Thermodynamic properties of spin-$\frac{1}{2}$ transverse XY chain with Dzyaloshinskii-Moriya interaction: Exact solution for correlated Lorentzian disorder

Oleg Derzhko† and Johannes Richter‡

†Institute for Condensed Matter Physics
1 Svientsitskii St., L’viv-11, 290011, Ukraine
‡Institut für Theoretische Physik, Universität Magdeburg
P.O. Box 4120, D-39016 Magdeburg, Germany

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Abstract

We extend the consideration of the spin-$\frac{1}{2}$ transverse XY chain with correlated Lorentzian disorder (Phys. Rev. B 55, 14298 (1997)) for the case of additional Dzyaloshinskii-Moriya interspin interaction. It is shown how the averaged density of states can be calculated exactly. Results are presented for the density of states and the transverse magnetization.

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Postal addresses:

Dr. Oleg Derzhko (corresponding author)
Institute for Condensed Matter Physics
1 Svientsitskii St., L’viv-11, 290011, Ukraine
Tel: (0322) 42 74 39
Fax: (0322) 76 19 78
E-mail: derzhko@icmp.lviv.ua

Prof. Johannes Richter
Institut für Theoretische Physik, Universität Magdeburg
P.O. Box 4120, D-39016 Magdeburg, Germany
Tel: (0049) 391 671 8841
Fax: (0049) 391 671 1217
E-mail: Johannes.Richter@Physik.Uni-Magdeburg.DE
Much work has been done since the famous paper by Lieb, Schultz and Mattis [1] to derive exact results for thermodynamics and spin correlations of one-dimensional spin-$\frac{1}{2}$ $XY$ models. Much less exact results were obtained for random versions of spin-$\frac{1}{2}$ $XY$ chains. One can mention here a group of papers dealing with random spin-$\frac{1}{2}$ $XY$ models using well-known Dyson’s and Lloyd’s models of disorder [2, 3, 4]. Recently the interest in random spin-$\frac{1}{2}$ $XY$ chains has been noticeably increased since they provide a laboratory for investigation of generic features of quantum phase transitions in disordered systems. As an example we refer to papers on renormalization group [5] and numerical [6] studies on random spin-$\frac{1}{2}$ transverse Ising chain.

In the present paper we continue the study started in Ref. [4] that concerns the spin-$\frac{1}{2}$ isotropic $XY$ chain with random Lorentzian exchange coupling $J_n$ and a transverse field $\Omega_n$ that depends linearly on the surrounding exchange couplings $J_{n-1}$ and $J_n$. Obviously, due to the relation between the transverse field and the random exchange couplings that is a model of correlated disorder. The Jordan-Wigner method [1] and the method elaborated by John and Schreiber [7] permitted to derive exactly the averaged density of states for such a model and as a result to study its thermodynamic properties. Apparently the most interesting result of introducing the correlated disorder is the appearance of the nonzero averaged transverse magnetization at zero averaged transverse field. Later this effect was checked numerically [8, 9]. In the present communication we shall extend the model introducing additional Dzyaloshinskii-Moriya interspin interaction. Spin-$\frac{1}{2}$ $XY$ chains with Dzyaloshinskii-Moriya interaction were studied in several papers [10, 11, 12, 13, 14] in which it was shown that they exhibit interesting thermodynamic and dynamic properties, which may be of interest for the understanding of the properties of some quasi-one-dimensional compounds (e.g. CsCuCl$_3$). It will be shown below that the Dzyaloshinskii-Moriya interaction may influence in specific manner the thermodynamic properties of a magnetic chain conditioned by correlated disorder.

Hereafter we consider isotropic $XY$ chain in a magnetic field along $z$ axis consisting of $N$ spins $\frac{1}{2}$. The Hamiltonian is defined by

$$H = \sum_{n=1}^{N} \Omega_n s^z_n + \sum_{n=1}^{N} J_n (s^x_n s^x_{n+1} + s^y_n s^y_{n+1}) + \sum_{n=1}^{N} D_n (s^x_n s^y_{n+1} - s^y_n s^x_{n+1}),$$

\[s^\alpha_{n+N} = s^\alpha_n.\]  

(1)

Besides the exchange coupling $J_n$ between the neighbouring sites $n$ and $n + 1$ an additional Dzyaloshinskii-Moriya interaction $D_n$ between these sites is introduced, i.e. a more general case than in Ref. [4] is considered.

In what follows we consider two models.

