Cosmological perturbation analysis in a scale invariant model of gravity

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Abstract
We consider a model for gravity that is invariant under global scale transformations. It includes one extra real scalar field coupled non-minimally to the gravity fields. In this model all the dimensionful parameters such as the gravitational constant and the cosmological constant etc are generated by a solution of the classical equations of motion. Hence, this solution provides a mechanism to break scale invariance. In this paper, we demonstrate the stability of such a solution against small perturbations in a flat FRW background by making a perturbative expansion around this solution and solving the resulting equations linear in the perturbations. This demonstrates the robustness of this symmetry breaking mechanism.

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1. Introduction

Scale invariance is an idea with quite a long history. The possibility of local scale invariance was first suggested by Weyl [1] in the 1920s. This subsequently led to considerable research effort by several physicists [2–7, 8–20]. Quantizing a scale invariant theory is considered problematic since scale invariance is anomalous in general. However, it has also been argued that in some cases it might be possible to preserve scale invariance in the full quantum field theory [18, 21–23]. Despite the difficulties scale invariance remains an interesting idea since it has the potential to resolve one of the greatest puzzles of physics, namely, the cosmological constant problem [18, 23–26].

A scale invariant theory contains no dimensionful parameter in the action. Hence, in any realistic theory scale invariance has to be broken in order to agree with observations. There
exist several mechanisms to break scale invariance [18, 27–32]. One such mechanism for breaking scale invariance is to assume the existence of a classical background cosmological solution [18, 28, 31, 32]. This may be demonstrated by including just one scalar field besides gravity. Consider the following action:

\[ S = \int d^4x \sqrt{-g} \left[ \frac{\beta \chi^2}{8} R + \frac{1}{2} D^\mu \chi D_\mu \chi - \frac{1}{4} \lambda \chi^4 \right] , \]  

(1)

where \( \chi \) is a real scalar field. This action is invariant under a global scale transformation,

\[ \chi \rightarrow \chi \Lambda, \quad x \rightarrow x/\Lambda. \]

(2)

Here, we have chosen the conventions followed in [33, 34] where the flat spacetime metric takes the form

\[ (1, -1, -1, -1), \]

and the curvature tensor and its contractions are defined as

\[ R_{\mu \nu \alpha \beta} = -\partial_\beta \Gamma^\rho_{\nu \alpha} + \partial_\alpha \Gamma^\rho_{\nu \beta} + \Gamma^\rho_{\nu \beta} \Gamma^\gamma_{\gamma \rho} - \Gamma^\gamma_{\nu \beta} \Gamma^\delta_{\gamma \rho}, \]

\[ R_{\rho \beta} = R^\alpha_{\rho \beta}, \quad R = R_{\rho \beta} g^{\rho \beta}. \]

(3)

Now, we seek a constant solution of the equation of motion for the scalar field \( \chi \), so that the term \( (\beta \chi^2/8) \) in the action generates the effective gravitational constant. Hence, we may drop the terms containing the derivatives of the scalar field in its equation of motion to obtain the following relation:

\[ \frac{\beta R}{4} = \lambda \chi^2. \]

(4)

Here, \( R \) represents the classical scale covariant curvature scalar, as defined in equation (3). We assume the FRW metric with the spatial curvature parameter \( k = 0 \) and scale parameter \( a(t) \), where \( t \) is the cosmic time. We obtain a solution to the classical equations,

\[ \chi = \chi_0 = \frac{M_{\text{PL}}}{\sqrt{2 \pi \beta}}, \]

(5)

where \( M_{\text{PL}} \) is the Planck mass. The FRW scale parameter is given by

\[ a(t) = a_0 e^{H_0 t}, \]

(6)

where \( H_0 \) is the Hubble parameter. This sets the curvature scalar as \( R = 12 H_0^2 \). Hence, we obtain the solution corresponding to the de Sitter spacetime. The background field \( \chi_0 \), the Hubble constant \( H_0 \) and the curvature scalar \( R \) are constants for this solution. One may of course obtain more realistic solutions by adding contributions from dark matter candidates as well as standard model particles [19]. One may also allow the scalar field \( \chi \) to be time dependent [32]. In such cases, \( R \) will no longer be independent of time. This will complicate our analysis but will not be conceptually different from the simple case we consider. Hence, in this paper we consider this simple case only.

