Evaluation of the Axial Vector Commutator Sum Rule
for Pion-Pion Scattering

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Abstract

We consider the sum rule proposed by one of us (SLA), obtained by taking the expectation value of an axial vector commutator in a state with one pion. The sum rule relates the pion decay constant to integrals of pion-pion cross sections, with one pion off the mass shell. We remark that recent data on pion-pion scattering allow a precise evaluation of the sum rule. We also discuss the related Adler–Weisberger sum rule (obtained by taking the expectation value of the same commutator in a state with one nucleon), especially in connection with the problem of extrapolation of the pion momentum off its mass shell. We find, with current data, that both the pion-pion and pion-nucleon sum rules are satisfied to better than six percent, and we give detailed estimates of the experimental and extrapolation errors in the closure discrepancies.
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1. Introduction.

In refs. 1, 2 a sum rule was obtained by taking matrix elements of the commutator,

$$[\chi_+(t), \chi_-(t)] = 2I_3,$$

between one-nucleon states. Here $\chi^{\pm}$ are the chiral charges,

$$\chi_{\pm}(x_0) = \int d^3 x \, A_\pm^0(x),$$

$I_3$ is the third component of isospin and the $A_\pm^\mu$ are axial currents. In terms of quarks,

$$A_\pm^\mu(x) = \bar{u}(x)\gamma^\mu \gamma_5 d(x), \quad A_\pm^\mu(y) = \bar{d}(y)\gamma^\mu \gamma_5 u(x).$$

One may note that, in QCD, the commutation relation (1.1) is an exact theorem, that follows from the global symmetries of the QCD Lagrangian; in fact, (1.1) may be obtained by integrating the local axial vector commutation relation

$$\delta(x^0 - y^0)[A_\pm^0(x), A_\nu^\mu(y)] = 2\delta_4(x - y)V_3^\nu(x)$$

with $V_3^\nu(x) = \frac{1}{2}(\bar{u}\gamma^\nu u - \bar{d}\gamma^\nu d)$ the third component of the vector (isospin) current. This commutation relation is exact in QCD, irrespective of the value of the $u, d$ quark masses.

The sum rule obtained in refs. 1, 2 relates the axial weak charge, $g_A$, to integrals over pion-nucleon cross sections, with the pion off-shell (and of zero mass),

$$1 - \frac{1}{g_A^2} = \frac{2M_N^2}{g_\pi^2 K_{NN\pi}(0)^2} \frac{1}{\pi} \int_0^\infty \frac{ds}{s-M_N^2} \left[ \sigma_{0+}^+(s) - \sigma_{0+}^-(s) \right].$$

Here, $\sigma_0^\pm$ are the total cross section for scattering of a zero mass $\pi^\pm$ on a proton, this latter on its mass shell (with averages over the spin of the nucleon implicit).

Later, in ref. 3, it was noted that one can also take matrix elements of (1.1) between one pion states. In this way one obtains the relation

$$\frac{2}{g_A^2} = \frac{2M_N^2}{g_\pi^2 K_{NN\pi}(0)^2} \frac{1}{\pi} \int_0^\infty \frac{ds}{s-M_N^2} \left[ \sigma_{0+}^-(s) - \sigma_{0+}^+(s) \right],$$

(1.3) the exact Goldberger–Treiman relation[4] (see Sect. 2 below for the meaning of this)

$$1 = \frac{M_N g_A}{g_\pi K_{NN\pi}(0) F_\pi}$$

was used to relate the $\chi^{\pm}$ to the pion field. In these formulas, $M_\pi = 139.57$ MeV is the (charged) pion mass, $M_N$ the nucleon mass, $g_A$ is the weak axial coupling, $g_\pi$ the on-shell strong $NN\pi$ coupling, and $K_{NN\pi}(0)$ the nucleon-nucleon-pion vertex, for a zero mass pion, normalized to $K_{NN\pi}(M_\pi^2) = 1$. $F_\pi$ is the pion decay constant.

One can rewrite (1.2), (1.3) in a form that is more convenient by expressing the constant prefactor in terms of $F_\pi$, corresponding to writing the PCAC definition of the pion field in terms of $F_\pi$,

$$\partial \cdot A_\pm = \sqrt{2} F_\pi M_\pi^2 \phi_\pm$$

(1.4) $\phi_\pm$ are the fields for $\pi^\pm$ and also expressing the sum rule in terms of scattering amplitudes, rather than cross sections. In this case we find that (1.2), (1.3) are replaced by the relations

$$g_A^2 = 1 + 8\pi F_\pi^2 \int_0^\infty \frac{ds}{(s-M_N^2)^2} \left\{ \text{Im } F_{\pi^+,-,\rho}(k^2 = 0; s, t = 0) - \text{Im } F_{\pi^+,+,-,\rho}(k^2 = 0; s, t = 0) \right\}$$

and

$$1 = 4\pi F_\pi^2 \int_0^\infty \frac{ds}{(s-M_N^2)^2} \left\{ \text{Im } F_{\pi^+,+,-}(k^2 = 0; s, t = 0) - \text{Im } F_{\pi^+,-,+}(k^2 = 0; s, t = 0) \right\}.$$
Here $F_{\pi^+,p}(k^2, s, t = 0)$ stands for the forward pion-proton scattering amplitude,

