POSITIVE SOLUTIONS OF NONLOCAL FRACTIONAL BOUNDARY VALUE PROBLEMS

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Abstract. The authors study a type of nonlinear fractional boundary value problem with nonlocal boundary conditions. An associated Green’s function is constructed. Then a criterion for the existence of at least one positive solution is obtained by using fixed point theory on cones.

1. Introduction. In this paper, we consider a boundary value problem (BVP) consisting of the fractional differential equation

\[-D_0^\alpha u + aD_0^\gamma u = f(t, u), \quad 0 < t < 1,\]

and the boundary condition (BC)

\[D_0^\beta u(0) = 0, \quad D_0^{\alpha - \gamma} u(1) = au(1),\]

where \(D_0^\alpha\) is the \(\alpha\)-th Riemann-Liouville fractional derivative of \(h : [0, 1] \to \mathbb{R}\) defined by

\[D_0^\alpha h(t) = \frac{1}{\Gamma(l - \alpha)} \frac{d^l}{dt^l} \int_0^t (t - s)^{l-\alpha-1} h(s)ds, \quad l = [\alpha] + 1,\]

whenever the right-hand side exists, \([\alpha]\) is the integer part of \(\alpha\), and \(\Gamma\) is the Gamma function.

We assume throughout this paper, and without further mention, that the following conditions hold:

(H1) \(1 < \gamma < \alpha \leq 2, \quad 0 \leq \beta < \alpha - \gamma, \) and \(0 \leq a < \Gamma(\alpha - \gamma + 1);\)

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(H2) $f \in C([0,1] \times \mathbb{R}_+, \mathbb{R}_+)$ with $\mathbb{R}_+ = [0, \infty)$.

Fractional differential equations have attracted extensive attention as they can be applied in various fields of science and engineering. We refer the reader to the monographs [9, 12] and the references therein for specific applications.

The existence of solutions or positive solutions of nonlinear fractional BVPs has been studied by many authors, for example, see [1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 16, 17] and the references therein. Fixed point theory on cones is an important tool in the study of nonlinear fractional BVPs. The existence of positive solutions of a BVP can be established by finding fixed points of an associated operator. The construction of such operators often involves the derivation of the Green’s functions and is a key step in this approach. The following lemma by Bai and Lü [3] is often used for this purpose.

**Lemma 1.1.** ([3, Lemma 2.2]) Assume that $\alpha > 0$ and $u \in C((0,1) \cap L(0,1)$ has an $\alpha$-th fractional derivative that belongs to $C((0,1) \cap L(0,1)$. Then

$$I_0^\alpha D_0^\alpha u(t) = u(t) + \sum_{i=1}^N C_i t^{\alpha-i}$$

for some $C_1, \ldots, C_N \in \mathbb{R}$, $N \in \mathbb{N}$, and $\alpha \leq N < \alpha + 1$, where $I_0^\alpha$ is the $\alpha$-th Riemann-Liouville integral of $h$ defined by

$$I_0^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds. \quad (4)$$

With this lemma, Bai and Lü [3] constructed the Green’s function for the BVP consisting of the equation

$$-D_0^\alpha u = 0 \quad (5)$$

and the BC

$$u(0) = u(1) = 0 \quad (6)$$

by first taking the $\alpha$-th integral of the equation

$$-D_0^\alpha u = h(t) \quad (7)$$

and then identifying the constants using BC (6). The reader is referred to the proof of [3, Lemma 2.3] for the details. This approach has also been employed to construct the Green’s functions for the problems consisting of Eq. (5) and one of the BCs

$$u^{(i)}(0) = D_0^\nu u(1) = 0, \ 0 \leq i \leq N-2, \ 1 \leq \nu \leq N-2,$$

or

$$u^{(i)}(0) = 0, \ u(1) = \int_0^1 h(t)u(s)ds, \ 0 \leq i \leq N-2,$$

where $N$ is defined as in Lemma 1.1. We refer the reader to [5, 6] for the details.

However, all BCs studied so far involved the BC $u(0) = 0$, which is crucial in the construction of the corresponding Green’s functions. In fact, by Lemma 1.1, a term with a negative power $C_N t^{\alpha-N}$ will appear after integrating Eq. (7). This term causes a singularity at $t = 0$ if $C_N \neq 0$. Hence, $u(0) = 0$ is used to ensure that $C_N = 0$.

Using the above idea together with spectral theory, the authors [7] studied the BVP consisting of the equation

$$-D_0^\alpha u + au = f(t, u), \quad (8)$$
and the BC
\[ u(0) = 0, \ u(1) = aD^\alpha_0+ u(1). \] (9)

Note that the condition \( u(0) = 0 \) was still involved.

