Linking Theorems of Local Semiflows on Complete Metric Spaces

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Abstract

In this paper we establish some linking theorems and prove mountain pass type results for local semiflows on separable complete metric spaces. Our results provide an alternative approach to detect the existence of compact invariant sets, without using the Conley index theory. They can also be applied to the variational problems of elliptic equations that may not satisfy the classical Palais-Smale Condition. As an example, the resonant problem of the nonautonomous parabolic equation $u_t - \Delta u - \mu u = f(u) + g(x, t)$ on a bounded domain is considered. The existence of recurrent solutions is proved via an appropriate linking theorem of semiflows under some Landesman-Laser type conditions. Another example is the elliptic equation $-\Delta u + a(x)u = f(x, u)$ on $\mathbb{R}^n$. We prove the existence of positive solutions by applying a mountain pass theorem of semiflows to the corresponding parabolic flow of the problem.

Keywords: Local semiflow, linking theorem, mountain pass theorem, resonant problem, elliptic equation.

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1 Introduction

Invariant sets are of particular interest in the theory of dynamical systems. This is because much of the long-term dynamics of a system is determined and described by such objects. Equilibrium points, periodic solutions, almost periodic solutions, homoclinic (heteroclinic) orbits and attractors are typical examples of compact invariant sets. It is therefore of great importance to detect the existence of invariant sets and locate their positions for a given dynamical system.

A powerful way to show the existence of invariant sets is to use the famous Ważewski’s Retract Theorem \[30, 31\]. Roughly speaking, the Ważewski’s Theorem states that for a given flow and a closed subset \(W\) of the phase space (a Ważewski set), if the exit set \(W^-\) of \(W\) is not a deformation retract of \(W\), then there exists a trajectory of the flow entirely contained in \(W\). Consequently we know that the invariant set of the flow in \(W\) is nonempty. The theorem turned out to be very useful in the study of asymptotic behavior of differential equations. Inspired by the Ważewski’s Theorem, C. Conley and his group developed an index theory for invariant sets in 1970s \[8\], which is now known as the Conley index theory. Because the Ważewski’s Theorem can be rephrased in terms of the Conley index, one can now prove the existence of invariant sets by directly using the Conley index theory. An infinite-dimensional version of the index theory can be found in the monograph \[21\] by Rybakowski, which can be successfully applied to partial differential equations.

A significant difference between the Conley index theory and the Ważewski’s Theorem is that the former possesses homotopy property. However, in spite of this elegant merit it is still one difficult task to compute the Conley index of an isolating neighborhood or to verify the non-triviality of the index. In general it is even not easy to find appropriate isolating neighborhoods or index pairs for higher dimensional systems.

In this present work, we try to develop an alternative approach for finding invariant sets of dynamical systems by using the basic theory of attractors, complementing the Ważewski’s Retract Theorem and the Conley index theory. Our main goal is to establish some linking theorems and mountain pass type results for local semiflows on separable complete metric spaces. These results are not parallel extensions of the classical Linking Theorem and Mountain Pass Lemma of variational functionals. As we will see in Sections 6 and 7, they not only enable us to examine the existence of compact invariant sets for dynamical systems, but
also provide a possible way to study variational problems of elliptic equations that may not satisfy the Palais-Smale Condition (P.S. Condition in short).

Now let us give a more detailed description of our work. Let $X$ be a separable complete metric space, and $G$ be a local semiflow on $X$. We will impose on $G$ the following conditions which are naturally satisfied by a large number of important examples in applications.

**(NS) (Nonsingularity)** Given $x \in X$, if $G(t)x$ is bounded on the maximal existence interval $[0, T_x)$, then $T_x = \infty$.

**(AC) (Asymptotic compactness)** For any sequences $x_n \in X$ and $t_n \in \mathbb{R}^1$ with $t_n \to +\infty$, if there exists a bounded set $B$ such that $G([0, t_n])x_n \subset B$ for all $n$, then the sequence of the end points $G(t_n)x_n$ has a convergent subsequence.

(For differential equations, the requirement (NS) is just to exclude black hole solutions owing to singularities in nonlinear terms.)

Let $N$ and $E$ be two closed subsets of $X$. If $E$ is an exit set of $N$, then we simply call $(N, E)$ a Ważewski pair. We are interested in the existence of invariant sets in $H := N \setminus E$. Since $H$ may be unbounded, to overcome the difficulties brought by possible explosion of the flow in $H$ and avoid assuming stronger compactness assumptions, we introduce a notion of stability at infinity:

**Definition 1.1 (Stability at infinity)** Let $A$ be a subset of $X$. $G$ is said to be stable in $A$ at infinity, if for any bounded set $B_0$, there exists a bounded set $B_1$ such that for any $x \in A \setminus B_1$ and $t > 0$ with $G([0, t])x \subset A$, it holds that

$$G([0, t])x \cap B_0 = \emptyset.$$ 

Then we establish some linking theorems. A typical one is as follows:

**Theorem 1.2** Assume that $G$ satisfies (NS) and (AC). Let $(N, E)$ be a Ważewski pair. Suppose that the following hypotheses are fulfilled:

1. $G$ is stable in $H := N \setminus E$ at infinity.

2. There exist a bounded closed set $L \subset N \setminus E$ and a closed set $Q \subset W := N \cup E$ such that $L$ and $Q$ link with respect to the family of maps

$$\Gamma := \{ h \in C(Q, W) : h|_S = id_S \}$$

for some $S \subset Q \cap E$ (see Figure 1.1).
Then $H$ contains a nonempty compact invariant set.

As direct consequences of linking theorems, one can immediately obtain some interesting mountain pass type results for semiflows. For instance, we have

**Theorem 1.3** Assume that $G$ satisfies $(NS)$ and $(AC)$. Let $(N, E)$ be a Ważewski pair. Suppose that the following hypotheses are fulfilled:

1. $G$ is stable in $H := N \setminus E$ at infinity.
2. $G$ has a local attractor $\mathcal{A}$ in $H$ with $\mathcal{A} \cap E = \emptyset$.
3. There exists a connected component $Q$ of $N$ such that $Q \cap \mathcal{A} \neq \emptyset \neq Q \cap E$.

Then $H$ contains a nonempty compact invariant set $M$ with $M \cap \mathcal{A} = \emptyset$.

The above results remain valid if we assume, instead of the stability condition at infinity, that $H$ is admissible (see Section 4 for the definition) and that $G$ does not explode in $H$. The admissibility of $H$ is in fact a compactness requirement on the flow in $H$. In the case when $H$ is bounded, it automatically follows from the asymptotic compactness of the flow. But in the case when $H$ is unbounded, this requirement becomes a dynamical version of the P.S. Condition for variational functionals, which can not be deduced from either the asymptotic compactness or the stability at infinity property of the flow in $H$. This can be seen from the Ważewski pair $(N, E)$ constructed in Section 6 in the proof of the main theorem therein, where $N$ is an unbounded closed set, and $E = \emptyset$. The semiflow is asymptotically compact and is stable in $N$ at infinity, whereas $N$ is not admissible.

The existence of bounded full solutions of nonautonomous systems is a classical topic in differential equations. When a system is dissipative, the existence of
bounded full solutions is a direct consequence of the existence of a global attractor (or pullback attractor). However, for a non-dissipative system this problem is far from being trivial. If the forcing term of the system is periodic, we can try to find periodic solutions of the system. This can be done by using some functional-analytic methods. The nonperiodic situation seems to be more complicated, and the functional-analytic methods may fail to work again. For second order scalar differential equations, bounded solutions can be obtained by using the classical phase-plane method [11] or upper and lower solutions method [16], where Landesman-Laser type conditions play crucial roles. Unfortunately these fruitful methods can hardly be applied to higher dimensional differential systems and partial differential equations. To deal with the general case, Ward [28, 29] and Prizzi [19] developed a topological approach by utilizing the Conley index theory. The basic idea is as follows. First, one proves that the autonomous averaged system possesses an isolated invariant set $K$ with a nontrivial Conley index. Then by homotopy property of the index, $K$ can be continued to a nearby isolated invariant set of the skew-product flow associated to the nonautonomous equation, provided the frequency of the oscillation is sufficiently large. As a result, one immediately obtains the existence of a bounded full solution of the original system.

In this paper, we consider the existence of recurrent solutions to the resonant problem of a nonautonomous parabolic equation

$$u_t - \Delta u - \mu u = f(u) + g(x, t), \quad x \in \Omega \tag{1.1}$$

associated with the homogeneous Dirichlet boundary condition, where $\Omega$ is a bounded domain in $\mathbb{R}^n$, and $\mu$ is an eigenvalue of the operator $A = -\Delta$. This problem may fall out of the scope of the theory developed in [19, 28], etc., as in general we do not know whether the autonomous average of the equation exists. The variational method does not seem to be suitable either, because of the lack of variational structures of the equation.

As an application of our theoretical results, here we study the problem by using an appropriate linking theorem of semiflows. Suppose that $f$ and $g$ satisfy some Landesman-Laser type conditions (see Section 6 for details). We prove that if $g$ is recurrent, then the problem has at least one recurrent solution.

As we have mentioned above, the framework developed in the current context also provides a possible way to investigate variational problems. As an example,
we consider the existence of positive solutions of the elliptic equation

\[- \Delta u + a(x)u = f(x, u)\]  \hspace{1cm} (1.2)

on \(\mathbb{R}^n (n \geq 3)\). This problem is closely related to finding standing wave solutions of nonlinear Schrödinger equations. Owing to the unboundedness of the domain, the usual Sobolev embeddings fail to be compact. This gives rise to technical difficulties in verifying the P.S. Condition of the corresponding variational functional and makes the problem interesting and challenging. During the past decades a considerable amount of research has been conducted towards the problem; see [1, 2, 3, 5, 9, 25, 27, 32] and the references cited therein.

Our main purpose here is not to seek hypotheses as weaker as possible to guarantee the existence of positive solutions of the equation (1.2), but to illustrate how the dynamical approach given here can be used to study elliptic problems without verifying the P.S. Condition of the variational functionals. The basic idea is as follows. First, we view (1.2) as a stationary problem of the heat equation

\[u_t - \Delta u + a(x)u = f(x, u), \quad x \in \mathbb{R}^n.\]  \hspace{1cm} (1.3)

Under some mild conditions it can be shown that the Cauchy problem of (1.3) has a unique local solution \(u(t)\) for each initial value \(u_0 \in H^1(\mathbb{R}^n)\). Moreover, if \(u_0\) is positive, then so is \(u(t)\) for each \(t \geq 0\). This allows us to define a local semiflow \(G\) on the space

\[X := \{v \in H^1(\mathbb{R}^n) : v(x) \geq 0 \text{ a.e. } x \in \mathbb{R}^n\}.

Then we prove that \(G\) is asymptotically compact and is stable at infinity between any two energy surfaces. Finally, by constructing an appropriate Ważewski pair \((N, E)\) via the level sets of the energy functional and applying a mountain pass theorem of semiflows to \(G\), we can immediately obtain the existence of nontrivial positive solutions.

It seems to be quite natural to solve elliptic problems via the corresponding parabolic flows. Typically, there are two approaches to follow. One is to apply the Conley index theory to the parabolic flows to obtain information about the solutions of the elliptic problems. See for instance [10, 21, 22], etc. The other is to use parabolic flows to construct deformations for the level sets of the variational functionals and then develop a corresponding variational theory [7]. However, both meet the difficulty that a solution of a parabolic equation with
superlinear nonlinearity may explode in finite time. To overcome this difficulty, it was assumed in [7] that the variational functional of the elliptic problem goes to \(-\infty\) along each solution of the parabolic equation that explodes in finite time. An important feature of our work is that we allow the parabolic flow to explode between two energy surfaces.

2 Preliminaries

Let \(X\) be a complete metric space with the metric \(d(\cdot, \cdot)\).

For a subset \(A\) of \(X\), the closure, interior and boundary of \(A\) are denoted by \(\overline{A}\), \(\text{int} A\) and \(\partial A\), respectively. A set \(U \subset X\) is called a neighborhood of \(A\), if \(\overline{A} \subset \text{int} U\). Given \(r > 0\), the \(r\)-neighborhood of \(A\) in \(X\) is defined to be the set

\[ B_X(A, r) := \{ y \in X : d(y, A) < r \}. \]

We will simply write \(B_X(A, r)\) as \(B(A, r)\) when there is no confusion.

Given two closed subsets \(N\) and \(E\) of \(X\), we denote by \(N/E\) the quotient space of \(N\) and \(E\) obtained by collapsing \(E \cap N\) to a single point \([E]\) in \(N\). It can be easily shown that in the case when \(N \cap E \neq \emptyset\), \(N/E\) can also be obtained by collapsing \(E\) to a single point in \(W := N \cup E\), namely,

\[ N/E \cong W/E. \]

Let \(f : \Omega \subset X \to \mathbb{R}^1\). For \(-\infty \leq a \leq b \leq \infty\), we denote by \(f^b_a\) the set \(\{a \leq f(x) \leq b\}\). In particular, for any \(c \in \mathbb{R}^1\), \(f^c := f^c_{-\infty}\) is called the \(c\)-level set of \(f\). Note also that \(f^c_c\) is precisely the surface \(\{f(x) = c\}\).

2.1 Local semiflows and invariant sets

**Definition 2.1** (Local semiflow) A local semiflow \(G\) on \(X\) is a continuous map from an open subset \(\mathcal{D}_G\) of \(\mathbb{R}^+ \times X\) to \(X\) that satisfies the following conditions:

1. For each \(x \in X\), there exists a positive number \(T_x \leq \infty\) such that

\[ (t, x) \in \mathcal{D}_G \Longleftrightarrow t \in [0, T_x). \] (2.1)

2. \(G(0, x) = x\) for all \(x \in X\).
(3) If \((t + s, x) \in D_G\), where \(t, s \in \mathbb{R}^+\), then 
\[
G(t + s, x) = G(t, G(s, x)).
\]

The set \(D_G\) and the interval \([0, T_x)\) in (2.1) are called the domain of \(G\) and the maximal existence interval of \(G(t, x)\), respectively.

In the case when \(D_G = \mathbb{R}^+ \times X\), we simply call \(G\) a global semiflow.

Let \(G\) be a local semiflow on \(X\). For convenience, hereafter we will rewrite \(G(t, x)\) as \(G(t)x\).

**Proposition 2.2** Let \(x \in X\). Then for any \(T < T_x\), there exists \(\delta_0 > 0\) such that 
\[
T < T_y, \quad \forall y \in B(x, \delta_0).
\]
Moreover, for any \(\varepsilon > 0\), there exists \(0 < \delta < \delta_0\) such that 
\[
d(G(t)y, G(t)x) < \varepsilon, \quad \forall t \in [0, T], y \in B(x, \delta).
\]

**Proof.** It is a simple consequence of the openness of \(D_G\) in \(\mathbb{R}^+ \times X\) and the continuity of \(G\). We omit the details of the argument. \(\square\)

Let \(I \subset \mathbb{R}^1\) be an interval. A solution (or trajectory) \(\gamma\) of \(G\) on \(I\) is a map \(\gamma : I \rightarrow X\) such that
\[
\gamma(t) = G(t-s)\gamma(s), \quad \forall s, t \in I, \ s \leq t.
\]
The set \(\text{orb}(\gamma) := \{\gamma(t) : t \in I\}\) is called the orbit of \(\gamma\).

A full solution means a solution defined on the whole line \(\mathbb{R}^1\).

**Lemma 2.3** Assume that \(G\) is asymptotically compact. Let \(x_n\) be a bounded sequence of \(X\), and \(t_n \rightarrow +\infty\). Suppose that there exists a bounded subset \(B\) of \(X\) such that \(G([0, t_n])x_n \subset B\) for all \(n\). Define 
\[
\gamma_n(t) = G(t + t_n/2)x_n, \quad t \in (-t_n/2, t_n/2).
\]
Then there exists a subsequence of \(\gamma_n\) converging to a bounded full solution \(\gamma\) of \(G\) uniformly on any compact interval.
**Proof.** Lemma 2.3 can be proved by using the continuity property and asymptotic compactness of $G$. Since the argument involved in the proof is quite standard in the theory of infinite dimensional dynamical systems, we omit the details. □

Let $M$ be a subset of $X$. $M$ is said to be *positively invariant* (resp. *invariant*), if $G(t)M \subset M$ (resp. $G(t)M = M$) for all $t \geq 0$.

**Remark 2.4** If $M$ is a nonempty invariant set, then by the definition of invariant sets one can easily verify that for each $x \in M$, there exists a solution $\gamma$ in $M$ defined on $(-\infty, 0]$ such that $\gamma(0) = x$. Further, if we assume that $T_x = \infty$ for each $x \in M$, then $\gamma$ can be extended to a full solution in $M$.

We denote by $\mathcal{C}(M)$ the union of compact invariant sets in $M$, i.e.,

$$\mathcal{C}(M) = \bigcup \{K \subset M : K \text{ is a compact invariant set of } G\}.$$ \hspace{1cm} (2.2)

Note that if $G$ is asymptotically compact, then $\mathcal{C}(M)$ is precisely the union of the orbits of all bounded full solutions of $G$ in $M$.

