MIXED SEGRE NUMBERS AND
INTEGRAL CLOSURE OF IDEALS

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ABSTRACT. We introduce mixed Segre numbers of ideals which generalize the notion of mixed multiplicities of ideals of finite colength and show how many results on mixed multiplicities can be extended to results on mixed Segre numbers. In particular, we give a necessary and sufficient condition in terms of these numbers for two ideals to have the same integral closure. Also, our theory yields a new proof of a generalization of Rees' theorem that links the integral closure of an ideal to its multiplicity. Finally, we give a quick application of our results to Whitney equisingularity.

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0. Introduction

In 1973, Bernard Teissier [T1] used the multiplicity of ideals that are primary to the maximal ideal \( m \) of a local ring to study the geometry of isolated hypersurface singularities. In fact, to study Whitney conditions, one has to understand the integral closure of the product of the Jacobian ideal and the maximal ideal. Now, Rees showed in his celebrated theorem that two \( m \)-primary ideals \( I \subset J \) of a formally equidimensional local ring have the same integral closures if and only if their multiplicities are equal; see e.g. [T1]. Teissier gave a formula that expresses the multiplicity of the product of \( m \) and an \( m \)-primary ideal in terms of some mixed multiplicities.

Later, in [T2], Teissier defined mixed multiplicities for any pair of \( m \)-primary ideals. And, in [T3], mixed multiplicities were used to give a numerical criterion for two \( m \)-primary ideals to have the same integral closure.

More recently, Terry Gaffney and the author [GG] defined Segre numbers of ideals to extend Teissier’s result on isolated hypersurface singularities to non–isolated ones. Segre numbers are a natural generalization of multiplicities of \( m \)-primary ideals. In fact, they allowed us to extend Rees’ theorem to arbitrary ideals. It seems natural to define also mixed Segre numbers for any pair of ideals, and to use them to find generalizations of Teissier’s results. Indeed, Corollary (4.4) of this work gives a necessary and sufficient criterion in terms of mixed Segre numbers for the integral closure of two ideals to coincide.

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Teissier also proved inequalities relating the multiplicity of the product of two ideals and mixed multiplicities. To prove this, Teissier showed that it is enough to prove such an inequality on a normal surface, and used then the negative definiteness of the intersection matrix of the components of the exceptional divisor of a resolution of singularities of the surface.

Following Teissier’s approach, we show that similar inequalities hold for Segre numbers. However, for such an inequality of Segre numbers of codimension \( k \) to hold, the ideals in question need to have the same behavior in lower codimension. This, in turn, can be ensured by a condition on the mixed Segre numbers of codimension less than \( k \). Again, the proof uses certain properties of the resolution of singularities of normal surfaces (see Section 3). Interestingly, the behaviour of the integral closure of an ideal in codimension \( k \) influences its behaviour in higher codimension; see the remark after Corollary (3.4).

Finally, Teissier used his product formula to derive Minkowski–type inequalities that relate the multiplicity of the product of the two ideals with their individual multiplicities. Similar product formulas are proven here in Section 4. Also, we extend Teissier’s result that these inequalities are equalities if and only if some powers of the two ideals have the same integral closure. However, for general ideals, we need to assume that the Segre cycles of one ideal satisfy a chain condition; see Theorem (4.8).

Finally, we give a new proof of the recent generalization [GG, Corollary (4.9)] of Rees’ theorem.

This paper shows that Segre numbers are a good generalization of multiplicity of ideals. Also, by using ‘induction on the codimension,’ most of the results for ideals of finite colength can be generalized without too many problems to ideals of lower codimension.

The paper is organized as follows. In Section 1, we review Teissier’s theory of mixed multiplicities. Using an easy combinatorial result (1.2), we generalize his results to arbitrary local rings of equidimensional analytic space germs.

Mixed Segre numbers are introduced in Section 2. Then, in Section 3 we study the case of ideals on surfaces. This is the model case which is used in Section 4 to study the general case. For this, we first review an alternative definition of Segre numbers and cycles, and then examine moving and fixed components of these cycles in more detail in (4.1) and (4.2). The criterion for two ideals to have the same integral closure is then given in Corollary (4.4). Then, we discuss product formulas and Minkowski-type inequalities. Finally, a first application to equisingularity theory is given in (4.11).

1. Mixed Multiplicities

(1.1) Setup. Let \((X, 0) \subseteq (\mathbb{C}^N, 0)\) be an analytic germ of pure dimension \( n \). We denote the maximal ideal of its local ring \( \mathcal{O}_{X, 0} \) at 0 by \( m \). Consider two \( m \)-primary ideals \( I_1, I_2 \) in this local ring. Then the multiplicity \( e(I_1) \) of \( I_1 \) equals the dimension of the \( \mathbb{C} \)-vector space

\[ \mathcal{O}_{X, 0}/(f_1, \ldots, f_n) \]
where \( f_1, \ldots, f_n \) are generic linear combinations of generators of \( I_1 \). Inspired by this theorem of Samuel, Teissier defined mixed multiplicities of two ideals as follows:

\[
e_{i,n-i}(I_1, I_2) = \dim O_{X,0}/(f_1, \ldots, f_i, g_1 \ldots, g_i)_{n-i},
\]

where the \( f_j \) are generic linear combinations of generators of \( I_1 \) and the \( g_k \) are generic linear combinations of generators of \( I_2 \). Clearly, this definition implies \( e_{i,n-i}(I_1, I_2) = e_{n-i,i}(I_2, I_1) \).

In [T2], Teissier showed that for normal \( X \) the following inequality holds.

\[
e_{i,n-i}(I_1, I_2)^n \leq e(I_1)^i e(I_2)^{n-i}. \tag{1.1.2}
\]

Furthermore, in [T3], he gave a criterion in terms of mixed multiplicities that determines when the integral closure of two \( m \)-primary ideals coincide. To prove a slight generalization of this result, we need the following observation.

**Lemma.** Let \((a_1, \ldots, a_k), (b_1, \ldots, b_k), (c_1, \ldots, c_k)\) \(k\)-tuples of nonnegative integers. Denote the sum of their elements by \(a, b, c\), respectively. Suppose that for \(i = 1, \ldots, k\) the inequality \(a_i^2 \leq b_i c_i\) obtains. Then \(a = b = c\) if, and only if, for \(i = 1, \ldots, k\) we have \(a_i = b_i = c_i\).

**Proof.** Consider the following inequalities.

\[
a = \sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} \sqrt{b_i c_i} \leq \sqrt{b} \sqrt{c}.
\]

The second inequality is the Cauchy–Schwarz inequality. It is an equality if, and only if, the tuples \(b\) and \(c\) are linearly dependent. In our situation, this is equivalent to \(b_i = c_i\). The claim follows.

We shall prove now Teissier’s result [T3, p.354] in a slightly more general situation.

**Theorem.** Let \((X, 0)\) be an equidimensional complex analytic space germ of dimension \(n\). Let \(I_1, I_2\) be two ideals in the local ring \(O_{X,0}\), primary with respect to the maximal ideal \(m\). Then their integral closures coincide if, and only if, we have

\[
e(I_1) = e_{n-1,1}(I_1, I_2) = \cdots = e_{n-1,n}(I_1, I_2) = e(I_2).
\]

**Proof.** Teissier showed that it is enough to prove the result for 2-dimensional germs. (In his reduction argument [T3, p.348] he assumes that \(X\) is Cohen–Macaulay. It is easy to see that this assumption is unnecessary.) In his original statement, Teissier also assumes that \(X\) is normal, and proves then the statement for normal surface germs using a result of Ramanujam [P] that expresses mixed multiplicities in terms of orders of vanishing along components of the exceptional set of a resolution of singularities of the surface germ.

