On the Grothendieck–Serre conjecture for classical groups

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Funding information
NSF, Grant/Award Number: DMS-1801951

Abstract
We prove some new cases of the Grothendieck–Serre conjecture for classical groups. This is based on a new construction of the Gersten–Witt complex for Witt groups of Azumaya algebras with involution on regular semilocal rings, with explicit second residue maps; the complex is shown to be exact when the ring is of dimension \( \leq 2 \) (or \( \leq 4 \), with additional hypotheses on the algebra with involution). Note that we do not assume that the ring contains a field.

MSC 2020
11E57 (primary), 11E39, 14L15, 16H05 (secondary)

INTRODUCTION

Let \( R \) be a regular local ring, and let \( G \) be a reductive group scheme over \( R \); let \( F \) be the field of fractions of \( R \). The Grothendieck–Serre conjecture states the following.

**Conjecture 0.1** (Grothendieck–Serre). *The restriction map*

\[
H^1_{\text{ét}}(R, G) \to H^1_{\text{ét}}(F, G)
\]

*has trivial kernel.*

In other words, it is conjectured that a \( G \)-torsor over \( R \) is trivial if it is trivial over \( F \).

The first forms of this conjecture — in special cases — appear in the works of Serre [45, Remark, p. 31] and Grothendieck [25, Remark 3, pp. 26-27]; it was first stated in the above form by
Grothendieck [26, Remark 1.11.a]. It is now proved when $R$ contains a field by Fedorov and Panin [14], and Panin [35]; see, for instance, [34, §5] for a detailed history of this conjecture.

The aim of the present paper is to prove some new cases of this conjecture, when $R$ does not necessarily contain a field. We assume that $2 \in R^\times$, and that $G$ is of classical type. More precisely, we prove the following.

**Theorem 0.2** (Theorems 9.7). Let $R$ be a regular semilocal domain with fraction field $F$, let $(A, \sigma)$ be an Azumaya $R$-algebra with involution, and let $U(A, \sigma)$ be the corresponding unitary group scheme. Suppose that one of the following hold:

1. $\dim R = 2$;
2. $\dim R \leq 4$ and $\text{ind } A$ is odd.

Then the restriction map $H^1_{\text{et}}(R, U(A, \sigma)) \to H^1_{\text{et}}(F, U(A, \sigma))$ has trivial kernel.

The same holds for the neutral connected component of $U(A, \sigma) \to \text{Spec } R$. We also prove a purity result for the Witt group of $(A, \sigma)$ under the same assumptions.

The proof uses the Gersten–Witt complex of Witt groups of hermitian and skew-hermitian forms over $(A, \sigma)$. In fact, we establish its exactness under the assumptions (1), (2) above (Theorems 9.4 and 9.6).

Recall that in the special case where $A = R$, a conjecture of Pardon [36, Conjecture A] predicts the existence of an exact cochain complex

$$0 \to W(R) \xrightarrow{d_{-1}} W(F) \xrightarrow{d_0} \bigoplus_{p \in R^{(1)}} W(k(p)) \to \cdots \to \bigoplus_{p \in R^{(d)}} W(k(p)) \to 0$$

called the Gersten–Witt complex of $R$; here, $R^{(e)}$ is the set of height-$e$ prime ideals, $k(p)$ is the fraction field of $R/p$, $W(R)$ is the Witt group of $R$ and $d$ is the Krull dimension of $R$. The map $d_{-1}$ is base-change from $R$ to $F$, and the $p$-component of $d_0$ is equivalent to the second residue map $\partial : W(F) \to W(k(p))$ (see [43, p. 209], for instance).

Constructions of a Gersten–Witt complex, applying to any regular domain $R$, were given by Pardon [37] and Balmer–Walter [6]. When $R$ is local, the exactness has been established when $\dim R \leq 4$ by Balmer–Walter [6] and when $R$ contains a field by Balmer–Gille–Panin–Walter [4] (see also [2]). The more general case of an Azumaya algebra with involution was considered by Gille [20], [21], who also proved its exactness when $R$ contains a field [22, Theorem 7.7]. See Balmer–Preeti [5, p. 3] and Jacobson [27, Theorem 3.8] for further positive results on the exactness.

We introduce a new construction of the Gersten–Witt complex of $\varepsilon$-hermitian forms over an Azumaya algebra with involution $(A, \sigma)$ (Section 2.1). In more detail, the complex that we construct is denoted by $\mathcal{G}W_{+}^{A,\sigma,\varepsilon}$ and takes the form

$$0 \to W_{\varepsilon}(A) \to W_{\varepsilon}(A_F) \to \bigoplus_{p \in R^{(1)}} W_{\varepsilon}(A(p)) \to \cdots \to \bigoplus_{p \in R^{(d)}} W_{\varepsilon}(A(p)) \to 0,$$

where $W_{\varepsilon}(A(p))$ denotes the Witt group of $\varepsilon$-hermitian forms over $A(p) := A \otimes k(p)$ which are valued in $A \otimes \bar{k}(p)$, and

$$\bar{k}(p) = \text{Ext}_{R_p}^{\dim R_p}(k(p), R_p) = \text{Hom}_{R_p}(\bigwedge_{R_p}^{\dim R_p}(p_p/p_p^2), k(p)).$$
(Since $k(p)$ is a 1-dimensional $k(p)$-vector space, the group $\tilde{W}_e(A(p))$ is isomorphic to the ordinary Witt group of $\varepsilon$-hermitian forms over $A(p)$, but the isomorphism is not canonical.) The $(-1)$-differential of $G\mathcal{W}_+^{A,\varepsilon}$ is base-change from $R$ to $F$, and the $(p,q)$-component of each of the other differentials is given as a generalized second residue map, see Section 2.1. This description is sufficiently explicit to allow hands-on computation of the differentials (Section 3), which facilitates many of our arguments.

The broad idea of the proof of Theorem 0.2 and the exactness of the Gersten–Witt complex is to use the 8-periodic exact sequence of Witt groups of $[18]$ (recalled in Section 6) in order to induct on the degree of $A$. The basis of the induction is the case $(A, \sigma) = (R, \text{id})$ established by Balmer, Preeti, and Walter $[6], [5]$.

Several supplementary results are required to make this approach work. First, we show that the Gersten–Witt complex is functorial in $(A, \sigma)$ (after dévissage) relative to base change and involution traces (Theorems 4.3 and 4.5), and hence compatible with the 8-periodic exact sequence of Witt groups of $[18]$ (Theorem 6.2). Second, we prove that the Gersten–Witt complex is exact at the last, that is, $d$th, term for all $(A, \sigma, \varepsilon)$ and semilocal $R$, and also at the 0-term if $\dim R \leq 2$ (Section 5). Third, we establish a version of the weak Springer Theorem on odd-degree base change for Azumaya algebras over semilocal rings (Corollary 7.4).

The paper is organized as follows: Section 1 is preliminary and recalls Azumaya algebras with involution, hermitian categories, and some homological algebra facts. In Section 2.1, we give our construction of the Gersten–Witt complex, and in Section 3 we explain how the differentials can be computed “by hand.” The construction is used in Sections 4 and 5 to establish various functoriality properties for the Gersten–Witt complex and the surjectivity of its last differential when $R$ is semilocal. Section 6 recalls the 8-periodic exact sequence of Witt groups of $[18]$ and shows that it is compatible with the Gersten–Witt complex. Applying the 8-periodic exact sequence to our situation is the subject matter of Sections 7 and 8. Finally, in Section 9, we put the machinery of the previous sections together to prove new cases of the Grothendieck–Serre conjecture and the exactness of the Gersten–Witt complex.

Notation. A ring means a commutative (unital) ring. Algebras are unital and associative, but not necessarily commutative. We assume that 2 is invertible in all rings and algebras.

Unless otherwise indicated, $R$ denotes a ring. By default, unadorned tensors, Hom-groups, and Ext-groups are taken over $R$. We will also suppress the base ring when tensoring or taking Hom-groups over a localization of $R$. An $R$-ring means a commutative $R$-algebra. An invertible $R$-module means a rank-1 projective $R$-module.

Given $p \in \text{Spec } R$, we let $k(p)$ denote the fraction field of $R/p$. The set of primes in $\text{Spec } R$ of codimension (that is, height) $e$ is denoted by $R^{(e)}$. The support of an $R$-module $M$ is denoted as $\text{supp}_R M$. If all primes in $\text{supp}_R M$ are of codimension at least $e$, then $M$ is said to be supported, or $R$-supported, in codimension $e$. An element $r \in R$ is said to act faithfully on $M$ if the map $x \mapsto xr : M \to M$ is injective; the set of all such elements is a multiplicative subset of $R$.

Let $S$ be an $R$-ring. Given $R$-modules $M, N$ and $f \in \text{Hom}(M,N)$, we write $M_S := M \otimes S$ and $f_S := f \otimes \text{id}_S \in \text{Hom}_S(M_S, N_S)$. When $S = k(p)$ for $p \in \text{Spec } R$, we write $M(p) = M_{k(p)}$ and $f(p) = f_{k(p)}$. In addition, we let $m(p)$ denote the image of $m \in M$ in $M(p)$.

For $M, N, f$ as before, we write $M^\vee = \text{Hom}(M, R)$ and $f^\vee = \text{Hom}(f, R)$.

Given an $R$-algebra $A$, its units and its center are denoted as $A^\times$ and $Z(A)$, respectively. Given $a \in A^\times$, we write $\text{Int}(a)$ for the inner automorphism $x \mapsto axa^{-1} : A \to A$. The category of finite
(that is, finitely generated) projective right $A$-modules is denoted as $\mathcal{P}(A)$. The category of all, respectively, finite, right $A$-modules is denoted as $\mathcal{M}(A)$, respectively, $\mathcal{M}_f(A)$. Unless otherwise indicated, $\text{Hom}_A$ and $\text{Ext}_A^*$ indicate Hom-groups and Ext-groups taken relative to the category of right $A$-modules. If $R$ is noetherian, $C$ is a multiplicative subset of $R$ and $M, N \in \mathcal{M}_f(A)$, then we shall freely identify $\text{Ext}_A^*(M, N)C^{-1}$ with $\text{Ext}_A^*(MC^{-1}, NC^{-1})$, see [39, Theorem 2.39].

In situations when an abelian group $M$ can be regarded as a module over multiple $R$-algebras, we shall sometimes write $M_A$ to denote “$M$, viewed as a right $A$-module.”

An $R$-algebra with involution means an $R$-algebra with an $R$-linear involution. Given an $R$-involution $\sigma : A \to A$ and $\epsilon \in \mu_2(R) := \{r \in R : r^2 = 1\}$, we let $S_\epsilon(A, \sigma) = \{a \in A : a = \epsilon a^\sigma\}$.

1 | PRELIMINARIES

1.1 | Azumaya algebras with involution

Given an Azumaya $R$-algebra $A$, the degree and the index of $A$ are denoted as $\deg A$ and $\text{ind} A$, respectively. When $R$ is connected, they can be regarded as integers. The Brauer class of $A$ is denoted by $[A]$, and the Brauer group of $R$ is denoted by $\text{Br} R$.

A finite étale $R$-algebra $S$ of rank 2 is called quadratic étale. In this case, $S$ admits a unique $R$-involution $\theta$ such that $R = S^\theta = \{s \in S : s^\theta = s\}$; it is given by $x^\theta = \text{Tr}_{S/R}(x) - x$ and called the standard $R$-involution of $S$.

In the sequel, we shall often consider $R$-algebras $A$ such that $A$ is Azumaya over $Z(A)$ and $Z(A)$ is finite étale over $R$. These turn out to be precisely the $R$-algebras which are separable and projective over $R$ [18, Proposition 1.1]. We therefore call such algebras separable projective. A module over a separable projective $R$-algebra is projective over the algebra if and only if it is projective over $R$ [41, Proposition 2.14].

Let $(A, \sigma)$ be an $R$-algebra with $(R$-linear$)$ involution. Following [18, §1.4] and [33, §4], we say that $(A, \sigma)$ is Azumaya over $R$ if $A$ is a separable projective $R$-algebra and $r \mapsto 1_A \cdot r$ defines an isomorphism $R \to Z(A)^{[\sigma]} = \{s \in Z(A) : s^\sigma = s\}$. In this case, provided that $R$ is connected, $Z(A)$ is either $R$ or a quadratic étale $R$-algebra.

If $(A, \sigma)$ is a separable projective $R$-algebra with involution, then $(A, \sigma)$ is Azumaya over $Z(A)^{[\sigma]}$ and $Z(A)^{[\sigma]}$ is finite étale over $R$ [18, Example 1.20]. Thus, if $S$ is a connected finite étale $R$-algebra and $\sigma : S \to S$ is a nontrivial $R$-involution, then $S$ is quadratic étale over $S^{[\sigma]}$, the standard $S^{[\sigma]}$-involution of $S$ is $\sigma$, and $S^{[\sigma]}$ is finite étale over $R$.

Suppose that $R$ is connected and let $(A, \sigma)$ be an Azumaya $R$-algebra with involution. If $Z(A) = R$, then the type of $\sigma$ can be either orthogonal or symplectic, and if $Z(A) \neq R$, then $\sigma$ is said to be of unitary type; see [18, §1.4]. Symplectic involutions can exist only when $\deg A$ is even. Given $\epsilon \in \mu_2(R) = \{\pm 1\}$, we define the type $(\sigma, \epsilon)$ to be the type of $\sigma$ if $\epsilon = 1$ and the opposite type of $\sigma$ if $\epsilon = -1$; here, orthogonal and symplectic types are opposite to each other and the unitary type is self-opposite.

These definitions extend to the non-connected case by considering the types as functions on $\text{Spec } R$ valued in $\{\text{orthogonal, symplectic, unitary}\}$; see [18, §1.4].

1.2 | Hermitian forms

We proceed with recalling some facts about hermitian spaces and Witt groups. See [43], [28], or [3] for an extensive discussion.
Let \((A, \sigma)\) be an \(R\)-algebra with involution and let \(Z\) be an \((A, A)\)-bimodule admitting a map \(\theta : Z \to Z\) satisfying \(x^{\theta} = x, (axb)^{\theta} = b^{\sigma}x^{\theta}a^{\sigma}\) and \(rx = xr\) for all \(x \in Z, a, b \in A\) and \(r \in R\), for example, \((Z, \theta) = (A, \sigma)\). Let \(\epsilon \in \mu_2(R)\). An \((Z, \theta)\)-valued \(\epsilon\)-hermitian space over \((A, \sigma)\) is a pair \((V, f)\) consisting of a right \(A\)-module \(V\) and a biadditive map \(f : V \times V \to Z\) satisfying \(f(a, b) = a^{\sigma}f(b, a)\) and \(f(x, y) = \epsilon f(y, x)^{\theta}\) for all \(x, y \in V, a, b \in A\). For example, when \((Z, \theta) = (A, \sigma)\) and \(V \in \mathcal{P}(A)\), this is just an \(\epsilon\)-hermitian space over \((A, \sigma)\) in the usual sense. Isometries of and orthogonal sums of \((Z, \theta)\)-valued \(\epsilon\)-hermitian spaces over \((A, \sigma)\) are defined in the usual way.

Let \(\hom_A^\epsilon(V, Z)\) denote \(\hom_A(V, Z)\) endowed with the right \(A\)-module structure given by \((\phi a)x = a^{\sigma} (\phi x)\) \((\phi \in \hom_A(V, Z), x \in V, a \in A)\). Then \(\hom_A^\epsilon(-, Z) : \mathcal{M}(A) \to \mathcal{M}(A)\) is a contravariant functor, and there is a natural transformation \(\omega_V : V \to \hom_A^\epsilon(\hom_A^\epsilon(V, Z), Z)\) given by \((\omega x)\phi = (\phi x)^{\theta}\). Every \((Z, \theta)\)-valued \(\epsilon\)-hermitian space \((V, f)\) gives rise to a right \(A\)-module map \(x \mapsto f(x, -) : V \to \hom_A^\epsilon(V, Z)\). We say that \((V, f)\) is unimodular if this map is bijective.

Let \(\mathcal{C}\) be a full additive subcategory of \(\mathcal{M}_1(A)\), for example, \(\mathcal{P}(A)\). Then \(\mathcal{C}\) inherits a exact category structure from \(\mathcal{M}(A)\). Suppose that

\((H1)\) \(\hom_A^\epsilon(-, Z)\) restricts to an exact functor \(\mathcal{C} \to \mathcal{C}\);

\((H2)\) \(\omega_V\) is an isomorphism for all \(V \in \mathcal{C}\).

Then \((\mathcal{C}, \hom_A^\epsilon(-, Z), \omega)\) is an exact hermitian category in the sense of [3, §1.1], and so we may consider its Witt group of \(\epsilon\)-hermitian forms, denoted as \(W_\epsilon(\mathcal{C})\). In more detail, let \(H^\epsilon(\mathcal{C})\) denote the category whose objects are unimodular \((Z, \theta)\)-valued \(\epsilon\)-hermitian spaces with underlying module in \(\mathcal{C}\) and whose morphisms are isometries. Then \(W_\epsilon(\mathcal{C})\) is the quotient of the Grothendieck group of \(H^\epsilon(\mathcal{C})\) (relative to orthogonal sum) by the subgroup generated by metabolic \(\epsilon\)-hermitian spaces. Here, \((V, f) \in H^\epsilon(\mathcal{C})\) is called metabolic (relative to \(\mathcal{C}\)) if there is a submodule \(M \subseteq V\) such that both \(M\) and \(V/M\) are in \(\mathcal{C}\) and \(M = M^\perp := \{x \in V : f(M, x) = 0\}\). The class represented by \((V, f) \in H^\epsilon(\mathcal{C})\) in \(W_\epsilon(\mathcal{C})\) is the Witt class of \((V, f)\); it is denoted by \([V, f]\) or \([f]\).

For the most part, we shall be concerned with the following two examples where \(\mathcal{C} = \mathcal{P}(A)\). A few examples with \(\mathcal{C} \neq \mathcal{P}(A)\) will be encountered in later sections.

**Example 1.1.** When \((Z, \theta) = (A, \sigma)\) and \(\mathcal{C} = \mathcal{P}(A)\), the category \(H^\epsilon(\mathcal{C})\) is just the category of unimodular \(\epsilon\)-hermitian spaces over \((A, \sigma)\) in the usual sense, for example, [18, §2]. In this case, we write \(H^\epsilon(A, \sigma)\) for \(H^\epsilon(\mathcal{C})\) and \(W_\epsilon(A, \sigma)\) for \(W_\epsilon(\mathcal{C})\).

**Example 1.2.** Let \(M\) be an invertible \(R\)-module and take \(Z = A \otimes M, \theta = \sigma \otimes \text{id}_M\) and \(\mathcal{C} = \mathcal{P}(A)\). Conditions \((H1)\) and \((H2)\) hold in this situation because they hold after localizing at every \(p \in \text{Spec} R\). The corresponding category and Witt group of \(\epsilon\)-hermitian forms are denoted as \(H^\epsilon(A, \sigma; M)\) and \(W_\epsilon(A, \sigma; M)\), respectively.

Continuing Example 1.2, let \(S\) be an \(R\)-ring. For every \((V, f) \in H^\epsilon(A, \sigma; M)\), let \(f_S : V_S \times V_S \to Z_S\) be the biadditive map determined by \(f_S(x \otimes s, y \otimes t) = f(x, y) \otimes st (x, y \in V, s, t \in S)\). It is easy to check that \((V_S, f_S) \in H^\epsilon(A_S, \sigma_S; M_S)\). The assignment \((V, f) \mapsto (V_S, f_S)\) extends to a functor \(H^\epsilon(A, \sigma; M) \to H^\epsilon(A_S, \sigma_S; M_S)\) by mapping every isometry \(\phi\) to \(\phi_S\). This in turn induces a group homomorphism \(W_\epsilon(A, \sigma; M) \to W_\epsilon(A_S, \sigma_S; M_S)\). When \(S = R_p\) for \(p \in \text{Spec} R\), we shall write \(f_S\) as \(f_p\).
1.3 | Some homological algebra

We record several facts about Cohen–Macaulay modules and the Koszul complex that will be needed in the sequel. See [11] for an extensive discussion. All rings in this subsection are noetherian.

**Proposition 1.3.** Let $R$ be a Cohen–Macaulay local ring with dualizing module $E$, let $M \in \mathcal{M}_f(R)$, and let $C$ be the set of elements in $R$ acting faithfully on $M$.

(i) If $M$ is supported in codimension $e$, then $\text{Ext}^i_R(M, E) = 0$ for all $i < e$, and $\text{Ext}^e_R(M, E)$ is $C$-torsion-free.

(ii) If $M$ is Cohen–Macaulay of dimension $e$, then $\text{Ext}^i_R(M, E) = 0$ for all $i \neq e$ and $\text{Ext}^e_R(M, E)$ is Cohen–Macaulay of dimension $e$.

**Proof.** Part (ii) is [11, Theorem 3.3.10(c)], and the first assertion of (i) follows from [11, Proposition 1.2.10, Theorem 2.1.2(b)] (applied with $M = E$ and $I = \text{Ann}_R M$). Finally, given $c \in C$, the short exact sequence $M \xrightarrow{c} M \rightarrow M/cM$ induces a short exact sequence $\text{Ext}^e_R(M/cM, E) \rightarrow \text{Ext}^e_R(M, E) \rightarrow \text{Ext}^e_R(M, E)$. Since $M/cM$ is supported in codimension $e + 1$, we have $\text{Ext}^e_R(M/cM, E) = 0$, so $c$ acts faithfully on $\text{Ext}^e_R(M, E)$. □

**Lemma 1.4.** Let $R$ be a Cohen–Macaulay ring. Then the set of zero divisors in $R$ is $\bigcup_{p \in \mathfrak{p} \in R(0)} \mathfrak{p}$. Moreover, every ideal of $R$ not contained in a minimal prime contains a regular element.

**Proof.** See [32, Theorems 6.1(ii), 17.3(i)] for the first statement. The second statement follows from the first and the Prime Avoidance Lemma [32, Exercise 16.8]. □

**Proposition 1.5.** Let $R$ be a 1-dimensional Cohen–Macaulay local ring with maximal ideal $\mathfrak{m}$, let $C$ denote the set of non-zero divisors in $R$, and let $M$ be a non-zero finite $R$-module. Then the following are equivalent:

(a) $M$ is $C$-torsion-free;

(b) $M$ is Cohen–Macaulay of dimension 1;

(c) $\mathfrak{m} \notin \text{Ass}_R M$.

**Proof.** (a) $\Rightarrow$ (b): By Lemma 1.4, there exists $r \in \mathfrak{m} \cap C$. This $r$ acts faithfully on $M$, so $\text{depth}_R M \geq 1$. Since $\dim R = 1$, this means that $M$ is Cohen–Macaulay of dimension 1.

(b) $\Rightarrow$ (c) is [46, Tag 0BUS] or [I, Theorem 2.12(a)].

(c) $\Rightarrow$ (a): Since $\dim R = 1$ and $R$ is local, (c) implies $\text{Ass}_R M \subseteq R(0)$, so by [46, Tag 00LD], elements of $R - (\bigcup_{p \in R(0)} p)$ act faithfully on $M$. Since $C \subseteq R - (\bigcup_{p \in R(0)} p)$, (a) follows. □

**Proposition 1.6.** Let $R$ be a Cohen–Macaulay local ring with dualizing module $E$, let $I \subseteq J$ be ideals of $R$ such that $R/I$ is of dimension $e$, and let $C$ denote the set of elements of $R$ acting faithfully on $R/I$. Assume $C \cap J \neq \emptyset$, so that $JC^{-1} = RC^{-1}$ and $\text{Ext}^e_R(J/I, E)C^{-1} = \text{Ext}^e_R(R/I, E)C^{-1}$. Then the natural maps from $\text{Ext}^e_R(R/I, E)$ and $\text{Ext}^e_R(J/I, E)$ to $\text{Ext}^e_R(R/I, E)C^{-1}$ are injective and, upon regarding the former as submodules of the latter, we have

$\text{Ext}^e_R(J/I, E) = \{ x \in \text{Ext}^e_R(R/I, E)C^{-1} : xJ \subseteq \text{Ext}^e_R(R/I, E) \}$. 


