ASYMPTOTIC DIMENSION OF RELATIVELY HYPERBOLIC GROUPS

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Abstract. Suppose that a finitely generated group $G$ is hyperbolic relative to a collection of subgroups $\{H_1, \ldots, H_m\}$. We prove that if each of the subgroups $H_1, \ldots, H_m$ has finite asymptotic dimension, then asymptotic dimension of $G$ is also finite.

1. Introduction

The notion of asymptotic dimension of a metric space was proposed by Gromov [13] as a large–scale analog of the Lebesgue covering dimension. Recall that a metric space $S$ has asymptotic dimension $\text{asdim } S \leq n$ if for any $r > 0$, there exists a covering

$$S = \bigcup_{\alpha \in A} U_{\alpha}$$

such that the sets $U_{\alpha}$ are uniformly bounded and no more than $n + 1$ elements of $\{U_{\alpha}\}_{\alpha \in A}$ meet any ball of radius $r$.

In the case of finitely generated groups endowed with word metrics, the question of finiteness of asymptotic dimension took on additional significance with a theorem of Yu stating that the Novikov Higher Signature Conjecture is true for manifolds $M$ such that $\text{asdim } \pi_1(M) < \infty$ [19]. Some other results concerning groups of finite asymptotic dimension can be found in [6, 9, 20]. Although, in general, $\text{asdim } G$ can be infinite for a finitely generated or even finitely presented group $G$, there are many classes of groups for which asymptotic dimension is known to be finite. For example, this is so for nilpotent groups, fundamental groups of finite graphs of groups where vertex groups have finite asymptotic dimension, hyperbolic groups, etc. (see [2, 3, 17] and references therein).

In the present paper we study the case of relatively hyperbolic groups. Originally the notion of relative hyperbolicity was suggested by Gromov in [13]. Since then it has been elaborated from various points of view [4, 10, 11, 16, 18]. We recall the definitions of relative hyperbolicity suggested in [16]. In the case of finitely generated groups this definition is equivalent to the definitions given in [4, 10, 11, 16, 18].

Let $G$ be a group, $\{H_{\lambda}\}_{\lambda \in \Lambda}$ a collection of subgroups of $G$, $X$ a subset of $G$. We say that $X$ is a relative generating set of $G$ with respect to $\{H_{\lambda}\}_{\lambda \in \Lambda}$ if $G$ is generated by $X$ together with the union of all $H_{\lambda}$. (For convenience, we always assume $X = X^{-1}$.) In this situation the group $G$ can be regarded as a quotient of the free product

$$F = (\ast_{\lambda \in \Lambda} H_{\lambda}) \ast F(X),$$

where $F(X)$ is the free group with the basis $X$. Let $\varepsilon$ denote the natural homomorphism $F \to G$. If $\text{Ker } \varepsilon$ is a normal closure of a subset $R \subseteq N$ in the group $F$, we say that $G$
has relative presentation
\[
\langle X, H_\lambda, \lambda \in \Lambda \mid R = 1, R \in \mathcal{R} \rangle.
\]
If \( \sharp X < \infty \) and \( \sharp \mathcal{R} < \infty \), the relative presentation (2) is said to be finite and the group \( G \) is said to be finitely presented relative to the collection of subgroups \( \{H_\lambda\}_{\lambda \in \Lambda} \).

Set
\[
\mathcal{H} = \bigsqcup_{\lambda \in \Lambda} (H_\lambda \setminus \{1\}).
\]
Given a word \( W \) in the alphabet \( X \cup \mathcal{H} \) such that \( W \) represents 1 in \( G \), there exists an expression
\[
W = F \prod_{i=1}^{k} f_i^{-1} R_i f_i
\]
with the equality in the group \( F \), where \( R_i, f_i \in \mathcal{R} \) and \( f_i \in F \) for \( i = 1, \ldots, k \). The smallest possible number \( k \) in a representation of the form (4) is called the relative area of \( W \) and is denoted by \( \text{Area}^{\text{rel}}(W) \).

**Definition 1.** A group \( G \) is hyperbolic relative to a collection of subgroups \( \{H_\lambda\}_{\lambda \in \Lambda} \) if \( G \) is finitely presented relative to \( \{H_\lambda\}_{\lambda \in \Lambda} \) and there is a constant \( L > 0 \) such that for any word \( W \) in \( X \cup \mathcal{H} \) representing the identity in \( G \), we have \( \text{Area}^{\text{rel}}(W) \leq L \|W\| \).

We note that the above definition does not require the group \( G \) and the subgroups \( H_\lambda \) to be finitely generated as well as the collection \( \{H_\lambda\}_{\lambda \in \Lambda} \) to be finite. However, in case \( G \) is generated by a finite set in the ordinary (non-relative) sense and is finitely presented relative to a collection of subgroups \( \{H_\lambda\}_{\lambda \in \Lambda} \), \( \Lambda \) is known to be finite and the subgroups \( H_\lambda \) are known to be finitely generated [16, Theorem 1.1].

The class of relatively hyperbolic groups includes many examples of interest such as fundamental groups of finite-volume Riemannian manifolds of pinched negative curvature [4, 11], geometrically finite convergence groups [18], small cancellation quotients of free products [16], fully residually free groups [7], etc. The main result of our paper is the following.

**Theorem 2.** Suppose that a finitely generated group \( G \) is hyperbolic relative to a (finite) collection of subgroups \( \{H_\lambda\}_{\lambda \in \Lambda} \) and each of the subgroups \( H_\lambda \) has finite asymptotic dimension. Then asymptotic dimension of \( G \) is finite.

