Running-phase state in a Josephson washboard potential

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We investigate the dynamics of the phase variable of an ideal underdamped Josephson junction in switching current experiments. These experiments have provided the first evidence for macroscopic quantum tunneling in large Josephson junctions and are currently used for state read-out of superconducting qubits. We calculate the shape of the resulting macroscopic wavepacket and find that the propagation of the wavepacket long enough after a switching event leads to an average voltage increasing linearly with time.

The dynamics of a large underdamped Josephson junction characterized by a capacitance $C$ and Josephson energy $E_J$ can be described by the motion of a particle in a washboard potential $U(\gamma) = E_J(1 - \cos \gamma) + \hbar \Phi_0 \gamma$. The particle has $C$ as the mass, the flux $\Phi_0 \gamma$ as the coordinate and the charge on the capacitor $Q$ as the canonically conjugate momentum. Here $\gamma$ is the phase difference of the superconducting order parameter across the junction and $\Phi_0 = \Phi_0/2\pi = \hbar/2e$ is the flux quanta divided by $2\pi$. Much attention has been given since the discovery of the Josephson effect to the switching dynamics of the junction in the thermal activation regime and in the macroscopic tunneling (MQT) regime. Surprisingly, while the description of the state of the junction before a switching event and calculations of the corresponding probability has been a topical issue for many decades, what happens with the quantum state of the junction after tunneling did not receive that much attention. It is argued that the junction ends up in a running-phase state, with the voltage $Q/C$ increasing until it becomes sufficiently large so that the transport could be done through quasiparticle excitations. However, a quantum mechanical description of this state is missing. Much of what we understand about the running-wave state, for instance the physics of the retrapping current, comes from assuming a quasi-classical dynamics.

In this paper we give an explicit formula for the macroscopic wavefunction of an ideally underdamped junction after a MQT switching event. If the switching probability is exponential, which is the case for all the theoretical models and also confirmed experimentally, one expects the following expression for the dynamics of the wavefunction

$$|\Psi(t)| = e^{-\Gamma t/2}e^{-i\omega_0 t}|\Psi_0| + |\Psi_{out}(t)|.$$ (1)

In this equation, $\Psi_0$ is the initial state, corresponding to a bound state inside one of the metastable wells, while $\Psi_{out}(t)$ is the wavefunction of the particle corresponding to states in the continuum, outside the well (Fig. 1). This expression gives indeed an exponentially decreasing probability for the particle to be inside the well, with lifetime $\Gamma^{-1}$. In the following, we are interested in the structure of $\Psi_{out}(t)$.

To solve this problem, the standard approach is to start with a wavepacket localized initially in one of the metastable wells, and then expand it and evolve it in the eigenfunctions of the full Hamiltonian. This procedure works for simple potentials, but even in these cases the solutions are complicated. Fortunately, unlike problems in scattering theory, in condensed matter the frequent situation is that we do not need an exact solution of the Schrödinger problem for tunneling, but rather we are interested in the most generic features of it. In most cases in solid state physics, tunneling is simply treated as a process that annihilates a particle on some mode of a solid and creates one on another mode. We will approach our problem in the same spirit. A good approximation in MQT is that no other state within the well is involved with the exception of the state with energy $\omega_0$ in which the system is prepared, $|\Psi_0\rangle$; therefore one can write a reduced Hamiltonian of the form

$$H = \hbar \omega_0 |\Psi_0\rangle \langle \Psi_0| + \int \hbar \epsilon |\psi_\epsilon\rangle \langle \psi_\epsilon| + \int d\epsilon [k(\omega_0, \epsilon)|\Psi_0\rangle \langle \psi_\epsilon| + k(\epsilon, \omega_0)|\psi_\epsilon\rangle \langle \Psi_0|],$$ (2)

where by $\{\psi_\epsilon\}$ we denote the continuum of eigenvectors outside the barrier. We then write the wavefunction in the form

$$|\Psi(t)| = a(t)e^{-i\omega_0 t}|\Psi_0\rangle + \int d\epsilon b(\epsilon, t)e^{-i\epsilon t}|\psi_\epsilon\rangle,$$ (3)

with $a(0) = 1$, $b(0) = 0$. Inserting this expression in the Schrödinger equation we get an integro-differential equation for $a(t)$. The Laplace transform of this equation reads

$$L[a](s) = \frac{1}{s + L[\chi](s)},$$ (4)
where
\[ \chi(t) = \frac{1}{\hbar^2} \int d\epsilon |k(\epsilon, \omega_0)|^2 e^{i(\omega_0 - \epsilon)t}. \]  
(5)