In this paper, we study BVP (1), (2). By comparing with BVP (8), (9), we see that Eq. (1) contains the term \( D^\alpha_0+ u \) instead of just \( u \), and BC (2) contains \( D^\beta_0+ u(0) = 0 \) instead of \( u(0) = 0 \). BVP (1), (2) can be viewed as an extension of the BVP consisting of the second order nonlinear equation
\[ -u'' + au' = f(t, u), \]
and the BC
\[ u(0) = 0, \ u'(1) = a, \]
or
\[ u'(0) = 0, \ u'(1) - au(1) = 0, \]
that has been studied extensively. We point out that when \( 0 \leq \beta < \beta^* < 1 \), the following BVP
\[ \begin{cases} -D^\alpha_0+ u = f(t, u), \\ D^\beta_0+ u(0) = 0, \ D^{\beta^*}_0+ u(1) = 0, \end{cases} \] (10)
is a special case of BVP (1), (2) with \( a = 0 \) and \( \gamma = \alpha - \beta^* \).

Using the ideas in [7, Theorem 2.1], we will construct the Green’s function as a uniformly convergent series of functions for the BVP consisting of the linear equation
\[ -D^\alpha_0+ u + aD^\gamma_0+ u = 0, \] (11)
and BC (2). Based on this, a theorem on the existence of positive solutions is established. It is interesting to note that the Green’s function obtained is independent of \( \beta \) as long as \( \beta \) satisfies (H1).

This paper is organized as follows: after this introduction, the main results and an example are presented in Section 2. All the proofs are given in Section 3.

2. Main results. The following functions and notation are needed in the presentation of our main results. Let
\[ G_0(t, s) = \begin{cases} (1 - s)^{\gamma-1}t^{\alpha-1} - (t - s)^{\alpha-1} & , \ 0 \leq s \leq t \leq 1, \\ (1 - s)^{\gamma-1}t^{\alpha-1} & , \ 0 \leq t \leq s \leq 1. \end{cases} \] (12)

For \( n \geq 1 \), set \( \Pi = \Pi_{i=1}^n \),
\[ G_n(t, s) = \int_0^1 \ldots \int_0^1 \Pi_{i=1}^n (t^{\tau_i^{-1}}(1 - \tau_i))^{\alpha-\gamma} G_0(t\Pi\tau_1, s) d\tau_1 \ldots d\tau_n, \]
and
\[ G(t, s) = \sum_{n=0}^{\infty} a^n G_n(t, s) \frac{1}{\Gamma(n(\alpha - \gamma))}. \] (13)

For convenience, we denote
\[ \hat{S}(s) = (1 - s)^{\gamma-1} \left( (1 - s)^{\alpha-\gamma} - 1 + s \right), \]
\[ L(t, s) = (\alpha - 1)\hat{S}(s) \sum_{n=0}^{\infty} \left[ \frac{a^n(\alpha-\gamma) + \alpha - 1}{\Gamma(n(\alpha - \gamma) + \alpha)} \left( 1 - \frac{\alpha t}{n(\alpha - \gamma) + \alpha} \right) \right], \] (14)
and
\[ U(s) = \sum_{n=0}^{\infty} \frac{a^n G_0(s, s)}{\Gamma^n(n(\alpha - \gamma) + 1)}. \quad (15) \]

Here we use the convention that \( 0^0 = 1 \) in \( (13)-(15) \). By a similar argument to the one used in the proof of \[7, Theorem 2.1\], we can prove that \( G \) and \( L \) as series of functions are uniformly convergent for \( (t, s) \in [0, 1] \times [0, 1] \) and \( U \) as a series is uniformly convergent for \( s \in [0, 1] \). In addition, for \( (t, s) \in [0, 1] \times [0, 1] \),

\[ 0 \leq L(t, s) \leq G(t, s) \leq U(s) < \infty \quad (16) \]

and \( L(t, s) > 0 \) on \((0, 1) \times (0, 1)\).

The following theorem gives the existence of the Green’s function for BVP (10), (2).

**Theorem 2.1.** The function \( G(t, s) \) defined by \( (13) \) is the Green’s function for BVP (11), (2).

**Remark 1.** (a) By \( (12) \) and \( (13) \), \( G \) is independent of \( \beta \) as long as \( \beta \) satisfies \((H1)\).

(b) It is easy to see that \( G_0(t) \) defined by \( (12) \) with \( \gamma = \alpha - \beta^* \) is the associated Green’s function for the BVP (10). This comes from the fact that BVP (10) is a special case of BVP (1), (2) where \( a = 0 \) and \( \gamma = \alpha - \beta^* \).

