Nonlocal Euler–Bernoulli beam theories with material nonlinearity and their application to single-walled carbon nanotubes

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1 Introduction

Nanobeams have immense potential applications in nanoelectromechanical systems (NEMS), for example, nanotube sensors [1] and resonators [2]. However, it is still an open question to model precisely the mechanical properties of nanobeams [3, 4]. There are two crucial characteristics in the mechanical properties of nanostructures. One is the small-scale effect in nanostructures that reveals the strong relationship between mechanical properties and geometric size [5–11]. Another is the nonlinear elastic property that determines the overall mechanical behavior of the beams [12–16]. The nonlinearity in elasticity can be introduced by the physical and the geometrical (large deformation). Researchers have expended considerable attention on the small-scale effect of nanobeams [17–21]. However, the material nonlinearity of nanobeams has been rarely considered by researchers [22]. Moreover, the nonlinearity of the thermal-electro-mechanical coupling in nanobeams is also required [23, 24].

Researchers have usually modified the classical continuum mechanics to capture the small-scale effect through three different paths. The first is the nonlocal stress gradient model [7]:

\[ 1 - (e_0 a)^2 \nabla^2 ] \sigma_{ij} = \tilde{\sigma}_{ij}. \]

The second is the strain gradient model [9, 25, 26]:

\[ \tilde{\sigma}_{ij} = E_{ijkl} [1 - (e_0 a)^2 \nabla^2 ] \varepsilon_{kl}. \]

And the last is surface stress model [27]. Here, \( \tilde{\sigma}_{ij} \) and \( \varepsilon_{ij} \) are the local stress...
and strain; $E_{ijkl}$ is the elastic stiffness; $\sigma_{ij}$ is the nonlocal stress; $e_0$ is a small-scale parameter; and $a$ is the material characteristic length which is the length of the carbon–carbon bond in SWCNTs. The stress or strain gradient models have been widely used to study carbon nanotubes (CNTs) and graphene [28–30]. Since the mechanical properties of graphene and CNTs have lacked a thorough understanding based on quantum mechanics [32], determining the scale parameter $e_0$ is still a controversial open question [5, 31].

For material nonlinearity, researchers have calculated nonlinear elastic parameters for some materials through the molecular dynamics (MD) and the density functional theory (DFT), for example, CNTs [16], graphene [14], and silicon nanowires [13]. However, because the material nonlinearity may complicate mechanical models of nanobeams, it has been neglected in most exciting studies. Nonetheless, several studies have shown that the strain’s cubic terms in the potential energy (corresponding to the quadratic terms in the stress–strain relationship) significantly affect the mechanical properties of graphene and SWCNTs [22, 33]. The accurate understanding of nanostructure’s mechanical properties is the basis of applications. Therefore, it is necessary to comprehensively consider both the small-scale effect and the material nonlinearity. The present paper will propose two Euler–Bernoulli theories, including the nonlocal effect and the material nonlinearity, to accurately characterize the nanobeam’s mechanical properties under different boundary conditions.

2 Mathematical models

We restrict our attention to slender beams in the present research. Hence, the Euler–Bernoulli hypothesis is employed [34]: the cross sections perpendicular to the centroid locus before deformation remain plane and perpendicular to the deformed locus, which implies no transverse shear strains. Under this hypothesis, the beam’s longitudinal (x-direction) strain component, $\varepsilon_{xx}$, is considered only, as shown in Fig. 1. Atomic calculations have shown that the potential energies of graphene and silicon materials contain the strain’s cubic nonlinear terms [12, 13]. Geometrically, an SWCNT can be viewed as a graphene sheet that has been rolled into a tube. Thus, the stress–strain relationship of big diameter SWCNTs may be consistent with graphene. This speculation was confirmed by MD simulations [16]. For simplicity, we assume that the strain is finite but small, so only cubic nonlinear terms of the potential energy are kept, and the local longitudinal stress can be expressed as $\sigma_{xx} = \sigma^0_{xx} + E\varepsilon_{xx} + D\varepsilon^2_{xx}$ [12, 15, 16, 22]. Here, $\sigma^0_{xx}$ is the initial prestress, and $E$ and $D$ are the second-order and third-order elastic coefficients, respectively. Following the nonlocal differential constitutive relationship, the nanobeam’s nonlocal constitutive relationship with the material nonlinearity is written as

$$\left[1 - \mu^2 \nabla^2\right] \sigma_{xx} = \sigma^0_{xx} + E\varepsilon_{xx} + D\varepsilon^2_{xx},$$  \hspace{1cm} (1)$$

here $\mu = e_0a$. For establishing models of nanobeams, it is necessary to give the strain–displacement relation according to different boundary constraints [34–37]. We consider two boundary conditions in this research. First, beams are extensible, such as clamped–clamped or hinged–hinged beams. Second, beams are inextensible, such as simply supported or clamped-free beams. The present paper will establish motion equations for two conditions.