The above Lemma will allow us to conclude the statement for non–normal surfaces from Teissier’s results. This is the only new ingredient.

We assume now that \(X\) is a surface germ. Let \(n : X \to X\) be its normalization, and assume that the preimage of 0 is formed by points \(x_1, \ldots, x_k\). Consider the
ideals $I_1^{(i)}$ and $I_2^{(i)}$ in the local rings $\mathcal{O}_{\bar{X}, x_i}$ induced by the pullbacks of the ideals on $X$. Then, any path $\phi: (\mathbb{C}, 0) \to (X, 0)$ lifts to a path $\bar{\phi}: (\mathbb{C}, 0) \to (\bar{X}, 0)$. Therefore, by the valuative criterion for integral closure, we have that $I_1$ and $I_2$ have the same integral closure if, and only if, for $i = 1 \ldots, k$ the integral closure of $I_1^{(i)}$ and $I_2^{(i)}$ coincides. By Teissier’s result, this happens if, and only if, we have

$$e(I_1^{(i)}) = e_1,1(I_1^{(i)}, I_2^{(i)}) = e(I_1^{(i)})$$

for all $i$. Using a projection formula for multiplicities [F, (4.3.6)] and (1.1.1), we see

$$e(I_1) = \sum_{i=1}^{k} e(I_1^{(i)}), \quad e(I_2) = \sum_{i=1}^{k} e(I_2^{(i)}), \quad e_{1,1}(I_1, I_2) = \sum_{i=1}^{k} e_{1,1}(I_1^{(i)}, I_2^{(i)}).$$

Our assumption, together with Lemma (1.2), imply that we can apply Teissier’s result to the normal surface $\bar{X}$ and the ideals induced by $I_1$ and $I_2$. This finishes the proof.

(1.4) Corollary. In the Setup (1.1), the following inequalities obtain for $i = 1, \ldots, n$.

$$e_{i,n-i}(I_1, I_2)^n \leq e(I_1)^i e(I_2)^{n-i}, \quad (1.4.1)$$

$$e(I_1 I_2)^{1/n} \leq e(I_1)^{1/n} + e(I_2)^{1/n}. \quad (1.4.2)$$

The second inequality is an equality if, and only if, there exist postive integers $a, b$ so that the integral closures of $I_1^a$, resp. $I_2^b$ coincide.

Proof. The first inequality follows, using Teissier’s arguments [T2], from the the following inequality in the local ring of a surface singularity:

$$e_{1,1}(I_1, I_2)^2 \leq e(I_1) e(I_2).$$

Now, with the notation of the above proof, we have

$$e_{1,1}(I_1, I_2)^2 = \left( \sum e_{1,1}(I_1^{(i)}, I_2^{(i)}) \right)^2 \leq \left( \sum \sqrt{e(I_1^{(i)}) e(I_2^{(i)})} \right)^2 \leq \left( \sum e(I_1^{(i)}) \right) \left( \sum e(I_2^{(i)}) \right).$$

Here, the first inequality is Teissier’s result, the second is the Cauchy–Schwarz inequality.

Finally, the second inequality (1.4.2) follows now from Teissier’s expansion formula for multiplicities. Furthermore, Teissier’s proof shows the last assertion if we replace his argument [T3, p. 354] by the above theorem.
2. Mixed Segre Numbers

(2.1) Setup. Let \((X, 0) \subseteq (\mathbb{C}^N, 0)\) be an equidimensional analytic germ of pure dimension \(n\). Recall from [GG, (2.1) and (2.2)] that polar varieties \(P_k(I, X)\) and Segre cycles \(\Lambda_{k+1}(I, X)\) of an ideal \(I\) of nowhere dense co-support on \(X\) are defined inductively for \(k = 0, \ldots, n-1\) as follows: \(P_0(I, X) = X\); for \(k \geq 1\) we define \(P_k(I, X)\) to be the closure of \(V(f|_{P_{k-1}(I, X)}) - V(I)\), where \(f\) is a general linear combination of generators of \(I\). The \(k\)th Segre cycle \(\Lambda_k(I, X)\) can be defined as the difference of cycles

\[ [V(f|_{P_{k-1}(I, X)})] - [P_k(I, X)]. \]

Although in general both, polar varieties and Segre cycles, depend on the choice of the general linear combinations of generators of \(I\), their multiplicities at 0 are well–defined and denoted by \(m_k(I, X)\) and \(e_k(I, X)\). We will often omit the \(X\) in these notations, and write, for example, \(m_k(I)\) for \(m_k(I, X)\).

For a subspace \(Y\) of \(X\) no component of which is contained in \(V(I)\), we define the \(k\)th Segre number \(e_k(I, Y)\) of \(I\) on \(Y\) to be \(e_k(I', Y)\), where \(I'\) is the ideal induced by \(I\) in the local ring of \(Y\) at 0.

We refer to [GG] for the general development of the theory of Segre numbers and cycles.

In this work, we will mainly use the following two properties of Segre numbers:

1. They are well–behaved with respect to polar varieties ([GG,(2.2.1)]), i.e. for \(i + j \leq n\) and \(i > 0\), we have

\[ e_{i+j}(I) = e_i(I, P_j(I)). \]

2. The one–codimensional Segre number can be computed on a plane section off 0 ([GG,(2.3)]): Let \(p : \mathbb{C}^N \to \mathbb{C}^{n-1}\) be a general linear projection, \(\epsilon\) a general point of \(\mathbb{C}^{n-1}\) close to 0, and \(H = p^{-1}(\epsilon)\). Then, we have

\[ e_1(I) = \sum_{x \in V(I) \cap H} e(I, (X \cap H, x)). \]

Now, let \(I_1, I_2\) be ideals in the local ring \(O_{X,0}\) of codimension at least one. In the following, we fix \(n\)–tuples \(f = (f_1, \ldots, f_n)\) and \(g = (g_1, \ldots, g_n)\) of linear combinations of generators of \(I_1\) and \(I_2\) respectively. For \(k = 1, \ldots, n\) we will denote the \(k\)–tuples \((f_1, \ldots, f_k)\) and \((g_1, \ldots, g_k)\) by \(f^{[k]}\) and \(g^{[k]}\). We denote the ideal generated by \(f^{[k]}\) by \(I_1^{[k]}\), and define \(I_2^{[k]}\) analogously.

We will first examine how many generic generators of an ideal are necessary to define polar varieties and Segre cycles.

(2.2) Proposition. In the above setup, outside the \(k\)–codimensional polar variety \(P_k(I_1)\) of \(I_1\), the ideal \(I_1^{[k]}\) is a reduction of \(I_1\).

