (8), we show that we have the inequality

\[
S((\Phi \otimes \Omega)(|e\rangle\langle e|)) \geq S((\Phi \otimes \text{Id})(|e\rangle\langle e|)) + \int_{\mathbb{R}} \pi(x)S(\Omega(|h_x\rangle\langle h_x|)) dx,
\] (12)

where \( \pi(x) = |\lambda(x)|^2 \), \( \Phi \) is a generalized dephasing channel, and \( \Omega \) is an arbitrary channel.

Quantum dephasing channel (7) is said to be a phase-damping channel if

\[
\lambda_{nm} = \lambda_{n-m}, \quad \overline{\lambda}_n = \lambda_{-n}
\] (13)

for some complex numbers \( (\lambda_n)_{n=0}^{N-1} \) that are the discrete Fourier transforms of a probability distribution \( (\pi_n)_{n=0}^{N-1} \) [10]. In [10], inequality (8) was obtained for a phase-damping channel \( \Phi \) in the case \( \dim H < +\infty \).

We extend the definition of the phase-damping channel to the infinite-dimensional space. Based upon Bochner’s theorem [11], we then give a complete classification of all phase-damping channels.

This paper is organized as follows. In Sec. 2, we prove some results about the entropy gain. As an application, we obtain inequalities of form (8) and (12) for all dephasing channels. In Sec. 3, we describe quantum phase-damping channels of form (13) and also their generalizations. The last section contains concluding remarks.

2. The entropy gain

We first derive a tighter bound on (3) with the role of the identity operator \( I \) played by the orthogonal projection \( P \) with the property \( \text{supp} \rho \subset \text{supp} P \).

\textbf{Proposition 1.} We suppose that for \( \rho \in \mathcal{S}(H) \), \( S(\rho) < +\infty \), there exists an orthogonal projection \( P \) such that

\[ \text{supp} \rho \subset \text{supp} P \]

and the strong limit

\[ \lim_{N \to +\infty} \sum_{j=1}^{N} V_j PV_j^* = \Omega(P) \]

exists. Then

\[ S(\Omega(\rho)) - S(\rho) \geq - \text{Tr}(\Omega(\rho) \log \Omega(P)) \].

399
Proof. We follow the method of proof in [2]. We choose an orthonormal basis \((e_k)\) spanning \(\text{supp} \ P\) such that the state \(\rho\), \(\text{supp} \ \rho \subset \text{supp} \ P\), can be represented in the form
\[
\rho = \sum_k \nu_k |e_k\rangle\langle e_k|.
\]
By hypothesis,
\[
S(\rho) = -\text{Tr}(\rho \log \rho) = \sum_k \nu_k (\log \nu_k) < +\infty.
\]
There then exists an operator \(F\) satisfying
\[
\text{Tr}(\rho F) < +\infty, \quad \text{Tr}(\exp(-\beta F)) < +\infty, \quad \beta > 0. \tag{14}
\]
Indeed, it suffices to set
\[
F = \sum_k \mu_k (-\log \nu_k) |e_k\rangle\langle e_k|,
\]
where \((\mu_k)\) are taken such that \(\mu_k \uparrow +\infty\) but \(\sum_k \mu_k \nu_k (-\log \nu_k)\) still converges. This allows defining a state \(\rho_\beta\) as
\[
\rho_\beta = P \exp(-\beta F) \frac{\text{Tr}(P \exp(-\beta F))}{\text{Tr}(P \exp(-\beta F))}. \tag{15}
\]
The monotonicity of the quantum relative entropy \(S(\rho||\sigma) = \text{Tr}(\rho(\log \rho - \log \sigma))\) leads to
\[
S(\Omega(\rho)||\Omega(\rho_\beta)) \leq S(\rho||\rho_\beta). \tag{16}
\]
On the other hand,
\[
S(\rho||\rho_\beta) = -S(\rho) + \beta \text{Tr}(\rho F) + \log \text{Tr}(P \exp(-\beta F)),
\]
and (16) consequently implies that
\[
\text{Tr}(\Omega(\rho)(-\log \Omega(\rho_\beta))) \leq S(\Omega(\rho)) - S(\rho) + \beta \text{Tr}(\rho F) + \log \text{Tr}(P \exp(-\beta F)). \tag{17}
\]
Substituting (15) in (17), we obtain
\[
S(\Omega(\rho)) - S(\rho) \geq \text{Tr}\{\Omega(\rho)(-\log \Omega(P \exp(-\beta F)))\} - \beta \text{Tr}(\rho F).
\]
Because \(P \exp(-\beta F) \leq P\), this implies \(\log \Phi(P \exp(-\beta F)) \leq \log \Phi(P)\) by the operator monotonicity of the function \(\log x\) on \(\mathbb{R}_+\). Taking the limit \(\beta \to 0\), we obtain the required result.

