Formula for the Mean Square Displacement Exponent of the Self-Avoiding Walk in 3, 4 and All Dimensions

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Abstract

This paper proves the formula

$$\nu(d) = \begin{cases} 
1 & \text{for } d = 1, \\
\max(\frac{1}{2}, \frac{1}{d} + \frac{1}{d}) & \text{for } d \geq 2
\end{cases}$$

for the root mean square displacement exponent $\nu(d)$ of the self-avoiding walk (SAW) in $\mathbb{Z}^d$, and thus, resolves some major long-standing open conjectures rooted in chemical physics (Flory (1949) [3]). The values $\nu(2) = 3/4$ and $\nu(4) = 1/2$ coincide with those that were believed on the basis of heuristic and “numerical evidence”. Perhaps surprisingly, there was no precise conjecture in dimension 3. Yet as early as in the 1980ies, Monte Carlo simulations produced a couple of confidence intervals for the exponent $\nu(3)$. This work is a follow-up to Hueter [7], which proves the result for $d = 2$ and lays out the fundamental building blocks for the analysis in all dimensions.

We consider (a) the point process of self-intersections defined via certain paths of length $n$ of the symmetric simple random walk in $\mathbb{Z}^d$ and (b) a “weakly self-avoiding cone process” relative to this point process in a certain “shape”. The asymptotic expected distance of the process in (b) can be calculated rather precisely as $n$ tends large and, if the point process has circular shape, can be shown to asymptotically equal (up to error terms) the one of the weakly SAW with parameter $\beta > 0$. From these results, a number of distance exponents are immediately collectable for the SAW as well. Our approach invokes the Palm distribution of the point process of self-intersections in a cone.

1 Introduction

This paper is a follow-up to Hueter [3] and establishes a formula for the root mean square displacement exponent of the self-avoiding walk in the $d$-dimensional hypercubic lattice for
all $d \geq 1$. This simple formula resolves the “puzzling” case $d = 3$ and confirms the longstanding open conjectures for $d = 2$ and $d = 4$ that originate in the work of Flory [3] in the 1940ies. The self-avoiding walk serves as a model for linear polymer molecules. Polymers are of interest to chemists and physicists and are the fundamental building schemes in biological systems. A polymer is a long chain of monomers which are joined to one another by chemical bonds. These polymer molecules arrange themselves randomly with the restriction of no overlap. This repelling force drives the polymers to be more diffusive than a simple random walk. Numerous stones wait to be uncovered from a mathematic ally rigorous point of view since very little is known about the 2-, 3- and 4-dimensional polymers or self-avoiding walk. At the other end, though, there is an unnumbered set of simulations and heuristic arguments and a zoo of “numerical artifacts” that lend themselves to a landscape of conjectures. The literature has devoted much attention to this theme. We refer the reader to other references for an overview (e.g. consult Madras and Slade [10]).

This paper presents some answers to the question on the average distance between the two ends of a long polymer. Our results, pertaining to the asymptotic expected distance of the weakly self-avoiding walk from its starting point up through a large step size, cover all dimensions $d$.

(Weakly) Self-Avoiding Walk. Consider the weakly self-avoiding walk in $\mathbb{Z}^d$ starting at the origin. More precisely, if $J_n = J_n(\cdot)$ denotes the number of self-intersections or the self-intersection local time (SILT) of a symmetric simple random walk $S_0 = 0, S_1, \ldots, S_n$ in the $d$-dimensional lattice starting at the origin, that is,

$$J_n = J_n(S_0, S_1, \ldots, S_n) = \sum_{0 \leq i < j \leq n} 1_{\{S_i = S_j\}}, \quad (1.1)$$

and if $\beta \geq 0$ denotes the self-intersection parameter, then the weakly self-avoiding walk is the stochastic process, induced by the probability measure

$$Q^\beta_n(\cdot) = \frac{\exp\{-\beta J_n(\cdot)\}}{\mathbb{E}\exp\{-\beta J_n(\cdot)\}}, \quad (1.2)$$

where $\mathbb{E}$ stands for the expectation relative to the random walk. In other words, $J_n = r$ self-intersections are penalized by the factor $\exp\{-\beta r\}$. The measure $Q^\beta_n$ may be looked at as a measure on the set of all simple random walks of length $n$ which weighs relative to the number of self-intersections. This restraint walk is also being called the Domb-Joyce model in the literature (see Lawler [3], p. 170) but differs from the discrete Edwards model, which is a related repelling walk (see Madras and Slade [10], p. 367 and Lawler [3], p. 172 for some background). While when setting $\beta = 0$ we recover the simple random walk (SRW), letting $\beta \to \infty$ well mimics the self-avoiding walk (SAW). The SAW in $\mathbb{Z}^d$ is a SRW-path of length $n$ without self-intersections. Thus, this walk visits each site of its path exactly once.

We shall investigate the expected distance of the weakly SAW from its starting point after $n$ steps, as measured by the Euclidean length and the root mean square displacement at the $n$-th step. Let $E^\beta = E^\beta_{Q^\beta_n}$ denote expectation under the measure $Q^\beta_n$, that is, expectation
wrt. to the weakly SAW. Thus, $E_0$ denotes expectation wrt. to the SRW. Also, write $S_n = (X_n^1, X_n^2, \ldots, X_n^d)$ for every integer $n \geq 0$. Objects of interest to us are the expectation $E_\beta$ of the distance

$$
\chi_n = \{ \sum_{k=1}^{d} (X_n^k)^2 \}^{1/2}
$$

of the walk from the starting point $0$, the mean square displacement $E_\beta \chi_n^2$, and the root mean square displacement $(E_\beta \chi_n^2)^{1/2}$ of the weakly SAW. Shorter, we shall write MSD and RMSD (for the latter two), respectively. The RMSD exponent of the weakly SAW and the SAW, respectively, may be defined by

$$
\nu_\beta(d) = \lim_{n \to \infty} \frac{\ln E_\beta(\chi_n^2)}{2 \ln n} \\
\nu_\infty(d) = \nu(d) = \lim_{n \to \infty} \lim_{\beta \to \infty} \frac{\ln E_\beta(\chi_n^2)}{2 \ln n}
$$

if the limits exist (otherwise we may regard the upper and lower exponents via $\lim \sup$ and $\lim \inf$). Moreover, define the numbers

$$
\mu = \mu(d) = 1 \quad \text{for } d = 1,
= \max\left(\frac{1}{2}, \frac{1}{4} + \frac{1}{d}\right) \quad \text{for } d \geq 2.
$$

Written out, $\mu(\cdot)$ takes the values

$$
1, \ 3/4, \ 7/12, \ 1/2, \ 1/2, \ldots.
$$

Next, we state our main results.

**Theorem 1** The exponents of the distance of the weakly self-avoiding walk with $\beta > 0$ and of the self-avoiding walk in $Z^d$ for $d \geq 1$ equal $\mu(d)$. Furthermore, there are some constants $0 < \rho_1(d) = \rho_1(d, \beta) \leq \rho_2(d) = \rho_2(d, \beta) < \infty$ such that

$$
\rho_1(d) \leq \lim_{n \to \infty} \inf n^{-\mu(d)} E_\beta(\chi_n) \leq \lim_{n \to \infty} \sup n^{-\mu(d)} E_\beta(\chi_n) \leq \rho_2(d),
$$

where $\rho_1(d)$ is uniform in $\beta$ for $d \geq 5$ and may depend on $\beta$ for $d \leq 4$ and $\rho_2(d)$ is uniform in $\beta$ for $d \leq 2$ and $d \geq 5$ and may depend on $\beta$ for $d = 3, 4$.

The proof is in Corollary $\|$ for $d \geq 2$ and in Theorem $\|$ for $d = 1$.

**Theorem 2** The weakly self-avoiding walk with $\beta > 0$ and the self-avoiding walk in $Z^d$ for $d \geq 1$ have

$$
\nu_\beta(d) = \nu(d) = \mu(d).
$$

Moreover, there are some constants $0 < \rho_3(d) = \rho_3(d, \beta) \leq \rho_4(d) = \rho_4(d, \beta) < \infty$ such that

$$
\rho_3(d) \leq \lim_{n \to \infty} \inf n^{-2\mu(d)} E_\beta(\chi_n^2) \leq \lim_{n \to \infty} \sup n^{-2\mu(d)} E_\beta(\chi_n^2) \leq \rho_4(d),
$$

where $\rho_3(d)$ is uniform in $\beta$ for $d \geq 5$ and may depend on $\beta$ for $d \leq 4$ and $\rho_4(d)$ is uniform in $\beta$ for $d \leq 2$ and $d \geq 5$ and may depend on $\beta$ for $d = 3, 4$. 
See Corollary 2 for a proof when $d \geq 2$ and Corollary 3 when $d = 1$. Hueter [7] proves the analogous results in the two-dimensional context. Theorem 2 settles a couple of major decades-old open conjectures that can be traced back to at least Flory’s work [3] in the 1940ies. It is hoped that our point of view will shed light onto the (weakly) SAW enough to bring to fruition solutions of other tantalizing problems on these and related objects. Whereas our results here and in [5] are novel for $d = 2, 3, 4$ and $\beta \in (0, \infty]$, the result on the RMSD exponent for the SAW for $d \geq 5$ is in Hara and Slade [5, 6] and the one on the RMSD exponent for the weakly SAW for $d = 1$ in Greven and den Hollander [4]. Of course, the result on the one-dimensional SAW is quite obvious. The former was investigated via the perturbation technique “lace expansion” and the latter via large deviation theory. Brydges and Spencer [2] establish that the scaling limit of the weakly SAW is Gaussian for sufficiently small $\beta > 0$ and $d \geq 5$.

