Quantum Monte Carlo treatment of elastic exciton-exciton scattering

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We calculate cross sections for low energy elastic exciton-exciton scattering within the effective mass approximation. Unlike previous theoretical approaches, we give a complete, non-perturbative treatment of the four-particle scattering problem. Diffusion Monte Carlo is used to calculate the essentially exact energies of scattering states, from which phase shifts are determined. For the case of equal-mass electrons and holes, which is equivalent to positronium-positronium scattering, we find \( a_s = 2.1a_x \) for scattering of singlet-excitons and \( a_s = 1.5a_x \) for triplet-excitons, where \( a_x \) is the excitonic radius. The spin dependence of the cross sections arises from the spatial exchange symmetry of the scattering wavefunctions. A significant triplet-triplet to singlet-singlet scattering process is found, which is similar to reported effects in recent experiments and theory for excitons in quantum wells. We also show that the scattering length can change sign and diverge for some values of the mass ratio \( m_h/m_e \), an effect not seen in previous perturbative treatments.

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I. INTRODUCTION

Excitons in semiconductors have been the subject of many experimental and theoretical investigations of Bose condensation. Low-energy exciton-exciton interactions are characterized by the exciton-exciton scattering length, \( a_x \), which determines the thermodynamics of a low density gas and is crucial for modeling the thermalization time of a dilute exciton gas. Despite its importance, the exciton-exciton scattering length is an elusive quantity, being difficult to measure experimentally or to estimate theoretically.

As is well-known in atomic physics, scattering lengths can be extremely sensitive to the details of the interactions between particles. In particular, the existence of a weakly bound or nearly bound state causes the scattering length to become quite large. Therefore, \textit{a priori} one should suspect that exciton-exciton scattering may be a very material dependent property of semiconductors. Reliable theoretical predictions of exciton-exciton scattering lengths require both a very accurate Hamiltonian for the semiconductor, and an accurate solution to the (four-particle) scattering problem. In this paper we provide an essentially exact solution to exciton-exciton scattering for a commonly used single-band effective mass Hamiltonian. This solution allows us to study three important questions: (1) how sensitive is the scattering length to the mass ratio \( m_e/m_h \), (2) how does the scattering length depend on spin states (singlet or triplet) of the scattering excitons, and (3) to what degree can inter-exciton exchange of electrons or holes cause excitons to scatter into different spin states? These calculations also serve as a benchmark for the single band limit of more complicated scattering Hamiltonians.

One experimental method for measuring the exciton scattering cross section is to look at line width broadening of the recombination spectra in a gas of excitons. Collisions between excitons increase the line width, causing the line width to depend on the exciton-exciton scattering rate, \( n \sigma v \), where \( n \) is the density and \( v \) is a typical exciton velocity. Extracting cross sections from a line width requires that (1) the density and velocity distribution are known, and (2) elastic scattering is the fastest process. As discussed below, \( \text{Cu}_2\text{O} \) is a good material for comparison to the model studied in this work. Snoke \textit{et al.} have performed such experiments on \( \text{Cu}_2\text{O} \) and have found a line width broadening that suggests an upper bound of \( 4a_x \) on the scattering length. Although our simulations do not exactly model \( \text{Cu}_2\text{O} \), we will compare our results to this value.

II. THEORETICAL BACKGROUND

Theoretical approaches to this problem start with the effective mass approximation, in which the system under consideration consists of two electrons, labeled 1 and 2, and two holes, labeled \( a \) and \( b \). The Hamiltonian is

\[
H = -\lambda_1 \nabla_1^2 - \lambda_2 \nabla_2^2 - \lambda_a \nabla_a^2 - \lambda_b \nabla_b^2 - r_{a1}^{-1} - r_{b2}^{-1} - r_{a2}^{-1} - r_{b1}^{-1} + r_{12}^{-1} + r_{ab}^{-1},
\]

