ON THE CHERN NUMBERS OF THE GENERALISED KUMMER VARIETIES

MARC A. NIEPER-WISSKIRCHEN

Abstract. Let $A^{[n]}$ denote the $2(n - 1)$-dimensional generalised Kummer variety constructed from the abelian surface $A$. Further, let $X$ be an arbitrary smooth projective surface with $\int_X c_1(X)^2 \neq 0$, and $X^{[k]}$ the Hilbert scheme of zero-dimensional subschemes of $X$ of length $k$. We give a formula which expresses the value of any complex genus on $A^{[n]}$ in terms of Chern numbers of the varieties $X^{[k]}$.

In [5] and [6] it is shown how to use Bott’s residue formula to effectively calculate the Chern numbers of the Hilbert schemes $(\mathbb{P}^2)^{[k]}$ of points on the projective plane. Since $\int_{\mathbb{P}^2} c_1(\mathbb{P}^2)^2 = 9 \neq 0$ we can use these numbers and our formula to calculate the Chern numbers of the generalised Kummer varieties.

A table with all Chern numbers of the generalised Kummer varieties $A^{[n]}$ for $n \leq 8$ is included.

1. Introduction

The two main series of irreducible holomorphic symplectic complex manifolds are Hilbert schemes of points on a K3 surface and generalised Kummer varieties invented by Beauville [1]. One can ask for their complex cobordism class tensored with $\mathbb{Q}$, which is given by the values of all their Chern numbers.

In [5] Ellingsrud, Göttsche and Lehn proved that the complex cobordism class of a Hilbert scheme $X^{[n]}$ of zero-dimensional subschemes of length $n$ on a smooth projective surface $X$ over the complex numbers depends only on the cobordism class of the surface $X$, i.e. on $c_1(X)^2$ and $c_2(X)$ (here and later on, top intersections on surfaces are to be understood as intersection numbers). They showed how this result can be used to calculate the Chern numbers of any such Hilbert scheme $X^{[n]}$ if one knows the Chern numbers of the varieties $(\mathbb{P}^2)^{[k]}$ and $(\mathbb{P} \times \mathbb{P})^{[k]}$, which in turn can be calculated by means of Bott’s residue formula.

Therefore, the Chern numbers of the Hilbert schemes of points on a K3 surface can be efficiently calculated though now explicit formula is known. These numbers can, for example, be used to check the conjecture of [3] about the elliptic genus of the Hilbert schemes $X^{[k]}$ where $X$ is a K3 surface (see [5]). In this note, we want to give a method for computing the Chern numbers of the $(n - 1)$-dimensional generalised Kummer variety $A^{[n]}$ for an abelian surface $A$ and for general $n$ as this has not appeared in the literature so far.

The $\chi_y$-genus of $A^{[n]}$ has been calculated by Göttsche and Soergel [8]. Expressing this genus in terms of Chern numbers by using the Hirzebruch-Riemann-Roch

Date: January 16, 2022.

1991 Mathematics Subject Classification. 14C05,14M99,14N10,14Q99.

I would like to thank Michael Britze and Daniel Huybrechts who discussed the ideas presented in this note with me. Without them it would have taken much more time to complete this paper.
formula gives us enough information to deduce the Chern numbers of \( A^{[n]} \) for \( n \leq 4 \).

Using the theory of Rozansky-Witten invariants, Sawon \[13\] produced a further relation that allowed him to compute all the Chern numbers for \( n \leq 5 \). The Chern numbers of \( A^{[6]} \) were calculated by M. Britze and the author in \[2\]. J. Sawon informed us that he also had computed these numbers. However, all these methods are not sufficient to compute the Chern numbers for \( n > 6 \).

We will give a closed formula describing the value of any complex genus on a generalised Kummer variety in terms of genera of the Hilbert schemes of points on a fixed surface \( X \) with \( c_1(X)^2 \neq 0 \). Since these are computable by means of Bott’s residue formula for \( X \) being the projective plane \( \mathbb{P}^2 \) we can compute the Chern numbers of \( A^{[n]} \) for any \( n \). We have done this for \( n \leq 8 \) (see appendix).

2. The generalised Kummer varieties

Let \( X \) be a smooth projective surface over the field of complex numbers. For every non-negative integer we denote by \( X^{[n]} \) the Hilbert scheme of zero-dimensional subschemes of \( X \) of length \( n \). By a result of Fogarty (\[7\]), this scheme is smooth and projective of dimension \( 2n \). It can be viewed as a resolution \( \rho : X^{[n]} \to X^{(n)} \) of the \( n \)-fold symmetric product \( X^{(n)} := X^n/\mathbb{S}_n \) of \( X \). The morphism \( \rho \), sending closed points, i.e. subschemes of \( X \), to their support counting multiplicities, is called the Hilbert-Chow morphism.

