AUTOMORPHISM GROUPS OVER A HYPERIMAGINARY

BYUNGHAN KIM AND HYOYoon LEE

ABSTRACT. In this paper we study the Lascar group over a hyperimaginary $e$. We verify that various results about the group over a real set still hold when the set is replaced by $e$. First of all, there is no written proof in the available literature that the group over $e$ is a topological group. We present an expository style proof of the fact, which even simplifies existing proofs for the real case. We further extend a result that the orbit equivalence relation under a closed subgroup of the Lascar group is type-definable. On the one hand, we correct errors appeared in [6, 5.1.14-15] and produce a counterexample. On the other, we extend Newelski’s Theorem in [12] that ‘a $G$-compact theory over a set has a uniform bound for the Lascar distances’ to the hyperimaginary context. Lastly, we supply a partial positive answer to a question raised in [4, 2.11], which is even a new result in the real context.

The Lascar (automorphism) group of a first-order complete theory and its quotient groups such as the Kim-Pillay group and the Shelah group have been central themes in contemporary model theory. The study on those groups enables us to develop Galois theoretic correspondence between the groups and their orbit-equivalence relations on a monster model such as Lascar types, Kim-Pillay types, and Shelah strong types. The notions of the Lascar group and its topology are introduced first by D. Lascar in [9] using ultraproducts. Later more favorable equivalent definition is suggested in [7] and [11], which is nowadays considered as a standard approach. However even a complete proof using the approach of the fundamental fact that the Lascar group is a topological group is not so well available. For example in [2], its proof is left to the readers, while the proof is not at all trivial. As far as we can see, only in [14], a detailed proof is written.

Aforementioned results are for the Lascar group over $\emptyset$, or more generally over a real set $A$. In this paper we study the Lascar group over a hyperimaginary $e$ and verify how results on the Lascar group over $A$ can be extended to the case when the set is replaced by $e$. Indeed this attempt was made in [8] (and rewritten in [6, §5.1]). However those contain some errors, and moreover a proof of that the Lascar group over $e$ is a topological group is also missing. In this paper we supply a proof of the fact in a detailed expository manner. Our proof is more direct and even simplifies that for the group over $\emptyset$ in [14]. We correct the mentioned errors in [6, 8], as well. In particular we correct the proof of that the orbit equivalence relation under a closed normal subgroup of the Lascar group over $e$ is type-definable over $e$. Moreover we extend Newelski’s Theorem in [12] to the hyperimaginary context. Namely we show that if $T$ is $G$-compact over $e$ then there is $n < \omega$ such that for

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any hyperimaginaries $b, c$, we have $b \equiv^L_c c$ iff the Lascar distance between $b$ and $c$ is $\leq n$. We also generalize the notions of relativized Lascar groups introduced in $[3]$, in the context of hyperimaginaries. Lastly, we prove a new result even in the real context, which is a partial positive answer to a question raised in $[4, 2.11]$. That is, if $T$ is G-compact over $e$, and $c$ has only finitely many conjugates $c_i$ ($i < m$) over $be$ such that $bc_i \not\equiv^L_{be} bc_j$ for $i < j < m$ then, for $p = tp(b/e)$ and $\mathcal{P} = tp(bc/e)$, the kernel of the canonical projection $\pi_n : \text{Gal}^n_{\mathcal{L}}(\mathcal{P}) \rightarrow \text{Gal}^n_{\mathcal{L}}(p)$ of the relativized Lascar groups is finite.

In Section 1, we introduce basic terminology for this paper and supply the mentioned detailed proof of that the Lascar group over $e$ is a quasi-compact topological group (Corollary 1.20).

In Section 2, we prove that the orbit equivalence relation under a closed subgroup of the Lascar group over $e$ is bounded and type-definable. In addition if the subgroup is normal then the equivalence relation is $e$-invariant (Corollary 2.4).

In Section 3, using results in Section 2 we study particular quotient groups of the Lascar group. Namely we investigate the Kim-Pillay group and the Shelah group, and corresponding notions of Kim-Pillay types and Shelah strong types, over a hyperimaginary. We point out an error occurred in $[8]$ (rewritten in $[6]$), by producing a counterexample.

In Section 4, using the approach in $[3]$ we prove that any type-definable Lascar type over $e$ has a finite diameter (Theorem 4.7), by which we can extend Newelski’s result (over $\emptyset$) to the class of theories being G-compact over $e$ (Corollary 4.8).

In Section 5, we introduce the notions of relativized Lascar groups over $e$, generalizing those for real tuples over $\emptyset$ in $[3]$. Then as mentioned above, we supply an answer covering the hyperimaginary case to a question in $[4]$ (Theorem 5.6). We also point out that results in $[4]$ can be extended to our hyperimaginary context (Corollary 5.8, Proposition 5.10, Theorem 5.12).

1. The Lascar Group over a Hyperimaginary

Throughout this paper, we fix a complete theory $T$ with a language $\mathcal{L}$ and we work in a $(\bar{\kappa}$-saturated) monster model $\mathcal{M}$ of $T$. We use standard terminology as in $[4]$ or $[6]$. For example, ‘bounded’ or ‘small’ sizes refer to cardinalities $< \bar{\kappa}$. By real sets or tuples, we mean small subsets of $\mathcal{M}$ or sequences from $\mathcal{M}$ of small lengths, respectively. Being ‘type-definable’ means being ‘type-definable over some real set.’ When $A$ is a small set, ‘$A$-invariant’ means ‘invariant under any automorphism of $\mathcal{M}$ fixing $A$ pointwise.’ As is well-known, an $A$-invariant type-definable (solution) set is $A$-type-definable. Recall that a hyperimaginary written as $b_{F} = b/F$ is an equivalence class of the real tuple $b$ of an $\emptyset$-type-definable equivalence relation $F$. Occasionally we may write hyperimaginaries using boldface letters such as $b, c$. The type $tp_z(b_{F}/c_{L})$ of $b_{F}$ over another hyperimaginary $c_{L}$ is a partial type $\exists z_1 z_2 (tp_{z_1 z_2}(bc) \land F(x, z_1) \land L(c, z_2))$, whose solution set is the set of automorphic images of $b_{F}$ over $c_{L}$. As usual $b_{F} \equiv_{c_{L}} d_{F}$ denotes $tp(b_{F}/c_{L}) = tp(d_{F}/c_{L})$. We say $b_{F}$ and $c_{L}$ are interdefinable or equivalent if, any automorphism $f$ fixes the class $b_{F}$ iff $f$ fixes the class $c_{L}$. Real or imaginary tuples in $\mathcal{M}^{eq}$ are examples of hyperimaginaries, but not every hyperimaginary is equivalent with an imaginary tuple. Notice that a sequence of hyperimaginaries $(d_{i}/F_{i} \mid i < \lambda)$ is interdefinable with a hyperimaginary $d/F$ where $d = (d_{i} \mid i < \lambda)$ and $F = \bigwedge_{i < \lambda} F_{i}(x_{i}^{0}, x_{i}^{1})$ such that $x_{k}^{k} \cap x_{l}^{l} = \emptyset$ iff $(k, i) \neq (l, j)$, where $k, l \in \{0, 1\}$. 

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For the rest of this paper, we fix $E$, an $\emptyset$-type-definable equivalence relation, $a$ a real (possibly infinite) tuple, and $e = a/E = a_E$ a hyperimaginary. For simplicity we may write a formula in a type, say $E(x,y)$ as $\varphi(x,y)$ (or $\varphi(a,b)$ for $\models E(a,b)$) which indeed means $\varphi(x',y')$ for some suitable finite subtypes $x', y'$ of $x, y$, respectively. In this section we show that the Lascar group over $e$ is a quasi-compact topological group (Corollary 1.20). In fact this is mentioned in [6, 5.1.3] without a proof. But we cannot find any stated proof in the literature showing that the proof for a real parameter (as in [14]) can go thru for a hyperimaginary parameter. We confirm this, largely by following ideas in the real case proof. But the proof here is more direct (for example Proposition 1.11, Theorem 1.19) and simplified.

**Definition 1.1.**

1. $\operatorname{Aut}_e(M) = \{ f \in \operatorname{Aut}(M) : f(e) = e \}$.
2. Let $e'$ be a hyperimaginary. We denote $e' \in \operatorname{dcl}(e)$ and say $e'$ is definable over $e$ if $f(e') = e'$ for all $f \in \operatorname{Aut}_e(M)$.
3. Likewise denote $e' \in \operatorname{bdd}(e)$ and say $e'$ is bounded over $e$ if \[ \{ f(e') \mid f \in \operatorname{Aut}_e(M) \} \] is bounded.
4. $\operatorname{Autf}_e(M)$ is the subgroup of $\operatorname{Aut}_e(M)$ generated by \[ \{ f \in \operatorname{Aut}_e(M) \mid f \in \operatorname{Aut}_M(M) \text{ for some model } M \text{ such that } e \in \operatorname{dcl}(M) \} \].

The following remarks will be freely used.

**Remark 1.2.**

1. A hyperimaginary $b_F$ is called countable if $|b|$ is countable. Recall that any hyperimaginary is interdefinable with a sequence of countable hyperimaginaries [5, Lemma 4.1.3]. Thus the definable closure of $e$, $\operatorname{dcl}(e)$ can be regarded as the set of all countable hyperimaginaries which are definable over $e$. Then $c_L$ is definable over $e$ iff $c_L$ is equivalent with a sequence of countable hyperimaginaries from $\operatorname{dcl}(e)$. Moreover the (bounded) set $\operatorname{dcl}(e)$ is interdefinable with a single hyperimaginary, as indicated at the end of the first paragraph of this section.
2. Likewise, the bounded closure of $e$, $\operatorname{bdd}(e)$ can be regarded as the set of all countable hyperimaginaries which are bounded over $e$. Again then $\operatorname{bdd}(e)$ is equivalent with a single hyperimaginary bounded over $e$.
3. It easily follows that $\operatorname{Autf}_e(M)$ is a normal subgroup of $\operatorname{Aut}_e(M)$.

**Definition 1.3.** $\operatorname{Gal}_L(M, e) = \operatorname{Aut}_e(M) / \operatorname{Autf}_e(M)$, and $\pi : \operatorname{Aut}_e(M) \to \operatorname{Gal}_L(M, e)$ is the canonical projection. For $f \in \operatorname{Aut}_e(M)$, $\overrightarrow{f}$ denotes $\pi(f) = f \cdot \operatorname{Autf}_e(M)$.

**Remark 1.4.**

1. [11] Lemma 1.6 The following are equivalent.
   a. $e \in \operatorname{dcl}(M)$.
   b. $E(x,a)$ is finitely satisfiable in $M$.
   c. $e \in \operatorname{bdd}(M)$.
2. Recall that hyperimaginaries $b_F, c_F$ have the same Lascar (strong) type over $e$, denoted by $b_F \equiv_L^e c_F$ or $\operatorname{Ltp}(b_F/e) = \operatorname{Ltp}(c_F/e)$, if there is $f \in \operatorname{Autf}_e(M)$ such that $f(b_F) = c_F$. 

As is well-known the following are equivalent (so $\equiv_\mathcal{L}$ does not depend on the choice of a monster model).

(a) $b_F \equiv_\mathcal{L} c_F$.
(b) There are $1 \leq n < \omega$, models $M_i (i < n)$ and reals $b_i (i \leq n)$ such that $e \in \text{dcl}(M_i), b = b_0$, $c = b_n$, and $b_i/F \equiv_{M_i} b_{i+1}/F$.
(c) There are $1 \leq n < \omega$, reals $b = b_0, \ldots, b_n = c$ such that for each $i < n$, $b_i/F$ and $b_{i+1}/F$ begin an $e$-indiscernible sequence.

Moreover $x_F \equiv_\mathcal{L} y_F$ is the finest $e$-invariant bounded equivalence relation coarser than $F$.

(3) For small models $M, N \prec M$ such that $e \in \text{dcl}(M), \text{dcl}(N)$, if $f(M) \equiv_N g(M)$ where $f, g \in \text{Aut}_e(M)$, then $\overline{f} = \overline{g}$ (see for example [6] Remark 5.1.2(1)).

We now point out the following.

**Theorem 1.5.** $\text{Gal}_L(M, e)$, up to isomorphism, does not depend on the choice of $M$.

The same proof for the real parameter case (e.g. [8] Remark 5.1.2]) would work, so we omit it.

**Definition 1.6.** We call $\text{Gal}_L(M, e)$ the *Lascar group* of $T$ over $e$. Due to Theorem 1.5 we omit $M$ and denote it by $\text{Gal}_L(T, e)$.

**Corollary 1.7.** $|\text{Gal}_L(T, e)| \leq 2^{|T|+|a|}$.

**Proof.** By Remark 1.2(3), any $\overline{f}$ is determined by any $M, N \models T$ such that $e \in \text{dcl}(M), \text{dcl}(N)$. Then $|\{\text{tp}(f(M)/N) : f \in \text{Aut}_e(M)\}| \leq 2^{|M|+|N|+|T|}$, and letting $M, N$ contain $a$ and have cardinality $|T|+|a|$, we see that $|\text{Gal}_L(T, e)| \leq 2^{|T|+|a|}$.