We now turn to considering a bipartite system \(H \otimes K\). Proposition 1 allows proving the following theorem.

**Theorem 1.** We suppose that \(\rho \in \mathcal{S}(H \otimes K)\) has the form
\[
\rho = \sum_{n,m} \lambda_{mn} |e_n\rangle\langle e_m| \otimes |h_n\rangle\langle h_m|, \tag{18}
\]
where \((e_n)\) is an orthonormal basis in \(H\), \(h_n \in K\) are unit vectors, and \((\lambda_{mn})\) is a positive-definite matrix. Then
\[
S((\text{Id} \otimes \Omega)(\rho)) \geq S(\rho) + \sum_n \pi_n S(\Omega(|h_n\rangle\langle h_n|)),
\]
where \(\pi_n = \lambda_{nn}\) and \(\Omega\) is an arbitrary quantum channel.
Proof. We suppose that a state $\rho \in \mathcal{S}(H \otimes K)$ has form (18). Following [10], we define the orthogonal projection $P$ by the formula

$$P = \sum_n |e_n\rangle\langle e_n| \otimes |h_n\rangle\langle h_n|.$$  \hspace{1cm} (19)

Consequently, $P\rho = \rho P = \rho$, and hence $\text{supp } \rho \subset \text{supp } P$.

Applying Proposition 1 to the state $\rho$ and the channel $\Phi = \text{Id} \otimes \Omega$, we obtain

$$S((\text{Id} \otimes \Omega)(\rho)) - S(\rho) \geq -\text{Tr}((\text{Id} \otimes \Omega)(\rho) \log(\text{Id} \otimes \Omega)(P)).$$

Further, by (19), the right-hand side of this inequality is represented as

$$\text{Tr}((\text{Id} \otimes \Omega)(\rho) \log(\text{Id} \otimes \Omega)(P)) = \text{Tr}\left(\sum_{n,m} \lambda_{nm} |e_n\rangle\langle e_m| \otimes \Omega(|h_n\rangle\langle h_m|) \log(\Omega(|h_k\rangle\langle h_k|))\right) = \text{Tr}\left(\sum_n \lambda_{nn} \Omega(|h_n\rangle\langle h_n|) \log(\Omega(|h_n\rangle\langle h_n|))\right).$$

The theorem is proved.

**Corollary 1.** We suppose that $\rho \in \mathcal{S}(H \otimes K)$ has the form

$$\rho = \sum_{n,m} \lambda_{nm} |e_n\rangle\langle e_m| \otimes |h_n\rangle\langle h_m|,$$ \hspace{1cm} (20)

where $(e_n)$ is an orthonormal basis in $H$, $h_n \in K$ are unit vectors, and $(\lambda_{mn})$ is a positive-definite matrix. Then

$$S((\text{Id} \otimes \Omega)(\rho)) - S(\rho) \geq S_{\Omega}(\text{Tr}_H(\rho)).$$

**Proof.** The proof follows immediately from the inequality

$$\sum_n \pi_n S(\Omega(|h_n\rangle\langle h_n|)) \geq S_{\Omega}(\sigma),$$

where $\sigma = \sum_n \pi_n |h_n\rangle\langle h_n|$. 

