EXISTENCE OF POSITIVE SOLUTIONS TO A SEMI-LINEAR ELLIPTIC SYSTEM WITH A SMALL ENERGY/CHARGE RATIO

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Abstract. We prove the existence of positive solutions to a system of $k$ non-linear elliptic equations corresponding to standing-wave $k$-uples solutions to a system of non-linear Klein-Gordon equations. Our solutions are characterised by a small energy/charge ratio, appropriately defined.

Introduction

Given the real numbers $0 < m_1 \leq m_2 \leq \cdots \leq m_k$, we show the existence of solutions to the non-linear elliptic system

\begin{equation}
-\Delta u_j + (m^2_j - \omega^2_j)u_j + \partial_z_i G(u) = 0, \quad 1 \leq j \leq k
\end{equation}

on the constraint

$$M_\sigma := \{(u, \omega) \in H \times \Sigma | C_j(u, \omega) = \sigma_j \}$$

for some $\sigma \in (0, +\infty)^k$. We used the notation

$$H := H^1(\mathbb{R}^n, \mathbb{R}^k), \quad \Sigma := [0, +\infty)^k.$$  

We also define

$$H_r := H^1_r(\mathbb{R}^n, \mathbb{R}^k), \quad M'_\sigma := M_\sigma \cap H_r$$

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where, by definition, $u \in H^1_r(\mathbb{R}^n, \mathbb{R}^k)$ if $u \in H^1(\mathbb{R}^n, \mathbb{R}^k)$ and $u^j(x) = u^j(y)$ if $|x| = |y|$, a.e.

for every $1 \leq j \leq k$. On the Hilbert spaces $H$ and $H_r$, we consider the norm induced by the scalar product

$$(u, v)_H := \sum_{j=1}^{k} (u_j, v_j)_{H^1}.$$ 

Solutions to $(E)$ with the variational characterisation above are interesting by several means: critical points of $E$ over $M_\sigma$ correspond to standing-wave $k$-uples solutions to the system of non-linear Klein-Gordon equations

$$(k\text{-NLKG}) \quad \partial_{tt} u_j - \Delta_x u_j + m_j^2 u_j + \partial_z G(u) = 0, \quad 1 \leq j \leq k$$

through the map

$$(u, \omega) \mapsto (e^{-i\omega_1 t} u_1(x), \ldots, e^{-i\omega_k t} u_k(x)).$$

Secondly, if we denote $H^1(\mathbb{R}^n, \mathbb{C}^k) \times L^2(\mathbb{R}^n, \mathbb{C}^k)$ by $X$, on solutions to $(k\text{-NLKG})$ the quantities

$$(\text{Energy}) \quad E : X \to \mathbb{R},$$

$$E(\phi, \phi_t) := \frac{1}{2} \int_{\mathbb{R}^n} |D\phi|^2 + |\phi_t|^2 + \frac{1}{2} \int_{\mathbb{R}^n} \sum_{j=1}^{k} (m_j^2 \phi_j^2 + 2k^{-1} G(\phi))$$

$$(\text{Charges}) \quad C_j : X \to \mathbb{R},$$

$$C_j(\phi, \phi_t) := -\text{Im} \int_{\mathbb{R}^n} \overline{\phi_j} \phi_t^j.$$ are constant (under the assumption $G(u) = G(|u_1|, \ldots, |u_k|)$) and

$$E(u, \omega) = E(u_1, \ldots, u_k, -i\omega_1 u_1, \ldots, -i\omega_k u_k)$$

$$C(u, \omega) = C(u_1, \ldots, u_k, -i\omega_1 u_1, \ldots, -i\omega_k u_k).$$

Such equalities turned out to be crucial to prove the orbital stability of standing-wave solutions to the scalar NLKG in [2], and to a coupled NLKG in [8]. Finally, according to [3], solutions $v$ to the scalar NLKG with initial datum $\Phi \in X$ such that the energy/charge ratio

$$\Lambda(\Phi) := \frac{E(\Phi)}{m C(\Phi)} < 1$$

have a non-dispersive property. We do not address in this work the orbital stability or dispersion. We use the notation

$$m := m_1, \quad H^*_r := H_r \setminus 0, \quad \Sigma_* := \Sigma \setminus 0.$$
and assume that $G$ is continuously differentiable and
\begin{align}
(A_0) \quad & G(z) = G(|z_1|, \ldots, |z_k|); \\
(A_1) \quad & F(z) := G(z) + \frac{1}{2} \sum_{j=1}^{k} m^2 z_j^2 \geq 0, \quad G(0) = 0; \\
(A_2) \quad & |DG(z)| \leq c(|z|^{p-1} + |z|^{q-1}), \quad 2 < p \leq q < \frac{2n}{n-2}; \\
(A_3) \quad & \alpha := \inf_{z \in \Sigma} \frac{F(z)}{|z|^2} < \frac{m^2}{2},
\end{align}

for every $1 \leq j \leq k$
\begin{align}
(A_4) \quad & \alpha_j := \inf_{\sum_{i \neq j} z_i^2 \neq 0} \frac{F(z)}{\sum_{i \neq j} z_i^2} > \alpha.
\end{align}