2.1 Model of extensible nanobeams

Here, we confine the problem by assuming that although the deflection of the beam is no longer small in comparison with its height, it is still small in comparison with the longitudinal dimension. Under this condition, the longitudinal deformation may mainly be induced by the transverse deformation if the two ends of a nanobeam cannot move along the x-axis, such as hinged–hinged or clamped–clamped beams. So the axial strain is [34]

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 - \varepsilon^2 \frac{\partial^2 w}{\partial x^2}. \hspace{1cm} (2)$$

Here $u$ and $w$ are axial displacements of the beam in the x and z directions, respectively, as shown in Fig. 1. Substituting Eq. (2) into the expression of local stress, $\sigma_{xx} = \sigma^0_{xx} + E\varepsilon_{xx} + D\varepsilon^2_{xx}$, gets the local axial force and bending moment as
Here $A = \pi dh$ and $I = \pi dh^3/8$ are the cross-sectional area and the inertia moment of thin-walled beams. $d$ is the diameter, and $h$ is the thickness; $N_0 = \int_A \sigma_{xx} dA$ is the initial axial load at the ends, as shown in Fig. 1. The quartic terms of $w$ can be neglected for slender beams [34], and notices $\frac{\partial u}{\partial x} = O(\frac{\partial w}{\partial x})^2$ [35], the force reduces to

$$\bar{N} = \int_A \sigma_{xx} dA = -N_0 + EA \left[ \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right]$$

$$+ ID \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + DA \left[ \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right]^2,$$

$$\bar{M} = \int_A \bar{\sigma}_{xx} dA = -EI \frac{\partial^2 w}{\partial x^2}
- 2ID \left[ \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] \frac{\partial^2 w}{\partial x^2}. \quad (4)$$

Here $A = \pi dh$ and $I = \pi dh^3/8$ are the cross-sectional area and the inertia moment of thin-walled beams. $d$ is the diameter, and $h$ is the thickness; $N_0 = \int_A \sigma_{xx} dA$ is the initial axial load at the ends, as shown in Fig. 1. The quartic terms of $w$ can be neglected for slender beams [34], and notices $\frac{\partial u}{\partial x} = O(\frac{\partial w}{\partial x})^2$ [35], the force reduces to

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$$+ ID \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + DA \left[ \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right]^2.$$

The motion equations with the extensible effect are [34]

$$\frac{\partial N}{\partial x} = m \frac{\partial^2 u}{\partial x^2}, \quad (6)$$

$$\frac{\partial^2 M}{\partial x^2} + N \frac{\partial^2 w}{\partial x^2} = m \frac{\partial^2 w}{\partial t^2} + F(x, t). \quad (7)$$

Here $m$ is the mass per unit length, $N$ and $M$ are the nonlocal axial force and the nonlocal moment, and $F(x, t)$ is the load. Taking into account Eqs. (4) and (5), the nonlocal constitutive Eq. (1) transforms into

$$M - \mu^2 \frac{\partial^2 M}{\partial x^2} = -EI \frac{\partial^2 w}{\partial x^2}
- 2ID \left[ \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] \frac{\partial^2 w}{\partial x^2}. \quad (8)$$

$$N - \mu^2 \frac{\partial^2 N}{\partial x^2} = -N_0 + ID \left( \frac{\partial^2 w}{\partial x^2} \right)^2$$
$$+ EA \left[ \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right]^2. \quad (9)$$

Substituting Eqs. (6) and (7) into Eqs. (8) and (9) gets

$$M - \mu^2 \left( m \frac{\partial^2 w}{\partial x^2} - N \frac{\partial^2 w}{\partial x^2} + \bar{F} \right) = -EI \frac{\partial^2 w}{\partial x^2}
- 2ID \left[ \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] \frac{\partial^2 w}{\partial x^2}. \quad (10)$$
Differentiating Eq. (10) twice with respect to $x$ and then substituting it into Eq. (7) gets

$$N - \mu^2 \frac{\partial}{\partial x} \left( m \frac{\partial^2 u}{\partial t^2} \right) = -N_0 + ID \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + EA \left[ \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right].$$

(11)

From Eq. (11), gets

$$N = \mu^2 \frac{\partial}{\partial x} \left( m \frac{\partial^2 u}{\partial t^2} \right) - N_0 + ID \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + EA \left[ \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right].$$

(13)

For slender beams, the longitudinal inertia term, $\frac{\partial^2 u}{\partial t^2}$, may be negligible. We can take inspiration from classical beam theory to show this [37]: because a slender beam with a small radius of gyration has $m \frac{\partial u}{\partial t^2} \ll EA \left[ \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right]$, we can ignore $\mu^2 \frac{\partial}{\partial x} \left( m \frac{\partial^2 u}{\partial t^2} \right)$ from Eq. (13) when $\mu^2 / \ell^2 < 1$. Substituting Eq. (13) into Eq. (12), and ignoring the inertia term, the lateral motion equation is as follows:

$$-EI \frac{\partial^4 w}{\partial x^4} + N \frac{\partial^2 w}{\partial x^2} - 2ID \frac{\partial^2 \omega}{\partial x^2} + \left\{ \left( \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right)^2 \right\} \frac{\partial^2 w}{\partial x^2}$$

$$+ \mu^2 \frac{\partial^2 \omega}{\partial x^2} \left( m \frac{\partial^2 w}{\partial x^2} - N \frac{\partial^2 w}{\partial x^2} + F \right) = m \frac{\partial^2 w}{\partial t^2} + F(x, t).$$

(12)

Equations (14) and (15) are the plane motion equations of nanobeams, in which we consider the nonlocal nonlinear constitutive and the extensible effect. The displacement boundary conditions of the two equations are the same as the classical beam theory. If we consider only bending motion, the inertia terms of Eq. (15) can be ignored [35, 36], and then Eq. (15) simplifies as

$$\frac{\partial^2 u}{\partial x^2} = -\frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 + \frac{1}{2} \left( \frac{\partial^2 w}{\partial x^2} \right)^2,$$

