ON SOME NONLINEAR EQUATION FROM THEORY OF THE FLOWS ON NETWORKS

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Abstract. Here we study the nonlinear hyperbolic equations of the type of equations from theory of the flows on networks, for which we prove the solvability theorem under the appropriate conditions and also investigate the behaviour of the solution.

1. Introduction

In this article we study one class of nonlinear hyperbolic equations, that in the one space dimension case, we can formulate in the form

\begin{equation}
\frac{\partial^2 u}{\partial t^2} - (f(u) \frac{\partial u}{\partial x})_x = g(u), \quad (t, x) \in R_+ \times (0, l), \ l > 0,
\end{equation}

\begin{equation}
u(0, x) = u_0(x), \ \frac{\partial u}{\partial t}(0, x) = u_1(x), \ u(t, 0) = u(t, l),
\end{equation}

where \(u_0(x), u_1(x)\) are known functions, \(f(\cdot), g(\cdot) : R \rightarrow R\) are continuous functions and \(l > 0\) is a number. The equation of type (1.1) describe mathematical model of the problem from theory of the flow in networks as is affirmed in articles [1], [2], [3], [4], [5], [6], [7], [9], [10] (e. g. Aw-Rascle equations, Antman–Cosserat model, etc.). As in the survey [4] is noted such a study can find application in accelerating missiles and space crafts, components of high-speed machinery, manipulator arm, microelectronic mechanical structures, components of bridges and other structural elements. Balance laws are hyperbolic partial differential equations that are commonly used to express the fundamental dynamics of open conservative systems (e.g. [3]). As the survey [4] possess of the sufficiently exact explanations of the significance of equations of such type therefore we not stop on this theme. It need to note that most often in these articles in which the being investigated problem descrebe the hyperbolic equation of second order as mathematical model, then for investigation the authors reduce it to the system of equations of first order. As it is explained in the cited above survey on the mathematical properties of the Antman–Cosserat model are similar to those of the first-order system associated with the nonlinear wave equation.

This article is organized as follows. In the section 2 we consider the class of the nonlinear hyperbolic equations of second order of such type that are arisen in the theory of flows on networks. In the section 3 we investigate the solvability of the considered problems and in the section 4 the behaviour of their solutions.
2. Formulation of the problem and the approach

Consider the following problem
\begin{align}
(2.1) & \quad u_{tt} - \nabla \cdot (f(u) \nabla u) = g(u), \quad (t, x) \in (0, T) \times \Omega, \quad T \in (0, \infty) \\
(2.2) & \quad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad u \big|_{(0,T) \times \partial \Omega} = 0,
\end{align}

where \( \Omega \subset \mathbb{R}^n, n \geq 1 \) is a bounded domain with sufficiently smooth boundary \( \partial \Omega \), \( T > 0 \) is arbitrary fixed number.

It is well known (\cite{8, 12, 13}) that under the conditions of this problem the operator \( -\Delta \) is a self-adjoint, positive operator densely defined in a Hilbert space \( H = L^2(\Omega) \) (and on \( H^1_0(\Omega) \equiv H^1(\Omega) \) with the norm \( \|v\|_{H^1_0} \equiv \|\nabla v\|_{L^2} \equiv \|\nabla v\|_{H^1_0}, \) see, e. g. \cite{12} \cite{15}) and \( f, g : \mathbb{R} \rightarrow \mathbb{R} \) are continuous functions.

For the investigation of this problem we will use the following approach
\begin{align}
(2.3) & \quad \left\langle u_{tt} - \Delta F(u), (\Delta)^{-1} u_t \right\rangle = \left\langle g(u), (\Delta)^{-1} u_t \right\rangle,
\end{align}

here \( F(r) = \int_0^r f(s) \, ds \) is a monotone function.

