Chiral Quark Dynamics in Dense Nuclear Matter†

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Abstract

We consider a new approach to the description of dense nuclear matter in the framework of chirally symmetric, quark-based hadron models. As previously in the Skyrme model, the dense environment is described in terms of hyperspherical cells of unit baryon number. The intrinsic curvature of these cells generates a new gauge interaction for the quark fields which mediates interactions with the ambient matter. We apply this approach to the Nambu-Jona-Lasinio (NJL) model, construct its curved-space quark propagator and solve the ladder Bethe-Salpeter equation for the pion. We find a high-density phase transition to chiral restoration, discuss the density dependence of the chiral order parameter and of the pion properties, and compare with results of the conventional chemical-potential approach. The new approach can additionally describe baryon-density-free cavities in the dense medium.
I. INTRODUCTION

Nuclear matter has been the subject of intense study over the last decades \[1\]. It represents in many ways the simplest nuclear many-body system and also plays significant roles in other areas of physics, ranging from relativistic heavy-ion collisions to the structure of dense stellar objects and to the evolution of the universe. In recent years, nuclear matter under extreme conditions, \textit{i.e.} at densities a couple of times beyond the saturation density and at finite temperature, became a particularly active research area.

Under these extreme conditions, the strong interactions and the hadron spectrum are expected to undergo qualitative changes and, in particular, transitions to new phases and vacua \[2\]. Lattice simulations \[3\] and different models \[4,5\] predict, for example, chiral symmetry restoration and deconfinement transitions. The new phases will probably still be complex and nonperturbative, with an excitation spectrum containing colorless bound states carrying hadron quantum numbers \[6\] and other collective modes, e.g. plasmons and plasminos \[7\]. Pion \[8\] and kaon \[9\] condensation and a vector-symmetric vacuum \[10,11\] have also been contemplated. Produced in finite volumes, some of the new vacua might furthermore have interesting coherent decay modes \[12,13\]. The study of all these phenomena promises new and direct insight into the elusive nonperturbative sector of the underlying theory, QCD.

Since the advent of relativistic heavy-ion accelerators at Brookhaven and CERN in 1986, matter under extreme conditions became also accessible in the laboratory. In central heavy-ion collisions very dense nuclear matter, albeit probably not close to equilibrium, can be produced. Fixed-target experiments with heavy beams, as \textit{e.g.} the now operational gold beam at the Brookhaven AGS and the planned lead beam at CERN’s SPS, will produce particularly large final state multiplicities of a few thousand charged particles per central collision, and accordingly very large baryo-chemical potentials.

Realistic calculations of nuclear matter properties at these high densities are difficult to perform. Direct information from the lattice cannot be expected soon, since a chemical
potential renders the euclidean action of QCD complex and conventional Monte-Carlo techniques inapplicable [14]. The traditional approaches, as for example Brueckner–Hartree–Fock and variational calculations, are based on effective interactions (e.g. from meson–exchange models [15]) between point-like nucleons. At high densities, however, this neglect of the nucleon structure is a serious shortcoming, since the chemical potential in compressed nuclear matter is of the order of or larger than the lowest nucleon excitation energies.

Additionally, density-induced modifications of the hadron structure, which recently became an active field of study [16], should be taken into account and will in turn affect the matter properties. Particularly dramatic changes in the hadron structure can be expected in the vicinity of phase transitions, where the ground state of the matter rearranges itself in a fundamental way. The chiral phase transition alters e.g. the hadron spectrum drastically, and the deconfinement transition transforms hadrons either into weakly bound states [6] or dissolves them completely into a quark-gluon plasma.

To account for the hadron structure and its modifications in high-density nuclear matter calculations is a challenging task. Indeed, a realistic treatment is beyond the reach of present capabilities. One can, however, attempt a more qualitative description in the framework of baryon models, based on the unit-cell method. This type of mean-field approach is familiar in solid state physics for the description of homogeneous or periodic media and becomes reliable at very high densities. Also, it is simple enough to be applicable in many existing baryon models.

Let us briefly recapitulate the basic concepts underlying the unit cell approach, which we will later generalize. After dividing the nuclear matter into identical cells of unit baryon number, periodic (or “twisted” periodic) boundary conditions are imposed on the cell, i.e. the physics on opposite surfaces is identified. As a consequence the cell looses its boundary and becomes self-contained, while the presence of the neighbors is reflected in its periodic images. In topological terms, the cell becomes a three-torus $S^1 \times S^1 \times S^1$.

The size and shape of the unit cell is found by minimizing its energy under variations of its geometry. In this way interactions with its surroundings determine via the variational
principle the detailed cell structure. This is exactly how one derives, e.g., the simple cubic unit cell of sodium chloride from the Coulomb interaction. Information about the neighbors and the mutual interactions is then encoded in one single, isolated cell. In the context of some baryon models, and soliton models in particular, the unit cell approach has the additional advantage of describing the density-dependent structure of the baryons and their mutual interactions in matter consistently, namely on the basis of the same underlying dynamics.

In the Skyrme model [17] – a widely studied soliton model based on an effective chiral meson lagrangian – a face-centered cubic (fcc) unit cell was found along these lines a few years ago [18]. More recently, Manton and Ruback pointed out that the cell energy could be lowered further by giving the unit cell an intrinsic curvature [19,20]. Their proposal can be regarded as an extension of the variational principle, which now allows all the geometrical properties, not only the shape, of the cell to be determined by the given dynamics. The optimal cell turned out to be a three-dimensional hypersphere \( S^3(L) \) of radius \( L \). Exclusively in this geometry the skyrmion can attain its absolute energy minimum, the Bogomolnyi bound [19].

The simple form of the optimal unit cell, \( S^3 \), is essentially determined by chiral symmetry. \( S^3 \) is invariant under the group \( SO(4) \), which is (locally) equivalent to the chiral group \( SU(2) \times SU(2) \). The pion fields therefore take values on \( S^3 \), and their lowest-energy configuration is attained if they are defined on another \( S^3 \), since the resulting map minimizes the deformation energy stored in the field gradients.

The most remarkable new feature of hyperspherical cells is their curvature, which induces additional dynamics for the fields. According to the unit cell concept, these interactions are ascribed to the dense environment. As in conventional unit cells, the cell boundary has disappeared. The periodicity in the three angular coordinates of \( S^3 \) can be regarded as a reflection of the neighboring, identical cells. The replacement of the surrounding cells and their boundaries by periodic coordinates is, as pointed out before, a generic feature of most unit cells and it supports the physical intuition in dealing with the hypersphere.

The hypersphere approach has by now been extensively applied in the Skyrme model.
Perhaps most remarkably, a chiral phase transition emerged at high densities, accompanied by parity doubling of the hadron spectrum and the disappearance of the Goldstone modes [21]. Also, strangeness condensates with distinct characteristics due to changes in the extended baryon structure have been found [22], and the vector limit [10] phase transition has been studied [11]. Furthermore, bag formation and dissolution [23], some properties of the loop expansion in the presence of a skyrmion [24] and its excitation spectrum [25] have been investigated. Skyrmion-like profiles on the hypersphere have also been derived from instantons [26], following an idea by Atiyah and Manton [27], and other soliton solutions have been constructed [28].

A remarkable and generic finding of these studies is that hypersphere calculations reproduce almost quantitatively the results of similar fcc array calculations, whenever those are available. This holds in particular for the chiral phase transition, its critical density, the density dependence of the order parameter and the energy density, and it established confidence in the new approach. At the same time the calculational effort is greatly reduced. Array calculations require the numerical solution of partial-differential field equations on a three-dimensional grid. Hypersphere calculations, on the other hand, involve at most the solution of an ordinary differential equation, as for isolated skyrmions. Some results can even be obtained analytically and allow a straightforward and transparent physical interpretation.

The rather successful and consistent results of this unorthodox approach call for both a better understanding of its physical foundations and for further exploration of its range of applicability. In the present paper we will take a first step in this direction by extending the hypersphere approach to quark-based hadron models, and to the Nambu-Jona-Lasinio (NJL) model [29] in particular. Some results in the chiral symmetry breaking sector have already been published in ref. [30].

The NJL model shares with the Skyrme model the underlying chiral symmetry, which led to the geometry of $S^3$. In many other respects, however, the NJL dynamics is different. In contrast to the purely mesonic degrees of freedom in Skyrme-like models, it is based solely on quark fields. Furthermore, it does not describe nucleons as solitons and it breaks chiral
symmetry dynamically. These differences can help to disentangle effects specific to the Skyrme model from more generic or even model-independent features of the hypersphere approach. In particular, they could help to clarify whether the specific geometrical and topological properties of the Skyrme model are indispensable for the effectiveness of the hypersphere approach.

The choice of $S^3$ unit cells even beyond the Skyrme model is additionally motivated by their maximal symmetry under all three-dimensional curved spaces. Hyperspherical cells can thus be considered as the simplest generalization of flat unit cells, and many of their characteristic features are indeed reminiscent of flat cells. This tendency to preserve qualitative properties of euclidean space is enhanced by the conformal equivalence between $S^3$ and $R^3$. The use of $S^3$ cells furthermore avoids potential problems with explicit chiral symmetry breaking by a cell boundary (as, for example, in bag models).

An alternative, more conventional approach to dense matter in the NJL model, in which the quarks are coupled to a homogeneous, external baryon number source, has been studied by Bernard et al. [4] and by others [31]. As in the Skyrme model, where the comparison with conventional array calculations served as a test for the hypersphere approach, we will profit from comparing our results to those of the chemical potential approach.

Let us add a couple of comments concerning the uses and limitations of our approach. Any attempt to describe nuclear matter in terms of unit cells, i.e. in a mean field framework, is bound to be qualitative at best [32]. Approximating the short- and medium-range nucleon-nucleon correlations by a strict long-range order and neglecting the nucleons kinetic energy are of course rather severe simplifications, except at very high densities such as in neutron stars, where nuclear matter might even crystallize.

On the other hand, one may expect that bulk features of nuclear matter are robust enough to survive these approximations, and that they can be revealed in the unit-cell framework. Global features and driving mechanisms of high-density phase transitions, for example, could fall into this category, and the alternative Fermi-gas treatment will fail to describe them. As already mentioned, the unit cell approach offers the so far unique possibility to account
consistently for the extended baryon structure, which certainly contributes qualitatively important physics at high densities.

The paper is organized as follows: In section II we establish the general framework for our investigation by implementing the quark fields into the curved unit cell. Section III deals with the specific NJL dynamics and the construction of its curved-cell quark propagator in Hartree-Fock approximation. The latter is a key step in our program. Its more technical parts are relegated to appendices A and B.

In section IV the chiral symmetry breaking sector and its density dependence is examined. We derive, in particular, the gap equation, which describes the dynamical quark mass generation, and the expression for the chiral order parameter. In section V the Bethe-Salpeter equation in the quark-antiquark channel is set up in the curved background, and its solution for the pion in ladder approximation is found.

From the explicit pion wave function we calculate in section VI the pion decay constant and its density dependence. In section VII the zero-density limits of the various calculated quantities are derived and compared with the standard flat-space NJL results. Section VIII discusses the quantitative aspects of our results and compares them with results of the chemical potential approach. Finally, section IX summarizes the paper and offers some conclusions and perspectives for future research.

**II. QUARKS IN THE CURVED UNIT CELL**

In this section we describe the implementation of quark fields into a curved unit cell and summarize the generic physical implications. This discussion is quite general and provides a framework for the adaptation of a large class of quark-based baryon models.

As motivated in the introduction, our unit cell has the geometry of a three-dimensional hypersphere, $S^3(L)$, of radius $L$. Accordingly, the flat Minkowski metric is replaced by the metric of $S^3(L) \times R$ ($R$ indicates the real, flat time dimension),

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - L^2[d\mu^2 + \sin^2 \mu(d\theta^2 + \sin^2 \theta d\phi^2)], \tag{1}$$
which we write in the polar coordinates of $\mathbb{R}^4$, i.e. $\{x^\mu\} = (t, \mu, \theta, \phi)$. Its intrinsic curvature leads to new interactions for the spinor fields, as it does, for example, in general relativity. In our context, however, these interactions will be ascribed to the dense environment.

The spatial section of \( \mathbb{I} \) has six symmetry generators\footnote{These $SO(4)$ generators form together with the time translation generator the Killing vectors of the metric \( \mathbb{I} \).}, the maximal number in a three-dimensional, curved space. Three of them are just the ordinary rotations in $\mathbb{R}^3$, while the remaining three are compact analogs of Lorentz boost generators and can be regarded as “generalized translations” in $S^3$. Combined with the usual translations in the flat time direction, the metric \( \mathbb{I} \) therefore possesses seven symmetries, only three less than the Minkowski metric of $\mathbb{R}^4$. This large symmetry group will be helpful both in establishing analogies with flat unit cells and in explicit calculations. Together with the conformal relation between the hypersphere and Minkowski space, it often leads to structural similarities in expressions for physical quantities.

In the Skyrme model, which contains only spin-0 pion fields, the change of the lagrangian to the new metric was sufficient to adapt the model to the hypersphere. Spinor fields in curved space, however, require the introduction of additional concepts. The straightforward generalization of the fields from representations of the Lorentz group to representations of the general linear group of (infinitesimal) coordinate transformations fails for fermions, since the linear group has no spinor representations.

