Wormhole and the Thermodynamic Arrow of Time

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In classical thermodynamics, heat cannot spontaneously pass from a colder system to a hotter system, which is called the thermodynamic arrow of time. However, if the initial states are entangled, the direction of the thermodynamic arrow of time may not be guaranteed. Here we take the thermofield double state at $0+1$ dimension as the initial state and assume its gravity duality to be the eternal black hole in AdS\textsubscript{2} space. We make the temperature difference between the two sides by changing the Hamiltonian. We turn on proper interaction between the two sides and calculate the changes in energy and entropy. The energy transfer, as well as the thermodynamic arrow of time, are mainly determined by the competition between two channels: thermal diffusion and anomalous heat flow. The former is not related to the wormhole and obeys the thermodynamic arrow of time; the latter is related to the wormhole and reverses the thermodynamic arrow of time, i.e. transfer energy from the colder side to the hotter side at the cost of entanglement consumption. Finally, we find that the thermal diffusion wins the competition, and the whole thermodynamic arrow of time has not been reversed.

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1. INTRODUCTION

As the experimental advances in the last two decades, it is possible to prepare and control the system with an increasing number of atoms, whose scale stretches across the gap between macroscopic thermodynamic and microcosmic quantum mechanics. Taking the former two theories as cornerstones, quantum thermodynamic tries to extend the framework of thermodynamic to finite size quantum system, non-equilibrium dynamics, strong coupling, and quantum information process [1].

The laws of thermodynamics constrain the evolution of a system coupled to heat baths. The second law can be derived when the initial correlation between the system and its heat baths is absent [2]. The correlation generated during the interaction between the system and the heat baths leads to non-negative entropy production. The thermodynamic arrow of time also plays a crucial role in the second law of thermodynamics, which refers to the phenomenon that heat will spontaneously pass from a hotter system to a colder system. Similarly, it can be derived if the two systems are uncorrelated initially [3].

The author in Ref. [3] argued that the second law, as well as the thermodynamic arrow of time, are emergent phenomena in the low entropy and low correlation environment. Furthermore, for the two systems with different temperatures and initial correlation, the thermodynamic arrow of time can be reversed for some proper interactions; heat pass from colder system to hotter system by consuming their correlation [3, 4], which is called anomalous heat flow, as illustrated in Fig. 1. The authors in Ref. [5] discussed the conditions for its presence. Such a phenomenon was observed in the experiment on two initially correlated spins [6] and quantum computer [7]. The reversion of the thermodynamic arrow of time may conflict with the original second law and urges a generalized second law in the presence of correlations [8].

It traces back to the relationship between work and correlation [9]. By acting unitary transformation on a state, one can lower down its energy and extract work. A state from which no work can be extracted is called passive, including all thermal states [10]. While, work can be extracted from the correlation between two systems, where each system is locally in a thermal state with the same temperature [11]. The anomalous heat flow is a refrigeration driven by the work stored in correlations [8]. Furthermore, the authors in Refs. [8, 12, 13] discussed the generalizations of the second law in the presence of correlation.

Similar issues have been being studied in gravity since two decades ago as well, while the famous AdS/CFT correspondence was discovered. The AdS/CFT correspondence is a realization of the holography principle, proposed by L. Susskind according to the surface law of black hole entropy. It conjectures that the degree of freedom of a gravitational system can be mapped to its boundary [19]. Furthermore, the AdS/CFT correspondence offers the dictionary between the quantum gravity in asymptotic Anti-de Sitter (AdS) space and the conformal field theory (CFT) on its boundary [20]. Notably, a large AdS-Schwarzschild black hole is dual to the CFT at finite temperature. The correspondence reveals that black hole thermodynamics is not only an analogy of the classical thermodynamic but also an equivalent description.

Another significant development in the AdS/CFT correspondence is the Ryu-Takayanagi (RT) formula, which conjectures that the von Neumann entropy of a subregion on the boundary is proportional to the area of the minimal homological surface in the bulk [21]. Furthermore, the black hole entropy is interpreted as the entanglement entropy between the two field theories on the boundaries of the maximal extension of the black hole, where is a wormhole connecting them.

The AdS/CFT correspondence also sheds light on the black hole information problem by utilizing the unitary of the dual boundary theory [22, 23]. Before the proposal of the AdS/CFT correspondence, D. Page embedded unitary in the process of black hole evaporation and study the entanglement between the remainder black hole and its Hawking radiation [24, 25]. Motivated by the Hartle-Hawking-Israel state [26] and the black hole complementarity [27], J. Maldacena conjectured that the eternal black hole is described by the thermo-field-double (TFD) state and try to resolve a part of information loss paradox [28]. Motivated by the RT formula, M. Van Raamsdonk conjectured the
correspondence between geometry and entanglement [29]. To resolve the firewall paradox [30], J. Maldacena and L. Susskind applied the idea to general entangled states and raised the ER=EPR conjecture where ER refers to Einstein-Roson bridge, and EPR refers to EPR pair, i.e. quantum entanglement [31].

The connection between the TFD state and the eternal black hole helps us to understand the complicated dynamics of boundary field theory from the gravity side. Quantum chaos, which diagonalized by out-of-time-order correlators (OTOCs), can be understood from the shock wave on black holes [32, 33]. The teleportation protocol or Hayden-Preskill protocol corresponds to constructing a traversable wormhole in an eternal black hole [34–41]. To make the wormhole traversable, one should turn on proper interactions between the two sides of the eternal black hole, where the interactions are equivalent to the measurements in the teleportation protocol. The evolution of the bi-systems with interactions and entanglement is an object of research in quantum thermodynamic.

We raise the following questions in the intersection between quantum thermodynamics and the AdS/CFT correspondence.

- Does anomalous heat flow exist in the systems which have holographic duality?
- If yes, will it challenge the law of black hole thermodynamic?

In previous works, anomalous heat flow was realized in weakly coupled or weakly chaotic systems. In this paper, we try to find a similar phenomenon in strongly coupled systems based on AdS$_2$/CFT$_1$ correspondence and ER=EPR conjecture. In Section 2, we review the relation between correlation and anomalous heat flow. In Section 3, we prepare a TFD state and the dual eternal black hole in Jackiw-Teitelboim (JT) gravity with different temperatures on each side. In Section 4, we show that work can be extracted from the wormhole. In Section 5, with proper interaction, we find two channels which mainly contribute to the energy transfer: the thermal diffusion via the boundary and the anomalous heat flow via the wormhole in the eternal black hole. The former is numerically larger than the latter. So we find two channels which mainly contribute to the energy transfer: the thermal diffusion via the boundary and the anomalous heat flow via the wormhole in the eternal black hole. The former is numerically larger than the latter. So the total thermodynamic arrow of time has not been reversed. In Section 6, we study the concomitant consumption of entanglement. Although we focus on the eternal black hole in the main text, in appendixes, similar issues are investigated for a product state in JT gravity and a TFD/product state in the Sachdev-Ye-Kitaev (SYK) model for comparison.

\section{Energy Transfer with Correlation}

\subsection{Thermodynamics for bi-systems}

Consider bi-systems labeled by $L$ and $R$. The state of the bi-systems is described by the total density matrix $\rho$, then the local density matrices of system $L$ and $R$ follow $\rho_\gamma = \text{Tr}_i \rho$, where $\gamma = L, R$ and $\bar{\gamma}$ is the complement of $\gamma$. The entropy of the bi-systems and local systems are defined as the von Neumann entropy $S_{LR} = -\text{Tr}[\rho \ln \rho]$ and $S_\gamma = -\text{Tr}[\rho_\gamma \ln \rho_\gamma]$. The two systems may be correlated. We use mutual information $I_{L,R} = S_L + S_R - S_{LR}$ to measure their correlation, where $I_{L,R} \geq 0$ because of the sub-additivity of entanglement entropy.

The local Hamiltonian of systems $L$ and $R$ are $H_L$ and $H_R$. Then the local energies are $E_\gamma = \text{Tr}[\rho_\gamma H_\gamma]$. We will consider the following process. Prepare an initial state $\rho(t_i)$ of the bi-systems at time $t_i$. Then we turn on an interaction $H_I$ between them for a while. After time $t_i$, $H_I$ is time independent. We define $E_I = \text{Tr}[\rho H_I]$ any time. In this paper, we will consider a weak interaction $H_I$ compared to the local Hamiltonian $H_\gamma$ such that the local energies of each local systems are well defined by the expectational valued of their local Hamiltonians. At time $t_f > t_i$, the state of the bi-systems becomes $\rho(t_f)$ through the unitary evolution with total Hamiltonian $H_{tot} = H_L + H_R + H_I$. So the entropy of the bi-systems $S_{LR}$ does not change. The change in entropies and energies are $\Delta S_\gamma = S_\gamma(t_f) - S_\gamma(t_i)$, $\Delta E_\gamma = E_\gamma(t_f) - E_\gamma(t_i)$ and $\Delta E_I = E_I(t_f) - E_I(t_i)$.

After time $t_i$, the total energy is conserved,

$$\Delta E_L + \Delta E_R + \Delta E_I = 0. \quad (2.1)$$
According to the first law of thermodynamics, the change in local energy can be divided into work $W$ and heat $Q$. We will adopt the definitions of work and heat in Ref. [13], and show them in Appendix A. For a specific model, we can calculate them accordingly.

### 2.2. Anomalous heat flow

We review some theoretical observations about the heat transfer in the presence of correlation in Refs. [3, 4]. Given a inverse temperature $\beta_\gamma = 1/T_\gamma$, one can define a local thermal state for a local Hamiltonian, namely $\tau_\gamma(\beta_\gamma) = Z_\gamma^{-1} \exp(-\beta_\gamma H_{\gamma})$. Now we specify the initial state $\rho(t_1)$ of the bi-systems at time $t_1$ such that the two local systems are individually thermal, i.e. $\rho_i(t_1) = \tau_i(\beta_i)$.

Note that, given a constant $\beta$, free energy $F = E - S/\beta$ is minimized by thermal state. As each local system is thermal initially, we have

$$\beta_L \Delta E_L \geq \Delta S_L, \quad \beta_R \Delta E_R \geq \Delta S_R.$$  \hspace{0.5cm} (2.2)

where $\beta_{L,R}$ are taken to be constant during the process, which are the inverse temperatures of the local thermal states at time $t_1$. We will call (2.2) as energy-entropy inequalities. When the inequalities are saturated, they are just the first law of entropy of each system. Combining the two inequality, we have

$$(\Delta E_L + \Delta E_R)(\beta_L + \beta_R) + (\Delta E_L - \Delta E_R)(\beta_L - \beta_R) \geq 2\Delta I_{L,R} = 2(\Delta S_L + \Delta S_R),$$  \hspace{0.5cm} (2.3)

The change in mutual information $\Delta I_{L,R} = \Delta S_L + \Delta S_R$ follows from the unitary evolution.

For an initial state which is a product states $\rho(t_1) = \rho_L(t_1) \otimes \rho_R(t_1)$, we have $I_{L,R}(t_1) = 0$. So $\Delta I_{L,R} \geq 0$, meaning that the interaction may induces entanglement. If we further consider the case of heat-contact, i.e. $\Delta E_L + \Delta E_R = 0$, one can easily obtain the thermodynamic arrow of time from inequality (2.3),

$$(\Delta E_L - \Delta E_R)(\beta_L - \beta_R) \geq 0.$$  \hspace{0.5cm} (2.4)

For example, we assume that system $R$ is hotter than system $L$, i.e. $\beta_L > \beta_R$. From the above inequality, we know $\Delta E_L = -\Delta E_R \geq 0$, which just means that heat can only pass from a hotter system to a colder system. The above inequality also holds once $\Delta E_L + \Delta E_R \leq 0$, which means that one cannot transfer energy from a colder system to a hotter system without positive work done on the bi-systems.

However, the above argument fails if the two systems are entangled initially. It is possible that the mutual information $I_{L,R}$ decreases. Then the l. h. s. of inequality (2.3) is not required to be non-negative any more. Especially for a pure initial state, $S_L = S_R$ always holds. Then $\Delta S_L = \Delta S_R$. Combining it with inequalities (2.2), we have $\Delta E_L + \Delta E_R \geq \Delta S_L(T_L + T_R)$. If $\Delta E_L + \Delta E_R \leq 0$ during the process, we find $\Delta I_{L,R} = 2\Delta S_L \leq 0$. So the reversal of inequality (2.4),

$$(\Delta E_L - \Delta E_R)(\beta_L - \beta_R) < 0$$  \hspace{0.5cm} (2.5)

is not forbidden anymore. Energy may pass from a colder system to a hotter system by consuming (quantum) correlation, which is called anomalous heat flow. No doubt that such a phenomenon highly depends on the state and the interaction. The reversal of the thermodynamic arrow of time during heat-contact was observed in the recent experiment on the two-spins system [6]. The time-reversal algorithms on a quantum computer were developed in Ref. [7].

In a word, we say that the thermodynamic arrow of time is reversed if

$$\Delta E_L + \Delta E_R \leq 0 \quad \text{and} \quad (\Delta E_L - \Delta E_R)(\beta_L - \beta_R) < 0.$$  \hspace{0.5cm} (2.6)

### 2.3. Extract work from correlation

No doubt that, the anomalous heat flow stems from (quantum) correlations, which plays an important role in quantum thermodynamics. One can extract work from correlation or create correlation by costing work [9, 11, 14–16]. The trade off between works and correlations explain the presence of anomalous heat flow. For a state $\rho$ and its Hamiltonian $H$, the work extracted from the system by acting unitary transformation $U$ is

$$W = \text{Tr}[\rho H] - \text{Tr}[U \rho U^\dagger H].$$  \hspace{0.5cm} (2.7)
For a Hamiltonian $H$, a state is called passive if and only if no positive work can be extracted from it, namely

$$W \leq 0, \quad \forall U. \tag{2.8}$$

Furthermore, it is proved in Refs. [17, 18] that a state $\pi$ is passive if and only if it is diagonal in the energy eigenbasis and its eigenvalues are monotonically decreasing with increasing energy, namely

$$\pi = \sum_n p_n |n\rangle \langle n| \quad \text{where} \quad p_n \geq p_m \text{ if } E_n \leq E_m. \tag{2.9}$$

Obviously, a thermal state $\tau = Z^{-1} e^{-\beta H}$ is a passive state of its Hamiltonian $H$. Furthermore, the outer product of thermal state $\tau^{\otimes n}$ is also a passive state of the $n$ copies of $H$ for any $n$. While an outer product of a passive state $\pi^{\otimes n}$ may not be a passive state of the $n$ copies of $H$.

The authors in Ref. [8] further explained the anomalous heat flow as the refrigeration driven by the work stored in correlations. Finally, they also modified the Clausius statement of the second law: no process is possible whose sole result is the transfer of heat from a cooler system to a hotter system, where the work potential stored in the correlations does not decrease.

3. A THERMO FIELD DOUBLE STATE AS AN INITIAL STATE

3.1. Balanced bi-systems and holography

Consider a local Hamiltonians $H$ of a local system, with eigenbasis $H |n\rangle = E_n |n\rangle$. Further, consider two copies of the local system labeled by $L$ and $R$ with Hamiltonian

$$H_0 = H_L + H_R, \quad H_L = H \otimes 1, \quad H_R = 1 \otimes H. \tag{3.1}$$

We call it balanced Hamiltonian since the local Hamiltonians are the same. We consider a state $|I\rangle = \sum_n |n\rangle_L |n\rangle_R$, where $|n\rangle_\gamma$ is the energy eigenstate in system $\gamma$. We prepare a TFD state at $t = 0$,

$$|\beta\rangle = \frac{1}{\sqrt{Z}} e^{-\beta H_0/4} |I\rangle = \frac{1}{\sqrt{Z}} \sum_n |n\rangle_L |n\rangle_R e^{-\beta E_n/2} \tag{3.2}$$

where $Z = \text{Tr} e^{-\beta H}$. The TFD state is not an eigenstate of $H_0$. We define $|\beta(t)\rangle = e^{-iH_0 t} |\beta\rangle$. Tracing out the system on one side, we obtain a thermal state on another side, $\rho_\gamma = \text{Tr}_\gamma |\beta(t)\rangle \langle \beta(t)| = Z^{-1} e^{-\beta H}$. Both of the inverse temperatures of local systems are $\beta$.

Throughout this paper, we consider $H$ to be a conformal field theory at $0 + 1$ dimension. We can also consider that the local Hamiltonian $H$ in Eq. (3.1) is the SYK model [42–44] in Appendix C. The TFD state in the two sites SYK model was constructed in Refs. [39, 40, 45]. The ways of preparing TFD states were discussed in Refs. [39, 47].

We consider an operator $O$ on the local system. We define

$$O_L = O^T \otimes 1, \quad O_R = 1 \otimes O, \tag{3.3}$$

where $O^T$ is the transpose of $O$ on the basis of energy eigenstates. The interaction picture for both sides are

$$O_L(t) = e^{iH_0 t} O_L e^{-iH_0 t} = e^{iH t} O^T e^{-iH t} \otimes 1, \quad O_R(t) = e^{iH_0 t} O_R e^{-iH_0 t} = 1 \otimes e^{iH t} O e^{-iH t}, \tag{3.4}$$

Due to the entanglement in the TFD state, the operators on both side are related by

$$O_L(t) |\beta\rangle = O_R \left(-t + \frac{\beta}{2}\right) |\beta\rangle. \tag{3.5}$$
We assume that the systems have a holographic duality of nearly-AdS. The bulk description of the TFD state is the Rindler patch in AdS2 space [48, 49]. For higher-dimensional CFT, one can further consider higher-dimensional generalization in the eternal black hole in AdS space. We consider that the AdS2 holography is described by dilaton gravity

\[ I_0 = -\frac{\phi_0}{16\pi G_N} \left[ \int dx^2 \sqrt{g} R + \int dx \sqrt{h} K \right] - \frac{1}{16\pi G_N} \left[ \int d^2x \sqrt{\bar{g}} \left( R(2) + 2 \int d\phi h \phi K \right) + \ldots \right] \]

The first term is a topological term. It does not control the dynamic and only contributes to extremal entropy \( S_0 = \phi_0/4G_N \). The second term is called Jackiw-Teitelboim (JT) theory. After integrating out dilaton field \( \phi \), one can fix the metric to be (E)AdS2. The remained d.o.f. is the boundary trajectory and the matter fields [49]. The dots refer to the expansion of the dilaton field from higher dimensional reduction, which is neglectable for \( \phi \ll \phi_0 \).