where \( \lambda = \hbar^2/2m \). The Hamiltonian has symmetry under exchange of electrons and exchange of holes, so eigenstates may be denoted by two exchange quantum num-
bers. The s-wave states are symmetric under exchange of excitons; a condition which is satisfied by states \( \phi^{++} \) and \( \phi^{--} \), where the \(+(-)\) signs refer to (anti)symmetry under exchange of electrons and holes, respectively. Although this Hamiltonian is a well-accepted model for exciton-exciton scattering, we should point out a few of its deficiencies. For small excitons, such as those in Cu2O, that have radii not much larger than the lattice spacing, non-parabolic terms in the kinetic energy and other corrections to the potential energy may be necessary. For many semiconductors, such as Si and Ge, the valence band is a mixture of three bands and cannot be described by a single parabolic band. In the case of Cu2O, the valence band is the parabolic spin-orbit split off band, and there are fewer complications. Interband exchange (virtual electron-hole recombination) is an important effect that has been neglected, and could be modeled by an additional spin-dependent potential term.

This Hamiltonian also describes a family of scattering processes for other particles, including hydrogen-hydrogen, positronium-positronium, and muonium-muonium scattering. The equal mass case is at an extreme (positronium scattering), where the Born-Oppenheimer approximation is the least applicable.

There have been several theoretical estimates of exciton-exciton scattering for bulk systems, and quantum wells as well as calculations on biexciton-biexciton scattering. Only the bulk, elastic scattering calculations are directly comparable to the results of this paper, but the techniques presented here could be generalized to the other theoretical problems. Also, the results presented here provide a benchmark for evaluating the approximations used in other theoretical treatments, and could lend insight into the reliability of the approximations in more complicated situations.

One standard theoretical approach is diagrammatic perturbation theory, as presented in the work of Keldysh and Kolosov and Haug and Hanamura. They estimate the exciton-exciton scattering matrix as arising from a single term, \( (k_1 + q, k_2 - q|H_{\text{int}}|k_1, k_2) \), where \( |k_1, k_2\rangle \) represents a state of two noninteracting excitons with momentum \( k_1 \) and \( k_2 \) and \( H_{\text{int}} \) is the inter-exciton Coulomb interaction. This method gives an estimate of \( a_x = \frac{\alpha}{m} a_x \) (independent of the mass ratio), where \( a_x = m_e^{-1} + m_h^{-1} \) is the exciton radius, but it is an uncontrolled approximation which may have limited validity in the low energy limit. One serious drawback of the method is that it does not include effects of the biexciton in the scattering. As we show later, biexciton vibrational states cause strong dependence of the scattering length on the mass ratio \( m_e/m_h \), which is not captured by the low order perturbation theory.

A second common approach was developed by Elkon-moss and Munchy and uses an effective exciton-exciton potential defined by \( V_{\text{eff}}(R) = \langle \phi_i(R)|H|\phi_i(R) \rangle \), where \( \phi_i(R) \) is the wavefunction for two free excitons a distance \( R \) apart. The effective potential \( V_{\text{eff}} \) arises from the Hartree term and is used in a two-particle central-field calculation. While an exciton-exciton scattering pseudopotential would be a very useful tool, this approximate form has some serious drawbacks. Among its deficiencies are a lack of correlation, no van der Waals attraction, a failure to reproduce biexciton states, and a vanishing interaction potential for \( m_e = m_h \). The cross sections calculated by this method are small and lack qualitative agreement with the results of the present work.

Some insight into exciton-exciton scattering can be gained by considering the bound states, biexcitons. Since the number of bound states, \( N_B \), enters in the phase shift at zero energy, \( \delta(0) = \pi N_B \), it is necessary that a good computation method for low energy scattering be able accurately calculate biexciton binding energies. For the mass ratios considered (and far beyond, including deuterium) the biexcitons cannot bind in the \( \phi^{--} \) states, so biexcitons in rotational \( s \) states always have \( \phi^{++} \) wavefunctions. Detailed theoretical descriptions of biexcitons can be found in Ref. [2]. The equal mass case was shown to have a bound biexciton by Hyllerass and Ore using a variational argument, and a better variational estimate of the binding energy was given by Brinkman, Rice and Bell who found \( E_B = 0.029 E_x \), where \( E_x = 0.5/(m_e^{-1} + m_h^{-1}) \) is the exciton binding energy. However, because of the importance of correlation energy, the latter variational treatment was missing half of the biexciton binding energy, as shown by diffusion Monte Carlo (DMC) calculations, which find \( E_B = 0.06404(4) E_x \). DMC is a quantum Monte Carlo (QMC) method that uses a random walk to project out the ground state wavefunction from a variational wavefunction, in order to stochastically sample the exact ground state energy. The success of DMC for calculating biexciton energies has been a motivation for its use in the present scattering calculations.

