Abstract

In this article a new $C^*$-algebra derived from the basic quantum variables: holonomies along paths and group-valued quantum flux operators in the framework of Loop Quantum Gravity is constructed. This development is based on the theory of cross-products and $C^*$-dynamical systems. In [9] the author has presented a set of actions of the flux group associated to a surface set on the analytic holonomy $C^*$-algebra, which define $C^*$-dynamical systems. These objects are used to define the holonomy-flux cross-product $C^*$-algebra associated to a surface set. Furthermore surface-preserving path- and graph-diffeomorphism-invariant states of the new $C^*$-algebra are analysed. Finally the holonomy-flux cross-product $C^*$-algebra is extended such that the graph-diffeomorphisms generate among other operators the holonomy-flux-graph-diffeomorphism cross-product $C^*$-algebra associated to a surface set.

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1 Introduction

In [8, 7] the quantum configuration variables, the quantum momentum variables and the spatial diffeomorphisms have been introduced briefly. These objects have been used to define the Weyl $C^*$-algebra for surfaces in [8, 7]. In this article the quantum variables are presented in section 2. The quantum configuration variables are holonomies along paths in a graph. The finite set of subgraphs of a graph forms a finite graph system. The configuration space is denoted by $\tilde{A}_T$ and is naturally identified with $G^{[T]}$, where $G$ is the structure group of a principal fibre bundle $P(\Sigma, G, \pi)$ and $|\Gamma|$ denotes the number of independent edges of the graph $\Gamma$. For generality it is assumed that, the structure group $G$ is a unimodular locally compact group. The quantum momentum variables are given by the group-valued quantum flux operators, which depend on a surface and a graph. For a certain surface set these operators form a group, which is called the flux group associated to a surface and a graph and which is denoted by $\tilde{G}_{S,\Gamma}$. For a certain fixed surface set this group is identified with $G^M$, where $M \leq |\Gamma|$.

The Weyl algebra of Quantum Geometry [5] or the Weyl algebra for surfaces [9, 7 Chapt. 6] are not the only $C^*$-algebras which will be constructed from the quantum configuration and momentum operators of the theory of Loop Quantum Gravity. The significant choice for a construction of the Weyl algebra for surfaces was the requirement of the group-valued quantum flux operators to be unitary Hilbert space operators. If this choice is not made, then the flux operators can be represented on a Hilbert space by the generalised group-valued quantum flux operators, which are given by the integrated representations of the flux group associated to a surface set. In this article even more general objects, which are given by algebra-valued functions depending on the flux group associated to a surface set, are introduced. The algebras are derived from these objects are cross-product $C^*$-algebras, which have been studied intensively by Williams [20], Hewitt and Ross [6] or Pedersen [15]. For a short overview refer to Blackadar [3].

Algebras derived from either quantum configuration or momentum variables

Let $G$ be a locally compact unimodular group. To start with consider the quantum momentum space, which contains all flux groups associated to surfaces. A certain flux group $\tilde{G}_{S,\Gamma}$ associated to a fixed surface set $\tilde{S}$ can be identified with $G^{[T]}$. Then the following algebras are studied in section 3. The convolution flux $*$-algebra $C(\tilde{G}_{S,\Gamma})$ associated to a surface set and a graph. This algebra is for in general a non-commutative $*$-algebra. Moreover, the flux group $C^*$-algebra $C^*(\tilde{G}_{S,\Gamma})$ associated to a surface set and a graph is derived from the generalised group-valued quantum flux operators. Finally, a particular cross-product $C^*$-algebra is given by the flux transformation group $C^*$-algebra $C^*(\tilde{G}_{S,\Gamma}, \tilde{G}_{S,\Gamma})$ associated to a surface set and a graph, which contains algebra-valued functions on the flux group $\tilde{G}_{S,\Gamma}$.

It is assumed that, the configuration space $\tilde{A}_T$ restricted to a fixed graph system $P_T$ is naturally identified with $G^{[T]}$. Then the convolution holonomy $*$-algebra $C(\tilde{A}_T)$ associated to a graph, the non-commutative holonomy $C^*$-algebra $C^*(\tilde{A}_T)$ associated to a graph and the heat-kernel holonomy $C^*$-algebra $C^*(\tilde{A}_T, \tilde{A}_T)$ associated to a graph is introduced. Note that, the analytic holonomy $C^*$-algebra $C(\tilde{A}_T)$ differs from the non-commutative holonomy $C^*$-algebra $C^*(\tilde{A}_T)$ in the multiplication operation and involution.

The construction of cross-products for the quantum configuration variables restricted to a graph is related to the observation, which has been noticed by Ashtekar and Lewandowski [2]. In the context of heat kernels the authors Lewandowski and Ashtekar [2, section 6.2] have presented an object, which can be understood as a generalised heat kernel representation $\pi^H_T$ of the non-commutative holonomy $C^*$-algebra $C^*(\tilde{A}_T)$ associated to a graph $\Gamma$ on the Hilbert space $\mathcal{H}_\Gamma := L^2(\tilde{A}_T, d\mu_\Gamma)$. This representation is given by

$$
\pi^H_T(\rho_\Gamma, \Gamma)\psi_\Gamma = \int_{\tilde{A}_\Gamma} d\mu_\Gamma(\hat{h}_\Gamma)\rho_\Gamma(\hat{h}_\Gamma^{-1}\hat{h}_\Gamma)\psi_\Gamma(\hat{h}_\Gamma)
= \rho_\Gamma(\Gamma) \psi_\Gamma
$$

(1)

where $\hat{h}_\Gamma, \hat{h}_\Gamma$ are two different holonomies along paths of a graph $\Gamma$, $\rho_\Gamma, \psi_\Gamma \in C^*(\tilde{A}_\Gamma)$ and $\psi_\Gamma \in \mathcal{H}_\Gamma$.
The inductive limit of the inductive family \( \{ C^*({\bar{A}}_\Gamma), \beta_{T',T} \} \) of \( C^* \)-algebras is called the non-commutative holonomy \( C^* \)-algebra \( C^*({\bar{A}}) \). Furthermore the inductive limit \( C^* \)-algebra of the inductive family \( \{ C^*({\bar{A}}_\Gamma, {\bar{A}}_\Gamma), \beta_{T',T} \} \) is called the heat-kernel holonomy \( C^* \)-algebra \( C^*({\bar{A}}, {\bar{A}}) \).

### Algebras derived from quantum configuration and momentum variables

In section 4, algebras are constructed from holonomy along paths and group-valued quantum fluxes by using \( C^* \)-dynamical systems. The concept of \( C^* \)-dynamical systems in LQG has been introduced in 3.4. In particular the analytic holonomy \( C^* \)-algebra restricted to a finite graph system \( P_T \) and an action \( \alpha \) of a certain flux group \( G_{S,\Gamma} \) associated to a surface set \( S \) on this algebra, form a \( C^* \)-dynamical system. In this article different holonomy-flux cross-product \( C^* \)-algebras associated to certain surface sets are constructed for these \( C^* \)-dynamical systems.

Then the holonomy-flux cross-product \( C^* \)-algebra \( C_0({\bar{A}}_\Gamma) \rtimes_{\alpha} G_{S,\Gamma} \) associated to a surface set and a graph contains \( C_0({\bar{A}}_\Gamma) \)-valued functions depending on the flux group \( G_{S,\Gamma} \). Let \( G \) be a compact group. Then there is an limit \( \bar{C}^* \)-algebra of the inductive family \( \{ C({\bar{A}}_\Gamma) \rtimes_{\alpha} G_{S,\Gamma}, \beta_{T',T} \} \), which is called the the holonomy-flux cross-product \( C^* \)-algebra \( C({\bar{A}}) \rtimes_{\alpha} G_{S} \) associated to a surface set.

Since there are many different \( C^* \)-dynamical systems presented in 4.3, there are a lot of different holonomy-flux cross-product \( C^* \)-algebras associated to suitable surface sets. These algebras are compared with the Weyl \( C^* \)-algebra for surfaces in section 3. In particular in theorem 1.3 it is proved that, the multiplier algebra of the holonomy-flux cross-product \( C^* \)-algebra associated to a certain surface set and a graph \( \Gamma \) contains all operators, which are contained in the other cross-product \( C^* \)-algebras associated to suitable surface sets and the graph \( \Gamma \), in the analytic holonomy \( C^* \)-algebra restricted to the finite graph system \( P_T \) and all Weyl elements for suitable surface sets and the graph \( \Gamma \).

All algebras presented in the previous paragraphs are constructed from the basic quantum variables, which are given by the holonomy along paths and the group-valued quantum fluxes. Hence they are possible algebras of a quantum theory of gravity.

### Simplified algebras derived from certain quantum configuration and momentum variables

If both quantum variables: the quantum configuration and momentum variables restricted to a fixed graph \( \Gamma \) and a fixed suitable surface set \( S \) are considered simultaneously, then the following simplifications can be studied.

In section 3.2 the flux transformation \( C^* \)-algebra associated to a surface set and a graph is presented. Similarly the flux group of a fixed graph \( \Gamma \) and a fixed suitable surface set \( S \) and the configuration space \( {\bar{A}}_\Gamma \) are identified with \( G^{[\Gamma]} \). Hence in both cases the cross-product \( C^* \)-algebras are simplified to \( C_0(G^{[\Gamma]}) \rtimes_{\alpha} G^{[\Gamma]} \).

Then it is verified in theorem 3.9 that the cross-product \( C^* \)-algebra \( C_0(G^{[\Gamma]}) \rtimes_{\alpha} G^{[\Gamma]} \) is Morita equivalent to the \( C^* \)-algebra of compact operators on the Hilbert space \( L^2(G^{[\Gamma]}, \mu_T) \), where \( \mu_T \) denotes the product of Haar measures. Therefore the representation theory of both \( C^* \)-algebras is the same and, hence, there is only one irreducible representation of the cross-product \( C^* \)-algebra up to unitary equivalence.

But this identification is only true for certain surface sets. The cross-product \( C^* \)-algebra is derived from the quantum momentum variables, which depend on different surface sets. In particular the flux group associated to a suitable surface set is identified with a product group \( G^M \) where \( M \leq |\Gamma| \). Then there exists a purely left (or right) action of \( G^M \) on the \( C^* \)-algebra \( C_0(G^{[\Gamma]}) \). For \( M < |\Gamma| \) a Morita equivalent \( C^* \)-algebra is not found in this project. In theorem 3.10 a Morita equivalent algebra for the \( C^* \)-algebra \( C_0(G^N) \rtimes_{\alpha} G^M \) whenever \( N < M \), is given.

In this article the general case of arbitrary surfaces is studied. Hence the quantum configuration and the momentum variables of the theory are manifestly distinguished from each other. The quantum configuration variables only depends on graphs and holonomy mappings, whereas the quantum momentum variables depend on graphs, maps from graphs to products of the structure group and the intersection behavior of the paths of the graphs and surfaces. But nevertheless the elements of the holonomy-flux cross-product \( C^* \)-algebra are understood as compact operators on the flux group associated to a surface set with values in the analytic holonomy \( C^* \)-algebra restricted to a graph, which are acting on the Hilbert space \( L^2({\bar{A}}_\Gamma, \mu_T) \).
States of algebras derived from certain quantum configuration and momentum variables

There exists several inductive limit holonomy-flux cross-product $C^*$-algebras, which are given by the inductive families of holonomy-flux cross-product $C^*$-algebras associated to graphs and a suitable surface set. The states on these algebras always depend on the choice of the surface set and, hence, they are not path- or graph-diffeomorphism invariant.

Algebras derived from quantum configuration and momentum variables and quantum spatial diffeomorphisms

In the last paragraphs new $C^*$-algebras of a special kind have been constructed. All these algebras are based on new operators, which are more general than group-valued quantum flux operators and which take in particular values in the analytic holonomy $C^*$-algebra. Until now the quantum diffeomorphisms are implemented only as automorphisms on these algebras. In section 5 one of the previous algebras is extended such that functions on the group of bisections of a finite graph system to the holonomy-flux cross-product $C^*$-algebra, form this new $C^*$-algebra.

The cross-product $C^*$-algebra construction is particularly based on $C^*$-dynamical systems. In the article [2], [7, section 6.2] it has been argued that, the action of the group of bisections of a finite graph system on the analytic holonomy $C^*$-algebra restricted to a finite graph system defines a $C^*$-dynamical system, too. Furthermore there is also an action of the group of certain bisections of a finite graph system on the holonomy-flux cross-product $C^*$-algebra associated to the surface set $\tilde{S}$ and a graph. These objects define another $C^*$-dynamical system and a new cross-product $C^*$-algebra, which is called the holonomy-flux-graph-diffeomorphism cross-product $C^*$-algebra.

There exists a covariant representation of this $C^*$-dynamical system on a Hilbert space. This pair is given by a unitary representation of the group of surface-orientation-preserving bisections of a finite graph system on the Hilbert space $\mathcal{H}_\Gamma$ and the multiplication representation $\Phi_M$ of the analytic holonomy $C^*$-algebra restricted to the finite graph system $\mathcal{P}_\Gamma$ on $\mathcal{H}_\Gamma$. The unitaries are called the unitary bisections of a finite graph system and surfaces in the project AQV. Then each unitary bisections of a finite graph system and surfaces is contained in the multiplier algebra of the holonomy-flux-graph-diffeomorphism cross-product $C^*$-algebra associated to a graph and the surface set. The remarkable point is that the multiplier algebra of the holonomy-flux cross-product $C^*$-algebra associated to a graph and the surface set does not contain these unitaries.

In general the multiplier algebra of the holonomy-flux cross-product $C^*$-algebra associated to a fixed surface set contains all operators of holonomy-flux cross-product $C^*$-algebra for other suitable surface sets, elements of the analytic holonomy $C^*$-algebra and all Weyl elements associated to other suitable surface sets. The Weyl $C^*$-algebra for surfaces contains elements of the analytic holonomy $C^*$-algebra and all Weyl elements. The multiplier algebra of the holonomy-flux cross-product $C^*$-algebra associated to the surface set $\tilde{S}$ contains the Weyl algebra for suitable surface sets. The Lie algebra-valued quantum flux operators and the right-invariant vector fields are affiliated with the holonomy-flux cross-product $C^*$-algebra, but they are not affiliated with the Weyl $C^*$-algebra for surfaces. For a detailed overview about the multiplier algebras and affiliated elements with the $C^*$-algebras of quantum variables refer to [2, table 11.6].

2 The basic quantum operators

2.1 Finite path groupoids and graph systems

Let $c : [0, 1] \to \Sigma$ be continuous curve in the domain $[0, 1]$, which is (piecewise) $C^k$-differentiable ($1 \leq k \leq \infty$), analytic ($k = \omega$) or semi-analytic ($k = \omega \omega$) in $[0, 1]$ and oriented such that the source vertex is $c(0) = s(c)$ and the target vertex is $c(1) = t(c)$. Moreover assume that, the range of each subinterval of the curve $c$ is a submanifold, which can be embedded in $\Sigma$. An edge is given by a reparametrisation invariant curve of class (piecewise) $C^k$. The maps $s_\Sigma, t_\Sigma : P\Sigma \to \Sigma$ where $P\Sigma$ is the path space are surjective maps and are called the source or target map.
A set of edges \( \{e_i\}_{i=1,...,N} \) is called **independent**, if the only intersections points of the edges are source \( s(e_i) \) or target \( t(e_i) \) target points. Composed edges are called **paths**. An **initial segment** of a path \( \gamma \) is a path \( \gamma_1 \) such that there exists another path \( \gamma_2 \) and \( \gamma = \gamma_1 \circ \gamma_2 \). The second element \( \gamma_2 \) is also called a **final segment** of the path \( \gamma \).

**Definition 2.1.** A **graph** \( \Gamma \) is a union of finitely many independent edges \( \{e_i\}_{i=1,...,N} \) for \( N \in \mathbb{N} \). The set \( \{e_1,...,e_N\} \) is called the **generating set** for \( \Gamma \). The number of edges of a graph is denoted by \( |\Gamma| \). The elements of the set \( V_{\Gamma} := \{s_{\Sigma}(e_k), t_{\Sigma}(e_k)\}_{k=1,...,N} \) of source and target points are called **vertices**.

A graph generates a finite path groupoid in the sense that, the set \( \mathcal{P}_{\Gamma} \Sigma \) contains all independent edges, their inverses and all possible compositions of edges. All the elements of \( \mathcal{P}_{\Gamma} \Sigma \) are called paths associated to a graph. Furthermore the surjective source and target maps \( s_{\Sigma} \) and \( t_{\Sigma} \) are restricted to the maps \( s, t : \mathcal{P}_{\Gamma} \Sigma \rightarrow V_{\Gamma} \), which are required to be surjective.

**Definition 2.2.** Let \( \Gamma \) be a graph. Then a **finite path groupoid** \( \mathcal{P}_{\Gamma} \Sigma \) over \( V_{\Gamma} \) is a pair \( (\mathcal{P}_{\Gamma} \Sigma, V_{\Gamma}) \) of finite sets equipped with the following structures:

\[
\begin{align*}
(i) \text{ two surjective maps } s, t : \mathcal{P}_{\Gamma} \Sigma \rightarrow V_{\Gamma}, \text{ which are called the source and target map,} \\
(ii) \text{ the set } \mathcal{P}_{\Gamma} \Sigma^2 := \{(\gamma_i, \gamma_j) \in \mathcal{P}_{\Gamma} \Sigma \times \mathcal{P}_{\Gamma} \Sigma : t(\gamma_i) = s(\gamma_j)\} \text{ of finitely many composable pairs of paths,} \\
(iii) \text{ the composition } \circ : \mathcal{P}_{\Gamma} \Sigma^2 \rightarrow \mathcal{P}_{\Gamma} \Sigma, \text{ where } (\gamma_i, \gamma_j) \mapsto \gamma_i \circ \gamma_j, \\
(iv) \text{ the inversion map } \gamma_i \mapsto \gamma_i^{-1} \text{ of a path,} \\
(v) \text{ the object inclusion map } i : V_{\Gamma} \rightarrow \mathcal{P}_{\Gamma} \Sigma \text{ and} \\
(vi) \mathcal{P}_{\Gamma} \Sigma \text{ is defined by the set } \mathcal{P}_{\Gamma} \Sigma \text{ modulo the algebraic equivalence relations generated by} \\
\gamma_i^{-1} \circ \gamma_i \simeq \mathbb{1}_{s(\gamma_i)} \text{ and } \gamma_i \circ \gamma_i^{-1} \simeq \mathbb{1}_{t(\gamma_i)} \quad (2)
\end{align*}
\]

Shortly write \( \mathcal{P}_{\Gamma} \Sigma \xrightarrow{s} V_{\Gamma} \).

Clearly, a graph \( \Gamma \) generates freely the paths in \( \mathcal{P}_{\Gamma} \Sigma \). Moreover the map \( s \times t : \mathcal{P}_{\Gamma} \Sigma \rightarrow V_{\Gamma} \times V_{\Gamma} \) defined by \( (s \times t)(\gamma) = (s(\gamma), t(\gamma)) \) for all \( \gamma \in \mathcal{P}_{\Gamma} \Sigma \) is assumed to be surjective (\( \mathcal{P}_{\Gamma} \Sigma \) over \( V_{\Gamma} \) is a transitive groupoid), too.

A general groupoid \( G \) over \( G^0 \) defines a small category where the set of morphisms is denoted in general by \( G \) and the set of objects is denoted by \( G^0 \). Hence in particular the path groupoid can be viewed as a category, since,

\[
\begin{align*}
\text{ - the set of morphisms is identified with } \mathcal{P}_{\Gamma} \Sigma, \\
\text{ - the set of objects is given by } V_{\Gamma} \text{ (the units)}
\end{align*}
\]

From the condition (2) it follows that, the path groupoid satisfies additionally

\[
\begin{align*}
(i) \text{ } s(\gamma_i \circ \gamma_j) = s(\gamma_i) \text{ and } t(\gamma_i \circ \gamma_j) = t(\gamma_j) \text{ for every } (\gamma_i, \gamma_j) \in \mathcal{P}_{\Gamma} \Sigma^2 \\
(ii) \text{ } s(v) = v = t(v) \text{ for every } v \in V_{\Gamma} \\
(iii) \text{ } \gamma \circ \mathbb{1}_{s(\gamma)} = \gamma = \mathbb{1}_{t(\gamma)} \circ \gamma \text{ for every } \gamma \in \mathcal{P}_{\Gamma} \Sigma \text{ and} \\
(iv) \text{ } \gamma \circ (\gamma_i \circ \gamma_j) = (\gamma \circ \gamma_i) \circ \gamma_j \\
(v) \text{ } \gamma \circ (\gamma_i^{-1} \circ \gamma_1) = \gamma_1 = (\gamma_1 \circ \gamma) \circ \gamma^{-1}
\end{align*}
\]

The condition [iii] implies that the vertices are units of the groupoid.
Definition 2.3. Denote the set of all finitely generated paths by

\[ \mathcal{P}_T \mathcal{\Sigma}^{(n)} := \{(\gamma_1, ..., \gamma_n) \in \mathcal{P}_T \times ... \mathcal{P}_T : (\gamma_i, \gamma_{i+1}) \in \mathcal{P}^{(2)}, 1 \leq i \leq n - 1 \} \]

The set of paths with source point \( v \in V_T \) is given by

\[ \mathcal{P}_T \mathcal{\Sigma}^v := s^{-1}(\{v\}) \]

The set of paths with target point \( v \in V_T \) is defined by

\[ \mathcal{P}_T \mathcal{\Sigma}_v := t^{-1}(\{v\}) \]

The set of paths with source point \( v \in V_T \) and target point \( u \in V_T \) is

\[ \mathcal{P}_T \mathcal{\Sigma}_{vu} := \mathcal{P}_T \mathcal{\Sigma}^v \cap \mathcal{P}_T \mathcal{\Sigma}_u \]

A graph \( \Gamma \) is said to be disconnected if it contains only mutually pairs \((\gamma_i, \gamma_j)\) of non-composable independent paths \( \gamma_i \) and \( \gamma_j \) for \( i \neq j \) and \( i, j = 1, ..., N \). In other words for all \( 1 \leq i, l \leq N \) it is true that \( s(\gamma_i) \neq t(\gamma_l) \) and \( t(\gamma_i) \neq s(\gamma_l) \) where \( i \neq l \) and \( \gamma_i, \gamma_l \in \Gamma \).