**Corollary 2.** Relation (8) holds for the dephasing channel $\Phi$ acting as

$$\Phi(|e_n\rangle\langle e_m|) = \lambda_{nm} |e_n\rangle\langle e_m|.$$

**Proof.** Given a unit vector $e \in H \otimes K$ and an orthonormal basis $(e_n)$ in $H$, there exist unit vectors $h_n \in K$ and complex numbers $\nu_n$, $\sum_n |\nu_n|^2 = 1$, such that

$$|e\rangle\langle e| = \sum_{n,m} \nu_n \bar{\nu}_m |e_n\rangle\langle e_m| \otimes |h_n\rangle\langle h_m|.$$ 

As a result, we obtain the state

$$\rho = (\Phi \otimes \text{Id})(|e\rangle\langle e|) = \sum_{n,m} \lambda_{nm} \nu_n \bar{\nu}_m |e_n\rangle\langle e_m| \otimes |h_n\rangle\langle h_m|,$$

satisfying the conditions in Theorem 1 if we replace $\lambda_{nm}$ with $\lambda_{nm} \nu_n \bar{\nu}_m$, which leads to the required result.
We now set \( H = L^2(\mathbb{R}) \) and consider the generalized eigenvectors \( |x\rangle \) of the position operator \( \hat{x} \) acting on \( H \) as
\[
(\hat{x} f)(x) = xf(x), \quad f(\cdot) \in H,
\]
such that
\[
\hat{x}|x\rangle = x|x\rangle, \quad x \in \mathbb{R}. \tag{21}
\]

Applying Proposition 1, we also prove the following statement.

**Theorem 2.** We suppose that \( \rho \in \mathcal{S}(H \otimes K) \) has the form
\[
\rho = \int_{\mathbb{R}^2} \lambda(x, y) |x\rangle \langle y| \otimes |h_x\rangle \langle h_y| \, dx \, dy, \tag{22}
\]
where \( h_x \in K \) are unit vectors and \( \lambda(x, y) \) is a positive-definite function. Then
\[
S((\text{Id} \otimes \Omega)(\rho)) \geq S(\rho) + \int_{\mathbb{R}} \pi(x) S(\Omega(|h_x\rangle \langle h_x|)) \, dx,
\]
where \( \pi(x) = \lambda(x, x) \), and \( \Omega \) is an arbitrary quantum channel.

**Proof.** We suppose that a state \( \rho \in \mathcal{S}(L^2(\mathbb{R}) \otimes K) \) has form (22). We define an orthogonal projection \( P \) by the formula
\[
P = \int_{\mathbb{R}} |x\rangle \langle x| \otimes |h_x\rangle \langle h_x| \, dx. \tag{23}
\]
It is straightforward to verify that \( \rho P = P \rho = \rho \). Therefore,
\[
\text{supp} \, \rho \subset \text{supp} \, P.
\]

Applying Proposition 1 to the state \( \rho \) and the channel \( \Phi = \text{Id} \otimes \Omega \), we obtain
\[
S((\text{Id} \otimes \Omega)(\rho)) - S(\rho) \geq -\text{Tr}((\text{Id} \otimes \Omega)(\rho) \log(\text{Id} \otimes \Omega)(P)).
\]

Taking (22) and (23) into account, for the right-hand side of this inequality, we obtain
\[
\text{Tr}((\text{Id} \otimes \Omega)(\rho) \log(\text{Id} \otimes \Omega)(P)) = \int_{\mathbb{R}^2} \lambda(x, y) |x\rangle \langle y| \otimes \Omega(|h_x\rangle \langle h_y|) \, dx \, dy \int_{\mathbb{R}} |z\rangle \langle z| \otimes \log(\Omega(|h_x\rangle \langle h_z|)) \, dz = \int_{\mathbb{R}} \lambda(x, x) \Omega(|h_x\rangle \langle h_x|) \log(\Omega(|h_x\rangle \langle h_x|)) \, dx.
\]

