Modular Symmetries in $Z_N$ Orbifold Compactified String Theories with Wilson Lines.

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ABSTRACT

Target space modular symmetries relevant to string loop threshold corrections are studied for $Z_N$ orbifold compactified string theories containing Wilson line background fields.
Orbifold compactifications of string theory \[1, 2\] possess various moduli, which are background fields corresponding to marginal deformations of the underlying conformal field theory including radii and angles of the underlying six dimensional torus. The spectrum of the states for an orbifold theory is invariant under certain discrete transformations on the moduli, together with the winding numbers and momenta, referred to as modular symmetries \[3-10\]. These modular symmetries also manifest themselves in the string loop threshold corrections \[11-16\] that are of crucial importance for unification of gauge coupling constants, though in this case it is only the transformations on the moduli associated with the fixed planes of twisted sectors of the theory that are relevant. It is these particular symmetries we shall focus on here.

In the case where there are no Wilson lines and in addition all twisted sector fixed planes are such that the 6-torus $T_6$ can be decomposed into a direct sum $T_2 \oplus T_4$ with the fixed plane lying in $T_2$, the group of modular symmetries is a product of $SL(2, Z)$ factors \[12\] one for each of the $T$ or $U$ moduli associated with the fixed planes. However, when there are twisted sectors with fixed planes that cannot be decomposed in this way the group of modular symmetries (relevant to the string loop threshold corrections) is in general a product of congruence subgroups \[17-19\] of $SL(2, Z)$. It is also known \[20, 21, 14\] that Wilson lines can break $SL(2, Z)$ modular symmetries. The present paper is directed towards finding modular symmetries relevant to string loop threshold corrections in orbifold theories with Wilson lines, but with the simplifying feature that all fixed planes of the twisted sectors allow the decomposition of the six torus discussed above.

In the presence of metric, antisymmetric tensor and Wilson line background fields, the momenta $P_L$ and $P_R$ for the left and right movers in the lattice basis\(^*\) take the form (see, for example, ref. \[20\])

\[ P_L = \left( \frac{m}{2} + (g - h)n - \frac{1}{2} A^i Cl, \quad l + An \right) \equiv \left( \tilde{P}_L, \quad l + An \right) \quad (1) \]

\(^*\) for string slope parameter $\alpha'$ taken to be $\frac{1}{2}$
and
\[ P_R = \left( \frac{m}{2} - (g + h)n - \frac{1}{2}A^tCA, 0 \right) \equiv \left( \tilde{P}_R, 0 \right) \quad (2) \]

where \( m \) and \( n \) are the momentum and winding number for the compact manifold, \( l \) is the momentum on the \( E_8 \times E_8' \) lattice, \( g, B, A \) are the metric, antisymmetric tensor and Wilson line background fields, \( C \) is the Cartan metric for the \( E_8 \times E_8' \) lattice, and
\[ h = B + \frac{1}{4}A^tCA \quad (3) \]

(Constant shifts in the \( E_8 \times E_8' \) lattice due to point group embeddings of Wilson lines are not relevant for these purposes [20] and have been suppressed throughout.)

If \( \theta \) generates the point group of the orbifold, it is the \( \theta^k \) twisted sectors with fixed planes that are relevant to string loop threshold corrections as mentioned earlier. The action of the point group on the winding numbers and momenta can always be written in the form [20, 21]
\[ u \rightarrow u' = Ru \quad (4) \]

where
\[ u = \begin{pmatrix} n \\ m \\ l \end{pmatrix} \quad (5) \]

and
\[ \mathcal{R} = \begin{pmatrix} Q & 0 & 0 \\ \alpha & Q^* & (1 - Q^*)A^tC \\ A(I - Q) & 0 & I \end{pmatrix} \quad (6) \]

with
\[ \alpha = \frac{1}{2}A^tCA(I - Q) + \frac{1}{2}(I - Q^*)A^tCA + 2Q^*\delta \quad (7) \]

where \( \delta \) is an antisymmetric integer matrix, which we shall take to be zero in what follows and \( * \) denotes the inverse transpose. In (6), the matrix \( Q \) defines the action
of the point group element $\theta$ on the basis vectors $e_a$ of the lattice of the six-torus

$$\theta : \ e_a^i \to e_b^i Q_{ba}$$  \hspace{1cm} (8)