In the other words we will understud the solution of this problem in the following sense

**Definition 1.** We will call a function
\begin{align}
\forall u \in C^0(0, T; L^p(\Omega)) \cap C^2(0, T; W^{-2,q}(\Omega) + H^{-1}(\Omega))
\end{align}
weak solution of problem \((2.2)-(2.3)\) if \( u \) satisfies the following equation
\begin{align}
\left\langle u_{tt} - \Delta F(u), w \right\rangle = \left\langle g(u), w \right\rangle
\end{align}
locally by a. e. \( t \in (0, T) \) for any \( w \in W^{2,p}(\Omega) \cap H^1_0(\Omega) \).

Consider the following conditions
1) let \( f, g : \mathbb{R} \rightarrow \mathbb{R} \) are a continuous functions and there are a numbers \( a_0, b_0, d > 0, a_1, b_1 \geq 0 \) and \( p > 2, 0 \leq 2p_0 \leq p \) such that the following inequations
\begin{align}
|F(r)| \leq a_0 |r|^{p-1} + a_1 |r|; \quad F(r) \cdot r \geq b_0 |r|^p + b_1 r^2; \quad |g(r)| \leq d |r|^{p_0},
\end{align}
hold for any \( r \in \mathbb{R} \), moreover \( g \) is continuous function (for example, \( f(r) = k_0 |r|^{p-2} - k_1 |r|^{p_1} + k_2, k_0 > 0, k_1, k_2 \geq 0, 0 \leq p_1 < p - 2 \), moreover \( k_1 = k_1 (k_0, k_2) \) and \( g(r) = d^{p_0} |r|^{p_0} \)).

It is well known (\cite{8, 12, 13}) that under the conditions of this problem the following problem
\begin{align}
(2.4) & \quad -\Delta v = w, \quad x \in \Omega \subset \mathbb{R}^n, \quad v \big|_{(0,T) \times \partial \Omega} = 0
\end{align}
is solvable for any \( w \in L^p(\Omega), p > 1 \) and has unique solution in \( W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega), \) i.e. the operator \( -\Delta : W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \rightarrow L^p(\Omega) \) is the isomorphism. Consequently if to set the denotation \( u \equiv -\Delta v \) then of the posed problem one can rewrite in the form
\begin{align}
-\Delta v_{tt} - \nabla \cdot (f(-\Delta v) \nabla (-\Delta v)) = g(-\Delta v), \quad (t, x) \in (0, T) \times \Omega,
\end{align}
or
\begin{align}
-\Delta v_{tt} - \Delta F(-\Delta v) = g(-\Delta v), \quad (t, x) \in (0, T) \times \Omega, \\
-\Delta v(0, x) = u_0(x), \quad -\Delta v_t(0, x) = u_1(x), \quad u \big|_{(0,T) \times \partial \Omega} = 0.
\end{align}
So from equation (2.3) we get

\[ \frac{1}{2} \frac{d}{dt} \| \nabla v_t \|^2_2 + \frac{d}{dt} \Phi (-\Delta v) = \langle g(-\Delta v), v_t \rangle \]

here \( \Phi \) is a non-negative functional and \( \Phi (u) = \int_0^1 \langle F(su), u \rangle \, ds \).

If to bear in mind of these conditions and condition 1 we get

\[ \frac{1}{2} \frac{d}{dt} \| \nabla v_t \|^2_2 + \frac{d}{dt} \Phi (-\Delta v) \leq \frac{1}{2} \| v_t \|^2_2 + \frac{1}{2} \| g(-\Delta v) \|^2_2 (t) \leq c \left[ \frac{1}{2} \| \nabla v_t \|^2_2 + \Phi (-\Delta v) \right] + \hat{d}, \]

here \( c > 0, \hat{d} \geq 0 \) are constants that independent of \( v \). Hence follows

\[ \| \nabla v_t \|^2_2 (t) + 2\Phi (-\Delta v) (t) \leq e^{cT} \left[ \| \nabla v_1 \|^2_2 + 2\Phi (u_0) \right] + \frac{\hat{d}}{c} (e^{cT} - 1). \]

by virtue of the Gronwall’s lemma.

