Mono-cross-anisotropic Metasurface Absorbing Boundary Condition (ABC)

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Abstract—We introduce the concept of Metasurface Absorbing-Boundary Condition (MS-ABC). This represents the first application of metasurface technology to computational electromagnetics. MS-ABCs have similar performance as previously reported surface ABCs, but they provide a richer description of absorption, that is independent from the incident wave, and deeper insight into the physics, through bianisotropic susceptibilities. The MS-ABC susceptibilities are synthesized using GSTC synthesis, from which the metasurface is found to be mono-cross-anisotropic, i.e. having only electric-to-magnetic and magnetic-to-electric coupling susceptibility tensors being non-zero, which is physically impossible, but numerically perfectly appropriate. Moreover, it is found that the metasurface is asymmetric, being amplifying in the opposite direction of its basic utilization as an ABC. The MS-ABC is implemented in a GSTC-FDTD scheme, which, for the first time, replaces the virtual grid concept by straightforward surface polarization currents residing on the regular Yee grid.

Index Terms—Absorbing Boundary Conditions (ABCs), metasurface ABCs, Perfectly Matched Layer (PML), computational electromagnetics, metasurface, Generalized Sheet Transition Conditions (GSTCs), bianisotropic medium, surface susceptibility tensors, thin absorber, Finite Difference Time Domain (FDTD).

I. INTRODUCTION

Perfect Matched Layers (PMLs) are the most robust Absorbing Boundary Conditions (ABCs) in computational electromagnetics [1]. Since their introduction under the split-field form by Bérenger in 1994 [2], they have evolved into several enhanced formulations, including the stretched-coordinate PMLs [3], the anisotropic uniaxial PMLs (UPMLs) [4], the complex frequency-shifted PMLs (CFS-PMLs) [5] and the convolutional PMLs (CPMLs) [6]. Despite currently being the gold standard in ABCs, PMLs are relatively difficult to implement and very computationally expensive (both time-wise and memory-wise), due to their volumetric nature. Alternative ABCs with similar absorbing performance but lower complexity and computational cost would therefore be highly desirable.

As physical volumetric absorbers [7],[8] may be replaced by ingenious physical film or metasurface absorbers [9]–[12], PMLs may, to some extent, be replaced by thin ABCs mimicking physical films or metasurfaces to solve their aforementioned issues. Such a strategy has been explored in multiple variations, including Multiple Absorbing Surface (MAS) conditions [13], Re-Radiating Boundary Conditions (rRBCs) [14] and Huygens ABCs [15]. Since PMLs are not competitive with PMLs in terms of absorption level [13]–[15], but represent both a fundamental concept and an easy-implementation approach of ABC in computational electromagnetics. Moreover, they may be combined with PMLs to enhance their performance [13],[16]. All these techniques leverage the Huygens principle to extinct the fields outside the computational domain. Specifically, they absorb the fields at the limits of the computational domain using a surface boundary formed by the equivalent electric and magnetic currents induced by the incident waves. The corresponding ABCs are thus expressed in terms of the incident fields. This represents a fundamental difference with PMLs, which are more conveniently expressed in terms of absorbing medium parameters, independently from the incident fields.

Here we present an alternative description of Huygens ABCs that, similarly to PMLs, is based on absorbing metasurface parameters, also independent from the incident fields. In this sense, this description is to surface boundaries what PMLs are to volume boundaries, and it makes the Huygens ABC concept stronger. Specifically, it provides a richer absorbing description (independence of the incident wave) and physical insight (in terms of bianisotropic susceptibilities). We shall then refer to those ABCs as Metasurface ABC (MS-ABC).

The challenge in the MS-ABC approach is that metasurfaces are generally sensitive to the wave incidence angle, polarization and frequency [17]–[19], and it is not straightforward to synthesize the metasurface realizing the desired Huygens ABC. However, we rigorously demonstrate, using Generalized Sheet Transition Conditions (GSTCs) [20]–[22], that this can be realized with a mono-cross-anisotropic metasurface. By “mono-cross-anisotropic,” we are here referring to a metasurface exhibiting only electric-magnetic and magnetic-electric surface susceptibility tensors ($\chi_{ee}$ and $\chi_{me}$) to the magnetic/electric field $B$ and $E$, respectively, with zero non-axi symmetric susceptibility tensors ($\chi_{mm}$ and $\chi_{em}$) that are zero.

Such a metasurface, as split-field PMLs [2], are not physical, as demonstrated in [21], but computationally perfectly appropriate. Moreover, we implement the MS-ABCs in the Finite-Difference Time-Domain (FDTD) scheme using surface parameters, independently from the incident fields.

This may be intuitively understood as follows. Consider the simplest case of pure bianisotropy ($D = (H \parallel B)$, which is a limiting case of anisotropy ($D = \xi \cdot H$ and $B = \xi \cdot E$). Here, the electric-magnetic coupling ($\xi = \chi_{em}$) and magnetic-electric coupling ($\xi = \chi_{me}$) result from current continuity in electrically small helices (with left or right handedness): since the nonlocal electric/magnetic field response ($D/\varepsilon$) results from loop currents induced by magnetic/electrical dipole moments ($m/p$), the preliminary (local) magnetic/electric response ($B = m/p$) is unavoidable. Here, we need a new name for referring to the numerical (unphysical) medium used in the ABC, which only includes electric-to-magnetic and magnetic-to-electric effects, hence the term “mono-cross-anisotropic”.

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electric and magnetic polarization currents, hence avoiding the introduction of virtual nodes as in \[25\]–\[27\], which leads to deeper physical insight and greater implementation simplicity.

The paper is organized as follows. Section II explains the concept of MS-ABC. Section III first derives the GSTCs in terms of polarization currents, next synthesizes the monoco- anisotropic metasurfaces realizing the desired MS-ABC, pointing out its fundamental asymmetry, and then describes its loss-gain properties in opposite directions. The FDTD implementation is developed in Sec. IV Finally, Sec. VI illustrates the proposed MS-ABC concept with a few numerical experiments. Conclusion are given in Sec. VI.

