Abstract

In this paper we give local and global parametric classifications of a class of Einstein submanifolds of Euclidean space. The highlight is for submanifolds of codimension two since in this case the assumptions are only of intrinsic nature.

Let \( f : M^n \to \mathbb{R}^{n+p} \) denote an isometric immersion of an \( n \)-dimensional Riemannian manifold into Euclidean space with codimension \( p \). The goal of this paper is to classify parametrically certain classes of these submanifolds for which \( M^n \) is an Einstein manifold of non-constant sectional curvature. Recall that a Riemannian manifold \( M^n \) is said to be \textit{Einstein} if its Ricci tensor is proportional to the metric, that is, if

\[
\text{Ric}_M(X,Y) = \rho \langle X,Y \rangle
\]

for any vector fields \( X, Y \in \mathfrak{X}(M) \) and some constant \( \rho \in \mathbb{R} \). Hence \( \rho \) is the (not normalized) constant Ricci curvature of \( M^n \).

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We focus on manifolds of dimension \( n \geq 4 \) since 3-dimensional Einstein manifolds have constant sectional curvature. Besides, we are interested in codimension \( p \geq 2 \) since, due to an observation by Cartan communicated by Thomas [19] in 1937 and the work of Fialkow [12] from 1938, we know that an Einstein hypersurface \( f : M^n \to \mathbb{R}^{n+1}, n \geq 3 \), is either flat or an open subset of a round sphere; see [9] for a proof. In particular, if \( M^n \) is a complete manifold, then the submanifold is either a cylinder over a complete plane curve or a round sphere.

At the present time, the knowledge on the subject of Euclidean Einstein submanifolds, except those with constant sectional curvature, is quite limited. This is because the assumption that a submanifold is Einstein is much weaker than having constant sectional curvature. In fact, as far as we know, until now the only classification result available under purely intrinsic assumptions is the aforementioned for hypersurfaces. Hence, in order to obtain a meaningful classification one has to require some strong additional hypothesis. For instance, a classification was obtained by Onti [16] under the extrinsic assumptions that the submanifold has parallel mean curvature vector and flat normal bundle.

It has been shown by Dajczer, Onti and Vlachos [10] that any Einstein Euclidean submanifold with flat normal bundle is locally holonomic. This means that the manifold carries a system of orthogonal coordinates such that the coordinate vector fields diagonalize the second fundamental form of the immersion at any point. This is a strong conclusion since, for instance, it allows to express additional assumptions on the submanifold in terms of partial differential equations in a system of special coordinates.

The attempt in [20] to obtain a classification of the Einstein submanifolds \( f : M^4 \to \mathbb{R}^6 \) with flat normal bundle only yielded all possible pointwise structures of the second fundamental form. But even in this rather special case there are many possibilities for the second fundamental form, as can be seen in the Appendix of this paper. Somehow, it is not a surprise that there is not a description given, outside of some very simple cases, of the actual isometric immersions that carry these second fundamental forms.

Warped products of Riemannian manifolds have been intensively used to construct interesting classes of Einstein manifolds. The class of those that have a surface as base was discussed in Chapter 9 of Besse [2]. The main purpose of this paper is to classify, locally or globally, the Euclidean Einstein submanifolds in codimension two that belong to this class. A classification is also given for higher codimension, but now under an extrinsic assumption.
Theorem 9.119 in [2], which is stated without a proof, claims a classification of the complete Einstein manifolds that are warped products of the form $M^n = L^2 \times_\varphi N^{n-2}$, $n \geq 4$. It is said that the manifold is either a Riemannian product (i.e., the warping function $\varphi$ is constant) or it belongs to one of four families being $L^2 = \mathbb{R}^2$ and $N^{n-2}$ an Einstein manifold. Some additional information about these examples can be seen in the Appendix of [5]. Although at the present time there is no proof in the literature, experts in the field believe that the claim in [2] is true.

The complete Einstein manifolds that fit our purposes are given next.

The Clifford tori. If $\rho > 0$ we have the Riemannian product of manifolds

$$M^n = \mathbb{S}^2(1/\sqrt{\rho}) \times \mathbb{S}^{n-2}(\sqrt{(n-3)/\rho})$$

where $\mathbb{S}^m(r)$ denotes an $m$-dimensional sphere of radius $r$.

The Ricci flat Generalized Schwarzschild metric. The examples a) in §9.118 of [2] are of the form

$$M^n = \mathbb{R}^2 \times_\varphi N^{n-2}$$

where $N^{n-2}$ is an Einstein manifold of Ricci curvature $\rho = n-3$. The precise description of the Ricci flat metric on $M^n$ is given in the next section. It was called in [2] and [5] the Generalized Schwarzschild metric since for $n = 4$ and $N^2 = \mathbb{S}^2(1)$ it has been named the Riemannian Schwarzschild metric in analogy to the well-known Lorentzian version.

Examples 2 given in the last section shows that $M^n = \mathbb{R}^2 \times_\varphi \mathbb{S}^{n-2}(1)$ endowed with a Generalized Schwarzschild metric can always be isometrically immersed in Euclidean space with codimension two as an $(n-2)$-rotational submanifold. Recall that an $(n-2)$-rotational submanifold $f : M^n \to \mathbb{R}^{n+p}$, $n \geq 3$, with axis $\mathbb{R}^{p+1}$ over a surface $g : L^2 \to \mathbb{R}^{p+2}$ is the $n$-dimensional submanifold generated by the orbits of the points of $g(L)$ (disjoint from $\mathbb{R}^{p+1}$) under the action of the subgroup $SO(n-1)$ of $SO(n+p)$ which keeps pointwise $\mathbb{R}^{p+1}$ invariant.

Throughout this paper, we denote by $F^m(\varepsilon)$, $m \geq 2$, an Einstein manifold with Ricci curvature $(m-1)\varepsilon$, $\varepsilon = 1, -1, 0$, (that is, with normalized Ricci curvature $\varepsilon$). This the case of the sphere $\mathbb{S}^m(1)$, the hyperbolic space $\mathbb{H}^m(-1)$ and the Euclidean space $\mathbb{R}^m$.

In the realm of complete manifolds the following is the main result of the present paper.
Theorem 1. Let \( M^n = L^2 \times \varphi F^{n-2}(\varepsilon) \), \( n \geq 5 \), be a complete simply connected Einstein manifold whose sectional curvature is not constant on any open subset. If \( F^{n-2}(\varepsilon) \) has constant sectional curvature and \( f : M^n \to \mathbb{R}^{n+2} \) is an isometric immersion, then one of the following holds:

(i) \( M^n = S^2(r_1) \times S^{n-2}(r_2) \) is a Clifford torus as in (1) and \( f \) is the product of inclusions \( S^{n_j}(r_j) \subset \mathbb{R}^{n_j+1} \), \( n_j = 2, n-2 \).

