We have considered D2-D8 model and obtain a numerical solution that exhibits spatially modulated phases corresponding to charge and spin density wave. We have analysed behaviour of the free energy density for different values of chemical potential and magnetic field.

* nishalrai10@gmail.com.
† subirkm@gmail.com.
1 Introduction

There are varieties of condensed matter systems that exhibit spontaneous breaking of spatial symmetry. In particular, some systems admit spatially modulated phases that correspond to breaking of the translation symmetry and symmetry of the lattice in the ground states. Common examples of such phases are charge and spin density waves [1,2]. Charge density waves were first predicted [3] for weakly coupled one dimensional system. Various metals admit spatially modulated charge density phases. They appear in strongly correlated systems as well and sometimes they are accompanied by spin density waves, as found in the striped phases of high $T_c$ cuprate superconductors [4]. They are characterized by competing orders, which is believed to play an important role in the rich phase structure of high $T_c$ superconductors. Such breaking of translation symmetry modifies the transport behaviours, e.g. showing unconventional behaviours of DC and AC conductivities.

Gauge/gravity duality offers a novel method to study the strongly correlated systems by translating problems in the strongly coupled d-dimensional field theory into phenomena of a weakly coupled gravity theory living in (d+1) dimension [5–8]. A number of works have appeared, where this duality has been applied for studying systems with broken symmetry. Solutions with explicitly broken symmetry can be obtained by introducing sources that depend on space, such as chemical potential. On the other hand, sometimes spatially homogeneous solutions in presence of large charge density or magnetic field develop instability leading to spatially modulated configurations. Systems with such instabilities arise in the context of holographic QCD as well as probe brane models. One of the mechanisms of such instability occurs in Maxwell-Chern-Simons theory at non-zero momentum once an electric field is turned on [9]. When coupled to gravity, RN-AdS black hole develops a similar instability in presence of an electric field, which depends on the non-zero momenta suggesting a phase transition to an inhomogeneous solution and thus signals spatially modulated solution in the dual theory [10].

Instabilities leading to spatially inhomogeneous solution have been studied in quite a number of different models in bottom-up approach [11–26]. Electrically charged AdS black hole with a neutral pseudoscalar was studied in four dimension [11] and with $SU(2) \times U(1)$ gauge theory was studied in five dimensions in [12]. In particular, analysis of linearized perturbations shows [12] that below a critical temperature, the black hole is unstable towards decay into a phase with non-zero momentum implying a spatially modulated phase. A full-fledged spatially modulated solution of it was obtained numerically in [13]. Subsequently, a black hole solution with spatially modulated horizon was obtained numerically in Einstein-Maxwell-pseudoscalar model with backreaction in [14]. On the other hand in magnetically charged black branes, with geometry AdS2 or AdS3 space-dependent instabilities were found in [15]. Similar momentum dependent instabilities were shown in a region of parameter space [16] for a magnetically charged AdS2 solution appearing in Einstein-Maxwell-dilaton model with hyperscaling violating geometry. [17] considered a geometry interpolating between AdS2 and AdS4 having anisotropic Lifshitz scaling and
hyperscaling violation in the intermediate with an electric field. Linearised analysis of instabilities this model predicts that its endpoint may lead to spatially modulated phase. This was further extended to two U(1) gauge fields \[18\] and found to admit a pair density wave, which involves intertwined order of a superconducting phase and charge density wave. Furthermore, Dirac equation was analyzed in this background \[19\] to obtain spectral density and Fermi surface and band gap were explored. Einstein-Maxwell-dilaton model was also considered in \[20\] leading to spatially inhomogeneous solution as a thermodynamically preferred phase. Similar axionic system was found to exhibit instability towards the formation of inhomogeneous stripes in the bulk and on the boundary \[21\]. Such solutions were further generalized to checkerboard solution which breaks translation symmetry spontaneously in two directions \[22\]. A higher derivative gravity model along with complex scalar and gauge field was considered in \[24\] giving rise to a solution spontaneously breaking translation symmetry, which corresponds to a charge density wave at T=0.

