Tomita–Takesaki Modular Theory

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Abstract

We provide an brief overview of Tomita–Takesaki modular theory and some of its applications to mathematical physics. This is an article commissioned by the Encyclopedia of Mathematical Physics, edited by J.-P. Francoise, G. Naber and T.S. Tsun, to be published by the Elsevier publishing house.

1 Basic Structure

The origins of Tomita–Takesaki modular theory lie in two unpublished papers of M. Tomita in 1967 and a slim volume [9] by M. Takesaki. It has developed into one of the most important tools in the theory of operator algebras and has found many applications in mathematical physics.

Though the modular theory has been formulated in a more general setting, it will be presented in the form in which it most often finds application in mathematical physics.¹

Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ containing a vector $\Omega$ which is cyclic and separating for $\mathcal{M}$. Define the operator $S_0$ on $H$ as follows:

$$S_0 A \Omega = A^* \Omega,$$

for all $A \in \mathcal{M}$.

This operator extends to a closed anti-linear operator $S$ defined on a dense subset of $\mathcal{H}$. Let $\Delta$ be the unique positive, self-adjoint operator and $J$ the unique anti-unitary operator occurring in the polar decomposition

$$S = J \Delta^{1/2} = \Delta^{-1/2} J.$$

$\Delta$ is called the modular operator and $J$ the modular conjugation (or modular involution) associated with the pair $(\mathcal{M}, \Omega)$. Note that $J^2$ is the identity operator and $J = J^*$. Moreover, the spectral calculus may be applied to $\Delta$ so that $\Delta^t$ is a unitary operator for each $t \in \mathbb{R}$ and $\{\Delta^t \mid t \in \mathbb{R}\}$ forms a strongly continuous unitary group. Let $\mathcal{M}'$ denote the set of all bounded linear operators on $\mathcal{H}$ which commute with all elements of $\mathcal{M}$. The modular theory begins with the following remarkable theorem.

¹The reader is referred to [6, 7, 8, 10] for generalizations, details and references concerning the material in the first four sections.
Theorem 1.1 Let $\mathcal{M}$ be a von Neumann algebra with a cyclic and separating vector $\Omega$. Then $J\Omega = \Omega = \Delta \Omega$ and the following equalities hold:

$$ J\mathcal{M}J = \mathcal{M}' \quad \text{and} \quad \Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M}, \quad \text{for all} \quad t \in \mathbb{R}. $$

Note that if one defines $F_0 A'\Omega = A'^*\Omega$, for all $A' \in \mathcal{M}'$, and takes its closure $F$, then one has the relations

$$ \Delta = FS, \quad \Delta^{-1} = SF, \quad F = J\Delta^{-1/2}. $$

2 Modular Automorphism Group

By Theorem 1.1 the unitaries $\Delta^{it}, t \in \mathbb{R}$, induce a one-parameter automorphism group $\{\sigma_t\}$ of $\mathcal{M}$ by

$$ \sigma_t(A) = \Delta^{it}A\Delta^{-it}, \quad A \in \mathcal{M}, \quad t \in \mathbb{R}. $$

This group is called the modular automorphism group of $\mathcal{M}$ (relative to $\Omega$). Let $\omega$ denote the faithful normal state on $\mathcal{M}$ induced by $\Omega$:

$$ \omega(A) = \frac{1}{\|\Omega\|^2} \langle \Omega, A\Omega \rangle, \quad A \in \mathcal{M}. $$

From Theorem 1.1 it follows that $\omega$ is invariant under $\{\sigma_t\}$, i.e. $\omega(\sigma_t(A)) = \omega(A)$ for all $A \in \mathcal{M}$ and $t \in \mathbb{R}$.

The modular automorphism group contains information about both $\mathcal{M}$ and $\omega$. For example, the modular automorphism group is an inner automorphism on $\mathcal{M}$ if and only if $\mathcal{M}$ is semi-finite. It is trivial if and only if $\omega$ is a tracial state on $\mathcal{M}$. Indeed, one has for any $B \in \mathcal{M}$ that $\sigma_t(B) = B$ for all $t \in \mathbb{R}$ if and only if $\omega(AB) = \omega(BA)$ for all $A \in \mathcal{M}$. Let $\mathcal{M}'$ denote the set of all such $B \in \mathcal{M}$.