This impasse can be circumvented \footnote{These $SO(4)$ generators form together with the time translation generator the Killing vectors of the metric \( \mathbb{I} \).} by defining the spinor fields in a local orthonormal basis, given by the vierbein fields $e^a(x)$. The $e^a$ are generated by linear, space-time dependent transformations of the coordinate basis, written in terms of the $4 \times 4$ matrix $e^a_{\mu}(x)$:

$$e^a(x) = e^a_{\mu}(x) dx^\mu, \quad a \in \{0, 1, 2, 3\}.$$  

(2)

The corresponding vector fields, which form the components of the gradient operator in the orthonormal basis, can be obtained from the inverse vierbein coefficients $e^a_{\mu}$, defined by
\[ e_a^\mu(x) e_b^\mu(x) = \delta_b^a. \] (3)

The \( e_b^\mu \) transform coordinate-vector components into Lorentz-vector components (in the local frame) and are obtained from the \( e_a^\mu \) by lowering the Latin index with the Minkowski metric \( \{\eta_{ab}\} = \text{diag}(1, -1, -1, -1) \) of the local frames and by raising the Greek index with the hypersphere metric \( g \). The gradients are then given by

\[ e_a(x) = e_a^\mu(x) \partial_\mu. \] (4)

Due to its orthonormality, the vierbein is simply related to the metric (1):

\[ g_{\mu\nu}(x) = \eta_{ab} e_a^\mu(x) e_b^\nu(x). \] (5)

In our context it is crucial to note that this decomposition of a given metric in terms of the vierbein frames is not unique. Indeed, an arbitrary, local Lorentz transformation of the Latin vierbein indices leaves the expression (5) unchanged and thus leads to the same metric. These Lorentz rotations consequently form a gauge group, which just changes the relative orientation convention (the "gauge") of the local frames, but leaves the physics (i.e. the metric) invariant. It generalizes the global Lorentz group of Minkowski space, and its spinor representations allow the definition of fermion fields in curved space. In order to preserve the local Lorentz invariance of the lagrangian, gradients of the quark fields have to be replaced by gauge-covariant derivatives,

\[ e_a^\mu(\partial_\mu + \omega_\mu) q \equiv (e_a + \omega_a) q, \] (6)

which transform homogeneously under the gauge group. In eq. (6) we have introduced a (4 \( \times \) 4-matrix-valued) gauge field, the spin connection

\[ \omega_a = e_a^\mu \omega_\mu \equiv -\frac{i}{4} \omega_a^{bc} \sigma_{bc}. \] (7)

(\( \sigma_{ab} = \frac{i}{2}[\gamma_a, \gamma_b] \)), which just compensates the change of the spinor field components due to different frame orientations at neighboring points by a spatial \( SO(3) \) rotation. In this paper,
we take $\omega$ to be the Levi-Civita connection, which is uniquely determined by the metric and the vierbein:

$$\omega_{bc}^a = -e^b_\nu e^c_\rho \left( \delta^\rho_\mu \partial^\nu - \Gamma ^\nu_{\rho\mu} \right) e_\mu^a. \quad (8)$$

The Christoffel connection $\Gamma$ is the corresponding gauge field of the local Lorentz group in the vector representation and can be directly obtained from the metric:

$$\Gamma _{\mu}{}^{\nu\rho} = \frac{1}{2} g_{\sigma\mu} \left( \partial^\nu g^{\rho\sigma} + \partial^\rho g^{\nu\sigma} - \partial^\sigma g^{\nu\rho} \right). \quad (9)$$

The explicit form of $\omega$ depends, of course, on the gauge. The group structure of $S^3$ implies the existence of a particularly useful choice, which we will refer to as the “Maurer-Cartan gauge”. In it, the spin connection takes a simple and, in particular, space-time independent form:

$$\omega_a = \frac{i}{4L} \epsilon_{abc} \sigma^{bc} \quad (a, b, c \in \{1, 2, 3\}) \quad \text{and} \quad \omega_o = 0. \quad (10)$$

$(\sigma_{ab} = \frac{i}{2}[\gamma_a, \gamma_b])$ More details about this gauge, in which all of the following calculations will be performed, can be found in appendix A.

In the context of nuclear matter unit-cells, it might at first be tempting to think of the spin connection as a covariantized chemical potential, since it acts as a constant, isoscalar vector interaction for the quarks and grows with density. Such an interpretation would, however, be misleading. Besides its gauge dependence, $\omega$ has a different Dirac-matrix structure and its time component vanishes identically in the matter rest frame.

Following the procedure outlined above, one can now adopt a general fermionic lagrangian from Minkowski space to the hypersphere. The corresponding action is obtained by integrating the lagrangian over $S^3(L) \times R$, i.e.

2The Levi-Civita connection is metric and torsion-free (see appendix A) and thus corresponds to the natural and minimal generalization of flat unit cells. In principle, one could use a more general spin connection, which would introduce a non-zero torsion into the unit cell. This issue might deserve future investigation.
\[ S = \int d\mu(x) \mathcal{L}(x). \] (11)

The integration measure \( d\mu(x) \) contains the jacobian \( \sqrt{-g} \), where \( g(x) = -L^6 \sin^4 \mu \sin^2 \theta \) is the determinant of the metric tensor (1):

\[
d\mu(x) = \sqrt{-g(x)} d^4x = L^3 \sin^2 \mu \sin \theta d\mu d\theta d\phi dt.
\] (12)

In order to complete the definition of the unit cell it remains to set its total baryon number to one. Since this is most conveniently done at the level of the quark propagator, we will reserve this last step for the next section.

Let us now summarize the model-independent physical implications of the curved unit cell for fermions. The novel aspects can be classified according to their either local or global nature. The local effects are due to the finite curvature: gradients are replaced by vierbein vector fields and a new gauge interaction emerges. However, as already pointed out in the introduction, there are also new features of global, i.e. topological character. Due to the compactness of the hypersphere, the fermion spectrum (see appendix A for its explicit form) will be discrete, with the energy scale of the excitations determined by the inverse radius of the cell. Furthermore, potentially symmetry-breaking cell boundaries are absent. The periodicity in the three (angular) spatial directions \( \mu, \theta \) and \( \phi \) is reminiscent of the periodic boundary conditions in flat unit cells and can be regarded as a reflection of the neighboring, identical cells. Quarks moving around the cell more than once are accordingly interpreted as entering a neighboring cell. This physical interpretation will become explicit in particular during the construction of the fermion propagator.

We conclude this section with a remark on the perturbative treatment of space curvature, which is frequently used in general relativity. While this approach often significantly simplifies explicit calculations, it will be insufficient for our purpose, for two reasons: First, the curvature of our unit cell is of the order of \( V^{-\frac{4}{3}} \), where \( V \) is the average volume occupied by one nucleon in nuclear matter, and thus becomes large at high densities. Secondly, a perturbative approximation to the metric disregards the global structure of the unit cell.
This is another serious shortcoming at high densities, where the cell becomes smaller and, as we will show in the next section, the quarks get lighter on the approach to chiral restoration. Under these circumstances the global features of the cell become important.

III. NJL MODEL AND QUARK PROPAGATOR ON $S^3 \times R$

Up to now our discussion established a generic framework for the study of fermionic models in hyperspherical unit cells. In order to proceed further and to arrive at quantitative results, we have to specify the dynamics of the quarks.

Many of the existing, chirally symmetric quark and quark-meson models are accessible in our approach. The Nambu-Jona-Lasinio (NJL) model \[29\], however, seems particularly well suited for a first, explorative study aimed at testing the use of hyperspherical unit cells beyond the Skyrme model. As already indicated in the introduction, the NJL dynamics is simple and contrasts the Skyrme model in many aspects, except for their common chiral symmetry. For example, the NJL model is entirely quark-based (in its modern form), it is not a soliton model\(^3\) and it breaks chiral symmetry dynamically.

In flat Minkowski space, the flavor-$SU(2)$ version of the NJL model \[29\] is based on the lagrangian

$$\mathcal{L} = \frac{i}{2} \left[ \overline{q} \gamma^\mu \partial_\mu q - (\partial_\mu \overline{q}) \gamma^\mu q \right] - m_0 \overline{q} q + g \left( \overline{q} \Gamma_a q \right) \left( \overline{q} \Gamma_a q \right). \quad (13)$$

The $q$’s denote Dirac-spinor quark fields, which are color triplets and isospin doublets. The scalar and pseudoscalar four-quark contact interactions resemble instanton-generated vertices of light quarks in QCD \[35\] and are written in a shorthand notation with the help of

\(^3\)In this paper we will consider only the original, fermionic version of the NJL model. A restricted bosonization of some generalized NJL lagrangians can support stable, skyrmion-like soliton solutions \[34\]. Their eventual investigation on $S^3$ could shed more light on connections between the hypersphere results in the Skyrme and NJL models.
the four matrices \( \Gamma^0 = 1 \gamma 1 \), \( \Gamma^i = i \gamma_5 \tau_i \) \( (i = 1, 2, 3) \). (The index \( a \in \{0, 1, 2, 3\} \) is implicitly summed over in the interaction term above.) In most of the following discussion we will set the small current quark masses \( m_o \) to zero.

Due to its dynamical symmetry breaking mechanism, simplicity and phenomenological successes, the NJL model is widely employed as an effective low-energy theory for QCD \[36,37\]. Recently, the original version of the NJL model has been generalized in a couple of directions, e.g. by adding vector and axial-vector couplings or by extensions to the \( SU(3) \)-flavor sector. These developments and many applications to meson and baryon physics, including some in hot and dense nuclear matter, are reviewed in refs. \[36,37\]. We will work with the original, “minimal” version of the model, however, since our aim is not a detailed phenomenological analysis, but rather the investigation of generic aspects of the new approach.

We begin our study by adapting the NJL lagrangian \[13\] according to the procedure established in the preceding section to the hyperspherical unit cell, \( i.e. \) we replace the flat Minkowski metric by eq. \[\text{(1)}\] and the gradients by gauge-covariant derivatives:

\[
\mathcal{L} = \frac{i}{2} \left[ \bar{q} \gamma^a (\epsilon_a + \omega_a) q - \bar{q} (\epsilon_a - \omega_a) \gamma^a q \right] - m_o \bar{q} q + g (\bar{q} \Gamma_a q) (\bar{q} \Gamma^a q). \tag{14}
\]

(The arrow indicates that the corresponding differential operator acts to the left.)

Most of our following analysis will be based on the standard Hartree-Fock approximation to the NJL dynamics \[29\]. As is well known, the quarks dynamically acquire already at this level a finite self-energy, if the coupling \( g \) exceeds a critical value. Due to the pointlike character of the interaction, the self-energy is space-time independent. The additional gauge interaction on the hypersphere preserves this property, since the spin connection is a non-propagating background field. In the presence of a finite baryon density, the self-energy acquires besides the standard scalar piece, which acts as a “constituent” mass, an additional vector part:

\[
\Sigma = \Sigma_s - \gamma_0 \Sigma_v. \tag{15}
\]
All the vacuum, quark and pion properties of interest in this paper can be conveniently calculated with the help of the NJL constituent quark propagator, which includes the self-energy [15]. The next and pivotal step of our program is therefore the derivation of an explicit expression for this propagator on $S^3 \times R$.

To this end, and at the densities of interest the crucial modifications in the quark dynamics due to the large curvature have to be taken into account exactly. While in general analytical expressions for propagators in curved spaces can rarely be found, the high symmetry of the hypersphere allows us to derive the quark propagator on $S^3 \times R$ exactly and in closed form. Our derivation starts from the definition of the propagator as the inverse of the gauge-covariant Dirac operator,

$$
[i \gamma^a (e_a + \omega_a) + \Sigma_v \gamma_0 - (m_0 + \Sigma_s)] \ S(x, y) = \delta^4(x, y).
$$

(16)

(The explicit form of the delta function on $S^3 \times R$ is given in appendix B.) We then specialize to the Maurer-Cartan gauge, separate $S$ into invariant scalar amplitudes and rewrite them such that the dependence on all but the geodesic coordinate of $S^3$ is removed. The remaining $\mu$ dependence can then be absorbed into redefined amplitudes, which finally allows the inversion to be performed by Fourier methods. The detailed calculation is described in appendix B.