In the particular case when \( G \) is hyperbolic relative to a collection of virtually nilpotent subgroups \( \{H_\lambda\}_{\lambda \in \Lambda} \), Theorem 2 was independently proved by Dahmani and Yaman [8]. However their method essentially uses the assumption about \( H_\lambda \)'s and can not be applied in the general case.

Recall that a group \( G \) is said to be weakly hyperbolic relative to a collection of subgroups \( \{H_\lambda\}_{\lambda \in \Lambda} \), if the Cayley graph \( \Gamma(G, X \cup \mathcal{H}) \) of \( G \) with respect to the generating set \( X \cup \mathcal{H} \) is hyperbolic, where \( X \) is a finite generating set of \( G \) modulo \( \{H_\lambda\}_{\lambda \in \Lambda} \) and \( \mathcal{H} \) is the set defined by (3). It is not hard to show that relative hyperbolicity implies weak relative hyperbolicity with respect to the same collection of subgroups. The converse is not true. For instance, \( G = \mathbb{Z} \times \mathbb{Z} \) is weakly hyperbolic but not hyperbolic relative to the multiples.

The main idea of the proof of Theorem 2 is based on exploring the weak relative hyperbolicity of \( G \). On the other hand, we also use certain additional arguments that do not follow from the hyperbolicity of \( \Gamma(G, X \cup \mathcal{H}) \). So the natural question is whether these arguments are essential.
In many particular cases, the weak hyperbolicity of $G$ relative to a finite collection of subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$ ensures the finiteness of $\text{asdim} \ G$ whenever $\text{asdim} \ H_\lambda$ is finite for all $\lambda \in \Lambda$. For example, if $H$ is a normal subgroup of $G$ and $G/H$ is an ordinary hyperbolic group, then $G$ is weakly hyperbolic relative to $H$ as $\Gamma(G,X \cup H)$ is quasi-isometric to the quotient group $G/H$. In these settings, $\text{asdim} \ H < \infty$ implies $\text{asdim} \ G < \infty$ by the result of Bell and Dranishnikov [3] stating that any extension of a group of finite asymptotic dimension by a group of finite asymptotic dimension has finite asymptotic dimension.

Another series of examples is provided by amalgamated free products and HNN–extensions (or, more generally, by fundamental groups of finite graphs of groups). Indeed any group of the form $G = H_1 \ast_{A=B} H_2$ is weakly hyperbolic relative to the collection $\{H_1,H_2\}$ [15]. In this case the finiteness of asymptotic dimensions of $H_1$ and $H_2$ implies $\text{asdim} \ G < \infty$ according to the main result of [2]. Similarly any HNN–extension $G = H \ast_A$ is weakly hyperbolic relative to $H$ [15]. And again $\text{asdim} \ G < \infty$ whenever $\text{asdim} \ H < \infty$ [2].

Note that, in general, the above–mentioned weakly relatively hyperbolic groups are not relatively hyperbolic with respect to the specified collections of subgroups. Generalizing these examples one may conjecture that if a group $G$ is weakly hyperbolic relative to a finite collection of subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$ and all subgroups $H_\lambda$ have finite asymptotic dimension, then asymptotic dimension of $G$ is finite. However this conjecture does not hold. To provide a counterexample we prove the following.

**Proposition 3.** There exists a finitely presented group $G$ and a finite collection of subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$ such that:

1) For any $\lambda \in \Lambda$, $H_\lambda$ is cyclic and hence $\text{asdim} \ H_\lambda \leq 1$.
2) The Cayley graph $\Gamma(G,X \cup \mathcal{H})$ has finite diameter; in particular, it is hyperbolic and thus $G$ is weakly hyperbolic relative to $\{H_\lambda\}_{\lambda \in \Lambda}$.
3) $\text{asdim} \ G = \infty$.

The paper is organized as follows. In the next section we collect all necessary definitions and results used in our paper. Sections 3 and 4 contain proofs of certain technical facts about relatively hyperbolic groups. The proofs of Theorem 2 and Proposition 3 are provided in Section 5 and 6 respectively.

2. Preliminaries

2.1. Some notation and conventions. Given a word $W$ in an alphabet $\mathcal{A}$, we denote by $||W||$ its length that is the number of letters in $W$. We also write $W \equiv V$ to express the letter for letter equality of words $W$ and $V$. For elements $g, t$ of a group $G$, $g^t$ denotes the element $t^{-1}gt$. Similarly $H^t$ denotes $t^{-1}Ht$ for subgroups $H \leq G$. Recall that a subset $X$ of a group $G$ is said to be symmetric if for any $x \in X$, we have $x^{-1} \in X$. In this paper all generating sets of groups under consideration are supposed to be symmetric.

Given a group $G$ generated by a (symmetric) set $\mathcal{A}$, the Cayley graph $\Gamma(G,\mathcal{A})$ of $G$ with respect to $\mathcal{A}$ is an oriented labelled 1–complex with the vertex set $V(\Gamma(G,\mathcal{A})) = G$ and the edge set $E(\Gamma(G,\mathcal{A})) = G \times \mathcal{A}$. An edge $e = (g,a)$ goes from the vertex $g$ to the vertex $ga$ and has label $\text{Lab}(e) \equiv a$. As usual, we denote the origin and the terminus of the edge $e$, i.e., the vertices $g$ and $ga$, by $e_-$ and $e_+$ respectively. Given a combinatorial path $p = e_1 e_2 \ldots e_k$ in the Cayley graph $\Gamma(G,\mathcal{A})$, where $e_1, e_2, \ldots, e_k \in E(\Gamma(G,\mathcal{A}))$, we denote by $\text{Lab}(p)$ its label. By definition,

$$\text{Lab}(p) \equiv \text{Lab}(e_1) \text{Lab}(e_2) \cdots \text{Lab}(e_k).$$
We also denote by $p_- = (e_1)_-$ and $p_+ = (e_k)_+$ the origin and the terminus of $p$ respectively. The length $l(p)$ of $p$ is the number of edges of $p$.