(ii) \( M^n = \mathbb{R}^2 \times \varphi S^{n-2}(1) \) is endowed with the Generalized Schwarzschild metric and \( f \) is an \( (n-2) \)-rotational submanifold given by Examples 2.

If \( M^n \) is not simply connected, then the composition of \( f \) with its universal cover has the same image as \( f \). A key ingredient in our proof of the above global theorem is a local classification result that characterizes a class of Einstein \( (n-2) \)-rotational submanifolds in codimension two with arbitrary Ricci curvature; see our Theorem 9 below.

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1 A class of Einstein manifolds

Warped products of Riemannian manifolds have been used to construct interesting classes of Einstein manifolds. The ones that have a surface as base are discussed in Chapter 9 of Besse [2]. A few facts from this book are exposed in this section.

Proposition 2. The warped product \( M^n = L^2 \times \varphi F^{n-2}(\varepsilon) \), \( n \geq 4 \), is an Einstein manifold of Ricci curvature \( \rho \) if and only if the warping function \( \varphi \in C^\infty(L) \) satisfies the equations

\[
(n-2)\text{Hess}\varphi = (K - \rho)\varphi I
\]

and

\[
\Delta \varphi + \frac{n-3}{\varphi}(\|\nabla\varphi\|^2 - \varepsilon) + \rho \varphi = 0
\]

where \( K \) is the Gaussian curvature of \( L^2 \) and \( \text{Hess}\varphi \) is the endomorphism of \( TL \) determined by the Hessian of \( \varphi \). Moreover, equations (2) and (3) for non-constant solutions \( \varphi > 0 \) are equivalent to the equation

\[
\text{Hess}\varphi = \frac{1}{2}(2\Delta \varphi + \frac{n-3}{\varphi}(\|\nabla\varphi\|^2 - \varepsilon) + \rho \varphi)I.
\]
Proof. Equations (2) and (3) are, respectively, a particular case of equations 9.107 b) and 9.107 c) in [2]. The equivalence with (4) was observed in §9.116 of [2].

The following result is Lemma 9.17 in [2] but can also be seen in [14] and [18]. It states that the existence of a non-constant solution of equation (1) implies that $L^2$ is locally an “intrinsic surface of rotation”, namely, it has a warped product structure.

**Proposition 3.** If equation (4) admits a non-constant solution $\varphi \in C^\infty(L)$ then there exist local coordinates $(t,u)$ on $L^2$ such that $\varphi = \varphi(t)$, the metric of $L^2$ has the form $ds^2 = dt^2 + \varphi'^2 du^2$ and $\text{Hess} \varphi = \varphi'' I$.

Equation (4) in the coordinates given by Proposition 3 has the form

$$2\varphi\varphi'' + (n - 3)(\varphi'^2 - \varepsilon) + \rho\varphi^2 = 0.$$  \hspace{1cm} (5)

Then, multiplying by $\varphi'\varphi^{n-4}$ and integrating gives that

$$\varphi'^2 = \varepsilon - \frac{\rho}{n - 1}\varphi^2 + \frac{c}{\varphi^{n-3}}$$ \hspace{1cm} (6)

where $c \in \mathbb{R}$ is a constant.

We summarize the above in the following statement.

**Proposition 4.** Let $M^n = L^2 \times_\varphi F^{n-2}(\varepsilon)$ be a warped product such that $\nabla \varphi \neq 0$ at any point. If $L^2$ in the coordinates $(t,u)$ carries the metric

$$ds^2 = dt^2 + \varphi'^2 du^2,$$ \hspace{1cm} (7)

where $\varphi$ is a solution of (6), then $M^n$ is an Einstein manifold with Ricci curvature $\rho$. Conversely, if $M^n$ is an Einstein manifold with Ricci curvature $\rho$ then there are local coordinates $(t,u)$ on $L^2$ such that the metric is as in (7) where $\varphi$ solves (6).

The Gauss curvature of $L^2$ endowed with the metric (7) is $K = -\varphi''/\varphi'$. It is easily seen that $L^2$ has constant Gauss curvature $K = \rho/(n - 1)$ if and only if $\varphi$ is a solution of equation (6) for $c = 0$. If this is the case and if $F^{n-2}(\varepsilon)$ has constant sectional curvature then it can be easily proved that $M^n = L^2 \times_\varphi F^{n-2}(\varepsilon)$ has constant sectional curvature $K$.  

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Remark 5. We observe that equation (5) is just equation (3). If willing to verify that any solution of equation (5) is also a solution of equation (2), take the derivative of (5) and then use that $K = -\varphi''/\varphi'$.

We now turn our attention to the case of Einstein manifolds with complete metrics in this way wrapping up the discussion started in the previous section. Recall that a warped product of two Riemannian manifolds is complete if and only if both factors are complete. A fact relevant to this paper is that a complete manifold can be represented by a warped product where one of the factor is not complete. For instance, this is the case of polar coordinates for Euclidean space.

The warped product $M^n = \mathbb{R}^2 \times \varphi F^{n-2}(1)$ is said to be endowed with the Ricci flat Generalized Schwarzschild metric when $\mathbb{R}^2$ has the rotationally invariant warped metric $ds^2 = dt^2 + \varphi^2(t)d\theta^2$ in polar coordinates being $\varphi \in C^\infty([0, +\infty))$ the unique positive solution of (6) (with $\rho = 0$) for a given initial condition and a given constant $c < 0$.

2 The local case

We first show that Einstein manifolds that belong to the class discussed above may admit local isometric immersions in Euclidean space as $(n-2)$-rotational submanifolds with codimension at least two.

Examples 1. Observe that an $(n-2)$-rotational submanifold $f: M^n \to \mathbb{R}^{n+p}$ can be parametrized as follows. The manifold $M^n$ is isometric to (an open subset of) a warped product $L^2 \times \varphi S^{n-2}(1)$ and there is an orthogonal splitting $\mathbb{R}^{p+2} = \mathbb{R}^{p+1} \oplus \text{span}\{e\}$, $\|e\| = 1$, such that the profile $g$ of $f$ has the form $g = (h, \varphi)$ where $h: L^2 \to \mathbb{R}^{p+1}$ and $\varphi = \langle g, e \rangle > 0$. Then

$$f(x, y) = (h(x), \varphi(x)\phi(y))$$

(8)

where $\phi: S^{n-2}(1) \to \mathbb{R}^{n-1}$ denotes the inclusion $S^{n-2}(1) \subset \mathbb{R}^{n-1}$.