The resulting Hartree-Fock propagator on $S^3 \times R$, in the chiral $\gamma$-matrix basis, reads

$$
S(x) \equiv S(x, 0) = e^{i(\vec{\Sigma} \hat{r} \frac{\theta}{r} + \Sigma_0 t)} \left[ S_0(\mu, t) \gamma_0 - S_1(\mu, t) \hat{r} \gamma - S_2(\mu, t) \right].
$$

(17)

The exponential factor, which contains the Dirac spin matrix $\Sigma$, is the spin parallel propagator and $\hat{r}(\theta, \phi)$ denotes the usual unit-vector of euclidean $R^3$ in spherical coordinates. The decomposition of the propagator into the three amplitudes

$^4\Sigma$ is the direct product of the $2 \times 2$ unit matrix and the standard Pauli matrices, see appendix B.
\( S_0(\mu, t) = -i\alpha (\sin \mu)^{-1} \sum_{n=-\infty}^{+\infty} (-1)^n [2I'_n + \tan \frac{\mu}{2} I_n], \) \( (18) \)

\( S_1(\mu, t) = i\frac{\alpha}{L} (\sin \mu)^{-1} \sum_{n=-\infty}^{+\infty} (-1)^n [2I''_n - \cot \frac{\mu}{2} I'_n], \) \( (19) \)

\( S_2(\mu, t) = m\alpha (\sin \mu)^{-1} \sum_{n=-\infty}^{+\infty} (-1)^n [2I'_n + \tan \frac{\mu}{2} I_n], \) \( (20) \)

\((\alpha \equiv 1/4\pi L^2)\) reflects the symmetry properties of \( S^3 \times R \). The \( S_i(\mu, t) \) are expressed as sums over all geodesic paths on which a quark can propagate between the pole \((y = 0)\) and the point \(x\). Since the cell is compact, these paths contain \(n\) full circles around the hypersphere. This becomes transparent by noting that all the integrals \(I_n \equiv I(\mu + 2n\pi, t)\) and their derivatives with respect to \(\mu\) \((t)\), denoted by primes (dots), are derived from the two-dimensional propagator

\[ I(\mu, t) = \int \frac{dk_0}{2\pi} \int \frac{dk}{2\pi} \frac{e^{i(k\mu L - k_0 t)}}{k_0^2 - k^2 - m^2 + i\epsilon} \] \( (21) \)

\((m = m_0 + \Sigma_s)\). The additional \(\mu\) dependence in equations \((18) - (20)\) just translates propagation along a flat “\(\mu\)-axis” into geodesic propagation in the \(\mu\) direction on the curved hypersphere.

The propagator \((17)\) describes quark propagation in a cell with zero net baryon number and consequently satisfies the usual Feynman vacuum boundary conditions. Its adaptation to the cell’s physical ground state with baryon number one is, however, straightforward, owing to the transparent implementation of the boundary conditions in eq. \((21)\).

The “Fermi sea” of three valence quarks in the cell affects the singularities of the integrand of \(I(\mu, t)\) inside the Fermi sphere both by the appearance of poles from positive energy holes and by the suppression of poles from Pauli-blocked states. Accordingly, the integrand in eq. \((21)\) has to be modified to

\[ e^{i(k\mu L - k_0 t)} \left[ \frac{1}{k_0^2 - \omega^2 + i\epsilon} + \frac{i\pi}{\omega} \delta(k_0 - \omega) \Theta(k_F - |k|) \right], \] \( (22) \)

where \(\omega = \sqrt{k^2 + m^2}\) and \(k_F\) is the Fermi momentum of the quarks. With the explicit expression for the propagator at hand, we are now ready to enter the discussion of the quark dynamics in the hyperspherical cell.
IV. DYNAMICAL CHIRAL SYMMETRY BREAKING

In the Skyrme model, the perhaps most important and unexpected result of the hypersphere approach was the prediction of a chiral restoration phase transition at high densities \([20,21]\). However, the restoration mechanism exploits specific topological and geometrical features and, in particular, the hedgehog structure of the skyrmion\(^5\). It is therefore not \textit{a priori} clear if and how hyperspherical unit cells can give rise to chiral restoration in other models.

In the NJL model, in particular, chiral symmetry breaking occurs dynamically, by quark-antiquark pair condensation in the vacuum \([29]\). At the same time the quarks become “dressed” by a virtual quark-antiquark cloud and acquire a “constituent” mass. This mechanism, analogous to the BCS mechanism for gap formation in superconductivity and perhaps mediated in a similar way by instantons in QCD \([38]\), is fundamentally different from its Skyrme model counterpart. In the present section we will examine how it is affected by a finite baryon density, described in terms of hyperspherical unit cells.

We base our study, following Nambu and Jona-Lasinio, on the Schwinger-Dyson equation

\[
\Sigma(x, y) = S_0^{-1}(x, y) - S^{-1}(x, y) \tag{23}
\]

for the quark self-energy, since the development of a constituent quark mass is a direct signature for chiral symmetry breaking. Equation (23) has the same form as in flat space and its derivation proceeds along the same standard lines \([39]\). Note, however, that both \(S\)

\(^5\)In the Skyrme model in flat space, chiral symmetry is broken by a nonlinear parametrization of the pion fields. On the hypersphere at high densities the ground state changes into a generalized hedgehog form. In addition to the coupling of isospin transformations and rotations of the standard hedgehog, it also couples the axial generators of the chiral group to the “translations” on the hypersphere. Projection onto the physical rotation \textit{and} translation eigenstates then reveals the restoration of the full chiral symmetry \([21]\).
and its free (i.e. \( g = 0 \)) counterpart \( S_0 \) contain the interactions with the curved background and, in particular, with the gauge field \( \omega \).

It is well known that nontrivial solutions of eq. (23), i.e. finite constituent masses, can only be generated nonperturbatively. Both the Hartree approximation, which becomes exact in the large-\( n_c \) limit of the NJL model (\( n_c \) is the number of colors of the quark fields), and the Hartree-Fock approximation are frequently used for this purpose. In the following, we will adopt the Hartree-Fock approximation to allow for a direct comparison with results of the conventional approach \([4]\). To find the Hartree-Fock solution of eq. (23), we start from the solution to \( O(g) \), i.e. to first order in the NJL interaction:

\[ \Sigma(x, y) = \Sigma(x) \delta^4(x, y), \tag{24} \]

with

\[ \Sigma(x) \equiv \Sigma = ig \{ 2\Gamma_a \text{tr} [S^-_0(x, x)\Gamma_a] - \Gamma_a [S^+_0(x, x) + S^-_0(x, x)] \Gamma_a \}. \tag{25} \]

Due to the pointlike character of the NJL interaction the self-energy is, as anticipated, space-time independent. For the same reason, both coincidence limits of the propagator (17) appear above,

\[ S^\pm_0(x, x) = S^\pm(0) = \lim_{x_0 \to 0^\pm} S(x_0, x = 0) = \tilde{S}^\pm_0 \gamma_0 - \tilde{S}_2, \tag{26} \]

and are independent of \( x \). Note that the amplitude \( S_1 \) in eq. (17) has to vanish in the coincidence limit, since it multiplies the unit distance vector.

Both \( S_0 \) and \( S_2 \) contain (quadratic and logarithmic) short distances singularities, which originate from the integral \( I(\mu, t) \), eq. (21), and its derivatives. We therefore multiply the corresponding integrands by a smoothed theta function

\[ \Theta_\epsilon(\Lambda - |k|) = \frac{1}{e^{\epsilon L(|k| - \Lambda)} + 1}, \tag{27} \]

\[ ^6 \text{Due to its negative mass dimension, the NJL coupling is not renormalizable. The introduced cutoff } \Lambda \text{ is therefore physical [10].} \]
which damps the contributions from the high-momentum region and thus regularizes the integrals. Note that cutting off the spatial part of the momentum integration is consistent with the symmetry properties of the metric and of the “Fermi sea” of valence quarks.

The representation of the propagator amplitudes as a sum over paths, eqs. (18) – (20), is sometimes inconvenient for practical purposes. At high densities, in particular, the unit cell radius $L$ is small and a large number of paths contribute. In appendix C, these sums are therefore transformed into equivalent mode sums over the discrete Dirac spectrum on $S^3 \times R$, with the energies

$$\omega_n = \sqrt{k_n^2 + m^2}, \quad k_n = \frac{2n + 1}{2L}, \quad n \in \{1, 2, 3, \ldots\}$$

and the corresponding degeneracies

$$D_n = 2n(n + 1)$$

of the $n$th quark level (for a given flavor and color), which emerges in the course of the transformation. (An alternative, more direct derivation of the spectrum is given in appendix A.)

In the spectral representation, the regularized coincidence limits of the propagator functions take the form (cf. appendix C)

$$\tilde{S}_0^\pm = \mp \frac{i}{4V} \sum_{n=1}^{\infty} D_n \Theta_\epsilon(\Lambda - k_n) \Theta(k_F - k_n),$$  \hspace{1cm} (30)

$$\tilde{S}_2 = \frac{im}{4V} \sum_{n=1}^{\infty} \frac{D_n}{\sqrt{k_n^2 + m^2}} \Theta_\epsilon(\Lambda - k_n) \Theta(k_n - k_F),$$  \hspace{1cm} (31)

where $V = 2\pi^2 L^3$ is the volume of the unit cell.

From the Dirac spectrum we can immediately read off the Fermi energy $\omega_F$, since all three valence quarks are contained in the lowest energy level:

$$\omega_F = \sqrt{k_F^2 + m^2}, \quad k_F = k_1 = \frac{3}{2L}. \hspace{1cm} (32)$$

Since this level is only partially filled, we have to prevent its remaining states from being counted in the above mode sums. To this end we associate a filling factor.
\[ f_n = \frac{N_n}{n_c n_f D_n} \]  

(33)

with each level, where \( N_n \) is the number of \textit{valence} quarks in the \( n \)th level, \textit{i.e.} \( f_1 = \frac{1}{8} \) and \( f_n = 0 \) for \( n > 1 \).

We are now ready to give explicit expressions for the self-energy, which properly take the valence quarks into account. Decomposing the Dirac-matrix structure of the self-energy in the rest frame into scalar and vector parts,

\[ \Sigma = \Sigma_s - \gamma_0 \Sigma_v, \]

(34)

as suggested by the symmetry properties and anticipated in the last section, and inserting the coincidence limit of the propagator into eq. (25), we obtain

\[
\Sigma_s = -4ig \left( 1 + 2n_c n_f \right) \tilde{S}_2 \\
= \frac{mg}{V} \left( 1 + 2n_c n_f \right) \sum_{n=1}^{\infty} \frac{D_n(1 - f_n)}{\omega_n} \Theta(\Lambda - k_n),
\]

(35)

\[
\Sigma_v = 4ig (\tilde{S}_0^+ + \tilde{S}_0^-) \\
= -\frac{2g}{V} \sum_{n=1}^{\infty} D_n f_n = -\frac{g}{V}.
\]

(36)

The above expressions exhibit clear analogies to their flat-space counterparts and allow a direct physical interpretation. The scalar self-energy receives contributions from virtual quark loops in all accessible levels of the Dirac sea down to the smooth cutoff \( \Lambda \). The three occupied states in the Fermi sea contribute with opposite sign. In both self-energies the finite-density effects, mediated by the curved cell, manifest themselves mainly through the altered quark momenta \( k_n \) and degeneracies \( D_n \), which replace the plane wave spectrum of flat space.

The vector part of the self-energy is entirely due to the three valence quarks and vanishes in an “empty” cell, even at finite ambient baryon density. This is in contrast to the chemical-potential approach of ref. [4], where \( \Sigma_v \) is necessarily finite at finite density. Furthermore, \( \Sigma_v \) does not depend on the value of the Fermi energy, but only on its density-independent relative position in the spectrum.

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We note in passing that the not explicitly \( n_c \)-dependent terms in the scalar self-energy (35) arise from exchange diagrams and become negligible in the large-\( n_c \) limit of the NJL model\(^7\). They are kept, however, in the Hartree-Fock approximation.

Since a constant self-energy just shifts the mass and energy scales of the quarks, the \( O(g) \) expressions in eqs. (35) and (36) have already the functional form of the Hartree-Fock solutions. We still have to impose the self-consistency condition, however, which requires the mass in the quark loop-propagator on the right-hand side of eq. (35) to be identical to the scalar part of the self-energy itself\(^8\) and leads to the gap equation

\[
m = \frac{mg}{V} (1 + 2n_c n_f) \sum_{\omega_n} \frac{D_n(1 - f_n)}{\omega_n} \Theta(\Lambda - k_n).
\]

(37)

At least at low densities and for sufficiently large coupling \( g \), we expect the perturbative vacuum to become unstable, as it does in flat space. The simultaneously generated quark mass can then be found as the nontrivial solution of eq. (37). The additional, trivial solution \( m = 0 \) does of course always exist.

With the self-consistent self-energies at hand, the Hartree-Fock propagator is now completely determined and vacuum expectation values of bilinear quark operators can be calculated. In the remainder of this section, we will specifically consider the vector and scalar quark condensates. The vector condensate, or equivalently the quark number density, provides a consistency check on our treatment of the valence quark sector, whereas the scalar condensate plays a central role in the discussion of dynamical chiral symmetry breaking and its density dependence.

\(^7\)At the physical \( n_c = 3 \) the exchange contributions are an order of magnitude smaller than the direct terms.