Definition 2.4. Let \( \Gamma \) be a graph. A subgraph \( \Gamma' \) of \( \Gamma \) is a given by a finite set of independent paths in \( \mathcal{P}_T \mathcal{\Sigma} \).

For example let \( \Gamma := \{\gamma_1, ..., \gamma_N\} \) then \( \Gamma' := \{\gamma_1 \circ \gamma_2, \gamma_3^{-1} \circ \gamma_4\} \) where \( \gamma_1 \circ \gamma_2, \gamma_3^{-1}, \gamma_4 \in \mathcal{P}_T \mathcal{\Sigma} \) is a subgraph of \( \Gamma \), whereas the set \( \{\gamma_1, \gamma_1 \circ \gamma_2\} \) is not a subgraph of \( \Gamma \). Notice if additionally \((\gamma_2, \gamma_4) \in \mathcal{P}_T \mathcal{\Sigma}^{(2)} \) holds, then \( \{\gamma_1, \gamma_3^{-1} \circ \gamma_2 \circ \gamma_4\} \) is a subgraph of \( \Gamma \), too. Moreover for \( \Gamma := \{\gamma\} \) the graph \( \Gamma^{-1} := \{\gamma^{-1}\} \) is a subgraph of \( \Gamma \). As well the graph \( \Gamma \) is a subgraph of \( \Gamma^{-1} \). A subgraph of \( \Gamma \) that is generated by compositions of some paths, which are not reversed in their orientation, of the set \( \{\gamma_1, ..., \gamma_N\} \) is called an orientation preserved subgraph of a graph. For example for \( \Gamma := \{\gamma_1, ..., \gamma_N\} \) orientation preserved subgraphs are given by \( \{\gamma_1 \circ \gamma_2\} \), \( \{\gamma_1, \gamma_2, \gamma_3\} \) or \( \{\gamma_{N-2} \circ \gamma_{N-1}\} \) if \( (\gamma_1, \gamma_2) \in \mathcal{P}_T \mathcal{\Sigma}^{(2)} \) and \( (\gamma_{N-2}, \gamma_{N-1}) \in \mathcal{P}_T \mathcal{\Sigma}^{(2)} \).

Definition 2.5. A finite graph system \( \mathcal{P}_T \mathcal{\Gamma} \) for \( \Gamma \) is a finite set of subgraphs of a graph \( \Gamma \). A finite graph system \( \mathcal{P}_T \mathcal{\Gamma} \) for \( \Gamma' \) is a finite graph subsystem of \( \mathcal{P}_T \mathcal{\Gamma} \) for \( \Gamma \) if the set \( \mathcal{P}_T \mathcal{\Gamma} \) is a subset of \( \mathcal{P}_T \mathcal{\Gamma} \) and \( \Gamma' \) is a subgraph of \( \Gamma \). Shortly write \( \mathcal{P}_T \mathcal{\Gamma} \preceq \mathcal{P}_T \mathcal{\Gamma} \).

A finite orientation preserved graph system \( \mathcal{P}_T \mathcal{\Gamma} \) for \( \Gamma \) is a finite set of orientation preserved subgraphs of a graph \( \Gamma \).

Recall that, a finite path groupoid is constructed from a graph \( \Gamma \), but a set of elements of the path groupoid need not be a graph again. For example let \( \Gamma := \{\gamma_1 \circ \gamma_2\} \) and \( \Gamma' := \{\gamma_1 \circ \gamma_3\} \), then \( \Gamma'' = \Gamma \cup \Gamma' \) is not a graph, since this set is not independent. Hence only appropriate unions of paths, which are elements of a fixed finite path groupoid, define graphs. The idea is to define a suitable action on elements of the path groupoid, which corresponds to an action of diffeomorphisms on the manifold \( \Sigma \). The action has to be transferred to graph systems. But the action of bisection, which is defined by the use of the groupoid multiplication, cannot easily generalised for graph systems.

Problem 2.1: Let \( \tilde{\Gamma} := \{\Gamma_i\}_{i=1,...,N} \) be a finite set such that each \( \Gamma_i \) is a set of not necessarily independent paths such that

(i) the set contains no loops and

(ii) each pair of paths satisfies one of the following conditions

- the paths intersect each other only in one vertex,
- the paths do not intersect each other or
- one path of the pair is a segment of the other path.

Then there is a map \( \circ : \tilde{\Gamma} \times \tilde{\Gamma} \rightarrow \tilde{\Gamma} \) of two elements \( \Gamma_1 \) and \( \Gamma_2 \) defined by

\[ \{\gamma_1, ..., \gamma_M\} \circ \{\tilde{\gamma}_1, ..., \tilde{\gamma}_M\} := \{\gamma_i \circ \tilde{\gamma}_j : \gamma_i = \tilde{\gamma}_j\}_{1 \leq i, j \leq M} \]
for $\Gamma_1 := \{\gamma_1, \ldots, \gamma_M\}, \Gamma_2 := \{\tilde{\gamma}_1, \ldots, \tilde{\gamma}_M\}$. Moreover define a map $-1 : \tilde{\Gamma} \to \Gamma$ by

$$\{\gamma_1, \ldots, \gamma_M\}^{-1} := \{\gamma_1^{-1}, \ldots, \gamma_M^{-1}\}$$

Then the following is derived

$$\{\gamma_1, \ldots, \gamma_M\} \circ \{\gamma_1^{-1}, \ldots, \gamma_M^{-1}\} = \left\{\gamma_i \circ \gamma_j^{-1} : t(\gamma_i) = t(\gamma_j) \right\}_{1 \leq i, j \leq M}$$

and

$$\{\gamma_1, \ldots, \gamma_M\} \circ \{\gamma_1^{-1}, \ldots, \gamma_M^{-1}\} = \left\{\gamma_i \circ \gamma_j^{-1} : t(\gamma_i) = t(\gamma_j) \text{ and } i \neq j \right\}_{1 \leq i, j \leq M}$$

and

$$\cup \{\mathbb{I}_{\gamma_j} \}_{1 \leq j \leq M}$$

The equality is true, if the set $\tilde{\Gamma}$ contains only graphs such that all paths are mutually non-composable. Consequently this does not define a well-defined multiplication map. Notice that, the same is discovered if a similar map and inversion operation are defined for a finite graph system $P_\Gamma$.

Consequently the property of paths being independent need not be dropped for the definition of a suitable multiplication and inversion operation. In fact the independence property is a necessary condition for the construction of the holonomy algebra for analytic paths. Only under this circumstance each analytic path is decomposed into a finite product of independent piecewise analytic paths again.

**Definition 2.6.** A finite path groupoid $P_{\Gamma, \Sigma}$ over $V_{\Gamma}$, is a finite path subgroupoid of $P_{\Gamma, \Sigma}$ over $V_{\Gamma}$ if the set $V_{\Gamma'}$ is contained in $V_{\Gamma}$ and the set $P_{\Gamma, \Sigma}$ is a subset of $P_{\Gamma, \Sigma}$. Shortly write $P_{\Gamma, \Sigma} \leq P_{\Gamma, \Sigma}$.

Clearly for a subgraph $\Gamma_1$ of a graph $\Gamma_2$, the associated path groupoid $P_{\Gamma_1, \Sigma}$ over $V_{\Gamma_1}$ is a subgroupoid of $P_{\Gamma_2, \Sigma}$ over $V_{\Gamma_2}$. This is a consequence of the fact that, each path in $P_{\Gamma_1, \Sigma}$ is a composition of paths or their inverses in $P_{\Gamma_2, \Sigma}$.

**Definition 2.7.** A family of finite path groupoids $\{P_{\Gamma_i, \Sigma}\}_{i=1, \ldots, \infty}$, which is a set of finite path groupoids $P_{\Gamma_i, \Sigma}$ over $V_{\Gamma_i}$, is said to be inductive if for any $P_{\Gamma_1, \Sigma}, P_{\Gamma_2, \Sigma}$ exists a $P_{\Gamma_3, \Sigma}$ such that $P_{\Gamma_1, \Sigma}, P_{\Gamma_2, \Sigma} \leq P_{\Gamma_3, \Sigma}$.

A family of graph systems $\{P_{\Gamma_i}\}_{i=1, \ldots, \infty}$, which is a set of finite path systems $P_{\Gamma_i}$, for $\Gamma_i$, is said to be inductive if for any $P_{\Gamma_1}, P_{\Gamma_2}$ exists a $P_{\Gamma_3}$ such that $P_{\Gamma_1}, P_{\Gamma_2} \leq P_{\Gamma_3}$.

**Definition 2.8.** Let $\{P_{\Gamma_i, \Sigma}\}_{i=1, \ldots, \infty}$ be an inductive family of path groupoids and $\{P_{\Gamma_i}\}_{i=1, \ldots, \infty}$ be an inductive family of graph systems.

The inductive limit path groupoid $P$ over $\Sigma$ of an inductive family of finite path groupoids such that $P := \lim_{i \to \infty} P_{\Gamma_i, \Sigma}$ is called the (algebraic) path groupoid $P \equiv \Sigma$.

Moreover there exists an inductive limit graph $\Gamma_\infty$ of an inductive family of graphs such that $\Gamma_\infty := \lim_{i \to \infty} \Gamma_i$.

The inductive limit graph system $P_{\Gamma_\infty}$ of an inductive family of graph systems such that $P_{\Gamma_\infty} := \lim_{i \to \infty} P_{\Gamma_i}$.

Assume that, the inductive limit $\Gamma_\infty$ of a inductive family of graphs is a graph, which consists of an infinite countable number of independent paths. The inductive limit $P_{\Gamma_\infty}$ of a inductive family $\{P_{\Gamma_i}\}$ of finite graph systems contains an infinite countable number of subgraphs of $\Gamma_\infty$ and each subgraph is a finite set of arbitrary independent paths in $\Sigma$.

### 2.2 Holonomy maps for finite path groupoids, graph systems and transformations

In section 2.1 the concept of finite path groupoids for analytic paths has been given. Now the holonomy maps are introduced for finite path groupoids and finite graph systems. The ideas are familiar with those presented by Thiemann [25]. But for example the finite graph systems have not been studied before. Ashtekar and Lewandowski [11] have defined the analytic holonomy $C^*$-algebra, which they have based on a finite set of independent loops. The loops are generalised for path groupoids and the independence requirement is implemented by the concept of finite graph systems.
2.2.1 Holonomy maps for finite path groupoids

Groupoid morphisms for finite path groupoids

Let $G_1 \xrightarrow{s_1}{t_1} G_1^0$, $G_2 \xrightarrow{s_2}{t_2} G_2^0$ be two arbitrary groupoids.

Definition 2.9. A groupoid morphism between two groupoids $G_1$ and $G_2$ consists of two maps $h : G_1 \to G_2$ and $s : G_1^0 \to G_2^0$ such that

$$(G1) \quad h(\gamma \circ \gamma') = h(\gamma)h'(\gamma') \text{ for all } (\gamma, \gamma') \in G_1^{(2)}$$

$$(G2) \quad s_2(h(\gamma)) = h(s_1(\gamma)), \quad t_2(h(\gamma)) = h(t_1(\gamma))$$

A strong groupoid morphism between two groupoids $G_1$ and $G_2$ additionally satisfies

$$(SG) \quad \text{for every pair } (h(\gamma), h(\gamma')) \in G_2^{(2)} \text{ it follows that } (\gamma, \gamma') \in G_1^{(2)}$$

Let $G$ be a Lie group. Then $G$ over $e_G$ is a groupoid, where the group multiplication $\cdot : G^2 \to G$ is defined for all elements $g_1, g_2, g \in G$ such that $g_1 \cdot g_2 = g$. A groupoid morphism between a finite path groupoid $\mathcal{P}_\Gamma$ to $G$ is given by the maps

$h_\Gamma : \mathcal{P}_\Gamma \to G$, $h_\Gamma : V_\Gamma \to e_G$

Clearly

$h_\Gamma(\gamma \circ \gamma') = h_\Gamma(\gamma)h_\Gamma(\gamma')$ for all $\gamma, \gamma' \in \mathcal{P}_\Gamma^{(2)}$

$s_\Gamma(h_\Gamma(\gamma)) = h_\Gamma(s_\mathcal{P}_\Gamma(\gamma))$, $t_\Gamma(h_\Gamma(\gamma)) = h_\Gamma(t_\mathcal{P}_\Gamma(\gamma))$ \hspace{1cm} (3)

But for an arbitrary pair $(h_\Gamma(\gamma_1), h_\Gamma(\gamma_2)) =: (g_1, g_2) \in G^{(2)}$ it does not follow that, $(\gamma_1, \gamma_2) \in \mathcal{P}_\Gamma^{(2)}$ is true. Hence $h_\Gamma$ is not a strong groupoid morphism.

Definition 2.10. Let $\mathcal{P}_\Gamma \rightrightarrows V_\Gamma$ be a finite path groupoid.

Two paths $\gamma$ and $\gamma'$ in $\mathcal{P}_\Gamma$ have the same-holonomy for all connections iff

$h_\Gamma(\gamma) = h_\Gamma(\gamma')$ for all $(h_\Gamma, h_\Gamma)$ groupoid morphisms

$h_\Gamma : \mathcal{P}_\Gamma \to G, h : V_\Gamma \to \{e_G\}$

Denote the relation by $\sim_{s.hol}$.

Lemma 2.11. The same-holonomy for all connections relation is an equivalence relation.

Notice that, the quotient of the finite path groupoid and the same-holonomy relation for all connections replace the hoop group, which has been used in [\[1\]].

Definition 2.12. Let $\mathcal{P}_\Gamma \rightrightarrows V_\Gamma$ be a finite path groupoid modulo same-holonomy for all connections equivalence.

A holonomy map for a finite path groupoid $\mathcal{P}_\Gamma$ over $V_\Gamma$ is a groupoid morphism consisting of the maps $(h_\Gamma, h_\Gamma)$, where $h_\Gamma : \mathcal{P}_\Gamma \to G, h_\Gamma : V_\Gamma \to \{e_G\}$. The set of all holonomy maps is abbreviated by $\text{Hom}(\mathcal{P}_\Gamma, G)$.

For a short notation observe the following. In further sections it is always assumed that, the finite path groupoid $\mathcal{P}_\Gamma \rightrightarrows V_\Gamma$ is considered modulo same-holonomy for all connections equivalence although it is not stated explicitly.
2.2.2 Holonomy maps for finite graph systems

Ashtekar and Lewandowski \([1]\) have presented the loop decomposition into a finite set of independent hoops (in the analytic category). This structure is replaced by a graph, since a graph is a set of independent edges. Notice that, the set of hoops that is generated by a finite set of independent hoops, is generalised to the set of finite graph systems. A finite path groupoid is generated by the set of edges, which defines a graph \(\Gamma\), but a set of elements of the path groupoid need not be a graph again. The appropriate notion for graphs constructed from sets of paths is the finite graph system, which is defined in section \(2.1\). Now the concept of holonomy maps is generalised for finite graph systems. Since the set, which is generated by a finite number of independent edges, contains paths that are composable, there are two possibilities to identify the image of the holonomy map for a finite graph system on a fixed graph with a subgroup of \(G^{\mathbb{F}}\). One way is to use the generating set of independent edges of a graph, which has been also used in \([1]\). On the other hand, it is also possible to identify each graph with a disconnected subgraph of a fixed graph, which is generated by a set of independent edges. Notice that, the author implements two situations. One case is given by a set of paths that can be composed further and the other case is related to paths that are not composable. This is necessary for the definition of an action of the flux operators. Precisely the identification of the image of the holonomy maps along these paths is necessary to define a well-defined action of a flux element on the configuration space. This issue has been studied in \([9, 7]\).

First of all consider a graph \(\Gamma\) that is generated by the set \(\{\gamma_1, \ldots, \gamma_N\}\) of edges. Then each subgraph of a graph \(\Gamma\) contain paths that are composition of edges in \(\{\gamma_1, \ldots, \gamma_N\}\) or inverse edges. For example the following set \(\Gamma' := \{\gamma_1 \circ \gamma_2 \circ \gamma_3, \gamma_4\}\) defines a subgraph of \(\Gamma := \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}\). Hence there is a natural identification available.

**Definition 2.13.** A subgraph \(\Gamma'\) of a graph \(\Gamma\) is always generated by a subset \(\{\gamma_1, \ldots, \gamma_M\}\) of the generating set \(\{\gamma_1, \ldots, \gamma_N\}\) of independent edges that generates the graph \(\Gamma\). Hence each subgraph is identified with a subset of \(\{\gamma_1, \ldots, \gamma_M\}\). This is called the **natural identification of subgraphs**.

**Example 2.1:** For example consider a subgraph \(\Gamma' := \{\gamma_1 \circ \gamma_2, \gamma_3 \circ \gamma_4, \ldots, \gamma_{M-1} \circ \gamma_M\}\), which is identified naturally with a set \(\{\gamma_1, \ldots, \gamma_M\}\). The set \(\{\gamma_1, \ldots, \gamma_M\}\) is a subset of \(\{\gamma_1, \ldots, \gamma_N\}\) where \(N = |\Gamma|\) and \(M \leq N\).

Another example is given by the graph \(\Gamma'' := \{\gamma_1, \gamma_2\}\) such that \(\gamma_2 = \gamma_1' \circ e_2\), then \(\Gamma''\) is identified naturally with \(\{\gamma_1, \gamma_1', \gamma_2\}\). This set is a subset of \(\{\gamma_1, \gamma_1', \gamma_2, \gamma_3, \ldots, \gamma_{N-1}\}\).

**Definition 2.14.** Let \(\Gamma\) be a graph, \(\mathcal{P}_\Gamma\) be the finite graph system. Let \(\Gamma' := \{\gamma_1, \ldots, \gamma_M\}\) be a subgraph of \(\Gamma\).

A **holonomy map for a finite graph system** \(\mathcal{P}_\Gamma\) is a given by a pair of maps \((\mathfrak{h}_\Gamma, h_\Gamma)\) such that there exists a holonomy map \(\mathfrak{h}_\Gamma : \mathcal{P}_\Gamma \to G^{\mathbb{F}}\), \(h_\Gamma : \{\gamma_1, \ldots, \gamma_M\} \to \{e_G\}\) for the finite path groupoid \(\mathcal{P}_\Sigma \cong V_\Gamma\) and

\[
\mathfrak{h}_\Gamma : \mathcal{P}_\Gamma \to G^{\mathbb{F}}, \quad h_\Gamma(\{\gamma_1, \ldots, \gamma_M\}) = (\mathfrak{h}_\Gamma(\gamma_1), \ldots, \mathfrak{h}_\Gamma(\gamma_M), e_G, \ldots, e_G)
\]

The set of all holonomy maps for the finite graph system is denoted by \(\text{Hom}(\mathcal{P}_\Gamma, G^{\mathbb{F}})\).

The image of a map \(\mathfrak{h}_\Gamma\) on each subgraph \(\Gamma'\) of the graph \(\Gamma\) is given by

\[
(\mathfrak{h}_\Gamma(\gamma_1), \ldots, \mathfrak{h}_\Gamma(\gamma_M), e_G, \ldots, e_G)
\]

is an element of \(G^{\mathbb{F}}\). The set of all images of maps on subgraphs of \(\Gamma\) is denoted by \(\bar{A}_\Gamma\).

The idea is now to study two different restrictions of the set \(\mathcal{P}_\Gamma\) of subgraphs. For a short notation of a "set of holonomy maps for a certain restricted set of subgraphs of a graph" in this article the following notions are introduced.

**Definition 2.15.** If the subset of all disconnected subgraphs of the finite graph system \(\mathcal{P}_\Gamma\) is considered, then the restriction of \(\bar{A}_\Gamma\), which is identified with \(G^{\mathbb{F}}\) appropriately, is called the **non-standard identification of the configuration space**. If the subset of all natural identified subgraphs of the finite graph system \(\mathcal{P}_\Gamma\) is considered, then the restriction of \(\bar{A}_\Gamma\), which is identified with \(G^{\mathbb{F}}\) appropriately, is called the **natural identification of the configuration space**.

