THE DIMENSION OF THE SPACE OF \( \mathbb{R} \)-PLACES OF CERTAIN RATIONAL FUNCTION FIELDS

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Abstract. We prove that the space \( M(K(x,y)) \) of \( \mathbb{R} \)-places of the field \( K(x,y) \) of rational functions of two variables with coefficients in a totally Archimedean field \( K \) has covering and integral dimensions \( \dim M(K(x,y)) = \dim_2 M(K(x,y)) = 2 \) and the cohomological dimension \( \dim_G M(K(x,y)) = 1 \) for any Abelian 2-divisible coefficient group \( G \).

1. Introduction

In this paper we study the topological structure and evaluate the topological dimensions of the spaces of \( \mathbb{R} \)-places of a field \( K \) and of its transcendental extensions \( K(x_1,\ldots,x_n) \) consisting of rational functions of \( n \) variables with coefficients in \( K \).

The shortest possible way to introduce \( \mathbb{R} \)-places on a field \( K \) is to define them as functions \( \chi : K \to \bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \) to the extended real line, preserving the arithmetic operations in the sense that \( \chi(0) = 0 \), \( \chi(1) = 1 \), \( \chi(x + y) = \chi(x) + \chi(y) \) and \( \chi(x \cdot y) = \chi(x) \cdot \chi(y) \) for all \( x,y \in K \), where \( + \) and \( \cdot \) are multivalued extensions of the addition and multiplication operations from \( \mathbb{R} \) to \( \bar{\mathbb{R}} \). By definition, for \( r,s \in \mathbb{R} \), \( r \oplus s = \{r+s\} \) if \( r+s \in \mathbb{R} \) is defined and \( r \odot s = \mathbb{R} \) if \( r+s \) is not defined, which happens if and only if \( r = s = \infty \), in which case \( \infty \oplus \infty = \mathbb{R} \). By analogy, we define \( r \odot s \) it equals the singleton \( \{r \cdot s\} \) if \( r \cdot s \) is defined and \( \mathbb{R} \) in the other case, which happens if and only if \( \{r,s\} = \{0,\infty\} \).

Historically, \( \mathbb{R} \)-places appeared from studying ordered fields. By an ordered field we understand a pair \( (K,P) \) consisting of a field \( K \) and a subset \( P \subseteq K \) called the positive cone of \( (K,P) \) such that \( P \) is an additively closed subgroup of index 2 of the multiplicative group of \( K \). There is a bijective correspondence between positive cones of \( K \) and linear orders compatible with addition and multiplication by positive elements. The set \( \{a \in K : a > 0\} \) is a positive cone, and the positive cone \( P \) generates a total order \( \leq \) on \( K \) defined by \( x \leq y \iff y - x \in P \). Each ordered field \( (K,P) \) has characteristic zero and hence contains the field \( \mathbb{Q} \) of rational numbers as a subfield. This fact allows us to define the Archimedean part

\[ A_P(K) = \{x \in K : \exists a,b \in \mathbb{Q} : a < x < b\} \]

of the ordered field \( (K,P) \) and also to define the canonical \( \mathbb{R} \)-place \( \chi_P : K \to \bar{\mathbb{R}} \) on \( K \) assigning \( \chi_P(x) = \infty \) to each \( x \in K \setminus A_P(K) \) and

\[ \chi_P(x) = \sup\{a \in \mathbb{Q} : a \leq x\} = \inf\{b \in \mathbb{Q} : b \geq x\} \in \mathbb{R} \]

to each \( x \in A_P(K) \). Here the supremum and infimum is taken in the ordered field \( \mathbb{R} \) of real numbers.

According to Theorems 1 and 6 of [12], a field \( K \) admits a \( \mathbb{R} \)-place if and only if it is orderable in the sense that it admits a total order. By [3], each \( \mathbb{R} \)-place \( \chi : K \to \bar{\mathbb{R}} \) on a field \( K \) is generated by a suitable total order \( \leq \) on \( K \).

For an orderable field \( K \) denote by \( \mathcal{X}(K) \) the space of total orders on \( K \) and by \( M(K) \) the space of \( \mathbb{R} \)-places on \( K \). The mentioned results [12] and [3] imply that the map

\[ \lambda : \mathcal{X}(K) \to M(K), \quad \lambda : \Delta \mapsto \chi_{\Delta}, \]

assigning to each total order \( \Delta \) on \( K \) the corresponding \( \mathbb{R} \)-place \( \chi_{\Delta} \) is surjective.

The spaces \( \mathcal{X}(K) \) and \( M(K) \) carry natural compact Hausdorff topologies. Namely, \( \mathcal{X}(K) \) carries the Harrison topology generated by the subbase consisting of the sets \( a^+ = \{P \in \mathcal{X}(K) : a \in P\} \) where \( a \in K \setminus \{0\} \). According to [7, 6.1], the space \( \mathcal{X}(K) \) endowed with the Harrison topology is

\begin{itemize}
  \item [2010 Mathematics Subject Classification.] 12F20; 12J15; 54F45; 55M10.
  \item Key words and phrases. space of \( \mathbb{R} \)-places, graphoid, dimension, cohomological dimension, extension dimension.
\end{itemize}
compact Hausdorff and zero-dimensional. By \[4\], each compact Hausdorff zero-dimensional space is homeomorphic to the space of orderings \(\mathcal{X}(K)\) of some field \(K\).

To introduce a natural topology on the space \(M(K)\) of \(\mathbb{R}\)-places of a field \(K\), first endow the extended real line \(\mathbb{R} = \mathbb{R} \cup \{\infty\}\) with the topology of one-point compactification of the real line \(\mathbb{R}\). It follows from the definition of \(\mathbb{R}\)-places that the space \(M(K)\) is a closed subspace of the compact Hausdorff space \(\mathbb{R}^K\) of all functions from \(K\) to \(\mathbb{R}\), endowed with the topology of Tychonoff product of the circles \(\mathbb{R}\). So, \(M(K)\) is a compact Hausdorff space, being a closed subspace of the compact Hausdorff space \(\mathbb{R}^K\).