Under the assumptions above, we can prove the following

**Theorem (Main).** There exists an open subset $\Omega \subset (0, +\infty)^k$ such that the infimum of $E$ is achieved on $M^r$ for every $\sigma \in \Omega$.

The technique we use is similar to the one adopted in [4] in the scalar case $k = 1$. Therein it is shown that if a minimising sequence $(u_n, \omega_n)$ for $E$ over $M^r$ is such that $\omega_n \to \omega < m$, then a subsequence of $(u_n)$ converges on $H^1$. The existence of such sequences is provided by the inequality
\begin{equation}
(2) \quad \inf_{H^r \times \Sigma} \Lambda < \inf_{H^r \times \Sigma^m} \Lambda
\end{equation}

where
$$
\Lambda(u, \omega) := \frac{E(u, \omega)}{C(u, \omega)}
$$

and
$$
\Sigma^m = \Sigma \cap \{z \geq m\}.
$$

In higher dimension, $\Sigma^m$ should be replaced by
$$
\Sigma^m := \bigcup_{j=1}^{k} \Sigma^m_j
$$

where
$$
\Sigma^m_j = \{z \in \Sigma \mid z_j \geq m_j\}.
$$

A direct attempt to prove the inequality (2) lead to minimise $\Lambda(u, \cdot)$ over the set $\Sigma^m$, whose boundary consists of $3^k - 1$ pieces each of them leading to a different condition on the non-linear term $F$. We believe that all these conditions include $(A_4)$.

So, rather than proving (2), we show in Lemma Coercive that when $\Lambda(u_n, \omega_n)$ converges to its infimum, each component of $\omega_n$ converges to $\sqrt{2\alpha} < m$. 

1. Properties of the functional $E$

We recall some properties of the functional $E$. We include the proof of them only for the sake of completeness, as they are similar to the scalar case [2].

**Proposition 1.** Suppose that $G$ fulfills the assumptions $(A_1)$ and $(A_2)$. Then, $E$ is continuously differentiable; if $\sigma \in (0, +\infty)^k$, then $E$ is coercive on $M_\sigma$.

**Proof.** The continuity and the differentiability of $E$ follows from analogous techniques used in theorems on bounded domains as [1, Theorem 2.2 and 2.6, p. 16,17]. For a detailed proof we also refer to [8, Proposition 2].

Let $(u, \omega) \in M_\sigma$ and set $E = E(u, \omega)$. By $(A_2)$, we have

$$\omega_i \leq \frac{2E}{\sigma_i}, \quad \|Du\|_{L^2}^2 \leq 2E. \quad (3)$$

By $(A_2)$ there exists $\varepsilon > 0$ such that

$$F(u) \geq m^2|u|^2/4, \text{ if } |u| \leq \varepsilon. \quad (4)$$

We have

$$E \geq \int_{|u| \geq \varepsilon} F(u) + \int_{|u| < \varepsilon} F(u).$$

From (4), it follows that

$$\|u\|_{L^2(|u|<\varepsilon)}^2 \leq 4E/m^2. \quad (5)$$

On the other hand, by the Sobolev inequality

$$\int_{|u| \geq \varepsilon} |u|^2 = \varepsilon^{2-2^*} \int_{|u| \geq \varepsilon} \varepsilon^{2^*-2} |u|^2 \leq \varepsilon^{2-2^*} \int_{|u| \geq \varepsilon} |u|^{2^*} \leq c^{2^*} \varepsilon^{2-2^*} \|Du\|_{L^2}^{2^*} \quad (6)$$

where $c$ is the constant in the proof of [6, Théorème IX.9,p. 165]. From (5) and (6)

$$\|u\|_{L^2}^2 \leq \frac{4E}{m^2} + 2c^{2^*} \varepsilon^{2-2^*} E.$$

Along with (3), we obtained that the sub-levels of $E$ are bounded, then $E$ is coercive. \hfill \qed

Hereafter, we assume that $\sigma_j > 0$ for every $1 \leq j \leq k$.