(16)

where $\lambda = DI / EA$. Integrating Eq. (16) with respect to $x$ gets

$$u = -\frac{1}{2} \int_0^x \left[ \left( \frac{\partial w}{\partial x} \right)^2 + 2\lambda \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \right] dx + C_1(t),$$

(17)

$$+ C_1(t)x + C_2(t).$$

Where $C_1$ and $C_2$ are functions of time $t$, which can be determined by imposing boundary conditions on $w$. For a beam with two unmovable ends [35, 36],

$$C_1(t) = \frac{1}{2} \int_0^l \left[ \left( \frac{\partial w}{\partial x} \right)^2 + 2\lambda \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \right] dx, \quad C_2 = 0.$$

(18)

Substituting Eq. (17) into Eq. (14) and omitting the quartic terms of $w$ gets

$$m \left( \frac{\partial^2 w}{\partial t^2} - \mu^2 \frac{\partial^4 w}{\partial x^4} \right) + C \frac{\partial w}{\partial t} + EI \frac{\partial^4 w}{\partial x^4} + N_0 \left( \frac{\partial^2 w}{\partial x^2} \right)^2 - 2ID \frac{\partial^2 \omega}{\partial x^2} \left( m \frac{\partial^2 w}{\partial x^2} - N \frac{\partial^2 w}{\partial x^2} + F \right)$$

$$= \frac{\partial^2 w}{\partial t^2} - \mu^2 m \frac{\partial^4 w}{\partial x^4}.$$

(19)

In Eq. (19), we add a linear damping term $C \frac{\partial w}{\partial t}$. For a hinged–hinged beam, the boundary conditions are [34–36]

$$C_1(t) = \frac{1}{2} \int_0^l \left[ \left( \frac{\partial w}{\partial x} \right)^2 + 2\lambda \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \right] dx, \quad C_2 = 0.$$
For simplicity, the $M$’s nonlinear term in Eq. (25) is ignored and the following is obtained:

\[
\frac{\partial^2 M}{\partial x^2} \approx \mu^2 \left[ \frac{\partial^2 w}{\partial x^2} + \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \frac{\partial^2 w}{\partial x^2} \right) \right] + N \frac{\partial^2 w}{\partial x^2} + \frac{m}{2} \frac{\partial}{\partial x} \left( \frac{\partial w}{\partial x} - \frac{\partial^2 w}{\partial x^2} \right) + m \frac{\partial w}{\partial x} \int_0^x \left( \frac{\partial^2 w}{\partial x^2} \right) ds + \mathcal{F}(x, t).
\]

Neglecting $m \partial^2 u/\partial t^2$, Eq. (24) can be simplified to

\[
\frac{\partial^2 N}{\partial x^2} = 0
\]

Substituting Eq. (21) into Eq. (1), the nonlocal constitutive relationship is transformed into

\[
N = -N_0 + m \frac{\partial^2 u}{\partial t^2}.
\]

Neglecting the nonlinear terms of $u$, the longitudinal motion equation can be obtained as [37]

\[
\frac{\partial N}{\partial x} = m \frac{\partial^2 u}{\partial t^2}.
\]

Since the strain of inextensible beams is different from the extensible beam, their transverse motion equations are different. According to the virtual work principle [34], we can get the motion equation as (the derivation details are in Appendix)

\[
\frac{\partial^2 M}{\partial x^2} + \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \frac{\partial^2 M}{\partial x^2} \right) + m \frac{\partial^2 w}{\partial x^2} = m \frac{\partial^2 w}{\partial x^2}
\]

\[
\frac{m}{2} \frac{\partial}{\partial x} \left( \frac{\partial w}{\partial x} - \frac{\partial^2 w}{\partial x^2} \right) + m \frac{\partial w}{\partial x} \int_0^x \left( \frac{\partial^2 w}{\partial x^2} \right) ds + \mathcal{F}(x, t).
\]

Differentiating Eq. (30) once and twice with respect to $x$ and then substituting the outcomes and Eq. (31) into Eq. (25) gets
In Eq. (32), the nonlinear terms of \( w \) only keep up to cubic terms. Because the nonlinear inertia terms can be omitted for slender beams [36], Eq. (32) can be simplified as

\[
\begin{align*}
\frac{\partial^2 w}{\partial t^2} - N_0 \left( \frac{\partial^2 w}{\partial x^2} - \mu \frac{\partial^4 w}{\partial x^4} \right) + \frac{\mu^2 N_0}{2} \frac{\partial}{\partial x} \left[ \left( \frac{\partial w}{\partial x} \right)^2 \right] \\
- \frac{E I}{\partial x^4} - N_0 \left( \frac{\partial^2 w}{\partial x^2} - \mu \frac{\partial^4 w}{\partial x^4} \right) + \frac{\mu^2 N_0}{2} \frac{\partial}{\partial x} \left[ \left( \frac{\partial w}{\partial x} \right)^2 \right]
\end{align*}
\]

\[
+ ID \left[ \left( \frac{\partial^2 w}{\partial x^2} \right)^3 - \mu^2 \frac{\partial^4 w}{\partial x^4} \frac{\partial^2 w}{\partial x^2} \right] = m \frac{\partial^2 w}{\partial t^2}
\]

\[
- \frac{\mu^2 m}{E A} \frac{\partial^2 w}{\partial x^2} \frac{\partial}{\partial x} \left[ \left( \frac{\partial w}{\partial x} \right)^2 \right] + \frac{m}{2} \frac{\partial}{\partial x} \mu^2 \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial t^2} \bigg|_0 \int_0^t \left[ \frac{\partial w}{\partial s} \right]^2 ds \bigg] ds
\]

\[
+ \mathbf{F} - \mu^2 \left\{ \frac{\partial^2 \mathbf{F}}{\partial x^2} + \frac{1}{2} \frac{\partial}{\partial x} \left[ \frac{\partial \mathbf{F}}{\partial x} \right]^2 \right\}.
\]

\[(32)\]