It turns out that the topology induced on \(M(K)\) by the product topology coincides with the quotient topology induced by the mapping \(\lambda : \mathcal{X}(K) \to M(K)\). This can be seen as follows. By \[10\], the sets
\[
U(a) = \{\chi \in M(K) : \chi(a) \in (0, \infty)\}, \quad a \in K,
\]
compose a sub-basis of the quotient topology on \(M(K)\). Since those sets are open in the product topology of \(M(K)\), the quotient topology is weaker than the product topology. Since the quotient topology is Hausdorff (see \[11, Cor.9.9\]) and the product topology is compact (so the weakest among Hausdorff topologies), both topologies on \(M(K)\) coincide.

The space \(M(K) \subset \mathbb{R}^K\) is metrizable if the field \(K\) is countable. The converse statement is not true as the uncountable field \(\mathbb{R}\) has trivial space of \(\mathbb{R}\)-places \(M(\mathbb{R}) = \{\text{id}\}\). The space of \(\mathbb{R}\)-places \(M(\mathbb{R}(x))\) of the field \(\mathbb{R}(x)\) is homeomorphic to the projective line \(\mathbb{R}\) while \(M(\mathbb{R}(x, y))\) is not metrizable, see \[14\].

In this paper we shall address the following general problem posed in \[2\].

\textbf{Problem 1.1.} \textit{Investigate the interplay between algebraic properties of a field \(K\) and topological properties of its space of \(\mathbb{R}\)-places \(M(K)\).}

We shall be mainly interested in the fields \(K(x_1, \ldots, x_n)\) of rational functions of \(n\) variables with coefficients in a subfield \(K \subset \mathbb{R}\). It is known that a field \(K\) is isomorphic to a subfield of \(\mathbb{R}\) if and only if \(K\) admits an Archimedean order, i.e., a total ordering \(P\) whose Archimedean part \(A_P(K)\) coincides with \(K\). This happens if and only if the corresponding \(\mathbb{R}\)-place \(\chi_P : K \to \mathbb{R}\) is injective if and only if \(\chi_P(K) \subset \mathbb{R}\). By \(M_A(K)\) we denote the space of injective \(\mathbb{R}\)-places on \(K\). Observe that \(M_A(K)\) coincides with the space of homomorphisms from \(K\) to the real line \(\mathbb{R}\).

A field \(K\) will be called \textit{totally Archimedean} if it is orderable and each total order on \(K\) is Archimedean. Such fields were introduced and characterized in \[16\]. Important examples of totally Archimedean fields are the fields \(\mathbb{Q}\) and \(\mathbb{R}\). For a totally Archimedean field \(K\) the quotient map \(\lambda : \mathcal{X}(K) \to M(K)\) is injective. In this case, the spaces \(M(K)\) and \(\mathcal{X}(K)\) are homeomorphic and hence the space \(M(K)\) is zero-dimensional.

In this paper we shall attack the following:

\textbf{Conjecture 1.2.} \textit{For any subfield \(K \subset \mathbb{R}\) and every natural number \(n\) the space \(M(K(x_1, \ldots, x_n))\) of \(\mathbb{R}\)-places of the field \(K(x_1, \ldots, x_n)\) has topological dimension \(\dim(K(x_1, \ldots, x_n)) \geq n\). If the field \(K\) is totally Archimedean, then \(\dim(K(x_1, \ldots, x_n)) = n\).}

For \(n = 1\) this conjecture was confirmed (in a stronger form) in \[10\]: \(\dim(M(K(x))) = 1\) for any (also non-Archimedean) real closed field \(K\). The main result of this paper is the following theorem confirming Conjecture \[1.2\] for \(n \leq 2\).

\textbf{Theorem 1.3.} \textit{For any field \(K\) admitting an Archimedean order, we get} \(\dim(M(K(x))) \geq 1\) \textit{and} \(\dim(M(K(x, y))) \geq 2\). \textit{If the field \(K\) is totally Archimedean, then} \(\dim(M(K(x))) = 1\) \textit{and} \(\dim(M(K(x, y))) = 2\).

Actually, Theorem \[1.3\] does not say all the truth about the dimension of the space \(M(K(x, y))\). It turns out that this space has covering topological dimension 2 but for any 2-divisible group \(G\) the cohomological dimension \(\dim_G M(K(x, y))\) is equal to 1! So, the space \(M(K(x, y))\) is a natural example of a compact space that is not dimensionally full-valued (which means that the cohomological dimensions of \(M(K(x, y))\) for various coefficient groups \(G\) do not coincide). A classical example of such a space is the Pontryagin surface, that is a surface with Möbius bands glued at each point of a countable dense subset, see \[5, 1.9\].

The covering and cohomological dimensions are partial cases of the extension dimension defined as follows, see \[6\]. We say that the \textit{extension dimension} of a topological space \(X\) does not exceed a topological space \(Y\) and write \(e\text{-dim}(X) \leq Y\) if each continuous map \(f : A \to Y\) defined on a closed
subspace $A$ of $X$ can be extended to a continuous map $\tilde{f}: X \to Y$. The classical Hurewicz-Wallman characterization of the covering dimension [9 1.9.3] says that $\dim(X) \leq n$ for a separable metric space $X$ if and only if $e\dim(X) \leq S^n$ where $S^n$ stands for the $n$-dimensional sphere. The sphere $S^n$ is an example of a Moore space $M(\mathbb{Z}, n)$ (whose reduced homology groups $H_k(S^n)$, $k \neq n$, are trivial except for the $n$-th group $H_n(S^n)$ which is isomorphic to $\mathbb{Z}$).

