Expansion and Flooding in Dynamic Random Networks with Node Churn

Luca Becchetti
Sapienza Università di Roma
Rome, Italy
becchetti@dis.uniroma1.it

Andrea Clementi
Università di Roma Tor Vergata
Rome, Italy
clementi@mat.uniroma2.it

Francesco Pasquale
Università di Roma Tor Vergata
Rome, Italy
pasquale@mat.uniroma2.it

Luca Trevisan
Università Bocconi
Milan, Italy
l.trevisan@unibocconi.it

Isabella Ziccardi
Università dell’Aquila
L’Aquila, Italy
isabella.ziccardi@graduate.univaq.it

Abstract

We study expansion and information diffusion in dynamic networks, that is in networks in which nodes and edges are continuously created and destroyed. We consider information diffusion by flooding, the process by which, once a node is informed, it broadcasts its information to all its neighbors.

We study models in which the network is sparse, meaning that it has $O(n)$ edges, where $n$ is the number of nodes, and in which edges are created randomly, rather than according to a carefully designed distributed algorithm. In our models, when a node is “born”, it connects to $d = O(1)$ random other nodes. An edge remains alive as long as both its endpoints do.

If no further edge creation takes place, we show that, although the network will have $\Omega(dn)$ isolated nodes, it is possible, with large constant probability, to inform a $1 - \exp(-\Omega(d))$ fraction of nodes in $O(\log n)$ time. Furthermore, the graph exhibits, at any given time, a “large-set expansion” property.

We also consider models with edge regeneration, in which if an edge $(v, w)$ chosen by $v$ at birth goes down because of the death of $w$, the edge is replaced by a fresh random edge $(v, z)$. In models with edge regeneration, we prove that the network is, with high probability, a vertex expander at any given time, and flooding takes $O(\log n)$ time.

The above results hold both for a simple but artificial streaming model of node churn, in which at each time step one node is born and the oldest node dies, and in a more realistic continuous-time model in which the time between births is Poisson and the lifetime of each node follows an exponential distribution.

Previous work on expansion and flooding studied models in which either the vertex set is fixed and only edges change with time or models in which edge generation occurs according to an algorithm. Our motivation for studying models with random edge generation is to go in the direction of models that may eventually capture the formation of social networks or peer-to-peer networks.

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1 Introduction

We study information diffusion in dynamic networks. We focus on flooding, the information diffusion process whereby each node, once informed, spreads the information to all its neighbors.

By dynamic networks we mean communication networks that change over time, in which nodes enter and leave the networks, and links between nodes are created and destroyed. Several networks in which information diffusion is of interest, such as social networks and peer-to-peer network, exhibit change over time.

Information diffusion in dynamic networks has been the focus of extensive previous work, surveyed in Section 2. We are interested in models that exhibit node churn (that is, in which nodes enter and exit the network over time) and in which edge creation occurs randomly, rather than being controlled by a sophisticated distributed algorithm. Our motivation is that a satisfactory modeling of network formation in social networks and peer-to-peer networks will have to satisfy both characteristics. As far as we are aware, information diffusion in dynamic networks with node churn and with uniformly random edge generation has not been studied before.

We made all other modeling choices as simple as possible, and we defined models with as few parameters as possible, in order to highlight qualitative features that we believe to be robust to different modelling choices. While our models are too simplified to predict all properties of realistic networks, one of our models (the Poisson model with edge regeneration that will be defined below) bears a certain resemblance of the way peer-to-peer networks such as bitcoin are formed.

We study models in which the network is sparse, meaning that it has \( O(n) \) edges, where \( n \) is the number of nodes. Specifically, when a node is “born,” it connects to \( d = O(1) \) random other nodes. We show that these dynamic random graphs maintain interesting expansion properties and that flooding informs all or most nodes (depending on details of the model) in \( O(\log n) \) time.

1.1 Modeling networks that change with time

To specify a dynamic network model we have to specify how nodes enter and exit the network, and how edges are generated and destroyed.

Modeling node churn. We initially study an unrealistic but very simple model of node churn: at each discrete time unit, one node enters the network, and each node is alive for precisely \( n \) time units. We refer to this as a streaming model of node churn. After the first \( n \) time units, the network has always exactly \( n \) nodes, and precisely one node is born and one node dies in each time unit. We then study a more realistic continuous-time model, in which the number of births within each time unit follows a Poisson distribution with mean \( \lambda \), and the lifetime of each node is independently distributed as an exponential distribution with parameter \( \mu \), so that the average lifetime of a node is \( 1/\mu \) and the average number of nodes in the network at any given time is \( \lambda/\mu \). In order to reduce the number of parameters, we assume that the time that it takes to send a message along an edge is the same, or is of the same order as, the typical time between node births, which is \( 1 \) in the streaming model and \( \lambda \) in the Poisson model. In order to have a consistent notation in the two models, we choose time units in the continuous model such that \( \lambda = 1 \), and we call \( n = 1/\mu \). With these conventions, the results that we prove in the streaming model are also true in this continuous time model, suggesting that the results have a certain robustness and that the streaming model, despite its simplicity, has some predictive power on the behavior of more realistic models.

Modeling edge creation and destruction. When a node enters the network, we assume that it connects to \( d = O(1) \) nodes chosen uniformly at random among those currently in the network. Once an edge \((u, v)\) is created, it remains active as long as both \( u \) and \( v \) are alive. We study two models: one without edge regeneration and one with edge regeneration. In the former edges are created only when a new node joins the network, in the latter a node creates its outgoing edges not only when it joins the network, but also every time it loses an outgoing edge due to one of its neighbors leaving the network, in order to keep its out-degree always equal to \( d \).

Although the assumption that a node can pick its neighbors uniformly at random among all nodes of the network is unrealistic in many scenarios, the edge creation and regeneration processes in our models resembles the way in which some unstructured peer-to-peer networks maintain a “random” topology. For example, each full-node of the Bitcoin network running the Bitcoin Core implementation has a “target out-degree value” and a “maximum in-degree value” (respectively \( 8 \) and \( 125 \), in the default configuration)
and it locally stores a large list of (ip addresses of) “active” nodes. Such list is initially started with nodes received in response to queries to some DNS seeds. Every time the number of current neighbors of a full-node is below the configured target value it tries to create new connections with nodes sampled from its list. The list stored by a full-node is periodically advertised to its neighbors and updated with the lists advertised by the neighbors. Hence, in the long run each full-node samples its out-neighbors from a list formed by a “sufficiently random” subset of all the nodes of the network.

1.2 Results and techniques

1.2.1 Informing most nodes in the models without edge regeneration

In the models without edge regeneration, we prove that, with high probability, at any given time, there are \( \Omega_d(n) \) isolated vertices in the network. A vertex \( v \) becomes isolated if all the \( d \) edges created at birth were to nodes that have meanwhile died, and \( v \) was never been chosen as neighbor by younger nodes. Because of the presence of such isolated nodes, broadcasting a message to all nodes is not possible, or at least it takes at least \( \Omega_d(n) \) time in the streaming model and \( \Omega_d(n \log n) \) time in the Poisson model. Furthermore, there is a constant probability that a broadcast dies out after reaching only \( O(1) \) vertices. There is, however, also a large constant probability (that tends to 1 as \( n/\exp(d/\log(n)) \) that a broadcast will reach, say, 90% of the nodes (in general, a constant fraction that tends to 1 as \( d \to \infty \)). This tradeoff is best possible because, as argued above, there are \( \Omega_d(n) \) isolated vertices that we will not be able to inform.

To prove this fast convergence we establish two results. One is that, in \( O(\log n) \) time, a broadcast reaches at least, say, \( n/10 \) nodes. To prove this, we argue that, while the number of informed nodes is less than \( n/10 \), there is a good probability that the number of informed nodes grows by a constant factor at each step (and the probability that the above condition fails after exactly \( t \) steps decreases exponentially with \( t \), so that we can take a union bound over all \( t \)). The basic idea of this proof is to apply the principle of deferred decision to the \( d \) edges chosen by each vertex, and assume that those edges are chosen after the vertex is informed, so that the “frontier” of newly informed vertices keeps growing. There are two difficulties with this approach. One is that older nodes are likely to have chosen neighbors that have meanwhile died, and so older nodes are unlikely to significantly contribute to the number of nodes that will be newly informed at the next step. The second difficulty is that a node may become informed by a message coming from one of the \( d \) neighbors chosen at birth, so that we cannot really apply deferred decision in the way that we would like.

To overcome these difficulties, we only consider nodes that are informed through special kinds of paths from the source node (this will undercount the number of informed nodes and make our result true for a stronger reason). Specifically, we define an “onion-skin” process that only considers paths that alternate between “young” nodes whose age is less than the median age and “old” nodes whose age is more than the median age. Furthermore, this process arbitrarily splits the \( d \) edges chosen by each node at birth into \( d/2 \) “type-A” edges and \( d/2 \) “type-B” edges, and only considers paths that, besides alternating between young nodes and old nodes, also alternate between type-A edges and type-B edges. With this restrictions and conventions in place, we can study what happens for every pair of consecutive steps by applying deferred decision.

As sketched above, we are able to show that we inform at least \( n/10 \) nodes after \( O(\log n) \) steps. To complete the argument, we show that, if \( d \) is a sufficiently large constant, all sets of at least \( n/10 \) vertices have constant vertex expansion, which leads to informing at least \( .9n \) nodes after another \( O(1) \) steps. Above, \( 1/10 \) can be replaced by \( \exp(-\Omega(d)) \). This tradeoff is best possible because, as argued above, there are \( \Omega_d(n) \) isolated vertices that we will not be able to inform.

1.2.2 Informing all nodes in the models with edge regeneration

In the model with edge regeneration, we show that the graph has, with high probability, constant vertex expansion at each time step. Despite the presence of node churn, this implies that broadcast reaches all nodes in \( O(\log n) \) steps.

In the streaming model, the proof of vertex expansion is similar to how expansion is generally proved in random graphs: we bound the probability that a fixed set of \( k \) vertices fails to have constant vertex expansion, then we take a union bound by multiplying by \( \binom{n}{k} \) and then by summing over \( k \). The only difficulty is in characterizing the probability that an edge exists between a pair of vertices \( u, v \), because such probability is a non-trivial function of the age of \( u \) and \( v \). Then, since in the streaming model
The node churn is limited and deterministic, we can easily exploit the vertex expansion to derive the logarithmic bound on the flooding time.

The analysis becomes considerably more technical in the Poisson model. The main difficulty is that, in order to compute the probability that an edge \((u,v)\) exists, we need to know the age of \(u\) and \(v\) and so we have to take a union bound over all subsets of vertices of all possible ages. But, at any given time, there are nodes of age up to \(n \log n\), and so we end up with \((n \log n)^k\) cases in our union bound for sets of size \(k\), while the probability that one such set is non-expanding is as high as \(1/(n^k)\) for sets that contain mostly young vertices. The point is that most of the \((n \log n)^k\) possible ways of choosing \(k\) nodes of all possible ages involve choices of several old nodes, which are unlikely to have all survived. In order to carefully account for the “demographics” of all possible sets of edges in our union bound, we look at the logarithm of the probability that a certain set fails to expand, interpret it as the KL divergence of two appropriately defined distributions, and then use inequalities about KL divergence. Moreover, some more technical care is required in the Poisson model to apply the above expansion property for bounding the flooding time. Indeed, the flooding analysis needs to cope with the presence of a random number of node insertion/deletions during every 1-hop message transmission.

1.3 Summary and roadmap

We consider four dynamic graph models, each corresponding to different choices as regards the process modelling node churn and the change in topology induced by departure of a node. For the former, we study both a streaming model of node churn and a more elaborate, continuous Poisson model. As for topology dynamics, we consider both the case in which a node’s departure simply determines failure of all incident edges, and a model in which all nodes maintain a constant degree, thus regenerating new edges to compensate for the loss of edges shared with nodes that died in the interim. Table 1 summarizes our positive and negative results and refers to the formal statements of results in theorems and lemmas below.

The paper is organized as follows. We provide more details on previous work in Section 2. In Section 3 we define streaming models of dynamic graphs, we state our results on the convergence of the flooding process, and we provide an overview of the proofs of such results. Our main technical contribution is the analysis via the onion-skin process in the model without edge regeneration, which is given in Subsection 3.1.2. In Section 4 we define the Poisson models of dynamic graphs, we state our results on the flooding process and provide an overview of the proofs. The main technical result, presented in Subsection 4.3.1, is to establish vertex expansion for the model with edge regeneration, using a notion of edge subset “demographics,” which is quantified via KL divergence. Section 5 provides some further overall remarks about our contribution and poses an open question. To highlight our major contributions and to keep Sections 3 and 4 reasonably short, all the omitted proofs of such sections are given in Sections 6 and 7, respectively. Finally, some mathematical tools we used in the analysis are given in the Appendix.
2 Related Work

A first, rough classification of dynamic graphs can be made according to an important feature: whether or not the set of nodes keeps the same along all the graph process. In the affirmative case, we have an edge-dynamic graph \( \{G_t = (V_t, E_t), t \geq 0\} \) where the topology dynamics defines the way the edges of a fixed set \( V \) of participant nodes change over time. For this class of dynamic graphs, several models, such as worst-case adversarial changes \([15, 16, 20]\) and Markovian evolving graphs \([6, 7]\), have been introduced, their basic connectivity properties have been derived, and, fundamental distributed tasks, such as broadcast and consensus, have been rigorously analyzed.

In contrast, much less analytical works are currently available when (even) the set of participant nodes can change over time. This class of dynamic graphs \( \{G_t = (V_t, E_t), t \geq 0\} \) are often called dynamic networks with churn \([3]\): in this framework, the specific graph dynamics describe both the node insertion/deletion rule for the time sequence \( V_t \) and the edge updating rule for the time sequence \( E_t \). The number of nodes that can join or leave the network at every round is called churn rate. For brevity’s sake, in what follows we will only describe those previous analytical results on dynamic networks with churn which are related to the models we studied in this paper. In particular, we mainly focus on previous work where some connectivity properties of a dynamic networks with churn have been rigorously proved.

As remarked in the Introduction, to the best of our knowledge, previous analytical studies focus on distributed algorithms that are suitably designed to maintain topologies having good connectivity properties.

Pandurangan et al. \([23]\) introduced a partially-distributed protocol that constructs and maintains a bounded-degree graph which relies on a centralized cache of a constant number of nodes. In more detail, their protocol ensures the network is connected, has logarithmic diameter, and has always bounded degree. The protocol manages a central cache which maintains a subset of the current set of vertices. When joining the network, a new node chooses a constant number of nodes in the cache. The insertion/deletion procedures for the central cache follows rather complex rules which take \( O(\log n) \) overhead and delays, w.h.p.

In \([12]\), Duchon et al presented ad-hoc protocols that maintain a given distribution of random graphs under an arbitrary sequence of vertex insertions and deletions. More in detail, given that the graph \( G_t \) is random uniform over the set of \( k \)-out-degree graphs with \( n \) nodes, they provide suitable distributed randomized protocols that can insert (respectively delete) a node such that the graph \( G_{t+1} \) at round \( t \) is again random uniform over the set of \( k \)-out-degree graphs with \( n + 1 \) (respectively, \( n - 1 \)) nodes. They do not assume a centralized knowledge of the whole graph but, instead, their protocol relies on some random primitives to sample arbitrary-sized subsets of nodes uniformly at random. For instance, once a new node \( u \) is inserted, a random subset of nodes is selected (thanks to one of such centralized primitives), and each of them is forced to delete one of its link and to deterministically connect to \( u \). The basic versions of their insertion/deletion procedures require each node to communicate with nodes at distance 2, while their more refined version (achieving optimal performance) require communications over longer paths.

An important and effective approach to keep a dynamic graph with churn having good expansion properties is based on the use of ID random walks. Roughly speaking, this approach let every participating node start \( k \) independent random walks of tokens containing its ID and all the other nodes collaborate to perform such random walks for enough time so that the token is well-mixed over the network. Once a token is mature, it can be used by any node that, in that step, needs a new edge by simply asking to connect to it. The probabilistic analysis then typically shows two main, correlated invariants: on one hand, the edge set, arising from the above random-walk process, form a random graph having good expansion properties. On the other hand, after a small number of steps, the random walks are well-mixed.

Cooper et Al \([8]\) consider two deterministic churn processes: in the first one, at every round a new node is inserted while no nodes leave the network, while, in the second process, the size \( n \) of the graph does never change since, at every round, a new node is inserted and the oldest node leaves the graph (this is in fact the streaming model we study in this paper). They provide a protocol where each node \( v \) starts \( c \cdot m \) independent random walks (containing the ID-label of \( v \)) until they are picked up, \( m \) at a time, by new nodes joining the network. The new node connects to the \( m \) peers that contributed the tokens it got. The resultant dynamic topology is shown to keep diameter \( O(\log n) \), and to be fault-tolerant against adversarial deletion of both edges and vertices. We remark that the tokens in the graph must be constantly circulated in order to ensure that they are well-mixed. Moreover, the rate at which new nodes can join the system is limited, as they must wait while the existing tokens mix before they can
use them.

Law and Siu [17] provide a distributed algorithm for maintaining a regular expander in the presence of limited number of insertions/deletions. The algorithm is based on a complex procedure that is able to sample uniformly at random from the space of all possible 2$d$-regular graphs formed by $d$ Hamiltonian circuits over the current set of alive nodes. They present possible distributed implementations of this sample procedure, the best of which, based on random walks, have $O(\log n)$ overhead and time delay. Such solutions cannot manage frequent node churn.

Further distributed algorithms with different approaches achieving $O(\log n)$ overhead and time delay in the case of slow node churn are proposed in [5, 13, 18, 24].

In [2], Augustine et al present an efficient randomized distributed protocol that guarantees the maintenance of a bounded degree topology that, with high probability, contains an expander subgraph whose set of vertices has size $n - o(n)$, where $n$ is the stable network size. This property is preserved despite the presence of a large oblivious adversarial churn rate — up to $O(n/\text{polylog}(n))$. In more detail, considering the node churn adopted in [3], i.e., an oblivious churn adversary that: can remove any set of nodes up to the churn limit in every round, and, at the same time, it should add (an equal amount of) nodes to the network with the following constraints. A new node should be connected to at least one existing node and the number of new nodes added to an existing node should not exceed a fixed constant (thus, all nodes have constant bounded degree).

The expander maintenance protocol is efficient even though it is rather complex and the local overhead for maintaining the topology is polylogarithmic in $n$. A complication of the protocol follows from the fact that, in order to prevent the growth of large clusters of nodes outside the expander subgraph, it uses special criteria to “refresh” the links of some nodes, even when the latter have not been involved by any edge deletion due to the node churn.

Recently, the flooding process has been analytically studied over dynamic graph models with churn in [4, 5]. Here, the authors consider the model analysed in [2], that we discussed above. Using the expansion property proved in [2], they show that, for any fixed churn rate $C(n) \leq n/\text{polylog}(n)$ managed by an oblivious worst-case adversary, there is a set $S$ of size $n - O(C(n))$ of nodes such that, if a source node in $S$ starts the flooding in round $t$, then all except $O(C(n))$ nodes get informed within round $t + O(\log(n/C(n))\log n)$, w.h.p.

Our models are inspired by the way some unstructured P2P networks maintain a “well-connected” topology, despite nodes joining and leaving the network, small average degree and almost fully decentralized network formation. For example, after an initial bootstrap in which they rely on DNS seeds for node discovery, full-nodes of the Bitcoin network [21] running the Bitcoin Core implementation turn to a fully-decentralized policy to regenerate their neighbors when their degree drops below the configured threshold [9]. This allows them to pick new neighbors essentially at random among all nodes of the network [25]. Notice also that the real topology of the Bitcoin network is hidden by the network formation protocol and discovering the real network structure has been recently an active subject of investigations [10, 22].

3 Warm-up: Preliminaries and the Streaming Model

We first recall the notion of vertex expansion of a graph.

**Definition 3.1 (Vertex expansion).** The vertex isoperimetric number $h_{\text{out}}(G)$ of a graph $G = (N, E)$ is

$$h_{\text{out}}(G) = \min_{0 \leq |S| \leq |N|/2} \frac{|\partial_{\text{out}}(S)|}{|S|},$$

where we used $\partial_{\text{out}}(S)$ for the outer boundary of $S$:

$$\partial_{\text{out}}(S) = \{v \in N \setminus S : \{u, v\} \in E \text{ for some } u \in S\}.$$

Given a constant $\varepsilon > 0$, a graph $G$ is a (vertex) $\varepsilon$-expander if $h_{\text{out}}(G) \geq \varepsilon$.

A dynamic graph $G$ is a sequence of graphs $G = (G_t = (N_t, E_t) : t \in \mathbb{N})$ where the sets of nodes and edges can change at any discrete round. If they can change randomly we call the corresponding random process a dynamic random graph. We call $G_t$ the snapshot of the dynamic graph at round $t$. For a set of nodes $S \subseteq N_t$, we denote with $\partial_{\text{out}}^t(S)$ the outer boundary $S$ in snapshot $G_t$; we omit superscript $t$ when it is clear from the context.
In this section we study two dynamic random graph models in which nodes join and leave the network according to a deterministic streaming, (see Definition 5.2 and edges are created randomly by nodes with low degree (see Definitions 4.4 and 4.14).

**Definition 3.2** (Streaming node churn). The set of nodes \(N_t\) evolves as follows: It starts with \(N_0 = \emptyset\); At each round \(t \geq 1\) a new node joins the network and it stays in the network for exactly \(n\) rounds (i.e., node joining at round \(t\) stays up to round \(t+n-1\), then it disappears. We say that a node has age \(k\) at round \(t\) if it joined the network at round \(t-k\). We say that a node \(u\) is older (respectively, younger) than a node \(v\) if it joined the network before (respectively, after) \(v\).

We are interested in estimating the time a message sent by a node takes to reach all (or a large fraction of) the nodes. To this end, we formalize the flooding process over a dynamic (random) graphs.

