Semileptonic decays $B/B_s \rightarrow (D^{(*)}, D_s^{(*)}) l\nu_l$ in the PQCD approach with the lattice QCD input*

Xue-Qing Hu(胡学卿$^{1,2,3}$) Su-Ping Jin(金苏平)$^{1,2}$ Zhen-Jun Xiao(肖振军)$^{1,2,3}$

$^1$Department of Physics and Institute of Theoretical Physics, Nanjing Normal University, Nanjing 210023, China
$^2$Jiangsu Key Laboratory for Numerical Simulation of Large Scale Complex Systems, Nanjing Normal University, Nanjing 210023, China

Abstract: We study the semileptonic $B/B_s \rightarrow (D^{(*)}, D_s^{(*)}) l\nu_l$ decays in the framework of the Standard Model (SM), by employing the perturbative QCD (PQCD) factorization formalism combined with the lattice QCD input for the relevant transition form factors. We calculate the branching ratios $\mathcal{B}(B_{(s)} \rightarrow D_{(s)}^{(*)} l\nu_l)$ with $l = (e, \mu, \tau)$, the ratios of the branching fractions $R(D^{(*)})$ and $R(D_s^{(*)})$, and the physical observables $P_{l}(D_{(s)}^{(*)})$, $F_{l}(D_{(s)}^{(*)})$ and $A_{FB}(\tau)$. The “PQCD+Lattice” predictions for $\mathcal{B}(B \rightarrow D^{(*)} l\nu_l)$ and $R(D^{(*)})$ agree with the available experimental measurements within errors. For the ratios $R(D_s)$ and $R(D_s')$, the "PQCD+Lattice" predictions agree with the other predictions. For $P_l(D^{(*)})$ and $F_l(D^{(*)})$, our theoretical predictions agree with the measured values within errors. Our theoretical predictions of the semileptonic $B/B_s$ decays considered could be tested in the near future by the LHCb and Belle II experiments.

Keywords: semileptonic decays, form factor, PQCD, $B_s$ meson

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1 Introduction

The studies of semileptonic decays of $B/B_s$ meson play an essential role in testing the Standard Model (SM) as well as in the search for new physics (NP) beyond SM, since the lepton flavor universality (LFU) can be examined in this kind of decay modes. LFU implies that all electroweak gauge bosons ($Z^0, \gamma$ and $W^\pm$) have equivalent couplings to the three generations of leptons, and the only difference arises from the mass differences: $m_e < m_\mu < m_\tau$. Therefore, if the experiments were to find signals of lepton flavor violation (LFV), that would be a true challenge for SM.

The first observation of the $R(D)$ and $R(D^*)$ anomalies in semileptonic decays $B \rightarrow D^{(*)} l\nu_l$ with $l = (e^-, \mu^-, \tau^-)$ by the BaBar collaboration in 2012 [1, 2] invoked intensive studies of the $B \rightarrow D^{(*)} l\nu_l$ decays in the SM framework [3–10] and various NP models [11–18]. With the measurements reported by the Belle and LHCb collaborations [19–25], however, the deviations between the measured $R(D)$ and $R(D^*)$ and the SM predictions become smaller [26–29]:

1) The latest Belle measurements [24] exhibit an excellent consistency with the averaged SM predictions [3–5, 29]:

$$R(D) = \begin{cases} 0.307 \pm 0.037 \text{(stat.)} \pm 0.016 \text{(syst.)}, & \text{Belle}[24] \ , \\ 0.299 \pm 0.003, & \text{SM}[29] \end{cases}$$

$$R(D^*) = \begin{cases} 0.283 \pm 0.018 \text{(stat.)} \pm 0.014 \text{(syst.)}, & \text{Belle}[24] \ , \\ 0.258 \pm 0.005, & \text{SM}[29] \end{cases}$$

since they are compatible within one standard deviation [24, 29].

2) The combined analysis of currently available measurements of $R(D)$ and $R(D^*)$, including the new Belle results [24], gives the following world averaged values:
As is well known, the central issue for calculating such semileptonic decays in the SM framework is the estimate of the values and shapes of the form factors \( F_{1,0}(q^2) \) for the \( B_{(s)} \to D_{(s)}^{(*)} \) transitions. However, calculations of these form factors are not easy and they cannot be estimated reliably in the whole range of \( q^2 \) carried by the lepton pair using a single method, and their extrapolation is indispensable. There are many traditional methods and approaches to estimate the form factors and provide predictions of the ratios \( R_{D^*} \), for example, the heavy quark effective theory (HQET) [4, 5], light cone sum rules (LCSR) [34–36], lattice QCD (LQCD) [37–41] and the perturbative QCD factorization approach (PQCD) [9, 10].

In recent years, the PQCD factorization approach has been used to study various kinds of semileptonic decays of \( B/B_s/B_c \) mesons, as given for example in Refs. [9, 10, 42–48]. However, like in many other theoretical approaches, the form factors evaluated using the PQCD approach are only reliable in the low \( q^2 \) region. Therefore, extrapolations must be done in order to cover the whole range of \( q^2 \) values of the form factors \( f_j(q^2) \). In order to improve the reliability of the values and shapes of the six form factors obtained in the PQCD approach, we include the lattice QCD result at the end point \( q_{\text{max}}^2 \) so that the extrapolation from the low \( q^2 \) region to the high \( q^2 \) region is more reliable.

In Refs. [9, 10], the \( B \to D^{(*)} \ell \nu \) decays were studied by employing the PQCD approach only [9], or with the inclusion of the lattice QCD input [10]. In Ref. [42], the \( B_s^0 \to D_s^{(*)} \ell \nu \) decays were studied by employing the PQCD approach without the lattice QCD input. In this paper, we study the semileptonic decays of \( B \) and \( B_s \) mesons simultaneously and focus on the following three tasks:

1. For the \( B_s \to D_s^{(*)} \) and \( B \to D^{(*)} \) transitions, we evaluate the relevant form factors in the region \( 0 < q^2 < m^2 \) using the PQCD factorization approach, and then include the lattice QCD result at the end point \( q_{\text{max}}^2 \) in the fitting process and extrapolate the form factors in the entire momentum region by employing the Bourrely-Caprini-Lellouch (BCL) parametrization [49, 50] instead

Fig. 1. (color online) Leading order Feynman diagrams for the semileptonic decays \( B_{(s)} \to D_{(s)}^{(*)} \ell \nu \) with \( l = (e, \mu, \tau) \) in the PQCD approach.

\[
R(D)^{\text{Exp}} = 0.340 \pm 0.027 \pm 0.013, \\
R(D^{(*)})^{\text{Exp}} = 0.295 \pm 0.011 \pm 0.008,
\]

so that the discrepancy decreases from the previous 3.8\( \sigma \) to 3.1\( \sigma \) with respect to the SM expectations [26, 27, 29].

(3) Besides the ratios \( R(D^{(*)}) \), other physical observables, such as the longitudinal polarization of the tau lepton \( P_L(D^*) \) and the fraction of the \( D^* \) longitudinal polarization \( F_L(D^*) \), were measured recently by the Belle collaboration [20, 21, 30]:

\[
P_L(D^*) = -0.38 \pm 0.51(\text{stat.})^{+0.21}_{-0.16}(\text{syst.})[20, 21],
\]

\[
F_L(D^*) = 0.60 \pm 0.08(\text{stat.}) \pm 0.04(\text{syst.})[30].
\]

They are compatible with the SM predictions \( P_L(D^*) = -0.497 \pm 0.013 \) [31], \( F_L(D^*) = 0.441(6) \) [32] and 0.457(10) [33] for the \( B \to D^* \ell \nu \) decays.

