HARDY SPACES AND THE SZEGŐ PROJECTION OF THE NON-SMOOTH WORM DOMAIN $D'_b$

ALESSANDRO MONGUZZI

Abstract. We define Hardy spaces $H^p(D'_b)$ on the non-smooth worm domain $D'_b = \{(z_1, z_2) \in \mathbb{C}^2 : |\text{Im} z_1 - \log |z_2|^\beta| < \frac{\pi}{2}, |\log |z_2|^\beta| < \frac{\pi}{2}\}$ and we prove a series of related results such as the existence of boundary values and a Fatou-type theorem (i.e. pointwise convergence to the boundary values). Thus, we study the Szegő projection operator $\tilde{S}$ and the Szegő kernel $K_{D'_b}$ associated. More precisely, if $H^p(\partial D'_b)$ denotes the space of functions which are boundary values for functions in $H^p(D'_b)$, we prove that the operator $\tilde{S}$ extends to a bounded linear operator

$$\tilde{S}: L^p(\partial D'_b) \to H^p(\partial D'_b)$$

for every $p \in (1, +\infty)$ and

$$\tilde{S}: W^{k,p}(\partial D'_b) \to W^{k,p}(\partial D'_b)$$

for every $k > 0$. Here $W^{k,p}$ denotes the Sobolev space of order $k$ and underlying $L^p$ norm. As a consequence of the $L^p$ boundedness of $\tilde{S}$, we prove that $H^p(D'_b) \cap C(\overline{D'_b})$ is a dense subspace of $H^p(D'_b)$.

1. Introduction and Main Results

Given a domain $\Omega \subseteq \mathbb{C}^n$, it is a classical problem to study the Hardy spaces of holomorphic functions and the Szegő projection operator associated to this domain. If $\rho$ is a defining function for $\Omega$, i.e. $\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0\}$, a standard way to define the Hardy spaces $H^p(\Omega)$, $p \in (1, \infty)$, is to consider a family of approximating subdomains $\Omega_\varepsilon = \{z \in \mathbb{C}^n : \rho(z) < -\varepsilon\}$ together with the growth condition

$$\|f\|_{H^p(\Omega)}^p := \sup_{\varepsilon > 0} \int_{\partial \Omega_\varepsilon} |F(\zeta)|^p d\sigma_\varepsilon,$$

where $d\sigma_\varepsilon$ is the euclidean measure induced on $b\Omega_\varepsilon$. Then,

$$H^p(\Omega) := \{F \text{ holomorphic in } \Omega : \|F\|_{H^p(\Omega)}^p < \infty\}.$$

Every function $F$ in $H^p(\Omega)$ admits a boundary value function $\tilde{F}$ and the linear space of these boundary value functions defines a closed subspace of $L^p(b\Omega)$ which we denote by $H^p(b\Omega)$. In the special case $p = 2$, the orthogonal projection

$$S_\Omega : L^2(b\Omega) \to H^2(b\Omega)$$

is called the Szegő projection operator associated to $\Omega$. We refer to [Ste72] for more details.

The geometry of the domain $\Omega$ affects the regularity of $S_\Omega$ and this problem has been extensively studied in the last 40 years. There is a number of results regarding the regularity of the Szegő projection in Sobolev scale for many classes of domains: strictly pseudoconvex domains [PS77], smooth bounded complete Reinhardt domains in...
C^n [Boa85, Str86], domains satisfying Catlin’s property [Boa87], complete Hartogs domains in C^2 [BCS88, BS99], domains of finite type in C^2 [NRSW89, BS99], domains that admit a defining function that is plurisubharmonic on the boundary [BS91] and convex domains of finite type in C^n [MS97]. We refer also to [Che91] and [Chr96] for some results regarding the behavior of the Szegő projection with respect to the real analyticity of functions.

We also have some results concerning the L^p regularity of the Szegő projection; in [Dia87] the problem is studied for a particular family of weakly pseudoconvex domains, in [MS97] the case of convex domains is threatened, while in [LS04] the authors deal with non-smooth, simply connected domains in the plane C. More recently, Lanzani and Stein announced in [LS13] some new results about the L^p regularity of the Szegő projection. They still deal with strictly pseudoconvex domains, but assuming only C^2 boundary regularity. We also cite [BL14] where a new definition of the Szegő kernel is suggested.

The smooth worm domain \( \mathcal{W} = \mathcal{W}_0 \) does not belong to any of the known situations. The domain \( \mathcal{W} \) was first introduced by Diederich and Fornæss in [DF77] as a counterexample to certain classical conjectures about the geometry of pseudoconvex domains. For \( \beta > \frac{\pi}{2} \), the worm domain is defined by

\[
\mathcal{W} = \{ (z_1, z_2) \in \mathbb{C}^2 : |z_1 - e^{i \log |z_2|^2}|^2 < 1 - \eta(\log |z_2|^2), z_2 \neq 0 \},
\]

where \( \eta \) is a smooth, even, convex, non-negative function on the real line, chosen so that \( \eta^{-1}(0) = [-\beta + \frac{\pi}{2}, \beta - \frac{\pi}{2}] \) and so that \( \mathcal{W} \) is bounded, smooth and pseudoconvex. We refer to [KP08b] for a history of the study of the worm domain. Diederich and Fornæss introduced this domain to provide an example of a smooth, bounded and pseudoconvex domain whose closure does not have a Stein neighbourhood basis. Nearly 15 years after its introduction, the interest in the worm domain has been renewed since it turned out to be a counterexample to other important conjectures. Starting from ground-breaking works of Kiselman [Kis91] and Barrett [Bar92], Christ [Chr96] finally proved that the Bergman projection \( P_{\mathcal{W}} \) of the worm domain, i.e. the orthogonal projection of \( L^2(\mathcal{W}) \) onto the closed subspace of holomorphic functions, does not map \( C^\infty(\bar{\mathcal{W}}) \) to \( C^\infty(\bar{\mathcal{W}}) \). Therefore, the worm domain is a smooth bounded pseudoconvex domain which does not satisfy the so-called Condition R. This conditions is closely related to the boundary regularity of biholomorphic mappings as it has been shown in works of Bell [Bel81] and Bell and Ligocka [BL14]. Due to the results of Christ’s, the Bergman projection of the worm domain has been extensively studied by many authors. We cite the recent papers [KP07, KP08a, KP08b, BS12, BEP14, KPS14] and the references therein. We remark that the Szegő projection can be considered a boundary analogue of the Bergman projection. Moreover, the regularity of the Szegő projection, at least in a certain setting, has been proved in [HPR15] to be closely linked to the regularity of the complex Green operator in analogy with the Bergman projection and the \( \bar{\partial} \)-Neumann operator.

Due to the lack of general results concerning the regularity of the Szegő projection of smooth bounded weakly pseudoconvex domains and the peculiar behavior of \( P_{\mathcal{W}} \), the study of the regularity of \( S_{\mathcal{W}} \) is an interesting starting point for research in this direction. The work presented here would like to be a first step for this investigation. In analogy with the Bergman case [Bar92, KP08b], we start studying a non-smooth model of the domain \( \mathcal{W} \), namely,

\[
D'_\beta = \{ (z_1, z_2) \in \mathbb{C}^2 : |\text{Im} z_1 - \log |z_2|^2| < \frac{\pi}{2}, |\log |z_2|^2| < \beta - \frac{\pi}{2} \}.
\]

The domain \( D'_\beta \) is rotationally invariant in the \( z_2 \) variable and it can represented in the plane \((\text{Im} z_1, \log |z_2|)\) as in Figure 1.
The Bergman projection of $\mathcal{W}$ is studied with the aid of another non-smooth model domain biholomorphically equivalent to $D'_\beta$, namely,

$$D'_\beta = \{(\zeta_1, \zeta_2) \in \mathbb{C}^2 : \Re(\zeta_1 e^{-i \log |\zeta_2|^2}) > 0, |\log |\zeta_2|^2| < \beta - \frac{\pi}{2}\}.$$ 

The analysis on $D'_\beta$ results to be easier than on $D_\beta$, but, in the Bergman setting, we can transfer the information we obtain on $D'_\beta$ to $D_\beta$ by means of the transformation rule for the Bergman projection under biholomorphic mappings. In the Szegő situation we lack such a tool, thus, at the moment, we only focus on the domain $D'_\beta$.

The feature which makes the analysis on $D'_\beta$ easier than on $D_\beta$ is the following. Let $z_1 \in \mathbb{C}$ such that $|\Im z_1| < \beta$ be fixed; then, as it is elementary to check, the set $D'_\beta(z_1) := \{z_2 \in \mathbb{C} : (z_1, z_2) \in D'_\beta\}$ is connected. This is not the case for the domain $D_\beta$ and the main difference between the two domains.

Notice that the domain $D'_\beta$ can be sliced in strips. More in detail, let us fix $z_2 \in \mathbb{C}$ such that $|\log |\zeta_2|^2| < \beta - \frac{\pi}{2}$; then, the set

$$D'_\beta(z_2) = \{z_1 \in \mathbb{C} : (z_1, z_2) \in D'_\beta\} = \{z_1 \in \mathbb{C} : |\Im z_1 - \log |\zeta_2|^2| < \frac{\pi}{2}\}$$

can be identified with a strip centered in $\log |\zeta_2|^2$ and width equals to $\pi$. All these characteristics will be reflected in our results. The rotationally invariance in the $z_2$-variable will allow us to use the theory of Fourier series, while the “strip-like” geometry in the $z_1$-variable will make the results for the Hardy spaces on a strip available.

In order to define Hardy spaces on $D'_\beta$ we need to establish a $H^p$-type growth condition for holomorphic functions on $D'_\beta$. Due to the geometry of $D'_\beta$, instead of considering a growth condition on copies of the topological boundary $bD'_\beta$, we decided to consider a growth condition on copies of the distinguished boundary $\partial D'_\beta$. This seems to be a natural choice in comparison with similar setting such as the polydisc. In detail, the distinguished boundary $\partial D'_\beta$ is the
set
\[ \partial D'_{\beta} = E_1 \cup E_2 \cup E_3 \cup E_4, \]
where
\[ E_1 = \left\{ (z_1, z_2) : \text{Im} z_1 = \beta, \log |z_2|^2 = \beta - \frac{\pi}{2} \right\}; \]
\[ E_2 = \left\{ (z_1, z_2) : \text{Im} z_1 = \beta - \pi, \log |z_2|^2 = \beta - \frac{\pi}{2} \right\}; \]
\[ E_3 = \left\{ (z_1, z_2) : \text{Im} z_1 = -\beta, \log |z_2|^2 = -\left( \beta - \frac{\pi}{2} \right) \right\}; \]
\[ E_4 = \left\{ (z_1, z_2) : \text{Im} z_1 = -\left( \beta - \pi \right), \log |z_2|^2 = -\left( \beta - \frac{\pi}{2} \right) \right\}. \]

For every \( p \in (1, \infty) \), we define the Hardy spaces \( H^p(D'_{\beta}) \) as the functional space
\[ H^p(D'_{\beta}) = \left\{ F \text{ holomorphic in } D'_{\beta} : \|F\|_{H^p(D'_{\beta})} = \sup_{(t,s) \in [0, \pi) \times [0, \beta - \pi]} \mathcal{L}_p F(t,s) < \infty \right\}, \]
where
\[ \mathcal{L}_p F(t,s) = \int_{\mathbb{R}} \int_{0}^{1} \left| F \left( x + i(t+s), e^{2\pi i t} \right) \right|^p d\theta dx + \int_{\mathbb{R}} \int_{0}^{1} \left| F \left( x - i(t+s), e^{-2\pi i t} \right) \right|^p d\theta dx + \int_{\mathbb{R}} \int_{0}^{1} \left| F \left( x + i(t-s), e^{2\pi i t} \right) \right|^p d\theta dx + \int_{\mathbb{R}} \int_{0}^{1} \left| F \left( x - i(t-s), e^{-2\pi i t} \right) \right|^p d\theta dx. \]

We emphasize that the domain \( D'_{\beta} \) is not a product domain, while, on the other hand, every component \( E_\ell \) of the distinguished boundary is and it can be identified with \( \mathbb{R} \times \mathbb{T} \).

The main results we obtain describe the good behavior of the Szegő projection associated to \( D'_{\beta} \) in term of Sobolev and \( L^p \) norms. Before we can state the theorems, we need a remark. As we already mentioned, the distinguished boundary \( \partial D'_{\beta} \) has 4 different components, thus when considering a function \( \varphi : \partial D'_{\beta} \to \mathbb{C} \) we actually mean a vector \( \varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4) \) where each function \( \varphi_\ell \) is considered as defined on the component \( E_\ell, \ell = 1, \ldots, 4 \) of the distinguished boundary. Recall again that each \( E_\ell \) can be identified with \( \mathbb{R} \times \mathbb{T} \).

