On Quasi-Hemi-Slant Riemannian Maps

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Abstract

In this paper, quasi-hemi-slant Riemannian maps from almost Hermitian manifolds onto Riemannian manifolds are introduced. The geometry of leaves of distributions that are involved in the definition of the submersion and quasi-hemi-slant Riemannian maps are studied. In addition, conditions for such distributions to be integrable and totally geodesic are obtained. Also, a necessary and sufficient condition for proper quasi-hemi-slant Riemannian maps to be totally geodesic is given. Moreover, structured concrete examples for this notion are given.

1. INTRODUCTION

A differentiable map $F$ between Riemannian manifolds $(N_1, g_1)$ and $(N_2, g_2)$ is said to be a Riemannian map if

$$g_2(F_*Z_1, F_*Z_2) = g_1(Z_1, Z_2), \text{ for } Z_1, Z_2 \in \Gamma(\ker F_*).$$

The theory of smooth maps between Riemannian manifolds plays a preeminent role in differential geometry and also in physics. It is useful for comparing geometric structures between the source manifolds and the target manifolds. A conspicuous property of Riemannian map provides the generalized eikonal equation $\|F_*\|^2 = \text{rank } F$ [1]. Since rank $F$ is an integer value function and $\|F_*\|^2$ is continuous function on the Riemannian manifold. Since energy density $2e(F) = \|F_*\|^2 = \text{rank } F$, i.e. density is quantized to integer if the Riemannian manifold is connected. In addition, complex manifolds are very useful tools for studying spacetime geometry [2]. In fact, Calabi-Yau manifolds and Teichmüller spaces are two interesting classes of Kähler manifold, which have applications in superstring theory [3] and in general relativity [4, 5]. Thus, the notion of Riemannian maps deserves through study from different perspectives.

In addition, O’Neills [6] and Gray [7] studied Riemannian submersions. Watson introduced almost Hermitian submersions as follows: A Riemannian submersion $F : (N_1, g_1, J_{N_1}) \rightarrow (N_2, g_2, J_{N_2})$ is said to be an almost Hermitian submersion if $F_*J_{N_1} = J_{N_2} F_*$ [8]. Watson also showed that, in most cases [8] and [9], each fiber and base manifold have the same kind of structure as the total space.

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After that, several kinds of Riemannian submersions were introduced and studied, some of them are like: contact-submersions [10], semi-slant and generic submersions [11, 12], semi-invariant $\xi^+\xi^-$-Riemannian submersions [13], hemi-slant submersions [14] etc. Sayar, Akyol and Prasad studied on bi slant submersions [15], and Prasad, Shukla and Kumar introduce quasi-bi slant submersions [16]. Recently, Longwap, Massamba and Homti introduce and study quasi-hemi slant Riemannian submersions which generalizes hemi-slant, semi-slant and semi-invariant Riemannian submersions [17]. It is well known that Riemannian submersion is a particular Riemannian map with $(\text{range } F_*)_\perp = \{0\}$, so we generalize the notion of quasi-hemi slant Riemannian submersions to quasi-hemi slant Riemannian maps in the present paper and study its geometry.

The notion of Riemannian map between Riemannian manifolds was introduced by Fischer [18]. Let $F : (N_1, g_1) \to (N_2, g_2)$ be a differentiable map with $0 < \text{rank } F_* < \min (m, n)$. If the kernel space of $F_*$ is denoted by $\ker F_*$, and the orthogonal complementary space of $\ker F_*$ is denoted by $(\ker F_*)^\perp$ in $T_{N_1}$, then

$$TN_1 = \ker F_* \oplus (\ker F_*)^\perp.$$  

Also, if the range of $F_*$ is denoted by range $F_*$, and for a point $q \in N_1$ the orthogonal complementary space of range $F_*|_{F(q)}$ is denoted by $(\text{range } F_*|_{F(q)})^\perp$ in $T_{F(q)}N_2$ then the tangent space $T_{F(q)}N_2$ has the following orthogonal decomposition:

$$T_{F(q)}N_2 = (\text{range } F_*|_{F(q)}) \oplus (\text{range } F_*|_{F(q)})^\perp.$$

A differentiable map $F : (N_1, g_1) \to (N_2, g_2)$ is called a Riemannian map at $q \in N_1$ if $F_*^h : (\ker F_*|_{q})^\perp \to (\text{range } F_*|_{F(q)})$ is linear isometry.

In this paper, we study the quasi-hemi-slant Riemannian maps from an almost Hermitian manifolds to Riemannian manifolds. In section 3, quasi-hemi-slant Riemannian maps are defined, and the geometry of leaves of distributions that are involved in the definition of such maps is studied. In addition, a necessary and sufficient condition for quasi-hemi-slant Riemannian maps to be totally geodesic is given. Finally, concrete examples for this setting are provided.

2. PRELIMINARIES

If $J$ is a $(1, 1)$ tensor field on an even-dimensional differentiable manifold $N_1$ such that

$$J^2 = -I$$  

(1)

then $(N_1, J)$ is said to be an almost complex manifold where $I$ is identity operator [19, 20]. Nijenhuis tensor $N$ of $J$ is described as:

$$N(X_1, X_2) = [JX_1, JX_2] - [X_1, X_2] - J[JX_1, X_2] - J[X_1, JX_2]$$  

(2)

for all $X_1, X_2 \in \Gamma(TN_1)$. If $N = 0$, then $N_1$ is said to be a complex manifold. If $g_1$ is a Riemannian metric on $N_1$ such that

$$g_1(JX_1, JX_2) = g_1(X_1, X_2), \text{ for all } X_1, X_2 \in \Gamma(TN_1)$$  

(3)

then $(N_1, g_1, J)$ is said to be an almost Hermitian manifold, and if $(\nabla_X J)X_2 = 0$ for all $X_1, X_2 \in \Gamma(TN_1)$ then $(N_1, g_1, J)$ is said to be a Kähler manifold where $\nabla$ is the Levi-Civita connection on $N_1$.