$$\pi^\pm(k) + N(p) \rightarrow \pi^\pm(k) + N(p)$$

with the nucleon $N(p)$ on its mass shell, $p^2 = M_N^2$, while the momentum of the pion is off-shell, $k^2 = 0$ in (1.5). Likewise, $F_{\pi^+,\pi^-}(k^2 = 0; s, t = 0)$ is the forward scattering amplitude for a $\pi^\pm(k)$ with momentum $k^2 = 0$ on a $\pi^-(p)$ on its mass shell, $p^2 = M_N^2$:

$$\pi^\pm(k) + \pi^-(p) \rightarrow \pi^\pm(k) + \pi^-(p).$$

In the form (1.6), the pionic sum rule is seen as a remarkable relation that allows one to calculate the pion decay constant in terms of (off-shell) pion-pion scattering amplitudes. In ref. 3, this sum rule for $\pi\pi$ scattering in the S0 wave was so scanty that it was only possible to note that fulfillment of the sum rule requires a strong $\pi\pi$ interaction in this S0 wave. The situation has improved dramatically at present; precise and reliable experimental data on the $\pi\pi$ and $\pi\pi$ scattering, and also on the question of extrapolation to the off-shell $\pi\pi$ scattering amplitudes, have appeared in the last years, which will allow us to present an evaluation of (1.3), (1.6) to a few percent accuracy.

The nucleon sum rule, known as the Adler–Weisberger sum rule, (1.2), (1.5) was already evaluated in refs. 1, 2, 3; however, we will present here a new calculation: the precision of the pionic sum rule is much less precise than that for $\pi\pi$ scattering in the S0 wave, and also on high energy scattering, have appeared in the last years, which

In the present paper we will only consider the sum rules (1.5), (1.6); but, by taking expectation values of (1.1) between kaon states one can get a sum rule identical to (1.5), replacing the $g_A^2$ in the left hand side by zero, the nucleon mass by the kaon mass, and pion-nucleon by pion-kaon amplitudes: e.g., for a $K^+$ state,

$$1 = 8\pi F_\pi^2 \int_{(M_K+M_N)^2}^{\infty} \frac{ds}{(s - M_K^2)^2} \left\{ \text{Im} F_{\pi^-,K^+}(k_\pi^2 = 0; s, t = 0) - \text{Im} F_{\pi^+,K^+}(k_\pi^2 = 0; s, t = 0) \right\}. \tag{1.7}$$

It is also possible to find a complementary relation to (1.7), but extrapolating in the mass of the kaon. Consider the currents $A_{K^+}^\mu = \bar{u}\gamma^\mu\gamma_5 s$, $A_{K^-}^\mu = \bar{s}\gamma^\mu\gamma_5 u$; $A_{K^0}^\mu = \bar{d}\gamma^\mu\gamma_5 s$, $A_{K^-}^\mu = \bar{s}\gamma^\mu\gamma_5 d$, and the corresponding chiral charges, $\chi_{K^+}, \chi_{K^-}; \chi_{K^0}, \chi_{K^-}$. We have

$$[\chi_{K^+}, \chi_{K^-}] - [\chi_{K^0}, \chi_{K^-}] = 2I_4.$$ 

Taking expectation values on a $\pi^-$ state we now find the sum rule

$$1 = 4\pi F_K^2 \int_{(M_K+M_N)^2}^{\infty} \frac{ds}{(s - M_K^2)^2} \left\{ \text{Im} F_{K^+,\pi^-}(k_K^2 = 0; s, t = 0) + F_{K^0,\pi^-}(k_K^2 = 0; s, t = 0) ight\}$$

$$- \text{Im} F_{K^-,\pi^-}(k_K^2 = 0; s, t = 0) - \text{Im} F_{K^0,\pi^-}(k_K^2 = 0; s, t = 0) \right\} \tag{1.8} \right.$$ 

$$= 8\pi F_K^2 \int_{(M_K+M_N)^2}^{\infty} \frac{ds}{(s - M_K^2)^2} \left\{ \text{Im} F_{K^+,\pi^-}(k_K^2 = 0; s, t = 0) - \text{Im} F_{K^+,\pi^+}(k_K^2 = 0; s, t = 0) \right\},$$

the last relation using isospin and charge conjugation invariance. We here leave the pion on its mass shell, and extrapolate in the kaon mass. Although there are some studies on pion-kaon scattering, variously using chiral perturbation theory, dispersive techniques and phenomenological information (see ref. 5 at low energies, and at high energies, the Regge calculations of ref. 10), the experimental information on $\pi K$ scattering amplitudes is much less precise than that for $\pi\pi$ scattering. However, having two different masses to extrapolate ($M_\pi$ and $M_K$) would perhaps give some insight on the matter of extrapolation: the only differences between (1.7) and (1.8) are $F_K = 113.00 \pm 1.06$ MeV instead of $F_\pi$, the different masses that are sent to zero, and the masses that appear in the denominators in the integrals.

Similarly, as discussed by Weisberger, ref. 5, by taking appropriate chiral charges in SU(3), and taking expectation values on nucleons one obtains relations involving kaon-nucleon scattering. The pion-pion, pion-kaon or pion-nucleon sum rules require only an extrapolation in the pion mass to yield an on-shell relation, while the kaon-nucleon (or kaon-pion, Eq. (1.8)] sum rule requires extrapolation in the much larger kaon mass to give a physical relation, which is therefore expected to be less precise than the ones studied here, but could also give information on the question of extrapolation on the meson masses. Nevertheless, and as already stated we will, in the present paper, concern ourselves only with the pion-pion and pion-nucleon
sum rules, for which good experimental data exist, and which require extrapolation only to \( M_{\pi}^2 \); leaving the rest for future work.