---

1In the work the holonomy map for a finite graph system and the holonomy map for a finite path groupoid is denoted by the same pair \((\mathfrak{h}_\Gamma, h_\Gamma)\).
A comment on the non-standard identification of $\tilde{A}_F$ is the following. If $\Gamma' := \{\gamma_1 \circ \gamma_2\}$ and $\Gamma'' := \{\gamma_2\}$ are two subgraphs of $\Gamma := \{\gamma_1, \gamma_2, \gamma_3\}$. The graph $\Gamma'$ is a subgraph of $\Gamma$. Then evaluation of a map $h_\Gamma$ on a subgraph $\Gamma'$ is given by

$$h_\Gamma(\Gamma') = (h_\Gamma(\gamma_1 \circ \gamma_2), h_\Gamma(s(\gamma_2)), h_\Gamma(s(\gamma_3))) = (h_\Gamma(\gamma_1)h_\Gamma(\gamma_2), e_G, e_G) \in G^3$$

and the holonomy map of the subgraph $\Gamma''$ of $\Gamma'$ is evaluated by

$$h_\Gamma(\Gamma'') = (h_\Gamma(s(\gamma_1)), h_\Gamma(s(\gamma_2))h_\Gamma(\gamma_2), h_\Gamma(s(\gamma_3))) = (h_\Gamma(\gamma_2), e_G, e_G) \in G^3$$

**Example 2.2:** Recall example 2.2[6]. For example for a subgraph $\Gamma' := \{\gamma_1 \circ \gamma_2, \gamma_3 \circ \gamma_4, \ldots, \gamma_{M-1} \circ \gamma_M\}$, which is naturally identified with a set $\{\gamma_1, \ldots, \gamma_M\}$. Then the holonomy map is evaluated at $\Gamma'$ such that

$$h_\Gamma(\Gamma') = (h_\Gamma(\gamma_1), h_\Gamma(\gamma_2), \ldots, h_\Gamma(\gamma_M), e_G, \ldots, e_G) \in G^N$$

where $N = |\Gamma|$. For example, let $\Gamma' := \{\gamma_1, \gamma_2\}$ such that $\gamma_2 = \gamma_1 \circ \gamma'_2$ and which is naturally identified with $\{\gamma_1, \gamma'_2\}$. Hence

$$h_\Gamma(\Gamma') = (h_\Gamma(\gamma_1), h_\Gamma(\gamma'_1), h_\Gamma(\gamma'_2), e_G, \ldots, e_G) \in G^N$$

is true.

Another example is given by the disconnected graph $\Gamma' := \{\gamma_1 \circ \gamma_2 \circ \gamma_3, \gamma_4\}$, which is a subgraph of $\Gamma := \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$. Then the non-standard identification is given by

$$h_\Gamma(\Gamma') = (h_\Gamma(\gamma_1 \circ \gamma_2 \circ \gamma_3), h_\Gamma(\gamma_4), e_G, e_G) \in G^4$$

If the natural identification is used, then $h_\Gamma(\Gamma')$ is identified with

$$(h_\Gamma(\gamma_1), h_\Gamma(\gamma_2), h_\Gamma(\gamma_3), h_\Gamma(\gamma_4)) \in G^4$$

Consider the following example. Let $\Gamma''' := \{\gamma_1, \alpha, \gamma_3\}$ be a graph such that

```
    γ₁
        ↘α
    γ₂  γ₃
```

Then notice the sets $\Gamma_1 := \{\gamma_1 \circ \alpha, \gamma_3\}$ and $\Gamma_2 := \{\gamma_1 \circ \alpha^{-1}, \gamma_3\}$. In the non-standard identification of the configuration space $\tilde{A}_{\Gamma'''}$, it is true that,

$$h_{\Gamma'''}(\Gamma_1) = (h_{\Gamma'''}(\gamma_1 \circ \alpha), h_{\Gamma'''}(\gamma_3), e_G, e_G) \in G^4$$

$$h_{\Gamma'''}(\Gamma_2) = (h_{\Gamma'''}(\gamma_1 \circ \alpha^{-1}), h_{\Gamma'''}(\gamma_3), e_G, e_G) \in G^4$$

holds. Whereas in the natural identification of $\tilde{A}_{\Gamma'''}$

$$h_{\Gamma'''}(\Gamma_1) = (h_{\Gamma'''}(\gamma_1), h_{\Gamma'''}(\alpha), h_{\Gamma'''}(\gamma_3), e_G) \in G^4$$

$$h_{\Gamma'''}(\Gamma_2) = (h_{\Gamma'''}(\gamma_1), h_{\Gamma'''}(\alpha^{-1}), h_{\Gamma'''}(\gamma_3), e_G) \in G^4$$

yields.

The equivalence class of similar or equivalent groupoid morphisms defined in definition ?? allows to define the following object. The set of images of all holonomy maps of a finite graph system modulo the similar or equivalent groupoid morphisms equivalence relation is denoted by $\tilde{A}_F/\tilde{\Theta}_F$.
2.3 The group-valued quantum flux operators associated to surfaces and graphs

Let $G$ be the structure group of a principal fibre bundle $P(\Sigma, G, \pi)$. Then the quantum flux operators, which are associated to a fixed surface $S$, are $G$-valued operators. For the construction of the quantum flux operator $\rho_S(\gamma)$ different maps from a graph $\Gamma$ to a direct product $G \times G$ are considered. This is related to the fact that, one distinguishes between paths that are ingoing and paths that are outgoing with respect to the surface orientation of $S$. If there are no intersection points of the surface $S$ and the source or target vertex of a path $\gamma$, then the map maps the path $\gamma$ to zero in both entries. For different surfaces or for a fixed surface different maps refer to different quantum flux operators.

**Definition 2.16.** Let $\hat{S}$ be a finite set $\{S_i\}$ of surfaces in $\Sigma$, which is closed under a flip of orientation of the surfaces. Let $\Gamma$ be a graph such that each path in $\Gamma$ satisfies one of the following conditions

- the path intersects each surface in $\hat{S}$ in the source vertex of the path and there are no other intersection points of the path and any surface contained in $\hat{S}$,
- the path intersects each surface in $\hat{S}$ in the target vertex of the path and there are no other intersection points of the path and any surface contained in $\hat{S}$,
- the path intersects each surface in $\hat{S}$ in the source and target vertex of the path and there are no other intersection points of the path and any surface contained in $\hat{S}$,
- the path does not intersect any surface $S$ contained in $\hat{S}$.

Finally let $\mathcal{P}_\Gamma$ denotes the finite graph system associated to $\Gamma$.

Then define the intersection functions $\iota_L : \hat{S} \times \Gamma \to \{\pm 1, 0\}$ such that

$$\iota_L(S, \gamma) := \begin{cases} 1 & \text{for a path } \gamma \text{ lying above and outgoing w.r.t. } S \\ -1 & \text{for a path } \gamma \text{ lying below and outgoing w.r.t. } S \\ 0 & \text{the path } \gamma \text{ is not outgoing w.r.t. } S \end{cases}$$

and the intersection functions $\iota_R : \hat{S} \times \Gamma \to \{\pm 1, 0\}$ such that

$$\iota_L(S, \gamma) := \begin{cases} -1 & \text{for a path } \gamma' \text{ lying above and ingoing w.r.t. } S \\ 1 & \text{for a path } \gamma' \text{ lying below and ingoing w.r.t. } S \\ 0 & \text{the path } \gamma' \text{ is not ingoing w.r.t. } S \end{cases}$$

whenever $S \in \hat{S}$ and $\gamma \in \Gamma$.

Define a map $o_L : \hat{S} \to G$ such that

$$o_L(S) = o_L(S^{-1})$$

whenever $S \in \hat{S}$ and $S^{-1}$ is the surface $S$ with reversed orientation. Denote the set of such maps by $\hat{o}_L$. Respectively the map $o_R : \hat{S} \to G$ such that

$$o_R(S) = o_R(S^{-1})$$

whenever $S \in \hat{S}$. Denote the set of such maps by $\hat{o}_R$. Moreover there is a map $o_L \times o_R : \hat{S} \to G \times G$ such that

$$(o_L, o_R)(S) = (o_L, o_R)(S^{-1})$$

whenever $S \in \hat{S}$. Denote the set of such maps by $\hat{o}$.

Then define the **group-valued quantum flux set for paths**

$$\mathcal{G}_{S, \Gamma} := \bigcup_{o_L, o_R \in \hat{o}} \bigcup_{S \in \hat{S}} \{(\rho^L, \rho^R) \in \text{Map}(\Gamma, G \times G) : \quad (\rho^L, \rho^R(\gamma)) := (o_L(S)^{\iota_L(S, \gamma)}, o_R(S)^{\iota_R(S, \gamma)})\}$$

where $\text{Map}(\Gamma, G \times G)$ denotes the set of all maps from the graph $\Gamma$ to the direct product $G \times G$. 

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Define the set of group-valued quantum fluxes for graphs

\[ G_{S, \Gamma} := \bigcup_{\sigma_L, \sigma_R \in \sigma} \bigcup_{S \in S} \{ \rho_{S, \Gamma} \in \text{Map}(P^*_{\Gamma}, \Gamma \times G) : \rho_{S, \Gamma} := \rho_S \times \ldots \times \rho_S \}
\]

where \( \rho_S(\gamma) := (s_L(\gamma, S), s_R(\gamma, S)) \),
\( \rho_S \in G_{S, \Gamma}, S \in \mathcal{S}, \gamma \in \Gamma \}\}

Notice if \( H \) is a closed subgroup of \( G \), then \( H_{S, \Gamma} \) can be defined in analogy to \( G_{S, \Gamma} \). In particular if the group \( H \) is replaced by the center \( Z(G) \) of the group \( G \), then the set \( G_{S, \Gamma} \) is replaced by \( Z(\mathcal{G}_{S, \gamma}) \), and \( G_{S, \Gamma} \) is changed to \( Z_{S, \Gamma} \).

Furthermore observe that, \((\nu_L \times \nu_R)S^{-1}, \gamma) = (\nu_L \times \nu_R)(S, \gamma)\) for every \( \gamma \in \Gamma \) holds. Remark that, the condition \( \rho^L(\gamma) = \rho^R(\gamma)^{-1} \) is not required.

**Example 2.3:** For example the following example can be analysed. Consider a graph \( \Gamma \) and two disjoint surface sets \( S \) and \( T \).

Then the elements of \( \mathcal{G}_{S, \Gamma} \) are for example the maps \( \rho^L_i \times \rho^R_i \) for \( i = 1, 2 \) such that

\[
\begin{align*}
\rho_1(\gamma) &:= (\rho^L_1, \rho^R_1)(\gamma) = (\sigma_L(S_1)^{\nu_L(S_1, \gamma)}, \sigma_R(S_1)^{\nu_R(S_1, \gamma)}) = (g_1, 0) \\
\rho_1(\tilde{\gamma}) &:= (\rho^L_1, \rho^R_1)(\tilde{\gamma}) = (\sigma_L(S_1)^{\nu_L(S_1, \tilde{\gamma})}, \sigma_R(S_1)^{\nu_R(S_1, \tilde{\gamma})}) = (g_1, 0) \\
\rho_2(\gamma) &:= (\rho^L_2, \rho^R_2)(\gamma) = (\sigma_L(S_2)^{\nu_L(S_2, \gamma)}, \sigma_R(S_2)^{\nu_R(S_2, \gamma)}) = (g_2, 0) \\
\rho_2(\tilde{\gamma}) &:= (\rho^L_2, \rho^R_2)(\tilde{\gamma}) = (\sigma_L(S_2)^{\nu_L(S_2, \tilde{\gamma})}, \sigma_R(S_2)^{\nu_R(S_2, \tilde{\gamma})}) = (g_2, 0) \\
\rho_3(\gamma) &:= (\rho^L_3, \rho^R_3)(\gamma) = (\sigma_L(S_3)^{\nu_L(S_3, \gamma)}, \sigma_R(S_3)^{\nu_R(S_3, \gamma)}) = (0, h_3^{-1}) \\
\rho_3(\tilde{\gamma}) &:= (\rho^L_3, \rho^R_3)(\tilde{\gamma}) = (\sigma_L(S_3)^{\nu_L(S_3, \tilde{\gamma})}, \sigma_R(S_3)^{\nu_R(S_3, \tilde{\gamma})}) = (0, h_3^{-1}) \\
\rho_4(\gamma) &:= (\rho^L_4, \rho^R_4)(\gamma) = (\sigma_L(S_4)^{\nu_L(S_4, \gamma)}, \sigma_R(S_4)^{\nu_R(S_4, \gamma)}) = (0, h_4) \\
\rho_4(\tilde{\gamma}) &:= (\rho^L_4, \rho^R_4)(\tilde{\gamma}) = (\sigma_L(S_4)^{\nu_L(S_4, \tilde{\gamma})}, \sigma_R(S_4)^{\nu_R(S_4, \tilde{\gamma})}) = (0, h_4)
\end{align*}
\]

This example shows that, the surfaces \( \{S_1, S_2\} \) are similar, whereas the surfaces \( \{T_1, T_2\} \) produce different signatures for different paths. Moreover the set of surfaces are chosen such that one component of the direct sum is always zero.

For a particular surface set \( S \), the following set is defined

\[
\bigcup_{\sigma_L, \sigma_R \in \sigma} \bigcup_{S \in S} \{ (\rho^L, \rho^R) \in \text{Map}(\Gamma, G \times G) : (\rho^L, \rho^R)(\gamma) := (\sigma_L(S)^{\nu_L(S, \gamma)}, 0) \}
\]

can be identified with

\[
\bigcup_{\sigma_L, \sigma_R \in \sigma} \bigcup_{S \in S} \{ \rho \in \text{Map}(\Gamma, G) : \rho(\gamma) := \sigma_L(S)^{\nu_L(S, \gamma)} \}
\]
The same is observed for another surface set $\tilde{T}$ and the set $\mathcal{G}_{T,\Gamma}$ is identifiable with

$$\bigcup_{\sigma_n \in \sigma_n} \bigcup_{T \in \tilde{T}} \left\{ \rho \in \text{Map}(\Gamma, G) : \rho(\gamma) := \sigma_{R(T)}(\Gamma, \gamma) \right\}$$

The intersection behavior of paths and surfaces plays a fundamental role in the definition of the flux operator. There are exceptional configurations of surfaces and paths in a graph. One of them is the following.

**Definition 2.17.** A surface $S$ has the **surface intersection property for a graph** $\Gamma$, if the surface intersects each path of $\Gamma$ once in the source or target vertex of the path and there are no other intersection points of $S$ and the path.

This is for example the case for the surface $S_1$ or the surface $S_3$, which are presented in example 2.3. Notice that in general, for the surface $S$ there are $N$ intersection points with $N$ paths of the graph. In the example the evaluated map $\rho_1(\gamma) = (g_1, 0) = \rho_1(\tilde{\gamma})$ for $\gamma, \tilde{\gamma} \in \Gamma$ if the surface $S_1$ is considered.

The property of a path lying above or below is not important for the definition of the surface intersection property for a surface. This indicates that the surface $S_4$ in the example 2.3 has the surface intersection property, too.

Let a surface $S$ does not have the surface intersection property for a graph $\Gamma$, which contains only one path $\gamma$. Then for example the path $\gamma$ intersects the surface $S$ in the source and target vertices such that the path lies above the surface $S$. Then the map $\rho^L \times \rho^R$ is evaluated for the path $\gamma$ by

$$(\rho^L \times \rho^R)(\gamma) = (g, h^{-1})$$

Hence simply speaking the surface intersection property reduces the components of the map $\rho^L \times \rho^R$, but for different paths to different components.

Now, consider a bunch of sets of surfaces such that for each surface there is only one intersection point.

**Definition 2.18.** A set $\tilde{S}$ of $N$ surfaces has the **surface intersection property for a graph** $\Gamma$ with $N$ independent edges, if it contains only surfaces, for which each path $\gamma_i$ of a graph $\Gamma$ intersects each surface $S_i$ only once in the same source or target vertex of the path $\gamma_i$, there are no other intersection points of each path $\gamma_i$ and each surface in $\tilde{S}$ and there is no other path $\gamma_j$ that intersects the surface $S_i$ for $i \neq j$ where $1 \leq i, j \leq N$.

Then for example consider the following configuration.

**Example 2.4:**

The sets $\{S_6, S_7\}$ or $\{S_5, S_8\}$ have the surface intersection property for the graph $\Gamma$. The images of a map $E$ is

$$\rho_5(\tilde{\gamma}) = (g_5, 0), \quad \rho_8(\gamma) = (0, h_8)$$
Note that simply speaking the property indicates that each map reduces to a component of $\rho^L \times \rho^R$.

A set of surfaces that has the surface intersection property for a graph is further specialised by restricting the choice to paths lying ingoing and below with respect to the surface orientations.

**Definition 2.19.** A set $\tilde{S}$ of $N$ surfaces has the *simple surface intersection property for a graph* $\Gamma$ with $N$ independent edges, if it contains only surfaces, for which each path $\gamma_i$ of a graph $\Gamma$ intersects only one surface $S_i$ only once in the target vertex of the path $\gamma_i$, the path $\gamma_i$ lies above and there are no other intersection points of each path $\gamma_i$ and each surface in $\tilde{S}$.

**Example 2.5:** Consider the following example.

\[ \begin{array}{c}
\tilde{S} \\
\gamma \\
\Gamma = \{\gamma, \tilde{\gamma}\} \\
S_{11} \\
S_{12}
\end{array} \]

The sets $\{S_9, S_{11}\}$ or $\{S_{10}, S_{12}\}$ have the simple surface intersection property for the graph $\Gamma$. Calculate $\rho_9(\tilde{\gamma}) = (0, h_9^{-1}), \quad \rho_{11}(\gamma) = (0, h_{11}^{-1})$.

In this case the set $G_{\tilde{S}, \Gamma}$ reduces to

\[
\bigcup_{\sigma_R \in \sigma_R} \bigcup_{S \in \tilde{S}} \left\{ \rho \in \text{Map}(\Gamma, g) : \rho(\gamma) := \sigma_R(S)^{-1} \text{ for } \gamma \cap S = t(\gamma) \right\}
\]

Notice that, the set $\Gamma \cap \tilde{S} = \{t(\gamma_i)\}$ for a surface $S_i \in \tilde{S}$ and $\gamma_i \cap S_j \cap S_i = \{\emptyset\}$ for a path $\gamma_i$ in $\Gamma$ and $i \neq j$.

On the other hand, there exists a set of surfaces such that each path of a graph intersects all surfaces of the set in the same vertex. This contradicts the assumption that each path of a graph intersects only one surface once.

**Definition 2.20.** Let $\Gamma$ be a graph that contains no loops.

A set $\tilde{S}$ of surfaces has the *same surface intersection property for a graph* $\Gamma$ iff each path $\gamma_i$ in $\Gamma$ intersects with all surfaces of $\tilde{S}$ in the same source vertex $v_i \in V_\Gamma$ ($i = 1, \ldots, N$), all paths are outgoing and lie below each surface $S \in \tilde{S}$ and there are no other intersection points of each path $\gamma_i$ and each surface in $\tilde{S}$.

A surface set $\tilde{S}$ has the *same right surface intersection property for a graph* $\Gamma$ iff each path $\gamma_i$ in $\Gamma$ intersects with all surfaces of $\tilde{S}$ in the same target vertex $v_i \in V_\Gamma$ ($i = 1, \ldots, N$), all paths are ingoing and lie above each surface $S \in \tilde{S}$ and there are no other intersection points of each path $\gamma_i$ and each surface in $\tilde{S}$.

Recall the example 2.3. Then the set $\{S_1, S_2\}$ has the same surface intersection property for the graph $\Gamma$.

Then the set $G_{\tilde{S}, \Gamma}$ reduces to

\[
\bigcup_{\sigma_L \in \sigma_L} \bigcup_{S \in \tilde{S}} \left\{ \rho \in \text{Map}(\Gamma, g) : \rho(\gamma) := \sigma_L(S)^{-1} \text{ for } \gamma \cap S = s(\gamma) \right\}
\]

Notice that, $\gamma \cap S_1 \cap \ldots \cap S_N = s(\gamma)$ for a path $\gamma$ in $\Gamma$ whereas $\Gamma \cap \tilde{S} = \{s(\gamma_i)\}_{1 \leq i \leq N}$. Clearly $\Gamma \cap S_i = s(\gamma_i)$ for a surface $S_i$ in $\tilde{S}$ holds. Simply speaking the physical intuition behind that is given by fluxes associated to different surfaces that should act on the same path.

A very special configuration is the following.
Definition 2.21. A set \( \mathcal{S} \) of surfaces has the same surface intersection property for a graph \( \Gamma \) containing only loops iff each loop \( \gamma_i \) in \( \Gamma \) intersects with all surfaces of \( \mathcal{S} \) in the same vertices \( s(\gamma_i) = t(\gamma_i) \) in \( V_T \) (\( i = 1, \ldots, N \)), all loops lie below each surface \( S \in \mathcal{S} \) and there are no other intersection points of each loop in \( \Gamma \) and each surface in \( \mathcal{S} \).

Notice that, both properties can be restated for other surface and path configurations. Hence a surface set have the simple or same surface intersection property for paths that are outgoing and lie above (or ingoing and below). The important fact is related to the question if the intersection vertices are the same for all surfaces or not.

Finally for the definition of the quantum flux operators notice the following objects.

Definition 2.22. The set of all images of maps in \( G_{S, \Gamma} \) for a fixed surface set \( \mathcal{S} \) and a fixed path \( \gamma \) in \( \Gamma \) is denoted by \( \mathcal{G}_{\mathcal{S}, \gamma} \).

The set of all finite products of images of maps in \( G_{S, \Gamma} \) for a fixed surface set \( \mathcal{S} \) and a fixed graph \( \Gamma \) is denoted by \( \mathcal{G}_{\mathcal{S}, \Gamma} \).

The product \( \cdot \) on \( \mathcal{G}_{\mathcal{S}, \Gamma} \) is given by

\[
\rho_{S, \Gamma}(\Gamma) \cdot \rho_{S, \Gamma}(\Gamma) = (\rho_{S_1}(\gamma_1) \cdot \rho_{S_2}(\gamma_1), \ldots, \rho_{S_N}(\gamma_N)) = (o_L(S_1)^{-1}o_L(S_2)^{-1}, \ldots, o_L(S_2)^{-1}o_L(S_1)^{-1}) = ((o_L(S_2)o_L(S_1))^{-1}, \ldots, (o_L(S_2)o_L(S_1))^{-1})
\]

Definition 2.23. Let \( S \) be a surface and \( \Gamma \) be a graph such that the only intersections of the graph and the surface in \( S \) are contained in the vertex set \( V_T \). Moreover let \( P_{\Gamma} \Sigma \equiv V_T \) be a finite path groupoid associated to \( \Gamma \).