Thus we get inequalities

(2.5) \[ \| \nabla v_t \|^2_2 (t) \leq e^{cT} \left[ \| \nabla v_1 \|^2_2 + 2\Phi (u_0) \right] + \frac{\hat{d}}{c} (e^{cT} - 1) \]

\[ \Phi (-\Delta v) (t) \leq e^{cT} \left[ \| \nabla v_1 \|^2_2 + 2\Phi (u_0) \right] + \frac{\hat{d}}{c} (e^{cT} - 1) \]

for every fixed \( T \in (0, \infty) \)

It not is difficult to see that if \( \hat{d} = 0 \) then occurs the inequation

\[ \| \nabla v_t \|^2_2 (t) + 2\Phi (u) (t) \leq e^{cT} \left[ \| \nabla v_1 \|^2_2 + 2\Phi (u_0) \right]. \]

3. Solvability of problem (2.1) - (2.2)

For the proof of the solvability of problem (2.1) - (2.2) we will use the approach of Galerkin. We need note that under condition 1 on the function \( g \) we not succeeded in proving the solvability of this problem. For this we assume the following severe constraint instead of the condition that is deduced in condition 1 on \( g \). Let function \( g : R \rightarrow R \) is the Lipschitz function, i.e. there is a such number \( k > 0 \) that the following inequality

(3.1) \[ |g(r) - g(s)| \leq d_0 |r - s| \]

holds for any \( r, s \in R \).

Let the system \( U \equiv \{ w_j (x) \}_{j=1}^{\infty} \) be a total system of the space

\[ W^{2, p} (\Omega) \cap H_0^1 (\Omega) \]

where \( w_j (x) \) be the sufficiently smooth functions. Using above approach we get the a priori estimations (2.3). We will seek out of the approximative solutions \( u_m (t, x) \) in the form

\[ (-\Delta)^{-1} u_m (t, x) = v_m (t, x) = \sum_{i=1}^{m} c_i (t) w_i (x) \text{ or } u_m (t) \in \text{span} \{ w_1, ..., w_m \} \]
as the solutions of the problem locally with respect to \( t \), where \( c_i(t) \) are as the unknown functions that will be defined as solutions of the following Cauchy problem for system of ODE

\[
\frac{d^2}{dt^2} \langle u_m, w_j \rangle - \langle F(u_m), \Delta w_j \rangle = \langle g(u_m), w_j \rangle, \quad j = 1, 2, ..., m
\]

\[u_m(0, x) = u_{0m}(x), \quad u_{tm}(0, x) = u_{1m}(x),\]

where \( u_{0m} \) and \( u_{1m} \) are contained in \( \text{span} \{w_1, ..., w_m\} \), \( m = 1, 2, ... \), moreover

\[u_{0m} \rightarrow u_0 \text{ in } H^1_0 \cap W^{1, p}(\Omega); \quad u_{1m} \rightarrow u_1 \text{ in } H \cap L^p(\Omega) \text{ at } m \not\to \infty.\]

Thus we obtain the following problem

\[
\frac{d^2}{dt^2} \langle u_m, w_j \rangle = \langle F(u_m), \Delta w_j \rangle + \langle g(u_m), w_j \rangle, \quad j = 1, 2, ..., m
\]

\[\langle u_m(t, x), w_j \rangle |_{t=0} = \langle u_{0m}(x), w_j \rangle, \quad \frac{d}{dt} \langle u_m(t, x), w_j \rangle |_{t=0} = \langle u_{1m}(x), w_j \rangle\]

that solvable on \( (0, T) \) for any \( m = 1, 2, ... \) and \( T > 0 \) by virtue of estimates \( 2.5 \).

