Lazy Restless Bandits for Decision Making
with Limited Observation Capability:
Applications in Wireless Networks

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Abstract

In this work we formulate the problem of restless multi-armed bandits with cumulative feedback and partially observable states. We call these bandits as lazy restless bandits (LRB) as they are slow in action and allow multiple system state transitions during every decision interval. Rewards for each action are state dependent. The states of arms are hidden from the decision maker. The goal of the decision maker is to choose one of the $M$ arms, at the beginning of each decision interval, such that long term cumulative reward is maximized.

This work is motivated from applications in wireless networks such as relay selection, opportunistic channel access and downlink scheduling under evolving channel conditions.

The Whittle index policy for solving LRB problem is analyzed. In course of doing so, various structural properties of the value functions are proved. Further, closed form index expressions are provided for two sets of special cases; for general cases, an algorithm for index computation is provided. A comparative study based on extensive numerical simulations is presented; the performances of Whittle index policy and myopic policy are compared with other policies such as uniform random, non-uniform random and round-robin.

I. INTRODUCTION

Wireless communication systems often operate in uncertain environments such as rapidly varying channel conditions, relative mobility of communicating nodes. Hence, decision making

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under uncertainty occurs in many applications, for example, the problem of relay selection [2], relay employment in wireless networks [3], opportunistic channel sensing and scheduling [4], [5], downlink scheduling in heterogeneous networks [6].

Let us look at the problem of relay selection in wireless networks. Consider a wireless relay network with a source(S), destination(D) and a set of M relays \( R_i \), \( 1 \leq i \leq M \). The links \( SR_i \) and \( R_i D \) operate on different frequencies. So there are \( M + 1 \) paths or links from source to destination including the direct SD link. The channel quality along each of these paths is time varying. However, the source is unable to observe the exact channel qualities. The time line is divided into intervals. One relay may be selected for use in each interval. A feedback in form of ACK/NACK is received by the source at the end of each interval signifying success or failure of the message transmission. The source has to plan the sequence of relays to be used such that the expected long term throughput is maximized. Thus, the problem involves sequential decision making, where each decision must take into account the information gathered till that instant in the form of ACKs or NACKs from previous transmissions (decisions). Similar sequential decision making is required in applications such as opportunistic scheduling in cognitive radio networks where the decision maker is a secondary user of the network, and downlink scheduling where the decision maker is a base station in a cellular network.

Assuming Markovian variation of channel quality, the above problems can be modeled using restless multi-armed bandits (RMAB). Each source-relay-destination link in a relay network corresponds to an arm; each link can be in one of finite set of states which represent the quality of that link. The reward from using a relay link is the throughput that depends on its state. Also, above scheduling problems can be modeled as RMAB, where each channel corresponds to an arm, state of an arm describes the channel quality and reward for arm play being the throughput that is state dependent. Markovian ON-OFF fading models have been used in literature for formulating these problems in order to account for temporal correlation of channel quality states during decision making [4]–[9]. Although an ON-OFF model is a lossy representation of fading channels, it aids in taking decisions which are inherently of ‘threshold type’. For example, while employing a relay the source might require the end to end signal to noise ratio to cross a certain threshold in order to ensure quality of service to the end user. Such a requirement might be abstractly captured by a two state channel model. We will now talk more about sequential decision problems and restless bandits in the following discussion.
A. Overview of Sequential Decision Problems

Often, sequential decision problems are modeled using Markov decision processes (MDP) \cite{10}, partially observable Markov decision processes (POMDP) \cite{11}, \cite{12} and multi-armed bandits (MAB) \cite{13}, \cite{14}. In these models, environment/system state transition and decision making occur at discrete time instants, uniformly spaced along the time line. The knowledge of system state at these instants, provides information that is necessary for decision making. This knowledge about the system state depends on the observation or feedback about state transition that occurs as a consequence of the previous decision. All the above models assume that, every state transition is either fully or partially observable by the decision maker. This form of information gathering by the decision maker about the consequence of its actions is imperative for all sequential decision models.

In this work we consider a scenario where the information gathering of the decision maker is not at par with the variation of system state. The instants of decision making are sparse compared to the instants of system state transition. The decision maker does not observe every state transition; instead, observation of the system takes place only when a decision needs to be made. This is due to the limited observation capability of the agent/decision maker. We refer to the information gathered by this form of observation as cumulative feedback; it represents the cumulative effect of a series of state transitions. In the multi-armed bandit setting we say that the bandit is lazy in gathering information.

B. Overview of Restless bandits

In a restless multi-armed bandit, each arm has a finite number of states. The play of an arm yields a reward that depends on that arm’s current state. At every decision epoch, the bandit can play a fixed number of arms simultaneously. The evolution of states depends on the bandit’s actions - choice of arms. Further, an arm is called restless as its state evolves even when it is not played. The restless multi-armed bandit (RMAB) was introduced by Whittle in his seminal work \cite{14}.

The model for each arm of an RMAB is an MDP or POMDP. A RMAB can be looked upon as an bunch of decision processes tied together with a joint constraint. In conventional RMAB models, decision processes corresponding to the arms have instantaneous feedback at the end of each decision interval. This means, there is at most one state transition during one play of arm (decision epoch). In the proposed model, the decision processes of arms have cumulative
feedback. This model allows multiple state transitions in one decision epoch. We call restless bandits with cumulative feedback as lazy restless bandits.

In case of an MDP the states are fully observable by the agent; feedback signal gives exact state information. In case of a POMDP the states are partially observable by the agent; feedback signal does not give exact state information. The agent uses the feedback to infer system state. This inference is in the form of a probability distribution over states and is called a belief vector. Further, some structural properties of value functions and optimal policies of MDPs and POMDPs provide insights into solving the RMAB problem. Some structural results for POMDPs can be found in [12], [15]–[17].

In [14], a heuristic index policy is proposed for RMAB, where states are assumed to be fully observable. This index policy is now referred to as Whittle index policy. The key ideas involved in obtaining Whittle index are as follows. 1) Lagrangian relaxation of original optimization problem. 2) This allows to solve single armed bandit problem where a subsidy is assigned for not playing arm. 3) This subsidy will play the role of an index. That is, the arm which requires largest subsidy in order to make not-playing profitable is chosen at each decision epoch.

Initial works focused on RMAB with fully observable states. In [14], Whittle had conjectured that the index policy is asymptotically optimal. This was later proved in [18]. Some variants of RMAB with partially observable states were discussed in [13], [19]–[21]. In later literature, RMAB with partially observable states was a major interest due to its applicability in communication networks, [4], [22], [23] and other areas [24], [25]. In some scenarios, Whittle index policy was shown to be nearly optimal, e.g., [4]. Recently, [9] showed asymptotic optimality of Whittle index policy for the downlink scheduling problem.

In general, finding the optimal solution to a restless multi-armed bandit problem is known to be PSPACE hard, [26]. So, myopic policy for solving RMAB has also been studied. It has been shown to be optimal for some scenarios, [5], [7], [8], [27].

RMABs have been used for various applications in multiple domains. Some specific specific applications include recommendation systems [25], sensor scheduling and target detection [28], multi-UAV routing for observing targets [24], stochastic network optimization [23]. Most models assume an instantaneous feedback and their main interest is to study the Whittle index or myopic policy. An alternative index policy called as marginal productivity index was studied by [22], [29]. Marginal productivity index here, is an extension of Whittle index with interpretations from marginal productivity theory used in economics.

A common assumption in the above literature that deals with optimality of Whittle index or
myopic policies is the full observability of arm states when played. That is, more information gathered by playing an arm allows more general inferences on performance of various policies. This additional information also makes computation of Whittle index expressions easier. In recent work of [30]–[32], hidden Markov restless multi-armed bandit has been studied and Whittle index policy is used. This model assumes that arm state is never fully observable but only binary signals corresponding to each state transition are observed. In these models, partial observability of states makes it cumbersome to prove indexability of arms and also to obtain close form index expressions. Recall that in the current work we allow multiple state transition in each decision interval; hence, information about arms states is even more sparser. This makes claiming indexability and deriving index expressions intractable in general. However, when the number of state transitions are quite large the tractability can be improved by making some assumptions (see Section III). In this work, we also provide results for any finite number of state transitions per decision interval.

C. Contributions

1) We formulate the problem of restless multi-armed bandits with cumulative feedback and hidden states. We call them as lazy restless bandits. This work is motivated from applications in wireless networks such as (1) relay selection or employment, (2) opportunistic channel sensing and access in cognitive networks and (3) scheduling in downlink heterogeneous networks. We assume that channel states are hidden from the decision maker unlike previous works where a channel’s state becomes exactly known when it is used [4], [6].

2) The proposed cumulative feedback model allows multiple system state transitions between consecutive decision epochs. This is unlike previous models which allow atmost one transition which is observable when the channels are used. Hence, our model better represents fast-fading channel conditions. We present the system model description in Section II where we formulate an optimization problem for lazy restless bandits.

3) We first consider the lazy single-armed bandit problem in Section III. We obtain the structural results for both positively and negatively correlated channel model. Further, we show that for single armed bandit the threshold type policy is optimal. Proving an optimal threshold policy is rendered cumbersome due to partially observability states. Hence, we need some restrictions on transition probabilities. Using these results, we show the Whittle indexability of lazy restless bandits under some restrictions on discount parameter.

4) Using threshold policy and other structural properties, we derive the closed form expression of Whittle index for two special cases in Section IV. Further, we present the Whittle index
computation algorithm for general settings. This algorithm is based on a two-timescales stochastic approximation scheme.

5) In Section V, we provide an extensive comparative study of the Whittle index policy with other policies such as myopic, uniform random, non-uniform random and round robin.

6) Finally, we conclude with a discussion about various aspects of the problem and some issues that are left unresolved, in Section VI.

II. MODEL DESCRIPTION AND PRELIMINARIES

Let us consider a lazy restless multi-armed bandit with $M$ independent arms. The timeline is divided into sessions that are indexed by $s$. Each arm represents a channel/link in a communication system. We model each channel using Gilbert-Elliot model. In this model each channel has two states, say, good (1) and bad (0). In any arbitrary session, each arm exists in one of the two states. $Y_m(s) \in \{0, 1\}$ denotes the state of arm $m$ at the beginning of session $s$. We assume that multiple state transitions occur during a given session. Let $K > 1$ be number of state transitions for each arm in a given session. The state of each arm evolves according to a Markov chain. $p_{i,j}^m$ represents the transition probability of arm $m$ from state $i$ to state $j$, $i, j \in \{0, 1\}$ and the corresponding transition probability matrix (TPM) is denoted by $P_m = [p_{i,j}^m]$. In a given session $s$, the decision maker plays one arm out of $M$ arms. $A_m(s)$ denotes the action corresponding to arm $m$ in session $s$. If arm $m$ is played in session $s$, then $A_m(s) = 1$ and $A_m(s) = 0$, otherwise. Since only one arm is played in a session, $\sum_{m=1}^{M} A_m(s) = 1$.