In general, the tunneling matrix element \( k(\omega_0, \epsilon) \) depends on the energies \( \epsilon \) and they are determined by the overlap of the left and right wavefunctions under the barrier \[5\]. We notice that since typically the lifetime of the metastable states is much larger than the oscillation period in the well (in other words the last term in the Hamiltonian is a perturbation), the states \( \{|\epsilon\rangle\} \) which contribute effectively to tunneling are located in a relatively small energy interval compared to the plasma oscillation frequency, therefore the shape of these states under the barrier is approximately identical. We can then take the tunneling matrix element as being a complex constant; but since we will be interested exclusively in the outgoing component, the relative phase between the wavefunction inside the well and that outside will not play any role. We have confirmed this assumption also by expanding the initial wavefunction in terms of the WKB solution of the washboard potential calculated in \[6\]. Therefore we take \( k = \hbar \sqrt{\Gamma/2\pi} \) real; we obtain \( \mathcal{L}[\chi](s) = \Gamma/2 \text{ constant} \), which turns out to be the decay probability of the system. Indeed, the inverse Laplace transform of Eq. (4) gives precisely the classical exponential decay law
\[ a(t) = e^{-\Gamma t/2}. \]  
(6)

The outgoing wavepacket becomes
\[ |\Psi_{\text{out}}(t)\rangle = -i \sqrt{\frac{\Gamma}{2\pi}} \int d\epsilon e^{(-\Gamma/2+i\omega_0)t - e^{-i\epsilon t}} |\psi^\pm_\epsilon\rangle. \]  
(7)

To conclude this derivation, we find that with the identification \( k = \hbar \sqrt{\Gamma/2\pi} \) the Hamiltonian Eq. (2) becomes a model Hamiltonian for decay in the continuum which can be solved exactly, with solution given by Eqs. (1) and (7). A similar type of model Hamiltonian has been obtained in \[7, 8\]. One can show, using the properties of the Lorentz distribution, that these wavefunctions are correctly normalized, as explained above.

Let us now single out one component of the wave \( \Psi_{\text{out}}(t) \), namely
\[ \Psi_{\text{\rightarrow out}} = i \sqrt{\frac{\Gamma}{2\pi}} \int d\epsilon e^{-i\epsilon t} \frac{e^{-\Gamma t}}{\Gamma/2 + i(\omega_0 - \epsilon)} |\psi^+_\epsilon\rangle. \]  
(8)

We first notice that the normalization of the total function \( |\Psi_{\text{out}}\rangle \) is such that \( \langle \Psi_{\text{out}} | \Psi_{\text{out}} \rangle = 1 - \exp(-\Gamma t) \), which reflects correctly the fact that the probability of finding the particle outside comes from an exponential decay law, while that of \( |\Psi_{\text{\rightarrow out}}\rangle \) is such that \( \langle \Psi_{\text{\rightarrow out}} | \Psi_{\text{\rightarrow out}} \rangle = 1 \). In the following we will see that \( |\Psi_{\text{\rightarrow out}}\rangle \) plays indeed a special role.

To move on, we notice that part of the expression for the outgoing phase contains a term which decays exponentially on a time scale \( \Gamma^{-1} \). These terms are associated with the fast components of the localized wavefunction which would escape first. Although a calculation that includes these terms is no doubt interesting, especially for the problem of non-exponential decay rates \[3\], in what follows we will regard them as transient oscillatory effects whose presence

![FIG. 1: Tunneling out of one of the metastable wells of the washboard potential.](image-url)
will be difficult to assess experimentally anyway, and we will neglect their contribution. In the WKB approximation, far enough from the classical turning point, the eigenvalues \{\ket{\psi_n}\} have the form (up to a normalization factor and constant phase factors due to matching to the region left of the classical turning point) of incoming and outgoing scattering states

$$\psi^\pm_c(\gamma) \approx \sqrt{\frac{e}{C\Phi_0 V_c(\gamma)}} \exp \left[ \pm i \frac{C}{\gamma} \int_{\gamma_0}^{\gamma} V_c(\phi) d\phi \right],$$

where

$$V_c(\gamma) = \frac{\hbar \omega_p}{\sqrt{2e}} \sqrt{\frac{2e e + \Gamma \gamma}{I_0}} + 1 - \cos \gamma. \tag{10}$$

The physical meaning of this voltage is that it corresponds to the classical energy accumulated on the capacitor when the phase difference across the junction is \(\gamma\) and the initial energy of the system is \(e\); indeed, \(CV^2_c(\gamma)/2 = \hbar e - U(\gamma)\).

