Verifying Programs Under Custom Application-Specific Execution Models

BRETT BOSTON, Massachusetts Institute of Technology
ZOE GONG, Massachusetts Institute of Technology
MICHAEL CARBIN, Massachusetts Institute of Technology

Researchers have recently designed a number of application-specific fault tolerance mechanisms that enable applications to either be naturally resilient to errors or include additional detection and correction steps that can bring the overall execution of an application back into an envelope for which an acceptable execution is eventually guaranteed. A major challenge to building an application that leverages these mechanisms, however, is to verify that the implementation satisfies the basic invariants that these mechanisms require—given a model of how faults may manifest during the application’s execution.

To this end we present Leto, an SMT based automatic verification system that enables developers to verify their applications with respect to a first-class execution model specification. Namely, Leto enables software and platform developers to programmatically specify the execution semantics of the underlying hardware system as well as verify assertions about the behavior of the application’s resulting execution. In this paper, we present the Leto programming language and its corresponding verification system. We also demonstrate Leto on several applications that leverage application-specific fault tolerance mechanisms.

1 INTRODUCTION

Due to the aggressive scaling of technology sizes in modern computer processor fabrication, modern processors have become more vulnerable to errors that result from natural variations in processor manufacturing, natural variations in transistor reliability as processors physically age over time, and natural variations in these processors’ operating environments (e.g., temperature variation and cosmic/environmental radiation) [Amarasinghe et al. 2009; Borkar 2005; Johnston 2000; Kurd et al. 2010; Mitra et al. 2005, 2006; Mukherjee et al. 2003; Shivakumar et al. 2002; Yim 2014].

Challenges. These systems encounter faults—anomalies in the underlying physical device—that produce errors—unanticipated or incorrect values that are visible to the program. Simple fault models include bit flips in the output of arithmetic, logical, and memory operations. These faults can be transient—occurring nondeterministically with the device eventually returning to correct behavior—or permanent—with the device never returning to correct behavior. A key challenge for building applications for these platforms is that reasoning about the reliability of these programs requires reasoning about the operation of the underlying execution model and its impact on the application’s behavior.

1.1 Application-Specific Fault Tolerance

In response to this increased error vulnerability, researchers have begun to expand on historical results for algorithm-based fault tolerance [Bronevetsky and de Supinski 2008; Hoemmen and Heroux 2011; Huang and Abraham 1984; Oboril et al. 2011; Roy-Chowdhury and Banerjee 1994, 1996; Sao et al. 2016; Sao and Vuduc 2013; Shantharam et al. 2012], alternatively application-specific fault tolerance, to identify new opportunities for low-overhead mechanisms that can steer an application’s execution to produce acceptable results: results that are within some tolerance of the result expected from a fully reliable execution. For example, application-specific fault tolerance techniques for linear algebra produce lightweight checksums that developers can use to validate that the computation produced the correct results. For some applications, these checksums are exact, enabling the exact error detection capabilities of dual-modular redundancy but with lower overhead. However, for other applications, these checksums either are not known to exist or, at best, compromise on their error coverage.
Other techniques include selective \( n \)-modular redundancy in which a developer either manually or with the support of a dynamic fault-injection tool identifies instructions or regions of code that do not need to be protected for the application to produce an acceptable result—as determined by an empirical evaluation [Carbin and Rinard 2010; Thomas and Pattabiraman 2016; Venkatagiri et al. 2016; Vishal Chandra Sharma 2016]. Another class of techniques is fault-tolerant algorithms that through the addition of algorithm-specific checking and correction code are tolerant to faults [Du et al. 2012; Hoemmen and Heroux 2011; Sao et al. 2016; Sao and Vuduc 2013].

**Challenges.** A major barrier to implementing these techniques is that their results either rely on empirical guarantees or—for self-stabilizing algorithms—hinge on the assumption that the fault model of the underlying computing substrate matches the assumptions of the algorithmic formalization.

### 1.2 Verified Application-Specific Fault Tolerance

To address these challenges we present Leto, an SMT based, automatic verification system that supports reasoning about unreliably executed programs. Leto enables a developer to verify their application-specific fault tolerance mechanism by providing tools to 1) programmatically specify the behavior of the computing substrate’s fault model and 2) verify *relational* assertions that relate the behavior of the unreliably executed program to that of the reliable execution. Specifically, Leto automatically weaves the behavior of the underlying hardware system—as given by a specification—into the execution of the main program. In addition, Leto’s program logic enables a developer to specify relational assertions that, for example, constrain the difference in results of the unreliable execution of the program from that of its reliable execution.

#### 1.2.1 First-Class Execution Models

Leto permits developers to programmatically specify a stateful execution model. Figure 1 presents a Leto specification for a first-class single-event upset (SEU) execution model with specifications for multiplication. An SEU model is a common fault model that application developers in the area of fault tolerance use to model the execution behavior of an application such that they can provide a variety of fault tolerance mechanisms [Chen et al. 2008; Yim et al. 2010]. The underlying assumption is that faults in a system (e.g., those due to cosmic radiation) occur with a probability such that at most one fault will occur during the execution of an application.

```plaintext
1 bool upset = false;
2 operator *(real x1, real x2)
3   ensures (result == x1 * x2);
4
5 operator *(real x1, real x2)
6   when (!upset)
7   modifies (upset)
8   ensures (upset);
```

Fig. 1. Unbounded SEU Execution Model

The model exports two versions of the multiplication operator. Line 3 specifies the standard reliable implementation of multiplication. The model denotes this fact with its ensures clause which asserts what must be true of the model’s state and outputs after execution of the operation. This operation specifically constrains the value of `result`—which represents the result of the operation—to equal \( x1 \times x2 \) where \( * \) has standard multiplication semantics.

Line 6 specifies an unreliable implementation of the multiplication operator. This implementation does not place any constraints on `result` and therefore permits unbounded errors.

**Stateful.** Because the model is an SEU model it must track whether a fault has occurred so that the model exposes at most one fault to the application. This model is therefore stateful and to achieve this semantics, the model includes a boolean valued state variable—`upset` (Line 1)—that records whether or not a fault has already occurred during the execution of the program. The model additionally predicates the availability of its unreliable operations by a guard. Specifically,
an operation’s guard is the optionally-specified boolean expression that occurs after the when keyword. The when clause for the unreliable version requires !upset indicating that the unreliable version is only enabled if a fault has yet to occur. Leto models the dynamic execution of the application such that the execution exposes only enabled operation implementations at a given program point.

1.2.2 Acceptability Properties. Leto enables developers to automatically verify the basic goal of an application-specific fault tolerance mechanism: ensure that the resulting application satisfies its acceptability properties [Carbin et al. 2012], such as its safety and accuracy.

Safety Properties: standard properties of the execution of the application that must be true of a single execution of the application. Such properties include, for example, memory safety and the assertion that the application returns results that are within some range. For example, a computation of a distance metric must—regardless of the accuracy of its results—return a value that is non-negative. In Leto, a developer specifies safety properties with the standard assert statements typically seen in verification systems.

Accuracy Properties: properties of the unreliable or relaxed execution of the application that relate its behavior and results to that of a reliably executed version. Accuracy properties are relational in that they relate values of the state of the program between its two semantic interpretations. For example, the assertion abs(x<o> - x<r>) < epsilon in Leto specifies that the difference in value of x between the program’s original, reliable execution (denoted by x<o>) and relaxed execution (denoted by x<r>) is at most epsilon.

Execution-Specific Properties. Given a first-class execution model, Leto also enables developers to refer to the state of the execution. For example, in many self-stabilizing iterative algorithms, the proof of convergence for the algorithm in the presence of faults requires reasoning about three cases: 1) the portion of the execution in which no fault has occurred, 2) the iteration on which a fault occurs (assuming an SEU model), and 3) the portion of the execution after the fault. Leto enables developers to verify such properties by exposing the state of the fault model into the program logic.

Asymmetric Relational Verification. Leto provides and implements an Asymmetric Relational Hoare Logic [Carbin et al. 2012] as its core program logic. An Asymmetric Relational Hoare Logic is a variant of the standard Hoare Logic that natively refers to the values of variables between two executions of the program. Leto’s use of a relational program logic serves two goals: 1) it gives a semantics to accuracy properties and 2) it enables tractable verification of safety properties. For example, proving the memory safety of an application outright can be challenging for many applications. However, application-specific fault tolerance mechanisms can typically be designed and deployed such that it is possible to verify that for any given array access or memory access, errors in the application do not interfere with the accessed address. Such properties are typically easier to verify for a protected application than verifying the safety of the memory access outright. Leto therefore enables developers to tractably verify a strong relative safety guarantee: if the original application satisfies all of the specified safety properties, then relaxed executions of the application with its deployed application-specific fault tolerance mechanisms also satisfy these safety properties.

1.3 Contributions
This paper presents the following contributions:
First-Class Execution Models. We present a language for specifying execution models that provides stateful, input-dependent selection of each operation’s implementation. We demonstrate numerous sample execution models.

Execution Model Refinement. We present a language construct for refining models into new models, permitting developers to create submodels that satisfy the specification of their supermodels. We describe the process by which refinement creates an interface between hardware vendors and software developers.

Programming Language and Semantics. We present language constructs that enable the developers to specify assertions that refer to the state of the execution model. These constructs enable a developer to, for example, specify the precise properties that self-stabilizing applications require to verify high-level convergence properties.

Program Logic and Verification Algorithm. Leto’s program logic enables developers to lower the overhead of verifying a standard safety property by enabling techniques that demonstrate the non-interference between the application’s faults and the validity of a property. Leto’s verification algorithm additionally automates this process through the inclusion of loop invariant inference.

Case Studies. We evaluate Leto on several self-correcting algorithms (Jacobi, Self-stabilizing Conjugate Gradient, Self-stabilizing Steepest Descent, and Self-correcting Connected Components) and demonstrate that it is possible to verify the key invariants required to prove that these algorithms’ self-stability guarantees hold for their implementations. We consider execution models that capture a range of substrates, including emerging hardware systems that bound potential error, emerging hardware security vulnerabilities (Rowhammer [Kim et al. 2014]), as well as standard fault modelling assumptions that expose unbounded errors to the application.

Leto’s contributions enable developers to specify and verify the rich properties seen in applications with application-specific fault tolerance mechanisms.

2 EXAMPLE EXECUTION MODELS

```cpp
vector<real> product(uint N,
   vector<real> x(N),
   vector<real> y(N))
{
   vector<real> result(N);
   for (uint i = 0; i < N; ++i)
   { result[i] = x[i] * y[i];
   }
   return result;
}
```

Figure 2 presents an implementation of a vector-vector product in Leto. It defines a function product that takes a size parameter N, a vector x of size N, a vector y of size N, and returns the element-wise product of x and y.

In Section 2.1 through Section 2.3 we introduce several execution models. We highlight the key features of Leto that these models use and demonstrate them on variants of the vector product function.

We use this specific example because vector products are a component of many numerical algorithms and exist in three of our four benchmarks as an intermediate calculation in matrix-vector products.

In Section 2.4 we present a technique in Leto for refining loosely defined models into stricter models. This technique ensures that these refined submodels satisfy the specifications of their respective supermodels.

2.1 Additive Single-Event Upset Execution Model

The additive single-event upset execution model in Figure 3 exports two versions of the multiplication operator. Line 4 specifies the standard reliable implementation of multiplication. The model
denotes this fact with its ensures clause which asserts what must be true of the model’s state and outputs after execution of the operation. This operation specifically constrains the value of result—which represents the result of the operation—to equal $x_1 \times x_2$.

Bounded Error. Leto enables developers to place bounds on the results of binary operations by using inequalities over the result variable. In addition, Line 7 specifies an unreliable implementation for multiplication. The semantics of this unreliable operator guarantees that even in the presence of an error, the result is within $\epsilon$ of the original result where $\epsilon$ is a programmer specified constant in the model on Line 1.

Fig. 3. Additive SEU Execution Model

2.1.1 Vector Product Under Additive SEU. Figure 4 presents a vector product implementation annotated to verify under the additive SEU execution model. All of the relaxation in this implementation occurs in the loop (Line 9) where the execution model may corrupt the value of $\text{result}[i]$. On Line 15 the implementation uses a relational assertion ($\text{assert}_r$) to verify that $\text{model}.\epsilon$ bounds the impact of these errors (Line 2). The notation $\text{model}.\epsilon$ is a first-class reference to the state of the execution model at that point in the program.

Verifying this vector product implementation relies on Leto’s specification capabilities to establish bounds on the error in the product. Leto’s two features that diverge from traditional programming languages are that developers can specify that some operations in the program may execute with an alternative semantics and—as consequence—write relational assertions that relate values between the standard, original execution and the alternative relaxed execution of the program.

Relaxed Execution. Leto exports custom operations by enabling developers to specify that an operation may execute according to the execution model specification (versus a standard implementation) by appending a dot to the operation as in the operation ‘$\cdot$’ (Line 12).

Bounded Error. Many self healing iterative algorithms for solving linear systems of equations experience increases in solve times proportional to the magnitude of errors they experience during

```
const real eps = ...;
bool upset = false;

operator*(real x1, real x2) ensures (result == x1 * x2);
operator*(real x1, real x2)
  when !upset modifies (upset)
  ensures upset &&
    x1*x2-eps < result < x1*x2+eps;
```

Fig. 4. Vector Product Under Additive SEU

```
property_r bounded_diff(vector<real> x, uint N) :
  \forall(uint i)((i < N) -> (abs(x[i] - x[i]) < model.\epsilon));

requires_r eq(N) && eq(x) && eq(y)
vector<real> product(uint N, vector<real> x(N), vector<real> y(N)) {
  vector<real> result(N);
  for (uint i = 0; i < N; ++i)
    invariant_r bounded_diff(result, i)
    { result[i] = x[i] \cdot y[i];
    }
  assert_r(bounded_diff(result, N));
  return result;
}
```

Fig. 4. Vector Product Under Additive SEU
execution. For example, the Jacobi iterative method has the property that the change in the number of iterations to converge after an error is bounded logarithmically by the magnitude of the error. Thus, by bounding the magnitude of errors, a developer using the Jacobi method can derive a static bound on the maximum impact errors can have on convergence time.

To verify that this vector product implementation has bounded error the developer uses the `assert_r` statement on Line 15 to assert \( \text{bounded\_diff}(\text{result}, N) \). `bounded\_diff` is a relational property application. A property is a hygienic macro to enable code reuse within loop invariants and assertions. We define the `bounded\_diff` property on Line 1. This property takes a vector \( x \) and a size \( N \) and enforces that for every index \( i \), \( \text{model.\_eps} \) bounds the error in \( x[i] \). \( x<o>[i] - x<r>[i] \) computes the error in \( x \) where \( x<o> \) refers to the value of \( x \) in the standard, original execution of the program and \( x<r> \) refers to the value of \( x \) in the relaxed execution. To facilitate the verification of this condition we also include it in the loop invariant on Line 9.

Another necessary component to verifying this assertion is that \( N, x, \) and \( y \) have the same values in both the original and relaxed executions. The implementation enforces this property through the relational function precondition (\( \text{requires\_r} \)) on Line 4. Leto expands terms of the form eq(\( x \)) to \( x<o> == x<r> \).

### 2.1.2 Verification Algorithm.
Leto provides an automated verification algorithm that performs relational forward symbolic execution to discharge assertions in the program. Namely, Leto traverses the program, building a logical characterization of the state of the program at each point and verifies that the resulting logical formula ensures that a given `assert` or `assert\_r` statement is valid. This approach also works in concert with the developer’s specification annotations; these include both function preconditions and loop invariants. Leto also provides support for automatic loop invariant inference, which can lower the annotation burden of the developer by automatically inferring additional loop invariants. In the vector product under additive SEU program Leto infers eq(\( N \)), eq(\( x \)), eq(\( y \)) and \( i <= N \), which is necessary to demonstrate that all of the vector accesses are in bounds.

### 2.2 Switchable Rowhammer Model

We present the switchable rowhammer execution model in Figure 5. This execution model simulates a Rowhammer attack that allows an attacker to selectively flip bits in DRAM by issuing frequent reads on DRAM rows [Kim et al. 2014].

Unlike the models we have presented thus far, this one enables developers to model memory errors. Additionally, it enables switchability, permitting the program to selectively disable errors to emulate selective Rowhammer protection techniques [Aweke et al. 2016].

### Memory Regions.
In addition to binary operators, Leto permits the specification of read and write behavior. Read and write specifications contain an additional \( \text{region} \) annotation allowing developers to partition their program variables into multiple memory regions with differing read and write characteristics.

Both operator specifications in Figure 5 govern writes to variables stored in the `unreliable` memory region. When Leto encounters an expression of the form \( v = e \) where \( v \) is in the memory region `unreliable`, Leto substitutes occurrences of \( dest \) in the model with \( v \) and occurrences of \( src \) with \( e \). Line 4 specifies a reliable write operator while Line 8 specifies a faulty write operator.
The faulty write operator permits errors that are larger than \( \text{eps} \), the result being that the system stores an erroneous value in the variable represented by \( \text{dest} \) and subsequent reads of that variable return the erroneous value.

**Switchability.** The \( \text{reliable} \) flag models the fact that it is possible to selectively enable Rowhammer protection techniques [Aweke et al. 2016] to trade performance for reliability. Specifically if \( \text{reliable} \) is set to \( \text{true} \), then the model does not generate errors. In addition to Rowhammer, this model can simulate scenarios that selectively use ECC-protected caches [Alameldeen et al. 2011; Kim et al. 2007; Yoon and Erez 2009] in conjunction with traditional caches.

2.2.1 **Vector Product Under Switchable Rowhammer.** Figure 6 presents a vector product implementation annotated to verify under the switchable rowhammer model. The relaxation in this implementation occurs in the loop (Line 10) where faulty writes may corrupt the value of \( \text{result}[i] \). On Line 16 the implementation verifies that the impact of errors is larger than \( \text{model.eps} \times \text{model.eps} \).

**Memory Regions.** Line 8 places the \( \text{result} \) vector in the \emph{unreliable} memory region. Thus, Leto will use the execution model specification of the \emph{write} operator on Line 13.

**Detectable Errors.** In some systems, large errors are always detectable as they produce invalid results that an application can easily check for. For example, in Self-Correcting Connected Components (SC-CC) [Sao et al. 2016]—an iterative algorithm for computing the connected subgraphs in a graph—a large error in the output vector produces references to nonexistent graph nodes. Thus, SC-CC detects and corrects all large errors by scanning through the output vector after each iteration for invalid entries and recomputing erroneous elements reliably.

The assertion on Line 16 uses the \emph{large_error} property on Line 1 to enforce that the impact of errors is sufficiency large to produce invalid values in \( \text{result} \). The \emph{large_error} property takes a vector \( x \) and a size \( N \) and ensures that for every index \( i \), \( x[i] \) is equal across both executions, or \( x<r>[i] \) contains a trivially invalid value.

To support this, the function precondition on Line 4 mandates that every element of \( x \) and \( y \) is bounded between 0 and \( \text{model.eps} \). This ensures that any non-erroneous value in \( \text{result}<r> \) is bounded between 0 and \( \text{model.eps} \times \text{model.eps} \). Therefore, if a program using this vector product function scans for elements that are out of this bound it will detect all errors. Such a
program could then correct these errors by setting the reliable flag in the switchable rowhammer model to true and recomputing the erroneous elements.

2.3 Multicycle Error Model

Figure 7 presents a multicycle error execution model. A multicycle error is an error state in which multiple consecutive instructions experience errors [Inoue et al. 2011]. This implementation permits a single multicycle error and tracks the state of this error through the use of model variables stuck and length. The stuck flag represents whether or not the system is currently experiencing a multicycle fault while the length variable indicates how many instructions the fault will continue for. We leave the length variable unbound, permitting the multicycle error to persist for an arbitrary number of operations. This implementation is compatible with the vector product implementation from the additive SEU example (Figure 3) and requires no modifications to verify.