2.1 The KMS-Condition

The modular automorphism group satisfies a condition which had already been used in mathematical physics to characterize equilibrium temperature states of quantum systems in statistical mechanics and field theory — the Kubo–Martin–Schwinger (KMS) condition. If $\mathcal{M}$ is a von Neumann algebra and $\{\alpha_t \mid t \in \mathbb{R}\}$ is a $\sigma$-weakly continuous one-parameter group of automorphisms of $\mathcal{M}$, then the state $\phi$ on $\mathcal{M}$ satisfies the KMS-condition at (inverse temperature) $\beta$ ($0 < \beta < \infty$) with respect to $\{\alpha_t\}$ if for any $A, B \in \mathcal{M}$ there exists a complex function $F_{A,B}(z)$ which is analytic on the strip $\{z \in \mathbb{C} \mid 0 < \text{Im} \, z < \beta\}$ and continuous on the closure of this strip such that

$$ F_{A,B}(t) = \phi(\alpha_t(A)B) \quad \text{and} \quad F_{A,B}(t + i\beta) = \phi(B\alpha_t(A)), $$

for all $t \in \mathbb{R}$. In this case, $\phi(\alpha_{i\beta}(A)B) = \phi(BA)$, for all $A, B$ in a $\sigma$-weakly dense, $\alpha$-invariant *-subalgebra of $\mathcal{M}$. Such KMS-states are $\alpha$-invariant, i.e. $\phi(\alpha_t(A)) = \phi(A)$, for all $A \in \mathcal{M}, \quad t \in \mathbb{R}$, and are stable and passive (cf. Chapter 5 in [3] and [5]).

Every faithful normal state satisfies the KMS-condition at value $\beta = 1$ (henceforth called the modular condition) with respect to the corresponding modular automorphism group.
Theorem 2.1 Let $\mathcal{M}$ be a von Neumann algebra with a cyclic and separating vector $\Omega$. Then the induced state $\omega$ on $\mathcal{M}$ satisfies the modular condition with respect to the modular automorphism group $\{\sigma_t | t \in \mathbb{R}\}$ associated to the pair $(\mathcal{M}, \Omega)$.

The modular automorphism group is therefore endowed with the analyticity associated with the KMS-condition, and this is a powerful tool in many applications of the modular theory to mathematical physics. In addition, the physical properties and interpretations of KMS-states are often invoked when applying modular theory to quantum physics.

Note that while the non-triviality of the modular automorphism group gives a measure of the non-tracial nature of the state, the KMS-condition for the modular automorphism group provides the missing link between the values $\omega(AB)$ and $\omega(BA)$, for all $A, B \in \mathcal{M}$ (hence the use of the term “modular”, as in the theory of integration on locally compact groups).

The modular condition is quite restrictive. Only the modular group can satisfy the modular condition for $(\mathcal{M}, \Omega)$, and the modular group for one state can satisfy the modular condition only in states differing from the original state by the action of an element in the center of $\mathcal{M}$.

Theorem 2.2 Let $\mathcal{M}$ be a von Neumann algebra with a cyclic and separating vector $\Omega$, and let $\{\sigma_t\}$ be the corresponding modular automorphism group. If the induced state $\omega$ satisfies the modular condition with respect to a group $\{\alpha_t\}$ of automorphisms of $\mathcal{M}$, then $\{\alpha_t\}$ must coincide with $\{\sigma_t\}$. Moreover, a normal state $\psi$ on $\mathcal{M}$ satisfies the modular condition with respect to $\{\sigma_t\}$ if and only if $\psi(\cdot) = \omega(h \cdot) = \omega(h^{1/2} \cdot h^{1/2})$ for some unique positive injective operator $h$ affiliated with the center of $\mathcal{M}$.

Hence, if $\mathcal{M}$ is a factor, two distinct states cannot share the same modular automorphism group. The relation between the modular automorphism groups for two different states will be described in more detail.

2.2 One Algebra and Two States

Consider a von Neumann algebra $\mathcal{M}$ with two cyclic and separating vectors $\Omega$ and $\Phi$, and denote by $\omega$ and $\phi$, respectively, the induced states on $\mathcal{M}$. Let $\{\sigma^\omega_t\}$ and $\{\sigma^\phi_t\}$ denote the corresponding modular groups. There is a general relation between the modular automorphism groups of these states.