At the end of each session a feedback is received by the decision maker in the form of ACK(1) or NACK(0) from the arm that is played. An ACK means a successful session and a NACK means a failed session. $Z_m(s) \in \{0, 1\}$ denotes the feedback signal that is obtained at end of session $s$ if arm $m$ is played in session $s$. This feedback is probabilistic, and we define $\rho_{m,i} := \Pr\{Z_m(s) = 1 \mid A_m(s) = 1, Y_m(s) = i\}$, $i \in \{0, 1\}$. We also assume that $\rho_{m,0} < \rho_{m,1}$. No feedback is obtained from an arm when it is not played.

A reward is accrued from arm $m$, if that arm is played in a session $s$. It depends on states of the arm, $Y_m(s)$. We denote $R_{m,i}$ as the average reward from arm $m$ if it is played and in a session beginning with state $i$. No reward is accrued if arm $m$ is not played.

We assume that the exact state of each arm is not observable by the decision maker. The decision maker maintains the belief about the state of each arm. Let $\pi_m(s)$ the probability that arm $m$ is in state 0 at the beginning of session $s$ given the history $H(s)$, where $H(s) = \{A(l), Z(l)\}_{1 \leq l < s}$. Thus $\pi_m(s) := \Pr(Y_m(s) = 0 \mid H(s))$. The belief $\pi_m(s)$ about arm $m$, is
Arm $m$ is not played in session $s$ ($A_m(s) = 0$)

Arm $m$ is played ($A_m(s) = 1$)

Fig. 1. The state transition probabilities, the reward, and the probability of ACK (1) being observed are illustrated above when the arm is not played. Also, the corresponding quantities are illustrated below when the arm is played.

updated by the decision maker at the end of every session $s$, based on the action taken $A_m(s)$ and feedback received $Z_m(s)$.

Let $\phi := \{\phi(s)\}_{s\geq 0}$ be the policy, where $\phi(s): H_s \to \{1, \cdots, M\}$ maps the history up to session $s$ to action of playing one of the $M$ arms. Let $A^\phi_m(s) = 1$, if $\phi(s) = m$, and $A^\phi_m(s) = 0$, if $\phi(s) \neq m$. The infinite horizon expected discounted reward under policy $\phi$ is given by

$$V^\phi(\pi) := E \left\{ \sum_{s=1}^{\infty} \beta^{s-1} \sum_{m=1}^{M} A^\phi_m(s) (\pi_m(s)R_{m,0} + (1 - \pi_m(s))R_{m,1}) \right\}. \quad (1)$$

Here, $\beta$ is discount parameter, $0 < \beta < 1$ and the initial belief $\pi = [\pi_1, \cdots, \pi_M]$, $\pi_m := \text{Pr}(Y_m(1) = 0)$. Our objective is to find the policy $\phi$ that maximizes $V^\phi(\pi)$ for all $\pi \in [0, 1]^M$.

In [14], Lagrangian relaxation of this problem is analyzed via introducing subsidy, it is payoff for not playing the arm. The approach to obtain the solution of the relaxed problem by first studying the single-armed restless bandit.

III. Single-armed Lazy Restless Bandit

We consider a subsidy $\eta$ is assigned if the arm is not played. As we are dealing with a single arm, we drop notation $m$ for convenience. In the view of subsidy $\eta$ one can reformulate problem in (1) for single-armed bandit as follows.

$$V^\phi(\pi) := E \left\{ \sum_{s=1}^{\infty} \beta^{s-1} \left( A^\phi(s) (\pi(s)R_0 + (1 - \pi(s))R_1) + \eta(1 - A^\phi(s)) \right) \right\}. \quad (2)$$

The goal is to find the policy $\phi$ that maximizes $V^\phi(\pi)$ for $\pi \in [0, 1]$, $\pi$ is the initial belief.
We now describe the belief update rule and it plays an important role in obtaining properties of the value function.

1) If a channel is used for transmission in session $s$ and ACK is received, i.e., $A(s) = 1$ and $Z(s) = 1$, then the belief at the beginning of session $s + 1$ is $\pi(s + 1) = \gamma_1(\pi(s))$. Here,

$$\gamma_1(\pi(s)) := \frac{(1 - \pi(s))\rho_1 p_{1,0} + \pi(s)\rho_0 p_{0,0}}{\rho_1 (1 - \pi(s)) + \rho_0 \pi(s)}.$$

2) If a channel is used for transmission in session $s$ and NACK is received, i.e., $A(s) = 1$ and $Z(s) = 0$, then the belief at the beginning of session $s + 1$ is $\pi(s + 1) = \gamma_0(\pi(s))$, where

$$\gamma_0(\pi(s)) := \frac{(1 - \pi(s))(1 - \rho_1)p_{1,0} + \pi(s)(1 - \rho_0)p_{0,0}}{(1 - \rho_1)(1 - \pi(s)) + (1 - \rho_0)\pi(s)}.$$

3) If a channel is not used for transmission, i.e., $A(s) = 0$, then the belief at the beginning of session $s + 1$ is $\pi(s + 1) = \gamma_2(\pi(s))$, where

$$\gamma_2(\pi(s)) := (p_{0,0} - p_{1,0})^K \pi + p_{1,0} \frac{(1 - (p_{0,0} - p_{1,0})^K)}{1 - (p_{0,0} - p_{1,0})^K}.$$ (3)

This is because the channel is evolving independently, after $K$ transitions of channel state, we obtain belief as given in the expression (3).

Note that whenever $p_{0,0} > p_{1,0}$, the arm is said to be positively correlated and when $p_{0,0} < p_{1,0}$, it is negatively correlated. While deriving the following results we assume that $\rho_0 < \rho_1$ and $R_0 < R_1$.

**Lemma 1:**

1) For fixed $\eta$, $V_S(\pi)$, $V_{NS}(\pi)$ and $V(\pi)$ are convex in $\pi$.

2) For fixed $\pi$, $V_S(\pi, \eta)$, $V_{NS}(\pi, \eta)$ and $V(\pi, \eta)$ are non-decreasing and convex in $\eta$.

The proof of this lemma is similar to [30, Lemma 2], due to space constraints we omit details of the proof.

We first provide structural results for positively correlated arm. Later we study the negatively correlated arm using similar procedures.

**A. Positively correlated arm**

**Lemma 2:** For positively correlated arm, i.e., $p_{0,0} > p_{1,0}$, the belief updates $\gamma_0(\pi)$, $\gamma_1(\pi)$ and $\gamma_2(\pi)$ are increasing in $\pi$. Further, $\gamma_1(\pi)$ and $\gamma_0(\pi)$ are convex and concave, respectively. Also, $p_{1,0} \leq \gamma_1(\pi) \leq \gamma_0(\pi) \leq p_{0,0}$.

The proof is straightforward; we omit it here due to space constraints.
The proof can be found in the Appendix B.

We next derive structural statistic for constructing such policies and the optimal value function can be determined by solving following dynamic program.

\[
V_S(\pi) = R_S(\pi) + \beta (\rho(\pi)V(\gamma_1(\pi)) + (1 - \rho(\pi))V(\gamma_0(\pi)))
\]

\[
V_{NS}(\pi) = \eta + \beta V(\gamma_2(\pi))
\]

\[
V(\pi) = \max\{V_S(\pi), V_{NS}(\pi)\}.
\]

(4)

Remark 1: We can see from the expression of \( \gamma_2 \) in Eqn. (3) that for fixed value of \( \pi \), as \( K \rightarrow \infty \), we get \( \gamma_2(\pi) \rightarrow q \), where \( q = \frac{p_{1,0}}{1 - (p_{0,0} - p_{1,0})} \). The rate of convergence of \( \gamma_2 \) to \( q \) depends on \( (p_{0,0} - p_{1,0}) \). This suggests that for large values of \( K \), we can approximate \( \gamma_2(\pi) \) with \( q \). If \( |p_{0,0} - p_{1,0}| \) is smaller, then \( K \) required for this approximation is small. In Table. I we present few examples where 1) \( |p_{0,0} - p_{1,0}| = 0.5 \), then \( \gamma_2(\pi) \approx q \) for \( K = 10 \), and 2) \( |p_{0,0} - p_{1,0}| = 0.2 \), then \( \gamma_2(\pi) \approx q \) for \( K = 5 \).

We now seek a stationary deterministic policy. From [15], [33], we know that \( \pi(s) \) is a sufficient statistic for constructing such policies and the optimal value function can be determined by solving following dynamic program.

Table I

| \( p_{0,0} \) | \( p_{1,0} \) | \( \rho_0 \) | \( \rho_1 \) | \( K \) | \( \gamma_2(\pi) \) | \( q \) |
|---|---|---|---|---|---|---|
| 0.9 | 0.4 | 0 | 0.95 | 10 | 0.80 | 0.8 |
| 0.95 | 0.45 | 0 | 0.95 | 10 | 0.9 | 0.9 |
| 0.8 | 0.3 | 0.2 | 0.95 | 10 | 0.6 | 0.6 |
| 0.8 | 0.6 | 0.2 | 0.95 | 5 | 0.75 | 0.75 |
| 0.5 | 0.3 | 0.1 | 0.9 | 5 | 0.375 | 0.375 |

Remark 1: We can see from the expression of \( \gamma_2 \) in Eqn. (3) that for fixed value of \( \pi \), as \( K \rightarrow \infty \), we get \( \gamma_2(\pi) \rightarrow q \), where \( q = \frac{p_{1,0}}{1 - (p_{0,0} - p_{1,0})} \). The rate of convergence of \( \gamma_2 \) to \( q \) depends on \( (p_{0,0} - p_{1,0}) \). This suggests that for large values of \( K \), we can approximate \( \gamma_2(\pi) \) with \( q \). If \( |p_{0,0} - p_{1,0}| \) is smaller, then \( K \) required for this approximation is small. In Table. I we present few examples where 1) \( |p_{0,0} - p_{1,0}| = 0.5 \), then \( \gamma_2(\pi) \approx q \) for \( K = 10 \), and 2) \( |p_{0,0} - p_{1,0}| = 0.2 \), then \( \gamma_2(\pi) \approx q \) for \( K = 5 \).

We now seek a stationary deterministic policy. From [15], [33], we know that \( \pi(s) \) is a sufficient statistic for constructing such policies and the optimal value function can be determined by solving following dynamic program.

\[
V_S(\pi) = R_S(\pi) + \beta (\rho(\pi)V(\gamma_1(\pi)) + (1 - \rho(\pi))V(\gamma_0(\pi)))
\]

\[
V_{NS}(\pi) = \eta + \beta V(\gamma_2(\pi))
\]

\[
V(\pi) = \max\{V_S(\pi), V_{NS}(\pi)\}.
\]

Here \( R_S(\pi) = \pi R_0 + (1 - \pi)R_1 \), and \( \rho(\pi) = \pi \rho_0 + (1 - \pi) \rho_1 \). \( V_S \) and \( V_{NS} \) are the action value functions for playing the arm and not-playing the arm, respectively. We next derive structural results for these value functions.

Lemma 3:

1) For a positively correlated arm with a fixed subsidy \( \eta, \beta \in (0, 1) \), the value functions \( V(\pi) \), \( V_S(\pi) \) and \( V_{NS}(\pi) \) are decreasing in \( \pi \).

