Discrete Painlevé system and the double scaling limit of the matrix model for irregular conformal block and gauge theory

H. Itoyama$^a$, T. Oota$^b$ and Katsuya Yano$^c$

$^a$Department of Mathematics and Physics, Graduate School of Science
Osaka City University

$^b$Osaka City University Advanced Mathematical Institute (OCAMI)
3-3-138, Sugimoto, Sumiyoshi-ku, Osaka, 558-8585, Japan

Abstract

We study the partition function of the matrix model of finite size that realizes the irregular conformal block for the case of the $\mathcal{N} = 2$ supersymmetric $SU(2)$ gauge theory with $N_f = 2$. This model has been obtained in [arXiv:1008.1861 [hep-th]] as the massive scaling limit of the $\beta$-deformed matrix model representing the conformal block. We point out that the model for the case of $\beta = 1$ can be recast into a unitary matrix model with log potential and show that it is exhibited as a discrete Painlevé system by the method of orthogonal polynomials. We derive the Painlevé II equation, taking the double scaling limit in the vicinity of the critical point which is the Argyres-Douglas type point of the corresponding spectral curve. By the 0d-4d dictionary, we obtain the time variable and the parameter of the double scaled theory respectively from the sum and the difference of the two mass parameters scaled to their critical values.

*e-mail: itoyama@sci.osaka-cu.ac.jp
†e-mail: toota@sci.osaka-cu.ac.jp
‡e-mail: yanok@sci.osaka-cu.ac.jp
Study of correlation functions in lower dimensional quantum field theory and statistical system has sometimes led us to a surprising occurrence of nonlinear differential/difference equations that they obey. In two dimensional physical systems, these equations are typically Painlevé equations that have attracted interest of both physicists and mathematicians, and that govern the scaling behavior of the systems. They first appeared in the study of Ising two point correlation functions, and take the form of Painlevé III [1, 2, 3, 4, 5].

The second such development was made in the context of two dimensional quantum gravity (2d gravity for short) and $c < 1$ non-critical strings [6]. For a review, see, for example, [7]. Equilateral triangulation of a two dimensional random surface generates the double infinite sum for its partition function by the number of triangles and by the number of holes of the discretized surface. Through dual Feynman diagrams, the partition function is recast into multiple integrals of a hermitean matrix of finite size $N$ and hence is the finite $N$ hermitean matrix model having a bare cosmological constant parameter. The method of orthogonal polynomials permits us to relate the partition function with a set of recursion relations. A specific set of recursion relations forms a system of difference equations called string equations, and a phenomenon of genus enhancement/resummation takes place in a limit commonly referred to as the double scaling limit where a difference operator is replaced by the corresponding derivative. Painlevé I equation has been exhibited this way as a universal equation governing the nonperturbative scaling function for the partition function containing all genus contributions.

In recent years, a certain class of $\beta$-deformed ensembles for matrix models containing log potentials have been serving as integral representations [8, 9, 10, 11, 12, 13, 14] of 2d conformal and irregular 2d conformal block [15, 16]. They in fact generate directly [13, 14] the expansion of the block in the form of the instanton expansion in accordance with the AGT correspondence [17]. The matrix model free energy $F$ is thus equal to the instanton part of the Seiberg-Witten prepotential $\mathcal{F}$ augmented by the higher genus contributions [18]: $F = \mathcal{F}$. For a review, see, for example [19]. In [20], Painlevé VI equation has been derived for a Fourier transform of the $c = 1$ conformal block with respect to the intermediate momentum. See [21, 22, 23, 24, 25, 26] for subsequent analyses.

In this letter, we will study the simplest prototypical case of the irregular block, namely the case of the $\mathcal{N} = 2$ supersymmetric $SU(2)$ gauge theory with the number of hypermultiplets $N_f = 2$ in the form of the matrix model integral representation derived in [14]. Unlike
our procedure is closer in spirit to that of 2d gravity in its unitary counterpart. We see that the finite $N$ system formulated by the orthogonal polynomials which we devise is already regarded as a discretized Painlevé system. We are able to take the double scaling limit of this system to its critical point to derive the Painlevé II equation for the scaling function. The “time” variable $t$ is obtained from the limit of the sum of the two hypermultiplet masses of the gauge theory to its critical value by the 0d-4d dictionary while the parameter $M$ in the equation from the limit of the difference of the two masses. Details of the derivation to our findings will be given elsewhere.

