Nodal auxiliary a posteriori error estimates

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Received: date / Accepted: date

Abstract We introduce and explain key relations between a posteriori error estimates and subspace correction methods viewed as preconditioners for problems in infinite dimensional Hilbert spaces. We set the stage using the Finite Element Exterior Calculus and Nodal Auxiliary Space Preconditioning. This framework provides a systematic way to derive explicit residual estimators and estimators based on local problems which are upper and lower bounds of the true error. We show the applications to discretizations of δd, curl-curl, grad-div, Hodge Laplacian problems, and linear elasticity with weak symmetry. We also provide a new regular decomposition for singularly perturbed H(d) norms and parameter-independent error estimators. The only ingredients needed are: well-posedness of the problem and the existence of regular decomposition on continuous level.

Mathematics Subject Classification (2010) MSC 65N30 · MSC 65N12

1 Introduction

Adaptive mesh refinement is an indispensable component in the design of competitive numerical models, as it can resolve singularities and sharp gradients in the solution and often can provide optimal discretizations with respect to
computational complexity and accuracy. Adaptivity is a broad and sophisticated research area that involves many technical ingredients. In this paper we focus on a posteriori error estimates, the building block of adaptive mesh refinement and error control.

Our objective is on deriving a posteriori error estimators by relating the error control in adaptive methods to operator preconditioning techniques \[29,35,61,67,68\]. This leads to simple and transparent analysis and to new a posteriori error estimation techniques. We choose to work within the Finite Element Exterior Calculus (FEEC) \[8,9,36\], because it is a general framework which allows for clear and concise presentation of the results.

For stationary problems, adaptive finite element methods (AFEMs) is a quite mature field, and we refer to the classical texts in adaptivity \[3,12,11,55,59,66\] for basic techniques and references. A posteriori error analysis of $H(\text{curl})$ and $H(\text{div})$, and in general $H(d)$, problems, where $d$ is an exterior derivative, has been a topic of intensive research in the last two decades. Related to our work is \[16\], where a posteriori error estimate for an eddy current curl-curl problem is proposed. The main tool there is a Helmholtz decomposition and the resulting estimator hinges on material parameters and convexity of the physical domain. An improved analysis of this estimator is presented in \[60\], which shows parameter-independent reliability of that estimator on Lipschitz domains with the help of local regular decomposition and regularized interpolation commuting with the gradient, curl and divergence. These works were further generalized using the FEEC language on de Rham complexes \[28\]. In this work, new estimators controlling the error of Hodge Laplacian in the mixed variational norm are derived based on global regular decomposition and commuting interpolation. Later several error estimators for decoupled errors of the discrete Hodge Laplacian could be found in \[26,44\].

To motivate the study of the relations between a posteriori error estimates and preconditioning, we make the following simple observation: both methods aim at developing “computable” approximations to the action of solution operators on some residuals living in an infinite- or finite-dimensional space (see \[45\] for an example of such techniques in the symmetric and positive definite case). It is, therefore, natural to find connections between these numerical techniques and explore such interaction to construct estimators. As it turns out in this way two-sided and robust a posteriori estimators can be derived for a wide class of partial differential equations posed on de Rham complexes.

To show that our a posteriori error estimators are reliable and efficient, we require two main ingredients: (1) solvability of the continuous problem (continuous inf-sup condition); and (2) existence of an $H^1$ regular decomposition \[17,18,37,50\] of vector fields (again, only on the continuous level). With these tools we are able to derive classical and new error estimators for any well-posed problems on de Rham complexes, including the positive definite curl-curl and grad-div problems, as well as indefinite Hodge Laplacian and linear elasticity. Moreover, we construct a new regular decomposition designed for singularly perturbed $H(d)$ norms. This decomposition suggests a new and robust two-sided error estimator for $H(d)$ problems w.r.t. material
parameters. As far as we know, such robust estimators even for $H({\text{curl}})$ and $H(d)$ problems could not be found in the literature.

The key ideas and tools borrowed from preconditioning theory include the Fictitious Space Lemma [54], and the methodology of Nodal Auxiliary Space/HX preconditioning [39]. Meanwhile, motivated by the preconditioning framework introduced in [39], we note that a posteriori error estimation is equivalent to preconditioning the Riesz representation $B_V$ of the underlying dual Hilbert space $V'$. With the help of $H^1$ regular decomposition and fictitious space lemma, $B_V$ is shown to be spectrally equivalent to an operator formed by Riesz representations of several $H^{-1}$ spaces. Therefore, estimating residuals in $V'$ reduces to a posteriori estimates of $H^{-1}$ residuals, which has been well established in the literature. The same approach works for $H(d)$ singularly perturbed problems by reducing it to a posteriori error estimates of the classical singularly perturbed $H^1$ reaction diffusion equation. Our framework directly generalizes to indefinite systems because the Riesz representation of underlying infinite-dimensional space in that case is simply the Cartesian product of representations of a few well-studied spaces.

Compared with $H({\text{curl}})$, a posteriori error analysis of $H({\text{div}})$ was initiated earlier because of its application in mixed methods of standard elliptic problems. Some early works in that direction could be found in [20,22] where mesh dependent norms, quasi-uniform meshes, and Helmholtz decomposition are used. Error estimators for controlling the $L^2$ flux error are presented in [4,23,40,41]. It is noted that a fully general error estimator and a convergent adaptive algorithm are derived in [25] for the positive definite $H({\text{div}})$ problem. We remark that residual estimators is not the whole story of a posteriori error estimation. Readers are referred to [2,13,14,15,19,24,30,48,59,66,69] and references therein for hierarchical basis, equilibrated residual, superconvergent recovery, equilibrated by reconstruction, and functional error estimators.

The rest of the paper is organized as follows. In Section 2 we give some standard notation and preliminaries and link preconditioning with error estimation. Section 3 introduces the basic FEEC notation and the minimum needed to set up finite element discretizations in the language of differential forms. Section 4 is devoted to the simplest case of error estimation for $H^{-1}$ residuals, the building block of our framework. Next, in Section 5 we give the construction of estimators for the abstract $H(d)$ problems, including singularly perturbed, curl-curl, grad-div problems. We extend such results to product of Hilbert spaces in Section 6 to present a posteriori estimation for mixed Hodge Laplacian and the linear elasticity with weakly imposed symmetry.

2 Abstract framework

2.1 Preliminaries

For an arbitrary Hilbert space $V$, let $(\cdot,\cdot)_V$ denote its inner product, $\Vert \cdot \Vert_V$ the $V$-norm, $V'$ the dual space of $V$, $\text{id}_V$ the identity mapping on $V$, and
\( \mathcal{B}_V : \mathcal{V}' \to \mathcal{V} \) be the Riesz representation of \( \mathcal{V}' \), namely,

\[
(v, \mathcal{B}_V r)_V = \langle r, v \rangle, \quad \forall r \in \mathcal{V}', \; \forall v \in \mathcal{V}.
\]

Here \( \langle \cdot, \cdot \rangle \) is the duality pairing between \( \mathcal{V}' \) and \( \mathcal{V} \). For a linear operator \( g : \mathcal{V}_1 \to \mathcal{V}_2 \), its adjoint is denoted as \( g^* : \mathcal{V}_2' \to \mathcal{V}_1' \). We use \( [\mathcal{V}]^i \) (resp. \( [\mathcal{V}]^{i \times j} \)) to denote the Hilbert space of vectors (resp. matrices with \( i \) rows and \( j \) columns) whose entries are contained in \( \mathcal{V} \).

Given \( f \in \mathcal{V}' \), consider the variational problem: Find \( u \in \mathcal{V} \) such that

\[
a(u, v) = \langle f, v \rangle, \quad \forall v \in \mathcal{V},
\]

(2.1)

We assume that the bilinear form \( a(\cdot, \cdot) \) is symmetric, bounded

\[
\sup_{0 \neq v \in \mathcal{V}} \sup_{0 \neq w \in \mathcal{V}} \frac{a(v, w)}{\|v\|_V \|w\|_V} := \overline{\alpha} < \infty,
\]

(2.2)

and satisfies the inf-sup condition, that is, the following holds

\[
\inf_{0 \neq v \in \mathcal{V}} \sup_{0 \neq w \in \mathcal{V}} \frac{a(v, w)}{\|v\|_V \|w\|_V} := \alpha > 0.
\]

(2.3)

The bilinear form \( a \) corresponds to a bounded isomorphism \( \mathcal{A} : \mathcal{V} \to \mathcal{V}' \) by

\[
\langle \mathcal{A} v, w \rangle := a(v, w), \quad \forall v, w \in \mathcal{V}.
\]

Hence (2.1) is equivalent to the operator equation

\[
\mathcal{A} u = f.
\]

(2.4)

Obviously (2.2) and (2.3) are equivalent to that the operator norms satisfy

\[
\|\mathcal{A}\|_{\mathcal{V} \to \mathcal{V}'} = \overline{\alpha}, \quad \|\mathcal{A}^{-1}\|_{\mathcal{V}' \to \mathcal{V}} = \frac{1}{\alpha}.
\]

(2.5)

Let us consider the approximation to (2.1) by restricting it to a subspace \( \mathcal{V}_h \subset \mathcal{V} \), namely: Find \( u_h \in \mathcal{V}_h \) such that

\[
a(u_h, v) = \langle f, v \rangle, \quad \forall v \in \mathcal{V}_h.
\]

(2.6)

We assume that the discrete analogue of (2.3) is satisfied on \( \mathcal{V}_h \), i.e.,

\[
\inf_{0 \neq v \in \mathcal{V}_h} \sup_{0 \neq w \in \mathcal{V}_h} \frac{a(v, w)}{\|v\|_V \|w\|_V} := \alpha_h > 0.
\]

(2.7)

It then follows from the Babuška–Brezzi theory that (2.6) admits a unique solution. In practice, \( \mathcal{V}_h \) could be a finite element space based on a triangulation of a space domain.
2.2 A posteriori error estimates by preconditioning

We are interested in a posteriori error estimates of the form

\[ C_1 \eta_h \leq \| u - u_h \|_V \leq C_2 \eta_h, \]

where \( \eta_h \) is computable and \( C_1, C_2 \) are absolute constants. In finite elements, \( \eta_h \) can be used for adaptive local mesh refinement. Let

\[ e := u - u_h \in V, \]
\[ \mathcal{R} := f - A u_h \in V'. \]

The variational formulation (2.6) implies

\[ a(e, v) = \langle \mathcal{R}, v \rangle, \quad \forall v \in V. \]

or \( A e = \mathcal{R} \) in the operator form.

A posteriori error estimation then is to bound the norm of the true error \( e = A^{-1} \mathcal{R} \) by computable error indicators. Direct computation of \( \| A^{-1} \mathcal{R} \|_V \) will be, in general, impossible or too expensive, since it requires to compute the action of \( A^{-1} \) on \( \mathcal{R} \). Approximating such action is known as preconditioning, see, e.g., [35,64,67,68] for the abstract frameworks on preconditioning. We now borrow some ideas from this field and extend our results for elliptic problems [45] on \( H^1 \) to a posteriori error estimation of the possibly indefinite system (2.6) with a more general space \( V \).

First, we need a bounded isomorphism (a preconditioner) \( B : V' \to V \). \( B \) is assumed to be symmetric and positive definite (SPD), i.e., \( \langle \cdot, B \cdot \rangle \) is an inner product on \( V' \). The \( B \) induced norm on \( V' \) is denoted by \( \| \cdot \|_B \). It is straightforward to show that \( \langle \cdot, B^{-1} \cdot \rangle \) is an inner product on \( V \) and the induced norm is denoted as \( \| \cdot \|_{B^{-1}} \). We say \( B \) is a preconditioner for \( A \) provided \( B A : V \to V \) is a bounded isomorphism, i.e.,

\[ \| B A \|_{B^{-1}} < \infty, \quad \|(B A)^{-1}\|_{B^{-1}} < \infty. \]

2.3 Estimating the residual

The next lemma shows that the existence of a preconditioner \( B \) naturally yields a two-sided estimate on \( \| e \|_{B^{-1}} \).