Theorem 2.3 There exists a $\sigma$-strongly continuous map $\mathbb{R} \ni t \mapsto U_t \in \mathcal{M}$ such that

1. $U_t$ is unitary, for all $t \in \mathbb{R}$;
2. $U_{t+s} = U_t \sigma^\omega_t(U_s)$, for all $s, t \in \mathbb{R}$;
3. $\sigma^\phi_t(A) = U_t \sigma^\omega_t(A) U_t^*$, for all $A \in \mathcal{M}$ and $t \in \mathbb{R}$.

The one-cocycle $\{U_t\}$ is commonly called the cocycle derivative of $\phi$ with respect to $\omega$ and one writes $U_t = (D\phi : D\omega)_t$. There is a chain rule for this derivative, as well: If $\phi, \psi$ and $\rho$ are faithful normal states on $\mathcal{M}$, then $(D\psi : D\phi)_t = (D\psi : D\rho)_t(D\rho : D\phi)_t$, for all $t \in \mathbb{R}$. More can be said about the cocycle derivative if the states satisfy any of the conditions in the following theorem.

Theorem 2.4 The following conditions are equivalent.

1. $\phi$ is $\{\sigma^\omega_t\}$-invariant;
(2) $\omega$ is $\{\sigma_t^\phi\}$-invariant;
(3) there exists a unique positive injective operator $h$ affiliated with $M^{\sigma^\omega} \cap M^{\sigma^\phi}$ such that $\omega(\cdot) = \phi(h \cdot) = \phi(h^{1/2} \cdot h^{1/2});$
(4) there exists a unique positive injective operator $h'$ affiliated with $M^{\sigma^\omega} \cap M^{\sigma^\phi}$ such that $\phi(\cdot) = \omega(h' \cdot) = \omega(h'^{1/2} \cdot h'^{1/2});$
(5) the norms of the linear functionals $\omega + i\phi$ and $\omega - i\phi$ are equal;
(6) $\sigma_t^\omega \sigma^\phi_s = \sigma^\phi_s \sigma_t^\omega$, for all $s, t \in \mathbb{R}$.

The conditions in Theorem 2.4 turn out to be equivalent to the cocycle derivative being a representation.

**Theorem 2.5** The cocycle $\{U_t\}$ intertwining $\{\sigma_t^\sigma\}$ with $\{\sigma_t^\phi\}$ is a group representation of the additive group of reals if and only if $\phi$ and $\omega$ satisfy the conditions in Theorem 2.4. In that case, $U(t) = h^{-it}$.

The operator $h' = h^{-1}$ in Theorem 2.4 is called the Radon–Nikodym derivative of $\phi$ with respect to $\omega$ and often denoted by $d\phi/d\omega$, due to the following result, which, if the algebra $M$ is abelian, is the well-known Radon–Nikodym Theorem from measure theory.

**Theorem 2.6** If $\phi$ and $\omega$ are normal positive linear functionals on $M$ such that $\phi(A) \leq \omega(A)$, for all positive elements $A \in M$, then there exists a unique element $h^{1/2} \in M$ such that $\phi(\cdot) = \omega(h^{1/2} \cdot h^{1/2})$ and $0 \leq h^{1/2} \leq 1$.

Though there is not sufficient space to treat the matter properly, the analogies with measure theory are not accidental. Indeed, any normal trace on a (finite) von Neumann algebra $M$ gives rise to a noncommutative integration theory in a natural manner. Modular theory affords an extension of this theory to the setting of faithful normal functionals $\eta$ on von Neumann algebras $M$ of any type, enabling the definition of noncommutative $L^p$ spaces, $L^p(M, \eta)$.

### 3 Modular Invariants and the Classification of von Neumann Algebras

As previously mentioned, the modular structure carries information about the algebra. This is best evidenced in the structure of type $III$ factors. As this theory is rather involved, only a sketch of some of the results can be given.