(a) Given $\rho \in \mathbb{R}$ choose an initial condition for $\varphi$ and a value for $c$ such that the right hand side of the differential equation (6) is positive and less than one. Then, on some open interval $I \subset \mathbb{R}$ there is a solution $\varphi \in C^\infty(I)$ of (6) that satisfies $0 < \varphi^2(t) < 1$. Endow $N^2 = I \times I_0$, where $I_0 \subset \mathbb{R}$ is also an open interval, with the warped metric

$$ds^2 = (1 - \varphi^2(t))dt^2 + \varphi^2(t)du^2.$$
Given \( p \geq 2 \) let \( h: U \to \mathbb{R}^{p+1} \) be an isometric immersion of an open subset \( U \) of \( N^2 \). The metric induced by the immersion \( g: U \to \mathbb{R}^{p+2} \) given by

\[
g(t, u) = (h(t, u), \varphi(t))
\]
is \( ds^2 = dt^2 + \varphi^2(t)du^2 \).

Now let \( f: M^n \to \mathbb{R}^{n+p} \) be the immersion \( f \) where \( M^n = U \times \varphi S^{n-2}(1) \). By Proposition 4 we have that \( f \) is an \((n-2)\)-rotational Einstein submanifold with Ricci curvature \( \rho \) whose profile is \( g = (h, \varphi) \).

Now assume that \( p = 2 \). The normal bundles of \( h \) and \( g \) are related by the orthogonal splitting

\[ N_g L = N_h L \oplus \text{span}\{\delta = (-\varphi' h_t, 1 - \varphi'^2)\}. \]

Since

\[
(\tilde{\nabla}_{\partial t} \delta)_{T_g L} = (-\varphi'' h_t - \varphi' h_{tt}, -2\varphi' \varphi'')_{T_g L} = -\varphi'' g_t
\]

and

\[
(\tilde{\nabla}_{\partial u} \delta)_{T_g L} = (-\varphi' h_{tu}, 0)_{T_g L} = -\varphi'' g_u,
\]

we have that the second fundamental form of \( g \) satisfies \( A^g_\delta = \varphi'' I \), thus \( g \) has flat normal bundle. On the other hand, from Lemma 2.4 of [11] (or just by computing the second fundamental form) we know that an \((n-2)\)-rotational submanifold has flat normal bundle if and only if the profile has this property. Hence, in this case \( f \) given by \( h \) has flat normal bundle.

(b) Assume that \( \varphi \in C^\infty(I) \) satisfies the stronger condition

\[
\varphi^2(t) + \varphi'^2(t) < 1, \quad t \in I.
\]

Let \( g: L^2 = I \times [0, 2\pi) \to \mathbb{R}^4 \) be the surface of rotation defined by

\[
g(t, \theta) = (\psi(t), \varphi'(t) \sin \theta, \varphi'(t) \cos \theta, \varphi(t))
\]

where \( \psi \in C^\infty(I) \) is determined by \( \psi'^2 = 1 - \varphi^2 - \varphi'^2 \). Then, we have that \( f: L^2 \times \varphi S^{n-2}(1) \to \mathbb{R}^{n+2} \) given by

\[
f(t, \theta, y) = (\psi(t), \varphi'(t) \sin \theta, \varphi'(t) \cos \theta, \varphi(t) \phi(y))
\]
is an \((n-2)\)-rotational Einstein submanifold with Ricci curvature \( \rho \) and flat normal bundle whose profile is \( g \).
A submanifold $f : M^n \to \mathbb{R}^{n+p}$, $n \geq 4$, is called \((n - 2)\)-umbilical if its tangent bundle carries a maximal \((n - 2)\)-dimensional umbilical distribution $\mathcal{U}$. Thus there exists a smooth non vanishing principal normal vector field $\eta$ such that

$$
\mathcal{U} = \{ X \in TM : \alpha(X, Y) = \langle X, Y \rangle \eta \text{ for all } Y \in TM \}
$$

where $\alpha : TM \times TM \to N_f M$ is the second fundamental form of $f$. Since $\dim \mathcal{U} \geq 2$ it is well-known that $\eta$ is a Dupin principal normal. This means that $\mathcal{U}$ is integrable being the leaves round spheres and $\eta$ is parallel in the normal connection along the leaves. The class of \((n - 2)\)-umbilical submanifolds has shown up in the literature in rather different geometric situations, for instance see [1], [6], [7] and [15].

The \((n - 2)\)-rotational submanifolds described earlier are the simplest examples of \((n - 2)\)-umbilical submanifolds. From either Lemma 6 in [6] or Proposition 10 in [7] we have the following characterization.

**Proposition 6.** Let $f : M^n \to \mathbb{R}^{n+p}$ be an \((n - 2)\)-umbilical submanifold. If the orthogonal distribution $\mathcal{U}^\perp$ to the umbilical distribution $\mathcal{U}$ is totally geodesic in $M^n$ then $f$ is locally an \((n - 2)\)-rotational submanifold.

Next we characterize a class of Einstein submanifolds with arbitrary codimension under an assumption of extrinsic nature.

**Theorem 7.** Let $f : M^n \to \mathbb{R}^{n+p}$, $n \geq 4$, be an \((n - 2)\)-umbilical isometric immersion of an Einstein manifold of Ricci curvature $\rho$ which does not have constant sectional curvature on any open subset. Then $p \geq 2$ and each point of an open dense subset of $M^n$ has an open neighborhood $V \subset M^n$ where one of the following holds:

(i) $V = U \times W$ is part of a Clifford torus and $f|_V = g \times i$ is the product of an isometric immersion $g : U \subset \mathbb{S}^2(1/\sqrt{\rho}) \to \mathbb{R}^{p+1}$ and the inclusion $i : W \subset \mathbb{S}^{n-2}(\sqrt{(n-3)/\rho}) \to \mathbb{R}^{n-1}$.

(ii) $f|_V$ is an \((n - 2)\)-rotational submanifold as in part (a) of Examples 1.

**Proof.** Since the dimension of the umbilical distribution in the assumption is maximal having this property, it follows from the Cartan-Fialkow result discussed in the introduction that $p \geq 2$. 

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We show next that the distribution $\mathcal{U}^\perp$ is totally geodesic in $M^n$. From the Gauss equation, we have

$$\text{Ric}(X, Y) = n\langle \alpha(X, Y), H \rangle - \sum_{i=1}^{n} \langle \alpha(X, X_i), \alpha(Y, X_i) \rangle$$

where \{$X_1, \ldots, X_n$\} is an orthonormal frame and $H = \frac{1}{n} \sum_{i=1}^{n} \alpha(X_i, X_i)$ the mean curvature vector field of the submanifold.

In the sequel, we denote $\alpha_{ij} = \alpha(e_i, e_j)$, $1 \leq i, j \leq 2$, where \{$e_1, e_2$\} is an orthonormal frame spanning $\mathcal{U}^\perp$.

