Abstract

We present a Kaluza-Klein vacuum solution which closely resembles the Taub-NUT magnetic monopole and we investigate its physical properties as viewed from four space-time dimensions. We show that the Taub-NUT Kaluza-Klein vacuum solution in five dimensions is a static magnetic monopole. We find that the four dimensional matter properties do not obey the equation of state of radiation and there is no event horizon. A comparison with the available magnetic monopole solutions and the issue of negative mass are discussed.

1 Introduction

One of the oldest ideas that unify gravity and electromagnetism is the theory of Kaluza and Klein which extends space-time to five dimensions \cite{1}. The physical motivation for this unification is that the vacuum solutions of the (4+1) Kaluza-Klein field equations reduce to the (3+1) Einstein field equations with effective matter and the curvature in (4+1) space induces matter in (3+1) dimensional space-time \cite{2}. With this idea, the four dimensional energy-momentum tensor is derived from the geometry of an exact five dimensional vacuum solution, and the properties of matter such as density and pressure as well as electromagnetic properties are determined by such a solution. In other words, the field equations of both electromagnetism and gravity can be obtained from the pure five-dimensional geometry.

Kaluza’s idea was that the universe has four spatial dimensions, and the extra dimension is compactified to form a circle so small as to be unobservable. Klein’s contribution was to make a reasonable physical basis for the compactification of the fifth dimension \cite{3}. This school of thinking later led to the eleven-dimensional supergravity theories in 1980s and to the “theory of everything” or ten-dimensional superstrings \cite{4}.

Many spherically symmetric solutions of Kaluza-Klein in ordinary space are investigated in \cite{5}-\cite{6}. In the work by Gross and Perry \cite{7} and Davidson and Owen \cite{8} some other solutions of the Kaluza-Klein equations are discussed. The (4+1) analogues of (3+1) Schwarzschild solution are among these solutions. However, since some of the solutions of Kaluza-Klein do not have event horizon of the type which exists in the Einstein’s theory, one needs not call them black holes \cite{9}.

In this paper, we present a vacuum solution of Kaluza-Klein theory in five-dimensional space-time which is closely related to the Taub-NUT metric. The Taub-NUT solution has many interesting features; it carries a new type of charge (NUT charge), which has topological origins and can be regarded as “gravitational magnetic charge”, so the solution is known in some other contexts as the Kaluza-Klein magnetic monopole \cite{10}. 

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The plan of this paper is as follows. In section 2, we briefly discuss the formalism of five-dimensional Kaluza-Klein theory and the effective four-dimensional Einstein-Maxwell equations. We will then present a Taub-NUT-like Kaluza-Klein solution and investigate its physical properties in four dimensions. In the last section we will draw our main conclusions.

2 Kaluza-Klein theory and the Taub-NUT solution

Kaluza (1921) and Klein (1926), used one extra dimension to unify gravity and electromagnetism in a theory which was basically five-dimensional general relativity [11]. The theoretical elegance of this idea is revealed by studying the vacuum solutions of Kaluza-Klein equations and the matter induced in the four-dimensional spacetime[12]. Thus, we are chiefly interested in the vacuum five-dimensional Einstein equations. For any vacuum solution, the energy-momentum tensor vanishes and thus \( \hat{G}_{AB} = 0 \) or, equivalently \( \hat{R}_{AB} = 0 \), where \( \hat{G}_{AB} \equiv \hat{R}_{AB} - \frac{1}{2} \hat{g}_{AB} \hat{R} \) is the Einstein tensor, \( \hat{R}_{AB} \) and \( \hat{g}_{AB} \) are the five-dimensional Ricci tensor and scalar, respectively. \( \hat{g}_{AB} \) is the metric tensor in five dimensions. Here, the indices \( A,B,... \) run over 0...4 [4].

