ANNULAR REPRESENTATIONS OF FREE PRODUCT CATEGORIES
Shamindra Kumar Ghosh, Corey Jones and B Madhav Reddy

Abstract. We provide a description of the annular representation category of the free product of two rigid C*-tensor categories.

1. Introduction

Rigid C*-tensor categories have become important in recent years as descriptors of generalized symmetries appearing in noncommutative analysis and mathematical physics. In operator algebras, they are closely connected to the standard invariants of finite index subfactors, and appear as the representation categories of compact quantum groups. In the world of physics, they describe the superselection sectors in algebraic quantum field theories, and the structure of local excitations in 2 dimensional topological phases of matter.

An important algebra associated to a rigid C*-tensor category $\mathcal{C}$ is the tube algebra $\mathcal{A}\mathcal{C}$, first introduced by Ocneanu [O]. In the fusion case, this algebra has long been known as a useful tool for understanding the Drinfeld center (see [I, M2]), while its importance in the case when $\mathcal{C}$ has infinitely many simple objects has recently emerged. The tube algebra admits a universal C*-algebra, hence has a well behaved representation category (see [GJ]). This category provides a useful way to describe the analytic properties of rigid C*-tensor categories, such as amenability, the Haagerup property, and property (T). These properties were first introduced by Popa in the context of subfactors ([P1, P2]) and generalized to rigid C*-tensor categories by Popa and Vaes ([PV]). By [PSV, Theorem 3.4], this category also provides a representation-theoretic characterization of the category $\mathcal{Z}$(Ind-$\mathcal{C}$), introduced by Neshveyev and Yamashita to provide a categorical understanding of analytic properties ([NY1]). In a different direction, the annular representation theory of Temperley-Lieb-Jones categories has proved very useful in the classification of small index subfactor planar algebras ([J2, JR, JMS]).

Unlike rigid C*-tensor categories themselves, whose underlying categorical structure is trivial due to semi-simplicity, the representation category of the tube algebra is a large W*-category, and is complicated to describe. Thus an important problem is to find concrete descriptions of these large representation categories in terms of representation categories of more familiar C*-algebras such as group C*-algebras. There are many procedures for producing new rigid C*-tensor categories from old ones, such as Deligne tensor product, equivariantization, $G$-graded extensions, etc. A natural question is, if we understand the annular structure of our starting categories, can we describe the annular representation category of the one we have produced?

One such procedure is the free product construction, due to Bisch and Jones. In this note, we will provide a decomposition of the category of annular representations of a free product category into the direct sum of four full W*-subcategories, where each component has an
illuminating description. To state the main result, first recall that $C_u^*(\mathcal{C})$ is a C$^*$-completion of the fusion algebra of $\mathcal{C}$ with respect to admissible representations. $\text{Rep}_+(\mathcal{AC})$ denotes the full subcategory of annular representations which contain the fusion algebra, viewed as a corner of $\mathcal{AC}$, in their kernels. Finally, for two rigid C$^*$-tensor categories $\mathcal{C}$ and $\mathcal{D}$, we let $W$ be the set of words whose letters are alternatively taken from $\text{Irr}(\mathcal{C}) \setminus \{[1]\}$ and $\text{Irr}(\mathcal{D}) \setminus \{[1]\}$ of even length with first letter coming from $\text{Irr}(\mathcal{C}) \setminus \{[1]\}$. We define an equivalence relation on the set $W$ by $w_1 \sim w_2$ if $w_1 = uv$ and $w_2 = vu$. Set $W_0 := W/\sim$. Then the main result of the paper is as follows:

**Theorem 1.1.** Let $\mathcal{C}$ and $\mathcal{D}$ be rigid C$^*$-tensor categories. Then as W$^*$-categories,

$$\text{Rep}(A(C \ast D)) \cong \text{Rep}(C_u^*(\mathcal{C}) \ast C_u^*(\mathcal{D})) \oplus \text{Rep}_+(\mathcal{AC}) \oplus \text{Rep}_+(\mathcal{AD}) \oplus \text{Rep}(\mathbb{Z}) \oplus W_0$$

We remark that this decomposition is *not* topological, in the sense that the first component $\text{Rep}(C_u^*(\mathcal{C}) \ast C_u^*(\mathcal{D}))$ is not necessarily closed. In particular, it is possible to have a net of representations from $\text{Rep}(C_u^*(\mathcal{C}) \ast C_u^*(\mathcal{D}))$ converging to a representation in $\text{Rep}_+(\mathcal{AC}) \oplus \text{Rep}_+(\mathcal{AD})$ in the Fell topology. We do not pursue this here, but it would interesting to give a general characterization of the topology for free products.

The outline of the paper is as follows. In the preliminaries section, we describe the construction of the free product of two rigid C$^*$-tensor categories, as well as the basics of annular representation theory. In the third section, we show that the representation category of the tube algebra is equivalent to the representation category of another annular algebra with a more convenient weight set. We then provide a combinatorial analysis of the annular algebra vector spaces. In the fourth section, we use these results to deduce the main result. Finally we briefly discuss some examples.

**Acknowledgements.** The authors would like to thank Dietmar Bisch, Mike Hartglass, David Penneys and Jyotishman Bhowmick for several useful discussions. A part of this work was completed during the trimester program on von Neumann algebras (during May-Aug, 2016) at Hausdorff Research Institute for Mathematics (HIM) and the authors would like to thank HIM for the opportunity. Corey Jones was supported by Discovery Projects Subfactors and symmetries DP140100732 and Low dimensional categories DP160103479 from the Australian Research Council.

2. Preliminaries

2.1. C$^*$-tensor categories. We refer the reader to [NT, LR] for detailed definitions and basic results concerning C$^*$-tensor categories, and in particular, rigid C$^*$-tensor categories.

Recall that a $*$-structure on a C-linear category $\mathcal{C}$ is a conjugate linear functor from $*: \mathcal{C} \rightarrow \mathcal{C}^{\text{op}}$ which fixes objects, and satisfies $* \circ * = \text{Id}_C$. Such a category will be called a $*$-category for short. A C$^*$-category is a $*$-category, such that the morphism spaces $\mathcal{C}(a,b)$ are equipped with Banach norms $|| \cdot ||_{a,b}$ satisfying $||f^*f||_{a,a} = ||ff^*||_{b,b} = ||f||_{a,b}^2$ for all $f \in \mathcal{C}(a,b)$. Note that this makes each $\mathcal{C}(a,a)$ into a C$^*$-algebra, and we further require that $f^*f \geq 0$ in the C$^*$-algebra $\mathcal{C}(a,a)$ for all $f \in \mathcal{C}(a,b)$. A W$^*$-category is a C$^*$-category such that each morphism space has a predual ([GLR]). We remark that although the norms on the spaces appear as additional structure, being a C$^*$ (or W$^*$)-category is actually a property
of a $\ast$-category. Indeed, one can take the semi-norms given by the spectral radius, and ask if they satisfy the conditions listed above. In particular it makes sense to say a $\ast$-category is a C* (or W*)-category without specifying extra structure.

An object $a$ in a C*-category is called simple if $C(a, a) = \mathbb{C}1_a$, and C is called semi-simple if it has (unitary) direct sums, (self-adjoint) sub-objects, and every object is isomorphic to the direct sum of finitely many simple objects. Note that in a semi-simple C*-category, all morphism spaces are finite dimensional, and so a semi-simple category is C* if and only if it is W*.

A C*-tensor category is a C*-category equipped with a bilinear functor $\otimes: C \times C \to C$ together with unitary associativity natural transformations (called the associators) satisfying the pentagon axioms, and a distinguished unit object $1 \in C$ with unitary unitor natural isomorphisms satisfying the triangle axioms. By MacLane’s strictness theorem we can (and usually do) assume our categories are strict, so that the associators and unitors are all identities. In particular, this makes it easy to write tensor equations, and apply the usual graphical calculus formalisms.

A C*-tensor category has duals if for every object $a \in C$, there exists an object $a^* \in C$ and maps $R \in C(1, a \otimes a^*)$ and $\overline{R} \in C(1, a^* \otimes 1)$ satisfying the duality equations:

$$(1_a \otimes R^*) \circ (\overline{R} \otimes 1_a) = 1_a \quad \text{and} \quad (R^* \otimes 1_{a^*}) \circ (1_{a^*} \otimes \overline{R}) = 1_{a^*}$$

**Definition 2.1.** A rigid C*-tensor category is a semi-simple C*-tensor category with duals and simple tensor unit.

Our definition of a rigid C*-tensor category is not universal, but is by far the most commonly studied.

We recall here that for a semi-simple tensor category $C$, the fusion algebra is the complex linear span of isomorphism classes of simple objects, with product given by the linear extension of $[a] \cdot [b] := \sum_{c \in \text{Irr}(C)} N^{c}_{ab}[c]$, where $N_{ab}^{c} = \dim (C(a \otimes b, c))$. This is an associative, unital algebra. When $C$, in addition, is rigid, there is a $\ast$-structure on this algebra, given by the conjugate linear extension of $[a]^* := [\overline{a}]$. This associative $\ast$-algebra is denoted $\text{Fus}(C)$.

We have the following large class of (not mutually exclusive) examples, which indicate their connections with other areas of mathematics and physics:

1. The category of bifinite Hilbert space bimodules of an II$_1$ factor.
2. $\text{Rep}(\mathbb{G})$ for $\mathbb{G}$ a compact (quantum) group.
3. The DHR category of a covariant net of von Neumann algebras.

The first example is actually universal, in the sense that every (countably generated) rigid C*-tensor category arises as a full subcategory of bimodules of the group von Neumann algebra $L\mathbb{F}_\infty$ ($[PS][BHP]$).

**2.2. Free product of categories.** In this subsection, we provide the definition of the free product of two semi-simple C*-tensor categories with simple tensor units. This notion, due to Bisch and Jones, arises from the free composition of finite index subfactors (see [BJ]). It also appears in the study of free products of compact quantum groups [W]. Our approach to free products closely follows the construction of Bisch and Jones as elaborated by [IMP], except we do not require duals in our categories.
To proceed with this construction, we will first define a certain $\mathbb{C}^*$-category involving the two given categories, and controlled by non-crossing partitions. The free product will be the resulting projection category.

Let $\mathcal{C}_+$ and $\mathcal{C}_-$ be two semi-simple $\mathbb{C}^*$-categories with simple tensor units $\mathbbm{1}_+$ and $\mathbbm{1}_-$ respectively. In our construction, we pick a strict model of $\mathcal{C}_\pm$. Let $\Sigma$ be the set of words with letters in $\text{Obj} \ \mathcal{C}_+ \cup \text{Obj} \ \mathcal{C}_-$.

Let $\mathcal{C}_+$ and $\mathcal{C}_-$ be two semi-simple $\mathbb{C}^*$-categories with simple tensor units $\mathbbm{1}_+$ and $\mathbbm{1}_-$ respectively. In our construction, we pick a strict model of $\mathcal{C}_\pm$. Let $\Sigma$ be the set of words with letters in $\text{Obj} \ \mathcal{C}_+ \cup \text{Obj} \ \mathcal{C}_-$.

For a word $\sigma \in \Sigma$, we associate the subword (whose letters are not necessarily adjacent) $\sigma_+ \in \text{Obj} \ \mathcal{C}_+$ (resp., $\sigma_- \in \text{Obj} \ \mathcal{C}_-$) consisting of all the letters in $\sigma$ coming from $\text{Obj} \ \mathcal{C}_+$ (resp., $\text{Obj} \ \mathcal{C}_-$). The object obtained by tensoring the letters in $\sigma_\pm$ will be denoted by $t(\sigma_\pm)$ with the convention $t(\emptyset) = 1_\pm$ where appropriate. For instance, if $\sigma = a_1^+ a_2^- a_3^+ a_4^- a_5^+$, then $\sigma_+ = a_1^+ a_3^+$, $t(\sigma_+) = a_1^+ \otimes a_3^+$, $\sigma_- = a_2^- a_4^- a_5^-$ and $t(\sigma_-) = a_2^- \otimes a_4^- \otimes a_5^-$. Let $\Sigma$ be the set of words with letters in $\text{Obj} \ \mathcal{C}_+ \cup \text{Obj} \ \mathcal{C}_-$.