See Appendix A for proof.

Lemma 4: For fixed subsidy \( \eta \), and \( p_{0,0} > p_{1,0} \). The difference in value function \( (V_S(\pi) - V_{NS}(\pi)) \) is decreasing in \( \pi \) for any of the following conditions

1) For large \( K \), i.e. \( \gamma_2(\pi) \approx q \).

2) For any \( K > 1 \), when, \( 0 < p_{0,0} - p_{1,0} < \frac{b}{5} \) and \( \beta \in (0, 1) \),

3) For any \( K > 1 \), when, \( \beta \in (0, b/5) \), where, \( b = \min\left\{ 1, \frac{R_1 - R_0}{\rho_1 - \rho_0} \right\} \).

The proof can be found in the Appendix B.
**Remark 2:**

- \( V_S(\pi) - V_{NS}(\pi) \) gives the advantage of playing over not playing, when the belief is \( \pi \).
- For large \( K \), the approximation \( \gamma_2(\pi) = q \) makes \( V_{NS}(\pi) \) independent of \( \pi \). Further, as \( V_S \) is decreasing in \( \pi \), the result is easy to claim.
- For an arbitrary \( K \), we claim a decreasing advantage of playing by imposing conditions on either the transition probabilities or the discount factor (see Appendix B2).

**B. Negatively correlated arm**

**Lemma 5:** For negatively correlated arm, i.e., \( p_{1,0} > p_{0,0} \), the belief updates \( \gamma_0(\pi), \gamma_1(\pi) \) are decreasing in \( \pi \). Further, \( \gamma_1(\pi) \) and \( \gamma_0(\pi) \) are concave and convex, respectively. Also, \( p_{0,0} \leq \gamma_0(\pi) \leq \gamma_1(\pi) \leq p_{1,0} \).

The proof is straightforward; we omit it here due to space constraints.

**Lemma 6:** For fixed subsidy \( \eta \), and \( p_{1,0} > p_{0,0} \). The difference in value function \( (V_S(\pi) - V_{NS}(\pi)) \) is decreasing in \( \pi \) under any of the following conditions

1) For any \( K > 1 \), when, \( 0 < p_{1,0} - p_{0,0} < \frac{\beta}{5} \) and \( \beta \in (0, 1) \),
2) For any \( K > 1 \), when, \( \beta \in (0, b/5) \), where, \( b = \min \left\{1, \frac{R_1-R_0}{\rho_1-\rho_0}\right\} \).

**Remark 3:**

- Note that for negatively correlated arms, \( V_S \) is not necessarily decreasing in \( \pi \), unlike their positively correlated counterparts. Hence, it is difficult to prove that \( V_S - V_{NS} \) is decreasing in \( \pi \), even for large \( K \), i.e. \( \gamma_2(\pi) \approx q \).
- However, the same procedure used for proving Lemma 4 works here; hence we omit the proof details.

The key ideas involved in proving this result are as follows.

1) We first bound the derivatives of \( V_S, V_{NS} \) and \( V \) w.r.t. \( \pi \), (see Lemma 8 in Appendix B2). This is also called as the Lipschitz property of the value functions.
2) Then, we show that the derivative of \( V_S - V_{NS} \) w.r.t. \( \pi \) is negative under the given conditions.

- One might consider the right partial derivatives at points where any of the functions are non-differentiable.

**C. Threshold policy and Indexability**

We now define a threshold type policy and we will show that the optimal policy is threshold type for single armed bandit.
**Definition 1:** A policy is called as a threshold type for single armed bandit if there exists $\pi_T \in [0, 1]$ such that an optimal action is to play the arm if $\pi \leq \pi_T$ and to not play the arm if $\pi \geq \pi_T$.

**Theorem 1:** For fixed subsidy $\eta$, $\beta \in (0, 1)$, the optimal policy for single-armed bandit is of a threshold type for each of the following conditions.

1) If $K$ is large i.e., $\gamma_2(\pi) \approx q$.
2) For any $K \geq 1$, if $0 < p_{0,0} - p_{1,0} < b/5$.
3) For any $K \geq 1$, if $0 < p_{1,0} - p_{0,0} < b/5$.
4) For any $\beta \in (0, b/5)$, where, $b = \min \left\{ 1, \frac{R_1 - R_0}{\rho_1 - \rho_0} \right\}$.

**Proof:** From the preceding Lemma 4 and Lemma 6, we know that $(V_s(\pi) - V_{NS}(\pi))$ is a decreasing in $\pi$. Further, $V_S(\pi)$ and $V_{NS}(\pi)$ are convex in $\pi$. This implies that there exists a either $\pi_T \in [0, 1]$ such that $V_S(\pi_T) = V_{NS}(\pi_T)$ or $V_S(\pi) > V_{NS}(\pi)$ for all $\pi$, or $V_S(\pi) < V_{NS}(\pi)$ for all $\pi$. This leads to desired result.

We here define the indexability and will show that a single-armed bandit is indexable. Using exact threshold-type policy result, we define the following.

$$\mathcal{P}_\beta(\eta) := \{\pi \in [0, 1] : V_S(\pi, \eta) \leq V_{NS}(\pi, \eta)\}.$$

It is a set of belief state $\pi$ for which the optimal action is to not to play the arm, i.e., $A(s) = 0$. From [14], we state the definition of indexability.

**Definition 2:** A single-armed restless bandit is indexable if $\mathcal{P}_\beta(\eta)$ is monotonically increases from $\emptyset$ to entire state space $[0, 1]$ as $\eta$ increases from $-\infty$ to $\infty$, i.e., $\mathcal{P}_\beta(\eta_1) \setminus \mathcal{P}_\beta(\eta_2) = \emptyset$ whenever $\eta_1 \leq \eta_2$.

To show indexability, we require to prove that a threshold $\pi_T$ as function of $\eta$ is monotonically increasing. We state the following lemma from [30].

**Lemma 7:** Let $\pi_T(\eta) = \inf\{\pi \in [0, 1] : V_S(\pi, \eta) = V_{NS}(\pi, \eta)\}$, if $\left.\frac{\partial V_S(\pi, \eta)}{\partial \eta}\right|_{\pi = \pi_T(\eta)} < \left.\frac{\partial V_{NS}(\pi, \eta)}{\partial \eta}\right|_{\pi = \pi_T(\eta)}$, then $\pi_T(\eta)$ is monotonically decreasing function of $\eta$.

Note that the value function may not be differentiable as function of $\eta$, in that case it should taken as right partial derivative. It exists due to convexity of value function in $\eta$ and rewards are bounded.

We now use Definition 2 and Lemma 7 to show that a single-armed restless bandit in our setting is indexable.

**Theorem 2:** If $\rho_0 < \rho_1$, and $\beta \in (0, 1/3)$, then, a single-armed restless bandit is indexable.

Proof can be found in the Appendix C.
Remark 4: We believe that the indexability result is true more generally, where, we do not require any assumption on $\beta$. This restriction on $\beta$ is required here because of difficulty in obtaining closed-form value function expression. But, for specific conditions such as $\rho_0 = 0, \rho_1 = 1$, and $K > 1$, we can derive the closed-form expressions of value functions and we can obtain conditions for indexability without any assumption on $\beta$.

IV. Whittle Index Calculations for Special Cases

We first define the Whittle index and later we provide index formula.

Definition 3 ([14]): If an indexable arm is in state $\pi$, its Whittle index $W(\pi)$ is

$$ W(\pi) = \inf\{\eta \in \mathbb{R} : V_{S,\beta}(\pi, \eta) = V_{NS,\beta}(\pi, \eta)\}. \tag{5} $$

In order to compute the index, we require to obtain the value function expressions at each threshold $\pi$ and solve it for subsidy $\eta$. Solving these equations, we get desired index for that $\pi$. The basic idea used for this computation is as follows. For a given belief $\pi$, assume that there is a threshold. Then, we know that for $\pi' > \pi$, $V(\pi') = V_{NS}(\pi')$ and for every $\pi' < \pi$, $V(\pi') = V_{S}(\pi')$. functions $\gamma$’s.

We provide expressions of Whittle index for positively correlated arms, i.e., $p_{0,0} > p_{1,0}$. Further, we do this for two special cases.

1) Arbitrary $K$, $\rho_0 = 0$ and $\rho_1 = 1$.

2) $K$ is large, i.e., $\gamma_2(\pi) \approx q$, $R_0 = \rho_0 = 0$, and $0 < R_1 = \rho_1 < 1$.

For other cases we provide an algorithm to compute the index. It is motivated from stochastic approximation algorithms. We consider four intervals, $A_1, A_2, A_3$, and $A_4$, as shown in Fig. 2, we compute the index for each interval separately. We make use of properties of $\gamma_0$, $\gamma_1$ and $\gamma_2$.

![Fig. 2. The different cases to calculate $W(\pi)$.

A. Whittle index for Example 1: arbitrary $K$, $p_{0,0} > p_{1,0}$, $\rho_0 = 0, \rho_1 = 1$.

1) For $\pi \in A_1$,

$$ W(\pi) = R_S(\pi) = R_1 + \pi(R_0 - R_1). $$
2) For $\pi \in A_2$, 
\[ W(\pi) = \frac{R_S(\pi)(1 - \beta)[1 - \beta(\pi - p_{1,0})]}{1 - \beta[1 + (1 - \beta)(\pi - p_{1,0})]} . \]

3) For $\pi \in A_3$, 
\[ W(\pi) = \frac{D(\pi) - \beta D(\gamma_2(\pi))}{1 + \beta B(\gamma_2(\pi)) - B(\pi)} . \]

where, 
\[ B(\pi) = \beta c[\pi(1 - b) + b] , \text{ and } D(\pi) = R_S(\pi) + \beta(1 - \pi)(a + bd + \pi d) \]
\[ a = \frac{R_S(p_{1,0})}{1 - \beta(1 - p_{1,0})}, \quad b = \frac{\beta p_{1,0}}{1 - \beta(1 - p_{1,0})}, \quad a_1 = \frac{\beta^t R_S(\gamma_2^t(p_{0,0}))}{1 - \beta^{t+1}\gamma_2^t(p_{0,0})}, \quad b_1 = \frac{\beta^{t+1}(1 - \gamma_2^t(p_{0,0}))}{1 - \beta^{t+1}\gamma_2^t(p_{0,0})} \]
\[ c = \frac{f}{1 - bb_1}, \quad d = \frac{a_1 + b_1a}{1 - bb_1}, \quad f = \frac{1 - \beta^t}{(1 - \beta)(1 - \beta^{t+1}\gamma_2^t(p_{0,0}))} . \]

4) For $\pi \in A_4$, 
\[ W(\pi) = m\pi + c_1 - \beta(m\gamma_2(\pi) + c_1); \quad m = \frac{R_0 - R_1}{1 - \beta(p_{0,0} - p_{1,0})}, \quad c_1 = \frac{R_1 + m\beta p_{1,0}}{1 - \beta} . \]

The derivations for above expressions can be found in Appendix D.