Proof. Let \( \{r_1, \ldots, r_n\} \) be a generating set for \( J \). Define \( f : (R/I)^n \to J/I \) by \( f(x_1, \ldots, x_n) = r_1x_1 + \cdots + r_nx_n \) and let \( K = \ker f \). The short exact sequence \( K \hookrightarrow (R/I)^n \to J/I \) and its localization at \( C \) give rise to a commutative diagram with exact rows:

\[
\begin{array}{ccc}
\text{Ext}^e_R(R/I, E)C^{-1} & \xrightarrow{f'} & \text{Ext}^e_R(R/I, E)C^{-1} \\
\uparrow & & \uparrow \\
\text{Ext}^e_R(J/I, E) & \xrightarrow{=} & \text{Ext}^e_R(R/I, E) \\
\uparrow & & \uparrow \\
\text{Ext}^e_R(K, E)C^{-1} & \xrightarrow{} & \text{Ext}^e_R(K, E)
\end{array}
\]

The vertical arrows are given by localization at \( C \), and are injective by Proposition 1.3(i). The same proposition also tells us that \( \text{Ext}^{e-1}_R(K, E) = 0 \) if \( e > 0 \), so \( f' \) is also injective. Since \( f' \) is given by \( f'(x) = (xr_1, \ldots, xr_n) \), the proposition is a consequence of a simple diagram chase.

\[\square\]

**Proposition 1.7.** Let \( R \) be a \( d \)-dimensional Cohen–Macaulay local ring with dualizing module \( E \), let \( A \) be a separable projective \( R \)-algebra, and let \( M \in \mathcal{M}_1(A) \). Suppose that \( M \) is a Cohen–Macaulay \( R \)-module of dimension \( e \). Then:

(i) \( \text{Ext}^i_A(M, A \otimes E) = 0 \) for all \( i \neq d - e \) and \( \text{Ext}^{d-e}_A(M, A \otimes E) \) is a Cohen–Macaulay \( R \)-module of dimension \( e \).

(ii) If \( e = d \), then the natural map \( M \to \text{Hom}^d_A(\text{Hom}_A(M, A \otimes E), A \otimes E) \) is an isomorphism. Here, \( \text{Hom}^d_A \) denotes the set of left \( A \)-module homomorphisms.

Proof. We prove (i) and (ii) together. Let \( R' \) denote a strict henselization of \( R \). Then \( R' \) is a Cohen–Macaulay local ring which is faithfully flat over \( R \) and has the same dimension as \( R \) [46, Tags 06LK, 07QM, 06LM]. By virtue of [11, Theorem 3.3.14(a)] and [46, Tag 04GP], \( E_{R'} \) is a dualizing module of \( R' \), and by [46, Tag 0338], \( M_{R'} \) is a Cohen–Macaulay \( R' \)-module of dimension \( e \) if and only if \( M \) is. Moreover, by [39, Theorem 2.39], there is a canonical isomorphism \( \text{Ext}^*_A(M, E) \otimes R' \cong \text{Ext}^*_A(M_{R'}, E_{R'}) \). We may therefore tensor \( M, A, \) and \( E \) with \( R' \) and assume \( R = R' \). In this case, \( A = \prod_{t=1}^{t} M_{n_t}(R) \) for some \( t \geq 0 \) and \( n_1, \ldots, n_t \in \mathbb{N} \). Working at each factor separately and using Morita equivalence, we may assume \( A = R \). This case is contained in [11, Theorem 3.3.10].

\[\square\]

**Corollary 1.8.** Let \( R \) be a Cohen–Macaulay ring of dimension \( e \) with dualizing module \( E \) and let \( (A, \sigma) \) be a separable projective \( R \)-algebra with involution. Set \( Z = A \otimes E, \theta = \sigma \otimes \text{id}_E \) and let \( \mathcal{C} \) denote the full subcategory of \( \mathcal{M}_1(A) \) consisting of \( A \)-modules which are Cohen–Macaulay \( R \)-modules of dimension \( e \). Then \( \mathcal{C} \) and \( (Z, \theta) \) satisfy conditions (H1) and (H2) of Section 1.2.

Proof. It is enough to check this after localizing at every \( p \), so we may assume that \( R \) is local. Observe that for every \( V \in \mathcal{M}(A) \), there is a natural isomorphism \( \psi \mapsto \theta \circ \psi : \text{Hom}^2_A(\text{Hom}_A^2(V, Z), Z) \to \text{Hom}^1_A(\text{Hom}_A(V, Z), Z) \) and under this isomorphism the map \( \omega_V \) is just the natural map \( V \to \text{Hom}_A^1(\text{Hom}_A(V, Z), Z) \). With this at hand, (H1) and (H2) follow from parts (i) and (ii) of Proposition 1.7, respectively.

\[\square\]

We finish with recalling the construction of the Koszul complex and several of its properties.
Let $E \in \mathcal{M}_f(R)$ and let $s \in E^\vee = \text{Hom}_R(E, R)$. The Koszul complex of $(E, s)$ is the chain complex $K = K(E, s) = (K_i, d_i)_{i \in \mathbb{Z}}$, where $K_i = \bigwedge^i_R E$ for $i \geq 0$, $K_i = 0$ for $i < 0$, and $d_i$ is given by

$$d_i(e_1 \wedge \cdots \wedge e_i) = \sum_{1 \leq j \leq i} (-1)^{j+1} s(e_j) \cdot e_1 \wedge \cdots \wedge \hat{e}_j \wedge \cdots \wedge e_i$$

for all $e_1, \ldots, e_i \in E$ ($\hat{e}_j$ means omitting $e_j$).

**Proposition 1.9.** Let $r_1, \ldots, r_n$ be a regular sequence in $R$, let $\mathfrak{a} = \sum i r_i R$, and let $K = K(E, s)$ where $E$ is a free $R$-module with basis $x_1, \ldots, x_n$ and $s$ is determined by $s(x_i) = r_i$ for all $i$. Then:

(i) $H^0(K) = R/\mathfrak{a}$ and $H^i(K) = 0$ for all $i \neq 0$;

(ii) $\mathfrak{a}/\mathfrak{a}^2$ is a free $R/\mathfrak{a}$-module with basis $\{r_1 + \mathfrak{a}^2, \ldots, r_n + \mathfrak{a}^2\}$;

(iii) There is a canonical isomorphism $\text{Ext}^n_R(R/\mathfrak{a}, R) \cong \text{Hom}(\bigwedge^n(\mathfrak{a}/\mathfrak{a}^2), R/\mathfrak{a})$.

**Proof.** (i) is [11, Corollary 1.6.14] and (ii) is [11, Theorem 1.1.8].

(iii) This is well known, but we recall the proof for later reference. By (i), $\text{Ext}^n_R(R/\mathfrak{a}, R)$ is the cokernel of $d^n_\vee: (\bigwedge^{n-1} E)^\vee \to (\bigwedge^n E)^\vee$. Let $\theta \in (\bigwedge^n E)^\vee$ denote the element mapping $x_1 \wedge \cdots \wedge x_n$ to 1. One readily checks that the image of $d^n_\vee$ is $a \theta$. Thus, $\text{coker}(d^n_\vee) \cong (\bigwedge^n E)^\vee / a(\bigwedge^n E)^\vee = \text{Hom}(\bigwedge^n E, R) \otimes (R/a) \cong \text{Hom}(\bigwedge^n(E/aE), R/a)$. By (ii), we may identify $E/aE$ with $a/a^2$ via $s: E \to R$, so $\text{Ext}^n_R(R/\mathfrak{a}, R) \cong \text{Hom}(\bigwedge^n(a/a^2), R/a)$. □

2 The Gersten–Witt Complex via Second Residue Maps

Henceforth, $R$ denotes a $d$-dimensional regular ring, $(A, \sigma)$ is a separable projective $R$-algebra with involution, for example, an Azumaya $R$-algebra with involution, and $\varepsilon \in \mu_2(R)$.

In this section, we give a new construction of the Gersten–Witt complex of $\varepsilon$-hermitian forms over $(A, \sigma)$, denoted as $\mathcal{GW}^{A, \sigma, \varepsilon}$ or $\mathcal{GW}^{A, \varepsilon}$, in which the differentials are defined using generalized second residue maps.

Given $e \geq 0$ and $p \in R^{(e)}$, write

$$\tilde{k}(p) := \text{Ext}_{R_p}^{\dim_{R_p} k(p)}(k(p), R_p) = \text{Hom}(\bigwedge_{R_p}^{\dim_{R_p} k(p)}(p_p^2 / p_p^2), k(p))$$

(see Proposition 1.9(iii) for second equality), and, using the notation of Example 1.2, set

$$\tilde{W}_\varepsilon(A(p)) = W_\varepsilon(A(p), \sigma(p); \tilde{k}(p)).$$

Since $\tilde{k}(p) \cong k(p)$ as $R$-modules, we actually have $\tilde{W}_\varepsilon(A(p)) \cong W_\varepsilon(A(p), \sigma(p))$, but this isomorphism is not canonical unless $e = 0$. (In contrast, for $e = 0$, we have $\tilde{k}(p) = \text{Hom}_{R_p}(R_p, k(p)) = \text{End}_{k(p)}(k(p)) = k(p)$, and so $\tilde{W}_\varepsilon(A(p)) = W_\varepsilon(A(p), \sigma(p))$.)

The cochain complex $\mathcal{GW}^{A, \sigma, \varepsilon}$ will take the form

$$0 \to \bigoplus_{p \in R^{(0)}} W_\varepsilon(A(p)) \xrightarrow{d_0} \bigoplus_{p \in R^{(1)}} W_\varepsilon(A(p)) \xrightarrow{d_1} \cdots \xrightarrow{d_{d-1}} \bigoplus_{p \in R^{(d)}} W_\varepsilon(A(p)) \to 0.$$
We will sometimes augment $GW^A,\sigma,\varepsilon$ with the additional map

$$d_{-1} : W_\varepsilon(A, \sigma) \to \bigoplus_{p \in R^{(0)}} W_\varepsilon(A(p)),$$

given by localizing at $p$ at the $p$-component. The resulting complex is the augmented Gersten–Witt complex of $(A, \sigma, \varepsilon)$, denoted as $GW^A/R,\sigma,\varepsilon$ or $GW^A,\sigma,\varepsilon$.

The true challenge in defining a Gersten–Witt complex is defining its differentials, and this shall be our concern in the remainder of this section. We shall first introduce a generalization of the classical second residue map for Witt groups (see [43, p. 209]), and then use it to define the differentials of $GW^A,\sigma,\varepsilon$. Section 3 will provide a method for explicit computation of the differentials.

Remark 2.1. In an unpublished work, Schmid [44] defined a Gersten–Witt complex when $A = R$ and $R$ is a localization of a finite-type ring over a field. The differentials in Schmid’s Gersten–Witt complex involve classical second residue maps, but also normalizations and traces, which are not present in our construction.

2.1 The residue map

Suppose that $R$ is regular local of dimension $e + 1$, let $q$ denote the maximal ideal of $R$, and let $I$ be an ideal of $R$ such that $S := R/I$ is a 1-dimensional Cohen–Macaulay ring, for example, any prime ideal $p \in R^{(e)}$. Let $C$ denote the set of elements of $R$ acting faithfully on $R/I$ and write

$$k(I) = SC^{-1} \quad \bar{k}(I) = \text{Ext}^{e}_{RC^{-1}}(SC^{-1}, RC^{-1})$$

$$A(I) = A \otimes k(I) \quad \bar{A}(I) = A \otimes \bar{k}(I)$$

$$\sigma(I) = \sigma \otimes \text{id}_{k(I)} \quad \bar{\sigma}(I) = \sigma \otimes \text{id}_{\bar{k}(I)}.$$

This agrees with our previous notation when $I = p$ for some $p \in R^{(e)}$. Note also that $k(I)$ is a 0-dimensional Cohen–Macaulay ring with a dualizing module $\bar{k}(I)$ [11, Theorems 3.3.7(b), 3.3.5].

We shall be concerned with $(\hat{A}(I), \hat{\sigma}(I))$-valued $\varepsilon$-hermitian forms over $(A(I), \sigma(I))$ in the sense of Section 1.2. By Corollary 1.8, conditions (H1) and (H2) of Section 1.2 hold for the category $\mathcal{M}_f(A(I))$, so we may define the associated category of unimodular $\varepsilon$-hermitian forms, denoted as $\bar{H}(A(I))$, and the associated Witt group, denoted as $\bar{W}_\varepsilon(A(I))$. Note that when $I = p \in R^{(e)}$, these are just $H^\varepsilon(A(p), \sigma(p); \bar{k}(p))$ and $\bar{W}_\varepsilon(A(p))$, respectively. We shall define a group homomorphism,

$$\partial_{I,q} = \partial^{A}_{I,q} : \bar{W}_\varepsilon(A(I)) \to \bar{W}_\varepsilon(A(q)),$$

called the $(I, q)$-residue map.

Let $m = q/I$ and write

$$\hat{S} = \text{Ext}^e_{S}(S, R) \quad \hat{m}^{-1} = \text{Ext}^e_{\hat{S}}(m, R)$$

$$\bar{A}_S = A \otimes \hat{S} \quad A\hat{m}^{-1} = A \otimes \hat{m}^{-1}.$$
Then $\tilde{S}$ is a dualizing module for $S$ [11, Theorem 3.3.7(b)]. By Proposition 1.6 and the fact $A$ that is a finite projective $R$-module, we may regard $\tilde{A}_S$ and $\tilde{m}^{-1}A$ as submodules of $\tilde{A}(I)$, and then

$$Am^{-1} = \{x \in \tilde{A}(I) : x m \subseteq \tilde{A}_S\}.$$

By Proposition 1.3, the short exact sequence $m \hookrightarrow S \rightarrow k(q)$ induces a short exact sequence $\tilde{S} \hookrightarrow \tilde{m}^{-1} \rightarrow \tilde{k}(q)$, and by tensoring with $A$, we get

$$\tilde{A}_S \hookrightarrow \tilde{A}m^{-1} \xrightarrow{T} \tilde{A}(q). \quad (2.1)$$

We denote the right map by $T_{I,A,A}$, dropping some or all of the subscripts when they are clear from the context.

Let $V \in \mathcal{M}_f(A(I))$. Following [39, p. 129], an $A$-lattice in $V$ is an $A$-submodule $U \subseteq V$ which is finitely generated over $S$ and satisfies $U \cdot k(I) = V$.

Let $(V, f) \in \tilde{\mathcal{H}} (A(I))$. Given an $A$-lattice $U$ in $V$, we write

$$U^f := \{x \in V : f(U, x) \subseteq \tilde{A}_S\}.$$

The following lemma was shown by the first and second authors when $S$ is a discrete valuation ring; see [7, Theorem 4.1] and its proof.

**Lemma 2.2.** In the previous notation:

(i) for every $A$-lattice $U$ in $V$, the set $U^f$ is an $A$-lattice in $V$ and $U = U^f f$;

(ii) there exists an $A$-lattice $U$ in $V$ such that $U^f m \subseteq U \subseteq U^f$.

**Proof.** (i) Recall from Section 1.2 that $\text{Hom}^g_A(U, \tilde{A}_S)$ denotes $\text{Hom}_A(U, \tilde{A}_S)$ endowed with the right $A$-module structure given by $(a \phi)x = a \hat{\phi}x$ ($a \in A$, $\phi \in \text{Hom}_A(U, \tilde{A}_S)$, $x \in U$).

Since $f$ is unimodular, the map $x \mapsto f(x, -) : U^f \rightarrow \text{Hom}^g_A(U, \tilde{A}_S)$ is an isomorphism of right $A_S$-modules. As $U$ is finitely generated over $A$ and $\tilde{A}_S$ is noetherian, this means that $U^f$ is finitely generated over $A_S$. Let $v \in V$. Then $f(U, v)$ is a left $A_S$-submodule of $\tilde{A}(I)$. Since $\tilde{A}_S$ is an $A$-lattice in $\tilde{A}(I)$, there exists $c \in C$ such that $f(U, v)c \subseteq \tilde{A}_S$, so $vc \in U^f$. As this holds for all $v \in V$, we have shown that $U^f \cdot k(I) = V$.

The unimodularity of $f$ also implies that $x \mapsto f(x, -) : U^{ff} \rightarrow \text{Hom}^g_A(U^f, \tilde{A}_S)$ is an isomorphism. Since $U^f \cong \text{Hom}^g_A(U, \tilde{A}_S)$, we get an isomorphism $U^{ff} \cong \text{Hom}^g_A(\text{Hom}^g_A(U, \tilde{A}_S), \tilde{A}_S)$. It is routine to check that the composition $U \rightarrow U^{ff} \rightarrow \text{Hom}^g_A(\text{Hom}^g_A(U^f, \tilde{A}_S), \tilde{A}_S)$ is the map $\varepsilon \cdot \omega_U$ of Section 1.2, defined using $(Z, \theta) = (\tilde{A}_S, \sigma \otimes \text{id}_S)$. By Corollary 1.8 and Proposition 1.5, $\omega_U$ is an isomorphism, so $U = U^{ff}$.

(ii) Choose an $A$-lattice $L$ in $V$. If $L \not\subseteq L^f$, replace $L$ with $L \cap L^f$. Since $LC^{-1} = V = L^f C^{-1}$, there exists $c \in C$ such that $L^f c \subseteq L$. This means that $\text{supp}_S(L^f / L) \subseteq \{m\}$, so $L = L^f$ there exists $t \in \mathbb{N} \cup \{0\}$ with $(L^f / L)m^t \neq 0$ and $(L^f / L)m^{t+1} = 0$ [46, Tag 00L6]. If $L = L^f$ or $t = 0$, take $U = L$. If not, then let $M = L + L^{f^t}$. We have $f(M, M) \subseteq \tilde{A}_S + f(L^f m^t, L^f m^t) = \tilde{A}_S + f(L^f m^{2t}, L^f) \subseteq \tilde{A}_S + f(L, L^f) = \tilde{A}_S$, so $M \subseteq M^f$ and $L \subseteq M \subseteq M^f \subseteq L^f$.

Replacing $L$ with $M$, we can repeat this process until we have $L^f m \subseteq L$. The number of iterations must be finite because $L^f$ is noetherian. \qed
We can now define $\delta_{I,\tilde{q}} : \tilde{W}_\varepsilon(A(I)) \to \tilde{W}_\varepsilon(A(q))$. Given $(V, f) \in \tilde{W}_\varepsilon(A(I))$, apply Lemma 2.2(ii) to choose an $A$-lattice $U$ in $V$ such that $U^f m \subseteq U \subseteq U^f$. Then $U^f // U \in P(A(q))$, and $f(U^f, U^f) \subseteq A\tilde{m}^{-1}$ because $f(U^f, U^f) m = f(U^f m, U^f) \subseteq \tilde{A}_S$. We may therefore define a biadditive map $\delta_{Uf} : U^f // U \times U^f // U \to \tilde{A}(q)$ by
\[
\delta_{Uf}(x+U, y+U) = T_{I,\tilde{q},A}(f(x, y)).
\]

We shall write $\delta f$ for $\delta_{Uf}$ when $U$ is clear from the context. Finally, set
\[
\delta_{I,\tilde{q}}[V, f] = [U^f // U, \delta f].
\]

This is well defined by the following lemma.

**Lemma 2.3.** In the previous notation, $(U^f // U, \delta f) \in \tilde{H}^f(A(q))$. Moreover, the Witt class $[U^f // U, \delta f] \in \tilde{W}_\varepsilon(A(q))$ depends only on the Witt class $[V, f]$ (and not on the choices of $V$, $f$, $U$).

**Proof.** Let $Z = \tilde{A}(I)/\tilde{A}_S = A \otimes (\tilde{k}(I)/S)$, $\theta = \sigma \otimes id_{\tilde{k}(I)/S}$ and let $\mathcal{C}$ denote the category of right $A_S$-modules of finite length. We shall verify at the end that conditions (H1) and (H2) of Section 1.2 hold for $(Z, \theta)$ and $\mathcal{C}$, so that we may consider the category $\tilde{W}_\varepsilon(\mathcal{C})$ and the associated Witt group $W_\varepsilon(\mathcal{C})$.

We identify $\tilde{A}(q)$ with $A\tilde{m}^{-1}/\tilde{A}_S \subseteq Z$ via $\tau$. Proposition 1.6 implies that $A\tilde{m}^{-1}/A_S$ is the $S$-socle of $Z$, and thus $\text{Hom}_A^2(P, \tilde{A}(q)) = \text{Hom}_A^2(P, Z)$ for every $P \in P(A(q))$. This allows us to regard every unimodular $(\tilde{A}(q), \tilde{\sigma}(q))$-valued $\varepsilon$-hermitian space with underlying module in $P(A(q))$ as an unimodular $(Z, \theta)$-valued $\varepsilon$-hermitian space. (More formally, $(P(A(q)), \text{Hom}_A^2(-, \tilde{A}(q)), \omega)$ is an exact hermitian subcategory of $(\mathcal{C}, \text{Hom}_A^2(-, Z), \omega)$.) Since $\mathcal{C}$ is abelian and its semisimple objects are the objects of $P(A(q))$, the Quebemann–Scharlau–Schulte theorem [38, Corollary 6.9, Theorem 6.10] tells us that the natural map $\tilde{W}_\varepsilon(A(q)) \to W_\varepsilon(\mathcal{C})$ is an isomorphism.

For every $A$-lattice $U \subseteq V$ with $U^f = U$, define $\delta_{Uf} : U^f // U \times U^f // U \to Z$ by $\delta_{Uf}(x+U, y+U) = f(x, y) + \tilde{A}_S$; this agrees with our previous definition of $\delta_{Uf}$ when $U^f \subseteq U$. It is enough to show that $(U^f // U, \delta_{Uf})$ lives in $\tilde{H}^f(\mathcal{C})$ and its Witt class in $W_\varepsilon(\mathcal{C})$ depends only on $[V, f]$.

That $\delta_{Uf}$ is $\varepsilon$-hermitian is straightforward. The exactness of $\text{Hom}_A^2(-, Z)$ means that $\text{length}_A M = \text{length}_A \text{Hom}_A^2(M, Z)$. Thus, in order to show that $\delta_{Uf}$ is unimodular, it is enough show that $x \mapsto \delta f(x, -) : U^f // U \to \text{Hom}_A^2(U^f // U, Z)$ is injective. This is the same as saying that radical of $\delta f$ is $0$, which follows readily from $U^f = U$ (Lemma 2.2(i)).

Next, assuming that $V$ and $f$ are fixed, we check that $[\delta_{Uf}]$ is independent of the $A$-lattice $U$. This is an adaptation of a classical argument [43, p. 204] (see also Lemma 6.1.4 and Theorem 5.3.4 in this source), to our more general situation. Let $U'$ be another $A$-lattice in $V$ with $U' \subseteq U^f$. We need to show that $[\delta_{Uf}] = [\delta_{U'f}]$. Since $U'' := U \cap U'$ is an $A$-lattice satisfying $U'' \subseteq U^f$ and $U'' \subseteq U, U'$, it is enough to consider the case where $U' \subseteq U$. Observe that $U' \subseteq U \subseteq U^f \subseteq U'$, and let $g = \delta_{Uf} \oplus (-\delta_{U'f})$. We claim that $L := \{(x+U, y+U') \in U^f // U \times U^f // U' : x - y \in U\}$ satisfies $L = L^\perp$ relative to $g$. Indeed, for all $x_1, x_2, y_1, y_2 \in U^f$ with $x_1 - y_1, x_2 - y_2 \in U$, we have
\[
g((x_1 + U, y_1 + U'), (x_2 + U, y_2 + U')) = f(x_1, x_2) - f(y_1, y_2) + \tilde{A}_S
\]
\[
= f(x_1, x_2 - y_2) + f(x_1 - y_1, y_2) + \tilde{A}_S = 0 + \tilde{A}_S,
\]
so \( g(L, L) = 0 \). On the other hand, if \( x \in U^f, \ y \in U'^f \) satisfy \( g(L, (x + U, y + U')) = 0 \), then \( f(z, x) - f(w, y) \in \mathcal{A}_S \) for all \( z, w \in U^f \) with \( z - w \in U \). Taking \( z = w \) shows that \( f(z, x - y) \in \mathcal{A}_S \) for all \( z \in U^f \), so \( x - y \in U^f \), whereas taking \( z = 0 \) and \( w \in U \) shows that \( f(U, y) \in \mathcal{A}_S \), so \( y \in U' \). Likewise, \( x \in U^f \) and we conclude that \( (x + U, y + U') \in L \), that is, \( L^\perp = L \). This means that \( [\partial_U f] \) is \( 0 \) in \( W_\epsilon(\mathcal{E}) \), or rather, \( [\partial_U f] = [\partial_U f'] \).