**Definition 2.2.** Let $\sigma, \tau \in \Sigma$. A $(\sigma, \tau)$-NCP consists of:

- a non-crossing partitioning of the letters in $\sigma$ and $\tau$ arranged at the bottom and on the top edges of a rectangle respectively moving from left to right, such that each partition block consists only of objects from $\mathcal{C}_+$ or only of objects $\mathcal{C}_-$,
- every block gives a pair of (possible empty) subwords of $\sigma$ and $\tau$, say, $(\sigma_0, \tau_0)$, where $\sigma_0$ (resp. $\tau_0$) consists of letters in the partition coming from $\sigma$ (resp. $\tau$). For each such block, seen as a rectangle with the bottom labeled by $\sigma_0$ and the top labeled by $\tau_0$, we choose a morphism from $t(\sigma_0)$ to $t(\tau_0)$ in the appropriate category.

We give an example of a $(\sigma, \tau)$-NCP in Figure 2.1 where $\sigma = a_1^+ a_2^- a_3^+ a_4^- a_5^+ a_6^+ a_7^- a_8^+$ and $\tau = b_1^+ b_2^- b_3^+ b_4^- b_5^+$ with $a_i^, b_j^ \in \mathcal{C}_\varepsilon$, $\varepsilon \in \{+, -\}$.

![Figure 2.1](image)

Here, the pair of subwords corresponding to the partition blocks are $\rho_1 = (a_1^+ a_2^+, b_1^+ b_4^+)$, $\rho_2 = (\emptyset, b_2^- b_3^+), \rho_3 = (a_3^+ a_4^+, b_2^-)$, $\rho_4 = (a_4^- a_5^-, \emptyset)$, and $\rho_5 = (a_5^+ a_7^+, \emptyset)$. Note that each letter of $\rho_i$ either belongs $\text{Obj} \ \mathcal{C}_+$ alone or $\text{Obj} \ \mathcal{C}_-$ alone, for every $i$ and each of $\rho_i$ is assigned a morphism from the corresponding category. For instance, all the letters of $\rho_3$ are objects of $\mathcal{C}_+$ and is assigned the morphism $f_3 \in \mathcal{C}_+(a_3^+ \otimes a_5^+, b_5^+)$. We denote the set of such $(\sigma, \tau)$-NCPs by $\text{NCP}(\sigma, \tau)$. Now, to every $T \in \text{NCP}(\sigma, \tau)$, we can associate unique $T_\pm \in \text{NCP}(\sigma_\pm, \tau_\pm)$ by deleting all blocks whose letters are labeled by the opposite sign.
Since all letters in $\sigma_\pm$ and $\tau_\pm$ come from either $C_+$ or $C_-$ only, the non-crossing partitions $T_\pm$ give rise to unique morphisms $Z_{T_\pm} \in C_\pm (t(\sigma_\pm), t(\tau_\pm))$ using the standard graphical calculus for monoidal categories.

So, for any $\sigma, \tau \in \Sigma$ and $T \in NC\mathcal{P}(\sigma, \tau)$, we have morphisms $Z_{T_\pm} \in C_\pm (t(\sigma_\pm), t(\tau_\pm))$. We write $Z_T := Z_{T_+} \otimes Z_{T_-} \in C_+ (t(\sigma_+), t(\tau_+)) \otimes C_- (t(\sigma_-), t(\tau_-))$. For example, for the NCP $T$ in Figure 2.1

$$Z_{T_+} = (f_1 \otimes f_3) \circ (1_{a_1^+} \otimes a_2^+ \otimes a_3^- \otimes f_5 \otimes 1_{a_2^-})$$ and $Z_{T_-} = f_2 \circ f_4$

We define the category $\mathcal{NCP}$ as follows:

- **Objects in $\mathcal{NCP}$** are given by $\Sigma$.

- **For** $\sigma, \tau \in \Sigma$, **the morphism space** is defined by

$$\mathcal{NCP}(\sigma, \tau) := \text{span} \{Z_T : T \in NC\mathcal{P}(\sigma, \tau)\} \subset C_+ (t(\sigma_+), t(\tau_+)) \otimes C_- (t(\sigma_-), t(\tau_-)).$$

Composition of morphisms is given by composing the tensor components, which is obviously bilinear, and associative. However, one needs to verify whether the morphism spaces of $\mathcal{NCP}$ are closed under such composition. Let $S \in NC\mathcal{P}(\sigma, \tau)$ and $T \in NC\mathcal{P}(\tau, \kappa)$. Consider the ‘composed’ rectangle obtained by gluing $T$ on the top of $S$ matching along the letters of $\tau$. The non-crossing partitions of $S$ and $T$ induce a non-crossing partition on the composed rectangle with $\sigma$ at the bottom and $\tau$ on the top; each partition is then labeled by composing the corresponding morphisms in $S$ and $T$. We call this $T \circ S \in NC\mathcal{P}(\sigma, \kappa)$.

In this process of composing two NCPs, we have ignored certain partitions of $S$ (staying only on its top) and $T$ (staying only at its bottom) which cancel each other and do not contribute towards the non-crossing partitioning of the composed rectangle. Since the tensor units $1_\pm$ are assumed to be simple, composing the morphisms associated to these partitions simply yield a scalar. Suppose $\lambda(T, S)$ denote the product of all such scalars. Then, $(Z_{T_+} \circ Z_{S_+}) \otimes (Z_{T_-} \circ Z_{S_-}) = \lambda(T, S) Z_{(T \circ S)_+} \otimes Z_{(T \circ S)_-} \in \mathcal{NCP}(\sigma, \kappa)$.

Clearly, $\mathcal{NCP}$ is a $\mathbb{C}$-linear category. There is also a $*$-structure given by applying $*$ on each of the tensor components. To see whether the morphism spaces of $\mathcal{NCP}$ is closed under $*$, we define an involution \(\left( NC\mathcal{P}(\sigma, \tau) \ni T \mapsto T^* \in NC\mathcal{P}(\tau, \sigma)\right)_{\sigma, \tau \in \Sigma}\) where we reflect $T$ about any horizontal line to obtain $T^*$ with a non-crossing partitioning and their corresponding morphisms being induced by the reflection of the initial partitioning and $*$ of the assigned morphisms in $T$ respectively.

Indeed, $Z_T^* = Z_{T^*} \in \mathcal{NCP}(\tau, \sigma)$ for all $T \in NC\mathcal{P}(\sigma, \tau)$. Thus, $\mathcal{NCP}$ is a $*$-category. Note that by construction, $\mathcal{NCP}$ is equipped with a canonical faithful $*$-functor to the Deligne tensor product $C_+ \boxtimes C_-$, which sends $\sigma$ to $\sigma_+ \boxtimes \sigma_- \in C_+ \boxtimes C_-$. Since $C_\pm$ are both semi-simple, the Deligne tensor product is again a $C^*$-category with finite dimensional morphism spaces. But any (not necessarily full) $*$-subcategory of a $C^*$-category with finite dimensional morphism spaces is easily seen to be $C^*$ itself. Since our canonical functor is faithful, this implies $\mathcal{NCP}$ is a $C^*$-category.

For the tensor structure, define $\sigma \otimes \tau$ as the concatenated word $\sigma \tau$. If $f = \sum a_i \otimes b_i \in \mathcal{NCP}(\sigma, \tau) \subset C_+ (\sigma_+, \tau_+) \otimes C_- (\sigma_-, \tau_-)$ and $g = \sum c_j \otimes d_j \in \mathcal{NCP}(\kappa, \nu) \subset C_+ (\kappa_+, \nu_+) \otimes C_- (\kappa_-, \nu_-)$, then

$$f \otimes g = \sum (a_i \otimes c_j) \otimes (b_i \otimes d_j) \in \mathcal{NCP}(\sigma \otimes \kappa, \tau \otimes \nu).$$
Proof. First we show that the objects \( C \) simple objects in \( C \) Proposition 2.4. Let \( \text{Irr}(\sigma) \) denote a choice of object from each isomorphism class of simple objects, such that the tensor units are chosen to represent their isomorphism class. Then \( \sigma \in \mathbb{N} \text{CP} \) see that \( \sigma \in \mathbb{N} \text{CP} \) are simple) which implies \( \sigma \in \mathbb{N} \text{CP} \) is isomorphic to a direct sum of simple objects in \( \Sigma \) such that the letters in each \( \Sigma \) come alternatively from \( C_+ \) and \( C_- \), and the partitions are non-crossing, the partition blocks should be of the form \( (a_1^{\text{bottom}}, a_1^{\text{top}}), (a_2^{\text{bottom}}, a_2^{\text{top}}), \ldots \), where \( \sigma = a_1 a_2 \ldots \). The assigned morphisms of these blocks are then scalars since \( a_i \)'s are simple. This says that \( \sigma \in \mathbb{N} \text{CP} \) is one-dimensional implying \( \sigma \) is simple for all \( \sigma \in \Sigma_0 \). Similar arguments will tell us that \( \mathbb{N} \text{CP}(\sigma, \tau) \) is zero for two distinct \( \sigma, \tau \in \Sigma_0 \).

We now show \( \Sigma_0 \) is complete, in the sense that any object \( \sigma \in \mathbb{N} \text{CP} \) is isomorphic to a direct sum of objects from \( \Sigma_0 \). Observe that if \( \sigma_1, \ldots, \sigma_n \in \Sigma \) such that the letters in each \( \sigma_i \) come from \( C_+ \) alone or \( C_- \) alone, then \( \sigma_1 \ldots \sigma_n \) is isomorphic to the word \( t(\sigma_1) \ldots t(\sigma_n) \). Moreover, a quick sketch of non-crossing partitions shows that \( \sigma \mathbb{1}_{\pm} \tau \cong \sigma \tau \). It is also easy to see that if \( a \cong b_1 \oplus b_2 \) in \( C_+ \) via decomposition isometries \( \nu_i \in C_+(b_i, a) \), then the word \( \sigma \tau \cong \sigma b_1 \tau \oplus \sigma b_2 \tau \) via decomposition isometries given by the \( (\sigma b_1 \tau, \sigma \tau) \)-non-crossing partitions \( T_i \) defined as follows: The underlying non-crossing partition has pairings which connect elements vertically, and for each block ending in \( \sigma \) or \( \tau \), we have the identity morphism, while the block connecting \( b_i \) with \( a \) is assigned the isometry \( \nu_i \). Taken together, these observations imply that any object can be decomposed as a finite direct sum of words in \( \Sigma_0 \).

\( C_+ * C_- \) inherits all the above properties from \( \mathbb{N} \text{CP} \). In particular, since every object \( \sigma \in \mathbb{N} \text{CP} \) is isomorphic to a direct sum of simple objects in \( \Sigma_0 \), this will be true for any
exists an object \((\sigma, 1_\sigma)\) for \(\sigma \in \Sigma_0\).

Thus to show that \(C_+ \ast C_-\) has direct sums, it suffices to show that for \(\sigma, \tau \in \Sigma_0\), there exists an object \((\sigma, 1_\sigma) \oplus (\tau, 1_\tau) \in C_+ \ast C_-\) satisfying direct condition.