Let us briefly recall the construction of the generalised Kummer varieties introduced by Beauville \[1\]. Let \( A \) be an abelian surface and \( n > 0 \). There is an obvious summation morphism \( A^{(n)} \to A \). We denote its composition with the Hilbert-Chow morphism \( \sigma : A^{[n]} \to A^{(n)} \) by \( \sigma : A^{[n]} \to A \).

Definition 1. The \( n \)th generalised Kummer variety \( A^{[n]} \) is the fibre of \( \sigma \) over \( 0 \in A \).

Beauville showed among other things the following property of the varieties \( A^{[n]} \):

Proposition 1. The \( n \)th generalised Kummer variety is a smooth projective holomorphic symplectic variety of dimension \( 2(n-1) \).

Proof. \[1\].

Since \( A \) acts on itself by translation there is also an induced operation of \( A \) on the Hilbert schemes \( A^{[n]} \). Let us denote the restriction of this operation to the generalised Kummer variety by \( \nu : A \times A^{[n]} \to A^{[n]} \). The following diagram is a cartesian one:

\[
\begin{array}{ccc}
A \times A^{[n]} & \xrightarrow{\nu} & A^{[n]} \\
\pi_A \downarrow & & \downarrow \sigma \\
A & \xrightarrow{n} & A.
\end{array}
\]

(1)

Here, \( n : A \to A, a \mapsto na \) is the (multiplication by \( n \))-morphism. It is a Galois covering of degree \( n^4 \). Therefore, also \( \nu \) is a Galois covering of degree \( n^4 \).

Next, we want to introduce certain line bundles on the Hilbert schemes and generalised Kummer varieties that are constructed from line bundles on the underlying surface:
Each line bundle \( L \) on a smooth projective surface \( X \) gives us a line bundle \( L_n \) on \( X^n \) in the following way: \( L_n := L^\otimes n \) is a \( \mathfrak{S}_n \)-invariant line bundle on the \( n \)-th product \( X^n \) of \( X \). Therefore, we can define the sheaf \( \mathcal{L}(n) := (\pi_\ast (L_n^\otimes))_{\mathfrak{S}_n} \) of \( \mathfrak{S}_n \)-invariant sections of \( \pi_\ast (L_n^\otimes) \) on \( X(n) \) where \( \pi : X^n \to X(n) \) is the canonical projection. The pull-back \( L_n := \rho^\ast \mathcal{L}(n) \) by the Hilbert-Chow morphism is a line bundle on \( X^n \). Note that \( \text{Pic}(X) \to \text{Pic}(X^n) \), \( L \mapsto L_n \) is a homomorphism of groups.

This construction has already appeared for example in [4] and [2]. If \( X \) is an abelian surface, we denote by \( L_{[n]} \) the restriction of \( L_n \) to the generalised Kummer variety \( X_{[n]} \subseteq X^n \). By using the seesaw principle (cf. [12]), it can be shown that \( \nu^* L_n = L_n^\otimes L_{[n]}^{(2)} \) (cf. [2]).

3. Complex genera in general

Let \( \Omega := \Omega^U \otimes \mathbb{Q} \) denote the complex cobordism ring. A complex genus \( \phi \) is a ring homomorphism \( \phi : \Omega \to R \) into any \( \mathbb{Q} \)-algebra \( R \). It is result of Milnor ([11]) that the complex cobordism ring is a polynomial ring in the cobordism classes of the complex projective spaces. The \( R \)-valued complex genera are in one-to-one correspondence with the formal power series \( f_\phi \in R[[x]] \) over \( R \) with constant coefficient 1. The correspondence is given as follows:

\[
\phi(X) = \int_X \prod_{i=1}^n f_\phi(\gamma_i),
\]

where \( X \) is any complex manifold of dimension \( n \) and \( \gamma_1, \ldots, \gamma_n \) are the Chern roots of \( X \). Therefore, the cobordism class of a manifold is determined by the values of its Chern numbers (take \( \phi = \text{id}_\Omega : \Omega \to \Omega \)). By Milnor’s result, the converse is also true, i.e. the cobordism class determines the Chern numbers.

Now, let us slightly generalise the notion of a genus.

