THE LIE ALGEBRA OF TYPE $G_2$ IS RATIONAL OVER ITS QUOTIENT BY THE ADJOINT ACTION

DAVE ANDERSON, MATHIEU FLORENCE, AND ZINOYY REICHSTEIN

Abstract. Let $G$ be a split simple group of type $G_2$ over a field $k$, and let $\mathfrak{g}$ be its Lie algebra. Answering a question of J.-L. Colliot-Thélène, B. Kunyavskiǐ, V. L. Popov, and Z. Reichstein, we show that the function field $k(\mathfrak{g})$ is generated by algebraically independent elements over the field of adjoint invariants $k(\mathfrak{g})^G$.

Résumé. Soit $G$ un groupe algébrique simple et déployé de type $G_2$ sur un corps $k$. Soit $\mathfrak{g}$ son algèbre de Lie. On démontre que le corps des fonctions $k(\mathfrak{g})$ est transcendant pur sur le corps $k(\mathfrak{g})^G$ des invariants adjoints. Ceci répond par l'affirmative à une question posée par J.-L. Colliot-Thélène, B. Kunyavskiǐ, V. L. Popov et Z. Reichstein.

I. Introduction. Let $G$ be a split connected reductive group over a field $k$ and let $\mathfrak{g}$ be the Lie algebra of $G$. We will be interested in the following natural question:

Question 1. Is the function field $k(\mathfrak{g})$ purely transcendental over the field of invariants $k(\mathfrak{g})^G$ for the adjoint action of $G$ on $\mathfrak{g}$? That is, can $k(\mathfrak{g})$ be generated over $k(\mathfrak{g})^G$ by algebraically independent elements?

In [5], the authors reduce this question to the case where $G$ is simple, and show that in the case of simple groups, the answer is affirmative for split groups of types $A_n$ and $C_n$, and negative for all other types except possibly for $G_2$. The standing assumption in [5] is that $\text{char}(k) = 0$, but here we work in arbitrary characteristic.

The purpose of this note is to settle Question 1 for the remaining case $G = G_2$.

Theorem 2. Let $k$ be an arbitrary field and $G$ be the simple split $k$-group of type $G_2$. Then $k(\mathfrak{g})$ is purely transcendental over $k(\mathfrak{g})^G$.

Under the same hypothesis, and also assuming $\text{char}(k) = 0$, it follows from Theorem 2 and [5, Theorem 4.10] that the field extension $k(G)/k(G)^G$ is also purely transcendental, where $G$ acts on itself by conjugation.

Apart from settling the last case left open in [5], we were motivated by the (still mysterious) connection between Question 1 and the Gelfand-Kirillov (GK) conjecture [9]. In this context $\text{char}(k) = 0$. A. Premet [11] recently showed that the GK conjecture fails for simple Lie algebras of any type other than $A_n$, $C_n$ and $G_2$. His paper relies on the negative results of [5] and their characteristic $p$ analogues ([11], see also [5, Theorem 6.3]). It is not known whether a positive answer to Question 1 for $\mathfrak{g}$ implies the GK conjecture for $\mathfrak{g}$. The GK conjecture has been proved for algebras of type $A_n$ (see [9]), but remains open for types $C_n$ and $G_2$. While Theorem 2 does not settle the GK conjecture for type $G_2$, it puts the remaining two open cases—for algebras of type $C_n$ and $G_2$—on equal footing vis-à-vis Question 1.

II. Twisting. Temporarily, let $W$ be a linear algebraic group over a field $k$. (In the sequel, $W$ will be the Weyl group of $G$; in particular, it will be finite and smooth.) We refer to [7, Section 3], [8, Section 2], or [5, Section 2] for details about the following facts.

D.A. was partially supported by NSF Grant DMS-0902967. Z.R. was partially supported by National Sciences and Engineering Research Council of Canada grant No. 250217-2012.
Let $X$ be a quasi-projective variety with a (right) $W$-action defined over $k$, and let $\zeta$ be a (left) $W$-torsor over $k$. The diagonal left action of $W$ on $X \times_{\text{Spec}(k)} \zeta$ (by $g.(x,z) = (xg^{-1}, gz)$) makes $X \times_{\text{Spec}(k)} \zeta$ into the total space of a $W$-torsor $X \times_{\text{Spec}(k)} \zeta \to B$. The base space $B$ of this torsor is usually called the \emph{twist} of $X$ by $\zeta$. We denote it by $\zeta X$.