\(^8\)Since our focus is on spontaneous chiral symmetry breaking, we neglect the small explicit breaking and specialize to the chiral limit, \( m_0 = 0 \). The generalization to a finite current mass is straightforward.
Let us start with the vector condensates. They are equal to the quark number densities in the cell, which add up to

\[
< u^\dagger u > + < d^\dagger d > = \frac{3}{V},
\]

(38)
corresponding to three valence quarks per cell. We further specialize to isosymmetric nuclear matter, so that the quark number densities of both up and down quarks become equal:

\[
< u^\dagger u > = < d^\dagger d > \equiv < q^\dagger q > = \frac{3}{2V}.
\]

(39)

To check our treatment of the valence quark sector, we now calculate \(< q^\dagger q >\) for comparison directly from the coincidence limit of the quark propagator, eq. (26):

\[
< q^\dagger q > = -i \text{tr}_{\gamma,c} \left[ \gamma_0 (S^- (0) - S(0)_{\rho=0}) \right] = -i \text{tr}_{\gamma,c} \left[ \gamma_0 \tilde{S}_0^- \right] = \frac{n_c}{V} D_1 f_1 = \frac{n_c}{2V}.
\]

(40)

As expected, the two results agree and thus confirm the correct pole structure of the propagator.

We now turn to the scalar quark condensate, which is the standard order parameter of the chiral phase transition. Again, it can be directly obtained from the propagator:

\[
< \bar{q}q > = -\lim_{t \to 0} \text{tr}_{\gamma,c} < Tq(x=0,t)\bar{q}(0) >= -i \text{tr}_{\gamma,c} \left[ S^- (0) \right] = -\frac{n_c m}{V} \sum_{n=1}^{\infty} \frac{D_n (1 - f_n)}{\omega_n} \Theta_\epsilon (\Lambda - k_n).
\]

(41)

Note that \(< \bar{q}q >\) is proportional to the quark mass, i.e. to the solution of the gap equation (37). Chiral symmetry breaking manifests itself, as anticipated, simultaneously in a finite quark condensate and a dynamically generated quark mass. If a nontrivial solution of the gap equation exists and if it is energetically favorable compared to the other solutions (see the discussion in section VIII), a quark condensate necessarily develops.

The gap equation (37) can be solved numerically and indeed yields, at low densities and for sufficiently large couplings, a finite quark mass. As expected, the mass decreases with density and finally vanishes, signaling the chiral restoration transition. The quantitative
solution of the gap equation as a function of density and the evaluation of the associated
free energy will be subject of section VIII, where we also discuss the behavior of the scalar
quark condensate and the order of the chiral phase transition. Section VIII furthermore
contains a comparison of our results with those of ref. [4].

V. THE PION WAVE FUNCTION

As a consequence of Goldstone’s theorem, the spontaneous breakdown of chiral symmetry
is accompanied by the appearance of an iso-triplet of (almost) massless pions. They emerge
in the NJL model as quark-antiquark pairs, tightly bound by the same interaction which
leads to quark condensation. In the present section we study this mechanism and its density
dependence in our framework.

Our strategy will be to generalize the original NJL approach of calculating the pion wave
function to the curved unit cell. Most of the new features encountered are related to the
changes in both the quark and pion spectra. (For the actual spectra of free fermions and
spin-0 bosons on $S^3 \times R$ see appendices A and D.) As a consequence, the Fourier transform
of the spatial coordinate dependence becomes impractical and our calculation will be done
in coordinate space.

Following NJL [29], we start from the connected four-point function in the quark-
antiquark channel,

$$S^{(4)}(x_1, y_1; y_2, x_2)_{\alpha\beta;\delta\gamma} = < 0| T \bar{q}_\alpha(x_1)q_\beta(y_1)q_\gamma(x_2)\bar{q}_\delta(y_2)|0 >_c $$

(42)

(flavor and color indices of the quarks are suppressed), with the aim to derive an equation for its pion pole contribution. As in flat space, the Green function [42] satisfies an
inhomogeneous Bethe-Salpeter equation [11],

$$S^{(4)}(x_1, y_1; y_2, x_2)_{\alpha\beta;\delta\gamma} = G_0(x_1, y_1; y_2, x_2)_{\alpha\beta;\delta\gamma} + \int d\mu(z_1) \int d\mu(z_2) \int d\mu(z_3) \int d\mu(z_4) 
\times G_0(x_1, y_1; z_2, z_1)_{\alpha\beta;\alpha'\beta'} K(z_1, z_2; z_3, z_4)_{\alpha\beta;\delta\gamma} S^{(4)}(z_3, z_4; y_2, x_2)_{\delta\gamma;\delta\gamma} $$

(43)
(the integration measure \(d\mu(z)\) was defined in eq. (12)), which can be derived either by summing the rescattering series of interactions specified by the \(\bar{q}q\)-irreducible vertex kernel \(K\), or formally from the generating functional of the NJL model. In eq. (43) we introduced the quark-antiquark propagator

\[ G_0(x_1, y_1; y_2, x_2)_{a\beta, \delta\gamma} = S(x_1, y_2)_{a\delta} S(x_2, y_1)^T_{\beta\gamma}, \quad (44) \]

where \(S\) is the quark propagator on \(S^3 \times R\).

Mesonic bound states appear as poles in eq. (43). In flat space one finds the corresponding intermediate state wave functions after a four-dimensional Fourier transform as the pole residua, which satisfy a homogeneous Bethe-Salpeter equation \[41\]. In general curved space-times the situation becomes more involved. Due to the lack of translational invariance (and consequently the absence of a natural global coordinate system, as provided by the cartesian coordinates in flat space), the Fourier transform would have to be replaced by a generalized spectral transform in accord with the symmetry properties of the metric\[9\].

Fortunately, however, the time direction of our metric is flat and we can isolate the pole piece by a Fourier transform of the time dependence only. Accordingly, we write the Bethe-Salpeter amplitude of the pion as

\[ <0|Tq_{\alpha}(x_1)\bar{q}_{\beta}(y_1)|\pi^a(p)> = \frac{e^{-i\omega_p X_0}}{\sqrt{2\omega_p V}} \chi^a(x_1, y_1; p)_{a\beta}, \quad (45) \]

where \(p = \{\omega_p, p\}\) stands for the set of four quantum numbers which replace the flat-space four-momentum vector of a free pion. The pion energy \(\omega_p\) and the conserved "momentum" quantum numbers, \(p = \{n, l, m\}\), specify the spatial state of the pion completely and are discussed in more detail in appendix D. The notation is intended to emphasize the similarities\[9\].

\[9\]In general curved space-times questions of principle regarding the consistent definition of particle and composite-particle concepts \[39\] can arise. None of these affect our metric, however, due to its time-like Killing vector.

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with the standard quantum numbers in flat space. The time coordinates of the center of mass are
\[ X^0_i = \frac{1}{2}(x^0_i + y^0_i), \quad i = 1, 2. \] (46)

The time dependence on the right-hand side of eq. (45) is entirely determined by time translation symmetry. For the same reason \( \chi^a \) depends only on time differences, which we will, however, not exhibit explicitly. Finally, the normalization of the Bethe-Salpeter amplitude agrees with the normalization of the one-pion states adopted in appendix D.

For the course of the following derivation it is convenient to introduce a finite pion mass, which allows the specialization to the pion rest frame. Later we will go back to the chiral limit. The bound state (pion) contribution to \( S^{(4)} \), obtained by inserting on-shell pion intermediate states into eq. (42), can now be written in terms of the Bethe-Salpeter amplitude and its conjugate:
\[ i \int \frac{dp_0}{2\pi} \sum_{p,a} \chi^a(x_1, y_1; p) \chi^a(x_2, y_2; p) \delta \gamma \mathcal{G}_0(x_1, y_1; z_2, z_1) \mathcal{K}(z_2, z_3; z_3, z_4) \chi^a(z_3, z_4; p) e^{-ip_0(X_1^0 - X_2^0)}/2. \] (47)

After inserting this contribution into eq. (43) and specializing the external momentum to the pion mass shell, we end up with the homogeneous Bethe-Salpeter equation for \( \chi^a \):
\[ e^{-ip_0X^0_1} \chi^a(x_1, y_1; p) \alpha^\beta = \int d\mu(z_1) \int d\mu(z_2) \int d\mu(z_3) \int d\mu(z_4) \times \mathcal{G}_0(x_1, y_1; z_2, z_1) \mathcal{K}(z_1, z_2; z_3, z_4) \chi^a(z_3, z_4; p) e^{-ip_0(z_3^0 - z_2^0)/2}. \] (48)

In order to proceed further, the interaction kernel \( K \) has to be specified. The standard ladder approximation to the Bethe-Salpeter equation is consistent with the Hartree-Fock approximation for the quark propagator from the last section. We thus adopt it here and choose \( K \) accordingly to contain just the tree-level contact interaction of the NJL model:
\[ K(z_1, z_2; z_3, z_4) = 2ig(\Gamma^A_{\alpha\beta} \Gamma^A_{\gamma\delta} - \Gamma^A_{\alpha\delta} \Gamma^A_{\gamma\beta}) \delta^4(z_1, z_2) \delta^4(z_2, z_3) \delta^4(z_3, z_4). \] (49)

As a consequence, eq. (48) reduces (from now on we use matrix notation in flavor, color and spinor space) to
In flat space this equation has been solved analytically by Nambu and Jona-Lasinio in their original paper [29]. The analytical solution owes its existence to the zero range of the interaction, and an analogous solution can thus be expected in the curved unit cell.

We will now derive this solution explicitly from an infinitesimal chiral transformation of the quark propagator. Writing the chiral generators as $T_5^a = \frac{i}{2} \gamma_5$ and using the explicit form of the self-energy (25) with the self-consistent Hartree-Fock propagator, we find

$$\{\Sigma(z), T_5^a\} = -2i g [\Gamma_A \text{tr} \{\Gamma_A \{S(z, z), T_5^a\}\} - \Gamma_A \{S(z, z), T_5^a\} \Gamma_A \} S(z, y),$$

(51)

This transformation behavior follows directly from the chiral invariance of the NJL interaction. (Note that the two coincidence limits $t \to \pm \infty$ of $S$ differ only in the vector part (cf. eq. (26)). Since the latter drops out of eq. (51), their further distinction becomes unnecessary.) Now we can write

$$\{S(x, y), T_5^a\} = \int d\mu(u) \int d\mu(v) S(x, u) \{S^{-1}(u, v), T_5^a\} S(v, y)$$

$$= -\int d\mu(z) S(x, z) \{\Sigma(z), T_5^a\} S(z, y)$$

$$= 2i g \int d\mu(z) S(x, z) [\Gamma_A \text{tr} \{\Gamma_A \{S(z, z), T_5^a\}\} - \Gamma_A \{S(z, z), T_5^a\} \Gamma_A \} S(z, y),$$

(52)

which has exactly the form of the Bethe-Salpeter equation (50) for zero-momentum pions in the chiral limit, with the solution

$$\chi^a(x, y) = \mathcal{N} \{S(x, y), T_5^a\} = 2 \mathcal{N} \Sigma_s \int d\mu(z) S(x, z) T_5^a S(z, y)$$

(53)

for the pion wave function. The normalization $\mathcal{N}$ will be fixed in the next section from the chiral Ward-Takahashi identity. Equation (53) remains a solution of the Bethe-Salpeter equation for a small, finite pion mass $m_\pi$, up to corrections of order $m_\pi^2$.

As the longest-wavelength excitations of the vacuum, the pions and their Bethe-Salpeter amplitude can be expected to depend strongly on the vacuum structure and its variations.
with density. In the next section we will study this issue further by calculating the pion decay constant and its density dependence, building on the results derived above.

VI. THE PION DECAY CONSTANT

A second key parameter in the discussion of spontaneous chiral symmetry breaking, besides the quark condensate, is the decay constant of the pion, $f_\pi$. The explicit expression for the Bethe-Salpeter amplitude from the last section allows us to adapt the standard method of Nambu and Jona-Lasinio to calculate $f_\pi$ in the curved unit cell. We start from its definition in terms of the axial current matrix element between the one-pion state and the vacuum,

$$<0|j_{5,\mu}^a(x)\pi^b(p)> = -f_\pi \delta^{ab} \partial_\mu \eta_p(x).$$  \hspace{1cm} (54)

The space-time dependence of eq. (54) follows entirely from the spatial $SO(4)$ symmetry of the hypersphere and from time translational invariance. It is contained in the eigenmodes $\eta$ of the Klein-Gordon equation on $S^3 \times R$, which generalize the usual plane waves of Minkowski space and are explicitly given and normalized in appendix D, where also the set of generalized momenta $p$ is defined.