The boundary conditions of Eq. (33) are the same as those of classical beams. For example, the conditions of simply supported beams are

\[
w(0, t) = w(l, t) = \frac{\partial^2 w}{\partial x^2}(0, t) = \frac{\partial^2 w}{\partial x^2}(l, t) = 0. \quad (34)
\]

Below, we will analyze Eqs. (19) and (33). The above two models can divide into four categories: the classical nonlinear model (CNM) for \( D = 0 \) and \( \mu = 0 \), the nonlinear constitutive model (NCM) for \( D \neq 0 \) and \( \mu = 0 \), the nonlocal nonlinear model (NNM) for \( D = 0 \) and \( \mu \neq 0 \), and the nonlocal nonlinear constitutive model (NNCM) for \( D \neq 0 \) and \( \mu \neq 0 \).

3 Solutions of models

It is convenient to introduce dimensionless variables in Eqs. (19) and (33). Let \( \bar{x} = x/l, \bar{w} = w/l, \bar{t} = t/\omega_0 \), and \( \omega_0^2 = \pi^4 E I (l^4 m)^{-1} \). It is assumed that the loads are uniform, namely \( \mathbf{F}(x) = \text{const} \), so \( \partial^2 \mathbf{F} / \partial x^2 = 0 \). Equation (19) for the extensible beams can be rewritten as

\[
\frac{\partial^2 \bar{w}}{\partial \bar{t}^2} \left( \bar{w} - \mu^2 \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} \right) + C \frac{\partial \bar{w}}{\partial \bar{t}} + 1 \frac{\partial^3 \bar{w}}{\partial \bar{x}^2 \partial \bar{t}}
\]

\[
+ \frac{N_0}{m_0 \mu l^2} \left( \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} - \mu^2 \frac{\partial^4 \bar{w}}{\partial \bar{x}^4} \right) - \frac{2 \lambda ID}{m_0 l^3} \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} \left( \frac{\partial^3 \bar{w}}{\partial \bar{x}^3} \right) - \frac{E A}{2m_0 l^3} \left( \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} - \mu^2 \frac{\partial^4 \bar{w}}{\partial \bar{x}^4} \right)
\]

\[
\int_0^1 \left[ \frac{\partial \bar{w}}{\partial \bar{x}} \right]^2 + 2 \lambda \left( \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} \right) d\bar{x} = \mathbf{F} \frac{m_0 l^3}{l}.
\]

\[(35)\]

Equation (33) for inextensible beams can be rewritten as

\[
\frac{\partial^2 \bar{w}}{\partial \bar{t}^2} - \mu^2 \frac{\partial^4 \bar{w}}{\partial \bar{x}^4} + C \frac{\partial \bar{w}}{\partial \bar{t}} + 1 \frac{\partial^3 \bar{w}}{\partial \bar{x}^2 \partial \bar{t}}
\]

\[
+ \frac{N_0}{m_0 l^2} \left( \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} - \mu^2 \frac{\partial^4 \bar{w}}{\partial \bar{x}^4} \right) - \frac{\mu^2 N_0}{2m_0 l^3} \frac{\partial}{\partial \bar{x}} \left( \frac{\partial \bar{w}}{\partial \bar{x}} \right)^2 \left( \frac{\partial^3 \bar{w}}{\partial \bar{x}^3} \right) + \frac{E I}{2m_0 l^3} \frac{\partial}{\partial \bar{x}} \left( \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} \right) \frac{\partial^2 \bar{w}}{\partial \bar{x}^2}
\]

\[
- \frac{DI}{m_0 l^3} \left[ \left( \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} \right)^3 - \mu^2 \frac{\partial^4 \bar{w}}{\partial \bar{x}^4} \left( \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} \right)^2 \right] = \mathbf{F} \frac{m_0 l^3}{l}.
\]

\[(36)\]

For simplicity, we use hinged–hinged and simply supported beams as examples to study the extensible and inextensible beams correspondingly. Hence, their normalized boundary conditions are identical:

\[
\bar{w}(0, \bar{t}) = \bar{w}(1, \bar{t}) = \bar{w}(0, \bar{t}) = \bar{w}(1, \bar{t}) = 0.
\]

\[(37)\]

It is challenging to solve nonlinear Eq. (35) or (36) accurately. There are two common approaches to solving a nonlinear partial differential equation approximately. The first method reduces the partial differential equation to nonlinear ordinary differential equations through the Galerkin method, and then the ordinary equations are solved through perturbation methods [34–37]. The second is to directly solve the nonlinear partial differential equation using...
perturbation methods. The direct multiscale method is an essential improvement of the classical perturbation methods. It has been widely used to solve nonlinear partial differential equations more accurately than the first methods [37]. Nayfeh [38–40], Luongo [41, 42], Lacarbonara [43–46] and their collaborators’ works may be substantial for the direct multiscale method. We will apply the Galerkin and the multiscale methods to solve the equations for simplicity. The two methods were widely applied to nonlinear equations of structures [18, 20–22, 47–50]. Under boundary conditions Eq. (37), the approximate solutions of Eqs. (35) and (36) can be written as

\[ \mathbf{w} = \sum_{n=1}^{\infty} \eta_n(t) \sin n\pi x. \]  

