Braided central elements

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Abstract

We present and study two families of polynomials with coefficients in the center of the universal enveloping algebra. These polynomials are analogues of a determinant and a characteristic polynomial of a certain non-commutative matrix, labeled by irreducible representations of $\mathfrak{gl}_n(\mathbb{C})$. The matrix is an image of the universal R-matrix of a Yangian of $\mathfrak{gl}_n(\mathbb{C})$ under certain representation. We compute the polynomials explicitly for $\mathfrak{gl}_2(\mathbb{C})$ and establish connections between the first family of polynomials and higher Capelli identities through some sort of plethysm.

1 Introduction

Generalizations of Capelli identities were studied in the series of works by M. Nazarov, A. Okounkov, A. Molev, G. Olshanskii, F. Knop, S. Sahi et al ([13], [11], [15], [17], [7], etc.). The identities are connected to shifted symmetric polynomials, satisfy some vanishing conditions and produce a linear basis of the center of the universal enveloping algebra of a Lie algebra. In [11], [13] these Capelli identities were studied through the theory of Yangians.

In this paper we introduce two families of central polynomials, which we believe to have similar links to the theory of representations of Yangian of $\mathfrak{gl}_n(\mathbb{C})$. Both series of polynomials are parameterized by dominant weights of $\mathfrak{gl}_n(\mathbb{C})$.

One of them can be interpreted as a family of determinants of some non-commutative matrix. We conjecture that these polynomials have coefficients in the center of the universal enveloping algebra and we prove this for representations of $\mathfrak{gl}_2(\mathbb{C})$. In Section 5 we connect ”determinants” to Capelli identities through some sort of plethysm.

The second family of central polynomials represents analogues of characteristic polynomials of certain non-commutative matrices. Their existence and centrality follows from the works of B. Kostant [8] and M. Gould [4]. We do not know yet the interpretation of these polynomials in terms of Yangians and relations to Capelli elements. In case of the vector representation, ”determinant” and ”characteristic polynomial” coincide – both can be obtained as an image of quantum determinant of the Yangian of $\mathfrak{gl}_n(\mathbb{C})$ under the evaluation map. In Section 4 we study the case of $\mathfrak{gl}_2(\mathbb{C})$, which illustrates that in general these polynomials are different.

Both series of central polynomials are produced by certain matrices with coefficients in the universal enveloping algebra. We call these matrices braided Casimir elements. These elements appear in different areas of representation theory. For example, in [8] braided Casimir elements were used to study tensor products of finite and infinite dimensional representations. In [16] it is proved that quantum family algebras, introduced by A. Kirillov in [9], are commutative if and only if they are generated by braided Casimir elements. In [4], [5] characteristic polynomials of braided Casimir elements were applied to calculate Wigner coefficients. In Section 5 we show that braided Casimir elements are images of the universal R-matrix of the Yangian of $\mathfrak{sl}_n(\mathbb{C})$ under certain representations.


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2 Definitions

Consider the universal enveloping algebra $U(\mathfrak{gl}_n(\mathbb{C}))$ of the general linear Lie algebra $\mathfrak{gl}_n(\mathbb{C})$. Let $Z(\mathfrak{gl}_n(\mathbb{C}))$ be the center of $U(\mathfrak{gl}_n(\mathbb{C}))$. Fix the basis $\{E_{ij}\}$ of $\mathfrak{gl}_n(\mathbb{C})$, which consists of standard unit matrices. We write the Casimir element of $U(\mathfrak{gl}_n(\mathbb{C}) \otimes U(\mathfrak{gl}_n(\mathbb{C}))$ as

$$\Omega = \sum_{i,j=1}^{n} E_{ij} \otimes E_{ji}.$$ 

The element $\Omega$ in the tensor square of $U(\mathfrak{gl}_n(\mathbb{C}))$ is closely related to the central element $t \in Z(\mathfrak{gl}_n(\mathbb{C}))$ defined by $t = \sum E_{ij} E_{ji}$ (here we mean multiplication in the universal enveloping algebra). Namely, let $\delta$ be the standard coproduct on $U(\mathfrak{gl}_n(\mathbb{C}))$, defined on the elements of $\mathfrak{gl}_n(\mathbb{C})$ as $\delta(x) = x \otimes 1 + 1 \otimes x$. Then

$$\Omega = \frac{1}{2} (\delta(t) - 1 \otimes t - t \otimes 1).$$

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_+$, for $i = 1, \ldots, n-1$, be a dominant weight of $\mathfrak{gl}_n(\mathbb{C})$. Denote by $\pi_\lambda$ be the corresponding irreducible rational $\mathfrak{gl}_n(\mathbb{C})$-representation and by $V_\lambda$ the space of this representation. We assume that $\dim V_\lambda = m + 1$. We construct from $\Omega$ a new object, an element of $U(\mathfrak{gl}_n(\mathbb{C})) \otimes \text{End}(V_\lambda)$, which we call braided Casimir element.

Definition. The braided Casimir element is defined by

$$\Omega_\lambda = E_{ij} \otimes \sum_{i,j=1}^{n} \pi_\lambda(E_{ji}).$$

We will think of $\Omega_\lambda$ as a matrix of size $(m + 1) \times (m + 1)$ with coefficients in $U(\mathfrak{gl}_n(\mathbb{C}))$. Hence, braided Casimir element comes from the central element $t$ of the universal enveloping algebra. It turns out that $\Omega_\lambda$ is a source of new central elements. Below we define two families of polynomials with coefficients in $Z(\mathfrak{gl}_n(\mathbb{C}))$, associated to $\Omega_\lambda$.