O’Neill’s tensors $T$ and $A$ are defined by
\[ A_{\varepsilon_1} \varepsilon_2 = H \nabla_{\varepsilon_1} \varepsilon + \nabla_{\varepsilon_1} H \varepsilon_2, \]  
(4)

\[ T_{\varepsilon_1} \varepsilon_2 = H \nabla_{\varepsilon_1} \varepsilon + \nabla_{\varepsilon_1} H \varepsilon_2 \]  
(5)

for any \( \varepsilon_1, \varepsilon_2 \in \Gamma(TN_1) \). From Equations (4) and (5), we have

\[ \nabla_{X_1} X_2 = T_{X_1} X_2 + \nabla_{X_1} X_2, \]  
(6)

\[ \nabla_{X_1} Z_1 = T_{X_1} Z_1 + H \nabla_{X_1} Z_1, \]  
(7)

\[ \nabla_{Z_1} X_1 = A_{Z_1} X_1 + \nabla_{Z_1} X_1, \]  
(8)

\[ \nabla_{Z_1} Z_2 = H \nabla_{Z_1} Z_2 + A_{Z_1} Z_2, \]  
(9)

for all \( X_1, X_2 \in \Gamma(\text{ker } F_\ast) \) and \( Z_1, Z_2 \in \Gamma(\text{ker } F_\ast)^\perp \), where \( H \nabla_{X_1} Z_1 = A_{Z_1} X_1 \), if \( Z_1 \) is basic. For \( q \in N_1 \), \( X_1 \in V_q \) and \( Z_1 \in H_q \) the linear operators

\[ A_{Z_1} \text{ and } T_{X_1} : T_q N_1 \rightarrow T_q N_1 \]

are skew-symmetric, that is

\[ g_1(A_{Z_1} \varepsilon_1, \varepsilon_2) = -g_1(\varepsilon_1, A_{Z_1} \varepsilon_2) \]  
and

\[ g_1(T_{X_1} \varepsilon_1, \varepsilon_2) = -g_1(\varepsilon_1, T_{X_1} \varepsilon_2) \]

for each \( \varepsilon_1, \varepsilon_2 \in T_q N_1 \).

Let \( F : (N_1, g_1) \rightarrow (N_2, g_2) \) be a smooth map. \( F \) is said to be a totally geodesic if

\( (VF_\ast)(X_1, X_2) = 0 \), for all \( X_1, X_2 \in \Gamma(TN_1) \).

The differential map \( F_\ast \) of \( F \) can be observed a section of the bundle \( \text{Hom} \ (TN_1, F^\ast TN_2) \rightarrow N_1 \), where \( F^\ast TN_2 \) is the bundle which has fibers \( (F^\ast TN_2)_x = T_{F(x)} N_2 \), has a connection \( V \) induced from the Riemannian connection \( V^{N_1} \) and the pullback connection. In addition, the second fundamental form of \( F \) is given by

\[ (VF_\ast)(X_1, X_2) = \nabla^{F_\ast}_{X_1} F_\ast(X_2) - F_\ast(\nabla^{N_1}_{X_1} X_2) \]  
(10)

for vector field \( X_1, X_2 \in \Gamma(TN_1) \), where \( \nabla^F \) is the pullback connection. Bi-harmonic Riemannian maps and the second fundamental form \( (VF_\ast)(U_1, U_2) \), for all \( U_1, U_2 \in \Gamma(\text{ker } F_\ast)^\perp \) of a Riemannian map has components in range \( F_\ast \), [21].

**Lemma 1.** Let \( F : (N_1, g_1) \rightarrow (N_2, g_2) \) be a Riemannian map. Then \( g_2((VF_\ast)(U_1, U_2), F_\ast(U_3)) = 0 \) for all \( U_1, U_2, U_3 \in \Gamma(\text{ker } F_\ast)^\perp \).

As a consequence of the above lemma, we get \( (VF_\ast)(U_1, U_2) \in \Gamma(\text{range } F_\ast)^\perp \), for all \( U_1, U_2 \in \Gamma(\text{ker } F_\ast)^\perp \).

Let \( F : (N_1, g_1, J) \rightarrow (N_2, g_2) \) be Riemannian map from an almost Hermitian manifold onto a Riemannian manifold.
F is said to be a semi-invariant Riemannian map if there is a distribution \( D_1 \subseteq \ker F \) such that
\[
\ker F = D_1 \oplus D_2, \quad J(D_1) = D_1,
\]
where \( D_1 \oplus D_2 \) is an orthogonal decomposition of \( \ker F \) \cite{1}. The complementary orthogonal subbundle to \( J(\ker F) \) in \( (\ker F)^\perp \) is denoted by \( \mu \). Thus, we get \( (\ker F)^\perp = J(D_2) \oplus \mu \). It is clear that \( \mu \) is an invariant subbundle.

If \( \ker F = D \oplus D_1 \oplus D_2, \quad J(D) = D, \quad JD_2 \subseteq (\ker F)^\perp \) the angle \( \theta \) between \( JZ \) and the space \( (D_1)_p \) is constant for any non-zero vector \( Z \) in \( (D_1)_p \) then \( F \) is said to be quasi-hemi-slant Riemannian map and the angle \( \theta \) is said to be the quasi-hemi-slant angle \cite{17}.

3. QUASI-HEMI-SLANT RIEAMANNIAN MAPS

Let \( F \) be quasi-hemi-slant Riemannian map from an almost Hermitian manifold \((N_1, g_1, J)\) onto a Riemannian manifold \((N_2, g_2)\). Thus, we get
\[
TN_1 = \ker F \oplus (\ker F)^\perp.
\]

Let \( P, Q \) and \( R \) be projection morphisms of \( \ker F \) onto \( D, D_1 \) and \( D_2 \) respectively. For any vector field \( X_1 \in \Gamma(\ker F) \), we put
\[
X_1 = PX_1 + QX_1 + RX_1. \tag{11}
\]

For all \( Z_1 \in \Gamma(\ker F) \), we get
\[
JZ_1 = \phi Z_1 + \omega Z_1 \tag{12}
\]
where \( \phi Z_1 \in \Gamma(\ker F) \) and \( \omega Z_1 \in \Gamma(\omega D_1 \oplus \omega D_2) \). The horizontal distribution \((\ker F)^\perp\) is decomposed as
\[
(\ker F)^\perp = \omega D_1 \oplus \omega D_2 \oplus \mu.
\]

