HOMOTOPY EQUIVALENCES BETWEEN \( p \)-SUBGROUP CATEGORIES

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Abstract. Let \( S^*_G \) be the Brown poset of nonidentity \( p \)-subgroups of the finite group \( G \) ordered by inclusion. Results of Bouc and Quillen show that \( S^*_G \) is homotopy equivalent to its subposets \( S^{+\mathrm{rad}}_G \) of nonidentity radical \( p \)-subgroups and \( S^{+\mathrm{eab}}_G \) of nonidentity elementary abelian \( p \)-subgroups. In this note we extend these results for the Brown poset of \( G \) to other categories of \( p \)-subgroups of \( G \) such as the \( p \)-fusion system of \( G \).

1. INTRODUCTION

Let \( p \) be a prime number and \( G \) a finite group of order divisible by \( p \). Let \( S^*_G \) denote the Brown poset of nonidentity \( p \)-subgroups of \( G \), \( S^{+\mathrm{rad}}_G \) the Bouc poset of nonidentity radical \( p \)-subgroups, and \( S^{+\mathrm{eab}}_G \) the Quillen poset of nonidentity elementary abelian \( p \)-subgroups. It was shown by Bouc [1] and by Quillen [16, Proposition 2.1] that the poset inclusions

\[
S^{+\mathrm{rad}}_G \hookrightarrow S^*_G, \quad S^{+\mathrm{eab}}_G \hookrightarrow S^*_G
\]

are homotopy equivalences. In this note we consider similar inclusions between other categories of \( p \)-subgroups of \( G \). The list of categories that we deal with can be found after the following theorem summarizing our main results.

Theorem 1.2. The following inclusion functors

\begin{enumerate}[(a)]
   \item \( S^{+\mathrm{rad}}_G \hookrightarrow S^*_G \), \( S^{+\mathrm{eab}}_G \hookrightarrow S^*_G \)
   \item \( T^{+\mathrm{rad}}_G \hookrightarrow T^*_G \), \( T^{+\mathrm{eab}}_G \hookrightarrow T^*_G \)
   \item \( F^{+\mathrm{eab}}_G \hookrightarrow F^*_G \)
   \item \( O^{+\mathrm{eab}}_G \hookrightarrow O^*_G \)
   \item \( L^{+\mathrm{eab}}_G \hookrightarrow L^*_G \)
\end{enumerate}

are homotopy equivalences.

The categories that we consider in Theorem 1.2 are

\begin{align*}
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\end{align*}
\[ S_G : \text{the poset of all p-subgroups of } G \text{ ordered by inclusion} \]
\[ T_G : \text{the transporter category of all p-subgroups of } G \]
\[ L_G : \text{the linking category of all p-subgroups } G \]
\[ F_G : \text{the fusion or Frobenius category of all p-subgroups of } G \]
\[ O_G : \text{the orbit category of all p-subgroups of } G \]
\[ \tilde{F}_G : \text{the exterior quotient of the Frobenius category } F_G \]

Also, for a fixed nonidentity p-subgroup \( P \), we consider
\[ F : \text{a Frobenius } P\text{-category over } P \]
\[ \tilde{F} : \text{the exterior quotient of } F \]
\[ L^{sc} : \text{the centric linking system associated to } F \]

If \( C \) is any of these categories, then
- \( C^* \) is the full subcategory of \( C \) generated by all nonidentity p-subgroups
- \( C^{+eab} \) is the full subcategory of \( C \) generated by all nonidentity elementary abelian p-subgroups
- \( C^{sc} \) is the full subcategory of \( C \) generated by all selfcentralizing p-subgroups
- \( C^{sc+rad} \) is the full subcategory of \( C \) generated by all selfcentralizing radical p-subgroups

where the terms selfcentralizing and radical are defined in Subsection 1.2.

**Definition 1.3.** A functor \( C \to D \) between categories \( C \) and \( D \) is a homotopy equivalence of categories if the induced map of classifying spaces \( BC \to BD \) is a homotopy equivalence of topological spaces.

An equivalence of categories is a homotopy equivalence as is any left or right adjoint functor.

The following, maybe somewhat unexpected, corollary follows immediately from items (c) and (e) of Theorem 1.2.

**Corollary 1.4.** The quotient functor \( F^* \to \tilde{F}^* \) is a homotopy equivalence.

The proofs of Theorem 1.2 and Corollary 1.4 are in Sections 5–9. Using Quillen’s Theorem A (Theorem 3.1) and Bouc’s theorem for EI-categories (Theorem 3.3) to show that the inclusion functors of Theorem 1.2 are homotopy equivalences, we have to prove contractibility of certain slice and coslice subcategories (Definition 2.2) of p-subgroup categories. Since we use Euler characteristics to help us in identifying these slices and coslices, Section 2 contains a short review of Euler characteristics for categories as defined by Leinster [12]. The reader, who wants a general impression of our method of proof, may take a glance at Section 5, where we re-establish the already known results that the inclusions of (1.1) are homotopy equivalences.

We find it curious that somehow the combinatorics of the Frobenius category \( F_G \) is able to identify the elementary abelian subgroups so some of the group theory is remembered in an unexpected way. Likewise, the orbit category \( O_G \) is able to identify the G-radical and the cyclic subgroups. Finally, the exterior quotient \( \tilde{F}^{sc} \) of an abstract Frobenius category \( F \) is able to identify the \( F \)-radical subgroups.

1.1. **Subgroup categories.** This subsection contains precise definitions of the p-subgroup categories occurring in this paper. By convention, maps act on elements from the right, and composition of morphisms is written in diagrammatic order. Likewise, functors act on categories from the right.

If \( a \) and \( b \) are objects in a category \( C \),
- \( C(a, b) \) is the set of \( C \)-morphisms with domain \( a \) and codomain \( b \)
- \( C(a) \) is the monoid of \( C \)-endomorphisms of \( a \)

The poset \( S_G \) is the set of all p-subgroups of \( G \) ordered by inclusion. In other words, \( S_G \) is the category whose objects are all p-subgroups of \( G \) with one morphism \( H \to K \) whenever \( H \leq K \) and no morphisms otherwise. The objects of the finite categories \( T_G, L_G, F_G, \tilde{F}_G, \) and \( O_G \) are again all p-subgroups of \( G \). For any two p-subgroups, \( H \) and \( K \), of \( G \), the morphism sets are
\[
\begin{align*}
T_G(H, K) &= N_G(H, K) \\
L_G(H, K) &= O^pC_G(H)\setminus N_G(H, K) \\
F_G(H, K) &= C_G(H)\setminus N_G(H, K) \\
O_G(H, K) &= N_G(H, K)/K \\
\tilde{F}_G(H, K) &= C_G(H)\setminus N_G(H, K)/K
\end{align*}
\]

Here \( N_G(H, K) = \{g \in G \mid H^g \leq K\} \) denotes the transporter set. Composition in any of these categories is induced from group multiplication in \( G \). The morphisms in \( F_G(H, K) \) are restrictions to \( H \) of inner
autmorphisms of $G$, $\mathcal{F}_G(H,K) = \text{Hom}_G(H,K)$, morphisms in $\mathcal{O}_G(H,K)$ are right $G$-maps $H \backslash G \to K \backslash G$, and morphisms in $\tilde{\mathcal{F}}_G(H,K)$ are $K$-conjugacy classes of restrictions to $H$ of inner automorphisms of $G$, $	ilde{\mathcal{F}}_G(H,K) = \mathcal{F}_G(H,K)/\text{Inn}(K) = \text{Hom}_G(H,K)/\text{Inn}(K)$. The automorphism groups in these categories of the p-subgroup $H$ of $G$ are

\[
\begin{align*}
\mathcal{S}_G(H) &= 1, \\
\mathcal{F}_G(H) &= C_G(H) \backslash N_G(H), \\
\mathcal{O}_G(H) &= N_G(H)/H, \\
\mathcal{L}_G(H) &= O_p C_G(H) \backslash N_G(H), \\
\tilde{\mathcal{F}}_G(H) &= C_G(H) \backslash N_G(H)/H,
\end{align*}
\]

where $O_p C_G(H)$ is the smallest normal subgroup of $C_G(H)$ of $p$-power index. The six categories $\mathcal{S}_G$, $\mathcal{T}_G$, $\mathcal{L}_G$, $\mathcal{F}_G$, $\mathcal{O}_G$, and $\tilde{\mathcal{F}}_G$ are related by a commutative diagram

\[
\begin{tikzcd}
\mathcal{S}_G & \mathcal{T}_G & \mathcal{L}_G & \mathcal{F}_G & \tilde{\mathcal{F}}_G \\
\mathcal{O}_G & & & & \\
\end{tikzcd}
\]

of one faithful and five full functors.

Fix a finite nonidentity $p$-group $P$. A Frobenius $P$-category or saturated fusion system over $P$ is a category whose objects are the subgroups of $P$ and whose morphisms satisfy a set of axioms [14, 4] that distill the properties of the Frobenius categories $\mathcal{F}_G$ coming from a group $G$. There are examples of abstract Frobenius $P$-categories $\mathcal{F}$ that are exotic in the sense that there is no finite group $G$ with $\mathcal{F}_G$ equivalent to $\mathcal{F}$. The exterior quotient or orbit category $\tilde{\mathcal{F}}$ of $\mathcal{F}$ is the category whose objects are the subgroups of $P$ and with morphism sets

\[
\tilde{\mathcal{F}}(H,K) = \mathcal{F}(H,K)/\mathcal{F}_K(K)
\]

consisting of $\mathcal{F}$-morphisms up to inner automorphisms of the codomain. Composition in $\mathcal{F}$ induces composition in its quotient category $\tilde{\mathcal{F}}$.