Associated to $A$ is the so-called word metric on $G$. More precisely, the length $|g|_A$ of an element $g \in G$ is defined to be the length of a shortest word in $A$ representing $g$ in $G$. This defines a metric on $G$ by $\text{dist}_A(f,g) = |f^{-1}g|_A$. We also denote by $\text{dist}_A$ the natural extension of the word metric to the Cayley graph $\Gamma(G,A)$.

2.2. Relatively hyperbolic groups. Recall that a metric space $M$ is $δ$–hyperbolic for some $δ \geq 0$ (or simply hyperbolic) if for any geodesic triangle in $M$, any side of the triangle belongs to the union of the closed $δ$–neighborhoods of the other two sides. A group $G$ is called (ordinary) hyperbolic if $G$ satisfies Definition 1 with respect to the trivial subgroup. An equivalent definition says that $G$ is hyperbolic if it is generated by a finite set $X$ and the Cayley graph $\Gamma(G,X)$ is a hyperbolic metric space. In the relative case these approaches are not equivalent, but we still have the following Theorem 1.7.

Lemma 4. Suppose that $G$ is a group hyperbolic relative to a collection of subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$. Let $X$ be a finite relative generating set of $G$ with respect to $\{H_\lambda\}_{\lambda \in \Lambda}$. Then the Cayley graph $\Gamma(G, X \cup H)$ of $G$ with respect to the generating set $X \cup H$ is a hyperbolic metric space.

Let us recall an auxiliary terminology introduced in [16], which plays an important role in our paper. Let $G$ be a group, $\{H_\lambda\}_{\lambda \in \Lambda}$ a collection of subgroups of $G$, $X$ a finite generating set of $G$ with respect to $\{H_\lambda\}_{\lambda \in \Lambda}$, $q$ a path in the Cayley graph $\Gamma(G, X \cup H)$. A subpath $p$ of $q$ is called an $H_\lambda$–component for some $\lambda \in \Lambda$ (or simply a component) of $q$, if the label of $p$ is a word in the alphabet $H_\lambda \setminus \{1\}$ and $p$ is not contained in a bigger subpath of $q$ with this property.

Two components $p_1, p_2$ of a path $q$ in $\Gamma(G, X \cup H)$ are called connected if they are $H_\lambda$–components for the same $\lambda \in \Lambda$ and there exists a path $c$ in $\Gamma(G, X \cup H)$ connecting a vertex of $p_1$ to a vertex of $p_2$ such that $\text{Lab}(c)$ entirely consists of letters from $H_\lambda$. In algebraic terms this means that all vertices of $p_1$ and $p_2$ belong to the same coset $gH_\lambda$ for a certain $g \in G$. Note that we can always assume $c$ to have length at most 1, as every nontrivial element of $H_\lambda$ is included in the set of generators. An $H_\lambda$–component $p$ of a path $q$ is called isolated if no distinct $H_\lambda$–component of $q$ is connected to $p$.

If the group $G$ is generated by the finite set $X$ in the ordinary (non–relative) sense, we have two word metrics $\text{dist}_X$ and $\text{dist}_{X \cup H}$ on $G$ associated to the generating sets $X$ and $X \cup H$ respectively. Each of these has its own advantage and disadvantage. Namely, if $G$ is hyperbolic relative to $\{H_\lambda\}_{\lambda \in \Lambda}$, then the metric space $(G, \text{dist}_{X \cup H})$ is hyperbolic, but in general is not locally finite. On the other hand, $(G, \text{dist}_X)$ is locally finite but usually is not hyperbolic. In the next two lemmas we consider these metric spaces together. The first result is a simplification of Lemma 2.27 from [16].

Lemma 5. Suppose that a group $G$ is generated by a finite set $X$ and is hyperbolic relative to a collection of subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$. Then there exists a constant $L > 0$ such that for any cycle $q$ in $\Gamma(G, X \cup H)$, any $\lambda \in \Lambda$, and any set of isolated $H_\lambda$–components $p_1, \ldots, p_k$ of $q$, we have

$$\sum_{i=1}^k \text{dist}_X((p_i)_-, (p_i)_+) \leq Ll(q).$$

Note that if $p$ is a geodesic path in $\Gamma(G, X \cup H)$, then each component of $p$ is isolated and consists of a single edge. The following lemma is a particular case of Theorem
3.23 from [16]. (In Farb’s approach [11], this is a part of the definition of a relatively hyperbolic group called the Bound ed Coset Penetration property.)

**Lemma 6.** Suppose that a group $G$ is generated by a finite set $X$ and is hyperbolic relative to a collection of subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$. Then for any $s \geq 0$, there exists a constant $\varepsilon = \varepsilon(s) \geq 0$ such that the following condition holds. Let $p_1, p_2$ be two geodesics in $\Gamma(G, X \cup H)$ such that

$$\max\{\text{dist}_X((p_1)_-, (p_2)_-), \text{dist}_X((p_1)_+, (p_2)_+)\} \leq s$$

and let $c$ be a component of $p_1$ such that $\text{dist}_X(c_-, c_+) \geq \varepsilon$. Then there is a component of $p_2$ connected to $c$.

