A NEW FORMULA FOR THE GENERATING FUNCTION OF THE NUMBERS OF SIMPLE GRAPHS

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Abstract. By using an approach of the invariant theory we obtain a new formula for the ordinary generating function of the numbers of the simple graphs with n nodes.

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1. Introduction. Let \( a_{n,i} \) be the number of simple graphs with \( n \) vertices and \( k \) edges. Let

\[
g_n(z) = \sum_{i=0}^{m} a_{n,i} z^i, \quad m = \binom{n}{2},
\]

be the ordinary generating function for the sequence \( \{a_{n,i}\} \), the OIES sequence A008406. For the small \( n \) we have

\[
g_1(z) = 1, \quad g_2(z) = 1 + z, \quad g_3(z) = 1 + z + z^2 + z^3, \quad g_4(z) = 1 + z + 2z^2 + 3z^3 + 2z^4 + z^5 + z^6.
\]

An expression for \( g_n(z) \) in terms of group cycle index was found by Harary in \cite{1}. The result is based on Polya’s efficient method for counting graphs, see \cite{2} and \cite{3}. Let \( G \) be a permutation group acting on the set \( [n] := \{1, 2, \ldots, n\} \). It is well known that each permutation \( \alpha \) in \( G \) can be written uniquely as a product of disjoint cycles. Let \( j_i(\alpha) \) be the number of cycles of length \( i, 1 \leq i \leq n \) in the disjoint cycle decomposition of \( \alpha \). Then the cycle index of \( G \) denoted \( Z(G, s_1, s_2, \ldots, s_n) \), is the polynomial in the variables \( s_1, s_2, \ldots, s_n \) defined by

\[
Z(G, s_1, s_2, \ldots, s_n) = \frac{1}{|G|} \sum_{\alpha \in G} \prod_{i=0}^{n} s_i^{j_i(\alpha)}.
\]

Denote by \( [n]^{(2)} \) the set of 2-subsets of \( [n] \). Let \( S_n \) be a permutation group on the set \( [n] \). The pair group of \( S_n \), denoted \( S_n^{(2)} \) is the permutation group induced by \( S_n \) which acts on \( [n]^{(2)} \). Specifically, each permutation \( \sigma \in S_n \) induces a permutation \( \sigma' \in S_n^{(2)} \) such that for every element \( \{i, j\} \in [n]^{(2)} \) we have \( \sigma'\{i, j\} = \{\sigma i, \sigma j\} \).

In \cite{1} F. Harary proved that the generating function \( g_n(z) \) is determined by substituting \( 1 + z^k \) for each variable \( s_k \) in the cycle index \( Z(S_n^{(2)}, s_1, s_2, \ldots, s_n) \). Symbolically

\[
g_n(z) = Z(S_n^{(2)}, 1 + z),
\]
where
\[ Z(S_n^{(2)}) = \frac{1}{n!} \sum_{j_1+j_2+\ldots+j_n=n} \frac{n!}{\prod_{k=1}^{k} k^{j_k+j_k!}} \prod_{r<t} (s_k s_{2k}^{k-1})^{j_{2k}} s_k^{k(j_k)} \prod_{r<t} s_{[r,t]}^{(r,t)^{j_r+j_t}}. \]

In the paper by an approach of the invariant theory we derive another formula for the generating function \( g_n(z) \).

Let \( \mathcal{V}_n \) be a vector space of weighted graphs on \( n \) vertices over the field \( \mathbb{K} \), \( \dim \mathcal{V}_n = m \). The group \( S_n^{(2)} \) acts naturally on \( \mathcal{V}_n \) by permutations of the basic vectors. Consider the corresponding action of the group \( S_n^{(2)} \) on the algebra of polynomial functions \( \mathbb{K}[\mathcal{V}_n] \) and let \( \mathbb{K}[\mathcal{V}_n]^{S_n^{(2)}} \) be the corresponding algebra of invariants. Let \( \mathcal{V}_n^0 \) be the set of simple graphs. The corresponding algebra of invariants \( \mathbb{K}[\mathcal{V}_n]^{S_n^{(2)}} \) is a finite-dimensional vector space and can be expanded into the direct sum of its subspaces:
\[ \mathbb{K}[\mathcal{V}_n]^0 = (\mathbb{K}[\mathcal{V}_n]^0)_0 + (\mathbb{K}[\mathcal{V}_n]^0)_1 + \cdots + (\mathbb{K}[\mathcal{V}_n]^0)_m. \]

In the paper we have proved that \( \dim(\mathbb{K}[\mathcal{V}_n]^0)_1 = a_{n,1} \). Thus the generating function \( g_n(z) \) coincides with the Poincaré series \( \mathcal{P}(\mathbb{K}[\mathcal{V}_n]^{S_n^{(2)}}(z), z) \) of the algebra invariants \( \mathbb{K}[\mathcal{V}_n]^{S_n^{(2)}} \).