II. MS-ABC CONCEPT

Figure 1 depicts the principle of the proposed Metasurface Absorbing Boundary Condition (MS-ABC). To be absorbing, the metasurface must transform the incident wave, \(\Psi_i\), into zero reflected and transmitted fields, i.e. process the incident wave in such a manner the total field on the incident and transmitted sides of it be \(\Psi^+ = \Psi_i\) and \(\Psi^- = 0\), respectively, as shown in Fig. 1(b). In the sequel, we shall assume that the metasurface is always placed in the plane \(\zeta = 0\) of the local coordinate system \((\xi, \upsilon, \zeta)\), so that the unit vector normal to the surface is \(\hat{n} = \zeta\), the fields \(\Psi^+\) and \(\Psi^-\) thus correspond to the half-spaces \(\zeta > 0\) and \(\zeta < 0\), that we will refer to as the forward (FWD) and backward (BWD) sides of the metasurface, respectively.

Whether such an operation in Fig. 1 is possible is a priori non trivial. However, the problem simplifies upon decomposition into a purely incident wave problem and a scattered wave (radiation) problem, as shown Figs. 1(b) and (c), respectively. This decomposition transforms the initial scattering problem into a radiation problem [Fig. 1(c)], with sources corresponding to the current induced on the metasurface by the incident wave [Fig. 1(b)], and reveals the type of metasurface equivalent sources required for absorption: the sources should be unidirectional radiators in the transmit side of the metasurface with phase opposite to that of the incident wave, \(\Psi_{st} = -\Psi_i\). Such radiation, as noted by many authors \[28\]–\[30\], can essentially be achieved by so-called Huygens sources.

A Huygens source is the perpendicular combination of collocated electric and magnetic dipoles with identical phases. Figure 2 plots the radiation patterns of mutually perpendicular electric and magnetic dipoles and their cross sections in the planes perpendicular to the source plane (\(\upsilon\zeta\)-plane) and in the directions of each electric dipole (\(\upsilon\zeta\)-plane) and magnetic dipole (\(\xi\zeta\)-plane), and pictorially shows how unidirectional radiation is achieved by the resulting Huygens source.

As we shall see shortly, the backward side will correspond to the interior of the computational domain, delimited by the surface ABC. Although it radiates mostly to the forward direction and exhibits a null in the direction perpendicular to its plane, the Huygens source still radiates non-negligible energy in the backward direction, with maxima at \(\alpha = \pm 60^\circ\), as seen in Fig. 2. Its back-to-front power ratio is \(\psi_{\text{BWD}}(\alpha = 60^\circ)/\psi_{\text{FWD}}(\alpha = 180^\circ) = 1.56 \times 10^{-2} \approx -36\) dB, and this ratio does not substantially decrease as the Huygens source is arrayed in the metasurface configuration.

The back-to-front power ratio is in fact the square of the reflection coefficient of the metasurface, according to the decomposition in Fig. 1 that corresponds to the following
perform the synthesis of the MS-ABC per se, and demonstrate Fig. 1(a). GSTCs. We shall first briefly recall the GSTCs, then
MS-ABC described in Sec. II according to the specification of
PML conditions (GSTCs), which connect the total fields (\(\Psi\), \(\Gamma\)) with the impressed electric and magnetic surface current densities (\(\mathbf{J}_{\text{s,imp}}\) and \(\mathbf{K}_{\text{s,imp}}\)), the electric surface current densities due to electric and magnetic polarization densities, \(\mathbf{J}_{\text{s,p}}\) and \(\mathbf{J}_{\text{s,m}}\), and the magnetic surface current densities due to electric and magnetic polarization densities, \(\mathbf{K}_{\text{s,p}}\) and \(\mathbf{K}_{\text{s,m}}\):

\[
\begin{align*}
\mathbf{J}_{\text{s,tot}} &= \mathbf{J}_{\text{s,imp}} + \mathbf{J}_{\text{s,p}} + \mathbf{J}_{\text{s,m}} \\
&= \mathbf{J}_{\text{s,imp}} + j\omega \mathbf{P}_s + \nabla \times \mathbf{M}_s \quad (\text{A/m}),
\end{align*}
\]

and

\[
\begin{align*}
\mathbf{K}_{\text{s,tot}} &= \mathbf{K}_{\text{s,imp}} + \mathbf{K}_{\text{s,m}} + \mathbf{K}_{\text{s,p}} \\
&= \mathbf{K}_{\text{s,imp}} + j\omega \mu_0 \mathbf{M}_s + \nabla \times (\mathbf{P}_s/\epsilon_0) \quad (\text{V/m}),
\end{align*}
\]

where \(\mathbf{P}_s\) and \(\mathbf{M}_s\) are surface polarization densities, measured in \(\text{C/m}\) and \(\text{A}\), respectively.

Applying the integral form of Maxwell equations at the metasurface sheet discontinuity yields, with the total currents in (2), and following the usual procedure for boundary conditions, yields

\[
\begin{align*}
\mathbf{n} \times \Delta \mathbf{H} &= \mathbf{J}_{\text{s,tot}}, \quad (3a) \\
\mathbf{n} \times \Delta \mathbf{E} &= -\mathbf{K}_{\text{s,tot}}, \quad (3b)
\end{align*}
\]

where

\[
\Delta \mathbf{H} = \mathbf{H}^+ - \mathbf{H}^- \quad \text{and} \quad \Delta \mathbf{E} = \mathbf{E}^+ - \mathbf{E}^-,
\]

where \(\mathbf{E}^±\) and \(\mathbf{H}^±\) are the total fields in the half-space sides \(\zeta > 0\) and \(\zeta < 0\) of the metasurface, respectively, and \(\mathbf{n}\) is the unit vector equal to \(\zeta\) [Fig. 1(a)]. Substituting now (2) into (3), and ignoring the impressed currents,\(^2\) yields

\[
\begin{align*}
\mathbf{n} \times \Delta \mathbf{H} &= j\omega \mathbf{P}_s - \mathbf{n} \times \nabla \mathbf{M}_{s,n}, \quad (4a) \\
\mathbf{n} \times \Delta \mathbf{E} &= -j\omega \mu_0 \mathbf{M}_{s||} + \nabla \mathbf{J}_\text{s,n}/\epsilon_0 \times \mathbf{n}, \quad (4b)
\end{align*}
\]