We now use the fact that for values of \(\gamma\) outside the well and far enough from the classical turning point the inequality \(|\hbar - \hbar \omega_0| \ll \hbar \omega_0 - U(\gamma)\) holds. We can therefore take \(V(\gamma) = V_0(\gamma) \approx V_0(\gamma)\) in the denominator of Eq. (10) and approximate the exponent as

$$V_0(\gamma) \approx V_0(\gamma) \left[ 1 + \frac{\varepsilon - \omega_0}{2(\omega_0 - \hbar^{-1} U(\gamma))} \right]. \tag{11}$$

With these approximations, using Eqs. (11) and (12) we can perform the integral over the angular frequencies \(\varepsilon\); as a result, the contribution of the in-going scattering states in zero, while the out-going scattering states build up a wavepacket of the form

$$\Psi_{\text{out}}(\gamma, t) = \frac{N}{\sqrt{V_0(\gamma)}} \exp \left[ \frac{i}{\hbar} \int_{\gamma_0}^{\gamma} C \tilde{V}(\phi) \Phi_0 d\phi - \left( i \omega_0 + \frac{\Gamma}{2} \right) \left( t - \int_{\gamma_0}^{\gamma} \Phi_0 d\phi \right) \right] \Theta \left[ t - \int_{\gamma_0}^{\gamma} \Phi_0 d\phi \right]. \tag{12}$$

Here \(N\) is a normalization factor which can be obtained through \(\int_{-\infty}^{\infty} d(\Phi_0\gamma)|\Psi_{\text{out}}(\gamma, t)|^2 = 1\) with the mention that we make a negligible error by extending the integral to \(-\infty\), i.e. before the well region (where the actual values are exponentially small). The voltage \(\tilde{V}(\gamma)\) is defined as \(\tilde{V}(\gamma) = V_0(\gamma) - \hbar \omega_0/CV_0(\gamma) \approx V_0(\gamma)\). It is interesting to see also what happens with the rest of the components of \(|\Psi_{\text{out}}\rangle\). Although they do contribute to the normalization as discussed before, they are decaying both in time and away from the barrier as \(\exp\left[-\frac{i}{2} \int_{\gamma_0}^{\gamma} V_0^{-1}(\phi) \Phi_0 d\phi \right]\) which, as we will see below, would give far from the barrier a factor of \(\exp(-\Gamma/2(t + \sqrt{2\gamma}/\omega_0))\). It is clear that these terms can be neglected starting roughly from a time \(\Gamma^{-1}\). The wavefunction Eq. (12) contains all the information about the dynamical evolution of the state of the circuit containing the Josephson junction, and it is the main result of this paper.

In a typical experiment, the voltage across the junction is monitored by a voltmeter at room temperature. A fundamental issue is to find a microscopic mechanism for the junction-voltmeter interaction and a suitable theory of quantum measurement that would model the collapse of the wavefunction; this is however beyond the scope of this paper. Still, Eqs. (12) and (13) give a quite clear qualitative picture of what happens: the particle rolls down the washboard potential with a quasi-classical speed given by energy conservation \(CV_0^2(\gamma)/2 = \hbar \omega_0 - U(\gamma)\). Quantum mechanics enters in the picture through the tunneling rate; we expect the results of the measurements to have a spread determined by \(\Gamma\). One can assume that the measurement projects the outgoing state onto eigenvalues of the voltage operator; therefore the probability of recording the value \(V\) at the moment \(t\) will be given by the standard quantum mechanics recipe

$$P(V, t) = \frac{1}{2\pi\hbar} \left| \int_{-\infty}^{\infty} d(\Phi_0\gamma) \Psi_{\text{out}}^*(\gamma, t) \exp(iV/C\gamma/2e) d\gamma \right|^2. \tag{13}$$

As an example, had the outside the well potential \(U\) been zero, we would have gotten for the charge \(CV\), by performing the integration in Eq. (12), a standard Cauchy-Breit-Wigner distribution centered around \(CV_0\) and full width at half maximum \(\Gamma\hbar/V_0\)

$$P(CV) = \frac{1}{\pi} \frac{\Gamma\hbar}{2V_0^2} \left( (CV - CV_0)^2 + \left( \frac{\Gamma\hbar}{2V_0} \right)^2 \right)^{-1}. \tag{14}$$