Generally, one can identify the \( \mu \nu \) part of \( \hat{g}_{AB} \) with \( g_{\mu \nu} \), which is the contravariant four-dimensional metric tensor, \( A_{\mu} \) with the electromagnetic potential and \( \phi \) as a scalar field. The correspondence between the above components is

\[
\hat{g}^{AB} = \begin{pmatrix}
g^{\mu \nu} & -\kappa A^{\mu} \\
-\kappa A^{\nu} & \kappa^2 A^{\sigma} A_{\sigma} + \phi^2
\end{pmatrix}
\]

where \( \kappa \) is a coupling constant for scaling the electromagnetic potential \( A^{\mu} \) and the indices \( \mu, \nu \) run over 0...3.

The five-dimensional field equations reduce to the four-dimensional field equations [13],[14]

\[
G_{\mu \nu} = \frac{\kappa^2 \phi^3}{2} T^{EM}_{\mu \nu} - \frac{1}{\phi} [\nabla_{\mu} (\partial_{\nu} \phi) - g_{\mu \nu} \square \phi],
\]

and

\[
\nabla^{\mu} F_{\mu \nu} = -3 \frac{\partial^{\mu} \phi}{\phi} F_{\mu \nu},
\]

\[
\square \phi = \frac{\kappa^2 \phi^3}{4} F_{\mu \nu} F^{\mu \nu},
\]

where \( G_{\mu \nu} \) is the Einstein tensor, \( T^{EM}_{\mu \nu} \equiv \frac{1}{4} g_{\mu \nu} F_{\rho \sigma} F^{\rho \sigma} - F_{\mu}^{\sigma} F_{\nu \sigma} \) is the electromagnetic energy-momentum tensor and the field strength \( F_{\mu \nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \). Knowing the five-dimensional metric, therefore, leads to a complete knowledge of the four-dimensional geometry, as well as the electromagnetic and scalar fields.

The Taub-NUT solution was first discovered by Taub (1951), and subsequently by Newman, Tamburino and Unti (1963) as a generalization of the Schwarzschild spacetime [15]. This solution is a single, non-radiating exact solution of Einstein’s field equations. The Taub-NUT solution is nowadays being involved in the context of higher-dimensional theories of semi-classical quantum gravity [16]. As an example, in the work by Gross and Perry [7] and of Sorkin [17], soliton solutions were obtained by embedding the Taub-NUT gravitational instanton inside the five dimensional Kaluza-Klein manifold[15]. Solitons are static, localized, and non-singular solutions
of nonlinear field equations which resemble particles. One such solution which obeys the Dirac quantization condition is considered in [7].

The Kaluza-Klein monopole of Gross and Perry is described by the following metric which is a generalization of the self-dual Euclidean Taub-NUT solution [7]

\[ ds^2 = -dt^2 + V(dx^2 + 4m(1 - \cos \theta)d\phi)^2 + \frac{1}{V}(dr^2 + r^2d\theta^2 + r^2 \sin^2 \theta d\phi^2), \]  

where

\[ \frac{1}{V} = 1 + \frac{4m}{r}. \]  

This solution has a coordinate singularity at \( r = 0 \) which is called NUT singularity. This can be absent if the coordinate \( x^5 \) is periodic with period \( 16\pi m = 2\pi R \), where \( R \) is the radius of the fifth dimension. Thus \( m = \frac{\sqrt{\pi G}}{2\pi} \) [18]. The gauge field \( A_\nu \) is given by \( A_\phi = 4m(1 - \cos \theta) \), and the magnetic field is \( B = \frac{4mr}{r^3} \), which is clearly that of a monopole and has a Dirac string singularity in the range \( r = 0 \) to \( \infty \). The magnetic charge of this monopole is \( g = \frac{m}{\sqrt{\pi G}} \) which has one unit of Dirac charge. In this model, the total magnetic flux is constant. For this solution, the soliton mass is determined to be \( M^2 = \frac{m_p^2}{16\alpha} \) where \( m_p \) is the Planck mass. Gross and Perry showed that the Kaluza-Klein theory can contain magnetic monopole solitons which would support the unified gauge theories and allow us for searching the physics of unification.