Model (i) — We assume the Dzyaloshinskii-Moriya interaction to be ordered, i.e. $D_n = D$, whereas the exchange couplings $J_n$ are independent random Lorentzian variables with the
probability distribution
\[ p(J_n) = \frac{1}{\pi} \frac{\Gamma}{(J_n - J_0)^2 + \Gamma^2}. \]  

(2)
The on-site transverse fields are determined by the formula
\[ \Omega_n - \Omega_0 = \frac{a}{2} (J_{n-1} + J_n - 2J_0) \]  

(3)
where \( a \) is real and \(| a | \geq 1\). Note that after putting \( D = 0 \) one obtains the model considered in Ref. [4].

Model (ii) — We assume the exchange coupling to be ordered, i.e. \( J_n = J \), whereas the \( D_n \) are independent random Lorentzian variables with the probability distribution
\[ p(D_n) = \frac{1}{\pi} \frac{\Gamma}{(D_n - D_0)^2 + \Gamma^2}. \]  

(4)
The on-site transverse fields are determined by the formula
\[ \Omega_n - \Omega_0 = \frac{a}{2} (D_{n-1} + D_n - 2D_0) \]  

(5)
where \( a \) is real and \(| a | \geq 1\).

With the help of the Jordan-Wigner transformation the Hamiltonian (1) can be rewritten as a Hamiltonian of non-interacting spinless fermions
\[ H = \sum_{n=1}^{N} \Omega_n \left( c_n^+ c_n - \frac{1}{2} \right) + \sum_{n=1}^{N} \left( \frac{J_n + iD_n}{2} c_n^+ c_{n+1} - \frac{J_n - iD_n}{2} c_n c_{n+1} \right) \]  

(6)
with cyclic boundary conditions. We omitted in (6) the boundary term that is not essential for the calculation of the thermodynamic properties [15]. Let us introduce the retarded and advanced temperature double-time Green functions \( G_{nm}^\pm(t) = \mp i\theta(\pm t) \langle \{ c_n(t), c_m^+ \} \rangle \), \( G_{nm}^\pm(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega t} G_{nm}^\pm(\omega \pm i\epsilon) \) that satisfy the set of equations
\[ (\omega \pm i\epsilon - \Omega_n) G_{nm}^\pm(\omega \pm i\epsilon) - \left[ \frac{J_{n-1} - D_{n-1}}{2} G_{n-1,m}^\pm(\omega \pm i\epsilon) + \frac{J_n + iD_n}{2} G_{n+1,m}^\pm(\omega \pm i\epsilon) \right] = \delta_{nm}. \]  

(7)
Our task is to evaluate the random-averaged Green functions since they yield the random-averaged density of states through the relation
\[ \rho(E) = \frac{1}{\pi} \text{Im} G_{nn}^\pm(E \pm i\epsilon). \]  

(8)
Having the independent Lorentzian random variables one may try to perform the random averaging of Eq. (7) with the help of contour integrals. However one must know the positions of the singularities of the Green functions in the planes of complex random variables. The latter information can be derived for the defined models on the basis of the Gershgorin criterion [16].
Consider at first spin model (i) described by Eqs. (1) - (3). Suppose that exchange couplings $J_n$ (and hence the transverse fields $\Omega_n$) are complex variables. As it follows from (7) the singularities of the matrix $G^\pm = || G_{nm}^\pm (\omega \pm i \epsilon) ||$ are determined by the zeros of the determinant of the matrix $A \pm i B^\pm$ where $A$ and $B^\pm$ are the Hermitian matrices given by

$$
A = \begin{pmatrix}
\omega - \text{Re}\Omega_1 & -\frac{1}{2}\text{Re}J_1 - \frac{i}{2}D & 0 & \ldots & -\frac{1}{2}\text{Re}J_N + \frac{i}{2}D \\
-\frac{1}{2}\text{Re}J_1 + \frac{i}{2}D & \omega - \text{Re}\Omega_2 & -\frac{1}{2}\text{Re}J_2 - \frac{i}{2}D & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
-\frac{1}{2}\text{Re}J_N - \frac{i}{2}D & 0 & 0 & \ldots & \omega - \text{Re}\Omega_N
\end{pmatrix}
$$

(9)

and

$$
B^\pm = \begin{pmatrix}
\epsilon \mp \text{Im}\Omega_1 & \mp \frac{1}{2}\text{Im}J_1 & 0 & \ldots & \mp \frac{1}{2}\text{Im}J_N \\
\mp \frac{1}{2}\text{Im}J_1 & \epsilon \mp \text{Im}\Omega_2 & \mp \frac{1}{2}\text{Im}J_2 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\mp \frac{1}{2}\text{Im}J_N & 0 & 0 & \ldots & \epsilon \mp \text{Im}\Omega_N
\end{pmatrix},
$$