Let \(\alpha_i\) and \(\varepsilon_j\) be the signs given by \(a_i \in C_{\alpha_i}\) and \(b_j \in C_{\tau_j}\). Consider \(\widehat{a}_i := a_i \oplus 1_{\alpha_i}\) implemented by the isometries \(u_i \in C_{\alpha_i}(a_i, \widehat{a}_i)\) and \(e_i \in C_{\alpha_i}(1_{\alpha_i}, \widehat{a}_i)\). Similarly, pick \(\widehat{b}_j := b_j \oplus 1_{\varepsilon_j}\) and implementing isometries \(v_j \in C_{\tau_j}(b_j, \widehat{b}_j)\) and \(f_j \in C_{\tau_j}(1_{\varepsilon_j}, \widehat{b}_j)\). Set

\[
\begin{aligned}
\tilde{\sigma} &:= \widehat{a}_1 \ldots \widehat{a}_m \text{ and } \sigma':= a_1 \ldots a_m 1_{\varepsilon_1} \ldots 1_{\varepsilon_n}, \\
\tilde{\tau} &:= \widehat{b}_1 \ldots \widehat{b}_n \text{ and } \tau':= 1_{\alpha_1} \ldots 1_{\alpha_m} b_1 \ldots b_n, \\
\gamma &:= \tilde{\sigma} \tilde{\tau}.
\end{aligned}
\]

We have already seen that \(\sigma' \cong \sigma\) and \(\tau' \cong \tau\) in \(\mathcal{NCP}\). Consider the isometries \(u := u_1 \otimes \cdots \otimes u_m \otimes 1_{\varepsilon_1} \otimes \cdots \otimes 1_{\varepsilon_n} \in \mathcal{NCP}(\sigma', \gamma)\) and \(v := 1_{\alpha_1} \otimes \cdots \otimes 1_{\alpha_m} \otimes v_1 \otimes \cdots \otimes v_n \in \mathcal{NCP}(\tau', \gamma)\). Note that projections \(uu^*\) and \(vv^*\) are mutually orthogonal in \(\mathcal{NCP}(\gamma, \gamma)\) (since \((u_i u_i^*, e_i e_i^*)\) and \((v_j v_j^*, f_j f_j^*)\) are pairs of mutually orthogonal projections). So, we have a projection in \(\mathcal{NCP}(\gamma, \gamma)\), namely \((uu^* + vv^*) \cong 1_{\sigma'} \oplus 1_{\tau'} \cong 1_{\sigma} \oplus 1_{\tau}\) in \(C_+ \ast C_-\).

\[\square\]

2.3. Annular representations. Now we recall from [CL], the definition and basic properties of annular algebras, and their representation categories associated to a rigid C*-tensor category \(\mathcal{C}\). For a simple object \(a \in \mathcal{C}\) and an arbitrary object \(b \in \mathcal{C}\), we naturally have an inner product on \(\mathcal{C}(a, b)\) given by \(g^* f = \langle f, g \rangle 1_a\). Let \(\text{Irr}(\mathcal{C})\) denote a set of representatives of isomorphism classes of simple objects in \(\mathcal{C}\). We assume that \(1 \in \text{Irr}(\mathcal{C})\) is chosen to represent its isomorphism class. Let \(\Lambda\) be any subset of the set representatives of isomorphism classes of all objects in \(\mathcal{C}\). Then the annular algebra with weight set \(\Lambda\) is defined as a vector space

\[
\mathcal{A} \Lambda := \bigoplus_{b, c \in \Lambda, a \in \text{Irr}(\mathcal{C})} \mathcal{C}(a \otimes b, c \otimes a).
\]

For \(f \in \mathcal{C}(a_1 \otimes b_1, b_2 \otimes a_1)\) and \(g \in \mathcal{C}(a_2 \otimes b_3, b_4 \otimes a_2)\), multiplication in \(\mathcal{A} \Lambda\) is given by

\[
f \cdot g := \delta_{b_1 = b_4} \sum_{c \in \Lambda} \sum_{u \in \text{onb}(\mathcal{C}(c, a_1 \otimes a_2))} (1_{b_2} \otimes u^*)(f \otimes 1_{b_2})(1_{a_1} \otimes g)(u \otimes 1_{b_3})
\]

where \(\text{onb}\) denotes an orthonormal basis with respect to the inner product defined above. This multiplication is associative and is independent of choice of representatives of isomorphism classes of simple objects and choice of \(\text{onb}\) in consideration. \(\mathcal{A} \Lambda\) has a \(*\)-structure, which we denote by \(#\), defined by

\[
f^# := (R_a^* \otimes 1_{b_1} \otimes 1_a)(1_{\bar{a}} \otimes f^* \otimes 1_{\bar{a}})(1_{\bar{a}} \otimes 1_{b_2} \otimes \bar{R}_a)
\]

for \(f \in \mathcal{C}(a \otimes b_1, b_2 \otimes a)\). The associative \(*\)-algebra \(\mathcal{A} \Lambda\) is unital if and only if \(\text{Irr}(\mathcal{C}) < \infty\). This algebra has a canonical trace defined by \(\Omega(f) := \delta_{a=c} \delta_{n=1} tr(f)\) for all \(f \in \mathcal{C}(a \otimes b, c \otimes a)\), where \(tr\) is the unnormalized categorical trace on \(\mathcal{C}(b, b)\), \(tr(f) := R_b^*(1_{\bar{a}} \otimes f)R_b = \bar{R}_b(f \otimes 1_b)\bar{R}_b\).

We denote the subspaces
\[ \mathcal{A} \Lambda_{a,c} := \mathcal{C}(a \otimes b, c \otimes a) \subset \mathcal{A} \Lambda \text{ and } \mathcal{A} \Lambda_{b,c} := \bigoplus_{a \in \text{Irr}(\mathcal{C})} \mathcal{A} \Lambda_{b,c}^a \subseteq \mathcal{A} \Lambda \]

The associative \(*\)-algebra \(\mathcal{A} \Lambda_{b,b}\) is called the weight \(b\) centralizer algebra. We call \(\mathcal{A} \Lambda_{1,1}\) the weight 0 centralizer algebra, primarily for historical reasons in connection with planar algebras. It turns out that the fusion algebra of \(\mathcal{C}\), \(\text{Fus}(\mathcal{C})\), is \(*\)-isomorphic to \(\mathcal{A} \Lambda_{1,1}/BD\) (See [GJ, Proposition 3.1]).

The annular category with weight set \(\Lambda\) is the category with objects space as \(\Lambda\) and the morphism space from \(b\) to \(c\) as \(\mathcal{A} \Lambda_{b,c}\). Composition is given by the multiplication defined above. Both the algebra as well as category are often denoted by \(\mathcal{A} \Lambda\). Since both of these essentially contain the same information, they are used interchangeably.

The tube algebra, \(\mathcal{A} \mathcal{C}\) is (by a slight abuse of notation) the annular algebra the weight set \(\text{Irr}(\mathcal{C})\). This algebra was first introduced by Ocneanu ([O]). A weight set \(\Lambda \subseteq \text{Irr}(\mathcal{C})\) is said to be full if every simple object is equivalent to subobject of some \(b \in \Lambda\). By [GJ, Proposition 3.5], any annular algebra with full weight set is strongly Morita equivalent to the tube algebra.

The representation category \(\text{Rep}(\mathcal{A} \Lambda)\) is the category of non-degenerate \(*\)-representations of \(\mathcal{A} \Lambda\) as bounded operators on a Hilbert space, with bounded intertwiners as morphisms. This is a \(W^*\)-category. By our above comments, whenever \(\Lambda\) is full, we have \(\text{Rep}(\mathcal{A} \Lambda) \cong \text{Rep}(\mathcal{A})\) as \(W^*\)-categories, and thus it makes sense to talk about the category of annular representations, which can be realized as the representation category of any annular algebra with full weight set. We shall see in Section 3 that the weight set can further be reduced in some cases without affecting the resulting category of annular representations.

One of the reasons these categories are nice is that the tube algebra (or any full annular algebra) admits a universal C*-algebra, \(C^*(\mathcal{A} \Lambda)\), such that \(\text{Rep}(\mathcal{A} \Lambda) \cong \text{Rep}(C^*(\mathcal{A} \Lambda))\), where the latter is the category of non-degenerate, continuous \(*\)-homomorphisms from the C*-algebra \(C^*(\mathcal{A} \Lambda)\) to \(\mathcal{B}(\mathcal{H})\). For example, this tells us that the category decomposes as a direct integral of factor representations.

One way to access this category is to understand the representation theory of the unital centralizer algebras \(\mathcal{A} \Lambda_{a,a}\). If \(a \in \Lambda\), a linear functional \(\phi : \mathcal{A} \Lambda_{a,a} \to \mathbb{C}\) with \(\phi(1) = 1\) is said to be weight \(a\) annular state, or an admissible state, if \(\phi(f^* \cdot f) \geq 0\) for every \(f \in \mathcal{A} \Lambda_{a,b}\) and \(b \in \Lambda\).

Using a GNS construction, each annular state gives a non-degenerate representation of \(\mathcal{A} \Lambda\) (see [GJ, Section 4]). Annular states provide a useful way of constructing representations of whole algebra by looking at representations of much smaller centralizer algebras or even subalgebras of the tube algebra. A representation \((\pi, \mathcal{H})\) of a centralizer algebra \(\mathcal{A} \Lambda_{a,a}\) is said to be admissible if there exists a representation \((\widetilde{\pi}, \widetilde{\mathcal{H}})\) of \(\mathcal{A} \Lambda\) such that \((\widetilde{\pi}, \widetilde{\mathcal{H}})|_{\mathcal{A} \Lambda_{a,a}}\) is unitarily equivalent to \((\pi, \mathcal{H})\). There are several equivalent conditions for a representation of centralizer algebra to be admissible. One such condition is that every vector state in \((\pi, \mathcal{H})\) is an annular (i.e. admissible) state. It turns out that we can construct a universal C*-algebra \(C_u^*(\mathcal{A}_{a,a})\) with respect to admissible representations, so that admissibility of \((\pi, \mathcal{H})\) is equivalent to saying that \((\pi, \mathcal{H})\) extends to a representation of \(C_u^*(\mathcal{A}_{a,a})\). This algebra is a corner of the universal C*-algebra of the entire tube algebra, so all the pieces fit together nicely.
3. Annular algebra of free product of categories

We will characterize the annular algebra of \( \mathcal{C} \ast \mathcal{D} \) where \( \mathcal{C} \) and \( \mathcal{D} \) are rigid, semi-simple C*-tensor categories with simple unit objects. We note that while providing definitions of the free product \( \mathcal{C}_\pm \) was more convenient to distinguish the two categories, while in this section, using \( \mathcal{C} \) and \( \mathcal{D} \) seems better. By [CL], the annular representation category can be obtained from representations of any annular algebra with respect to any full weight set in \( \text{Obj}(\mathcal{C} \ast \mathcal{D}) \) (in particular, a set of representatives of the isomorphism classes of simple objects).

However, in our case, we can actually work with a smaller, non-full weight set, and still capture the entire category. To describe this weight set, let \( I_C \) (respectively \( I_D \)) be a set of representatives of the isomorphism classes of simple objects in \( \mathcal{C} \) (respectively \( \mathcal{D} \)) excluding the isomorphism class of the unit object. Recall that the set of words (including the empty one) with letters coming alternatively from \( I_C \) and \( I_D \) is in bijective correspondence with the set of isomorphism classes of simple objects \( \text{Irr}(\mathcal{C} \ast \mathcal{D}) \), where the empty word corresponds to the tensor unit in \( \mathcal{C} \ast \mathcal{D} \). We define \( W \) to be the subset of these words with \textit{strictly positive and even} length, such that the first letter comes from \( I_C \). We will say a positive length word is a \( \mathcal{C} \ast \mathcal{D} \) word if it starts with a letter of \( \mathcal{C} \) and ends with a letter of \( \mathcal{D} \), and extend this terminology in the obvious way. We define the weight set \( \Lambda := \{ \emptyset \} \cup I_C \cup I_D \cup W \), which we note is not full. Indeed, the alternating words of odd length and the alternating words of even length starting with a letter from \( I_D \) do not appear in \( \Lambda \). Nevertheless, we have the following result:

**Lemma 3.1.** \( \text{Rep}(\mathcal{A} \Lambda) \) and the representation category of the tube algebra \( \mathcal{A} \) of \( \mathcal{C} \ast \mathcal{D} \), are unitarily equivalent as linear *-categories.

