Some Remarks on the Best Approximation Rate of Certain Trigonometric Series

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ABSTRACT. The main object of the present paper is to give a complete result regarding the best approximation rate of certain trigonometric series in general complex valued continuous function space under a new condition which gives an essential generalization to $O$-regularly varying quasimonotonicity. An application is present in Section 3.

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§1. Introduction

As generalizations to monotonicity condition, various quasimonotonicity conditions including so-called $O$-regularly varying quasimonotonicity are introduced and applied to convergence of trigonometric (Fourier) series. Interested readers could check references such as Nurcombe [3], and Xie and Zhou [8] for uniform convergence and Stanovevic [4, 5], and Xie and Zhou [7] for $L^1$ convergence.

Let $C_{2\pi}$ be the space of all real or complex valued continuous functions $f(x)$ of period $2\pi$ with norm

$$\|f\| = \max_{-\infty < x < \infty} |f(x)|,$$

and $E_n(f)$ the best approximation by trigonometric polynomials of degree $n$. Given a trigonometric series $\sum_{k=-\infty}^{\infty} c_k e^{ikx} := \lim_{n \to \infty} \sum_{k=-n}^{n} c_k e^{ikx}$, write

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} =: \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx}$$

at any point $x$ where the series converges. Denote its $n$th partial sum $S_n(f, x)$ by

$$\sum_{k=-n}^{n} \hat{f}(k) e^{ikx}.$$

For a sequence $\{c_n\}_{n=0}^{\infty}$, let

$$\Delta c_n = c_n - c_{n+1}.$$

A non-decreasing positive sequence $\{R(n)\}_{n=1}^{\infty}$ is said to be $O$-regularly varying if

$$\limsup_{n \to \infty} \frac{R(2n)}{R(n)} < \infty.$$

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2In some papers, this requirement is written as that for some $\lambda > 1$, $\limsup_{n \to \infty} R([\lambda n])/R(n) < \infty$. It is just a pure decoration.
A complex sequence \( \{c_n\}_{n=0}^{\infty} \) is \( O \)-regularly varying quasimonotone in complex sense if for some \( \theta_0 \in [0, \pi/2) \) and some \( O \)-regularly varying sequence \( \{R(n)\} \) the sequence

\[
\Delta \frac{c_n}{R(n)} \in K(\theta_0) := \{z : |\arg z| \leq \theta_0\}, \quad n = 1, 2, \cdots.
\]

Evidently, if \( \{c_n\} \) is a real sequence, then the \( O \)-regularly varying quasimonotonicity becomes

\[
\Delta \frac{c_n}{R(n)} \geq 0, \quad n = 1, 2, \cdots,
\]

which was used in many works to generalize the regularly varying quasimonotone condition and, in particular, the quasimonotone condition\(^3\).

In 1992, without proof, Belov [1] announced the following result:

**Theorem B.** Let \( \{a_n\} \) and \( \{b_n\} \) be quasimonotone sequences satisfying \( \sum_{n=0}^{\infty} a_n < \infty \) and \( \lim_{n \to \infty} nb_n = 0 \). Define

\[
f(x) = \sum_{n=0}^{\infty} a_n \cos nx, \quad g(x) = \sum_{n=1}^{\infty} b_n \sin nx.
\]

Then \( f, g \in C_{2\pi} \) have the following estimates:

\[
E_n(f) \approx \max_{1 \leq k \leq n} ka_{n+k} + \sum_{k=2n+1}^{\infty} a_k,
\]

and

\[
E_n(g) \approx \max_{k \geq 1} kb_{n+k}.
\]

It is essentially generalized recently by Xiao, Xie and Zhou [6] for general complex valued trigonometric series (thus a proof is given to Theorem B) as follows.