**Proposition 2.** Let $(u_n, \omega_n) \subset M'_\sigma$ be a Palais-Smale sequence and $\omega_n \to \omega$ such that $\omega_i < m_i$. Then $(u_n)$ has a converging subsequence.

**Proof.** By Proposition 1, $(u_n)$ is bounded. Thus, by [5, Theorem A.1], we can suppose that

$$u_n^j \to u_j \text{ in } H^1_p, \quad u_n^j \to u_j \text{ in } L^p \cap L^q \quad (7)$$
for every $1 \leq j \leq k$. Because $(u_n, \omega_n)$ is a Palais-Smale sequence, there are
\begin{equation}
(\lambda_n) \subset \mathbb{R}, \quad (v_n, \eta_n) \subset H^r \times \mathbb{R}^k
\end{equation}
such that
\begin{equation}
DE(u_n, \omega_n) = \sum_{j=1}^{k} \lambda_n^j DC_j(u_n, \omega_n) + (v_n, \eta_n), \quad (v_n, \eta_n) \to 0.
\end{equation}
We multiply (8) by $(0, e_j) \in \{0\} \times \mathbb{R}^k$ and obtain
\begin{equation}
\omega_n^j \| u_n^j \|_{L^2}^2 = \lambda_n^j \| u_n^j \|_{L^2}^2 + \eta_n^j
\end{equation}
whence
\begin{equation}
\lambda_n^j = \omega_n^j - \frac{\eta_n^j \omega_n^j}{\sigma_j}.
\end{equation}
We multiply (8) by $(\phi, 0) \in H^r \times \{0\}$ and obtain
\begin{equation}
s\sum_{j=1}^{k} (Du_n^j, D\phi_j)_{L^2} + m_n^2(u_n^j, \phi_j)_{L^2} + \int_{\mathbb{R}^n} DG(u_n) \cdot \phi
\end{equation}
\begin{equation}
+ \sum_{j=1}^{k} (\omega_n^j)^2(u_n^j, \phi_j)_{L^2} - 2 \sum_{j=1}^{k} \lambda_n^j \omega_n^j(u_n^j, \phi_j)_{L^2} = (v_n, \phi)_H
\end{equation}
which, by (9), becomes
\begin{equation}
s\sum_{j=1}^{k} (Du_n^j, D\phi_j)_{L^2} + m_n^2(u_n^j, \phi_j)_{L^2} + \int_{\mathbb{R}^n} DG(u_n) \cdot \phi
\end{equation}
\begin{equation}
- \sum_{j=1}^{k} (\omega_n^j)^2(u_n^j, \phi_j)_{L^2} = (v_n, \phi)_H - 2 \sum_{j=1}^{k} \lambda_n^j \omega_n^j(u_n^j, \phi_j)_{L^2}.
\end{equation}
From $(A_1)$, (10) can be written as
\begin{equation}
s\sum_{j=1}^{k} (Du_n^j, D\phi_j)_{L^2} + (m_n^2 - \omega_n^j)(u_n^j, \phi_j)_{L^2}
\end{equation}
\begin{equation}
= (v_n, \phi)_H - \sum_{j=1}^{k} \beta_n^j(u_n^j, \phi_j)_{L^2}
\end{equation}
\begin{equation}
- \int_{\mathbb{R}^n} (DG(u_n) - DG(u)) \cdot (u_n - u)
\end{equation}
where
\begin{equation}
\beta_n^j := \left( \omega_n^j - (\omega_n^j)^2 - \frac{2m_n^j \omega_n^j}{\sigma_j} \right) \to 0, \quad 1 \leq j \leq k.
\end{equation}
Given a pair of integers \((n, m)\), taking the difference of the equations, (11\(_n\)) and (11\(_m\)) with \(\phi = u_n - u_m\), we obtain

\[
\sum_{j=1}^{k} \|Du_n^j - Du_m^j\|_{L^2}^2 + (m_j^2 - \omega_j^2 + \beta_n^j + \beta_m^j)\|u_n^j - u_m^j\|_{L^2}^2
\]

\[
= (v_n - v_m, u_n - u_m)_H - \int_{\mathbb{R}^n} \left( DG(u_n) - DG(u_m) \right) \cdot (u_n - u_m).
\]