The partition functions of the $\beta$-deformed matrix models which directly generate \cite{13,14} the instanton expansion of the four-dimensional $\mathcal{N} = 2$ $SU(2)$ gauge theories with $N_f$ fundamental matters can be generically presented as

$$Z^{(N_f)} = N(N_f) \prod_{I=1}^{N} \int_{\mathcal{C}^{(N_f)}_I} dw_I \Delta(w)^{2\beta} \exp \left( \sqrt{\beta} \sum_{I=1}^{N} W^{(N_f)}(w_I) \right). \quad (1)$$

Here, $N(N_f)$ is a normalization factor, $\Delta(w) = \prod_{I<J}(w_I - w_J)$ the Vandermonde determinant, and $\mathcal{C}^{(N_f)}_I$ certain integration contours below.

For $N_f = 4$, namely, the case of 2d conformal block, the potential $W^{(4)}(w)$ is given by the three-Penner (logarithmic) potential

$$W^{(4)}(w) = \alpha_1 \log(w) + \alpha_2 \log(w - q_0) + \alpha_3 \log(w - 1). \quad (2)$$

Let $N = N_L + N_R$. The $N_L$ contours $\mathcal{C}^{(4)}_I$ ($1 \leq I \leq N_L$) are taken to be the interval $[0, q_0]$ and the remaining $N_R$ contours $\mathcal{C}^{(4)}_J$ ($N_L + 1 \leq J \leq N$) are chosen as $[1, \infty]$. This corresponds to the four-point conformal block of the two-dimensional conformal field theory with $c = 1 - 6 Q_E^2$, $Q_E \equiv \sqrt{\beta} - 1/\sqrt{\beta}$:

$$Z^{(4)} = \langle V_{\alpha_1}(0)V_{\alpha_2}(q_0)V_{\alpha_3}(1)V_{\alpha_4}(\infty) \rangle. \quad (3)$$

The $\beta$-deformed matrix model for $N_f = 4$ contains seven parameters $\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_4$, $\beta$, $N_L$, $N_R$ undergoing one constraint (the momentum conservation)

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 2\sqrt{\beta}N = 2 Q_E. \quad (4)$$

These are transcribed into six unconstrained 4d parameters of $N_f = 4$ $SU(2)$ gauge theory

$$\frac{\epsilon_1}{g_s}, \frac{a}{g_s}, \frac{m_1}{g_s}, \frac{m_2}{g_s}, \frac{m_3}{g_s}, \frac{m_4}{g_s},$$

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by the 0d-4d dictionary \cite{[13]}:

\[ \sqrt{\beta} N_L = \frac{a - m_2}{g_s}, \]

\[ \sqrt{\beta} N_R = \frac{-a + m_1}{g_s}, \]

\[ \alpha_1 = \frac{1}{g_s} (m_2 - m_4 + \epsilon), \]

\[ \alpha_2 = \frac{1}{g_s} (m_2 + m_4), \]

\[ \alpha_3 = \frac{1}{g_s} (m_1 + m_3), \]

\[ \alpha_4 = \frac{1}{g_s} (m_1 - m_3 + \epsilon). \]

The omega background parameters \( \epsilon_{1,2} \) are related to \( \beta \) as \( \epsilon_1 = \sqrt{\beta} g_s \) and \( \epsilon_2 = -g_s/\sqrt{\beta} \). Hence \( g_s^2 = -\epsilon_1 \epsilon_2 \) and \( \epsilon \equiv \epsilon_1 + \epsilon_2 = Q_E g_s \). The cross ratio \( q_0 \) is identified with the exponentiated ultraviolet gauge coupling constant \( q_0 \equiv e^{i \pi \tau_0}, \tau_0 \equiv (\theta_0/\pi) + 8 \pi i/g_s^2 \).