B. Whittle index for Example 2: Large $K$, $\gamma_2(\pi) \approx q$.

Here, we assume that $p_{0,0} > p_{1,0}$, and $K$ is large, that is $\gamma_2(\pi) \approx q$, $R_0 = \rho_0 = 0$, and $0 < R_1 = \rho_1 < 1$.

The index formula for each interval is given as follow.

1) For $\pi \in A_1$, the Whittle index $W(\pi) = \rho(\pi)$.

2) For $\pi \in A_2$, we consider following cases.

   a) if $\gamma_0(p_{1,0}) \geq \pi$, then Whittle index is
   \[ W(\pi) = \frac{\rho(\pi)}{1 - \beta(\rho(p_{1,0}) - \rho(\pi))} . \]

   b) if $\gamma_0(p_{1,0}) < \pi$ but $\gamma_0^2(p_{1,0}) \geq \pi$ then Whittle index $W(\pi) = \frac{\rho(\pi)}{C_1}$. Here,
   \[ C_1 = 1 - \beta(\rho(p_{1,0}) - \rho(\pi)) - \beta^2(\rho(\gamma_0(p_{1,0}) - \rho(\pi))) + \beta^3(\rho(\gamma_0(p_{1,0}))\rho(p_{1,0})) . \]

3) For $\pi \in A_3$, obtaining index is tedious, and this has to be computed numerically by using Algorithm D.

4) For $\pi \in A_4$ the Whittle index is.

\[ W(\pi) = m\pi(1 - \beta(p_{0,0} - p_{1,0})) + (1 - \beta)c - \beta p_{1,0}m, \]

where \[ m = \frac{-\rho_1}{1 - \beta(p_{0,0} - p_{1,0})}, \quad \text{and} \quad c = \frac{\rho_1 + \beta p_{0,0} - p_{1,0}}{1 - \beta} . \]

The derivations for above expression can be found in Appendix E.
C. Algorithm for Whittle index computation

We now present an algorithm for computing Whittle index in a general case. Here, for a given \( \pi \in [0, 1] \), assume that it is the threshold and compute index \( W(\pi) \). Start at \( t = 0 \) with an initial subsidy \( \eta_0 \) and run the value iteration algorithm to compute action value functions \( V_S(\pi, \eta_0) \) and \( V_{NS}(\pi, \eta_0) \). The subsidy \( \eta_0 \) incremented or decremented proportionally with the difference \( V_S(\pi, \eta_0) - V_{NS}(\pi, \eta_0) \) and a learning parameter \( \alpha \), as follows

\[
\eta_{t+1} = \eta_t + \alpha(V_S(\pi, \eta_t) - V_{NS}(\pi, \eta_t)).
\]

The algorithm terminates when the difference \(|V_S(\pi, \eta_t) - V_{NS}(\pi, \eta_t)| < h\), where \( h \) is the tolerance limit. See Algorithm 1 below for details. Here, we use two timescales, one for updating the subsidy and the other for updating value functions. The \( \alpha \) parameter is chosen such that, subsidy \( \eta_t \) is updated at a slower timescale compared to the value iteration algorithm that computes \( V_S(\pi, \eta_t) \) and \( V_{NS}(\pi, \eta_t) \). This is a two-timescales stochastic approximation algorithm and is based on similar schemes studied in [34], [35]. In [34, Chapter 6], the convergence of a two-timescales stochastic approximation algorithm was discussed.

\begin{algorithm}
\caption{Whittle index computation for single arm}
\begin{algorithmic}
\State \textbf{Input:} Reward values \( R_0, R_1 \); Initial subsidy \( \eta_0 \), tolerance \( h \), step size \( \alpha \).
\State \textbf{Output:} Whittle’s index \( W(\pi) \)
\For {\( \pi \in [0, 1] \)}
\State \( \eta_t \leftarrow \eta_0 \);
\While {\( |V_S(\pi, \eta_t) - V_{NS}(\pi, \eta_t)| > h \)}
\State \( \eta_{t+1} = \eta_t + \alpha(V_S(\pi, \eta_t) - V_{NS}(\pi, \eta_t)) \);
\State \( t = t + 1 \);
\State compute \( V_S(\pi, \eta_t), V_{NS}(\pi, \eta_t) \);
\EndWhile
\EndFor
\State \textbf{return} \( W(\pi) \leftarrow \eta_t \);
\end{algorithmic}
\end{algorithm}

V. Numerical Simulations and Discussion

We now present a few numerical examples and compare different policies that are used to solve partially observable RMAB. The policies that compared are 1) Whittle index policy (WI)—plays the arm with highest Whittle index in each session, 2) myopic policy (MP)—plays arm with highest immediate expected reward in each session, 3) uniformly random (UR), 4) non-uniform
random (NUR)– plays arm randomly with distribution derived from current belief and 5) round robin (RR)– plays arm in round robin order.

MATLAB was used for performing simulations. In these simulations, the arms start in a random state with a given initial belief about the state of the arm. In each session one arm is played according to the given policy of study. Reward from the played arm is accumulated stored at the end of each session. Later, these rewards are averaged over \( L \) iterations (sample paths of states).

We shall compare the discounted cumulative rewards obtained from each of the policies as a function of session number. Another parameter of interest while comparing various policies is the arm choice fraction which is defined as follows. Let \( 1_{m,s,l} \) be the indicator variable if arm \( m \) is played in session \( s \), and \( l \)th iteration. Then 
\[
N_{m,l} := \frac{1}{S_{\text{max}}} \sum_{s=1}^{S_{\text{max}}} 1_{m,s,l},
\]
where \( S_{\text{max}} \) number of sessions for which simulations are performed. Further, this fraction is averaged over \( L \) iterations. We call this as the choice fraction of arm \( m \) corresponding to the policy under study.

We illustrate five numerical examples and use discount parameter \( \beta = 0.99 \). In first three examples, we assume that \( K \) is large, i.e., \( \gamma_2(\pi) \approx q \); and use \( R_{m,0} = \rho_{m,0} = 0 \), \( R_{m,1} = \rho_{m,1} \) for \( 1 \leq m \leq M \). For the last two examples we have a more general setting. We compare \% value gain of various policies with uniform random policy as the baseline.

### A. Example-1 : Arms with similar reward structure and stationary behaviour

In this scenario, all the arms have identical reward from play of that arm and \( K \) is large. Also, all the arms have same \( q_m = 0.45 \), except for arm 9, i.e. \( q_9 = 0.4 \). We use following set of parameters: \( \rho_0 = 0 \), \( \rho_1 = 0.9 \), \( p_{0,0} = [0.45, 0.5, 0.51, 0.57, 0.63, 0.66, 0.69, 0.75, 0.78, 0.87] \) and \( p_{1,0} = [0.45, 0.41, 0.4, 0.35, 0.3, 0.28, 0.25, 0.2, 0.15, 0.1] \).

In Fig. (3a) we can see the discounted cumulative reward as function of session number, plotted for various policies. As shown, the discounted cumulative reward obtained by Whittle index policy (WI) is higher than that of the myopic policy (MP). We also observe that WI and MP yield higher discounted cumulative reward compared to that of random and round robin policies. In Fig. (3b), we can see arm choice fractions of all arms under different policies. Notice the tendency of Whittle index policy to prefer a smaller subset of arms, \{9, 10\} as compared to other policies. This behavior of WI might be because it accounts for future rewards through the action value functions. This is also determined by channel characteristics, i.e., \( p_{0,0} \), and \( p_{1,0} \), where we observe that the difference \( (p_{0,0} - p_{1,0}) \) is very large for arms 9 and 10 compared to other arms. Myopic policy, plays arm 9 more frequently than other arms. This is because belief
a) Discounted cumulative reward  

b) Arm choice fraction

Fig. 3. Example-1: a) discounted cumulative rewards as function of sessions for different policies and b) arm choice fraction for each arm with different policies. This is plotted for identical reward for all arms $\rho_1 = 0.9$ and identical $q_m = 0.45$ for $1 \leq m \leq M, m \neq 9$ and $q_9 = 0.4$.

about arm 9 reaches state $q_9 = 0.4$ when it is not played, while belief about other arms that are not played reach $q = 0.45$. Recall that immediate expected reward from each session is state dependent and is decreasing in state ($\pi$). Since rewards are identical for all arms, myopic policy plays arm 9. Another reason for myopic policy to play arm 9 is that $(p_{0,0} - p_{1,0})$ is large. If arm 9 is played and session is successful then state reaches to $p_{1,0} = 0.15$, that means it is more likely to be in good state and hence it will be played again. Whereas for other policies all the arms are played equally often and hence it leads to smaller discounted cumulative reward.

**TABLE II**

**Example-1:** % Value gain relative to uniform-random policy for different initial beliefs $\pi(1)$.

| $\pi(1) =$ | $p_{10}$ | $q$ | $p_{00}$ | random |
|-----------|----------|-----|----------|--------|
| WP        | 34.0     | 33.88 | 34.49    | 31.92  |
| MP        | 25.28    | 25.26 | 25.61    | 23.91  |
| NUR       | 1.17     | 1.61  | 1.75     | 0.88   |
| RR        | 1.0      | 0.76  | 0.33     | 0.16   |

Table II gives the performances of various algorithms for different values of initial beliefs. Notice, the slight decrease in gain for all policies in case of random initial beliefs. This might be caused by initial beliefs which are outside the region $[p_{1,0}, p_{0,0}]$. If the initial belief falls in this region, it stays it stays here; if it falls outside, it is later “pulled” inside this region by the $\gamma$’s.
Fig. 4. Example-2: a) The discounted cumulative reward versus session number for different policies and b) arm choice fraction for each arm with different policies. This is plotted for identical reward for all arms $\rho_i = 0.9$ but different $q_m$ for each arm.

### B. Example-2: Arms with identical reward structure and nonidentical behavior

In this example we consider that all arms has identical reward structure $\rho_0 = 0, \rho_1 = 0.9$, but, different values of $q_m$. Also, $K$ is large. We use, $p_{0,0} = [0.5, 0.45, 0.45, 0.78, 0.6, 0.6, 0.7, 0.7, 0.4, 0.45]$, $p_{1,0} = [0.41, 0.4, 0.35, 0.15, 0.55, 0.5, 0.6, 0.3, 0.25]$, with $q = [0.45, 0.42, 0.38, 0.40, 0.57, 0.55, 0.62, 0.66, 0.33, 0.31]$. In Fig. 4a) we can see the discounted cumulative reward as function of session number, plotted for various policies. Again, as expected WI policy obtains higher discounted cumulative reward than all other policies. In this case, the myopic policy also performs well and has a cumulative reward comparable to WI policy. We also observe that non-uniform random policy gives better performance compared to round robin and uniform random policy. In Fig. 4b), we can see arm choice fractions for all arms under various policies. Once again, Whittle index policy shows the tendency of preferring a smaller subset of arms, $\{4, 10\}$. On the other hand, myopic policy prefers arms $\{9, 10\}$. Other policies plays all the arms equally. The behavior of Whittle index policy is determined by channel characteristics, i.e., $p_{0,0}$, and $p_{1,0}$, where we observe that for arm 4 has highest difference in $(p_{0,0} - p_{1,0})$, arm 10 has smallest value of $p_{1,0}$ and $(p_{0,0} - p_{1,0}) = 0.2$. Here, the index depends on these parameters and it accounts for future rewards through the action value functions. Here, the behavior of myopic policy depends on value of $q$ and observe that arm 9 and 10 has least value of $q$. Thus MP plays arms 9 and 10 most frequently compare to other arms. For other policies all the arms are played equally often.