**Notation.** Given a function \( \psi \in C_0^\infty(\mathbb{R} \times \mathbb{T}) \), we denote with \( \mathcal{F}_R \psi(\xi, \tilde{\gamma}) \) the Fourier transform of \( F \) in the first variable and the \( j \)th Fourier coefficient in the second, i.e.
\[ \mathcal{F}_R \psi(\xi, \tilde{\gamma}) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{0}^{1} \psi(x, \gamma) e^{-ix \xi} e^{-2\pi i \tilde{\gamma} t} d\gamma dx. \]

Let \( k > 0 \), then the Sobolev space \( W^{k,p}(\partial D'_{\beta}) \) is defined as
\[ W^{k,p}(\partial D'_{\beta}) = \{ \varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4) : \|\varphi\|_{W^{k,p}(\partial D'_{\beta})} = \sum_{\ell=1}^{4} \|\varphi_\ell\|_{W^{k,p}(\mathbb{R} \times \mathbb{T})} < \infty \}, \]
where
\[ \|\varphi_\ell\|_{W^{k,p}(\mathbb{R} \times \mathbb{T})} = \int_{\mathbb{R} \times \mathbb{T}} \left| \sum_{j \in \mathbb{Z}} e^{2\pi i j t} \mathcal{F}_R^{-1} \left[ \left( 1 + j^2 + (\cdot)^2 \right)^{k/2} \mathcal{F}_R \varphi_\ell(\cdot, j) \right](x) \right|^p dxd\tilde{t}. \]

Let \( p \in (1, \infty) \), then
\[ L^p(\partial D'_{\beta}) = \{ \varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4) : \|\varphi\|_{L^p(\partial D'_{\beta})} = \sum_{\ell=1}^{4} \|\varphi_\ell\|_{L^p(\mathbb{R} \times \mathbb{T})} < \infty \}. \]

The space \( H^2(D'_{\beta}) \) turns out to be a reproducing kernel Hilbert space (see, for instance, [Aro30]) and its reproducing kernel \( K_{D'_{\beta}} \) induces a Hilbert space orthogonal projection, the so-called Szegő projection,
\[ S : L^2(\partial D'_{\beta}) \to H^2(\partial D'_{\beta}). \]
where $H^2(\partial D'_\beta)$ is the closed subspaces of $L^2(\partial D'_\beta)$ consisting of functions that are boundary values for functions in $H^2(D'_\beta)$. Thus, we study the mapping properties of the operator $\tilde{S}$ in $L^p$ and Sobolev scale. The main results we prove are the following.

**Theorem 1.1.** The Szegő projection $\tilde{S}$ extends to a linear bounded operator

$$
\tilde{S} : L^p(\partial D'_\beta) \to H^p(\partial D'_\beta)
$$

for every $p \in (1, \infty)$.

**Theorem 1.2.** The Szegő projection $\tilde{S}$ extends to a linear bounded operator

$$
\tilde{S} : W^{k,p}(\partial D'_\beta) \to W^{k,p}(\partial D'_\beta)
$$

for every $k > 0$.

Besides these theorems, we carefully study the spaces $H^p(D'_\beta)$ proving a series of results such as a Fatou type theorem (i.e., pointwise convergence to the boundary values), a Paley–Wiener type theorem for the space $H^2(D'_\beta)$ and a nice decomposition for the spaces $H^p(D'_\beta)$.

The paper is organized in the following way. In section 2 we recall some results concerning the Hardy spaces on a symmetric strip. The boundedness results of the singular integrals which arise in this context are consequence of the standard theory of Calderón-Zygmund convolution operators, but, to the best of the author’s knowledge, they do not appear explicitly in the literature. Therefore, we give some hints for the proofs since we perform some computations which will be used in the sections that follow. In section 3 we study in detail the Hilbert space $H^2(D'_\beta)$. In Section 4 we study the spaces $H^p(D'_\beta)$ and we prove Theorem 1.1 and Theorem 1.2.

Unless specified, we will use standard and self-explanatory notation. We will denote by $C$, possibly with subscripts, a constant that may change from place to place.

This paper is part of the author’s doctoral dissertation, written under the supervision of Prof. Marco M. Peloso at Università degli Studi di Milano. The author extends his gratitude to Professor Peloso.

2. HARDY SPACES FOR A SYMMETRIC STRIP

In the Introduction we mentioned that the non-smooth worm domain $D'_\beta$ can be sliced in strips. This feature of $D'_\beta$ will be fundamental in the development of the Hardy spaces $H^p(D'_\beta)$ since it will allow us to use the theory of Hardy spaces on a strip. Hence, we recall here some results concerning the $H^p(S\beta)$ spaces where $S\beta$ is the symmetric strip

$$
S\beta = \{x + iy \in \mathbb{C} : |y| < \beta\}.
$$

The results contained in this section are well-known. The boundedness results of the singular integrals which arise in this context are consequence of the standard theory of Calderón-Zygmund convolution operators. Some of these results are contained in [BK07] and [Sed75], nevertheless, for the reader’s convenience, we include here some details. For full details, we refer also to [Mon].
For every \( p \in (1, \infty) \), the Hardy space for the strip \( S_\beta \) is the functional space

\[
H^p(S_\beta) = \left\{ f \text{ holomorphic in } S_\beta : \|f\|_{H^p(S_\beta)} < \infty \right\},
\]

where

\[
\|f\|_{H^p(S_\beta)} = \sup_{0 \leq y < \beta} \left[ \int_{\mathbb{R}} |f(x + iy)|^p + \int_{\mathbb{R}} |f(x - iy)|^p \, dy \right].
\]

By Mean Value Theorem, it is immediate to prove that

\[
\sup_{z \in K} |f(z)| \leq C_K \|f\|_{H^p(S_\beta)}
\]

where \( K \) is a compact subset of \( S_\beta \).

Now, we recall the well-known Paley–Wiener Theorem for a strip, which relates the growth of a holomorphic function in a strip with the growth of the Fourier transform of its restriction to the real line. We refer to [PWS7] for the proof.

**Theorem 2.1.** (Paley–Wiener Theorem for a strip) Let \( f_0 \) in \( L^2(\mathbb{R}) \). Then the following are equivalent:

(i) \( f_0 \) is the restriction to the real line of a function \( F \) holomorphic in the strip \( S_\beta \) such that

\[
\sup_{|y| < \beta} \int_{\mathbb{R}} |F(x + iy)|^2 \, dy < \infty;
\]

(ii) \( e^{\beta|\xi|} \hat{f}_0 \in L^2(\mathbb{R}) \).

Moreover, the following relationship holds

\[
F(z) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}_0(\xi) e^{iz\xi} \, d\xi = \mathcal{F}^{-1}[e^{-\text{Im} z(\cdot)} \hat{f}_0](\text{Re } z).
\]

Since \( \partial S_\beta \) has two boundary components and each of these components can be identified with the real line, the notation \( L^p(\partial S_\beta) \) denotes the space of functions \( \varphi = (\varphi_+, \varphi_-) \) such that

\[
\|\varphi\|_{L^p(\partial S_\beta)}^p = \int_{\mathbb{R}} |\varphi_+(x)|^p \, dx + \int_{\mathbb{R}} |\varphi_-(x)|^p \, dx < \infty.
\]

We use the notation \( \varphi_\pm \) since we think of \( \varphi_+ \) as a function defined on the upper boundary of the strip \( S_\beta \) and of \( \varphi_- \) as a function defined on the lower boundary.

The Paley–Wiener Theorem guarantees that the function \( \tilde{f}_\kappa(x + \kappa i\beta) := \mathcal{F}^{-1}[e^{-\kappa \text{Im}(\cdot)} \hat{f}_0](x) \) is well-defined for \( \kappa \in \{+, -\} \), therefore we can endow \( H^2(S_\beta) \) with the inner product

\[
\langle f, g \rangle_{H^2(S_\beta)} := \langle \tilde{f}, \tilde{g} \rangle_{L^2(\partial S_\beta)}.
\]

The space \( H^2(S_\beta) \) is a reproducing kernel Hilbert spaces with respect to this inner product. Hence, from (2.2) and the Paley–Wiener Theorem, we obtain the following result.

**Theorem 2.2.** The reproducing kernel of the Hardy space \( H^2(S_\beta) \) is the function

\[
K_{S_\beta}(w, z) = \frac{1}{4\pi} \int_{\mathbb{R}} \frac{e^{i(w-x-\kappa i\beta)}}{\text{Ch}[2\beta\xi]} \, d\xi = \frac{1}{2\beta \text{Ch}[\frac{1}{2\beta}] |w-z|}.
\]
Moreover, for all \( f \in H^2(S) \),
\[
\lim_{y \to \pm \beta} f(\cdot + iy) = f_{\pm}
\]
where the limit holds in \( L^2(\mathbb{R}) \) and for almost every \( x \) in \( \mathbb{R} \).

The integration against the kernel \( K_{S \beta} \) induces an operator which can be continuously extended to \( L^p(\partial S) \) for every \( p \in (1, \infty) \).

**Theorem 2.3.** Let \( \varphi = (\varphi_+, \varphi_-) \) be a function in \( L^2(\partial S) \cap L^p(\partial S) \), \( p \in (1, \infty) \) and consider the operator \( \varphi \mapsto S\varphi \)
\[
S\varphi(z) := \int_{\mathbb{R}} \varphi_+(x) K_{S \beta}(z, x + i\beta) \, dx + \int_{\mathbb{R}} \varphi_-(x) K_{S \beta}(z, x - i\beta) \, dx.
\]

Then, the operator \( \varphi \mapsto S\varphi \) extends to a bounded linear operator \( S : L^p(\partial S) \to H^p(S) \).

**Proof.** For future reference, we observe that for a function \( f \in L^2(\partial S) \cap L^p(\partial S) \) it holds
\[
(2.4) \quad S\varphi(z) = \mathcal{F}^{-1} \left[ e^{-\frac{\text{Im}z \pm \beta}{2}} \right] (\text{Re} z) + \mathcal{F}^{-1} \left[ e^{-\frac{\text{Im}z - \beta}{2}} \right] (\text{Re} z).
\]
The \( L^p \) boundedness of the operator \( S \) easily follows from Mihlin’s multipliers theorem (see e.g. [Gra08, Chapter 5]). \(\square\)

More can be proved; namely, given \( \varphi = (\varphi_+, \varphi_-) \) in \( L^p(\partial S) \), define the function \( \tilde{\varphi} = (\tilde{\varphi}_+, \tilde{\varphi}_-) \) by
\[
\tilde{\varphi}_+(x + i\beta) = \mathcal{F}^{-1} \left[ e^{-\frac{2|\beta|}{2}} \right] (x) + \mathcal{F}^{-1} \left[ \frac{\varphi_+}{2 \text{Ch}[2|\beta|]} \right] (x),
\]
\[
\tilde{\varphi}_-(x + i\beta) = \mathcal{F}^{-1} \left[ \frac{\varphi_-}{2 \text{Ch}[2|\beta|]} \right] (x) + \mathcal{F}^{-1} \left[ e^{\frac{2|\beta|}{2}} \right] (x).
\]

Consider now the operator \( \varphi \mapsto \tilde{\varphi} \) and define
\[
H^p(\partial S) = \{ \varphi = (\varphi_+, \varphi_-) \in L^p(\partial S) : \exists f \in H^p(S) \text{ s.t. } \tilde{f}_+ = \varphi_+, \tilde{f}_- = \varphi_- \}.
\]
Then, the following theorem holds.

**Theorem 2.4.** The operator \( \tilde{S} \) extends to a bounded linear operator
\[
\tilde{S} : L^p(\partial S) \to H^p(\partial S)
\]
\[
\varphi \mapsto \tilde{\varphi}
\]
for every \( p \in (1, \infty) \). Moreover,
\[
\lim_{y \to \pm \beta^p} S\varphi(\cdot + iy) = \tilde{\varphi}
\]
where the limits are in \( L^p(\mathbb{R}) \) and pointwise almost everywhere in \( \mathbb{R} \).

**Proof.** The boundedness of the operator \( \tilde{S} \) is immediately obtained by means of Mihlin’s multipliers theorem.