A couple of Monte Carlo simulations were performed as early as in the 1980ies to estimate the RMSD exponents for the SAW (for more references and details on this, see Madras and Sokal [11]). The produced 95%-confidence intervals appear to center around the value 0.59... and would suggest a value slightly larger than $7/12 = 0.58333...$. On another historical note, an earlier estimate was the Flory estimate 0.6. Just for $d = 4$, a logarithmic correction associated with $E_{\beta} \chi_{\beta}^2$ is being predicted (visit e.g. Lawler [9], p. 167). We point out that our results leave space for such a correction for $d = 3, 4$. Indeed, the expressions that we derive for both the mean square displacement and the expected distance of the weakly SAW in this paper are bounded by constants that depend on $\beta$ as $\beta \to \infty$. In order to exchange limits as $\beta \to \infty$ and as $n \to \infty$ of $E_{\beta} \chi_{\beta}^2/n^{2\mu(d)}$ as is necessary to obtain the MSD of the SAW, we would need to know how $\beta$ and $n$ are related, in other words, in how far $\beta$ is bounded by some function in $n$ and vice versa. Such a relationship would allow to translate the bounds of $E_{\beta} \chi_{\beta}^2/n^{2\mu(d)}$ in terms of $\beta$ into bounds in $n$, and thus, into some correction factors to $n^{2\mu(d)}$. Nevertheless, none of this is needed to extend the values of the distance and MSD exponents of the weakly SAW to the SAW. Also, while our results are not new for $d = 1$ and $d \geq 5$, our approach provides an alternative proof.

Perhaps surprisingly, one and the same approach – the one employed in this work – suffices to handle all dimensions $d \geq 1$. With a bit of extra work, the results for $d \geq 3$ follow from the analysis for the case $d = 2$, whereas the case $d = 1$ has a different touch. Therefore, the latter dimension is dealt with in a separate section (Section 6). In dimension 4, an interesting twist occurs to the expression for the expected distance of the (weakly) SAW, and thus, for the distance exponent as well. This expression carries two significant terms, one of which is dominating in dimensions 2 through 3, the other of which is dominating in dimensions $d \geq 5$ and may be identified as the term that resembles the contribution which we would obtain for the SRW. In dimension 4, both terms compete with each other. While the SRW-term wins for all $\beta > 0$ below a certain threshold, it is not clear which of both terms dominates for large $\beta$. In this sense, for instance, dimension 4 is more intriguing than dimension 3 despite the fact that the RMSD exponent is the same as for the SRW. Hence,
in dimension 4, the exponent 1/2 arises for different reasons than it occurs in dimensions 5 and higher.

Our strategy of proof is to regard a process which is penalized according to the number of self-intersections of the random walk that we see in one direction and to compare its expected distance to the one of the weakly SAW. For this purpose, we will spread a fixed collection of rays that emanate from the origin, each of which describes a cone. Some of these cones will carry a more typical number of self-intersections than others – typical will mean of order $\sqrt{n}$. Furthermore, the event that a cone is less typical will depend on the realized SRW-path. For $d \geq 3$, the space becomes large in the sense that the cones which contribute most of the self-intersections of the weakly SAW have cardinality of order strictly less than the order of the total number of cones (see (4.20)).

We will invoke the Palm distribution of the point process of self-intersections, defined via certain paths of length $n$ of the symmetric SRW in $\mathbb{Z}^d$, in a cone to introduce a “weakly self-avoiding cone process” relative to this point process when in a certain “shape”. The asymptotic expected distance of this process can be calculated rather precisely as $n$ tends large and, if the point process has circular shape, can be shown to asymptotically equal (up to error terms) the one of the weakly SAW with parameter $\beta > 0$. Then we collect analogous results on the mean square displacement of the weakly SAW. From these results, upon some considerations on uniform bounds and estimates in $\beta$ as $\beta \to \infty$, the distance and the MSD exponents of the SAW immediately derive.

The paper is organized as follows. Section 2 specifies the SRW-paths that are significant from a weakly SAW’s point of view. Section 3 makes a connection between Palm distributions and the random walk and recalls the notions of shape of the underlying point process and of a weakly self-avoiding cone process. Section 4 calculates some asymptotic mean distances of this process and links those to the ones of the weakly SAW. Section 5 discusses the transfer of the distance and MSD exponents to the SAW. Some remarks on the transitions $\beta \to \infty$ and $\beta \to 0$ end Section 5. Finally, Section 6 takes care of the one-dimensional setup.

## 2 SILT that is Typical for the Weakly SAW

Most of the remainder of the paper will be devoted to studying the weakly SAW. We shall exploit the information that is contained in the intersections that are discouraged but not forbidden as for the SAW. In low dimensions, the weakly SAW pays attention to the SRW-paths that exhibit a smaller number of self-intersections than is expected for the SRW. This effect is most emphasized in dimension 1. Paths that have about $E_0 J_n$ self-intersections are not important from the perspective of a weakly SAW. While a weakly SAW-path of length $n$ will turn out to have expected SILT of order $n$ in all dimensions, the SRW is forced to intersect itself more frequently, at least in dimensions 1 and 2. We begin to review the average $E_0 J_n$ for the SRW and to derive the range for $J_n$ that is significant from the point
of view of the weakly SAW.

A favorite exercise in a probability course is as follows. By invoking the Fubini theorem and the Local Central Limit theorem, we obtain for all sufficiently large even \( n \),

\[
\mathbb{E}_0 J_n = \sum_{0 \leq i < j \leq n} P_0(S_i = S_j)
\]

\[
= (1 + o(1)) \sum_{0 \leq i < j \leq n/2} 2 \left( \frac{d}{2\pi(j - i)} \right)^{d/2}
\]

\[
= (1 + o(1)) \begin{cases} 
\frac{2}{3\pi^{d/2}} n^{3/2} & d = 1, \\
\frac{1}{d} \ln n & d = 2, \\
c_d n & d \geq 3
\end{cases}
\]

for some positive finite constants \( c_d \), where we used the \( o(\cdot) \) notation, that is, write \( f(n) = o(g(n)) \) as \( n \to \infty \) for two real-valued functions \( f \) and \( g \) if \( \lim_{n \to \infty} f(n)/g(n) = 0 \).

Now, let \( \omega_0 = \omega_0(d) \) denote the logarithm of the connective constant or the exponent of the number of SAW-paths, in other words, the exponential rate at which the cardinality of the set of all SAW-paths \( S_0 = 0, S_1, \ldots, S_n \) (with \( J_n = 0 \)) up through time \( n \) grows in \( n \). Observe that \( \omega_0(1) = 0 \) and \( d \leq e^{\omega_0(d)} \leq 2d - 1 \) for all \( d \). The upper bound \( 2d - 1 \) for \( e^{\omega_0} \) may be seen by counting all paths of length \( n \) that do not return to the most recently visited point (clearly, an overestimate), whereas the lower bound \( d \) for \( e^{\omega_0} \) may be seen by counting all paths of length \( n \) that take only positive steps in both coordinates, for example for \( d = 2 \), i.e. move only north or east, say.

The two subsequent Propositions restate the results in Propositions 1 and 2 in Hueter [7], Section 2, without proofs. Since the arguments of proof are carried out in the time space, as opposed to the state space, they as well apply for \( d \neq 2 \). When reading the proofs of Hueter [7], written for \( d = 2 \), the reader might want to replace the number ‘4’, the number of nearest neighboring sites of each lattice site, by \( 2d \) and rely on \( \omega_0(d) \), as just described, rather than \( \omega_0(2) \).

The idea to prove the upper bound is that it suffices to find a subset of SRW-paths that contributes strictly more to \( \mathbb{E}_0 \exp\{-\beta J_n\} \) than the set of paths with \( J_n > Bn \) for all \( B > B_* \) and some suitable positive finite constant \( B_* \). In fact, the set of all self-avoiding paths satisfies this requirement. It is enough to derive a lower bound for \( P_0(J_n = 0) = \mathbb{E}_0(\exp\{-\beta J_n\} 1_{\{J_n = 0\}}) \) and to see for which \( B_* \) it is strictly larger than \( \exp\{-\beta B_* n\} > \mathbb{E}_0(\exp\{-\beta J_n\} 1_{\{J_n > B_* n\}}) \).

**Proposition 1 (Upper Bound for \( J_n \))** Let \( d \geq 1, \beta > 0 \), and let \( \omega_0(d) \) denote the exponent of the number of self-avoiding walks. Then for every \( B > B_* = B_*(d) = (\ln(2d) - \omega_0(d))/\beta > 0 \) and every integer \( n \geq 0 \),

\[
\mathbb{E}_0(\exp\{-\beta J_n\} 1_{\{J_n > Bn\}}) < \mathbb{E}_0(\exp\{-\beta J_n\} 1_{\{J_n = 0\}}),
\]
in particular, as \( n \to \infty \),
\[
E_0(e^{-\beta J_n} 1_{\{J_n > B_n\}}) = o(E_0(e^{-\beta J_n} 1_{\{J_n = 0\}})).
\]

**Proof.** The proof is in Hueter [7], Proposition 1.

Hence, we may restrict our attention to the SRW-paths that exhibit \( J_n \leq n B_* \). On the other hand, the paths with \( J_n \) of order less than \( n \) are not significant either.

**Proposition 2 (Lower Bound for \( J_n \))** Let \( d \geq 1 \) and \( \beta > 0 \). There is some \( \zeta_*(\beta) > 0 \) that can be made precise ([7], Proposition 2) such that for \( b_* = \zeta_*(\beta) / \beta > 0 \), for every \( \delta > 0 \) and every \( b < b_* \), as \( n \to \infty \),
\[
E_0(e^{-\beta J_n} 1_{\{J_n \leq n^{1-\delta}\}}) = o(E_0(e^{-\beta J_n} 1_{\{J_n < bn\}})).
\]

**Proof.** The proof is carried out in Hueter [7], Proposition 2.

The reasoning to prove the lower bound is that it is sufficient to identify a subset of SRW-paths that contributes strictly more to \( E_0 \exp\{-\beta J_n\} \) than the set of paths with \( J_n \leq n^{1-\delta} \). The latter set is modified by introducing of order \( n \) repetitions of steps to each SRW-path, which gives rise to a set of paths that have \( J_n \leq bn \) for \( b < b_* \), whose size is of strictly larger exponential order. Then \( b_* \) can be chosen suitably small.