2. The Goldberger–Treiman relation

Although the Goldberger–Treiman relation\(^4\) is not the object of this paper, we say a few words about it as we will use some information connected with it. We first have what may be called the exact or off-shell Goldberger–Treiman relation,

\[
\frac{M_N g_A}{K_{NN\pi}(0) g_{\pi} F_\pi} = 1. \tag{2.1}
\]

This involves the pion-nucleon coupling, with the pion off its mass shell (with momentum \( k^2 = 0 \)). One can define the quantity (the G.T. discrepancy)

\[
\Delta_{G.T.} \equiv 1 - \frac{M_N g_A}{g_{\pi} F_\pi}, \tag{2.2}
\]

which measures the validity of the approximation \( K_{NN\pi}(0) \approx K_{NN\pi}(M_{\pi}^2) = 1 \) that is used to get the on-shell Goldberger–Treiman relation \( F_{\pi} = M_N g_A / g_{\pi} \).

With the present values \( g_A = 1.2695 \pm 0.0029, F_\pi = 92.42 \pm 0.26 \) (both from the Particle Data Tables\(^6\)) and with \( M_N = 938.9 \) (average n-p mass) and taking the on-shell pion nucleon coupling constant\(^7\) \( g_{\pi} = 13.2 \pm 0.2 \), the relation (2.2) equals

\[
\Delta_{G.T.} = 0.023 \pm 0.016, \tag{2.3}
\]

so the effect of approximating \( K_{NN\pi}(0) \) by unity appears to be small. This corrects most of the mismatch studied by Pagels and Zepeda,\(^8\) which turns out to be largely due to an underestimated \( g_A \) (a possibility that they actually considered); the remainder in (2.3) can easily be attributed, as was done in ref. 8, to the contribution of the \( \pi(1300) \) resonance to \( K_{NN\pi}(0) \), expected to be of \( O[M_{\pi}^2/M_{\pi(1300)}^2] \sim 1\% \).

One can look at the off-shell Goldberger–Treiman relation in a different way, as providing the value of the quantity \( K_{NN\pi}(0) \): (2.1) tells us that

\[
K_{NN\pi}(0) = \frac{M_N g_A}{F_\pi} = 0.977 \pm 0.015. \tag{2.4}
\]

In fact, from the careful analysis of Pagels and Zepeda, it follows that most of the deviation of \( K_{NN\pi}(0) \) from unity is due to the fact that, since the pion is off-shell, and the corresponding Green’s function is not amputated, the \( N N\pi \) vertex must contain a factor \( M_{\pi}^2 \Pi(0) \), where \( \Pi(k^2) \) is the pion propagator normalized to \( 1 = (M_{\pi}^2 - k^2) \Pi(k^2)|_{k^2=M_{\pi}^2} \). Thus, one expects

\[
K_{NN\pi}(0) \approx M_{\pi}^2 \Pi(0), \tag{2.5}
\]

which will play a role in the extrapolation discussion for the sum rules below. In fact, the same factor \( M_{\pi}^2 \Pi(0) \) will appear in the sum rules because, both in the pion-pion and pion-nucleon cases, the propagator corresponding to the off-shell pion line is not amputated. Use of \( \Delta_{G.T.} \) to estimate off-shell extrapolation corrections has also been discussed by Dominguez.\(^8\)