III. METHOD: QUANTUM MONTE CARLO CALCULATION OF SCATTERING

A R-Matrix approach and scattering boundary conditions

The R-Matrix approach to scattering is to examine the standing waves of the system. As shown by Carlson, Pandharipande, and Wiringa and Alhassid and Koonin by fixing nodes in the standing waves the scattering problem may be cast as a stationary state problem suitable for QMC methods. For an elastic scattering process, we label the distance between the products by \( R \), and the reduced mass of the products by \( \mu_r \). In exciton-exciton scattering there is a subtlety in the definition of \( R \) due to inter-exciton exchange, which we will address below in our discussion of the exciton-exciton scattering wavefunctions. Nonetheless, for large separation \( R \), the relative motion of the products is free-particle like, so the
many-body wavefunction depends on \( R \) as
\[
\phi \propto \sin[kR - \frac{1}{2}l\pi + \delta_l(k)],
\]
where \( l \) is the relative angular momentum, \( k \) is the scattering momentum, and \( \delta_l(k) \) is the phase shift. If we constrain the wavefunction to have a node at a large excited state separation \( R_n \), we find a discrete energy spectrum \( E_\alpha(R_n) \), which may be computed by ground state or excited state methods, such as DMC. Each choice of \( R_n \) gives a spectrum of states \( \phi_\alpha, \alpha = 1, 2, \ldots \), with energies \( E_\alpha \) that determine values of \( \delta_l(k) \),
\[
\delta_l(k_\alpha) = -k_\alpha R_n + \frac{1}{2}l\pi + \alpha\pi,
\]
where \( k_\alpha = \sqrt{2m_\alpha E_\alpha} \). The scattering matrix elements are determined by the phase shifts,
\[
S_l(k) = \frac{1}{2ik} \exp[2i\delta_l(k)] - 1.
\]
Carlson has also proposed fixing the logarithmic derivative of the wavefunction at the boundary instead of setting wavefunction to zero,\textsuperscript{[20]}
\[
\frac{\hat{S} \cdot \nabla_R \phi}{\phi} \bigg|_{R=R_b} = \beta,
\]
were \( \hat{S} \) is the normal to the boundary surface, at a fixed radius \( R_b \), and \( \beta \) parameters the boundary condition. This formulation has the advantages of separating the choice of simulation size \( R_b \) (subject to \( R_b \) lying in the asymptotic region), from the sampling of energy, which is handled by varying \( \beta \), and is particularly well suited for finding the scattering length. The application to VMC calculations is straightforward, but preserving the boundary condition in DMC calculations requires a method of images.\textsuperscript{[21]} The results presented here do not use the logarithmic-derivative boundary condition.

B Calculation of excited states

The use of excited states \( \phi_\alpha(\alpha > 1) \) is necessary when there is a bound state and, more generally, when the scattering state being studied has its first node before the asymptotic region is reached. We use a QMC method to calculate excited states developed by Ceperley and Bertini,\textsuperscript{[22]} to adapt VMC and DMC methods for a Hilbert space of several low energy wavefunctions. A set of \( m \) trial wavefunctions is chosen, \( f_\alpha, \alpha = 1, \ldots, m \). The generalized eigenvalue equation to be solved is\textsuperscript{[23]}
\[
\sum_{\beta=1}^{m} [H_{\alpha\beta}(t) - \Lambda_k(t) N_{\alpha\beta}] d_{\beta}(t) = 0,
\]
where \( d_k(t) \) is the \( k \)th eigenvector with eigenvalue \( \Lambda_k(t) \) and the matrices \( N \) and \( H \) are the overlap and Hamiltonian matrices in our trial basis, given by
\[
N_{\alpha\beta}(t) = \int dR_1dR_2 f_\alpha^*(R_2) e^{-tH} f_\beta(R_1),
\]
\[
H_{\alpha\beta}(t) = \int dR_1dR_2 H f_\alpha^*(R_2) e^{-tH} f_\beta(R_1).
\]
The parameter \( t \) is the projection time. The eigenvalues \( \Lambda_k(t) \) are energy eigenvalues \( E_k \) within the Hilbert space spanned by the projected trial functions \( \{e^{-tH/2} f_\alpha\} \), and approach the exact energy eigenvalues in the limit of large \( t \).