The first family of polynomials is provided by the following proposition.

Let $\mathfrak{g}$ be a simple Lie algebra with a linear basis $\{I_\alpha\}$. Let $\{I^\alpha\}$ be the dual basis with respect to Killing form. Consider an element $\omega$ of $U(\mathfrak{g}) \otimes U(\mathfrak{g})$, defined similarly by $\omega = \sum_\alpha I_\alpha \otimes I^\alpha$. For any irreducible representation $\pi$ of $\mathfrak{g}$ put $\omega_\pi = (\text{id} \otimes \pi)(\omega)$.

Proposition 2.1. ([3], [2]) There exists a polynomial

$$p_\pi(u) = \sum_{k=0}^{m} z_k u^k$$

with coefficients $z_k$ in the center of $U(\mathfrak{g})$ such that $p_\pi(\omega_\pi) = 0$.

The polynomial $p_\pi(u)$ can be chosen so that $\deg p_\pi(u) = \dim \pi$.

Corollary 2.2. For any dominant weight $\lambda$ of $\mathfrak{gl}_n(\mathbb{C})$ there exists a polynomial

$$P_\lambda(u) = \sum_{k=0}^{m} z_k u^k$$

with coefficients $z_k \in Z(\mathfrak{gl}_n(\mathbb{C}))$, such that $P_\lambda(\Omega_\lambda) = 0$. 

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Proof. Choose a basis in \( \mathfrak{sl}_n(\mathbb{C}) \) which consists of matrices \( E_{ij} \) for \( i \neq j \) and \( H_i = E_{ii} - E_{i+1,i+1} \), \( (i = 1, \ldots, n - 1) \). The dual basis consists of matrices

\[
E'_{ij} = E_{ji} \quad \text{and} \quad H_i^* = (E_{11} + \ldots + E_{ii}) - \frac{i}{n}(E_{11} + \ldots + E_{nn}).
\]

The element \( \omega \) for \( \mathfrak{sl}_n(\mathbb{C}) \) has the form

\[
\omega = \sum_{i \neq j} E_{ij} \otimes E_{ji} + \sum_i H_i \otimes H_i^* = \sum_{ij} E_{ij} \otimes E_{ji} - \frac{1}{n}(\Delta_1 \otimes \Delta_1) = \Omega - \frac{1}{n}(\Delta_1 \otimes \Delta_1),
\]

where \( \Delta_1 = (E_{11} + \cdots + E_{nn}) \) is a central element of \( U(\mathfrak{gl}_n(\mathbb{C})) \). Consider a dominant weight \( \lambda' \) of \( \mathfrak{sl}_n(\mathbb{C}) \), defined as \( \lambda' = (\lambda_1 - \lambda_m, \ldots, \lambda_{m-1} - \lambda_m, 0) \). Then

\[
\omega_{\pi_{\lambda'}} = \Omega_{\lambda} - \frac{d}{n} \Delta_1 \otimes \text{id}
\]

with \( d = \sum \lambda_i \), and the polynomial \( p_{\lambda}(u - \frac{d}{n} \Delta_1) \) from Proposition 2.1 annihilates \( \Omega_{\lambda} \).

We will call \( P_{\lambda}(u) \) characteristic polynomial of \( \Omega_{\lambda} \).

Next we define the second family of polynomials \( D_{\lambda}(u) \), which we call shifted determinants.

Let \( A \) be an element of \( \mathcal{A} \otimes \text{End}(\mathbb{C}^{m+1}) \), where \( \mathcal{A} \) is a non-commutative algebra, let \( V = \mathbb{C}^{m+1} \) be an \((m+1)\)-dimensional vector space. We again think of \( A \) as a non-commutative matrix of size \((m+1) \times (m+1)\) with coefficients \( A_{ij} \in \mathcal{A} \).

Definition. The (column)-determinant of \( A \) is the following element of \( \mathcal{A} \):

\[
\det(A) = \sum_{\sigma \in S_{m+1}} (-1)^{\sigma} A_{\sigma(1)1} A_{\sigma(2)2} \cdots A_{\sigma(m+1)(m+1)}.
\]

(2.1)

Here the sum is taken over all elements \( \sigma \) of the symmetric group \( S_{m+1} \) and \((-1)^{\sigma}\) is the sign of the permutation \( \sigma \).

Put \( \Omega_{\lambda}(u) = \Omega_{\lambda} + u \otimes \text{id} \). Define \( L \) as a diagonal matrix of the size \((m+1) \times (m+1)\) of the form:

\[
L = \text{diag}(m, m-1, \ldots, 0).
\]

Definition. The shifted determinant of \( \Omega_{\lambda}(u) \) is the column-determinant \( \det(\Omega_{\lambda}(u) - L) \). We will use notation \( D_{\lambda}(u) \) for this polynomial with coefficients in \( U(\mathfrak{gl}_n(\mathbb{C})) \):

\[
D_{\lambda}(u) = \det(\Omega_{\lambda}(u) - L).
\]

Conjecture 2.3. For any dominant weight \( \lambda \) there exists a basis of the vector space \( V_{\lambda} \) such that the polynomial \( D_{\lambda}(u) \) has coefficients in the center \( Z(\mathfrak{gl}_n(\mathbb{C})) \).