We now show that \([\partial_U f] \) is \( 0 \) whenever \((V, f)\) is metabolic. Since we assume that \( 2 \in R \times \), \([28, \text{Proposition I.3.7.1}]\) tells us that there is \( V_1 \in \mathcal{P}(A(p)) \) such that \( V \cong V_1 \times \text{Hom}_{\mathcal{A}}(V_1, \tilde{A}(p)) \) and under this isomorphism, \( f \) is given by \( f((x, \phi), (x', \phi')) = \phi x' + \varepsilon(\phi' x)\tilde{\sigma}(p) \). Identify \( V \) with \( V_1 \times \text{Hom}_{\mathcal{A}}(V_1, \tilde{A}(p)) \). Let \( U_1 \) be an \( \tilde{A}_S \)-lattice in \( V_1 \), and regard \( U'_1 := \text{Hom}(U_1, \tilde{A}_S) \) as an \( \tilde{A}_S \)-lattice in \( V'_1 := \text{Hom}_{\mathcal{A}}(V_1, \tilde{A}(p)) \). Then \( f \) restricts to a \( \tilde{A}_S \)-valued \( \mathcal{A}_S \)-bilinear pairing on \( U := U_1 \times U'_1 \). Written in matrix form, the map \( x \mapsto f(x, -) : U \to \text{Hom}_{\mathcal{A}}(U, \tilde{A}_S) \), denoted as \( \hat{f} \), is \([0 \ id_{U_1'} \gamma_{U_1} 0] \), where \( \omega_{U_1} \) is the natural map \( U_1 \to \text{Hom}_{\mathcal{A}}(\text{Hom}_{\mathcal{A}}(U_1, \tilde{A}_S), \tilde{A}_S) \) considered in Section 1.2. By Corollary 1.8 and 1.5 (applied with \( S \) in place of \( R \) and \( e = 1 \)), \( \omega_{U_1} \) is an isomorphism, and thus so is \( \hat{f} \). This implies readily that \( Uf = U \), so \([Uf/U, \partial f] = 0 \) in \( W_\epsilon(\mathcal{E}) \).

It remains to establish (H1) and (H2) for \( \mathcal{C} \) and \((Z, \theta)\). Condition (H1) will follow if we show that \( \text{Ext}^1_{\mathcal{A}_S}(M, Z) = 0 \) for all \( M \in \mathcal{C} \). It is enough to check this when \( M \) is semisimple, and hence for \( M = A(q) \). Since \( \text{Ext}^1_{\mathcal{A}_S}(A(q), Z) \cong \text{Ext}^1_{\mathcal{A}}(k(q), \tilde{k}(I)/\tilde{S}) \otimes A [39, \text{Theorem 2.39}] \), we are reduced into showing that \( \text{Ext}^1_{\mathcal{A}}(k(q), \tilde{k}(I)/\tilde{S}) = 0 \). To that end, it is enough to show that \( \text{Ext}^1_{\mathcal{A}}(k(q), \tilde{k}(I)) = 0 \) and \( \text{Ext}^2_{\mathcal{A}}(k(q), S) = 0 \). The first claim holds because \( \text{Ext}^1_{\mathcal{A}}(k(q), \tilde{k}(I)) \) is a \( C \)-divisible module annihilated by \( q \), and the second follows from Proposition 1.3(ii). Now that \( \text{Hom}_{\mathcal{A}}(-, Z) \) is exact on \( \mathcal{C} \), we can induct on the length of \( M \in \mathcal{C} \) in order to show that \( \omega_M \) is an isomorphism. When \( M \) is simple, this holds because \( M \in \mathcal{F}(A(q)) \) and \( \text{Hom}_{\mathcal{A}}(M, Z) = \text{Hom}_{\mathcal{A}}(M, \tilde{A}(q)) \), so we are done.

We shall see in Example 3.3 that when \( S = R/I \) is a discrete valuation ring, \( A = R \) and \( \varepsilon = 1 \), the map \( \partial_{I,q} \) is equivalent to the classical second residue map.

### 2.2 Differentials

We retain our original setting where \( R \) is a regular ring of dimension \( d \). For every \( e \geq 0, p \in R^{(e)} \) and \( q \in R^{(e+1)} \), define \( \partial_{p,q} = \partial_{A} : \tilde{W}_\epsilon(A(p)) \to \tilde{W}(A(q)) \) to be the \((p, q)\)-residue map \( \partial_{p,q} \) of Section 2.1 if \( p \subseteq q \), and 0 otherwise. We define the \( e \)th differential of \( \mathcal{G}W^{A,\sigma,\varepsilon} \) to be

\[
\partial_e := \sum_{p \in R^{(e)}} \sum_{q \in R^{(e+1)}} \partial_{p,q} : \mathcal{G}W^{A,\sigma,\varepsilon}_e \to \mathcal{G}W^{A,\sigma,\varepsilon}_{e+1}.
\]

By the following lemma, the inner sum is finite when evaluated on a Witt class.

**Lemma 2.4.** Let \( p \in R^{(e)} \) and \((V, f) \in \tilde{H}(A(p)) \). Then there exist only finitely many ideals \( q \in R^{(e+1)} \) such that \( \partial_{p,q}[V, f] \neq 0 \).
Proof. We may assume \( e < \dim R \). Write \( S = R/\mathfrak{p} \) (beware that \( S \) is not local) and \( \tilde{S} = \text{Ext}_R^e(S,R) \). We may still consider \( A \)-lattices \( U \) in \( V \) and define the \( A \)-lattice \( U^f \) as in Section 2.1 \((U^f)^f = U \) is no longer guaranteed).

Let \( q \in R^{e+1} \) be an ideal containing \( \mathfrak{p} \), and identify \( \tilde{S}_q \) with \( \text{Ext}_R^e(S_q,R_q) \). We claim that \((U^f)_q = (U_q)^f\), where the right-hand side is defined as in Section 2.1 relative to the ideals \( I = \mathfrak{p}_q \) and \( \mathfrak{q}_q \). Indeed, recall from the proof of Lemma 2.2 that \( x \mapsto f(x,−) \) defines an \( S \)-isomorphism \( U^f \to \text{Hom}_A(U,A \otimes \tilde{S}) \), and likewise, \((U_q)^f \cong \text{Hom}_A(U_q,A \otimes \tilde{S}_q) \). It is routine to check that under these isomorphisms, the natural isomorphism \( \text{Hom}_A(U,  \tilde{S} \otimes A) \cong \text{Hom}_A(U_q,  \tilde{S}_q \otimes A) \) corresponds to the natural map \((U^f)_q \to (U_q)^f\), hence our claim.

Choose an \( A \)-lattice \( U \) in \( V \), and replace it with \( U \cap U^f \) to assume \( U \subseteq U^f \). By the previous paragraph, we have \((U_q)^f = (U_q)_q = U_q \) for every \( q \in \text{Spec } R \) containing \( \mathfrak{p} \) with \( q \not\in \text{supp } R(U^f/U) \).

Thus, \( \partial_{\mathfrak{p},q}[V,f] = 0 \) for all \( q \not\in \text{supp } R(U^f/U) \cap R^{e+1} \). Choose \( s \in S \setminus \{0\} \) such that \( U^f \subseteq U \).

Then \( U^f/U \) is a finite \( S/sS \)-module. Since \( \dim S/sS \leq \dim S - 1 \), the set \( \text{supp } R(U^f/U) \cap R^{e+1} \) is finite, hence the lemma.

It remains to check that \( \mathcal{G}^{A,e}_+ \) is a cochain complex, that is, \( d_{e+1} \circ d_e = 0 \) for all \( e \geq -1 \). This is the content of the next theorem.

**Theorem 2.5.** In the previous notation, \( d_{e+1} \circ d_e = 0 \) for all \( e \geq -1 \).

The proof is somewhat technical. Given \( M \in \mathcal{M}(R) \), we shall abbreviate \( A \otimes M \) to \( A M \) for brevity. We first prove the following lemma.

**Lemma 2.6.** Assume that \( R \) is a regular local ring of dimension \( e + 1 \) \((e \geq 0)\) with maximal ideal \( q \). Let \( I_2 \subseteq I_1 \) be two ideals of \( R \) such that both \( S_1 = R/I_1 \) and \( S_2 = R/I_2 \) are Cohen–Macaulay of dimension 1. For \( i = 1, 2 \), define \( C_i, m_i, \tilde{S}_i, \tilde{m}_i^{-1} \) similarly to \( C, m, \tilde{S}, \tilde{m}^{-1} \) in Section 2.1. Then:

(i) \( C_2 \subseteq C_1 \), and for every finite \( C_1 \)-torsion-free \( S_1 \)-module \( M \), the map \( MC_1^{-1} \to MC_2^{-1} \) is an isomorphism.

Let \( t = t_{I_1,I_2} \) denote the composition

\[
  t_{I_1,I_2} : \tilde{A}(I_1) = A\tilde{S}_1C_1^{-1} = A\tilde{S}_1C_2^{-1} \to A\tilde{S}_2C_1^{-1} = \tilde{A}(I_2)
\]

induced by (i) and the quotient map \( R/I_2 \to R/I_1 \).

(ii) The map \( t : \tilde{A}(I_1) \to \tilde{A}(I_2) \) is injective, \( \text{im}(t) = \text{Ann } \tilde{A}(I_2)I_1 \), and we have \( A\tilde{S}_2 \cap t(\tilde{A}(I_1)) = t(A\tilde{S}_1) \) and \( A\tilde{m}^{-1}_2 \cap t(\tilde{A}(I_1)) = t(A\tilde{m}^{-1}_1) \) as subsets of \( \tilde{A}(I_2) \).

(iii) For every \((V,f) \in \mathcal{H}^e(A(I_1))\), we have \((V,t \circ f) \in \mathcal{H}^e(A(I_2))\) and

\[
  \partial_{I_2,q}[V,t \circ f] = \partial_{I_1,q}[V,f].
\]

**Proof.** (i) That \( C_2 \subseteq C_1 \) is a consequence of Lemma 1.4. The same proposition also implies that \( S_2C_1^{-1} \) is 0-dimensional, so it is artinian. This means that \( \text{length }_{MC_2^{-1}} MC_1^{-1} < \infty \). Since every element of \( C_1 \) acts faithfully on \( M \), it follows that \( MC_1^{-1} \) is \( C_1 \)-divisible, and \( MC_2^{-1} \to MC_2^{-1}C_1^{-1} = MC_1^{-1} \) is an isomorphism.
(ii) Let \( r_1, \ldots, r_t \in R \) be generators of \( I_1 \). Define \( f : S'_2 \to S_2 \) by \( f(x_1, \ldots, x_t) = \sum_{i=1}^t r_i x_i \) and let \( K = \ker f \). Then we have an exact sequence

\[
0 \to K \to S'_2 \xrightarrow{f} S_2 \to S_1 \to 0,
\]

which, thanks to Proposition 1.5, consists of 1-dimensional Cohen–Macaulay \( R \)-modules. Since \( \text{Ext}^e_R(−, R) \) induces a duality on the latter [11, Theorem 3.3.10(c)], and since \( A \) is flat over \( R \), we have an exact sequence of \( R \)-modules

\[
0 \to A \tilde{S}_1 \to A \tilde{S}_2 \to (A \tilde{S}_2)^f
\]

in which \( f' \) is given by \( f'(x) = (r_1 x, \ldots, r_t x) \).

By localizing (2.2) at \( C_2 \), we see that \( \iota : A\tilde{I}_1 \to A\tilde{I}_2 \) is injective and \( \text{im}(\iota) = \text{ann} A\tilde{I}_1 \).

Finally, if \( x \in \iota(A\tilde{m}_1^{-1}) \), then \( xI_1 = 0 \) and \( x m_2 \subseteq \iota(A\tilde{S}_1) \subseteq A\tilde{S}_2 \), so \( x \in A\tilde{m}_2^{-1} \cap \text{Ann} A\tilde{I}_1 = A\tilde{m}_2^{-1} \cap \iota(A\tilde{I}_1) \). Conversely, if \( x \in A\tilde{m}_2^{-1} \cap \iota(A\tilde{I}_1) \), then there is \( y \in A\tilde{I}_1 \) with \( \iota(y) = x \). We have \( x m_2 \subseteq A\tilde{S}_2 \cap \iota(A\tilde{I}_1) = (A\tilde{S}_2)_{\iota} \).

(iii) That \( \iota \circ f \) is unimodular follows easily from \( \iota(A\tilde{I}_1) = \text{Ann} A\tilde{I}_1 \) and \( V I_1 = 0 \).

By Lemma 2.2, there exists an \( A \)-lattice \( U \) in \( V \) with \( U^f \subseteq U \subseteq U^f \). Thanks to (ii) and the fact that \( U \cdot I_1 = 0 \), we have \( U^f \circ f = U^f \), so \( \partial_{2,q} [V, \iota \circ f] \) is represented by \( g : U^f / U \times U^f / U \to k(q) \) given by \( g(x + U, y + U) = T_{2,q}(\iota(f(x, y))) \). It is therefore enough to show that \( T_{2,q,A} \circ \iota | A\tilde{m}_1^{-1} = T_{1,q,A} \). This follows by applying \( \text{Ext}^e_R(−, R) \otimes A \) to the following morphism of short exact sequences:

\[
\begin{array}{ccc}
\mathcal{m}_2 & \longrightarrow & S_2 \\
\downarrow & & \downarrow \\
\mathcal{m}_1 & \longrightarrow & S_1 \\
\end{array}
\]

\[
\begin{array}{ccc}
& & k(q) \\
\mathcal{m}_2 & \longrightarrow & S_2 \\
\downarrow & & \downarrow \\
\mathcal{m}_1 & \longrightarrow & S_1 \\
\end{array}
\]

\[
\begin{array}{ccc}
& & k(q) \\
\end{array}
\]

\[\square\]

**Proof of Theorem 2.5.** That \( d_0 \circ d_{-1} = 0 \) is straightforward and is left to the reader. We will prove that \( d_{e+1} \circ d_e = 0 \) for all \( e > 0 \). This amounts to showing that for all \( t \in R(e) \), \( q \in R(e+2) \) with \( t \subseteq q \) and \( (V, f) \in H^e(A(t)) \), we have \( \sum_p \delta_{p,q} \delta_{t,p} [V, f] = 0 \), where the sum is taken over the set of primes lying strictly between \( t \) and \( q \). To that end, we may base-change from \( R \) to \( R_\mathcal{q} \) and assume that \( R \) is regular local of dimension \( e + 2 \) and \( q \) is its maximal ideal.

The set of height-\( i \) primes containing a given ideal \( J \) will be denoted as \( R_{i\mathcal{J}} \). Given \( M \in \mathcal{M}(R) \), we shall abbreviate

\[
M^{[i]} := \text{Ext}^e_R(\mathcal{M}(R), M),
\]

for example, \( \mathcal{K}(t) = (R/t)^{[e]} \).

**Step 1.** Since \( R \) is regular and \( \text{hgt} t = e \), there is a regular sequence \( r_1, \ldots, r_e \in t \) [32, Theorem 17.4(i)]. Write \( J = r_1 R + \cdots + r_e R \), and let \( D \) denote the set of elements of \( R \) acting faithfully on \( R/J \). Then \( R/J \) is a complete intersection local ring of dimension \( 2 = (e + 2) - e \), hence Cohen–Macaulay with dualizing module \((R/J)^{[e]}\).
Proposition 1.3(i) and the flatness of $A$ over $R$ imply that for every ideal $J' \subseteq R$ with $J \subseteq J'$ and $J' \cap D \neq \emptyset$, the natural map $A(J'/J)[e] \to A(J'/J)[e]D^{-1} = A(R/J)[e]D^{-1}$ is injective. We shall therefore freely regard $A(J'/J)[e]$ as a subset of $A(R/J)[e]D^{-1}$. If $J''$ is another ideal satisfying $J \subseteq J''$ and $J' \cap D \neq \emptyset$, then $A(J'/J)[e]D^{-1} \subseteq A(J''/J)[e]D^{-1}$ whenever $J'' \subseteq J'$. Note that $J' \cap D \neq \emptyset$ holds whenever $J'/J$ is not contained in a minimal prime of $R/J$ (Lemma 1.4).

By Lemma 2.6(iii), for every $\mathfrak{p} \in R^{(e+1)}$ containing $\mathfrak{t}$, we have $\frac{\partial}{\partial \mathfrak{p}, \mathfrak{p}} : A(R/\mathfrak{t})[e] \to A(R/J)[e]D^{-1}$ (this is independent of $\mathfrak{p}$, in fact). Replacing $f$ with $\mathfrak{t} \circ f$, we are reduced into proving

$$\sum_{\mathfrak{p}} \partial_{\mathfrak{p}, \mathfrak{q}} \partial_{J, \mathfrak{p}}[f] = 0,$$

where the sum is taken over $R^{(e+1)}$ (note that if $\mathfrak{t} \not\subseteq \mathfrak{p}$, then $V_{\mathfrak{p}} = 0$ and $\partial_{J, \mathfrak{p}}[f] = 0$).

Step 2. Let us call an $A$-submodule $L \subseteq V$ an $A$-lattice (in $V$) if $L$ is finitely generated and $LD^{-1} = V$. In this case, define

$$L^f = \{x \in V : f(L, x) \subseteq A(R/J)[e]D^{-1}\}.$$

As in the proof of Lemma 2.2, $L^f$ is also an $A$-lattice in $V$.

Choose an $A$-lattice $L$ in $V$. Replacing $L$ with $L \cap L^f$, we may assume that $L \subseteq L^f$. Since $L^f$ is a noetherian $R$-module, there is $L' \subseteq L^f$ maximal with respect to the property that $L' \subseteq L'^f$. Replace $L$ with $L'$. Then every $A$-lattice $M$ containing $L$ and satisfying $M \subseteq M^f$ must be equal to $L$. In particular, $L^f = L$.

Since $LD^{-1} = V = L^f D^{-1}$, there is some $r \in D \cap q$ such that $L^f r \subseteq L$. Fix such $r$, let

$$I = J + Jr = Jr_1 + \cdots + Jr_e + Jr,$$

and let $C$ denote the set of elements of $R$ acting faithfully on $R/I$. Since $r_1, \ldots, r_e, r$ is a regular sequence, $R/I$ is a 1-dimensional complete intersection local ring with dualizing module $(R/I)^{e+1}$.

As in the proof of Lemma 2.4, the set

$$(LC^{-1})^f = \{x \in V : f(LC^{-1}, x) \subseteq A(R/J)[e]C^{-1}\}$$

coincides with $L^f C^{-1}$. Since $L^f C^{-1}r \subseteq LC^{-1}$, we have $f(L^f C^{-1}, L^f C^{-1})r \subseteq A(R/J)[e]C^{-1}$, or rather, $f(L^f C^{-1}, L^f C^{-1}) \subseteq A(R/J)[e]C^{-1} \cdot r^{-1}$. By Proposition 1.6, this means that $f(L^f C^{-1}, L^f C^{-1}) \subseteq A(I/J)[e]C^{-1}$. Writing $U = L^f C^{-1}/LC^{-1}$, we may now define $g : U \times U \to \tilde{A}(I) = A(R/I)[e]C^{-1}$ by

$$g(x + LC^{-1}, y + LC^{-1}) = T_{J,I}(f(x, y)),$$

where

$$T_{J,I} : A(I/J)[e]C^{-1} \to A(R/I)[e+1]C^{-1}$$

is induced by $I/J \leftarrow R/J \rightarrow R/I$. 
Step 3. Let $I_0$ denote the radical of $I$. Since $L/\ell L \subseteq L$ and $R$ is noetherian, there is some $t \in \mathbb{N} \cup \{0\}$ such that $L/I_0^{t+1} \subseteq L$. If $t > 1$, then one readily checks that $f(L + L/I_0^{t-2}, L + L/I_0^{t-1}) \subseteq A(R/J)[e]$, or rather, $L + L/I_0^{t-2} \subseteq (L + L/I_0^{t-1})^f$. By the maximality of $L$ (Step 2), this means that $L/I_0^{t+1} \subseteq L$. Continuing in this manner, we find that $L/I_0 \subseteq L$.

As in the proof of Lemma 2.4, for every $p \in R_{(e+1)}$, we have $(L_p)[f] = (L_p)$. If $p \supseteq I$, then $(L_p)[f] = (L_p)[I_0] \subseteq L_p$, and thus $\partial_{L,p}[f]$ is represented by $((L_p[p)/L_p, \partial_{L,p} f)$. On the other hand, if $p \nsubseteq I$, then $L_p = L$ and $\partial_{L,p} f = 0$.

Since $U = L/C^{-1}/LC^{-1}$ is a module over theartinian ring $(R/I)C^{-1}$, the natural map $U \to \prod_{p \in R_{(e+1)}} U_p = \prod_{p \in R_{(e+1)}} (L_p)/L_p$ is an isomorphism. We claim that under this isomorphism,

$$g = \bigoplus_{p \in R_{(e+1)}} (t_{p,[L_p]} \circ \partial_{L,p} f).$$

(2.3)

(In particular, $g : U \times U \to A(R/I)^{e+1}C^{-1}$ is unimodular.) Provided that this holds, Lemma 2.6(iii) tells us that

$$\partial_{L,q}[g] = \sum_{p \in R_{(e+1)}} \partial_{q,p} \partial_{L,p} f,$$

so we are reduced into proving that $\partial_{q,p} [U, g] = 0$.

We now prove (2.3). Let $p \in R_{(e+1)}$. Since $UI_0 = 0$, the image of the (unique) $R$-module section of $U \to U_p$ is the set of $p$-torsion elements in $U$. Let $x, y$ be elements in that set, and let $x, y \in L/C^{-1}$ be lifts of $x, y$. Then $y \subseteq LC^{-1}$, and so $f(x, y)p = f(x, y) \subseteq A(R/J)[e]C^{-1}$. By Proposition 1.6, this means that $f(x, y) \in A(p/J)[e]C^{-1}$, and thus $g(x, y) = T_{f,l}(f(x, y))$. On the other hand, $(t_{p,[L_p]} \circ \partial_{L,p} f)(x, y) = t_{p,[L_p]} T_{f,l} f(x, y)$, so it is enough to check that $T_{f,l}$ and $t_{p,[L_p]} \circ \partial_{L,p} f$ agree on $A(p/J)[e]C^{-1}$. This follows from the commutative diagram on the right, which is induced by the commutative diagram on the left.

Step 4. Write $M = L/\ell L$. We claim that $M$ is either 0 or a 1-dimensional Cohen–Macaulay $R$-module. By Proposition 1.5, it is enough to show that $q \notin Ass_R M$. Suppose that $\bar{x} \in M$ is annihilated by $q$, and let $x \in L$ be a lift of $x$. Then $xq \in L$, so $f(x, x)q = f(x, xq) \subseteq A(R/J)[e]$. By Proposition 1.6, this means that $f(x, x) \in A(q/J)[e]$. Since both $(R/q)[e]$ and $(R/q)[e]^{-1}$ are 0 (Proposition 1.3), the map $(R/J)[e] \to (q/J)[e]$ is an isomorphism, and $f(x, x) \in A(R/J)[e]$. This means that $f(L + xA, L + xA) \subseteq (R/J)[e]$, so $L + xA = L$ by the maximality of $L$ (Step 2). We conclude that $\bar{x} = 0$.