(i) For $X = Y = e_i$, we obtain

$$\rho + \|\alpha_{ii}\|^2 + \|\alpha_{ij}\|^2 = n\langle \alpha_{ii}, H \rangle, \ i \neq j.$$  

Equivalently, we have

$$\rho + \|\alpha_{ii}\|^2 + \|\alpha_{ij}\|^2 = \langle \alpha_{ii}, \alpha_{ii} + \alpha_{jj} + (n - 2)\eta \rangle, \ i \neq j,$$

where $\eta$ is the principal normal vector field. Hence

$$\rho - K(\mathcal{U}^\perp) = (n - 2)\langle \alpha_{ii}, \eta \rangle \ (9)$$

where $K(\mathcal{U}^\perp)$ denotes the sectional curvature. In particular, we have

$$\langle \alpha_{11}, \eta \rangle = \langle \alpha_{22}, \eta \rangle. \ (10)$$

Then, since the frame \{$e_1, e_2$\} is arbitrary, we also have

$$\langle \alpha_{12}, \eta \rangle = 0. \ (11)$$

(ii) For unit vectors $X = Y \in \mathcal{U}$, we obtain

$$\rho + \|\eta\|^2 = n\langle \eta, H \rangle.$$  

Equivalently, using (10) we have

$$\rho - (n - 3)\|\eta\|^2 = 2\langle \alpha_{ii}, \eta \rangle, \ 1 \leq i \leq 2. \ (12)$$

Next we use the Codazzi equation

$$\nabla^\perp_X \alpha(Y, Z) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z) = \nabla^\perp_Y \alpha(X, Z) - \alpha(\nabla_Y X, Z) - \alpha(X, \nabla_Y Z).$$
For \( Y = e_i, Z = e_j \) with \( i \neq j \) and \( X \in \mathcal{U} \), we obtain
\[
\nabla_X^\perp \alpha_{ij} - \alpha(\nabla_X e_i, e_j) - \alpha(e_i, \nabla_X e_j) = \nabla_{e_i}^\perp \alpha(X, e_j) - \alpha(\nabla_{e_i} X, e_j) - \alpha(X, \nabla_{e_i} e_j).
\]
Equivalently, we have
\[
\nabla_X^\perp \alpha_{ij} + \langle \nabla_X e_i, e_j \rangle (\alpha_{ii} - \alpha_{jj}) = \langle X, \nabla_{e_i} e_j \rangle \alpha_{ij} + \langle X, \nabla_{e_i} e_j \rangle (\alpha_{jj} - \eta).
\]
Taking the inner product with \( \eta \) and using (10), (11) and that \( \nabla_X^\perp \eta = 0 \) gives
\[
\langle \nabla_{e_i} e_j, X \rangle (\langle \alpha_{jj}, \eta \rangle - \|\eta\|^2) = 0, \quad i \neq j.
\]
Then using (12) yields
\[
(\rho - (n - 1)\|\eta\|^2) \langle \nabla_{e_i} e_j, X \rangle = 0, \quad i \neq j.
\]
(13)

We claim that \( \rho - (n - 1)\|\eta\|^2 \neq 0 \) on an open dense subset. Otherwise, there is an open subset of \( M^n \) where \( \|\eta\|^2 = \rho/(n - 1) \). We obtain from (9) and (12) that
\[
K(U^\perp) = \frac{\rho}{n - 1}.
\]
But this is easily seen to imply that \( M^n \) has constant sectional curvature on an open subset, and that is a contradiction. It follows from (13) that
\[
\langle \nabla_{e_i} e_j, X \rangle = 0, \quad i \neq j,
\]
(14)
for any \( X \in \mathcal{U} \).

For \( Y = Z = e_i \) and \( X \in \mathcal{U} \), we obtain
\[
\nabla_X^\perp \alpha_{ii} - 2\alpha(\nabla_X e_i, e_i) = \nabla_{e_i}^\perp \alpha(e_i, X) - \alpha(\nabla_{e_i} X, e_i) - \alpha(\nabla_{e_i} e_i, X).
\]
Equivalently, we have
\[
\nabla_X^\perp \alpha_{ii} - 2\langle \nabla_X e_i, e_j \rangle \alpha_{ij} = \langle \nabla_{e_i} e_i, X \rangle (\alpha_{ii} - \eta) + \langle \nabla_{e_i} e_j, X \rangle \alpha_{ij}, \quad i \neq j.
\]
Using (14) we obtain
\[
\nabla_X^\perp \alpha_{ii} - 2\langle \nabla_X e_i, e_j \rangle \alpha_{ij} = \langle \nabla_{e_i} e_i, X \rangle (\alpha_{ii} - \eta), \quad i \neq j.
\]
Taking the inner product with \( \eta \) and using (11) gives
\[
\langle \nabla_X^\perp \alpha_{ii}, \eta \rangle = (\langle \alpha_{ii}, \eta \rangle - \|\eta\|^2) \langle \nabla_{e_i} e_i, X \rangle, \quad 1 \leq i \leq 2.
\]

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On the other hand, from (12) and since $\nabla_X \eta = 0$ we have

$$\langle \nabla_X \alpha_{ii}, \eta \rangle = 0, \quad 1 \leq i \leq 2.$$  

Using (12) we obtain

$$(\rho - (n - 1)\|\eta\|^2)\langle \nabla e_i e_i, X \rangle = 0, \quad 1 \leq i \leq 2,$$

and therefore

$$\langle \nabla e_i e_i, X \rangle = 0, \quad 1 \leq i \leq 2,$$

(15)

for any $X \in \mathcal{U}$. That the distribution $\mathcal{U}^\perp$ is totally geodesic follows from (14) and (15).

From the above we have that Proposition 6 applies. Hence $M^n$ is part of a warped product $L^2 \times_\varphi S^{n-2}(1)$. For simplicity, we assume that either $\varphi$ is constant or that $\nabla \varphi \neq 0$ on $M^n$. In the former case, we obtain from (2) that the Gauss curvature of $L^2$ is the constant $K = \rho$. Since a Riemannian product of manifolds of constant sectional curvature $M_{c_1}^p \times M_{c_2}^{n-p}$ is an Einstein manifold if and only if only

$$(p - 1)c_1 = (n - p - 1)c_2,$$

(16)

we have from (8) that we are in case (i) of the statement. In the latter case, it follows from Proposition 4 that we are in case (ii).

**Remark 8.** If $f : M^n \to \mathbb{R}^{n+p}$ is an $(n - 1)$-umbilical Einstein submanifold then $M^n$ has constant sectional curvature. This can be proved by a simpler version of the above computation or just by using that a locally conformally flat Einstein manifold has constant sectional curvature.

The following is the main result of this section. Observe that only assumptions of intrinsic nature are required.

**Theorem 9.** Let $M^n = L^2 \times_\varphi F^{n-2}(\varepsilon), \ n \geq 5,$ be an Einstein manifold such that neither the warping function nor the sectional curvature are constant on open subsets of $L^2$ and $M^n$, respectively. If $F^{n-2}(\varepsilon)$ has constant sectional curvature then any isometric immersion $f : M^n \to \mathbb{R}^{n+2}$ is an $(n - 2)$-rotational submanifold as in part (a) of Examples 1 along connected components of an open dense subset of $M^n$.

