COMBINATORIAL RECIPROCITY THEOREMS FOR GENERALIZED
PERMUTAHEDRA, HYPERGRAPHS, AND PRUNED INSIDE-OUT
POLYTOPES

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Abstract. Generalized permutahedra are a class of polytopes with many interesting combinatorial subclasses. We introduce pruned inside-out polytopes, a generalization of inside-out polytopes introduced by Beck–Zaslavsky (2006), which have many applications such as recovering the famous reciprocity result for graph colorings by Stanley. We study the integer point count of pruned inside-out polytopes by applying classical Ehrhart polynomials and Ehrhart–Macdonald reciprocity. This yields a geometric perspective on and a generalization of a combinatorial reciprocity theorem for generalized permutahedra by Aguiar–Ardila (2017) and Billera–Jia–Reiner (2009). Applying this reciprocity theorem to hypergraphic polytopes allows us to give an arguably simpler proof of a recent combinatorial reciprocity theorem for hypergraph colorings by Aval–Karaboghossian–Tanasa (2020). Our proof relies, aside from the reciprocity for generalized permutahedra, only on elementary geometric and combinatorial properties of hypergraphs and their associated polytopes.

1. Introduction

Generalized permutahedra are an interesting class of polytopes containing numerous subclasses of polytopes defined via combinatorial structures, such as graphic zonotopes, hypergraphic polytopes (Minkowski sums of simplices), simplicial complex polytopes, matroid polytopes, associahedra, and nestohedra. Generalized permutahedra themselves are closely related to submodular functions, which have applications in optimization.

A combinatorial reciprocity theorem can be described as a result that relates two classes of combinatorial objects via their enumeration problems (see, e.g., [Sta74, BS18]). For example, the number of proper $m$-colorings of a graph $g = (I, E)$ agrees with a polynomial $\chi(g)(m)$ of degree $d = |I|$ for positive integers $m \in \mathbb{Z}_{>0}$, and $(-1)^d \chi(g)(-m)$ counts the number of pairs of compatible acyclic orientations and $m$-colorings of the graph $g$ [Sta73]. For precise definitions see Section 4.2 below.

One of our main results is a combinatorial reciprocity theorem for generalized permutahedra counting integral directions with $k$-dimensional maximal faces:

**Theorem 4.4.** For a generalized permutahedron $\mathcal{P} \subset \mathbb{R}^d$ and $k = 0, \ldots, d - 1$,

$$\chi_{d,k}(\mathcal{P})(m) := \# \{ y \in [m]^d : y\text{-maximum face } \mathcal{P}_y \text{ is a } k\text{-face} \}
$$

agrees with a polynomial of degree $d - k$, and

$$(-1)^{d-k} \chi_{d,k}(\mathcal{P})(-m) = \sum_{y \in [m]^d} \# (k\text{-faces of } \mathcal{P}_y).$$

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We will use integer point counting in dissected and dilated cubes to prove this result.

The special case of this theorem for \( k = 0 \), i.e., generic directions, was obtained by Aguiar and Ardila [AA17], and earlier by Billera, Jia, and Reiner [BJR09] in a slightly different language. As shown for some examples in [AA17, Section 18] the application of such a result to the various subclasses of generalized permutahedra yields already known combinatorial reciprocity theorems for their related combinatorial structures such as matroid polynomials [BJR09], Bergmann polynomials of matroids and Stanley’s famous reciprocity theorem for graph colorings [Sta73].

Aguiar and Ardila develop a Hopf monoid structure on the species of generalized permutahedra, work with polynomial invariants defined by characters, and apply their antipode formula to get the combinatorial interpretation of the reciprocity result for generalized permutahedra for \( k = 0 \) (Theorem 4.1) [AA17, Sections 16, 17]. The approach in [BJR09] is similar to the one by Aguiar and Ardila. Billera, Jia, and Reiner use Hopf algebras of matroids and quasisymmetric functions, as well as a multivariate generating function as isomorphism invariants of matroids. The reciprocity providing ingredient is again the antipode of a Hopf algebra together with Stanley’s reciprocity for \( P \)-partitions [BJR09, Sections 6 and 9].

We give a different, a geometric perspective. In order to prove Theorem 4.4 we apply Ehrhart–Macdonald reciprocity to pruned inside-out polytopes. A pruned inside-out polytope \( Q \setminus \bigcup N^{\text{co}1} \) consist of the points that lie inside a polytope \( Q \) but not in the codimension one cones \( N^{\text{co}1} \) of a complete polyhedral fan \( N \). This is a generalization of inside-out polytopes introduced by Beck and Zaslavsky [BZ06b]. An inside-out polytope \( Q \setminus H \) consists of the points in a polytope \( Q \) but off the hyperplanes in the arrangement \( H \). We think of the codimension-one cones defining a pruned inside-out polytope as pruned hyperplanes, hence the name. One of the many applications of inside-out polytopes [BZ06c, BZ06a, BZ10, BS18] is yet a different proof of Stanley’s reciprocity result for graph colorings [Sta73].

Recently, Aval, Karaboghossian, and Tanasa presented a reciprocity theorem for hypergraph colorings [AKT20], generalizing Stanley’s result for graph colorings. A main tool in the paper is a Hopf monoid structure on hypergraphs defined in [AA17, Section 20.1] and the associated basic polynomial invariant. However, they do not use the antipode as reciprocity inducing element, but rather technical computations involving Bernoulli numbers.

In Section 4.2 we show how the reciprocity theorem for hypergraph colorings in [AKT20] is a consequence of the reciprocity for generalized permutahedra. Our main tool is a vertex description of hypergraphic polytopes in terms of acyclic orientations of hypergraphs (Proposition 4.8). This allows us to give an arguably simpler proof of Aval, Karaboghossian, and Tanasa’s result on hypergraph colorings.

As spelled out in [AA17, Sections 21–25] and [AKT20, Section 4] hypergraphs and hypergraphic polytopes contain a number of interesting combinatorial subclasses such as simple hypergraphs, graphs, simplicial complexes, building sets, set partitions, and paths, together with their associated polytopes such as graphical zonotopes, simplicial complex polytopes, nestohedra, and graph associahedra.

The paper is organized as follows: In Section 2 we recall basic notions from discrete geometry, set the notation, and review some facts about standard and generalized permutahedra. Secondly we give a very brief overview of Ehrhart theory. In Section 3 we introduce the notion of pruned inside-out polytopes, define two counting functions on pruned inside-out polytopes, and derive (quasi-)polynomiality and reciprocity results. Section 4 provides two applications of the results in Section 3; first, to generalized permutahedra, giving a new geometric perspective on reciprocity theorems in [BJR09, AA17] and, moreover, presenting a generalized
version for arbitrary face dimensions. Lastly, we apply the reciprocity theorem for generalized permutahedra to the subclass of hypergraphic polytopes giving an elementary combinatorial and geometric proof of the reciprocity thereorem for hypergraph colorings in [AKT20].

2. Preliminaries

In this paper we will consider geometric results developed in a Hopf-algebraic context, and interpret and extend them from an discrete-geometric perspective. We will use two different notations for real vector spaces, one with an unordered basis $I$ and one with the ordered basis consisting of the canonical unit vectors. The unordered basis is particularly convenient in the Hopf–algebraic setting while the standard notation is easier to handle in discrete-geometric situations.

For a non-empty finite set $I$ let $\mathbb{R}^I$ be the real vector space with distinguished, unordered basis $I$. We will denote the elements $i \in I$ with $b_i$ when we want to distinguish the elements $i$ in the set $I$ from the corresponding basis vector $b_i$ in the vector space $\mathbb{R}^I$. Moreover, we identify an element $\sum_{i \in I} x_i b_i$ in the vector space $\mathbb{R}^I$ with the tupel $(x_i)_{i \in I}$ for $x_i \in \mathbb{R}$, so

$$\mathbb{R}^I = \{(x_i)_{i \in I} : x_i \in \mathbb{R}\} = \left\{ \sum_{i \in I} x_i b_i : x_i \in \mathbb{R} \right\}.$$ 

For the disjoint union $I = S \sqcup T$ of two finite sets $S, T$ the equality $\mathbb{R}^S \times \mathbb{R}^T = \mathbb{R}^I = \mathbb{R}^T \times \mathbb{R}^S$ holds, which is handy in combinatorial contexts.

In some sections we will switch to the standard basis notation. That is, for a finite set $I$ with $|I| := d$ we will identify $\mathbb{R}^d \cong \mathbb{R}^I$ by fixing a bijection $\sigma : I \to [d] := \{1, \ldots, d\}$. Via this bijection we may also assume $I = [d]$.

We recall some basic notions from polytopes; for more detailled information consult, e.g., [Zie98, Grü03]. A polyhedron $\mathcal{P} \subset \mathbb{R}^d$ is the intersection of finitely many halfspaces. If the intersection is bounded it is called a polytope and can equivalently be described as the convex hull of finitely many points in $\mathbb{R}^d$. A (polyhedral) cone $N$ is a polyhedron such that for $x \in N$ the point $\lambda x$ is again contained in $N$ for every $\lambda \in \mathbb{R}_{\geq 0}$. A supporting hyperplane $H$ of a polyhedron $\mathcal{P}$ is a hyperplane such that the polyhedron is contained in one of the closed halfspaces. The intersection of a polyhedron $\mathcal{P}$ with a supporting hyperplane $H$ is a face $F = H \cap \mathcal{P}$. The dimension $\dim(\mathcal{P})$ (resp. $\dim(F)$) of a polyhedron $\mathcal{P}$ (resp. face $F$) is the dimension of the affine hull of the polytope $\mathcal{P}$ (resp. face $F$), 0-dimensional faces are called vertices and $(\dim(\mathcal{P}) - 1)$-dimensional faces are called facets. The codimension $\text{codim}(F)$ of an polyhedron $F$ is the difference between the dimension of the ambient space and the dimension of the polyhedron $\dim(F)$. A polyhedron $\mathcal{P}$ is a rational polyhedron, if all its facet defining hyperplanes $H$ can be described as $H = \{ x \in \mathbb{R}^d : \mathbf{a} \cdot x = b \}$ for some $\mathbf{a} \in \mathbb{Z}^d$ and $b \in \mathbb{Z}$.