This is known to be true in the case of vector representation and it is proved below in Section 4 for representations of \( \mathfrak{gl}_n(\mathbb{C}) \).

There is another way to define the same determinant. Let \( A_1, \ldots, A_s \) be a set of matrices of size \((m+1) \times (m+1)\) with coefficients in some associative (non-commutative) algebra \( \mathcal{A} \). Let \( \mu \) be the multiplication in \( \mathcal{A} \). Consider an element of \( \mathcal{A} \otimes \text{End}(V) \)

\[
\Lambda^*(A_1 \otimes \cdots \otimes A_s) = (\mu^{(s)} \otimes \text{Asym}_s)(A_1 \otimes \cdots \otimes A_s),
\]

where \( \text{Asym}_s = \frac{1}{s!} \sum_{\sigma \in S_{s}} (-1)^{\sigma} \sigma \). By Young’s construction, the antisymmetrizer can be realized as an element of \( \text{End}(V^{\otimes s}) \).
Lemma 2.4. (cf [12].) For $s = m + 1 = \dim V$

$$A^{m+1}(A_1 \otimes \cdots \otimes A_{m+1}) = \alpha(A_1, \ldots, A_{m+1}) \otimes \text{Asym}_{m+1}, \tag{2.2}$$

where $\alpha(A_1, \ldots, A_{m+1}) \in A$.

$$\alpha(A_1, \ldots, A_{m+1}) = \sum_{\sigma \in S_{m+1}} (-1)^{\sigma} [A_1]_{\sigma(1), 1} \cdots [A_{m+1}]_{\sigma(m+1), m+1},$$

$[A_k]_{i,j}$ - matrix elements of $A_k$.

**Proof.** Let $\{e_i\}, (i = 1, \ldots, m + 1)$, be a basis of $V$. Observe that $\text{Asym}_{m+1}$ is a one-dimensional projector to

$$v = \frac{1}{(m+1)!} \sum_{\sigma \in S_{m+1}} (-1)^{\sigma} e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(m+1)}.$$

We apply $\Lambda^m(A_1 \otimes \cdots \otimes A_{m+1})$ to $e_1 \otimes \cdots \otimes e_{m+1} \in V^\otimes m$:

$$\Lambda^m(A_1 \otimes \cdots \otimes A_{m+1})(e_1 \otimes \cdots \otimes e_m)$$

$$= \sum_{i_1, \ldots, i_k} (A_1)_{i_1, 1} \cdots (A_{m+1})_{i_{m+1}, m+1} \text{Asym}_{m+1} (e_{i_1} \otimes \cdots \otimes e_{i_{m+1}}). \tag{2.3}$$

The vector $\text{Asym}_{m+1} (e_{i_1} \otimes \cdots \otimes e_{i_{m+1}}) \neq 0$ only if all indices $\{i_1, \ldots, i_{m+1}\}$ are pairwise distinct. In this case denote by $\sigma$ be a permutation defined by $\sigma(k) = i_k$. Then

$$\text{Asym}_{m+1} (e_{i_1} \otimes \cdots \otimes e_{i_{m+1}}) = (-1)^{\sigma} v,$$

and (2.3) gives

$$\Lambda^m(A_1 \otimes \cdots \otimes A_{m+1}) (e_1 \otimes \cdots \otimes e_{m+1}) = \alpha(A_1, \ldots, A_{m+1}) v$$

$$= \alpha(A_1, \ldots, A_{m+1}) \text{Asym}_{m+1} (e_1 \otimes \cdots \otimes e_{m+1}).$$

□

**Remark.** Note that

$$\alpha(A, \ldots, A) = \det(A),$$

$$\alpha(\Omega_\lambda(u - m), \ldots, \Omega_\lambda(u)) = D_\lambda(u).$$

3 Yangian of $\mathfrak{gl}_n(\mathbb{C})$ and Casimir element

In this section we would like to recall the case of vector representation and its connection to Yangians. This example serves as an inspiration for the rest of the project. Let us recall some definitions (2, 12).

**Definition.** The Yangian $\text{Y}(n)$ for $\mathfrak{gl}_n(\mathbb{C})$ is a unital associative algebra over $\mathbb{C}$ with countably many generators $\{t_{ij}^{(r)}\}, r = 1, 2, \ldots, 1 \leq i, j \leq n$ and the defining relations

$$[t_{ij}^{(r+1)}, t_{kl}^{(s)}] - [t_{ij}^{(r)}, t_{kl}^{(s+1)}] = t_{kj}^{(r)}t_{il}^{(s)} - t_{kj}^{(s)}t_{il}^{(r)},$$

where $r, s = 0,1, 2, \ldots$ and $t_{ij}^{(0)} = \delta_{ij}$. 

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The same set of defining relations can be combined into one equation, sometimes called RTT relation. Namely, denote by \( T(u) = (t_{ij}(u))_{i,j=1}^n \) the matrix with coefficients \( t_{ij}(u) \) which is a formal power series of generators of \( Y(n) \):

\[
t_{ij}(u) = \delta_{ij} + \sum_{k=1}^{\infty} \frac{t_{ij}^{(k)}}{u^k}.
\]

For \( P = \sum E_{ij} \otimes E_{ji} \), the permutation matrix of \( \mathbb{C}^n \otimes \mathbb{C}^n \), define the Yang matrix

\[
R(u) = 1 - \frac{P}{u}.
\]

\( R(u) \) is a rational function with values in \( \text{End} \, \mathbb{C}^n \otimes \text{End} \, \mathbb{C}^n \).