It is clear from the above points that although the anomalies of \( R(D^{(*)}) \) have become less serious recently, there is still a sizeable discrepancy with respect to the SM expectations, and that they should be investigated with complementary and more precise measurements in order to draw a conclusion. In addition to the \( B \to D^{(*)} \ell \nu \) decays, the \( B_s \to D_s^{(*)} \ell \nu \) decay mode is one of the best choices for cross examination. Systematic theoretical studies and experimental measurements of the semileptonic decays of \( B_s \) meson are therefore very important and should be performed promptly.

As illustrated by the Feynman diagrams in Fig. 1, the \( B_s \to D_s^{(*)} \ell \nu \) decays are closely related to the \( B \to D^{(*)} \ell \nu \) decays via the \( SU(3) \) flavor symmetry. They are all controlled by the same \( b \to c \ell \nu \) transitions at the quark level, but with a different spectator quark: \( (u, d) \) or \( s \) quark. As a consequence, in the limit of the \( SU(3) \) flavor symmetry, these two sets of semileptonic decays must have very similar properties. If the current anomalies in \( B \to D^{(*)} \ell \nu \) decays are indeed induced by contributions from NP, they must appear in the \( B_s \to D_s^{(*)} \ell \nu \) decays. As is well known, the central issue for calculating the entire momentum region by employing the Bourrely-Caprini-Lellouch (BCL) parametrization [49, 50] instead
of the pole models used in Refs. [9, 10, 42].

(2) In addition to the calculations of the branching ratios and the ratios \( R(D_0^{(*)}) \) and \( R(D^{(*)}) \), we also calculate three physical observables (which were not considered in [9, 10, 42]) for the decays of \( B \) and \( B_s \) mesons: the longitudinal polarization of the tau lepton \( \gamma \tau (D_0^{(*)}) \), the fraction of \( D_0^{(*)} \) longitudinal polarization \( F_L(D_0^{(*)}) \), and the forward-backward asymmetry of the tau lepton \( \gamma \tau \) in the ordinary PQCD approach and the “PQCD + Lattice” approach (i.e. the PQCD approach with the inclusion of the lattice QCD input for the form factors).

(3) We present our theoretical predictions, compare them with the available experimental measurements, or the theoretical predictions obtained using other different theories or models.

The paper is organized as follows: In Sec. 2, we briefly review the kinematics of the \( B^0 \to D^{(*)} l \nu \) decays. The calculations of the form factors for the \( B^0 \to D^{(*)} \) transitions are then given. We use the lattice QCD result at \( \alpha^2 \) given by the HPQCD group [41] as input in our extrapolation. Explicit expressions for the differential decay rates and additional physical observables are also given in Sec. 2. In Sec. 3, we present the theoretical predictions of the other physical observables obtained using the PQCD approach, the “PQCD+Lattice” approach, and some other models. A short summary is given in the final section.

2 Theoretical framework

2.1 Kinematics and the wave functions

In the PQCD approach, the tree-level Feynman diagrams for \( B_{(s)} \to D^{(*)} l \nu \) decays\(^1\) are shown in Fig. 1. We define the \( B_{(s)} \) meson momentum as \( p_1 \), the \( D_{(s)}^{(*)} \) meson momentum as \( p_2 \), and the polarization vectors \( \epsilon_{L,T} \) in the \( D_{(s)}^{(*)} \) at \( B_{(s)} \) rest frames as in Ref. [51].

\[
p_1 = \frac{m_{B_{(s)}}}{\sqrt{2}} (1, 1, 0_z, 0), \quad p_2 = \frac{m_{D_{(s)}}}{\sqrt{2}} (q^+, q^-, 0, 0_z),
\]

\[
\epsilon_L = \frac{1}{\sqrt{2}} (q^+, -q^-, 0_z), \quad \epsilon_T = (0, 0, 1),
\]

where \( \epsilon_L \) and \( \epsilon_T \) denote the longitudinal and transverse polarization of \( (D^+, D^0) \) mesons, respectively. The parameter \( q^+ \) and \( r \) are defined as:

\[
q^+ = q + \frac{q^2}{m_{D_{(s)}}}, \quad \frac{r}{m_{B_{(s)}}} = \frac{m_{D_{(s)}}}{m_{B_{(s)}}},
\]

where \( q = p_1 - p_2 \) is the momentum of the lepton pair. The momenta of the spectator quarks in \( B_{(s)} \) and \( D_{(s)}^{(*)} \) mesons are parametrized as

\[
k_1 = \frac{m_{B_{(s)}}}{\sqrt{2}} (0, x_1, k_{1z}), \quad k_2 = \frac{r \cdot m_{B_{(s)}}}{\sqrt{2}} (x_2 \eta^+, x_2 \eta^-, k_{2z}).
\]

where \( x_1 \) and \( x_2 \) are the fraction of the momentum carried by the light spectator quark in the initial \( B/B_s \) meson and the final state meson \( D^{(*)}/D^{(*)} \), respectively.

For the \( B/B_s \) meson wave functions, we use the functions from Refs. [51, 52].

\[
\Phi_{B_{(s)}}(x, b) = \frac{i}{\sqrt{2N_c}} (\phi_{B_{(s)}} + m_{B_{(s)}}) \phi_{B_{(s)}}(x, b),
\]

\[
\phi_{B_{(s)}}(x, b) = \frac{m_{B_{(s)}}}{2} \exp \left[ -\frac{x^2 \cdot m_{B_{(s)}}^2}{2 \alpha_b^2} - \frac{1}{2} (\alpha_{B_{(s)}} \cdot b)^2 \right].
\]

The normalization factor \( N_{B_{(s)}} \) depends on the values of the parameter \( \alpha_{B_{(s)}} \) and decay constant \( f_{B_{(s)}} \), through the normalization relation: \( \int d^4 x (\phi_{B_{(s)}}(x, b = 0) = f_{B_{(s)}}/(2 \sqrt{6}) \). In order to estimate the uncertainties of theoretical predictions, we set the shape parameter \( \omega_B = 0.40 \pm 0.04 \) GeV and \( \omega_{B_s} = 0.50 \pm 0.05 \) GeV.

For \( D_{(s)} \) and \( D_{(s)}^{(*)} \) mesons, we use the same wave functions as in Ref. [53]

\[
\Phi_{D_{(s)}}(x, b) = \frac{i}{\sqrt{6}} \gamma_5 (\phi_{D_{(s)}} + m_{D_{(s)}}) \phi_{D_{(s)}}(x),
\]

\[
\phi_{D_{(s)}}(x, b) = \frac{m_{D_{(s)}}}{2} \exp \left[ -\frac{x^2 \cdot m_{D_{(s)}}^2}{2 \alpha_b^2} - \frac{1}{2} (\alpha_{D_{(s)}} \cdot b)^2 \right].
\]

with the distribution amplitudes

\[
\phi_{D_{(s)}}(x) = \frac{f_{D_{(s)}}}{2 \sqrt{6}} \cdot 6x(1-x) \left[ 1 + C_{D_{(s)}}(1 - 2x) \right] \times \exp \left[ -\frac{\omega_{D_{(s)}}^2 b^2}{2} \right].
\]

where we set the parameters \( C_D = C_{D_s} = C_{D_s} = C_{D_s} = 0.5 \) and \( \omega = 0.1 \). From the heavy quark limit, we assume that

\[
\phi_{D^+} = \phi_{D}, \quad \phi_{D^0} = \phi_{D^0}, \quad \phi_{D^-} = \phi_{D^-}, \quad \phi_{D_s^0} = \phi_{D_s^0}.
\]