**Theorem XZ.** Let \( \{\hat{f}(n)\}_{n=0}^{\infty} \) and \( \{\hat{f}(n) + \hat{f}(-n)\}_{n=1}^{\infty} \) be complex \( O \)-regularly varying quasimonotone sequences, and

\[
f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}.
\]

Then \( f \in C_{2\pi} \) if and only if

\[
\sum_{n=1}^{\infty} |\hat{f}(n) + \hat{f}(-n)| < \infty
\]

and

\[
\lim_{n \to \infty} n\hat{f}(n) = 0.
\]

\(^{3}\) A real sequence \( \{b_n\}_{n=0}^{\infty} \) is defined to be quasimonotone if, for some \( \alpha \geq 0 \), the sequence \( \{b_n/n^\alpha\} \) is non-increasing.
Furthermore, if \( f \in C_{2\pi} \), then

\[
E_n(f) \approx \max_{1 \leq k \leq n} \left( |\hat{f}(n + k)| + |\hat{f}(-n - k)| \right) + \max_{k \geq 2n+1} k |\hat{f}(k) - \hat{f}(-k)| + \sum_{k=2n+1}^{\infty} |\hat{f}(k) + \hat{f}(-k)|.
\]

We make a quick remark here. We notice that, in the complex valued function space, the assumption that both \( \{\hat{f}(n)\}_{n=0}^{\infty} \) and \( \{\hat{f}(-n)\}_{n=1}^{\infty} \) are complex \( O \)-regularly varying quasimonotone is a convenient one, but is almost trivial, since it is almost the same as the condition in real case. Thus people usually use one side quasimonotonicity with some kind balance condition in considering those problems. Theorem XZ reflects this kind of thinking.

Motivated by an idea in Leindler [2], our work [11] introduced a new condition, and we present here a revised form as follows:

**Definition.** Let \( c = \{c_n\}_{n=1}^{\infty} \) be a sequence satisfying \( c_n \in K(\theta_1) \) for some \( \theta_1 \in [0, 2\pi) \) and \( n = 1, 2, \cdots \). If there is a natural number \( N_0 \) such that

\[
\sum_{n=m}^{2m} |\Delta c_n| \leq M(c) \max_{m \leq n < m+N_0} |c_n| \quad (1)
\]

holds for all \( m = 1, 2, \cdots \), where \( M(c) \) indicates a positive constant only depending upon \( c \), then we say that the sequence \( c \) belongs to class GBV.

We recall the following results.

**Lemma 1.** Suppose a complex sequence \( \{c_n\} \) is \( O \)-regularly varying quasimonotone, then there is a positive constant \( M \) depending upon \( \theta_0 \) only such that

\[
|c_n| \leq M \text{Re} c_n, \quad n = 1, 2, \cdots,
\]

or in other words, \( c_n \in K(\theta_1) \) for some \( \theta_1 \in [0, \pi/2) \) and \( n = 1, 2, \cdots \).

The argument exactly follows from Xie and Zhou [8, Lemma 1], and the condition \( \lim_{n \to \infty} c_n = 0 \) there can be cancelled however.

**Lemma 2.** Let \( \{c_n\} \) be any given complex \( O \)-regularly varying quasimonotone sequence. Then \( \{c_n\} \) satisfies (1) for \( N_0 = 1 \).

The proof can be copied from Zhou and Le [11, Theorem 3] with omitting the condition \( \lim_{n \to \infty} c_n = 0 \) there.

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4As usual, \( A_n \approx B_n \) means that there is a positive constant \( C > 0 \) independent of \( n \) such that \( C^{-1}B_n \leq A_n \leq CB_n \).
We obviously see that, from Lemma 1 and Lemma 2, if \( c = \{c_n\}_{n=1}^{\infty} \) is any given complex \( O \)-regularly varying quasimonotone sequence, then \( \{c_n\} \in \text{GBV} \). On the other hand, since the converse is not true (see [11]), the class \( \text{GBV} \) gives an essential and explicit generalization to the class of \( O \)-regularly varying quasimonotone sequences.

From \( \sum_{k=1}^{\infty} k^{-\alpha} \sin 2^k x =: \sum_{n=1}^{\infty} b_n \sin nx \), \( \alpha > 1 \), we can clearly see that, for any \( \epsilon > 0 \), \( n^\epsilon b_n \to \infty \), \( n \to \infty \), therefore, the condition (1), in general sense, cannot be further generalized.

Throughout the paper, \( C \) denotes a positive constant (which is independent of \( n \) and \( x \in [0, 2\pi] \)) not necessarily the same at each occurrence. In some specific cases, we also use \( M(c) \) to indicate a positive constant only depending upon the sequence \( c \).

§2. Results and Proofs

We present the main result of the paper.