The matrices \( N \) and \( H \) are sampled with random walks, using a guiding function \( \psi \) which must be positive everywhere. The guiding function must have significant overlap with all basis functions, and should be optimized to decrease the variance of the sampled matrices. At each step \( i+1 \) in the random walk, the coordinates \( R_{i+1} \) of the particles are updated using
\[
R_{i+1} = R_i + \tau \lambda \psi^{-1} \nabla \psi(R_i) + (2\tau \lambda)^{1/2} \chi_i,
\]
where \( \chi_i \) is a normally distributed random variate with zero mean and unit variance and \( \tau \) is the time step. In the limit of small \( \tau \), Eq. (9) describes a process for sampling \( \psi^2(R) \).

The matrix element of \( e^{-tH} \) is estimated by integrating the local energy of the guiding function, \( E_{L\psi}(R) = \psi^{-1}(R) H \psi(R) \), along the random walk,
\[
W_{n,n+k} = e^{-\tau \sum_{j=n}^{n+k-1} \psi^2(R_j) + \psi^2(R_{j+1})}.
\]
The estimators for matrices \( N \) and \( H \) are
\[
n_{\alpha\beta}(k\tau) = \frac{1}{p} \sum_{i=1}^{p} F_\alpha^*(R_i) W_{i,i+k} F_\beta(R_{i+k}),
\]
\[
h_{\alpha\beta}(k\tau) = \frac{1}{p} \sum_{i=1}^{p} F_\alpha^*(R_i) W_{i,i+k} F_\beta(R_{i+k}) E_{L\beta}(R_{i+k})
\]
where \( F_\alpha = f_\alpha/\psi \) and \( E_{L\beta} = f_\beta^{-1}(R) H f_\beta(R) \) are the local energies of the trial basis states.

C Form of exciton-exciton scattering wavefunctions

We now discuss the form for the exciton scattering functions \( f_\alpha^+ \) and \( f_\alpha^- \). As mentioned before, interexciton exchange of particles complicates the definition of exciton-exciton separation. There are two configurations for well-separated excitons, as shown in Fig. 2. Configuration I has the electrons and hole paired as \( 1a, 2b \); and Configuration II as \( 1b, 2a \). We choose wavefunctions \( f_{ib} \)
and $f^\alpha_{II}$ to represent these states,

\[
\begin{align*}
    f^\alpha_I &= e^{-\gamma r_1} e^{-\gamma r_2} U_\alpha(r_{3b-2b}) \\
    f^\alpha_{II} &= e^{-\gamma r_1} e^{-\gamma r_2} U_\alpha(r_{1b-2a})
\end{align*}
\]

where $\gamma, c_f, d_f, c_g, d_g, c_h, d_h$, and parameters in the function $U_\alpha(r)$ are variational. These wavefunctions represent two excitons in a relative s-wave state. Since these are not eigenstates of the exchange operator for electrons $\hat{P}_e$ or holes $\hat{P}_h$, we take linear combinations of the two for our trial wavefunctions, $f^\alpha_{II} = f^\alpha_I + f^\alpha_{II}$ and $f^\alpha_{II} = f^\alpha_I - f^\alpha_{II}$. For large separation of excitons, the exponential factors prohibit both configurations from simultaneously contributing to the wavefunction. Thus, a node can be approximated in the scattering wavefunction by simply requiring that $U_\alpha(r)$ be zero for all $r > R_\alpha$. The error introduced by this approximation is of order $\exp(-2\alpha R_\alpha)$, and is another limit on the use of small values for $R_\alpha$. Since we only do calculations for low energy scattering, large $R_\alpha$, the lack of a well-defined exciton-exciton separation distance for short distances does not matter.