2.2 Form factors in the PQCD approach

The form factors of the \( B_{(s)} \to D_{(s)}^{(*)} \) transition are defined in the same form as in Ref. [54]

\[
\langle D_{(s)}(p_2) l \bar{c}(0) \gamma_{\mu}(b_0) B_{(s)}(p_1) \rangle = \left( p_{1z} + p_{2z} \right) \mu - \frac{m_{B_{(s)}} - m_{D_{(s)}}}{q^2} \frac{\mu_j}{q_j} \cdot \left( F_+(q^2) + \frac{m_{B_{(s)}} - m_{D_{(s)}}}{q^2} q_j \right) F_0(q^2). \]

In order to cancel the poles at \( q^2 = 0 \), \( F_+(0) \) should be

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1) Throughout this paper the symbol \( B_{(s)} \) describes both \( B(B_s, B_s) \) and \( B \) mesons.
equal to $F_0(0)$. For convenience, we also define the auxiliary form factors $f_1(q^2)$ and $f_2(q^2)$,
\[
(D_{(s)}(p_2)|\bar{c}(0)\gamma_{\mu}b(0)|B_{(s)}(p_1)) = f_1(q^2)p_{1\mu} + f_2(q^2)p_{2\mu},
\]
(17)
which are related to $F_+(q^2)$ and $F_0(q^2)$ by the following relations,
\[
F_+(q^2) = \frac{1}{2} \left[ f_1(q^2) + f_2(q^2) \right],
\]
(18)
\[
F_0(q^2) = \frac{1}{2} f_1(q^2) \left[ 1 + \frac{q^2}{m_{B_{(s)}}^2 - m_{D_{(s)}}^2} \right] + \frac{1}{2} f_2(q^2) \left[ 1 - \frac{q^2}{m_{B_{(s)}}^2 - m_{D_{(s)}}^2} \right].
\]
(19)

For vector meson $D_{(s)}^*$ in the final state, the form factors for $B_{(s)} \rightarrow D_{(s)}^*$ transitions are $V(q^2)$ and $A_{0,1,2}(q^2)$, which are given in the following form [54]:
\[
(D_{(s)}^*(p_2)|\bar{c}(0)\gamma_{\mu}b(0)|B_{(s)}(p_1)) = \frac{2iV(q^2)}{m_{B_{(s)}} + m_{D_{(s)}}} \epsilon_{\mu\nu\rho\sigma} c_{\nu} \rho (p_1)^{\rho} (p_2)^{\sigma},
\]
(20)
\[
(D_{(s)}^*(p_2)|\bar{c}(0)\gamma_{\mu}\gamma_5 b(0)|B_{(s)}(p_1)) = 2m_{D_{(s)}} A_0(q^2) \frac{e^+ \cdot q}{q^2} q_{\mu} + (m_{B_{(s)}} + m_{D_{(s)}}) A_1(q^2) \left( \epsilon_{\mu} - \frac{e^+ \cdot q}{q^2} q_{\mu} \right)
- A_2(q^2) \frac{e^+ \cdot q}{m_{B_{(s)}} + m_{D_{(s)}}} \left( (p_1 + p_2)_\mu - \frac{m_{B_{(s)}}^2 - m_{D_{(s)}}^2}{q^2} q_{\mu} \right).
\]
(21)

The analytical expressions for the form factors mentioned above are the following:
\[
f_1(q^2) = 8\pi m_{B_{(s)}}^2 C_F \int dx_1 dx_2 \int b_1 b_1 b_2 b_2 \phi_{B_{(s)}}(x_1, b_1) \phi_{D_{(s)}}(x_2, b_2) \left[ 2r (1 - r x_2) \right] \cdot H_1(t_1)
\]
\[
+ \left[ 2 r (r_c - r) + x_1 r \right] \left[ 2 + 2 \eta + \sqrt{\eta^2 - 1} - \frac{2 \eta}{\sqrt{\eta^2 - 1}} + \frac{\eta^2}{\sqrt{\eta^2 - 1}} \right] \cdot H_2(t_2)
\]
(22)
\[
f_2(q^2) = 8\pi m_{B_{(s)}}^2 C_F \int dx_1 dx_2 \int b_1 b_1 b_2 b_2 \phi_{B_{(s)}}(x_1, b_1) \phi_{D_{(s)}}(x_2, b_2)
\]
\[
\times \left\{ \left[ 2 - 4 x_2 r (1 - \eta) \right] \cdot H_1(t_1) + \left[ 4 r - 2 r_c - x_1 + \frac{x_1}{\sqrt{\eta^2 - 1}} (2 - \eta) \right] \cdot H_2(t_2) \right\},
\]
(23)
where the mass ratios are $r_c = m_c / m_B$, $r = m_{D_{(s)}} / m_{B_{(s)}}$.
\[
V(q^2) = 8\pi m_{B_{(s)}}^2 C_F \int dx_1 dx_2 \int b_1 b_1 b_2 b_2 \phi_{B_{(s)}}(x_1, b_1) \phi_T^{T_{(s)}}(x_2, b_2) \cdot (1 + r) \left[ 1 - r x_2 \right] \cdot H_1(t_1)
\]
\[
+ \left[ r^2 + r_c + \frac{x_1}{2} + \frac{\eta x_1}{2 \sqrt{\eta^2 - 1}} + \frac{r x_1}{2 \sqrt{\eta^2 - 1}} \left( 1 - 2 \eta \sqrt{\eta^2 - 1} \right) \right] \cdot H_2(t_2)
\]
(24)
\[
A_0(q^2) = 8\pi m_{B_{(s)}}^2 C_F \int dx_1 dx_2 \int b_1 b_1 b_2 b_2 \phi_{B_{(s)}}(x_1, b_1) \phi_T^{T_{(s)}}(x_2, b_2) \left[ 1 + r - r x_2 (2 + 2 r - 2 \eta) \right] \cdot H_1(t_1)
\]
\[
+ \left[ r^2 + r_c + \frac{x_1}{2} + \frac{\eta x_1}{2 \sqrt{\eta^2 - 1}} + \frac{r x_1}{2 \sqrt{\eta^2 - 1}} \left( 1 - 2 \eta \sqrt{\eta^2 - 1} \right) \right] \cdot H_2(t_2)
\]
(25)
\[
A_1(q^2) = 8\pi m_{B_{(s)}}^2 C_F \int dx_1 dx_2 \int b_1 b_1 b_2 b_2 \phi_{B_{(s)}}(x_1, b_1) \phi_T^{T_{(s)}}(x_2, b_2) \cdot \frac{r}{1 + r}
\]
\[
\times \left\{ [1 + \eta (1 - r x_2 (1 - 2 r + \eta (2 + 2 r - 2 \eta))] \cdot H_1(t_1)
\right.
\]
\[
+ \left. \left[ r + r_c (1 - \eta) - \eta r^2 + x_1 \eta^2 - \frac{x_1}{2} (\eta + r) + x_1 \left( \eta - \frac{1}{2} \right) \sqrt{\eta^2 - 1} \right] \cdot H_2(t_2) \right\},
\]
(26)
\[
A_2(q^2) = \frac{(1 + r)^2 (\eta - r)}{2r (\eta^2 - 1)} \cdot A_1(q^2) - 8\pi m_{B_{(s)}}^2 C_F \int dx_1 dx_2 \int b_1 b_1 b_2 b_2 \phi_{B_{(s)}}(x_1, b_1) \phi_T^{T_{(s)}}(x_2, b_2) \cdot \frac{1 + r}{\eta^2 - 1}
\]
\[
\times \left\{ \left[ 1 + \eta (1 - r x_2 (1 - 2 r + \eta (2 + 2 r - 2 \eta))] \cdot H_1(t_1)
\right.
\right.
\]
\[
+ \left. \left[ r + r_c (1 - \eta) - \eta r^2 + x_1 \eta^2 - \frac{x_1}{2} (\eta + r) + x_1 \left( \eta - \frac{1}{2} \right) \sqrt{\eta^2 - 1} \right] \cdot H_2(t_2) \right\},
\]
(27)
where the color factor is $C_F = 4/3$, the mass ratios are $r_c = m_c/m_{B_{0,s}}$, $r = m_D/m_{B_{0,s}}$, and the functions $H_i(t_i)$ are in the following form