**Lemma 2.1** We have the following two sided bound

\[ \| B A \|_{B^{-1}} \| \mathcal{R} \|_B \leq \| e \|_{B^{-1}} \leq \|(B A)^{-1}\|_{B^{-1}} \| \mathcal{R} \|_B. \]

**Proof** Using the relation \( e = A^{-1} \mathcal{R} \), we have

\[ \| e \|_{B^{-1}} = \| A^{-1} B^{-1} B \mathcal{R} \|_{B^{-1}} \leq \|(B A)^{-1}\|_{B^{-1}} \| B \mathcal{R} \|_{B^{-1}}. \]

On the other hand,

\[ \| \mathcal{R} \|_B = \| B A e \|_{B^{-1}} \leq \| B A \|_{B^{-1}} \| e \|_{B^{-1}}. \]

The proof is complete. \( \square \)
Hence we obtain an error estimator provided the norm of the residual $\|R\|_B$ can be efficiently approximated. It turns out that the fictitious space lemma (see [54]) is a natural way to estimate the efficiency of a preconditioner and will be a powerful tool in the current framework. We state this result next.

**Lemma 2.2 (Fictitious Space Lemma)** Let $\bar{V}$ be a Hilbert space, $\bar{B} : \bar{V}' \to \bar{V}$ be a SPD operator. Assume $\Pi : \bar{V} \to V$ is a surjective linear operator, and

- For any $\bar{v} \in \bar{V}$, it holds that $\|\Pi \bar{v}\|_{B^{-1}} \leq c_0 \|\bar{v}\|_{B^{-1}}$,
- For each $v \in V$, there exists $\bar{v} \in \bar{V}$ such that $\Pi \bar{v} = v$, $\|\bar{v}\|_{B^{-1}} \leq c_1 \|v\|_{B^{-1}}$.

Then for $\tilde{B} := \Pi B \Pi^*$ we have

$c_0^{-2} \langle r, \tilde{B} r \rangle \leq \langle r, B r \rangle \leq c_1^2 \langle r, \tilde{B} r \rangle, \quad \forall r \in \mathcal{V}.$

In our framework, the space $\bar{V}$ is formed by simple auxiliary spaces such that $\bar{B}$ could be efficiently inverted.

Clearly, the Riesz representation $B_V$ is a SPD operator satisfying $\langle B_V^{-1} \cdot, \cdot \rangle = (\cdot, \cdot)_V$ and $\|\cdot\|_{B_V^{-1}} = \|\cdot\|_V$. Due to the boundedness (2.2) and the inf-sup condition (2.3), $B_V$ is a preconditioner for $A$, i.e.,

$$\|B_V A\|_V = \alpha, \quad \| (B_V A)^{-1} \|_V = \alpha^{-1},$$

respectively. This fact is discovered by Mardal and Winther [49] and leads to a unified framework of preconditioning for linear systems arising from PDE discretization. For our purpose, Lemma 2.1 with $B = B_V$ given above yields

$$\alpha^{-1} \|R\|_{B_V} \leq \|e\|_V \leq \alpha^{-1} \|R\|_{B_V}. \quad (2.7)$$

### 3 Finite Element Exterior Calculus

In what follows, we shall focus on the $H(d)$ space of differential forms which includes the classical $H(\text{curl})$ and $H(\text{div})$ spaces as special examples. For differential forms and exterior calculus, we adopt the notation in [8,9].

#### 3.1 Continuous de Rham complex

Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain, $\Gamma \subseteq \partial \Omega$ be a closed set. We assume the pair $(\Omega, \Gamma)$ is admissible in the sense of [32,16]. Given a subset $\Omega_0 \subseteq \Omega$, let $\langle \cdot, \cdot \rangle_{\Omega_0}$ denote the $L^2(\Omega_0)$ inner product on $\Omega_0$ and $\|\cdot\|_{\Omega_0}$ the $L^2(\Omega_0)$ norm. For convenience, $\langle \cdot, \cdot \rangle_\Omega$ and $\|\cdot\|_\Omega$ simplify as $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively.

Let $dx^i$ be the $i$-th cotangent vector in $\mathbb{R}^n$ and $\wedge$ denote the wedge product of tensors. We start with $A^k(\Omega)$, the space of smooth differential $k$-forms on $\Omega$, i.e.,

$$v \in A^k(\Omega) \iff v = \sum_{\alpha} v_\alpha dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}, \quad v_\alpha \in C^\infty(\Omega).$$
where the summation is taken over the set of increasing multi-indices \( \alpha = (\alpha_1, \ldots, \alpha_k) \) with \( 1 \leq \alpha_1 < \cdots < \alpha_k \leq n \). In what follows, we say \( \{v_{\alpha}\} \) are coefficients of \( v \in A^k(\Omega) \). The \( L^2 \) norm and \( H^1 \) semi-norm on \( \Omega \) are naturally defined coefficient-wise as
\[
\|v\|_2 = \sum_\alpha \|v_\alpha\|^2, \quad |v|_{H^1}^2 = \|\nabla v\|^2 = \sum_\alpha \|\nabla v_\alpha\|^2.
\]

Other Sobolev norms could be defined in an analogous fashion.

Given an \( (n-1) \)-dimensional set \( \gamma \subset \partial \Omega \), we use \( \text{tr} = \text{tr}_\gamma \) to denote the trace of differential forms on \( \gamma \), which is the pullback of the inclusion \( \gamma \hookrightarrow \partial \Omega \).

For each index \( k \), there exists exterior derivative \( d_k : A^k(\Omega) \to A^{k+1}(\Omega) \).

Let \( L^2 A^k(\Omega) \) the space of \( k \)-forms with \( L^2 \) coefficients. The derivative \( d_k \) could be extended as a densely defined, unbounded operator \( d_k : L^2 A^k(\Omega) \to L^2 A^{k+1}(\Omega) \) and we have \( d_k \circ d_{k+1} + d_{k+1} \circ d_k = 0 \).

The \( H(d) \) spaces of \( k \)-forms are
\[
H A^k(\Omega) = \{ v \in L^2 A^k(\Omega) : d_k v \in L^2 A^{k+1}(\Omega) \},
\]
\[
\mathcal{V}^k = \{ v \in H A^k(\Omega) : \text{tr} v = 0 \text{ on } \Gamma \}.
\]

It is noted that \( d_0 \) could be identified with the usual gradient and \( \mathcal{V}^0 \) is
\[
\mathcal{H}_\Gamma := \{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma \}.
\]

In addition, \( \mathcal{V}^1 \) and \( \mathcal{V}^2 \) in \( \mathbb{R}^3 \) could be identified with \( H(\text{curl}) \) and \( H(\text{div}) \) spaces, respectively.

The de Rham complex is as follows
\[
\mathcal{V}^0 \xrightarrow{d_0} \mathcal{V}^1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-2}} \mathcal{V}^{n-1} \xrightarrow{d_{n-1}} \mathcal{V}^n.
\]

Let \( \mathcal{F}^k = N(d_k) \) denote the kernel of \( d_k \), \( \mathcal{B}^k = R(d_{k-1}) \) the range of \( d_{k-1} \), \( \parallel \) the operation of taking \( L^2 \) orthogonal complement, and \( \mathcal{F}^k \cap \mathcal{B}^{k-1} \) be the space of harmonic \( k \)-forms. Since \( d_k \) has a closed range (see, e.g., \([9,32]\)), we have the Hodge decomposition
\[
\mathcal{V}^k = d_{k-1} \mathcal{V}^{k-1} \oplus \mathcal{F}^k \oplus \mathcal{H}^k \cap \mathcal{B}^{k-1},
\]
where \( \oplus \) is the \( L^2 \) direct sum, and Poicaré inequality
\[
\|v\| \leq c_P \|d_k v\|, \quad v \in \mathcal{H}^k \cap \mathcal{B}^{k-1},
\]
where \( c_P \) depends on \( \Omega, \Gamma \).

Let \( H^1 A^k(\Omega) \) denote the space of \( k \)-forms with \( H^1 \) coefficients. Assuming \( \Gamma = \emptyset \) or \( \Gamma = \partial \Omega \) and \( \Omega \) is convex or has smooth boundary, it is well known that \( \mathcal{F}^k \) and \( \mathcal{H}^k \cap \mathcal{B}^{k-1} \) are continuously embedded in \( H^1 A^k(\Omega) \) (see \([5,8,31]\)), i.e.,
\[
|q|_{H^1} \leq C_q \|q\|, \quad \forall q \in \mathcal{F}^k,
\]
\[
|z|_{H^1} \leq C_z (\|z\| + \|d_k z\|), \quad \forall z \in \mathcal{H}^k \cap \mathcal{B}^{k-1},
\]
where \( C_q, C_z \) depend on \( \Omega, \Gamma \).
Let $\star : L^2 A^k(\Omega) \to L^2 A^{n-k}(\Omega)$ denote the Hodge star. The $L^2$ adjoint of $d_{k-1}$ is the exterior coderivative $\delta_k : L^2 A^k(\Omega) \to L^2 A^{k-1}(\Omega)$ satisfying $\star \delta_k = (-1)^k d_{n-k}\star$. For any subdomain $\Omega_0 \subseteq \Omega$ and $v \in H^1 A^k(\Omega_0)$, $w \in H^1 A^{k+1}(\Omega_0)$, we shall frequently use the Stokes formula

$$(d_k v, w)_{\Omega_0} = (v, \delta_{k+1} w)_{\Omega_0} + \int_{\partial \Omega_0} \text{tr} v \wedge \text{tr} w. \quad (3.5)$$

### 3.2 Discrete complex and proxy vector fields

Let domain $\Omega$ be partitioned into a conforming and simplicial triangulation $T_h$ aligned with $\Gamma$. We use $S_h$ to denote the set of interior faces as well as boundary faces in $\partial \Omega \setminus \Gamma$. The mesh size functions $h$ and $h_s$ are defined as

$$h|_T = h_T := \text{diam} T, \quad \forall T \in T_h,$$
$$h_s|_S = h_S := \text{diam} S, \quad \forall S \in S_h.$$

Let $\nu$ denote a piecewise constant vector on $S_h$ such that $\nu|_S = \nu_S$ is a unit normal to $S \in S_h$. For a boundary face $S \subseteq \partial \Omega$, $\nu_S$ is chosen to be outward pointing. Let $(\cdot, \cdot)_S$ and $\| \cdot \|_S$ denote the $L^2$ inner product and norm on the skeleton $S_h$, respectively. The partition $T_h$ is assumed to be shape regular in the sense that

$$\max_{T \in T_h} \frac{h_T}{h} = C_{\text{shape}} < \infty,$$

where $p_T$ and $p_s$ denote the radii of the circumscribed and inscribed spheres of an element $T$, respectively.