Line 6 describes a reliable multiplication implementation that the model may use before the fault occurs (!stuck) and after it ends (length == 0).

```
1 const real eps = ...;
2 bool stuck = false;
3 uint length;
4 @label (reliable)
5 operator*(real x1, real x2)
6 when !stuck || length == 0
7 ensures result == x1 * x2;
8
9 @label (unreliable)
10 operator*(real x1, real x2)
11 when length > 0
12 modifies (stuck, length)
13 ensures stuck &&
14 length == old(length) - 1 &&
15 x1*x2-eps < result < x1*x2+eps;
```

Fig. 7. Multicycle Error Execution Model

Line 11 encodes an operator that the model may use during, or to begin a multicycle error. The model may substitute these operators so long as a multicycle error has not occurred and resolved before the current instruction (length > 0). This operator sets stuck, decrements length, and constrains result to be within eps of the original result.

Together, these two operators ensure that at some point the system may be stuck experiencing faults on all multiplications, but after length multiplications it will unstick and execution will be reliable.

Temporal Variable References. To support more complex models than one can express with boolean flags, Leto allows developers to differentiate between variable state before and after each operation. By default, all variables in ensures clauses refer to their state after the operator executes. However, by wrapping the variable in old() the wrapped variable will instead refer to its state before the operator executes. Temporal variable references through the old keyword is in the spirit of similar constructs in ESC/Java [Flanagan and Leino 2001], Spec# [Barnett et al. 2004], and many other verification systems. The multicycle error model uses this feature to track the length of a multicycle event on Line 15 by asserting that the length variable after the multiplication is one less than its value prior to the multiplication.

Named Operators. To enable developers to refine models, Leto supports optional labels on operators. Refinement is the process by which developers may create submodels that satisfy the specification of supermodels. It enables developers to construct a lattice of such models such that programs verified under an abstract execution model will also verify under specialized versions of that model. Line 5 gives the reliable label to the first operator while Line 10 gives the unreliable label to the second operator.

2.3.1 Vector Product Under Multicycle. Leto verifies the vector product implementation from the additive SEU example (Figure 4) under this model with no modifications to the implementation.
thus demonstrating Leto’s high automatic reasoning power, as well as the modular characteristic of Leto’s execution models.

2.4 Refinement

To enable an parallel development process in which developers may successively build their hardware, models, and programs in tandem, Leto supports execution model refinement. Refinement enables developers to construct a lattice of models such that more precise submodels satisfy the specification of less precise supermodels.

By verifying refinement, Leto guarantees that programs verified under an abstract execution model will also verify under specialized versions of that model. Therefore, refinement separates model elaboration from program verification. That is, it provides an interface between hardware vendors and software developers. Software developers may verify their programs under loosely defined execution models, while hardware vendors may provide detailed models that are true to the underlying semantics of their hardware. As long as software developers use Leto to verify that these precise models refine their loose models, Leto guarantees that their programs will run as expected on the hardware vendor’s product.

Figure 8 presents an SEU model refined from the multicycle model. Line 1 indicates that this model is a refinement of the multicycle model. Leto supports multiple refinement, allowing submodels to refine any number of supermodels. Line 2 and Line 3 import the reliable and unreliable operators from the multicycle model by name. This makes these operators available to the submodel. Developers may add additional operators by indicating that they refine a named operator in each supermodel. Leto checks that this submodel operator refines each supermodel operator by verifying that the when clause of the submodel operator logically implies the when clause of each supermodel operator and the ensures clause of the submodel operator logically implies the ensures clause of each supermodel operator. Every additional operator in a submodel must explicitly refine some named operator in each supermodel. When using multiple refinement, any imported operators from one model must also explicitly refine an operator in every other supermodel.

Figure 9 presents the core of Leto’s programming language. Leto provides a general-purpose imperative language that includes specification primitives (e.g., requires) in the spirit of ESC/Java [Flanagan and Leino 2001], Boogie [Barnett et al. 2005], Eiffel [Meyer 1992], and Spec# [Barnett et al. 2004] to support verifying applications.
writes may have a custom semantics according to the execution model.

### Data Types

The language includes primitive data types ($\tau$) of (signed and unsigned) integers, reals, and booleans as well as vectors/matrices of these types. A developer can use the @region annotation to state the variable is allocated in a named memory region, $r$, for which reads and writes may have a custom semantics according to the execution model.

### Expressions

Leto includes standard numerical operations, comparison, and logical expressions, along with dotted notations (e.g., $x +. y$) that communicate that the operation may have a custom semantics as specified in the execution model.

### Memory Operations

Leto supports reads from and writes to variables, including values of both primitive and array/matrix type. Reads and writes to variables allocated in a designated memory region operate with the semantics as given in the program’s execution model.

### Assertions and Assumptions

Leto also enables developers to specify assertions and assumptions on the state of the program. Leto’s language includes both standard assert statements and assume statements (with their traditional meaning). Each such statement can use a quantified boolean expression, $P$, that quantifies over the value of variable (e.g., the index of an array/matrix). A relational assertion statement, assert_r, uses a quantified relational boolean expression, $P_r$, that specifies a relationship between the original and relaxed executions to verify.

### Control flow

Leto’s language includes standard control constructs, such as sequential composition, if statements, while, and for statements. For while and for statements, a developer can specify loop invariants to support verification via the syntax invariant_r (unary loop invariant) and invariant_r (relational loop invariant). A loop invariant specifies a property that must be true on entry to the loop, as well as at the end of each loop iteration. Loop invariants are a key to verifying applications that contain loops because automatically inferring loop invariants is undecidable in general. Therefore, a developer may need to specify additional loop invariants when Leto’s loop invariant inference procedure is insufficient.

### Execution Model

An execution model $M$ consists of a set of state variables $x^+$ and operation specifications ($O^+$). Each operation specification ($O$) specifies 1) the target operator for the specification, 2) a list of variables as parameters to the specification, and 3) a set of clauses. A clause is either a when clause, which guards the execution of the specification with a predicate $P$, an ensures...
clause, which establishes a relationship on the output of the specification given the inputs to the specification and fault model’s state variables, or a modifies clause, that specifies which of the model’s state variables changes as a result of using the operation. Predicates consist of standard operations over standard expressions with the addition of the distinguished result variable, which captures the result of the specification’s execution.

### 3.1 Dynamic Semantics

We next present an abbreviated dynamic semantics of Leto’s language. We expand on these semantics in Appendix C and Appendix D. We formalize the semantics via a lowered syntax of the Leto language that includes registers (Figure 10). We assume a standard compilation process that translates high-level Leto to this lowered language. Additionally, to support our formalization of Leto’s program logic on this representation, we extend the predicates $P$ and $P_r$ with registers into $P^*$ and $P_r^*$, respectively.

#### 3.1.1 Preliminaries

Leto’s semantics models an abstract machine that includes a frame, a heap, and an execution model state. Leto allocates memory for program variables (both scalar and array) in the heap. A frame serves two roles: 1) a frame maps a program variable to the address of the location in the heap. A frame serves two roles: 1) a frame maps a program variable to the address of the memory region allocated for that variable in the heap, and 2) a frame maps a register to its current value in the program. The model state stores the values for state variables within the execution model.

**Frames, Heaps, Model States, Environments.** A frame, $\sigma \in \Sigma = \text{Var} \cup \text{Reg} \rightarrow \text{Int}_{N}$, is a finite map from variables and registers to N-bit integers. A heap, $h \in H = \text{Loc} \rightarrow \text{Int}_{N}$, is a finite map from locations ($n \in \text{Loc} \subset \text{Int}_{N}$) to N-bit integer values. A region map, $\theta \in \Theta = \text{Loc} \rightarrow \text{Region}$, is a finite map from locations to memory regions. A model state, $m \in M = \text{Var} \rightarrow \text{Int}_{N}$, is a finite map from model state variables to N-bit integer values. An environment, $e \in E = \Sigma \times H \times \Theta \times M$, is a tuple consisting of a frame, a heap, a region map, and a model state. An execution model specification, $\mu \subseteq \text{Op} \times \text{list}(\text{Var}) \times \text{set}(\text{Var}) \times P \times P$, is a relation consisting of tuples of an operation $op \in \text{Op}$, a list of input variables, a list of modified variables, and two unary logical predicates representing the when and ensures clauses of the operation.

**Initialization.** For clarity of presentation, we assume a compilation and execution model in which memory locations for program variables are allocated and the corresponding mapping in the frame are done prior to execution of the program (similar in form to C-style declarations).

#### 3.1.2 Execution Model Semantics

Here we provide an abbreviated presentation of the execution model relation $\langle m, op, \langle \text{args} \rangle \rangle \parallel_{\mu} \langle n, m' \rangle$. The relation states that given the arguments, $\text{args}$ to an operation $op$, evaluation of the operation from the model state $m$ yields a result $n$ and a new model state $m'$ under the execution model specification $\mu$.

\[
\begin{align*}
\text{F-binop} & \quad \mu([\oplus, [x_1, x_2], X, P_w, P_e]) \quad \forall x_m \in X \cdot \text{fresh}(x'_m) \quad m[x_1 \mapsto n_1][x_2 \mapsto n_2] \models P_w \\
& \quad m'[x_1 \mapsto n_1][x_2 \mapsto n_2][\forall x_m \in X \cdot x_m \mapsto x'_m][\text{result} \mapsto n_3] \models P_e \quad \text{dom}(m) = \text{dom}(m')
\end{align*}
\]

The [F-binop] rule specifies the meaning of this relation for binary operations. This relation states that the value of an operation $\oplus$ given a tuple of input values $(n_1, n_2)$ and an execution model state $m$ evaluates to value $n_3$ and a new model state $m'$. The rule relies on the relation $\mu(op, vlist, X, P_w, P_e)$ which specifies the list of argument names, $vlist$, the set of modified variables...
X, the precondition \( P_w \), and the postcondition \( P_e \) for the operation \( op \) in the developer-provided execution model. The set of modified variables is the union of the modifies clauses in the operation’s specification. The precondition of an operation is the conjunction of the when clauses in the operation’s specification. The postcondition of an operation is the conjunction of the ensures clauses in the operation’s specification.

The semantics of the model relation non-deterministically selects an operation specification, result value, and output model state subject to the constraint that: 1) the current model state satisfies the precondition (after the inputs to the operation have been appropriately assigned into the model state), 2) the output model state satisfies the postcondition (after the inputs, modified variables, and result value have been appropriately assigned into the model state), and 3) the domains of the input and output state are the same.

Because of the uniformity of the execution model specification, the semantics for other operations (e.g., reads and writes) is similar with the sole distinction being the number of arguments passed to the operation. For clarity of presentation, we elide the presentation of rules for those operations, but we provide them in Appendix D, Figure 22.

### 3.1.3 Language Semantics

We next present the non-deterministic small-step transition relation \( \langle s, \epsilon \rangle \xrightarrow{\mu} \langle s', \epsilon' \rangle \) of a Leto program. The relation states that execution of statement \( s \) from the environment \( \epsilon \) takes one step yielding the statement \( s' \) and environment \( \epsilon' \) under the execution model specification \( \mu \). The semantics of the statements are largely similar to that of traditional approaches except for the ability to the statements to encounter faults. Broadly, we categorize Leto’s instructions into four categories: register instructions, assertions, memory instructions, and control flow.

#### Register Instructions

The rules [ASSIGN] and [BINOP] specify the semantics of two of Leto’s register manipulation instructions. [ASSIGN] defines the semantics of assigning an integer value to a register, \( r = n \). This has the expected semantics updating the value of \( r \) within the current frame with the value \( n \). Of note is that register assignment executes fully reliably without faults.

[BINOP] specifies the semantics of a register only binary operation, \( r = r_1 \oplus r_2 \). Note that reads of the input registers execute fully reliably. The result of the operation is \( n_3 \), which is the value of the operation given the semantics of that operation’s execution model when executed from the model state \( m \) on parameters \( n_1 \) and \( n_2 \). Executing the execution model may change the values of the execution model’s state variables. Therefore, the instruction evaluates to an instruction that assigns \( n_3 \) to the destination register and evaluates with a environment that consists of the unmodified frame, the unmodified heap, and the modified execution model state. Note that by virtue of the fact that both the frame and heap are unmodified, faults in register instructions cannot modify the contents or organization of memory.

#### Assertions

The assert and assume statements have standard semantics, yielding a skip and continuing the execution of the program if their conditions are satisfied. For either of these statements, if their conditions evaluate to false, then execution yields fail denoting that execution has
failed and become stuck in error. We provide the rules for both assert and assume in Appendix C, Figure 20.

\[
\begin{align*}
\text{READ} & \quad a = \sigma(x) \quad n = h(a) \quad q = \theta(a) \quad \langle m, \text{read}, (n, q) \rangle \downarrow_{\mu} \langle n', m' \rangle \\
\text{WRITE} & \quad n_{\text{old}} = h(a) \quad n_{\text{new}} = \sigma(r) \quad q = \theta(a) \quad \langle m, \text{write}, (n_{\text{old}}, n_{\text{new}}, q) \rangle \downarrow_{\mu} \langle n_r, m' \rangle
\end{align*}
\]

**Memory Instructions.** The rules [READ] and [WRITE] specify the semantics of two of Leto’s memory manipulation instructions. [READ] defines the semantics of reading the value of a program variable \(x\) from its corresponding memory location: \(r = x\). The rule fetches the program variable’s memory address from the frame, reads the value of the memory location \(n = h(a)\) and the region the memory location belongs to \(q = \theta(a)\) and then executes the execution model with the program variable’s current value in memory and the memory region it resides in as parameters. The execution model non-deterministically yields a result \(n'\) that the rule uses to complete its implementing by issuing an assignment to the register.

[WRITE] defines the semantics of writing the value of a register to memory. The rule reads the value of the memory location to record the old value of the memory location, reads the value of the input register, fetches the region the memory location corresponds to, and then executes the execution model with these values as parameters. The execution model yields a new value \(n_r\) that the rule then assigns to the value the program variable.

**Control Flow.** The rules for control flow have standard semantics. An important note is that the semantics of these statements is such that the transfer of control from one instruction to another always executes reliably and, therefore, faults do not introduce control flow errors into the program. This modeling assumption is consistent with standard fault injection and reliability analysis models [Sampson et al. 2011; Vishal Chandra Sharma 2016]. We provide the control flow rules in Appendix C, Figure 20.

**Big-Step Semantics.** To support the formalization in the remainder of the paper, we introduce the big-step relation \(\langle s, \epsilon \rangle \Downarrow_{m} v \subseteq S \times E \times V\) where \(V ::= E \mid \text{fail} E\) such that \(\langle s, \epsilon \rangle \Downarrow\) is the reflexive transitive closure of \(\rightarrow_{m}\) that yields the environment \(\epsilon\) if execution ends successfully in a skip statement or yields the pair fail \(\epsilon\) when the execution ends in a failure. We also introduce the big-step relation \(\langle s, \epsilon \rangle \Downarrow v \subseteq S \times E \times V\) where \(\langle s, \epsilon \rangle \Downarrow \equiv \langle s, \epsilon \rangle \Downarrow_{\rho} v\) where \(\rho\) denotes a fully reliable fault model where the only implementations exposed for each operation are fully reliable implementations.

4 PROGRAM LOGIC

Leto’s program logic is a relational program logic in that it relates relaxed executions of the program to its original, reliable execution. A key idea behind our development is the separation of the rules into a part that solely characterizes the reliable execution of the program, (the Left Rules), a part that solely characterizes the relaxed execution (the Right Rules), and a part the characterizes the lockstep execution of the reliable and relaxed execution (the Lockstep Rules). The result is an Asymmetric Relational Hoare Logic that characterizes the two interpretations of the program.
Figure 9 presents our language syntax, including the syntax of our assertion language. Assertions include standard quantified boolean predicates, $P^*$, with the standard semantic function $[[P^*]] \in \mathcal{P}(E)$ that gives the denotation of $P^*$ as the set of environments that satisfy the predicate. Assertions also include quantified relational boolean predicates, $P^ r$, with the semantic function $[[P^ r]] \in \mathcal{P}(E \times E)$ that gives $P^ r$ the meaning of the set of pairs of environments that satisfy the predicate. In our standard convention, the first environment of the pair corresponds to the state of the reliable execution whereas the second environment corresponds to that of the relaxed execution.

Auxiliary Definitions. To support the formalization in the remainder of the paper we define the auxiliary notation $\text{inj}_t(\cdot)$ where $t \in \{o, r\}$ implements an injection for standard unary predicates into a relational domain. For $t = o$, the definition injects a predicate into the domain of the reliable execution of the program whereas when $t = r$, the definition injects a predicate into the domain of the relaxed execution.

### 4.1 Preliminaries

**Assertion Logic Syntax and Semantics.** Figure 9 presents our language syntax, including the syntax of our assertion language. Assertions include standard quantified boolean predicates, $P^*$, with the standard semantic function $[[P^*]] \in \mathcal{P}(E)$ that gives the denotation of $P^*$ as the set of environments that satisfy the predicate. Assertions also include quantified relational boolean predicates, $P^ r$, with the semantic function $[[P^ r]] \in \mathcal{P}(E \times E)$ that gives $P^ r$ the meaning of the set of pairs of environments that satisfy the predicate. In our standard convention, the first environment of the pair corresponds to the state of the reliable execution whereas the second environment corresponds to that of the relaxed execution.

**Auxiliary Definitions.** To support the formalization in the remainder of the paper we define the auxiliary notation $\text{inj}_t(\cdot)$ where $t \in \{o, r\}$ implements an injection for standard unary predicates into a relational domain. For $t = o$, the definition injects a predicate into the domain of the reliable execution of the program whereas when $t = r$, the definition injects a predicate into the domain of the relaxed execution.

### 4.2 Proof Rules

Figures 11 and 12 provide an abbreviated presentation of the rules of our program logic. We present the remainder of the rules in Appendix A. We have partitioned the presentation into two parts: 1) the Left Rules and Right Rules for primitive statements and 2) the Lockstep Rules.
**Left Rules.** The Left Rules, which we denote by the judgment \( \tau I \{ P_r^* \} s \{ Q_r^* \} \), characterize the behavior of the reliable execution of the statement \( s \). The denotation of the judgment is that if \( (e_1, e_2) \models P_r^* \) and \( \langle s, e_1 \rangle \downarrow e'_1 \), then \( (e'_1, e_2) \models Q_r^* \). Namely, given a proof in the Left Rules, for a pair of environments satisfying the precondition of the proof, then if a reliable execution of \( s \) terminates, then the resulting environment pair satisfies the proof’s postcondition.

**Right Rules.** The right rules, which we denote by the judgment \( \mu \tau R \{ P_r^* \} s \{ Q_r^* \} \), characterize the behavior of the relaxed execution of \( s \) under a fault model specification \( \mu \). The denotation of the judgment is similar to that of the Left Rules: if \( (e_1, e_2) \models P_r^* \), \( \langle s, e_1 \rangle \not\downarrow e'_2 \), then \( (e_1, e'_2) \models Q_r^* \). Namely, given a proof in the right rules, for a pair of environments satisfying the proof’s precondition, then if execution of \( s \) under the fault model specification \( \mu \) terminates, then the resulting environment pair satisfies the proof’s postcondition.

**Lockstep Rules.** The Lockstep Rules together constitute the main top-level judgment of the logic reasons about relations between the two semantics as they proceed in lockstep: \( \mu \tau \{ P_r^* \} s \{ Q_r^* \} \). The denotation is that if \( (e_1, e_2) \models P_r^* \), \( \langle s, e_1 \rangle \not\downarrow e'_1 \), and \( \langle s, e_2 \rangle \not\downarrow e'_2 \), then \( (e'_1, e'_2) \models Q_r^* \).

4.2.1 **Left and Right Rules.**

**Register Assignment.** The rules [ASSIGN-L] and [ASSIGN-R] capture the semantics of the register assignment statement, \( r = n \) in the lowered language. In the reliable execution, the rule [ASSIGN-L] captures the semantics of the assignment statement via the standard backward characterization of assignment as seen in standard Hoare logic [Hoare 1969]. The major distinction between a standard presentation and the presentation here is that the substitution replaces the injected form of the register \( r \) in the postcondition of the statement. The expression \( inj_{\mu}(r) \) denotes the value of \( r \) in the reliable version of the program. For the relaxed execution, the rule [ASSIGN-R] captures the semantics by substituting for \( inj_{\mu}(r) \), which denotes the value of \( r \) in the relaxed execution. We note that given these results, assignment is reliable in both the reliable and relaxed executions with the primary distinction being which environment is modified (either that corresponding to the reliable execution or that of the relaxed execution).