If $M$ is a type $III$ algebra, then its crossed product $N = M \ltimes_{\sigma^\omega} \mathbb{R}$ relative to the modular automorphism group of any faithful normal state $\omega$ on $M$ is a type $II_\infty$ algebra with a faithful semifinite normal trace $\tau$ such that $\tau \circ \theta_t = e^{-t} \tau$, $t \in \mathbb{R}$, where $\theta$ is the dual of $\sigma^\omega$ on $N$. Moreover, the algebra $M$ is isomorphic to the cross product $N \ltimes_{\theta} \mathbb{R}$, and this decomposition is unique in a very strong sense. This structure theorem entails the existence of important algebraic invariants for $M$, which has many consequences, one of which is made explicit here.

If $\omega$ is a faithful normal state of a von Neumann algebra $M$ induced by $\Omega$, let $\Delta_\omega$ denote the modular operator associated to $(M, \Omega)$ and $\text{sp} \Delta_\omega$ denote the spectrum of $\Delta_\omega$. The intersection

$$S'(M) = \cap \text{sp} \Delta_\omega$$

over all faithful normal states $\omega$ of $M$ is an algebraic invariant of $M$. 


Theorem 3.1 Let $\mathcal{M}$ be a factor acting on a separable Hilbert space. If $\mathcal{M}$ is of type III, then $0 \in S'(\mathcal{M})$; otherwise, $S'(\mathcal{M}) = \{0,1\}$ if $\mathcal{M}$ is of type $I_{\infty}$ or $II_{\infty}$ and $S'(\mathcal{M}) = \{1\}$ if not. Let $\mathcal{M}$ now be a factor of type III.

(i) $\mathcal{M}$ is of type $III_\lambda$, $0 < \lambda < 1$, if and only if $S'(\mathcal{M}) = \{0\} \cup \{\lambda^n \mid n \in \mathbb{Z}\}$.

(ii) $\mathcal{M}$ is of type $III_0$ if and only if $S'(\mathcal{M}) = \{0,1\}$.

(iii) $\mathcal{M}$ is of type $III_1$ if and only if $S'(\mathcal{M}) = \{0, \infty\}$.

In certain physically relevant situations, the spectra of the modular operators of all faithful normal states coincide, so that Theorem 3.1 entails that it suffices to compute the spectrum of any conveniently chosen modular operator in order to determine the type of $\mathcal{M}$. In other such situations, there are distinguished states $\omega$ such that $S'(\mathcal{M}) = sp \Delta_\omega$. One such example is provided by asymptotically abelian systems. A von Neumann algebra $\mathcal{M}$ is said to be asymptotically abelian if there exists a sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ of automorphisms of $\mathcal{M}$ such that the limit of $\{\alpha_n(B) - \alpha_n(B)A\}_{n \in \mathbb{N}}$ in the strong operator topology is zero, for all $A, B \in \mathcal{M}$. If the state $\omega$ is $\alpha_n$-invariant, for all $n \in \mathbb{N}$, then $sp \Delta_\omega$ is contained in $sp \Delta_\phi$, for all faithful normal states $\phi$ on $\mathcal{M}$, so that $S'(\mathcal{M}) = sp \Delta_\omega$. If, moreover, $sp \Delta_\omega = [0, \infty)$, then $sp \Delta_\omega = sp \Delta_\phi$, for all $\phi$ as described.

4 Self-Dual Cones

Let $j : \mathcal{M} \to \mathcal{M}'$ denote the antilinear $^*$-isomorphism defined by $j(A) = JAJ$, $A \in \mathcal{M}$. The natural positive cone $\mathcal{P}_+$ associated with the pair $(\mathcal{M}, \Omega)$ is defined as the closure in $\mathcal{H}$ of the set of vectors

$$\{Aj(A)\Omega \mid A \in \mathcal{M}\}.$$ 

Let $\mathcal{M}_+$ denote the set of all positive elements of $\mathcal{M}$. The following theorem collects the main attributes of the natural cone.

Theorem 4.1 (1) $\mathcal{P}_+^\circ$ coincides with the closure in $\mathcal{H}$ of the set $\{\Delta^{1/4}A\Omega \mid A \in \mathcal{M}_+\}$.

(2) $\Delta^{it}\mathcal{P}_+^\circ = \mathcal{P}_+^\circ$ for all $t \in \mathbb{R}$.