**Corollary 3.** We suppose that \( \rho \in \mathcal{S}(H \otimes K) \) has the form
\[
\rho = \int_{\mathbb{R}^2} \lambda(x, y) |x\rangle \langle y| \otimes |h_x\rangle \langle h_y| \, dx \, dy,
\]
where \( h_x \in K \) are unit vectors and \( \lambda(x, y) \) is a positive-definite function. Then
\[
S((\text{Id} \otimes \Omega)(\rho)) - S(\rho) \geq S(\text{Tr}_H(\rho)),
\]
where \( \Omega \) is an arbitrary quantum channel.
Proof. The proof follows immediately from the inequality
\[ \int_{\mathbb{R}} \pi(x) S(\Omega(|h_x\rangle\langle h_x|)) \, dx \geq S_\Omega(\sigma), \]
where \( \sigma = \int_{\mathbb{R}} \pi(x)|h_x\rangle\langle h_x| \, dx. \)

Corollary 4. Relation (12) is satisfied for the generalized dephasing channel \( \Phi \) acting as
\[ \Phi(|x\rangle\langle y|) = \lambda(x, y)|x\rangle\langle y| \]
and an arbitrary channel \( \Omega. \)

Proof. Given a unit vector \( e \in L^2(\mathbb{R}) \otimes K, \) there exists a measurable function \( x \rightarrow h_x \) acting from the real line \( \mathbb{R} \) to unit vectors \( h_x \in K \) and a function \( \nu(x), \int_{\mathbb{R}} |\nu(x)|^2 \, dx = 1, \) such that
\[ |e\rangle\langle e| = \int_{\mathbb{R}^2} \nu(x)\tilde{\nu}(y)|x\rangle\langle y| \otimes |h_x\rangle\langle h_y| \, dx \, dy. \] (24)

Applying the generalized dephasing channel \( \Phi \) to (24), we obtain the state
\[ \rho = (\Phi \otimes \text{Id})(|e\rangle\langle e|) = \int_{\mathbb{R}^2} \lambda(x, y)\nu(x)\tilde{\nu}(y)|x\rangle\langle y| \otimes |h_x\rangle\langle h_y| \, dx \, dy. \] (25)

It satisfies the condition of Theorem 2 if we replace \( \lambda(x, y) \) with \( \lambda(x, y)\nu(x)\tilde{\nu}(y). \)

3. Quantum phase-damping channels

We recall that a quantum dephasing channel \( \Phi \) defined by formula (7) is said to be a phase-damping channel if condition (13) is satisfied. Analogously, starting from a generalized dephasing channel \( \Phi \) defined by (10), we can introduce a generalized phase-damping channel if the condition
\[ \lambda(x, y) = \lambda(x - y), \quad \lambda(-x) = \tilde{\lambda}(x), \] (26)
is satisfied. Because the kernels \( \lambda_{nm} \) in (7) and \( \lambda(x, y) \) in (10) are positive definite, so are the functions \( \lambda_n \) and \( \lambda(x), \) i.e.,
\[ \sum_{n, m} \lambda_{n-m}c_n\bar{c}_m \geq 0, \quad \sum_{n, m} \lambda(x_n - x_m)c_n\bar{c}_m \geq 0 \]
for any choice of \( c_n \in \mathbb{C} \) and \( x_n \in \mathbb{R} \) and \( 1 \leq n \leq N < +\infty. \) Hence, we can use the following theorem (Bochner’s theorem [11]) to classify all phase-damping channels.

Theorem 3. We suppose that \( f \) is a positive-definite function on a locally compact Abelian group \( G \) normalized by the condition \( f(e) = 1. \) Then there exists a unique probability measure \( \mu \) on the dual group \( \hat{G} \) such that
\[ f(g) = \int_{\hat{G}} \langle \hat{h}, g \rangle \, d\mu(\hat{h}). \]

We consider three possible cases corresponding to \( G = \mathbb{Z}_N, G = \mathbb{Z}, \) and \( G = \mathbb{R}. \)
3.1. Finite dimension. We suppose that the Hilbert space $H$ has a finite dimension, $\dim H = N < +\infty$. In accordance with (7),

$$\Phi(\langle e_n | e_m \rangle) = \lambda_{m-n} |e_n \rangle \langle e_m|, \quad 0 \leq n, m < N.$$ 

Then the following theorem holds.