$$H_i(t_i) = h_i(x_1, x_2, b_1, b_2) \cdot \alpha_i(t_i) \exp[-S_{ab}(t_i)], \quad \text{for } i = 1, 2, 3.$$  

(28)

The explicit expressions for the hard functions $h_i(x_1, x_2, b_1, b_2)$, hard scales $t_i$ and the Sudakov factors $[-S_{ab}(t_i)]$ are given in the Appendix.

### 2.3 Lattice input at $q^2_{\text{max}}$

The lattice QCD has advantages in calculating the form factors for large $q^2$, and it is generally believed that the lattice QCD predictions close to or at $q^2_{\text{max}}$ are reliable. In this work, we make use of the lattice QCD predictions of all relevant form factors at the endpoint $q^2_{\text{max}}$ as the additional input in the fitting process, so as to make the extrapolation of these form factors from the low $q^2$ region to $q^2_{\text{max}}$ more reliable.

The form factors used in the lattice QCD are parametrized as follows [55, 56]:

$$\langle D_{ij} | V^{\alpha \beta} b_{B_{ij}} | D_{kl} \rangle = \sqrt{m_{B_{ij}} m_{D_{kl}} \epsilon^{i} \epsilon^{j} \epsilon^{k} \epsilon^{l}} h_{ij}(w)(v + u')^{\alpha'} h_{kl}(w)(v - u')^{\beta'},$$

$$\langle D_{ij} | A^{\alpha \beta} b_{B_{ij}} | D_{kl} \rangle = \sqrt{m_{B_{ij}} m_{D_{kl}} \epsilon^{i} \epsilon^{j} \epsilon^{k} \epsilon^{l}} [g^{\alpha \beta}(1 + w) h_{ij}(w)$$

$$- u'(v^{\alpha'} h_{ij}(w) + u^{\beta'} h_{ij}(w))],$$

(29)

where $v = p_{B_{ij}}/m_{B_{ij}}, u' = p_{D_{kl}}/m_{D_{kl}}$, the velocity transfer is $w = v \cdot u'$, and $\epsilon$ is the polarization vector of the $D_{ij}$ meson. With a simple transformation, we can relate them to the form factors used in our work:

$$F_{+}(q^2) = \frac{1}{2 \sqrt{r}} [1 + w] h_{ij}(w) - (1 - r) h_{ij}(w),$$

$$F_{0}(q^2) = \sqrt{r} \left[ 1 + w \right] h_{ij}(w) - \frac{w - 1}{1 - r} h_{ij}(w),$$

(30)

where $w = (m_{B_{ij}}^2 + m_{D_{kl}}^2 - q^2)/(2m_{B_{ij}} m_{D_{kl}})$, and

$$V(q^2) = \frac{1 + r}{2 \sqrt{r}} h_{ij}(w),$$

$$A_0(q^2) = \frac{1}{2 \sqrt{r}} [1 + w] h_{ij}(w),$$

$$A_1(q^2) = \frac{1 + r}{2} [1 + w] h_{ij}(w),$$

$$A_2(q^2) = \frac{1 + r}{2} [r h_{ij}(w) + h_{ij}(w)],$$

where $w = (m_{B_{ij}}^2 + m_{D_{kl}}^2 - q^2)/(2m_{B_{ij}} m_{D_{kl}})$.

At the end point $q^2 = q^2_{\text{max}}$, we have

$$h_{ij}(1) = h_{ij}(1) = h_{ij}(1), \quad h_{ij}(1) = 0, \quad h_{ij}(1) = \mathcal{F}(1),$$

$$w = 1,$$  

$$\left[ h_{ij}(1) = \frac{1}{1 + r} h_{ij}(1) \right] = \mathcal{G}(1).$$

(32)

Therefore, the relevant form factors at the endpoint $q^2_{\text{max}}$ become:

$$F_{+}(q^2_{\text{max}}) = \frac{1 + r}{2 \sqrt{r}} \mathcal{G}(1),$$

$$V(q^2_{\text{max}}) = A_0(q^2_{\text{max}}) = A_2(q^2_{\text{max}}) = \frac{1}{A_1(q^2_{\text{max}})} = \frac{1 + r}{2 \sqrt{r}} \mathcal{F}(1).$$

(33)

Using the formulas above and including the lattice input [41, 57]

$$\mathcal{G}^{B-D}(1) = 1.033 \pm 0.095,$$

$$\mathcal{F}^{B-D}(1) = 0.895 \pm 0.010 \pm 0.024,$$

$$\mathcal{G}^{B_s-D_s}(1) = 1.052 \pm 0.046, \mathcal{F}^{B_s-D_s}(1) = 0.883 \pm 0.012 \pm 0.028,$$

with the relation between $F_0$ and $F_+$ near the endpoint $q^2_{\text{max}}$ evaluated in Ref. [57]:

$$F_0^{B-D}/F_+^{B-D} = 0.73 \pm 0.04, \quad F_0^{B_s-D_s}/F_+^{B_s-D_s} = 0.77 \pm 0.02,$$

(35)

we find the following values of the relevant form factors at the endpoint $q^2_{\text{max}}$:

$$F_0^{B-D}(q^2_{\text{max}}) = 0.86 \pm 0.08, \quad F_0^{B_s-D_s}(q^2_{\text{max}}) = 0.91 \pm 0.05,$$

$$F_+^{B-D}(q^2_{\text{max}}) = 1.17 \pm 0.10, \quad F_+^{B_s-D_s}(q^2_{\text{max}}) = 1.19 \pm 0.05,$$

$$V^{B-D}(q^2_{\text{max}}) = 1.01 \pm 0.05, \quad V^{B_s-D_s}(q^2_{\text{max}}) = 0.98 \pm 0.05,$$

$$A_0^{B-D}(q^2_{\text{max}}) = 1.01 \pm 0.05, \quad A_0^{B_s-D_s}(q^2_{\text{max}}) = 0.98 \pm 0.05,$$

$$A_1^{B-D}(q^2_{\text{max}}) = 0.80 \pm 0.04, \quad A_1^{B_s-D_s}(q^2_{\text{max}}) = 0.79 \pm 0.04,$$

$$A_2^{B-D}(q^2_{\text{max}}) = 1.01 \pm 0.05, \quad A_2^{B_s-D_s}(q^2_{\text{max}}) = 0.98 \pm 0.05.$$

(36)

The uncertainty of the form factors comes from the errors of $\mathcal{G}(1)$, given in Eq. (34), is around 5% to 10%, while the uncertainty of $\mathcal{F}(1)$ is around 2%. We set conservatively a common error of 5% for the form factors $V/A_{1,2}$ in Eq. (36), which takes into account the small variations of the central values of $\mathcal{G}(1)$ and $\mathcal{F}(1)$ in recent years [41, 55–57].