About the limits, we do not include the details of the proof in full generality, but we give the general idea of the proof in a simplified situation. Namely, we prove the theorem for a function \( \varphi = (\varphi_+, 0) \) in \( L^p(S) \) meaning that
\( \varphi_- \equiv 0 \). Instead of computing the limit for \( y \to \beta^- \), we compute the equivalent limit \( \lim_{\epsilon \to 0^+} S \varphi | + i(\beta - \epsilon) \rangle \), where

\[
S \varphi [x + i(\beta - \epsilon)] = \int_{\mathbb{R}} e^{-i(\beta - \epsilon) \xi} \overline{\varphi_-(\xi)} \frac{d\xi}{2 \text{Ch}[2|\beta|]} dy = \frac{1}{2\beta} \int_{\mathbb{R}} \overline{\varphi_+(x-y)} \text{Ch} \left[ \frac{\beta}{2|\beta|} \right] (y + i(2\beta - \epsilon)) dy
\]

\[
= \frac{1}{2\beta} \int_{\mathbb{R}} \varphi_+(x-y) \text{Ch} \left[ \frac{\beta}{2|\beta|} \right] \left( \frac{\beta}{2|\beta|} \right)^2 \frac{d\xi}{\text{Sh}[\frac{2|\beta|}{\pi}] + \sin^2 \frac{\beta}{|\beta|}} dy - i \frac{1}{2\beta} \int_{\mathbb{R}} \varphi_+(x-y) \text{Sh} \left[ \frac{\beta}{2|\beta|} \right] \cos \frac{\beta}{|\beta|} \frac{d\xi}{\text{Sh}[\frac{2|\beta|}{\pi}] + \sin^2 \frac{\beta}{|\beta|}} dy
\]

(\text{2.5})

Thus, we can study the kernels \( K_\epsilon \) and \( \tilde{K}_\epsilon \) separately. It is not hard to prove that the family of functions \( \{ \tilde{K}_\epsilon \} \) is a summability kernel, while the operator associated to the kernel \( \tilde{K}_\epsilon \) can be studied comparing it to the singular integral operator \( T \) defined on Schwartz functions by

\[
T g(x) = \lim_{\epsilon \to 0^+} \int_{|\xi| < \epsilon} \frac{g(x-y)}{\text{Sh}[\frac{2|\beta|}{\pi}]} dy.
\]

The conclusion follows now by the classical theory of Calderón-Zygmund singular integral operators. \( \square \)

**Remark 2.5.** We conclude this section with a remark concerning the continuity of functions in \( H^p(S_\beta) \).

Suppose that \( \varphi \in C_0^\infty (\partial S_\beta) \), i.e. \( \varphi_+ \) and \( \varphi_- \) belong to \( C_0^\infty (\mathbb{R}) \), then \( S \varphi \) belongs to \( H^p(S_\beta) \cap C(S_\beta) \), that is \( S \varphi \) is continuous up to the boundary of \( S_\beta \). This fact easily follows by dominated convergence from (\text{2.4}).

3. HARDY SPACES ON \( D_\beta^\prime \): THE L^2-THEORY

In this section we study in detail the Hardy space \( H^2(D_\beta^\prime) \) according to this plan:

- we first decompose \( H^2(D_\beta^\prime) \) as direct sum of subspaces \( \mathcal{H}^2 \) using the rotational invariance in the second variable and the theory of Fourier series (Proposition \text{3.3});

- using such a decomposition we show that each \( F \in H^2(D_\beta^\prime) \) admits boundary values in \( L^2(\partial D_\beta^\prime) \) (Proposition \text{3.8});

- we identify the inner product in \( H^2(D_\beta^\prime) \) as an \( L^2 \) inner product on the distinguished boundary (Proposition \text{3.8}) obtaining that the decomposition of \( H^2(D_\beta^\prime) \) is an orthogonal decomposition;

- we define the Szegö projection operator and we formulate a Paley-Wiener Theorem for the domain \( D_\beta^\prime \) (Theorem \text{3.14});

- we prove the Sobolev regularity of the Szegö projection (Theorem \text{1.2}).

We start proving a results which is actually true for every \( p \in (1, \infty) \). Using only the definition of \( H^p(D_\beta^\prime) \), we can immediately prove that every function \( F \) in \( H^p(D_\beta^\prime) \) admits a boundary value function \( \tilde{F} = (\tilde{F}_1, \tilde{F}_2, \tilde{F}_3, \tilde{F}_4) \) in \( L^p(\partial D_\beta^\prime) \) at least in a weak-* sense.

We define a family of functions \( F_{1,s} \) in \( L^p(\partial D_\beta^\prime) \), hence \( F_{1,s} = (F_{1,s,1}, F_{1,s,2}, F_{1,s,3}, F_{1,s,4}) \) by restricting \( F \) to copies of the distinguished boundary \( \partial D_\beta^\prime \) inside the domain \( D_\beta^\prime \). Namely, given a function \( F \) in \( H^p(D_\beta^\prime) \), \( p \in (1, \infty) \), for every \( (t,s) \in [0, \frac{\beta}{2}) \times [0, \beta - \frac{\beta}{2}) \), we define

\[
F_{1,s,1}(\zeta_1, \zeta_2) := F \left( \text{Re} \zeta_1 + i \frac{s-f}{\beta}, \text{Im} \zeta_1, e^{-i(\beta - \frac{\beta}{2})} \zeta_2 \right); \quad F_{1,s,2}(\zeta_1, \zeta_2) := F \left( \text{Re} \zeta_1 + i \frac{s-f}{\beta}, \text{Im} \zeta_1, e^{-i(\beta - \frac{\beta}{2})} \zeta_2 \right);
\]

\[
F_{3,s}(\zeta_1, \zeta_2) := F \left( \text{Re} \zeta_1 + i \frac{s-f}{\beta}, \text{Im} \zeta_1, e^{i(\beta - \frac{\beta}{2})} \zeta_2 \right). \quad F_{3,s,3}(\zeta_1, \zeta_2) := F \left( \text{Re} \zeta_1 + i \frac{s-f}{\beta}, \text{Im} \zeta_1, e^{i(\beta - \frac{\beta}{2})} \zeta_2 \right).
\]
The following proposition is elementary.

**Proposition 3.1.** Let $F$ be a function in $H^p(D'_β)$, $p \in (1, \infty)$. Then, the following facts hold:

(i) there exists a subsequence $F^{(r,s)}_n$ which admits a weak-$*$ limit $\tilde{F}$ in $L^p(\partial D'_β)$;

(ii) for every compact subset $K$ of $D'_β$, the estimate

$$
\sup_{(z_1, z_2) \in K} |F(z_1, z_2)| \leq C_K \|F\|_{H^p}^p
$$

holds.

We now focus on the space $H^2(D'_β)$; we prove that $H^2(D'_β)$ admits a nice decomposition which allows to describe explicitly its reproducing kernel.

**Theorem 3.2.** The Hardy space $H^2(D'_β)$ admits an orthogonal decomposition

$$
H^2(D'_β) = \bigoplus_{j \in \mathbb{Z}} \mathcal{H}_j^2,
$$

where each $\mathcal{H}_j^2$ is the subspace of $H^2(D'_β)$ defined as

$$
\mathcal{H}_j^2 = \{F \in H^2(D'_β) : F(z_1, e^{iθ}z_2) = e^{iθj}F(z_1, z_2)\}.
$$

Moreover, each subspace $\mathcal{H}_j^2$ is isometric to the Hardy space of the strip $H^2(S_β)$ equipped with a weighted norm depending on $j$.

The proof of Theorem 3.2 will follow from a series of results that we state and prove separately for the reader’s convenience.

Adapting a decomposition introduced by Barrett [Bar92] (see also [KP08a]), using the $z_2$ rotationally invariance of $D'_β$ and the connectedness of the set $D'_β(z_1) = \{z_2 \in C(z_1, z_2) \in D'_β\}$ for every fixed $z_1$, we obtain the following proposition.

**Proposition 3.3.** Let $F \in H^2(D'_β)$. Then,

$$
F(z_1, z_2) = \sum_{j \in \mathbb{Z}} \int_0^1 F(z_1, e^{2πiθ}z_2)e^{-2πiθj} dθ
$$

$$
= \sum_{j \in \mathbb{Z}} F_j(z_1, z_2) = \sum_{j \in \mathbb{Z}} f_j(z_1)z_2^j,
$$

where the series converges pointwise for every $(z_1, z_2) \in D'_β$ and each $f_j$ belongs to the Hardy space $H^2(S_β)$.

Since each function $f_j$ belongs to the Hardy space $H^2(S_β)$, all the results contained in the previous section are available. In particular, we know that the each function $f_j$ admits a boundary value function $\tilde{f}_j$ in $L^2(\partial S_β)$.

We remark also that the connectedness of the set $D'_β(z_1) = \{z_2 \in C : (z_1, z_2) \in D'_β\}$ for every fixed $z_1$ has a primary role since it permits to split the variables in each function $F_j$.

By the Paley–Wiener Theorem for the strip, the $H^2(D'_β)$ norm of each function $F_j$ in the sum (3.3) is easily computed.
Proposition 3.4. Let \( F_j(z_1, z_2) = f_j(z_1)z_2^2 \) be a function in \( H^2_j \), \( j \in \mathbb{Z} \). Then,
\[
\|F_j\|^2_{H^2_j(D'_B)} = \left[ e^{i(\beta - \frac{\pi}{2})} \|f_j| + i(\beta - \frac{\pi}{2})\|_{H^2_j(S^2)} + e^{-i(\beta - \frac{\pi}{2})} \|f_j| - i(\beta - \frac{\pi}{2})\|_{H^2_j(S^2)}^2 \right] = \frac{2}{\pi} \int_{\mathbb{R}} |\widetilde{f}_j(0)(\xi)^2 \text{Ch}(\pi \xi) \text{Ch}(\sqrt{2\beta - \pi}((\xi - \frac{j}{2})) \, d\xi.
\]
In particular,
\[
\sup_{(t,s)} L_2 F_j(t,s) = \lim_{(t,s) \to (\frac{\pi}{2}, \beta - \frac{\pi}{2})} L_2 F_j(t,s).
\]

Remark 3.5. Notice that, for every \( j \) fixed, the quantity
\[
(3.4) \quad \|f_j\|^2_{H^2_j(S^2)} = \|\widetilde{f}_j\|^2_{L^2(\partial S^2)} := \frac{2}{\pi} \int_{\mathbb{R}} |\widetilde{f}_j(0)(\xi)^2 \text{Ch}(\pi \xi) \text{Ch}(\sqrt{2\beta - \pi}((\xi - \frac{j}{2})) \, d\xi
\]
defines a norm on \( H^2(S^2) \) equivalent to the standard one. In conclusion, the previous proposition shows that \( F_j \mapsto \widetilde{f}_j \)
is an isometry between \( H^2_j \) and \( L^2(\partial S^2) \). This proves the second part of Theorem 3.2.

Proposition 3.6. Let \( F \) be a function in \( H^2(D'_B) \). Then
\[
\|F\|^2_{H^2(D'_B)} = \sup_{(t,s) \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} L_2 F_j(t,s) = \sum_{j \in \mathbb{Z}} \sup_{(t,s)} L_2 F_j(t,s) = \sum_{j \in \mathbb{Z}} \|F\|^2_{H^2_j(D'_B)},
\]
where the supremum is taken for \( (t,s) \) varying in \( [0, \frac{\pi}{2}] \times [0, \beta - \frac{\pi}{2}] \).

Proof. We already know that \( \|F\|^2_{H^2(D'_B)} = \sup_{(t,s)} \sum_{j \in \mathbb{Z}} L_2 F_j(t,s) \); it trivially follows from the orthogonality of trigonometric monomials. We would like to prove that it is possible to switch the supremum with the sum, i.e.
\[
\sup_{(t,s)} \sum_{j \in \mathbb{Z}} L_2 F_j(t,s) = \sum_{j \in \mathbb{Z}} \sup_{(t,s)} L_2 F_j(t,s).
\]
Since we know from Proposition 3.4 that \( \sup_{(t,s)} L_2 F_j(t,s) = \lim_{(t,s) \to (\frac{\pi}{2}, \beta - \frac{\pi}{2})} L_2 F_j(t,s) \), we can conclude using monotone convergence.

\[
\square
\]

Remark 3.7. From Proposition 3.3 and Proposition 3.6, it is easily deduced that the series \( \sum_{j \in \mathbb{Z}} F_j \) converges not only pointwise, but also in norm. That is,
\[
\|F - \sum_{j=-N}^{N} F_j\|_{H^2(D'_B)} \to 0
\]
as \( N \) tends to \(+\infty\).

Finally, we are able to prove that a function \( F \in H^2(D'_B) \) admits boundary values in \( L^2(\partial D'_B) \). Let \( F_{t,s} \) be the function defined in Proposition 3.1. Then, we have the following result.