We will consider \( M \) free scalar with the same mass fields \( \{ \chi_i \} \) in the bulk. So the theory have \( SO(M) \) symmetry of rotation \( \chi_i \rightarrow \sum_j G_{ij} \chi_j \) where \( G \in SO(M) \). In this paper, we will frequently exchange these scalar fields, which is a subgroup of the \( SO(M) \) symmetry.

The coordinates of EAdS2 and AdS2 used in this paper are

\[
\begin{align*}
    ds^2 &= \frac{d\mu^2 + dz^2}{z^2} = \sinh^2 \rho d\varphi^2 + d\rho^2, \\
    ds^2 &= -d\nu^2 + dz^2 \quad \text{where (}\mu, z\text{) and (}\nu, z\text{) are Poincare coordinates, (}\phi, \rho\text{) and (}\psi, \rho\text{) are Rindler coordinates and (}\theta, \sigma\text{) is conformal coordinate. We will first work on Euclidean time. One can impose boundary conditions}
\end{align*}
\]

\[
\begin{align*}
    d\sigma^2 &= \frac{1}{c^2} d\tau^2, \quad \sigma = \frac{\bar{\phi}}{\epsilon}.
\end{align*}
\]

The cutoff is imposed to prevent dilaton field \( \phi \) from exceeding \( \phi_0 \). So \( \bar{\phi}/\phi_0 \) is related to the UV cutoff \( \epsilon \) of the dual boundary theory, namely \( \bar{\phi}/\phi_0 = \alpha_\epsilon \) with a order 1 constant \( \alpha_\epsilon. \) One can parameterize the boundary as \( (\mu(\tau), z(\tau)) \), where \( z(\tau) \) is determined by the boundary condition

\[ z = \epsilon \mu'(\tau) + O(\epsilon^3). \]

From the JT term in EAdS2, one can obtain the effective action of reparameterization \( \mu(\tau) \)

\[ I_{\text{eff}} = -C \int d\tau \{\mu, \tau\} = \frac{C}{2} \int_0^{2\pi} d\tau \left( \epsilon'' \varphi^2 - \epsilon^2 \right) + O(\epsilon^3), \]

\[ \{\mu, \tau\} = \left( \frac{\mu''(\tau)}{\mu'(\tau)} \right)' - \frac{\mu''(\tau)^2}{2\mu'(\tau)^2}, \quad \mu = \tan \frac{\varphi}{2}, \quad \varphi = \tau + \epsilon(\tau), \quad C = \frac{\bar{\phi}}{8\pi G_N}, \quad \beta = 2\pi. \]

where \( \{\mu, \tau\} \) is the Schwarzian derivative. Coefficient \( C \) has the dimension of time. The dependence on \( \beta \) can be recovered by dimensional analysis. Those \( SL(2, R) \) zero modes \( \varepsilon = 1, e^{i\tau}, e^{-i\tau} \) should be excluded in the functional integrate on \( \epsilon \). One can obtain the correlation of reparameterization modes

\[ \langle \varepsilon(\tau) \varepsilon(0) \rangle = \frac{1}{2\pi C} \left[ -\frac{1}{2}(|\tau| - \pi)^2 + (|\tau| - \pi) \sin |\tau| + c_1 + c_2 \cos \tau \right], \]

where arbitrary coefficients \( \{c_1, c_2\} \) come from the redundancy of \( SL(2, R) \) zero modes.

For a free scalar field \( \chi \) in the bulk, the correlation of its dual operator is modulated by reparameterization modes. The sources \( \dot{J}(\mu) \) in \( \mu \) are related to the source \( J(\tau) \) in \( u \) by

\[ \chi \sim z^{1-\Delta} \dot{J}(\mu) \sim (\epsilon \mu'(\tau))^{1-\Delta} \dot{J}(\mu) = \epsilon^{1-\Delta} J(\tau) \quad \Rightarrow \quad \dot{J}(\mu) = \mu'(\tau)^{\Delta-1} J(\tau). \]
So the effective action after we integrate out the bulk field $\chi$ is

$$\begin{align*}
-I_M &= \int d\mu^1 d\mu^2 \hat{J}(\mu_1) \hat{J}(\mu_2) = \int d\tau_1 d\tau_2 J(\tau_1) J(\tau_2) \left[ \frac{\mu_1'(\tau_1) \mu_2'(\tau_2)}{\mu_1(\tau_1) - \mu_2(\tau_2)} \right]^2 \\
&= \int d\tau_1 d\tau_2 J(\tau_1) J(\tau_2) \left( \frac{2}{2 \sin \frac{\tau_1}{2} \sin \frac{\tau_2}{2}} \right)^2 (1 + B(\tau_1, \tau_2) + C(\tau_1, \tau_2) + O(\varepsilon^3)),
\end{align*}$$

(3.14)

where $\tau_{12} = \tau_1 - \tau_2$, $\varepsilon_1 = \varepsilon(\tau_1)$. The operator $O$ has been normalized such that the common factor $D_\Delta = \frac{(\Delta - 1/2)\Gamma(\Delta)}{\sqrt{\pi} \Gamma(\Delta - 1/2)}$ does not appear in the correlation function. After integrate out $\varepsilon$,

$$\ln \langle e^{-I_M} \rangle = \int d\tau_1 d\tau_2 \frac{J(\tau_1) J(\tau_2)}{2 \sin \frac{\tau_1}{2} \sin \frac{\tau_2}{2}} \langle C(\tau_1, \tau_2) \rangle + \frac{1}{2} \int d\tau_1 d\tau_2 d\tau_3 d\tau_4 \frac{J(\tau_1) J(\tau_2) J(\tau_3) J(\tau_4)}{2 \sin \frac{\tau_1}{2} \sin \frac{\tau_2}{2}} \langle B(\tau_1, \tau_2) B(\tau_3, \tau_4) \rangle + O(C^{-2}),$$

(3.17)

$$\langle C(\tau_1, \tau_2) \rangle = -\frac{\Delta}{8\pi C} \left( \sin \frac{\tau_{12}}{2} \right)^{-2} \left( \frac{\tau^2_{12} + 2\pi |\tau_{12}|}{(\tau^2_{12} - 2\pi |\tau_{12}|)} \right) + \frac{\Delta^2}{4\pi C} \left( \frac{\tau_{12}}{\tan \frac{\tau_{12}}{2}} - 2 \right) \left( \frac{\tau^2_{12} - 2\pi |\tau_{12}|}{\tau_{12} \tan \frac{\tau_{12}}{2}} \right),$$

(3.18)

where the bracket $\langle \ldots \rangle = \int \frac{d\nu}{SL(2, R)} \cdots e^{-I_M}$ and Eq. (3.12) is used. The specific express of $\langle B(\tau_1, \tau_2) B(\tau_3, \tau_4) \rangle$ is too tedious to be shown here, which does not only depend on $\tau_{12}$ and $\tau_{34}$ for ordering $\tau_1 > \tau_3 > \tau_2 > \tau_4$.

The real time correlator can be obtained by Wick rotation $\tau \rightarrow iT$. The two boundaries of AdS$_2$ space satisfy the boundary conditions

$$d s^2_{\bar{\partial}, L} = - \frac{1}{\varepsilon^2} dt^2, \quad \phi_{\bar{\partial}, L} = \frac{\bar{\phi}}{\varepsilon},$$

$$d s^2_{\bar{\partial}, R} = - \frac{1}{\varepsilon^2} dt^2, \quad \phi_{\bar{\partial}, R} = \frac{\bar{\phi}}{\varepsilon},$$

(3.19)

The boundaries are parameterized by $\{\nu_L(t), \nu_R(t)\}$ in Poincare coordinate or $\{\psi_L(t), \psi_R(t)\}$ in Rindler coordinate, whose effective action is

$$I_{\text{eff}} = -C \int dt_L \{\nu_L(t), t_L\} - C \int dt_R \{\nu_R(t), t_R\} = -C \int dt_L \left\{ -\coth \frac{\psi_L}{2}, t_L \right\} - C \int dt_R \left\{ \tanh \frac{\psi_R}{2}, t_R \right\}.$$

(3.20)

The SL$(2, R)$ symmetry of parameterization $\{\psi_L(t), \psi_R(t)\}$ is

$$\psi_L \rightarrow \psi_L + \varepsilon_0 + \varepsilon_1 e^{\psi_L} + \varepsilon_2 e^{-\psi_L}, \quad \psi_R \rightarrow \psi_R - \varepsilon_0 + \varepsilon_1 e^{-\psi_R} + \varepsilon_2 e^{\psi_R}.$$

(3.21)

Without the excitation of scalar fields, those three SL$(2, R)$ charges are required to vanish for the consistence of the effective action. Then the equations of motion are automatically satisfied. Corresponding to the TFD state $|\beta(t)\rangle$, the solution is

$$\psi_L = \frac{2\pi}{\beta} t_L, \quad \psi_R = \frac{2\pi}{\beta} t_R, \quad t_L = t_R = t.$$

(3.22)

The correlator in bi-systems can be obtained by doing analytical continuation $t \rightarrow t \pm i\beta_\perp$. For example, the two points function at tree level in real time are

$$\langle O_L(t_1) O_L(t_2) \rangle = \langle O_R(t_1) O_R(t_2) \rangle = \text{Tr}[O(t_1) O(t_2)] y^2 \left( 2i \sinh \frac{t_{12}}{2} \right)^{-2\Delta},$$

(3.23)

$$\langle O_L(t_1) O_R(t_2) \rangle = \text{Tr}[O(-t_1) O(t_2)] y^2 \left( 2 \cosh \frac{t_{12}}{2} \right)^{-2\Delta},$$

(3.24)
where, within the trace,

\[ O(t) = e^{iHt}Oe^{-iHt}, \quad (3.25) \]
\[ y = Z^{-1/2}e^{-\beta H/2}. \quad (3.26) \]

### 3.2. Unbalanced bi-systems and holography

Now we make the Hamiltonian of the bi-systems unbalanced

\[ \tilde{H}_0 = H_L + \tilde{H}_R, \quad H_L = H \otimes 1, \quad \tilde{H}_R = 1 \otimes \lambda H. \quad (3.27) \]

where the Hamiltonian of system \( R \) is multiplied by a constant \( \lambda \). While, we still consider the TFD state \(|\beta\rangle \) in Eq. (3.2) at time \( t = 0 \), which is prepared by imaginary time evolution with \( H_0 \) rather than \( \tilde{H}_0 \). While, \(|\beta\rangle \) evolves with \( H_0 \) along real time, \(|\tilde{\beta}(t)\rangle = e^{-i\tilde{H}_0t}|\beta\rangle = |\beta(1+\lambda t)\rangle \). Trace it by part, we have \( \rho_\gamma = Tr_\gamma |\tilde{\beta}(t)\rangle \langle \tilde{\beta}(t)| = \frac{1}{2}e^{-\beta H} = \frac{1}{2}e^{-(\beta/\lambda)\lambda H} \).

Thus \( \beta_L = \beta \) and \( \beta_R = \beta/\lambda \). Without loss of generality, we set \( \lambda \geq 1 \) through this paper. So system \( R \) is hotter than system \( L \) or they have the same temperature, i.e. \( \beta_L \geq \beta_R \). Effectively, multiplying the Hamiltonian of system \( R \) by \( \lambda \) is equivalent to redefining the time of system \( R \) as \( \tilde{t} = t/\lambda \). Such a fact will be reflected below.

We still consider the same operator \( O \) at \( t = 0 \). Since we have change the Hamiltonian, the time evolution of \( O \) is changed as well. Through this paper, \( \tilde{O}(t) \) refers to the time evolution of \( O \) with \( \tilde{H}_0 \) and \( O(t) \) refers to the time evolution of \( O \) with \( H_0 \). They are related by

\[ \tilde{O}_L(t) = e^{i\tilde{H}_0t}O_Le^{-i\tilde{H}_0t} = e^{iH_0t}OTe^{-iH_0t} \otimes 1 = O_L(t), \]
\[ \tilde{O}_R(t) = e^{i\tilde{H}_0t}ORe^{-i\tilde{H}_0t} = 1 \otimes e^{i\lambda Ht}Oe^{-i\lambda Ht} = O_R(\lambda t). \quad (3.28) \]

For system \( R \), the evolution with \( \tilde{H}_0 \) by time \( t \) is just the evolution with \( H_0 \) by time \( \lambda t \). Take correlator as an example,

\[ \langle \tilde{O}_L(t_1)\tilde{O}_R(t_2) \rangle = \langle e^{iH_0t_1}OTe^{-iH_0t_1} \otimes e^{i\lambda Ht_2}Oe^{-i\lambda Ht_2} \rangle = \langle O_L(t_1)O_R(\lambda t_2) \rangle, \quad (3.29) \]

where \( \langle ... \rangle = \langle \beta | ... | \beta \rangle \) for short. By using the relation between two evolutions, we can map the observables under the evolution with \( \tilde{H}_0 \) to the that under the evolution with \( H_0 \). This trick will be frequently used in later calculation.

We can further discuss the effect on the gravity side under the change of Hamiltonian. Corresponding to the evolution with unbalanced \( \tilde{H}_0 \) by time \( t \), the conditions of the two boundaries in \( \text{AdS}_2 \) become

\[ ds^2_{\partial,L} = -\frac{1}{\epsilon^2}dt^2, \quad \phi_{\partial,L} = \frac{\tilde{\phi}}{\epsilon}, \]
\[ ds^2_{\partial,R} = -\frac{\lambda^2}{\epsilon^2}dt^2, \quad \phi_{\partial,R} = \frac{\tilde{\phi}/\lambda}{\epsilon/\lambda}, \quad (3.30) \]

where the UV cutoff of system \( R \) becomes \( \epsilon/\lambda \). Let the boundaries corresponding to the evolution with unbalanced \( \tilde{H}_0 \) to be parameterized by \{\( \tilde{v}_L(t), \tilde{v}_R(t) \)\} or \{\( \tilde{\psi}_L(t), \tilde{\psi}_R(t) \)\}. The evolution with unbalanced \( \tilde{H}_0 \) can inherit the bulk description from the evolution with \( H_0 \) according to Eq. (3.28) and

\[ \tilde{v}_L(t) = v_L(t), \quad \tilde{v}_R(t) = v_R(\lambda t), \]
\[ \tilde{\psi}_L(t) = \psi_L(t), \quad \tilde{\psi}_R(t) = \psi_R(\lambda t). \quad (3.31) \]

Then the effective action becomes

\[ I_{eff} = -C \int dt \{ \tilde{v}_L(t) - C \int dt \tilde{v}_R(t) \} = -C \int dt \left\{ -\coth \frac{\tilde{\psi}_L}{2}, t \right\} = C \int dt \left\{ \tanh \frac{\tilde{\psi}_R}{2}, t \right\}. \quad (3.32) \]

Corresponding to the TFD state \(|\tilde{\beta}(t)\rangle \), the saddle point solution of an eternal black hole with unbalanced \( \tilde{H}_0 \) is

\[ \tilde{\psi}_L = \frac{2\pi}{\beta}t, \quad \tilde{\psi}_R = \frac{2\pi\lambda}{\beta}t. \quad (3.33) \]
FIG. 2. Two boundaries $\tilde{\upsilon}_L(t)$ and $\tilde{\upsilon}_R(t)$ with $\lambda = 2$ in conformal coordinate $(\sigma, \theta)$. Systems $L, R$ are denoted by blue curve and orange curve. Parameterizations are shown by the graduation line on the curves. Dashed lines indicate the insertion of interaction.

As expected, the black hole $R$ is hotter than the black hole $L$. Although the parameterizations are unbalanced, the classical location of the boundary is balanced, as shown in Fig. 2.

The source $\tilde{J}_R$ of system $R$ can be identified as follows

$$O_R(\lambda t) = \tilde{O}_R(t) \quad \text{and} \quad \int dt J_R(t) O_R(t) = \int dt \tilde{J}_R(t) \tilde{O}_R(t) \Rightarrow \lambda J_R(\lambda t) = \tilde{J}_R(t)$$

(3.34)

which is consistent with the correlation (3.29), since

$$\int dt_1 dt_2 J_L(t_1) J_R(t_2) \left[ \frac{\nu'_L(t_1) \nu'_R(t_2)}{(\nu_L(t_1) - \nu_R(t_2))^2} \right]^\Delta = \int dt_1 dt_2 \tilde{J}_L(t_1) \tilde{J}_R(t_2) \left[ \frac{\nu'_L(t_1) \nu'_R(\lambda t_2)}{(\nu_L(t_1) - \nu_R(\lambda t_2))^2} \right]^\Delta.$$  

(3.35)

Compared with dictionary (3.13), the dictionary for $\tilde{J}_R$ is

$$\chi \sim z^{1-\Delta} \tilde{J}_R(\nu_R) \sim (\epsilon \nu'_R(\lambda t)/\lambda)^{1-\Delta} \tilde{J}_R(\nu_R) = \epsilon^{1-\Delta} J_R(\lambda t) = \epsilon^{1-\Delta} \lambda^{-1} \tilde{J}_R(t)$$

(3.36)

rather than $\chi \sim (\epsilon/\lambda)^{1-\Delta} \tilde{J}_R(\tau_R)$ and $\tilde{J}_R(\nu_R) = \nu'_R(t)^{1-\Delta} \tilde{J}_R(t)$. It means that the holographic dictionary of $\tilde{O}_R(t)$ is different from the one of $O_R(t)$.

One may wonder whether any physics will be changed by redefining the time. Indeed, no physics have been changed so far. However, physics will be changed if we couple the two systems.

### 3.3. Product states as a comparison

As a comparison, we still adopt the unbalanced Hamiltonian $\tilde{H}_0$ in Eq. (3.27) but replaced the initial state by product state

$$\rho = \tau(\beta) \otimes \tau(\beta/\lambda).$$  

(3.37)

which is the same as TFD state (3.2) locally. The interaction picture (3.28) also holds.

The Hamiltonian $\tilde{H}$ of conformal field theory is gapless. Considering a non-trivial Hamiltonian $\tilde{H}_0 \neq 0$, we find that product state (3.37) is a passive state if and only if $\lambda = 1$. The analysis is as follows. The probability and energy of the energy eigenstate $|n\rangle_L |m\rangle_R$ are $p_{nm} = (Z_L Z_R)^{-1} \exp(-\beta (E_n + E_m))$ and $E_{nm} = E_n + \lambda E_m$. When $\lambda = 1$, $p_{nm} \propto \exp(-\beta E_{nm})$, then $\rho$ satisfies condition (2.9). Recall that local Hamiltonian $\tilde{H}_\gamma$ here is gapless. When $\lambda > 1$, we can choose $m$ and $m'$ such that $E_m - E_{m'} = \delta > 0$, and then choose $n$ and $n'$ such that $\lambda \delta > E_{n'} - E_n > \delta$. We find $E_{nm} > E_m$, $p_{nm} > p_{nm'}$, and then $\rho$ does not satisfy condition (2.9).