A set of ordinary differential equations can be obtained through the Galerkin truncation [34, 36]: Eq. (38) is substituted into Eqs. (35) or (36) and then \( \sin(n\pi x) \) is multiplied by both sides of the equations and integrated in the interval \([0, 1]\). We only take the first term of Eq. (38) and let \( \eta_1 = \eta \), so have

\[ m_j \ddot{\eta} + c_j \dot{\eta} + k_j \eta + d_j \eta^3 = f, \quad j = 1, 2. \]  

Here \( j = 1 \) indicates the hinged–hinged beams, and \( j = 2 \) indicates the simply supported beams. The parameters in Eq. (39) are

\[ m_1 = m_2 = 1 + \frac{\pi^2 \mu^2}{L^2}, \quad c_1 = c_2 = \frac{C}{m_0}, \]
\[ k_1 = k_2 = 1 - \frac{N_0 \mu^2}{m_0 L^2} \left( 1 + \frac{\pi^2 \mu^2}{L^2} \right), \]
\[ d_1 = \frac{\pi^4 EA}{4m_0 L^2} \left[ 1 + \frac{2\pi^2 \mu}{L^2} \right] - \frac{3\pi^4 JD}{4m_0 L^2} + \frac{\pi^4 EA^2}{4m_0 L^2} \left[ 1 + \frac{2\pi^2 \mu}{L^2} \right], \]
\[ d_2 = -\frac{3\pi^4 \mu N_0}{8m_0 L^2} + \frac{\pi^6 E}{6m_0 L^2} + \frac{3\pi^4 D}{4m_0 L^2} \left[ 1 + \frac{\pi^2 \mu^2}{L^2} \right], \]
\[ F = \frac{4F}{\pi m_0 L^2}. \]  

Equation (40) indicates that the linear coefficients in the two models are the same, but the nonlinear coefficients are different. Moreover, the nonlocal effect and the material nonlinearity are coupled in nonlinear terms. \( d_1 \) demonstrates that the influence of the material nonlinearity will decrease accompanying the increase of the beam length, as hinged–hinged SWCNTs shown in Fig. 2. If the nonlocal effect is neglected, the material nonlinearity’s influence on \( d_2 \) is independent of the beam length due to \( \omega_0^2 \delta^4 = \pi^4 EI/m \), as simply supported SWCNTs shown in Sect. 4.2.

The nonlinear terms, which are induced by the finite deformations and the nonlocal effect, produce a hardening effect in the two models due to \( \mu^2 > 0 \). Conversely, the nonlinear terms induced by the material nonlinearity have a softening effect in two models due to \( D < 0 \) for SWCNTs. Interestingly, the hardening effect is smaller for hinged–hinged SWCNTs than the softening effect in small-length tubes, as shown in Fig. 2. The following discussions will show that this competitive relationship between the softening and the hardening impacts the mechanical behavior of SWCNTs significantly.

Neglecting the inertia and damping terms in Eq. (39), static bending deformations of the middle points of beams are obtained as

\[ k_j \eta + d_j \eta^3 = F, \quad j = 1, 2. \]  

We will research the mechanical behaviors of hinged–hinged beams and simply supported beams according to Eqs. (39) and (41), respectively.

We rewrite Eq. (39) as

\[ \ddot{\eta} + c_j \dot{\eta} + \alpha \eta + \beta \eta^3 = \ddot{f}, \quad j = 1, 2, \]  

here \( \alpha = c_j/m_j, \quad \phi = k_1/m_j, \quad \beta = d_j/m_j, \quad \ddot{f} = \ddot{f}_j = F/m_j \). The multiple scale method [36] will be applied to solve Eq. (42). Let \( \tau_1 = \varepsilon c_j, \quad \ddot{f} = \varepsilon^3 f \cos(\Omega t) \), it gives

\[ \ddot{\eta} + \eta + \phi \eta + \beta \eta^3 = \varepsilon^3 f \cos(\Omega t), \quad j = 1, 2. \]  

Suppose

\[ \eta(t; \varepsilon) = \omega_1(T_0, T_2) + \varepsilon^3 \eta_3(T_0, T_2), \]  

here \( T_j = \varepsilon^j t, j = 0, 2 \). We substitute Eq. (44) into Eq. (43) and then equate the coefficients of \( \varepsilon \) and \( \varepsilon^3 \) on both sides,

\[ \varepsilon : D_0 \eta_1 + \omega_0^2 \eta_1 = 0, \]  
\[ \varepsilon^3 : D_3 \eta_3 + \omega_3^2 \eta_3 = -2D_0D_2 \eta_1 \]
\[ -2c_jD_0 \eta_1 - \beta_2 \eta_3^3 + \frac{1}{2} f \exp(i\Omega t), \]  

where \( D_0 = d/dT_0, \quad D_j = d/dT_j, \quad j = 1, 2 \) and \( D_0^3 = d^3/dT_0^3 \). The solution of Eq. (45) is
\[ \eta_1 = A(T_2) \exp(i\omega_j T_0) + CC, \quad (47) \]

here \( CC \) means the complex conjugate. Substituting Eq. (47) into Eq. (46) gets

\[
D_0^2 \eta_3 + \omega_j^2 \eta_3 = \frac{1}{2} f \exp(i\Omega T_0) - [i2\omega(A' + c_j A) + 3d_j \mathbf{A}^2 \exp(i\omega_j T_0) + \text{NST}].
\quad (48)
\]