**Definition 3.3** (Flooding). Let \(G = \{G_t = (N_t, E_t) : t \in \mathbb{N}\}\) be a dynamic (random) graph. The flooding process over \(G\) starting at time \(t_0\) from the source node \(v_0 \in N_{t_0}\) is the sequence of (random) sets of nodes \(\{I_t : t \in \mathbb{N}\}\) where, for all \(t < t_0\), \(I_t = \emptyset\) (in this paper we will assume that \(I_0\) contains the node joining the network at round \(t_0\)) and, for every \(t \geq t_0\), \(I_t\) contains all nodes in \(N_t\) that were neighbor of some node in \(I_{t-1}\) in the snapshot \(G_{t-1}\), i.e.,

\[
I_t = (I_{t-1} \cup \partial_{\text{out}}^{-1}(I_{t-1})) \cap N_t.
\]

We say that \(I_t\) is the subset of informed nodes at round \(t\). We say that the flooding completes the broadcast if a round \(t\) exists such that \(I_t \supseteq N_{t-1} \cap N_t\) and, in this case, the number of rounds \(t-t_0\) is the flooding time of the source message.

### 3.1 Streaming graphs without edge regeneration

In this section, we study the streaming model SDG where edges are created only when a new node joins the network. We first show that, for constant \(d\) and for any given round, the corresponding random snapshot of the dynamic graph has a linear fraction of isolated nodes, w.h.p. Moreover, we show that flooding fails with constant probability. On the other hand, this model still affords a weaker notion of epidemic process. In particular, in Subsection 3.1.2 we show that, with constant probability, flooding can still inform a large, constant fraction of the nodes within a time interval of size \(O(\log n)\).

**Definition 3.4** (Streaming graphs without edge regeneration). A Streaming Dynamic Graph (for short, SDG) \(G(n,d)\) is a dynamic random graph \(\{G_t = (N_t, E_t) : t \in \mathbb{N}\}\) where the set of nodes \(N_t\) evolves according to Definition 3.2 while the set of edges \(E_t\) according to the following topology dynamics

1. When a new node appears, it creates \(d\) independent connections, each one with a node chosen uniformly at random among the nodes in the network.
2. When a node dies, all its incident edges disappear.

**Remark.** The considered graphs are always undirected. However, given any active node \(v\), our analysis will need to distinguish between out-edges from \(v\), i.e., those requested by \(v\), and the in-edges, i.e., the ones due to the requests from other nodes and accepted by \(v\).

**Preliminary properties.** It is possible to prove that the expected degree of each node in a snapshot \(G_t = (V_t, E_t)\) of a SDG \(G(n,d)\) is \(d\) (see Lemma 6.1 in Subsection 6.1.1), for any \(t \geq n\). Thus, the expected number of edges in the graph is \(nd/2\).

It is well-known that a static random graph in which each node chooses \(d\) random neighbors is a \(\Theta(1)\)-expander, w.h.p., for any choice of the parameter \(d \geq 3\) (see Lemma 5.4 in Section 5 of the Appendix). In the next lemma we instead show that this is not the case for the SDG model: w.h.p., there can be a linear fraction of isolated nodes at every time steps. Informally speaking, this fact is essentially due to the presence of “older” nodes that have good chance to see all their out-edges disappear and, at the same time, to get no in-edges from younger nodes. The formal argument (which is given in Subsection 6.1.2) to prove this intuitive fact requires the use of the method of bounded difference to manage the correlations among the random variables, each one indicating whether a given node gets isolated or not.

**Lemma 3.5** (Isolated nodes). For every positive constant \(d\) and for every sufficiently large \(n\), let \(\{G_t = (N_t, E_t) : t \in \mathbb{N}\}\) be an SDG sampled from \(G(n,d)\). For every fixed \(t \geq n\), w.h.p. the number of isolated nodes in \(G_t\) is at least \(\frac{d}{6}ne^{-2d}\). Moreover, w.h.p., each of these nodes will remain isolated across its entire lifetime.
3.1.1 Expansion properties

As Lemma 3.5 suggests, we have no generalized expansion properties in the SDG model. Still, we can prove a weaker expansion property, which only applies to sufficiently large subsets of the vertices. This property is crucial in proving our positive result about flooding in this model and it is stated in Lemma 3.6 below.

**Lemma 3.6** (Expansion of large subsets). For every constant \(d \geq 20\) and for every sufficiently large \(n\), let \(\{G_t = (N_t, E_t) : t \in \mathbb{N}\}\) be an SDG sampled from \(G(n, d)\). For every fixed \(t \geq n\), w.h.p. the snapshot \(G_t\) satisfies the following:

\[
\min_{S \subseteq N_t, n e^{-d/10} \leq |S| \leq n/2} \frac{\partial_{\text{out}}(S)}{|S|} \geq 0.1.
\]

The proof’s idea of the above property is to show that any two disjoint sets \(S, T \subseteq N_t\), with \(n e^{-d/10} \leq |S| \leq n/2\) and \(|T| = 0.1|S|\), such that \(\partial_{\text{out}}(S) \subseteq T\), exist with negligible probability and then apply a union bound over all possible pairs \(S, T \subseteq N_t\). To get the first fact, we derive a suitable argument that takes care about the younger/older relationship between any pair of nodes. We then exploit the fact that any node \(u\) has probability \(1/n\) to send a request to a node \(v\) older than him. The full proof is given in Subsection 3.1.3.

3.1.2 Flooding

We begin with a negative result about flooding in the SDG model. We recall that Lemma 3.5 shows the existence of a linear fraction of nodes that keep isolated for all their respective lifetime. This is the key-ingredient in proving the following fact (the full proof is given in Subsection 3.1.4).

**Theorem 3.7** (Flooding). For every positive constant \(d\), for every sufficiently large \(n\), and for every fixed \(t_0 \geq n\), the flooding process over an SDG sampled from \(G(n, d)\) starting at \(t_0\) satisfies the following two statements:

1. With probability \(\Omega(e^{-d^2})\), for every \(t \geq t_0\), \(I_t\) contains at most \(d + 1\) nodes;
2. W.h.p. The flooding time is \(\Omega(dn)\).

On the other hand, there is a large constant probability that a broadcast will reach a large fraction of nodes within \(O(\log n)\) time.

**Theorem 3.8** (Flooding completes for a large fraction of nodes). For every constant \(d > 200\), for every sufficiently large \(n\) and for every fixed \(t_0 \geq n\), there is a \(\tau = O(\log n / \log d + d)\), such that the flooding process over an SDG sampled from \(G(n, d)\) starting at \(t_0\) satisfies the following:

\[
\Pr \left( |I_{t_0 + \tau}| \geq (1 - e^{-d/10})n \right) \geq 1 - 4e^{-d/100} - o(1),
\]

As remarked in Subsection 3.2.1, the proof of the above result is one of our major technical contributions: for this reason, in what follows, we provide its description.

**Proof of Theorem 3.8**

The proof consists of two steps. Assuming the source node \(s\) joined the network in round \(t_0\), we first show (Lemma 3.9) that, with probability at least \(1 - 4e^{-d/100}\), a restriction of the true topology dynamics establishes a bipartite graph which i) contains \(s\), ii) only connects nodes with ages in the interval \(\{1, \ldots, n/2\}\) to nodes with ages in the interval \(\{n/2 + 1, \ldots, n - \log n\}\), iii) has diameter \(O(\log n)\), iv) includes at least \(2n/d\) nodes. This is enough to prove that, with probability \(1 - 4e^{-d/100}\), \(2n/d\) nodes are informed at time \(t_0 + \tau_1\), where \(\tau_1 = O(\log n)\).

The second step consists in showing (Lemma 3.12) that, thanks to the expansion properties established in Lemma 3.6, once \(2n/d\) nodes have been informed, at least \((1 - e^{-d/10})n\) nodes will become informed within a constant number \(\tau_2 = \Theta(d)\) of additional steps, w.h.p.

Overall, the above two steps prove that within time \(t_0 + \tau_1 + \tau_2\), at least \((1 - e^{-d/10})n\) nodes have been informed, with probability at least \(1 - 4e^{-d/100} - o(1)\). We begin with the first part, corresponding to the following lemma.
Lemma 3.9 (Flooding completes for a large fraction of nodes, phase 1). Under the hypotheses of Theorem 3.8, there is a $\tau_1 = O(\log n / \log d)$ such that
\[
\Pr \left( |H_{t_0 + \tau_1}| \geq \frac{2n}{d} \right) \geq 1 - 4e^{-\frac{n}{100}}.
\] (1)

Proof. We begin by defining the following subsets of $N_{t_0}$:
- the set of the young nodes: $Y = \{ v \in N_{t_0} \mid v$ has life $l$ with $2 \leq l < \frac{d}{2} \}$
- the set of the old nodes: $O = \{ v \in N_{t_0} \mid v$ has life $l$ with $\frac{d}{2} \leq l \leq n - \log n \}$
- the set of the very old nodes: $\hat{O} = N_{t_0} - (Y \cup O) = \{ v \in N_{t_0} \mid v$ has life $l$ with $n - \log n < l \leq n \}$

To prove (1) we show that $G_{t_0} = (N_{t_0}, E_{t_0})$ contains a bipartite subgraph with logarithmic diameter, containing the informed node $s$ and such that i) links are established only between nodes in $Y$ and in $O$ and ii) it contains no very old node. The graph in question is the result of the onion-skin process described below.

The onion-skin process. The iterative process we consider operates in phases, each consisting of two steps. Starting from $s$, the onion-skin process builds a connected, bipartite graph, corresponding to alternating paths in which young nodes only connect to old ones. In particular, each realization of this process generates a subset of the edges generated by the original topology dynamics. Moreover, each iteration of the process corresponds to a partial flooding in the original graph, in which a new layer of informed nodes is added to the subset of already informed ones, hence the term onion-skin. Flooding is partial since i) the network uses a subset of the edges that would be present in the original graph.

In the following, we denote by $Y_k \subseteq Y$ and $O_k \subseteq O$ the subsets of young and old nodes that are informed by the end of phase $k$, respectively. In the remainder, we let $O_{-1} = \emptyset$ for notational convenience.

| Onion-skin process |
|--------------------|
| **Phase 0:** $Y_0 = \{ s \}$; $O_0$ is obtained as follows: $s$ establishes $d$ links. We let $O_0 \subset O$ denote the subset of old nodes that are destinations of these links. Links with endpoints in $Y$ or $O$ are discarded; |
| **Phase $k \geq 1$:** $Y_k$ and $O_k$ are iteratively obtained as follows: |
| *Step 1.* Each node in $Y - Y_{k-1}$ establishes $d/2$ links. More precisely: |
| $Y_k - Y_{k-1} = \{ v \in Y - Y_{k-1} \mid v$ connects to $O_{k-1}$ by a request $i \in \{ \frac{d}{2} + 1, \ldots, d \} \}$ |
| Links to nodes not belonging to $O$ are discarded; |
| *Step 2.* Each node in $Y_k - Y_{k-1}$ establishes $d/2$ links to nodes in $O - O_{k-1}$. More precisely: |
| $O_k - O_{k-1} = \{ v \in O - O_{k-1} \mid$ some $w \in Y_k$ connects to $v$ by a request $i \in \{ 1, \ldots, \frac{d}{2} \} \}$ |
| Links to nodes not belonging to $O$ are discarded. |

A couple remarks are in order. It is clear that the links in $E_{t_0}$ can be established in any order, as long as they are created from younger nodes towards older ones. As a consequence, each realization of the onion-skin process produces a subset of $E_{t_0}$. In particular, i) nodes in $O$ and $\hat{O}$ do not create any links, though they can still be the targets of links originating from $Y$; ii) a node $v$ in $Y$ released at time $\ell$ ($\leq t_0$) creates $d$ links, with possible destinations the nodes released in the interval $[\ell, t_0]$, but only links with destinations in $O$ are retained, the others are discarded.

The next claim states that, at each step, the sets of informed nodes $Y_k \subseteq Y$ and $O_k \subseteq O$ grow by a constant factor $d/20$. It analyzes Phase 0 and the generic Phase $k$ separately and it is proved in Subsection 6.1.5.

Claim 3.10. The following holds for Phase 0,
\[
\Pr \left( |O_0| \geq \frac{d}{20} \right) \geq 1 - e^{-d/100}.
\]
In the generic phase \( k \geq 1 \), if \( |Y_{k-1}| \leq n/d \) and \( |O_{k-1}| \leq n/d \),

\[
\Pr \left( |Y_k - Y_{k-1}| > \frac{d}{20} y \mid |O_{k-1} - O_{k-2}| \geq y \right) \geq 1 - e^{-yd/100}
\]

\[
\Pr \left( |O_k - O_{k-1}| > \frac{d}{20} x \mid |Y_k - Y_{k-1}| \geq x \right) \geq 1 - e^{-dx/100}.
\]

Then, from the above claim and using the chain rule, we get that, for each \( k \geq 0 \),

\[
\Pr (|O_k - O_{k-1}| \geq a_{2k+1}) \geq \prod_{i=0}^{2k} \left( 1 - e^{-a_i(d/100)} \right) \quad \text{and} \quad \Pr (|Y_k - Y_{k-1}| \geq a_{2k}) \geq \prod_{i=0}^{2k} \left( 1 - e^{-a_i(d/100)} \right),
\]

where \( a_k = \left( \frac{d}{20} \right)^k \) and as long as \( a_{2k} \) and \( a_{2k+1} \) are smaller than \( n/d \). Then, after some \( \tau_1 = \log n / \log d \) rounds, we get \( |Y_{t_0 + \tau_1}| \geq n/d \) and \( |O_{t_0 + \tau_1}| \geq n/d \), with probability at least

\[
c = \prod_{i=0}^{\infty} \left( 1 - e^{-a_i(d/100)} \right) > 1 - 4e^{-100}.
\]

In Subsection 6.1.0 using standard calculus, we prove the following claim, which concludes the proof.

**Claim 3.11.** For each \( d > 200 \), if \( a_i = (d/20)^i \),

\[
c = \prod_{i=0}^{\infty} \left( 1 - e^{-a_i(d/100)} \right) \geq 1 - 4e^{-100}.
\]

**Lemma 3.12** (Flooding completes for a large fraction of nodes, phase 2). Under the hypotheses of Theorem 3.8, a constant \( \tau_2 = \Theta(d) \) exists such that, for \( \tau_1 = \mathcal{O}(\log n / \log d) \) (as in Lemma 3.9) we have:

\[
\Pr \left( |I_{t_0 + \tau_1 + \tau_2}| \geq (1 - e^{-d/10})n \right) \geq 1 - 4e^{-d/100} - o(1).
\]

The proof of the above lemma, which is given in Subsection 6.1.0, heavily relies on the expansion properties of large subsets proven in Lemma 3.9. In more detail, we first observe that Lemma 3.9 implies \( |I_{t_0 + \tau_1}| \geq 2n/d \). We can then inductively apply the expansion property stated by Lemma 3.9 to the set of informed nodes \( I_t \), for each \( t \geq t_0 + \tau_1 \), until the size of this subset becomes \( n/2 \). After that, the expansion property and the consequent inductive argument is instead applied to the set of non-informed nodes. The process ends when the size of the set of non-informed nodes falls below \( \leq ne^{-d/10} \), since at that point we can no longer apply Lemma 3.9. We notice that, in the whole proof, the oldest \( \tau_2 \) nodes in \( N_{t_0 + \tau_1} \) are never considered, since they all die within the next \( \tau_2 \) steps.

### 3.2 Streaming graphs with edge regeneration

We now consider a variant of the streaming dynamic graph model where a node creates its outgoing links not only when it joins the network, but also every time it looses an outgoing link due to one of its neighbors leaving the network. In this model, at every round \( t \) the snapshot \( G_t \) is a sparse random graph having exactly \( dn \) edges.

**Definition 3.13** (Streaming graphs with edge regeneration). A Streaming Dynamic Graph with edge Regeneration (for short, SDGR) \( G(n, d) \) is a dynamic random graph \( \{ G_t = (N_t, E_t) : t \in \mathbb{N} \} \) where the set of nodes \( N_t \) evolves according to Definition 3.2, while the set of edges \( E_t \) evolves according to the following topology dynamics:

1. When a new node appears, it creates \( d \) independent connections, each one with a node chosen uniformly at random among the nodes in the network.
2. When a node dies, all its incident edges disappear.
3. When a node has one of its \( d \) outgoing edges disappearing, it creates a new connection with a node chosen uniformly at random among all nodes in the network.
**Preliminary properties.** We will prove that the streaming model with edge regeneration yields snapshots having good vertex expansion. To derive the expansion properties we first prove a bound on the edge probability. Informally, we show that, despite the presence of nodes of different ages, making the edge distribution non uniform, the probability that a fixed node chooses any other active node in the network is still $O(1/n)$.

For constant $d \geq 20$ and for sufficiently large $n$, let $\{G_t = (N_t, E_t) : t \in \mathbb{N}\}$ be an SDG sampled from $G(n,d)$. For every fixed $t \geq n$, w.h.p. the snapshot $G_t$

**Lemma 3.14.** For every $d \geq 1$ and for every sufficiently large $n$, let $\{G_t = (N_t, E_t) : t \in \mathbb{N}\}$ be an SDGR sampled from $G(n,d)$. For every fixed $t \geq n$, consider the snapshot $G_t$. Let $k \leq t-1$ and let $u$ be the node having age $k+1$. Then, if another node $v$ in $N_t$ is born before $u$, the probability that a single request of $u$ has destination $v$ is

$$\frac{1}{n-1} \left(1 + \frac{1}{n-1} \right)^k,$$

while, if $v$ is born after $u$, the probability that a single request of $u$ has destination $v$ is always $\leq \frac{1}{n-1}$.

The almost-uniformity of the destination distribution stated in the lemma above is essentially due to the fact that, in the streaming model, every node node has lifetime $n$ and, hence, it has at most $n$ chances to be chosen as destination along the regeneration process. In formula, this yields, in the worst case, the extra factor $(1 + 1/n)^{O(n)}$ in (2). The full proof of the lemma is given in Subsection 3.2.1

### 3.2.1 Expansion properties

In this subsection, we show that, for a sufficiently large constant $d$, the streaming model with edge regeneration yields snapshots having good vertex expansion.

For constant $d \geq 20$ and for sufficiently large $n$, let $\{G_t = (N_t, E_t) : t \in \mathbb{N}\}$ be an SDG sampled from $G(n,d)$. For every fixed $t \geq n$, w.h.p. the snapshot $G_t$

**Theorem 3.15 (Expansion).** For every $d \geq 14$ and for every sufficiently large $n$, let $\{G_t = (N_t, E_t) : t \in \mathbb{N}\}$ be an SDGR sampled from $G(n,d)$. Then, w.h.p., for every fixed $t \geq n$, the snapshot $G_t$ is an $\varepsilon$-expander with parameter $\varepsilon \geq 0.1$.

The full proof of the above result is given in Subsection 3.2.2, while an overview is given below. The proof is divided into two parts: the expansion for the big-size sets (with size in the range $[n/4, n/2]$) and the expansion for small-size subsets (with size in the range $[1, n/4]$). As for the first case, the analysis is identical to that of Lemma 3.6 for the SDG model. To analyze the expansion of small-size subsets, we show that, for every pair of vertex subset $S$, with $|S| \leq n/4$ and $T$, with $S \cap T = \emptyset$ and $|T| = 0.1|S|$, the event “all the out-neighbors of $S$ are in $T$”, i.e. $A_{S,T} = \{\partial_{out}(S) \subseteq T\}$, does happen with negligible probability. To give an upper bound on $Pr(A_{S,T})$, we observe that $A_{S,T}$ is bounded by the event that each link request of every node in $S$ must have destination in $S \cup T$. Thanks to Lemma 3.14, for any pair of subset $S$ and $T$, we can derive the following bound

$$Pr(A_{S,T}) \leq \left(\frac{\varepsilon}{n-1} \cdot |S \cup T|\right)^{d[S]}.$$

Since $|S| \leq n/4$, using standard calculus, we show the above equation offers a sufficiently small bound. The theorem then follows from an union bound over all possible pairs $S, T \subseteq N_t$.

### 3.2.2 Flooding

An important consequence of the expansion property we prove in Theorem 3.15 is that the flooding process over the SDGR model is fast and reaches all nodes of the network. The proof of this fact for this streaming model is a simple adaptation of the expansion argument which is typically used in dynamic graph models with no node churn (see, for example, [7]). The deterministic and limited node churn has in fact a negligible impact in the analysis, only. The proof is given in Subsection 3.2.3.

**Theorem 3.16 (Flooding).** For every $d \geq 21$, for every sufficiently large $n$, and for every fixed $t_0 \geq n$, w.h.p. the flooding process over an SDGR sampled from $G(n,d)$ starting at $t_0$ has completion time $\mathcal{O}(\log n)$.
4 The Poisson Model

A continuous dynamic graph $G$ is a continuous family of graphs $G = \{G_t = (N_t, E_t) : t \in \mathbb{R}^+\}$ where the sets of nodes and edges can change at any time $t \in \mathbb{R}^+$. As in the discrete case, we call $G_t$ the snapshot of the dynamic graph at time $t$ and, for a set of nodes $S \subseteq N_t$, we denote with $\partial^\text{out}_t(S)$ the outer boundary of $S$ in snapshot $G_t$ and we omit superscript $t$ when it is clear from context.

In this section we study expansion properties and flooding over two continuous-time dynamic graph models in which nodes’ arrivals follow a Poisson process and their lifetimes obey an exponential distribution.

Definition 4.1 (Poisson node churn). Initially $N_0 = \emptyset$. Node arrivals in $N_t$ follow a Poisson process with mean $\lambda$. Moreover, once a node joins the network, its lifetime has exponential distribution with parameter $\mu$.

While the definition of flooding is straightforward in the discrete case (Definition 3.3), where we assume that the sets of nodes and edges can change and all the neighbors of an informed node gets informed in one unit of time, in the continuous case we need to specify how the time it takes a message to flow from a node to its neighbors and the changes in the topology of the graph relate to each other. Since we want to preserve in the model the fact that a message takes one unit of time to flow from an informed node to its neighbors, the most natural way to define the flooding process in a continuous setting would be the following “asynchronous” version.