**Theorem 4.** The complex numbers $(\lambda_n)$ are the discrete Fourier transforms of a probability distribution $(\pi_n)_{n=0}^{N-1}$ determined by the formula

$$\lambda_n = \sum_{k=0}^{N-1} e^{2\pi n k / N} \pi_k.$$ 

Moreover, there exists a unitary operator $U : H \rightarrow H$ and an orthonormal basis $(f_n)$ in $H$ such that

$$Uf_n = f_{n+1} \mod N, \quad 0 \leq n \leq N - 1,$$

and

$$\Phi(\rho) = \sum_{n=0}^{N-1} \pi_n U^n \rho U^{*n}, \quad \rho \in \mathcal{S}(H).$$

**Proof.** It is more convenient here to give a direct proof without using Bochner’s theorem. We consider the operator $T$ acting in $\mathbb{C}^N$ by the formula

$$(T\nu)_m = \sum_{n=0}^{N-1} \lambda_{m-n} \nu_n,$$

where $\nu = (\nu_0, \ldots, \nu_{N-1}) \in \mathbb{C}^N$. Because $\lambda_n$ is a positive-definite function, we obtain

$$(\nu, T\nu) \geq 0.$$ (27)

Taking the Parseval equality for the discrete Fourier transform into account, we find that (27) results in

$$\sum_{n,m=0}^{N-1} \pi_n |\nu_n|^2 \geq 0$$

for any choice of complex numbers $(\nu_n)$. This implies that $\pi_n \geq 0$. On the other hand,

$$\sum_{n=0}^{N-1} \pi_n = \lambda_0 = 1.$$ 

We then consider the unitary operator $U$ acting in $H$ as

$$U|e_n\rangle = e^{2\pi n i / N} |e_n\rangle, \quad 0 \leq n < N.$$ (28)

We define the orthonormal basis $(f_n)$ in $H$ by the formula

$$|f_n\rangle = \sum_{m=0}^{N-1} e^{2\pi n m i / N} |e_m\rangle.$$ 

Applying operator (28) to the vectors $(f_n)$, we obtain

$$U|f_n\rangle = \sum_{m=0}^{N-1} e^{2\pi (n+1) m i / N} |e_m\rangle = |f_{n+1}\rangle.$$
3.2. Infinite dimension. We now let \( \dim H = +\infty \). We fix an orthonormal basis \((e_n)\) and consider a quantum phase-damping channel \( \Phi \) defined by (2).

**Theorem 5.** The complex numbers \((\lambda_n)\) are the Fourier transforms of a probability measure \( \mu \) on the unit circle \( T \) such that
\[
\lambda_n = \int_T e^{2\pi nti} d\mu(t).
\]
Moreover, there exists a unitary representation \( t \to U_t \) of the multiplicative group \( T \) in \( H \) such that
\[
\Phi(\rho) = \int_T U_t \rho U_t^* d\mu(t), \quad \rho \in \mathcal{S}(H).
\]

**Proof.** By Bochner’s theorem, there exists a probability measure \( \mu \) such that
\[
\lambda_n = \int_T e^{2\pi nti} d\mu(t).
\]
We define a unitary representation \( T \ni t \to U_t \) by the formula
\[
U_t |e_n\rangle = e^{2\pi nti} |e_n\rangle, \quad n \in \mathbb{Z}.
\]
We consider a quantum channel \( \tilde{\Phi} \) of the form
\[
\tilde{\Phi}(\rho) = \int_T U_t \rho U_t^* d\mu(t), \quad \rho \in \mathcal{S}(H).
\]
It follows that
\[
\tilde{\Phi}(|e_n\rangle \langle e_m|) = \int_T e^{2\pi(n-m)ti} d\mu(t) = \lambda_{n-m} |e_n\rangle \langle e_m|.
\]
Hence, \( \tilde{\Phi} = \Phi \).