With the Green’s function \( G \) given in Theorem 2.1, we may apply fixed point theory on cones to establish criteria for the existence of positive solutions of BVP (1), (2).

Let \( C[0, 1] \) be the Banach space of continuous functions on \([0, 1] \) with the standard maximum norm \( \|u\| \). Then we obtain the following result.

**Theorem 2.2.** If there exist \( 0 \leq r_* < r^* < \infty \) such that

\[ f(t, x) \leq \left( \int_0^1 U(s)ds \right)^{-1} r^*, \quad (t, x) \in [0, 1] \times [0, r^*], \]

and

\[ f(t, x) \geq \left( \min_{t \in [1/4, 3/4]} \int_0^1 L(t, s)ds \right)^{-1} r_*, \quad (t, x) \in [0, 1] \times [0, r_*], \]

then BVP (1), (2) has at least one positive solution \( u \in C[0, 1] \) with \( r_* \leq ||u|| \leq r^* \).

**Remark 2.** The adoption of the subinterval \([1/4, 3/4]\) of \([0, 1]\) in \( (18) \) is only for convenience; it guarantees that \( \left( \min_{t \in [1/4, 3/4]} \int_0^1 L(t, s)ds \right)^{-1} \) is well defined. Actually, it can be replaced by any closed interval \([\delta_1, \delta_2]\) with \( 0 < \delta_1 < \delta_2 < 1 \). Furthermore, when \( a > 0 \), it is easy to verify that \( L(t, s) \neq 0 \) on \([\delta_1, 1] \times [0, 1]\). Hence, \([\delta_1, 1]\) can replace \([1/4, 3/4]\) in \( (18) \).

To illustrate the applicability of our result, let us consider the following example. Let \( \alpha, \gamma, \) and \( a \) satisfy \((H1)\), and \( U(s) \) be defined by \( (15) \).

**Example 1.** The BVP

\[ \begin{cases} -D_{0+}^\alpha u + aD_{0+}^\gamma u = (u - b)^2, \\ u^\beta(0) = 0, \quad D_{0+}^{\gamma+}u(1) = au(1), \end{cases} \]

(19)
has at least one positive solution if $0 < b < 2(\int_0^1 U(s)ds)^{-1}$. To see this, let $f(t, x) = f(x) = (x - b)^2$ and
\[
 r^* = 2^{-1} \left[ 2b + \left( \int_0^1 U(s)ds \right)^{-1} + \sqrt{4b \left( \int_0^1 U(s)ds \right)^{-1} + \left( \int_0^1 U(s)ds \right)^{-2}} \right].
\]
Clearly $(r^* - b)^2 = \left( \int_0^1 U(s)ds \right)^{-1} r^*$. We claim that $(x - b)^2 \leq \left( \int_0^1 U(s)ds \right)^{-1} r^*$ on $[0, r^*]$. In fact, all we need to show is $r^* \geq 2b$, i.e.,
\[
 2b + \left( \int_0^1 U(s)ds \right)^{-1} + \sqrt{4b \left( \int_0^1 U(s)ds \right)^{-1} + \left( \int_0^1 U(s)ds \right)^{-2}} \geq 4b
\]
or
\[
 \sqrt{4b \int_0^1 U(s)ds + 1} \geq 2b \int_0^1 U(s)ds - 1. \tag{20}
\]
When $0 < b \leq \left( 2 \int_0^1 U(s)ds \right)^{-1}$, $2b \int_0^1 U(s)ds - 1 \leq 0$. Hence (20) holds. When $(2 \int_0^1 U(s)ds)^{-1} < b < 2(\int_0^1 U(s)ds)^{-1}$, we have
\[
 4b \int_0^1 U(s)ds + 1 \geq \left( 2b \int_0^1 U(s)ds - 1 \right)^2.
\]
Hence, (20) also holds. Therefore, (17) holds on $[0, r^*]$. Since $f(0) > 0$, by the continuity of $f$, there exists a sufficient small $r^* < b$ such that (18) holds on $[0, r^*]$. By Theorem 2.2, BVP (19) has at least one positive solution.

3. Proofs. The following two lemmas on fractional integrals and inverse operators will be used in the proof of Theorem 2.1. We refer the reader to [9, Property 2.1 and Lemma 2.3] and [7, Lemma 3.2] for details of the proofs.

