C*-algebras of Boolean inverse monoids – traces and invariant means

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Abstract

To a Boolean inverse monoid $S$ we associate a universal C*-algebra $C^*(S)$ and show that it is equal to Exel’s tight C*-algebra of $S$. We then show that any invariant mean on $S$ (in the sense of Kudryavtseva, Lawson, Lenz and Resende) gives rise to a trace on $C^*(S)$, and vice-versa, under a condition on $S$ equivalent to the underlying groupoid being Hausdorff. Under certain mild conditions, the space of traces of $C^*(S)$ is shown to be isomorphic to the space of invariant means of $S$. We then use many known results about traces of C*-algebras to draw conclusions about invariant means on Boolean inverse monoids; in particular we quote a result of Blackadar to show that any metrizable Choquet simplex arises as the space of invariant means for some AF inverse monoid $S$.

1 Introduction

This article is the continuation of our study of the relationship between inverse semigroups and C*-algebras. An inverse semigroup is a semigroup $S$ for which every element $s \in S$ has a unique “inverse” $s^*$ in the sense that

$$ss^*s = s \text{ and } s^*ss^* = s^*.$$ 

An important subsemigroup of any inverse semigroup is its set of idempotents $E(S) = \{e \in S \mid e^2 = e\} = \{s^*s \mid s \in S\}$. Any set of partial isometries closed under product and involution inside a C*-algebra is an inverse semigroup, and its set of idempotents forms a commuting set of projections. Many C*-algebras $A$ have been profitably studied in the following way:

1. identify a generating inverse semigroup $S$,
2. write down an abstract characterization of $S$,
3. show that $A$ is universal for some class of representations of $S$.

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We say “some class” above because typically considering all representations (as in the construction of Paterson [Pat99]) gives us a larger C*-algebra than we started with. For example, consider the multiplicative semigroup inside the Cuntz algebra $O_2$ generated by the two canonical generators $s_0$ and $s_1$; in semigroup literature this is usually denoted $P_2$ and called the \textit{polycyclic monoid} of order 2. The C*-algebra which is universal for all representations of $P_2$ is $T_2$, the Toeplitz extension of $O_2$. In an effort to arrive back at the original C*-algebra in cases such as this, Exel defined the notion of \textit{tight} representations [Exe08], and showed that the universal C*-algebras for tight representations of $P_2$ is $O_2$.

See [Sta15b], [Sta15a], [EP14b], [EP14a], [EGS12], [MOP15] for other examples of this approach.

Another approach to this issue is to instead alter the inverse semigroup $S$. An inverse semigroup carries with it a natural order structure, and when an inverse semigroup $S$ is represented in a C*-algebra $A$, two elements $s,t \in S$, which did not have a lowest upper bound in $S$, may have one inside $A$. So, from $P_2$, Lawson and Scott [LS14, Proposition 3.32] constructed a new inverse semigroup $C_2$, called the \textit{Cuntz inverse monoid}, by adding to $P_2$ all possible joins of compatible elements ($s,t$ are compatible if $s^*t, st^* \in E(S)$).

The above is an example of a \textit{Boolean inverse monoid}, and the goal of this paper is to define universal C*-algebras for such monoids and study them. A Boolean inverse monoid is an inverse semigroup which contains joins of all finite compatible sets of elements and whose idempotent set is a Boolean algebra. To properly represent a Boolean inverse monoid $S$, one reasons, then one should insist that the join of two compatible $s,t \in S$ be sent to the join of the images of $s$ and $t$. We prove in Proposition 3.3 that such a representation is necessarily a tight representation, and so we obtain that the universal C*-algebra of a Boolean inverse monoid is exactly its tight C*-algebra, Theorem 3.5. This is the starting point of our study, as the universal tight C*-algebra can be realized as the C*-algebra of an ample groupoid.

The main inspiration of this paper is [KLLR16] which defines and studies \textit{invariant means} on Boolean inverse monoids. An invariant mean is a function $\mu : E(S) \to [0, \infty)$ such that $\mu(e \lor f) = \mu(e) + \mu(f)$ when $e$ and $f$ are orthogonal, and such that $\mu(ss^*) = \mu(s^*s)$ for all $s \in S$. If one thinks of the idempotents as clopen sets in the Stone space of the Boolean algebra $E(S)$, such a function has the flavour of an invariant measure or a trace. We make this precise in Section 4: as long as $S$ satisfies a condition which guarantees that the induced groupoid is Hausdorff (which we call condition (H)), every invariant mean on $S$ gives rise to a trace on $C^*(S)$ (Proposition 4.5) and every trace on $C^*(S)$ gives rise to an invariant mean on $S$ (Proposition 4.6). This becomes a one-to-one correspondence if we assume that the associated groupoid $G_{\text{tight}}(S)$ is principal and amenable (Theorem 4.12). We also prove that, whether $G_{\text{tight}}(S)$ is principal and amenable or not, there is an affine isomorphism between the space of invariant means on $S$ and the space of $G_{\text{tight}}(S)$-invariant measures on its unit space (Proposition 4.10).

In the final section, we apply our results to examples of interest. We study the \textit{AF inverse monoids} in detail – these are Boolean inverse monoids arising from Bratteli diagrams in much the same way as AF C*-algebras. As it should be, given a Bratteli diagram, the C*-algebra of its Boolean inverse monoid is isomorphic to the AF algebra it determines (Theorem 5.1). From this we can conclude, using the results of Section 4 and the seminal result of Blackadar [Bla80], that any Choquet simplex arises as the space of invariant
means for some Boolean inverse monoid. We go on to consider two examples where there is typically only one invariant mean, those being self-similar groups and aperiodic tilings.

2 Preliminaries and notation

We will use the following general notation. If $X$ is a set and $U \subset X$, let $\text{Id}_U$ denote the map from $U$ to $U$ which fixes every point, and let $1_U$ denote the characteristic function on $U$, i.e. $1_U : X \to \mathbb{C}$ defined by $1_U(x) = 1$ if $x \in U$ and $1_U(x) = 0$ if $x \notin U$. If $F$ is a finite subset of $X$, we write $F \subset \text{fin} X$.

2.1 Inverse semigroups

An inverse semigroup is a semigroup $S$ such that for all $s \in S$, there is a unique element $s^* \in S$ such that $ss^*s = s$, $s^*ss = s^*$. The element $s^*$ is called the inverse of $s$. All inverse semigroups in this paper are assumed to be discrete and countable. For $s,t \in S$, one has $(s^*)^* = s$ and $(st)^* = t^*s^*$. Although not implied by the definition, we will always assume that inverse semigroups have a 0 element, that is, an element such that $0s = s0 = 0$ for all $s \in S$.

An inverse semigroup with identity is called an inverse monoid. Even though we call $s^*$ the inverse of $s$, we need not have $ss^* = 1$, although it is always true that $(ss^*)^2 = ss^*ss^* = ss^*$, i.e. $ss^*$ (and $s^*s$ for that matter) is an idempotent. We denote the set of all idempotents in $S$ by $E(S) = \{ e \in S \mid e^2 = e \}$.

It is a nontrivial fact that if $S$ is an inverse semigroup, then $E(S)$ is closed under multiplication and commutative. It is also clear that if $e \in E(S)$, then $e^* = e$.

Let $X$ be a set, and let

$$\mathcal{I}(X) = \{ f : U \to V \mid U, V \subset X, f \text{ bijective} \}.$$ 

Then $\mathcal{I}(X)$ is an inverse monoid with the operation of composition on the largest possible domain, and inverse given by function inverse; this is called the symmetric inverse monoid on $X$. Every idempotent $e$ in $\mathcal{I}(X)$ is given by $\text{Id}_U$ for some $U \subset X$. The function $\text{Id}_X$ is the identity for $\mathcal{I}(X)$, and the empty function is the 0 element for $\mathcal{I}(X)$. The fundamental Wagner-Preston theorem states that every inverse semigroup is embeddable in $\mathcal{I}(X)$ for some set $X$ – one can think of this as analogous to the Cayley theorem for groups.

Every inverse semigroup carries a natural order structure: for $s,t \in S$ we say $s \leq t$ if and only if $ts^*s = s$, which is also equivalent to $ss^*t = s$. For elements $e,f \in E(S)$, we have $e \leq f$ if and only if $ef = e$. As usual, for $s,t \in S$, the join (or least upper bound) of $s$ and $t$ will be denoted $s \vee t$ (if it exists), and the meet (or greatest lower bound) of $s$ and $t$ will be denoted $s \wedge t$ (if it exists). For $A \subset S$, we let $A^\uparrow = \{ t \in S \mid s \leq t \text{ for some } s \in A \}$ and $A^\downarrow = \{ t \in S \mid t \leq s \text{ for some } s \in A \}$. 

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If \( s, t \in S \), then we say \( s \) and \( t \) are compatible if \( s^*t, st^* \in E(S) \), and a set \( F \subset \text{fin} \ S \) is called compatible if all pairs of elements of \( F \) are compatible.

**Definition 2.1.** An inverse semigroup \( S \) is called distributive if whenever we have a compatible set \( F \subset \text{fin} \ S \), then \( \bigvee_{s \in F} s \) exists in \( S \), and for all \( t \in S \) we have

\[
t \left( \bigvee_{s \in F} s \right) = \bigvee_{s \in F} ts \quad \text{and} \quad \left( \bigvee_{s \in F} s \right) t = \bigvee_{s \in F} st.
\]

In the natural partial order, the idempotents form a meet semilattice, which is to say that any two elements \( e, f \in E(S) \) have a meet, namely \( ef \). If \( C \subset X \subset E(S) \), we say that \( C \) is a cover of \( X \) if for all \( x \in X \) there exists \( c \in C \) such that \( cx \neq 0 \).

In a distributive inverse semigroup each pair of idempotents has a join in addition to the meet mentioned above, but in general \( E(S) \) will not have relative complements and so in general will not be a Boolean algebra. The case where \( E(S) \) is a Boolean algebra is the subject of the present paper.

**Definition 2.2.** A Boolean inverse monoid is a distributive inverse monoid \( S \) with the property that \( E(S) \) is a Boolean algebra, that is, for every \( e \in E(S) \) there exists \( e^\perp \in E(S) \) such that \( ee^\perp = 0 \), \( e \lor e^\perp = 1 \), and the operations \( \lor, \land, \perp \) satisfy the laws of a Boolean algebra [GH09 Chapter 2].

**Example 2.3.** Perhaps the best way to think about the order structure and related concepts above is by describing them on \( I(X) \), which turns out to be a Boolean inverse monoid. Firstly, for \( g, h \in I(X) \), \( g \leq h \) if and only if \( h \) extends \( g \) as a function. In \( I(X) \), two functions \( f \) and \( g \) are compatible if they agree on the intersection of their domains and their inverses agree on the intersection of their ranges. In such a situation, one can form the join \( f \lor g \) which is the union of the two functions; this will again be an element of \( I(X) \). Composing \( h \in I(X) \) with \( f \lor g \) will be the same as \( hf \lor hg \). Finally, \( E(I(X)) = \{ \text{Id}_U \mid U \subset X \} \) is a Boolean algebra (isomorphic to the Boolean algebra of all subsets of \( X \)) with \( \text{Id}_U^\perp = \text{Id}_{U^c} \).

### 2.2 Étale groupoids

A groupoid is a small category where every arrow is invertible. If \( G \) is a groupoid, the set of elements \( \gamma \gamma^{-1} \) is denoted \( G^{(0)} \) and is called the set of units of \( G \). The maps \( r : G \to G^{(0)} \) and \( d : G \to G^{(0)} \) defined by \( r(\gamma) = \gamma \gamma^{-1} \) and \( d(\gamma) = \gamma^{-1} \gamma \) are called the range and source maps, respectively.