### C. Example-3: Positively and negatively correlated arms

In this example, we consider a more generic parameter set for which no index expressions are available. This set consists both of positively and negatively correlated arms unlike other
is varied from decrease in cumulative throughput is expected. Even for small value of random policy as the baseline. We observe that Whittle index and myopic policy have high gain below. In Table III, we described the relative gain percentage of different policies with uniform channel depends on channel characteristics $p$. For first 10 channels, $K$ is assumed to be large, thus $\gamma_2(\pi) \approx q$. For next 5 channels, $K$ is varied from 1 to 10. Further, we assumed $R_{m,0} = \rho_{m,0} = 0$. Other parameters are given below. In Table III we described the relative gain percentage of different policies with uniform random policy as the baseline. We observe that Whittle index and myopic policy have high gain for small value of $K$ as compared to $K = 10$. This is because for $K = 10$, the channel is not correlated, but it is memory less, i.e., $\gamma_{2,m}(\pi) = q_m$ for all $m = 1, 2, \cdots, M$. Hence, decrease in cumulative throughput is expected. Even for small $K$, if channel is not played for large number of sessions, $\gamma_{2,m}(\pi) = q_m$ for all $m = 1, 2, \cdots, M$. The number of plays of a channel depends on channel characteristcs $p_{0,0}$ and $p_{1,0}$. This determines the gain in throughput value for different policies. $p_{0,0} = [0.5, 0.45, 0.45, 0.78, 0.6, 0.6, 0.7, 0.4, 0.45, 0.5, 0.6, 0.7, 0.5, 0.35]$. 

D. Example-4: Effect of multiple state transitions $K$.

In this example, we illustrate the effect of $K$ in Table III, where we used $M = 15$ channels. For first 10 channels, $K$ is assumed to be large, thus $\gamma_2(\pi) \approx q$. For next 5 channels, $K$ is varied from 1 to 10. Further, we assumed $R_{m,0} = \rho_{m,0} = 0$. Other parameters are given below. In Table III we described the relative gain percentage of different policies with uniform random policy as the baseline. We observe that Whittle index and myopic policy have high gain for small value of $K$ as compared to $K = 10$. This is because for $K = 10$, the channel is not correlated, but it is memory less, i.e., $\gamma_{2,m}(\pi) = q_m$ for all $m = 1, 2, \cdots, M$. Hence, decrease in cumulative throughput is expected. Even for small $K$, if channel is not played for large number of sessions, $\gamma_{2,m}(\pi) = q_m$ for all $m = 1, 2, \cdots, M$. The number of plays of a channel depends on channel characteristcs $p_{0,0}$ and $p_{1,0}$. This determines the gain in throughput value for different policies. $p_{0,0} = [0.5, 0.45, 0.45, 0.78, 0.6, 0.6, 0.7, 0.4, 0.45, 0.5, 0.6, 0.7, 0.5, 0.35]$.
Fig. 6. Example-4: a) The discounted cumulative reward versus session number for different policies and b) arm choice fraction for each arm with different policies. This is plotted for $K = 3$.

TABLE III

|          | $K = 1$ | $K = 2$ | $K = 3$ | $K = 4$ | $K = 5$ | $K = 10$ |
|----------|---------|---------|---------|---------|---------|---------|
| WP       | 41.87   | 38.37   | 37.47   | 37.85   | 37.30   | 36.39   |
| MP       | 27.82   | 25.65   | 24.70   | 25.60   | 25.73   | 25.33   |
| NUR      | 3.66    | 2.66    | 2.67    | 3.03    | 2.85    | 2.13    |
| RR       | 0.24    | 0.50    | −1.3    | 1.16    | −1.08   | 0.42    |

$p_{1,0} = [0.41, 0.4, 0.35, 0.15, 0.55, 0.5, 0.5, 0.6, 0.3, 0.25, 0.2, 0.2, 0.3, 0.25]$, 

$p_1 = [0.9, 0.8, 0.8, 0.8, 0.9, 0.9, 0.9, 0.9, 0.8, 0.7, 1, 1, 1, 1]$, 

$R_1 = [0.9, 0.8, 0.8, 0.8, 0.9, 0.9, 0.9, 0.9, 0.8, 0.7, 0.6, 0.7, 0.85, 0.6, 0.7]$.

E. Example-5: Effect of inaccurate estimates of $K$.

In Example-5, we simulate a situation where the decision maker does not know the exact values of $K$, and proceeds with an inaccurate estimate $K_e$ for all arms. We use the same parameters as Example-4, except for the values of $K$. For arms 1 to 10 the value $K = 20$, while, for arms 11 to 15 respectively have $K$ values $1, 2, 3, 4, 5$. However, as these values are unknown, the value $K_e$ is used in decision making. Table IV shows the relative gains of the policies when different inaccurate estimates $K_e$ are used for decision making. There are minor changes (1% − 4%) in the performances of policies compared to those in Table III.

VI. DISCUSSION AND CONCLUSION

We first present a discussion on various aspects of our problem.
TABLE IV
EXAMPLE 5: RELATIVE VALUE GAINS (%) WITH INACCURATE ESTIMATE $K_e$ OF THE NUMBER OF STATE TRANSITIONS.
INITIAL BELIEF - RANDOM.

|         | $K_e = 1$ | $K_e = 2$ | $K_e = 3$ | $K_e = 4$ | $K_e = 5$ | $K_e = 10$ |
|---------|-----------|-----------|-----------|-----------|-----------|------------|
| WP      | 35.35     | 36.08     | 35.85     | 37.85     | 36.01     | 37.18      |
| MP      | 25.23     | 25.69     | 24.43     | 26.36     | 25.10     | 25.85      |
| NUR     | 3.68      | 3.64      | 2.57      | 3.8       | 1.58      | 2.05       |
| RR      | 1.41      | 0.00      | -0.95     | 1.53      | -0.81     | -3.2       |

A. Discussion

1) Markovain representation of fading channels: The proposed model is aimed at Markovian fading channels. Commonly used channel fading models such as Rayleigh fading and the resultant exponential SNR distribution, can be represented as finite state Markov chains (FSMC) [36]–[38]. One method involved is to partition the fading coefficients value range such that, the duration spent in each state is the same, say $\tau$. This $\tau$ depends on parameters of the fading distribution of the channel. Suppose that, each of $M$ different channels is represented using an $n$-state FSMC. If the number of states $n$ and the length of interval $\tau$ is fixed, then the FSMC representation of heterogeneous channels may threaten the validity of the conventional RMAB model assumption—one state transition per decision interval. Our proposed model remedies this problem by allowing multiple state transitions ($K$) per decision interval and also, arms with different values of $K$. The authors plan to take up the case of random $K$ drawn from a known distribution in future. A survey of the basic principles of representing fading channels using FSMCs is given in [39].

2) Whittle index policy gain in moderately sized systems: In our numerical study, we presented numerical examples with moderate size systems in terms of the number of arms, say $M = 10, 15$. From these examples, we observed that index policy performs better than other policies; but, it difficult to say about the optimality of the index policy. However, the Whittle index policy has been shown to be asymptotically optimal for large systems, see [6], [40], where both the number of arms $N$ and the arms to be played per decision interval are large. The analysis is usually done by using fluid approximation technique. The analysis provided in Section III is valid for systems of all sizes with $M \geq 1$ and $K \geq 1$.

3) Approximation causes memory loss: In numerical examples 1 and 2, we have assumed that $K$ is large; and made use of the approximation $\gamma_{2,m}(\pi) \approx q_m$, for all $m$. Here, the belief update
When not playing the arm, becomes independent of prior belief $\pi$. That is, the history of actions and observations described by sufficiently by statistic $\pi$, is forgotten. Hence, we say that approximation causes memory loss. On the other hand, see numerical examples 4 and 5, where some arms have large $K$ and others have small $K$. In these examples, Whittle index policy performs better than in examples $1-3$. It is possible that, memory loss due to approximation is detrimental to gains obtained by Whittle index policy. This is because the index policy which tries to maximize conditional expectation of long term rewards, loses valuable historical information along the way. Hence, when approximation is used, its performance is lowered and becomes comparable to myopic policy. Further, notice that our problem of (lazy restless bandit) with $K$ state transitions and cumulative feedback cannot be reduced equivalently to the conventional restless bandit with instantaneous feedback by replacing transition matrix $P_m$ with $P^K_m$. This is because, in our case $P^K_m$ tends to $Q_m$ as $K$ increases, where $Q_m$ is the stationary transition matrix of arm $m$. Now, for large $K$, using $Q_m$ in the place of $P_m$ changes all the belief update functions $\gamma_0$, $\gamma_1$ and $\gamma_2$, leading to loss of important channel information descriptors.

4) Trade off between information gain and computational load: Note that we can observe the state of an arm when it is played for $\rho_0 = 0$ and $\rho_1 = 1$. Hence, we could obtain a closed form expression for Whittle index. In case of $\rho_0 = 0$ and $0 < \rho_1 < 1$, when arm is played and ACK is observed, state of arm is known and otherwise the state is not known. Even then, we are able to compute the index expression except for one belief region. Thus, the less information gained from playing an arm, the more difficulty in obtaining closed form expression for index. We devised an index computation Algorithm 1 when decision maker has less information about state of the arm even after it is played, i.e., $0 < \rho_0, \rho_1 < 1$. Index computation algorithms based on stochastic approximation schemes are often computationally taxing because index needs to be computed offline and stored. Also, the accuracy of index computation algorithm depends on tolerance level $h$. A smaller tolerance level $h$ requires larger computation time and vice-versa. Thus, there is often a trade off between the information gain and computational load. In Example—3, we considered a parameter set for which closed form index expressions are not available; hence index computation was done using Algorithm 1.

5) Complexity of online implementation: Two cases arise during online implementation of the model. The first is when closed form index expressions are available. In this case, the decision maker for an $M$-armed restless bandit would require to make $O(M)$ computations every session. The second case is when closed form index expressions are not available. Here, the decision maker uses index values that were computed offline and stored for every point on a ‘grid’ $G$
over a subspace of Markov matrices in $\mathbb{R}^{2 \times 2}$. For $M$ arms, it takes initial computations of $O(M \log |G|)$ for searching the appropriate matrices. Further, for each of these matrices, there are index values for different points on a belief grid $B$. Hence, in every session the decision maker needs $O(M \log |B|)$ computations. Notice that the complexity depends on the fineness of the grids. An important study would be to look into change of decisions with slight change of transition probabilities.