Here, we introduce a metric which is a vacuum five dimensional solution, having some properties in common with the monopole of Gross and Perry, while some other properties being different. The proposed static metric is given by

\[ ds^2_{(5)} = -dt^2 + (1 - \frac{2m}{r}) (dr^2 + r^2d\theta^2 + r^2 \sin^2 \theta d\phi^2) + \left( \frac{4m^2}{1 - \frac{2m}{r}} \right) (d\psi + Q \cos \theta d\phi)^2. \]  

Here, the extra coordinate is represented by \( \psi \) and \( Q \) is a constant. The coordinates take on the usual values with range \( r \geq 0 \), \( 0 \leq \theta \leq \pi \), \( 0 \leq \phi \leq 2\pi \) and \( 0 \leq \psi \leq 2\pi \). In general, the above metric does not have a zero Ricci tensor, except when \( Q = \pm 1 \). The non-vanishing components of the Ricci tensor in five dimensions are given by

\[ R_{22} = \frac{-2m^2(Q^2 - 1)}{(-r + 2m)^2}, \]
\[ R_{33} = \frac{2m^2[4m^2Q^2 \cos^2 \theta - (r^2 + 4m^2 - 4rm) \sin^2 \theta][-1 + Q^2]}{(-r + 2m)^4}, \]
\[ R_{34} = \frac{8m^4Q \cos \theta(Q^2 - 1)}{(-r + 2m)^4}, \]
\[ R_{44} = \frac{8m^4(Q^2 - 1)}{(-r + 2m)^4}. \]

By setting \( Q = \pm 1 \), the metric will satisfy the vacuum Einstein equations \( R_{AB} = 0 \) in five dimensions. Since we are interested in a Ricci flat solution we set \( Q = 1 \) from now on. The Killing vectors associated with metric (6) with \( Q = 1 \) are given by

\[ K_0 = (1, 0, 0, 0, 0), \quad K_1 = (0, 0, 0, 0, 1), \quad K_2 = (0, 0, 0, 1, 0), \]
\[ K_3 = (0, 0, -\sin \phi, -\cot \theta \cos \phi, \csc \theta \cos \phi), \]
\[ K_4 = (0, 0, \cos \phi, -\cot \theta \sin \phi, \csc \theta \sin \phi), \]
which are the same as in the Taub-NUT space discussed in [19], where the authors studied spinning particles in the Taub-NUT space. The gauge field, $A_\mu$ and the scalar field $\phi$ deduced from the metric (6) are

$$A_\phi = \frac{\cos \theta}{\kappa},$$  

(9)

and

$$\phi^2 = \frac{4m^2}{1 - \frac{2m}{r}},$$  

(10)

respectively.

Thus the only non-vanishing component for the electromagnetic tensor field is

$$F_{\theta\phi} = -F_{\phi\theta} = -\frac{\sin \theta}{\kappa},$$  

(11)

which corresponds to a radial magnetic field. The scalar field equals $\phi_0^2 = 4m^2$ as $r \to \infty$ so that the second part of equation (2) becomes zero. Thus we have

$$G_{\mu\nu} = \frac{\kappa^2 \phi_0^2 T_{EM}}{2} = 8\pi G T_{\mu\nu}, \quad \text{as} \quad r \to \infty,$$  

(12)

where we have put the speed of light $c$ equal to 1. By comparing this equation with the ordinary Einstein equation, we obtain $\frac{\kappa^2 \phi_0^2}{2} = 8\pi G$, thus the constant $\kappa$ equals $\kappa = \frac{m}{\sqrt{\pi G}}$.

The total magnetic flux through any spherical surface centered at the origin is calculated as

$$\Phi_B = \oint _S F = \oint _S F_{\mu\nu} d\Sigma^{\mu\nu} = \frac{2\pi}{\kappa},$$  

(13)

which shows that the flux of the magnetic monopole is constant (i.e. we have a singular magnetic charge). The total magnetic charge is given by

$$Q_M = \frac{m}{2\sqrt{\pi G}}.$$  

(14)

Equation (14) can be considered as a constraint between the constants.