1.2. Selfcentralizing and radical subgroups. For ease of reference we state here the definitions of $G$- and $\mathcal{F}$-selfcentralizing and $G$- and $\mathcal{F}$-radical subgroups as these concepts are used throughout this paper. As usual, $O_pK$ is the greatest normal $p$-subgroup of the finite group $K$ [9, Chp 6.3].

**Definition 1.5.** [14, 4.8.1] [4, Definition A.3] [1, Proposition 4] The $p$-subgroup $H$ of $G$ is

- $G$-selfcentralizing if the center $Z(H)$ of $H$ is a Sylow $p$-subgroup of the centralizer $C_G(H)$ of $H$;
- $G$-radical if $O_pO_G(H) = 1$, or, equivalently, $H = O_pN_G(H)$.

**Definition 1.6.** [14, 4.8] [4, Definition A.9] An object $H$ of $\mathcal{F}$ is

- $\mathcal{F}$-selfcentralizing if $C_{\mathcal{F}}(H^\varphi) \leq H^\varphi$ for every $\mathcal{F}$-morphism $\varphi \in \mathcal{F}(H,P)$ with domain $H$
- $\mathcal{F}$-radical if $O_p\tilde{\mathcal{F}}(H) = 1$

Every $G$-selfcentralizing subgroup of $G$ is nontrivial, and every $\mathcal{F}$-selfcentralizing subgroup of $P$ is nontrivial.

Let $P$ be a Sylow $p$-subgroup of $G$ and $\mathcal{F}$ the full subcategory of $\mathcal{F}_G$ with objects all subgroups of $P$. For every subgroup $H$ of $P$

\[
H \text{ is } \mathcal{F}\text{-selfcentralizing } \iff \text{ H is } G\text{-selfcentralizing}
\]

$H$ is $\mathcal{F}$-selfcentralizing and $\mathcal{F}$-radical $\implies$ $H$ is $G$-radical

and the second implication can not be reversed.

According to Quillen [16, Proposition 2.4] we have that

\[
\text{\begin{small}
(1.7) \quad S^K_P \text{ is noncontractible } \implies O_pK = 1
\end{small}}
\]

for any finite group $K$. In the present context, this means that

\[
\text{\begin{small}
(1.8) \quad S_{G_0(H)} \text{ is noncontractible } \implies H \text{ is } G\text{-radical}
\end{small}}
\]

\[
\text{\begin{small}
(1.9) \quad S_{F(H)} \text{ is noncontractible } \implies H \text{ is } \mathcal{F}\text{-radical}
\end{small}}
\]

---

\[\text{\footnotesize
1\text{Define a product } H \backslash G \times N_G(H,K)/K \to K \backslash G \text{ by } Hg \cdot xK = Kx^{-1}g. \text{ This product is well-defined because } Hhg \cdot xkK = Kk^{-1}x^{-1}hg = Kx^{-1}hg = Khk^{-1}x^{-1}g = Kx^{-1}g = Hg \cdot xK \text{ when } h \in H, x \in N_G(H,K), k \in K. \text{ Right multiplication with } xK \text{ is clearly a right } G\text{-map } H \backslash G \to K \backslash G. \text{ If } y \in N_G(K,L) \text{ then } xy \in N_G(H,L) \text{ and } (Hg \cdot xK) \cdot yL = Kx^{-1}g \cdot yL = Ly^{-1}x^{-1}g = L(\text{xy})^{-1}g = Hg \cdot (\text{xy}L).} \]
Properties (1.8) and (1.9) will be very important in the proof of Theorem 1.2. (Quillen conjectures in [16, Conjecture 2.9] that the reverse implication of (1.7) is true. If Quillen’s conjecture holds, then also the reverse implications of (1.8) and (1.9) are true.)

2. Weightings and and coweightings for EI-categories

Let \( C \) be a finite category and \([C]\) the set of isomorphism classes of its objects. In this section we show that we can use strict coslice categories to define weightings in the sense of Leinster [12].

**Definition 2.1.** [12, Definitions 1.10, 2.2] A weighting for \( C \) is a function \( k^x_\bullet : \text{Ob}(C) \to \mathbb{Q} \) so that

\[ \forall a \in \text{Ob}(C) : \sum_{b \in \text{Ob}(C)} |C(a, b)| k^x_b = 1 \]

and a coweighting for \( C \) is a function \( k^C_\bullet : \text{Ob}(C) \to \mathbb{Q} \) so that

\[ \forall b \in \text{Ob}(C) : \sum_{a \in \text{Ob}(C)} k^C_a |C(a, b)| = 1 \]

If \( C \) has both a weighting and a coweighting, then the rational number

\[ \sum_{a \in \text{Ob}(C)} k^a_\bullet = \chi(C) = \sum_{b \in \text{Ob}(C)} k^b_\bullet \]

is the Euler characteristic of \( C \). The reduced Euler characteristic of \( C \) is \( \bar{\chi}(C) = \chi(C) - 1 \).

A general finite category may not admit a weighting or a coweighting or it may have several weightings or coweightings and in that case the Euler characteristic is independent of the choice of weighting or coweighting.

**Definition 2.2** (Coslice and slice categories). Let \( C \) be a category, \( A \) a full subcategory, and \( x, y \) objects of \( C \).

- \( x/A \) is the category of \( C \)-morphisms from \( x \) to an object of \( A \) (the coslice of \( A \) under \( x \))
- \( A/y \) is the category of \( C \)-morphisms from an object of \( A \) to \( y \) (the slice of \( A \) over \( y \))
- \( x//A \) is the full subcategory of \( x/A \) with objects all nonisomorphisms from \( x \) to an object of \( A \).
- \( A//y \) is the full subcategory of \( A/y \) with objects all nonisomorphisms from an object of \( A \) to \( y \).

We write

\[ \text{supp}(\bullet/A) = \{ x \in \text{Ob}(C) \mid x/A \text{ is noncontractible} \} \quad \text{supp}(A/\bullet) = \{ y \in \text{Ob}(C) \mid A/y \text{ is noncontractible} \} \]

\[ \text{supp}(\bullet//A) = \{ x \in \text{Ob}(C) \mid x//A \text{ is noncontractible} \} \quad \text{supp}(A//\bullet) = \{ y \in \text{Ob}(C) \mid A//y \text{ is noncontractible} \} \]

for the supports of the coslice functors \( \bullet/A, \bullet//A : C^{op} \to \text{CAT} \) and slice functors \( A/\bullet, A//\bullet : C \to \text{CAT} \).

The notation \( A//\bullet \) and \( \bullet//A \) is taken from [10, p 269].

**Lemma 2.3.** Let \( C \) be any finite category admitting a weighting \( k^x_\bullet : \text{Ob}(C) \to \mathbb{Q} \). Let \( a \) be any object of \( C \). The function

\[ k^{a/c}_\bullet = k^{\text{cod}(\bullet)}_C : \text{Ob}(a/C) \to \mathbb{Q}, \quad k^{a/c}_{a/b} = k^b_c \]

is a weighting for the coslice \( a/C \) of \( C \) under \( a \).

**Proof.** The set of objects of \( a/C \), which is the set of \( C \)-morphisms with domain \( a \), is partitioned

\[ \text{Ob}(a/C) = \coprod_{b \in \text{Ob}(C)} C(a, b) \]

according to codomains. Also, for any \( C \)-morphism \( \varphi \in C(a, b) \) with codomain \( b \) and any \( C \)-object \( c \), the set of \( C \)-morphisms from \( b \) to \( c \) is partitioned

\[ C(b, c) = \coprod_{\psi \in C(b, c)} (a/C)(\varphi, \psi) \]

into \( a/C \)-morphism sets with domain \( \varphi \). The computation

\[ \sum_{\psi \in \text{Ob}(a/C)} |a/C(\varphi, \psi)| k^{\text{cod}(\psi)}_C = \sum_{b \in \text{Ob}(C)} \sum_{\psi \in \text{Ob}(a/C)} |a/C(\varphi, \psi)| k^{\text{cod}(\psi)}_C = \sum_{b \in \text{Ob}(C)} |C(\text{cod}(\varphi), b)| k^b_c = 1 \]
shows that the function \(k^{\text{cod}}_C(\bullet)\) is a weighting on \(a/C\).

Here is a variation \([11, 2.4]\) of Definition 2.1. For any object \(a\) of \(C\), write \([a] \in [C]\) for the set of objects isomorphic to \(a\). Viewing \([C]\) as a skeletal category, the matrix for \([C]\) is the quadratic \(((|C| \times |C|))-\text{matrix}\) with entries \([C)(([a], [b]) = |C(a, b)|, [a], [b] \in [C]\). A \textit{weighting} for \([C]\) is a function \(k^\bullet_C : [C] \to \mathbb{Q}\) so that
\[
\forall [a] \in [C] : \sum_{[b] \in [C]} |C|(([a], [b])|k^\bullet_{[b]}_C = 1
\]
and a coweighting for \([C]\) is a function \(k^\bullet_{[b]} : [C] \to \mathbb{Q}\) so that
\[
\forall [a] \in [C] : \sum_{[b] \in [C]} k^\bullet_{[a]}([a], [b]) = 1
\]
The category \(C\) has a weighting \(k^\bullet_C\) if and only its set of isomorphism classes of objects \([C]\) has a weighting \(k^\bullet_{[C]}\). The relation between the two weightings are
\[
k^\bullet_{[b]} = \sum_{y \in [b]} k^\bullet_{[a]}, \quad k^\bullet_{[b]} = |[b]|^{-1} k^\bullet_{[b]}_{[C]}, \quad b \in \text{Ob}(C), [b] \in [C]
\]
Similarly, \(C\) has a coweighting if and only if \([C]\) has a coweighting. This construction in fact provides a bijection between (co)weightings for \(C\) that are constant on isomorphism classes of objects and (co)weightings for \([C]\).