The \( N_f = 3 \) limit of the gauge theory is taken by \( m_4 \to \infty \) with \( \Lambda_3 \equiv 4 q_0 m_4 \) fixed. By the above dictionary (6), this corresponds to the \( q_0 \to 0 \) limit with \( 2 q_{03} \equiv q_0 (-\alpha_1 + \alpha_2) \) and \( \alpha_{1+2} \equiv \alpha_1 + \alpha_2 \) fixed. The parameter \( q_{03} \) is related to the dynamical mass scale \( \Lambda_3 \) of the \( N_f = 3 \) theory by \( q_{03} = \Lambda_3/(4 g_s) \). Also, \( \alpha_{1+2} = (2 m_2 + \epsilon)/g_s \). The constraint (4) reduces to

\[ \alpha_{1+2} + \alpha_3 + \alpha_4 + 2\sqrt{\beta} N = 2 Q_E. \]

(7)

In the \( N_f = 3 \) limit, the potential \( W^{(3)}(w) \) for the “irregular matrix model” consists of two logarithmic terms and an inverse power term:

\[ W^{(3)}(w) = \alpha_{1+2} \log w + \alpha_3 \log(w - 1) - \frac{q_{03}}{w}. \]

(8)

For an explicit form of integration contours \( C_f^{(3)} \), see [14].

We can take the \( N_f = 2 \) limit in the gauge theory subsequently after the \( N_f = 3 \) limit by \( m_3 \to \infty \) with the dynamical scale \( \Lambda_2 \equiv (m_3 \Lambda_3)^{1/2} \) fixed. In the matrix model, this corresponds to the \( q_{03} \to 0 \) limit with \( q_{02}^2 \equiv (1/2)q_{03} (\alpha_3 - \alpha_4) \) and \( \alpha_{3+4} \equiv \alpha_3 + \alpha_4 \) fixed. The momentum conservation (7) becomes

\[ \alpha_{1+2} + \alpha_{3+4} + 2\sqrt{\beta} N = 2 Q_E. \]

(9)

Now \( \alpha_{3+4} = (2 m_1 + \epsilon)/g_s \) and \( q_{02} = \Lambda_2/(2 g_s) \). The potential of the resultant \( N_f = 2 \) irregular matrix model takes the following form:

\[ W^{(2)}(w) = \alpha_{3+4} \log w - q_{02} \left( w + \frac{1}{w} \right). \]

(10)

\footnote{Here in comparison to [13], we have renamed the mass parameters as \( m_1^{[13]} = m_4, m_3^{[13]} = m_1, m_4^{[13]} = m_3 \), such that the ordering of masses subsequently sent to infinity is natural.}
In dealing with matrix model partition functions in general, one needs to predecide whether filling fractions are explicitly specified or not as the number of integrations lying in different contours. It is argued in [25] (also implicit in [20]) that these two distinct cases are related to each other by a version of Fourier transform. We will transplant their discussion at \( N_f = 4 \), \( 0 \) here at \( N_f = 2 \).

In general, let
\[
Z_s(N) = \sum_n s^n Z(N - n, n). \tag{11}
\]
Here \( Z(N - n, n) \) is the object which we have been discussing up to now and \( Z_s(N) \) is the object which we will study from now on for the derivation of Painlevé system, the point of view of which is in accordance with the current literature. For simplicity, we set \( s = 1 \) and we have one less parameters at hand from now on.

Moreover, it has been argued [27, 28, 29] that two-cut hermitean matrix model and the unitary matrix model share the critical properties. Therefore, the planar scaling or the double scaling limit of our irregular models would lie in the same universality class which the unitary matrix model belongs to.