We introduce some standard notations. For any vector space \( V \) and any element \( S \) of \( \text{End} \, V \), we define an element \( S_k \) of \( \text{End} (\mathbb{C}^n \otimes \text{End} (\mathbb{C}^n) \otimes m) \) by

\[
S_k = 1 \otimes (k-1) \otimes S \otimes 1 \otimes (m-k).
\]

In particular, we write

\[
T_k(u) = \sum_{ij} t_{ij}(u) \otimes (E_{ij})_k \in Y(n) \otimes \text{End}(\mathbb{C}^n)^\otimes m.
\]

Let \( S \) be an element of \( \text{End} V \otimes \text{End} V \). Using the abbreviated notation \( S = S(1) \otimes S(2) \), we define an element \( S_{ij} \) of \( \text{End}(\mathbb{C}^n)^\otimes m \) by

\[
S_{ij} = 1 \otimes (i-1) \otimes S(1) \otimes 1 \otimes (j-i-1) \otimes S(2) \otimes 1 \otimes (m-j-i).
\]

**Definition.** The Yangian \( Y(n) \) of \( \mathfrak{gl}_n(\mathbb{C}) \) is an associative unital algebra over \( \mathbb{C} \) with the set of generators \( \{ t_{ij}^{(k)} \} \) which satisfy the equation

\[
R(u - v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u - v).
\]

(This is an equation in \( Y(n) \otimes \text{End}(\mathbb{C}^n)^\otimes 2) \).

Yangian \( Y(n) \) is an example of infinite-dimensional quantum group. It has a remarkable group-like central element, which is called *quantum determinant* of \( Y(n) \).

**Definition.** Quantum determinant \( \text{qdet} \, T(u) \) is a formal series with coefficients in \( Y(n) \) defined by

\[
\text{qdet} \, T(u) = \sum_{\sigma \in S_n} (-1)^\sigma t_{1\sigma(1)}(u - n + 1) \ldots t_{n\sigma(n)}(u).
\]

Let \( V \) be a \( \mathfrak{gl}_n(\mathbb{C}) \)-module. The action \( \pi \) of \( \mathfrak{gl}_n(\mathbb{C}) \) on \( V \) can be extended to the action of the Yangian \( Y(n) \). Usually it is done by the means of evaluation map which is a homomorphism of algebras \( \text{ev} : Y(n) \to U(\mathfrak{gl}_n(\mathbb{C})) \). By definition,

\[
\text{ev} \cdot t_{ij}(u) = \delta_{ij} + \frac{E_{ij}}{u},
\]

It is clear that

\[
\text{ev} \cdot T(u) = 1 + \frac{\Omega^+}{u},
\]
where $\top$ is for the matrix transposition. Along with (ev) we will consider another homomorphism $\mathbf{ev}: Y(n) \rightarrow U(\mathfrak{gl}_n(\mathbb{C}))$, which is defined by

$$\mathbf{ev} \cdot t_{ij}(u) = \delta_i^j - \frac{E_{ji}}{u}, \quad \mathbf{ev} \cdot T(u) = 1 - \frac{\Omega}{u}.$$ 

In Section 5 we discuss these two maps in more details.

One can see that

$$\mathbf{ev}(q \det T(u)) = \frac{D_{\lambda_0}(u)}{u(u - 1) \ldots (u - n + 1)};$$

where $D_{\lambda_0}(u)$ is the "shifted" determinant for the vector representation $\lambda_0 = (1, 0, \ldots, 0)$:

$$D_{\lambda_0}(u) = \det \begin{pmatrix} E_{11} + u - n + 1 & E_{21} & \cdots & E_{n,1} \\ E_{12} & E_{22} + u - n + 2 & \cdots & E_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ E_{1n} & E_{2n} & \cdots & E_{n,n + u} \end{pmatrix}.$$ 

**Proposition 3.1.** (\cite{[14], [10].}) $D_{\lambda_0}(u)$ is a polynomial with coefficients in the center of $U(\mathfrak{gl}(n))$. Moreover, for vector representation $D_{\lambda_0}(-u) = P_{\lambda_0}(u)$, the shifted determinant provides the characteristic polynomial of the corresponding braided Casimir element $\Omega_{\lambda_0}$.

As we will see in Section 4, these two polynomials do not coincide in general.