This method for calculating scattering properties is very sensitive to the energy spectra $\{E_\alpha(R_\alpha)\}$. To get accurate energies, we do not try to construct and optimize elaborate variational wavefunctions, but rather use DMC to project the energy from trial wavefunctions of the form given in Eq. (13). The coefficients $\gamma, c_f, c_g$, and $c_h$ are chosen to obey the cusp conditions on the wavefunction for small particle separations. The s-wave envelope functions $U_\alpha(r)$ are taken as solutions to an empirical exciton-exciton scattering potentials,

\[
V(r) = \begin{cases} 
-V_0 \left(1 - \frac{3r^2}{4d^2}\right) ; & r \leq d, \\
-V_0 & ; r > d,
\end{cases}
\]

where $V_0$ and $d$ have been self-consistently fit to approximate the energy spectrum of the four particle scattering states. We take the guiding function $\psi$ to have the same form as the $f^{++}$ wavefunctions with $U_\psi(r) = \{c e^{-r^2/8} + \sum \alpha d_\alpha U_\alpha^2(r)\}^{1/2}$. The parameters $\{c, d_\alpha\}$ are chosen to bias sampling towards the collision: $\{c = 0, d_1 = 2, d_2 = d_3 = d_4 = 1\}$ for $\phi^{++}$ states and $\{c = 2, d_\alpha = 1\}$ for $\phi^{--}$ states. To check for convergence of the energies in DMC, we plot the energy difference, $E_{DMC} - E_{VMC}$, as a function of projection time in Fig. 2. We see convergence after a projection time of $3 E^{-1}$.

We thus find two energy spectra for each value of $R_\alpha$, as shown in Figs. 3(a) and 3(b). The spectra for the symmetric states $\phi^{++}$, show a clearly bound bie exciton state, as shown in Fig. 3(a). The antisymmetric states $\phi^{--}$, as shown in Fig. 3(b), have no bound state. The binding energy of the bie exciton is $E_B = -0.0642(3) E_x$, in agreement with other ground state calculations, and is insensitive to the position of the node $R_N$ because it is localized. In contrast, the delocalized scattering states are quite sensitive to $R_N$, and their dependence on $R_N$ is a measure of the elastic scattering matrix elements.

**D Calculation of phase shifts**

Using Eq. (13), we determine scattering phase shifts, $\delta^{++}(k)$ and $\delta^{--}(k)$, which are shown in Fig. 4 for the equal-mass case. The $k = 0$ limits show that there
FIG. 3: Energy spectra as a function of nodal position $R_N$, for (a) symmetric $\phi^{++}$ states, and (b) antisymmetric $\phi^{--}$ states, with $m_e = m_h$. The lowest energy curve in (a) is the biexciton with binding energy $E_B = 0.0642(3)$. These functions $E(R_N)$ determine the phase shifts $\delta^{++}$ and $\delta^{--}$, by the relationship in Eq. (8).

is one bound symmetric state $\phi^{++}$ and no bound antisymmetric states $\phi^{--}$. The slope at $k = 0$ is related to the scattering length $a_s$ by $\delta'(k) = -a_s$. From a cubic polynomial fit to the data, with coefficients given in Table II, we find $a = 4.57(2)a_x$ for the symmetric state, and $a = 1.51(3)$ for the antisymmetric state. These model values are consistent with the measured upper bound $4a_x$ found in Cu$_2$O.

TABLE I: Coefficients for polynomial fit to the low energy part of the phase shift functions for the case $m_e = m_h$.

| $\delta^{++}$ | $\delta^{--}$ |
|-------------|-------------|
| $c_0$ | $c_1$ | $c_2$ | $c_3$ |
| $-5.547(21)$ | $2.995(70)$ | $-0.829(54)$ |
| $0$ | $-1.512(29)$ | $-0.138(90)$ | $0.216(64)$ |

FIG. 4: Phase shifts $\delta(k)$ for the two s-wave scattering states, for $m_e = m_h$, calculate using Eq. (8) and the data from Fig. 3