Next, we claim that $L/\ell L$ is a 2-dimensional Cohen–Macaulay $R$-module. Since the element $r$ of Step 2 acts faithfully on $L/\ell L$, it is enough to show that $L/\ell L/r$ is a 1-dimensional Cohen–Macaulay $R$-module, which again amounts to showing $q \notin Ass_R (L/\ell L)/r$. Let $\bar{x} \in L/\ell L/r$ be an element annihilated by $q$ and let $x \in L$ be a lift of $\bar{x}$. Then $q \subseteq L, r$, hence $f(L, xr^{-1})q = f(L, xr^{-1})q \subseteq$
As in the previous paragraph, this means that \( f(L, xr^{-1}) \subseteq A(R/J)^{[e]} \) and \( xr^{-1} \in L^f \).

Thus, \( x \in L^f r \) and \( \overline{x} = 0 \).

**Step 5.** By Step 4 and Proposition 1.5, \( M \) is \( C \)-torsion-free, so we may regard \( M \) as an \( A \)-lattice in \( U \). We will prove that \( M^g = M \), and thus \( \partial_{I,g} \cdot [g] = 0 \).

We first prove that \( M \subseteq M^g \), or equivalently, \( g(M, M) \subseteq A(R/I)^{[e+1]} \). Consider the diagram on the right, which is induced by the commutative diagram on the left:

\[
\begin{array}{cccc}
I/J & q/J & A(I/J)^{[e]} & A(q/I)^{[e+1]} & A(q/J)^{[e+1]} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
I/J & R/J & A(R/I)^{[e+1]} & \tilde{A}(q) & \tilde{A}(q) \\
\downarrow & \downarrow & \downarrow & \downarrow \\
k(q) & k(q) & k(q) & k(q) & k(q)
\end{array}
\]

The top row of the right diagram is exact in the middle, so \( \tau_{I,q} \circ \tau_{J,I} |_{A(I/J)^{[e]}} = 0 \), or rather, \( \tau_{J,I}(A(I/J)^{[e]}) \subseteq \ker \tau_{I,q} = A(R/I)^{[e+1]} \) (see (2.1)). Since \( \tau(I^f, L^f) I = \tau(L^f J, L^f) \subseteq \tau(L, L^f) \subseteq A(R/J)^{[e]} \), we have \( \tau(L^f, L^f) \subseteq A(I/J)^{[e]} \) (Proposition 1.6), and it follows that \( g(M, M) \subseteq A(R/I)^{[e+1]} \).

**Step 6.** It remains to show that \( y \mapsto g(y, -) : M \to Hom_A(M, A(R/I)^{[e]}) \), denoted as \( \varphi \), is an isomorphism. Consider the following diagram of \( R \)-modules:

\[
\begin{array}{cccc}
L & \xrightarrow{x \mapsto f(x, -)} & Hom_{A/AJ}(L^f, A(R/J)^{[e]}) & \\
\downarrow & \downarrow & \downarrow & \\
L^f & \xrightarrow{x \mapsto f(x, -)} & Hom_{A/AJ}(L, A(R/J)^{[e]}) & u \\
\downarrow & \downarrow & \downarrow & \\
M & \xrightarrow{\varphi} & Hom_A(M, A(R/I)^{[e+1]}) & \xrightarrow{u} Ext^1_{A/AJ}(M, A(R/J)^{[e]}) \\
\end{array}
\]

The right column is induced by \( L \hookrightarrow L^f \to M \), so it is exact in the middle, and the map \( u \) is induced by \( (R/J)^{[e]} \hookrightarrow (I/J)^{[e]} \to (R/J)^{[e+1]} \) (we have \( (R/I)^{[e]} = 0 \) and \( (R/J)^{[e+1]} = 0 \) by Proposition 1.3(ii)). The top and middle horizontal maps are bijective because \( f \) is unimodular and \( L^f f = L \), and the top rectangle clearly commutes. Thus, in order to show that \( \varphi \) is an isomorphism, it is enough to show the following:

(i) \( u \) is onto,
(ii) \( v \) is bijective,
(iii) the bottom rectangle commutes.

By Proposition 1.7(i) and Step 4, we have \( Ext^1_{A/AJ}(L^f, A(R/J)^{[e]}) = 0 \), so \( u \) is onto. We have \( I/J = r(R/J) \), so for similar reasons, \( Hom_{A/AJ}(M, A(I/J)^{[e]}) \cong Hom_{A/AJ}(M, A(R/J)^{[e]}) = 0 \) and \( v \) is injective. To show that \( v \) is surjective, it is enough to check that \( Ext^1_{A/AJ}(M, A(R/J)^{[e]}) \to Ext^1_{A/AJ}(M, A(I/J)^{[e]}) \) is the zero map. Observe that \( I/J \hookrightarrow R/J \to R/I \) is isomorphic to \( R/J \to \).
This means that the morphism $\text{Ext}^1_{\mathcal{A}/\mathcal{A}_J}(\mathcal{M}, A(R/J)^{[e]}) \to \text{Ext}^1_{\mathcal{A}/\mathcal{A}_J}(\mathcal{M}, A(I/J)^{[e]})$ is isomorphic to $r : \text{Ext}^1_{\mathcal{A}/\mathcal{A}_J}(\mathcal{M}, A(R/J)^{[e]}) \to \text{Ext}^1_{\mathcal{A}/\mathcal{A}_J}(\mathcal{M}, A(R/J)^{[e]})$, which is the zero map because $Mr = 0$. We conclude that (i) and (ii) hold.

In order to prove (iii), we interpret elements of $\text{Ext}^1_{\mathcal{A}/\mathcal{A}_J}(\mathcal{M}, A(R/J)^{[e]})$ as $\mathcal{A}$-module extensions of $\mathcal{M}$ by $A(R/J)^{[e]}$. Let $x \in L^r$, and let

$$X = \{(u, m) \in A(I/J)^{[e]} \times M : T_{f,J}(u) = g(x + L, m), \}$$

$$Y = (A(R/J)^{[e]} \times L^f) / \{(f(x, \ell), -\ell) | \ell \in L\}.$$

One readily checks that the extensions of corresponding to the two possible images of $x$ in $\text{Ext}^1_{\mathcal{A}}(\mathcal{M}, A(R/J)^{[e]})$ are

$$A(R/J)^{[e]} \xrightarrow{\alpha} X \xrightarrow{\beta} M \quad \text{and} \quad A(R/J)^{[e]} \xrightarrow{\gamma} Y \xrightarrow{\delta} M,$$

where $\alpha(u) = (u, 0), \beta(u, m) = m, \gamma(u) = (u, 0)$ and $\delta(u, \ell) = \ell + L$, and $(u, \ell)$ denotes the image of $(u, \ell) \in A(R/J)^{[e]} \times L^f$ in $Y$. Define $\psi : Y \to X$ by

$$\psi(u, \ell) = (u + f(x, \ell), \ell + L).$$

It is easy to check that $\psi$ is well-defined and defines a morphism between the extensions. Moreover, $\psi$ is an isomorphism by the Five Lemma. This completes the proof of (iii), so we have established the theorem. 

**Proposition 2.7.** The complex $GW_{n,R}^{\mathbb{Z},1}$ is isomorphic to the Gersten–Witt complex of $R$ defined by Balmer and Walter in [6].

**Proof.** The proposition follows from the fact that $\partial_{p,q} : \tilde{W}(k(p)) \to \tilde{W}(k(q))$ ($p \in R^{(e)}, q \in R^{(e+1)}$) is equivalent to the corresponding map in [6, Proposition 8.5(b)], a fact that we now prove. By localizing at $q$, we may assume that $R$ is local and $\mathfrak{m}$ is its maximal ideal.

Let $(V, f) \in \mathcal{H}^{1}(k(p))$. Choose a regular sequence $r_1, \ldots, r_e \in \mathfrak{p}$ (use [32, Theorem 17.4(i)]), let $I = r_1R + \cdots + r_eR$, and define $S$ and $\tilde{S}$ as in Section-2.1. By Lemma 2.6, $\partial_{p,q}[f] = \partial_{i,q}[t_i \circ f]$, so we may replace $f$ with $t_i \circ f$. Choose any $\mathcal{A}$-lattice $U$ in $V$ with $U \subseteq U^f$, set $Z = \tilde{k}(I)/\tilde{S}$, and define $\partial_{U,f} : U^f/U \times U^f/U \to Z$ as in the proof of Lemma 2.3. Let $\varphi$ denote the isomorphism $x \mapsto \partial_{U,f}(x, -) : U^f/U \to \text{Hom}(U^f/U, Z)$. One readily checks that upon identifying $U^f$ and $U$ with $\text{Hom}(U, \tilde{S})$ and $\text{Hom}(U^f, \tilde{S})$ via $y \mapsto f(y, -)$, the map $\varphi$ fits into the commutative diagram

\begin{equation}
\begin{array}{ccc}
U & \to & \text{Hom}(U, \tilde{S}) \\
& & \downarrow \varphi \\
\text{Hom}(\text{Hom}(U, \tilde{S}), \tilde{S}) & \to & \text{Hom}(U, \tilde{S}) \\
& & \downarrow \psi \\
& & \text{Ext}^1(U^f/U, \tilde{S})
\end{array}
\end{equation}

(2.4)
in which $v$ is induced by $\tilde{S} \hookrightarrow \tilde{k}(I) \to Z$ and $u$ is induced by the top row. The commutativity is shown as in Step 6 of the proof of Theorem 2.5. The map $v$ is an isomorphism because $\text{Hom}(U^f/U, \tilde{k}(I))$ and $\text{Ext}^1(U^f/U, \tilde{k}(I))$ are both annihilated by some power of $q$ and $C$-divisible, hence $0$ $(q \cap C \neq \emptyset$ by Lemma 1.4).

For every $i$-dimensional Cohen–Macaulay $S$-module $M$, there is a canonical isomorphism $\text{Ext}^i(M, \tilde{S}) \cong \text{Ext}^{i+e}(M, R)$. To see this, let $P_0 \to \cdots \to P_0 \to S$ and $Q_{e+i} \to \cdots \to Q_0 \to M$ be $R$-projective resolutions with $P_0 = R$ (their lengths are justified by [11, Theorem 1.3.3]). By Proposition 1.3(ii), $P_0^e \to \cdots \to P_0^e \to \tilde{S}$ is an $R$-projective resolution of $\tilde{S}$. Consider the induced double complex:

$$
\begin{array}{ccc}
0 & 0 & 0 \\
\cdots \to \text{Hom}(Q_{i}, P_{0}^e) \to \cdots & \text{Hom}(Q_{e+i}, P_{0}^e) \to \text{Ext}^{e+i}(M, P_{0}^e) \to 0 \\
\cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
0 & 0 & 0
\end{array}
$$

All rows except the bottom one and all columns except the right one are exact. Standard diagram chasing therefore implies that $\text{Ext}^i(M, \tilde{S}) = (\ker h')/(\im h)$ is isomorphic to $\ker w$. By taking $P_0$ to be the Koszul complex associated to the regular sequence $r_1, \ldots, r_e$ (Proposition 1.9), we see that $w$ is in fact 0, because $MI = 0$. Thus, $\ker w = \text{Ext}^{e+i}(M, P_0^e) \cong \text{Ext}^{e+i}(M, R)$ and $\text{Ext}_S^i(M, \tilde{S}) \cong \text{Ext}^{e+i}(M, R)$.

Now, after replacing $f$ with $(-1)^{\frac{(e-1)(e-2)}{2}} f$, the diagram (2.4) (without the middle term on the right column) is naturally isomorphic to diagram (35) in [6] when $U$ is a free $R/p$-module. This means that, up to a sign depending on $e$, the map $\delta_{p,q}$ is equivalent to the map in [6, Proposition 8.5(b)].

\[\square\]

**Remark 2.8.** Suppose that $(A, \sigma)$ is an Azumaya $R$-algebra with unitary involution. Then $S := Z(A)$ is a quadratic étale $R$-algebra and $\sigma |_S$ is the standard $R$-involution on $S$. If $p \in R^{(e)}$ splits in $S$, then $S(p) \cong k(p) \times k(p)$ and $(A(p), \sigma(p)) \cong (B \times B^{op}, (x, y^{op}) \mapsto (y, x^{op}))$ for some central simple $k(p)$-algebra $B$. As a result, $\mathcal{W}_c^e(A(p)) = 0$ [18, Example 2.4] and $p$ does not contribute to $\mathcal{G}W_{c}^{A, \sigma}$. In particular, if $S = R \times R$, then $\mathcal{G}W_{c}^{A, \sigma}$ is the zero complex.

We finish this section by showing that the isomorphism class of $\mathcal{G}W_{c}^{A, R, \sigma, \varepsilon}$ is independent of $R$. This will usually be used to replace $R$ with $Z(A)^{[e]}$ in order to assume that $(A, \sigma)$ is Azumaya over $R$.

**Theorem 2.9.** Let $R'$ be a regular ring and suppose that $A$ is equipped with an $R'$-algebra structure such that $(A, \sigma)$ is a separable projective $R'$-algebra with involution. Then $\mathcal{G}W_{c}^{A, R, \sigma, \varepsilon} \cong \mathcal{G}W_{c}^{A, R', \sigma, \varepsilon}$. 

Proof. Since the structure morphism $R' \to A$ factors through $Z(A)[σ]$, it is enough to prove the theorem when $R' = Z(A)[σ]$. In particular, we may assume that $R'$ is a finite étale $R$-algebra. Under this additional assumption, we shall define an isomorphism $ψ = ψ_{R,R'} : GW_{+}^{A/R,σ,ζ} \cong GW_{+}^{A'R',σ,ζ}$ as follows.

Let $ψ_{-1}$ be the identity map $W(σ) \to W(A,σ)$. To define $ψ_e$ for $e \geq 0$, let $φ : Spec R' \to Spec R$ denote the morphism corresponding to the structure map $R \to R'$. Since $R'$ is finite étale over $R$, the fibers of $φ$ are finite, $φ(R'(e)) \subseteq R(e)$ for all $e \geq 0$, and $𝔭 R' = \bigcap 𝔮 R' \subseteq R(𝔭)$ for all $𝔭 R'$. Let $ψ = ψ_{φ,R,R'} : W(A(𝔭)) \to W(A(𝔭))$ be the identity map $W(𝔭) \to W(𝔭)$.

To define $ψ_e$ for $e \geq 0$, let $φ : Spec R' \to Spec R$ denote the morphism corresponding to the structure map $R \to R'$. Since $R'$ is finite étale over $R$, the fibers of $φ$ are finite, $φ(R'(e)) \subseteq R(e)$ for all $e \geq 0$, and $𝔭 R' = \bigcap 𝔮 R' \subseteq R(𝔭)$ for all $𝔭 R'$. Let $ψ = ψ_{φ,R,R'} : W(A(𝔭)) \to W(A(𝔭))$ be the identity map $W(𝔭) \to W(𝔭)$.

Finally, let

$$ψ_e = \sum_{p \in R(e)} \sum_{p \in p^{-1}(p)} ψ_{p,q} : \bigoplus_{p \in R(e)} W(A(p)) \to \bigoplus_{p \in R(e)} W(A(p)).$$

In order to show that $ψ_e$ is an isomorphism, it is enough to check that the sum $\sum_{p \in Av^{-1}(p)} \psi_{p,q} : W(A(p)) \to \bigoplus_{p \in p^{-1}(p)} W(A(p))$ is an isomorphism. This follows readily from the fact that $R'(p) = \prod_{p \in Av^{-1}(p)} k'(p)$, which means that the natural map $V \to \prod_{p \in p^{-1}(p)} V_p$ is an isomorphism of $R'(p)$-modules for every $R'(p)$-module $V$, and likewise for $A(p)$-modules.

It remains to check that $ψ = (ψ_e)_{e \geq -1}$ is compatible with the differentials of $GW_{+}^{A/R,σ,ζ}$ and $GW_{+}^{A'R',σ,ζ}$. Then we need to check that $d'_e \circ ψ_e = ψ_{e+1} \circ d_e$ for all $e \geq -1$. We leave the straightforward case where $e = -1$ to the reader and assume $e \geq 0$. In this case, it is enough to show that for all $p \in R(e)$ and $Q \in R(e+1)$, we have $\sum_{q} q_p \circ ψ_{q,p} = q_{Q} \circ ψ_{q',p'}$, where $q = i(Q)$ and the sum is taken over all $p \in p^{-1}(p)$ contained in $Q$. We assume that $p \subseteq q$ otherwise both sides are 0. We may further assume that $R$ is local and $q$ is its maximal ideal.

Fix $p \in R(e)$ and $(V, f) \in H^e(A(p))$. Write $I = p R'_e$. Then $R'_e / I$ is Cohen–Macaulay [46, Tag 025Q], so we may consider $δ_{l,Q}$. We claim that

$$δ_{l,Q}[f] = (δ_{p,q}[f])_{Q}. \hspace{1cm} (6)$$

Indeed, choose an $A$-lattice $U$ in $V$ with $U^f q \subseteq U \subseteq U^f$. As in the proof of Lemma 2.4, we have $(U^f_q) = (U^f)_{Q}$. Since $U^f_q$ is an $R'_e / I$-module and $U^f_q Q = U^f_q R'_e q \subseteq U^f_q$, it follows that $δ_{l,Q}[f]$ is represented by $(U^f_q / U^f, δ f_q) = (U^f / U, δ f)_Q$, hence our claim.

Let $Q \in R(e+1)$ and let $T = \{ p \in φ^{-1}(p) : p \subseteq Q \}$. Then the natural maps $V_q \to \prod_{p \in T} V_q$ and $A(p)_{Q} \to \prod_{p \in T} A(p)_{q}$ are isomorphisms. Given $p \in T$, the (unique) $A$-module section of $A(p)_{q} \to A(p)_{q}$ is just the map $q_{Q} \circ φ_{p}$. By part (iii) of the lemma, $δ_{l,Q}[f] = \sum_{q \in Q} δ_{p,q}[f]$. Thanks to (6), this means that $ψ_{q,Q} δ_{p,q}[f] = \sum_{q \in Q} ψ_{q,p,φ}[f]$, which is what we need to show. □
3 | COMPUTING RESIDUE MAPS

Suppose $R, (A, \sigma)$ and $\varepsilon$ are as in Section 2.1. Let $e \in \mathbb{N} \cup \{0\}$, $p \in R^{(e)}$, $q \in R^{(e+1)}$ and assume $p \subseteq q$. We now show that the definition of $\partial_{p,q} : W_\varepsilon(A(p)) \rightarrow W_\varepsilon(A(q))$, see Section 2.2, can be made even more explicit, to the extent of allowing hands-on computations. This will give an elementary method to compute the differentials of $GW^{A,\sigma,\varepsilon}$, which will be used in the sequel.

To that end, we may assume that $R$ is local and $q$ is its maximal ideal. In fact, we shall further require that $R/p$ is a complete intersection ring, because the general case can be deduced from this special case, see Remark 3.5.

Similarly to Section 2.1, we shall work in greater generality, replacing $p$ with any ideal $I$ of $R$ such that $S := R/I$ is a 1-dimensional complete intersection ring. Define $C$, $m$, $k(I)$, $\tilde{S}$, $\tilde{m}$ and $\tilde{k}(I)$ as in Section 2.1. By Proposition 1.9, $I/I^2$ is a free $S$-module of rank $e$, and

$$S \cong \text{Hom}(\wedge^e(I/I^2), S)$$

canonicaly. In particular, $S$ is a free $S$-module of rank 1. For the sake of brevity, given a regular sequence $r_1, \ldots, r_e \in R$ generating $I$ and some $s \in R$, we shall write $[r_1 \wedge \cdots \wedge r_e \mapsto s]$ to denote the unique element of $S$ sending $(r_1 + I^2) \wedge \cdots \wedge (r_e + I^2)$ to $s + I$. The same convention will be applied to other complete intersection ideals, for example, $IC^{-1} \subseteq RC^{-1}$ and $q \subseteq R$.

Let us fix a regular sequence $\alpha_1, \ldots, \alpha_e \in R$ generating $I$ and a regular sequence $\beta_1, \ldots, \beta_{e+1} \in R$ generating $q$. Then $S$ is generated by $[\alpha_1 \wedge \cdots \wedge \alpha_e \mapsto 1]$. Since $I \subseteq q$, we can find elements $\gamma_{ji} \in R$ for all $i \leq j \leq e + 1$ such that $\sum_j \beta_j \gamma_{ji} = \alpha_i$ for all $i$. In addition, since $qC^{-1} = RC^{-1}$ (Lemma 1.4), there exist $\xi_1, \ldots, \xi_{e+1} \in RC^{-1}$ such that $\sum_j \xi_j \beta_j = 1$.

**Proposition 3.1.** Let $d \in RC^{-1}$ denote the determinant of the matrix

$$\begin{bmatrix}
\xi_1 & \gamma_{11} & \cdots & \gamma_{1e} \\
\xi_2 & \gamma_{21} & \cdots & \gamma_{2e} \\
\vdots & \vdots & \ddots & \vdots \\
\xi_{e+1} & \gamma_{(e+1)1} & \cdots & \gamma_{(e+1)e}
\end{bmatrix}$$

Then $\tilde{m}^{-1} = \tilde{S} + [\alpha_1 \wedge \cdots \wedge \alpha_n \mapsto d]S$ and the map $\tau_{I,q,R} : \tilde{m}^{-1} \rightarrow \tilde{k}(q)$ (see (2.1)) sends $\tilde{S}$ to 0 and $[\alpha_1 \wedge \cdots \wedge \alpha_n \mapsto d]$ to $[\beta_1 \wedge \cdots \wedge \beta_{e+1} \mapsto 1] \in \tilde{k}(q)$.

**Proof.** We work in the bounded derived category of $\mathcal{M}(R)$, denoted as $D$. Recall that $M^\vee$ denotes $\text{Hom}(M, R)$. If $P \in D$, we write the $i$th differential of $P$ as $d^P_i$, and set $P^\vee = (P_{-i}, (d^P_{-i})^\vee)_{i \in \mathbb{Z}}$ and $\Delta^P = T^n(P^\vee)$. We define $P^\vee$ and $\Delta^P$ similarly, by dualizing relative to $RC^{-1}$.

Let $E$ be a free $R$-module with basis $x_1, \ldots, x_e$ and let $s \in E^\vee$ be the $R$-module homomorphism sending $x_i$ to $\alpha_i$ for all $i$. Let $F$ be a free $R$-module with basis $y_1, \ldots, y_{e+1}$ and let $t \in F^\vee$ be the $R$-module homomorphism sending $y_j$ to $\beta_j$ for all $j$. Let $L$ and $K$ denote the Koszul complexes of $(E, s)$ and $(F, t)$, respectively, see Section 1.3. Then $H_0(E) = R/I = S$ and $H_0(F) = R/q = k(q)$.

Let $\gamma : E \rightarrow F$ denote the $R$-homomorphism determined by $\gamma x_i = \sum_j y_j \gamma_{ji}$. Then $t = s \circ \gamma$, hence $\gamma$ determines a morphism $L \rightarrow K$, which is also denoted as $\gamma$. Explicitly, $\gamma_i : L_i \rightarrow K_i$ is just $\gamma^A_i : \wedge^i E \rightarrow \wedge^i F$ (and $\gamma^A_0 = 0$ if $i < 0$). The induced map $H_0(\gamma) : S \rightarrow k(q)$ is the quotient map.
Let $G$ denote the cone of $\gamma : L \to K$. Then we have a distinguished triangle

$$T^{-1}G \xrightarrow{u} L \xrightarrow{\gamma} K \xrightarrow{v} G$$

in $D$, where $G_i = E_{i-1} \oplus F_i$ (viewed as column vectors) and

$$d_i^G = \begin{bmatrix} -d_{i-1}L & 0 \\ \gamma \wedge (i-1) & d_iK \end{bmatrix}, \quad u_i = \begin{bmatrix} \text{id}_L \\ 0 \end{bmatrix}, \quad v_i = \begin{bmatrix} 0 \\ \text{id}_K \end{bmatrix}.$$

The cochain complex $T^{-1}G$ is quasi-isomorphic to a projective resolution of $m$ (the image of $H_0(u) : H_0(T^{-1}G) \to H_0(L) = S$ is precisely $m$), hence $H_0(\Delta_0 T^{-1}G) = H_0(\Delta_{e+1}G)$ can be identified canonically with $\hat{m}^{-1} = \text{Ext}_{R}^{1}(m, R)$. Moreover, the maps $S \to \hat{m}^{-1}$ and $\tau f_i : \hat{m}^{-1} \to \hat{k}(q)$ are just $H_0(\Delta e u)$ and $H_0(\Delta e+1 v)$, respectively.