Let us identify the elements of the group \( S_n^{(2)} \) with permutation \( m \times m \) matrices and denote \( I_m \) the identity \( m \times m \) matrix. In the paper we offer the following formula for the generating function \( g_n(z) \):
\[ g_n(z) = \frac{1}{n!} \sum_{\alpha \in S_n^{(2)}} \frac{\det(I_m - \alpha \cdot z^2)}{\det(I_m - \alpha \cdot z)}. \]

Also for the generating function \( m_n(z) \) of multigraphs on \( n \) vertices we prove that
\[ m_n(z) = \frac{1}{n!} \sum_{\alpha \in S_n^{(2)}} \frac{1}{\det(I_m - \alpha \cdot z)}. \]

### 2. Algebra of invariants of simple graphs.

Let \( \mathbb{K} \) be a field of characteristic zero. Denote by \( \mathcal{V}_n \) the set of undirected graphs on the vertices \( \{1, \ldots, n\} \) and whose edges are weighted in \( \mathbb{K} \). A simple graph is a graph with weights in \( \{0, 1\} \) and a multigraph is a graph with weights in \( \mathbb{K} \). For any pair \( \{i, j\} \) let \( e_{\{i,j\}} \) be the simple graph with one single edge \( \{i, j\} \) and let \( g_{\{i,j\}} e_{\{i,j\}} \) be the graph with one single edge \( \{i, j\} \) and with the weight \( g_{\{i,j\}} \in \mathbb{K} \). The set \( \mathcal{V}_n \) is the vector space with the basis \( \{e_{\{1,2\}}, e_{\{1,3\}}, \ldots, e_{\{n-1,n\}}\} \) of dimension \( m = \binom{n}{2} \). Indeed, any graph can be written uniquely as a sum \( \sum g_{\{i,j\}} e_{\{i,j\}} \). Let \( \mathcal{V}_n^* \) be the dual space with dual basis generated by the linear functions \( x_{\{i,j\}} \) for which \( x_{\{i,j\}}(e_{\{k,l\}}) = \delta_{ik}\delta_{jl} \). The symmetric group \( S_n \) acts on \( \mathcal{V}_n \) and on \( \mathcal{V}_n^* \) by
\[ \sigma e_{\{i,j\}} = e_{\{\sigma(i), \sigma(j)\}}, \sigma^{-1} x_{\{i,j\}} = x_{\{\sigma(i), \sigma(j)\}}. \]

Let us expand the action on the algebra of polynomial functions \( \mathbb{K}[\mathcal{V}_n] = \mathbb{K}[\{x_{\{i,j\}}\}] \).

We say that a polynomial function \( f \in \mathbb{K}[x_{\{i,j\}}] \) of \( m \) variables \( x_{\{i,j\}} \) is a \( S_n \)-invariant if \( \sigma f = f \) for all \( \sigma \in S_n \). The \( S_n \)-invariants form a subalgebra \( \mathbb{K}[\mathcal{V}_n]^{S_n} \) which is called the algebra of invariants of the vector space of the weighted graphs in \( n \) vertices. It is clear that there is an isomorphism \( \mathbb{K}[\mathcal{V}_n]^{S_n} \cong \mathbb{K}[x_{\{i,j\}}]^{S_n} \).