$E$ Extracting spin dependence from spatial symmetries

Single excitons can be in either a spin triplet or spin singlet energy eigenstates. (These states are degenerate in our model, but are split by electron-hole exchange interaction in more realistic models.) During collisions, however, the spin of individual excitons is not a conserved quantity — the excitons can change their internal spin states during collision. To relate our scattering states $\phi^{++}$ and $\phi^{--}$ to excitonic spin states, we invoke the antisymmetry of the total (spatial plus spin) fermionic wavefunction. Fermionic antisymmetry implies that the states $\phi^{++}$ and $\phi^{--}$ have spin eigenstates $|s00\rangle_{eh}$ and $|s11\rangle_{eh}$, where the notation $|seeh\rangle_{eh}$ stands for total spin $s$, total electron spin $s_e$ and total hole spin $s_h$. We denote the experimentally relevant "excitonic spin basis" as $|s1s2\rangle_{ex}$, where the total spin is $s$ and the spins on the scattering excitons are $s_1$ and $s_2$, which take the values 0 for singlet excitons and 1 for triplet excitons. Table II lists matrix elements for the change of basis $|ss_eeh\rangle_{eh} \rightarrow |s1s2\rangle_{ex}$. The states $|00\rangle_{eh}$ and $|s11\rangle_{eh}$ act as channels for s-wave exciton-exciton scattering. The matrix elements for scattering through these channels are:

$$s^{++}(k) = _{eh}(000; k|s|000; k)_{eh} = \frac{e^{2ik^{++}(k)} - 1}{2ik}, \quad (15)$$

$$s^{--}(k) = _{eh}(S11; k|s|S11; k)_{eh} = \frac{e^{2ik^{--}(k)} - 1}{2ik}, \quad (16)$$

The collision of two excitons with well defined initial and final spin states can be determined by decomposing the scattering events into the two channels, $s^{++}(k)$ and $s^{--}(k)$. Using the change of basis matrix (Table II), we find $s(k) = \alpha^+_s s^{++}(k) + \alpha^- s^{--}(k)$, where the coefficients $\alpha^+_s$ and $\alpha^-_s$ for all non-zero s-wave scattering processes are given in Table II. The s-wave scattering cross sections are given by $\sigma(k) = 8\pi |s(k)|^2$, where there is a factor of two enhancement due to the identical particle statistics. The spin-dependent cross sections take the form,

$$\sigma(k) = \frac{8\pi}{k^2} [c^{++} \sin^2 \delta^{++} + c^{--} \sin^2 \delta^{--} - c^{++} \sin^2 (\delta^{++} - \delta^{--})], \quad (17)$$
TABLE III: All non-zero s-wave scattering process. Initial and final states are denoted by \((s_1 s_2)_{ex}\), where \(s\) is the total spin and \(s_1\) and \(s_2\) are the individual exciton spins. The coefficients \(\alpha_+, \alpha_-\) are for computing the scattering matrix elements, and the coefficients \(c_{++}, c_{--}, c_{+-}\) are for computing cross sections. The last column list the ratio of the scattering length \(a_s\) to the exciton radius \(a_e\) for the case \(m_e = m_h\).

| Process | \(\alpha_+\) | \(\alpha_-\) | \(c_{++}\) | \(c_{--}\) | \(c_{+-}\) | \(a_s/a_e\) |
|---------|-------------|-------------|-------------|-------------|-------------|-------------|
| \((000)_{ex} \to (000)_{ex}\) | \(\frac{3}{2}\) | \(\frac{3}{2}\) | \(\frac{7}{4}\) | \(-\frac{1}{4}\) | 2.128(27) |
| \((011)_{ex} \to (011)_{ex}\) | \(-\frac{3}{2}\) | \(\frac{3}{2}\) | \(\frac{7}{4}\) | \(-\frac{1}{4}\) | 3.759(22) |
| \((011)_{ex} \to (000)_{ex}\) | \(-\frac{3}{2}\) | \(\frac{3}{2}\) | 0 | 0 | -1.411(17) |
| \((000)_{ex} \to (011)_{ex}\) | \(-\frac{3}{2}\) | \(\frac{3}{2}\) | 0 | 0 | -1.411(17) |
| \((110)_{ex} \to (110)_{ex}\) | 0 | 0 | 0 | 0 | 0.706(14) |
| \((211)_{ex} \to (211)_{ex}\) | 0 | 0 | 0 | 0 | 1.512(29) |