**Remark 10.** Implicit in the above statement there is a rigidity fact. Namely, any local isometric deformation of the submanifold $f$ as above is necessarily the result of a local isometric deformation of the surface $h$ in $\mathbb{R}^3$. 11
Let $V^k \subset \mathbb{R}^k$, $k \geq 1$, denote the half space
\[ V^k = \{ x \in \mathbb{R}^k : \sigma(x) = \langle x, e \rangle > 0 \} \]
where $e \in \mathbb{R}^k$ is a unit vector. A warped product representation of the Euclidean space $\mathbb{R}^m$, $m = k + \ell$ and $\ell \geq 2$, is the explicitly constructible isometry $\psi : V^k \times \sigma \mathbb{S}^\ell(r) \to \mathbb{R}^m$ onto $\mathbb{R}^m$ up to a subspace $\mathbb{R}^{k-1} \subset \mathbb{R}^m$. The warping function is a linear function $\sigma(x) = \langle x, v \rangle$ where $\|v\| = 1$. By the warped product of isometric immersions $h_1 : L^p \to V^k$ and $h_2 : N^q \to \mathbb{S}^\ell(r)$ determined by the warped product representation $\psi$ of $\mathbb{R}^m$ we mean the isometric immersion $f = \psi \circ (h_1 \times h_2) : L^p \times_{\sigma \circ h_1} N^q \to \mathbb{R}^m$. Notice that $\sigma \circ h_1 \in C^\infty(L^p)$ is just a coordinate function of $h_1$.

**Lemma 11.** Assume that the warped product $M^n = L^2 \times_{\varphi} N^{n-2}$, $n \geq 5$, of Riemannian manifolds has the following properties: (a) $L^2$ does not contain an open subset of points of constant Gauss curvature and (b) $M^n$ does not admit a local isometric immersion into $\mathbb{R}^{n+1}$. Given an isometric immersion $f : M^n \to \mathbb{R}^{n+2}$, there exists an open dense subset of $M^n$ each of whose points lies in an open product neighborhood $U = L_0^2 \times N_0^{n-2} \subset M^n$ such that $f|_U$ is a warped product of isometric immersions with respect to a warped product representation $\psi : V^{2+k_1} \times_{\sigma} \mathbb{S}^{n-2+k_2} \to \mathbb{R}^{n+2}$, $V^{2+k_1} \subset \mathbb{R}^{2+k_1}$, where either $k_1 = k_2 = 1$ or $k_1 = 2$ and $k_2 = 0$.

**Proof.** It follows from the assumption $(b)$ that $M^n$ does not contain an open subset of flat points. Then Theorem 14 in [8] applies and, according to that result, there are three disjoint possibilities named there $(i)$ through $(iii)$. But our assumption $(a)$ excludes the case $k_1 = 0$ in $(i)$ as well as $(iii)$ whereas the assumption $(b)$ excludes $(ii)$.

**Remark 12.** That $k_1 = 2$, $k_2 = 0$ just says that $f|_U$ is an $(n-2)$-rotational submanifold with profile $h_1$. 

Proof of Theorem 9. We have from Proposition 4 that \( L^2 \) admits local coordinates \((t, u)\) such that its metric has the form \( ds^2 = dt^2 + \varphi^2 du^2 \) being \( \varphi \) a solution of equation (6) where \( c \neq 0 \) since \( L^2 \) has no constant Gauss curvature. Therefore, according to Lemma 11 it remains to show that the case \( k_1 = k_2 = 1 \) there cannot occur. If otherwise, we have \( h_1 = (h^0, \varphi); L^2 \rightarrow \mathbb{R}^3 \) where \( h^0 = h^0(t, u) \) is a local parametrization of \( \mathbb{R}^2 \) in an open neighborhood of any point in the open dense subset where \( \varphi' \neq 0 \).

Thus, there is an orthonormal frame \( \{e_1, e_2\} \) such that

\[
h^0_t = \sqrt{1 - \varphi'^2} e_1 \quad \text{and} \quad h^0_u = \varphi' e_2.
\]

From \( \nabla_{\partial u} h^0_t = \nabla_{\partial t} h^0_u \) we derive that \( e_j = e_j(u), \ j = 1, 2, \) and that

\[
\langle \nabla_{\partial u} e_1, e_2 \rangle = \frac{\varphi''}{\sqrt{1 - \varphi'^2}}.
\]

It follows that

\[
h^0_{tt} = -\frac{\varphi' \varphi''}{\sqrt{1 - \varphi'^2}} e_1 \quad \text{and} \quad h^0_{tu} = \varphi'' e_2.
\]

Hence

\[
h^0_{ttu} = -\frac{\varphi' \varphi''}{\sqrt{1 - \varphi'^2}} \nabla_{\partial u} e_1 = -\frac{\varphi' \varphi''^2}{1 - \varphi'^2} e_2 \quad \text{and} \quad h^0_{tut} = \varphi''' e_2.
\]

Thus, we have that

\[
\varphi''' + \frac{\varphi' \varphi''^2}{1 - \varphi'^2} = 0
\]

or, equivalently, that

\[
(\varphi''/\sqrt{1 - \varphi'^2})' = 0.
\]

Then \( \varphi = (1/a) \sin at, \ 0 \neq a \in \mathbb{R} \), and this is a contradiction since \( \varphi \) is not a solution of equation (6) with \( c \neq 0 \).

We have shown that \( k_1 = 2, k_2 = 0 \) and, in particular, that we have that \( F^{n-2}(\varepsilon) = S^{n-2}(1) \). \qed

Theorem 9 does not hold if we allow \( M^n \) to have constant sectional curvature. In this situation, we can still exclude the case (iii) in Theorem 14 of
since the manifold there does not carry a foliation by \((n - 2)\)-dimensional spheres. Taking a second derivative of \([8]\) and using that \(K = -\varphi'''/\varphi'\) it follows that \(K = \rho/(n - 1)\). By Theorem 14 of \([8]\) there are two possible situations for which there are non-rotational submanifolds. We give next examples for both cases in the notation of Lemma 11.

(1) We have \(L^2 = \mathbb{R}^2\) endowed with Cartesian coordinates \((t, u)\), \(\varphi(t) = t\) and \(h_2: F^{n-2}(1) \to S^n(r)\) is any non-totally geodesic submanifold. For instance, an appropriate product of spheres. Then \(f(M)\) is a Ricci flat submanifold but it is not flat.

(2) \(\mathbb{R}^{n+2} = \mathbb{R}^3 \times_\sigma S^{n-1}(r)\) where \(\sigma = \langle x, e \rangle\) and \(r = (n - 4)/(n - 3)\). Let \(L^2 \subset \mathbb{R}^2\) be endowed with the coordinates \((t, u)\) and the metric induced by \(h_1: L^2 \to \mathbb{R}^3\) given below. Then let \(h_2: F^{n-2}(1) \to S^{n-1}(r)\) be a torus being

\[
F^{n-2}(1) = S^m(r_1) \times S^{n-m-2}(r_2), \quad 2 \leq m \leq n - 4,
\]

where \(r_1 = \sqrt{(m - 1)/(n - 3)}\) and \(r_2 = \sqrt{(n - m - 3)/(n - 3)}\).