\(^2\)We assume here that there is no impressed source within the metasurface.
Equations (4) with (5c) form the GSTCs. The polarizations in (5c) may be expressed in terms of the bianisotropic constitutive relations

$$P = \varepsilon_0 \tilde{\chi}_{ee} E_{av} + \sqrt{\mu_0 \varepsilon_0} \tilde{\chi}_{en} H_{av},$$  

$$M = \sqrt{\varepsilon_0 / \mu_0 \varepsilon_{nm}} E_{av} + \tilde{\chi}_{mm} H_{av},$$

involving the surface susceptibility tensors

$$\tilde{\chi}_{ab} = \begin{bmatrix} \chi_{ee} & \chi_{em} \\ \chi_{me} & \chi_{mm} \end{bmatrix},$$

with \((a,b) = (e,e), (e,m), (m,e)\) and \((m,m)\), and

$$E_{av} = \frac{E^+ + E^-}{2} \quad \text{and} \quad H_{av} = \frac{H^+ + H^-}{2}.$$  

### B. Synthesis of the MS-ABC Susceptibilities

As illustrated in Fig. 3, the tangential parts of the polarization densities are sufficient to produce the required fields [Fig. 1(a)]. Therefore, we assume that \(P_z = M_z = 0\), which eliminates the second terms in the right-hand side of (4). Substituting (5) into the resulting equations yields the GSTCs

$$\hat{n} \times \Delta \mathbf{H} = j \omega \varepsilon_0 \tilde{\chi}_{ee} E_{av} + j k_0 \chi_{en} H_{av},$$  

$$\Delta E \times \hat{n} = j k_0 \chi_{me} E_{av} + j \omega \mu_0 \chi_{mm} H_{av},$$

where the four \(3 \times 3\) tensors (6) have reduced to the transpose tensors

$$\tilde{\chi}_{ab} = \begin{bmatrix} \chi_{ee} & \chi_{em} \\ \chi_{me} & \chi_{mm} \end{bmatrix}.$$  

The off-diagonal terms of \(\tilde{\chi}_{ee}\) and \(\tilde{\chi}_{mm}\) and the diagonal terms of \(\tilde{\chi}_{em}\) and \(\tilde{\chi}_{me}\) induce field rotation. Consider physical gyrotropic structures, which can be perfectly transparent, there is no a priori reason why gyrotropy would provide any benefit in terms of isolation. Therefore, and for maximal simplicity, we heuristically set these terms to zeros, i.e.

$$\tilde{\chi}_{ee,mn} = 0, \quad \tilde{\chi}_{em,mm} = 0, \quad \tilde{\chi}_{em,me} = 0, \quad \tilde{\chi}_{mm,me} = 0,$$

which leaves out only 8 out of the initial \(4 \times 4 = 16\) components in (9).

We shall now apply these GSTCs to the four walls of the computational domain, which is represented in Fig. 5.

Let us start with the bottom MS-ABC wall, where \(\hat{n} = \hat{\zeta} = \hat{y}, \hat{n} = \hat{\xi} = \hat{x}\) and \(\hat{\xi} = \hat{\zeta}\), so that Eq. (1) becomes

$$\hat{\zeta} \times \Delta \mathbf{H} = j \omega \varepsilon_0 \tilde{\chi}_{ee} E_{av} + j k_0 \chi_{en} H_{av},$$  

$$\Delta \mathbf{E} \times \hat{\zeta} = j k_0 \chi_{me} E_{av} + j \omega \mu_0 \chi_{mm} H_{av}.$$  

We shall derive the MS-ABC only for TM fields, with arbitrary incident wave angle \(\phi\) on the metasurface, the TE case being of course analogous. In this case, which retains only 2 of the 4 remaining susceptibility components, Eq. (11) simplifies to

$$\Delta H_z = j \omega \varepsilon_0 \chi_{ee} E_{av} + j k_0 \chi_{em} H_{av},$$  

$$\Delta E_x = j k_0 \chi_{me} E_{av} + j \omega \mu_0 \chi_{mm} H_{av},$$

where

$$\Delta H_z = [(1 - R) - T]H_{av}^1,$$

$$H_{av} = [(1 - R) + T]H_{av}^2,$$

$$\Delta E_x = [(1 + R) - T]E_{av}^x,$$

$$E_{av} = [(1 + R) + T]E_{av}^x,$$

with \(T\) and \(R\) being the transmission and reflection coefficients, respectively. Inserting (13) into (12) and using the relation \(E_{av}^x/H_{av}^1 = -\eta \sin(\phi)\) yields

$$(1 - R) - T = -j \omega \varepsilon_0 \chi_{ee} [(1 + R) + T] \eta \sin(\phi)/2,$$

$$(1 + R) - T = j k_0 \chi_{em} [(1 + R) + T]/2,$$

$$-j \omega \mu_0 \chi_{mm} [(1 - R) + T]/(2\eta \sin(\phi)).$$

Solving this equation for \(R\) and \(T\) yields

$$R = \frac{j \omega \varepsilon_0 \chi_{ee} \eta \sin(\phi) - j k_0 \chi_{em} + j k_0 \chi_{ee} - j \omega \mu_0 \chi_{mm} \eta \sin(\phi) - \frac{\eta \sin(\phi)}{2}}{2 - j \omega \varepsilon_0 \chi_{ee} \eta \sin(\phi) - \frac{\eta \sin(\phi)}{2}} = \frac{-k_0 \chi_{mm} \eta \sin(\phi) - \frac{\eta \sin(\phi)}{2}}{2 - j \omega \varepsilon_0 \chi_{ee} \eta \sin(\phi)}.$$  

$$T = \frac{-2 - j k_0 \chi_{em} - j k_0 \chi_{me} - j k_0 \chi_{ee} - j k_0 \chi_{mm} - \frac{\eta \sin(\phi)}{2}}{2 - j \omega \varepsilon_0 \chi_{ee} \eta \sin(\phi) - \frac{\eta \sin(\phi)}{2}} = \frac{k_0 \chi_{mm} \eta \sin(\phi) + \frac{\eta \sin(\phi)}{2}}{2 - j \omega \varepsilon_0 \chi_{ee} \eta \sin(\phi)}.$$  