For a non-trivial abelian group $G$ the cohomological dimension $\dim_G(X)$ of a compact space $X$ coincides with the smallest non-negative number $n$ such that $e\dim(X) \leq K(G, n)$ where $K(G, n)$ is the Eilenberg-MacLane complex of $G$ (this is a CW-complex having all homotopy groups trivial except for the $n$-th homotopy group $\pi_n(K(G, n))$ which is isomorphic to $G$). If no such $n$ exists, then we put $\dim_G(X) = \infty$. It is known that $\dim_G(X) \leq \dim(X)$ for each abelian group $G$ and $\dim(X) = \dim_\mathbb{Z}(X)$ for any finite-dimensional compact space $X$. On the other hand, the famous Pontryagin surface $\Pi_2$ has covering dimension $\dim_\mathbb{Z}(\Pi_2) = 2$ and cohomological dimension $\dim_G(\Pi_2) = 1$ for any 2-divisible abelian group $G$, see [5 1.9]. A group $G$ is called 2-divisible if for each $x \in G$ there is $y \in G$ with $y^2 = x$. Surprisingly, but for any totally Archimedean field $K$ the space $M(K(x, y))$ has the same pathological dimension properties:

**Theorem 1.4.** For any totally Archimedean field $K$ the space of $\mathbb{R}$-places $M(K(x, y))$ has integral cohomological dimension $\dim_\mathbb{Z}(M(K(x, y))) = 2$ and the cohomological dimension $\dim_G(M(K(x, y))) = 1$ for any non-trivial 2-divisible Abelian group $G$.

Theorems [1 3] and [1 4] will be proved in Section [3] after some preliminary work made in Section [2].

2. Graphoids and spaces of $\mathbb{R}$-places

In this section we shall discuss the interplay between spaces of $\mathbb{R}$-places and graphoids. The notion of a graphoid has topological nature and can be defined for any family $\mathcal{F}$ of partial functions between topological spaces.

By a partial function between topological spaces $X, Y$ we understand a continuous function $f: \text{dom}(f) \to Y$ defined on a subspace $\text{dom}(f)$ of the space $X$. Its graphoid $\Gamma(f)$ is the closure of its graph $\Gamma(f) = \{(x, f(x)) : x \in \text{dom}(f)\}$ in the Cartesian product $X \times Y$. The graphoid $\tilde{\Gamma}(f)$ determines a multi-valued extension $\tilde{f}: X \to Y$ of $f$ whose graph $\Gamma(\tilde{f}) = \{(x, y) \in X \times Y : y \in f(x)\}$ coincides with the graphoid $\Gamma(f)$ of $f$. The multi-valued function $\tilde{f}: X \to Y$ assigns to each point $x \in X$ the closed subset $f(x) = \{y \in Y : (x, y) \in \Gamma(f)\}$ of the space $Y$.

For a finite family $\mathcal{F}$ of partial functions between topological spaces $X, Y$ we define the graphoid $\Gamma(\mathcal{F})$ of $\mathcal{F}$ as the graphoid of the “vector” function

$$ F : \text{dom}(\mathcal{F}) \to Y^\mathcal{F}, \quad \mathcal{F} : x \mapsto (f(x))_{f \in \mathcal{F}}, $$

defined on the subset $\text{dom}(\mathcal{F}) = \bigcap_{f \in \mathcal{F}} \text{dom}(f)$.

For an arbitrary family $\mathcal{F}$ of partial functions between $X$ and $Y$ we define its graphoid $\tilde{\Gamma}(\mathcal{F})$ as the intersection

$$ \tilde{\Gamma}(\mathcal{F}) = \bigcap \{\text{pr}_E^{-1}(\tilde{\Gamma}(\mathcal{E})) : \mathcal{E} \subset \mathcal{F}, \quad |\mathcal{E}| < \infty\} \subset X \times Y^\mathcal{F} $$

where for $\mathcal{E} \subset \mathcal{F}$

$$ \text{pr}_E : X \times Y^\mathcal{F} \to X \times Y^\mathcal{E}, \quad \text{pr}_E : (x, (y_f)_{f \in \mathcal{E}}) \mapsto (x, (y_f)_{f \in \mathcal{E}}), $$

denotes the natural projection.

The following lemma describing the structure of the graphoid $\tilde{\Gamma}(\mathcal{F})$ easily follows from the definition of $\tilde{\Gamma}(\mathcal{F})$.

**Lemma 2.1.** The graphoid $\tilde{\Gamma}(\mathcal{F})$ consists of all points $(x, (y_f)_{f \in \mathcal{F}}) \in X \times Y^\mathcal{F}$ such that for any finite subfamily $\mathcal{E} \subset \mathcal{F}$ and neighborhoods $O(x) \subset X$ and $O(y_f) \subset Y$ of the points $x$ and $y_f$, $f \in \mathcal{E}$, there is a point $x' \in O(x) \cap \text{dom}(\mathcal{E})$ such that $f(x') \in O(y_f)$ for all $f \in \mathcal{E}$.

Now we consider the graphoids in the context of rational functions of $n$ variables. To shorten notation, we shall denote the $n$-tuple $(x_1, \ldots, x_n)$ by $\vec{x}$. So, $K(\vec{x})$ will denote the field $K(x_1, \ldots, x_n)$ of rational functions of $n$ variables with coefficients in a field $K$.

Observe that each rational function $f \in \mathbb{R}(\vec{x})$, written as an irreducible fraction $f = \frac{p}{q}$ of two polynomials $p, q \in \mathbb{R}(\vec{x})$, can be thought as a partial function $\text{dom}(f) \to \mathbb{R}$ defined on the open dense subset $\text{dom}(f) = \mathbb{R}^n \setminus (p^{-1}(0) \cap q^{-1}(0))$ of the $n$-dimensional torus $\mathbb{R}^n$. 
Now we see that any family of rational functions $\mathcal{F} \subset \mathbb{R}(\vec{x})$ can be considered as a family of partial functions whose graphoid $\overline{\Gamma}(\mathcal{F}) \subset \mathbb{R}^n \times \mathbb{R}^\mathcal{F}$ is a well-defined closed subset of the compact Hausdorff space $\mathbb{R}^n \times \mathbb{R}^\mathcal{F}$.