3. Calculation of the sum rule on pions

3.1. The sum rules

We present here a sketch of the derivation of the sum rules, for ease of reference; more details may be found in refs. 1, 2, 3. We will treat in detail the pionic case derived in ref. 3, and indicate the modifications necessary for the nucleon case.
We first take the expectation value of the commutator (1.1) between physical π⁻ states, and introduce a sum over a complete set of states: we find
\[
\langle \pi^-(p') | 2 F_3 | \pi^-(p) \rangle = -2 \times 2 p_0 \delta(p - p') &= \int \frac{d^3q}{2q_0} \sum_{\text{INT}} \langle \pi^-(p') | \chi_+(t) | q; \text{INT} \rangle \langle q; \text{INT} | \chi_-(t) | \pi^-(p) \rangle - (\leftrightarrow),
\]
(3.1)
where by \( | q; \text{INT} \rangle \) we denote a physical intermediate state with total momentum \( q \) and internal degrees of freedom “INT”. Next, using the PCAC definition (1.4), one can relate
\[
\langle q; \text{INT} | \chi_{\pm} (x_0) | \pi^-(p) \rangle = \frac{-\sqrt{2} i}{p_0 - q_0} F_\pi M_\pi^2 \langle q; \text{INT} | \int d^3 x \, \phi_\pm (x) | \pi^-(p) \rangle
\]
with \( \phi_\pm (x) \) the field operator for \( \pi^\pm \). Working in the infinite momentum frame \( (p \rightarrow \infty) \) gives the sum rule
\[
1 = 2 \pi F_\pi^2 \int ds \frac{1}{(s - M_\pi^2)^2} \sum_{\text{INT}} \left\{ | F(\pi^+(k^2 = 0), \pi^-(p) \rightarrow q; \text{INT})|^2 - | F(\pi^-(k^2 = 0), \pi^-(p) \rightarrow q; \text{INT})|^2 \right\};
\]
(3.2a)
and
\[
F(\pi^\pm(k^2 = 0), \pi^-(p) \rightarrow q; \text{INT}) = (2\pi)^{5/2} \langle q; \text{INT} | M_\pi^2 \phi_\pm (0) | \pi^-(p) \rangle
\]
(3.2b)
is the amplitude for a pseudoscalar current, with virtual four momentum \( k \), \( k^2 = 0 \), to scatter off a physical \( \pi^- \), \( p^2 = M_\pi^2 \), into the physical intermediate state \( | q; \text{INT} \rangle \). Using extended unitarity, this may be written as the sum rule (1.6), which we repeat here in the form of a discrepancy, \( \Delta_\pi = 0 \), with
\[
\Delta_\pi \equiv 4 \pi F_\pi^2 \int_{M_\pi^2}^\infty \frac{ds}{(s - M_\pi^2)^2} \left\{ \text{Im} \, F_{\pi^+, \pi^-}(k^2 = 0; s, t = 0) - \text{Im} \, F_{\pi^-, \pi^+}(k^2 = 0; s, t = 0) \right\} - 1.
\]
(3.3)
In these formulas, the \( F_{\pi^\pm, \pi^-}(k^2 = 0; s, t = 0) \) are the forward scattering amplitudes for an off-shell pion with zero mass. For an on-shell pion, the corresponding scattering amplitude is obtained by replacing, in (3.2b),
\[
F(\pi^\pm(k^2 = 0), \pi^-(p) \rightarrow q; \text{INT}) &= (2\pi)^{5/2} \langle q; \text{INT} | M_\pi^2 \phi_\pm (0) | \pi^-(p) \rangle \bigg|_{k^2 = 0} - (2\pi)^{5/2} \langle q; \text{INT} | (M_\pi^2 + \Box \phi_\pm (0)) | \pi^-(p) \rangle \bigg|_{k^2 = M_\pi^2}
\]
(3.4)
and using unitarity to perform the sum over intermediate states.
In QCD one expects the mass scale for the internal dynamics to be given by a parameter \( \mu_0 \) of the order of the parameter \( \Lambda \sim 0.4 \text{ GeV} \) or the rho resonance mass, \( M_\rho \); thus, to an error \( M_\pi^2 / \mu_0^2 \sim 10\% \) or smaller, we can relate (3.3) to physical quantities by approximating the off-shell scattering amplitudes by the physical scattering amplitudes as in (3.4).
The sum rule (1.2), (1.5) on nucleons is derived in a similar manner. The only differences are the replacement of the pion mass by the nucleon mass in the denominator corresponding to (3.3), the different isospin of the proton (that results in a factor \(-1/2 \) with respect to that for the \( \pi^- \)) and that, due to the existence of the neutron intermediate state, we find the extra term proportional to \( g_A^2 \) in (1.5) because the proton-neutron matrix element of the divergence \( \langle p | (\partial \cdot A) | n \rangle \) is proportional to \( g_A \). The pion-kaon and kaon-pion sum rules are also similar to the pion-pion one.

3.2. The pion sum rule in the on-shell approximation

In the form (3.3), the sum rule is an exact theorem, following from the commutation relation (1.1) and the definition (1.4); but, of course, direct comparison with experiment is precluded by the fact that the amplitudes that appear in (3.3) involve a pion off its mass shell. As stated in the previous subsection, a first approximation is obtained by neglecting the fact that (3.3) is defined for off-shell pseudoscalar currents, i.e., working in the approximation (3.4). The sum rule is then,
1 \approx 4 \pi F_\pi^2 \int_{4M^2_\pi}^{\infty} \frac{ds}{(s - M^2_\pi)^2} \left\{ \text{Im} F_\pi^-(s) - \text{Im} F_\pi^+(s) \right\} \tag{3.5}

with $F_\pm(s)$ physical, forward $\pi^\pm\pi^-$ scattering amplitudes. One can write these scattering amplitudes in terms of amplitudes with well-defined isospin in the $t$ or $s$ channels,

$$F_\pi^-(s) - F_\pi^+(s) = F(I_s=1) = \frac{1}{3} F(I_s=0) + \frac{1}{2} F(I_s=1) - \frac{5}{6} F(I_s=0), \tag{3.6}$$

the $F$ are normalized so that, for pions on their mass shell, and in the elastic region, one has

$$F(I_s=0) = \frac{2}{\pi k} \sum_l (2l + 1) \sin \delta_l^{(I_s)} e^{i\eta_l^{(I_s)}}. \tag{3.7}$$

The factor 2 in front of the right hand side is due to identity of the particles and $k = \frac{1}{2} \sqrt{s - 4M^2_\pi}$ is the center of mass momentum, for physical pions.