**Definition 2.5** (*Minimal set*) A compact invariant set $M$ is said to be minimal, if it does not contain any proper nonempty compact invariant subset.

### 2.2 Limit sets and attractors

Let $G$ be a given local semiflow on $X$. From now on we always assume that $G$ is *nonsingular and asymptotically compact*.

#### 2.2.1 Limit sets and attractors

For a subset $M$ of $X$, the *\(\omega\)-limit set* $\omega(M)$ of $M$ is defined as

$$\omega(M) = \{y \in X : \exists x_n \in M \text{ and } t_n \to +\infty \text{ such that } G(t_n)x_n \to y\}.$$ 

If $\gamma$ is a solution on $(a, \infty)$, then the *\(\omega\)-limit set* of $\gamma$ is defined to be the set

$$\omega(\gamma) = \{y \in X : \exists t_n \to +\infty \text{ such that } \gamma(t_n) \to y\}.$$ 

Similarly, we define the *\(\alpha\)-limit set* of a solution $\gamma$ on $(-\infty, a)$ as

$$\alpha(\gamma) = \{y \in X : \exists t_n \to -\infty \text{ such that } \gamma(t_n) \to y\}.$$
Definition 2.6 (Attraction) Let $A, B \subset X$. We say that $A$ attracts $B$, if

1. $T_x = \infty$ for all $x \in B$, and
2. for any $\varepsilon > 0$, there exists $t_0 > 0$ such that

$$G(t)B \subset B(A, \varepsilon), \quad \forall t > t_0.$$ 

The following results about limit sets are quite fundamental in the theory of dynamical systems; see [12, 23], etc.

Proposition 2.7 Let $M \subset X$. Assume that $T_x = \infty$ for all $x \in M$, and that $G([a, \infty))M$ is bounded for some $a > 0$. Then $\omega(M)$ is a nonempty compact invariant set of $G$ that attracts $M$.

Proposition 2.8 Let $\gamma$ be a bounded solution on $(a, \infty)$ (resp. $(\infty, a)$). Then $\omega(\gamma)$ (resp. $\alpha(\gamma)$) is a nonempty compact invariant set of $G$.

Definition 2.9 (Attractor) A compact invariant set $A \subset X$ is said to be an attractor of $G$, if it attracts a neighborhood $U$ of itself.

The region of attraction $\Omega(A)$ of an attractor $A$ is defined as

$$\Omega(A) = \{x \in X : A \text{ attracts } x\}.$$ 

An attractor $A$ is said to be a global attractor, if $\Omega(A) = X$.

Proposition 2.10 The region of attraction $\Omega := \Omega(A)$ of an attractor $A$ is open.

Proof. By definition we know that $A$ attracts a neighborhood $U$. We may assume that $U$ is open. Let $x \in \Omega$. Then $G(T)x \in U$ for some $T > 0$. Therefore, by Proposition 2.2 there exists $r > 0$ such that $T_y > T$ for all $y \in B(x, r)$. Moreover,

$$G(T)y \in U, \quad \forall y \in B(x, r).$$

It then follows that $B(x, r) \subset \Omega$. Hence $\Omega$ is open. □

Remark 2.11 It is trivial to check that an attractor $A$ attracts each compact subset of $\Omega(A)$. As a simple consequence, we know that the global attractor in the terminology of Definition 2.9, if exists, is necessarily unique.

Concerning the existence of attractors, we have
Proposition 2.12 Suppose that there exists a bounded closed set $M \subset X$ which attracts a neighborhood $U$ of itself. Then the semiflow $G$ has an attractor $A$ contained in $M$.

Proof. We may assume that $U$ is bounded. Then one can easily check that $A = \omega(U)$ is an attractor with $A \subset M$. □

Definition 2.13 (Stability) A subset $M$ of $X$ is said to be stable, if for any neighborhood $V$ of $M$, there exists a neighborhood $U$ of $M$ such that $T_x = \infty$ for all $x \in U$, and moreover,

$$G(\mathbb{R}^+)U \subset V.$$  

Proposition 2.14 Let $A$ be a nonempty compact invariant set. Assume that $A$ is stable, and that there exists a neighborhood $W$ of $A$ such that $A$ attracts each point $x \in W$ (hence $T_x = \infty$ for all $x \in W$). Then $A$ is an attractor.

Proof. We only need to show that $\omega(U) = A$ for some neighborhood $U$ of $A$.

First, by the stability of $A$ we deduce that there exists a neighborhood $U$ of $A$ such that $G(t)U \subset W$ for all $t \geq 0$, where $W$ is the neighborhood of $A$ given in the proposition. We may assume that $W$ is bounded. Hence by Proposition 2.7, $\omega(U)$ is a nonempty compact invariant set. We prove that $\omega(U) = A$.

Since $A \subset U$ and $A$ is invariant, it is clear that $\omega(U) \supset A$. Thus to prove what we desired, it suffices to check that $\omega(U) \subset A$. Suppose the contrary. Then there would exist $y \in \omega(U)$ such that $d(y,A) = \delta > 0$. Let $\gamma$ be a solution on $(-\infty,0]$ contained in $\omega(U)$ with $\gamma(0) = y$. By Proposition 2.8, $\alpha(\gamma) \subset W$ and is a nonempty compact invariant set. If $\alpha(\gamma) \cap A = \emptyset$, then since both $\alpha(\gamma)$ and $A$ are compact, we deduce that $d(\alpha(\gamma),A) > 0$. This contradicts the attraction property of $A$. Hence $\alpha(\gamma) \cap A \neq \emptyset$. It then follows that there exists a sequence $t_n \to -\infty$ such that $d(\gamma(t_n),A) \to 0$. However, this contradicts the stability property of $A$, as $d(y,A) = \delta > 0$ and $G(-t_n)\gamma(t_n) = y$ for all $n$. Therefore we conclude that $\omega(U) \subset A$. □

2.2.2 Lyapunov functions of attractors

Let $A$ be an attractor of $G$ with the region of attraction $\Omega = \Omega(A)$.

Definition 2.15 A continuous function $\phi: \Omega \to \mathbb{R}^+$ is called a Lyapunov function of $A$, if $\phi(x) \equiv 0$ on $A$ and

$$\phi(G(t)x) < \phi(x), \quad \forall \ x \in \Omega \setminus A, \ t > 0.$$
Proposition 2.16 Each attractor $A$ has a Lyapunov $\phi$ on $\Omega$ with
\[
\phi(x) \geq d(x, A), \quad \forall x \in \Omega.
\] (2.3)

Proof. The procedure of the construction of $\phi$ is the same as in [14] (pp. 226). Here we give a sketch for the reader’s convenience. First, define
\[
\xi(x) = \sup_{t \geq 0} d(G(t)x, A), \quad x \in \Omega.
\]
Then by the attraction property of $A$ and the continuity of $G$, it is not hard to check the continuity of the function $\xi$. Clearly
\[
\xi(x) \geq d(G(0)x, A) = d(x, A), \quad \forall x \in \Omega,
\]
and $\xi(x) \equiv 0$ on $A$. A basic property of $\xi$ is that it is decreasing along each solution of $G$ in $\Omega$.

Now set
\[
\phi(x) = \xi(x) + \int_0^\infty e^{-t}\xi(G(t)), \quad x \in \Omega.
\]
Then $\phi$ is a Lyapunov function of $A$ satisfying (2.3). □

2.3 Bebutov’s dynamical system and recurrent functions

Let $C = C(\mathbb{R}^1; X)$ be the space that consists of all continuous functions from $\mathbb{R}^1$ to $X$. $C$ is equipped with the metric $\varrho = \varrho(\cdot, \cdot)$ defined as follows:
\[
\varrho(u, v) = \sum_{n=1}^{\infty} \frac{1}{2^n} \max_{|t| \leq n} \frac{d(u(t), v(t))}{1 + \max_{|t| \leq n} d(u(t), v(t))}, \quad \forall u, v \in C.
\]
It is well known that this metric yields the compact-open topology on $C$. Hence for convenience in statement, we call $\varrho$ the compact-open metric.

Let $\theta = \theta_t$ be the translation operator on $C$ defined by
\[
\theta_t u = u(t + \cdot), \quad \forall u \in C.
\]
Then $\theta$ is a dynamical system on $C$, which is usually known as the Bebutov’s dynamical system [23].

For a function $u \in C$, the hull $\mathcal{H}_\varrho(u)$ of $u$ in $C$ is defined to be the closure of the set $\{\theta_t u : \ t \in \mathbb{R}^1\}$ in $C$, namely,
\[
\mathcal{H}_\varrho(u) = \{\theta_t u : \ t \in \mathbb{R}^1\}.
\]

Now we introduce the concept of recurrence (in the sense of Birkhoff) with respect to Bebutov’s dynamical system.
Definition 2.17 (Recurrence) A function \( u \in \mathcal{C} \) is said to be recurrent, if it satisfies the following conditions:

1. The hull \( \mathcal{H}_\varphi(u) \) of \( u \) is compact.

2. For any \( \varepsilon > 0 \), there exists \( l > 0 \) such that for any interval \( J \subset \mathbb{R}_1 \) of length \( l \), one can find a \( \tau \in J \) such that \( \varrho(\theta_{\tau}u, u) < \varepsilon \).

Lemma 2.18 \([23]\) \( u \in \mathcal{C} \) is recurrent if and only if \( \mathcal{H}_\varphi(u) \) is minimal under the Bebutov’s dynamical system.

3 The Quotient Flow Lemma

From now on we also assume that the space \( X \) is separable. Let \( G \) be a given local semiflow on \( X \) which is nonsingular and asymptotically compact. In this section we prove a fundamental result concerning quotient flows induced by \( G \) on some quotient spaces related to Ważewski pairs.

3.1 Ważewski pairs and the one-point expansion of the phase space

In this subsection we introduce some basic notions and define the one-point expansion \( X^* \) of the phase space \( X \).

3.1.1 Ważewski pairs

Let \( A \) and \( B \) be two subsets of \( X \). \( B \) is said to be \( A \)-invariant, if for each \( x \in A \cap B \) and \( t > 0 \),

\[
G([0,t))x \subset A \implies G([0,t))x \subset B.
\]

For each \( x \in A \), we denote by \( t_A(x) \) the largest time such that \( G(t)x \) stays in \( A \) before \( t_A(x) \), i.e.,

\[
t_A(x) = \sup\{t \geq 0 : G([0,t])x \subset A\}. \quad (3.1)
\]

Let \( N \) and \( E \) be two closed subsets of \( X \). \( E \) is called an exit set of \( N \), if it is \( N \)-invariant, and moreover, for any \( x \in N \) with \( t_N(x) < T_x \), there exists \( t \leq t_N(x) \) such that \( G(t)x \in E \).
Note that an exit set may be empty. For example, if $N$ is positively invariant, then $E = \emptyset$ is an exit set of $N$.

**Definition 3.1** (Ważewski pair) Let $N$, $E$ be two closed subsets of $X$. $(N, E)$ is called a Ważewski pair of $G$, if $E$ is an exit set of $N$.

### 3.1.2 The one-point expansion of $X$

**Definition 3.2** (One-point expansion) Pick a new element $* \notin X$. Then the one-point expansion $X^*$ of $X$ is defined to be the space $X^* = X \cup \{ * \}$ equipped with the topology $\mathcal{U} = \mathcal{U}_X \cup \mathcal{N}(*)$. Here $\mathcal{U}_X$ is the family of all open subsets of $X$, and $\mathcal{N}(*)$ is the family of open neighborhoods of $*$ defined as follows:

$$\mathcal{N}(*) = \{ X^* \setminus B : B \text{ is a bounded closed subset of } X \}.$$

**Remark 3.3** We infer from the Appendix that $X^*$ is completely metrizable, that is, there exists a metric $\rho = \rho(\cdot, \cdot)$ on $X^*$ such that $X^*$ is a complete metric space. It can be easily seen that $X^*$ is separable as well.

**Remark 3.4** The topology of the original space $X$ is precisely the one induced by the topology of $X^*$ on $X$. Hence one can simply regard $X$ as a subspace of $X^*$ (from the point of view of topology).

**Remark 3.5** If $B$ is a closed set in $X^*$ with $* \notin B$, then $V := X^* \setminus B$ is an open neighborhood of $*$ in $X^*$. By the definition of the topology of $X^*$, it follows that $B$ is a bounded closed set in $X$.

Conversely, each bounded closed set $B$ in $X$ is closed in $X^*$.

### 3.2 The Quotient Flow Lemma

Let $(N, E)$ be a Ważewski pair of $G$ in $X$. Set

$$N = N \cup \{ * \}, \quad E = E \cup \{ * \}.$$

Then both $N$ and $E$ are closed in $X^*$. Consider the quotient space $N/E$. We infer from the Appendix that $N/E$ is completely metrizable. Let

$$\Pi : \mathcal{W} := N \cup E \rightarrow N/E$$
be the quotient map. For any $x \in \mathbb{N}$ and $A \subset \mathbb{N}$, we simply write $\Pi(x)$ and $\Pi(A)$ as $[x]$ and $[A]$, respectively.

Define the quotient flow $\tilde{G}$ of $G$ on $\mathbb{N}/E$ as follows. If $u = [E]$, then
\[
\tilde{G}(t)[E] \equiv [E] \quad \text{for } t \in \mathbb{R}^+;
\] (3.2)
and if $u = [x]$ with $x \in \mathbb{N} \setminus E = \mathbb{N} \setminus E$, then
\[
\tilde{G}(t)u = \begin{cases} 
[G(t)x], & 0 \leq t < t_H(x); \\
[E], & t \geq t_H(x) 
\end{cases}
\] (3.3)

Here (and hereafter) $H := \overline{\mathbb{N} \setminus E}$ denotes the closure of $\mathbb{N} \setminus E$ in $X$.

Since $E$ is $\mathbb{N}$-invariant, it is easy to see that $\tilde{G}$ is well defined with $[E]$ being an equilibrium point.

**Remark 3.6** One can replace “$t_H(x)$” in the definition of $\tilde{G}$ by “$t_{\mathbb{N} \setminus E}(x)$”.

Now let us state and prove the main result in this section.

**Lemma 3.7** (Quotient Flow Lemma) Assume that $G$ is stable in $H$ at infinity. Suppose that $C(H)$ is compact, and that $C(H) \cap E = \emptyset$, where $C(H)$ is the union of the orbits of all bounded full solutions in $H$.

Then the quotient flow $\tilde{G}$ defined by (3.2) and (3.3) is continuous and asymptotically compact. Furthermore, the equilibrium point $[E]$ is an attractor of $\tilde{G}$.

**Proof.** We split the argument into several steps.

**Step 1.** $[E]$ is stable.

Let $\tilde{V}$ be an arbitrary open neighborhood of $[E]$ in $\mathbb{N}/E$. We need to show that there exists a neighborhood $\tilde{U}$ of $[E]$ such that
\[
\tilde{G}(t)\tilde{U} \subset \tilde{V}, \quad \forall t \geq 0.
\] (3.4)

Take an open neighborhood $V$ of $E$ in $X^*$ such that $\tilde{V} = [V \setminus \tilde{W}]$, where $\tilde{W} = \mathbb{N} \cup E$. Let $V = V \setminus \{\ast\}$. Then $V$ is open in $X$. By the definition of the topology of $X^*$, we deduce that $H \setminus V$ is a bounded subset of $X$. Hence by the stability of $G$ in $H$ at infinity, there exists a bounded closed subset $B_1$ of $X$ such that
\[
G(t)x \not\in H \setminus V, \quad \forall t \in [0, \tau_x), \quad \forall x \in H \setminus B_1,
\] (3.5)
where $\tau_x = t_H(x)$.

In what follows we show that for each $y \in E$, there exists $r_y > 0$ such that

$$G([0, \tau_x]) x \subset V \quad (3.6)$$

for all $x \in B(y, r_y)$. Two cases may occur.

Case 1). “$G([0, T_y]) y \subset H$.” In this case we claim that $G([0, T_y]) y$ is necessarily unbounded. Suppose the contrary. Then since $G$ is nonsingular, one should have $T_y = \infty$. Further by Proposition 2.7, $\omega(y)$ is a nonempty compact invariant set. Clearly $\omega(y) \subset H$. On the other hand, we infer from the $N$-invariance property of $E$ that $G([0, T_y]) y \subset E$, and consequently $\omega(y) \subset E$. This contradicts the assumption “$C(H) \cap E = \emptyset$” and proves our claim.