4 **Case of $\mathfrak{gl}_2(\mathbb{C})$**

Here we prove the centrality of polynomials $D_\lambda(u)$ and compare $D_\lambda(u)$ and $P_\lambda(u)$ for irreducible representations of $\mathfrak{gl}_2(\mathbb{C})$. The center $Z(\mathfrak{gl}_2(\mathbb{C}))$ of the universal enveloping algebra $U(\mathfrak{gl}_2(\mathbb{C}))$ is generated by two elements:

$$\Delta_1 = E_{11} + E_{22}, \quad \Delta_2 = (E_{11} - 1)E_{22} - E_{12}E_{21}.$$ 

Let $\lambda = (\lambda_1 \geq \lambda_2)$ be a dominant weight. Put $m = \lambda_1 - \lambda_2$, $d = \lambda_1 + \lambda_2$. Then $\dim V_\lambda = m + 1$ and $\Omega_\lambda$ is a "tridiagonal" matrix: all entries $[\Omega_\lambda]_{ij}$ of the matrix $\Omega_\lambda$ are zeros except

$$[\Omega_\lambda]_{k,k} = (\lambda_1 - k + 1)E_{11} + (\lambda_2 + k - 1)E_{22}, \quad k = 1, \ldots, m + 1,$$

$$[\Omega_\lambda]_{k,k+1} = (m + 1 - k)E_{21}, \quad k = 1, \ldots, m,$$

$$[\Omega_\lambda]_{k+1,k} = kE_{12}, \quad k = 1, \ldots, m.$$ 

**Proposition 4.1.** a) Polynomial $D_\lambda(u)$ is central.

b) Let $\mu = (\mu_1 \geq \mu_2)$ be another dominant weight of $\mathfrak{gl}_n(\mathbb{C})$. The image of $D_\lambda(u)$ under Harish - Chandra isomorphism $\chi$ is the following function of $\mu$:

$$\chi(D_\lambda(u)) = \prod_{k=0}^{m} (u + (\lambda_1 - k)\mu_1 + (\lambda_2 + k)\mu_2 - k). \quad (4.1)$$

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Lemma 4.2. The determinants $I^{(k)}$ satisfy the recursion relation
\[ I^{(k+1)} = a_k I^{(k)} - c_k b_k I^{(k-1)} \quad (k = 2, ..., n - 1) \] (4.2)
with initial conditions $I^{(1)} = a_0$, $I^{(2)} = a_1 a_0 - c_1 b_1$.

Lemma 4.3. If $X(a, b, c)$ is a tri-diagonal matrix with coefficients $\{a_i, b_j, c_k\}$ and $X(a', b', c')$ is another tri-diagonal matrix with coefficients $\{a'_i, b'_j, c'_k\}$ with the property
\[ a_i = a'_i, \quad c_j b_j = c'_j b'_j \]
for all $i = 0, ..., n$, $j = 1, ..., n$, then $\det X(a, b, c) = \det X(a', b', c')$.

Proof. Follows from the recursion relation. \qed

We apply these observations to compute the determinant of the matrix $X(a, b, c) = \Omega_{\lambda}(u) - L$ with parameters
\[
\begin{align*}
    a_k &= (\lambda_1 + k - m)E_{11} + (\lambda_2 + m - k)E_{22} + u - k, \quad k = 0, \ldots, m, \\
    c_k &= (m - k + 1)E_{12}, \quad b_k = kE_{21}, \quad k = 1, \ldots, m.
\end{align*}
\]

The following obvious lemma allows to reduce the determinant of non-commutative matrix $(\Omega_{\lambda}(u) - L)$ to a determinant of a matrix with commutative coefficients.

Lemma 4.4. The subalgebra of $U(gl_2(\mathbb{C}))$ generated by $\{E_{11}, E_{22}, (E_{12} E_{21})\}$ is commutative.

Put $h = E_{11} - E_{22}$, $a = E_{12} E_{21}$. Due to Lemma 4.3 the tri-diagonal matrix $X(a', b', c')$ with coefficients
\[
\begin{align*}
    a'_k &= a_k = \lambda_1 E_{11} + \lambda_2 E_{22} + u - m + (k - m)(h - 1), \quad k = 0, \ldots, m, \\
    b'_k &= k a, \quad c'_k = (m - k + 1), \quad k = 1, \ldots, m,
\end{align*}
\]
has the same determinant as $(\Omega_{\lambda}(u) - L)$. By Lemma 4.4 $X(a', b', c')$ has commutative coefficients. Hence $\det (\Omega_{\lambda}(u) - L)$ equals $\det (\lambda_1 E_{11} + \lambda_2 E_{22} + u - m + A_m)$ where $A_m$ is the following matrix:
\[
A_m = \begin{pmatrix}
0 & m a & 0 & \ldots & 0 & 0 \\
1 & -(h - 1) & (m - 1)a & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & (1 - m)(h - 1) & a \\
0 & 0 & 0 & \ldots & m & -m(h - 1)
\end{pmatrix}
\]
with $h$ and $a$ as above.
Lemma 4.5.

\[
\det A_m = \prod_{k=0}^{m} \left( \frac{-m(h-1)}{2} + \frac{(m-2k)}{2} \sqrt{(h-1)^2 + 4a} \right).
\]

Proof. By Lemma 4.3 \( \det A_m = (h-1)^{m+1} \det A'_m \) with

\[
A'_m = \begin{pmatrix}
0 & ms & 0 & \ldots & 0 & 0 \\
1(s-1) & -1 & (m-1)s & \ldots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ldots & 1-m & s \\
0 & 0 & 0 & \ldots & m(s-1) & -m
\end{pmatrix},
\]

and \( s \) is such that \( s(s-1) = a/(h-1)^2 \). This reduces to

\[
s = \frac{1 \pm \sqrt{1 + 4a/(h-1)^2}}{2}. \tag{4.3}
\]

The determinant of \( A'_m \) is a variant of Sylvester determinant ([1], [6], [18]). It equals

\[
\det A'_m = \prod_{k=0}^{m} ((m-2k)s - m + k).
\]

With \( s \) as in (4.3) we have:

\[
(h-1)((m-2k)s - m + k) = \frac{-m(h-1)}{2} + \frac{(m-2k)}{2} \sqrt{(h-1)^2 + 4a}
\]

and lemma follows. Note that both values of \( s \) give the same value of \( \det A_m \).