In this case, we can use the precise determinations of the pion-pion scattering amplitudes, obtained fitting experimental data, that have been found recently\cite{9,10} thanks to the availability of very precise data: on the low energy $S_0$ wave from kaon decays and, for the $P$ wave, from determinations of the pion form factor.\cite{11} One finds, for the contributions of the various waves to the right hand side of (3.5) for energy below 1420 MeV,\cite{9}

$$S_0; \frac{s^{1/2}}{2} \leq 932 \text{ MeV} : \quad 0.408 \pm 0.013$$
$$S_0; 932 \leq \frac{s^{1/2}}{2} \leq 1420 \text{ MeV} : \quad 0.043$$
$$D_0 : \quad 0.097 \pm 0.003$$
$$P : \quad 0.403 \pm 0.003$$
$$F : \quad 0.0016$$
$$S_2 : \quad -0.090 \pm 0.005$$
$$D_2 : \quad -0.0023; \tag{3.8a}$$

errors are only given for the more significant pieces. The results are similar to those already obtained in ref. 3 (although, of course, now much more precise) for the contributions of $P$, $D_0$, $S_2$ waves. What is new is the contribution of the $S_0$ wave, which turns out to be the most important of all, thus confirming the prediction in ref. 3 that an important $S_0$ wave contribution is needed to saturate the sum rule.

For the (Regge) contribution above 1420 MeV we use the Regge formula

$$\text{Im} F^{(1)}(s) = (1.22 \pm 0.14)(s/1 \text{ GeV}^2)^{0.42}$$

and then find\cite{10}

$$\frac{s^{1/2}}{2} > 1420 \text{ MeV} : \quad 0.167 \pm 0.017. \tag{3.8b}$$

Altogether, the right hand side of (3.5) now reads

$$1.027 \pm 0.022, \quad \text{i.e.,} \quad \Delta_\pi = 0.027 \pm 0.022. \tag{3.9}$$

The error is due to the experimental errors in the pion-pion scattering amplitudes in (3.8). Therefore, we only have a discrepancy of $(2.7 \pm 2.2)\%$ in the fulfillment of the sum rule in this approximation.

A few words may be said on the smallness of the error in (3.9). This is due to the fact that, as stated, recent experimental results have allowed us to get very precise fits to data at low energy. Moreover, the larger contributions to the final result come from independent sources, so one can add their errors quadratically. And, finally, all the large contributions ($S_0$, $P$, $D_0$ waves and Regge region) are positive: only the relatively small $S_2$ and $D_2$ contributions produce cancellations. As we will see, the situation is less favourable for the sum rule on nucleons, where large cancellations take place.
3.3. Extrapolation

To improve the evaluation using (3.5) one can calculate by taking the recipe of ref. 3 for extrapolation, which takes into account threshold kinematic effects by replacing (3.7), in the elastic region, by

$$F_0^{(I_e)} = \frac{4s^{1/2}}{\pi k} \sum_l (2l + 1) \left[\frac{k^{(0)}}{k}\right]^{2l} \sin \delta_l^{(I_e)} e^{i\delta_l^{(I_e)}},$$

(3.10)

where $k^{(0)} = (s - M_N^2)/2s^{1/2}$ is the center of mass momentum for a pion of zero mass incident on a target pion that is on mass shell. This does not take into account the effects of the extrapolation at high energies, $s^{1/2} > 1.42$ GeV, likely below the 1% level, which we neglect. We have verified that this recipe works in a model calculation in which the interactions are generated by effective Lagrangians

$$g_\rho(\vec{\phi} \times \partial_\mu \vec{\phi}) \rho_\mu, \quad g_\sigma(\vec{\phi} \partial_\mu \phi) \sigma_{\mu\nu}, \cdots$$

coupling pions to various resonances $[\rho, f_2(1275), \ldots]$. In these models, the off-shell correction is valid not only at threshold but throughout the resonance region, which justifies our using it here in the elastic region.

The recipe in (3.10) then gives a discrepancy

$$\Delta_\pi = 0.069 \pm 0.023.$$  

(3.11)

This deteriorates the sum rule, which seems to imply that the dynamical effects of the extrapolation are not negligible. In fact, this could be expected; the replacement (3.10) does not affect the S waves, in particular the S0 wave, which is the one that contributes most to the sum rule.

A possible way to estimate at least part of the dynamical correction is to assume it to be universal, and take it from the Goldberger–Treiman relation: thus multiplying the right hand side of (3.11) by the factor $K_{NN\pi}^2(0) = 0.955 \pm 0.03$, as was done in ref. 3; see also ref. 8. A motivation for this was already given when we discussed the Goldberger–Treiman relation (end of Sect. 2); the motivation for this in ref. 3 was the observation that in a field theory of pions and nucleons, a zero mass pion must couple to a physical pion through a virtual nucleon loop, and so the factor $K_{NN\pi}^2(0)$ should be present (of course, both motivations are not exclusive, but complementary). In terms of QCD, an analogous observation is that an off-shell pion couples to both nucleons and pions through a coupling to a single quark line, suggesting that a universal off-shell factor may be present. It would be of interest to pursue this idea further within a QCD framework; see also the discussion in Sect. 5 below.

Including this factor $K_{NN\pi}^2(0)$ improves the fulfillment of the sum rule to

$$\Delta_\pi = 0.021 \pm 0.023,$$  

(3.12)

an agreement as good as one can wish. However, as we will see in the case of the nucleon sum rule, following the same procedure of including a $K_{NN\pi}^2(0)$ factor depreciates, rather than improves, agreement with experiment. Hence a more conservative procedure is to take the difference between (3.11) and (3.12) as a measure of the uncertainty in the extrapolation procedure and thus write for the discrepancy, Eq. (3.3b),

$$\Delta_\pi = 0.021 \pm 0.023 \pm 0.048 \text{ (Exp.)} \pm 0.048 \text{ (Extr.)},$$  

(3.13)

showing explicitly the error arising from experimental errors in the pion-pion amplitudes and the estimated error of the extrapolation procedure. This will be our final result for the sum rule on pions.