2.3. **Asymptotic dimension.** We also recall some properties of asymptotic dimension used in our paper. The first property is quite obvious and immediately follows from the definition.

**Lemma 7.** If $M_1$ is a metric space and $M_2 \subseteq M_1$ is a subspace endowed with the induced metric, then $\text{asdim } M_2 \leq \text{asdim } M_1$.

Recall that a map $\alpha : M_1 \to M_2$ from a metric space $M_1$ to a metric space $M_2$ is called a quasi–isometry, if there are constants $\lambda > 0$, $c \geq 0$, $\varepsilon \geq 0$ such that the following conditions hold:

1. For any two points $x, y \in M_1$, we have

$$\frac{1}{\lambda} \text{dist}_{M_2}(\alpha(x), \alpha(y)) - c \leq \text{dist}_{M_1}(x, y) \leq \lambda \text{dist}_{M_2}(\alpha(x), \alpha(y)) + c.$$

2. The image $\alpha(M_1)$ is $\varepsilon$–dense in $M_2$, that is, for any $z \in M_2$ there exists $x \in M_1$ such that $\text{dist}_{M_2}(\alpha(x), z) \leq \varepsilon$.

Two metric spaces $M_1$ and $M_2$ are called quasi–isometric, if there exists a quasi–isometry from $M_1$ to $M_2$. It is not hard to check that this is an equivalence relation. The lemma below is also quite obvious (see, for example, [17, Sec. 9.1]).

**Lemma 8.** If $M_1$ and $M_2$ are quasi–isometric, then $\text{asdim } M_1 = \text{asdim } M_2$.

The next two results were proved by Bell and Dranishnikov [1].

**Lemma 9.** Let $M = M' \cup M''$ be a metric space. Then

$$\text{asdim } M = \max\{\text{asdim } M', \text{asdim } M''\}.$$

As pointed out in [1], Lemma 9 can be generalized to certain infinite unions. More precisely, one says that a collection of spaces $\{M_\alpha\}_{\alpha \in \Lambda}$ has asymptotic dimension $\leq n$ uniformly, if for any $r > 0$, there exist coverings $U_\alpha$ of $M_\alpha$ and a constant $d$ such that for all $\alpha$, all $U \in U_\alpha$ have diameter at most $d$ and any ball of radius $r$ in $M_\alpha$ intersects at most $n + 1$ elements of the covering $U_\alpha$. Recall also that two subsets $A, B$ in a metric space are called $s$–separated for some $s > 0$ if $\text{dist}(a, b) \geq s$ for all $a \in A$, $b \in B$.

**Lemma 10.** Suppose that $M$ is a metric space and $M = \bigcup \alpha M_\alpha$, where $M_\alpha$ have asymptotic dimension at most $n$ uniformly. Suppose also that for any $s > 0$, there is $Y_s \in M$ such that $\text{asdim } Y_s \leq n$ and the sets $M_\alpha \setminus Y_s$ (fixed $s$, varying $\alpha$) are $s$–separated. Then $\text{asdim } M \leq n$.

Finally, let $G$ be a group acting on a matric space $M$. Given $x \in M$ and $R \geq 0$, we define an $R$–quasi–stabilizer of $x$ by

$$W_R(x) = \{g \in G \mid \text{dist}(x, gx) \leq R\}.$$

The lemma below appears as Theorem 2 in [1].
Applying Lemma 5 we obtain the inequality

Consequently,

\[ u \] of the assumption that \( p \)

is isolated component of \( q \) the cycle \( H \) (see Fig. 1).

Proof. By Lemma 6, there is a component \( e \) such that \( \varepsilon \) \( \Lambda < \infty \).

We keep the notation for Cayley graphs, word metrics, etc., introduced in the previous section. Speaking on asymptotic dimension of subsets of the group \( G \) we always mean the asymptotic dimension associated to the metric \( \text{dist}_X \).

We begin with an auxiliary result.

**Lemma 11.** Suppose that a finitely generated group \( G \) acts by isometries on a metric space \( M \) such that \( \text{asdim} \, M < m \) and for some \( x \in M \), \( \text{asdim} \, \text{WR}(x) \leq n \). Then \( \text{asdim} \, G \leq (m + 1)(n + 1) - 1 \).

3. **Asymptotic dimension of relative balls**

Throughout the rest of the paper, \( G \) denotes a group that is generated by a finite set \( X \) and is hyperbolic relative to a collection of subgroups \( \{H_\lambda\}_{\lambda \in \Lambda} \), where \( \sharp \Lambda < \infty \).

Let \( \lambda \) be a finitely generated group \( G \), \( X \) is a word in the alphabet \( \{1, \ldots, n\} \) and is hyperbolic relative to a collection of subgroups \( \{H_\lambda\}_{\lambda \in \Lambda} \), where \( \sharp \Lambda < \infty \).

We also denote by \( u \) a path connecting \( (p_1)_+ \) to \( (p_2)_+ \) such that \( \text{Lab} \, (u) \) is a word in the alphabet \( X \) and \( l(u) \leq s \). Consider the cycle \( q = cu[(p_2)_+, t_+] \), where \( [(p_2)_+, t_+] \) is a segment of \( p_2^{-1} \). Note that \( e_2 \) is an isolated component of \( q \). Indeed, \( e_2 \) is isolated in \( [(p_2)_+, t_+] \) since \( p_2 \) is geodesic. Further \( e_2 \) is not connected to \( c \). Indeed otherwise \( e_2 \) is connected to \( t \) that contradicts the assumption that \( p_2 \) is geodesic again. Finally \( e_2 \) can not be connected to a component of \( u \) as \( \text{Lab} \, (u) \) is a word in \( X \) and thus \( u \) contains no components at all.