**Arithmetic Operation.** The rules [BINOP-L] and [BINOP-R] give the semantics of binary arithmetic operations on registers: \( r = r_1 \oplus r_2 \). For the reliable execution, [BINOP-L] relies on the backwards characterization of assignment as seen in [ASSIGN-L] to substitute the value \( r \) in the reliable execution of the program with the value of the arithmetic operation \( inj_{\mu}(r_1 \oplus r_2) \). For the relaxed execution, [BINOP-R], augments the traditional backwards characterization to include the potentially unreliable execution of the binary operation.

**Assert.** The rules [ASSERT-L] and [ASSERT-R] give the semantics of assertion statements. There is a major distinction between the role of assertion statements between the reliable and relaxed execution of the program. Specifically, while the logic requires that the condition of an assert statement is verified in the relaxed execution, the condition of an assert statement in the reliable execution does not need to be verified; it is instead assumed. The major design point is that Leto enables a developer to use a variety of means (e.g., testing, verification, or code review) to validate an assertion in the original program and transfer that reasoning to the verification process for the relaxed execution. To achieve this design, the Left rule for assertions assumes the validity of the assertion whereas the Right rule asserts. Although the assert and assume have the same semantics in the reliable and relaxed executions, the intentions of the statements differ. Specifically, if a developer places an assert in the program, the assumption is that they have used other means to evaluate the validity of that assertion in the reliable execution (potentially including other verification systems). An assume statement, however, does not carry that intention.
Assume. The rules [ASSUME-L] and [ASSUME-R] give the semantics of assume statements. The primary distinction for assume statements is that while assume statements have their standard semantics in the reliable execution of the program (no proof obligation is required), assume statements do in fact require a proof obligation in the relaxed semantics. The semantics of an assume statement in the relaxed semantics is therefore the same as that of an assert statement. The rationale behind this design is that as part of the verification of the relaxed execution we must verify that faults do not interfere with the reasoning behind an assumption.

Control Flow. For clarity of presentation we have elided the left and right rules control flow because the rules adhere to the standard formalization as seen in traditional Hoare logic. The only distinction between these rules and their standard implementation is that they operate over relational predicates.

4.2.2 Lockstep Rules. To support the lock step rules, we first present the [STAGE] rule, which joins the Left Rules and Right Rules.

\[
\text{STAGE} \quad \mu \vdash \{ P^* \} s_1 \{ R^*_r \} \quad \mu \vdash \{ P^* \} s_2 \{ Q^*_r \} \\
\mu \vdash \{ P^*_r \} s_1 \sim s_2 \{ Q^*_r \}
\]

Stage. The rule [STAGE] gives a semantics to a pair of statements \( s_1 \) and \( s_2 \) for which the goal is to characterize the behavior when the reliable execution executes \( s_1 \) and the relaxed execution executes \( s_2 \). The specific composition we have chosen for this rule is to apply the Left Rules for \( s_1 \) before applying the Right Rules to \( s_2 \). Namely, the rule first applies the Left Rule for \( s_1 \), yielding a new predicate \( R'_r \), before then applying the Right Rule for \( s_2 \) to \( R'^*_r \). The rule [SPLIT] provides a rationale for this specific composition. The rule [INVERSE-STAGE] has the opposite semantics, applying the Right Rule for \( s_2 \), yielding a new predicate \( R'^*_r \), before then applying the Left Rule for \( s_1 \) to \( R'^*_r \). The rule [IF] provides a rationale for this composition during nonlockstep execution.

Split. The rule [SPLIT] gives a semantics to individual statements in the lockstep semantics. The rule relies on the [STAGE] rule to apply the left rules for the statement before applying the right rules. This design forces a specific composition of the rules in order to achieve more tractable verification. For example, for a statement \( \text{assert } r \), this rule will first apply the left rule for assertions, which can be used to derive \( r<o> = true \). Note that this derivation occurs by assumption as the logic assumes the validity of assertions in the reliable execution. Next, the rule requires the proof to establish that \( r<r> = true \). If, for example, the predicate \( r<o> = r<r> \) is in the context, then this proof obligation is easily established.

If. The rule [IF] gives the semantics of if statements. The rule considers all cases of the execution of the statement. Specifically, the reliable and relaxed executions may proceed in lockstep or they may diverge by proceeding down different branches. The logic captures this divergence by leveraging the inverse staging rule to apply the Right Rules for the branch on which the relaxed execution has taken before applying the Left Rules for the one which the reliable version has taken. Again, this forces a specific methodology for reasoning about the programs in that the logic extracts the full availability of assertions that may exist on the branch that the relaxed execution takes before proceeding with the reliable execution.

4.3 Properties

Leto’s program logic ensures two basic properties of Leto programs: preservation and progress. The preservation property states the partial correctness of the logic (but does not not establish termination—and therefore total correctness). The progress property establishes that the relaxed
execution of a program verified with Leto satisfies all of its `assert` and `assume` statements—provided that all reliable executions of the program also satisfy the program’s `assert` and `assume` statements. We state these properties formally below and provide proofs of these theorems in Appendix B.

**Theorem 4.1 (Preservation).**

If \( \mu \models \{ P^r \} \ s \ { Q^r \} \) and \( (\epsilon_1, \epsilon_2) \models P^r \) and \( (s, \epsilon_1) \Downarrow \epsilon'_1 \) and \( (s, \epsilon_2) \Downarrow \epsilon'_2 \), then \( (\epsilon'_1, \epsilon'_2) \models Q^r \)

Leto’s preservation property states that given a proof in the program logic of a program \( s \), for all pairs of environments \((\epsilon_1, \epsilon_2)\) that satisfy the proof’s precondition, if the executions of \( s \) under both the reliable semantics and the relaxed semantics terminate in a pair of environments \((\epsilon'_1, \epsilon'_2)\), then this pair of environments satisfies the proof’s postcondition.

**Theorem 4.2 (Progress).**

If \( \mu \models \{ P^r \} \ s \ { Q^r \} \) and \( (\epsilon_1, \epsilon_2) \models P^r \) and \( (s, \epsilon_1) \Downarrow \epsilon'_1 \) and \( (s, \epsilon_2) \Downarrow \epsilon'_2 \) and \( \mu \models \epsilon'_2 \), then \(-\text{failed}(\epsilon'_2) \) where \( \text{failed}(\langle \text{fail}, \epsilon \rangle) = \text{true} \)

Leto’s progress property states that given a proof in the program logic of a program \( s \), for all pairs of environments \((\epsilon_1, \epsilon_2)\) that satisfy the proof’s precondition, if the reliable execution of \( s \) terminates successfully, then if the relaxed execution of \( s \) under \( \mu \) terminates, then it does not terminate in an error.

5 IMPLEMENTATION

Leto’s verification algorithm performs forward symbolic execution to discharge verification conditions generated by `assert`, `assert_t`, `invariant`, and `invariant_r` statements in the program. The algorithm directly implements the Hoare-style relational program logic from Section 4. We present a detailed description of the algorithm in Appendix E. We also present a description of our loop invariant inference algorithm in Appendix F. The inference algorithm is modeled after Houdini [Flanagan and Leino 2001], and its base design is to infer common equivalence relations between programs.

Leto generates constraints to be solved by Microsoft’s Z3 SMT solver [De Moura and Bjørner 2008]. Our system makes use of Z3’s real, int, and bool types as well as uninterpreted functions for arrays/matrices. As such, our system does not necessarily generate a set of constraints for which Z3 is complete. The practical impact of this design is that it is possible for Z3 to be unable to verify valid constraints. However, we have been able to successfully verify critical fault tolerance properties for several applications as presented in the following section.

6 CASE STUDY: JACOBI ITERATIVE METHOD

Figure 14 presents an implementation of the Jacobi iterative method, alternatively Jacobi, in Leto. The Jacobi iterative method is an algorithm for solving a system of linear equations. Specifically, given a matrix of coefficients \( A \) and a vector \( b \) of intercepts, the algorithm computes a solution vector, \( x \), where \( A \cdot x = b \). The algorithm works iteratively by computing successive approximations of \( x \). For a system of two equations (where \( A \) is a 2x2 matrix and both b and x are of length two), Jacobi uses the solution vector from the previous iteration, \( x^k \), to produce the solution vector for the current iteration, \( x^{k+1} \), using the following approximation scheme:

\[
x^{k+1}_0 = \frac{(b_0 - A_{0,1} \cdot x^{k}_1)}{A_{0,0}} \\
x^{k+1}_1 = \frac{(b_1 - A_{1,0} \cdot x^{k}_0)}{A_{1,1}}
\]

In words, for a given coordinate \( x_i \), Jacobi approximates \( x_i^{k+1} \), by substituting the values \( x_j^k \), where \( i \neq j \), into the linear equation for \( i \), and solving for \( x_i^{k+1} \).
Modulo floating-point rounding error, Jacobi converges to the correct \( x \) as the number of iterations goes to infinity.

**Fault Tolerance.** Jacobi is *naturally self-stabilizing*. Specifically, given an execution platform in which faults do not modify the contents of \( A \), then Jacobi is in a *valid state* at the end of each iteration: if no additional faults occur during its execution, then Jacobi will converge to the correct solution.

To understand this property intuitively, if an iteration produces an incorrect solution vector, then the subsequent execution of the computation is equivalent to having started the computation from scratch with the produced vector as the initial starting point. Moreover, Jacobi enjoys the nice result that the change in the number of iterations required to converge from the new starting point is bounded logarithmically by the magnitude of the error in the solution vector.

Verifying Jacobi for a given execution platform therefore poses two challenges: 1) verifying that faults only affect the value of \( x \) and 2) identifying a bound on the number of added iterations in the presence of a fault. Note that the latter determination not only serves as important information for understanding if the implementation will meet the developer’s convergence requirements, but it also serves the practical purpose of setting the maximum number of iterations such that a faulty execution will produce a result that is at least as good as a fully reliable execution.

### 6.1 Multiplicative Single Event Upset Model

We verify Jacobi under the multiplicative SEU model we present in Figure 13. The model exports three versions of the multiplication operator. Line 4 specifies the standard reliable implementation of multiplication. Lines 7 and 15 each additionally specify an unreliable implementation for the case where \( x_1 \times x_2 \) is positive or negative respectively. Each implementation models potentially unreliable multipliers that are protected by truncated error correction [Sullivan and Swartzlander 2012, 2013]. Specifically, the semantics of each unreliable version provides the guarantee that even in the presence of an error, the result differs by at most \( E_{\text{REL}} \times 100\% \) of the original result [Sullivan and Swartzlander 2012, 2013], where \( E_{\text{REL}} \) is a constant the model specifies on Line 1.

```
const real E_REL = ...;
bool upset = false;
operator *(real x1, real x2)
ensures (result == x1 * x2);
operator *(real x1, real x2)
when !upset && (0 < x1 * x2)
modifies (upset)
ensures upset &&
((1 - E_REL) * x1 * x2 <=
result <=
(1 + E_REL) * x1 * x2);
operator *(real x1, real x2)
when !upset && (x1 * x2 < 0)
modifies (upset)
ensures upset &&
((1 + E_REL) * x1 * x2 <=
result <=
(1 - E_REL) * x1 * x2);
```

Fig. 13. Multiplicative SEU Execution Model

### 6.2 Jacobi Implementation

The overall architecture of the implementation in Figure 14 is that the outer loop on Line 4 computes and stores the solution vector for the current iteration into \( \text{next}_x \). At the end of each iteration, the implementation updates \( x \) by copying \( \text{next}_x \) into \( x \). The second loop on Line 8 iterates through each \( x_i \) (stored at \( x[i] \)), sums the other terms in the \( i \)th equation using the third loop (Line 14) into \( \text{sum} \), and then computes \( x[i] \) as the value \( b[i] - \text{sum}/A[i][i] \). We discuss the definitions of the properties \( \text{sig} \) and \( \text{bounded_diff} \) below.

All of the relaxation in Jacobi occurs in the third loop (Line 14), where the execution model may corrupt the value of \( \text{sum} \). The implementation first performs a relaxed multiplication (Line 17),
uint N; int iters;
matrix<real> A(N,N); vector<real> b(N); vector<real> x(N)
for (; 0 <= iters; --iters) invariant_r !model.upset -> eq(x)
{
    vector<real> next_x(N);
    for (uint i = 0; i < N; ++i)
        invariant_r !model.upset -> eq(next_x)
        invariant_r bounded_diff(N, next_x)
        { 
            real sum = 0;
            for (uint j = 0; j < N; ++j) invariant_r sig(sum)
            { 
                if (i != j) {
                    real delta = A[i][j] *. x[j];
                    if (E/model.E_REL-E <= abs(delta)) { 
                        delta = A[i][j] * x[j];
                    }
                }
            } 
            sum = sum + delta;
            } 
            real num = b[i] - sum;
            next_x[i] = num / A[i][i];
        }
    x = next_x;
    assert_r eq(A);
    assert_r(bounded_diff(N, x));
}

Fig. 14. Jacobi Iterative Method.

then dynamically checks that the error in this multiplication could not have exceeded the statically set bound \( E \) given a relative error with a maximum percentage deviation from the correct value of model.E_REL (Line 19). The notation model.E_REL is a first-class reference to the state of the execution model at that point in the program. If the value of \( \text{abs}(A[i][j] \times x[j]) \) is less than \( E / \text{model.E_REL} \), then a relative error cannot exceed the magnitude of the absolute error bound \( E \) because the quantity \( A[i][j] \times x[j] \) is small. However, checking this property requires the use of a reliable multiplication, so we instead perform a relaxed multiplication and conservatively approximate this property by instead checking that \( \text{abs}(\text{delta}) \) is less than \( E / \text{model.E_REL} - E \). If it was possible for an error to exceed this bound, the algorithm repeats the multiplication reliably (Line 20).

6.3 Specification

The verified Jacobi implementation relies on Leto’s specification capabilities to establish self-stability and verify a convergence bound.

**Self-Stability.** To verify that this Jacobi implementation is self-stabilizing the developer uses the assert_r statement on Line 32 to assert eq(A), which denotes that \( A \) has the same value in both the original and relaxed execution.

This property therefore asserts that faults do not disturb the matrix of coefficients and therefore precludes any execution models that may disturb the contents of \( A \).
**Convergence Bound.** Jacobi also enjoys a bound on the additional number of iterations added to its execution given a fault. Specifically, \( \Delta_c = O(\log_T \left( \frac{1}{(N \times \text{EPS})^2} \right)) \) where \( \Delta_c \) is the number of additional iterations in the relaxed execution, \( N \) is the size of the \( x \) vector, \( \text{EPS} \) is the maximum perturbation in each element of the solution vector due to a fault in an iteration, and \( T \) is a value between 0 and 1 representing the magnitude of the non-diagonal elements of \( A \) relative to the magnitude of the diagonal elements of \( A \). We specify that the maximum perturbation is bounded by \( \text{EPS} \) with the `assert_r` on Line 33, which asserts that `bounded_diff(N, x)`. The property `bounded_diff` states that if an upset occurs during an execution of the outer loop, then the error in each element of \( x \) is bounded. For clarity of presentation, we elide the full specification of `bounded_diff`.

### 6.4 Verification Approach

To verify Jacobi, the developer needs to provide a set of loop invariants that structure the proof.

**Outer Loop.** The outermost loop on Line 4 has a single invariant specified by the developer: `!model.upset -> eq(x)`. Given our target execution model in Figure 13, this invariant therefore states that if a fault has yet to occur, then \( x \) is equivalent between both the original and relaxed executions. This invariant follows because in the absence of a fault, Jacobi is a deterministic computation for which any two executions (the original and relaxed execution) that start from the same state compute the same result. By default, Leto models the two executions as starting from the same state. Therefore, all variables initialized in Line 1 and Line 2 have the same values between the two executions.

**Middle Loop.** The middle loop on Line 8 has two developer-specified invariants. The first invariant states the behavior of the loop if a fault has yet to occur in the program. In this case, `next_x` is the same in both the original and relaxed executions (i.e., `eq(next_x)`).

The second invariant is a key step towards one main proof goal: `bounded_diff(N, next_x)`, which states that if no fault has occurred previously, then the error in each position of `next_x` is bounded if a fault occurs on this iteration. Because the next step of the algorithm directly assigns `next_x` to `x`, this must be an invariant of the middle loop.

**Inner Loop.** The second invariant of the middle loop is critical for verifying the innermost loop. The invariant `sig(sum)` verifies that if an error occurs on this inner loop iteration, then `sum<r>` is within \( E \) of `sum<o>`. Otherwise, `eq(sum)`. This property is true because if a fault did not occur on the previous calculation of \( x \) in the outer loops (as provided by the third invariant of the middle loop) then at most one fault may happen during the calculation of each `delta` that contributes to `sum` and the error in each `delta` is bounded by \( E \). Leto verifies that the unreliable multiplication in combination with the conservative check on Line 19 establishes this fact.

### 7 EVALUATION

We next present our results from using Leto to implement and verify several self-stabilizing and self-correcting algorithms.

#### 7.1 Benchmarks and Properties

Figure 15 presents for each benchmark (Column 1) the execution model we verified under (Column 2), and the number of lines of code it contains (Column 3).

**Jacobi Iterative Method.** We verify the Jacobi benchmark as presented in Section 6 under a multiplicative (SEU) error model (Figure 13).
| Benchmark | Execution Model    | LOC | Manual Annotations | Invariants Inferred |
|-----------|--------------------|-----|--------------------|--------------------|
| Jacobi    | Multiplicative SEU | 51  | 16                 | 30                 |
| SS-CG     | Additive SEU       | 167 | 22                 | 36                 |
| SS-SD     | Unbounded SEU      | 57  | 9                  | 0                  |
| SC-CC     | Switchable Rowhammer | 89  | 38                 | 42                 |

Fig. 15. Benchmark Verification Effort

| Benchmark | Time (s) | Memory Usage (kbytes) | Constraints Generated |
|-----------|----------|-----------------------|-----------------------|
| Jacobi    | 37.79    | 36132                 | 12473                 |
| SS-CG     | 4.24     | 37836                 | 11707                 |
| SS-SD     | 0.19     | 25440                 | 420                  |
| SC-CC     | 158.77   | 199428                | 4200                 |

Fig. 16. Benchmark Runtime Characteristics

**Self-Stabilizing Steepest Descent (SS-SD).** SS-SD is another iterative linear system of equations solver that employs a periodic, reliable correction step [Sao and Vuduc 2013] to repair the state of the program. We verify that a developer can correctly implement the correction step using instruction duplication (i.e., dual modular redundancy (DMR)) under an unbounded SEU execution model (Figure 1). We present the full benchmark in Appendix I.

**Self-Stabilizing Conjugate Gradient Descent (SS-CG).** SS-CG is an iterative linear system of equations solver that employs a periodic, reliable correction step to repair the program state in the presence of faults [Sao and Vuduc 2013]. We verify under an additive SEU error model (Figure 3) that errors are sufficiently small such that the algorithm does not diverge. We also verify that the correction step can be correctly implemented using instruction duplication. We present a full description of the SS-CG in Appendix H.

**Self-Correcting Connected-Components (SC-CC).** SC-CC is an iterative algorithm for computing the connected subgraphs in a graph where each iteration consists of a faulty initial computation step followed by a correction step [Sao et al. 2016]. We verify that each iteration computes the correct result under a Rowhammer [Kim et al. 2014] error model that allows for an unbounded number of faulty writes to storage. We specifically verify that the implementation detects and corrects all errors. We present the full benchmark in Appendix G.

### 7.2 Verification Effort

Figure 15 also presents the annotation burden Leto imposes on the programmer. For each benchmark, we present the number of manual annotations (Column 4) and the number of automatically inferred loop invariants (Column 5). Manual annotations include loop invariants, assertions, and function requirements. We consider each conjunct a separate annotation when counting inferred invariants and manual annotations.

**Results.** We significantly reduce the number of invariants we must provide using inference in all but one benchmark. In half of the cases we infer more invariants than we provide. We infer no invariants for SS-SD as Z3 very quickly runs out of memory on our machine and therefore we must disable inference on all loops in that benchmark. We believe that we could resolve this issue by monitoring the memory usage of the Z3 subprocess, killing the process if it consumes too much, and falling back on our weak inference algorithm.
**Runtime Characteristics.** Figure 16 presents also the runtime performance characteristics of the Leto C++ implementation. We ran our experiments on an Intel i5-5200U CPU clocked at 2.20GHz with 8 GB of RAM. For each benchmark we present the time it took to run in seconds (Column 2), the maximum memory usage in kilobytes (Column 3), and the number of constraints generated for use with Z3 (Column 4).