Let $L'$ and $G'$ denote the localizations of $L$ and $G$ at $C$, respectively. Then $L'$ is a projective resolution of the $RC^{-1}$-module $k(I)$. Let $c = \sum_i y_i \xi_i \in FC^{-1}$. For $i \geq 0$, define $c_i : \wedge^i EC^{-1} \to \wedge^{i+1} FC^{-1}$ by $c(z) = c \wedge \gamma \wedge z$. Let $f_i : L_i C^{-1} \to G_{i+1} C^{-1}$ denote $[\text{id}]$ if $0 \leq i \leq e$ and the zero map otherwise. It is routine to check that $f = (f_i)_{i \in \mathbb{Z}}$ is a morphism from $L'$ to $T^{-1}G'$ (use $s_{RC^{-1}}(c) = 1$ and $\gamma \wedge (i-1) d_i L = d_i K \gamma \wedge i$).

Let $u' = u_{RC^{-1}} : T^{-1}G' \to L'$. Then $u'$ is a quasi-isomorphism, and since $u' \circ f = \text{id}_{L'}$, we see that $H_0(\Delta' u')$ is the inverse of $H_0(\Delta' u') : \hat{S}C^{-1} \to \hat{m}^{-1}C^{-1}$. Thus, writing the natural morphism $\Delta_{e+1}G \to \Delta'_{e+1}G'$ as $\tau$, the image of $H_0(\Delta' u' \circ \tau)$ in $H_0(\Delta' L') = \hat{k}(I)$ is the copy of $\hat{m}^{-1}$ in $\hat{k}(I)$.

The morphisms $\Delta' u' \circ \tau$ and $\Delta_{e+1} v$ are illustrated, in degrees 0 and 1, in the following diagram (the top row has degree 1 and the bottom row is degree 0),

\[
\begin{array}{c}
\Lambda^e F^{\vee} \xrightarrow{[0 \; \text{id}]} \Lambda^{e-1} E^{\vee} \oplus \Lambda^e F^{\vee} \\
\Lambda^{e+1} P^{\vee} \xrightarrow{[0 \; \text{id}]} \Lambda^{e+1} F^{\vee} \\
\end{array}
\]

where $(* \circ \tau)$ is $(-1)^{e+1} \begin{bmatrix} (\Lambda^e F)^{\vee} & (\Lambda^{e-1} E)^{\vee} \\ (\Lambda^{e+1} F)^{\vee} & (\Lambda^{e+1} P)^{\vee} \end{bmatrix}$. As noted in the proof of Proposition 1.9(iii), the isomorphism $H_0(\Delta' L') = \text{coker}(d_{e+1}^{\vee})^{\vee} \cong \hat{k}(I)$ sends $[x_1 \wedge \cdots \wedge x_e] \mapsto 1$ to $[\alpha_1 \wedge \cdots \wedge \alpha_e] \mapsto 1$, and the isomorphism $H_0(\Delta_{e+1} K) = \text{coker}(d_{e+1}^{\vee})^{\vee} \cong \hat{k}(q)$ sends $[y_1 \wedge \cdots \wedge y_{e+1}] \mapsto 1$ to $[\beta_1 \wedge \cdots \wedge \beta_{e+1}] \mapsto 1$. It is routine to check that $c_i^{\vee}$ maps $[y_1 \wedge \cdots \wedge y_{e+1}] \mapsto 1$ to $[x_1 \wedge \cdots \wedge x_e] \mapsto d$. The theorem follows readily from this observation and the bottom row of (3.1).

In the remainder of this section, we show how Proposition 3.1 can be applied to compute $\vartheta_{p,q}$ in various situations. Given $a_1, \ldots, a_n \in \hat{A}(p)$ with $\varepsilon a_i^{\vartheta(p)} = a_i$ for all $i$, we write $(a_1, \ldots, a_n)_{\hat{A}(p)}$ to denote the $(\hat{A}(p), \sigma(p))$-valued $\varepsilon$-hermitian form on $A(p)^n$ given by $(x_i, y_i) \mapsto \sum_i x_i^2 a_i y_i$. The Witt equivalence relation is denoted as $\sim$. The ring $R$ is assumed to be regular local of dimension $e + 1$ if not otherwise indicated.
**Example 3.2.** Suppose that $R/p$ is regular, and hence a discrete valuation ring. Then there exists $\alpha_{e+1} \in R$ such that $\alpha_1, \ldots, \alpha_e, \alpha_{e+1}$ generate $q$. Using this generating set, we can take $\gamma_{ji} = \delta_{ji}$, $\xi_1 = \ldots = \xi_e = 0$ and $\xi_{e+1} = \alpha_{e+1}^{-1}$. By Proposition 3.1, $\tilde{m}^{-1}$ is the $S$-submodule of $\tilde{k}(p)$ generated by $[\alpha_1 \wedge \ldots \wedge \alpha_e \mapsto \alpha_{e+1}^{-1}]$ (this module already contains $\tilde{S}$), and $T : \tilde{m}^{-1} \to \tilde{k}(q)$ maps $[\alpha_1 \wedge \ldots \wedge \alpha_e \mapsto \alpha_{e+1}^{-1}]$ to $(-1)^e [\alpha_1 \wedge \ldots \wedge \alpha_{e+1} \mapsto 1]$.

We apply this to describe $\partial_p \tilde{m}^{-1}$ in the case $((A, \sigma, e) = (R, \text{id}_R, 1)$. It is enough to evaluate $\partial_{p,q} f_b$, where $f_b := ([\alpha_1 \wedge \ldots \wedge \alpha_e \mapsto b])_{k(p)}$ and $b \in R^\times$.

Since $S$ is a discrete valuation ring with uniformizer $\alpha_{e+1}(p)$, we can write $b \equiv \alpha_{n+1}^r \pmod{p}$ with $n \in \mathbb{Z}$ and $r \in R^\times$. We claim that, up to Witt equivalence,\

$$\partial_{p,q} ([\alpha_1 \wedge \ldots \wedge \alpha_e \mapsto \alpha_{e+1}^n r])_{k(p)} = \begin{cases} 0 & n \in 2 \mathbb{Z} \\ (-1)^e [\alpha_1 \wedge \ldots \wedge \alpha_{e+1} \mapsto r]_{k(q)} & n \not\in 2 \mathbb{Z}. \end{cases}$$

Indeed, write $f := f_b$, let $m = \lfloor \frac{n}{2} \rfloor$, and consider the submodule $U = \alpha_{e+1}^{-m}(p)S$ of $V = k(p)$. One readily checks that $U^f = \alpha_{e+1}^{-m-1}(p)S = U$ if $n$ is even and $U^f = \alpha_{e+1}^{-m-2}(p)S = \alpha_{e+1}^{-1}(p)U$ if $n$ is odd. Thus, $\partial f$ is the zero form when $n$ even. On the other hand, when $n$ is odd, $U^f / U$ is a one-dimensional $k(q)$-vector space generated by the image of $\alpha_{e+1}^{-m-2}(p)$, and we have

$$\partial f([\alpha_1 \wedge \ldots \wedge \alpha_e \mapsto \alpha_{e+1}^{n-2} r])_{k(q)} = (-1)^e [\alpha_1 \wedge \ldots \wedge \alpha_{e+1} \mapsto r],$$

so $\partial_{p,q} f_b \sim \langle (-1)^e [\alpha_1 \wedge \ldots \wedge \alpha_{e+1} \mapsto r] \rangle_{k(q)}$.

To illustrate this formula, let $F$ be a field, let $R = F[x, y]$, and consider the ideals $a = xR$, $b = yR$, and $c = xR + yR$. Then one can similarly check that all other components of $d_0$, respectively, $d_1$, vanish on $\langle xy \rangle_{k(0)}$, respectively, $\langle [x \mapsto y] \rangle_{k(a)}$ and $\langle [y \mapsto x] \rangle_{k(b)}$. Thus, $d_0, d_0 \langle xy \rangle_{k(0)} \sim \langle [x \wedge y \mapsto 1], [x \wedge y \mapsto 1] \rangle_{k(c)} \sim 0$, as one would expect.

**Example 3.3.** We continue to assume that $R/p$ is regular as in Example 3.2, and choose $\alpha_{e+1} \in R$ such that $\alpha_1, \ldots, \alpha_{e+1}$ generate $q$. Recall that $(A, \sigma)$ is a separable projective $R$-algebra with involution. We write $m^{-1} = \{ r \in k(p) : mr \subseteq S \}$ and $Am^{-1} = A \otimes m^{-1} = \{ a \in A(p) : a m \subseteq A_S \}$.

Let us identify $k(p)$ with $k(p)$ by sending $[\alpha_1 \wedge \ldots \wedge \alpha_e \mapsto 1]$ to 1 and $\tilde{k}(p)$ with $m^{-1}/S$ by sending $[\alpha_1 \wedge \ldots \wedge \alpha_{e+1} \mapsto 1]$ to $\alpha_{e+1}^{-1} + S$. Then $\tilde{S}$ and $\tilde{m}^{-1}$ correspond to $S$ and $m^{-1}$, respectively, and we have induced identifications $\tilde{A}_S = A_S$ and $\tilde{A}(q) = A \otimes (m^{-1}/S) = Am^{-1}/A_S$. By Example 3.2, the map $T_A : Am^{-1} \to Am^{-1}/A_S$ of (2.1) is just $(-1)^e$ times the quotient map. Note that the identifications just made are independent of $\alpha_{e+1}$, so they are canonical when $e = 0$. 
Write $K = k(p)$, $k = k(q)$ and $\tilde{k} = m^{-1}/S$. Then $\tilde{W}_\varepsilon(A(p)) = W_\varepsilon(A_K, \sigma_K)$, $\tilde{W}_\varepsilon(A(q)) = \tilde{W}_\varepsilon(A_K, \sigma_K; \tilde{k})$ (notation as in Example 1.2), and $\tilde{\partial}_{p,q} : \tilde{W}_\varepsilon(A_K, \sigma_K) \to \tilde{W}_\varepsilon(A_K, \sigma_K; \tilde{k})$ can be described as follows: Given $(V, f) \in \tilde{H}^*(A_K, \sigma_K)$, choose an $A$-lattice $U \subseteq V$ such that $U^f \subseteq U \subseteq U^f$, and define $\tilde{\partial}_{p,q}[V, f] = [U^f / U, \tilde{\partial}f]$, where $\tilde{\partial}f$ is given by $\tilde{\partial}f(x + U, y + U) = (-1)^e f(x, y) + A_S$ (here we identify $A\tilde{k}^{-1}/A_S$ with $A \otimes \tilde{k}$).

Taking $(A, \sigma, \varepsilon) = (R, \text{id}_R, 1)$ and fixing an isomorphism $\tilde{k}^{-1}/S \cong k$, the induced map $W(K) \to \tilde{W}(k) \cong W(k)$ is (up to sign) the usual second residue map, see [43, Definition 6.2.5], for instance.

The previous paragraphs also imply that $(-1)^e \cdot \tilde{\partial}_A^{\tilde{k}}$ can be identified non-canonically with $\tilde{\partial}_S^0$ when $S = R/\mathfrak{p}$ is regular.

Example 3.4. Let $F$ be a field, let $R$ denote the localization of $F[x, y]$ at the ideal generated by $x$ and $y$, and let $p = (x^2 - y^5)R$ and $q = xR + yR$. Then $R/p$ is a complete intersection ring which is not regular. Suppose $(A, \sigma, \varepsilon) = (R, \text{id}_R, 1)$. We apply Proposition 3.1 to compute $\tilde{\partial}_{p,q}[x^2 - y^5 \mapsto xy]\rangle_k(q)$.

To that end, we identify $\tilde{k}(q)$ with $k(q)$ by mapping $[x^2 - y^5 \mapsto 1]$ to $1$ and $\tilde{k}(q)$ with $k(q)$ by mapping $[x^1 \cdot y \mapsto 1]$ to $1$. It is also convenient to identify $k(q)$ with $F[z]$ via $x \mapsto z^5$, $y \mapsto z^2$; the ring $S = R/\mathfrak{p}$ corresponds to the localization of $F[z^2, z^5]$ at the ideal $M$ generated by $\{z^2, z^5\}$.

Take $\alpha_1 = x^2 - y^5$, $\beta_1 = x$, $\beta_2 = y$, $\gamma_1 = x$, $\gamma_2 = -y^4$, $\xi_1 = x^{-1}$, and $\xi_2 = 0$. Then, under the previous identifications, Proposition 3.1 asserts that $\tilde{m}^{-1} = S + S(-y^4x^{-1}) = S + Sz^3 = F[z^2, z^3]_M$, and $\mathcal{T} : \tilde{m}^{-1} \to k(q)$ maps $S$ to $0$ and $-z^3 = -y^4x^{-1}(p)$ to $1$.

Writing $f = \langle xy \rangle_{k(q)} = \langle z^7 \rangle_{k(q)}$ and $U = z^{-1}F[z]_M$, it is routine to check that $U^f = \{u \in k(z) : z^6F[z]u \subseteq S\} = z^{-2}F[z]_M$. Thus, $U^f / U$ is a simple $S$-module generated by $z^{-2} + U$. Since $\tilde{\partial}f(z^{-2} + U, z^{-2} + U) = \mathcal{T}(z^3) = -1$, we conclude that $\tilde{\partial}_{p,q}[xy]_{k(q)} \sim (-1)^{k(q)}$.

Remark 3.5. Let $R$ be a regular local ring of dimension $e + 1$, let $\mathfrak{q}$ denote its maximal ideal, and let $\mathfrak{p} \in R^{(e)}$. We do not assume that $R/\mathfrak{p}$ is a complete intersection ring. Given $(V, f) \in \tilde{H}^*(A(p))$, we can use Proposition 3.1 to describe $\tilde{\partial}_{p,q}[V, f]$ as follows. By [11, Theorem 2.12(b)], $\mathfrak{p}$ contains a regular sequence $\alpha_1, \ldots, \alpha_e$. Write $I = \alpha_1R + \cdots + \alpha_eR$. Then $R/I$ is a complete intersection ring of dimension 1. Thanks to Lemma 2.6(iii), we have

$$\tilde{\partial}_{p,q}[V, f] = \tilde{\partial}_{I,q}[V, t_{p,I} \circ f],$$

and the right-hand side can be computed using Proposition 3.1.

This approach requires a description of $t_{p,I} : \tilde{k}(p) \to \tilde{k}(I)$, which is given as follows. Defining $C$ as in Section 2.1, let $\pi_1, \ldots, \pi_e \in \mathfrak{p}C^{-1}$ be a regular sequence generating $\mathfrak{p}C^{-1}$ in $RC^{-1}$. Since $I \subseteq \mathfrak{p}$, we can write $\alpha_i = \sum_j \pi_j u_{ji}$ for some $\{u_{ji}\}_{i,j} \subseteq RC^{-1}$. Then $t_{p,I}$ is determined by

$$t_{p,I}[\pi_1 \wedge \cdots \wedge \pi_e \mapsto 1] = \det(u_{ij}) \cdot [\alpha_1 \wedge \cdots \wedge \alpha_e \mapsto 1].$$

Indeed, let $s : R^e \to R$, $t : R^e \to R$ and $u : R^e \to R^e$ be given by $s(r_1, \ldots, r_e) = \sum_i \alpha_i r_i$, $t(r_1, \ldots, r_e) = \sum_i \pi_i r_i$ and $u(r_1, \ldots, r_e) = (\sum_i u_{i1} r_i, \ldots, \sum_i u_{ie} r_i)$. As in the proof of Proposition 3.1, $u$ determines a morphism $u = (u^{(i)})_{i \geq 0} : K(R^n, s) \to K(R^n, t)$ such that $H^0(u)$ is the quotient map $RC^{-1}/IC^{-1} \to RC^{-1}/\mathfrak{p}C^{-1}$. The assertion now follows by examining $(u^{(e)})^\vee : (K(R^n, t)^e)^\vee \to (K(R^n, s)^e)^\vee$ and using Proposition 1.9(iii).
4 | Functoriality

In this section, we prove that the Gersten–Witt complex (see Section 2.1) is compatible with a number of natural operations, for example, base change of $(A, \sigma)$. We stress that the elementary proofs are possible thanks to the new construction of the Gersten–Witt complex in Section 2.1.

Throughout, $(A, \sigma)$ and $(B, \tau)$ denote $R$-algebras with involution, $\varepsilon \in \mu_2(R)$, and $M$ is an invertible $R$-module. Ultimately, we will specialize to the case where $R$ is regular and $A$ and $B$ are separable projective over $R$. Recall [30, §2B] that every $V \in \mathcal{P}(A)$ admits a finite dual basis, that is, a collection $\{(x_i, \phi_i)\}_{i=1}^n \subseteq V \times \text{Hom}_A(V, A)$ such that $x = \sum_i x_i \cdot \phi_i x$ for all $x \in V$.

Remark 4.1. Regard $\mathcal{P}(A)$ and $\mathcal{P}(B)$ as exact hermitian categories using the involutions $\sigma$ and $\tau$ (see Section 1.2 and Example 1.1). The operations that we consider in this section — base change, involution trace, and $e$-transfer — are induced by exact hermitian functors between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ in the sense of Balmer [3]. These functors also respect the $R$-codimension filtration on the bounded derived categories of $\mathcal{P}(A)$ and $\mathcal{P}(B)$, and therefore induce a morphism between the Gersten–Witt complexes of $A$ and $B$ à la Gille [20], [21]. This morphism agrees with the respective operation in degrees $-1$ and $0$, but it is a priori not clear that the same holds in degrees $\geq 1$. The compatibility in all degrees, which we establish in this section for the Gersten–Witt complex of Section 2.1, will be important later on.

4.1 | Base change

Let $\rho : (A, \sigma) \to (B, \tau)$ be a morphism of $R$-algebras with involution. There is a base change functor $\rho^* : \mathcal{H}^\varepsilon(\mathcal{P}(A), M) \to \mathcal{H}^\varepsilon(\mathcal{P}(B), M)$ (notation as in Example 1.2) given on objects by $\rho^*(V, f) = (V \otimes_A B, \rho f)$, where $\rho f$ is determined by

$$\rho f(x \otimes b, x' \otimes b') = b' \cdot (\rho \otimes \text{id}_M)(f(x, x')) \cdot b \quad (x, x' \in V, b, b' \in B),$$

and on morphisms by $\rho^* \varphi = \varphi \otimes_A \text{id}_B$. The hermitian form $\rho f$ is unimodular by the following lemma, which also tells us that that $\rho$ induces a group homomorphism $W_\varepsilon(A, \sigma; M) \to W_\varepsilon(B, \tau; M)$. The latter is also denoted as $\rho^*$.

Lemma 4.2. In the previous notation, $\rho^*(V, f) \in \mathcal{H}^\varepsilon(B, \tau)$. If $(V, f)$ is metabolic, then so is $\rho^*(V, f)$.

Proof. The first statement follows by a simple adaptation of the argument in [28, I.7]. If $L \subseteq V$ is an $A$-submodule satisfying $V = L + L'$ and $V/L \in \mathcal{P}(A)$, then there exists another $A$-submodule $L'$ such that $V = L \oplus L'$ and $\rho f(L \otimes_A B, L \otimes_A B) = 0$. Since $V \otimes_A B = (L \otimes_A B) \oplus (L' \otimes_A B)$, the unimodularity of $\rho f$ forces $(L \otimes_A B)^\perp f(L') = L \otimes_A B$, so $\rho f$ is metabolic. \hfill \Box

Theorem 4.3. Suppose that $R$ is regular and that $A$ and $B$ are separable projective $R$-algebras. Then $\rho : (A, \sigma) \to (B, \tau)$ induces a morphism of cochain complexes $\rho = (\rho_i)_{i \in \mathbb{Z}} : \mathcal{GW}_+^{A, \sigma, \varepsilon} \to \mathcal{GW}_+^{B, \tau, \varepsilon}$ with $\rho_{-1} = \rho^*$ and $\rho_e = \bigoplus_{p \in R(\varepsilon)} \rho(p)_e$.

Proof. Let $d_e$ and $d'_e$ denote the $e$th differentials of $\mathcal{GW}_+^{A, \sigma, \varepsilon}$ and $\mathcal{GW}_+^{B, \tau, \varepsilon}$, respectively. We need to show that $d'_e \circ \rho_e = \rho_{e+1} \circ d_e$ for all $e \geq -1$. The case $e = -1$ is clear because $\rho^*$ is compatible
with localization. Proving the case $e > 0$ amounts to showing that $\vartheta^B_{p,q} \circ \rho(q) = \rho(q) \circ \vartheta^A_{p,q}$ for all $p \in R^{(e)}$ and $q \in R^{(e+1)}$ with $p \subseteq q$. To that end, we may assume that $R$ is local and $q$ is its maximal ideal. We then apply the notation of Section 2.1, taking $I = p$.

Let $(V, f) \in \tilde{H}^\varepsilon(A(p))$ and let $U$ be an $A$-lattice in $V$ such that $U^f \subseteq U \subseteq U^f$. Since $A$ is separable over $R$, the algebra $B$ is finite projective as a left (or right) $A$-module (see Section 1.1). Thus, we may and shall view $U$ as an $A$-lattice in $V$ such that $U^f \subseteq U \subseteq U^f$. Since $A$ is separable over $R$, the algebra $B$ is finite projective as a left (or right) $A$-module (see Section 1.1). Thus, we may and shall view $U$ as an $A$-lattice in $V$ such that $U^f \subseteq U \subseteq U^f$.

Let $g = \rho(q) f$. Provided that $(U \otimes A B) g = U f \otimes A B$, it is easy to see that the natural isomorphism

$$(U f \otimes A B) / (U \otimes A B) \cong (U f / U) \otimes A(\mathfrak{m}) B(\mathfrak{m})$$

is an isometry from $\vartheta g$ to $\rho(\mathfrak{m})(\vartheta f)$, which is exactly what we want. We finish by showing that $(U \otimes A B) g = U f \otimes A B$.

That $(U \otimes A B) g \supseteq U f \otimes A B$ is straightforward, so we turn to show the converse. We noted earlier that $B$ is finite projective as an $A$-module, so it has a finite dual basis $\{ (b_i, \phi_i) \}_{i=1}^n \subseteq B \times \text{Hom}_A(B, A)$. For all $i \in \{1, \ldots, n\}$, let $b'_i = b_i(p) \in B(p)$ and $\phi'_i = \phi_i \otimes \text{id}(p) \in \text{Hom}_A(B(p), \tilde{A}(p))$. Then $\phi'_i(B_S) \subseteq \tilde{A}_S$ and $b = \sum_i b'_i \cdot \phi'_i b$ for all $b \in B(p)$. Let $v \in (U \otimes A B)^g$. Since $f$ is unimodular, for all $i \in \{1, \ldots, n\}$, there exists $v_i \in V$ such that $f(v_i, x) = \phi'_i(g(v, x \otimes 1))$ for all $x \in V$. Since $g(v, U \otimes 1) \subseteq B_S$, we have $\phi'_i(g(v, U \otimes 1)) \subseteq \tilde{A}_S$, hence $v_i \in U^f$. Now, for all $x \in V$ and $b \in B(p)$, we have $g(v, x \otimes b) = \sum_i b'_i \cdot \phi'_i(g(v, x \otimes 1))b = \sum_i b'_i \cdot f(v_i, x)b = g(\sum_i v_i \otimes b'_i x \otimes b)$, so $v = \sum_i v_i \otimes b'_i \in U f \otimes A B$.

\[\square\]

### 4.2 Involution traces

Throughout this subsection, we assume that $B$ is an $R$-subalgebra of $A$ ($\sigma|_B \neq \tau$ is possible), and $A$ is a finite projective right $B$-module; the latter is automatic if $A$ and $B$ are separable projective over $R$ (see Section 1.1). We further let $\gamma \in \mu_2(R)$.