Here the prime denotes the derivative with respect to \( T_2 \) and NST denotes nonsecular terms [36]. When the load’s frequency \( \Omega \) approaches the nanobeam’s modal frequency \( \omega_j \) (the primary resonance), the beam will appear a relatively large amplitude response. Under this condition, let \( \Omega = \omega_j + \varepsilon^2 \sigma \), so the solvable conditions of Eq. (48) are

\[
-i2\omega(A' + c_j A) - 3d_j \mathbf{A}^2 \mathbf{A} + \frac{1}{2} f \exp(i\sigma T_2) = 0,
\quad j = 1, 2.
\quad (49)
\]

Letting \( A = (\sigma/2) \exp(i\beta) \) and substituting it into Eq. (49) and then separating the real and the imaginary part gets

\[
\begin{align*}
dx' &= -c_j x + \frac{f}{2\omega_j} \sin \gamma, \\
dx'' &= \sigma x - \frac{3d_j}{8\omega_j} x^3 + \frac{f}{2\omega_j} \cos \gamma,
\end{align*}
\quad (50)
\]

here \( \gamma = \sigma T_2 - \beta \). Steady-state motions occur when \( x' = x'' = 0 \), which corresponds to the singular points of Eq. (50). The steady-state solutions can be obtained from the following algebraic equations [36]

\[
\begin{align*}
c^2 + \left( \sigma - \frac{3d_j}{8\omega_j} x^2 \right) &= \frac{f^2}{4\omega_j^2}, \quad j = 1, 2. \\
\end{align*}
\quad (51)
\]

The stability of the steady-state solutions is judged by investigating the nature of the singular points of Eq. (50). Let \( a = a_0 + a_1 \) and \( \gamma = \gamma_0 + \gamma_1 \), and substitute them into Eq. (50), expand for small \( a_1 \) and \( \gamma_1 \), and keep linear terms in \( a_1 \) and \( \gamma_1 \); then the following is obtained:

\[
\begin{align*}
x''_1 &= -c_j x_1 + \frac{\gamma_1 f \cos \gamma_0}{2\omega_j}, \\
x''_1 &= -\left( \frac{f \cos \gamma_0}{2\omega_j \gamma_0^2} + \frac{3d_j \gamma_0}{4} \right) x_1 - \frac{\gamma_1 f \sin \gamma_0}{2\omega_j \gamma_0}.
\end{align*}
\quad (52)
\]

Here it is used in Eq. (52) that \( a_0 \) and \( \gamma_0 \) are the singular points of Eq. (50). The stability of steady-state motions depends on the coefficient matrix’s eigenvalues of Eq. (52). If the real parts of eigenvalues are greater than zero, the solutions are unstable [36]. Hence, the steady-state motions are unstable if

\[
\begin{align*}
c^2 + \left( \sigma - \frac{3d_j}{8\omega_j} x^2 \right) &= \frac{f^2}{4\omega_j^2}, \quad j = 1, 2. \\
\end{align*}
\quad (53)
\]

Equation (53) indicates that the nonlinear term affects the stability of the steady-state solutions. Therefore, both the nonlocal effect and the material nonlinearity affect the stability of the solutions. We use solid lines to represent the stable solutions and dashed lines to represent the unstable solutions in the response curves in the next section.
4 Results and discussion

It is challenging to obtain the mechanical properties of nanobeams through experiments, and the experimental results have significant differences \cite{17, 51}. Molecular dynamics (MD) calculations of SWCNTs show a significant gap among the results \cite{52, 53}. Furthermore, MD calculations also confirm that the calculated results are consistent with the models that include the nonlocal effect \cite{9, 31} or the material nonlinearity \cite{16}. The present theories comprehensively consider nonlocal effects and physical nonlinearity, which provides a basis to fit the mechanical parameters of nanotubes through experiments or MD calculations.

This section uses (15, 15) SWCNTs to demonstrate the differences between the four theories. The physical and geometrical parameters come from Refs. \cite{15} and \cite{22}, and they are shown in Table 1. Using the parameters in Table 1, we get \( \lambda = -1.027 \times 10^{-18} \) nm\(^2\). The damping parameters \( \bar{\nu}_1 = \bar{\nu}_2 = 0.01 \) and \( N_0 = 2 \times 10^{-8} \) N are used in the present study for simplicity.

It is mentioned in the introduction that determining the nonlocal parameter \( \kappa_0 \) of SWCNTs is an open problem. This problem may come from the ambiguous understanding of the mechanical properties of one-atom-thick nanostructures \cite{32, 54}. The nonlocal parameter obtained by atomic calculations has also been confusing \cite{17, 31, 55}. However, if the vibration frequencies of CNTs are in the terahertz range, a conservative estimate is \( \kappa_0 < 2 \) nm \cite{5, 55}. In the present research, we take the SWCNT’s scale coefficient \( \mu = \kappa_0 \alpha = 1 \) nm. This value is bigger than most values in the existing studies on SWCNTs \cite{5}. This big coefficient may help compare the material nonlinearity and the nonlocal effect.