2.1. Generalized permutahedra. We define the standard permutahedron $\pi_I$ as the convex hull of the $d!$ permutations of the point $(1, 2, \ldots, d)$ where $|I| = d$, that is, the standard permutahedron $\pi_I$ is defined by

$$\pi_I := \text{conv} \left\{ (x_i)_{i \in I} \in \mathbb{R}^I : (x_i)_{i \in I} = [d] \right\} \subset \mathbb{R}^I.$$ 

$^1$The definition of standard permutahedron is not consistent within literature, e.g., Postnikov defines the standard permutahedron in a more general way: as the convex hull of all the points obtained by permuting the coordinates of an arbitrary point [Pos09, Definition 2.1].
Figure 1 shows some examples. Note that the standard permutahedron is of dimension $|I| - 1$ since all vertices are contained in a hyperplane with constant coordinate sum. In our definition, standard permutahedra are integer polytopes.

The standard permutahedron can equivalently be described as the Minkowski sum of line segments:

$$\pi_I = \sum_{\{i,j\}} \Delta_{\{i,j\}},$$

where the sum is over all two-element subsets of $I$ and $\Delta_{\{i,j\}} := \text{conv}\{b_i, b_j\}$. This implies, in particular, that standard permutahedra are zonotopes. The facet description of the standard permutahedron is given by

$$\sum_{i \in I} x_i = |I| + (|I| - 1) + \cdots + 1 = \frac{|I|(|I| + 1)}{2}$$

$$\sum_{i \in T} x_i \leq |I| + (|I| - 1) + \cdots + (|I| - |T| + 1) \text{ for all } T \subseteq I.$$

Moreover, every face of the standard permutahedron can be described combinatorially by compositions, for details see, e.g., [AA17, Section 4.1].

Let $(\mathbb{R}I)^*$ be the dual vector space to $\mathbb{R}I$. We identify

$$(\mathbb{R}I)^* = \mathbb{R}^I := \{\text{maps } y : I \to \mathbb{R}\}$$

and call the elements $y \in \mathbb{R}^I$ directions. Directions $y \in \mathbb{R}^I$ act as linear functionals on elements $x = \sum_{i \in I} x_i b_i \in \mathbb{R}I$ via

$$y\left(\sum_{i \in I} x_i b_i\right) = \sum_{i \in I} x_i y(i).$$

As for the primal vector space $\mathbb{R}I$ we can employ a bijection $\sigma : I \to [d]$ to identify the dual vector spaces $\mathbb{R}^I \simeq (\mathbb{R}^d)^*$. We will also exploit that primal and dual vector spaces are isomorphic.

For a direction $y \in \mathbb{R}^I$ we define the $y$-maximal face $\mathcal{P}_y$ of a polytope $\mathcal{P}$ by

$$\mathcal{P}_y := \{x \in \mathcal{P} : y(x) \geq y(x') \text{ for all } x' \in \mathcal{P}\}.$$
Figure 2. The standard permutahedron $\pi\{a,b,c\}$ (left) and the normal fan in $(\mathbb{R}\{a,b,c\})^*$ (right), where the intersection line is the normal cone $N_{\pi\{a,b,c\}}(\pi\{a,b,c\})$, the half hyperplanes are the normal cones of the edges and the full-dimensional cones are the normal cones of the vertices of $\pi\{a,b,c\}$.

For a face $F$ of a polytope $P$ define the open and closed normal cone $N^\circ_P(F)$ and $N_P(F)$ to be the set of all directions that (strictly) maximize $F$ in $P$, that is,

$$N^\circ_P(F) := \{ y \in \mathbb{R}^I : Py = F \}$$

$$N_P(F) := \{ y \in \mathbb{R}^I : Py \supseteq F \}.$$  

Collecting the normal cones $N_P(F)$ of all faces $F$ of a polytope $P$ defines the normal fan $N(P) := \{ N_P(F) : F$ a face of $P \}$.

See Figure 2 for an example. The following is straightforward.

Lemma 2.1. For a face $F$ of a polytope $P \subset \mathbb{R}^d$ with dimension $\dim(F) = k$ the dimension of the normal cone is given by $\dim(N_P(F)) = d - k = \text{codim}(F)$. For another face $G$ of the polytope $P$ we have $F \subseteq G$ if and only if $N_P(F) \supseteq N_P(G)$.

The normal fan of the standard permutahedron has a nice description via the braid arrangement $B_I$, the hyperplane arrangement consisting of the finite set of hyperplanes $H_{ij} := \{ x \in \mathbb{R}^I : x_i = x_j \}$ for $i, j \in I, i \neq j$. See Figure 2b for the example $B_{\{a,b,c\}}$. The connected components of $\mathbb{R}^I \setminus \bigcup B_I$ are the (open) regions of the arrangement. The closed regions of the braid arrangement are the topological closures of the open regions. They are polyhedral cones and their faces are the faces of the braid arrangement, also called braid cones. For more details about concepts on hyperplane arrangements see, for example, [Sta07]. The faces of the braid arrangement $B_I$ form the braid fan and the normal fan $N(\pi_I)$ of the standard permutahedron $\pi_I$ is precisely the braid fan (see, for example, [AA17, Section 4]).

We say a fan $\mathcal{N}$ is a coarsening of another fan $\mathcal{N}'$ if every cone in $\mathcal{N}$ is the union of some cones in $\mathcal{N}'$. A polytope $P \subset \mathbb{R}^I$ is a generalized permutahedron if its normal fan $N(P)$ is a coarsening of the normal fan $N(\pi_I)$ of the standard permutahedron $\pi_I$, that is, it is a coarsening of the fan induced by the braid arrangement $B_I$. There are several equivalent definitions of generalized permutahedra (see, e.g., [PRW08, CL20, AA17, Pos09]).
Figure 3. A generalization \( \mathcal{P} \) of the standard permutahedron \( \pi_{\{a,b,c\}} \): here the upright edge was moved outwards until it degenerated to a vertex. The normal cone of that “new” vertex is the union of the normal cones of the “old” degenerated edge and its adjacent vertices.

Since the normal fan \( \mathcal{N}(\mathcal{P} + \mathcal{Q}) \) of the Minkowski sum \( \mathcal{P} + \mathcal{Q} \) of two polytopes \( \mathcal{P} \) and \( \mathcal{Q} \) is the common refinement of the two normal fans \( \mathcal{N}(\mathcal{P}) \) and \( \mathcal{N}(\mathcal{Q}) \) [Zie98, Proposition 7.12], generalized permutahedra are the (weak) Minkowski summands of standard permutahedra. That is, \( \mathcal{P} \subset \mathbb{R}^d \) is a generalized permutahedron if and only if there exists a polytope \( \mathcal{Q} \subset \mathbb{R}^d \) and a real scalar \( \lambda > 0 \) such that \( \mathcal{P} + \mathcal{Q} = \lambda \pi_{\{d\}} \).

One picturesque way of defining generalized permutahedra is by deforming standard permutahedra by parallel shifts of facets. This deformation maintains the normal fan until a face degenerates, i.e., at least two vertices are merged into one vertex. In that case the corresponding normal cones of the vertices are glued together. One example can be seen in Figure 3, where the top right edge degenerated and the two corresponding neighboring full-dimensional cones were combined. A formal description of these deformations and a detailed proof of equivalence can be found in [PRW08, Appendix].

Finally, generalized permutahedra can be uniquely described as the base polytopes of submodular functions \( z : 2^I \to \mathbb{R} \) with \( z(\emptyset) = 0 \). That is, the polytope \( \mathcal{P} \) is a generalized permutahedron if and only if there exists a unique submodular function \( z : 2^I \to \mathbb{R} \) with \( z(\emptyset) = 0 \) such that

\[
\mathcal{P} = \left\{ x \in \mathbb{R}^I : \sum_{i \in I} x_i = z(I) \quad \text{and} \quad \sum_{i \in T} x_i \leq z(T) \quad \text{for} \quad T \subseteq I \right\},
\]

see, e.g., [CL20, Theorem 3.11 and 3.17]. For the sake of completeness and the convenience of the reader we include a self-contained proof of the well-known equivalence of the definitions of generalized permutahedra through braid fan coarsenings and submodular functions in the appendix (Theorem A.2). We do not claim the proof to be neither new nor original, but hard to find in the literature.

2.2. Ehrhart theory. For a polytope \( \mathcal{Q} \subset \mathbb{R}^d \) and a positive integer \( t \in \mathbb{Z}_{>0} \) we define the \( t^{th} \) dilate of \( \mathcal{Q} \) as

\[
t\mathcal{Q} := \left\{ x \in \mathbb{R}^d : \frac{1}{t} x \in \mathcal{Q} \right\} = \left\{ tx \in \mathbb{R}^d : x \in \mathcal{Q} \right\}.
\]
The Ehrhart counting function $\text{Ehr}_Q(t)$ counts the number of integer points in the $t^{\text{th}}$ dilate of the polytope $Q$:

$$\text{Ehr}_Q(t) := \# \left( \frac{1}{t} \mathbb{Z}^d \cap Q \right) = \# \left( \mathbb{Z}^d \cap tQ \right).$$

See Figure 4 for an example. Recall that a rational (resp. integral) polytope has vertices with rational (resp. integral) coordinates. We call the least common multiple of the denominators of all coordinates of all vertices of a rational polytope the denominator of $Q$.

**Theorem 2.2** (Ehrhart’s theorem [Ehr62]). For a rational polytope $Q$ of dimension $d$ the Ehrhart counting function $\text{Ehr}_Q(t)$ agrees with a quasipolynomial of degree $d$ and period dividing the denominator of $Q$ for all $t \in \mathbb{Z}_{>0}$.

For an integer polytope $Q$ Ehrhart’s theorem implies that the Ehrhart counting function $\text{Ehr}_Q$ is a polynomial. Therefore it is often called the Ehrhart polynomial. The following reciprocity theorem was conjectured and proved for various special cases by Eugéne Ehrhart and proved by Ian G. Macdonald. It is the foundation for the results in this paper.

**Theorem 2.3** (Ehrhart–Macdonald reciprocity [Mac71]). Let $Q \subset \mathbb{R}^d$ be a rational polytope and $t \in \mathbb{Z}_{>0}$. Then

$$(-1)^{\dim Q} \text{Ehr}_Q(-t) = \text{Ehr}_{Q^\circ}(t) := \# \left( \mathbb{Z}^d \cap tQ^\circ \right).$$

where $Q^\circ$ is the (relative) interior of the polytope $Q$.