### 2.4 Extrapolation and differential decay rates

We know that the form factors calculated in the PQCD approach are only reliable in the low $q^2$ region. In order to cover the whole momentum range $m_T^2 < q^2 < q^2_{\text{max}}$ we have to perform an extrapolation. In order to improve the reliability of the extrapolation, we also use the values of the form factors at the endpoint $q^2_{\text{max}}$ evaluated in lattice QCD.

In the previous works [9, 42], we used the pole model parametrization [58] to perform the fit. In this work, we use the Bourrely-Caprini-Lellouch (BCL) parametrization [49] instead, and match it to the lattice input to improve the reliability of the extrapolation. We use our PQCD predictions of the form factors $f(q^2)$ at six anti points in the interval $0 < q^2 < m_T^2$, and the lattice QCD...

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value as an additional input at the endpoint $q^2_{\text{max}}$. Analogous to Ref. [50], we consider only the first two terms of the series in parameter $z$:

$$f(z) = \frac{1}{1 - t/m_R^2} \sum_{k=0}^{1} a_k^* z^k(t, t_0)$$

$$= \frac{1}{1 - t/m_R^2} \left( \alpha_0^* + \alpha_1^* \frac{\sqrt{t_+ - t} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - t} + \sqrt{t_+ - t_0}} \right),$$

(37)

with

$$0 \leq t_0 = t_+ \left( 1 - \frac{t - t_0}{t_+} \right) \leq t_-, \quad (38)$$

where $t = q^2$, $t_+ = (m_B^2 \pm m_{D^*}^2)^2$, and $m_R$ are the masses of the low-lying resonances. The optimized value of $t_0$ is chosen to be the same as in Ref. [50]. Since the choice of $m_R$ depends on the charged currents of semileptonic decays, i.e. of the $b \rightarrow c \ell \nu$ or the $b \rightarrow u \ell \nu$ transitions, we use the same set of $m_R$ as in Ref. [50], where the $B \rightarrow (q_\ell, J/\psi) \ell \nu$ decays have the same quark level $b \rightarrow c \ell \nu$ transitions as in this paper.

With these extrapolations, we have access to the full momentum dependence of the form factors, and the branching ratios of the semileptonic decays $B_{(s)} \rightarrow D_{(s)}^* \ell \nu$ can be calculated. The quark level transition in these semileptonic decays is $b \rightarrow c \ell \nu$ with the effective Hamiltonian [59]

$$\mathcal{H}_{b \rightarrow c \ell \nu} = \frac{G_F}{\sqrt{2}} V_{cb} \bar{c} \gamma_\mu (1 - \gamma_5) b \cdot \bar{\ell} \gamma^\mu (1 - \gamma_5) \nu_\ell,$$  (39)

where $G_F = 1.16637 \times 10^{-5}$ GeV$^{-2}$ is the Fermi constant. The differential decay rates of the decay mode $B_{(s)} \rightarrow D_{(s)}^* \ell \nu$ can be expressed as [60]

$$\frac{\text{d} \Gamma(B_{(s)} \rightarrow D_{(s)}^* \ell \nu)}{\text{d} q^2} = \frac{G_F^2 |V_{cb}|^2}{192 \pi^3 m_B^9} \left( 1 - \frac{m_\ell^2}{q^2} \right)^2 \frac{\lambda^{1/2}(q^2)}{2q^2} \left[ 3 m_\ell^2 \lambda(q^2) A_3^0(q^2) 

+ \frac{m_\ell^2 + 2q^2}{4m_D^2} \left( m_{B^*}^2 - m_{D^*}^2 - q^2 \right) \left( m_{B^*} + m_{D^*} \right) A_2(q^2) \frac{\lambda(q^2) A_3^2(q^2)}{m_{B^*} + m_{D^*}} \right],$$

(40)

$$\frac{\text{d} \Gamma(B_{(s)} \rightarrow D_{(s)}^* \ell \nu)}{\text{d} q^2} = \frac{G_F^2 |V_{cb}|^2}{192 \pi^3 m_B^9} \left( 1 - \frac{m_\ell^2}{q^2} \right)^2 \frac{\lambda^{1/2}(q^2)}{2q^2} \left[ \frac{V^2(q^2)}{m_{B^*} + m_{D^*}} + \frac{m_{B^*} + m_{D^*}}{2q^2} \right].$$

(41)

(42)

With the phase space factor $\lambda(q^2) = (m_B^2 + m_{D^*}^2 - q^2)^2 - 4m_B^2 m_{D^*}^2$, the total differential decay width is defined as:

$$\frac{\text{d} \Gamma}{\text{d} q^2} = \frac{\text{d} \Gamma_L}{\text{d} q^2} + \frac{\text{d} \Gamma_T}{\text{d} q^2}.$$  \hspace{1cm} (43)

2.5 Additional physical observables $P_\tau$, $F_{\perp}(D_{(s)}^*)$ and $A_{FB}(\tau)$

Besides the decay rates and the ratios $R(X)$, there are three additional physical observables for the $B/B_s$ semileptonic decays: the longitudinal polarization of the tau lepton $P_\tau$, the fraction of $D_{(s)}^*$ longitudinal polarization $F_{\perp}(D_{(s)}^*)$ and the forward-backward asymmetry of the tau lepton $A_{FB}(\tau)$. These physical observables are sensitive to new physics beyond SM [2, 61, 62].

As for the definitions of these physical observables, we follow Refs. [31, 63, 64]. For the $\tau$ longitudinal polarization, for example, the authors of Refs. [31, 63] used the $Q$ rest frame, where the spatial components of the momentum transfer $Q = p_B - p_M$ vanish. In this coordinate system the direction of the $B$ momenta is along the $z$ axis, and the $\tau$ momentum lies in the $x-z$ plane. Here, we use the same definition:

$$P_\tau(D_{(s)}^*) = \frac{\Gamma_+ (D_{(s)}^*) - \Gamma_-(D_{(s)}^*)}{\Gamma_+ (D_{(s)}^*) + \Gamma_-(D_{(s)}^*)},$$

(44)

where $\Gamma_\pm (D_{(s)}^*)$ denotes the decay rate of $B_{(s)} \rightarrow D_{(s)}^* \tau \bar{\nu}_\tau$ with the $\tau$ lepton helicity of $\pm 1/2$. The explicit expressions for $\Gamma_\pm$ in SM can be found in the Appendix of Ref. [64]. For $B_{(s)} \rightarrow D_{(s)}^* \tau \bar{\nu}_\tau$ decays, we find

$$\frac{\text{d} \Gamma_+}{\text{d} q^2} = \frac{G_F^2 |V_{cb}|^2}{192 \pi^3 m_B^9} q^2 \sqrt{\lambda(q^2)} \left( 1 - \frac{m_\ell^2}{q^2} \right)^2 \frac{m_\ell^2}{2q^2} \left( H_{V_0} + 3H_{V_1}^2 \right),$$

(45)

$$\frac{\text{d} \Gamma_-}{\text{d} q^2} = \frac{G_F^2 |V_{cb}|^2}{192 \pi^3 m_B^9} q^2 \sqrt{\lambda(q^2)} \left( 1 - \frac{m_\ell^2}{q^2} \right)^2 \left( H_{V_0}^2 \right).$$

(46)