For convenience, we introduce a new set of variables:
\[ \{x_1, x_2, \ldots, x_m\} = \{x_{\{1,2\}}, x_{\{1,3\}}, \ldots, x_{\{n-1,n\}}\}. \]
Then the action of $S_n$ on the set \{$x_{\{1,2\}}, x_{\{1,3\}}, \ldots, x_{\{n-1,n\}}$\} induces its action of the pair group $S_n^{(2)}$ on the set \{$x_1, x_2, \ldots, x_m$\}. We have

$$\mathbb{K}[x_{\{i,j\}}]^{S_n} \cong \mathbb{K}[x_1, x_2, \ldots, x_m]^{S_n^{(2)}}.$$ 

In this notation any graph can be written in the way

$$g_1 e_1 + g_2 e_2 + \cdots + g_m e_m, g_i \in \mathbb{K},$$

where $e_s$ is the edge which connect the vertices \{i', j'\} if the pair \{i', j'\} has got the number $s$. Thus, in this case the old variable $x_{\{i', j'\}}$ corresponds to the new variable $x_s$.

Since for the simple graphs all its weights are 0, 1 then the reduction of the algebra $\mathbb{K}[\mathcal{V}_n]^{S_n}$ on the set of simple graphs has a simple structure.

Denote by $\mathcal{V}_n^0$ the set of all simple graphs on $n$ vertices:

$$\mathcal{V}_n^0 = \left\{ \sum_{i=0}^m g_i e_i \mid g_i \in \{0, 1\} \right\} \subset \mathcal{V}_n,$$

The corresponding subalgebra of polynomial function $\mathbb{K}[\mathcal{V}_n^0] \subset \mathbb{K}[\mathcal{V}_n]$ is generated by polynomial functions $x_i$ which on every simple graph only take values 1 or 0.

Let us consider the ideal $I_m = (x_1^2 - x_1, x_2^2 - x_2, \ldots, x_m^2 - x_m)$ in the algebra $\mathbb{K}[\mathcal{V}_n] = \mathbb{K}[x_1, x_2, \ldots, x_m]$. The following statement golds:

**Theorem 1.**

(i) \quad $\mathbb{K}[\mathcal{V}_n^0] \cong \mathbb{K}[x_1, x_2, \ldots, x_m]/I_m$,

(ii) \quad $\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}} \cong \mathbb{K}[x_1, x_2, \ldots, x_m]^{S_n^{(2)}}/I_m$.

**Proof.** On the ring of polynomial functions $\mathbb{K}[x_1, x_2, \ldots, x_m]$ let us introduce a binary relation $\sim$: $f \sim g$ if $f = g$, considered as functions from $\{0, 1\}^m$ to $\mathbb{K}$. Obviously that $x_i^p \sim x_i$, for all $p \geq 1$. Define an endomorphism $\gamma : \mathbb{K}[\mathcal{V}_n] \to \mathbb{K}[\mathcal{V}_n^0]$ by the way:

$$\gamma(x_i^p) = x_i.$$

It is clear that the kernel of the endomorphism is exactly the ideal $I_m$. Then

$$\mathbb{K}[\mathcal{V}_n^0] \cong \mathbb{K}[x_1, x_2, \ldots, x_m]/I_m.$$ 

Note that the algebra $\mathbb{K}[\mathcal{V}_n^0]$ is a finite dimensional vector space of the dimension $2^m$ with the basis

$$1, x_1, x_2, \ldots, x_n, x_1 x_2, x_1 x_2, \ldots, x_{m-1} x_m, \ldots, x_1 x_2 \cdots x_m.$$ 

(iii) It is enough to prove that $\gamma$ commutes with the action of the group $S_n^{(2)}$. Without lost of generality it is sufficient to check on the monomials. For arbitrary element $\sigma \in S_n^{(2)}$ and for arbitrary monomial $x_1^{k_1} x_2^{k_2} \cdots x_s^{k_s}, s \leq m$ we have

$$\gamma(\sigma^{-1}(x_1^{k_1} x_2^{k_2} \cdots x_s^{k_s})) = \gamma(x_1^{k_1(\sigma(1))} x_2^{k_2(\sigma(2))} \cdots x_s^{k_s(\sigma(s))}) = x_1^{k_1(\sigma(1))} x_2^{k_2(\sigma(2))} \cdots x_s^{k_s(\sigma(s))} =$$

$$= \sigma^{-1}(x_1 x_2 \cdots x_s) = \sigma^{-1}(\gamma(x_1^{k_1} x_2^{k_2} \cdots x_s^{k_s})).$$