(10)

respectively. John and Schreiber noticed that if all eigenvalues of $B^\pm$ are positive then $\det(A \pm i B^\pm) \neq 0$ [7]. On the other hand for any eigenvalue $\lambda$ of the matrix $B^\pm$ (10) the Gershgorin criterion after making use of Eq. (3) guarantees that at least one of the inequalities

$$
|\epsilon \mp \frac{a}{2} (\text{Im}J_{n-1} + \text{Im}J_n) - \lambda| \leq \frac{1}{2} |\text{Im}J_{n-1}| + \frac{1}{2} |\text{Im}J_n|,
$$

(11)

is satisfied. From (11) it immediately follows that the retarded (advanced) Green function does not have poles for $\text{Im}J_n < 0$ ($\text{Im}J_n > 0$) if $a \geq 1$ and for $\text{Im}J_n > 0$ ($\text{Im}J_n < 0$) if $a \leq -1$. Noting that $F(\ldots, \Omega_n, J_n, \ldots) = F(\ldots, \Omega_n - i\Gamma, J_n - i\Gamma, \ldots)$ if $F(\ldots, \Omega_n, J_n, \ldots)$ does not have poles in lower half-planes $J_n$ and $F(\ldots, \Omega_n, J_n, \ldots) = F(\ldots, \Omega_n + i\Gamma, J_n + i\Gamma, \ldots)$ if $F(\ldots, \Omega_n, J_n, \ldots)$ does not have poles in upper half-planes $J_n$ one finds the following result of averaging the set of equations (7)

$$
- \left[ J_0 - iD \mp i \text{sgn}(a) \Gamma \frac{G^\pm_{n-1,m}(\omega)}{G^\pm_{n-1,m}(\omega)} + J_0 + iD \mp i \text{sgn}(a) \Gamma \frac{G^\pm_{n+1,m}(\omega)}{G^\pm_{n+1,m}(\omega)} \right] = \delta_{nm}.
$$

(12)

The obtained equations (12) possess translational symmetry and proceeding further in standard manner one obtains

$$
\rho(E) = \frac{1}{\pi} \sqrt{\frac{A^2 + B^2 - A}{2(A^2 + B^2)}},
$$

$$
A = (E - \Omega_0)^2 + (1 - |a|^2)\Gamma^2 - J_0^2 - D^2,
$$

$$
B = 2\Gamma |\epsilon a (E - \Omega_0) + \text{sgn}(a) J_0|.
$$

(13)
Consider now spin model (ii) described by Eqs. (1), (4), (5). Certainly we may repeat the whole calculation once more obtaining as a result $\rho(E)$ for this model. However there is a relationship between the models (i) and (ii) that immediately yields thermodynamics of the latter model if it is known for the former one. Namely, consider the following rotations of spin axes around $z$ axis

$$(s_n^z)' = \exp \left[-i\frac{\pi(n-1)}{2}s_n^z\right] s_n^z \exp \left[i\frac{\pi(n-1)}{2}s_n^z\right]. \tag{14}$$

One immediately finds that the Hamiltonian (1) arises as a result of transformations (14) applied to a Hamiltonian of form (1), however, with the exchange couplings $D_n$ and the Dzyaloshinskii-Moriya interactions $-J_n$. Therefore it becomes evident that the density of states (13) after the replacement $J_0 \to D_0$, $D^2 \to J^2$ transforms into the density of states for the model (ii). Hence it is sufficient in what follows to consider only the spin model (i) defined by (1) - (3).

Let us discuss the obtained density of magnon states (13). It can be straightforwardly checked that (13) covers in the particular case $D = 0$ the result derived in Ref. [4]. In the limit of diagonal disorder $\Gamma \to 0$, $|a| = \gamma = \text{const}$ Eq. (13) reproduces the density of states for spin-$\frac{1}{2}$ isotropic XY chain with Dzyaloshinskii-Moriya interaction in a random Lorentzian transverse field with the mean value $\Omega_0$ and the width of distribution $\gamma$ [14]. The density of states (13) remains the same after the simultaneous change of signs of $J_0$ and $a$; hereafter we choose $J_0 > 0$.

Let us remind how the density of states is influenced by correlated disorder in case of $D = 0$ (for details see [4]). For $|a| \approx 1$ the disorder causes a smearing out of mainly one edge of the magnon band (which one depends on the sign of $a$). As a result we have $\int_{-\infty}^{0} dE\overline{\rho(E)} \neq \int_{0}^{\infty} dE\overline{\rho(E)}$ at $\Omega_0 = 0$ that leads to the appearance of a nonzero averaged transverse magnetization $\overline{m_z} = -\frac{1}{2} \int_{-\infty}^{\infty} dE\overline{\rho(E)} \tanh \frac{BE}{2}$ at zero averaged transverse field $\Omega_0$. With an increase of $|a|$ the symmetry of the non-random case is recovered, i.e., both edges of the magnon band become smeared out in a symmetric way, the numbers of states $\int_{-\infty}^{0} dE\overline{\rho(E)}$ and $\int_{0}^{\infty} dE\overline{\rho(E)}$ at $\Omega_0 = 0$ become equal to each other, and $\overline{m_z} = 0$ at $\Omega_0 = 0$.