A function $\pi : A \rightarrow B$ is called an involution $\gamma$-trace (relative to $\sigma$ and $\tau$) if:

- (T1) $\pi$ is additive and $\pi(b_1 \sigma a b_2) = b_1 \pi(a) b_2$ for all $b_1, b_2 \in B, a \in A$;
- (T2) $\pi \circ \sigma = \gamma \tau \circ \pi$;
- (T3) the map $a \mapsto [x \mapsto \pi(ax)] : A \rightarrow \text{Hom}_B(A, B)$ is an isomorphism.

Note that condition (T3) is equivalent to the unimodularity of the $\gamma$-hermitian form $(a, a') \mapsto \pi(\sigma a' a) : A \times A \rightarrow B$ over $(B, \tau)$. Under the additional assumptions $\tau = \sigma|_A$ and $\gamma = 1$, the map $\pi$ is an involution trace in the sense of [28, I.7.2.4].

Every involution $\gamma$-trace $\pi : A \rightarrow B$ induces a functor $\pi_* : H^\varepsilon(A, \sigma; M) \rightarrow H^\varepsilon(B, \tau; M)$ given on objects by $\pi_* (V, f) = (V_B, \pi f)$, where $\pi f = (\pi \otimes \text{id}_M) \circ f$, and by $\pi_* \varphi = \varphi$ on morphisms. This is well defined by the following lemma, which also tells us that we have an induced group homomorphism $\pi_* : W_\varepsilon(A, \sigma; M) \rightarrow W_{\gamma \varepsilon}(B, \tau; M)$.

**Lemma 4.4.** In the previous notation, $\pi_* (V, f) \in H^\varepsilon(B, \tau; M)$. If $(V, f)$ is metabolic, then so is $\pi_* (V, f)$.

**Proof.** We have $V_B \in \mathcal{P}(B)$ because $A_B \in \mathcal{P}(B)$ and $V$ is an $A$-module summand of $A^n$ for some $n$. That $\pi f$ is a $(B \otimes M, \tau \otimes \text{id}_M)$-valued $\gamma \varepsilon$-hermitian form over $(B, \tau)$ follows readily from (T1) and (T2).

Write $f$ for $x \mapsto f(x, -) : V \rightarrow \text{Hom}_A(V, A)$, and define $\tilde{\pi} f : V_B \rightarrow \text{Hom}_B(V_B, B \otimes M)$ similarly. Then $\tilde{\pi} f = j\gamma \circ \tilde{f}$, where $j\gamma : \text{Hom}_A(V, A \otimes M) \rightarrow \text{Hom}_B(V_B, B \otimes M)$ is given by $j\gamma(\phi) = (\pi \otimes \text{id}_M) \circ \phi$. Thus, in order to show that $\pi f$ is unimodular, it is enough to show that $j\gamma$ is an iso-
morphism. It is routine to check that \( j_V \) is natural in \( V \). Since \( V \) is a summand of \( (A \otimes M)^n \) for some \( n \in \mathbb{N} \), it is enough to consider the case \( V = A \otimes M \). Since the map \( \phi \mapsto \phi \otimes \text{id}_M : \text{Hom}_A(U, W) \to \text{Hom}_A(U \otimes M, W \otimes M) \) is an isomorphism for all \( U, W \in \mathcal{P}(A) \) ([19, Theorem 1.3.26], \( \text{End}(M) = R \)), and likewise for \( B \)-modules, we are reduced to proving that \( \phi \mapsto \pi \circ \phi : \text{Hom}_A(A, A) \to \text{Hom}_B(A_B, B) \) is an isomorphism. This holds because the composition of this map with the isomorphism \( A \to \text{End}_A(A) \) given by sending \( a \in A \) to left multiplication by \( a \) is the map considered in (T3).

The second assertion is shown exactly as in the proof of Lemma 4.2.

If \( S \) is an \( R \)-ring, then \( \pi_S : A_S \to B_S \) is also a \( \gamma \)-involution trace relative to \( \sigma_S \) and \( \tau_S \) (use [19, Corollary 1.3.27] to check (T3)).

**Theorem 4.5.** Keeping the previous notation, suppose that \( R \) is regular and \( A \) and \( B \) are separable projective over \( R \). Then \( \pi \) induces a morphism of cochain complexes \( \pi = (\pi_i)_{i \in \mathbb{Z}} : \mathcal{G} \mathcal{W}^+_A,\sigma,\varepsilon \to \mathcal{G} \mathcal{W}^{B,\gamma,\varepsilon}_+ \) with \( \pi_{-1} = \pi_* \) and \( \pi_e = \bigoplus_{p \in R(e)} \pi(p)^* \) for \( e \geq 0 \).

**Proof.** As in the proof of Theorem 4.3, we may assume that \( R \) is local of dimension \( e + 1 \) with maximal ideal \( \mathfrak{m} \), and the proof reduces to showing that \( \pi(\mathfrak{m})^* \circ \partial_A \mathfrak{m} = \partial_B \mathfrak{m} \circ \pi(\mathfrak{m})^* \) for all \( \mathfrak{m} \in R(e) \).

We use the notation of Section 2.1 with \( I = \mathfrak{m} \).

Let \((V, f) \in \mathcal{H}^A(A(\mathfrak{m}))\) and let \( g = \pi(\mathfrak{m}) f \). Choose an \( A \)-lattice \( U \) in \( V \) with \( U f \mathfrak{m} \subseteq U \subseteq U f \).

Provided that \( U g = U f \), it is easy to see that \( \mathfrak{m} = \pi(\mathfrak{m}) (\partial f) \), which would finish the proof.

It is clear that \( U f \subseteq U^g \). To see the converse, let \( \{\alpha_i, \phi_i\}_{i=1}^n \in A \) such that \( \pi(\alpha_i) x = \phi_i x \) for all \( x \in A \), hence \( \sum \alpha_i \pi(\alpha_i) x = x \). Let \( \pi = \pi \otimes \text{id}_A(\mathfrak{m}) \). Then \( \sum \alpha_i \pi(\alpha_i) x = x \) for all \( x \in \mathcal{A}(\mathfrak{m}) \).

Now, if \( u \in U^g \), then for all \( x \in U \), we have \( f(x, u) = \sum \alpha_i \pi(\alpha_i) f(x, u) = \sum \alpha_i \pi(\alpha_i) (x c_i, u) = \sum \alpha_i g(x c_i, u) \in \sum \alpha_i B_S \subseteq A_S \), so \( u \in U f \). □

**Example 4.6.** Let \( u \in A^X \cap S(H(A, \sigma)) \) and let \( \text{Int}(u) \) denote the inner automorphism \( a \mapsto u a u^{-1} : A \to A \). Then \( \tau = \text{Int}(u) \circ \sigma \) is an involution \( \gamma \)-trace relative to \( \sigma \) and \( \tau \). We write \((\pi_u)_* \) as \( u_* \), and \( \pi_u f \) as \( u f \) when \((V, f) \in \mathcal{H}(A, \sigma; M) \).

The functor \( u_* : \mathcal{H}(A, \sigma; M) \to \mathcal{H}(A, \tau; M) \) is called \( u \)-conjugation. It is clearly an equivalence, the inverse being \((u^{-1})_* \), so the induced map on the corresponding Witt groups is an isomorphism.

By Theorem 4.5, if \( R \) is regular and \( A \) is separable projective over \( R \), then \( u \)-conjugation induces an isomorphism \( u_* : \mathcal{G} \mathcal{W}^+_A,\sigma,\varepsilon \to \mathcal{G} \mathcal{W}^+_A,\text{Int}(u) \circ \sigma,\gamma,\varepsilon \).

### 4.3 Hermitian Morita equivalence

We show that the Gersten–Witt complex is compatible with a special kind of hermitian Morita equivalence.

Let \( e \in A \) be an idempotent satisfying \( e^2 = e \) and \( AeA = A \). Put \( A_e = eAe \). Then \( \sigma \) restricts to an involution \( \sigma_e : A_e \to A_e \). Since \( A_e \) is an \( R \)-summand of \( A \), we may and shall regard \( A_e \otimes M \) as a subset of \( A \otimes M \).

Following [18, §2.7] and [15, Proposition 2.5], define the \( e \)-transfer functor \( e_* : \mathcal{H}(A, \sigma; M) \to \mathcal{H}(A_e, \sigma_e; M) \) by setting \( e_*(V, f) = (Ve, f_e := f|_{Ve \times Ve}) \) for objects and \( e_* \varphi = \varphi|_{Ve} \) for every morphism \( \varphi : (V, f) \to (V', f') \).
Lemma 4.7. $e_* : \mathcal{H}^\ell(A, \sigma; M) \to \mathcal{H}^\ell(A_e, \sigma_e; M)$ is an equivalence of categories taking metabolic spaces to metabolic spaces.

Proof. Since we assume $2 \in R^\times$, the proof of [15, Proposition 2.5] applies to our situation with the following modification: Replace $\phi$ in op. cit. with the natural transformation $\iota_V : \text{Hom}_\sigma(A(V, A \otimes M) \cdot e = \text{Hom}_{A_e}(V_e, A_e \otimes M)$ given by $\iota_V(\psi) = \psi|_{V_e}$; this is an isomorphism by Morita Theory [30, Example 18.30].

The isomorphism $W_\epsilon(A, \sigma; M) \to W_\epsilon(A_e, \sigma_e; M)$ induced by $e_*$ will also be denoted by $e_*$. 

Theorem 4.8. In the previous notation, suppose that $R$ is regular and $A$ is separable projective over $R$. Then $e$-transfer induces an isomorphism of cochain complexes $e = (e_i)_{i \in \mathbb{Z}} : \mathcal{GW}^A, \sigma, \epsilon_+ \to \mathcal{GW}^{A_e}, \sigma_e, \epsilon_+$ with $e_{-1} = e_*$ and $e_{\ell} = \bigoplus_{p \in R(\ell)} e(p)_*$ for $\ell \geq 0$.

Proof. As in the proof of Theorem 4.3, we may assume that $R$ is local of dimension $\ell + 1$ with maximal ideal $q$, and the proof reduces to showing that $e(q)_* \circ \delta^A_{p,q} = \delta^{A_e}_{p,q} \circ e(p)_*$ for all $p \in R(\ell)$.

We use the notation of Section 2.1 with $I = p$.

Let $(V, f) \in \bar{H}^\ell(A(p))$ and let $U$ be an $A$-lattice in $V$ such that $U/\mathfrak{m} \subseteq U \subseteq U_f$. Then $Ue$ is an $A_e$-lattice in $Ve$. Write $g = f_e$. Provided that $(Ve)^e = U_f e$, the natural map $(U_f/\mathfrak{m})e \to U_f e/ue$ is an isometry from $(\delta f)_e$ to $(\delta g)_e$, and therefore $e(q)_* \delta^A_{p,q}[V, f] = \delta^{A_e}_{p,q}[Ve, f_e]$. 

That $U_f e \subseteq (Ve)^e$ is straightforward. Conversely, if $v \in (Ve)^e$, then $f(U, v) = f(U Ae, v) = A \cdot f(ue, v) = A \cdot g(ue, v) \subseteq \bar{A}_S$, so $v \in U_f \cap Ve = U_f e$. 

5 | SURJECTIVITY OF THE LAST DIFFERENTIAL

Throughout this section, $R$ is a regular ring, $(A, \sigma)$ is an Azumaya $R$-algebra with involution, and $\epsilon \in \mu_2(R)$. We show that some cohomologies of $\mathcal{GW}^{A, \sigma, \epsilon}_+$ vanish under certain assumptions. Our first and main result of this kind is as follows.

Theorem 5.1. If $R$ is semilocal of dimension $d$, then $H^d(\mathcal{GW}^{A, \sigma, \epsilon}_+) = 0$.

As in Section 4, the new construction of $\mathcal{GW}^{A, \sigma, \epsilon}_+$ in Section 2.1 enables us to give a proof based on classical methods. A different proof was given independently by Gille in [23, Theorem 8.4].

We begin with the following well-known lemma.

Lemma 5.2. Let $R$ be a regular semilocal ring of dimension $d > 1$ and let $q \in R^{(d)}$. Then there exists $p \in R^{(d-1)}$ contained in $q$ such that $R/p$ is a discrete valuation ring. In particular, $q$ is the only height-$d$ prime ideal containing $p$.

Lemma 5.3. Let $(A, \sigma)$ be a central simple algebra with involution over a field $F$ such that $Z(A) \neq F \times F$. If $\sigma$ is symplectic, we also require that $\text{ind} A$ is even. Then every unimodular $1$-hermitian space over $(A, \sigma)$ (see Example 1.1) is isomorphic to an orthogonal sum of $1$-hermitian spaces of $A$-length $1$.

Proof. By [18, Proposition 1.27], there exists a primitive idempotent $u \in A$ with $u^2 = u$. Let $B = uAu$ and let $\tau = \sigma|_B$. Since $A$ is a simple artinian ring, $B$ is a division ring and $AuA = A$. As
explained in Section 4.3, we have an equivalence of categories \( \mathcal{H}^1(A, \sigma) \rightarrow \mathcal{H}^1(B, \tau) \), which is easily seen to respect orthogonal sums and preserve length. It is therefore enough to prove the claim when \( A \) is a division ring. This this is well known, see [43, Theorem 7.6.3], for instance. □

**Proof of Theorem 5.1. Step 1.** Since \( R \) is regular semilocal, it is a finite product of regular domains. By working over each factor separately, we may assume that \( R \) is a domain.

Let \( d = \dim R \). The theorem is clear if \( d = 0 \), so assume \( d > 0 \).

The case \( d > 1 \) can be reduced to the case \( d = 1 \) as follows: By Lemma 5.2, for every \( q \in R^{(d)} \), there is \( p \in R^{(d-1)} \) such that \( R/p \) is a discrete valuation ring and \( q \) is the only height-\( d \) prime containing \( p \). As a result, the theorem will follow if we verify that \( \partial_{p, q} : \tilde{W}(A(p)) \rightarrow \tilde{W}(A(q)) \) is surjective for all such \( p \) and \( q \). By the last paragraph of Example 3.3, we may replace \( R, p, q, A \) with \( R/p, p/q, p/q, A_R/p \) and reduce into proving the surjectivity of \( \partial_{0, m} : W(A_K) \rightarrow \tilde{W}(A(m)) \) when \( R \) is a discrete valuation ring with maximal ideal \( m \) and fraction field \( K \).

Note that the case \( d = 1 \) cannot be similarly reduced to the case where \( R \) is a discrete valuation ring.

To conclude the previous discussion, we may assume that \( R \) is a semilocal Dedekind domain. Write \( K = \text{Frac}(R) \). For \( m \in \text{Max} R \), we abbreviate \( \partial_{0, m} \) to \( \partial_{m} \). We identify \( \tilde{k}(m) \) with \( m^{-1}/R_m \cong m^{-1}/R \) and \( \tilde{A}(m) \) with \( A_m m^{-1}/A_m \cong A^{-1}/A \) as in Example 3.3. The map \( T = T_{0, m, A} \) is just the quotient map \( A_m m^{-1} \rightarrow A_m m^{-1}/A_m \).

**Step 2.** Fix some \( m \in \text{Max} R \) and let \((W, g) \in \tilde{W}(A(m))\). It is enough to construct \((V, f) \in \mathcal{H}^2(A_K, \sigma_K)\) such that \( \partial_{m}[V, f] = [W, g] \) and \( \partial_q[V, f] = 0 \) for all \( q \in \text{Max} R - \{m\} \).

If \([A(m)] = 0\) and \((\sigma(m), \varepsilon(m))\) is symplectic (see Section 1.1), or \( Z(A(m)) \cong k(m) \times k(m) \), then \( W_{\varepsilon}(A(m)) = 0 \) ([18, Example 2.4, Proposition 6.8(ii)], for instance) and we can take \( f = 0 \). We therefore exclude these cases until the end of the proof. As the order of \([A(m)]\) in \( \text{Br} Z(A(m)) \) divides \( \text{id} A(m) \), this means that \( \text{id} A(m) \) is even when \((\sigma(m), \varepsilon(m))\) is symplectic. Consequently, \( \deg A \) is even when \((\sigma, \varepsilon)\) is symplectic.

We may assume that \( \varepsilon = 1 \). Indeed, if \( \varepsilon = -1 \), then by [18, Lemma 1.26], there exists \( u \in S_{-1}(A, \sigma) \cap A^\times \). Applying \( v\)-conjugation (Example 4.6), we may replace \( \sigma \) and \( g \) with \( \text{Int}(v) \circ \sigma \) and \( v g \), and assume that \( \varepsilon = 1 \). The type of \((\sigma, \varepsilon)\) is unchanged by this transition [18, Corollary 1.22(i)].

Now, by Lemma 5.3, \((W, g)\) is the orthogonal sum of 1-hermitian spaces with a simple underlying module. It is therefore enough to consider the case where \( W \) is a simple \( A(m) \)-module.

By [18, Proposition 1.27], there exists a \( \sigma \)-invariant primitive idempotent \( u \in A(m) \). We may assume that \( W = u A(m) \). Writing \( a = g(u, u) \in u(A m^{-1}/A)u \), we have \( g(x, y) = g(xu, uy) = x^\sigma ay \) for all \( x, y \in u A(m) \). Fixing a generator \( \pi \) to \( m \), it is also easy to see that the image of \( a \) under \( x + A \rightarrow \pi x + m A : u(A m^{-1}/A)u \rightarrow u(A A m)u \) must be in \((u(A A m)u)^x \); otherwise, \( g \) would not be unimodular.

It is well known that \( u \in A / A m \) can be lifted to an idempotent \( u_1 \in A / A m^2 \). By [18, Lemma 1.28], \( u_1 \) can be chosen so that \( u_1^2 = u_1 \). Choose some \( a_1 \in u_1(A m^{-1}/A)u_1 \) projecting onto \( a \). Replacing \( a_1 \) with \( \frac{1}{2}(a_1 + a_1^\sigma) \), we may assume that \( a_1^\sigma = a_1 \). By [18, Lemma 1.26], there exists a \( \sigma \)-invariant element \( b \in ((1 - u)(A/m)(1 - u))^x \). Then \( a_2 := a_1 + b \in A m^{-1}/A m \) satisfies \( a_2^\sigma = a_2 \).

For all \( q \in \text{Max} R - \{m\} \), the natural map \( A m^{-1}/A m^{-1} q \rightarrow (A m^{-1})q / (A m^{-1})q q = A q / q A q \) is an isomorphism. This allows us to apply the Chinese Remainder Theorem and choose \( c \in A m^{-1} \) projecting onto \( a_2 \) and such that the image of \( c \) in \( A q / q A q \) is 1 for all \( q \in \text{Max} R - \{m\} \). Replacing \( c \) with \( \frac{1}{2}(c + c^\sigma) \), we may assume that \( c^\sigma = c \).
We claim that \( c \in A_K^\times, c^{-1} \in A_m, u \cdot c^{-1}(m) = 0 \) in \( A(m) \) and \( A_m/c^{-1}A_m \) is a simple \( A(m) \)-module. It is enough to check this after base-changing to the completion of \( R \) at \( m \), so we assume that \( R \) is a complete discrete valuation ring until the end of the paragraph. In this case, \( u_1 \) can be lifted to an idempotent \( u_2 \in A \) [39, Theorem 6.18]. Let \( c' = \pi u_2 + (1 - u_2) \in A \). Then, working in the \( A \)-bimodule \( A_K/Am \), we have \( cc' + Am = \pi a_1 + b \in A(m)^\times \). Thus, \( cc' \in A^\times \) and it follows that \( c \in A_K^\times \) and \( c^{-1} \in c'A^\times \subseteq A \). Moreover, we have \( c^{-1}A = c'A \), so \( A/c^{-1}A = A/c'A \cong uA(m) \), which is a simple \( A(m) \)-module. Finally, \( u_2c^{-1} \) can be lifted to an idempotent \( u_2 \in A \) [39, Theorem 6.18]. Let \( c' = \pi u_2 + (1 - u_2) \in A \). Then, working in the \( A \)-bimodule \( A_K/A_p \), we have \( c'c + A_p = \pi a_1 + b \in A(m)^\times \). Thus, \( cc' \in A^\times \) and it follows that \( c \in A_K^\times \) and \( c^{-1} \in c'A^\times \subseteq A \). Moreover, we have \( c^{-1}A = c'A \), so \( A/c^{-1}A = A/c'A \cong uA(m) \), which is a simple \( A(m) \)-module. Finally, \( u_2c^{-1} \) can be lifted to an idempotent \( u_2 \in A \) [39, Theorem 6.18]. Let \( c' = \pi u_2 + (1 - u_2) \in A \). Then, working in the \( A \)-bimodule \( A_K/A_p \), we have \( c'c + A_p = \pi a_1 + b \in A(m)^\times \). Thus, \( cc' \in A^\times \) and it follows that \( c \in A_K^\times \) and \( c^{-1} \in c'A^\times \subseteq A \). Moreover, we have \( c^{-1}A = c'A \), so \( A/c^{-1}A = A/c'A \cong uA(m) \), which is a simple \( A(m) \)-module. Finally, \( u_2c^{-1} \) can be lifted to an idempotent \( u_2 \in A \) [39, Theorem 6.18].

Let \( f : A_K \times A_K \to A_K \) be given by \( f(\gamma, \delta) = \gamma \sigma(c) \delta \). Since \( \pi \gamma \in A^\times \), we have \( (A_K, f) \in \mathcal{H}^1(A_K, \sigma_K) \). Let \( U = c^{-1}A_p \). Using \( c\sigma = c \), one readily checks that \( Uf \subset V : f(U, \gamma) \subseteq A_p \). Since \( Uf/\gamma \) is a simple \( A_p \)-module, \( \delta_p[A_K, f] \) is represented by the \( 1 \)-hermitian space \( (Uf/\gamma, \delta f) \), where

\[
\delta f(x + c^{-1}A_m, y + c^{-1}A_m) = x^\sigma cy + A_m = (x^\sigma + A_m)a(y + A_m).
\]

Since \( u \cdot c^{-1}(m) = 0 \) and \( uau = a \), it follows that \( (x + c^{-1}A_m) \mapsto u \cdot x(m) : A_m/c^{-1}A_m \to uA(m) \) defines an isometry from \( (Uf/\gamma, \delta f) \) to \( (W, g) \) (it is invertible because it is surjective and the source and target are simple).

To finish, note that for all \( q \in \text{Max}(R - \{m\}) \), we have \( c \in A_q^\times \) because \( c + qA_q = 1 + qA_q \). Computing \( \delta_q[A_K, f] \) using \( U = c^{-1}A_p = A_p \), we get \( Uf = A_p = U \), so \( \delta_q[A_K, f] = 0 \).

\[\text{Lemma 5.4. Assume that } R \text{ is a regular domain with fraction field } K. \text{ Let } (V, f) \in \mathcal{H}^1(A_K, \sigma_K) \text{ and } p \in R^{(1)} \text{. If } \delta_{0,p}[V, f] = 0 \text{, then there exists } (U, g) \in \mathcal{H}^1(A_p, \sigma_p) \text{ such that } (U_K, g_K) \cong (V, f) \text{.} \]

**Proof.** Since \( \delta_{0,p}[V, f] = 0 \), there exists an \( A_p \)-lattice \( U \subset V \) such that \( Uf/\gamma \subset U \subset Uf \) and \( Uf/\gamma, \delta f = 0 \). By [18, Theorem 2.8(ii)] (for instance), \( \delta f \) is metabolic. Let \( L \) be an \( A_p \)-submodule of \( Uf/\gamma \) such that \( L = L^\perp := \{v \in Uf/\gamma : \delta f(Uf/\gamma, v) = 0\} \). Then \( L = M/\gamma \) for some \( A_p \)-module \( M \) lying between \( U \) and \( Uf \). That \( L^\perp = L \) implies readily that \( M/\gamma = M \). Since \( R_p \) is a discrete valuation ring, \( M \in \mathcal{P}(A_p) \), so the restriction of \( f \) to \( M \times M \) — call it \( g \) — is a unimodular \( \varepsilon \)-hermitian form over \( (A_p, \sigma_p) \). Since \( (M_K, g_K) \cong (V, f) \), we are done.