Here \( \mu \) is an invariant distribution of \( \omega D_1 \oplus \omega D_2 \) in \( (\ker F)^\perp \). From Equations (11) and (12), we have
\[
JX_1 = J(PX_1) + J(QX_1) + J(RX_1)
= \phi (PX_1) + \omega (PX_1) + \phi (QX_1) + \omega (QX_1) + \phi (RX_1) + \omega (RX_1).
\]

Since \( JD = D \), we have \( \omega PX_1 = 0 \) and \( \phi (RX_1) = 0 \). Thus, we get
\[
JX_1 = \phi (PX_1) + \phi QX_1 + \omega QX_1 + \omega RX_1.
\]

Hence we get the below decomposition
\[
J(\ker F) = D \oplus \phi (D_1) \oplus (\omega D_1 \oplus \omega D_2)
\]

where \( \oplus \) denotes orthogonal direct sum. Further, let \( X_1 \in \Gamma(D_1) \) and \( X_2 \in \Gamma(D_2) \). Then
\[
g_1(X_1, X_2) = 0.
\]
From above equation, we have
\[ g_1(JX_1, X_2) = -g_1(X_1, JX_2) = 0. \]

Now, consider
\[ g_1(\phi X_1, X_2) = g_1(JX_1 - \omega X_1, X_2) = g_1(JX_1, X_2). \]

Similarly, we have \( g_1(X_1, \phi X_2) = 0. \)

Let \( V_1 \in \Gamma(D) \) and \( V_2 \in \Gamma(D_1). \) Then we have
\[ g_1(\phi V_1, V_2) = g_1(JV_1 - \omega V_1, V_2) = g_1(JV_1, V_2) = -g_1(V_1, JV_2) = 0 \]
as \( D \) is invariant i.e., \( JV_1 \in \Gamma(D). \)

Similarly, for \( Z_1 \in \Gamma(D) \) and \( Z_2 \in \Gamma(D_2), \) we obtain \( g_1(\phi Z_2, Z_1) = 0. \) From above equations, we have
\[ g_1(\phi Y_1, \phi Y_2) = 0 \] and \( g_1(\omega Y_1, \omega Y_2) = 0 \)
for all \( Y_1 \in \Gamma(D_1) \) and \( Y_2 \in \Gamma(D_2). \) Since \( \omega D_1 \subseteq (\ker F_*)^\perp, \omega D_2 \subseteq (\ker F_*)^\perp. \) So we can write
\[ (\ker F_*)^\perp = \omega D_1 \oplus \omega D_2 \oplus \mathcal{V} \]
where \( \mathcal{V} \) is orthogonal complement of \( (\omega D_1 \oplus \omega D_2) \) in \( (\ker F_*)^\perp. \) For any \( X_1 \in \Gamma(\ker F)^\perp, \) we get
\[ JX_1 = BX_1 + CX_1. \tag{13} \]
where \( BX_1 \in \Gamma(\ker F_*) \) and \( CX_1 \in \Gamma(\mathcal{V}). \)

**Lemma 2.** If \( F \) is a quasi-hemi-slant Riemannian map then we have
\[ \phi^2 V_1 + B_0 V_1 = -V_1, \omega \phi V_1 + C_0 V_1 = 0, \]
\[ \omega B V_2 + C^2 V_2 = -V_2, \phi B V_2 + B C V_2 = 0 \]
for all \( V_1 \in \Gamma(\ker F_*) \) and \( V_2 \in \Gamma(\ker F_*)^\perp. \)

**Proof.** The desired results are obtained by using Equations (1), (12) and (13).

Evidence of the following result is the same as given in [1], so we will skip the proof.

**Lemma 3.** If \( F \) is a quasi-hemi-slant Riemannian map then we have
i) \( \phi^2 V_1 = -(\cos^2 \theta_1) V_1, \)
ii) \( g_1(\phi V_1, \phi V_2) = \cos^2 \theta_1 g_1(V_1, V_2), \)
iii) \( g_1(\omega V_1, \omega V_2) = \sin^2 \theta_1 g_1(V_1, V_2), \)
for all \( V_1, V_2 \in \Gamma(D_1). \)
From now on we will denote a quasi-hemi-slant Riemannian map from a Kähler manifold \((N_1, g_1, J)\) onto a Riemannian manifold \((N_2, g_2)\) by \(F\).

**Lemma 4.** If \(F\) is a quasi-hemi-slant Riemannian map then, we have

\[
\nabla X_1 \phi X_2 + \mathcal{T}_{X_1} \omega X_2 = B\mathcal{T}_{X_1} X_2 + \omega \nabla X_1 X_2,
\]

\[
\mathcal{T}_{X_1} \phi X_2 + \mathcal{H}\nabla X_1 \omega X_2 = C\mathcal{T}_{X_1} X_2 + \omega \nabla X_1 X_2,
\]

\[
\nabla X_1 BZ_1 + \mathcal{T}_{X_1} CZ_1 = \phi \mathcal{T}_{X_1} Z_1 + B\mathcal{H}\nabla X_1 Z_1,
\]

\[
\mathcal{T}_{X_1} BZ_1 + \mathcal{H}\nabla X_1 CZ_1 = \omega \mathcal{T}_{X_1} Z_1 + C\mathcal{H}\nabla X_1 Z_1.
\]

\[
\nabla Z_1 \phi X_1 + \mathcal{A}Z_1 \omega X_1 = B\mathcal{A}Z_1 X_1 + \phi \nabla Z_1 X_1,
\]

\[
\mathcal{A}Z_1 \phi X_1 + \mathcal{H}\nabla Z_1 \omega X_1 = \omega \mathcal{H}\nabla Z_1 X_1 + C\mathcal{A}Z_1 X_1,
\]

\[
\nabla Z_1 BZ_2 + \mathcal{A}Z_1 CZ_2 = B\mathcal{H}\nabla Z_1 Z_2 + \phi \mathcal{A}Z_1 Z_2,
\]

\[
\mathcal{A}Z_1 BZ_2 + \mathcal{H}\nabla Z_1 CZ_2 = \omega \mathcal{A}Z_1 Z_2 + C\mathcal{H}\nabla Z_1 Z_2,
\]

for any \(X_1, X_2 \in \Gamma(\ker F_\ast)\) and \(Z_1, Z_2 \in \Gamma(\ker F_\ast)^\perp\).