The boundary conditions for the $\theta^k$ twisted sector require

$$P_L = \theta^k P_L, \quad P_R = \theta^k P_R$$  \hspace{1cm} (9)

which in terms of the matrix $R$ are the conditions

$$R^k u = u.$$  \hspace{1cm} (10)

For convenience let the fixed plane of $\theta^k$ be the first complex plane. Then, $Q^k$ is block diagonal with the $2 \times 2$ identity matrix as its leading block. The solution of (10) is

$$(I - Q^k)n = 0$$  \hspace{1cm} (11)

and

$$(I - Q^{*k})\hat{p} = 0$$  \hspace{1cm} (12)

where

$$\hat{p} = m - \frac{1}{2} A^t C An - A^t Cl.$$  \hspace{1cm} (13)

Consequently, $n$ and $\hat{p}$ can only take non-zero entries for their first two components.

If we use the variables $n$, $\hat{p}$ and $l + An$, then the problem is two dimensional so far as $n$ and $\hat{p}$ are concerned. It is thus convenient to define

$$u_\perp = \begin{pmatrix} n \\ \hat{p} \end{pmatrix}$$  \hspace{1cm} (14)

In this basis, the action of the point group element $\theta$ is simply

$$u_\perp \to u'_\perp = R_\perp u_\perp,$$  \hspace{1cm} (15)
with $l + An$ left invariant by the point group, where

$$R_\perp = \begin{pmatrix} Q & 0 \\ 0 & Q^* \end{pmatrix}$$  \hspace{1cm} (16)$$

In (14) and (16), $n$ and $\hat{p}$ are now understood to be restricted to their first 2 components and $Q$ to the $2 \times 2$ block that acts on the first complex plane (corresponding to the fixed plane in the $\theta^k$ twisted sector.)

In terms of the variables $n$, $\hat{p}$ and $l + An$, the momenta $P_L$ and $P_R$ take the form

$$P_L \equiv \left( \tilde{P}_L, \ l + An \right) = \left( \hat{p} \frac{1}{2} + (g - B)n, \ l + An \right)$$  \hspace{1cm} (17)$$

and

$$P_R \equiv \left( \tilde{P}_R, \ 0 \right) = \left( \hat{p} \frac{1}{2} - (g + B)n, \ 0 \right)$$  \hspace{1cm} (18)$$

where $g$ and $B$ are now understood to be restricted to the $2 \times 2$ block that acts on the first complex plane. The components $\tilde{P}_L$ and $\tilde{P}_R$ on the compact manifold are just as for the case of a 2 dimensional orbifold without Wilson lines except for the replacement of $m$ by $\hat{p}$. The world sheet momentum $P$ is given by

$$P = \frac{1}{2} \tilde{P}_L^t g^{-1} \tilde{P}_L + \frac{1}{2} (l + An)^t C(l + An) - \frac{1}{2} \tilde{P}_R^t g^{-1} \tilde{P}_R$$

$$= \frac{1}{2} u_\perp^t \eta u_\perp + \frac{1}{2} (l + An)^t C(l + An)$$  \hspace{1cm} (19)$$

where

$$\eta = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}$$  \hspace{1cm} (20)$$

and the Hamiltonian $H$ is given by

$$H = \frac{1}{2} \tilde{P}_L^t g^{-1} \tilde{P}_L + \frac{1}{2} (l + An)^t C(l + An) + \frac{1}{2} \tilde{P}_R^t g^{-1} \tilde{P}_R$$

$$= \frac{1}{2} u_\perp^t \Xi u_\perp + \frac{1}{2} (l + An)^t C(l + An)$$  \hspace{1cm} (21)$$

where
\[ \Xi = \begin{pmatrix} 2(g - B)g^{-1}(g + B) &Bg^{-1} \\ -g^{-1}B &\frac{1}{2}g^{-1} \end{pmatrix} \] (22)

We look for modular symmetries that act only on the components \( \tilde{P}_L \) and \( \tilde{P}_R \) associated with the compact manifold, leave the momentum \( l + An \) associated with the internal \( E_8 \times E'_8 \) degrees of freedom invariant and leave the Wilson line \( A \) invariant.