If we know the algebra of invariants $\mathbb{K}[\mathcal{V}_n]^{S_n^{(2)}}$ then we are able to find the algebra of invariants $\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}}$ of simple graphs. Indeed, if the invariants $f_1, f_2, \ldots, f_s$ generate the algebra $\mathbb{K}[\mathcal{V}_n]^{S_n^{(2)}}$ then the surjectivity of $\gamma$ implies that the invariants $\gamma(f_1), \gamma(f_2), \ldots, \gamma(f_s)$ generate the algebra $\mathbb{K}[\mathcal{V}_n^0]^{S_n^{(2)}}$.
Example. Let us consider the case \( n = 4 \). The algebra of invariants \( \mathbb{K}[\mathcal{V}_n]^{S_4(2)} \) is well known, see \cite{4}, and its minimal generating system consists of the following 9 invariants:

\[
R(x_1) = \frac{1}{6}(x_1 + x_2 + x_3 + x_4 + x_5 + x_6), R(x_1^2) = \frac{1}{6}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2),
\]

\[
R(x_1 x_6) = \frac{1}{3}(x_1 x_6 + x_2 x_5 + x_3 x_4), R(x_1^3) = \frac{1}{6}(x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3),
\]

\[
24R(x_1^2 x_2) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_1 + x_3^2 x_2 + x_1^2 x_5 + x_4^2 x_1 +
\]

\[
\quad + x_4^2 x_5 + x_5^2 x_1 + x_5^2 x_4 + x_2^2 x_4 + x_2^2 x_6 + x_4^2 x_2 + x_4^2 x_6 + x_6^2 x_2 + x_6^2 x_4 + x_3^2 x_5 +
\]

\[
\quad + x_3^2 x_6 + x_5^2 x_3 + x_5^2 x_6 + x_6^2 x_3 + x_6^2 x_5,
\]

\[
R(x_1 x_2 x_3) = \frac{1}{4}(x_1 x_2 x_3 + x_1 x_5 x_4 + x_2 x_6 x_4 + x_3 x_6 x_5),
\]

\[
R(x_1^4) = \frac{1}{6}(x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_5^4 + x_6^4), R(x_1^5) = \frac{1}{6}(x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 + x_6^5),
\]

\[
24R(x_1^3 x_2) = x_1^3 x_2 + x_1^3 x_3 + x_2^3 x_1 + x_2^3 x_3 + x_3^3 x_1 + x_3^3 x_2 + x_1^3 x_4 + x_3^3 x_5 + x_4^3 x_1 +
\]

\[
\quad + x_4^3 x_5 + x_5^3 x_1 + x_5^3 x_4 + x_3^3 x_4 + x_2^3 x_5 + x_4^3 x_2 + x_4^3 x_6 + x_6^3 x_2 + x_6^3 x_4 + x_3^3 x_5 +
\]

\[
\quad + x_3^3 x_6 + x_5^3 x_3 + x_5^3 x_6 + x_6^3 x_3 + x_6^2 x_5.
\]

Here

\[
R = \frac{1}{n!} \sum_{g \in S_n} g
\]

is the Reinfeldts group action averaging operator which is a projector from \( \mathbb{K}[\mathcal{V}_n] \) into \( \mathbb{K}[\mathcal{V}_n]^{S_4(2)} \). We have that

\[
\gamma(R(x_1^5)) = \gamma(R(x_1^4)) = \gamma(R(x_1^3)) = \gamma(R(x_1^2)) = R(x_1), \gamma(R(x_1^2 x_2)) = R(x_1 x_2).
\]

Therefore, the algebra of invariants \( \mathbb{K}[\mathcal{V}_n]^{S_4(2)} \) of simple graphs on \( n \) vertices is generated by the 4 invariants:

\[
R(x_1), R(x_1 x_6), R(x_1 x_2), R(x_1 x_2 x_3).
\]

So far, the algebra of invariants \( \mathbb{K}[\mathcal{V}_n]^{S_n(2)} \) is calculated only for \( n \leq 5 \), see \cite{5}.