**Proof.** Clearly, the restriction functor \( \text{Res} : \text{Rep}(\mathcal{A}) \longrightarrow \text{Rep}(\mathcal{A} \Lambda) \) is a linear *-functor. We begin by showing that \( \text{Res} \) is essentially surjective.

Given a representation \( (\pi, V) \) of \( \mathcal{A} \Lambda \) and \( w \in \text{Irr}(\mathcal{C} \ast \mathcal{D}) \), we consider the vector space \( \bigoplus_{v \in \Lambda} \{ A_{v,w} \otimes V_v \} \). We define a sesquilinear form \( \langle \cdot, \cdot \rangle \) on this vector space by \( \langle y_1 \otimes \xi_1, y_2 \otimes \xi_2 \rangle_w := \langle \pi(y_2 \# \cdot y_1) \xi_1, \xi_2 \rangle_{v_2} \), where \( y_i \in A_{v_i,w} \) and \( \xi_i \in V_{v_i} \).

We first want to show that \( \langle x, x \rangle_w \geq 0 \) for any vector \( x = \sum_{i=1}^n y_i \otimes \xi_i \). But we have

\[
\langle x, x \rangle_w = \langle T \xi, \xi \rangle, \text{ where } T = \left( \pi(y_i^\# \cdot y_j) \right)_{i,j} : \bigoplus_{i=1}^n V_{v_i} \longrightarrow \bigoplus_{i=1}^n V_{v_i}, \text{ and } \xi = (\xi_i)_{i} \in \bigoplus_{i=1}^n V_{v_i}.
\]

If \( w \in \Lambda \), then \( T \) is clearly a positive operator and hence we have non-negativity of \( \langle x, x \rangle_w \). Suppose now that \( w \) has even length and its first letter is in \( I_D \), say \( w = d_1 c_1 d_2 c_2 \ldots d_k c_k \). Consider the word \( w' = c_1 d_2 c_2 \ldots d_k c_k d_1 \in \Lambda \). Let \( \rho \in A_{w',w} \) be the canonical rotation unitary. Then, for any \( y \in A_{v,w} \), there is a unique \( y' \in A_{v,w'} \) such that \( y = \rho \cdot y' \). Thus we have

\[
T = \left( \pi(y_i^\# \cdot y_j) \right)_{i,j} = \left( \pi((\rho \cdot y_i')^\# \cdot (\rho \cdot y_j')) \right)_{i,j} = \left( \pi(y_i'^\# \cdot y_j') \right)_{i,j},
\]

hence positivity follows from the previous case. Defining \( \overline{\Lambda} \) to be the union of \( \Lambda \) and the set of words of even length (regardless of starting character), we have just shown positivity for weights in \( \overline{\Lambda} \).
Now suppose $w$ has odd length; say $w = a_{-k} \ldots a_{-1}a_0a_1 \ldots a_k$. Note that the $a_{2l}$’s are either all in $I_C$ or all in $I_D$, and similarly for the odd letters. Now define the word $w' = a_0a_1 \ldots a_k a_{-k} \ldots a_{-1}$. This word no longer represents an isomorphism class of simple object, however the object it represents is isomorphic to a direct sum of simple objects, all of which have even length, i.e., $w' \cong \otimes_s u_s$, where $u_s \in \Lambda$. Let $p_s \in (C \ast D)(u_s, w')$ be isometries such that $\sum_s p_s p_s^* = 1_{w'}$ (which automatically implies $p_s p_t = \delta_{s,0}1_{u_s}$).

Let $A_{Obj}$ denote the annular algebra whose weight set consists of all isomorphism classes of objects in $C \ast D$, and pick any rotation $\rho \in A_{Obj}_{w',w}$ (which is automatically unitary). Then any element $A_{\rho}$ in $A_{Obj}$ can be written $A_{\rho} = \sum_s \rho \cdot p_s \cdot y_{s,i}$, where $y_{s,i} := p_s \cdot \rho^\# \cdot y_i \in A_{\rho_{v_i,u_s}}$.

Observe that

$$T = \left( \pi(y_i^\# \cdot y_j) \right)_{i,j} = \left( \pi \left( \left[ \sum_s \rho \cdot p_s \cdot y_{s,i}^\# \cdot y_{s,i} \right] \cdot \left[ \sum_t \rho \cdot p_t \cdot y_{t,j} \right] \right) \right)_{i,j}$$

which is positive as all $u_s$’s are in $A$ and hence our argument is complete.

Now that we have shown $\langle x, x \rangle_w \geq 0$, we can define $Ind(V)_w$ as the Hilbert space obtained by the completion of the quotient of our vector space over the null space of the inner product. Before quotienting and completing, our vector space has the obvious action of $A$. Our above argument shows that $\langle \pi(\cdot)x, x \rangle_w$ is a positive annular functional. Thus by [GJ, Lemma 4.4], we have a well-defined, bounded, *-action of the tube algebra $A$ on $Ind(V)$. It is now easy to verify that $Res \circ Ind(V) \cong V$ via the interwinder defined by sending $\sum_t y_i \otimes \xi_t$ to $\pi(y_i)\xi_i$.

Now to prove that $Res$ is fully faithful, we first claim that any representation $(\theta, H) \in \text{Rep}(A)$ is generated by $\bigcup_{w \in \Lambda} H_w$. We need to check

$$H_w^0 := \text{span} \{ \theta(x)\xi : x \in A_{w,w}, \xi \in H_v, v \in \Lambda \}$$

is dense in $H_w$ for all $w \in \text{Irr}(C \ast D) \setminus \Lambda$; we will, in fact, show $H_w^0 = H_w$. Now, $w \in \text{Irr}(C \ast D) \setminus \Lambda$ implies $|w| \geq 2$. Suppose $w$ is of $D$-$C$ type, so that $w = ud$ for some $u$ of $C$-$C$ type. We have the unitary rotation

$$\rho := 1_d \otimes 1_u \otimes 1_d \in (C \ast D) \langle dw', wd \rangle = A_{w',w}^d \subset A_{w',w},$$

where $w' = ud \in \Lambda$, whose $\theta$-action implements a unitary from $H_{w'}$ to $H_w$; so, $H_w^0 = H_w$.

The remaining elements of $\text{Irr}(C \ast D) \setminus \Lambda$ are words of types $C$-$C$ or $D$-$D$ type, which necessarily have odd length $\geq 3$. Consider such a $w$, say $w = a_{-k} \ldots a_{-1}a_0a_1 \ldots a_k$. As above, the even $a_i$’s are either all in $I_C$ or all in $I_D$. Let $w' := a_0a_1 \ldots a_k \otimes a_{-k} \ldots a_{-1}$ or $a_1 \ldots a_k \otimes a_{-k} \ldots a_{-1}a_0$ depending on whether $a_0 \in I_C$ or $I_D$, and $\rho'$ be the rotation unitary from $w$ to $w'$. Note that $w'$ may no longer be simple; however, it decomposes into a direct sum of simple objects all of which either have even length or lie in $\Lambda$ (using the fusion rule). Suppose $w' \cong \oplus_i u_i$ is the simple object decomposition. Let $p_i \in (C \ast D)(u_i, w')$ be isometries such that $\sum_i p_i p_i^* = 1_{w'}$. Set $x_i := (\rho')^\# \cdot p_i \in A_{u_i,w}$. Clearly, $\sum_i x_i \cdot x_i^\# = 1_{w'}$ (in $A_{w,w}$).

Since the $u_i$’s belong to $\Lambda$, any $\xi \in H_w$ can be expressed as $\sum_i \theta(x_i)[\theta(x_i^\#)] \xi \in H_w^0$. 10
Thus our claim that any representation is generated by the Λ weight spaces is proven. This immediately implies that the restriction functor is faithful. It also shows that Res is full. Indeed, consider a morphism \( f : \text{Res}(\pi, \mathcal{H}) \to \text{Res}(\gamma, \mathcal{K}) \) in \( \text{Rep}(\mathcal{A} \Lambda) \). For \( w \in \text{Irr}(\mathcal{C} \ast \mathcal{D}) \setminus \Lambda \), if an \( \mathcal{A} \)-linear extension of \( f \) exists we see that \( f(\sum \pi(y_i)\xi_i) = \sum \gamma(y_i)f(\xi_i) \), for \( y_i \in \mathcal{A}_{v,w} \), \( v \in \Lambda \), and \( \xi \in \mathcal{H}_v \). Indeed, this will serve as a definition of the extension, but we must show it is well defined. Suppose \( \sum \pi(y_i)\xi_i = 0 \). Then for any fixed \( j \), \( \sum \pi(y_j^# \cdot y_i)\xi_i = 0 \). Since \( y_j^# \cdot y_i \in \mathcal{A} \Lambda \), we have

\[
\sum_{i,j} \langle \gamma(y_i)f(\xi_i), \gamma(y_j)f(\xi_j) \rangle_k = \langle \gamma(y_j^# \cdot y_i)f(\xi_i), f(\xi_j) \rangle_k
\]

\[
= \sum_{j} \sum_{i} \langle \pi(y_j^# \cdot y_i)\xi_i, f^*f(\xi_j) \rangle_{\mathcal{H}} = 0
\]

It is easy to see that the extension of \( f \) remains bounded. This concludes the proof. \( \square \)

We proceed to the study of the \( * \)-algebra \( \mathcal{A} \Lambda \). We divide this into subsections corresponding to the length (denoted by \( |\cdot| \) of the words in \( \Lambda \)). Since the empty word (that is, zero length word) stands for the tensor unit of \( \mathcal{C} \ast \mathcal{D} \), the centralizer algebra \( \mathcal{A} \Lambda_{0,0} \) is isomorphic to the fusion algebra, and we will be able to describe admissible representations of these in terms of representations of free product \( \mathcal{C} \ast \mathcal{D} \)-algebras. Thus in this section, we will focus on the structure of \( \mathcal{A} \Lambda_{v,w} \) for words \( v, w \in \Lambda \) of positive length. By \( \mathcal{A} \mathcal{C} \) (resp., \( \mathcal{A} \mathcal{D} \)) we mean the tube algebra/category of \( \mathcal{C} \) (resp. \( \mathcal{D} \)).

3.1. Words of length at least 2. Define a relation on \( \mathcal{W} \) by \( w_1 \sim w_2 \) if and only if \( w_1 = uv \) and \( w_2 = vu \) for some subwords \( u, v \). Clearly, \( \sim \) defines an equivalence relation on \( \mathcal{W} \). Obviously if \( w_1 \sim w_2 \), then \( |w_1| = |w_2| \).

**Lemma 3.2.** For \( w_1, w_2 \in \mathcal{W} \), \( \mathcal{A} \Lambda_{w_1,w_2} \neq \{0\} \) if and only if \( w_1 \sim w_2 \).

**Proof.** Suppose \( w_1 \sim w_2 \) so that \( w_1 = uv \) and \( w_2 = vu \). Consider the rotation \( \rho := (1_v \otimes \overline{R}_u)(R_u^* \otimes 1_v) \in (\mathcal{C} \ast \mathcal{D})(\overline{u}w_1, w_2\overline{u}) \subseteq \mathcal{A}_{w_1,w_2} \) for any standard solution \( (R_u, \overline{R}_u) \) to the conjugate equation for \( (u, \overline{u}) \). It is non-zero (since it is unitary) and hence \( \mathcal{A} \Lambda_{w_1,w_2} \neq \{0\} \).