We obtain from calculations above

\[
\det (\Omega_{\lambda}(u) - L) = \prod_{k=0}^{m} \left( u + \frac{d}{2}(E_{11} + E_{22}) - \frac{m}{2} + \frac{(m-2k)}{2} \left( (E_{11} - E_{22} - 1)^2 + 4E_{12}E_{21} \right)^{\frac{1}{2}} \right). \tag{4.4}
\]

Observe that \( ((E_{11} - E_{22} - 1)^2 + 4E_{12}E_{21})^{\frac{1}{2}} = ((\Delta_1 - 1)^2 - 4\Delta_2)^{\frac{1}{2}} \), and finally we get

\[
D_{\lambda}(u) = \prod_{k=0}^{m} \left( u + \frac{d\Delta_1}{2} - \frac{m}{2} + \frac{(m-2k)}{2} \left( (\Delta_1 - 1)^2 - 4\Delta_2 \right)^{\frac{1}{2}} \right). \tag{4.5}
\]

The quantity in (4.5) has coefficients in \( W_{\tau}\)-extension of \( Z(\mathfrak{gl}_2(\mathbb{C})) \), where \( W_{\tau} \) is the translated Weyl group. But it is easy to see that after expanding the product, we get a polynomial in \( u \) with coefficients in \( Z(\mathfrak{gl}_n(\mathbb{C})) \). We proved the first part of the proposition.

b) The images of the generators of \( Z(\mathfrak{gl}_n(\mathbb{C})) \) under Harish-Chandra homomorphism \( \chi \) are

\[
\chi(\Delta_1) = \mu_1 + \mu_2, \quad \chi(\Delta_2) = \mu_1(\mu_2 - 1).
\]

This together with (4.5) implies (4.1).

Proposition 4.6. The image of the polynomial \( P_{\lambda}(u) \) under Harish-Chandra homomorphism is

\[
\chi(P_{\lambda}(u)) = \prod_{k=0}^{m} \left( -u + (\lambda_1 - k)\mu_1 + (\lambda_2 + k)\mu_2 - k(m + 1 - k) \right). \tag{4.6}
\]
This follows from the formula for characteristic polynomial for \( \omega_n \), the braided Casimir element of \( \mathfrak{sl}_2(\mathbb{C}) \). It is proved in [16].

Despite the fact that in general \( D_\lambda(u) \) and \( P_\lambda(u) \) are different, we believe that still they are closely related. We hope to find the source of the link between these two polynomials in representation theory of Yangian \( Y(n) \). Some steps towards this goal are presented in the next section.

5 Connection with Yangian \( Y(\mathfrak{gl}_N(\mathbb{C})) \)

As we saw in the Section 3 the Casimir element \( \Omega \) has a natural interpretation as an image of the matrix of generators of the Yangian under the map \( \text{ev} \). Here we give some similar interpretation of the braided Casimir element. Though, in this interpretation the element \( \Omega_\lambda \) arises not as an element of \( U(\mathfrak{gl}_n(\mathbb{C})) \otimes \text{End } V_\lambda \), but as an element of \( \text{End } \mathbb{C}^n \otimes \text{End } V_\lambda \).

Let \( \mathcal{R}(u) \) be the universal R-matrix of the Yangian of \( \mathfrak{sl}_n(\mathbb{C}) \) (see [2]). It is a formal power series in \( u^{-1} \) with coefficients in \( Y(n) \otimes Y(n) \).

Let \( \pi \) be a \( \mathfrak{gl}_n(\mathbb{C}) \)-representation in some space \( V \). Using the maps \( (\text{ev}) \) and \( (\hat{\text{ev}}) \) we can construct two actions \( \pi \) and \( \hat{\pi} \) of \( Y(n) \) on \( V \):

\[
\pi \cdot y = (\pi \circ \text{ev}) \cdot y \quad \text{and} \quad \hat{\pi} \cdot y = (\pi \circ \hat{\text{ev}}) \cdot y
\]

We need both actions to make the correspondence between the images of the universal R-matrix, braided Casimir element, and Capelli polynomials.

**Remark.** It is well-known that the universal R-matrix satisfies the quantum Yang-Baxter equation with spectral parameter:

\[
\mathcal{R}^{12}(u-v)\mathcal{R}^{13}(u-w)\mathcal{R}^{23}(v-w) = \mathcal{R}^{23}(v-w)\mathcal{R}^{13}(u-w)\mathcal{R}^{12}(u-v).
\]

(5.1)

Let \( \hat{\pi} \) be some representation of \( Y(n) \), extended in some way from the representation \( \pi \) of \( U(\mathfrak{gl}_n(\mathbb{C})) \). After the application of \( \hat{\pi} \otimes \hat{\pi} \otimes \text{id} \) to both sides of (5.1) we get exactly the RTT-relation for the matrix \( T(u) = (\hat{\pi} \otimes \text{id})\mathcal{R}(u) \) of elements in \( Y(n) \) with the defining R-matrix \( (\hat{\pi} \otimes \hat{\pi})\mathcal{R}(u) \). We would like to stay with the tradition and we would like to get in the case of \( \hat{\pi} \), extended from the vector representation of \( \mathfrak{gl}_n(\mathbb{C}) \), the Young R-matrix \( 1 + \frac{P}{u} \) and the defining relation of \( Y(n) \). Observe that for any \( \hat{\pi} \)

\[
(\text{id} \otimes \hat{\pi})T(u) = (\hat{\pi} \otimes \hat{\pi})\mathcal{R}(u).
\]