**Definition 1.8.** Let $M, N \prec M$ such that $e \in \text{dcl}(M), \text{dcl}(N)$. Define $S_{M,e}(N) = S_M(N) := \{\text{tp}(f(M)/N) : f \in \text{Aut}_e(M)\}$.

We give the relative Stone topology on $S_M(N)$: As usual, let $S_x(N) = \{p(x) : |x| = |M| \text{ and } p \text{ is a complete type over } N\}$ be the Stone space (that is, the basic open sets are of the form $[\varphi] = \{p \in S_x(N) : \varphi \in p\}$ for $\varphi \in \mathcal{L}(N)$). Then $S_M(N)$ is a subspace of $S_x(N)$. We remark as follows that $S_M(N)$ is a compact (i.e. quasi-compact and Hausdorff) space.

**Remark 1.9.** Let $r(x, a)$ be a partial type over $a$ that describes $\text{tp}_x(M/e)$. Since $e \in \text{dcl}(N)$, we have $r_0(x, N) := \text{tp}(a/N) = E(x, a)$. Hence $\text{tp}_x(M/e)$ can also be described by

$$r'(x, N) = 3g(r(x, y) \land r_0(y, N)).$$

Thus $S_M(N) = \{p \in S_x(N) : r'(x, N) \subseteq p\}$ is a closed (so compact) subspace of the compact space $S_x(N)$.

Due to Remark 1.2(3), there are well-defined maps $\mu, \nu$

- $\text{Aut}_e(M) \xrightarrow{\sim} S_M(N)$
- $\text{Gal}_L(T, e)$

such that $\mu = \pi \circ \nu$. We give $\text{Gal}_L(T, e)$ the quotient topology with respect to $\nu$. Notice that $S_M(N) \xrightarrow{\nu} \text{Gal}_L(T, e)$ implies $\text{Gal}_L(T, e)$ is quasi-compact (but not necessarily Hausdorff).
We point out another independence.

**Proposition 1.10.** The topology on $\text{Gal}_L(T, e)$ does not depend on the choice of $M$ and $N$, i.e. any pair of models $M'$ and $N'$ such that $e \in \text{dcl}(M'), \text{dcl}(N')$ induces the same topology.

Again the proof is the same as the real parameter case, so we omit.

Now we begin to show that $\text{Gal}_L(T, e)$ is a topological group with the equipped topology. Our proof of this is more direct than that in [14]. Especially, we do not appeal to the argument extending the language. Recall that a group equipped with a topology is called a topological group if the group operations $G \times G \to G : (x, y) \mapsto xy$, $G \to G : x \mapsto x^{-1}$ are continuous.

Thanks to Proposition 1.10, from now on, we fix a small model $M$ containing $a$ and work with the relative compact Stone space $S_{M,e}(M) = S_M(M)$. As said, we have the following maps:

$$\text{Aut}_e(M) \xrightarrow{\mu} S_M(M) \xrightarrow{\nu} \text{Gal}_L(T, e)$$

$$f \mapsto p = \text{tp}(f(M)/M) \mapsto \nu(p) = f$$

**Proposition 1.11.** The inverse operation on $\text{Gal}_L(T, e)$ is continuous.

**Proof.** Let $C$ be a closed subset of $\text{Gal}_L(T, e)$. We want to show $B = \{f \in \text{Gal}_L(T, e) | f = C\}$ is closed. Now assume $\nu^{-1}(C) \subseteq S_M(M)$ is type-defined by say $\Phi(y)$ over $M$. Consider a type over $M$

$$\Psi(x) = \exists y ((xM = y \land \Phi(y)).$$

Let $B_0 = \{p(x) \in S_M(M) | \Psi(x) \subseteq p(x)\}$, which is closed.

It suffices to see that $B_0 = \nu^{-1}(B)$: Let $q = \text{tp}(M_0/M) \in B_0$. Then there is $M_1$ such that $M_0 M \equiv M_1$ and $\text{tp}(M_1/M) \in \nu^{-1}(C)$. Now there is $f \in \text{Aut}_e(M)$ such that $f(MM_1) = M_0$. Thus $f^{-1}(M) = M_1$ and $f^{-1}(B) \subseteq C$. Therefore $f(B) \subseteq B_0$, and $q \in \nu^{-1}(B)$ since $q = \text{tp}(f(M)/M)$.

Conversely, let $q = \text{tp}(f(M)/M) \in \nu^{-1}(B)$. Then $\text{tp}(f(M)/M) \in \nu^{-1}(C)$, thus $f^{-1}(M) \models \Phi(y)$. Also, clearly $f(M)M \equiv Mf^{-1}(M)$, hence $q = \text{tp}(f(M)/M) \in B_0$. \qed

Showing the product map of $\text{Gal}_L(T, e)$ is continuous is more complicated and we need preparatory work. Recall that $x_0x_1 \ldots x_n \equiv_e x_{i_0} \ldots x_{i_n}$ means $\exists y((x_0 \ldots x_n a \equiv x_{i_0} \ldots x_{i_n}, y) \land E(y, a)).$

**Definition 1.12.** Let $\Gamma(x_0, x_1)$ be a partial type over $a$ saying there is an $e$-indiscernible sequence beginning with $x_0, x_1$, that is

$$\exists x_2x_3 \ldots (x_0x_1 \ldots x_n \equiv_e x_{i_0} \ldots x_{i_n} | i_0 < \cdots < i_n \in \omega, 1 \leq n).$$

We will freely use the following.

**Remark 1.13.**

1. We can assume that each formula in $\Gamma$ is reflexive and symmetric since $\Gamma$ itself is so, and $\Gamma$ is closed under finite conjuction of formulas.
2. Due to Remark 1.4(2), $x \equiv_e y$ is the transitive closure of $\Gamma(x, y)$.
(3) Assume $\delta \in \Gamma$. Then there is $n \in \omega$ such that there exists at most length $n$ antichain $c_0, \ldots, c_n$ of $\delta$ (i.e. $\neg \delta(c_i, c_j)$ for $i < j \leq n$): Otherwise by Ramsey, there is an $a$-indiscernible sequence $I = \{c_i \mid i < \omega\}$ such that $\neg \delta(c_0, c_1)$ holds. Then since $I$ is $e$-indiscernible as well $\Gamma(c_0, c_1)$ must hold, a contradiction.

Remark 1.14. For $p_i \in S_M(M) (i = 0, 1)$, we write $p_0 \approx p_1$ if $\nu(p_0) = \nu(p_1)$. Then $p_0 \approx p_1$ iff $M_0 \equiv^e_M M_1$ for any (some) $M_i \models p_i$.

Proof. Assume $M_i \models p_i$. Then there are $f_i \in \text{Aut}_e(M)$ such that $f_i(M) = M_i$. Now assume $p_0 \approx p_1$, so $\bar{f}_0 = \bar{f}_1$. Hence there is $g \in \text{Aut}_e(M)$ such that $f_0 = gf_1$, so $M_0 = f_0(M) = g(f_1(M)) = g(M_1)$. Therefore $M_0 \equiv^e_M M_1$.

Conversely assume $M_0 \equiv^e_M M_1$. Then there is $g \in \text{Aut}_e(M)$ such that $M_0 = g(M_1)$, so $\text{tp}(f_0(M)/M) = \text{tp}(g(f_1(M))/M)$. Thus $\bar{f}_0 = g\bar{f}_1 = \bar{f}_1$ by Remark 1.13, and $\nu(p_0) = \nu(p_1)$. □

Remark & Definition 1.15.

(1) Let

$$S_{M^2}(M) := \{\text{tp}(f(M)g(M)/M) \mid f, g \in \text{Aut}_e(M)\}.$$

Then the map $r : S_{M^2}(M) \to S_M(M) \times S_M(M)$ sending $\text{tp}(M_0M_1/M)$ to $(\text{tp}(M_0/M), \text{tp}(M_1/M))$ is a closed map since it is a continuous map between compact spaces.

(2) We define a binary relation $D$ on $S_M(M)$ by, $D(p, q)$ holds iff there are $M' \models p$, $M'' \models q$ such that $\models \Gamma(M', M'')$, i.e. $D(x, y)$ is type-defined by

$$\text{tp}_x(M/e) \land \text{tp}_y(M/e) \land \Gamma(x, y).$$

Then by Remark 1.13 and 1.14, $p \approx q$ is the transitive closure of $D$. Now by (1), $D$ is a closed subset of $S_M(M) \times S_M(M)$.

(3) For $p \in S_M(M)$, we let $D[p] = \{q \in S_M(M) : D(p, q)\}$, i.e. $D[p]$ is type-defined by $\exists y(p(y) \land \Gamma(x, y))$. It is closed so compact in $S_M(M)$. For $q \in D[p]$, $\nu(p) = \nu(q)$.

(4) For $\delta(x', y') = \delta(x, y', a) \in \Gamma$, let

$$D_\delta[p] = \{q \in S_M(M) \mid \exists y(p(y) \land \delta(x', y')) \subseteq q(x)\}.$$

Note that $p \in D[p] = \bigcap_{\delta \in \Gamma} D_\delta[p]$. Hence by compactness for any open $O \subseteq S_M(M)$, $D[p] \subseteq O$ iff for some $\delta \in \Gamma$, $D_\delta[p] \subseteq O$.

Lemma 1.16. Let $p \in S_M(M)$. For any $\delta \in \Gamma$, $p \in \text{int}(D_\delta[p])$.

Proof. As in Remark 1.13, there is a maximal antichain $\langle M_0, \ldots, M_n \rangle$ of $\delta$ in $p$, so that for any $M' \models p$, there is $i$ such that $\models \delta(M_i, M')$. Then $p(y) = \bigvee_{i=0}^n \delta(M_i, y)$, so there is $\psi \in p$ such that $\psi(y) = \bigvee_{i=0}^n \delta(M_i, y)$. Note that $p \in [\psi]$, which is open. We claim $[\psi] \subseteq D_\delta[p]$; For any $q \in [\psi]$, since $\psi \models \bigvee_{i=0}^n \delta(M_i, y)$, if $M' \models q$, then there is $j$ satisfying $\delta(M_j, M')$. Thus $q \in D_\delta[p]$, so $p \in \text{int}(D_\delta[p])$. □

The following lemma and corollary will play a critical role as in [13]. We modify/simplify the proof.

Lemma 1.17. Assume $W(\subseteq S_M(M))$ is closed under $\approx$, i.e. $W = \nu^{-1}(\nu(W))$. Let $O = \text{int}(W)$, and let $U = \{p \in S_M(M) \mid D[p] \subseteq O\}(\subseteq O)$.

(1) For $p \in O$, the following are equivalent.

(a) $\nu^{-1}(\nu(p)) \subseteq O$, i.e. $q \in O$ for any $q \approx p$.  

Theorem 1.19. The multiplication on $M$ type-defined over $\Gamma$.

Proof. (1) By Remark 1.15(2),(3),(4), we have (a)$\Rightarrow$(b)$\Rightarrow$(c). Assume (b) and (c). We will show (a).

(1) By Remark 1.15(2),(3),(4), we have (a)$\Rightarrow$(b)$\Rightarrow$(c). Assume (b) and (c). We will show (a).

Proof of Claim. (1) is by the definition of $\Phi(z)$. We now let $B := \{\bar{\varphi} \in C \mid \varphi \in \Phi(z)\}$.

Claim.

(1) $B_0$ is $\approx$-closed, i.e. if $(p_0,q_0) \in B_0$ and $q_i \approx p_i (i = 0,1)$ then $(q_0,q_1) \in B_0$.

(2) $B_0$ is closed in $S_M(M)^2$.

Proof of Claim. (1) is clear by the definition of $B_0$.

(2) Consider a type over $M$

$$\Psi(x,y) = \exists z (z \equiv M y \wedge x \equiv y \wedge \Phi(z)).$$

Let $B_1 \subseteq S_Mz(M)$ be type-defined by $\Psi$, i.e. $B_1 = \{p(x,y) \in S_Mz(M) \mid \Psi(x,y) \subseteq p(x,y)\}$, which is closed in $S_Mz(M)$.

Due to Remark 1.15(1), it suffices to show that $B_0 = r(B_1)$; Let $tp(M_0M_1/M) \in B_1$. Hence $M \equiv M_1$ and there is $M_2$ such that $M_0M_2 \equiv M_1M_2$ and $tp(M_2/M) \in B_1$.
\[\nu^{-1}(C). \] Thus there are \(f, g \in \text{Aut}_e(M)\) such that \(M_1 = g(M)\) and \(M_0M_2 = f(MM_1)\). Then \(fg(M) = f(M_1) = M_2\). Therefore \(fg \in C\), and \((\text{tp}(M_0/M), \text{tp}(M_1/M)) = (\text{tp}(f(M)/M), \text{tp}(g(M)/M)) \in B_0\) as wanted.