3.4. Connection with chiral perturbation theory

Consider the forward dispersion relation for the (physical) amplitude $F^{(I_e=1)}(s)$,

$$F^{(I_e=1)}(s) = \frac{2s - 4M_N^2}{\pi} \int_{M_N^2}^{\infty} ds' \frac{\text{Im} F^{(I_e=1)}(s')}{(s' - s)(s' + s - 4M_N^2)}.$$  

1 Actually, we make the corresponding replacement up to the Regge region, $s^{1/2} \simeq 1.42$ GeV.
Evaluating it at threshold, and writing \( F(I_t=1)(4M_P^2) \) in terms of scattering lengths, we find the so-called (first) Olsson sum rule\^[12] \[ which \ is \ thus \ an \ exact \ consequence \ of \ the \ dispersion \ relation, \ independent \ of \ the \ current \ algebra \ commutator \ (1.1): \]

\[
2a_0^{(0)} - 5a_0^{(2)} = 3M_\pi \int_{4M_P^2}^{\infty} ds \frac{\text{Im} F(I_t=1)(s)}{s(s - 4M_P^2)}.
\]

(3.14)

One can evaluate the scattering lengths to lowest order (l.o.) in chiral perturbation theory, i.e., to lowest order in \( M_P^2 \) (actually, this is strictly equivalent to the old Weinberg\^[13] evaluation) finding

\[
2a_0^{(0)} - 5a_0^{(2)} \bigg|_{l.o.} = \frac{3M_\pi}{4\pi F_\pi^2}
\]

and, substituting in (3.14), the l.o. relation

\[
1 = 4\pi F_\pi^2 \int_{4M_P^2}^{\infty} ds \frac{\text{Im} F(I_t=1)(s)}{s(s - 4M_P^2)} \bigg|_{l.o.}
\]

(3.15)

To l.o. (which implies \( M_P^2 \approx 0 \)) Eq. (3.15) is equivalent to (1.6) since

\[
\int_{4M_P^2}^{\infty} ds \frac{\text{Im} F(I_t=1)(s)}{s(s - 4M_P^2)} \bigg|_{l.o.} \approx \int_{4M_P^2}^{\infty} ds \frac{\text{Im} F(I_t=1)(s)}{(s - M_P^2)^2} \bigg|_{l.o.} \approx \int_{4M_P^2}^{\infty} ds \frac{\text{Im} F(I_t=1)(k^2 = 0; s, t = 0)}{(s - M_P^2)^2} \bigg|_{l.o.}
\]

The same conclusion is reached if we use the Froissart–Gribov representation of the P wave scattering length.

Of course, (3.15) differs in status from (1.6) as the latter is valid exactly, whereas the l.o. expression \( 2a_0^{(0)} - 5a_0^{(2)} \bigg|_{l.o.} = 3M_\pi/4\pi F_\pi^2 \) is known to have large loop corrections.\^[14]

Another connection with results based on chiral perturbation theory is obtained by remarking that, using chiral methods to two loops, plus analyticity (in the form of Roy equations) and extra experimental information, has led Colangelo, Gasser and Leutwyler\^[15] to propose parametrizations of the S0, S2 and P waves at low energy consistent with these requirements. Substituting them into Eq. (3.5) gives the result \( \Delta_P = 0.046 \), to be compared with what we found using experimental data in Eq. (3.9), \( 0.027 \pm 0.022 \). Likewise, implementing off-shell corrections as in (3.10) gives \( \Delta_P = 0.086 \) \{to be now compared with 0.069 \pm 0.023 from Eq. (3.11)\} and, finally, applying the correction deduced from the off-shell Goldberger–Treiman relation under the universality assumption gives \( \Delta_P = 0.033 \), a number which is also similar to what was found using experimental \( \pi\pi \) data in Eq. (3.12), viz., \( 0.021 \pm 0.023 \).\^[2]

We emphasize that it is not our intention in this paper to give a detailed review of either chiral perturbation theory or the Olsson sum rule, which are substantial topics in their own right. We are only interested in elucidating the connection between the Olsson relation and the pion-pion sum rule (1.6), which, as far as we know, can only be established to leading order in chiral perturbation theory. Tests of the Olsson relation by itself have been made extensively in the literature; see, for example, refs. 14, 15 for evaluations using two-loop chiral perturbation theory, and the papers of Kamiński, Peláez and Ynduráin and of Peláez and Ynduráin in ref. 9 for calculations using experimental data prior to 2006. The Olsson relation has also been verified including more recent data (those of the NA48/2 collaboration, ref. 11) as in our evaluation of the pion-pion sum rule here, by Kamiński, Peláez, and Ynduráin (unpublished work in progress).