The product of two thermal states is dual to two black holes without a wormhole connecting them [29, 50]. We will call them disconnected black holes. In AdS$_2$ holography, we should consider the union of two AdS$_2$ spaces, each of which controlled by a theory of JT gravity. The disconnected black holes are described by two Rindler patches of the two AdS$_2$ spaces, as discussed in Appendix B. The wormhole in each Rindler path is caused by the purification of the corresponding thermal state and has nothing to do with the entanglement between system $L$ and system $R$. 
4. Extract Work from Wormhole

From the ER=EPR conjecture, for an entangled bi-systems \( LR \), it is conjectured that there is a wormhole connecting them. The wormhole in the eternal black hole geometrically represents the entanglement between system \( L \) and system \( R \), each of which is in a thermal state. From quantum thermodynamics, as work can be extracted from correlation, it is natural to expect that work can be extracted from wormhole as well.

We take the TFD state as the initial state and turn on the interaction \( H_I \) at time \( t_i \). After time \( t_i \), the total Hamiltonian is \( H_{tot} = \hat{H}_0 + H_I \). In interaction picture, the state at time \( t_f > t_i \) is

\[
|\tilde{\beta}_I(t_f)\rangle = \mathcal{T}\exp\left(-i\int_{t_i}^{t_f} dt \tilde{H}_I(t)\right)|\beta\rangle,
\]

where \( \tilde{H}_I(t) \) is the interaction picture of \( H_I \) under the evolution with \( \tilde{H}_0 \) and \( \mathcal{T} \) is time ordering.

To study the work extracted from the eternal black hole, in this section, we consider the interaction

\[
H_I = gO_LO_R. \tag{4.2}
\]

Such an interaction is also introduced in literatures to couple two spacetimes \([50]\) or construct traversable wormhole for \( g < 0 \) \([35, 36, 39]\). For balanced \( \tilde{H}_0 \), the energy changes at the first-order perturbation of interaction (4.2) have been calculated in Refs. \([35, 36]\). While here, we revisit them from the viewpoint of quantum thermodynamics. Due to the consideration of weak interaction \( H_I \), we should consider a small \( g \), such that the perturbation theory holds. For unbalanced \( \tilde{H}_0 \), the energy changes are

\[
\begin{align*}
\Delta E_L &= \langle \tilde{\beta}_I(t_f) | H_L | \tilde{\beta}_I(t_f) \rangle - \langle H_L \rangle = \Delta E^{(1)}_L + \Delta E^{(2)}_L + O(g^3), \tag{4.3a} \\
\Delta E_R &= \langle \tilde{\beta}_I(t_f) | \tilde{H}_R | \tilde{\beta}_I(t_f) \rangle - \langle \tilde{H}_R \rangle = \Delta E^{(1)}_R + \Delta E^{(2)}_R + O(g^3), \tag{4.3b} \\
\Delta E_I &= \langle \tilde{\beta}_I(t_f) | \tilde{H}_I(t_f) | \tilde{\beta}_I(t_f) \rangle - \langle \tilde{H}_I(t_i) \rangle = \Delta E^{(1)}_I + \Delta E^{(2)}_I + O(g^3), \tag{4.3c}
\end{align*}
\]

where the superscribe in \( \Delta E^{(i)} \) denote the energy change at \( O(g^i) \). During \( t_i \leq t \leq t_f \), energy conservation holds at each order of \( g \). By virtue of the entanglement, the work extracted at \( O(g) \) is

\[
W^{(1)} = \Delta E^{(1)}_I = g \left(\langle O_L(t_f)O_R(\lambda t_f) \rangle - \langle O_L(t_i)O_R(\lambda t_i) \rangle\right). \tag{4.4}
\]

According to Eq. (3.24) at tree level, \( \text{sgn}(W^{(1)}) = -\text{sgn}(g)\text{sgn}(t_f + t_i) \), which can be positive. As explained in Ref. \([36]\), interaction (4.2) imposes a potential energy \( \langle \tilde{H}_I(t) \rangle = g \langle O_L(t)O_R(\lambda t) \rangle \sim ge^{-m(t)} \) and does work on each system, where \( l(t) \) is the distant between the location of the two operator \( O_L(t) \) and \( O_R(\lambda t) \) in AdS_2 space, and \( m \) is the mass of dual scalar field \( \chi \).

The first law of entropy holds at first order perturbation, then

\[
\beta \Delta E^{(1)}_L = \Delta S^{(1)}_L = \Delta S^{(1)}_R = (\beta/\lambda) \Delta E^{(1)}_R. \tag{4.5}
\]

Combining it with Eqs. (2.1)(4.4), we have

\[
\begin{align*}
\Delta E^{(1)}_L &= -\frac{1}{1+\lambda} W^{(1)}, \quad \Delta E^{(1)}_R = -\frac{\lambda}{1+\lambda} W^{(1)}, \quad \Delta S^{(1)}_L = \Delta S^{(1)}_R = -\frac{1}{1+\lambda} \beta W^{(1)}. \tag{4.6}
\end{align*}
\]

Then we find such work is extracted from correlation

\[
W^{(1)} = -\frac{1}{2} \left( \frac{1}{\beta_L} + \frac{1}{\beta_R} \right) \Delta I^{(1)}_{L,R}. \tag{4.7}
\]

We can alternatively calculate the change in the von Neumann entropy from replica trick in Section 6.

According to the definitions of the work and heat of the local system in Appendix A, we find that \( \Delta E^{(1)}_I \) is the change in the binding energy during the process. The work \( W^{(1)} \) is extracted through the binding energy in Eq. (4.4). The local energy change in the way of heat, \( \Delta E^{(1)}_I = \Delta Q^{(1)}_I \), and the work done on the local system vanishes, \( \Delta W^{(1)}_I = 0 \).

As a comparison, we try to extract work from the disconnected black holes, as calculated in Appendix B. While, without a wormhole, \( \Delta E^{(1)}_I \propto \langle \hat{O}(t) \rangle = 0 \), so no work can be extracted at \( O(g) \). We further go to \( O(g^2) \) and find that \( \Delta E^{(2)}_L + \Delta E^{(2)}_R \geq 0 \), so no positive work can be extracted via such interaction. It is expectable when \( \lambda = 1 \), since the disconnected black holes form a passive state, while the eternal black hole does not. However, when \( \lambda > 1 \), the disconnected black holes do not form a passive state anymore, as explained below Eq. (3.37). The fail of extracting positive work results from the unsatisfactory form of the interaction.
5. ANOMALOUS HEAT FLOW VIA WORMHOLE

5.1. Interaction

In this section, we continue to consider the unbalanced \( \tilde{H}_0 \) and the TFD state. As explained in the previous section, the appearance of anomalous heat flow is highly dependent on the form of interaction term \([5]\). To look for the anomalous heat flow in the eternal black hole, we will alternatively consider the interaction

\[
H_I = \frac{1}{\sqrt{2}}g(V_L W_R - W_L V_R),
\]

(5.1)

where two scalar operators \( V \) and \( W \) are dual to two scalar fields \( \chi_V = \chi_1 \) and \( \chi_W = \chi_2 \) separately in the bulk for \( M = 2 \). The scaling dimensions of \( V \) and \( W \) are the same. Due to the \( SO(M) \) symmetry of Eq. (3.6), any Green functions of \( V \) and \( W \) are invariant under exchange \( V \leftrightarrow W \). Similar interaction was appeared in thermo field dynamic \([51]\) \(^1\). One may consider other kinds of interaction, such as Eq. (4.2) or

\[
H_I = gV_L W_R
\]

(5.2)

or

\[
H_I = \frac{1}{\sqrt{2}}g(V_L W_R + W_L V_R).
\]

(5.3)

Since \( V \) and \( W \) are dual to different scalar fields, \( \langle V_L(t_1) W_R(t_2) \rangle = \langle W_L(t_1) V_R(t_2) \rangle = 0 \). So the effect of each of Eqs. (5.1)(5.2)(5.3) at \( O(g) \) vanishes. We have to look at the contribution at \( O(g^2) \). Actually, even if it does not vanish, \( e.g. \) for the interaction (4.2), the first law of entropy leads to Eq. (4.5) as shown before. Then we have

\[
\text{sgn}((\Delta E^{(1)}_L - \Delta E^{(1)}_R)(\beta_L - \beta_R)) = -\text{sgn}(\Delta E^{(1)}_L + \Delta E^{(1)}_R).
\]

(5.4)

Comparing it with criterion (2.6), we find that the arrow of time will not be reversed at first order. This statement holds for other form of interaction and higher dimensional generalization. While it does not hold for mixed initial state. So we design interaction (5.1) to kill the effect of first order perturbation.

At second-order perturbation, different interactions will generate different sets of Feynman diagrams. As one will see, the sets of diagrams generated by interactions (5.2) and (5.3) are contained in the set of diagrams generated by interaction (5.1). Furthermore, the set of diagrams generated by interaction (5.1) are contained in the set of diagrams generated by interaction (4.2). We do not observe any anomalous heat flow with interactions (4.2)(5.2)(5.3).

5.2. Early time

We will consider interaction (5.1) and go to \( O(g^2) \). The change in the local energies \( \Delta E_\gamma \) and interaction energy \( \Delta E_I \) from time \( t_i \) to time \( t_f \) at \( O(g^2) \) are

\[
\Delta E_L \approx \int_{t_i}^{t_f} dt \int_{t_i}^{t_i} dt' K_L(t,t'),
\]

\[
K_L(t,t') = (-i)^3g^2 \left( \langle [\hat{V}_L(t) W_R(\lambda t'), V_L(t') W_R(\lambda t')] \rangle - \langle [\hat{V}_L(t) W_R(\lambda t), V_L(t') V_R(\lambda t')] \rangle \right),
\]

(5.5a)

\[
\Delta E_R \approx \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt' K_R(t,t'),
\]

\[
K_R(t,t') = (-i)^3g^2 \lambda \left( \langle [V_L(t) \hat{W}_R(\lambda t), V_L(t') W_R(\lambda t')] \rangle - \langle [V_L(t) W_R(\lambda t'), V_L(t') W_R(\lambda t')] \rangle \right),
\]

(5.5b)

\[
\Delta E_I \approx \int_{t_i}^{t_f} dt K_I(t_f,t),
\]

\[
K_I(t,t') = (-i)g^2 \left( \langle [V_L(t) W_R(\lambda t), V_L(t') W_R(\lambda t')] \rangle - \langle [V_L(t) W_R(\lambda t), W_L(t') V_R(\lambda t')] \rangle \right).
\]

(5.5c)

\(^1\) Thank Sang Pyo Kim for pointing out this point.
We have used the $SO(M)$ symmetry to exchange $V \leftrightarrow W$. Since all the following results are of $O(g^2)$, we have omitted the superscript in $\Delta E^{(2)}$.s. The first (second) term in each integral kernel of Eq. (5.5) come from the diagonal (cross) terms in $H_I^2$. If we choose interaction (5.2), the second term in each integral kernel of Eq. (5.5) disappears. If we choose interaction (5.3), the second term in each integral kernel of Eq. (5.5) has an inverse sign. The following discussion corresponding to interaction (5.1) can be easily translated into the case of interactions (5.2) and (5.3).

The two terms in each integral kernel of Eq. (5.5) are denoted as $(K_r(t,t'))_{VWVW}$ and $(K_r(t,t'))_{VWVW}$ where $\gamma = L, R, I$. They are related to the following two commutators

\begin{align}
C_{VWVW} &= \langle [V_L(t_1)W_R(t_2), V_L(t_3)W_R(t_4)] \rangle = 2\text{ImTr}[V(-t_3)V(-t_1)gW(t_2)W(t_4)y] \\
&\approx (C_{VWVW})_{\text{discon.}} + (C_{VWVW})_{\text{t-channel}} + O(C^{-2}), \quad (5.6a) \\
C_{VWVW} &= \langle [V_L(t_1)W_R(t_2), W_L(t_3)V_R(t_4)] \rangle = 2\text{ImTr}[W(-t_3)V(-t_1)yW(t_2)V(t_4)y] \\
&\approx (C_{VWVW})_{\text{u-channel}} + O(C^{-2}), \quad \{t_1, t_2, t_3, t_4\} \approx \{t, \lambda t, t', \lambda t'\}, \quad t' < t \quad (5.6b)
\end{align}

and their derivative. $C_{VWVW}$ is transformed into $VVWV$-order four points functions and $C_{VWVW}$ is transformed into $VVVV$-order four points functions at finite temperature.

Those diagrams contributing to the commutators (5.5) stems from the causal connection between time $t$ and time $t'$, which are shown in Fig. 3. Since there are two operators on each side, (dis)connected diagrams will (not) go across the wormhole.

The first commutator (5.6a) contain the diagrams in Figs. 3(a), 3(b) and 3(c) up to $O(C^{-2})$. The tree diagram in Figs. 3(a) and its one-loop correction in Fig. 3(b) are disconnected and not go across the wormhole. They are due to the propagation of the two scalar fields $\chi_V, \chi_W$ in each side separately. So, they are functions of $t_{13}$ and $t_{24}$, which are summed into

\begin{align}
(C_{VWVW})_{\text{discon.}} &= -i2^3\sin(2\pi \Delta)\left(4\sinh\frac{t_{13}}{2} \sinh\frac{t_{24}}{2}\right)^{-2\Delta} (1 + O(C^{-1})) \quad (5.7)
\end{align}

It has an important contribution on the kernels at short time $t - t' < t_d/2$, where $t_d = \beta/2\pi$ is the dissipation time. It suffers from UV divergence once $0 < \Delta$. However, when $\Delta < 1/4$, the integral $(\Delta E_t)_{\text{discon.}}$ contributed by $(C_{VWVW})_{\text{discon.}}$ is finite. The kernel $\langle KL(t,t') - KR(t,t')\rangle_{\text{discon.}}$ converges when $\Delta < 1/4$. We will focus on the case of $0 < \Delta < 1/4$. Of course, one can also introduce a small separation $\epsilon$ on imaginary time to regularize the UV behavior, such as $\langle V(t_1)V(t_2) \rangle \rightarrow \langle V(t_1 - i\epsilon)V(t_2) \rangle$. It acts as the UV cutoff of the boundary theory which is dual to JT gravity and satisfies $\epsilon \ll \beta$. However, the integrals $(\Delta E_L - \Delta E_R)_{\text{discon.}}$ and $(\Delta E_t)_{\text{discon.}}$ are not UV-sensitive. They only have some shifts if we change $\epsilon$ and keep finite if we send $\epsilon \rightarrow 0$. We illustrate the energy changes contributed by these disconnected diagrams 3(a) and 3(b) in Fig. 4. After the dissipation time, $(\Delta E_t(t,t'))_{\text{discon.}}$ is saturated, while energy starts to diffuse from hotter system $R$ to colder system $L$ via the channel of disconnected diagrams. Further numerical calculation shows that the energy changes contributed by these disconnected diagrams never satisfy criterion (2.6). It is predictable since these disconnected diagrams are not related to the wormhole. They can also appear in the interaction between two disconnected black holes in Appendix B, where the thermodynamic arrow of time should not be reversed.
The connected diagram in Fig. 3(c) is called t-channel, which goes across the wormhole. The interpretation of the t-channel is that: the energy-momentum tensors of the two scalar fields $\chi_Y, \chi_W$ on each side are correlated with each other by the propagation of virtual graviton through the wormhole. The propagation of virtual graviton just reflects the energy correlation between the two sides of TFD state, $H_L |\beta\rangle = H_R |\bar{\beta}\rangle$. The t-channel contributes to the commutator $C_{VWWV}$ with

$$ (C_{VWWV})_{t\text{-channel}} = -i 2^3 \sin(2\pi \Delta) \left(4 \sinh \frac{t_{13}}{2} \sinh \frac{t_{24}}{2}\right) - \frac{\Delta^2}{2\pi C} \left( t_{13} \coth \frac{t_{13}}{2} - 2 \right) \left( t_{24} \coth \frac{t_{24}}{2} - 2 \right). $$

(5.8)

It also depends on $t_{13}$ and $t_{24}$ only, but it is free from UV divergence once $\Delta < 1$. It is always $O(C^{-1})$, so it is smaller than the contribution of the tree diagram in Fig. 3(a). Since $t_{13} \geq 0, t_{24} \geq 0$ and $0 < \Delta < 1/4$, we find $(-i)(C_{VWWV})_{t\text{-channel}} \leq 0$, then $(\Delta E_L + \Delta E_R)_{t\text{-channel}} = -(\Delta E_I)_{t\text{-channel}} \geq 0$, which does not satisfy criterion (2.6).

Thus, the first commutator (5.6a) is a function of $t_{13}$ and $t_{24}$, which describe the process without signal passing the wormhole since the original wormhole in the eternal black hole is not traversable. The energy change contributed by the first commutator (5.6a) obeys the thermodynamic arrow of time.

The second commutator (5.6b) is an OTOC. It leads to the diagram in Fig. 3(d) at $O(C^{-1})$, which is called u-channel. It plays an important role to quantum chaos and traversable wormhole [35, 36]. Its corresponding process in the bulk is that two particles scatter near the horizon by exchanging graviton and then reach the opposite boundaries. In contrast, without the exchange of graviton, the diagram does not contribute to Eq. (5.6) since the wormhole is not traversable. The u-channel contributes to commutator $C_{VWWV}$ with

$$ (C_{VWWV})_{u\text{-channel}} = -i 2^3 \left(4 \cosh \frac{t_{2} + t_{3}}{2} \cosh \frac{t_{1} + t_{4}}{2}\right) - \frac{\Delta^2}{C} \left( \frac{\sinh \frac{t_{14} + t_{23}}{2}}{\cosh \frac{t_{2} + t_{3}}{2}} \cosh \frac{t_{1} + t_{4}}{2} + \frac{t_{14} + t_{23}}{2} \tanh \frac{t_{1} + t_{4}}{2}\right) $$

(5.9)

Due to the suppression of $C^{-1}$, it is visible only within the chaos region, $t' \approx -t$ and $t_{d}/2 \ll t < t_{*}/2$, where $t_{*} = \frac{\beta}{2\pi} \log \frac{2C}{\beta}$ is the scrambling time. Specifically, let $\lambda = 1, t_1 = t_2 = t > 0$ and $t_3 = t_4 = t' = -t$, it becomes

$$ (C_{VWWV})_{u\text{-channel}} = -i 2^{3-4\lambda} \Delta^2 C^{-1} \sinh(2t) $$

(5.10)

which will exponentially grows at early time $t_{d}/2 \ll t < t_{*}/2$. While, at late time $t \gg t_{*}/2$, the approximation in Eq. (5.9) at $O(C^{-1})$ is unreliable, because of the strong backreaction on the metric, which will be investigated in the next section. If we let $t_i = -t_f$ and $\lambda \approx 1$ in Eq. (5.5), the u-channel will contribute exponentially grows on the energy changes at early time $t_{d}/2 \ll t < t_{*}/2$, whose leading behaviors are

$$ (\Delta E_L + \Delta E_R)_{u\text{-channel}} = - (\Delta E_I)_{u\text{-channel}} \sim -g^2 C^{-1} e^{2t_f} + O((\lambda - 1)^{1}), $$

(5.11)

$$ (\Delta E_L - \Delta E_R)_{u\text{-channel}} \sim - (\lambda - 1) g^2 C^{-1} e^{2t_f} + O((\lambda - 1)^{2}), $$

(5.12)

both of which are negative and satisfy criterion (2.6). It means that the u-channel with the interaction (5.1) contribute to anomalous heat flow, at lead at early time, when $\lambda \approx 1$. Our numerical calculation shows that, for general $\lambda > 1$, such anomalous heat flow also appear. We illustrate kernels contributed by the u-channel in Fig. 5 for some parameters.