The ABC requires \(R = T = 0\) and \(R, T \neq f(\phi)\). The latter condition is ensured by setting the susceptibilities multiplying \(\sin(\phi)\), i.e. \(\chi_{ee} = \chi_{mm} = 0\) in (15). The former condition is then satisfied if the remaining numerators vanish, i.e. \(\chi_{em} = \chi_{me} = \frac{2}{jk}\). As a final result,

$$\chi_{ee} = \chi_{mm} = 0,$$

$$\chi_{em} = \chi_{me} = \frac{2}{jk}.$$  

These MS-ABC susceptibilities characterize the metasurface structure, which exists independently from the incident wave, and these expressions are therefore also independent from the incident wave. This provides an incidence-independent
the sake of paper consistency. Its divergence reads

\[
\nabla \cdot \langle \mathbf{S} \rangle = -\frac{1}{4} j \omega \left[ 2 k_0 E_x^\ast \left( -\frac{2}{j k_0} - \frac{2}{j k_0} \right) H_z \right],
\]

(20)

whose sign can be evaluated upon replacing \(E_x\) and \(H_z\) by the proper fields found with the help of Fig. 5. When the incident wave propagates from the inside to the outside of the computational domain, as in practice, Eq. (20) becomes

\[
\nabla \cdot \langle \mathbf{S} \rangle = -2 \omega \frac{E_z^2 \sin(\phi)}{\eta} < 0
\]

(21)

which means the metasurface is lossy, as illustrated by the green arrow in Fig. 6 and consistently with the synthesis in Sec. III-E. In contrast, if the incident wave would propagate from the outside to the inside of the computational domain, then Eq. (20) would become

\[
\nabla \cdot \langle \mathbf{S} \rangle = 2 \omega \frac{E_z^2 \sin(\phi)}{\eta} > 0
\]

(22)

indicating gain, as illustrated by the red arrow in Fig. 6.

The MS-ABC is thus asymmetric, with absorption in one direction and amplification in the opposite direction. In a real code, some amount of transmission from the computational domain across the MS-ABC is inevitable, of course, and the transmitted wave could in principle be amplified upon reflection from the terminations of the ABC. Fortunately, the amplification of the reflected wave is equal to the absorption of the transmitted wave, so no instability occurs.

Since the susceptibilities of the MS-ABC are purely mono-cross-anisotropic, the first two terms in the right-hand side of (19) vanish. Substituting, for TM-case top wall, Eq. (18) into (19) yields

\[
\nabla \cdot \langle \mathbf{S} \rangle = -\frac{1}{4} j \omega \left[ \frac{\mu_0}{\eta} \left( \chi^{xx} - \chi^{zz} \right) \right] \mathbf{E} \cdot \mathbf{E}^\ast
\]

(21)

The susceptibilities of the left and right MS-ABCs are found in a similar fashion. Table II summarizes the results for TM polarization.

The TE polarization results are immediately inferred by duality, as listed in Tab. III.

### C. MS-ABC Loss-Gain Property

The general time-averaged Poynting vector in a bianisotropic medium in terms of susceptibility tensors has been presented in [13], and is concisely derived in Appendix A for the sake of paper consistency. Its divergence reads

\[
\nabla \cdot \langle \mathbf{S} \rangle = -\frac{1}{4} j \omega \left[ \epsilon_0 \mathbf{E} \cdot \left( \frac{\chi^{xx}}{\eta} \mathbf{E}^\ast + \frac{\chi^{yy}}{\eta} \mathbf{E}^\ast \right) \right]
\]

(19)

where \(\langle \cdot \rangle\) denotes the time average operation, \(\ast\) the conjugate and \(\dagger\) the transpose conjugate. The physical interpretation of (19) is that the amount of loss or gain that an electromagnetic field experiences when propagating through a volume of a given medium, related to the difference of electromagnetic energy entering and exiting that volume.

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it requires no virtual grid plane, and more insightful, as the surface currents represent the currents physically induced on the metasurface.

We start with Maxwell equations in their integral form,

\[
\begin{align*}
\int \mathbf{H} \cdot d\mathbf{l} &= \varepsilon_0 \int \frac{\partial \mathbf{E}}{\partial t} \cdot ds + \iint (\mathbf{J}_{sp} + \mathbf{J}_{sm}) \cdot \delta(\zeta) ds, \\
\int \mathbf{E} \cdot d\mathbf{l} &= -\mu_0 \int \frac{\partial \mathbf{H}}{\partial t} \cdot ds - \iint (\mathbf{K}_{sp} + \mathbf{K}_{sm}) \cdot \delta(\zeta) ds,
\end{align*}
\]

(23a)

(23b)

where the polarization surface currents, given in \(2\), will be computed from the metasurface susceptibilities given in Tabs. I and II for the TM and TE cases, respectively.

The integral equations (23) are discretized in the Yee cell as shown in Fig. 7. Because of the staggered electric and magnetic field grid, the equivalent electric surface polarization currents and magnetic surface polarization currents do not lie in the same plane but are spaced by half a lattice cell, which corresponds to a non-zero sheet approximation, characteristic to GSTC-FDTD [34].

![Fig. 7. FDTD unit cell for the implementation of proposed MS-ABCs. (a) Yee field locations. (b) Equivalent polarization current locations.](image)

**A. Top and Bottom Boundaries**

For the \(p\) case, corresponding to the nonzero field components \((H_z, E_x, E_y)\), the update equations deriving from (23) for the tangential components, \(E_x\) and \(H_z\), read, assuming that the metasurface is located in the \((i, j\alpha)\) plane with \(\alpha = b, t\) as shown in Fig. 8.

\[
E_n^{n-1}(i, j\alpha) = E_n^{n-1}(i, j\alpha) + \frac{\Delta t}{\varepsilon_0} \left[ \frac{H_z^{n-1/2}(i, j\alpha) - H_z^{n-1/2}(i, j\alpha - 1)}{\Delta x} \right. \\
- \left. \frac{\Delta t}{\varepsilon_0} J_x^{n-1/2}(i, j\alpha) \right] + \frac{\Delta t}{\mu_0} \left[ \frac{E_y(i, j\alpha + 1) - E_y(i, j\alpha)}{\Delta y} \right. \\
- \left. \frac{E_y(i, j\alpha + 1) - E_y(i, j\alpha)}{\Delta y} \right],
\]

(24a)

and

\[
H_z^{n+1/2}(i, j\alpha) = H_z^{n-1/2}(i, j\alpha) + \frac{\Delta t}{\mu_0} \left[ \frac{E_y(i, j\alpha + 1) - E_y(i, j\alpha)}{\Delta y} \right. \\
- \left. \frac{E_y(i, j\alpha + 1) - E_y(i, j\alpha)}{\Delta y} \right],
\]

(24b)

In these relations, the currents are given by (3), which read here

\[
\pm \mathbf{\hat{y}} \times \Delta(\hat{\mathbf{k}} \mathbf{H}_z) = \mathbf{\hat{x}} J_x, \tag{25a}
\]

\[
\pm \mathbf{\hat{y}} \times \Delta(\hat{\mathbf{k}} E_x) = -\hat{\mathbf{k}} K_z, \tag{25b}
\]

where the positive and minus sign hold for the bottom and top boundaries, respectively.