The following theorem is a consequence of a purity theorem of Colliot-Thélène and Sansuc [12, Corollaire 2.5] and Lemma 5.4. (Colliot-Thélène and Sansuc consider only the case \( (A, \sigma, \varepsilon) = (R, \text{id}_R, 1) \), but their proof applies, essentially verbatim, to Azumaya algebras with involution; use [41, Proposition 2.14]. See also [1, §4] for relevant generalizations of patching arguments from [12, §2].)

\[\text{Theorem 5.5. If } \dim R \leq 2 \text{, then } H^0(GW^{A,\sigma,\varepsilon}_+ = 0 \text{.} \]

6 | AN OCTAGON OF WITT GROUPS

In [24, §6], Grenier and Mahmoudi associated with a central simple algebra and some auxiliary data an 8-periodic cochain complex — octagon, for short — of Witt groups; special cases have been observed before, for example, [9, Appendix] and [31]. This was extended to Azumaya algebras in [18]. In this section, we recall the octagon’s construction and prove that it is compatible with the Gersten–Witt complex.
Following [18, §3.1], suppose that:

(G1) \((A, \sigma)\) is an Azumaya \(R\)-algebra with involution;
(G2) \(\varepsilon \in \mu_2(R)\);
(G3) \(\lambda, \mu \in A^\times\) and satisfy \(\lambda^2 = -\lambda, \mu^2 = -\mu, \lambda \mu = -\mu \lambda, \lambda^2 \in Z(A)\).

Now define the following:

(N1) \(S = Z(A)\);
(N2) \(B\) is the commutant of \(\lambda\) in \(A\);
(N3) \(T = Z(B)\);
(N4) \(\tau_1 := \sigma|_B\);
(N5) \(\tau_2 := \text{Int}(\mu^{-1}) \circ \sigma|_B\) (that is, \(x^{\tau_2} = \mu^{-1} x \sigma x \mu\)).

It is shown in [18, §3.1] that \(A\) is an Azumaya \(S\)-algebra, \(B\) is an Azumaya \(T\)-algebra, \(T\) is a quadratic étale \(S\)-algebra, \(\deg A = \frac{1}{2} \deg B\) (if \(\deg A\) is constant), \(\mu B = B \mu\), and \(A = B \oplus \mu B\). Using the last fact:

(N6) \(\pi, \pi' : A \to B\) are defined by \(\pi(b_1 + \mu b_2) = b_1\) and \(\pi'(b_1 + \mu b_2) = b_2\) for all \(b_1, b_2 \in B\).

Note that \(\pi\) is an involution \(1\)-trace from \((A, \sigma)\) to \((B, \tau_1)\) and \(\pi'\) is an involution \((-1)\)-trace from \((A, \sigma)\) to \((B, \tau_2)\); condition (T3) of Section 4.2 is verified in the proof of [18, Lemma 3.3].

Write \(\iota\) for the inclusion morphism \(B \to A\). We shall view \(\iota\) as morphism of \(R\)-algebras with involution from \((B, \tau_1)\) to \((A, \sigma)\), or from \((B, \tau_2)\) to \((A, \sigma)\), where \(\sigma_2 := \text{Int}((\lambda \mu)^{-1}) \circ \sigma\). Recall from Section 4.1 that \(\iota_*\) denotes the induced map between the respective Witt groups. We write

\[
\begin{align*}
\rho_* := \iota_* \circ \lambda_* &: W_e(B, \tau_1) \to W_{-e}(A, \sigma), \\
\rho'_* := (\lambda \mu)_* \circ \iota_* &: W_e(B, \tau_2) \to W_{-e}(A, \sigma),
\end{align*}
\]

where \(\lambda_*\) and \((\lambda \mu)_*\) denote \(\lambda\)-conjugation and \(\lambda \mu\)-conjugation, respectively; see Example 4.6.

The octagon of Witt groups associated with the data (G1)–(G3) is:

\[
\begin{array}{c}
W_e(A, \sigma) \xrightarrow{\pi_*} W_e(B, \tau_1) \xrightarrow{\rho_*} W_{-e}(A, \sigma) \xrightarrow{\pi'_*} W_e(B, \tau_2) \\
W_{-e}(B, \tau_2) \xleftarrow{\pi'_*} W_e(A, \sigma) \xleftarrow{\rho_*} W_{-e}(B, \tau_1) \xleftarrow{\pi_*} W_{-e}(A, \sigma)
\end{array}
\]

**Theorem 6.1** [18, Theorem 3.4, Proposition 3.5]. The octagon (6.1) is a cochain complex. If \(R\) is semilocal, then it is exact.

**Theorem 6.2.** Suppose that \(R\) is regular. Then there is an octagon of cochain complexes:

\[
\begin{array}{c}
GW^{A/R, \sigma, \varepsilon}_+ \xrightarrow{\pi_*} GW^{R/R, \tau_1, \varepsilon}_+ \xrightarrow{\rho_*} GW^{A/R, \sigma, -\varepsilon}_+ \xrightarrow{\pi'_*} GW^{B/R, \tau_2, \varepsilon}_+ \\
GW^{B/R, \tau_2, -\varepsilon}_+ \xleftarrow{\pi'_*} GW^{A/R, \sigma, \varepsilon}_+ \xleftarrow{\rho_*} GW^{R/R, \tau_1, -\varepsilon}_+ \xleftarrow{\pi_*} GW^{A/R, \sigma, -\varepsilon}_+
\end{array}
\]
It is a cochain complex of cochain complexes, and its e-level is exact for all e ≥ 0. If R is semilocal, then all its levels are exact.

Proof. The existence of the octagon follows from Theorem 4.3, Theorem 4.5, and Example 4.6. By Theorem 6.1, the (−1)-level of the octagon is a cochain complex, which is also exact when R is semilocal. To see that the e-level is exact for e ≥ 0, fix isomorphisms $\tilde{k}(p) \to k(p)$ for all $p \in R^{(e)}$ and use them to identify the p-component of the e-level of the octagon with the octagon of Witt groups associated to $A(p), \sigma(p), \mu(p), \lambda(p)$. The latter is exact by Theorem 6.1 (or, alternatively, [24, Corollary 6.1]). □

Remark 6.3. In general, $(B, \tau_i)$ $(i = 1, 2)$ is not always Azumaya over $R$. However, it is Azumaya over $R_i := Z(B)[\tau_i]$, which is a finite étale $R$-algebra (see Section 1.1), and by Theorem 2.9, we have $GW_{+}^{B/R_i, \tau_i, \varepsilon} \cong GW_{+}^{B/R, \tau_i, \varepsilon}$.

Overriding previous notation, let $S$ be a quadratic étale $R$-algebra, let $\theta$ be its standard $R$-involution (see Section 1.1), and suppose that there exists $\lambda \in S$ such that $\{1, \lambda\}$ is an $R$-basis of $S$ and $\lambda^2 \in R^\times$; such $\lambda$ always exists if $R$ is semilocal [18, Lemma 1.19]. Write $Tr = Tr_{S/R}$. It is easy to check that $Tr : S \to R$ is an involution 1-trace relative to both $(\theta, id_R)$ and $(id_S, id_R)$. Writing $\iota$ for the inclusion map $R \to S$, consider the sequence

$$0 \to W_1(S, \theta) \xrightarrow{Tr_\iota} W_1(R, id) \xrightarrow{\lambda_\iota} W_1(S, id) \xrightarrow{Tr_\iota} W_1(R, id) \xrightarrow{\lambda_\iota} W_{-1}(S, \theta) \to 0.$$

This is in fact a special case of the octagon (6.1); see [18, Corollary 8.3] and its proof. Thus, the sequence is a cochain complex and it is exact when $R$ is semilocal. Arguing as in the proof of Theorem 6.2, we get the following theorem.

**Theorem 6.4.** In the previous notation, when $R$ is regular, there exists a 5-term cochain complex of cochain complexes

$$0 \to GW_{+}^{S, \theta, 1} \xrightarrow{Tr_\iota} GW_{+}^{R, id, 1} \xrightarrow{\lambda_\iota} GW_{+}^{S, id, 1} \xrightarrow{Tr_\iota} GW_{+}^{R, id, 1} \xrightarrow{\lambda_\iota} GW_{+}^{S, \theta, -1} \to 0.$$

The e-level of the complex is exact for all e ≥ 0. If R is semilocal, then all levels are exact.

7 | SPRINGER’S THEOREM ON ODD-RANK EXTENSIONS

We recall the Scharlau transfer, which is a special kind of involution 1-trace (see Section 4.2), and establish a version of the weak Springer Theorem on odd-rank extensions for algebras with involution.

As before, $(A, \sigma)$ denotes an R-algebra with involution. Recall that an R-algebra $S$ is called monogenic if there exists $x \in S$ and $n \in \mathbb{N}$ such that $\{1, x, \ldots, x^{n-1}\}$ is an R-basis of $S$, or equivalently, if $S \cong R[X]/(f)$ for some monic polynomial $f$.

Let $S$ be a monogenic R-algebra of odd rank $n$ and let $x \in S$ be an element such that $\{1, x, \ldots, x^{n-1}\}$ is an R-basis of $S$. Denote the inclusion $A \to A_S$ by $\rho$. The Scharlau transfer is the
map $\pi = \pi_X : A_S \to A$ defined by

$$\pi(a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}) = a_0$$

for all $a_0, \ldots, a_{n-1} \in A$ (Scharlau [42] considered the case $A = R$). Arguing as in the proof of [8, Proposition 1.2] or [13, Proposition 2.1], one sees that $\pi$ is an involution 1-trace from $(A_S, \sigma_S)$ to $(A, \sigma)$ in the sense of Section 4.2, and the following version of the weak Springer Theorem holds.

**Proposition 7.1.** In the previous notation, the composition

$$W_\varepsilon(A, \sigma) \xrightarrow{\rho_*} W_\varepsilon(A_S, \sigma_S) \xrightarrow{\pi_*} W_\varepsilon(A, \sigma)$$

is the identity. In particular, the base change map $\rho_* : W_\varepsilon(A, \sigma) \to W_\varepsilon(A_S, \sigma_S)$ is injective.

We would like to have an analogue of Proposition 7.1 when $S$ is a finite étale $R$-algebra of odd rank. Such algebras are not monogenic in general. When $R$ is semilocal, we achieve this by showing that every odd-rank finite étale $R$-algebra can be embedded in an odd-rank monogenic étale $R$-algebra.

We begin with the following lemma, which will also be needed in Section 8. The standard proof is omitted.

**Lemma 7.2.** Assume that $R$ is semilocal. Let $M \in P(R)$ and let $m_1, \ldots, m_n \in M$.

(i) If $m_1(m), \ldots, m_n(m)$ form a $k(m)$-basis to $M(m)$ for all $m \in \text{Max } R$, then $m_1, \ldots, m_n$ form an $R$-basis to $M$.

(ii) If $m_1(m), \ldots, m_n(m)$ are $k(m)$-linearly independent in $M(m)$ for all $m \in \text{Max } R$, then $m_1, \ldots, m_n$ are $R$-linearly independent in $M$ and $\sum m_i R$ is a summand of $M$.

**Proposition 7.3.** Suppose that $R$ is semilocal, let $S$ be a finite étale $R$-algebra of constant rank and let $c \in \mathbb{N}$. Then there exists an odd-rank finite étale $R$-algebra $T$ such that $S \otimes T$ is monogenic and $|S \otimes T/M| \geq c$ for all $m \in \text{Max } (S \otimes T)$.

**Proof.** Suppose first that $R$ is a finite field of cardinality $q$. Then $S = F_1 \times \cdots \times F_t$, where $F_1, \ldots, F_t$ are finite $R$-fields. Let $\ell$ be a prime number and let $T$ be the field with $q^\ell$ elements. We claim that $T$ satisfies the requirements of the lemma for all sufficiently large $\ell$. To see this, write $n_i := \dim_R F_i$ (1 $\leq$ $i$ $\leq$ $t$). Observe first that if $\ell > \max\{n_1, \ldots, n_t\}$, then $F_i \otimes T$ is an $R$-field of dimension $n_i \ell$. For all $\ell$ large enough, we shall have $|F_i \otimes T| \geq c$. Choose a monic polynomial $g_i \in R[X]$ such that $F_i \otimes T \cong R[X]/(g_i)$. It is well known that the number of monic prime polynomials of degree $n_i \ell$ over $R$ is $(n_i \ell)^{-1}q^{n_i \ell} + O(q^{n_i \ell}/2)$. Thus, for every $\ell$ sufficiently large, the polynomials $g_1, \ldots, g_t$ can be chosen to be distinct (even when $n_1, \ldots, n_t$ are not distinct). Let $f = \prod g_i$. By the Chinese Remainder Theorem, $S \otimes T = \prod F_i \otimes T \cong \prod R[X]/(g_i) \cong R[X]/(f)$, so $S \otimes T$ is monogenic.

Suppose now that $R$ is an arbitrary semilocal ring with maximal ideals $m_1, \ldots, m_s$. For every $i \in \{1, \ldots, s\}$ such that $k(m_i)$ is finite, use the previous paragraph to choose a prime number $\ell_i$ and a $k(m_i)$-field $T_i$ of dimension $\ell_i$ such that $S \otimes T_i$ is a monogenic $k(m_i)$-algebra which cannot surject onto a field of cardinality less than $c$. Choose also a monic polynomial $h_i \in k(m_i)[X]$ such
that \( T_i \cong k(m_i)[X]/(h_i) \). Enlarging \( \ell_i \), if needed, we may assume that \( \ell_i \) is odd and independent of \( i \); write \( \ell = \ell_i \). If \( k(m_i) \) is infinite for all \( i \), take \( \ell = 1 \).

For every \( i \in \{1, \ldots, s\} \) such that \( k(m_i) \) is infinite, choose an arbitrary separable polynomial \( h_i \in k(m_i)[X] \) of degree \( \ell_i \) and set \( T_i = k(m_i)[X]/(h_i) \). Then \( S \otimes T_i \) is étale over the infinite field \( k(m_i) \), hence monogenic (see [16, §4], for instance).

By the Chinese Remainder Theorem, there exists a monic polynomial \( h \in R[X] \) such that \( \deg h = \ell \) and \( h(m_i) = h_i \) for all \( i \in \{1, \ldots, s\} \). Let \( T = R[X]/(h) \). Then \( T(m_i) = T_i \) is étale over \( k(m_i) \) for all \( i \); hence, \( T \) is a finite étale \( R \)-algebra of rank \( \ell \). Furthermore, \( (S \otimes T)(m_i) \cong S \otimes T_i \) is monogenic for all \( i \). Write \( m = \text{rank}_R S \) and choose \( x_i \in (S \otimes T)(m_i) \) such that \( \{1, x_i, \ldots, x_i^{\ell m-1}\} \) is a \( k(m_i) \)-basis of \( (S \otimes T)(m_i) \). By the Chinese Remainder Theorem, there exists \( x \in S \otimes T \) such that \( x(m_i) = x_i \) for all \( i \), and by Lemma 7.2(i), \( \{1, x, \ldots, x^{\ell m-1}\} \) is an \( R \)-basis for \( S \otimes T \).

\[ \square \]

**Corollary 7.4.** Let \((A, \sigma)\) be an algebra with involution over a semilocal ring \( R \) and let \( S \) be a finite étale \( R \)-algebra of odd rank. Then the base change map

\[ \rho_* : W_\varepsilon(A, \sigma) \to W_\varepsilon(A_S, \sigma_S) \]

admits a splitting. In particular, this map is injective.

If, in addition, \( R \) is regular and \((A, \sigma)\) is Azumaya over \( R \), then the base change morphism \( \rho_* : GW_+^{A/R, \sigma, \varepsilon} \to GW_+^{A_S/R, \sigma_S, \varepsilon} \) of Theorem 4.3 admits a splitting in the category of abelian cochain complexes.

**Proof.** We may assume that \( R \) is connected; otherwise, write \( R \) as a finite product of connected rings and work over each factor separately. Then \( \text{rank}_R S \) is constant. By Proposition 7.3, there exists an odd-rank finite étale \( R \)-algebra \( T \) such that \( S \otimes T \) is monogenic. By Proposition 7.1, the composition

\[ W_\varepsilon(A, \sigma) \to W_\varepsilon(A_S, \sigma_S) \to W_\varepsilon(A_S \otimes T, \sigma_{S \otimes T}) \overset{\pi_*}{\longrightarrow} W_\varepsilon(A, \sigma) \]

is the identity, hence the first part of the corollary. The second part follows from Theorems 4.3 and 4.5. \[ \square \]

## 8 APPLICATION THE OCTAGON AFTER ODD-RANK EXTENSIONS

Let \((A, \sigma)\) be an Azumaya \( R \)-algebra with involution. In order to apply the octagon of Section 6, one needs to find elements \( \lambda, \mu \in A^\times \) satisfying condition (G3). In this section, we shall see that after applying operations such as conjugation (Example 4.6), \( e \)-transfer (see Section 4.3) and tensoring with an odd-rank finite étale \( R \)-algebra, we can guarantee the existence of \( \lambda \) and \( \mu \).

Some proofs in this section use Galois theory of commutative rings. We refer the reader to [19, Chapter 12] for the necessary definitions and an extensive discussion.

**Lemma 8.1.** Assume that \( R \) is semilocal and let \( A \) be a separable projective \( R \)-algebra with center \( S \). Suppose that \( \ell := \text{rank}_R S \) and \( n := \deg A \) are constant. Then there exists a finite étale \( R \)-algebra \( T \) such that \( A_T \cong M_n(S_T) \) as \( S_T \)-algebras.
Proof. By [41, Proposition 2.18], there is a finite étale $R$-algebra $T_0$ such that $S_{T_0} \cong T_0 \times \cdots \times T_0$ ($\ell$ times). There is a corresponding decomposition $A_{T_0} = A_1 \times \cdots \times A_\ell$ where each $A_i$ is Azumaya of degree $n$ over $T_0$. By [28, III.5.1.17] and [19, Theorem 7.4.4], for each $i \in \{1, \ldots, \ell\}$, there is a finite étale $T_0$-algebra $T_i$ such that $A_i \otimes_{T_0} T_i \cong M_n(T_i)$ (the first source assumes that $R$ is local, but the proof also applies in the semilocal case by Lemma 7.2(ii)). Take $T = T_1 \otimes_{T_0} \cdots \otimes_{T_0} T_\ell$. \hfill $\square$

**Lemma 8.2.** Let $A$ be an Azumaya $R$-algebra and let $S$ be a finite étale $R$-algebra such that $[A_S] = 0$ in $\text{Br} S$. Then $\text{ind} A \mid \text{rank}_R S$.

Proof. By [19, Theorem 7.4.3], there exists $B \in [A]$ such that $\text{deg} B = \text{rank}_R S$, hence the lemma. \hfill $\square$

**Lemma 8.3.** Suppose that $R$ is connected semilocal and let $S$ be a connected finite étale $R$-algebra of rank $2^n$ ($n \geq 1$). Then there is a connected odd-rank finite étale $R$-algebra $T$ such that $S_T$ is connected and contains a quadratic étale $T$-subalgebra.

Proof. The proof goes by a standard argument similar to the case of fields using Galois theory of fields. Use [19, Theorems 12.5.4, 12.6.3] and [18, Lemma 1.3]. \hfill $\square$

**Lemma 8.4.** Suppose that $R$ is semilocal and let $(A, \sigma)$ be an Azumaya $R$-algebra with involution such that $\sigma$ is orthogonal or unitary. Assume that $n := \text{deg} A$ is constant and let $S = Z(A)$. Then there exists an odd-rank finite étale $R$-algebra $T$ and $x \in S_1(A_T, \sigma_T)$ such that $S_T[x]$ is a finite étale $S_T$-algebra of rank $n$.

Proof. By Proposition 7.3 (applied with $S = R$), there exists an odd-rank finite étale $R$-algebra $T$ such that $|T/\mathfrak{m}| > n$ for all $\mathfrak{m} \in \text{Max} T$. We may replace $R, A, \sigma$ with $T, A_T, \sigma_T$ and assume that $|k(\mathfrak{m})| > n$ for all $\mathfrak{m} \in \text{Max} R$.

Suppose first that $R$ is a field. By [10, Theorem 4.1], $A$ contains a finite étale $R$-subalgebra $E$ of rank $n$ fixed pointwise by $\sigma$ (here we need $\sigma$ to be orthogonal or unitary). By [17, Corollary 4.2] (see also [29, Theorem 6.3]) and our assumption that $|R| > n$, the $R$-algebra $E$ is monogenic, say $E = R + xR + \cdots + x^{n-1}R$. It is therefore enough to show that $1, x, \ldots, x^{n-1}$ are linearly independent over $S$, or equivalently, that $\dim_R ES = n \cdot \dim_R S$. This is clear if $S = R$. Otherwise, $S$ is quadratic étale over $R$, so there is $\lambda \in S^x$ such that $\{1, \lambda\}$ is an $R$-basis of $S$ and $\lambda^2 = -\lambda$. Since $E \cap \lambda E \subseteq S_1(A, \sigma) \cap S_1(A, \sigma) = 0$, we have $ES = E \oplus \lambda E$, hence our claim.

The case where $R$ is a general semilocal ring can be deduced from the previous paragraph using the Chinese Remainder Theorem and Lemma 7.2(ii). \hfill $\square$

The following generalization of the Skolem–Noether theorem is known to experts. We include a proof for the sake of completeness.

**Theorem 8.5.** Suppose that $R$ is connected semilocal, let $A$ be an Azumaya $R$-algebra, and let $B$ be a separable projective subalgebra of $A$ with connected center. Then any $R$-algebra homomorphism $\phi : B \to A$ is the restriction of an inner automorphism of $A$.

Proof. Let $S = Z(B)$. View $A$ as a right $A \otimes B^{\text{op}}$-module by setting $a \star (a' \otimes b^{\text{op}}) = ba a'$ and let $M$ denote $A$ with the right $A \otimes B^{\text{op}}$-module structure given by $a \star (a' \otimes b^{\text{op}}) = \phi(b) a a'$. Since $R$ and
S are connected, \( \text{rank}_S A_{\otimes B_{\text{op}}} = \text{rank}_S A \otimes_{\text{op}} M \), and since \( A \otimes B_{\text{op}} \) is separable over \( R \), we have \( A, M \in \mathcal{P}(A \otimes B_{\text{op}}) \) (see Section 1.1). Thus, by [18, Lemma 1.24], there is an \( A \otimes B_{\text{op}} \)-module isomorphism \( \psi : A \to M \). Let \( a = \psi(1) \). Then for all \( b \in B \), we have \( \varphi(b)a = \psi(1) \ast (1 \otimes b_{\text{op}}) = \psi(1 \ast (1 \otimes b_{\text{op}})) = \psi(b) = \psi(1)b = ab \). It is easy to see that \( a \in A^X \), so \( \varphi(b) = aba^{-1} \). □

**Theorem 8.6.** Suppose that \( R \) is connected semilocal. Let \((A, \sigma)\) be an Azumaya \( R \)-algebra with involution and let \( S = \mathcal{Z}(A) \). Let \( B \) be a separable projective \( S \)-subalgebra of \( A \) with connected center \( T \), and let \( \tau : B \to B \) be an involution such that \( \tau|_S = \sigma|_S \). Then there exists \( \varepsilon \in \{ \pm 1 \} \) and \( a \in A \times \cap \mathfrak{m} \) such that \( \tau = \text{Int}(a) \circ \sigma |_B \). When \( \tau|_T \neq \text{id}_T \), one can take any prescribed \( \varepsilon \in \{ \pm 1 \} \).

**Proof.** Suppose first that \( T = \mathcal{Z}_A(B) \). By Theorem 8.5, \( \sigma \circ \tau \) is the restriction of an inner automorphism of \( A \). This implies that there is \( x \in A^X \) such that \( \tau = \text{Int}(x) \circ \sigma |_B \). Set \( t = x \sigma x^{-1} \).