The set \( G^{(2)} = \{ (\gamma, \eta) \in G^2 \mid r(\eta) = d(\gamma) \} \) is called the set of composable pairs. A topological groupoid is a groupoid \( G \) which is a topological space and for which the inverse map from \( G \) to \( G \) and the product from \( G^{(2)} \) to \( G \) are both continuous (where in the latter, the topology on \( G^{(2)} \) is the product topology inherited from \( G^2 \)).

We say that a topological groupoid \( G \) is étale if it is locally compact, second countable, \( G^{(0)} \) is Hausdorff, and the maps \( r \) and \( d \) are both local homeomorphisms. Note that an étale groupoid need not be Hausdorff. If \( G \) is étale, then \( G^{(0)} \) is open, and \( G \) is Hausdorff if and only if \( G^{(0)} \) is closed (see for example [EP14b Proposition 3.10]).
For \( x \in \mathcal{G}^{(0)} \), let \( \mathcal{G}(x) = \{ \gamma \in \mathcal{G} \mid r(\gamma) = d(\gamma) = x \} \) – this is a group, and is called the isotropy group at \( x \). A groupoid \( \mathcal{G} \) is said to be principal if all the isotropy groups are trivial, and a topological groupoid is said to be essentially principal if the points with trivial isotropy groups are dense in \( \mathcal{G}^{(0)} \). A topological groupoid is said to be minimal if for all \( x \in \mathcal{G}^{(0)} \), the set \( O_\mathcal{G}(x) = r(d^{-1}(x)) \) is dense in \( \mathcal{G}^{(0)} \) (the set \( O_\mathcal{G}(x) \) is called the orbit of \( x \)).

If \( \mathcal{G} \) is an étale groupoid, an open set \( U \subset \mathcal{G} \) is called a bisection if \( r|_U \) and \( d|_U \) are both injective (and hence homeomorphisms). The set of all bisections is denoted \( \mathcal{G}^{op} \), and is a distributive inverse semigroup when given the operations of setwise product and inverse. We say that \( \mathcal{G} \) is ample if the set of compact bisections forms a basis for the topology on \( \mathcal{G} \). The set of compact bisections is called the ample semigroup of \( \mathcal{G} \), denoted \( \mathcal{G}^a \), and is also a distributive inverse subsemigroup of \( \mathcal{G}^{op} \) [LL13, Lemma 3.14]. Since \( \mathcal{G} \) is second countable, \( \mathcal{G}^a \) must be countable [Exc10, Corollary 4.3]. If \( \mathcal{G}^{(0)} \) is compact, then the idempotent set of \( \mathcal{G}^a \) is the set of all clopen sets in \( \mathcal{G}^{(0)} \), and so \( \mathcal{G}^a \) is a Boolean inverse monoid.

To an étale groupoid \( \mathcal{G} \) one can associate C*-algebras through the theory developed by Renault [Ren80]. Let \( C_c(\mathcal{G}) \) denote the linear space of continuous compactly supported functions on \( \mathcal{G} \). Then \( C_c(\mathcal{G}) \) becomes a *-algebra with product and involution given by

\[
fg(\gamma) = \sum_{\gamma_1 \gamma_2 = \gamma} f(\gamma_1)g(\gamma_2), \quad f^*(\gamma) = \overline{f(\gamma^{-1})}.
\]

From this one can produce two C*-algebras \( C^*(\mathcal{G}) \) and \( C^*_{red}(\mathcal{G}) \) (called the C*-algebra of \( \mathcal{G} \) and the reduced C*-algebra of \( \mathcal{G} \), respectively) by completing \( C_c(\mathcal{G}) \) in certain norms, see [Ren80, Definitions 1.12 and 2.8]. There is always a surjective *-homomorphism \( \Lambda : C^*(\mathcal{G}) \to C^*_{red}(\mathcal{G}) \), and if \( \Lambda \) is an isomorphism we say that \( \mathcal{G} \) satisfies weak containment. If \( \mathcal{G} \) is amenable [ADR00], then \( \mathcal{G} \) satisfies weak containment. There is an example of a case where \( \Lambda \) is an isomorphism for a nonamenable groupoid [Wil15], but under some conditions on \( \mathcal{G} \) one has that weak containment and amenability are equivalent, see [AD16b, Theorem B].

Let \( \mathcal{G} \) be an ample étale groupoid. Both C*-algebras contain \( C_c(\mathcal{G}) \), and hence if \( U \) is a compact bisection, \( 1_U \) is an element of both C*-algebras. Hence we have a map \( \pi : \mathcal{G}^a \to C^*(\mathcal{G}) \) given by \( \pi(U) = 1_U \). This map satisfies \( \pi(UV) = \pi(U)\pi(V) \) and \( \pi(0) = 0 \), in other words, \( \pi \) is a representation of the inverse semigroup \( \mathcal{G}^a \) [Exc10].

### 2.3 The tight groupoid of an inverse semigroup

Let \( S \) be an inverse semigroup. A filter in \( E(S) \) is a nonempty subset \( \xi \subset E(S) \) such that

1. \( 0 \notin \xi \),
2. \( e, f \in \xi \) implies that \( ef \in \xi \), and
3. \( e \in \xi, e \leq f \) implies \( f \in \xi \).

The set of filters is denoted \( \tilde{E}_0(S) \), and can be viewed as a subspace of \( \{0, 1\}^{E(S)} \). For \( X, Y \subset \text{fin} \ E(S) \), let

\[
U(X, Y) = \{ \xi \in \tilde{E}_0(S) \mid X \subset \xi, Y \cap \xi = \emptyset \}.
\]
sets of this form are clopen and generate the topology on \( \hat{E}_0(S) \) as \( X \) and \( Y \) vary over all the finite subsets of \( E(S) \). With this topology, \( \hat{E}_0(S) \) is called the spectrum of \( E(S) \).

A filter is called an ultrafilter if it is not properly contained in any other filter. The set of all ultrafilters is denoted \( \hat{E}_\infty(S) \). As a subspace of \( \hat{E}_0(S) \), \( \hat{E}_\infty(S) \) may not be closed. Let \( \hat{E}_{\text{tight}}(S) \) denote the closure of \( \hat{E}_\infty(S) \) in \( \hat{E}_0(S) \) – this is called the tight spectrum of \( E(S) \). Of course, when \( E(S) \) is a Boolean algebra, \( \hat{E}_{\text{tight}}(S) = \hat{E}_\infty(S) \) by Stone duality [GH09, Chapter 34].

An action of an inverse semigroup \( S \) on a locally compact space \( X \) is a semigroup homomorphism \( \alpha : S \to \mathcal{I}(X) \) such that

1. \( \alpha_s \) is continuous for all \( s \in S \),
2. the domain of \( \alpha_s \) is open for each \( s \in S \), and
3. the union of the domains of the \( \alpha_s \) is equal to \( X \).

If \( \alpha \) is an action of \( S \) on \( X \), we write \( \alpha : S \curvearrowright X \). The above implies that \( \alpha_{s^*} = \alpha_s^{-1} \), and so each \( \alpha_s \) is a homeomorphism. For each \( e \in E(S) \), the map \( \alpha_e \) is the identity on some open subset \( D^e_s \), and one easily sees that the domain of \( \alpha_s \) is \( D^\alpha_s \) and the range of \( \alpha_s \) is \( D^{s*}_s \), that is

\[
\alpha_s : D^\alpha_s \to D^{s*}_s.
\]

There is a natural action \( \theta \) of \( \alpha : S \curvearrowright X \). Let \( \Theta(s,x) = \{(s,x) \in S \times X | x \in D^\alpha_s \} \), and put an equivalence relation \( \sim \) on this set by saying that \( (s,x) \sim (t,y) \) if and only if \( x = y \) and there exists some \( e \in E(S) \) such that \( se = te \) and \( x \in D^e_\theta s \). The set of equivalence classes is denoted

\[
\mathcal{G}(\alpha) = \{[s,x] | s \in S, x \in X \}
\]

and becomes a groupoid when given the operations

\[
[s,x]^{-1} = [s^*, \alpha_s(x)], \quad d([s,x]) = x, \quad r([s,x]) = \alpha_s(x), \quad [t, \alpha_s(x)][s,x] = [ts,x].
\]

This is called the groupoid of germs of \( \alpha \). Note that above we are making the identification of the unit space with \( X \), because \( [e,x] = [f,x] \) for any \( e, f \in E(S) \) with \( x \in D^e_\alpha, D^f_\alpha \). For \( s \in S \) and open set \( U \subset D^\alpha_{s^*} \) we let

\[
\Theta(s,U) = \{[s,x] | x \in U \}
\]

and endow \( \mathcal{G}(\alpha) \) with the topology generated by such sets. With this topology \( \mathcal{G}(\alpha) \) is an étale groupoid, sets of the above type are bisections, and if \( X \) is totally disconnected \( \mathcal{G}(\alpha) \) is ample.

Let \( \theta : S \curvearrowright \hat{E}_{\text{tight}}(S) \) be the standard action, and define

\[
\mathcal{G}_{\text{tight}}(S) = \mathcal{G}(\theta).
\]
This is called the *tight groupoid* of $S$. This was defined first in [Exe08] and studied extensively in [EP14b].

Let $\mathcal{G}$ be an ample étale groupoid, and consider the Boolean inverse monoid $G^a$. By work of Exel [Exe10] if one uses the above procedure to produce a groupoid from $G^a$, one ends up with exactly $\mathcal{G}$. In symbols,

$$G^{\text{tight}}(G^a) \cong \mathcal{G}$$

for any ample étale groupoid $G$.

In particular,

$$G^{\text{tight}}(G^{\text{tight}}(S^a)) \cong G^{\text{tight}}(S)$$

for all inverse semigroups $S$.

This result can be made categorical [LL13, Theorem 3.26], and has been generalized to cases where the space of units is not even Hausdorff. This duality between Boolean inverse semigroups and ample étale groupoids falls under the broader program of noncommutative Stone duality, see [LL13] for more details.

## 3 C*-algebras of Boolean inverse monoids

In this section we describe the tight C*-algebra of a general inverse monoid, define the C*-algebra of a Boolean inverse monoid, and show that these two notions coincide for Boolean inverse monoids.

If $S$ is an inverse monoid, then a *representation* of $S$ in a unital C*-algebra $A$ is a map $\pi : S \to A$ such that $\pi(0) = 0$ and $\pi(st) = \pi(s)\pi(t)$. If $\pi$ is a representation, then $C^*(\pi(E(S)))$ is a commutative C*-algebra. Let

$$\mathcal{B}_\pi = \{ e \in C^*(\pi(E(S))) \mid e^2 = e = e^* \}$$

Then this set is a Boolean algebra with operations

$$e \wedge f = ef, \quad e \vee f = e + f - ef, \quad e^\perp = 1 - e.$$

We will be interested in a subclass of representations of $S$. Take $X, Y \subset \text{fin} E(S)$, and define

$$E(S)^{X,Y} = \{ e \in E(S) \mid e \leq x \text{ for all } x \in X, ey = 0 \text{ for all } y \in Y \}$$

We say that a representation $\pi : S \to A$ with $A$ unital is *tight* if for all $X, Y, Z \subset \text{fin} E(S)$ where $Z$ is a cover of $E(S)^{X,Y}$, we have the equation

$$\bigvee_{z \in Z} \pi(z) = \prod_{x \in X} \pi(x) \prod_{y \in Y} (1 - \pi(y)).$$

The *tight C*-algebra of $S$, denoted $C^*_{\text{tight}}(S)$, is then the universal unital C*-algebra generated by one element for each element of $S$ subject to the relations that guarantee that the standard map from $S$ to $C^*_{\text{tight}}(S)$ is tight. The above was all defined in [Exe08] and the interested reader is directed there for the details. It is a fact that $C^*_{\text{tight}}(S) \cong C^*(G^{\text{tight}}(S))$ where the latter is the full groupoid C*-algebra.
If $S$ has the additional structure of being a Boolean inverse monoid, then we might wonder what extra properties $\pi$ should have, in particular, what is the notion of a “join” of two partial isometries in a C*-algebra?