But for the vector representation \( \pi_0 \) of \( \mathfrak{gl}_n(\mathbb{C}) \) and the matrix \( T(u) \) of generators of \( Y(n) \) we have

\[
(\text{id} \otimes \pi_0)T(u) = 1 + \frac{P^\top}{u}, \quad (\text{id} \otimes \hat{\pi}_0)T(u) = 1 - \frac{P}{u} = R(u).
\]

Hence, to be consistent in definitions, we have to extend the representations from \( U(\mathfrak{gl}_n(\mathbb{C})) \) to \( Y(n) \) by the map \( (\hat{\text{ev}}) \):

\[
\hat{\pi} = \hat{\pi}, \quad R(u) = f_0(u)(\pi_0 \otimes \pi_0)\mathcal{R}(u), \quad T(u) = (\pi_0 \otimes \text{id})\mathcal{R}(u),
\]

(5.2)

for some complex-valued rational function \( f_0(u) \).
Proposition 5.1. Let $\lambda_0 = (1, 0, \ldots, 0)$ be the highest weight of the vector representation $\pi_0$ of $\mathfrak{gl}_n(\mathbb{C})$, let $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0)$ be a partition (a dominant weight of $\mathfrak{sl}_n(\mathbb{C})$). Let $\tilde{\pi}_0, \tilde{\pi}_\lambda$ be the representations of $Y(n)$, extended from $\mathfrak{gl}_n(\mathbb{C})$ by the map $\tilde{e} \tilde{v}$. Then

$$(\tilde{\pi}_0 \otimes \tilde{\pi}_\lambda) R(u) = f(u) \left( 1 - \frac{\Omega_\lambda}{u} \right),$$

where $f(u)$ is some complex-valued rational function of $u$ (which depends on $\lambda$).

Proof. In general, this follows from the Corollary 3.7 in [13]. We repeat the steps which would lead to the proposition to avoid confusion in notations. Let $V_0 = \mathbb{C}^n$ be the space of the vector representation $\pi_0$ with the highest weight $\lambda_0$. By the means of the map $\tilde{e} \tilde{v}$ it becomes $Y(n)$-module. Moreover, there exists a family of $Y(n)$-modules $\{V_0(a)\}_{a \in \mathbb{C}}$. The action $\tilde{\pi}_0(a)$ of $Y(n)$ on $V_0(a)$ is defined as follows:

$$\tilde{\pi}_0(a) \cdot T(u) = \tilde{\pi}_0 \cdot T(u - a),$$

where $T(u)$ is the matrix of generators of $Y(n)$.

For two dominant weights $\nu$ and $\lambda$ of $\mathfrak{sl}_n(\mathbb{C})$ put $R_{\nu, \lambda}(u) = (\tilde{\pi}_\nu \otimes \tilde{\pi}_\lambda) R(u)$. By the discussion above, the image of the universal $R$-matrix under the representation $\tilde{\pi}_0 \otimes \tilde{\pi}_0$ is proportional to the Yang $R$-matrix. It is also well-known that the universal $R$-matrix satisfies

$$(id \otimes \Delta)(R(u)) = R_{12}(u) R_{13}(u),$$

(5.3)

where $\Delta$ is a coproduct in $Y(n)$. Let $\lambda$ be a partition of $M$ and let $\{c_1, \ldots, c_M\}$ be the set of contents of the standard Young tableau of the shape $\lambda$. Using (5.3) we can expand the action of the universal $R$-matrix to the $Y(n)$-module $W = V_0(0) \otimes V_0(c_1) \otimes \cdots \otimes V_0(c_M)$, $i = 1, \ldots, M$. This action is

$$\tilde{\pi}_0 \otimes \tilde{\pi}_0(c_1) \otimes \ldots \otimes \tilde{\pi}_0(c_M) R(u) = f(u) \prod_{k=1}^M \left( 1 - \frac{P_{1, k+1}}{u - c_k} \right),$$

(5.4)

with $f(u)$ – some rational complex-valued function of $u$. Observe that $W = (V_0)^{\otimes (M+1)}$ as $\mathfrak{gl}_n(\mathbb{C})$-module. Using the Young symmetrizer $F_\lambda$ we construct the element $(id \otimes F_\lambda)$ of $\text{End}(W)$, which is a projector from $W$ to $V_0 \otimes V_\lambda$. We apply this projector to (5.4) to obtain $R_{\lambda_0, \lambda}(u)$. By Proposition 2.12 in [13], the following equality holds:

$$\prod_{k=1}^M \left( 1 - \frac{P_{1, k+1}}{u - c_k} \right) (id \otimes F_\lambda) = \left( 1 - \sum_{k=1}^M \frac{P_{1, k+1}}{u} \right) (id \otimes F_\lambda).$$

(5.5)

But with the standard coproduct $\delta$ in $U(\mathfrak{gl}_n(\mathbb{C}))$ we obtain:

$$\Omega_\lambda = \sum_{ij} E_{ij} \otimes \pi_\lambda(E_{ji}) = \left( \sum_{ij} E_{ij} \otimes \delta^{(M)}(E_{ji}) \right) (id \otimes F_\lambda) = \left( \sum_{l=1}^M P_{1, l+1} \right) (id \otimes F_\lambda),$$

(5.6)

and we get

$$R_{\lambda_0, \lambda}(u) = f(u) \left( (id \otimes F_\lambda) - \frac{\Omega_\lambda}{u} \right).$$

The operator $(id \otimes F_\lambda) - \frac{\Omega_\lambda}{u}$ acts as $1 - \frac{\Omega_\lambda}{u}$ on the subspace $V_0 \otimes V_\lambda \subset W$. 