Figs. 1a, 1b demonstrate the changes in the behaviour of the averaged density of states $\overline{\rho(E)}$ versus $E - \Omega_0$ for $\Gamma = 1$, $a = \pm 1.01$, $J_0 = 1$ for three different strengths of the Dzyaloshinskii-Moriya interaction $D = 0$, $D = 1$, $D = 2$. It can be seen that an additional Dzyaloshinskii-Moriya interspin interaction 1) increases the width of the smoothed magnon band; 2) leads to the recovering of the symmetry with respect to the change $E - \Omega_0 \to -(E - \Omega_0)$. Thus the increase of the Dzyaloshinskii-Moriya interaction leads to the decrease of the nonzero value of $\overline{m_z}$ at $\Omega_0 = 0$ (Figs. 1c, 1d).

In Fig. 2 we depicted the influence of an increase of the averaged exchange coupling $J_0$ at fixed $D = 0$. Similarly to the previous case one observes an increasing of the band width, however, in contrast to the previous case the density of states remains not symmetric with respect to the change $E - \Omega_0 \to -(E - \Omega_0)$ (Figs. 2a, 2b) and as a result the model exhibits
a noticeable nonzero value of $m_z$ at $\Omega_0 = 0$ (Figs. 2c, 2d). The difference in the behaviour of the density of states with increasing $D$ or $J_0$ is not surprising since $J_0$ and $D$ enter in a different way into (13).

To summarize, we have studied the spin-$\frac{1}{2}$ transverse isotropic XY chain in the presence of correlated Lorentzian disorder. Going beyond the results given in Ref. [4] we include in the model the Dzyaloshinskii-Moriya interaction. The assumption of correlated disorder allows the exact calculation of the averaged density of states $\overline{\rho(E)}$. The exact formula (13) for $\overline{\rho(E)}$ is the main result of the paper. Based on this formula one can calculate in a simple way exactly the thermodynamic properties like entropy, specific heat, transverse magnetization and static transverse linear susceptibility (see for details [4]). In that sense the presented random quantum spin model may serve as a reference model to study the interplay of disorder and quantum effects. In particular, we find that the Dzyaloshinskii-Moriya interaction may lead to a decrease of the nonzero averaged transverse magnetization at zero averaged transverse field that appears due to correlated disorder. It is known [10, 11, 12, 13] that in the non-random case the Dzyaloshinskii-Moriya interaction leads to spectacular changes in the spin correlations and their dynamics. However, the rigorous consideration of correlated disorder in this paper is restricted to thermodynamic quantities based on the density of states. The effect of the Dzyaloshinskii-Moriya interaction on the spin correlations and their dynamics in the presence of correlated disorder may be studied numerically [17].

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Figure 1: The density of states (described by Eq. (13)) (Figs. 1a, 1b) and the transverse magnetization $-\mathbf{m}_z$ versus $\Omega_0$ at $\beta = 1000$ (Figs. 1c, 1d) at fixed $J_0 = 1$, $\Gamma = 1$ and $a = -1.01$ (Figs. 1a, 1c) or $a = 1.01$ (Figs. 1b, 1d). The short-dashed curves correspond to $D = 0$, long-dashed curves to $D = 1$ and the solid curves to $D = 2$. 
Figure 2: The density of states (described by Eq. (13)) (Figs. 2a, 2b) and the transverse magnetization $-\overline{m_z}$ versus $\Omega_0$ at $\beta = 1000$ (Figs. 2c, 2d) at fixed $D = 0$, $\Gamma = 1$ and $a = -1.01$ (Figs. 2a, 2c) or $a = 1.01$ (Figs. 2b, 2d). The short-dashed curves correspond to $J_0 = 1$, long-dashed curves to $J_0 = 1.5$ and the solid curves to $J_0 = 2$. 
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FIG. 2. The density of states (described by Eq. (13)) (Figs. 2a, 2b) and the transverse magnetization $-m_z$ versus $\Omega_0$ at $\beta = 1000$ (Figs. 2c, 2d) at fixed $D = 0$, $\Gamma = 1$ and $a = -1.01$ (Figs. 2a, 2c) or $a = 1.01$ (Figs. 2b, 2d). The short-dashed curves correspond to $J_0 = 1$, long-dashed curves to $J_0 = 1.5$ and the solid curves to $J_0 = 2$. 