8 RELATED WORK

Analyzing Approximate Computation. Researchers have developed a number of programming systems that enable developers to reason about approximate computations: computations for which the underlying execution substrate (e.g., the programming system and/or hardware system) augments the behavior of the application to produce approximate results. For example, EnerJ [Sampson et al. 2011] and FlexJava [Park et al. 2015] enable developers to demonstrate non-interference between approximate computations and critical parts of the computation that should not be modified. Meola and Walker propose a sub-structural logic for reasoning about fault tolerant programs [Meola and Walker 2010]. Their logic enables the proof system to count the number of faults that have occurred and therefore reason about properties that may hold for one model but not another. In contrast to all of these approaches, Leto provides a more expressive and unconstrained logic that supports verifying complicated relational properties of the application.

The relaxed programming model [Carbin et al. 2012] enables developers to prove both safety and accuracy programs for relaxed computations by hand using a Coq library. In contrast, Leto automates many aspects of the proof. Further, Leto’s first-class execution models enable the developer to automatically weave in an execution model—which may impact many operations in the program—whereas the relaxed programming framework requires that operations be modelled individually by hand.

He et al. [He et al. 2016] leverage the symdiff framework [Lahiri et al. 2012a] to verify instances of approximate programs. However, it does not enable a developer to integrate an execution model without directly changing the application’s logic. Leto enables modular specification of execution models.

Relational Hoare Logic. Researchers have proposed a number of relational Hoare Logics and verification systems to support verifying relational properties of programs [Barthe et al. 2011; Benton 2004; Carbin et al. 2012; Lahiri et al. 2012b; Sousa and Dillig 2016]. The verification algorithms produced by Sousa and Dillig [Sousa and Dillig 2016] and Lahiri et al. [Lahiri et al. 2012b] demonstrate that it is possible to automatically compose proofs for relational verification. Leto’s verification system differs from that of CHL in that 1) the semantics of the two program executions are asymmetric and 2) Leto attempts to verify with a specific program composition strategy that matches the types of proofs that are seen in practice for approximate and unreliablely executed programs. Namely, although the semantics of the two executions of the program differ, their structure is typically identical and therefore assert and assumes can often be matched to enable maximum reuse of assumed properties of the reliable execution during the verification of the relaxed.

Type Systems for Self-Stabilization. Self-Stabilizing Java provides developers with a type system and analysis that enables a developer to prove that any corrupted state of the program exits the system in a finite amount of time. Leto’s logic (versus the information-flow type system of Self-Stabilizing Java) enables developers to specify the richer invariants that need to be true of emerging algorithms for self-stability. For example, instead of verifying that corrupted state leaves the system within bounded time, Leto enables a developer to verify that the corruption in the program’s state is small enough that the algorithm’s correction steps will work as designed.
**Fault Rate Analysis.** We have driven the design and implementation of Leto by the anticipated fault rates, abstract fault models, and resilience tools exported by the computer architecture, high-performance computing, and fault tolerance communities. Specifically, soft fault rates have led major organizations – such as Intel [Kurd et al. 2010; Mitra et al. 2005, 2006; Mukherjee et al. 2003], Google [Yim 2014], NASA [Johnston 2000], DOE [Snir et al. 2014], and DARPA [Amarasinghe et al. 2009] – to express concern over such faults. Leto is the first system – to our knowledge – to enable automated verification for these faults.

The assumption of instruction-level arithmetic errors is the most common model for building 1) application-specific fault analyses and mechanisms [Bronevetsky and de Supinski 2008; Hoemmen and Heroux 2011; Huang and Abraham 1984; Oboril et al. 2011; Roy-Chowdhury and Banerjee 1994, 1996; Sao et al. 2016; Sao and Vuduc 2013; Shantharam et al. 2012], 2) software-level fault tolerance analyses and mechanisms [Li et al. 2016; Reis et al. 2005; Santini et al. 2017; Wei and Pattabiraman 2012; Yim et al. 2011], 3) micro-architectural resilience analyses and mechanisms [Austin 1999; Lu 1982; Meixner et al. 2007], and 4) circuit-level resilience analyses/mechanisms [Bowman et al. 2009, 2011; Kelin et al. 2010; Lilja et al. 2013; Quinn et al. 2015a,b; Turowski et al. 2015].

Mitra et al. have found that combinational logic faults account for 11% of all soft errors [Mitra et al. 2005]. In addition, soft error rates, including combinational faults, are expected to increase as chips continue grow in the number of transistors [Mitra et al. 2005; Shivakumar et al. 2002]. These trends have inspired a variety of different contributions, including modeling the propagation of transient faults [Chen and Tahoori 2012; Omana et al. 2003], analyzing the rate of combinational soft faults [Buchner et al. 1997; Rao et al. 2007; Wang and Xie 2011; Zhang and Shanbhag 2006], analyzing the impact of combinational soft faults [Rajaraman et al. 2006], and correcting combinational logic faults [Mitra et al. 2006].

9 CONCLUSION

Emerging computational platforms are increasingly vulnerable to errors. Future computations designed to execute on these platforms must therefore be designed to be fault tolerant and naturally resilient to error. We present a verification system, Leto, that facilitates the verification of application-specific fault tolerance mechanisms under programmer-specified execution models. As these proofs frequently relate a faulty execution to a fault-free one, Leto provides assertions that enable the developer to specify and verify expressions that relate the semantics of both executions. First-class execution models permit developers to convey information about the class of faults they expect their computational platforms to experience. By giving developers tools to verify relational invariants under first-class execution models, we enable developers to verify the self-stability of their programs.

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Fig. 17. Left and Right Rules for Primitive Statements

A ADDITIONAL PROOF RULES
Figures 17, 18, and 19 expand on the proof rules we presented in Section 4.

A.1 Additional Left and Right Rules
Figure 17 presents the remainder of our left and right rules.

Read. The rules [READ-L] and [READ-R] give the semantics of reads from memory. The left rule [READ-L] mimics the behavior of [READ-L] with the primary differing being that it substitutes the value of a register, r, for a local variable x. The right rule, [READ-R], on the other hand more closely resembles [BINOP-R] in that it models the potentially unreliable execution of the read from memory.

Write. The rules [WRITE-L] and [WRITE-R] give the semantics of writes to memory. These rules are analogous to [READ-L] and [READ-R], except modified in their exact implementation to captures writes to memory.

Relational Assert. The rules [RELATIONAL-ASSERT-L] and [RELATIONAL-ASSERT-R] give the semantics of relational assertion statements. These statements are similar to my normal assert statements, but they take relational predicates as arguments rather than standard predicates. The logic requires that the condition of a relational assertion is verified in relaxed execution, but in reliable execution the logic treats these statements as no-ops.

A.2 Left and Right Rules for Control Flow
Figure 18 presents the left and right rules for control flow statements. With the exception of [WHILE-L], the rules adhere to the standard formalization as seen in traditional Hoare logic. The only distinction between these rules and their standard implementation is that they operate over relational predicates.
A.3 Additional Lockstep Rules

Figure 19 presents the remainder of our lockstep control flow and structure rules.

\textbf{Weakening.} The rule \texttt{[weak]} gives the standard semantics for weakening as found in the standard Hoare logic with the distinction that it operates on relational predicates.

\textbf{While.} The rule \texttt{[while]} gives the semantics of \texttt{while} statements. The rule is similar in design to the rule for \texttt{if} statements in that it must also consider cases in which the control flow of the two executions diverge. The rule first considers the case where the two executions proceed in lockstep by both executing an iteration of the loop. The next two cases leverage the left and right rules to consider the cases when:

- The relaxed execution halts, but the reliable execution executes an additional iteration.
- The reliable execution halts, but the relaxed execution executes an additional iteration, respectively.

\section{Proofs of Properties}

Leto’s program logic ensures two basic properties of Leto programs: \textit{preservation} and \textit{progress}. 

Lemma B.1 (Left Preservation).

If \( \tau_1 \{ P_r \} s \{ Q_r \} \) and \( (\epsilon_1, \epsilon_2) \vdash P_r \) and \( \langle s, \epsilon_1 \rangle \Downarrow \epsilon'_1 \), then \( (\epsilon'_1, \epsilon_2) \vdash Q_r \).

Leto’s left preservation property states that given a proof in the left rules, for a pair of environments satisfying the precondition of the proof, then if a reliable execution of \( s \) terminates, then the resulting environment pair satisfies the proof’s postcondition.

Proof. By induction on the lemma statement:

- **Assign-l.** Stepping \( \langle s, \epsilon_1 \rangle \) produces an environment \( \epsilon'_1 \) which differs from \( \epsilon_1 \) only in that \( r \) is mapped to \( n \) in \( \sigma \). As \( \epsilon_1 \models Q_r[n/\text{inj}_\mu(r)] \), \( \epsilon'_1 \) trivially satisfies \( Q_r^* \). Additionally, as the precondition contains no substitutions on \( \text{inj}_\mu(r) \), \( \epsilon_2 \) trivially satisfies \( Q_r^* \).
- **Binop-l.** Similar to Assign-l.
- **Read-l.** Similar to Assign-l.
- **Write-l.** Similar to Assign-l.
- **Assert-l.** Stepping \( \langle s, \epsilon_1 \rangle \) by definition adds \( r \) to \( \epsilon_1 \) to form \( \epsilon'_1 \). Therefore, \( (\epsilon'_1, \epsilon_2) \models \text{inj}_\mu(r) \).
- **Assume-l.** Similar to Assert-l.
- **Relational-Assert-l.** Relational assertions in left mode do nothing.
- **Seq-l.** We start with inversion on \( \tau_1 \{ P_r \} s \{ Q_r \} \), which yields \( \tau_1 \{ P'_r \} s_1 \{ R'_r \} \) and \( \tau_1 \{ R'_r \} s_2 \{ Q'_r \} \). Applying the induction hypothesis to \( \tau_1 \{ P'_r \} s_1 \{ R'_r \} \) yields \( (\epsilon'_1, \epsilon_2) \models R'_r \). Applying the induction hypothesis to \( \tau_1 \{ R'_r \} s_2 \{ Q'_r \} \) yields \( (\epsilon'_1, \epsilon_2) \models Q_r^* \).
- **While-l.** Stepping \( \langle s, \epsilon_1 \rangle \) by definition adds \( P_r^* \) to \( \epsilon_1 \) to form \( \epsilon'_1 \). Therefore, \( (\epsilon'_1, \epsilon_2) \models P_r^* \).
- **If-l.** First, we perform inversion on \( s \). Then, we destruct \( b \). In the case where \( b \) is true, we apply the induction hypothesis to \( \tau_1 \{ P_r^* \land b \} \) \( s \{ Q_r^* \} \). In the case where \( b \) is false, we apply the induction hypothesis to \( \tau_1 \{ P_r^* \land \neg b \} \) \( s \{ Q_r^* \} \).

Lemma B.2 (Right Preservation).

If \( \mu \vdash_r \{ P_r \} s \{ Q_r \} \) and \( (\epsilon_1, \epsilon_2) \models P_r \) and \( \langle s, \epsilon_2 \rangle \Downarrow \mu \epsilon'_2 \), then \( (\epsilon_1, \epsilon'_2) \models Q_r^* \).

Leto’s right preservation property states that given a proof in the right rules, for a pair of environments satisfying the precondition of the proof, then if execution of \( s \) under the execution model specification \( \mu \) terminates, then the resulting environment pair satisfies the proof’s postcondition.

Proof. By induction on the lemma statement:

- **Assign-r.** Stepping \( \langle s, \epsilon_2 \rangle \) produces an environment \( \epsilon'_2 \) which differs from \( \epsilon_2 \) only in that \( r \) is mapped to \( n \) in \( \sigma \). As \( \epsilon_1 \models Q_r[n/\text{inj}_\mu(r)] \), \( \epsilon'_1 \) trivially satisfies \( Q_r^* \). Additionally, as the precondition contains no substitutions on \( \text{inj}_\mu(r) \), \( \epsilon_1 \) trivially satisfies \( Q_r^* \).
- **Binop-r.** Stepping \( \langle s, \epsilon_2 \rangle \) produces an environment \( \epsilon'_2 \) which differs from \( \epsilon_2 \) only in that \( r \) is mapped to \( r' \) in \( \sigma \). The restrictions placed on \( r' \) in \( Q_r'^* \) codify the operator substitution routine from \( \text{f-Binop} \) in the operational semantics. Therefore, the runtime always sets \( r \) in a way such that the resulting environment satisfies \( Q_r^* \). Additionally, as the precondition contains no substitutions on \( \text{inj}_\mu(r) \), \( \epsilon_1 \) trivially satisfies \( Q_r^* \).
- **Read-r.** Similar to Binop-r.
- **Write-r.** Similar to Binop-r.
- **Assert-r.** Stepping \( \langle s, \epsilon_2 \rangle \) does not modify \( \epsilon_2 \). Therefore, \( \epsilon_2 = \epsilon'_2 \). Since the precondition and postcondition are identical, \( (\epsilon_1, \epsilon'_2) \models r \).
- **Assume-r.** Similar to Assert-r.
- **Relational-Assert-r.** Similar to Assert-r.

□
• **SEQ-R.** We start with inversion on \( \tau_r \{ P_r^* \} s \{ Q_r^* \} \), which yields \( \tau_r \{ P_r^* \} s_1 \{ R_r^* \} \) and \( \tau_r \{ R_r^* \} s_2 \{ Q_r^* \} \). Applying the induction hypothesis to \( \tau_r \{ P_r^* \} s_1 \{ R_r^* \} \) yields \( \langle \epsilon_1, \epsilon'_2 \rangle \models R_r^* \). Applying the induction hypothesis to \( \tau_r \{ R_r^* \} s_2 \{ Q_r^* \} \) yields \( \langle \epsilon_1, \epsilon'_2 \rangle \models Q_r^* \).

• **WHILE-R.** First, we perform inversion on \( s \). Then, we destruct \( b \). In the case where \( b \) is true, we apply the induction hypothesis to \( \tau_r \{ P_r^* \wedge b \} s \{ Q_r^* \} \). This proves that \( P_r^* \) holds after each loop iteration. Upon exiting the loop, \( \neg b \) trivially holds. In the case where \( b \) is false, the loop does not run and therefore \( \langle \epsilon_1, \epsilon'_2 \rangle \) trivially satisfies \( P_r^* \wedge \neg b \).

• **IF-R.** First, we perform inversion on \( s \). Then, we destruct \( b \). In the case where \( b \) is true, we apply the induction hypothesis to \( \tau_r \{ P_r^* \wedge b \} s \{ Q_r^* \} \). In the case where \( b \) is false, we apply the induction hypothesis to \( \tau_r \{ P_r^* \wedge \neg b \} s \{ Q_r^* \} \).

\[ \square \]

**Theorem B.3 (Preservation).**

*If \( \mu \models \{ P_r^* \} s \{ Q_r^* \} \) and \( \langle \epsilon_1, \epsilon_2 \rangle \models P_r^* \) and \( \langle s, \epsilon_1 \rangle \Downarrow \epsilon'_1 \) and \( \langle s, \epsilon_2 \rangle \Downarrow \mu \epsilon'_2 \), then \( \langle \epsilon_1, \epsilon'_2 \rangle \models Q_r^* \).*

Let’s preservation property states that given a proof in the program logic of a program \( s \), for all pairs of environments \( \langle \epsilon_1, \epsilon_2 \rangle \) that satisfy the proof’s precondition, if the executions of \( s \) under both the reliable semantics and the relaxed semantics terminate in a pair of environments \( \langle \epsilon'_1, \epsilon'_2 \rangle \), then this pair of environments satisfies the proof’s postcondition. Note that this states the partial correctness of the logic and does not establish termination (and therefore total correctness).

**Proof.** By induction on the theorem statement:

• **STAGE.** We first perform inversion on \( s \). Then, we apply Theorem B.1 to \( \tau_l \{ P_r^* \} s_1 \{ R_r^* \} \) and Theorem B.2 to \( \tau_r \{ R_r^* \} s_2 \{ Q_r^* \} \).

• **INVERSE-STAGE.** We first perform inversion on \( s \). Then, we apply Theorem B.2 to \( \tau_r \{ P_r^* \} s_2 \{ R_r^* \} \) and Theorem B.1 to \( \tau_l \{ R_r^* \} s_1 \{ Q_r^* \} \).

• **SPLIT.** We first perform inversion on \( s \), followed by inversion on \( \mu \models \{ P_r^* \} s \sim s \{ Q_r^* \} \). Then, we apply Theorem B.1 to \( \tau_l \{ P_r^* \} s \{ R_r^* \} \) and Theorem B.2 to \( \tau_r \{ R_r^* \} s \{ Q_r^* \} \).

• **SEQ.** We start with inversion on \( \mu \models \{ P_r^* \} s \{ Q_r^* \} \), which yields \( \mu \models \{ P_r^* \} s_1 \{ R_r^* \} \) and \( \mu \models \{ R_r^* \} s_2 \{ Q_r^* \} \). Applying the induction hypothesis to \( \mu \models \{ P_r^* \} s_1 \{ R_r^* \} \) yields \( \langle \epsilon'_1, \epsilon'_2 \rangle \models R_r^* \). Applying the induction hypothesis to \( \mu \models \{ R_r^* \} s_2 \{ Q_r^* \} \) yields \( \langle \epsilon'_1, \epsilon'_2 \rangle \models Q_r^* \).

• **WEAK.** We start with inversion on \( \mu \models \{ P_r^* \} s \{ Q_r^* \} \). Applying the induction hypothesis to \( \mu \models \{ P_r^* \} s \{ Q_r^* \} \) yields \( \langle \epsilon'_1, \epsilon'_2 \rangle \models Q_r^* \). Since \( Q_r^* \models Q_r^* \), \( \langle \epsilon'_1, \epsilon'_2 \rangle \models Q_r^* \).

• **IF.** We begin with inversion on \( s \). Then, we destruct \( b \). In the case where \( \neg inj_{o}(b) \land inj_j(b) \), we apply the induction hypothesis to \( \mu \models \{ P_r^* \wedge inj_{o}(b) \land inj_j(b) \} s_1 \{ Q_r^* \} \). In the case where \( \neg inj_{o}(b) \land \neg inj_j(b) \), we apply the induction hypothesis to \( \mu \models \{ P_r^* \wedge \neg inj_{o}(b) \land \neg inj_j(b) \} s_2 \sim s_1 \{ Q_r^* \} \).

In the case where \( inj_{o}(b) \land \neg inj_j(b) \), we apply the induction hypothesis to \( \mu \models \{ P_r^* \wedge inj_{o}(b) \land \neg inj_j(b) \} s_1 \sim s_2 \{ Q_r^* \} \).

In the case where \( \neg inj_{o}(b) \land \neg inj_j(b) \), we apply the induction hypothesis to \( \mu \models \{ P_r^* \wedge \neg inj_{o}(b) \land \neg inj_j(b) \} s_2 \{ Q_r^* \} \).

• **WHILE.** First, we perform inversion on \( s \). Then, we destruct \( b \). In the case where \( inj_j(b) \land inj_j(b) \), we apply the induction hypothesis to \( \mu \models \{ R_r^* \wedge inj_{o}(b) \land \neg inj_j(b) \} s \{ R_r^* \} \). This proves that \( R_r^* \) holds after each loop iteration. Upon exiting the loop, \( \neg inj_{o}(b) \land \neg inj_j(b) \) trivially holds. The cases where \( \neg inj_{o}(b) \land \neg inj_j(b) \) and \( inj_j(b) \land \neg inj_j(b) \) are similar to the previous case. In the case where \( \neg inj_{o}(b) \land \neg inj_j(b) \), the loop does not run in either execution and therefore \( \langle \epsilon'_1, \epsilon'_2 \rangle \) trivially satisfies the postcondition.
Lemma B.4 (Right Progress).
If $\mu \vdash \{ P^r_r \} s \{ Q^r_r \}$ and $(\varepsilon_1, \varepsilon_2) \models P^r_r$ and $(s, \varepsilon_2) \not\models \varepsilon'_2$, then $\neg \text{failed}(\varepsilon'_2)$ where $\text{failed}(\langle \text{fail}, \varepsilon \rangle) = \text{true}$.