6) The case of large state spaces: The RMAB with partially observable states is generally analyzed in literature for two state model. However, in many applications larger number of states might be required for accurate representation of communication channels. In such cases, the proposed model can be applied using state aggregation. That is, if a channel has $n$ states, the states 1 to $n_0 < n$ are mapped to state 0 and states $n_0 + 1$ to $n$ are mapped to state 1 of the corresponding arm. Further, limited work is available on multi-state model for partially observable RMAB, see [40]. For a multi-state single armed bandit, the optimal policy of the corresponding POMDP is not threshold type. Hence, in [40], the POMDP is modified by making steady state approximations; the modified POMDP has optimal policy of threshold type. The approximations in [40] are valid for large number of arms and large number of states. The applicability of this analysis while allowing multiple state transitions per session, needs careful study by considering the “memory loss” that may be caused together by both steady state approximations and multiple state transitions.

B. Conclusion

In this work, the problem of restless multi-armed bandits with cumulative feedback and partially observable states was formulated. Such bandits are called lazy restless bandits. This model allows multiple state transitions between consecutive decision epochs. Hence, it is directly applicable to sequential decision problems in wireless networks with fast fading channel conditions.

Some directions for future work include, a network level performance analysis of the Whittle index policy for solving problems like relay selection considering various aspects such as path loss, interference, etc. Also, restless bandits with constrained or intermittently available arms would make a useful study.
APPENDIX

A. Proof of Lemma 3

The proof is done by induction technique. Assume that $V_n(\pi)$ is non increasing in $\pi$. Let $\pi' > \pi$ and consider playing the arm is optimal. Then

$$V_{n+1}(\pi) = R_S(\pi) + \beta [\rho(\pi)V_n(\gamma_1(\pi)) + (1 - \rho(\pi))V_n(\gamma_0(\pi))]$$

Here $R_S(\pi) = \pi R_0 + (1 - \pi) R_1$. Note that $R_S(\pi)$ is decreasing in $\pi$, i.e. $R_S(\pi') < R_S(\pi)$ whenever $\pi' > \pi$. Hence we get

$$V_{n+1}(\pi) \geq R_S(\pi') + \beta [\rho(\pi)V_n(\gamma_1(\pi)) + (1 - \rho(\pi))V_n(\gamma_0(\pi))].$$

(6)

From our assumptions $p_{00} > p_{10}$ and $\rho_1 > \rho_0$, we get a stochastic ordering ($\leq_s$) on observation probabilities, i.e., $[1 - \rho(\pi'), \rho(\pi')]^T \leq_s [1 - \rho(\pi), \rho(\pi)]^T$. Also, $\gamma_0(\pi) \geq \gamma_1(\pi)$; and as $V_n(\pi)$ is decreasing in $\pi$, $V_n(\gamma_0(\pi)) \leq V_n(\gamma_1(\pi))$. Then, using a property of stochastic ordering \[15\] Lemma 1.1] along with (6), we obtain

$$V_{n+1}(\pi) \geq R_S(\pi') + \beta [\rho(\pi')V_n(\gamma_1(\pi)) + (1 - \rho(\pi'))V_n(\gamma_0(\pi))].$$

Now that $\gamma_0, \gamma_1$ are increasing in $\pi$ and $V_n$ is decreasing in $\pi$, we have

$$V_{n+1}(\pi) \geq R_S(\pi') + \beta [\rho(\pi')V_n(\gamma_1(\pi')) + (1 - \rho(\pi'))V_n(\gamma_0(\pi'))] \geq V_{n+1}(\pi').$$

This is true for every $n$. From \[33\], we know $V_n(\pi) \to V(\pi)$ as $n \to \infty$. Thus $V(\pi)$ is decreasing in $\pi$. Similarly, when not playing the arm is optimal, it is clear that $V_{n+1}(\pi) = \eta + V_n(\gamma_2(\pi)) \geq V_{n+1}(\pi')$, for $\pi' > \pi$, as $\gamma_2$ is increasing in $\pi$ for positively correlated arms. Likewise, the same stochastic ordering argument for showing $V_S$ is decreasing in $\pi$. \qed

B. Proof of Lemma 4

1) Part 1) - For large $K$: Let $d(\pi) := V_S(\pi) - V_{NS}(\pi)$. We want to prove that $d(\pi)$ decreasing in $\pi$. This implies that we need to show $V_S(\pi) - V_{NS}(\pi) < V_S(\pi') - V_{NS}(\pi')$, whenever $\pi > \pi'$. That is to show $V_S(\pi) - V_S(\pi') < V_{NS}(\pi) - V_{NS}(\pi')$.

In our setting $V_{NS}(\pi) - V_{NS}(\pi') = 0$ whenever $\gamma_2(\pi) = q$ and this is true for large values of $k$. We know for positive correlated arms, $V_S(\pi) - V_S(\pi') < 0$, as $V_S$ is decreasing in $\pi$. Hence, the claim follows. \qed
2) Parts 2), 3) - for any $K > 1$: To prove $V_S - V_{NS}$ is decreasing with $\pi$, we need the following result.

**Lemma 8:** The functions $\frac{\partial V_S(\pi)}{\partial \pi}$, $\frac{\partial V_1(\pi)}{\partial \pi}$, and $\frac{\partial V_{NS}(\pi)}{\partial \pi}$ are convex, $|\frac{\partial V_S(\pi)}{\partial \pi}| \leq \kappa c(\rho_1 - \rho_0)$, when, $\beta < \frac{1 + b}{4}$ or $0 < \rho_{0,0} - \rho_{1,0} < \frac{1 + b}{4}$. Here $\kappa = \frac{1}{1 - \beta \rho_{0,0} - \rho_{1,0}}$, $b = \min\{1, \frac{R_1 - R_0}{\rho_1 - \rho_0}\}$ and $c = \max\{1, \frac{R_1 - R_0}{\rho_1 - \rho_0}\}$.

**Proof 1:** We prove this by induction. We provide the proof for the case $\rho_{0,0} > \rho_{1,0}$, i.e., positively correlated arms. The same procedure also works for $\rho_{0,0} < \rho_{1,0}$, i.e., negatively correlated arms. Also, notice that $\kappa \geq 1$.

1) $V_{S,1} = R_S(\pi)$, $V_{NS,1} = \eta$ and so, $V_1(\pi) = \max\{R_S(\pi), \eta\}$. Clearly, as all the functions are convex, $|\frac{\partial V_1(\pi)}{\partial \pi}| \leq \kappa c(\rho_1 - \rho_0)$.

2) Assume $|\frac{\partial V_1(\pi)}{\partial \pi}| < \kappa c(\rho_1 - \rho_0)$.

3) Now,

$$
V_{S,n+1}(\pi) = R_S(\pi) + \beta(\rho(\pi)V_n(\gamma_1(\pi)) + (1 - \rho(\pi))V_n(\gamma_0(\pi)))
$$

$$
V_{NS,n+1}(\pi) = \eta + \beta V_n(\gamma_2(\pi))
$$

$$
V_{n+1}(\pi) = \max\{V_{S,n+1}(\pi), V_{NS,n+1}(\pi)\}.
$$

(7)

$$
\frac{\partial V_{S,n+1}(\pi)}{\partial \pi} = (R_0 - R_1) + \beta(\rho_1 - \rho_0)[V_n(\gamma_0(\pi)) - V_n(\gamma_1(\pi))] + \beta \rho(\pi) \frac{\partial V_n(\gamma_1(\pi))}{\partial \pi} \gamma_1'(\pi)
$$

$$
+ \beta(1 - \rho(\pi)) \frac{\partial V_n(\gamma_0(\pi))}{\partial \pi} \gamma_0'(\pi).
$$

(8)

Substituting $\gamma_1'(\pi) = \frac{\rho_1 \rho_0 (\rho_{0,0} - \rho_{1,0})}{(\rho(\pi)^2)}$, $\gamma_0'(\pi) = \frac{(1 - \rho_1)(1 - \rho_0)(\rho_{0,0} - \rho_{1,0})}{(1 - \rho(\pi))}$,

$$
\frac{\partial V_{S,n+1}(\pi)}{\partial \pi} = (R_0 - R_1) + \beta(\rho_1 - \rho_0)[V_n(\gamma_0(\pi)) - V_n(\gamma_1(\pi))] + \beta \rho(\pi) \frac{\partial V_n(\gamma_1(\pi))}{\partial \pi} \rho_1 \rho_0 (\rho_{0,0} - \rho_{1,0})
$$

$$
+ \beta \rho(\pi) \frac{\partial V_n(\gamma_0(\pi))}{\partial \pi} (1 - \rho_1)(1 - \rho_0)(\rho_{0,0} - \rho_{1,0})
$$

(9)

4) Bound: We know that $\rho(\pi) \in [\rho_0, \rho_1]$ and $1 - \rho(\pi) \in [1 - \rho_1, 1 - \rho_0]$. Substituting in above equation, we have

$$
\frac{\partial V_{S,n+1}(\pi)}{\partial \pi} \leq (R_0 - R_1) + \beta(\rho_1 - \rho_0)[V_n(\gamma_0(\pi)) - V_n(\gamma_1(\pi))]
$$

$$
+ \beta \rho(\pi) \frac{\partial V_n(\gamma_1(\pi))}{\partial \pi} \rho_1 (\rho_{0,0} - \rho_{1,0}) + \beta \rho(\pi) \frac{\partial V_n(\gamma_0(\pi))}{\partial \pi} (1 - \rho_0)(\rho_{0,0} - \rho_{1,0}).
$$

From the assumption in Step 2), we can conclude that $V_n(\gamma_0(\pi)) - V_n(\gamma_1(\pi)) \leq \kappa c(\rho_1 - \rho_0)(\gamma_0(\pi) - \gamma_1(\pi))$. Further we have $\gamma_0(\pi) - \gamma_1(\pi) \leq p_{0,0} - p_{1,0}$. Using this, we have

$$
\frac{\partial V_{S,n+1}(\pi)}{\partial \pi} \leq (R_0 - R_1) + \beta(\rho_1 - \rho_0)^2 \kappa c(\rho_{0,0} - \rho_{1,0})
$$

$$
+ \beta \kappa c(\rho_{0,0} - \rho_{1,0}) p_1 (\rho_1 - \rho_0) + \beta \kappa c(\rho_{0,0} - \rho_{1,0})(1 - \rho_0)(\rho_1 - \rho_0)
$$

$$
\leq (R_0 - R_1) + (\rho_1 - \rho_0)(\kappa c(\rho_{0,0} - \rho_{1,0})[1 + 2(\rho_1 - \rho_0)])
$$

$$
\leq (R_0 - R_1) + (\rho_1 - \rho_0)[3\kappa c(\rho_{0,0} - \rho_{1,0})]$$
Rewriting the R.H.S. of the above inequality, we obtain
\[ \frac{\partial V_{S,n+1}(\pi)}{\partial \pi} \leq \kappa c(\rho_1 - \rho_0) \{ -b + 4\beta(p_{0,0} - p_{1,0}) \} \] (10)
where, \( b = \min \left\{ 1, \frac{R_1 - R_0}{\rho_1 - \rho_0} \right\} \) and \( c = \max \left\{ 1, \frac{R_1 - R_0}{\rho_1 - \rho_0} \right\} \). If \( \beta < \frac{(1+b)}{4} \) or \( 0 < p_{0,0} - p_{1,0} < \frac{(1+b)}{4} \), then, \( |-b + 4\beta(p_{0,0} - p_{1,0})| \leq 1 \). Now, it follows that \( \left| \frac{\partial V_{S,n+1}(\pi)}{\partial \pi} \right| \leq \kappa c(\rho_1 - \rho_0) \). Hence, by principle of induction, the claim is true for all \( n > 0 \). By the property of the value function that \( \lim_{n \to \infty} V_{S,n}(\pi) = V_S(\pi) \), it follows that \( \left| \frac{\partial V_S(\pi)}{\partial \pi} \right| < \kappa c(\rho_1 - \rho_0) \).