Let $A$ be a C*-algebra, and suppose that $S$ is a Boolean inverse monoid of partial isometries in $A$. If we have $s, t \in S$ and $s^*, st^* \in E(S)$, then

$$tt^* s = tt^* ss^* s = ss^* tt^* s = s(s^* t)^* (s^* t)^* = ss^* t$$

and consider the element $a_{s,t} := s + t - ss^* t = s + t - tt^* s$, this is a partial isometry with range $a_{ss^* tt^*}$ and support $a_{s^* s, t^* t}$. A short calculation shows that $a_{s,t}$ is the least upper bound for $s$ and $t$ in the natural partial order, and so $a_{s,t} = s \lor t$. It is also straightforward that $r(s \lor t) = rs \lor rt$ for all $r, s, t \in S$. This leads us to the following definitions.

**Definition 3.1.** Let $S$ be a Boolean inverse monoid. A Boolean inverse monoid representation of $S$ in a unital C*-algebra $A$ is a map $\pi : S \to A$ such that

1. $\pi(0) = 0$,
2. $\pi(st) = \pi(s)\pi(t)$ for all $s, t \in S$, and
3. $\pi(s \lor t) = \pi(s) + \pi(t) - \pi(ss^* t)$ for all compatible $s, t \in S$.

**Definition 3.2.** Let $S$ be a Boolean inverse monoid. Then the universal C*-algebra of $S$, denoted $C^*(S)$, is defined to be the universal unital C*-algebra generated by one element for each element of $S$ subject to the relations which say that the standard map of $S$ into $C^*(S)$ is a Boolean inverse monoid representation. The map $\pi_u$ which takes an element $s$ to its corresponding element in $C^*(S)$ will be called the universal Boolean inverse monoid representation of $S$, and we will sometimes use the notation $\delta_s := \pi_u(s)$.

The theory of tight representations was originally developed to deal with representing inverse semigroups (inside which joins may not exist) inside C*-algebras, because in a C*-algebra one always has the join of two commuting projections. It should come as no surprise then that once we are dealing with an inverse semigroup where we can take joins, the representations which respect joins end up being exactly the tight representations, see [DM14, Corollary 2.3]. This is what we prove in the next proposition.

**Proposition 3.3.** Let $S$ be a Boolean inverse monoid. Then a map $\pi : S \to A$ is a Boolean inverse monoid representation of $S$ if and only if $\pi$ is a tight representation.

*Proof.* Suppose that $\pi$ is a Boolean inverse monoid representation of $S$. Then when restricted to $E(S)$, $\pi$ is a Boolean algebra homomorphism into $\mathcal{B}_\pi$, and so by [Exe08, Proposition 11.9], $\pi$ is a tight representation.

On the other hand, suppose that $\pi$ is a tight representation, and first suppose that $e, f \in E(S)$. Then the set $\{e, f\}$ is a cover for $E(S) \setminus \{e \lor f\}$, so

$$\pi(e) \lor \pi(f) = \pi(e \lor f).$$

Now let $s, t \in S$ be compatible, so that $s^* t = t^* s$ and $st^* = ts^*$ are both idempotents, and we have

$$s^* st^* t = s^* ts^* t = s^* t.$$
Since \((s \lor t)\star (s \lor t) = s \star s \lor t \star t\), we have
\[
\pi(s \lor t) = \pi(s \lor t)\pi(s \star s \lor t \star t)
= \pi(s \lor t)\pi(s^*s) + \pi(t \star t) - \pi(s^*stt)
= \pi(s \star st) + \pi(ts^t \lor t) - \pi(st^t \lor ts^s)
= \pi(s + \pi(t) - \pi(s^t))
\]
where the last line follows from the facts that
\[st^* \leq s, ts^* \leq t \text{ and } ts^* = st^* = ss^t = tt^*s.\]

We have the following consequence of the proof of the above proposition.

**Corollary 3.4.** Let \(S\) be a Boolean inverse monoid. Then a map \(\pi : S \rightarrow A\) is a Boolean monoid representation of \(S\) if and only if it is a representation and for all \(e, f \in E(S)\) we have \(\pi(e \lor f) = \pi(e) + \pi(f) - \pi(ef)\).

We now have the following.

**Theorem 3.5.** Let \(S\) be a Boolean inverse monoid. Then
\[
C^*(S) \cong C^*_\text{tight}(S) \cong C^*(\mathcal{G}_\text{tight}(S)).
\]

In what follows, we will be studying traces on \(C^*\)-algebras arising from Boolean inverse monoids. However, many of our examples will actually arise from inverse monoids which are not distributive, and so the Boolean inverse monoid in question will actually be \(\mathcal{G}_\text{tight}(S)^a\), see [1]. The map from \(S\) to \(\mathcal{G}_\text{tight}(S)^a\) defined by
\[
s \mapsto \Theta(s, D^9_{ss^*})
\]
may fail to be injective, and so we cannot say that a given inverse monoid can be embedded in a Boolean inverse monoid. The obstruction arises from the following situation: suppose \(S\) is an inverse semigroup and that we have \(e, f \in E(S)\) such that \(e \leq f\) and for all \(0 \neq k \leq f\) we have \(ek \neq 0\), in other words, \(\{e\}\) is a cover for \(\{f\}\). In such a situation, we say that \(e\) is dense in \(f\) and by [2] we must have that \(\pi(e) = \pi(f)\) (see also [Exe09, Proposition 11.11]). For most of our examples, we will be considering inverse semigroups which have faithful tight representations, though we consider one which does not.

We close this section by recording some consequences of Theorem 3.5. The tight groupoid and tight \(C^*\)-algebra of an inverse semigroup were extensively studied in [EPT14b], where they gave conditions on \(S\) which imply that \(C^*_\text{tight}(S)\) is simple and purely infinite. We first recall some definitions from [EPT14b]

**Definition 3.6.** Let \(S\) be an inverse semigroup, let \(s \in S\) and \(e \leq s^*s\). Then we say that

1. \(e\) is fixed by \(s\) if \(se = e\), and

\[\text{This is the terminology used in [Exe08, Definition 11.10] and [Exe09], though in [LS14, Section 6.3] such an } e \text{ is called essential in } f.\]
2. $e$ is weakly fixed by $s$ if for all $0 \neq f \leq e$, $fsfs^* \neq 0$.

Denote by $J_s := \{ e \in E(S) \mid se = e \}$ the set of all fixed idempotents for $s \in S$. We note that an inverse semigroup for which $J_s = \{0\}$ for all $s \not\in E(S)$ is called $E^*$-unitary.

**Theorem 3.7.** Let $S$ be an inverse semigroup. Then

1. $\mathcal{G}_{\text{tight}}(S)$ is Hausdorff if and only if $J_s$ has a finite cover for all $s \in S$. [EP14b, Theorem 3.16]

2. If $\mathcal{G}_{\text{tight}}(S)$ is Hausdorff, then $\mathcal{G}_{\text{tight}}(S)$ is essentially principal if and only if for every $s \in S$ and every $e \in E(S)$ weakly fixed by $s$, there exists a finite cover for $\{e\}$ by fixed idempotents. [EP14b, Theorem 4.10]

3. $\mathcal{G}_{\text{tight}}(S)$ is minimal if and only if for every nonzero $e, f \in E(S)$, there exist $F \subset \text{fin} S$ such that $\{e\}$ is a cover for $\{e\}$. [EP14b, Theorem 5.5]

We translate the above to the case where $S$ is a Boolean inverse monoid.

**Proposition 3.8.** Let $S$ be a Boolean inverse monoid. Then

1. $\mathcal{G}_{\text{tight}}(S)$ is Hausdorff if and only if for all $s \in S$, there exists an idempotent $e_s$ with $se_s = e_s$ such that if $e$ is fixed by $s$, then $e \leq e_s$.

2. If $\mathcal{G}_{\text{tight}}(S)$ is Hausdorff, then $\mathcal{G}_{\text{tight}}(S)$ is essentially principal if and only if for every $s \in S$, $e$ weakly fixed by $s$ implies $e$ is fixed by $s$.

3. $\mathcal{G}_{\text{tight}}(S)$ is minimal if and only if for every nonzero $e, f \in E(S)$, there exist $F \subset \text{fin} S$ such that $e \leq \bigvee_{s \in F} sfs^*$.

**Proof.** Statements 2 and 3 are easy consequences of taking the joins of the finite covers mentioned. Statement 1 is central to what follows, and is proven in Lemma [4.2]

If an étale groupoid $\mathcal{G}$ is Hausdorff, then $C^*(\mathcal{G})$ is simple if and only if $\mathcal{G}$ is essentially principal, minimal, and satisfies weak containment, see [BCFS14] (also see [ES15] for a discussion of amenability of groupoids associated to inverse semigroups).

### 4 Invariant means and traces

In this section we consider invariant means on Boolean inverse monoids, and show that such functions always give rise to traces on the associated $C^*$-algebras. This definition is from [KLLR16].

**Definition 4.1.** Let $S$ be a Boolean inverse monoid. A function $\mu : E(S) \rightarrow [0, \infty)$ will be called an invariant mean if

1. $\mu(s^*s) = \mu(ss^*)$ for all $s \in S$

2. $\mu(e \lor f) = \mu(e) + \mu(f)$ for all $e, f \in E(S)$ with $ef = 0$.

If in addition $\mu(1) = 1$, we call $\mu$ a normalized invariant mean. An invariant mean $\mu$ will be called faithful if $\mu(e) = 0$ implies $e = 0$. We will denote by $M(S)$ the affine space of all normalized invariant means on $S$. 

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We make an important assumption on the Boolean inverse monoids we consider here. This assumption is equivalent to the groupoid $\mathcal{G}_{\text{tight}}(S)$ being Hausdorff [EP14b, Theorem 3.16]².

For every $s \in S$, the set $\mathcal{J}_s = \{ e \in E(S) \mid se = e \}$ admits a finite cover. (H)

The next lemma records straightforward consequences of condition (H) when $S$ happens to be a Boolean inverse monoid.

**Lemma 4.2.** Let $S$ be Boolean inverse monoid which satisfies condition (H). Then,

1. for each $s \in S$ there is an idempotent $e_s$ such that for any finite cover $C$ of $\mathcal{J}_s$,
   \[ e_s = \bigvee_{c \in C} c. \quad (3) \]

   and $\mathcal{J}_s = \mathcal{J}_{e_s}$,

2. $e_{s^*s} = e_s$ for all $s \in S$,

3. $e_{st} \leq ss^*, t^*t$ for all $s, t \in S$, and

4. $e_{st^*t}e_{t^*r} \leq e_{s^*r}$ for all $s, t, r \in S$.