Leto’s right progress property states that given a proof in the right rules of a program $s$, for all pairs of environments $(\varepsilon_1, \varepsilon_2)$ that satisfy the proof’s precondition, then if the relaxed execution of $s$ under $\mu$ terminates, then it does not terminate in an error. The right progress property establishes that a Leto’s program of right rules satisfies all of its assert and assume statements.

Proof. By induction on the theorem statement:
- ASSIGN-R. Assignment cannot fail.
- BINOP-R. We start with inversion on $s$, giving us the fact that the dynamic semantics is always able to perform an operator substitution $(\exists([x_1, x_2], X, P^w_\mu, P^e_e) \in \mu(\oplus) \cdot \text{inj}_r(P^w_\mu[r_1/x_1][r_2/x_2]), so binary operations cannot fail.
- READ-R. We start with inversion on $s$, giving us the fact that the dynamic semantics is always able to perform an operator substitution $(\exists([x_{\text{mem}}], X, P^w_\mu, P^e_e) \in \mu(\oplus) \cdot \text{inj}_r(P^w_\mu[x/x_{\text{mem}}]), so reads cannot fail.
- WRITE-R. We start with inversion on $s$, giving us the fact that the dynamic semantics is always able to perform an operator substitution $(\exists([x_1, x_2], X, P^w_\mu, P^e_e) \in \mu(\oplus) \cdot \text{inj}_r(P^w_\mu[x/x_{\text{mem}}]), so writes cannot fail.
- ASSERT-R. Stepping $(s, \varepsilon_2)$ produces the environment $\varepsilon'_2$ where $\varepsilon_2 = \varepsilon'_2$. Since $\neg \text{failed}(\varepsilon_2)$, $\neg \text{failed}(\varepsilon'_2)$ trivially holds.
- ASSUME-R. Similar to ASSERT-R.
- RELATIONAL-ASSERT-R. Similar to ASSERT-R.
- SEQ-R. We start with inversion on $\tau_r \{ P^r_r \} s \{ Q^r_r \}$, which yields $\tau_r \{ P^r_r \} s_1 \{ R^r_r \}$ and $\tau_r \{ R^r_r \} s_2 \{ Q^r_r \}$. Applying Theorem B.2 to $\tau_r \{ P^r_r \} s_1 \{ R^r_r \}$ yields $(\varepsilon_1, \varepsilon'_2) \models R^r_r$. Applying the induction hypothesis to $\tau_r \{ P^r_r \} s_1 \{ R^r_r \}$ provides $\neg \text{failed}(\varepsilon'_2)$. Applying the induction hypothesis to $\tau_r \{ R^r_r \} s_2 \{ Q^r_r \}$ yields $\neg \text{failed}(\varepsilon'_2)$.
- WHILE-R. First, we perform inversion on $s$. Then, we destruct $b$. In the case where $b$ is true, we apply the induction hypothesis to $\tau_r \{ P^r_r \land b \} s \{ P^r_r \}$.
In the case where $b$ is false, the loop does not run and therefore $(\varepsilon_2 = \varepsilon'_2)$ trivially satisfies $\neg \text{failed}(\varepsilon_2)$.
Lastly, since $(\varepsilon_1, \varepsilon_2) \models P^r_r$, the invariant check before the loop cannot fail.
- IF-R. First, we perform inversion on $s$. Then, we destruct $b$. In the case where $b$ is true, we apply the induction hypothesis to $\tau_r \{ P^r_r \land b \} s \{ Q^r_r \}$. In the case where $b$ is false, we apply the induction hypothesis to $\tau_r \{ P^r_r \land \neg b \} s \{ Q^r_r \}$.

Theorem B.5 (Progress).
If $\mu \vdash \{ P^r_r \} s \{ Q^r_r \}$ and $(\varepsilon_1, \varepsilon_2) \models P^r_r$ and $(s, \varepsilon_1) \not\models \varepsilon'_1$ and $(s, \varepsilon_2) \not\models \varepsilon'_2$, then $\neg \text{failed}(\varepsilon'_2)$ where $\text{failed}(\langle \text{fail}, \varepsilon \rangle) = \text{true}$.

Leto’s progress property states that given a proof in the program logic of a program $s$, for all pairs of environments $(\varepsilon_1, \varepsilon_2)$ that satisfy the proof’s precondition, if the reliable execution of $s$ terminates successfully, then if the relaxed execution of $s$ under $\mu$ terminates, then it does not terminate in an error. The progress property establishes that a Leto’s program satisfies all of its assert and assume statements – provided that all reliable executions of the program also satisfy the program’s assert and assume statements.
Proof. By induction on the theorem statement:

- **Stage.** We first perform inversion on \( s \). Then, we apply Theorem B.1 to \( \vdash_I \{ P^r \} s_1 \{ R^r \} \) and Theorem B.4 to \( \vdash_r \{ R^r \} s_2 \{ Q^r \} \).

- **Inverse-stage.** We first perform inversion on \( s \). Then, we apply Theorem B.4 to \( \vdash_r \{ P^r \} s_2 \{ R^r \} \) and Theorem B.1 to \( \vdash_I \{ R^r \} s_1 \{ Q^r \} \).

- **Split.** We first perform inversion on \( s \), followed by inversion on \( \mu \vdash \{ P^r \} s \sim s \{ Q^r \} \). Then, we apply Theorem B.1 to \( \vdash_I \{ P^r \} s_1 \{ R^r \} \). Finally, we apply Theorem B.4 to \( \vdash_r \{ R^r \} s_2 \{ Q^r \} \).

- **Seq.** We start with inversion on \( \mu \vdash \{ P^r \} s \{ Q^r \} \), which yields \( \mu \vdash \{ P^r \} s_1 \{ R^r \} \) and \( \mu \vdash \{ R^r \} s_2 \{ Q^r \} \). Applying the induction hypothesis to \( \mu \vdash \{ P^r \} s_1 \{ R^r \} \) yields \( \neg \text{failed}(\varepsilon'_s) \). Applying the induction hypothesis to \( \mu \vdash \{ R^r \} s_2 \{ Q^r \} \) yields \( \neg \text{failed}(\varepsilon'_s) \).

- **Weak.** We start with inversion on \( \mu \vdash \{ P^r \} s \{ Q^r \} \). Applying the induction hypothesis to \( \mu \vdash \{ P^r \} s \{ Q^r \} \) yields \( \neg \text{failed}(\varepsilon'_s) \). Since \( P^r \models P^r \) and \( Q^r \models Q^r \), \( \neg \text{failed}(\varepsilon'_s) \) holds.

- **If.** We begin with inversion on \( s \). Then, we destruct \( b \). In the case where \( \text{inj}_o(b) \land \text{inj}_r(b) \), we apply the induction hypothesis to \( \mu \vdash \{ P^r \land \text{inj}_o(b) \land \text{inj}_r(b) \} s_1 \{ Q^r \} \). In the case where \( \neg \text{inj}_o(b) \land \neg \text{inj}_r(b) \), we apply the induction hypothesis to \( \mu \vdash \{ P^r \land \neg \text{inj}_o(b) \land \neg \text{inj}_r(b) \} s_2 \sim_r s_1 \{ Q^r \} \).

In the case where \( \text{inj}_o(b) \land \neg \text{inj}_r(b) \), we apply the induction hypothesis to \( \mu \vdash \{ P^r \land \text{inj}_o(b) \land \neg \text{inj}_r(b) \} s_1 \sim_r s_2 \{ Q^r \} \).

In the case where \( \neg \text{inj}_o(b) \land \neg \text{inj}_r(b) \), we apply the induction hypothesis to \( \mu \vdash \{ P^r \land \neg \text{inj}_o(b) \land \neg \text{inj}_r(b) \} s_2 \{ Q^r \} \).

- **While.** First, we perform inversion on \( s \). Then, we destruct \( b \). In the case where \( \text{inj}_o(b) \land \text{inj}_r(b) \), we apply the induction hypothesis to \( \mu \vdash \{ R^r \land \text{inj}_o(b) \land \text{inj}_r(b) \} s \{ R^r \} \). In the case where \( \text{inj}_o(b) \land \neg \text{inj}_r(b) \), taking a step from \( \vdash_I \{ R^r \land \text{inj}_o(b) \land \neg \text{inj}_r(b) \} s \{ R^r \} \) yields \( \varepsilon'_o \) where \( \varepsilon'_o = \varepsilon_o \). Since \( \neg \text{failed}(\varepsilon_o) \), \( \neg \text{failed}(\varepsilon'_o) \) holds as well.

In the \( \neg \text{inj}_o(b) \land \text{inj}_r(b) \) case, we apply Theorem B.4 to \( \mu \vdash_r \{ R^r \land \neg \text{inj}_o(b) \land \text{inj}_r(b) \} s \{ R^r \} \).

In the case where \( \neg \text{inj}_o(b) \land \neg \text{inj}_r(b) \), the loop doesn’t run and therefore \( \varepsilon = \varepsilon'_o \), so \( \neg \text{failed}(\varepsilon'_o) \) trivially holds.

Lastly, since \( (\varepsilon_1, \varepsilon_2) \models R^r \), the invariant check before the loop cannot fail.

\[ \square \]

C DYNAMIC SEMANTICS

Figure 20 presents an abbreviated dynamic semantics of Leto’s language. We present an extended semantics in Appendix D. Leto’s semantics models an abstract machine that includes a frame, a heap, and an execution model state. Leto allocates memory for program variables (both scalar and array) in the heap. A frame serves two roles: 1) a frame maps a program variable to the address of the memory region allocated for that variable in the heap, and 2) a frame maps a register to its current value in the program. The model state stores the values for state variables within the execution model.
C.2 Execution Model Semantics

Figure 20 provides an abbreviated presentation of the execution model relation \( \langle m, op, (\text{args}) \rangle \models_\mu \langle n, m' \rangle \). The relation states that given the arguments, args to an operation op, evaluation of the
operation from the model state \( m \) yields a result \( n \) and a new model state \( m' \) under the execution model specification \( \mu \).

\[
\begin{align*}
\mu(\oplus, [x_1, x_2], X, P_w, P_e) & \quad \forall x_m \in X \cdot \text{fresh}(x'_m) \quad m[x_1 \mapsto n_1][x_2 \mapsto n_2] \models P_w \\
& \quad m'[x_1 \mapsto n_1][x_2 \mapsto n_2][\forall x_m \in X \cdot x_m \mapsto x'_m][\text{result} \mapsto n_3] \models P_e \\
& \quad \text{dom}(m) = \text{dom}(m')
\end{align*}
\]

The \([F\text{-BINOP}]\) rule specifies the meaning of this relation for binary operations. This relation states that the value of an operation \( \oplus \) given a tuple of input values \((n_1, n_2)\) and an execution model state \( m \) evaluates to value \( n_3 \) and a new model state \( m' \). The rule relies on the relation \( \mu(op, vlist, X, P_w, P_e) \) which specifies the list of argument names, \( vlist \), the set of modified variables \( X \), the \textit{precondition} \( P_w \), and the \textit{postcondition} \( P_e \) for the operation \( op \) in the developer-provided execution model. The set of modified variables is the union of the \texttt{modified} clauses in the operation’s specification. The precondition of an operation is the conjunction of the \texttt{when} clauses in the operation’s specification. The postcondition of an operation is the conjunction of the \texttt{ensures} clauses in the operation’s specification.

The semantics of the model relation non-deterministically selects an operation specification, result value, and output model state subject to the constraint that: 1) the current model state satisfies the precondition (after the inputs to the operation have been appropriately assigned into the model state), 2) the output model state satisfies the postcondition (after the inputs, modified variables, and result value have been appropriately assigned into the model state), and 3) the domains of the input and output state are the same.

Because of the uniformity of the execution model specification, the semantics for other operations (e.g., reads and writes) is similar with the sole distinction being the number of arguments passed to the operation. For clarity of presentation, we elide the presentation of rules for those operations, but we provide them in Appendix D, Figure 22.

### C.3 Language Semantics

Figure 20 presents the non-deterministic small-step transition relation \( \langle s, \epsilon \rangle \xrightarrow{\mu} \langle s', \epsilon' \rangle \) of a Leto program. The relation states that execution of statement \( s \) from the environment \( \epsilon \) takes one step yielding the statement \( s' \) and environment \( \epsilon' \) under the execution model specification \( \mu \). The semantics of the statements are largely similar to that of traditional approaches except for the ability to the statements to encounter faults. Broadly, we categorize Leto’s instructions into four categories: \textit{register instructions}, \textit{memory instructions}, \textit{assertions}, and \textit{control flow}.

**Register Instructions.** The rules \([\text{ASSIGN}]\) and \([\text{BINOP}]\) specify the semantics of two of Leto’s register manipulation instructions. \([\text{ASSIGN}]\) defines the semantics of assigning an integer value to a register, \( r = n \). This has the expected semantics updating the value of \( r \) within the current frame with the value \( n \). Of note is that register assignment executes fully reliably without faults.

\([\text{BINOP}]\) specifies the semantics of a register only binary operation, \( r = r_1 \oplus r_2 \). Note that reads of the input registers execute fully reliably. The result of the operation is \( n_3 \), which is the value of the operation given the semantics of that operation’s execution model when executed from the model state \( m \) on parameters \( n_1 \) and \( n_2 \). Executing the execution model may change the values of the execution model’s state variables. Therefore, the instruction evaluates to an instruction that assigns \( n_3 \) to the destination register and evaluates with an environment that consists of the unmodified frame, the unmodified heap, and the modified execution model state. Note that by virtue of the fact that both the frame and heap are unmodified, faults in register instructions cannot modify the contents or organization of memory. This modeling choice is consistent with standard fault modeling approaches.
**Memory Instructions.** The rules [READ] and [WRITE] specify the semantics of two of Leto’s memory manipulation instructions. [READ] defines the semantics of reading the value of a program variable \( x \) from its corresponding memory location: \( r = x \). The rule fetches the program variable’s memory address from the frame, reads the value of the memory location \( n = h(a) \) and the region the memory location belongs to \( q = \theta(a) \) and then executes the execution model with the program variable’s current value in memory and the memory region it resides in as parameters. The execution model non-deterministically yields a result \( n' \) that the rule uses to complete its implementing by issuing an assignment to the register.

[WRITE] defines the semantics of writing the value of a register to memory. The rule reads the value of the memory location to record the old value of the memory location, reads the value of the input register, fetches the region the memory location corresponds to, and then executes the execution model with these values as parameters. The execution model yields a new value \( n_r \) that the rule then assigns to the value the program variable.

**Assertions.** The rules [ASSERT-T] and [ASSUME-T] specify the semantics of assert and assume statements, respectively. These statements have standard semantics, yielding a skip and continuing the execution of the program if their conditions are satisfied. For either of these statements, if their conditions evaluate to false, then execution yields fail denoting that execution has failed and become stuck in error.

**Control Flow.** The rules for control flow ([IF-T], [IF-F], [SEQ1], [SEQ2], [WHILE-F], and [WHILE-T]) have standard semantics. An important note is that the semantics of these statements is such that the transfer of control from one instruction to another always executes reliably and, therefore, faults do not introduce control flow errors into the program. This modeling assumption is consistent with standard fault injection and reliability analysis models [Vishal Chandra Sharma 2016].

### D FULL SEMANTICS
In this appendix we present the full dynamic semantics of the Leto language. We have elided the rules presented in Figure 20 as they remain unchanged. A preprocessing pass performs the following actions:

- It places all variables without a label into a reliable memory region.
- It flattens multidimensional vectors into single dimensional vectors.
- It inlines function calls.

The alloc function we use in DECLARE and DECLARE-ARRAY takes a mapping from variables to addresses \( \sigma \) and an integer \( n \) and returns the first addresses in a contiguous block of \( n \) unmapped addresses in \( \sigma \).

Figures 21 and 22 expand on the operational semantics we present in Figure 20.

### E VERIFICATION ALGORITHM
Figures 23, and 24 present the core of Leto’s verification algorithm. The algorithm performs forward symbolic execution to discharge verification conditions generated by assert, assert_t, invariant, and invariant_r statements in the program. The algorithm directly implements the Hoare-style relational program logic from Section 4.

**Preliminaries.** We denote Leto’s verification algorithm by the function \( \Psi \), which takes as input a statement \( s \), a logical predicate \( \sigma \), a model specification \( m \), and a verification mode \( c \). The statement \( s \) is the statement to be verified, \( \sigma \) is the symbolic context under which the verification algorithm is invoked, \( m \) computes the symbolic representation for a operation in the execution model.
The control value $c \in C = \{\text{lock}, \text{left}, \text{right}\}$ determines whether or not the algorithm is performing verification in lockstep mode, left mode, or right mode, respectively. When performing verification in lockstep, the algorithm models the original and relaxed execution as each executing an instruction one at a time. In this mode, the algorithm is able to demonstrate an easy correspondence between the two executions that therefore enables the algorithm to, for example, transfer assumed properties of the original execution over to verify the relaxed execution. For both left mode and right mode, the algorithm assumes the two executions have diverged and, therefore, that there is no simple correspondence between the two executions. In left mode, the algorithm symbolically evaluates the original execution of the program, ignoring the verification conditions of the required for the relaxed execution. In right mode, the algorithm symbolically evaluates the relaxed execution of the program and checks the verification conditions that are required of the relaxed execution.

The function $vexp(e, m, c)$ maps a standard unary expression $e$ to a constraint that represents the resulting value in either the original or relaxed execution. For example, $vexp$ maps a variable reference $x$ to either the variable reference $x<\sigma>$ or $x<\tau>$ if $c$ equals left or right, respectively.
The function returns a constraint because the expression may reference an operation suffixed with a period, denoting that the operation has a custom semantics. The constraint characterizes the non-deterministic choice of the operation’s implementation. The function uses the model specification $m$ to compute the symbolic representation for these operations.

The function $\text{vbexp}(b, m, c)$ maps a standard unary boolean expression $b$ to a constraint. Its operation is similar to that of $\text{vexp}$.

**Assignment.** For an assignment statement $x = e$, the algorithm maps the $e$ to an appropriate relational expression for both the original and relaxed execution by creating the constraint that $x$ in the original (relaxed) execution has the value $e$. The algorithm then uses the $\text{join}()$ function to return a result. The $\text{join}()$ function joins two constraints into a conjunction depending on the value of $c$. If $c = \text{lock}$ – denoting that the algorithm is modelling the lockstep execution of both the original and relaxed executions – then the $\text{join}$ includes both constraints. If $c = \text{left}$ or $c = \text{right}$ – denoting that the algorithm is modeling the original or relaxed execution, respectively – then $\text{join}$ includes only the first or second constraint, respectively.

**Assume and Assert.** The algorithm verifies both $\text{assert}$ and $\text{assume}$ using the same logical approach. The algorithm first generates the verification conditions for both the original and relaxed executions, namely that the statement’s boolean expression $b_v$ is true ($\sigma_o$ and $\sigma_r$, respectively). The algorithm next considers two cases. In lockstep mode, the algorithm verifies that the current context $\sigma$ extended with the $\text{assumption}$ that the assertion or assumption is true in the original execution implies that the verification condition holds. The function $\text{Verify}(\sigma_1, \sigma_2)$ verifies that $\sigma_1$ implies $\sigma_2$ (Leto specifically uses an SMT solver to do so) and halts the execution of the algorithm if the implication does not hold or the solver is unable to demonstrate that it holds. In right mode, the algorithm directly verifies that the current context implies the verification condition. The insight is that unlike in lockstep mode, the algorithm must verify the relaxed execution independently of the original execution and, therefore, the algorithm cannot leverage the assumption that the assertion or assumption is valid. In the last step, the algorithm returns the join of the two verification conditions.
Fig. 23. Verification Algorithm (sans Control Flow)

Relational Assert. The algorithm verifies relational assertions under the current context. If verification fails, then the verification procedure halts. If verification succeeds, then the algorithm appends the assertion to the context and returns the result.

If. Figure 24 presents the algorithm’s implementation for if statement verification. The algorithm has a different implementation for each of the verification modes:

- **Lockstep.** In lockstep mode, the algorithm verifies and generates a symbolic representation for four different scenarios: the case when 1) the original execution and relaxed execution both take the true branch of the statement, represented by $\sigma_1$, 2) the original execution and relaxed execution both take the false branch of the statement, $\sigma_2$, 3) the original execution takes the true branch and the relaxed execution takes the false branch, $\sigma_4$, and 4) the original execution takes the false branch and the relaxed execution takes the true branch, $\sigma_6$.