Spatially inhomogeneous models are analysed in top-down approach as well. In \[27\] D7 brane probe in the background of D3 brane with a black hole embedding was studied using linearised perturbation and it was found that above a critical density the system develops instability, which is similar to charge density and spin density wave state. A similar study for D2-D8 model appeared in \[28\]. At high enough charge density, the instabilities developed in homogeneous state of D3-D7 leads to a spatially modulated configuration, which is a CDW and SDW \[29\]. The phase space of the parameters and nature of the transition was also studied. Analysis of AC and DC conductivities of this model appeared in \[30\] and it was found that by applying an electric field the stripes slide. It was also shown that for large chemical potential the ground state will be a combination of CDW and SDW. Field theory with spontaneous stripe order was considered in holographic setup \[31\] and DC and AC conductivities with a small applied electric field was studied. It was shown in \[32\] that spontaneously broken spatially modulated ground state with explicit symmetry breaking in the form of ionic lattice etc. gives rise to pinning and study direct and optical conductivity.

As mentioned above, charge density wave along with spin density waves occur in the striped phases of high T\textsubscript{c} cuprate superconductors \[4\]. The aim of the present work is to study a probe D2-D8 brane model in top-down approach, which admits both charge density and spin density waves. Analysis of linearised fluctuations of D2-D8 model showed \[28\] that once the charge density of the homogeneous black hole phase is greater than some critical value it is unstable leading to spatially modulated configuration. We have obtained spatially modulated numerical solutions, and from analysis of the free energy, we find that it becomes thermodynamically preferred phase when chemical potential crosses a critical value. In this spatially modulated solutions, The bulk fields are coupled with one another and from the study of the asymptotic behavior of the gauge field and a certain scalar field we find it is a mixture of both charge and spin density waves. Nature of the phase transition turns out to be second order. We have studied behavior of the free energy in
the presence of a magnetic field and find there is a non-zero critical value that minimizes the free energy.

The article is structured as follows. In the next section we review D2-D8 model. Section 3 consists of the discussion of the partial differential equations and the numerical method employed to solve it. The results are given in section 4 and we conclude with a discussion in section 5.

2 D2-D8 model

We consider a probe D8 brane embedded in the background of N D2 branes [28]. In the limit of large $N$ and large ’t Hooft coupling $g_sN \gg 1$, the holographic field theory dual is super Yang-Mills (SYM) in (2 + 1) dimensions along with charged fermions following from low energy degrees of freedom of bi-fundamental strings and can serve as a dual model for applications in condensed matter.

The ten-dimensional near-horizon metric for a D2 brane is given by [28]

$$ds^2 = L_0^2[r^{5/2}(-f(r)dt^2 + dx^2 + dy^2) + r^{-5/2}(\frac{dr^2}{f(r)} + r^2dS_6^2)], \quad (2.1)$$

where $L_0^5 = 6\pi^2g_sN\ell_s^5$ and $f(r) = 1 - \frac{rT}{r^5}$. The metric on the internal sphere $S^6$ is written as

$$dS_6^2 = d\psi^2 + \sin^2\psi(d\theta_1^2 + \sin^2\theta_1d\phi_1^2) + \cos^2\psi(d\xi^2 + \sin^2\xi d\theta_2^2 + \sin^2\xi \sin^2\theta_2 d\phi_2^2). \quad (2.2)$$

The ranges of the angular coordinates are given by $0 \leq \psi \leq \pi/2$, $0 \leq \xi, \theta_1, \theta_2 \leq \pi$, $0 \leq \phi_1, \phi_2 \leq 2\pi$. the dilaton is given by $e^\phi = g_s(\frac{T}{L_0})^{-5/4}$. The 5-form potential is given by $C(5) = c(\psi)L_0^5dS_2\wedge dS_3$, where $dS_2$ and $dS_3$ are volume form on the $S^2$ and $S^3$ respectively. The coefficient $c(\psi)$ depends on the angular variable and is given as $c(\psi) = \frac{5}{8}(\sin\psi - \frac{1}{6}\sin(3\psi) - \frac{1}{10}\sin(5\psi))$. In the following we set $L_0 = 1$.