It is easy to see that if $\zeta$ is trivial then $\zeta X$ is $k$-isomorphic to $X$. Hence, $\zeta X$ is a $k$-form of $X$, i.e., $X$ and $\zeta X$ become isomorphic over an algebraic closure of $k$.

The twisting construction is functorial in $X$: a $W$-equivariant morphism $X \to Y$ (or rational map $X \dashrightarrow Y$) induces a $k$-morphism $\zeta X \to \zeta Y$ (resp., rational map $\zeta X \dashrightarrow \zeta Y$).

III. The split group of type $G_2$. We fix notation and briefly review the basic facts, referring to [13], [1], or [2] for more details. Over any field $k$, a simple split group $G$ of type $G_2$ has a faithful seven-dimensional representation $V$. Following [2, (3.11)], one can fix a basis $f_1, \ldots, f_7$, with dual basis $X_1, \ldots, X_7$, so that $G$ preserves the nonsingular quadratic norm $N = X_1X_7 + X_2X_6 + X_3X_5 + X_4^2$. (See [1, §6.1] for the case $\text{char}(k) = 2$. In this case $V$ is not irreducible, since the subspace spanned by $f_4$ is invariant; the quotient $V/(k \cdot f_4)$ is the minimal irreducible representation. However, irreducibility will not be necessary in our context.) The corresponding embedding $G \hookrightarrow \text{GL}_7$ yields a split maximal torus and Borel subgroup $T \subset B \subset G$, by intersecting with diagonal and upper-triangular matrices. Explicitly, the maximal torus is

\begin{equation}
T = \text{diag}(t_1, t_2, t_1^{-1}t_2, 1, t_1^{-1}t_2^{-1}, t_2^{-1}, t_1^{-1}) ;
\end{equation}

cf. [2, Lemma 3.13].

The Weyl group $W = N(T)/T$ is isomorphic to the dihedral group with 12 elements, and the surjection $N(T) \to W$ splits. The inclusion $G \hookrightarrow \text{GL}_7$ thus gives rise to an inclusion $N(T) = T \times W \hookrightarrow D \rtimes_S 7$, where $D \subset \text{GL}_7$ is the subgroup of diagonal matrices. On the level of the dual basis $X_1, \ldots, X_7$, we obtain an isomorphism $W \cong S_3 \times S_2$ realized as follows: $S_3$ permutes the three ordered pairs $(X_1, X_7), (X_2, X_6)$ and $(X_3, X_5)$, and $S_2$ exchanges the two ordered triples $(X_1, X_5, X_6)$ and $(X_7, X_3, X_2)$. The variable $X_4$ is fixed by $W$. For details, see [2, §A.3].

The subgroup $P \subset G$ stabilizing the isotropic line spanned by $f_1$ is a maximal standard parabolic, and the corresponding homogeneous space $P\backslash G$ is isomorphic to the five-dimensional quadric $Q \subset \mathbb{P}(V)$ defined by the vanishing of the norm, i.e., by the equation

\begin{equation}
X_1X_7 + X_2X_6 + X_3X_5 + X_4^2 = 0 .
\end{equation}

Note that the quadric $Q$ is endowed with an action of $T$. An easy tangent space computation shows that $P$ is smooth regardless of the characteristic of $k$.

Lemma 3. The group $P$ is special, i.e., $H^1(l/P) = \{1\}$ for every field extension $l/k$. Moreover, $P$ is rational, as a variety over $k$.

Proof. Since the split group of type $G_2$ is defined over the prime field, we may replace $k$ by the prime field for the purpose of proving this lemma, and in particular, we can assume $k$ is perfect. We begin by briefly recalling a construction of Chevalley [4]. The isotropic line $E_1 \subset V$ stabilized by $P$ is spanned by $f_1$, and $P$ also preserves an isotropic 3-space $E_3$ spanned by $f_1, f_2, f_3$; see, e.g., [2, §2.2]. There is a corresponding map $P \to \text{GL}(E_3/E_1) \cong \text{GL}_2$, which is a split surjection thanks to the block matrix described in [10, p. 13] as the image of \textquotedblleft $B$\textquotedblright in $\text{GL}_7$. The kernel is unipotent, and we have a split exact sequence corresponding to the Levi decomposition:

\begin{equation}
1 \to R_u(P) \to P \to \text{GL}_2 \to 1 .
\end{equation}

Combining the exact sequence in cohomology induced by (3) with the fact that both $R_u(P)$ and $\text{GL}_2$ are special (see [12, pp. 122 and 128]), shows that $P$ is special.
Since $P$ is isomorphic to $R_u(P) \times \text{GL}_2$ as a variety over $k$, and $P$ is smooth, so is $R_u(P)$. A smooth connected unipotent group over a perfect field is rational [6, IV, §2(3.10)]: therefore $R_u(P)$ is $k$-rational, and so is $P$. \hfill \Box