For $B_{(s)} \rightarrow D_{(s)}^* \tau \bar{\nu}_\tau$ decays, we have
The lepton forward-backward asymmetry $A_{FB}(\tau)$ is more complicated since it is defined in the $\tau\bar{\nu}$ rest frame. The explicit expression is [64]:

$$A_{FB} = \frac{\int \frac{d\tau}{d\cos\theta} d\cos\theta - \int \frac{d\tau}{d\cos\theta} d\cos\theta}{\int \frac{d\tau}{d\cos\theta} d\cos\theta} = \int b_0(q^2) dq^2,$$

where $\theta$ is the angle between the 3-momenta of $\tau$ and the initial $B$ or $B_s$, in the $\tau\bar{\nu}$ rest frame. The function $b_0(q^2)$ is the angular coefficient which can be written as [64]:

$$b_0^{(D)}(q^2) = \frac{G_F^2 V_{cb}^2}{128\pi^3 m_{B_s}^3} q^2 \sqrt{\Lambda(q^2)} \left( 1 - \frac{m_{B_s}^2}{q^2} \right)^2 \left( \frac{1}{2} H_{V;+}^2 + m_{D_s}^2 (H_{V;0}^2 + H_{V;t}^2) \right).$$

With the above definitions and formulas, we are now ready to present our numerical results and a phenomenological analysis.

3 Numerical results and discussion

In the numerical calculations, we use the following parameters (masses and decay constants are in units of GeV) [28, 29, 65, 66]:

$m_B = 5.28, \quad m_{B_s} = 5.367, \quad m_{D_s} = 1.865, \quad m_{D_s} = 1.870, \quad m_{D_s} = 2.007, \quad m_{D_s} = 2.010, \quad m_{D_s} = 1.968, \quad m_{D_s} = 2.112, \quad m_{D_s} = 1.777, \quad m_{D_s} = 1.275 V_7^{0.025, -0.035}, \quad f_B = 0.212, \quad f_{D_s} = 0.249, \quad f_{D_s} = 0.187, \quad f_{D_s} = 0.191, \quad f_{D_s} = 0.227, \quad |V_{cb}| = (42.2 \pm 0.8) \times 10^{-3}, \quad \tau_B = 1.638 \text{ps}, \quad \tau_{B_s} = 1.520 \text{ps}, \quad \tau_{B_s} = 1.509 \text{ps}, \quad f_D = (1.078 \pm 0.036) f_{D_s}, \quad f_{D_s} = (1.087 \pm 0.020) f_{D_s}, \quad \Lambda_{MS}^{(4s)} = 0.287.$

3.1 Form factors

It is obvious that the theoretical predictions of the differential decay rates and other physical observables for the considered semileptonic $B/B_s$ meson decays strongly depend on the form factors $F_{c,s}(q^2), V(q^2)$ and $A_{0,1,2}(q^2)$. Specifically, $F_{c,s}(q^2)$ controls the process $B_{c,s} \rightarrow D_{c,s} (l V)$, while $V(q^2)$ and $A_{0,1,2}(q^2)$ play key roles in the process $B_{c,s} \rightarrow D_{c,s} (l V)$. The value of these form factors at $q^2 = 0$ and their $q^2$ dependence in the range $0 < q^2 \leq q_{\text{max}}^2$ contain a lot of information about the relevant physical processes.

In Refs. [9, 51, 53], the applicability of the PQCD approach to $(B \rightarrow D^{(*)})$ transitions was examined, and it was shown that the PQCD approach with the inclusion of the Sudakov effects is applicable to the semileptonic decays.
$B \to D^{(*)} V_1$ in the low $q^2$ region. Therefore, we present our predictions of the form factors at $q^2 = 0$ in Table 1.

It can be seen from Table 1 that the form factors for $B \to D^{(*)}$ and $B_s \to D_s^{(*)}$ transitions are very similar at $q^2 = 0$, which implies that the effect of $SU(3)$ flavor symmetry breaking is small, less than 10%. In order to cover the whole $q^2$ region, an extrapolation of the form factors has to be made from $q^2 = 0$ up to $q^2 = q^2_{\text{max}}$. In this work, we make the extrapolation using two different methods.

| Form Factor | $B(0)$ | $F(0)$ | $V(0)$ | $A_0(0)$ | $A_1(0)$ | $A_2(0)$ |
|-------------|-------|------|------|-------|-------|-------|
| $B^+ \to D^+$ | $0.53\pm0.10$ | $0.54\pm0.10$ | $0.52\pm0.10$ | $0.54\pm0.09$ | $0.55\pm0.10$ | $0.53\pm0.11$ |
| $B^0 \to D^+$ | $0.54\pm0.10$ | $0.64\pm0.11$ | $0.65\pm0.11$ | $0.48\pm0.09$ | $0.50\pm0.09$ | $0.50\pm0.09$ |
| $B_s^0 \to D_s^+$ | $0.52\pm0.10$ | $0.49\pm0.08$ | $0.50\pm0.09$ | $0.52\pm0.08$ | $0.55\pm0.10$ | $0.53\pm0.11$ |

(1) The first method is analogous to that used in Refs. [9, 42]. We first calculate the form factors at several points in the lower region $0 \leq q^2 \leq m^2_c$ using the expressions in Eqs. (22)-(27) and the definitions in Eqs. (18), (19). In the fitting process, we use the BCL parametrization formula given in Eqs. (37), (38) instead of the pole model parametrization used in Refs. [9, 42].

(2) The second method is the “PQCD+Lattice” method, similar to Ref. [10]. Since the lattice QCD results for the form factors are reliable and accurate at $q^2 = 0$, we take the lattice QCD predictions of the form factors at the endpoint $q^2_{\text{max}}$ as an additional input in the fitting process. In order to match the lattice input, we use the BCL parametrization [49, 50] for extrapolation.

In Figs. 2 and 3, we show the theoretical predictions of the $q^2$ dependence of the six form factors for $B^+ \to (D^+, D^0)$ and $B_s \to (D_s^+, D_s^0)$ transitions, obtained by employing the PQCD approach (blue solid curves) and the “PQCD + Lattice” method (red dashed curves). The blue and red bands show the theoretical uncertainties. As a comparison, we also show the central values of the form factors obtained using the traditional pole model in the PQCD approach (“PQCD + Lattice” method). For the $B_s \to D_s^*$ transition, we found a very similar result.

(3) The difference between the theoretical predictions of the central values of the form factors obtained using the traditional pole model and the BCL model is very small in the region $q^2 < 8$ GeV$^2$. The largest differences in the region near $q^2_{\text{max}}$ are $\Delta F_+ (11.55) = 0.59$ (i.e., about 31% of $F_+ (11.55) = 1.89$ in the pole model), and $\Delta V (10.59) = 0.36$ (i.e., about 19% of $V (10.59) = 1.89$ in the pole model) for the $B_s \to D_s^{(*)}$ transition, as illustrated in Fig. 3.