We now show that equation (3) is satisfied explicitly in four dimensions. We first note that

$$\Box \phi = \frac{4m^3}{r^4(1 - \frac{2m}{r})^{\frac{3}{2}}},$$  

(15)

The value of the scalar $F_{\mu\nu}F^{\mu\nu}$ is easily calculated to be

$$F_{\mu\nu}F^{\mu\nu} = \frac{2}{\kappa^2 r^4 (1 - \frac{2m}{r})^2},$$  

(16)

and therefore

$$\frac{\kappa^2 \phi^3}{4} F_{\mu\nu}F^{\mu\nu} = \frac{4m^3}{r^4(1 - \frac{2m}{r})^{\frac{3}{2}}},$$  

(17)

which together with (15) shows that equation (3) is satisfied.

The four dimensional metric deduced from equation (6) leads to the following asymptotically flat spacetime:

$$ds^2_{(4)} = -dt^2 + \left(1 - \frac{2m}{r}\right) dr^2 + r^2 \left(1 - \frac{2m}{r}\right) \left[d\theta^2 + \sin^2 \theta d\phi^2\right].$$  

(18)
Let us calculate the surface area of a $S^2$ hypersurface at constant $t$ and $r$ to clarify the nature of the $r = 2m$ singularity

$$A(r) = \int \sqrt{g^{(2)}} dx^2 = \int r^2 \left(1 - \frac{2m}{r}\right) \sin \theta d\theta d\phi = 4\pi r^2 \left(1 - \frac{2m}{r}\right). \quad (19)$$

It can be seen that the surface area $A(r)$ becomes zero at $r = 2m$, showing that the space shrinks to a point at $r = 2m$. This clarifies that the $r = 2m$ hypersurface is not an event horizon. For the case $r > 2m$ the signature of the metric is proper $(-, +, +, +)$ but for the range $r < 2m$ the signature of the metric will be improper and non-Lorentzian $(-, -, -, -)$, thus the patch $r < 2m$ is excluded from the spacetime. From now on, this spacetime is only considered in the range $r \geq 2m$. For this metric, the Ricci scalar and the Kretschmann invariant $K$ are given by

$$R = \frac{6m^2}{r^4 (1 - \frac{2m}{r})^3}, \quad (20)$$

and

$$K = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \frac{m^2}{(r - 2m)^5} \left[\frac{2}{r - 2m} + \frac{2r - 3m}{r^2}\right], \quad (21)$$

since the range $0 < r < 2m$ is omitted from the spacetime, there is only one curvature singularity at $r = 2m$.

The metric (18) can be transformed into the following form, using the radial coordinate transformation $\tilde{r}^2 = r^2 \left(1 - \frac{2m}{r}\right)$:

$$ds^2 (t) = -dt^2 + \frac{\tilde{r}^4}{(m^2 + \tilde{r}^2)(m + \sqrt{m^2 + \tilde{r}^2})^2} d\tilde{r}^2 + \tilde{r}^2 d\Omega^2. \quad (22)$$

While the metric (18) is singular at $r = 0$ and $r = 2m$, the transformed metric is singular only at $\tilde{r}$ which is a curvature singularity.

The properties of the induced matter associated with the above metric can be gained by the equation (3). The components of the energy-momentum tensor can be easily calculated for the metric (6). We find that the effective source is like a fluid with an anisotropic pressure. At sufficiently large $r$, the energy-momentum components will tend to zero. The trace of the energy-momentum tensor does not vanish generally, which shows that the effective matter field around the singularity can not be considered as an ultra-relativistic quantum field in contrast to the Kaluza-Klein solitons described in [21]. The gravitational mass can be obtained from the energy-momentum tensor components according to

$$M_g (r) \equiv \int \left(T^{(2)}_0 - T^1_1 - T^2_2 - T^3_3\right) \sqrt{g^{(3)}} dV_3, \quad (23)$$

where $g^{(3)}$ is the determinant of the 3-metric and $dV_3$ is a 3D volume element. One would expect the mass of the monopole be independent of the sign of the magnetic charge, since the electromagnetic energy density depends on the magnetic field squared and independent of its direction. Using the components of the $T^{\mu\nu}$ make the value of integrand zero so that

$$M_g (r) = 0, \quad (24)$$

meaning that the gravitational mass vanishes everywhere. The conserved Komar mass which is defined as [22]

$$M_g (r) = -\frac{1}{8\pi} \oint_S \nabla^\mu K^\nu dS_{\mu\nu}, \quad (25)$$
also leads to

\[ M_g(r) = 0, \quad (26) \]

which is in agreement with the gravitational mass.