3. The Poincaré series of the algebra \( \mathbb{K}[\mathcal{V}_n]^{S_n(2)} \). Let us consider the algebra \( \mathbb{K}[\mathcal{V}_n] \) as a vector space. Then the following decomposition into the direct sum of its subspaces holds:

\[
\mathbb{K}[\mathcal{V}_n] = (\mathbb{K}[\mathcal{V}_n])_0 + (\mathbb{K}[\mathcal{V}_n])_1 + \cdots + (\mathbb{K}[\mathcal{V}_n])_m,
\]

where \((\mathbb{K}[\mathcal{V}_n])_i\) is the vector space generated by the elements

\[
x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_m^{\varepsilon_m}, \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_m = i, \quad \text{where} \quad \varepsilon_k = 0 \text{ or } \varepsilon_k = 1.
\]

Also, for the algebra \( \mathbb{K}[\mathcal{V}_n]^{S_n(2)} \) the decomposition holds:

\[
\mathbb{K}[\mathcal{V}_n]^{S_n(2)} = (\mathbb{K}[\mathcal{V}_n]^{S_n(2)})_0 + (\mathbb{K}[\mathcal{V}_n]^{S_n(2)})_1 + \cdots + (\mathbb{K}[\mathcal{V}_n]^{S_n(2)})_m.
\]

Since, the Reynolds operator is a projector which save the degree of a polynomial then the component \((\mathbb{K}[\mathcal{V}_n]^{S_n(2)})_i\) is generated by the following elements

\[
R(x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_m^{\varepsilon_m}).
\]

Particularly, we have that \((\mathbb{K}[\mathcal{V}_n]^{S_n(2)})_0 = \mathbb{K}\). Also, the component \((\mathbb{K}[\mathcal{V}_n]^{S_n(2)})_m\) has the dimension 1 and it is generated by the polynomial \(R(x_1 x_2 \cdots x_m) = x_1 x_2 \cdots x_m\).
Let us now give an interpretation of \( \dim(\mathbb{K}[V_n^0]_{S_n^{(2)}}) \) in terms of the graph theory. The following important theorem holds.

**Theorem 2.** The dimension \( \dim(\mathbb{K}[V_n^0]_{S_n^{(2)}}) \) equal to the number of non-isomorphic simple graphs with \( n \) vertices and \( i \) edges.

**Proof.** The \( S_n^{(2)} \)-module \( (\mathbb{K}[V_n^0]/I_m)_i \) is generated by the monomials
\[
x_1^{\varepsilon_1}x_2^{\varepsilon_2} \cdots x_m^{\varepsilon_m}, \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_m = i, \varepsilon_k = 0 \text{ or } \varepsilon_k = 1.
\]
Since the group \( S_n^{(2)} \) is finite then the module \( (\mathbb{K}[V_n^0]/I_m)_i \) is decomposed into the direct sum of its irreducible \( S_n^{(2)} \)-submodules:
\[
(\mathbb{K}[V_n^0]_{S_n^{(2)}})_i = M_1 \oplus M_2 \oplus \cdots \oplus M_p.
\]
Each of these submodules has a basis generated by the monomials of the form \( x_1^{\varepsilon_1}x_2^{\varepsilon_2} \cdots x_m^{\varepsilon_m} \). Let us choose for each \( M_i \) the corresponding basis monomials \( m_1, m_2, \ldots, m_p \) and consider the invariants \( R(m_1), R(m_2), \ldots, R(m_p) \). By the construction they are different and linearly independent. Therefore the component \( (\mathbb{K}[V_n^0]_{S_n^{(2)}})_i \) is the sum of one-dimensional \( S_n^{(2)} \)-submodules
\[
(\mathbb{K}[V_n^0]_{S_n^{(2)}})_i = \langle R(m_1) \rangle + \langle R(m_2) \rangle + \cdots + \langle R(m_p) \rangle,
\]
and \( \dim(\mathbb{K}[V_n^0]_{S_n^{(2)}})_i = p \) for some \( p \). To each of monomial \( m_1, m_2, \ldots, m_p \) assign a simple graph in the following way: if \( m_i = x_1^{\varepsilon_1}x_2^{\varepsilon_2} \cdots x_m^{\varepsilon_m} \) then the corresponding simple graph has the form
\[
G_{m_i} = \varepsilon_1 e_1 + \varepsilon_2 e_2 + \cdots + \varepsilon_m e_m.
\]
Since the monomials \( m_1, m_2, \ldots, m_p \) belong to the different irreducible \( S_n^{(2)} \)-modules then \( G_{m_i} \) are non-isomorphic and they exhausted all the possible classes of isomorphic classes of isomorphic graphs with \( n \) vertices and \( i \) edges.