An EI-category is a category where all endomorphisms are isomorphisms (automorphisms). Because the matrix for \([C]\) can be arranged to be upper triangular with positive diagonal entries, any finite EI-category \(C\) has a \textit{unique} weighting and a \textit{unique} coweighting that are constant on isomorphism classes of objects \([12, \text{Lemma 1.3}, \text{Theorem 1.4}, \text{Lemma 1.12}]\).

A full subcategory \(\mathcal{I}\) of a category \(C\) is a \textit{left ideal} if any \(\mathcal{C}\)-morphism whose domain is an object of \(\mathcal{I}\) is an \(\mathcal{I}\)-morphism. For instance, if \(C\) is an EI-category and \(a\) an object of \(C\) then \(a/\mathcal{C}\) is a left ideal in \(a/C\) by \([12, \text{Lemma 1.3}]\).

\textbf{Theorem 2.6.} Let \(C\) be a finite EI-category, and let \(k^\bullet_C\) and \(k^\bullet_{[C]}\) be the weighting and the coweighting on \(C\) that are constant on isomorphism classes of objects of \(C\). Then
\[
k^\bullet_C = -\frac{\chi(a//C)}{|[a]| |C(a)|}, \quad k^\bullet_{[b]} = -\frac{\chi(C//b)}{|[b]| |C(b)|}, \quad a, b \in \text{Ob}(C),
\]
and the Euler characteristic of \(C\) is
\[
\sum_{a \in [C]} -\frac{\chi(a//C)}{|C(a)|} = \chi(C) = \sum_{b \in [C]} -\frac{\chi(C//b)}{|C(b)|}
\]
where the sums run over the set \([C]\) of isomorphism classes of objects of \(C\).

\textit{Proof.} We shall only prove the statement about the weighting since the statement about the coweighting is entirely dual. Since \(C\) is a finite EI-category, the coslice categories \(a/C\) and \(a//C\) are also finite EI-categories. Thus they admits weightings and coweightings. Since \(a/\mathcal{C}\) is a left ideal in \(a/C\), the weighting for \(a/C\) from Lemma 2.3 restricts to a weighting for \(a//C\) \([11, \text{Remark 2.6}]\). The category \(a/C\) has an initial element, so it is contractible and has Euler characteristic 1. Therefore
\[
1 = \sum_{\varphi \in \text{Ob}(a/C)} k^\text{cod}(\varphi) = |[a]| |C(a)| k^\bullet_C + \sum_{\varphi \in \text{Ob}(a//C)} k^\text{cod}(\varphi) = |[a]| |C(a)| k^\bullet_C + \chi(a//C)
\]
because the weighting \(k^\bullet_C\) is assumed to be constant on the isomorphism class \([a]\) of \(a\).

The rational functions
\[
k^\bullet_C([a]) = -\frac{\chi(a//C)}{|C(a)|}, \quad k^\bullet_{[b]}([b]) = -\frac{\chi(C//b)}{|C(b)|}, \quad [a], [b] \in [\text{Ob}(C)],
\]
are the weighting and the coweighting for \([C]\), respectively.

In case \(S\) is a poset, we sometimes write \(a \leq S, a < S, S < b, S \leq b\) for \(a/S, a//S, S/b, S//b\), respectively. Using this notation, the last part of Theorem 2.6 takes the following form.
Corollary 2.7. The Euler characteristic of a finite poset $S$ is the sum

$$
\sum_{a \in \text{Ob}(S)} -\widetilde{\chi}(a \lessdot S) = \chi(S) = \sum_{b \in \text{Ob}(S)} -\widetilde{\chi}(S \lessdot b)
$$

of the opposite of the local reduced Euler characteristics.

This reproduces a well-known result from the combinatorial theory of posets.

3. Homotopy equivalences between categories

The famous Quillen’s Theorem A provides a sufficient criterion for a functor between two categories to be a homotopy equivalence. We quote here Quillen’s Theorem A not in its full generality but for the special case of inclusions between categories.

Theorem 3.1 (Quillen’s Theorem A for categories). [15, Theorem A] Let $C$ be a category and $A$ a full subcategory. The inclusion $A \hookrightarrow C$ is a homotopy equivalence if either $\text{supp}(\bullet/A)$ or $\text{supp}(A/\bullet)$ is empty.

We also quote a perhaps less well-known result of Bouc providing a sufficient condition for an inclusion of posets to be a homotopy equivalence.

Theorem 3.2 (Bouc’s theorem for posets). [1] Let $S$ be a finite poset and $A$ a subposet. The inclusion $A \hookrightarrow S$ is a homotopy equivalence if either $\text{supp}(\bullet/S)$ or $\text{supp}(S/\bullet)$ is contained in $\text{Ob}(A)$.

In this section we generalize Bouc’s theorem for poset inclusions to finite EI-category inclusions.

Theorem 3.3 (Bouc’s theorem for finite EI-categories). Let $C$ be a finite EI-category and $A$ a full subcategory that is closed under isomorphisms. The inclusion of $A \hookrightarrow C$ is a homotopy equivalence if either $\text{supp}(\bullet/C)$ or $\text{supp}(C/\bullet)$ is contained in $\text{Ob}(A)$.

Proof. Assume that $\text{Ob}(A)$ contains the support $\text{supp}(\bullet/C)$ of the functor $\bullet/C$. The claim is that the inclusion functor $\iota: A \rightarrow C$ is a homotopy equivalence. It suffices to show that the coslice $x/A$ of $A$ is contractible for every object $x$ of $C$ (Theorem 3.1).

For any object $x$ of $C$ define the height of $x$, $\text{ht}(x)$, to be the maximal length of any path

$$
x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_h = x
$$

of nonisomorphisms in $C$ terminating at $x$. The height of $x$ is finite since there are no circuits in paths of nonisomorphisms [12, Lemma 1.3]. If there is a nonisomorphism from $x_0$ to $x_1$, then $\text{ht}(x_0) < \text{ht}(x_1)$. Define $\text{ht}(C)$ to be the maximal height of any object of $C$.

Suppose that $x$ is an object of $C$ of maximal height, $\text{ht}(C)$. Then $x/C$ is the empty category because there is no nonisomorphism from $x$ to any object of $C$. The empty category is not contractible, so $x \in \text{supp}(x/C) \subset \text{Ob}(A)$ is an object of $A$. Then $x/A$ is contractible with the identity of $x$ as an initial element.

Let now $x$ be any object of $C$ such that the coslice $y/A$ of $A$ is contractible for all objects $y$ of height greater than $\text{ht}(x)$. Then the functor

$$
x/C \hookrightarrow A \rightarrow x/C
$$

is a homotopy equivalence by Theorem 3.1 because the category

$$
(x \rightarrow y)/(x/C) = y/A
$$

is contractible for every object $x \rightarrow y$ of $x/C$. In case $x$ is an object of $A$, $x/A$ is contractible as before. In case $x$ is not an object of $A$, $x/A = x/C$ because there can be no isomorphism from $x$ to an object of $A$ as $A$ is closed under isomorphisms. We now have

$$
x/A = x/C \simeq x/C
$$

and $x/C$ is contractible since $x \notin \text{supp}(\bullet/C)$. Thus $x/A$ is also contractible.

By finite downward induction on $\text{ht}(x)$ we see that $x/A$ is contractible for all objects $x$ of $C$. □
4. Subgroup categories for \( p \)-groups

For any small category \( C \) and any set \( D \subseteq \text{Ob}(C) \) of objects of \( C \), we let \( C^D \) denote the full subcategory of \( C \) generated by the objects in the set \( D \). For instance, if \( H \leq K \) are \( p \)-subgroups of \( G \), then \( \mathcal{F}_G^{[H,K]} \) denotes the full subcategory of \( \mathcal{F}_G \) with objects the set of all subgroups \( L \) of \( G \) for which \( H \leq L \leq K \).

In the following lemma we consider

\[
\begin{align*}
\mathcal{S}_p^{(1,P)} & : \text{the poset of nonidentity and proper subgroups of } P \\
\mathcal{O}_p^{(1,P)} & : \text{the full subcategory of } \mathcal{O}_p \text{ with objects all proper subgroups of } P \\
\mathcal{F}_p^{(1,P)} & : \text{the full subcategory of } \mathcal{F}_p \text{ with objects all nonidentity and proper subgroups of } P
\end{align*}
\]

for \( P \) a nonidentity \( p \)-group. We write \( \mu \) for the Möbius function of the poset \( \mathcal{S}_p \) [18, §3.7], and we abbreviate \( \mu(1,K) \) to \( \mu(K) \) for any subgroup \( K \) of \( P \).