Consider the transformation

\[ u_\perp \to \Omega^{-1}_\perp u_\perp, \] (23)

where \( \Omega_\perp \) is a matrix with integer entries. In order to obtain a well-defined transformation on the integral components of \( m, n \) and \( l \), it is necessary to place restrictions on the integers occurring in \( \Omega_\perp \). We return to this point shortly. For \( \Omega_\perp \) to be consistent with level matching it must leave the world sheet momentum \( P \) invariant and so must satisfy

\[ \Omega^t_\perp \eta_\perp \Omega_\perp = \eta_\perp. \] (24)

The solutions \( \Omega_\perp \) of (24) form an \( O(2,2,\mathbb{Z}) \) group isomorphic to \( SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z}) \times \mathbb{Z}_2 \times \mathbb{Z}_2 \) generated by \( \Omega_T, \Omega_U, \Omega_1 \) and \( \Omega_2 \) where

\[ \Omega_T = \begin{pmatrix} aI &cJ \\ -bJ &dI \end{pmatrix} \] (25)

with

\[ J = \begin{pmatrix} 0 &1 \\ -1 &0 \end{pmatrix} \] (26)

and \( a, b, c \) and \( d \) are integers satisfying

\[ ad - bc = 1 \] (27)
and
\[ \Omega_U = \begin{pmatrix} F & 0 \\ 0 & F^* \end{pmatrix} \] (28)

with
\[ F = \begin{pmatrix} d' & b' \\ c' & a' \end{pmatrix} \] (29)

and \( a', b', c' \) and \( d' \) are integers satisfying
\[ a'd' - b'c' = 1 \] (30)

also
\[ \Omega_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \] (31)

and
\[ \Omega_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \] (32)

The modular symmetries must be compatible with the point group so that
\[ \Omega_\perp R_\perp = R_\perp \Omega_\perp \] (33)

For \( Z_N \) orbifolds generated by point group element \( \theta \), fixed planes only occur in \( \theta^k \) twisted sectors with \( k = 2, 3 \) or \( 4 \). In other words, the action of the point group in that complex plane is \( Z_2, Z_3 \) or \( Z_4 \). Requiring also that the six- torus \( T_6 \) can be decomposed as direct sum \( T_2 \oplus T_4 \) with the fixed plane lying in \( T_2 \), fixed planes with \( Z_2 \) point group occur for the \( Z_6 - II - d, Z_8 - II - b \) and \( Z_{12} - II \) orbifolds, in the notation of ref. [22], fixed planes with \( Z_3 \) point group occur for the \( Z_3, Z_6 - I \),
$Z_6 - II - b - c - d$ and $Z_{12} - II - b$ orbifolds, and fixed planes with $Z_4$ point group for the $Z_8 - I$ orbifold. For the cases of $Z_2$, $Z_3$ and $Z_4$ fixed planes $Q$ in (16) takes the form

$$Q = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad \text{and } Q = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \quad (34)$$

respectively. Then, (33) is satisfied by both $\Omega_T$ and $\Omega_U$ for $Z_2$ fixed planes, but by only $\Omega_T$ for $Z_3$ and $Z_4$ fixed planes. In the $n$, $\hat{p}$ basis, the moduli associated with fixed planes are constrained by

$$Q^t g Q = g \quad (35)$$

and

$$Q^t B Q = B \quad (36)$$

for consistency with point group, as in the case without Wilson lines [23] except that $g$ and $B$ are restricted to the $2 \times 2$ blocks acting on the fixed plane. If we define the moduli $T$ and $U$ as in the case without Wilson lines [12]

$$T = T_1 + iT_2 = 2 \left( B_{12} + i \sqrt{\det g} \right) \quad (37)$$

and

$$U = U_1 + iU_2 = \frac{1}{g_{11}} \left( g_{12} + i \sqrt{\det g} \right). \quad (38)$$