4. The Adler–Weisberger sum rule

4.1. An approximate calculation

The Adler–Weisberger sum rule (1.2), (1.5), has been evaluated in a number of papers (besides the original ones); a recent article is ref. 17. If we approximate the amplitudes in (1.5) by the scattering amplitudes for

\^[2] The numbers 0.046, 0.086 and 0.033 were kindly communicated to us by H. Leutwyler\^[16] as an independent check on the evaluation of Sect. 3.2, using different methods.
a physical pion, $k^2 = M^2$, then we get the sum rule

$$ g_A^2 \simeq 1 + 8\pi f^2 \int_{(M_N + M_p)^2}^{\infty} \frac{ds}{(s - M^2_N)} \left\{ \text{Im} F_{\pi^+}(s) - \text{Im} F_{\pi^-}(s) \right\} $$  \hspace{1cm} (4.1) 

with $F_{\pi^\pm}(s)$ the physical, forward $\pi^\pm p$ scattering amplitudes, normalized so that the pion-proton cross sections are

$$ \sigma_{\pi^\pm p}(s) = \frac{4\pi^2}{\lambda^{1/2}(s, M^2_N, M^2_N)} \text{Im} F_{\pi^\pm}(s), $$

with the function $\lambda(a, b, c)$ as defined below. In terms of $s$-channel isospin amplitudes, we have

$$ F_{\pi^+}(s) - F_{\pi^-}(s) = \frac{2}{3} \left\{ F^{(I_s=3/2)} - F^{(I_s=1/2)} \right\} $$

where the partial wave expansion in the elastic region is

$$ F^{(I)}(s) = \frac{2s^{1/2}}{\pi k} \sum_l \left\{ (l + 1) \sin \delta^{(I)}_l e^{i\delta^{(I)}_l} + l \sin \delta^{(I)}_l e^{i\delta^{(I)}_l} \right\}; $$ \hspace{1cm} (4.2) 

the phase shifts $\delta^{(I)}_{l\pm}$ correspond to isospin $I$, orbital angular momentum $l$, and total angular momentum $j = l \pm \frac{1}{2}$. The center of mass momentum is

$$ k = \sqrt{\frac{\lambda(s, M^2_N, M^2_N)}{s}}, \quad \lambda(a, b, c) = [a^2 - (b + c)^2][a^2 - (b - c)^2]. $$

The high energy part, $s^{1/2} > 2374$ MeV of the integral in (4.1) is easily evaluated with the fit to $\pi N$ cross sections in ref. 10: Regge formulas are good approximations for kinetic energies above 1 GeV. We will use the Regge formula

$$ \text{Im} F_{\pi^+}(s) - \text{Im} F_{\pi^-}(s) = (0.42 \pm 0.04)(s/1 \text{ GeV}^2)^{0.42}, $$

and integrate this from $s^{1/2} = 2374$ MeV to infinity. This would give a result of $-0.091$. Alternatively, we can use numerical values of the cross sections given in ref. 18 from $s^{1/2} = 2374$ MeV to $s^{1/2} = 3004$ MeV, and use the Regge formula above this latter energy. This would give $-0.087$. We consider this last number to be the more reliable one and then write, taking errors into account,

$$ 8\pi f^2 \int_{(2374 \text{ MeV})^2}^{\infty} \frac{ds}{(s - M^2_N)^2} \left\{ \text{Im} F_{\pi^+}(s) - \text{Im} F_{\pi^-}(s) \right\} = -0.087 \pm 0.006. $$  \hspace{1cm} (4.3a) 

On the other hand, numerical evaluation of the low energy piece, using the numerical cross sections collected in ref. 18 gives

$$ 8\pi f^2 \int_{(M_N + M_p)^2}^{(2374 \text{ MeV})^2} \frac{ds}{(s - M^2_N)^2} \left\{ \text{Im} F_{\pi^+}(s) - \text{Im} F_{\pi^-}(s) \right\} = 0.460 \pm 0.024. $$  \hspace{1cm} (4.3b) 

This large error is due to the fact that the number in the right hand side is the difference between two large numbers: specifically,

$$ 0.46 = 1.71 \text{ [from } \pi^+\text{]} - 1.25 \text{ [from } \pi^-\text{].} $$ \hspace{1cm} (4.4) 

Substituting the value of $g_A$ the sum rule (4.1) reads

$$ 1.612 \pm 0.006 = 1 + 0.373 \pm 0.025. $$ \hspace{1cm} (4.5) 

We may define a discrepancy $\Delta_A\,_{W}$ as the difference between $g_A^2$ and the right hand side of Eq. (1.5), and express the result in (4.5) as a largish mismatch,

$$ \Delta_A\,_{W} = 0.239 \pm 0.025. $$ 

In the present case, there are various substantial cancellations: as already stated, there are cancellations between the $\pi^+p$ and $\pi^-p$ cross sections at low energy, but there is also a cancellation between the
low energy region and higher energy \((s^{1/2} > 1.390\text{ GeV})\) contributions. For example, if we only integrated up to and including the \(\Delta(3,3)\) resonance region, we would have obtained

\[
8\pi f_{\pi}^2 \int_{(M_N + M_\pi)^2}^{(1390\text{ MeV})^2} \frac{ds}{(s - M_N^2)^2} \left\{ \text{Im} F_{\pi^+}(s) - \text{Im} F_{\pi^-}(s) \right\} = 0.70.
\]

These cancellations amplify the errors in the sum rule, and indicate that the effects of the extrapolation to a zero mass pion, which affect mostly low energies, are now very important, as already remarked in refs. 1 and 3.