By covariantly differentiating eq. (54) one obtains the hypersphere equivalent of the PCAC relation between the vacuum and one-pion states,

$$<0|\nabla_\mu j_{5,\mu}^a(x)\pi^b(p)> = -f_\pi \delta^{ab} g^{-\frac{1}{2}} \partial_\mu g^{\frac{1}{2}} g_{\mu\nu} \partial^\nu \eta_p(x) = f_\pi m_\pi^2 \delta^{ab} \eta_p(x).$$  \hspace{1cm} (55)

In the derivation of eq. (55) we used the free Klein-Gordon equation (D2) and the standard definition of the covariant derivative of a vector field,

$$\nabla_\mu A_\nu(x) \equiv \partial_\mu A_\nu(x) + \Gamma_\nu^{\mu\rho} A_\rho(x),$$  \hspace{1cm} (56)

which contains the Christoffel connection introduced in eq. (9). The relation to the Laplace-Beltrami operator is immediately established with the help of the identity $\Gamma_\nu^{\mu\nu} = g^{-\frac{1}{2}}(\partial_\mu g^\frac{1}{2})$. We further introduce the axial-vector and pseudoscalar current densities of the NJL model,
\[ j_5^{a,\mu}(x) = \bar{q}(x)\gamma_\mu\gamma_5\frac{\tau^a}{2}q(x), \quad j_5^a(x) = \bar{q}(x)i\gamma_5\frac{\tau^a}{2}q(x), \quad (57) \]

which are connected by the divergence of the axial current,

\[ \nabla^\mu j_5^{a,\mu}(x) = 2m_0 j_5^a(x). \quad (58) \]

The relation between the pion decay constant and the Bethe-Salpeter amplitude, eq. (45), is established through the axial current matrix element (54), which can be expressed in terms of the coincidence limit of \( \chi^a \):

\[ <0|j_5^{a,\mu}(x)|\pi^b(p)> = -\frac{e^{-i\omega_p\chi^0}}{\sqrt{2\omega_pV}}\text{tr}[\gamma_\mu\gamma_5\frac{\tau^a}{2}\chi^b(x,x;p)]. \quad (59) \]

(The trace is over Dirac, flavor and color indices.)

Before calculating \( f_\pi \) from this relation, we have to fix the normalization of the pion wave function. To this end we consider the quark-antiquark three-point functions of the current densities (57),

\[ G^{a,\mu}_5(z,x,y) = <0|Tj_5^{a,\mu}(z)q(x)\bar{q}(y)|0>, \quad (60) \]

\[ G^a_5(z,x,y) = <0|Tj_5^a(z)q(x)\bar{q}(y)|0>, \quad (61) \]

and derive with the help of eq. (58) the chiral Ward-Takahashi identity

\[ \nabla_{(z)}^\mu G^{a,\mu}_5(z,x,y) = 2m_0 G^a_5(z,x,y) - T^a_5 S(z,y)\delta^4(x,z) - S(x,z)\delta^4(z,y) T^a_5. \quad (62) \]

Integrating over \( z \), the surface term on the left-hand side vanishes (recall that the pions have a finite mass) and we obtain

\[ 2m_0 \int d\mu(z) G^a_5(z,x,y) = \{S(x,y), T^a_5\}. \quad (63) \]

The left-hand side receives contributions from one-pion intermediate states, which can be expressed in terms of the Bethe-Salpeter amplitude as

\[ 2m_0 \int d\mu(z) G^a_5(z,x,y) = -if_\pi\chi^a(x,y), \quad (64) \]
and, comparing eq. (53) with eqs. (63) and (64), we read off the normalization constant

$$N = \frac{i}{f_\pi}. \quad (65)$$

With the normalization fixed, we can now equate the time components of the two expressions for the axial current matrix element, eqs. (54) and (59), in the pion rest frame $p = \{0, 0, 0\}$ with (cf. appendix D)

$$\eta_{(0,0,0)}(x) = e^{-im_\pi t} \sqrt{2m_\pi V}, \quad (66)$$

to obtain an equation for the pion decay constant in terms of the Bethe-Salpeter amplitude:

$$im_\pi f_\pi = -\text{tr} \left[ \gamma_0 \gamma_5 \frac{\tau^3}{2} \chi^3(x, x; p = m_\pi) \right]. \quad (67)$$

After inserting the explicit solution in the pion rest frame, eq. (53), we have

$$im_\pi f_\pi^2 = \Sigma_s \int d\mu(x) e^{-im_\pi t} \text{tr} \left[ \gamma_0 \gamma_5 \frac{\tau^3}{2} S(0, x) \gamma_5 \tau^3 S(x, 0) \right]. \quad (68)$$

We now use the Hartree-Fock Dirac propagator on $S^3 \times R$ from section III to evaluate the right hand side further. The color and flavor traces in eq. (68) produce a trivial factor $n_c n_f$ and the complete trace (over color, flavor and Dirac indices) becomes

$$\text{tr}[\gamma_0 \gamma_5 \frac{\tau^3}{2} S(0, x) \gamma_5 \tau^3 S(x, 0)] = 2n_c n_f \left[ S_0(\mu, -t)S_2(\mu, t) - S_0(\mu, t)S_2(\mu, -t) \right]$$

$$= -4n_c n_f S_0(\mu, t)S_2(\mu, t) = \frac{2i}{m} \frac{\partial}{\partial t} S_2^2(\mu, t) \quad (69)$$

($m$ is the dynamical quark mass), where we made use of the time-reversal properties of the propagator functions,

$$S_0(\mu, -t) = -S_0(\mu, t), \quad S_2(\mu, -t) = S_2(\mu, t), \quad (70)$$

and of the identity

$$S_0(\mu, t) = \frac{-i}{m} \frac{\partial}{\partial t} S_2(\mu, t), \quad (71)$$

which follows directly from the explicit expressions for $S_0$ and $S_2$ in appendix B. After performing the two trivial angular integrations in the measure eq. (12), and a partial integration in the time coordinate, the equation for the decay constant simplifies to
\[ f^2_\pi = -4i\pi n_c n_f L^3 \int_{-\infty}^{\infty} dt e^{-im_\pi t} \int_0^{2\pi} d\mu \sin^2 \mu \, S^2_2(\mu, t). \]  

(72)

We now use the spectral representation of the propagator function \( S_2 \), eq. (C2), to write

\[
\int_{-\infty}^{\infty} dt e^{-im_\pi t} \int_0^{2\pi} d\mu \sin^2 \mu \, S^2_2(\mu, t) = \left( \frac{im}{4V} \right)^2 \sum_{n,m=1}^{\infty} \int_0^{2\pi} d\mu \, s_n(\mu) s_m(\mu) \int_{-\infty}^{\infty} dt \frac{e^{-(i\omega_n + i\omega_m + \epsilon)|t|}}{\omega_n \omega_m},
\]

(73)

where

\[ s_n(\mu) = 2k_n L \sin(k_n L \mu) - \cos(k_n L \mu) \tan \frac{\mu}{2}. \]  

(75)

The integral over \( \mu \) can be done explicitly,

\[ \int_0^{2\pi} d\mu \, s_n(\mu) s_m(\mu) = 2\pi \delta_{nm} D_n, \]

(76)

with the quark level degeneracies \( D_n \) given in eq. (29). It remains to combine the above equations, to regularize the remaining mode sum consistently with the gap equation (by using the same regulator, eq. (27)) and to introduce the filling factors of section IV to account for the valence quarks. Our final expression for the pion decay constant then becomes

\[ f^2_\pi = \frac{3m^2}{2V} \sum_{n=1}^{\infty} \frac{D_n(1 - f_n)}{\omega_n^3} \Theta(\Lambda - k_n). \]  

(77)

A comparison of this result with the flat-space expression in ref. [4] shows similarities analogous to those between the expressions for the quark condensate. In particular, \( f_\pi \) gets contributions from the quark Dirac sea with the same functional dependence as in [4]. The main difference to the flat-space result is again due to the new quark spectrum, which replaces the plane wave spectrum of euclidean space.

We will continue the discussion of eq. (77) and its density dependence in section VIII. There we will also present results from the numerical evaluation and a comparison with the corresponding results of the chemical-potential approach.
VII. THE ZERO-DENSITY LIMIT

At low baryon densities the volume of the unit cell becomes large and its curvature goes to zero. In the low-density limit, or equivalently in the large-$L$ limit, one therefore expects to recover the results of the flat-space NJL model. In the present section we will perform this limit explicitly, both as a consistency check on our results and to illustrate how the quark spectrum and its degeneracies turn into the plane wave form.

Let us start by considering the $L \to \infty$ limit of the scalar self-energy, eq. (35). In this limit the Fermi momentum goes to zero and contributions from quark propagation around the whole sphere are suppressed. Therefore it is convenient to use the representation of the propagator in terms of a sum over paths. Up to contributions from complete circles around the hypersphere we obtain

\[
\Sigma_s = -\frac{g m}{4 \pi^2 L^2} (1 + 2 n_c n_f) \int_{k_F}^{\infty} dk \frac{1 - 4 k^2 L^2}{\omega} \Theta(\Lambda - k)
\]

\[
\to \frac{g m}{\pi^2} (1 + 2 n_c n_f) \int_{0}^{\infty} dk \frac{k^2}{\omega} \Theta(\Lambda - k),
\]

which is indeed the original flat space result \[29\]. Similarly, the vector part of the self-energy becomes

\[
\Sigma_v = \frac{g}{2 \pi^2 L^2} \int_{0}^{k_F} dk (1 - 4 k^2 L^2)
\]

\[
\to 0,
\]

and goes, as expected, to zero in the large-$L$ limit.

To check the zero-density limit of $f_\pi^2$ is a bit more subtle, since the corresponding loop contains two quark propagators instead of the coincidence limit of only one. We start from eq. (72) and use the representation (B23) for the propagator function $S_2$. Neglecting again additional complete turns around the hypersphere, we have

\textsuperscript{10}Recall that the Fermi momentum sets the baryon number in the cell to one.
\[ f_\pi^2 = -4i\pi n_c n_f (\alpha m)^2 L^3 \int_0^\infty dt \int_0^{2\pi} d\mu [2I_0'(\mu, t) + \tan \frac{\mu}{2} I_0(\mu, t)]^2 \]

\[ = -\frac{in_c n_f m^2}{4\pi L} \int \frac{dk_0}{2\pi} \int_\Lambda^\Lambda \frac{dk}{2\pi} \int_\Lambda^\Lambda \frac{dq}{2\pi} \int_0^{2\pi} d\mu e^{iL(k+q)\mu} \left[ 2i k L + \tan \frac{\mu}{2} \right] \left[ 2i q L + \tan \frac{\mu}{2} \right] \] (82)

In the second line the time integration, which leads to a delta function \( \delta(k_0 - q_0) \), and the consecutive \( q_0 \) integration are already performed. Note that the range of the \( \mu \) integration in the above expression can be changed to \([-2\pi, 0]\) without changing the integral, if we simultaneously change the signs of the integration variables \( k \) and \( q \). We can thus replace half of the \( \mu \)-integral above by an integral with the alternative, equivalent range. After furthermore scaling \( \mu \) to \( \tilde{\mu} = \mu L \), the \( L \)-dependent part of the above expression becomes

\[ \frac{1}{2L} \int_\Lambda^\Lambda \frac{dk}{2\pi} \int_\Lambda^\Lambda \frac{dq}{2\pi} \frac{d\mu}{2\pi} e^{i(k+q)\tilde{\mu}} \left[ 2i k L + \tan \frac{\tilde{\mu}}{2} \right] \left[ 2i q L + \tan \frac{\tilde{\mu}}{2} \right], \] (83)

which, in the large-\( L \) limit, reduces to

\[ \int_\Lambda^\Lambda \frac{dk}{2\pi} \int_\Lambda^\Lambda \frac{dq}{2\pi} \frac{d\pi}{2\pi} \delta(k + q)[-4kqL^2]. \] (84)

Inserting this in the expression for \( f_\pi^2 \) above, we finally get

\[ f_\pi^2 \rightarrow -in_c n_f m^2 \int_\Lambda^\Lambda \frac{dk}{2\pi} k^2 \int \frac{dk_0}{2\pi} \frac{1}{[k_0^2 - k^2 - m^2 + i\epsilon][k_0^2 - k^2 - m^2 + i\epsilon]} \]

\[ = \frac{n_c n_f m^2}{4\pi^2} \int_0^\Lambda \frac{dk}{k^2} \frac{1}{\omega^3}, \] (85)

(\( \omega = \sqrt{k^2 + m^2} \)), which agrees with the flat-space result [29].

The large-\( L \) limit thus reproduces, as expected, the standard zero-density quantities of the NJL model. The transition of the physics from the high- to the low-density regime can be traced to the differences in the quark propagation in small and large hyperspheres. At high densities geodesic paths with any number of turns around the sphere contribute and their sum builds up to the equivalent sums over the quark spectrum on \( S^3 \times R \) (cf. appendix C). In the large-\( L \) limit, on the other hand, only the direct path is relevant and yields the standard flat-space propagator.

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Before embarking on the quantitative discussion of our results, we fix the coupling $g$ and the cutoff $\Lambda$ of the NJL model at $g = 4.08$ GeV$^{-2}$ and $\Lambda = 700$ MeV. These values have been found in ref. [42] to best reproduce the phenomenological values of the quark condensate and of the pion decay constant in the vacuum. We adopt this choice to allow for a direct comparison between our results and those of ref. [4], and to ensure a phenomenologically acceptable zero-density limit of both the quark condensate and of $f_\pi$. Nevertheless, given the simplicity of the minimal NJL model and the sensitivity to the values of $\Lambda$ and $g$ already noted in ref. [4], our numerical results should be considered as qualitative.

The above model parameters could in principle be density dependent. Since we are mainly interested in new, qualitative features of the hypersphere formulation, however, we will follow the standard treatment and keep them constant. Finally, the diffuseness parameter in the regulator, eq. (27), is set to $\epsilon = 2.4$. This choice compromises between a rapid convergence of the mode sums and a sufficiently smooth behavior of the observables at high densities.