Let $U_0 = X \setminus B_1$. We fix a positive number $s < T_y$ such that $G(s)y \in U_0$. Then there exists $r_y > 0$ such that $G(t)x$ exists on $[0, s]$ for all $x \in B(y, r_y)$. Note that $G([0, s]) y \subset E \subset V$. Since $U_0$ and $V$ are open in $X$, by the continuity properties of $G$, one can restrict $r_y$ sufficiently small such that

$$G(s)x \in U_0, \quad G([0, s]) x \subset V \quad (3.7)$$

for all $x \in B(y, r_y)$. Now let $x \in B(y, r_y)$. If $s \geq \tau_x$, the second relation in (3.7) readily implies (3.6). Thus we assume that $s < \tau_x := t_H(x)$. Then by the first relation in (3.7), we have

$$G(s)x \in H \cap U_0 = H \setminus B_1.$$ 

It follows by (3.5) that

$$G(t)x = G(t-s)G(s)x \in V, \quad \forall t \in [s, \tau_x).$$

Combining this with (3.7) we immediately conclude that $G(t)x \in V$ for all $t \in [0, \tau_x)$. Hence (3.6) holds true.

Case 2). “$G(t)y \not\subset H$ for some $t < T_y$.” Let $\tau_y = t_H(y)$. Then by the definition of $t_H(y)$ we necessarily have $\tau_y < T_y \leq \infty$. Moreover, there exists a sequence of positive numbers $\delta_n \to 0$ such that $G(\tau_y + \delta_n) \not\in H$ for all $n$. Note that $G(\tau_y)y \in H$. Since $E$ is $N$-invariant and $y \in E$, we see that $G([0, \tau_y]) y \subset E \subset V$.

Fix a $\delta_n > 0$ sufficiently small so that

$$G([0, T]) y \subset V,$$
where \( T = \tau_y + \delta_n \) (\( T < T_y \)). Then as in \( 3.7 \), there exists \( r_y > 0 \) such that for all \( x \in B(y, r_y) \), we have \( T_x > T \) and that
\[
G([0, T])x \subset V. \tag{3.8}
\]
Noticing that \( H \) is closed in \( X \) and that \( G(T)y \notin H \), we can restrict \( r_y \) so that
\[
G(T)x \notin H, \quad \forall x \in B(y, r_y). \tag{3.9}
\]
Then \( \tau_x < T \) for all \( x \in B(y, r_y) \). Now \( 3.6 \) follows from \( 3.8 \).

Set \( U_1 = \bigcup_{y \in E} B(y, r_y) \). Clearly \( U_1 \) is an open neighborhood of \( E \) in \( X \). Recalling that \( H \cap U_0 = H \setminus B_1 \), by \( 3.5 \) and \( 3.6 \) we deduce that
\[
G([0, \tau_x])x \subset H \cap V, \quad \forall x \in H \cap U, \tag{3.10}
\]
where \( U = U_0 \cup U_1 \). Note that \( U_0 \cup \{*\} \) is an open neighborhood of the point \(*\) in \( X^* \). Therefore \( \bar{U} := U \cup \{*\} \) is an open neighborhood of \( E \) in \( X^* \). Consequently \( \bar{U} = [U \cap W] \) is a neighborhood of \([E]\).

Let \( u \in \bar{U} \). If \( u = [E] \), then \( \tilde{G}(t)[E] \equiv [E] \in \tilde{V} \) for \( t \geq 0 \). Assume that \( u \neq [E] \). Then there exists \( x \in H \cap U \) such that \( u = [x] \). By \( 3.10 \) we see that \( \tilde{G}(t)[x] \subset \tilde{V} \) for \( t < \tau_x \). On the other hand, if \( t \geq \tau_x \), we infer from the definition of \( \tilde{G} \) that \( \tilde{G}(t)[x] = [E] \). Hence \( \tilde{G}(t)[x] \in \tilde{V} \) for all \( t \geq 0 \). In conclusion, we have \( \tilde{G}(t) \bar{U} \subset \tilde{V} \) for all \( t \geq 0 \). This finishes the proof of \( 3.4 \).

**Step 2.** \( \tilde{G} \) is continuous.

Let \((t, u) \in \mathbb{R}^+ \times (\mathbb{N}/E) \). If \( u = [E] \), the continuity of \( \tilde{G} \) at \((t, u)\) follows from the stability of \([E]\). Hence we only consider the case when \( u = [x] \) for some \( x \in N \setminus E \).

If \( \tilde{G}(t)u \neq [E] \), then by the definition of \( \tilde{G} \), we have \( G(t)x \in N \setminus E \). Consequently \( G(s)x \in N \setminus E \) for all \( s \in [0, t] \). Since \( G([0, t])x \) is compact and \( E \) is closed in \( X \), we can pick a \( \delta > 0 \) sufficiently small so that \( G(s)x \in N \setminus E \) for all \( s \in [0, t + 2\delta] \). Further by Proposition \( 2.2 \) we deduce that there exists \( \varepsilon > 0 \) such that \( G(s)y \) exists on \([0, t + \delta]\) for all \( y \in B(x, \varepsilon) \) with
\[
G([0, t + \delta])y \cap E = \emptyset.
\]
Thus for \( y \in B(x, \varepsilon) \cap W := N \setminus E \), one has
\[
\tilde{G}(s)[y] = [G(s)y], \quad s \in [0, t + \delta].
\]
The continuity of $\tilde{G}$ at $(t, u)$ then follows from that of $G$ at $(t, x)$.

Now assume that $\tilde{G}(t)u = [E]$. Let $\eta_x = t_{N\setminus E}(x)$. We infer from the definition of $t_{N\setminus E}(x)$ (see (3.1)) and $\tilde{G}(t)u = [E]$ that $\eta_x \leq t$. Note that $\tilde{G}(\eta_x)u = [E]$. Given any open neighborhood $\tilde{V}$ of $[E]$, by the stability of $[E]$ one can find an open neighborhood $\tilde{U}$ of $[E]$ such that

$$\tilde{G}(s)\tilde{U} \subset \tilde{V}, \quad \forall s \geq 0. \quad (3.11)$$

Pick two open neighborhoods $U$ and $V$ of $E$ in $X^*$ such that

$$\tilde{U} = [U \cap W], \quad \tilde{V} = [V \cap W].$$

Then since $\tilde{G}(\eta_x)u = [E]$, we have either $G(\eta_x)x \in E$, or $\eta_x = T_x$ with

$$G(s)x \to * \text{ in } X^* \text{ as } s \to \eta_x.$$

In any case one can take a positive number $\tau < \eta_x$ such that $G(\tau)x \in U$. Choose a number $\varepsilon > 0$ sufficiently small such that $G(s)y$ exists on $[0, \tau]$ with $G(\tau)y \in U$ for all $y \in B(x, \varepsilon)$. Then

$$\tilde{G}(\tau)[y] \in \tilde{U}, \quad \forall y \in B(x, \varepsilon) \cap W := B_\varepsilon.$$

Combining this with (3.11), it yields that

$$\tilde{G}(s)[y] \in \tilde{V}, \quad \forall s \geq \tau, \ y \in B_\varepsilon. \quad (3.12)$$

Fix a number $\delta > 0$ with $t - \delta > \tau$. Then by (3.12) one concludes that

$$\tilde{G}([t - \delta, \infty))[B_\varepsilon] \subset \tilde{V}.$$

This completes the proof of the continuity of $\tilde{G}$ at $(t, u)$, as $\tilde{V}$ is arbitrary and $[t - \delta, \infty) \times [B_\varepsilon]$ is a neighborhood of $(t, u)$ in $\mathbb{R}^+ \times (N/E)$.

**Step 3.** $[E]$ attracts each point in a neighborhood $\tilde{U}$ of itself.

Because $M := \mathcal{C}(H)$ is compact and that $M \cap E = \emptyset$, we deduce that $[M]$ is a compact subset of $N/E$ with $[E] \notin [M]$. Take a closed neighborhood $\tilde{V}$ of $[E]$ in $N/E$ so that

$$[M] \cap \tilde{V} = \emptyset. \quad (3.13)$$

By the stability of $[E]$ one can find a neighborhood $\tilde{U}$ of $[E]$ such that $\tilde{G}(t)\tilde{U} \subset \tilde{V}$ for all $t \geq 0$. We prove that $[E]$ attracts each point $u \in \tilde{U}$. For this purpose, it suffices to verify that for any neighborhood $\tilde{O}$ of $[E]$, one has

$$\tilde{G}(t)u \in \tilde{O}, \quad (3.14)$$
provided $t$ is sufficiently large.

It can be assumed that $\tilde{\Theta} \subset \tilde{V}$. If $u = [E]$, then we are done. Thus we assume that $u = [x]$ for some $x \not\in E$. Take a neighborhood $\tilde{V}_0$ of $[E]$ such that $\tilde{G}(t)\tilde{V}_0 \in \tilde{\Theta}$ for all $t \geq 0$. We show that $\tilde{G}(t_0)u \in \tilde{V}_0$ for some $t_0 > 0$. Consequently $\tilde{G}(t)u \in \tilde{\Theta}$ for all $t \geq t_0$, and (3.14) follows.

We argue by contradiction and suppose that

$$\tilde{G}(t)u \notin \tilde{V} \setminus \tilde{V}_0, \quad \forall t \geq 0. \quad (3.15)$$

Then by the definition of $\tilde{G}$ we deduce that $G(t)x$ exists on $[0, \infty)$. Pick two neighborhoods $V$ and $V_0$ of $E$ so that $[V \cap W] = \tilde{V}$, $[V_0 \cap W] = \tilde{V}_0$. Set

$$B = \Pi^{-1}\left(\tilde{V} \setminus \tilde{V}_0\right) = (V \setminus V_0) \cap W,$$

where $\Pi$ is the quotient map from $W := N \cup E$ to $N/E$. Clearly $B \subset X$. By the definition of the neighborhoods of $\ast$, one easily sees that $B$ is also bounded in $X$. On the other hand, (3.15) implies that $G(t)x \in B$ for all $t \geq 0$. By Proposition 2.7 it then follows that $\omega(x)$ is a nonempty compact invariant set of $G$. Since $M$ is the maximal compact invariant set of $G$ in $H$, we necessarily have $\omega(x) \subset M$. However, recalling that $\tilde{V}$ is closed and $\tilde{G}(t)u = [G(t)x] \notin \tilde{V}$ for all $t \geq 0$, one concludes that $[\omega(x)] \subset \tilde{V}$. This contradicts (3.13).

**Step 4.** $\tilde{G}$ is asymptotically compact.

We show that for any sequences $u_n \in N/E$ and $t_n \to +\infty$, the sequence $\tilde{G}(t_n)u_n$ has a convergent subsequence. There are two possibilities.

1. There exists a subsequence $n_k$ of $n$ and a sequence $s_{n_k}$ with $s_{n_k} \in [0, t_{n_k}]$ for each $k$ such that $\tilde{G}(s_{n_k})u_{n_k} \to [E]$.

In this case one easily deduces by the stability of $[E]$ that $\tilde{G}(t_{n_k})u_{n_k} \to [E]$.

2. There exist a neighborhood $\tilde{U}$ of $[E]$ and a number $n_0 > 0$ such that

$$\tilde{G}([0, t_n])u_n \cap \tilde{U} = \emptyset, \quad \forall n > n_0. \quad (3.16)$$

Let $u_n = [x_n]$ (for $n > n_0$). Pick a neighborhood $U$ of $E$ such that $[U \cap W] = \tilde{U}$. Then by the definition of $\tilde{G}$ and (3.16), $G([0, t_n])x_n \cap U = \emptyset$ for all $n > n_0$, that is,

$$G([0, t_n])x_n \subset B := X \setminus U.$$

Since $U$ is a neighborhood of $\ast$, $B$ is a bounded subset of $X$. Hence by the asymptotic compactness of $G$ in $X$, we conclude that $G(t_n)x_n$ has a convergent subsequence $G(t_{n_k})x_{n_k}$. Consequently $\tilde{G}(t_{n_k})u_{n_k}$ converges in $N/E$. 

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Step 5. \([E]\) is an attractor of \(\tilde{G}\).

The conclusion follows from Proposition 2.14 and what we have just proved above.

The proof of the lemma is completed. □

4 Linking Theorems of Local Semiflows

In this section we establish some existence results on compact invariant sets of local semiflows, extending the well-known Linking Theorem and Mountain Pass Lemma of variational functionals to dynamical systems.

As in Section 3, let \(X\) be a separable complete metric space, and \(G\) be a local semiflow on \(X\) which is always assumed to be nonsingular and asymptotically compact.

4.1 Linking theorems of local semiflows

We first recall the definition of linking.

**Definition 4.1** (Linking) Let \(L\) and \(Q\) be two closed subsets of \(X\), and \(\Gamma\) be a family of continuous maps from \(Q\) to \(X\). We say that \(L\) and \(Q\) link with respect to \(\Gamma\), if

\[ L \cap h(Q) \neq \emptyset, \quad \forall h \in \Gamma. \]

Now let us state and prove our linking theorems on semiflows.

Let \((N, E)\) be a given Ważewski pair, and let \(H := \overline{N \setminus E}\) be the closure of \(N \setminus E\) in \(X\). Set \(W = N \cup E\).

**Theorem 4.2** Assume that \(G\) is stable in \(H\) at infinity. Suppose that there exists a bounded closed set \(L \subset N\) with \(L \cap E = \emptyset\) such that for any bounded subset \(B\) of \(X\), one can find a closed set \(Q \subset W\) and a subset \(S\) of \(Q\) with \(S \cap B \subset E\) such that \(L\) and \(Q\) link with respect to the family of maps

\[ \Gamma := \{ h \in C(Q, W) : h|_S = \text{id}_S \}. \quad (4.1) \]

Then \(C(H) \neq \emptyset\).
Remark 4.3 Note that if there exists a closed set $Q \subset W$ such that $L$ and $Q$ link with respect to the family of maps in (4.1) for some $S \subset Q \cap E$, then the hypothesis on linking in the above theorem is fulfilled. Accordingly the theorem can be rephrased as follows.

**Theorem 4.4** Assume that $G$ is stable in $H$ at infinity. Suppose that there exist a bounded closed set $L \subset N$ with $L \cap E = \emptyset$ and a closed set $Q \subset W$ such that $L$ and $Q$ link with respect to the family of maps $\Gamma$ in (4.1) for some $S \subset Q \cap E$. Then $C(H) \neq \emptyset$.

**Proof of Theorem 4.2** We argue by contradiction and suppose

$$C(H) = \emptyset. \quad (4.2)$$

Let $X^* := X \cup \{\ast\}$ be the one-point expansion of $X$, and let $N = N \cup \{\ast\}$, $E = E \cup \{\ast\}$. Then the quotient flow $\tilde{G}$ on $N/E$ defined by (3.2) and (3.3) is continuous. Furthermore, $[E]$ is an attractor of $\tilde{G}$. We claim that

$$\Omega([E]) = N/E, \quad (4.3)$$

where $\Omega([E])$ is the region of attraction of $[E]$, and hence $[E]$ is the global attractor of $\tilde{G}$ in the terminology of Definition 2.9. Indeed, if (4.3) fails to be true, then $R := (N/E) \setminus \Omega([E])$ is a nonempty closed subset of $N/E$ which is positively invariant under $\tilde{G}$. (Recall that the region of attraction of an attractor is open.) Let $R = \Pi^{-1}(R)$, where $\Pi$ is the quotient map from $\mathcal{W} := N \cup E$ to $N/E$. Then $R$ is closed in $\mathcal{W}$. Since $\mathcal{W}$ is closed in $X^*$, $R$ is closed in $X^*$ as well. Clearly $R \cap E = \emptyset$. Thus by Remark 3.5 we deduce that $R$ is a bounded closed subset of $X$. Noticing that $R \subset N \setminus E$, by the positive invariance of $\mathcal{R}$ and the definition of $\tilde{G}$ one can easily see that $R$ is positively invariant under the semiflow $G$. It follows by Proposition 2.7 that $\omega(x)$ is a nonempty compact invariant set contained in $R$ for any $x \in R$. This contradicts (4.2) and completes the proof of (4.3).

Let $L$ be the bounded closed set in the theorem. We infer from Remark 3.5 that $L$ is closed in $X^*$. Thereby $\mathcal{W} \setminus L$ is (relatively) open in $\mathcal{W}$. Consequently $[\mathcal{W} \setminus L]$ is an open subset of $N/E$. On the other hand, it is easy to verify that

$$[L] = (N/E) \setminus [\mathcal{W} \setminus L].$$

Thus we find that $[L]$ is closed in $N/E$. It is clear that $[E] \not\in [L]$.
Let \( \varrho(\cdot, \cdot) \) be a complete metric on \( \mathbb{N}/E \). By virtue of Proposition 2.16 the global attractor \([E]\) has a Lyapunov function \( \phi \) on \( \mathbb{N}/E \) with
\[
\phi(u) \geq \varrho(u, [E]), \quad \forall u \in \mathbb{N}/E.
\] (4.4)

Pick a number \( \delta \) with
\[
0 < \delta < \varrho([E], [L]) := \inf_{u \in [L]} \varrho([E], u).
\]

Then (4.4) implies that
\[
\phi^\delta \cap [L] = \emptyset,
\] (4.5)

where \( \phi^\delta \) is the \( \delta \)-level set of \( \phi \).