Observe that for any finite subfamily $\mathcal{F} \subset \mathbb{R}(\vec{x})$ the subset $\text{dom}(\mathcal{F}) = \bigcap_{f \in \mathcal{F}} \text{dom}(f)$ is open and dense in $\mathbb{R}^n$. Thus the graphoid $\overline{\Gamma}(\mathcal{F}) \subset \mathbb{R}^n \times \mathbb{R}^\mathcal{F}$ projects surjectively onto the $n$-torus $\mathbb{R}^n$. The same fact is true for any family $\mathcal{F} \subset \mathbb{R}(\vec{x})$: its graphoid $\overline{\Gamma}(\mathcal{F})$ projects surjectively onto the $n$-torus $\mathbb{R}^n$.

It turns out that for a subfield $\mathcal{F} \subset \mathbb{R}(\vec{x})$, containing $\mathbb{Q}(\vec{x})$, the graphoid $\overline{\Gamma}(\mathcal{F})$ can be identified with a subspace of the space of $\mathbb{R}$-places $M(\mathcal{F})$.

**Theorem 2.2.** Let $\mathcal{F} \supset \mathbb{Q}(\vec{x})$ be a subfield of the field $\mathbb{R}(\vec{x})$.

1. Each point $\gamma = (\vec{a}, (y_f)_{f \in \mathcal{F}})$ of the graphoid $\overline{\Gamma}(\mathcal{F}) \subset \mathbb{R}^n \times \mathbb{R}^\mathcal{F}$ determines an $\mathbb{R}$-place $\delta_\gamma : \mathcal{F} \to \mathbb{R}$, $\delta_\gamma : f \mapsto y_f$.

To each rational function $f \in \mathcal{F}$ this $\mathbb{R}$-place assigns a point $\delta_\gamma(f) \in \overline{f}(\vec{a})$ where $\overline{f} : \mathbb{R}^n \to \mathbb{R}$ is the multivalued extension of $f$ whose graph $\overline{\Gamma}(\overline{f})$ coincides with the graphoid $\Gamma(\overline{f})$ of $f$.

2. The map $\delta : \overline{\Gamma}(\mathcal{F}) \to M(\mathcal{F})$, $\delta : \gamma \mapsto \delta_\gamma$, is a topological embedding.

3. If $\mathcal{F} = \mathbb{K}(\vec{x})$ for some subfield $\mathbb{K} \subset \mathbb{R}$, then $\delta(\overline{\Gamma}(\mathcal{F})) = \{ \chi \in M(\mathcal{F}) : \chi | \mathbb{K} = \text{id} \}$.

**Proof.** 1. Fix a point $\gamma = (\vec{a}, (y_f)_{f \in \mathcal{F}}) \in \overline{\Gamma}(\mathcal{F}) \subset \mathbb{R}^n \times \mathbb{R}^\mathcal{F}$ and consider the function $\delta_\gamma : \mathcal{F} \to \mathbb{R}$, $\delta_\gamma : f \mapsto b_f$.

Given any rational function $f \in \mathcal{F}$ consider its graphoid $\Gamma(f)$, which is equal to the closure of its graph $\{(\vec{x}, f(\vec{x})) : \vec{x} \in \text{dom}(f)\}$ in $\mathbb{R}^n \times \mathbb{R}$. Next, consider the projection $\text{pr}_f : \mathbb{R}^n \times \mathbb{R}^\mathcal{F} \to \mathbb{R}^n \times \mathbb{R}$, $\text{pr}_f : (\vec{x}, (y_f)_{f \in \mathcal{F}}) \mapsto (\vec{x}, y_f)$

and observe that $\text{pr}_f(\gamma) = (\vec{a}, b_f) \in \Gamma(f) = \Gamma(\overline{f})$ by the definition of the graphoid $\overline{\Gamma}(\mathcal{F})$. Consequently, $\delta_\gamma(f) = b_f \in \overline{f}(\vec{a})$.

In particular, $\delta_\gamma(x_i) = x_i(\vec{a}) = a_i$ for all $i \leq n$. Here $a_i$ denotes the $i$-th coordinate of the vector $\vec{a} = (a_1, \ldots, a_n)$. Also for any constant function $c \in \mathcal{F}$ we get $\delta_\gamma(c) = \overline{c}(\vec{a}) = c$. In particular, $\delta_\gamma(0) = 0$ and $\delta_\gamma(1) = 1$.

To show that $\delta_\gamma$ is an $\mathbb{R}$-place on the field $\mathcal{F}$, it remains to check that $\delta_\gamma(f + g) = \delta_\gamma(f) + \delta_\gamma(g)$ and $\delta_\gamma(f \cdot g) = \delta_\gamma(f) \cdot \delta_\gamma(g)$ for any rational functions $f, g \in \mathcal{F}$. Consider the finite subfamily $\mathcal{E} = \{f, g, f + g, f \cdot g\} \subset \mathcal{F}$, its graph

$\Gamma(\mathcal{E}) = \{(\vec{x}, (y_e)_{e \in \mathcal{E}}) \in \text{dom}(\mathcal{E}) \times \mathbb{R}^\mathcal{E} : \forall e \in \mathcal{E} : y_e = e(\vec{x})\}$

and its graphoid $\overline{\Gamma(\mathcal{E})} = \overline{\Gamma(\mathcal{E})} \subset \mathbb{R}^n \times \mathbb{R}^\mathcal{E}$. Observe that for any point $(\vec{x}, (y_e)_{e \in \mathcal{E}}) \in \Gamma(\mathcal{E})$ we get

$y_{f+g} = (f+g)(\vec{x}) = f(\vec{x}) + g(\vec{x}) = y_f + y_g$

and similarly $y_{fg} = y_f \cdot y_g$.

Consequently, $\overline{\Gamma(\mathcal{E})} \subset \mathbb{R}^n \times Y$ where $Y = \{(y_e)_{e \in \mathcal{E}} \in \mathbb{R}^\mathcal{E} : y_{f+g} = y_f + y_g, y_{fg} = y_f \cdot y_g\}$. Observe that the closure of the set $Y$ in $\mathbb{R}^\mathcal{E}$ coincides with the subset

$\overline{Y} = \{(y_e)_{e \in \mathcal{E}} \in \mathbb{R}^\mathcal{E} : y_{f+g} \in y_f \oplus y_g, y_{fg} = y_f \odot y_g\}$.

