Approximate Solution of Delay Integral Equations via Functions of two Variable

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Abstract: In this work, we adopt B. spline function of two variables basis for solving linear multi-dimensional delay Volterra integral equations nonhomogeneous of the second type. We employ the two methods to approximate solution via B. spline function of two variables basis that yields linear system. Some examples are given, the results shown in tables and figures. These methods are very effective, convenient and overcome the difficulty of traditional methods. We solve this problem with the assistance of Matlab18.

1. Introduction

Integral equations are used as mathematical models for many physical situations, and integral equations also occur as reformulations of other mathematical problems. The solutions of integral equations have a major role in the field of science and engineering [1]. The theory and application of integral equations are important subject within applied mathematics. The most recent kind of equation that worth studying is the delay integral equation. These equations have many applications like a model to explain the observed periodic out breaks of certain infection diseases. Another application is electromagnetic inverse scattering. The delay arguments are consistent with the real phenomena which make the models more realistic for simulation. Finding the approximate or exact solutions of multi-dimensional delay integral equations is an important task. Save in a limited number, there difficulty in finding the analytical solutions of multi-dimensional delay integral equations. Therefore, there have been attempts to develop new methods for obtaining analytical solutions which reasonably approximate the exact solutions. Delay integral equations allow mathematicians and engineers better modeling of a wide class of systems with anomalous dynamic behavior and a better understanding of the facets of both physical phenomena and artificial processes.

The multi-dimensional delay integral equations are equations in which the unknown function appears under two integral signs and dependents only on two independent variables [1,4,6].

The general form of linear multi-dimensional Volterra delay integral equation is:

\[ h(x,y)u(x,y) = g(x,y) + \int_a^d \int_a^b k(x,y,s,t) \ u(s - \tau_1, t - \tau_2) \ ds \ dt \]  \hspace{1cm} (1)

where \( h \) is known function of \( x \) and \( y \) and \( k \) is known function of \( x \) and \( y \) called kernel of the integral equation and \( g \) is a given function often called the driving term, \( \{a,c\} \) are known scalars, \( \{b,d\} \) are known functions of \( x \) and \( y \) and \( \tau_1, \tau_2 \) are known positive numbers and \( u \) is the unknown that must be determined. Eq.(1) is classified as first kind if \( h(x,y) = 0 \), otherwise \( h(x,y) = 1 \) it is called second kind.
However, several numbers of algorithms for solving linear multi-dimensional linear Volterra delay integral equations have been investigated. M. Abdelkawy and A. Amin [7] use Jacobi collocation approximation for solving multi-dimensional volterra integral equations. Reza Abazari and Adem KILÇMAN [8] adopt RDTM and comparison with DTM to study of two-dimensional Volterra integral equations. H. Brunner and J. Kaithn[9] employ collocation and iterated collocation to solve two-dimensional Volterra integral equations.

The aim of this paper, to show how the approximate methods which are based on the expansion method can be used to solve (LMDVIEs) obtain approximate solutions via B. spline polynomials as basis functions.

2. A new formulation of B. spline functions of two variable \( B_{ij}^{n+m}(x,y) \) and the properties

B. spline is a piecewise polynomial approximation. Schoenberdy first introduced the B. spline in 1949. This function used to find approximated solution of one dimensional linear Volterra integral differential equations [2]. Moreover, this function is used to solve one dimensional delay differential equation [5]. In this paper we used B. spline of two variables \( B_{ij}^{n+m}(x,y) \) to find approximated solution of the two dimensional linear Volterra delay integral equation.

\[
B_{ij}^{n+m}(x,y) = \binom{n}{i} \binom{m}{j} x^i y^j (1 - x)^{n-i} (1 - y)^{m-j}, \quad 0 \leq x \leq n, 0 \leq y \leq m
\]

where \( \binom{n}{i} \binom{m}{j} \) are the degree of polynomials and \( i, j \) are the index of polynomials and \( x, y \) are variables.