Define a function \( \tau_u \) on \( \mathbb{N}/E \) as
\[
\tau_u = \inf \{ t : \tilde{G}(t)u \in \phi^\delta \}, \quad u \in \mathbb{N}/E.
\]

For each \( u \in (\mathbb{N}/E)\setminus\phi^\delta \), \( \tau_u \) is precisely the unique moment such that \( \phi \left( \tilde{G}(\tau_u)u \right) = \delta \). The global attraction of \([E]\) implies that \( \tau_u < \infty \) for all \( u \in \mathbb{N}/E \). It is also clear that \( \tau_u = 0 \) for all \( u \in \phi^\delta \). By a standard argument as in the proof of Theorem 3.6 in [14], it can be shown that \( \tau_u \) is continuous in \( u \).

\( \Pi^{-1}(\phi^\delta) \) is closed in \( \mathbb{W} \). Hence it is also closed in \( X^* \). Let \( F = \Pi^{-1}(\phi^\delta) \setminus \{*\} \). Then \( F \) is a closed neighborhood of \( E \) in \( W := \mathbb{N} \cup E \); see Figure 4.1.

![Figure 4.1: F is a neighborhood of E and W \setminus F is bounded](image)

Define
\[
t_x = \inf \{ t : G(t)x \in F \}, \quad x \in W.
\]

By the definitions of \( \tilde{G} \) and \( F \) we find that \( t_x = \tau_{[x]} \) for all \( x \in W \). Hence
\[
t_x < \infty, \quad \forall x \in W.
\] (4.6)
Moreover, \( t_x \) is continuous in \( x \).

Now we define a global semiflow \( \Phi \) on \( W \) as follows: if \( x \in F \), then

\[
\Phi(t)x \equiv x, \quad t \geq 0;
\]

and if \( x \in W \setminus F \), we define

\[
\Phi(t)x = \begin{cases} 
G(t)x, & t < t_x; \\
G(t_x)x, & t \geq t_x.
\end{cases}
\]

Since both \( G \) and \( t_x \) are continuous, it can be easily seen that \( \Phi \) is continuous in \((t, x)\) on \( \mathbb{R}^+ \times W \).

Let \( \tilde{\Phi} = \pi \circ \Phi \) be the quotient flow induced by \( \Phi \) on \( N/F \), where \( \pi : W \to N/F \) is the quotient map. Then \( \tilde{\Phi} \) is continuous. We claim that for any sequences \( u_n \in \tilde{\Phi} \) and \( t_n \to +\infty \), the sequence \( \tilde{\Phi}(t_n)u_n \) is precompact. To see this, we may assume that there exists \( n_0 \) such that \( \tilde{\Phi}(t_n)u_n \neq [F] \) for all \( n > n_0 \) (otherwise we are done). Then for each \( n > n_0 \), there exists \( x_n \in W \setminus F \) such that \( u_n = [x_n] \), and \( \Phi(t_n)x_n \in W \setminus F \). Further by the definition of \( \Phi \), we see that \( G([0, t_n])x_n \subset W \setminus F \).

We claim that \( W \setminus F \) is bounded in \( X \). Indeed, since \( \phi^\delta \) is a closed neighborhood of \([E]\) in \( N/E \), the set \( \mathbb{F} := F \cup \{\ast\} = \Pi^{-1}(\phi^\delta) \) is a closed neighborhood of \( \ast \) in \( \mathbb{W} \). Choose a closed neighborhood \( \mathbb{V} \) of \( \ast \) in \( X^* \) so that \( \mathbb{F} = \mathbb{V} \cap \mathbb{W} \). Then

\[
W \setminus F = \mathbb{W} \setminus \mathbb{F} = \mathbb{W} \setminus \mathbb{V} \subset X^* \setminus \mathbb{V},
\]

and the boundedness of \( W \setminus F \) in \( X \) immediately follows from the definition of the topology of \( X^* \).

Now by the asymptotic compactness of \( G \), we deduce that \( G(t_n)x_n \) (and hence \( \Phi(t_n)x_n \)) has a subsequence converging in \( X \). Consequently \( \tilde{\Phi}(t_n)u_n \) has a subsequence that converges in \( N/F \).

Set \( A = w(N/F) \). By the compactness property of \( \tilde{\Phi} \), it can be shown that \( A \) is a nonempty compact invariant set of \( \tilde{\Phi} \), which is actually the global attractor of \( \tilde{\Phi} \) in \( N/F \). We prove that \( A \neq \{[F]\} \).

Let \( B = W \setminus F \). We deduce by the hypotheses of the theorem that there exists a closed set \( Q \subset W \) such that \( L \) and \( Q \) link with respect to the family of maps \( \Gamma \) in \( (4.1) \) for some \( S \subset Q \) with \( S \cap B \subset E \). Note that

\[
S = (S \cap B) \cup (S \cap F) \subset E \cup F = F.
\]

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(See Figure 4.1.) For each $t \geq 0$ fixed, since $\Phi(t)|_F = \text{id}_F$, we have $\Phi(t) \in \Gamma$. Hence

$$L \cap \Phi(t)Q \neq \emptyset, \quad \forall t \geq 0. \quad (4.7)$$

It follows that $[L] \cap \tilde{\Phi}(t)[Q] \neq \emptyset$ for all $t \geq 0$, where $[L] = \pi(L)$, and $[Q] = \pi(Q)$. Since $L$ is closed and $L \cap F = \emptyset$ (see (4.5)), $[L]$ is a closed subset of $N/F$. Thus we deduce that $\omega([Q]) \cap [L] \neq \emptyset$. As a result, $A \cap [L] \neq \emptyset$. On the other hand, $L \cap F = \emptyset$ implies that $[F] \notin [L]$. Hence $A \neq \{[F]\}$.

In what follows we show that $[F]$ is also the global attractor of $\tilde{\Phi}$, thus obtain a contradiction and complete the proof of the theorem.

First, we infer from the definition of $\Phi$ that $\tilde{\Phi}(t_n)[x] = [F]$ for all $x \in W$. (4.6) then implies that $[F]$ attracts each point $u \in N/F$. Now we verify that $[F]$ is stable. Suppose the contrary. Then there exits an open neighborhood $\tilde{U}$ of $[F]$ and a sequence $u_n \in N/F$ with $u_n \rightarrow [F]$ such that for each $n$, there exists $t_n > 0$ such that $\tilde{\Phi}(t_n)u_n \notin \tilde{U}$. We claim that $t_n \rightarrow \infty$. Otherwise, $t_n$ would have a bounded subsequence $t_{n_k}$. It can be assumed that $t_{n_k} \rightarrow T$. Then

$$[F] = \tilde{\Phi}(T)[F] = \lim_{k \rightarrow \infty} \tilde{\Phi}(t_{n_k})u_{n_k} \notin \tilde{U}.$$  

This leads to a contradiction and proves our claim. Let $u_n = [x_n]$, where $x_n \in B := W \setminus F$. Then $\Phi(t_n)x_n \in B$. Consequently $\Phi(t)x_n \in B$ for all $t \in [0,t_n]$. Further by the definition of $\Phi$, we see that

$$G(t)x_n \in B, \quad \forall t \in [0,t_n].$$

Therefore by Lemma 2.3 and the boundedness of $B$, $G$ has a full solution $\gamma$ contained in the closure $\overline{B}$ of $B$ in $X$. Noticing that both $\omega(\gamma)$ and $\alpha(\gamma)$ are nonempty compact invariant sets in $H$, by (4.2) we immediately obtain a contradiction. Hence $[F]$ is stable. Thanks to Proposition 2.14, we finally conclude that $[F]$ is the global attractor of $\tilde{\Phi}$. □

### 4.2 A linking theorem under the admissibility condition

In this subsection we give a linking theorem for local semiflows under the conditions of admissibility and nonexplosion.

**Definition 4.5** \[21\] (Admissibility) A subset $A$ of $X$ is said to be admissible (with respect to $G$), if for any sequences $x_n \in A$ and $t_n \rightarrow \infty$ with $G([0,t_n])x_n \subset A$, the sequence of end points $G(t_n)x_n$ has a convergent subsequence.
Remark 4.6 One can easily verify that if $A \subset X$ is admissible, then $\mathcal{C}(A)$ is compact.

It is worth noticing that each bounded subset $B$ of $X$ is admissible (owing to the asymptotic compactness of $G$).

Definition 4.7 We say that $G$ does not explode in $A$, if $T_x = \infty$ whenever $x \in A$ and $G([0,T_x])x \subset A$.

Now assume $(N,E)$ is a Ważewski pair. Let $H := N \setminus E$ be the closure of $N \setminus E$ in $X$. Set $W = N \cup E$.

Theorem 4.8 Assume that $H$ is admissible, and that $G$ does not explode in $H$. Suppose that there exist nonempty closed sets $L \subset N$ and $Q \subset W$ with $L \cap E = \emptyset$ such that $L$ and $Q$ link with respect to

$$\Gamma := \{ h \in C(Q,W) : h|_S = \text{id}_S \}$$

for some $S \subset Q \cap E$. Then $\mathcal{C}(H) \neq \emptyset$.

Proof. We only give a skeleton of the proof.

As in the proof of Theorem 4.2, we argue by contradiction and suppose that $\mathcal{C}(H) = \emptyset$. Let $\pi : W \to N/E$ be the quotient map. Rewrite $\pi(A)$ as $[A]$ for any subset $A$ of $W$. Consider the quotient flow $\tilde{G}$ on $N/E$ defined as follows:

$$\tilde{G}(t)[E] = \begin{cases} [G(t)x], & \text{if } t < t_H(x); \\ [E], & \text{if } t \geq t_H(x). \end{cases}$$

By a similar argument as in the proof of Lemma 3.7, it can be shown that $\tilde{G}$ is continuous. Furthermore, $[E]$ is an attractor of $\tilde{G}$. (The details of the argument are far less involved than those in the proof of Lemma 3.7)

Since $L \cap E = \emptyset$, $[E] \not\in [L]$. Let $\phi$ be a Lyapunov function of $[E]$. Take a positive number $\delta > 0$ sufficiently small so that $[L] \cap \phi^\delta = \emptyset$. Set $F = \pi^{-1}(\phi^\delta)$. Then by repeating the same argument as in the proof of Theorem 4.2 below with minor modifications, we can immediately obtain a contradiction. The only difference is that in the current situation, the sets $N \setminus F$ and $L$ may be unbounded, hence one needs to use a stronger compactness condition (namely the admissibility) to replace the asymptotic compactness of $G$. \qed
4.3 Mountain pass theorems

In this subsection we prove some mountain pass type results for local semiflows. They are actually particular cases of the linking theorems.

Let \((N, E)\) be a given Ważewski pair, and let \(H := N \setminus E\) be the closure of \(N \setminus E\) in \(X\).

**Theorem 4.9** Assume that \(G\) is stable in \(H\) at infinity, and that the following hypotheses are fulfilled:

1. \(G\) has a positively invariant bounded closed set \(K \subset N\) with \(K \cap E = \emptyset\).
2. There exists a connected component \(Q\) of \(N\) such that
   \[Q \cap K \neq \emptyset \neq Q \cap E.\] (4.8)

Then \(N \setminus (E \cup K)\) contains a nonempty compact invariant set.

**Proof.** Set \(F = E \cup K\). Then by the positive invariance of \(K\), \(F\) is an exit set of \(N\). Let \(W = N \cup E\). Since \(K\) is bounded and \(K \cap E = \emptyset\), we can find a bounded open neighborhood \(U\) of \(K\) such that \(U \cap E = \emptyset\). Define
\[L = \partial U \cap W, \quad S = Q \cap F.\]

We show that \(L\) and \(Q\) link with respect to the family of maps
\[\Gamma := \{h \in C(Q, W) : h|_S = \text{id}_S\}.\]

Let \(h \in \Gamma\). We need to verify
\[h(Q) \cap L \neq \emptyset.\] (4.9)

As \(h(Q) \subset W\), to prove (4.9) it suffices to check that \(h(Q) \cap \partial U \neq \emptyset\). We argue by contradiction and suppose that \(h(Q) \cap \partial U = \emptyset\). Then
\[h(Q) \subset U_0 \cup U_1,\] (4.10)

where \(U_0 = U \cap W\), and \(U_1 = W \setminus \overline{U}_0\). Note that both \(U_0\) and \(U_1\) are (relatively) open in \(W\). Furthermore, \(U_0 \cap U_1 = \emptyset\). On the other hand, by (4.8) and the choice of \(U\) we see that
\[h(Q) \cap U_0 \supset h(Q) \cap K \supset h(S) \cap K = S \cap K = Q \cap K \neq \emptyset,\] (4.11)
\[ h(Q) \cap U_1 \supset h(Q) \cap E \supset h(S) \cap E = S \cap E = Q \cap E \neq \emptyset. \quad (4.12) \]

Equations (4.10)-(4.12) contradict the connectedness of \( h(Q) \) (recall that \( Q \) is connected).

Now the conclusion of the theorem follows from Theorem 4.4. □

A particular but important case of Theorem 4.9 is the one when \( K \) is an attractor of \( G \) contained in \( N \). Specifically, we have

**Theorem 4.10** Assume that \( G \) is stable in \( H \) at infinity, and that the following hypotheses are fulfilled:

1. \( G \) has an attractor \( A \subset N \) with \( A \cap E = \emptyset \).

2. There exists a connected component \( Q \) of \( N \) such that \( Q \cap A \neq \emptyset \neq Q \cap E \).

Then \( H \) contains a nonempty compact invariant set \( M \) with \( M \cap A = \emptyset \).

**Proof.** Let \( \phi \) be a Lyapunov function of \( A \) defined on the region of attraction \( \Omega(A) \) of \( A \). Take a \( \delta > 0 \) suitably small so that \( \phi^\delta \) is bounded with \( \phi^\delta \cap E = \emptyset \). Then \( K := \phi^\delta \cap N \) is a bounded subset of \( N \) that is positively invariant. Thanks to Theorem 4.9, we conclude that \( N \setminus (E \cup K) \) contains a nonempty compact invariant set \( M \). Clearly \( M \cap A = \emptyset \). □

## 5 Linking theorems of Semiflows with Lyapunov Functions

In this section we consider semiflows with Lyapunov functions. One will see that we can establish some results fully analogous to those for variational functionals.

### 5.1 Linking theorems and mountain pass theorems

Let \( X \) and \( G \) be as in Section 4.

A function \( \phi \in C(X) \) is said to be a Lyapunov function of \( G \) on \( X \), if \( \phi(G(t)x) \) is nonincreasing for any \( x \in X \). Throughout this subsection, we always assume that \( G \) has a Lyapunov function \( \phi \). It is well known that for any bounded solution \( \gamma \) of \( G \) on \( (a, \infty) \) (resp. \( (-\infty, a) \)),

\[ \phi(x) \equiv \text{const.}, \quad x \in \omega(\gamma) \text{ (resp. } \alpha(\gamma) \text{).} \]
Define the *LaSalle set* $\mathcal{E}$ of $G$ (with respect to $\phi$) as

$$
\mathcal{E} = \bigcup \{ \text{orb}(\gamma) : \gamma \text{ is a full solution with } \phi(\gamma(t)) \equiv \text{const.} \}.
$$

In general, the set that consists of orbits of all bounded full solutions of a system may be very large and complicated. But the LaSalle set can be small and simple, as in the case of a gradient system whose LaSalle set consists of precisely the equilibrium points of the system.

**Theorem 5.1** Assume that for any $a > 0$, $G$ is stable in $\phi_a := \{-a \leq \phi(x) \leq a\}$ at infinity. Suppose that there exist closed subsets $L$ and $Q$ of $X$ as well as a closed subset $S$ of $Q$ such that

(A1) $L$ and $Q$ link with respect to

$$
\Gamma := \{ h \in C(Q, X) : h|_S = \text{id}_S \}; \quad (5.1)
$$

(A2) $L \cap \phi^a$ is bounded for any $a > 0$, and

$$
\beta := \inf_{x \in L} \phi(x) > \sup_{x \in S} \phi(x) := \alpha.
$$

Define a number $c$ as

$$
c = \inf_{h \in \Gamma} \sup_{x \in h(Q)} \phi(x). \quad (5.2)
$$

Then the following assertions hold:

(1) $\beta \leq c \leq \infty$.

(2) If $c < \infty$, then $\phi_{\beta-\varepsilon}$ contains a nonempty compact invariant set $K \subset \mathcal{E}$ for any $\varepsilon \in (0, \beta - \alpha)$.