3.2 Branching ratios and the ratio of Brs

Inserting the obtained form factors in the differential decay rates given in Eqs. (40)-(43), it is straightforward to integrate over the range $m^2_c \leq q^2 \leq (m_{B_s} - m_{D_s})^2$. For the four semileptonic decays $B_s^0 \to D_s^{(*)} \tau^+ \nu_{\tau}$ and $B_s \to D_s^{(*)} l^+ v_l$ with $l^+ = (e^+, \mu^+, \tau^+)$, the theoretical predictions of the branching ratios are the following:

\[
\mathcal{B}(B^+_s \to D_s^{(*)} \tau^+ \nu_{\tau}) = \begin{cases} 
0.72^{+0.32}_{-0.23} (\omega_{B_s}) & \pm 0.03 (V_{cb}) \pm 0.02 (m_c), \\
0.63^{+0.17}_{-0.13} (\omega_{B_s}) & \pm 0.03 (V_{cb}) \pm 0.02 (m_c),
\end{cases} 
\text{PQCD},
\]

\[
\mathcal{B}(B^+_s \to D_s^{(*)} l^+ v_l) = \begin{cases} 
1.97^{+0.88}_{-0.65} (\omega_{B_s}) & \pm 0.08 (V_{cb}) \pm 0.03 (m_c), \\
1.84^{+0.76}_{-0.50} (\omega_{B_s}) & \pm 0.08 (V_{cb}) \pm 0.03 (m_c),
\end{cases} 
\text{PQCD} + \text{Lattice},
\]

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where the theoretical errors come mainly from the uncertainties of the input parameters \( \omega_R = 0.50 \pm 0.05 \) GeV, \( |V_{cb}| = (42.2 \pm 0.8) \times 10^{-3} \) and \( m_c = 1.275^{+0.025}_{-0.035} \) GeV. Errors from the uncertainties of the decay constants of the initial \( B_s \) and the final \( D_s^{(*)} \) mesons are small and have been neglected. In Table 2, we list our PQCD and “PQCD + Lattice” predictions of the branching ratios of the semileptonic decays of \( B_s^0 \), where the total errors are obtained by adding the individual errors in quadrature. As a comparison, we also show predictions of some other theoretical approaches in the SM framework. Unfortunately, there are no experimental results available at present. In Table 3, we show the theoretical predictions of the ratios
of the branching fractions \( R(D_s) \) and \( R(D_s') \) defined in the same way as \( R(D^{(*)}) \):

\[
R(D_s) = \frac{B(B_s^0 \to D_s^- \tau^+ \nu_\tau)}{B(B_s^0 \to D_s^- \ell^+ \nu_\ell)} = \begin{cases} 0.365^{+0.09}_{-0.012} & \text{PQCD,} \\ 0.341^{+0.024}_{-0.025} & \text{PQCD + Lattice,} \end{cases}
\]

\[
R(D_s') = \frac{B(B_s^0 \to D_s'^- \tau^+ \nu_\tau)}{B(B_s^0 \to D_s'^- \ell^+ \nu_\ell)} = \begin{cases} 0.287^{+0.008}_{-0.011} & \text{PQCD,} \\ 0.271^{+0.015}_{-0.016} & \text{PQCD + Lattice,} \end{cases}
\]

where \( \ell \) denotes an electron or a muon.

Following the same procedure as for \( B_s \) decays, we calculate the branching ratios and the ratios \( R(D^{(*)}) \) for \( B \to (D, D^*) (\ell^+ \nu_\ell, \tau^+ \nu_\tau) \) decays. The numerical results are listed in Table 4 for the branching ratios and in Table 5 for the ratios \( R(D) \) and \( R(D^*) \). In the estimate of the errors, \( \omega_B = 0.40 \pm 0.04 \) GeV was used.

From Tables 2–5, the following points can be noted:

1. For the semileptonic decays \( B \to D^{(*)} \ell^+ \nu_\ell \) with \( \ell = (e, \mu, \tau) \), the “PQCD+Lattice” predictions of the branching ratios and the ratios \( R(D^{(*)}) \) agree with the available experimental data within errors. This can be viewed as evidence of the reliability of the “PQCD+Lattice” method for calculating semileptonic decays of \( B \) meson.

2. For the four \( B_s \) decay modes and the eight \( B \) decay modes, the “PQCD+Lattice” predictions of the
branching ratios are generally smaller than the conventional PQCD predictions, but the differences are relatively small, less than 20%. The predictions also agree with the previous PQCD predictions given in Refs. [9, 10, 42] within the still large theoretical uncertainties.

(3) For the ratios \( R(D_s) \) and \( R(D_s') \), the theoretical errors largely cancel, and the PQCD and "PQCD+Lattice" predictions have a very small error, only about 5%. These predictions could be tested in the near future by the LHCb and Belle-II experiments.

(4) For the branching ratios, the predictions of the different theoretical approaches can be somewhat different for the same decay mode, but they still agree within errors, since the theoretical uncertainties are large. For the ratios \( R(D) \) and \( R(D') \), our "PQCD+Lattice" predictions agree with the average of the SM predictions obtained by HQET and the lattice QCD input for the form factors [3–6, 29].

(5) Comparing the two sets of ratios \( (R(D), R(D')) \) and \( (R(D'), R(D')) \), listed in Tables 3 and 5, one can see that the SU(3) flavor symmetry is well preserved in the PQCD and the "PQCD+Lattice" approaches. This point could also be tested by experiments.

3.3 \( P_F(D_s^+), F_L(D_s^+) \) and \( A_F(\tau) \)

Besides the branching ratios and the ratios \( R(X) \), we consider three additional physical observables: the longitudinal polarization of the tau lepton \( P_F(D_s^+) \), the fraction of \( D_s^+ \), longitudinal polarization \( F_L(D_s^+) \) and the forward-backward asymmetry of the tau lepton \( A_F(\tau) \). These quantities could be measured by the LHCb and Belle experiments, and may be sensitive to new physics [2, 31,
61–63]. Investigation of these physical observables may provide new clues for understanding the $R(D^{(*)})$ puzzle, and it is therefore necessary and interesting to calculate them.

In Refs. [31–33], these three physical observables were calculated in the SM framework, and possible new physics effects were studied. We calculate these observables in the PQCD and “PQCD+Lattice” approaches using Eqs. (44)-(59), and show the numerical predictions of $P_f(D_{(s)}^{(*)})$, $F_L(D_{(s)}^{(*)})$, and $A_{FB}(\tau)$ in Table 6. The measured values of $P_f(D^*)$ and $F_L(D^*)$ [20, 21, 30] given in Eqs. (4), (5) are also listed in Table 6. As a comparison, we also show the SM predictions of these physical observables given in Refs. [31, 32].

From the results listed in Eqs. (4), (5) and in Table 6, the following points can be noted:

1. The uncertainties of the theoretical predictions of the physical observables $P_f(D_{(s)}^{(*)})$, $F_L(D_{(s)}^{(*)})$, and $A_{FB}(\tau)$ are very small compared to the branching ratios, since the theoretical uncertainties largely cancel in the ratios.

2. The PQCD and “PQCD+Lattice” predictions of the physical observables $P_f(D_{(s)}^{(*)})$, $F_L(D_{(s)}^{(*)})$, and $A_{FB}(\tau)$ for $B_{(s)} \to D_{(s)}^{(*)} \ell^+\nu_\ell$ decays are very similar: the difference for a fixed decay mode is less than 5%. For $A_{FB}(\tau)$ of the $B_{(s)} \to D_{(s)}^{(*)} \ell^+\nu_\ell$ decays, however, the difference is about 40%. The reason is the definition of the angular coefficient $b^{\ell}(q^2)$, where the term $(H_{V,V}^2 - H_{V,V}^2)$ in Eq. (59) can be moderately changed by the high $q^2$ behavior of the form factors in the PQCD and “PQCD+Lattice” approaches. Clearly, a precise experimental measurement of the forward-backward asymmetry $A_{FB}(\tau)$ in the $B_{(s)} \to D_{(s)}^{(*)} \ell^+\nu_\ell$ decays could be of great help to test and improve the factorization models.