Definition 4.2. (“Asynchronous” Flooding) Let $G = \{G_t = (N_t, E_t) : t \in \mathbb{R}^+\}$ be a dynamic (random) graph. The flooding process over $G$ starting at time $t_0$ from vertex $v_0 \in N_{t_0}$ is the sequence of (random) sets of nodes $\{I_t : t \in \mathbb{R}^+\}$ where, $I_t = \emptyset$ for all $t < t_0$, $I_{t_0} = \{v_0\}$ (in this paper we will thus assume that $I_0$ contains the node joining the network at round $t_0$) and, for every $t \geq t_0$, $I_t$ contains all nodes in $N_t$ that were neighbor of some node in $I_{t-1}$ in the snapshot $G_{t-1}$, in addition to all previously informed nodes

$$I_t = \left( \bigcup_{\nu < t} I_\nu \right) \cup \partial^{\text{out}}_{t-1}(I_{t-1}) \cap N_t.$$

We say that the nodes in $I_t$ are informed at time $t$. We say that the flooding completes the broadcast if a time $t$ exists such that $I_t \supseteq N_t$, in this case the time $t - t_0$ is the flooding time of the source message.

In order to analyze the process of Definition 4.2, it will be convenient to define the discretized process below, in which nodes are informed only at discrete times.

Definition 4.3. (“Discretized” Flooding) Let $G = \{G_t = (N_t, E_t) : t \in \mathbb{R}^+\}$ be a continuous dynamic (random) graph. The flooding process over $G$ starting at time $t_0 \in \mathbb{R}^+$ from vertex $v_0 \in N_{t_0}$ is the sequence of (random) sets of nodes $\{I_t : t \in \mathbb{N}\}$ where, $I_t = \emptyset$ for all $t < t_0$, $I_{t_0} = \{v_0\}$ and, for every $t$ of the form $t_0 + m$ with integer $m$, $I_t$ contains all nodes in $I_{t-1}$ that did not die in the time interval $(t-1,t)$ and all nodes in $N_t$ that have been neighbor of some node in $I_{t-1}$ for the whole time interval $(t-1,t)$:

$$I_t = \left( I_{t-1} \cup \partial^{\text{out}}_{t-1}(I_{t-1} \cap N_t) \right) \cap N_t.$$

We say that the nodes in $I_t$ are informed at round $t$. We say that the flooding completes the broadcast if a round $t$ exists such that $I_t \supseteq N_t$, in this case the time $t - t_0$ is the flooding time of the source message.

The discretized process, which is artificial and is defined only for the purpose of the analysis, can be thought of as the asynchronous modified in such a way that an informed node waits until a discrete time before sending messages. Thus, the convergence of the discretized flooding can only be slower than the convergence of the asynchronous flooding, and any upper bound that we prove on the convergence time of the former will apply to the latter.

Our negative results, however, also apply to Definition 4.2.

4.1 Poisson node churning

In this subsection, we present useful properties of Poisson dynamic graphs that only depend on the random node churn process and therefore apply to both variants of the model, i.e., with and without edge regeneration.

We remark that, according to Definition 4.1 above, the time interval between two consecutive node arrivals is an exponential random variable of parameter $\lambda$, while the number of nodes joining the network
in a time interval of duration $\tau$ is a Poisson random variable with expectation $\tau \cdot \lambda$. We finally note that the stochastic continuous process $\{N_t : t \in \mathbb{R}^+\}$ is clearly a continuous Markov Process.

A first important fact our analysis relies on is that we can bound the number of active nodes at every time. In particular, it is easy to show that $E[|N_t|] \to \lambda/\mu$ and, moreover, we have the following bound in concentration.

**Lemma 4.4** (Pandurangan et al. [23] - Number of nodes in the network). For every pair of parameters $\lambda$ and $\mu$ such that $n = \lambda/\mu$ is sufficiently large, consider the Poisson node churn $\{N_t : t \in \mathbb{R}^+\}$ in Definition 4.1. Then, for every fixed real $t \geq 3n$, w.h.p. $|N_t| = \Theta(n)$ and, more precisely,

$$\Pr\left( 0.9n \leq |N_t| \leq 1.1n \right) \geq 1 - 2e^{-\sqrt{n}}.$$

Leveraging Lemma 4.4, our analysis of the Poisson considers the setting $\lambda = 1$ without loss of generality. In the remainder, we define the key parameter $n = \frac{1}{\mu}$ representing the “expected” size of the network. Moreover, since the probability that two or more churn events occur at the same time is zero, the points of change of the dynamic graph yield a discrete-time sequence of events. In particular, we can observe and prove properties of the dynamic graph only when one event changing the graph occurs, namely, the arrival of a new node or the death of an existing one.

**Definition 4.5.** Let $\{N_t : t \in \mathbb{R}^+\}$ be a Poisson node churn as in Definition 4.1. Define the infinite sequence of random variables steps (also called rounds) $\{T_r : r \in \mathbb{N}\}$ (with parameters $\lambda$ and $\mu$) as follows:

$$T_0 = 0 \quad \text{and} \quad T_{r+1} = \inf\{t > T_r : N_t \neq N_{T_r}\}, \quad \text{for } r = 0, 1, 2, \ldots.$$

It is worth mentioning that, since the Poisson stochastic process $\{N_t : t \in \mathbb{R}^+\}$ is a continuous Markov process, the above defined stochastic process $\{N_{T_r} : r \in \mathbb{N}\}$ consistently is a discrete Markov chain.

Thanks to Theorem C.3 in the Appendix, we can easily find the law of the random variables that define the time steps at which new events occur. The proof of the next lemma is given in Subsection 7.1.1.

**Lemma 4.6** (Jump process). The stochastic process $\{N_{T_r}, r \in \mathbb{N}\}$ in Definition 4.5 is a discrete Markov chain where, for every fixed integer $r \geq 0$ and for every integer $N \geq 0$, conditional to the event “$|N_{T_r}| = N$”, $T_{r+1}$ is a random variable of exponential distribution with parameter $N\mu + \lambda$. Moreover,

$$\Pr\left(|N_{T_{r+1}}| = |N_{T_r}| - 1 \mid |N_{T_r}| = N\right) = N\mu \over N\mu + \lambda,$$

$$\Pr\left(|N_{T_{r+1}}| = |N_{T_r}| + 1 \mid |N_{T_r}| = N\right) = \frac{\lambda}{N\mu + \lambda}.$$ 

Finally, for every fixed node $v \in N_{T_r}$, the probability that the decreasing of $N_{T_r}$ is due to the death of $v$ is

$$\Pr\left(v \notin N_{T_{r+1}} \mid v \in N_{T_r}, |N_{T_r}| = N\right) = \frac{\mu}{N\mu + \lambda}.$$

The next lemma shows that the probability of the next event being a node arrival or death is close to $1/2$ since, for large enough $r$, $|N_t|$ is w.h.p. close to $n$. The proof is deferred to Subsection 7.1.2.

**Lemma 4.7.** For every sufficiently large $n$, consider the Markov chain $\{N_{T_r}, r \in \mathbb{N}\}$ in Definition 4.5 with parameters $\lambda = 1$ and $\mu = 1/n$. Then, for every fixed integer $r \geq n \log n$,

$$0.47 \leq \Pr\left(|N_{T_{r+1}}| = |N_{T_r}| - 1\right) \leq 0.53 \quad \text{and} \quad 0.47 \leq \Pr\left(|N_{T_{r+1}}| = |N_{T_r}| + 1\right) \leq 0.53.$$ 

Moreover, if $v \in N_{T_r}$,

$$\frac{1}{2.2n} \leq \Pr\left(v \notin N_{T_{r+1}} \mid v \in N_{T_r}\right) \leq \frac{1}{1.8n}.$$ 

The next lemma provides a useful bound on the lifetime of any node in the network. The proof is given in Subsection 7.1.3.

**Lemma 4.8** (Lifetime of the nodes). For every sufficiently large $n$, consider the Markov chain $\{N_{T_r}, r \in \mathbb{N}\}$ in Definition 4.5 with parameters $\lambda = 1$ and $\mu = 1/n$. Then, for every fixed integer $r \geq 7n \log n$, with probability at least $1 - 2/n^{2.1}$, each node in $N_{T_r}$ was born after step $T_{r-7n \log n}$.
4.2 Poisson graphs without edge regeneration

We consider two variants of dynamic graphs with node churns governed by Poisson processes that mirror the two dynamics in Definitions 3.4 and 3.13. In this subsection, we consider the first variant, in which new edges are created only when a new node joins the network.

**Definition 4.9** (Poisson dynamic graphs without edge regeneration). A Poisson Dynamic Graph without edge regeneration (for short, PDG) \( \mathcal{G}(\lambda, \mu, d) \) is a continuous dynamic random graph \( \{G_t = (N_t, E_t) : t \in \mathbb{R}^+\} \) where the set of nodes \( N_t \) evolves according to Definition 4.1 while the set of edges \( E_t \) according to the following topology dynamics:

1. When a new node appears, it creates \( d \) independent connections, each one with a node chosen uniformly at random among the nodes in the network.
2. When a node dies, all its incident edges disappear.

**Preliminary properties.** Similarly to the streaming model, the Poisson model without edge regeneration may result in the presence of a linear fraction of isolated nodes. The proof of this negative result proceeds along the same lines as the case of the streaming model (Lemma 4.9). In more detail, we leverage Lemma 4.1 and Lemma 4.8 to characterize the random churn. The full proof is given in Subsection 7.2.2.

**Lemma 4.10** (Isolated nodes). For every positive constant \( d \) and for every sufficiently large \( n \), let \( \{G_t = (N_t, E_t) : t \in \mathbb{R}^+\} \) be a PDG sampled from \( \mathcal{G}(\lambda, \mu, d) \) with \( \lambda = 1 \) and \( \mu = 1/n \). For every fixed integer \( r \geq 7n \log n \), w.h.p. the number of isolated nodes in \( G_{Tr} \) is at least \( \frac{1}{18} ne^{-2d} \). Moreover, w.h.p., each of these nodes will remain isolated across its entire lifetime.

4.2.1 Expansion properties

The lemma that follows highlights weak expansion properties of the Poisson model without edge regeneration. In particular, we show that, for any sufficiently large \( t \), all subsets of \( N_t \) including a sufficiently large, constant fraction of the nodes exhibit good expansion properties.

**Lemma 4.11** (Expansion of large subsets). For every constant \( d \geq 20 \) and for every sufficiently large \( n \), let \( \{G_t = (N_t, E_t) : t \in \mathbb{R}^+\} \) be a PDG sampled from \( \mathcal{G}(\lambda, \mu, d) \) with \( \lambda = 1 \) and \( \mu = 1/n \). Then, for every fixed integer \( r = \Omega(n \log n) \), with probability at least \( 1 - 2/n^2 \), the snapshot \( G_{Tr} \) satisfies

\[
\min_{|S| \geq 0.1|N_{Tr}|/2} \frac{|\partial_{out}(S)|}{|S|} \geq 0.1.
\]

The full proof of the above lemma is given in Subsection 7.2.2 and it is based on the following idea. For any fixed pair of subsets \( S \) and \( T \) with \( |T| = 0.1|S| \), we first observe that the event \( A_{S,T} \) (defined as in the analogous proof in the SDG model, Lemma 4.6) implies that there is no outgoing link from \( S \) to the subset \( P = N_{Tr} - S - T \), and, using Lemma 4.1, we know that \( |P| \geq 0.9n - 1.1s \), w.h.p. By using a simple counting argument, we show that each of at least half of all the edges in the cut \( E(S, T) \) have probability \( \geq 1/\Theta(n) \) to belong to \( E_{Tr} \). Then, since \( S \) and \( P \) are large, the number of such potential edges is large enough to apply a standard union bound to all possible subset pairs \( S,T \subseteq N_{Tr} \).

4.2.2 Flooding

The negative result of Lemma 4.10 implies that the flooding process has non-negligible chances to fail in rapidly informing the entire network. Its proof, which is given in Subsection 7.2.3, relies on the presence of isolated nodes and uses the original Definition 4.2, adapting the argument we used in the proof of Theorem 4.7 (for the streaming model) to take care about the presence of nodes having random lifetime that can be of length \( \Theta(n \log n) \).

**Theorem 4.12** (Flooding). For every positive constant \( d \), for every sufficiently large \( n \) and for every fixed \( r_0 \geq 7n \log n \), the flooding process over a PDG sampled from \( \mathcal{G}(\lambda, \mu, d) \) with \( \lambda = 1, \mu = 1/n \) and starting at \( t_0 = T_{r_0} \) satisfies the following two properties:

1. With probability \( \Omega(e^{-d}) \), for every \( t \geq t_0 \), \( I_t \) contains at most \( d + 1 \) nodes;
2. W.h.p., the flooding time is \( \Omega_d(n) \).
We next complement the negative results above by showing that, following the arrival of an informed node at some time \( t \), a fraction \( 1 - e^{-\Omega(d)} \) of the vertices of the network will become informed within the following \( O(\log n) \) flooding steps, with probability \( 1 - e^{-\Omega(d)} \).

**Theorem 4.13** (Flooding completes for a large fraction of nodes). For every constant \( d \geq 1152 \), for every sufficiently large \( n \) and for every fixed \( r_0 \geq 7n \log n \), there is a \( \tau = \Theta(\log n / \log d + d) \), such that the flooding process over a PDG sampled from \( G(\lambda, \mu, d) \) with \( \lambda = 1 \), \( \mu = 1/n \) and starting at \( t_0 = T_{r_0} \), satisfies the following:

\[
\Pr \left( \left| I_{t_0 + \tau} \right| \geq (1 - e^{-\frac{2}{n}})|N_{t_0 + \tau}| \right) \geq 1 - 2e^{-\frac{576}{n}} - o(1).
\]

The proof of Theorem 4.13 is presented in Subsection 7.2.4 and proceeds along lines similar to those of Section 3.1.2, though with some important differences, which we briefly discuss below. It should be noted that, in order to account for the fact that a live node might die at any point of a given flooding interval, the proof of Theorem 4.13 uses the discretized version of the flooding process described by Definition 4.1, which clearly provides a worst case scenario when we are interested in proving lower bounds on the extent and upper bounds on the speed of flooding.

We first show that, starting with an informed node \( s \) joining the network at time \( t_0 \), with probability \( 1 - 2e^{-\frac{576}{n}} - o(1) \), at least \( \frac{n}{10} \) nodes are informed at time \( t_0 + \tau_1 \), where \( \tau_1 = \Theta(\log n / \log d) \). To prove a similar result in the streaming model, we considered a subset of the vertices, inducing a topology that remained unchanged within an interval of interest of logarithmic size. This way, the proof boiled down to proving diameter properties of this induced subgraph, which we did by introducing the onion-skin process. Unfortunately, this approach does not trivially carry over to the Poisson model, since every node that is in the network at any given time \( t \) has some probability of dying within each time unit. To address this issue, we define a variant of the onion-skin process, in which i) flooding proceeds alongside edge creation, in the sense that a node only establishes its links upon becoming informed (deferred decisions), ii) each newly informed node tosses a coin to decide whether or not it is going to die before time \( t_0 + \tau_1 \). In order to consider a worst-case scenario, if a node dies before time \( t_0 + \tau_1 \), the node leaves the network immediately upon being reached in the flooding process, without generating any links or informing any neighbours. We also leverage two facts, namely, since we are considering an overall flooding interval spanning a logarithmic number of steps, the number of nodes joining the network in this interval is itself at most logarithmic, while the probability of a node that is alive at time \( t_0 \) to die before time \( t_0 + \Theta(\log n) \) is \( O(\log n / n) \). This allows us to prove that, with probability at least \( 1 - 2e^{-\frac{576}{n}} - o(1) \), at least a constant fraction of the nodes are informed within time \( t_0 + \tau_1 \). The second step is similar to the case of the streaming model, leveraging expansion of large sets and, in particular, Lemma 4.11 above. In particular, if at least \( n/10 \) are informed, Lemma 4.11 allows to show that, with high probability, a further, constant number \( \tau_2 = \Theta(d) \) of flooding steps suffice to reach a fraction \( 1 - e^{-\frac{2}{n}} \) of the nodes. Choosing \( \tau = \tau_1 + \tau_2 \) allows to prove that within time \( t_0 + \Theta(\log n / \log d + d) \), a fraction at least \( 1 - e^{-\frac{2}{n}} \) of the nodes is informed, with probability at least \( 1 - 2e^{-\frac{576}{n}} - o(1) \).

### 4.3 Poisson graphs with edge regeneration

We now model graph dynamics where an active node replaces each of its \( d \) outgoing edges that will be deleted.

**Definition 4.14** (Poisson dynamic random graphs with edge regeneration). A Poisson Dynamic Graph with edge Regeneration \( G(\lambda, \mu, d) \) (for short, PDGR) is a continuous dynamic random graph \( \{G_t = (N_t, E_t) : t \in \mathbb{R}^+ \} \) where the set of nodes \( N_t \) evolves according to Definition 3.1, while the set of edges \( E_t \) evolves according to the following topology dynamics:

1. When a new node appears, it creates \( d \) independent connections, each one with a node chosen uniformly at random among the nodes in the network.
2. When a node dies, all its incident edges disappear.
3. When a node has one of its \( d \) outgoing edges disappearing, it creates a new connection with a node chosen uniformly at random among all the nodes in the network.

**Preliminary properties.** Similarly to the approach we adopted for the streaming model, our first technical step is to provide an upper bound on the probability that a fixed node chooses any other active node in the network as destination of one of its \( d \) requests. However, things in this setting get
more complicated essentially because of the presence of “very old” nodes (i.e. those nodes having age \(\omega(n)\)). Indeed, such old nodes can be selected as destination of a link request from a younger node with probability \(\omega(1/n)\). The next lemma formalizes this fact as function of the age of the nodes. The proof follows the same approach we used to get Lemma 6.14 and it is given in Subsection 7.3.3.

**Lemma 4.15.** For every constant \(d \geq 20\) and for every sufficiently large \(n\), let \(\{G_i = (N_i, E_i) : t \in \mathbb{R}^+\}\) be a PDGR sampled from \(G(\lambda, \mu, d)\) with \(\lambda = 1\) and \(\mu = 1/n\). Then, for every fixed integer \(r = \Omega(n \log n)\), consider the snapshot \(G_{T_r}\). Let \(u \in N_{T_r}\) be the node born in round \(T_{r-1}\) for some integer \(i \leq r\). Then, if another node \(v \in N_{T_r}\) is born before \(u\), the probability that a single request of \(u\) has destination \(v\) is at most

\[
\frac{1}{0.8n} \left(1 + \frac{i}{1.7n}\right).
\]

While, if \(v\) is born after \(u\), the probability that a single request of \(u\) has destination \(v\) is always \(\leq \frac{1}{1.8n}\).

#### 4.3.1 Expansion properties

The expansion property satisfied by the Poisson model with edge regeneration can be stated as follows.

**Theorem 4.16 (Expansion).** For every constant \(d \geq 35\) and for every sufficiently large \(n\), let \(\{G_i = (N_i, E_i) : t \in \mathbb{R}^+\}\) be a PDGR sampled from \(G(\lambda, \mu, d)\) with \(\lambda = 1\) and \(\mu = 1/n\). Then, for every fixed integer \(r \geq 7n \log n\), w.h.p. the snapshot \(G_{T_r}\) is an \(\varepsilon\)-expander with parameter \(\varepsilon \geq 0.1\).

The proof proceeds analyzing three different size ranges of the vertex subset \(S \subseteq N_{T_r}\), the expansion of which has to be shown.

**Expansion of small subsets.**

**Lemma 4.17** (Expansion of small subsets). Under the hypothesis of Theorem 4.16, for subsets \(S\) of \(N_{T_r}\), with probability of at least \(1 - 2/n^2\),

\[
\min_{0 \leq |S| \leq n/\log^2 n} \frac{|\partial_{out}(S)|}{|S|} \geq 0.1.
\]

The proof of the above lemma adapts the argument we used in the proof of Lemma 6.14 for the streaming model, and it is given in the Subsection 7.3.2.

**Expansion of middle-size subsets.** The second case deals with subsets of size in the range \(n/\log^2 n \leq |S| \leq n/14\) and its analysis definitely represents one of the key technical contributions of this paper. Indeed, departing from the first case, the presence of a large number of subsets in this range does not allow to use any rough worst-case counting argument: for instance, assuming that all nodes in the considered subset \(S\) have age \(\Theta(n \log n)\) and applying the corresponding edge-probability bound given by (3) would lead to a useless, too large union bound for the probability of non-expansion for some subset \(S\).

In few words, to cope with this technical issue, we need to partition and classify the subsets \(S\) and \(T\) according to their age profile. More in detail, we first define a sequence of \(\Theta(\log n)\) slices of possible nodes ages and then we provide an effective age profile of each subset \(S\) (and \(T\)) depending on how large its intersection is with each of these slices. Thanks to the properties of the exponential distributions of the life of every node in the Poisson model (see (1) in Lemma 4.7), we show that the existence of a given subset in a given time has a probability that essentially depends on its profile. Roughly speaking, the more is the number of old nodes in \(S\), the less is the probability of the presence of \(S\) in \(N_{T_r}\).

Then, combining this profiling with a more refined use of the parameterized bound on the edge probability in (3), we get a mathematical expression (see (23)) that, in turn, we show to be dominated by the KL divergence of two suitably defined probability distributions. Finally, our target probability bound, stated in the next lemma, is obtained by the standard KL divergence inequality (see Theorem A.3). The arguments above allow us to prove the following result.

**Lemma 4.18** (Expansion of middle-size subsets). Under the hypothesis of Theorem 4.16, for subsets \(S\) of \(N_{T_r}\), with probability of at least \(1 - 2/n^2\),

\[
\min_{n/\log^2 n \leq |S| \leq n/14} \frac{|\partial_{out}(S)|}{|S|} \geq 0.1.
\]
Indeed, if we know that, in a given round, the node \( v \) we know that the death of one node in one single round is not independent of the death of the others.