Consequently with use of the known procedure (\cite{11, 16, 14}) we obtain, \( \nabla v_{mt} \in C^0(0, T; H(\Omega)), \Delta v_m \in C^0(0, T; L^p(\Omega)) \) and \( u_m \in C^0(0, T; L^p(\Omega)), u_{mt} \in C^0(0, T; H^{-1}(\Omega)) \), moreover they are contained in the bounded subset of these spaces. Thus from (3.3) we get

\[u_{mtt} \in C^0(0, T; \left(W^{2, p}(\Omega) \cap W_0^{1, p}(\Omega)\right)^* + H^{-1}(\Omega)).\]

So for the sequence of the approximate solutions we have: \( \{u_m\}_{m=1}^\infty \) is contained in the bounded subset of the space

\[C^0(0, T; L^p(\Omega)) \cap C^2(0, T; W^{-2, q}(\Omega) + H^{-1}(\Omega))\]

and the sequence \( \{v_m\}_{m=1}^\infty \) is contained in the bounded subset of the space

\[C^0(0, T; W^{2, p}(\Omega) \cap H^1_0(\Omega)) \cap C^1(0, T; H_0^1(\Omega)) \cap C^2(0, T; L^q(\Omega)).\]

Then the sequence \( \{v_m\}_{m=1}^\infty \) has a precompact subset in the space

\[C^1(0, T; [W^{2, p}(\Omega), L^q(\Omega)]_2)\]

by virtue of the known interpolation theory (see, \cite{13}), and consequently in the space \( C^1(0, T; H^1_0(\Omega)) \) as the imbedding \([W^{2, p}(\Omega), L^q(\Omega)]_2 \subseteq H^1(\Omega)\) holds.

Thus for us it is remained to show the following: if the sequence

\[\{u_m\}_{m=1}^\infty \subset C^0(0, T; L^p(\Omega)) \cap C^2(0, T; W^{-2, q}(\Omega) + H^{-1}(\Omega))\]

is weakly converging to \( u \) in this space and \( \{F(u_m)\}_{m=1}^\infty \) and \( \{g(u_m)\}_{m=1}^\infty \) have an weakly converging subsequence to \( \eta \) in \( H \) and to \( \theta \) in \( L^q(\Omega) \) (\( q = \frac{p}{p-1} \)) respectively, for a. e. \( t \in (0, T) \) then \( \eta = F(u) \) and \( \theta = g(u) \). (Here and in what follows for brevity we don’t changing of indexes of subsequences.)

In the beginning we will show the equation \( \theta = g(u) \). Let the sequence \( \{u_m\}_{m=1}^\infty \) is such as above mentioned and \( -\Delta v_m = u_m \). Then in condition (3.1) for the operator

\[g : C^0(0, T; L^p(\Omega)) \subset C^0(0, T; H) \rightarrow C^0(0, T; H)\]

we have

\[\langle g(u_m), w_j \rangle \rightarrow \langle \theta, w_j \rangle \text{ for } \forall j : j = 1, 2, ...\]
and also
\[ \langle g(u_m), z \rangle \rightarrow \langle \theta, z \rangle \quad \text{for } \forall z \in W^{2,p}(\Omega) \subset H(\Omega). \]

Therefore we consider the expression \( g(u_m), v_m \) under the assumption that \( u_m \rightarrow u \) in \( L^p(\Omega) \subset H \) and \( v_m \rightarrow v \) in \( H^1(\Omega) \) and \( g(u_m) \rightarrow \theta \) in the corresponding spaces. In order to prove that \( \langle g(u_m), v_m \rangle \) is the Cauchy sequence we carry out the following estimations

\[
|\langle g(u_m), v_m \rangle - \langle g(u_{m+k}), v_{m+k} \rangle| \leq |\langle g(u_m) - g(u_{m+k}), v_m \rangle| + |\langle g(u_{m+k}), v_m - v_{m+k} \rangle| \leq |\langle g(u_m) - g(u_{m+k}), v_m \rangle| + |\langle g(u_{m+k}), v_m - v_{m+k} \rangle| \leq d_0 (|u_m - u_{m+k}| \cdot |v_m|) + |\langle g(u_{m+k}), v_m - v_{m+k} \rangle| \]

(3.4)

that shows the correctness of this statement as the right side converges to zero with respect to \( m \rightarrow \infty \). If to take account of the above assumption we can conduct the estimation of such type for the expression \( |\langle g(u_m), v_m \rangle - (g(u), v)\rangle \), as \( g(u) \) is defined, then we obtain that equation \( \theta = g(u) \) holds, i.e. \( g(u_m) \rightarrow g(u) \) in \( H \).