Considering again the case of a junction with a washboard potential \(U(\gamma)\), we notice that a good approximation is \(U(\gamma) \approx EJ\gamma\). This comes from the fact that switching is typically observed at values of the bias current close to
the critical current of the junction, as well as from the observation that for times larger than $\Gamma^{-1}$ the wavepacket is concentrated at large values of $\gamma \gg 1$, in which case the $\cos \gamma$ term in the potential is negligible. In other words, the particle gets soon so fast that the "speed bumps" created by the Josephson effect are not slowing it down significantly. This can be checked \textit{a posteriori}. A first observation is that the relevant quantity for the dynamics of the center of the wavepacket is the argument of the $\Theta$ function; the condition that this argument vanishes sets the maximum value of $|\Psi_{\text{out}}|^2$ and gives a phase $\omega_p / 2 \gg 1$ for $t$ larger than $\Gamma^{-1}$. A legitimate concern is whether the wavefunction does not spread faster than it moves downwards. This is not the case, as we will see below: the spread of the wavefunction increases linearly with time, while the average coordinate (phase) is advancing as $t^2$. With these observations, the normalization constant can be calculated, and the outgoing wavepacket becomes

$$
\psi_{\text{out}}(\gamma, t) = \sqrt{\frac{\Gamma}{\sqrt{2\pi}\Phi_{0}\omega_p}} \exp \left[ i \frac{\hbar\omega_p \gamma^{3/2}}{6\sqrt{2}E_c} - \left( \frac{\omega_0}{\omega_p} + \frac{\Gamma}{2\omega_p} \right) \left( t\omega_p - \sqrt{2}\gamma \right) \right] \Theta \left( t\omega_p - \sqrt{2}\gamma \right). \tag{15}
$$

A plot of the wavefunction is given in Fig. 2.

![Fig. 2](image-url)

**FIG. 2**: Contourplot $(\gamma, t)$ of the modulus of the running-phase wavefunction Eq. (15) for $\omega_p = 60\Gamma = 30$ GHz.

The average phase (flux) corresponding to this wavepacket can be obtained

$$
\langle \gamma \rangle(t) = \omega_p^2 \left( t^2 / 2 - \Gamma^{-1}t + \Gamma^{-2} \right), \tag{16}
$$

and we notice that the dominant term is quadratic in $t$. The spread of the flux variable is given by (we keep only the dominant term here)

$$
\sqrt{\langle \gamma^2 \rangle(t) - \langle \gamma \rangle^2(t)} = \omega_p^2 \Gamma^{-1}t. \tag{17}
$$

To get the average voltage we can use Ehrenfest theorem; we obtain

$$
\langle V(t) \rangle = \Phi_{0}\omega_p^2 t - \Gamma^{-1}. \tag{18}
$$

The dominant term for the voltage is linear in time and satisfies the classical energy conservation $C \langle V \rangle^2(t)/2 = \hbar\omega_0 - U(\langle \gamma \rangle(t)) \approx I_0(\langle \gamma \rangle(t)).$

Let us now analyze what happens in typical switching current experiments, as they are done now in the context of superconducting qubits [9]: the bias current of the junction is increased fast to a value that allows tunneling, it is kept there for a time $0 < \tau < \Gamma^{-1}$, then it is lowered to a value that suppresses tunneling. This value has to be large enough so that the experimentalist can get a reliable reading of voltage on the quasiparticle branch if the junction
the average phase and voltage on the state Eq. (20)

\[ |\Psi_{\text{out}}(t)\rangle = -i\sqrt{\frac{\Gamma}{2\pi}} \int d\epsilon \frac{e^{(-\Gamma/2+i\omega_0)\tau}e^{-i\omega_2 t - e^{-i\epsilon t}}} {\Gamma/2 + i(\omega_0 - \epsilon)} |\psi^\pm\rangle, \]  