We will consider the D8 brane to fill up $t, x$ and $y$ directions, wrapped $S^2$ and $S^3$ and extended in $r$ directions. The embedding then is given by $\psi$ as a function of the radial coordinate $r$ and $x$. In order to stabilize the embedding, we have turned on a magnetic field on $S^2$ given by $2\pi\alpha'F_{\theta_1\phi_1} = L_0^2b\sin\phi_1$ [28]. In addition to that we have turned on gauge potential $a_\mu$ along the worldvolume as well as the radial direction $a_{t,x,y,r}(r, x)$, which in general depends on the world volume directions as well as radial direction. We have also turned on a constant magnetic field $h$ along the $xy$ direction, $2\pi\alpha'F_{xy} = h$. In what follows, we will choose a radial gauge and work with $a_r = 0$.

We will replace the radial variable $r$ into a compact variable $u$, which ranges between 0 to 1 and will scale $x$ and $a_\mu$ to avoid explicit dependence of $r_T$ as follows.

$$r = \frac{r_T}{u}, \quad a_\mu = r_T\tilde{a}_\mu, \quad x_\mu = \tilde{x}_\mu r_T^{-3/2}, \quad \tilde{b} = b\sqrt{r_T}. \quad (2.3)$$
The action consists of a Dirac-Born-Infeld term and a Chern-Simons term, which are given in terms of the rescaled variables, as follows.

\[
S = S_{DBI} + S_{CS},
\]

\[
S_{DBI} = -T_s \int d^3 x e^{-\phi} \sqrt{-\det(g_{\mu\nu} + 2\pi \alpha' F_{\mu\nu})}
= -N r^2 \int dud\hat{x} \frac{1}{u^2} \sqrt{A(A_1 + A_2 + A_4)},
\]

where \( N = 8\pi^3 T_s V_{1,1} \), \( V_{1,1} \) being the volume of spacetime in \( t \) and \( y \) direction. \( A \), \( A_1, A_2 \) and \( A_3 \) are defined as follows:

\[
A = \cos^6 \psi \left( \sin^4 \psi + \frac{b^2}{u} \right),
\]

\[
A_1 = \frac{1}{u^5} \left[ 1 + u^2 f \psi_u^2 - u^4 \tilde{a}_{u}^2 + u^4 f \tilde{a}_{\tilde{u}}^2 + f u^4 \tilde{a}_{\tilde{u}}^2 \right],
\]

\[
A_2 = \frac{\psi_x^2}{u^2} - \frac{\tilde{a}_{ou}^2}{f} + \tilde{a}_{\tilde{u}}^2,
\]

\[
A_3 = -u^2 \tilde{a}_{0u} \psi_x^2 - u^2 \tilde{a}_{\tilde{u}} \psi_u^2 + u^2 f \tilde{a}_{\tilde{u}} \psi_x^2 - u^4 \tilde{a}_{0u} \tilde{a}_{\tilde{u}}^2 + u^4 f \tilde{a}_{\tilde{u}} \psi_u^2 - u^4 \tilde{a}_{0u} \tilde{a}_{\tilde{u}}^2
+ 2u^2 \tilde{a}_{ou} \tilde{a}_{\tilde{u}} \psi_x \psi_u + 2u^4 \tilde{a}_{ou} \tilde{a}_{\tilde{u}} \tilde{a}_{0u} \tilde{a}_{\tilde{u}} - 2u^2 f \tilde{a}_{\tilde{u}} \tilde{a}_{\tilde{u}} \psi_x \psi_u.
\]

In order to keep expressions compact we have used \( \psi_u = \frac{\partial \psi}{\partial u} \), \( \tilde{a}_{0\tilde{u}} = \frac{\partial \tilde{a}_{\tilde{u}}}{\partial \tilde{u}} \), etc. As can be seen through a straightforward analysis, \( \tilde{a}_{\tilde{u}} \) appears through a term \((\partial_u \tilde{a}_{\tilde{u}})^2/f \) in \( A_1 \), does not mix with other fields and thus decouples from the rest of the system. So we have dropped \( \tilde{a}_{\tilde{u}} \).