Then define the set for a fixed surface \( S \) by

\[
\text{Map}_S(P_{\Gamma} \Sigma, G \times G) := \bigcup_{o_L \circ \rho_R \in \mathcal{G}_S} \bigcup_{S \in \mathcal{S}} \{ (\rho^L, \rho^R) \in \text{Map}(P_{\Gamma} \Sigma, G \times G) : (\rho^L, \rho^R)(\gamma) := (o_L(S)^{\rho_L(S, \gamma)}, o_R(S)^{\rho_R(S, \gamma)}) \}
\]

Then the quantum flux operators are elements of the following group.

Proposition 2.24. Let \( \mathcal{S} \) a set of surfaces and \( \Gamma \) be a fixed graph, which contains no loops, such that the set \( \mathcal{S} \) has the same surface intersection property for the graph \( \Gamma \).

The set \( \mathcal{G}_{\mathcal{S}, \Gamma} \) has the structure of a group.

The group \( \mathcal{G}_{\mathcal{S}, \Gamma} \) is called the flux group associated a path and a finite set of surfaces.

Proof: This follows easily from the observation that in this case \( \mathcal{G}_{\mathcal{S}, \Gamma} \) reduces to

\[
\bigcup_{o_L \circ \rho_R \in \mathcal{G}_S} \bigcup_{S \in \mathcal{S}} \{ \rho^L \in \text{Map}(\Gamma, G) : \rho^L(\gamma) := o_L(S)^{-1} \text{ for } \gamma \cap S = s(\gamma) \}
\]

There always exists a map \( \rho_{S, \Gamma}^L \in \mathcal{G}_{\mathcal{S}, \Gamma} \) such that the following equation defines a multiplication operation

\[
\rho_{S, \Gamma}^L(\gamma) \cdot \rho_{S, \Gamma}^L(\gamma) = g_1g_2 := \rho_{S, \Gamma}^L(\gamma) \in \mathcal{G}_{\mathcal{S}, \Gamma}
\]

with inverse \((\rho_{S, \Gamma}^L(\gamma))^{-1}\) such that

\[
\rho_{S, \Gamma}^L(\gamma) \cdot (\rho_{S, \Gamma}^L(\gamma))^{-1} = (\rho_{S, \Gamma}^L(\gamma))^{-1} \cdot \rho_{S, \Gamma}^L(\gamma) = e_G \quad \forall \gamma \in \Gamma
\]
Notice that for a loop \( \alpha \) an element \( \rho_S(\alpha) \in G_{S,\gamma} \) is defined by
\[
\rho_S(\alpha) := (\rho_S^L \times \rho_S^R)(\alpha) = (g, h) \in G^2
\]
In the case of a path \( \gamma' \) that intersects a surface \( S \) in the source and target vertex there is also an element \( \rho_S(\gamma') \in G_{S,\gamma} \) defined by
\[
\rho_S(\gamma') := (\rho_S^L \times \rho_S^R)(\gamma') = (g, h) \in G^2
\]

**Proposition 2.25.** Let \( S \) be a set of surfaces and \( \Gamma \) be a fixed graph, which contains no loops, such that the set \( S \) has the same surface intersection property for the graph \( \Gamma \). Let \( P^r_\Gamma \) be a finite orientation preserved graph system such that the set \( S \).

The set \( G_{S,\Gamma} \) has the structure of a group.

The set \( G_{S,\Gamma} \) is called the **flux group associated a graph and a finite set of surfaces**.

**Proof:** This follows from the observation that the set \( G_{S,\Gamma} \) is identified with
\[
\bigcup_{S \in \mathcal{S}} \bigcup_{\rho \in \mathcal{P}^r_\mathcal{F}} \left\{ \rho_{S,\Gamma} \in \text{Map}(P^r_\Gamma, G^{1|E}^r) : \quad \rho_{S,\Gamma} := \rho_{S} \times \cdots \times \rho_{S} \right\}
\]
where \( \rho_{S}(\gamma) := o_{L}(S)^{-1}, \rho_{S} \in G_{S,\Gamma}, S \in \mathcal{S}, \gamma \in \Gamma \)

Let \( S \) be a surface set having the same intersection property for a fixed graph \( \Gamma := \{ \gamma_1, \ldots, \gamma_N \} \). Then for two surfaces \( S_1, S_2 \) contained in \( S \) define
\[
\rho_{S_1,\Gamma}(\Gamma) \cdot \rho_{S_2,\Gamma}(\Gamma) = (\rho_{S_1}(\gamma_1) \cdot \rho_{S_2}(\gamma_1), \ldots, \rho_{S_1}(\gamma_N) \cdot \rho_{S_2}(\gamma_N))
\]
\[
= (g_{S_1} \cdot \ldots \cdot g_{S_1}, g_{S_2} \cdot \ldots \cdot g_{S_2}) = (g_{S_1} g_{S_2} \cdot \ldots \cdot g_{S_1} g_{S_2})
\]
where \( \Gamma = \{ \gamma_1, \ldots, \gamma_N \} \). Note that, since the maps \( o_L \) are arbitrary maps from \( S \) to \( G \), it is assumed that the maps satisfy \( o_L(S_i) := g_{S_i}^{-1} \in G \) for \( i = 1, 2 \). Clearly this is related to in this particular case of the graph \( \Gamma \) and can be generalised.

The inverse operation is given by
\[
(\rho_{S,\Gamma}(\Gamma))^{-1} = ((\rho_{S}(\gamma_1))^{-1}, \ldots, (\rho_{S}(\gamma_N))^{-1})
\]
where \( N = |\Gamma| \) and \( \rho_{S} \in G_{S,\gamma} \) for \( S \in \mathcal{S} \). Since it is true that
\[
\rho_{S,\Gamma}(\Gamma) \cdot \rho_{S,\Gamma}(\Gamma)^{-1} = (g_{S} \cdot \ldots \cdot g_{S}, g_{S}^{-1} \cdot \ldots \cdot g_{S}^{-1})
\]
\[
= (\rho_{S}(\gamma_1) \cdot \rho_{S}(\gamma_1)^{-1}, \ldots, \rho_{S}(\gamma_N) \cdot \rho_{S}(\gamma_N)^{-1})
\]
\[
= (g_{S} g_{S}^{-1} \cdot \ldots \cdot g_{S} g_{S}^{-1}) = (e_G, \ldots, e_G)
\]
yields.

Notice that, it is not defined that
\[
\rho_{S_1,\Gamma}(\Gamma) \bullet \rho_{S_2,\Gamma}(\Gamma)
\]
\[
= (o_L(S_2))^{-1} o_L(S_1)^{-1}, \ldots, o_L(S_2)^{-1} o_L(S_1)^{-1} = ((o_L(S_1) o_L(S_2))^{-1}, \ldots, (o_L(S_1) o_L(S_2))^{-1})
\]
\[
= \rho_{S_1,\Gamma}(\Gamma)
\]
is true. Moreover observe that, if all subgraphs of a finite orientation preserved graph system are identified, then \( G_{S,\Gamma}^{\prime} \) is a subgroup of \( G_{S,\Gamma} \) for all subgraphs \( \Gamma' \) in \( P_\Gamma \). If \( G \) is assumed to be a compact Lie group, then the flux group \( G_{S,\Gamma} \) is called the Lie flux group.

There is another group, if another surface set is considered.
Proposition 2.26. Let \( \tilde{T} \) be a set of surfaces and \( \Gamma \) be a fixed graph such that the set \( \tilde{T} \) has the simple surface intersection property for the graph \( \Gamma \). Let \( \mathcal{P}_{\Gamma}^{\tilde{T}} \) be a finite orientation preserved graph system.

The set \( \tilde{G}_{T,\Gamma} \) has the structure of a group.

The same arguments using the identification of \( \tilde{G}_{\tilde{T},\Gamma} \) with \( \bigcup_{\sigma \in \sigma_n} \{ \rho_{T,\Gamma} \in \text{Map}(\mathcal{P}_{\tilde{T}}^{\tilde{T}}, G^{|E_{\Gamma}|}) : \rho_{T,\Gamma} := \rho_{T_1} \times \cdots \rho_{T_N} \}
\)

where \( \rho_{T_i}(\gamma) := o_{R(T_i)}^{-1}, \rho_{T_i} \in \tilde{G}_{\tilde{T},\Gamma}, T_i \in \tilde{T}, \gamma \in \Gamma \)

which is given by

\[
\rho_{T_1,\Gamma}(\Gamma) \cdot \cdots \cdot \rho_{T_N,\Gamma}(\Gamma) = (\rho_{T_1}(\gamma_1)e_{G}, \cdots, e_{G}) \cdot \cdots \cdot (\cdots e_{G} \cdot \rho_{T_N}(\gamma_N)e_{G})
\]

\[
= (\rho_{T_1}^1(\gamma_1), \cdots, \rho_{T_N}^1(\gamma_N)) = (g_1, \cdots, g_N) \in G^N
\]

Then the multiplication operation is presented by

\[
\rho_{T,\Gamma}^1(\Gamma) \cdot \rho_{T,\Gamma}^2(\Gamma) = (\rho_{T_1}^1(\gamma_1) \cdot \rho_{T_1}^2(\gamma_1), \cdots, \rho_{T_N}^1(\gamma_N) \cdot \rho_{T_N}^2(\gamma_N))
\]

\[
= (g_{1,1}, \cdots, g_{1,N}) \cdot (g_{2,1}, \cdots, g_{2,N}) = (g_{1,1}g_{2,1}, \cdots, g_{1,N}g_{2,N}) \in G^N
\]

where \( \Gamma = \{\gamma_1, \cdots, \gamma_N\} \).

It is also possible that, the fluxes are located only in a vertex and do not depend on ingoing or outgoing, above or below orientation properties.

Definition 2.27. Let \( \mathcal{P}_{\Gamma} \) be a finite graph groupoid associated to a graph \( \Gamma \) and let \( N \) be the number of edges of the graph \( \Gamma \).

Define the set of maps

\[
G_{\Gamma}^{\text{loc}} := \{ g_{\Gamma} \in \text{Map}(\mathcal{P}_{\Gamma}, G^{|\Gamma|}) : g_{\Gamma} := g_1^1 \circ s \times \cdots \times g_N^N \circ s \}
\]

\[
g_{\Gamma}^i \in \text{Map}(\Gamma, G)
\]

Then \( G_{\Gamma}^{\text{loc}} \) is the set of all images of maps in \( G_{\Gamma}^{\text{loc}} \) for all graphs in \( \mathcal{P}_{\Gamma} \) and \( G_{\Gamma}^{\text{loc}} \) is called the local flux group associated a finite graph system.

3 The flux and flux transformation group, n.c. and heat-kernel holonomy \( C^*\)-algebra

In this section new algebras are constructed from either the quantum configuration or the quantum momentum variables of LQG. In the following the focus is based on the quantum momentum variables, which are given by the group-valued quantum flux operators associated to surfaces and paths.

3.1 The flux group \( C^*\)-algebra associated to graphs and a surface set

Let \( C(G) \) be the convolution *-algebra of continuous functions \( C_c(G) \) on a locally compact unimodular group \( G \) equipped with the convolution product, an inversion and supremum norm.

Recall that a surface \( S \) has the same surface intersection property for a graph \( \Gamma \), if each path of \( \Gamma \) intersect the surface \( S \) exactly once in a source vertex of the path and the path is outgoing and lies below.
Corollary 3.1. Let $S$ be a surface with same surface intersection property for a finite graph system associated to a graph $\Gamma$. Let $G$ be a unimodular locally compact group and let $\hat{G}_{\Gamma}$ be the flux group.

Then the convolution flux $\ast$-algebra $C(\hat{G}_{\Gamma})$ associated to a surface and a graph $\Gamma$ is defined by the following product

$$
(f_1 \ast f_2)(\rho_{S,\Gamma}(\Gamma)) = (f_1 \ast f_2)(\rho_{S}(\gamma_1), ..., \rho_{S}(\gamma_N))
$$

$$
= \int_{G_{\Gamma}} f_1(\rho_{S}(\gamma_1)\hat{\rho}(\gamma_1)^{-1}, ..., \rho_{S}(\gamma_N)\hat{\rho}(\gamma_N)^{-1})f_2(\hat{\rho}(\gamma_1), ..., \hat{\rho}(\gamma_N))
$$

$$
d\mu_{S,\Gamma}(\rho_{S}(\gamma_1), ..., \rho_{S}(\gamma_N))
$$

$$
= \int_{G_{\Gamma}} f_1(\rho_{S,\Gamma}(\Gamma)\hat{\rho}(\Gamma)^{-1})f_2(\hat{\rho}(\Gamma))d\mu_{S,\Gamma}(\hat{\rho}(\Gamma))
$$

for $\Gamma = \{\gamma_1, ..., \gamma_N\}$ and where $\rho_{S,\Gamma} \in G_{\Gamma}$ and $\rho_{S} \in G_{S,\Gamma}$ which reduces to

$$
(f_1 \ast f_2)(\rho_{S,\Gamma}(\Gamma)) = (f_1 \ast f_2)(\rho_{S}(\gamma_i)) = \int_{G_{\Gamma}} f_1(\rho_{S}(\gamma_i)\hat{\rho}(\gamma_i)^{-1})f_2(\hat{\rho}(\gamma_i))d\mu_{S,\Gamma}(\hat{\rho}(\gamma_i))
$$

for any $i = 1, ..., N$ and $\Gamma' = \{\gamma_i\} \in \mathcal{P}_\Gamma$, the involution

$$
f_{\Gamma}(\rho_{S,\Gamma}(\Gamma'))^\ast := f_{\Gamma}(\rho_{S,\Gamma}(\Gamma'))^{-1}
$$

for any $i = 1, ..., N$ and equipped with the supremum norm.

Set $\Gamma' := \{\gamma_i\}$. Remark that, if all paths $\gamma_i$ are ingoing and above, then the product reads

$$
(f_1 \ast f_2)(\rho_{S,\Gamma}(\Gamma')) = \int_{\mathcal{G}} f_1(gS\hat{g}^{-1})f_2(\hat{g})d\mu(\hat{g})
$$

(4)

otherwise

$$
(f_1 \ast f_2)(\rho_{S,\Gamma}(\Gamma')) = \int_{\mathcal{G}} f_1(gS^{-1}\hat{g})f_2(\hat{g})d\mu(\hat{g})
$$

(5)

This implies that, only for one surface the structure is identified with $C(G)$. The convolution algebra $C(\hat{G}_{\Gamma})$ is defined similarly to the one defined in corollary 3.1 for a surface set $\hat{S}$ with same surface intersection property for a finite graph system associated to a graph $\Gamma$.

Recall that, a set $\hat{S}$ of $N$ surfaces has the simple surface intersection property for a graph $\Gamma$ with $N$ independent edges, if it contain only surfaces, for which each path $\gamma_i$ of a graph $\Gamma$ intersects only one surface $S_i$ only once in the target vertex of the path $\gamma_i$, the path $\gamma_i$ lies above and there are no other intersection points of each path $\gamma_i$ and each surface in $\hat{S}$. Then the convolution algebra can be defined as follows.

Corollary 3.2. Let $\hat{S} := \{S_i\}_{1 \leq i \leq N}$ be a set of surfaces with simple surface intersection property for a finite graph system associated to a graph $\Gamma$.

Then the convolution flux $\ast$-algebra $C(\hat{G}_{\Gamma})$ associated to a surface set and a graph $\Gamma$ is defined by the following product

$$
(f_1 \ast f_2)(\rho_{S_i}(\gamma_1), ..., \rho_{S_N}(\gamma_N))
$$

$$
= \int_{\mathcal{G}} f_1(\rho_{S_i}(\gamma_1)\hat{\rho}(\gamma_1)^{-1}, ..., \rho_{S_N}(\gamma_N)\hat{\rho}(\gamma_N)^{-1})f_2(\hat{\rho}(\gamma_1), ..., \hat{\rho}(\gamma_N))d\mu(\hat{\rho}(\Gamma))
$$

where $\rho_{S,\Gamma} \in G_{\Gamma}$, $\rho_{S} \in G_{S,\Gamma}$ for $i = 1, ..., N$, $\rho_{S,\Gamma}(\Gamma) := (\rho_{S_i}(\gamma_1), ..., \rho_{S_N}(\gamma_N))$, the involution is defined by

$$
f_{\Gamma}(\rho_{S_i}(\gamma_1), ..., \rho_{S_N}(\gamma_N))^\ast := f_{\Gamma}(\rho_{S_i}(\gamma_1)^{-1}, ..., \rho_{S_N}(\gamma_N)^{-1})
$$

and equipped with the supremum norm.

Clearly $\hat{G}_{\Gamma}$ is identified with $G^N$ for $N$ being the number of independent paths in $\Gamma$ such that each of the path $\gamma_i$ intersects a surface $S_i$. 

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The convolution algebra $\mathcal{C}(\tilde{G}_{S,\Gamma})$ is also studied for other situations as far as the surface set $\tilde{S}$ has one of the surface intersection properties, which have been given in section 2.3.

The dual space $C_0(\tilde{G}_{S,\Gamma})^*$ is identified by the Riesz-Markov theorem with the Banach space of bounded complex Baire measures on $\tilde{G}_{S,\Gamma}$. Moreover, each Baire measure has a unique extension to a regular Borel measure on $\tilde{G}_{S,\Gamma}$. The Banach space of all regular Borel measures is denoted by $\mathbf{M}(\tilde{G}_{S,\Gamma})$. There is a convolution multiplication

$$\int_{\tilde{G}_{S,\Gamma}} f(\rho_{S,\Gamma}(\Gamma')) d(\mu \ast \nu)(\rho_{S,\Gamma}(\Gamma'))$$

$$= \int_{\tilde{G}_{S,\Gamma}} \int_{\tilde{G}_{S,\Gamma}} f(\rho_{S,\Gamma}(\Gamma') \rho_{S,\Gamma}(\Gamma')) d\mu(\rho_{S,\Gamma}(\Gamma')) d\nu(\rho_{S,\Gamma}(\Gamma'))$$

(6)

where $\rho_{S,\Gamma} \in \tilde{G}_{S,\Gamma}$, $\rho_i \in \tilde{G}_{S,\Gamma}$ for $i = 1, \ldots, N$, $\rho_{S,\Gamma}(\Gamma') := (\rho_{S,\Gamma}(\gamma_1), \ldots, \rho_{S,\Gamma}(\gamma_M))$, $\Gamma' := \{\gamma_1, \ldots, \gamma_M\}$, $\mu, \nu \in \mathbf{M}(\tilde{G}_{S,\Gamma})$ and $f \in C_0(\tilde{G}_{S,\Gamma})$ and an inversion

$$\int_{\tilde{G}_{S,\Gamma}} f(\rho_{S,\Gamma}(\Gamma')) d\mu(\rho_{S,\Gamma}(\Gamma')) = \int_{\tilde{G}_{S,\Gamma}} \tilde{f}(\rho_{S,\Gamma}(\Gamma'))^{-1} d\mu(\rho_{S,\Gamma}(\Gamma'))$$

(7)

which transfers $\mathbf{M}(\tilde{G}_{S,\Gamma})$ to a Banach $^*$-algebra. Then restrict $\mathbf{M}(\tilde{G}_{S,\Gamma})$ to the norm closed subspace consisting of measures absolutely continuous w.r.t. the Haar measure $\mu_{S,\Gamma}$, which is identified with $L^1(\tilde{G}_{S,\Gamma})$ by $d\mu(\rho_{S,\Gamma}(\Gamma')) = f(\rho_{S,\Gamma}(\Gamma')) d\mu_{S,\Gamma}(\rho_{S,\Gamma}(\Gamma'))$ for $f \in L^1(\tilde{G}_{S,\Gamma})$.

**Corollary 3.3.** Let $\tilde{S} := \{S_i\}_{1 \leq i \leq N}$ be a set of surfaces with same surface intersection property for a finite graph system associated to a graph $\Gamma$.

The Banach $^*$-algebra $L^1(\tilde{G}_{S,\Gamma}, \mu_{S,\Gamma})$ is the continuous extension of $\mathcal{C}(\tilde{G}_{S,\Gamma})$ in the $L^1$-norm.

There is a non-degenerate $^*$-representation $\pi_0$ of $L^1(\tilde{G}_{S,\Gamma}, \mu_{S,\Gamma})$ on the Hilbert space $\mathcal{H}_\Gamma = L^2(\tilde{G}_{S,\Gamma}, \mu_{S,\Gamma})$, which is of the form

$$\pi_0(f_\Gamma) := \int_{\tilde{G}_{S,\Gamma}} f_\Gamma(\rho_{S,\Gamma}(\gamma_1), \ldots, \rho_{S,\Gamma}(\gamma_N)) d\mu_{S,\Gamma}(\rho_{S,\Gamma}(\gamma_1), \ldots, \rho_{S,\Gamma}(\gamma_N))$$

(8)

for $f_\Gamma \in L^1(\tilde{G}_{S,\Gamma}, \mu_{S,\Gamma})$ (defined in the sense of a Bochner integral).