Consequently, $\text{pr}_E(\overline{\Gamma(\mathcal{F})}) \subset \overline{\Gamma(\mathcal{E})} \subset \mathbb{R}^n \times \overline{Y}$ which implies the desired inclusions

$\delta_\gamma(f + g) = b_{f+g} \in b_f \oplus b_g = \delta_\gamma(f) \oplus \delta_\gamma(g)$

and

$\delta_\gamma(f \cdot g) = b_{fg} \in b_f \odot b_g = \delta_\gamma(f) \odot \delta_\gamma(g)$.

2. It is easy to see that the map $\delta : \overline{\Gamma}(\mathcal{F}) \to M(\mathcal{F})$, $\delta : \gamma \mapsto \delta_\gamma$, is continuous. Let us show that it is injective. Take two distinct points $\gamma = (\vec{a}, (b_f)_{f \in \mathcal{F}})$ and $\gamma' = (\vec{a}', (b'_f)_{f \in \mathcal{F}})$ in the graphoid $\overline{\Gamma}(\mathcal{F})$. Then either $b_f \neq b'_f$ for some $f \in \mathcal{F}$ or $a_i \neq a'_i$ for some $i \leq n$. 
If $b_f \neq b'_f$ for some $f$, then $\delta_\gamma(f) = b_f \neq b'_f = \delta_{\gamma'}(f)$ and hence $\delta_\gamma \neq \delta_{\gamma'}$. If $a_i \neq a'_i$ for some $i \leq n$, then for the monomial $x_i$, we get $\delta_\gamma(x_i) = x_i(\bar{a}) = a_i \neq a'_i = x_i(\bar{a}') = \delta_{\gamma'}(x_i)$ and again $\delta_\gamma \neq \delta_{\gamma'}$.

Therefore, the continuous map $\delta : \bar{\Gamma}(\mathcal{F}) \to M(\mathcal{F})$ is injective. Since the space $\bar{\Gamma}(\mathcal{F})$ is compact and $M(\mathcal{F})$ is Hausdorff, the map $\delta$ is a topological embedding.

3. Assume that $\mathcal{F} = \mathbb{K}(\bar{x})$ for some subfield $\mathbb{K}$ of $\mathbb{R}$. Then inclusion $\delta(\bar{\Gamma}(\mathcal{F})) \subset \{ \chi \in M(\mathcal{F}) : \chi|\mathbb{K} = \text{id} \}$ follows from the statement (1). To prove the reverse inclusion we shall apply the Tarski-Seidenberg Transfer Principle [15]. This Principle says that for two real closed extensions $R_1$, $R_2$ of an ordered field $\mathbb{K}$, a finite system of inequalities between polynomials with coefficients in $\mathbb{K}$ has a solution in $R_1$ if and only if it has a solution in the field $R_2$.

Fix an $\mathbb{R}$-place $\chi : \mathcal{F} \to \mathbb{R}$ such that $\chi|\mathbb{K} = \text{id}$. By [3], the $\mathbb{R}$-place $\chi$ is induced by some total ordering $P$ of the field $\mathcal{F}$. Taking into account that $\chi|\mathbb{K} = \text{id}$ is the identity $\mathbb{R}$-place on the field $\mathbb{K}$, we conclude that the orders on $\mathbb{K}$ induced from the ordered fields $(\mathcal{F}, P)$ and $(\mathbb{R}, \mathbb{R}_+)$ coincide. Let $\bar{\mathbb{K}}$ be the relative algebraic closure of $\mathbb{K}$ in the real closed field $\mathbb{R}$ and $\bar{\mathcal{F}}$ be a real closure of the ordered field $(\mathcal{F}, P)$. The Uniqueness Theorem [13, XI.2] for real closures guarantees that $\bar{\mathbb{K}}$ can be identified with the real closure of $\mathbb{K}$ in the field $\bar{\mathcal{F}}$. By Theorem 6 of [12], the $\mathbb{R}$-place $\chi$ extends to a unique $\mathbb{R}$-place $\bar{\chi} : \bar{\mathcal{F}} \to \mathbb{R}$. The $\mathbb{R}$-place $\bar{\chi}|\bar{\mathbb{K}}$, being a unique $\mathbb{R}$-place on the real closed field $\bar{\mathbb{K}}$, coincides with the identity $\mathbb{R}$-place id : $\bar{\mathbb{K}} \to \mathbb{R}$.

For every $i \leq n$ let $a_i = \chi(x_i)$, $\bar{a} = (a_1, \ldots, a_n)$, and $b_f = \chi(f)$ for $f \in \mathcal{F}$. The inclusion $\chi \in \delta(\bar{\Gamma}(\mathcal{F}))$ will be proved as soon as we check that the point $\gamma = (\bar{a}, (b_f)_{f \in \mathcal{F}}) \in \mathbb{R}^n \times \bar{\mathcal{F}}$ belongs to the graphoid $\bar{\Gamma}(\mathcal{F})$. This will follow from Lemma 2.14 as soon as for any finite subfamily $\mathcal{E} \subset \mathcal{F}$, a neighborhood $O(\bar{a}) \subset \mathbb{R}^n$ of the point $\bar{a} = (a_1, \ldots, a_n)$ and neighborhoods $O(b_f) \subset \mathbb{R}$ of the points $b_f, f \in \mathcal{E}$, we find a vector $\bar{z} = (z_1, \ldots, z_n) \in O(\bar{a}) \cap \text{dom}(\mathcal{E})$ such that $f(\bar{z}) \in O(b_f)$ for all $f \in \mathcal{E}$.

We lose no generality assuming that $\{x_1, \ldots, x_n\} \subset \mathcal{E}$ and $\prod_{i=1}^n O(\chi(x_i)) \subset O(\bar{a})$.