The Chern-Simons term is given by

\[
S_{CS} = -\frac{T_s}{2} (2\pi \alpha')^2 \int C^{(5)} \wedge F \wedge F,
\]

\[
= -Nr^2 \int du \, du \hat{x} \, c(\psi) (\tilde{a}_{ou} \tilde{a}_{\tilde{u}} - \tilde{a}_{\tilde{u}} \tilde{a}_{\tilde{u}}).
\]

Therefore, the full action can be written as

\[
S_{DBI} + S_{CS} = -Nr^2 \int dud\hat{x} \frac{1}{u^2} [\sqrt{A(A_1 + A_2 + A_4)} + u^2 c(\psi)(\tilde{a}_{ou} \tilde{a}_{\tilde{u}} - \tilde{a}_{\tilde{u}} \tilde{a}_{\tilde{u}})].
\]

The equations of motion are obtained from this action by taking variation of the fields, \( \psi(\hat{x}, u) \), \( \tilde{a}_{0}(\hat{x}, u) \) and \( \tilde{a}_{\tilde{u}}(\hat{x}, u) \). They give rise to nonlinear partial differential equations in \( \hat{x} \) and \( u \) and are difficult to solve. Since the equations are quite long we have not given the explicit expressions. The action is invariant under \( \hat{x} \to -\hat{x} \) and the solutions also share this symmetry. Furthermore, in absence of magnetic field there is a reflection symmetry \( \hat{x} \to \frac{1}{2} - \hat{x} \).
The boundary conditions following from physical considerations. At the ultraviolet limit, given by \( u = 0 \) the boundary conditions are given by

\[
\psi(\hat{x}, 0) = \psi_\infty, \quad \partial_u \psi(\hat{x}, 0) = m_\psi,
\]

\[
\hat{a}_0(\hat{x}, 0) = \mu, \quad \hat{a}_y(x, 0) = h \hat{x}, \tag{2.8}
\]

where \( \mu \) is the chemical potential and we have turned on a constant magnetic field \( h \) in the \( xy \)-plane of the world volume. \( \psi_\infty \) is the asymptotic value of the field \( \psi \) at \( u \to 0 \), which we have chosen to be constant and \( m_\psi \) represents mass of the fermion.

At the infrared limit, \( u \to 1 \) the boundary conditions follow from the regularity of the solutions. Since \( A_2 \) contains a term \( (\partial_y \hat{a}_0)^2 / f \) and since \( f \) vanishes at horizon, \( \partial_y \hat{a}_0 \) has to vanish over there implying \( \hat{a}_0 \) is a constant. Furthermore, the constant needs to vanish so that \( \hat{a} \) is a well-defined one-form at the horizon. Therefore, we impose

\[
\hat{a}_0(\hat{x}, 1) = 0. \tag{2.9}
\]

In addition, we have expanded the equations of motion corresponding to \( \psi, a_0 \) and \( a_y \) around the horizon \( u = 1 \) and consider the leading order equations in the expansions.

Since we are looking for a spatially modulated solution we impose periodic boundary condition along with \( \hat{x} \) direction for all the fields:

\[
\psi(\hat{x} + L) = \psi(\hat{x}), \quad \hat{a}_0(\hat{x} + L) = \hat{a}_0(\hat{x}), \quad \hat{a}_y(\hat{x} + L) = \hat{a}_y(\hat{x}). \tag{2.10}
\]

One can obtain the asymptotic behavior of the various fields considering \( u \to 0 \) limit. From the equations of motion, it turns out that the asymptotic behavior of \( a_0 \) and \( \psi \) can be given by,

\[
a_0(\hat{x}, u) = \mu + d(\hat{x}) u^2 + \ldots, \quad \psi(\hat{x}, u) = \psi_\infty + m_\psi u - c_\psi(\hat{x}) u^3 + \ldots. \tag{2.11}
\]

In the expansion of \( a_0 \), apart from the leading term which is given by the chemical potential \( \mu \), we have identified coefficient in the subleading term \( d(\hat{x}) \) with the charge density function in the boundary field theory. Since we have considered a spatially modulated solution, this has a nontrivial dependence on \( \hat{x} \) and we define average of the charge density over the period by

\[
<d> = \frac{1}{L} \int_0^L d(\hat{x}) \, d\hat{x}. \tag{2.12}
\]

The amplitude of the charge density wave can be obtained by considering Max\((d(\hat{x}) - <d>)\). Similarly, we have identified \([28]\) the coefficient of \( u^3 \) in the asymptotic expansion of the field \( \psi \) with the fermion bilinear in the dual field theory and Max\((c_\psi(\hat{x}))\) will play a similar role in the context of spin density wave in the boundary field theory.