Let us recall that the ordinary generating function for the sequence \( \dim(\mathbb{K}[V_n^0]_{S_n^{(2)}})_i \)
\[
\mathcal{P}(\mathbb{K}[V_n^0]_{S_n^{(2)}}, z) = \sum_{i=0}^{m} \dim(\mathbb{K}[V_n^0]_{S_n^{(2)}})_i \cdot z^i.
\]
is called the Poincaré series of the algebra \( \mathbb{K}[V_n^0]_{S_n^{(2)}} \).

The Theorem 2 implies that
\[
g_n(z) = \mathcal{P}(\mathbb{K}[V_n^0]_{S_n^{(2)}}, z).
\]
In the following theorem we derived explicit formulas for the series \( \mathcal{P}(\mathbb{K}[V_n^0]_{S_n^{(2)}}, z) \) and \( \mathcal{P}(\mathbb{K}[V_n]_{S_n^{(2)}}, z) \).

**Theorem 3.** Let the group \( S_n^{(2)} \) be realized as \( m \times m \) matrices. Then
\[
(i) \quad \mathcal{P}(\mathbb{K}[V_n^0]_{S_n^{(2)}}, z) = \frac{1}{n!} \sum_{\alpha \in S_n^{(2)}} \frac{\det(1_m - \alpha \cdot z^2)}{\det(1_m - \alpha \cdot z)},
\]
\[
(ii) \quad \mathcal{P}(\mathbb{K}[V_n]_{S_n^{(2)}}, z) = \frac{1}{n!} \sum_{\alpha \in S_n^{(2)}} \frac{1}{\det(1_m - \alpha \cdot z)}.
\]
here \( 1_m \) is the unit \( m \times m \) matrix.
Proof. (i) The vector space $\mathbb{K}[V^0_{n}]_1$ has the basis $\langle x_1, x_2, \ldots, x_m \rangle$ and a permutation $\alpha \in S_n^{(2)}$ acts on the $(\mathbb{K}[V^0_{n}]_1)$ by permutation of the basis vectors. Denote this linear operator by $A_\alpha$.

Let us expand this operator on the component $(\mathbb{K}[V^0_{n}]^{S_n^{(2)}})_k$ as endomorphism and denote it by $A_\alpha^{(k)}$. Since $A_\alpha^{(k)}$ is endomorphism and acts as a permutation of the basis vectors of $(\mathbb{K}[V^0_{n}]^{S_n^{(2)}})_k$ then the action of the operator $A_\alpha^{(k)}$ is defined correctly.

Let a permutation $\alpha$ be written uniquely as a product of disjoint cycles and let $j_i(\alpha)$ be the number of cycles of length $i$ in the disjoint cycle decomposition of $\alpha$.

Now find the track of the operator $A_\alpha^{(k)}$.

**Lemma 1.**

$$\text{Tr}(A^{(i)}_\alpha) = \sum_{\beta_1 + 2\beta_2 + \cdots + m\beta_m = i} \binom{(j_1(\alpha))}{\beta_1} \binom{(j_2(\alpha))}{\beta_2} \cdots \binom{(j_i(\alpha))}{\beta_i}.$$

**Proof.** Since the operator $A_\alpha^{(i)}$ acts by permutations of the basis vectors of the vector space $(\mathbb{K}[V^0_{n}]_i)$ that its track equal to the numbers of its fixed point.

For $i = 1$ we have $(\mathbb{K}[V^0_{n}]_1) = \langle x_1, x_2, \ldots, x_m \rangle$ and $A_\alpha^{(1)}(x_s) = x_{\alpha^{-1}(s)}$. Thus $\text{Tr}(A^{(1)}_\alpha) = j_1(\alpha)$.