We consider the Arnold–Falk–Winther element space $\mathcal{S}_{k}[9]$,

$$\mathcal{P}_r A^k(T_h) = \{ v \in HA^k(\Omega) : v|_T \in \mathcal{P}_r A^k(T), \forall T \in T_h \},$$
$$\mathcal{P}_{r+1} A^k(T_h) = \mathcal{P}_r A^k(T_h) + \epsilon \mathcal{P}_r A^k(T_h),$$

where $\mathcal{P}_r A^k(T)$ is the space of $k$-forms on $T$ with polynomial coefficients of degree no greater than $r$, and $\epsilon$ is the interior product w.r.t. the coordinate position vector $x = (x_1, \ldots, x_n)$. Let $\mathcal{V}_h \subseteq \mathcal{V}^k$ be

$$\mathcal{V}_h^k = \begin{cases} \mathcal{P}_r A^k(T_h, \Gamma) := \mathcal{P}_r A^k(T_h) \cap \mathcal{V}^k, \\ \mathcal{P}_{r+1} A^k(T_h, \Gamma) := \mathcal{P}_{r+1} A^k(T_h) \cap \mathcal{V}^k. \end{cases} \quad (3.6)$$

We have a discrete de Rham complex

$$\mathcal{V}_h^0 \xrightarrow{d_n} \mathcal{V}_h^1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-2}} \mathcal{V}_h^{n-1} \xrightarrow{d_{n-1}} \mathcal{V}_h^n$$

provided $d_k \mathcal{V}_h^k \subseteq \mathcal{V}_h^{k+1}$ holds for each $k$. The lowest order discrete complex is based on $\mathcal{V}_h^1 = \mathcal{V}_h^{k,0} := \mathcal{P}_1 A^k(T_h, \Gamma)$, the space of $k$-forms with incomplete piecewise linear coefficients.
In \( \mathbb{R}^3 \), the \( H(\text{curl}) \) and \( H(\text{div}) \) spaces are

\[
\mathcal{V}^c = \{ v \in [L^2(\Omega)]^3 : \nabla \times v \in [L^2(\Omega)]^3, \ v \times v = 0 \text{ on } \Gamma \}, \\
\mathcal{V}^d = \{ v \in [L^2(\Omega)]^3 : \nabla \cdot v \in L^2(\Omega), \ v \cdot v = 0 \text{ on } \Gamma \}.
\]

Using proxy vector fields as in \cite{8}, we identify the de Rham complex \((3.1)\) \((n = 3)\) with a classical sequence

\[
\begin{aligned}
\mathcal{V}^0 & \rightarrow \mathcal{V}^1 \rightarrow \mathcal{V}^2 \rightarrow \mathcal{V}^3 \\
\mathcal{H}_h & \rightarrow \nabla \mathcal{V}^c \rightarrow \nabla \times \mathcal{V}^d \rightarrow \nabla \cdot \mathcal{V}^d \rightarrow L^2(\Omega)
\end{aligned}
\]

Let \( \mathcal{V}_h^c \subset \mathcal{V}^c \) be a Nédélec edge element space, and \( \mathcal{V}_h^d \subset \mathcal{V}^d \) be a Raviart–Thomas–Nédélec or Brezzi–Douglas–Marini finite element space, see, e.g., \cite{21, 50, 52, 53, 58}. We use to identify the de Rham complex \((3.1)\) \(\mathcal{V}^0 \rightarrow H^1(\Omega) \rightarrow \mathcal{V}^c \rightarrow \mathcal{V}^d \rightarrow L^2(\Omega)\) provided \(C_1 \sim \mathcal{V}_h^d \sim C_2\) with \(C_3\) being a generic constant dependent only on \(\Omega, \Gamma, \Omega_{\text{shape}}\), and given physical parameters \(\Omega, \Gamma, \Omega_{\text{shape}}\) we say.

The building block of our theory is \( \mathcal{H} \), a closed subspace of \( H^1(\Omega) \). In this section, we make the natural choice

\[
\mathcal{H} = H^k := \{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma \}.
\]

We note that \( H^k \) with \(0 \leq k \leq n - 1\) is equivalent to the Cartesian product of \( \binom{n}{k} \) copies of the scalar-valued space \(\mathcal{H}_f\). The \(\mathcal{H}\) inner product is

\[
(v, w)_\mathcal{H} = (v, w) + (\nabla v, \nabla w), \quad \forall v, w \in \mathcal{H}.
\]

Let \(\mathcal{A}_\mathcal{H} = B^{-1} : \mathcal{H} \rightarrow \mathcal{H}'\) be the operator associated with \((\cdot, \cdot)_\mathcal{H}\). We shall make use of the subspace \( H^k_\mathcal{H} \subset H^k \) of \(k\)-forms with continuous and piecewise linear coefficients.

Throughout the rest of this paper, we adopt the notation \( C_1 \lesssim C_2 \) (resp. \( C_1 \preceq C_2 \)) provided \( C_1 \leq C_2 \) with \( C_3 \) being a generic constant dependent only on \(\Omega, \Gamma, C_{\text{shape}}\), and given physical parameters \(\Omega, \Gamma, C_{\text{shape}}\). We say

\[
\begin{aligned}
C_1 \approx C_2 & \quad \text{provided } C_1 \lesssim C_2, C_2 \lesssim C_1; \quad \text{and } C_1 \simeq C_2 \quad \text{provided } C_1 \preceq C_2 C_2 \text{ and } C_2 \preceq C_2 C_1 \text{ with } C_3, C_4 \text{ being generic constants dependent only on } \Omega, \Gamma.
\end{aligned}
\]

This notation generalizes to SPD operators in an obvious way.
4.1 Residual estimator

The estimation of \( \|r\|_{B_h} = \langle r, B_H r \rangle^{\frac{1}{2}} \) is well understood and essential to a posteriori estimates of AFEMs for second order elliptic and Stokes equations, see, e.g., [27]. We consider \( r \in H' \) with the following \( L^2 \) representation

\[
\langle r, v \rangle = (R(r), v) + (J(r), tr v)_{S_h}, \quad \forall v \in H,
\]

where \( R(r) \in L^2(\Omega) \) and \( J(r) \in L^2(S_h) \) is piecewise \( L^2 \) on the \((n-1)\)-dimensional skeleton \( S_h \). The Riesz representor of \( r \) is \( e(r) \in H \) such that

\[
(e(r), v)_H = \langle r, v \rangle, \quad \forall v \in H.
\]

To estimate the dual norm \( \|r\|_{B_h} \), it is necessary to construct a regularized interpolation onto the lowest order finite element space with local approximation property. We present such an interpolation in the next lemma. The details of the construction and the proof are given in the appendix. For \( T \in T_h \) (resp. \( S \in S_h \)), Let \( \Omega_T^k \) (resp. \( \Omega_S^k \)) denote the union of elements (including \( T \)) sharing at least one \( k \)-dimensional simplex with \( T \) (resp. \( S \)) in \( T_h \).

**Lemma 4.1** For \( 0 \leq k \leq n-1 \), there exists \( \Pi_h^k : H^k \to P^{-1}_k(T_h, \Gamma) \) such that for all \( v \in H^k \) and \( T \in T_h \), \( S \in S_h \),

\[
\|\Pi_h^k v\|_T \approx \|v\|_{\Omega_T^k},
\]

\[
\|v - \Pi_h^k v\|_T \approx h_T |v|_{H^1(\Omega_T^k)},
\]

\[
\|\text{tr}(v - \Pi_h^k v)\|_S \approx h_S^\frac{n}{2} |v|_{H^1(\Omega_S^k)}.
\]

If the lowest order space \( V_h^{k,0} \) is not under consideration, we have that \( V_h^k \) contains the space of continuous and piecewise linear polynomials \( H^{k,1}_h \). Then it suffices to use the coefficient-wise Clément interpolation (see [27]), \( \Pi_h^k : L^2A^k(\Omega) \to H^{k,1}_h \), such that

\[
\|h^{-\frac{1}{2}}(v - \Pi_h^k v)\| + \|h^{-\frac{1}{2}}(v - \Pi_h^k v)\|_{S_h} \approx |v|_{H^1(\Omega)}
\]

for all \( v \in H^k \). In addition, \( \Pi_h^k \) coincides with \( \Pi_h^0 \) when \( k = 0 \).

Let \( Q_h \) (resp. \( Q_h^k \)) denote the \( L^2 \) projection onto the space of piecewise polynomials, not necessarily continuous, and of fixed degree on \( T_h \) (resp. \( S_h \)). The data oscillation is

\[
\text{osc}_h(r) := \|h(R(r) - Q_h R(r))\| + \|h^{\frac{1}{2}}(J(r) - Q_h^k J(r))\|_{S_h}.
\]

The next lemma is proved using standard arguments in a posteriori error estimation in the space \( H^1(\Omega) \). For clarity we sketch its proof. The assumption \( r \in (V_h^{k,0})^\prime \) is necessary because the residual \( r \) always acts on the lowest order space \( V_h^{k,0} \subset V^k \), which is not contained in \( H^{k,1}_h \) if \( k \geq 1 \).
Lemma 4.2 For $0 \leq k \leq n - 1$, let $r \in \mathcal{H}' \cap (V_{h}^{k,0})'$ be in the form (4.1) and assume $\langle r, v_{h} \rangle = 0$ for $v_{h} \in V_{h}^{k,0}$. Then we have

$$\|hR(r)\| + \|h^{\frac{k}{2}}J(r)\|_{S_{h}} - \text{osc}_{h}(r) \lesssim \|r\|_{B_{h}} \lesssim \|hR(r)\| + \|h^{\frac{k}{2}}J(r)\|_{S_{h}}.$$ 

Proof It follows from the Cauchy–Schwarz inequality and Lemma 4.1 that

$$\|r\|_{B_{h}} = \sup_{v \in \mathcal{H}} \langle r, v \rangle = \sup_{v \in \mathcal{H}} \langle r, v - \Pi_{h}^{k}v \rangle \leq (\|hR(r)\|^{2} + \|h^{\frac{k}{2}}J(r)\|_{S_{h}}^{2})^{\frac{1}{2}} \times (\|h^{-1}(v - \Pi_{h}^{k}v)\|^{2} + \|h^{-\frac{k}{2}}\text{tr}(v - \Pi_{h}^{k}v)\|_{S_{h}}^{2})^{\frac{1}{2}}.$$ 

Then using (4.3), we prove the upper bound. The lower bound follows from (4.2) with the bubble function technique explained in [65]. □

4.2 Local Dirichlet problems

Let $\{x_{i}\}_{i=1}^{N}$ denote the set of vertices in $\mathcal{T}_{h}$, and $\phi_{i}$ be the continuous piecewise linear function that takes the value 1 at $x_{i}$ and 0 at other vertices for each $x_{i}$. For $1 \leq i \leq N$, we define $\Omega_{i} := \text{supp} \phi_{i}$ and have

$$\sum_{i=1}^{N} \phi_{i}(x) = 1, \quad \|\nabla \phi_{i}\|_{L^{\infty}(\Omega)} \simeq (\text{diam} \Omega_{i})^{-1}. \quad (4.5)$$

Let

$$\mathcal{H}_{i}^{k} := \{ v \in \mathcal{H} : v = 0 \text{ on } \Omega \setminus \Omega_{i} \}$$

and $\mathcal{I}_{h} : \mathcal{H}_{i}^{k} \hookrightarrow \mathcal{H}, \mathcal{I}_{i} : \mathcal{H}^{k} \hookrightarrow \mathcal{H}$ denote inclusions. Let $A_{h} = \mathcal{I}_{h}^{*}A_{\mathcal{H}}\mathcal{I}_{h} : \mathcal{H}_{i}^{k} \to (\mathcal{H}_{i}^{k})'$ and $A_{i} = \mathcal{I}_{i}^{*}A_{\mathcal{H}}\mathcal{I}_{i} : \mathcal{H}^{k} \to (\mathcal{H}^{k})'$ with $1 \leq i \leq N$. We take the product space $\mathcal{H} = \mathcal{H}_{1}^{k} \times \mathcal{H}_{1} \times \cdots \times \mathcal{H}_{N}$ and define $\Pi = (\mathcal{I}_{h}, \mathcal{I}_{1}, \ldots, \mathcal{I}_{N}) : \mathcal{H} \to \mathcal{H}$ as

$$\Pi(v_{h}, v_{1}, \ldots, v_{N}) = v_{h} + v_{1} + \cdots + v_{N}, \quad v_{h} \in \mathcal{H}_{h}^{k,1}, \quad v_{i} \in \mathcal{H}_{i}^{k}.$$ 

Using the shape regularity of $\mathcal{T}_{h}$ and Cauchy–Schwarz inequality, we have

$$\|\Pi \hat{v}\|_{\mathcal{H}} \lesssim \|\hat{v}\|_{\mathcal{R}}, \quad \forall \hat{v} \in \mathcal{H}.$$ 

For each $v \in \mathcal{H}$, let $v_{h} = \Pi_{h}^{k}v, v_{i} = (v - v_{h})\phi_{i}$, and $\hat{v} = (v_{h}, v_{1}, \ldots, v_{N})$. It follows from (4.4) and (4.5) that

$$\Pi \hat{v} = v, \quad \|\hat{v}\|_{\mathcal{R}} \lesssim \|v\|_{\mathcal{H}}.$$
Therefore taking $\tilde{B} = B_{\mathcal{H}} = \text{diag}(A_{h}^{-1}, A_{1}^{-1}, \ldots, A_{N}^{-1}) : \mathcal{H} \to \mathcal{H}$ and $B = B_{\mathcal{H}}$ in Lemma 2.2 respectively, the two-level additive Schwarz preconditioner

\[ B_{H}^{a} := I_{h} A_{h}^{-1} I_{h}^{*} + \sum_{i=1}^{N} I_{i} A_{i}^{-1} I_{i}^{*}. \]

is shown to be spectrally equivalent to $B_{H}$, i.e.,

\[ \langle r, B_{H} r \rangle \simeq \langle r, B_{H}^{a} r \rangle = \langle I_{h}^{*} r, A_{h}^{-1} I_{h}^{*} r \rangle + N \sum_{i=1}^{N} \langle I_{i}^{*} r, A_{i}^{-1} I_{i}^{*} r \rangle, \quad \forall r \in \mathcal{H}. \]  

(4.6)

Hence we obtain the implicit error indicator based on local problems.