In order to show the equation \( \eta = F(u) \) we will use the monotonicity condition of \( F \), i.e. for any \( v, w \in C^0(0, T; W^{2,p}(\Omega)) \cap C^2(0, T; L^p(\Omega)) \) occurs the following inequality

\[ \langle -\Delta F(-\Delta v) + F(-\Delta \bar{v}), v - \bar{v} \rangle \geq 0 \]

and if rewrite it for \( u_m = -\Delta v_m \) and \( \bar{u} = -\Delta \bar{v} \) then we have

\[ \langle (F(u_m) - F(\bar{u})), u_m - \bar{u} \rangle \geq 0. \]

It is not difficult to see that the following convergence takes place

\[
\frac{d}{dt} \langle u_{mt}, w_j \rangle - \langle \Delta F(u_m), w_j \rangle - \langle g(u_m), w_j \rangle \rightarrow \frac{d}{dt} \langle u_t, w_j \rangle - \langle \Delta \eta, w_j \rangle - \langle \theta, w_j \rangle, \quad \forall w_j
\]

then

(3.5)

\[
\frac{d^2}{dt^2} \langle u, w \rangle - \langle \Delta \eta, w \rangle = \langle g(u), w \rangle, \quad \forall w \in H_0^1 \cap W^{2,p}(\Omega)
\]

for a.e. \( t \in (0, T) \) by virtue of the obtained above equation \( \theta = g(u) \), consequently

\[ u_{tt} - \Delta \eta = g(u), \quad \text{in the sense of } H^{-1} + W^{-2,q}(\Omega) \]

for a.e. \( t \in (0, T) \).

Let us apply monotonicity of \( F \)

\[ 0 \leq \langle (F(u_m) - F(\bar{u}), u_m - \bar{u}) = -\langle \Delta F(u_m) + \Delta F(\bar{u}), v_m - \bar{v} \rangle = -\langle \Delta F(u_m), v_m \rangle + \langle F(u_m), \bar{v} \rangle + \langle \Delta F(\bar{u}), v_m \rangle = \]

(here \( \bar{u} = -\Delta \bar{v}, \bar{v} \in H_0^1 \cap W^{2,p}(\Omega) \)) for that use equation (3.5) we have

\[
-\langle g(u_m), \bar{v} \rangle + \frac{d^2}{dt^2} \langle u_m, \bar{v} \rangle + \langle F(u_m), u_m \rangle + \langle \Delta F(\bar{u}), v_m - \bar{v} \rangle \Rightarrow \]

from here we get

\[ 0 \leq -\langle g(u), \bar{v} \rangle + \frac{d^2}{dt^2} \langle u, \bar{v} \rangle + \langle F(u_m), u_m \rangle + \langle \Delta F(\bar{u}), v - \bar{v} \rangle \]
by pass to the limit with respect to \( m \): \( m \rightarrow \infty \) and if to take account the following known inequation

\[
\int_\Omega \liminf_{m} (F(u_m)u_m) \, dx \leq \langle \eta, u \rangle,
\]

by the Fatou’s lemma, more exactly

\[
\int_\Omega \liminf_{m} (F(-\Delta v_m)(-\Delta v_m)) \, dx \leq \langle \eta, u \rangle
\]

as \( \langle -\Delta F(u_m), v_m \rangle = \langle F(-\Delta v_m), -\Delta v_m \rangle \). Then with use of this inequation and (3.5) we get

\[
0 \leq -\langle g(u), \bar{v} \rangle + \frac{d^2}{ds^2} \langle \Delta v, \bar{v} \rangle - \langle \Delta \eta, v \rangle - \langle \Delta F(\bar{u}), v - \bar{v} \rangle = \langle -\Delta \eta, v - \bar{v} \rangle - \langle -\Delta F(\bar{u}), v - \bar{v} \rangle = \langle \eta - F(\bar{u}), u - \bar{u} \rangle.
\]

Hence we obtain the equation \( \eta = F(u) \) by virtue of arbitrariness of \( \bar{u} = -\Delta \bar{v} \).