(19)

with normalization \( \langle \Psi_{\text{out}}(t) | \Psi_{\text{out}}(t) \rangle = 1 - e^{-\Gamma\tau} \). Now, for \( t-\tau > \Gamma^{-1} \) we can see that the outgoing function consists of two consecutive (separated by the time \( \tau \)) and dephased (with \( \omega_0\tau \)) outgoing wavepackets with the structure of \( |\Psi_{\text{out}}(t)\rangle \) which propagate at the same speed across the phase coordinate \( \gamma \). The second wavepacket, which has a probability amplitude smaller by a factor of \( e^{-\Gamma/2\tau} \), results from the waves localized near the barrier during the time \( \tau \) when tunneling was in progress. After integration over energy, we get

\[ |\Psi_{\text{out}}(t)\rangle = e^{-(\Gamma/2+i\omega_0)\tau} |\Psi_{\text{in}}(t-\tau)\rangle - |\Psi_{\text{in}}^-(t)\rangle, \]  

(20)

where \( |\Psi_{\text{in}}^-(t)\rangle \) is given by Eqs. (12) and (15). To check that the normalization \( \langle \Psi_{\text{out}}(t) | \Psi_{\text{out}}(t) \rangle = 1 - e^{-\Gamma\tau} \) remains valid, we notice that in the region of overlap of the two wavepackets, which coincides with the domain where \( |\Psi_{\text{out}}^-(t-\tau)\rangle \) is finite \( t-\tau > \int_{\tau_0}^{\tau} d\Phi V_0^{-1}(\phi)d\phi \), there exists a very simple relation between them: \( |\Psi_{\text{out}}(t)\rangle = \exp[-i\omega_0\tau - \Gamma\tau/2] |\Psi_{\text{in}}^-(t-\tau)\rangle \). Using this property and the previous expressions Eqs. (18) and (19), we can calculate the average phase and voltage on the state Eq. (18)

\[ \langle \gamma(t) \rangle = \frac{1}{2} \omega_0^2 \left[ 1 - e^{-\Gamma\tau} \right] t^2 - e^{-\Gamma\tau} \omega_0^2 \tau t - \frac{1}{2} e^{-\Gamma\tau} \omega_p^2, \]  

(21)

and

\[ \langle V(t) \rangle = \Phi_0 \omega_p^2 t \left( 1 - e^{-\Gamma\tau} \right) - \Phi_0 \omega_p^2 e^{-\Gamma\tau}. \]  

(22)

In Fig. 3 we present a plot of the average voltage as a function of the time \( \tau \). We see that for values of \( \tau \) of the same order or larger than the lifetime \( \Gamma^{-1} \) the average voltage at \( t \) flattens, reflecting the fact that the junction has switched, as in the case of Eq. (18).

![FIG. 3: The average voltage as a function of the time \( \tau \) for \( \omega_p = 30 \text{ GHz} \), \( \Gamma = 1 \text{ ns} \) and \( t = 10 \text{ ns} \).](image)

For designing an experiment to test these predictions, several remarks should be made. In the case of real junctions, the Josephson energy and the plasma frequency can be reduced by using a SQUID configuration and by adding capacitors in parallel with the junctions. This makes the time evolution of the switching state slower and therefore easier to detect. An important limitation on time comes from the fact that as soon as the voltage reaches the quasiparticle branch (at twice the value of the gap) our analysis is not valid. The other limitation is technological: even with a good dilution refrigerator, thermalizing the junction is very difficult at low temperatures. With a good high-power refrigerator with base temperature of about 5 mK, we assume an optimistic value of 10 mK for the effective temperature of the electrons. This temperature corresponds to a crossover angular frequency of 8.66 GHz between
the MQT and the thermal activation transition. A plasma frequency of $\omega_p = 30$ GHz (zero bias current) will thus keep us safely in the MQT regime when the current is raised up to about half a percent close to the critical current, according to the formula that gives the plasma oscillation frequency at a finite bias current \cite{1, 2}. For Nb, with gap of 1.4 meV, this corresponds to a time of approximately 10 ns, as given by Eq. \cite{15}. A voltage increase on this timescale can be detected with standard experimental techniques Suppose now that we choose to work at currents about 5% less than the critical current. We still have to satisfy the condition $t > \Gamma^{-1}$; an inspection of the formula that gives the tunneling rate for underdamped junctions (see \textit{e.g.} \cite{2}) shows that switching rates of about 500 MHz and more (with the restriction $\Gamma \ll \omega_p$) can be achieved for $E_J/\omega_p$ of the order of 30, values which can be obtained easily with large junctions.

G. S. P. was supported by an EU Marie Curie Fellowship (HPMF-CT-2002-01893); this work is also part of the SQUBIT-2 project (IST-1999-10673), the Academy of Finland TULE No.7205476, and the Center of Excellence in Condensed Matter and Nuclear Physics at the University of Jyväskylä.

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