Since the system of partial differential equations following from the action is quite difficult to solve we have used pseudospectral method. We have expanded each function
along $u$ direction in terms of Chebyshev polynomial for the region $0 \leq u \leq 1$. In the $x$ direction we have expanded each function in terms of Fourier series. The expansions of the different fields are given by

$$
\psi(\hat{x}, u) = \sum_{j=0}^{N_u-1} \sum_{k=0}^{N_x-1} \psi[j, k] T_j(2u - 1) \cos \frac{2\pi k \hat{x}}{L},
$$

$$
\tilde{a}_0(\hat{x}, u) = \sum_{j=0}^{N_u-1} \sum_{k=0}^{N_x-1} a_0[j, k] T_j(2u - 1) \cos \frac{2\pi k \hat{x}}{L}, \tag{2.13}
$$

$$
\tilde{a}_y(\hat{x}, u) = \sum_{j=0}^{N_u-1} \sum_{k=0}^{N_x-1} \psi[j, k] T_j(2u - 1) \cos \frac{2\pi k \hat{x}}{L}.
$$

We have chosen the grid of collocation points as consisting of $N_x + 1$ points along $\hat{x}$ direction from $\hat{x} = 0$ to $\hat{x} = L$ evenly distributed over the range and $N_u + 1$ points from $u = 0$ to $u = 1$ forming the Gauss-Lobatto grid. Substituting the expansions in the equations of motion and boundary conditions and evaluating them at the collocation points reduces the system of partial differential equations to a set of algebraic equations of the coefficients in the expansion (2.13).

We have incorporated the first and third equations of boundary conditions at UV (2.8) in the choice of the coefficients and thus reducing the number of variables. The rest of the boundary conditions are evaluated at the collocation points at $u = 0$ and $u = 1$ accordingly and the three equations of motion following from the action are evaluated at all the collocation points from the grid. We have chosen the number of collocation points in such a way so as to have the number of variables equal to the number of algebraic equations and solve them numerically.

For the numerical solution of the algebraic equations we have used the Newton-Rhapson method. This set of algebraic equations has an additional difficulty stemming from the fact that it involves square root due to the DBI part of the action. For a generic choice of initial value, it leads to complex solution. One has to choose the initial value judiciously so as to get real solutions for the coefficients.

### 3 Spatially modulated solution

In this section, we will consider spatially inhomogeneous solutions of the equations of motion following from the action of D2-D8 model (2.7), which consists of three partial differential equations and four boundary conditions. Furthermore, they are being coupled partial differential equations it is difficult to obtain an analytical solution. We have used the pseudospectral method as elaborated in the last section and obtain solutions to these partial differential equations numerically. Regarding the various parameters in the boundary conditions we have made the following choices. In accordance with the linearized perturbation analysis (2.8) we have set the value of the field $\psi$ at the boundary
$u = 0$ to be $\psi_\infty = 0$. The parameter $m_\psi$, that appears in the second of the equation (2.8) represents mass of the fermions and for the sake of simplicity, we have chosen $m_\psi = 0$, setting fermions to be massless. We set $b = 1$ and consider appropriate variations in $\mu$, $L$ and $h$ as mentioned in the following.

We begin with the condition that the magnetic field is absent and set $h = 0$. Using numerical procedure, we look for a spatially inhomogeneous solution by varying $\mu$ and $L$. It turns out that there are a number of branches of solutions and we have considered a specific branch of solution.