**Definition 2.** Let \( \phi \) be a complex genus. For a complex manifold \( X \) together with a line bundle \( L \) on \( X \) we define

\[
\phi(X, L) := \int_X c_1(L) \prod_{i=1}^n f_\phi(\gamma_i)
\]

as the genus \( \phi \) of the pair \((X, L)\).

**Remark 1.** Obviously, \( \phi(X, \mathcal{O}_X) = \phi(X) \).

**Example 1.** If \( \text{td}(X) \) denotes the Todd genus of \( X \), and \( \chi(X, L) \) the holomorphic Euler characteristic of the line bundle \( L \) on \( X \), we have by the Hirzebruch-Riemann-Roch theorem that

\[
\text{td}(X) = \chi(X, L).
\]

The genera of pairs \((X, L)\) have the following properties, which follow directly from the appropriate properties of Chern classes/roots.

**Proposition 2.** Let \( \phi : \Omega \to R \) be any complex genus with values in \( R \). We have...
1. \( \phi(X \times Y, L \boxtimes M) = \phi(X, L) \phi(Y, M) \) for two complex manifolds \( X \) and \( Y \) together with a line bundle \( L \) resp. \( M \).

2. \( \phi(X, \nu^*L) = \deg(\nu) \phi(Y, L) \) for any Galois covering \( \nu : X \to Y \) and any line bundle \( L \) on \( Y \).

Furthermore, any genus gives us a deformed genus in the following sense:

**Definition 3.** Let \( \phi \) be a complex genus with values in the \( \mathbb{Q} \)-algebra \( R \). By \( \phi_t \) we denote the genus with values in \( R[t] \) given by

\[
\phi_t(X) := \int_X \prod_{i=1}^{n} (f_{\phi}(\gamma_i) e^{t\gamma_i})
\]

for any complex manifold \( X \).

**Remark 2.** We have \( \phi_n(X) = \phi(X, K_X^{-n}) \) for any integer \( n \) where \( K_X \) is the canonical line bundle on \( X \).

4. **Complex genera of Hilbert schemes of points on surfaces**

In this section we want to cite some of the results of [4] and give some corollaries which will be used later on.

Let \( X \) be a smooth projective surface. Following [4], we define

\[
H_X := \sum_{n=0}^{\infty} [X^{[n]}] z^n
\]

as an invertible element in the formal power series ring \( \Omega[[z]] \). Analogously we define

\[
K := \sum_{n=1}^{\infty} [A^{[n]}] z^n
\]

in \( \Omega[[z]] \) where \( A \) is any abelian surface. The cobordism class does not depend on the choice of \( A \) since the generalised Kummer varieties deform with \( A \). We can reformulate our task to determine the Chern numbers of the generalised Kummer varieties by asking: What is the value \( \phi(K) \in R[[z]] \) for any complex genus \( \phi : \Omega \to R \).

There are various tautological bundles on \( X^{[n]} \) (cf. [11]). The construction is as follows: Since \( X^{[n]} \) represents a functor there is a universal family \( \Xi_n \subseteq X^{[n]} \times X \). Let us denote by \( \mathcal{O}_n \) its structure sheaf. For any locally free sheaf \( F \) on \( X \) we define the sheaf \( F^{[n]} := p_*(\mathcal{O}_n \otimes q^*F) \) on \( X^{[n]} \), where \( p : X^{[n]} \times X \to X^{[n]} \) and \( q : X^{[n]} \times X \to X \) are the canonical projections.

The following lemma is a generalization of Theorem 4.2 in [4] for line bundles.

**Lemma 1.** Let \( k \) be a nonnegative integer, \( m_1, \ldots, m_k \in \mathbb{Z} \), and \( \phi : \Omega \to R \) be a genus. Then there exist uniquely determined universal power series \( A_{i,j} \in R[[z]], 1 \leq i \leq j \leq k, \) and \( B_1, \ldots, B_k \in R[[z]], \) and \( C, D \in R[[z]] \) depending only on \( \phi \) and \( m_1, \ldots, m_k \) such that for every smooth projective surface \( X \) and line bundles
L_1, \ldots, L_k on X we have

\begin{equation}
(9) \quad \sum_{n=0}^{\infty} \phi \left( X^{[n]}, \det(L_1^{[n]})^{m_1} \otimes \cdots \otimes \det(L_k^{[n]})^{m_k} \right) z^n
= \exp \left( \sum_{1 \leq i \leq j, 1 \leq k} c_{1}(L_1) c_{1}(L_j) A_{ij} + \sum_{i=1}^{k} c_{1}(L_i) c_{1}(X) B_i + c_1(X)^2 C + c_2(X) D \right).
\end{equation}

Proof. First note that for k = 1 the statement of the theorem is just Theorem 4.2 of [1] for the case of line bundles with Ψ (in the notation of [1]) being the Chern character of the m_1^{th} power of the determinant.