Also we can assume that for each function $f \in \mathcal{E}$ the neighborhood $O(b_f)$ is of the form

- $[\alpha_f, \beta_f]$ for some rational numbers $\alpha_f < \beta_f$ if $b_f \in \mathbb{R}$, and
- $\mathbb{R} \setminus [\alpha_f, \beta_f]$ for some rational numbers $\alpha_f < \beta_f$ if $b_f = \infty$.

Write each rational function $f \in \mathcal{E}$ as an irreducible fraction $f = \frac{p_f}{q_f}$ of two polynomials $p_f, q_f \in \mathbb{K}(\bar{x})$. Replacing the polynomials $p_f$ and $q_f$ by $-p_f$ and $-q_f$, if necessary, we can assume that $q_f > 0$ in the ordered field $\bar{\mathcal{F}}$.

Write the finite set $\mathcal{E}$ as the union $\mathcal{E} = \mathcal{E}_- \cup \mathcal{E}_0 \cup \mathcal{E}_+$ where

$\mathcal{E}_0 = \{ f \in \mathcal{E} : \chi(f) \in \mathbb{R} \}$,
$\mathcal{E}_- = \{ f \in \mathcal{E} : \chi(f) = \infty, f < 0 \text{ in } \bar{\mathcal{F}} \}$,
$\mathcal{E}_+ = \{ f \in \mathcal{E} : \chi(f) = \infty, f > 0 \text{ in } \bar{\mathcal{F}} \}$.

To each $f \in \mathcal{E}_0$ we shall assign a system of two polynomial inequalities that has a solution in the field $\bar{\mathcal{F}}$. Observe that the inclusion $b_f \in O(b_f) = [\alpha_f, \beta_f]$ implies that $\alpha_f < \chi(\frac{p_f}{q_f}) = \frac{\chi(p_f)}{\chi(q_f)} < \beta_f$. Since the $\mathbb{R}$-place $\bar{\chi}$ is generated by the total order of the real closed field $\bar{\mathcal{F}}$, these inequalities are equivalent to the inequalities $\alpha_f < \frac{p_f}{q_f} < \beta_f$ holding in the ordered field $\bar{\mathcal{F}}$. Since $q_f > 0$, the latter inequalities are equivalent to $\alpha_f q_f < p_f < \beta_f q_f$. It follows that the vector $\bar{x} = (x_1, \ldots, x_n) \in \bar{\mathcal{F}}^n$ is a solution of the system

$$\alpha_f q_f(\bar{x}) < p_f(\bar{x}) < \beta_f q_f(\bar{x})$$

in the real closed field $\bar{\mathcal{F}}$.

Next, consider the case of a function $f \in \mathcal{E}_+$. Since $\chi(\frac{p_f}{q_f}) = \chi(f) = \infty$ and $f > 0$, we get $\beta_f q_f < p_f$ in $\bar{\mathcal{F}}$ and hence the inequality

$$\beta_f q_f(\bar{x}) < p_f(\bar{x})$$

has solution $\bar{x} = (x_1, \ldots, x_n)$ in $\bar{\mathcal{F}}$. By the same reasoning, for every $f \in \mathcal{E}_-$ the inequality

$$p_f(\bar{x}) < \alpha_f q_f(\bar{x})$$

has solution in $\bar{\mathcal{F}}$. 
Therefore, the system of the inequalities
\[
\begin{cases}
q_f(\vec{x}) > 0 & \text{for all } f \in \mathcal{E} \\
\alpha_f q_f(\vec{x}) < p_f(\vec{x}) < \beta_f q_f(\vec{x}) & \text{for all } f \in \mathcal{E}_0 \\
\beta_f q_f(\vec{x}) < p_f(\vec{x}) & \text{for all } f \in \mathcal{E}_+ \\
p_f(\vec{x}) < \alpha_f q_f(\vec{x}) & \text{for all } f \in \mathcal{E}_-
\end{cases}
\]
has solution $\vec{x} = (x_1, \ldots, x_n)$ in the real closed field $\hat{\mathbb{F}}$. By the Tarski-Seidenberg Transfer Principle [15, 11.2.2], this system has a solution in the real closed field $\hat{\mathbb{K}} \subseteq \mathbb{R}$. Using the continuity of the polynomials $p_f, q_f, f \in \mathcal{E}$, we can find a solution $\vec{z}$ of this system in the dense subset $(\hat{\mathbb{K}} \cap \text{dom}(\mathcal{E}))^n$ of $\mathbb{K}^n$. The choice of the inequalities from the system guarantees that $\vec{z} \in \prod_{i=1}^n O(\chi(x_i)) \cap \text{dom}(\mathcal{E})^n \subseteq O(a) \cap \text{dom}(\mathcal{E})^n$ and $f(\vec{z}) = \frac{p_f(\vec{z})}{q_f(\vec{z})} \in O(b_f)$ for all $f \in \mathcal{E}$.

Theorem 2.2 will help us to analyze the structure of certain fibers of the restriction operator $\rho_K : M(K(\vec{x})) \rightarrow M(K)$, $\rho_K : \chi \mapsto \chi|K$.

**Proposition 2.3.** Take any field $K$ with an injective $\mathbb{R}$-place $\varphi : K \rightarrow \mathbb{R}$. Then the fiber $\rho_K^{-1}(\varphi) \subseteq M(K(\vec{x}))$ can be identified with the graphoid $\hat{\Gamma}(\mathcal{F})$, where $\mathcal{F} = \mathbb{K}(\vec{x})$ for $\mathbb{K} = \varphi(K) \subseteq \mathbb{R}$.

**Proof.** The $\mathbb{R}$-place $\varphi : K \rightarrow \mathbb{R}$, being injective, is an isomorphism of the fields $K$ and $\mathbb{K}$. This isomorphism extends to a unique isomorphism $\Phi : K(\vec{x}) \rightarrow \mathbb{K}(\vec{x})$ such that $\Phi(x_i) = x_i$ for all $i \leq n$, where $\vec{x} = (x_1, \ldots, x_n)$.