We remark that the same proof with slight adjustments applies when the bound \( n^{1-\delta} \) in the statement of Proposition 2 is replaced by \( n q_n \), where \( q_n \to 0 \) arbitrarily slowly as \( n \to \infty \). Hence, the set of paths with \( J_n \in [0, n q_n] \) contributes to \( E_0(e^{-\beta J_n}) \) or to the \( k \)-th moments \( E_{\beta}(\chi_n^k) \) only in a negligible fashion in the sense that the contribution is \( o(E_0(e^{-\beta J_n})) \) or \( o(E_{\beta}(\chi_n^k)) \), respectively, as \( n \) tends large (in fact, this error term is exponentially smaller, as the proof of Proposition 2 indicates). As a consequence of Propositions 2 and 2, for all that follows, we may neglect to keep track of those error terms and assume that
\[
J_n \in [b_1 n, b_2 n]
\]
for all sufficiently large \( n \) and for some constants \( 0 < b_1 < b_2 < \infty \) such that \( \beta b_2 \) is a positive number independent of \( \beta \) and \( \beta b_1 \) may depend on \( \beta \) in such a way that \( \beta b_1 \) tends to zero as \( \beta \to \infty \). Observe that comparing (2.1) and (2.2) along with the observations in the last paragraph reveals that, for \( d \leq 2 \), the expectation \( E_0 J_n \) is of larger order in \( n \) than \( E_{\beta} J_n \), whereas for \( d \geq 3 \) and \( \beta \) below a certain threshold, \( E_0 J_n < E_{\beta} J_n \).

### 3 Point Process of Self-Intersections and Cones

Throughout this section, we will assume that \( d > 1 \), and for the rest of the paper, we shall omit discussion of the obvious case \( \beta = 0 \). If we let \( X_n^1, X_n^2, \ldots, X_n^d \) denote the coordinate
processes of the SRW, that is, \( S_n = (X_n^1, \ldots, X_n^d) \) for every integer \( n \geq 0 \), define the distance
\[
\chi_n = |S_n| = \left\{ \sum_{k=1}^{d} (X_n^k)^2 \right\}^{1/2}
\] or the root of the square displacement of the walk from the starting point \( 0 \). Furthermore, let \( P_{\chi_n} \) denote the probability distribution of the distance \( \chi_n \) of the SRW.

As in Hueter [7], we shall rely on a process which is intimately related to the weakly SAW. Let us think about asymptotically calculating the expected distance of the SRW after \( n \) steps from the starting point (for which process, though, the calculation is more straightforward). One possible route involves approximating the SRW by means of a Brownian motion and controlling the entailed errors, which may be sketched in the following way. Rely on the Local Central Limit theorem and rewrite the density of the approximating Brownian motion to the SRW in polar coordinates. Then the asymptotic expected distance of the SRW is calculated via integrations over the radial part and the angle. In case of the weakly self-avoiding process, the penalizing weight takes into consideration the number of self-intersections in a direction, that is, near the line that passes through the starting point and the endpoint of the SRW-path. Part of our strategy will consist in relating the expected distance of this newly-defined process with the one of the weakly SAW and in finding bounds on the expected distance of the former process by

(a) keeping track of the radial part of the SRW, penalized by the SILT in a certain cone,
(b) by integrating out over all lines in \( V \).

3.1. Point Process of Self-Intersections and Cones. The next subsection will utilize Palm distributions of the point process of self-intersection points of the SRW with \( J_n \in [b_1n, b_2n] \) to define typical penalizing weights within certain classes of cones. Palm distributions help answer questions dealing with properties of a point process, viewed from a typical random geometric object that is defined via the point process. As a simple example we could explore the mean number of points of a point process in the plane whose nearest neighbors are all at distance at least \( r \).

Let us, however, first recall the notation set in Hueter [7] to describe the point process of self-intersections and its associated cones. Let \( \Phi = \Phi_n = \{x_1, x_2, \ldots\} \) denote the point process of self-intersection points of the SRW in \( \mathbb{Z}^d \) when \( J_n \in [b_1n, b_2n] \). Note that \( |\Phi| \in [b_1n, b_2n] \) and \( \Phi \) depends on \( n \), \( b_1 \), and \( b_2 \), thus, on \( \beta \). We allow the points \( x_i \) of \( \Phi \) to have multiplicity and count such a point exactly as many times as there are self-intersections of the SRW at \( x_i \). This random sequence of points \( \Phi \) in \( \mathbb{Z}^d \) may also be interpreted as a random measure. Note that \( E_0 \Phi \) is \( \sigma \)-finite. Let \( N_\Phi \) denote the set of all point sequences, generated by \( \Phi \), \( N_\Phi \) the point process \( \sigma \)-algebra generated by \( N_\Phi \), and \( \varphi \in N_\Phi \) denote a realization of \( \Phi \). Formally, \( \Phi \) is a measurable mapping from the underlying probability space into \( (N_\Phi, N_\Phi) \) that induces a distribution on \( (N_\Phi, N_\Phi) \), the distribution \( P_\Phi \) of the point process \( \Phi \). By virtue of the \( \sigma \)-finiteness of \( E_0 \Phi \), \( P_\Phi \) is a probability measure. Also, let \( E_\Phi \)
and a certain number of points x of lines in V with the convention that if equality dist(x, R) for each 0 ≤ L.

We will postpone determining the cardinality |V| of V to the proofs of Propositions 1 and 2, which will be the only relevant fact about V to retain. Next, for any L ∈ V, let the “cone” C_L be defined by

\[ C_L = \{ x_i \in \Phi : \text{dist}(x_i, L) \leq \text{dist}(x_i, L') \text{ for all } L \neq L' \in V \} \tag{3.2} \]

with the convention that if equality \( \text{dist}(x_i, L) = \text{dist}(x_i, L') \) holds for two lines L and L' and a certain number of points \( x_i \), then half of them will be assigned to \( C_L \) and the other half to \( C_{L'} \). Note that no point of \( \Phi \) belongs to more than one \( C_L \) and each point to exactly one \( C_L \). Thus, |\( C_L \)| equals the SILT of the SRW \( S_n \) in a cone at the origin that contains the line L. Once the lines are selected for \( V \), we may classify them according to the SILT that their cones carry. For any constants 0 < \( a_1 < a_2 < \infty \), for any suitably small \( \delta > 0 \), and for each 0 ≤ r ≤ 1, define the random sets

\[
\begin{align*}
\mathcal{L}_{1/2} &= \mathcal{L}_{1/2}(\Phi) = \{ L \in V : 2|C_L| \in [a_1 n^{1/2}, a_2 n^{1/2}] \} \tag{3.3} \\
\mathcal{L}_{1/2+} &= \mathcal{L}_{1/2+}(\Phi) = \{ L \in V : 2|C_L| \in [a_1 n^{1/2-\delta}, a_2 n^{1/2+\delta}] \} \\
\mathcal{L}_{-} &= \mathcal{L}_{-}(\Phi) = \{ L \in V : 2|C_L| \in (0, a_1 n^{1/2-\delta}) \} \\
\mathcal{L}_{+} &= \mathcal{L}_{+}(\Phi) = \{ L \in V : 2|C_L| \in (a_2 n^{1/2+\delta}, 2b_2 n) \} \\
\mathcal{L}_{r} &= \mathcal{L}_{r}(\Phi) = \{ L \in V : 2|C_L| \in [a_1 n^r, a_2 n^r] \} \\
\mathcal{L}_{\emptyset} &= \mathcal{L}_{\emptyset}(\Phi) = \{ L \in V : |C_L| = 0 \},
\end{align*}
\]

which depend on \( a_1, a_2 \), and \( V \). We will choose \( a_1 \) and \( a_2 \) such that \( a_1 \beta \) and \( a_2 \beta \) are positive numbers which are independent of \( \beta \) and \( n \).

3.2. Weakly Self-Avoiding Cone Process relative to r-Shaped \( \Phi \). If \( h : \mathbb{R} \times N_{\Phi} \to \mathbb{R}_+ \) denotes a nonnegative measurable real-valued function and \( \mathcal{L}_+(\Phi) \) denotes any subset of lines in \( V \), then since \( E_0 \Phi \) is \( \sigma \)-finite, we may disintegrate relative to the probability
measure \( P_\Phi \),
\[
E_\Phi \left( \sum_{L \in \mathcal{L}(\Phi)} h(L, \Phi) \right) = \int \sum_{L \in \mathcal{L}(\varphi)} h(L, \varphi) dP_\Phi(\varphi) \quad (3.4)
\]
(visit also Kallenberg [3], p. 83, and Stoyan, Kendall, and Mecke [13], p. 99). For a discussion of some examples of Palm distributions of \( P_\Phi \), the reader is referred to the Appendix in Hueter [7].

Observe that the conditional distribution \( P_{\Phi|\chi_n} \) of the point process \( \Phi \), given \( \chi_n \), is a function of \( \chi_n \) and depends on condition [2.2], as explained earlier, so as to produce realizations that exhibit \( J_n \in [b_1 n, b_2 n] \). Apply formula (3.4) with
\[
h(L, \Phi) = \frac{\exp\{-\beta |\mathcal{C}_L|\}}{|\mathcal{L}(\Phi)|}, \quad (3.5)
\]
with \( P_{\Phi|\chi_n}(\varphi|x) \) in place of \( P_\Phi(\varphi) \), and \( \mathcal{L}_r = \mathcal{L} \subset \mathcal{L}_1 \subset \mathcal{V} \) to define the numbers \( a_x = a_x(\mathcal{L}) \) by
\[
\exp\{-\beta a_x n^r/2\} = E_{\Phi|\chi_n}( |\mathcal{L}(\Phi)|^{-1} \sum_{L \in \mathcal{L}(\Phi)} e^{-\beta |\mathcal{C}_L|} |\chi_n = x|) \quad (3.6)
\]
\[
= \int_{\mathbb{R}^d} |\mathcal{L}(\varphi)|^{-1} \sum_{L \in \mathcal{L}(\varphi)} e^{-\beta |\mathcal{C}_L|} dP_{\Phi|\chi_n}(\varphi|x)
\]
for \( 0 \leq x \leq n \), where we set \( \sum_{L \in \mathcal{L}} = 0 \) if \( \mathcal{L} = \emptyset \). For example, if we set \( \mathcal{L} = \mathcal{L}_1/2 \) and \( r = 1/2 \), then conditioned on the event \( \chi_n = x \), the number \( a_x(\mathcal{L}_1/2)n^{1/2}/2 \) may be interpreted as “typical” SILT relative to the lines in \( \mathcal{L}_1/2 \), equivalently, \( \exp\{-\beta a_x n^{1/2}/2\} \) represents a “typical” penalizing factor with respect to \( \mathcal{L}_1/2 \), provided that \( \chi_n = x \). Taking expectation, we arrive at the expected “typical” penalizing factor
\[
E_0(e^{-\beta J_n^{\mathcal{L}}}) = E_0(\exp\{-\beta a_x n^r/2\}). \quad (3.7)
\]
In the same manner, we calculate
\[
E_0(\chi_n e^{-\beta J_n^{\mathcal{L}}}) = E_0(\chi_n E_{\Phi|\chi_n}( |\mathcal{L}(\Phi)|^{-1} \sum_{L \in \mathcal{L}(\Phi)} e^{-\beta |\mathcal{C}_L|} |\chi_n = x|)). \quad (3.8)
\]
The proofs of Propositions [4] and [5] below (see also Definition [2]) will shed light on the issue of this particular choice of penalizing weight. It will turn out that a crucial role will be played by variants of the quotient \( E_0(\chi_n e^{-\beta J_n^{\mathcal{L}_1/2}})/E_0(e^{-\beta J_n^{\mathcal{L}_1/2}}) \).