**Lemma 4.3** Let $r \in \mathcal{H}$ and $\langle r, v_{h} \rangle = 0 \forall v_{h} \in \mathcal{H}_{k}^{1}$. We have

\[ \langle r, B_{H} r \rangle \simeq \sum_{i=1}^{N} \| \eta_{i} \|_{H^{1}(\Omega)}^{2}, \]

where $\eta_{i} \in \mathcal{H}_{k}^{1}$ solves

\[ (\eta_{i}, \phi) + (\nabla \eta_{i}, \nabla \phi) = \langle r, \phi \rangle, \quad \forall \phi \in \mathcal{H}_{k}^{1}. \]

**Proof** Let $\eta_{i} = A_{i}^{-1} I_{i}^{*} r$ in (4.6), which solves the local problem in the lemma. Then noticing $I_{h}^{*} r = 0$ and using (4.6) and $\langle I_{i}^{*} r, A_{i}^{-1} I_{i}^{*} r \rangle = \| \eta_{i} \|_{H^{1}(\Omega)}^{2}$, we complete the proof. \qed

The error estimator in Lemma 4.3 was first proposed in [10] and explained using operator preconditioning in [45]. Here we derive it from a new point of view, utilizing Lemma 2.2 (the fictitious space lemma). To make the local problem in Lemma 4.3 solvable in practice, one could replace $\mathcal{H}_{k}^{1}$ with a (piece-wise) polynomial subspace $\tilde{\mathcal{H}}_{k}^{1}$. It can be shown that the estimator based on the discrete local subspace is still a two-sided bound of the residual provided $\tilde{\mathcal{H}}_{k}^{1}$ contains suitable bubble functions, see, e.g., [45,65].

### 4.3 Robust error estimators in $H(\text{grad})$

Let $\varepsilon$ and $\kappa$ be positive constants. The singularly perturbed $H(\text{grad})$ problem is: Find $u \in \mathcal{H}_{\mathcal{F}}$ such that

\[ \varepsilon (\nabla u, \nabla v) + \kappa (u, v) = (f, v), \quad \forall v \in \mathcal{H}_{\mathcal{F}}. \]  

(4.7)

A posteriori analysis of finite element methods for (4.7) is well established, see, e.g., [66]. Here we consider the weighted $H^{1}$ space of $k$-forms $\mathcal{H}_{w}^{k}$, which consists of the same elements in $\mathcal{H}^{k}$ but equipped with the inner product and norm

\[ (v, \phi)_{w}^{k} := \epsilon (\nabla v, \nabla \phi) + \kappa (v, \phi), \]

\[ \| v \|_{w}^{2} := \varepsilon \| \nabla v \|^{2} + \kappa \| v \|^{2}, \quad \forall v, \phi \in \mathcal{H}_{w}^{k}. \]
We assume $J(r)$ is a piecewise polynomial on $S_h$ because it is the case for singularly perturbed problem. The weighted data oscillation is

$$\text{osc}_k(r) := \|\bar{h}(R(r) - Q_h R(r))\|.$$

The following lemma is essential for a posteriori error analysis of singularly perturbed $H(d)$ problems. The weighted mesh size functions and the technique in the following proof were used in [66] for reaction diffusion equations.

**Lemma 4.4** Let $\bar{h} = \min\{\varepsilon^{-\frac{1}{2}} h, \kappa^{-\frac{1}{2}}\}$ and $\bar{h}_s = \min\{\varepsilon^{-\frac{1}{2}} h_s, \kappa^{-\frac{1}{2}}\}$. Let $r \in (H^k_h)^\prime \cap (V^{k,0}_h)^\prime$ be in the form (4.1) and assume $(r, v_h) = 0$ for $v_h \in V^{k,0}_h$. Then we have

$$\|\bar{h} R(r)\| + \|\varepsilon^{-\frac{1}{4}} \bar{h}^{\frac{1}{2}} J(r)\|_{S_h} - \text{osc}_k(r) \lesssim \|r\|_{S_{\kappa h}} \lesssim \|\bar{h} R(r)\| + \|\varepsilon^{-\frac{1}{4}} \bar{h}^{\frac{1}{2}} J(r)\|_{S_h}.$$

**Proof** Using (4.3a) and (4.3b), we have for $v \in H^k$,

$$\|v - \Pi^k_h v\|_T \lesssim h_T \|\nabla v\|_{\Omega_T^k} \leq \varepsilon^{-\frac{1}{2}} h_T \|v\|_{H^k(\Omega_T)},$$

$$\|v - \Pi^k_h v\|_T \lesssim \|v\|_{H^1(\Omega_T)} \leq \kappa^{-\frac{1}{2}} \|v\|_{H^k(\Omega_T^k)}.$$  
(4.8)

Therefore it holds that

$$\|v - \Pi^k_h v\|_T \lesssim \bar{h}_T \|v\|_{H^k(\Omega_T^k)}.$$  

Similarly, it follows from the trace inequality

$$\|\text{tr} \phi\|_S^2 \lesssim h_S^{-1} \|\phi\|_{T}^2 + \|\phi\|_T \|\nabla \phi\|_T, \quad \forall \phi \in H^1 A^k(T)$$  
(4.9)

with $S \subset \partial T$ and (4.8) that

$$\|\text{tr}(v - \Pi^k_h v)\|_S \lesssim \varepsilon^{-\frac{1}{4}} \bar{h}^{\frac{1}{2}}_S \|v\|_{H^k(\Omega_T^k)}.$$  
(4.10)

The rest of the proof follows from (4.8), (4.10) and the arguments used in the proof of Lemma 4.2. □

Robust equilibrated estimators for (4.7) have been established in e.g., [1,62].

**Remark 1** In addition to error indicators provided by Lemmata 4.2, 4.3 and 4.4, any a posteriori error estimate that estimates $H^{-1}$-norm of a residual can be used in the current framework. For instance, equilibrated residual estimators [2,14,51] are able to yield very tight bounds on the residual.
5 A posteriori estimates in $H(d)$

In many important applications, the space $V$ in (2.1) is more complicated than $H_{T}$ or $H^{k}$, the standard $H(\text{grad})$ space. Let $W^{-}, W$ be auxiliary Hilbert spaces that are connected with $V$ via bounded linear operators $D : W^{-} \to V$ and $I : W \to V$. We shall choose $W, W^{-}$ in appropriate ways such that the Riesz representation operators $B_{W}, B_{W^{-}}$ are amenable to a posteriori error analysis. The next corollary is a direct consequence of Lemma 2.2 and very useful for preconditioning $B_{V}$.

**Corollary 5.1** Assume for any $v \in V$, there exist $w^{-} \in W^{-}$ and $w \in W$, such that

$$v = Dw^{-} + Iw,$$

$$\|w^{-}\|_{W^{-}}^{2} + \|w\|_{W}^{2} \leq C_{\text{stab}}\|v\|_{V}^{2},$$

where $C_{\text{stab}}$ is a constant. Let

$$B_{a} = DB_{W^{-}}D^{\ast} + IB_{W}I^{\ast} : V' \to V.$$

Then we have

$$\left(\|D\|^{2} + \|I\|^{2}\right)\langle r, B_{a}r \rangle \leq \langle r, B_{a}r \rangle \leq C_{\text{stab}}\langle r, B_{a}r \rangle, \quad \forall r \in V'.$$

**Proof** In Lemma 2.2 we take $\tilde{V} = W^{-} \times W$, $\Pi : \tilde{V} \to V$ as $\Pi(w^{-}, w) = Dw^{-} + Iw$ and complete the proof. $\square$

The stable splitting in Corollary 5.1 is motivated by the regular and Helmholtz decomposition, see, e.g., [7, 28, 37]. Typically, when applying the result from Corollary 5.1 $D$ will be a differential operator and $I$ will be a natural inclusion.

5.1 Standard $H(d)$ problems

Let

$$\varepsilon, \kappa \in L^{\infty}(\Omega) \cap H^{1}(T_{h}), \quad f \in H^{1}A^{k}(T_{h}).$$

For $1 \leq k \leq n - 1$, we assume the following problem is well-posed

$$\delta_{k+1}(\varepsilon d_{k}u) + \kappa u = f, \quad \text{in } \Omega,$$

$$\text{tr}u = 0, \quad \text{on } T.$$

The variational problem is: Find $u \in V^{k}$ such that

$$(\varepsilon d_{k}u, d_{k}v) + (\kappa u, v) = (f, v), \quad \forall v \in V^{k}. \quad (5.1)$$

The discrete problem seeks $u_{h} \in V^{k}_{h}$ such that

$$(\varepsilon d_{k}u_{h}, d_{k}v) + (\kappa u_{h}, v) = (f, v), \quad \forall v \in V^{k}_{h}. \quad (5.2)$$
Theorem 5.1. Therefore, we obtain the spectral equivalence in Corollary 5.1, where the stable splitting assumption is readily confirmed by $W$ where $C$

For Theorem 5.1 (Regular Decomposition) The next theorem states the regular decomposition, see [18, 28, 34, 37, 56].

It then follows from (2.7) that

$$
\|u - u_h\|_{V_h}^2 \approx \langle R_h, B_{V} R_h \rangle
$$

(5.3)

The next theorem states the regular decomposition, see [18, 28, 34, 37, 56].

Theorem 5.1 (Regular Decomposition) For $1 \leq k \leq n - 1$ and any $v \in V^k$, there exist $\varphi \in H^{k-1}$, $z \in H^k$, such that

$$
v = d_{k-1} \varphi + z,
$$

$$
\|z\|_{H^1} \leq C_{\text{reg}} \|d_k v\|,
$$

$$
\|\varphi\|_{H^1} \leq C_{\text{reg}} \|v\|_{V^k},
$$

where $C_{\text{reg}}$ is a constant dependent on $\Omega$, $\Gamma$.