**Lemma 4.1.** Let \( P \) be a nonidentity \( p \)-subgroup. Then

(a) \( \bar{\chi}(\mathcal{S}_p^{(1,P)}) = \mu(P), \bar{\chi}(\mathcal{F}_p^{(1,P)}) = \mu(P)/|P : Z(P)| = \bar{\chi}(\mathcal{F}_p^{(1,P)}), \) and \( \chi(\mathcal{O}_p^{(1,P)}) = p^{-1} \) if \( P \) is cyclic and \( \chi(\mathcal{O}_p^{(1,P)}) = 1 \) if \( P \) is noncyclic.

(b) \( \mathcal{S}_p^{(1,P)} \) is noncontractible \( \iff \) \( P \) is elementary abelian

(c) \( \mathcal{O}_p^{(1,P)} \) is homotopy equivalent to \( \mathcal{O}_V^{(1,V)} \) where \( V = P/\Phi(P) \) is the Frattini quotient of \( P \).

(d) \( \mathcal{F}_p^{(1,P)} \) is noncontractible \( \iff \) \( P \) is elementary abelian

**Proof.** (a) It is well known that \( \bar{\chi}(\mathcal{S}_p^{(1,P)}) = \bar{\chi}(1,P) = \mu(P) \) [18, 3.8.5, 3.8.6] [11, 2.3]. The formulas for \( \bar{\chi}(\mathcal{F}_p^{(1,P)}) \) and \( \chi(\mathcal{O}_p^{(1,P)}) \) follow from [11, Remark 2.6, Example 3.7, Theorem 7.7, Theorem 4.1]: If \( k_\bullet \) is a coweighting for \( \mathcal{F}_p^{(1,P)} \) then

\[
1 = \chi(\mathcal{F}_p^{(1,P)}) = \chi(\mathcal{F}_p^{(1,P)}) + k_\mu = \chi(\mathcal{F}_p^{(1,P)}) + \frac{-\mu(P)}{|P : Z(P)|}
\]

because \( \chi(\mathcal{F}_p^{(1,P)}) \) is contractible, with \( P \) as final element, containing the right ideal \( \chi(\mathcal{F}_p^{(1,P)}) \). Similarly, If \( k_\mu \) is a coweighting for \( \mathcal{O}_p \) then

\[
1 = \chi(\mathcal{O}_p) = \chi(\mathcal{O}_p^{(1,P)}) + k_\mu = \chi(\mathcal{O}_p^{(1,P)}) + \begin{cases} 1 - \frac{1}{p} & \text{if } P \text{ is cyclic} \\ 0 & \text{if } P \text{ is not cyclic} \end{cases}
\]

and the expression for the Euler characteristic of \( \mathcal{O}_p^{(1,P)} \) follows.

(b) If \( P \) is elementary abelian, the poset \( \mathcal{S}_p^{(1,P)} \) is noncontractible because its reduced Euler characteristic is nonzero according to (a). If \( P \) is not elementary abelian, the Frattini subgroup \( \Phi(P) \) is nontrivial [9, Chp 5, Theorem 1.3]. There are adjoint functors

\[
\begin{array}{c}
\mathcal{S}_p^{(1,P)} \\
\xrightarrow{L} \mathcal{S}_p^{(\Phi(P),P)} \\
\xrightarrow{R} \mathcal{S}_p^{(1,P)}
\end{array}
\]

where \( QL = Q\Phi(P) \) and \( QR = Q \) for \( Q \leq P \). Observe that \( Q \hookrightarrow P \iff Q\Phi(P) \hookrightarrow P \) because the Frattini subgroup is the group of nongenerators of \( P \). The poset on the right, \( \mathcal{S}_p^{(1,P)} \), is contractible with the trivial group as an initial object. The poset on the left, \( \mathcal{S}_p^{(1,P)} \), is therefore also contractible. Alternatively, the natural transformations \( Q \leq Q\Phi(P) \geq \Phi(P), 1 \leq Q \leq P \), define a homotopy from the identity of \( \mathcal{S}_p^{(1,P)} \) to a constant map.

(c) There are functors of categories

\[
\begin{array}{c}
\mathcal{O}_p^{(1,P)} \\
\xrightarrow{L} \mathcal{O}_p^{(\Phi(P),P)} \\
\xrightarrow{U} \mathcal{O}_p^{(1,P/\Phi(P))}
\end{array}
\]

where \( R \) and \( L \) are adjoint functors and \( U \) is an isomorphism. The functors \( R \) and \( L \) are given by \( QL = Q\Phi(P) \) and \( QR = Q \) for \( Q \leq P \). The category in the middle, \( \mathcal{O}_p^{(\Phi(P),P)} \), is isomorphic to the category \( \mathcal{O}_p^{(1,P/\Phi(P))} \).

To see this, observe that all supergroups of the Frattini subgroup \( \Phi(P) \), or just \( |P,P| \), are normal, so that \( \mathcal{O}_p(Q_1,Q_2) = P/Q_2 = \frac{P/\Phi(P)}{Q_2/\Phi(P)} = \mathcal{O}_p/\Phi(P)(Q_1/\Phi(P),Q_2/\Phi(P)) \) when \( Q_1 \) and \( Q_2 \) both contain \( \Phi(P) \).
(d) If \( P \) is elementary abelian, then \( \tilde{F}_P = S_P \) and \( \tilde{F}^{(1,P)}_P = S^{(1,P)}_P \), is noncontractible by (b). If \( P \) is not elementary abelian, the Frattini subgroup \( \Phi(P) \) is a nontrivial normal subgroup and so is its intersection with the center \( Z(P) \) of \( P \) \([17, 5.2.1]\). There are adjoint equivalences of categories

\[
\tilde{F}^{(1,P)}_P \xrightarrow{\ell} \tilde{F}^{(\Phi(P) \cap Z(P),P)}_P \xrightarrow{\ell} \tilde{F}^{(1,P)}_P
\]

where \( QL = Q\Phi(P) \) and \( QR = Q \) for \( Q \nsubseteq P \). The category to the right, \( \tilde{F}^{(1,P)}_P \), is contractible because it has the trivial group as an initial object. The category to the left, \( \tilde{F}^{(1,P)}_P \), is therefore also contractible. \( \square \)

One might be led by Lemma 4.1.(a) to suspect that, for any nonidentity \( p \)-group \( P \),

\[
\mathcal{O}^{(1,P)}_P \text{ is noncontractible } \implies P \text{ is cyclic}
\]

or, equivalently, for any nonidentity elementary abelian \( p \)-group \( V \),

\[
\mathcal{O}^{(1,V)}_V \text{ is noncontractible } \implies \text{rank}(V) = 1
\]

To see that these two statements are equivalent, recall that the Frattini quotient of \( P \) is cyclic precisely when \( P \) itself is cyclic \([9, Chp. 5, Corollary 1.2]\) and use Lemma 4.1.(c). However, Example 4.2 demonstrates that these statements are false.

**Example 4.2.** Let \( V = C^r_p \) be the elementary abelian \( p \)-group of rank \( r \geq 1 \). The objects of the category \( \mathcal{O}^{(1,V)}_V \) are the proper subgroups of \( V \), and the set of morphisms from \( H \leq K \) to \( K \leq V \) is

\[
\mathcal{O}^{(1,V)}_V(H,K) = \begin{cases} V/K & \text{if } H \leq K \\ \emptyset & \text{otherwise} \end{cases}
\]

with composition in this category induced from composition in the abelian group \( V \).

If the rank \( r = 1 \), then the category \( \mathcal{O}^{(1,V)}_V = \mathcal{O}^{(1)}_V \) is the cyclic group \( V \), which is not contractible.

Let us now explore the category \( \mathcal{O}^{(1,V)}_V \) in case where the rank \( r > 1 \). There is an obvious functor

\[
\pi: \mathcal{O}^{(1,V)}_V \to S^{(1,V)}_V
\]

to the poset of proper subgroups of \( V \). For any proper subgroup \( K \) of \( V \), the \( \pi \)-slice over \( K \) is \( \pi/K = \mathcal{O}^{(1,K)}_V \). There is an adjunction

\[
\mathcal{O}^{(1,K)}_V \xrightarrow{LR} \mathcal{O}^{(K)}_V, \quad RL \subseteq 1_{\mathcal{O}^{(K)}_V}, \quad 1_{\mathcal{O}^{(1,K)}_V} \xrightarrow{\eta} LR,
\]

where \( HL = K \) and \( KR = K \). The functor \( R \) includes the full subcategory of \( \mathcal{O}^{(1,V)}_V \) with \( K \) as its only object into the full subcategory of all subgroups of \( K \). The functor \( L \) is the projection \( \mathcal{O}^{(1,V)}_V(H_1,H_2) = V/H_2 \to V/K = \mathcal{O}^{(1,V)}_V(K,K), H_1 \leq H_2 \leq K \). Thus the category \( \mathcal{O}^{(1,K)}_V \) is homotopy equivalent to the category \( \mathcal{O}^{(1,V)}_V \) which is the group \( V/K \). The composite functor spectral sequence \([8, pp 155–157]\) \([7, Proof of Proposition 2.3]\)

\[
E^2_{st} = H_s(\mathcal{O}^{(1,V)}_V; H_t(V/\bullet; F_p)) \Rightarrow H_{s+t}(\mathcal{O}^{(1,V)}_V; F_p)
\]