However, in Eq. (5.5), we should sum over all the diagrams. We estimation their scale, as concluded in Tab. 1. Specially, the energy change contributed by the u-channel is order $C^{-1}$ before the dissipation time $t_d$, while it reaches
FIG. 5. Kernels \((K_L(t,t') + K_R(t,t'))_{u-channel}\) and \((K_L(t,t') - K_R(t,t'))_{u-channel}\) contributed by the u-channel at early time, where \(\Delta = 1/6, \ C = 10^5, \ \beta = 2\pi, \ \lambda = 2\) and then \(t_*/2 = (\ln C)/2 \approx 5.7\).

| diagrams | (dis)connected | the thermodynamic arrow of time | scale of energy change |
|----------|----------------|-------------------------------|------------------------|
| tree     | disconnected   | obey                          | 1                      |
| one-loop | disconnected   | obey                          | \(C^{-1}\)              |
| t-channel| connected      | obey                          | \(C^{-1}\)              |
| u-channel| connected      | reverse at early time         | from \(C^{-1}\) to \(C^{-1} e^{t*} \sim 1\) |

TABLE I. Comparing those diagrams in Fig. 3.

order \(C^{-1} e^{t*} \sim 1\) near the scrambling time \(t_*\). First, the wormhole supports two channels, the t-channel, and the u-channel, whose contribution to the energy changes should be dominated by the u-channel near the scrambling time. So, with interaction \((5.1)\), the wormhole supports an anomalous heat flow near the scrambling time. Second, since the contributions of the tree diagram and the u-channel are in the same order, to determine the whole thermodynamic arrow of time, we should compare the contribution of the two diagrams numerically, as shown in Fig. 6. We find that the contribution of the tree diagram is larger than the one of the u-channel at early time. So we conclude that the whole thermodynamic arrow of time has not been reversed at early time.

5.3. late time

At late time, one should consider higher order \(C^{-1}\) correction on \(C_{VWWV}\) beyond the u-channel. By taking the limit \(t \to \infty, \ C \to \infty\) while keeping \(C^{-1} e^{t}\) finite, one can calculate \(C_{VWWV}\) in eikonal approximation \([33, 49]\)

\[
C_{VWWV} = -2i \langle V_L(t_1) V_R(t_4) \rangle \langle W_R(t_2) W_L(t_3) \rangle \text{Im} \mathcal{F}
\]

\[
\mathcal{F} = \left( \frac{i}{\zeta} \right)^{2\Delta} U \left( 2\Delta, 1, \frac{i}{\zeta} \right), \quad \zeta = \frac{1}{8C} \frac{e^{(t_1+t_4-t_2-t_3)/2}}{\cosh \frac{a+t_4}{2} \cosh \frac{a+t_3}{2}}
\]

\[
t_1 \approx t_2/\lambda \approx t, \quad t_3 \approx t_4/\lambda \approx t', \quad t' < 0 < t
\]

FIG. 6. \(\Delta E_L \pm \Delta E_R\) contributed by different diagrams in Table I as functions of \(t_f\), where the fifth line “eikonal” refers to the contribution of \(C_{VWWV}\) in Eq. \((5.13)\). Parameters are chosen to be \(\Delta = 1/6, \ C = 10^5, \ \beta = 2\pi, \ \lambda = 2, \ \epsilon = 0.2\) and \(t_1 = -15\). The contributions of the one-loop correction and the t-channel are too small to be visible.
where the two points functions are given in Eq. (3.24) and the confluent hypergeometric function $U(a, 1, x) = \Gamma(a) \int_0^\infty \frac{s^{a-1} e^{-sx}}{(1+s)^x} ds$ is used. In real time, $\zeta > 0$ always holds, then $\text{Im} F > 0$ always holds. Eq. (5.13) is negligible before the early time. It agrees with the u-channel in Eq. (5.9) well at early time. While it decays to zero at late time. So its amplitude is maximized at the time scale of the scrambling time. More precisely, $\text{Im} F$ is maximal at $\zeta = \zeta^*$, whose dependence on $\Delta$ is shown in Fig. 7. It corresponds to the time scale on $(t, t')$ as $e^{(\lambda+1)t} 2C (e^{t' + \lambda t} + 1) (e^{\lambda t' + t} + 1) \approx \zeta^*$. (5.14)

The time dependence of the two points functions in Eq. (3.24) also affect the time of the maximal amplitude of $C_{VWWV}$. Basically, when $\lambda \approx 1$, it is maximized at the time scale (5.14); when $\lambda \gg 1$, it is maximized at the time scale which is smaller than (5.14).

Beside, there are two lines in the plane of $(t, t')$ where the two points functions in Eq. (5.13) are maximized respectively

$$\lambda t + t' = 0, \quad t + \lambda t' = 0.$$ (5.15) (5.16)

The intersections of Eqs. (5.14)(5.15) and (5.14)(5.16) are approximately

$$(t, t') = (t_a, -\lambda t_a), \quad t_a \approx \frac{1}{1 + \lambda} t_s,$$ (5.17)

$$(t, t') = (t_b, -t_b/\lambda), \quad t_b \approx \frac{\lambda}{1 + \lambda} t_s.$$ (5.18)

The two time scales $t_a$ and $t_b$ can characterize the process of energy change.

In Fig. 8, we show the energy changes $(\Delta E_L)_{VWWV}$ and their kernels $(K_\gamma(t, t'))_{VWWV}$ contributed by $C_{VWWV}$ in Eq. (5.13) including the late time. We can consider the process with initially time $t_i \ll -t_s$ and observe the energy changes contributed by $C_{VWWV}$ when $t_f$ goes from $t_i$ to $+\infty$. We will find several periods. All the energy changes in the following list refer to the contribution by $C_{VWWV}$ only.

1. When $t_f \ll t_a$, these energy changes are negligible, since the system hardly scrambles.

2. When $t_f \sim t_a$, the amplitude of commutator $C_{VWWV}$ exponentially grows. Both $(\Delta E_L)_{VWWV}$ and $(\Delta E_R)_{VWWV}$ synchronously decrease and $(\Delta E_I)_{VWWV}$ increase. During this time, $| (\Delta E_L - \Delta E_R)_{VWWV} |$ is still small.

3. When $t_a \ll t_f \ll t_b$, $(\Delta E_L)_{VWWV}$ keeps dropping, while $(\Delta E_R)_{VWWV}$ in turn increases. Then $(\Delta E_L - \Delta E_R)_{VWWV}$ start to drop and system $R$ keeps absorbing the energy from system $L$, which is quite apparent in Figs. 8(b) and 8(c). $(\Delta E_I)_{VWWV}$ reaches its maximum in this period.

4. When $t_f \sim t_b$, $(\Delta E_L)_{VWWV}$ in turn increases and $(\Delta E_R)_{VWWV}$ increases, whose speeds are low when $\lambda \gg 1$. $E_I$ decreases.

5. When $t_b \ll t_f$, all the energies go to some constants, because of the quasinormal decay of commutator $C_{VWWV}$ at late time [49]. Especially, $(\Delta E_I)_{VWWV}$ is forced to vanish. While the net energy transfer $(\Delta E_L - \Delta E_R)_{VWWV}$ reaches a negative constant.
During the whole process contributed by $C_{VWWV}$, energy transfers from colder system $L$ to hotter system $R$, which exhibits the anomalous heat flow via the wormhole. Note that the final magnitude of the anomalous heat flow decreases when $\lambda$ increases from 1, which agrees with the analysis of Renyi entropy in the next section.

Similar to the $u$-channel in the previous subsection, the amplitude of $C_{VWWV}$ is order 1. Numerically, it is usually smaller than the contribution of the disconnected-tree diagram, as shown in Fig. 6. In other words, the thermal diffusion is stronger than the anomalous heat flow. So the total thermodynamic arrow of time cannot be reversed even at late time.

According to Appendix A, we find that the work done on the local systems vanish as well because one point functions $\langle V \rangle$ and three points functions $\langle VVW \rangle$ vanish. So the energies change in the way of heat. It agrees with our terminologies, “thermal diffusion” and “anomalous heat flow”.

Finally, in Appendix C, we show that such anomalous heat flow via wormhole can be constructed in the SYK model as well since its breaking of conformal symmetry at large $N$ limit is also described by Schwarzian action at low energy.

**5.4. An Interpretation from Boundary Theory**

According to the AdS/CFT correspondence, the dual system on the boundary are strongly coupled and highly chaotic. It goes beyond the ordinary realization of anomalous heat flow in weakly coupled or weakly chaotic models [3, 4, 6] and is relatively difficult to be understood from boundary theory. Here we try to explain. Recall that energy
changes highly rely on the form of interaction, which must be taken into consideration. For simplicity, we assume \( g > 0 \).

We first consider interaction \( H_I = gV_LW_R \). So only the disconnected diagrams present. We will explain the behaviors of kernels \( (K_{\gamma}(t_2,t_1)_{\text{discon.}}) \) for \( \gamma = L, R, I \), as shown in Fig. 4, which corresponds to the energy change after the instantaneous deformation \( \Delta H(t) = H_I(\delta(t - t_1) + \delta(t - t_2)) \). At time \( t_2 \), we turn on a potential \( gV_LW_R \) and then immediately turn it off. It creates an EPR pair, which is a superposition of all the combinations tending to lower the value of the potential \( g \langle V_LW_R \rangle \). However, it will not change \( E_L + E_R \), since \( \langle V_LW_R \rangle \) initially vanishes on quantum average. In other words, the first order perturbation vanishes. The EPR pair will interact with other particles on their own sides and then dissipate when time goes forward, characterized by a dissipation time \( t_d \) which scales as the inverse temperature \( \beta_\gamma \) on each side. At time \( t_2 > t_1 \), we instantaneously turn on the potential \( gV_LW_R \) again. The accumulation of the tendency of the EPR pair has lowered \( g \langle V_LW_R \rangle \) and leads to \( (K_{\gamma}(t_2,t_1)_{\text{discon.}}) < 0 \). However, when \( t_2 - t_1 \gg t_d \), the dying EPR pair is unable to lower \( g \langle V_LW_R \rangle \), so that \( g \langle V_LW_R \rangle \) as well as \( (K_{\gamma}(t_2,t_1)_{\text{discon.}}) \) return to zero. While \( (K_{\gamma}(t_2,t_1)_{\text{discon.}}) \) is proportional to the power of \( gV_LW_R \) done on system \( \gamma \) at time \( t_2 \). When \( t_2 \) goes from \( t_1 \) to infinity, the value \( g \langle V_LW_R \rangle(t_2) \) decrease first and then return to zero, such that its power done on system \( \gamma \) is positive first and is negative later, which agrees with the tendency of \( (K_{\gamma}(t_2,t_1)_{\text{discon.}}) \). The hotter the system is, the faster the decay of the EPR pair. So \( (K_{R}(t_2,t_1)_{\text{discon.}}) \) becomes negative earlier than \( (K_{L}(t_2,t_1)_{\text{discon.}}) \).

For the sustained interaction \( H_I = gV_LW_R \), since \( (\Delta E_t)_{\text{discon.}} \) does not change after \( t_d \), the difference between the temperatures leads to a net energy transfer from system \( R \) to system \( L \).

Now we consider interaction \( H_I = g(V_LW_R - W_LV_R)/\sqrt{2} \). The analysis of the contribution of the disconnected diagrams are the same as before. We will focus on kernels \( (K_{\gamma}(t_2,t_1))_{\text{VWVV}} \) for \( \gamma = L, R, I \), as shown in Fig. 8, whose main contribution is near \( t_2 \sim -t_1 = t \). We first analyze \( (K_I(t,-t))_{\text{VWVV}} \) when \( \lambda = 1 \). We find

\[
C_{\text{VWVV}}(t,t,-t,-t) = \text{Tr}[y \{W(t), V(-t)\} y \{W(t), V(-t)\}],
\]

\[
(K_I(t,-t))_{\text{VWVV}} = \frac{i}{\hbar}g^2C_{\text{VWVV}}(t,t,-t,-t) \xrightarrow{\hbar \to 0} -\frac{g^2}{2} \langle\langle \{W(t)^2, V(-t)^2\}\rangle\rangle_p,
\]

where we have restored \( \hbar \) and taken the classical limit \( \hbar \to 0 \). So \( C_{\text{VWVV}}(t,t,-t,-t) \) is just the cross-correlation between fluctuation and dissipation [52]. \( \{...,\} \) is the Poisson bracket. \( \langle\langle \cdot \rangle\rangle \) is classical ensemble average. From the classical limit, we know that \( (K_I(t,-t))_{\text{VWVV}} \) characterize the dissipation of fluctuations, which can be understood from linear respond.

Consider a classical and highly chaotic system containing \( N \) canonical coordinates \( \{x_i\} \) \( i = 1,...,N \) with zero average value \( \langle \langle x_i \rangle \rangle = 0 \). The classical model in Ref. [36] is a good example, as discussed in Appendix D. At time \( t_1 \), we add an instantaneous potential \( \Delta H(t) = gx^2 \delta(t - t_1) \), which give the system a momentum along the direction of reducing the fluctuation \( \langle \langle x_i^2 \rangle \rangle \). At \( t_2 > t_1 \), we measure the fluctuation \( \langle \langle x_i^2 \rangle \rangle \). \( g \langle \langle x_2(t_2)^2, x_1(t_1)^2 \rangle \rangle_p \) characterize the change of the fluctuation \( \langle \langle x_2^2 \rangle \rangle \) at time \( t_2 \) due to the instantaneous potential at time \( t_1 \), which is the classical interpretation of Eq. (5.20) if we identify \( x_1 = V, x_2 = W \). Since the system is highly chaotic, the fluctuation of \( x_i \) is sourced from the noise given by the other \( (N-1) \) canonical coordinates. When the fluctuation of \( x_1 \) is suppressed by the instantaneous potential at time \( t_1 \), we expect that the fluctuation of other \( x_i \neq 1 \) will be suppressed later due to their interactions with \( x_1 \). If we only focus on one of them, such as \( x_2 \), the respond is split into \( N \) parts. So \( g \langle \langle x_2(t_2)^2, x_1(t_1)^2 \rangle \rangle_p \) is negative and is suppressed by a \( N^{-1} \) factor. Furthermore, since the system is chaotic, such respond should exponentially grow along time \( t_2 - t_1 \). Finally, when \( t_2 \) goes to infinity, the system should reach a new equilibrium and lose its memory. The final value of fluctuation \( \langle \langle x_2^2 \rangle \rangle \) is only dependent on the final equilibrium state which is determined by the energy of the system after the perturbation \( \Delta H(t) \). While, the energy at \( t_2 \) does not change at linear order, i.e. \( \langle \langle \{H, gx_1(t_1)^2\}\rangle \rangle_p \propto \partial_t \langle \langle x_1(t_1)^2 \rangle \rangle = 0 \). So linear respond \( g \langle \langle x_2(t_2)^2, x_1(t_1)^2 \rangle \rangle_p \) should decay to zero, whose time scale scales as the inverted temperature. We summarize the behavior from classical approximation

\[
(K_I(t,-t))_{\text{VWVV}} \approx \left\{ \begin{array}{ll} \frac{g^2N^{-1}e^{\lambda(t-t)}}{2}, & \text{early time} \\ \frac{g^2e^{-c_1/t}/\beta}{2}, & \text{late time} \end{array} \right.,
\]

where coefficients \( \alpha_1, c_1 > 0 \) and Lyapunov exponent \( \lambda_L \ll 2\pi/\beta \) for highly chaotic system. The behavior of \( (K_{L,R}(t,-t))_{\text{VWVV}} \) can be approximated by the time derivative of \( -(K_I(t,-t))_{\text{VWVV}} \). So \( (K_{L,R}(t,-t))_{\text{VWVV}} \) is negative at early time, becomes positive at later time and vanishes finally. The classical approximation here agrees with the results in JT gravity (5.11).

Now we consider the case of \( \lambda \ll 1 \). The general behaviors of \( (K_{\gamma}(t,-t))_{\text{VWVV}} \) for \( \gamma = L, R, I \) do not change much. The main difference between \( (K_L(t,-t))_{\text{VWVV}} \) and \( (K_R(t,-t))_{\text{VWVV}} \) is in their time scales. The dissipation time and the scrambling time of system \( R \) are brought forward due to the \( \lambda \) in the temporal argument of Eq. (5.5). So \( (K_R(t,-t))_{\text{VWVV}} \) becomes positive earlier than \( (K_L(t,-t))_{\text{VWVV}} \), as shown in Fig. 8(a). These shorter time
scales advance the dissipation of fluctuation in system $R$. Recall that $(\Delta E_f)_{VWWV}$ must return to zero finally. So the integral effect is that system $R$ obtains energy while system $L$ loses energy.