**Proof.** (1) is almost obvious, so we only consider the second assertion (2).

Let $\varepsilon \in (0, \beta - \alpha)$ be given arbitrary. Set

$$
N = \phi^{c+\varepsilon}, \quad E = \phi^{\beta-\varepsilon}.
$$

Then $S \subset E$. Note that $G$ is stable in $\bar{N} \setminus E = \phi^{c+\varepsilon}_{\beta-\varepsilon}$ at infinity. By the definition of $\beta$, it is also clear that $L \cap E = \emptyset$. 
We infer from the definition of the number $c$ that there exists $h_0 \in \Gamma$ such that $h_0(Q) \subset N$. Let $L' = L \cap N$, and $Q' = h_0(Q)$. Then by (A2) it can be seen that $L'$ is bounded. We claim that $L'$ and $Q'$ link with respect to

$$\Gamma' := \{h \in C(Q', N) : h|_S = \text{id}_S\}.$$  

(5.3)

Indeed, if $h \in \Gamma'$ then

$$h(Q') = h(h_0(Q)) = (h \circ h_0)(Q).$$

Noticing that $h \circ h_0 \in C(Q, N)$ and $h \circ h_0|_S = \text{id}_S$, by (A1) one deduces that

$$L' \cap h(Q') = L \cap (h \circ h_0)(Q) \neq \emptyset.$$  

This justifies our claim.

Theorem 4.2 asserts that $\phi^{c+\varepsilon}_\alpha$ contains a nonempty compact invariant set $M$. Take an $x \in M$. Then $K := \omega(x)$ is a compact invariant set with $K \subset E$. □

The following theorem is fully analogous to the Linking Theorem for variational functionals.

**Theorem 5.2** Assume that $\phi^a_\alpha$ is admissible for any $a > 0$. Let $L$ and $Q$ be two closed subsets of $X$. Suppose that there exists a subset $S$ of $Q$ such that $L$ and $Q$ link with respect to the family of maps $\Gamma$ in (5.1), and that

$$\beta := \inf_{x \in L} \phi(x) > \sup_{x \in S} \phi(x) := \alpha.$$  

Define a number $c$ as in (5.2). Then the following assertions hold true:

(1) $\beta \leq c \leq \infty$.

(2) If $c < \infty$, then $\phi^c_\varepsilon$ contains a nonempty compact invariant set $K \subset E$.

**Proof.** We only prove the second assertion (2).

Suppose $c < \infty$. Fix an $\varepsilon_0 > 0$ with $c - \varepsilon_0 > \alpha$. Since $H_0 := \phi^{c+\varepsilon_0}_c$ is admissible, we infer from Remark 4.6 that $C(H_0)$ is compact. Therefore to prove (2), it suffices to show that $\phi^{c+\varepsilon}_c$ contains a nonempty compact invariant set $K_\varepsilon \subset E$ for any $0 < \varepsilon < \varepsilon_0$.

Let $N = \phi^{c+\varepsilon}_c$, and $E = \phi^{c-\varepsilon}_c$. Then both $N$ and $E$ are positively invariant. Set $L' = \phi^{\infty}_{c-\varepsilon}$. Clearly $L' \cap E = \emptyset$. By the definition of the number $c$, there
exists \( h_0 \in \Gamma \) such that \( h_0(Q) \subset N \). Let \( Q' = h_0(Q) \). We claim that \( L' \) and \( Q' \) link with respect to the family of maps \( \Gamma' \) defined as in (5.3). Indeed, if \( h \in \Gamma' \), then \( h(Q') = h \circ h_0(Q) \). Noticing that \( h \circ h_0 \in \Gamma \), we have

\[
\sup_{x \in h(Q')} \phi(x) = \sup_{x \in h \circ h_0(Q)} \phi(x) \geq c > c - \varepsilon/2.
\]

Hence \( L' \cap h(Q') \neq \emptyset \), which proves our claim.

Thanks to Theorem 4.8, one immediately concludes that \( \phi_{c+\varepsilon}^\Gamma \) contains a nonempty compact invariant set \( M_\varepsilon \). Take an \( x \in M_\varepsilon \). Then \( K_\varepsilon := \omega(x) \subset \mathcal{E} \). □

The following mountain pass theorems are direct consequences of Theorems 5.1 and 5.2.

**Theorem 5.3** Assume that for any \( a > 0 \), \( G \) is stable in \( \phi_a^{-a} \) at infinity. Let \( \Omega \) be a bounded open subset of \( X \), and \( Q \) be a connected closed subset of \( X \). Suppose that there exist \( x_0, x_1 \in Q \) with \( x_0 \in \Omega \) and \( x_1 \not\in \overline{\Omega} \) such that

\[
\phi(x_0), \phi(x_1) < \inf_{x \in \partial \Omega} \phi(x) := \beta. \tag{5.4}
\]

Define a number \( c \) as follows:

\[
c = \inf_{h \in \Gamma} \sup_{x \in h(Q)} \phi(x), \tag{5.5}
\]

where

\[
\Gamma = \{ h \in C(Q, X) : h(x_i) = x_i, \ i = 0, 1 \}. \tag{5.6}
\]

Then \( \beta \leq c \leq \infty \). If \( c < \infty \), \( \phi_c^{c+\varepsilon} \) contains a nonempty compact invariant set \( K \subset \mathcal{E} \) for any \( \varepsilon > 0 \).

**Proof.** By the connectedness of \( Q \) and (5.4), one can easily verify that \( \partial \Omega \) and \( Q \) link with respect to \( \Gamma \), and the conclusions follow immediately from Theorem 5.1 □

**Theorem 5.4** Assume \( \phi_a^{-a} \) is admissible for any \( a > 0 \). Let \( \Omega \) be a bounded open subset of \( X \), and \( Q \) be a connected closed subset of \( X \). Suppose that there exist \( x_0, x_1 \in Q \) with \( x_0 \in \Omega \) and \( x_1 \not\in \overline{\Omega} \) such that (5.4) holds. Define a number \( c \) as in (5.5) with \( \Gamma \) being the same as in (5.6).

Then \( \beta \leq c \leq \infty \). If \( c < \infty \), \( \phi_c^c \) contains a nonempty compact invariant set \( K \subset \mathcal{E} \).

**Proof.** As in the proof of Theorem 5.3, we know that \( \partial \Omega \) and \( Q \) link with respect to \( \Gamma \). The conclusions then follow from Theorem 5.2 □
5.2 Some remarks on variational functionals

In this subsection we give some remarks that may help us to have a better understanding on the relationship between variational functionals and dynamical systems.

Let $X$ be a Banach space, and let $J \in C^1(X)$ be a given functional on $X$. Denote by $\mathcal{K}$ the set of all critical points of $J$.

A local semiflow $G$ on $X$ is said to be a descending flow of $J$, if $J$ is a Lyapunov function of $G$ on $X$ with

$$\mathcal{E} = \mathcal{K},$$

where $\mathcal{E}$ is the LaSalle set of $G$ with respect to $J$.

Suppose that $J$ has a descending flow $G$. Then as in Theorems 5.1-5.4 we can derive a nonempty compact invariant set $K \subset \mathcal{E}$ of $G$ by applying an appropriate linking theorem or mountain pass theorem of local semiflows. Further by (5.7) one concludes that $K \subset \mathcal{K}$, thus asserting the existence of critical points of $J$.

In applications, the descending flows of a functional can be obtained by different approaches. For instance, when dealing with variational problems of elliptic partial differential equations, one can use either parabolic flows or pseudo-gradient flows. In many cases parabolic flows are naturally asymptotically compact. Therefore if we employ the parabolic flow of an elliptic equation to study the variational problem of the equation, then instead of examining the P.S. Condition of the variational functional, we can try to verify the stability property at infinity of the flow between two level surfaces of the functional (see Section 7).

For general variational problems, parabolic flows may not be available. Fortunately pseudo-gradient flows can always be constructed. It is interesting to note that, the Linking Theorem and Mountain Pass Lemma of variational functionals can be derived by directly applying the results on semiflows presented in this work to the pseudo-gradient flows of the functionals. To see this, let us first recall the definition of a pseudo-gradient flow.

According to Struwe [24], there exists a vector field $V$ on $X$ which is locally Lipschitz on $X \setminus \mathcal{K}$, such that

(V1) $V(x) \leq 2 \min\{||J'(x)||, 1\}$, and

(V2) $\langle J'(x), V(x) \rangle > \min\{||J'(x)||, 1\}||J'(x)||$

for all $x \in X$, where $\langle \cdot, \cdot \rangle$ denotes the dual product between $X$ and its dual space.
\[ x_t = -V(x), \quad x(0) = x_0. \]  \hfill (5.8)

The solutions of (5.8) define a continuous global semiflow \( G \) on \( X \). \( G \) is called the \textit{pseudo-gradient flow} of \( J \) on \( X \).

Let \( x = x(t) \) be the solution of (5.8). It is trivial to verify that
\[ \frac{d}{dt} J(x) = -\langle J'(x), V(x) \rangle \leq - \min\{||J'(x)||, 1\} ||J'(x)||. \]  \hfill (5.9)

A sequence \( x_n \in X \) is called a \textit{P.S. sequence}, if \( |J(x_n)| \leq C < \infty \) for all \( n \), and \( J'(x_n) \to 0 \) as \( n \to \infty \). We say that \( J \) satisfies the \textit{P.S. Condition}, if any P.S. sequence has a convergent subsequence in \( X \).

Now as an example, let us give a dynamical proof for the Linking Theorem of variational functionals.

**Theorem 5.5** Assume that \( J \) satisfies the P.C. Condition. Let \( L \) and \( Q \) be two closed subsets of \( X \). Suppose that there exists a subset \( S \) of \( Q \) such that
\begin{enumerate}
  \item \( L \) and \( Q \) link with respect to the family of maps \( \Gamma \) in (5.6);
  \item \( \beta := \inf_{x \in L} \phi(x) > \sup_{x \in S} \phi(x) := \alpha \).
\end{enumerate}

Define a number \( c \) as in (5.2).

Then \( \beta \leq c \leq \infty \). If \( c < \infty \), \( J_{\varepsilon}^c \) contains at least one critical point.

**Proof.** It is obvious that \( c \geq \beta \). Suppose \( c < \infty \). Fix an \( \varepsilon_0 > 0 \) with \( c - \varepsilon_0 > \alpha \). Then by the P.S. Condition, \( K \cap H_{\varepsilon_0} \) is compact. Here \( H_{\varepsilon} = J_{\varepsilon-c-\varepsilon}^c \) for any \( \varepsilon > 0 \). Thus to prove the theorem, it suffices to check that \( K \cap H_{\varepsilon} \neq \emptyset \) for any \( 0 < \varepsilon < \varepsilon_0 \).

We argue by contradiction and suppose that \( K \cap H_{\varepsilon} = \emptyset \) for some \( \varepsilon > 0 \). Then by the P.S. Condition one can easily verify that
\[ ||J'(x)|| \geq \delta > 0, \quad \forall x \in H_{\varepsilon}. \]  \hfill (5.10)

Let \( G \) be the pseudo-gradient flow of \( J \) defined as above. By virtue of (5.9) and (5.10), it can be seen that if a sequence \( x_n \in H_{\varepsilon} \) and \( t_n \geq 0 \) are such that \( G([0, t_n])x_n \subset H_{\varepsilon} \) for all \( n \), then the sequence \( t_n \) is necessarily bounded. Consequently \( H_{\varepsilon} \) is admissible for the pseudo-gradient flow \( G \). Repeating the same argument as in the proof of Theorem 5.2 with \( \phi \) therein replaced by \( J \) and applying Theorem 4.8, we immediately deduce that \( H_{\varepsilon} \) contains at least one critical point of \( J \). This leads to a contradiction and completes the proof. \( \square \)
Remark 5.6 If we drop the P.S. Condition from the theorem, then by slightly modifying the above argument, one can prove the existence of a P.S. sequence $x_n$ with $J(x_n) \to c$ as $n \to \infty$.

6 A Resonant Problem: Existence of Recurrent Solutions

As an application of linking theorems of dynamical systems, in this section we consider the resonant problem:

$$
\begin{cases}
    u_t - \Delta u - \mu u = f(u) + g(x,t), & x \in \Omega; \\
    u(x,t) = 0, & x \in \partial \Omega,
\end{cases}
$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$, $\mu$ is an eigenvalue of the Laplace operator $A = -\Delta$ associated with the homogenous Dirichlet boundary condition. Our main goal is to prove the existence of recurrent solutions under the following Landesman-Laser type conditions:

(F1) $f$ is a $C^1$ bounded function on $\mathbb{R}^1$, and

$$
\lim_{s \to +\infty} \inf f(s) := \overline{f} > 0, \quad \lim_{s \to -\infty} \sup f(s) := \underline{f} < 0.
$$

(G1) $g \in C(\overline{\Omega} \times \mathbb{R}^1)$, and

$$
-\overline{f} < \inf_{\Omega \times \mathbb{R}^1} g(x,t) \leq \max_{\Omega \times \mathbb{R}^1} g(x,t) < \underline{f}.
$$

6.1 Mathematical setting and the main result

Let $H = L^2(\Omega)$, and $V = H_0^1(\Omega)$. Denote by $(\cdot, \cdot)$ and $|\cdot|$ the usual inner product and norm on $H$, respectively. The norm $||\cdot||$ on $V$ is defined as

$$
||u|| = \left( \int_{\Omega} |\nabla u|^2 dx \right)^{1/2}, \quad u \in V.
$$

For notational simplicity, in this section we use $B_H(R)$ and $B_V(R)$ to denote the balls in $H$ and $V$ with radius $R$ centered at $0$, respectively.

Denote by $A$ the operator $-\Delta$ associated with the homogenous Dirichlet boundary condition. Let $L = A - \mu$. Then the space $H$ can be decomposed
into the orthogonal direct sum of its subspaces $H^-, H^0$ and $H^+$ corresponding to the negative, zero and positive eigenvalues of $L$, respectively. It is well known that $H^-$ and $H^0$ are of finite-dimensional.

Let

$$V^\sigma = V \cap H^\sigma, \quad \sigma \in \{0, +, -\} := \mathcal{I}.$$  

Then by the finite dimensionality of $H^0$ and $H^-$, we know that $V^-$ and $V^0$ coincide with $H^-$ and $H^0$, respectively. It also holds that

$$V = V^- \oplus V^0 \oplus V^+.$$  

Denote by $P^\sigma$ ($\sigma \in \mathcal{I}$) the projection operator from $V$ to $V^\sigma$.

The problem (6.1) can be rewritten as an abstract evolution equation in $V$:

$$u_t + Lu = f(u) + g(t), \quad (6.2)$$

where $g(t) = g(\cdot, t)$. Our main result in this section is the following theorem.

**Theorem 6.1** Suppose that the conditions $(F1)$ and $(G1)$ are satisfied. Then if $g$ is a recurrent function in $\mathcal{C} = C(\mathbb{R}^1; C(\Omega))$, the equation (6.2) has at least one recurrent solution $u \in C(\mathbb{R}^1; V)$.

Before proving Theorem 6.1 let us first do some auxiliary work.

### 6.2 Positively invariant sets

In this section we discuss positive invariance property of the level sets of the functional $J$ on $V$ defined by

$$J(u) = \frac{1}{2} (||u||^2 - \mu |u|^2) - \int_\Omega F(u)dx, \quad u \in V,$$

where $F(s) = \int_0^s f(t)dt$. More precisely, we will show that $J^c$ is positively invariant with respect to the system (6.6)-(6.7) provided $c > 0$ is sufficiently large.

**Lemma 6.2** Let $u^+ = P^+ u$. Then

$$J(u) \to +\infty \implies ||u^+|| \to \infty.$$
Proof. One easily verifies by (F1) that there exists $c_0 > 0$ such that
\[ F(s) \geq \kappa |s| - c_0, \quad \forall \ s \in \mathbb{R}^1, \tag{6.3} \]
where $\kappa = \frac{1}{2} \min \{f, \overline{f}\}$. Consequently $\int_{\Omega} F(u) dx$ is bounded from below on $V$. Now assume that $J(u) \to +\infty$. Then by the definition of $J$, we necessarily have $||u||^2 - \mu |u|^2 \to \infty$. On the other hand, simple computations show that
\[ ||u||^2 - \mu |u|^2 = (||u^+||^2 - \mu |u^+|^2) + (||u^-||^2 - \mu |u^-|^2), \tag{6.4} \]
where $u^- = P^- u$. Since the largest eigenvalue of $A$ on $V^-$ is less than $\mu$, we deduce that $||u^-||^2 - \mu |u^-|^2 \leq 0$ for all $u \in V$. Hence by (6.4) we see that $||u^+||^2 - \mu |u^+|^2 \to \infty$. This completes the proof of the lemma. □

Lemma 6.3 The following assertions hold:

1. $J(u) \to \infty$ as $u \in V^+$ and $||u|| \to \infty$.
2. $J(u) \to -\infty$ as $u \in W := V^- \bigoplus V^0$ and $||u|| \to \infty$.

Proof. See Chang [6].

Lemma 6.4 $|Lu| \to \infty$ as $u \in D(A)$ and $||u^+|| \to \infty$.