**Property (1)** [3] The B. spline of degree \( n + m \) in terms of the power basis is given by the following formula: \( B_{ij}^{n+m}(x,y) = \sum_{k=1}^{n} \sum_{l=1}^{m} (-1)^{k-l} (-1)^{1-l} \binom{k}{i} \binom{l}{j} \binom{l}{j} x^k y^l \)

**Property (2)** [3] The first derivative of \( B_{ij}^{n+m}(x,y) \) polynomial with respect to \( x \) is a polynomial of degree \((n + m + 1)\) and is given by:

\[
\frac{\partial B_{ij}^{n+m}(x,y)}{\partial x} = n \left( B_{i-1}^{n-1}(x) - B_{i}^{n-1}(x) \right) B_{j}^{m}(y).
\]

**Property (3)** [3] The first derivative of \( B_{ij}^{n+m}(x,y) \) polynomial with respect to \( y \) is a polynomial of degree \((n + m + 1)\) and is given by:

\[
\frac{\partial B_{ij}^{n+m}(x,y)}{\partial y} = m \left( B_{j-1}^{m-1}(y) - B_{j}^{m-1}(y) \right) B_{i}^{n}(x)
\]

**Property (4)**[3] The first derivative of \( B_{ij}^{n+m}(x,y) \) polynomial with respect to \((x,y)\) is a polynomial of degree \((n + m - 2)\) and is given by:

\[
\frac{\partial^2 B_{ij}^{n+m}(x,y)}{\partial x \partial y} = nm \left( B_{i-1}^{n-1}(x) - B_{i}^{n-1}(x) \right) \left( B_{j-1}^{m-1}(y) - B_{j}^{m-1}(y) \right)
\]

3. Expansion Methods:

The expansion method is one of the famous methods used to find an approximate solutions of the multi-dimensional integral equations, [1,2,4,6].

To illustrate that, consider multi-dimensional linear Volterra delay equation

\[
u(x,y) = g(x,y) + \int_{c}^{d} \int_{a}^{b} k(x,y,s,t) \, u(s - \tau_{1}, t - \tau_{2}) \, ds \, dt
\]

Approximate the solution \( u \) of the equation as:

\[
u(x,y) = \sum_{k=1}^{N} c_{k} B_{k}(x,y)
\]

By substituting Eq. (3) into Eq. (2) one can get

\[
\sum_{k=1}^{N} c_{k} \left[ B_{k}(x,y) - \int_{c}^{d} \int_{a}^{b} k(x,y,s,t) B_{k}(s - \tau_{1}, t - \tau_{2}) \, ds \, dt \right] - g(x,y) = \epsilon(x,y, c_{1}, c_{2}, ..., c_{N})
\]

Hence the error function \( \epsilon(x,y, c_{1}, c_{2}, ..., c_{N}) \) will depends on \( x,y \) and on the unknown’s coefficients \( c_{1}, c_{2}, ..., c_{N} \). The expansion method (collocation and Galerkin’s) depends on finding the coefficients \( c_{1}, c_{2}, ..., c_{N} \) in which the error function is minimized.
4. B. spline solving multi-dimensional linear Volterra delay integral equation using collocation method:

Consider multi-dimensional linear Volterra delay integral equation

\[ u(x, y) = g(x, y) + \int_{a}^{d} \int_{a}^{b} k(x, y, s, t) u(s - \tau_1, t - \tau_2) \, ds \, dt \]

(5)

The collocation method requires that \( \epsilon(x_i, y_j) = 0 \) where \( (x_i, y_j) \in D \) and \( D = \{(x, y) | a \leq x \leq b, \ c \leq y \leq d\} \) for all \( l = 1, 2, ..., M \), \( j = 1, 2, ..., M \)

Therefore

\[ \sum_{k=1}^{N} c_k B_k(x, y) = g(x, y), \quad i = 1, 2, ..., N, \quad j = 1, 2, ..., M \]

(6)

We can obtain system of linear equations with unknown \( c_1, c_2, ..., c_N \) which can be solved by using mat lab language to find approximate solution.

5. B. spline solving multi-dimensional linear Volterra delay integral equation using Gelarkin’s method:

This method is based on approximating the unknown function \( u \) and substituting it into Eq. (2) to give following equation:

\[ \sum_{k=1}^{N} c_k B_k(x, y) = g(x, y) - \sum_{k=1}^{N} c_k \int_{a}^{d} \int_{a}^{b} k(x, y, s, t) B_k(s - \tau_1, t - \tau_2) \, ds \, dt = \epsilon(x, y, c_1, c_2, ..., c_N) \]

(7)

Gelarkin’s method established N-conditions which are necessary for determining N-coefficients \( c_k \) appeared in the above equation, by making the error \( \epsilon(x, y, c_1, c_2, ..., c_N) \) orthogonal to \( N \) given linearly independent functions \( \varphi_{i}(x, y), \varphi_{2}(x, y), ..., \varphi_{N}(x, y) \).

\[ \int_{a}^{d} \int_{a}^{b} \varphi_{j}(x, y) \, \epsilon(x, y, c_1, c_2, ..., c_N) \, dx \, dy = 0, \quad j = 1, 2, ..., N \]

(8)