Theorem 4.2 of Ellingsrud, Göttsche and Lehn and the proof presented by them can be easily generalised for more than one bundle, i.e. for k > 1. Therefore, our lemma as a specialization of this generalization is proven. \qed

From the lemma we conclude the following:

Proposition 3. Let φ : Ω → R be a genus. Then there exist uniquely determined universal power series A_φ, B_φ, C_φ, D_φ ∈ R[[z]] depending only on φ such that for every smooth projective surface X together with a line bundle L on it we have

\begin{equation}
(10) \quad \phi(H_{X,L}) := \sum_{n=0}^{\infty} \phi \left( X^{[n]}, L_n \right) z^n
= \exp \left( c_1(L)^2 A_\phi + c_1(L) c_1(X) B_\phi + c_1(X)^2 C_\phi + c_2(X) D_\phi \right).
\end{equation}

Proof. As noted in section 5 of [1], we have

\begin{equation}
(11) \quad L_n = \det(L)_n = \det(L^{[n]}) \otimes \det(\mathcal{O}_X^{[n]})^{-1}.
\end{equation}

Therefore by the previous lemma,

\begin{equation}
(12) \quad \phi(H_{X,L}) = \sum_{n=0}^{\infty} \phi \left( X^{[n]}, \det(L^{[n]}) \otimes \det(\mathcal{O}_X^{[n]})^{-1} \right)
= \exp(c_1(L)^2 A_{11} + c_1(L) c_1(O_X) A_{12} + c_1(O_X)^2 A_{22} + c_1(L) c_1(X) B_1 + c_1(O_X) c_1(X) B_2 + c_1(X)^2 C + c_2(X) D)
\end{equation}

for certain power series A_{i,j}, B_i, C, D independent of X and L. Since c_1(O_X) = 0 this proves the proposition with A_φ = A_{11}, B_φ = B_1, C_φ = C and D_φ = D. \qed

It is possible to express the power series A_φ in terms of genera of Hilbert schemes of points on surfaces:

Proposition 4. Let φ : Ω → R be any genus. For every smooth projective surface X,

\begin{equation}
(13) \quad c_1(X)^2 A_\phi = \frac{1}{2} \ln \frac{\phi(H_X) \phi(-1)(H_X)}{\phi(H_X)^2}.
\end{equation}

Proof. In [1] it is proven that the canonical bundle of X^{[n]} is K_n where K denotes the canonical bundle on X.
Using this we have by proposition 3 that
\[
\ln \phi_m(H_X) = \ln \sum_{n=0}^{\infty} \phi_m(X^{[n]}) z^n = \ln \sum_{n=0}^{\infty} \phi(X^{[n]}, K_n^{-m}) z^n
\]
\[
= m^2 A_\phi c_1(K)^2 - mB_\phi c_1(K)c_1(X) + C_\phi c_1(X)^2 + D_\phi c_2(X)
\]
\[
= (m^2 A_\phi + mB_\phi + C_\phi)c_1(X)^2 + D_\phi c_2(X),
\]
for all integers \(m\), which proves the proposition. \(\square\)

5. Complex genera of the generalised Kummer varieties

In this section we will relate the (generalised) complex genera of Beauville’s generalised Kummer variety to the complex genera of Hilbert schemes of points on surfaces, which we studied in the previous section.

The first step in this direction is the following:

**Proposition 5.** Let \(\phi : \Omega \to R\) be a complex genus with values in the \(\mathbb{Q}\)-algebra \(R\). For every abelian surface \(A\) together with a line bundle \(L\) on it we have
\[
c_1(L)^2 \phi(A^{[n]}, L^{[n]}) = 2n^2 \phi(A^{[n]}, L_n)
\]
for all positive integers \(n\).