**Proof.** Using Equations (3), (6), (7), (8), (9), (12) and (13), we get the lemma completely.

Now, we define

\[
(\nabla X_1 \phi)X_2 = \nabla X_1 \phi X_2 - \phi \nabla X_1 X_2,
\]

\[
(\nabla X_1 \omega)X_2 = \mathcal{H}\nabla X_1 \omega X_2 - \omega \nabla X_1 X_2,
\]

\[
(\nabla Z_1 C)Z_2 = \mathcal{H}\nabla Z_1 CZ_2 - C\mathcal{H}\nabla Z_1 Z_2,
\]

\[
(\nabla Z_1 B)Z_2 = \mathcal{H}\nabla Z_1 BZ_2 - B\mathcal{H}\nabla Z_1 Z_2
\]

for any \(X_1, X_2 \in \Gamma(\ker F_\ast)\) and \(Z_1, Z_2 \in \Gamma(\ker F_\ast)^\perp\).

**Lemma 5.** If \(F\) is a quasi-hemi-slant Riemannian map then, we have

\[
(\nabla X_1 \phi)X_2 = B\mathcal{T}_{X_1} X_2 - \mathcal{T}_{X_1} \omega X_2,
\]

\[
(\nabla X_1 \omega)X_2 = C\mathcal{T}_{X_1} X_2 - \mathcal{T}_{X_1} \phi X_2,
\]

\[
(\nabla Z_1 C)Z_2 = \omega \mathcal{A}Z_1 Z_2 - \mathcal{A}Z_1 BZ_2,
\]

\[
(\nabla Z_1 B)Z_2 = \phi \mathcal{A}Z_1 Z_2 - \mathcal{A}Z_1 CZ_2,
\]

for any vectors \(X_1, X_2 \in \Gamma(\ker F_\ast)\) and \(Z_1, Z_2 \in \Gamma(\ker F_\ast)^\perp\).
**Proof.** The proof is straightforward, so we omit its proof.

If $\phi$ and $\omega$ are parallel with respect to $\nabla$ on $N_1$ respectively, then

$$BT_{X_1}X_2 = T_{X_1}\omega X_2 \quad \text{and} \quad CT_{X_1}X_2 = T_{X_1}\phi X_2$$

for any $X_1, X_2 \in \Gamma(TN_1)$.

**Theorem 1.** $D$ is integrable if and only if

$$g_1(T_{X_2}JX_1 - T_{X_1}JX_2, \omega QZ_1 + \omega RZ_1) = g_1(\nabla_{X_1}JX_2 - \nabla_{X_2}JX_1, \phi QZ_1)$$

for all $X_1, X_2 \in \Gamma(D)$ and $Z_1 \in \Gamma(D_1 \oplus D_2)$.

**Proof.** For all $X_1, X_2 \in \Gamma(D)$, $Z_1 \in \Gamma(D_1 \oplus D_2)$ and $Z_2 \in (\ker F_1)^\perp$, since $[X_1, X_2] \in (\ker F_1)$, we have $g_1([X_1, X_2], Z_2) = 0$. Thus $D$ is integrable $\iff g_1([X_1, X_2], Z_1) = 0$. Now, using Equations (2), (3), (6), (7), (11), (12) and (13), we have

$$g_1([X_1, X_2], Z_1) = g_1(J\nabla_{X_1}JX_2, JZ_1) - g_1(J\nabla_{X_2}JX_1, JZ_1) = g_1(\nabla_{X_1}JX_2, JZ_1) - g_1(\nabla_{X_2}JX_1, JZ_1) = g_1(T_{X_1}JX_2 - T_{X_2}JX_1, \omega QZ_1 + \omega RZ_1) - g_1(\nabla_{X_1}JX_2 - \nabla_{X_2}JX_1, \phi QZ_1).$$

**Theorem 2.** $D_1$ is integrable if and only if

$$g_1(T_{Z_1}\phi Z_2 - T_{Z_2}\phi Z_1, V_1) = g_1(T_{Z_1}\omega Z_2 - T_{Z_2}\omega Z_1, \phi PV_1) + g_1(\nabla Z_1\omega Z_2 - \nabla Z_2\omega Z_1, \omega RV_1)$$

for all $Z_1, Z_2 \in \Gamma(D_1)$ and $V_1 \in \Gamma(D_1 \oplus D_2)$.

**Proof.** For all $Z_1, Z_2 \in \Gamma(D)$ and $V_1 \in \Gamma(D_1 \oplus D_2)$ and $V_2 \in (\ker F_2)^\perp$, since $[Z_1, Z_2] \in (\ker F_2)$, we have $g_1([Z_1, Z_2], V_2) = 0$. Thus $D_1$ is integrable $\iff g_1([Z_1, Z_2], V_1) = 0$. Using Equations (2), (3), (6), (7), (11), (12), (13) and the Lemma 4, we have

$$g_1([Z_1, Z_2], V_1) = g_1(\nabla_{Z_1}JZ_2, JV_1) - g_1(\nabla_{Z_2}JZ_1, JV_1) = g_1(\nabla_{Z_1}JZ_2, JV_1) + g_1(\nabla_{Z_1}\omega Z_2, JV_1) - g_1(\nabla_{Z_2}\omega Z_1, JV_1) - g_1(\nabla_{Z_2}\omega Z_1, JV_1) = \cos^2\theta g_1(\nabla_{Z_1}Z_2, V_1) - \cos^2\theta g_1(\nabla_{Z_1}Z_2, V_1) - g_1(T_{Z_1}\omega Z_2 - T_{Z_2}\omega Z_1, V_1) + g_1(\nabla Z_1\omega Z_2 + T_{Z_1}\omega Z_2, JV_1 + \omega RV_1) - g_1(\nabla Z_2\omega Z_1 + T_{Z_2}\omega Z_1, JV_1 + \omega RV_1).