4.1 Hinged–hinged SWCNTs

It can be found from Eq. (40) that the nonlinear parameters are related to the beam length. We first focus on the length’s effect on \( d_1 \), as shown in Fig. 2. The figure and Eq. (40) show that \( d_1 \) in the CNM and the NNM is positive for (15, 15) SWCNTs. This indicates that the geometrical nonlinear terms, which are induced by finite deformations and the nonlocal effect, produce a hardening effect for the SWCNTs. However, when the material nonlinearity appears in the models (NCM and NNCM), \( d_1 \) will change from positive to negative with the length’s decrease, as shown in Fig. 2. This means that the nonlinear terms in Eq. (39) change from a soft spring to a hard spring. Nonlinear springs significantly influence the mechanical properties of macrostructures \cite{35–50}.

Here, we take \( l = 8 \) nm and \( l = 6 \) nm as examples to study the differences between four theories. Since the nonlinear coefficients in the four models are all greater than zero for \( l = 8 \) nm, the SWCNT is a hard spring system (\( d_1 > 0 \)). However, for \( l = 6 \) nm, the models with material nonlinearity (NCM and NNCM) are soft spring systems (\( d_1 < 0 \)), while other models (CNM and NNM) are still hard spring systems (\( d_1 > 0 \)). The static load–deformation curves can be obtained from Eq. (41) with \( j = 1 \), as shown in Figs. 3 and 4. The two figures show that the softening effect remarkably increases the deformation amplitudes of the static bending.

Further, the material nonlinearity may significantly impact the SWCNT’s vibrations under the primary resonance, as shown in Figs. 5, 6 and 7 obtained from Eq. (51) with \( j = 1 \). For example, four load–response curves at \( \sigma = 10 \) are significantly different between \( l = 8 \) and \( l = 6 \), as shown in Figs. 5 and 6. These two figures show that the material nonlinearity in short tubes is more prominent than in long tubes. Therefore, ignoring the material nonlinearity may lead to evident errors for short SWCNTs, as shown in Figs. 7 and 8. The material nonlinearity produces the softening effect in the NCM and NNMC for \( l = 6 \), and the softening effect makes their frequency–response curves deviate to the left. On the contrary, the nonlinear terms in the CNM and the NNM are hard springs. This skews the frequency–response curves to

### Table 1 Physical and geometrical parameters of (15, 15) SWCNTs

| \( d \) (nm) | \( h \) (nm) | \( I \) (nm\(^4\)) | \( A \) (nm\(^2\)) | \( E \) (TPa) | \( D \) (TPa) | \( m \) (kg/m) |
|---|---|---|---|---|---|---|
| 2.034 | 0.34 | 1.115 | 2.171 | 1 | – 2 | \( 4.86 \times 10^{-15} \) |
the right, as shown in Fig. 7. We calculated Eq. (42) with \( j = 1 \) numerically through the Runge–Kutta method for \((l, r, f) = (6, 5, 5)\). The results show that ignoring the material nonlinearity may seriously underestimate the vibration amplitudes, as shown in Fig. 8. The numerical calculations also demonstrate the accuracy of perturbation solutions.

4.2 Simply supported SWCNTs

Equation (40) shows that the linear parameters of a hinged–hinged beam are the same as these of a simply supported beam. So we will demonstrate the influence of tube length on the nonlinear coefficient \( d_2 \) for simply supported SWCNTs, as shown in Fig. 9. It is found in Fig. 9 and Eq. (40) that the material...
nonlinearity makes $d_2 < 0$ for the NCM and NNCM. This indicates that both the nonlocal effect and the material nonlinearity have a stiffness softening effect on simply supported SWCNTs. The softening makes static bending deformations of the NNM and the NNCM significantly larger than those of the NCM and the NNM when the SWCNTs are subjected to big loads, as shown in Fig. 10 obtained by Eq. (41) with $j = 2$. Furthermore, the material nonlinearity more significantly affects the dynamical behaviors of SWCNTs under the primary resonance. For example, the response amplitudes of the NCM and the NNCM have jumps accompanying the load’s changes, while the CNM and NNM do not produce amplitude jumps for $(l, \sigma) = (6, -5)$, as shown in Fig. 11 obtained by Eq. (51) with $j = 2$. Similar to the hinged–hinged
Fig. 9  Nonlinear coefficients $d_2$ as functions of simply supported beam lengths $l$

![Graph of $d_2$ vs. $l$]

Fig. 10  Static deformations as functions of loads for simply supported beams at $l = 8$

![Graph of static deformations vs. load]

Fig. 11  Amplitudes of response as functions of the amplitudes of loads for simply supported beams at $(l, \sigma) = (6, -5)$

![Graph of response amplitudes vs. load amplitudes]
beams, the frequency–response curves of the NCM and the NNCM have maximum vibration amplitudes in $\Omega < \omega_2$. In contrast, the maximum vibration amplitudes of the CNM and the NNM appear in $\Omega > \omega_2$, as shown in Fig. 12. The above results indicate that one may obtain incorrect results if the material nonlinearity or nonlocal effect is neglected. We implement numerical calculations of Eq. (42) with $j = 2$ through the Runge–Kutta method to check the perturbation solutions Eq. (51). The numerical simulations confirm the accuracy of the analytical solutions, as shown in Fig. 13.