where \(c_{++} = \alpha_+ (\alpha_+ + \alpha_-)\), \(c_{--} = \alpha_- (\alpha_+ + \alpha_-)\), and \(c_{+-} = \alpha_+ \alpha_-\) depend on the initial and final spin states and are tabulated in Table III. The scattering lengths \(a_s\) are given by \(a_s = -\alpha_+ \tilde{\delta}_+^0(0) - \alpha_- \tilde{\delta}_-^0(0)\), where the derivatives of the phase shifts are determined from the linear coefficients in Table III.

IV. RESULTS

A Spin dependence of scattering cross sections for the equal mass case

The calculated spin-dependent scattering lengths for the case \(m_e = m_h\) are presented in the last column of Table III. These are the low-energy limit the cross-sections shown in Fig. 5. In Fig. 5 we have plotted the s-wave scattering cross sections versus scattering momentum for the case \(m_e = m_h\) and all non-zero spin configurations. Fig. 5(a) shows scattering of two singlet-excitons. Scattering of two triplet-excitons is shown in Fig. 5(b), where the solid line represents the spin aligned \(S = 2\) state, and the dashed line represents the \(S = 0\) state. The \(S = 0\) state scatters particularly strongly because it has a large contribution from the \(s^{++}\) channel, which is enhanced by the weakly bound bie exciton. Triplet-excitons in a relative \(S = 1\) state are spatially antisymmetric and thus have no s-wave scattering. We show s-wave scattering of triplet-excitons from singlet-excitons in Fig. 5(c). This state has two distinguishable excitons, and can scatter by both s-wave and p-wave processes. As can be seen in Table III, the only contribution to the cross section is from the weaker \(s^{--}\) channel. The coefficient for s-wave scattering is particularly small because only half the scattering process is symmetric (s-wave), and there is an additional factor of one-half which cancels the identical particle factor.

There is also an triplet- to singlet-exciton conversion cross section given in Fig. 5(d). Although this is an inelastic process in experimental situations, it conserves energy according to our model Hamiltonian because we do not have an explicit interband spin coupling. The conversion of two triplet-excitons to two singlet-excitons can be understood as an inter-exciton exchange of a pair of electrons (or holes). Since the spins of the individual excitons do not correspond to symmetries of the Hamiltonian, they need not remain constant during scattering. This conversion process is a physical consequence of the two inequivalent scattering channels \(s^{++}\) and \(s^{--}\). This effect has been reported in experimental and theoretical work on exciton scattering in quantum wells.

FIG. 5: All non-zero s-wave cross sections, Eq. (17), for exciton-exciton scattering with \(m_e = m_h\), for the processes: (a) singlet-singlet \(\leftrightarrow\) singlet-singlet, (b) triplet-triplet \(\leftrightarrow\) triplet-triplet for total spin \(s = 0\) (dashed line) and \(s = 2\) (solid line), (c) triplet-singlet \(\leftrightarrow\) triplet-singlet with \(s = 1\), and (d) triplet-triplet \(\leftrightarrow\) singlet-singlet and singlet-singlet \(\leftrightarrow\) triplet-triplet, both with \(s = 1\).
FIG. 6: The scattering lengths $a^{++}_s$ and $a^{--}_s$ as a function of the electron-hole mass ratio. The divergence in $a^{++}_s$ near $m_h/m_e = 15$ is due to the appearance of second bound biexciton state. Solid lines are a guide to the eye.