So, if we denote \( i \) as the node that joined the network at round \( T_{r-1} \) (i.e. the node has age of \( i \) rounds), conditioning to \( L_r \),

\[
N_{T_r} \subseteq \{1, 2, 3, \ldots, 7n \log n\}.
\]

As in the previous analysis of small subsets, we have to show that (conditioning to \( L_r \)) any event occurs, and, so, no one of the other nodes dies. Moreover, if we know that one node does not die

we can use the bound (4) in Lemma 4.7. However, according to the definition of \( \text{event "Each of the } k \text{ considered nodes. The last inequality in (9) follows from the binomial inequality. So, thanks to (9) and to the memoryless property of the exponential distribution,}

\[
\Pr (S, T \subseteq N_{T_r} | L_r) \leq \prod_{i \in S \cup T} \left( 1 - \frac{1}{2.2n} \right)^i \leq \prod_{i \in S \cup T} e^{-i/2.2n},
\]

where, in (10) we used the fact that, from (9), each node contributes in the product with a factor \( 1 - 1/(2.2n) \) for each round of its life. Since each node chooses the destination of its out-edges independently of the other nodes, we can place (8) and (10) into (7), and obtain

\[
\Pr (A_{S,T} | L_r) \leq \prod_{i \in S \cup T} e^{-i/2.2n} \cdot \prod_{i \in S} \min \left\{ 1, \left[ \frac{|S \cup T|}{0.8n} \left( 1 + \frac{i}{1.7n} \right) \right]^d \right\}.
\]
For each set \( R \subseteq N_{T_r} \), we define the sequence \((K_R^1, \ldots, K_R^L)\) (where \( L = 7 \log n \)), whose goal is to classify the nodes of the set according to their age profile:

- \( K_R^1 = |R \cap \{1, 2, \ldots, n\}| \)
- \( K_R^2 = |R \cap \{n + 1, \ldots, 2n\}| \)
- \( \ldots \)
- \( K_R^L = |R \cap \{(L - 1)n + 1, \ldots, Ln\}|. \)

Notice that, if \(|R| = r\) and \( K_R^1 = r_1, \ldots, K_R^L = r_L \), then it must holds \( \sum_{m=1}^L r_m = r \). For each set \( R \subseteq N_{T_r} \), we denote the vector of random variables \((K_R^1, \ldots, K_R^L)\) as \( K^R \). According to this definition, by setting \( k = (k_1, \ldots, k_L) \) and \( h = (h_1, \ldots, h_L) \), we can rewrite (9) as follows:

\[
\Pr \left( \min_{n/\log^2 n \leq |S| \leq n/14} \frac{|\partial_{out}(S)|}{|S|} \leq 0.1 \right) \leq \sum_{k=n/\log^2 n}^{n/14} \sum_{k_1 + \cdots + k_L = k} \Pr \left( A_{S,T} \text{ s.t. } K^S = k, K^T = h \mid L_r \right) + \frac{1}{n^r}.
\]

Indeed, we have to sum over all the possible size \( k = n/\log^2 n, \ldots, n/14 \) of the set \( S \), all the possible vectors \( k \) and \( h \) whose sum of the elements is equal to \( k \) and \( 0.1k \), respectively (i.e. the characterization of the age profiles of \( S \) and \( T \) with \(|S| = k\) and \(|T| = 0.1|S| = 0.1k\)), and, finally, over all the possible sets \( S, T \) characterized by \( K^S = k \) and \( K^T = h \), respectively.

From (11), we get

\[
\Pr \left( A_{S,T} \text{ s.t. } K^S = k, K^T = h \mid L_r \right) \leq p(k, h) \leq \sum_{S,T} \Pr \left( A_{S,T} \text{ s.t. } K^S = k, K^T = h \right) \leq n(k, h) \cdot p(k, h). \tag{15}
\]

We place (14) and (13) into (15), and, since \(|S \cup T| = 1.1k\), then:

\[
s(k, h) \leq \prod_{m=1}^L \left( \binom{n}{h_m} e^{-0.4(m-1)h_m} \cdot \binom{n}{k_m} e^{-0.4(m-1)k_m} \min \left\{ 1, \frac{1.1k(1 + 0.6m)}{0.8n} \right\} \right)^{dk_m}. \tag{16}
\]

The next step is to prove that \( s(k, h) \leq 2^{-0.15k} \) and, to this aim, we split \( s(k, h) \) in two factors, \( s_1(k, h) \) and \( s_2(k, h) \):

\[
s_1(k, h) = \prod_{m=1}^L \left( \binom{n}{h_m} e^{-0.4(m-1)h_m} \right); \quad s_2(k, h) = \prod_{m=1}^L \left( \binom{n}{k_m} e^{-0.4(m-1)k_m} \min \left\{ 1, \frac{1.1k(1 + 0.6m)}{0.8n} \right\} \right)^{dk_m}. \tag{16}
\]

To give an upper bound on \( s(k, h) \), we provide separate upper bounds for \( \log(s_1(k, h)) \) and \( \log(s_2(k, h)) \). In particular, we want to show that

\[
\log(s(k, h)) \leq -0.15k, \tag{16}
\]
which implies that
\[
s(k, h) \leq 2^{-0.15k}. \tag{17}
\]

We will start bounding \(\log(s_1(k, h))\). Using \(\binom{n}{k} \leq \left(\frac{ne}{k}\right)^k\),
\[
\log(s_1(k, h)) \leq \sum_{m=1}^{L} h_m \log \left( \frac{n}{h_m} e^{-0.4m+1.4} \right). \tag{18}
\]

Since \(\log(x)\) is a concave function, we can apply Jensen’s inequality. In detail, for any concave function \(\varphi\), numbers \(x_1, \ldots, x_L\) in its domain, and positive weights \(a_1, \ldots, a_L\), it holds that
\[
\sum_{m=1}^{L} a_m \varphi(x_m) \leq \varphi\left( \sum_{m=1}^{L} a_m x_m \right).
\]

So, taking \(a_m = h_m x_m = \frac{n}{h_m} e^{-0.4m+1.4}\) and recalling that \(\sum_{m=1}^{L} h_m = 0.1k\), we obtain
\[
\sum_{m=1}^{L} h_m \log \left( \frac{n}{h_m} e^{-0.4m+1.4} \right) \leq \log \left( n \sum_{m=1}^{L} e^{-0.4m+1.4} \right). \tag{19}
\]

Since \(\sum_{m=1}^{L} e^{-0.4m+1.4} \leq 7\), combining (18), (19) and since \(k \leq n/14\), we get
\[
\log(s_1(k, h)) \leq 0.1k \log \left( \frac{7n}{0.1k} \right) \leq k \log \left( \frac{n}{7k} \right), \tag{20}
\]

where the last inequality follows by a simple calculation.

As for \(\log(s_2(k, h))\),
\[
\log(s_2(k, h)) \leq \sum_{m=1}^{L} k_m \log \left( \frac{n}{7k} \cdot \frac{n \cdot e^{-0.4(m-1)}}{k_m} \left( \min \left\{ 1, \frac{1.1k(0.6m + 1)}{0.8n} \right\} \right)^d \right) - k \log \left( \frac{n}{7k} \right). \tag{21}
\]

Then, since \(\log(s(k, h)) = \log(s_1(k, h)) + \log(s_2(k, h))\), from (20) and (21),
\[
\log(s(k, h)) \leq \sum_{m=1}^{L} k_m \log \left( \frac{0.6n^2}{k \cdot k_m} e^{-0.4m} \left( \min \left\{ 1, \frac{1.1k(0.6m + 1)}{0.8n} \right\} \right)^d \right).
\]

So, from the above inequality,
\[
- \frac{\log(s(k, h))}{k} \geq \sum_{m=1}^{L} k_m \log \left( \frac{k_m 9}{k \cdot k_m} 10^2 e^{-0.4m} \left( \min \left\{ 1, \frac{1.1k(0.6m + 1)}{0.8n} \right\} \right)^d \right) + \log(10/9). \tag{22}
\]

Now, notice that, if we prove that
\[
\sum_{m=1}^{L} k_m \log \left( \frac{k_m 9}{k \cdot k_m} 10^2 e^{-0.4m} \left( \min \left\{ 1, \frac{1.1k(0.6m + 1)}{0.8n} \right\} \right)^d \right) \geq 0, \tag{23}
\]

then, from (22), we would get (10), since \(\log(10/9) \geq 0.15\).

So, we want to prove (23). Thanks to the KL divergence inequality (see Theorem (3)), it is sufficient to show that the following functions are density mass functions over \(\{1, 2, \ldots, L\}\):
\[
p_m = \frac{k_m}{k} \text{ and } q_m = \frac{10}{9} \cdot \frac{0.6n^2}{k^2} e^{-0.4m} \min \left\{ 1, \frac{1.1k(0.6m + 1)}{0.8n} \right\}^d.
\]

Notice that \(\sum_{m=1}^{L} p_m = 1\), and
\[
\sum_{m=1}^{L} q_m \leq 1.1 \left( \frac{1.1}{0.8} \right)^2 \left( \frac{9}{10} \right)^{d-3} + 10 \left( \frac{0.6n^2}{k^2} e^{-0.36}\right) \leq 1,
\]
where the last inequality holds we taking \( d \) large enough (\( d \geq 30 \)) and \( k \leq \frac{n}{14} \). So, we have proved that \( q_m \) and \( p_m \) are density mass functions over \( \{1, 2, \ldots, L\} \) and so, thanks to Theorem \( \ref{lem3} \), \( \ref{thm1} \) holds and implies \( \ref{thm2} \).

By placing \( \ref{thm1} \) in \( \ref{lem3} \) and using \( \ref{thm2} \),

\[
\Pr \left( \min_{n/\log^2 n \leq |S| \leq n/14} \frac{\|\partial_{out}(S)\|}{|S|} \leq 0.1 \right) \leq \sum_{k=n/\log^2 n}^{n/14} \sum_{k_1+\ldots+k_L=k \atop h_1+\ldots+h_L=0.1k} s(k, h) + \frac{1}{n^2} \leq \frac{2}{n^2},
\]

where the last inequality holds since the number of integral sequences \( k_1, \ldots, k_L \) that sum up \( k \) is bounded by \( \binom{k+L}{L} \) (and the same holds for \( h_m \)), and hence, from simple calculations and recalling that \( L = 7n \log n \),

\[
\sum_{k=n/\log^2 n}^{n/14} \sum_{k_1+\ldots+k_L=k \atop h_1+\ldots+h_L=0.1k} s(k, h) \leq \sum_{k=n/\log^2 n}^{n/14} \binom{L + 0.1k}{L} \binom{L + k}{L} 2^{-0.15k} \leq \frac{1}{n^2}.
\]

\( \square \)

**Expansion of big subsets.** The last case of our analysis of the vertex expansion of the PDG model considers subsets of big size \( |S| \geq n/14 \). It analysis is much simpler than the that of the previous case and proceeds exactly as the proof of Lemma \( \ref{lem4} \) about the expansion of large subsets in the PDG model. Indeed, in both the PDG and PDGR models, we use the fact that any node \( u \in N_{T_0} \) chooses any fixed older node \( v \in N_{T_0} \) with probability \( \geq 1/1.1n \) (thanks to Lemma \( \ref{lem4} \)). The proof is omitted since it is identical to that of Lemma \( \ref{lem4} \) in Subsection \( \ref{subsec:lem4} \).

**Lemma 4.19 (Expansion of large subsets).** Under the hypothesis of Theorem \( \ref{thm1} \) for subsets \( S \) of \( N_{T_0} \), with probability of at least \( 1 - 2/n^2 \),

\[
\min_{n/14 \leq |S| \leq |N_{T_0}|/2} \frac{\|\partial_{out}(S)\|}{|S|} \geq 0.1.
\]

### 4.3.2 Flooding

We now consider the flooding process over the PDGR model introduced in Definition \( \ref{def3} \). The vertex expansion property we derived in Theorem \( \ref{thm1} \) is here exploited to obtain a logarithmic bound on the time required by this process to inform all the nodes of the graph. Notice that, according to the considered topology dynamics, if there is a time in which all the alive nodes are informed, then every successive snapshot of the dynamic graph will have all its nodes informed as well, w.h.p.

**Theorem 4.20 (Flooding).** For every constant \( d \geq 35 \), for every sufficiently large \( n \) and for every fixed \( r_0 \geq 7n \log n \), consider the flooding process over a PDGR sampled from \( \mathcal{G}(\lambda, \mu, d) \), with \( \lambda = 1 \) and \( \mu = 1/n \), and starting at \( t_0 = T_{r_0} \). Then, w.h.p., the flooding time is \( \mathcal{O}(\log n) \).

As remarked before, in dynamic networks without node churn, it has already been shown \( \ref{lem4} \) that the good vertex expansion of every snapshot implies fast flooding time (see for instance \( \ref{lem4} \)). In the Poisson models, the presence of random node churn requires to consider some new technical issues. Indeed, once we observe the set of informed nodes \( I_t \) at a given snapshot \( G_t = (N_t, E_t) \), the expansion of \( I_t \) refers to topology \( E_t \) while the 1-hop message transmissions take one unit of time. So, during this time interval, some topology changes may take place affecting the expansion observed at time \( t \). To cope with this issue, our analysis splits the process into three consecutive phases and prove they all have logarithmic length, w.h.p. The details of this approach are given in Subsection \( \ref{subsec:lem4} \).

### 5 Overall Remarks and Open Questions

We studied two models of fully-random dynamic networks with node churns. We analysed their expansion properties and gave results about the performances of the flooding process. We essentially show that such important aspects depends on the specific adopted topology dynamic, namely, on whether or not, edge regeneration takes place along the time process.
While our models are too simplified to predict all properties of realistic networks, the Poisson model with edge regeneration bears a certain similarity to the way peer-to-peer networks such as Bitcoin are formed. In particular, although the random choices over the current node set \( N_t \) the nodes make to establish connections is not the connection mechanism adopted in standard Bitcoin implementations, the set of IP addresses of the active full-nodes of the Bitcoin network can be easily discovered by a crawler (see, e.g., [25]). This implies that, potentially, nodes can implement a good approximation of the fully-random strategy by picking random elements from such on-line table.

We see an interesting future research direction related to our work. The topology dynamics we considered yield sparse graphs at every round, however, the maximum node degree can be of magnitude \( O(\log n) \). For some real applications this bound is too large, and finding natural, fully-random topology dynamics that yield bounded-degree snapshots of good expansion properties is a challenging issue which has strong theoretical and practical motivations [1, 3, 19].

6 Omitted Proofs for the Streaming Model

In this section we present the proofs of the results we obtained for the streaming model.

6.1 Omitted Proofs for the streaming model without edge regeneration

6.1.1 Lemma 6.1

We first observe that the expected degree of each node in this graph is \( d \). Thus, the expected number of edges in the graph \( nd/2 \).

Lemma 6.1 (Expected degree). Let \( G_t = (N_t, E_t) \) be the snapshot of a SDG \( G(n, d) \). Then, for any \( t \geq n \), every node in \( N_t \) has expected degree \( d \).

Proof. We first fix \( t \geq n \), and let \( v_1, \ldots, v_n \) be the nodes n the network at round \( t \), where \( v_i \) is the node with age \( i \). We define the following Bernoulli random variable, for each \( i, j \in [n] \) with \( i < j \) (\( v_i \) joined the network after \( v_j \))

\[
z^{(k)}(v_i, v_j) = \begin{cases} 1 & \text{if the node } v_i \text{ at the time of its arrival has connected its } k-\text{th request to } v_j \\ 0 & \text{otherwise} \end{cases}
\]

We notice that for each \( k \) and \( i, j \) s.t. \( i < j \) the random variables \( z^{(k)}(v_i, v_j) \) are independent. We indicate with \( \Delta^i_t \) the degree of the node \( v_i \) at time \( t \), where \( i \in [n] \). Then, we can say that \( \Delta^i_t \) is the random variable where

\[
\Delta^i_t = \sum_{k=1}^{d} \left( \sum_{j=1}^{i-1} z^{(k)}(v_j, v_i) + \sum_{j=i+1}^{n} z^{(k)}(v_i, v_j) \right)
\]

and, since for each \( i, j \) with \( i < j \) and \( k \) we have that \( \Pr(z^{(k)}(v_i, v_j) = 1) = 1/(n-1) \), it holds that for each \( t \geq n \) and \( i = 1, \ldots, n \)

\[
\mathbb{E} [\Delta^i_t] = d.
\]

6.1.2 Proof of Lemma 3.5

Let \( \varepsilon \) be an arbitrary value with \( 0 < \varepsilon \leq 1/3 \). We first define the set \( H \) of the oldest \( \varepsilon n \) nodes in \( N_t \). We now define the random variable

\[
X = \{ \text{number of nodes in } H \text{ that are isolated at round } t \text{ and for the rest of their lifetime} \}.
\]

Our goal is to show that, w.h.p., \( X \geq \frac{1}{2} \varepsilon n e^{-2d} \). First, to evaluate the expectation of \( X \), we introduce the following random variables, for each \( v \in N_t \):

\[
\Delta^i_v = \{ \text{maximum in-degree of the node } v \text{ for all its lifetime} \};
\]

\[
\Delta^i_v = \{ \text{out-degree of the node } v \text{ at round } t \}.
\]
We show that any two disjoint sets 

\[ X = \sum_{v \in H} 1_{\{\Delta^i_v = 0\}} \cdot 1_{\{\Delta^o_v = 0\}}. \]  

(24)

Since each node sends its requests independently of the others, we get

\[ \mathbb{E}[X] = \sum_{v \in H} \Pr(\Delta^i_v = 0) \cdot \Pr(\Delta^o_v = 0). \]  

(25)

The probability that a node \( v \in H \) has always in-degree 0, i.e. it does not receive any request by other nodes for all its lifetime is

\[ \Pr(\Delta^i_v = 0) = \left(1 - \frac{1}{n}\right)^{nd} \geq e^{-3d/2}. \]

The probability that node \( v \in H \) has no out-edges in the current round is

\[ \Pr(\Delta^o_v = 0) = (1 - \varepsilon)^d \geq e^{-d/2}, \]

since \( \varepsilon \leq 1/3 \). So, from (25) and since \( |H| = \varepsilon n \)

\[ \mathbb{E}[X] \geq \varepsilon n e^{-2d}. \]

Observe that, in (24), \( X \) is not a sum of independent random variables, so to get a concentration result, we use the method of bounded differences. We introduce the random variables \( Y_j^d \), returning the index of the node to which the node \( v \) sends its \( j \)-th request. Notice that the random variables \( \{Y_j^d : v \in \mathcal{N}_t \cup \mathcal{N}_{t+1} \cdots \cup \mathcal{N}_{t+\varepsilon n}, j \in [d]\} \) are independent, and they represent the destination of the requests of the nodes in the network and of the \( \varepsilon n \) nodes that will join the network after time \( t \). Then, considering the vector \( Y \) of these random variables, we can easily express \( X \) as a function of \( Y \),

\[ X = f(Y). \]

Moreover, if any two vectors \( Y \) and \( Y' \) differs only in one coordinate, it holds \( |f(Y) - f(Y')| \leq 2 \). Indeed, in the worst case, an isolated node can change its destination from a leaving node to another isolated node: so, the number of isolated nodes decreases (or increases) by at most 2 units. By applying Theorem A.2 we get that, if \( \mu \) is a lower bound to \( \mathbb{E}[X] \) and \( M > 0 \),

\[ \Pr(X \leq \mu - M) \leq e^{-\frac{2M^2}{\mathbb{E}[X] + \varepsilon n^2}}. \]

Hence, we can fix \( \mu = \varepsilon n e^{-2d} \) and \( M = \frac{1}{2}\varepsilon n e^{-2d} \), and get

\[ \Pr\left(X \leq \frac{1}{2}\varepsilon n e^{-2d}\right) \leq e^{-n \frac{2e^{-4d}}{\mathbb{E}[X] + \varepsilon n^2}}. \]

Finally, the lemma is proved by setting \( \varepsilon = 1/3 \).

### 6.1.3 Proof of Lemma 3.6

We show that any two disjoint sets \( S, T \subseteq \mathcal{N}_t \), with \( \varepsilon n^{-d/10} \leq |S| \leq n/2 \) and \( |T| = 0.1 |S| \), such that \( \partial_{out}(S) \subseteq T \), exist with negligible probability. If we denote

\[ A_{S,T} = \{\partial_{out}(S) \subseteq T\} \]

we have that

\[ \Pr\left(\min_{\varepsilon n^{-d/10} \leq |S| \leq n/2} \frac{|\partial_{out}(S)|}{|S|} \leq 0.1 \right) \leq \sum_{\varepsilon n^{-d/10} \leq |S| \leq n/2} \Pr(A_{S,T}). \]  

(26)

To upper bound \( \Pr(A_{S,T}) \) we define \( P \) as the set \( P = \mathcal{N}_t - S - T \), and notice that the event \( A_{S,T} \) is like saying that between \( S \) and \( P \) there are no edges. Clearly, it holds that

\[ \{|(a, b) \text{ s.t. } a \in S, b \in P\} = |S| \cdot |P|. \]

Two cases may arise:
1. |{(a, b) s.t. a ∈ S, b ∈ P and a is younger than b}| ≥ |S| · |P|/2;
2. |{(a, b) s.t. a ∈ S, b ∈ P and b is younger than a}| ≥ |S| · |P|/2.

As for the first case, for each a ∈ S, we define N_a as the number of nodes in P older then a. We get that ∑_{a ∈ S} N_a ≥ |S| · |P|/2. Since a fixed request of a node u has probability 1/n to get a node v older than him as destination (this is in fact the probability that u connects to v when u joins the network), we get

\[
\Pr(A_{S,T}) \leq \prod_{a \in S} \left(1 - \frac{N_a}{n}\right)^d \leq e^{-d \sum_{a \in S} N_a/n} \leq e^{-d |S| \cdot |P|/2n}.
\]

(27)

As for the second case, we get the same bound above by proceeding with a similar argument. Then, from (26) and (27),

\[
\Pr\left(\min_{n-4^{d/10} \leq |S| \leq n/2} \frac{|\partial_{out}(S)|}{|S|} \leq 0.1\right) \leq \sum_{s=n-4^{d/10}}^{n/2} \binom{n}{s} (n-s) e^{-ds \frac{n-1.1s}{n^4}} \leq \frac{1}{n^4},
\]

where the last inequality holds for large enough n and for any d ≥ 20. It easily follows by bounding each binomial coefficient with the \(\binom{n}{k} \leq \left(\frac{en}{k}\right)^k\) and by standard calculations.