Proposition 3.8. Let \( F(z_1, z_2) = \sum_{j \in \mathbb{Z}} f_j(z_1)z_2^2 \) be a function in \( H^2(D'_B) \). For \( (\zeta_1, \zeta_2) \in \partial D'_B \) define
\[
\bar{F}(\zeta_1, \zeta_2) := \sum_{j \in \mathbb{Z}} \bar{f}_j(\zeta_1, \zeta_2) = \sum_{j \in \mathbb{Z}} \bar{f}_j(\zeta_1) \zeta_2^2.
\]
Then \( F_{t,s} \to \bar{F} \) in \( L^2(\partial D'_B) \) as \( (t,s) \to (\frac{\pi}{2}, \beta - \frac{\pi}{2}) \).
Proof. Theorem 3.6 guarantees that $\tilde{F}$ is well defined. We want to prove that
\[ \int_{\partial D'_b} |\tilde{F}(\zeta_1, \zeta_2) - F_{t,s}(\zeta_1, \zeta_2)|^2 \to 0 \]
as $(t,s) \to (\frac{\beta}{2}, \beta - \frac{\pi}{2})$. Since $F$ is in $H^2(D'_b)$, it holds
\[ \|\tilde{F} - F_{t,s}\|_{L^2(\partial D'_b)}^2 = \sum_{j \in \mathbb{Z}} \|\tilde{F}_j - (F_{t,s})_j\|_{L^2(\partial D'_b)}^2 < \infty. \]
Moreover, $\|\tilde{F}_j - (F_{t,s})_j\|_{L^2(\partial D'_b)}^2 \to 0$ as $(t,s) \to (\frac{\beta}{2}, \beta - \frac{\pi}{2})$. By monotone convergence for decreasing sequences, we can switch the sum and the limit obtaining
\[ \lim_{(t,s) \to (\frac{\beta}{2}, \beta - \frac{\pi}{2})} \|\tilde{F} - F_{t,s}\|_{L^2(\partial D'_b)}^2 = \sum_{j \in \mathbb{Z}} \lim_{(t,s) \to (\frac{\beta}{2}, \beta - \frac{\pi}{2})} \|\tilde{F}_j - (F_{t,s})_j\|_{L^2(\partial D'_b)}^2 = 0. \]
The conclusion follows. \(\square\)

Thus, we proved that a given function $F(z_1, z_2) = \sum_{j \in \mathbb{Z}} f_j(z_1)z_2^j$ admits a boundary value function $\tilde{F}(\zeta_1, \zeta_2) = \sum_{j \in \mathbb{Z}} \tilde{f}_j(\zeta_1)\zeta_2^j$ in $L^2(\partial D'_b)$. Moreover, as expected, the identity
\begin{equation}
(3.5) \quad \|F\|_{H^2(D'_b)} = \|\tilde{F}\|_{L^2(\partial S_b)}
\end{equation}
holds.

As in the case of the strip, we identify the inner product in $H^2(D'_b)$ as an $L^2$ inner product on the distinguished boundary. Namely, given $F, G$ in $H^2(D'_b)$, we define
\begin{equation}
(3.6) \quad \langle F, G \rangle_{H^2(D'_b)} := \langle \tilde{F}, \tilde{G} \rangle_{L^2(\partial D'_b)} = \frac{1}{4} \int_{E_4} \tilde{F}(\zeta_1, \zeta_2)\overline{G(\zeta_1, \zeta_2)} \, d\zeta_1 d\zeta_2.
\end{equation}
The decomposition (3.7) is an orthogonal decomposition with respect to this inner product and Theorem 3.2 is finally proved.

3.1. The Szegő kernel and projection of $D'_b$. Before investigating the reproducing kernel $K_{D'_b}$ of $H^2(D'_b)$, we investigate the reproducing kernels of the subspaces $H^2_j$. The particular structure of each $H^2_j$ and Proposition 3.4 allow us to look for the kernels of the spaces $H^2_j(S_b)$.

Proposition 3.9. The reproducing kernel of $H^2_j(S_b)$ is the function
\[ k_j(z_1, z_2) = \frac{1}{8\pi} \int_{\mathbb{R}} \frac{e^{i(z_1 - \pi)\xi}}{\text{Ch}(\pi\xi)\text{Ch}((2\beta - \pi)(\xi - \frac{1}{2}))} \, d\xi. \]

Proof. Given $z_2$ in $S_b$, by Remark 3.4 we have
\[ f(z_2) = \frac{2}{\pi} \int_{\mathbb{R}} f_0(\xi)\hat{k}_{j,0}(\xi, z_2) \text{Ch}(\pi\xi)\text{Ch}((2\beta - \pi)(\xi - \frac{1}{2})) \, d\xi = \frac{1}{2\pi} \int_{\mathbb{R}} f_0(\xi)e^{i\xi z_2} \, d\xi, \]
where the last equality holds since $f$ belongs to $H^2(S_b)$. It follows
\[ \hat{k}_{j,0}(\xi, z_2) = \frac{1}{4\pi} \frac{e^{-i\pi\xi}}{\text{Ch}(\pi\xi)\text{Ch}((2\beta - \pi)(\xi - \frac{1}{2}))}. \]
Proof. Finally, we prove the proposition supposing that 

\[ j \leq \sum_{j \in \mathbb{Z}} \frac{w_j}{8\pi} \int \frac{e^{j(\zeta - \pi)^2}}{C_h(\pi \xi) C_h((2\beta - \pi)(\xi - \frac{\pi}{2}))} d\xi, \]

(3.7)

where the series converges in $H^2(D'_{\beta'})$ for every fixed $(z_1, z_2)$ in $D'_{\beta'}$.

If we fix a compact subset $K$ in $D'_{\beta'}$ we have a stronger convergence for the series which defines $K_{D'_{\beta'}}$.

**Proposition 3.10.** Let us consider $K_{D'_{\beta'}} (z, \zeta) = K_{D'_{\beta'}} ([z_1, z_2], (\zeta_1, \zeta_2)]$ where $(\zeta_1, \zeta_2) \in \partial D'_{\beta'}$ and $(z_1, z_2)$ varies in a compact set $K \subseteq D'_{\beta'}$. Then,

\[ \sum_{j \in \mathbb{Z}} \sup_{z, \zeta \in K} |k_j(z_1, \zeta_1)z_2^2z_2| < \infty \]

**Proof.** We prove the proposition supposing that $(\zeta_1, \zeta_2)$ is in $E_1$. The general case will follow analogously. In order to estimate the size of $k_j$, suppose for the moment that $j < 0$. Then,

\[ |k_j(z_1, \zeta_1)| = |k_j(z_1, x + i\beta)| \leq \int_{-\infty}^{\pi/2} \frac{e^{-|\zeta_1 + \beta|^2}}{C_h(\pi \xi) C_h((2\beta - \pi)(\xi - \frac{\pi}{2}))} d\xi \]

\[ = \left( \int_{-\infty}^{\pi/2} + \int_{\pi/2}^{0} + \int_{0}^{+\infty} \right) \frac{e^{-|\zeta_1 + \beta|^2}}{C_h(\pi \xi) C_h((2\beta - \pi)(\xi - \frac{\pi}{2}))} d\xi. \]

It follows that

\[ \int_{-\infty}^{\pi/2} \frac{e^{-|\zeta_1 + \beta|^2}}{C_h(\pi \xi) C_h((2\beta - \pi)(\xi - \frac{\pi}{2}))} d\xi \approx \int_{-\infty}^{\pi/2} \frac{e^{-|\zeta_1 + \beta|^2}}{e^{-\pi \xi} e^{-(2\beta - \pi)(\xi - \frac{\pi}{2})}} d\xi = \frac{e^{-j(\beta - \frac{\pi}{2})}}{\beta - \text{Im} z_1}; \]

\[ \int_{\pi/2}^{0} \frac{e^{-|\zeta_1 + \beta|^2}}{C_h(\pi \xi) C_h((2\beta - \pi)(\xi - \frac{\pi}{2}))} d\xi \approx \int_{0}^{\pi/2} \frac{e^{-|\zeta_1 + \beta|^2}}{e^{-\pi \xi} e^{-(2\beta - \pi)(\xi - \frac{\pi}{2})}} d\xi = \frac{e^{j(\beta - \frac{\pi}{2})}}{\text{Im} z_1 + 3\beta}; \]

\[ \int_{0}^{+\infty} \frac{e^{-|\zeta_1 + \beta|^2}}{C_h(\pi \xi) C_h((2\beta - \pi)(\xi - \frac{\pi}{2}))} d\xi \approx \int_{0}^{+\infty} \frac{e^{-|\zeta_1 + \beta|^2}}{e^{-\pi \xi} e^{-(2\beta - \pi)(\xi - \frac{\pi}{2})}} d\xi = \frac{e^{-j(\beta - \frac{\pi}{2})}}{\text{Im} z_1 + 3\beta}. \]

Notice that all these estimates do not depend on Re $\zeta_1$ and the term $\frac{e^{-j(\beta - \frac{\pi}{2})}}{\beta - \text{Im} z_1}$ is not singular when $\text{Im} z_1 + 3\beta - 2\pi \rightarrow 0$. Finally,

\[ \sum_{j < 0} |z_2| e^{j(\beta - \frac{\pi}{2})} |k_j(z_1, x + i\beta)| \leq C \sum_{j < 0} \left[ \frac{e^{j|z_2|^2 + \frac{\pi}{2} - \text{Im} z_1}}{\beta - \text{Im} z_1} + \frac{e^{j|z_2|^2 - \text{Im} z_1 + \frac{\pi}{2}}}{\text{Im} z_1 + 3\beta - 2\pi} + \frac{e^{j|z_2|^2 + \log |z_2|^2 + \frac{\pi}{2}}}{\text{Im} z_1 + 3\beta} \right]. \]
and it is immediate to see that we get a uniform bound for \((z_1, z_2) \in K\). Analogous computations prove the estimate for the sum over positive \(j\)'s.

We want to prove now that the integration against the kernel \(K_{D'_b}\) not only reproduces function in \(H^2(D'_b)\), but actually produces functions in \(H^2(D'_b)\).

**Proposition 3.11.** Let \(\varphi\) be a function in \(L^2(\partial D'_b)\). Then, the function

\[
S\varphi(z_1, z_2) := \langle \varphi, K_{D'_b}((\cdot, \cdot), (z_1, z_2)) \rangle_{L^2(\partial D'_b)}
\]

is in \(H^2(D'_b)\). Moreover,

\[
\|S\varphi\|_{H^2(D'_b)} \leq \|\varphi\|_{L^2(\partial D'_b)}.
\]

**Proof.** It is sufficient to prove the theorem for a function in \(L^2(\partial D'_b)\) of the form \(\varphi = (\varphi_1, 0, 0, 0)\). The results for a general function \(\varphi\) will follow by linearity. Therefore, by Plancherel’s theorem,

\[
\|
\varphi\|^2_{L^2(\partial D'_b)} = \frac{1}{2\pi} \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}} |\mathcal{F}\varphi(\xi, \tilde{j})|^2 \, d\xi.
\]

The holomorphicity of \(S\varphi\) is obtained using Proposition [3.10] and Morera’s theorem. It remains to prove that \(S\varphi\) satisfies the \(H^2\) growth condition. Thus,

\[
S\varphi(u + iv, re^{2\pi i r}) = \int_{\mathbb{R}} \int_0^1 \varphi_1(x, \theta) \sum_{j \in \mathbb{Z}} k_j(u + iv, x + ij\beta) e^{2\pi i j\beta - \frac{\pi}{2}} e^{-2\pi i j\theta} \, d\theta dx
\]

\[
= \frac{1}{4} \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}} e^{2\pi i j\beta - \frac{\pi}{2}} e^{2\pi i j\theta} \left[ \frac{e^{-((\pi + \beta)(\xi - \tilde{j}))}}{\text{Ch}[\pi \xi] \text{Ch}[(\pi - \beta)(\xi + \tilde{j})]} \right] (u).
\]

Hence,

\[
\int_{\mathbb{R}} \int_0^1 \left| S\varphi(u + iv, re^{2\pi i r}) \right|^2 d\varphi du = \frac{1}{8\pi} \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}} \left| \frac{e^{-((\pi + \beta)(\xi - \tilde{j}))} e^{((\pi + \beta)(\xi + \tilde{j}))}}{\text{Ch}[\pi \xi] \text{Ch}[(\pi - \beta)(\xi + \tilde{j})]} \right|^2 d\xi
\]

\[
\leq \frac{1}{8\pi} \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}} \left| \mathcal{F}\varphi(\xi, \tilde{j}) \right|^2 d\xi,
\]

By analogous computations we estimate the other three terms of the \(H^2\) growth condition and we conclude by taking the supremum for \((t, s) \in [\frac{\pi}{2}, \pi] \times [0, \beta - \frac{\pi}{2}]\) we conclude. □

**Remark 3.12.** We report for completeness the explicit expression of \(S\varphi\) given a general initial data \(\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)\) in \(L^2(\partial D'_b)\). Let \((u + iv, r^{2\pi i r})\) in \(D'_b\), then

\[
S\varphi(u + iv, re^{2\pi i r}) = \frac{1}{4} \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}} e^{2\pi i j\beta - \frac{\pi}{2}} e^{2\pi i j\theta} \left[ \frac{e^{-((\pi + \beta)(\xi - \tilde{j}))}}{\text{Ch}[\pi \xi] \text{Ch}[(\pi - \beta)(\xi + \tilde{j})]} \right] (u)
\]

\[
+ \frac{1}{4} \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}} e^{2\pi i j\beta - \frac{\pi}{2}} e^{2\pi i j\theta} \left[ \frac{e^{-((\pi - \beta)(\xi - \tilde{j}))}}{\text{Ch}[(\pi - \beta)(\xi + \tilde{j})]} \right] (u)
\]

\[
+ \frac{1}{4} \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}} e^{2\pi i j\beta - \frac{\pi}{2}} e^{2\pi i j\theta} \left[ \frac{e^{-((\pi + \beta)(\xi - \tilde{j}))}}{\text{Ch}[(\pi + \beta)(\xi + \tilde{j})]} \right] (u)
\]

\[
+ \frac{1}{4} \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}} e^{2\pi i j\beta - \frac{\pi}{2}} e^{2\pi i j\theta} \left[ \frac{e^{-((\pi - \beta)(\xi - \tilde{j}))}}{\text{Ch}[(\pi - \beta)(\xi + \tilde{j})]} \right] (u)
\]
Proposition 3.13. Let \( \phi \) be a function in \( L^2(\partial D^\beta_0) \). Then,

\[
\lim_{(s,t) \to (\beta - \frac{3}{2}, \frac{3}{2})} \| [S\phi]_{s,t} - \psi \|_{L^2(\partial D^\beta_0)} = \lim_{(s,t) \to (\beta - \frac{3}{2}, \frac{3}{2})} \sum_{l=1}^{4} \| [S\phi]_{l,s} - \psi_{l} \|_{L^2(E_l)} = 0
\]

In particular, \( \tilde{S\phi} = \psi \).