- **Left.** In left mode, the algorithm need only generate a symbolic representation for the original execution. The algorithm achieves this by conjoining the results of recursive calls to $\Psi$ on $s_1$ and $s_2$ given the current context.

- **Right.** In right mode, the algorithm need only generate a symbolic representation and discharge the verification conditions for the relaxed execution. Similar to that of left mode, algorithm achieves this by conjoining the results of recursive calls to $\Psi$ on $s_1$ and $s_2$.

For performance, Leto only considers scenarios that are potentially viable. The Check($\sigma_1$, $\sigma_2$) function returns true if a satisfying assignment for $\sigma_1 \rightarrow \sigma_2$ may exist and false otherwise. If Check is able to prove that such an implication cannot exist, then Leto does not recurse on that execution scenario.
function $\Psi(\text{if } (b) \{s_1\} \text{ else } \{s_2\}, \sigma, m, c) =
\begin{align*}
\sigma_o & \leftarrow \text{vbexp}(b, m, \text{left}), \quad \sigma_r \leftarrow \text{vbexp}(b, m, \text{right}) \\
\text{if } (c == \text{lock}) \text{ then} & \\
\quad \text{if Check}(\sigma, \sigma_o \land \sigma_r) \text{ then} & \\
\qquad \sigma_1 & \leftarrow \Psi(s_1, \sigma \land \sigma_o \land \sigma_r, \text{lock}) \\
\quad \text{else } \sigma_1 & \leftarrow \sigma \\
\text{end if} & \\
\text{else } & \\
\quad \text{if Check}(\sigma, \neg\sigma_o \land \neg\sigma_r) \text{ then} & \\
\qquad \sigma_2 & \leftarrow \Psi(s_1, \sigma \land \neg\sigma_o \land \neg\sigma_r, \text{lock}) \\
\quad \text{else } \sigma_2 & \leftarrow \sigma \\
\text{end if} & \\
\text{end if} & \\
\text{if Check}(\sigma, \sigma_o \land \neg\sigma_r) \text{ then} & \\
\quad \sigma_3 & \leftarrow \Psi(s_1, \sigma \land \sigma_o \land \neg\sigma_r, \text{left}) \\
\quad \sigma_4 & \leftarrow \Psi(s_2, \sigma \land \sigma_3 \land \sigma_o \land \neg\sigma_r, \text{right}) \\
\text{else } \sigma_4 & \leftarrow \sigma \\
\text{end if} & \\
\text{else if } (c == \text{left}) \text{ then} & \\
\quad \text{if Check}(\sigma, \sigma_o) \text{ then } \sigma_1 & \leftarrow \Psi(s_1, \sigma \land \sigma_o, \text{left}) \\
\quad \text{else } \sigma_1 & \leftarrow \sigma \\
\text{end if} & \\
\text{else if } (c == \text{right}) \text{ then} & \\
\quad \text{if Check}(\sigma, \sigma_r) \text{ then } \sigma_1 & \leftarrow \Psi(s_1, \sigma \land \sigma_r, \text{right}) \\
\quad \text{else } \sigma_1 & \leftarrow \sigma \\
\text{end if} & \\
\text{end if} & \\
\text{return } (\sigma_1 \land \sigma_2 \land \sigma_4 \land \sigma_6) & \\
\text{else if } (c == \text{lock}) \text{ then} & \\
\quad \text{if Check}(\sigma, \sigma_o) \text{ then } \sigma_1 & \leftarrow \Psi(s_1, \sigma \land \sigma_o, \text{left}) \\
\quad \text{else } \sigma_1 & \leftarrow \sigma \\
\text{end if} & \\
\text{else if } (c == \text{right}) \text{ then} & \\
\quad \text{if Check}(\sigma, \sigma_r) \text{ then } \sigma_1 & \leftarrow \Psi(s_1, \sigma \land \sigma_r, \text{right}) \\
\quad \text{else } \sigma_1 & \leftarrow \sigma \\
\text{end if} & \\
\text{end if} & \\
\text{return } (\sigma_1 \land \sigma_2) & \\
\text{end function} &
\end{align*}

Fig. 24. If statement verification algorithm
and present Leto's loop invariant inference algorithm. Leto uses Hou dini-style loop invariant inference to reduce the annotation burden on the program-

the function Check is identical to the Verify function except that:

Preliminaries. We denote a modified version of Leto’s verification algorithm by the function $\Psi'$, which is identical to $\Psi$ except that it ignores calls to the Verify function.

The value $r \in R = \{ \text{sat}, \text{unsat}, \text{unknown} \}$ represents the response from the SMT solver Leto uses. $\text{sat}$ indicates that a satisfying assignment exists for all variables in the predicate. $\text{unsat}$ indicates that no satisfying assignment exists for all variables in the predicate. $\text{unknown}$ indicates that the SMT solver cannot determine whether a satisfying assignment exists.

The function Check is identical to the Verify function except that:
1: function WEAKINF(while (b) (b_r) (b_r) {s_b}, σ, m, c, b_{cv}, b_{cr}, b_{pv}, b_{pr}) =
2:     match b_{cv} do
3:         b_h :: b'_{cv}:
4:             if (c == lock) then
5:                 p_o ← vbexp(b_{u}, m, left)) :: vbexp(b_{h}, m, left)
6:                 p_r ← vbexp(b_{u}, m, right)) :: vbexp(b_{h}, m, right)
7:                 (r_1, b_{f u 1}, b_{fr 1}) ← Check(σ :: p_o, p_r)
8:                 σ_c ← σ :: p_o :: p_r :: vbexp(b_{pv}, m, left)) :: vbexp(b_{pr}, m, right)
9:                 (r_2, b_{f u 2}, b_{fr 2}) ← Check(Ψ'(s_b, σ_c :: σ_o :: σ_r, m, lock), p_o :: p_r)
10:                (r_3, b_{f u 3}, b_{fr 3}) ← Check(Ψ'(s_b, σ_c :: σ_o :: σ_r, m, right), p_r)
11:            if (r_1 == unsat ∧ r_2 == unsat ∧ r_3 == unsat) then
12:                return WEAKINF(while (b) (b_{h} :: b_{u}) (b_{r}) {s_b}, σ, m, c, b'_{cv}, b_{cr}, b_{pv}, b_{pr})
13:            else return WEAKINF(while (b) (b_{u}) (b_{r}) {s_b}, σ, m, c, b_{cv}, b_{cr}, b_{pv}, b_{pr})
14:         end if
15:     else if (c == right) then
16:         p ← vbexp(b_{u}, m, right)) :: vbexp(b_{h}, m, right)) :: vbexp(b_{pv}, m, right)
17:         (r_1, b_{f u 1}, b_{fr 1}) ← Check(σ :: p, p)
18:         (r_2, b_{f u 2}, b_{fr 2}) ← Check(Ψ'(s_b, σ :: p :: σ_r :: p_{pv}, m, right), p)
19:            if (r_1 == unsat ∧ r_2 == unsat) then
20:                return WEAKINF(while (b) (b_{h} :: b_{u}) (b_{r}) {s_b}, σ, m, c, b'_{cv}, b_{cr}, b_{pv}, b_{pr})
21:            else return WEAKINF(while (b) (b_{u}) (b_{r}) {s_b}, σ, m, c, b_{cv}, b_{cr}, b_{pv}, b_{pr})
22:         end if
23:     end if
24: []:
25: match b_{cr} do
26:     []: return (b_{u}, b_{r})
27:         b_h :: b'_{cr}:
28:             if (c == lock) then
29:                 p_o ← vbexp(b_{u}, m, left)) :: vbexp(b_{u}, m, right)
30:                 (r_1, b_{f u 1}, b_{fr 1}) ← Check(σ :: p_o, b_{h} :: p_r)
31:                 σ_c ← σ :: p_o :: p_r :: b_{h} :: vbexp(b_{pv}, m, left)) :: vbexp(b_{pr}, m, right)
32:                 (r_2, b_{f u 2}, b_{fr 2}) ← Check(Ψ'(s_b, σ_c :: σ_o :: σ_r, m, lock), p_o :: p_r :: b_{h} :: b_{h})
33:                (r_3, b_{f u 3}, b_{fr 3}) ← Check(Ψ'(s_b, σ_c :: σ_o :: σ_r, m, right), p_r :: b_{h} :: b_{h})
34:            if (r_1 == unsat ∧ r_2 == unsat ∧ r_3 == unsat) then
35:                return WEAKINF(while (b) (b_{u}) (b_{r}) {s_b}, σ, m, c, b_{cv}, b'_{cr}, b_{pv}, b_{pr})
36:            else return WEAKINF(while (b) (b_{u}) (b_{r}) {s_b}, σ, m, c, b_{cv}, b_{cr}, b_{pv}, b_{pr})
37:         end if
38:     else if (c == right) then
39:         p ← vbexp(b_{u}, m, right)) :: vbexp(b_{u}, m, right)
40:         (r_1, b_{f u 1}, b_{fr 1}) ← Check(σ :: p :: b_{h} :: b_{h})
41:         (r_2, b_{f u 2}, b_{fr 2}) ← Check(Ψ'(s_b, σ :: p :: b_{r} :: b_{h} :: σ_r :: p_{pv}, m, right), p :: b_{r} :: b_{h})
42:            if (r_1 == unsat ∧ r_2 == unsat) then
43:                return WEAKINF(while (b) (b_{u}) (b_{r}) {s_b}, σ, m, c, b_{cv}, b_{cr}, b_{pv}, b_{pr})
44:            else return WEAKINF(while (b) (b_{u}) (b_{r}) {s_b}, σ, m, c, b_{cv}, b_{cr}, b_{pv}, b_{pr})
45:         end if
46:     end if
47: end match
48: end match
49: end function

Fig. 26. Weak Inference Algorithm
• Check(\(\sigma_1, \sigma_2\)) does not halt execution if \(\sigma_1\) does not imply \(\sigma_2\).
• Check returns a tuple consisting of:
  – An SMT result \(r \in R\).
  – A set of false conjuncts from the loop invariant \(b_v\).
  – A set of false conjuncts from the relational invariant \(b_r\).

**Strong Inference.** Figure 25 presents the strong inference algorithm. Before checking a loop, Leto assembles the candidate invariants

\[
b_v \equiv p_v \\
\]

\[
b_r \equiv p_r \land \left( \bigwedge_{x \in \text{vars}} x_r = x_o \right)
\]

where \(p_v\) is the invariant for the immediate parent loop or function, \(p_r\) is the relational invariant for the immediate parent loop or function, and \(\text{vars}\) is the set of program variables currently in scope.

Leto replaces the programmer provided invariants in the loop with \(b_v\) and \(b_r\) and invokes the \(\text{INF}\) function. It also provides the \(\text{INF}\) function with the programmer provided invariant as \(b_{p_v}\) and the programmer provided relational invariant as \(b_{p_r}\). Leto uses these invariants as assumptions during the inference process. The algorithm has a different behavior for each of the verification modes:

• **Lockstep.** The beginning of the lockstep algorithm (Lines 4 through 8) is similar to the lockstep case for while loop verification, but we’ve replaced all invocations of Verify with invocations of Check and all applications of \(\Psi\) with applications of \(\Psi'\).

  The algorithm proceeds in three possible ways based on the results of the Check function:
  – If any of the three Check results is unknown, then Leto falls back on its weak inference algorithm (Line 10).
  – If any of the three Check results is sat, the \(\text{INF}\) function recurses with false conjuncts removed from \(b_v\) and \(b_r\) (Lines 12 through 14).
  – If all three of the Check results are unsat, then the algorithm has converged on a set of invariants and returns \((b_v, b_r)\) (Line 16).

• **Left.** In left mode Leto does no invariant inference due to the fact that Leto does not verify loop invariants in left mode.

• **Right.** The beginning of the right algorithm (Lines 19 through 21) is similar to the right case for while loop verification, but we’ve replaced all invocations of Verify with invocations of Check and all applications of \(\Psi\) with applications of \(\Psi'\).

  The algorithm proceeds in three possible ways based on the results of the Check function:
  – If any of the two Check results is unknown, then Leto falls back on its weak inference algorithm (Line 23).
  – If any of the two Check results is sat, the \(\text{INF}\) function recurses with false conjuncts removed from \(b_v\) and \(b_r\) (Lines 25 through 27).
  – If both of the Check results are unsat, then the algorithm has converged on a set a invariants and returns \((b_v, b_r)\) (Line 29).

**Weak Inference.** Figure 26 presents the weak inference algorithm. Leto falls back on this algorithm when any call to the SMT solver returns unknown. While the strong inference algorithm iteratively prunes a set of candidate invariants, the weak inference algorithm builds up a set of invariants one at a time from a set of candidates. This is inherently weaker than the strong inference algorithm as it cannot always infer invariants that depend on other invariants. The \text{WEAKINF}
function takes these candidates as parameters (\(b_{cv}\) for standard invariants and \(b_{cr}\) for relational invariants) in addition to the loop to perform inference over and the programmer provided invariants for that loop.

The weak inference algorithm operates in three stages:

- **Standard invariant inference (Lines 4 through 23).** Leto adds the head of the standard candidate invariant list (\(b_h\)) to the loop invariant then proceeds differently depending on the verification mode:
  - **Lockstep.** The beginning of the lockstep algorithm (Lines 5 through 10) is similar to the lockstep case for while loop verification, but we’ve replaced all invocations of Verify with invocations of Check and all applications of \(\Psi\) with applications of \(\Psi'\) and added the head of the candidate invariant list at each step.
    The algorithm proceeds in two possible ways based on the results of the Check function:
    - If all three of the Check results are \textit{unsat}, then the algorithm recurses with \(b_h\) appended to \(b_v\) and the tail of \(b_{cv}\) as the candidate invariant list (Line 12).
    - If any of the three Check results are not \textit{unsat}, then the algorithm discards the candidate invariant and recurses (Line 13).
  - **Left.** In left mode Leto does no invariant inference due to the fact that Leto does not verify loop invariants in left mode.
  - **Right.** The beginning of the right algorithm (Lines 16 through 18) is similar to the right case for while loop verification, but we’ve replaced all invocations of Verify with invocations of Check and all applications of \(\Psi\) with applications of \(\Psi'\) and added the head of the candidate invariant list at each step.
    The algorithm proceeds in two possible ways based on the results of the Check function:
    - If both of the Check results are \textit{unsat}, then the algorithm recurses with \(b_h\) appended to \(b_v\) and the tail of \(b_{cv}\) as the candidate invariant list (Line 20).
    - If any of the two Check results are not \textit{unsat}, then the algorithm discards the candidate invariant and recurses (Line 21).

- **Relational invariant inference (Lines 28 through 46).** After exhausting the standard candidate invariant list, Leto iterates through the relational candidate invariant list. This process is identical to the previous stage but uses \(b_{cr}\) in place of \(b_{cv}\).

- **Base case (Line 26).** When no candidate invariants remain, \textsc{WeakInf} returns the pair of invariants \((b_v, b_r)\).

G SELF-CORRECTING CONNECTED COMPONENTS

Figure 27 presents an implementation of self-correcting connected components (SC-CC) [Sao et al. 2016], an iterative algorithm that computes the connected components of an input graph. A connected component is a subgraph in which every pair of vertices in the subgraph is connected through some path, but no vertex is connected to another vertex that is not also in the subgraph.

The standard connected components algorithm begins by constructing a vector \(CC^0\) and initializing this vector such that \(\forall v, CC^0[v] = v\). Then, on iteration \(i\) for each node \(v\) the algorithm looks up the value of each of \(v\’s\) neighbors in \(CC^{i-1}\) and sets \(CC^i[v]\) to the minimum of its neighbors and \(CC^{i-1}[v]\). In other words,

\[
CC^i[v] = \min_{j \in \mathcal{N}(v)} CC^{i-1}[j] \quad (1)
\]

where \(\mathcal{N}(v)\) is the union of \(v\) and the neighbors of node \(v\). The algorithm iterates this process until no elements in \(CC\) are updated at which point it has converged.
Self-correcting connected components adds an additional step of checking $CC^i$ after each iteration to verify that it is valid and has not been corrupted by memory errors. If SC-CC detects an error at $CC^i[v]$, it repeats the computation for node $v$ with reliably backed storage.

Our implementation allows errors when writing $CC^i$ so long as the errors are sufficiently large. Therefore, we consider $CC^i$ to be valid if $\forall v. 0 \leq CC^i[v] < |V|$ and in all other cases SC-CC corrects the invalid positions. When this property holds, then after each iteration $CC^{<o>} = CC^{<r>}$, even though intermediate values may differ during faulty execution.

The original SC-CC algorithm described by Sao et al. contains an additional data structure $P^*$ and permits a larger class of errors than this implementation does. However, this flexibility comes at a cost: the original algorithm is not guaranteed to converge. As such, we modified the algorithm to prove strong convergence properties that the original does not provide.

**Self Correction.** SC-CC is self correcting. This means that given some valid state, SC-CC can correct errors encountered during each iteration. In this case, if an error occurs at iteration $i$, SC-CC can correct $CC^i$ using data from $CC^{i-1}$. Therefore, SC-CC always stores $CC$ for the previous iteration correctly. This is weaker than self-stabilizing algorithms which may correct themselves from any state and do not rely on certain state elements remaining uncorrupted.

### G.1 SC-CC Implementation

The overall structure of the SC-CC implementation is as follows:

- **Initialization.** The $cc$ function takes a description of a graph in the form of an adjacency matrix ($adj$). It then declares and initializes $CC$, which holds the result of the previous iteration. It also declares $next_CC$, which holds the result of the current iteration, in the unreliable memory region.

- **Outer while loop (Line 9)** The outer while loop computes the next iteration of $CC$. It converges when the algorithm makes no changes to $CC$ over the course of a single iteration.

- **Faulty step (Line 15)** The faulty step computes Equation 1 element-wise over $next_CC$. The inner loop allows errors during writes to $next_CC$, which SC-CC will correct in the correction step. Prior to the entrance of the outer loop, SC-CC sets the $model.reliable$ flag to false to permit errors (Line 14).

- **Correction step (Line 31)** The correction step detects and corrects errors in $next_CC$. First, it sets the $model.reliable$ flag to true to prevent further errors in this iteration (Line 30). If $next_CC$ contains an error at index $v$, the implementation reliably computes $next_CC[v]$ using Equation 1. After correcting $next_CC$, the implementation sets $CC$ equal to $next_CC$ and begins the next iteration.

**Constants and Properties.** SC-CC uses the following constants and properties, found in Figure 28:

- **$max_N$ (Line 1).** This constant bounds the maximum number of nodes an input graph may contain.

- **$vec_bound$ (Line 3).** This property takes a vector $V$ and an index $i$ and stipulates that $\forall j < i<o>. \ V<o>[j] \leq j$.

- **$large_error_r$ (Line 6).** This property takes a vector $V$ and an index $i$ and asserts that $\forall j < i<r>. \ V<r>[j] = V<o>[j] \land j < V<r>[j]$.