### 6.1.4 Proof of Theorem 3.7

Let s_0 be the source node of the flooding process. We bound the probability that s_0 has all its out-edges towards d nodes that are isolated at round t_0 and keep so for the rest of their lifetime. Define the events A = "the source node s_0 connects to d nodes that are isolated at round t_0 and for the rest of their lifetime" and B = "there are at least \(\frac{2d}{n}\) nodes in N_t_0 that are isolated at time t_0 and for the rest of their lifetime". From Lemma 3.5.

\[
\Pr(A) \geq \Pr(B) \Pr(A \mid B) \geq \frac{1}{2} \cdot \left(\frac{e^{-2d/6}}{6}\right)^d.
\]

(28)

Let s_1, ..., s_d be the d out-neighbors of s_0. Notice that the events A imply that s_0, s_1, ..., s_d are isolated and after n rounds all of them will leave the network. Then, from (28),

\[
\Pr(I_{t_0+\tau = 0}) \geq \Pr(A) \geq \frac{1}{2} \cdot \left(\frac{e^{-2d/6}}{6}\right)^d.
\]

Finally, as for the stated linear bound on the flooding time τ, we observe this is an easy consequence of Lemma 3.5. Indeed, in the network there are at least \(\frac{2d}{n}e^{-2d/6}\) isolated nodes that will remain isolated for the rest of their lifetime: to have all the nodes informed, we have to wait that these nodes leave the network, and so τ = \(\Omega_d(n)\).

### 6.1.5 Proof of Claim 3.10

As for the claim for Phase 0, the proof of the first inequality of the claim proceeds as follows. For each i = 1, ..., d and for every v ∈ N_{t_0} that joined the network at time \(\tau \leq t_0\), we define the variable A_{V}^{(i)} \in N_{\tau} as follows:

\[
A_{V}^{(i)} = w \text{ if } w ∈ N_{\tau} \text{ is the destination of the } i\text{-th link of } v.
\]

(29)

Assume w ∈ O. We have:

\[
\Pr(3i \in [d] : A_{V}^{(i)} = w) = 1 - \left(1 - \frac{1}{n}\right)^d,
\]

which implies

\[
\mathbb{E}[|O_0|] \geq |O| \left(1 - \left(1 - \frac{1}{n}\right)^d\right) \geq |O| \frac{d}{2n} \geq \frac{d}{5}.
\]

where the last equality follows since |O| = n/2 − log n. We next bound the probability that O_0 is smaller than d/10. To this purpose, we cannot simply apply a Chernoff bound to the binary variables that describe whether or not a node w ∈ O was the recipient of at least one link originating from s, since these are not independent. We instead resort to Theorem A.2 In particular, we consider the random
variables $A^{(1)}_i, \ldots, A^{(d)}_s$ and we define the function $f(A^{(1)}_i, \ldots, A^{(d)}_s) = |O_0|$. Clearly, $f$ is well-defined for each possible realization of the $A^{(i)}_s$. Moreover, $f$ satisfies the Lipschitz condition with values $\beta_1 = \cdots = \beta_d = 1$, since changing the destination of one link can affect the value of $|O_0|$ by at most 1. We can thus apply Theorem A.2 to obtain:

$$\Pr \left( |O_0| < \frac{d}{10} \right) \leq \Pr \left( f < \mathbb{E} [f] - \frac{d}{10} \right) \leq e^{-d/50}.$$  

To uniform the results, in the claim we give a weaker bound following from the bound above.

As for the first inequality of the generic phase $k \geq 1$, we proceed as follows. For each $i = 1, \ldots, d$, for each node $v \in N_{t_0}$ and for each set $A \subseteq N_{t_0}$, we define the Bernoulli random variable $R_{v,A}$ as follows:

$$R_{v,A} = \begin{cases} 1 & \text{if } x \geq 1 \text{ links in } \{\frac{n}{2}, \ldots, d\} \text{ from } v \text{ have destination in } A \\ 0 & \text{otherwise} \end{cases}$$

We remark that

**Fact 6.2.** If $v \in Y - Y_{k-1}$ establishes a link with destination $w \in O_{k-1}$ in phase $k$, then $w \not\in O_{k-2}$.

**Proof.** If this were the case, the definition of the onion-skin process would imply $v \in Y_j$, for some $j \leq k - 1$, a contradiction.

From the fact above and from definition of $Y_k - Y_{k-1}$ given above we have:

$$|Y_k - Y_{k-1}| = \sum_{v \in Y - Y_{k-1}} R_{v,O_{k-1} - O_{k-2}}.$$  

Moreover, for each $v \in N_{t_0}$

$$\Pr (R_{v,O_{k-1} - O_{k-2}} = 1 | \ |O_{k-1} - O_{k-2}| \geq y) \geq 1 - \left(1 - \frac{y}{n}\right)^{\frac{d}{2}},$$

whence

$$\mathbb{E} [Y_k - Y_{k-1} | \ |O_{k-1} - O_{k-2}| \geq y] \geq |Y - Y_{k-1}| \left(1 - \left(1 - \frac{y}{n}\right)^{\frac{d}{2}}\right).$$

Since $y \leq n/d$, we have:

$$\mathbb{E} [Y_k - Y_{k-1} | \ |O_{k-1} - O_{k-2}| \geq y] \geq |Y - Y_{k-1}| \frac{yd}{4n} \geq \frac{yd}{10},$$

where in the last inequality we used the assumption that $|Y_{k-1}| \leq n/d$. The $R_{v,O_{k-1} - O_{k-2}}$’s are independent, we can therefore apply Chernoff’s Bound (Theorem A.1 in the Appendix) to obtain

$$\Pr \left( |Y_k - Y_{k-1}| \leq \frac{yd}{20} | \ |O_{k-1} - O_{k-2}| \geq y \right) \leq e^{-yd/100}.$$  

As for the last inequality of the claim, we proceed similarly to the case $k = 0$. In this setting, the following fact holds.

**Fact 6.3.** If $v \in Y_k$ establishes a link with destination $w \in O - O_{k-1}$ in phase $k$, then $w \not\in Y_{k-1}$.

**Proof.** If this were the case, the definition of the onion-skin process would imply $w \in O_j$ for some $j \leq k - 1$, a contradiction. 

Assume $w \in O$. Recalling the definition of the random variables $A^{(i)}_v$ in (24), we have

$$\Pr \left( \exists i \in \left[ \frac{d}{2} \right], \exists v \in Y_k - Y_{k-1} : A^{(i)}_v = w | |Y_k - Y_{k-1}| \geq x \right) = 1 - \left(1 - \frac{1}{n}\right)^{\frac{d}{2}}.$$  

From the fact above and from the definition of $|O_k - O_{k-1}|$ in the onion-skin process we have

$$\mathbb{E} [|O_k - O_{k-1}| | |Y_k - Y_{k-1}| \geq x] = |O - O_{k-1}| \left(1 - \left(1 - \frac{1}{n}\right)^{\frac{d}{2}}\right).$$
Since $x \leq n/d$, we have
\[
E[|O_k - O_{k-1}| \mid |Y_k - Y_{k-1}| \geq x] \geq |O - O_{k-1}| \frac{dx}{4n} \geq \frac{xd}{10},
\]
where the last inequality follows from the fact that $|O| = n/2 - \log n$ and $|O_{k-1}| \leq n/d$. However, in this case, as in the proof of the case $k = 0$, we cannot simply apply a Chernoff bound to the binary random variables that describe whether or not a node $w \in N$ was the recipient of at least one link in $\{1, \ldots, d/2\}$ originating from one node $v \in Y_k$, since these are not independent. We will use instead the method of bounded differences (Theorem A.2). In particular, we consider the random variables $A_t^{(i)}$, $i \in \{d/2, \ldots, d\}$, and we define the function $g$ depending on these variables which returns $|O_k - O_{k-1}|$. Clearly, $g$ is well-defined for each possible realization of the $A_t^{(i)}$’s and satisfies the Lipschitz condition with values $\beta_1 = \cdots = \frac{\beta_d}{2}$, $|Y_k - Y_{k-1}| = 1$, since changing the destination of one link can affect the value of $|O_k - O_{k-1}|$ of at most 1. We can thus apply Theorem A.2 to obtain
\[
\Pr\left(|O_k - O_{k-1}| \geq \frac{dx}{20} \mid |Y_k - Y_{k-1}| \geq x\right) \leq e^{-dx/100}.
\]

6.1.6 Proof of Claim 3.11

From $a_t = (d/2)^i$ we have:
\[
\log c = \log \left(\prod_{i=0}^{\infty} \left(1 - e^{-(d/20)^i/d(100)}\right)\right) = \sum_{i=0}^{\infty} \log \left(1 - e^{-(d/20)^i/d(100)}\right) = -\sum_{i=0}^{\infty} \log \left(\frac{1}{1 - e^{-(d/20)^i/d(100)}}\right).
\]
Moreover, since $\log \left(\frac{1}{1-x}\right) \leq 2x$ for each $x \leq 1$,
\[
\sum_{i=0}^{\infty} \log \left(\frac{1}{1 - e^{-(d/20)^i/d(100)}}\right) \leq \sum_{i=0}^{\infty} 2e^{-(d/20)^i/d(100)} = 2e^{-(d/100)} + \sum_{i=1}^{\infty} 2e^{-(d/20)^i/d(100)}
\]
\[
< 2e^{-(d/100)} + 2e^{-(d/100)} \sum_{i=1}^{\infty} e^{-(d/20)^i} < 4e^{-(d/100)}.
\]
The third inequality holds since $d > 200$, which implies $\frac{d}{100} \left(\frac{d}{20}\right)^i > \frac{d}{100} + \left(\frac{d}{20}\right)^i$ for $i \geq 1$, while the last inequality follows since the double exponential is dominated by a simple one, summing to a constant not exceeding 1. This in turn implies So, from (30) and (31) we have that
\[
\log c \geq -4e^{-(d/100)},
\]
whence:
\[
c \geq e^{-4e^{-(d/100)}} > 1 - 4e^{-x},
\]
where the last inequality follows since $e^{-x} > 1 - x$ for $x \geq 0$.

6.1.7 Proof of Lemma 3.12

We recall that, for Lemma 3.11, $|I_{t_0 + \tau_1}| \geq 2n/d$. In this proof, we show that the size of the set of informed nodes grows by a constant factor at each step, reaching size $(1 - e^{-d/10})n$ in $\tau_2$ steps. In our analysis, we can fix $\tau_2 = \Theta(d)$. In the whole proof, we will not consider the oldest $\tau_2$ nodes in $N_{t_0 + \tau_1}$, because they will die in the next $\tau_2$ steps. We call the set of such nodes $V$, and let $V_t$ be the intersection between $V$ and $N_t$. To show that the set of informed nodes grows at each step by a constant factor, it is sufficient to notice that, for $d \geq 20$, the graph $G_t = (N_t, E_t)$ is an expander for sets of large size w.h.p. (see Lemma 3.11), i.e.,
\[
\min_{n e^{-d/10} \lesssim |S| \lesssim n/2} \frac{|\partial_{\text{out}}(S)|}{|S|} \geq 0.1 .
\]
Indeed, the size of the set of informed nodes $I_t$, for $t \geq t_0 + \tau_1$, grows by a constant factor at each step, as long as $|I_t| \leq n/2$. We have from the definition of flooding that $I_{t+1} = (I_t \cup \partial_{\text{out}}(I_t)) \cap N_t$ and since, at
each step, at most 1 informed node leaves the network, \(|I_{t+1}| \geq |\partial_{\text{out}}(I_t)| + |I_t| - 1\). Since \(|I_{t_0 + \tau_1}| \geq n/d\), from (32) follows that for each \(t \geq t_0 + \tau_1\)

\[|I_{t+1}| \geq 1.1|I_t| - 1,\]

as long as \(|I_t| \leq n/2\). This consideration implies that in \(\tau_2 = \Theta(\log d)\) steps, we will have \(|I_{t_0 + \tau_1 + \tau_2}| \geq n/2\).

Now, we consider \(S_t = N_t - I_t\) as the set of non-informed nodes in the graph at time \(t\), and we will show that the size of this set decreases by a constant factor at each step, as long as \(|S_t| \geq ne^{-d/10}\). First of all, we notice that \(\partial_{\text{out}}(S_{t+1}) \subseteq (S_t - S_{t+1}) \cap N_t\), because \(\partial_{\text{out}}(S_{t+1}) \subseteq I_{t+1}\) are nodes reachable in one edge from the non-informed nodes, and so they were not informed in the previous time. Since, at each step, at most 1 node joins the network, we have that \(|S_t| - |S_{t+1}| + 1 \geq |\partial_{\text{out}}(S_{t+1})|\). Since \(|S_{t_0 + \tau_1 + \tau_2}| \leq n/2\), from (32) follows that for each \(t \geq t_0 + \tau_1 + \tau_2\), \(|\partial_{\text{out}}(S_{t+1})| \geq 0.1|S_{t+1}|\), as long as \(|S_{t+1}| \geq ne^{-d/10}\), so,

\[|S_{t+1}| \leq \frac{1}{1.1}(|S_t| + 1).\]

This consideration implies that in \(\tau_2 = \Theta(d)\) steps, we will have \(|S_{t_0 + \tau_1 + \tau_2}| \leq ne^{-d/10}\). Then, conditional at the event \(|I_{t_0 + \tau_1}| \geq n/d\), w.h.p. \(|I_{t_0 + \tau_1 + \tau_2}| \geq n(1 - e^{-d/10})\). Since from Lemma 3.9 \(\Pr(|I_{t_0 + \tau_1}| \leq n/d) \geq 1 - 4e^{-d/100}\), the lemma is proved.

### 6.2 Omitted Proofs for the streaming model with edge regeneration

#### 6.2.1 Proof of Lemma 3.14

If \(u\) is younger than \(v\), then the request of \(u\) can choose \(v\) only if some previous neighbor of \(u\) leave the network: the probability of this event is clearly \(\leq \frac{1}{n-1}\).

If, instead, \(u\) is older than \(v\), then the analysis leading to (2) needs to take care of more chances \(u\) has to get \(v\) as destination of its request because of the edge-regeneration process. The argument we adopt here for the streaming process is also a good warm-up for the more complex Poisson process we analyze in Subsection 13.

According to the SDGR model, when a node leaves the network, all its incident edges are removed and, in the same step, if an edge of a fixed request of \(u\) fails, \(u\) instantly reassigns it by choosing its destination uniform at random (with replacement) over the current set of nodes: in this proof, this action

\[v\] and, in the same step, if an edge of a fixed request of \(u\) in Subsection 4.3.

\(u\) and then

\[u\] are mutually independent. So, \(\Pr(v\) is the \(u\)-request destination) = \(\sum_{i=1}^{k+1} \Pr(v\) is the \(u\)-request destination in the \(i\)-th assignment) .

Since there are \(\binom{k}{i-1}\) ways to choose the \(i - 1\) destinations of the request of \(u\) before the choice of \(v\), the probability of getting a fixed subset of \(i\) nodes in the reassignments is \(\left(\frac{1}{n-1}\right)^i\), and such choices in each round are mutually independent. So,

\[\Pr(v\) is the \(u\)-request destination in the \(i\)-th assignment) = \(\binom{k}{i-1} \left(\frac{1}{n-1}\right)^i\).

From (33) and (34) and by Newton’s Binomial Theorem

\[\Pr(v\) is the \(u\)-request destination in the \(i\)-th assignment) = \(\sum_{i=1}^{k+1} \binom{k}{i-1} \left(\frac{1}{n-1}\right)^i = \sum_{j=0}^{k} \binom{k}{j} \left(\frac{1}{n-1}\right)^{j+1} = \frac{1}{n-1} \left(1 + \frac{1}{n-1}\right)^k\).

#### 6.2.2 Proof of Theorem 3.15

The theorem is consequence of the next two lemmas. The first one shows the claimed expansion for subsets of size \(\leq n/4\).
Lemma 6.4 (Expansion of “small” subsets). Under the hypothesis of Theorem 3.15 for subsets $S$ of $N_t$, it holds
\[
\min_{0 \leq |S| \leq n/4} \frac{|\partial_{\text{out}}(S)|}{|S|} \geq 0.1,
\]
with probability at least $1 - 1/n^4$.

Proof. We proceed similarly to the proof of the expansion for big subset in the SDG model (Lemma 3.6). To prove the lemma, it is sufficient to show that any two disjoint sets $S,T \subseteq N_t$, with $|S| \leq n/4$ and $|T| = 0.1|S|$, such that $\partial_{\text{out}}(S) \subseteq T$, exist with negligible probability. For any $S$ and any $T \subseteq N_t - S$, we define $A_{S,T}$ as in Lemma 3.6
\[
A_{S,T} = \{\partial_{\text{out}}(S) \subseteq T\}.
\]
Therefore,
\[
\Pr\left(\min_{0 \leq |S| \leq n/4} \frac{|\partial_{\text{out}}(S)|}{|S|} \leq 0.1\right) \leq \sum_{|S| \leq n/4} \sum_{|T| = 0.1|S|} \Pr( A_{S,T}) .
\]
(35)
The quantity $\Pr( A_{S,T})$ is upper bounded by the probability that each request of the nodes in $S$ has destination in $S \cup T$. From Lemma 6.14 we know that a request of a node $u$ with age $k + 1$ has probability at most $1/n - 1$ to have a node $v$ younger than $u$ as destination and probability $1/n - 1 \left(1 + \frac{1}{n - 1}\right)^k$ to have a node $v$ older than $u$ as destination. Since $k \leq n - 1$, the probability that a single request of $u$ has an arbitrary, fixed node $v$ as destination is at most $1/n - 1$. Since to have $\partial_{\text{out}}(S) \subseteq T$, each request of $u \in S$ must have destination in $S \cup T$, it holds
\[
\Pr( A_{S,T}) \leq \left(\frac{e}{n - 1} \cdot |S \cup T|\right)^{d|S|}.
\]
(36)
So, from (35) and (36) we have that
\[
\Pr\left(\min_{0 \leq |S| \leq n/4} \frac{|\partial_{\text{out}}(S)|}{|S|} \leq 0.1\right) \leq \sum_{s=1}^{n/4} \sum_{s=1}^{n/4} \left(\frac{n}{s}\right) \left(\frac{n - s}{0.1s}\right) \left(\frac{1.1s \cdot e}{n - 1}\right)^{ds} \leq \frac{1}{n^4} .
\]
(37)
By standard calculus, it can be proved that, for $d \geq 21$, the equation above is upper bounded by $1/n^4$. This is obtained by bounding each binomial coefficient in (37) with the bound (10) and by computing the derivative of the function $f(s)$ (representing each term of the sum), obtaining that each of these terms attained its maximum at the "boundaries", i.e. or in $s = 1$ or in $s = n/4$. □

The second lemma provides the expansion property for subsets of size $\geq n/4$. Its proof is omitted since it is identical to that of Lemma 5.6 about the expansion of large subsets in the SDG model.

The only difference between the two proofs is that now we need to consider sets of size in the range $[n/4,n/2]$, and so we get a weaker condition on the value of $d$.

Lemma 6.5 (Expansion of large subsets). Under the hypothesis of Theorem 3.15 for subsets $S$ of $N_t$, it holds
\[
\min_{n/4 \leq |S| \leq n/2} \frac{|\partial_{\text{out}}(S)|}{|S|} \geq 0.1 ,
\]
with probability of at least $1 - 1/n^4$.

6.2.3 Proof of Theorem 3.16
We first show that the size of the set of informed nodes $I_t$, for $t \geq t_0$, grows by a constant factor at each step, as long as $|I_t| \leq n/2$. From the definition of flooding it holds $I_{t+1} = (I_t \cup \partial_{\text{out}}(I_t)) \cap N_t$ and, since, at each round, one single node leaves the network, $|I_{t+1}| \geq |\partial_{\text{out}}(I_t)| + |I_t| - 1$. Since $d \geq 21$, Theorem 3.15 implies that the graph $G_t$ is an $(1/10)$-expander, w.h.p., and so, as long as $|I_t| \leq n/2$, w.h.p.
\[
|I_{t+1}| \geq 1.1|I_t| - 1 .
\]
Then, $\tau_1 = O(\log n)$ exists such that $|I_{\tau_1 + t_0}| \geq n/2$, w.h.p.
To reach $n$ informed nodes, from time $\tau_1 + t_0$, we consider the set of non-informed nodes in the network at each time $t \geq \tau_1 + t_0$, i.e. $S_t = N_t - I_t$, and we next show that the size of this set decreases by a constant factor at each step. Notice that, since every node $v$ in $\partial_{\text{out}}^+ (S_{t+1}) \subseteq I_{t+1}$ is reachable in 1-hop from the set of non-informed nodes at time $t + 1$, $v$ was not informed at time $t$. This implies that $\partial_{\text{out}}^+ (S_{t+1}) \subseteq (S_t - S_{t+1}) \cap N_{t+1}$. Since, at each step, one single node joins the network, we have that $|S_t| - |S_{t+1}| + 1 \geq |\partial_{\text{out}}^+ (S_{t+1})|$. Since $S_{t_0 + \tau_1} \leq n/2$, from the expansion of the graph $G_{t+1}$ (Theorem 3.15) it holds w.h.p. that, for each $t \geq t_0 + \tau_1$,

$$|S_{t+1}| \leq \frac{1}{1.1} (|S_t| + 1).$$

The above equation implies that a time $\tau_2 = O(\log n)$ exists such that $|S_{t_0 + \tau_1 + \tau_2}| < 1$.

7 Omitted Proofs for the Poisson Model

7.1 Omitted proofs for the preliminary properties

7.1.1 Proof of Lemma 4.6

It is sufficient to apply Theorem C.5 in Appendix to the exponential random variables that represent the time arrival of a new node and the lifetime of the node in the network, for each node $v \in N_{T_n}$.

7.1.2 Proof of Lemma 4.7

The lemma easily follows from Lemma 4.6 and from the concentration of the nodes (Lemma 4.4). We first define, for each $r \geq n \log n$, the following event

$$C_r = \{|N_{T_r}| \in [0.9n, 1.1n]\}$$

and, for Lemma 4.6, we have that $\Pr(C_r) \geq 1 - 1/n^2$.