Now suppose \( \mathcal{A} \Lambda_{w_1,w_2} \neq \{0\} \) and without loss of generality, let \( w_1 \neq w_2 \). Then there exists \( v \in \text{Irr}(\mathcal{C} \ast \mathcal{D}) \) (of length, say, \( m > 0 \)) such that \( \mathcal{A} \Lambda_{w_1,w_2}^v \neq \{0\} \). Suppose \( m \) is odd. Then \( v \) is either of \( \mathcal{C} \)-\( \mathcal{C} \) type or \( \mathcal{D} \)-\( \mathcal{D} \) type. If \( v \) is of \( \mathcal{C} \)-\( \mathcal{C} \) type (resp. \( \mathcal{D} \)-\( \mathcal{D} \) type), then \( w_2v \) (resp. \( vw_1 \)) is simple and is of odd length, whereas \( vw_1 \) (resp. \( w_2v \)) is not simple and any simple subobject will be of length strictly smaller than that of \( vw_1 \). Hence \( \mathcal{A} \Lambda_{w_1,w_2}^v = \{0\} \) which is a contradiction. So \( m \) cannot be odd.

Thus \( m \) must be even, so \( v \) can be of \( \mathcal{C} \)-\( \mathcal{D} \) or \( \mathcal{D} \)-\( \mathcal{C} \) type. It is enough to consider the case where \( v \) is of \( \mathcal{C} \)-\( \mathcal{D} \) type, since the other case will follow by taking \( \# \). As \( w_1, w_2 \in \mathcal{W} \), \( vw_1 \) and \( w_2v \) are simple. Therefore, \( \mathcal{A} \Lambda_{w_1,w_2}^v \neq \{0\} \) implies the equality

\[
vw_1 = w_2v
\]

In particular, we see that \( w_1 \) and \( w_2 \) have the same length, say \( n \).
If $m = n$, then Equation 3.1 implies $w_1 = v = w_2$ which is not possible by assumption. Suppose $m < n$. By Equation 3.1 there exists a word $u$ such that $w_2 = vu$. So, $vw_1 = vv_1$ implying $w_1 = w$, and thus $w_1 \sim w_2$.

We are left with the case when $m > n$. Equation 3.1 tells us that $v$ starts with the subword $w_2$; say $v = w_2v'$. Plugging this into Equation 3.1 we get $v'w_1 = w_2v'$. Note that $|v'| = n - m$. If length of $v'$ is not less than or equal to $n$, then we repeat the above argument with $v'$. Since $|v'| < |v|$, we will eventually find some tail-end subword of $v$, say $v_0$, such that $v_0w_1 = w_2v_0$ with $|v_0| \leq n$. Then we apply the previous cases. \hfill $\square$

Using similar techniques, we also have the following lemma:

**Lemma 3.3.** Let $w \in W$. For any $v \in \Lambda \setminus W$, $A_{v,w} = \{0\}$.

**Proof.** First we consider the case $v = \emptyset$. In general, $A_{\emptyset,w} \neq \{0\}$ implies that $w$ is an object in the adjoint sub-category of $C \ast D$, or in other words, $w$ is isomorphic to a sub-object of $u\bar{u}$ for some simple object $u \in C \ast D$. If $u$ is length 0, then obviously $|w| = 0$, a contradiction. If $u$ has length greater than or equal to 1, as every word that appears as a sub-object of $v\bar{v}$ is of $C$-$C$ or $D$-$D$ type, $w$ cannot be a sub-object of $u\bar{u}$, which implies that $A_{\emptyset,w} = \{0\}$.

Now we consider the case $v$ has length 1. First assume $v \in C$. If $A_{v,w} \neq \{0\}$, then there is some word $u$ so that $(C \ast D)(uw, wu) \neq \{0\}$, which is equivalent to $(C \ast D)(v\bar{u}, \bar{w}u) \neq \{0\}$. First suppose $|u|$ is odd. If it is of $C$-$C$ type, then $wu$ is simple, and $uv$ is isomorphic to a direct sum of simple objects each of which have length strictly smaller than the length of $wu$, so the morphism space must be 0. Similarly if $u$ is of $D$-$D$ type, then so is $\bar{u}$, and our hypothesis implies $(C \ast D)(v\bar{u}, \bar{w}u) \neq \{0\}$. In this case, both words are simple, but $|v\bar{u}| < |\bar{w}u|$, and thus the morphism space must be $\{0\}$.

Thus we are left to consider the case when $|u|$ is even. If $u$ is $C$-$D$ type, then $wu$ is simple, and the length is strictly greater than the length of any subobject of $uv$ (since $|v| = 1$) a contradiction. If $u$ is $D$-$C$ type, then $\bar{w}u$ is simple with length strictly greater than the length of any simple sub-object of $v\bar{u}$.

The case with $v \in D$ is entirely analogous. \hfill $\square$

**Lemma 3.4.** For $w \in W$, the centralizer algebra $A\Lambda_{w,w}$ is isomorphic to the group algebra $\mathbb{C}[\mathbb{Z}]$ as $\ast$-algebras.

**Proof.** Let $v$ be a subword of $w$ such that $w = v^k = v v \ldots v$, for largest possible positive integer $k$. We will say that $w$ is **maximally periodic with respect to** $v$. Note that $v$ must be of $C$-$D$ type. Consider the (unitary) rotation

$$ \rho_{w,w}^v := 1_{v^k+1} \in (C \ast D)(vw, vw) = A\Lambda_{w,w}^v $$

whose inverse is given by

$$ (\rho_{w,w}^v)^\# = (1_{v^{k-1}} \otimes \overline{R_v})(R_v^* \otimes 1_{v^{k-1}}) \in (C \ast D)(\overline{vw}, w\overline{v}) = A\Lambda_{w,w}^{\overline{v}} $$

for any standard solution $(R_v, \overline{R_v})$ of the conjugate equation for $(v, \overline{v})$.

Note that for any $n \in \mathbb{Z}$, $(\rho_{w,w}^v)^n \in A\Lambda_{w,w}^{vn}$ with the convention $v^{-1} = \bar{v}$ and $v^0 := 1$. Thus, $\{(\rho_{w,w}^v)^n : n \in \mathbb{Z}\}$ is an orthogonal sequence in $A\Lambda_{w,w}$ with respect to the canonical
trace. Hence, we have an injective homomorphism from $\mathbb{C}[Z]$ to $\mathcal{A}\Lambda$ sending the generator of $Z$, which we denote $g$, to $\rho_{w,w}^n$. It remains to show that the homomorphism is surjective.

We now claim that if $u \in \text{Irr}(\mathcal{C} \rtimes \mathcal{D})$, then $\mathcal{A}\Lambda_{w,w} = (\mathcal{C} \rtimes \mathcal{D})(uw, wu) \neq \{0\}$ if and only if $u = v^n$ for some $n \in \mathbb{Z}$.

By the same argument as in proof of “if” part of Lemma 3.2, it is easy to deduce that $u$ must be any one of $\mathcal{C} \rtimes \mathcal{D}$ or $\mathcal{D} \rtimes \mathcal{C}$ types if $\mathcal{A}\Lambda_{w,w} = (\mathcal{C} \rtimes \mathcal{D})(uw, wu) \neq \{0\}$. It suffices to consider the case of $\mathcal{C} \rtimes \mathcal{D}$ type $u$, since the other case will follow from this by applying $\#$.

Since both $u$ and $w$ are of $\mathcal{C} \rtimes \mathcal{D}$ type, both $uw$ and $wu$ are simple, $(\mathcal{C} \rtimes \mathcal{D})(uw, wu) \neq \{0\}$ implies $uw = wu$. Now, consider the bi-infinite word $\ldots uwuuuwu \ldots$. If $m = |u|$ and $n = |w|$, then by the commutation of $u$ and $w$, we may conclude that the infinite word is both $m$- and $n$-periodic, and thereby, $l := \gcd(m, n)$-periodic. So, there exists a word $v'$ of length $l$ such that both $u$ and $w$ are integral powers of $v'$. Since $w$ is maximally periodic with respect to $v$, $|v| \leq |v'|$, which will then imply that $v'$ is an integral power of $v$. Hence, $u$ is an integral power of $v$.

We will be done if we can show $\mathcal{A}\Lambda_{w,w}^n = \mathbb{C}\rho_{w,w}^n$ for $n \in \mathbb{Z}$. Again, it is enough to show for $n \geq 0$ since the other cases follow by taking $\#$. If $n \geq 0$, however, then $\mathcal{A}\Lambda_{w,w}^n = (\mathcal{C} \rtimes \mathcal{D})(v^{k+n}, v^{k+n})$ is one-dimensional (by simplicity of $v^{k+n}$).

Via the inclusion $W \subset \Lambda$, we may consider $\mathcal{A}W$ as a *-subalgebra of $\mathcal{A}\Lambda$. In fact, by Lemma 3.3, we see that $\mathcal{A}W$ is actually a summand of $\mathcal{A}\Lambda$. The above lemma now allows us to identify $\mathcal{A}W$. Let $W_0 = W/\sim$, the set of equivalence classes of words in $W$ modulo the cyclic relation $\sim$ defined in the beginning of this section. Recall that $M_n(\mathbb{C})$ denotes the algebra of $n \times n$ matrices.

**Corollary 3.5.** $\mathcal{A}W$ is a direct summand of the algebra $\mathcal{A}\Lambda$. Moreover, as *-algebras

$$\mathcal{A}W \cong \bigoplus_{[w] \in W_0} M_{|w|}(\mathbb{C}) \otimes \mathbb{C}[Z].$$

**Proof.** As explained above, the first statement follows from Lemma 3.3.

For the second one, we pick a representative $w \in [w] \in W_0$. Then for any other $v \in [w]$, it is clear from Lemma 3.4 that $\mathcal{A}W_{w,v} \cong \mathbb{C}[Z]$ as a vector space, where $Z$ is identified with powers of unitary rotation operators $\sigma_v \in \mathcal{A}\Lambda_{w,v}$ for all $v \in [w]$. Note that $\mathcal{A}W_{w,v} = \{0\}$ for $v \notin [w]$ by Lemma 3.2.

The required isomorphism is given by the map defined for $w_1, w_2 \in [w]$ and $x \in \mathcal{A}\Lambda_{w_1,w_2}$ by

$$x \mapsto E_{w_1,w_2} \otimes \sigma_{w_2} x \sigma_{w_1}^\# \in M_{|w|}(\mathbb{C}) \otimes \mathcal{A}\Lambda_{w_1,w_2} \cong M_{|w|}(\mathbb{C}) \otimes \mathbb{C}[Z].$$

\[\square\]

**3.2. Words of length 1.** For a rigid $\mathcal{C}^\ast$-tensor category $\mathcal{C}$, we let $S(\mathcal{C}) := \{[a] \in \text{Irr}(\mathcal{C}) : N_a^\circ \neq 0 \text{ for some } [b] \in \text{Irr}(\mathcal{C})\}$. $S(\mathcal{C})$ tensor generates the adjoint subcategory of $\mathcal{C}$, which is the trivial graded component with respect to the universal grading group, but in general $S(\mathcal{C})$ gives a proper subset of the simple objects in the adjoint subcategory.

**Lemma 3.6.** Let $w \in I_\mathcal{C}$. Then $\mathcal{A}\Lambda_{\emptyset,w} \neq \{0\}$ if and only if $w$ belongs to $S(\mathcal{C})$. The same holds replacing $\mathcal{C}$ with $\mathcal{D}$. 

13
Proof. Suppose \( w \in S(C) \). then there is a simple \( v \) such that \( \{0\} \neq (C \ast D)(v, wv) = \mathcal{A}_\Lambda w \) implying, \( \mathcal{A}_\Lambda \neq \{0\} \).