From now on, we restrict ourselves to the case \( \beta = 1 \). Hence \( \epsilon_1 = -\epsilon_2 = g_s \) and \( \epsilon = 0 \). Let us consider the following unitary matrix model
\[
Z_U(N) = \frac{(-1)^{(1/2)N(N-1)}}{N!} \left( \prod_{I=1}^{N} \oint \frac{dw_I}{2\pi i} \right) \Delta(w)^2 \exp \left( \sum_{I=1}^{N} W(w_I) \right)
= \frac{1}{N!} \left( \prod_{I=1}^{N} \oint \frac{dw_I}{2\pi i w_I} \right) \Delta(w) \Delta(w^{-1}) \exp \left( \sum_{I=1}^{N} W_U(w_I) \right), \tag{12}
\]
where the potential is given by \( W_U(w) = W(w) + N \log w \). In particular, we study the \( N_f = 2 \) case for simplicity:
\[
W_U(w) = W^{(2)}(w) + N \log w = -q_{02} \left( w + \frac{1}{w} \right) + M \log w, \tag{13}
\]
where \( M \equiv \alpha_{3+4} + N = (m_1 - m_2)/g_s \). In order for the contour integrals to be well-defined, we assume that \( M \) is an integer. In [14], the original integration path on the real axis of the \( N_f = 4 \) model was deformed into a contour in the complex plane by an analytic continuation to avoid the singularity which is induced by the \( N_f = 3, 2 \) potential in the limit. In \( N_f = 2 \) case, the contour derived is the one wrapping the positive real axis from the origin to infinity.
When $M$ is an integer, this contour becomes a closed circle around the origin. Our $N_f = 2$ model is in fact equivalent to the above unitary matrix model.

The unitary matrix model can be solved \[30, 31, 32\] by the method of orthogonal polynomials \[33, 34\]. Let us use the monic orthogonal polynomials \[31, 32\]:

$$\oint \frac{dw}{2\pi i} e^{W_U(w)} p_n(w) \tilde{p}_m(w^{-1}) = h_n \delta_{n,m},$$

$$p_n(w) = w^n + \sum_{k=0}^{n-1} A^{(n)}_k w^k, \quad \tilde{p}_n(w^{-1}) = w^{-n} + \sum_{k=0}^{n-1} B^{(n)}_k w^{-k}.\quad (15)$$

In \[31, 32\], $M = 0$ case was considered with $\tilde{p}_n(w) = p_n(w)$ ($B^{(n)}_k = A^{(n)}_k$).

Through explicit computation, we have found that the moments of this model are given by the modified Bessel functions up to phase factors. Let

$$K^{(n)}_{\nu} \equiv \det(I_{j-i+\nu}(1/g_s))_{1 \leq i, j \leq n}, \quad (\nu \in \mathbb{C}; n = 0, 1, 2, \cdots),\quad (16)$$

where $I_{\nu}(z)$ is the modified Bessel function of the first kind and $g_s \equiv 1/(2q_0) = g_s/A_2$. The normalization constants of the orthogonal polynomials are given by $h_n = (-1)^M K^{(n+1)}_M / K^{(n)}_M$. In particular, $h_0 = (-1)^M I_M(1/g_s)$. For notational simplicity, let $A_n \equiv p_n(0) = A^{(n)}_0$, $B_n \equiv \tilde{p}_n(0) = B^{(n)}_0$. These constants are respectively given by $A_n = K^{(n)}_{M+1}/K^{(n)}_M$, $B_n = K^{(n)}_{M-1}/K^{(n)}_M$.

The partition function (12) can be written in terms of these objects:

$$Z^{U(N)} = (-1)^{MN} K^{(N)}_M = \prod_{k=0}^{N-1} h_k = h_0^N \prod_{j=1}^{N-1} (1 - A_j B_j)^{N-j}.\quad (17)$$

The string equations lead to the following recursion relations for $A_n$ and $B_n$:

$$A_{n+1} = -A_{n-1} + \frac{2 n g_s A_n}{1 - A_n B_n}, \quad B_{n+1} = -B_{n-1} + \frac{2 n g_s B_n}{1 - A_n B_n},$$

$$A_n B_{n+1} - A_{n+1} B_n = 2 M g_s.\quad (19)$$

With the initial conditions $A_0 = B_0 = 1$, and

$$A_1 = \frac{I_{M+1}(1/g_s)}{I_M(1/g_s)}, \quad B_1 = \frac{I_{M-1}(1/g_s)}{I_M(1/g_s)},\quad (20)$$