The solution generates both the gravitational constant and an effective cosmological constant. It is interesting that an effective cosmological constant is generated, despite the fact that scale invariance forbids this term in the action. This might be indicative of the fundamental nature of this parameter [35]. We have argued in earlier papers that this mechanism for breaking scale invariance is different from spontaneous symmetry breaking. In the latter case, one is interested in the behavior of the ground state of the Hamiltonian of matter fields under the symmetry transformation. Hence, one makes a quantum expansion around the minimum of the potential. In the present case, we seek a time-dependent solution and the minimum of the potential is not directly relevant. The time dependence of the solution comes from the scale parameter, \( a(t) \). In earlier papers, we have called this phenomenon cosmological symmetry breaking [18, 31] in order to emphasize its difference from spontaneous symmetry breaking. The important question now is whether such a solution is stable or not. In this
paper, we investigate this question. We demonstrate the stability of this solution under small perturbations and hence show that it can consistently break scale invariance.

Such de Sitter or approximately de Sitter solutions have also been discussed in the case of general scalar–tensor theories. Several papers have addressed the issue of stability of these solutions [36–38] as well as computed the power spectrum of perturbations [39–45]. The power spectrum is useful if one assumes that this solution is applicable to inflation. The theory we consider is obtained as a special case of these scalar–tensor theories by imposing the constraint that the action must be invariant under scale transformations. The background de Sitter solution plays a crucial role in our case since it provides a mechanism to break scale symmetry. Indeed, it is difficult to find alternate methods to break this symmetry. For example, this symmetry can be broken spontaneously only if we arbitrarily set some terms to zero in the action [22]. Since no symmetry prohibits such terms, these have to be tuned to zero at each order in perturbation theory. In contrast, our mechanism does not require any such constraint. Due to the special role played by the de Sitter solution in our model, testing the stability of this solution is crucial in our case. If the solution is found to be stable, then it establishes the cosmological symmetry breaking as a robust mechanism to break scale and possibly other symmetries.

We may also directly use the results of the stability analysis of the de Sitter solutions in general scalar–tensor theories and apply these to our special case. A general criterion for testing stability is given, for example, in [36, 37]. We find that our results are consistent with this condition, as discussed in section 2.

The background de Sitter solution in our scale invariant model may be applicable to inflation or to dark energy. However, in this paper we are not interested in cosmological applications of this model and only in demonstrating that the solution is stable.

In the rest of this paper we shall use the conformal time $\eta$ instead of $t$. Hence, the spatially flat FRW metric becomes

$$g_{\mu\nu} = a^2(\eta)(1, -1, -1, -1).$$  \hspace{1cm} (7)

In terms of $\eta$ Einstein’s equations, at the leading order, may be written as

$$\left(\frac{a'}{a}\right)^2 = \frac{\lambda \chi_0^2 a^2}{3\beta},$$

$$\left(\frac{a''}{a}\right) = 2 \left(\frac{\lambda \chi_0^2 a^2}{3\beta}\right).$$  \hspace{1cm} (8)

Here, the primes represent derivatives with respect to $\eta$.

2. Perturbations

In this section, we study the perturbations to the leading order solution. As already mentioned we shall restrict ourselves to small perturbations. The perturbed metric may be expressed as

$$g_{\mu\nu} = a^2 \left(1 + 2A \frac{\partial_i B + M_i}{\partial_i B + M} \right),$$

$$\partial_i V - \partial_i M_i = 0; \quad \partial_i P_{ij} = 0; \quad P_{ij} = 0.$$  \hspace{1cm} (10)

where $a = a(\eta)$ is the scale factor of the universe in terms of the conformal time, $D_i = \left(\partial_i \partial_j - \frac{1}{2} \delta_{ij} \nabla^2\right)$, $A$, $B$, $E$ and $\psi$ are the scalar perturbations, $V_i$ and $M_i$ are the vector perturbations and $P_{ij}$ stands for the pure tensor perturbations. The pure vector and the tensor parts satisfy the following constraints [46]:

$$\partial_i V_i = \partial_i M_i = 0; \quad \partial_i P_{ij} = 0; \quad P_{ij} = 0.$$  \hspace{1cm} (10)
The inverse metric becomes
\[
g^{\mu \nu} = \frac{1}{a^2} \left( 1 - 2A \frac{\partial_i B + M_i}{\partial_i B + M_i} \right) \delta_{ij} + D_{ij} E - \partial_j V_i - \partial_i V_j - P_{ij},
\]
where we have neglected all the terms beyond the first order in perturbations. We also expand the field \( \chi \) such that
\[
\chi = \chi_0 + \hat{\chi},
\]
where \( \chi_0 \) represents the solution at the leading order and \( \hat{\chi} \) a small perturbation.

We express the Einstein equation as
\[
G_{\mu \nu} = T_{\mu \nu},
\]
where
\[
G_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R,
\]
\[
T_{\mu \nu} = -\chi^2 \left( g_{\mu \nu} \right) - \frac{4}{\beta} \left( g^{\rho \sigma} \left( \partial_\rho \chi \right) \left( \partial_\sigma \chi \right) - \frac{1}{4} \lambda \chi^4 \right) g_{\mu \nu} - \frac{4}{\beta} \chi^2 \left( \partial_\mu \chi \right) \left( \partial_\nu \chi \right).
\]

Using the decomposition theorem \([45, 47, 48]\), we can now decompose the Einstein equation into scalar, vector and tensor modes and treat their perturbations separately as these modes evolve independently \([49–51]\). The tensor modes are gauge invariant \([52]\) and so gauge fixing is not required. However, in both the cases of the scalar and the vector modes we need to fix a gauge to get a physical solution.

2.1. Scalar modes

For the scalar modes we choose the longitudinal (conformal Newtonian) gauge \([53–55]\), \( B = E = 0 \). At the first order in the perturbations, the components of the Einstein tensor are given by
\[
\delta G_{00} = -2\nabla^2 \psi + 6\frac{a'}{a} \psi',
\]
\[
\delta G_{0i} = -2\frac{a'}{a} \partial_i A - 2\partial_i \psi',
\]
\[
\delta G_{ij} = \left( -\nabla^2 \left( A - \psi \right) - 2\frac{a'}{a} A' - 4\frac{a'}{a} \psi' - 2\psi'' - \frac{2\lambda \chi^2 a^2}{\beta} \left( A + \psi \right) \right) \delta_{ij}
\]
\[
+ \partial_i \partial_j \left( A - \psi \right).
\]

The corresponding components of the energy–momentum tensor are
\[
\delta T_{00} = -2\nabla^2 \left( \frac{\hat{\psi}}{\chi_0} \right) + 6 \left( \frac{a'}{a} \right) \left( \frac{\hat{\psi}}{\chi_0} \right) - \frac{2\lambda \chi^2 a^2}{\beta} \left( A + \frac{\hat{\psi}}{\chi_0} \right),
\]
\[
\delta T_{0i} = 2 \left( \frac{a'}{a} \right) \partial_i \left( \frac{\hat{\psi}}{\chi_0} \right) - 2\partial_i \left( \frac{\hat{\psi}}{\chi_0} \right).
\]
\[ \delta T_{ij} = \left[ 2\nabla^2 \left( \frac{\dot{\chi}}{\chi_0} \right) - 2 \left( \frac{a'}{a} \right) \left( \frac{\ddot{\chi}}{\chi_0} \right) - 2 \left( \frac{\dot{\chi}''}{\chi_0} \right) + \frac{2\lambda \chi_0^2 a^2}{\beta} \left( \frac{\dot{\chi}}{\chi_0} - \psi \right) \right] \delta_{ij} \]

- \frac{2}{a} \dot{a} \partial_j \left( \frac{\dot{\chi}}{\chi_0} \right). \quad (21)

Comparing all the components of the Einstein tensor and the energy–momentum tensor we obtain

- \nabla^2 \psi + 3 \frac{a'}{a} \psi' = -\nabla^2 \left( \frac{\dot{\chi}}{\chi_0} \right) + 3 \left( \frac{a'}{a} \right) \left( \frac{\dot{\chi}}{\chi_0} \right) - \frac{\lambda \chi_0^2 a^2}{\beta} \left( A + \frac{\dot{\chi}}{\chi_0} \right), \quad (22)