A representative solution for $\psi$, $\hat{a}_0$ and $\hat{a}_y$ are given in 1a, 1b and 2a respectively. As one can observe, all the three fields $\psi$, $\hat{a}_0$ and $\hat{a}_y$ are spatially modulated in the $\hat{x}$ direction. Among them, both $\psi$ and $\hat{a}_y$ have periodicity $L$ as imposed by the boundary condition and they are $\pi/2$ out of phase with each other. The spatial modulation of the gauge field $\hat{a}_0$ is manifested, once we subtract the homogeneous part and consider $\Delta \hat{a}_0 = \hat{a}_0(\hat{x}, u) - \hat{a}_0(0, u)$ as shown in 2b. Unlike the other two fields, it turns out to have periodicity to be $L/2$.

The solutions share the symmetry $x \rightarrow -x$ and $x \rightarrow L/2 - x$.

![Figure 1: Plot of $\psi$ and $a_0$ vs. $\hat{x}$ and $u$](image)

While solving the equations, the periodicity $L$ can be chosen manually. However, it turns out that with $\mu$ and $b$ fixed, the free energy depends on the period $L$ and for a given $\mu$ there is a length $L$ for which the free energy shows a minimum. In order to see that, consider the system with fixed $\mu$ as the grand canonical ensemble and identify the free energy density with the Euclidean action evaluated on-shell. The free energy is given by

$$F = \frac{1}{N} S_{Euclidean}(\psi, a_0, a_y)|_{on-shell}. \quad (3.1)$$

For each solution we have evaluated the free energy from (3.1), subtract the free energy for the homogeneous solution from this and divide it by $L$ to obtain the free energy density.

We have varied the period $L$ in small increments and numerically solved the equations along with boundary conditions for a range of values of $L$. We have plotted the free energy
density $F/L$ vs $\hat{k} = \frac{2\pi}{L}$ for zero magnetic field in Fig. 3a for several values of the chemical potential $\mu$. Within the range of $k$, it shows a minimum at around certain values of $\hat{k} = \hat{k}_0$ depending on $\mu$ and for higher or lower values of $\hat{k}$ the free energy will increase. Since $\hat{k} = \hat{k}_0$ corresponds to a minimum energy configuration and the corresponding free energy is less than that of the homogeneous solution, the spatially modulated configuration with wave number $\hat{k} = \hat{k}_0$ is thermodynamically preferred. Forcing the system to have a higher or lower wave number will cost additional energy and will be thermodynamically unstable.

Next, we have studied the behavior of the system with a variation of chemical potential. However, since there are multiple branches of solutions we wanted to focus on a single branch. For a specific branch of solution we expect that at the critical value of chemical
potential both the amplitude of the charge density wave and spin density wave given by 
\[ \text{Max}(d(x)) - \langle d \rangle \] and \( \text{Max}(c_\psi) \) respectively should vanish and around that point they 
should have a smooth behavior. We have considered the branch for which, \( \hat{k} = a_0 - a_1\mu \), 
with \( a_0 \sim 2.13 \) and \( a_1 \sim 4.77 \).

The free energy density is plotted against \( \mu \) in Fig.4a. As one can observe the free 
energy density increases with decreasing \( \mu \) and at a critical value of chemical potential 
\( \mu = \mu_c \), which turns out to be \( \mu_c = 1.09656 \) for the present choice of parameters, the free 
energy density of the inhomogeneous solution equals to that of the homogeneous solution 
indicating a phase transition. For \( \mu > \mu_c \) the spatially modulated solution is thermo-
dynamically preferred, while for \( \mu < \mu_c \) the homogeneous solution is thermodynamically 
preferred. From the behavior of the plot, we conclude that the nature of the phase tran-
sition is second order. We have tried a numerical fit for the data obtained for free energy 
density as a function of \( \mu \). A quadratic expression turns out to be 
\[ F/L = a_1 (\mu - \mu_c)^2. \]
From the fit numerical value of \( a_1 \) turns out to be \( a_1 = -6.53576 \). The charge density for 
this system in grand canonical ensemble is defined to be 
\[ \rho = -\frac{\partial F}{\partial \mu}. \] (3.2)

From the plot of the free energy density vs. \( \mu \) we obtain plot of the charge density \( \rho \) vs. \( \mu \) 
in Fig.4b using (3.2). As one can see it increases linearly with \( \mu \) and vanishes at \( \mu = \mu_c \).