Although we obtained the total mass of the monopole using some known methods in four dimensional gravity, but the issue of energy in five dimensions is a subtle one. Three approaches to the calculation of energy in five dimensional gravity are discussed in Bombelli et al. [23]. The first approach is the direct application of the action principle. The second is the ADM energy corresponding to an initial-value formulation of the Kaluza-Klein theory, and the third approach is the Hamiltonian functional formulation appropriate to a covariant phase space. These authors show that the three approaches are equivalent and they have used them to calculate the mass of the Kaluza-Klein monopole. It turns out that for the Kaluza-Klein monopole in isotropic coordinates gets the asymptotic form

\[ ds^2 = -(1 + 2m/r)^{-1/2} dt^2 + (1 + 2m/r)^{1/2} (dr^2 + r^2 d\Omega^2), \quad (27) \]

and the energy (mass) of the monopole becomes [20]

\[ E = \frac{m^2 G}{2}. \quad (28) \]

In order for this mass to be positive, the mass term in the metric should appear as \(+ 2m/r\) with the mass parameter \(m\) positive. The case of negative mass term \(-2m/r\) is considered in Gibbons and Manton [24]. The negative choice (like our choice in this paper), first of all, leads to a singularity at finite \(r\) which we discussed before. According to [24], this singularity is unimportant since it is well inside regions of interest. We showed that in the case of our metric, this singularity is in fact spatially point-like (not an event horizon or a hypersurface of finite area) and the region inside \(r = 2m\) was excluded from the space-time since it had improper signature.

Let us follow Bombelli et al. [23] to calculate the 5D total energy of the solution, by using

\[ E = -(2k)^{-1} \int \omega^a t^a dS_{ab}, \quad (29) \]

where \(t^a = \partial/\partial t\) is the timelike Killing vector, \(\omega^a \equiv g^{mn} \Gamma^a_{mn} - g^{am} \Gamma^m_{mn}\), and the integration is performed at the \(r \to \infty\) limit. We therefore obtain

\[ E = (2k)^{-1} \oint d\theta d\phi d\psi 2mr^2 (1 - \frac{2m}{r}) \sin \theta \omega^a \nabla_a r, \quad (30) \]

where

\[ \omega^a \nabla_a r = (\nabla_a - \nabla_a) \nabla^a r - \nabla_b \nabla^b r + g^{bc} \nabla_b \nabla_c r. \quad (31) \]

Note that the derivatives \(\nabla\) is taken at the \(r \to \infty\) asymptotic. Simple calculation yields

\[ E = \lim_{r \to \infty} -\frac{4m(4m^2 - 5mr + r^2)}{2m - r}. \quad (32) \]

One observes that using this approach, \(E\) diverges, unless \(m = 0\), in which case the total 5D energy vanishes.
3 Conclusion

Inspired by the Taub-NUT solution, we introduced a Kaluza-Klein vacuum solution in $5D$, which described a magnetic monopole in $(3 + 1)D$ spacetime. The gravitational mass of the solution was shown to vanish. The fluid supporting the four dimensional space-time was shown to differ from that of an ultra-relativistic fluid, in contrast with the work by Wesson and Leon\cite{21}. The pressure is anisotropic in both works. The issue of negative mass term and other definitions of total energy in five dimensions were also discussed, and in particular the procedure of \cite{23} was used to calculate the 5D total energy. This was shown to diverge, unless the mass term in the metric vanishes.

The main conclusion of this paper is that the presented Taub-NUT Klauza-Klein solution contains a singular magnetic monopole with vanishing gravitational mass. We calculated the magnetic charge and showed that the total magnetic flux of the monopole through any spherical surface centered at the origin was constant, indicating that the monopole is singular and there is no extended magnetized fluid. The gravitational mass was derived in two different ways and was shown to vanish using both definitions. It was pointed out that the singularity which appears at finite $r$ is neither a horizon, nor a surface of finite, non-vanishing surface area. In a more appropriate coordinate system, this was shown to be curvature singularity at $\tilde{r} = 0$.

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