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2 In \[30\], orthogonal polynomials of different type have been introduced to solve the unitary matrix model.
the remaining constants $A_n$ and $B_n$ are completely characterized by the recursion relations (18), (19). When $M = 0$ with $B_n = A_n$, the recursion relation (18) is the discrete Painlevé II equation [31]. Moreover, let $x_n = A_{n+1}/A_n$ and $y_n = B_{n+1}/B_n$. Then from the recursion relations (18) and (19), one can show that these variables respectively satisfy the alternate discrete Painlevé II equations [35, 36]:

$$\frac{2(n+1)g_s}{1 + xx_{n+1}} + \frac{2ng_s}{1 + xx_{n-1}} = -x_n + \frac{1}{x_n} + 2ngs - 2Mg_s, \quad (21)$$

$$\frac{2(n+1)g_s}{1 + yy_{n+1}} + \frac{2ng_s}{1 + yy_{n-1}} = -y_n + \frac{1}{y_n} + 2ngs + 2Mg_s. \quad (22)$$

Note that our solutions

$$x_n = \frac{K_{M+1}^{(n+1)} K_M^{(n)}}{K_M^{(n+1)} K_M^{(n)}}, \quad y_n = \frac{K_{M-1}^{(n+1)} K_M^{(n)}}{K_M^{(n+1)} K_{M-1}^{(n)}} \quad (23)$$

belong to a class of the Casorati determinant solutions to the alt-dPII considered in [36]. Furthermore, the partition function $Z_U^{(N)} = (−1)^{MN}K_M^{(N)}$ is the $\tau$-function of the alt-dPII equation. It is well known that the alt-dPII equation is closely related to the (differential) Painlevé III equation (PIII$_1$ or PIII($D_{6}^{(1)}$)). Following [37, (4.26)], let us introduce a function of $t = 1/g_s^2$ by

$$\sigma(t) := -t \frac{d}{dt} \log(e^{-t/4 tM^2/4 K_M^{(N)}}). \quad (24)$$

Then, $\sigma(t)$ satisfies the $\sigma$-form of the Painlevé III equation

$$(t\sigma')^2 - v_1 v_2 (\sigma')^2 + \sigma'(4\sigma' - 1)(\sigma - t\sigma') - \frac{1}{64} (v_1 - v_2)^2 = 0, \quad (25)$$

with

$$v_1 = -M + N = -\frac{2m_1}{g_s}, \quad v_2 = M + N = -\frac{2m_2}{g_s}. \quad (26)$$

Since we are considering the $N_f = 2$ case, it is natural to appear PIII$_1$ [24]. The Bäcklund transformations of the PIII$_1$ form the affine Weyl group of type $(2A_1)^{(1)}$. The translation subgroup generates the alt-dPII equation. It generates integer shifts of parameters $v_{1,2}$. In terms of the gauge theory parameters, it corresponds to constant shifts of mass parameters $m_{1,2}$.

Let $A_n = R_n D_n$ and $B_n = R_n / D_n$. The partition function (17) becomes

$$Z_U^{(N)} = h_0^N \prod_{j=1}^{N-1} (1 - R_j^2)^{N-j}. \quad (27)$$
Eliminating $D_n$ from (18) and (19), we obtain the recursion relation for $R^2_n$:

$$(1 - R^2_n)\left(\sqrt{R^2_n R^2_{n+1} + M^2 g_s^2} + \sqrt{R^2_n R^2_{n-1} + M^2 g_s^2}\right) = 2 n g_s R^2_n. \quad (28)$$

This is equivalent to

$$0 = \eta^2_n \left[ \xi^2_n (1 - \xi_n)^2 - \eta^2_n \xi^2_n + \zeta^2 (1 - \xi_n)^2 \right]$$

$$+ \frac{1}{2} \eta^2_n \xi_n (1 - \xi_n)^2 (\xi_{n+1} - 2 \xi_n + \xi_{n-1}) - \frac{1}{16} (1 - \xi_n)^4 (\xi_{n+1} - \xi_{n-1})^2,$$

where $\xi_n \equiv R^2_n$, $\eta_n \equiv n g_s$, $\zeta \equiv M g_s$.