Since \( p_2 \) is geodesic, we have

\[ l([(p_2)_+, t_+]) \leq l(c) + l(u) \leq s + 1. \]

Consequently,

\[ l(q) = l([(p_2)_+, t_+]) + l(c) + l(u) \leq 2(s + 1). \]

Applying Lemma 11 we obtain the inequality

\[ \text{dist}_X((e_2)_-, (e_2)_+) \leq Ll(q) \leq 2L(s + 1), \]
which contradicts \([5]\). The lemma is proved. \(\square\)

Denote by \(B(n)\) the ball in \(G\) centered at 1 of radius \(n\) with respect to the metric \(\text{dist}_{X \cup \mathcal{H}}\), i.e.,
\[
B(n) = \{ g \in G \mid |g|_{X \cup \mathcal{H}} \leq n \}.
\]
We have already pointed out that \(B(n)\) is not necessarily finite. The main result of this section is the following.

**Lemma 13.** Suppose that all subgroups \(H_\lambda, \lambda \in \Lambda\), have asymptotic dimension at most \(d\). Then for any \(n \in \mathbb{N}\), we have \(\text{asdim} \, B(n) \leq d\).

**Proof.** We proceed by induction on \(n\). For \(n = 1\), we have \(B(1) = X \cup \bigcup_{\lambda \in \Lambda} H_\lambda\).

Since \(X\) and \(\Lambda\) are finite, the inequality \(\text{asdim} \, B(1) \leq d\) follows from Lemma \([9]\).

Now assume \(n > 1\). Then clearly
\[
B(n) = \left( \bigcup_{\lambda \in \Lambda} B(n-1)H_\lambda \right) \cup \left( \bigcup_{x \in X} B(n-1)x \right).
\]

Note that the identity map \(B(n-1) \to B(n-1)\) induces a quasi–isometry from \(B(n-1)\) to \(B(n-1)x\). Hence,
\[
\text{asdim} \left( \bigcup_{x \in X} B(n-1)x \right) \leq d
\]
according to Lemma \([8]\) Lemma \([9]\) and the inductive assumption. Thus it remains to show that
\[
(6) \quad \text{asdim} \, B(n-1)H_\lambda < d
\]
for any \(\lambda \in \Lambda\).

Throughout the rest of the proof we fix an arbitrary \(\lambda \in \Lambda\) and denote by \(R(n-1)\) the subset of \(B(n-1)\) such that for any \(b \in B(n-1)\), we have \(bH_\lambda = gH_\lambda\) for a certain \(g \in R(n-1)\) and \(gH_\lambda \neq fH_\lambda\) whenever \(f, g\) are different elements of \(R(n-1)\). Obviously we have
\[
B(n-1)H_\lambda = \bigcup_{g \in R(n-1)} gH_\lambda.
\]

Let us fix some \(s > 0\) and set
\[
T_s = \{ g \in G \mid |g|_X \leq \max \{ \varepsilon, 2L(s+1) \} \},
\]
where \(\varepsilon = \varepsilon(s)\) and \(L\) are the constants from Lemma \([8]\) and Lemma \([5]\) respectively. Further we define
\[
Y_s = B(n-1)T_s.
\]

Since \(\sharp T_s < \infty\), we have \(\text{asdim} \, Y_s = \text{asdim} \, B(n-1) \leq d\). Let us show that the sets \(gH_\lambda \setminus Y_s, g \in R(n-1)\), are \(s\)-separated.

Suppose that \(x \in g_1H_\lambda \setminus Y_s, y \in g_2H_\lambda \setminus Y_s\) for different \(g_1, g_2 \in R(n-1)\). Then \(x = g_1h_1, y = g_2h_2\) for some \(h_1, h_2 \in H_\lambda \setminus T_s\). Assume that \(\text{dist}_X(x, y) \leq s\). Let \(A_i, i = 1, 2\), denote a shortest word in \(X \cup \mathcal{H}\) representing \(g_i\) in \(G\). Let also \(p_i, i = 1, 2\), denote the path in \(\Gamma(G, X \cup \mathcal{H})\) such that \((p_i)_- = 1\) and \(\text{Lab}(p_i) = A_ih_i\). Clearly \(p_i\) is geodesic in \(\Gamma(G, X \cup \mathcal{H})\). Indeed otherwise we would have
\[
|g_ih_i|_{X \cup \mathcal{H}} = \text{dist}_{X \cup \mathcal{H}}((p_i)_-, (p_i)_+) < l(p_i) = |A_i| + 1 = |g_i|_{X \cup \mathcal{H}} + 1 = n
\]
and hence \( g_ih_i \in B(n-1) \subseteq Y_s \) that contradicts our assumption. Note also that

\[
dist_X((p_1)_+, (p_2)_+) = dist_X(x, y) \leq s.
\]

As \( h_i \notin T_s \), we have \( |h_i|_X > \max\{\varepsilon, 2L(s+1)\} \) for \( i = 1, 2 \). Therefore, the \( H_\lambda \)-components of \( p_1 \) and \( p_2 \) labelled \( h_1 \) and \( h_2 \) respectively are connected by Lemma 12. This means that \( g_1H_\lambda = g_2H_\lambda \) contradicts the choice of \( R(n-1) \).