For the converse, let \((\text{tp}(f(M)/M), \text{tp}(g(M)/M)) \in B_0\), so that \((\overline{f}, \overline{g}) \in B\) by definition of \(B_0\). Hence \(\overline{fg} \in C\). Now \(f(M)fg(M) \equiv e\ M_0g(M) \land M \equiv e\ g(M) \land \Phi(fg(M))\), and \(\text{tp}(f(M)g(M)/M) \in B_1\).

We are ready to show \(B\) is closed in \(G^2\). Let \((\overline{f_0}, \overline{f_1}) \in G^2 \setminus B\), and let \(W = \{(p, q) \in S_M(M)^2 \mid \nu(p) = \overline{f_0}, \nu(q) = \overline{f_1}\}\). Hence \(B_0 \cap W = \emptyset\) and again \(W\) is \(\approx\)-closed. Fix \((p_0, p_1) \in W\). Now we have \(D[p_0] \times D[p_1] \subseteq W\) (Remark [1.18 (3)]). Since both \(B_0\) and \(D[p_0] \times D[p_1]\) are closed so compact in \(S_M(M)^2\), by purely topological arguments, there are open neighborhoods \(U_i(\subseteq S_M(M))\) of \(D[p_i]\) \((i = 0, 1)\) such that \(B_0 \cap (U_0 \times U_1) = \emptyset\). Hence by Lemma [1.18] \(\nu(p_i) = f_i \in O_i := \text{int}(\nu(U_i)) \subseteq G\), so \((\overline{f_0}, \overline{f_1}) \in O_0 \times O_1\). Now since \(B_0 \cap (U_0 \times U_1) = \emptyset\), by Claim (1) \(B_0 \cap (\nu^{-1}(\nu(U_0))) \times \nu^{-1}(\nu(U_1))) = \emptyset\) as well. Hence we must have \(B \cap (O_0 \times O_1) = \emptyset\). Therefore \(B\) is closed in \(G^2\).

**Corollary 1.20.** The Lascar group over a hyperimaginary \(e\), \(\text{Gal}_L(T, e)\) is a quasi-compact topological group.

**Proof.** By Proposition [1.11] and Theorem [1.19] \(\square\)

### 2. Type-definability of orbit equivalence relations

Given a subgroup \(H \leq \text{Aut}_e(M)\), we write \(x \equiv_e^H y\) for \(|x| = |y| = \alpha\) to denote the orbit equivalence relation on \(M^\alpha\) under \(H\), that is for \(b, c \in M^\alpha\), \(b \equiv_e^H c\) iff there is \(h \in H\) such that \(c = h(b)\). More generally if \(F\) is an \(\emptyset\)-type-definable equivalence relation on \(M^\alpha\), then \(x_F \equiv_e^H y_F\) is the orbit equivalence relation on \(M^\alpha/F\) under \(H\), which is coarser than \(F\).

In this section we aim to show that given closed \(H' \leq \text{Gal}_L(T, e)\) and \(H = \pi^{-1}(H') \leq \text{Aut}_e(M)\), \(x_F \equiv_e^H y_F\) is a type-definable bounded equivalence relation; and in addition if \(H' \leq \text{Gal}_L(T, e)\) then \(x_F \equiv^e_F y_F\) is \(e\)-invariant (Corollary [2.4]). This result is claimed to be proved in [8] (restated in [6] Lemma 5.1.6(1)) but the proof contains an error: There for each complete hyperimaginary type \(p\) over a hyperimaginary, \(\Psi_p(x, y)\) type-defines a bounded equivalence relation on the solution set of \(p\), but the description after this there need not work to extend \(\Psi_p(x, y)\) to the whole monster model keeping it type-definable and bounded.

When \(e = \emptyset\) and \(H' = \{\text{id}\} \leq \text{Gal}_L(T)\), a correct proof using an ultraproduct argument is given in [11] Lemma 4.18]. Here we supply a direct proof (not appealing to the ultraproduct method) of the general case result while we still utilize ideas in [11].

We recall the following folklore first.

**Fact 2.1.** Let \(F\) be a type-definable equivalence relation on \(M^\alpha\) which is \(e\)-invariant. Then there is an \(\emptyset\)-type-definable equivalence relation \(F'\) such that for any \(b \in M^\alpha\), \(bF = bF'\) and \(b/\text{Aut}_e(M)\) are interdefinable over \(e\).

**Proof.** Since \(F\) is \(e(= a/E)\)-invariant, \(F\) is type-definable over \(a\) by say \(F(x, y; a)\). Then for \(p(x) = \text{tp}(a)\), we put
\[F'(x, y) = (F(x, y; z) \land E(z, w) \land p(z) \land p(w)) \lor \exists z = w.\]
It is not hard to check that \(F'\) satisfies the statement. \(\square\)
The equivalence of (1) and (3) of the following proposition is proved in [11, Lemma 1.9] when $e = \emptyset$. The argument is essentially the same, but modifications are made to handle the hyperimaginary parameter.

**Proposition 2.2.** Let $b$ be any small tuple in $M$, and let $H \leq \text{Aut}_e(M)$. The following are equivalent.

1. $H = \text{Aut}_{b\cdot e}(M)$ for some $e$-invariant type-definable equivalence relation $F$.
2. $H = \text{Aut}_{(ba/L)\cdot e}(M)$ for some $\emptyset$-type-definable equivalence relation $L$ such that $ba/L \in \text{dcl}(be)$.
3. $\text{Aut}_{be}(M) \leq H$ and the orbit of $b$ under $H$ is type-definable.

**Proof.** Let $X$ be the orbit of $b$ under $H$.

(1) $\Rightarrow$ (2): This follows from Fact 244.

(2) $\Rightarrow$ (3): Since $ba/L \in \text{dcl}(be)$, we have $\text{Aut}_{be}(M) \leq H$. Moreover $X$ is type-defined by $\exists y(L(xy, ba) \land xy \equiv_e ba$).

(3) $\Rightarrow$ (1): Since $X$ is invariant under $H$, it is also invariant under $\text{Aut}_{be}(M)$, so type-definable by some $be$-invariant partial type $\Phi(x, b, a)$ (*). We define $F^*(x, y) \equiv \exists z(\Phi(x, y, z) \land \text{tp}_x(b/e) \land \text{tp}_y(ba/e))$.

**Claim.** $F^*(x, y)$ is an equivalence relation on the solution set of $\text{tp}(b/e)$.

**Proof of Claim.** Let $d \equiv_e b$ be given, hence $d = f(b)$ for some $f \in \text{Aut}_e(M)$. Then $\models \Phi(b, b, a)$ implies $\models \Phi(d, d, f(a))$ and so $F^*(d, d)$ holds.

To show symmetry, assume $F^*(c, d)$ holds for some $c, d \models \text{tp}(b/e)$. Thus there is $a'$ such that $da' \equiv_e ba$ and $\Phi(c, d, a')$ holds. Hence there also is $c'$ such that $cd \equiv_e c'c$, and for some $g \in \text{Aut}_e(M)$ we have $g(da') = c'b$ and $\Phi(c', b, a''$ holds. Note that $ba \equiv_e ba''$ and as said in (*) above, $\Phi(x, b, a''$ also type-defines $X$. Hence $c' \in X$ and there is $f \in H$ such that $c' = f(b)$. Note that $f^{-1}(b) \in X$ as well and $\Phi(f^{-1}(b), b, a')$ holds. Hence by applying $g^{-1}f \in \text{Aut}_e(M)$ we have $\Phi(d, c, a''$ where $a''' = g^{-1}f(a'')$ and $ca''' \equiv_e ba$. Therefore we have $F^*(d, c)$.

For transitivity, assume $F^*(u, v)$ and $F^*(v, w)$. Hence there are $a_0, a_1$ such that $u'v'a_2ba \equiv_e uv_0va_1$ and $\Phi(u, v, a_0, \Phi(v, w, a_1)$ hold. Now there is $u'v'a_2$ such that $u'v'a_2ba \equiv_e uv_0va_1$. Hence $\Phi(u', v', a_2)$ and $\Phi(v', b, a)$ hold, and $v' \in X$. Thus there is $h \in H$ such that $h(b) = v'$, and we have $\Phi(u''(h), b, a_3)$ where $u'' = h^{-1}(u')$, $a_3 = h^{-1}(a_2)$. Notice that $ba_3 \equiv_e v'a_2 \equiv_e ba$, and hence again by (*), we have $u'' \in X$. Therefore there is $k \in H$ such that $k(b) = u'' = h^{-1}(u')$, and $u' = h.k(b) \in X$. Hence $\Phi(u', b, a)$ and $\Phi(u, w, a_1)$ hold. Thus $F^*(u, w)$ follows because $ba \equiv_e wa_1$.

Define $F(x, y) \equiv F^*(x, y) \lor x = y$, so that $F$ is a type-definable equivalence relation on $M|^{\exists!}$. Notice that $F$ is $e$-invariant. It remains to show $H = \text{Aut}_{b\cdot e}(M)$.

If $h \in H$, then $h(b) \in X$ and $\Phi(h(b), b, a)$ holds. Thus $F(h(b), b)$. Conversely, assume $h \in \text{Aut}_{b\cdot e}(M)$. Then $F(h(b), b)$, so $F^*(h(b), b)$ and $\Phi(h(b), b, a')$ hold where $ba' \equiv_e ba$. Hence again by (*), we have $h(b) \in X$ and there is $g \in H$ such that $h(b) = g(b)$. Then $g^{-1}.h \in \text{Aut}_{be}(M) \leq H$ since $g^{-1}.h(b) = b$. Therefore $h \in H$. 

□
Proposition 2.3. Let $H \leq \text{Aut}_{e}(M)$ and let $H' = \pi(H) \leq \text{Gal}_{L}(T, e)$. Then $H'$ is closed in $\text{Gal}_{L}(T, e)$ and $H = \pi^{-1}(H')$, if and only if $H = \text{Aut}_{e'}(M)$ for some hyperimaginary $e'$ in $\text{bdd}(e)$.

Proof. (⇒): Recall that (before Proposition 2.2) we have fixed $M \models T$ such that $a \in M$. Note that $\text{Aut}_{M}(M) = \text{Aut}_{M}(e)(M) \leq \text{Aut}_{e}(M) \leq H$, and since $H'$ is closed, $\{h(M) : h \in H\}$ is type-definable over $M$. Thus by Proposition 2.2, $H = \text{Aut}_{M}(e)(M)$ for some $\emptyset$-type-definable equivalence relation $F$.

It remains to show that $M_{F} \in \text{bdd}(e)$: Notice that due to Corollary 1.7, $[\text{Aut}_{e}(M) : H] = \lambda$ is small. Thus there is $\{f_{i} \in \text{Aut}_{e}(M) : i < \lambda\}$ such that $\text{Aut}_{e}(M) = \bigsqcup_{i < \lambda} f_{i} \cdot H$. But for all $g, h \in \text{Aut}_{e}(M)$, if $g \cdot H = h \cdot H$, then $h^{-1} g \in H$ and hence $g(M_{F}) = h(M_{F})$.

$(\Leftarrow)$: Without loss of generality, say $e' = a'/E'$ where $a' \in M$. Note that for $q(x') = tp(a'/e)$, since $e' \in \text{bdd}(e)$, $F(x', y') := (q(x') \wedge q(y') \wedge E'(x', y')) \vee (\neg q(x') \wedge \neg q(y'))$

is an $e$-invariant bounded equivalence relation on $M^{[a']}$. Then due to the last statement of Remark 1.2 with real $x, y$ of arity $[a']$, we have $\text{Aut}_{e}(M) \leq H$. Thus $\pi^{-1}(H') = H$. Moreover, $H = \{f \in \text{Aut}_{e}(M) : f(a') \models E'(x', a')\}$, so $\nu^{-1}(H') = \mu_{\pi}(H') = \{p(x) \in S_{M}(M) : E'(x', a') \subseteq p(x)\}$ where $x' \subseteq x$, which is closed.

Corollary 2.4. Let $H' \leq \text{Gal}_{L}(T, e)$ be closed, and let $F$ be an $\emptyset$-type-definable equivalence relation. Then for $H = \pi^{-1}(H')$, $x_{F} \equiv_{e}^{H} y_{F}$ is equivalent to $x_{F} \equiv_{e}^{e'} y_{F}$ for some hyperimaginary $e' \in \text{bdd}(e)$, and hence $x_{F} \equiv_{e}^{H} y_{F}$ is an $e'$-invariant type-definable bounded equivalence relation. Especially, if $H' \leq \text{Gal}_{L}(T, e)$, then $x_{F} \equiv_{e}^{H} y_{F}$ is $e$-invariant.

Proof. Due to Proposition 2.3, $H = \text{Aut}_{e}(M)$ for some hyperimaginary $e' = d_{F} \in \text{bdd}(e)$. Hence the $H$-orbit equivalence relation $x_{F} \equiv_{e}^{H} y_{F}$ is simply $x_{F} \equiv_{e}^{e'} y_{F}$ which is type-definable, bounded (since $x_{F} \equiv_{e}^{e} y_{F}$ implies $x_{F} \equiv_{e}^{e'} y_{F}$), and $e'$-invariant. In addition if $H \leq \text{Aut}_{e}(M)$ then it easily follows that $x_{F} \equiv_{e}^{H} y_{F}$ is $e$-invariant.