Proof. Suppose \(I_1\) is generated by \(m + 1\) elements, and consider the blowups \(B = \text{Bl}_{I_1}X \subseteq X \times \mathbb{P}^m\) and \(B' = \text{Bl}_{I_1^{[k]}}X \subseteq X \times \mathbb{P}^{k-1}\). Then, the identity map on \(X\) induces a map

\[ B = (B \cap (X \times K)) \to B'. \]
induced by a central projection \( \pi : \mathbb{P}^m - K \to \mathbb{P}^{k-1} \) with \((m-k)\)-dimensional center \( K \). Now, as \( f \) is formed by generic linear combinations, the center \( K \) is in general position with respect to \( X \). So, a standard argument shows that \( B \cap (X \times K) \) is of minimal dimension \( n - k \). The image of this intersection in \( X \) equals \( P^f_k(I_1) \).

Note that if \( x \) is point of \( X \) where the fibre \( B(x) \) of \( B \) over \( x \) is of dimension bigger than \( k \), then \( B(x) \cap K \) is non-empty. Hence, the point \( x \) is contained in the mentioned polar variety.

Also, if \( x \) is outside the polar variety, the induced projection \( B(x) \to \mathbb{P}^{k-1} \) is finite. Otherwise, there would be a curve \( C \) in \( B(x) \) that is mapped to one point \( l \) in \( \mathbb{P}^{k-1} \). Now, \( \pi^{-1}(l) \) is isomorphic to \( \mathbb{C}^{m-k+1} = \mathbb{P}^{m-k+1} - K \). As \( B(x) \) is closed, the closure \( \overline{C} \subseteq \mathbb{P}^{m-k+1} \) is also contained in \( B(x) \), and, by Bezout’s theorem, intersects \( K \). This is a contradiction to the assumption on \( x \).

Therefore, the induced projection \( \text{Bl}_{I_1}(X - P^f_k(I_1)) \to \text{Bl}_{I_1^{[k]}}(X - P^f_k(I_1)) \) is finite and generically one-to-one. So, by a well-known characterization of integral dependence (see e.g. [T1, Ch.0, 0.4, p.288]), the result follows.

\[ \text{(2.3) Corollary. In the above setup, we have the equalities:} \]

\[ P^f_k(I_1) = P^{f[k+1]}_k(I_1^{[k+1]}), \quad \Lambda^f_k(I_1) = \Lambda^{f[k+1]}_k(I_1^{[k+1]}). \]

**Proof.** The integral closures of \( I_1 \) and \( I_1^{[k+1]} \) coincide outside \( P^f_{k+1}(I_1) \), therefore the desired equalities hold on \( X - P^f_{k+1}(I_1) \). Now, the \((k+1)\)-codimensional polar variety, which was cut out, is nowhere dense in the \( k \)-dimensional objects in question, so the equality remains when passing to the closure.

Looking back into the definition, we see that the element \( f_{k+1} \) is used in the formation of \( \Lambda_k(I) \) only to distinguish components of \( P_k(I) \) and \( \Lambda_k(I) \) in the cycle \( V(f_k|P_{k-1}(I)) \). So, if we know the support of \( V(I_1) \cap P_{k-1}(I) \), then \( f_{k+1} \) is not needed to define the Segre cycle of codimension \( k \). This observation will lead us to the correct definition of mixed Segre cycles.

\[ \text{(2.4) (Mixed Segre numbers). Before we define mixed Segre numbers in general,} \]

we consider the special case of 2-codimensional mixed Segre cycles and numbers. The first naive approach, following Teissier’s definition in the case of ideals of finite colength, would be to consider the Segre cycle of codimension 2 of \( I_1^{[1]} + I_2^{[1]} \). However, as we have seen in Corollary (2.2), we generally need three elements to define the Segre cycles of codimension 2, unless we give ourselves some set that will play the role of the support of \( I \) as in the above observation. So, we define the mixed Segre cycle as follows: Let \( h_1 \) and \( h_2 \) be two generic linear combinations of \( f_1 \) and \( g_1 \). Define \( P^{1,1}_1(I_1, I_2) \) to be the closure of \( V(h_1) - V(I_1 + I_2) \), and \( P^{1,1}_2(I_1, I_2) \) to be the closure of \( V(h_2|P^{1,1}_1(I_1, I_2)) - V(I_1 + I_2) \). Finally, define the Segre cycle \( \Lambda^{1,1}_2(I_1, I_2) \) to be the part of the 2-codimensional cycle \( V(h_2|P^{1,1}_1(I_1, I_2)) \) supported in \( V(I_1 + I_2) \).

In general, for \( k = 1, \ldots, n \), and positive integers \( i, j \), we define the mixed Segre number of codimension \( k \) as follows: Let \( h \) be a \( k \)-tuple of generic linear combinations of the elements \( f_1, \ldots, f_i, g_1, \ldots, g_j \). Then, we define inductively \( P^{i,j}_k(I_1, I_2) \) to be the closure of \( V(h_k|P^{i,j}_{k-1}(I_1, I_2)) - V(I_1 + I_2) \) and \( \Lambda^{i,j}_k(I_1, I_2) \) to be the part of the cycle \( V(h_k|P^{i,j}_{k-1}(I_1, I_2)) \) supported in \( V(I_1 + I_2) \). Finally, let \( e^{i,j}_k(I_1, I_2) \) be its multiplicity.
We also define \( e^k_0(I_1, I_2) := e_k(I_1) \) and \( e^{0,k}_0(I_1, I_2) := e_k(I_2) \).

In particular, for ideals of finite colength our definition of mixed multiplicities coincide with Teissier’s.

3. The Surface Case

We will first discuss ideals on a normal surface \( X \). We will also use classical results on resolutions of normal surfaces, which we are going to review.

(3.1) (Resolutions of normal surfaces). An outline of the proofs and more references can be found in Fulton’s book [F, Ex. 2.4.4, p.39 and Ex. 7.1.16, p.125].

Let \((X,0)\) be a normal surface, and \( \pi : \tilde{X} \to (X,0) \) a resolution of singularities. Denote the irreducible components of the exceptional fibre \( \pi^{-1}(0) \) by \( E_1, \ldots, E_r \).

Then, for an irreducible curve \( C \subset (X,0) \), there exist unique positive \( a_j \in \mathbb{Q} \) so that for all \( i = 1, \ldots, r \) the equality of zero–cycles in \( \pi^{-1}(0) \) obtains:

\[
\tilde{C} \cdot E_i + \sum_{j=1}^r a_j (E_j \cdot E_i) = 0.
\]

Here, \( \tilde{C} \) is the strict transform of \( C \) by \( \pi \). We define

\[
C' = \tilde{C} + \sum_{j=1}^r a_j E_j.
\]

Then, if \( D \) is a divisor on \( X \), we have \([D'] = [\pi^* D]\).

The main ingredient for the proof is the negative definiteness of the intersection matrix \( \{(E_i, E_j)\}_{i,j} \), where \((E_i, E_j)\) denotes the degree of the intersection product \( E_i \cdot E_j \). This degree is well–defined as the intersection is supported in the complete fibre \( \pi^{-1}(0) \) of \( \tilde{X} \) over \( 0 \).

We will extend the notion of strict transforms to cycles on \( X \) by linearity; again the strict transform of a cycle \( S \) by \( \pi \) will be denoted by \( \tilde{S} \).

For the proof of inequality of mixed Segre numbers on a surface germ, we will need the following modified version of the Cauchy–Schwarz inequality.