Thus the sets \( gH_\lambda \setminus Y_s, g \in R(n-1) \), are \( s \)-separated. To complete the proof of (4) it remains to apply Lemma 10. \( \square \)

### 4. Geodesic triangles in \( \Gamma(G, X \cup \mathcal{H}) \)

Let \( \Delta = \Delta(x, y, z) \) be a geodesic triangle in a metric space with vertices \( x, y, z \). The triangle inequality tells us that there exist (unique) points \( a \in [y, z] \), \( b \in [z, x] \), and \( c \in [x, y] \) such that \( dist(x, b) = dist(x, c) \), \( dist(y, a) = dist(y, c) \), and \( dist(z, a) = dist(z, b) \). Recall that two points \( u \in [x, b] \) and \( v \in [x, c] \) are conjugate if \( dist(u, v) = dist(x, v) \). In the same way one defines conjugate points on pairs of segments \( [y, a] \), \( [y, c] \) and \( [z, a] \), \( [z, b] \). The triangle \( \Delta \) is said to be \( \xi \)-thin if \( dist(u, v) \leq \xi \) for any two conjugate points \( u, v \in \Delta \).

The following observation is quite obvious. We leave the proof to the reader.

**Lemma 14.** Let \( \Delta(x, y, z) \) be a geodesic triangle in a metric space. Suppose that \( u \in [x, y] \), \( v \in [x, z] \), \( dist(x, u) = dist(x, v) \), and

\[
dist(u, y) + dist(v, z) \geq dist(y, z).
\]

Then \( u \) and \( v \) are conjugate.

The next lemma provides an equivalent definition of hyperbolicity. (see, for example, [5], Ch. III.H, Prop. 1.17).

**Lemma 15.** A geodesic metric space \( M \) is hyperbolic if and only if there exists \( \xi \geq 0 \) such that every geodesic triangle in \( M \) is \( \xi \)-thin.

In particular, any geodesic triangle in \( \Gamma(G, X \cup \mathcal{H}) \) is \( \xi \)-thin for some \( \xi = \xi(G) \). For our goals we need a stronger result stating that geodesic triangles in \( \Gamma(G, X \cup \mathcal{H}) \) are thin with respect to the metric \( dist_X \) associated to the finite generating set \( X \). To simplify our exposition we do not prove this result in the full generality and restrict ourselves to the following particular case, which is sufficient for our goal.

**Lemma 16.** There exist constants \( \rho, \sigma > 0 \) having the following property. Let \( \Delta \) be a triangle with vertices \( x, y, z \), whose sides \( [x, y], [y, z], [x, z] \) are geodesics in \( \Gamma(G, X \cup \mathcal{H}) \). Suppose that \( u \) and \( v \) are vertices on \( [x, y] \) and \( [x, z] \) respectively such that

\[
dist_X(x, u) = dist_X(x, v)
\]

and

\[
dist_X(u, y) + dist_X(v, z) \geq dist_X(y, z) + \sigma.
\]

Then

\[
dist_X(u, v) \leq \rho.
\]

**Proof.** We set

\[
\sigma = 5\xi,
\]

...
where $\xi$ is the constant provided by Lemma 15 for $M = \Gamma(G, X \cup H)$. Note that $u$ and $v$ are conjugate according to Lemma 14 and the inequality $\eqref{en5}$. We denote by $p$ a geodesic in $\Gamma$ such that $p_- = u$, $p_+ = v$. By the choice of $\xi$ we have

\begin{equation}
\ell(p) \leq \xi. \tag{11}
\end{equation}

Further let $u_1 \in [x, u]$, $v_1 \in [x, v]$ be the vertices chosen as follows. If $\dist_{X \cup H}(x, u_1) = \dist_{X \cup H}(x, v_1) < 2\xi$, we set $u_1 = v_1 = x$. Otherwise $u_1$, $v_1$ are uniquely defined by the equality

\begin{equation}
\dist_{X \cup H}(u_1, u) = \dist_{X \cup H}(v_1, v) = 2\xi. \tag{12}
\end{equation}

Obviously $u_1$ and $v_1$ are conjugate. Similarly let $u_2$ and $v_2$ be the vertices on the segments $[u, y]$ and $[v, z]$ respectively such that

\begin{equation}
\dist_{X \cup H}(u_2, u) = \dist_{X \cup H}(v_2, v) = 2\xi. \tag{13}
\end{equation}

Note that such vertices always exist. Indeed the inequalities $\eqref{en5}$ and $\eqref{en11}$ imply

\[
\dist_{X \cup H}(u, y) \geq \frac{1}{2} (\dist_{X \cup H}(u, y) + \dist_{X \cup H}(v, z) - \dist_{X \cup H}(y, z) - \dist_{X \cup H}(u, v)) \\
\geq \frac{1}{2} (\sigma - \xi) = 2\xi
\]

and similarly $\dist_{X \cup H}(v, z) \geq 2\xi$. The vertices $u_2$ and $v_2$ are conjugate by Lemma 14 since

\[
\dist_{X \cup H}(u_2, y) + \dist_{X \cup H}(v_2, z) = \dist_{X \cup H}(u, y) + \dist_{X \cup H}(v, z) - 4\xi \\
\geq \dist_{X \cup H}(y, z)
\]

according to $\eqref{en5}$ and $\eqref{en11}$. We denote by $o_1$ and $o_2$ geodesic paths in $\Gamma(G, X \cup H)$ such that $(o_1)_- = u_1$, $(o_1)_+ = v_1$, $(o_2)_- = u_2$, $(o_2)_+ = v_2$. By Lemma 16 we have

\begin{equation}
\ell(o_i) \leq \xi, \quad i = 1, 2. \tag{14}
\end{equation}

Let us consider an arbitrary $H_\lambda$-component $s$ of $p$ for some $\lambda \in \Lambda$. In order to obtain an upper bound on the distance $\dist_{X}(s_-, s_+)$, we are going to show that $s$ is an isolated component in at least one of the cycles

\[\ell(o_i) \leq \xi, \quad i = 1, 2.\]