Proof. We only prove explicitly that

\[
\| [S\phi]_{l,s} - \psi_{l} \|_{L^2(E_l)} \to 0
\]

for a simpler function \( \phi \) of the form \( \phi = (\phi_1, 0, 0, 0) \) in \( L^2(\partial D^\beta_0) \). The complete proof for a general function \( \phi \) is obtained with similar arguments. We have

\[
\| S\phi(\cdot + i(s+t))e^{2\pi i(\cdot)} - \psi(\cdot + i\beta, e^{\frac{i}{2}(\beta - \frac{3}{2})}) e^{2\pi i(\cdot)} \|_{L^2(\mathbb{R} \times T)} =
\]

\[
= \frac{1}{8\pi} \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}} \left| \mathcal{F}_\mathbb{R} \phi_1(\xi, \bar{\xi}) e^{-(i\beta+\frac{\beta}{2})(\xi - \frac{3}{2})} - e^{-(\xi - \frac{3}{2})} e^{-\pi \xi} - e^{-(2\beta - \pi)\xi} e^{-\pi \xi} \right|^2 d\xi
\]

\[
\leq \frac{1}{8\pi} \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}} \left| \mathcal{F}_\mathbb{R} \phi_1(\xi, \bar{\xi}) \right|^2 d\xi
\]

< \infty.

By dominated convergence, we can conclude. \( \square \)

Let us define

\[ H^2(\partial D^\beta_0) := \{ \phi = (\phi_1, \phi_2, \phi_3, \phi_4) \in L^2(\partial D^\beta_0) : \exists F \in H^2(D^\beta_0) \text{ s.t. } \phi = \tilde{F} \}. \]

We conclude this section stating a Paley–Wiener type of result.
Theorem 3.14. (Paley–Wiener Theorem for $D'_p$) Let $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ be a function in $L^2(\partial D'_p)$. Then, $\varphi$ is in $H^2(\partial D'_p)$ if and only if there exists a sequence of functions $\{g_j\}$ such that

$$
\sum_{j \in \mathbb{Z}} \int_{\mathbb{R}} |g_j(\xi)|^2 \text{Ch}[\pi \xi] \text{Ch}[(2\beta - \pi)(\xi - \frac{j}{2})] \, d\xi < \infty
$$

and

$$
\varphi_1(x + i\beta, e^{\frac{1}{2}(\beta - \frac{x}{2})} e^{2\pi i j}) = \sum_{j \in \mathbb{Z}} \varphi_{1,j}(x + i\beta) e^{\frac{1}{2}(\beta - \frac{x}{2})} e^{2\pi i j};
$$

$$
\varphi_2(x + i(\beta - \pi), e^{\frac{1}{2}(\beta - \frac{x}{2})} e^{2\pi i j}) = \sum_{j \in \mathbb{Z}} \varphi_{2,j}(x + i(\beta - \pi)) e^{\frac{1}{2}(\beta - \frac{x}{2})} e^{2\pi i j};
$$

$$
\varphi_3(x - i\beta, e^{-\frac{1}{2}(\beta - \frac{x}{2})} e^{2\pi i j}) = \sum_{j \in \mathbb{Z}} \varphi_{3,j}(x - i\beta) e^{-\frac{1}{2}(\beta - \frac{x}{2})} e^{2\pi i j};
$$

$$
\varphi_4(x - i(\beta - \pi), e^{-\frac{1}{2}(\beta - \frac{x}{2})} e^{2\pi i j}) = \sum_{j \in \mathbb{Z}} \varphi_{4,j}(x - i(\beta - \pi)) e^{-\frac{1}{2}(\beta - \frac{x}{2})} e^{2\pi i j},
$$

where, for every $j \in \mathbb{Z}$,

$$
\varphi_{1,j}(x + i\beta) = \mathcal{F}^{-1}_{\mathbb{R}} \left[ e^{-\beta(\cdot)} g_j(\cdot) \right](x);
\varphi_{2,j}(x + i(\beta - \pi)) = \mathcal{F}^{-1}_{\mathbb{R}} \left[ e^{-(\beta - \pi)(\cdot)} g_j(\cdot) \right](x);
\varphi_{3,j}(x - i\beta) = \mathcal{F}^{-1}_{\mathbb{R}} \left[ e^{\beta(\cdot)} g_j(\cdot) \right](x);
\varphi_{4,j}(x - i(\beta - \pi)) = \mathcal{F}^{-1}_{\mathbb{R}} \left[ e^{-(\beta - \pi)(\cdot)} g_j(\cdot) \right](x).
$$

Moreover,

$$
\left(3.11\right)
S\varphi(u + iv, re^{2\pi i j}) = \sum_{j \in \mathbb{Z}} r^j e^{2\pi i j} \mathcal{F}^{-1}_{\mathbb{R}} \left[ e^{-v(\cdot)} g_j(\cdot) \right](u).
$$

Proof. Suppose that $\varphi$ belongs to $H^2(\partial D'_p)$. Then, the conclusion follows from Theorem 3.2 and the Paley–Wiener Theorem for a strip. Conversely, let $\{g_j\}$ be a sequence which defines $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ as in the hypothesis. It follows that $S\varphi$ belongs to $H^2(D'_p)$ and the formula in Definition 3.10 guarantees that $S\varphi_1 = \varphi_1$. Analogously it can be proved for $k = 2, 3, 4$. The proof is complete. \qed

4. HARDY SPACES ON $D'_p$: THE $L^p$-THEORY

In this section we extend the results we have seen so far to the case $p \in (1, \infty)$. In detail,

- we show that the Szegő projection can be realized as a composition of simpler operators we are able to study and we prove our main result Theorem 1.1;
- we prove that the space $H^p(D'_p)$, $p \in (1, \infty)$, admits a decomposition analogous to (3.1) for the case $p = 2$ (Proposition 3.3);
- we prove a Fatou-type theorem; that is, we prove that an appropriate restriction of a function $F$ in $H^p(D'_p)$, $p \in (1, \infty)$, converges to its boundary value function $F$ pointwise almost everywhere (Theorem 4.12).

For a function $\varphi$ in $L^p(\partial D'_p)$ of the form $\varphi = (\varphi_1, 0, 0, 0)$, we recall that the formulas in Definition 3.10 and 3.9 reduce to

$$
\left(4.1\right)
\widetilde{S\varphi}_1(x + i\beta, e^{\frac{1}{2}(\beta - \frac{x}{2})} e^{2\pi i j}) = \frac{1}{4} \sum_{j \in \mathbb{Z}} e^{2\pi i j} \mathcal{F}^{-1}_{\mathbb{R}} \left[ e^{-\pi(\cdot)} \text{Ch}[\pi(\cdot)] \text{Ch}[(2\beta - \pi)(\cdot - \frac{j}{2})] \right](x);
$$
We observe that the operators \( \varphi \mapsto \widetilde{S}\varphi \) and \( \varphi \mapsto S\varphi_{\ell, s} \) are well-defined for functions \( \varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4) \) where each \( \varphi_{\ell, s} \), \( \ell = 1, \ldots, 4 \) is of the form
\[
\varphi_{\ell, s}(x, y) = \sum_{|j| < N} \varphi_{\ell}(x, j)e^{2\pi ijy} \quad \text{with } \varphi_{\ell}(x, j) \in C_0^\infty(\mathbb{R}) \text{ for every } j \text{ and the sum is over a finite number of } j's. \text{ Moreover, the set of functions } \varphi \text{ of such a form is dense in } L^p(\partial D_0').
\]

**Proposition 4.1.** Let \( \varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4) \) be a function in \( L^p(\partial D_0') \). Then, for every \( p \in (1, \infty) \),
\[
\|S_{\ell, s}\varphi\|_{L^p(\mathbb{R}^2 \times \mathbb{T})} \leq C_p\|\varphi\|_{L^p(\partial D_0')},
\]
where the constant \( C_p \) does not depend on \( y \) and \( s \).

**Proof.** By density and linearity it suffices to prove the theorem for a function \( \varphi \) of the form \( \varphi = (\varphi_1, 0, 0, 0) \) where \( \varphi_1(x, y) = \sum_{j=-N}^N \varphi_1(x, j)e^{2\pi ijy} \) and each function \( \varphi_1(\cdot, j), j = 1, \ldots, N \) is in \( C_0^\infty(\mathbb{R}) \). Then,
\[
S_{\ell, s}\varphi(x, y) = [\lambda_{\ell, s} \circ \lambda_{s}]\varphi(x, y),
\]
where
\[
\lambda_{s}\varphi(x, y) = \frac{1}{2\pi} \int_{\mathbb{R}} \sum_{j=-N}^N e^{2\pi ijy} \frac{e^{-(\beta - \frac{2\pi}{\xi} + s)(\xi - \frac{j}{2})}}{4 \text{Ch}[2\beta - \pi](\xi - \frac{j}{2})} F_{\text{Re}}\varphi_1(\xi, j)e^{it\xi} d\xi
\]
(4.4)
and
\[
\lambda'_{s}\varphi(x, y) = \frac{1}{2\pi} \int_{\mathbb{R}} \sum_{j=-N}^N e^{2\pi ijy} m_s(\xi - \frac{j}{2}) F_{\text{Re}}\varphi_1(\xi, j)e^{it\xi} d\xi
\]
(4.5)
We recall that \( y \) and \( s \) are such that \( (x + iy, e^{ix}e^{2\pi ijy}) \) is in \( D_0' \), thus \(|s| \in (0, \beta - \frac{\pi}{2}) \) and \(|y - s| \in (0, \frac{\pi}{2}) \). Then, by Mihlin’s multipliers theorem, it is not hard to prove that \( m'_{s, \ell, s} \) is a multiplier of \( L^p(\mathbb{R}) \) for every \( p \in (1, \infty) \) with norm independent of \( y \) and \( s \). Thus the operator \( \lambda'_{s, \ell} \) extends to a bounded linear operator \( L^p(\mathbb{R}^2 \times \mathbb{T}) \rightarrow L^p(\mathbb{R}^2 \times \mathbb{T}) \) for every \( p \in (1, \infty) \). About \( \lambda_{s} \), we have
\[
\lambda_{s}\varphi(x, y) = \frac{1}{2\pi} \int_{\mathbb{R}} \sum_{j=-N}^N e^{2\pi ijy} m_s(\xi + \frac{j}{2}) F_{\text{Re}}\varphi_1(\xi + \frac{j}{2}, j)e^{it\xi} d\xi
\]
(4.6)
Therefore, by a change of variables and the periodicity of the exponential function,
\[
\int_{\mathbb{R} \times T} |\lambda\varphi(x,\gamma)|^p dx d\gamma = \int_0^1 \int_{\mathbb{R}} \left( \frac{1}{2\pi} \int_{\mathbb{R}} m_\lambda(\xi) \sum_{j=-N}^N e^{2\pi i j(\cdot, j)} \left[ e^{-i\xi(t, j)} \varphi_1(t, j) \right] e^{i\xi} d\xi \right)^p dx d\gamma
\]
\[
= \int_0^1 \int_{\mathbb{R}} \left( \frac{1}{2\pi} \int_{\mathbb{R}} m_\lambda(\xi) \sum_{j=-N}^N e^{-i\xi(t, j)} \varphi_1(t, j) e^{2\pi i j(\cdot, j)} \right) e^{i\xi} d\xi \right)^p dx d\gamma.
\]
Again, by Mihlin’s multipliers theorem, we obtain that \(m_\lambda\) is a multiplier of \(L^p(\mathbb{R})\) for every \(p \in (1, \infty)\) with norm independent of \(s\). Therefore, if we prove that the function \(\sum_{j=-N}^N e^{-i\xi(t, j)} \varphi_1(t, j)e^{2\pi i j(\cdot, j)}\) is in \(L^p(\mathbb{R} \times T)\), we will obtain the \(L^p\) boundedness of the operator \(\lambda\). By a change of variables and the periodicity of the exponential function, we have
\[
\int_{\mathbb{R}} \int_0^1 \left( \sum_{j=-N}^N e^{-i\xi(t, j)} \varphi_1(t, j)e^{2\pi i j(\cdot, j)} \right)^p dt d\gamma = \int_{\mathbb{R}} \int_0^1 \left( \sum_{j=-N}^N \varphi_1(t, j)e^{2\pi i j(\cdot, j)} \right)^p dt d\gamma
\]
\[
= \|\varphi\|_{L^p(\partial D'_b)}^p < \infty.
\]
Finally, we conclude the proof exploiting the boundedness of the operators \(\lambda\) and \(\lambda'_\gamma\).