Case \(K = 1\): Take \(h_1(u, t) = (\cos t \sin u, \cos t \cos u, \sin t)\).

Case \(K = 0\): Take \(h_1(u, t) = (\sin u, \cos u, t)\).

Finally, we observe that a similar construction for \(K = -1\) does not work.

3 The complete case

We first show that \(M^n = \mathbb{R}^2 \times_\varphi S^{n-2}(1)\) endowed with the Generalized Schwarzschild metric can be realized parametrically as a rotational submanifold in codimension two with flat normal bundle. The case when the dimension is \(n = 5\) turns out to be special in the sense that the parametrization is completely explicit.

Examples 2. We make use of the construction of submanifolds given in part (b) of Examples 1. Let \(\varphi \in C^\infty([0, +\infty))\) be the unique positive solution of

\[
\varphi^2 = 1 + \frac{c}{\varphi^{n-3}}
\]

such that \(\varphi(0) = (n - 3)/2\) and \(c = -((n - 3)/2)^{n-3}\). From

\[
\varphi^2 + \varphi'''^2 = 1 - ((n - 3)/2\varphi)^{n-3} + ((n - 3)/2\varphi)^{2(n-2)}
\]
we have
\[ \varphi'^2(t) + \varphi''^2(t) < 1 \text{ if and only if } \varphi(t) > (n - 3)/2. \]
Since \( \varphi'(t) \geq 0 \) the inequality holds true for \( t > 0 \).

We have that \( M^n = \mathbb{R}^2 \times \varphi \mathbb{S}^{n-2}(1) = [0, +\infty) \times \varphi \mathbb{S}^1(1) \times \varphi \mathbb{S}^{n-2}(1) \) is endowed with the Generalized Schwarzschild metric. Let \( h: \mathbb{R}^2 \to \mathbb{R}^3 \) be any surface isometric to the surface of rotation
\[ h_0(t, \theta) = (\psi(t), \varphi'(t) \sin \theta, \varphi'(t) \cos \theta). \]
Then the isometric immersion \( f: M^n \to \mathbb{R}^{n+2} \) given by
\[ f(t, \theta, y) = (h(t, \theta), \varphi(t) \phi(y)) \quad (17) \]
is an \((n - 2)\)-rotational submanifold with flat normal bundle.

We recall that by the classical result of Bour any surface of rotation in \( \mathbb{R}^3 \) admits a 2-parameter family of isometric helicoidal surfaces; for instance see Lemma 2.3 of [3].

For dimension \( n = 5 \) the above yields a totally explicit parametrization of the submanifold. In fact, in this case the solutions of equation (5) are
\[ \varphi(t) = \sqrt{t^2 - c}, \quad t^2 > c \text{ and } c \in \mathbb{R}. \]
In particular, taking \( \varphi(t) = \sqrt{t^2 + 1} \) in (17) we obtain explicitly parametrized complete Ricci flat Einstein submanifolds.

**Remark 13.** Solving (6) for some dimensions, namely, for \( n = 4, 7 \text{ and } 9 \) leads to elliptic integrals and hence the solutions can be expressed in terms of elliptic functions whereas in other cases the integrals are hyperelliptic.

**Proposition 14.** Let \( M^n = L^2 \times \varphi F^{n-2}(1), n \geq 4, \) be a complete Einstein manifold such that \( L^2 \) is non-compact and \( \varphi \in C^\infty(L) \) is non-constant. If there is an isometric immersion \( g: L^2 \to \mathbb{R}^m, m \geq 4, \) such that \( \varphi = \langle g, e \rangle \) where \( e \in \mathbb{R}^m, \| e \| = 1 \), then the following facts hold:

(i) \( M^n \) is Ricci flat.

(ii) If \( L^2 \) is simply connected then \( M^n = \mathbb{R}^2 \times \varphi F^{n-2}(1) \) is endowed with the Generalized Schwarzschild metric.
Proof. First observe that \( \rho \leq 0 \). In fact, if otherwise \( M^n \) is compact, and hence also \( L^2 \) would be compact which has been excluded.

Notice that \( \nabla \varphi = e^\top \) gives \( \| \nabla \varphi \| \leq 1 \). We argue that \( \rho = 0 \). Taking traces in (2) gives

\[
(n - 2)\Delta \varphi = 2(K - \rho)\varphi. \tag{18}
\]

Then combining with (3) yields

\[
2K + (n - 4)\rho = (n - 2)(n - 3)\frac{1 - \| \nabla \varphi \|^2}{\varphi^2}. \tag{19}
\]

Suppose that \( \rho < 0 \). We have from (19) that

\[
2K \geq (4 - n)\rho. \tag{20}
\]

Thus, it follows for \( n \geq 5 \) that \( L^2 \) is compact and this is a contradiction.

It remains to consider the case \( n = 4 \). We have that

\[
\nabla \| \nabla \varphi \|^2 = \frac{n - 3}{\varphi}(1 - \| \nabla \varphi \|^2)\nabla \varphi - \rho \varphi \nabla \varphi \tag{21}
\]

on the subset

\[
L_0 = \{ x \in L^2 : \nabla \varphi(x) \neq 0 \}.
\]

In fact, in the coordinates of Proposition 3 we have that (21) is equation (5). We obtain from (20) that \( L^2 \) is parabolic. It follows from (21) that

\[
\nabla (\varphi(\| \nabla \varphi \|^2 - 1) + (\rho/3)\varphi^3) = 0 \tag{22}
\]

on \( L_0 \). Therefore, on each connected component of the subset \( L_0 \) we have that

\[
\| \nabla \varphi \|^2 = 1 - \frac{\rho}{3} \varphi^2 + \frac{C}{\varphi} \tag{23}
\]

for some constant \( C < 0 \) such that \( \varphi^3 \leq C/\rho \). On the other hand, we obtain from (22) that the set int\( \{ x \in L^2 : \nabla \varphi(x) = 0 \} \) is empty. Hence (23) holds on all of \( L^2 \). In particular, we have that \( \varphi \) is bounded. It follows from (18) and (20) that \( \Delta \varphi > 0 \). Since \( L^2 \) is parabolic then \( \varphi \) is constant, and this is a contradiction to (18).
Assume that $L^2$ is simply connected. Since $\rho = 0$, then from (2), (3), (18) and (19) we have that

$$Hess \varphi = \frac{1}{n-2} K \varphi I,$$

(24)

$$\Delta \varphi = \frac{n-3}{\varphi} (1 - \|\nabla \varphi\|^2) \geq 0,$$

(25)

$$2K = (n-2)(n-3) \frac{1 - \|\nabla \varphi\|^2}{\varphi^2} \geq 0.$$

(26)

In particular, since $K \geq 0$ then $L^2 = \mathbb{R}^2$.