Thus from the assumption \(\omega_j < m_j\) and (12), there exists \(c_0 > 0\) such that

\[
\|u_n - u_m\|_H^2 \leq c_0 \sum_{j=1}^{k} \left( \|Du_n^j - Du_m^j\|_{L^2}^2 + (m_j^2 - \omega_j^2 + \beta_n^j + \beta_m^j)\|u_n^j - u_m^j\|_{L^2}^2 \right)
\]

and

\[
(v_n - v_m, u_n - u_m)_H \leq \|u_n - u_m\|_H (\gamma_n + \gamma_m)
\]

where

\[
\gamma_n := \|v_n\|_H \to 0.
\]

We have

\[
\int_{\mathbb{R}^n} \left( \sum_{j=1}^{k} \left( \|u_n^j - u_m^j\|_{L^p} + \|u_n^j - u_m^j\|_{L^q} \right) \right)
\]

it is convenient to estimate each of the two summand of the inequality above as follows: by [6, Corollaire IX.10, p. 165]

\[
\|u_n^j - u_m^j\|_{L^p} = \|u_n^j - u_m^j\|_{L^2}^{1/2} \|u_n^j - u_m^j\|_{L^2}^{1/2}
\]

\[
\leq \|u_n^j - u_m^j\|_{L^2}^{1/2} \|u_n^j - u_m^j\|_{L^2}^{1/2}
\]

\[
\leq \delta_{n,m}^{p}\|u_n^j - u_m^j\|_{L^2}^{1/2}
\]

where

\[
\delta_{n,m}^p := \max_{1 \leq j \leq k} \|u_n^j - u_m^j\|_{L^2}^{1/2}
\]

is infinitesimal by (7). By the H"older inequality, we have

\[
\sum_{j=1}^{k} \|u_n^j - u_m^j\|_{L^2}^{1/2} \leq k^{4/3} \|u_n - u_m\|_{H^1}^{1/4}
\]

whence

\[
\int_{\mathbb{R}^n} \left( DG(u_n) - DG(u_m) \right) \cdot (u_n - u_m)
\]

\[
\leq k^{4/3} (\delta_{n,m}^p + \delta_{n,m}^q) \|u_n - u_m\|_{H^1}^{1/4}.
\]
Now, putting together (14, 15, 18) we obtain
\[(c_0)^{-1}\|u_n - u_m\|_{H}^{7/8} \leq c_1(\gamma_n + \gamma_m) + \delta_n + \delta_m\]
where
\[c_1 = \sup_{n,m} \left( \sum_{j=1}^{k} \|u_n^j - u_m^j\|_{H^1}^2 \right)^{3/4}.\]
Then each of \((u_n^j)\) is a Cauchy sequence in \(H^1\) for every \(1 \leq j \leq k\), thus converges to \(v_j \in H^1\). From (7), \(v_j = u_j\), thus \(u_n \to u\) in \(H_\tau\). □

2. Properties of \(\Lambda\)

We define the following energy/charge ratio
\[\Lambda(u, \omega) := \frac{E(u, \omega)}{\sum_{j=1}^{k} C_j(u, \omega)}\]
and introduce the notation
\[a(u) := \frac{1}{2} \int_{\mathbb{R}^n} |Du|^2 + \int_{\mathbb{R}^n} F(u), \quad b_j(u) := \int_{\mathbb{R}^n} u_j^2;\]
If we fix \(u \in H_\ast\), we have the smooth function defined on \(\Sigma_\ast\)
\[\Lambda(u, \cdot) : \Sigma \to \mathbb{R}, \quad \omega \mapsto \Lambda(u, \omega) = \frac{1}{2} \cdot \frac{2a(u) + \sum_{j=1}^{k} b_j(u)\omega^2}{\sum_{j=1}^{k} b_j(u)\omega_j}.
\]
It is not hard to check, arguing by induction on \(k\), that the following properties hold for \(\Lambda(u, \cdot)\):

(i) is non-negative and achieves its infimum in a (unique) interior point lying on the principal diagonal. We denote this point by \(\omega(u)\) and each of its components by \(\xi(u)\);
(ii) there holds
\[\Lambda(u, \omega(u)) = \xi(u), \quad \xi(u)^2 = \frac{2a(u)}{\sum_{j=1}^{k} b_j(u)}.
\]