**Proof.** To show the first statement, we need to show that any two covers give the same join. If $\mathcal{J}_s = \{0\}$, there is nothing to do. So suppose that $0 \neq e \in \mathcal{J}_s$, suppose that $C$ is a cover for $\mathcal{J}_s$, and let $e_C = \bigvee_{c \in C} c$. Indeed, the element $ee_C^\perp$ must be in $\mathcal{J}_s$, and since it is orthogonal to all elements of $C$ and $C$ is a cover, $ee_C^\perp$ must be 0. Hence we have

\[ e = ee_C \lor ee_C^\perp = ee_C \]

and so $e \leq e_C$. Now if $K$ is another cover for $\mathcal{J}_s$ with join $e_K$ and $k \in K$, we must have that $k \leq e_C$, and so $e_K \leq e_C$. Since the argument is symmetric, we have proven the first statement.

To prove the second statement, suppose $e \in \mathcal{J}_s$ then we have

\[ ses^* = es^* = (se)^* = e \]

and so

\[ s^*e = s^*(ses^*) = es^*ss^* = es^* = (se)^* = e \]

and again by symmetry we have $\mathcal{J}_s = \mathcal{J}_{s^*}$, and so $e_s = e_{s^*}$.

To prove the third statement, we notice

\[ ss^*e_{st} = ss^*se_{st} = stse_{st} = e_{st} \]

\[ e_{st^*t} = se_{st}^*t = st^*tse_{st} = stse_{st} = e_{st}. \]

For the fourth statement, we calculate (using 2)

\[ e_{st^*t}e_{t^*r} = s^*te_{s^*t}e_{t^*r} = s^*t^*rse_{s^*t}e_{t^*r} = s^*t^*rte_{s^*t}e_{t^*r} = s^*rte_{s^*t}e_{t^*r} = s^*rse_{s^*t}e_{t^*r} \]

hence $e_{st^*t}e_{t^*r} \leq s^*r$ and so $e_{s^*t}e_{t^*r} \leq e_{s^*r}$. \qed

²In [Sta15a], we define condition (H) for another class of semigroups, namely the right LCM semigroups. Right LCM semigroups and inverse semigroups are related, but the intersection of their classes is empty (because right LCM semigroups are left cancellative and we assume that our inverse semigroups have a zero element). We note that a right LCM semigroup $P$ satisfies condition (H) in the sense of [Sta15a] if and only if its left inverse hull $I(P)$ satisfies condition [H] in the sense of the above.
In what will be a crucial step to obtaining a trace from an invariant mean, we now obtain a relation between $e_{st}$ and $e_{ts}$.

**Lemma 4.3.** Let $S$ be Boolean inverse monoid which satisfies condition (H). Then for all $s, t \in S$, we have that $s^*e_{st}s = e_{ts}$.

**Proof.** Suppose that $e \in J_{ts}$. Then $tse = e$, and so

$$(st)ses^* = ses^*$$

hence $ses^* \in J_{st}$. If $C$ is a cover of $J_{st}$ and $f \in J_{ts}$, there must exist $c \in J_{st}$ such that $c(sfs^*) \neq 0$. Hence

$$css^*sfs^* \neq 0$$
$$ss^*csf \neq 0$$
$$s^*csf \neq 0$$

and so we see that $s^*Cs$ is a cover for $J_{ts}$. By Lemma 4.2

$$e_{ts} = \bigvee_{c \in C} s^*cs = s^* \left( \bigvee_{c \in C} c \right) s = s^*e_{st}s.$$  

The above together with Lemma 4.2 implies that for all $s, t \in S$ and all $\mu \in M(S)$, we have $\mu(e_{st}) = \mu(e_{ts})$.

**Definition 4.4.** Let $A$ be a C*-algebra. A bounded linear functional $\tau : A \to \mathbb{C}$ is called a **trace** if

1. $\tau(a^*a) \geq 0$ for all $a \in A$,
2. $\tau(ab) = \tau(ba)$ for all $a, b \in A$.

A trace $\tau$ is said to be **faithful** if $\tau(a^*a) > 0$ for all $a \neq 0$. A trace $\tau$ on a unital C*-algebra is called a **tracial state** if $\tau(1) = 1$. The set of all tracial states of a C*-algebra $A$ is denoted $T(A)$.

We are now able to define a trace on $C^*(S)$ for each $\mu \in M(S)$.

**Proposition 4.5.** Let $S$ be Boolean inverse monoid which satisfies condition (H), and let $\mu \in M(S)$. Then there is a trace $\tau_\mu$ on $C^*(S)$ such that

$$\tau_\mu(\delta_s) = \mu(e_s) \quad \text{for all } s \in S.$$  

If $\mu$ is faithful, the $\tau_\mu$ is a faithful trace.
Proof. We define $\tau_\mu$ to be as above on the generators $\delta_s$ of $C^*(S)$, and extend it to $B := \text{span}\{\delta_s \mid s \in S\}$, a dense $*$-subalgebra of $C^*(S)$.

We first show that $\tau_\mu(\delta_s \delta_t) = \tau_\mu(\delta_t \delta_s)$. Indeed, by Lemmas 4.2 and 4.3 we have

$$
\tau_\mu(\delta_s \delta_t) = \mu(e_{st}) = \mu(e_{st} s^* s) = \mu((e_{st} s^*) (e_{st} s)) = \mu((e_{st} s^*) (e_{st} s)) = \mu(s^* e_{st}) = \mu(e_{ts}) = \tau_\mu(\delta_t \delta_s).
$$

Since $\tau_\mu$ is extended linearly to $B$, we have that $\tau_\mu(ab) = \tau_\mu(ba)$ for all $a, b \in B$.

Let $F$ be a finite index set and take $x = \sum_{i \in F} a_i \delta_s$ in $B$. We will show that $\tau_\mu(x^* x) \geq 0$. For $i, j \in F$, we let $e_{ij} = e_{s_i s_j}$ and note that $e_{ij} = e_{ji}$. We calculate:

$$
x^* x = \left(\sum_{s \in S} \overline{a}_s \delta_{s_i}^*\right) \left(\sum_{j \in F} a_j \delta_{s_j}\right)
= \sum_{i, j \in F} \overline{a}_i a_j \delta_{s_i}^* \delta_{s_j}
\tau_\mu(x^* x) = \sum_{i, j \in F} \overline{a}_i a_j \mu(e_{ij})
= \sum_{i \in F} |a_i|^2 \mu(e_{ii}) + \sum_{i, j \in F, i \neq j} (\overline{a}_i a_j + \overline{a}_j a_i) \mu(e_{ij}).
$$

We will show that this sum is positive by using an orthogonal decomposition of the $e_{ij}$. Let $F^2_{\neq} = \{\{i, j\} \subset F \mid i \neq j\}$, and let $D(F^2_{\neq}) = \{(A, B) \mid A \cup B = F^2_{\neq}, A \cap B = \emptyset\}$. For $a = \{i, j\} \in F^2_{\neq}$, let $e_a = e_{ij}$. We have

$$
e_{ij} = e_{ij} \bigvee_{(A, B) \in D(F^2_{\neq})} \left(\prod_{a \in A, b \in B} e_a e_b^\perp\right)
$$

where the join is an orthogonal join. Of course, the above is only nonzero when $\{i, j\} \in A$. We also notice that

$$
e_{ii} \geq \bigvee_{i \in \cup A} \left(\prod_{a \in A, b \in B} e_a e_b^\perp\right)
$$

and so $\tau(x^* x)$ is larger than a linear combination of terms of the form $\mu(\prod_{a \in A, b \in B} e_a e_b^\perp)$ for partitions $(A, B)$ of $F^2_{\neq}$:

$$
\tau_\mu(x^* x) \geq \sum_{(A, B) \in D(F^2_{\neq})} \left[\left(\sum_{i \in \cup A} |a_i|^2 + \sum_{a = \{j, k\} \in A} (\overline{a}_j a_j + \overline{a}_k a_k)\right) \mu \left(\prod_{a \in A, b \in B} e_a e_b^\perp\right)\right]
$$

(4)

If a term $\prod_{a \in A, b \in B} e_a e_b^\perp$ is not zero, then we claim that the relation

$$
i \sim j \text{ if and only if } i = j \text{ or } \{i, j\} \in A
$$

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is an equivalence relation on $\cup A$. Indeed, suppose that $i, j, k \in \cup A$ are all pairwise nonequal and \{i, j\}, \{j, k\} $ \in A$. By Lemma 4.2.4, $e_{ij}e_{jk} \leq e_{ik}$ and since the product is nonzero, we must have that \{i, k\} $ \in A$. Writing $[\cup A]$ for the set of equivalence classes, we have

$$
\sum_{i \in \cup A} \left| a_i \right|^2 + \sum_{a = \{j, k\} \in A} (\overline{a_i}a_j + \overline{a_j}a_i) = \sum_{C \in [\cup A]} \left( \sum_{i \in C} \left| a_i \right|^2 + \sum_{i, j \in C, i \neq j} (\overline{a_i}a_j + \overline{a_j}a_i) \right)
$$

$$
= \sum_{C \in [\cup A]} \left| \sum_{i \in C} a_i \right|^2.
$$

Hence, $\tau_\mu(x^*x) \geq 0$, and $\tau_\mu$ is positive on $B$. Hence, $\tau_\mu$ extends to a trace on $C^*(S)$. The above calculation shows that if $\mu$ is faithful, then so is $\tau_\mu$.

We now show that given a trace on $C^*(S)$ we can construct an invariant mean on $S$.

**Proposition 4.6.** Let $S$ be Boolean inverse monoid which satisfies condition (H), let $\pi_u : S \to C^*(S)$ be the universal Boolean monoid representation of $S$, and take $\tau \in \mathcal{T}(C^*(A))$. Then the map $\mu_\tau : E(S) \to [0, \infty)$ defined by

$$
\mu_\tau(e) = \tau(\pi_u(e)) = \tau(\delta_e)
$$

is a normalized invariant mean on $S$. If $\tau$ is faithful then so is $\mu_\tau$.

**Proof.** That $\mu_\tau$ takes positive values follows from $\tau$ being positive. We have

$$
\mu_\tau(s^*s) = \tau(\pi_u(s^*)\pi_u(s)) = \tau(\pi_u(s^*)\pi_u(s)) = \tau(\pi_u(s^*)\pi_u(s)) = \mu_\tau(s^*s).
$$

Also, if $e, f \in E(S)$ with $ef = 0$, then

$$
\mu_\tau(e \vee f) = \tau(\pi_u(e \vee f)) = \tau(\pi_u(e) + \pi(f)) = \tau(\pi_u(e)) + \tau(\pi_u(f)) = \mu_\tau(e) + \mu_\tau(f).
$$

If $\tau$ is faithful, $\tau(\delta_e) > 0$ because $\delta_e$ is positive and nonzero, and so $\mu_\tau$ is faithful.

**Proposition 4.7.** Let $S$ be Boolean inverse monoid which satisfies condition (H). Then the map

$$
\mu \mapsto \tau_\mu \mapsto \mu_\tau
$$

is an affine automorphism of $M(S)$.

**Proof.** This is immediate, since both maps in question are clearly affine maps, and if $\mu \in M(S)$ and $e \in E(S)$ we have

$$
\mu_{\tau_\mu}(e) = \tau_\mu(\pi(e)) = \tau_\mu(\delta_e) = \mu(e).
$$
Given the above, one might wonder under which circumstances we have that $T(C^*(S)) \cong M(S)$. This is not true in the general situation – take for example $S$ to be the group $\mathbb{Z}_2 = \{1, -1\}$ with a zero element adjoined – this is a Boolean inverse monoid. Here $M(S)$ consists of one element, namely the function which takes the value 1 on 1 and the value 0 on the zero element. The C*-algebra of $S$ is the group C*-algebra of $\mathbb{Z}_2$, which is isomorphic to $\mathbb{C}^2$, a C*-algebra with many traces (taking the dot product of an element of $\mathbb{C}^2$ with any nonnegative vector whose entries add to 1 determines a normalized trace on $\mathbb{C}^2$).