3. Pruned inside-out polytopes and Ehrhart theory

In [BZ06b] Beck and Zaslavsky develop the notion of an inside-out polytope, that is, a polytope dissected by hyperplanes. Counting integer point in a polytope but off certain hyperplanes turns out to be a useful tool to derive (quasi-)polynomiality results and reciprocity laws for various applications such as graph colorings [Sta73] and signed graph colorings, composition of integers, nowhere-zero flows on graphs and signed graphs, antimagic labellings, as well as magic, semimagic, and magilatin squares [BZ06c, BZ06a, BZ10].

In this section we introduce a generalization of inside-out polytopes, which we call pruned inside-out polytopes and develop some Ehrhart-theoretic results. We then apply these results in Section 4 to generalized permutahedra and their subclass of hypergraphic polytopes.

Let $\mathcal{N}$ be a complete fan in $\mathbb{R}^d$, that is, a family of polyhedral cones such that
(i) every non-empty face of a cone $N \in \mathcal{N}$ is also contained in $\mathcal{N}$,
(ii) the intersection of two cones in $\mathcal{N}$ is a face of both cones,
(iii) the union of the cones in the fan $\mathcal{N}$ covers the ambient space $\mathbb{R}^d$, i.e.,

$$
\bigcup_{N \in \mathcal{N}} N = \mathbb{R}^d.
$$

For an introduction to complete fans consult, e.g., [Zie98, Section 7.1]. A fan is called rational if its cones $N \in \mathcal{N}$ are generated by rational vectors. For a complete fan $\mathcal{N}$ in $\mathbb{R}^d$ we define the codimension-one fan $\mathcal{N}^{\text{co 1}}$ in $\mathbb{R}^d$ to contain the cones in $\mathcal{N}$ with codimension $\geq 1$, that is, all but the full-dimensional cones in $\mathcal{N}$:

$$
\mathcal{N}^{\text{co 1}} := \{ N \in \mathcal{N} : \dim N \leq d - 1 \}.
$$

We think of the codimension-one fan as a pruned hyperplane arrangement, since cones of codimension one can be seen as parts of hyperplanes. In the case of a normal fan $\mathcal{N}(\mathcal{P})$ of a polytope $\mathcal{P}$ this amounts to

$$
\mathcal{N}^{\text{co 1}}(\mathcal{P}) = \mathcal{N}(\mathcal{P}) \setminus \{ N_{\mathcal{P}}(v) : v \text{ vertex of } \mathcal{P} \}
= \{ N_{\mathcal{P}}(F) : F \text{ face of } \mathcal{P} \text{ with } \dim(F) \geq 1 \}.
$$

For a polytope $\mathcal{Q} \subset \mathbb{R}^d$ and a complete fan $\mathcal{N}$ in $\mathbb{R}^d$ we call

$$
\mathcal{Q} \setminus \left( \bigcup_{N \in \mathcal{N}} N^{\text{co 1}} \right) = \bigcup_{N \in \mathcal{N}, N \text{ full-dimensional}} (\mathcal{Q} \cap N^{\circ})
$$

a pruned inside-out polytope and we call the connected components in $\mathcal{Q} \setminus (\bigcup N^{\text{co 1}})$ regions. So, a pruned inside-out polytope $\mathcal{Q} \setminus (\bigcup N^{\text{co 1}})$ is the disjoint union of its regions $\mathcal{Q} \cap N^{\circ}$, where $N^{\circ}$ is an open full-dimensional cone in $\mathcal{N}$. We will mostly consider open pruned inside-out polytopes $\mathcal{Q}^{\circ} \setminus (\bigcup N^{\text{co 1}})$, which decompose into disjoint open polytopes, the regions. See Figure 5 for examples. A pruned inside-out polytope is rational if the topological closures of all its regions are rational polytopes.

**Lemma 3.1.** Let $\mathcal{Q} \subset \mathbb{R}^d$ be a rational polytope and $\mathcal{N}$ be a rational complete fan in $\mathbb{R}^d$. Then the pruned inside-out polytope $\mathcal{Q} \setminus (\bigcup N^{\text{co 1}})$ is rational.

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This is still a fan, but it is not complete anymore, i.e., condition (iii) in the above definition is not fulfilled.
For a positive integer $t \in \mathbb{Z}_{>0}$ we define the inner pruned Ehrhart function as
\[
\text{In}_{Q, \mathcal{N}^{\text{co}1}}(t) := \# \left( \frac{1}{t} \mathbb{Z}^d \cap \left( Q \setminus \left( \bigcup \mathcal{N}^{\text{co}1} \right) \right) \right) = \# \left( \mathbb{Z}^d \cap t \cdot \left( Q \setminus \left( \bigcup \mathcal{N}^{\text{co}1} \right) \right) \right),
\]
where
\[
t \cdot \left( Q \setminus \left( \bigcup \mathcal{N}^{\text{co}1} \right) \right) := t \cdot Q \setminus \left( \bigcup t \cdot \mathcal{N}^{\text{co}1} \right)
:= \{ ty \in \mathbb{R}^d : y \in Q \} \setminus \{ ty \in \mathbb{R}^d : y \in \mathcal{N}, \text{ for some } \mathcal{N} \in \mathcal{N}^{\text{co}1} \}.
\]
See Figures 6a and 6b for illustrations.

**Lemma 3.2.** For a polytope $Q \subset \mathbb{R}^d$ and a complete normal fan $\mathcal{N}$ in $\mathbb{R}^d$,
\[
\text{In}_{Q, \mathcal{N}^{\text{co}1}} = \sum_{i=1}^{k} \text{Ehr}_R^i(t)
\]
where $R_i^o$ are the open regions of the pruned inside-out polytope $Q^o \setminus \left( \bigcup \mathcal{N}^{\text{co}1} \right)$.

We define a second counting function for pruned inside-out polytopes, the cumulative pruned Ehrhart function $\text{Ex}_{Q, \mathcal{N}^{\text{co}1}}(\mathbb{Z}^d)$, for a positive integer $t \in \mathbb{Z}_{>0}$ as
\[
\text{Ex}_{Q, \mathcal{N}^{\text{co}1}}(t) := \sum_{y \in \mathbb{Z}^d} \text{mult}_{Q, \mathcal{N}^{\text{co}1}}(y) = \sum_{y \in \mathbb{Z}^d} \text{mult}_{(t, Q, t \cdot \mathcal{N}^{\text{co}1})}(y),
\]
where
\[
\text{mult}_{Q, \mathcal{N}^{\text{co}1}}(y) := \begin{cases} 
\# (\text{closed full-dimensional normal cones in } \mathcal{N} \text{ containing } y) & \text{if } y \in Q, \\
0 & \text{else.}
\end{cases}
\]
See Figure 6c for an illustration.

**Lemma 3.3.** For a polytope $Q \subset \mathbb{R}^d$ and a complete fan $\mathcal{N}$ in $\mathbb{R}^d$,
\[
\text{Ex}_{Q, \mathcal{N}^{\text{co}1}}(t) = \sum_{i=1}^{k} \text{Ehr}_{\overline{R_i}}(t),
\]
where $\overline{R_i}$ are the topological closures of the regions $R_i$ of the pruned inside-out polytope $Q \setminus \left( \bigcup \mathcal{N}^{\text{co}1} \right)$.

**Figure 6.** Inner and cumulative pruned Ehrhart functions illustrated.
Theorem 3.4. Let $Q \subset \mathbb{R}^d$ be a rational polytope of dimension $d$ and $N$ a rational complete fan in $\mathbb{R}^d$. Then the inner pruned Ehrhart function $\In_{Q^oN^{\co 1}}(t)$ and the cumulative pruned Ehrhart function $\Ex_{Q^oN^{\co 1}}(t)$ agree with quasipolynomials in $t$ of degree $d$ and $(−1)^d \In_{Q^oN^{\co 1}}(−t) = \Ex_{Q^oN^{\co 1}}(t)$.

Proof. As in Lemma 3.2 we decompose the pruned inside-out polytope $Q \setminus (\bigcup N^{\co 1})$ into its regions and then use classical Ehrhart theory and Ehrhart–Macdonald reciprocity. If $R_1, \ldots, R_k$ are the regions of $Q^o \setminus (\bigcup N^{\co 1}) = \bigcup_{i=1}^k R_i^0$ is the disjoint union of the open polytopes $R_i^0, \ldots, R_i^k$, so

$$\In_{Q^oN^{\co 1}}(t) = \sum_{i=1}^k \Ehr_{R_i^0}(t).$$

We can apply Ehrhart’s Theorem 2.2 to $\Ehr_{R_i^0}(t)$ for $i = 1, \ldots, k$ so the counting function $\In_{Q^oN^{\co 1}}(t)$ is a sum of quasipolynomials, which is again a quasipolynomial.

For the second part of the claim we use Ehrhart–Macdonald reciprocity (Theorem 2.3) and compute

$$\In_{Q^oN^{\co 1}}(t) = \sum_{i=1}^k (-1)^d \Ehr_{R_i}(t) = (-1)^d \Ex_{Q^oN^{\co 1}}(−t),$$

where the last equality follows from Lemma 3.3. □

Remark 3.5. In the case that the polytope $Q$ and the complete fan intersect such that all the closed regions $\overline{R} = Q \cap N$ of the pruned inside-out polytope $Q \setminus (\bigcup N^{\co 1})$ are integer polytopes, the counting functions $\In_{Q^oN^{\co 1}}(t)$ and $\Ex_{Q^oN^{\co 1}}(t)$ agree with a polynomial of degree $d$, by Theorem 2.2 and Theorem 2.3. We will use this fact in the proof of Theorem 4.2.

4. Applications

In the first part of this section we apply the theory from Section 3 to give a new perspective on a reciprocity theorem for generalized permutahedra (Theorem 4.2) developed by Ardila and Aguiar [AA17, ], who use a Hopf monoid structure on the vector species of generalized permutahedra and their antipode formula to derive the reciprocity result as a basic polynomial invariant (Theorem 4.1). Moreover, we employ pruned inside-out polytopes to prove an extension of the reciprocity result for generalized permutahedra to arbitrary face dimensions (Theorem 4.4).

