Many Exact Solutions of the Nonlinear KPP Equation Using the Bäcklund Transformation of the Riccati Equation

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Abstract

The Bäcklund transformation of the Riccati equation is applied in this article to construct traveling wave solutions for the nonlinear Kolmogorov-Petrovskii-Piskunov (KPP) equation. Solitons, trigonometric and rational solutions of this equation are obtained. This transformation is straightforward and concise. It gives much more general results than the well-known results obtained by other methods. With the aid of Maple, some graphical representations for some results are presented by choosing suitable values of parameters.

Keywords

Exact traveling wave solutions, Bäcklund transformation of the Riccati equation, Kolmogorov-Petrovskii-Piskunov equation, Soliton solutions, Trigonometric solutions, Rational solutions

Mathematics Subject Classification: 35K99, 35P05, 35P99, 35C05.

Introduction

The investigation of exact traveling wave solutions to nonlinear PDEs plays an important role in the study of nonlinear physical phenomena. Nonlinear wave phenomena appears in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics and geochemistry. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations. In recent decades, many effective methods have been established to obtain exact solutions of nonlinear PDEs, such as the inverse scattering transform [1], the Hirota method [2], the truncated Painlevé expansion method [3], the Bäcklund transform method [1,4-8], the simplest equation method [9,10], the Weierstrass elliptic function method [11], the Jacobi elliptic function method [12-14], the tanh-function method [15,16], the \((G'/G)\)-expansion method [17-22], the modified simple equation method [23-26], the Kudryashov method [27-29], the multiple exp-function algorithm method [30,31], the transformed rational function method [32], the Frobenius decomposition technique [33], the local fractional variation iteration method [34], the local fractional series expansion method [35], the \((G'/G, 1/G)\)-expansion method [36-40], the generalized Riccati equation mapping method [41-43], the test function method [44,45] and so on.

The objective of this article is to use the Bäcklund transformation of the Riccati equation to construct new exact traveling wave solutions of the following nonlinear Kolmogorov-Petrovskii-Piskunov (KPP) equation [22,26,46].

\[
    u_t - u_{xx} + \mu u + \gamma u^2 + \delta u^3 = 0, \tag{1.1}
\]
Where $\mu, \nu, \delta$ are real constants? Eq. (1.1) includes the Fisher equation, Huxley equation, Burgers-Huxley equation, Chaffee-Infante equation and Fitzhugh-Nagumo equation as special cases. Recently, Feng, et al. [22] have discussed Eq. (1.1) using the $(G'/G)$-expansion method and found its exact solutions, while Zayed et al. [26,46] have applied two methods via the modified simple equation method and the Riccati equation method combined with the $(G'/G)$-expansion method respectively, to Eq. (1.1) and determined the exact traveling wave solutions of it.

This paper is organized as follows: In Sec. 2, the description of the Bäcklund transformation of the Riccati equation is given. In Sec. 3, we use the given method described in Sec. 2, to find traveling wave solutions of the nonlinear KPP equation. In Sec. 4, physical explanations of some results are presented. In Sec. 5, some conclusions are obtained.

Description of the Bäcklund Transformation of the Riccati Equation

Suppose that we have the following nonlinear PDE:

$$F(u, u_x, u_{xx}, u_{xxx}, \ldots) = 0,$$

(2.1)

Where $F$ is a polynomial in $u(x,t)$ and its partial derivatives, in which the highest order derivatives and the nonlinear terms are involved. In the following, we give the main steps of this method [5,47]:

**Step 1**

Using the wave transformation.

$$u(x,t) = u(\xi), \quad \xi = kx + \omega t,$$

(2.2)

Where $k, \omega$ are constants, to reduce Eq. (2.1) to the following ODE:

$$P(u, u_x, u_{xx}, \ldots) = 0,$$

(2.3)

Where $P$ is a polynomial in $u(\xi)$ and its total derivatives while $\gamma = \frac{d}{d\xi}$.