In the planar limit $(\xi_n, \eta_n, \zeta) \to (\xi, \eta, \zeta)$, the second line of (29) is ignored and the three roots out of four in the resulting quartic equation in $\xi$ become degenerate to zero at $\eta = \pm 1, \zeta = 0$, where we take the continuum limit. In fact, setting $\xi = a^2 u$, $\eta = \pm 1 - (1/2) a^2 t$, $\zeta = \pm a^3 M$, we obtain at $\mathcal{O}(a^6)$

$$\pm t = 2u - \frac{M^2}{a^2}. \quad (30)$$

Eq. (29) also becomes the defining relation of an algebraic variety. With the introduction of the homogeneous coordinates $(\mathcal{X} : \mathcal{Y} : \mathcal{Z} : \mathcal{W}) = (\xi : \eta : \zeta : 1)$ of the three-dimensional complex projective space $\mathbb{P}^3$, this algebraic variety is the union of the hyperplane $\mathcal{Y} = 0$ (with multiplicity two) and the singular K3 surface

$$- \mathcal{Y}^2 \mathcal{X}^2 + \mathcal{X}^2 (\mathcal{X} - \mathcal{W})^2 + (\mathcal{X} - \mathcal{W})^2 \mathcal{Z}^2 = 0. \quad (31)$$

The singular loci of this surface consist of three spheres whose intersections are represented by the $A_3$ Dynkin diagram. We are unaware of further geometrical interpretation.

In order to present the critical behavior at the planar and the double scaling limit better, let us rewrite the potential (13) as

$$W_{U}(w) = \frac{N}{\tilde{S}} \left\{ -\frac{1}{2} \left( w + \frac{1}{w} \right) + \frac{m_1 - m_2}{\Lambda_2} \log w \right\}. \quad (32)$$

Here, $\tilde{S} \equiv N g_s = g_s N/\Lambda_2 = -(m_1 + m_2)/\Lambda_2$ is the parameter we fine tune to $\pm 1$, and is the counterpart of the bare cosmological constant in 2d gravity. Also note that $\zeta = (m_1 - m_2)/(\tilde{S} \Lambda_2) = \mathcal{O}(a^3)$ and the two masses are fine tuned to be equal in this limit. It is easy to see what this critical point corresponds to in the Seiberg-Witten curve \[38, 39\] (quartic one), which is the spectral curve obtained from the planar loop equation/Virasoro constraints
Omitting the standard procedure of this derivation, the curve \((y(z), z)\), where the resolvent \(\omega(z) = \lim_{N \to \infty} \left( g_s \sum_{I=1}^{N} 1/(z - w_I) \right)\) lies, is given by

\[
y(z) \equiv \omega(z) + \frac{W^{(2)}/(z)}{2},
\]

\[
y^2 = \frac{\Lambda_2^2}{16z^4} \left( 1 + \frac{8m_1}{\Lambda_2} z + \frac{16u}{\Lambda_2^2} z^2 + \frac{8m_2}{\Lambda_2} z^3 + z^4 \right).
\]

Here, we have used (33) and the residue relation of the resolvent at \(z = \infty\). We have parametrized the coefficient of \(z^2\) by the coordinate of the moduli space of the curve. Clearly, at our critical point \(m_1/\Lambda_2 = m_2/\Lambda_2 = \mp 1/2\), this genus one curve shrinks to a point at \(u/\Lambda_2^2 = 3/8\). Our limit is, therefore, the limit to the Argyres-Douglas point [43, 44, 45].