3. Strong types over a hyperimaginary

In this section we reconfirm results in [6] 5.1.6-18] (excerpted from [8]), while correcting errors in [6] 5.1.14,15] and supplying a counterexample to [6] 5.1.15].

We begin to equip a topology on $\text{Aut}_{e}(M)$ by pointwise convergence; basic open sets are of the form $O_{u, v} = \{f \in \text{Aut}_{e}(M) : f(u) = v\}$ where $u, v$ are some finite real tuples. The proof of [14] Lemma 29] can go through in the hyperimaginary context;

Proposition 3.1. The projection map $\pi : \text{Aut}_{e}(M) \rightarrow \text{Gal}_{L}(T, e)$ is continuous.

Proof. Let $U$ be an open subset of $\text{Gal}_{L}(T, e)$ and $\overline{F} = f \cdot \text{Aut}_{e}(M) \subseteq U$. Since $\nu$ is a quotient map, $\nu^{-1}(U)$ is open, so there is a basic open set $V_{\varphi(x')} = \{p \in S_{M}(M) : \varphi(x') \subseteq p\} \subseteq \nu^{-1}(U)$ such that $tp(f(M)/M) \in V_{\varphi(x')}$. Let $u \in M$ be the finite tuple corresponding to the finite tuple $x'$ of variables. Then $\mu^{-1}(V_{\varphi(x')}) = \{g \in \text{Aut}_{e}(M) : g(u) \models \varphi(x')\}$ contains $f$ and $f(u) \models \varphi(x')$. Notice that the basic open set $O_{u, f(u)} = \{h \in \text{Aut}_{e}(M) : h(u) = f(u)\}$ contains $f$ and is contained in $\{g \in \text{Aut}_{e}(M) : g(u) \models \varphi(x')\} = \mu^{-1}(V_{\varphi(x')})$, implying that $\pi^{-1}(U)$ is open. \(\square\)
Corollary 3.2. Let $H'$ be a closed subgroup of $\text{Gal}_L(T, e)$, $H = \pi^{-1}(H') \leq \text{Aut}_e(\mathcal{M})$ and let $b, c$ be any small real tuples. Then

$$H = \{ f \in \text{Aut}_e(\mathcal{M}) : f \text{ fixes all the } \equiv_{e}^{H} \text{-classes of any hyperimaginaries}\}$$

$$= \{ f \in \text{Aut}_e(\mathcal{M}) : f \text{ fixes all the } \equiv_{e}^{H} \text{-classes of any finite real tuples}\}.$$ 

Moreover, $b \equiv_{e}^{H} c$ iff $b' \equiv_{e}^{H} c'$ where $b', c'$ are corresponding finite subtuples of $b, c$ respectively.

Proof. For the equalities, clearly it suffices to show the second equality. Notice that by Proposition 3.1 $H$ is closed in $\text{Aut}_e(\mathcal{M})$. Thus $f \in H$ iff for every basic open set $O \subseteq \text{Aut}_e(\mathcal{M})$ containing $f$, $O \cap H$ is nonempty. Recall that every $O$ is of the form $O_{u,v}$ for finite $u, v$ and $f \in O_{u,v}$ iff $f(u) = v$. Thus $f \in H$ iff for each finite real $u$, there is $g \in H$ such that $g(u) = f(u)$. But it is equivalent to say that $u \equiv_{e}^{H} f(u)$ for every finite real tuple $u$.

The last statement follows by compactness, since Corollary 2.4 says $x \equiv_{e}^{H} y$ is equivalent to $x \equiv_{e'}^{\text{bdd}(e)} y$ for some $e' \in \text{bdd}(e)$. \hfill $\square$

We now define and characterize the KP(Kim-Pillay)-type over a hyperimaginary.

Definition 3.3. 

1. Denote $$\text{Aut}_{\text{KP}}(\mathcal{M}, e) = \pi^{-1}(\overline{\{\text{id}\}})$$ where $\overline{\{\text{id}\}}$ is the (topological) closure of the identity in $\text{Gal}_L(T, e)$.

2. We let $\equiv_{e}^{\text{KP}}$ be the orbit equivalence relation under $\text{Aut}_{\text{KP}}(\mathcal{M}, e)$. Namely, for hyperimaginaries $b_F, c_F$,

$$b_F \equiv_{e}^{\text{KP}} c_F \text{ iff } b_F \equiv_{e}^{H} c_F$$

where $(\text{Aut}_{e}(\mathcal{M}) \leq H = \text{Aut}_{\text{KP}}(\mathcal{M}, e)(\leq \text{Aut}_{e}(\mathcal{M})))$. We call the equivalence class $b_F/\equiv_{e}^{\text{KP}}$ the KP-type of $b_F$ over $e$. Obviously $b_F \equiv_{e}^{H} c_F$ implies $b_F \equiv_{e}^{\text{KP}} c_F$.

Remark 3.4. Since $\text{Gal}_L(T, e)$ is a topological group, $\overline{\{\text{id}\}}$ is a normal closed subgroup of $\text{Gal}_L(T, e)$ (see [5]). We denote

$$\text{Gal}_{\text{KP}}(T, e) := \text{Gal}_L(T, e)/\overline{\{\text{id}\}} = \text{Aut}_{e}(\mathcal{M})/\text{Aut}_{\text{KP}}(\mathcal{M}, e).$$

Hence $\text{Gal}_{\text{KP}}(T, e)$ is a compact topological group.

Now we characterize $\equiv_{e}^{\text{KP}}$ and find equivalent conditions. Some arguments are from [6] Section 5.1.

Proposition 3.5. Let $F$ be an $0$-type-definable equivalence relation.

1. $x_F \equiv_{e}^{\text{KP}} y_F$ is an $e$-invariant type-definable bounded equivalence relation which is coarser than $F$.

2. $x_F \equiv_{e}^{\text{KP}} y_F$ is the finest among the $e$-invariant type-definable bounded equivalence relations which are coarser than $F(x, y)$.

Proof. (1) This follows directly from Corollary 2.4 and Remark 3.1.

(2) Let $L$ be any $e$-invariant type-definable bounded equivalence relation coarser than $F$ and assume that $b_F \equiv_{e}^{\text{KP}} c_F$. It suffices to show that $L(b, c)$. Now by Fact 2.1 there is a hyperimaginary $ba/L'$ such that $ba/L'$ and $b_L$ are interdefinable over $e$. Thus $ba/L' \in \text{bdd}(e)$ and by Proposition 2.3 $\pi(\text{Aut}_{(ba/L')e}(\mathcal{M}))$
is closed in $\text{Gal}_L(T,e)$ and $\pi^{-1}(\pi(\text{Aut}_{(ba/L')}e(M))) = \text{Aut}_{(ba/L')}e(M)$. Hence $\text{Aut}_{KP}(M,e) \leq \text{Aut}_{(ba/L')}e(M)$. Now $b_F \equiv_{e}^{KP} c_F$ implies there is $f \in \text{Aut}_{(ba/L')}e(M)$ such that $f(b_F) = c_F$ and this $f$ fixes $b_L$ by the interdefinability over $e$, thus we have $L(b,f(b))$ and $F(f(b),c)$, resulting $L(b,c)$ (since $L$ is coarser than $F$). □

**Proposition 3.6.** Let $b_F,c_F$ be hyperimaginaries. The following are equivalent.

1. $b_F \equiv_{e}^{KP} c_F$.
2. $b_F \equiv_{b}^{\text{bdd}(e)} c_F$.
3. For any $e$-invariant type-definable equivalence relation $L$ which is coarser than $F$, if $b_L$ has boundedly many $e$-conjugates, then $L(b,c)$ holds.
4. $bc \models \exists z(z \equiv_{e}^{KP} x \land F(z,y))$.

Therefore it follows $\text{Aut}_{KP}(M,e) = \text{Aut}_{b}^{\text{bdd}(e)}(M)$.

**Proof.** (1) $\Rightarrow$ (2): Notice that $x_F \equiv_{b}^{\text{bdd}(e)} y_F$ is $e$-invariant, type-definable (see Remark 3.6), $b_F \equiv_{e}^{KP} c_F$ implies $b_F \equiv_{b}^{\text{bdd}(e)} c_F$.

(2) $\Rightarrow$ (3): Assume (2) and the conditions for $L$ in (3). By Fact 3.1 there is a hyperimaginary $ba/L'$ such that $ba/L'$ and $b_L$ are interdefinable over $e$. Then $ba/L' \in \text{bdd}(e)$, thus by (2), there is $f \in \text{Aut}_{(ba/L')}e(M)$ such that $f(b_F) = c_F$. Then $F(f(b),c)$ holds, thus $L(f(b),c)$. But $f \in \text{Aut}_{(ba/L')}e(M)$, hence fixes $b_L$, thus $L(b,f(b))$ and by transitivity $L(b,c)$ holds.

(3) $\Rightarrow$ (1): By Proposition 3.6, $x_F \equiv_{e}^{KP} y_F$ satisfies all conditions of $L$ in (3). Thus (3) implies that $b_F \equiv_{e}^{KP} c_F$.

(1) $\iff$ (4): Easy to check.

Now the last equality follows from Corollary 3.2. □

We now recall the notion of $G$-compactness and its equivalent conditions in the context of hyperimaginaries.

**Definition 3.7.** $T$ is called $G$-compact over $e$ if the trivial identity subgroup is closed in $\text{Gal}_L(T,e)$.

Therefore by the general facts on compact groups (see [5]), $T$ is $G$-compact over $e$ iff $\text{Gal}_L(T,e)$ is Hausdorff (so compact) iff $\text{Aut}_{e}(M) = \text{Aut}_{KP}(M,e)$ iff $\text{Gal}_L(T,e)$ and $\text{Gal}_{KP}(T,e)$ are isomorphic as topological groups.

**Proposition 3.8.** The following are equivalent.

1. $T$ is $G$-compact over $e$.
2. For any hyperimaginaries $b_F$ and $c_F$, $b_F \equiv_{e}^{KP} c_F$ iff $b_F \equiv_{e}^{KP} c_F$.
3. For any small real tuples $b$ and $c$, $b \equiv_{e}^{KP} c$ iff $b \equiv_{e}^{KP} c$.
4. For any $\emptyset$-type-definable equivalence relation $F$, $x_F \equiv_{e}^{KP} y_F$ is type-definable.

**Proof.** (1) $\Rightarrow$ (2): By (1), $\text{Aut}_{KP}(M,e) = \text{Aut}_{e}(M)$, thus the orbit equivalence relations $\equiv_{e}^{\text{Aut}_{KP}(M,e)}$ and $\equiv_{e}^{\text{Aut}_{e}(M)}$ coincide for any hyperimaginaries.

(2) $\Rightarrow$ (3): Trivial by letting $F$ the equality.

(3) $\Rightarrow$ (1): Recall that $M \models T$ and $a \in M$. Now let $f \in \text{Aut}_{KP}(M,e)$. Assuming (3), in particular we have $M \equiv_{e}^{ KP} f(M)$, thus there is $g \in \text{Aut}_{e}(M)$ such that $g(M) = f(M)$. Then by Remark 3.1(3), it follows that $f \in \text{Aut}_{e}(M)$. Hence $\text{Aut}_{KP}(M,e) = \text{Aut}_{e}(M)$, and (1) follows.
(2) ⇒ (4): Clear by Proposition 3.5(1).

(4) ⇒ (2): We already know that $x_F \equiv^e y_F$ is an $e$-invariant bounded equivalence relation coarser than $F$ (Remark 1.3(2)). Then assuming (4), by Proposition 3.5(2), we have (2). □

As is well-known any simple theory is $G$-compact over a hyperimaginary (see [6]), while non $G$-compact theories (over $\emptyset$) are presented in [2] and [13]. In [2], an example shows the KP-type and the Lascar type of a finite tuple over $\emptyset$ can be distinct; and another shows KP-types and Lascar types over $\emptyset$ of all finite tuples are the same while they are distinct for an infinite tuple. In [13, Section 3.2], an example of $G$-compact theory $T$ over $\emptyset$ is given, while it is no longer $G$-compact after naming real parameters.

Next we define and characterize the (Shelah) strong type over a hyperimaginary.

Definition 3.9.

(1) $\text{Gal}_L^0(T,e)$ denotes the connected component of the identity in $\text{Gal}_L(T,e)$.

(2) $\text{Autf}_S(M,e) := \pi^{-1}(\text{Gal}_L^0(T,e))$.

(3) Two hyperimaginaries $b_F$ and $c_F$ are said to have the same (Shelah) strong type if there is $f \in \text{Autf}_S(M,e)$ such that $f(b_F) = c_F$, denoted by $b_F \equiv^s_e c_F$. Since $\text{Autf}_{KP}(M,e) \leq \text{Autf}_S(M,e)$, $b_F \equiv^K_P c_F$ implies $b_F \equiv^s_e c_F$.