**Step 2**

Assume that Eq. (2.3) has the formal solution.

$$u(\xi) = \sum_{i=0}^{N} a_i \psi^i(\xi),$$

(2.4)

Where $a_i$ are constants to be determined, such that $a_N \neq 0$ or $a_N = 0$, while $\psi^i(\xi)$ comes from the following Bäcklund transformation

$$\psi^i(\xi) = \frac{-bB + A\phi(\xi) + A^i B^i \phi(\xi)}{A^i + B^i \phi(\xi)},$$

(2.5)

Where $b, A, B$ are constants with $B \neq 0$, while $\phi(\xi)$ satisfies the Riccati equation:

$$\phi(\xi) = b + \phi(\xi)^2.$$  

(2.6)

It is well-known [16] that Eq. (2.6) has the following solutions:

(i) If $b < 0$, then

$$\phi(\xi) = -\sqrt{-b} \tan(\sqrt{-b} \xi), \text{ or } \phi(\xi) = -\sqrt{-b} \cot(\sqrt{-b} \xi).$$

(ii) If $b > 0$, then

$$\phi(\xi) = \sqrt{b} \tan(\sqrt{b} \xi), \text{ or } \phi(\xi) = \sqrt{b} \cot(\sqrt{b} \xi).$$

(iii) If $b = 0$, then

$$\phi(\xi) = -\frac{1}{\xi}.$$  

**Step 3**

We determine the positive integer $N$ in (2.4) by using the homogeneous balance between the highest-order derivatives and the nonlinear terms in Eq. (2.3). More precisely we define the degree of $u(\xi)$ as $D[u(\xi)] = N$ which gives rise to the degree of other expressions as follows:

$$D\left(\frac{d^nu}{dx^n}\right) = N + q,$$

(2.7)

Therefore, we can get the value of $N$ in (2.4). In some nonlinear equations the balance number $N$ is not a positive integer. In this case, we make the following transformations [47]:

(a) When $N = \frac{a}{p}$ where $\frac{a}{p}$ is a fraction in the lowest terms, we let

$$u(\xi) = v^{\frac{1}{a}}(\xi).$$

(2.8)

Then substituting (2.8) into (2.3) to get a new equation in the new function $v(\xi)$ with a positive integer balance number.

(b) When $N$ is a negative number, we let

$$u(\xi) = v^\gamma(\xi).$$

(2.9)

and substituting (2.9) into (2.3) to get a new equation in the new function $v(\xi)$ with a positive integer balance number.

**Step 4**

We substitute (2.4) along with Eq. (2.6) into Eq. (2.3), collect all the terms with the same powers of $\psi^i(\xi)$ and set them to zero, we obtain a system of algebraic equations, which can be solved by Maple to get the values of $a_i$ and $k, \omega$. Consequently, we obtain the exact traveling wave solutions of Eq. (2.1).

Finally if $B = 0$, then the above method reduces to the well-known modified extended tanh-function method [16].
An Application

In this section, we will apply the method described in Sec. 2 to find the exact traveling wave solutions of the nonlinear KPP equation (1.1). To this end, we use the wave transformation (2.2) to reduce Eq. (1.1) to the following ODE:

\[ \omega u''(\xi) - k^2 u'(\xi) + mu(\xi) + gu^2(\xi) + \delta u(\xi) = 0. \]  

(3.1)

By balancing \( u' \) with \( u' \) in Eq. (3.1), we get \( N = 1 \). Consequently, we have the formal solution

\[ u(\xi) = a_0 + a_1\Psi(\xi) + a_n\Psi^{-n}(\xi), \]  

(3.2)

Where \( a_0, a_1, a_n \) are constants to be determined, such that \( a_1 \neq 0 \) or \( a_1 \neq 0 \) while \( \Psi(\xi) \) is given by (2.5).

Now, substituting (3.2) along with Eqs. (2.5) and (2.6) into (3.1), collecting the coefficients of \( \Psi(\xi) \), \( i = 0, 1, \ldots, 6 \) and setting them to zero, we get a system of algebraic equations. Solving this system of algebraic equations with aid of Maple, we have the following results:

**Result 1**

\[ b = \frac{4\mu\delta - \gamma^2}{8\kappa^2}, \quad \omega = \frac{\gamma k}{\sqrt{2\delta}}, \quad a_{-1} = 0, \quad a_0 = -\frac{\gamma}{2\delta}, \quad a_1 = \frac{\sqrt{2}k}{\delta}, \]  

(3.3)

Provided that \( \delta > 0 \).