For $i = 2$ let us find out the number of fixed points of the operator $A_\alpha^{(2)}$ which acts on the vector space $(\mathbb{K}[V^0_{n}]_2)$ with the basis vectors $x_i x_j, i < j$. An arbitrary pair of fixed points of the operator $A_\alpha$ form one fixed point for the operator $A_\alpha^{(2)}$. Thus we get $(j_2(\alpha))$ such points. Also, every transposition define one fixed point. Therefore

$$\text{Tr}(A_\alpha^{(2)}) = \binom{(j_1(\alpha))}{2} + j_2(\alpha).$$

All $j_1(\alpha)$ fixed points of the permutation $\alpha$ generates $(j_1(\alpha))$ fixed points of the operator $A_\alpha^{(3)}$. Every fixed point of $A_\alpha$ together with $j_2(\alpha)$ transposition generate one fixed point of $A_\alpha^{(3)}$. At last, each any of 3-cycle of $\alpha$ generates one fixed point for $A_\alpha^{(3)}$. Then

$$\text{Tr}(A_\alpha^{(3)}) = \binom{(j_1(\alpha))}{3} + j_1(\alpha)j_2(\alpha) + j_3(\alpha).$$

Analogously

$$\text{Tr}(A_\alpha^{(4)}) = \binom{(\alpha_1)}{4} + \binom{(\alpha_1)}{2}\alpha_2 + \binom{(\alpha_2)}{2} + \alpha_1\alpha_3 + \alpha_4.$$

In the general case any partition of $i$

$$\beta_1 + 2\beta_2 + \cdots + m\beta_m = i.$$

generates

$$\binom{(j_1(\alpha))}{\beta_1} \binom{(j_2(\alpha))}{\beta_2} \cdots \binom{(j_m(\alpha))}{\beta_m}$$

fixed points of the operator $A_\alpha^{(i)}$. Therefore

$$\text{Tr}(A_\alpha^{(i)}) = \sum_{\beta_1 + 2\beta_2 + \cdots + m\beta_m = i} \binom{(j_1(\alpha))}{\beta_1} \binom{(j_2(\alpha))}{\beta_2} \cdots \binom{(j_m(\alpha))}{\beta_m}.$$

\[\square\]

**Lemma 2.**

$$\sum_{i=0}^{m} \text{Tr}(A^{(i)}_\alpha)z^i = (1 + z)^{j_1(\alpha)}(1 + z^2)^{j_2(\alpha)} \cdots (1 + z^m)^{j_m(\alpha)}.$$
Proof. We have

\[
\sum_{i=0}^{m} \text{Tr}(A_{\alpha}^{(i)}) z^i = \sum_{\beta_1 + 2\beta_2 + \cdots + m\beta_m = i} \left( \frac{j_1(\alpha)}{\beta_1} \right) \left( \frac{j_2(\alpha)}{\beta_2} \right) \cdots \left( \frac{j_m(\alpha)}{\beta_i} \right) z^i =
\]

\[
= \sum_{\beta_1 + 2\beta_2 + \cdots + m\beta_m = i} \left( \frac{j_1(\alpha)}{\beta_1} \right) \left( \frac{j_2(\alpha)}{\beta_2} \right) \cdots \left( \frac{j_m(\alpha)}{\beta_i} \right) z^{\beta_1 + 2\beta_2 + \cdots + m\beta_m} =
\]

\[
= \sum_{\beta_1 + 2\beta_2 + \cdots + m\beta_m = i} \left( \frac{j_1(\alpha)}{\beta_1} \right) z^{\beta_1} \left( \frac{j_2(\alpha)}{\beta_2} \right) (z^2)^{\beta_2} \cdots \left( \frac{j_m(\alpha)}{\beta_i} \right) (z^m)^{\beta_m} =
\]

\[
= \left( \sum_{\beta_1 = 0}^{\sum_{i=1}^{m} j_i(\alpha)} \frac{j_1(\alpha)}{\beta_1} \right) \left( \sum_{\beta_2 = 0}^{j_2(\alpha)} \frac{j_2(\alpha)}{\beta_2} \right) (z^2)^{\beta_2} \cdots \left( \sum_{\beta_m = 0}^{j_m(\alpha)} \frac{j_m(\alpha)}{\beta_m} \right) (z^m)^{\beta_m} =
\]

\[
= (1 + z)^{j_1(\alpha)} (1 + z^2)^{j_2(\alpha)} \cdots (1 + z^m)^{j_m(\alpha)}.
\]

Lemma 3.

\[
\sum_{i=0}^{m} \text{Tr}(A_{\alpha}^{(i)}) z^i = \frac{\det(1_m - A_{\alpha} \cdot z^2)}{\det(1_m - A_{\alpha} \cdot z)}.
\]