Theorem 1. Let \( \{\hat{f}(n)\}_{n=0}^{\infty} \in \text{GBV} \) and \( \{\hat{f}(n) + \hat{f}(-n)\}_{n=1}^{\infty} \in \text{GBV} \), and

\[
\hat{f}(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}.
\]

Then \( f \in C_{2\pi} \) if and only if

\[
\sum_{n=1}^{\infty} |\hat{f}(n) + \hat{f}(-n)| < \infty \quad (2)
\]

and

\[
\lim_{n \to \infty} n \hat{f}(n) = 0. \quad (3)
\]

Furthermore, if \( f \in C_{2\pi} \), then

\[
E_n(f) \approx \max_{1 \leq k \leq n} k \left( |\hat{f}(n + k)| + |\hat{f}(-n - k)| \right) + \max_{k \geq 2n+1} k \left| \hat{f}(k) - \hat{f}(-k) \right|
\]

\[
+ \sum_{k=2n+1}^{\infty} |\hat{f}(k) + \hat{f}(-k)|.
\]

First we establish several lemmas.

Lemma 3 (Xie and Zhou [8, Lemma 2]). Let \( \{\hat{f}(n)\} \) satisfy

\[
\hat{f}(n) + \hat{f}(-n) \in K(\theta_1), \quad n = 1, 2, \ldots,
\]
for some $\theta_1 \in [0, \pi/2)$. Then $f \in C_{2\pi}$ implies that

$$\sum_{n=1}^{\infty} |\hat{f}(n) + \hat{f}(-n)| < \infty.$$  

**Lemma 4.** Let $\{\hat{f}(n)\}_{n=0}^{\infty} \in \text{GBV}$ and $\{\hat{f}(n) + \hat{f}(-n)\}_{n=1}^{\infty} \in \text{GBV}$. Suppose $f \in C_{2\pi}$, then

$$\max_{k \geq 1} k|\hat{f}(\pm(n + k))| = O(E_n(f)).$$  

**Proof.** Let $t_n^*(x)$ be the trigonometric polynomials of best approximation of degree $n$, then from an easy equality

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |e^{\pm i(n+1)x} \left(\sum_{k=0}^{N-1} e^{\pm ikx}\right)|^2 \, dx = N$$

we get

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x) - t_n^*(x))e^{\pm i(n+1)x} \left(\sum_{k=0}^{N-1} e^{\pm ikx}\right) \, dx \right| \leq NE_n(f).$$

Also, the left integral within the absolute value symbols of the above inequality equals to

$$\sum_{k=1}^{N} \left( k \hat{f}(\mp(n + k)) + (N - k) \hat{f}(\mp(n + N + k)) \right),$$

then

$$NE_n(f) \geq \left| \sum_{k=1}^{N} \left( k \hat{f}(\mp(n + k)) + (N - k) \hat{f}(\mp(n + N + k)) \right) \right|$$

$$\geq \text{Re} \left( \sum_{k=1}^{N} \left( k \hat{f}(\mp(n + k)) + (N - k) \hat{f}(\mp(n + N + k)) \right) \right).$$

Thus by the definition of GBV,

$$NE_n(f) \geq \sum_{k=1}^{N} \left( k \text{Re} \hat{f}(n + k) + (N - k) \text{Re} \hat{f}(n + N + k) \right)$$

$$\geq \sum_{k=1}^{N} k \text{Re} \hat{f}(n + k), \quad (4)$$

and

$$2NE_n(f) \geq \sum_{k=1}^{N} k \left( \text{Re} \hat{f}(n + k) + \text{Re} \hat{f}(-n - k) \right). \quad (5)$$