In the second part of this section we show how a reciprocity theorem for hypergraph colorings by Aval, Karaboghossian, and Tanasa [AKT20] can be seen as a special case of the above reciprocity for generalized permutahedra. While Aval et al again use Hopf monoid theoretic and rather technical methods, we present a proof that, assuming the generalized permutahedra reciprocity, uses only elementary geometric and combinatorial arguments.

4.1. Generalized permutahedra. For a Hopf monoid on the ground set $I$, a character, and an element $x$ in the Hopf monoid, there is a polynomial invariant $\chi_I(x)(m)$. Using the antipode $s_I$ of the Hopf monoid one obtains the reciprocity relation

$$\chi_I(x)(−m) = \chi_I(s_I(x))(m)$$

which gives an interpretation for negative integers [AA17, Section 16]. In [AA17] Ardila and Aguiar define a Hopf monoid structure on the species of generalized permutahedra and
then obtain combinatorial formulas for the polynomial invariant \( \chi I(x)(m) \) and \( \chi I(x)(-m) \) for \( m \in \mathbb{Z}_{>0} \) using the basic character, which takes values in \( \{0, 1\} \).

**Theorem 4.1 ([AA17, Propositions 17.3 and 17.4]).** At a positive integer \( m \in \mathbb{Z}_{>0} \) the basic polynomial invariant \( \chi I \) of a generalized permutahedron \( P \subset \mathbb{R}^I \) is given by

\[
\chi I(P)(m) = \# (P\text{-generic directions } y: I \to [m])
\]

and

\[
(-1)^{|I|} \chi I(P)(-m) = \sum_{y: I \to [m]} \# (\text{vertices of } P_y).
\]

This result was obtained earlier but stated differently by Billera, Jia, and Reiner using a similar Hopf-algebraic approach (using the antipode) on quasisymmetric functions and matroids [BJR09, Theorem 9.2. (v)].

We now restate Theorem 4.1 in a slightly different language and prove it using Ehrhart theory. Fixing a bijection \( \sigma: I \to [d] \) we can identify \( \mathbb{R}^I \simeq \mathbb{R}^d \) as well as \( \mathbb{R}^I \simeq (\mathbb{R}^d)^* \). We will consider the cube

\[
[1, m]^d := \{ x \in \mathbb{R}^d : 1 \leq x_i \leq m \text{ for } i = 1, \ldots, d \} \subset \mathbb{R}^d
\]

and intersect it with the integer lattice:

\[
[1, m]^d \cap \mathbb{Z}^d = \{ x \in \mathbb{R}^d : x_i \in \{1, \ldots, m\} \text{ for } i = 1, \ldots, d \} = \{1, \ldots, m\}^d = [m]^d.
\]

The same holds in the dual space \((\mathbb{R}^d)^* \simeq \mathbb{R}^I \). Now, a direction \( y: I \to [m] \in \mathbb{R}^I \) can be identified with an integer point \( y \) in the cube \( \{1, \ldots, m\}^d = [m]^d \) in the dual space. In contrast to [BJR09, AA17] we will prove Theorem 4.2 without using any Hopf-algebraic method. Our proof gives a geometric point of view by counting integer points in pruned inside-out cubes. See Figure 7. While we will extend Theorem 4.2 (and our proof) in Theorem 4.4 below we provide a self-contained proof here to present a flavor of our method.
Theorem 4.2 (Theorem 4.1 restated). Let $\mathcal{P} \subset \mathbb{R}^d$ be a generalized permutahedron and $m \in \mathbb{Z}_{>0}$. Then

$$\chi_d(\mathcal{P})(m) := \# \left( \mathcal{P}\text{-generic directions } y \in (\mathbb{R}^d)^* \text{ with } y \in [m]^d \right)$$

agrees with a polynomial in $m$ of degree $d$. Moreover,

$$(-1)^d \chi_d(\mathcal{P})(-m) = \sum_{y \in [m]^d} \# \text{(vertices of } \mathcal{P}_y).$$

Proof. We will argue in the dual space $(\mathbb{R}^d)^*$ and its integer lattice; to simplify notation we will not always explicitly point that out. Let us recall that $y \in (\mathbb{R}^d)^*$ being $\mathcal{P}$-generic means that the $y$-maximal face of $\mathcal{P}$ is a vertex, that is, $y$ is contained in a full-dimensional cone of the normal fan $\mathcal{N}(\mathcal{P})$. So the direction $y$ is not contained in any cone $N$ in the codimension-one fan $\mathcal{N}(\mathcal{P})^{\text{co}1}$. Hence,

$$\chi_d(\mathcal{P})(m) = \# \left( \mathcal{P}\text{-generic directions } y \in (\mathbb{R}^d)^* \text{ with } y \in [m]^d \right)$$

$$= \# \{ y \in [1, m]^d \cap \mathbb{Z}^d : \text{ } y\text{-maximal face } \mathcal{P}_y \text{ is a vertex} \}$$

$$= \# \{ y \in [1, m]^d \cap \mathbb{Z}^d : y \in N \in \mathcal{N}(\mathcal{P}) \text{ with } N \text{ full-dimensional} \}$$

$$= \# \{ y \in [1, m]^d \cap \mathbb{Z}^d : y \notin N \text{ for all } N \in \mathcal{N}(\mathcal{P}) \text{ with codimension } \geq 1 \}$$

$$= \# \left( [1, m]^d \setminus \bigcup \mathcal{N}(\mathcal{P})^{\text{co}1} \right) \cap \mathbb{Z}^d$$

$$= \text{In}_{(0,1)^d, \mathcal{N}(\mathcal{P})^{\text{co}1}}(m + 1),$$

where we use in the last line that $[1, m]^d \cap \mathbb{Z}^d = (0, m + 1)^d \cap \mathbb{Z}^d = (m + 1) \cdot (0, 1)^d \cap \mathbb{Z}^d$. With Lemma 4.3 (below) we know that the unit cube and the normal fan $\mathcal{N}(\mathcal{P})$ intersect producing integer regions. Therefore, using Theorem 3.4 and Remark 3.5, polynomiality of $\chi_d(\mathcal{P})(m)$ follows. With the above equality and Theorem 3.4 at hand, we compute

$$(-1)^d \chi_d(\mathcal{P})(-m) = (-1)^d \text{In}_{(0,1)^d, \mathcal{N}(\mathcal{P})^{\text{co}1}}(-m + 1)$$

$$= (-1)^d \text{In}_{(0,1)^d, \mathcal{N}(\mathcal{P})^{\text{co}1}}(-(m - 1))$$

$$= \text{Ex}_{[0,1]^d, \mathcal{N}(\mathcal{P})^{\text{co}1}}(m - 1)$$

$$= \sum_{y \in \mathbb{Z}^d} \text{mult}_{[0,1]^d, \mathcal{N}(\mathcal{P})^{\text{co}1}}(y)$$

$$= \sum_{y \in \mathbb{Z}^d} \text{mult}_{[0,m-1]^d, \mathcal{N}(\mathcal{P})^{\text{co}1}}(y),$$

where we use that the polytope $\mathcal{P}$ is a generalized permutahedron, which implies that $\mathcal{N}(\mathcal{P})$ is a coarsening of the braid arrangement. The braid arrangement contains the line $L = \lambda(1, \ldots, 1)$. Therefore, the fans $\mathcal{N}(\mathcal{P})$ and $\mathcal{N}(\mathcal{P})^{\text{co}1}$ are invariant under scaling and translations by vectors.
in the lineality space $L$. So,
\[
(-1)^d \chi_d(P)(-m) = \sum_{y \in \mathbb{Z}^d} \text{mult}_{[1,m]^d, N(P)_{co1}}(y)
\]
\[
= \sum_{y \in [1,m]^d \cap \mathbb{Z}^d} \# \text{(closed full-dimensional normal cones that contain } y) \]
\[
= \sum_{y \in [m]^d} \# \text{(closed normal cones of vertices that contain } y) \]
\[
= \sum_{y \in [m]^d} \# \text{(vertices of } P_y) ,
\]
where we make use of Lemma 2.1. \qed

**Lemma 4.3.** The unit cube and braid cones intersect “nicely”, i.e., the intersections of the unit cube $[0,1]$ and the braid cones $B_{T_1,...,T_k}$ are integer polytopes.

**Proof.** It is enough to consider the full-dimensional braid cones corresponding to a permutation $\sigma$. These intersections are order polytopes of chains and order polytopes are known to be $0/1$-polytopes [Sta86]. \qed

We can extend Theorem 4.2 above to faces of arbitrary dimension.

**Theorem 4.4.** For a generalized permutahedron $P \subset \mathbb{R}^d$ and $k = 0, \ldots, d-1$,
\[
\chi_{d,k}(P)(m) := \# \{ y \in [m]^d : y \text{-maximum face } P_y \text{ is a } k\text{-face} \}
\]
agrees with a polynomial of degree $d - k$, and
\[
(-1)^{d-k} \chi_{d,k}(P)(-m) = \sum_{y \in [m]^d} \# \text{(}k\text{-faces of } P_y) .
\]

Before we prove the theorem we extend the notion of codimension-one fans to arbitrary dimensions by defining the **codimension-$k$ fan** $N^{co\,k}$ as
\[
N^{co\,k} := \{ N \in N(P) : \text{codim}(N) \geq k \},
\]
that is, for a polytope $P$,
\[
N(P)^{co\,k} = \{ N_P(F) : F \text{ a face of } P \text{ with } \text{dim}(F) \geq k \} .
\]

For a polytope $Q \subset \mathbb{R}^d$ and $k \geq 0$ we define the **k-pruned inside-out polytope** as
\[
\left( Q \cap \bigcup N^{co\,k} \right) \setminus \left( \bigcup N^{co\,k+1} \right) = Q \cap \bigcup \left\{ N^\circ : N \in N^{co\,k} \right\} .
\]
Note this is consistent with the notation in the beginning of this section. As before, for a polytope $Q \subset \mathbb{R}^d$ the open $k$-pruned inside-out polytope $\left( Q^\circ \cap \bigcup N^{co\,k} \right) \setminus \left( \bigcup N^{co\,k+1} \right)$ is the disjoint union of relatively open $(d-k)$-dimensional polytopes, namely, the intersection of $Q^\circ$ with the relatively open cones in $N$ of codimension $k$. 
**Proof of Theorem 4.4.** We compute
\[ \chi_{d,k}(P)(m) = \# \{ y \in [1, m]^d \cap \mathbb{Z}^d : \text{y-maximum face } P_y \text{ is a } k\text{-face} \} \]
\[ = \# \left( \left( \mathbb{Z}^d \cap (0, m + 1)^d \right) \cap \bigcup_{N \in \mathcal{N}(P)} \bigcup_{\dim N = d-k} (N^\circ \cap (0, m + 1)) \cap \mathbb{Z}^d \right) \]
\[ = \sum_{N \in \mathcal{N}(P), \dim N = d-k} \operatorname{Ehr}_{N^\circ \cap (0,1)}(m + 1). \]

Using again Lemma 4.3 and Ehrhart’s Theorem 2.2 we obtain polynomiality for \( \chi_{d,k}(P)(m) \).