6. ENTROPY CHANGE

6.1. Replica Trick

We will calculate the entropy change at time $t_f$ after we turn on an interaction $H_I$ at time $t_i$. We will work on imaginary time first, then do a Wick rotation $\tau \to it$ and obtain the entropy change as a function of real time. For the simplicity of notation, we will use imaginary time evolution as follows

$$O(\tau) = e^{H\tau}Oe^{-H\tau}. \quad (6.1)$$

Since the bi-systems are in a pure state, the entropies of both sides are the same. So we will only calculate the entropy of system $R$. Consider the density matrix of system $R$

$$\rho_R(t_f) = \text{Tr}_L \left| \tilde{\beta}_I(t_f) \right\rangle \langle \tilde{\beta}_I(t_f) \right|.$$

where $\left| \tilde{\beta}_I(t_f) \right\rangle$ is given in Eq. (4.1). The Renyi entropy and von Neumann entropy of system $R$ can be written as

$$S_n = \frac{\ln \text{Tr}[\rho_R(t_f)^n]}{1 - n} = \frac{\ln Z_n - n \ln Z}{1 - n}, \quad S = -\text{Tr}[\rho_R(t_f) \ln \rho_R(t_f)] = -\partial_n \ln Z_n + \ln Z|_{n \rightarrow 1}, \quad (6.3)$$

where $Z_n$ is the $n$-replicated partition function and the twisted boundary conditions are applied on system $R$ at time $t_f$. We define a twist operator $X_n$, which cyclically permutes the replica index. We can also write the Renyi entropy of system $R$ as

$$e^{(1-n)S_n} = Z_n/Z^n = \text{Tr}[\rho_R(t_f)^\otimes n X_n]$$

$$= \langle \beta | \otimes^n \{ (\mathbb{I} \otimes X_n) | \tilde{\beta}_I(t_f) \rangle \rangle$$

$$= \langle \beta | \otimes^n \left\{ \mathcal{T} \exp \left( -i \int_{t_i}^{t_f} dt \tilde{H}_I(it) \right) \right\} (\mathbb{I} \otimes X_n) \left\{ \mathcal{T} \exp \left( -i \int_{t_i}^{t_f} dt \tilde{H}_I(it) \right) \right\}^\otimes |\beta | \otimes^n \quad (6.4a)$$

where $C_C$ is the contour ordering. Contour $C^-$ goes from $it_i + 0^-$ to $it_f$ and contour $C^+$ goes from $it_f$ to $it_i + 0^+$, namely

$$C^- = \{ z \in \mathbb{Z} | z = (it_i + 0^-)(1 - \kappa) + it_f \kappa, \kappa \in [0, 1] \}, \quad (6.5a)$$

$$C^+ = \{ z \in \mathbb{Z} | z = it_f(1 - \kappa) + (it_i + 0^+)\kappa, \kappa \in [0, 1] \}, \quad (6.5b)$$

whose directions in real time are opposite. In $\mathbb{I} \otimes X_n$, $I$ acts on the $n$ copies of system $L$, denoted as systems $L^{\otimes n}$, and $X_n$ acts on systems $R^{\otimes n}$.

Eq. (6.4d) can help us to write down the path integral on closed time contour. Recall that an operator of system $R$ ($L$) acting on the TFD state can be transformed into a corresponding operator of system $L$ ($R$) acting on the TFD state. Those operators on the left hand side of $\mathbb{I} \otimes X_n$ can go through the channels of systems $L^{\otimes n}$, reach the right
hand side of $X_n$ and finally become the operators of systems $R^\otimes n$. Taking $H_I = V_L W_R$ as an example, we have

$$
\langle \beta \rangle^\otimes n \left\{ \mathcal{T}_C \exp \left[ -\sum_{a=0}^{n-1} \int_{C^+} d\tau V_{L,a}(\tau) W_{R,a}(\lambda \tau) \right] \right\} (I \otimes X_n) \left\{ \mathcal{T}_C \exp \left[ -\sum_{a=0}^{n-1} \int_{C^-} d\tau V_{L,a}(\tau) W_{R,a}(\lambda \tau) \right] \right\} |\beta\rangle^\otimes n
$$

(6.6a)

$$
= \text{Tr} \left[ \rho^\otimes n X_n \mathcal{T}_C \exp \left\{ -\sum_{a=0}^{n-1} \left[ \int_{C^-} d\tau W_a(\lambda \tau) V_a \left( -\frac{\beta}{2} - \tau \right) + \int_{C^+} d\tau V_a(\beta - \tau) W_a(\lambda \tau) \right] \right\} \right]
$$

(6.6b)

$$
= \left\langle \mathcal{T}_C \exp \left\{ -\sum_{a=0}^{n-1} \left[ \int_{C^-} d\tau W_a(\beta + \lambda \tau) V_a(\beta - \tau) + \int_{C^+} d\tau V_a(\beta - \tau) W_a(\beta + \lambda \tau) \right] \right\} X_n \right\rangle^\otimes n,
$$

(6.6c)

$$
= \left\langle \mathcal{T}_C \exp \left\{ -\sum_{a=0}^{n-1} \int_C d\tau_1 d\tau_2 \sigma_+(\tau_1, \tau_2) V_a(\tau_1) W_a(\tau_2) \right\} X_n \right\rangle^\otimes n,
$$

(6.6d)

$$
= \langle \mathcal{T}_C \exp (-\Delta I_n) \rangle^\otimes n,
$$

(6.6e)

where $\langle ... \rangle^\otimes n = \text{Tr}[\rho^\otimes n]$ and

$$
\sigma_+(\tau_1, \tau_2) = \int_{C^-} d\tau \delta(\tau_1 - (\frac{\beta}{2} - \tau)) \delta(\tau_2 - (\beta + \lambda \tau)) + \int_{C^+} d\tau \delta(\tau_1 - (\frac{\beta}{2} - \tau)) \delta(\tau_2 - \lambda \tau)
$$

(6.7)

We let the replica index $a$ begins at 0 for later convenience. The deformation $\Delta I_n$ is non-local. The action is evaluated on time contour $C$ where the imaginary time goes from 0 to $\beta$, as shown in Fig. 9(a), which is covered by a $n$-sheets manifold. Due to the contour ordering, the integral along $C^+$ is always on the left hand side of the integral along $C^-$. The twist operator $X_n$ imposes the following twisted boundary conduction

$$
V_a(0^-) = V_{a+1}(0^+), \quad W_a(0^-) = W_{a+1}(0^+), \quad a \sim a + n.
$$

(6.8)

When $n = 1$, we have $W(\beta + \tau) = W(\tau)$ and $\Delta I_1 = 0$, which agrees with the unitary evolution of TFD state. One can further unfold the $n$-sheets manifold in Eq. (6.6) and write the path integral into

$$
\left\langle \mathcal{T}_n \exp \left\{ -\sum_{a=0}^{n-1} \left[ \int_{C^-} d\tau W((a + 1)\beta + \lambda \tau) V((a + 1)\beta - \tau) + \int_{C^+} d\tau V((a + 1)\beta - \tau) W((a + 1)\beta + \lambda \tau) \right] \right\} \right\rangle^\otimes_{n\beta},
$$

(6.9)

where $\langle ... \rangle_{n\beta} = Z^{-n} \text{Tr}[e^{-n\beta H}]$ and

$$
V(\alpha \beta + \tau) = V_a(\tau), \quad W(\alpha \beta + \tau) = W_a(\tau).
$$

(6.10)

The action is evaluated on the unfold contour $C_n$, where the imaginary time goes from 0 to $n\beta$, as shown in Fig. 9(b). Similar approaches appeared in Refs. [41, 45, 46].

### 6.2. Entropy at Equilibrium

We will first calculate the entropy at equilibrium by using replica trick. For the Euclidean parameterizations $\{\varphi_a(\tau)\}$ with $n$-sheets in hyperbolic dish, the twisted boundary conditions are

$$
\varphi_a(\beta) = \varphi_{a+1}(0^-), \quad a \sim a + n.
$$

(6.11)

It is convenient to introduce a global parameterization $f(\tau)$ which satisfies

$$
f(\tau + a \beta) = \varphi_a(\tau).
$$

(6.12)

Without $H_I$, the original replicated effective action $I_{n,0}$ can be written as a function of $f(\tau)$

$$
I_{n,0} = -C \sum_{a=0}^{n-1} \int_0^\beta d\tau \left\{ \tan \frac{\varphi_a}{2}, \tau \right\} = -C \int_0^{n\beta} d\tau \left\{ \tan \frac{f}{2}, \tau \right\}.
$$

(6.13)
The saddle point solution is

$$f_c = \frac{2\pi\tau}{n\beta}. \quad (6.14)$$

Then we have

$$\ln Z_n = S_0 + \frac{2\pi^2 C}{n\beta}, \quad S_n = S_0 + \frac{2\pi^2 C(n+1)}{n\beta}, \quad S = S_0 + \frac{4\pi^2 C}{\beta}, \quad (6.15)$$

for nearly extremal case, where $S_0 = \phi_0/4G$ is the extremal entropy contributed by the topological term in Eq. (3.6) [53]. So $C/S_0 = \bar{\phi}/2\pi\phi_0 = \alpha\epsilon/2\pi$.

### 6.3. The entanglement changed by extracting work

Things become complicated when interaction $H_I$ is turned on. The change in Renyi entropy $\Delta S_n$ is proportional to the change in replicated generating functional $\Delta \ln Z_n$, since $\Delta \ln Z = 0$ under unitary evolution. In perturbation theory, it is reduced to the Feynman diagrams expansion of deformation $\Delta I_n$, like (6.9), on the saddle point background (6.14). We have

$$\Delta S_n = \frac{\Delta \ln Z_n}{1-n} = \frac{\ln \langle e^{-\Delta I_n} \rangle}{1-n} = \frac{-\langle \Delta I_n \rangle}{1-n} + \cdots \quad (6.16)$$

Recall that we have extracted work from the TFD state by applying interaction $H_I = gO_L O_R$ in Section 4. As a warm-up, we will derive $\Delta S_n$ at $O(g)$. From Eq. (6.16), we have

$$\langle \Delta S_n^{(1)} \rangle = \frac{\text{ind}}{1-n} \int_{t_i}^{t_f} dt \left[ G_{n\beta} \left( \frac{\beta}{2} + i(t + \lambda t) \right) - G_{n\beta} \left( \frac{\beta}{2} - i(t + \lambda t) \right) \right], \quad (6.17)$$

$$G_{n\beta}(\tau) = \text{Tr}[O(\tau)O\rho^n] = 2 \left( \frac{n\beta}{\pi} \sin \frac{\pi\tau}{n\beta} \right)^{-2\Delta} + O(C^{-1}), \quad 0 < \tau < n\beta \quad (6.18)$$

where $G_{n\beta}(\tau)$ is the Euclidean two points function at inverse temperature $n\beta$. Then the von Neumann entropy change is

$$\Delta S^{(1)} = \frac{2g\beta}{\lambda+1} \left( \frac{\beta}{\pi} \cosh \left( \frac{\pi(\lambda+1)t}{\beta} \right) \right)^{-2\Delta} \left|_{t_i}^{t_f} \right| + O(C^{-1}) \quad (6.19)$$

which agrees with Eq. (4.5) derived from the first law of entropy. Then the consumption of entanglement by extraction work in Eq. (4.7) is justified.
6.4. The entanglement changed by anomalous heat flow

We will calculate the entropy change due to the interaction in Section 5. Due to the special form of interaction (5.1), the result of the first order perturbation vanishes, and we have to consider the second order perturbation. The insertions of two \( H_I \)'s on the \( n \)-sheets time contour have various possibilities. Both time-order and out-of-time-order correlation functions will appear. Similar to Eq. (6.6), we can write down the deformation \( \Delta I_n \) on the original replicated effective action (6.13) due to interaction \( H_I = g (V_L W_R - W_L V_R) / \sqrt{2} \),

\[
\Delta I_n = - \frac{g^2}{\sqrt{2}} \sum_{a=0}^{n-1} \int C d\tau_1 d\tau_2 \sigma_+(\tau_1, \tau_2) [V_a(\tau_1)W_a(\tau_2) - W_a(\tau_1)V_a(\tau_2)] - I_{M,a}
\]  

(6.20a)

\[
= \frac{g^2}{\sqrt{2}} \sum_{a,b=0}^{n-1} \int C d\tau_1 d\tau_2 d\tau_3 d\tau_4 \sigma_+(\tau_1, \tau_2) \sigma_+(\tau_3, \tau_4) [G_{ab}^\sigma(\tau_1, \tau_3)G_{ab}^\sigma(\tau_2, \tau_4) - G_{ab}^\sigma(\tau_1, \tau_4)G_{ab}^\sigma(\tau_2, \tau_3)]
\]  

(6.20b)

\[
= \frac{g^2}{\sqrt{2}} \sum_{a,b=0}^{n-1} \int C d\tau_1 d\tau_2 d\tau_3 d\tau_4 \sigma_+(\tau_1, \tau_2) \sigma_+(\tau_3, \tau_4) [G^I(a_\beta + \tau_1, b_\beta + \tau_3)G^I(a_\beta + \tau_2, b_\beta + \tau_4) - G^I(a_\beta + \tau_1, b_\beta + \tau_4)G^I(a_\beta + \tau_2, b_\beta + \tau_3)]
\]  

(6.20c)

where \( \sigma_+(\tau_1, \tau_2) \) is defined in Eq. (6.7) and

\[
G_{ab}^\sigma(\tau_1, \tau_2) = \langle T_E V_a(\tau_1) W_b(\tau_2) \rangle = \langle T_E W_a(\tau_1) V_b(\tau_2) \rangle = G^I(a_\beta + \tau_1, b_\beta + \tau_2)
\]  

(6.21)

\[
G^I(\tau_1, \tau_2) = 2 \left[ \frac{f'(\tau_1)f'(\tau_2)}{2 \sin \frac{f(\tau_1)-f(\tau_2)}{2}} \right] \Delta
\]  

(6.22)

\( I_{M,a} \) is the action of the matter fields on the \( a \)-sheet. In Eq. (6.20b), we further integrate out the matter fields in \( I_{M,a} \) which gives the correlation function \( G_{ab}^\sigma \). Similar to the situation of energy change at \( O(g^2) \), the first (second) term in Eq. (6.20b) comes from the diagonal (cross) terms in \( H_I^2 \). In Eq. (6.20c), we use the global parameterization (6.12). Contours \( C^- \) and \( C^+ \) in Eq. (6.7) are defined in Eq. (6.5). According to Eq. (6.13), the reparameterization modes around the saddle point solution \( f_\epsilon = 2\pi\tau/n_\beta \) is the same as the original modes in Eq. (3.11) except that the inverse temperature is \( n_\beta \) now. We can integrate them out and obtain the expression of the four points function \( \langle G^I(\tau_1, \tau_2)G^I(\tau_3, \tau_4) \rangle \) up to \( O(C^{-1}) \), which is the same as the original one derived from the generating functional (3.17) except that the inverse temperature is \( n_\beta \). After integrating out the four times \( \{\tau_1, \tau_2, \tau_3, \tau_4\} \) with those delta functions, we obtain the change in Renyi entropy with the form

\[
\Delta S_n = \frac{\langle -\Delta I_n \rangle}{1 - n} + O(g^4) \approx \int_{t_1}^{t_2} dt \int_{t_1}^{t_2} dt' Y_n(t, t')
\]  

(6.23)

We have used perturbation theory at \( O(g^2) \) at large \( C \) limit. The double integrals on real time come from the integral along contours \( C^\pm \) in Eq. (6.7). The integral kernel \( Y_n(t, t') \) is real and enjoys the symmetric

\[
Y_n(t, t') = Y_n(t', t) = Y_n(-t, -t').
\]  

(6.24)

It contains two kinds of four points functions \( \langle V(\tau_1)V(\tau_2)W(\tau_3)W(\tau_4) \rangle_{n_\beta} (\tau_1 > \tau_2 > \tau_3 > \tau_4) \) and \( \langle V(\tau_1)W(\tau_3)V(\tau_2)W(\tau_4) \rangle_{n_\beta} (\tau_1 > \tau_3 > \tau_2 > \tau_4) \). We can group those terms in \( Y_n(t', t) \) into two parts accordingly, denoted as \( Y_n(t', t') \rangle_{VVWW} \) and \( Y_n(t', t) \rangle_{VWVW} \). The former is suffered from UV divergence when \( \tau_1 \rightarrow \tau_2 \) or \( \tau_3 \rightarrow \tau_4 \). So we will introduce a small separation \( \epsilon \) to regularized it, such as \( \langle V(\tau_1 + \epsilon)V(\tau_2)W(\tau_3 + \epsilon)W(\tau_4) \rangle \). While the latter is free from UV divergence since the situation of \( \tau_1 \rightarrow \tau_2 \) or \( \tau_3 \rightarrow \tau_4 \) never happen in Eq. (6.20).

Four points functions will be calculated in JT gravity. We will calculate \( \langle VVWW \rangle \) by integrating out the reparameterization modes at \( O(C^{-1}) \) and calculate \( \langle VVWW \rangle \) by using eikonal approximation. One can alternatively calculate \( \langle VVWW \rangle \) by integrating out the reparameterization modes at \( O(C^{-1}) \). The result is closed to the result from eikonal approximation at early time while it becomes unreliable at late time. For \( VVWW \), we have the property

\[
\langle V(it_1)V(it_3)V(it_2)W(it_4) \rangle_{n_\beta} = \langle W(it_4 + n_\beta)V(it_1)W(it_3)V(it_2) \rangle_{n_\beta} = \langle V(it_4 + n_\beta)W(it_1)V(it_3)W(it_2) \rangle_{n_\beta},
\]  

(6.25)
FIG. 10. $Y_n(t,t')$ in the region of $-|t| < t' < |t|$ for $n = 2,3,4$, $g = 1$, $\beta = 2\pi$, $\Delta = 1/6$, $C = 10^5$, $\epsilon = 0.2$ and $\lambda = 1$. The rest can be recovered by using the symmetry (6.24).

FIG. 11. Five kinds of regions in $(t,t')$ plane. (1) UV Sensitive Region. (2) Conformal Region. (3) Dissipation Region. (4a&4b) Early Time Chaos Region. (5a&5b) Later Time Chaos Region.

FIG. 12. $(Y_2(t,t'))_{VWWW}$ and $(Y_2(t,t'))_{VWWW}$, where $g = 1$, $\beta = 2\pi$, $\Delta = 1/6$, $C = 10^5$, $\epsilon = 0.2$ and $\lambda = 1$.

FIG. 13. (a) $Y_2(t,t)$ and (b) $Y_2(t,-t)$ as functions of $t$, where $g = 1$, $\beta = 2\pi$, $\Delta = 1/6$, $C = 10^5$, $\epsilon = 1,0.5,0.2,0.1$ and $\lambda = 1$.