(3.2) Lemma. Let \( u, v, w \) be elements of a real vector space \( V \), and assume \( \langle , \rangle \) is a positive definite bilinear form on \( V \). Then, if \( \langle u, w \rangle \geq \langle v, w \rangle \geq 0 \) holds, we have the following inequality

\[
\langle u + w, v \rangle^2 \leq \langle u + w, u \rangle \langle v + w, v \rangle.
\]

Proof. Let \( a = u - v \), and consider the function

\[
f(t) = \langle v + ta + w, v + ta \rangle \langle v + w, v \rangle - \langle v + ta + w, v \rangle^2.
\]

Clearly, \( f(0) = 0 \); we are going to show that its derivative \( f'(t) \) is positive for \( t \geq 0 \). This implies the desired inequality.

Now, an easy computation shows

\[
f'(t) = 2t(\langle a, a \rangle \langle w, v \rangle + \langle a, a \rangle \langle v, v \rangle - \langle a, v \rangle^2)
+ \langle a, w \rangle \langle v + w, v \rangle.
\]

Now, the Cauchy–Schwarz inequality together with the assumptions implies \( f'(t) \geq 0 \). This finishes the proof.
(3.3) Proposition. Assume that $X$ is a normal surface and the equality of one-cycles

$$[V(I_1)]_1 = [V(I_2)]_1$$

hold, where $[Y]_1$ denotes the one-cycle formed by the one-dimensional components of the cycle $[Y]$. Then, the following inequality obtains:

$$e_2^{1,1}(I_1, I_2)^2 \leq e_2(I_1) e_2(I_2).$$

Proof. First, we claim the inequality $e_2^{1,1}(I_1, I_2) \leq e(I_2, P_1(I_1))$. In fact, consider a generic linear combination $h = f_1 + t g_1$ of the generators of $I_1^{[1]} + I_2^{[1]}$. Then, we have

$$I_1^{[1]} + I_2^{[1]}|P_1 h(I_1^{[1]} + I_2^{[1]}) = I_2^{[1]}|P_1 h(I_1^{[1]} + I_2^{[1]}),$$

as $f_1|P_1 h(I_1^{[1]} + I_2^{[1]}) = -t g_1|P_1 h(I_1^{[1]} + I_2^{[1]}).$ Next, consider the space

$$Y = V(f_1 + t g_1) - V(I_1) \times \mathbb{C} \subset X \times \mathbb{C}$$

where $t$ is now the coordinate on $\mathbb{C}$. Then, as we have the equality of cycles

$$[V(I_1)] = [V(I_2)] = [V(I_1^{[1]} + I_2^{[1]})],$$

the fibre of $Y$ over $t = 0$ equals $P_1(I_1)$, while the fibre over non-zero $t$ equals $P_1 f_1 + t g_1(I_1^{[1]} + I_2^{[1]})$; see [GG, (4.3)] for a proof of this. So, by the upper semicontinuity of the multiplicity, and as $g_1$ was chosen generically, we have

$$e_2^{1,1}(I_1, I_2) = e(I_2^{[1]}, P_1 h(I_1^{[1]} + I_2^{[1]})) \leq e(I_2^{[1]}, P_1(I_1)) = e(I_2, P_1(I_1)).$$

Next, consider a resolution of singularities $\pi : \tilde{X} \to X$ as above with the additional property that $\pi^* I_1$ and $\pi^* I_2$ are invertible sheaves, e.g. a resolution of singularities of the blowup of $X$ along $I_1 I_2$. Note that by the normality of $X$, this resolution is still an isomorphism outside the exceptional fibre. In fact, as $X$ is normal, it is smooth outside 0, and so at a point $p$ outside 0 the ideal induced by $I_1$ in $\mathcal{O}_{X,p}$ is principal. Hence, the blowup of $(X, p)$ along $I_1$ is isomorphic to $(X, p)$. Also, we assume that the strict transforms of $P_1(I_1)$ and $\Lambda_1(I_1)$ in $\tilde{X}$ don’t meet. This can be achieved by replacing $\tilde{X}$ by a successive blowup of finitely many points in $\tilde{X}$.

Then, by the above, there exist positive numbers $u_i, v_i, w_i \in \mathbb{Q}$ so that we have for all $j$

$$\left(\Lambda_1(I_1) + \sum w_i E_i, E_j\right) = 0,$$

$$\left(P_1(I_1) + \sum u_i E_i, E_j\right) = 0$$

$$\left(P_1(I_2) + \sum v_i E_i, E_j\right) = 0.$$

Furthermore, consider the divisor on $X$ given by $f_1 = 0$ with associated Weil divisor $[P_1(I_1)] + \Lambda_1(I_1)$. Then, by the above we have $u_i + w_i = \operatorname{ord}_{E_i} \pi^* f_1$. 
Now, by the projection formula and the assumed properties of $\bar{X}$, we have

$$e_2(I_1) = e(\pi^* I_1, P_1(I_1)) = (\Lambda(I_1)^- + \sum (\text{ord}_{E_i} \pi^* (I_1) E_i, P_1(I_1)) = (\sum (\text{ord}_{E_i} \pi^* (I_1) E_i, P_1(I_1)) = -\sum (u_i + w_i) u_j (E_i, E_j).$$

For the last equality, we used that the pullback of $I_1$ is invertible; hence its order of vanishing along $E_i$ equals the order of vanishing of $f_1$ along $E_i$.

Similarly, we obtain

$$e_2(I_2) = -\sum (v_i + w_i) v_j (E_i, E_j)$$

and

$$e(I_2, P_1(I_1)) = -\sum \sum (v_i + w_i) u_j (E_i, E_j).$$

Hence, it is enough to show the inequality $\langle v + w, u \rangle^2 \leq \langle u + w, u \rangle \langle v + w, v \rangle$, where $\langle , \rangle$ denotes the positive definite bilinear form given by the matrix $\{-E_i, E_j\}_{i,j}$.

After exchanging $I_1$ and $I_2$, we may assume $\langle u, w \rangle \geq \langle v, w \rangle$. Hence, to apply the above lemma, it remains to check $\langle v, w \rangle \geq 0$. But, we have

$$\langle v, w \rangle = (\sum v_i E_i, \Lambda(I_1)^-) > 0,$$

as $(\Lambda(I_1)^-, E_i) \geq 0$ for all $i$. Furthermore, as $\Lambda(I_1)$ is a cycle through the origin, its strict transform intersects at least one component of the exceptional fibre. Hence, for at least one index $i$, the intersection number is non-zero. This finishes the proof.

**(3.4) Corollary.** In the situation of the above Proposition (3.3), the integral closures of $I_1$ and $I_2$ coincide if and only if the equalities

$$e_2(I_1) = e_2^{1,1} (I_1, I_2) = e_2(I_2)$$

obtain.

**Proof.** If the two ideals have the same integral closure, then the equalities of 2–codimensional Segre numbers clearly obtain. So, assume now that the equalities hold. If both, $I_1$ and $I_2$, are of finite colength, this is a result of Teissier [T3, p.354]. So, we consider now the case where $I_1$ and $I_2$ are of codimension one; in other words, their one–dimensional Segre cycles are not 0. We also assume that at least one of the two–codimensional Segre numbers in question is not zero. Otherwise, both ideals are principal, and then, by the assumptions, obviously equal.