Proof. Simple computations show that
\[ |Lu|^2 = |Lu^+|^2 + |Lu^-|^2 \geq |Lu^+|^2. \]
We observe that
\[ |Lu^+|^2 = |Au^+|^2 - 2\mu (Au^+, u^+) + \mu^2 |u^+|^2 \geq |Au^+|^2 - 2\mu |Au^+||u^+| + \mu^2 |u^+|^2 = |Au^+|^2 (1 - 2\lambda + \lambda^2). \]
where $\lambda = |u^+|/|Au^+|$. Denote by $\mu^+$ the smallest eigenvalue of $A$ restricted on $H^+$. Then $\mu^+ > \mu$, and hence
\[ \lambda = \frac{\mu |u^+|}{|Au^+|} \leq \frac{\mu}{\mu^+} < 1. \tag{6.5} \]
Since $1 - 2s + s^2 = 0$ if and only if $s = 1$, there exists $\delta > 0$ such that $1 - 2s + s^2 \geq \delta$ for all $s \in \mathbb{R}$ with $|s| \geq \mu/\mu^+$. It then follows from (6.5) that $1 - 2\lambda + \lambda^2 \geq \delta$ for all $u \in D(A)$. Therefore

$$|Lu|^2 \geq |Lu^+|^2 \geq \delta|Au^+|^2 \geq \delta \mu^+ ||u^+||^2, \quad \forall u \in D(A),$$

from which we immediately conclude that $|Lu| \to \infty$ as $||u^+|| \to \infty$. □

Let $\Sigma = \mathcal{H}_\phi(g)$ be the hull of $g$ in the space $\mathcal{C} = C(\mathbb{R}, C(\Omega))$ equipped with the compact-open metric $\varrho$. Consider the initial value problem:

$$u_t + Lu = f(u) + p(t), \quad \text{ (6.6)}$$
$$u(0) = u_0, \quad \text{ (6.7)}$$

where $p \in \Sigma$, and $u_0 \in V$. By the basic theory on evolution equations, we know that the problem has a unique global solution $u = u(t; p, u_0)$ with

$$u \in C([0, \infty); V) \cap C^1((0, \infty), H),$$
$$u(t) \in D(A), \quad \forall t > 0.$$

By a very standard argument it can be easily shown that $u(t; p, u_0)$ is continuous in $(t, p, u_0)$ as a map from $\mathbb{R}^+ \times \Sigma \times V$ to $V$.

**Lemma 6.5** There exists $c_1 > 0$ (independent of $p$ and $u_0$) such that $J(u(t))$ is decreasing in $t$ for any solution $u(t) := u(t; p, u_0)$ of (6.6)-(6.7) in $J_\infty^c$.

**Remark 6.6** Lemma 6.5 implies that if $c > c_1$, then $J^c$ is positively invariant with respect to the system (6.6)-(6.7). Specifically, $u(t; p, u_0) \in J^c$ for all $t \geq 0$ and $p \in \Sigma$ whenever $u_0 \in J^c$.

**Proof of Lemma 6.5.** We infer from (G1) that

$$|p(t)| = \left( \int_\Omega p^2(x, t)dx \right)^{1/2} \leq M|\Omega|^{1/2}, \quad \forall t \in \mathbb{R}, \ p \in \Sigma,$$

where $M = \sup_{s \in \mathbb{R}} |f(s)|$, and $|\Omega|$ denotes the measure of $\Omega$. By virtue of Lemmas 6.2 and 6.4, it is easy to deduce that there exists $c_1 > 0$ such that

$$|Lv| > 3M|\Omega|^{1/2}, \quad \forall v \in D(A) \cap J_\infty^{c_1}, \quad \text{ (6.8)}$$
Let \( u(t) := u(t; p, u_0) \) be a solution of the system (6.6)-(6.7) with \( u(t) \in J_{c_1}^\infty \) for \( t \in [0, T) \). We show that

\[
\frac{d}{dt} J(u(t)) < 0, \quad \forall t \in (0, T),
\]

thus proving the lemma.

Taking the inner product of the equation (6.6) in \( H \) with \( Lu - f(u) \), it yields

\[
\frac{d}{dt} J(u) = -|Lu - f(u)|^2 + (p, Lu - f(u)) \leq -\frac{1}{2}|Lu - f(u)|^2 + \frac{1}{2}|p|^2.
\]

By (6.8) we have

\[
|Lu - f(u)| \geq |Lu| - |f(u)| \geq |Lu| - M|\Omega|^{1/2} > 2M|\Omega|^{1/2}.
\]

Hence

\[
\frac{d}{dt} J(u) \leq -\frac{1}{2}|Lu - f(u)|^2 + \frac{1}{2}|p|^2 < -\frac{3}{2}M|\Omega|^{1/2}
\]

for \( t \in (0, T) \). This justifies (6.9). \( \square \)

### 6.3 The stability property of the problem at infinity

Now we discuss stability property of the system (6.6)-(6.7) at infinity. Given a function \( w \) on \( \Omega \), we denote by \( w_\pm \) the positive and negative parts of \( w \), respectively,

\[
w_\pm(x) = \max(0, \pm w(x)), \quad x \in \Omega.
\]

Then \( w = w_+ - w_- \).

We first give a simple result concerning the nonlinear term.

**Lemma 6.7** Suppose \( f \) satisfies (F1). Then for any \( R, \varepsilon > 0 \), there exists \( s_0 > 0 \) such that

\[
\int_\Omega f(u + sw) w \, dx \geq \int_\Omega (\overline{f}w_+ + \overline{f}w_-) \, dx - \varepsilon \quad (6.10)
\]

for all \( s \geq s_0 \), \( u \in \overline{B}_1(R) \) and \( w \in \overline{B}_1(1) \).

**Proof.** Let

\[
I = \int_\Omega f(u + sw) w \, dx - \int_\Omega (\overline{f}w_+ + \overline{f}w_-) \, dx.
\]
Since \( w = w_+ - w_- \), we can rewrite \( I \) as \( I = I_+ - I_- \), where
\[
I_+ = \int_{\Omega} \left( f(u + sw) - \overline{f} \right) w_+ dx, \quad I_- = \int_{\Omega} \left( f(u + sw) + f \right) w_- dx.
\]

In what follows, let us estimate \( I_+ \) for \( u \in \overline{B}_H(R) \) and \( w \in \overline{B}_H(1) \).

We observe that
\[
R^2 \geq \int_{\Omega} |u|^2 dx \geq \int_{\{|u| \geq \sigma\}} |u|^2 dx \geq \sigma^2 |\{|u| \geq \sigma\}|,
\]
from which it can be easily seen that \(|\{|u| \geq \sigma\}| \to 0 \text{ as } \sigma \to \infty \) uniformly with respect to \( u \in \overline{B}_H(R) \). Therefore one can pick a \( \sigma > 0 \) sufficiently large so that
\[
|\{|u| \geq \sigma\}|^{1/2} < \delta := \varepsilon/8||f||(|\Omega| + 1), \quad \forall u \in \overline{B}_H(R), \quad (6.11)
\]
where \(||f|| = \sup_{s \in \mathbb{R}} |f(s)|\).

For each \( u \in \overline{B}_H(R) \) and \( w \in \overline{B}_H(1) \), let
\[
D = D_{u,w} := \{|u| < \sigma\} \cap \{w_+ > \delta\}.
\]
Then \( \Omega = D \cup \{|u| \geq \sigma\} \cup \{w_+ \leq \delta\} \). Hence
\[
I_+ \geq \int_{D} \left( f(u + sw) - \overline{f} \right) w_+ dx - \int_{\{|u| \geq \sigma\}} |f(u + sw) - \overline{f}| w_+ dx
- \int_{\{w_+ \leq \delta\}} |f(u + sw) - \overline{f}| w_+ dx
\geq \int_{D} \left( f(u + sw) - \overline{f} \right) w_+ dx - 2||f|| \left( \int_{\{|u| \geq \sigma\}} w_+ dx + \int_{\{w_+ \leq \delta\}} w_+ dx \right).
\]

Note that
\[
\int_{\{|u| \geq \sigma\}} w_+ dx \leq \left( \int_{\{|u| \geq \sigma\}} w_+^2 dx \right)^{1/2} |\{|u| \geq \sigma\}|^{1/2}
\leq (\text{by (6.11)}) \leq |w| \delta \leq \delta.
\]
It is also obvious that
\[
\int_{\{w_+ \leq \delta\}} w_+ dx \leq |\Omega| \delta.
\]
Thereby
\[
I_+ \geq \int_{D} \left( f(u + sw) - \overline{f} \right) w_+ dx - 2||f||(|\Omega| + 1)\delta
\geq \int_{D} \left( f(u + sw_+) - \overline{f} \right) w_+ dx - \frac{\varepsilon}{4}. \quad (6.12)
\]
Since \( z + s\eta \to +\infty \) (as \( s \to +\infty \)) uniformly with respect to \( z \in [-\sigma, \sigma] \) and \( \eta \geq \delta \), there exists \( s_1 > 0 \) (depending only upon \( \sigma, \delta \) and \( f \)) such that if \( s > s_1 \),

\[
f(z + s\eta) - \overline{f} \geq -\frac{\varepsilon}{4|\Omega|^{1/2}}, \quad \forall z \in [-\sigma, \sigma], \; \eta \geq \delta.
\]

Now suppose that \( s > s_1 \). Then by the definition of \( D \), we have

\[
\int_D (f(u + sw) - \overline{f}) \, w_+ \, dx \geq -\frac{\varepsilon}{4|\Omega|^{1/2}} \int_D w_+ \, dx \geq -\frac{\varepsilon}{4} |D|^{1/2} (\int_D |w|^2 \, dx)^{1/2} \geq -\frac{\varepsilon}{4}.
\]

It then follows from (6.12) that

\[
I_+ \geq \int_D (f(u + sw) - \overline{f}) \, w_+ \, dx - \frac{\varepsilon}{4} > -\frac{\varepsilon}{2}.
\]

Similarly it can be shown that there exists \( s_2 > 0 \) (independent of \( u \) and \( w \)) such that \( I_- < \frac{\varepsilon}{2} \), provided \( s > s_2 \). Set \( s_0 = \max\{s_1, s_2\} \). Then if \( s > s_0 \),

\[
I \geq I_+ - I_- > -\frac{\varepsilon}{2} - \frac{\varepsilon}{2} = -\varepsilon
\]

for all \( u \in \overline{B}_H(R) \) and \( w \in \overline{B}_H(1) \). This completes the proof of the lemma. \( \Box \)

**Lemma 6.8** There exist \( \lambda, \rho_1 > 0 \) (independent of \( p \) and \( u_0 \)) such that for any solution \( u = u(t) \) of (6.6)-(6.7), we have

\[
||u^+(t)||^2 \leq ||u_0^+||^2 e^{-2\lambda t} + \rho_1^2 (1 - e^{-2\lambda t}), \quad \forall t \geq 0.
\]

**Proof.** Taking the inner product of the equation (6.6) with \( Au^+ \) in \( H \), it yields

\[
\frac{1}{2} \frac{d}{dt} ||u^+||^2 + |Au^+|^2 \leq \mu ||u^+||^2 + (Au^+, f(u) + p).
\]

Recalling that \( f \) and \( g \) are bounded, one finds that

\[
(Au^+, f(u) + p) \leq \varepsilon |Au^+|^2 + C_\varepsilon
\]

for any \( \varepsilon > 0 \), where \( C_\varepsilon \) is a positive constant depending only upon \( \varepsilon \) and the upper bounds of \( ||f|| \) and \( ||g|| \). Thus we have

\[
\frac{1}{2} \frac{d}{dt} ||u^+||^2 + (1 - \varepsilon)|Au^+|^2 \leq \mu ||u^+||^2 + C_\varepsilon. \tag{6.14}
\]
Note that $|Au^+|^2 \geq \mu^+||u^+||^2$, where $\mu^+$ the smallest eigenvalue of $A$ in $V^+$. Fix an $\varepsilon > 0$ sufficiently small so that $(1 - \varepsilon)\mu^+ > \mu$. Then by (6.14),

$$\frac{1}{2} \frac{d}{dt} ||u^+||^2 \leq -\lambda||u^+||^2 + C_\varepsilon,$$

where $\lambda = (1 - \varepsilon)\mu^+ - \mu > 0$. Now the conclusion follows immediately from the classical Gronwall Lemma. □

Lemma 6.9 Let $c_1$ and $\rho_1$ be the positive numbers given in Lemmas 6.5 and 6.8, respectively. Then for any $c > c_1$ and $\rho > \rho_1$, the set

$$N_{c,\rho} := \{v \in V : ||P^+v|| \leq \rho, J(v) \leq c\}$$

is positively invariant with respect to the system (6.6)-(6.7). Moreover, for any $R > 0$, there exists $R_1 > R$ such that for any $u_0 \in N_{c,\rho}$ with $||u_0|| > R_1$,

$$||u(t)|| > R, \quad \forall t \geq 0,$$

(6.15)

where $u(t) := u(t;p,u_0)$ is the solution of (6.6)-(6.7).

Proof. The positive invariance of $N_{c,\rho}$ follows from Remark 6.6 and Lemma 6.8.

For $v \in V$, we write $v^\pm = P^\pm v$, and $v^0 = P^0 v$. Then by the definition of $N_{c,\rho}$,

$$||u^+(t)|| \leq \rho, \quad \forall t \geq 0$$

(6.16)

for any solution $u = u(t) := u(t;p,u_0)$ of (6.6)-(6.7) with $u_0 \in N_{c,\rho}$. Thus to prove (6.15), it suffices to show that for any $R > 0$, there exists $R_1 > R$ such that if $||w_0|| > R_1$, then

$$||w(t)|| > R, \quad \forall t \geq 0,$$

(6.17)

where $w = u^- + u^0$, and $w_0 = u_0^- + u_0^0$.

We multiply the equation (6.6) with $w$ and integrate over $\Omega$ to obtain that

$$\frac{1}{2} \frac{d}{dt} |w|^2 + ||u^-||^2 = \mu ||u^-||^2 + (f(u) + p, w).$$

(6.18)

If dim$V^- \geq 1$, we denote by $\mu^-$ the largest eigenvalue of $A$ restricted on $V^-$. Then $||u^-||^2 \leq \mu^- ||u^-||^2$. Thereby (recall $\mu^- < \mu$)

$$\frac{1}{2} \frac{d}{dt} |w|^2 \geq (\mu - \mu^-)||u^-||^2 + (f(u) + p, w) \geq (f(u) + p, w).$$

(6.19)
Note that if \( \dim V^- = 0 \), then \( u^- = 0 \). By (6.18) we see that (6.19) readily holds.

Since the space \( W := V^- \oplus V^0 \) is finite dimensional, all the norms on \( W \) are equivalent. Thus we deduce that

\[
m := \min \{|v|_{L^1(\Omega)} : v \in W \cap \partial B_H(1)\} > 0.
\]

By (G1) there exists \( \delta > 0 \) such that

\[
\bar{f} + p(x,t) \geq \delta, \quad f - p(x,t) \geq \delta
\]

for all \( x \in \overline{\Omega} \) and \( t \in \mathbb{R}^1 \). Thanks to Lemma 6.7, there exists \( s_0 > 0 \) (depending only upon \( \rho \)) such that if \( s > s_0 \), then

\[
(f(h + sv), v) = \int_{\Omega} f(h + sv)v \, dx \geq \int_{\Omega} (\bar{f} v_+ + f v_-) \, dx - \frac{1}{2} m \delta
\]

for all \( h \in \overline{B}_H(\rho) \) and \( v \in \overline{B}_H(1) \). We rewrite \( w \) as \( w = sv \), where \( s = |w| \). Then \( v \in \partial B_H(1) \). By (6.16) and (6.21) one deduces that

\[
(f(u) + p, w) = s \left[ (f(u^+ + sv), v) + (pv, v) \right] \\
\geq s \left[ \left( \int_{\Omega} (\bar{f} v_+ + f v_-) \, dx - \frac{1}{2} m \delta \right) + \int_{\Omega} (pv_+ - pv_-) \, dx \right] \\
= s \left[ \int_{\Omega} \left( (\bar{f} + p)v_+ + (f - p)v_- \right) \, dx - \frac{1}{2} m \delta \right].
\]

Since

\[
\int_{\Omega} \left( (\bar{f} + p)v^+ + (f - p)v^- \right) \, dx - \frac{1}{2} m \delta \\
\geq \delta \int_{\Omega} |v| \, dx - \frac{1}{2} m \delta \geq \delta \int_{\Omega} |v| \, dx \geq \frac{1}{2} m \delta,
\]

we have

\[
(f(u) + p, w) \geq \frac{1}{2} m \delta s = \frac{1}{2} m \delta |w|.
\]

Combining this with (6.19), it yields that

\[
\frac{d}{dt} |w(t)|^2 \geq m \delta |w(t)|
\]

as long as \( |w(t)| > s_0 \). Recalling that \( || \cdot || \) and \( | \cdot | \) are equivalent norms on \( W \), we immediately confirm the validity of the conclusion in (6.17). □
6.4 The proof of the main result

Let \( \Sigma = \mathcal{H}_\theta (g) \) be the hull of \( g \) in the space \( \mathcal{C} = C(\mathbb{R}^1, C(\Omega)) \) (equipped with the compact-open metric \( \varrho \)), and \( \theta \) be the Bebutov's dynamical system on \( \Sigma \). Define a global semiflow \( G \) on \( X := \Sigma \times V \) as follows:

\[
G(t)(p, u) = (\theta_t p, u(t; p, u)), \quad \forall (p, u) \in X, \ t \geq 0,
\]

where \( u(t) = u(t; p, u) \) is the solution of (6.6)-(6.7) with \( u_0 = u \). \( G \) is usually called the skew-product flow of the system (6.6)-(6.7). We have

**Lemma 6.10** \( G \) is asymptotically compact on \( X \).

**Proof.** It suffices to check that for any sequences \( (p_k, v_k) \in X \) and \( t_k \to +\infty \), if there exists \( R > 0 \) such that

\[
||u_k(t)|| \leq R, \quad \forall t \in [0, t_k]
\]

for all \( k \), where \( u_k(t) = u(t; p_k, v_k) \), then the sequence \( u_k(t_k) \) has a convergent subsequence in \( V \). This can be done as follows.