Lemma 3.1. Let $\lambda > 0$, $\eta > 0$, and $m \in \mathbb{N}$. Then
(a) $I_{0+}^\lambda I_{0+}^\eta u(t) = I_{0+}^{\lambda+\eta} u(t)$ holds on $[0, 1]$ for $u \in L(0, 1)$ if $\lambda + \eta > 1$;
(b) $I_{0+}^\lambda(t^{\eta-1}) = \Gamma(\eta)t^{\eta+\lambda-1}/\Gamma(\eta + \lambda)$; moreover, if $\lambda > 1$, then $(I_{0+}^\lambda)^m(t^{\eta-1}) = I_{0+}^{\lambda m}(t^{\eta-1})$;
(c) $D_{0+}^\lambda t^{\lambda-1} = \Gamma(\lambda)t^{\lambda-\eta-1}/\Gamma(\lambda - \eta)$ if $\lambda > \eta$;
(d) $D_{0+}^\lambda I_{0+}^\lambda u(t) = I_{0+}^{\lambda-\eta} u(t)$.

Lemma 3.2. Let $I$ be the identity operator. Assume $0 \leq a < \Gamma(\alpha - \gamma + 1)$. Then $I - aI_{0+}^{\alpha-\gamma}$ is invertible and
\[
 (I - aI_{0+}^{\alpha-\gamma})^{-1} = \sum_{n=0}^{\infty}(aI_{0+}^{\alpha-\gamma})^n,
\]
where $(aI_{0+}^{\alpha-\gamma})^0 = I$ and $(aI_{0+}^{\alpha-\gamma})^n u = aI_{0+}^{\alpha-\gamma}(aI_{0+}^{\alpha-\gamma})^{n-1} u$ for $u \in C[0, 1]$ and $n \in \mathbb{N}$.

Proof of Theorem 2.1. For any $h \in C[0, 1]$, let $u$ be a solution of the BVP consisting of the equation
\[
 -D_{0+}^\alpha u + aD_{0+}^\gamma u = h(t), \quad 0 < t < 1, \tag{21}
\]
and BC (2). Taking the $\alpha$-th integral of both sides of (21), we have

$$-I_0^\alpha D_0^\alpha u + aI_0^\alpha D_0^\alpha u = I_0^\alpha h.$$  \hfill (22)

By Lemma 1.1,

$$I_0^\alpha D_0^\alpha u = u + \bar{C}_1 t^{\alpha-1} + \bar{C}_2 t^{\alpha-2}$$ \hfill (23)

and

$$I_0^\gamma D_0^\gamma u = u + \bar{C}_3 t^{\gamma-1} + \bar{C}_4 t^{\gamma-2},$$

where $\bar{C}_i$, $i = 1, \ldots, 4$, are some constants. By Lemma 3.1 (a) and (b),

$$I_0^\alpha D_0^\alpha u = I_0^\alpha \gamma (I_0^\gamma D_0^\gamma u) = I_0^\alpha \gamma (u + \bar{C}_3 t^{\gamma-1} + \bar{C}_4 t^{\gamma-2})$$

$$= I_0^\alpha \gamma u + \bar{C}_3 \frac{\Gamma(\gamma)}{\Gamma(\alpha)} t^{\alpha-1} + \bar{C}_4 \frac{\Gamma(\gamma - 1)}{\Gamma(\alpha - 1)} t^{\alpha-2}.$$ \hfill (24)

Hence, by (22)–(24),

$$(I - aI_0^\alpha)^{-\gamma} u(t) = -I_0^\alpha + C_1 t^{\alpha-1} + C_2 t^{\alpha-2}$$ \hfill (25)

for some constants $C_1$ and $C_2$. Then, (25) and Lemma 3.1 (c) and (d) imply

$$\left(D_0^\beta - aI_0^\alpha \gamma - \beta \right) u(t) = -I_0^\alpha \gamma h(t)$$

$$+ C_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha - \beta)} t^{\alpha-\beta-1} + C_2 \frac{\Gamma(\alpha - 1)}{\Gamma(\alpha - \beta - 1)} t^{\alpha-\beta-2}.$$ \hfill (26)

Since $\alpha - \beta - 2 < 0$, the first condition in (2) implies $C_2 = 0$. Hence,

$$(I - aI_0^\alpha)^{-\gamma} u(t) = -I_0^\alpha \gamma h(t) + C_1 t^{\alpha-1}$$ \hfill (27)

and

$$\left(D_0^\beta - aI_0^\alpha \gamma - \beta \right) u(t) = -I_0^\alpha \gamma h(t) + C_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha - \beta)} t^{\alpha-\beta-1}.$$ \hfill (28)

Taking the $(\alpha - \gamma - \beta)$-th derivative on both sides of (27) and again using Lemma 3.1 (c), we have

$$D_0^{\alpha-\gamma} u(t) - au(t) = -I_0^{\alpha-\gamma} h(t) + C_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha - \beta)} \frac{\Gamma(\alpha - \beta)}{\Gamma(\gamma)} t^{\gamma-1}.$$ \hfill (29)