Now we will show that the function \( u(t, x) \) satisfies of the initial conditions and for this we will consider the following equation

\[
\langle u_{mt}, v_m \rangle(t) = \int_0^t \langle u_{ms}, v_m \rangle \, ds + \int_0^t \langle u_{ms}, v_m \rangle \, ds + \langle u_{0m}, v_0m \rangle
\]

for \( t \in (0, T) \) and \( u_m = -\Delta v_m \), that is equivalent to the equation

\[
(3.6) \quad \langle \nabla v_{mt}, \nabla v_m \rangle(t) = \int_0^t \langle \nabla v_{ms}, \nabla v_m \rangle \, ds + \int_0^t \langle \nabla v_{ms}, \nabla v_m \rangle \, ds + \langle \nabla v_{1m}, \nabla v_{0m} \rangle.
\]

From obtained a priori estimations follow the boundedness of the right side of (3.6), consequently we get the boundedness of the left side of (3.6) for any \( t \in (0, T) \). Thus one can pass to the limit by \( t \rightarrow 0 \) by virtue of the a priori estimates. Really as \( \{v_m\}_{m=1}^{\infty} \subset C^0(0, T; W^{2,p}(\Omega)) \cap C^2(0, T; L^q(\Omega)) \) and bounded in this space we get: the right side is bounded as all terms in the left side are bounded in respective spaces, therefore one can pass to limit with respect to \( m \) as here \( v_{mt} \) are continuous with respect to \( t \) for any \( m \) then \( v_{mt} \) strongly converges to \( v_t \) in \( H \) and \( \Delta v_m \) weakly converges to \( \Delta v \) in \( L^p(\Omega) \).

Consequently is proved the following result.

**Theorem 1.** Let \( u_0 \in H^1_0 \cap W^{1,p}(\Omega) \), \( u_1 \in L^p(\Omega) \) and that there are functions \( v_0 \in H^1_0 \cap W^{2,p}(\Omega) \), \( v_1 \in H^1_0 \cap W^{1,p}(\Omega) \) such that \(-\Delta v_k = u_k, k = 0, 1\). Let \( f : R \rightarrow R \) and \( g : R \rightarrow R \) are a continuous functions such that \( F(u) = \int f(r) \, dr \) is a monotone operator and satisfies the following inequalities

\[
|F(r)| \leq a_0 |r|^{p-1} + a_1 |r|; \quad F(r) \cdot r \geq b_0 |r|^p + b_1 r^2
\]

for \( r \in R \), and \( g \) satisfies inequation (3.7) for \( r, s \in R \), where \( a_0, b_0, d_0 > 0 \), \( a_1, b_1 \geq 0 \) and \( p > 1 \) are a numbers.