associated to the functor \( \pi \) provides information about the homology groups of the category \( \mathcal{O}^{(1,V)}_V \). Here, we write \( H_s(\mathcal{O}^{(1,V)}_V; H_t(V/\bullet)) \) for the \( s \)-th left derived of the functor colim \( H_t(V/\bullet) \). In concrete terms, these groups are the homology groups of the normalized chain complex \([13, Theorem VIII.6.1]\) of the simplicial abelian group \( \bigoplus H_t(V/\bullet) \) \([2, XII.5.5]\),

\[
0 \leftarrow \bigoplus_{0 \leq L_0 < V} H_t(V/L_0) \xrightarrow{\partial_1} \bigoplus_{0 \leq L_0 \leq L_1 < V} H_t(V/L_0) \xrightarrow{\partial_2} \cdots \xrightarrow{\partial_s} \bigoplus_{0 \leq L_0 \leq \cdots \leq L_s < V} H_t(V/L_0) \xrightarrow{\partial_{s+1}} \cdots
\]

with boundary homomorphism \( \partial_s \) is defined by deleting single entries of the \( s \)-flag \( L_0 < L_1 < \cdots < L_s \) and applying \( H_t(V/L_0) \to H_t(V/L_1) \) in case of deletion of the first entry. This chain complex is trivial in degrees \( > r-1 \) so that the spectral sequence (4.3) is concentrated in the vertical band \( 0 \leq s \leq r-1 \).
Take $r = 2$ and $p = 2$ and consider the category $O_{V}^{1,V}$ where $V$ is the Klein 4-group. The objects of $O_{V}^{1,V}$ are the identity subgroup, $\{0\}$, and three subgroups, $L_1$, $L_2$, and $L_3$, of order 2. The category $O_{V}^{1,V}$ is

\[
\begin{array}{ccc}
V/L_1 & \overset{V/L_2}{\leftrightarrow} & V/L_3 \\
\downarrow & \downarrow & \downarrow \\
L_1 & \overset{L_2}{\leftrightarrow} & L_3 \\
\downarrow & \downarrow & \downarrow \\
V/L_1 & \overset{V/L_2}{\leftrightarrow} & V/L_3 \\
\downarrow & \downarrow & \downarrow \\
0 & \overset{1}{\leftrightarrow} & V \\
\end{array}
\]

\[\zeta = \begin{pmatrix} 4 & 2 & 2 & 2 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad k^G_0 = (1/4, 1/4, 1/4, 1/4) \quad \chi(O_{V}^{1,V}) = 1
\]

with composition induced from addition in the abelian group $V$. The first quadrant spectral sequence (4.3) is concentrated on the two vertical lines $s = 0$ and $s = 1$ so that all differentials are trivial. The groups $E_{0t} = E_{0\infty}$ and $E_{1t} = E_{\infty}$ are the homology groups of the normalized simplicial replacement chain complex

\[\cdots \rightarrow 0 \rightarrow H_t(V/L_1) \oplus H_t(V/L_2) \oplus H_t(V/L_3) \rightarrow H_t(V/0) \oplus H_t(V/0) \oplus H_t(V/0) \rightarrow 0 \rightarrow \cdots\]

concentrated in degrees 0 and 1. Since $H_t(V/0)$ has dimension $t + 1$, $H_t(V/L_i)$, $i = 1, 2, 3$, is 1-dimensional, the term $E_{t,1}^\infty$ has dimension at least $2t - 1$, and consequently $\dim F_j H_{t+1}(O_{V}^{1,V}; F_2) \geq 2t - 1$ for all degrees $t \geq 1$.

The above argument is easily seen to work for any prime $p$ and we conclude that $\dim F_p H_{t+1}(O_{V}^{1,V}; F_p) \geq pt - 1$ for all degrees $t \geq 1$ when the rank $r = 2$. Thus $O_{V}^{1,V}$ is noncontractible when $V$ has rank $r = 2$.

Here are a few remarks about the spectral sequence (4.3) for arbitrary prime $p$ and rank $r \geq 2$. When $t = 0$, $E_{00}^2 = H_s(S_{G}^{1,V}; F_p)$, so that $E_{00}^2 = F_p$ and $E_{0s}^2 = 0$ for $s > 0$, as $O_{V}^{1,V}$ is contractible. When $t > 0$, we conjecture, based on computer calculations, that $E_{st}^2 = 0$ except for $s = r - 1$. We have not been able to prove this conjecture.

5. Brown posets and transporter categories

Let $G$ be a finite group of order divisible by $p$ and $S_G$ the poset of $p$-subgroups of $G$. The Brown poset for $G$ is the subposet $S^*_G = S_{G}^{1,G}$ of nonidentity $p$-subgroups of $G$. The results of this section are not new.

**Theorem 5.1.** [1] [16] The inclusions and $S_{G}^{r+rad} \hookrightarrow S^*_G$, $S_{G}^{sfc+rad} \hookrightarrow S^*_G$, and $S_{G}^{r+rad} \hookrightarrow S^*_G$ are homotopy equivalences.

**Proof.** The two expressions, from [11, Theorem 1.3.(1)] and Theorem 2.6, for the weighting for $S^*_G$,

\[-\bar{\chi}(S_{O_G(H)}) = k^H_S = -\bar{\chi}(H/\langle S_G^r \rangle)\]

show that the categories $H/\langle S_G \rangle$ and $S_{O_G(H)}^*$ have identical Euler characteristics. Indeed, for any nonidentity $p$-subgroup $H$ of $G$, there are functors

\[H/\langle S_G \rangle \overset{r_H}{\longrightarrow} S_{O_G(H)}^*, \quad H/\langle S_G \rangle \overset{r_H}{\longleftarrow} S_{O_G(H)}^* \]

given by $K_{rH} = \hat{N}_K(H)/H$ for all $p$-subgroups $K$ of $H$ and $K_{iH} = K$ when $K = K/H$ and $H \leq K \leq \hat{N}_G(H)$ ([16, Lemma 6.1]). The composite functor $i_Hr_H$ is the identity of $S_{O_G(H)}^*$ and there is a natural transformation from $K_{rH}i_H = \hat{N}_K(H)$ to the identity functor of $H/\langle S_G \rangle$. This shows that these functors are homotopy equivalences of categories. By property (1.8),

\[\supp(\bullet/\langle S_G \rangle) = \{H \in \text{Ob}(S_G^r) \mid S_{O_G(H)}^* \text{ is noncontractible}\} \subset \{H \in \text{Ob}(S_G^r) \mid H \text{ is } G\text{-radical}\} = \text{Ob}(S_{G}^{r+rad})\]

and Bouc’s Theorem 3.3 shows that the inclusion of $S_{G}^{r+rad}$ into $S_G^*$ is a homotopy equivalence.

Since any $p$-subgroup of a $G$-selfcentralizing $p$-subgroup is itself $G$-selfcentralizing, $H/\langle S_G^{sfc} \rangle = H/\langle S_G^r \rangle$ for any $G$-selfcentralizing $p$-subgroup $H$ of $G$. By property (1.8)

\[\supp(\bullet/\langle S_G^{sfc} \rangle) \subset \text{Ob}(S_{G}^{sfc+rad})\]

and the inclusion of $S_{G}^{r+rad}$ into $S_G^{sfc}$ is a homotopy equivalence by Bouc’s Theorem 3.3.
The two expressions, from [11, Theorem 1.1.(1)] and Theorem 2.6, for the coweighting for $S_G^*$

$$-\tilde{\chi}(S_K^{(1,K)}) = k^K = -\tilde{\chi}(S_G^*/K)$$

show that the categories $S_G^*/K$ and $S_K^{(1,K)}$ have identical Euler characteristics. Indeed, they are identical! By Lemma 4.1.(b),

$$\text{supp}(S_G^*/\star) = \{k \in \text{Ob}(S_G^*) \mid K \text{ is elementary abelian} \} = \text{Ob}(S_G^{+\text{eab}})$$

and Bouc’s Theorem 3.3 shows that the inclusion of $S_G^{+\text{eab}}$ into $S_G^*$ is a homotopy equivalence. \hfill \Box

**Example 5.2.** If $G = C_2 \times \Sigma_3$ and $p = 2$, then $S_G^{\text{sc}}$ is a discrete poset consisting of the 3 Sylow 2-subgroups, while $S_G^{+\text{rad}}$ is contractible since $O_2 G = C_2$ is nontrivial. Thus the inclusion $S_G^{\text{sc}} \rightarrow S_G^{+\text{rad}}$ is not a homotopy equivalence.

The following proposition points out that the largest normal $p$-subgroup is the smallest $G$-radical $p$-subgroup. It implies that the poset $S_G^{+\text{rad}}$ has a least element in case $O_p G$ is nontrivial. (We thank Andy Chermak for the proof.)

**Proposition 5.3.** Any $G$-radical $p$-subgroup of $G$ contains the $G$-radical $p$-subgroup $O_p G$.

*Proof.* It is clear that $O_p G$ is a normal $G$-radical $p$-subgroup. Let $H$ be a $p$-subgroup of $G$ not containing $O_p G$. The normalizer of $H$ in the $p$-subgroup $(O_p G)H$ is normal in $N_G(H)$ for any element of $G$ normalizing $H$ normalizes $(O_p G)H$. Since $N_{(O_p G)H}(H)$ is a normal $p$-subgroup of $N_G(H)$ strictly larger than $H$, the $p$-subgroup $H$ is not $G$-radical. \hfill \Box

Let $T_G$ be the transporter category of $p$-subgroups of $G$.