(3) $J\Phi = \Phi$ for all $\Phi \in \mathcal{P}_+^\circ$.

(4) $Aj(A)\mathcal{P}_+^\circ \subset \mathcal{P}_+^\circ$ for all $A \in \mathcal{M}$.

(5) $\mathcal{P}_+^\circ$ is a pointed, self-dual cone whose linear span coincides with $\mathcal{H}$.

(6) If $\Phi \in \mathcal{P}_+^\circ$, then $\Phi$ is cyclic for $\mathcal{M}$ if and only if $\Phi$ is separating for $\mathcal{M}$.

(7) If $\Phi \in \mathcal{P}_+^\circ$ is cyclic, and hence separating, for $\mathcal{M}$, then the modular conjugation and the natural cone associated with the pair $(\mathcal{M}, \Phi)$ coincide with $J$ and $\mathcal{P}_+^\circ$, respectively.

(8) For every normal positive linear functional $\phi$ on $\mathcal{M}$ there exists a unique vector $\Phi_\phi \in \mathcal{P}_+^\circ$ such that $\phi(A) = \langle \Phi_\phi, A\Phi_\phi \rangle$, for all $A \in \mathcal{M}$.

In fact, the algebras $\mathcal{M}$ and $\mathcal{M}'$ are uniquely characterized by the natural cone $\mathcal{P}_+^\circ$ [4]. In light of (8), if $\alpha$ is an automorphism of $\mathcal{M}$, then

$$V(\alpha)\Phi_\phi = \Phi_{\phi^\alpha^{-1}}$$

defines an isometric operator on $\mathcal{P}_+^\circ$, which by (5) extends to a unitary operator on $\mathcal{H}$. The map $\alpha \mapsto V(\alpha)$ defines a unitary representation of the group of automorphisms $\text{Aut}(\mathcal{M})$ on $\mathcal{M}$ in such a manner that $V(\alpha)AV(\alpha)^{-1} = \alpha(A)$ for all $A \in \mathcal{M}$ and $\alpha \in \text{Aut}(\mathcal{M})$. Indeed, one has the following.
**Theorem 4.2** Let \( \mathcal{M} \) be a von Neumann algebra with a cyclic and separating vector \( \Omega \). The group \( \mathcal{V} \) of all unitaries \( V \) satisfying
\[
V \mathcal{M} V^* = \mathcal{M} , \quad VJ V^* = J , \quad VP^2 = P^2
\]
is isomorphic to \( \text{Aut}(\mathcal{M}) \) under the above map \( \alpha \mapsto V(\alpha) \), which is called the standard implementation of \( \text{Aut}(\mathcal{M}) \).

Often of particular physical interest are (anti-)automorphisms of \( \mathcal{M} \) leaving \( \omega \) invariant. They can only be implemented by (anti-)unitaries which leave the pair \( (\mathcal{M}, \Omega) \) invariant. In fact, if \( U \) is a unitary or anti-unitary operator satisfying \( U\Omega = \Omega \) and \( U\mathcal{M}U^* = \mathcal{M} \), then \( U \) commutes with both \( J \) and \( \Delta \).

## 5 Two Algebras and One State

Motivated by applications to quantum field theory, the study of the modular structures associated with one state and more than one von Neumann algebra has begun (see the review paper [2] for references and details). Let \( \mathcal{N} \subset \mathcal{M} \) be von Neumann algebras with a common cyclic and separating vector \( \Omega \). \( \Delta_N, J_N \) and \( \Delta_M, J_M \) will denote the corresponding modular objects. The structure \( (\mathcal{M}, \mathcal{N}, \Omega) \) is called a \( \pm \)-half-sided modular inclusion if \( \Delta_{\mathcal{M}}^{it}\mathcal{N}\Delta_{\mathcal{M}}^{-it} \subset \mathcal{N} \), for all \( \pm t \geq 0 \).

**Theorem 5.1** Let \( \mathcal{M} \) be a von Neumann algebra with cyclic and separating vector \( \Omega \). The following are equivalent.

(i) There exists a proper subalgebra \( \mathcal{N} \subset \mathcal{M} \) such that \( (\mathcal{M}, \mathcal{N}, \Omega) \) is a \( \mp \)-half-sided modular inclusion.