Then problems (2.1)-(2.2) possess, at least, one weak solution \( u(t, x) \) in the sense of Definition (2.1) that belongs to the space \( C^0(0, T; L^p(\Omega)) \cap C^2(0, T; W^{2,q}(\Omega) + H^{-1}(\Omega)) \) for every fixed number \( T \in (0, \infty) \).
Remark 1. It should be noted that by using (2.4) one can reformulate of the
considered problem in the following form: let \( g (u) \equiv g (t, x) \) is given function
\[
- \Delta (v_{tt} + F (-\Delta v)) = g (t, x) \equiv -\Delta \tilde{g}, \quad (t, x) \in (0, T) \times \Omega,
\]
\[
- \Delta v (0, x) = u_0 (x) = -\Delta v_0 (x),
\]
\[
- \Delta v_t (0, x) = u_1 (x) = -\Delta v_1 (x), \quad \Delta v \big| _{(0, T) \times \partial \Omega} = 0.
\]
In the other words we get
\[
(3.7) \quad - \Delta (v_{tt} + F (-\Delta v) - \tilde{g}) = 0, \quad (t, x) \in (0, T) \times \Omega,
\]
\[
\text{hence one can obtain the following equivalent problem if } F \text{ is the homogeneous operator}
\]
\[
v_{tt} + F (-\Delta v) = \tilde{g}, \quad (t, x) \in (0, T) \times \Omega,
\]
\[
v (0, x) = v_0 (x), \quad v_t (0, x) = v_1 (x), \quad v \big| _{(0, T) \times \partial \Omega} = -\Delta v \big| _{(0, T) \times \partial \Omega} = 0
\]
since if equation (3.7) possess a solution then the expression \( v_{tt} + F (-\Delta v) - \tilde{g} \) is a
harmonic function for any \( t \) and also satisfies the homogeneous boundary condition.
\[
\text{In this case we get, that the considered problem is equivalent to the problem}
\]
\[
v_{tt} + F (-\Delta v) = \tilde{g} (t, x) \text{ with the mixed conditions as above.}
\]

4. Behaviour of the solution of the problem \((2.1)-(2.2)\)

Now we introduce the function \( E (t) = \| \nabla w \|^2 \|_H (t) \) and consider this function
on the solution of problem \((2.1)-(2.2)\) and assume \( g \) satisfies inequation \( | g (r) |^2 \leq d_1 \Phi (r) \) for any \( r \in R \).

So we will study the problem
\[
u_{tt} - \Delta F (u) = g (u), \quad (t, x) \in R_+ \times \Omega,
\]
\[
u (0, x) = u_0 (x), \quad u_t (0, x) = u_1 (x), \quad u \big| _{R_+ \times \partial \Omega} = 0
\]
for which behaving as above we get the equation
\[
\| \nabla v_t \|^2 (t) + 2 \Phi (u) (t) \leq \| \nabla v_1 \|^2 + 2 \Phi (u_0) + \int_0^t \left[ \| \nabla v_s \|^2 + \| g (u) \|^2 \right] (s) ds
\]
using the condition on \( g (u) \) we have
\[
\| \nabla v_t \|^2 (t) + 2 \Phi (u) (t) \leq \| \nabla v_1 \|^2 + 2 \Phi (u_0) + \tilde{d} \int_0^t \left[ \| \nabla v_s \|^2 + 2 \Phi (u) \right] (s) ds.
\]
Hence follows
\[
(4.1) \quad \| \nabla v_t \|^2 (t) \leq \frac{1}{\tilde{d}} \left[ e^{\tilde{d}t} \left( 1 + \tilde{d} \right) - 1 \right] \left( \| \nabla v_1 \|^2 + 2 \Phi (u_0) \right) - 2 \Phi (u) (t).
\]