With Ehrhart–Macdonald reciprocity (Theorem 2.3) we compute
\[ (-1)^{d-k} \chi_{d,k}(P)(-m) = (-1)^{d-k} \sum_{N \in \mathcal{N}(P), \dim N = d-k} \operatorname{Ehr}_{N^\circ \cap (0,1)}(-m + 1) \]
\[ = \sum_{N \in \mathcal{N}(P), \dim N = d-k} (-1)^{d-k} \operatorname{Ehr}_{N^\circ \cap (0,1)}(-(m - 1)) \]
\[ = \sum_{N \in \mathcal{N}(P), \dim N = d-k} \operatorname{Ehr}_{N \cap [0,1]}(m - 1) \]
\[ = \sum_{N \in \mathcal{N}(P), \dim N = d-k} \# (N \cap [0, m - 1]^d \cap \mathbb{Z}^d). \]

Here we use, as in the proof of Theorem 4.2, that the normal fan of a generalized permutahedron is a coarsened braid fan and therefore is invariant under scaling and shifts by \( \lambda(1, \ldots, 1) \) for \( \lambda \in \mathbb{R} \). So,
\[ (-1)^{d-k} \chi_{d,k}(P)(-m) = \sum_{N \in \mathcal{N}(P), \dim N = d-k} \# \left( N \cap [1, m]^d \cap \mathbb{Z}^d \right) \]
\[ = \sum_{y \in [1,m]^d \cap \mathbb{Z}^d} \# ((d - k)\text{-dimensional cones } N \in \mathcal{N}(P) \text{ that contain } y) \]
\[ = \sum_{y \in [m]^d} \# (k\text{-faces of } P_y), \]
applying Lemma 2.1 in the last equality. \( \square \)

**Remark 4.5.** We observe that we used the following properties of generalized permutahedra in the proofs of Theorem 4.2 and Theorem 4.4

(i) the intersections of the unit cube and the normal fans form integer pruned inside-out polytopes,
(ii) the normal fans have a lineality space containing the line \( \{\lambda(1, \ldots, 1) : \lambda \in \mathbb{R}\} \).

It would be interesting to find more examples of polytope classes with normal fans providing these two properties. The first property (i) can be weakened to rational intersections leading to a quasipolynomiality result. Considering normal fans not containing the lineality space
λ(1, ⋯, 1) produces a similar but more complicated statement, since the shift of the cube $[0, m - 1]^d$ to the cube $[1, m]^d$ cannot be performed in general.

4.2. Hypergraphs and their polytopes. Aval, Karaboghossian, and Tanasa use a Hopf-theoretic ansatz similar to that of Ardila and Aguiar to derive a reciprocity theorem for hypergraph colorings [AKT20]. They define a basic polynomial invariant on hypergraphs and give combinatorial interpretations. However, they do not use the antipode to get reciprocity, but rather technical computations involving Bernoulli numbers.3

We give another perspective and proof by applying Theorem 4.1 (reciprocity for generalized permutahedra) and exploiting geometric and combinatorial properties of the hypergraph and its associated polytope.

A hypergraph $h = (I, E)$ is a pair of a finite set $I$ of nodes and a finite multiset $E$ of non-empty subsets $e \subseteq I$ called hyperedges. Note that we allow multiple edges and edges consisting of only one node. For simplicity we will often assume without loss of generality that the node set $I$ equals $\{1, \ldots, d\} = [d]$ for $d = |I|$, since all the claims in this section are invariant under relabeling the set $I$. In a similar fashion we might switch back and forth between the two vector space notations $\mathbb{R}I \simeq \mathbb{R}^d$ and $\mathbb{R}I \simeq (\mathbb{R}^d)^*$. For every hypergraph $h$ we define the corresponding hypergraphic polytope $\mathcal{P}(h) \subset \mathbb{R}I$ as the following Minkowski sum of simplices:

$$\mathcal{P}(h) = \sum_{e \in E} \Delta_e \subset \mathbb{R}I$$

where

$$\Delta_e = \text{conv}\{b_i : i \in e\}, \quad \text{for a hyperedge } e \subseteq I$$

and $b_i$ are the basis vectors for $i \in I$. An example is depicted in Figure 8. Hypergraphic polytopes have been studied (sometimes as Minkowski sum of simplices) such as in [Agn17, BBM19].

Recall that generalized permutahedra can be uniquely described as base polytopes of submodular functions $z: 2^I \to \mathbb{R}$ with $z(\emptyset) = 0$.

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3There seems to be a polytopal approach by Alexander Postnikov, mentioned in [AKT20, Acknowledgments] and on http://math.mit.edu/~apost/courses/18.218_2016/ (Lecture 19. W 03/16/2016), but to the best of our knowledge no reference is available.

4We decided to use the less common term nodes for hypergraphs to distinguish them from the vertices of a polytope.
Proposition 4.6 ([Pos09, Proposition 6.3.]). For a hypergraph \( h = (I, E) \) and its hypergraphic polytope \( P(h) \), the function \( z: 2^I \to \mathbb{R} \) defined by

\[
z(T) := \sum_{e \in E \cap T \neq \emptyset} 1 = \# \{ \text{hyperedges in } h \text{ that intersect } T \} \quad \text{for } T \subseteq I
\]

is a submodular function with \( z(\emptyset) = 0 \) and

\[
P(h) = \left\{ x \in \mathbb{R}^I : \sum_{i \in I} x_i = z(I) \quad \text{and} \quad \sum_{i \in T} x_i \leq z(T) \quad \text{for } T \subseteq I \right\}.
\]

Hence, hypergraphic polytopes are generalized permutahedra.

Remark 4.7. Postnikov uses a different convention for the facet description of a generalized permutahedron:

\[
P(z) := \left\{ (t_1, \ldots, t_d) \in \mathbb{R}^d : \sum_{i=1}^d t_i = z([d]), \sum_{i \in J} t_i \geq z(J), \text{ for } J \subseteq [d] \right\}.
\]

This results in a differing formulation of Proposition 4.6 which is nevertheless equivalent.

For an interesting characterization when a submodular function gives rise to a hypergraphic polytope see [AA17, Proposition 19.4.]

The vertices of graphic polytopes are described by the acyclic orientations of the corresponding graph [Zas91, Corollary 4.2]. We will give an analogous statement and proof for hypergraphic polytopes. In order to do so we need the subsequent definitions following\(^5\) [AKT20]. A heading \( \sigma \) of a hypergraph \( h = (I, E) \) is a map \( \sigma: E \to I \) such that for every hyperedge \( e \in E \) we have \( \sigma(e) \in e \). In other words the heading \( \sigma \) picks for every hyperedge \( e \) a node \( i = \sigma(e) \in e \) within that hyperedge. We will call that node \( \sigma(e) \) the head of the hyperedge \( e \). An oriented cycle in a heading \( \sigma \) of a hypergraph \( h \) is a sequence \( e_1, \ldots, e_\ell \) of hyperedges such that

\[
\sigma(e_1) \in e_2 \setminus \sigma(e_2) \\
\sigma(e_2) \in e_3 \setminus \sigma(e_3) \\
\vdots
\\
\sigma(e_{\ell-1}) \in e_{\ell-1} \setminus \sigma(e_{\ell-1}) \\
\sigma(e_\ell) \in e_1 \setminus \sigma(e_1).
\]

A heading \( \sigma \) of a hypergraph \( h \) is called acyclic if it does not contain any oriented cycle. See Figure 9 for some examples. Note that the notions of heading and acyclic here are different from the notions of orientation in [BBM19] and [RR12, Rus13].

The following description of the vertices of the hypergraphic polytope in terms of acyclic orientations plays a central role in the remainder of this paper. This result (Proposition 4.8) is stated without proof in [CF18], where Cardinal and Felsner give a graph-theoretic proof for the vertex description of graphic polytopes. We adapt and generalize the proof idea to the hypergraphic setting.

\(^5\)Some of the definitions are also mentioned by Postnikov (http://math.mit.edu/~apost/courses/18.218_2016/ Problem set 2, Problem 6).

\(^6\)We have chosen to call this generalization of orientations heading to distinguish it from other definitions of orientations for hypergraphs.
Proposition 4.8. For a hypergraph $h = (I, E)$ the hypergraphic polytope $P(h)$ can be described as

$$P(h) = \text{conv}\{ \delta(\sigma) \in \mathbb{R}I : \sigma \text{ is an acyclic heading of } h\}$$

where

$$\delta(\sigma)_i = |\sigma^{-1}(i)| \text{ for } i \in I,$$

i.e., $\delta(\sigma) \in \mathbb{R}I$ is the vector of in-degrees of the nodes $i \in I$ in the heading $\sigma$.

Proof. This proof generalizes the proof idea for graphs presented in [CF18]. Since the convex hull and the Minkowski sum commute,

$$P(h) = \sum_{e \in E} \text{conv}\{ b_i : i \in e \} = \text{conv} \sum_{e \in E} \{ b_i : i \in e \}.$$

Every point in the convex hull on the right-hand side is the vector of in-degrees of the nodes for some heading $\sigma$. Indeed, choosing some $b_i$ in every summand corresponds to choosing $i \in e$ as the head for the hyperedge $e$, and vice versa.