One can still obtain this isomorphism using the following.

**Definition 4.8.** Let $\mathcal{G}$ be an étale groupoid. A regular Borel probability measure $\nu$ on $\mathcal{G}^{(0)}$ is called $\mathcal{G}$-invariant if for every bisection $U$ one has that $\nu(r(U)) = \nu(d(U))$. The affine space of all regular $\mathcal{G}$-invariant Borel probability measures is denoted $IM(\mathcal{G})$.

**Theorem 4.9.** Let $\mathcal{G}$ be a Hausdorff principal étale groupoid with compact unit space. Then

$$T(C^*_\text{red}(\mathcal{G})) \cong IM(\mathcal{G})$$

For $\tau \in T(C^*_\text{red}(\mathcal{G}))$ the image of $\tau$ under the above isomorphism is the regular Borel probability measure $\nu$ whose existence is guaranteed by the Riesz representation theorem applied to the positive linear functional on $C(\mathcal{G}^{(0)})$ given by restricting $\tau$.

For a proof, see [Put, Theorem 3.4.5].

For us, the groupoid $\mathcal{G}_\text{tight}(S)$ satisfies all of the conditions in Theorem 4.9 except possibly for being principal. Also note that in the general case, $C^*_\text{red}(\mathcal{G}_\text{tight})$ may not be isomorphic to $C^*(S)$. So if we restrict our attention to Boolean inverse monoids which have principal tight groupoids and for which $C^*_\text{red}(\mathcal{G}_\text{tight}) \cong C^*(S)$, we can obtain the desired isomorphism. While this may seem like a restrictive set of assumptions, they are all satisfied for the examples we consider here.

**Proposition 4.10.** Let $S$ be Boolean inverse monoid which satisfies condition (H), and suppose $\nu \in IM(\mathcal{G}_\text{tight}(S))$. Then the map $\eta_\nu : E(S) \rightarrow [0, \infty)$ defined by

$$\eta_\nu(e) = \nu(D^{\theta}_e)$$

is a normalized invariant mean on $S$. The map that sends $\nu \mapsto \eta_\nu$ is an affine isomorphism of $IM(\mathcal{G}_\text{tight}(S))$ and $M(S)$.

**Proof.** That $\eta_\nu(s^*s) = \eta_\nu(ss^*)$ follows from invariance of $\nu$ applied to the bisection $\Theta(s, D_{s^*s})$, and that $\eta_\nu$ is additive over orthogonal joins follows from the fact that $\nu$ is a measure. This map is clearly affine. Suppose that $\eta_\nu = \eta_\kappa$ for $\nu, \kappa \in IM(\mathcal{G}_\text{tight}(S))$. Then $\nu, \kappa$ agree on all sets of the form $D^{\theta}_e$, and since these sets generate the topology on $\widehat{E}_\text{tight}(S)$, $\nu$ and $\kappa$ agree on all open sets. Since they are regular Borel probability measures they must be equal, and so $\nu \mapsto \eta_\nu$ is injective.

To get surjectivity, let $\mu$ be an invariant mean, and let $\tau_\mu$ be as in Proposition 4.5. Then restricting $\tau_\mu$ to $C(\widehat{E}_\text{tight}(S))$ and invoking the Riesz representation theorem gives us a regular invariant probability measure $\nu$ on $\widehat{E}_\text{tight}(S)$, and we must have $\eta_\nu = \mu$. \qed

**Corollary 4.11.** Let $\mathcal{G}$ be an ample Hausdorff étale groupoid. Then $IM(\mathcal{G}) \cong M(\mathcal{G}^0)$.
So the invariant means on the ample semigroup of a Hausdorff étale groupoid are in one-to-one correspondence with the $\mathcal{G}$-invariant measures.

**Theorem 4.12.** Let $S$ be Boolean inverse monoid which satisfies condition (H). Suppose that $\mathcal{G}_{\text{tight}}(S)$ is principal, and that $C^*_\text{red}(\mathcal{G}_{\text{tight}}(S)) \cong C^*(S)$. Then

$$T(C^*(S)) \cong M(S)$$

via the map which sends $\tau$ to $\mu_\tau$ as in Proposition 4.6. In addition, both are isomorphic to $IM(\mathcal{G}_{\text{tight}}(S))$. 

*Proof.* This follows from Theorem 4.9 and Proposition 4.10. 

There are many results in the literature concerning traces which now apply to our situation.

**Corollary 4.13.** Let $S$ be Boolean inverse monoid which satisfies condition (H). If $S$ admits a faithful invariant mean, then $C^*(S)$ is stably finite.

*Proof.* If $\mu$ is a faithful invariant mean, then after normalizing one obtains a faithful trace on $C^*(S)$ by Proposition 4.5. Now the result is standard, see for example [LLR00, Exercise 5.2].

**Corollary 4.14.** Let $S$ be Boolean inverse monoid which satisfies condition (H). If $C^*(S)$ is stably finite and exact, then $S$ has an invariant mean.

*Proof.* This is a consequence of the celebrated result of Haagerup [Haa91] when applied to Proposition 4.6.

For the undefined terms above, we direct the interested reader to [BO08]. We also note that exactness of $C^*(S)$ has recently been considered in [Li16] and [AD16a].

## 5 Examples

### 5.1 AF inverse monoids

This is a class of Boolean inverse monoids introduced in [LS14] motivated by the construction of AF C*-algebras from Bratteli diagrams.

A *Bratteli diagram* is an infinite directed graph $B = (V, E, r, s)$ such that

1. $V$ can be written as a disjoint union of finite sets $V = \cup_{n \geq 0} V_n$
2. $V_0$ consists of one element $v_0$, called the *root*,
3. for all edges $e \in E$, $s(e) \in V_i$ implies that $r(e) \in V_{i+1}$ for all $i \geq 0$, and
4. for all $i \geq 1$ and all $v \in V_i$, both $r^{-1}(v)$ and $s^{-1}(v)$ are finite and nonempty.
We also denote \( s^{-1}(V_i) := E_i \), so that \( E = \bigcup_{n \geq 0} E_n \). Let \( E^* \) be the set of all finite paths in \( B \), including the vertices (treated as paths of length zero). For \( v, w \in V \cup E \), let \( vE^* \) denote all the paths starting with \( v \), let \( E^*w \) be all the paths ending with \( w \), and let \( vE^*w \) be all the paths starting with \( v \) and ending with \( w \).

Given a Bratteli diagram \( B = (V, E, r, s) \) we construct a C*-algebra as follows. We let

\[
A_0 = \mathbb{C}
\]

\[
A_1 = \bigoplus_{v \in V_1} \mathbb{M}|r^{-1}(v)|,
\]

and define \( k_1(v) = |r^{-1}(v)| \) for all \( v \in V_1 \). For an integer \( i > 1 \) and \( v \in V_i \), let

\[
k_i(v) = \sum_{\gamma \in r^{-1}(v)} k_{i-1}(s(\gamma)).
\]

Define

\[
A_i = \bigoplus_{v \in V_i} \mathbb{M}_{k_i(v)}
\]

Now for all \( i \geq 0 \), one can embed \( A_i \hookrightarrow A_{i+1} \) by viewing, for each \( v \in V_{i+1} \)

\[
\bigoplus_{\gamma \in r^{-1}(v)} \mathbb{M}_{k_i(s(\gamma))} \subset \mathbb{M}_{k_{i+1}(v)}
\]

where the algebras in the direct sum are orthogonal summands along the diagonal in \( \mathbb{M}_{k_{i+1}(v)} \). So \( A_0 \hookrightarrow A_1 \hookrightarrow A_1 \hookrightarrow \cdots \) can be viewed as an increasing union of finite dimensional C*-algebra, all of which can be realized as subalgebras of \( \mathcal{B}(\mathcal{H}) \) for the same \( \mathcal{H} \), and so we can form the norm closure of the union

\[
A_B := \bigcup_{n \geq 0} A_n.
\]

The C*-algebra is what is known as an \( AF \) algebra, and every unital AF algebra arises this way from some Bratteli diagram.

The AF algebra \( A_B \) can always be described as the C*-algebra of a principal groupoid derived from \( B \), see \cite{Ren80} and \cite{RE06}. We reproduce this construction here. Let \( X_B \) denote the set of all infinite paths in \( B \) which start at the root. When given the product topology from the discrete topologies on the \( E_n \), this is a compact Hausdorff totally disconnected space. For \( \alpha \in v_0E^* \), we let \( C(\alpha) = \{ x \in X_B \mid x_i = \alpha_i \text{ for all } i = 0, \ldots, |\alpha| - 1 \} \). Sets of this form are clopen and form a basis for the topology on \( X_B \). For \( n \in \mathbb{N} \), let

\[
\mathcal{R}_B^{(n)} = \{(x, y) \in X \times X \mid x_i = y_i \text{ for all } i \geq n + 1 \}
\]

so a pair of infinite paths \( (x, y) \) is in \( \mathcal{R}_B^{(n)} \) if and only if \( x \) and \( y \) agree after the vertices on level \( n \). Clearly, \( \mathcal{R}_B^{(n)} \subset \mathcal{R}_B^{(n+1)} \), and so we can form their union

\[
\mathcal{R}_B = \bigcup_{n \in \mathbb{N}} \mathcal{R}_B^{(n)}.
\]
This is an equivalence relation, known as *tail equivalence* on $X_B$. For $v \in V \setminus \{v_0\}$ and $\alpha, \beta \in v_0 E^* v$, define

$$C(\alpha, \beta) = \{(x, y) \in \mathcal{R}_B \mid x \in C(\alpha), y \in C(\beta)\}$$

sets of this type form a basis for a topology on $\mathcal{R}_B$, and with this topology $\mathcal{R}_B$ is a principal Hausdorff étale groupoid with unit space identified with $X_B$, and

$$C^*(\mathcal{R}_B) \cong C^*_\text{red}(\mathcal{R}_B) \cong A_B.$$

In [LS14], a Boolean inverse monoid is constructed from a Bratteli diagram, mirroring the above construction. We will present this Boolean inverse monoid in a slightly way which may be enlightening. Let $B = (V, E, r, s)$ be a Bratteli diagram. Let $S_0$ be the Boolean inverse monoid (in fact, Boolean algebra) $\{0, 1\}$. For each $i \geq 1$, let

$$S_i = \bigoplus_{v \in V_i} \mathcal{I}(v_0 E^* v)$$

where as in Section 2.1, $\mathcal{I}(X)$ denotes the set of partially defined bijections on $X$.