4.2. Extrapolation

To perform the extrapolation, we repeat the method used for the case of the sum rule on pions, and replace the expression (4.2) for the scattering amplitude by

\[
F_0^{(J)}(s) = \frac{2s^{1/2}}{\pi k} \sum_l \left[ \frac{k^{(0)}}{k} \right]^{2l} \left\{ (l + 1) \sin \delta_{l+} e^{i\delta_{l+}} + l \sin \delta_{l-} e^{i\delta_{l-}} \right\}
\]

with \(k^{(0)}\) the momentum for an incident pion of zero mass,

\[
k^{(0)} = \frac{1}{2} \sqrt{\frac{\lambda(s, 0, M_N^2)}{s}}.
\]

We integrate this in the elastic region, \(s^{1/2} \lesssim 1.5\text{ GeV}\), using the parametrizations of ref. 19. These parametrizations have been obtained by fitting up to energies of, respectively, 1.3 GeV and 1.38 GeV; however, we have verified that they continue to fit the experimental cross sections up to \(\sim 150\text{ MeV}\) above their nominal maximum. Between these energies and 1.9 GeV, one can use a resonance saturation model, with the resonance parameters taken from the PDT\(^6\) plus a background, estimated as the tail of the lower energy resonances. Above 1.9 GeV, there are not well determined values for the resonance parameters and, moreover, a resonance model will cease to be valid as one is entering the Regge regime. Fortunately, one can likely neglect the effects of the extrapolation above such energy, and so this will be done here.

The results one gets for the extrapolation correction are

\[
0.225: \text{ up to } 1460\text{ MeV}; \quad -0.065: \text{ resonances, from } 1460\text{ MeV}; \quad 0.005: \text{ background, from } 1460\text{ MeV}.
\]

Adding this and considering as an error estimate the variation of the result if we vary the matching point of the parametrizations and the resonance model from 1460 to 1420 MeV or 1520 MeV, we get a correction of 0.165 ± 0.009, and hence the sum rule becomes

\[
1.612 \pm 0.006 = 1 + (0.373 \pm 0.025) + (0.165 \pm 0.009) = 1.538 \pm 0.034.
\]

We have added the errors linearly, as they are clearly correlated. The results show reasonable fulfillment of the Adler–Weisberger sum rule, \(\Delta_{A,W.} = 0.074 \pm 0.034\).

If we include a global correction, as we did in the pionic case, multiplying the r.h.s. of (4.9) by the factor \(K_{NN\pi}^2(0) = 0.955 \pm 0.03\), the sum rule now deteriorates to

\[
1.612 \pm 0.006 = 1 + [(0.373 \pm 0.025) + (0.165 \pm 0.009)] \times (0.955 \pm 0.030) = 1.514 \pm 0.038.
\]

We can now write our final result, as we did for the pionic case, as (4.10), adding as an extra error the difference between (4.10) and (4.9):

\[
\Delta_{A,W.} = 0.098 \pm 0.006 \left[ g_A^2 \right] \pm 0.031 \text{ [Exp.]} \pm 0.035 \text{ [Extr.]}\]

To compare with the results obtained in refs. 1 and 3, we note that these papers did not multiply through by a factor of \(g_A^2 = 1.612\), and included the \(K_{NN\pi}^2(0)\) extrapolation factor. Thus, the relevant number to use is \((0.514 \pm 0.038)/1.612 = 0.319 \pm 0.024\). The comparable number in refs. 1 and 3 is \((4M_N^2/g_A^2)(R_1 + R_2 + R_3) = 0.254 + 0.155 - 0.061 = 0.348\), with a roughly estimated error of \(\pm 0.025\). Hence our current evaluation, and the 1965 evaluation of refs. 1 and 3, are in satisfactory agreement. This should come as no surprise, since good pion-nucleon cross sections were already available in 1965.
has changed dramatically since then, and is the motivation for the present paper, is the status of data on pion-pion scattering.

4.3. Connection with lowest order chiral perturbation theory

As we did for the pion case, the Adler–Weisberger relation can be connected with (lowest order) chiral perturbation theory, by comparing e.g., (4.1) with the forward dispersion relation for the πp scattering amplitude for exchange of isospin unity at threshold, evaluating the amplitude at threshold in terms of the scattering lengths combination \( a_3 - a_1 \), and calculating the latter in terms of \( F_\pi \), as in the Tomozawa–Weinberg articles.\(^{[21]}\) Details of this may be found in ref. 17.

5. Comments

We have shown that with current data, the pion-pion sum rule, as well as the pion-nucleon one, is satisfied to better than six percent. To improve on this precision, it will be necessary to have an improved understanding of dynamical extrapolation corrections that account for the appearance of a zero mass, off-shell pion in the sum rules. We make two remarks in this regard. The first is that extrapolation of the incident pion to zero mass is not the same as taking the chiral limit of QCD, since the target pion or nucleon, and all intermediate state particles, remain on mass shell. The second is that while for a generic pion interpolating field the results of this extrapolation are not well-defined, the sum rules involve a very specific choice of pion interpolating field: the divergence of the axial vector current, which is a well defined entity in QCD, as are the on shell pion-pion and pion-nucleon scattering amplitudes. Thus, the question of estimating extrapolation corrections, that are needed for a very accurate comparison of the sum rules with experiment, is a well-posed one in QCD. Modern lattice methods may permit improvement on the estimates that have been used here and in refs. 1 and 3.

In this respect, it is amusing to remark that, unlike the situation in 1965, the precision of the pionic sum rule is now greater than that of the pion-nucleon one. This is very likely due to the fact that the latter involves small differences of large numbers, so any small alteration is amplified here.

\(^{3}\) This is generally known as the Goldberger–Miyazawa–Oehme sum rule.\(^{[20]}\)
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