Therefor

\[ \int_{a}^{d} \int_{a}^{b} \varphi_{j}(\sum_{k=1}^{N} c_k B_k(x, y)) - \int_{a}^{d} \int_{a}^{b} k(x, y, s, t) B_k(s - \tau_1, t - \tau_2) \, ds \, dt \, g(x, y)) \, dx \, dy = 0 \]

(9)

Where

\[ u_k(x, y) = B_k(x, y) - \int_{a}^{d} \int_{a}^{b} k(x, y, s, t) B_k(s - \tau_1, t - \tau_2) \, ds \, dt \]

Then

\[ \sum_{k=1}^{N} c_k \int_{a}^{d} \int_{a}^{b} \varphi_{j}(x, y) u_k(x, y) \, dx \, dy = \int_{a}^{d} \int_{a}^{b} \varphi_{j}(x, y) g(x, y) \, dx \, dy \]

(10)

By evaluating the above equation for each \( j = 1, 2, ..., N \) one can get a system of \( N \) linear equations with \( N \) unknowns \( c_1, c_2, ..., c_N \) which can be solved by any suitable method.

6. Numerical Examples

Example1: Consider the multi-dimensional linear Volterra delay integral equation of first kind:

\[ \frac{1}{2} x^3 + \frac{1}{2} x^3 y^2 + 2 y^2 x^2 = \int_{0}^{y} \int_{0}^{x} u(s - 2, t) \, ds \, dt \]

and the exact solution is \( u(x, y) = x + y \). with \( N = 10 \), \( h = 0.1 \) and \( x_i = ih, \ y_i = ih, \ i = 0, 1, ..., N \).

| \( x \) | \( y \) | Exact solution | Collection with (B. spline) | Gelarkin’s with (B. spline) |
|-------|-------|----------------|-----------------------------|-----------------------------|
| 0.0   | 0.0   | 0              | 0                           | 0                           |
| 0.1   | 0.1   | 0.2            | 0.2                         | 0.2                         |
| 0.2   | 0.2   | 0.4            | 0.4                         | 0.4                         |
| 0.3   | 0.3   | 0.6            | 0.6                         | 0.6                         |
| 0.4   | 0.4   | 0.8            | 0.8                         | 0.8                         |
| 0.5   | 0.5   | 1.0            | 1.0                         | 1.0                         |
| 0.6   | 0.6   | 1.2            | 1.2                         | 1.2                         |
| 0.7   | 0.7   | 1.4            | 1.4                         | 1.4                         |
| 0.8   | 0.8   | 1.6            | 1.6                         | 1.6                         |
| 0.9   | 0.9   | 1.8            | 1.8                         | 1.8                         |
| 1     | 1     | 2              | 2                           | 2                           |

Table (1)

Presents a comparison the exact solution and approximate solution
Example 2: Consider the (LMDVIEs): 
\[ u(x, y) = x + y - \frac{1}{2}x^2y - \frac{1}{2}xy^2 + xy + \int_0^y \int_0^x xy \, u(s - 1, t) \, ds \, dt \]
and the exact solution \( u(x, y) = x + y \).

With \( N = 10 \), \( h = 0.1 \) and \( x_i = ih \), \( y_i = ih \), \( i = 0, 1, ..., N \).

| \( x \) | \( y \) | Exact solution | Collection with (B. spline) | Gelarkin’s with (B. spline) |
|-------|-------|----------------|---------------------------|---------------------------|
| 0.0   | 0.0   | 0              | 0                         | 0                         |
| 0.1   | 0.1   | 0.2            | 0.2                       | 0.2                       |
| 0.2   | 0.2   | 0.4            | 0.4                       | 0.4                       |
| 0.3   | 0.3   | 0.6            | 0.6                       | 0.6                       |
| 0.4   | 0.4   | 0.8            | 0.8                       | 0.8                       |
| 0.5   | 0.5   | 1.0            | 1.0                       | 1.0                       |
| 0.6   | 0.6   | 1.2            | 1.2                       | 1.2                       |
| 0.7   | 0.7   | 1.4            | 1.4                       | 1.4                       |
| 0.8   | 0.8   | 1.6            | 1.6                       | 1.6                       |
| 0.9   | 0.9   | 1.8            | 1.8                       | 1.8                       |
| 1     | 1     | 2              | 2                         | 2                         |
7. Conclusions:
1. The expansion method with the aid of B. spline polynomials as a basis functions to compute the approximate solutions for multi-dimensional linear Volterra delay integral equations and B. spline polynomial gave accurate results.
2. A disadvantage of the B. spline function is its dependence up a free parameter $n$.

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