Now suppose \( \mathcal{A}_\Lambda \neq \{0\} \). Choose \( v \in \text{Irr}(C \ast D) \setminus \{1\} \) such that \( \mathcal{A}_\Lambda (v, wv) \neq \{0\} \). By arguments as in the proof of Lemma 3.2, one can see that \( v \) must be of \( C \ast D \) type for the morphism space to be non-zero. Let \( v = cv \) with \( c \in I_C \). If \( v' = 1 \), then we are done. Suppose \( |v'| \geq 1 \); so, \( v' \) starts in \( I_D \). Consider \( \mathcal{A}_\Lambda (v, wv) \neq \{0\} \). Since \( \{0\} \neq (C \ast D)(v, wv) \neq \{0\} \). Since \( \mathcal{A}_\Lambda \neq \{0\} \) implies both \( c \) and \( d \) are non-trivial and is of \( D \)-type (since \( v' \) is simple). Thus, for all \( i \geq 1 \), \( cu_i \bar{c} \) is simple and is of length greater than 1, implying \( (C \ast D)(v, cu_i \bar{c}) = \{0\} \). Since \( \{0\} \neq (C \ast D)(v, wv) \neq \{0\} \), we must have \( \mathcal{C}(c\bar{c}, w) = (C \ast D)(c\bar{c}, w) = \{0\} \). So \( w \in S(C) \).

For the statement of the next lemma, for \( c \in I_C \), note that since \( C \) is a full subcategory of \( C \ast D \), we can view \( \mathcal{A}_C \subseteq \mathcal{A}_D \). Similarly for \( d \in I_D \).

Lemma 3.7. If \( c \in I_C \subseteq \Lambda \) and \( d \in I_D \subseteq \Lambda \), then \( \mathcal{A}_C \neq \{0\} \) if and only if \( c \in S(C) \) and \( d \in S(D) \). Furthermore \( \mathcal{A}_C = \mathcal{A}_D \cdot \mathcal{A}_\emptyset \cdot \mathcal{A}_C \).

Proof. If \( c \in I_C \) and \( d \in I_D \), choose \( a \in I_C \) and \( b \in I_D \) such that \( c \) and \( d \) are subobjects of \( aa \) and \( bb \) in \( C \) and \( D \) respectively. Let \( 0 \neq y_1 \in C(ac, a) \), \( 0 \neq y_2 \in D(b, db) \). Note that \( (y_2 \otimes 1_{(1)})(1_{(2)} \otimes y_1) \in (C \ast D)(bac, dba) = \mathcal{A}_{C,D} \subset \mathcal{A}_C \) is nonzero.

Conversely, let \( \mathcal{A}_C \neq \{0\} \). Then there exists a non-unit simple object \( v \in \text{Irr}(C \ast D) \) such that \( (C \ast D)(v, dv) = \mathcal{A}_C \neq \{0\} \). If \( v \) is of \( C \)-type (resp. \( D \)-type), then \( (C \ast D)(v, dv) = \{0\} \) as \( dv \) is simple of \( C \)-type, and any simple subobject of \( v \) in \( C \ast D \) has length smaller than that of \( dv \) (resp. \( vc \)). Now suppose \( v \) is of \( C \)-type; then, both \( vc \) and \( dv \) are simple with the same length but are of different types, hence \( (C \ast D)(v, dv) = \{0\} \). Thus \( v \) can only be of \( D \)-type. Also since \( v \neq 1 \), length of \( v \) is at least 2.

Let \( v = d'v'c' \), where \( d' \in I_D \), \( c' \in I_C \) and \( v' \in \text{Irr}(C \ast D) \) is either trivial or \( C \)-type. Consider \( \bar{v}d'v = c'v' \bar{d}'d'd'v'c' \). If \( \bar{d}'d' \) does not contain \( 1 \) as a subobject, then length of every simple subobject of \( \bar{v}d'v \) is strictly greater than 1, and thereby \( \mathcal{C}(wv_1, wv_2) \cong (C \ast D)(wv_1, wv_2) = \{0\} \) which is a contradiction. Thus, \( 1 \) appears as a subobject of \( d' \bar{d}d' \) and hence \( d \in S(D) \). Similarly, by considering \( v \bar{c} \), one may deduce that \( c \in S(C) \).

For the last part, let \( v = d'v'c' \) be as above. Then \( vc = d'v'c'c \) and \( dv = dd'v'c' \). Since \( v' \) is a word of \( C \)-type of length at least 2 whose letters are all simple, by the definition of the free product category, any morphism \( x \in (C \ast D)(vc, dv) \) factorizes as \( x_1 \otimes 1_{v'} \otimes x_2 \), where \( x_1 \in D(d', dd') \) and \( x_2 \in C(c', cc') \). The result then follows.

Lemma 3.8. Suppose \( c_1, c_2 \in I_C \). If \( v \in \text{Irr}(C \ast D) \) and \( |v| \geq 1 \), then the space \( \mathcal{A}_\Lambda (v) \neq \{0\} \) implies \( v \) is of \( C \)-type. Furthermore, we have

(i) If \( |v| = 1 \), then \( v \in I_C \) and \( \mathcal{A}_\Lambda (v) = \mathcal{A}_C \).

(ii) If \( |v| \geq 2 \) then \( \mathcal{A}_\Lambda (v) \neq \{0\} \) implies both \( c_1 \) and \( c_2 \) lie in \( S(C) \). Furthermore, \( \mathcal{A}_\Lambda (v) = \mathcal{A}_C \).

The same statement holds, replacing \( C \) with \( D \).

Proof. Let \( c_1, c_2 \in I_C \). And suppose \( \mathcal{A}_\Lambda (v) \neq \{0\} \), for \( |v| \geq 1 \).
If \( v \) is of \( \mathcal{C} \cdot \mathcal{D} \) type or \( \mathcal{D} \cdot \mathcal{C} \) type, then \( vc_1 \) (respectively, \( c_2v \)) is simple, and any simple sub-object of \( c_2v \) (respectively \( vc_1 \)) will have length strictly smaller than that of \( vc_1 \) (respectively \( c_2v \)). Hence \( \mathcal{A} \Lambda^v_{c_1, c_2} = (\mathcal{C} \cdot \mathcal{D}) (vc_1, c_2v) = \{0\} \). Again, we can rule out \( v \) being \( \mathcal{D} \cdot \mathcal{D} \) type by comparison of the two simple objects \( vc_1 \) and \( c_2v \), which cannot be equal since one starts with \( \mathcal{D} \) while the other starts with \( \mathcal{C} \).

For (i), note that for \( |v| = 1 \) and \( \mathcal{A} \Lambda^v_{c_1, c_2} \neq \{0\} \), we must have \( v \in I^v_{\mathcal{C}} \) and in this case we see that \( \mathcal{A} \Lambda^v_{c_1, c_2} = (\mathcal{C} \cdot \mathcal{D}) (vc_1, c_2v) = \mathcal{C}(c_1, c_2v) = \mathcal{A} \Lambda_{c_1, c_2}^v \).

For (ii), suppose we have \( \mathcal{A} \Lambda^v_{c_1, c_2} \neq \{0\} \), with \( |v| \geq 2 \). By the first part of the lemma, \( v \) is of \( \mathcal{C} \cdot \mathcal{C} \) type, and hence we have \( v = c_1'v'c_2' \), where \( v' \) is a simple word of \( \mathcal{D} \cdot \mathcal{D} \) type of length \( n \geq 1 \). Thus we see that for any \( x \in (\mathcal{C} \cdot \mathcal{D}) (vc_1, c_2v) = (\mathcal{C} \cdot \mathcal{D}) (c_1'v'c_2'c_1, c_2c_1'v'c_2) \), from the definition of the free product category we must have \( x_1 \in \mathcal{C}(c_1', c_2c_1) \) and \( x_2 \in \mathcal{C}(c_2c_1, c_2') \) so that \( x \) factorizes as \( x = x_1 \otimes 1_{v'} \otimes x_2 \). This gives us (ii).

\[ \square \]

4. ANNULAR REPRESENTATIONS OF FREE PRODUCT OF CATEGORIES

Let \( \mathcal{C} \) be an arbitrary rigid \( C^* \)-tensor category, and \( \Gamma \subseteq [\text{Obj} \mathcal{C}] \) be an arbitrary weight set containing \( 1 \), which is sufficiently full to generate a universal \( C^* \)-algebra. Consider the ideal \( J_{\mathcal{G}} := \mathcal{A} \Gamma \cdot \mathcal{A} \Gamma_{1,1} \cdot \mathcal{A} \Gamma \) in \( \mathcal{A} \Gamma \) generated by \( \mathcal{A} \Gamma_{1,1} \). In the particular case of \( \Gamma = \text{Irr} \mathcal{C} \), we write \( \mathcal{I} \mathcal{C}_0 \) for \( J_{\mathcal{G}} \).

Any bounded \( * \)-representation of \( J_{\mathcal{G}} \) defines a bounded \( * \)-representation of \( \mathcal{A} \Gamma \). In fact, the induction functor \( \text{Ind}_{\mathcal{G}} : \text{Rep}(J_{\mathcal{G}}) \rightarrow \text{Rep}(\mathcal{A} \Gamma) \) is a fully faithful functor, and its image defines the full subcategory \( \text{Rep}_{0}(\mathcal{A} \Gamma) \) of representations generated by their weight \( 1 \) space. Furthermore, \( \text{Rep}_{0}(\mathcal{A} \Gamma) \) is precisely the category of admissible representations of the fusion algebra with respect to \( \Gamma \).

Consider the \( W^* \)-category \( \text{Rep}_{+}(\mathcal{A} \Gamma) := \text{Rep}(\mathcal{A} \Gamma / J_{\mathcal{G}}) \) of representations of \( \mathcal{A} \Gamma \) which contain \( J_{\mathcal{G}} \) in their kernel. \( \text{Rep}_{+}(\mathcal{A} \Gamma) \) is referred to as the category of higher weight representations. It consists of precisely the representations of \( \mathcal{A} \Gamma \) such that the projection \( p_1 \in \mathcal{A} \Gamma_{1,1} \) acts by \( 0 \).

Then, for any non-degenerate \( * \)-representation of \( (\pi, \mathcal{H}) \in \text{Rep}(\mathcal{A} \Gamma) \), we can decompose \( \mathcal{H} \) as direct sum of subrepresentations \( \mathcal{H}_0 \oplus \mathcal{H}^+_0 \), where \( \mathcal{H}_0 := [\pi(J_{\mathcal{G}}) \mathcal{H}] \) and \( \mathcal{H}^+_0 \) is its orthogonal complement. We can view \( \mathcal{H}_0 \in \text{Rep}_0(\mathcal{A} \Gamma) \) and \( \mathcal{H}^+_0 \in \text{Rep}_{+}(\mathcal{A} \Gamma) \). Any representation of \( J_{\mathcal{G}} \) and any representation of \( \mathcal{A} \Gamma_{+} \) are disjoint as representations of \( \mathcal{A} \Gamma \). This discussion gives us the following proposition:

**Proposition 4.1.** For any sufficiently full weight set, \( \text{Rep}(\mathcal{A} \Gamma) \cong \text{Rep}_0(\mathcal{A} \Gamma) \oplus \text{Rep}_{+}(\mathcal{A} \Gamma) \).

Thus, the problem of understanding \( \text{Rep}(\mathcal{A} \Lambda) \) decomposes into the problem of understanding the admissible representations of the fusion algebra, and the higher weight structure. In the particular case of free products, what we will see is that the weight 0 part is controlled by a free product \( C^* \)-algebra, while the higher weight parts can be read off in terms of the higher weight parts of \( \mathcal{C} \) and \( \mathcal{D} \). There are also some additional copies of the category \( \text{Rep}(\mathbb{Z}) \) that appear at higher weights.

We first turn our attention to the weight 0 case. Let \( \text{Fus}(\mathcal{C}) \) be the fusion algebra of \( \mathcal{C} \) with the distinguished basis \( \text{Irr}(\mathcal{C}) \). Recall there exists a universal \( C^* \)-algebra completion of the fusion algebra, denoted by \( C^*_u(\mathcal{C}) \), first introduced by Popa and Vaes [PV], which is universal.
with respect to *admissible representations.* In [CGJ], it was shown that \( \mathcal{A}C_{1,1} \cong \text{Fus}(\mathcal{C}) \) and admissible representations are precisely those that induce bounded \(*\)-representations of the tube algebra, and thus \( C_u^*(\mathcal{C}) \) can be viewed as the weight 0 corner (or centralizer algebra) of the universal \( C^* \)-algebra of the tube algebra.