The last proposition allows us to prove that the operator \(S\) extends to a continuous operator with respect to the \(L^p\) norm.

**Theorem 4.2.** For every \(p \in (1, \infty)\), the operator \(S\) extends to a bounded linear operator
\[
S : L^p(\partial D'_b) \to H^p(D'_b).
\]

**Proof.** Suppose that \(\varphi = (\varphi_1, 0, 0, 0)\) is a function in \(L^p(\partial D'_b) \cap L^2(\partial D'_b)\). Then, Proposition 3.11 assures that \(S\varphi\) is holomorphic on \(D'_b\). Moreover,
\[
\|S\varphi\|_{L^p(\partial D'_b)}^p = \sup_{(t, s) \in [0, 1/2] \times [0, b]} \|L_p S\varphi(t, s)\|
\]
\[
= \sup_{(t, s)} \left[ \|S_{s+t, s} \varphi\|_{L^p(\mathbb{R} \times T)}^p + \|S_{s-t, s} \varphi\|_{L^p(\mathbb{R} \times T)}^p + \|S_{-(s+t), -s} \varphi\|_{L^p(\mathbb{R} \times T)}^p + \|S_{-(s-t), -s} \varphi\|_{L^p(\mathbb{R} \times T)}^p \right]
\]
\[
\leq C_p \|\varphi\|_{L^p(\partial D'_b)}^p
\]
with \(C_p\) independent of \(t\) and \(s\) thanks to Proposition 4.1. Thus, we proved the theorem when \(\varphi\) is in \(L^p(\partial D'_b) \cap L^2(\partial D'_b)\). By density we obtain the proof for a general function \(\varphi\) in \(L^p(\partial D'_b)\).

It remains to prove that \(S\varphi\) admits a boundary value function \(\tilde{S}\varphi\). In order to keep the length of this work contained, we prove explicitly that \(\tilde{S}\varphi\) is a boundary value function for \(S\varphi\) on the component \(E_1\) of the distinguished boundary \(\partial D'_b\). Using a similar strategy it is possible to prove the analogous result for the other three components of the distinguished boundary.

**Theorem 4.3.** Let \(\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)\) be a function in \(L^p(\partial D'_b)\). Then, for every \(p \in (1, \infty)\),
\[
\lim_{(t, s) \to (1/2, b)} \|S_{s+t, s} \varphi - \tilde{S}\varphi_1\|_{L^p(\mathbb{R} \times T)} = 0.
\]
The proof of the theorem will follow from a series of results that we state and prove separately. Let us fix some notation. Given \( \varphi = (\varphi_1,0,0) \), we define

\[
T_{s,t} \varphi(x,\gamma) := \left[ \hat{S}^{(s)} \varphi_t - S_{s+t,t} \varphi_t \right](x,\gamma)
\]

\[
= \sum_{j \in \mathbb{Z}} e^{2\pi i j T} \mathcal{F}_R^{-1} \left[ e^{-(2\beta-\pi)(\xi - \frac{j}{2})} e^{-\pi(\cdot)} e^{-i(\beta \cdot + 1)(\xi - \frac{j}{2})} \mathcal{F}_R \varphi_t(\cdot,\gamma) \right](x)
\]

\[
= \sum_{j \in \mathbb{Z}} e^{2\pi i j T} \mathcal{F}_R^{-1} \left[ m^I_{s,t}(\cdot,j) \mathcal{F}_R \varphi_t(\cdot,\gamma) \right](x) + \sum_{j \in \mathbb{Z}} e^{2\pi i j T} \mathcal{F}_R^{-1} \left[ m^H_{s,t}(\cdot,j) \mathcal{F}_R \varphi_t(\cdot,\gamma) \right](x)
\]

(4.8)

\[
= T^I_{s,t} \varphi(x,\gamma) + T^H_{s,t} \varphi(x,\gamma),
\]

where

\[
m^I_{s,t}(\xi,j) = \frac{1}{8} \left[ e^{-\pi \xi} e^{-i\frac{\xi}{2} - i\frac{j}{2}} \right] \left[ e^{-(2\beta-\pi)(\xi - \frac{j}{2})} e^{-\pi(\cdot)} e^{-i(\beta \cdot + 1)(\xi - \frac{j}{2})} \right] \left[ \frac{\text{Ch}[\pi(\xi - \frac{j}{2})]}{\text{Ch}[2\beta - \pi \xi]} \right]
\]

\[
m^H_{s,t}(\xi,j) = \frac{1}{8} \left[ e^{-\pi \xi} e^{-i\frac{\xi}{2} + i\frac{j}{2}} \right] \left[ e^{-(2\beta-\pi)(\xi - \frac{j}{2})} e^{-\pi(\cdot)} e^{-i(\beta \cdot + 1)(\xi - \frac{j}{2})} \right] \left[ \frac{\text{Ch}[\pi(\xi - \frac{j}{2})]}{\text{Ch}[2\beta - \pi \xi]} \right]
\]

Thus, the operator \( T^I_{s,t} \) can be seen as a composition of two operators, that is,

(4.9)

\[
T^I_{s,t} \varphi(x,\gamma) = [\Lambda_s \circ \Xi_t] \varphi(x,\gamma),
\]

where \( \Lambda_s \) and \( \Xi_t \) are defined by

\[
\Lambda_s \varphi(x,\gamma) := \sum_{j \in \mathbb{Z}} \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-\pi \xi}}{\text{Ch}[\pi(\xi - \frac{j}{2})]} \mathcal{F}_R \varphi_t(\xi,\gamma) e^{i\xi} d\xi;
\]

\[
\Xi_t \varphi(x,\gamma) := \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-\pi \xi}}{\text{Ch}[\pi(\xi - \frac{j}{2})]} \mathcal{F}_R \varphi_t(\xi,\gamma) e^{i\xi} d\xi.
\]

Similarly, for the operator \( T^H_{s,t} \) we have

(4.10)

\[
T^H_{s,t} \varphi(x,\gamma) = [\Lambda'_s \circ \Xi'_t] \varphi(x,\gamma),
\]

where the operators \( \Lambda'_s \) and \( \Xi'_t \) are defined by

\[
\Lambda'_s \varphi(x,\gamma) := \sum_{j \in \mathbb{Z}} \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-\pi \xi}}{\text{Ch}[\pi(\xi - \frac{j}{2})]} \mathcal{F}_R \varphi_t(\xi,\gamma) e^{i\xi} d\xi;
\]

\[
\Xi'_t \varphi(x,\gamma) := \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-\pi \xi}}{\text{Ch}[\pi(\xi - \frac{j}{2})]} \mathcal{F}_R \varphi_t(\xi,\gamma) e^{i\xi} d\xi.
\]

So, in order to obtain information on the mapping properties of the operator \( T_{s,t} \), we study the operators \( \Lambda_s, \Xi_t, \Lambda'_s \) and \( \Xi'_t \) separately. The realization of \( T_{s,t} \) as composition of these operators is particularly effective since the parameters \( t \) and \( s \) become, in some sense, independent.

**Proposition 4.4.** The operator \( \Lambda_s \) extends to a bounded linear operator

\[
\Lambda_s : L^p(\mathbb{R} \times \mathbb{T}) \to L^p(\mathbb{R} \times \mathbb{T})
\]
for every \( p \in (1, \infty) \). Moreover,

\[
\sup_{\lambda \in [0, \beta - \frac{4}{3}]} \| \Lambda_\gamma \|_p < \infty.
\]

**Proof.** By density it suffices to prove the theorem for a function \( g \) of the form \( g(x, \gamma) = \sum_{j=N}^{N} g(x, j)e^{2\pi i j/\gamma} \) and each \( g(\cdot, j) \) is in \( C_0^\infty(\mathbb{R}) \). Then, similarly to the proof of Proposition 4.1 for the operator \( \Lambda_\gamma \), we obtain

\[
\int_\mathbb{R} \int_0^1 |\Lambda_\gamma g(x, \gamma)|^p \, d\gamma \, dx = \int_\mathbb{R} \int_0^1 \left[ \frac{m^{1,2}(\xi)}{2\pi} \mathcal{F} \left[ \sum_{j=-N}^{N} e^{2\pi i j/\gamma}(\cdot)e^{2\pi i j/\gamma}(\xi)e^{i\xi x} \right] \right]^p \, d\gamma \, dx.
\]

By Mihlin’s condition, we obtain that the function

\[
m^{1,2}(\xi) = \frac{e^{-|2\beta - \pi|\xi} + e^{-|\beta - \frac{4}{3} + i\xi}}{\text{Ch}(|2\beta - \pi|\xi)}
\]

identifies a multiplier operator that is bounded on \( L^p(\mathbb{R}) \) for every \( p \in (1, \infty) \) and that satisfies (4.11). Notice also that the function \( \sum_{j=-N}^{N} e^{2\pi i j/\gamma}g(x, j)e^{2\pi i j/\gamma} \) is in \( L^p(\mathbb{R} \times T) \).

Finally, by Fubini’s theorem,

\[
\int_\mathbb{R} \int_0^1 |\Lambda_\gamma g(x, \gamma)|^p \, d\gamma \, dx = \int_\mathbb{R} \int_0^1 \left[ \frac{m^{1,2}(\xi)}{2\pi} \mathcal{F} \left[ \sum_{j=-N}^{N} e^{2\pi i j/\gamma}(\cdot)e^{2\pi i j/\gamma}(\xi)e^{i\xi x} \right] \right]^p \, d\gamma \, dx
\]

\[
\leq C_p \int_\mathbb{R} \int_0^1 \left| \sum_{j=-N}^{N} e^{2\pi i j/\gamma}g(x, j)e^{2\pi i j/\gamma} \right|^p \, d\gamma \, dx
\]

\[
= C_p \int_\mathbb{R} \int_0^1 \left| \sum_{j=-N}^{N} g(x, j)e^{2\pi i j/\gamma} \right|^p \, d\gamma \, dx.
\]

\[ \square \]

**Proposition 4.5.** The operator \( \Xi'_t \) extends to a bounded linear operator

\[ \Xi'_t : L^p(\mathbb{R} \times T) \to L^p(\mathbb{R} \times T) \]

for every \( p \in (1, \infty) \). Moreover,

\[
\sup_{t \in [0, \frac{\pi}{2A}]} \| \Xi'_t \|_p < \infty.
\]

**Proof.** By Mihlin’s condition we obtain that the function \( m^{1,1}(\xi) \) is a \( L^p(\mathbb{R}) \) multiplier for every \( p \in (1, \infty) \) which satisfies (4.12). By Fubini’s theorem we conclude. \[ \square \]

**Proposition 4.6.** The operator \( \Xi_t \) extends to a bounded linear operator

\[ \Xi_t : L^p(\mathbb{R} \times T) \to L^p(\mathbb{R} \times T) \]

for every \( p \in (1, \infty) \). Moreover,

\[
\sup_{t \in [0, \frac{\pi}{2A}]} \| \Xi_t \|_p < \infty
\]

and

\[
\lim_{t \to \frac{\pi}{2A}} \| \Xi_t g \|_{L^p(\mathbb{R} \times T)} = 0
\]
Moreover, we prove explicitly only that for every function $g$ in $L^p(\mathbb{R} \times \mathbb{T})$.