(ii) There exists a unitary group \( \{U(t)\} \) with positive generator such that
\[
U(t)\mathcal{M}U(t)^{-1} \subset \mathcal{M} \quad \text{for all} \quad \pm t \geq 0 , \quad \text{and} \quad U(t)\Omega = \Omega \quad \text{for all} \quad t \in \mathbb{R} .
\]

Moreover, if these conditions are satisfied, then the following relations must hold:
\[
\Delta_{\mathcal{M}}^{it}U(s)\Delta_{\mathcal{M}}^{-it} = \Delta_{\mathcal{N}}^{it}U(s)\Delta_{\mathcal{N}}^{-it} = U(e^{\mp 2\pi t}s)
\]
and
\[
J_M U(s) J_M = J_N U(s) J_N = U(-s) ,
\]
for all \( s, t \in \mathbb{R} \). In addition, \( \mathcal{N} = U(\pm 1)\mathcal{M}U(\pm 1)^{-1} \), and if \( \mathcal{M} \) is a factor, it must be type \( III_1 \).

The richness of this structure is further suggested by the next theorem.

**Theorem 5.2** (a) Let \( (\mathcal{M}, \mathcal{N}_1, \Omega) \) and \( (\mathcal{M}, \mathcal{N}_2, \Omega) \) be \(-\)-half-sided, resp. \(+\)-half-sided, modular inclusions satisfying the condition \( J_{\mathcal{N}_1} J_{\mathcal{N}_2} = J_{\mathcal{M}} J_{\mathcal{N}_2} J_{\mathcal{N}_1} J_{\mathcal{M}} \). Then the modular unitaries \( \Delta_{\mathcal{M}}^{iu}, \Delta_{\mathcal{N}_1}^{iu}, \Delta_{\mathcal{N}_2}^{iu}, s, t, u \in \mathbb{R} \), generate a faithful continuous unitary representation of the identity component of the group of isometries of two-dimensional Minkowski space.

(b) Let \( \mathcal{M}, \mathcal{N}, \mathcal{N} \cap \mathcal{M} \) be von Neumann algebras with a common cyclic and separating vector \( \Omega \). If \( (\mathcal{N}, \mathcal{M}, N \cap \mathcal{M}) \) and \( (\mathcal{N}, \mathcal{M}, N \cap \mathcal{M}) \) are \(-\)-half-sided, resp. \(+\)-half-sided, modular inclusions such that \( J_{\mathcal{N}} J_{\mathcal{M}} J_{\mathcal{N}} = \mathcal{M} \), then the modular units \( \Delta_{\mathcal{M}}^{iu}, \Delta_{\mathcal{N}}^{iu}, \Delta_{\mathcal{N} \cap \mathcal{M}}^{iu}, s, t, u \in \mathbb{R} \), generate a faithful continuous unitary representation of \( SL(2, \mathbb{R})/\mathbb{Z}_2 \).
This has led to a further useful notion. If $N \subset M$ and $\Omega$ is cyclic for $N \cap M$, then $(M, N, \Omega)$ is said to be a $\pm$-modular intersection if both $(M, M \cap N, \Omega)$ and $(N, M \cap N, \Omega)$ are $\pm$-half-sided modular inclusions and

$$J_N[\lim_{t \to \pm \infty} \Delta^i_N(t) \Delta^{-i,t}_N]J_N = \lim_{t \to \pm \infty} \Delta^i_M(t) \Delta^{-i,t}_N,$$

where the existence of the strong operator limits is assured by the preceding assumptions. An example of the utility of this structure is the following theorem.

**Theorem 5.3** Let $N, M, L$ be von Neumann algebras with a common cyclic and separating vector $\Omega$. If $(M, N, \Omega)$ and $(N', L, \Omega)$ are $-\pm$-modular intersections and $(M, L, \Omega)$ is a $+\pm$-modular intersection, then the unitaries $\Delta^s_M, \Delta^u_M, \Delta^s_L, \Delta^u_L, s, t, u \in \mathbb{R}$, generate a faithful continuous unitary representation of $SO^+(1, 2)$.

These results and their extensions to larger numbers of algebras were developed for application in algebraic quantum field theory, but one may anticipate that half-sided modular inclusions will find wider use. Modular theory has also been fruitfully applied in the theory of inclusions $N \subset M$ of properly infinite algebras with finite or infinite index.