We are now ready to discuss our numerical results. The non-trivial solution of the gap equation (37), i.e. the dynamical quark mass $m$, and the negative cube root of the corresponding quark condensate, $-<\bar{q}q>^{1/3}$, are shown as a function of density in fig. 1. Both quantities decrease monotonically with density and finally go to zero at the density $\rho_c$.

The vanishing quark condensate does of course not necessarily imply a second-order phase transition to chiral restoration. Recall that the condensate shown in fig.1 was calculated by using the nontrivial solution of the gap equation in the quark propagator. It is not a priori clear, however, that this solution remains physical, i.e. the one of lowest free energy, if the baryon density increases.

Indeed, it was claimed in ref. [43] that in a certain parameter range of the standard NJL model, close to the chiral limit, the trivial solution becomes the global minimum of the free energy before the nontrivial solution reaches zero. This result, obtained in the chemical-
potential approach, would imply a first order phase transition to chiral restoration. However, studies in more realistic versions of the NJL model found a second-order transition for the whole phenomenologically acceptable parameter range [44].

To determine the order of the phase transition in our approach, we monitor the extrema of the free energy density

$$\Omega(V, m) = \frac{gn_f}{n_c}(1 + 2n_cn_f) <\bar{q}q>^2 - \frac{gn_f}{n_c} <q^\dagger q>^2 - \frac{n_cn_f}{V} \sum_{1}^{\infty} D_n(1 - f_n) \omega_n \Theta(\Lambda - k_n)$$

as a function of the cell volume or, equivalently, of the baryon density. The last term in eq. (86) is just the regulated sum over the energies of the occupied quark states, while the first two terms are required to avoid double-counting of the vacuum energy [45]. It is straightforward to check that the extrema of eq. (86) under variations of the quark mass are, as expected, the solutions of the gap equation (37).

The dependence of $\Omega/V$ on the quark mass is plotted in fig. 2 for four baryon densities between $\rho = 0$ and $\rho = \rho_c$. These curves demonstrate that only two solutions of the gap equation exist at all densities up to $\rho_c$: the trivial one at $m = 0$ and the nontrivial one with finite $m$. In particular, the nontrivial (trivial) solution remains the minimum (maximum) of the free energy density up to $\rho_c$, where both extrema merge. The phase transition is therefore of second order.

Closer inspection of the free energy density reveals that this result is, in contrast to the chemical potential approach, independent of the model parameters. Indeed, it is straightforward to show that, as long as the nontrivial solution exists, it remains the minimum of the free energy density. This robust second-order phase transition, which actually persists for a much larger class of NJL models, is a welcome result, since the specific choice of the lagrangian is somewhat arbitrary. It also insures that a potential density dependence of the model parameters would not affect the order of the phase transition.

For comparison, the quark condensate as given in ref. [4], i.e. calculated on the basis of the nontrivial solution in the chemical-potential approach, is also shown in fig. 1. There may exist parameter regions in which the transition of ref. [4] becomes actually first order.
in the chiral limit. Introducing a current quark mass would in this case restore the second order transition without significant effects on the curve and the approximate critical density. In order to compare the density dependence and the critical densities of the second-order transitions, we therefore reproduce the original plots of ref. [4] in fig. 1.

Both approaches yield a very similar behavior of the order parameter with density. This is reminiscent of the analogous situation in the Skyrme model, where close similarities between hypersphere and array results first demonstrated the use of hypersphere calculations.

In fig. 3 we plot the density dependence of the pion decay constant and again, for comparison, the same quantity as obtained in the chemical potential approach. $f_\pi$ decreases with density and goes to zero at the chiral restoration density. This is expected, since the pion has to decouple when the chiral condensate disappears. In the hyperspherical cell $f_\pi$ decreases initially somewhat faster then in the chemical potential approach, but goes to zero at almost the same critical density.

Figure 4 shows the density dependence of both the quark condensate and the pion decay constant in the absence of valence quarks in the cell. This situation corresponds to setting $k_F = 0$ or, equivalently, to setting all $f_n = 0$. Even in this case both the condensate and the decay constant decrease and eventually vanish. This happens, however, at a considerably higher density. The absence of baryon number sources inside the cell delays the transition to chiral restoration, as one would expect.

The valence-quark free cell can be interpreted as a region of reduced baryon density in the nuclear medium. While the interaction with the surrounding matter is still mediated by the shape and curvature of the $B = 0$ cell, it is felt only by the quark vacuum in the interior. Such $B = 0$ cavities can therefore neither be studied in the chemical-potential approach to the NJL model nor in the Skyrme model. In the latter, a unit of baryon number in every cell is crucial for the chirally restored phase to exist, at least in semiclassical approximation. In the standard approach to the NJL model at finite density, on the other hand, a homogeneous baryon density distribution is built in from the beginning.

Due to the fixed relation between the ambient baryon density and the radius of all cells,
however, the size of the $B = 0$ cavity cannot be varied at a given density. Such baryon-density free bubbles with delayed chiral restoration will therefore very likely not describe an equilibrium situation, but they might be related to a transient bubble phase in heavy-ion reactions, where the chirally broken and restored phases coexist.
IX. SUMMARY AND CONCLUSIONS

In this paper we have generalized a new approach to dense nuclear matter in the framework of extended hadron models. Originally developed for the Skyrme model, it divides the nuclear medium into cells of unit baryon number, which have the form of a 3-dimensional hypersphere \( S^3 \). The new and crucial feature of these cells is their intrinsic curvature, which mediates interactions with the ambient matter.

In order to explore the use of this approach beyond the Skyrme model, we first extend it to quark-based baryon models in general and then apply it specifically to the Nambu-Jona-Lasinio model, by investigating vacuum, constituent-quark and pion properties as well as their density dependence. The presence of fermions in the curved cell requires the introduction of additional concepts, under which a new, isoscalar gauge interaction for the quarks is the most important.

We find a high-density phase transition, which restores the chiral symmetry of the vacuum by decondensing quark-antiquark pairs. At the same critical density the pions decouple, as signaled by the vanishing of the pion decay constant, which monotonically decreases with density. The phase transition is, for a large class of generalized NJL models, robustly of second order and independent of the model parameters. Furthermore, delayed chiral restoration takes place even inside of valence–quark–free cavities in the dense medium. In this case it is solely driven by the interaction of the quark vacuum with the cell curvature.

Remarkably, we find close similarities between the density dependence of our results and those of the conventional chemical-potential approach. The behavior of the constituent quark mass and of the quark condensate with density, for example, is (in the phenomenologically reasonable NJL parameter range) in both cases almost identical. This similarity may seem at first surprising, since the two approaches are based on rather different descriptions of the dense environment. At least qualitatively, however, it is easy to understand. Recall that the energy gain from the Dirac sea, filled with massive as compared to massless quarks, leads at zero baryon density and for sufficiently large coupling to the development of a nontrivial
vacuum. In the chemical potential approach, this energy gain gets with increasing density more and more compensated by the occupied states in the Fermi sea, since they count with opposite sign in the spectral sum of the gap equation. At the critical density, it becomes energetically favorable for the quarks to be massless, and the chiral phase transition occurs.

On the hypersphere the mechanism is different in principle, but similar in effect. The occupation numbers of the valence quark levels are here density-independent, but the spectrum itself changes in a characteristic way. Due to the stronger localization in the smaller high-density cells, the quark momenta grow inversely with the cell size. This leads to a reduced number of states in the regularized Dirac sea, with the same consequences as in the chemical potential approach.

The overall increase of the quark momenta at larger densities is an effect common to all compact unit cells. The specific and detailed change of the quark spectrum, however, and therefore the quantitative density dependence of the results, is governed by the particular form of the unit cell. The close agreement with the chemical-potential approach thus supports the choice of $S^3$ as the cell geometry. An analogous agreement between results of the conventional and the hypersphere approach was observed in the Skyrme model.

Experience gained from hypersphere calculations in different models can help to disentangle features specific to the Skyrme model from more general or even model-independent aspects of the hypersphere approach. In this respect our results indicate, for example, that the presence of a soliton is not required for the hypersphere approach to be effective. In particular, the specific topological properties of the Skyrme soliton and its hedgehog form are not indispensable. Since the winding number of the skyrmion plays an important role in its hypersphere behavior, this was not a priori obvious. Furthermore, a non-zero baryon number (i.e. winding number) in the cell and the corresponding hedgehog structure are necessary requirements for chiral restoration in the Skyrme model, whereas this condition is suspended in the NJL model, as the results in the $B = 0$ cell show. The NJL results also indicate that the hypersphere approach is compatible with different mechanisms for spontaneous chiral symmetry breaking.
The application of the hypersphere approach has often practical advantages, which may, however, vary in different models. In the Skyrme model, for example, $S^3$ calculations require drastically less numerical effort than conventional unit-cell calculations. While this particular benefit does not translate to the NJL case (since the numerical requirements of the conventional (chemical–potential) approach are already moderate), the hypersphere approach to the NJL model can be applied in realms not easily accessible by other methods. The discussed low-density bubbles in nuclear matter with delayed chiral restoration can e.g. neither be studied in the Skyrme model nor in the chemical-potential approach to the NJL model. Another profitable application of the hypersphere approach to the NJL model is encountered in the study of nuclear matter at finite temperature. While the Skyrme model calculation meets with rather involved technical difficulties in this situation even on the hypersphere, it can be straightforwardly addressed in the NJL case [50].

We regard the above results as promising indications for the hypersphere approach to be useful beyond the Skyrme model. Nevertheless, many important and interesting questions, also on a more fundamental level, remain open. It would, for example, be useful to know if hyperspherical and flat unit cells can be explicitly related to each other, at least approximately. A hint in this direction comes from the work of ref. [51], where matter at a fixed baryon density is almost identically described in hyperspherical cells with both baryon number and volume doubled, as a first step towards the limit of a skyrmion matter configuration in flat space. One might also hope to make contact with older speculations [52] about a description of chiral dynamics in terms of space curvature.

The theoretical framework developed in the present paper can straightforwardly be adapted to other chirally symmetric quark and hybrid models, as for example to bag models [53] and to non-topological soliton models [54]. It might also be interesting to study the relations between the Skyrme and NJL models, which can be established by bosonization of the latter [34, 13], in the hypersphere formulation. Finally, our work gives an explicit example for symmetry breaking and restoration due to space-time curvature in a fermionic theory, and as such may be also useful in other areas of physics. Related questions are, for
example, presently discussed in a cosmological context [55].

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APPENDIX A: MAURER-CARTAN BASIS AND DIRAC SPECTRUM ON $S^3 \times R$

The generalization of Lorentz (Poincaré) tensor fields to curved space-times is straightforward. One simply replaces the Lorentz group by the group of general coordinate transformations, which becomes locally (i.e. in the tangent spaces of the manifold) the full linear group $GL(R, 4)$. The generalized tensor fields form its representations.

The situation is more involved for fields of half-integer spin, because the linear group has no double-valued (i.e. spinor) representations. For this reason, spinors have to be introduced with respect to a local, orthonormal basis, the vierbein $e_a$ (in the (co)tangent space of space-time). As discussed in the introduction, the metric and therefore the physics is independent of the arbitrary orientation of these frames, which can be locally changed by a Lorentz gauge group. The fermion fields form the spinor representations of this local Lorentz group, which acts in its defining representation on the Latin index of the vielbein.

For explicit calculations we have to choose an appropriate vielbein on $S^3 \times R$, i.e. we have to fix the gauge of the local Lorentz group. The spatial hypersphere has the geometry of the group manifold of $SU(2)$, scaled to radius $L$. This suggests the rescaled Maurer-Cartan vector fields of $SU(2)$ as the natural choice for the spatial components of the vielbein:

$$e_a = \frac{L}{2i} \mathrm{Tr} [\tau_a (\partial_i h) h^{-1}] \ g^{ij} \partial_j \quad (a, i = 1, 2, 3),$$

(A1)

where we parametrized the elements of $SU(2)$ in polar coordinates as

$$h(x) = \cos \mu + i \tau_a \hat{r}^a(\theta, \phi) \sin \mu.$$

(A2)

11This local definition of the spinors can in general not be consistently extended to the manifold as a whole. $S^3 \times R$, however, as a group manifold, is parallelizable. In technical terms this implies the vanishing of its first two Stiefel-Whitney classes, which guarantees the existence of a global spin structure.

12To be specific, we use the right-invariant Maurer-Cartan fields. This is purely a matter of convention.
\( \hat{\mathbf{r}}^a(\theta, \phi) \) is the unit vector in \( R^3 \). Complemented by the time component \( e_o = \partial_t \) we obtain an orthonormal vierbein on \( S^3(L) \times R \), from which the metric eq. \( (I) \) can be recovered with the help of eq. \( (3) \). The simplest way to derive the corresponding Levi-Civita spin connection is to solve Cartan’s first structure equation \( [33] \) in the same “Maurer-Cartan gauge”. The result is

\[
\omega_a = \frac{i}{4L} \epsilon_{abc} \sigma^c \quad (a, b, c \in \{1, 2, 3\}) \quad \text{and} \quad \omega_o = 0. \quad (A3)
\]

(\( \sigma_{ab} = \frac{i}{2} [\gamma_a, \gamma_b] \)) Note that the \( \omega_a \) are space-time independent and essentially given by the structure constants of \( SU(2) \), which is the main advantage of using the Maurer-Cartan vielbein.