- \frac{a'}{a} a \partial_i A - \partial_i \psi' = \left( \frac{a'}{a} \right) \partial_i \left( \frac{\dot{\chi}}{\chi_0} \right) - \partial_i \left( \frac{\dot{\chi}'}{\chi_0} \right). \quad (23)

- \nabla^2 (A - \psi) - 3 \frac{a'}{a} A' - 6 \frac{a'}{a} \psi' - 3 \psi'' = 2\nabla^2 \left( \frac{\dot{\chi}}{\chi_0} \right) - 3 \left[ \left( \frac{a'}{a} \right) \left( \frac{\dot{\chi}'}{\chi_0} \right) + \left( \frac{\dot{\chi}''}{\chi_0} \right) - \frac{\lambda \chi_0^2 a^2}{\beta} \left( \frac{\dot{\chi}}{\chi_0} - \psi \right) + \frac{\lambda \chi_0^2 a^2}{\beta} (A + \psi) \right]. \quad (24)

\[ \partial_i \partial_j (A - \psi) = -2 \partial_i \partial_j \left( \frac{\dot{\chi}}{\chi_0} \right). \quad (25) \]

Since we seek solutions whose spatial dependence is equal to \( e^{i\vec{k} \cdot \vec{x}} \), from equation (25) we obtain

- \[ A - \psi = -2 \left( \frac{\dot{\chi}}{\chi_0} \right). \quad (26) \]

Using equation (8) we find

- \[ \frac{a'}{a} = -\frac{1}{\eta + C}. \quad (27) \]

where \( C \) is a constant. Using equations (22) and (26) we obtain the following solutions for \( \xi \), defined as \( \xi = \psi - \frac{\dot{\chi}}{\chi_0} \):

- \[ \xi = D(\eta + C) e^{\xi \left( \eta + C_0 \right)} \quad (28) \]

or \( \xi = 0 \), \quad (29)

where \( D \) is an arbitrary constant. Since the non-zero solution does not satisfy equations (23) and (24), \( \xi = 0 \) is the only simultaneous solution to equations (22)–(25). This implies

- \[ A = -\psi = -\frac{\dot{\chi}}{\chi_0}. \quad (30) \]

The background scale factor \( a(\eta) \) may be obtained by solving equation (8),

- \[ a(\eta) = -\frac{1}{H_0(\eta + C)}. \quad (31) \]

Without loss of generality we may set \( a(\eta = 0) = 1 \). This gives

- \[ C = \frac{1}{H_0} = -\eta_0. \quad (32) \]
As $a \to \infty$ for $\eta \to \eta_0$, the maximum value of $\eta$ is $\eta_0 = 1/H_0$.

The equation of motion of $\hat{\chi}$ is given by

$$- (1 + 2\beta) \nabla^2 \left( \frac{\hat{\chi}}{\chi_0} \right) - \frac{\beta}{2} \nabla^2 A + (1 + 3\beta) \left( \frac{\hat{\chi}''}{\chi_0} \right) + \frac{3\beta}{2} \left( A'' \right)$$

$$+ (2 + 9\beta) \left( \frac{a'}{a} \right) \chi_0^2 \left( \frac{\hat{\chi}'}{\chi_0} \right) + 6\beta \left( \frac{a'}{a} \right) A' + 2\lambda \chi_0^2 a^2 \left( A + \left( \frac{\hat{\chi}}{\chi_0} \right) \right) = 0.$$  \hspace{1cm} (33)

By using equation (30) this equation reduces to

$$\nabla^2 A - A'' - 2 \left( \frac{a'}{a} \right) A' = 0.$$  \hspace{1cm} (34)

Let $A(\vec{x}, \eta) = \sigma_k(\eta) e^{i \vec{k} \cdot \vec{x}}$. We find

$$\sigma_k''(\eta) + 2\sigma_k'(\eta) \frac{\eta_0 - \eta}{\eta_0} = -k^2 \sigma_k(\eta).$$  \hspace{1cm} (35)