**Definition 1 (Φ or V are r-shaped)** Let \( \rho > 0 \) be suitably small. We say that \( \mathcal{L}_r \) contributes (to \( J_n \)) essentially if
\[
\sum_{L \in \mathcal{L}_r} |\mathcal{C}_L| \geq 1/2 J_n^{1-\rho}.
\]
In this case, we say that \( \mathcal{V} \) and \( \Phi \) are r-shaped or have shape \( r \). In particular, when \( r = 1/2 \), then we say that \( \mathcal{V} \) and \( \Phi \) have circular shape or are circular. The convention is that multiple shapes are allowed, that is, \( \Phi \) may simultaneously have shape 1/2 and shape 3/4.
Remarks.

(1) For our purposes and later calculations, it is not necessary that the lines contributing essentially, as explained in Definition 1, have exact SILT of order \(n^r\) in the sense that the real value \(r\) is hit precisely. Instead, it suffices to replace \(L^r\) by \(L^r\star = \{L \in V : 2|C_L| \in [a_1n^r, a_2n^r+\delta]\}\) for \(\delta > 0\), and to ultimately let \(\delta \to 0\) in the obtained results (because \(\delta > 0\) was arbitrary). Hence, when applying Definition 1, we will think of \(L^r\star\) rather than \(L^r\) and refer to \(\sum_{L \in L^r} \geq \frac{1}{2} J_n^{1-\rho}\). (3.9)

With this meaning, it is obvious that, for sufficiently large \(n\), there must be \(0 \leq r \leq 1\) such that the set \(L^r\star\) contributes essentially, and thus, the shape of \(\Phi\) and \(V\) is well-defined. Nevertheless, for the sake of not complicating our presentation, we shall not write \(L^r\star\) and not use the extension in (3.9) but simply write \(L^r\).

(2) We might as well choose \(J_n\tau_n/2\) with \(\tau_n \to 0\) arbitrarily slowly as \(n \to \infty\) in place of \(J_n^{1-\rho}/2\) in the defining inequality for the shape of \(\Phi\). There is nothing special about the choice above.

Next, if \(L^r\) contributes essentially then, by (2.2) and (3.3),

\[
\frac{b_1}{a_2} n^{1-r-\rho} \leq |L^r| \leq \frac{2b_2}{a_1} n^{1-r}.
\] (3.10)

It is apparent that the upper bound in (3.10) holds even when \(\Phi\) is not \(r\)-shaped. Since we choose \(a_1\) and \(a_2\) such that \(\beta a_1\) and \(\beta a_2\) are independent of \(\beta\), it follows that \(b_2/a_1\) is independent of \(\beta\).

**Definition 2 (Weakly self-avoiding cone process relative to \(r\)-shaped \(V\))** Define a weakly self-avoiding cone process relative to \(V\) in shape \(r\) by some \(d\)-dimensional process whose radial part is induced by the probability measure

\[
Q_n^{\beta,V,r} = \frac{\exp\{-\beta|C_L|\}}{E_0 \exp\{-\beta J_n^L\}}
\] (3.11)

on the set of SRW-paths of length \(n\) if \(V\) has shape \(r\), where \(L\) denotes the line through the origin and the endpoint of the SRW after \(n\) steps. Moreover, the expectation \(E_{\beta,V,L^r} = E_{Q_n^{\beta,V,r}}\) relative to the radial part is calculated as in (3.6) followed by (3.7) with \(L = L^r\).

Let \(E_{\beta,V,\star(r)}\) denote expectation of the \(d\)-dimensional weakly self-avoiding cone process relative to \(V\) in shape \(r\). In particular, we write \(E_{\beta,V,\star} = E_{\beta,V,\star(1/2)}\). Thus, the definition of this process depends on the choice of \(V\) and on \(\Phi\). Note that there is no unique such process since only the distribution of the radial component of the process is prescribed and not even the distribution on the lines in \(V\) is specified. Consequently, there will be several ways to choose the set \(V\). Importantly though, the shape carries much information.
4 Expected Distances

We turn to a technical lemma that engages a condition and a couple more definitions. The main players in this condition are bounded numbers $a_x$ that depend on $x > 0$ and will be substituted by the numbers $a_x(L_r)$, especially, $a_x(L_{1/2})$, shortly. Recall that the latter are bounded in $x$ and that we assumed that there is some number $\zeta > 0$, independent of $\beta$, so that $\beta a_x(L_r) \geq \zeta$ for every $0 \leq x \leq n$. Define

$$
\mu_x = (\beta a_x)^{1/2} n^{3/4}
$$

(4.1)

$$
q(x) = \exp\{-\beta a_x n^{1/2}\}
$$

(4.2)

for every $n \geq 0$, $\beta > 0$, and $x$ in $[0, n]$. Since $a_x$ is bounded in $x$, for suitably small $\varepsilon \geq 0$ and for $\gamma > 0$, we may define

$$
r_1 = r_1(\varepsilon, \gamma) = \sup\{x \in [0, n] : x \leq \gamma \mu n^{-\varepsilon}\}
$$

$$
r_2 = r_2(\gamma) = \sup\{x \in [0, n] : x \leq \gamma \mu\}.
$$

(4.3)

Thus, $r_2(\gamma) = r_1(0, \gamma)$.

**Condition D.** For any suitably small $\varepsilon \geq 0$, there exist some $\gamma > 0$ and $\rho_* > 0$ such that

$$
\int_{r_1}^{n} x q(x) dP_{\chi_n}(x) = \rho_n \int_{0}^{r_1} x q(x) dP_{\chi_n}(x)
$$

(4.4)

with $\rho_n \geq \rho_*$ for all sufficiently large $n$.

Note that if $\int_{0}^{r_2} x q(x) dP_{\chi_n}(x) = o(\int_{r_1}^{n} x q(x) dP_{\chi_n}(x))$ as $n \to \infty$, then $\varepsilon = 0$ and $\rho_n \to \infty$. In addition, observe that, in light of the expression in (4.2) for $q(x)$, Condition D guarantees that $a_x$ not be constant in $x$ and $\beta > 0$. Throughout the paper, we shall be careful about whether constants in $n$ and/or $x$ depend on $\beta$ or not and indicate this.

In the next result, drawn from Hueter [7], the $a_x$ are some general numbers that obey the stated assumptions.

**Lemma 1 ([7], Lemma 1)** *(Exponent of Expected Radial Distance equals 3/4)* Let $\beta > 0$ and $d \geq 1$. Assume that the $a_x$ are bounded numbers that depend on $x$, are such that there is some number $\zeta > 0$ so that $\beta a_x \geq \zeta$ for every $0 \leq x \leq n$, and that satisfy Condition D in (4.4) for some $\varepsilon \geq 0$ and $\gamma > 0$. Define

$$
I_n = \int_{0}^{n} x q(x) dP_{\chi_n}(x)
$$

(4.5)

$$
g(n) = \int_{0}^{n} (a_x)^{1/2} q(x) dP_{\chi_n}(x),
$$

where $q(x)$ is defined in (4.2). Then there are some constants $M < \infty$ and $c(\rho_*) > 0$ (both independent of $\beta$) such that as $n \to \infty$,

$$
\gamma c(\rho_*) \beta^{1/2} n^{3/4-\varepsilon} (1 + o(1)) \leq \frac{I_n}{g(n)} \leq M \beta^{1/2} n^{3/4} (1 + o(1)).
$$

(4.6)
Proof. We do not reproduce the proof that is presented in Hueter \cite{Hueter}, Lemma 1, in two dimensions.

The next result is as well borrowed from Hueter \cite{Hueter}.

**Lemma 2 (\cite{Hueter}, Lemma 2) (The $a_x(L_{1/2})$ satisfy Condition D)** Let $d > 1$. If $\Phi$ has circular shape for sufficiently large $n$, then the $a_x(L_{1/2})$, defined in (3.6) when $L = L_{1/2}$ and $r = 1/2$ satisfy Condition D in (4.4) for $\epsilon = 0$ and $\gamma > 0$, independent of $\beta$ as $\beta \to \infty$.

Proof. The idea of proof is to violate Condition D and to take this assumption to a contradiction to the one that $\Phi$ be circular. The proof is omitted here and can be found in Hueter \cite{Hueter}, Lemma 2. It runs in parallel with the proof of Lemma 3 that we will present in Section 6.

**Proposition 3 (\cite{Hueter}, Proposition 3) (Expected Distance Along Cones with Order $n^{1/2}$ SILT)** Let $d > 1$ and $\beta > 0$. There are some constants $0 < \gamma_s \leq M < \infty$ (independent of $\beta$ as $\beta \to \infty$ and $M$ independent of $\beta > 0$ as well) such that as $n \to \infty$,

$$E_0(\chi_{n} e^{-\beta J_{n}^{L_{1/2}}}) = K(n) n^{3/4} \beta^{1/2} g(n)(1 + o(1))$$

for $\gamma_s \leq K(n) \leq M$, where $g(n)$ was defined in (4.5).