- **$large_error_r_inclusive$ (Line 10).** This property takes a vector $V$, an index $from$, an index to and asserts that $\forall i. (from<r> \leq i < to<r>) \rightarrow V<r>[i] = V<o>[i] \lor i < V<r>[i]$.
requires N < max_N requires_r eq(adj)
vector<uint> cc(uint N, matrix<uint> adj(N, N)) {
    vector<uint> CC(N);
    @region(unreliable) vector<uint> next_CC(N);
    for (uint v = 0; v < N; ++v) invariant_r vec_bound(CC, v) {CC[v] = v;}
    uint N_s = N;

    @noinf while (0 < N_s)
    invariant N < max_N
    invariant_r vec_bound(CC, N)
    invariant_r eq(N) && eq(adj) && eq(N_s) && eq(CC) {
        next_CC = CC; N_s = 0; model.reliable = false;
        for (uint v = 0; v < N; ++v)
            invariant_r vec_bound(next_CC, N)
            invariant_r large_error_r(next_CC, N)
            invariant_r ∀(uint fi)((v<o><fi<N<o>>) -> next_CC<o>[fi]==CC<o>[fi])
            invariant_r outer_spec(v<o>, N<o>, next_CC<o>, CC<o>, adj<o>) {
                for (uint j = 0; j < N; ++j)
                    invariant v < N && N < max_N
                    invariant_r ∀(uint fi)((v<o><fi<N<o>>) -> next_CC<o>[fi]==CC<o>[fi])
                    invariant_r inner_spec(j<o>, v<o>, next_CC<o>, CC<o>, adj<o>) {
                        if (CC[j] < next_CC[v] && next_CC[v] <= v && adj[v][j] == 1) {
                            next_CC[v] = CC[j];
                        }
                    }
            }
    }
    model.reliable = true;
    @noinf @label(outer_correction) for (uint v = 0; v < N; ++v)
    invariant_r outer_spec(v<r>, N<r>, next_CC<r>, CC<r>, adj<r>)
    invariant_r eq(N) && eq(CC) && eq(adj) && eq(v) && eq(N_s)
    invariant_r forall(uint fi)(((fi < v<r>) ->
        (next_CC<r>[fi] == next_CC<o>[fi])))
    invariant_r vec_bound(next_CC, N)
    invariant_r model.reliable
    invariant_r large_error_r_inclusive(next_CC, N)
    invariant_r outer_spec(N<o>, N<o>, next_CC<o>, CC<o>, adj<o>) {
        if (v < corrected_next_CC[v]) {
            next_CC[v] = CC[v];
            for (uint j = 0; j < N; ++j)
                invariant v < N && v < outer_correction[next_CC[v]]
                invariant_r inner_spec(j<r>, v<r>, next_CC<r> CC<r>, adj<r>)
                invariant_r large_error_r_exclusive(next_CC, v, N) {
                    if (CC[j] < next_CC[v] && adj[v][j] == 1) {next_CC[v] = CC[j];}
                }
        }
        if (next_CC[v] < CC[v]) (++N_s;)
    }
    CC = corrected_next_CC;
    return CC;
}

Fig. 27. Self-Correcting Connected Components
\begin{verbatim}
const real max_N = ..;

property_r vec_bound (vector<uint> V, uint i) :
  forall (uint j)((j < i<o >) \rightarrow (V<o>[j] <= j));

property_r large_error_r (vector<uint> V, uint i) :
  forall (uint j)((j < i<r>) \rightarrow (V<r>[j] == V<o>[j] || j < V<r>[j]));

property_r large_error_r_inclusive (vector<uint> V, uint from, uint to) :
  forall (uint j)((from<r> <= j < to<r>) \rightarrow (V<r>[j] == V<o>[j] || j < V<r>[j]));

property_r large_error_r_exclusive (vector<uint> V, uint from, uint to) :
  forall (uint j)((from<r> < j < to<r>) \rightarrow (V<r>[j] == V<o>[j] || j < V<r>[j]));

property_r outer_spec (uint to, uint N, vector<uint> next_CC, vector<uint> CC, matrix<uint> adj) :
  forall (uint fi)((fi < to) \rightarrow
    (forall (uint fj)((fj < N && adj[fi][fj] == 1) \rightarrow
      next_CC[fi] <= CC[fj]) &&
      next_CC[fi] <= CC[fi] &&
      (exists (uint ej)(next_CC[fi] == CC[ej] && ej < N &&
        adj[fi][ej] == 1) ||
      next_CC[fi] == CC[fi])));

property_r inner_spec (uint to, uint v, uint N, vector<uint> next_CC, vector<uint> CC, matrix<uint> adj) :
  forall (uint fi)((fi < to && adj[v][fi] == 1) \rightarrow
    next_CC[v] <= CC[fi]) &&
    next_CC[v] <= CC[v] &&
    (exists (uint ei)(next_CC[v] == CC[ei] && ei < N &&
      adj[v][ei] == 1) ||
    next_CC[v] == CC[v]);
\end{verbatim}

Fig. 28. Constant and Properties for Self-Correcting Connected Components

- **large_error_r_exclusive** (Line 14). This property takes a vector \(V\), an index \(\text{from}\) to an index \(\text{to}\) and asserts that \(\forall i. (\text{from}<r> < i < \text{to}<r>) \rightarrow V<r>[i] = V<o>[i] \lor i < V<r>[i]\).

- **outer_spec** (Line 18). This property takes an index \(\text{to}\), a size \(N\), a vector \(\text{next}_\text{CC}\), a vector \(\text{CC}\), and an adjacency matrix \(\text{adj}\). It ensures that every element of \(\text{next}_\text{CC}\) from index 0 to index \(\text{to}\) (exclusive) satisfies Equation 1.

- **inner_spec** (Line 28). This property takes an index \(\text{to}\), an index \(v\), a size \(N\), a vector \(\text{next}_\text{CC}\), a vector \(\text{CC}\), and an adjacency matrix \(\text{adj}\). It ensures that \(\text{next}_\text{CC}[v]\) satisfies Equation 1 up to the neighbor at index \(\text{to}\). That is,

\[
\text{next}_\text{CC}[v] = \min_{j \in \mathcal{N}(v, \text{to})} \text{CC}[j]
\]

where \(\mathcal{N}(v, \text{to})\) is the union of \(v\) and the neighbors of node \(v\) up to (but not including) nodes with the id \(\text{to}\).

**Error Model.** I evaluate SC-CC under the switchable rowhammer fault model from Figure 5. This model provides two implementations for the write operator in the memory region unreliable. The first implementation is fully reliable while the second allows for errors so long as they are larger than the programmer specified constant \text{min_error}.
I use this fault model that allows only faulty writes because it captures all possible rowhammer attacks over high order bits in the elements of next_CC. A rowhammer attack allows an attacker to selectively flip bits in DRAM by issuing frequent reads on DRAM rows surrounding the row under attack [Kim et al. 2014]. This drains capacitors in the attacked row and therefore permanently flips bits in that row. As empty capacitors may indicate a 0 or 1 depending on location on the chip, this attack does not always flip bits from 1 to 0. Although researchers have devised protections to address rowhammer attacks [Kim et al. 2015, 2014], the JDEC Solid State Technology Association did not include these protections in the DDR4 standard [Association et al. 2012] and Mark Lanteigne has demonstrated rowhammer attacks on some DDR4 memory [Lanteigne 2016].

Given that some regions of memory may be more vulnerable to rowhammer attacks than others [Kim et al. 2014], I place next_CC in an unreliable region prone to attacks (Line 4) and all other variables in reliable memory.

G.2 Specification
We use Leto’s specification abilities to verify the error detection, self-correction, and convergence properties of SC-CC.

Perfect Error Detection. To verify that this SC-CC implementation detects all errors, the developer must verify that \( N < \text{max}_N \), where \( N \) is the number of nodes in the input graph and \( \text{max}_N \) is a programmer specified value bounding the maximum graph size a calling function may provide. The developer must also ensure that \( \text{max}_N \) is less than the magnitude of the largest error they expect to encounter. This property ensures that any error will be larger than \( N \), which in turn ensures that SC-CC detects all errors as errors result in trivially invalid values. SC-CC specifies this property as a unary prerequisite to calling the cc function on Line 1. It also specifies this property as unary invariants that must hold before and after each loop iteration on Lines 10 and 22. Although not explicitly stated, Leto infers this invariant for the loop on Line 15.

Error detection also requires that impact of errors on next_CC\( r > \) is large enough to be detected. Specifically, it is necessary that

\[
\forall i. \text{next}_\text{CC}\langle r \rangle[i] > i \lor \text{next}_\text{CC}\langle r \rangle == \text{next}_\text{CC}\langle o \rangle.
\]

This property ensures that for every element \( e \) of next_CC\( r > \) at index \( i \), \( e \) is in one of two states:

- **Detectable Error.** When \( e > i \), \( e \) violates the property that every element of CC does not increase from its initialization value. The check in the conditional on Line 40 of the correction step trivially detects this error.
- **Equality.** When \( e = \text{next}_\text{CC}\langle o \rangle[i] \), \( e \) is correctly computed.

Note that \( e \) can not be in a state where it contains an undetectable error. Loop invariants on Lines 17 and 38 (\( \text{large}_\text{error}_r(\text{next}_\text{CC}, N) \)) ensure this property.

The final property SC-CC requires to ensure that it has perfect error detection is eq(adj). This property asserts that the graphs both executions operate over are equivalent, and unable to experience errors. SC-CC enforces this property by specifying it as a requirement to call the cc function on Line 1. On all loops it either explicitly specify eq(adj), or Leto infers it.

Self-Correction. To verify that SC-CC is self correcting, the implementation first verifies that every element of next_CC\( o > \) satisfies Equation 1. It capture this with the outer_spec property application on Line 19 and the inner_spec property application on Line 24. Both of these properties capture the semantics of the min operator. After exiting inner faulty loop (\( j == N \), inner_spec(\( j<o>, v<o>, \text{next}_\text{CC}<o>, \text{CC}<o>, \text{adj}<o> \)) is equivalent to Equation 1 at index \( v \). Similarly, after exiting the outer loop (\( v == N \), the application outer_spec(\( v<o>, N<o>, \text{next}_\text{CC}<o>, \)
CC<o>, adj<o>) is equivalent to Equation 1 at all indices, or

\[ \forall v. CC^t[v] = \min_{j \in N(v)} CC^{t-1}[j]. \]

The correction step applies the same two properties (outer_spec and inner_spec) in the same fashion to specify that next_CC<r> also satisfies Equation 1 at all indices.

SC-CC passes the specification of next_CC<o> into the correction loop on Line 39 where Leto combines it with the specification of next_CC<r> to prove that

\[ \forall i < v<r>. \text{next}_CC<r>[i] == \text{next}_CC<o>[i], \]

stated on Line 34.

Finally, with the assignment CC = next_CC Leto can verify the outer loop invariant eq(CC) (Line 12) thus proving that SC-CC is self correcting.

**Convergence Equality.** Given that SC-CC is self correcting and detects all errors, it is trivial to see that both executions converge in the same number of iterations. That is, the outer while loop (Line 9) must run in lockstep. To demonstrate this fact to Leto SC-CC uses the eq(N_s) invariant on Lines 12 and 33.

N_s itself is updated in two places:

- **Line 14.** SC-CC sets N_s to 0 at the top of the outer loop. As N_s is stored in reliable memory, it is clear that N_s<o> == N_s<r>.
- **Line 49.** SC-CC increments N_s at this location if next_CC[v] < CC[v]. From the surrounding loop invariants Leto knows that

\[ \text{next}_cc<o>[v<o>] == \text{next}_CC<r>[v<r>] \]

and

\[ \text{CC}<o>[v<o>] == \text{CC}<r>[v<r>] \]

so the if statement must execute in lockstep. Therefore, if N_s was equal across both executions before the if statement, then it is equal afterwards as the implementations performs the increment to N_s reliably.

Leto realizes that N_s is equal across both executions after both assignments and therefore eq(N_s) must hold in both loop invariants. This forces the outer while loop into a lockstep execution and proves that convergence time is equal under relaxed and reliable execution semantics.

**G.3 Verification Approach**

I next demonstrate how the developer works with Leto to verify that the implementation meets these specifications.

**Precondition.** The verification algorithm begins with the preconditions on cc. The pre-condition stipulates that N be less than max_N and that the graph be equal across both executions (eq(adj)). Leto adds these preconditions as assumptions to its context.

**Initialization.** On Line 6 SC-CC initializes the CC vector such that \( \forall v. \text{CC}[v] = v \). The loop invariant verifies that all elements with index v in CC are between 0 and v. This property is critical in the detection of errors and must hold for CC after each iteration of the connected components algorithm.
**Outer Loop.** The loop on Line 9 runs the iterative portion of the algorithm. The loop enforces the critical invariants vec_bound(CC, N), which ensures that ∀v. CC[v] ≤ v, and eq(CC). It also contains the invariant eq(N_s), which ensures that the loop runs in lockstep. Lastly, it enforces the invariants eq(N) and eq(adj) which ensure that the input graph is identical across both executions.

The loop also sports the @noinf annotation, which disables inference over this loop. SC-CC disables inference on this loop because the inference algorithm’s time complexity is exponential in the depth of nested loops.

**Faulty Middle Loop.** Verification then proceeds to the faulty middle loop on Line 15. This loop contains the following invariants:

- **vec_bound(next_CC, N).** This invariant enforces that elements of next_CC<o> are bounded by their respective indices. This fact is important to pass on to the correction step as it implies that the original execution never runs the inner correction loop.
- **large_error_r(next_CC, N).** This invariant enforces that errors to next_CC<r> are large enough to be detected. It enables the implementation to detect and correct all errors during the correction step.
- **forall(uint fi)((v<o> <= fi < N<o>) -> next_CC<o>[fi] == CC<o>[fi]).** This invariant states that elements in next_CC that the implementation hasn’t yet updated are still equal to CC. This is necessary as it communicates to Leto that if an element is not updated on this iteration, then it is already the minimum of its neighbors.
- **outer_spec(v<o>, N<o>, next_CC<o>, CC<o>, adj<o>).** This invariant specifies the contents of next_CC<o>. I use it to pass information about next_CC<o> on to the correction step.

**Faulty Inner Loop.** Verification then continues to the loop on Line 21, where it encounters the following invariants:

- **v < N.** This invariant bounds v so that Leto knows that the vector accesses within this loop are in bounds.
- **N < max_N.** This invariant bounds the maximum size of the graph so that Leto knows that errors are larger than the maximum graph size and are therefore detectable.
- **forall(uint fi)((v<o> < fi < N<o>) -> next_CC<o>[fi] == CC<o>[fi]).** This invariant serves to pass on that although the implementation may have altered next_CC<o>[v<o>], the other elements that Leto previously knew were equal to CC<o> in the middle faulty loop are still equal to CC<o>.
- **inner_spec(j<o>, v<o>, next_CC<o>, CC<o>, adj<o>).** This invariant specifies the contents of next_CC<o>[v<o>]. It serves only to pass this information on to the outer loop so Leto may verify the outer_spec invariant in that loop.

**Correction Middle Loop.** Verification next proceeds to the loop on Line 31. This loop contains the following notable invariants:

- **outer_spec(v<r>, N<r>, next_CC<r>, CC<r>, adj<r>).** This invariant specifies the contents of next_CC<r>. Leto combines it with the other outer_spec invariant to verify next_CC<o> == next_CC<r> after exiting the loop.
- **eq(v).** This invariant states that v<o> == v<r>. It informs Leto that this loop must run in lockstep.
- **forall(uint fi)(((0 <= fi < v<r>) -> (next_CC<r>[fi] == next_CC<o>[fi]))).** This invariant states that elements in next_CC that the implementation has updated are
equal across both executions. This allows Leto to prove $eq(CC)$ at the end of the outer while loop.

- **vec_bound(next_CC, N)**. This invariant enforces that elements of next_CC_o are bounded by their respective indices. This implies that the original execution never runs the inner correction loop. Therefore, Leto prunes any paths in which the original execution runs through the inner loop.

- **model.reliable**. This invariant enforces that the correction step runs reliably.

- **large_error_r_inclusive(next_CC, v, N)**. This invariant enforces that errors to next_CC_r are large enough to be detected. It enables the implementation to detect and correct all errors during the course of this loop.

- **outer_spec(N_o, N_o, next_CC_o, CC_o, adj_o)**. This invariant passes in the specification for next_CC_o from the faulty loop. Leto combines it with specifications over next_CC_r to prove that next_CC_o[v_o] is equivalent to next_CC_r[v_r] at the end of the current loop iteration.

**Correction Inner Loop.** Verification then continues to the loop on Line 42 which contains three developer specified invariants:

- **v < N**. This invariant bounds v so that Leto knows that the vector accesses within this loop are in bounds.

- **v < outer_correction[next_CC[v]]**. The syntax outer_correction[next_CC[v]] refers to the value of next_CC[v] at the top of the loop labeled outer_correction. Therefore, this invariant states that v is less than next_CC[v] at the top of the outer correction loop. The conditional that contains this loop implies this invariant.

- **large_error_r_exclusive(next_CC, v, N)**. This invariant enforces that errors to next_CC_r are large enough to be detected. It enables the implementation to detect and correct all errors during the course of this loop.

- **inner_spec(j_r, v_r, corrected_next_CC_r, CC_r, adj_r)**. This invariant specifies the contents of corrected_next_CC_r[v_r]. It serves only to pass this information on to the outer loop so that Leto may verify the outer_spec in that loop.

### H SELF-STABILIZING CONJUGATE GRADIENT DESCENT

We use Leto to verify an implementation of self-stabilizing conjugate gradient (SS-CG) [Sao and Vuduc 2013] and present relevant snippets required for verification below. Conjugate gradient descent is another method for solving linear systems of equations. However, unlike Jacobi, the standard conjugate gradient method is sensitive to errors that may corrupt internal state variables and, therefore, is not naturally self-stabilizing. SS-CG employs a periodic correction step to recalculate appropriate values for internal state variables from the current estimated solution vector $x$ and the input matrix $A$. 

50
The standard conjugate gradient descent algorithm computes the next iteration’s variables as follows:

\[ q_i = A p_i \]  
\[ \alpha_i = \frac{r_i^T r_i}{p_i^T q} \]  
\[ x_{i+1} = x_i + \alpha p_i \]  
\[ r_{i+1} = r_i - \alpha q_i \]  
\[ \beta = \frac{r_{i+1}^T r_{i+1}}{r_i^T r_i} \]  
\[ p_{i+1} = r_{i+1} + \beta p_i \]  

SS-CG adds a periodic correction step to repair state variables that may have been corrupted by errors. Unlike the previous steps, the repair step must be computed reliably. This repair step computes:

\[ r_i' = A x_i \]  
\[ q_i = A p_i \]  
\[ r_i = b - r_i' \]  
\[ \alpha_i = \frac{r_i^T p_i}{p_i^T q} \]  
\[ x_{i+1} = x_i + \alpha p_i \]  
\[ r_{i+1} = r_i - \alpha q_i \]  
\[ \beta = -\frac{r_{i+1}^T q_i}{p_i^T q} \]  
\[ p_{i+1} = r_{i+1} + \beta p_i \]  

We use Leto to verify two properties of SS-CG that are necessary for self-stability under the SEU model we present in Figure 1:

- **Reliable Correction Step.** We verify that it is possible to correct errors even when the matrix vector products in Equations 8 and 9 may experience faults. To accomplish this, the implementation uses dual modular redundancy (DMR) to duplicate arithmetic instructions and repeat the correction step until the result of both sets of instructions agree with each other.

- **Correctable Errors.** We verify that errors in the matrix vector product from Equation 2 are sufficiently small. Specifically, SS-CG requires that if an element of \( q \) is corrupted by \( \epsilon \), then

\[ \epsilon^2 < \max_{i,j} (A[i][j])^2. \]

We present the code snippet for the reliable correction step in Figure 29 and the snippet for correctable errors in Figure 30. We have included the full implementation in Appendix J.

**H.1 Implementation**

**H.1.1 Reliable Correction Step.** The SS-CG correction step we present in Figure 29 operates over the following pre-existing variables:

- **A.** \( A \) is a matrix of coefficients.
property_r dmr_eq(vector<real> x1, vector<real> x2, vector<real> sx) :
    model.upset -> (x1<r> == sx && x2<r> == sx);

property_r dmr_imp(vector<real> x1, vector<real> x2, vector<real> sx) :
    (x1<r> == x2<r>) -> (x1<r> == sx);

vector<real> r2(N), q2(N);
specvar vector<real> spec_r(N), spec_q(N);
r = r2 = spec_r = q = q2 = spec_q = zeros;
bool not_run = true;

@noinf while (not_run || r != r2 || q != q2)
  invariant_r dmr_eq(r, r2, spec_r)
  invariant_r dmr_eq(q, q2, spec_q)
  invariant_r dmr_imp(r, r2, spec_r)
  invariant_r dmr_imp(q, q2, spec_q)
  { not_run = false;
    for (uint i = 0; i < N; ++i)
      for (uint j = 0; j < N; ++j)
        { real tmp = A[i][j] *. x[j];
          real tmp2 = A[i][j] *. x[j];
          specvar real spec_tmp = A[i][j] * x[j];
          r[i] = r[i] +. tmp;
          r2[i] = r2[i] +. tmp2;
          spec_r[i] = spec_r[i] + spec_tmp;
          tmp = A[i][j] *. p[j];
          tmp2 = A[i][j] *. p[j];
          spec_tmp = A[i][j] * p[j];
          q[i] = q[i] +. tmp;
          q2[i] = q2[i] +. tmp2;
          spec_q[i] = spec_q[i] + spec_tmp;
        }
    }

assert_r (!outer_while[model.upset] -> (r<r> == spec_r));
assert_r (!outer_while[model.upset] -> (q<r> == spec_q));

Fig. 29. SS-CG Correction Step

- x. x is a solution vector.
- r. r holds the residual of the current iteration.
- p and q. p and q are vectors of loop carried state.