![Fig. 8. Integration of the MS-ABCs in the FDTD grid, which is terminated by a backing impedance boundary.](image)

Substituting the field differences in the bottom version of (25) with their expressions in (12), and discretizing the resulting expressions, yields the bottom metasurface equivalent currents

\[
J_x^{n-1/2}(i, j_b) = j\omega\varepsilon_0 \chi_{xx}^{av} E_{x, av} + jk_0 \chi_{xz}^{av} H_{z, av} \\
= 2H_x^{n-1/2}(i, j_b),
\]

(26a)

and

\[
K_z^{n}(i, j_b) = jk_0 \chi_{z}^{av} E_{x, av} + j\omega\mu_0 \chi_{zz}^{av} H_{z, av} \\
= 2E_x^{av}(i, j_b),
\]

(26b)

where the second equalities have been obtained by substituting
the susceptibilities given in Tab. I and the third equalities by spatial averaging to fit the Yee scheme.

Similarly, the equivalent currents for the top metasurface are found as

$$J_n^{n-1/2}(i, j_1) = -[H_z^{n-1/2}(i, j_1 + 1) + H_z^{n-1/2}(i, j_1 - 1)],$$  
(27a)

and

$$K_z^{n}(i, j_1) = -[E_y^{n}(i, j_1 + 1) + E_y^{n}(i, j_1 - 1)].$$  
(27b)

The final update equations of the MS-ABCs are obtained by substituting (26) or (27) into (24).

Let us compare these results with the Multiple Absorbing Surface (MAS) conditions [14], the re-radiating boundary conditions (rRBCs) [14] and the Huygens ABCs [15]. In these approaches, the equivalent surface currents are $J_x^{n-1/2}(i, j_1) = -H_z^{n-3/2}(i, j_1 - 1)$ and $K_y^{n}(i, j_1) = -E_y^{n-1}(i, j_1 - 1)$, and thus somewhat differ from the MS-ABCs in (27). However, once they have been inserted into (26), the terms $H_z^{n-1/2}(i, j_1 + 1)$ and $E_y^{n}(i, j_1 + 1)$ are set to zero, because they are located at the exterior of the computational domain. As a result, only the terms $-H_z^{n-1/2}(i, j_1 - 1)$ and $-E_y^{n}(i, j_1 - 1)$, which are a one-time-step-shifted versions of those mentioned above for MAS, rRBC and Huygens ABCs, survive. Thus, the proposed MS-ABCs are essentially equivalent, in terms of performance, to the MAS, rRBC and Huygens ABCs. Their main interest being, as pointed in Sec. III, is that they provide an alternative formulation with richer absorbing description (independence of the incident wave) and physical insight (in terms of bianisotropic susceptibilities), which forms the contents of Sec. III.

### B. Left and Right Boundaries

As shown in Fig. 3, the left and right absorbing metasurfaces are located at the grid planes $(i_α, j)$ with $α = l, r$, respectively. The related update equations are found from (23) following the same procedure as in Sec. IV-A. They read

$$E_y^{n}(i_α, j) = E_y^{n-1}(i_α, j) + \frac{∆t}{ε_0} \left[ \frac{-H_z^{n-1/2}(i_α, j)}{∆x} - \frac{-J_y^{n-1/2}(i_α, j)}{∆x} \right]$$  
(28a)

and

$$H_z^{n+1/2}(i_α, j) = -H_z^{n-1/2}(i_α, j) + \frac{∆t}{µ_0} \left[ \frac{E_y^{n}(i_α, j + 1)}{∆y} - \frac{E_y^{n}(i_α, j - 1)}{∆y} \right] - \frac{∆t}{µ_0} \frac{M_y^{n}(i_α, j)}{∆x}.$$  
(28b)

where the surface currents are given by (5) with $\hat{\eta} = ± \hat{x}$, i.e.

$$± \hat{x} × ∆(\hat{\eta} H_z) = \hat{\eta} J_y,$$  
(29a)

$$± \hat{x} × ∆(\hat{\eta} E_y) = -\hat{\eta} K_z,$$  
(29b)

with the positive and minus sign holding for the left and right boundaries, respectively.

Using the susceptibilities in Tab. I one obtains then surface currents on the left side absorbing metasurface as

$$J_y^{n-1/2}(i_1, j) = -2H_z^{n-1/2}$$

$$= -[H_z^{n-1/2}(i_1 + 1, j) + H_z^{n-1/2}(i_1 - 1, j)],$$  
(30a)

and

$$M_z^{n}(i_1, j) = -2E_y^{n}(i_1, j),$$  
(30b)

while the surface currents on the right side absorbing surface as

$$J_y^{n-1/2}(i_r, j) = 2H_z^{n-1/2}$$

$$= [H_z^{n-1/2}(i_r + 1, j) + H_z^{n-1/2}(i_r - 1, j)],$$  
(31a)

and

$$M_z^{n}(i_r, j) = 2E_y^{n}(i_r, j).$$  
(31b)

And the final update MS-ABC equations for left and right boundaries are obtained upon substituting (30) or (31) into (28).

### C. Backing Impedance Boundary

As shown in Sec. III-C waves transmitted from the exterior to the interior of the computational domain would be amplified, which could affect the performance of the ABC if they were backed by perfect electric conditions. Therefore, instead of adopting such backing conditions, as most ABCs do, we use here an impedance boundary, that further absorbs normally incident waves, as will be shown in Sec. V.

The impedance boundary has the form

$$E = η[\hat{n} × H]$$  
(32)

where $\hat{n}$ is the normal unit vector at the boundary surface pointed to the computational region. The implementation of such a boundary in FDTD is described in [35].