IV. Proof of Theorem 2. Keep the notation of the previous section. By a $W$-model (of $k(Q)^T$), we mean a quasi-projective $k$-variety $Y$, endowed with a right action of $W$, together with a dominant $W$-equivariant $k$-rational map $Q \to Y$ which, on the level of function fields, identifies $k(Y)$ with $k(Q)^T$. Such a map $Q \to Y$ is called a ($W$-equivariant) rational quotient map. A $W$-model is unique up to a $W$-equivariant birational isomorphism; we will construct an explicit one below.

We reduce Theorem 2 to a statement about rationality of a twisted $W$-model, in two steps. The first holds for general split connected semisimple groups $G$.

Proposition 4. Let $G$ be a split connected semisimple group over $k$, with split maximal $k$-torus $T$. Let $K = k(t)^W$, $L = k(t)$, and let $Q$ be the $W$-torsor corresponding to the field extension $L/K$. If the twisted variety $G\!/T_K$ is rational over $K$, then $k(g)$ is purely transcendental over $k(g)^G$.

Proof. Consider the $(G \times W)$-equivariant morphism

$$f: G/T \times_{\text{Spec}(k)} t \to g$$

given by $(\pi, t) \mapsto \text{Ad}(a)t$, where $t$ is the Lie algebra of $T$, $\pi \in G/T$ is the class of $a \in G$, modulo $T$. Here $G$ acts on $G/T \times t$ by translations on the first factor (and trivially on $t$), and via the adjoint representation on $g$. The Weyl group $W$ naturally acts on $t$ and $G/T$ (on the right), diagonally on $G/T \times t$, and trivially on $g$.

The image of $f$ contains the semisimple locus in $g$, so $f$ is dominant and induces an inclusion $f^*: k(g) \hookrightarrow k(G/T \times t)$. Clearly $f^* k(g) \subset k(G/T \times t)^W$. We will show that in fact

$$f^* k(g) = k(G/T \times t)^W.$$  

Write $\overline{k}$ for an algebraic closure of $k$, and note that the preimage of a $\overline{k}$-point of $g$ in general position is a single $W$-orbit in $(G/T \times t)_{\overline{k}}$. To establish (4), it remains to check that $f$ is smooth at a general point $(g, x)$ of $G/T \times t$. (In particular, when char($k) = 0$ nothing more is needed.) To carry out this calculation, we may assume without loss of generality that $k$ is algebraically closed and (since $f$ is $G$-equivariant) $g = 1$. Since $\dim(G/T \times t) = \dim(g)$, it suffices to show that the differential

$$df: T_{(1, x)}(G/T \times t) \to T_x(g)$$

is surjective, for any regular semisimple element $x \in t$. Equivalently, we want to show that $[x, g] + t = g$. Since $x$ is regular, we have $\dim([x, g]) + \dim t = \dim g$. Thus it remains to show that $[x, g] \cap t = 0$. To see this, suppose $[x, y] \in t$ for some $y \in g$. Since $x$ is semisimple, we can write $y = \sum_{i=1}^r y_{\lambda_i}$, where $y_{\lambda_i}$ is an eigenvector for $\text{ad}(x)$ with eigenvalue $\lambda_i$. If $\lambda_1, \ldots, \lambda_r$ are distinct. Then $[x, y] = \sum_{i=1}^r \lambda_i y_{\lambda_i} \in t$ is an eigenvector for $\text{ad}(x)$ with eigenvalue 0. Remembering that eigenvectors of $\text{ad}(x)$ with distinct eigenvalues are linearly independent, we conclude that $[x, y] = 0$. This completes the proof of (4).