3. For $P_f(D^*)$ and $F_L(D^*)$, our theoretical predictions agree with the measured data within errors, partially due to the still large experimental errors. For all decay modes considered, the PQCD and “PQCD+Lattice” predictions of the three physical observables are consistent with the other approaches in the SM framework. We expect that these physical observables could be measured with high precision in the future by the LHCb and Belle-II experiments, which could help to test the theoretical models or approaches.

Table 5. PQCD and “PQCD+Lattice” predictions of the ratios $R(D)$ and $R(D^*)$. As a comparison, the previous “PQCD+Lattice” predictions given in Ref. [10], the average of the SM predictions given in Ref. [29], measured values reported by the BaBar, Belle and LHCb collaborations [1, 24, 25], and the world averaged results from HFLAV group [29] are also shown.

| Ratios | PQCD | PQCD+Lattice | PQCD[10] | SM[29] | BaBar[1] | Belle[24] | LHCb[25] | HFLAV[29] |
|--------|------|--------------|----------|--------|----------|-----------|-----------|-----------|
| $R(D)$ | 0.376^{+0.011}_{-0.012} | 0.324^{+0.020}_{-0.022} | 0.337(38) | 0.299(3) | 0.440(72) | 0.307(40) | – | 0.340(30) |
| $R(D^*)$ | 0.288^{+0.008}_{-0.010} | 0.272^{+0.014}_{-0.014} | 0.269(21) | 0.258(5) | 0.332(30) | 0.283(23) | 0.291(35) | 0.295(14) |

Table 6. PQCD and “PQCD+Lattice” predictions of $P_f(D_{(s)}^{(*)})$, $F_L(D_{(s)}^{(*)})$, and $A_{FB}(\tau)$. The SM predictions and the measured values are also listed.

| Observable | Approach | $B^0 \to D^-\tau^+\nu_\tau$ | $B_s^0 \to D_s^-\tau^+\nu_\tau$ | $B^0 \to D^{-}\tau^-\nu_\tau$ | $B_s^0 \to D_s^{-}\tau^-\nu_\tau$ |
|-----------|----------|---------------------|---------------------|---------------------|---------------------|
| $P_f(D_{(s)}^{(*)})$ | PQCD | 0.32(1) | 0.31(1) | – | – |
| | PQCD+Lat. | 0.30(1) | 0.30(1) | – | – |
| | SM[31] | – | – | –0.497(13) | – |
| | SM[32] | 0.325(3) | – | –0.508(4) | – |
| | Belle[20] | – | – | –0.38 ± 0.51^{+0.12}_{-0.10} | – |
| $F_L(D_{(s)}^{(*)})$ | PQCD | – | – | 0.42(1) | 0.42(1) |
| | PQCD+Lat. | – | – | 0.43(1) | 0.43(1) |
| | SM[33] | – | – | 0.457(10) | – |
| | SM[32] | – | – | 0.441(6) | – |
| | Belle[30] | – | – | 0.60 ± 0.08 ± 0.04 | – |
| $A_{FB}(\tau)$ | PQCD | 0.35(1) | 0.36(1) | –0.085(2) | –0.083(2) |
| | PQCD+Lat. | 0.36(1) | 0.36(1) | –0.054(2) | –0.050(2) |
| | SM[32] | 0.361(1) | – | –0.084(13) | – |

4 Summary

In this paper, we studied the semileptonic decays $B_{(s)} \to D_{(s)}^{(*)}\ell^+\nu_\ell$ in the SM framework by employing the conventional PQCD factorization approach and the “PQCD+Lattice” approach. In the second approach, we took into account the lattice QCD results of the relevant form factors as input for the extrapolation from the low $q^2$
to the endpoint $q^2_{\text{max}}$. We calculated the form factors $F_{0,+}(q^2)$, $V(q^2)$ and $A_{0,1,2}(q^2)$ of the $B_i \to D^{(*)}_i$ transitions, provided the theoretical predictions of the branching ratios of the $B_i/B_s$ semileptonic decays and of the ratios $R(D^{(*)})$ and $R(D^{(*)}_s)$. In addition, we also gave our theoretical predictions of the additional physical observables: the longitudinal polarization of the tau lepton $P_L(D^{(*)}_s)$, the fraction of $D^{(*)}_s$ longitudinal polarization $F_L(D^{(*)}_s)$ and the forward-backward asymmetry of the tau lepton $A_{FB}(\tau)$.

From the numerical calculations and phenomenological analysis we noted the following points:

1. For the twelve $B_i/B_s$ semileptonic decay modes considered, the "PQCD+Lattice" predictions of the branching ratios are generally smaller than the conventional PQCD predictions, but the differences are relatively small, less than 20%. The "PQCD+Lattice" predictions of the branching ratios and the ratios $R(D^{(*)})$ agree with the available experimental measurements within errors.

2. For the ratios $R(D_s)$ and $R(D'_s)$, the PQCD and "PQCD+Lattice" predictions are the following:

$$R(D_s) = \begin{cases} 0.365^{+0.009}_{-0.012} & \text{PQCD,} \\ 0.341^{+0.024}_{-0.023} & \text{PQCD+Lattice,} \end{cases}$$

3. For most of the observables $P_L(D^{(*)}_s)$, $F_L(D^{(*)}_s)$ and $A_{FB}(\tau)$, the PQCD and "PQCD+Lattice" predictions agree, and the difference is less than 5%. The relatively large difference of $A_{FB}(\tau)$ in the $B_i \to D^{(*)}_s \tau^+\nu_\tau$ decays can be reasonably explained.

4. For $P_L(D^*)$ and $F_L(D^*)$, our theoretical predictions agree with the measurements within errors. For all decay modes considered, the PQCD and "PQCD+Lattice" predictions of these physical observables are consistent with the results of the other approaches in the SM framework. The LHCb and Belle-II experiments could help to test the above theoretical predictions.

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### Appendix: Relevant functions

In this appendix, we present the explicit expressions for some functions which have already appeared in the previous sections. The hard functions $h_1$ and $h_2$ come form the Fourier transform and can be written as:

$$h_1(x_1, x_2, b_1, b_2) = K_0(b_1 b_2) \int_0^{x_1} d\bar{r} \int_0^{x_2} d\bar{t} \sigma(x_1 \bar{r}, \bar{t}; b_1 b_2),$$

$$h_2(x_1, x_2, b_1, b_2) = K_0(b_1 b_2) \int_0^{x_2} d\bar{r} \int_{x_1}^{x_2} d\bar{t} \sigma(x_2 \bar{r}, \bar{t}; b_1 b_2),$$

where $K_0$ and $K_0$ are the modified Bessel functions, and

$$a_1 = m_{B_i} \sqrt{s_{B_i}}, \quad a_2 = m_{B_i} \sqrt{s_{B_i} r^2 + r^2}, \quad b_1 = b_2 = m_{B_i} \sqrt{s_{B_i} r^2}.$$  

The threshold resummation factor $S_{\sigma}(x_1)$ is adopted from Ref. [74]:

$$S_{\sigma} = \frac{2^{1+2c} (3/2 + c)}{\sqrt{\Gamma(1+c)}} (x_1 x_2) c,$$

and here we use $c = 0.3$.

The factor $\exp[-S_{\sigma}(t)]$ contains the Sudakov logarithmic corrections and the renormalization group evolution effects of the wave functions and the hard scattering amplitude with $S_{\sigma}(t) = S_{B}(t) + S_{MB}(t)$.

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