Proof. Since the proof is rather short, we present it here again. Observe that, in view of Lemma 2, the $a_x = a_x(L_{1/2})$ satisfy Condition D in (4.4) for $\epsilon = 0$ and $\gamma > 0$. Combining the observations preceding (3.8) together with (3.6) and (3.7) and Lemma 1 leads to, as $n \to \infty$,

$$E_0(\chi_{n} e^{-\beta J_{n}^{L_{1/2}}}) = E_0(\chi_{n} E_{\Phi|\chi_{n}}( |L_{1/2}(\Phi)|^{-1} \sum_{L \in L_{1/2}(\Phi)} e^{-\beta |c_L|} |\chi_{n}| = x))$$

$$= \int_{0}^{n} x E_{\Phi|\chi_{n}}( |L_{1/2}(\Phi)|^{-1} \sum_{L \in L_{1/2}(\Phi)} e^{-\beta |c_L|} |\chi_{n}| = x) dP_{\chi_{n}}(x)$$

$$= \int_{0}^{n} x \left( \int_{Z^d} |L_{1/2}(\varphi)|^{-1} \sum_{L \in L_{1/2}(\varphi)} e^{-\beta |c_L|} dP_{\Phi|\chi_{n}}(\varphi|x) \right) dP_{\chi_{n}}(x)$$

$$= \int_{0}^{n} x \exp\{-\beta a_x n^{1/2}/2\} dP_{\chi_{n}}(x)$$

$$= \int_{0}^{n} x q(x) dP_{\chi_{n}}(x)$$

$$= K(n) n^{3/4} \beta^{1/2} g(n)(1 + o(1))$$

for $\gamma_s \leq K(n) \leq M$, where to obtain the last two lines of the display, we apply Lemma 1 with $\gamma_s = \gamma c(\rho_s)$, $\epsilon = 0$, and with the $a_x$ being bounded and such that there is some number.
\( \zeta > 0 \) so that \( \beta a_x \geq \zeta \) for every \( 0 \leq x \leq n \). These two properties of \( a_x \) may be seen as follows. First, since, by (3.3), \( 2|C_L|/n^{1/2} \) is in \([a_1, a_2]\), the average of the exponential terms \( \exp\{-\beta|C_L|\} \) over all lines in \( L_{1/2}(\varphi) \) may be rewritten as \( \exp\{-\beta a_x n^{1/2}/2\} \), say, for some number \( a_x \in [a_1, a_2] \), depending on \( x \). In particular, the \( a_x \) are bounded. Additionally, we assumed (remark following display (3.3)) that \( a_1 \beta \) is a positive number independent of \( \beta \), thus, there is some number \( \zeta > 0 \) so that \( \beta a_x \geq \zeta \) for all \( x \). This completes our proof. \( \square \)

We collect two propositions and key ingredients to our main results.

**Proposition 4 (Upper Bound for \( E_\beta \chi_n \))** Let \( \beta > 0 \) and \( d > 1 \). There is some constant \( M_*(d) = M_*(d, \beta) < \infty \) (made precise below) such that as \( n \to \infty \),

\[
E_\beta(\chi_n) \leq M_*(d) (1 + o(1)) \max(n^{1/4+1/d}, n^{1/2}),
\]

where \( M_*(d) \) is uniform in \( \beta \) for \( d = 2 \) and \( d \geq 5 \) and may depend on \( \beta \) for \( d = 3, 4 \).

**Proof.** It will suffice to prove that, for \( \mathcal{V} \) in circular shape, as \( n \to \infty \),

1. \( E_{\beta, \mathcal{V}, \ast}(\chi_n) \leq M_*(d) (1 + o(1)) \max(n^{1/4+1/d}, n^{1/2}) \) for \( M_*(d) < \infty \) and
2. \( E_\beta(\chi_n) \leq E_{\beta, \mathcal{V}, \ast}(\chi_n) (1 + o(1)) \).

**Part (I).** Assume that \( \mathcal{V} \) is 1/2-shaped for all sufficiently large \( n \). First, we fix the size of \( \mathcal{V} \). Since, up to scaling by a factor between 1 and \( \sqrt{d} \), any direction in \( \mathbb{R}^d \) is the same for the weakly self-avoiding cone process relative to \( \Phi \), we choose the lines in \( \mathcal{V} \) uniformly distributed over some \( d \)-sphere such that there are of order \( n^{1/d} \) lines along each side of the smallest \( d \)-cube that contains the \( d \)-sphere. In other words, \( |\mathcal{V}| \) is of order \( n^{(d-1)/d} = n^{1-1/d} \), say, \( |\mathcal{V}| = v_n n^{1-1/d} \) for \( v_1(\beta) \leq v_n \leq v_2 \), for all sufficiently large \( n \), where, in view of (3.10), we can choose the two constants \( 0 < v_1(\beta) \leq v_2 < \infty \) so that \( v_2 \) is independent of \( n \) and \( \beta \) for each \( \beta > 0 \) and \( v_1 \) is independent of \( n \) but may depend on \( \beta \), even as \( \beta \to \infty \).

Next, a consequence of the arguments in the proof of Proposition 4 in Hueter [7] is that as \( n \to \infty \),

\[
E_{\beta, \mathcal{V}, \ast}(\chi_n) \leq P_\Phi(L \in \mathcal{L}_{\ast/2}) \max_{\mathcal{L} \subseteq \mathcal{L}_{1/2}} \frac{E_0(\chi_n e^{-\beta J_n^\xi})}{E_0(e^{-\beta J_n^{\xi/2}})} (1 + o(1))
+ P_\Phi(L \in \mathcal{L}_0) \frac{E_0(\chi_n e^{-\beta J_n^\xi})}{E_0(e^{-\beta J_n^{\xi/2}})},
\]

where \( \mathcal{L}_0 \) and \( \mathcal{L}_{1/2} \) are defined in (3.3) and \( E_0(\chi_n e^{-\beta J_n^\xi}) \) is to be understood in the sense of definitions (3.7) and (3.8). Furthermore, we have seen in Hueter [7], proof of Proposition 4, that as \( n \to \infty \),

\[
\max_{\mathcal{L} \subseteq \mathcal{L}_{1/2}} \frac{E_0(\chi_n e^{-\beta J_n^\xi})}{E_0(e^{-\beta J_n^{\xi/2}})} \leq M (1 + o(1)) (\beta a_2)^{1/2} n^{3/4},
\]

(4.11)
where $\beta a_2$ and $M$ are finite constants that do not depend on $\beta$ (uniform in $\beta$). Let us devote a moment to bound the last term in (4.10). A routine exercise yields that as $n \to \infty$,

$$\frac{E_0(\chi_n e^{-\beta J_{n}^{\ell,\beta}^a})}{E_0(e^{-\beta J_{n}^{\ell,\beta}^a})} = \left(\frac{2}{\pi}\right)^{1/2} n^{1/2}(1 + o(1)).$$

(4.12)

The probability $P_\Phi(L \in \mathcal{L})$ may be interpreted as a Palm probability, that is,

$$P_\Phi(L \in \mathcal{L}) = \frac{E_\Phi \sum_{L \in \mathcal{V}} 1_{\mathcal{L}}(L)}{|\mathcal{V}|} = \frac{E_\Phi |\mathcal{L}|}{|\mathcal{V}|}.$$  

(4.13)

Therefore, since $v_1 n^{1-1/d} \leq |\mathcal{V}|$, by virtue of (3.10),

$$P_\Phi(L \in \mathcal{L}_{1/2}) \leq \frac{2b_2}{a_1 v_1} n^{1/d - 1/2},$$

(4.14)

where $2b_2/(a_1 v_1)$ may depend on $\beta$ since $v_1$ does. Hence, (4.10), (4.11), (4.12), and (4.14) can be summarized as

$$E_{\beta, V, \gamma}(\chi_n) \leq M_*(d) (1 + o(1)) \max(n^{1/4 + 1/d}, n^{1/2})$$

(4.15)

as $n \to \infty$ for $M_*(d) = M(\beta a_2)^{1/2}(2b_2/(a_1 v_1)) + (2/\pi)^{1/2} < \infty$ for $d = 3, 4$, $M_*(d) = M(\beta a_2)^{1/2}$ for $d = 2$, and $M_*(d) = (2/\pi)^{1/2}$ for $d \geq 5$. In summary, $M_*(d)$ is uniform in $\beta$ for $d = 2$ and $d \geq 5$ but may depend on $\beta$, even as $\beta \to \infty$, for $d = 3, 4$. This completes the verification of (I) along with the asymptotic evaluation of its righthand side.

**Part (II).** Demonstrating (II) will finish our proof. This portion is much as given in Hueter [3], part (II) of the proof of Proposition 4. We sketch an outline. Recall that $E_{\beta, V, \gamma}(r)$ denotes expectation of the $d$-dimensional weakly self-avoiding cone process relative to $V$ in shape $r$. In order to compare $E_{\beta, V, \gamma}(r)(\chi_n)$ and $E_\beta(\chi_n)$, the strategy will be to show that, for fixed $J_n \in [b_1 n, b_2 n]$, the number of SRW-paths with $J_n$ whose point process $\Phi$ is $r$-shaped is larger than the number of SRW-paths with $J_n$ whose point process $\Phi$ is $s$-shaped (but not $r$-shaped) for $1/2 \leq r < s$. We will continue to show that $\mathcal{L}_s$ for $0 \leq s < 1/2$ plays a negligible role as well. In other words, most SRW-paths that satisfy (2.2) arise from a $\Phi$ that is 1/2-shaped. Finally, we shall compare the centers of mass of the weakly self-avoiding cone process and the weakly SAW.