For the functional \( E (t) = \| \nabla v \|^2 \|_H (t) \) we have
\[
E' (t) = 2 \langle \nabla v_t, \nabla v \rangle \leq \| \nabla v_t \|^2 (t) + \| \nabla v \|^2 (t)
\]
using here inequation (4.1)
\[ E'(t) \leq E(t) - 2\Phi(u)(t) + \frac{1}{d} \left[ e^{\hat{d}t} (1 + \hat{d}d) - 1 \right] \left( \| \nabla v_1 \|^2 + 2\Phi(u_0) \right) \]
and the condition on \( F \) (consequently, on \( \Phi \))
\[ E'(t) \leq E(t) - c \| -\Delta v \|^p \left( \hat{d}\right) + \frac{1}{d} \left[ e^{\hat{d}t} (1 + \hat{d}d) - 1 \right] \left( \| \nabla v_1 \|^2 + 2\Phi(u_0) \right) \leq E(t) - cE^{\hat{d}}(t) + \frac{1}{d} \left[ e^{\hat{d}t} (1 + \hat{d}d) - 1 \right] \left( \| \nabla v_1 \|^2(0) + 2\Phi(-\Delta v_0) \right) \implies \]
and at last we get
\[ E'(t) \leq E(t) - cE^{\hat{d}}(t) + C_1 (v_0, v_1) e^{\hat{d}t} - C_2 (v_0, v_1) \]
by virtue of the condition \( \Phi(r) \geq c_0 |r|^p \) and of the continuity of embeddings \( L^p(\Omega) \subset L^2(\Omega), \ W^{2,p}(\Omega) \subset \subset W^{1,p}(\Omega) \), where \( C_j (v_0, v_1) > 0 \) (\( j = 1, 2 \)) are constants.

So we have the Cauchy problem for differential inequality
\[ y'(t) \leq y(t) - cy^r(t) + C_1 e^{\hat{d}t} - C_2, \quad y(0) = \| \nabla v_0 \|^2. \]
One can replace problem (4.2) with the following problem in order to investigate of the behaviour of the solution of considered problem
\[ y'(t) \leq y(t) - cy^r(t) + C_1 e^{\hat{d}t} - C_2, \quad y(0) = \| \nabla v_0 \|^2 \]
as \( \hat{d} > 0 \). Inequation (4.2) one can rewrite in the form
\[ (y(t) + IC(v_0, v_1))^r \leq y(t) + IC(v_0, v_1) - \varepsilon |y(t) + IC(v_0, v_1)|^r, \]
where \( l > 1 \) is a number and \( \varepsilon = \varepsilon(c, l, r) > 0 \) is sufficiently small number and \( C = C(\hat{d}, T, C_1, C_2) \). Then solving this problem we get
\[ y(t) + IC(v_0, v_1) \leq \left[ e^{(1-r)t} (\nabla y_0 + IC(v_0, v_1))^{1-r} + \varepsilon (1 - e^{(1-r)t}) \right]^{\frac{1}{1-r}} \]
or
\[ E(t) \leq \left[ e^{(1-r)t} \left( \| \nabla v_0 \|^2_H + IC(v_0, v_1) \right)^{1-r} + \varepsilon (1 - e^{(1-r)t}) \right]^{\frac{1}{1-r}} - IC(v_0, v_1) \]
\[ \| \nabla v \|^2_H(t) \leq \left[ 1 + \varepsilon \left( \| \nabla v_0 \|^2_H + IC(v_0, v_1) \right)^{r-1} (e^{(r-1)t} - 1) \right]^{\frac{1}{r-1}} - IC(v_0, v_1). \]
Here the right side is greater than zero, because \( \varepsilon \leq \frac{1}{r(r-1)} \) and \( 2r > p > 2 \). It is necessary to note here the dependence on \( T \) of the behaviour of the solution is essentially that follows from the received last problem.

Thus is proved the result

**Lemma 1.** Let \( u_0 \in H^1_0 \cap W^{1,p}(\Omega) \), \( u_1 \in L^p(\Omega) \) and that there are functions \( v_0 \in H^1_0 \cap W^{2,p}(\Omega) \), \( v_1 \in H^1_0 \cap W^{1,p}(\Omega) \) such that \( -\Delta v_k = u_k, k = 0, 1 \). Then the function \( v(t,x) \), defined by the solution \( u(t,x) \) of problem (2.1)-(2.2), for any \( t \in (0,T) \) belong in ball \( B^{H^1_0 \cap W^{1,p}(\Omega)}_{R_T}(0) \subset H^1_0 \cap W^{1,p}(\Omega) \) depending from the initial values \( (u_0, u_1) \in H^1_0 \cap W^{1,p}(\Omega) \times L^p(\Omega), \) here \( R_T = R_T (u_0, u_1, p, T) > 0. \)
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