By the second condition in (2) and (4), we have

$$C_1 = \frac{1}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\gamma-1} h(s) ds.$$ \hfill (30)

Then, by (4), (26), and (28),

$$(I - aI_0^{\alpha-\gamma}) u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} h(s) ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\gamma-1} t^{\alpha-1} h(s) ds = \int_0^1 G_0(t, s) h(s) ds,$$
where $G_0$ is defined by (12). Hence, by Lemma 3.2 and a similar argument to [7, Theorem 2.1],

$$
\begin{align*}
u(t) &= (I - aI^{\alpha-\gamma})^{-1} \left( \int_0^1 G_0(t, s) h(s) ds \right) \\
&= \sum_{n=0}^{\infty} (aI^{\alpha-\gamma})^n \left( \int_0^1 G_0(t, s) h(s) ds \right) = \int_0^1 G(t, s) h(s) ds.
\end{align*}
$$

□

In order to prove Theorem 2.2, we let $X$ be a Banach space and $P \subset X$ be a cone in $X$. For $r > 0$, define

$$
P_r = \{ u \in P : \|u\| < r \} \quad \text{and} \quad \partial P_r = \{ u \in P : \|u\| = r \}.
$$

For an operator $T : P_r \to P$, let $i(T, P_r, P)$ be the fixed point index of $T$ on $P_r$ with respect to $P$. The following well-known lemmas on the fixed point index will be used in the proof of Theorem 2.2. We refer the reader to [4, 8] or [15, page 529, items A2, A3] for details.

Lemma 3.3. Assume that $T : P_r \to P$ is a completely continuous operator such that $Tu \neq u$ for $u \in \partial P_r$.

(a) If $\|Tu\| \leq \|u\|$ for $u \in \partial P_r$, then $i(T, P_r, P) = 1$.

(b) If $\|Tu\| \geq \|u\|$ for $u \in \partial P_r$, then $i(T, P_r, P) = 0$.

Lemma 3.4. Let $0 < r_1 < r_2$ satisfy

$$
i(T, P_{r_1}, P) = 0 \quad \text{and} \quad i(T, P_{r_2}, P) = 1,
$$

or

$$
i(T, P_{r_1}, P) = 1 \quad \text{and} \quad i(T, P_{r_2}, P) = 0.
$$

Then $T$ has a fixed point in $P_{r_2} \setminus P_{r_1}$.

In the sequel, let $X = C[0, 1]$ and $P$ be the cone in $X$ given by

$$
P = \{ u \in C[0, 1] : u(t) \geq 0 \text{ on } [0, 1] \}.
$$

Define $T : P \to X$ by

$$
(Tu)(t) = \int_0^1 G(t, s) f(s, u(s)) ds.
$$

(29)

It is clear that $u(t)$ is a solution of BVP (1), (2) if $u \in P$ is a fixed point of $T$. By the same argument as [7, Lemma 4.3], we have that $TP \subset P$ and $T$ is completely continuous.

Proof of Theorem 2.2. For any $u \in \partial P_{r^*}$, we have $\|u\| = r^*$ and $u(t) \leq r^*$ on $[0, 1]$. By (29), (17), and (16),

$$
(Tu)(t) = \int_0^1 G(t, s) f(s, u(s)) ds \\
\leq \left( \int_0^1 U(s) ds \right)^{-1} r^* \int_0^1 U(s) ds = r^*.
$$

Hence, $\|Tu\| \leq \|u\|$, so by Lemma 3.3 (a), $i(T, P_{r^*}, P) = 1$. 


For any $u \in \partial P_{r^*}$, we have $\|u\| = r^*$ and $u(t) \leq r^*$ on $[0, 1]$. For $t \in [1/4, 3/4]$, by (29), (18), and (16),
\[
(Tu)(t) = \int_0^1 G(t,s) f(s, u(s))ds \geq \int_0^1 L(t,s) f(s, u(s))ds \\
\geq \left( \min_{t \in [1/4, 3/4]} \int_0^1 L(t,s)ds \right)^{-1} r^* \int_0^1 L(t,s)ds = r^*.
\]
Thus, $\|Tu\| \geq \|u\|$, so by Lemma 3.3 (b), $i(T, P_{r^*}, P) = 0$.

By Lemma 3.4, $T$ has a fixed point $u$ in $P$ with $r^* \leq \|u\| \leq r^*$. Hence, BVP (1), (2) has at least one positive solution $u(t)$. \hfill $\square$

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