### V. NUMERICAL RESULTS

This section conducts several numerical experiments to validate our implementation of the MS-ABC based FDTD scheme, and to verify the MS-ABC asymmetry property predicted in Sec. III-C.

#### A. Implementation Validation

We consider the problem of a magnetic line source (p-polarization problem) asymmetrically positioned and radiating a modulated Gaussian pulse in a rectangular computational domain.
Figure 9 plots the fields computed using the MS-ABC, for a computational lattice cell size of \( \Delta x = \Delta y = \lambda/200 \) to ensure a sufficiently good approximation of the metasurface sheet model. It may be observed that fields radiated by the line source are very well absorbed, as expected.

Figure 10 compares the performance of the MS-ABCs with PMLs and open boundaries. The results are almost perfectly superimposed.

We do not perform a detailed performance study of the MS-ABC, since we know from Sec. IV-A that it is essentially identical to that of the Multiple Absorbing Surface (MAS) conditions, the re-radiating boundary conditions (rRBCs) and the Huygens ABCs.

**B. Asymmetry Verification**

We now investigate the metasurface asymmetry predicted in Sec. III-C. For this purpose, we compare the field waveforms at two different points, one within the computational domain (O1) and one beyond it (O2), between the MS-ABC and the impedance boundary, as shown in Fig. 11 in a 1D-FDTD simulation.

The result is plotted in Fig. 12. The strong attenuation of the field at O2 confirms the absorbing effect of the MS-ABC. From these curves, the absorption level of the MS-ABS is found to be in the order of 50 dB. The field propagated across the MS-ABC is seen to be essentially the time-derivative of the field incident on it; this is an effect of discretization, that has been described in.

Let us now replace the backing impedance boundary by a perfect electric conductor boundary, as shown in Fig. 13, to see if the the predicted inverse amplification indeed occurs (Sec.III-C). The result is plotted in Fig. 14. It shows that the incident pulse first traverses the MS-ABC, experiencing the time-derivation, is then reflected by the perfect electric conductor (with expected phase reversal), and finally traverses the MS-ABC in the opposite direction, where it is amplified back to its original level. This confirms that the MS-ABC in the opposite direction from the attenuation direction amplifies the wave by the same amount as it attenuates it in the first direction, hence confirming the theory of Sec.III-C.
We have introduced Metasurface Absorbing Boundary Conditions (MS-ABCs). MS-ABCs have similar performance as previously reported surface ABCs, such as the Multiple absorbing surfaces (MAS) conditions [13], the re-radiating boundary conditions (rRBCs) [14] and the Huygens ABCs [15], but they provide a richer description of absorption, where the surface is independent from the incident wave, and deeper insight into the physics, through bianisotropic susceptibilities. The synthesized MS-ABC is mono-cross-anisotropic and asymmetric. MS-ABCs are implemented in FDTD in the form of of equivalent surface electric and magnetic polarization currents.

VI. CONCLUSION

We have introduced Metasurface Absorbing Boundary Conditions (MS-ABCs). MS-ABCs have similar performance as previously reported surface ABCs, such as the Multiple absorbing surfaces (MAS) conditions [13], the re-radiating boundary conditions (rRBCs) [14] and the Huygens ABCs [15], but they provide a richer description of absorption, where the surface is independent from the incident wave, and deeper insight into the physics, through bianisotropic susceptibilities. The synthesized MS-ABC is mono-cross-anisotropic and asymmetric. MS-ABCs are implemented in FDTD in the form of of equivalent surface electric and magnetic polarization currents.