The isomorphism $\varphi : K \rightarrow \mathbb{K}$ induces a homeomorphism $M\varphi : M(\mathbb{K}) \rightarrow M(K)$ which assigns to each $\mathbb{R}$-place $\chi : \mathbb{K} \rightarrow \mathbb{R}$ the $\mathbb{R}$-place $\chi \circ \varphi : K \rightarrow \mathbb{R}$. In the same way the isomorphism $\Phi$ induces a homeomorphism $M\Phi : M(\mathbb{K}(\vec{x})) \rightarrow M(K(\vec{x}))$. Now look at the commutative diagram

\[
\begin{array}{ccc}
M(K(\vec{x})) & \xrightarrow{M\Phi} & M(\mathbb{K}(\vec{x})) \\
\downarrow \rho_K & & \downarrow \rho_K \\
M(K) & \xrightarrow{M\varphi} & M(\mathbb{K}) \\
\end{array}
\]

Here $\delta : \hat{\Gamma}(\mathbb{K}(\vec{x})) \rightarrow M(\mathbb{K}(\vec{x}))$ is the embedding defined in Theorem 2.2 which implies that $\rho_K^{-1}(\text{id}) = \delta(\hat{\Gamma}(\mathbb{K}(\vec{x})))$. Since the maps $M\varphi$ and $M\Phi$ are homeomorphisms, we conclude that the composition $M\Phi \circ \delta$ maps homeomorphically the graphoid $\hat{\Gamma}(\mathbb{K}(\vec{x}))$ onto the fiber $\rho_K^{-1}(\varphi)$.

\section{3. Extension dimension of the space $M(K(\vec{x}))$}

In this section we shall evaluate the extension dimension of the space of $\mathbb{R}$-places $M(K(\vec{x}))$ of the field $K(\vec{x}) = K(x_1, \ldots, x_n)$ of rational functions of $n$ variables with coefficients in a field $K$.

We shall say that a topological space $X$ is an absolute neighborhood extensor for compacta (briefly, an ANE) if each continuous map $f : B \rightarrow Y$ defined on a closed subspace $B$ of a compact Hausdorff space $X$ can be extended to a continuous map $\hat{f} : A \rightarrow Y$ defined on a neighborhood $A$ of $B$ in $X$.

We recall that a topological space $X$ has extension dimension $\text{e-dim} \ X \leq Y$ if each continuous map $f : B \rightarrow Y$ defined on a closed subspace $B$ of $X$ admits a continuous extension $\hat{f} : X \rightarrow Y$.

**Theorem 3.1.** For a (totally Archimedean) field $K$ the space of $\mathbb{R}$-places of the field $K(\vec{x})$ has extension dimension $\text{e-dim} \ M(K(\vec{x})) \leq Y$ for some ANE-space $Y$ (if and) only if for each isomorphic copy $\mathbb{K} \subset \mathbb{R}$ of the field $K$ the graphoid $\hat{\Gamma}(\mathbb{K}(\vec{x}))$ has extension dimension $\text{e-dim} \ \hat{\Gamma}(\mathbb{K}(\vec{x})) \leq Y$.

**Proof.** To prove the “only if” part, assume that $\text{e-dim} \ M(K(\vec{x})) \leq Y$ for some space $Y$. Given any subfield $\mathbb{K} \subset \mathbb{R}$, isomorphic to $K$, we need to check that $\text{e-dim} \ \hat{\Gamma}(\mathbb{K}(\vec{x})) \leq Y$. Fix any isomorphism $\varphi : K \rightarrow \mathbb{K}$ and observe that it is an injective $\mathbb{R}$-place on $K$. By Proposition 2.3 the graphoid $\hat{\Gamma}(\mathbb{K}(\vec{x}))$ of the function family $\mathbb{K}(\vec{x})$ is homeomorphic to a subspace of the space $M(K(\vec{x}))$. Because of that $\text{e-dim} \ M(K(\vec{x})) \leq Y$ implies $\text{e-dim} \ \hat{\Gamma}(\mathbb{K}(\vec{x})) \leq Y$. 

The “if” part holds under the assumption that the field $K$ is totally Archimedean and the space $Y$ is an ANE. Assume that for each isomorphic copy $\mathbb{K} \subset \mathbb{R}$ of the field $K$ the graphoid $\Gamma(\mathbb{K}(\vec{x}))$ has extension dimension $\text{e-dim} \Gamma(\mathbb{K}(\vec{x})) \leq Y$. Since the field $K$ is totally Archimedean, each $\mathbb{R}$-place $\chi : K \to \mathbb{R}$ is injective, has image in $\mathbb{R}$ and is generated by a unique total order on $X$ (defined as $x < y$ if $\chi(x) < \chi(y)$). This means that the quotient map $\lambda : \mathcal{X}(K) \to M(K)$ is injective and hence is a homeomorphism. Since the space $\mathcal{X}(K)$ is zero-dimensional, so is the space $M(K) = M_A(X)$.

Now consider the restriction operator $\rho_K : M(K(\vec{x})) \to M(K)$, $\rho_K : \chi \mapsto \chi|K$. By Proposition 2.8 for each $\mathbb{R}$-place $\varphi \in M_A(K) = M(K)$ the fiber $\rho_K^{-1}(\varphi)$ is homeomorphic to the graphoid $\Gamma(\mathbb{K}(\vec{x}))$ of the family $\mathbb{K}(\vec{x})$ of rational functions of $n$ variables with coefficients in the subfield $\mathbb{K} = \varphi(K)$ of $\mathbb{R}$.

Our assumption on the extension dimension of $\Gamma(\mathbb{K}(\vec{x}))$ implies that $\text{e-dim} \rho_K^{-1}(\varphi) \leq Y$. The following lemma implies that $\text{e-dim} M(K(\vec{x})) \leq Y$.

**Lemma 3.2.** Let $\rho : X \to Z$ be a continuous map from a compact Hausdorff space $X$ onto a zero-dimensional compact Hausdorff space $Z$. The space $X$ has extension dimension $\text{e-dim} X \leq Y$ for some ANE-space $Y$ if and only if for each $z \in Z$ the fiber $\rho^{-1}(z)$ has extension dimension $\text{e-dim} f^{-1}(z) \leq Y$.