We will first show the upper bound for the first inequality in (3). We notice that, for each $r \geq n \log n$ we have

$$\Pr (|N_{T_r+1}| = |N_{T_r}| - 1) = \Pr (|N_{T_r+1}| = |N_{T_r}| - 1 \mid C_r) \Pr (C_r) + \Pr (|N_{T_r+1}| = |N_{T_r}| - 1 \mid C_r^C) \Pr (C_r^C)$$

and we have that, for the law of total probability, and from the equation above

$$\Pr (|N_{T_r+1}| = |N_{T_r}| - 1) \leq \sum_{N=0.9n}^{1.1n} \Pr (|N_{T_r+1}| = |N_{T_r}| - 1 \mid |N_{T_r}| = N) \Pr (|N_{T_r}| = N \mid C_r) + \frac{1}{n^2}.\]

For Lemma 4.6 we have that $\Pr (|N_{T_r+1}| = |N_{T_r}| - 1 \mid |N_{T_r}| = N) = (N/n)/(N/n + 1)$, so, from the inequality above, we get

$$\Pr (|N_{T_r+1}| = |N_{T_r}| - 1) \leq \sum_{N=0.9n}^{1.1n} \frac{N}{N + n} \Pr (|N_{T_r}| = N \mid C_r) + \frac{1}{n^2} \leq 0.53.$$

To show the lower bound, we utilize the upper bound above, getting

$$\Pr (|N_{T_r+1}| = |N_{T_r}| + 1) = 1 - \Pr (|N_{T_r+1}| = |N_{T_r}| - 1) \geq 1 - 0.53 = 0.47.$$

By a similar argument, we can show also the other inequalities in the statement of the lemma.

7.1.3 Proof of Lemma 4.8

Let $r \geq 7n \log n$. We know from Lemma 4.3 that $|N_{T_r}| \in [0.9n, 1.1n]$ with probability at least $1 - 1/n^2$; we call this event $C_{T_r}$. For Lemma 4.6 and for the memoryless property of the exponential distribution, we have that

$$\Pr (v \in N_{T_r} \mid v \in N_{T_{r-7n \log n}}) \leq \left(1 - \frac{1}{2.2n}\right)^{7n \log n} \leq e^{-3.1 \log n} = \frac{1}{n^{3.1}}$$

(38)
Now we want to prove that every node in $N_T$ has joined the network after time $T_m - 7n \log n$ and, to do that, we have to do an union bound over all the nodes in the network. To doing that, we have to know how nodes in the network they are. So, for the law of total probability and for the concentration
\[
\Pr(\text{there exists a node } v \in N_T \text{ born before } T_m - 7n \log n) \leq \Pr(\text{there exists a node } v \in N_T \text{ born before } T_m - 7n \log n \mid C_T) + \frac{1}{n^2}
\]
and, since $C_T$, guarantees that the nodes in the network at time $T_r$ are at most $1.1n$, from equation (38) we get the lemma.

### 7.2 Omitted proofs for the Poisson model without edge regeneration

#### 7.2.1 Proof of Lemma 4.10

Let $r \geq 7n \log n$. We define the following event
\[
L_r = \{\text{each node in } N_T \text{ is born after time } T_m - 7n \log n \cap \{|N_T| \in [0.9n, 1.1n] \text{ with } i = r - 7n \log n, \ldots, r\}
\]
From Lemma 4.4 and Lemma 4.5 we have that $\Pr(L_r) \geq 1 - 1/n^2$. From the event $L_r$ follows that, when each node in $N_T$ joined the network, the network was composed by at least $0.9n$ nodes and at most $1.1n$ nodes.

Let $\epsilon$ be an arbitrary value with $0 < \epsilon \leq 1/3$: we consider the set of the $\epsilon n$ oldest nodes in $N_T$, and we call that set $H$. We define the following random variables
\[
A = \{v \in H \mid v \text{ has lifetime of at most } 2n \text{ rounds}\}
\]
Recalling the equation 41 of Lemma 4.7, which gives a bound on the lifetime (in rounds) of a node, we can apply a standard concentration argument (Theorem A.1) getting that $|A| \geq \epsilon n/2$ w.h.p. Similarly to the proof of Lemma 3.5 we first define the random variable below
\[
X = \{\text{number of nodes in } A \text{ that are isolated at time } T_r \text{ and for the rest of their lifetime}\}.
\]
The purpose is to show that, w.h.p., $X \geq \frac{1}{3} \epsilon ne^{-2d}$. As in Lemma 5.3 we will utilize the method of bounded differences, expressing $X$ as a function of $2n \cdot d$ independent random variables. First, to calculate the expectation of $X$, we introduce the following random variables, for each $v \in N_T$:
\[
\Delta_{in}^v = \{\text{maximum in-degree of the node } v \text{ for all its lifetime}\},
\]
\[
\Delta_{out}^v = \{\text{out-degree of the node } v \text{ at time } T_r\}.
\]
Since, if $\Delta_{out}^v = 0$, then the node $v$ will have not out-edges from round $t$ to the rest of its lifetime, we can then write $X$ as a function of $\Delta_{in}^v$ and $\Delta_{out}^v$, with $v \in A$:
\[
X = \sum_{v \in A} \mathbb{1}_{\Delta_{in}^v = 0} \mathbb{1}_{\Delta_{out}^v = 0}.
\]
As each node sends its requests independently to the others, we have that
\[
E[X \mid L_r] = \sum_{v \in A} \Pr(\Delta_{in}^v = 0 \mid L_r) \Pr(\Delta_{out}^v = 0 \mid L_r).
\]
(39)
The probability of a node $v \in N_T$ to have in-degree 0 for all their lifetime, conditional to $L_r$, is $\Pr(\Delta_{in}^v = 0 \mid L_r) = (1 - \frac{1}{0.3n})^{2n} \geq e^{-3d}$, since each node $v \in A$ has lifetime of at most $2n$ rounds. The probability of a node $v \in A$ to have out-edges in the current round (conditional to $L_r$) is $\Pr(\Delta_{out}^v = 0 \mid L_r) = (1 - \frac{1}{0.3n})^d \geq e^{-d/2}$, since $\epsilon \leq 1/3$. So, since $|A| \geq \epsilon n/2$ w.h.p. it follows from (39) that
\[
E[X \mid C] \geq \frac{\epsilon n}{3} e^{-2d}.
\]
We now introduce the random variables $Y_j$, returning the index of the node to which the node $v$ send its $j$-th request. The random variables $\{Y_j : v \in N_T, \cup N_{T+1}, \ldots, \cup N_{T+2n}, j \in \{1, \ldots, d\}\}$ are independent. We take $Y$ as the vector of these random variables. Apparently, we can express $X$ as a function of $Y$,
\[
X = f(Y).
\]
Moreover, if the vectors $\mathbf{Y}$ and $\mathbf{Y}'$ differs only in one coordinate, we have that $|f(\mathbf{Y}) - f(\mathbf{Y}')| \leq 2$. This is because, in the worst case, an isolated node can change its destination from a dead node to an other isolated node: so, the number of isolated nodes decrease (or increase) of only 2 units. So, applying Theorem 4.2 we have that, if $\mu$ is a lower bound to $\mathbb{E}[X | C_t]$, 

$$
\Pr(X \leq \mu - M | L_r) \leq e^{-\frac{2\mu^2}{4\mu^2}}.
$$

So, taking $\mu = \frac{1}{6} \epsilon ne^{-2d}$ and $M = \frac{1}{6} \epsilon ne^{-2d}$ we get that

$$
\Pr \left( X \leq \frac{1}{6} \epsilon ne^{-2d} | L_r \right) \leq e^{-n^{2} \epsilon^{-2d}}.
$$

(40)

So, the number of isolated nodes is w.h.p. $X \geq \frac{1}{6} \epsilon ne^{-2d}$. This is because for the law of total probability and from (10), if $n$ is large enough

$$
\Pr \left( X \leq \frac{1}{6} \epsilon ne^{-d} \right) = \Pr \left( X \leq \frac{1}{6} \epsilon ne^{-d} | L_r \right) + \frac{1}{n^2} \leq e^{-n^{2} \epsilon^{-2d}} + \frac{1}{n^2} \leq \frac{2}{n^2}.
$$

Taking $\epsilon = 1/3$ we get the lemma.

7.2.2 Proof of Lemma 4.11

We proceed as in the proof of Lemma 4.6 for the SDG model. We show that any two disjoint sets $S, T \subseteq N_{T_r}$, such that $ne^{-d/20} \leq |S| \leq |N_{T_r}|/2$, $|T| = 0.1|S|$, and $\partial_{out}(S) \subseteq T$, exist with negligible probability. For any $S$ and $T \subseteq N_{T_r} - S$ we again consider $A_{S,T} = \{ \partial_{out}(S) \subseteq T \}$. Consider the event $L_r = \{ \text{each node in } N_{T_r} \text{ is born after time } T_r-7n \log n \} \cap \{ |N_{T_r}| \in [0.9n, 1.1n] \}$ with $i = r-7n \log n$, . . . , $r$, and notice that Lemma 4.8 and Theorem 4.4 imply $\Pr(L_r) \geq 1 - 1/n^2$. Thus, from the law of total probability,

$$
\Pr \left( \min_{ne^{-d/20} \leq |S| \leq n/2} \frac{\partial_{out}(S)}{|S|} \leq 0.1 \right) \leq \sum_{ne^{-d/20} \leq |S| \leq |N_{T_r}|/2} \Pr(A_{S,T} | L_r) + \frac{1}{n^2},
$$

(41)

To upper bound $\Pr(A_{S,T} | L_r)$ we define $P$ as the set $P = N_{T_r} - S - T$ and notice that $|P| \geq 0.9n - 1.1s$. The event $A_{S,T}$ implies that all the edges coming from $S$ must go to $T$: this is equivalent to say that there are no edges between $S$ and $P$. Since

$$
|\{(a,b) \mid a \in S, b \in P\}| = |S| \cdot |P|,
$$

two cases may arise: either

1. $|\{(a,b) \mid a \in S, b \in P, a \text{ younger then } b\}| \geq |S| \cdot |P|/2$, or

2. $|\{(a,b) \mid a \in S, b \in P, b \text{ younger then } a\}| \geq |S| \cdot |P|/2$.

For each $a \in S$, let $N_a$ be the number of nodes in $P$ older than $a$. In the first case, we clearly have that $\sum_{a \in S} N_a \geq |S| \cdot |P|/2$. We can prove that

$$
\Pr(A_{S,T} | L_r) \leq \prod_{a \in S} \left( 1 - \frac{N_a}{1.1n} \right)^d \leq e^{-d \sum_{a \in S} N_a/(1.1n)} \leq e^{-d|S| \cdot |P|/2.2n}.
$$

(42)

Indeed, as for the first inequality above, for each $a \in S$, we considered the probability that a fixed request of node $a$ does not choose any node in $P$ which is older than $a$ and we used the fact that, conditional to the event $L_r$, the probability that a node $a$ chooses any fixed older node $v \in N_{T_r}$ is $\geq 1/1.1n$ (thanks to the event "$a$ chooses $v$ when it joins the network"). Using a symmetric argument, we get the same claim for the second case. Hence, placing (42) into (41),

$$
\Pr \left( \min_{ne^{-d/20} \leq |S| \leq n/2} \frac{\partial_{out}(S)}{|S|} \leq 0.1 \right) = \sum_{s=ne^{-d/20}}^{n/2} \frac{1.1n}{s} \left( 1.1n - s \right) \left( \frac{s}{0.1n} \right) e^{-ds \frac{s-0.1n}{1.1n}} + \frac{1}{n^2} \leq \frac{2}{n^2},
$$

where the last inequality holds for a large enough $n$ and for any $d \geq 20$. It can be easily proved by bounding each binomial coefficient with the bound $\binom{\frac{1.1n}{s}}{s} \leq \left( \frac{1.1e}{s} \right)^s$ and by standard calculation.
7.2.3 Proof of Theorem 4.12

We begin by noting that the proof of Theorem 4.12 uses the original Definition 4.2 of asynchronous flooding. Let $s_0$ be the source node that joins the network at time $t_0 = T_0$. Consider the event

$$C_{r_0}^{v_0+n} = \{ |N_{r_0}| \in [0.9n, 1.1n] \text{ with } i = r_0, \ldots, r_0 + n \},$$

and notice that Lemma 4.4 implies that $Pr(C_{r_0}^{v_0+n}) > 1 - 1/n^2$. We then consider the events $A$ and $B$ we defined in the proof of Lemma 5.5. $A = \text{the source node } s_0 \text{ has all its out-edges to nodes that are isolated at time } t_0 \text{ and for the rest of their lifetime}$ and $B = \text{there are at least } \frac{1}{18}n \text{ nodes in } N_{r_0} \text{ that are isolated at time } t_0 \text{ and for the rest of their lifetime}$. Observe that Lemma 4.10 implies

$$Pr(A \mid C_{r_0}^{v_0+n}) \geq Pr(A \cap B \mid C_{r_0}^{v_0+n}) = Pr(A \mid B, C_{r_0}^{v_0+n}) Pr(B \mid C_{r_0}^{v_0+n}) \geq \frac{1}{2} \left( \frac{e^{-2d}}{18 \cdot 1.1} \right)^d. \quad (43)$$

We define the event

$$E = \{ \text{the node } s_0 \text{ will not get any in-edges for all its lifetime} \}. \quad (44)$$

Notice that the event $A \cap E$ imply that all the informed nodes $s_0, s_1, \ldots, s_d$ are isolated for all their lifetime, and so:

$$Pr(|I_t| \leq d + 1 \text{ for all } t \geq t_0) \geq Pr(A \cap E). \quad (45)$$

Let $D_{s_0}$ the random variable which indicates the lifetime (in rounds) of the node $s_0$. We recall that, for any two events $P, Q$ it holds $Pr(P \cap Q) \geq Pr(P) + Pr(Q) - 1$. Since the life of each nodes follows an exponential random variable of parameter $1/n$, we have that, if $n$ is large enough,

$$Pr(D_{s_0} \leq n \mid C_{r_0}^{v_0+n}) \geq Pr(D_{s_0} \leq n, C_{r_0}^{v_0+n}) Pr(C_{r_0}^{v_0+n}) \geq (1 - e^{-1} + \frac{1}{n^2}) \left( 1 - \frac{1}{n^2} \right) \geq \frac{1 - e^{-1}}{2}. \quad (46)$$

Moreover,

$$Pr(E \mid C_{r_0}^{v_0+n}) \geq Pr(E \mid C_{r_0}^{v_0+n}, D_{s_0} \leq n) Pr(D_{s_0} \leq n \mid C_{r_0}^{v_0+n}) \quad (47)$$

Since each edge chooses its destination uniform at random among the nodes in the network, we have that

$$Pr(E \mid C_{r_0}^{v_0+n}, D_{s_0} \leq n) \geq \left( 1 - \frac{1}{0.9n} \right)^{dn} \geq e^{-2d} \quad (48)$$

Replacing and (47) and (46) in (45) we get

$$Pr(E \mid C_{r_0}^{v_0+n}) \geq \frac{(1 - e^{-1})e^{-2d}}{2} \quad (49)$$

Finally, from (47), since $A$ and $E$ are independent and because of (48) and (49)

$$Pr(|I_t| \leq d + 1 \text{ for all } t \geq t_0) \geq Pr(A \cap E \mid C_{r_0}^{v_0+n}) Pr(C_{r_0}^{v_0+n}) \geq \frac{(1 - e^{-1})e^{-2d}}{8}, \quad (e^{-2d}/20) \geq c(d).$$

Finally, as for the stated linear bound on the flooding time $\tau$, we observe this is an easy consequence of Lemma 4.10. Indeed, in the network there are at least $\frac{1}{18}n$ isolated nodes that will remain isolated for the rest of their lifetime: to have all the nodes informed, we have to wait that these nodes leave the network, and so $\tau = \Omega_d(n)$.

7.2.4 Proof of Theorem 4.13

We begin by reminding the reader that, in order to account for the fact that a live node might die at any point of a given flooding interval, the proof of Theorem 4.13 uses the discretized version of the flooding process described by Definition 4.3 which clearly provides a worst case scenario when we are interested in proving lower bounds on the extent and upper bounds on the speed of flooding. The first observation is that, without edge regeneration, the distribution of links created by a node $v$ is uniform over the set of nodes that were in the network as $v$ joined. In particular, this distribution does not depend on past history of the network as is the case in the model with edge regeneration, where death of a node triggers reallocation of incoming links. We begin by showing a number of preliminary facts that will be useful in the remainder of this proof. The first is the following lemma, which is just a variant of Lemma 4.8 that is of easier use here.
Lemma 7.1 (Nodes’ lifetimes). Let $G_t = (N_t, E_t)$ be a $G(n, d)$ Poisson random graph. If $n$ is large enough, for every $t \geq 4n \log n$, each node in $N_t$ has life $\leq 4n \log n$ with probability at least $1 - 1/n^2$.

Proof. We condition on the event $\mathcal{E} = (|N_{t-4 \log n}| \leq 2n)$. From Lemma 4.4

$$\Pr(|N_{t-4 \log n}| \leq 2n) \geq 1 - \frac{1}{2n^2}.$$ Next, for every $i \in N_t$ we define a binary variable $L_i$, such that $L_i = 1$ if $i$ is alive at time $t$. Note that, conditioned on the event $\mathcal{E}$, each $\Pr(L_i = 1)$ is exponential with parameter $\mu = 1/n$, hence:

$$\Pr(L_i = 1 \mid \mathcal{E}) = e^{-\frac{\mu t}{n}} = \frac{1}{n^t},$$

which implies:

$$\Pr(\exists i \in N_{t-4 \log n} : L_i = 1 \mid \mathcal{E}) \leq \frac{2}{n^t}.$$ Denote by $\mathcal{O}$ the event that there exists a node with age higher than $4n \log n$ at time $t$. We have:

$$\Pr(\mathcal{O}) \leq \Pr(\exists i \in N_{t-4 \log n} : L_i = 1 \mid \mathcal{E}) \Pr(\mathcal{E}) + \Pr(\neg \mathcal{E}) \leq \frac{1}{n^t}.$$ 

\[\square\]

The following fact is instead a simple corollary of Lemma 4.4

Fact 7.2. With probability at least $1 - 1/n^2$, $0.9n \leq N_t \leq 1.1n$, for every $t \in [t_0 - n^2, t_0]$.

Now, assume $N_{t_0} = m$ and recall that $m \in [0.9n, 1.1n]$ from Fact 7.2. Next, we consider the number of nodes that die in the interval $[t_0, t_0 + \log n]$.

Lemma 7.3. With probability at least $1 - 1/n^2$, at most $4 \log n$ nodes die in the interval $[t_0, t_0 + \log n]$.

Proof. Consider the generic node $i \in N_{t_0}$. We define a binary variable $L_i(\tau)$, such that $L_i(\tau) = 1$ if $i$ is alive at time $t_0 + \tau$, $L_i(\tau) = 0$ otherwise. Clearly, since the death process follows the exponential distribution with parameter $\mu = 1/n$ and is thus memoryless, we have:

$$\Pr(L_i(\tau) = 1) = e^{-\frac{\tau}{n}} \geq 1 - \frac{\tau}{n}.$$ Denote by $Z$ the number of nodes that die in the interval $[t_0, t_0 + a \log n]$. Setting $\tau = a \log n$ we immediately have:

$$\mathbb{E}[Z] \leq m - m + \frac{m \log n}{n} \leq 1.1 \log n,$$

where we used $m \leq 1.1n$ with probability $1 - 1/n^2$. Finally, since $Z = \sum_{i \in N_{t_0}} L_i(\log n)$ and since the $L_i(\log n)$’s are independent, a simple application of Chernoff’s bound yields:

$$\Pr(Z > 4 \log n) \leq e^{-1.1 \log n},$$

which proves our claim. \[\square\]

Finally, we bound the number of nodes’ arrivals in an interval of logarithmic duration starting at time $t_0$.

Lemma 7.4. With probability at least $1 - 1/n^2$, at most $4 \log n$ nodes join the network in the interval $[t_0, t_0 + \log n]$.

Proof. Nodes enter the system according to a Poisson process with rate 1 in each step, which corresponds to a Poisson process with rate $4 \log n$ over an interval of $4 \log n$ steps. This fact and tail bounds for the Poisson distribution imply the claim. \[\square\]
We next prove that, starting from $s$ at time $t_0$, with probability $1 - 2e^{-\frac{m}{n^2}} - o(1)$ we reach a fraction $1 - e^{-\frac{m}{n^2}}$ of the nodes that are in the network at time $t_0 + T$, where $T = O(\log n / \log d + d)$. To this purpose, we apply the “onion skin” technique as we did in Section 5.1.2, although with a number of more or less significant changes. To begin, differently from the SDG model, we consider all nodes that are in the system when the informed node joins the network at time $t_0$. On the other hand, we completely disregard nodes that were born in the interval $[t_0, t_0 + \log n]$ (i.e., we don’t count them as hits, though we keep into account that they remove probability mass from destinations in $N_{t_0}$), since their number is negligible with high probability from Lemma 7.4. Conversely, we do need to consider nodes that die in the interval $[t_0, t_0 + \log n]$, since their failure might in principle significantly affect flooding. Removing them at the onset or at the end may be tricky, since in both cases we need to argue that their removal has no significant topological effects (we cannot simply assume an adversarial removal). To sidestep these challenges, we remove each of them with probability $\log n/n$ upon receiving the information for the first time, thus without any chance of informing other nodes of the network, which clearly is a worst-case scenario.

**The onion-skin process.** Denote by $S$ the set of nodes that were in the network at time $t_0$ and let $m = |S|$. From Fact 7.2 we know that $m \in [0.9n, 1.1n]$ with probability at least $1 - 1/n^2$. We now build a map $h : S \to [m]$, so that for $v \in S$, $h(v) = i$ if $v$ is the $i$-th youngest node in the system at time $t$. Note that $h(s) = 1$ by definition. We next define $Y = \{v \in N_t : h(v) \leq m/2\}$ as the subset of young nodes and $O = \{v \in N_t : h(v) \geq m/2 + 1\}$ as the subset of old nodes.

Starting from $s$, the onion-skin process builds a connected, bipartite graph, so that young nodes are only connected to old ones. In particular, each realization of this process generates a subset of the edges generated by the original topology dynamics. Moreover, each iteration of the process corresponds to a partial flooding in the original graph. Flooding is partial since i) the network uses a subset of the edges that would be present in the original graph and ii) every newly informed node tosses a coin and dies, with probability equal to the overall probability of dying in the interval $[t_0, t_0 + \log n]$.