By the same argument as (4), we also have

$$2NE_{2n}(f) \geq \sum_{k=1}^{2N} k \text{Re} \hat{f}(2n + k). \quad (4')$$
Fix sufficient large $n$, assume
\[
\max_{N/2+jN_0 \leq k < N/2 + (j+1)N_0} |\hat{f}(n+k)| = |\hat{f}(n+k_j)|, \quad j = 0, 1, \ldots, \lfloor N/(2N_0) \rfloor - 1,
\]
\[
n + N/2 \leq n + N/2 + jN_0 \leq n + k_j < n + N/2 + (j+1)N_0 \leq n + N.
\]
From condition (1), we get for $0 \leq j \leq \lfloor N/(2N_0) \rfloor - 1$,
\[
|\hat{f}(n+N)| = \left| \sum_{k=n+N}^{2n+N+2jN_0-1} \Delta \hat{f}(k) + \hat{f}(2n+N+2jN_0) \right|
\leq \sum_{n+N/2+jN_0 \leq k \leq 2n+N+2jN_0} |\Delta \hat{f}(k)| + |\hat{f}(2n+N+2jN_0)|
\leq M(\hat{f})(|\hat{f}(n+k_j)| + |\hat{f}(2n+N+2jN_0)|)
\leq M(\hat{f}, \theta_1) \left( \text{Re} \hat{f}(n+k_j) + \text{Re} \hat{f}(2n+N+2jN_0) \right),
\]
therefore, together with (4) and (4'), we deduce that
\[
3NE_n(f) \geq NE_n(f) + 2NE_2n(f) \geq \sum_{k=1}^{N} k \text{Re} \hat{f}(n+k) + \sum_{k=1}^{2N} k \text{Re} \hat{f}(2n+k)
\geq \sum_{j=0}^{\lfloor N/(2N_0) \rfloor - 1} j \text{Re} \hat{f}(n+k_j) + \sum_{j=0}^{\lfloor N/(2N_0) \rfloor - 1} (N+2jN_0) \text{Re} \hat{f}(2n+N+2jN_0)
\geq C \sum_{j=0}^{\lfloor N/(2N_0) \rfloor - 1} j \left( \text{Re} \hat{f}(n+k_j) + \text{Re} \hat{f}(2n+N+2jN_0) \right)
\geq C \sum_{j=0}^{\lfloor N/(2N_0) \rfloor - 1} j |\hat{f}(n+N)| \geq C |\hat{f}(n+N)| \sum_{j=0}^{\lfloor N/(2N_0) \rfloor - 1} (N/2 + jN_0)
\geq CN_0^{-1}N^2 |\hat{f}(n+N)|.
\]
Finally, we achieve that, for $N \geq 1$,
\[
N|\hat{f}(n+N)| \leq CE_n(f),
\]
or in other words,
\[
\max_{k \geq 1} k |\hat{f}(n+k)| \leq CE_n(f).
\]
At the same time, starting from (5), since \( \{\hat{f}(n) + \hat{f}(-n)\}_{n=1}^{\infty} \in \text{GBV} \), a similar argument leads to that
\[
\max_{k \geq 1} k |\hat{f}(n+k) + \hat{f}(-n-k)| = O(E_n(f)).
\]
Thus
\[
\max_{k \geq 1} k|\hat{f}(-n - k)| \leq \max_{k \geq 1} k|\hat{f}(n + k)| + \hat{f}(-n - k)| + \max_{k \geq 1} k|\hat{f}(n + k)| = O(E_n(f)).
\]

Lemma 4 is completed.

As an important application, we write the following corollary of Lemma 4 as a theorem.

**Theorem 2.** Let \( \{\hat{f}(n)\}_{n=0}^{\infty} \in \text{GBV} \) and \( \{\hat{f}(n) + \hat{f}(-n)\}_{n=1}^{\infty} \in \text{GBV} \). If \( f \in C_{2\pi} \), then
\[
\lim_{n \to \infty} n|\hat{f}(\pm n)| = 0.
\]

**Corollary.** Let \( \{b_n\}_{n=1}^{\infty} \in \text{GBV} \) be a real sequence. Suppose \( g(x) = \sum_{n=1}^{\infty} b_n \sin nx \in C_{2\pi} \), then
\[
\lim_{n \to \infty} nb_n = 0.
\]

**Lemma 5 (Xiao, Xie and Zhou [6, Lemma 4]).** Let \( f \in C_{2\pi} \), \( \hat{f}(n) + \hat{f}(-n) \in K(\theta_1), n = 1, 2, \ldots, \) for some \( 0 \leq \theta_1 < \pi/2 \). Then
\[
\sum_{k=2n+1}^{\infty} |\hat{f}(k) + \hat{f}(-k)| = O(E_n(f)).
\]

**Lemma 6.** Let \( \{\hat{f}(n)\}_{n=0}^{\infty} \in \text{GBV} \) and \( \{\hat{f}(n) + \hat{f}(-n)\}_{n=1}^{\infty} \in \text{GBV} \), then
\[
\left| \sum_{k=1}^{n} \hat{f}(\pm(n + k)) \sin kx \right| = O\left( \max_{1 \leq k \leq n} k \left( |\hat{f}(n + k)| + |\hat{f}(-n - k)| \right) \right)
\]
holds uniformly for any \( x \in [0, \pi] \).