Notice that, the Banach $^*$-algebra $L^1(\tilde{G}_{S,\Gamma}, \mu_{S,\Gamma})$ has an approximate unit. Then for a $^*$-representation $\pi_0$ of $L^1(\tilde{G}_{S,\Gamma}, \mu_{S,\Gamma})$ on $\mathcal{H}_\Gamma$ exists a GNS-triple $(\mathcal{H}_\Gamma, \pi_0, \Omega_0)$ and an associated state $\omega_0$ [19] section 8.6. Furthermore, there is a left regular unitary representation $U^N_{\cal L}$ of $\tilde{G}_{S,\Gamma}$ on $\mathcal{H}_\Gamma$ associated to an action $\alpha^N_{\cal L}$ [9] Lemma 3.4 and Lemma 3.16 or [7] Lemma 6.1.4 and Lemma 6.1.16. Then observe that, for $f_\Gamma \in L^1(\tilde{G}_{S,\Gamma}, \mu_{S,\Gamma})$ and $\rho^N_{S,\Gamma}, \tilde{\rho}^N_{S,\Gamma} \in \tilde{G}_{S,\Gamma}$ the unitary $U^N_{\cal L}$ satisfies

$$U^N_{\cal L}(\rho^N_{S,\Gamma}) \pi_0(\rho_{S,\Gamma}(\Gamma')) \Omega_0$$

$$= \int_{\tilde{G}_{S,\Gamma}} d\mu_{\tilde{S},\Gamma}(\rho^N_{S,\Gamma}(\Gamma')) U^N_{\cal L}(\rho^N_{S,\Gamma}) \rho_{S,\Gamma}(\Gamma')) f_\Gamma'(\Omega_0)$$

$$= \int_{\tilde{G}_{S,\Gamma}} d\mu_{\tilde{S},\Gamma}(\rho_{S,\Gamma}(\gamma_1), \ldots, \rho_{S,\Gamma}(\gamma_N)) f_\Gamma'(\rho_{S,\Gamma}(\gamma_1) \rho_{S,\Gamma}(\gamma_1), \ldots, \rho_{S,\Gamma}(\gamma_N) \rho_{S,\Gamma}(\gamma_N)) \Omega_0$$

$$= \pi_0(\alpha^N_{\cal L}(\rho_{S,\Gamma}(\gamma_1), \Gamma')) \Omega_0$$

whenever $\rho^N_{S,\Gamma}(\Gamma') := (\rho_{S,\Gamma}(\gamma_1), \ldots, \rho_{S,\Gamma}(\gamma_N))$ and $\Omega_0$ is the cyclic vector. This implies

$$\omega_0(\alpha^N_{\cal L}(\rho_{S,\Gamma}(\gamma_1), f_\Gamma') = \omega_0(\Gamma f_\Gamma')$$

(10)

and hence that the state $\omega_0$ on the Banach $^*$-algebra $L^1(\tilde{G}_{S,\Gamma}, \mu_{S,\Gamma})$ associated to the representation $\pi_0$ is $\tilde{G}_{S,\Gamma}$-invariant.

The same is true if all paths in $\Gamma$ intersect in vertices of the set $V_{\Gamma}$ with a surface $S$ such that all paths are outgoing and lie below the surface $S$ and the unitaries $U^N_{\cal L}(\tilde{\rho}_{S,\Gamma}(\Gamma'))$ are analysed. Clearly this can be also studied for other situations presented in [9] Section 3.1 or [7] Section 6.1.
Notice that, the Banach *-algebra $L^1(\bar{G}_{S,\Gamma})$ is generated by all Dirac point measures \( \delta(\rho_{S,\Gamma}(\Gamma')) : \rho_{S,\Gamma}(\Gamma') \in \bar{G}_{S,\Gamma} \) such that

\[
\delta(\rho_{S,\Gamma}(\Gamma')) \ast \delta(\hat{\rho}_{S,\Gamma}(\Gamma')) = \delta(\rho_{S,\Gamma}(\Gamma') \hat{\rho}_{S,\Gamma}(\Gamma'))
\]

\[
\delta^*(\rho_{S,\Gamma}(\Gamma')) = \delta(\rho_{S,\Gamma}(\Gamma')^{-1})
\]

Moreover, recognize that,

\[
(\delta(\rho_{S,\Gamma}(\Gamma')) \ast \hat{f}_T)(\hat{\rho}_{S,\Gamma}) = f_T(\rho_{S,\Gamma}(\Gamma')^{-1}) \rho_{S,\Gamma}(\Gamma')
\]

\[
(f_T \ast \delta(\rho_{S,\Gamma}(\Gamma')))(\hat{\rho}_{S,\Gamma}) = f_T(\rho_{S,\Gamma}(\Gamma') \rho_{S,\Gamma}^{-1}(\Gamma'))
\]

yields for all \( f_T \in L^1(\bar{G}_{S,\Gamma}, \rho_{S,\Gamma}) \) and \( \rho_{S,\Gamma} \in \bar{G}_{S,\Gamma} \).

Observe that, for \( A = \sum_{i=1}^{n} a_i \delta(\rho_{S,\Gamma}(\Gamma')) \in L^1(\bar{G}_{S,\Gamma}) \) and \( \hat{S} := \{S_i\}_{1 \leq i \leq N} \), there is a state \( \hat{\omega}_0 \) on \( L^1(\bar{G}_{S,\Gamma}) \) such that

\[
\hat{\omega}_0(A^*A) = \sum_{n,m} a_n a_m \hat{\omega}_0(\delta(\rho_{S_n,\Gamma}(\Gamma')) \delta(\rho_{S_m,\Gamma}(\Gamma')))
\]

\[
= \sum_{n,m} a_n a_m \hat{\omega}_0(\delta(\rho_{S_n,\Gamma}(\Gamma')^{-1}) \rho_{S_m,\Gamma}(\Gamma'))
\]

(11)

Moreover, for an action \( \alpha \) of \( \bar{G}_{S,\Gamma} \) on \( L^1(\bar{G}_{S,\Gamma}) \) the action is automorphic and point-norm continuous. The state is defined by

\[
\hat{\omega}_0(\delta(\rho_{S,\Gamma}(\Gamma')) := \begin{cases} 
  1 & \text{for } \rho_{S,\Gamma}(\Gamma') = e_G \\
  0 & \text{for } \rho_{S,\Gamma}(\Gamma') \neq e_G
\end{cases}
\]

Derive

\[
\hat{\omega}_0(\alpha(\hat{\rho}_{S,\Gamma}) \delta(\rho_{S_n,\Gamma}(\Gamma')))) = \hat{\omega}_0(\delta(\hat{\rho}_{S_n,\Gamma}(\Gamma') \rho_{S_n,\Gamma}(\Gamma')^{-1}))
\]

\[
= \hat{\omega}_0(\delta(\rho_{S_n,\Gamma}(\Gamma')))
\]

(12)

and conclude that, the state \( \hat{\omega}_0 \) is \( \bar{G}_{S,\Gamma} \)-invariant.

**Definition 3.4.** Let the surface \( S \) has the same surface intersection property for a graph \( \Gamma \), let \( \hat{S} \) be a set of surfaces \( S_1, ..., S_N \) having the same surface intersection property for a graph \( \Gamma \).

The **generalised group-valued quantum flux operator for a surface** \( S \) is given by the following non-degenerate representation \( \pi_{S,\Gamma} \) of \( L^1(\bar{G}_{S,\Gamma}, \mu_{S,\Gamma}) \) on the Hilbert space \( L^2(\bar{G}_{S,\Gamma}, \mu_{S,\Gamma}) \), which satisfies \( \|\pi_{S,\Gamma}(f_T)\|_2 \leq \|f_T\|_1 \) and is defined as a \( L^2(\bar{G}_{S,\Gamma}, \mu_{S,\Gamma}) \)-valued Bochner integral

\[
\pi_{S,\Gamma}(f_T)\psi_T := \int_{\bar{G}_{S,\Gamma}} d\mu_{S,\Gamma}(\rho_{S,\Gamma}(\Gamma)) f_T(\rho_{S,\Gamma}(\Gamma)) U(\rho_{S,\Gamma}(\Gamma)) \psi_T \quad \text{for } f_T \in L^1(\bar{G}_{S,\Gamma}, \mu_{S,\Gamma})
\]

and a weakly continuous unitary representation \( U \) of \( \bar{G}_{S,\Gamma} \) acting on a vector \( \psi_T \) in \( L^2(\bar{G}_{S,\Gamma}, \mu_{S,\Gamma}) \) is considered.

The **generalised group-valued quantum flux operator for a set of surfaces** \( \hat{S} \) is given by the following non-degenerate representation \( \pi_{\hat{S},\Gamma} \) of \( L^1(\bar{G}_{\hat{S},\Gamma}, \mu_{\hat{S},\Gamma}) \) on the Hilbert space \( L^2(\bar{G}_{\hat{S},\Gamma}, \mu_{\hat{S},\Gamma}) \), which satisfies \( \|\pi_{\hat{S},\Gamma}(f_T)\|_2 \leq \|f_T\|_1 \) and is defined as a \( L^2(\bar{G}_{\hat{S},\Gamma}, \mu_{\hat{S},\Gamma}) \)-valued Bochner integral

\[
\pi_{\hat{S},\Gamma}(f_T)\psi_T := \int_{\bar{G}_{\hat{S},\Gamma}} d\mu_{\hat{S},\Gamma}(\rho_{\hat{S},\Gamma}(\Gamma_1), ..., \rho_{\hat{S},\Gamma}(\Gamma_N)) f_T(\rho_{\hat{S},\Gamma}(\Gamma_1), ..., \rho_{\hat{S},\Gamma}(\Gamma_N)) U_{\hat{S},\Gamma}(\rho_{\hat{S},\Gamma}(\Gamma_1), ..., \rho_{\hat{S},\Gamma}(\Gamma_N)) \psi_T(\hat{\rho}_{\hat{S},\Gamma}(\Gamma_1), ..., \hat{\rho}_{\hat{S},\Gamma}(\Gamma_N))
\]

whenever \( f_T \in L^1(\bar{G}_{\hat{S},\Gamma}, \mu_{\hat{S},\Gamma}) \) and a weakly continuous unitary representation \( U \) of \( \bar{G}_{\hat{S},\Gamma} \) acting on a vector \( \psi_T \) in \( L^2(\bar{G}_{\hat{S},\Gamma}, \mu_{\hat{S},\Gamma}) \) is considered.
It is easy to show that, for example the representation associated to a left regular representation $U^N_L$ of $\hat{G}_{s,\Gamma}$ on $L^2(\hat{G}_{s,\Gamma}, \mu_{s,\Gamma})$ fulfill
\[
\Phi_M(f_T)\psi_T(\hat{\rho}_S(\gamma_1),...,\hat{\rho}_S(\gamma_N))
= \int_{\hat{G}_{s,\Gamma}} d\mu_{s,\Gamma}(\rho_S(\gamma_1),...,\rho_S(\gamma_N)) f_T(\rho_S(\gamma_1),...,\rho_S(\gamma_N))
\]
\[
U^N_L(\rho_S(\gamma_1),...,\rho_S(\gamma_N))\psi_T(\hat{\rho}_S(\gamma_1),...,\hat{\rho}_S(\gamma_N))
\]
\[
= \int_{\hat{G}_{s,\Gamma}} d\mu_{s,\Gamma}(\rho_S(\gamma_1),...,\rho_S(\gamma_N)) f_T(\rho_S(\gamma_1),...,\rho_S(\gamma_N))
\]
\[
\psi_T(\rho_S(\gamma_1)^{-1}\hat{\rho}_S(\gamma_1),...,\rho_S(\gamma_N)^{-1}\hat{\rho}_S(\gamma_N))
= f_T * \psi_T \text{ for } \psi_T \in L^2(\hat{G}_{s,\Gamma},\mu_{s,\Gamma})
\] (13)

It is a $^*$-representation on the Hilbert space $L^2(\hat{G}_{s,\Gamma},\mu_{s,\Gamma})$, since it is true that,
\[
\Phi_M(f_1^* f_2^*)\psi_T = \Phi_M(f_1^*)\Phi_M(f_2^*)\psi_T
\]
\[
\Phi_M(\lambda_1 f_1^* + \lambda_2 f_2^*)\psi_T = \lambda_1 \Phi_M(f_1^*)\psi_T + \lambda_2 \Phi_M(f_2^*)\psi_T
\]
\[
\Phi_M(f_2^*)\psi_T = \Phi_M(f_T)^* \psi_T
\] (14)
yields whenever $f_T, f_1^*, f_2^* \in L^1(\hat{G}_{s,\Gamma},\mu_{s,\Gamma})$ and $\lambda_1, \lambda_2 \in \mathbb{C}$.

The representation associated to the right regular representation $U^N_R$ of $\hat{G}_{s,\Gamma}$ on $L^2(\hat{G}_{s,\Gamma},\mu_{s,\Gamma})$ is equivalent to
\[
\Phi_M(f_T)\psi_T(\hat{\rho}_S(\gamma_1),...,\hat{\rho}_S(\gamma_N))
= \int_{\hat{G}_{s,\Gamma}} d\mu_{s,\Gamma}(\rho_S(\gamma_1)) f_T(\rho_S(\gamma_1),...,\rho_S(\gamma_N))
\]
\[
U^N_R(\rho_S(\gamma_1))\psi_T(\hat{\rho}_S(\gamma_1),...,\hat{\rho}_S(\gamma_N))
\]
\[
= \int_{\hat{G}_{s,\Gamma}} d\mu_{s,\Gamma}(\rho_S(\gamma_1)) f_T(\rho_S(\gamma_1),...,\rho_S(\gamma_N))
\]
\[
\psi_T(\rho_S(\gamma_1)\hat{\rho}_S(\gamma_1),...,\rho_S(\gamma_N)\hat{\rho}_S(\gamma_N))
= f_T * \psi_T \text{ for } \psi_T \in H_{t,\Gamma}
\] (15)

Clearly there is a representation of $L^1(\hat{G}_{s,\Gamma},\mu_{s,\Gamma})$, which correponds to the situation of all paths intersecting with one surface $S$ and such that all paths are outgoing and lie below the surface $S$, on the Hilbert space $L^2(\hat{G}_{s,\Gamma},\mu_{s,\Gamma})$. The representation is illustrated by
\[
\pi_{\Gamma,\gamma_1}(f_T)\psi_T(\hat{\rho}_S(\gamma_1),...,\hat{\rho}_S(\gamma_N))
= \int_{\hat{G}} d\mu(\rho_S(\gamma_1),...,\rho_S(\gamma_N)) f_T(\rho_S(\gamma_1),...,\rho_S(\gamma_N))
\]
\[
U^N_L(\rho_S(\gamma_1),...,\rho_S(\gamma_N))\psi_T(\hat{\rho}_S(\gamma_1),...,\hat{\rho}_S(\gamma_N))
\]
\[
= \int_{\hat{G}} d\mu(\rho_S(\gamma_1),...,\rho_S(\gamma_N)) f_T(\rho_S(\gamma_1),...,\rho_S(\gamma_N))
\]
\[
\psi_T(\rho_S(\gamma_1)^{-1}\hat{\rho}_S(\gamma_1),...,\rho_S(\gamma_N)^{-1}\hat{\rho}_S(\gamma_N))
= f_T * \psi_T \text{ for } \psi_T \in \mathfrak{H}_{\Gamma}
\] (16)

for any $i = 1,...,N$ and where all surfaces $S_i$ are elements of the surface set $\hat{S}$.

Another representation of $L^1(\hat{G}_{s,\Gamma},\mu_{s,\Gamma})$ on the Hilbert space $L^2(\hat{G}_{s,\Gamma},\mu_{s,\Gamma})$ is defined by
\[
\pi_{\Gamma,\gamma_1}(f_T)\psi_T(\hat{\rho}_S(\gamma_1),...,\hat{\rho}_S(\gamma_N))
= \int_{\hat{G}} d\mu_{s,\Gamma}(\rho_S(\gamma_1),...,\rho_S(\gamma_N)) f_T(\rho_S(\gamma_1),...,\rho_S(\gamma_N))
\]
\[
U^N_L(\rho_S(\gamma_1),...,\rho_S(\gamma_N))\psi_T(\hat{\rho}_S(\gamma_1),...,\hat{\rho}_S(\gamma_N))
\]
\[
= \int_{\hat{G}} d\mu_{s,\Gamma}(\rho_S(\gamma_1)) f_T(\rho_S(\gamma_1))\psi_T(\rho_S(\gamma_1)\hat{\rho}_S(\gamma_1))
= f_T * \psi_T \text{ for } \psi_T \in L^2(\hat{G}_{s,\Gamma},\mu_{s,\Gamma})
\] (17)
Moreover, a general representation $\pi_{S,\Gamma}$ is a faithful regular $^\ast$-representation of $C_\ast^r(\bar{G}_{S,\Gamma})$ in $L^2(\bar{G}_{S,\Gamma}, \mu_{S,\Gamma})$. It is a faithful representation, since from $fr \ast \psi_T = 0$ it is deducible that, $fr = 0$ holds. The left and the right regular representations $U^L_{\bar{G}}$ and $U^R_{\bar{G}}$ are unitarily equivalent, hence the generalised representations $\pi_{\bar{G},1}$ and $\pi_{\bar{G},2}$ are unitarily equivalent, too.

**Definition 3.5.** Let $S$ be a surface and $\tilde{S}$ be a set of surfaces such that $S$ and $\tilde{S}$ have the same surface intersection property for a graph $\Gamma$.

The reduced flux group $C^\ast$-algebra $C_\ast^r(\bar{G}_{S,\Gamma})$ for a surface $S$ or $C^\ast_r(\bar{G}_{\tilde{S},\Gamma})$ for a set $\tilde{S}$ of surfaces is defined as the closure of $L^1(\bar{G}_{S,\Gamma}, \mu_{S,\Gamma})$, or respectively $L^1(\bar{G}_{\tilde{S},\Gamma}, \mu_{\tilde{S},\Gamma})$, in the norm $\|fr\|_r := \|\pi_{S,\Gamma}(fr)\|_2$ or $\|fr\| := \|\pi_{\tilde{S},\Gamma}(fr)\|_2$.

In fact all continuous unitary representations $U$ of the flux group $\bar{G}_{S,\Gamma}$ on $L^2(\bar{G}_{S,\Gamma}, \mu_{S,\Gamma})$ give a non-degenerate representation $\pi_{S,\Gamma}$ of $L^1(\bar{G}_{S,\Gamma}, \mu_{S,\Gamma})$. Each representation is given by

$$
\pi_{S,\Gamma}(fr) := \int_{\bar{G}_{S,\Gamma}} d\mu_{S,\Gamma}(\rho_{S,\Gamma}(\Gamma))fr(\rho_{S,\Gamma}(\Gamma))U(\rho_{S,\Gamma}(\Gamma))
$$

**Definition 3.6.** Let $S$ be a surface and $\tilde{S}$ be a set of surfaces such that $S$ and $\tilde{S}$ have the same surface intersection property for a graph $\Gamma$.

The flux group $C^\ast$-algebras $C^\ast_r(\bar{G}_{S,\Gamma})$ for a surface $S$ or $C^\ast_r(\bar{G}_{\tilde{S},\Gamma})$ for a set $\tilde{S}$ of surfaces is the closure of $L^1(\bar{G}_{S,\Gamma}, \mu_{S,\Gamma})$ or respectively $L^1(\bar{G}_{\tilde{S},\Gamma}, \mu_{\tilde{S},\Gamma})$ in the norm $\|fr\| = \sup_{\Gamma} \|\pi_{\Gamma}(fr)\|_2$ where the supremum is taken over all non-degenerate $L^1$-norm decreasing $^\ast$-representations of $L^1(\bar{G}_{S,\Gamma}, \mu_{S,\Gamma})$, or respectively all representations $\pi$ of the form (18), where $U$ is a continuous unitary representation (one representative of each equivalence class) of the flux group $\bar{G}_{S,\Gamma}$ on a Hilbert space.

**Remark 3.7.** In the case of a (second countable) compact group $G$ the structures above are well known. Let $\bar{G}$ be the unitary dual consisting of all unitary equivalence classes of irreducible, continuous and unitary and therefore finite-dimensional representations $\pi_{s,\gamma_i}$ of $G$ w.r.t. a graph $\Gamma := \{\gamma_i\}$ on a finite dimensional Hilbert space $H_{s,\gamma_i}$. Notice that, every element of $\bar{G}$ is one-dimensional, iff $G$ is commutative. The dual $\bar{G}$ is discrete and countable. The set $\bar{G}$ is finite, iff $G$ is finite. The finite-dimensional representation $U_{s,\gamma_i}$ is equivalent to the left-regular representation $U_L : G \to U(L^2(G))$.