This equation is consistent with the condition given in [36, 37], expressed in terms of conformal time. It corresponds to the limiting case of equation (30) of [37], where we use the equality sign. This is just the equation of a damped harmonic oscillator with the damping force increasing with time. We point out that the maximum value of the conformal time $\eta$ is $\eta_0$. As $\eta \to \eta_0$, the background scale factor approaches infinity. Hence, the damping force is always positive and we expect no unstable modes. The solution, $\sigma_k(\eta)$, is given by

$$\sigma_k(\eta) = C_1 [\sin k(\eta - \eta_0) - k(\eta - \eta_0) \cos k(\eta - \eta_0)]$$

$$- C_2 [\cos k(\eta - \eta_0) + k(\eta - \eta_0) \sin k(\eta - \eta_0)],$$  \hspace{1cm} (36)

where $k = |\vec{k}|$ and $C_1$ and $C_2$ are the constants. The solution for some representative values of $k$ is shown in figure 1. As expected we do not find any unstable modes. We do, however, find a mode of zero frequency which arises if $k = |\vec{k}| = 0$. In this case we find a solution, $\sigma_0 = \text{constant}$, which is independent of $\eta$.

Our results show that the classical de Sitter solution, which breaks scale invariance, is stable under small perturbations. Hence, we have demonstrated that the cosmological symmetry breaking mechanism consistently breaks scale invariance. This is the main new result of our paper.

2.2. Vector modes

We next consider the vector perturbations. For simplicity, we rewrite the Einstein equation in the following form:

$$R_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^a.$$  \hspace{1cm} (37)

The $j - i$ components of equation (37) give

$$\partial_j \left[ V_{ij}'' + 2\frac{a'}{a} V_{ij}' - M_i' - 2\frac{a'}{a} M_i \right] = 0$$  \hspace{1cm} (38)

and $0 - i$ components may be written as

$$\nabla^2 (M_i - V_i') = 0.$$  \hspace{1cm} (39)

Here, we have used equation (10). We make a gauge choice as $M_i = 0$ which implies $\nabla^2 V_i = 0$. From equation (38), we obtain

$$\partial_j \left[ V_{ij}'' + 2\frac{a'}{a} V_{ij}' \right] = 0.$$  \hspace{1cm} (40)

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As we seek solutions whose spatial dependence is given by \( \exp(i \vec{k} \cdot \vec{x}) \) this implies

\[
V_i'' - \frac{2}{\eta + C} V_i' = 0,
\]

where we have used equation (27). Hence, the solution of the vector mode perturbation is

\[
V_i = C_i^V \left( \eta - \eta_0 \right)^3 e^{i \vec{k} \cdot \vec{x}}.
\]

where \( C_i^V \) is a constant vector. This implies that as \( \eta \to \eta_0 \), the vector perturbation dies off as \( (1 - \eta/\eta_0)^3 \).

2.3. Tensor modes

We finally consider the tensor modes. As already mentioned, we do not need to make any gauge choice to solve for the tensor mode. Using equation (10), the equation for the tensor modes can be written as

\[
\nabla^2 P_{ij} - 2 \frac{d'}{d} P_{ij}' - P_{ij}'' = 0.
\]

Comparing this with equation (34) we see that \( P_{ij} \) satisfies the same equation as \( A \). Hence, the solution for the tensor modes is the same as that of the scalar perturbations.

3. Conclusion

We have considered a scale invariant model of gravity including an extra real scalar field. The scaling symmetry is broken by a mechanism which has some similarity to spontaneous symmetry breaking. In this case, a time-dependent solution of the classical equations of motion breaks the symmetry and generates all the dimensionful parameters of the theory such
as the gravitational constant and an effective cosmological constant. In this paper, we have analyzed the stability of such a solution against small perturbations. We have shown that the symmetry breaking solution is stable against small perturbations, irrespective of their scalar, vector or tensor nature. However, the exact behavior depends on the nature of the perturbation. The scalar and tensor perturbations show damped oscillations as a function of conformal time $\eta$. The vector perturbation, however, decays monotonically with conformal time. Our results establish the robustness of this mechanism to break scale invariance.

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