**Proposition 5.4.** The inclusions and $T_G^{+\text{rad}} \hookrightarrow T_G^*$, $T_G^{\text{sc}+\text{rad}} \hookrightarrow T_G^{\text{sc}}$, and $T_G^{+\text{eab}} \hookrightarrow T_G^*$ are homotopy equivalences.

*Proof.* The two expressions, from [11, Theorem 1.3.(2)] and Theorem 2.6, for the weighting on $|T_G^*|$, 

$$\frac{-\tilde{\chi}(S_{O_p G}(H))}{|T_G(H)|} = k^{[H]} = \frac{-\tilde{\chi}(H//T_G^*)}{|T_G(H)|}$$

lead to homotopy equivalences $H//T_G^* \simeq S_{O_p G}(H)$ for any object $H$ of $T_G^*$. Property 1.8 and Bouc’s Theorem 3.3 now imply that the inclusion functors $T_G^{+\text{rad}} \hookrightarrow T_G^*$, $T_G^{\text{sc}+\text{rad}} \hookrightarrow T_G^{\text{sc}}$ are homotopy equivalences.

The two expressions, from [11, Theorem 1.1.(2)] and Lemma 4.1.(a) together with Theorem 2.6, for the coweighting on $|T_G^*|$, 

$$\frac{-\tilde{\chi}(S_K^{(1,K)})}{|T_G(K)|} = k^{[K]} = \frac{-\tilde{\chi}(T_G^*/K)}{|T_G(K)|}$$

lead to homotopy equivalence $T_G^*/K \simeq S_K^{(1,K)}$ for any object $K$ of $T_G^*$. Lemma 4.1.(b) and Bouc’s Theorem 3.3 now imply that the inclusion functor $T_G^{+\text{eab}} \hookrightarrow T_G^*$ is a homotopy equivalence. \hfill \Box

Suppose that $C$ is a small category and $X, Y : C \rightarrow \text{CAT}$ are functors with values in the category $\text{CAT}$ of small categories. If there is a natural transformation from $X$ to $Y$ with components $X(c) \rightarrow Y(c)$, $c \in \text{Ob}(C)$, that are all homotopy equivalences, then the induced functor $\int_C X \rightarrow \int_C Y$ of Grothendieck constructions is a homotopy equivalence. This follows from Thomason’s homotopy colimit theorem [19] and homotopy invariance of the homotopy colimit [2, Ch. XII, §4, Homotopy Lemma 4.2]. As the inclusions of Theorem 5.1 are $G$-equivariant inclusions of $G$-categories and $T_G^* = (S_G^*)_{hG}$ etc we obtain an alternative proof of Proposition 5.4. Similarly, if $O_p G$ is nontrivial, there is a homotopy equivalence $G \hookrightarrow T_G^{+\text{rad}}$ induced from the $G$-equivariant homotopy equivalence $* \hookrightarrow S_G^{+\text{rad}}$ of Proposition 5.3.
6. Frobenius categories

Let $P$ be a finite $p$-group and $F$ a Frobenius $P$-category.

**FACTS:**
- All morphisms in $F$ are monomorphisms
- The categories $F^*/K$ and $F^*//K$ are thin
- The coweighting for $F^*$ vanishes off the elementary abelian subgroups [11, Theorem 7.5]

A category is thin if there is at most one morphism between any two objects.

**Proposition 6.1.** The inclusion $F^*+eab \to F^*$ is a homotopy equivalence.

**Proof.** The two expressions, from [11, Theorem 7.5] and from Lemma 4.1.(a) together with Theorem 2.6, for the coweighting on $[F^*]$,

$$\frac{-\chi(S^{(1,K)}_K)}{|F^*(K)|} = k_{[F^*]}^{[F^*/K]} = \frac{-\chi(F^*//K)}{|F^*(K)|}$$

show that $S^{(1,K)}_K$ and $F^*//K$ have identical Euler characteristics. And indeed, there are functors

$$F^*/K \xrightarrow{r_K} S^{(1,K)}_K, \quad F^*//K \xrightarrow{i_K} S^{(1,K)}_K$$

The functor $r_K$ takes $\varphi \in F^*(H,K)$ to its image $H^\varphi$ in $K$. The functor $i_K$ takes $H \leq K$ to the inclusion $H \hookrightarrow K$ of $H$ into $K$. Obviously, $i_K r_K$ is the identity functor of $S^{(1,K)}_K$, and there is a natural transformation from the identity functor to the endofunctor $(H \xrightarrow{r_K} K) r_K i_K = (H^\varphi \hookrightarrow K)$ of $F^*//K$. This shows that $r_K$ and $i_K$ are homotopy equivalences between $F^*/K$ and $S^{(1,K)}_K$. Their restrictions are homotopy equivalences between $F^*/K$ and $S^{(1,K)}_K$. By Lemma 4.1.(b),

$$\text{supp}(F^*//\bullet) = \text{Ob}(F^*+eab)$$

and Bouc’s Theorem 3.3 shows that the inclusion of $F^*+eab$ into $F^*$ is a homotopy equivalence. □

In the course of the proof of Proposition 6.1 we saw that the homotopy type of the category $F^*/K$ of $F^*$-nonisomorphisms to $K$ depends only on $K$, not on $F$.

We know of no formula for the weighting of a general Frobenius category $F$. Even though there is an explicit formula in [11, Theorem 1.3.(3)] for the weighting of the Frobenius category $F_G$ associated to a finite group $G$, we have not been able to determine the support of this weighting or describe the categories $H//F_G$.

7. Orbit categories

Let $G$ be a finite group of order divisible by $p$ and $O_G$ the orbit category of $p$-subgroups of $G$.

**FACTS:**
- The trivial subgroup is not initial in $O_G$
- All morphisms in $O_G$ are epimorphisms
- The categories $H/O_G$ and $H//O_G$ are thin
- The weighting for $O_G$ vanishes off the $G$-radical $p$-subgroups of $G$ [11, Proposition 3.14]
- The coweighting for $O_G$ vanishes off the cyclic $p$-subgroups [11, Theorem 4.1]

**Theorem 7.1.** The inclusions $O_G^{rad} \hookrightarrow O_G$, $O_G^{rad+} \hookrightarrow O_G$, and $O_G^{rc+} \hookrightarrow O_G^{rc}$ are homotopy equivalences.

**Proof.** The two expressions, from [11, Equation (3.15)] and Theorem 2.6, for the weighting for $[O_G]$

$$\frac{-\chi(S^{(1,G)}_{O_G(H)})}{|O_G(H)|} = k_{[O_G]}^{[H]} = \frac{-\chi(H//O_G)}{|O_G(H)|}$$

show that $S^{(1,G)}_{O_G(H)}$ and $H//O_G$ have identical Euler characteristics. Indeed, for any nonidentity $p$-subgroup $H$ of $G$, there are functors

$$r_H: H/O_G \to S^{(1,G)}_{O_G(H)}, \quad r_H: H//O_G \to S^{(1,G)}_{O_G(H)}$$

The functor $r_H$ takes $gK \in O_G(H,K) = N_G(H,K)/K$ to the subgroup $N_{gK}(H)/H$ of $O_G(H) = N_G(H)/H$. Let $L$ be a $p$-subgroup such that $H \leq L \leq N_G(H)$ and let $\overline{L} = L/H$ be the image of $L$ in $N_G(H)/H = O_G(H)$. 

The category \( \mathcal{T}/r_H \) is the full subcategory of \( \mathcal{O}_G/H \) generated by all morphisms \( gK \in \mathcal{O}_G(H, K) \) such that \( L \leq N_{gK}(H) \). The inclusion of \( H \) into \( L \) is an object of \( \mathcal{T}/r_H \) as \( L = N_H(H) \). Note that the morphism \( gK : H \to K \) extends to a morphism \( gK : L \to K \) because \( L^g \leq N_{gK}(H)^g = N_K(H^g) \leq K \). This is thus a morphism

\[
\begin{array}{c}
H \\
\downarrow^{gK} \\
K
\end{array}
\]

in \( \mathcal{T}/r_H \). This shows that the inclusion \( H \hookrightarrow L \) is an initial object of \( \mathcal{T}/r_H \). By Quillen’s Theorem A (Theorem 3.1), the functor \( r_H \) is a homotopy equivalence from \( H/\mathcal{O}_G \) to \( \mathcal{S}_{\mathcal{O}_G(H)} \). The same argument shows that \( r_H \) restricts to a homotopy equivalence from \( H/\mathcal{O}_G \) to \( \mathcal{S}_{\mathcal{O}_G(H)}^\ast \). By property (1.8),

\[
\text{supp}(\bullet/\mathcal{O}_G) \subset \text{Ob}(\mathcal{O}_G^{\text{rad}})
\]

and Bouc’s Theorem 3.3 shows that the inclusion of \( \mathcal{O}_G^{\text{rad}} \) into \( \mathcal{O}_G \) is a homotopy equivalence.