**Proof.** The cases \( x = 0 \) and \( x = \pi \) are trivial. Suppose \( 0 < x < \pi \). When \( 0 < x \leq \pi/n \), by the inequality \( |\sin x| \leq |x| \) we have
\[
\left| \sum_{k=1}^{n} \hat{f}(n + k) \sin kx \right| \leq \frac{\pi}{n} \sum_{k=1}^{n} k|\hat{f}(n + k)| \leq \pi \max_{1 \leq k \leq n} k|\hat{f}(n + k)|.
\]

On the other hand, in case \( \pi/n < x \), we find an natural number \( m < n \) such that \( m \leq \pi/x < m + 1 \), then
\[
\left| \sum_{k=1}^{n} \hat{f}(n + k) \sin kx \right| \leq \frac{\pi}{m} \sum_{k=1}^{m} k|\hat{f}(n + k)| + \left| \sum_{k=m+1}^{n} \hat{f}(n + k) \sin kx \right| =: |I_1| + |I_2|.
\]

It is clear that
\[
|I_1| \leq \pi \max_{1 \leq k \leq m} k|\hat{f}(n + k)|.
\]

\(^{5}\)It means \( b_n \geq 0 \) and satisfies condition (1).
By Abel transformation,

\[ I_2 = \sum_{k=m+1}^{n-N_0-1} \Delta \hat{f}(n+k) \sum_{j=1}^{k} \sin jx + \hat{f}(2n-N_0) \sum_{j=1}^{n-N_0} \sin jx - \hat{f}(n+m+1) \sum_{j=1}^{m} \sin jx + \sum_{k=n-N_0+1}^{n} \hat{f}(n+k) \sin kx, \]

so that\(^6\)

\[ |I_2| \leq (m+1) \sum_{k=m+1}^{n-N_0-1} |\Delta \hat{f}(n+k)| + C \max_{m<k \leq n} k|\hat{f}(n+k)| + N_0 \max_{n-N_0<k \leq n} |\hat{f}(n+k)|. \]

But by the condition of Lemma 6, taking a natural number \(l\) such that \(2^l(m+1) \leq n-N_0-1 < 2^{l+1}(m+1)\), and setting

\[ \max_{2^j(m+1) \leq k < 2^{j+1}(m+1)+N_0} |\hat{f}(n+k)| = |\hat{f}(n+k_j)|, \]

we then have\(^7\)

\[ \sum_{k=m+1}^{n-N_0-1} |\Delta \hat{f}(n+k)| \leq \sum_{j=0}^{l+1} \sum_{k=2^j(m+1)}^{2^{j+1}(m+1)-1} |\Delta \hat{f}(n+k)| \leq M(\hat{f}) \sum_{j=0}^{l+1} |\hat{f}(n+k_j)| \]

\[ \leq C \max_{1 \leq k \leq n} k|\hat{f}(n+k)| \sum_{j=0}^{l+1} k_j^{-1} \leq C(m+1)^{-1} \max_{1 \leq k \leq n} k|\hat{f}(n+k)| \sum_{j=0}^{\infty} 2^{-j} \]

\[ \leq C(m+1)^{-1} \max_{1 \leq k \leq n} k|\hat{f}(n+k)|. \]

Therefore, with the above estimates, we get

\[ |I_2| \leq C \max_{1 \leq k \leq n} k|\hat{f}(n+k)|. \]

Combining with (6) and (7), we have

\[ \left| \sum_{k=1}^{n} \hat{f}(n+k) \sin kx \right| = O \left( \max_{1 \leq k \leq n} k|\hat{f}(n+k)| \right). \]

Now write

\[ \sum_{k=1}^{n} \hat{f}(-n-k) \sin kx = \sum_{k=1}^{n} \left( \hat{f}(n+k) + \hat{f}(-n-k) \right) \sin kx - \sum_{k=1}^{n} \hat{f}(n+k) \sin kx. \]

\(^6\)Note that \(\sum_{j=1}^{k} \sin jx = O(x^{-1}) = O((m+1))\) for \(m \leq \pi/x < m+1\).