3.3. The generalized phase-damping channel. We now consider the case \( H = L^2(\mathbb{R}) \). The generalized quantum phase-damping channel is then defined by the formula
\[
\Phi(|x\rangle \langle y|) = \lambda(x - y) |x\rangle \langle y|. \quad (29)
\]

**Theorem 6.** The function \( \lambda(x) \) in (29) is the Fourier transform of a probability measure \( \mu \) on the line \( \mathbb{R} \):
\[
\lambda(x) = \int_\mathbb{R} e^{ixy} d\mu(y).
\]
Moreover, there exists a strong continuous one-parameter group of unitary operators \( t \to U_t \), \( U_0 = I \), such that
\[
\Phi(\rho) = \int_\mathbb{R} U_t \rho U_t^* d\mu(t), \quad \rho \in \mathcal{S}(H).
\]

**Proof.** By Bochner’s theorem, there exists a probability measure \( \mu \) on the line \( \mathbb{R} \) such that
\[
\lambda(x) = \int_\mathbb{R} e^{ixy} d\mu(y).
\]
We define a strong continuous group of unitary operators \( (U_t) \) by the formula
\[
(U_t \psi)(x) = e^{itx} \psi(x), \quad \psi \in H.
\]
We consider a quantum channel $\tilde{\Phi}$ of the form

$$\tilde{\Phi}(\rho) = \int_{\mathbb{R}} U_t \rho U_t^* d\mu(t), \quad \rho \in \mathcal{S}(H).$$

It follows that for $\psi, \phi, \xi \in H$, we have

$$\langle \tilde{\Phi}(\langle \psi | \phi \rangle \xi)(x) \rangle = \int_{\mathbb{R}^2} e^{it(x-y)} \psi(x) \phi(y) \xi(y) dy d\mu(t) = \int_{\mathbb{R}} \lambda(x-y) \psi(x) \phi(y) \xi(y) dy.$$

This means that $\tilde{\Phi} = \Phi$.

4. Conclusion

We have derived a nontrivial lower bound on the entropy gain under the action of an arbitrary quantum channel affecting only one part of the system for a class of bipartite quantum states (Corollaries 1 and 3). Based on this result, we bounded the output entropy for the tensor product of the dephasing quantum channel and an arbitrary channel from below (Corollaries 2 and 4). Finally, we introduced a classification of quantum phase-damping channels resulting as special cases of the dephasing channels (Theorems 4–6).

Acknowledgments. The author is grateful to all participants of the seminar “Quantum Probability, Statistics, Information” at the Steklov Mathematical Institute for the useful discussions. The author especially thanks A. S. Holevo and S. Mancini, whose careful reading and remarks allowed improving the exposition.

This work is supported by the Russian Science Foundation (Grant No. 14-21-00162).

REFERENCES

1. R. Alicki, “Isotropic quantum spin channels and additivity questions,” arXiv:quant-ph/0402080v1 (2004).
2. A. S. Kholevo, Dokl. Math., 82, 730–731 (2010).
3. A. S. Holevo and M. E. Shirokov, Commun. Math. Phys., 249, 417–430 (2004); arXiv:quant-ph/0306196v2 (2003).
4. M. B. Ruskai, “Some open problems in quantum information theory,” arXiv:0708.1902v1 [quant-ph] (2007).
5. I. Devetak, M. Junge, C. King, and M. B. Ruskai, Commun. Math. Phys., 266, 37–63 (2006).
6. I. Devetak and P. W. Shor, Commun. Math. Phys., 256, 287–303 (2005); arXiv:quant-ph/0311131v1 (2003).
7. A. S. Holevo, Theory Probab. Appl., 51, 92–100 (2007).
8. G. G. Amosov and S. Mancini, Quantum Inf. Comput., 9, 594–609 (2009).
9. A. S. Holevo, Problems Inform. Transmission, 44, 171–184 (2008).
10. G. G. Amosov, Problems Inform. Transmission, 49, 224–231 (2013).
11. W. Rudin, Fourier Analysis on Groups, Wiley, New York (1990).