Then for $Z_2$ fixed planes, both $T$ and $U$ are consistent with the point group but for $Z_3$ and $Z_4$ fixed planes only $T$ survives as a continuous modulus and $U$ takes the fixed values

$$U = -\frac{1}{2} (1 + i\sqrt{3}), \quad \text{and } U = -\frac{1}{2} (1 + i) \quad (39)$$

for $Z_3$ and $Z_4$ respectively. The $\Xi_{\perp}$ for the Hamiltonian in (21) and (22) may be
written in terms of the moduli as

$$\Xi_\perp = \frac{1}{T_2 U_2} \begin{pmatrix} W & X \\ Y & Z \end{pmatrix}$$  \hspace{1cm} (40)

with

$$W = |T|^2 \begin{pmatrix} 1 & U_1 \\ U_1 & |U|^2 \end{pmatrix}$$  \hspace{1cm} (41)

$$X = Y^t = T_1 \begin{pmatrix} -U_1 & 1 \\ -|U|^2 & U_1 \end{pmatrix}$$  \hspace{1cm} (42)

$$Z = \begin{pmatrix} |U|^2 & -U_1 \\ -U_1 & 1 \end{pmatrix}$$  \hspace{1cm} (43)

Modular transformations $\Omega_\perp$ that leave the Hamiltonian invariant act on the matrix $\Xi_\perp$ as

$$\Xi_\perp \rightarrow \Omega_\perp^t \Xi_\perp \Omega_\perp.$$  \hspace{1cm} (44)

It can then be seen that the modular transformations $\Omega_T$ induce the transformation on the modulus $T$,

$$T \rightarrow \frac{aT + b}{cT + d},$$  \hspace{1cm} (45)

and, in the case of $Z_2$ fixed planes, the modular transformation on the modulus $U$,

$$U \rightarrow \frac{a'U + b'}{c'U + d'},$$  \hspace{1cm} (46)

As mentioned earlier, it is necessary to put restrictions on the integers occurring in $\Omega_\perp$ in order to insure that (23) together with

$$l + An \rightarrow l + An$$  \hspace{1cm} (47)

provide a well defined transformation on the integral components of $m$, $n$ and $l$. 

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In detail, for $\Omega_U$, these conditions are

$$A(I - F^{-1}) \in Z$$  \hspace{1cm} (48)$$

and

$$A^tCA - \frac{1}{2}A^tCAF^{-1} - \frac{1}{2}F^tA^tCA \in Z$$  \hspace{1cm} (49)$$

For $\Omega_T$, the corresponding conditions are

$$cA \in Z$$  \hspace{1cm} (50)$$

$$\frac{c}{2}A^tCA \in Z$$  \hspace{1cm} (51)$$

$$cAJA^tC \in Z$$  \hspace{1cm} (52)$$

$$(1 - d)A - \frac{c}{2}AJA^tCA \in Z$$  \hspace{1cm} (53)$$

$$(1 - a)CA + \frac{c}{2}CAJA^tCA \in Z$$  \hspace{1cm} (54)$$

and

$$(1 - \frac{a}{2} - \frac{d}{2})A^tCA - \frac{c}{4}A^tCAJA^tCA \in Z$$  \hspace{1cm} (55)$$

The integers $a'$, $b'$, $c'$ and $d'$ in (29) are then constrained by (48) and (49), and the integers $a$, $b$, $c$ and $d$ in (25) are constrained by (50)-(55), in a way that depends on the choice of Wilson line $A$. 