Since \( \mathcal{O}_G^{\text{cyc}} \) and \( \mathcal{O}_G^{\text{rad}} \) are left ideals in \( \mathcal{O}_G \), \( H/\mathcal{O}_G^{\text{cyc}} = H/\mathcal{O}_G \) and \( H/\mathcal{O}_G^{\text{rad}} = H/\mathcal{O}_G \) for any nonidentity, respectively, \( G \)-selfcentralizing \( p \)-subgroup \( H \) of \( G \). By property (1.8),

\[
\text{supp}(\bullet/\mathcal{O}_G) \subset \text{Ob}(\mathcal{O}_G^{\ast+\text{rad}}), \quad \text{supp}(\bullet/\mathcal{O}_G^{\text{cyc}}) \subset \text{Ob}(\mathcal{O}_G^{\ast\text{cyc}+\text{rad}})
\]

and Bouc’s Theorem 3.3 shows that the two inclusion functors \( \mathcal{O}_G^{\ast+\text{rad}} \to \mathcal{O}_G^{\ast} \) and \( \mathcal{O}_G^{\ast\text{cyc}+\text{rad}} \to \mathcal{O}_G^{\ast\text{cyc}} \) are homotopy equivalences.

The two expressions, from [11, Theorem 4.1] and from Lemma 4.1.(a) together with Theorem 2.6, for the coweighting for \([\mathcal{O}_G]\),

\[
-\chi(\mathcal{O}_G^{[1,K]})
\]

show that the categories \( \mathcal{O}_K^{[1,K]} \) and \( \mathcal{O}_G/\mathcal{K} \) have identical Euler characteristics for any object \( K \) of \( \mathcal{O}_G \). In fact, there are equivalences of categories

\[
i_K : \mathcal{O}_K \to \mathcal{O}_G/\mathcal{K}, \quad i_K : \mathcal{O}_K^{[1,K]} \to \mathcal{O}_G/\mathcal{K}
\]

On objects, \( Hi_K = 1K \in N_G(H, K)/K = \mathcal{O}_G(H, K) \), for any subgroup \( H \) of \( K \). We observe that there is an obvious identification of morphism sets,

\[
\mathcal{O}_K(H_1, H_2) = (\mathcal{O}_G/\mathcal{K})(H_1 i_K, H_2 i_K)
\]

and we use this identification to define \( i_K \) on morphism sets. By construction, \( i_K \) is full and faithful, and as it is also essentially surjective on objects, \( i_K \) is an equivalence of categories. Lemma 4.1.(c) tells us that

\[
\mathcal{O}_K^{[1,K]} \simeq \mathcal{O}_V^{[1,V(K)]}, \quad V(K) = K/\Phi(K),
\]

only depends on the Frattini quotient \( V(K) \) of \( K \). Therefore,

\[
\text{supp}(\mathcal{O}_G/\bullet) = \{ K \in \text{Ob}(\mathcal{O}_G) \mid \mathcal{O}_V^{[1,V(K)]} \text{ is noncontractible} \}
\]

and Bouc’s Theorem 3.3 shows that the inclusion of the full subcategory of \( \mathcal{O}_G \) generated by this support into \( \mathcal{O}_G \) is a homotopy equivalence. However, it is not clear what \( p \)-subgroups of \( G \) are in the support. Certainly, all cyclic subgroups of \( G \) are in the support but there could very well be other subgroups as well.

**Example 7.2.** When \( V = C_2 \times C_2 \) is Klein’s 4-group, \( \mathcal{O}_V^{\text{cyc}} \to \mathcal{O}_V \) is not an equivalence for \( \mathcal{O}_V^{\text{cyc}} = \mathcal{O}_V^{[1,V]} \) is not contractible (Example 4.2) while \( \mathcal{O}_V \) is contractible (it has \( V \) as a terminal element).
8. Exterior quotients of Frobenius categories

Let \( P \) be a nonidentity finite \( p \)-group, \( \mathcal{F} \) a Frobenius \( P \)-category, and \( \bar{\mathcal{F}} \) the exterior quotient of \( \mathcal{F} \) [14, 1.3, 2.6, 4.8].

**FACTS:**
- All morphisms in \( \bar{\mathcal{F}}_{sfc} \) are epimorphisms [14, Corollary 4.9]
- The categories \( H/\bar{\mathcal{F}}_{sfc} \) are thin
- The coweighting for \( \bar{\mathcal{F}}^* \) vanishes off the elementary abelian subgroups
- The weighting for \( \bar{\mathcal{F}}_{sfc} \) vanishes off the \( \mathcal{F} \)-radical subgroups
- \( \mathcal{F}^* \) and \( \bar{\mathcal{F}}^* \) have identical coweights

**Theorem 8.1.** The inclusions \( \bar{\mathcal{F}}_{sfc+rad} \to \bar{\mathcal{F}}_{sfc} \) and \( \bar{\mathcal{F}}^{*+eab} \to \bar{\mathcal{F}}^* \) are homotopy equivalences.

**Proof.** The two expressions, from [11, Proposition 8.5] and Theorem 2.6, for the weighting for \( \bar{\mathcal{F}}_{sfc} \),

\[
\frac{-\bar{\chi}(S^*_{\bar{\mathcal{F}}_{sfc}^G(H)})}{|\bar{\mathcal{F}}_{sfc}^G(H)|} = k[H]_{\bar{\mathcal{F}}_{sfc}^G} = \frac{-\bar{\chi}(H/\bar{\mathcal{F}}_{sfc}^G)}{|\bar{\mathcal{F}}_{sfc}^G(H)|}
\]

show that the strict under-category \( H/\bar{\mathcal{F}}_{sfc} \) and the poset \( S^*_{\bar{\mathcal{F}}_{sfc}^G(H)} \) have identical Euler characteristic for any object \( H \) of \( H/\bar{\mathcal{F}}_{sfc} \). In fact, they homotopy equivalent.

Let \( H \) be any \( \mathcal{F} \)-selfcentralizing object of \( \bar{\mathcal{F}} \). We claim that there are homotopy equivalences

\[
(8.2) \quad r_H : H/\bar{\mathcal{F}} \to S_{\bar{\mathcal{F}}(H)}, \quad r_H : H/\bar{\mathcal{F}} \to S_{\bar{\mathcal{F}}(H)}^*
\]

There is no restriction in also assuming that \( H \) is fully normalized in \( \bar{\mathcal{F}} \). The functor \( r_H \) takes the object \( \varphi \mathcal{F}_K(K) \in \bar{\mathcal{F}}(H,K) = \mathcal{F}(H,K)/\mathcal{F}_K(K) \) of \( H/\bar{\mathcal{F}} \) to \( r_H(\varphi) = \varphi \bar{\mathcal{F}}_K(H^\varphi) \). Note that this is well-defined even though \( \varphi \) is only defined up to conjugacy in \( K \). The group

\[
\bar{\mathcal{F}}_K(H^\varphi) = C_K(H^\varphi)/N_K(H^\varphi)/H^\varphi = Z(H^\varphi)/N_K(H^\varphi)/H^\varphi = N_K(H^\varphi)/H^\varphi
\]

and the isomorphic group \( r_H(\varphi \mathcal{F}_K(K)) = \varphi \bar{\mathcal{F}}_K(H^\varphi) \) are related by the commutative diagram

\[
\begin{array}{ccc}
H & \xrightarrow{\varphi} & H^\varphi \\
\| & \| & \| \\
\varphi \bar{\mathcal{F}}_K(H^\varphi) & \xrightarrow{\varphi} & \bar{\mathcal{F}}_K(H^\varphi) = N_K(H^\varphi)/H^\varphi \\
\| & \| & \| \\
H & \cong & H^\varphi
\end{array}
\]

It is clear that \( \varphi \bar{\mathcal{F}}_K(H^\varphi) \) is indeed a \( p \)-subgroup of \( \bar{\mathcal{F}}(H) \) and that \( r_H(\varphi_1) \leq r_H(\varphi_2) \) whenever there is a \( \bar{\mathcal{F}} \)-morphism

\[
\varphi : \mathcal{F}_K(K_1) \to \mathcal{F}_K(K_2)
\]

under \( H \). Thus \( r_H \) is a functor. We now want to use Quillen’s Theorem A to show that \( r_H \) is an equivalence of categories.

Let \( \mathcal{T} \) be a \( p \)-subgroup of \( \bar{\mathcal{F}}(H) = \mathcal{F}(H)/\mathcal{F}_H(H) \). We may assume that \( \mathcal{T} \) is contained in the Sylow \( p \)-subgroup \( \bar{\mathcal{F}}_p(H) = N_P(H)/H \) of \( \bar{\mathcal{F}}(H) \). There is a unique \( p \)-subgroup \( L \in [P, N_P(H)] \) such that \( \mathcal{T} = L/H \). The category \( \mathcal{T}/r_H \) is the full subcategory of \( H/\bar{\mathcal{F}} \) generated by all objects \( \varphi \mathcal{F}_K(K) \in \bar{\mathcal{F}}(H,K) \) such that \( \mathcal{T}^\varphi \leq \bar{\mathcal{F}}_K(H^\varphi) = N_K(H^\varphi)/H^\varphi \), or, equivalently, \( L^\varphi \leq N_K(H^\varphi) \). Here is an attempt

\[
\begin{array}{ccc}
H & \xrightarrow{\varphi} & H^\varphi \\
L/H & \xrightarrow{\varphi} & N_K(H^\varphi)/H^\varphi \\
H & \cong & H^\varphi
\end{array}
\]
to visualize this relation. The inclusion \( i^H_L : H \hookrightarrow L \) of \( H \) into \( L \) represents a morphism in \( \bar{F}(H, L) \) and an object of \( \mathcal{T}_r/H \) because \( \mathcal{T} \) is contained in \( r_H(F_L(L)) = N_L(H)/H = L/H = \mathcal{T} \). By Lemma 8.3 there is an extension in \( \bar{F} \) of \( \varphi : H \rightarrow K \)

\[
\begin{array}{ccc}
H & \xrightarrow{\varphi} & K \\
\downarrow i^H_L & & \downarrow \\
L & \xrightarrow{=} & K
\end{array}
\]

to a morphism \( L \rightarrow K \). We have now shown that \( i^H_L F_L(L) \) is an initial object of \( \mathcal{T}_r/H \) for any object \( \mathcal{T} \) of \( S_{\bar{F}(H)} \). According to Quillen’s Theorem A (Theorem 3.1), \( r_H \) is a homotopy equivalence of categories.