First we note that $s$ cannot be connected to a component of $o_1$ or $o_2$. Indeed if, for example, $s$ is connected to a component $t$ of $o_1$ (see Fig. 2), then $\dist_{X \cup H}(t_-, s_-) \leq 1$.\n
\begin{figure}[ht]
\centering
\includegraphics[width=\textwidth]{asymptotic_dim_2}
\caption{Figure 2.}
\end{figure}
Taking into account (14) we obtain
\[ \text{dist}_{X \cup H}(u_1, u) \leq \text{dist}_{X \cup H}(u_1, t_-) + \text{dist}_{X \cup H}(t_-, s_-) + \text{dist}_{X \cup H}(s_-, u) \leq (l(o_1) - 1) + 1 + (l(p) - 1) < 2\xi. \]

According to the choice of \( u_1 \) and \( v_1 \), this means \( u_1 = v_1 = x \). However, in this case \( o_1 \) is trivial and can not contain components. A similar argument shows that \( s \) can not be connected to a component of \( o_2 \).

Further assume that there are edges \( a \) and \( b \) of the cycles \( c_1 \) and \( c_2 \) respectively labelled by elements of \( H_\lambda \) such that \( a \neq s \), \( b \neq s \), and \( a_\pm \) and \( b_\pm \) are connected to \( s_\pm \) by paths of lengths at most 1 labelled by elements of \( H_\lambda \). As shown above, \( a \), \( b \) can not belong to \( o_1 \) or \( o_2 \). Since \( p \) is geodesic, \( a \) and \( b \) can not belong to \( [x, y] \) (or \( [x, z] \)) simultaneously. Thus the only possibility is \( a \in [x, y] \), \( b \in [x, z] \) (or conversely). For definiteness, we assume \( a \in [x, y] \), \( b \in [x, z] \) (see Fig. 3). In this case we have
\[ \text{dist}_{X \cup H}(x, v) \leq \text{dist}_{X \cup H}(x, b_+) - 1 \leq \text{dist}_{X \cup H}(x, a_-) + \text{dist}_{X \cup H}(a_-, b_+) - 1 \leq \text{dist}_{X \cup H}(x, u) \]

contradictory (17).

Thus each component of \( p \) is isolated in at least one of the cycles \( c_1 \), \( c_2 \). Combining (11)–(14) yields \( l(c_i) \leq 6\xi \) for \( i = 1, 2 \). Applying Lemma 5 we obtain the inequality \( \text{dist}_{X}(s_-, s_+) \leq 6L\xi \) for any component \( s \) of \( p \). Without loss of generality we may assume \( 6L\xi \geq 1 \). Thus,
\[ \text{dist}_{X}(u, v) \leq 6L\xi l(p) \leq 6L\xi^2 \]
and the inequality (9) holds for \( \rho = 6L\xi^2 \). \( \square \)

5. Proof of the main theorem

We begin by proving the following result, which seems to be of independent interest. We stress that it does not follow from known results concerning asymptotic dimensions of hyperbolic graphs since, in general, the graph \( \Gamma(G, X \cup H) \) is not locally finite.

**Theorem 17.** The Cayley graph \( \Gamma(G, X \cup H) \) has finite asymptotic dimension (with respect to the metric \( \text{dist}_{X \cup H} \)).
Proof. For every $r > 0$, we construct a covering of $\Gamma(G, X \cup H)$ as follows. Let

$$A_k = \{g \in G \mid 2kr \leq |g| \leq 2(k+1)r\}$$

and

$$S_k = \{g \in G \mid |g| = 2kr\},$$

where $k = 0, 1, \ldots$. To each element $g \in A_k$, $k = 1, 2, \ldots$, we associate an arbitrary geodesic $\gamma_g$ in the Cayley graph $\Gamma(G, X \cup H)$ connecting $g$ to $1$, and denote by $t_g$ the vertex $\gamma_g \cap S_{k-1}$. Consider the collection

$$U(r) = \{U_k(x) \mid k \in \mathbb{N}, x \in S_{k-1}\},$$

where $U_k(x) = \{g \in A_k \mid t_g = x\}$.

Obviously the collection $U(r) \cup \{A_0\}$ covers $G$. Further if $g_1, g_2 \in U_k(x)$ for some $k \in \mathbb{N}, x \in X_k$, we have

$$\text{dist}_{X \cup H}(g_1, g_2) \leq \text{dist}_{X \cup H}(g_1, x) + \text{dist}_{X \cup H}(x, g_2) \leq 4r.$$