From (3.1), (3.2) and (3.3), we deduce the exact traveling solutions of Eq. (1.1) as follows:

If \( b < 0 \), then we have the solutions

\[ u(\xi) = \frac{-\gamma}{2\delta} + \frac{\sqrt{2}k}{\sqrt{\delta}} \left[ -B \left( \frac{u_{1/2}(\xi)}{\nu_{1/2}(\xi)} + \frac{u_{1/2}(\xi)^3}{\nu_{1/2}(\xi)^3} \right) \right]. \]  

(3.4)

\[ \text{or} \]

\[ u(\xi) = \frac{-\gamma}{2\delta} + \frac{\sqrt{2}k}{\sqrt{\delta}} \left[ -B \left( \frac{u_{1/2}(\xi)}{\nu_{1/2}(\xi)} + \frac{u_{1/2}(\xi)^3}{\nu_{1/2}(\xi)^3} \right) \right]. \]  

(3.5)

Provided that \( 4\mu\delta - \gamma^2 < 0 \).

If \( b > 0 \) then we have the solutions

\[ u(\xi) = \frac{-\gamma}{2\delta} + \frac{\sqrt{2}k}{\sqrt{\delta}} \left[ -B \left( \frac{u_{1/2}(\xi)}{\nu_{1/2}(\xi)} + \frac{u_{1/2}(\xi)^3}{\nu_{1/2}(\xi)^3} \right) \right]. \]  

(3.6)

Or

\[ u(\xi) = \frac{-\gamma}{2\delta} + \frac{\sqrt{2}k}{\sqrt{\delta}} \left[ -B \left( \frac{u_{1/2}(\xi)}{\nu_{1/2}(\xi)} + \frac{u_{1/2}(\xi)^3}{\nu_{1/2}(\xi)^3} \right) \right]. \]  

(3.7)

Provided that \( 4\mu\delta - \gamma^2 > 0 \).

If \( b = 0 \), then we get \( \mu = \frac{\gamma^2}{4\delta} \) and we have the rational solution

\[ u(\xi) = \frac{-\gamma}{2\delta} - \frac{\sqrt{2}k}{\delta} \left( \frac{A}{\xi - B} \right), \]  

(3.8)

Where \( \xi = k \left( x + \frac{\gamma^2}{4\delta} t \right) \).

**Result 2**

\[ b = k, \quad k = \frac{4B}{2\delta + \gamma^2}, \quad \omega = \frac{\gamma}{4\delta} \left( \frac{2\delta + \gamma^2}{4\delta} \right), \quad a_{-1} = \frac{-\gamma}{2\delta}, \quad a_0 = \frac{4B}{2\delta + \gamma^2}, \]  

(3.9)

Provided that \( \delta > 0 \).

From (3.1), (3.2) and (3.9), we deduce the exact traveling solutions of Eq. (1.1) as follows:

If \( b < 0 \), then we have the solutions

\[ u(\xi) = \frac{-\gamma}{2\delta} \left( \frac{A + B}{A - B} \right) \]  

(3.10)

or

\[ u(\xi) = \frac{-\gamma}{2\delta} \left( \frac{A - B}{A + B} \right) \]  

(3.11)

If \( b > 0 \) then we have the solutions

\[ u(\xi) = \frac{-\gamma}{2\delta} \left( \frac{A + B}{A - B} \right) \]  

(3.12)

or

\[ u(\xi) = \frac{-\gamma}{2\delta} \left( \frac{A - B}{A + B} \right) \]  

(3.13)

Where \( \xi = \frac{\gamma^2}{4\delta} x + \frac{2\delta + \gamma^2}{4\delta} t \).

**Result 3**

\[ b = 2\mu\delta - \gamma^2 + \gamma \sqrt{\gamma^2 - 4\mu\delta}, \quad \omega = \frac{-k}{2\delta} \left( \gamma + 3 \sqrt{\gamma^2 - 4\mu\delta} \right), \]  

(3.14)

\[ a_{-1} = 0, \quad a_0 = -\frac{\gamma + \sqrt{\gamma^2 - 4\mu\delta}}{4\delta}, \quad a_1 = \frac{\sqrt{2}k}{\delta}, \]  

Provided that \( 4\mu\delta - \gamma^2 > 0 \) and \( \delta > 0 \).

From (3.1), (3.2) and (3.14), we deduce the exact traveling solutions of Eq. (1.1) as follows:

If \( b < 0 \) then we have the solutions

\[ u(\xi) = \frac{-\gamma}{2\delta} \left( \frac{A + B}{A - B} \right) \]  

(3.15)

or

\[ u(\xi) = \frac{-\gamma}{2\delta} \left( \frac{A - B}{A + B} \right) \]  

(3.16)

under the constraint condition \( 2\mu\delta - \gamma^2 + \gamma \sqrt{\gamma^2 - 4\mu\delta} < 0 \).