We now show that it is enough to take the convex hull of in-degree vectors for acyclic headings. That is, we show that $\delta(\sigma)$ is a vertex of $P(h)$ if and only if the heading $\sigma$ is acyclic.

First, consider a heading containing an oriented cycle $e_1, \ldots, e_\ell$. Then

$$\sigma(e_1) \in e_2 \setminus \sigma(e_2), \ldots, \sigma(e_\ell) \in e_1 \setminus \sigma(e_1)$$

holds. We will construct new headings $\sigma^*_1, \ldots, \sigma^*_\ell$ such that their vectors of in-degrees $\delta(\sigma^*_1), \ldots, \delta(\sigma^*_\ell)$ convex combine the vector of in-degrees $\delta(\sigma)$ of the original heading $\sigma$. The idea is to define the new heading $\sigma^*_j$ by changing the orientation of the hyperedge $e_j$ in the cycle, see Figure 10. Let

$$\sigma^*_1(e) := \begin{cases} \sigma(e_\ell) & \text{if } e = e_1 \\ \sigma(e) & \text{otherwise.} \end{cases}$$

and

$$\sigma^*_j(e) := \begin{cases} \sigma(e_{j-1}) & \text{if } e = e_j \\ \sigma(e) & \text{otherwise} \end{cases} \text{ for } j = 2, \ldots, \ell.$$

Then

$$\delta(\sigma) = \sum_{j=1}^{\ell} \frac{1}{\ell} \delta(\sigma^*_j).$$

Indeed, for every node $\eta$ not being a head of a hyperedge in the cycle we have $\delta(\sigma^*_j)_\eta = \delta(\sigma)_\eta$, and for every node $\eta$ that is the head of some hyperedge in the cycle, i.e., $\eta = \sigma(e_k)$ for some
Figure 10. An oriented cycle $e_1, \ldots, e_\ell$ with heading $\sigma$ (top) and the new headings $\sigma_1^*, \ldots, \sigma_\ell^*$ on the edges $e_1, \ldots, e_\ell$ (below).

For $k \in \{1, \ldots, \ell\}$, we have

$$
\sum_{j=1}^{\ell} \frac{1}{\ell} \delta(\sigma_j^*)_\eta = \sum_{\substack{j=1 \\ j \neq k-1}}^{\ell} \frac{1}{\ell} \delta(\sigma)_\eta + \frac{1}{\ell} \delta(\sigma)_{\eta + 1} + \frac{1}{\ell} \delta(\sigma)_{\eta - 1} = \frac{1}{\ell} \delta(\sigma),
$$

where $k-1$ is taken modulo $\ell$. Therefore, the vector of in-degrees $\delta(\sigma)$ of a non-acyclic heading $\sigma$ cannot be a vertex.

Now, let $\sigma$ be an acyclic heading and let us assume there are headings $\sigma_1^*, \ldots, \sigma_\ell^*$ and scalars $0 \leq \lambda_1, \ldots, \lambda_\ell \in \mathbb{R}$ such that

$$
\delta(\sigma) = \sum_{j=1}^{\ell} \lambda_j \delta(\sigma_j^*) \quad \text{and} \quad \sum_{j=1}^{\ell} \lambda_i = 1.
$$

First note that hyperedges $e$ with cardinality $|e| = 1$ have only one possible heading (the one choosing the only node in the hyperedge as head) and those edges do not appear in oriented cycles. Hence they are irrelevant when it comes to deciding whether an heading is acyclic or not. Therefore we delete all singleton hyperedges and adjust the values in $\delta(\sigma)$ as well as in $\delta(\sigma_1^*), \ldots, \delta(\sigma_\ell^*)$, i.e., for a node $i$ contained in $k$ singleton hyperedges, we have for every heading $\delta(\sigma^*_i) \geq k$ and we can substract $k$.

Since the heading $\sigma$ is acyclic and we deleted all singleton hyperedges, there exists at least one source $s \in I$ with $\delta(s)_s = 0$. From Equation (4.1) it follows that $\delta(\sigma_j^*)_s = 0$ for all $j = 1, \ldots, \ell$. So, for the node $s$ all the headings are identical. We proceed by first deleting the source $s$ in all hyperedges, then deleting all hyperedges $e$ with cardinality $|e| = 1$, and adjusting the entries in $\delta(\sigma), \delta(\sigma_1^*), \ldots, \delta(\sigma_\ell^*)$. After finitely many iterations (the node set $I$ is finite) we get $\delta(\sigma)_i = \delta(\sigma_j^*)_i$ for every node $i \in I$ and all $j = 1, \ldots, \ell$ and the in-degree
A coloring of a hypergraph is a map \( c: I \rightarrow [m] \) that assigns a color \( c(i) \in [m] \) to every node \( i \in I \). A node \( i \in e \in E \) is called a maximal node in the hyperedge \( e \) for the coloring \( c \) if the color \( c(i) \) is maximal among the colors in the hyperedge \( e \), that is, \( c(i) = \max_{j \in e} c(j) \). The color \( \max_{j \in e} c(j) \) is called the maximal color. A coloring \( c: I \rightarrow [m] \) of a hypergraph \( h = (I, E) \) is called proper if every hyperedge \( e \in E \) contains a unique maximal node \( i \in e \). This definition of a proper coloring is the same as, e.g., in [AKT20], but different from the ones in [EH66, BTV15, BDK12, AH05]. A coloring \( c: I \rightarrow [m] \) and a heading \( \sigma: E \rightarrow I \) of a hypergraph \( h = (I, E) \) are said to be compatible if \( c(\sigma(e)) = \max_{j \in e} c(j) \), i.e., if the head \( \sigma(e) \) of a hyperedge \( e \) has maximal color. See Figure 11 for some examples.

Remark 4.9. The statement in Proposition 4.8 is also implicitly contained in [BBM19, Theorem 2.18]. Benedetti, Bergeron, and Machacek define orientations in a different and more general manner. Their definition includes the one presented here as special cases. Theorem 2.18 in [BBM19] gives an involved combinatorial characterization for all faces of the hypergraph polytope derived using a Hopf monoid structure for hypergraphs (different from that of [AA17]).

Remark 4.10. Considering usual graphs, the above definitions of (proper) colorings, (acyclic) headings and compatible pairs for hypergraphs specialize to those commonly used for graphs. In the same way the following Theorem 4.11 and Theorem 4.12 generalize Stanley’s reciprocity theorem for chromatic polynomials of graphs [Sta73].

Theorem 4.11 ([AKT20, Theorem 18]). For a hypergraph \( h = (I, E) \) with \( |I| =: d \) and a positive integer \( m \in \mathbb{Z}_{>0} \),
\[
\chi_d(h)(m) := \#(\text{proper colorings of } h \text{ with } m \text{ colors})
\]
agrees with a polynomial in \( m \) of degree \( d \).

Proof. Without loss of generality we assume \( I = [d] \). For a hypergraph \( h = (I, E) \) we consider its corresponding hypergraphic polytope \( \mathcal{P}(h) \) and since \( \mathcal{P}(h) \) is a generalized permutahedron (Proposition 4.6) we can apply Theorem 4.1. Hence we need to show
\[
\#(\mathcal{P}(h))\text{-generic directions } y \in [m]^d = \#(\text{ proper colorings of } h \text{ with } m \text{ colors})
\]
We do so via a bijection. For \( y \in [m]^d \) we define the coloring \( c_y(i) := y_i \) for \( i = 1, \ldots, d \) and vice versa, for a coloring \( c: I \rightarrow [m] \) define \( y^c \in [m]^d \) by \( y^c_i := c(i) \).
It is left to show that a direction \( y \in [m]^d \) is \( \mathcal{P}(h) \)-generic if and only if the coloring \( c_y \) is proper. Recall \( y \in \mathbb{R}^d \) is \( \mathcal{P}(h) \)-generic if the maximal face \( (\mathcal{P}(h))_y \) in direction \( y \) is a vertex. Linear functionals and Minkowski sums commute (see, e.g., [BS18, Lemma 7.5.1]), so

\[
(\mathcal{P}(h))_y = \left( \sum_{e \in E} \Delta_e \right)_y = \sum_{e \in E} (\Delta_e)_y. \tag{4.2}
\]

Since the Minkowski sum is a point if and only if every summand is a point, the direction \( (4.2) \)
\( \mathcal{P}(h) \)-generic if and only if it is \( \Delta_e \)-generic for every hyperedge \( e \in E \). Finally, the direction \( y \) is \( \Delta_e \)-generic if and only if \( (\Delta_e)_y \) is a vertex. Recall that \( \Delta_e = \text{conv}\{b_i : i \in e\} \) is the convex hull of standard basis vectors \( b_i \), so

\[
(\Delta_e)_y = \text{conv} \left\{ b_i : i \in e, \ y(i) = \max_{j \in e} y(j) \right\}. \tag{4.3}
\]

Therefore \( (\mathcal{P}(h))_y \) is a vertex, if and only if for every hyperedge \( e \) the direction \( y \) has a unique maximal value among the entries \( y(i) \) with \( i \in e \). The last statement is equivalent to the coloring \( c_y \) having a unique maximal node, i.e., being proper. In summary, for a positive integer \( m \in \mathbb{Z}_{>0} \)

\[
\chi_d(h)(m) = \# \text{ (proper colorings of } h \text{ with } m \text{ colors)} = \# \left( \mathcal{P}(h) \text{-generic directions } y \in [m]^d \right) = \chi_d(\mathcal{P}(h))(m)
\]

which is a polynomial in \( m \) of degree \( d \).

\[ \square \]

**Theorem 4.12** ([AKT20, Theorem 24]). Let \( h = (I, E) \) be a hypergraph and \( m \in \mathbb{Z}_{>0} \) a positive integer. Then

\[
(-1)^d \chi_d(h)(-m) = \# \text{(compatible pairs of acyclic headings of } h \text{ and colorings of } h \text{ with } m \text{ colors).}
\]

In particular, the number of acyclic headings of \( h \) equals \((-1)^d \chi_d(h)(-1)\).

Note that the colorings do not need to be proper here.