\(^7\)Note that any \(k_j \leq n\).
Applying the above known estimate, by noting that both \( \{\hat{f}(n)\}_{n=0}^\infty \in \text{GBV} \) and \( \{\hat{f}(n) + \hat{f}(-n)\}_{n=1}^\infty \in \text{GBV} \), we get

\[
\left| \sum_{k=1}^n \hat{f}(-n-k) \sin kx \right| = O \left( \max_{1 \leq k \leq n} k \left| \hat{f}(n+k) + \hat{f}(-n-k) \right| \right) \\
= O \left( \max_{1 \leq k \leq n} k \left( \left| \hat{f}(n+k) \right| + \left| \hat{f}(-n-k) \right| \right) \right).
\]

Lemma 6 is proved.

Similarly, we can establish the following

**Lemma 7.** Let \( f \in C_{2\pi} \), \( \{\hat{f}(n)\}_{n=0}^\infty \in \text{GBV} \) and \( \{\hat{f}(n) + \hat{f}(-n)\}_{n=1}^\infty \in \text{GBV} \), then

\[
\left| \sum_{k=m}^{\infty} \hat{f}(\pm k) \sin kx \right| = O \left( \max_{k \geq m} \left( \left| \hat{f}(k) \right| + \left| \hat{f}(-k) \right| \right) \right)
\]

holds uniformly for any \( x \in [0, \pi] \) and any \( m \geq 1 \).

**Proof.** Write

\[ I(x) = \sum_{k=m}^{\infty} \hat{f}(k) \sin kx. \]

Noting for \( x = 0 \) and \( x = \pi \) that \( I(x) = 0 \), we may restrict \( x \) within \((0, \pi)\) without loss. Take \( N = [1/x] \) and set

\[ I(x) = \sum_{k=m}^{N-1} \hat{f}(k) \sin kx + \sum_{k=N}^{\infty} \hat{f}(k) \sin kx =: J_1(x) + J_2(x). \]

Now

\[ |J_1(x)| \leq \sum_{k=m}^{N-1} |\hat{f}(k)| |\sin kx| \leq x \sum_{k=m}^{N-1} k |\hat{f}(k)| \]

\[ < x(N - 1) \epsilon_m \leq \epsilon_m \]

follows from \( N = [1/x] \), where \( \epsilon_m = \max k |\hat{f}(k)| \). By Abel’s transformation, similar to the proof of Lemma 6,

\[ |J_2(x)| = \left| \sum_{k=N}^{\infty} \Delta \hat{f}(k) \sum_{v=1}^{k} \sin vx - \hat{f}(N) \sum_{v=1}^{N-1} \sin vx \right| \]

\[ \leq \sum_{k=N}^{\infty} |\Delta \hat{f}(k)| \left| \sum_{v=1}^{k} \sin vx \right| + |\hat{f}(N)| \left| \sum_{v=1}^{N-1} \sin vx \right| \]

\[ \text{When } N \leq m, \text{ the same argument as in estimating } J_2 \text{ can be applied to deal with } I(x) = \sum_{k=m}^{\infty} \hat{f}(k) \sin kx \text{ directly.} \]
 Altogether, we have 
\[ |I(x)| \leq C \epsilon_m, \]
that is the required result for \( \sum_{k=m}^{\infty} \hat{f}(k) \sin kx \). Since \( \hat{f}(-k) = \hat{f}(k) + \hat{f}(-k) - \hat{f}(k) \), by the condition, we can deduce the required result for \( \sum_{k=m}^{\infty} \hat{f}(-k) \sin kx \). Lemma 7 is completed.

**Proof of Theorem 1. Necessity.** Suppose \( f \in C_{2\pi} \). From Lemma 4, (3) clearly holds, while by Lemma 5, we see

\[ \sum_{k=2n+1}^{\infty} \sum_{k=2n+1}^{\infty} |\hat{f}(k) + \hat{f}(-k)| = O(E_n(f)), \]

thus (2) holds.