Via the inclusions of \( \mathcal{C} \) and \( \mathcal{D} \) into \( \mathcal{C} \ast \mathcal{D} \), \( \text{Fus}(\mathcal{C} \ast \mathcal{D}) \) contains the fusion algebras \( \text{Fus}(\mathcal{C}) \) and \( \text{Fus}(\mathcal{D}) \) as unital \(*\)-subalgebras. Indeed, we have a canonical \(*\)-algebra isomorphism between \( \text{Fus}(\mathcal{C} \ast \mathcal{D}) \) and the (algebraic) free product \( \text{Fus}(\mathcal{C}) \ast \text{Fus}(\mathcal{D}) \).

We briefly recall the definition of (universal) free product of \( C^* \)-algebras:

**Definition 4.2.** If \( A_1 \) and \( A_2 \) are unital \( C^* \)-algebras, a free product is a unital \( C^* \)-algebra \( A_1 \ast A_2 \), together with unital \(*\)-homomorphisms \( \iota_i : A_i \to A_1 \ast A_2 \) satisfying the following universal property: for any unital \( C^* \)-algebra \( C \) and unital \(*\)-homomorphisms \( \gamma_i : A_i \to C \) there exists a unique \(*\)-homomorphism \( \gamma : A_1 \ast A_2 \to C \) such that \( (\gamma_1 \ast \gamma_2) \circ \iota_i = \gamma_i \).

Any two free products of two \( C^* \)-algebras are \(*\)-isomorphic if they exist by the universal property. Furthermore, free products do exist, so it makes sense to talk about the free product \( C^* \)-algebra, which we will denote by \( A_1 \ast A_2 \).

The main result of this section is the following:

**Proposition 4.3.** \( C_u^*(\mathcal{C} \ast \mathcal{D}) \cong C_u^*(\mathcal{C}) \ast C_u^*(\mathcal{D}) \).

To prove this, we already know that \( \mathcal{A}C_{2,1} \), \( \mathcal{A}D_{2,1} \) and \( \mathcal{A}\Lambda_{0,0} \) are isomorphic to the fusion algebras \( \text{Fus}(\mathcal{C}) \), \( \text{Fus}(\mathcal{D}) \) and \( \text{Fus}(\mathcal{C} \ast \mathcal{D}) \cong \text{Fus}(\mathcal{C}) \ast \text{Fus}(\mathcal{D}) \) respectively. Using these isomorphisms, any representation of the weight zero centralizer algebra \( \mathcal{A}\Lambda_{0,0} \) can also be viewed as representations of \( \mathcal{A}C_{1,1} \) and \( \mathcal{A}D_{1,1} \) by restricting \( \pi \) to the corresponding subalgebras. We have the following lemma:

**Lemma 4.4.** A representation \((\pi, \mathcal{H})\) of \( \text{Fus}(\mathcal{C} \ast \mathcal{D}) \) is admissible if and only if its restrictions \((\pi^c, \mathcal{H})\) and \((\pi^d, \mathcal{H})\) to \( \text{Fus}(\mathcal{C}) \) and \( \text{Fus}(\mathcal{D}) \) are admissible respectively.

**Proof.** If \((\pi, \mathcal{H})\) be admissible then, \((\pi^c, \mathcal{H})\) and \((\pi^d, \mathcal{H})\) are clearly admissible.

Suppose \((\pi^c, \mathcal{H})\) and \((\pi^d, \mathcal{H})\) are admissible. Set \( \hat{\mathcal{H}}_w := \mathcal{A}\Lambda_{0,w} \otimes \mathcal{H} \) for \( w \in \Lambda \). By Lemma 3.3 and Lemma 3.6 \( \hat{\mathcal{H}}_w \) is nonzero only when \( w = \emptyset \) or \( w \) has length 1 and is in \( \text{S}(\mathcal{C}) \cup \text{S}(\mathcal{D}) \). As usual, we define a sesquilinear form on \( \hat{\mathcal{H}}_w \) by

\[
\langle y_1 \otimes \xi_1, y_2 \otimes \xi_2 \rangle_w := \langle \pi(y_2^\# \cdot y_1) \xi_1, \xi_2 \rangle
\]

for \( y_1, y_2 \in \mathcal{A}\Lambda_{0,w} \) and \( \xi_1, \xi_2 \in \mathcal{H} \).

By the definition of admissibility and [CGJ], it suffices to show that this form is positive semi-definite. Further, it is enough to show

\[
\sum_{i,j=1}^n \langle \pi(x^\#_j \cdot x_i) \xi_i, \xi_j \rangle \geq 0
\]

for \( x_i \in \mathcal{A}\Lambda_{0,w}^{v_i}, v_i \in \text{Irr}(\mathcal{C} \ast \mathcal{D}), \xi_i \in \mathcal{H} \). When \( w = \emptyset \), the sum becomes \( \sum_{i=1}^n \| \pi(x_i) \xi_i \|_{\mathcal{H}}^2 \geq 0 \).

It remains to consider the case \( w \in \text{S}(\mathcal{C}) \cup \text{S}(\mathcal{D}) \). Suppose \( w \in \text{S}(\mathcal{C}) \). In order to have \( \mathcal{A}\Lambda_{0,w} = (\mathcal{C} \ast \mathcal{D}) (v, wv) \) nonzero, \( v_i \) must be one of \( \mathcal{C} \) or \( \mathcal{C} \) type. Let \( v_i = c_i u_i \) where
$c_i \in I_C$ and $u_i$ is either $\emptyset$ or of $D-C$ or $D-D$ type. Note that $wv_i = wc_i u_i$. As $w \in C$, any morphism $x_i \in (C * D) (c_i u_i, wc_i u_i)$ is of the form $x_i = z_i \otimes 1_{u_i}$, where $z_i \in C(c_i, wc_i)$.

One may express this in another useful way: $x_i = z_i 1_{u_i}$ where we view $z_i \in AC^G_{1,w} \subset A\Lambda_{\emptyset,w}$, and $1_{u_i} \in A\Lambda^a_{\emptyset,0}$. Setting $\zeta_i := \pi(1_{u_i}) \zeta_i$, $1 \leq i \leq n$, we have

$$\sum_{i,j=1}^n \langle \pi(x_j^* \cdot x_i) \zeta_i, \zeta_j \rangle = \sum_{i,j=1}^n \langle \pi(z_j^* \cdot z_i) \zeta_i, \zeta_j \rangle \geq 0$$

where the last inequality follows from admissibility of $(\pi^c, \mathcal{H})$. An entirely analogous argument holds for the case $w \in S(D)$. \hfill \Box

**Proof of Proposition 4.3.** Let $i_C$ (resp., $i_D$) be the canonical $*$-inclusion of $Fus(C)$ (resp., $Fus(D)$) into $Fus(C * D)$.

If $(\pi, \mathcal{H})$ is any admissible representation of $Fus(C * D)$, then $(\pi \circ i_C, \mathcal{H})$ and $(\pi \circ i_D, \mathcal{H})$ are admissible representations of $Fus(C)$ and $Fus(D)$ respectively by Lemma 4.4. Therefore, for any $x \in Fus(C)$,

$$||i_C(x)||_{\pi} = ||x||_{\pi \circ i_C} \leq ||x||_{C^*_u(C)} .$$

By the definition of the universal norm,

$$||i_C(x)||_{C^*_u(C * D)} = \sup_{\pi'} ||i_C(x)||_{\pi'}$$

where the supremum is taken over all admissible representations of $Fus(C * D)$. Thus the map $i_C$ extend to $*$-homomorphisms $i_C : C^*_u(C) \to C^*_u(C * D)$. The same argument applies to $D$, yielding an extension $i_D : C^*_u(D) \to C^*_u(C * D)$.

Let $A$ be any C*-algebra with $*$-homomorphisms $\gamma_C : C^*_u(C) \to A$ and $\gamma_D : C^*_u(D) \to A$. By the universal property of free product of ordinary $*$-algebras, there is a unique $*$-homomorphism $h : Fus(C * D) \to A$ such that $h \circ i_C = \gamma_C|_{Fus(C)}$ and $h \circ i_D = \gamma_D|_{Fus(D)}$. By density of the fusion algebras in their universal C*-algebras, to conclude the proof it suffices to show that $h$ extends to a $*$-homomorphism $\gamma_C * \gamma_D : C^*_u(C * D) \to A$, which is equivalent to showing $||h(x)||_A \leq ||x||_{C^*_u(C * D)}$.

Without loss of generality, assume $A \subset B(K)$ for some Hilbert space $K$. Since $||\gamma_C(y)||_A \leq ||y||_{C^*_u(C)}$ for every $y \in Fus(C)$, $(\gamma_C|_{Fus(C)}, K)$ is admissible and similarly, $(\gamma_D|_{Fus(D)}, K)$ is also admissible. Thus, by Lemma 4.4 $(h, K)$ is an admissible representation of $Fus(C * D)$. Therefore, $||x||_h = ||h(x)||_A \leq ||x||_{C^*_u(C * D)}$. \hfill \Box

This immediately implies the following corollary:

**Corollary 4.5.** The category of $Rep_0(A\Lambda)$ is equivalent as a $W^*$-category to $Rep(C^*_u(C) \ast C^*_u(D))$.

On one hand, it is well known that representation categories of free product algebras are wild and uncontrollable, and thus this answer for describing $Rep_0(A\Lambda)$ is somewhat unsatisfactory, compared to descriptions of other representation categories such as $Rep(ATLJ)$ ([GJ]). On the other hand, there are a plethora of ways to produce examples of representations of free products, so these categories are quite flexible. For example, given two states $\psi, \phi$ on C*-algebras $A$ and $B$, one can construct the free convolution state $\psi \ast \phi$ on the
C*-algebra $A \ast B$ (\cite{GJ}). Alternatively one simply has to take a representation of $A$ and one of $B$, and identify their underlying Hilbert space.

We now move on to describing the higher weight categories, which, depending on $C$ and $D$, can be more manageable. As described in the beginning of the section $\text{Rep}_+(A\Lambda) = \text{Rep}(A\Lambda / \mathcal{J}\Lambda_0)$. We have the following lemma:

**Lemma 4.6.** As $*$-algebras, $A\Lambda / \mathcal{J}\Lambda_0 \cong AC / \mathcal{J}C_0 \oplus AD / \mathcal{J}D_0 \oplus AW$.

**Proof.** Recall that $A\Lambda \cong A[\Lambda \setminus W] \oplus AW$. From Lemma 3.3, we see that $\mathcal{J}\Lambda_0 \subseteq A[\Lambda \setminus W]$, and thus

$$A\Lambda / \mathcal{J}\Lambda_0 \cong A[\Lambda \setminus W] / \mathcal{J}\Lambda_0 \oplus AW$$

Thus we consider the spaces $A\Lambda_{w_1, w_2}^v$ with $w_1, w_2 \in S(C) \cup S(D)$, and $v \in \text{Irr}(C \ast D)$. By Lemma 3.7 and Lemma 3.8, the image of these spaces under the quotient is 0 unless $w_1$ and $w_2$ are either both in $S(C)$ and $v \in \text{Irr}(C)$ or both $w_1$ and $w_2$ are in $S(D)$ and $v \in \text{Irr}(D)$. Since $\mathcal{J}C_0, \mathcal{J}D_0 \subseteq \mathcal{J}\Lambda_0$, it is now clear that the quotient map assembles into an isomorphism $A[\Lambda \setminus W] / \mathcal{J}\Lambda_0 \cong AC / \mathcal{J}C_0 \oplus AD / \mathcal{J}D_0$, concluding the proof.