If $v \in V_{i+1}$ and $\gamma \in r^{-1}(v)$ then one can view $\mathcal{I}(v_0 E^* \gamma)$ as a subset of $\mathcal{I}(v_0 E^* v)$, and if $\eta \in r^{-1}(v)$ with $\gamma \neq \eta$, $\mathcal{I}(v_0 E^* \gamma)$ and $\mathcal{I}(v_0 E^* \eta)$ are orthogonal. Furthermore, $\mathcal{I}(v_0 E^* \gamma)$ can be identified with $\mathcal{I}(v_0 E^* s(\gamma))$ Hence the direct sum over $r^{-1}(v)$ can be embedded into $\mathcal{I}(v_0 E^* v)$:

$$\bigoplus_{\gamma \in r^{-1}(v)} \mathcal{I}(v_0 E^* s(\gamma)) \hookrightarrow \mathcal{I}(v_0 E^* v). \quad (6)$$

This allows us to embed $S_i \hookrightarrow S_{i+1}$

$$\bigoplus_{v \in V_i} \mathcal{I}(v_0 E^* v) \hookrightarrow \bigoplus_{w \in V_{i+1}} \mathcal{I}(v_0 E^* w)$$

where an element $\phi$ in a summand $\mathcal{I}(v_0 E^* v)$ gets sent to $|s^{-1}(v)|$ summands on the right, one for each $\gamma \in s^{-1}(v)$: $\phi$ will be sent to the summand inside $\mathcal{I}(v_0 E^* s(\gamma))$ corresponding to $v$ in left hand side of the embedding from (6). We then define

$$I(B) = \lim_{\rightarrow} (S_i \hookrightarrow S_{i+1})$$

This is a Boolean inverse monoid [LS14, Lemma 3.13]. As a set $I(B)$ is the union of all the $S_i$, viewed as an increasing union via the identifications above. In [LS14, Remark 6.5], it is stated that the groupoid one obtains from $I(B)$ (i.e., $\mathcal{G}_{\text{tight}}(I(B))$) is exactly tail equivalence. We provide the details of that informal discussion there.

We will describe the ultrafilters in $E(I(B))$, a Boolean algebra. For $v \in V_i$ and a path $\alpha \in v_0 E^* v$, let $e_\alpha = \text{Id}_{\{\alpha\}} \in \mathcal{I}(v_0 E^* v)$. As $v$ ranges over all of $V_i$ and $\alpha$ ranges over all of $v_0 E^* v$, these idempotents form a orthogonal decomposition of the identity of $I(B)$. Hence, given an ultrafilter $\xi$ and $i > 0$ there exists one and only one path, say $\alpha^{(i)}_\xi$ ending at level $i$ with $e_{\alpha^{(i)}_\xi} \in \xi$. Furthermore, if $j > i$, we must have that $\alpha^{(j)}_\xi$ is a prefix of $\alpha^{(i)}_\xi$, because products in an ultrafilter cannot be zero. So for $x \in X_B$, if we define

$$\xi_x = \{e_\alpha \mid \alpha \text{ is a prefix of } x\}$$
then we have that
\[ \widehat{E}_\infty(I(B)) = \{ \xi_x \mid x \in X_B \} \]

By [EP14b, Proposition 2.6], the set
\[ \{ \lambda(\{ e_\alpha \}, \emptyset) \mid \alpha \text{ is a prefix of } x \} \]
is a neighborhood basis for \( \xi_x \). The map \( \lambda : \lambda : X_B \rightarrow \widehat{E}_\infty(I(B)) \) given by \( \lambda(x) = \xi_x \) is a bijection, and since \( \lambda(\{ e_\alpha \}, \emptyset) = \lambda(C(\alpha)) \), it is a homeomorphism. If \( \phi \in S_i \) such that \( \phi^* \phi \in \xi_x \), then we must have that one component of \( \phi \) is in \( I(v_0E^*r(x_i)) \), and we must have that
\[ \theta_{\phi}(\xi_x) = \xi_{\phi(x_0x_1...x_i)x_{i+1}x_{i+2}...} \] (7)

Finally, we claim that \( R_B \) is isomorphic to \( G_{\text{tight}}(I(B)) \). We define a map
\[ \Phi : G_{\text{tight}}(I(B)) \rightarrow R_B \]
\[ \Phi([\phi, \xi_x]) \mapsto (\phi(x_0x_1...x_i)x_{i+1}x_{i+2}...), \]
where \( \phi \text{ and } x \text{ are as in (7)} \). If \( \Phi([\phi, \xi_x]) = \Phi([\psi, \xi_y]) \), then clearly we must have \( \xi_x = \xi_y \). We must also have that \( \phi, \psi \in S_i \), and \( \phi e_{0x_0x_1...x_i} = \psi e_{0x_0x_1...x_i} \), hence \( [\phi, \xi_x] = [\psi, \xi_y] \). It is straightforward to verify that \( \Phi \) is surjective and bicontinuous, and so \( R_B \cong G_{\text{tight}}(I(B)) \).

Since they are both étale, their C*-algebras must be isomorphic. Hence with the above discussion, we have proven the following.

**Theorem 5.1.** Let \( B \) be a Bratteli diagram. Then
\[ C^*(I(B)) \cong A_B. \]

Furthermore, every unital AF algebra is isomorphic to the universal C*-algebra of a Boolean inverse monoid of the form \( I(B) \) for some \( B \).

Recall that a compact convex metrizable subset \( X \) of a locally convex space is a **Choquet simplex** if and only if for each \( x \in X \) there exists a unique measure \( \nu \) concentrated on the extreme points of \( X \) for which \( x \) is the center of gravity of \( X \) for \( \nu \) [Phe01]. Now we can use the following seminal result of Blackadar to make a statement about the set of normalized invariant means for AF inverse monoids.

**Theorem 5.2.** (Blackadar, see [Bla80, Theorem 3.10]) Let \( \Delta \) be any metrizable Choquet simplex. Then there exists a unital simple AF algebra \( A \) such that \( T(A) \) is affinely isomorphic to \( \Delta \).

**Corollary 5.3.** Let \( \Delta \) be any metrizable Choquet simplex. Then there exists an AF inverse monoid \( S \) such that \( M(S) \) is affinely isomorphic to \( \Delta \).

**Proof.** This result follows from Theorem 4.12 because \( G_{\text{tight}}(S) \) is Hausdorff, amenable, principal for every AF inverse monoid \( S \). \( \square \)
5.2 The $3 \times 3$ matrices

This example is a subexample of the previous example, but it will illustrate how we approach the following two examples.

Let $\mathcal{I}_3$ denote the symmetric inverse monoid on the three element set $\{1, 2, 3\}$. This is a Boolean inverse monoid which satisfies condition (H), and we define a map $\pi : \mathcal{I}_3 \to \mathbb{M}_3$ by saying that

$$\pi(\phi)_{ij} = \begin{cases} 1 & \text{if } \phi(j) = i \\ 0 & \text{otherwise.} \end{cases}$$

Then it is straightforward to verify that $\pi$ is in fact the universal Boolean inverse monoid representation of $\mathcal{I}_3$.

Now instead consider the subset $R_3 \subset \mathcal{I}_3$ consisting of the identity, the empty function, and all functions with domain consisting of one element. Then $R_3$ is an inverse monoid, and $\pi(R_3)$ is the set of all matrix units together with the identity matrix and zero matrix. When restricted to $R_3$, $\pi$ is the universal tight representation of $R_3$. Hence $C^*_\text{tight}(R_3) \cong C^*(\mathcal{I}_3) \cong \mathbb{M}_3$.

There is only one invariant mean $\mu$ on $\mathcal{I}_3$ – for an idempotent $\text{Id}_U \in \mathcal{I}_3$, we have $\mu(\text{Id}_U) = \frac{1}{3} |U|$. The tight groupoid of $R_3$ is the equivalence relation $\{1, 2, 3\} \times \{1, 2, 3\}$, which is principal – we also have that $G^\text{tight}(R_3) \cong \mathcal{I}_3$. The unique invariant mean on $\mathcal{I}_3$ is identified with the unique normalized trace on $\mathbb{M}_3$.

Our last two examples follow this mold, where we have an inverse monoid $\mathcal{S}$ which generates a $C^*$-algebra $C^*_\text{tight}(\mathcal{S})$, and we relate the traces of $C^*_\text{tight}(\mathcal{S})$ to the invariant means of $G^\text{tight}(\mathcal{S})$.

5.3 Self-similar groups

Let $X$ be a finite set, let $G$ be a group, and let $X^*$ denote the set of all words in elements of $X$, including an empty word $\varnothing$. Let $X^\omega$ denote the Cantor set of one-sided infinite words in $X$, with the product topology of the discrete topology on $X$. For $\alpha \in X^*$, let $C(\alpha) = \{\alpha x \mid x \in X^\omega\}$ – sets of this type are called cylinder sets and form a clopen basis for the topology on $X$.

Suppose that we have a faithful length-preserving action of $G$ on $X^*$, with $(g, \alpha) \mapsto g \cdot \alpha$, such that for all $g \in G$, $x \in X$ there exists a unique element of $G$, denoted $g|_x$, such that for all $\alpha \in X^*$

$$g(x\alpha) = (g \cdot x)(g|_x \cdot \alpha).$$

In this case, the pair $(G, X)$ is called a self-similar group. The map $G \times X \to G$, $(g, x) \mapsto g|_x$ is called the restriction and extends to $G \times X^*$ via the formula

$$g|_{\alpha_1 \ldots \alpha_n} = g|_{\alpha_1} |_{\alpha_2} \cdots |_{\alpha_n}$$

and this restriction has the property that for $\alpha, \beta \in X^*$, we have

$$g(\alpha\beta) = (g \cdot \alpha)(g|_\alpha \cdot \beta).$$

The action of $G$ on $X^*$ extends to an action of $G$ on $X^\omega$ given by

$$g \cdot (x_1x_2x_3\ldots) = (g \cdot x_1)(g|_{x_1} \cdot x_2)(g|_{x_1x_2} \cdot x_3)\ldots$$
In [Nek09], Nekrashevych associates a C*-algebra to \((G, X)\), denoted \(O_{G,X}\), which is the universal C*-algebra generated by a set of isometries \(\{s_x\}_{x \in X}\) and a unitary representation \(\{u_g\}_{g \in G}\) satisfying

(i) \(s_x^* s_y = 0\) if \(x \neq y\),
(ii) \(\sum_{x \in X} s_x s_x^* = 1\),
(iii) \(u_g s_x = s_{g \cdot x} u_{g|_x}\).

One can also express \(O_{G,X}\) as the tight C*-algebra of an inverse semigroup. Let

\[ S_{G,X} = \{ (\alpha, g, \beta) \mid \alpha, \beta \in X^*, g \in G \} \cup \{0\}. \]

This set becomes an inverse semigroup when given the operation

\[
(\alpha, g, \beta) (\gamma, h, \nu) = \begin{cases} 
(\alpha (g \cdot \gamma'), g|_{\gamma'}, h, \nu), & \text{if } \gamma = \beta \gamma', \\
(\alpha, g(h^{-1}|_{\gamma'})^{-1}, \nu(h^{-1} \cdot \beta')), & \text{if } \beta = \gamma \beta', \\
0 & \text{otherwise}
\end{cases}
\]

with

\[(\alpha, g, \beta)^* = (\beta, g^{-1}, \alpha).\]

Here, \(E(S_{X,G}) = \{ (\alpha, 1_G, \alpha) \mid \alpha \in X^* \}\), and the tight spectrum \(\hat{E}_{\text{tight}}(S_{G,X})\) is homeomorphic \(X^\omega\) by the identification

\[ x \in X^\omega \mapsto \{ (\alpha, 1_G, \alpha) \in E(S_{G,X}) \mid \alpha \text{ is a prefix of } x \} \in \hat{E}_{\text{tight}}(S_{G,X}). \]

If \(\theta\) is the standard action of \(S_{G,X}\) on \(\hat{E}_{\text{tight}}(S_{G,X})\), then \(D^\theta_{(\alpha,1_G,\alpha)} = C(\alpha)\). If \(s = (\alpha, g, \beta) \in S_{X,G}\), then

\[
\theta_s : C(\alpha) \rightarrow C(\beta) \quad \theta_s(\alpha x) = \beta(g \cdot x)
\]

It is shown in [EP14a] that \(O_{G,X}\) is isomorphic to \(C^*_\text{tight}(S_{G,X})\).