It is easy to see $k(G/T \times t)^G = k(t)^W$. Summarizing, $f^*$ induces a diagram

$$\begin{array}{ccc}
\text{Spec}(k) \times_{k(t)^W} k(g) & \longrightarrow & k(g) \\
\downarrow & & \downarrow \\
\text{Spec}(k) \times_{k(t)^W} k(g) & \longrightarrow & k(g)^G.
\end{array}$$

\hfill \Box

□
where the top row is the \( G \)-equivariant isomorphism (4), and the bottom row is obtained from the top by taking \( G \)-invariants. Note that

\[
k(G/T \times_{\Spec(k)} t) \simeq K((G/T)_K \times_{\Spec(K)} \Spec L),
\]

where \( \simeq \) denotes a \( G \)-equivariant isomorphism of fields. (Recall that \( G \) acts trivially on \( t \) and hence also on \( L \) and \( K \).) Thus the field extension on the left side of our diagram can be rewritten as \( K(\zeta(G_K/T_K))/K \), where \( \zeta \) is the \( W \)-torsor \( \Spec(L) \rightarrow \Spec(K) \). By assumption, this field extension is purely transcendental; the diagram shows it is isomorphic to \( k(g)/k(g)^G \). \( \square \)

For the second reduction, we return to the assumptions of Section III.

**Proposition 5.** Let \( G \) be a split simple group of type \( G_2 \), with maximal torus \( T \) and Weyl group \( W \), and let \( Q \) be the quadric defined in Section III. Suppose that for a given \( W \)-model \( Y \) of \( k(Q)^T \), and for some \( W \)-torsor \( \zeta \) over some field \( K/k \), the twisted variety \( \zeta(Y_K) \) is rational over \( K \). Then the twisted variety \( \zeta(G_K/T_K) \) is rational over \( K \).

**Proof.** For the purpose of this proof, we may view \( K \) as a new base field and replace it with \( k \).

We claim that the left action of \( P \) on \( G/T \) is generically free. Since \( G \) has trivial center, the (characteristic-free) argument at the beginning of the proof of [5, Lemma 9.1] shows that in order to establish this claim it suffices to show that the right \( T \)-action on \( Q = P^7G \) is generically free. The latter action, given by restricting the linear action (1) of \( T \) on \( \mathbb{P}^6 \) to the quadric \( Q \) given by (2), is clearly generically free.

Let \( Y \) be a \( W \)-model. The \( W \)-equivariant rational map \( G/T \dashrightarrow Y \) induced by the projection \( G \rightarrow P\backslash G = Q \) is a rational quotient map for the left \( P \)-action on \( G/T \); cf. [5, p. 458]. Since the \( P \)-action is generically free, this map is a \( P \)-torsor over the generic point of \( Y \); see [3, Theorem 4.7]. By the functoriality of the twisting operation, after twisting by a \( W \)-torsor \( \zeta \), we obtain a rational map \( \zeta(G/T) \dashrightarrow \zeta Y \), which is a \( P \)-torsor over the generic point of \( \zeta Y \). This torsor has a rational section, since \( P \) is special; see Lemma 3. In particular, \( \zeta(G/T) \) is \( k \)-birationally isomorphic to \( P \times \zeta Y \). Since \( P \) is \( k \)-rational (once again, by Lemma 3), \( \zeta(G/T) \) is rational over \( \zeta Y \). Since \( \zeta Y \) is rational over \( k \), we conclude that so is \( \zeta(G/T) \), as desired. \( \square \)

It remains to show that the hypothesis of Proposition 5 holds. As before, we may replace the field \( K \) with \( k \). The following lemma completes the proof of Theorem 2.

**Lemma 6.** Let \( Y \) be a \( W \)-model. The twisted variety \( \zeta Y \) is rational over \( k \), for every \( W \)-torsor \( \zeta \) over \( k \).

**Proof.** We begin by constructing an explicit \( W \)-model. The affine open subset \( Q^{\text{aff}} = \{ x_1x_7 + x_2x_6 + x_3x_5 + 1 = 0 \} \subset \mathbb{A}^6 \) (where \( X_4 \neq 0 \)) is \( N(T) \)-invariant. Here the affine coordinates on \( \mathbb{A}^6 \) are \( x_i := X_i/X_4 \), for \( i \neq 4 \). The field of rational functions invariant for the \( T \)-action on \( Q^{\text{aff}} \) is \( k(y_1, y_2, y_3, z_1, z_2) \), where the variables

\[
y_1 = x_1x_7, \quad y_2 = x_2x_6, \quad y_3 = x_3x_5, \quad z_1 = x_1x_5x_6, \quad \text{and} \quad z_2 = x_2x_3x_7
\]