The equalities imply that the inequality in Proposition (3.3) is an equality. Consider now the derivative $f'(t)$ of the function $f(t)$ in the proof of Lemma (3.2) that was used to prove this equality. It has to be identically to 0 as, by the assumptions, $f(0) = f(1)$ and $f'(t) \geq 0$. As we have seen in the above proof, the product $\langle w, v \rangle$ is positive. Also, by the classical Cauchy–Schwarz inequality, $\langle a, a \rangle \langle v, v \rangle - \langle a, v \rangle^2$ is positive. Hence $\langle a, a \rangle = 0$, and so $a = 0$. Hence, by a characterization of integral closure [LT], the two ideals have the same integral closure. This finishes the proof.

\[\square\]
Note that, for ideals of codimension one, in order to prove that the equalities imply that the ideals have the same integral closure, we only used the fact that the assumed equalities imply that the inequality in Proposition (3.3) is actually an equality. This should be contrasted to Teissier’s results for ideals of finite co-length: In this case, if the inequality is an equality, we can only conclude that some powers of the ideals have the same integral closure.

The stronger result for ideals of codimension one comes from the assumption that we assumed the ideals to have the same integral closure outside 0. As the proof of the inequality of Proposition (3.3) showed, the behavior outside 0 influences their behavior at 0. We will use this observation later on to derive some results on Minkowski–type inequalities.

From the above proposition and the corollary, we can also derive similar results for arbitrary surfaces.

(3.5) Corollary. Let \((X, 0)\) be an arbitrary surface germ, and \(I_1, J\) be two ideals of codimension at least one in the local ring of \(X\) at 0. Assume that the integral closure of \(I_1\) and \(I_2\) at points of \(X\) outside 0 coincide (that is for suitably chosen representatives.) Then the inequality

\[
e_1^{1,1}(I_1, I_2) e_2^2(I_1, I_2) \leq e_2(I_1) e_2^2(I_2)
\]

obtains. Furthermore, \(\bar{I}_1 = \bar{I}_2\) if and only if we have the equality

\[
e_2(I_1) = e_2^{1,1}(I_1, I_2) = e_2(I_2).
\]

Proof. Again, by (1.3), it is enough to prove this ideals of codimension one. Consider the normalization \(n : (\bar{X}, S) \to (X, 0)\) where \(S\) is a finite set of points. Then, the integral closures of \(I_1\) and \(I_2\) coincide if and only if the integral closures of the pullbacks \(n^* I_1\) and \(n^* I_2\) coincide. Furthermore, by the above Proposition (3.3), at each point \(x\) of \(S\) we have an inequality

\[
e_2^{1,1}(n^* I_1, n^* I_2, (\bar{X}, x))^2 \leq e_2(n^* I_1, (\bar{X}, x)) e_2(n^* I_2, (\bar{X}, x)).
\]

Hence, using Cauchy–Schwarz inequality, we get the desired inequality. Also, we see, using Lemma (1.2), that the equality in the statement of the corollary obtains if and only if the above equality holds at each point \(x\) of \(S\). So, the claim follows from the above proposition.

Finally, we can also give a numerical criterion for the integral closure of \(I\) and \(I_2\) to coincide outside 0:

(3.6) Corollary. Let \((X, 0)\) be an arbitrary surface germ, and \(I_1, I_2\) be two ideals of codimension at least one in the local ring of \(X\) at 0. Then \(\bar{I}_1 = \bar{I}_2\) if and only if the following equalities obtain:

\[
e_2(I_1) = e_2^{1,1}(I_1, I_2) = e_2(I_2) \quad \text{and} \quad e_1(I_1) = e_1^{1,1}(I_1, I_2) = e_1(I_2)
\]

Proof. We are going to show that the second equality implies that \(I_1\) and \(I_2\) have the same integral closure outside 0. By symmetry, it is enough to show that \(I_2\) is integrally dependent on \(I_1\) outside 0. So, consider the normalized blowup
For a generic hyperplane $H$ is no smaller than the order of $b^*I_2$ along $C$ is no smaller than the order of $D$ along $C$. This is equivalent to the following claim: For a generic hyperplane $H$ off 0 that intersects $b(C)$ transversally, the ideal $I_2\cap H$ is integrally dependent on $I_1\cap H$. In fact, as $H$ is chosen generically, its preimage $b^{-1}(H)$ intersects $C$ transversally, and the order of $b^*I_2$ along $C$ equals the order of $b^*(I_2|H)$ along $b^{-1}(H)\cap C$. The latter is a component of the exceptional divisor of the blowup of $X\cap H$ along $I_1|X\cap H$.

Now, $(X\cap H, V(I_1)\cap H)$ is a multi-curve germ, and so, by Rees Theorem, it is enough to show that at each point of $V(I_1)\cap H$ the multiplicities of the ideals induced by $I_1$ and $I_1+I_2$ in the local ring of $X\cap H$ coincide. Furthermore, again as we’re working on a curve, both $I_1$ and $I_2$ have a reduction generated by one element. So, $I_1^{[1]}+I_2^{[1]}$ is a reduction of $I_1+I_2$. Now, at a point $x\in V(I_1+I_2)\cap H \subseteq V(I_1)\cap H$, we have
\[ e(I_1+I_2, (X\cap H, x)) \leq e(I_1, (X\cap H, x)). \]
So, as $e_1(I_1) = e(I_1, X\cap H)$ and similarly for $I_1+I_2$, the assumptions imply the equality of sets $V(I_1)\cap H = V(I_1+I_2)\cap H$, and, furthermore, at each point of this set the above inequality is actually an equality. This finishes the proof.

4. The general case

We now return to the setup (2.1) to study mixed Segre numbers in the general case. We do this by reducing it again to the the surface case; however, we will have to deal now with non–local phenomena. The first lemma is a key–ingredient.

For the following, we need an alternative description of the Segre cycles, discussed in [GG, (2.1)]. We review this briefly and discuss the phenomena of moving components in more detail.

(4.1) (Fixed and moving parts of Segre cycles). In the Setup (2.1), consider the blowup
\[ \text{Bl}_X X \subset X \times \mathbb{P}^m, \]
where $m+1$ is the number of a set of generators of $I$, with exceptional divisor $D$. Denote the part of $D$ formed by components mapping to 0 by $D^0$, and the part formed by the other components by $D^{X-0}$. A hyperplane $H \subset \mathbb{P}^m$ gives rise, via pullback by the map $\text{Bl}_X X \to \mathbb{P}^m$ induced by the projection onto the second factor, to a Cartier divisor on the blowup, which we denote again by $H$. Intersecting with $H$ represents the first Chern class of the canonical line bundle of the blowup.

Then, let $H_1, \ldots, H_{n-1}$ be generic hyperplanes in the projective space. Consider the cycle
\[ b_*(H_1 \cap \cdots \cap H_k \cap D^{X-0}) \]
in $X$, where $b$ is the canonical map of the blowup. It is a Segre cycle of $I$ of codimension $k+1$. Finally, the degree of the 0–cycle
\[ H_1 \cap \cdots \cap H_{n-1} \cap D^0 \]
equals the top Segre number $e_n(I)$. 