The process unfolds over a suitable number $k$ of phases, with $k = O(\log n / \log d)$. Each phase corresponds to 2 flooding rounds, each consisting of two steps. In the following, we denote by $Y_k \subseteq Y$ and $O_k \subseteq O$ the subsets of young and old nodes that are informed by the end of phase $k$, respectively. In the remainder, we let $O_{-1} = \emptyset$ for notational convenience and, without loss of generality, we use the interval $[d]$ to number the links established by each vertex.

**Onion-skin process (extended version):**

**Phase 0:** $Y_0 = \{s\}; O_0$ is obtained as follows:

1. $s$ establishes $d$ links. We let $Z_0 \subseteq O$ denote the subset of old nodes that are destinations of these links. Links with endpoints in $Y$ are discarded;

2. Let $R$ the subset obtained by removing each vertex in $Z_0$ (and the just established links) independently, with probability $\log n/n$. $O_0 = Z_0 - R$;

**Phase $k \geq 1$:** (two flooding steps)

1.a Each node in $Y - Y_{k-1}$ establishes links $\{1, \ldots, d/2\}$. Denote by $W_k \subseteq Y - Y_{k-1}$ the subset of nodes with at least one link to nodes in $O_{k-1} - O_{k-2}$. Links to nodes in $Y$ are again discarded;

1.b Let $R$ the subset obtained by removing each vertex in $W_k$ (and the just established links) independently, with probability $\log n/n$. $Y_k - Y_{k-1} = W_k - R$;

2.a Each node in $Y_k - Y_{k-1}$ establishes links $\{d/2 + 1, \ldots, d\}$. Denote by $Z_k$ the subset of nodes in $O - O_{k-1}$ that are reached by at least one such link. Links to nodes in $Y$ are discarded;

2.b Let $R$ the subset obtained by removing each node in $Z_k$ (and the links just established) independently, with probability $\log n/n$. $O_k - O_{k-1} = Z_k - R$;
A couple remarks are in order. First of all, we are using the principle of deferred decisions, delaying decision as to the establishment of a link \((u, v)\) to the moment one of its endpoints is informed in the flooding process. On the other hand, this means that the probability that the \(j\)-th link originating from \(u\) has \(v\) as destination is equal to \(1/N_T\), if \(v\)'s arrival in the network corresponds to the \(r\)-th event. We use Lemma \ref{lem:2} and Fact \ref{fact:2} to claim that, with probability at least \(1 - 1/n^2\) the above probability fell in the interval \([1/16, 1/9n]\) for every node in the network at time \(t_0\). Second, the aforementioned probabilities do not depend on nodes that joined or left the network in the interval \([t_0, t_0 + \log n]\). Accordingly, all events we consider in the remainder of this proof are conditioned on \(\{N_t \in [0.9n, 1.1n], \forall t \in [t_0 - n^2, t_0]\}\) and on the ages of all nodes alive at time \(t_0\) belonging to the interval \([1, \ldots, 4n \log n]\). We omit these conditionings for ease of notation.

We next analyze Phase 0 and the generic Phase \(k\) separately. To this purpose, we use the same random variables we defined in the proof of Theorem \ref{thm:3.7}, whose definition is repeated here to make the proof self-contained. For each \(i = 1, \ldots, d\), for each node \(v \in N_{t_0}\) and for each set \(A \subseteq N_{t_0}\), we define the Bernoulli random variable \(R_{v, A}\) as follows:

\[
R_{v, A} = \begin{cases} 
1 & \text{if } x \geq 1 \text{ links in } \{d/2, \ldots, d\} \text{ from } v \text{ have destination in } A \\
0 & \text{otherwise}
\end{cases}
\]

Moreover, for each \(i = 1, \ldots, d\) and for every \(v \in N_{t_0}\) that joined the network at time \(t\), we define the variable \(A_{i}^{[v]} \in N_{t}\) as follows: \(A_{i}^{[v]} = w\) if \(w \in N_{t}\) is the destination of the \(i\)-th link of \(v\).

**Analysis of phase 0.** For phase 0 we begin with the following claim.

**Claim 7.5.** The following holds at the end of phase 0:

\[
\Pr \left( |O_0| \geq \frac{d}{16} \right) \geq \left( 1 - \frac{2\log n}{n} \right) \left( 1 - e^{-\frac{d}{12}} \right).
\]

**Proof.** We begin with step 1. Assume \(v \in O\). We have:

\[
\Pr \left( \exists i \in [d]: A_{i}^{[v]} = v \right) = 1 - \left( 1 - \frac{1}{m} \right)^d,
\]

which implies

\[
E |Z_0| = |O| \left( 1 - \left( 1 - \frac{1}{m} \right)^d \right) \geq |O_0| \frac{d}{2m} = \frac{d}{4},
\]

where the last equality follows since \(|O| = m/2\). We next bound the probability that \(Z_0\) is smaller than \(d/8\). To this purpose, we cannot simply apply a Chernoff bound to the binary variables that describe whether or not a node \(v \in O\) was the recipient of at least one link originating from \(s\), since these are not independent. We instead resort to Theorem \ref{thm:A.2} In particular, similarly to what we did in the proof of Theorem \ref{thm:3.7} we define the function \(f(A_{1}^{[s]}, \ldots, A_{d}^{[s]} = |Z_0|\). Clearly, \(f\) is well-defined and it satisfies the Lipschitz condition with values \(\beta_1 = \cdots = \beta_d = 1\), since changing the destination of one link can affect the value of \(|Z_0|\) by at most 1. We can thus apply Theorem \ref{thm:A.2} to obtain:

\[
\Pr \left( |Z_0| < \frac{d}{8} \right) \leq \Pr \left( |Z_0| < E |Z_0| - \frac{d}{8} \right) \leq e^{-\frac{d}{12}}.
\]

We next obtain \(O_0\). If \(|Z_0| = x\) and \(R\) is the number of nodes in \(Z_0\) that are removed, we have:

\[
E[R] \leq \frac{ax \log n}{n}.
\]

Applying Markov’s inequality immediately yields

\[
\Pr \left( R \geq \frac{x}{2} \right) \leq 2 \frac{\log n}{n},
\]

whence the thesis immediately follows. \(\square\)
Analysis of Phase $k$ - step 1. We next examine the number of nodes in $Y - Y_{k-1}$ that connect to nodes in $|O_{k-1} - O_{k-2}|$ using their links belonging to the subset $\{d/2 + 1, \ldots, d\}$.

Claim 7.6. Assume that $|O_{k-1} - O_{k-2}| = y$ and $|Y_{k-1}| \leq m/10$. For sufficiently large $n$, the following holds at the end of phase $k$:

\[
\Pr \left( |Y_k - Y_{k-1}| \geq \frac{y d}{48} \mid |O_{k-1} - O_{k-2}| = y \right) \geq \left( 1 - \frac{2 \log n}{n} \right) \left( 1 - e^{-\frac{n d}{44d}} \right), \quad y \leq \frac{1.1n}{d}
\]

\[
\Pr \left( |Y_k - Y_{k-1}| \geq \frac{m}{20} \mid |O_{k-1} - O_{k-2}| = y \right) \geq \left( 1 - \frac{2 \log n}{n} \right) \left( 1 - e^{-\frac{n d}{20d}} \right), \quad y > \frac{1.1n}{d}.
\]

Proof. It should be noted that i) no links originating from nodes in $Y - Y_{k-1}$ have been established so far and ii) at this point we know that, for each $u \in Y - Y_{k-1}$, none of its links in $\{d/2 + 1, \ldots, d\}$ had destination in $O_{k-2}$, thus the probability of pointing to some vertex in $O_{k-1} - O_{k-2}$ can only be magnified. With these premises, if $u \in Y - Y_{k-1}$ we have:

\[
\Pr \left( R_{v, O_{k-1} - O_{k-2}} = 1 \mid |O_{k-1} - O_{k-2}| = y \right) = 1 - \left( 1 - \frac{y d}{\ell} \right)^{\frac{d}{\ell}} > 1 - e^{-\frac{n d}{44d}},
\]

where $\ell \in [0.9n, 1.1n]$ denotes the number of nodes when $u$ joined the network. If $y \leq 1.1n/d$ we have $\frac{y d}{\ell} < 0.61$, which implies $e^{-\frac{n d}{44d}} < \frac{2 y d}{\ell}$, whence:

\[
\Pr \left( R_{v, O_{k-1} - O_{k-2}} = 1 \mid |O_{k-1} - O_{k-2}| = y \right) > \frac{y d}{\ell} > \frac{y d}{3.3n}.
\]

As a consequence, if $|Y_{k-1}| \leq m/10$:

\[
E[|W_k| \mid |O_{k-1} - O_{k-2}| = y] > |Y - Y_{k-1}| \frac{y d}{3.3n} \geq \frac{2 m y d}{3.3n} > \frac{y d}{12},
\]

where in the last inequality we used $m \geq 0.9n$. On the other hand, the $R_{v, O_{k-1} - O_{k-2}}$’s are independent. We can thus apply a standard Chernoff bound to obtain:

\[
\Pr \left( W_k < \frac{y d}{24} \mid |O_{k-1} - O_{k-2}| = y \right) \leq e^{-\frac{n d}{24}}.
\]

We next consider step 1.b. Assume $|W_k| = x$. If $R$ denotes the number of vertices removed from $W_k$ we can argue exactly like for the analysis of step 2 of Phase 0 to obtain:

\[
\Pr \left( |Y_k - Y_{k-1}| \geq \frac{y d}{48} \mid |O_{k-1} - O_{k-2}| = y \right) \geq \left( 1 - \frac{2 \log n}{n} \right) \left( 1 - e^{-\frac{n d}{44d}} \right).
\]

Conversely, if $y > 1.1n/d$ we have $\frac{y d}{\ell} > 1/2$, so that

\[
\Pr \left( R_{v, O_{k-1} - O_{k-2}} = 1 \mid |O_{k-1} - O_{k-2}| = y \right) > 1 - e^{-1/2}.
\]

In this case:

\[
E[|W_k| \mid |O_{k-1} - O_{k-2}| = y] \geq |Y - Y_{k-1}| \left( 1 - \frac{1}{\sqrt{e}} \right) \geq \frac{2 m}{5} \left( 1 - \frac{1}{\sqrt{e}} \right) > 0.78 \frac{m}{5},
\]

which is both larger than $(2/15)m$ and $(2/15)n$ (the latter follows since $m \geq 0.9n$). We therefore have:

\[
\Pr \left( |W_k| \geq \frac{m}{10} \mid |O_{k-1} - O_{k-2}| = y \right) \geq \Pr \left( |W_k| \geq \frac{3}{4} E[|W_k|] \mid |O_{k-1} - O_{k-2}| = y \right) \leq e^{-\frac{n d}{24}}.
\]

Hence, arguing as we did before, we conclude:

\[
\Pr \left( |Y_k - Y_{k-1}| \geq \frac{m}{20} \mid |O_{k-1} - O_{k-2}| = y \right) \geq \left( 1 - \frac{2 \log n}{n} \right) \left( 1 - e^{-\frac{n d}{20d}} \right).
\]
Analysis of Phase $k$ - step 2. In this case, we are interested in nodes from $Y_k - Y_{k-1}$ that connect to nodes in $O - O_{k-1}$ using their first $d/2$ links.

**Claim 7.7.** Assume $|Y_k - Y_{k-1}| = x$ and $|O_{k-1}| \leq m/10$. For sufficiently large $n$, the following holds at the end of phase $k$:

$$\Pr \left( |O_k - O_{k-1}| \geq \frac{x d}{48} \mid |Y_k - Y_{k-1}| = x \right) \geq \left( 1 - \frac{2 \log n}{n} \right) \left( 1 - e^{-\frac{x d}{n}} \right), \quad x \leq \frac{1.1 n}{d},$$

$$\Pr \left( |O_k - O_{k-1}| \geq \frac{m}{20} \mid |Y_k - Y_{k-1}| = x \right) \geq \left( 1 - \frac{2 \log n}{n} \right) \left( 1 - e^{-\frac{x d}{\sqrt{n}}} \right), \quad x > \frac{1.1 n}{d}.$$

**Proof.** Assume $v \in O - O_{k-1}$. We have:

$$\Pr \left( \exists u \in Y_k - Y_{k-1}, \exists i \in [d/2] : A_u^{(i)} = v \mid |Y_k - Y_{k-1}| = x \right) \geq 1 - \left( 1 - \frac{1}{1.1 n} \right)^d > 1 - e^{-\frac{x d}{1.1 n}}.$$

As a consequence:

$$E[|Z_k| \mid |Y_k - Y_{k-1}| = x] > |O - O_{k-1}|(1 - e^{-\frac{x d}{1.1 n}}).$$

We consider two cases, as we did in the proof of Claim 7.6. If $x \leq \frac{1.1 n}{d}$ we have:

$$E[|Z_k| \mid |Y_k - Y_{k-1}| = x] \geq |O - O_{k-1}| \frac{x d}{3.3 n} \geq \frac{2 m x d}{5 \cdot 3.3 n} > \frac{x d}{12},$$

where similarly to Claim 7.6 we used $|O_{k-1}| \leq m/10$ and $m \geq 0.9 n$. where the second inequality follows since i) $|O| = m/2$ and ii) we are assuming $|Y_{k-1}| \leq m/10$. Again and differently from Claim 7.6 we cannot simply concentrate, since the events $\{A_u^{(i)} = v\}$ are negatively correlated as $v$ varies over $O - O_{k-1}$. We again resort to Theorem A.2. In this case, we have $d/2$ links that are established (independently of each other) from vertices in $Y_k - Y_{k-1}$. Consider the $x d/2$ variables $\{A_u^{(i)}\}_{u \in Y_k - Y_{k-1}, i \in [d/2]}$. The domain of $A_u^{(i)}$ is the set $N_t$ if $u$ joined the system at time $t$, where $0.9 n \leq |N_t| \leq 1.1 n$ with high probability, from Lemma 7.4 and Fact 7.2. We then define the function $f(\{A_u^{(i)}\}_{u \in Y_k - Y_{k-1}, i \in [d/2]} = |Z_k|$. Like in the analysis of step 1 of Phase 0, we note that $f$ satisfies the Lipschitz condition with constants $\beta_1 = \cdots = \beta_{\frac{d}{2}} = 1$. We can thus apply Theorem A.2 to obtain:

$$\Pr \left( |Z_k| < \frac{x d}{24} \mid |Y_k - Y_{k-1}| = x \right) \leq \Pr \left( |Z_k| < E[|Z_k| \mid |Y_k - Y_{k-1}| = x] - \frac{x d}{24} \mid |Y_k - Y_{k-1}| = x \right) \leq e^{-\frac{x d}{12}}.$$

Finally, we remove nodes from $Z_k$ exactly as we did in Phase 0 and in step 1.b of Phase $k$. The analysis proceeds exactly the same, so that we can conclude:

$$\Pr \left( |O_k - O_{k-1}| \geq \frac{x d}{48} \mid |Y_k - Y_{k-1}| = x \right) \geq \left( 1 - \frac{2 \log n}{n} \right) \left( 1 - e^{-\frac{x d}{n}} \right).$$

Conversely, if $x > 1.1 n/d$ we have $\frac{x d}{d} > \frac{x d}{n}$, so that:

$$\Pr \left( \exists u \in Y_k - Y_{k-1}, \exists i \in [d/2] : A_u^{(i)} = v \mid |Y_k - Y_{k-1}| = x \right) > 1 - e^{-1/2},$$

whence:

$$E[|Z_k| \mid |Y_k - Y_{k-1}| = x] \geq |O - O_{k-1}| \left( 1 - \frac{1}{\sqrt{e}} \right) \geq \frac{2 m}{5} \left( 1 - \frac{1}{\sqrt{e}} \right) > \frac{2}{15} m.$$

Next, application of Theorem A.2 yields in this case:

$$\Pr \left( |Z_k| < \frac{m}{10} \mid |Y_k - Y_{k-1}| = x \right) \leq \Pr \left( |Z_k| < E[|Z_k|] - \frac{m}{10} \mid |Y_k - Y_{k-1}| = x \right) \leq e^{-\frac{m^2}{100}} \leq e^{-\frac{m}{100}} < e^{-\sqrt{n}},$$

where the third inequality follows from $x \leq |Y| \leq m/2$, while the fourth follows since $d$ is a constant and $m \geq 0.9 n$, so it holds for sufficiently large $n$. Finally, proceeding like in Claim 7.3 (analysis of step 1.b) we obtain:

$$\Pr \left( |O_k - O_{k-1}| \geq \frac{m}{20} \mid |Y_k - Y_{k-1}| = x \right) \geq \left( 1 - \frac{2 \log n}{n} \right) \left( 1 - e^{-\sqrt{n}} \right).$$

$\square$
Finally, Claims 7.5, 7.6 and 7.7 imply the following result.

**Lemma 7.8** (Flooding completes, part 1). For sufficiently large and constant $d \geq 1152$, there exist a constant $c > 0$ and $k = O(\log n / \log d)$, such that:

\[
\Pr \left( |Y_k| \geq \left( \frac{m}{20} \right) \cap |O_k| \geq \left( \frac{m}{20} \right) \right) \geq 1 - 2e^{-\gamma} - o(1).
\]

**Proof.** Consider the generic $i$-th phase and assume i) $|Y_{i-1}|, |O_{i-1}| < 1.1n/d \leq n/10$ (i.e., $d \geq 11$) and ii) $|O_{i-1} - O_{i-1}| \geq \left( \frac{d}{48} \right)^{2i}$. Then, Claims 7.5, 7.6 and 7.7 imply that, conditioned to i) and ii) we have:

\[
\Pr \left( |Y_i - Y_{i-1}| \geq \left( \frac{d}{48} \right)^{2i} \cap |O_i - O_{i-1}| \geq \left( \frac{d}{48} \right)^{2i+1} \right) \geq \left( 1 - \frac{2 \log n}{n} \right)^2 \left( 1 - e^{-\gamma} \right)^{2i} \left( 1 - e^{-\gamma} \right)^{2i+1}.
\]

(49)

Since $|Y_i| \geq |Y_i - Y_{i-1}|$ and $|O_i| \geq |O_i - O_{i-1}|$, using the chain rule and (49), for

\[
k \geq \frac{\log \left( \frac{d}{48} \right)}{2\log \left( \frac{d}{48} \right)},
\]

at the end of phase $k$ we have:

\[
\Pr \left( \left| Y_k \right| \geq \left( \frac{m}{20} \right) \cap \left| O_k \right| \geq \left( \frac{m}{20} \right) \right) \geq \left( 1 - e^{-\gamma} \right)^2 \left( 1 - \frac{2 \log n}{n} \right)^{2k+1} \left( 1 - e^{-\gamma} \right)^{2k+1} \left( 1 - e^{-\gamma} \right)^{2k+1} \prod_{i=1}^{2k+1} \left( 1 - e^{-\gamma} \right)^{2k+1}.
\]

Let $P = \prod_{i=1}^{2k+1} \left( 1 - e^{-\gamma} \right)^{2k+1}$. We next show that $P \geq c$ for a suitable constant $c$. This is equivalent to showing that $-\log P \leq c'$, where $c' = \log \frac{d}{48}$, i.e., for $d$ a sufficiently large, absolute constant. The fifth inequality follows since $\frac{d}{48} \geq \frac{d}{576}$ whenever $d/576 \geq 2$, while the last inequality follows since the double exponential is dominated by a single one, summing to a constant not exceeding 1. Proceeding like in the final steps of Claim 3.12, we obtain $P > 1 - 2e^{-\gamma}$.

\[
\Pr \left( \left| Y_k \right| \geq \left( \frac{m}{20} \right) \cap \left| O_k \right| \geq \left( \frac{m}{20} \right) \right) \geq \left( 1 - e^{-\gamma} \right)^2 \left( 1 - \frac{2 \log n}{n} \right)^{2k+1} \left( 1 - e^{-\gamma} \right)^{2k+1} P,
\]

the proof follows for sufficiently large $n$. \qed

**Information spreading via the expansion of large subsets.** We finally show that, if at least $m/10$ nodes become informed (which occurs with probability at least $1 - 2e^{-\gamma} - o(1)$) within $O(\log n / \log d)$ flooding steps from Lemma 7.5, then at least $(1 - e^{-\gamma})m$ nodes become informed within a further, constant number of flooding steps, w.h.p.

To prove this, we leverage Lemma 4.11 and we proceed along the same lines as Theorem 4.8 and in particular Lemma 3.12, albeit with the following difference: in each flooding step, we need to account for the fact that a node that was present at time $t_0$ might die before being informed (or right upon being informed) and thus be unable to contribute to the flooding process. We have the following

**Lemma 7.9** (Flooding completes, part 2). Under the hypotheses of Theorem 4.8 for some $\tau_2 = O(d)$ and for $\tau_1 = O(\log n / \log d)$ as in Lemma 7.5, we have:

\[
\Pr \left( I_{t_0 + \tau_1 + \tau_2} \geq (1 - e^{-\gamma})m \right) \geq 1 - 2e^{-\gamma} - o(1).
\]
Proof. The proof proceeds along the very same lines as Lemma 3.12, hence we only discuss the differences.

If we start with \( n/10 \) informed nodes, in each flooding step the set of informed nodes increases by a constant factor, exactly like in the proof of Lemma 3.12, but this time we use Lemma 4.11 for the expansion. The main difference is that, this time, each newly informed node has a chance to die. Assume \( S \) is the set of the newly informed nodes at the end of the generic expansion/flooding step. Since we are interested in a constant number of rounds, we can handle nodes’ deaths in a simplified way with respect to what we did in the proof of Lemma 7.8. In particular, with high probability, at most \( 4 \log n \) nodes are removed from \( S \) in worst-case fashion, but this still implies that, with high probability, \(|S| - 4 \log n\) new nodes have been informed, thus the set of informed nodes has increased by a constant factor, since \(|S| = \Omega(n)\). If we iterate over a sufficiently large, constant number \( \tau_2 = \mathcal{O}(d) \) of flooding steps, we can conclude, like in the proof of Lemma 3.12, that \((1 - e^{-\frac{\tau_2}{n}})m\) nodes have been informed within time \( t_0 + \tau_1 + \tau_2 \). \( \square \)

This last step concludes the proof of Theorem 7.13.