Now, we have

$$\sin^2\theta g_1([Z_1, Z_2], V_1) = g_1(T_{Z_1}\omega Z_2 - T_{Z_2}\omega Z_1, JV_1) + g_1(\nabla Z_1\omega Z_2 - \nabla Z_2\omega Z_1, \omega RV_1) - g_1(T_{Z_1}\omega Z_2 - T_{Z_2}\omega Z_1, V_1).$$
which proofs the assertion.

**Theorem 3.** $D_2$ is always integrable.

**Theorem 4.** $(\text{ker} F_\ast)^{-1}$ is integrable if and only if

$$g_1(\nabla_X_1 B_2 - \nabla_X_2 B_1, \phi Z_1) = -g_2(F_\ast(CX_2), (VF_\ast)(X_1, \phi Z_1)) + g_2(F_\ast(CX_1), (VF_\ast)(X_2, \phi Z_1)),$$

$$g_1(A X_1 B_2 - A X_2 B_1, \omega Q Z_2) = g_2((VF_\ast)(X_1, CX_2), F_\ast(\omega Q Z_2)) + g_2((VF_\ast)(X_2, CX_1), F_\ast(\omega Q Z_2)),$$

$$g_1(A X_1 B_2 - A X_2 B_1, \omega Q Z_3) = g_2((VF_\ast)(X_1, CX_2), F_\ast(\omega Q Z_3)) + g_2((VF_\ast)(X_2, CX_1), F_\ast(\omega Q Z_3)),$$

for all $X_1, X_2 \in \Gamma(\ker F_\ast)^{-1}$, $Z_1 \in \Gamma(D)$, $Z_2 \in \Gamma(D_1)$ and $Z_3 \in \Gamma(D_3)$.

**Proof.** For $X_1, X_2 \in \Gamma(\ker F_\ast)^{-1}$, $Z_1 \in \Gamma(D)$, $Z_2 \in \Gamma(D_1)$ and $Z_3 \in \Gamma(D_3)$ and using Equations (2), (3), (8), (12) and (13), we have

$$g_1([X_1, X_2]), Z_1 = g_1(\nabla_X_1 \phi X_2, \phi Z_1) - g_1(\nabla_X_2 \phi X_1, \phi Z_1) = g_1(\nabla_X_1 B_2 - \nabla_X_2 B_1, \phi Z_1) - g_1(CX_2, \nabla_X_1 \phi Z_1) + g_1(CX_1, \nabla_X_2 \phi Z_1).$$

Using Equation (10), we get

$$g_1([X_1, X_2]), Z_1 = g_1(\nabla_X_1 B_2 - \nabla_X_2 B_1, \phi Z_1) + g_2(F_\ast(CX_2), (VF_\ast)(X_1, \phi Z_1)) - g_2(F_\ast(CX_1), (VF_\ast)(X_2, \phi Z_1)).$$

From Equations (2), (3), (8), (9), (11), (12), (13) and the Lemma 4, we obtain

$$g_1([X_1, X_2]), Z_2 = g_1(\phi \nabla_X_1 X_2, \phi Q Z_2) + g_1(\phi \nabla_X_2 X_2, \omega Q Z_2) - g_1(\phi \nabla_X_2 X_1, \phi Q Z_2) - g_1(\phi \nabla_X_2 X_1, \omega Q Z_2) = \cos^2 \theta g_1([X_1, X_2], Z_2) - g_1(\nabla_X_1 X_2, \omega \phi Q Z_2) + g_1(\nabla_X_2 X_1, \omega \phi Q Z_2) + g_1(\nabla_X_1 B_2, \omega Q Z_2) + g_1(\nabla_X_1 C X_2, \omega Q Z_2) + g_1(\nabla_X_2 B_1, \omega Q Z_2) - g_1(\nabla_X_2 C X_1, \omega Q Z_2).$$

Using Equation (10), we have

$$\sin^2 \theta g_1([X_1, X_2], Z_2) = g_1(A X_1 B_2 - A X_2 B_1, \omega Q Z_2) - g_2((VF_\ast)(X_1, CX_2), F_\ast(\omega Q Z_2)) + g_2((VF_\ast)(X_2, CX_1), F_\ast(\omega Q Z_2)).$$

Similarly, we get

$$\sin^2 \theta g_1([X_1, X_2], Z_3) = g_1(A X_1 B_2 - A X_2 B_1, \omega Q Z_3) - g_2((VF_\ast)(X_1, CX_2), F_\ast(\omega Q Z_3)) + g_2((VF_\ast)(X_2, CX_1), F_\ast(\omega Q Z_3)).$$

**Theorem 5.** $(\ker F_\ast)^{-1}$ is totally geodesic if and only if

$$g_1(A X_1 X_2, P Z_1 + \cos^2 \theta Q Z_1) = g_1(\mathcal{H} \nabla_X_1 X_2, \omega \phi P Z_1 + \omega \phi Q Z_1) - g_1(A X_1 B_2 + \mathcal{H} \nabla_X_1 C X_2, \omega Q Z_1 + \omega R Z_1).$$
for all $X_1, X_2 \in \Gamma(\ker F_\ast)^\perp$ and $Z_1 \in \Gamma(\ker F_\ast)$.

**Proof.** For all $X_1, X_2 \in \Gamma(\ker F_\ast)^\perp$ and $Z_1 \in \Gamma(\ker F_\ast)$ and using Equations (2), (3), (8), (9), (11), (12), (13) and the Lemma 4, we have

\[
g_1(\nabla_{X_1} X_2, Z_1) = g_1(J\nabla_{X_1} X_2, JZ_1)
\]

\[
= - g_1(\nabla_{X_1} X_2, \phi^2 PZ_1 + \omega \phi PZ_1 + \omega \phi QZ_1) + g_1(\nabla_{X_1} BX_2, \omega QZ_1 + \omega RZ_1) + g_1(\nabla_{X_1} CX_2, \omega QZ_1 + \omega RZ_1)
\]

\[
= g_1(AX_1 PZ_1 + \cos^2 \theta_1 QZ_1) - g_1(H\nabla_{X_1} X_2, \omega \phi PZ_1 + \omega \phi QZ_1) + g_1(AX_1 BX_2, \omega QZ_1 + \omega RZ_1)
\]

\[
+ g_1(H\nabla_{X_1} CX_2, \omega QZ_1 + \omega RZ_1)
\]

which shows our assertion.