are subject to the relations \( y_1 + y_2 + y_3 + 1 = 0 \) and \( y_1y_2y_3 = z_1z_2 \). Thus we may choose as a \( W \)-model the affine subvariety \( \Lambda_1 \) of \( \mathbb{A}^5 \) given by these two equations, where \( W = S_2 \times S_3 \) acts on the coordinates as follows: \( S_2 \) permutes \( z_1, z_2 \), and \( S_3 \) permutes \( y_1, y_2, y_3 \). (Recall the \( W \)-action defined in Section III, and note that the field \( k(Q) \) is recovered by adjoining the classes of variables \( x_1 \) and \( x_2 \).) We claim that \( \Lambda_1 \) is \( W \)-equivariantly birationally isomorphic to

\[
\begin{align*}
\Lambda_2 &= \{ (Y_1 : Y_2 : Y_3 : Z_0 : Z_1 : Z_2) : Y_1 + Y_2 + Y_3 + Z_0 = 0 \text{ and } Y_1Y_2Y_3 = Z_1Z_2Z_0 \} \subset \mathbb{P}^5, \\
\Lambda_3 &= \{ (Y_1 : Y_2 : Y_3 : Z_1 : Z_2) : Y_1Y_2Y_3 + (Y_1 + Y_2 + Y_3)Z_1Z_2 = 0 \} \subset \mathbb{P}^4, \text{ and} \\
\Lambda_4 &= \{ (Y_1 : Y_2 : Y_3 : Z_1 : Z_2) : Z_1Z_2 + Y_2Y_3 + Y_1Y_3 + Y_1Y_2 = 0 \} \subset \mathbb{P}^4,
\end{align*}
\]
where $W$ acts on the projective coordinates $Y_1, Y_2, Y_3, Z_1, Z_2, Z_0$ as follows: $S_2$ permutes $Z_1, Z_2, S_3$ permutes $Y_1, Y_2, Y_3$, and every element of $W$ fixes $Z_0$. Note that $\Lambda_2 \subset \mathbb{P}^5$ is the projective closure of $\Lambda_1 \subset \mathbb{A}^5$; hence, using $\simeq$ to denote $W$-equivariant birational equivalence, we have $\Lambda_1 \simeq \Lambda_2$. The isomorphism $\Lambda_2 \simeq \Lambda_3$ is obtained by eliminating $Z_0$ from the system of equations defining $\Lambda_2$. Finally, the isomorphism $\Lambda_3 \simeq \Lambda_4$ comes from the Cremona transformation $\mathbb{P}^4 \dashrightarrow \mathbb{P}^4$ given by $Y_i \rightarrow 1/Y_i$ and $Z_j \rightarrow 1/Z_j$ for $i = 1, 2, 3$ and $j = 1, 2$.

Let $\zeta$ be a $W$-torsor over $k$. It remains to be shown that $\zeta \Lambda_4$ is $k$-rational. Since $\Lambda_4$ is a $W$-equivariant quadric hypersurface in $\mathbb{P}^4$, and the $W$-action on $\mathbb{P}^4$ is induced by a linear representation $W \rightarrow \text{GL}_5$, Hilbert’s Theorem 90 tells us that $\zeta \mathbb{P}^4$ is $k$-isomorphic to $\mathbb{P}^4$, and $\zeta \Lambda_4$ is isomorphic to a quadric hypersurface in $\mathbb{P}^4$ defined over $k$; see [7, Lemma 10.1]. It is easily checked that $\Lambda_4$ is smooth over $k$, and therefore so is $\zeta \Lambda_4$. The zero-cycle of degree 3 given by $(1 : 0 : 0 : 0 : 0) + (0 : 0 : 1 : 0 : 0)$ in $\Lambda_4$ is $W$-invariant, so it defines a zero-cycle of degree 3 in $\zeta \Lambda_4$. By Springer’s theorem, the smooth quadric $\zeta \Lambda_4$ has a $k$-rational point, hence is $k$-rational.

**Acknowledgement.** We are grateful to J.-L. Colliot-Thélène for stimulating conversations.

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**Instituto Nacional de Matemática Pura e Aplicada**, Rio de Janeiro, RJ 22460-320 Brasil

*E-mail address:* dave@impa.br

**Institut de Mathématiques de Jussieu**, Université Paris 6, place Jussieu, 75005 Paris, France

*E-mail address:* florence@math.jussieu.fr

**Department of Mathematics, University of British Columbia**, BC, Canada V6T 1Z2

*E-mail address:* reichst@math.ubc.ca