We show that the universal tight representation of \(S_{G,X}\) is faithful. This will be accomplished if we can show that the map from \(S_{G,X}\) to \(G^\text{tight}(S_{G,X})\) given by

\[ s \mapsto \Theta(s, D^\theta_{s^* s}) \]

is injective. If \(s = (\alpha, g, \beta)\), then

\[
\Theta(s, D^\theta_{s^* s}) = \{ [(\alpha, g, \beta), \beta x] \mid x \in X^\omega \}.
\]

It is straightforward that \(r(\Theta(s, D^\theta_{s^* s})) = C(\beta)\) and \(d(\Theta(s, D^\theta_{s^* s})) = C(\alpha)\). Suppose we have another element \(t = (\gamma, h, \eta)\) such that \(\Theta(s, D^\theta_{s^* s}) = \Theta(t, D^\theta_{t^* t})\). Since these two bisections are equal, their ranges (resp. sources) must be equal, so \(C(\beta) = C(\eta)\) (resp. \(C(\alpha) = C(\gamma)\)). Hence, \(\alpha = \gamma\) and \(\beta = \eta\). Since \(r\) and \(d\) are both bijective on these slices, we must have that for all \(\beta x \in C(\beta)\), \(\alpha(g \cdot x) = \alpha(h \cdot x)\). Hence for all \(x \in X^\omega\), we must have that \(g \cdot x = h \cdot x\).

The action of \(G\) on \(X^*\) is faithful, so the induced action of \(G\) on \(X^\omega\) is also faithful, hence \(g = h\) and so \(t = s\).
As it stands, the Boolean inverse monoid $G_{\text{tight}}(S_{G,X})^a$ cannot have any invariant means. This because the subalgebra of $O_{G,X}$ generated by $\{s_x \mid x \in X\}$ is isomorphic to the Cuntz algebra $O_{|X|}$, and a trace on $O_{G,X}$ would have to restrict to a trace on $O_{|X|}$, which is purely infinite and hence has no traces.

To justify the inclusion of this example in this paper about invariant means, we restrict to an inverse subsemigroup of $S_{G,X}$ whose corresponding ample semigroup will admit an invariant mean. Let

$$S_{G,X}^\infty = \{(\alpha, g, \beta) \in S_{G,X} \mid |\alpha| = |\beta|\} \cup \{0\}.$$ 

One can easily verify that this is closed under product and involution, and so is an inverse subsemigroup of $S_{G,X}$, with the same set of idempotents as $S_{G,X}$. If $\alpha, \beta \in X^*$, $|\alpha| = |\beta|$, and $g \in G$, then

$$(\alpha, g, \beta)^*(\alpha, g, \beta) = (\beta, 1_G, \beta), \quad (\alpha, g, \beta)(\alpha, g, \beta)^* = (\alpha, 1_G, \alpha).$$

If $\mu$ were an invariant mean on $G_{\text{tight}}(S_{G,X})^a$, then we would have to have, for all $\alpha, \beta \in X^*$ and $|\alpha| = |\beta|$, $\mu(C(\alpha)) = \mu(C(\beta))$. Moreover, for a given length $n$, the set $\{C(\alpha) \mid |\alpha| = n\}$ forms a disjoint partition of $X^\omega$, and so we must have

$$\mu(C(\alpha)) = |X|^{-|\alpha|}. \tag{8}$$

Any clopen subset of $X^\omega$ it must be a finite disjoint union of cylinders. Hence the map $\mu$ on $E(G_{\text{tight}}(S_{G,X})^a)$ determined by (8) is an invariant mean, and is in fact the unique invariant mean on $G_{\text{tight}}(S_{G,X})^a$.

In the general case, it is possible for $G_{\text{tight}}(S_{G,X})$ to be neither Hausdorff nor principal. We now give an explicit example where we get a unique trace to go along with our unique invariant mean.

**Example 5.4. (The 2-odometer)**

Let $X = \{0, 1\}$, let $Z = \langle z \rangle$ be the group of integers with identity $e$ written multiplicatively. The 2-odometer is the self-similar group $(Z, X)$ determined by

$$z \cdot 0 = 1 \quad z|_0 = e$$

$$z \cdot 1 = 0 \quad z|_1 = z.$$ 

If one views a word $\alpha \in X^*$ as a binary number (written backwards), then $z \cdot \alpha$ is the same as 1 added to the binary number for $\alpha$, truncated to the length of $\alpha$ if needed. If such truncation is not needed, $z|_\alpha = e$, but if truncation is needed, $z|_\alpha = z$.

The action of $Z$ on $\{0, 1\}^\omega$ induced by the 2-odometer is the familiar Cantor minimal system of the same name. For $x \in \{0, 1\}^\omega$ we have

$$z \cdot x = \begin{cases} 000 \cdots & \text{if } x_i = 1 \text{ for all } i \\ 00 \cdots 01x_{i+1}x_{i+2} \cdots & \text{if } x_i = 1 \text{ and } x_j = 0 \text{ for all } j < i \end{cases}$$

This action of $Z$ is free (i.e. $z^n \cdot x = x$ implies $n = 0$) and minimal (i.e. the set $\{z^n \cdot x \mid n \in Z\}$ is dense in $\{0, 1\}^\omega$ for all $x \in \{0, 1\}^\omega$).
Lemma 5.5. The groupoid of germs \( \mathcal{G}_{\text{tight}}(S_{Z,X}^\omega) \) is principal.

Proof. Take \( x, y \in \{0,1\}^\omega \) and suppose that we have \( \alpha, \beta \in \{0,1\}^* \) with \( |\alpha| = |\beta| \) and \( n \in \mathbb{Z} \) such that \( [(\alpha, z^n, \beta), x] \in \mathcal{G}_{\text{tight}}(S_{Z,X}^\omega) \) and \( r([(\alpha, z^n, \beta), x]) = y \). This implies that \( x = \beta v \) for some \( v \in \{0,1\}^\omega \), and that \( y = \alpha(z^n \cdot v) \). Suppose we can find another germ from \( x \) to \( y \), that is, suppose we have \( \gamma, \eta \in \{0,1\}^* \) with \( |\gamma| = |\eta| \) and \( m \in \mathbb{Z} \) such that \( [(\gamma, z^m, \eta), x] \in \mathcal{G}_{\text{tight}}(S_{Z,X}^\omega) \) and \( r([(\gamma, z^m, \eta), x]) = y \). Again we can conclude that \( x = \eta u \) for some \( u \in \{0,1\}^\omega \), and that \( y = \gamma(z^m \cdot u) \). There are two cases.

Suppose first that \( \beta = \eta \delta \) for some \( \delta \in \{0,1\}^* \). Then \( \eta \delta v = x = \eta u \), and so \( \delta v = u \). We also have \( \alpha(z^n \cdot v) = y = \gamma(z^m \cdot u) \). Because \( |\alpha| = |\beta| \geq |\eta| = |\gamma| \), this implies that there exists \( \nu \in \{0,1\}^* \) with \( |\nu| = |\delta| \) and \( \alpha = \gamma \nu \). Hence \( \nu(z^n \cdot v) = z^m \cdot u = (z^m \cdot \delta) z^m_{\delta} \cdot v \), which gives us that \( \nu = z^m \cdot \delta \) and \( z^n \cdot v = z^m_{\delta} \cdot v \), and since the action on \( \{0,1\}^\omega \) is free we have \( z^n = z^m_{\delta} \).

So we have that \( x \in C(\beta) = D_{(\beta,e,\beta)}^\beta \), and we calculate

\[
(\gamma, z^n, \eta)(\beta, e, \beta) = (\gamma(z^m \cdot \delta), z^m_{\delta}, \beta) = (\gamma \nu, z^n, \beta) = (\alpha, z^n, \beta) = (\alpha, z^n, \beta)(\beta, e, \beta)
\]

where the first equality is by the definition of the product. Hence \( [(\alpha, z^n, \beta), x] = [(\gamma, z^n, \eta), x] \). The case where \( \beta \) is shorter than \( \eta \) is similar. Hence, \( \mathcal{G}_{\text{tight}}(S_{Z,X}^\omega) \) is principal. \( \square \)

It is routine to check that \( S_{Z,X}^\omega \) satisfies condition [H] (in fact, it is \( E^* \)-unitary, see [ES16, Example 3.4]). The groupoid \( \mathcal{G}_{\text{tight}}(S_{Z,X}^\omega) \) is amenable, see [ADR00, Proposition 5.1.1] and [EP13, Corollary 10.18]. Hence Theorem 4.12 applies, and there is only one normalized trace on \( C^*(S_{Z,X}^\omega) \), the one arising from the invariant mean.

As the observant reader is no doubt aware at this point, \( C^*(S_{Z,X}^\omega) \) is nothing more than the crossed product \( C((\{0,1\}^\omega) \rtimes \mathbb{Z} \) arising from the usual odometer action [Nek04, Theorem 7.2], which has a unique normalized trace due to the dynamical system \( (\{0,1\}^\omega, \mathbb{Z}) \) having a unique invariant measure (given by \( \mathbb{B} \)).

5.4 Aperiodic tilings

We close with another example where the traces on the relevant \( C^* \)-algebras are known beforehand, and hence give us invariant means.

A tile is a closed subset of \( \mathbb{R}^d \) homeomorphic to the closed unit ball. A partial tiling is a collection of tiles in \( \mathbb{R}^d \) with pairwise disjoint interiors, and the support of a partial tiling is the union of its tiles. A patch is a finite partial tiling, and a tiling is a partial tiling with support equal to \( \mathbb{R}^d \). If \( P \) is a partial tiling and \( U \subset \mathbb{R}^d \), then let \( P(U) \) be the partial tiling of all tiles in \( P \) which intersect \( U \). A tiling \( T \) is called aperiodic if \( T + x \neq T \) for all \( 0 \neq x \in \mathbb{R}^d \).

Let \( T \) be a tiling. We form an inverse semigroup \( S_T \) from \( T \) as follows. For a patch \( P \subset T \) and tiles \( t_1, t_2 \in P \) we call the triple \( (t_1, P, t_2) \) a doubly pointed patch. We put an equivalence relation on such triples, by saying that \( (t_1, P, t_2) \sim (r_1, Q, r_2) \) if there exists a vector \( x \in \mathbb{R}^d \) such that \( (t_1 + x, P + x, t_2 + x) = (r_1, Q, r_2) \), and let \( [t_1, P, t_2] \) denote the equivalence class of such a triple – this is referred to a doubly pointed patch class. Let

\[
S_T = \{ [t_1, P, t_2] \mid (t_1, P, t_2) \text{ is a patch class} \} \cup \{0\}
\]
be the set of all doubly pointed patch classes together with a zero element. If \([t_1, P, t_2], [r_1, Q, r_2]\)
are two elements of \(S_T\), we let
\[
[t_1, P, t_2][r_1, Q, r_2] = \begin{cases} 
[t_1, P \cup Q', r_2'] & \text{if there exists } (r_1', Q', r_2') \in [r_1, Q, r_2] \text{ such that } r_1' = t_2 \text{ and } P \cup Q' \text{ is a patch in } T + x \\
0 & \text{for some } x \in \mathbb{R}^d 
\end{cases}
\]
and define all products involving 0 to be 0. Also, let \([t_1, P, t_2] * [r_1, Q, r_2] = [t_2, P, t_1]\). With these
operations, \(S_T\) is an inverse semigroup. This inverse semigroup was defined by Kellendonk [Kel97], and is \(E^*\)-unitary.