An easy dimension count shows that for a component $C$ of $D$, the cycle
\[ b_*(H_1 \cap \cdots \cap H_k \cap C) \]
is non–zero if and only if either $b(C)$ is of codimension $k+1$ in $X$, or the image $b(C)$
is of lower codimension and the fibre of $C$ over 0 has dimension bigger than $k$. In
the second case, the non–zero cycle associated to $C$ is called a moving component
of the Segre cycle $\Lambda_{k+1}(I)$. Indeed, it depends on the choice of the hyperplanes $H_i$.

Now, consider a generic linear projection $p : \mathbb{C}^n \to \mathbb{C}^{n-k-1}$ and a general point
$\varepsilon$ in its target close to 0. We will call the plane $L = p^{-1}(\varepsilon)$ of codimension $n-k-1$
a generic $(n-k-1)$–codimensional plane off 0. We consider a small representative
of $X$, again denoted by $X$, the space $X \cap L$, and the ideal sheaf induced by a
representative of $I$. If all these representatives are chosen small enough, the plane
$L$ intersects $\Lambda_{k+1}(I)$ transversally, and the degree of the intersection equals $e_{k+1}(I)$.

As $L$ was chosen generically, its preimage in $\text{Bl}_L X$ also intersects the exceptional
divisor transversally. In fact, $b^{-1}(L)$ is isomorphic to the blowup of $X \cap L$ along the
ideal induced by $I$, and $D \cap L$ is the exceptional divisor of this blowup. So, if
$C$ is a component of $D$ that maps to a subset of codimension $k+1$, it induced the
component $C \cap L$ of the exceptional divisor of the blowup of $X \cap L$. Also, as the
intersection is transversal and the canonical line bundle of $\text{Bl}_L X$ restricts to the
one of the blowup of $X \cap L$, the degree of the 0–cycle
\[ H_1 \cap \cdots \cap H_k \cap C \cap b^{-1}(L) \]
equals the multiplicity of the cycle
\[ b_*(H_1 \cap \cdots \cap H_k \cap C) \]
of codimension $k+1$ in $X$. If we now sum over all such components, we get
\[ \sum_{x \in X \cap L} e_{k+1}(I, (X \cap L, x)) = e_{k+1}(I)_f; \quad (4.1.1) \]
where $e_{k+1}(I)_f$ is the contribution to $e_{k+1}(I)$ coming from the fixed components of
the Segre cycle $\Lambda_{k+1}(I)$; we call it the fixed part of $e_{k+1}(I)$. Note that the summand
on the left–hand side is non–zero only at finitely many points.

We denote the contribution to $e_{k+1}(I)$ coming from the moving components of
the Segre cycle $\Lambda_{k+1}(I)$ by $e_{k+1}(I)_m$, the moving part of $e_{k+1}(I)$. Then, we have
\[ e_{k+1}(I) = e_{k+1}(I)_m + \sum_{x \in X \cap L} e_{k+1}(I, (X \cap L, x)). \quad (4.1.2) \]
Note that the moving part cannot computed locally at points of $X \cap L$, because a
component $C$ of $D$ that gives rise to a moving component of the Segre cycle maps
to a subset of $X$ of codimension at most $k$. So, the map
\[ C \cap b^{-1}(L) \to X \cap L \]
has fibres of dimension at most $k-1$. So, they don’t contribute to $e_{k+1}(I, (X \cap L, x))$
at any point of $x$ of $X \cap L$.

As the moving part of the Segre cycles of codimension $k$ come from components
of the exceptional divisor that map to subsets of smaller codimension, controlling
these components should be enough to control the moving part of the Segre cycle
of codimension $k$. This is in fact true as the next lemma shows.

As we will see in Proposition (4.3), the two assumptions in the following lemma
are very closely related.
(4.2) Lemma. Assume that the integral closures of $I_1$ and $I_2$ coincide outside a subset of codimension $k$ and $e_i(I_1) = e_i^{1-1}(I_1, I_2) = e_i^{1-2,2}(I_1, I_2)$ for $i = 2, \ldots, k-1$. Then, we have

$$e_k(I_1)_m = e_k^{k-1,1}(I_1, I_2)_m = e_k^{k-2,2}(I_1, I_2)_m.$$ 

Proof. Consider the family

$$Y = P_{k-1}^{k-1,1}(I_1, tI_2; X \times \mathbb{C}) \rightarrow \mathbb{C},$$

where $t$ denotes the coordinate on $\mathbb{C}$. Then, by [GG, Proof of Corollary (4.5)], the assumptions imply that the fibre of $Y$ over 0 equals $P_{k-1}(I_1, X)$. Now, consider a generic plane $L$ of codimension $n-k$ off 0 and the germ $\bar{Y}$ of $Y \cap (L \times \mathbb{C})$ along the moving part of $Y \cap (L \times \mathbb{C}) \cap V(I_1 + tI_2)$, i.e. the part of this intersection that varies with the tuple which is used to construct the polar variety. Now, by assumption, the ideal $I_1^{[k-1]} + I_2^{[1]}$ generates a reduction of $I_1|L$ and $I_2|L$. Furthermore, by construction of $\bar{Y}$, we have

$$V(I_1^{[k-1]} + I_2^{[1]}|\bar{Y}) = V(I_2^{[1]}|\bar{Y}).$$

Hence, the degree of $V(g_1|\bar{Y}(0))$ equals $e_k(I_1)_m$, and, for non-zero $t$, the degree of $V(g_1|\bar{Y}(t))$ equals $e_k^{k-1,1}(I_1, I_2)_m$. Hence, ‘conservation of number’ yields the first equality; see [F, Prop. 10.2, p. 180]. The second equality follows from an analogous argument applied to

$$Y = P_{k-1}^{k-2,2}(I_1^{[k-1]} + I_2^{[1]}, tI_1^{[2]}, X \times \mathbb{C}) \rightarrow \mathbb{C}.$$ 

(4.3) Proposition. In the Setup (2.1), let $k$ be an integer so that $2 \leq k \leq n$ and assume that the equalities $e_1(I_1) = e_1^{1-1}(I_1, I_2) = e_1(I_2)$ hold, and for $j = 2, \ldots, k-1$ we have

$$e_j(I_1) = e_j^{j-1,1}(I_1, I_2) = e_j^{j-2,2}(I_1, I_2)$$

and

$$e_j^{2j-2}(I_1, I_2) = e_j^{1j-1}(I_1, I_2) = e_j(I_2).$$

Then, the integral closure of $I_1$ and $I_2$ coincide outside a subset of codimension $k$. Also, the following inequalities of mixed Segre numbers of codimension $k$ obtain

$$e_k^{k-1,1}(I_1, I_2)^2 \leq e_k(I_1)e_k^{k-2,2}(I_1, I_2)$$

and

$$e_k^{1k-1}(I_1, I_2)^2 \leq e_k^{2k-2}(I_1, I_2)e_k(I_2).$$

Proof. We will proof the assertion by induction on $k$. First, for any $k$, the proof of Corollary (3.6) carries over to the general case and shows that the first assumed equality implies that $\bar{I}_1$ and $\bar{I}_2$ coincide outside a subset of codimension two.