**Proposition 3.** \(\inf_{H_\ast} \xi = \sqrt{2\alpha}\).

**Proof.** That the right member is not greater than the left one, follows from the definition of \(\alpha\). In fact,
\[\xi(u)^2 = \frac{\int_{\mathbb{R}^n} |Du|^2 + 2\int_{\mathbb{R}^n} F(u)}{\sum_{j=1}^{k} b_j(u)} \geq \frac{\int_{\mathbb{R}^n} |Du|^2 + 2\alpha\int_{\mathbb{R}^n} |u|^2}{\int_{\mathbb{R}^n} |u|^2} \geq 2\alpha,\]
where in the last inequality we neglected the gradient terms. In order to prove the opposite inequality, we define
\[u_R(x) = \begin{cases} 
  z & \text{if } |x| \leq R \\
  (1 + R - |x|)z & \text{if } R \leq |x| \leq R + 1 \\
  0 & \text{if } |x| \geq R + 1.
\end{cases}\]
where \( z \in \Sigma \) is an arbitrary point and \( R > 0 \). We compute its gradient
\[
Du^j_R(x) = \begin{cases} 
0 & \text{if } |x| \leq R \text{ or } |x| \geq R + 1 \\
-\frac{z_j x}{|x|} & \text{otherwise.}
\end{cases}
\]
By standard computations, we have
\[
\|u^j_R\|_{L^2} = \mu(B_1) R^n |z_j|^2 + O(R^{n-1}),
\]
\[
\|Du^j_R\|_{L^2} = O(R^{n-2}),
\]
where \( B_1 \) is the unit ball of \( \mathbb{R}^n \) and \( \mu(B_1) \) is its Lebesgue measure.

Then,
\[
\xi(u_R)^2 = 2\mu(B_1) R^n F(z) + o(R^n) = o(1) + \frac{2F(z)}{|z|^2}.
\]
Taking the limit as \( R \to +\infty \), we obtain
\[
\inf_{H^*_\omega} \xi^2 \leq \frac{2F(z)}{|z|^2}
\]
for every \( z \in \Sigma \). Because \( z \) was chosen arbitrarily, we obtain the conclusion.

Looking at the behaviour of \( \Lambda(u, \cdot) \), one can easily deduce that sequences converging to the minimum value converge to the minimum point. The next lemma exploits the uniform behaviour of \( \Lambda \) on \( u \).

**Lemma** (Coercive). For every \( \varepsilon > 0 \) there exists \( \eta \) such that
\[
\Lambda(u, \omega) < \sqrt{2\alpha + \eta}
\]
implies
\[
|\omega_j - \sqrt{2\alpha}| < \varepsilon
\]

**Proof.** For every \( 1 \leq j \leq k \) and \( u \in H_* \), we define
\[
B_j(u) = \frac{b_j(u)}{\sum_{j=1}^k b_j(u)}.
\]
We divide the proof in three steps.

**Step 1.** We show that if \( k \geq 2 \) and \( \eta \) is small enough, there exists \( \delta_0 \in (0, 1) \) such that
\[
B_j(u) \in (\delta_0, 1 - \delta_0).
\]
It is useful to define \( \alpha_* := \min \{\alpha_j \mid 1 \leq j \leq k\} \). Due to \((A_4)\) we have \( \alpha < \alpha_* \). By property (ii) of \( \Lambda \)
\[
\sqrt{2\alpha + \eta} > \Lambda(u, \omega) \geq \xi(u);
\]
we fix $1 \leq j \leq k$. We have
\[
\xi(u)^2 = \frac{\|Du\|_{L^2}^2 + 2 \int_{\mathbb{R}^n} F(u)}{\sum_{j=1}^k b_j(u)} = \frac{\|Du\|_{L^2}^2 + 2 \int_{\mathbb{R}^n} F(u)}{\sum_{j \neq s} b_j(u)} \cdot \frac{1}{1 + B_j(u)} \geq \frac{2 \alpha_j}{1 + B_j(u)}
\]
where in the last inequality we neglected the gradient terms and used the notation of the assumption $(A_4)$. From (20) and the inequality above, we obtain
\[
\sqrt{2 \alpha + \eta} > \frac{\sqrt{2 \alpha_j}}{\sqrt{1 + B_j(u)}}
\]
whence
\[
(21) \quad B_j(u) > \frac{2 \alpha_j}{(\sqrt{2 \alpha + \eta})^2} - 1 \geq \frac{2 \alpha_s}{(\sqrt{2 \alpha + \eta})^2} - 1 =: \delta_0.
\]
Thus, if $\delta_0 > 0$, the obtain a bound from below for $B_j(u)$. Thus, we require
\[
(22) \quad \eta < \sqrt{2 \alpha_s} - \sqrt{2 \alpha}
\]
which gives $B_j(u) > \delta_0$ for every $1 \leq j \leq k$. Because
\[
\sum_{j=1}^k B_j(u) = 1
\]
it follows that
\[
B_j(u) = 1 - \sum_{h \neq j} B_h(u) \leq 1 - (k - 1) \delta_0 \leq 1 - \delta_0.
\]