**Proof.** The “only if” trivially follows from the definition of extension dimension. To prove the “if” part, assume that each fiber of $\rho$ has extension dimension $\leq Y$. To prove that $\text{e-dim} X \leq Y$, fix a continuous map $f : B \to Y$ defined on a closed subspace $B$ of $X$. For each point $z \in Z$ consider the fiber $X_z = \rho^{-1}(z) \subset X$ of the map $\rho$. Since $\text{e-dim} X_z \leq Y$, the map $f|B \cap X_z$ admits a continuous extension $f_z : X_z \to Y$. Consider the map $\tilde{f}_z : Z \times X \to Y$ defined by $\tilde{f}_z|X = f_z$ and $\tilde{f}_z|B = f$. Since $Y$ is an ANE-space, the map $\tilde{f}_z$ admits a continuous extension $\tilde{f}_z : A_z \to Y$ defined on a open neighborhood $A_z$ of $X_z \cup B$ in $X$. Since the space $X$ is compact and $Z$ is Hausdorff, the map $\rho$ is closed. Consequently, the set $f(X \setminus A_z)$ is closed in $Z$ and its complement $O_z = Z \setminus f(X \setminus A_z)$ is an open neighborhood of $z$ in $Z$. Since the space $Z$ is compact and zero-dimensional, the open cover $\{O_z : z \in Z\}$ of $Z$ can be refined by a finite disjoint open cover $U$. For every set $U \in U$ choose a point $z \in Z$ with $U \subset O_z$ and put $f_U = \tilde{f}_z|\rho^{-1}(U)$. It follows that the map $f_U$ is a continuous extension of the map $f|B \cap \rho^{-1}(U)$. Then the maps $f_U$, $U \in U$, compose a required continuous extension $\hat{f} = \bigcup_{U \in U} f_U : X \to Y$ of the map $f$.

By Hurewicz–Wallman Theorem [9, 1.9.3], a compact Hausdorff space $X$ has covering topological dimension $\dim X \leq d$ for some $d \in \omega$ if and only if $\text{e-dim} X \leq S^d$ where $S^d$ stands for the $d$-dimensional sphere. Because of that Theorem 3.1 implies:

**Corollary 3.3.** For a (totally Archimedean) field $K$ the space of $\mathbb{R}$-places of the field $K(\vec{x})$ has dimension $\dim M(K(\vec{x})) \leq d$ for some $d \in \omega$ (if and only if for each isomorphic copy $\mathbb{K} \subset \mathbb{R}$ of the field $K$ the graphoid $\Gamma(\mathbb{K}(\vec{x}))$ has dimension $\dim \Gamma(\mathbb{K}(\vec{x})) \leq d$).

Let us recall [5] for an Abelian group $G$ a compact Hausdorff space $X$ has cohomological dimension $\dim_G X \leq d$ for some $d \in \omega$ if and only if $\text{e-dim} X \leq K(G, d)$ (see Introduction). This fact combined with Theorem 3.1 implies:

**Corollary 3.4.** For a (totally Archimedean) field $K$ the space of $\mathbb{R}$-places of the field $K(\vec{x})$ has dimension $\dim_G M(K(\vec{x})) \leq d$ for some $d \in \omega$ and some Abelian group $G$ (if and only if for each isomorphic copy $\mathbb{K} \subset \mathbb{R}$ of the field $K$ the graphoid $\Gamma(\mathbb{K}(\vec{x}))$ has dimension $\dim_G \Gamma(\mathbb{K}(\vec{x})) \leq d$).

Now we see that Theorems 1.3 and 1.4 follows from Corollaries 3.3, 3.4 and the following deep result [1] about the (cohomological) dimension of the graphoids.

**Theorem 3.5 (Banakh–Potyatynyk).**

1. For any subfamily $\mathcal{F} \subset \mathbb{R}(x)$ the graphoid $\Gamma(\mathcal{F})$ is homeomorphic to the extended real line $\mathbb{R}$ and hence has dimension $\dim \Gamma(\mathcal{F}) = 1$.
2. For any subfamily $\mathcal{F} \subset \mathbb{R}(x, y)$ the graphoid $\Gamma(\mathcal{F})$ has dimensions $\dim \Gamma(\mathcal{F}) = \dim_G \Gamma(\mathcal{F}) = 2$.
3. For any subfamily $\mathcal{F} \subset \mathbb{R}(x, y)$ containing the rational functions $\frac{a}{b-x}$, $a, b \in \mathbb{Q}$, the graphoid $\Gamma(\mathcal{F})$ has cohomological dimension $\dim_G \Gamma(\mathcal{F}) = 1$ for any non-trivial 2-divisible Abelian group $G$.

In light of Corollaries 3.3 and 3.4 the following problem arises naturally.

**Problem 3.6.** Let $\mathbb{K}, \mathbb{F} \subset \mathbb{R}$ be two isomorphic copies of a (totally Archimedean) field $K$. Are the graphoids $\Gamma(\mathbb{K}(x, y))$ and $\Gamma(\mathbb{F}(x, y))$ homeomorphic?
Remark 3.7. In light of this question it is interesting to remark that a totally Archimedean field can have distinct isomorphic copies in $\mathbb{R}$. A suitable example can be constructed as follows. Take the polynomial $f(x) = x^4 - 5x^2 + 2$. This polynomial is irreducible over $\mathbb{Q}$ and has four real roots. The Galois group of $f$ is the dihedral group with 8 elements, so the degree of the splitting field of $f$ over $\mathbb{Q}$ is 8. Therefore, for every root $\alpha$ of $f$ there is another root $\beta$ such that $\beta \notin \mathbb{Q}(\alpha)$. It follows that $\mathbb{Q}(\alpha)$ and $\mathbb{Q}(\beta)$ are isomorphic, but not equal, totally Archimedean subfields of $\mathbb{R}$.

4. Acknowledgements

The authors would like to thank Andrzej Śladek for providing the example in Remark 3.7.

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