There exists an isomorphisms between Hilbert spaces such that

$$
H_{\Gamma} := L^2_{\gamma_i}(G) \cong L^2_{\gamma_i}(\bar{G}) := H_{\Gamma} = \bigoplus_{s \in \bar{G}} M_{d_{s,\gamma_i}}(\mathbb{C})
$$

where $d_{s,\gamma_i}$ is the dimension of $s$ in $\bar{G}$, given by the unitary Plancherel transform $F : L^2_{\gamma_i}(G) \to L^2_{\gamma_i}(\bar{G})$ with

$$
\hat{\psi}_\gamma(s) := (F\psi_{s,\gamma})(s) = \sqrt{d_{s,\gamma_i}} \int_G d\mu(\rho_{s,\gamma})(\rho_{s,\gamma}(\gamma_i))\psi_{s,\gamma}(\rho_{s,\gamma}(\gamma_i))
$$

where $\rho_{s,\gamma}(\gamma_i) \in \bar{G}_{S,\Gamma}$ is identified with $G$ if $\tilde{S} := \{S_i\}_{1 \leq i \leq N}$ has the same intersection property for $\Gamma$. The inverse transform is given by

$$
F^{-1}\hat{\psi}_\gamma(s) := \sum_{s \in \bar{G}} \sqrt{\dim \pi_{s,\gamma}} \text{tr}(\hat{\psi}_\gamma(s)U_{s,\gamma}(\rho_{s,\gamma}(\gamma_i))^*)
$$

Clearly if $\psi_L \in L^2_{\gamma_i}(G)$ and $\hat{\psi}_L \in L^2_{\gamma_i}(\bar{G})$ it is true that,

$$
\int_G |\psi_L(\rho_{s,\gamma}(\gamma_i))|^2 d\mu(\rho_{s,\gamma}(\gamma_i)) = \sum_{s \in \bar{G}} (\dim \pi_{s,\gamma})|\text{tr}(\hat{\psi}_L(s)\hat{\psi}_L(s)^*) |
$$

holds. Let $\Gamma$ be equivalent to $\gamma_i$ and $S$ has the same intersection property for $\Gamma$. The representation $\pi_{S,\Gamma}$ of the $C^\ast$-algebra $C^\ast_r(\bar{G}_{S,\Gamma})$ on the Hilbert space $H_{\Gamma} := L^2(\bar{G}_{S,\gamma_i}, \mu_{s,\gamma_i})$ is given for a path $\gamma$ that intersects $S$ such that the path is outgoing and lies below by

$$
\pi_{S,\Gamma}(fr)\psi_T := \int_G d\mu_{S,\gamma}(\rho_{S,\gamma}(\gamma))fr(\rho_{S,\gamma}(\gamma))U_{s,\gamma}(\rho_{S,\gamma}(\gamma))\psi_T(\rho_{S,\gamma}(\gamma))
$$

\[2\] A representation $(\pi, \mathcal{H})$ of a $C^\ast$-algebra $\mathcal{A}$ of the form (3.4) is called regular iff the unitary representation $U$ of a locally compact group $G$ is weak operator continuous on $\mathcal{H}$.

\[3\] A norm $\|\cdot\|$ of $\mathcal{A}$ is called $L^1$-norm decreasing if $\|\pi_{\Gamma}(f)\|_2 \leq \|f\|$ for all $f \in \mathcal{A}$.
for $\psi_\Gamma \in \mathcal{H}_\Gamma$. Notice that, for an abelian (locally) compact flux group $\hat{G}_{S,\Gamma}$ there is an isomorphism $F : C^*_r(\hat{G}_{S,\Gamma}) \to C_0(\hat{G}_{S,\Gamma})$ given by

$$F(f_\Gamma)(s) := \int_G d \mu_{S,\gamma}(\rho_S(\gamma)) f_\Gamma(\rho_S(\gamma)) U_{s,\Gamma}(\rho_S(\gamma))$$

which is called the generalised Fourier transform. The set of characters is denoted by $\hat{\hat{G}}_{S,\Gamma}$.

**Example 3.1:** For an abelian locally compact group $G$ the group algebra $C^*(G)$ coincides with $C^*_r(G)$. This is true, since for $s \in \hat{G}$ the representation $\pi_s$ of $G$ on $L^2(G)$ coincides with $\hat{f}(s) \in \mathbb{C}$ and consequently the norm $\| \cdot \|_r$ and $\| \cdot \|$ are the same.

Moreover, since $\mathbb{R}$ and $\hat{\mathbb{R}}$ are equal, there are the following isomorphisms

$$C_0(\mathbb{R}) \simeq C^*(\mathbb{R}) \simeq C^*_r(\mathbb{R})$$

Notice that, this statement generalises for an abelian locally compact group $G$. There is an isomorphism $C^*(G)$ and $C(\hat{G})$.

For a general locally compact group $\hat{G}_{S,\Gamma}$ it is true that,

$$C^*_r(\hat{G}_{S,\Gamma}) := \pi_{S,\Gamma}(C^*(\hat{G}_{S,\Gamma})) \simeq C^*(\hat{G}_{S,\Gamma}) \setminus \ker(\pi_{S,\Gamma})$$

holds. Therefore a Lie group is called amenable, if $C^*(\hat{G}_{S,\Gamma})$ coincides with $C^*_r(\hat{G}_{S,\Gamma})$ and hence iff $\pi_{S,\Gamma}$ is faithful. Since for locally compact groups, the representation $\pi_{S,\Gamma}$ is always faithful, these groups are always amenable.

**Proposition 3.8.** Let $S$ be a surface with the same surface intersection property for a graph $\Gamma$.

For a compact Lie group $G$ the flux group $C^*$-algebras for surface $S$ and a graph $\Gamma := \{\gamma\}$ is given by

$$C^*_r(\hat{G}_{S,\Gamma}) \simeq C^*(\hat{G}_{S,\Gamma}) \simeq \bigoplus_{\pi_s \in \hat{G}} M_{d_{s,\Gamma}}(\mathbb{C}) =: M_{\Gamma}$$

or

$$C^*_r(\hat{G}_{S,\Gamma}) \simeq C^*(\hat{G}_{S,\Gamma}) \simeq \bigoplus_{\pi_{s,\Gamma} \in \hat{G}_{S,\Gamma}} M_{d_{s,\Gamma}}(\mathbb{C})$$

and, hence, $\hat{G}_{S,\Gamma}$ is amenable.

**Proof:** This is due to the remark 3.7.

### 3.2 The flux transformation group $C^*$-algebra associated to graphs and a surface set

In the general theory for arbitrary locally compact groups the left regular representation $\pi_{T,1}$ of $G_{S,\Gamma}$ is defined by $\pi_{T,1}(f_T) \psi_T := f_T * \psi_T$ for $f_T \in L^1(G_{S,\Gamma}, \mu_{S,\Gamma})$ on the Hilbert space $L^2(G_{S,\Gamma}, \mu_{S,\Gamma})$. The operator $\pi_{T,1}(f_T)$ is compact for every $f_T \in L^1(G_{S,\Gamma}, \mu_{S,\Gamma})$. The set of functions $\mathcal{C}(G_{S,\Gamma}, \hat{G}_{S,\Gamma})$ for a locally compact group $G$ is a linear subspace of $\mathcal{C}(\hat{G}_{S,\Gamma}, C_0(\hat{G}_{S,\Gamma}))$.

**Theorem 3.9.** [26, Theorem 4.24] (Generalised Stone- von Neumann theorem):

Let $S$ be a set of surfaces with the simple surface intersection property for a graph $\Gamma$.

Let $G$ be a locally compact amimodular group, $G_{S,\Gamma}$ be the flux group and let $U$ be a continuous, irreducible and unitary representation of $\hat{G}_{S,\Gamma}$ on $L^2(G_{S,\Gamma}, \mu_{S,\Gamma}) =: \mathcal{H}_\Gamma$. Hence $U \in \text{Rep}(\hat{G}_{S,\Gamma}, \mathcal{K}(\mathcal{H}_\Gamma))$.  

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Let \( C_0(\bar{G}_{S,Γ}) \) be the \( C^* \)-algebra of continuous functions vanishing at infinity on \( \bar{G}_{S,Γ} \) with a pointwise multiplication and sup-norm and let \( Φ_M \) is the multiplication representation of \( C_0(\bar{G}_{S,Γ}) \) on \( H_Γ \). Therefore \( Φ_M \in \text{Mor}(C_0(\bar{G}_{S,Γ}), L(H_Γ)) \).

Then the linear map \( π_1 : \mathcal{C}(\bar{G}_{S,Γ}, C_0(\bar{G}_{S,Γ})) \to \mathcal{L}(L^2(\bar{G}_{S,Γ}, \mu_{S,Γ})) \) of the form
\[
(π_1(F_Γ)ψ_Γ)(ψ_S(γ_1), ..., ψ_S(γ_N)) \\
:= \int_{\bar{G}_{S,Γ}} dμ_{S,Γ}(ψ_S(γ_1), ..., ψ_S(γ_N))\mu(ψ_S(γ_1), ..., ψ_S(γ_N))
\]
is a faithful and irreducible representation of the convolution \( * \)-algebra \( \mathcal{C}(\bar{G}_{S,Γ}, C_0(\bar{G}_{S,Γ})) \) of continuous functions \( \bar{G}_{S,Γ} \to C_0(\bar{G}_{S,Γ}) \) with compact support acting on the Hilbert space \( L^2(\bar{G}_{S,Γ}, \mu_{S,Γ}) \). The convolution \( * \)-algebra \( \mathcal{C}(\bar{G}_{S,Γ}, C_0(\bar{G}_{S,Γ})) \) is equipped with a norm \( \| \|_1 \) such that its completion is given by the Banach \( * \)-algebra \( L^1(\bar{G}_{S,Γ}, C_0(\bar{G}_{S,Γ})) \). Consequently \( π_1 \in \text{Rep}(L^1(\bar{G}_{S,Γ}, C_0(\bar{G}_{S,Γ})), \mathcal{L}(H_Γ)) \).

Set \( \| F_Γ \|_u := \sup_{ψ_Γ} \| π_1(F_Γ) \| \), where the supremum is taken over all non-degenerate \( L^1 \)-norm decreasing \( * \)-representations \( π_1 \) of the Banach \( * \)-algebra \( L^1(\bar{G}_{S,Γ}, C_0(\bar{G}_{S,Γ})) \), or respectively over all representations \( π_1 \) of the form \( \Phi_M \) where \( (Γ, U_Γ) \) is a covariant Hilbert space representation of the \( C^* \)-dynamical system \( (C_0(\bar{G}_{S,Γ}), \alpha_{L,Γ}, G_{S,Γ}) \).

Then the range of the closure of \( L^1(\bar{G}_{S,Γ}, C_0(\bar{G}_{S,Γ})) \) w.r.t. the norm \( \| \|_u \) is called the \textbf{flux transformation group} \( C^*(\bar{G}_{S,Γ}, G_{S,Γ}) \) for a set \( S \) of surfaces and a graph \( Γ \).

Moreover, \( C^*(\bar{G}_{S,Γ}, G_{S,Γ}) \) is isomorphic to the \( C^* \)-algebra \( \mathcal{K}(L^2(\bar{G}_{S,Γ}, \mu_{S,Γ})) \) of compact operators. The \( C^* \)-algebras \( C^*(G_{S,Γ}, G_{S,Γ}) \) and \( \mathcal{K}(L^2(G_{S,Γ}, \mu_{S,Γ})) \) are Morita equivalent \( C^* \)-algebras.

\textbf{Proof} : Step A.: Existence of the flux transformation group algebra for a graph

The convolution \( * \)-algebra \( C(\bar{G}_{S,Γ}, C_0(\bar{G}_{S,Γ})) \) is given by the convolution product
\[
(F_1^1 * F_2^2)(ψ_S(γ_1), ..., ψ_S(γ_N)) = \int_{\bar{G}_{S,Γ}} dμ(ψ_S(γ_1), ..., ψ_S(γ_N))F_1^1(ψ_S(γ_1), ..., ψ_S(γ_N))F_2^2(ψ_S(γ_1), ..., ψ_S(γ_N))
\]
and involutive
\[
F_Γ(ψ_S(γ_1), ..., ψ_S(γ_N)) = F(ψ_S(γ_1), ..., ψ_S(γ_N))
\]

Eq IPPH the convolution \( * \)-algebra \( C(\bar{G}_{S,Γ}, C_0(\bar{G}_{S,Γ})) \) with the \( \| \|_1 \)-norm, which is defined by
\[
\| F_Γ \|_1 = \int_{\bar{G}_{S,Γ}} dμ(ψ_S(γ_1), ..., ψ_S(γ_N)) \sup_{(ψ_S(γ_1), ..., ψ_S(γ_N)) \in G_{S,Γ}} |F_Γ(ψ_S(γ_1), ..., ψ_S(γ_N), ψ_S(γ_1), ..., ψ_S(γ_N))|
\]
and complete the algebra to the Banach \( * \)-algebra \( L^1(\bar{G}_{S,Γ}, C_0(\bar{G}_{S,Γ})) \).

Set \( H_Γ := L^2(\bar{G}_{S,Γ}, C_0(\bar{G}_{S,Γ})) \). Assume that the surface set has the simple surface property for a graph \( Γ \) and all paths lie below and are outgoing. Let \( U^N_L \in \text{Rep}(\bar{G}_{S,Γ}, H_Γ) \), \( F_Γ \in \mathcal{C}(\bar{G}_{S,Γ}, \bar{G}_{S,Γ}) \). Then the map
\[
π^{N,L}_{F_Γ}(ψ_Γ)(ψ_S(γ_1), ..., ψ_S(γ_N)) \\
:= \int_{\bar{G}_{S,Γ}} dμ(ψ_S(γ_1))...dμ(ψ_S(γ_N))F_Γ(ψ_S(γ_1), ..., ψ_S(γ_N))U^N_L(ψ_S(γ_1), ..., ψ_S(γ_N))ψ_Γ(ψ_S(γ_1), ..., ψ_S(γ_N))
\]
is a \( \| \| \)-norm decreasing if \( π_1(f) \leq \| f \|_1 \) for all \( f \in Ψ \).

\footnote{A norm \( \| \| \) of \( Ψ \) is called \( L^1 \)-norm decreasing if \( \| π_1(f) \| \leq \| f \|_1 \) for all \( f \in Ψ \).}
defines a $^*$-homomorphism $\pi_I : C(\hat{G}_{S,\Gamma}, \hat{G}_{S,\Gamma}) \to \mathcal{L}(H_\Gamma)$, which is extended to a $^*$-homomorphism from $C(\hat{G}_{S,\Gamma}, C_0(\hat{G}_{S,\Gamma}))$ to $\mathcal{L}(H_\Gamma)$. Therefore this defines a $^*$-representation of $C(\hat{G}_{S,\Gamma}, C_0(\hat{G}_{S,\Gamma}))$. Furthermore, it extends to a $^*$-representation of $L^1(\hat{G}_{S,\Gamma}, C_0(\hat{G}_{S,\Gamma}))$ on $H_\Gamma$. The representation is faithful, since from 

$$\pi_I^N(F_I)\psi_T = F_I \ast \psi_T = 0$$

it follows that, $F_I = 0$ holds. Clearly this investigation carries over for arbitrary surface sets, which have the simple surface intersection property for $\Gamma$.

**Step B.: Isomorphism between $C^*(\hat{G}_{S,\Gamma}, \hat{G}_{S,\Gamma})$ and $\mathcal{K}(L^2(\hat{G}_{S,\Gamma}, \mu_{S,\Gamma}))$**

Secondly $\pi_I(F_I)$ is Hilbert-Schmidt if $\|\pi_I(F_I)\|_2 < \infty$, which is verified by the following computation

$$\|\pi_I^N(F_I)\|_2^2 = \int_{\hat{G}_{S,\Gamma}} \mu_{S,\Gamma}(\Gamma) \int_{\hat{G}_{S,\Gamma}} d\mu_{S,\Gamma}(\Gamma)$$

where $F_I(\rho_{S,1}(\gamma_1), ..., \rho_{S,N}(\gamma_N); \rho_{S,1}(\gamma_1)^{-1} \rho_{S,N}(\gamma_N))$ is finite for every $F_I \in C^*(\hat{G}_{S,\Gamma}, \hat{G}_{S,\Gamma})$. Consequently $\pi_I^N(\hat{C}(\hat{G}_{S,\Gamma}, \hat{G}_{S,\Gamma}))$ is a subset of the Hilbert Schmidt class $\mathcal{K}_{HS}(L^2(\hat{G}_{S,\Gamma}))$, which is a dense subspace (w.r.t. the usual operator norm) of the $C^*$-algebra $\mathcal{K}(L^2(\hat{G}_{S,\Gamma}))$ of compact operators. Hence the closure of $\pi_I^N(\hat{C}(\hat{G}_{S,\Gamma}, \hat{G}_{S,\Gamma}))$ is equivalent to $\mathcal{K}(L^2(\hat{G}_{S,\Gamma}))$ in the operator norm and equality of the $C^*$-algebra $\pi_I^N(\hat{C}(\hat{G}_{S,\Gamma}, C_0(\hat{G}_{S,\Gamma}))$ and $\mathcal{K}(L^2(\hat{G}_{S,\Gamma}))$ is due to the fact that, $\pi_I^N$ is faithful.

**Step C.: All non-degenerate representations of $L^1(\hat{G}_{S,\Gamma}, C_0(\hat{G}_{S,\Gamma}))$ are unitarily equivalent to $\pi_I^N$**

To show that, there is an isomorphism between the categories of representations of $C^*(\hat{G}_{S,\Gamma}, \hat{G}_{S,\Gamma})$ and $\mathcal{K}(L^2(\hat{G}_{S,\Gamma}))$, which is isomorphic to the representations of $C$, on a Hilbert space. This is equivalent to the property of $C^*(\hat{G}_{S,\Gamma}, \hat{G}_{S,\Gamma})$ and $\mathcal{K}(L^2(\hat{G}_{S,\Gamma}))$ being Morita equivalent $C^*$-algebras.

**Step 1.: two pre-$C^*$-algebras $\mathfrak{A}_\Gamma, \mathfrak{B}$ and a full pre-Hilbert $\mathfrak{B}$-module $\mathcal{E}_\Gamma$**

Assume that, the surface set has the simple surface property for a graph $\Gamma$ and all paths lie below and are outgoing. Let $U^N:\mathcal{E}_\Gamma \in \text{Rep}(\hat{G}_{S,\Gamma}, \mathcal{K}(H_\Gamma))$.

Consider the pre-$C^*$-algebras $\mathfrak{A}_\Gamma = C(\hat{G}_{S,\Gamma}, \hat{G}_{S,\Gamma})$ and $\mathfrak{B} = \mathfrak{C}$. Moreover, let $\mathcal{E}_\Gamma = C_c(\hat{G}_{S,\Gamma})$ be a full pre-Hilbert $\mathfrak{C}$-module, which is defined by the $\mathfrak{C}$-action $\pi_R$ on $C_c(\hat{G}_{S,\Gamma})$, i.e. $\pi_R(\lambda)\psi_T = \psi_T \lambda$, and the inner product

$$\langle \psi_T, \phi_T \rangle_\mathfrak{C} := \langle \psi_T, \phi_T \rangle_2$$

**Step 2.: full right Hilbert $\mathfrak{B}$-module $\mathcal{E}_\Gamma$**

The completion of $\mathcal{E}_\Gamma$ is a Hilbert $\mathfrak{C}$-module.

**Step 3.: left-action of $\mathfrak{A}_\Gamma$ on $\mathcal{E}_\Gamma$ s.t. $\mathcal{E}_\Gamma$ is a full left pre-Hilbert $\mathfrak{A}_\Gamma$-module**

The left-action of $\mathfrak{A}_\Gamma$ on $\mathcal{E}_\Gamma$ is defined by $F_I \psi_T := \pi_I^N(F_I)\psi_T$ and therefore

$$\pi_I^N(F_I^*)\psi_T = \int_{\hat{G}_{S,\Gamma}} \mu_{S,\Gamma}(\Gamma) d\mu_{S,\Gamma}(\Gamma) F_I(\rho_{S,1}(\gamma_1), ..., \rho_{S,N}(\gamma_N) \rho_{S,1}(\gamma_1)^{-1} \rho_{S,N}(\gamma_N))$$

and for $F_I^* := \pi_I^N(F_I^*)\psi_T$

$$\pi_I^N(F_I^*)\psi_T = \int_{\hat{G}_{S,\Gamma}} \mu_{S,\Gamma}(\Gamma) d\mu_{S,\Gamma}(\Gamma) F_I(\rho_{S,1}(\gamma_1)^{-1}, ..., \rho_{S,N}(\gamma_N))$$

with

$$F_I(\rho_{S,1}(\gamma_1) \rho_{S,1}(\gamma_1)^{-1}, ..., \rho_{S,N}(\gamma_N)^{-1} \rho_{S,N}(\gamma_N)^{-1}) = \int_{\hat{G}_{S,\Gamma}} \mu_{S,\Gamma}(\Gamma) d\mu_{S,\Gamma}(\Gamma)$$
and there is a \( C(\tilde{G}_{S,T} \times \tilde{G}_{S,T}) \)-valued inner product on \( C_c(\tilde{G}_{S,T}) \) given by
\[
\langle \phi_T, \varphi_T \rangle_c = \int_{\tilde{G}_{S,T}} d\mu_{\tilde{G}_{S,T}}(\tilde{\rho}_{S,T}^{(\Gamma)}) \int_{\tilde{G}_{S,T}} d\mu_{\tilde{G}_{S,T}}(\tilde{\rho}_{S,T}^{(\Gamma)}) \phi_T(\tilde{\rho}_{S,T}^{(\Gamma)}) \tilde{\rho}_{S,T}^{(\Gamma)}(\Gamma) \varphi_T(\tilde{\rho}_{S,T}^{(\Gamma)}) \tilde{\rho}_{S,T}^{(\Gamma)}(\Gamma) \nu_{\tilde{G}_{S,T}}(\tilde{\rho}_{S,T}^{(\Gamma)})
\]
\[
:= \phi_T(\tilde{\rho}_{S,T}^{(\Gamma)}) \tilde{\rho}_{S,T}^{(\Gamma)}(\Gamma) \varphi_T(\tilde{\rho}_{S,T}^{(\Gamma)}) \tilde{\rho}_{S,T}^{(\Gamma)}(\Gamma) \nu_{\tilde{G}_{S,T}}(\tilde{\rho}_{S,T}^{(\Gamma)})
\]
Notice \( C(\tilde{G}_{S,T} \times \tilde{G}_{S,T}) \subset C(\tilde{G}_{S,T}, \tilde{G}_{S,T}) \). Consequently \( C_c(\tilde{G}_{S,T}) \) is a full pre-Hilbert \( \mathfrak{A}_T \)-module.

**Step 4.** full left Hilbert \( \mathfrak{A}_T \)-module \( \mathfrak{E}_T \)

The completion of \( \mathfrak{E}_T \) is a Hilbert \( C^*(\tilde{G}_{S,T}, \tilde{G}_{S,T}) \)-module.