Remark 3.10. Note that $\text{Gal}_L^0(T,e)$ is a normal closed subgroup of $\text{Gal}_L(T,e)$ [5] and $\equiv^s_e$ is the orbit equivalence relation $\equiv^\text{Aut}_e^{\text{at}(M,e)}$, so $\equiv^s_e$ is type-definable over $e$ by Corollary 2.4. We denote

$\text{Gals}(T,e) := \text{Gal}_L^0(T,e)/\text{Gal}_L^0(T,e) = \text{Aut}_e(M)/\text{Autf}_S(M,e)$.

Thus $\text{Gals}(T,e)$ is a profinite (i.e. compact and totally disconnected) topological group. Notice that $\text{Gal}_L^0(T,e)$ is the intersection of all closed (normal) subgroups of finite indices in $\text{Gal}_L(T,e)$, since such an intersection is the identity for a profinite group [5].

Now in order to characterize $\equiv^s_e$ in the context of hyperimaginaries, we define the algebraic closure of $e$.

Remark & Definition 3.11.

(1) For a hyperimaginary $e'$, denote $e' \in \text{acl}(e)$ and say $e'$ is algebraic over $e$ if $\{f(e') \mid f \in \text{Autf}_e(M)\}$ is finite.

(2) As in Remark 1.2, the algebraic closure of $e$, $\text{acl}(e)$ can be regarded as a bounded set of countable hyperimaginaries, which is interdefinable with a single hyperimaginary $b_F \in \text{bdd}(e)$ (but possibly $b_F \notin \text{acl}(e)$).

(3) Notice that given $d_i/L_i \in \text{acl}(e)$ ($i \leq n$), as pointed out at the end of the first paragraph of Section 1, $(d_0/L_0, \ldots, d_n/L_n)$ is equivalent to a single $d_L \in \text{acl}(e)$. Hence by compactness, for any hyperimaginaries $b_F$ and $c_F$,

$b_F \equiv_{\text{acl}(e)} c_F$ iff $b_F \equiv_{d_L} c_F$ for any $d_L \in \text{acl}(e)$.

The following clarifies Propositions 5.1.14 and 5.1.17 in [8]. There, the proof of 5.1.14(1)⇒(2) with the hyperimaginary parameter need not work, and (1),(5) should be deleted out (we will supply a counterexample).

Proposition 3.12.

(1) $\text{Autf}_S(M,e) = \text{Aut}_{\text{acl}(e)}(M)$. 
(2) Let $b_F, c_F$ be hyperimaginaries. The following are equivalent.

(a) $b_F \equiv_e c_F$.

(b) $b_F \equiv_{acl(e)} c_F$.

(c) For any $e$-invariant type-definable equivalence relation $L$ coarser than $F$, if $b_L$ has finitely many conjugates over $e$, then $L(b, c)$ holds.

(d) $bc = \exists \bar{z}(\equiv_e x \land F(z, y))$.

Proof. (1) We claim first that

$$\text{Gal}_L^0(T, e) = \bigcap \{ \pi(\text{Aut}_{d, L, e}(M)) \mid d_L \in acl(e) \} :$$

Let $d_L \in acl(e)$ where $d_L$ is a hyperimaginary. Let $d_L^0, \ldots, d_L^n$ be all the conjugates of $d_L$ over $e$. Then any $f \in \text{Aut}_e(M)$ permutes the set $\{d_L^0, \ldots, d_L^n\}$. Hence it follows that $\text{Aut}_{d, L, e}(M)$ has a finite index in $\text{Aut}_e(M)$. Thus (due to Proposition 2.3) $\text{Aut}_{d, L, e}(M) = \pi^{-1}(\pi(\text{Aut}_{d, L, e}(M)))$ and $\pi(\text{Aut}_{d, L, e}(M))$ is a closed subgroup of finite index in $\text{Gal}_L(T, e)$. Then as in Remark 3.11 we have $\text{Gal}_L^0(T, e) \leq \pi(\text{Aut}_{d, L, e}(M))$.

Conversely, given a normal closed subgroup $H' \leq \text{Gal}_L(T, e)$ of finite index, and $H := \pi^{-1}(H')$, Proposition 2.3 says $H' = \pi(\text{Aut}_{b_F, e}(M))$ for some $b_F \in \text{bdd}(e)$. But since $H'$ is of finite index, so is $H = \text{Aut}_{b_F, e}(M)$ in $\text{Aut}_e(M)$, and we must have $b_F \in acl(e)$. Thus the claim follows from Remark 3.11.

Therefore

$$\text{Aut}_S(M, e) = \pi^{-1}(\text{Gal}_L^0(T, e)) = \pi^{-1}(\bigcap \{ \pi(\text{Aut}_{d, L, e}(M)) \mid d_L \in acl(e) \}) = \bigcap \{ \text{Aut}_{d, L, e}(M) \mid d_L \in acl(e) \} = \text{Aut}_{acl(e)}(M),$$

where the last equality follows by Remark 3.11(3).

(2)(a) $\iff$ (b): It follows from (1).

(b) $\Rightarrow$ (c): Suppose that $L$ is an $e$-invariant type-definable equivalence relation coarser than $F$, and $b_L$ has finitely many conjugates over $e$. By Fact 2.1 there is a hyperimaginary $ba/L'$ such that $ba/L'$ and $b_L$ are interdefinable over $e$. Then $ba/L' \in acl(e)$. Now by (b), there is $f \in \text{Aut}_{acl(e)}(M)$ such that $f(b_F) = c_F$, so $F(f(b), c)$. This $f$ fixes $ba/L'$ and $e$, in turn, $f$ fixes $b_L$. Thus $L(f(b), b)$, and since $L$ is coarser than $F$, we have $L(b, c)$.

(c) $\Rightarrow$ (b): Assume (c). Let a hyperimaginary $d_L \in acl(e)$ (containing $\text{bdd}(e)$) and let $\{d_i/L : i \in I\}$ be the set of all $e$-conjugates of $d_L$, where $I$ is a finite set and $d_i \equiv_e d$ for each $i \in I$. Due to Remark 3.11(3), it suffices to show $b_F \equiv_{d_L} c_F$.

Let $p(x, d) := tp(b_F/d_L) = \exists z_1 z_2 (tp_{z_1 z_2} (bd) \land F(x, z_1) \land L(z_2, d))$. Put $J$ the maximal subset of $I$ such that $(p(x, d) \in \{p(x, d_i) : i \in J\}$ is realized by $b$. Then denote $d_j = \{d_i : i \in J\}, w_J = \{w_i : i \in J\}$. and let

$$p'(x, w_J, a) := tp_{w_J}(d_j/e) \land \bigwedge_{j \in J} p(x, w_i)$$

$$= \exists z (w_J z = d_j a \land E(z, a)) \land \bigwedge_{j \in J} \exists y_i z_i = bd(F(x, y_i) \land L(z_i, w_i)).$$

Claim.

$L'(x, y; a) := \exists w_J (p'(x, w_J; a) \land p'(y, w_J; a)) \lor F(x, y)$ is an $e$-invariant type-definable equivalence relation coarser than $F$.
Proof of Claim. We check \( L' \) is an equivalence relation: It is clearly reflexive and symmetric. For transitivity, say \( L'(b_1, b_2) \) and \( L'(b_2, b_3) \) holds.

(i) If \( F(b_1, b_2) \) and \( F(b_2, b_3) \), then \( F(b_1, b_3) \), so nothing to prove.

(ii) If \( F(b_1, b_2) \) and \( b_2 b_3 \), then \( b_1 b_2 \), so we must have \( \forall u_f(x, w, a) \) and \( \forall u_f(y, w, a) \), since \( b_1 = b_2 = b_3 \), we have \( b_1 b_3 \). By (3), \( b_1 b_3 \) has more than \( b_2 b_3 \)-many conjugates. Thus \( b_1 b_3 \) is \( \equiv \) to \( b_2 b_3 \).

(iii) If there are \( u_f(x, w, a) \) such that \( b_2 u_f(x, w, a) \) and \( b_2 v_f(x, w, a) \), then in particular \( b_2 u_f \) and \( b_2 v_f \) both satisfy \( p(x, w, a) \). Thus \( b_2 u_f \) and \( b_2 v_f \) both satisfy \( p(x, w, a) \). But we must have \( \{ u_i / L | i \in J \} \neq \{ v_i / L | i \in J \} \), contradicting the maximality of \( J \).

Then due to the definition of \( p' \), we have \( b_1 b_2 u_f \) and \( b_2 v_f \). By (3), \( b_1 b_2 \) has more than \( b_2 \)-many conjugates. Thus \( b_1 b_2 \) is \( \equiv \) to \( b_2 \).

\( L' \) is clearly coarser than \( F \) and since \( L' \) is \( e \)-invariant, \( L' \) is also \( e \)-invariant. Claim is proved.

We now check that \( b_L \) has finitely many \( e \)-conjugates. Notice that \( \{ d_i / L : i \in J \} \) has finitely many \( e \)-conjugates, say the number is \( m_0 \). But by the definition of \( L' \), \( b_L \) cannot have more than \( m_0 \)-many conjugates.

Therefore by Claim and (c), we have \( L'(b, c) \). Thus \( F(b, c) \) or \( bc \) is isomorphic to \( \exists w_f(x, w, a) \). The former case, of course \( b_F \equiv_{d_L} c_F \). The latter case, say there is \( d' \) such that \( p'(b, d', a) \) and \( p'(c, d', a) \) hold. Note that we already have \( p'(b, d, a) \), thus as argued in (iii) above, we must have \( \{ d_i / L | i \in J \} = \{ d'_i / L | i \in J \} \). But since \( p(x, d) \in \{ p(x, d_i) : i \in J \} \), we have \( c = p(x, d) \) and so \( b_F \equiv_{d_L} c_F \), as wanted.

(a) \( \Leftrightarrow \) (d): Easy to check.

We now state several criterion for KP-types and strong types over \( e \) being equal, reconfirming [6] 5.1.18. Note that, \( \text{Gal}_L(T, e) = \{ \text{id} \} \) iff \( \text{Gal}_{K_P}(T, e) \) is profinite iff \( \text{Aut}_{K_P}(M, e)(M) = \text{Aut}_{S}(M, e) \) iff \( \text{Gal}_{S}(T, e) \) and \( \text{Gal}_S(T, e) \) are isomorphic as topological groups [5].

Proposition 3.13. The following are equivalent.

1. \( \text{Aut}_{K_P}(M, e) = \text{Aut}_{S}(M, e) \).
2. For any hyperimaginaries \( b \) and \( c_F \), \( b_F \equiv_{K_P} c_F \) iff \( b_F \equiv_e c_F \).
3. For any real small tuples \( b \) and \( c \), \( b \equiv_{K_P} c \) iff \( b \equiv_e c \).
4. For any real finite tuples \( b \) and \( c \), \( b \equiv_{K_P} c \) iff \( b \equiv_e c \).
5. \( \text{acl}(e) \) and \( \text{bdd}(e) \) are interdefinable.

Proof. (1) \( \Rightarrow \) (2): Directly follows by the definition of \( \equiv_{K_P} \) and \( \equiv_e \).

(2) \( \Rightarrow \) (3): Clear by letting \( F \) the equality.

(3) \( \Rightarrow \) (1): Use the same method as the proof of (3) \( \Rightarrow \) (1) in Proposition 3.8 and work with \( M \models T \) such that \( e \in \text{dcl}(M) \). Let \( f \in \text{Aut}_{S}(M, e) \). Then \( M \equiv_e f(M) \), then \( g \in \text{Aut}_{K_P}(M, e) \) such that \( g(M) = f(M) \). By Remark 3.4, there is \( h \in \text{Aut}_{P}(M) \leq \text{Aut}_{K_P}(M, e) \) such that \( h g = f \), thus \( f \in \text{Aut}_{K_P}(M, e) \). Hence we get \( \text{Aut}_{K_P}(M, e) = \text{Aut}_{S}(M, e) \).

(3) \( \Rightarrow \) (4): Trivial.

(4) \( \Rightarrow \) (3): Let \( b \) and \( c \) be any real small tuples such that \( b \equiv_e c \) and let \( b_0, c_0 \) be any corresponding finite tuples respectively. Then by (4), \( b_0 \equiv_{K_P} c_0 \). Thus by Corollary 3.2 we have \( b \equiv_{K_P} c \).
We finish this section with an investigation of acl\(^{eq}\) and acl. Recall that acl\(^{eq}\)(\(e\)) := \(\{e\} \cup (acl(e) \cap M^{eq})\) is the eq-algebraic closure of \(e\), where as usual \(M^{eq}\) is the set of all imaginary elements (equivalence classes of \(\emptyset\)-definable equivalence relations) of \(M\).