Theorem 5.1 suggests a preconditioner for $B_{V^k}$. Let $V = V^k$, $W = H^{k-1}$, $W = H^k$, $D = d_{k-1} : H^{k-1} \to V^k$, and $I = I_{H^k} : H^k \hookrightarrow V^k$ be the inclusion in Corollary 5.1 where the stable splitting assumption is readily confirmed by Theorem 5.1. Therefore, we obtain the spectral equivalence

$$
B_{V^k} \simeq d_{k-1} B_{H^{k-1}} d_{k-1}^* + I_{H^k} B_{H^k} I_{H^k}^*.
$$

(5.4)

Collecting (5.3) and (5.4), the nodal auxiliary a posteriori error estimate for (5.2) reads

$$
\|u - u_h\|_{V_h} \approx (d_{k-1}^* R_h, B_{H^{k-1}} d_{k-1}^* R_h) + (I_{H^k}^* R_h, B_{H^k} I_{H^k}^* R_h).
$$

(5.5)

In what follows, we use results in Section 4 to estimate the $H^{-1}$ residuals $d_{k-1}^* R_h$ and $I_{H^k}^* R_h$. For each $T \in T_h$ and $S \in S_h$, let

$$
R_1^k|_T = \delta_k (f - \kappa u_h)|_T,
$$

$$
J_1^k|_S = \left[ \text{tr} \ast (f - \kappa u_h) \right]|_S,
$$

$$
R_2^k|_T = \delta_{k+1} (\varepsilon d_k u_h - \kappa u_h)|_T,
$$

$$
J_2^k|_S = -\left[ \text{tr} \ast \varepsilon d_k u_h \right]|_S.
$$

(5.6)

Using $d_k \circ d_{k-1} = 0$ and the Stokes formula (3.5) element-wise, we obtain the $L^2$ representation of $d_{k-1}^* R_h \in (H^{k-1})'$

$$
\langle d_{k-1}^* R_h, v \rangle = \langle R_h, d_{k-1} v \rangle = \langle R_1^k, v \rangle + \langle J_1^k, \text{tr} v \rangle|_S, \quad \forall v \in H^{k-1}.
$$

(5.7)

Similarly, the $L^2$ representation of $I_{H^k}^* R_h \in (H^{k})'$ is

$$
\langle I_{H^k}^* R_h, v \rangle = \langle R_h, v \rangle = \langle R_2^k, v \rangle + \langle J_2^k, \text{tr} v \rangle|_S, \quad \forall v \in H^k.
$$

(5.8)

Collecting previous results, we obtain the first main result.
where  \( \eta \)  estimators in Theorems 5.2 and 5.3 deteriorate as  \( k \rightarrow 0 \), \( \epsilon \) being (5.1) with continuity and inf-sup constants of (5.1) are not uniformly bounded. We note if  \( \epsilon \) (elliptic equation (4.7). To derive robust a posteriori estimates, it is necessary

Theorem 5.2 For \( 1 \leq k \leq n-1 \), we have

\[
\| u - u_h \|_{V^k} \lesssim \| hR^k_1 \| + \| h^{\frac{1}{2}} J^k_1 \|_{S_h} + \| hR^k_2 \| + \| h^{\frac{1}{2}} J^k_2 \|_{S_h},
\]

and

\[
\| hR^k_1 \| + \| h^{\frac{1}{2}} J^k_1 \|_{S_h} + \| hR^k_2 \| + \| h^{\frac{1}{2}} J^k_2 \|_{S_h} \lesssim \| u - u_h \|_{V^k} + \text{osc}_h(d^k_{k-1} \mathcal{R}^k) + \text{osc}_h(T^*_h \mathcal{R}^k).
\]

Proof Using  \( d^k_{k-1} V^k_{h_1} \subset V^k_{h_1} \subset V^k_h \) and (5.2), we have \( \langle d^k_{k-1} \mathcal{R}^k, v_h \rangle = 0 \) \( \forall v_h \in V^k_{h-1} \). Meanwhile (5.2) implies \( T^*_h \mathcal{R}^k \) vanishes in \( V^k_{h_1} \). It then follows from Lemma 12 that

\[
\| hR^k_1 \| + \| h^{\frac{1}{2}} J^k_1 \|_{S_h} + \text{osc}_h(d^k_{k-1} \mathcal{R}^k) \lesssim \| d^k_{k-1} \mathcal{R}^k \|_{B_{k-1}} \lesssim \| hR^k_1 \| + \| h^{\frac{1}{2}} J^k_1 \|_{S_h},
\]

\[
\| hR^k_2 \| + \| h^{\frac{1}{2}} J^k_2 \|_{S_h} - \text{osc}_h(T^*_h \mathcal{R}^k) \lesssim \| T^*_h \mathcal{R}^k \|_{B_{h_k}} \lesssim \| hR^k_2 \| + \| h^{\frac{1}{2}} J^k_2 \|_{S_h}.
\]

Combining previous inequalities with (5.5) completes the proof. \( \square \)

Recall \( \mathcal{H}^k_i := \{ v \in \mathcal{H}^k : v = 0 \text{ on } \partial \Omega \} \) and note that \( \mathcal{H}^k_{h_k} \subset \mathcal{H}^k_1 \) if \( V^k_h \neq \mathcal{P}_0 \mathcal{A}^k(\mathcal{T}_h, \Gamma) \). As a consequence of (5.5) and Lemma 13 we obtain a new implicit error indicator for the Hodge Laplacian based on solving local Dirichlet problems.

Theorem 5.3 For \( 1 \leq k \leq n-1 \), Let \( V^k_h \neq \mathcal{P}_0 \mathcal{A}^k(\mathcal{T}_h, \Gamma) \). Then we have

\[
\| u - u_h \|_{V^k} \approx \sum_{i=1}^{N} \left( \| \eta_i \|_{\mathcal{H}^1(\Omega)} + \| \zeta_i \|_{\mathcal{H}^1(\Omega)} \right),
\]

where \( \eta_i \in \mathcal{H}^k_{i-1} \) solves

\[
(\eta_i, \phi) + (\nabla \eta_i, \nabla \phi) = (f - \kappa u_h, d_{k-1} \phi), \quad \forall \phi \in \mathcal{H}^k_{i-1},
\]

and \( \zeta_i \in \mathcal{H}^k_i \) solves

\[
(\zeta_i, \phi) + (\nabla \zeta_i, \nabla \phi) = (f, \phi) - (\epsilon d_h u_h, d_h \phi) - (\kappa u_h, \phi), \quad \forall \phi \in \mathcal{H}^k_i.
\]

5.2 Singularly perturbed \( H(d) \) problems

In this subsection, we focus on the case when \( \epsilon \) and \( \kappa \) are positive constants. If \( \epsilon \ll 1 \) or \( \kappa \gg 1 \), then (5.1) becomes singularly perturbed. In this case, the estimators in Theorems 5.2 and 5.3 deteriorate as \( \epsilon \rightarrow 0 \) or \( \kappa \rightarrow \infty \) because the continuity and inf-sup constants of (5.1) are not uniformly bounded. We note that (5.1) with \( k = 0 \) reduces to the classical singularly perturbed second order elliptic equation (4.7). To derive robust a posteriori estimates, it is necessary
to use the weighted space $V^k_w$, which consists of the same elements in $V^k$, but is equipped with the weighted inner product

$$(v, \phi)_{V^k_w} = \varepsilon(d_kv, d_k\phi) + \kappa(v, \phi), \quad \forall v, \phi \in V^k_w.$$ 

As a result, we obtain that the continuity and stability constants of (5.1) equal 1, i.e., we have

$$\|u - u_h\|^2_{V^k_w} = (R^k, B_{V^k_w} R^k).$$

(5.9)

Let $\kappa H^{k-1}$ consist of the same elements in $H^{k-1}$ and be equipped with the inner product $\kappa(\cdot, \cdot)_{H^{k-1}}$. Let $I_{H^k_w} : H^k_w \rightarrow V^k_w$ be the inclusion. Using Theorem 5.1 and following the same analysis for (5.4), it is straightforward to check that the uniform spectral equivalence holds if $\kappa \leq \varepsilon$:

$$B_{V^k_w} \simeq d_{k-1} B_{\kappa H^{k-1}} d_{k-1}^* + I_{H^k_w} B_{H^k_w} I_{H^k_w}^*.$$ 

(5.10)

However, the regular decomposition is not suitable for the singularly perturbed case ($\kappa \gg \varepsilon$). Alternatively, we present a new stable decomposition of $V^k$.

**Theorem 5.4 (Robust Regular Decomposition)** For $1 \leq k \leq n - 1$ and any $v \in V^k$, there exist $\varphi \in H^{k-1}$ and $z \in H^k$, such that

$$v = d_{k-1} \varphi + z,$$

$$\|\varphi\|_{H^1} + \|z\| \leq C_{rh} \|v\|,$$

$$|z|_{H^1} \leq C_{rh} \|v\|_{V^k},$$

where $C_{rh}$ is a constant depending on $\Omega$, $\Gamma$.

In contrast to the classical regular decomposition, the $L^2$ norm of component $z$ in Theorem 5.4 is controlled by $\|v\|$, which is crucial for studying the dominant reaction case, i.e. $\kappa \gg 1$.

**Remark 2** The HX preconditioner in [39] is shown to be spectrally equivalent to the discrete $H^k_w$ norm and robust w.r.t. $\kappa$ in $\mathbb{R}^3$. However, that analysis is based on the assumption that the space domain is 2-regular, due to required regularity of the components in the Helmholtz decomposition. In this regard, we point out that the new regular decomposition from Theorem 5.4 also helps to remove such regularity assumption in the HX preconditioner.

In [33], the HX preconditioner is generalized to the H(d) problem. Then Theorem 5.4 indeed helps to construct a robust preconditioner for the discrete $H^k_w$ norm in arbitrary dimensional space $\mathbb{R}^n$.

In Corollary 5.1 we take $V = V^k_w$, $W = \kappa H^{k-1}$, $W = H^k_w$, $D = d_{k-1}$, and $I = I_{H^k_w}$ to be the inclusion. When $\kappa \geq \varepsilon$, Theorem 5.4 implies that the assumption in Corollary 5.1 holds with $\|D\|$, $\|I\|$, $C_{stab}$ independent of $\kappa, \varepsilon$. Therefore the same uniform spectral equivalence follows if $\kappa \geq \varepsilon$

$$B_{V^k_w} \simeq d_{k-1} B_{\kappa H^{k-1}} d_{k-1}^* + I_{H^k_w} B_{H^k_w} I_{H^k_w}^*.$$ 

(5.11)
Proof. It follows from Lemma 4.4 that

\[ \| u - u_h \|_{V_h}^2 \simeq \kappa^{-1} \| d_{k-1}^r \|_{B_h}^2 + \| I_{k}^{*,h} \|_{B_h}^2, \]  

(5.12)

Note that the \( L^2 \) representation of \( I_{k}^{*,h} \) is the same as \( I_{k}^{*,h} \) in (5.6). We are now in a position to present the second main result.

**Theorem 5.5** For \( 1 \leq k \leq n - 1 \), it holds that

\[ \| u - u_h \|_{V_h} \leq \| \kappa^{-\frac{1}{2}} h R_k^1 \| + \| \kappa^{-\frac{1}{2}} h J_1^k \|_{S_h} + \| \kappa^{-\frac{1}{2}} h J_2^k \|_{S_h} + \| \kappa^{-\frac{1}{2}} \varepsilon \| \kappa^{\frac{1}{2}} \| B_h \|_{S_h}, \]

and

\[ \| \kappa^{-\frac{1}{2}} h R_k^1 \| + \| \kappa^{-\frac{1}{2}} h J_2^k \|_{S_h} + \| \kappa^{-\frac{1}{2}} \varepsilon \| \kappa^{\frac{1}{2}} \| B_h \|_{S_h} \]

\[ \leq \| I_{k}^{*,h} \|_{B_h} \| B_h \|_{S_h} \]

(5.13)

\[ \leq \| I_{k}^{*,h} \|_{B_h} \| B_h \|_{S_h} \]

(5.14)

\[ \leq \| I_{k}^{*,h} \|_{B_h} \| B_h \|_{S_h} \]

(5.15)

Proof. It follows from Lemma 1.4 that

\[ \| h R_k^1 \| + \| \varepsilon \| \kappa^{\frac{1}{2}} \| B_h \|_{S_h} \]

\[ \leq \| I_{k}^{*,h} \|_{B_h} \| B_h \|_{S_h} \]

(5.16)

We complete the proof using (5.12), the above estimate, and the bound for \( \| d_{k-1}^r \|_{B_h} \) in the proof of Theorem 5.2. \( \square \)

The H(d) singularly perturbed problem has not been investigated and the estimator in Theorem 5.5 is new. Even a posteriori estimates in the special case \( k = 1, 2 \) in \( \mathbb{R}^3 \) could not be found in the literature.