First, \( u_k \) satisfies the equation

\[
\frac{d}{dt} u_k + Au_k = h_k(t), \quad t \in [0, t_k],
\]

where \( h_k = \mu u_k + f(u_k) + p_k \). Owing to (6.22) and the boundedness of \( f \) and \( g \), there exists \( C > 0 \) such that \( \max_{t \in [0, t_k]} |h_k(t)| \leq C \) for all \( k \geq 1 \). Further by utilizing some quite standard argument on parabolic equations (see e.g. [20, 26]), it can be easily shown that there exist \( t_0 > 0 \) and \( C' > 0 \) (depending only upon \( R \) and \( C \)) such that

\[
|Au_k(t)| \leq C', \quad \forall t \in [t_0, t_k]
\]

whenever \( t_k > t_0 \). The conclusion of the lemma then follows from the compactness of the embedding \( D(A) \hookrightarrow V \). □

We are now in a position to prove Theorem 6.1.

**Proof of Theorem 6.1** Let \( \beta = \inf_{u \in V^+} J(u) \). Then by Lemma 6.3 (1), there exits \( \rho > \rho_1 \) such that

\[
\beta = \inf \{ J(u) : u \in V^+, ||u|| \leq \rho \}.
\]
We also infer from Lemma 6.3 (2) that \( J(u) \) is bounded from above on \( W := V - \bigoplus V^0 \). Set

\[
c = c_1 + \beta + \sup_{u \in W} J(u),
\]

where \( c_1 \) is the positive number in Lemma 6.9. Define

\[
\mathcal{N} = \Sigma \times N_{c,\rho}, \quad \mathcal{L} = \Sigma \times L,
\]

where \( L = V^+ \cap N_{c,\rho} \). Then it follows from Lemma 6.9 that \( \mathcal{N} \) is positively invariant under the skew-product flow \( G \). By Lemma 6.3 (1) and the definition of \( c \), it is easy to see that \( \mathcal{L} \) is a bounded nonempty closed subset of \( \mathcal{N} \).

For any \( k > 0 \), let

\[
W_k = \{ v \in W : ||v|| \leq k \}.
\]

Then \( W_k \subset W \subset N_{c,\rho} \). By some classical results on linking (see e.g. Struwe [24]), we know that \( V^+ \) and \( W_k \) link with respect to \( \Gamma := \{ h \in C(W_k, N_{c,\rho}) : h|_{\partial W_k} = \text{id}_{\partial W_k} \} \), where \( \partial W_k := \{ v \in W : ||v|| = k \} \) is the boundary of \( W_k \) in \( W \). Since \( h(W_k) \subset N_{c,\rho} \) for each \( h \in \Gamma \), this amounts to say that \( L \) and \( W_k \) link with respect to \( \Gamma \).

Set \( Q = \Sigma \times W_k \), and \( S = \Sigma \times \partial W_k \). Clearly \( Q \subset \mathcal{N} \). We show that \( \mathcal{L} \) and \( Q \) link with respect to the family of maps

\[
\mathcal{T} := \{ h \in C(Q, \mathcal{N}) : h|_S = \text{id}_S \}.
\]

Let \( h \in \mathcal{T} \). We write

\[
h(p, u) = (h_1(p, u), h_2(p, u)), \quad \forall (p, u) \in Q,
\]

where \( h_1(p, u) \in \Sigma \), and \( h_2(p, u) \in V \). Fix a \( p \in \Sigma \). Then \( h_2(p, \cdot) \in \Gamma \). Hence

\[
h_2(p, W_k) \cap L \neq \emptyset.
\]

Thus there exists \( v \in W_k \) such that \( h_2(p, v) \in L \). Consequently

\[
h(p, v) = (h_1(p, v), h_2(p, v)) \in \Sigma \times L = \mathcal{L}.
\]

Therefore \( h(Q) \cap \mathcal{L} \neq \emptyset \).

Let \( \mathcal{E} = \emptyset \). Then \( (\mathcal{N}, \mathcal{E}) \) is a Ważewski pair. We infer from Lemma 6.9 that \( G \) is stable in \( \mathcal{N} = \overline{\mathcal{N}} \setminus \mathcal{E} \) at infinity. For any bounded closed subset \( \mathcal{B} \) of \( \mathcal{N} \),
we can pick a \( k > 0 \) sufficiently large so that \( P_V B \cap \partial W_k = \emptyset \), where \( P_V \) denotes the projection operator from \( \mathcal{X} \) to \( V \). Then
\[
S \cap B = (\Sigma \times \partial W_k) \cap B = \emptyset \subset \mathcal{E}.
\]

Since \( L \) and \( Q \) link with respect to \( \mathcal{X} \), we see that the hypothesis on linking in Theorem 4.2 is fulfilled. Hence one concludes that \( G \) has a nonempty compact invariant set \( K \) in \( \mathcal{N} \).

The remaining part of the argument is quite standard. We give the details for the reader’s convenience.

Take a \((p, w) \in K\). Let \( \gamma(t) = (\theta_t p, v(t)) \) be a full solution of the skew-product flow \( G \) in \( K \) with \( \gamma(0) = (p, w) \). Then one can easily verify that \( v = v(t) \) is a full solution of the equation (6.6). Consequently \( \theta_{\tau} v \) is a full solution of (6.6) with \( p \) therein replaced by \( \theta_{\tau} p \).

By a similar argument as in the verification of the asymptotic compactness of \( G \), it can be shown that \( v(t) \) is bounded in \( D(A) \). Further by the equation (6.6) we see that \( v_t \in L^\infty(\mathbb{R}^1, H) \). It then follows that \( v \) is equi-continuous in \( H \) on \( \mathbb{R}^1 \). On the other hand,
\[
||v(t + h) - v(t)||^2 = (A(v(t + h) - v(t)), v(t + h) - v(t)) \\
\leq (|Av(t + h)| + |Av(t)||v(t + h) - v(t)|).
\]

We thereby deduce that \( v \) is equi-continuous in \( V \) on \( \mathbb{R}^1 \). Since \( v \) takes values in a compact subset of \( V \), by the classical Arzela-Ascoli Theorem, the hull \( \mathcal{H}_{\mathcal{B}_1}(v) \) of \( v \) in \( \mathcal{B}_1 := C(\mathbb{R}^1, V) \) (equipped with the compact-open metric) is compact.

Denote by \( \mathcal{K} \) the closure of \( \{(\theta_{\tau} p, \theta_{\tau} v) : \tau \in \mathbb{R}^1\} \) in \( \Sigma \times \mathcal{H}_{\mathcal{B}_1}(v) \). Then \( \mathcal{K} \) is invariant under the system \( \Theta \) defined by
\[
\Theta_t(q, h) = (\theta_t q, \theta_t h), \quad \forall (q, h) \in \Sigma \times \mathcal{H}_{\mathcal{B}_1}(v).
\]

Invoking a recurrence theorem due to Birkhoff and Bebutov (see e.g. [23]), \( \mathcal{K} \) contains a nonempty compact minimal invariant set \( \mathcal{M} \). Let
\[
\Sigma_0 = \{ q \in \Sigma : \text{there exists } w \in \mathcal{H}_{\mathcal{B}_1}(v) \text{ such that } (q, w) \in \mathcal{M} \}.
\]

It is trivial to check that \( \Sigma_0 \) is a compact invariant subset of \( \Sigma \). Because \( \Sigma \) is minimal, we necessarily have \( \Sigma_0 = \Sigma \). It then follows by the definition of \( \Sigma_0 \) that \( (g, u) \in \mathcal{M} \) for some \( u \in \mathcal{H}_{\mathcal{B}_1}(v) \). By the minimality of \( \mathcal{M} \) one can easily verify
that $H_{\phi_1}(u)$ is minimal. Thanks to Lemma 2.18 we deduce that $u$ is a recurrent function.

We show that $u$ is a full solution of the problem (6.2), hence completes the proof of the theorem. Indeed, by the definition of $\mathcal{K}$ and the fact that $(g, u) \in \mathcal{M} \subset \mathcal{K}$, there exists a sequence $\tau_k \in \mathbb{R}^1$ such that

$$(p_k, v_k) := (\theta_{\tau_k} p, \theta_{\tau_k} v) \to (g, u) \text{ (in } \Sigma \times H_{\phi_1}(v)).$$

Since each $v_k$ solves the equation $v_t + L v = f(v) + p_k(t)$ on $\mathbb{R}^1$, passing to the limit one immediately concludes that $u$ is a full solution of (6.2). □

7 Positive Solutions of an Elliptic PDE on $\mathbb{R}^n$

As another example illustrating the application of our theoretical results, we consider the existence of positive solutions of the elliptic equation

$$-\Delta u + a(x)u = f(x, u) \quad (7.1)$$

on $\mathbb{R}^n$ ($n \geq 3$). Such problems are closely related to finding standing wave solutions of nonlinear Schrödinger equations, and have attracted much attention in the past decades. Our main purpose here is not to pursue hypotheses that are as weaker as possible to guarantee the existence of positive solutions for (7.1), but to demonstrate how the dynamical approach developed here can be used to study these problems via parabolic flows.

We assume that $a(x)$ and $f(x, s)$ are continuous functions, and moreover, that $f(x, s)$ is continuously differentiable in $s$ for each fixed $x \in \mathbb{R}^n$. We have

**Theorem 7.1** Suppose that $a$ and $f$ satisfy the following conditions:

(A1)$^\circ$ There exist $0 < a_0 < a_1 < \infty$ such that

$$a_0 \leq a(x) \leq a_1, \quad \forall x \in \mathbb{R}^n.$$

(F1)$^\circ$ There exist a positive number $\gamma < \min\left(\frac{2}{n-2}, 1\right)$ and a nonnegative function $b \in L^{p_\gamma}(\mathbb{R}^n)$, where $p_\gamma = \frac{2n}{2 - \gamma(n-2)}$, such that

$$|f_s'(x, s)| \leq b(x)|s|^\gamma, \quad \forall x \in \mathbb{R}^n, \ s \in \mathbb{R}^1.$$
There exists an open subset Ω of \(\mathbb{R}^n\) such that
\[
\lim_{s \to \pm \infty} \frac{f(x, s)}{s} = +\infty \tag{7.2}
\]
uniformly with respect to \(x \in \overline{\Omega}\).

There exists a positive number \(\mu > 2\) such that
\[
0 \leq \mu F(x, s) \leq f(x, s)s, \quad \forall x \in \mathbb{R}^n, \ s \in \mathbb{R}^1.
\]

Then the equation (7.1) has at least one nontrivial positive solution \(u\).

To prove Theorem 7.1, let us first make a discussion on the parabolic flow of the equation.

### 7.1 The stability of the parabolic flow at infinity

Let \(H = L^2(\mathbb{R}^n)\), and \(V = H^1(\mathbb{R}^n)\). Denote by \(|\cdot|\) the usual norm on \(H\), and define the norm \(||\cdot||\) on \(V\) as follows:
\[
||u|| = \left( \int_{\mathbb{R}^n} |\nabla u|^2 \, dx + \int_{\mathbb{R}^n} a(x)|u|^2 \, dx \right)^{1/2}, \quad \forall u \in V.
\]
It is well known that \(||\cdot||\) is equivalent to the usual one. We use \(|\cdot|_q\) to denote the norm of \(L^q(\mathbb{R}^n)\) \((q \geq 1)\).

Consider the parabolic equation:
\[
u_t - \Delta u + a(x)u = f(x, u), \quad x \in \mathbb{R}^n, \ t > 0. \tag{7.3}
\]

Define the Nemitski operator \(\tilde{f} : V \to H\) as follows: \(\forall u \in V\),
\[
\tilde{f}(u)(x) = f(x, u(x)), \quad x \in \mathbb{R}^n.
\]
By \((F1)\) and the Sobolev embedding \(V \hookrightarrow L^{2^*}(\mathbb{R}^n)\) \((where \(2^* = 2n/(n - 2)\)), one can easily verify that \(\tilde{f}\) is well defined. Moreover, \(\tilde{f}\) is locally Lipschitz continuous. The Cauchy problem of the equation can be reformulated as an abstract one:
\[
u_t + Lu = \tilde{f}(u), \quad u(0) = u_0, \tag{7.4}
\]
where \(Lu = \Delta u + a(x)u\). Thanks to the general theory on evolution equations in Banach spaces (see Henry [13]), (7.4) has a unique local solution \(u(t; u_0)\) that
exists on a maximal existence interval \([0, T_{u_0})\) for each \(u_0 \in V\). Moreover, \(u(t; u_0)\) is continuous in \((t, u_0)\). Set

\[
G(t)u_0 = u(t; u_0), \quad u_0 \in V, \ t \in [0, T_{u_0}).
\]

\(G\) is a local semiflow on \(V\), which is called the parabolic flow of (7.1).

**Lemma 7.2** \(G\) is asymptotically compact.

**Proof.** See Prizzi [18], Theorem 2.4. □

Note that \(G\) has a natural Lyapunov function \(J\) on \(V\) defined as follows:

\[
J(u) = \frac{1}{2}||u||^2 - \int_{\mathbb{R}^n} F(x, u)dx, \quad u \in V,
\]

where \(F(x, s) = \int_0^s f(x, \tau)d\tau\). We have

**Lemma 7.3** For any \(c > 0\), \(G\) is stable in \(J^- c\) at infinity.

**Proof.** We need to prove that for any \(R > 0\), there exists \(R_1 > R\) such that for any \(u_0 \in J^- c\) and \(\tau > 0\) with \(||u_0|| > R_1\) and \(G([0, \tau])u_0 \subset J^- c\), it holds that

\[
||G(t)u_0|| > R, \quad \forall t \in [0, \tau]. \quad (7.5)
\]

Let \(u(t) = G(t)u_0\). We first show that there exists \(R_0 > 0\) such that

\[
\frac{d}{dt}||u||^2 \bigg|_{t=s} \geq 2\mu c \quad (7.6)
\]

whenever \(u(s) \in J^- c\) and \(|u(s)| \geq R_0\). Multiplying the equation (7.3) with \(u = u(t)\) and integrating over \(\mathbb{R}^n\), one finds that

\[
\frac{1}{2} \frac{d}{dt}||u||^2 + ||u||^2 = \int_{\mathbb{R}^n} f(x, u)u dx \geq (\text{by (F3)}) \geq \mu \int_{\mathbb{R}^n} F(x, u) dx. \quad (7.7)
\]

Noticing that \(\int_{\mathbb{R}^n} F(x, u)dx = \frac{1}{2}||u||^2 - J(u)\), we deduce that

\[
\frac{d}{dt}||u||^2 \geq (\mu - 2)||u||^2 - 2\mu J(u) \geq (\mu - 2)a_0||u||^2 - 2\mu J(u). \quad (7.8)
\]

Set \(R_0 = 2\sqrt{\mu c/(\mu - 2)a_0}\). Then if \(u(s) \in J^- c\) and \(|u(s)| \geq R_0\), we have

\[
\frac{d}{dt}||u||^2 \bigg|_{t=s} \geq (\mu - 2)a_0R_0^2 - 2\mu c \geq 2\mu c.
\]
This completes the proof of (7.6).

We proceed to the proof of (7.5). It can be assumed that

\[ R > \max \left( R_0, \sqrt{12c} \right). \quad (7.9) \]

We argue by contradiction and suppose that (7.5) fails to be true. Then for each \( k \geq 1 \), one can find a \( v_k \in J_c - c \) and \( t_k > 0 \) with \( ||v_k|| > 2kR \) such that

\[ G([0, t_k])v_k \subset J_c - c, \quad ||G(t_k)v_k|| = R. \quad (7.10) \]

One may assume \( ||v_k|| = 2kR \) and that

\[ R \leq ||G(t)v_k|| \leq 2kR \quad (7.11) \]

for all \( t \in [0, t_k] \). Otherwise, let

\[ s_k = \inf \{ 0 \leq s \leq t_k : G(t)v_k < 2kR \text{ for } t \in (s, t_k) \}. \]

Then \( ||G(s_k)v_k|| = 2kR \); moreover, (7.11) holds for all \( t \in [s_k, t_k] \). Hence we can use \( v'_k := G(s_k)v_k \) and \( t'_k := t_k - s_k \) to replace \( v_k \) and \( t_k \), respectively.