**Step 4.1:** \( \mathfrak{A}_T \)-\( \mathfrak{B} \)-imprimitivity bimodule \( \mathfrak{E}_T \)

**Step 4.2:**

\[
\langle \psi_T, F_T \varphi_T \rangle_c = \int_{\tilde{G}_{S,T}} d\mu_{\tilde{G}_{S,T}}(\tilde{\rho}_{S,T}^{(\Gamma)}) \int_{\tilde{G}_{S,T}} d\mu_{\tilde{G}_{S,T}}(\tilde{\rho}_{S,T}^{(\Gamma)}) \psi_T(\tilde{\rho}_{S,T}^{(\Gamma)}) \tilde{\rho}_{S,T}^{(\Gamma)}(\Gamma) \varphi_T(\tilde{\rho}_{S,T}^{(\Gamma)}) \tilde{\rho}_{S,T}^{(\Gamma)}(\Gamma) \nu_{\tilde{G}_{S,T}}(\tilde{\rho}_{S,T}^{(\Gamma)})
\]
for \( F_T \in C(\tilde{G}_{S,T}, \tilde{G}_{S,T}) \), \( \psi_T, \varphi_T \in C_c(\tilde{G}_{S,T}) \) and
\[
\langle \lambda \psi_T, \phi_T \rangle_{C_c(\tilde{G}_{S,T}, \tilde{G}_{S,T})} = \langle \psi_T, \lambda^* \phi_T \rangle_{C_c(\tilde{G}_{S,T}, \tilde{G}_{S,T})} = \langle \psi_T, \overline{\phi_T} \rangle_{C_c(\tilde{G}_{S,T}, \tilde{G}_{S,T})}
\]
for \( \lambda \in \mathbb{C} \) and \( \psi_T, \varphi_T \in C_c(\tilde{G}_{S,T}) \).

**Step 5.** Morita equivalence

Hence conclude that, the \( C^* \)-algebras \( C^*(\tilde{G}_{S,T}, \tilde{G}_{S,T}) \) and \( \mathbb{C} \) are Morita equivalent. Moreover, for two Morita equivalent \( C^* \)-algebras there is a bijective correspondence between the non-degenerate representations of those two \( C^* \)-algebras. Consequently all irreducible representations of the \( * \)-algebra \( C(\tilde{G}_{S,T}, C_0(\tilde{G}_{S,T})) \) are unitarily equivalent to \( \pi_{T,L}^N \). Clearly for different unitarily inequivalent irreducible representations of \( \tilde{G}_{S,T} \), there are different inequivalent irreducible representations of \( C(\tilde{G}_{S,T}, C_0(\tilde{G}_{S,T})) \), which corresponds, therefore, to possible superselections of the system. Remark that, every non-degenerate representation of the compact operators \( \mathcal{K}(\mathcal{H}_T) \) is equivalent to a direct sum of copies of the identity representation. Hence it follows that, every non-degenerate representation of \( C^*(\tilde{G}_{S,T}, \tilde{G}_{S,T}) \) is equivalent to a direct sum of copies of \( \pi_{T,L}^N : \Phi_M \times U^{\mathbb{N}}_L \), where \( \Phi_M \) is the multiplication representation of \( C_0(\tilde{G}_{S,T}) \) on \( \mathcal{H}_T \).
To summarise the Generalised Stone-von Neumann theorem 3.9 states that, there is a bijective correspondence strongly continuous unitary representations of a group $\hat{G}_{S,\Gamma}$ on the $C^*$-algebra $\mathcal{L}(\mathcal{H}_\Gamma)$ and elements of $\text{Mor}(C^*(\hat{G}_{S,\Gamma}, \hat{G}_{S,\Gamma}), \mathcal{K}(\mathcal{H}_\Gamma))$. This correspondence preserves direct sums and irreducibility.

Furthermore, all unitary representations of $\hat{G}_{S,\Gamma}$ on $C_0(A_\Gamma)$ for suitable surface sets $\hat{S}$ and $\hat{S}$, which have the simple surface property for $\Gamma$, are naturally elements of the multiplier algebra $M(C^*(\hat{G}_{S,\Gamma}, \hat{G}_{S,\Gamma}))$. Or equivalently all unitary representations of $\hat{G}_{S,\Gamma}$ for a surface $S$ having the same surface intersection property for $\Gamma$ are naturally elements of the multiplier algebra $M(C^*(\hat{G}_{S,\Gamma}, \hat{G}_{S,\Gamma}))$. Clearly the closed linear span $\{U(\rho_{S,\Gamma}(\Gamma)) : \rho_{S,\Gamma}(\Gamma) \in \hat{G}_{S,\Gamma}\}$ of all unitary representations of $\hat{G}_{S,\Gamma}$ on the $C^*$-algebra $C(\hat{G}_{S,\Gamma})$ forms a $C^*$-subalgebra of $M(C^*(\hat{G}_{S,\Gamma}, \hat{G}_{S,\Gamma}))$.

In the next investigations the question is what happen if different surface sets are used for the construction of the flux transformation group $C^*$-algebra. In particular is there a generalised von Neumann theorem available?

For a simplification the following identifications are used. The flux group $\hat{G}_{S,\Gamma}$ is identified with $G^N$. Then the following coset spaces (or space of orbits) are defined by the sets

$$G^N/G := \{(\rho_S(\gamma_1)\rho_S(\gamma_2), \ldots, \rho_S(\gamma_N)) : \rho_S \in \hat{G}_{S,\Gamma}, \rho_S \in \hat{G}_{S,\Gamma},$$

$$\rho_S(\gamma_i) = \rho_S(\gamma_j) = g_S \in G; 1 \leq i, j \leq N\}$$

$$G^N \setminus G := \{(\rho_S(\gamma_1)\rho_S(\gamma_2), \ldots, \rho_S(\gamma_N)) : \rho_S \in \hat{G}_{S,\Gamma}, \rho_S \in \hat{G}_{S,\Gamma},$$

$$\rho_S(\gamma_i) = \rho_S(\gamma_j) = g_S \in G; 1 \leq i, j \leq N\}$$

whenever $\hat{S}$ is a surface set with simple surface intersection property for $\Gamma$ and $S$ has the same surface intersection property for $\Gamma$.

The space $G^N/G^2$ is identified with $G^2/G^2 \times G^{N-2}$, which is given by

$$G^2/G^2 \times G^{N-2} := \{(\rho_S(\gamma_1)\rho_S(\gamma_2), \ldots, \rho_S(\gamma_N)) : \rho_S \in \hat{G}_{S,\Gamma}, \rho_S \in \hat{G}_{S,\Gamma}, \forall l = 1, 2; i = 1, \ldots, N \}$$

$$= G^{N-2}$$

whenever $\hat{S}$ is a surface set with simple surface intersection property for $\Gamma$ and $\hat{S} := \{S_1, S_2\}$ has the simple surface intersection property for $\{\gamma_1, \gamma_2\}$. The space $G^2/G \times G^{N-2}$ is derivable as

$$G^2/G \times G^{N-2} := \{(\rho_S(\gamma_1)\rho_S(\gamma_2), \ldots, \rho_S(\gamma_N)) : \rho_S \in \hat{G}_{S,\Gamma}, \forall i = 1, \ldots, N \}$$

whenever $\hat{S}$ is a surface set with simple surface intersection property for $\Gamma$.

Or more general define

$$G^N/G^{N-M} = G^{N-M}/G^{N-M} \times G^M$$

$$= \{(\rho_S(\gamma_1)\rho_S(\gamma_2), \ldots, \rho_S(\gamma_N)) : \rho_S \in \hat{G}_{S,\Gamma}, \rho_S \in \hat{G}_{S,\Gamma}, \rho_S(\gamma_1), \ldots, \rho_S(\gamma_N) \in G^{N-M}\}$$

or

$$G^{N-M}/G \times G^M$$

$$= \{(\rho_S(\gamma_1)\rho_S(\gamma_2), \ldots, \rho_S(\gamma_N)) : \rho_S \in \hat{G}_{S,\Gamma}, \rho_S \in \hat{G}_{S,\Gamma}, \rho_S(\gamma_1), \ldots, \rho_S(\gamma_N) \in G^{N-M} \}$$

for suitable surface sets $\hat{S}$ and $\hat{S}$ and a surface $S$. Hence the coset $G^N/G^{N-1}$ of a group $G^N$ and a closed subgroup $G^{N-1}$ is the set

$$G^N/G^{N-1} = G^{N-1}/G^{N-1} \times G$$

$$= \{(\rho_S(\gamma_1)\rho_S(\gamma_2), \ldots, \rho_S(\gamma_N)) : \rho_S \in \hat{G}_{S,\Gamma}, \rho_S \in \hat{G}_{S,\Gamma}, \rho_S(\gamma_1), \ldots, \rho_S(\gamma_N) \in G^{N-1}\}$$

for suitable surface sets $\hat{S}$ and $\hat{S}$. For suitable surface sets $\hat{S}$, $\hat{S}$ and a graph $\Gamma$ the following theorem is derivable.
Theorem 3.10. It is true that:

(i) the algebras $C_0(G^N/G^{N-1}) \times G^N$ and $C^*(G^{N-1})$ are Morita equivalent $C^*$-algebras (for $N > 1$).

(ii) The algebras $C_0(G^N/G^{N-M}) \times G^N$ and $C^*(G^{N-M})$ are Morita equivalent $C^*$-algebras (for $N > 1$ and $1 \leq M < N$)

Proof : In the following the case (ii) is considered.

Step 1.: two pre-$C^*$-algebras $\mathfrak{A}_G, \mathfrak{B}_G$ and a full pre-Hilbert $\mathfrak{B}$-module $\mathcal{E}_G$.
Set $N$ be equivalent to $|\Gamma|$ for a graph $\Gamma$. Let $\mathfrak{A}_G = C(G^N,G^N/G^{N-1})$ be the dense subalgebra of $C^*(G^N,G^N/G^{N-1})$ such that $C(G^N,G^N/G^{N-1})$ is a pre-$C^*$-algebra. Similarly let $\mathfrak{B}_G = C(G^{N-1})$ be a dense subalgebra of $C^*(G^{N-1})$ such that $C(G^{N-1})$ is a pre-$C^*$-algebra. Identify $G^N/G^{N-1}$ with $G$.

The full pre-Hilbert $C(G^{N-1})$-module is given by $C_c(G^N)$ and the right action $\pi_G(f) := \pi_G(f_N)\psi_T$, which is of the form

$$
\pi_G(f)\psi_T := \int_{G^{N-1}} \int_{G^N} d\rho_S(\gamma) \int_{G^N} d\rho_S(\gamma_1, \ldots, \rho_S(\gamma_N)) \psi_T(\rho_S(\gamma_1), \ldots, \rho_S(\gamma_N))
$$

$$
\pi_G(f_N)\psi_T := \int_{G^{N-1}} d\rho_S(\gamma_1, \ldots, \rho_S(\gamma_N)) \int_{G^N} d\rho_S(\gamma_1, \ldots, \rho_S(\gamma_N)) \psi_T(\rho_S(\gamma_1), \ldots, \rho_S(\gamma_N))
$$

for $\psi_T \in C_c(G^N)$ and $f_N \in C(G^{N-1})$. The $C(G^{N-1})$-valued product on $C_c(G^N)$ is given by

$$
\langle \psi_T, \phi_T \rangle_{C(G^{N-1})} := \int_{G^N} d\rho_S(\gamma_1, \ldots, \rho_S(\gamma_N)) \int_{G^N} d\rho_S(\gamma_1, \ldots, \rho_S(\gamma_N)) \phi_T(\rho_S(\gamma_1), \ldots, \rho_S(\gamma_N))
$$

Step 2.: full right Hilbert $\mathfrak{B}$-module $\mathcal{E}_G$.

The completion of $C_c(G^N)$ is a Hilbert $C(G^{N-1})$-module.

Step 3.: left-action $\pi_L$ of $\mathfrak{B}_G$ on $\mathcal{E}_G$ s.t. $\mathcal{E}_G$ is a full left pre-Hilbert

Then there is a pre-Hilbert $C(G^N, G)$-module is given by $C_c(G^N)$ and the left action $f_G \psi_T := \pi_L(f_N)\psi_T$, which is of the form

$$
\pi_L(f_N)\psi_T := \int_{G^N} d\rho_S(\gamma_1, \ldots, \rho_S(\gamma_N)) \int_{G^N} d\rho_S(\gamma_1, \ldots, \rho_S(\gamma_N)) \psi_T(\rho_S(\gamma_1), \ldots, \rho_S(\gamma_N))
$$

$$
\pi_L(f_N)\psi_T := \int_{G^N} d\rho_S(\gamma_1, \ldots, \rho_S(\gamma_N)) \int_{G^N} d\rho_S(\gamma_1, \ldots, \rho_S(\gamma_N)) \psi_T(\rho_S(\gamma_1), \ldots, \rho_S(\gamma_N))
$$

where $\rho_S(\gamma_i) = \rho_S(\gamma_j)$ for $i, j = 1, \ldots, N$ and

$$
\pi_L(f_N)\psi_T := \int_{G^N} d\rho_S(\gamma_1, \ldots, \rho_S(\gamma_N)) \int_{G^N} d\rho_S(\gamma_1, \ldots, \rho_S(\gamma_N)) \psi_T(\rho_S(\gamma_1), \ldots, \rho_S(\gamma_N))
$$

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for \( F_T \in \mathcal{C}(G^N, G) \) and \( \psi_T \in \mathcal{C}_c(G^N) \). The \( \mathcal{C}(G^N, G) \)-valued inner product on \( \mathcal{C}_c(G^N) \) is equal to

\[
\langle \psi_T, \phi_T \rangle_{\mathcal{C}(G^N, G)} := \int_{G^N} \mu(\hat{\rho}_S(\gamma_1), \ldots, \hat{\rho}_S(\gamma_N))\psi_T(\rho_S(\gamma_1)^{-1}\hat{\rho}_S(\gamma_1), \ldots, \rho_S(\gamma_N)^{-1}\hat{\rho}_S(\gamma_N)) \frac{\phi_T(\hat{\rho}_S(\gamma_1), \ldots, \rho_S(\gamma_N)^{-1}\hat{\rho}_S(\gamma_N))}{\phi_T(\rho_S(\gamma_1), \ldots, \rho_S(\gamma_N)^{-1}\hat{\rho}_S(\gamma_N))}
\]

for \( \psi_T, \phi_T \in \mathcal{C}_c(G^N) \).

**Step 4.** \( \mathfrak{A}_T - \mathfrak{B}_T \)-imprimitivity bimodule \( \mathcal{E}_T \)

**Step 4.1:**

\[
\langle \psi_T F_T \phi_T \rangle_{\mathcal{C}(G^{N-1})} = \langle \psi_T, \pi_R(F_T) \phi_T \rangle_{\mathcal{C}(G^{N-1})}
\]

\[
= \int_G \mu(\rho_S(\gamma_1)) \int_{G^N} \mu(\hat{\rho}_S(\gamma_1), \ldots, \hat{\rho}_S(\gamma_N)) \int_{G^N} \mu(\rho_S(\gamma_1), \ldots, \rho_S(\gamma_N)) \psi_T(\rho_S(\gamma_1)^{-1}\hat{\rho}_S(\gamma_1), \ldots, \rho_S(\gamma_N)^{-1}\hat{\rho}_S(\gamma_N)) \frac{\phi_T(\rho_S(\gamma_1)^{-1}\hat{\rho}_S(\gamma_1), \ldots, \rho_S(\gamma_N)^{-1}\hat{\rho}_S(\gamma_N))}{\phi_T(\rho_S(\gamma_1), \ldots, \rho_S(\gamma_N)^{-1}\hat{\rho}_S(\gamma_N))}
\]

and

\[
\langle \psi_T, F_T \phi_T \rangle_{\mathcal{C}(G^{N-1})} = \langle \psi_T, \pi_L(F_T) \phi_T \rangle_{\mathcal{C}(G^{N-1})}
\]

\[
= \int_G \mu(\rho_S(\gamma_1)) \int_{G^N} \mu(\hat{\rho}_S(\gamma_1), \ldots, \hat{\rho}_S(\gamma_N)) \int_{G^N} \mu(\rho_S(\gamma_1), \ldots, \rho_S(\gamma_N)) \psi_T(\rho_S(\gamma_1)^{-1}\hat{\rho}_S(\gamma_1), \ldots, \rho_S(\gamma_N)^{-1}\hat{\rho}_S(\gamma_N)) \frac{\phi_T(\rho_S(\gamma_1)^{-1}\hat{\rho}_S(\gamma_1), \ldots, \rho_S(\gamma_N)^{-1}\hat{\rho}_S(\gamma_N))}{\phi_T(\rho_S(\gamma_1), \ldots, \rho_S(\gamma_N)^{-1}\hat{\rho}_S(\gamma_N))}
\]

\[
\langle \psi_T, F_T \phi_T \rangle_{\mathcal{C}(G^{N-1})} = \frac{\psi_T(\rho_S(\gamma_1)^{-1}\hat{\rho}_S(\gamma_1), \ldots, \rho_S(\gamma_N)^{-1}\hat{\rho}_S(\gamma_N)) F_T(\rho_S(\gamma_1)^{-1}\hat{\rho}_S(\gamma_1), \ldots, \rho_S(\gamma_N)^{-1}\hat{\rho}_S(\gamma_N))}{\phi_T(\rho_S(\gamma_1), \ldots, \rho_S(\gamma_N)^{-1}\hat{\rho}_S(\gamma_N))}
\]

\[
= \langle \pi_L(F_T^*) \psi_T, \phi_T \rangle_{\mathcal{C}(G^{N-1})}
\]

\[
= \langle F_T^* \psi_T, \phi_T \rangle_{\mathcal{C}(G^{N-1})}
\]
Step 4.2:
The following is true
\[ \phi_T(\psi_T, \varphi_T)(G^{N-1}) = \pi_R(\phi_T, \varphi_T)(C(G^{N-1})) \phi_T \]
\[ = \int_{G^{N-1}} \mu(\rho S_1, \ldots, \rho S_{N-1}) \int_{G^N} \mu(\hat{\rho} S_1, \ldots, \hat{\rho} S_{N}) \]
\[ \phi_T(\hat{\rho} S_1(\gamma), \ldots, \hat{\rho} S_{N-1}(\gamma), \hat{\rho} S_{N}(\gamma)) \phi_T(\hat{\rho} S_1(\gamma), \ldots, \hat{\rho} S_{N}(\gamma)) \]
\[ \phi_T(\hat{\rho} S_1(\gamma), \ldots, \hat{\rho} S_{N-1}(\gamma), \hat{\rho} S_{N}(\gamma)) \phi_T(\hat{\rho} S_1(\gamma), \ldots, \hat{\rho} S_{N}(\gamma)) \]
\[ = \int_{G^{N-1}} \mu(\rho S_1, \ldots, \rho S_{N-1}) \int_{G^N} \mu(\hat{\rho} S_1, \ldots, \hat{\rho} S_{N}) \]
\[ \phi_T(\hat{\rho} S_1(\gamma), \ldots, \hat{\rho} S_{N-1}(\gamma), \hat{\rho} S_{N}(\gamma)) \phi_T(\hat{\rho} S_1(\gamma), \ldots, \hat{\rho} S_{N}(\gamma)) \]
\[ \phi_T(\hat{\rho} S_1(\gamma), \ldots, \hat{\rho} S_{N-1}(\gamma), \hat{\rho} S_{N}(\gamma)) \phi_T(\hat{\rho} S_1(\gamma), \ldots, \hat{\rho} S_{N}(\gamma)) \]
for \( \phi_T, \psi_T, \varphi_T \in C_c(G^N) \). Then
\[ \phi_T(\psi_T, \varphi_T)(G^{N-1}) = \langle \phi_T, \psi_T \rangle_{C(G^N)} \varphi_T \]
since the properties of the surfaces and paths force the identity
\[ \int_{G^{N-1}} \mu(\rho S_1, \ldots, \rho S_{N-1}) \]
\[ \phi_T(\hat{\rho} S_1(\gamma), \ldots, \hat{\rho} S_{N-1}(\gamma), \hat{\rho} S_{N}(\gamma)) \phi_T(\hat{\rho} S_1(\gamma), \ldots, \hat{\rho} S_{N}(\gamma)) \]
\[ \phi_T(\hat{\rho} S_1(\gamma), \ldots, \hat{\rho} S_{N-1}(\gamma), \hat{\rho} S_{N}(\gamma)) \phi_T(\hat{\rho} S_1(\gamma), \ldots, \hat{\rho} S_{N}(\gamma)) \]
\[ = \int_{G^N} \mu(\rho S_1, \ldots, \rho S_{N}) \]
\[ \phi_T(\hat{\rho} S_1(\gamma), \ldots, \hat{\rho} S_{N-1}(\gamma), \hat{\rho} S_{N}(\gamma)) \phi_T(\hat{\rho} S_1(\gamma), \ldots, \hat{\rho} S_{N}(\gamma)) \]
\[ \phi_T(\hat{\rho} S_1(\gamma), \ldots, \hat{\rho} S_{N-1}(\gamma), \hat{\rho} S_{N}(\gamma)) \phi_T(\hat{\rho} S_1(\gamma), \ldots, \hat{\rho} S_{N}(\gamma)) \]
The case (i) is derivable for the dense subalgebra \( \mathfrak{M} := C(G^N, G^M) \) of \( C^*(G^N, G^M) \) such that \( C(G^N, G^M) \) is a pre-\( C^* \)-algebra. Similarly, let \( \mathfrak{M}_R := C(G^{N-M}, G^M) \) be a dense subalgebra of \( C^*(G^{N-M}, G^M) \) such that \( C(G^{N-M}) \) is a pre-\( C^* \)-algebra. Then \( C_c(G^N) \) is a full left Hilbert \( C(G^N, G^M) \)-module or full right Hilbert \( C(G^{N-M}) \)-module. Moreover, \( C_c(G^N) \) is a \( \mathfrak{M} \)-\( \mathfrak{M}_R \)-imprimitivity bimodule.