Finally, we recall that $W_0$ is the set of cyclic equivalence classes of words in $W$, and note that $\text{Rep}(AW) \cong \text{Rep}(Z \oplus W)$. The above results imply Theorem 1.1, which is the main result of this article.

5. Examples

In this section, we apply the main result to several examples. First, we show how this matches another known result.

**Example 5.1. Free products of group categories.** In particular, for any countable group $G$, we consider the rigid C*-tensor category $\text{Hilb}_{f.d.}(G)$ of finite dimensional $G$-graded Hilbert spaces. Let $\Lambda$ denote the set of conjugacy classes of $G$. For each $\lambda \in \Lambda$ we can define $C_\lambda(G)$ to be the centralizer subgroup of some element $g \in \lambda$. We note that different choices of $g \in \Lambda$ yield conjugate subgroups, and so $C_\lambda(G)$ is well defined up to isomorphism. Then, from \cite{GJ}, the category of annular representations

$$\text{Rep}(A) \cong \bigoplus_{\lambda \in \Lambda} \text{Rep}(C_\lambda(G))$$

Now, for any two countable groups $G$ and $H$, it’s easy to see that $\text{Hilb}_{f.d.}(G) \ast \text{Hilb}_{f.d.}(H)$ is equivalent as a C*-tensor category to $\text{Hilb}_{f.d.}(G \ast H)$. Thus we can compare our result for $\text{Hilb}_{f.d.}(G) \ast \text{Hilb}_{f.d.}(H)$ to the above result for $\text{Hilb}_{f.d.}(G \ast H)$.

Since $C_\ast^u(\text{Hilb}_{f.d.}(G))$ is isomorphic to the universal group C*-algebra $C_\ast^u(G)$, and $C_\ast^u(G \ast H) \cong C_\ast^u(G) \ast C_\ast^u(H)$, we can identify the first component in the main theorem (Theorem 1.1) with $\text{Rep}(G \ast H)$.

Note that there is always distinguished conjugacy class $[1] \in \Lambda$, the conjugacy class of the unit 1. We have $C_{[1]}(G) = G$. It is easy to see that

$$\text{Rep}_+(A\text{Hilb}_{f.d.}(G)) \cong \bigoplus_{\lambda \in \Lambda \setminus [1]} C_\lambda(G)$$
This helps us identify the second two components, while the last component needs no identification.

Now, consider the group \(G \ast H\). This group has 4 types of conjugacy classes: \([\{1\}],[g] : g \in G\}, \{[h] : h \in H\} and \{[g_1h_1 \cdots g_nh_k] : g_i \in G, h_i \in H, k \geq 1\}. It is also easy to see that \(C_{[1]}(G \ast H) = G \ast H, C_{[g]}(G \ast H) = G, C_{[h]}(G \ast H) = H\) and \(C_{[g_1h_1 \cdots g_nh_k]} = \{(g_1h_1 \cdots g_nh_k)^n : n \in \mathbb{Z}\} \cong \mathbb{Z}\). It is now easy to see the equivalence of the two descriptions.

**Example 5.2. Fuss-Catalan representations.** Bisch and Jones introduced the Fuss-Catalan subfactor planar algebras \(\mathcal{FC}(\alpha, \beta)\), where \(\alpha, \beta \in \{2\cos(\frac{n}{3}) : n \geq 3\} \cup [2, \infty)\) \([BJ]\). These planar algebras are universal for intermediate subfactors. For a subfactor planar algebra, the category of affine annular representations in the sense of Jones-Reznikoff \([JR]\) is equivalent to the category of annular representations of the even part of the subfactor (see, for example, \([DGG]\) Remark 3.6] or \([NY2]\) Corollary 4.4]). The even part of the Fuss-Catalan can be realized as a full subcategory of the free product category \(\mathcal{T \mathcal{L}}(\alpha) \ast \mathcal{T \mathcal{L}}(\beta)\). In particular, if \(a \in \mathcal{T \mathcal{L}}(\alpha)\) is the standard tensor generating object with dimension \(\alpha\) and \(b \in \mathcal{T \mathcal{L}}(\beta)\) is the standard tensor generating object with dimension \(\beta\), then the full subcategory generated by \(abba \in \mathcal{T \mathcal{L}}(\alpha) \ast \mathcal{T \mathcal{L}}(\beta)\) is equivalent to the even part of \(\mathcal{FC}(\alpha, \beta)\). Thus to determine the annular representation category of \(\mathcal{FC}(\alpha, \beta)\), it suffices to determine the annular representations of the full subcategory \(\mathcal{T \mathcal{L}}(\alpha) \ast \mathcal{T \mathcal{L}}(\beta)\) generated by \(abba\). Let \(\mathcal{T \mathcal{L}}(\alpha)\) denote the adjoint subcategory, generated by \(aa\). This can also be realized as the even part of the usual Temperley-Lieb-Jones subfactor planar algebras.

We recall briefly that two rigid \(C^\ast\)-tensor categories \(\mathcal{C}\) and \(\mathcal{D}\) are weakly Morita equivalent if there is a rigid \(C^\ast\)-2 category with two objects 0 and 1, such that the tensor category \(\text{End}(0) \cong \mathcal{C}\) and the tensor category \(\text{End}(1) \cong \mathcal{D}\) (see \([NY2]\) for further details). The two even parts of a subfactor planar algebra are weakly Morita equivalent, but weak Morita equivalence is more general. If we have two full subcategories of a tensor category, to show they are weakly Morita equivalent, it suffices to find an object \(x \in \mathcal{C}\) so that \(\mathcal{F}x\) tensor generates one and \(\mathcal{F}_x\) tensor generates the other, since one can, using the usual subfactor approach, construct a rigid \(C^\ast\)-2 category whose two even parts are as desired. We apply this in the free product case to obtain the following proposition:

**Proposition 5.3.** The tensor category generated by \(abba\) is weakly Morita equivalent to \(\mathcal{T \mathcal{L}}(\alpha) \ast \mathcal{T \mathcal{L}}(\beta)\).

**Proof.** It suffices to find an object \(x \in \mathcal{T \mathcal{L}}(\alpha) \ast \mathcal{T \mathcal{L}}(\beta)\) such that \(\langle x \mathcal{F} \rangle = \langle abba \rangle\) and \(\langle \mathcal{F}_x \rangle = \mathcal{T \mathcal{L}}(\alpha) \ast \mathcal{T \mathcal{L}}(\beta)\). Choose \(x := abba\). Then since both \(aa\) and \(bb\) contain the tensor unit as a subobject, we see \(\langle abba \rangle = \langle abba \rangle\). On the other hand, \(bbaabb\) contains \(aa\) and \(bb\) as subobjects, and so clearly \(\langle bbaabb \rangle = \langle aa, bb \rangle\). \(\square\)

Again, by \([DGG]\) Remark 3.6] or \([NY2]\) Corollary 4.4], the above proposition implies the following:

**Corollary 5.4.** The category of affine annular representations of the subfactor planar algebra \(\mathcal{FC}(\alpha, \beta)\) is equivalent as a \(W^\ast\)-category to the annular representation category of \(\mathcal{T \mathcal{L}}(\alpha) \ast \mathcal{T \mathcal{L}}(\beta)\).

This category \(\mathcal{T \mathcal{L}}(\alpha)\) is fully described in \([JR]\), and thus combining those results with ours leads to a description of the representations of Fuss-Catalan categories.
References

[A] D. Avitzour, *Free product of C*-algebras*. Trans. Amer. Math. Soc., 271-2, pp-423-435, 1982.
[BHP] A. Brothier, M. Hartglass, D. Penneys, *Rigid C*-tensor categories of bimodules over interpolated free group factors*. J. Math. Phys., 53, pp. 123525, 2012.
[BJ] D. Bisch, V.F.R. Jones, *Algebras associated to intermediate subfactors*. Inv. Math., 128-1, pp. 89-157, 1997.
[DGG] P. Das, S. Ghosh, V. P. Gupta, *Drinfeld center of planar algebra*. Int. J. Math., 25-8, pp. 1450076, 2014.
[GJ] S. Ghosh, C. Jones, *Annular representation theory for rigid C*-tensor categories*. J. Funct. Anal., 270-4, pp. 1537-1584, 2016.
[GLR] P. Ghez, R. Lima, J.E. Roberts, *W*-categories*. Pacific J. Math., 120-1, pp. 79-109, 1985.
[I] M. Izumi, *The structure of sectors associated with the Longo-Rehren inclusion I. General Theory*. Commun. Math. Phys, 213-1, pp.127-179, 1999.
[IMP] M. Izumi, S. Morrison, D. Penneys, *Quotients of A2 ∗ T2*. Canad. J. Math., 68-5, pp. 999–1022, 2016.
[J1] V.F.R. Jones, *Planar algebras I*. arXiv:math/9909027v1, 1999.
[J2] V.F.R. Jones, *The annular structure of subfactors*. Essays on geometry and related topics, Vol.1,2. Monogr. Enseign. Math., 38, pp.401-463, 2001.
[JMS] V.F.R. Jones, S. Morrison, N. Snyder, *The classification of subfactors of index at most 5*. Bull. Am. Math. Soc., 51-2, pp. 277-327, 2014.
[JR] V.F.R. Jones, S. Reznikoff, *Hilbert space representations of the annular Temperley-Lieb algebra*. Pacific J. Math., 228-2, pp. 219-248, 2006.
[LR] R. Longo, J.E. Roberts, *A theory of dimension*. K-theory, 11-2, pp. 103-159, 1997.
[M1] M. Mueger, *From subfactors to categories and topology I: Frobenius algebras and Morita equivalence of tensor categories*. J. Pure Appl. Algebra, 180-1, pp.81-157, 2003.
[M2] M. Mueger, *From subfactors to categories and topology II: The quantum double of tensor categories and subfactors*. J. Pure Appl. Algebra, 180-1, pp.159-219, 2003.
[NT] S. Neshveyev, M. Tuset, *Compact quantum groups and their representation categories*. SMF, 20, Specialized courses, 2013.
[NY1] S. Neshveyev, M. Yamashita, *Drinfeld center and representation theory for monoidal categories*. Commun. Math. Phys., 345-1, pp. 385-434, 2016.
[NY2] S. Neshveyev, M. Yamashita, *A few remarks on the tube algebra of a monoidal categories*. to appear Proc. Edinb. Math. Soc, arXiv:1511.06332v4.
[O] A. Ocneanu, *Chirality for operator algebras*. Subfactors, ed. by H. Araki, et al., World Scientific, pp. 39-63, 1994.
[P1] S. Popa, *Amenability in the theory of subfactors*. Operator Algebras and Quantum Field Theory, pp. 199-211, 1996.
[P2] S. Popa, *Some properties of the symmetric enveloping algebra of a subfactor, with applications to amenability and property T*. Doc. Math., 4, pp. 665-744, 1999.
[PS] S. Popa, D. Shlyakhtenko, *Universal properties of L∞ in subfactor theory*. Acta. Math., 191, pp. 225-257, 2003.
[PSV] S. Popa, D. Shlyakhtenko, S. Vaes, *Cohomology and L2-Betti numbers for subfactors and quasi-regular inclusions*. to appear in Int. Math. Res. Not., arXiv:1511.07329v3.
[PV] S. Popa, S. Vaes, *Representation theory for subfactors, λ-lattices, and C*-tensor categories*. Comm. Math. Phys. 340-3, pp. 1239-1280, 2015.
[W] S. Wang, *Free products of compact quantum groups*. Comm. Math. Phys. 167- 3, pp. 671-692, 1995.

Stat-Math Unit, Indian Statistical Institute, Kolkata, INDIA
E-mail address: shami@isical.ac.in

Australian National University, Mathematical Sciences Institute, Canberra, AUSTRALIA
E-mail address: cormjones88@gmail.com