**Theorem 6.** \(\ker F_\ast\) is parallel if and only if

\[
g_1(T_{X_1} PX_2, X_3) + \cos^2 \theta_1 g_1(T_{X_1} QX_2, X_3) = g_1(H\nabla_{X_1} \phi PX_2, X_3) + g_1(H\nabla_{X_1} \phi QX_2, X_3)
\]

\[
- g_1(H\nabla_{X_1} \phi QX_2 + H\nabla_{X_1} \phi RX_2, CX_3) + g_1(T_{X_1} \phi QX_2 + T_{X_1} \phi RX_2, BX_3)
\]

for all $X_1, X_2 \in \Gamma(\ker F_\ast)^\perp$ and $Z_1 \in \Gamma(\ker F_\ast)^\perp$.

**Proof.** For all $X_1, X_2 \in \Gamma(\ker F_\ast)^\perp$ and $X_3 \in \Gamma(\ker F_\ast)^\perp$, using Equations (2), (3), (8), (9), (11), (12), (13) and the Lemma 4, we have

\[
g_1(\nabla_{X_1} X_2, X_3) = g_1(J\nabla_{X_1} X_2, JX_3)
\]

\[
= g_1(\nabla_{X_1} \phi PX_2, JX_3) + g_1(\nabla_{X_1} \phi QX_2, JX_3) + g_1(\nabla_{X_1} \phi QX_2, JX_3) + g_1(\nabla_{X_1} \phi RX_2, JX_3)
\]

\[
= g_1(T_{X_1} PX_2, X_3) + \cos^2 \theta_1 g_1(T_{X_1} QX_2, X_3) - g_1(H\nabla_{X_1} \phi PX_2, X_3) - g_1(H\nabla_{X_1} \phi QX_2, X_3)
\]

\[
+ g_1(H\nabla_{X_1} \phi QX_2 + H\nabla_{X_1} \phi RX_2, CX_3) + g_1(T_{X_1} \phi QX_2 + T_{X_1} \phi RX_2, BX_3)
\]

which completes the proof.

**Theorem 7.** \(D\) is parallel if and only if

\[
g_1(T_{X_1} JPX_2, \omega QZ_1 + \omega RZ_1) = - g_1(\nabla_{X_1} JPX_2, \phi Z_1)
\]

and

\[
g_1(T_{X_1} JPX_2, CZ_2) = - g_1(\nabla_{X_1} JPX_2, BZ_2)
\]

for all $X_1, X_2 \in \Gamma(D)$, $Z_1 \in \Gamma(D_1 \oplus D_2)^\perp$ and $Z_2 \in \Gamma(\ker F_\ast)^\perp$.

**Proof.** For all $X_1, X_2 \in \Gamma(D)$, $Z_1 \in \Gamma(D_1 \oplus D_2)^\perp$ and $Z_2 \in \Gamma(\ker F_\ast)^\perp$, using Equations (2), (3), (7), (11), (12) and (13), we have
\[ g_1 (V_{X_1} X_2, Z_i) = g_1 (V_{X_1} JX_2, JZ_i) \]
\[ = g_1 (V_{X_1} JPX_2, JQZ_1 + JRZ_i) \]
\[ = g_1 (T_{X_1} \phi PX_2, \omega QZ_1 + \omega RZ_i) + g_1 (V_{X_1} \phi PX_2, \phi QZ_1). \]

Using equations (2), (3), (7), (11) and (13), we obtain
\[ g_1 (V_{X_1} X_2, Z_i) = g_1 (V_{X_1} JX_2, JZ_2) \]
\[ = g_1 (V_{X_1} JPX_2, BZ_2 + CZ_2) \]
\[ = g_1 (V_{X_1} JPX_2, BZ_2) + g_1 (T_{X_1} JPX_2, CZ_2) \]

which completes the assertion.

**Theorem 8.** \( D_1 \) is parallel if and only if
\[ g_1 (T_{Z_1} \omega \phi Z_2, X_1) = g_1 (T_{Z_1} \omega Z_2, \phi PX_1) + g_1 (H_{V_{Z_1}} \omega Z_2, \omega RX_1) \]
and
\[ g_1 (H_{V_{Z_1}} \omega \phi Z_2, X_2) = g_1 (H_{V_{Z_1}} \omega Z_2, CX_2) + g_1 (T_{Z_1} \omega Z_2, BX_2) \]
for all \( Z_1, Z_2 \in \Gamma (D_1), X_1 \in \Gamma (D \oplus D_2) \) and \( X_2 \in \Gamma (\ker F_\ast)^\perp. \)

**Proof.** For all \( Z_1, Z_2 \in \Gamma (D_1), X_1 \in \Gamma (D \oplus D_2) \) and \( X_2 \in \Gamma (\ker F_\ast)^\perp, \) using Equations (2), (3), (8), (11), (13) and the Lemma 4, we have
\[ g_1 (V_{Z_1} Z_2, X_1) = g_1 (V_{Z_1} JZ_2, JX_1) \]
\[ = g_1 (V_{Z_1} \phi Z_2, JX_1) + g_1 (V_{Z_1} \omega Z_2, JX_1) \]
\[ = \cos^2 \theta_1 g_1 (V_{Z_1} Z_2, X_1) - g_1 (T_{Z_1} \omega \phi Z_2, X_1) + g_1 (T_{Z_1} \omega Z_2, \phi PX_1) + g_1 (H_{V_{Z_1}} \omega Z_2, \omega RX_1). \]

That is,
\[ \sin^2 \theta_1 g_1 (V_{Z_1} Z_2, X_1) = - g_1 (T_{Z_1} \omega \phi Z_2, X_1) + g_1 (T_{Z_1} \omega Z_2, JPX_1) + g_1 (H_{V_{Z_1}} \omega Z_2, \omega RX_1). \]

From Equations (2), (3), (8), (12), (13) and the Lemma 4, we have
\[ g_1 (V_{Z_1} Z_2, X_2) = g_1 (V_{Z_1} JZ_2, JX_2) = g_1 (V_{Z_1} \phi Z_2, JX_2) + g_1 (V_{Z_1} \omega Z_2, JX_2) \]
\[ = \cos^2 \theta_1 g_1 (V_{Z_1} Z_2, X_2) - g_1 (H_{V_{Z_1}} \omega \phi Z_2, X_2) + g_1 (H_{V_{Z_1}} \omega Z_2, CX_2) + g_1 (T_{Z_1} \omega Z_2, BX_2). \]

So, we have
\[ \sin^2 \theta_1 g_1 (V_{Z_1} Z_2, X_2) = - g_1 (H_{V_{Z_1}} \omega \phi Z_2, X_2) + g_1 (H_{V_{Z_1}} \omega Z_2, CX_2) + g_1 (T_{Z_1} \omega Z_2, BX_2), \]
which completes the proof.