5) Similarly,
\[ \frac{\partial V_{NS,n+1}(\pi)}{\partial \pi} = \beta \frac{\partial V_{NS,n}(\gamma_2(\pi))}{\partial (\gamma_2(\pi))} \gamma'_2(\pi) \leq \beta \kappa c(\rho_1 - \rho_0)(p_{0,0} - p_{1,0})^K \leq \kappa c(\rho_1 - \rho_0). \] (11)
Hence, by principle of induction, the claim is true for all \( n > 0 \). By the property of value function that \( \lim_{n \to \infty} V_{NS,n}(\pi) = V_{NS}(\pi) \), if follows that \( \left| \frac{\partial V_{NS}(\pi)}{\partial \pi} \right| < \kappa c(\rho_1 - \rho_0) \). □

Now, consider \( d(\pi) = V_S(\pi) - V_{NS}(\pi) \). It is enough to show \( \frac{\partial d(\pi)}{\partial \pi} < 0 \).

From (10) and (11), we have
\[ \frac{\partial V_S(\pi)}{\partial \pi} \leq \kappa c(\rho_1 - \rho_0) \{ -b + 4\beta(p_{0,0} - p_{1,0}) \} \]
\[ \frac{\partial V_{NS}(\pi)}{\partial \pi} \geq -\beta \kappa c(\rho_1 - \rho_0)|p_{0,0} - p_{1,0}|^K. \]
\[ \frac{\partial d(\pi)}{\partial \pi} \leq \kappa c(\rho_1 - \rho_0) \{ -b + 4\beta(p_{0,0} - p_{1,0}) \} + \beta \kappa c(\rho_1 - \rho_0)|p_{0,0} - p_{1,0}|^K \]
\[ \leq \kappa c(\rho_1 - \rho_0) \{ -b + 5\beta(p_{0,0} - p_{1,0}) \} \]

In the R.H.S of above inequality, \( \{ -b + 5\beta(p_{0,0} - p_{1,0}) \} < 0 \) when, \( 0 < p_{0,0} - p_{1,0} < \frac{b}{5} \) or \( \beta < \frac{b}{5} \). Also, \( \frac{b}{5} < \frac{1+b}{4} \), \( \forall b > 0 \). Hence, under these conditions \( V_S - V_{NS} \) is decreasing in \( \pi \). □

C. Proof of Theorem 2

Using induction technique, one can obtain the following inequalities.
\[ \left| \frac{\partial V_S(\pi, \eta)}{\partial \eta} \right|, \left| \frac{\partial V_S(\pi, \eta)}{\partial \eta} \right|, \left| \frac{\partial V_{NS}(\pi, \eta)}{\partial \eta} \right| \leq \frac{1}{1 - \beta} \]

Also,
\[ \frac{\partial V_S(\pi, \eta)}{\partial \eta} = \beta \left[ \rho(\pi) \frac{\partial V(\gamma_1(\pi), \eta)}{\partial \eta} + (1 - \rho(\pi)) \frac{\partial V(\gamma_0(\pi), \eta)}{\partial \eta} \right] \]
\[ \frac{\partial V_{NS}(\pi, \eta)}{\partial \eta} = 1 + \beta \frac{\partial V(q, \eta)}{\partial \eta}. \]

Now taking differences
\[ \frac{\partial V_{NS}(\pi, \eta)}{\partial \eta} - \frac{\partial V_S(\pi, \eta)}{\partial \eta} = 1 + \beta \frac{\partial V(q, \eta)}{\partial \eta} - \beta \left[ \rho(\pi) \frac{\partial V(\gamma_1(\pi), \eta)}{\partial \eta} + (1 - \rho(\pi)) \frac{\partial V(\gamma_0(\pi), \eta)}{\partial \eta} \right] \]
From Lemma [7] we require the above difference to be nonnegative at $\pi_T(\eta)$. This reduces to following expression.

$$\left[\rho(\pi)\frac{\partial V(\gamma_1(\pi), \eta)}{\partial \eta} + (1 - \rho(\pi))\frac{\partial V(\gamma_0(\pi), \eta)}{\partial \eta}\right] - \frac{\partial V(q, \eta)}{\partial \eta} < \frac{1}{\beta}. \quad (12)$$

Note that we can provide upper bound on LHS of above expression and it is upper bounded by $2/(1 - \beta)$. If $\beta < 1/3$, Eqn. (12) is satisfied. $\pi_T(\eta)$ is decreasing in $\eta$. Thus indexability claim follows. □

\[D. \text{ Index computation for arbitrary } K, \ p_{0,0} > p_{1,0}, \ p_0 = 0, \ p_1 = 1.\]

1) For $\pi \in A_1$, $V_S(\pi) = R_S(\pi) + \beta(1 - \pi)V(p_{1,0}) + \beta \pi V(p_{0,0})$. Assuming the threshold is at $\pi$, we have $V(p_{1,0}) = V_{NS}(p_{1,0})$ and $V(p_{0,0}) = V_{NS}(p_{0,0})$. Further, for any $\pi < q$, $\gamma_2(\pi) > \pi$.

$$V_{NS}(\pi) = \eta + \beta V(\gamma_2(\pi)) = \eta + \beta V_{NS}(\gamma_2(\pi)) = \eta + \beta(\eta + \beta V_{NS}(\gamma_2^2(\pi)))$$

$$\Rightarrow V_{NS}(\pi) = \eta(1 + \beta + \beta^2 + \ldots + \beta^{t-1}) + \beta^t V_{NS}(\gamma_2^t(\pi)).$$

As $t \to \infty$, $\gamma_2^t(\pi) \to q$. And it follows that $V_{NS}(\pi) = \frac{\eta}{1-\beta}$. Further, putting above equations together we have $V_S(\pi) = R_S(\pi) + \beta \pi V(p_{0,0})$. Because $\pi$ is the threshold, the subsidy $\eta$ required such that $V_S(\pi) = V_{NS}(\pi)$ is the index $W(\pi)$. Hence, we get $W(\pi) = R_S(\pi)$.

2) For $\pi \in A_2$, Assume that the threshold is at $\pi$. Hence, $V(p_{1,0}) = V_S(p_{1,0})$ and $V(p_{0,0}) = V_{NS}(p_{0,0})$. And $V_S(\pi) = R_S(\pi) + \beta(1 - \pi)V_S(p_{1,0}) + \beta \pi V_{NS}(p_{0,0})$. Further,

$$V_S(p_{1,0}) = R_S(p_{1,0}) + \beta(1 - p_{1,0})V_S(p_{1,0}) + \beta p_{1,0} V_{NS}(p_{0,0})$$

$$V_S(p_{1,0}) = \frac{R_S(p_{1,0})}{1 - \beta(1 - p_{1,0})} + \frac{\beta p_{1,0}}{1 - \beta(1 - p_{1,0})} V_{NS}(p_{0,0}) = a + b V_{NS}(p_{0,0}).$$

Now, just as in the previous interval, it can easily be shown for any $\pi \in A_2$, that, $V_{NS}(\pi) = \frac{\eta}{1-\beta}$. Now, $V_S(p_{1,0}) = a + \frac{b\eta}{1-\beta}$. Then, we have

$$V_S(\pi) = R_S(\pi) + \beta(1 - \pi)\left(a + \frac{b\eta}{1-\beta}\right) + \beta \frac{\eta}{1-\beta}.$$

Equating $V_S$ and $V_{NS}$, we get $W(\pi) = \frac{(1-\beta)[R_S(\pi) + \beta(1-\pi)a]}{1-\beta[\pi+(1-\pi)b]}$.

3) For $\pi \in A_3$,

$$V_S(\pi) = R_S(\pi) + \beta(1 - \pi)V_S(p_{1,0}) + \beta \pi V_{NS}(p_{0,0})$$

$$V_S(p_{1,0}) = R_S(p_{1,0}) + \beta(1 - p_{1,0})V_S(p_{1,0}) + \beta p_{1,0} V_{NS}(p_{0,0}).$$

Hence,

$$V_S(p_{1,0}) = \frac{R_S(p_{1,0})}{1 - \beta(1 - p_{1,0})} + \frac{\beta p_{1,0}}{1 - \beta(1 - p_{1,0})} V_{NS}(p_{0,0}) = a + b V_{NS}(p_{0,0}).$$
Further,

\[
V_{NS}(p_{0,0}) = \eta + \beta V(\gamma_2(p_{0,0})) = \eta(1 + \beta + \ldots + \beta^{t-1}) + \beta^t V(\gamma_2^t(p_{0,0})) \\
= \eta \frac{1 - \beta^t}{1 - \beta} + \beta^t V(\gamma_2^t(p_{0,0})) \equiv \eta e + \beta^t V(\gamma_2^t(p_{0,0}))
\]

(13)

where, \( t = \inf \{ l \geq 1 : \gamma_2^l(p_{0,0}) \leq \pi \} \). Then, \( V(\gamma_2^t(p_{0,0})) = V_S(\gamma_2^t(p_{0,0})) \). Now,

\[
V_S(\gamma_2^t(p_{0,0})) = R_S(\gamma_2^t(p_{0,0})) + \beta(1 - \gamma_2^t(p_{0,0}))V_S(p_{1,0}) + \beta \gamma_2^t(p_{0,0})V_{NS}(p_{0,0})
\]

Let \( \gamma_2^t(p_{0,0}) \equiv g_2 \). Now,

\[
V_{NS}(p_{0,0}) = \eta e + \beta^t[R_S(g_2) + \beta(1 - g_2)V_S(p_{1,0}) + \beta g_2V_{NS}(p_{0,0})]
\]

\[
= \eta e \frac{1 - \beta^{t+1}g_2}{1 - \beta^t} + \beta^tR_S(g_2) \frac{1 - \beta^{t+1}g_2}{1 - \beta^t} + \beta^{t+1}(1 - g_2) \frac{1 - \beta^{t+1}g_2}{1 - \beta^t} V_S(p_{1,0}) \equiv \eta f + a_1 + b_1V_S(p_{1,0})
\]

Using above equations,

\[
V_{NS}(p_{0,0}) = \eta f + a_1 + b_1V_S(p_{1,0}) = \eta f + a_1 + b_1[a + bV_{NS}(p_{0,0})]
\]

\[
V_{NS}(p_{0,0}) = \frac{a_1 + b_1a}{1 - \beta b_1} + \frac{\eta f}{1 - \beta b_1} \equiv \eta c + d.
\]

\[
V_S(p_{1,0}) = a + b(\eta c + d) = \eta bc + a + bd.
\]

Using above equations, it follows that

\[
V_{NS}(\pi) = \eta + \beta V_S(\gamma_2(\pi)) = \eta + \beta[\eta B(\gamma_2(\pi)) + D(\gamma_2(\pi))]
\]

Equating \( V_S(\pi) \) and \( V_{NS}(\pi) \), we get, \( W(\pi) = \frac{D(\pi) - D(\gamma_2(\pi))}{1 + \beta[B(\gamma_2(\pi)) - B(\pi)]} \).