Suppose there exists a finite set \(P\) of tiles each of which contain the origin in the interior such that for all \(t \in T\), there exists \(x_t \in \mathbb{R}^d\) and \(p \in P\) such that \(t = p + x_t\). In this case, \(P\) is called a set of prototiles for \(T\). By possibly adding labels, we may assume that \(x_t\) and \(p\) are unique – we call \(x_t\) the puncture of \(t\). Consider the set
\[
X_T = \{T - x_t \mid t \in T\}
\]
and put a metric on \(X_T\) by setting
\[
d(T_1, T_2) = \inf \{1, \epsilon \mid T_1(B_{1/\epsilon}(0)) = T_1(B_{1/\epsilon}(0))\}
\]
and let \(\Omega_{\text{punc}}\) denote the completion of \(X_T\) in this metric. One can show that all elements of \(\Omega_{\text{punc}}\) are tilings consisting of translates of \(P\) which contain an element of \(P\) and that the metric above extends to the same metric on \(\Omega_{\text{punc}}\) – this is called the punctured hull of \(T\).

We make the following assumptions on \(T\):

1. \(T\) has finite local complexity if for any \(r > 0\), there are only finitely many patches in \(T\) with supports having outer radius less than \(r\), up to translational equivalence.
2. \(T\) is repetitive if for every patch \(P \subset T\), there exists \(R > 0\) such that every ball of radius \(R\) in \(\mathbb{R}^d\) contains a translate of \(P\).
3. \(T\) is strongly aperiodic if all elements of \(\Omega_{\text{punc}}\) are aperiodic.

In this case \(\Omega_{\text{punc}}\) is homeomorphic to the Cantor set. For a patch \(P \subset T\) and tile \(t \in P\), let
\[
U(P, t) = \{T' \in \Omega_{\text{punc}} \mid P - x_t \subset T'\}
\]
Then these sets are clopen in \(\Omega_{\text{punc}}\) and generate the topology. Let
\[
\mathcal{R}_{\text{punc}} = \{(T_1, T_1 + x) \in \Omega_{\text{punc}} \times \Omega_{\text{punc}} \mid x \in \mathbb{R}^d\}
\]
and view this equivalence relation as a principal groupoid. Endow it with the topology inherited by viewing it as a subspace of \(\Omega_{\text{punc}} \times \mathbb{R}^d\). For a patch \(P \subset T\) and \(t_1, t_2 \in P\), let
\[
V(t_1, P, t_2) = \{(T_1, T_2) \in \mathcal{R}_{\text{punc}} \mid T_1 \in U(P, t_1), T_2 = T_1 + x_{t_1} - x_{t_2}\}
\]
Then these sets are compact open bisections in \(\mathcal{R}_{\text{punc}}\), and generate the topology on \(\mathcal{R}_{\text{punc}}\). This groupoid is Hausdorff, étale, ample, and amenable [PS99]. The \(C^*\)-algebra of \(\mathcal{R}_{\text{punc}}\)
Figure 1: In the Robinson triangles version of the Penrose tiling, each triangle is always next to a similar triangle with which it forms a rhomb. Let $P$ be the dark gray patch, and let $P'$ be the patch with the lighter gray tiles added. Then for any dark gray tile $t$, $U(P, t) = U(P', t)$
was defined by Kellendonk in [Kel95] (denoted there $A_T$) and studied further in [KP00], [Put00], [Put10], [Phi05], [Sta14].

We proved in [EGS12, Theorem 3] that $G_{\text{tight}}(S_T) \cong \mathcal{R}_{\text{punc}}$ – the universal tight representation of $S_T$ maps $[t_1, P, t_2]$ to the characteristic function of $V(t_1, P, t_2)$. It is interesting to note that in this case that the universal tight representation may not be faithful. Suppose that we could find $P \subset P'$, both patches in $T$, and that $P + x \subset T$ can only happen if $P' + x \subset T$. Then for a tile $t \in P$, the two idempotents $[t, P, t], [t, P', t]$ are different elements in $S_T$, but are both mapped to the characteristic function of $U(P, t) = U(P', t)$ under the universal tight representation – $[t, P', t]$ is dense in $[t, P, t]$, see Figure 1.

The C*-algebra $A_T$ can be seen as the C*-algebra of a Boolean inverse monoid, namely $G_{\text{tight}}(S_T)^a$ – one could then rightly call this the **Boolean inverse monoid associated to $T$**. The traces of $A_T$ are already well-studied, see [KP00], [Put00]. Often, as is the case with the Penrose tiling, there is a unique trace, see [Put00].

**Theorem 5.6.** Let $T$ be a tiling which satisfies conditions 1–3 above, and let $G_{\text{tight}}(S_T)^a$ be the Boolean inverse monoid associated to $T$. Then $M(G_{\text{tight}}(S_T)^a) \cong T(A_T) \cong IM(\mathcal{R}_{\text{punc}})$.

**References**

[AD16a] Claire Anantharaman-Delaroche. Exact Groupoids. *arXiv:1605.05117*, May 2016.

[AD16b] Claire Anantharaman-Delaroche. Some remarks about the weak containment property for groupoids and semigroups. *arXiv:1604.01724*, April 2016.

[ADR00] Claire Anantharaman-Delaroche and Jean Renault. *Amenable Groupoids*. Monographie de l’Enseignement mathématique. L’Enseignement Mathématique, 2000.

[BCFS14] Jonathan Brown, Lisa Orloff Clark, Cynthia Farthing, and Aidan Sims. Simplicity of algebras associated to étale groupoids. *Semigroup Forum*, 88(2):433–452, 2014.

[Bla80] Bruce E Blackadar. Traces on simple AF C-algebras. *Journal of Functional Analysis*, 38(2):156 – 168, 1980.

[BO08] N.P. Brown and N. Ozawa. *C*-algebras and Finite-dimensional Approximations*. Graduate studies in mathematics. American Mathematical Soc., 2008.

[DM14] Allan P. Donsig and David Milan. Joins and covers in inverse semigroups and tight C*-algebras. *Bulletin of the Australian Mathematical Society*, 90:121–133, 8 2014.

[EGS12] Ruy Exel, Daniel Gonçalves, and Charles Starling. The tiling C*-algebra viewed as a tight inverse semigroup algebra. *Semigroup Forum*, 84:229–240, 2012.

[EP13] Ruy Exel and Enrique Pardo. Graphs, groups and self-similarity. *arXiv:1307.1120*, July 2013.

[EP14a] Ruy Exel and Enrique Pardo. Self-similar graphs, a unified treatment of Katsura and Nekrashevych C*-algebras. *arXiv:1409.1107*, September 2014.

[EP14b] Ruy Exel and Enrique Pardo. The tight groupoid of an inverse semigroup. *arXiv:1408.5278*, August 2014.
[ES15] Ruy Exel and Charles Starling. Amenable actions of inverse semigroups. *Ergodic Theory Dynam. Systems*, To appear 2015.

[ES16] Ruy Exel and Charles Starling. Self-similar graph C*-algebras and partial crossed products. *J. Operator Theory*, To appear 2016.

[Exe08] Ruy Exel. Inverse semigroups and combinatorial C*-algebras. *Bull. Braz. Math. Soc. (N.S.)*, 39(2):191–313, 2008.

[Exe09] Ruy Exel. Tight representations of semilattices and inverse semigroups. *Semigroup Forum*, 79(1):159–182, 2009.

[Exe10] Ruy Exel. Reconstructing a totally disconnected groupoid from its ample semigroup. *Proc. Amer. Math. Soc.*, 138(8):2991–3001, 2010.

[GH09] S. Givant and P. Halmos. *Introduction to Boolean algebras*. Undergraduate Texts in Mathematics. Springer, 2009.

[Haa91] U. Haagerup. Quasitraces on exact C*-algebras are traces. *arXiv:1307.1120*, March 1991.

[Kel95] Johannes Kellendonk. Noncommutative geometry of tilings and gap labelling. *Rev. Math. Phys.*, 7(7):1133–1180, 1995.

[Kel97] Johannes Kellendonk. The local structure of tilings and their integer group of coinvariants. *Comm. Math. Phys.*, 187(1):115–157, 1997.

[KLLR16] Ganna Kudryavtseva, Mark V. Lawson, Daniel H. Lenz, and Pedro Resende. Invariant means on boolean inverse monoids. *Semigroup Forum*, 92(1):77–101, 2016.

[KP00] Johannes Kellendonk and Ian F. Putnam. Tilings, C*-algebras, and K-theory. In *Directions in mathematical quasicrystals*, volume 13 of *CRM Monogr. Ser.*, pages 177–206. Amer. Math. Soc., Providence, RI, 2000.

[Li16] X. Li. Partial transformation groupoids attached to graphs and semigroups. *arXiv:1603.09165*, March 2016.

[LL13] Mark V. Lawson and Daniel H. Lenz. Pseudogroups and their étale groupoids. *Advances in Mathematics*, 244(0):117 – 170, 2013.

[LLR00] F. Larsen, N. J. Lausten, and M. Rørdam. *An Introduction to K-Theory for C*-Algebras*. London Mathematical Society Student Texts. Cambridge University Press, 2000.

[LS14] Mark V. Lawson and Phil Scott. AF inverse monoids and the structure of countable MV-algebras. *arXiv:1408.1231*, 2014.

[MOP15] T. Meier Carlsen, E. Ortega, and E. Pardo. C*-algebras associated to Boolean dynamical systems. *arXiv:1510.06718*, October 2015.

[Nek04] Volodymyr Nekrashevych. Cuntz-Pimsner algebras of group actions. *J. Operator Theory*, 52:223–249, 2004.

[Nek09] Volodymyr Nekrashevych. C*-algebras and self-similar groups. *J. reine angew. Math*, 630:59–123, 2009.

[Pat99] Alan Paterson. *Groupoids, inverse semigroups, and their operator algebras*. Birkhäuser, 1999.
[Phe01] R. Phelps. *Lectures on Choquet’s Theorem*. Lecture notes in mathematics. Springer-Verlag Berlin Heidelberg, 2001.

[Phi05] N. Christopher Phillips. Crossed products of the Cantor set by free minimal actions of $\mathbb{Z}^d$. *Comm. Math. Phys.*, 256(1):1–42, 2005.

[PS99] Ian F Putnam and Jack Spielberg. The structure of C*-algebras associated with hyperbolic dynamical systems. *Journal of Functional Analysis*, 163(2):279 – 299, 1999.

[Put] Ian Putnam. Lecture Notes on C*-algebras. http://www.math.uvic.ca/faculty/putnam/ln/C*-algebras.pdf.

[Put00] Ian F. Putnam. The ordered K-theory of C*-algebras associated with substitution tilings. *Comm. Math. Phys.*, 214(3):593–605, 2000.

[Put10] Ian F. Putnam. Non-commutative methods for the K-theory of C*-algebras of aperiodic patterns from cut-and-project systems. *Comm. Math. Phys.*, 294(3):703–729, 2010.

[RE06] J. Renault R. Exel. AF-algebras and the tail-equivalence relation on Bratteli diagrams. *Proceedings of the American Mathematical Society*, 134(1):193–206, 2006.

[Ren80] Jean Renault. A *groupoid approach to C*-algebras*, volume 793 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.

[Sta14] Charles Starling. Finite symmetry group actions on substitution tiling C*-algebras. *Münster J. of Math*, 7:381–412, 2014.

[Sta15a] Charles Starling. Boundary quotients of C*-algebras of right LCM semigroups. *Journal of Functional Analysis*, 268(11):3326 – 3356, 2015.

[Sta15b] Charles Starling. Inverse semigroups associated to subshifts. arXiv:1505.01766 [math.OA], 2015.

[Wil15] Rufus Willett. A non-amenable groupoid whose maximal and reduced C*-algebras are the same. arXiv:1504.05615 [math.OA], 2015.

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