Amplitude of the charge density wave, given by \( \text{Max}(d(x)) - \langle d \rangle \) is plotted against 
the chemical potential \( \mu \) in Fig. 5b. Its behavior near \( \mu_c \) is linear and can be approximated 
by \( 2.76824(\mu - \mu_c) \). We have also plotted the maximum of \( c_\psi \) vs. \( \mu \) in Fig.5a, which can be 
considered to be related to the amplitude of spin density wave [29]. Its behaviour near \( \mu_c \) 
goes as \( 2.76629(\mu - \mu_c) \) unlike [29], where it was found to have a square root dependence. 
Comparing the plots of \( \text{Max}(d(x)) - \langle d \rangle \) and \( \text{Max}(c_\psi) \) one can see charge density wave 
and spin density wave are of the same order in this case, unlike the result obtained in the 
case of D3-D7 model [29], where spin density wave dominates.

Next, we have turned on the magnetic field \( h \) and find the spatially modulated solutions 
for different values of \( L \), as in the case of \( h = 0 \). In the presence of magnetic field, however, 
\( x \to L/2-x \) symmetry is broken. We have considered a range of magnetic field from \( h = 0 \) 
to \( h = 0.7 \). the free energy density shows a minimum at \( \hat{k} = \hat{k}_0 \). This is very different 
from similar studies [29] where free energy increases with magnetic field monotonically.

4 Discussion

In this work, we have considered a probe D2-D8 brane model and solving non-linear partial 
differential equations following from the action along with the boundary conditions we 
have shown that the equations admit spatially modulated solution. Plotting the free 
energy density vs. the wave number \( \hat{k} = \frac{2\pi}{L} \) we find, within the range of parameters, 
it shows a minimum at certain \( \hat{k} = \hat{k}_0 \), indicating spatially modulated solutions are
thermodynamically preferred for the chemical potential $\mu > \mu_c$. From the behaviour of charge density and $c_\psi$, it turns out that this spatially modulated solution involves both spin density wave and charge density wave. The behavior of the free energy with the variation of chemical potential indicates a second order phase transition occurring at $\mu = \mu_c$. We have also studied the free energy with variation of the magnetic field.

Similar spatially modulated solutions have been obtained for Einstein–Maxwell-dilaton or axion systems in the bottom up approach [14,20,22], which are dual to charge/current density wave and for D3-D7 model in the top-down approach [29], which is a mixture of spin density wave and charge density wave. In that respect, this solution has more similarity to that obtained in D3-D7 model. However, unlike the D3-D7 model in the present case, we find amplitudes of both are comparable. While studying the behaviour of the free energy density with the variation of magnetic field we find that the free energy density shows a minimum at certain value of the magnetic field and thus indicating that stability increases in the presence of a weak magnetic field.
This work can be further extended in several directions. Study of direct conductivity and thermoelectric properties, their dependence on temperature and other parameters may enable one to connect it to the experimental observations and provides a better theoretical understanding.

Another direction is to study fermionic responses of this system. Some work has appeared for striped solution obtained in Einstein-Maxwell-dilaton model with two $U(1)$ gauge fields \cite{19} in bottom-up approach. Solving the Dirac equation in the background with the present solution one can study the Fermi surfaces, the conditions when Fermi surface will form or spectrum will have a gap and parameters determining its properties.

One can also investigate the role of symmetry breaking mechanisms in the present context. For spontaneous symmetry breaking it gives rise to Goldstone modes, which may lead to sliding of the stripes under a weak electric field. In addition, if we incorporate explicit symmetry breaking, the Goldstone modes are lifted and the stripes remain pinned. However, for large enough electric fields such solutions may lead to a non-linear behavior of conductivity \cite{31,32} which can be studied in the present context. Finally, in the present work, we have taken probe approximation and neglected the back reaction of the gravity. One can incorporate the back reaction by considering the solution of Einstein equation as well to obtain a spatially modulated solution in the present set up. We hope to report some of these in the near future.

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