## 6 Applications in Quantum Theory

Tomita–Takesaki theory has found many applications in quantum field theory and quantum statistical mechanics. As previously mentioned, the modular automorphism group satisfies the KMS-condition, a property of physical significance in the quantum theory of many-particle systems, which includes quantum statistical mechanics and quantum field theory. In such settings it occurs that for a suitable algebra of observables $M$ and state $\omega$ an automorphism group $\{\sigma_{\beta t}\}$ representing the time evolution of the system satisfies the modular condition. Hence, on the one hand, $\{\sigma_{\beta t}\}$ is the modular automorphism group of the pair $(M, \Omega)$, and, on the other hand, $\omega$ is an equilibrium state at inverse temperature $\beta$, with all the consequences which both of these facts have.

But it has become increasingly clear that the modular objects $\Delta^\beta, J$, of certain algebras of observables and states encode additional physical information. In 1975, it was discovered that if one considers the algebras of observables associated with a finite-component quantum field theory satisfying the Wightman axioms, then the modular objects associated with the vacuum state and algebras of observables localized in certain wedge-shaped regions in Minkowski space have geometric content. In fact, the unitary group $\{\Delta^\beta\}$ implements the group of Lorentz boosts leaving the wedge region invariant (this property is now called modular covariance), and the modular involution $J$ implements the spacetime reflection about the edge of the wedge, along with a charge conjugation. This discovery caused some intense research activity.²

### 6.1 Positive Energy

In quantum physics the time development of the system is often represented by a strongly continuous group $\{U(t) = e^{itH} \mid t \in \mathbb{R}\}$ of unitary operators, and the generator $H$ is interpreted as the total energy of the system. There is a link between modular structure

²See [1, 2, 5] for further details and references.
and positive energy, which has found many applications in quantum field theory. This result was crucial in the development of Theorem 5.1 and was motivated by the 1975 discovery mentioned above, now commonly called the Bisognano–Wichmann Theorem.

**Theorem 6.1** Let $\mathcal{M}$ be a von Neumann algebra with a cyclic and separating vector $\Omega$, and let $\{U(t)\}$ be a continuous unitary group satisfying $U(t)\mathcal{M}U(-t) \subset \mathcal{M}$, for all $t \geq 0$. Then any two of the following conditions imply the third.

1. $U(t) = e^{itH}$, with $H \geq 0$;
2. $U(t)\Omega = \Omega$, for all $t \in \mathbb{R}$;
3. $\Delta^{it}U(s)\Delta^{-it} = U(e^{-2\pi t}s)$ and $JU(s)J = U(-s)$, for all $s, t \in \mathbb{R}$.

### 6.2 Modular Nuclearity and Phase Space Properties

Modular theory can be used to express physically meaningful properties of quantum “phase spaces” by a condition of compactness or nuclearity of certain maps. In its initial form, the condition was formulated in terms of the Hamiltonian, the global energy operator of theories in Minkowski space. The above indications that the modular operators carry information about the energy of the system were reinforced when it was shown that a formulation in terms of modular operators was essentially equivalent.

Let $O_1 \subset O_2$ be nonempty bounded open subregions of Minkowski space with corresponding algebras of observables $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)$ in a vacuum representation with vacuum vector $\Omega$, and let $\Delta$ be the modular operator associated with $(\mathcal{A}(O_2), \Omega)$ (by the Reeh–Schlieder Theorem, $\Omega$ is cyclic and separating for $\mathcal{A}(O_2)$). For each $\lambda \in (0, 1/2)$ define the mapping $\Xi_\lambda : \mathcal{A}(O_1) \to \mathcal{H}$ by $\Xi_\lambda(A) = \Delta^{\lambda}A\Omega$. The compactness of any one of these mappings implies the compactness of all of the others. Moreover, the $l^p$ (nuclear) norms of these mappings are interrelated and provide a measure of the number of local degrees of freedom of the system. Suitable conditions on the maps in terms of these norms entail the strong statistical independence condition called the split property. Conversely, the split property implies the compactness of all of these maps. Moreover, the existence of equilibrium temperature states on the global algebra of observables can be derived from suitable conditions on these norms in the vacuum sector.