**Proof.** We will make use of (2). Recall that \(\nu\) is a Galois covering of degree \(n^4\). By proposition 3 we have
\[
\phi(A, L^n)\phi(A^{[n]}, L^{[n]}) = \phi(A \times A^{[n]}, L^n \boxtimes L^{[n]})
\]
\[
= \phi(A \times A^{[n]}, \nu^* L_n) = n^4 \phi(A^{[n]}, L_n),
\]
which proves the theorem once we have shown that \(\phi(A, L^n) = \frac{n^2}{2} c_1(L)^2\). This follows from the fact that the Chern classes of an abelian surface are trivial:
\[
\phi(A, L^n) = \int_A f_\phi(\gamma_1)f_\phi(\gamma_2)e^{c_1(L^n)} = \int_A \frac{c_1(L^n)^2}{2} = \frac{n^2}{2} c_1(L)^2,
\]
where we used that \(f_\phi\) is a power series with constant coefficient 1. \(\square\)

In [2] M. Britze and the author expressed the (holomorphic) Euler characteristic of the line bundle \(L^{[n]}\) in terms of the Euler characteristic of \(L\) in order to deduce a formula for the Euler characteristic of an arbitrary line bundle \(M\) on \(A^{[n]}\) as a polynomial in the Beauville-Bogomolov quadratic form of \(c_1(M)\). By using the analogous expression of the Euler characteristic of the line bundle \(L_n\) on \(A^{[n]}\) (see [3]) we get the mentioned result of [2] as a corollary of the previous theorem:

**Corollary 1 ([2]).** The holomorphic Euler characteristic of the line bundle \(L^{[n]}\) on \(A^{[n]}\) is given by
\[
\chi(A^{[n]}, L^{[n]}) = n \left( \frac{\chi(A, L) + n - 1}{n-1} \right).
\]

**Proof.** By lemma 5.1 of [3] we have
\[
\chi(A^{[n]}, L_n) = \left( \frac{\chi(A, L) + n - 1}{n} \right).
\]
Using this, the corollary follows from the proposition applied to the case for \(\phi\) being the Todd genus (remember example [3]). Also note that \(\chi(A, L) = \frac{1}{2} c_1(L)^2\) by the Hirzebruch-Riemann-Roch formula. \(\square\)
If we are interested in the usual genera of the generalised Kummer varieties, i.e. the genera of the pairs \((A^{[n]}, O_{A^{[n]}})\), we can’t use proposition 5 directly since for \(L = O_A\) it just states \(0 = 0\).

However, it is still possible to make use of the proposition. We have to look at all generalised Kummer varieties at the same time. Doing so we get the following main result of this work:

**Theorem 1.** Let \(\phi : \Omega \to R\) be a complex genus with values in the \(\mathbb{Q}\)-algebra \(R\). For every smooth projective surface \(X\) with \(\int_X c_1(X)^2 \neq 0\),

\[
\phi(K) = \frac{1}{c_1(X)^2} \left( \frac{d}{dz} \right)^2 \ln \frac{\phi_1(H_X) \phi_{-1}(H_X)}{\phi(H_X)^2}.
\]

**Proof.** Let \(L\) be any line bundle on \(A\). We have

\[
c_1(L)^2 \sum_{n=1}^{\infty} \phi(A^{[n]}, L^{[n]}) z^n = 2 \sum_{n=1}^{\infty} n^2 \phi(A^{[n]}, L_n) z^n = 2 \left( \frac{d}{dz} \right)^2 \phi(H_A, L)
\]

\[
= 2 \left( \frac{d}{dz} \right)^2 \exp \left( c_1(L)^2 A_\phi + c_1(L)c_1(A)B_\phi + c_1(A)^2 C_\phi + c_2(A)D_\phi \right)
\]

\[
= 2 \left( \frac{d}{dz} \right)^2 \exp \left( c_1(L)^2 A_\phi \right) = 2 \left( \frac{d}{dz} \right)^2 c_1(L)^2 A_\phi + O \left( (c_1(L)^2)^2 \right),
\]

which together with proposition 4 proves the theorem, since there are line bundles on \(A\) with \(c_1(L) \neq 0\). \(\square\)

**Remark 3.** Of course, everything still holds true if we replace the abelian surface \(A\) from which we constructed the generalised Kummer varieties, by an arbitrary complex torus of dimension two.
APPENDIX A. THE CHERN NUMBERS OF THE GENERALISED KUMMER VARIETIES OF DIMENSION UP TO 14