**(a) $\Phi$ prefers circular shape.** Fix $J_n$ (and assume that $J_n \in [b_1 n, b_2 n]$). Partition the interval $[1/2, 1]$ into $R$ subintervals of equal length, that is, let $1/2 = r_0 < r_1 < r_2 < \ldots < r_R = 1$. We are interested in comparing the number of SRW-paths with $J_n$ whose point process $\Phi$ is $r_{k-1}$-shaped to the number of SRW-paths with $J_n$ whose point process $\Phi$ is $r_k$-shaped (but not $r_{k-1}$-shaped). For this purpose, we shall give an inductive argument over $k$. Pick a SRW-path $\gamma$ of length $n$ with $J_n$ whose point process $\Phi$ has shape $r_k$. We will show that (i) associated with $\gamma$, there is a large set $F_\gamma$ of SRW-paths whose realizations of $\Phi$ have shape $r_{k-1}$, and (ii) two sets $F_\gamma$ and $F_{\gamma'}$ are disjoint for $\gamma \neq \gamma'$. To see this, we **cut and paste** the path $\gamma$ as follows. Let $P_\gamma$ denote the smallest parallelepiped that contains
the path $\gamma$ and let $l_\gamma$ denote the largest integer less than or equal to the length of the longest side of $P_\gamma$. Divide $P_\gamma$ into sub-parallelepipeds whose sides are parallel to the sides of $P_\gamma$ by partitioning the two longest sides of $P_\gamma$ into $n_f$ subintervals in the same fashion whose endpoints are vertices of the integer lattice and by connecting the two endpoints of the subintervals that are opposite to each other on the two sides. Shift each of the sub-parallelepipeds including the SRW-subpaths contained by a definite amount between 1 and $K$ ($K$: some constant) along one of the directions of the shorter sides of $P_\gamma$ and reconnect the SRW-subpaths where they were disconnected. In doing this, the shifts are chosen such that the new path $\gamma'$ will have shape $r_{k-1}$ and the total number of connections needed to reconnect those subpaths equals a number $C_n$ that is constant in $k$. Observe that such a choice of shifts exists. When walking through the new path $\tilde{\gamma}$, because of the necessary extra steps to reconnect the subpaths, the last several steps of $\gamma$ will be ignored. Note that this latter number of steps is independent of $k$. Hence, if the pieces to reconnect are self-avoiding, then $J_n$ is no larger after this cut-and-paste procedure than before. This is always possible for otherwise we shift apart the sub-parallelepipeds such that they are sufficiently separated from each other. Now, either we choose the reconnecting pieces such that $J_n$ is preserved or we “shift back” (along the direction of the long sides of the parallelepiped) some or all of the sub-parallelepipeds so that any two parallelepiped overlap sufficiently to preserve $J_n$ and then reconnect the SRW-subpaths where they were disconnected. Again, we shift in such a fashion that the total number of connections needed to reconnect the subpaths equals $C_n$. The number of these newly constructed paths in $F_\gamma$ grows at least at the order that the number of ways does to choose $n_f$ locations (to shift) among $l_\gamma$ sites, which is a number larger than 1 for all large enough $n$. Hence, the number of SRW-paths with $J_n$ whose point process $\Phi$ is $r_{k-1}$-shaped is larger than the number of SRW-paths with $J_n$ whose point process $\Phi$ is $r_k$-shaped. Since this argument can be made for every $1 \leq k \leq R$ and the number of SRW-paths with $J_n$ whose point process $\Phi$ has shape $r_R = 1$ is at least 1, it follows that the number of SRW-paths with $J_n$ whose point process $\Phi$ is $r$-shaped is maximal for $r = 1/2$.

(b) It suffices to consider shapes $r$ with $r \geq 1/2$. This passage is identical to part (II)(b) in the proof of Proposition 4 in HUETER [7] and omitted here.

The considerations in (a) above also imply that both probability distributions decay exponentially fast around their centers of mass. Combining this observation with the fact that the shape of $\Phi$ relates the SILT of the weakly SAW to the one of the weakly self-avoiding cone process provides that the two probability distributions asymptotically have the same centers of mass (up to error terms). Together with these, the upshot of above passages (a) and (b) is that, in comparing $E_\beta(\chi_n)$ to $E_{\beta,V,*}(r)(\chi_n)$ for $0 \leq r \leq 1$, it is enough to choose $r = 1/2$ and to study the expected distance of the weakly self-avoiding cone process relative to $\Phi$ when in circular shape. Hence, in particular, we are led to

$$E_\beta(\chi_n) \leq E_{\beta,V,*}(\chi_n)(1 + o(1))$$
as \( n \to \infty \). This accomplishes the proof of (II), and thus, ends the proof.

\[\Box\]

**Proposition 5 (Lower Bound for \( E_\beta \chi_n \))** Let \( \beta > 0 \) and \( d > 1 \). There is a constant \( m(d) = m(d, \beta) > 0 \) (made precise below) such that as \( n \to \infty \),

\[
E_\beta(\chi_n) \geq m(d) (1 + o(1)) \max(n^{1/4+1/d}, n^{1/2}),
\]

where \( m(d) \) is uniform in \( \beta \) for \( d \geq 5 \) and may depend on \( \beta \) for \( d = 2, 3, 4 \).

**Proof.** Fix \( \rho > 0 \) and \( \delta > 0 \). From the reasoning in Hueter [7], proof of Proposition 5, and part (II)(b) in the proof of Proposition 4, we collect

\[
E_{\beta, \mathcal{V}, *}(\chi_n) \leq E_\beta(\chi_n)
\]

and that there are two competing terms involved to bound \( E_{\beta, \mathcal{V}, *}(\chi_n) \) from below. The dimension \( d \) decides which one is maximal. (Hence, for the rest of the proof, we may assume that \( \Phi \) has circular shape.) Namely, we have, as \( n \to \infty \),

\[
E_{\beta, \mathcal{V}, *}(\chi_n) \geq (1 + o(1)) \max \left[ P_{\Phi}(L \in \mathcal{L}_{1/2}) \frac{\min_{\mathcal{L} \subset \mathcal{L}_{1/2}} E_0(\chi_n e^{-\beta J_n^{L}})}{E_0(e^{-\beta J_n^{L_{1/2}}})}, \right.
\]

\[
\left. P_{\Phi}(L \in \mathcal{L}_{-} \cup \mathcal{L}_{\emptyset}) \frac{\min_{\mathcal{L} \subset \mathcal{L}_{-} \cup \mathcal{L}_{\emptyset}} E_0(\chi_n e^{-\beta J_n^{L}})}{E_0(e^{-\beta J_n^{L_{-} \cup \mathcal{L}_{\emptyset}}})} \right]
\]

\[
\geq (1 + o(1)) \max \left[ P_{\Phi}(L \in \mathcal{L}_{1/2}) \frac{\min_{\mathcal{L} \subset \mathcal{L}_{1/2}} E_0(\chi_n e^{-\beta J_n^{L}})}{E_0(e^{-\beta J_n^{L_{1/2}}})}, \right.
\]

\[
\left. P_{\Phi}(L \in \mathcal{L}_{-} \cup \mathcal{L}_{\emptyset}) \frac{E_0(\chi_n e^{-\beta J_n^{L_{\emptyset}}})}{E_0(e^{-\beta J_n^{L_{\emptyset}}})} \right],
\]

where \( \mathcal{L}_{-} \) and \( \mathcal{L}_{\emptyset} \) are defined in [7,3] and the minimum \( \min_{\mathcal{L} \subset \mathcal{L}_{1/2}} \) is over subsets \( \mathcal{L} \subset \mathcal{L}_{1/2} \) that form a subset of \( \mathcal{V} \) that is circular for sufficiently large \( n \). Moreover, in Hueter [7], proof of Proposition 4, we arrived at

\[
\frac{\min_{\mathcal{L} \subset \mathcal{L}_{1/2}} E_0(\chi_n e^{-\beta J_n^{L}})}{E_0(e^{-\beta J_n^{L_{1/2}}})} \geq (1 + o(1)) \gamma_* (\beta a_1)^{1/2} n^{3/4}
\]

(4.18)

as \( n \to \infty \), where \( \gamma_* \) is independent of \( \beta \) as \( \beta \to \infty \) and \( \beta a_1 \) is a positive number that is independent of all \( \beta > 0 \). The fact that we assumed \( \Phi \) to be \( 1/2 \)-shaped together with [3,10] for \( r = 1/2 \) yields \( |\mathcal{L}_{1/2}| \geq (b_1/a_2) n^{1/2-\rho} \). As we pointed out at the outset of the proof of Proposition 4, part (I), we can choose \( |\mathcal{V}| = v_n n^{-1/d} \), where \( v_n \leq v_2 \) for all sufficiently large \( n \) and \( v_2 \) is independent of \( \beta \). Consequently, we end up with

\[
P_{\Phi}(L \in \mathcal{L}_{1/2}) = \frac{E_\Phi \sum_{L \in \mathcal{V}} 1_{\mathcal{L}_{1/2}}(L)}{|\mathcal{V}|} > m_* n^{1/d-1/2-\rho}
\]

(4.19)
for $m_\ast = b_1/(v_2a_2) > 0$ and every sufficiently large $n$. Note that $m_\ast$ may depend on $\beta$, even as $\beta \to \infty$, since $b_1/a_2$ does.

Furthermore, recall $L_+$ and $L_{1/2\pm}$ from (1.3). It is obvious that $|L_- \cup L_0| = |V| - |L_{1/2\pm} \cup L_+|$. We claim that $|L_{1/2\pm} \cup L_+| \leq (R + 1)(2b_2/a_1)n^{1/2} = R'n^{1/2}$, where $R' = (R + 1)(2b_2/a_1)$ is a finite constant, independent of $n$. To see this, fix some integer $R > 0$ and partition the interval $[1/2 - 1/2R, 1]$, into $R + 1$ subintervals of equal lengths, that is, $1/2 - 1/2R = r_0 < r_1 < \ldots < r_R = 1 - 1/2R < r_{R+1} = 1$ with $r_{k+1} = r_k + 1/2R$ for all $k$. Consider the sets $L_{r_\ast} = \{ L \in V : 2|C_L| \in [a_1n^{r},a_2n^{r+\delta}]\}$ for $\delta = 1/2R$ and $r = r_0,r_1,\ldots,r_R$. We have $L_{1/2\pm} \cup L_+ \subset \bigcup_{k=0}^{R}L_{r_\ast}$. Therefore, in view of (3.10) for $r \geq 1/2 - 1/2R$, we find $|L_{1/2\pm} \cup L_+| \leq (R + 1)(2b_2/a_1)n^{1/2+1/2R}$.

Hence, when $d \geq 3$, we can choose $R$ suitably large such that we obtain $|L_{1/2\pm} \cup L_+| = o(|V|)$ as $n \to \infty$ (since $|V| = v_n n^{-1/d}$). Whence, when $d \geq 3$, as $n \to \infty$,

$$
P_\Phi(L \in L_- \cup L_0) = \frac{|V| - |L_{1/2\pm} \cup L_+|}{|V|} \geq (1 - o(1)). \tag{4.20}$$

Finally, in light of (1.12), (4.17), (4.18), (4.19), and (4.20), we obtain, as $n \to \infty$,

$$
E_\beta(\chi_n) \geq m(d) (1 + o(1)) \max(n^{1/4+1/d-\rho}, n^{1/2}) \tag{4.21}
$$

for $m(d) = \max(m_\ast,\gamma_\ast(\beta a_1)^{1/2}, (2/\pi)^{1/2})$ for $d = 2, 3, 4$ and $m(d) = (2/\pi)^{1/2}$ for $d \geq 5$.