**Remark 3.14.** We point out that in any \(T\), acl(\(\emptyset\)) and acl\(^{eq}\)(\(\emptyset\)) are interdefinable over \(\emptyset\). This follows from Proposition 3.12(1) and the fact that Aut\(_{acl^{eq}\emptyset}\)(\(M\)) = Aut\(_{eq}\)(\(M\)) \[^{14}\] Theorem 21(2)], or one can directly show it by compactness: If a hyperimaginary \(c/F\) has finitely many conjugates over \(\emptyset\), then the union of its conjugate classes is \(\emptyset\)-type-defined, say by \(\Phi(x)\). In \(\Phi(x)\), \(F\) is relatively \(\emptyset\)-definable with finitely many classes. Hence by compactness one can find a formula \(\delta(x) \in \Phi(x)\) and an \(\emptyset\)-definable finite equivalence relation \(F'\) on \(\delta(x)\) such that \(c/F\) and the imaginary \(c/F'\) are interdefinable.

However, contrary to \[^{6}\] Corollary 5.1.15, in general acl(\(e\)) and acl\(^{eq}\)(\(e\)) need not be interdefinable. As said before Proposition 3.12, the error occurred there due to the incorrect proof of \[^{6}\] 5.1.14(1)⇒(2)]. An example presented in \[^{4}\] for another purpose supplies a counterexample. Consider the following 2-sorted model:

\[
M = (\langle M_1, S_1, \{g_{1/n}^i \mid n \geq 1\}, M_2, S_2, \{g_{1/n}^i \mid n \geq 1\}, \delta \rangle)
\]

where

1. \(M_1\) and \(M_2\) are unit circles centered at origins of two disjoint (real) planes.
2. \(S_i\) is a ternary relation on \(M_i\), defined by \(S_i(b, c, d)\) holds iff \(b, c\) and \(d\) are in clockwise-order.
3. \(g_{1/n}^i\) is a unary function on \(M_i\) such that \(g_{1/n}^i(b) = \text{rotation of } b\text{ by } 2\pi/n\text{-radians clockwise.}
4. \(\delta : M_1 \to M_2\) is the double covering, i.e. \(\delta(\cos t, \sin t) = (\cos 2t, \sin 2t)\).
5. Let \(M\) be a monster model of Th(\(M\)), and let \(M_1\) and \(M_2\) be the two sorts of \(M\).

In \[^{3}\] Theorems 5.8 and 5.9, it is shown that Th(\(M_1\)) has weak elimination of imaginaries (that is, for any imaginary element \(e\) there is a real tuple \(b\) such that \(e \in \text{dcl}(b)\) and \(b \in \text{acl}(e)\)), and quantifier elimination. We point out that similar proofs, which we omit, yield the same results.

**Fact 3.15.** Th(\(M\)) has quantifier elimination and weak elimination of imaginaries.

Now for \(i = 1, 2\), we let \(E_i(x, y)\) iff \(x\) and \(y\) in \(M_i\) are infinitesimally close, i.e.

\[
E_i(x, y) := \bigwedge_{1 < n} (S_i(x, y, g_{1/n}^i(x)) \lor S_i(y, x, g_{1/n}^i(y))),
\]

which is an \(\emptyset\)-type-definable equivalence relation. Let \(c \in M_2, c_1, c_2 \in M_1\) where \(\delta(c_1) = \delta(c_2) = c\), so that \(c_1, c_2\) are antipodal to each other. Notice that \(c_1/E_1\) and \(c_2/E_1\) are conjugates over \(c/E_2\), so that \(c_1/E_1, c_2/E_1 \in \text{acl}(c/E_2)\). Then the following implies that acl(\(c/E_2\)) and acl\(^{eq}\)(\(c/E_2\)) are not interdefinable.

**Claim.** acl\(^{eq}\)(\(c/E_2\)) and \(c/E_2\) are interdefinable in \(M\).

**Proof.** To lead a contradiction suppose that there are distinct imaginaries \(d_1, d_2 \in \text{acl}(c/E_2)\) such that \(d_1 \equiv_{c/E_2} d_2\). Now weak elimination of imaginaries of Th(\(M\)) implies that acl\(^{eq}\)(\(d_1, d_2\)) and \(B := \{b \in M \mid b \in \text{acl}(d_1, d_2)\}\) are interdefinable (*). In particular, \(B \subseteq \text{acl}(c/E_2)\cap M\). However, due to quantifier elimination,
for any infinitesimally close $b, b' \in M$ ($i = 1, 2$), there is $f \in \text{Aut}_{c/E_b}(M)$ sending $b$ to $b'$. Hence indeed $B = \emptyset$, which contradicts (*).

4. Diameter

In [12], Newelski showed that a type-definable Lascar strong type over a real set $A$ has a finite diameter, which implies that $T$ is $G$-compact over $A$ if and only if there exists a uniform $n < \omega$ bounding the diameters of any real tuples over $A$. The proof uses intricate analyses of open subsets of the type-definable set. Then in [13], using the notion of c-free sets, a shorter and direct proof showing the same result is given. Here by applying arguments in [13], we prove that the same holds for hyperimaginaries. We cite basic definitions below.

Let $\Phi(x) = \Phi(x, b)$ be a partial type over $b$. Assume that for any $c, d \models \Phi(x)$ there is $f \in \text{Aut}(M)$ such that $f(c) = d$ while $\Phi(x, b) \leftrightarrow \Phi(x, f(b))$, i.e. $\Phi(M) = f(\Phi(M))$ (*). Note that any two realizations of $\Phi(x)$ have the same type over $\emptyset$.

**Definition 4.1.**

1. A formula $\psi(x, d)$ is said to be c-free over $\Phi$ if there are $f_0, \ldots, f_n \in \text{Aut}(M)$ for some $n < \omega$ such that

$$\Phi(x) \models \bigvee_{i \leq n} \psi(x, f_i(d))$$

and $f_i(\Phi(M)) = \Phi(M)$ for every $i \leq n$.

2. A type $q(x)$ is said to be c-free over $\Phi$ if for any $\varphi(x) \in \mathcal{L}(M)$ such that $q(x) \vdash \varphi(x)$, $\varphi(x)$ is c-free over $\Phi$.

3. A type $q(x)$ is said to be weakly c-free over $\Phi$ if for any $\psi(x) \in \mathcal{L}(M)$ with $q(x) \vdash \psi(x)$, there is a non-c-free formula $\phi(x) \in \mathcal{L}(M)$ over $\Phi$ such that $\psi(x) \lor \phi(x)$ is c-free over $\Phi$.

**Definition 4.2.** We put

$$P^\Phi := \{q(x, y) \in S(\emptyset) \mid q(x, y) \cup \Phi(x) \cup \Phi(y) \text{ is consistent}\},$$

and

$$P^\Phi_{\text{wcf}} := \{q(x, y) \in P^\Phi \mid q(d, y) \text{ is weakly c-free over } \Phi \text{ for any (some) } d \models \Phi\}.$$

The following is the key result from [13] which we will use.

**Fact 4.3.**

1. $P^\Phi$ and $P^\Phi_{\text{wcf}}$ are closed and non-empty in $S_{x,y}(\emptyset)$.

2. Let $D \neq \emptyset$ be a relatively open subset of $P^\Phi_{\text{wcf}}$. Then there are $d_0, \ldots, d_k \models \Phi$ such that for any $d \models \Phi$, there is $d' \models \Phi$ holding

(a) $\text{tp}(d, d') \in D$ and

(b) $\text{tp}(d_i, d') \in D$ for some $i \leq k$.

Now we begin to prove our hyperimaginary context result Theorem 4.7. Assume that given an $\emptyset$-type-definable equivalence relation $F$ and real $b$, the Lascar strong type of $b_F$ over $e$ is type-definable. That is there is a partial type $\pi_0(x)$ (over $ab$) such that $d \models \pi_0(x)$ iff $d_F \equiv^L_b b_F$.

**Definition 4.4.** Recall that we say the Lascar distance between $b_F$ and $d_F$ over $e$ is at most $n \in \omega$, denoted by $d_n(b_F, d_F) \leq n$, if there are $b = b_0, b_1, \ldots, b_n = d \in M$ such that $b_i/F$, $b_{i+1}/F$ begin an $e$-indiscernible sequence for each $i < n$. 


n. If \( d_\varepsilon(b_F, d_F) \leq n \) for some \( n < \omega \), then we let \( d_\varepsilon(b_F, d_F) := \min \{ m < \omega \mid d_\varepsilon(b_F, d_F) \leq m \} \). The (Lascar) diameter of \( b_F \) over \( \varepsilon \) is \( \max \{ d_\varepsilon(b_F, d_F) \mid b_F \equiv_\varepsilon d_F \} \) if exists. Otherwise, it is \( \infty \).

**Remark 4.5.**

1. By Remark [4.2], \( b_F \equiv_\varepsilon d_F \) iff \( d_\varepsilon(b_F, d_F) \leq n \) for some \( n < \omega \).
2. For \( r(x, y) := \text{tp}(ba) \), we let
   \[
   \pi(x, y) \equiv \pi_0(x) \land r(x, y) \land E(y, a),
   \]
   a type over \( ab \). If \( dc, d'c' \models \pi(x, y) \), then there is \( f \in \text{Aut}(M) \) such that \( f(dc) = d'c' \). Then since \( f(e) = e \) and \( f(\pi_0(M)) = \pi_0(M) \), it follows that \( f(\pi(M)) = \pi(M) \). Therefore \( \pi(x, y) \) satisfies the assumption (*) in the beginning of this section. Now as in Definition 4.2, we let
   \[
   P'(xy, x'y') := P^\pi \text{ and } P_{\text{wcf}}(xy, x'y') := P_{\text{wcf}}^\pi,
   \]
   and we can apply Fact 4.3 to \( P \) and \( P_{\text{wcf}} \).
3. Recall that \( (b_i/F \mid i < \omega) \) is \( e \)-indiscernible iff there are \( a' \equiv a \) with \( E(a, a') \) and \( b'_i \mid i < \omega \) such that \( (b'_i \mid i < \omega) \) is \( a' \)-indiscernible.
4. Clearly there is a type \( \Lambda (x_0, x_1, z) \) over \( \emptyset \) having \( x_0, x_1 \) begin a \( z \)-indiscernible sequence. Using (3) and \( \Lambda \), we can \( \emptyset \)-type-define a relation \( \Phi_n(xy, x'y') \) saying that \( (\text{tp}_{xy}(ba) =) r(x, y), r(x', y'), E(y, y') \) and \( d_{y/E}(x_F, x_{F'}) \leq n \).

**Fact 4.6 (Baire Category Theorem).**

1. A topological space \( X \) is a Baire space iff whenever the union of countably many closed subsets of \( X \) has an interior point, at least one of the closed subsets must have an interior point.
2. Any locally compact Hausdorff space is a Baire space.

We are ready to prove our goal.

**Theorem 4.7.** Type-definable Lascar strong type of a hyperimaginary over a hyperimaginary has a finite diameter.

**Proof.** We will show there is \( m \in \omega \) such that for any \( u_F \) with \( u_F \equiv_b b_F \), we have \( d_\varepsilon(u_F, b_F) \leq m \). As pointed out in Fact 4.3(1), \( P \) is closed in \( S_{xy,x'y'}(\emptyset) \). Indeed, \( P(xy, x'y') \) is type-defined by
   \[
   \exists z w z' w' (xy x'y' \equiv zwz' w' \land \pi(z, w) \land \pi(z', w')),
   \]
   which is \( \emptyset \)-invariant, so \( \emptyset \)-type-definable.

**Claim.** \( P(xy, x'y') \subseteq \bigcup_{i<\omega} [\Phi_i(xy, x'y')] \).

**Proof of Claim.** Suppose that \( \text{tp}(d_0\phi_0d_0'\phi_0') \in P \). Hence there is \( d cd'c' = d_0\phi_0d_0'\phi_0' \) such that \( \pi(d, c) \) and \( \pi(d', c') \) hold. It suffices to show that \( \Phi_n(dc, d'c') \) for some \( n \). Notice that \( dc \equiv ba \equiv d'c' \) and \( c E = e(= a_E) = c_E' \) hold. Also, since \( d_F \equiv_\varepsilon d_F' (=_e b_F) \), \( \Phi_n(dc, d'c') \) must holds for some \( n \) as described in Remark 4.5. \( \square \)
Now we let \( X_i = P \cap [\Phi_i(xy,x'y')] \subseteq S_{xy,x'y'}(\emptyset) \), so that \( P = \bigcup_{i<\omega} X_i \) by Claim. Note that both \( X_i \) and \( P_{wcf} \) are closed (Fact 1.3), so \( P_{wcf} = \bigcup_{i<\omega}(P_{wcf} \cap X_i) \) is compact Hausdorff. Then by Baire Category Theorem, there is \( n<\omega \) such that the interior of \( P_{wcf} \cap X_n \) in \( P_{wcf} \) is nonempty, so we can apply Fact 1.3. Namely there are \( u_0 v_0,\ldots,u_k v_k \models \pi \) such that for any real \( uv \models \pi \), there is \( u'v' \models \pi \) such that

(a) \( \text{tp}(uv, u'v') \in X_n \) and

(b) \( \text{tp}(u_0 v_0, u'v') \in X_n \) for some \( i_0 \leq k \).