Similarly as above, we get the following theorem:

**Theorem 9.** $D_2$ is parallel if and only if

\[ g_1(\mathcal{H}(\nabla X, \omega RX_2, \omega QZ_1)) = - g_1(\nabla X, \omega RX_2, \phi PZ_1 + \phi QZ_1) \]

and

\[ g_1(\mathcal{H}(\nabla X, \omega RX_2, CZ_2)) = - g_1(\nabla X, \omega RX_2, BZ_2) \]

for all $X_1, X_2 \in \Gamma(D_2)$, $Z_1 \in \Gamma(D \oplus D_1)$ and $Z_2 \in \Gamma(\ker F^*)$.

**Proof.** For all $X_1, X_2 \in \Gamma(D_2)$, $Z_1 \in \Gamma(D \oplus D_1)$ and $Z_2 \in \Gamma(\ker F^*)$, using Equations (2), (3), (8), (11) and (12), we have

\[ g_1(\nabla X_1 X_2, Z_1) = g_1(\nabla X_1 JX_2, JZ_1) \]

\[ = g_1(\nabla X_1, \omega RX_2, \phi PZ_1 + \phi QZ_1 + \omega QZ_1) \]

\[ = g_1(\nabla X_1, \omega RX_2, \phi PZ_1 + \phi QZ_1) + g_1(\mathcal{H}(\nabla X_1, \omega RX_2, \omega QZ_1)) \]

Using Equations (2), (3), (8), (11) and (13), we have

\[ g_1(\nabla X_1 X_2, Z_2) = g_1(\nabla X_1 JX_2, JZ_2) \]

\[ = g_1(\nabla X_1, \omega RX_2, BZ_2 + CZ_2) \]

\[ = g_1(\nabla X_1, \omega RX_2, BZ_2) + g_1(\mathcal{H}(\nabla X_1, \omega RX_2, CZ_2)) \]

which shows our assertion.

**Theorem 10.** $F$ is a totally geodesic map if and only if

\[ g_1(\mathcal{H}(Z_1 PZ_2 + \cos^2 \theta_1 \mathcal{H}Z_1 QZ_2 - \mathcal{H}(\nabla V_1, \omega PZ_2 - \mathcal{H}(\nabla V_1, \omega PZ_2, V_1) = g_1(\mathcal{H}(Z_1 PZ_2 + \mathcal{H}(\nabla V_1, \omega PZ_2, B V_1) \]

\[ + g_1(\mathcal{H}(V_1, \omega PZ_2 + \mathcal{H}(\nabla V_1, \omega PZ_2, V_1) \]

and

\[ g_1(\mathcal{H}(V_1, PZ_4 + \cos^2 \theta_1 A_1 V_1 QZ_1 - \mathcal{H}(\nabla V_1, \omega PZ_4 - \mathcal{H}(\nabla V_1, \omega PZ_4, V_2) = g_1(\mathcal{H}(V_1, \omega PZ_4 + A_1 V_1 \omega RZ_1, B V_2) \]

\[ + g_1(\mathcal{H}(V_1, \omega PZ_4 + \mathcal{H}(\nabla V_1, \omega RZ_1, CV_2) \]

for all $Z_1, Z_2 \in \Gamma(\ker F^*)$ and $V_1, V_2 \in \Gamma(\ker F^*)$.

**Proof.** For $F$ is a Riemannian map, we have

\[ (VF_*) (V_1, V_2) = 0 \]
for all $V_1, V_2 \in \Gamma(\ker F_\ast)$. For all $Z_1, Z_2 \in \Gamma(\ker F_\ast)$ and $V_1, V_2 \in \Gamma(\ker F_\ast)$, using Equations (2), (3), (7), (8), (10), (11), (12), (13) and the Lemma 4, we have

$$g_2 ((VF_\ast)(Z_1, Z_2), F_\ast(V_1)) = -g_1 (\nabla_{Z_1} Z_2, V_1)$$

$$= -g_1 (\nabla_{Z_1} JZ_2, JV_1)$$

$$= -g_1 (\nabla_{Z_1} JPZ_2, JV_1) - g_1 (\nabla_{Z_1} JQZ_2, JV_1) - g_1 (\nabla_{Z_1} JRZ_2, JV_1)$$

$$= -g_1 (\nabla_{Z_1} \phi PZ_2, JV_1) - g_1 (\nabla_{Z_1} \phi QZ_2, JV_1) - g_1 (\nabla_{Z_1} \phi RZ_2, JV_1)$$

$$= -g_1 (\nabla_{Z_1} \phi PZ_2 + \cos^2 \theta_1 \nabla_{Z_1} QZ_2 - \nabla_{Z_1} \phi QZ_2 - \nabla_{Z_1} \phi RZ_2, JV_1) = -g_1 (\nabla_{Z_1} \phi QZ_2 + \nabla_{Z_1} \phi RZ_2, JV_1)$$

$$- g_1 (\nabla_{Z_1} \phi QZ_2 + \nabla_{Z_1} \phi RZ_2, JV_1).$$

Similarly, from Equations (2), (3), (7), (8), (10), (11), (12), (13) and the Lemma 4, we get

$$g_2 ((VF_\ast)(V_1, Z_1), F_\ast(V_2)) = -g_1 (\nabla_{V_1} Z_1, V_2)$$

$$= -g_1 (\nabla_{V_1} JPZ_1 + JV_2) - g_1 (\nabla_{V_1} JQZ_1, JV_2) - g_1 (\nabla_{V_1} JRZ_1, JV_2)$$

$$= -g_1 (\nabla_{V_1} \phi PZ_1, JV_2) - g_1 (\nabla_{V_1} \phi QZ_1, JV_2) - g_1 (\nabla_{V_1} \phi RZ_1, JV_2)$$

$$= -g_1 (A_{V_1} \phi PZ_1 + \cos^2 \theta_1 A_{V_1} QZ_1 - \nabla_{V_1} \phi PZ_1 - \nabla_{V_1} \phi QZ_1, JV_2) = -g_1 (A_{V_1} \phi QZ_1 + A_{V_1} \phi RZ_1, CV_2)$$

which completes the proof.