**B Mass dependence of scattering lengths**

The dependence of the cross sections on mass may be numerically studied by our methods. In Fig. 6 we show our calculated $a^{++}_s$ and $a^{--}_s$ as a function of mass ratio, $m_h/m_e$. We find that the scattering length is remarkably insensitive to the mass ratio for $m_e/m_h < 10$ (corresponding to a wide range of semiconductors), but then diverges near $m_e/m_h = 15$. This feature is lost in previously published theoretical treatments of exciton-exciton scattering. The divergence in $a^{++}_s$ near $m_h/m_e = 15$ is due to the acquisition of a biexciton vibrational state. For $m_e = m_h$ the biexciton has no bound excited states, but a H$_2$ molecule has 15 bound vibrational states. Our calculations have shown the first of these appears near $m_h/m_e = 15$, with dramatic effects on the scattering length. The $a^{--}_s$ curve is relatively featureless because there are are no bound antisymmetric states in this range. We interpret the upward drift of $a^{--}_s$ for larger mass ratios as a systematic error due to difficulties in projecting states in excitonic systems with very different electron and hole masses. The heavy particle determines the projection time ($\sim m^{-1} R^{-2}$) while the light particle determines the diffusion time step $\tau$. The difficulty in handling large mass ratios make the method (as presented here) complementary to calculations that use the Born-Oppenheimer approximation.

It is important to realize that similar relationships must exist between the scattering length and other material parameters, such as the Luttinger-Kohn parameters describing realistic hole states, external strain, and spin-orbit coupling, to name a few. Theoretical studies of such effects will need similar high-accuracy scattering calculations, but applied to more accurate Hamiltonians, and are an area for future research.

**V. CONCLUSION**

To summarize, we have shown that there are several significant elastic scattering processes for excitons, and have given numerically exact values for a widely used theoretical model. We find strong triplet-triplet and singlet-singlet scattering, with weaker triplet-singlet scattering and triplet-triplet to singlet-singlet conversion processes. Scattering is relatively insensitive to the mass ratio for $m_e/m_h \lesssim 10$, but becomes very sensitive and actually diverges near $m_e/m_h \approx 15$. DMC has been found be a good tool for this four-particle excited state calculation, since the detection of weakly bound states requires very accurate evaluation of the correlation energy.

This computational approach should be extended in many ways. The extension to higher angular momentum states would give important corrections at higher scattering energies. Application to biexciton-biexciton scattering are possible, but would be a bit more difficult because the scattering wavefunction would then have to describe eight interacting particles. Most importantly, the method should be adapted to better Hamiltonians so that the sensitivity of the scattering length to material properties for a wide range of materials can be studied. This approach should be quite useful for quantum well problems, which have similarities to the bulk problem study here, but have many more experimental parameters that can affect exciton-exciton interaction.

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**REFERENCES**

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1 D. W. Snoke, J. P. Wolfe, and A. Mysyrowicz, Phys. Rev. B 41, 11171 (1990).

2 D. W. Snoke, D. Braun, and M. Cardona, Phys. Rev. B 44, 2991 (1991).

3 H. Haug, Z. Physik B 24, 351 (1976).

4 S. G. Elkomoss and G. Munschy, J. Phys. Chem. Solids 42, 1 (1981).

5 S. G. Elkomoss and G. Munschy, J. Phys. Chem. Solids 45, 345 (1981).
For example, the location of the scattering length divergence at $m_e/m_h \approx 15$, reported later in this article, is extremely sensitive to an accurate treatment of correlation energy, since it relies on the appearance of a second, weakly bound biexciton state at that mass ratio.

A. Quattropani and J. J. Forney, Il Nuovo Cimento 39, 569 (1977).

E. A. Hyllerass and A. Ore, Phys. Rev. 71, 43 (1947).

W. F. Brinkman, T. M. Rice, and B. Bell, Phys. Rev. B 8, 1570 (1973).

M. A. Lee, P. Vashishta, and R. K. Kalia, Phys. Rev. Lett. 51, 2422 (1983).

D. Bressanini, M. Mella, and G. Morosi, Phys. Rev. A 55, 200 (1997).

J. Carlson, V. R. Pandharipande, and R. B. Wiringa, Nuclear Physics A424, 47 (1984).

Y. Alhassid and S. E. Koonin, Ann. Phys. 155, 108 (1984).

J. Carlson and R. Shiavilla, Rev. Mod. Phys. 70, 743 (1998).

J. Carlson, private communication.

D. M. Ceperley and B. Bernu, J. Chem. Phys. 89, 6316 (1988).

T. Amand, D. Robart, X. Marie, M. Brousseau, P. Le Jeune and J. Barrau, Phys. Rev. B 55, 9880 (1997).