We have used \([\star]\) to compute all Chern numbers of the generalised Kummer varieties of dimension up to fourteen. Our results are as follows:

| Chern number | Evaluated on \(A^{[n]}\) | Chern number | Evaluated on \(A^{[n]}\) |
|--------------|----------------------------|--------------|----------------------------|
| \(c^2\)     | 24                         | \(c^2\)     | 421414305792               |
| \(c^2\)     | 756                        | \(c_2^3\)   | 149664301056               |
| \(c_4\)     | 108                        | \(c_3^2\)   | 53149827072                |
| \(c_4^2\)   | 30208                      | \(c_2^3\)   | 18874417152                |
| \(c_2 c_4\) | 6784                       | \(c_4^2\)   | 24230756352                |
| \(c_6\)     | 448                        | \(c_2^2 c_6\) | 8610545664                |
| \(c_2^2\)   | 1470000                    | \(c_6^2\)   | 3059945472                 |
| \(c_2^2 c_4\)| 405000                    | \(c_2^2\)   | 1397121024                 |
| \(c_4^2\)   | 111750                     | \(c_6^3\)   | 1914077184                 |
| \(c_2 c_6\) | 37500                      | \(c_2 c_4 c_8\)| 681332736               |
| \(c_8\)     | 750                        | \(c_6 c_8\) | 110853120                 |
| \(c_2^3\)   | 84478464                   | \(c_2^2 c_10\)| 71909376                |
| \(c_2^2 c_4\)| 26220672                   | \(c_4 c_10\) | 25700352                  |
| \(c_2 c_4^2\)| 8141472                    | \(c_2 c_{12}\)| 1198080               |
| \(c_2^2 c_6\)| 3141504                    | \(c_{14}\)   | 7680                      |
| \(c_4 c_6\) | 979776                      | \(c_2 c_10\) | 441784                   |
| \(c_2 c_8\) | 142560                     | \(c_{12}\)   | 2744                      |
| \(c_{10}\)  | 2592                       |              |                           |

It is remarkable fact that all Chern numbers of the varieties \(A^{[n]}\) with \(n \leq 8\) are positive and divisible by \(n^3\). As the known Chern numbers of Hilbert schemes of points on K3 surfaces are also positive one can wonder if, given a compact Hyperkähler manifold \(X\), all Chern numbers of \(X\) are positive.

REFERENCES

[1] Arnaud Beauville. Variétés Kähleriennes dont la première classe de Chern est nulle. J. Differential Geom., 18(4):755–782, 1983.
[2] Michael Britze and Marc A. Nieper. Hirzebruch-Riemann-Roch formulae on irreducible symplectic Kähler manifolds. arXiv:math.AG/0101062.
[3] Robert Dijkgraaf, Gregory Moore, Erik Verlinde, and Herman Verlinde. Elliptic genera of symmetric products and second quantized strings. Comm. Math. Phys., 185:197–201, 1997.
[4] Geir Ellingsrud, Lothar Göttsche, and Manfred Lehn. On the cobordism class of the Hilbert scheme of a surface. J. Algebraic Geom., 10(1):81–100, 2001.
[5] Geir Ellingsrud and Stein Arild Strømme. On the homology of the Hilbert scheme of points in the plane. *Invent. Math.*, 87:343–352, 1987.

[6] Geir Ellingsrud and Stein Arild Stromme. Bott’s formula and enumerative geometry. *J. Amer. Math. Soc.*, 9:175–193, 1996.

[7] John Fogarty. Algebraic families on an algebraic surface. *Amer. J. Math.*, 90:511–521, 1968.

[8] Lothar Göttsche and Wolfgang Soergel. Perverse sheaves and the cohomology of Hilbert schemes of smooth algebraic surfaces. *Math. Ann.*, 296(2):235–245, 1993.

[9] Daniel Huybrechts and Manfred Lehn. *The Geometry of Moduli Spaces of Sheaves*, volume E31 of *Aspects of Mathematics*. Friedr. Vieweg and Sohn Verlagsgesellschaft mbH, Braunschweig/Wiesbaden, 1997.

[10] Manfred Lehn. Chern classes of tautological sheaves on Hilbert schemes of points on surfaces. *Invent. Math.*, 136:157–207, 1999.

[11] John Milnor. On the cobordism ring $\Omega^*$ and a complex analogue. *Amer. J. Math.*, 82:505–521, 1960.

[12] David Mumford. *Abelian varieties*. Published for the Tata Institute of Fundamental Research, Bombay, 1970. Tata Institute of Fundamental Research Studies in Mathematics, No. 5.

[13] Justin Sawon. Rozansky-Witten invariants of hyperkähler manifolds. PhD thesis, University of Cambridge, October 1999.

Mathematisches Institut der Univ. zu Köln, Weyertal 86–90, 50931 Köln, Germany
E-mail address: mnieper@mi.uni-koeln.de