REFERENCES

[1] A. Taflove and S. C. Hagness, Computational Electrodymanics: The Finite-Difference Time-Domain Method. Artech house, 2005.
[2] J.-P. Bérenger, “A perfectly matched layer for the absorption of electromagnetic waves,” J. Comput. Phys., vol. 114, no. 2, pp. 185–200, 1994.
[3] W. C. Chew and W. H. Weedon, “A 3D perfectly matched medium from modified maxwell’s equations with stretched coordinates,” Microw. Opt. Technol. Lett., vol. 7, no. 13, pp. 599–604, 1994.
[4] Z. Z. Sacks, D. M. Kingsland, R. Lee, and J.-F. Lee, “A perfectly matched anisotropic absorber for use as an absorbing boundary condition,” IEEE Trans. Antennas Propag., vol. 43, no. 12, pp. 1460–1463, 1995.
[5] M. Kuzuoglu and R. Mittra, “Frequency dependence of the constitutive parameters of causal perfectly matched anisotropic absorbers,” IEEE Microwave Guided wave lett., vol. 6, no. 12, pp. 447–449, 1996.
[6] J. A. Roden and S. D. Gedney, “Convolutional PML (CPML): An efficient FDTD implementation of the CFS-PML for arbitrary media,” Microw. Opt. Technol. Lett., vol. 27, no. 5, pp. 334–338, 2000.
[7] W. Dallenbach and W. Kleinsteuber, “Reflection and absorption of decimeter-waves by plane dielectric layers,” Hochfrequenztechnik und Elektroakustik, vol. 51, no. 5, pp. 152–156, 1998.
[8] W. W. Salisbury, “Absorbing boundary for electromagnetic waves,” Jun. 10 1952, US Patent 2,599,944.
[9] N. Engheta, “Thin absorbing screens using metamaterial surfaces,” in IEEE AP-S Int. Antennas Propag. (APS) Conf., vol. 2, IEEE, 2002, pp. 392–395.
[10] N. I. Landy, S. Sajuyigbe, J. Mock, D. Smith, and W. Padilla, “Perfect metamaterial absorber,” Phys. Rev. Lett., vol. 100, no. 20, p. 207402, 2008.
[11] F. Bilotti, A. Toscano, K. B. Alici, E. Ozbay, and L. Vegni, “Design of miniaturized narrowband absorbers based on resonant-magnetic inclusions,” IEEE Trans. Electromagn. Compat., vol. 53, no. 1, pp. 63–72, 2011.
[12] Y. Ra’Di, C. Simovski, and S. Tretyakov, “Thin perfect absorbers for electromagnetic waves: theory, design, and realizations,” Phys. Rev. Appl., vol. 3, no. 3, p. 037001, 2015.
[13] I. W. Sudjariata, “An absorbing boundary condition for FDTD truncation using multiple absorbing surfaces,” IEEE Trans. Antennas Propag., vol. 51, no. 12, pp. 3268–3275, 2003.
[14] R. Diaz and I. Scherbako, “A simple stackable re-radiating boundary condition (rRBC) for FDTD,” IEEE Antennas Propag. Magazine, vol. 46, no. 1, pp. 124–130, 2004.
[15] J.-P. Bérenger, “On the Huygens absorbing boundary conditions for electromagnetics,” J. Comput. Phys., vol. 226, no. 1, pp. 354–378, 2007.
[16] F. Costen and J. P. Bérenger, “Implementation of the huygens absorbing boundary condition in corner regions,” IEEE Trans. Electromagn. Compat., vol. 54, no. 2, pp. 367–374, 2012.
[17] C. L. Holloway, E. F. Kuester, J. A. Gordon, J. O’Hara, J. Booth, and D. R. Smith, “An overview of the theory and applications of metasurfaces: The two-dimensional equivalents of metamaterials,” IEEE Trans. Antennas Propag., vol. 54, no. 2, pp. 10–35, 2012.
[18] K. Achouri and C. Caloz, “Design, concepts and applications of electromagnetic metasurfaces,” arXiv preprint, arXiv:1712.00618, 2017.
[19] S. Tretyakov, Analytical Modeling in Applied Electromagnetics. Artech House, 2003.
[20] M. Idenen, “Straightforward derivation of boundary conditions on sheet simulating an anisotropic thin layer,” Electron. Lett., vol. 24, no. 11, pp. 663–665, 1988.
[21] E. F. Kuester, M. A. Mohamed, M. Piket-May, and C. L. Holloway, “Averaged transition conditions for electromagnetic fields at a metafi lm,” IEEE Trans. Antennas Propag., vol. 51, no. 10, pp. 2641–2651, 2003.
[22] K. Achouri, M. A. Salem, and C. Caloz, “General metasurface synthesis based on susceptibility tensors,” IEEE Trans. Antennas Propag., vol. 63, no. 7, pp. 2977–2991, 2015.
[23] M. Albooyeh, V. Asadchy, R. Alaee, S. Hashemi, M. Yazdi, M. Mirmoosa, C. Rockstuhl, C. Simovski, and S. Tretyakov, “Purely bianisotropic scatterers,” Phys. Rev. E, vol. 94, no. 24, p. 245426, 2016.
[24] J. A. Kong, “Theory of electromagnetic waves,” New York, Wiley-Interscience, 1975. 348 p.
[25] M. Albooyeh, V. Asadchy, R. Alaee, M. Yazdi, S. Hashemi, C. Rockstuhl, and S. Tretyakov, “Averaged transition conditions for electromagnetic fields at a metasurface,” IEEE Trans. Antennas Propag., vol. 54, no. 2, pp. 367–374, 2012.
[26] Y. Vahabzadeh and C. Caloz, “Finite-element modeling of metasurfaces with generalized sheet transition conditions,” IEEE Trans. Antennas Propag., vol. 65, no. 5, pp. 2413–2420, 2017.
[27] Y. Vahabzadeh and C. Caloz, “GSTC-based simulation of metasurfaces in finite difference techniques,” in 2015 IEEE Int. Symp. on Antennas Propag. (APSURSI), June 2016, pp. 373–374.
[28] Y. Vahabzadeh, N. Chamanara, and C. Caloz, “Generalized sheet transition condition FDTD simulation of metasurface,” IEEE Trans. Antennas Propag., 2017.
[29] A. Chlavin, “A new antenna feed having equal e - and h-plane patterns,” Trans. of the IRE Professional Group on Antennas Propag., vol. 2, no. 3, pp. 113–119, July 1954.
A. Poynting Vector in a Bi-anisotropic Medium

Assuming the time-harmonic dependence $e^{j\omega t}$ and the presence of electric currents $J$ and magnetic currents $K$, Maxwell equations take the form

\begin{align}
\nabla \times \mathbf{E} &= -j\omega \mathbf{B} - \mathbf{K}, \\
\nabla \times \mathbf{H} &= j\omega \mathbf{D} + \mathbf{J},
\end{align}

where the fields are related to the electric polarization density $\mathbf{P}$ and the magnetic polarization density $\mathbf{M}$ through the constitutive relations

\begin{align}
\mathbf{D} &= \varepsilon_0 \mathbf{E} + \mathbf{P}, \quad \text{with} \quad \mathbf{P} = \varepsilon_0 \chi_{ee} \mathbf{E} + \sqrt{\mu_0 \varepsilon_0} \chi_{mm} \mathbf{H}, \\
\mathbf{B} &= \mu_0 (\mathbf{H} + \mathbf{M}), \quad \text{with} \quad \mathbf{M} = \sqrt{\varepsilon_0 / \mu_0} \chi_{me} \mathbf{E} + \chi_{mm} \mathbf{H}.
\end{align}

Dot-multiplying (33a) with $\mathbf{H}$ and dot-multiplying (33b) with $\mathbf{E}$, and subtracting the latter result from the former result yields

\begin{align}
\mathbf{H} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot (\nabla \times \mathbf{H}) &= \nabla \cdot (\mathbf{E} \times \mathbf{H}) \\
&= -\mathbf{H} \cdot \mathbf{K} - \mathbf{E} \cdot \mathbf{J} - j\omega \mathbf{H} \cdot \mathbf{B} - j\omega \mathbf{E} \cdot \mathbf{D}.
\end{align}

Writing the left-hand side of the second equality in (35) in terms of the Poynting vector,

\begin{align}
\mathbf{S} = \mathbf{E} \times \mathbf{H},
\end{align}

yields

\begin{align}
\nabla \cdot \mathbf{S} &= -\mathbf{H} \cdot \mathbf{K} - \mathbf{E} \cdot \mathbf{J} - j\omega \mathbf{H} \cdot \mathbf{B} - j\omega \mathbf{E} \cdot \mathbf{D} \\
&= I_K + I_J + I_M + I_P,
\end{align}

whose time average reads

\begin{align}
\nabla \cdot \langle \mathbf{S} \rangle &= [\text{Re}(I_K) + \text{Re}(I_J) + \text{Re}(I_M) + \text{Re}(I_P)] \\
&= -\frac{1}{2} \mathbf{H}^* \cdot \mathbf{K} - \frac{1}{2} \mathbf{E}^* \cdot \mathbf{J} - \frac{1}{2} j\omega \mathbf{H}^* \cdot \mathbf{B} - \frac{1}{2} j\omega \mathbf{E}^* \cdot \mathbf{D}.
\end{align}