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Acceptable Wilson lines must satisfy [20, 21, 24]

\[ A(I - Q) \in Z \]

and

\[ \frac{1}{2} A^t C A (I - Q) + \frac{1}{2} (I - Q^*) A^t C A \in Z \]  \tag{56}

as can be seen from (6) and (7), with \( \delta = 0 \) For the case of \( Z_2 \) fixed plane, a simple example is provided by

\[ A^t = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \]

where we have for simplicity assumed that the embedding is entirely in the first \( E_8 \). Only the first 2 rows of \( A^t \) have been displayed since the other 4 rows can not contribute to the non-zero components of \( \hat{p} \). For this choice of Wilson lines, eqns. (50)-(55) imply that the modular symmetries \( \Omega_T \) on the \( T \) modulus are restricted to the subgroup of \( SL(2, Z) \) characterized by

\[ c = 0 \quad (\text{mod}\ 2), \quad a, d = 1 \quad (\text{mod}\ 2). \]  \tag{57}

On the other hand, (48) and (49) require the modular transformations \( \Omega_U \) on the \( U \) modulus to be restricted to a subgroup of \( SL(2, Z) \) satisfying the constraints

\[ a', d' = 0 \quad (\text{mod}\ 2), \quad b', c' = 1 \quad (\text{mod}\ 2) \]  \tag{58}

or

\[ a', d' = 1 \quad (\text{mod}\ 2), \quad b', c' = 0 \quad (\text{mod}\ 2). \]  \tag{59}

For the case of a \( Z_3 \) fixed plane, a simple example is

\[ A^t = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \]  \tag{60}

Then there is only a \( T \) modulus and the modular transformations \( \Omega_T \) are limited
to the subgroup of $SL(2, Z)$ determined by

$$c = 0 \ (mod \ 3), \quad a, \ d = 1 \ (mod \ 3) \quad (61)$$

The main conclusion we can draw about the presence of Wilson lines in orbifold models, is that they break some of the duality symmetries associated with the moduli of invariant planes. In the specific examples considered above we can see that the $T \rightarrow -1/T$ and $U \rightarrow -1/U$ transformations are broken. A simpler model with this property is compactification on a 1-dimensional orbicircle. The procedure for analysing the duality symmetries of this model including Wilson lines, is as given above for the more realistic case. As before, we look for modular symmetries that act only the components of $P_L$ and $P_R$ associated with the orbicircle, and leave the internal $l + An$ momentum and the Wilson line $A$ invariant. The world sheet momentum $P$ and Hamiltonian $H$ are given by eqns (19) and (21) with

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (62)$$

An orbicircle of radius $R$ is constructed from a circle of the same radius, by modding out by a ($Z_2$-valued ) reflection symmetry. The matrix $\Xi$ in this case is given by

$$\Xi = \begin{pmatrix} 2R^2 & 0 \\ 0 & \frac{1}{2R^2} \end{pmatrix} \quad (63)$$

Duality symmetries leaving $P$ invariant take the form

$$\Omega = \begin{pmatrix} 0 & \rho \\ \frac{1}{\rho} & 0 \end{pmatrix} \quad (64)$$

where $\rho$ is a real parameter. From the constraints on $\rho$, ( analogous to those listed
in eqns (50) - (55) one may easily deduce that

\[ \frac{1}{\rho} \in \mathbb{Z} \]  \hspace{1cm} (65)

and

\[ \frac{1}{2\rho}(A^t C A) \in \mathbb{Z} \]  \hspace{1cm} (66)

The constraint (65) fixes \( \rho = 1 \). On the other hand since the matrix \( Q = -1 \) in this case, \( A^t C A \) is half-integer valued in general, so constraint (66) is not satisfied. In fact the only solution to (65) and (66) is when \( A = 0 \) i.e. the case without Wilson lines, which corresponds to the usual \( 2R^2 \to 1/2R^2 \) duality symmetry. Therefore, even in this relatively simple example, we can see how Wilson lines break the ‘stringy’ duality that exchanges the radius with inverse radius.

In conclusion, we have found a class of target space modular symmetries relevant to string loop threshold corrections where only the components of the momenta \( P_L \) and \( P_R \) associated with the compact manifold transform while the components in the \( E_8 \times E_8' \) internal space and the Wilson lines are left invariant. These symmetries are subgroups of \( SL(2, \mathbb{Z}) \) acting on the moduli \( T \) and \( U \) with the specific subgroup being determined by the form of the Wilson lines.

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