If \( b > 0 \) then we have the solutions

\[ u(\xi) = \frac{-\gamma}{2\delta} \left( \frac{A + B}{A - B} \right) \]  

(3.17)
or

\[ u(\xi) = -\frac{\gamma}{\delta} + \frac{\sqrt{-4\mu\delta}}{4\delta} \left( -B + A - \sqrt{B + 2AB + A^2} \right) \left( \frac{1}{2\sqrt{\sqrt{B + 2AB + A^2}}} \right) \left( \xi - \frac{\sqrt{-4\mu\delta}}{2\delta} t \right) \]  
(3.18)

under the constraint condition \(2\mu\delta - \gamma^2 + \gamma\sqrt{-4\mu\delta} > 0\), where \( \xi = k\left(x - \frac{\sqrt{-4\mu\delta}}{2\delta} t\right) \).

**Result 4**

\[ b = b, \mu = \frac{8\delta}{4\delta} - \frac{\gamma}{\delta} , \omega = \frac{-\gamma}{\delta} \]  
(3.19)

Provided that \(\delta > 0\).

From (3.1), (3.2) and (3.19), we deduce the exact traveling wave solutions of Eq. (1.1) as follows:

If \(b < 0\) then we have the solutions

\[ u(\xi) = -\frac{\gamma}{\delta} + \frac{\sqrt{2\delta b}}{\delta} \left( -Bh - A\sqrt{B}\tan\left(\sqrt{-\delta}\xi\right) \right) \]  
(3.20)

or

\[ u(\xi) = -\frac{\gamma}{\delta} + \frac{\sqrt{2\delta b}}{\delta} \left( -Bh - A\sqrt{B}\coth\left(\sqrt{-\delta}\xi\right) \right) \]  
(3.21)

If \(b > 0\) then we have the solutions

\[ u(\xi) = -\frac{\gamma}{\delta} + \frac{\sqrt{2\delta b}}{\delta} \left( B + \sqrt{B}\tan\left(\sqrt{-\delta}\xi\right) \right) \]  
(3.22)

or

\[ u(\xi) = -\frac{\gamma}{\delta} + \frac{\sqrt{2\delta b}}{\delta} \left( B + \sqrt{B}\coth\left(\sqrt{-\delta}\xi\right) \right) \]  
(3.23)

where \(\xi = k\left(x - \frac{\gamma}{\sqrt{-\delta}} t\right)\).

**Result 5**

\[ b = \frac{2\mu + 2\omega}{4\delta} + \frac{\omega}{\delta} - \frac{2\mu\omega}{4\delta} \]  
(3.24)

Provided that \(\mu + 2\omega a_1 < 0\).

Then from (3.1), (3.2) and (3.24), we have the exact traveling solutions:

\[ u(\xi) = a_1 \frac{\sqrt{2(\mu + 2\omega)}}{4\delta} \left( B - A - B\sqrt{B + 2AB + A^2} \right) \left( \frac{1}{2\sqrt{B + 2AB + A^2}} \right) \left( \xi - \frac{\sqrt{\mu + 2\omega}}{\sqrt{\mu + 2\omega}} t \right) \]  
(3.25)

or

\[ u(\xi) = a_1 \frac{\sqrt{2(\mu + 2\omega)}}{4\delta} \left( B - A - B\sqrt{B + 2AB + A^2} \right) \left( \frac{1}{2\sqrt{B + 2AB + A^2}} \right) \left( \xi - \frac{\sqrt{\mu + 2\omega}}{\sqrt{\mu + 2\omega}} t \right) \]  
(3.26)

where \(\xi = k\left(x - \frac{2\mu + 2\omega}{\sqrt{\mu + 2\omega}} t\right)\).

**Result 6**

\[ b = \frac{A^2(2\gamma + 2\sqrt{4\mu\delta}) + 12\mu\delta - \gamma^2}{4B(4\mu\delta - \gamma)} \]  
(3.27)

Provided that \(\delta > 0, \mu > 0\) and \(4\mu\delta - \gamma^2 > 0\).