FIG. 14. Two OTOCs in $(n = 3)$-sheets time contour. The red points and the blue points represent the operators on different types.
where we have exchanged operators $V$ and $W$ according to the $SO(M)$ symmetry of Eq. (3.6). While the eikonal approximation on $\langle V(t_1)W(t_3)V(t_2)W(t_4)\rangle$ in (6.57) of [49] does not maintain the above property. So we adjust the formula accordingly as follows

$$\langle V(it_1)W(it_3)V(it_2)W(it_4)\rangle_{n\beta} = \left(\frac{1}{z}\right)^{2\Delta} U\left(2\Delta, 1, \frac{1}{z}\right), \quad z = \frac{2 \sinh(\pi t_3 + t_4 - t_1 - t_2)}{2 \sinh(\pi t_3) \sinh(\pi t_4)}$$

where we have replaced the exponent by $2 \sinh$ which is valid for both $\text{Re}[t_3 + t_4 - t_1 - t_2] > 0$ and $\text{Re}[t_3 + t_4 - t_1 - t_2] < 0$.

Those four points functions in kernel $Y_n(t, t')$ for general $n$ and times are rather complicate. So we are unable to take the summation $\sum_{n=0}^{n-1}$ in Eq. (6.20) and find the expression of kernel $Y_n(t, t')$ as a analytical function of $n$. It prevents us from obtaining the change in von Neumann entropy $\Delta S$. However, we still can look at $\Delta S_n$ for integer $n \geq 2$ and describe the change in the entanglement between the two systems.

We first analysis the case of $\lambda = 1$. The configuration of $Y_n(t, t')$ is shown in Fig. 10. We also separately draw the configuration of $(Y_n(t, t'))_{VVWW}$ and $(Y_n(t, t'))_{VWWW}$ in Fig. 12. We can find that finally $(Y_n(t, t'))_{VVWW}$ goes to zero due to the quasinormal decay of OTOCs at late time. The characteristic time scales are UV cutoff $\epsilon$, dissipation time $t_d = \frac{\beta}{2\pi}$ and scrambling time $t_s = \frac{\beta}{2\pi} \ln \frac{2\pi C}{\beta}$. We divides the $(t, t')$ plane into several regions according to these time scales, as shown in Fig. 11.

(1) UV Sensitive Region, $|t - t'| \ll \epsilon$ and $|t + t'| \ll t_s$:

The kernel regularized by separation $\epsilon$ is UV sensitive. In Fig. 13, we show the dependence on the choice of separation $\epsilon$. We find that $Y_n(t, t') > 0$ when $t' \rightarrow t$ after such regularization. Such positivity in the UV time scale also appears even the wormhole is absent in Appendix B and the two sites SYK model in Appendix C, where the latter has a known UV completion. The increase of Renyi entropy during the interaction between product states keep the Renyi mutual information positive, although the positivity of Renyi mutual information does not always hold for general states [54]. In Fig. 13, we further find that when $\epsilon$ becomes smaller, the value of $Y_2(t, t')$ in the UV time scale has a positive shift.

(2) Conformal Region, $\epsilon \ll |t - t'| \ll nt_d$ and $|t + t'| \ll t_s$:

The four points functions are factorized into conformal two points functions, such that $Y_n(t, t') \sim -|t - t'|^{-4\Delta} < 0$.
(3) **Dissipation Region**, \(nt_d \ll |t-t'|\) and \(|tt'| \sim 0\):
These four points functions are factorized into two points functions at finite temperature, such that the kernel is dominated by dissipation \(Y_n(t,0) \sim -e^{-2\Delta|t|/nt_d} < 0\) and \(Y_n(0,t') \sim -e^{-2\Delta|t'|/nt_d} < 0\).

(4a) **Early Time Chaos Region A**, \(nt_d \ll |t| < nt_*\) and \(t \sim t'\):
The kernel contains important terms \(-C^{-1}|t|^{-4\Delta} \exp \frac{2\pi t}{nt_d}\). So the Renyi entropy \(S_n\) will exponentially decreases with an exponent \(2\pi t/nt\) as well. It is contributed by the OTOCs in the first term of Eq. (6.20c) as well as the cross terms in \(H_2^2\), as illustrated in Fig. 14(a). Such behavior appears during general interactions between the two sides of the TFD state, as long as the interaction contains a term like \(V_L W_R\) or \(O_L O_R\).

(4b) **Early Time Chaos Region B**, \(nt_d \ll |t| < nt_*\) and \(t \sim -t'\):
The kernel is dominated by terms \(-C^{-1}|t|^{-4\Delta} \exp \frac{2\pi t}{nt_d}\) and the Renyi entropy \(S_n\) exponentially decreases with an exponent \(2\pi t/nt\) as well. It is contributed by the OTOCs in the second term of Eq. (6.20c) as well as the cross terms in \(H_2^2\), as illustrated in Fig. 14(b). So it will not appear if we adopt \(H_I = gV_L W_R\). Such decrease in the entropy kernel corresponds to the anomalous heat flow, since both of them appear near \(t \sim -t'\) and source from the cross terms in \(H_2^2\).

(5a) **Late Time Chaos Region A**, \(|t| > nt_*\) with \(t \sim t'\):
The kernel saturates at a constant, which is contributed by both \(\epsilon\)-regularized conformal two points functions and OTOCs, as shown in Fig. 13(a).

(5b) **Late Time Chaos Region B**, \(|t| > nt_*\) with \(t \sim -t'\):
Due to the late time decay of \((Y_n(t,t'))_{VWVW}\), the kernel goes to a negative constant

\[
\lim_{t \rightarrow \infty} Y_n(t,-t) = -\frac{2n}{n-1} \sum_{a=1}^{n-1} \left( V(a\beta-it)V(\frac{\beta}{2}-it)W((a+\frac{1}{2})\beta+it)W(it) \right)_{n\beta}
\]

\[
= -\frac{8n}{n-1} \left( n \beta \frac{\pi}{\sqrt{2\pi}} \sum_{a=1}^{n-1} \cos \left( \frac{\pi a}{n} \right) - \cos \left( \frac{2\pi a}{n} \right) \right)^{-2\Delta} + O(C^{-1})
\]

(6.27)

without UV sensitivity, as shown in Fig. 13(b).

Interestingly, if we first take the late time limit such that \((Y_n(t,t'))_{VWVW}\) dies out as shown in Fig. 12, and then take the small \(\Delta\) limit, we find

\[
\lim_{\Delta \rightarrow 0} \lim_{t \rightarrow \infty} Y_n(t,t') = -8n.
\]

(6.28)

which is negative at \(n = 1\). Then the von Neumann entropy will also decrease at late time and small \(\Delta\) limit.

The above two kinds of exponential decrease at early time rely on the existence of wormhole. In Appendix B, we replace the eternal black hole by two disconnected black holes, where the wormhole is absent. We calculate the change in Renyi entropy after turning on the interaction, where the former three regions (1)(2)(3) exist, while the chaos regions (4)(5) disappear and are replaced by dissipation (3).

We discuss the case of general \(\lambda\). \(Y_n(t,t')\) for different \(\lambda\) are shown in Fig. 15. \(Y_n(t,t)\) reaches a constant at late time which is independent from \(\lambda\). \(Y_n(t,-t)\) approaches zero from below at late time once \(\lambda \neq 1\), since those four points functions contributing to Eq. (6.27) finally die out due to the mismatch between \(t\) and \(\lambda t\). We integrate the kernel and calculate \(\Delta S_2\) as a function of time, as shown in Fig. 16. In the case of \(t_f \leq 0\), \(\Delta S_2\) momentarily increases near \(t_f = 0\) due to the positive contribution in UV sensitive region (1). However in other period of time, it generally decreases. Specially, in the case of \(-nt_* < t_i \leq 0\), the negative contribution mainly come from chaos regions (4a)(5a). In the case of \(t_f < -nt_*\), the negative contribution mainly come from chaos regions (4)(5). We also check the Renyi entropy for higher \(n\) and similar behaviors are observed.

Recall that the anomalous heat flow in Section 2 appears when \(t_f \sim t_*\) in the case of \(t_i < t_*\) and \(\lambda > 1\), basically due to \((K_L(t,t') - K_R(t,t'))_{VWVW} < 0\) near \(t \sim -t' \sim t_*\). Such agreements on time scale and sign support our expectation that anomalous heat flow appears with the consumption of entanglement. Furthermore, when \(\lambda\) increases, the magnitude of the anomalous heat flow become smaller as shown in Fig. 8. It agrees with the suppress of \(|Y_n(t,t')|\) in chaos region (4b)(5b) and the suppress of \(|\Delta S_2|\) near \(t_f \sim t_*\), as shown in Fig. 15(c) and Fig. 16(a).

In Appendix C, we calculate the entropy changes for the initial TFD state in the two sites SYK model at early time. Regions (1)(2)(3)(4a)(4b) also appears. In region (1), at the large-\(q\) limit, we can get rid of the cutoff dependence. In region (4b), the kernel exponentially decreases as well, which agrees with the presence of the anomalous heat flow in the SYK model. While in region (4a), the kernel exponentially increases, which is different from the case of JT gravity.
7. CONCLUSION AND OUTLOOK

Quantum entanglement can be taken as the resource to extract work or transfer energy. In this paper, we extract the work and study the thermodynamic arrow of time in the presence of wormhole in the AdS/CFT correspondence and the SYK model, where the field theory description is strongly coupled and highly chaotic. The initial state is taken to be the eternal black hole in JT gravity or the TFD state in the two sites SYK model. Such kind of states has two faces: it is highly entangled and local thermal. When some proper interactions between the two sides are turned on, we discover the anomalous heat flow, which transfers the energy from the colder side to the hotter side and further investigate its concomitant consumption of entanglement. Due to the property of the local thermal state, besides the anomalous heat flow, other channels allow the energy diffuses from the hotter side to the colder side. The former effect reverses the thermodynamic arrow of time while the latter effect obeys it. Finally, within the setup in this paper, the latter effect wins the game at the perturbation theory. So far, we do not need to generalize the laws of black hole thermodynamics in the presence of wormhole.

The following issues deserve further investigated in the future.

- The anomalous heat flow in the bulk picture. The energy-momentum tensor will carry the energy transfer in the bulk, which is desired to be solved after the interaction is turned on. It might reveal the specific role of the wormhole in the presence of the anomalous heat flow.

- Go beyond perturbation theory. Since $I_{n,0} \sim N$ and $\Delta I_n \sim g^2$ in the main text, we have taken the large $N$ limit and the small $g$ limit to support the perturbation theory. If we change the scale of $\Delta I_n$ by replacing $g^2 \rightarrow Ng^2$, then $I_{n,0}$ and $\Delta I_n$ will have the same scale. One should solve the full backreaction, which is similar to the eternal traversable wormhole [39] and the geometrical interpretation of the twist operators in the generalized SYK model [45]. However, it is difficult to handle the highly non-local term in $I_{n,0}$ here. One may have to consider other interaction $H_f$ to maintain the solvability and the anomalous heat flow simultaneously.

- The possibility of reversing the total thermodynamic arrow of time. We scan the reasonable parameters space satisfying $\epsilon < \beta < C$ in JT gravity but do not observe the reversion of the total thermodynamic arrow of time. Those disconnected diagrams universally appear for general interactions as long as it contain the term $V_L W_R$. It means that the TFD state is a local thermal state on each side, and thermal diffusion may be unavoidable.

- About black hole evaporation near the Page time. Black hole evaporation is a non-equilibrium process, where energy transfers from a black hole of finite temperature to empty space of zero temperature [25]. At the Page time, the Hawking radiation and the remainder black hole are maximally entangled. It is argued that, after some quantum manipulations on the Hawking radiation, the total system can be transformed into a TFD state [31]. However, even at this time, it does not means that the two local systems have the same temperature. So the TFD state with unbalanced Hamiltonian considered here captures both of the entanglement and the temperature gradient between the remaining black hole and the Hawking radiation after the manipulations, where the former corresponds to the system $R$ and the latter corresponds to system $L$. Interactions between them can trigger subsequent evaporation. The calculation in this paper offers a framework to estimate the changes in energy and entropy of the evaporating black hole near the Page time. If the interaction is $H_f = V_L W_R$, according to the analysis of Sections 5 and 6, energy will transfer into the cold space, and the Renyi entropy will decrease at late time, which agree with the tendency of the Page curve after the Page time [25].

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\[ \text{Although } \Delta S_2 \text{ in Fig. 16(b) corresponds to interaction } H_f = g(V_L W_R - W_L V_R), \text{ the result corresponding to } H_f = gV_L W_R \text{ is almost the same, since the second term of Eq. (6.20) is unimportant when } t_i = 0. \]
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Appendix A: Define work and heat

We review the definitions of work and heat in Ref. [13], which treat the two systems equally. One can divide the state of bi-systems into non-correlated part and correlated part at time $t$,

$$\rho(t) = \rho_L(t) \otimes \rho_R(t) + \chi(t).$$

(A.1)

The infinitesimal change in local energy can be divided into work and heat

$$dE_{\gamma}(t) = dW_{\gamma}(t) + dQ_{\gamma}(t), \quad \gamma = L, R.$$

(A.2)

By introducing two auxiliary parameters $\alpha_L$ and $\alpha_R = 1 - \alpha_L$, one can rearranges the total Hamiltonians as

$$H_{\text{tot}} = H_{L}^{\text{eff}}(t) + H_{R}^{\text{eff}}(t) + H_{I}^{\text{eff}}(t),$$

(A.3)

$$H_{\gamma}^{\text{eff}}(t) = H_{\gamma} + \text{Tr}_{\gamma}[\rho_{\gamma}(t)H_I] - \alpha_{\gamma}\text{Tr}[\rho_L(t) \otimes \rho_R(t)H_I].$$

(A.4)

One can define the so-called binding energy according to the correlation and the interaction

$$U_I = \text{Tr}[\chi(t)H_I^{\text{eff}}(t)].$$

(A.5)

Work and heat are defined as

$$dW_L(t) = -dW_R(t) = -\alpha_L\text{Tr}[d\rho_L(t) \otimes \rho_R(t)H_I] + \alpha_R\text{Tr}[\rho_L(t) \otimes d\rho_R(t)H_I],$$

(A.6a)

$$dQ_{\gamma}(t) = -i\text{Tr}[\chi(t)[H_{\gamma}^{\text{eff}}(t), H_I]] dt,$$

(A.6b)

which satisfy

$$dW_L(t) + dW_R(t) = 0, \quad dQ_L(t) + dQ_R(t) + dU_I(t) = 0.$$  

(A.7)

The two parameters $\alpha_L, \alpha_R$ should be determined by the relation between the pseudo-temperature and the equilibrium temperature in Ref. [13]. However, in the models of this paper, the term $\text{Tr}[ho_L(t) \otimes \rho_R(t')H_I]$ vanishes at the first and second orders perturbation. So, the effect of the two parameters and the work done on the local systems vanish.

Appendix B: Interaction between two disconnected black holes

We consider the unbalanced Hamiltonian $\tilde{H}_0$ and prepare a product of thermal state $\rho = \tau(\beta) \otimes \tau(\beta/\lambda)$ in $CFT_L \otimes CFT_R$, which is dual to two disconnected black holes living in two disconnected AdS$_2$ spaces [50]. Similarly, we consider that in each space there is dialton-gravity described by JT term and some free matter fields

$$I_0 = I_L + I_R,$$

$$I_\gamma = -\frac{1}{16\pi G_N}\left[\int d^2x \sqrt{g}\phi_\gamma(R_\gamma + 2) + 2\int d\sqrt{h}\phi_\gamma,_{\partial}K_\gamma\right] - \int d^2x \sqrt{g}\left[(\partial_\gamma)^2 + m^2\partial_\gamma^2\right], \quad \gamma = L, R$$

(B.1)

where the topology terms are omitted. As shown in the main text, the dynamic of the system $R$ in unbalanced $\tilde{H}_0$ can be obtained by redefined the time of the system $R$ in balanced $H_0$. The boundary conductions are

$$ds^2_{\psi, L} = \frac{1}{\epsilon^2}d\tau^2, \quad \phi_{\psi, L} = \frac{\tilde{\phi}}{\epsilon},$$

$$ds^2_{\psi, R} = \frac{\lambda^2}{\epsilon^2}d\tau^2, \quad \phi_{\psi, R} = \frac{\tilde{\phi}/\lambda}{\epsilon/\lambda},$$

(B.2)
Furthermore, at low energy limit, each $S_i$ with scalar source $J_{i,j}$ is reduced to a Schwarzian action plus the generating function

$$I_{eff} = - C \int d\tau_1 \{ \mu_L, \tau_1 \} - \int d\tau_1 d\tau_2 J_L(\tau_1) J_L(\tau_2) \left[ \frac{\mu'_L(\tau_1) \mu'_L(\tau_2)}{(\mu_L(\tau_1) - \mu_L(\tau_2))^2} \right]^\Delta$$

$$- \frac{C}{\lambda} \int d\tau_1 \{ \mu_R, \tau_1 \} - \int d\tau_1 d\tau_2 J_R(\tau_1) J_R(\tau_2) \left[ \frac{\mu'_R(\lambda \tau_1) \mu'_R(\lambda \tau_2)}{(\mu_R(\lambda \tau_1) - \mu_R(\lambda \tau_2))^2} \right]^\Delta$$

(B.3)

When $J_j = 0$, the on-shell solutions with different temperatures are

$$\mu_L = \tan \frac{\pi \tau}{\beta}, \quad \mu_R = \tan \frac{\pi \lambda \tau}{\beta}.$$  

(B.4)

Real time is obtain by Wick rotation $\mu \to i v$ and $\tau \to i t$. So far, the $LL$ and $RR$ correlations here are the same as those of the TFD state in the main text.

After turning on the interaction $H_I = g O_L O_R$, we find that the $O(g^2)$ contribution vanishes because of $\langle O_L O_R \rangle = 0$. At $O(g^4)$, since the four point function is factorized $\langle O_L O_R O_L O_R \rangle = \langle O_L O_L \rangle \langle O_R O_R \rangle$, the energy changes are contributed by the diagrams 3(a) and 3(b), whose kernel are

$$K_L(t, t') = - G_\beta(-i\Delta t \lambda) G'_\beta(-i\Delta t) - G_\beta(i\Delta t \lambda) G'_\beta(i\Delta t),$$

(B.5)

$$K_R(t, t') = - G_\beta(-i\Delta t \lambda) G'_\beta(-i\Delta t) - G_\beta(i\Delta t \lambda) G'_\beta(i\Delta t).$$

(B.6)

Our calculation in the main text already tell us that $\Delta E_L + \Delta E_R \geq 0$ and $\Delta E_L - \Delta E_R \geq 0$. Same things happen if we alternatively consider interaction $H_I = g V_L W_R$ or $H_I = g(W_L V_R - W_R V_L)/\sqrt{2}$. Since $\langle VW \rangle = 0$, in both of cases, we find $\langle H_1 H_I \rangle = g^2 (V_L W_L) (W_R W_R)$ and only diagrams 3(a) and 3(b) are left.