Thus for every $r$, $U(r)$ is uniformly bounded. Finally let $\sigma$ denote the constant provided by Lemma \[10\] and let $r \geq \sigma$. We consider a ball $B(a, r) = \{g \in G \mid \text{dist}_{X \cup H}(g, a) \leq r\}$, where $a \in G$, and assume that $B(a, r) \cap U_k(x) \neq \emptyset$ for some $k \in \mathbb{N}, x \in S_{k-1}$. Let $y \in B(a, r) \cap U_k(x)$. Then

$$\text{dist}_{X \cup H}(1, a) \geq \text{dist}_{X \cup H}(1, y) - \text{dist}_{X \cup H}(y, a) \geq 2kr - r.$$
We fix an arbitrary geodesic $\gamma$ in $\Gamma(G, X \cup \mathcal{H})$ going from $a$ to $1$ and denote by $s$ the vertex on $\gamma$ such that $\text{dist}_{X \cup \mathcal{H}}(1, s) = 2kr - 2r$. Observe that
\[
\text{dist}_{X \cup \mathcal{H}}(x, y) + \text{dist}_{X \cup \mathcal{H}}(s, a) \geq 3r > \text{dist}_{X \cup \mathcal{H}}(y, a) + \sigma.
\]
Therefore, by Lemma 16 $\text{dist}_X(x, s) \leq \rho$. Thus for any fixed $k \in \mathbb{N}$, $B(a, r)$ meets at most $\mu$ subsets $U_k(x)$, where $\mu = \sharp \{g \in G, \ | g|_X \leq \rho\}$. Since $B(a, r)$ intersects at most three annuli $A_k$, we have asdim $\Gamma(G, X \cup \mathcal{H}) \leq 3\mu$. \hfill $\Box$

Now we are ready to prove the main result of our paper.

**Proof of Theorem 2** The group $G$ acts on $\Gamma(G, X \cup \mathcal{H})$ by left multiplication. Obviously the $R$–quasi–stabilizer $W_R(1)$ coincides with the ball $B(R)$ of radius $R$ with respect to $\text{dist}_{X \cup \mathcal{H}}$ centered at the identity. It remains to combine Theorem 17, Lemma 13, and Lemma 14. \hfill $\Box$

6. **Boundedly generated groups of infinite asymptotic dimension**

Recall that a group $G$ is called *boundedly generated* if there are elements $x_1, \ldots, x_n$ of $G$ such any $g \in G$ can be represented in the form
\[
g = x_1^{\alpha_1} \cdots x_n^{\alpha_n}
\]
for some $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}$. This obviously implies that the Cayley graph $\Gamma(G, X \cup \mathcal{H})$ has finite diameter for any generating set $X$ and $H_\lambda = \langle x_\lambda \rangle, \lambda \in \{1, \ldots, n\}$. Thus Proposition 3 is a corollary of the following result.

**Proposition 18.** There exists a finitely presented boundedly generated group of infinite asymptotic dimension.

**Proof.** Recall that a group is called *universal* if it contains an isomorphic copy of any recursively presented group. The existence of finitely presented universal groups was first proved by Higman (see [14, Ch. IV, Theorem 7.3]). Let $U$ denote a finitely presented universal group generated by a finite (symmetric) set $X$.

We set
\[
G_1 = U \ast \langle a_1 \rangle \ast \cdots \ast \langle a_n \rangle.
\]
Let us enumerate all words $\{w_1, w_2, \ldots\}$ in the alphabet $X$ and consider the set
\[
\mathcal{R} = \{w_i^{-1}a_i^i \cdots a_n^i, \ i \in \mathbb{N}\}.
\]
It is easy to see that $\mathcal{R}$ satisfies the $C'(\lambda)$ small cancellation condition over the free product (15), where $\lambda \to 0$ as $n \to \infty$. (For the definition we refer the reader to [14, Ch. V, Sec. 9].) In particular, if $n$ is big enough, $\mathcal{R}$ satisfies $C'(1/6)$ over the free product (15) and hence $U$ embeds into the quotient group
\[
G_2 = \langle G_1 \mid R = 1, R \in \mathcal{R} \rangle
\]
(see [14, Ch. V, Corollary 9.4]).

Note that the presentation (16) is recursive. As $U$ is universal, it contains an isomorphic copy of $G_2$. Let $\alpha: G_2 \to U$ be the monomorphism that maps $G_2$ to its copy in $U \leq G_2$. By $G$ we denote the ascending HNN–extension
\[
G = \langle G_2, t \mid g^t = \alpha(g), g \in G_2 \rangle.
\]

Let $g$ be an arbitrary element of $G$. Observe that the subgroup
\[
N = \bigcup_{j=1}^{\infty} t^jG_2t^{-j} = \bigcup_{j=1}^{\infty} t^jUt^{-j}
\]
is normal in $G$ and thus the kernel of the natural homomorphism $G \to \langle t \rangle$ coincides with $N$. Therefore there exist $j, k \in \mathbb{N}$ such that $t^{-j}(gt^k)t^j \in U$. Furthermore, any element of $U$ can be represented as a product $a_1^j \cdots a_n^j$ for a certain $i$ according to the relations $R = 1$, $R \in \mathcal{R}$. Hence any element $g \in G$ can be represented as

$$g = t^{\alpha_1} a_1^i \cdots a_n^i t^{\alpha_2}$$

for some $\alpha_1, \alpha_2, i \in \mathbb{Z}$. In particular, the group $G$ is boundedly generated.

One can also observe that $G$ is finitely presented. Indeed expanding the presentation (17) we obtain

$$G = \langle a_1, \ldots, a_n, t, U \mid R = 1, R \in \mathcal{R}, g^t = \alpha(g), g \in G_2 \rangle.$$  

Since $(G_2)^i \leq U$ and words from the set $\mathcal{R}$ involve elements of $G$ only, all relations of the form $R = 1, R \in \mathcal{R}$, follow from the relations of the group $U$. Thus we can omit the set of relations $R = 1, R \in \mathcal{R}$ in (18) and get a finite presentation for $G$. Finally we notice that $\text{asdim } G = \infty$ since any universal group contains an isomorphic copy of $\mathbb{Z}^m$ for all $m \in \mathbb{N}$ and $\text{asdim } \mathbb{Z}^m = m$. □

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