Assume now that the assertion is proven for $k-1$. Then, by the above lemma, we have the equalities

$$e_k(I_1)_m = e_k^{k-1,1}(I_1, I_2)_m = e_k^{k-2,2}(I_1, I_2)_m.$$
Hence, it is enough to show the claim for the multigerm of the intersection of $X$ with a generic $(n-k)$–codimensional plane $L$ off 0 along the fixed parts of the Segre cycles of codimension $k$ in question. Now, if $h$ is a $(k-1)$–tuple of generic linear combinations of generators of $I_1^{[k-1]} + I_2^{[1]}$, the polar curve $P_{k-1}^{1,1}(I_1, I_2)$ equals the closure of $V(h_1, \ldots, h_{k-1}) - V(I_1 + I_2)$. But, by assumption, the integral closures of the two ideals coincide outside a set of dimension 0. Hence, the underlying sets of $V(I_1 + I_2)$ and $V(I_1)$ are equal. Also, without changing $V(h_1, \ldots, h_{k-1})$, we may assume that $h_2, \ldots, h_{k-1}$ are generic linear combinations of $f^{[k-1]}$ only. Furthermore, after changing $f$, we can write $h_2, \ldots, h_{k-1}$ as generic linear combinations of $f^{[k-2]}$. Hence, the above polar curve equals $P_{1,1}^0(I_1, I_2; P_{k-2}(I_1))$. A similar argument shows

$$P_{k-1}^{k-2,2}(I_1, I_2) = P_1(I_2, P_{k-2}(I_2)).$$

Hence, the first inequality follows from our earlier result (3.5) on surfaces. Also, the same result implies that the integral closures of the restrictions of $I_1$ and $I_2$ to $P_{k-2}(I_1)$ coincide. This implies already that $I_2$ is integrally dependent on $I_1$ by a result of Teissier [T3, Prop. (2), p.349]. (Teissier assumes in his proof that $I_1$ is of finite colength and that $X$ is Cohen-Macaulay, but it is easy to see that these assumptions are unnecessary.) Then, the claim follows by symmetry.

The following Corollary follows immediately from the previous proposition.

**Corollary.** In the Setup (2.1), the ideals $I_1$ and $I_2$ have the same integral closure if and only if the equalities $e_1(I_1) = e_1^{1,1}(I_1, I_2) = e_1(I_2)$ hold, and for $j = 2, \ldots, n$ we have

$$e_j(I_1) = e_j^{j-1,1}(I_1, I_2) = e_j^{j-2,2}(I_1, I_2) \quad \text{and} \quad e_j^{2j-2}(I_1, I_2) = e_j^{1j-1}(I_1, I_2) = e_j(I_2).$$

**Example.** Let $X = (\mathbb{C}^3, 0)$ with coordinates $x, y, z$. Consider the ideals $I_1 = (z)$ and $I_2 = (xz, yz, z^2)$ in $\mathcal{O}_{X,0}$. Clearly, the two ideals coincide outside 0. Also, we have $e_1(I_1) = e_1(I_2) = e_1^{1,1}(I_1, I_2)$. However, the polar surface of $I_1$ is empty, while the polar surface of $I_2$ isn’t. In particular, we have $e_2(I_1) = 0$ and $e_2(I_2) = 1$. So, the Segre numbers of codimension 2 already distinguish $I_1$ and $I_2$, even though they’re only different in codimension 3. In fact, the 2–codimensional Segre cycle $\Lambda_2(I_2)$ only has a moving component which is caused by the behavior of the ideal at 0.

Risler and Teissier [T1, Ch.1.2, p.302] proved a product formula expressing the multiplicity of two ideals of finite colength as a linear combination of the mixed multiplicities of the two ideals with binomial coefficients. As the top Segre number $e_n(I)$ has a length theoretic interpretation (see [GG, (3.7)] and [KT, (3.5)]), one can derive the following product formula.

**Proposition.** In the setup (2.1), the following product formula obtains.

$$e_n(I_1 I_2) = \sum_{i=0}^n e_n^{i,n-i}(I_1, I_2).$$
Proof. The length theoretic interpretation of the top Segre number $e_n(J)$ of an ideal $J \subseteq O_{X,0}$ shows that it equals the multiplicity of the ideal $J_k = JO_{X,0}$ induced by $J$ in the $k$th infinitesimal neighborhood of $0$ in $X$, for sufficiently big $k$. Now, the ideal $J_k$ is of finite colength, as any ideal in an Artinian ring. Hence, we can apply the result of Risler and Teissier. This yields the formula.

(4.6) Corollary. Under the assumptions of Proposition (4.3), we have the following product formula:

$$e_k(I_1I_2) = \sum_{i=0}^{k} e_{i,k-i}(I_1, I_2).$$

Proof. As we have seen in the proof of Proposition (4.3), the assumptions imply that the moving part of $e_{i,k-i}(I_1, I_2)$ is independent of $i$. Furthermore, the same argument shows the moving part $e_k(I_1I_2)_m$ equals $e_k(I_1^2)_m$. Now, the underlying sets of the blowups of $X$ along $I_1$ and along $I_2^2$ are equal. Only, the exceptional divisor of the second blowup is twice the divisor of the first blowup, and similarly for the first Chern classes of the canonical bundles. Hence, from the alternative description of Segre cycles discussed in (4.1), we get

$$e_k(I_2^2) = 2^k e_k(I_1).$$

Also, as the divisors of the two blowups have the same underlying set, the same relation holds for the moving parts of these numbers:

$$e_k(I_2^2)_m = 2^k e_k(I_1)_m.$$ 

Hence, the claimed product formula holds for the moving parts.

For the fixed parts, we intersect $X$ with a generic plane $L$ of codimension $n - k$ off $0$. Then, it is enough to show the equality locally at points of $X \cap L$. This, in turn, follows from the above proposition.

(4.7) Corollary. Under the assumptions of Proposition (4.3), we have the following Minkowski–type formula:

$$e_k(I_1I_2)^{1/k} \leq e_k(I_1)^{1/k} + e_k(I_2)^{1/k}.$$ 

Proof. The proof follows almost exactly the proof of Teissier [T2, pp.39–42]. Only, we need to change his induction slightly. Instead of considering $V(f_1) \subset X$ we need to consider $P_{f_1}^1(I_1) \subset X$. Then, by the same argument as in the proof of Proposition (4.3), we have

$$e_{i,k-i}(I_1, I_2) = e_{i-1,k-i}^{1}(I_1, I_2; P_{f_1}^1(I_1)).$$

The claim then follows from this modification of Teissier’s proof.

Teissier [T3,Thm. 1] also showed that for ideals of finite colength this Minkowski–type inequality is an equality if, and only if, there exist positive integers $a$ and $b$ so that the integral closures of $I_1^a$ and $I_2^b$ coincide. It is not possible to prove a similar result in general, rather one would get a similar result with different powers in different codimensions.

On the other hand, if the two ideals define subsets of $X$ of the same dimension, and the components of Segre cycles of one of them form chains, then we can derive an analogous result, using the observation after Corollary (3.4).
(4.8) Theorem. Let \((X, 0)\) be an equidimensional analytic germ of dimension \(n\) and \(I_1, I_2\) ideals in its local ring at 0. Assume that \(V(I_1)\) and \(V(I_2)\) are of both of codimension \(k\). Furthermore, assume
\[
|\Lambda_k(I_1)| \supset |\Lambda_{k+1}(I_1)| \supset \cdots \supset |\Lambda_n(I_1)|.
\]

(1) If \(I_1\) and \(I_2\) are of codimension at least two, then the equality
\[
e_k(I_1 I_2)^{1/k} = e_k(I_1)^{1/k} + e_k(I_2)^{1/k}
\]
holds for \(k = 2, \ldots, n\) if and only if there exist positive integers \(a\) and \(b\) so that the integral closures of \(I_1^a\) and \(I_2^b\) coincide.