The overall structure of the SS-CG correction step is as follows:

- **Initialization (Line 7).** Initialization declares the following variables:
  - r2 and q2. The algorithm computes r2 according to Equation 8 and q2 according to Equation 9. It then uses r2 and q2 to verify that it has correctly computed r and q respectively.
  - spec_r and spec_q. spec_r and spec_q are specification variables that also compute Equation 8 and Equation 9 respectively. Unlike r, r2, q, and q2, the implementation computes these specification variables reliably.
  - Outer while loop (Line 12). The outer while loop repeats the correction step until r == r2 and q == q2.
\begin{verbatim}
const real M = ...;

property_r sqr_lt(vector<real> v, uint i) : 
  ((v<r>[i<r>] - v<o>[i<o>]) * (v<r>[i<r>] - v<o>[i<o>])) < M;

for (uint i = 0; i < N; ++i) {
  q[i] = 0;
  @label(inner_err)
  for (int j = 0; j < N; ++j)
    invariant_r (!model.upset && eq(p)) -> q<r>[i<r>] == q<o>[i<o>]
    { 
      real tmp = A[i][j] *. p[j];
      q[i] = q[i] +. tmp;
      assert_r((!inner_err[model.upset] && eq(p)) -> sqr_lt(q, i));
    }
}
\end{verbatim}

Fig. 30. SS-CG Faulty Matrix Vector Product

- Middle for loop (Line 20). The middle for loop computes Equation 8 element-wise for \(r\), \(r_2\), and spec\(_r\) and Equation 9 element-wise for \(q\), \(q_2\), and spec\(_q\).
- Inner for loop (Line 22). The inner for loop computes the matrix vector products:

\[
\begin{align*}
r &= A \times x \\
q &= A \times p
\end{align*}
\]

It computes \(r_2\) and spec\(_r\) similarly to \(r\), and \(q_2\) and spec\(_q\) similarly to \(q\). The algorithm permits errors in the computation of \(r\), \(r_2\), \(q\), and \(q_2\), but not in spec\(_r\) or spec\(_q\).

Properties. The SS-CG correction step implementation uses the following properties, found in Figure 29:

- dmr\_eq (Line 1). This property asserts that if an upset has not occurred, then \(x_1\) and \(x_2\) are both equal to the specification variable \(sx\).
- dmr\_imp (Line 4). This property asserts that if \(x_1\) and \(x_2\) are equal, then \(x_1\) is equal to the specification variable \(sx\).

H.1.2 Faulty Matrix Vector Product. The SS-CG faulty matrix vector product I present in Figure 30 operates over the following pre-existing variables:

- \(A\). \(A\) is a matrix of coefficients.
- \(p\) and \(q\). \(p\) and \(q\) are vectors of loop carried state.

The structure of the SS-CG faulty matrix vector product is as follows:

- Outer for loop (Line 6). The outer for loop computes Equation 2 element-wise over \(q\).
- Inner for loop (Line 11). The inner for loop computes the unreliable matrix vector product \(q = A \times p\).

Constants and Properties. The SS-CG faulty matrix vector product uses the following constants properties, found in Figure 30:
• **M (Line 1).** \( M \) represents the maximum square error permissible in a single element of \( q \). The developer must set it according to the formula
\[
 M < \min_{a \in A} \left( \max_{(i,j)} (a[i][j])^2 \right)
\]
where \( A \) is the set of \( A \) input matrices the developer expects to run our implementation over.

• **sqr_lt (Line 3).** \texttt{sqr}\_\texttt{lt} takes a vector \( v \), and index \( i \), and ensures that the square error of \( v[i] \) is strictly less than \( M \). In other words, it mandates that
\[
(v^{\text{r}}[i^{\text{r}}] - v^{\text{o}}[i^{\text{o}}])^2 < M.
\]

\subsection*{H.2 Specification}

We use Leto’s specification abilities to verify the error correction and small errors properties of SS-CG.

**Error Correction.** Using DMR, the correction step corrects \( r \) and \( q \) even in the presence of errors. SS-CG enforces this property through the assertions on Lines 41 and 42 of Figure 29. As the algorithm computes \texttt{spec\_r} and \texttt{spec\_q} correctly, Leto knows that if \( r^{\text{r}} == \texttt{spec\_r} \) and \( q^{\text{r}} == \texttt{spec\_q} \), then the system has computed \( r^{\text{r}} \) and \( q^{\text{r}} \) correctly.

The \texttt{dmr}\_\texttt{imp} and \texttt{dmr}\_\texttt{eq} property applications in the outer while loop invariants (Lines 13 through 16) pass the information Leto needs to verify this assertion out of the loop. Leto infers these invariants for the inner loops, thus allowing the outer loop to verify that the invariant holds after the modifications the inner loop performs.

**Correctable Errors.** Under SEU, SS-CG requires that if an element of \( q \) is corrupted by \( \epsilon \), then
\[
\epsilon^2 < \max_{(i,j)} (A[i][j])^2.
\]

SS-CG enforces this property through the assertion on Line 17 of Figure 30. The invariant on Line 12 enforces the complementary invariant that if no error occurred, then \( q^{\text{o}}[i^{\text{o}}] \) is equal to \( q^{\text{o}}[i^{\text{o}}] \). \texttt{eq(p)} guards both of these constraints because the implementation verifies this section in isolation without specifying the global properties of \( p \). However, if no upset occurred prior to the start of this section then \texttt{eq(p)} trivially holds before and throughout as this snippet does not modify \( p \).

\subsection*{H.3 Verification Approach}

Next, we demonstrate how the developer works with Leto to verify that the implementation meets these specifications.

**H.3.1 Correction Step.**

**Outer Loop.** Verification begins with the loop on Line 12 of Figure 29. This loop contains the following invariants:

• **\texttt{dmr}\_\texttt{eq}(r, r2, \texttt{spec\_r}).** This invariant enforces that in the absence of errors during a loop iteration, \( r^{\text{r}} \) is equal to \( \texttt{spec\_r} \). That is, if no errors occur then \( r \) is correct. This fact is important to pass on so that Leto may verify the assertions on Line 41.

• **\texttt{dmr}\_\texttt{eq}(q, q2, \texttt{spec\_q}).** This invariant enforces that in the absence of errors during a loop iteration, \( q^{\text{r}} \) is equal to \( \texttt{spec\_q} \). That is, if no errors occur then \( q \) is correct. This fact is important to pass on so that Leto may verify the assertions on Line 42.
• \texttt{dmr\_imp(r, r2, spec\_r)}. This invariant states that if the duplicated \( r \) variables are equal to each other, then \( r \) is also equal to \( \text{spec\_r} \). Combining this with the loop condition, Leto knows that \( r<r> == \text{spec\_r} \) after exiting the loop.

• \texttt{dmr\_imp(q, q2, spec\_q)}. This invariant states that if the duplicated \( q \) variables are equal to each other, then \( q \) is also equal to \( \text{spec\_q} \). Combining this with the loop condition, Leto knows that \( q<r> == \text{spec\_q} \) after exiting the loop.

**Assertions.** Verification concludes with the assertions on Lines 41 and 42:

- \( \neg \text{outer\_while[\text{model.upset}]} \rightarrow (r<r> == \text{spec\_r}) \). This assertion verifies that if no upset occurred prior to entering the loop containing the outer while loop, then the correction step computed \( r \) correctly.

- \( \neg \text{outer\_while[\text{model.upset}]} \rightarrow (q<r> == \text{spec\_q}) \). This assertion verifies that if no upset occurred prior to entering the loop containing the outer while loop, then the correction step computed \( q \) correctly.

**H.3.2 Faulty Matrix Vector Product.**

**Outer Loop.** Verification begins with the loop on Line 6 of Figure 30. This loop contains no invariants.

**Inner Loop.** Verification then proceeds to the inner loop on Line 11. This loop contains the following invariants:

- \((!\text{model.upset} \&\& \text{eq(p)}) \rightarrow q<r>[i<r>] == q<o>[i<o>]\). This invariant ensures that if no upset occurred and \( p<o> == p<r> \), then \( q[i] \) is equal across both executions.

**Assertion.** Verification concludes with the assertion on Line 17. This assertion verifies that if no error had occurred prior to the top of the inner loop, and \( p<o> == p<r> \), then the square difference between \( q<o> \) and \( q<r> \) is less than \( M \). This ensures that errors are sufficiently small to be correctable. That is, it ensures that the error in \( q \) satisfies Equation 16.

I SELF-STABILIZING STEEPEST DESCENT CORRECTION STEP

Figure 31 presents an implementation of the correction step from self-stabilizing steepest descent (SS-SD) [Sao and Vuduc 2013]. SS-SD is an iterative algorithm that computes the solution to a linear system of equations. It takes as input a matrix of coefficients \( A \), a vector \( b \) of intercepts, and returns an approximate solution vector \( x \) such that \( A \times x \approx b \). On each iteration, steepest descent uses \( r_i, x_i, \) and \( A \) to compute \( r_{i+1} \) and \( x_{i+1} \) as follows:

\[
q_i = Ar_i \tag{17}
\]
\[
\alpha_i = \frac{r_i^T r_i}{r_i^T q_i} \tag{18}
\]
\[
x_{i+1} = x_i + \alpha_i r_i \tag{19}
\]
\[
r_{i+1} = r_i - \alpha_i q_i \tag{20}
\]

SS-SD adds a periodic correction step to repair the residual \( r \) of any errors it may have incurred. This repair step computes

\[
r_i = b - Ax. \tag{21}
\]

Unlike Equation 18 through Equation 20, SS-SD requires that Equation 21 is performed reliably. Therefore, the implementation verifies that it is possible to correct errors under the SEU execution model from Figure 1 even when the error correction step may experience errors. To accomplish
The overall structure of the SS-SD correction step is as follows:

- **Initialization.** The `correct_sd` function takes a matrix of coefficients $A$, a vector of intercepts $b$, and a solution vector $x$. It then declares:
  - $r$. $r$ holds the residual that will be returned.

I.1 SS-SD Correction Step Implementation

This, we use dual modular redundancy (DMR) to duplicate arithmetic instructions and repeat the correction step until the result of both sets of instructions agree with each other.
property_r upset(vector<real> r, vector<real> r2, vector<real> spec_r,
       vector<real> Ax, vector<real> Ax2, vector<real> spec_Ax):
  (model.upset ->
   ((r<r> == spec_r && Ax<r> == spec_Ax) ||
    (r2<r> == spec_r && Ax2<r> == spec_Ax)))
  && (!model.upset ->
    (r<r> == spec_r && r2<r> == spec_r &&
     Ax<r> == spec_Ax && Ax2<r> == spec_Ax));

Fig. 32. SS-SD Properties

- r2. The function computes r2 according to Equation 21. The algorithm uses r2 to verify that it has correctly computed r.
- spec_r. spec_r is a specification variable that also computes Equation 21. Unlike r and r2, the implementation computes spec_r correctly.

• Outer while loop (Line 12). The outer while loop repeats the correction step until r == r2. This loop also sets model.upset to false (Line 15) to permit at most one fault per correction iteration.

• Middle for loop (Line 22). The middle for loop computes Equation 21 element-wise for r, r2, and spec_r. The algorithm permits errors in the computation of r and r2 but not spec_r.

• Inner for loop (Line 26). The inner for loop computes the matrix vector product Ax = A*x. It computes Ax2 and spec_Ax similarly. The algorithm permits errors in the computation of Ax and Ax2, but not in spec_Ax.

Properties. The SS-SD correction step implementation uses the following properties, found in Figure 32:

• upset (Line 1). This property consists of two conjuncts:
  - The first conjunct states that if there was an upset since the start of the outer loop then at least one of the duplicated computations is correct. That is, r<r> == spec_r && Ax<r> == spec_Ax or r2<r> == spec_r && Ax2<r> == spec_Ax.
  - The second conjunct states that if no upset has occurred since the loop of the outer loop then both sets of duplicated instructions are correct.

1.2 Specification
Leto’s specification abilities verify the error correction property of SS-SD’s correction step.

Error Correction. Using DMR, the correction step corrects r even in the presence of errors. SS-SD enforces this property through the assertion on Line 45. As the algorithm computes spec_r correctly, if r<r> == spec_r, then r<r> is the correct residual.

The invariant on the outer while loop passes the information needed to verify this assertion out of the loop. In turn, this invariant relies on the upset invariant in the middle loop (Line 23), which itself relies on the upset invariant in the inner loop (Line 27).

1.3 Verification Approach
Next, we demonstrate how the developer works with Leto to verify that the implementation meets these specifications.

• Outer Loop. Verification begins with the loop on Line 12. This loop contains the following invariants:
• \( r^{<r>} = r^{2<rl>} \rightarrow r^{<r>} = \text{spec}_r \). This invariant states that if the duplicated \( r \) variables are equal to each other, then \( r \) is also equal to \( \text{spec}_r \). Therefore, Leto knows that \( r^{<r>} = \text{spec}_r \) after exiting the loop.

**Middle Loop.** Verification then proceeds to the middle loop on Line 22. This loop contains the following invariants:

- \( \text{upset}(r, r2, \text{spec}_r, Ax, Ax2, \text{spec}_Ax) \). This invariant enforces that in the event of an error during this outer loop iteration, the algorithm computes at least one of set of instructions (\( r \) and \( Ax \) or \( r2 \) and \( Ax2 \)) correctly. Otherwise, the algorithm computes both sets correctly. It is an invariant in the middle loop because it must hold at the top of the inner loop.

**Inner Loop.** Verification then proceeds to the inner loop on Line 26. This loop contains the following invariants:

- \( \text{upset}(r, r2, \text{spec}_r, Ax, Ax2, \text{spec}_Ax) \). This invariant enforces that in the event of an error during this outer loop iteration, the algorithm computes at least one of set of instructions (\( r \) and \( Ax \) or \( r2 \) and \( Ax2 \)) correctly. Otherwise, the algorithm computes both sets correctly. This invariant captures information about the impacts of errors on variables modified in the inner loop and passes this information back to the middle loop.

**Assertion.** Verification concludes with the assertion on Line 45. This assertion verifies that \( r^{<r>} = \text{spec}_r \), which enforces that \( r^{<r>} \) is the correct residual and it satisfies Equation 21.

### J FULL SELF-STABILIZING CONJUGATE GRADIENT DESCENT IMPLEMENTATION

We present our full implementation for self-stabilizing conjugate gradient descent below. We use the @noinf annotation regularly in this benchmark on loops that do not contribute to the final verification result to increase performance.

```plaintext
const real SQR_MIN_MAX_AIJ = 2;

property_r sqr_lt (vector<real> v, uint i) : ((q<r>[i<r>] - q<o>[i<o>]) * (q<r>[i<r>] - q<o>[i<o>])) < SQR_MIN_MAX_AIJ;

property_r dmr_eq (vector<real> x1, vector<real> x2, vector<real> spec_x) :
!model.upset -> x1<r> == spec_x && x2<r> == spec_x;

property_r dmr_imp (vector<real> x1, vector<real> x2, vector<real> spec_x) :
(x1<r> == x2<r>) -> (x1<r> == spec_x);

r_requires eq(N) && eq(M) && eq(F) && eq(A)
vector<real> ss_cg (uint N, uint M, uint F, matrix<real> A(N, N), vector<real> b(N), vector<real> x(N)) {
  vector<real> r(N), p(N), q(N), next_x(N), next_r(N), next_p(N)
  real alpha, beta, tmp, tmp2, num, denom;
  uint man_mod;

  vector<real> zeros(N);
  @noinf for (uint i = 0; i < N; ++i) { zeros[i] = 0 ;

```
uint it = 0;

@noinf for (uint i = 0; i < N; ++i)
{
    tmp = 0;
    @noinf for (uint j = 0; j < N; ++j)
    {
        tmp = tmp + A[i][j] * x[i];
    }
    r[i] = b[i] - tmp;
}

p = r;

@noinf @label(outer_while)
while (it < M)
    invariant_r eq(A) && eq(it) && eq(M) && eq(N) && eq(man_mod) && eq(F)
{
    if (man_mod == F)
    {
        vector<real> r2(N), q2(N);
        specvar vector<real> spec_r(N), spec_q(N);
        r = r2 = spec_r = q = q2 = spec_q = zeros;
        bool not_run = true;
        @noinf while (not_run || r != r2 || q != q2)
            invariant_r dmr_eq(r, r2, spec_r)
            invariant_r dmr_eq(q, q2, spec_q)
            invariant_r dmr_imp(r, r2, spec_r)
            invariant_r dmr_imp(q, q2, spec_q)
        {
            not_run = false;
            for (uint i = 0; i < N; ++i)
            {
                for (uint j = 0; j < N; ++j)
                {
                    tmp = A[i][j] * x[j];
                    tmp2 = A[i][j] * x[j];
                    specvar real spec_tmp = A[i][j] * x[j];
                    r2[i] = r2[i] + tmp2;
                    spec_r[i] = spec_r[i] + spec_tmp;
                }
            }
            tmp = A[i][j] * p[j];
            tmp2 = A[i][j] * p[j];
            spec_tmp = A[i][j] * p[j];
            q[i] = q[i] + tmp;
            q2[i] = q2[i] + tmp2;
            spec_q[i] = spec_q[i] + spec_tmp;
        }
    }
    assert_r(!outer_while[model.upset] -> r<r> == spec_r);
    assert_r(!outer_while[model.upset] -> q<r> == spec_q);

    @noinf for (uint i = 0; i < N; ++i) { r[i] = b[i] - r[i]; }

    num = 0;
    denom = 0;
    @noinf for (uint i = 0; i < N; ++i)
    {
        tmp = r[i] * p[i];
        num = num + tmp;
tmp = p[i] * q[i];
    denom = denom + tmp;
}
alpha = num / denom;

@noinf for (i = 0; i < N; ++i)
{
    tmp = alpha * p[i];
    next_x[i] = x[i] + tmp;
    tmp = alpha * q[i];
    next_r[i] = r[i] - tmp;
}

num = 0;
denom = 0;
@noinf for (uint i = 0; i < N; ++i)
{
    tmp = -next_r[i];
    tmp = tmp * q[i];
    num = num + tmp;
    tmp = p[i] * q[i];
    denom = denom + tmp;
}
beta = num / denom;

@noinf for (i = 0; i < N; ++i)
{
    tmp = beta * p[i];
    next_p[i] = next_r[i] + tmp;
}
else {
    for (uint i = 0; i < N; ++i)
    {
        q[i] = 0;
        @label(inner_err)
        for (uint j = 0; j < N; ++j)
            invariant_r (model.upset == false && eq(p)) ->
                q<r>[i<r>] == q<o>[i<o>]
            {
                tmp = A[i][j] *. p[j];
                q[i] = q[i] +. tmp;
                assert_r((! inner_err[ model.upset] && eq(p)) -> sqr_lt(q, i));
            }
    }
}
num = 0;
denom = 0;
@noinf for (uint i = 0; i < N; ++i)
{
    tmp = r[i] * r[i];
    num = num + tmp;
    tmp = p[i] * q[i];
    denom = denom + tmp;
}
alpha = num / denom;

@noinf for (i = 0; i < N; ++i)
{
    tmp = alpha * p[i];
    next_x[i] = x[i] + tmp;
    tmp = alpha * q[i];
    next_r[i] = r[i] - tmp;
}
num = 0;
denom = 0;
@noinf for (uint i = 0; i < N; ++i)
{
tmp = next_r[i] * next_r[i];
num = num + tmp;
tmp = r[i] * r[i];
denom = denom + tmp;
}
beta = num / denom;

@noinf for (i = 0; i < N; ++i)
{
tmp = beta * p[i];
next_p[i] = next_r[i] + tmp;
}
++it;
p = next_p;
x = next_x;
r = next_r;
++man_mod;
if (man_mod == M)
{
man_mod = 0;
}
return x;