**Sufficiency.** It can be deduced from Zhou and Le [11, Theorem 1] with a minor revision.

Assume \( f \in C_{2\pi} \) now. By Lemma 4 and Lemma 5, we see

\[ \max_{k \geq 1} k|\hat{f}(\pm(n + k))| + \sum_{k=2n+1}^{\infty} |\hat{f}(k) + \hat{f}(-k)| = O(E_n(f)). \]

On the other hand, rewrite \( f(x) \) as

\[
\begin{align*}
  f(x) &= \sum_{k=-2n}^{2n} \hat{f}(k)e^{ikx} + i \sum_{k=2n+1}^{\infty} \left( \hat{f}(k) - \hat{f}(-k) \right) \sin kx \\
  &\quad + \frac{1}{2} \sum_{k=2n+1}^{\infty} \left( \hat{f}(k) + \hat{f}(-k) \right) \left( e^{ikx} + e^{-ikx} \right),
\end{align*}
\]

then we have

\[
E_n(f) \leq \left\| \sum_{k=1}^{n} \left( \hat{f}(n + k)e^{i(n+k)x} + \hat{f}(-n - k)e^{-i(n+k)x} \right) \right\| \\
- \left\| \sum_{k=1}^{n} \left( \hat{f}(n + k)e^{i(n-k)x} + \hat{f}(-n - k)e^{i(n+k)x} \right) \right\| \\
+ \left\| \sum_{k=2n+1}^{\infty} \left( \hat{f}(k) - \hat{f}(-k) \right) \sin kx \right\| + \left\| \sum_{k=2n+1}^{\infty} \left( \hat{f}(k) + \hat{f}(-k) \right) \sin kx \right\| \\
\leq \left\| \sum_{k=1}^{n} \left( \hat{f}(n + k) \sin kx + \hat{f}(-n - k) \sin(-kx) \right) \right\| + \left\| \sum_{k=2n+1}^{\infty} \left( \hat{f}(k) - \hat{f}(-k) \right) \sin kx \right\|. 
\]
Applying Lemma 6 immediately yields that
\[
\left\| \sum_{k=1}^{n} \left( \hat{f}(n+k) \sin kx + \hat{f}(-n-k) \sin(-kx) \right) \right\| \leq \max_{1 \leq k \leq n} k \left( |\hat{f}(n+k)| + |\hat{f}(-n-k)| \right),
\]
while by noting that \( \hat{f}(k) - \hat{f}(-k) = 2\hat{f}(k) - (\hat{f}(k) + \hat{f}(-k)) \) and both \( \{\hat{f}(n)\}_{n=0}^{\infty} \in GBV \) and \( \{\hat{f}(n) + \hat{f}(-n)\}_{n=1}^{\infty} \in GBV \), we deduce that
\[
\left\| \sum_{k=2n+1}^{\infty} \left( \hat{f}(k) - \hat{f}(-k) \right) \sin kx \right\| = O \left( 2 \max_{k \geq 2n+1} k |\hat{f}(k)| + \max_{k \geq 2n+1} k |\hat{f}(k) + \hat{f}(-k)| \right)
\]
by Lemma 7. Suppose \( \max_{k \geq 2n+1} |\hat{f}(k)| = m_0 |\hat{f}(m_0)| \). Take sufficient large \( n \) such that \( 4n - 2N_0 \geq 3n + 1 \). Assume that \( 2n+1 \leq m_0 \leq 4n \). Then in view of (1), by a standard technique used in the present paper, one has
\[
|\hat{f}(m_0)| \leq \sum_{k=m_0}^{4n-2N_0} |\Delta \hat{f}(k)| + |\hat{f}(4n - 2N_0 + 1)|
\]
\[
\leq \sum_{k=2n-N_0}^{4n-2N_0} |\Delta \hat{f}(k)| + |\hat{f}(4n - 2N_0 + 1)|
\]
\[
\leq C \left( \max_{n-N_0 \leq k \leq n} |\hat{f}(n+k)| + |\hat{f}(4n - 2N_0 + 1)| \right).
\]
Hence,
\[
m_0 |\hat{f}(m_0)| \leq C \left( \max_{1 \leq k \leq n} k |\hat{f}(n+k)| + (4n - 2N_0 + 1) |\hat{f}(4n - 2N_0 + 1)| \right). \tag{8}
\]
Meanwhile, for \( m_0 \geq 4n + 1 \),
\[
m_0 |\hat{f}(m_0)| \leq \frac{1}{2} m_0 |\hat{f}(m_0) - \hat{f}(-m_0)| + \frac{1}{2} m_0 |\hat{f}(m_0) + \hat{f}(-m_0)|
\]
\[
\leq \max_{k \geq 2n+1} k |\hat{f}(k) - \hat{f}(-k)| + O \left( \sum_{k=2n+1}^{\infty} |\hat{f}(k) + \hat{f}(-k)| \right),
\]
where the last inequality follows from \( m_0 \geq 4n + 1 \) and the following calculation: Assume that
\[
\max_{(m_0+1)/2 + jN_0 \leq k < (m_0+1)/2 + (j+1)N_0} |\hat{f}(k) + \hat{f}(-k)| = |\hat{f}(k_j) + \hat{f}(-k_j)|,
\]
\[j = 0, 1, \ldots, [(m_0 + 1)/(2N_0)] - 1,\]
2n + 1 \leq (m_0 + 1)/2 + jN_0 \leq k_j < (m_0 + 1)/2 + (j + 1)N_0 < m_0 + 1.