In particular, \( e = v_E = v'_E = v_i/E \) for all \( i \leq k \), and \( d_e(u_F, u'_F), d_e(u_{i_0}/F, u'_F) \leq n \) (*).

Now there is \( m_0 \in \omega \) such that \( d_e(u_i/F, u_j/F) \leq m_0 \) for all \( i, j \leq k \), since \( u_i/F \equiv^e u_j/F \). We show that \( m = 4n + m_0 \) is the bound in (1) above.

Suppose that \( u_F \equiv^e b_F \). Since there is \( f \in \text{Autf}_e(M) \) such that \( u_F = f(b_F) \), without loss of generality we can assume \( u = f(b) \). Hence there is \( v \) such that \( uv \equiv ba \) and \( \pi(u,v) \) hold. Since \( \pi(b,a) \) holds as well, by (a)(b) above, there are \( u',v' \models \pi \) such that

(1) \( \text{tp}(ba, b'a') \), \( \text{tp}(uv, u'v') \) \( \in X_n \), and

(2) \( \text{tp}(u_j v_j, b'a') \), \( \text{tp}(u_l v_l, u'v') \) \( \in X_n \) for some \( j, l \leq k \).

Therefore by the same reason in (*) above,

\[
\begin{align*}
d_e(u_F,b_F) &\leq d_e(u_F,u'_F) + d_e(u'_F,u_l/F) + d_e(u_l/F,u_j/F) \\
&\quad + d_e(u_j/F,b'_F) + d_e(b'_F,b_F) \\
&\leq n + n + m_0 + n + n = 4n + m_0.
\end{align*}
\]

\[\Box\]

In addition to Proposition 3.8, we now supply more conditions equivalent to \( T \) being \( G \)-compact over \( e \). In particular, by applying Theorem 4.7 we extend Newelski’s result on the uniform finite bound for the diameters of real tuples, mentioned in the beginning of this section, to the hyperimaginary context.

**Corollary 4.8.** The following are equivalent.

1. \( T \) is \( G \)-compact over \( e \).
2. For any hyperimaginary \( b_F \), there is a partial type \( \Psi(x) \) (over \( ab \)) such that \( c \models \Psi(x) \) iff \( c_F \equiv^L_e b_F \).
3. For any hyperimaginary \( b_F \), there is \( n<\omega \) such that for any \( c_F \equiv^L_e b_F \), we have \( d_e(b_F,c_F) \leq n \).
4. There is a uniform \( n<\omega \) such that for every pair of hyperimaginaries \( b_F \) and \( c_F \), \( b_F \equiv^L_e c_F \) iff \( d_e(b_F,c_F) \leq n \).

**Proof.** (1) \( \Rightarrow \) (2), (4) \( \Rightarrow \) (1): By Proposition 3.8(4).

(2) \( \Rightarrow \) (3): By Theorem 4.7.

(3) \( \Rightarrow \) (4): Assume (3). To lead a contradiction, suppose not (4), that is for each \( n \) there is a pair of hyperimaginaries \( b_n/F_n \), \( c_n/F_n \) such that \( b_n/F_n \equiv^L_e c_n/F_n \) but \( d_e(b_n/F_n,c_n/F_n) \not> n \) (*).

Without loss of generality we can assume \( b_n \equiv^L_e c_n \). Consider \( b = (b_n \mid n<\omega) \), and let \( F = \bigcap_n F_n \). As pointed out in the first paragraph of Section 1, \( b_F \) is a single hyperimaginary. Then by (3), there must be \( m \) such that for any real \( d \) with \( |d| = |b| \), we have \( b_F \equiv^L_e d_F \) iff \( d_e(b_F,d_F) \leq m \). Now since \( b_m \equiv^L_e c_m \), there is \( f \in \text{Autf}_e(M) \)
such that \( f(b_m) = c_m \). We now let \( c_F = f(b_F) \) where \( c = (c'_n \mid n < \omega) \) and \( c'_n = f(b_n) \) (so \( c_m = c'_m \)). Thus \( b_F \equiv^L_e c_F \), and \( d_e(b_F, c_F) \leq m \). In particular \( d_e(b_m/F_m, c_m/F_m) \leq m \), contradicting (*).

\[
\square
\]

5. Relativized Lascar groups

In [3], the notions of relativized Lascar groups for real types are introduced. In this last section, we generalize the definitions for the hyperimaginary types, and supply a partial positive answer (Theorem 5.6) to a question raised in [4], which is even a new result in the real context. We will use Proposition 3.8 in proving the result. Throughout this section we fix a hyperimaginary \( b_F \) and \( p = \text{tp}(b_F/e) \).

**Definition 5.1.**

1. \( \text{Aut}(p) = \text{Aut}_e(p) := \{ f \mid p(M) \mid f \in \text{Aut}_e(M) \} \).
2. For a cardinal \( \lambda > 0 \), \( \text{Aut}^\lambda(p) = \text{Aut}_e^\lambda(p) := \{ f \in \text{Aut}(p) \mid \text{ for any } \overline{b_F} = (b_i/F)_{i < \lambda} \text{ such that } b_i/F \models p, \overline{b_F} \equiv^L_e f(\overline{b_F}) \} \).
3. \( \text{Aut}^{\text{fix}}(p) = \text{Aut}_e^{\text{fix}}(p) := \{ f \in \text{Aut}(p) \mid \text{ for any } \lambda \text{ and } \overline{b_F} = (b_i/F)_{i < \lambda} \text{ such that } b_i/F \models p, \overline{b_F} \equiv^L_e f(\overline{b_F}) \} \).
4. \( \text{Gal}_1^L(p) = \text{Aut}(p)/\text{Aut}^L(p) \).
5. \( \text{Gal}_1^{\text{fix}}(p) = \text{Aut}(p)/\text{Aut}^{\text{fix}}(p) \).

**Remark 5.2.**

1. Obviously, if \( f \in \text{Aut}_e(M) \) then \( f \mid p(M) \in \text{Aut}^L(p) \).
2. One can easily check that \( \text{Aut}^L(p), \text{Aut}^{\text{fix}}(p) \) are normal subgroups of \( \text{Aut}(p) \).
3. There is an example in [3] that \( \text{Gal}_1^L(p) \) and \( \text{Gal}_1^2(p) \) are distinct for \( p \in S(\emptyset) \).

We point out the following basic facts which can be proved by the same arguments in [3]. (A detailed proof may be found in [4].)

**Fact 5.3.**

1. \( \text{Aut}^{\text{fix}}(p) = \text{Aut}^{\text{fix}}_e(p) \), so \( \text{Gal}_1^{\text{fix}}(p) = \text{Gal}_1^2(p) \).
2. Up to isomorphism, the group structure of \( \text{Gal}_1^L(p) \) is independent of the choice of a monster model.
3. Recall that \( e \in \text{acl}(M) \). Then \( \nu' : S_M(M) \rightarrow \text{Gal}_1^L(p) \) defined by \( \nu'(\text{tp}(f(M)/M)) = (f \mid p(M)) \cdot \text{Aut}^L(p) \) is well-defined and \( \text{Gal}_1^L(p) \) is a quasi-compact topological group with the quotient topology given by \( \nu' \), which is independent of the choice of \( M \).

We use a boldface letter \( b \) to denote the fixed hyperimaginary \( b_F \), so the type \( p(x) = \text{tp}(b/e) \). Now let \( c = c_L \) be a hyperimaginary such that \( c \in \text{acl}(be) \). Say \( c_0(=c_0/L), \cdots, c_{m-1} = c_{m-1}/L \) are all the distinct \( be \)-conjugates of \( c \).

**We assume that \( bc, b^i \neq^L_e bc_j \) for any distinct \( i, j < m \) (1) until Theorem 5.6.**

Put

\[
\overline{p}(xy) = \text{tp}(bc/e) \equiv \exists z_1 z_2 w(\text{tp}_{z_1 z_2 w}(bca) \land F(x, z_1) \land L(y, z_2) \land E(a, w)).
\]

In the rest, \( bc \) with attached small scripts refers to a realization of \( \overline{p}(x, y) \). Moreover for example, given \( b'c' \), the corresponding plain letter \( b'c' \) refers to a real tuple such
that \( b^f = b^f / F \) and \( c^f = c^f / L \), so that \( b^f c^f \) is a real realization of \( \overline{p}(x,y) \) and vice versa. Notice that \( b^f c^f \equiv_e b c \), while \( b^f \not\equiv_e b c \) need not hold.

For each \( b' \models p \), we fix \( c_0^{b'} \cdots c_{m-1}^{b'} \) which is the image of \( c_0 \cdots c_{m-1} \) under some \( e \)-automorphism sending \( b \) to \( b' \). We may just write \( b^f c^f \) to refer to some \( b^f c^f (j < m) \). As usual we write \( x_{<n} \) to denote the sequence of variables \( x_0 \cdots x_{n-1} \) and we use similar abbreviations for finite sequences of hyperimaginaries.

**Notation 5.4.**

1. For \( T \) \( G \)-compact over \( e \), we write \( E^l(\_,-,a) \) to denote an \( e \)-invariant type over \( a \) defining the Lascar equivalence over \( e \) of hyperimaginaries (see Proposition 5.3). We may assume that \( E^l \) is closed under finite conjunctions and every formula \( \phi(x,y,a) \in E^l \) is reflexive and symmetric.

2. \( \pi_n : Gal_n^L(\mathcal{P}) \to Gal_n^L(p) \) is the natural projection. Namely \( \pi_n((f \upharpoonright \mathcal{P}(M)) : \text{Autf}^n(\mathcal{P})) = (f \upharpoonright p(M)) : \text{Autf}^n(p) \), which clearly is well-defined.

3. \( K_n \) is the kernel of \( \pi_n : Gal_n^L(\mathcal{P}) \to Gal_n^L(p) \). We write \( \overline{f} \) to denote \( f : \text{Autf}^n(\mathcal{P}) \in Gal_n^L(\mathcal{P}) \).

In [3], it is asked whether \( K_n \) is finite for finite \( n > 0 \) (when both \( b, e \) are real) if \( T \) is \( G \)-compact over \( e \). In this section we positively answer the question under the assumption (**) (even for hyperimaginaries).

**Lemma 5.5.** Assume that \( T \) is \( G \)-compact over \( e \) and that (**). Then given positive \( n < \omega \), there is a formula

\[
\alpha_n((xy)_{<n},(x'y')_{<n},a) \in E^l((xFyL)_{<n},(x'Fy'L)_{<n},a)
\]

such that if

\[
\models (\bigwedge_{i<n} \overline{p}(b^f_i, c^f_i) \land \overline{p}(b'^f_i, c'^f_i)) \land b'^f_{<n} \models E_{<e} b'^f_{<n} \land \alpha_n((b^f c^f)_{<n},(b'^f c'^f)_{<n},a),
\]

then \( \models E^l((b^f c^f)_{<n},(b'^f c'^f)_{<n},a) \).

**Proof.** To lead a contradiction, suppose that there is no such \( \alpha_n \). Then for each \( \alpha \in E^l((xFyL)_{<n},(x'Fy'L)_{<n},a) \), there are \( b^f c^f, b'^f c'^f \models \overline{p} (i < n) \) such that

\[
\models b'^f_{<n} \models E_{<e} b'^f_{<n} \land \alpha((b^f c^f)_{<n},(b'^f c'^f)_{<n},a) \land E^l((b^f c^f)_{<n},(b'^f c'^f)_{<n},a)
\]

where for example,

\[
(b^f c^f)_{<n} = (b^f c^f_i | i < n) \text{ and } (b'^f c'^f)_{<n} = (b'^f c'^f_i | i < n).
\]

Now since \( b'^f_{<n} \models E_{<e} b'^f_{<n} \), there are conjugates \( b^f, c^f \) \((i < n)\) satisfying

\[
\models E^l((b^f c^f)_{<n},(b'^f c'^f)_{<n},a),
\]

so that \( (c^f_i | i < n) \neq (c^f_i | i < n) \). Note now that since \( -L(c_j, c_{\ell}) \) for each pair \( j < \ell(< m) \), there is \( \psi_{J}(y,z) \in L(y,z) \subseteq L \) such that \( -\psi_{J}(c_j, c_{\ell}) \). Put \( \psi(y,z) \equiv \bigvee_{j<\ell<m} -\psi_{J}(y,z) \). Then we must have \((c^f_i, c^f_i | i < n) \models \Psi(y_i, z_i | i < n) \)

\[
\Psi(y_i, z_i | i < n) := \exists z'(y,z')_{<n}(\bigwedge_{i<n} L(y_i, y'_i) \land \bigwedge_{i<n} L(z_i, z'_i) \land \bigvee_{i<n} \psi(y_i, z'_i)).
\]

In conclusion, the following type \( \bigwedge_{i<n}(\overline{p}(x_i, y_i) \land \overline{p}(x'_i, y'_i)) \land E^l((xFyL)_{<n},(x'Fy'L)_{<n},a) \land E^l((xFzL)_{<n},(x'Fy'L)_{<n},a) \land \Psi((yz)_{<n})
\]
of variables \((xy)_{<n}, (x'y')_{<n}\) is finitely satisfiable, witnessed by \((b'c')_{<n}, (b''c''_i)_{<n}\). Hence by compactness, there are realizations \((bc)_{<n}\) and \((b'c')_{<n}\) of \((xy)_L\) and \((xyz)_L\) of the type, respectively. Thus we have \((bc)_{<n} \equiv^L_e (b'c')_{<n}\), which implies \(bc_i \equiv^L_e b'_i c'_i\) for all \(i < n\). But since \(|\Psi((c')_{<n})|\), there must be some \(i_0 < n\) such that \(c_{i_0} \neq c'_i\), contradicting our assumption (†).

\[\text{□} \]

**Theorem 5.6.** If \(T\) is \(G\)-compact over \(e\) and (†) holds, then \(K_n\) is finite for each finite \(n \geq 1\).