Proof. Let \(\lambda_1, \lambda_2, \ldots, \lambda_m\) be the eigenvalues of the operator \(A_{\alpha}\). Since \(A_{\alpha}^m\) is the identity matrix then all eigenvalues \(\lambda_i\) are roots of unity of orders \(j_1(\alpha), j_2(\alpha), \ldots, j_m(\alpha)\). Therefore the characteristic polynomial of the operator \(A_{\alpha}\) has the form

\[
\det(1_m - A_{\alpha} z) = (1 - \lambda_1 z)(1 - \lambda_2 z) \cdots (1 - \lambda_n z) = (1 - z)^{j_1(\alpha)} (1 - z^2)^{j_2(\alpha)} \cdots (1 - z^m)^{j_m(\alpha)}.
\]

Now

\[
\sum_{i=0}^{m} \text{Tr}(A_{\alpha}^{(i)}) z^i = (1 + z)^{j_1(\alpha)} (1 + z^2)^{j_2(\alpha)} \cdots (1 + z^m)^{j_m(\alpha)} =
\]

\[
= \frac{(1 - z)^{j_1(\alpha)}}{(1 - z)^{j_1(\alpha)}} \frac{(1 - z^4)^{j_2(\alpha)}}{(1 - z^2)^{j_2(\alpha)}} \cdots \frac{(1 - z^{2m})^{j_m(\alpha)}}{(1 - z^m)^{j_m(\alpha)}} = \frac{\det(1_m - A_{\alpha} \cdot z^2)}{\det(1_m - A_{\alpha} \cdot z)}.
\]

Lemma 4.

\[
\dim(\mathbb{K}[\mathcal{V}^0_n]_{S^{(2)}_n})_i = \frac{1}{n!} \sum_{\alpha \in G} \text{Tr}(A_{\alpha}^{(i)}).
\]

Proof. The dimension of the subspace \((\mathbb{K}[\mathcal{V}^0_n]_{S^{(2)}_n})_i\) is equal to the number of eigenvectors that correspond to the eigenvalue 1 and which are common eigenvectors for all operators \(A_{\alpha}^{(i)}\). Consider the average matrix

\[
P^{(i)} = \frac{1}{|G|} \sum_{g \in G} A_{\alpha}^{(i)}.
\]

Since the Reynolds operator is a projector from \((\mathbb{K}[\mathcal{V}^0_n])_i\) into \((\mathbb{K}[\mathcal{V}^0_n]_{S^{(2)}_n})_i\), then it has the only eigenvalues 1 and 0. Therefore the dimension of the space \((\mathbb{K}[\mathcal{V}^0_n]_{S^{(2)}_n})_i\) is equal to the track of the matrix \(P^{(i)}\).
Taking into account the lemmas stated above we have

$$\mathcal{P}(\mathbb{K}[V^0_n]S_n^{(2)}, z) = \sum_{i=0}^{m} \dim(\mathbb{K}[V^0_n]S_n^{(2)})_i z^i = \frac{1}{n!} \sum_{i=0}^{m} \left( \sum_{\alpha \in S_n^{(2)}} \text{Tr}(A^{(i)}_{\alpha}) \right) z^i =$$

$$= \frac{1}{n!} \sum_{g \in S_n^{(2)}} \left( \sum_{i=0}^{m} \text{Tr}(A^{(i)}_{g}) z^i \right) = \frac{1}{n!} \sum_{\alpha \in S_n^{(2)}} \frac{\det(1_m - \alpha \cdot z^2)}{\det(1_m - \alpha \cdot z)}.$$

(ii) It is the Molien formula for the Poincaré series of the algebra invariants of the group $S_n^{(2)}$. 

REFERENCES

[1] Harary F., The number of linear, directed, rooted, and connected graphs, Trans. Amer. Math. Soc. 78 (1955), 445-463
[2] Pólya, G. Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen, Acta Math 68, 145-254(1937)
[3] Harary F., Palmer E., Graphical Enumeration. Academic Press, New York, 1973, 271 p.
[4] Aslaksen H., Chan S.-P., Gulliksen T., Invariants of $S_4$ and the Shape of Sets of Vectors, Appl. Algebra Eng. Commun. Comput. 7, No.1, 53-57 (1996).
[5] Thiéry N., Algebraic invariants of graphs; a study based on computer exploration. ACM SIGSAM Bulletin, 2000, 34 (3), pp.9-20.

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