Let \( u_k(t) = G(t)v_k \). Define

\[ \tau_k = \max \{ \tau > 0 : ||u_k(t)|| \geq kR \text{ for } t \in [0, \tau] \}. \]

Then \( ||u_k(\tau_k)|| = kR \). We claim that

\[ |u_k(t)| \leq c_0 := (1 + 1/\sqrt{a_0}) R, \quad \forall t \in [0, \tau_k]. \quad (7.12) \]

Indeed, if \( |u_k(t')| > c_0 ( > R > R_0) \) for some \( t' > 0 \), then by (7.6) we necessarily have \( |u_k(t)| > c_0 \) for all \( t \in [t', t_k] \). Further by (A1) and the definition of the norm \( || \cdot || \), we find that

\[ ||u_k(t_k)|| \geq \sqrt{a_0}|u_k(t_k)| > \sqrt{a_0}c_0 > R. \]

This contradicts (7.10).

In what follows we give an estimate for \( \tau_k \). If we multiply the equation (7.3) with \( Lu_k \) and integrate over \( \mathbb{R}^n \), it gives

\[ \frac{1}{2} \frac{d}{dt} ||u_k||^2 + |Lu_k|^2 = (f(u_k), Lu_k) \geq -\frac{1}{2} |Lu_k|^2 - \frac{1}{2} |\tilde{f}(u_k)|^2. \]
Hence
\[ \frac{d}{dt} ||u_k||^2 \geq -3|Lu_k|^2 - |\tilde{f}(u_k)|^2. \]  
(7.13)

Multiplying (7.3) with \(-(Lu_k - \tilde{f}(u_k))\) and integrating over \(\Omega\), it yields
\[ -\frac{d}{dt} J(u_k) = |Lu_k - \tilde{f}(u_k)|^2 = |Lu_k|^2 + |\tilde{f}(u_k)|^2 - 2(Lu_k, \tilde{f}(u_k)). \]  
(7.14)

Since
\[ 2(Lu_k, \tilde{f}(u_k)) \leq \frac{1}{2} |Lu_k|^2 + 2|\tilde{f}(u_k)|^2, \]
by (7.14) it holds that
\[ |Lu_k|^2 \leq 2|\tilde{f}(u_k)|^2 - 2\frac{d}{dt} J(u_k). \]
Combining this with (7.13) we obtain that
\[ \frac{d}{dt} ||u_k||^2 \geq -7|\tilde{f}(u_k)|^2 + 6\frac{d}{dt} J(u_k). \]  
(7.15)

We infer from (F1)° that
\[ |f(x, s)| \leq \frac{b(x)}{\gamma + 1}|s|^\gamma, \quad \forall x \in \mathbb{R}^n, \ s \in \mathbb{R}^1. \]  
(7.16)

Using (7.16) and the Hölder inequality, it is easy to deduce that
\[ |\tilde{f}(u_k)|^2 \leq \frac{1}{(\gamma + 1)^2} \int_{\mathbb{R}^n} b^2(x)|u_k|^\beta dx \leq \frac{|b|^2_{p_\gamma}}{(\gamma + 1)^2} \left( \int_{\mathbb{R}^n} |u_k|^{2^\ast} dx \right)^{1/q'}, \]
where \(q' = p_\gamma/(p_\gamma - 2)\), and \(\beta = 2(\gamma + 1)\). The Sobolev embedding \(V \hookrightarrow L^{2^\ast}(\mathbb{R}^n)\) then implies that
\[ |\tilde{f}(u_k)|^2 \leq c_1 ||u_k||^{\beta}. \]
Thus by (7.15) it follows that
\[ 7c_1 ||u_k||^{\beta} \geq -\frac{d}{dt} ||u_k||^2 + 6\frac{d}{dt} J(u_k). \]  
(7.17)

Integrating (7.17) from 0 to \(\tau_k\), one finds that
\[ 7c_1 \int_0^{\tau_k} ||u_k||^{\beta} dt \geq (||u_k(0)||^2 - ||u_k(\tau_k)||^2) + 6(J(u_k(\tau_k)) - J(u_k(0))) \]
\[ = 3k^2 R^2 + 6(J(u_k(\tau_k)) - J(u_k(0))) \]
\[ \geq 3k^2 R^2 - 12c \geq (by \ (7.9)) \geq 2k^2 R^2. \]
Therefore
\[ 7c_1(2kR)^\beta \tau_k \geq 7c_1 \int_0^{\tau_k} |u_k|^\beta \, dt \geq 2k^2 R^2. \]

Hence
\[ \tau_k \geq c_2 k^{2-\beta}, \quad \forall \, k \geq 1, \]
where \( c_2 > 0 \) depends only upon \( c_1 \) and \( R \).

Let \( s_k = \min \{1, \tau_k\} \). As \( \beta := 2(\gamma + 1) < 4 \), we conclude that
\[ \int_0^{s_k} |u_k(t)|^2 \, dt \geq k^2 R^2 s_k \geq R^2 k^2 \min \{1, c_2 k^{2-\beta}\} \to \infty \]
as \( k \to \infty \). On the other hand, we infer from (7.8) that
\[ \frac{d}{dt} |u_k|^2 \geq (\mu - 2) |u_k|^2 - 2\mu J(u) \geq (\mu - 2) |u_k|^2 - 2\mu c. \]

Integrating the inequality from 0 to \( s_k \), it yields
\[ \int_0^{s_k} |u_k(t)|^2 \, dt \leq \frac{1}{(\mu - 2)} \left( |u_k(0)|^2 + |u_k(s_k)|^2 \right) + \frac{2\mu c}{(\mu - 2)} s_k \leq c_3 + c_4, \]
where \( c_3 = \frac{2c_0^2}{(\mu - 2)} \), and \( c_4 = \frac{2\mu c}{(\mu - 2)} \). This leads to a contradiction. □

7.2 The proof of the main result

We are now in a position to prove Theorem 7.1.

Proof of Theorem 7.1. We will prove the theorem by applying an appropriate mountain pass theorem of local semiflows to the parabolic flow \( G \) of the equation. For this purpose, let us first demonstrate the mountain pass geometry of the Lyapunov function \( J \) of the flow. The argument involved here seems to be quite standard in the variational theory. We give the details for the reader’s convenience.

By (F3)° and (7.16) we deduce that
\[ \int_{\mathbb{R}^n} F(x, u) \, dx \leq \frac{1}{\mu} \int_{\mathbb{R}^n} f(x, u) u \, dx \leq \frac{1}{(\gamma + 1)\mu} \int_{\mathbb{R}^n} b(x)|u|^{\gamma+2} \, dx. \]

Let \( q_\gamma = p_\gamma / (p_\gamma - 1) \). Then \( q_\gamma < n / (n - 1) \) (recall that \( p_\gamma > n \)). Hence \( (\gamma + 2)q_\gamma < 2^* \). By virtue of the Hölder inequality and the Sobolev embedding, one has
\[ \int_{\mathbb{R}^n} F(x, u) \, dx \leq \frac{|b|_{p_\gamma}}{(\gamma + 1)\mu} \left( \int_{\mathbb{R}^n} |u|^{(\gamma + 2)q_\gamma} \, dx \right)^{1/q_\gamma} \leq c_5 ||u||^{\gamma+2}. \]
It then follows from the definition of $J$ that

$$J(u) \geq \frac{1}{2}||u||^2 - c_5||u||^{\gamma+2}, \quad \forall u \in V.$$ 

Taking $\rho = (4c_5)^{-1/\gamma}$, one concludes that

$$J(u) \geq \frac{1}{4}\rho^2 > 0, \quad \forall u \in \partial B_V(\rho),$$

where $B_V(\rho) = \{u \in V : ||u|| < \rho\}$. 

Let

$$X = \{u \in V : u(x) \geq 0 \text{ a.e. } x \in \mathbb{R}^n\}.$$ 

It is easy to see that $X$ is a closed subset of $V$. By the comparison principle of parabolic equations (see e.g. [4]), we know that if $u_0 \geq 0$ then $u(t) = G(t)u_0 \in X$ for all $t \in [0, T_{u_0})$. Hence $X$ is positively invariant under $G$.

By (F2) there exists an open subset $\Omega$ of $\mathbb{R}^n$ such that (7.2) holds. We may assume that $\Omega$ is bounded. Denote $\lambda_1$ the first eigenvalue of the operator $-\Delta$ on $\Omega$ associated with the homogenous Dirichlet boundary condition, and let $w_1$ be the corresponding eigenvector. It is well known that $w_1 > 0$ in $\Omega$. We extend $w_1$ to a function on $\mathbb{R}^n$ by setting $w_1(x) = 0$ for $x \in \mathbb{R}^n \setminus \Omega$. Then $w_1 \in X$. Observe that

$$\varphi(s) := J(sw_1) = \frac{\lambda_1}{2}s^2|w_1|^2 + s^2 \int_{\Omega} a(x)|w_1|^2 dx - \int_{\Omega} F(x, sw_1) dx.$$ 

By a very standard argument as in [24] (pp. 102-103) or [6], it can be shown that $\varphi(s) \to -\infty$ as $s \to +\infty$. Therefore one can pick an $s_1 > 0$ such that $J(s_1w_1) \leq 0$.

Set $U = X \cap B_V(\rho)$, and let

$$Q = \{sw_1 : s \in [0, s_1]\}, \quad S = \{0, s_1w_1\}.$$ 

Let

$$\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) = s_1w_1\}.$$ 

Define

$$\beta = \inf_{u \in U} J(u), \quad c = \inf_{\gamma \in \Gamma} \sup_{t \in [0, 1]} J(\gamma(t)).$$ 

Clearly $\beta > 0$. Thanks to Theorem 5.3 we deduce that $c \geq \beta$. Moreover, for any $\varepsilon > 0$ with $\varepsilon < \beta/2$, $J_{\beta-\varepsilon}^{c+\varepsilon}$ contains at least one critical point $u \in X$. $u$ is precisely a positive solution of the equation. \(\square\)
Remark 7.4 It has long been recognized that dynamical methods can be very useful in the study of variational problems. For example, in [15] Liu and Sun obtained some nice results on the existence of at least four critical points for variational functionals by developing a dynamical method via positively invariant sets of descending flows.

Appendix: Complete Metrizability of Spaces

In this part we make some discussions on complete metrizability of topological spaces. Specifically, we show that the one-point expansion $X^*$ of a separable complete metric space $X$ and all the quotient spaces introduced in Sections 3 and 4 are completely metrizable.

Let $X$ be a topological space. We say that $X$ is metrizable, if there exists a metric $\rho = \rho(\cdot, \cdot)$ on $X$ such that the topology of $X$ is precisely the one induced by $\rho$. If $(X, \rho)$ is also complete, then we say that $X$ is completely metrizable.

The following Urysohn Metrization Theorem is well-known and can be found in many text books on general topologies.

**Theorem A1.** If $X$ is normal and second countable, then it is metrizable.

There is also an important theorem on complete metrizability due to Z. Frolik and A. V. Arhangel’skii. Before stating the theorem explicitly, let us first recall some basic notions in the theory of general topology.

A family $\mathcal{F}$ of nonempty subsets of $X$ is said to be a filter base on $X$, if it satisfies the following conditions:

1. $\mathcal{F} \neq \emptyset$, and $\emptyset \notin \mathcal{F}$.
2. For any $A, B \in \mathcal{F}$, there exists $C \in \mathcal{F}$ such that $C \subset A \cap B$.

A filter base $\mathcal{F}$ necessarily enjoys the finite intersection property, namely, the intersection of any finite number of elements of $\mathcal{F}$ is non-void.

To avoid confusion, in the remaining part we use $\text{Cl}_X A$ to denote the closure of $A \subset X$ in $X$.

A sequence of covers $\mathcal{U}_n$ of $X$ is called complete, if whenever $\mathcal{F}$ is a filter base on $X$ such that each $\mathcal{U}_n$ has an element $U_n$ containing some $F \in \mathcal{F}$, then

$$\bigcap_{F \in \mathcal{F}} \text{Cl}_X F \neq \emptyset.$$  

**Theorem A2.** [17] A metrizable space $X$ is completely metrizable if and only if it has a complete sequence of open covers $\mathcal{U}_n$.  

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Now suppose that $X$ is a complete metric space with metric $d = d(\cdot, \cdot)$. Let $X^*$ be the one-point expansion of $X$ introduced in Section 3. We have

**Theorem A3.** If $X$ is separable, then $X^*$ is completely metrizable.

**Proof.** If $X$ is separable, then it is normal and second countable. Let $\mathscr{B}$ be a countable topology base of $X$. Fix a point $x_0 \in X$. Then by the definition of the topology of $X^*$, it can be easily seen that $\mathscr{B} \cup \{V_k\}_{k=1}^\infty$ is a countable topology base of $X^*$, where

$$V_k = X^* \setminus \text{Cl}_{X^*}B_{X}(x_0, k).$$

It is trivial to verify that $X^*$ is normal, i.e., every two disjoint closed sets of $X^*$ have disjoint open neighborhoods. Hence by Theorem A1, there exists a metric $\varrho = \varrho(\cdot, \cdot)$ on $X^*$ such that the topology of $X^*$ can be induced by $\varrho$.

There remains to prove that $X^*$ is completely metrizable. For this purpose, it suffices to show that $X^*$ has a complete sequence of open covers.

We infer from the completeness of $X$ and Theorem A2 that $X$ has a complete sequence of open covers $\mathcal{U}_n$. It can be assumed that $\mathcal{U}_{n+1}$ is a refinement of $\mathcal{U}_n$; otherwise, one can use the sequence $\mathcal{U}'_n$ to replace $\mathcal{U}_n$, where

$$\mathcal{U}'_n = \bigwedge_{j=1}^n U_j : U_j \in \mathcal{U}_j.$$

Hence for each fixed $m \geq 1$, the sequence $\mathcal{U}_n (n \geq m)$ of open covers is complete as well.

Define a sequence of open covers $\mathcal{V}_n$ of $X^*$ as follows:

$$\mathcal{V}_n = \mathcal{U}_n \cup \{G_n\}, \quad n = 1, 2, \ldots,$$

where $G_n := B_{X^*}(\ast, \frac{1}{n})$ is the ball in $X^*$ centered at $\ast$ with radius $\frac{1}{n}$ (with respect to the metric $\varrho$ mentioned above). We show that this sequence is complete, thus completes the proof of the theorem.

Let $\mathcal{F}^*$ be a filter base on $X^*$ which is such that each $\mathcal{V}_n$ has an element $V_n$ containing some $F \in \mathcal{F}^*$. We need to verify that $\bigcap_{F \in \mathcal{F}^*} \text{Cl}_{X^*}F \neq \emptyset$.

If $\ast \in \text{Cl}_{X^*}F$ for each $F \in \mathcal{F}^*$, then $\ast \in \bigcap_{F \in \mathcal{F}^*} \text{Cl}_{X^*}F$, and we are done. Thus we assume that $\ast \notin \text{Cl}_{X^*}F_0$ for some $F_0 \in \mathcal{F}^*$. Then

$$\text{Cl}_{X^*}F_0 \cap \text{Cl}_{X^*}G_n = \emptyset$$

for all $n$ sufficiently large. Pick an $m \geq 1$ so that (2) holds for all $n \geq m$. Set

$$\mathcal{F} = \{F \cap F_0 : F \in \mathcal{F}^*\}.$$
Then $\mathcal{F}$ is a filter base on $X^*$. As $\ast \not\in F'$ for all $F' \in \mathcal{F}$, $\mathcal{F}$ is also a filter base on $X$. By the assumption on $\mathcal{F}^*$, each $\mathcal{V}_n$ has an element $V_n$ containing some $F_n \in \mathcal{F}^*$. Hence $V_n$ contains $F_n' := F_n \cap F_0 \in \mathcal{F}$. In consideration of (2), this amounts to say that for each $n \geq m$, $\mathcal{V}_n$ has an element $U_n$ containing $F_n'$.

Because the sequence of covers $\mathcal{V}_n \ (n \geq m)$ of $X$ is complete, by Theorem A2 we deduce that $\bigcap_{F' \in \mathcal{F}^*} \text{Cl}_X F' \neq \emptyset$. Therefore

$$\bigcap_{F' \in \mathcal{F}^*} \text{Cl}_X F' \supset \bigcap_{F' \in \mathcal{F}^*} \text{Cl}_X (F \cap F_0) = \bigcap_{F' \in \mathcal{F}^*} \text{Cl}_X F' \supset \bigcap_{F' \in \mathcal{F}} \text{Cl}_X F' \neq \emptyset.$$

The proof is finished. $\square$

**Remark A4.** Assume that $X$ is a separable complete metric space. Then for any closed subsets $A$ and $B$ of $X$, by a similar argument as above, it can be shown that the quotient space $A/B$ is completely metrizable. Since the one-point expansion $X^*$ of $X$ is also separable, we conclude that all the quotient spaces introduced in Sections 3 and 4 are completely metrizable.

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