We shall thus consider the real parts of the four terms in the right-hand side of (37). Let us start with the first two terms. The currents definitions

\begin{align}
\mathbf{J} &= \overline{\sigma}_e \cdot \mathbf{E}, \\
\mathbf{K} &= \overline{\sigma}_m \cdot \mathbf{H},
\end{align}

and representation of the conductivity tensors in these definitions in terms of imaginary susceptibility tensors in Maxwell equations, leads to the following relations

\begin{align}
\overline{\sigma}_e &= -\omega_0 \text{Im}(\overline{\chi}_{ee}) = \frac{j\omega_0 \chi_{ee}}{2} (\overline{\chi}_{ee} - \overline{\chi}_{ee}^*), \\
\overline{\sigma}_m &= -\omega_0 \text{Im}(\overline{\chi}_{mm}) = \frac{j\omega_0 \chi_{mm}}{2} (\overline{\chi}_{mm} - \overline{\chi}_{mm}^*),
\end{align}

Inserting the second equalities of (40) into (39), and substituting the result into (37) yields

\begin{align}
\text{Re}(I_K) &= -\frac{j\omega_0}{4} \mathbf{H}^* \cdot (\overline{\chi}_{mm} - \overline{\chi}_{mm}^*) \cdot \mathbf{H}, \\
\text{Re}(I_J) &= -\frac{j\omega_0}{4} \mathbf{E}^* \cdot (\overline{\chi}_{ee} - \overline{\chi}_{ee}^*) \cdot \mathbf{E}.
\end{align}

Let us now consider the last two terms in (37). Using the constitutive relationships (34), their time average becomes

\begin{align}
\text{Re}(I_M) &= -\frac{j\omega_0}{4} \left[ \mathbf{H}^* \cdot (\mathbf{B} - (\mathbf{H} \cdot \mathbf{B}) \mathbf{B}) \right] \\
&= -\frac{j\omega_0}{4} \left[ \mathbf{H}^* \cdot \mathbf{H} + \sqrt{\varepsilon_0 / \mu_0} \chi_{me} \mathbf{E} + \sqrt{\mu_0 \varepsilon_0} \chi_{mm} \mathbf{H} \right] \\
&= -\frac{j\omega_0}{4} \left[ \mathbf{H}^* \cdot (\mathbf{H} + \sqrt{\varepsilon_0 / \mu_0} \chi_{me} \mathbf{E} + \sqrt{\mu_0 \varepsilon_0} \chi_{mm} \mathbf{H}) \right] \\
&= -\frac{j\omega_0}{4} \left[ \mathbf{H}^* \cdot (\overline{\chi}_{mm} - \overline{\chi}_{mm}^*) \cdot \mathbf{H} - 2\varepsilon_0 \chi_{mm} \mathbf{E} \cdot \mathbf{H} / \eta_0 \right]
\end{align}

and

\begin{align}
\text{Re}(I_P) &= -\frac{j\omega_0}{4} \left[ \mathbf{E}^* \cdot (\mathbf{D} - (\mathbf{E} \cdot \mathbf{D}) \mathbf{D}) \right] \\
&= -\frac{j\omega_0}{4} \left[ \mathbf{E}^* \cdot \mathbf{E} + (\varepsilon_0 \overline{\chi}_{ee} \mathbf{E} + \sqrt{\mu_0 \varepsilon_0} \chi_{mm} \mathbf{H}) \right] \\
&= -\frac{j\omega_0}{4} \left[ \mathbf{E}^* \cdot (\overline{\chi}_{ee} - \overline{\chi}_{ee}^*) \cdot \mathbf{E} + 2\varepsilon_0 \chi_{mm} \mathbf{E} \cdot \mathbf{H} \right]
\end{align}

Combining now the four terms (41), (42), (43) and (44) of (38), rearranging and simplifying yields

\begin{align}
\nabla \cdot \langle \mathbf{S} \rangle &= \text{Re}(I_K) + \text{Re}(I_J) + \text{Re}(I_M) + \text{Re}(I_P) \\
&= -\frac{j\omega_0}{4} \left[ (\varepsilon_0 \mathbf{E}^* \cdot (2\overline{\chi}_{ee} - \overline{\chi}_{ee}^*) \cdot \mathbf{E} + \mu_0 \mathbf{H}^* \cdot (2\overline{\chi}_{mm} - \overline{\chi}_{mm}^*) \cdot \mathbf{H} \\
+ 2\varepsilon_0 \chi_{mm} \mathbf{E} \cdot \mathbf{H} / \eta_0 \right]
\end{align}

The time-averaged divergence of the Poynting vector is related to the difference of the electromagnetic energy entering and exiting the volume, i.e. the medium, over which it is integrated. Thus, a lossless passive medium has zero average Poynting vector (equal exiting and entering energies), i.e. $\nabla \cdot \langle \mathbf{S} \rangle = 0$, a lossy passive medium has a negative average Poynting vector (smaller exiting energy than entering energy), i.e. $\nabla \cdot \langle \mathbf{S} \rangle < 0$, and an active medium has a positive average Poynting vector (larger exiting than entering entering), i.e. $\nabla \cdot \langle \mathbf{S} \rangle > 0$, as illustrated in Fig. 15. Thus, the lossless/lossy passive/active nature of a metasurface can be simply determined by substituting its tensor $\overline{\chi}_{ee}$, $\overline{\chi}_{mm}$, $\overline{\chi}_{me}$ and $\overline{\chi}_{mm}$ into (45), and considering the resulting sign according to Fig. 15.
\( \nabla \cdot \langle \mathbf{S} \rangle = 0 \) (a)
\( \nabla \cdot \langle \mathbf{S} \rangle < 0 \) (b)
\( \nabla \cdot \langle \mathbf{S} \rangle > 0 \) (c)

Fig. 15. Time-averaged divergence of the Poynting vector. (a) Passive lossless medium. (b) Passive lossy medium. (c) Active medium.