We summarize to say that $m(d)$ is uniform in $\beta$ for $d \geq 5$ but may depend on $\beta$, even as $\beta \to \infty$, for $d = 2, 3, 4$. Since $\rho > 0$ was arbitrary, the announced lower bound for $E_\beta(\chi_n)$ is an immediate consequence. \hfill $\Box$

## 5 Distance Exponents of the Self-Avoiding Walk

We recall the numbers $\mu(d) = \max(1/4 + 1/d, 1/2)$ for every integer $d > 1$ from (1.5). A number of arguments towards uniform bounds in $\beta$ as $\beta \to \infty$ together with Propositions 4 and 5 will demonstrate that the values of the distance exponents extend to the SAW.

**Corollary 1** Let $\beta > 0$ and $d > 1$. There are some constants $0 < \rho_1(d) = \rho_1(d, \beta) \leq \rho_2(d) = \rho_2(d, \beta) < \infty$ such that

$$
\rho_1(d) \leq \liminf_{n \to \infty} n^{-\mu(d)} E_\beta(\chi_n) \leq \limsup_{n \to \infty} n^{-\mu(d)} E_\beta(\chi_n) \leq \rho_2(d),
$$

where $\rho_1(d)$ is uniform in $\beta$ for $d \geq 5$ and may depend on $\beta$ for $d \leq 4$ and $\rho_2(d)$ is uniform in $\beta$ for $d = 2$ and $d \geq 5$ and may depend on $\beta$ for $d = 3, 4$. In particular, the self-avoiding walk in $\mathbf{Z}^d$ for $d \geq 2$ has distance exponent $\max(1/4 + 1/d, 1/2)$.

**Proof.** The statements on the weakly SAW are immediate consequences of Propositions 4 and 5. The reasoning to verify that the SAW has the same distance exponents in $\mathbf{Z}^d$
for \( d \geq 2 \) as the weakly SAW is identical to the one presented in Hueter [7], proof of Corollary 1. The part to establish the lower bound \( 3/4 \) for the distance exponent of the SAW was slightly more involved. For \( d = 3,4 \), in fact, we apply those lines of arguments to both the upper and lower bounds for the distance exponent of the SAW. The details are omitted here.

This completes the proof of Theorem 1. The next result accomplishes Theorem 2.

**Corollary 2** Let \( \beta > 0 \) and \( d > 1 \). There are some constants \( 0 < \rho_3(d) = \rho_3(d, \beta) \leq \rho_4(d) = \rho_4(d, \beta) < \infty \) such that

\[
\rho_3(d) \leq \liminf_{n \to \infty} n^{-2\mu(d)} \mathbb{E}_\beta (\chi_n^2) \leq \limsup_{n \to \infty} n^{-2\mu(d)} \mathbb{E}_\beta (\chi_n^2) \leq \rho_4(d),
\]

where \( \rho_3(d) \) is uniform in \( \beta \) for \( d \geq 5 \) and may depend on \( \beta \) for \( d \leq 4 \) and \( \rho_4(d) \) is uniform in \( \beta \) for \( d = 2 \) and \( d \geq 5 \) and may depend on \( \beta \) for \( d = 3,4 \). In particular, the MSD exponent of the SAW in \( \mathbb{Z}^d \) for \( d \geq 2 \) equals \( 2 \max(1/4 + 1/d, 1/2) \).

**Proof.** We shall argue in the setting of the weakly SAW and point out that parallel lines to the ones given before allow to extend the results about the MSD exponent to the SAW. First, the lower bound follows from the inequality \( \mathbb{E}_\beta (\chi_n^2) \geq (\mathbb{E}_\beta (\chi_n))^2 \) and Corollary 1.

Hence, it is enough to verify the upper bound. For this purpose, in light of (4.10), it suffices to consider shapes \( s \) of \( \Phi \) for \( s \geq 1/2 \) in the case when \( 2 \leq d \leq 4 \), and, to collect the contribution that would come from the second moment of the distance of the SRW for \( d \geq 5 \) (compare also to (4.12)). For \( 2 \leq d \leq 4 \), we remark the following. It follows from refined considerations of those in Proposition 4, part (II), that the number of SRW-paths that have circular shape is exponentially larger (in \( n \)) than the number of SRW-paths that have shape \( s > 1/2 \). Also, the expected penalizing weight is exponentially smaller for \( L_s \) and \( s > 1/2 \) and decays exponentially fast in \( n \). Combining both of these observations implies that \( Q_n^\beta \) decays exponentially fast in \( n \) around \( \mathbb{E}_\beta (\chi_n) \). Exponential decay holds for \( d \geq 5 \), too. In other words, for every \( \epsilon > 0 \), if we write \( M_\epsilon = (\rho_2(d))^1+\epsilon \), we have

\[
Q_n^\beta (\chi_n > M_\epsilon) \leq Q_n^\beta (\chi_n \leq M_\epsilon) \exp\{-\kappa(n, \epsilon)\}
\]

where \( \kappa(n, \epsilon) \to \infty \) as \( n \to \infty \). Hence, since \( \chi_n^2 \) is bounded for every \( n \), we have as \( n \to \infty \),

\[
\mathbb{E}_\beta (\chi_n^2) = (1 + o(1)) \sum_{k=0}^{M_\epsilon^2} Q_n^\beta (\chi_n^2 \geq k) \leq (1 + o(1)) (M_\epsilon^2 + 1).
\]

Since \( \epsilon > 0 \) was arbitrary, we let \( \rho_4(d) = \rho_2(d)^2 \) to wind up with the required result for \( d \geq 2 \). 

\( \square \)
Theorem 3 Let $\beta > 0$ and let $R_n$ denote the radius of the convex hull of the SRW-path $S_0, S_1, \ldots, S_n$ in $\mathbb{Z}^d$ for $d \geq 2$. Then $R_n$ satisfies all statements in Corollaries 1 and 2 with $\chi_n$ replaced by $R_n$.

Proof. We are interested in the maximal distance of $S_0, S_1, \ldots, S_n$ along any line rather than the distance of the position of $S_n$ from the starting point. The reflection principle gives the upper bound $\frac{dP_{R_n}(x)}{dx} \leq 2\frac{dP_{\chi_n}(x)}{dx}$ whereas the lower bound $\frac{dP_{R_n}(x)}{dx} \geq \frac{dP_{\chi_n}(x)}{dx}$ is readily apparent. The results now follow. \hfill $\Box$

Remark (Transition $\beta \to 0$). For $d \leq 3$, the transition $\beta \to 0$ looks dramatically different from the transition $\beta \to \infty$. The principal reasons are reflected upon the expressions in (4.10) and (4.17). Let us briefly glance at what happens when $\beta = 0$. In that fictive case (since the results were proved under the assumption $\beta > 0$), both terms in (4.10) are of asymptotic order $n^{-1/2}$, and so is the term in (4.17). Since this is drastically different from the case $\beta > 0$, at least when $d = 1, 2, 3$, in which case the asymptotic order in $n$ of the largest term is $n^\mu(d) \gg n^{1/2}$, we observe a discontinuity of the expected distance measures and distance exponents of the weakly SAW as $\beta \to 0$ for $d \leq 3$. In contrast, the case $\beta \to \infty$ behaves as any case for fixed $\beta$.

6 One-Dimensional Weakly SAW

This paragraph handles the case $d = 1$. We note in passing that the one-dimensional SAW is not interesting. Recall the numbers $a_x(L_1)$ from (3.6) with $r = 1$. They take values in $[a_1, a_2]$. For the remainder of the paper, since $d = 1$, we assume that $a_1 = b_1$ and $a_2 = b_2$. Recall that $a_1 \beta$ depends on $\beta$ while $a_2 \beta$ does not. Define

$$\mu_x(1) = (\beta a_x)^{1/2} n$$
$$q_1(x) = \exp\{-\beta \frac{a_x}{2} n\}$$

for every $n \geq 0$, $\beta > 0$, and $x$ in $[0, n]$. Similarly as in (4.3), for suitably small $\varepsilon \geq 0$ and for $\gamma > 0$, we may define

$$\hat{r}_1 = \hat{r}_1(\varepsilon, \gamma) = \sup\{x \in [0, n] : x \leq \gamma \mu_x(1)n^{-\varepsilon}\}$$
$$\hat{r}_2 = \hat{r}_2(\gamma) = \hat{r}_1(0, \gamma).$$

Then the numbers $a_x(L_1)$, defined in (3.6) when $r = 1$, satisfy a hypothesis analogous to Condition D in (4.4).

Lemma 3 (Condition satisfied by $a_x(L_1)$) Let $d = 1$. Then the $a_x(L_1)$ obey the following condition for every $\varepsilon \geq 0$:
Condition $\tilde{D}$. For any suitably small $\varepsilon \geq 0$, there exist some $\gamma > 0$ and $\omega_* > 0$ such that

$$\int_{r_1}^n x q_1(x) \, dP_{\chi_n}(x) = \omega_n \int_0^{\hat{r}_1} x q_1(x) \, dP_{\chi_n}(x)$$

with $\omega_n \geq \omega_*$ for all sufficiently large $n$.

Proof. Showing this statement requires no more than a small number of modifications to the proof of Lemma 2 in Huetzer [3] and can do without the assumption that $\Phi$ be circular. Fix some suitably small $\varepsilon > 0$. Let us invoke the notation that we introduced in the proof of Lemma 1, that is, write $E_0(\chi_n e^{-\beta J^1_n}) = I_n = J_1(n) + J_2(n) + J_3(n)$, and in the same spirit, $E_0(e^{-\beta J^1_n}) = \tilde{J}_1(n) + \tilde{J}_2(n) + \tilde{J}_3(n)$, where the $r_i$ for $i = 1, 2, 3$ are replaced by the $\hat{r}_i$.

We need to show that there is some $\omega_* > 0$ so that $J_2(n) + J_3(n) = \omega_* J_1(n)$ with $\omega_n \geq \omega_*$ for all sufficiently large $n > 0$. We first show that $J_2(n) + J_3(n) \neq o(J_1(n))$ as $n \to \infty$.