### 5.3 Examples

Using the identifications (3.7) and (3.8), problems (5.1) and (5.2) with \( k = 1, n = 3 \) translate into:

Find \( u \in V_h^c \) and \( u_h \in V_h^c \) such that

\[ (\varepsilon \nabla u, \nabla \cdot v) + (\kappa u, v) = (f, v), \quad \forall v \in V_h^c, \]  

(5.13a)

\[ (\varepsilon \nabla u_h, \nabla \cdot v_h) + (\kappa u_h, v_h) = (f_h, v_h), \quad \forall v_h \in V_h^c. \]  

(5.13b)

Similarly, (5.1) and (5.2) with \( k = 2, n = 3 \) is to find \( u \in V_h^d \) and \( u_h \in V_h^d \) such that

\[ (\varepsilon \nabla \cdot u, \nabla \cdot v) + (\kappa u, v) = (f, v), \quad \forall v \in V_h^d, \]  

(5.14a)

\[ (\varepsilon \nabla \cdot u_h, \nabla \cdot v_h) + (\kappa u_h, v_h) = (f_h, v_h), \quad \forall v_h \in V_h^d. \]  

(5.14b)

The element residuals and face jumps in (5.6) are \( (k = 1) \)

\[ R_1^{k} \big|_{T} = -\nabla \cdot (f - \kappa u_h) \big|_{T}, \]

\[ J_1^{k} \big|_{S} = \| f - \kappa u_h \|_{S} \cdot u_{S}, \]

\[ R_2^{k} \big|_{T} = (f - \nabla \times (\varepsilon \nabla u_h) - \kappa u_h) \big|_{T}, \]

\[ J_2^{k} \big|_{S} = -\| \varepsilon \nabla u_h \|_{S} \cdot u_{S}. \]

(5.15)
and \((k = 2)\)
\[
\begin{align*}
R_1^2 |T| &= \nabla \times (f - \kappa u_h)|T|, \\
J_1^2 |S| &= \| f - \kappa u_h \|_{S} \times \nu_{S}, \\
R_2^2 |T| &= (f + \nabla (\varepsilon \cdot u_h) - \kappa u_h)|T|, \\
J_2^2 |S| &= \| \varepsilon \cdot u_h \|_{S}.
\end{align*}
\] (5.16)

Let \(\text{osc}_h\) be a generic data oscillation. Then Theorem 5.2 with \(k = 1\) and \(k = 2\) yields residual estimators for the Maxwell equation (5.13)
\[
\| u - u_h \|_{H(\text{curl})} \lesssim \| R_1^1 \|_{S_h} + \| h \frac{1}{2} J_1^1 \|_{S_h},
\]
and the grad-div problem (5.14)
\[
\| u - u_h \|_{H(\text{div})} \lesssim \| h R_2^1 \|_{S_h} + \| h \frac{1}{2} J_2^1 \|_{S_h},
\]
respectively. As mentioned in the introduction, the above two estimators are first presented in [16, 25]. Let
\[
H_i := \{ v \in H_T : v = 0 \text{ on } \Omega \setminus \Omega_i \}.
\]
We note that \(H^k_i\) defined in Section 4 is the Cartesian product of \(\binom{n_k}{n}\) copies of \(H_i\). Theorem 5.3 with \(k = 1\) yields a new implicit error estimator for the Maxwell equation
\[
\| u - u_h \|_{H(\text{curl})} \approx \sum_{i=1}^{N} \left( \| \eta_i \|_{H^1(\Omega)}^2 + \| \zeta_i \|_{H^1(\Omega)}^2 \right),
\]
where \(\eta_i \in H_i\) and \(\zeta_i \in \mathcal{H}^{i3}\) solve
\[
\begin{align*}
(\eta_i, \psi) + (\nabla \eta_i, \nabla \psi) &= (f - \kappa u_h, \nabla \psi), \quad \forall \psi \in H_i, \\
(\zeta_i, \phi) + (\nabla \zeta_i, \nabla \phi) &= (f - \kappa u_h, \phi) - (\varepsilon \nabla \times u_h, \nabla \times \phi), \quad \forall \phi \in \mathcal{H}^{i3}.
\end{align*}
\]
Other implicit a posteriori error estimates for Maxwell equations could be found in [42, 57]. Similarly, for the problem (5.14), the error estimator is
\[
\| u - u_h \|_{H(\text{div})} \approx \sum_{i=1}^{N} \left( \| \eta_i \|_{H^1(\Omega)}^2 + \| \zeta_i \|_{H^1(\Omega)}^2 \right),
\]
where \(\eta_i \in \mathcal{H}^{i3}\) and \(\zeta_i \in \mathcal{H}^{i3}\) solve
\[
\begin{align*}
(\eta_i, \psi) + (\nabla \eta_i, \nabla \psi) &= (f - \kappa u_h, \nabla \times \psi), \quad \forall \psi \in \mathcal{H}^{i3}, \\
(\zeta_i, \phi) + (\nabla \zeta_i, \nabla \phi) &= (f - \kappa u_h, \phi) - (\varepsilon \nabla \cdot u_h, \nabla \cdot \phi), \quad \forall \phi \in \mathcal{H}^{i3}.
\end{align*}
\]
5.3.1 Robust estimators for $H(\text{curl})$ and $H(\text{div})$

To close this section, we discuss the application of the robust estimator in Theorem 5.5. With the identifications (5.15) and (5.16) explained above, Theorem 5.5 with $k = 1, n = 3$ gives a new robust residual estimator for the Maxwell equation (5.13), and Theorem 5.5 with $k = 2, n = 3$ yields another robust estimator for the grad-div (5.14) problem. Those two estimators are uniform w.r.t. $\varepsilon$ and $\kappa$.

In [60], for the Maxwell equation (5.13), Schöberl used innovative commuting extension, smoothing, and interpolating operators to derive a residual estimator that is a uniform upper bound of the true error. However, the lower bound in [60] seems to be influenced by $\varepsilon$ and $\kappa$.

6 Applications to saddle point systems

In this section, we apply the theory developed in Sections 2, 4, 5 to several important saddle point systems. In particular, the space $V$ in (2.1) is chosen as a Cartesian product of several Hilbert spaces.

6.1 Hodge Laplace equation

Given an index $1 \leq k \leq n$, we assume the harmonic space $\mathcal{H}^k = \{0\}$ is trivial for simplicity. The Hodge Laplace equation with $f \in L^2 \Lambda^k(\Omega)$ under the mixed boundary condition is

\[
\begin{align*}
\sigma - \delta_k u &= 0, \quad \text{in } \Omega, \\
\partial_{k-1} \sigma + \delta_{k+1} d_k u &= f, \quad \text{in } \Omega, \\
\text{tr} u &= 0, \quad \text{on } \Gamma, \\
\text{tr} \ast u &= 0, \quad \text{on } \partial \Omega \setminus \Gamma.
\end{align*}
\]

The derivative $d_n$ vanishes as in the classical notation. The variational Hodge Laplacian problem seeks $(\sigma, u) \in V^{k-1} \times V^k$ such that

\[
\begin{align*}
(\sigma, \tau) - (d_{k-1} \tau, u) &= 0, \quad \tau \in V^{k-1}, \\
(d_{k-1} \sigma, v) + (d_k u, d_k v) &= (f, v), \quad v \in V^k.
\end{align*}
\] (6.1)

The corresponding discrete problem is: Find $(\sigma_h, u_h) \in V_h^{k-1} \times V_h^k$ such that

\[
\begin{align*}
(\sigma_h, \tau) - (d_{k-1} \tau, u_h) &= 0, \quad \tau \in V_h^{k-1}, \\
(d_{k-1} \sigma_h, v) + (d_k u_h, d_k v) &= (f, v), \quad v \in V_h^k.
\end{align*}
\] (6.2)

The well-posedness of (6.1) and (6.2) are confirmed in [9]. It follows from (5.4) that the Riesz representation $B_{V^{k-1} \times V^k}$ with $1 \leq k \leq n - 1$ decouples as

\[
B_{V^{k-1} \times V^k} = B_{V_{k-1}} \times B_{V_k} \\
\simeq (d_{k-2} B_{H^{k-2}} d_{k-2}^* + I_{H^{k-1}} B_{H^{k-1}} I_{H^{k-1}}) \\
\times (d_{k-1} B_{H^{k-1}} d_{k-1}^* + I_{H^k} B_{H^k} I_{H^k}).
\] (6.3)
When \( k = n \), \( V^n \) is simply \( L^2 A^n(\Omega) \) and
\[
\mathcal{B}_{V_{k-1}V^n} \simeq (d_{n-2} \mathcal{B}_{H^{n-2}d_{n-2}^*} + \mathcal{I}_{H^{n-1}} \mathcal{B}_{H^{n-1}d_{n-1}^*}) \times \text{id}_{V^n}.
\]
(6.4)

The residual \( \mathcal{R} = (\mathcal{R}_\sigma, \mathcal{R}_u) \in (V_k') \times (V_k') \) is given as
\[
\langle \mathcal{R}_\sigma, \tau \rangle = -(\sigma_h, \tau) + (d_{k-1} \tau, u_h), \quad \forall \tau \in V^{k-1},
\]
\[
\langle \mathcal{R}_u, v \rangle = (f, v) - (d_{k-1} \sigma_h, v) - (d_k u_h, d_k v), \quad \forall v \in V^k.
\]
(6.5)

Using (2.7) and (6.3), it is shown that the true error when \( 1 \leq k \leq n - 1 \) could be controlled by four \( H^{k-1} \) residuals
\[
\|\sigma - \sigma_h\|_{V^{k-1}}^2 + \|u - u_h\|_{V^k}^2 \simeq \langle \mathcal{R}_\sigma, \mathcal{B} V_{k-1} \mathcal{R}_\sigma \rangle + \langle \mathcal{R}_u, \mathcal{B} V_k \mathcal{R}_u \rangle
\]
\[
= (d_{k-2} \mathcal{R}_\sigma, \mathcal{B} V_{k-2} d_{k-2}^* \mathcal{R}_\sigma) + \langle \mathcal{I}_{H^{k-1}} \mathcal{R}_\sigma, \mathcal{B} H^{k-1} \mathcal{I}_{H^{k-1}} \mathcal{R}_\sigma \rangle
\]
\[
+ (d_{k-1} \mathcal{R}_u, \mathcal{B} H^{k-1} d_{k-1}^* \mathcal{R}_u) + \langle \mathcal{I}_{H^k} \mathcal{R}_u, \mathcal{B} H^k \mathcal{I}_{H^k} \mathcal{R}_u \rangle.
\]
(6.6)

Here for convenience, any quantity with index \( k = -1 \) vanishes. Similarly, in the case \( k = n \), (6.4) implies that
\[
\|\sigma - \sigma_h\|_{V^{n-1}}^2 + \|u - u_h\|_{V^n}^2 \simeq \langle \mathcal{R}_\sigma, \mathcal{B} V_{n-1} \mathcal{R}_\sigma \rangle + \langle \mathcal{R}_u, \mathcal{B} V_n \mathcal{R}_u \rangle + \|\mathcal{R}_u\|^2,
\]
(6.7)

where \( \mathcal{R}_u = f - d_{n-1} \sigma_h \in L^2 A^n(\Omega) \) in this case. Using (6.5) and the Stokes formula (3.5) on each element, it follows that the \( L^2 \) representations of the four residuals in (6.6) are
\[
\langle d_{k-2} \mathcal{R}_\sigma, \xi \rangle = (R^k_{H,1,1} \xi + (J^k_{H,1,1} \xi) S_h, \quad \forall \xi \in H^{k-2},
\]
\[
\langle \mathcal{I}_{H^{k-1}} \mathcal{R}_\sigma, \tau \rangle = (R^k_{H,2,1} \tau + (J^k_{H,2,1} \tau) S_h, \quad \forall \tau \in H^{k-1},
\]
\[
\langle d_{k-1} \mathcal{R}_u, \tau \rangle = (R^k_{H,3,1} \tau + (J^k_{H,3,1} \tau) S_h, \quad \forall \tau \in H^{k-1},
\]
\[
\langle \mathcal{I}_{H^k} \mathcal{R}_u, \tau \rangle = (R^k_{H,4,1} \tau + (J^k_{H,4,1} \tau) S_h, \quad \forall \tau \in H^k,
\]
where the element residuals and jump faces are
\[
R^k_{H,1,1} \tau = -d_{k-1} \sigma_h \tau, \quad J^k_{H,1,1} S = -[\text{tr} S],
\]
\[
R^k_{H,2,1} \tau = (-\sigma_h + \delta_k u_h) \tau, \quad J^k_{H,2,1} S = [\text{tr} u_h],
\]
\[
R^k_{H,3,1} \tau = \delta_k (f - d_{k-1} \sigma_h) \tau, \quad J^k_{H,3,1} S = [\text{tr} (f - d_{k-1} \sigma_h)],
\]
\[
R^k_{H,4,1} \tau = (f - d_{k-1} \sigma_h - \delta_k d_k u_h) \tau, \quad J^k_{H,4,1} S = -[\text{tr} d_k u_h],
\]
for all \( T \in T_h, S \in S_h \). Therefore estimating the four residuals in (6.6) and (6.7) by Lemma 4.2, we obtain the residual estimator for the Hodge Laplacian.