The Dirac operator on \( S^3 \times R \), which appears in the free part of the NJL lagrangian, eq. \( (14) \), and to which we add a constant vector self-energy (we do not include the scalar part of the self-energy, since it will later be absorbed in the mass term), is

\[
i\mathcal{D} = [i\gamma^a (e_a + \omega_a) + \Sigma v \gamma_0]
\]

and now takes the simple form \( (A4) \)

\[
i\mathcal{D} = \begin{pmatrix}
0 & \Delta_+ \\
\Delta_- & 0
\end{pmatrix},
\]

\( (A5) \)

\( ^{13} \)A fixed vierbein does in general not determine the corresponding spin connection completely. We take our cell to be torsion-free, however, and further require the spin connection to be compatible with the metric (metricity condition), so that it is uniquely given by the first Cartan structure equation.

\( ^{14} \)We use the chiral representation of the \( \gamma \)-matrices:

\[
\gamma^0 \equiv \beta = \begin{pmatrix}
0 & -I \\
-I & 0
\end{pmatrix}, \quad \bar{\gamma} = \begin{pmatrix}
0 & \bar{\sigma} \\
-\bar{\sigma} & 0
\end{pmatrix}, \quad \bar{\alpha} = \begin{pmatrix}
\bar{\sigma} & 0 \\
0 & -\bar{\sigma}
\end{pmatrix}, \quad \gamma_5 = \begin{pmatrix}
I & 0 \\
0 & -I
\end{pmatrix}
\]
in terms of the operators

\[ \Delta_\pm = -i \partial_t - \Sigma_v \pm (i \vec{\sigma} \cdot \vec{e} + \frac{3}{2L}). \]  
(A6)

(Arrows indicate vectors in the spatial, euclidean subspace of the local vielbein.)

For the interpretation of our results it will be useful to derive the energy spectrum of
the corresponding Dirac equation. Let us therefore look for the static solutions

\[ \psi_n(\mathbf{x}, t) = \psi_n(\mathbf{x}) e^{-i \tilde{\omega}_n t} \]  
(A7)
of the Dirac equation

\[ (i \gamma \cdot D - m) \psi_n = \begin{pmatrix} -m & \Delta_+ \\ \Delta_- & -m \end{pmatrix} \begin{pmatrix} \psi^{(1)} \\ \psi^{(2)} \end{pmatrix} = 0, \]  
(A8)

which has the equivalent Hamiltonian form

\[ [-i \vec{\alpha} (\vec{e} - \vec{\omega}) + \beta m] \psi_n(\mathbf{x}) = \omega_n \psi_n(\mathbf{x}), \]  
(A9)

where \( \omega_n \equiv \tilde{\omega}_n + \Sigma_v \). (The constant vector self-energy just shifts the origin of the energy scale.) Now we can again exploit the properties of \( S^3(1) \) as the group manifold of \( SU(2) \). The invariant vector fields of \( S^3(L) \) are (as for every Lie group) simply the group generators \( \hat{L}_a \), rescaled to radius \( L \):

\[ e_a = \frac{2}{iL} \hat{L}_a, \]  
(A10)

where the \( \hat{L}_a \) satisfy the usual angular momentum commutation relations

\[ [\hat{L}_a, \hat{L}_b] = i \epsilon_{abc} \hat{L}_c. \]  
(A11)

The appearance of a “spin-orbit” term in

\[ \Delta_\pm \psi^a = \{-\tilde{\omega} - \Sigma_v \pm \frac{1}{L} (4 \vec{S} \cdot \vec{\bar{L}} + \frac{3}{2})\} \psi^a \quad a = 1, 2 \]  
(A12)

(\( \vec{S} \) is the spin operator \( \vec{\sigma}/2 \)) then suggests to diagonalize \( \Delta_\pm \) by taking \( \psi^a \) in the coupled basis with \( \vec{J} = \vec{L} + \vec{S} \), so that
\[
\bar{S}L \psi^a_{j,m} = \frac{1}{2} [j(j+1) - l(l+1) - \frac{3}{4}] \psi^a_{j,m}
\]

\[
= \begin{cases}
\frac{1}{2} l \psi^a_{j,m} & \text{for } j = l + \frac{1}{2}, \quad l = \{0, 1, 2, \ldots\} \\
-\frac{1}{2} (l+1) \psi^a_{j,m} & \text{for } j = l - \frac{1}{2}, \quad l = \{1, 2, 3, \ldots\}.
\end{cases}
\] (A13)

Inserting eq. (A13) into the Dirac equation and combining the resulting two coupled equations for the upper and lower spinor components one obtains

\[
\omega_n^2 = m^2 + \frac{1}{L^2} \times \begin{cases}
(2l + \frac{3}{2})^2 & \text{for } j = l + \frac{1}{2} \\
(-2l - \frac{1}{2})^2 & \text{for } j = l - \frac{1}{2}
\end{cases}
\] (A14)

which determines the energy eigenvalues \(\omega_n\). All properties of the spectrum can be read off directly from eq. (A14): each pair of levels with \(j = \frac{n}{2}, l = \frac{n-1}{2}\) and \(j = \frac{n-1}{2}, l = \frac{n}{2}\), \(n = \{1, 2, \ldots\}\) contains

\[
D_n = 2n(n+1)
\] (A15)

degenerate states with momentum \(k_n\) and energy \(\omega_n\),

\[
k_n = \frac{2n + 1}{2L}, \quad \omega_n^2 = k_n^2 + m^2.
\] (A16)

Finally, the total number of states up to the \(N\)th level sums up to

\[
\sum_1^N D_n = \frac{2}{3} N(N^2 + 3N + 2).
\] (A17)

This spectrum agrees with existing derivations in the literature, obtained by various different methods, see e.g. \cite{46} and also \cite{47} for the massless case. Let us finally mention that the existence of a time-like Killing vector \((\epsilon^0 = \partial_t)\) in our cell permits a unique definition of in- and out-states (and their Fock spaces) on the basis of this spectrum.

**APPENDIX B: THE FERMION PROPAGATOR ON \(S^3 \times R\)**

In this appendix we construct the quark propagator \(S(x)\) on \(S^3 \times R\) for a Dirac operator of the form discussed in appendix A, i.e. including a constant self-energy (34). \(S(x)\) is the solution of
\((i\mathcal{D} - m)S(x, y) = \delta^4(x, y), \tag{B1}\)

with \(i\mathcal{D}\) given in eq. (A4). The scalar part of \(\Sigma\) is combined with the current quark mass in the mass term \(m = m_0 + \Sigma_s\), the vector part is absorbed in the covariant derivative and Feynman boundary-conditions are implied. (We will later generalize the boundary conditions to take the presence of valence quarks into account.)

The four-dimensional delta function is generalized to curved space by demanding

\[
\int d\mu(x) \delta^4(x, y) f(x) = f(y), \tag{B2}\]

so that

\[
\delta^4(x, y) \equiv (-g)^{-1/2} \prod_{\mu=0}^{3} \delta(x^\mu - y^\mu), \tag{B3}\]

up to the sum over periodic paths (see below). \((g = -L^6 \sin^4 \mu \sin^2 \theta\) is the determinant of the metric.)

Due to the compactness of the unit cell, the free propagation of a quark between \(x\) and \(y\) can proceed either directly (on a geodesic) or via an arbitrary number of intermediate turns around the hypersphere. We can therefore consider the geodesic coordinates as non-periodic real numbers and count the number of windings of a given path in multiples of \(2\pi\). The propagator then becomes a sum of the propagators for each of these paths. (In order not to complicate the following equations unnecessarily, we will suppress these periodic sums until they play an active role in the derivation.) The above procedure ensures that the full propagator is periodic in the coordinates, as required by the physical situation, and allows to write its Fourier transform in terms of continuous momenta. Although the Fourier integrals can be transformed into equivalent sums over the discrete quark spectrum in the compact space (see appendix C), the former representation turns out to be more convenient for the derivation of the propagator.

To derive the Dirac propagator explicitly, we start from the representation

\[
S(x, y) = -(i\mathcal{D} + m)G(x, y), \tag{B4}\]
which defines the block-diagonal $4 \times 4$ matrix

$$G(x, y) = \begin{pmatrix} \bar{G}(x, y) & 0 \\ 0 & G(x, y) \end{pmatrix}, \quad (B5)$$

a Greens function of the iterated Dirac operator. By inserting eq. (B4) into the defining equation (B1) for $S(x, y)$, we then obtain

$$(-\Delta_+ \Delta_- + m^2) \bar{G}(x, y) = \delta^4(x, y), \quad (B6)$$

with $\Delta_{\pm}$ given in eq. (A6), and

$$\Delta_+ \Delta_- = (i \partial_t + \Sigma_v)^2 + \bar{e}e - \frac{i}{L} \bar{\sigma} \bar{e} - \frac{9}{4L^2}. \quad (B7)$$

(The derivatives act on $x$.) Note that $\bar{e}e$ is the Laplace-Beltrami operator on $S^3$:

$$\bar{e}e = g^{-\frac{1}{2}} \partial_i g^{\frac{1}{2}} g^{ij} \partial_j$$

$$= \frac{\partial^2}{\partial \mu^2} + 2 \cot \mu \frac{\partial}{\partial \mu} + \frac{1}{\sin^2 \mu} \left( \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \mu \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}. \quad (B8)$$

The $2 \times 2$ matrix structure of $\bar{G}(x, y)$ can be expanded in the Pauli-matrix basis,

$$\bar{G}(x, y) = G_1(x, y) + \bar{\sigma} \bar{e} G_2(x, y), \quad (B9)$$

where $G_1, G_2$ are scalar functions.

As in Minkowski space, the required invariance of the vacuum under the symmetries of the metric leads to considerable simplifications in the space-time dependence of the propagator. Due to the symmetry group of the spatial hypersphere, $SO(4)$, free propagation between two points is equivalent to propagation of the same geodesic distance from an arbitrarily fixed pole\[15]\). Taking the pole as the origin of the polar coordinate system on $S^3$,\[15]\)

\[15\]This can be made explicit with the help of the seven Killing vectors of $S^3 \times R$, which generate the isometry group. They replace the ten Killing vectors of the Poincaré group in flat Minkowski space.
the geodesic distance simply becomes $\mu L$. Furthermore, the propagator will depend only on time differences $t$ (the reference time is chosen to be zero), due to the invariance of both metric and dynamics under time translations.

After inserting the decomposition (B9) of $G$ into eq. (B6), we obtain two coupled equations for $G_{1,2}$:

$$
\left( -(i\partial_t + \Sigma v)^2 - \vec{e}\vec{e} + \frac{9}{4L^2} + m^2 \right) G_1(\mu, t) + \frac{i}{L} \vec{e}\vec{e} G_2(\mu, t) = \delta^4(x) \tag{B10}
$$

and

$$
\left( -(i\partial_t + \Sigma v)^2 - \vec{e}\vec{e} + \frac{1}{4L^2} + m^2 \right) G_2(\mu, t) + \frac{i}{L} G_1(\mu, t) = 0. \tag{B11}
$$

(In the above derivation we used the Lie bracket relation $[e_a, e_b] = \frac{2}{L} e_a^c e_c$, see eqs. (A10), (A11).) The remaining $\theta$ and $\phi$ dependence in eq. (B10) is removed by integrating the first one over both angles with their corresponding measure. (Eq. (B11) is already independent of $\theta$ and $\phi$). The remaining explicit $\mu$ dependence can then be absorbed into the functions $\tilde{G}_{1,2}(\mu, t) \equiv \sin \mu G_{1,2}(\mu, t)$ by writing the relevant part of $\vec{e}\vec{e}$ as $L^{-2} \sin^{-1} \mu (1 + \frac{d^2}{d\mu^2}) \sin \mu$ and the $\mu$-dependent part of the delta function as $-\sin^{-1} \mu \delta'(\mu)$ (a prime indicates a derivative with respect to $\mu$). The Fourier transforms of $\tilde{G}_{1,2}$ then satisfy simple algebraic equations,

$$
\left( (k_0 + \Sigma v)^2 - k^2 - m^2 - \frac{5}{4L^2} \right) \tilde{G}_1(k_0, k) + \frac{i}{L} \left( k^2 - \frac{1}{L^2} \right) \tilde{G}_2(k_0, k) = \frac{ik}{2\pi L}
$$

$$
\left( (k_0 + \Sigma v)^2 - k^2 - m^2 + \frac{3}{4L^2} \right) \tilde{G}_2(k_0, k) - \frac{i}{L} \tilde{G}_1(k_0, k) = 0, \tag{B12}
$$

which have the solutions (we introduce the abbreviations $k_\pm = k \pm \frac{1}{2L}$, $\delta = (k_0 + \Sigma v)^2 - m^2$)

$$
\tilde{G}_2(k_0, k) = -\frac{1}{4\pi L} \left( \frac{1}{\delta - k_+^2} - \frac{1}{\delta - k_-^2} \right) \tag{B13}
$$

and

$$
\tilde{G}_1(k_0, k) = -iL \left( \delta - k^2 + \frac{3}{4L^2} \right) \tilde{G}_2(k_0, k). \tag{B14}
$$