4) For \( \pi \in A_1 \), \( V_S(\pi) = R_S(\pi) + \beta(1 - \pi)V_S(p_{1,0}) + \beta \pi V_S(p_{0,0}) \). We need to compute \( V_S(p_{1,0}) \) and \( V_S(p_{0,0}) \). In this case the optimal action for the \( \pi \) is not sample the arm once and later sample the arm always. Similarly if the initial action is to sample the arm and later the optimal action is to sample the arm always. This behavior can be observed from the operation \( \gamma_0(\pi) \), which is smaller than \( p_{0,0} \). Then, one can easily show by induction that, \( V_S(\pi) \) is linear in \( \pi \) with slope \( m \) and intercept \( c_1 \) as mentioned earlier. That is, \( V_S(p_{1,0}) = mp_{1,0} + c_1 \) and \( V_S(q) = mq + c_1 \). Now,

\[
V_S(\pi) = R_S(\pi) + \beta(1 - \pi)(mp_{1,0} + c_1) + \beta \pi(mp_{0,0} + c_1)
\]

\[
= \pi [m \beta(p_{0,0} - p_{1,0}) + R_0 - R_1] + [R_1 + \beta(c_1 + mp_{1,0})] = m \pi + c_1
\]

Simplifying above equations, we have \( m = \frac{R_0 - R_1}{1 - \beta(p_{0,0} - p_{1,0})} \) and \( c_1 = \frac{R_1 + m \beta p_{1,0}}{1 - \beta} \).

Further, \( V_{NS}(\pi) = \eta + \beta V_S(\gamma_2(\pi)) = \eta + \beta[m \gamma_2(\pi) + c_1] \). Equating \( V_S(\pi) \) and \( V_{NS}(\pi) \), we have \( W(\pi) = m \pi + c_1 - \beta(m \gamma_2(\pi) + c_1) \).
E. Index Computation for large $K$, $\gamma_2(\pi) \approx q$, and $R_0 = \rho_0 = 0, R_1 = \rho_1 < 1$.

1) We first compute index for $\pi \in A_1$. We have following expressions, $V_S(\pi, \eta) = \rho(\pi) + \beta \rho(\pi)V(\pi_1(\pi)) + \beta(1 - \rho(\pi))V(\pi_0(\pi))$ and $V_{NS}(\pi, \eta) = \eta + \beta V(q)$.

Assuming $\pi$ is a threshold, if have $V_S(\pi, \eta) = V_{NS}(\pi, \eta)$ because of the threshold type policy, then, $\eta$ is called the index $W(\pi)$. Solving for $\eta$ we can obtain index. For this case, we note that $q > \pi$ and hence $V(q) = V_{NS}(q) = \eta + \beta V(q)$, and $V(q) = \frac{\eta}{1 - \beta}$. This implies that $V_{NS}(\pi) = \frac{n}{1 - \beta}$. Also, note that $V(\gamma_1(\pi)) = V_{NS}(\gamma_1(\pi))$, and $V(\gamma_0(\pi)) = V_{NS}(\gamma_0(\pi))$. Hence $V_S(\pi) = \rho(\pi) + \beta \frac{n}{1 - \beta}$, and $V_{NS}(\pi) = \frac{n}{1 - \beta}$. By equating these expressions and solving for $\eta$ we obtain $W(\pi) = \rho(\pi)$.

2) For $\pi \in A_2$ we have $V_S(\pi, \eta) = V_{NS}(\pi, \eta)$ and solving for $\eta$ we can obtain index. We also use condition $\rho_0 = 0$. This implies that $\gamma_1(\pi) = p_1$.

We first show that $V(q) = \frac{n}{1 - \beta}$ in this case. We observe that $V(q) = \max\{V_S(q), V_{NS}(q)\} = V_{NS}(q)$. After simplification we obtain $V(q) = \frac{n}{1 - \beta}$.

a) We first derive index for $\gamma_0(p_{1,0}) \geq \pi$. Since $p_{1,0} < \pi$, we have $V(p_{1,0}) = V_S(p_{1,0})$.

Hence we first require to compute $V_S(p_{1,0})$. We also have $\gamma_0(p_{1,0}) \geq \pi$ and hence, $V(\gamma_0(p_{1,0})) = V_{NS}(\gamma_0(p_{1,0})) = \frac{n}{1 - \beta}$. Then we can write $V_S(p_{1,0})$ as follows.

$$V_S(p_{1,0}) = \rho(p_{1,0}) + \beta \rho(p_{1,0})V(\pi_1) + \beta(1 - \rho(p_{1,0}))V_{NS}(\gamma_0(p_{1,0}))$$

$$= \frac{\rho(p_{1,0})}{1 - \beta \rho(p_{1,0})} + \frac{\beta(1 - \rho(p_{1,0}))}{1 - \beta} \frac{n}{1 - \beta}.$$ 

Then we can rewrite $V_S(\pi)$ in the following way.

$$V_S(\pi) = \rho(\pi) + \beta \rho(\pi) \left[\frac{\rho(p_{1,0})}{1 - \beta \rho(p_{1,0})} + \frac{\beta(1 - \rho(p_{1,0}))}{1 - \beta} \frac{n}{1 - \beta}\right] + \beta(1 - \rho(\pi)) \frac{n}{1 - \beta}.$$ 

After simplification we get

$$V_S(\pi) = \rho(\pi) + \beta(1 - \rho(\pi)) \frac{n}{1 - \beta} + \beta \rho(\pi) \left[\rho(p_{1,0})(1 - \beta) + \beta(1 - \rho(p_{1,0}))\frac{n}{1 - \beta}\right].$$

Further, it is easy to show $V_{NS}(\pi) = \frac{n}{1 - \beta}$. Equating $V_S(\pi)$ and $V_{NS}(\pi)$, we obtain

$$\rho(\pi) + \beta \rho(\pi) \rho(p_{1,0})(1 - \beta) = \frac{-\beta(1 - \rho(p_{1,0}))n}{(1 - \beta \rho(p_{1,0}))(1 - \beta)} + \beta(1 - \rho(\pi)) \frac{n}{1 - \beta} + \frac{n}{1 - \beta}.$$ 

After simplification, we get

$$\rho(\pi) + \frac{\beta \rho(\pi) \rho(p_{1,0})(1 - \beta)}{(1 - \beta \rho(p_{1,0}))} = \frac{n}{1 - \beta} \left[\frac{1 - \beta(1 - \rho(p_{1,0}))}{(1 - \beta \rho(p_{1,0}))} - \beta(1 - \rho(\pi))\right].$$

Further simplification, we get $W(\pi)$.

b) We now consider the case of $\gamma_0(p_{1,0}) < \pi$ but $\gamma_0^2(p_{1,0}) \geq \pi$. From our assumptions, we can obtain the following expressions.

$$V(\gamma_0(p_{1,0})) = V_S(\gamma_0(p_{1,0})), \quad V(\gamma_0^2(p_{1,0})) = V_{NS}(\gamma_0^2(p_{1,0})) = \frac{n}{1 - \beta}.$$
Thus we can write

\[ V_S(\gamma_0(p_{1,0})) = \rho(\gamma_0(p_{1,0})) + \beta \rho(\gamma_0(p_{1,0})) V(p_{1,0}) + \beta(1 - \rho(\gamma_0(p_{1,0}))) V(\gamma_0^2(p_{1,0})). \]

After substituting the value of \( V(\gamma_0^2(p_{1,0})) \) we get

\[ V_S(\gamma_0(p_{1,0})) = \rho(\gamma_0(p_{1,0})) + \beta \rho(\gamma_0(p_{1,0})) V(p_{1,0}) + \beta(1 - \rho(\gamma_0(p_{1,0}))) \frac{\eta}{1 - \beta}. \]  

(14)

We can compute \( V(p_{1,0}) = V_S(p_{1,0}) \) and obtain following expression.

\[ V_S(p_{1,0}) = \rho(p_{1,0}) + \beta \rho(p_{1,0}) V_S(p_{1,0}) + \beta(1 - \rho(p_{1,0})) V_S(\gamma_0(p_{1,0})). \]  

(15)

After substituting Eqn. (14) in Eqn. (15), we obtain

\[ V_S(p_{1,0}) = \rho(p_{1,0}) + \beta \rho(p_{1,0}) V_S(p_{1,0}) + \beta(1 - \rho(p_{1,0})) \left[ \rho(\gamma_0(p_{1,0})) + \beta \rho(\gamma_0(p_{1,0})) V(p_{1,0}) \right. \\
\left. + \beta(1 - \rho(\gamma_0(p_{1,0}))) \frac{\eta}{1 - \beta} \right]. \]  

(16)

After solving Eqn. (16) we can obtain expression of \( V_S(p_{1,0}). \)

\[ V_S(p_{1,0}) = \frac{\rho(p_{1,0}) + \beta(1 - \rho(p_{1,0})) \rho(\gamma_0(p_{1,0})) + \beta^2 \rho(\gamma_0(p_{1,0})) \frac{\eta}{1 - \beta}}{1 - \beta \rho(p_{1,0}) - \beta^2 \rho(\gamma_0(p_{1,0})), \rho(\gamma_0(p_{1,0})) = 1 - \rho(\gamma_0(p_{1,0})).} \]

(17)

We now rewrite \( V_S(\pi). \) \( V_S(\pi) = \rho(\pi) + \beta \rho(\pi) V_S(p_{1,0}) + \beta(1 - \rho(\pi)) V_{NS}(\gamma_0(\pi)). \)

Equating \( V_S(\pi) \) and \( V_{NS}(\pi) \) and solving for \( \eta \), we obtain the required \( W(\pi) \).

3) For \( \pi \in A_3 \), computing index expression is non-trivial because it is tedious to compute the value function expression for \( V_S(p_{1,0}). \) Hence, we use algorithm [1].

4) For \( \pi \in A_4 \), To obtain value function expressions \( V_S(\pi) \) and \( V_{NS}(\pi) \) have to obtain expressions for \( V_S(p_{1,0}) \) and \( V_S(q) \). As shown before, in this interval, \( V_S(\pi) \) is linear in \( \pi \), slope \( m \) and intercept \( c \) as mentioned earlier. That is, \( V_S(p_{1,0}) = mp_{1,0} + c \) and \( V_S(q) = mq + c. \) Therefore, \( V_{NS}(\pi) = \eta + \beta(\eta + c) \), and \( V_S(\pi) = \rho(\pi) + \beta \rho(\pi)(mq + c) + \beta(1 - \rho(\pi))(m\gamma_0(\pi) + c). \) Equating \( V_S(\pi) \) and \( V_{NS}(\pi) \) and solving for \( \eta \) we get required \( W(\pi). \)  

\[ \square \]

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