7.3 Proofs for the Poisson model with edge regeneration

7.3.1 Proof of Lemma 4.15

We define the following event avoiding the use of

\[ A_{u,v} = \{ \text{a fixed request of } u \text{ has destination } v \text{ at time } T_r \} . \]

Notice that we avoid to index the specific request since the considered graph process is perfectly symmetric w.r.t. the \( d \) random requests of every node. We first bound the probability that a fixed request of \( u \) has destination \( v \) when \( v \) is younger than \( u \). Calling \( L_r \) the event

\[ L_r = \{ \text{each node in } N_{T_r} \text{ is born after time } T_r - r \log n \} \cap \{ |N_{T_i}| \in [0.9n, 1.1n] \text{ with } i = r - 7 \log n, \ldots, r \} \]

from Lemma 4.4 and Lemma 4.8 we get that \( \text{Pr}(L_r) \geq 1 - 1/n^2 \). We notice that the event \( L_r \) means that, when each node in \( N_{T_r} \) joined the network, the network was composed by at least \( 0.9n \) nodes and at most \( 1.1n \) nodes. From the law of total probability, we have

\[ \text{Pr}(A_{u,v}) \leq \text{Pr}(A_{u,v} | L_r) + \frac{1}{n^2} \leq \frac{1}{0.9n} + \frac{1}{n^2} \leq \frac{1}{0.8n} , \]

where \( \text{Pr}(A_{u,v} | L_r) \leq 1/0.9n \) since \( u \) can choose \( v \) only after a death of one of its neighbours, being \( v \) younger than \( u \).

We now analyze the case in which \( v \) is older than \( u \), where \( u \) is born at step \( T_{r-i} \). For the law of total probability,

\[ \text{Pr}(A_{u,v}) \leq \text{Pr}(A_{u,v} | L_r) + \frac{1}{n^2} . \quad (50) \]

So, the next step is to evaluate \( \text{Pr}(A_{u,v} | L_r) \). For each \( k \geq 1 \) and \( w \in N_{T_k} \), define the following event:

\[ D_{w,k} = \{ w \text{ dies at time } T_k \} . \]

To bound \( \text{Pr}(D_{w,k} | L_r) \), for each \( k = r - i, \ldots, r \) and \( w \in N_{T_k} \), we use Lemma 4.4 to get \( \text{Pr}(D_{w,k}) \leq 1/(1.8n) \), and, hence, for the Bayes’ rule,

\[ \text{Pr}(D_{w,k} | L_r) = \frac{\text{Pr}(D_{w,k} \cap L_r)}{\text{Pr}(L_r)} \leq \frac{\text{Pr}(D_{w,k})}{1 - 1/n^2} = \frac{1/1.8n}{1 - 1/n^2} \leq \frac{1}{1.7n} . \quad (51) \]

Now, for each \( j = r - i, \ldots, r \), define the following events

\[ A_{u,v}^j = \{ \text{a fixed request of } u \text{ connects to } v \text{ at time } T_j \} , \]

and write \( A_{u,v} = \bigcup_{j=r-i}^{r} A_{u,v}^j \). Notice that there is some difference between the probability distribution of \( A_{u,v}^{r-i} \) and that of \( A_{u,v}^j \) for each \( j > r - i \). Indeed, it holds that

\[ \text{Pr}(A_{u,v}^{r-i} | L_r) \leq \frac{1}{0.9n} , \quad (52) \]

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since this is the probability that the request of \( u \) has destination \( v \) at the time of \( u \)'s arrival (since \( v \) is older than \( u \)). On the other hand, for each \( j = r - i + 1, \ldots, r \), thanks to the memoryless property of the exponential distribution,

\[
\Pr(A_{u,v}^1 \mid L_r) \leq \frac{1}{1.7n} \cdot \frac{1}{0.9n}.
\]

(53)

The above bound holds since any fixed request of \( u \) can choose \( v \) as destination at round \( T_j \) only if, at round \( T_j - 1, u \) is not connected to \( v \). So, the first factor in the r.h.s. of (53) is an upper bound on the probability that, at time \( T_j - 1, u \) is not connected to \( v \). The second factor, \( 1/(1.7n) \), is the upper bound on the probability (conditional to \( L_r \) from (51)) that the node to which \( u \) is connected dies at time \( T_j \). Moreover, \( 1/(0.9n) \) is the probability, conditional to \( L_r \), that the request of \( u \) connects to \( v \) at time \( T_j \), if its neighbour is died at time \( T_j \).

So, recalling that \( A_{u,v} = \bigcup_{j=r-i}^r A_{u,v}^j \), from (52) and (53),

\[
\Pr(A_{u,v} \mid L_r) \leq \sum_{j=r-i}^r \Pr(A_{u,v}^j \mid L_r) \leq \frac{1}{0.9n} \left(1 + \frac{i}{1.7n}\right).
\]

(54)

Finally, since conditional to \( L_r \) we have that \( i \leq 7n \log n \), using (54) into (50), the proof is completed.

7.3.2 Proof of Lemma 4.17

We proceed as in the proof of the analogous lemma in the SDGR model (Lemma 6.4): we want to show that two disjoint sets \( S, T \subseteq N_t \), with \(|S| \leq n/\log^2 n \) and \(|T| = 0.1|S| \), such that \( \partial_{out}(S) \subseteq T \), exist with negligible probability.

We recall the definition

\[
A_{S,T} = \{\partial_{out}(S) \subseteq T\}.
\]

Then, as for the event

\[
L_r = \{\text{each node in } N_{T_r} \text{ is born after time } T_{r-7n \log n} \cap \{|N_{T_r}| \in [0.9n, 1.1n] \},
\]

from Lemma 4.4 and Lemma 4.8, we obtain \( \Pr(L_r) \geq 1 - 1/n^2 \). So, for the law of total probability,

\[
\Pr\left(\min_{0 \leq |S| \leq n/\log^2 n} \frac{|\partial_{out}(S)|}{|S|} \leq 0.1\right) \leq \sum_{|S| \leq n/\log^2 n} \Pr(A_{S,T} \mid L_r) + \frac{1}{n^2}.
\]

(55)

The next step of the proof is to upper bound \( \Pr(A_{S,T} \mid L_r) \). From Lemma 4.15 since \( L_r \) implies that all the active nodes were born after time \( T_{r-7n \log n} \),

\[
\Pr(A_{S,T} \mid L_r) \leq \left(\frac{|S \cup T|}{0.8n} \left(1 + \frac{7n \log n}{1.7n}\right)^d|S|\right)^{|S \cup T|} \leq \left(\frac{|S \cup T|}{0.8n} (1 + 5n \log n)^d|S|\right).
\]

(56)

Notice that, since \(|S| \leq n/\log^2 n \), the above equation offers a sufficiently small bound. So, combining (56) with (55), we obtain

\[
\Pr\left(\min_{0 \leq |S| \leq n/\log^2 n} \frac{|\partial_{out}(S)|}{|S|} \leq 0.1\right) \leq \sum_{s=1}^{n/\log^2 n} \left(\frac{1.1n}{s}\left(1.1n - s\right)\right)^s \left(\frac{1.1s}{0.8n}\right)^{1.1s} \left(1 + 5n \log n\right)^d s + \frac{1}{n^2}.
\]

(57)

In the equation above, we bounded each binomial coefficient with the inequality \( \binom{n}{k} \leq \left(\frac{ne}{k}\right)^k \) for each \( k \leq n \) and \( n \geq 2 \). Then, by calculating the derivative of the function \( f(s) \) that represents each term of the sum, we derive that each of such terms reaches its maximum at the extremes, i.e. in \( s = 1 \) or in \( s = n/\log^2 n \). So, we get that the sum in (57), if \( d \geq 35 \), is bounded by \( 2/n^2 \).

7.3.3 Proof of Theorem 4.20

As remarked in Subsection 4, the discretized version of the flooding process (Definition 4.3) is always slower than the original one (Definition 4.2), so we can analyze the former version along three consecutive phases.

Phase 1: The Bootstrap. The first phase lasts until the source information reaches a subset of size \( n^\epsilon \), for some constant \( \epsilon < 1 \) (in our analysis we fix \( \epsilon = 1/10 \)).
Lemma 7.10 (Phase 1: The Bootstrap). Under the hypotheses of Theorem 4.20 there is a \( \tau_1 = \mathcal{O}(\log n) \) such that, w.h.p.
\[
|I_{t_0+\tau_1}| \geq n^{1/10}.
\]

Proof. Let \( I_t \) be the set of informed nodes, with \( t \geq t_0 = T_\tau \) and \( |I_t| \leq n^{1/10} \). We want to prove that, w.h.p., \( I_t = I_t \cap N_{t+1} \), i.e. all the nodes in \( I_t \) survive for a time interval equal to 1. Since the life of a node follows an exponential distribution of parameter 1/n, and since \( |I_t| \leq n^{1/10} \), this event has probability \( e^{-|I_t|/n} \geq 1 - 1/n^{9/10} \). According to Definition 4.3 we let \( I_{t+1} = (I_t \cup \partial^t_{\text{out}}(I_t \cap N_{t+1})) \cap N_{t+1} \) be the set of informed nodes at time \( t + 1 \). Since the graph \( G_t \) is an expander of parameter 0.1, w.h.p. (Theorem 4.16) and since \( I_t \cap N_{t+1} = I_t \) w.h.p., it holds w.h.p.
\[
|\partial^t_{\text{out}}(I_t \cap N_{t+1})| \geq 0.1|I_t|.
\]
Since \( |I_t| \leq n^{1/10} \), all the nodes in \( \partial^t_{\text{out}}(I_t \cap N_{t+1}) \) survive for a time interval equal to 1 with probability \( e^{-0.1|I_t|/n} \geq 1 - 1/n^{9/10} \), so w.h.p.
\[
|I_{t+1}| \geq |(\partial^t_{\text{out}}(I_t \cap N_{t+1})) \cap N_{t+1}| \geq 1.1|I_t|.
\]
It follows that, after a phase of length \( \tau_1 = \mathcal{O}(\log n) \), we get \( |I_{t_0+\tau_1}| \geq n^{1/10} \), w.h.p. \( \square \)

Phase 2: Exponential growth of the informed nodes. In the next lemma, we show that, after the bootstrap, the flooding process yields an exponential increase of the number of informed nodes until it reaching half of the nodes in the network.

Lemma 7.11 (Phase 2). Under the same hypotheses of Theorem 4.20 there is a \( \tau_2 = \mathcal{O}(\log n) \) such that, for \( \tau_1 = \mathcal{O}(\log n) \) (as in Lemma 7.10), w.h.p.
\[
|I_{t_0+\tau_1+\tau_2}| \geq \frac{|N_{t_0+\tau_1+\tau_2}|}{2}.
\]

Proof. Observe first that in any interval of time equal to 1, w.h.p. at most \( 2\log n \) nodes leave the network. Indeed, the number of nodes that leave the network in the time interval \([t, t+1]\) is a random variable
\[
D = \sum_{v \in N_t} D_v,
\]
where each \( D_v \) is a Bernoulli random variable, such that \( \Pr(D_v = 1) = 1 - e^{-1/n} \), which indicates if the node \( v \in N_t \) leaves the network before \( t + 1 \). So, from Lemma 4.4 and the Chernoff Bound (Theorem A.1),
\[
\Pr(D \geq 2\log n) \leq \frac{1}{n^{1/3}}.
\]
We recall that the set of infected nodes at time \( t + 1 \) is \( I_{t+1} = (I_t \cup \partial^t_{\text{out}}(I_t \cap N_{t+1})) \cap N_{t+1} \). Since we have shown that, in the interval \([t, t+1]\), at most \( 2\log n \) nodes leave the network w.h.p. and since the graph \( G_t \) is a expander with parameter 0.1 w.h.p. (Theorem 4.16), it holds, w.h.p.,
\[
|I_{t+1}| \geq |I_t| + 0.1(|I_t| - 2\log n) - 2\log n.
\]
So, for each \( t \geq t_0 + \tau_1 \), since \( |I_{t_0+\tau_1}| \geq n^{1/10} \), w.h.p.
\[
|I_{t+1}| \geq 1.09|I_t|.
\]
We thus have an exponential growth of the set of the informed nodes and, so, there exists \( \tau_2 = \mathcal{O}(\log n) \) such that \( |I_{t_0+\tau_1+\tau_2}| \geq |N_{t_0+\tau_1+\tau_2}|/2 \), w.h.p. \( \square \)

Phase 3: Exponential decrease of the non-informed nodes. The analysis of this phase considers the subset \( S_t \subseteq N_t \) of the non-informed nodes. More precisely, we prove that \( S_{t+1} \) w.h.p. decreases by a constant factor despite the node churn.

Lemma 7.12 (Phase 3). Under the same hypotheses of Theorem 4.20 there is a \( \tau_3 = \mathcal{O}(\log n) \) such that, for \( \tau_1 = \mathcal{O}(\log n) \) (as in Lemma 7.10) and \( \tau_2 = \mathcal{O}(\log n) \) (as in Lemma 7.11), we have w.h.p.
\[
I_{t_0+\tau_1+\tau_2+\tau_3} = N_{t_0+\tau_1+\tau_2+\tau_3}.
\]
Proof. To prove this lemma, we will consider the set $S_t \subseteq N_t$ of non-informed nodes at time $t$, i.e. $S_t = N_t - I_t$. Notice that, since every node $v$ in $\partial_{out}^{t+1}(S_{t+1}) \subseteq I_{t+1}$ is reachable in 1-hop to the set of non-informed nodes at time $t + 1$, $v$ was not informed at time $t$. This implies that

$$\partial_{out}^{t+1}(S_{t+1}) \subseteq (S_t - S_{t+1}) \cap N_{t+1}.$$  

Consider the random variable $J_t$ that indicates the number of nodes that join the network in the time interval $[t, t+1]$. Then, the above consideration implies that

$$|S_t| - |S_{t+1}| + J_{t,t+1} = |\partial_{out}^{t+1}(S_{t+1})|. $$

Since $J_{t,t+1}$ is a Poisson random variable of parameter 1, for the tail bound for the Poisson distribution (Theorem [C.4]), $J_{t,t+1} \leq \log n$, w.h.p. Since $|S_{t_0 + \tau_1 + \tau_2}| \leq |N_{t_0 + \tau_1 + \tau_2}|/2$ w.h.p., from the expansion of the graph $G_{t+1}$ (Theorem [4.16]), it holds w.h.p. that, for each $t \geq t_0 + \tau_1 + \tau_2$,

$$|S_{t+1}| \leq 1 \frac{1}{1.1} (|S_t| + \log n).$$

Then, a time $\tau'_3 = O(\log n)$ exists such that $|S_{t_0 + \tau_1 + \tau_2 + \tau'_3}| \leq \log^2 n$.

After the process reaches the above small number of non-informed nodes, we consider the set of non-informed nodes at time $t$ without including the set of nodes that join the network after time $t_0 + \tau_1 + \tau_2 + \tau'_3$; we call the latter set as $S^*_t$, for each $t \geq t_0 + \tau_1 + \tau_2 + \tau'_3$. As in the first part of the proof, we get that $\partial_{out}^{t+1}(S^*_{t+1}) \subseteq (S^*_t - S_{t+1}) \cap N_{t+1}$. Since the graph $G_{t+1}$ is an expander w.h.p. (Theorem [4.16]), we have that w.h.p.

$$|S^*_{t+1}| \leq 1 \frac{1}{1.1} |S^*_t|.$$  

Since $|S^*_{t_0 + \tau_1 + \tau_2 + \tau'_3}| \leq \log^2 n$, there is a $\tau_3 = O(\log n)$ such that, $|S^*_{t_0 + \tau_1 + \tau_2 + \tau'_3}| < 1$ w.h.p.

In conclusion, let $J^*_{\tau'_3, \tau_3}$ be the number of nodes that join the network from time $t_0 + \tau_1 + \tau_2 + \tau'_3$ to time $t_0 + \tau_1 + \tau_2 + \tau_3$. Since the arrival of the nodes during an interval of length $\tau_3 - \tau'_3$ is a Poisson process of mean $\tau_3 - \tau'_3$, for the tail bound for the Poisson distribution (Theorem [C.4]), w.h.p. $J^*_{\tau'_3, \tau_3} = O(\log n)$. Moreover, from Lemma [4.4] each of these new nodes connect to the set of informed nodes with probability at least $(1 - (2\log^2 n/n)^3)(1 - 1/n^3)$. Moreover, each informed node to which the new nodes have connected survive for the 1-hop transmission with probability $e^{-1/n} \geq 1 - \frac{1}{n}$. So, each node that joins the network after time $t_0 + \tau_1 + \tau_2 + \tau'_3$ gets informed within time $t_0 + \tau_1 + \tau_2 + \tau_3$, w.h.p.

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Appendix

A Mathematical tools

Theorem A.1 (Chernoff Bound, [11]). Let $X_1, \ldots, X_n$ be independent Poisson trials such that $\Pr(X_i = 1) = p_i$. Let $X = \sum_{i=1}^n X_i$, $\mu = \mathbb{E}[X]$ and suppose $\mu_L \leq \mu \leq \mu_H$. Then, for all $0 < \varepsilon \leq 1$ the following Chernoff bounds hold

\[
\begin{align*}
\Pr(X \geq (1 + \varepsilon)\mu_H) \leq e^{-\frac{\varepsilon^2}{3} \mu_H} \\
\Pr(X \leq (1 - \varepsilon)\mu_L) \leq e^{-\frac{\varepsilon^2}{3} \mu_L}.
\end{align*}
\]

Theorem A.2 (Method of bounded differences, [11]). Let $Y = (Y_1, \ldots, Y_m)$ be independent random variables, with $Y_j$ taking values in the set $A_j$. Suppose the real-valued function $f$ defined on $\prod_j A_j$ satisfies the Lipschitz condition with coefficients $\beta_j$, i.e.

\[|f(y) - f(y')| \leq \beta_j\]

whenever vectors $y, y'$ differs only in the $j$-th coordinate. Then, for any $M > 0$, it holds that

\[
\Pr(f(Y) \geq \mathbb{E}[f(Y)] + M) \leq e^{-\frac{2M^2}{\sum_j \beta_j^2}},
\]

and

\[
\Pr(f(Y) \leq \mathbb{E}[f(Y)] - M) \leq e^{-\frac{2M^2}{\sum_j \beta_j^2}}.
\]

Theorem A.3 (Kullback-Leibler divergence inequality). Let $p_m$ and $q_m$ be two discrete probability mass functions, with $m \in \{1, \ldots, L\}$. We have that

\[
\sum_{r=1}^L p_m \log_2 \left( \frac{p_m}{q_m} \right) \geq 0.
\]

B Static Random Graphs

Lemma B.1. The static random graph in which each node picks $d$ random neighbors is a $\Theta(1)$-expander w.h.p., for each $d \geq 3$.

Proof. We consider the static random graph $G = (N, E)$ and $S \subseteq N$ a subset of the nodes with $|S| = s$. Let $T \subseteq N \setminus S$ be an arbitrary set disjointed from $S$, with $|T| = 0.1s$. In this model we know that an edge starting from a node $v$ has destination $u$ with probability $\frac{1}{n-1}$. We know that the probability that the edges from $S$ are in $S \cup T$

\[
\binom{|S \cup T|}{n-1} d_{S}.
\]

So, the probability that the outer boundary of $S$ is at most in $T$ is

\[
\Pr(\partial_{out}(S) \subseteq T) \leq \left( \frac{1.1s}{n-1} \right)^d s.
\]

From an union bound over all the set $T$ disjointed with $S$ and with $|T| = 0.1s$, all the set $S$ with $s$ elements and all the possible sizes $s = 1, \ldots, n/2$ of $s$ we get

\[
\Pr(G \text{ is not an expander}) \leq \sum_{s=1}^{n/2} \binom{n}{s} (n - k) \left( \frac{1.1s}{n-1} \right)^d s.
\]

From standard calculus, it can be proved that, for $d \geq 3$, the equation above is upper bounded by $1/n^{d-2}$. This is obtained by bounding each binomial coefficient with the bound $\binom{n}{k} \leq \left( \frac{e}{k} \right)^k$ and by computing the derivative of the function $f(s)$ (representing each term of the sum), obtaining that each of these terms attained its maximum in $s = 1$ or in $s = n/2$.

\[\square\]
C Useful Tools for the Poisson Models

Definition C.1 (Counting process). The stochastic process \{X(t), t \geq 0\} is said to be a counting process if \(X(t)\) represents the total number of events which have occurred up to time \(t\).

Definition C.2. Let \{X(t), t \geq 0\} be a counting process. It is a Poisson process if

1. \(X(0) = 0\);
2. \(X(t)\) has independent increments;
3. The number of events in any interval of length \(t\) has a Poisson distribution with mean \(\lambda t\). That is, for all \(s, t \geq 0\)
   \[\Pr(X(t + s) - X(s) = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}, \quad n \geq 0.\]

Theorem C.3. Let \{X(t), t \geq 0\} be a Poisson process. Then, given \(X(t) = n\), the \(n\) arrival times \(S_1, \ldots, S_n\) have the same distribution as the order statistics corresponding to \(n\) independent random variables uniformly distributed in the interval \((0, t)\).

Theorem C.4 (Tail bound for the Poisson distribution). Let \(X\) have a Poisson distribution with mean \(\lambda\). Then, for each \(\varepsilon > 0\),
   \[\Pr(|X - \lambda| \geq \varepsilon) \leq 2e^{-\frac{\varepsilon^2}{2\lambda}}.\]

Theorem C.5 ([14]). Let \(I\) be a countable set and let \(T_k, k \in I\), be independent exponential random variables of parameter \(q_k\). Let \(0 < q = \sum_{k \in I} q_k \leq \infty\). Set \(T = \inf_k T_k\). Then this infimum is attained at a unique random value \(K\) of \(k\), with probability 1. Moreover, \(T\) and \(K\) are independent with \(T\) exponential of parameter \(q\) and \(\Pr(K = k) = q_k/q\).