Let us consider the double scaling limit of (29). Let \(x \equiv n/N, a^3 \equiv 1/N\) and

\[
\eta_n = \tilde{S} x = 1 - (1/2)a^2 t, \quad \zeta = a^3 \tilde{S} M,
\]

\[
\xi(x) = \xi(n/N) = \xi_n = a^2 u(t).
\]

Here, we have taken the upper sign without losing generality. With these scaling ansatze, the double scaling limit is defined as the \(N \to \infty\) (\(a \to 0\)) limit while simultaneously sending \(\tilde{S}\) to its critical value 1 by (35). The original ’t Hooft expansion parameter \(1/N\) gets dressed by the combination which is kept finite in this limit:

\[
\kappa \equiv \frac{1}{N} \frac{1}{(1 - \tilde{S})^{1+\gamma}}, \quad \gamma = -1
\]

with \(\gamma\) being the susceptibility of the system. This last point can be checked from the free energy \(F\) computation from (27):

\[
F = - \lim_{N \to \infty} \frac{\log Z_{U(N)}}{N^2} \sim - \int_0^1 dx \left( 1 - x \right) \log \left( 1 - \xi(x) \right) \sim \left( 1 - \tilde{S} \right)^3 = \left( 1 - \tilde{S} \right)^{2-\gamma}.
\]

In the double scaling limit, the string equation (29) turns into the Painlevé II equation

\[
u'' = \left( \frac{(u')^2}{2u} + u^2 - \frac{1}{2} t u - \frac{M^2}{2u} \right).
\]

It is noteworthy that the parameter \(M\) in the original model survives the limit. We can convert (39) into standard form as follows. By using \(p_u \equiv -u'/u\), this equation (39) can be

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\[3\text{Here, we work in the same planar scaling limit as 24.}\]
written as a Hamilton system with the Hamiltonian
\[
H_{\text{II}}(u, p_u; t) = -\frac{1}{2} p_u^2 u + \frac{1}{2} u^2 - \frac{1}{2} t u + \frac{M^2}{2 u}. \tag{40}
\]
By a canonical transformation \((u, p_u) \rightarrow (v, p_v)\) with \(u = -p_v\) and \(p_u = v + (M/p_v)\), this Hamiltonian becomes
\[
H_{\text{II}} = \frac{1}{2} p_v^2 + \frac{1}{2} (v^2 + t) p_v + M v, \tag{41}
\]
and \(v = v(t)\) obeys the following form of the Painlevé II equation:
\[
v'' = \frac{1}{2} v^3 + \frac{1}{2} t v + \left(\frac{1}{2} - M\right). \tag{42}
\]
When there is no logarithmic potential \((M = 0)\), the appearance of the Painlevé II equation in the unitary matrix model was shown in [31, 32].

Note that (41) is the non-autonomous Hamiltonian for the Painlevé II equation [46]. The Bäcklund transformations for (41) are generated by [47]
\[
s_1(v) = v + \frac{2 M}{p_v}, \quad s_1(p_v) = p_v, \quad s_1(M) = -M, \\
\pi(v) = -v, \quad \pi(p_v) = -p_v - v^2 - t, \quad \pi(M) = 1 - M. \tag{43}
\]
The restriction of \(M\) to being an integer is compatible with these transformations.

We remark that these Bäcklund transformations form the affine Weyl group of type \(A_1^{(1)}\) and the translation \(T = s_1 \pi\) generates the alternate discrete Painlevé I equation [35]. Explicitly, let \(v_n = T^n(v)\) and \(p_n = T^n(p_v)\) \((n \in \mathbb{Z})\). Using (43), we obtain a discrete dynamical system for these variables:
\[
v_{n+1} + v_n = \frac{2(M + n)}{p_n}, \quad p_n + p_{n-1} = -v_n^2 - t. \tag{44}
\]
By removing \(p_n\), we find the following form of the alt-dPI:
\[
\frac{2(M + n)}{v_n + v_{n+1}} + \frac{2(M + n - 1)}{v_n + v_{n-1}} = v_n^2 + t. \tag{45}
\]

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