Since \( \tau \) is an involution, for all \( b \in B \), we have \( b = b \tau \tau = (x \sigma x^{-1})^{-1}b(x \sigma x^{-1}) \), so \( t \in \mathcal{Z}_A(B) = T \). Furthermore, \( t \tau t = x \sigma x^{-1}t = x(x \sigma x^{-1}) = t \sigma t = \varepsilon x \). Since \( t \tau t = \varepsilon t \), we can take \( a = t \sigma x \).

Now assume that \( B \) is arbitrary, and let \( B' = \mathcal{Z}_A(T) \) and \( C = \mathcal{Z}_B(B) \). Then \( B \otimes_T C \cong B' \) via \( b \otimes c \mapsto bc \) and \( [B'] = [A \otimes_S T] \) in \( \text{Br}_T \), hence \([C] = [A \otimes_S T] - [B]\). Note that both \( A \otimes_S T \) and \( B \) carry involutions restricting to \( \tau|_T \) on the center (for \( A \otimes_S T \), take \( \sigma \otimes_S (\tau|_T) \)). Thus, theorems of Saltman [40, Theorems 3.1b, 4.4b] imply that \( C \) also admits an involution \( \theta \) with \( \theta|_T = \tau|_T \). Now apply the previous paragraphs to \( B' \cong B \otimes_T C \) and the involution \( \tau \otimes_T \theta \). □

We are now ready to prove that the octagon of Section 6 can be applied after an odd-degree extension.

**Theorem 8.7.** Suppose that \( R \) is a regular semilocal domain, let \((A, \sigma)\) be an Azumaya \( R \)-algebra with involution, and let \( \varepsilon \in \{ \pm 1 \} \). Then there exist a connected odd-rank finite étale \( R \)-algebra \( R_1 \), an Azumaya \( R_1 \)-algebra with involution \((A_1, \sigma_1)\) and \( \varepsilon_1 \in \{ \pm 1 \} \) such that

(i) \([A_1] = [A_1] \) in \( \text{Br}_R \) and \((\sigma, \varepsilon)\) has the same type as \((\sigma_1, \varepsilon_1)\) (see Section 1.1);

(ii) \( G^\lambda_{A_1, \sigma_1, \varepsilon_1} \) is isomorphic to a summand of \( G^\lambda_{A_1, \sigma_1, \varepsilon_1} \);

and at least one of the following hold:

(iii-1) \( Z(A_1) \cong R_1 \times R_1 \);

(iii-2) \( \text{deg} A_1 = 1 \);

(iii-3) \( \text{ind} A_1 = \text{deg} A_1 \) is a power of 2 dividing \( \text{ind} A \) and there exist \( \lambda, \mu \in A^X_1 \) such that \( \lambda^2 \in R_1^X, \lambda^\sigma_1 = -\lambda, \mu^\sigma_1 = -\mu \) and \( \lambda \mu = -\mu \lambda \).

**Proof.** Write \( S = \mathcal{Z}(A), \varepsilon = \text{rank}_R S \in \{ 1, 2 \} \), and \( n = \text{deg} A \).

We may assume throughout that \( \sigma \) is orthogonal or unitary. Indeed, if \( \sigma \) is symplectic, choose \( u \in S_{-1}(A, \sigma) \cap A^X \) (use [18, Lemma 1.26]) and replace \( \sigma, \varepsilon \) with \( \text{Int}(u) \circ \sigma, -\varepsilon \). This does not affect the isomorphism class of \( G^\lambda_{A, \sigma, \varepsilon} \) (Example 4.6) or the type of \((\sigma, \varepsilon)\) [18, Corollary 1.22(i)].
We prove the theorem by induction on $n = \deg A$. The case $n = 1$ is clear, so assume that $n > 1$ and the theorem holds for Azumaya algebras of degree smaller than $n$. Note if $T$ is an odd-rank connected finite étale $R$-algebra, then Corollary 7.4 allows us to replace $R, A, \sigma$ with $T, A_T, \sigma_T$.

**Step 1.** We first show that the induction hypothesis implies the theorem if at least one of the following conditions fail:

1. $S$ is connected,
2. $\text{ind } A = \deg A$,
3. $A$ contains no non-trivial idempotents.

Indeed, if $S$ is not connected, then (iii-1) holds for $A_1 = A$ by \cite[Lemma 1.16]{18}.

Next, if $\text{ind } A < \deg A$, then by \cite[Theorem 1.30]{18}, there exists a full idempotent $e \in A$ such that $\sigma^e = e$ and $\deg eAe = \text{ind } A$ (here we need $\sigma$ to be non-symplectic). By Theorem 4.8, $GW^{A,\sigma,e}_* \cong GW^{eAe,\sigma,e}_*$, and since $\deg eAe = \text{ind } A \mid \deg A$, we may apply the induction hypothesis to $(eAe, \sigma_e, e)$ and finish. The type of $(\sigma, e)$ remains unchanged by \cite[Corollary 1.22(ii)]{18}.

If $S$ is connected and $A$ contains a non-trivial idempotent $e$, then $eAe$ is an Azumaya $S$-algebra with $[eAe] = [A]$ and $\deg eAe < \deg A$ \cite[Corollary 1.12]{18}, so $\text{ind } A < \deg A$ and we can proceed as in the previous paragraph.

**Step 2.** We claim that the theorem holds if $\deg A$ is not a power of 2.

Indeed, by Lemma 8.1, there exists a finite étale $R$-algebra $T$ such that $A_T \cong M_n(S_T)$ as $S_T$-algebras. By \cite[Theorem 12.6.1]{19}, there exists a finite group $G$ such that $T$ can be embedded in a $G$-Galois $R$-algebra. Replace $T$ with this $G$-Galois algebra. Let $P$ be a 2-Sylow subgroup of $G$ and write $E := T^P$. Then $E$ is a finite étale $R$-algebra of rank $|G/P|$, which is odd. Furthermore, $T$ is a $P$-Galois $E$-algebra such that $[(A_E \otimes E) T] = [A_T] = 0$. By Lemma 8.2, $\text{ind } A_E \mid \text{rank } E T = |P|$, so $\text{ind } A_E$ is a power of 2.

Write $E$ as a product of connected finite étale $R$-algebras. At least one of these algebras has odd $R$-rank. Replacing $E$ with that algebra, we may assume that $E$ is connected. As explained above, we may now replace $R, A, \sigma$ with $E, A_E, \sigma_E$ to assume that $\text{ind } A$ is a power of 2. Since we assumed that $\deg A$ is not a power of 2, we have $\text{ind } A < \deg A$ and the theorem holds by Step 1.

**Step 3.** By Lemma 8.4, there exists an odd-rank finite étale $R$-algebra $T$ and $x \in S_1(A_T, \sigma_T)$ such that $L := S_T[x]$ is a finite étale $S_T$-algebra satisfying $\text{rank } S_T L = \deg A_T$. If $T$ is not connected, express it as a product of connected $R$-algebras and replace $T$ with one of the odd-rank factors $T_1$ and $x$ with its image in $A_{T_1}$. We may replace $R, A, \sigma$ with $T, A_T, \sigma_T$. By Steps 1 and 2, we may assume that $\deg A$ is a power of 2 greater than 1 and $A$ contains no non-trivial idempotents. In particular, all commutative $R$-subalgebras of $A$ are connected.

Let $M = L^{[\sigma]}$. If $S = R$, then $M = L$. Otherwise, $\sigma|_L \neq \text{id}_L$, hence $L$ is quadratic étale over $M$ and $M$ is finite étale over $R$ (see Section 1.1). In any case, $\text{rank } M L = \text{rank } R S$. Since $M$ and $S$ are connected,

$$\text{rank } R M \cdot \text{rank } M L = \text{rank } R L = \text{rank } R S \cdot \text{rank } S L = \text{rank } R S \cdot \deg A,$$

so $\text{rank } R M = \deg A$ is a power of 2 greater than 1.

By Lemma 8.3, there exists a connected odd-rank finite étale $R$-algebra $E$ such that $M_E$ contains a quadratic étale $E$-algebra $Q$. We may replace $R, A, \sigma$ with $E, A_E, \sigma_E$ and, thanks to Steps 1 and 2, continue to assume that $\deg A$ is an even power of 2 and $A$ contains no non-trivial idempotents.

We claim that the map $q \otimes s \mapsto q s : Q \otimes S \to Q \cdot S$ is an isomorphism. This is immediate if $S = R$, so assume that $S$ is quadratic étale over $R$. Note first that $QS$ is an epimorphic
image of $Q \otimes S$, hence separable over $R$. By [19, Proposition 4.3.6], $QS$ is also projective over $R$. This means that $\ker(Q \otimes S \to QS)$ is a projective $R$-module of rank $\text{rank}_R Q \otimes S - \text{rank}_R QS$, so we need to show that $\text{rank}_R Q \otimes S = \text{rank}_R QS$. Clearly, $\text{rank}_R QS \leq \text{rank}_R Q \otimes S = 4$. On the other hand, $QS \neq S$ because $S \subseteq S_1(S, \sigma) = R$, so $\text{rank}_R QS = \text{rank}_S QS - \text{rank}_R S \geq 2 \cdot 2 = 4$, forcing $\text{rank}_R QS = 4 = \text{rank}_R Q \otimes S$.

We identify $Q \otimes S$ with $QS$ henceforth.

Step 4. By [18, Lemma 1.19], there exists $\lambda \in Q$ such that $Q = R \oplus \lambda R$ and $\lambda^2 \in R^\times$. We have $\lambda^\sigma = \lambda$ because $Q \subseteq M$. Let $\vartheta$ denote the standard $R$-involution of $Q$ and let $\tau := \vartheta \otimes (\sigma|_S) : QS \to QS$. Then $\tau$ is an involution of $QS$ agreeing with $\sigma$ on $S$ and satisfying $\lambda^\tau = -\lambda$.

By Theorem 8.6, there exists $\mu \in S^{-1}(A, \sigma) \cap A^\times$ such that $\text{Int}(\mu) \circ \sigma|_S = \tau$. Let $\sigma_1 = \text{Int}(\mu) \circ \sigma$ and $\epsilon_1 = -\epsilon$. Then

\[
\begin{align*}
\lambda^{\sigma_1} &= \lambda^\tau = -\lambda, \\
\mu^{\sigma_1} &= \mu \mu^\sigma \mu^{-1} = -\mu, \\
\mu \lambda \mu^{-1} &= \mu \lambda^\sigma \mu^{-1} = \lambda^{\sigma_1} = -\lambda,
\end{align*}
\]

and we have established (iii-3) with $A_1 = A$. By Example 4.6, $GW^{A,\sigma,\epsilon}_+ \cong GW^{A_1,\sigma_1,\epsilon_1}_+$, and the type of $(\sigma_1, \epsilon_1)$ is the same as the type of $(\sigma, \epsilon)$ by [18, Corollary 1.22(i)], so (i) and (ii) also hold.

Remark 8.8. Without condition (ii), Theorem 8.7 holds under the milder assumption that $R$ is connected semilocal.

9 | THE GROTHENDIECK–SERRE CONJECTURE AND EXACTNESS OF THE GERSTEN–WITT COMPLEX IN DIMENSION 2

Let $R$ denote a regular ring, let $(A, \sigma)$ be an Azumaya $R$-algebra with involution and let $\epsilon \in \mu_2(R)$. In this section, we put the machinery of the previous sections to exhibit new cases where $GW^{A,\sigma,\epsilon}_+$ is exact, and as a consequence, verify some open cases of the Grothendieck–Serre conjecture. We achieve this by appealing to a theorem of Balmer, Preeti, and Walter.

Theorem 9.1 (Balmer, Preeti, Walter). $GW^{R,\text{id}_{R},\epsilon}_+$ is exact when $R$ is regular semilocal of dimension $\leq 4$.

Proof. We first note that by Proposition 2.7, the complex $GW^{R,\text{id}_{R},1}_+$ is isomorphic to the Gersten–Witt complex of $R$ defined in [6].

The case $\epsilon = 1$ was verified by Balmer and Walter when $R$ is local [6, Corollary 10.4], and Balmer and Preeti [5, p. 3] showed that the assumption on $R$ can be relaxed to $R$ being semilocal.

The case $\epsilon = -1$ is vacuous because $GW^{R,\text{id}_{R},-1}_+$ is the zero complex.

The fact that Theorem 9.1 applies only in dimension $\leq 4$ is the reason why our results require a similar assumption on $\dim R$ — extending it to higher dimensional rings will result in similar improvements to some of our main results. The precise formulation of this principle is the content of Theorems 9.3 and 9.5.
Lemma 9.2. Consider a double cochain complex $A_{\cdot,\cdot}$ of abelian groups (partially illustrated below).

Suppose that there exist $s, t \in \mathbb{N}$ such that:

1. $A_{t,-t} = 0$ and $A_{-s,s} = 0$;
2. the rows are exact at $A_{i,-i}$ for $-s < i < t$;
3. the columns are exact at $A_{1,0}, A_{2,-1}, \ldots, A_{t,1-t}$ and $A_{-1,0}, A_{-2,1}, \ldots, A_{-s,s-1}$ (these places are indicated by boxes in the illustration).

Then the 0-column is exact at $A_{0,0}$.

Proof. The proof is by diagram chasing. Throughout, subscripts of elements indicate the row in which they live. We write $h$ for the horizontal maps in the diagram and $v$ for the vertical maps.

Let $a_0 \in A_{0,0}$ be an element such that $va_0 = 0$. Define elements $a_{-n} \in A_{n,-n}$ for $n \in \{1, \ldots, t\}$ satisfying

$$va_{-n} = ha_{1-n}.$$ 

as follows: Assuming that $a_{-n}$ has been defined, we have $vha_{-n} = hha_{1-n} = 0$ if $n > 0$ and $vha_{-n} = hva_{-n} = 0$ if $n = 0$. Use the exactness of the columns at $A_{n+1,-n}$ to choose $a_{-n-1} \in A_{n+1,-n-1}$ such that $va_{-n-1} = ha_{n+1-n}$.

Since $A_{t,-t} = 0$, we must have $a_{-t} = 0$. Set $b_{-t} := 0 \in A_{t-1,-t}$ and $b_{-t-1} := 0 \in A_{t,-t-1}$. For $n \in \{t-1, \ldots, 0\}$, define elements $b_{-n} \in A_{n-1,-n}$ satisfying

$$hb_{-n} = a_{-n} - vb_{-n-1}$$

inductively as follows: Assuming that $b_{-n}$ has been defined, we have $h(a_{1-n} - vb_{-n}) = ha_{1-n} - vhb_{-n} = ha_{1-n} - v(a_{-n} - vb_{-n-1}) = ha_{1-n} - va_{-n} = 0$. By the exactness of the rows at $A_{1-n,1-n}$, there exists $b_{1-n} \in A_{n-2,1-n}$ such that $hb_{1-n} = a_{1-n} - vb_{-n}$.

Write $c_0 := b_0$ and observe that $vhc_0 = va_0 - vb_{-1} = 0$. For $n \in \{1, \ldots, s\}$, we define elements $c_n \in A_{-n-1,n}$ satisfying

$$hc_n = vc_{n-1}.$$ 

by induction. Assuming $c_{n-1}$ has been defined, we have $hvc_{n-1} = vvc_{n-2} = 0$ if $n > 1$ and $hvc_{n-1} = vhc_0 = 0$ if $n = 1$. By the exactness of the rows at $A_{n,n}$, there exists $c_n \in A_{-n-1,n}$ such that $hc_n = vc_{n-1}$.
Since $A_{s,s} = 0$, we have $c_s = 0$. Let $d_s := 0 \in A_{s-2,s}$ and $d_{s-1} := 0 \in A_{s-1,s-1}$. For $n \in \{s-2, \ldots, -1\}$, define $d_n \in A_{n-2,n}$ satisfying $vd_n = c_n + 1 - hd_n + 1$

as follows: Assuming $d_{n+1}$ has been defined, we have $v(c_{n+1} - hd_{n+1}) = v c_{n+1} - hd_{n+1}$

Thus, by the exactness of the columns at $A_{n-2,n+1}$, there exists $d_n \in A_{n-2,n}$ such that $vd_n = c_n + 1 - hd_{n+1}$.

Finally, note that $vd_{-1} = c_0 - hd_0$, and $hc_0 = hb_0 = a_0 - vb_{-1}$. Thus, $v(b_{-1} + hd_{-1}) = vb_{-1} + h(c_0 - hd_0) = vb_{-1} + a_0 - vb_{-1} = a_0$, which is what we want.

**Theorem 9.3.** Assume that $R$ is regular semilocal and $\text{ind } A$ is odd. If $GW_{+}^{R_1, id, 1}$ is exact for every finite étale $R$-algebra $R_1$, then $GW_{+}^{A, \sigma, \varepsilon}$ is exact.

**Proof.** By writing $R$ as a product of connected rings and working over each factor separately, we may assume that $R$ is a domain. Write $S = Z(A)$. By Theorem 8.7, we may assume that $S = R \times R$ or $\text{deg } A = 1$, that is, $A = S$. In the former case $GW_{+}^{A, \sigma, \varepsilon} = 0$ (Remark 2.8) and there is nothing to prove, so assume $A = S$. If $S = R$, then we are done by assumption. It remains to consider the case where $S$ is a quadratic étale $R$-algebra and $\sigma$ is its standard $R$-involution. By Theorem 6.4, we have an exact sequence of cochain complexes

$$0 \to GW_{+}^{S, \sigma, 1} \to GW_{+}^{R, id, 1} \to GW_{+}^{S, id, 1} \to GW_{+}^{R, id, 1} \to GW_{+}^{S, \sigma, -1} \to 0,$$

which we view as a double cochain complex. By assumption, $GW_{+}^{R, id, 1}$ and $GW_{+}^{S, id, 1}$ are exact. Furthermore, by [18, Lemma 1.26], there exists $u \in S_{-1}(S, \sigma) \cap S^{\times}$, which induces an isomorphism $u_* : GW_{+}^{S, \sigma, 1} \cong GW_{+}^{S, \sigma, -1}$ by Example 4.6.

We now use induction on $i \in \mathbb{Z}$ to show that $H^i(GW_{+}^{S, \sigma, 1}) \cong H^i(GW_{+}^{S, \sigma, -1}) = 0$. Assuming $H^i(GW_{+}^{S, \sigma, 1}) \cong H^i(GW_{+}^{S, \sigma, -1}) = 0$ has been established, we get $H^{i+1}(GW_{+}^{S, \sigma, 1}) = 0$ by applying Lemma 9.2.

**Theorem 9.4.** If $R$ is regular semilocal of dimension $\leq 4$ and $\text{ind } A$ is odd, then $GW_{+}^{A, \sigma, \varepsilon}$ is exact.

**Proof.** This follows from Theorems 9.3 and 9.1.

**Theorem 9.5.** Assume that $R$ is a regular semilocal domain. If for every connected finite étale $R$-algebra $R_1$, every Azumaya $R_1$-algebra with involution $(B, \tau)$ with $\text{deg } B \mid \text{deg } A$, and every $i \in \mathbb{Z}$, we have $H^{2i}(GW_{+}^{B, \tau, \pm 1}) = 0$ and $H^i(GW_{+}^{R_1, id, 1}) = 0$, then $GW_{+}^{A, \sigma, \varepsilon}$ is exact.

**Proof.** We prove the theorem by induction on $\text{deg } A$. The case $\text{deg } A = 1$ holds by assumption, so assume that $\text{deg } A > 1$ and the theorem holds for all Azumaya algebras with involution of degree smaller than $\text{deg } A$.

By Theorem 8.7 and the induction hypothesis, we may assume that $Z(A) = R \times R$, or $\text{ind } A = \text{deg } A$ is a power of 2 and there exist $\lambda$ and $\mu$ as in Section 6. In the first case, we have $GW_{+}^{A, \sigma, \varepsilon} = 0$ (Remark 2.8), so we only need to treat the second case.
Define \( B, \tau_1, \tau_2 \) as in Section 6. By Theorem 6.2, we have an exact 8-periodic sequence of cochain complexes

\[ \cdots \to \mathcal{W}^B_{+,-\varepsilon} \xrightarrow{\rho'_+} \mathcal{W}^{A,\sigma,\varepsilon}_{+} \xrightarrow{\pi'_+} \mathcal{W}^B_{+,-\varepsilon} \xrightarrow{\rho'_+} \mathcal{W}^{A,\sigma,\varepsilon}_{+} \xrightarrow{\pi'_+} \mathcal{W}^B_{+,-\varepsilon} \xrightarrow{\rho'_+} \cdots, \]

which we view as a double cochain complex. The columns \( \mathcal{W}^B_{+,-\varepsilon} \) and \( \mathcal{W}^B_{+,\varepsilon} \) are exact by the induction hypothesis (deg \( B = \frac{1}{2} \) deg \( A \), Remark 6.3), and by assumption, \( H^2(\mathcal{W}^{A,\sigma,\varepsilon}_{+}) = 0 \) for all \( i \in \mathbb{Z} \). Now, by Lemma 9.2, we also have \( H^2+i(\mathcal{W}^{A,\sigma,\varepsilon}_{+}) = 0 \) for all \( i \in \mathbb{Z} \), so \( \mathcal{W}^{A,\sigma,\varepsilon}_{+} \) is exact.

**Theorem 9.6.** If \( R \) is regular semilocal of dimension \( \leq 2 \), then \( \mathcal{W}^{A,\sigma,\varepsilon}_{+} \) is exact.

**Proof.** As in the proof of Theorem 9.3, we may assume that \( R \) is a domain. Let \( R_1 \) be a finite étale \( R \)-algebra and let \( (B, \tau) \) be an Azumaya \( R_1 \)-algebra with involution. Then \( \mathcal{W}_{+}^{R_1,\text{id},1} \) is exact by Theorem 9.1 and \( H^0(\mathcal{W}_{+}^{B,\tau,\pm 1}) = H^2(\mathcal{W}_{+}^{B,\tau,\pm 1}) = 0 \) by Theorems 5.1 and 5.5. The corollary therefore follows from Theorem 9.5.

We use the previous theorems to establish new cases of the Grothendieck–Serre conjecture (see the Introduction) and prove a purity result for Witt groups of hermitian forms. Given an Azumaya \( R \)-algebra with involution \( (A, \sigma) \), recall that \( U(A, \sigma) \to \text{Spec} \, R \) denotes the group \( R \)-scheme of \( \sigma \)-unitary elements in \( A \) and \( U^0(A, \sigma) \to \text{Spec} \, R \) is its neutral connected component (see [18, §2.5]).

**Theorem 9.7.** Let \( R \) be a regular semilocal domain with fraction field \( F \), let \( (A, \sigma) \) be an Azumaya \( R \)-algebra with involution, and assume that one of the following hold:

1. \( \dim R = 2 \);
2. \( \dim R \leq 4 \) and \( \text{ind } A \) is odd.

Then:

i. The restriction map \( H^1_{\text{et}}(R, U(A, \sigma)) \to H^1_{\text{et}}(F, U(A, \sigma)) \) has trivial kernel. Likewise for \( U^0(A, \sigma) \).

ii. \( \text{im}(W_\varepsilon(A, \sigma) \to W_\varepsilon(A_F, \sigma_F)) = \bigcap_{p \in \mathbb{R}^{(1)}} \text{im}(W_\varepsilon(A_p, \sigma_p) \to W_\varepsilon(A_F, \sigma_F)) \).

**Proof.** Part (i) follows from \( H^{-1}(\mathcal{W}^{A,\sigma,\varepsilon}_{+}) = 0 \) and [18, Proposition 8.7]. Part (ii) follows from \( H^0(\mathcal{W}^{A,\sigma,\varepsilon}_{+}) = 0 \) and Lemma 5.4.

**Acknowledgements**

We are grateful to Stefan Gille and Paul Balmer for several useful correspondences. We also thank the anonymous referees for their comments and suggestions. The third author was partially supported by the NSF grant DMS-1801951.

**Journal Information**

The *Journal of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.
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