**Step 2.** If $\Lambda(u, \omega) < \sqrt{2 \alpha + \eta}$, then $\omega$ is bounded from above. If $\eta$ is chosen as in (22) and $k \geq 2$ then
\[
\Lambda(u, \omega) \geq \frac{\sum_{j=1}^k B_j \omega_j^2}{2 \sum_{j=1}^k B_j \omega_j} \geq \frac{\delta_0}{2(1 - \delta_0)} \cdot \sum_{j=1}^k \omega_j^2.
\]
Thus,
\[
\sum_{j=1}^k \omega_j^2 \leq 2C_0 \cdot \sum_{j=1}^k \omega_j
\]
where
\[
C_0 := \frac{(\sqrt{2 \alpha + \eta})(1 - \delta_0)}{\delta_0}
\]
Thus,
\[
(23) \quad \omega_j < C_0(1 + \sqrt{k}), \quad 1 \leq j \leq k.
\]
When $k = 1$, \[
\sqrt{2 \alpha + \eta} > \Lambda(u, \omega) \geq \omega/2
\]
thus,
\[
\omega < 2(\sqrt{2\alpha} + \eta).
\]

Step 3. We conclude the proof of the lemma. When \( k \geq 2 \),
\[
\eta \geq \Lambda(u, \omega) - \Lambda(u, \omega(u)) = \Lambda(u, \omega) - \xi(u)
\]
\[
= \frac{1}{2} \left( \xi^2 + \sum_{j=1}^{k} B_j \omega_j^2 - 2 \sum_{j=1}^{k} B_j \omega_j \xi \right)
\]
\[
= \frac{1}{2} \sum_{j=1}^{k} B_j (\omega_j - \xi)^2
\]
\[
= \frac{1}{2} \sum_{j=1}^{k} B_j \omega_j
\]
\[
\geq \frac{\delta_0}{2(1 - \delta_0) C_0 (\sqrt{k} + 1)} \sum_{j=1}^{k} (\omega_j - \xi)^2
\]
the last inequality follows from the bounds on \( \omega \) (23) and on \( B_j \) from Step 1 and Step 2. Thus,
\[
\frac{2\eta(1 - \delta_0)^2(\sqrt{k} + 1)(\sqrt{2\alpha} + \eta)}{\delta_0^2} > (\omega_j - \xi)^2.
\]
Because \( \xi < \sqrt{2\alpha} + \eta \),
\[
(25) \quad |\omega_j - \sqrt{2\alpha}| < \sqrt{\eta} \left( \sqrt{\eta} + \frac{1 - \delta_0}{\delta_0} \cdot \left( 2(\sqrt{2\alpha} + \eta)(\sqrt{k} + 1) \right)^{1/2} \right)
\]
for every \( 1 \leq j \leq k \). Because the term on the right member of the inequality above is \( O(\sqrt{\eta}) \), the proof is complete when \( k \geq 2 \). When \( k = 1 \), by (24)
\[
\eta > \Lambda(u, \omega) - \xi(u) = \frac{1}{2\omega} (\omega - \xi)^2 \geq \frac{1}{4(\sqrt{2\alpha} + \eta)} (\omega - \xi)^2
\]
then
\[
|\omega - \xi| < 2 \left( \eta(\sqrt{2\alpha} + \eta) \right)^{1/2}
\]
whence
\[
(26) \quad |\omega - \sqrt{2\alpha}| < \sqrt{\eta} \left( \sqrt{\eta} + 2 \left( \sqrt{2\alpha} + \eta \right)^{1/2} \right)
\]
\[
\square
\]