**Proof.** We follow the same idea as in the previous proof, that is, we use

\[
(-1)^d \chi_d(h)(-m) = (-1)^d \chi_d(\mathcal{P}(h))(-m) = \sum_{y \in [m]^d} \# \text{ (vertices of } \mathcal{P}(h)_y \text{)}
\]

and need to show

\[
\sum_{y \in [m]^d} \# \text{ (vertices of } \mathcal{P}(h)_y \text{)} = \sum_{c \text{ m-coloring of } h} \# \text{ (acyclic headings of } h \text{ compatible to } c) \cdot
\]

We use the same bijection between \( m \)-colorings \( c_y \) of \( h \) and directions \( y^c \in [m]^d \) as above. It is left show that for every direction \( y \in [m]^d \) the number of vertices of the maximal face \( (\mathcal{P}(h))_y \) in direction \( y \) equals the number of acyclic headings of \( h \) compatible to the coloring \( c_y \) defined by the direction \( y \). We compute the \( y \)-maximum faces as in Equations (4.2) and (4.3):

\[
\mathcal{P}(h)_y = \left( \sum_{e \in E} \Delta_e \right)_y = \sum_{e \in E} (\Delta_e)_y = \sum_{e \in E} \text{conv} \left\{ b_i \in \mathbb{R}^I : i \in e, \ y(i) = \max_{j \in e} y(j) \right\}. \tag{4.4}
\]

From Equation (4.4) we can see that a vertex of \( \mathcal{P}(h)_y \) corresponds to choosing for every hyperedge \( e \in E \) one of the nodes \( i \in e \) with maximal entry \( y(i) \), i.e., maximal color \( c_y(i) \).
This is, by definition, the same as constructing a compatible heading for the coloring $c_y$. We know by Proposition 4.8 that vertices correspond to acyclic headings. Hence, vertices of $P(h)_y$ correspond to acyclic headings compatible to the coloring $c_y$. Vice versa, for a coloring $c$ the compatible acyclic headings are those with heads of hyperedges having a maximal coloring. That is, these acyclic headings correspond to those vertices, that are vertices of the maximum face $P(h)_y$ in direction $y^c$.

We would like to emphasize that this Section 4.2 relies, assuming Theorem 4.2, only on elementary discrete geometric and combinatorial observations about hypergraphs and there polytopes. This part is independent from the approach chosen to prove Theorem 4.2, Ehrhart-theoretic or Hopf-algebraic, and does not need involved technical computations as in [AKT20].

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**Appendix A. Characterization of generalized permutahedra by submodular functions**

Recall that generalized permutahedra are those polytopes that have a coarsening of the braid fan as normal fan. We first give a description of braid cones, then we recap the definition of submodular functions and finally proof the bijective relation between generalized permutahedra submodular function $z : 2^I \to \mathbb{R}$ with $z(\emptyset) = 0$.

A composition of a finite set $I$ is an ordered sequence $(T_1, \ldots, T_k)$ of disjoint non-empty subsets $T_a \subseteq I$ such that $I = T_1 \cup \cdots \cup T_k$. Let $1_T$ for some $T \subseteq I$ be the 0/1-vector with entries equal to one for indizes in the subset $T$ and zero otherwise.

**Lemma A.1.** The faces of the braid arrangement $B_I$, also called braid cones, can be described uniquely by compositions $I = T_1 \cup \cdots \cup T_k$:

$$B_{T_1, \ldots, T_k} := \{ y \in \mathbb{R}^I : y(i) = y(j) \text{ for all } i, j \in T_a, y(i) \geq y(j) \text{ for } i \in T_a, j \in T_b \text{ and } a < b \}$$

$$= \text{cone}\{1_{T_1}, 1_{T_1 \cup T_2}, \ldots, 1_{T_1 \cup \cdots \cup T_{k-1}}\} + \text{span}\{1_I\}$$

with

$$\dim B_{T_1, \ldots, T_k} = k,$$

where $1_T$ for some subset $T \subseteq I$ is the 0/1-vector with entries equal to one for indices in the subset $T$ and zero otherwise.

A set function $z : 2^I \to \mathbb{R}$ is called submodular if for all $A, B \subseteq I$

$$z(A) + z(B) \geq z(A \cup B) + z(A \cap B).$$

We define the base polytope $P(z)$ of a submodular function $z : 2^I \to \mathbb{R}$ by

$$P(z) := \{ x \in \mathbb{R}^I : \sum_{i \in I} x_i = z(I) \text{ and } \sum_{i \in A} x_i \leq z(A) \text{ for all } A \subseteq I \}.$$
To simplify the proof of Theorem A.2 we will use the following notation
\[ x(A) := \sum_{i \in A} x_i \quad \text{for } A \subseteq I. \]

With that notation at hand we can write the definition of the base polytopes as
\[ P(z) = \{ x \in \mathbb{R}^I : x(I) = z(I) \text{ and } x(A) \leq z(A) \text{ for all } A \subseteq I \}. \]

It can be checked that the standard permutahedron \( \pi_I \) is the base polytope of the submodular function
\[ z(A) := |I| + (|I| - 1) + \cdots + (|I| - |A| + 1). \]

**Theorem A.2.** A polytope \( P \) is a generalized permutahedron if and only if it is the base polytope \( P(z) \) of a submodular function \( z : 2^I \to \mathbb{R} \) with \( z(\emptyset) = 0 \).

**Proof.** For a submodular function \( z : 2^I \to \mathbb{R} \) we show that \( P(z) \) is a generalized permutahedron by showing that every braid cone \( B_{T_1, \ldots, T_k} \subset \mathbb{R}^I \) is contained in a normal cone of \( P(z) \). Since \( P(z) \) is contained in the hyperplane \( \{ x \in \mathbb{R}^I : x(I) = z(I) \} \) the normal cone \( N_{P(z)}(P(z)) \) contains the line spanned by \( 1_I \in \mathbb{R}^I \), hence every normal cone of \( P(z) \) contains that line.

The following part of the proof relies on [FT83]. Fujishige and Tomizawa show under which conditions a greedy-like algorithm gives an optimal solution in the base polytope of a submodular function on a general distributive lattice. We adapt the proof to our special case.

Let \( B_{T_1, \ldots, T_k} \subset \mathbb{R}^I \) a braid cone. Choose a maximal chain \( C : \emptyset = C_0 \subset \cdots \subset C_n = I \) in the boolean lattice \( 2^I \) such that \( T_1, T_1 \cup T_2, \ldots, T_1 \cup \cdots \cup T_k \) are sets in the chain \( C \). Then
\[ |C_j \setminus C_{j-1}| = 1 \]
for \( j = 1, \ldots, n := |I| \) and we define a linear ordering on \( I \) by \( i_j := C_j \setminus C_{j-1} \in I \) for \( j = 1, \ldots, n \). Now, consider the point \( \tilde{x} \in \mathbb{R}^I \) defined by
\[ (A.1) \quad \tilde{x}_{i_j} := z(C_j) - z(C_{j-1}) \quad \text{für } j = 1, \ldots, n. \]

We will show
(i) that \( \tilde{x}(C_j) = z(C_j) \) for \( j = 1, \ldots, n \), and that the point \( \tilde{x} \) lies in \( P(z) \),
(ii) that \( \tilde{x} \) is maximal for all directions in the braid cone \( B_{T_1, \ldots, T_k} \).

Then it follows that the braid cone \( B_{T_1, \ldots, T_k} \) is contained in the normal cone \( N_{P(z)}(F) \), where \( F \) is a face containing \( \tilde{x} \).

For \( j = 1, \ldots, n \) we compute
\[ \tilde{x}(C_j) = \sum_{l=1}^j \tilde{x}_{i_l} = \sum_{l=1}^j (z(C_l) - z(C_{l-1})) = z(C_j), \]
in particular, \( \tilde{x}(I) = z(I) \). We show by induction on the cardinality \( |A| \) of a subset \( A \subseteq I \) that \( \tilde{x}(A) \leq z(A) \).

For the empty set we have \( 0 = \tilde{x}(\emptyset) = z(\emptyset) \).

For an arbitrary set \( A \subseteq I \) let \( j^* \) be the minimal index such that \( A \subseteq C_{j^*} \) and define the element \( i^* := A \setminus C_{j^*-1} \in I \).

We compute using the induction hypothesis, Equation (A.1), and submodularity of \( z \) together with \( A \setminus \{ i^* \} = A \cap C_{j^*-1} \) and \( C_{j^*} = A \cup C_{j^*-1} \):
\[ \tilde{x}(A) = \tilde{x}(\{ i^* \}) + \tilde{x}(A \setminus \{ i^* \}) \leq \tilde{x}(\{ i^* \}) + z(A \setminus \{ i^* \}) \]
\[ = z(C_{j^*}) - z(C_{j^*-1}) + z(A \setminus \{ i^* \}) \leq z(A). \]

Hence, \( \tilde{x} \in P(z) \).
Now, choose an arbitrary direction $y \in \mathcal{B}_{T_1, \ldots, T_k}$. By Lemma A.1 $y(i) = y(i')$ for $i, i' \in T_l$ so we can set $\hat{y}_l := y(i)$ for $i \in T_l$ and $l = 1, \ldots, k$. Moreover, $\hat{y}_l \geq \hat{y}_{l+1}$. For a point $x \in P(z)$ compute:

\[
y(\hat{x}) - y(x) = \sum_{i \in I} \hat{x}_i y(i) - \sum_{i \in I} x_i y(i) = \sum_{l=1}^k \hat{y}_l (\hat{x}(T_l) - x(T_l)) \]

\[
= \sum_{l=1}^k \left( \hat{y}_l (\hat{x}(T_1 \cup \cdots \cup T_l) - x(T_1 \cup \cdots \cup T_l)) - \hat{y}_l (\hat{x}(T_1 \cup \cdots \cup T_{l-1}) - x(T_1 \cup \cdots \cup T_{l-1})) \right)
\]

\[
= \sum_{l=1}^{k-1} (\hat{y}_l - \hat{y}_{l+1})(\hat{x}(T_1 \cup \cdots \cup T_l) - x(T_1 \cup \cdots \cup T_l)) + \hat{y}_k (\hat{x}(I) - x(I))
\]

\[
= \sum_{l=1}^{k-1} (\hat{y}_l - \hat{y}_{l+1}) \left( \sum_{I \cup \cdots \cup T_l \geq 0} (z(T_1 \cup \cdots \cup T_l) - x(T_1 \cup \cdots \cup T_l)) \right) \geq 0,
\]

where we use in the last equality, that the sets $T_1, T_2, \ldots, T_1 \cup \cdots \cup T_k$ are contained in the chain $C_1 \cup \cdots \cup C_k = I$ and that we already know $\hat{x}(C_j) = z(C_j)$ for $j = 1, \ldots, n$. Since the computation in A.2 is independent from the actual values of the direction $y \in \mathcal{B}_{T_1, \ldots, T_k}$, the inequality $y(\hat{x}) \geq y(x)$ holds for every direction $y \in \mathcal{B}_{T_1, \ldots, T_k}$. So the braid cone $\mathcal{B}_{T_1, \ldots, T_k}$ is contained in the normal cone $N_{P(z)}(F)$, where $F$ is a face containing $\hat{x}$. Hence, $P(z)$ is a generalized permutahedron.