Since the functor \( r_H \) takes nonisomorphisms \( \varphi F_K(K) \in \bar{F}(H, K) \subset \text{Ob}(\bar{F}/\bar{F}) \) to nonidentity \( p \)-subgroups of \( \bar{F}(H) \), it restricts to a functor \( r_H : H//F \rightarrow S^*_H(H) \) of \( H//F \) into the Brown poset of the automorphism group of \( H \). But since \( \mathcal{T}_r/H \) is contractible for any nonidentity \( p \)-subgroup \( \mathcal{T} \) of \( \bar{F}(H) \), we already know that the restricted functor \( r_H \) is a homotopy equivalence of categories. By property (1.9),

\[
\text{supp}(\bullet//\bar{F}e^\text{fc}) \subset \text{Ob}(\bar{F}e^\text{fc+rad})
\]

and Bouc’s Theorem 3.3 shows that the inclusion of \( \bar{F}e^\text{fc+rad} \) into \( \bar{F}e^\text{fc} \) is a homotopy equivalence.

The two expressions, from [11, Theorem 7.7] and from Lemma 4.1.(a) together with Theorem 2.6, for the coweighting for \( \bar{F}^* \),

\[
\frac{-\chi(\bar{F}^1(K))}{|\bar{F}^*(K)|} = i^{[\bar{F}^*]}_{[K]} = \frac{-\chi(\bar{F}^*/K)}{|\bar{F}^*(K)|}
\]

show that \( \bar{F}1^{(1,K)} \) and \( \bar{F}^*/K \) have identical Euler characteristics for any object \( K \) of \( \bar{F}^* \). In fact, there are equivalences of categories

\[
i_K : \bar{F}^1(K) \rightarrow \bar{F}^*/K, \quad i_K : \bar{F}^1(K) \rightarrow \bar{F}^*/K
\]

On objects, \( Hi_K = i^H_K K \in \bar{F}^*(H, K) \), the class of the inclusion \( i^H_K \in \bar{F}^*(H, K) \) of \( H \) into \( K \), for any subgroup \( H \) of \( K \). Observe that there is an obvious identification of morphism sets

\[
\bar{F}^*_K(H_1, H_2) = (\bar{F}^*/K)(H_1i_K, H_2i_K)
\]

which defines the effect of the functor \( i_K \) on morphism sets in \( \bar{F}^*_K \). The functor \( r_H \) is a full and faithful functor by construction. It is also easily seen to be essentially surjective on objects. Thus \( i_K \) is an equivalence of categories. This applies to both above versions of \( i_K \).

By Lemma 4.1.(d),

\[
\text{supp}(\bar{F}^*/\bullet) \subset \text{Ob}(\bar{F}^*+e^\text{ab})
\]

and Bouc’s Theorem 3.3 shows that the inclusion of \( \bar{F}^*+e^\text{ab} \) into \( \bar{F}^* \) is a homotopy equivalence. \( \square \)

**Lemma 8.3.** Let \( H, N, \) and \( K \) be objects of \( F \) such that \( H \) is \( F \)-selfcentralizing and \( H \leq N \leq N_P(H) \). An \( F \)-morphism \( \varphi : H \rightarrow K \) extends to an \( F \)-morphism \( \psi : N \rightarrow K \) if and only if \( \bar{F}_N(H)H = \bar{F}_K(H)H \).

**Proof.** Since \( H \) is \( F \)-selfcentralizing, the same is true of \( H^\varphi \) and thus \( H^\psi \) is fully centralized in \( F \) [14, 4.8]. By the Extension Axiom for Frobenius \( P \)-categories, \( \varphi : H \rightarrow K \) extends to a morphism \( \rho : N \rightarrow P \) [14, 2.10.1]. We claim that \( (x)\rho \in K \) for all \( x \in N \). By assumption, there is some \( y \in K \) such that conjugation with \( (x)\rho \) and with \( y \) has the same effect on \( H^\varphi \). This means that \( (x)\rho y^{-1} \in C_P(H^\varphi) \leq Z(H^\varphi) \leq H^\varphi \leq K \), and thus \( (x)\rho \in K \). The corestriction \( \psi = K|\rho : N \rightarrow K \) of \( \psi : N \rightarrow P \) extends \( \varphi : H \rightarrow K \). \( \square \)

Consequently,

\[
\bar{F}(N, K) = \bar{F}(H, K)\bar{F}_N(H)
\]

under the assumptions of Lemma 8.3. (This is a reformulation of [6, Proposition 2.4]).

**Proof of Corollary 1.4.** This corollary follows immediately from the commutative diagram

\[
\begin{array}{ccc}
\bar{F}^* & \xrightarrow{\simeq} & \bar{F}^* \\
\downarrow & & \downarrow \\
\bar{F}^*+e^\text{ab} & \xrightarrow{\simeq} & \bar{F}^*+e^\text{ab}
\end{array}
\]
where the vertical morphisms are homotopy equivalences. □

9. LINKING CATEGORIES

Let $L^sfc$ be the centric linking system associated to a Frobenius $P$-category $F$ [4, Definition 1.7]. FACTS:

- All morphisms in $L^sfc$ are monomorphisms and epimorphisms [14, Proposition 24.2]
- The weighting for $L^sfc$ vanishes off the $F$-radical subgroups [11, Proposition 8.5]

**Theorem 9.1.** The inclusion functor $L^{sfc+rad} \to L^{sfc}$ is a homotopy equivalence.

**Proof.** Let $H$ be a $F$-selfcentralizing object of $F$. The functor $\bar{\pi} : L^{sfc} \to \bar{F}^{sfc}$ is bijective on objects and $\{K\}$-to-1 for on morphism sets $L^{sfc}(H, K) \to \bar{F}^{sfc}(H, K)$ with domain $K \in \text{Ob}(F^{sfc})$. $K = L^sfc(K) \leq L(K)$ acts freely from the right on $L(H, K)$ with quotient $L^{sfc}(H, K)/K = \bar{F}^{sfc}(H, K)$ [4, Lemma 1.10]. This implies that if $\varphi_1 \in L(H, K_1)$, $\varphi_2 \in L(H, K_2)$, and the commutative $\bar{F}$-diagram to the right has a solution

\[
\begin{array}{ccc}
H & \xrightarrow{\bar{\pi}} & \bar{F} \\
\varphi_1 & \downarrow & (\varphi_1)\pi \\
K_1 & \xrightarrow{\pi} & K_2 \\
\end{array}
\]

then the commutative $L$-diagram to the left has a unique solution [4, Lemma 1.10]. Consider the functor $H/\bar{\pi} : H/L^{sfc} \to H/\bar{F}^{sfc}$ induced by the functor $\bar{\pi} : L^{sfc} \to \bar{F}^{sfc}$. The above considerations mean that any $\varphi \in L(H, K) \subset \text{Ob}(H/L)$ is initial in the category $(\varphi)\bar{\pi}/H/\bar{\pi}$. By Quillen’s Theorem A (Theorem 3.1), $H/\bar{\pi}$ is a homotopy equivalence.

Restricting to the nonisomorphisms we get a homotopy equivalence $H/\bar{\pi} : H//L \to H//\bar{F}$. Compose these homotopy equivalences with the homotopy equivalences of (8.2) to get homotopy equivalences

\[
H/L^{sfc} \to S_{\bar{F}(H)} \quad H//L^{sfc} \to S^*_\bar{F}(H)
\]

By property (1.9),

\[
\text{supp}(\bullet//L^{sfc}) \subset \text{Ob}(L^{sfc+rad})
\]

and Bouc’s Theorem 3.3 shows that the inclusion of $L^{sfc+rad}$ into $L^{sfc}$ is a homotopy equivalence. □

The proof of the next proposition is similar to that of Proposition 6.1.

**Proposition 9.3.** The inclusion $L^{sfc+rad}_G \to L^{sfc}_G$ is a homotopy equivalence.

**Proof.** The two expressions, from [11, Theorem 1.1.(2)], and Lemma 4.1.(a) and Theorem 2.6, for the coweighting for $[L^sfc_G]$

\[
\frac{-\chi(S^{1, K}_K)}{|L^sfc_G(K)|} = k[L^sfc_G/K] = \frac{-\chi(L^sfc_G//K)}{|L^sfc_G(K)|}
\]

show that $S^{1, K}_K$ and $L^sfc_G//K$ have identical Euler characteristics for any object $K$ of $L^sfc_G$. In fact they are homotopy equivalent as we see in much the same way as in the proof of Proposition 6.1. The proof now follows from Bouc’s Theorem 3.3 because $\text{supp}(L^sfc_G//K) \subset \text{Ob}(L^{sfc+rad}_G)$ by Lemma 4.1.(b). □

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