**Proof.** The boundedness of $\Xi_\gamma$ follows once again by Mihlin’s condition, while the limit is computed as in Theorem 4.3 for the strip $S_{\beta}^\frac{1}{2}$.

**Proposition 4.7.** The operator $\Lambda'_{\psi}$ extends to a bounded linear operator

$$\Lambda'_{\psi} : L^p(\mathbb{R} \times \mathbb{T}) \to L^p(\mathbb{R} \times \mathbb{T})$$

for every $p \in (1, \infty)$. Moreover,

$$\sup_{x \in [0,\beta-\frac{1}{2}]} \|\Lambda'_{\psi}f\|_p < \infty,$$

and

$$\lim_{s \to \beta-\frac{1}{2}} \|\Lambda'_{\psi}g\|_{L^p(\mathbb{R} \times \mathbb{T})} = 0$$

for every function $g$ in $L^p(\mathbb{R} \times \mathbb{T})$.

**Proof.** The proof follows similarly as the proofs of Proposition 4.4 and Proposition 4.6.

Now, using Propositions 4.4, 4.5, 4.6 and 4.7 we obtain the proof of Theorem 4.3 for a function $\varphi = (\varphi_1, 0, 0, 0)$. The proof for a general function $\varphi$ follows with similar arguments. Moreover, Theorem 4.2 and Theorem 4.3 prove Theorem 1.1.

### 4.1. Sobolev regularity

Finally, we prove Theorem 1.2.

**Proof.** It suffices to prove the theorem for a function of the form $\varphi = (\varphi_1, 0, 0, 0)$. For such a function $\varphi$, it holds

$$\overline{S\varphi}_1(x + i\beta, e^{\frac{i}{2}(\beta-x)} e^{2\pi j \gamma}) = \frac{1}{4} \sum_{\alpha \in \mathbb{Z}} e^{2\pi i j \gamma} \mathcal{F}_{\alpha}^{-1} \left[ e^{-(2\beta-x)(\beta-x)} e^{-\pi j \gamma} \mathcal{F}_{\alpha} \varphi_1(:, \hat{j}) \right] (x).$$

Moreover, we prove explicitly only that $\|\overline{S\varphi}_1\|_{W^{k,p}(\mathbb{R} \times \mathbb{T})} \leq \|\varphi\|_{W^{k,p}(\partial D'_{\beta})}$. With similar arguments, it is then possible to prove $\|\overline{S\varphi}\|_{W^{k,p}(\partial D'_{\beta})} \leq \|\varphi\|_{W^{k,p}(\partial D'_{\beta})}$.

We notice that

$$\sum_{\alpha \in \mathbb{Z}} e^{2\pi i j \gamma} \mathcal{F}_{\alpha}^{-1} \left[ 1 + j^2 + (\cdot)^2 \right] \mathcal{F}_{\alpha} \varphi_1(:, \hat{j}) (x) = \sum_{\alpha \in \mathbb{Z}} e^{2\pi i j \gamma} \mathcal{F}_{\alpha}^{-1} \left[ \frac{1 + j^2 + (\cdot)^2}{\operatorname{Ch}[\pi] \operatorname{Ch}[(2\beta-x)(\cdot - \frac{1}{2})]} \right] \mathcal{F}_{\alpha} \varphi_1(:, \hat{j}) (x)$$

$$= \overline{S\varphi}^\delta (x, \gamma),$$

where $\varphi^\delta = (\varphi_1^\delta, 0, 0, 0)$ with

$$\varphi_1^\delta (x, \gamma) = \sum_{\alpha \in \mathbb{Z}} e^{2\pi i j \gamma} \mathcal{F}_{\alpha}^{-1} \left[ 1 + j^2 + (\cdot)^2 \right] \mathcal{F}_{\alpha} \varphi_1 (:, \hat{j}) (x).$$

Thus,

$$\|\overline{S\varphi}_1\|_{W^{k,p}(E_1)}^p = \|\overline{S\varphi}^\delta\|_{L^p(E_1)}^p$$

and the conclusion follows from the $L^p$ boundedness of the operator $\overline{S}$. 

\qed
4.2. A decomposition of $H^p(D'_b)$. In this section we prove that the the space $H^p(D'_b)$ admits for every $p \in (1, \infty)$ a decomposition

\begin{equation}
H^p(D'_b) = \bigoplus_{j \in \mathbb{Z}} \mathcal{H}^p_j
\end{equation}

analogously to (3.1) for the case $p = 2$. We recall that, for every $j \in \mathbb{Z}$,

\[ \mathcal{H}^p_j = \left\{ F \in H^p(D'_b) : F(z_1, e^{2\pi i j z_2}) = e^{2\pi i j} F(z_1, z_2) \right\}. \]

Thus, we will prove that given a function $F$ in $H^p(D'_b)$, there exist functions $F_j$'s such that

\[ \lim_{N \to \infty} \left\| F - \sum_{j=-N}^{N} F_j \right\|_{H^p(D'_b)} = 0, \]

where each function $F_j$ belongs to $\mathcal{H}^p_j$.

We begin proving this result for functions which belong to the range of the operator $S$. Once again, without losing generality, we prove the result using simplified initial data. The general result will follow by linearity. Given a function $\varphi = (\varphi_1, 0, 0, 0)$ in $L^p(\partial D'_b)$, we define

\[ S_N \varphi(x + iy, e^{2\pi i j y}) := \sum_{j=-N}^{N} e^{2\pi i j y} \frac{1}{4\pi i} \left[ e^{-(\beta + \frac{1}{2} + \gamma)}(\hat{\varphi}_{1}(\cdot, j)) \right](x) \]

Notice that each function $S_j \varphi$ trivially belongs to $\mathcal{H}^p_j$.

**Proposition 4.8.** Let $\varphi = (\varphi_1, 0, 0, 0)$ be a function in $L^p(D'_b)$, $p \in (1, \infty)$. Then,

\[ \lim_{N \to \infty} \left\| S \varphi - S_N \varphi \right\|_{H^p(D'_b)} = 0. \]

**Proof.** For almost every function $x \in \mathbb{R}$, the function $\varphi_1(x, \cdot)$ is in $L^p(\mathbb{R})$. Thus, the $L^p$ convergence of one-dimensional Fourier series guarantees that

\[ \lim_{N \to \infty} \int_{0}^{1} \left| \varphi_1(x, y) - \varphi_1^{(N)}(x, y) \right|^p dy = 0, \]

where $\varphi_1^{(N)}(x, y) = \sum_{j=-N}^{N} \varphi_1(x, j) e^{2\pi i j y}$. By dominated convergence, we can conclude that

\[ \lim_{N \to \infty} \int_{\mathbb{R}} \int_{0}^{1} \left| \varphi_1(x, y) - \varphi_1^{(N)}(x, y) \right|^p d\gamma dx = \lim_{N \to \infty} \int_{\mathbb{R}} \int_{0}^{1} \left| \varphi_1(x, y) - \varphi_1^{(N)}(x, y) \right|^p d\gamma dx = 0. \]

Thus we can conclude that $\| \varphi - \varphi^{(N)} \|_{L^p(\partial D'_b)} \to 0$ as $N$ tends to $+\infty$. where $\varphi^{(N)} = (\varphi_1^{(N)}, 0, 0, 0)$. By definition, it holds

\[ S^{(N)} \varphi(x + iy, e^{2\pi i j y}) := \sum_{j=-N}^{N} e^{2\pi i j y} \frac{1}{4\pi i} \left[ e^{-(\beta + \frac{1}{2} + \gamma)}(\hat{\varphi}_{1}(\cdot, j)) \right](x) \]

\[ = S_N \varphi(x + iy, e^{2\pi i j y}) \]
3.1. Proof.

Finally, using estimates \((4.6)\), we get

\[
\lim_{N \to \infty} \|S\varphi - S_{(N)}\|_{H^p(D^p)} = \lim_{N \to \infty} \|S\varphi - S\varphi^{(N)}\|_{H^p(D^p)} \\
\leq C_p \lim_{N \to \infty} \|\varphi - \varphi^{(N)}\|_{L^p(\partial D^p)} = 0.
\]

The proof is complete. \(\square\)

So far we proved that every function which is in the range of \(S\) admits a decomposition \(S\varphi = \sum_{j \in \mathbb{Z}} S_j \varphi\) where the equality is meant in \(H^p(D^p)\) and each \(S_j \varphi\) belongs to \(\mathcal{H}_p\). To obtain \((4.14)\) it remains to prove that the operator \(S\) is surjective on \(H^p(D^p)\). We already know this the case for the case \(p = 2\); the general case \(p \in (1, \infty)\) will follow as a corollary of the following result.

**Proposition 4.9.** For every \(p\) in \((1, \infty)\), we have

\[
H^2(D^p) \cap H^p(D^p) = H^p(D^p).
\]

**Proof.** For every \(\varepsilon > 0\) and \(z_1 \in S^p\) consider the function

\[
G^\varepsilon(z_1) = \frac{1}{1 + \varepsilon|2z + z_1|}.
\]

It can be proved that, for every fixed \(\varepsilon > 0\), the function \(F \cdot G^\varepsilon\) is in \(H^2(D^p) \cap H^p(D^p)\), thus \(FG^\varepsilon = S|FG^\varepsilon|\). Notice that \(G^\varepsilon\) admits a continuous extension to \(\overline{D^p}\), therefore \(FG^\varepsilon = \tilde{F}G^\varepsilon\), where \(\tilde{F}\) is the weak-* limit of \(F\) (see Proposition 3.1). Now,

\[
\lim_{\varepsilon \to 0^+} \|F - FG^\varepsilon\|_{H^p(D^p)} \leq \lim_{\varepsilon \to 0^+} \sup_{\varepsilon \to 0^+} \int_0^1 \int_R |(F - FG^\varepsilon)[x + i(s + t), \varepsilon^2 e^{2\pi i t}]|^p dxdy + \lim_{\varepsilon \to 0^+} \sup_{\varepsilon \to 0^+} \int_0^1 \int_R |(F - FG^\varepsilon)[x + i(s + t), \varepsilon^2 e^{2\pi i t}]|^p dxdy.
\]

We focus on one of these term; the computation for the other terms is similar. Therefore,

\[
\lim_{\varepsilon \to 0^+} \sup_{\varepsilon \to 0^+} \int_0^1 \int_R |(F - FG^\varepsilon)[x + i(s + t), \varepsilon^2 e^{2\pi i t}]|^p dxdy
\]

\[
= \lim_{\varepsilon \to 0^+} \sup_{\varepsilon \to 0^+} \int_0^1 \int_R |F[x + i(s + t), \varepsilon^2 e^{2\pi i t}] [1 - G^\varepsilon[x + i(s + t)]]|^p dxdy
\]

\[
\leq \lim_{\varepsilon \to 0^+} \sup_{\varepsilon \to 0^+} \int_0^1 \int_R |F[x + i(s + t), \varepsilon^2 e^{2\pi i t}][G^\varepsilon - G^\varepsilon][x + i(s + t)]|^p dxdy
\]

\[
= \lim_{\varepsilon \to 0^+} \sup_{\varepsilon \to 0^+} \int_0^1 \int_R |S\tilde{F}([G^\varepsilon - G^\varepsilon])|^p dxdy
\]

\[
\leq \lim_{\varepsilon \to 0^+} \sup_{\varepsilon \to 0^+} \|S\tilde{F}([G^\varepsilon - G^\varepsilon])\|_{H^p(D^p)}
\]

\[
\leq C_p \lim_{\varepsilon \to 0^+} \sup_{\varepsilon \to 0^+} \|\tilde{F}([G^\varepsilon - G^\varepsilon])\|_{L^p(\partial D^p)}
\]
where in the last two lines we used the boundedness of the operator $S$ and the dominated convergence theorem. The proof is complete. \hfill $\square$

**Corollary 4.10.** Let $F$ be a function in $H^p(D'_p)$, $p \in (1, \infty)$. Then, there exists a function $\varphi$ in $L^p(\partial D'_p)$ such that $F = S\varphi$.

**Remark 4.11.** Theorem 4.3 shows that every function in the range of $S$ tends to its boundary values in norm. The previous corollary allows to conclude that this is true for every element of $H^p(D'_p)$, $p \in (1, \infty)$.