The assumption in Lemma 4.2 is verified in the same way as in Theorem 5.2.

**Theorem 6.1** For \( 1 \leq k \leq n - 1 \), we have
\[
\|\sigma - \sigma_h\|_{V^{k-1}} + \|u - u_h\|_{V^k} \lesssim \sum_{i=1}^4 \left( h_{R^k_{H,1,1}} + h_{J^k_{H,1,1}} \right) S_h,
\]
and
\[ \sum_{i=1}^{4} (\|hR_{H,i}^k\| + \|h^{\frac{1}{2}}J_{H,i}^k\|) \lesssim \|\sigma - \sigma_h\|_{V^{k-1}} + \|u - u_h\|_{V^k} + \text{osc}_h(d_{k-2}^n\mathcal{R}_\sigma) + \text{osc}_h(I_{H,k-1}^n\mathcal{R}_\sigma) + \text{osc}_h(d_{k-3}^n\mathcal{R}_u) + \text{osc}_h(I_{H,k}^n\mathcal{R}_u). \]

For \( k = n \), we have
\[ \|\sigma - \sigma_h\|_{V^{n-1}} + \|u - u_h\|_{V^n} \lesssim \sum_{i=1}^{2} (\|hR_{H,i}^n\| + \|h^{\frac{1}{2}}J_{H,i}^n\|) + \|f - d_{n-1}\sigma_h\| \]

and
\[ \sum_{i=1}^{2} (\|hR_{H,i}^n\| + \|h^{\frac{1}{2}}J_{H,i}^n\|) + \|f - d_{n-1}\sigma_h\| \]
\[ \lesssim \|\sigma - \sigma_h\|_{V^{n-1}} + \|u - u_h\|_{V^n} + \text{osc}_h(d_{k-2}^n\mathcal{R}_\sigma) + \text{osc}_h(I_{H,n-1}^n\mathcal{R}_\sigma). \]

The residual estimator in Theorem 6.1 was first derived in [28] using commuting regularized interpolation, which could be avoided in our framework. In addition, a new implicit error estimator follows from (6.6), (6.7), Lemma 4.3, and \( H_{h}^{k,1} \triangleq V_h \) if \( V_h^{k} \neq P_0\Lambda^k(T_h, \Gamma) \).

**Theorem 6.2** Let \( V_h^{k} \neq P_0\Lambda^k(T_h, \Gamma) \) with \( 1 \leq k \leq n - 1 \). Then for \( 1 \leq k \leq n - 1 \), we have
\[ \|\sigma - \sigma_h\|_{V^{k-1}}^2 + \|u - u_h\|_{V^k}^2 \]
\[ \lesssim \sum_{i=1}^{N} \left( \|\eta_i^\sigma\|_{H^1(\Omega_i)}^2 + \|\zeta_i^\sigma\|_{H^1(\Omega_i)}^2 + \|\eta_i^u\|_{H^1(\Omega_i)}^2 + \|\zeta_i^u\|_{H^1(\Omega_i)}^2 \right), \]
where \( \eta_i^\sigma \in H_{i}^{k-2}, \zeta_i^\sigma, \eta_i^u \in H_{i}^{k-1}, \zeta_i^u \in H_{i}^{k} \) solve
\[ (\eta_i^\sigma, \xi) + (\nabla \eta_i^\sigma, \nabla \xi) = -(\sigma_h, d_{k-2}\xi), \]
\[ (\zeta_i^\sigma, \tau) + (\nabla \zeta_i^\sigma, \nabla \tau) = -(\sigma_h, \tau) + (d_{k-1}\tau, u_h), \]
\[ (\eta_i^u, v) + (\nabla \eta_i^u, \nabla v) = (f, d_{k-1}\tau) - (d_{k-1}\sigma_h, d_{k-1}\tau), \]
\[ (\zeta_i^u, v) + (\nabla \zeta_i^u, \nabla v) = (f, v) - (d_{k-1}\sigma_h, v) - (d_{k}\sigma_h, d_{k}\tau) \]
for all \( \xi \in H_{i}^{k-2}, \tau \in H_{i}^{k-1}, v \in H_{i}^{k} \). For \( k = n \) we have
\[ \|\sigma - \sigma_h\|_{V^{n-1}}^2 + \|u - u_h\|_{V^n}^2 \lesssim \sum_{i=1}^{N} \left( \|\eta_i^\sigma\|_{H^1(\Omega_i)}^2 + \|\zeta_i^\sigma\|_{H^1(\Omega_i)}^2 \right) + \|f - d_{n-1}\sigma_h\|^2. \]
6.2 Elasticity with weakly imposed symmetry

In this subsection, we consider the linear elasticity using weakly symmetric tensors in $\mathbb{R}^3$. Let $\Gamma \neq \emptyset$, $\Sigma = [V^d]^3 \subseteq [H(\text{div}, \Omega)]^3$, and $U = Q = [L^2(\Omega)]^3$. The space $\Sigma$ consists of matrix-valued functions whose rows are contained in $V^d$. Let $\lambda, \mu \in L^\infty(\Omega)$ be the Lamé parameters. The elasticity compliance tensor is

$$A \tau = \frac{1}{2\mu} \left( \tau - \frac{\lambda}{2\mu + 3\lambda} (\text{Tr} \tau) I \right), \quad \tau \in \Sigma,$$

where $\text{Tr} \tau$ denotes the trace of the matrix $\tau$, and $I \in \mathbb{R}^{3 \times 3}$ is the identity matrix. Let $\text{div}$ denote the row-wise divergence operator. Define $\text{skw} \tau = (\tau_{23} - \tau_{32}, \tau_{31} - \tau_{13}, \tau_{12} - \tau_{21})$ to be the operation of taking the skew-symmetric part of matrices. The $L^2$-adjoint of $\text{skw}$ is defined for $q = (q_1, q_2, q_3)$ as

$$\text{skw}^* q = \begin{pmatrix} 0 & q_3 & -q_2 \\ -q_3 & 0 & q_1 \\ q_2 & -q_1 & 0 \end{pmatrix}.$$

The variational formulation of linear elasticity with weakly symmetric tensor is: Find $(\sigma, u, p) \in \Sigma \times U \times Q$ such that

$$(A \sigma, \tau) + (\text{div} \tau, u) + (\text{skw} \tau, p) = 0, \quad \tau \in \Sigma,$$

$$(\text{div} \sigma, v) + (\text{skw} \sigma, q) = (f, v), \quad (v, q) \in U \times Q. \quad (6.8)$$

The stable discretization $(\sigma_h, u_h, p_h) \in \Sigma_h \times U_h \times Q_h \subset \Sigma \times U \times Q$ for (6.8) has been well-established in the literature [6, 63]

$$(A \sigma_h, \tau_h) + (\text{div} \tau_h, u_h) + (\text{skw} \tau_h, p_h) = 0, \quad \tau_h \in \Sigma_h,$$

$$(\text{div} \sigma_h, v_h) + (\text{skw} \sigma_h, q) = (f, v), \quad (v, q) \in U_h \times Q_h. \quad (6.9)$$

For instance, $\Sigma_h \times U_h = [V^d_h]^3 \times [L^2_h]^3$ could be the Brezzi–Douglas–Marini pair and $Q_h = U_h$. The residual of (6.9) consists of

$$\langle R \Sigma, \tau \rangle = -(A \sigma_h, \tau) - (\text{div} \tau_h, u_h) - (\text{skw} \tau_h, p_h), \quad \forall \tau \in \Sigma,$$

$$R_U = f - \text{div} \sigma_h, \quad R_Q = -\text{skw} \sigma_h. \quad (6.10)$$

The error-residual relation of (6.9) reads

$$\| \sigma - \sigma_h \|^2_{\Sigma} + \| u - u_h \|^2_U + \| p - p_h \|^2_Q \approx \langle R \Sigma \times R_U \times R_Q, B_{\Sigma \times U \times Q} R \Sigma \times R_U \times R_Q \rangle, \quad (6.11)$$

where the constants hidden in the equivalence are independent of $\lambda$, see, [17].

With the identification [57, 6.3] (k = n - 1, n = 3), we have

$$B_{V^d} \simeq \nabla \times B_{[H^1]^3} (\nabla \times + I_{[H^1]^3} B_{[H^1]^3} I_{[H^1]^3}), \quad (6.12)$$
where $\nabla \times : [H_T]^3 \to \mathcal{V}^d$ and $\mathcal{I}_{[H_T]^3} : [H_T]^3 \to \mathcal{V}^d$ is the inclusion. It follows from (6.11), (6.12) and

$$B_{\Sigma \times U \times Q} = B_\Sigma \times B_U \times B_Q = [B_{U \times Q}]^3 \times \text{id}_U \times \text{id}_Q,$$

(6.13)

that

$$\|\sigma - \sigma_h\|_\Sigma^2 + \|u - u_h\|_U^2 + \|p - p_h\|_Q^2 \approx \|((\nabla \times)^* \mathcal{R}_\Sigma\|_{B_{U \times Q}^2} + \|\mathcal{I}_{[H_T]^3}^* \mathcal{R}_\mathcal{S}\|_{B_{U \times Q}^2} + \|\mathcal{R}_U\| + \|\mathcal{R}_Q\|^2.$$ (6.14)

Here $\nabla \times : [H_T]^3 \times 3 \to \Sigma$ is the row-wise curl in (6.14). Using (6.10) and integration-by-parts, we have

$$\langle (\nabla \times)^* \mathcal{R}_\Sigma, \xi \rangle = \langle R^E_T, \xi \rangle + \langle J^E_T, \xi \times \nu \rangle_{S_h}, \forall \xi \in [H_T]^3,$$

$$\langle \mathcal{I}_{[H_T]^3}^* \mathcal{R}_\mathcal{S}, \tau \rangle = \langle R^E_T, \tau \rangle + \langle J^E_T, \tau \cdot \nu \rangle_{S_h}, \forall \tau \in [H_T]^3,$$

where for all $T \in \mathcal{T}_h$ and $S \in S_h$,

$$R^E_T|_T = -\nabla \times (A\sigma_h + \text{skw} p_h)|_T, J^E_T|_T = -[A\sigma_h + \text{skw}^* p_h] \times \nu|_S,$$

$$R^E_T|_T = (-A\sigma_h + \nabla u_h - \text{skw}^* p_h)|_T, J^E_T|_S = -[u_h]|_S.$$

A combination of (6.14) and Lemma 1.2 yields a residual estimator.

Theorem 6.3 There exist $C_{E,1} > 0$, $C_{E,2} > 0$ dependent only on $\mu$, $\Omega$, $\Gamma$ such that

$$\|\sigma - \sigma_h\|_\Sigma + \|u - u_h\|_U + \|p - p_h\|_Q \leq C_{E,1} \left\{ \sum_{i=1}^2 \left( \|hR^E_i\| + \|h^2 J^E_i\|_{S_h} + \|\mathcal{R}_U\| + \|\mathcal{R}_Q\| \right) \right\},$$

and

$$C_{E,2} \left\{ \sum_{i=1}^2 \left( \|hR^E_i\| + \|h^2 J^E_i\|_{S_h} + \|\mathcal{R}_U\| + \|\mathcal{R}_Q\| \right) \right\} \leq \|\sigma - \sigma_h\|_\Sigma + \|u - u_h\|_U + \|p - p_h\|_Q + \text{osc}_h((\nabla \times)^* \mathcal{R}_\Sigma) + \text{osc}_h(\mathcal{I}_{[H_T]^3}^* \mathcal{R}_\mathcal{S}).$$

The same estimator could be found in [47] under the assumption that the domain is convex. The work [43] derives an equilibrated estimator for (6.9) with guaranteed upper bound in two dimension.