**Fact 1:** There exists a constant $c < 0$ such that

$$\|\nabla \varphi\|^2 = 1 + \frac{c}{\varphi^{n-3}}$$

(27)

on $L^2$. Moreover, the following equations hold on $L^2$:

1. $K = -\frac{(n-2)(n-3)c}{2\varphi^{n-1}}$,  
2. $\Delta \varphi = -\frac{(n-3)c}{\varphi^{n-2}}$,  
3. $Hess \varphi = -\frac{(n-3)c}{2\varphi^{n-2}} I$.

We have from (21) that

$$\nabla (\varphi^{n-3} (\|\nabla \varphi\|^2 - 1)) = 0.$$

on $L_0$. Therefore, we have that (27) holds for some constant on connected components of $L_0$. Moreover, it holds trivially on connected components of $\text{int}\{x \in L^2 : \nabla \varphi(x) = 0\}$. Thus, by continuity (27) holds on all of $L^2$.

Equations (i), (ii) and (iii) follow from (24), (25), (26) and (27). Finally, we argue that $c < 0$. If otherwise and since $K \geq 0$, we have from (i) that $c = 0$. Then (i) and (ii) give that $K = 0$ and that $\varphi$ is a positive harmonic function. It follows from Liouville’s theorem that $\varphi$ is constant, and this is a contradiction due to (27).

**Fact 2:** We have that (i) $\inf \varphi = (-c)^{1/(n-3)} > 0$, (ii) $\sup \varphi = +\infty$, (iii) $K$ is bounded and (iv) $\inf K = 0$. Moreover, a point $p_0$ is a critical point of $\varphi$ if and only if

$$\varphi(p_0) = \inf \varphi = (-c)^{1/(n-3)}.$$

(28)

Since $\varphi > 0$, by the well-known Omori-Yau maximum principle there exists a sequence of points $\{x_m\}_{m \in \mathbb{N}}$ in $L^2$ such that

$$\lim_{m \to \infty} \varphi(x_m) = \inf \varphi, \text{ and } \lim_{m \to \infty} \|\nabla \varphi\|(x_m) = 0.$$
From (27) we have
\[ 1 + \frac{c}{\inf \varphi^{n-3}} = 0, \]
and this is (i). To prove (ii) suppose that sup \( \varphi < +\infty \). Since \( L^2 \) is parabolic and \( \varphi \) is subharmonic due to part (ii) of Fact 1, then \( \varphi \) is constant, a contradiction by (ii) of Fact 1. From (i) of Fact 1 we have
\[ 0 \leq K = -\frac{(n-2)(n-3)c}{2\varphi^{n-1}} \leq -\frac{(n-2)(n-3)c}{2\inf \varphi^{n-1}} \]
which proves (iii). Since sup \( \varphi = +\infty \) the equality part above proves (iv). Finally, it follows from (27) that (28) holds at a critical point.

Fact 3: The function \( \varphi : L^2 \to (0, +\infty) \) has at most one critical point.

Let \( p_0 \) be a critical of \( \varphi \). By (28) \( p_0 \) is a global minimum of \( \varphi \). Let \( \gamma : \mathbb{R} \to L^2 \) be a geodesic parametrized by arc-length such that \( \gamma(0) = p_0 \) and \( \gamma(\ell) = p \). Now consider the function \( u = \varphi \circ \gamma : \mathbb{R} \to (0, +\infty) \). Then
\[ \min_{s \in \mathbb{R}} u(s) = u(0). \]
From (iii) of Fact 1 we have that
\[ \nabla \dot{\gamma} \nabla u = -\frac{(n-3)c}{2u^{n-2}} \dot{\gamma}. \]
Then
\[ \dot{\gamma} \langle \nabla u, \dot{\gamma} \rangle = \langle \nabla \dot{\gamma} \nabla u, \dot{\gamma} \rangle = -\frac{(n-3)c}{2u^{n-2}} \]
and thus
\[ \ddot{u} = -\frac{(n-3)c}{2u^{n-2}} > 0. \]
Thus \( \dot{u} \) is monotone increasing which implies for \( s < 0 \) that \( \dot{u}(s) < \dot{u}(0) = 0 \) and for \( s > 0 \) that \( \dot{u}(s) > \dot{u}(0) = 0 \). Hence \( t = 0 \) is the only point where \( u \) attains its minimum. Thus \( p_0 \) is the only point of \( \gamma \) that \( \varphi \) attains its minimum or, equivalently, the only point of \( L^2 \) where \( \nabla \varphi \) vanishes.

Let \( p_0 \in L^2 \) be the necessarily unique possible critical point of \( \varphi \). Then on either \( L^2 \) or \( L^2 \setminus \{p_0\} \) we define \( e_1 = \nabla \varphi/\|\nabla \varphi\| \) and \( e_2 = J(e_1) \) where \( J \) is the complex structure of \( L^2 \). From (iii) in Fact 1, we have
\[ X(\|\nabla \varphi\|)e_1 + \|\nabla \varphi\| \nabla_X e_1 = -\frac{(n-3)c}{2\varphi^{n-2}} X. \]
for any $X \in TL$. It follows that

$$\nabla e_1 e_1 = 0, \quad e_1(\|\nabla \varphi\|) = -\frac{(n-3)c}{2\varphi^{n-2}}, \quad e_2(\|\nabla \varphi\|) = 0$$

and

$$\|\nabla \varphi\| \langle \nabla e_2 e_1, e_2 \rangle = -\frac{(n-3)c}{2\varphi^{n-2}}.$$  

Then

$$\langle e_1, \|\nabla \varphi\| e_2 \rangle = \|\nabla \varphi\| (\nabla e_1 e_2 - \nabla e_2 e_1) + e_1(\|\nabla \varphi\|) e_2 = 0.$$  

Hence, there exist coordinates $(t, u)$ such that

$$\partial/\partial t = e_1 \quad \text{and} \quad \partial/\partial u = \|\nabla \varphi\| e_2$$

and in these coordinates the metric of $L^2$ is $ds^2 = dt^2 + \|\nabla \varphi\|^2 du^2$ where $\frac{\partial}{\partial u}\|\nabla \varphi\|^2 = 0$. Thus, we can write

$$\varphi'^2 = 1 + \frac{c}{\varphi^{n-3}}. \quad (29)$$

Let $\phi_1(x, t)$ and $\phi_2(x, t)$ be the one-parameter groups of diffeomorphisms generated by $\nabla \varphi$ and $J(\nabla \varphi)$, respectively. Since $\|\nabla \varphi\| = \|J(\nabla \varphi)\| < 1$, it follows that $\phi_1(x, t)$ and $\phi_2(x, t)$ are defined for all $x \in L^2$ and $t \in \mathbb{R}_+$ or $\mathbb{R}$ according to the existence or not of a critical point of $\varphi$. Hence $(t, u)$ are either Cartesian or polar coordinates in $\mathbb{R}^2$.