(2) If \(I_1\) and \(I_2\) are of codimension one, then the equalities \(e_1(I_1) = e_1(I_1, I_2) = e_1(I_2)\) and
\[
e_k(I_1 I_2)^{1/k} = e_k(I_1)^{1/k} + e_k(I_2)^{1/k}
\]
hold for \(k = 2, \ldots, n\) if and only if the integral closures of \(I_1\) and \(I_2\) coincide.

Proof. We will only show that the Minkowski–type equalities implies the claimed equalities of integral closures. The proof of the other direction follows easily from the product formula (4.6).

In case (1), all components of the exceptional divisor of the blowup of \(X\) along \(I_1\) map to subsets of \(X\) of codimension at least \(k\). It follows that the Segre cycle \(\Lambda_k(I_1)\) has no moving components, and analogously for \(I_2\). Hence, the Segre numbers of codimension \(k\) can be computed locally at points of the intersection \(X \cap L\), where \(L\) is a generic plane of codimension \(n - k\) off 0. Now, Teissier showed [T3, p.354] that the assumed Minkowski–type equality imply
\[
a^k e_k^{k,0}(I_1, I_2) = a^{k-1} b e_k^{k-1,1}(I_1, I_2) = \cdots = b^k e_k^{0,k}(I_1, I_2).
\]
Furthermore, a similar result to Lemma (1.2) holds for the \(1/k\)–norm. Hence, by Corollary (1.4), we can conclude that analogous equalities hold locally at points of \(X \cap L\). Now, at a point \(x\) of \(X \cap L\) the ideals \(I_1|(X \cap L, x)\) and \(I_2|(X \cap L, x)\) are of finite colength. So, Teissier’s arguments imply
\[
I_1^a|X \cap L = I_2^b|X \cap L.
\]
Now, replacing \(I_1\) by \(I_1^a\) and \(I_2\) by \(I_2^b\), we may assume that the integral closure of \(I_1\) and \(I_2\) coincide outside a subset of codimension \(k + 1\).

In case (2), the argument of Corollary (3.6) apply again and show that the integral closure of \(I_1\) and \(I_2\) coincide outside a subset of codimension 2.

The following arguments apply now to both cases. We proceed by induction on \(k\). So, assume now that \(I_1\) and \(I_2\) coincide outside a subset of codimension \(k - 1\). Then, by the same arguments as in the proof of Lemma (4.2) and Proposition (4.3), it is enough to show the claim for multi surface–germs. Furthermore, using Lemma (1.2) once more, it suffices to consider normal surface germs. Furthermore, the assumptions imply that the reduction to the surface case yields ideals of codimension one. This case has been discussed already in Corollary (3.4); see also the remark after the Corollary.

If we assume that one of the ideals contains the other, we can drop the ‘chain–assumption’ and get the generalization of Rees’ theorem [GG, (4.9)].
Here, \( J \) maximal ideal in the local ring \( \mathcal{O}_{\text{off} 0} \), we have 
\[ e_k(I_1) = e_k(I_2) \]
holds for \( k = 1, \ldots, n \).

**Proof.** If the integral closures coincide, then, as the Segre numbers of an ideal only depend on its integral closure, the desired equalities hold. So assume now that the equalities of Segre numbers hold.

First, by the upper semi–continuity of multiplicities applied to the ideals induced by \( I_1 \) and \( I_2 \) at points of \( X \cap L \) where \( L \) is a generic plane of codimension \( n - 1 \) off \( 0 \), we have 
\[ e_1(I_2) = e_1(I_1 + I_2) \leq e_1(I_1) \]
Hence, the assumptions imply that the three numbers are in fact equal. Now, the remainder of the proof follows exactly Teissier’s original proof [T3, p.355] for ideals of finite colength: He shows that the assumptions imply that the Minkowski–type equalities of Theorem (4.8) hold. Then, as in the proof of the above theorem, we show, by induction on the codimension \( k \), that the two ideals have the same integral closure outside a subset of codimension \( k \). For the induction step, again we can reduce to the case of surfaces. If the ideals we end up with are of codimension one, then the result follows from Corollary (3.4). And if they are of finite colength, we can apply Teissier’s original result. This finishes the proof.

(4.10) **An application in equisingularity theory.** Consider two analytic function germs \( f_0, f_1 : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0) \) that define reduced hypersurfaces \( X(0) \) and \( X(1) \). We view them as the fibres over \( 0 \) and \( 1 \) of the family of hypersurfaces \( X \subseteq (\mathbb{C}^{n+1}, 0) \times \mathbb{C} \) defined by 
\[ F(z, t) = f_0(z) + t(f_1(z) - f_0(z)) \]
where \( t \) is a coordinate of the parameter axis \( \mathbb{C} \). We denote the singular locus of \( X \) by \( S(X) \). Let \( m \) be the maximal ideal in the local ring \( \mathcal{O}_{n+1} \) of \( \mathbb{C}^{n+1} \) at \( 0 \).

In the theory of contact equivalence one defines the tangent space to the contact equivalence class of a function \( f \) on \( (\mathbb{C}^{n+1}, 0) \) to be the ideal 
\[ T_k(f) = m J(f) + (f) \mathcal{O}_{n+1}. \]
Here, \( J(f) \) is the Jacobian ideal generated by the partial derivatives of \( f \). Gaffney [G1, Prop. 3.3] showed that the integral closure of this tangent space controls some aspects of Whitney equisingularity of the family \( X \):

The pair \((X - S(X), 0 \times \mathbb{C})\) satisfies the Whitney conditions at \((0,0)\) and \((0,1)\) if the integral closures of \( T_k(f_0) \) and \( T_k(f_1) \) coincide.

Using this result and our result on mixed Segre numbers we can give a criterion for \( X(0) \) and \( X(1) \) to be members of a Whitney equisingular family of hypersurfaces.

(4.11) **Proposition.** In the above setup (4.10), assume that for \( k = 2, \ldots, n + 1 \) the equalities 
\[ e_k(T_k(f_0)) = e_k^{k-1,1}(T_k(f_0), T_k(f_1)) = e_k^{k-2,2}(T_k(f_0), T_k(f_1)) \]
and 
\[ e_k(T_k(f_1)) = e_k^{k-1,1}(T_k(f_1), T_k(f_0)) = e_k^{k-2,2}(T_k(f_1), T_k(f_0)) \]
obtain. Then, there is an Zariski–open subset \( U \) of \( \mathbb{C} \) containing the points \( 0,1 \) so that the smooth part of \( X(U) = X \cap (U \times (\mathbb{C}^{n+1}, 0)) \) is Whitney regular along \( 0 \times U \).

**Proof.** Gaffney’s result together with Corollary (4.4) shows that the smooth part of \( X \) is Whitney regular along the parameter axis at the points \((0,0)\) and \((0,1)\). Also, as the Whitney–conditions hold generically, the existence of \( U \) as in the claim follows. This finishes the proof. \( \square \)
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