**Proof.** We start with a claim. We work with the formula \(\alpha_n\) obtained in Lemma 5.3.

**Claim 1.** There is \(\phi_n((xy)_{<n}, (x'y')_{<n}, a) \in E^L((xy)_L)_{<n}, (x'y')_{<n}, a)\) such that for any \(a_0, a_1, a_2 \models q(z) := tp(a/e)\),

\[
\phi_n((xy)_{<n}, (x'y')_{<n}, a_0) \land \phi_n((xy)_{<n}, (uv)_{<n}, a_1) \land \phi_n((uv)_{<n}, (x'y')_{<n}, a_2) \\
\models \alpha_n((xy)_{<n}, (x'y')_{<n}, a).
\]

**Proof of Claim 1.** Suppose not the claim, that is, for each \(\phi((xy)_{<n}, (x'y')_{<n}, a) \in E^L((xy)_{<n}, (x'y')_{<n}, a)\), there are \(a_0, a_1, a_2 \models q\) such that

\[
\phi((xy)_{<n}, (x'y')_{<n}, a_0) \land \phi((x'y')_{<n}, (uv)_{<n}, a_1) \land \phi((uv)_{<n}, (x'y')_{<n}, a_2) \\
\land \neg \alpha_n((xy)_{<n}, (x'y')_{<n}, a)
\]

is consistent. Then by compactness,

\[
q(z_0) \land q(z_1) \land q(z_2) \land E^L((xy)_{<n}, (x'y')_{<n}, z_0) \land E^L((x'y')_{<n}, (uv)_{<n}, z_1) \\
\land E^L((uv)_{<n}, (x'y')_{<n}, z_2) \land \neg \alpha_n((xy)_{<n}, (x'y')_{<n}, a)
\]

is consistent. But \(E^L((xy)_{<n}, (x'y')_{<n}, a)\) is \(e\)-invariant, thus in fact we get

\[
(xy)_{<n} \equiv^L_e (x'y')_{<n} \land \neg \alpha_n((xy)_{<n}, (x'y')_{<n}, a),
\]

which is impossible since \(\alpha_n((xy)_{<n}, (x'y')_{<n}, a) \in E^L((xy)_{<n}, (x'y')_{<n}, a)\).

\[\text{□} \]

Let \(\{(b'e')_{<n} | \ell \in I\}\) be a (small) set of all the representatives of \(\equiv^L_e\)-classes of realizations of \(\bigwedge_{i<n} \overline{p}(x_i, y_i)\). Thus for any \((b'e')_{<n} \models \bigwedge_{i<n} \overline{p}(x_i, y_i)\), there is \(\ell \in I\) such that \(|\overline{E}((b'e')_{<n}, (b'e')_{<n}, a)|\), so \(\phi_n((b'e')_{<n}, (b'e')_{<n}, a)\) a fortiori. Hence by compactness there are \((b'e'_{<n})_{<n}, \ldots, (b'^{k-1}e^{k-1})_{<n}\) such that

\[
\bigwedge_{i<n} \overline{p}(x_i, y_i) \land \bigvee_{\ell<k} \phi_n((xy)_{<n}, (b'_i c'_i | i < n), a). \quad (*)
\]

(Here some of \(b'e\)'s (\(\ell < k\)) could be the same.) Recall that for \(i < n\), \(b'_0, \ldots, b'_{m-1}\) are the fixed \(b'e\)-conjugates of \(c'_{i_0}\). Hence indeed \(c'_i = c''_{i_{i_0}}\) for some \(i_{i_0} < n\).

Now let \(\overline{f} \in K_n\). Thus \(f(b'e')_{<n} \equiv^L_e b'_e\). Then for each \(\ell < k\), there is \(i_\ell \in \{m\}_m\) such that \(f((b'e')_{<n}) \equiv^L_e (b'_i c'_{i_\ell}(i) | i < n)\). Moreover if \(\overline{f} = \overline{g}\) then \(f((b'e')_{<n}) \equiv^L_e g((b'e')_{<n})\). Hence due to our assumption (†), the mappings \(\ell < k \mapsto i_\ell\) for \(f\) and \(g\) must be the same. Therefore by the following claim we can conclude that \(|K_n|\) is finite \((m^n)_k = m^{nk}\), since there are only \(m^n\)-many choices of \(i_\ell\) for each \(\ell < k\).

**Claim 2.** Let \(\overline{f}, \overline{g} \in K_n\). Suppose that the described mappings \(\ell \mapsto i_\ell\) for all \(\ell < k\) above for \(\overline{f}\) and \(\overline{g}\) are the same. Then \(\overline{f} = \overline{g}\).
Now since $\phi_n$ is inconsistent. Assume $b$ over by $(\ast \ast)$. Hence there is $\phi_n((b')_n, (b')_n) \wedge (\ast \ast)$. Thus
\[
\phi_n(f((b')_n), f((b')_n), f(a)) \text{ and } \phi_n(g((b')_n), g((b')_n), g(a)).
\]

Now due to the supposition in Claim 2, we have
\[
f((b')_n) = \prod_{i<n} (b'_i)^{\phi_{i+1}(i)} \equiv g((b')_n).
\]
Hence there is $h \in \text{Aut}_f(M)$ such that $f((b')_n) = hg((b')_n)$. Then $f((b')_n) = hg((b')_n)$ may not hold, while $\phi_n(f((b')_n), hg((b')_n), (b')_n) \equiv (\ast \ast)$. Moreover by $(\ast \ast)$, we have $\phi_n(hg((b')_n), hg((b')_n), (b')_n)$.

Thus by Claim 1 (with $\phi_n$ being symmetric), it follows that
\[
\models \alpha_n(f((b')_n), hg((b')_n), (b')_n).
\]
Now since $\mathcal{F}, \mathcal{H} \in K_n$, we have $f((b')_n) \equiv_{e}^1 h g((b')_n) \equiv_{e}^1 b'$. Hence Lemma 5.5 implies that $f((b')_n) \equiv_{e}^1 h g((b')_n) \equiv_{e}^1 g((b')_n)$. Since $(b')_n$ is an arbitrary realization of $\bigwedge_{i<n} \beta(x_i, y_i)$, we conclude that $\mathcal{F} = \mathcal{H}$.

In the remaining section we confirm that two real case results on $K_1$ in [4] can be extended, by following essentially the same proofs, to the hyperimaginary context (Corollary 5.8 Proposition 5.10).

We now remove the assumption $(\dagger)$, and recall that $c = c_0, \cdots, c_{m-1}$ are all the distinct $bc$-conjugates of $c$. Possibly reordering them, we assume that $bc_0, \cdots, bc_{m_0-1} (1 \leq m_0 \leq m)$ are the representatives of all the distinct $e$-classes in $p = tp(bc/e)$ with the first coordinate $b$.

**Lemma 5.7.** If $T$ is $G$-compact over $e$, then there is a formula
\[
\alpha(x, y, x', y', a) \in E^L(x_F y_L, x_F y'_L, a)
\]
such that if
\[
\models \mathcal{F}(b', c') \wedge \mathcal{P}(b'', c'') \wedge b' \equiv_{e}^1 b'' \wedge \alpha(b', c', b'', a),
\]
then $\models E^L(b', c', b''', a)$.

**Proof.** If $m_0 = 1$ then any formula in $E^L$ would work. Hence we assume $m_0 > 1$.

Now the type
\[
E^L(x_F y_L, x'_F y'_L, a) \wedge \exists z z w' w'(z_F w_L y'_L, w'_L, a) \equiv_{e} x_F y_L, x'_F y'_L
\]
\[
\wedge \bigvee_{0 \leq i < j < m_0} E^L(z_F w_L, b_{c_i} a) \wedge E^L(z_F w'_L, b_{c_j} a)
\]
is inconsistent. Thus there is $\alpha(x, y, x', y', a) \in E^L(x_F y_L, x'_F y'_L, a)$ such that
\[
\alpha(x, y, x', y', a) \wedge \exists z z w' w'(z_F w_L, z'_F w'_L, a) \equiv_{e} x_F y_L, x'_F y'_L
\]
\[
\wedge \bigvee_{0 \leq i < j < m_0} E^L(z_F w_L, b_{c_i} a) \wedge E^L(z_F w'_L, b_{c_j} a)
\]
is inconsistent. Assume $b'c'b''c''$ $\models \mathcal{F}(xy) \wedge \mathcal{P}(x'y') \wedge \alpha(x, y, x', y', a) \wedge x_F \equiv_{e} x_F$. Notice that there is $c''$ such that $b'c'b''c'' \equiv_{e} b_c'c''$. Thus $b' \equiv_{e} b$. Now $b_{c_i} \equiv_{e} b_{c_i}$ and $b'c''$ $\equiv_{e} b_{c_j}$ for some $i, j < m_0$. But by the choice of $\alpha$, it must be $i = j$ and hence $b'c' \equiv_{e} b''c''$. 

\[\Box\]
The following can be obtained by exactly the same proof of Theorem 5.6 with \( n = 1 \) (in this case (†) is needless) and \( \alpha \) in Lemma 5.7 which we will not repeat.

**Corollary 5.8.** If \( T \) is \( G \)-compact over \( e \), then \( K_1 \) is finite.

**Remark 5.9.** Assume that \( \text{Gal}^1_{e}(p) \) is abelian. Let \( f \in \text{Aut}_e(p) \) and let \( b_0 \models p \). If \( f(b_0) \equiv^L_e b_0 \), then for any \( b' \models p \), \( f(b') \equiv^L_e b' \).

**Proof.** There is \( g \in \text{Aut}_e(p) \) such that \( g(b_0) = b' \). Now since \( \text{Gal}^1_{e}(p) \) is abelian, \( f(b') = f(g(b_0)) \equiv^L_e g(f(b_0)) \equiv^L_e g(b_0) = b' \). \( \square \)

**Proposition 5.10.** If \( \text{Gal}^1_{e}(\mathfrak{P}) \) is abelian, then \( |K_1| = m_0 \).

**Proof.** Let \( \mathfrak{f}, \mathfrak{g} \in \text{Gal}^1_{e}(\mathfrak{P}) \) and assume \( f(bc) \equiv^L_e g(bc) \). Then there is \( h \in \text{Aut}_{e}(M) \) such that \( hf(bc) = g(bc) \), thus \( g^{-1}hf(bc) = bc \). By Remark 5.9 with \( \mathfrak{g} \), we have \( g^{-1}hf \in \text{Aut}_{e}(\mathfrak{P}) \). Hence \( \mathfrak{f} = \mathfrak{g} \). In addition if \( \mathfrak{f}, \mathfrak{g} \in K_1 \) then \( f(b) \equiv^L_e g(b) \equiv^L_e b \). Therefore we get \( |K_1| = m_0 \), since there are only \( m_0 \)-many distinct \( \equiv^L_e \)-classes in \( \mathfrak{P} \) with the first coordinate Lascar equivalent to \( b \) over \( e \). \( \square \)

Now several observations in \[3\] automatically follow in our context. For example, if \( T \) is \( G \)-compact over \( e \) then \( \pi_n \) is a covering homomorphism when (†) holds, or \( n = 1 \). We point out Theorem 5.12 as well, which relies on a purely compact group theoretical result from \[3\].

**Fact 5.11.** Let \( G \) be an abelian compact connected topological group, and let \( F \) be a finite subgroup of \( G \). Then \( G \) and \( G/F \) are isomorphic as topological groups.

**Theorem 5.12.** Assume \( T \) is \( G \)-compact over \( e \); and \( \text{acl}(e) \) and \( e \) are interdefinable. If \( n = 1 \) or (†) holds; and \( \text{Gal}^n_{e}(\mathfrak{P}) \) is abelian, then it is isomorphic to \( \text{Gal}^1_{e}(p) \) as topological groups.

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Department of Mathematics, Yonsei University, Seoul, Korea
Email address: bkim@yonsei.ac.kr

Department of Mathematics, Yonsei University, Seoul, Korea
Email address: alternative@yonsei.ac.kr