To conclude the proof, we argue that the case of Cartesian coordinates cannot occur due to the completeness of $L^2$. In fact, if these coordinates occur we would have from (29) that

$$\lim_{t \to -\infty} \int_t^{t_0} \frac{\varphi'(u)du}{\sqrt{1 + c\varphi^{3-n}(u)}} = +\infty.$$  

On the other hand, we have

$$\int_t^{t_0} \frac{\varphi'(u)du}{\sqrt{1 + c\varphi^{3-n}(u)}} = \int_{\varphi(t)}^{\varphi(t_0)} \frac{dx}{\sqrt{1 + c x^{3-n}}} = A \int_{\varphi(t)}^{\sqrt{1 + c\varphi^{3-n}(t)}} \frac{dy}{\sqrt{1 + c x^{3-n}}} (1 - y^2)^{\frac{n-4}{n-3}}$$

where $A = 2(-c)^{\frac{1}{n-3}}/(n-3)$ and $y = \sqrt{1 + c x^{3-n}}$, hence $y \in (0, 1)$. But then

$$\lim_{t \to -\infty} \int_t^{t_0} \frac{\varphi'(u)du}{\sqrt{1 + c\varphi^{3-n}(u)}} = A \int_0^{\sqrt{1 + c\varphi^{3-n}(t_0)}} \frac{dy}{(1 - y^2)^{\frac{n-4}{n-3}}} < +\infty,$$

and this is a contradiction. \hfill \Box
Corollary 15. Let \( f : M^n \rightarrow \mathbb{R}^{n+p}, \ n \geq 4, \) be a complete \((n-2)\)-rotational submanifold. If \( M^n \) is an Einstein manifold with Ricci curvature \( \rho \), then one of the following holds:

(i) We have that \( \rho > 0 \) and \( M^n = L^2 \times S^{n-2}(\sqrt{(n-3)/\rho}) \) where \( L^2 \) is either \( S^2(1/\sqrt{\rho}) \) or the projective space \( \mathbb{RP}^2(1/\sqrt{\rho}) \).

(ii) The universal cover \( \mathbb{R}^2 \times \varphi S^{n-2}(1) \) of \( M^n \) is endowed with the Generalized Schwarzschild metric.

Proof. By assumption we have that \( M^n = L^2 \times \varphi S^{n-2}(1) \) where \( L^2 \) is complete. If \( L^2 \) is compact then also \( M^n \) is compact, and from either \([13]\) or Theorem 1.2 of \([1]\) it follows that \( \varphi \) is constant. Then from \([2]\) and \([16]\) we obtain \( K = \rho > 0 \), and hence \( L^2 \) is as in part (i) of the statement. If \( L^2 \) is non-compact the proof follows from Proposition \([14]\). \( \Box \)

Proof of Theorem \([1]\) If \( \nabla \varphi = 0 \) on an open subset it follows easily from Theorem 17 in \([8]\) that \( f \) is locally as in part (i), hence it is \((n-2)\)-rotational. On the other hand, from Proposition \([3]\) we have that in a neighborhood of any point of \( L^2 \) where \( \nabla \varphi \neq 0 \) there are local coordinates \((t, u)\) such that \( \varphi = \varphi(t) \) satisfies \([6]\). It follows from Theorem \([9]\) that \( f \) is locally an \((n-2)\)-rotational submanifold along an open dense subset \( U \) of \( M^n \). Since the manifold is complete and the dimension of the umbilical distribution is minimal, then it is well known (see \([17]\)) that on \( U \) the umbilical leaves are complete spheres, and hence \( f \) is globally an \((n-2)\)-rotational submanifold. The proof now follows from Corollary \([15]\). \( \Box \)

Example 16. Let \( M^n = \mathbb{R}^2 \times \varphi N^{n-2} \) be endowed with the Generalized Schwarzschild metric where

\[
N^{n-2} = S^m(r_1) \times S^{n-m-2}(r_2), \quad 2 \leq m \leq n-3,
\]

for \( r_1 = (m-1)/(n-4) \) and \( r_2 = (n-m-3)/(n-4) \). Then \( f : M^n \rightarrow \mathbb{R}^{n+3} \) given by

\[
f(t, \theta, y) = (h(t, \theta), \varphi(t)j(y)),
\]

where \( h \) is as in Examples 2 and \( j : N^{n-2} \rightarrow \mathbb{R}^n \) the inclusion, is an isometric immersion that is not rotational.

Problem: In respect to Theorem \([1]\) notice that for dimension \( n = 4 \) the classification remains open.
4 Appendix

Let \( f: M^4 \to \mathbb{R}^6 \) be an isometric immersion with flat normal of an Einstein manifold. What makes this case special is that a four dimensional Riemannian manifold is Einstein if and only if its sectional curvature tensor satisfies that \( K(X,Y) = K(Z,W) \) for any orthonormal tangent vectors \( X, Y, Z, W \). Besides submanifolds of constant sectional curvature or products of space forms, it was observed in [20] that the possibilities for the second fundamental form of \( f \) are as follows. There exists an orthonormal frame \( \{\xi_1, \xi_2\} \) such that the corresponding shape operators with respect to a tangent frame that diagonalizes the second fundamental form have the form:

\[
A_{\xi_1} = \begin{bmatrix}
a & 0 & 0 & 0 \\
0 & b & 0 & 0 \\
0 & 0 & c & 0 \\
0 & 0 & 0 & d
\end{bmatrix}, \quad A_{\xi_2} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & p & 0 & 0 \\
0 & 0 & q & 0 \\
0 & 0 & 0 & r
\end{bmatrix}
\]

where \( a \neq 0 \) and

\[
pq = ad - bc, \quad pr = ac - bd, \quad qr = ab - cd,
\]

\[
(ba - cd)(ca - bd)(da - bc) > 0.
\]

If we drop the last condition, we have

\[
A_{\xi_1} = \begin{bmatrix}
a & 0 & 0 & 0 \\
0 & \epsilon a & 0 & 0 \\
0 & 0 & b & 0 \\
0 & 0 & 0 & \epsilon b
\end{bmatrix}, \quad A_{\xi_2} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & p & 0 \\
0 & 0 & 0 & q
\end{bmatrix}
\]

where \( a \neq 0, \ pq = \epsilon(a^2 - b^2) \) and \( \epsilon = \pm 1 \). The submanifolds in the case \( \epsilon = 1 \) are the ones given by part \( (ii) \) of Theorem 7 when \( p = 2 \) and the surface \( g: L^2 \to \mathbb{R}^4 \) has flat normal bundle. On the other hand, making use of the Codazzi equations one can show that the case \( \epsilon = -1 \) does not represent the second fundamental form of any actual submanifold.

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