In the following, we will calculate entropy change by using replica trick. Now the entropy change of system $L$ may not be the same as the one of system $R$ since the total system is not a pure state anymore. We will use the parameterization $\mu_{j,a} = \tan \frac{\varphi_{j,a}}{2} \lambda$ and $\varphi_{j,a} \sim \varphi_{j,a} + 2\pi$. Similar to the method in Section 6, Renyi entropy $S_{n,R}$ of system $R$ can be written as

$$e^{(1-n)S_{n,R}} = \text{Tr} \left[ T_C \exp \left( - \int_{C^-} d\tau \hat{H}_I(\tau) \right) \right] \otimes \left( I \otimes \gamma_X \right)$$

(B.7a)

$$= \text{Tr} \left[ (\rho \otimes \rho)^{\otimes n} \left( T_C \exp \left( - \int_{C^-} d\tau \hat{H}_I(\tau + \beta) \right) - \int_{C^+} d\tau \hat{H}_I(\tau) \right) \right] \otimes \left( I \otimes \gamma_X \right)$$

(B.7b)

$$= \text{Tr} \left[ (\rho \otimes \rho)^{\otimes n} T_C \exp \left( - \sum_a \int_{C^+} d\tau_1 d\tau_2 \sigma(\tau_1, \tau_2) V_{L,a}(\tau_1) W_{R,a}(\tau_2) \right) \right]$$

(B.7c)

where, at the last step, we take the interaction term to be $H_I = g V_L W_R$ and

$$\sigma(\tau_1, \tau_2) = \int_{C^-} d\tau \delta(\tau_1 - (\beta + \tau)) \delta(\tau_2 - (\beta + \lambda \tau)) + \int_{C^+} d\tau \delta(\tau_1 - \tau) \delta(\tau_2 - \lambda \tau),$$

(B.8)

countors $C_\pm$ are defined in Eq. (6.5) and contour $G$ goes from 0 to $\beta$ through $C_\pi$, as shown in Fig. 17. The twisted boundary conductions are only applied on system $R$, leading to boundary conditions $O_{L,a}(\beta) = O_{L,a}(0^-)$ and $O_{R,a}(\beta) = O_{R,a+1}(0^-)$ for any scalar operator $O$ in the path integral of action $I_n = I_{n,0} + \Delta I_n$. The undeformed replicated effective action $I_{n,0}$ can be written as

$$I_{n,0} = - C \sum_{a=0}^{n-1} \int_0^\beta d\tau \left\{ \tan \frac{\varphi_{L,a}}{2}, \tau \right\} - C \sum_{a=0}^{n-1} \int_0^\beta d\tau \left\{ \tan \frac{\varphi_{R,a}}{2}, \tau \right\}$$

(B.9a)

$$= - C \sum_{a=0}^{n-1} \int_0^\beta d\tau \left\{ \tan \frac{\varphi_{L,a}}{2}, \tau \right\} - C \sum_{a=0}^{n-1} \int_0^\beta d\tau \left\{ \tan \frac{f_{R,a}}{2}, \tau \right\}$$

(B.9b)

where $f_R(\tau + a \beta) = \varphi_{R,a}(\tau)$. The saddle point solution is $\phi_{L,a,c} = 2\pi \tau / \beta$, $f_{R,c} = 2\pi \tau / n \beta$. The deformation on
effective action up to $O(g^2)$ is

$$-\Delta I_n = -g \sum_{a=0}^{n-1} \int \mathcal{L} \left( \tau_1, \tau_2, V_{L,a}(\tau_1)V_{R,a}(\tau_2) - I_{M,L,a} - I_{M,R,a} \right)$$

$$= \frac{g^2}{2} \sum_{a=0}^{n-1} \int \mathcal{L} \left( \tau_1, \tau_2, V_{L,a}(\tau_1)V_{R,a}(\tau_2) \right)$$

$$= \frac{g^2}{2} \sum_{a=0}^{n-1} \int \mathcal{L} \left( \tau_1, \tau_2, V_{L,a}(\tau_1)V_{R,a}(\tau_2), G_{\alpha\beta}^{(\alpha)}(\tau_1, \tau_2)G_{\gamma\delta}^{(\beta)}(\tau_1, \tau_2) \right)$$

$$= \frac{ng^2}{2} \int \mathcal{L} \left( \tau_1, \tau_2, V_{L,a}(\tau_1)V_{R,a}(\tau_2), G_{\alpha\beta}^{(\alpha)}(\tau_1, \tau_2)G_{\gamma\delta}^{(\beta)}(\tau_1, \tau_2) \right)$$

where $G_{\alpha\beta}^{(\alpha)}$, $G_{\gamma\delta}^{(\beta)}$, $G_{ij}$ are defined in Eqs. (6.21) and (6.22). $I_{M,L,a}$ is the action of the matter fields in the $\gamma$ AdS$_2$ space on the $a$-th sheet. In Eq. (B.10b), we have used $G_{\alpha\beta}^{(\alpha)}(\tau_1, \tau_2) = \delta_{\alpha\beta}G_{\gamma\delta}^{(\beta)}(\tau_1, \tau_2)$, since the different sheets of system $L$ are disconnected. In Eq. (B.10d), we have used the translational symmetry on $\tau$ and let $G_{\alpha\beta}^{(\alpha)}(\tau_1, \tau_2) = G_{\gamma\delta}^{(\beta)}(\tau_1, \tau_2)$. If the interaction is $H_I = g(V_L W_R - W_L V_R)/\sqrt{2}$, we have

$$-\Delta I_n = \frac{g^2}{\sqrt{2}} \sum_{a=0}^{n-1} \int \mathcal{L} \left( \tau_1, \tau_2, V_{L,a}(\tau_1)V_{R,a}(\tau_2) - I_{M,L,a} - I_{M,R,a} \right)$$

$$= \frac{g^2}{2} \sum_{a=0}^{n-1} \int \mathcal{L} \left( \tau_1, \tau_2, V_{L,a}(\tau_1)V_{R,a}(\tau_2), G_{\alpha\beta}^{(\alpha)}(\tau_1, \tau_2)G_{\gamma\delta}^{(\beta)}(\tau_1, \tau_2) \right)$$

and the final result is the same. The calculation of Renyi entropy $S_{n,L}$ of system $L$ is parallelized to above calculation.

Finally, the change in Renyi entropy at $O(g^2)$ is a double integral (6.23) as well. Since the $G_{\alpha\beta}$ and $G_{ij}$ in Eq. (B.10d) are uncorrelated with each other, the integral kernel $Y_{n,L}(\tau, \tau')$ is factorized into

$$Y_{n,L}(\tau, \tau') = \frac{n}{n-1} \left( G_{\alpha\beta}(\epsilon + i\Delta t)G_{\alpha\beta}(\epsilon + i\Delta t) - G_{\beta\gamma}(\beta - \epsilon + i\Delta t)G_{\beta\delta}(\beta - i\Delta t) \right)$$

$$+ (\Delta t \leftrightarrow -\Delta t), \quad \Delta t = t - t'$$

where $G_{\alpha\beta}$ is given in Eq. (6.18). $Y_{n,L}(\tau, \tau')$ has the similar UV sensitivities as the eternal black hole in Section 6. Above formula is analytical in $n$, so we can take the $n \to 1$ limit and obtain the change in the von Neumann entropy, whose kernels are

$$Y_{n,L}(\tau, \tau') = \frac{n}{n-1} \left( G_{\alpha\beta}(\epsilon + i\Delta t)G_{\alpha\beta}(\epsilon + i\Delta t) - G_{\beta\gamma}(\beta - \epsilon + i\Delta t)G_{\beta\delta}(\beta - i\Delta t) \right)$$

$$+ (\Delta t \leftrightarrow -\Delta t), \quad \Delta t = t - t'$$

The configurations of kernel $Y_{n}(\tau, \tau')$ are shown in Fig. 18. By comparing Eq. (B.5) with Eq. (B.14) or comparing Fig. 4 with Fig. 18, we find that the energy-entropy inequalities (2.2) are saturated at $O(g^2)$ for the initial product state.

### Appendix C: Energy and Entropy in the two sites SYK Model

#### 1. TFD state in the two sites SYK model

We will investigate similar phenomenon in the two sites SYK model in parallel. The local Hamiltonian of the SYK model is a random $q$-body interaction within $N$ Majorana fermions [42]

$$H = i^{q/2} \sum_{1 \leq j_1 < j_2 < \ldots < j_q \leq N} J_{j_1j_2\ldots j_q} \psi^{j_1} \psi^{j_2} \ldots \psi^{j_q},$$

$$\langle J_{j_1j_2\ldots j_q}^2 \rangle = \frac{2^{q-1} J_{j_1j_2\ldots j_q}^2 (q-1)!}{qNq-1}$$ (no sum)
Consider the balanced Hamiltonian \[ [39, 45] \]

where \( q \geq 4 \) and \( N \) is even. The effective action is \([43]\)

\[
\frac{I_0}{N} = \frac{1}{2} \ln \det(\partial_\tau - \Sigma) - \frac{1}{2} \int d\tau_1 d\tau_2 \left[ \Sigma(\tau_1, \tau_2)G(\tau_1, \tau_2) - \frac{J^2}{q} G(\tau_1, \tau_2)^q \right],
\]

where \( G(\tau_1, \tau_2) = \frac{1}{N} \sum_i \langle T_\tau \psi^i(\tau_1)\psi^i(\tau_2) \rangle \). At the limit \( N \gg \beta J \gg 1 \), the saddle point solution is \( G_c(\tau_1, \tau_2) = b \left( \frac{\Delta}{\pi} \sin \frac{\pi(\tau_1 - \tau_2)}{\beta} \right)^{-2\Delta} \) where scaling dimension \( \Delta = [\phi_1] = 1/q \) and coefficient \( b^q \pi = (\frac{\Delta}{2} - \Delta) \tan(\pi\Delta) \). The extremal entropy is \( S_0 = \alpha_0 N \) where the constant \( \alpha_0 \) is a function of \( q \) \([43]\). The leading UV correction is reparameterization \( G(\tau_1, \tau_2) = [f'(\tau_1)f'(\tau_2)]^{\Delta} G_c(f(\tau_1), f(\tau_2)) \) controlled by Schwarzian derivative

\[
I_{\text{eff}} = -\frac{N\alpha_S}{J} \int d\tau \{ f, \tau \}
\]

where \( J = \sqrt{q} J^{(1-q)/2} \) and the constant \( \alpha_S \) is determined by numerical calculation \([43, 44]\). The \( SL(2, R) \) symmetry of the ground state is \( f_L \to \frac{a_{fL} + b}{e_f + d}, f_R \to \frac{a_{fR} + b}{e_f + d} \), with \( ad - bc = 1 \). We can give a correspondence between the parameters in JT gravity and the SYK model by matching their effective actions and the extremal entropy

\[
\frac{1}{q} = \Delta, \quad \frac{N\alpha_S}{J} = C, \quad \frac{\alpha_S}{\alpha_0 J} = \frac{\alpha_c \epsilon}{2\pi}
\]

We consider two sites SYK\(_L\) and SYK\(_R\) consisted of two copies of \( N \) Majorana fermions \( \{ \psi^i_L \} \) and \( \{ \psi^i_R \} \). We first consider the balanced Hamiltonian \([39, 45]\)

\[
H_0 = H_L + H_R, \quad H_L = H \otimes 1, \quad H_R = 1 \otimes H^T,
\]

where

\[
\Delta S_n(t_i, t_f) = \int_{t_i}^{t_f} dt \left( T_\tau \psi^i(\tau)\psi^i(\tau) \right)
\]

and \( n = 1, 2, 3, 4 \).

The configuration of \( Y_n(t, t') \) as a function of \( t - t' \) and \( \Delta S_n \) as a function of \( t_f - t_i \) for the product state in JT gravity, where \( n = 1, 2, 3, 4, q = 1, \beta = 2\pi, \Delta = 1/6, C = 10^5, \epsilon = 0.2 \) and \( \lambda = 2 \). Solid (Dashed) lines correspond to system \( L \) (\( R \)). The entropy changes for the product state in the two sites SYK model are the same.
where the transpose is taken on a basis \{ |\Psi\rangle \}. Those coupling \( J_{ij} \) in \( H_L \) and \( H_R \) are the same. So the disordering of \( SYK_L \) and \( SYK_R \) are correlated. Given such a basis \{ |\Psi\rangle \}, we construe state

\[
|I\rangle = \sum_{\psi} |\Psi\rangle_L |\Psi\rangle_R,
\]

which satisfies \((H_L - H_R) |I\rangle = 0\) automatically. We specify the basis by imposing \[45\]

\[
(\psi^j_L + i\psi^j_R) |I\rangle = 0, \quad j = 1, 2, ..., N.
\]

One can find \( H^T = H \) in this basis. The TFD state at \( t = 0 \) is obtained from the Euclidean evolution of \( |I\rangle \)

\[
|\beta\rangle \equiv Z_{\beta}^{-1/2} e^{-\beta(H_L + H_R)/2} |I\rangle.
\]

By using Eq. (C.8), the two points function evaluated on the TFD state can be transformed into \( \langle \beta | \psi^j_L (t_1) \psi^k_R (t_2) |\beta\rangle = i \text{Tr}[\gamma \psi^j (-t_1) y \psi^k (t_2)] \), where \( y = Z^{-1/2} e^{-\beta H/2} \) and \( \text{Tr}[O] = \text{Tr}[O_L] \).

Similar to the main text, we further consider the unbalanced Hamiltonian

\[
\hat{H}_0 = H_L + \hat{H}_R, \quad \hat{H}_R = \lambda H_R,
\]

on the two sites SYK model. The TFD state which evolves with \( \hat{H}_0 \) is \[\beta(t)\] = \( e^{-i\hat{H}_0 t} |\beta\rangle \). It is equivalent to redefine the time of \( SYK_R \).

## 2. Anomalous Heat Flow

To construe the anomalous heat flow in the two sites SYK model, we turn on the following interaction after \( t_i \),

\[
H_I = i \sum_{j,k=1}^N g_{j,k} \psi^j_L \psi^k_R,
\]

\[
g_{j,k} = g_{k,j}, \quad \langle g_{j,k} g_{l,m} \rangle = N^{-2} g^2 (\delta_{jl} \delta_{km} + \delta_{jm} \delta_{kl}).
\]

At large \( N \) limit, the energy changes at \( t_f \) are

\[
\Delta E_L = -i g^2 N^{-2} \sum_{j,k} \int_{t_i}^{t_f} dt \int_{t_i}^{t} dt' \langle \left[ \psi^j_L (t) \psi^k_R (\lambda t), \psi^j_L (t') \psi^k_R (\lambda t') \right] \rangle + \langle \left[ \psi^j_L (t) \psi^k_R (\lambda t), \psi^k_R (t') \psi^j_L (\lambda t') \right] \rangle
\]

\[
= 2g^2 N^{-2} \text{Im} \sum_{j,k} \int_{t_i}^{t_f} dt \int_{t_i}^{t} dt' \text{Tr}[\gamma \psi^j (-t') \psi^k (\lambda t') y \psi^k (\lambda t) y] + \text{Tr}[\psi^k (-t') \psi^j (\lambda t) y \psi^k (\lambda t) y],
\]

\[
\Delta E_R = -i g^2 N^{-2} \lambda \sum_{j,k} \int_{t_i}^{t_f} dt \int_{t_i}^{t} dt' \langle \left[ \psi^j_L (t) \psi^k_R (\lambda t), \psi^j_L (t') \psi^k_R (\lambda t') \right] \rangle + \langle \left[ \psi^j_L (t) \psi^k_R (\lambda t), \psi^k_R (t') \psi^j_L (\lambda t') \right] \rangle
\]

\[
= 2g^2 N^{-2} \lambda \text{Im} \sum_{j,k} \int_{t_i}^{t_f} dt \int_{t_i}^{t} dt' \text{Tr}[\gamma \psi^j (-t') \psi^k (\lambda t') y \psi^k (\lambda t) y] + \text{Tr}[\psi^k (-t') \psi^j (\lambda t) y \psi^k (\lambda t) y],
\]

\[
\Delta E_I = -i g^2 N^{-2} \sum_{j,k} \int_{t_i}^{t_f} dt \langle \left[ \psi^j_L (t_f) \psi^k_R (\lambda t_f), \psi^j_L (t_f) \psi^k_R (\lambda t_f) \right] \rangle + \langle \left[ \psi^j_L (t_f) \psi^k_R (\lambda t_f), \psi^k_R (t_f) \psi^j_L (\lambda t_f) \right] \rangle
\]

\[
= 2g^2 N^{-2} \text{Im} \sum_{j,k} \int_{t_i}^{t_f} dt \text{Tr}[\gamma \psi^j (-t_f) \psi^k (\lambda t_f) y \psi^k (\lambda t_f) y] + \text{Tr}[\psi^k (-t_f) \psi^j (\lambda t_f) y \psi^k (\lambda t_f) y],
\]

where \( \psi^j_L (t) = e^{itH_L} \psi^j_L e^{-itH_L}, (\gamma = L, R), \psi^j (t) = e^{itH} \psi^j e^{-itH} \) and \( y = Z^{-1/2} e^{-\beta H/2} \). Above equations have the same form as Eq. (5.5) \[3\]. The disconnected four points functions are fixed by conformal symmetry and the leading

\[3\] Exchanging the two fermions in the second term will give a minus sign.
contribution of the connected four points functions at early time are fixed by the effective action (C.4). So the energies change at early time here are the same as these in nearly AdS$_2$. The anomalous heat flow in the SYK model is contributed by the second term in Eq. (C.13).

If we consider an initial product state in the two sites SYK model and turn on the same interaction (C.11), similar to Section B, only the first term in Eq. (C.13) left.