In the present paper, we consider the nonlocal effect and the material nonlinearity in the bending deformations of SWCNTs. Recent studies have shown that geometric nonlinearity may cause coupled vibrations between bending and radial deformations through a resonance [56–58]. Therefore, we speculate that the material nonlinearity may also induce some coupled vibrations if the nonlocal shell theories apply to SWCNTs. In the present study, we consider only the mechanical properties of CNTs based on the modified continuum mechanics theory. However, it is also necessary to develop atomic calculation methods that can better model the mechanical properties of CNTs at the atomic scale. For example, the atomistic–continuum approach, in which the higher-order Cauchy–Born rule has been used, can simulate the bending deflections of graphene with geometrical and material nonlinearities [59]. Furthermore, bridging between the atomic scale and the effective continuum has been carried out by parameterizing the continuum elastic energy and determining the parameters using unit cell atomistic simulations over a range of deformation magnitudes [60]. We also note that a larger diameter SWCNT’s stress–strain relationship is similar to that of graphene in this paper. Physically, the covalent bonds are almost the same between a large-diameter CNTs and graphene, so their mechanical properties may rarely change [32, 33]. However, the difference in

**Fig. 12** Frequency–response curves for simply supported beams at $(l, f) = (6, 2)$

![Frequency–response curves for simply supported beams](image1)

**Fig. 13** Time series of hinged–hinged beams for $(l, \sigma, f) = (6, -4, 2)$ at an initial value $(\eta, \dot{\eta}) = (0.1, 0)$

![Time series of hinged–hinged beams](image2)
covalent bonds between CNTs and graphene will become significant when the tube diameters decrease. This makes the mechanical properties of CNTs dependent on the tube diameters [61, 62]. Determining the relationship between mechanical properties and CNT’s diameter requires further research. Although the mechanical properties of CNTs have been studied since 30 years ago, they still need to be further explored theoretically and experimentally.

5 Conclusions

In the present study, we combine the nonlocal effect and the material nonlinearity to suggest two new Euler–Bernoulli theories for nanobeams. The integral-partial differential equation models the extensible nanobeams, and the partial differential equation models the inextensible nanobeams. The results indicate that the material nonlinearity softens the SWCNT’s stiffness. When the material nonlinearity and nonlocal effect are considered simultaneously, they will more significantly impact the mechanical properties of the nanobeams. Noticeable mistakes may appear if one neglects the material nonlinearity or the nonlocal effect. The main results are as follows:

(1) The nonlinear terms combined material nonlinearity and nonlocal effect appear in the beam models, and their influence increases with decreasing beam length.

(2) The material nonlinearity has a softening stiffness effect on the mechanical deformation of SWCNTs due to the third-order elastic coefficient $D<0$. This stiffness softening effect in inextensible beams is more significant than that of extensible beams, and this softening effect increases with decreasing beam length.

(3) Neglecting material nonlinearities may turn the vibration equations with the hard-spring into ones with the soft-spring. This may induce significant differences or qualitative mistakes in vibrational behaviors.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Data availability The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

Appendix: Derivation of Eq. (25)

The principle of virtual work for the present dynamical problem is written as [34]

$$
\int_{t_1}^{t_2} \left\{ \iint_{V} \sigma_{xx} \delta e_{xx} dxdydz - \delta \iint_{V} \frac{\rho}{2} \left( \frac{\partial u}{\partial t} \right)^2 + \left( \frac{\partial w}{\partial t} \right)^2 \right\} dxdydz - \int_0^l F \delta w dx - N \delta u \right\} dt = 0.
$$

(A1)

here $\rho$ is the density of the beam. For an inextensional beam, the longitudinal deformation $u$ is mainly induced by the transverse deformation $w$ and can be written as [37]: $\partial u/\partial x \approx -(\partial w/\partial x)^2/2$. Integrating this equation with respect to $x$ and using the boundary condition $u = 0$ at $x = 0$, we have

$$
u = -\frac{x}{\int_0^x \left( \frac{\partial w}{\partial s} \right)^2/2 \right] ds. \text{Therefore, the virtual work of the axial load and the longitudinal velocity is}
$$

$$
N \delta u = -N \delta \int_0^x \frac{1}{2} \left( \frac{\partial w}{\partial s} \right)^2 \right) ds \right)\right\}.
$$

(A2)

$$
\frac{\partial u}{\partial t} \approx -\frac{1}{2} \frac{\partial}{\partial t} \int_0^s \left( \frac{\partial w}{\partial s} \right)^2 ds.
$$

(A3)

Substituting Eqs. (A2), (A3) and Eq. (21) into Eq. (A1), and considering $M = \int_A \sigma_{xx} dA$, $N = \int_A \sigma_{xx} dA$, we have
By performing complex but straightforward calculations, including integrations by parts on Eq. (A4), we have
\[ -\int_{t_i}^{t_f} \left\{ \int_{0}^{l} \left[ \frac{\partial^2 w}{\partial x^2} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] dx \right\} dt + \delta \int_{0}^{l} \left\{ \frac{1}{4} \left( \frac{\partial w}{\partial t} \right)^2 \right\} ds + \left( \frac{\partial w}{\partial t} \right)^2 \right\} \int_{0}^{l} \int_{0}^{\delta} \delta \omega dx + N \delta \left[ \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] dt = 0. \]

(A4)

Equation (25) can be obtained by moving the inertia and load terms to the right side of Eq. (A6).

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