After transforming back to space-time and imposing Feynman boundary conditions, we get

$$
\tilde{G}_1(\mu, t) = \frac{1}{4\pi L^2} \left( 2 \cos \frac{\mu}{2} I'(\mu, t) + \sin \frac{\mu}{2} I(\mu, t) \right) e^{i\Sigma_v t} \tag{B15}
$$
\[ \bar{G}_2(\mu, t) = \frac{i}{2\pi L} \sin \frac{\mu}{2} e^{i\Sigma v t} I(\mu, t) \tag{B16} \]

in terms of the integral
\[ I(\mu, t) = \int \frac{dk_0}{2\pi} \int \frac{dk}{2\pi} \frac{e^{(k_0 \mu L-k_0 t)}}{k_0^2 - k^2 - m^2 + i\epsilon}, \tag{B17} \]

which is the scalar Feynman propagator in two-dimensional Minkowski space. The ultraviolet regularization of this integral and the change in the pole structure of the integrand due to valence quarks will be considered in appendix C. The full \((2 \times 2)\) propagator can now be obtained by inserting eqs. (B15) and (B16) (divided by \(\sin \mu\)) into eq. (B9) and carrying out the periodic sum:
\[ \bar{G}(x) = \alpha (\sin \mu)^{-1} e^{i(\vec{\Sigma} \hat{r} + \Sigma v t)} \sum_{n=-\infty}^{+\infty} (-1)^n [2I'_n + \tan \frac{\mu}{2} I_n], \tag{B18} \]

where \(I_n(\mu, t) \equiv I(\mu + 2n\pi, t)\), the prime represents a derivative with respect to \(\mu\) and \(\alpha \equiv 1/4\pi L^2\). It remains to evaluate the expression (B4) for the spinor propagator, using the block anti-diagonal form of the Dirac operator given in eq. (A5). As an intermediate step, we calculate
\[ \Delta_{\pm} \bar{G}(x) = \alpha (\sin \mu)^{-1} e^{i(\vec{\Sigma} \hat{r} + \Sigma v t)} \sum_{n=-\infty}^{+\infty} (-1)^n [2I'_n + \tan \frac{\mu}{2} I_n]
\[
\pm \frac{\vec{\Sigma}}{L} (2I''_n - \cot \frac{\mu}{2} I'_n)], \tag{B19} \]

where the dot represents a time derivative. Inserted into (B4), we obtain the quark propagator explicitly. It can be brought into a familiar form by expanding the Dirac-matrix structure in the \(\gamma\)-matrix basis:
\[ S(x) = e^{i(\vec{\Sigma} \hat{r} + \Sigma v t)} [S_0(\mu, t) \gamma_0 - S_1(\mu, t) \hat{r} \vec{\gamma} - S_2(\mu, t)]. \tag{B20} \]

\((\vec{\Sigma} \) is the Dirac spin matrix with the Pauli matrices on the diagonal and should be distinguished from the self-energy.) The three invariant scalar amplitudes \(S_i\) are defined as

\[ I(\mu, t) \] can be expressed in terms of a Hankel function.
\[
S_0(\mu, t) = -i \alpha (\sin \mu)^{-1} \sum_{n=\infty}^{+\infty} (-1)^n [2I_n' + \tan \frac{\mu}{2} I_n], \\ (B21)
\]
\[
S_1(\mu, t) = i \frac{\alpha}{L} (\sin \mu)^{-1} \sum_{n=\infty}^{+\infty} (-1)^n [2I_n'' - \cot \frac{\mu}{2} I_n], \\ (B22)
\]
\[
S_2(\mu, t) = m \alpha (\sin \mu)^{-1} \sum_{n=\infty}^{+\infty} (-1)^n [2I_n' + \tan \frac{\mu}{2} I_n]. \\ (B23)
\]

Eqs. (B20) – (B23) are the central result of this appendix. The closed, analytical form of the fermion propagator on \( S^3 \times \mathbb{R} \) may be useful also beyond the context of the present paper.

The sums over all geodesic paths connecting the pole with \( x \) ensure, as mentioned above, the required spatial periodicity of the propagator. This can now be easily checked explicitly. The \( I_n(\mu, t) \) and their derivatives, appearing in eqs. (B21) – (B23), satisfy
\[
I_n(\mu + 2\pi, t) = I_{n+1}(\mu, t). \\ (B24)
\]

Inserted into the expression for the quark propagator, the expected periodicity in \( \mu \) follows immediately:
\[
S(\mu + 2\pi, \theta, \phi, t) = (-1)^n e^{i\Sigma \hat{r}} S(\mu, \theta, \phi, t) = S(\mu, \theta, \phi, t). \\ (B25)
\]

**APPENDIX C: SPECTRAL REPRESENTATION AND COINCIDENCE LIMITS**

The sums over geodesic paths in the propagator functions (B21) – (B23) can be rewritten alternatively as sums over the spectrum of the Dirac operator. For many applications the spectral representation, which we will now derive, is more transparent and convenient. In numerical calculations it is particularly useful at high densities, where the cell is small and paths which circle many times around the sphere contribute significantly.

The \( n \) dependence of the integrals \( I_n \), defined after eq. (B18), and their derivatives resides exclusively in the factors \((-1)^n \exp(2i\pi nkL)\). Exchanging the order of the sums over \( n \) and the integrals over \( k \) and using the identity

\[
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\]
\[
I_n(\mu + 2\pi, t) = I_{n+1}(\mu, t).
\]
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we can replace the sums over geodesic paths in the propagator by mode sums over the discrete quark momenta, eq. (A10). After performing the now trivial $k$ integrals this leads to the spectral representation. Recall that the integrals over continuous $k$ in the compact cell appeared since we took $\mu$ effectively in the interval $\mu \in [0, \infty]$, which allowed the use of continuous Fourier methods in the derivation of the propagator. Consequently, we had to add contributions from any number of (physically indistinguishable) geodesic circles. The two equivalent ways of representing the compactness of the unit cell are simply related by eq. (C1).

For the propagator functions $S_0$ and $S_2$ we now get the spectral representations

$$S_2(\mu, t) = \frac{m\alpha}{2\pi L} (\sin \mu)^{-1} \sum_{n=-\infty}^{\infty} \int \frac{dk_0}{2\pi} \frac{e^{i(k_n L \mu - k_0 t)}}{k_0^2 - k_n^2 - m^2 + i\epsilon} \left[ 2ik_n L + \tan \frac{\mu}{2} \right]$$

and

$$S_0(\mu, t) = -\frac{i}{m} \partial_t S_2(\mu, t)$$

and obtain

$$S^\pm(0) = \lim_{x_0 \to 0^\pm} S(x_0, x = 0) = \tilde{S}_{0}^\pm \gamma_0 - \tilde{S}_2$$

The evaluation of the constituent quark self-energies in section IV requires the regularized coincidence limits of $S$. The time-ordering ambiguity of the propagator in the coincidence limit is resolved as usual by referring to the order in which the fields appear in the interaction lagrangian. Accordingly, we define

$$S^\pm(0) = \lim_{x_0 \to 0^\pm} S(x_0, x = 0) = \tilde{S}_{0}^\pm \gamma_0 - \tilde{S}_2$$

and obtain

$$\tilde{S}_{0}^\pm = \frac{i}{4V} \sum_{n=1}^{\infty} (\mp 1 + f_n) D_n \Theta(\Lambda - k_n),$$

$$\tilde{S}_2 = \frac{im}{4V} \sum_{n=1}^{\infty} \frac{D_n(1 - f_n)}{\sqrt{k_n^2 + m^2}} \Theta(\Lambda - k_n).$$
Here we have additionally accounted for the presence of the valence quarks by adding the filling factor terms introduced in section IV. We recognize again the appearance of the quark spectral properties, which can alternatively be derived directly from the Dirac equation (see appendix A).

**APPENDIX D: THE FREE SPIN-0 FIELD ON $S^3 \times R$**

As a prerequisite for our discussion in sections V and VI, we will derive here the mode decomposition of the free pion field (or any real spin-0 field, in general) in the hyperspherical unit cell. We denote the set of three spatial coordinates by $(x)$ and the three generalized momentum quantum numbers (see below) by $(k)$, and we suppress the trivial isospin indices.

Let us now expand the free field in the $S^3 \times R$ background as

$$\phi(x) = \sum_k \left( \eta_k(x) a_k + \eta_k^*(x) a_k^\dagger \right)$$

in terms of the complete set of solutions $\eta_k$ of the free Klein-Gordon equation on $S^3 \times R$, which generalize the usual plane waves of Minkowski space:

$$(\Box + m^2) \eta_k(x) = \left( \frac{\partial^2}{\partial t^2} - \vec{e} \vec{e} + m^2 \right) \eta_k(x) = 0.$$  

($\vec{e} \vec{e}$ is the Laplace-Beltrami operator defined in eq. (B8).) The $\eta$’s are chosen to be orthonormal in the generalized scalar product

$$(\eta_k, \eta_{k'}) = -i \int_{S^3} d\mu(x) \eta_k(x) \frac{\partial}{\partial t} \eta_{k'}^*(x) = \delta_{k,k'}.$$  

Note that the integral above does not include the time direction and is restricted to the spatial hypersphere. One easily shows that this definition of the scalar product is time independent. In order to find the explicit form of the modes $\eta_k$, we first separate the time dependence,

17Note that we consider the minimally coupled case. In general an additional term proportional to the Ricci scalar of the metric can be added to the Klein-Gordon equation.
\[ \eta_k(x) = u_k(x) \frac{e^{-i\omega_k t}}{\sqrt{2\omega_k}}, \]  

(D4)

and then solve the remaining static eigenvalue equation of the Laplace-Beltrami operator:

\[ \tilde{e} \tilde{e} u_k(x) = -\kappa_k^2 u_k(x) \equiv (m^2 - \omega_k^2) u_k(x), \]  

(D5)

where we defined \( \omega_k^2 = \kappa_k^2 + m^2 \). Separating further the dependence on the spatial coordinates, the general solution of eq. \( \text{(D3)} \) can be found to be

\[ u_k(x) = \frac{1}{\sqrt{L^3 h_n^{(l+1)}}} \sin \mu \, C_n^{(l+1)}(\cos \mu) \, Y_{lm}(\theta, \phi). \]  

(D6)

Here the \( C_n^{(\alpha)} \) are ultra-spherical Gegenbauer polynomials \( [48] \), the \( Y_{lm} \) are spherical harmonics and the \( h_n^{(\alpha)} \) are normalization constants defined by

\[ h_n^{(\alpha)} = \frac{\pi^{(1-2\alpha)}}{n!(n+\alpha)\Gamma^2(\alpha)} \text{ for } \alpha \neq 0. \]  

(D7)

The set \( k \) is now specified by the three quantum numbers \( k = \{n, l, m\} \) within the range

\[ n = 0, 1, 2, ... \]
\[ l = 0, 1, 2, ... \]
\[ m = -l, -l + 1, ..., l - 1, l. \]  

(D8)

The eigenvalue corresponding to the eigenmode \( u_k \) is

\[ \kappa_k^2 = \kappa_{nl}^2 = \frac{1}{L^2} (n + l)(n + l + 2). \]  

(D9)

The static modes are orthogonal on \( S^3 \) and normalized to

\[ \int_{S^3} d\mu(x) \, u_k(x)u_k^*(x) = \delta_{k,k'}, \]  

(D10)

so that the time-dependent modes \( \eta_k \) satisfy the orthonormality relation eq. \( \text{(D3)} \). The standard equal-time commutation relations

\[ [\Phi(x, t), \Pi(x', t)] = i\delta^3(x, x') \]  

(D11)
(Π = \partial_t \Phi is the canonical momentum conjugate to \Phi, \delta^i is the spatial part of the delta function defined in appendix B) then insure the norm of the one-particle states:

\[ [a_k, a^\dagger_{k'}] = \delta_{k,k'}, \]
\[ [a_k, a_{k'}] = [a^\dagger_k, a^\dagger_{k'}] = 0, \tag{D12} \]

so that

\[ <0|\phi^a(x)|\pi^b(k)> = \delta^{ab}\eta_k(x). \tag{D13} \]
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FIG. 1. $<\bar{q}q>^{1/3}$ as a function of density a) in the hyperspherical unit cell (solid line) and for comparison b) in the chemical potential approach of ref. [4] (dashed line). Also shown is the constituent quark mass (dotted line).

FIG. 2. The free energy density (in units of $10^{10} MeV^4$) as a function of the quark mass for four values of the baryon density.

FIG. 3. $f_\pi$ as a function of density a) in the hyperspherical unit cell (solid line) and for comparison b) in the chemical potential approach of ref. [4] (dotted line).

FIG. 4. $<\bar{q}q>^{1/3}$ and $f_\pi$ (in MeV) as a function of density in the $B = 0$ cell, which contains no valence quarks. For comparison, the same curves in the $B = 1$ cell from figs 1 and 3 are also shown (dotted line).