From (3.1), (3.2) and (3.27), we deduce the exact traveling solutions of Eq. (1.1) as follows:

If \(b < 0\) then we have the solutions

\[ u(\xi) = -\frac{\gamma}{\delta} + \frac{A\left(\gamma + 2\sqrt{4\mu\delta}\right)}{2Bb} \left( -Bb - A\sqrt{B}\tan\left(\sqrt{-\delta}\xi\right) \right) \]  
(3.28)

or

\[ u(\xi) = -\frac{\gamma}{\delta} + \frac{A\left(\gamma + 2\sqrt{4\mu\delta}\right)}{2Bb} \left( -Bb - A\sqrt{B}\coth\left(\sqrt{-\delta}\xi\right) \right) \]  
(3.29)

If \(b > 0\), then we have the solutions

\[ u(\xi) = \frac{\gamma}{\delta} + \frac{A\left(\gamma + 2\sqrt{4\mu\delta}\right)}{2Bb} \left( -Bb + A\sqrt{B}\tan\left(\sqrt{-\delta}\xi\right) \right) \]  
(3.30)

Or

\[ u(\xi) = \frac{\gamma}{\delta} + \frac{A\left(\gamma + 2\sqrt{4\mu\delta}\right)}{2Bb} \left( -Bb + A\sqrt{B}\coth\left(\sqrt{-\delta}\xi\right) \right) \]  
(3.31)

Where \(\xi = \frac{-B(\mu - 2\omega)}{2\delta\left(\gamma + 2\sqrt{4\mu\delta}\right)} t\).

**Result 7**

\[ b = b, \mu = \frac{\omega}{4\delta} + \frac{\omega}{\delta} - \frac{2\mu\omega}{4\delta} \]  
(3.32)

Provided that \(A^2 - 3bB^2 \neq 0\) and \(3A^2 - bB^2 \neq 0\).

From (3.1), (3.2) and (3.32), we deduce the exact traveling solutions of Eq. (1.1) as follows:

If \(b < 0\) then we have the solutions

\[ u(\xi) = \frac{-A\left(\mu + \omega\right) + \omega}{A\left(\gamma + 3\mu\delta\right)} \left( -Bb - A\sqrt{B}\tan\left(\sqrt{-\delta}\xi\right) \right) \]  
(3.33)

or

\[ u(\xi) = \frac{-A\left(\mu + \omega\right) + \omega}{A\left(\gamma + 3\mu\delta\right)} \left( -Bb - A\sqrt{B}\coth\left(\sqrt{-\delta}\xi\right) \right) \]  
(3.34)

If \(b > 0\), then we have the solutions

\[ u(\xi) = \frac{-A\left(\mu + \omega\right) + \omega}{A\left(\gamma + 3\mu\delta\right)} \left( -Bb - A\sqrt{B}\tan\left(\sqrt{-\delta}\xi\right) \right) \]  
(3.35)

Or

\[ u(\xi) = \frac{-A\left(\mu + \omega\right) + \omega}{A\left(\gamma + 3\mu\delta\right)} \left( -Bb - A\sqrt{B}\coth\left(\sqrt{-\delta}\xi\right) \right) \]  
(3.36)

Where \(\xi = \frac{-B(\mu + 2\omega)}{2\delta\left(\gamma + 3\mu\delta\right)} t\).

**Physical Explanations of Our Obtained Solutions**

The obtained exact traveling wave solutions for the nonlinear KPP equation (1.1) are hyperbolic, trigonometric and rational. In this section, we have presented
some graphs of the exact solutions constructed by taking suitable values Figure 1, Figure 2, Figure 3 and Figure 4 as it illustrates some of our results obtained in this article. To this end, we select some special values of the obtained parameters, for example, in some of the hyperbolic solutions (3.4), (3.5) and the trigonometric solutions (3.12), (3.13) of the nonlinear KPP equation (1.1) with \(B = k = \delta = \gamma = 1\), \(-10 < x, t < 1\), respectively.

**Conclusions**

In this article, we have employed the Bäcklund transformation of the Riccati equation to obtain exact traveling wave solutions of the nonlinear Kolmogorov-Petrovskii-Piskunov (KPP) equation (1.1). On comparing our results in this paper with the well-known results obtained in [22, 26, 46] we deduce that our results in this article are new and are not published elsewhere. Further, all solutions obtained in this article have been checked with the Maple by putting them back into the original equations. Finally, the proposed method in this article can be applied to many other nonlinear PDEs in mathematical physics, which will be done in forthcoming papers.

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