□
6 Capelli elements.

Next we would like to show how Capelli elements fit into this picture and to connect the polynomial $D_\lambda(u)$ to these elements by some sort of plethysm. Let $S = \sum_{ij} E_{ij} \otimes E_{ij}$. Following [15], define an element $S_\lambda(u)$ of $U(\mathfrak{gl}_n(\mathbb{C}) \otimes \text{End} V_\lambda$ by

$$S_\lambda(u) = ((S_{12} - u - c_1) \ldots (S_{1M+1} - u - c_M)) (id \otimes F_\lambda).$$

Then

$$c_\lambda(u) = tr (S_\lambda(u))$$

is the Capelli polynomial, associated to $\lambda$. It has coefficients in $Z(\mathfrak{gl}_n(\mathbb{C}))$. The theory of Capelli elements is developed in full in the papers, mentioned in the Introduction. Here we would like to observe the following facts:

1) $S_\lambda(u)$ is proportional to $(\pi_0 \otimes \tilde{\pi}) R(u)$ (the first representation is extended by evaluation map (ev), and the second one by (ev)).

2) Let $\phi: \mathfrak{gl}_n(\mathbb{C}) \to \mathfrak{gl}_n(\mathbb{C})$ be an automorphism, defined by $\phi(X) = -X^\top$. Put $\pi_\lambda^\ast = \pi_\lambda \circ \phi$. Then

$$\Omega_\lambda^\ast(u) = u \prod_{k=1}^M \left( \frac{-1}{u + c_k} \right) S_\lambda(u). \quad (6.1)$$

Indeed, from (5.6) and (5.5) we get

$$\Omega_\lambda(u) = u \prod_{k=1}^M \left( 1 + \frac{P_{1,k+1}}{u + c_k} \right) (id \otimes F_\lambda). \quad (6.2)$$

Observe that $(\phi \otimes \pi_\lambda) \Omega = id \otimes (\pi_\lambda \circ \phi) \Omega$, so

$$\Omega_\lambda^\ast(u) = u \prod_{k=1}^M \left( 1 - \frac{S_{1,k+1}}{u + c_k} \right) (id \otimes F_\lambda). \quad (6.3)$$

and (6.1) follows.

The representation $X \to -(\pi_\lambda^\ast(X))^\top$, $X \in \mathfrak{gl}_n(\mathbb{C})$ is isomorphic to $\pi_\lambda$ (it has the same set of weights). Thus we can write in some basis

$$\Omega_\lambda^\top(u) = -\Omega_\lambda^\ast(-u) \quad (6.4)$$

Combining (6.4) and (6.2) we prove that the shifted determinant $D_\lambda(u)$ is the trace of the composition of two Young symmetrizers, applied to several copies of shifted matrix $S$. In other words, $D_\lambda(u)$ is a result of some sort of plethysm of (6.1).

Proposition 6.1. Let $\lambda \vdash M$, $\text{dim} V_\lambda = (m + 1)$. Then

$$D_\lambda(u) = tr \left( \prod_{s=0}^m (u - s) \prod_{k=1}^M \left( \frac{S_{1,sM+k+1}(u + s - m - c_k)}{u - s - c_k} \right) (id \otimes F_\lambda^{(m+1)} \cdot \text{Asym}_{m+1}) \right) \quad (6.5)$$

Proof. Recall from Section 2 that

$$\alpha(\Omega_\lambda(u - m), \ldots, \Omega_\lambda(u)) = D_\lambda(u).$$

Hence,

\[ D_\lambda = \text{tr} \left( \Lambda^{m+1}(\Omega_\lambda(u-m), \ldots, \Omega_\lambda(u)) \right) \]
\[ = \text{tr} \left( (\Omega_\lambda^{-1}(u-m))_{12} \ldots (\Omega_\lambda^{-1}(u))_{1_{m+2}} \text{Asym}_{m+1} \right) \]
\[ = (-1)^{m+1} \text{tr} \left( (\Omega_\lambda^{-1}(-u+m))_{12} \ldots (\Omega_\lambda^{-1}(-u))_{1_{m+2}} \text{Asym}_{m+1} \right) \]
\[ = \text{tr} \left( \prod_{s=0}^{m} (u-s) \prod_{k=1}^{M} \frac{1}{(u-s-c_k)} \left( S_1, S_{M+k+1}(u+s-m-c_k) \right) \right) \left( \text{id} \otimes \mathcal{F}_\lambda^{\otimes(m+1)} \cdot \text{Asym}_{m+1} \right) \]

We hope that the plethysm relation (6.5) and the connection of traces of \( \Omega_\lambda(u) \) to Capelli elements will allow to prove Conjecture 2.3 in general.

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