Recall that $H^1_h = H^1_h \cap H_T$ is the subspace of continuous and scalar-valued piecewise linear polynomials. Finally we present an implicit error estimator using (6.14) and a vector-valued version of Lemma 4.3.

Theorem 6.4 Assume $\Sigma_h \supset [H^1_h]^3 \times 3$. Then we have

$$\|\sigma - \sigma_h\|_\Sigma^2 + \|u - u_h\|_U^2 + \|p - p_h\|_Q^2 \approx \sum_{i=1}^N \left( \|\eta_i\|_{H^1(\Omega_i)}^2 + \|\zeta_i\|_{H^1(\Omega_i)}^2 \right) + \|f - \text{div}\sigma_h\|^2 + \|\text{skw}\sigma_h\|^2,$$
where $\eta_i \in [H_i]^{3 \times 3}$, $\zeta_i \in [H_i]^{3 \times 3}$ solve

$$(\eta_i, \xi) + (\nabla \eta_i, \nabla \xi) = -(A \sigma_h, \nabla \times \xi) - (\text{skw} \nabla \times \xi, p_h), \quad \forall \xi \in [H_i]^{3 \times 3},$$

$$(\zeta_i, \tau) + (\nabla \zeta_i, \nabla \tau) = -(A \sigma_h, \tau) - (\text{div} \tau, u_h) - (\text{skw} \tau, p_h), \quad \forall \tau \in [H_i]^{3 \times 3}.$$

**Appendix**

We construct $P_h^k$ in Lemma 4.1 following the same idea in [27,61]. However, we regularize $v \in H^1 A^k(\Omega)$ on $n$-dimensional elements except for degrees of freedom related to $\Gamma$. As a consequence, the interpolation is $L^2$ bounded in $H^k$, which is essential for deriving error estimators for singularly perturbed problems.

**Proof of Lemma 4.1**

Let $\{\Delta_i\}_{i=1}^M$ be the set of $k$-dimensional simplexes in $\mathcal{T}_h$. Given a sufficiently smooth $k$-form $v$, the degrees of freedom of $P_0^{-1}A^k(\mathcal{T}_h)$ consists of $\int_{\Delta_i} \text{tr} v$ with $1 \leq i \leq M$. Let $\{\phi_i\}_{i=1}^M$ be the corresponding dual basis of $V^k_0$. Each $\Delta_i$ is assigned with an $n$- or $(n-1)$-dimensional simplex $\sigma_i \supset \Delta_i$ in $\mathcal{T}_h$. In particular, let $\sigma_i$ be an element in $\mathcal{T}_h$ for $\Delta_i \not\subset \Gamma$ and be a face in $\Gamma$ for $\Delta_i \subset \Gamma$. Let $Q_{\sigma_i}$ denote the $L^2$ projection onto $P_0 A^k(\sigma_i)$. The interpolation operator $P_h^k : H^1 A^k(\Omega) \to P_0^{-1}A^k(\mathcal{T}_h)$ is defined as

$$P_h^k v = \sum_{i=1}^M \left( \int_{\Delta_i} \text{tr} Q_{\sigma_i} v \right) \phi_i.$$  

By construction, we have for $T \in \mathcal{T}_h$,

1. $v \in P_0 A^k(\Omega_T^k) \implies P_h^k v = v$ on $T$,
2. $\text{tr} T v = 0 \implies \text{tr} T P_h^k v = 0$ and $\|P_h^k v\|_T \leq \|v\|_{\Omega_T^k}$.

The approximation in Lemma 4.3 follows from the Bramble–Hilbert lemma and that $P_h^k$ preserves constants locally. □

**Proof of Theorem 5.4**

Using the Hodge decomposition (3.2) and the Poincaré inequality (3.3), we have

$$v = d_{k-1} \varphi_0 + z_0,$$

where $\varphi_0 \in \mathcal{V}^{k-1} \subset V^{k-1}$, $z_0 \in \mathcal{N}^k \oplus \mathcal{Z}^{k, \perp}$, and

$$\|\varphi_0\| \leq \|d_{k-1} \varphi_0\| \leq \|v\|,$$

$$\|z_0\| \leq \|v\|, \quad \|d_k z_0\| = \|d_k v\|.$$

(A.2)
We shall extend $z_0$ to a slightly larger domain with smooth boundary. In fact, there exist a Lipschitz domain $\Omega^e \supset \Omega$, an extension operator $E^k : \mathcal{V}^k \to H\Lambda^k(\Omega^e)$, and an outer Lipschitz neighborhood $\Omega^e \subset \Omega^e$ of $\Gamma$ (see, e.g., [34, 46, 60]), such that for all $v \in \mathcal{V}^k$,

$$
E^k v|_{\Omega} = v, \quad E^k v|_{\Omega^e} = 0, \\
\|E^k v\|_{\Omega^e} \lesssim \|v\|, \quad d_k E^k v = E^{k+1} d_k v. \tag{A.3}
$$

Without loss of generality, we assume the boundary of $\Omega^e$ is smooth by restricting $E^k v$ to a slightly smaller but smooth subdomain of $\Omega^e$. We refer the reader to Figure A.1 for sketch of the domains involved in the construction.

Now, let $\tilde{z} = E^k z_0$. It follows from (A.3) that

$$
\|\tilde{z}\|_{\Omega^e} \lesssim \|z_0\|, \quad \|d\tilde{z}\|_{\Omega^e} \lesssim \|d_k z_0\|, \tag{A.4a}
$$

$\tilde{z} = 0$ on $\Omega^e$. \tag{A.4b}

Using the Hodge decomposition on $\Omega^e$ gives that

$$
\tilde{z} = d_{k-1} \tilde{\psi} + \tilde{w} \quad \text{on} \quad \Omega^e, \tag{A.5}
$$

where $\tilde{\psi} \in N(d_{k-1}|_{\Omega^e})^\perp$ and $\tilde{w} \in R(d_{k-1}|_{\Omega^e})^\perp$. Since $\partial \Omega^e$ is smooth and [34], we have $\tilde{\psi} \in H^1 A^{k-1}(\Omega^e)$, $\tilde{w} \in H^1 A^k(\Omega^e)$, and

$$
\|\tilde{\psi}\|_{H^1(\Omega^e)} \lesssim \|\tilde{z}\|_{\Omega^e}, \quad \|\tilde{w}\|_{\Omega^e} \lesssim \|\tilde{z}\|_{\Omega^e}, \quad \|d_k \tilde{z}\|_{\Omega^e}. \tag{A.6}
$$

If $\Gamma = \emptyset$, then we complete the proof by taking $\psi = \varphi_0 + \tilde{\psi}|_{\Omega} \in \mathcal{V}^{k-1}$ and $z = \tilde{z}|_{\Omega} \in \mathcal{H}^k$. In general, $\tilde{\psi}$, $\tilde{w}$ must be modified to satisfy the homogeneous boundary condition on $\Gamma$. Using (A.4b), it follows that $d_{k-1} \tilde{\psi} = -\tilde{w} \in \mathcal{V}^{k-1}$. 

**Fig. A.1** An extended domain $\Omega^e \supset \Omega$ and an outer neighborhood $\Omega^e$ of $\Gamma$. 

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$H^1 A^k(\Omega_T)$ in the subdomain $\Omega_T$. Then by the universal extension theorem in [38], we can extend $\tilde{\varphi}|_{\Omega_T} \in H^1 A^k(\Omega_T)$ with $d_{k-1}\tilde{\varphi}|_{\Omega_T} \in H^1 A^{k-1}(\Omega_T)$ to obtain $\tilde{\varphi} \in H^1 A^{k-1}(\mathbb{R}^n)$ with $d_{k-1}\tilde{\varphi} \in H^1 A^{k-1}(\mathbb{R}^n)$, and

\begin{align*}
|\tilde{\varphi}| + |d_{k-1}\tilde{\varphi}| & \leq |\tilde{\varphi}|_{\Omega_T} + |\tilde{w}|_{\Omega_T}, \\
|\tilde{\varphi}|_{H^1} + |d_{k-1}\tilde{\varphi}|_{H^1} & \leq |\tilde{\varphi}|_{H^1(\Omega_T)} + |\tilde{w}|_{H^1(\Omega_T)}.
\end{align*}

(A.7)

Using (A.1) and (A.5), we rewrite $v$ for $x \in \Omega$ as

\begin{equation}
\begin{aligned}
 v &= d_{k-1}\varphi_1 + z, \\
 \varphi_1 &= \varphi_0 + \tilde{\varphi}|_{\Omega} - \tilde{\varphi}|_{\Omega_T} \in \mathcal{Y}^{k-1}, \\
 z &= d_{k-1}\varphi_1|_{\Omega} + \tilde{w}|_{\Omega} \in \mathcal{H}^k.
\end{aligned}
\end{equation}

(A.8)

where $\varphi_1 = \varphi_0 + \tilde{\varphi}|_{\Omega} - \tilde{\varphi}|_{\Omega_T} \in \mathcal{Y}^{k-1}$, $z = d_{k-1}\varphi_1|_{\Omega} + \tilde{w}|_{\Omega} \in \mathcal{H}^k$. Here we have $z = 0$ on $\Gamma$ because $d_{k-1}\varphi_1 + \tilde{w} = 0$ on $\Omega_T$, $\Gamma \subset \partial\Omega_T$, and $d_{k-1}\varphi = d_{k-1}\tilde{\varphi}$ on $\Gamma$. Collecting previous bounds (A.2), (A.4a), (A.6), (A.7) then shows that

\begin{align*}
|\varphi_1|_{\mathcal{Y}^{k-1}} + |z| & \leq |v|, \\
|z|_{H^1} & \leq |v| + |d_k v|.
\end{align*}

Finally, the regular decomposition of $\varphi_1$ implies that there exists $\varphi \in \mathcal{H}^{k-1}$ satisfying

\begin{equation}
\begin{aligned}
 d_{k-1}\varphi &= d_{k-1}\varphi_1, \\
 |\varphi|_{H^1} & \leq |d_{k-1}\varphi_1| \leq |v|.
\end{aligned}
\end{equation}

Replacing $\varphi_1$ with $\varphi$ in (A.8) completes the proof. \hfill \square

If the space of harmonic forms $\mathcal{H}^k = \{0\}$ is trivial, the component $z$ in Theorem 5.4 can be chosen such that

\begin{align*}
|z| & \leq |v|, \\
|z|_{H^1} & \leq |d_k v|.
\end{align*}

In the previous proof for $z_0 \in \mathcal{H}^k \oplus \mathcal{Z}^{k-1}$, let us consider the decomposition $z_0 = q + w_0$ where $q \in \mathcal{H}^k$, $w_0 \in \mathcal{Z}^{k-1}$. If, in the arguments in this proof, we replace $z_0$ with $w_0$, then we arrive at the following corollary, which might be useful for the analysis of other methods and techniques.

**Corollary A.1** For any $v \in \mathcal{V}^k$, there exist $z \in \mathcal{H}^k$ such that

\begin{equation}
\begin{aligned}
 d_k v &= d_k z, \\
 |z| & \leq |v|, \\
|z|_{H^1} & \leq |d_k v|.
\end{aligned}
\end{equation}

**Remark 3** Unlike the pure Dirichlet ($\Gamma = \partial\Omega$) or Neumann ($\Gamma = \emptyset$) boundary condition, the structure of $\mathcal{H}^k$ under the mixed boundary condition is dependent on the relative homology of the pair $(\Omega, \Gamma)$. It turns out that $\mathcal{H}^k$ could be nontrivial even if $\Omega$ is star-shaped, see, e.g., [32,46].
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