For the opposite implication let $P$ be a generalized permutahedron. We will define a submodular function $z_P$ and show that $P = P(z_P)$. Since the generalized permutahedron $P$ is contained in the hyperplane with constant coordinate sum, the following set function is well defined:

\[
z_P(I) := \sum_{i \in I} x_i \quad \text{for } x \in P
\]

\[
z_P(A) := \max_{x \in P} \left( \sum_{i \in A} x_i \right) \quad \text{for } A \subseteq I.
\]

We can immediately deduce that $z(\emptyset) = 0$ and $P \subseteq P(z_P)$.

First we show that $z_P$ is submodular. For arbitrary $A, B \subseteq I$ find a chain $C:\emptyset = C_0 \subseteq C_1 \subseteq \cdots \subseteq C_k = I$ in the Boolean lattice $2^I$ that contains $A \cap B$ and $A \cup B$. We set $T_i := C_i \setminus C_{i-1}$ for $i = 1, \ldots, k$ and consider the braid cone $\mathcal{B}_{T_1, \ldots, T_k} = \text{cone}\{1_{T_1}, \ldots, 1_{T_k \cup \cdots \cup T_{k-1}}\} + \text{span}\{1_I\}$. Then there exists a face $F$ of $P$ such that the normal cone $N_P(F)$ contains the braid cone $\mathcal{B}_{T_1, \ldots, T_k}$ and in particular every point $x \in F$ is maximal in the direction $1_{A \cap B}, 1_{A \cup B} \in \mathcal{B}_{T_1, \ldots, T_k}$. Then,

\[
z_P(A) + z_P(B) \geq x(A) + x(B) = x(A \cup B) + x(A \cap B) = z_P(A \cup B) + z_P(A \cap B)
\]

and $z_P$ is submodular.

Now it is left to show that $P \supseteq P(z_P)$. The main idea for this part of the proof can be found in [DF10]. For the sake of a contradiction, let us assume there is a point $u \in P(z_P) \setminus P$. 

Then there exists a separating hyperplane \( H_{t,c} := \{ x \in \mathbb{R}^I : t(x) = c \} \) such that
\[
t(u) = \sum_{i \in I} t_i u_i > c \quad \text{and} \quad t(p) = \sum_{i \in I} t_i p_i \leq c \quad \text{for all} \ p \in \mathcal{P}
\]
Now choose a braid cone \( \mathcal{B}_{T_1, \ldots, T_k} \) such that \( t \in \mathcal{B}_{T_1, \ldots, T_k} \) and set again \( \hat{t}_l := t_i \) for \( i \in T_l \), \( l = 1, \ldots, k \). For points \( q \) in the \( t \)-maximal face \( F := \mathcal{P}_t \) we know by the definition of \( z \) that \( q(T_1 \cup \cdots \cup T_l) = z(T_1 \cup \cdots \cup T_l) \) for \( l = 1, \ldots, k \). Using telescoping sums we compute
\[
t(u) = \sum_{i \in I} t_i u_i > c \geq t \cdot q = \sum_{i \in I} t_i q_i = \sum_{l=1}^{k} \hat{t}_l q(T_l)
\]
\[
= \hat{t}_k q(T_1 \cup \cdots \cup T_k) + \sum_{l=k-1}^{1} \left( \hat{t}_l - \hat{t}_{l+1} \right) q(T_1 \cup \cdots \cup T_l)
\]
\[
= \hat{t}_k z(T_1 \cup \cdots \cup T_k) + \sum_{l=k-1}^{1} \left( \hat{t}_l - \hat{t}_{l+1} \right) z(T_1 \cup \cdots \cup T_l)
\]
\[
\geq \hat{t}_k u(T_1 \cup \cdots \cup T_k) + \sum_{l=k-1}^{1} \left( \hat{t}_l - \hat{t}_{l+1} \right) u(T_1 \cup \cdots \cup T_l)
\]
\[
= \sum_{l=1}^{k} \hat{t}_l u(T_l) = \sum_{i \in I} t_i u_i = t(u).
\]
That is a contradiction and completes the proof. \( \square \)

References

[AA17] Marcelo Aguiar and Federico Ardila. Hopf monoids and generalized permutahedra. 2017. arXiv:1709.07504.

[Agn17] Geir Agnarsson. On a special class of hyper-permutahedra. Electron. J. Combin., 24(3):article P3.46, 25 pp, 2017.

[AH05] Geir Agnarsson and Magnús M. Halldórsson. Strong colorings of hypergraphs. In Giuseppe Persiano and Roberto Solis-Oba, editors, Approximation and Online Algorithms, pages 253–266, Berlin, Heidelberg, 2005. Springer.

[AKT20] Jean-Christophe Aval, Théo Karaboghossian, and Adrian Tanasa. The Hopf monoid of hypergraphs and its sub-monoids: Basic invariant and reciprocity theorem. Electron. J. Combin., pages article P1.34, pp23, 2020.

[BBM19] Carolina Benedetti, Nathel Bergeron, and John Machacek. Hypergraphic polytopes: Combinatorial properties and antipode. J. Comb., 10(3):515–544, 2019.

[BDK12] Felix Breuer, Aaron Dall, and Martina Kubitzke. Hypergraph coloring complexes. Discrete Math., 312(16):2407–2420, 2012.

[BJR09] Louis J. Billera, Ning Jia, and Victor Reiner. A quasisymmetric function for matroids. Eur. J. Combin., 30(8):1727–1757, 2009.

[BS18] Matthias Beck and Raman Sanyal. Combinatorial Reciprocity Theorems: An Invitation To Enumerative Geometric Combinatorics. American Mathematical Society, Providence, R.I., 2018.

[BTV15] Csilla Bujtás, Zsolt Tuza, and Vitaly Voloshin. Hypergraph colouring. In Lowell W. Beineke and Robin J. Wilson, editors, Topics in Chromatic Graph Theory, pages 230–254. Cambridge University Press, Cambridge, 2015.

[BZ06a] Matthias Beck and Thomas Zaslavsky. An enumerative geometry for magic and magilatin labellings. Ann. Comb., 10(4):395–413, 2006.

[BZ06b] Matthias Beck and Thomas Zaslavsky. Inside-out polytopes. Adv. Math., 205(1):134–162, 2006.
COMBINATORIAL RECIPROCITY THEOREMS AND PRUNED INSIDE-OUT POLYTOPES

[BZ06c] Matthias Beck and Thomas Zaslavsky. The number of nowhere-zero flows on graphs and signed graphs. *J. Combin. Theory Ser. B*, 96(6):901–918, 2006.

[BZ10] Matthias Beck and Thomas Zaslavsky. Six little squares and how their numbers grow. 2010. arXiv:1004.0282.

[CF18] Jean Cardinal and Stefan Felsner. Notes on Hypergraphic Polytopes. Unpublished, 2018.

[CL20] Federico Castillo and Fu Liu. Deformation cones of nested braid fans. *Int. Math. Res. Notices*, 2020.

[DF10] Harm Derksen and Alex Fink. Valuative invariants for polymatroids. *Adv. Math.*, 225(4):1840–1892, 2010.

[EH66] P. Erdős and A. Hajnal. On chromatic number of graphs and set-systems. *Acta Math Acad Sci H.*, 17(1-2):61–99, 1966.

[Ehr62] Eugène Ehrhart. Sur les polyèdres rationnels homothétiques à n dimensions. *C. R. Hebd. Seances Acad. Sci.*, 254:616–618, 1962.

[FT83] Satoru Fujishige and Nobuaki Tomizawa. A note on submodular functions on distributive lattices. *J. Oper. Res. Soc. Japan*, 26(4):309–318, 1983.

[Grü03] Branko Grünbaum. *Convex Polytopes*. Springer New York, New York, 2003.

[Mac71] Ian G. Macdonald. Polynomials associated with finite cell-complexes. *J. London Math. Soc.*, s2-4(1):181–192, 1971.

[Pos09] Alexander Postnikov. Permutohedra, associahedra, and beyond. *Int. Math. Res. Notices*, (6):1026–1106, 2009.

[PRW08] Alex Postnikov, Victor Reiner, and Lauren Williams. Faces of generalized permutohedra. *Doc. Math.*, 13:207–273, 2008.

[RR12] Nathan Reff and Lucas J. Rusnak. An oriented hypergraphic approach to algebraic graph theory. *Linear Algebra Appl.*, 437(9):2262–2270, 2012.

[Rus13] Lucas J. Rusnak. Oriented hypergraphs: Introduction and balance. *Electron. J. Combin.*, 20(3):article P48, 29 pp, 2013.

[Sta73] Richard P. Stanley. Acyclic orientations of graphs. *Discrete Math.*, 5(2):171–178, 1973.

[Sta74] Richard P. Stanley. Combinatorial reciprocity theorems. *Adv. Math.*, 14(2):194–253, 1974.

[Sta86] Richard P. Stanley. Two poset polytopes. *Discrete Comput. Geom.*, 1(1):9–23, 1986.

[Sta07] Richard P. Stanley. An introduction to hyperplane arrangements. In Ezra Miller, Reiner, Victor, and Sturmfels, Bernd, editors, *Geometric Combinatorics*, pages 389–496. American MathSoc, Providence, R.I., 2007.

[Zas91] Thomas Zaslavsky. Orientation of signed graphs. *Eur. J. Combin.*, 12(4):361–375, 1991.

[Zie98] Günter M. Ziegler. *Lectures on Polytopes*. Springer, New York, 1998.