For a moment, let us suppose in contrast that $J_2(n) + J_3(n) = o(J_1(n))$ as $n \to \infty$ so as to take this claim to a contradiction. Thus, $J_2(n) = o(J_1(n))$ and $J_3(n) = o(J_1(n))$ as $n \to \infty$. It would follow that $I_n = J_1(n)(1 + o(1))$ as $n \to \infty$ and $\sum_{i=1}^3 \tilde{J}_i(n) = \tilde{J}_1(n)(1 + o(1))$. The probability measure $Q_n^{3,V,1}$ induces a one-dimensional process $W_n$ which has expectation $E_{\beta,V,L_1}(\chi_n)$, call it $E_{\beta,V,L_1}(\chi_n)$. Associate $W_n$ with the numbers $a_x(L_1)$.

In view of the exponential form of the integrand of $I_n$, our assumption would imply that there is a number $z_n = z$ in $[0, \hat{r}_1]$ that enjoys the property

$$\frac{E_0(\chi_n e^{-\beta J^1_n})}{E_0(e^{-\beta J^1_n})} = \frac{I_n}{E_0(\chi_n e^{-\beta J^1_n})} = (1 + o(1)) z$$

as $n \to \infty$. In that event, the function $a_x$ is minimal at $z = z_n$, that is $a_x = \inf_{0 \leq y \leq \hat{r}_1} a_y$ for all sufficiently large $n$. This may be seen as follows. Define $k_1(x) = \exp\{-x^2 + \mu_x(1)^2/(2n)\}$, let $a_0 > 0$ and let $0 < \tau \leq a_0$ be some arbitrarily small number. If $a_{x_1} = a_0$ and $a_{x_2} = a_0 - \tau \geq 0$ for $0 \leq x_1, x_2 \leq \hat{r}_1$, then it follows that $k_1(x_1) < k_1(x_2)$ for all sufficiently large $n$.

Now, for some suitably small $\tau = \tau(\beta) > 0$, define the set

$$S_\tau = \{x \in [0, \hat{r}_1] : a_x > a_x + \tau\}.$$

Consider a modified process $\tilde{W}_n$ that is associated with numbers $\hat{a}_x$ with $\hat{a}_x = a_x$ for $x \in [0, \hat{r}_1] \setminus S_\tau$, $\hat{a}_x = a_x + \tau$ for $x \in S_\tau$, and $\hat{a}_x = a_x + a(n)$ for $\hat{r}_1 < x \leq n$, where $a(n) > 0$ is some suitable number, chosen so as to preserve the distribution of $J_n$. Thus, $\hat{a}_x \leq a_x + \tau$ for $x \in [0, \hat{r}_1]$. Observe that the modified process $\tilde{W}_n$ has the same expectation $E_{\beta,V,L_1}(\chi_n) = E_{\beta,V,L_1}(\chi_n)$ as the process $W_n$ since, firstly, $q_1(x)$ in (5.2) was decreased on $(\hat{r}_1, n]$; and thus, $J_2(n) + J_3(n)$ and the corresponding part $\tilde{J}_2(n) + \tilde{J}_3(n)$ of the integral in the denominator of $E_{\beta,V,L_1}(\chi_n)$ were both decreased, and secondly, $J_1(n)$ is as before in view of (5.3). Note that adding a constant number of self-intersections to all realizations of this underlying weakly self-avoiding process $\tilde{W}_n$ does not change its probability measure.
Subtract the number $a_z$ from $\tilde{a}_x$ for every $0 \leq x \leq n$, that is, let $\tilde{a}_x = \tilde{a}_x - a_z \geq 0$ for every $0 \leq x \leq n$. Thus, $\tilde{a}_x \leq \tau$ for $x \in [0, \tilde{r}_1]$ and $\tilde{a}_x$ is suitable on $[\tilde{r}_1, n]$. The gotten process $\tilde{W}_n$ associated with the numbers $\tilde{a}_x$ has expectation $E_{\tilde{\beta}, \tilde{\gamma}, \tilde{\chi}, 1}(\chi_n) = E_{\tilde{\beta}, \tilde{\gamma}, \tilde{\chi}, 1}(\chi_n)$, too, the same as do $W_n$ and $\tilde{W}_n$. Since we may choose $\tau < b_1$, we arrive at $\tilde{a}_x \leq \tau < b_1$. But this contradicts condition (2.2). We conclude that $J_2(n) + J_3(n) / o(J_1(n))$ as $n \to \infty$. Since $\varepsilon > 0$ was arbitrary, it follows that, for every $\varepsilon > 0$, $J_2(n) + J_3(n) / o(J_1(n))$ as $n \to \infty$.

It remains to be shown that there is no subsequence $n_k$ such that $J_2(n_k) + J_3(n_k) / o(J_1(n_k))$ as $k \to \infty$. From this it will follow that there is some number $\omega_n > 0$ that bounds $\omega_n$ from below with $n$. But the same point can be made as explained above when $n$ is replaced by $n_k$ everywhere. Whence, we conclude that (6.4) is satisfied for every $\varepsilon > 0$. Hence, it must hold for $\varepsilon = 0$. Note that this implies that $\tilde{r}_1 = \tilde{r}_2$ and $J_2(n) = 0$.

Finally, we remark that $\gamma > 0$ may be chosen uniformly over $\beta$ as $\beta \to \infty$. This can be seen as follows. Any of the asymptotic statements in a variant of Lemma 4 and in the above lines of proof depend on expressions, for example, of the form $\tilde{\beta} / n$. Hence, if $N(\beta)$ is a threshold so that, for all $n \geq N(\beta)$, a given expression in $n$ differs from its corresponding limiting expression by at most $\varepsilon$ (some fixed $\varepsilon$), it follows that $N(\beta') \leq N(\beta)$ for $\beta < \beta'$. As a consequence of the fact that $\gamma > 0$ may be chosen uniformly in $n$, the choice of $\gamma$ is uniformly over $\beta > 0$ as $\beta \to \infty$ (yet not as $\beta \to 0$.) This accomplishes the proof.

**Theorem 4** Let $\beta > 0$ and $d = 1$. Then there is a constant $m_1(\beta) > 0$ that may depend on $\beta$ such that as $n \to \infty$,

$$E_{\tilde{\beta}}(\chi_n) = (1 + o(1)) M_1(n) n$$

for $m_1(\beta) \leq M_1(n) \leq 1$.

**Proof.** Clearly, it is sufficient to verify the lower bound for $E_{\tilde{\beta}}(\chi_n)$. Analogous lines to the one to obtain (4.17) lead to

$$E_{\tilde{\beta}}(\chi_n) \geq (1 + o(1)) P_{\Phi}(L \in \mathcal{L}_1) \frac{E_0(\chi_n e^{-\beta J_n^{\varepsilon_1}})}{E_0(e^{-\beta J_n^{\varepsilon_1}})}$$

(6.6)

as $n \to \infty$. First, observe that $P_{\Phi}(L \in \mathcal{L}_1) \geq 1/2$ (there are only two half-lines that emanate from the origin in $\mathbb{Z}$). Next, writing

$$I_n(1) = \int_0^n x q_1(x) dP_{\chi_n}(x)$$

$$g_1(n) = \int_0^n (a_x)^{1/2} q_1(x) dP_{\chi_n}(x)$$

and proceeding as to prove Lemma 1 in HUETER 6 in connection with Lemma 8 with $\varepsilon = 0$, we collect, as $n \to \infty$,

$$I_n(1) = K_1(n) \beta^{1/2} g_1(n) n (1 + o(1))$$

(6.7)
for $0 < \gamma c(\omega) \leq K_1(n)$, where the positive constant $\gamma$ arises in Condition $\tilde{D}$ stated in Lemma 3 and $c(\omega)$ is a positive constant, independent of $\beta$. Both, $\gamma$ and $c(\omega)$ may be chosen independently of $\beta$ as $\beta \to \infty$, as we reasoned earlier. In parallel to the arguments in proving Proposition 3 together with the estimate in (6.7), we obtain, as $n \to \infty$,

$$
E_0(\chi_n e^{-\beta J_{n}^{1/2}}) = E_0(\chi_n E_{\Phi|\chi_n}(|L_1(\Phi)|^{-1} \sum_{L \in L_1(\Phi)} e^{-\beta |C_L|} |\chi_n = x}))
$$

$$
= \int_{0}^{n} x E_{\Phi|\chi_n}(|L_1(\Phi)|^{-1} \sum_{L \in L_1(\Phi)} e^{-\beta |C_L|} |\chi_n = x}) dP_{\chi_n}(x)
$$

$$
= \int_{0}^{n} x (\int_{Z} |L_1(\varphi)|^{-1} \sum_{L \in L_1(\varphi)} e^{-\beta |C_L|} dP_{\Phi|\chi_n}(\varphi|x)) dP_{\chi_n}(x)
$$

$$
= \int_{0}^{n} x \exp\{-\beta a_n x/2\} dP_{\chi_n}(x)
$$

$$
= \int_{0}^{n} x q_1(x) dP_{\chi_n}(x)
$$

$$
= K_1(n) \beta^{1/2} g_1(n) n (1 + o(1))
$$

for some $0 < \gamma_1 \leq K_1(n)$, where $\gamma_1 > 0$ is independent of $\beta$ as $\beta \to \infty$. Additionally, upon a similar but easier exercise, we gain

$$
E_0(e^{-\beta J_{n}^{1/2}}) = \int_{0}^{n} q_1(x) dP_{\chi_n}(x)
$$

$$
= h_1(n).
$$

Note that $g_1(n)/h_1(n) \geq (a_1)^{1/2} = (b_1)^{1/2}$. Hence, putting these two or three pieces together, along with (6.2), yields as $n \to \infty$,

$$
E_{\beta}(\chi_n) \geq (1 + o(1)) \frac{1}{2} \frac{E_0(\chi_n e^{-\beta J_{n}^{1/2}})}{E_0(e^{-\beta J_{n}^{1/2}})}
$$

$$
\geq (1 + o(1)) m_1(\beta) n
$$

with $m_1(\beta) = (\beta b_1)^{1/2} \gamma_1/2 > 0$, possibly depending on $\beta$. This ends the proof.

Since $EY^2 \geq (EY)^2$ for any random variable $Y$, we have the following

**Corollary 3** Let $\beta > 0$ and $d = 1$. Then there is a constant $m_2(\beta) > 0$ that may depend on $\beta$ such that as $n \to \infty$,

$$
E_{\beta}(\chi_n^2) = (1 + o(1)) M_2(n) n
$$

for $m_2(\beta) \leq M_2(n) \leq 1.$
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