Proof of the Theorem Main. Let \((u', \omega')\) be such that
\[
\Lambda(u', \omega') < \sqrt{2\alpha} + \eta
\]
where \( \eta \) is chosen in such a way that the right term in (25) (for \( k \geq 2 \)) or (26) (when \( k = 1 \)) is not greater than
\[
\frac{1}{2} (m - \sqrt{2\alpha}).
\]
We define
\[ \sigma_j := \omega_j' \int_{\mathbb{R}^n} (u'_j)^2. \]
Clearly \((u', \omega') \in M_{r, \sigma}^r\). Now, let us take a minimizing sequence \((u_n, \omega_n)\) of \(E\) over \(M_{r, \sigma}^r\). By the Ekeland’s theorem [11, Theorem 5.1, p. 48], we can suppose that \((u_n, \omega_n)\) is a Palais-Smale sequence. Then, there exists \(n_0 \in \mathbb{N}\) such that
\[ \Lambda(u_n, \omega_n) \leq \Lambda(u', \omega') = \Lambda(u', \omega') < \sqrt{2\alpha} + \eta. \]
if \(n \geq n_0\). Thus
\[ \Lambda(u_n, \omega_n) < \sqrt{2\alpha} + \eta, \quad n \geq n_0. \]
By the preceding lemma, we have
\[ |\omega_n' - \sqrt{2\alpha}| < \frac{1}{2}(m - \sqrt{2\alpha}); \]
up extract a subsequence from \((\omega_n')\), we can suppose that each of the \((\omega_n')\) converge to some \(\omega_j\). Therefore
\[ m - \omega_j = m - \sqrt{2\alpha} + \sqrt{2\alpha} - \omega_j \geq \frac{1}{2}(m - \sqrt{2\alpha}) > 0. \]
By Proposition 2, we obtain that \(E\) achieves its infimum on \(M_{\sigma}\). Finally, we observe that the subset of \((0, +\infty)^k\)
\[ \Omega := \left\{ \sigma \in (0, +\infty)^k \mid \frac{f(\sigma)}{\sum_{j=1}^{k} \sigma_j} < \sqrt{2\alpha} + \eta \right\} \]
is open. In fact, let \(\sigma_0 \in \Omega\) and \((u_0, \omega_0)\) be a minimiser of \(E\) over \(M_{\sigma_0}\). Thus,
\[ \Lambda(u_0, \omega_0) < \sqrt{2\alpha} + \eta. \]
Given an arbitrary \(\sigma\), we define
\[ \omega'_\sigma := \omega'_0 \frac{\sigma_j}{\sigma_0}. \]
Using the continuity of \(\Lambda\) on \(\omega\), it can be showed that
\[ \Lambda(u_0, \omega_\sigma) = \Lambda(u_0, \omega_0) + O(|\sigma - \sigma_0|). \]
Thus, if \(|\sigma - \sigma_0|\) is small enough,
\[ \Lambda(u, \omega_\sigma) < \sqrt{2\alpha} + \eta \]
which concludes the proof. \(\square\)

**Corollary.** There exists \(\eta_0\) such that, for every \(\eta < \eta_0\) there exists \((u_\eta, \omega_\eta)\) such that \(u_\eta\) is a solution to (E)
\[ \Lambda(u_\eta, \omega_\eta) < \sqrt{2\alpha} + \eta, \quad \omega'_\eta - \sqrt{2\alpha} < \eta \]
for every \(1 \leq j \leq k\).

SOLUTIONS WITH A SMALL ENERGY/CHARGE RATIO
Proof. The existence of \((u_\eta, \omega_\eta)\) follow from Theorem Main. All we need to prove is that \(u_\eta > 0\) and solves the elliptic system in (E). So, let \(\sigma \in (0, \infty)^k\) be as in Theorem Main and \((u, \omega) \in M^r_\sigma\) a minimiser of \(E\) over \(M^r_\sigma\). From \((A_\eta)\),

\[
(v, \omega) := (|u_1|, \ldots, |u_k|, \omega)
\]

is also a minimiser of \(E\) over \(M^r_\sigma\) and, thus, a constrained critical point. There is a natural action of the orthogonal group \(O(n, \mathbb{R})\) on 

\[
O(n) \times H^1(\mathbb{R}^n, \mathbb{R}^k) \times [0, +\infty)^k \to H^1(\mathbb{R}^n, \mathbb{R}^k) \times [0, +\infty)^k
\]

\[
(G, u, \omega) \mapsto G \cdot (u, \omega) := (u(Gx), \omega)
\]

this action restricts to \(M_\sigma\) and the set of fixed point is \(M^r_\sigma\). Moreover, \(E\) is invariant for the action

\[
E(u, \omega) = E(u(Gx), \omega).
\]

By the symmetric criticality principle [10, §0], \((u, \omega)\) is a critical point of \(E\) over \(M_\sigma\). Thus, each of the equations in (E) can be written as

\[
- \Delta v_j + c_j(x)v_j = 0
\]

where

\[
c_j(x) = \begin{cases} 
    m_j^2 - \omega_j^2 + \frac{\partial_j G(v)}{v_j} & \text{if } v_j(x) \neq 0 \\
    m_j^2 - \omega_j^2 & \text{if } v_j(x) = 0.
\end{cases}
\]