From condition (1), we get for 0 \leq j \leq \lceil (m_0 + 1)/(2N_0) \rceil - 1,

\[
|\hat{f}(m_0) + \hat{f}(-m_0)| \leq \sum_{(m_0+1)/2+jN_0 \leq k \leq m_0+1+2jN_0} |\Delta(\hat{f}(k) + \hat{f}(-k))| + |\hat{f}(m_0 + 1 + 2jN_0 + 1) + \hat{f}(-m_0 - 1 - 2jN_0 - 1)|
\leq M(\hat{f})|\hat{f}(k_j) + \hat{f}(-k_j)| + |\hat{f}(m_0 + 1 + 2jN_0 + 1) + \hat{f}(-m_0 - 1 - 2jN_0 - 1)|,
\]

thus

\[
m_0|\hat{f}(m_0) + \hat{f}(-m_0)| \leq CM(\hat{f}) \sum_{k = 2n + 1}^{\infty} |\hat{f}(k) + \hat{f}(-k)|.
\]

By the same technique, for the factor appearing in (8), we also have (note 4n - 2N_0 \geq 3n + 1)

\[
(4n - 2N_0 + 1) |\hat{f}(4n - 2N_0 + 1)| \leq \max_{k \geq 2n+1} k |\hat{f}(k) - \hat{f}(-k)| + O\left( \sum_{k = 2n + 1}^{\infty} |\hat{f}(k) + \hat{f}(-k)| \right).
\]

From the above discussion, we see that

\[
\max_{k \geq 2n + 1} k |\hat{f}(k)| \leq C \max_{1 \leq k \leq n} k |\hat{f}(n + k)| + \max_{k \geq 2n + 1} k |\hat{f}(k) - \hat{f}(-k)| + O\left( \sum_{k = 2n + 1}^{\infty} |\hat{f}(k) + \hat{f}(-k)| \right)
\]

holds in any case. With condition that \(\hat{f}(k) + \hat{f}(-k) \in \text{GBV}\), by a similar way we can also get

\[
\max_{k \geq 2n + 1} k |\hat{f}(k) + \hat{f}(-k)| \leq C \max_{1 \leq k \leq n} k \left( |\hat{f}(n + k)| + |\hat{f}(-n - k)| \right) + O\left( \sum_{k = 2n + 1}^{\infty} |\hat{f}(k) + \hat{f}(-k)| \right).
\]

Combining all the above estimates, we have completed the proof of Theorem 1.

§3. An Application

Let

\[
f(x) = \sum_{n=0}^{\infty} \hat{f}(n) \cos nx.
\]

The following theorem is an interesting application to a hard problem in classical Fourier analysis. Interested readers could check Zhou [10] or Xie and Zhou [9, p.112], for example.
**Theorem 3.** Let \( \{\hat{f}(n)\} \in \text{GBV} \) be a real sequence. If \( f \in C_{2\pi} \) and
\[
\sum_{k=n+1}^{2n} \hat{f}(k) = O\left( \max_{1 \leq k \leq n} k\hat{f}(n+k) \right),
\]
then
\[
\|f - S_n(f)\| = O(E_n(f)),
\]
where \( S_n(f) \) is the \( n \)th Fourier partial sum of \( f \).

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