**Remark 4.12.** Proposition 4.8 and Corollary 4.10 together prove the decomposition (4.14).

### 4.3. Pointwise convergence

We conclude this section proving a Fatou-type theorem. We prove that an appropriate restriction of a function $F$ in $H^p(D'_p)$, $p \in (1, \infty)$, converges to its boundary value function also pointwise almost everywhere. As usual, we prove our results in a simplified situation. The general case follows by linearity. Let $\varphi = (\varphi_+, 0, 0, 0)$ be a function in $L^p(\partial D'_p)$, then we proved that

$$
\lim_{(t, s) \to \left(\frac{1}{2}, \beta_0 - \frac{2}{\pi} \right)} \int_{\mathbb{R}} \int_{0}^{1} \left| S\varphi_{\pi}(s + t), e^{i2\pi t} \right| \, dt \, dx = 0.
$$

In general, to prove a pointwise convergence result, we expect that we need to put some restrictions on the parameters $t$ and $s$. For example, also in the simpler case of the polydisc $D^2(0, 1) = D(0, 1) \times D(0, 1)$, we are able to prove the almost everywhere existence of the pointwise radial limit

$$
\lim_{(r_1, r_2) \to (1, 1)} G(r_1 e^{2i\pi \theta}, r_2 e^{2i\pi \gamma})
$$

for a function $G$ in $H^p(D^2)$ under the hypothesis that the ratio $\frac{1 - r_1}{1 - r_2}$ is bounded (see, for example, [Rud69] Chapter 2, Section 2.3).

At the moment, we are able to prove a pointwise convergence result which depends only on one parameter. It would be interesting to determine a larger approach region to the distinguished boundary $\partial D'_p$.

We need the following lemma which is not hard to prove using the results contained in Section 2.

**Lemma 4.13.** Let $S_{\beta} = \{z = x + iy \in \mathbb{C} : |y| < \beta\}$. Let $\varphi = (\varphi_+, \varphi_-)$ be a function in $L^p(\partial S_{\beta})$, $p \in (1, \infty)$. Then, the function

$$
S_{\beta} \varphi(x, \gamma) = \mathcal{F}^{-1} \left[ e^{-(y + \beta)(\cdot)} \varphi_+ + e^{-(y - \beta)(\cdot)} \varphi_- \right]
$$

belongs to $H^p(S_{\beta})$ for every integer $j$.

Now we prove our result of pointwise convergence.

**Theorem 4.14.** Let $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ be a function in $L^p(\partial D'_p)$, $p \in (1, \infty)$. Then,

$$
\lim_{t \to \beta} S\varphi_{\pi}(x + it, e^{i\beta_0(\beta_0 - \frac{2}{\pi})} e^{2i\pi t}) = S\varphi_{\pi}(x + i\beta, e^{i\beta(\beta - \frac{2}{\pi})} e^{2i\pi t})
$$

for almost every $(x, \gamma) \in \mathbb{R} \times T$. 

Proof. We prove the theorem for $\varphi = (\varphi_1, 0, 0, 0)$. By (4.13), we want to prove that

$$
L(x, \gamma) = \left| \sum_{j \in \mathbb{Z}} e^{2\pi i j} \mathcal{F}_R^{-1} \left[ \frac{e^{-2\beta(\cdot)} e^{i(\beta-\frac{x}{2})} - e^{-(\beta + i\gamma)(\cdot)} e^{i(\beta-\frac{x}{2})(1 + \frac{1}{t})}}{4 \text{Ch}[\pi(\cdot)] \text{Ch}[2\beta - \pi](\cdot - \frac{x}{2})} \right] \mathcal{F}_R \varphi_1(\cdot, j) \right|(x)
$$

for almost every $(x, \gamma) \in \mathbb{R} \times T$ when $t$ tends to $\beta^-$. Let $\epsilon > 0$ be fixed. Then,

$$
\left\{ (x, \gamma) \in \mathbb{R} \times T : \limsup_{t \to \beta^-} \left| L_t^{(\varphi)}(x, \gamma) - \varphi \right| > \epsilon \right\} \leq \sum_{j \in \mathbb{Z}} \left\{ (x, \gamma) \in \mathbb{R} \times T : \limsup_{t \to \beta^-} \left| L_t^{(\varphi_j)}(x, \gamma) - \varphi_j \right| > \alpha_j \right\},
$$

where the $\alpha_j$'s are positive and $\sum j \in \mathbb{Z} \alpha_j = \epsilon$. We claim that the sets in the right-hand side of the previous inequality are all of measure zero. Following the proof of Theorem 4.3, we obtain that

$$
(4.16) \lim_{t \to \beta^-} \| S_t^{(\varphi)} \|_{L^p(\mathbb{R} \times T)} = 0.
$$

Therefore, it is enough to prove the existence of the pointwise limit

$$
\lim_{t \to \beta^-} e^{2\pi i t} \mathcal{F}_R^{-1} \left[ \frac{e^{-(\beta + i\gamma)(\cdot)} e^{i(\beta-\frac{x}{2})(1 + \frac{1}{t})}}{4 \text{Ch}[\pi(\cdot)] \text{Ch}[2\beta - \pi](\cdot - \frac{x}{2})} \right] \mathcal{F}_R \varphi_1(\cdot, j)
$$

for almost every $(x, \gamma) \in \mathbb{R} \times T$. To prove this, it is sufficient to prove that

$$
\lim_{t \to \beta^-} \mathcal{F}_R^{-1} \left[ \frac{e^{-(\beta + i\gamma)(\cdot)} \mathcal{F}_R G(\cdot)}{4 \text{Ch}[\pi(\cdot)] \text{Ch}[2\beta - \pi](\cdot - \frac{x}{2})} \right]
$$

exists for almost every $x \in \mathbb{R}$ and for every function $G$ in $L^p(\mathbb{R})$, $p \in (1, \infty)$. The existence of this last limit follows immediately from the lemma and Theorem 2.4.

Analogously we can prove the pointwise convergence of $S\varphi$ to the other components of $\partial D'_p$.

Remark 4.15. We proved the previous theorem for functions that belong to the range of the operator $S$. From Proposition 4.10, we can conclude that the result is true for every function in $H^p(D'_p)$, $p \in (1, \infty)$.

4.4. A density result. In this section we use the $L^p$ boundedness of the operator $\tilde{S}$ to prove a density results regarding the space $H^p(D'_p)$, $p \in (1, \infty)$.

Theorem 4.16. Let $p \in (1, \infty)$. Then,

$$
H^p(D'_p) \cap \text{C}(\mathcal{D}_p') = H^p(D'_p).
$$

Proof. It is enough to prove that given a function $\varphi = (\varphi_1, 0, 0, 0)$ where $\varphi_1(x, \gamma) = \sum_{j \in \mathbb{Z}} \varphi_1(x, j) e^{2\pi i j} \mathcal{F}_R^{-1} \left[ \frac{e^{-(\beta + i\gamma)(\cdot)} e^{i(\beta-\frac{x}{2})(1 + \frac{1}{t})}}{4 \text{Ch}[\pi(\cdot)] \text{Ch}[2\beta - \pi](\cdot - \frac{x}{2})} \right] \mathcal{F}_R \varphi_1(\cdot, j) (x)$

and $S\varphi$ is continuous up to the boundary of $\partial D'_p$ since each term of the sum is thanks to Lemma 4.13 and Remark 2.5. The proof is complete.
REFERENCES

[Aro50] N. Aronszajn. Theory of reproducing kernels. Trans. Amer. Math. Soc., 68:337–404, 1950.

[Bar92] D. E. Barrett. Behavior of the Bergman projection on the Diederich-Fornaess worm. Acta Math., 168(1-2):1–10, 1992.

[BCS88] Harold P. Boas, So-Chin Chen, and Emil J. Straube. Exact regularity of the Bergman and Szegő projections on domains with partially transverse symmetries. Manuscripta Math., 62(4):467–475, 1988.

[Bel81] S. R. Bell. Biharmonic mappings and the ¯∂-problem. Ann. of Math. (2), 114(1):103–113, 1981.

[BE8] D. E. Barrett, D. Ehsani, and M. M. Peloso. Regularity of projection operators attached to worm domains. ArXiv e-prints, August 2014.

[BL07] A. Bakan and S. Kajiser. Hardy spaces for the strip. J. Math. Anal. Appl., 333(1):347–364, 2007.

[BL80] S. Bell and E. Ligocka. A simplification and extension of Fefferman’s theorem on biholomorphic mappings. Invent. Math., 75(3):283–289, 1980.

[BL14] David Barrett and Lina Lee. On the Szegő metric. J. Geom. Anal., 24(1):104–117, 2014.

[Boa85] Harold P. Boas. Regularity of the Szegő projection in weakly pseudoconvex domains. Indiana Univ. Math. J., 34(1):217–223, 1985.

[Boa87] Harold P. Boas. The Szegő projection: Sobolev estimates in regular domains. Trans. Amer. Math. Soc., 300(1):109–132, 1987.

[BS89] Harold P. Boas and Emil J. Straube. Complete Hartogs domains in $\mathbb{C}^2$ have regular Bergman and Szegő projections. Math. Z., 201(3):441–454, 1989.

[BS91] H. P. Boas and E. J. Straube. Sobolev estimates for the complex Green operator on a class of weakly pseudoconvex boundaries. Comm. Partial Differential Equations, 16(10):1573–1582, 1991.

[BS12] D. E. Barrett and S. Şahutoğlu. Irregularity of the Bergman projection on worm domains in $\mathbb{C}^n$. Michigan Math. J., 61(1):187–198, 2012.

[Chen91] So-Chin Chen. Real analytic regularity of the Szegő projection on circular domains. Pacific J. Math., 148(2):225–235, 1991.

[Chr96a] M. Christ. Global $C^\infty$ irregularity of the ¯∂-Neumann problem for worm domains. J. Amer. Math. Soc., 9(4):1171–1185, 1996.

[Chr96b] M. Christ. The Szegő projection need not preserve global analyticity. Ann. of Math. (2), 143(2):301–330, 1996.

[DF77] K. Diederich and J. E. Fornaess. Pseudoconvex domains: an example with nontrivial Nebenhülle. Math. Ann., 225(3):275–292, 1977.

[Dia87] Katharine Perkins Diaz. The Szegő kernel as a singular integral kernel on a family of weakly pseudoconvex domains. Trans. Amer. Math. Soc., 304(1):141–170, 1987.

[Gra08] L. Grafakos. Classical Fourier analysis, volume 249 of Graduate Texts in Mathematics. Springer, New York, second edition, 2008.

[HPR15] P. S. Harrington, M. M. Peloso, and A. S. Raich. Regularity equivalence of the Szegő projection and the complex Green operator. Proc. Amer. Math. Soc., 143(1):353–367, 2015.

[Kis91] C. O. Kiselman. A study of the Bergman projection in certain Hartogs domains. In Several complex variables and complex geometry, Part 3 (Santa Cruz, CA, 1989), volume 52 of Proc. Sympos. Pure Math., pages 219–231. Amer. Math. Soc., Providence, RI, 1991.

[KPS14] S. G. Krantz and M. M. Peloso. The Bergman kernel and projection on the unbounded worm domain. Electron. Res. Announc. Math. Sci., 14:35–41 (electronic), 2007.

[LS04] Loredana Lanzani and Elias M. Stein. Szegő and Bergman projections on non-smooth planar domains. J. Geom. Anal., 14(1):63–86, 2004.

[LS13] L. Lanzani and E. M. Stein. Cauchy-type integrals in several complex variables. Bull. Math. Sci., 3(2):241–285, 2013.

[Mon] A. Monguzzi. On the regularity of singular integrals operators on complex domains. PhD thesis, Università degli Studi di Milano, 2015.

[MS97] J. D. McNeal and E. M. Stein. The Szegő projection on convex domains. Math. Z., 224(4):519–553, 1997.

[NRSW89] A. Nagel, J.-P. Rosay, E. M. Stein, and S. Wainger. Estimates for the Bergman and Szegő kernels in $\mathbb{C}^2$. Ann. of Math. (2), 129(1):113–149, 1989.

[PS77] D. H. Phong and E. M. Stein. Estimates for the Bergman and Szegő projections on strongly pseudo-convex domains. Duke Math. J., 44(3):695–704, 1977.
[PW87] R. E. A. C. Paley and N. Wiener. *Fourier transforms in the complex domain*, volume 19 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1987. Reprint of the 1934 original.

[Rud69] Walter Rudin. *Function theory in polydiscs*. W. A. Benjamin, Inc., New York-Amsterdam, 1969.

[Sed75] A. M. Sedleckiı. An equivalent definition of the $H^p$ spaces in the half-plane, and some applications. *Mat. Sb. (N.S.)*, 96(138):75–82, 167, 1975.

[Ste72] E. M. Stein. *Boundary behavior of holomorphic functions of several complex variables*. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1972. Mathematical Notes, No. 11.

[Str86] Emil J. Straube. Exact regularity of Bergman, Szegő and Sobolev space projections in nonpseudoconvex domains. *Math. Z.*, 192(1):117–128, 1986.

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ STATALE DI MILANO, VIA C. SALDINI 50, 20133 MILAN, ITALY

E-mail address: alessandro.monguzzi@unimi.it