To calculate the change in Renyi entropy $\Delta S_n$, we apply replica trick on system $R$. We will alternately use imaginary time $\psi(\tau) = e^{\tau H} \langle \psi e^{-\tau H} \rangle$ below. Similar to Eq. (6.6), for interaction (C.11), we have

$$e^{(1-n)S_n} = \langle \beta \rangle \sum_b \left\{ \mathcal{T}_C \exp \left[ -i \sum_{a, jk} \int_{c_-} d\tau g_{jk} \psi^j_{L,a}(\tau) \psi^k_{R,a}(\lambda \tau) \right] \right\} (\mathbb{I} \otimes X_n) \left\{ \mathcal{T}_C \exp \left[ -i \sum_{a, jk} \int_{c_-} d\tau g_{jk} \psi^j_{L,a}(\tau) \psi^k_{R,a}(\lambda \tau) \right] \right\} |\beta\rangle \otimes^n
$$

$$= \text{Tr} \left[ \rho^n X_n \mathcal{T}_C \exp \left\{ \sum_{a, jk} \left[ \int_{c_-} d\tau g_{jk} \psi^j_{a}(\lambda \tau) \psi^k_{a}(\lambda \tau) \right] - \int_{c_-} d\tau g_{jk} \psi^j_{a}(\lambda \tau) \psi^k_{a}(\lambda \tau) - \int_{c_-} d\tau g_{jk} \psi^j_{a}(\lambda \tau) \psi^k_{a}(\lambda \tau) \right] \right\} X_n \right\} \otimes^n
$$

where

$$\sigma_-(\tau_1, \tau_2) = \int_{c_-} d\tau \delta(\tau_1 - (\frac{\beta}{2} - \tau)) \delta(\tau_2 - (\beta + \lambda \tau)) - \int_{c_-} d\tau \delta(\tau_1 - (\frac{\beta}{2} - \tau)) \delta(\tau_2 - \lambda \tau).$$

In the finally expression, twist operator $X_n$ is located at a time $\tau_*$ which is infinitesimally lesser than 0. It imposes the relations

$$\psi^j_{a}(\tau_*) = \psi^j_{a+1}(\tau_*), \quad \forall a = 0, 1, \ldots, n - 1, \quad (\psi^j_{n} = \psi^j_{0}).$$

We write down the replicated action, integrate out the disorder and introduce the bi-local field $G_{ab}(\tau_1, \tau_2) = N^{-1} \sum_j \left\{ \mathcal{T}_E \psi^j_{a}(\tau_1) \psi^j_{b}(\tau_2) \right\}$ and Legendre multiplier $\Sigma_{ab}(\tau_1, \tau_2)$,

$$Z_n = \int \prod_{j, a} \mathcal{D}\psi^j_{a} e^{-I_n[\psi]} = \int \prod_{a} \mathcal{D}G_{ab} \mathcal{D}\Sigma_{ab} e^{-I_n[G, \Sigma]},$$

(C.17)
and

\[ I_{\alpha \beta} = \sum_{\tau} \int d\tau \left[ -\frac{1}{2} \sum_j \psi_j^a(\tau) \frac{\partial}{\partial \tau} \psi_j^a(\tau) + i \frac{J}{2} \sum_{j_1 < j_2 < \ldots < j_n} J_{j_1 j_2 \ldots j_n} \psi_{j_1}^a(\tau) \psi_{j_2}^a(\tau) \ldots \psi_{j_n}^a(\tau) \right] \]

where

\[ I_n = \sum_{a=0}^{n-1} \int_{I_c} d\tau \left[ -\frac{1}{2} \sum_j \psi_j^a(\tau) \frac{\partial}{\partial \tau} \psi_j^a(\tau) + i \frac{J}{2} \sum_{j_1 < j_2 < \ldots < j_n} J_{j_1 j_2 \ldots j_n} \psi_{j_1}^a(\tau) \psi_{j_2}^a(\tau) \ldots \psi_{j_n}^a(\tau) \right] \]

\[ - \sum_{a,jk} \int_{I_c} d\tau_1 d\tau_2 \sigma_{-(\tau_1, \tau_2)} g_{jk} \psi_a^j(\tau_1) \psi_k^j(\tau_2) \]

\[ \sum_{a,j} \int_{I_c} d\tau \psi_a^j(\tau) \frac{\partial}{\partial \tau} \psi_a^j(\tau) + \frac{J^2}{2qNq-1} \sum_{a,b,j_1 < j_2 < \ldots < j_4} \int_{I_c} d\tau_1 d\tau_2 \psi_a^j(\tau_1) \psi_b^j(\tau_2) \psi_a^j(\tau_1) \psi_b^j(\tau_2) \ldots \psi_a^j(\tau_1) \psi_b^j(\tau_2) \]

\[ + \frac{g^2}{2N^2} \sum_{a,b,jk} \int_{I_c} d\tau_1 d\tau_2 d\tau_3 d\tau_4 \sigma_{-(\tau_1, \tau_2)} \sigma_{-(\tau_3, \tau_4)} \left[ \psi_a^j(\tau_1) \psi_b^j(\tau_2) \psi_a^j(\tau_3) \psi_b^j(\tau_4) + \psi_a^j(\tau_1) \psi_b^j(\tau_2) \psi_b^j(\tau_3) \psi_a^j(\tau_4) \right] \]

\[ = \frac{N}{2} \ln \det(\partial_{\beta} - \Sigma_{ab}) - \frac{N}{2} \sum_{a,b} \int_{I_c} d\tau_1 d\tau_2 \left[ \Sigma_{ab}(\tau_1, \tau_2) G_{ab}(\tau_1, \tau_2) - \frac{J^2}{q} G_{ab}(\tau_1, \tau_2)^q \right] \]

\[ + \frac{g^2}{2N^2} \sum_{a,b,jk} \int_{I_c} d\tau_1 d\tau_2 d\tau_3 d\tau_4 \sigma_{-(\tau_1, \tau_2)} \sigma_{-(\tau_3, \tau_4)} \left[ -G_{ab}(\tau_1, \tau_2) G_{ab}(\tau_3, \tau_4) + G_{ab}(\tau_1, \tau_4) G_{ab}(\tau_2, \tau_3) \right] \]

\[ = - I_{n,0} - \Delta I_n, \]

(C.18a)

(C.18b)

(C.18c)

(C.18d)

where \( \Delta I_n \) is a highly non-local deformation on the original bi-local action \( I_{n,0} \). Relation (C.16) leads to the twisted boundary conditions in the path integral

\[ \psi_a(\beta) = \psi_a^1(0^-), \quad \psi_{a+1}(\beta) = -\psi_0^1(0^-) \quad \text{for} \quad a = 0, 1, \ldots, n - 2, \]

\[ G_{ab}(\beta, \tau) = G_{a+1b}(0^-, \tau), \quad G_{ab}(n-1, \beta, \tau) = -G_{ab}(0^-, \tau) \quad \text{for} \quad a = 0, 1, \ldots, n - 2, \quad b, \forall b. \]

\[ G_{ab}(\tau, \beta) = G_{a(b+1)}(\tau, 0^-), \quad G_{a(n-1)}(\tau, \beta) = G_{a(n-1)}(\tau, 0^-) \quad \text{for} \quad b = 0, 1, \ldots, n - 2, \quad \forall a. \]

Similar conditions arise in Refs. [45, 55]. Similar to Eq. (6.21), we introduce a global fermion \( \psi_j(\tau) \) and global bi-local fields \( G(\tau_1, \tau_2), \Sigma(\tau_1, \tau_2) \) by

\[ \psi_a^j(\tau) = \psi^j(a\beta + \tau), \quad G_{ab}(\tau_1, \tau_2) = G(a\beta + \tau_1, b\beta + \tau_2), \quad \Sigma_{ab}(\tau_1, \tau_2) = \Sigma(a\beta + \tau_1, b\beta + \tau_2). \]

According to Eq. (C.19), these global fields are continuous in \( (0, n\beta) \) and anti-periodic

\[ \psi_j(a\beta) = -\psi_j(0^-), \quad G(a\beta, \tau) = -G(0^-, \tau), \quad G(\tau, n\beta) = -G(0^-, \tau), \quad \Sigma(n\beta, \tau) = -\Sigma(0^-, \tau), \quad \Sigma(\tau, n\beta) = -\Sigma(\tau, 0^-). \]

Then we have

\[ - I_{n,0} = \frac{N}{2} \ln \det(\partial_{\beta} - \Sigma) - \frac{N}{2} \int_{C_n} d\tau_1 d\tau_2 \left[ \Sigma(\tau_1, \tau_2) G(\tau_1, \tau_2) - \frac{J^2}{q} G(\tau_1, \tau_2)^q \right], \]

\[ - \Delta I_n = \frac{g^2}{2} \sum_{a,b} \int_{C_n} d\tau_1 d\tau_2 d\tau_3 d\tau_4 \sigma_{-(\tau_1, \tau_2)} \sigma_{-(\tau_3, \tau_4)} \left[ -G(a\beta + \tau_1, b\beta + \tau_3) G(\tau_1, \tau_2) G(\tau_3, \tau_4) + G(a\beta + \tau_1, b\beta + \tau_4) G(\tau_2, \tau_3) \right], \]

where \( C_n \) is the unfold time contour in Fig. 9 and the \( \tau \) in the integral goes from 0 to \( n\beta \). Eq. (C.22a) is just the action of the SYK model at inverted temperature \( n\beta \), with the saddle point solution and effective action in the same forms when \( \beta J \gg 1 \). Considering the reparameterization modes in Eq. (C.22a) will lead to Schwarzian derivative as the case of JT gravity in Eq. (6.13). If we integrate out the four times in Eq. (C.22b), expand the product into eight terms and sort the time ordering of fermions, we will find that the deformation \( \Delta I \) in SYK model is different from Eq. (6.20) in JT gravity, where some terms have inverted signs. Perturbation theory (6.16) at \( O(g^2) \) is applicable at large \( N \) limit. The kernel \( Y_n(t, t') \) is consisted of four points functions \( \sum_{jk} \langle \psi \psi \psi \psi \rangle \) and \( \sum_{jk} \langle \psi \psi \psi \psi \psi \psi \psi \psi \rangle \), which is controlled by effective action (C.4) as well. So, at early time, we will apply the four points functions \( \langle VVWW \rangle \) and \( \langle VVVV \rangle \) from JT gravity but take care of their signs when evaluating \( Y_n(t, t') \). The parameters of the SYK model and JT gravity can be related by using relation (C.5). The configuration of kernel \( Y_n(t, t') \) at early time is
FIG. 19. A part of $Y_n(t, t')$ for an initial TFD state in the two sites SYK model. There parameters are $n = 2, 3, 4$, $g = 1$, $\beta = 2\pi$, $\Delta = 1/6$, $C = 10^7$, $\epsilon = 0.2$ and $\lambda = 1$, which can be translated into the parameters of the SYK model $q, J, N$ according to relation (C.5). The rest part of the configuration can be recovered by using the symmetry (6.24).

shown in Fig. 19. Kernel $Y_n(t, t')$ exponentially increase at early time chaos region A ($t \sim t'$), contributed by the first term in Eq. (C.22b). Kernel $Y_n(t, t')$ exponentially decrease at early time chaos region B ($t \sim -t'$), contributed by the second term in Eq. (C.22b). Its decrease agrees with the fact that the anomalous heat flow in the SYK model is also contributed by the second term in Eq. (C.13).

For an initial product state of the two sites SYK model, we calculate the entropy changes in the way of Appendix B and obtain

$$e^{(1-n)S_{n,R}} = \text{Tr} \left[ (\rho \otimes \rho)^{\otimes n} \mathcal{T}_C \exp \left( -i \sum_{a,j,k} \int_{\mathcal{C}} d\tau_1 d\tau_2 \sigma(\tau_1, \tau_2) g_{jk} \psi^j_{L,a}(\tau_1) \psi^k_{R,a}(\tau_2) \right) (\mathbb{I} \otimes X_n) \right]$$  \hspace{1cm} (C.23a)

$$= \int \prod_{\gamma,a,j} D\psi^j_{\gamma,a} \exp(-I_{n,0} - \Delta I_n),$$  \hspace{1cm} (C.23b)

with boundary conditions $\psi_{L,a}(\beta) = -\psi_{L,a}(0^-)$ for ($a = 0, 1, ..., n - 1$), $\psi_{R,a}(\beta) = \psi_{R,a+1}(0^-)$ for ($a = 0, 1, ..., n - 2$) and $\psi_{R,n-1}(\beta) = -\psi_{R,0}(0^-)$. We can transform $I_{n,0}$ into the effective action (B.9) and transform

$$- \Delta I_n
\begin{align*}
= & -i \sum_{a,j,k} \int_{\mathcal{C}} d\tau_1 d\tau_2 \sigma(\tau_1, \tau_2) g_{jk} \psi^j_{L,a}(\tau_1) \psi^k_{R,a}(\tau_2) \\
= & - \frac{g^2}{2N^2} \sum_{a,b,j,k} \int_{\mathcal{C}} d\tau_1 d\tau_2 d\tau_3 d\tau_4 \sigma(\tau_1, \tau_2) \sigma(\tau_3, \tau_4) \left[ \psi^j_{L,a}(\tau_1) \psi^k_{R,a}(\tau_2) \psi^j_{L,b}(\tau_3) \psi^k_{R,b}(\tau_4) + \psi^j_{L,a}(\tau_1) \psi^k_{R,a}(\tau_2) \psi^k_{L,b}(\tau_3) \psi^j_{R,b}(\tau_4) \right] \\
= & \frac{g^2}{2} \sum_a \int_{\mathcal{C}} d\tau_1 d\tau_2 d\tau_3 d\tau_4 \sigma(\tau_1, \tau_2) \sigma(\tau_3, \tau_4) G_{L,aa}(\tau_1, \tau_2) G_{R,aa}(\tau_2, \tau_4).
\end{align*}$$  \hspace{1cm} (C.24)

The final result is the same as deformation (B.10) in JT gravity. So the energy-entropy inequalities (2.2) are the saturated as well.

3. Large $q$ limit and UV sensitive region

In the main part of this paper, we only use the correlators with nearly conformal symmetry, which is only valid in the IR. Due to the lack of the UV part of the boundary theory which is dual to JT gravity in the bulk, we are unable to give a detail description on the change in energy and entropy in the UV sensitive region. Thanks to the fact that the UV fixed point of the SYK model is known as free fermions, we can calculate the UV behaviors. Here, we will
consider the large $q$ limit. The Euclidean Green function with $1/q$ expansion is [43]

$$G(\tau) = \frac{1}{2} \text{sgn}(\tau) \left[ 1 + \frac{2}{q} \ln \left( \cos \frac{\pi \nu(\beta)}{2} - \frac{|\tau|}{\beta} \right) \right] + \cdots,$$

(C.27)

where the logarithm is required to be of order 1 and $\nu(\beta)$ is determined by

$$\beta J = \frac{\pi \nu(\beta)}{\cos \frac{\pi \nu(\beta)}{2}}.$$

(C.28)

In (C.27), the second term, as a correction, should be smaller than the first term. So we should keep the real time small and focus on UV sensitive region. Then we can assume that four points functions are factorized into the product of two points functions. We turn on the interaction (C.11) at time $t_i$ and calculate the changes in energy and entropy at time $t_f$.

For an initial product state in the two sides SYK model, we use the method of Section B. The energy-entropy inequalities (2.2) are saturated. We illustrate the entropy change in Fig. 20. The kernels of both entropies and energies are positive within the period where approximation (C.27) is valid. When $\lambda \geq 1$, the increase in energy agrees with the fact that the product of thermal states is a passive state; the increase in entropy agrees with the fact that interaction increases entanglement for an initial product state. When $\lambda > 1$, as shown in Fig. 20, we find that the entropies obtained in the way of the large $q$ limit increase as well. It agrees with the behaviors of the Renyi entropy in the UV sensitive region obtained in the way of UV regularization in Fig. 18.

For an initial TFD state in the two sides SYK model, we use Eqs. (C.13)-(C.22)(6.23) to calculate the changes in energy and entropy. The energy changes are the same as the case of product state because of the factorization of four points functions. While the entropy changes are different because of the initial entanglement. So the energy-entropy inequalities (2.2) may not be saturated any more. We illustrate the result in Fig. 21.
Appendix D: Classical model

We will try to discuss the phenomenon analogous to anomalous heat flow in the classical model of [36]. Consider a local Hamiltonian

$$H(\{x_i\} , \{p_i\}) = \frac{1}{2} \sum_{i}^{N} p_i^2 + \sum_{a}^{N} \left( \sum_{ij}^{N} x_i J_{aij} x_j \right)^2,$$  \hspace{1cm} (D.1)

where $J_{aij}$ is random coupling satisfying $\langle J_{aij}^2 \rangle = N^{-1}$. Such model is non-integrable and expected to exhibit classical chaos.

Consider two copies of phase space $(x^L_i, p^L_i)$ and $(x^R_i, p^R_i)$ where $i = 1, 2, ..., N$ and their total Hamiltonian

$$H_0 = H_L + H_R = H(\{x^L_i\}, \{p^L_i\}) + \lambda H(\{x^R_i\}, \{p^R_i\}).$$ \hspace{1cm} (D.2)

Analogous to the TFD state, we prepare an ensemble at $t = 0$ each state of which satisfies

$$x^L_i = x^R_i, \quad p^L_i = -p^R_i,$$ \hspace{1cm} (D.3)

and distributes according to Gibb distribution $\rho(\{x_i\}, \{p_i\}) \sim e^{-\beta H(\{x_i\}, \{p_i\})}$. Each states evolved by $H_0$ will satisfy

$$x^L_i(\lambda t) = x^R_i(-t), \quad p^L_i(\lambda t) = -p^R_i(-t).$$

To trigger heat flow, at time $t_i$, we turn on the interaction

$$H_I = \sum_{ij} g_{ij} x^L_i x^R_j,$$ \hspace{1cm} (D.4)

where $g_{ij}$ is random coupling satisfying $g_{ij} = -g_{ji}$ and $\langle g_{ij} g_{kl} \rangle = N^{-2} g^2 (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk})$ for $i \neq j, k \neq l$.

Similarly, by using the classical interaction picture, we can estimate the energy change at time $t_f$ with second order perturbation

$$\Delta E_I = \int_{t_i}^{t_f} dt \langle\langle H_I(t_f), H_I(t)\rangle\rangle = \int_{t_i}^{t_f} dt K_I(t_f, t),$$ \hspace{1cm} (D.5a)

$$K_I(t, t') = \frac{g^2}{N^2} \sum_{i \neq j} \langle\langle x^L_i(t)x^R_j(\lambda t), x^L_j(t')x^R_j(\lambda t')\rangle\rangle - \langle\langle x^L_i(t)x^R_j(\lambda t), x^L_j(t')x^R_j(\lambda t')\rangle\rangle$$ \hspace{1cm} (D.5b)

$$= (K_I(t, -t))_{ijij} + (K_I(t, -t))_{ijji},$$ \hspace{1cm} (D.5c)

whose form is the same as Eq. (5.5c). When $t = -t'$ and $\lambda = 1$, $(K_I(t, -t))_{ijji}$ can be written as

$$\langle\langle x_i(t)^2, x_j(-t)^2\rangle\rangle = -\frac{g^2}{2N^2} \sum_{i \neq j} \langle\langle x_i(t)^2, x_j(-t)^2\rangle\rangle,$$ \hspace{1cm} (D.6)

which agrees with the classical limit (5.20).