From \((A_2)\)

\[
|c_j(x)| \leq m_j^2 - \omega_j^2 + c (|v_j|^{p-2} + |v_j|^{q-2}).
\]

Thus, for every bounded domain \(V \subset \mathbb{R}^n\), \(c_j \in L^\infty(V)\), because \(v_j\) is continuous. Then, we can apply the maximum principle to the elliptic equation (27) (for example, [7, Lemma 1,p. 556]) and conclude that \(v_j > 0\) on \(V\). Because this holds for every \(V, v_j > 0\) on \(\mathbb{R}^n\). Hence \(u_\eta\) has a sign for every \(\eta\). Up to adjusting the signs of \(u^*_\eta\), \((u, \omega)\) is the sought solution to (E).

Some remarks are in order.

Concentration of minimising sequences. If we add the requirement

\[
(A_5) \quad \int_{\mathbb{R}^n} F(u_1^*, \ldots, u_k^*) \leq \int_{\mathbb{R}^n} F(u)
\]

where \(u^*_j\) denotes the decreasing rearrangement of \(u_j\), then minimisers of \(E\) over \(M^r_\sigma\) are minimisers of \(E\) over \(M_\sigma\). We define

\[
I(\sigma) := \inf_{M_\sigma} E.
\]

Moreover, if for every minimiser \((u, \omega)\) there holds

\[
(A_6) \quad \lim E(u_1(\cdot + y^1_n), u_2(\cdot + y^2_n), \ldots, u_k(\cdot + y^k_n), \omega) > E(u, \omega)
\]
if $|y_n^j - y_n^h|$ is not bounded for some $j \neq h$, then it is natural to expect the sub-additivity property of $I$, that is

$$I(\sigma) < I(\sigma') + I(\sigma - \sigma')$$

for every $\sigma'$ such that $\sigma' \neq \sigma$ and $\sigma'_j \leq \sigma$ for every $1 \leq j \leq k$. Thus, by means of the concentration-compactness Lemma, it would follow that a minimising sequence exhibits a concentration behaviour.

**Some example of non-linearity.** It might be surprising the fact that in our solutions all the frequencies tend to converge in the interval $(\sqrt{2\alpha}, m)$ regardless of the relations between $m_j$ and $m_h$ for $j \neq h$. This follows from the assumption ($A_4$): when the non-linearity $G$ does not have coupling terms, that is

$$(30) \quad G(z) = G_1(z_1) + \cdots + G_k(z_k)$$

the system (E) reduces to $k$ scalar elliptic equations

$$-\Delta u_j + (m_j^2 - \omega_j^2)u_j + G_j'(u_j) = 0$$

each of them can be solved separately as in [4] or [2] in order to obtain positive solutions. By the Derrick-Pohozaev identity and the maximum principle it follows

$$m_j > \omega_j > \sqrt{2\alpha}, \quad 1 \leq j \leq k.$$ 

So, if $G$ is as in (30), the frequencies $\omega_j$ have a different behaviour from the one proved in Theorem Main, where

$$\sqrt{2\alpha} < \omega_j < m, \quad 1 \leq j \leq k.$$ 

In fact, a non-linearity as in (30) does not satisfy the assumption ($A_4$): given $z \neq (0, \ldots, 0)$, we have

$$\frac{F(z)}{|z|^2} \geq \frac{\sum_{j=1}^k \alpha_j |z_j|^2}{|z|^2} \geq \min_{1 \leq j \leq k} \alpha_j.$$

Also, it is more simple to treat each equation of the case (30) separately, using the result of [2] or the theorem when $k = 1$.

Some non-linearities $G$ satisfying assumptions ($A_1$–$A_4$) are given by

$$(z_1 z_2)^p - (z_2 z_3)^p - (z_1 z_3)^p + |z|^q, \quad z \in \Sigma$$

$G(z) := G(|z_1|, |z_2|)$

when $k = 2, N = 3$ and

$$1 < p, \quad 2p < q < 5.$$ 

When $k = 3, N = 3$, we can define

$$(z_1 z_2 z_3)^p - (z_2 z_3)^p - (z_1 z_3)^p - (z_1 z_2 z_3)^p + |z|^q, \quad z \in \Sigma$$

$G(z) := G(|z_1|, |z_2|, |z_3|).$
and

\[ 2 < 2p_i < q < 5, \text{ for } 1 \leq i \leq 3 \]
\[ 3 < 3p_4 < q. \]

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