The conceptual advantage of the modular compactness and nuclearity conditions compared to their original Hamiltonian form lies in the fact that they are meaningful also for quantum systems in curved space–times, where global energy operators (i.e. generators corresponding to global timelike Killing vector fields) need not exist.

### 6.3 Modular Position and Quantum Field Theory

The characterization of the relative “geometric” position of algebras based on the notions of modular inclusion and modular intersection was directly motivated by the Bisognano–Wichmann Theorem. Observable algebras associated with suitably chosen wedge regions in Minkowski space provided examples whose essential structure could be abstracted for more general application, resulting in the notions presented in Section 5.

Theorem 5.2(b) has been used to construct from two algebras and the indicated half-sided modular inclusions a conformal quantum field theory on the circle (compactified light ray) with positive energy. And since the chiral part of a conformal quantum field model in two spacetime dimensions naturally yields such half-sided modular inclusions, studying the inclusions in Theorem 5.2(b) is equivalent to studying such field theories. Theorems
5.2(a) and 5.3 and their generalizations to inclusions involving up to 6 algebras have been employed to construct Poincaré covariant nets of observable algebras (the algebraic form of quantum field theories) satisfying the spectrum condition on $d + 1$-dimensional Minkowski space for $d = 1, 2, 3$. Conversely, such quantum field theories naturally yield such systems of algebras.

This intimate relation would seem to open up the possibility of constructing interacting quantum field theories from a limited number of modular inclusions/intersections.

### 6.4 Geometric Modular Action

The fact that the modular objects in quantum field theory associated with wedge-shaped regions and the vacuum state in Minkowski space have geometric significance ("geometric modular action") was originally discovered in the framework of the Wightman axioms. As algebraic quantum field theory (AQFT) does not rely on the concept of Wightman fields, it was natural to ask (1) when does geometric modular action hold in AQFT and (2) which physically relevant consequences follow from this feature?

There are two approaches to the study of geometric modular action. In the first, attention is focused on modular covariance, expressed in terms of the modular groups associated with wedge algebras and the vacuum state in Minkowski space. Modular covariance has been proven to obtain in conformally invariant AQFT, in any massive theory satisfying asymptotic completeness, and also in the presence of other, physically natural assumptions. To mention only three of its consequences, both the Spin–Statistics Theorem and the PCT Theorem, as well as the existence of a continuous unitary representation of the Poincaré group acting covariantly upon the observable algebras and satisfying the spectrum condition follow from modular covariance.

In a second approach to geometric modular action the modular involutions are the primary focus. Here, no a priori connection between the modular objects and isometries of the space–time is assumed. The central assumption, given the state vector $\Omega$ and the von Neumann algebras of localized observables $\{A(O)\}$ on the space–time, is that there exists a family $W$ of subsets of the space–time such that $J_{W_1} R(W_2) J_{W_1} \in \{R(W) \mid W \in W\}$, for every $W_1, W_2 \in W$. This condition makes no explicit appeal to isometries or other special attributes and is thus applicable in principle to quantum field theories on general curved space–times.

It has been shown for certain space–times, including Minkowski space, that under certain additional technical assumptions, the modular involutions encode enough information to determine the dynamics of the theory, the isometry group of the space–time, and a continuous unitary representation of the isometry group which acts covariantly upon the observables and leaves the state invariant. In certain cases including Minkowski space, it is even possible to derive the space–time itself from the group $J$ generated by the modular involutions $\{J_W \mid W \in W\}$.

The modular unitaries $\Delta^W_t$ enter in this approach through a condition which is designed to assure the stability of the theory, namely that $\Delta^W_t \in J$, for all $t \in \mathbb{R}$ and $W \in W$. In Minkowski space this additional condition entails that the derived representation of the Poincaré group satisfies the spectrum condition.
6.5 Further Applications

As previously observed, through the close connection to the KMS condition, modular theory enters naturally into the equilibrium thermodynamics of many-body systems. But in recent work on the theory of nonequilibrium thermodynamics it also plays a role in making mathematical sense of the notion of quantum systems in local thermodynamic equilibrium. Modular theory has also proved to be of utility in recent developments in the theory of superselection rules and their attendant sectors, charges and charge-carrying fields.

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