4. EXAMPLE

Let $(x_1, x_2, \ldots, x_{2n-1}, x_{2n})$ be coordinates on Euclidean space $\mathbb{R}^{2n}$. An almost complex structure $J$ on $\mathbb{R}^{2n}$ is defined by

$$J = \left( \begin{array}{cc}
\frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + \cdots + a_{2n-1} \frac{\partial}{\partial x_{2n-1}} + a_{2n} \frac{\partial}{\partial x_{2n}} \\
-a_2 \frac{\partial}{\partial x_1} + a_1 \frac{\partial}{\partial x_2} + \cdots - a_{2n-1} \frac{\partial}{\partial x_{2n-1}} + a_{2n} \frac{\partial}{\partial x_{2n}}
\end{array} \right)$$

where $a_1, a_2, \ldots, a_{2n}$ are $C^\infty$ functions defined on $\mathbb{R}^{2n}$. This notation will use throughout this section.

Example 1. Let $(\mathbb{R}^{14}, g_{14}, J)$ be an almost Hermitian manifold as defined above. $F: \mathbb{R}^{14} \rightarrow \mathbb{R}^8$ is defined by

$$F(x_1, x_2, \ldots, x_{14}) = (x_3 \sin \alpha + x_5 \cos \alpha, x_6, x_7, x_{10}, a, b, x_{13}, x_{14})$$

where $\theta_1 \in (0, \frac{\pi}{2})$ and $a, b \in \mathbb{R}$. Then $F$ is a quasi-hemi-slant Riemannian map (where rank $F_\ast = 6$) such that
\[ X_1 = \frac{\partial}{\partial x_1}, X_2 = \frac{\partial}{\partial x_2}, X_3 = \cos \alpha \frac{\partial}{\partial x_3} - \sin \alpha \frac{\partial}{\partial x_5}, X_4 = \frac{\partial}{\partial x_4}, X_5 = \frac{\partial}{\partial x_6}, X_6 = \frac{\partial}{\partial x_7}, X_7 = \frac{\partial}{\partial x_8}, X_8 = \frac{\partial}{\partial x_9}, \]

\[ \ker F_* = D \oplus D_1 \oplus D_2 \]

where

\[ D = \langle X_1 = \frac{\partial}{\partial x_1}, X_2 = \frac{\partial}{\partial x_2}, X_3 = \frac{\partial}{\partial x_3}, X_4 = \frac{\partial}{\partial x_4}, X_5 = \frac{\partial}{\partial x_6}, X_6 = \frac{\partial}{\partial x_7}, X_7 = \frac{\partial}{\partial x_8}, X_8 = \frac{\partial}{\partial x_9} \rangle, \]

\[ D_1 = \langle X_3 = \cos \alpha \frac{\partial}{\partial x_3} - \sin \alpha \frac{\partial}{\partial x_5}, X_4 = \frac{\partial}{\partial x_4} \rangle, \]

\[ D_2 = \langle X_5 = \frac{\partial}{\partial x_4}, X_6 = \frac{\partial}{\partial x_9} \rangle, \]

and

\[ (\ker F_*)^\perp = \left< \frac{\partial}{\partial x_6}, \sin \alpha \frac{\partial}{\partial x_3} + \cos \alpha \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8}, \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_{13}}, \frac{\partial}{\partial x_{14}} \right> \]

which \( D = \text{Span} \{ X_1, X_2, X_3, X_8 \} \) is invariant, \( D_1 = \text{Span} \{ X_3, X_4 \} \) is slant with slant angle \( \theta_1 = \alpha \) and \( D_2 = \text{Span} \{ X_5, X_6 \} \) is anti-invariant.

**Example 2.** Let \((\mathbb{R}^{12}, g_{12}, J)\) be an almost Hermitian manifold as defined above. \( F: \mathbb{R}^{12} \rightarrow \mathbb{R}^8 \) is defined by

\[ F(x_1, x_2, \ldots, x_{12}) = (x_1, x_2, c, x_5, \frac{x_7 + \sqrt{3} x_9}{2}, x_{10}, d, x_{12}) \]

where \( \theta_1 \in (0, \frac{\pi}{2}) \) and \( c, d \in \mathbb{R} \). Then \( F \) is a quasi-hemi-slant Riemannian map (where \( \text{rank} F_* = 6 \)) such that

\[ X_1 = \frac{\partial}{\partial x_3}, X_2 = \frac{\partial}{\partial x_4}, X_3 = \frac{\partial}{\partial x_6}, X_4 = \frac{\partial}{\partial x_7}, X_5 = \frac{\partial}{\partial x_8}, X_6 = \frac{\partial}{\partial x_9}, \]

\[ \ker F_* = \text{D} \oplus D_1 \oplus D_2, \]

where

\[ D = \langle X_1 = \frac{\partial}{\partial x_3}, X_2 = \frac{\partial}{\partial x_4} \rangle, \]

\[ D_1 = \langle X_4 = \frac{\partial}{\partial x_4}, X_5 = \frac{\partial}{\partial x_8} \rangle, \]

\[ D_2 = \langle X_6 = \frac{\partial}{\partial x_9} \rangle, \]

and

\[ (\ker F_*)^\perp = \left< \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8}, \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_{12}} \right> \]

which \( D = \text{span} \{ X_1, X_2 \} \) is invariant, \( D_1 = \text{span} \{ X_4, X_5 \} \) is slant with slant angle \( \theta_1 = \frac{\pi}{6} \) and \( D_1 = \text{span} \{ X_3, X_6 \} \) is anti-invariant.
CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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