3.3 The non-commutative holonomy and the heat-kernel holonomy \( C^* \)-algebra for graphs and a surface set

Assume that, the configuration set \( \mathcal{A}_\Gamma \) of generalised connections is naturally identified with \( G^{|\Gamma|} \). Consider the convolution algebra \( C(\mathcal{A}_\Gamma) \). Observe that, the convolution product is for example for a graph \( \Gamma := \{ \gamma \} \)
defined by
\[(f_\Gamma \ast k_\Gamma)(h_\Gamma(\gamma')) = \int_{\mathcal{A}_\Gamma} d\mu_\Gamma(g_\Gamma(\gamma')) f_\Gamma(g_\Gamma(\gamma')) k_\Gamma(g_\Gamma(\gamma')^{-1} h_\Gamma(\gamma'))\]

The non-commutative holonomy C*-algebra for a graph is given by the object $C^*(\bar{\mathcal{A}}_\Gamma)$ and reduces in the case of a compact Lie group $G$ to the following object.

**Remark 3.11.** In the case of a compact group $G$ the holonomy algebra $C^*(\bar{\mathcal{A}}_\Gamma)$ for a graph $\Gamma$ is equivalent to a C*-algebra of matrices.

The new algebra is given by the infinite matrix algebra
\[
M_\Gamma := \bigotimes_{\gamma \in \Gamma} \bigoplus_{\pi_{s,\gamma} \in \hat{G}} M_{d_{s,\gamma}}(\mathbb{C}),
\]
where $\hat{G}$ is the dual of $G$, $\pi_{s,\gamma}$ is a representation of $G$ associated to a path $\gamma$, and $d_{s,\gamma}$ is the dimension of the representation $\pi_{s,\gamma}$. Finally, the inductive limit of a increasing family of matrix algebras $M_{\Gamma_i}$ associated to graphs is considered.

For an inductive family $\{\Gamma_i\}_i$ of graphs there is an injective ∗-homomorphism $\hat{\beta}_{\Gamma_i}: C^*(\bar{\mathcal{A}}_{\Gamma_i}) \to C^*(\bar{\mathcal{A}}_{\Gamma_i})$ for all $\mathcal{P}_\Gamma \leq \mathcal{P}_{\Gamma_i}$. This ∗-homomorphism is for example given for a subgraph $\Gamma := \{\gamma\}$ of $\Gamma' := \{\gamma \circ \gamma'\}$ by
\[
(\hat{\beta}_{\Gamma_i}(f_{\Gamma_i})(h_{\Gamma_i}(\gamma))) := f_{\Gamma'}(h_{\Gamma'}(\gamma \circ \gamma'))
\]

Consequently, there exists an inductive family of C*-algebras $\{(C^*(\bar{\mathcal{A}}_{\Gamma_i}), \hat{\beta}_{\Gamma_i}) : \mathcal{P}_\Gamma \leq \mathcal{P}_{\Gamma_i}\}$.

Clearly, a similar algebra for the flux group and the flux transformation group C*-algebra is constructed for generalised connections. In this case the holonomy transformation group C*-algebra $C^*(\bar{\mathcal{A}}_{\Gamma}, \bar{\mathcal{A}}_{\Gamma})$ is called the heat-kernel holonomy C*-algebra.

## 4 The holonomy-flux cross-product C*-algebra for surface sets

After the considerations of algebras generated by either quantum configuration or quantum momentum variables, algebras generated by both quantum variables simultaneously is studied in this section.

There is no particular holonomy-flux cross-product C*-algebra generated by all group-valued quantum flux operators and certain functions depending on holonomies along paths. But there exists a bunch of holonomy-flux cross-product C*-algebra associated to a finite graph system and many different suitable surface sets. These algebras are developed in section 4.1. The existence of this variety is the consequence of the following facts.

The group-valued quantum flux operators associated to certain surfaces and a graph $\Gamma$ form the flux group associated to a surface set $S$ and a graph $\Gamma$. These elements are implemented as point-norm continuous and automorphic actions on the analytic holonomy C*-algebra $C_0(\bar{\mathcal{A}}_{\Gamma})$ restricted to the finite orientation preserved graph system $\mathcal{P}_\Gamma^e$. For a short notation the analytic holonomy C*-algebra $C_0(\bar{\mathcal{A}}_{\Gamma})$ is abreviated by the term analytic holonomy C*-algebra associated to the graph $\Gamma$. It is assumed that, the configuration space is naturally identified with $C[G]$. Then the elements of the flux group are represented as unitary operators on the Hilbert space $H_\Gamma$, which is given by $L^2(\bar{\mathcal{A}}_{\Gamma}, \mu_\Gamma)$.

For each automorphic action of a certain flux group, which has been presented in [9 Section 3.1], [7 Section 6.1], a holonomy-flux cross-product C*-algebra is constructed. Precisely, an automorphic action $\alpha$ of the flux group $G_{\mathcal{S},\Gamma}$ on $C_0(\bar{\mathcal{A}}_{\Gamma})$ defines a holonomy-flux cross-product C*-algebra associated to the graph $\Gamma$ and the surface set $S$. This C*-algebra is denoted by $C_\Gamma(\bar{\mathcal{A}}_{\Gamma}) \rtimes_\alpha G_{\mathcal{S},\Gamma}$.

There are many different possible actions of flux groups depending on a surface or a surface set. For example in [9 Lemma 3.16], [7 Lemma 6.1.16] there is the point-norm continuous automorphic action $\alpha_{L}^1$ of the flux
group $\tilde{G}_{S,G}$ associated to one suitable surface $S$ on the analytic holonomy $C^*$-algebra $C_0(\tilde{\mathcal{A}}_\Gamma)$. Moreover, the action $\alpha^N_\mathcal{L}$ is defined for a flux group associated to a set $\tilde{S}$ of surfaces, which has the simple surface intersection property for a finite orientation preserved graph system associated to a graph $\Gamma$. In the following these two actions are often used.

Finally, there is an algebra, which unifies all cross-product algebras associated to a graph and different suitable sets of surfaces. This algebra is given by the multiplier algebra of the cross-product algebra $C_0(\tilde{\mathcal{A}}_\Gamma) \rtimes_{\alpha^N_\mathcal{L}} G_{S,G}$ of a certain surface set $\tilde{S}$. In theorem 4.12 it is proven that this algebra contains the cross-product $C^*$-algebra associated to the graph $\Gamma$ and suitable surface sets and every Weyl element, which is obtained by the unitary representation of flux groups associated to the graph and suitable surface sets.

The inductive limit of the inductive families of holonomy-flux cross-product $C^*$-algebras is studied in section 4.2. There the inductive limit $C^*$-algebra is derived from the inductive limit of $C^*$-algebras restricted to finite orientation preserved graph systems. This algebra is called the holonomy-flux cross-product $C^*$-algebra (of a special surface configuration $\tilde{S}$).

### 4.1 The holonomy-flux cross-product $C^*$-algebra associated to a graph and a surface set

For the development of such a cross-product algebra generated by holonomies along paths and quantum fluxes the following Banach $^*$-algebra is fundamental.

**Definition 4.1.** Let $\tilde{S}$ be a set of surfaces with same surface intersection property for a finite orientation preserved graph system associated to a graph $\Gamma$ with $N$ independent edges. Furthermore, let $(\tilde{G}_{S,G}, C_0(\tilde{\mathcal{A}}_\Gamma), \alpha)$ be a $C^*$-dynamical system defined by a point-norm continuous automorphic flux action $\alpha$ of $\tilde{G}_{S,G}$ on the analytic holonomy $C^*$-algebra $C_0(\tilde{\mathcal{A}}_\Gamma)$ associated to a graph $\Gamma$.

The space $L^1(\tilde{G}_{S,G}, C_0(\tilde{\mathcal{A}}_\Gamma), \alpha)$ consists of all measurable functions $F_\Gamma : \tilde{G}_{S,G} \to C_0(\tilde{\mathcal{A}}_\Gamma)$ for which

$$
\|F_\Gamma\|_1 := \int_{\tilde{G}_{S,G}} d\mu_{S,G}(\rho S_1(\gamma_1), \ldots, \rho S_N(\gamma_1)) \|F_\Gamma(\rho S_1(\gamma_1), \ldots, \rho S_N(\gamma_1))\|_2 < \infty
$$

yields whenever $\rho S \in G_{S,G}$.

**Proposition 4.2.** Let $\tilde{S}$ be a set of surfaces with simple surface intersection property for a finite orientation preserved graph system associated to a graph $\Gamma$. Furthermore, let $(\tilde{G}_{S,G}, C_0(\tilde{\mathcal{A}}_\Gamma), \alpha^N_\mathcal{L})$ be a $C^*$-dynamical system where $\alpha^N_\mathcal{L} \in \text{Act}(\tilde{G}_{S,G}, C_0(\tilde{\mathcal{A}}_\Gamma))$.

Then the multiplication operation between functions in $L^1(\tilde{G}_{S,G}, C_0(\tilde{\mathcal{A}}_\Gamma), \alpha^N_\mathcal{L})$:

$$(F_\Gamma \ast \tilde{F}_\Gamma)(\tilde{\rho} S_1(\gamma_1), \ldots, \tilde{\rho} S_N(\gamma_N))$$

$$= \int_{\tilde{G}_{S,G}} d\mu_{S,G}(\rho S_1(\gamma_1))$$

$$F_\Gamma(\rho S_1(\gamma_1)) \left(\alpha^N_\mathcal{L}(\rho S_1(\gamma_1)) (F_\Gamma^+)(\rho S_1(\gamma_1)^{-1} \tilde{\rho} S_1(\gamma_1), \ldots, \rho S_N(\gamma_1)^{-1} \tilde{\rho} S_N(\gamma_N))\right)$$

whenever $\rho S_1(\gamma_1) = (\rho S_1(\gamma_1), \ldots, \rho S_N(\gamma_N)) \in G_{S,G}$, $\rho S_i, \tilde{\rho} S_i \in G_{S,G}$

the involution on $L^1(\tilde{G}_{S,G}, C_0(\tilde{\mathcal{A}}_\Gamma), \alpha^N_\mathcal{L})$:

$$F_\Gamma(\rho S_1(\gamma_1), \ldots, \rho S_N(\gamma_1))^* = \left(\alpha^N_\mathcal{L}(\rho S_1(\gamma_1)) (F_\Gamma^+)^*\right) (\rho S_1(\gamma_1)^{-1}, \ldots, \rho S_N(\gamma_1)^{-1})$$

where the involution $^\dagger$ on $C_0(\tilde{\mathcal{A}}_\Gamma)$ is given by

$$F_\Gamma(\rho S_1(\gamma_1)^{-1}, \ldots, \rho S_N(\gamma_1)^{-1}) := \bar{F_\Gamma}(\rho S_1(\gamma_1)^{-1}, \ldots, \rho S_N(\gamma_1)^{-1})$$

turn $L^1(\tilde{G}_{S,G}, C_0(\tilde{\mathcal{A}}_\Gamma), \alpha^N_\mathcal{L})$ into a Banach $^*$-algebra.
In particular let $S$ be a surface having the same surface intersection property for a finite orientation preserved graph system associated to a graph $\Gamma$. Then the action $\alpha^1_L$ is defined in [9, Lemma 3.11] or [7, Lemma 6.1.11] for a graph $\Gamma$ and the convolution product reads

\[
(F_T \ast \tilde{F}_T)(\tilde{\rho}_S(\gamma_i)) = \int_G d\mu(\rho_S(\gamma_i)) F_T(\rho_S(\gamma_i)) \left( \alpha^1_L(\rho_S(\gamma_i)) \right) (\rho_S^{-1}(\gamma_i)\tilde{\rho}_S(\gamma_i))
\]

\[= \int_G d\mu(\rho_S(\gamma_i)) F_T(\rho_S(\gamma_i); \gamma_1, \ldots, \gamma_N) \tilde{F}_T(\rho_S^{-1}(\gamma_i)\tilde{\rho}_S(\gamma_i); \rho_S(\gamma_i)\gamma_1, \ldots, \rho_S(\gamma_i)\gamma_N))
\]

for any $i = 1, \ldots, N$. Since it is true that, $\rho_S(\gamma_i) = \rho_S(\gamma_j) = g_S \in G$ yields for all $i, j = 1, \ldots, N$. Clearly, this correspondence is given in one direction by the fact that, the representation $\alpha^1_L$ is used whenever $\rho_S(\gamma_i) = \rho_S(\gamma_j) = g_S \in G$ yields for all $i, j = 1, \ldots, N$. Hence, for a redefined convolution product and involution the $^\ast$-Banach algebra $L^1(G_{\hat{S},\Gamma}, C(\hat{\alpha}_T), \alpha^1_L)$ exists. Moreover, it is also possible to construct the $^\ast$-Banach algebras $L^1(G_{\hat{S},\Gamma}, C(\hat{\alpha}_T), \alpha^1_L)$ for $1 \leq M \leq N$.

Indeed, there are a lot of different Banach $^\ast$-algebras depending on the choice of the set of surfaces $\hat{S}$. Let $\hat{S}$ has the same surface intersection property for a graph $\Gamma$ such that each path $\gamma_i$, that intersect the surface $S_i$, lie above and ingoing w.r.t. the surface orientation of $S_i$. There are no other intersection points of each path $\gamma_i$ with any other surface $S_j$ where $i \neq j$. Then for the map $F_T : G_{\hat{S},\Gamma} \to C(\hat{\alpha}_T)$ write for the image of this function $F_T(\rho_S(\gamma_1), \ldots, \rho_S(\gamma_N)) = F_T(\rho_S(\gamma_1), \ldots, \rho_S(\gamma_N); \gamma_1, \ldots, \gamma_N)$ and derive

\[
(F_T \ast \tilde{F}_T)(\tilde{\rho}_S(\gamma_1), \ldots, \tilde{\rho}_S(\gamma_N)) = \int_{G_{\hat{S},\Gamma}} d\mu(\rho_S(\gamma)) F_T(\rho_S(\gamma)) \left( \alpha^1_L(\rho_S(\gamma)) \right) (\rho_S^{-1}(\gamma_1)\tilde{\rho}_S(\gamma_1), \ldots, \rho_S^{-1}(\gamma_N)\tilde{\rho}_S(\gamma_N))
\]

Hence for a redefined convolution product and involution the $^\ast$-Banach algebras $L^1(G_{\hat{S},\Gamma}, C(\hat{\alpha}_T), \alpha^1_L)$ for $1 \leq M \leq N$ are studied. Furthermore, it is also possible to construct the $^\ast$-Banach algebras $L^1(G_{\hat{S},\Gamma}, C(\hat{\alpha}_T), \alpha^1_L)$ for $1 \leq M \leq N$ and other algebras of that form for a modified convolution product, which is given in general by

\[
(F_T \ast \tilde{F}_T)(\tilde{\rho}_S(\gamma)) = \int_{G_{\hat{S},\Gamma}} d\mu(\rho_S(\gamma)) F_T(\rho_S(\gamma)) \left( \alpha^1_L(\rho_S(\gamma)) \right) (\rho_S^{-1}(\gamma)\tilde{\rho}_S(\gamma))
\]

whenever $\rho_S(\gamma) = (\rho_S(\gamma_1), \ldots, \rho_S(\gamma_N)), \rho_S(\gamma) \in G_{\hat{S},\Gamma}$ and a modified involution

\[
F_T^{\ast}(\rho_S(\gamma)) = \alpha^1_L(\rho_S(\gamma)) \left( F_T^{\ast}(\rho_S(\gamma)^{-1}) \right)
\]

is used whenever $\alpha = Act(G_{\hat{S},\Gamma}, C(\hat{\alpha}_T))$. Hence, for all well-defined $C^\ast$-dynamical system $(G_{\hat{S},\Gamma}, \hat{\alpha}, \alpha)$ there exists a general Banach $^\ast$-algebra $L^1(G_{\hat{S},\Gamma}, \hat{\alpha}, \alpha)$.

**Theorem 4.3.** Let $S$ be a set of surfaces with simple surface intersection property for a finite orientation preserved graph system associated to a graph $\Gamma$. Furthermore, let $(G_{\hat{S},\Gamma}, C(\hat{\alpha}_T), \alpha^N_L)$ be a $C^\ast$-dynamical system where $\alpha^N_L \in Act(G_{\hat{S},\Gamma}, C(\hat{\alpha}_T))$.

There is a bijective correspondence between non-degenerate $L^1$-norm decreasing $^\ast$-representations $\pi$ of the Banach $^\ast$-algebra $L^1(G_{\hat{S},\Gamma}, C(\hat{\alpha}_T), \alpha^N_L)$ and covariant representations $(\Phi_M, U^N_L)$ of the $C^\ast$-dynamical system $(G_{\hat{S},\Gamma}, C(\hat{\alpha}_T), \alpha^N_L)$ in $L(H_T)$.

This correspondence is given in one direction by the fact that, the representation $\pi^N_{I,L} = L^1(G_{\hat{S},\Gamma}, C(\hat{\alpha}_T), \alpha^N_L)$ is defined by a covariant pair $(\Phi_M, U^N_L)$ via

\[
\pi^N_{I,L}(F_T) \psi_T := \int_{G_{\hat{S},\Gamma}} d\mu(\rho_S(\gamma)) \Phi_M(F_T(\rho_S(\gamma))) U^N_L(\rho_S(\gamma)) \psi_T
\]

whenever $\rho_S(\gamma) \in G_{\hat{S},\Gamma}, F_T \in L^1(G_{\hat{S},\Gamma}, C(\hat{\alpha}_T), \alpha^N_L)$ and $\psi_T \in H_T$. 

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The other direction is given by the definition of the covariant pair \((\Phi_M, U^N_L)\) through the maps
\[
F_T : \rho^N_{S,\Gamma} \mapsto F_T(\rho^N_{S,\Gamma})\text{ and }
\alpha^N_L(\rho^N_{S,\Gamma})(F_T) : \rho^N_{S,\Gamma} \mapsto \left(\alpha^N_L(\rho^N_{S,\Gamma})(F_T)\right) \left(L((\hat{\rho}^N_{S,\Gamma})^{-1})(\rho^N_{S,\Gamma})\right)
\]
such that
\[
U^N_L(\rho^N_{S,\Gamma})\pi^N_{I,T}(F_T)\Omega := \pi^N_{I,T} \left(\alpha^N_L(\rho^N_{S,\Gamma})(F_T)\right)\Omega
\]
\[
\Phi_M(f_T)\pi^N_{I,T}(F_T)\Omega := \pi^N_{I,T}(f_T F_T)\Omega
\]
where \(\Omega\) denotes a cyclic vector for \(\pi^N_{I,L}(C(\hat{G}_{S,\Gamma}, C_0(\hat{A}_T)))\), \(f_T \in C_0(\hat{A}_T)\), \(\rho_{S,\Gamma}^N, \hat{\rho}^N_{S,\Gamma} \in \hat{G}_{S,\Gamma}\) and \(F_T \in L^1(\hat{G}_{S,\Gamma}, C_0(\hat{A}_T), \alpha^N_L)\). This bijection preserves unitary equivalence, direct sums and irreducibility.

The reduced holonomy-flux group \(C^*-\text{algebra} \ G^*_r(\hat{G}_{S,\Gamma}, C_0(\hat{A}_T))\) associated to a graph \(\Gamma\) and a set \(S\) of surfaces is defined as the norm-closure of \(L^1(\hat{G}_{S,\Gamma}, C_0(\hat{A}_T), \alpha^N_L)\) with respect to the norm
\[
\|\pi^N_{I,L}(F_T)\| := \|\pi^N_{I,L}(F_T)\|_2.
\]

With no doubt there are a big bunch of reduced holonomy-flux group \(C^*-\text{algebra} \ G^*_r(\hat{G}_{S,\Gamma}, C_0(\hat{A}_T))\) for different graph systems and different sets of surfaces.

**Definition 4.4.** Let \(S\) be a set of surfaces with simple surface intersection property for a finite orientation preserved graph system associated to a graph \(\Gamma\).

Then the Weyl-integrated holonomy-flux representation w.r.t. a finite orientation preserved graph system associated to a graph \(\Gamma\) and a set \(S\) of surfaces is given by
\[
\pi^{I,\Gamma}_{E(S)}(F_T)\psi_T = \int_{\hat{G}_{S,\Gamma}} \mathrm{d}\mu_{S,G}(\rho_{S,G}(\Gamma))\Phi_M \left(\pi^L_{I,T}(\rho_{S,G}(\Gamma))\right) U(\rho_{S,G}(\Gamma))\psi_T
\]
for \(F_T \in C^*(\hat{G}_{S,\Gamma}, C_0(\hat{A}_T))\), \(\rho_{S,G}(\Gamma) \in \hat{G}_{S,\Gamma}\), \(U \in \text{Rep}\left(\hat{G}_{S,\Gamma}, \mathcal{C}(L^2(\hat{A}_T, \mu_{\Gamma}))\right)\) and \(\psi_T \in L^2(\hat{A}_T, \mu_{\Gamma})\). The Weyl-integrated holonomy-flux representation \(\pi^{I,\Gamma}_{E(S)}\) is a *-representation of the \(C^*-\text{algebra} \ G^*_r(\hat{G}_{S,\Gamma}, C_0(\hat{A}_T))\) with a norm inherited from the representations \(\pi^{I,\Gamma}_{E(S)}\) on \(L^2(\hat{A}_T, \mu_{\Gamma})\). The representation \(\pi^{I,\Gamma}_{E(S)}\) is also denoted by \(\Phi_M \times U\).

**Proposition 4.5.** Let \(S\) be a set of surfaces with simple surface intersection property for a finite orientation preserved graph system associated to a graph \(\Gamma\). Furthermore, let \((\hat{G}_{S,\Gamma}, C_0(\hat{A}_T), \alpha^N_L)\) is a \(C^*-\text{dynamical system.}\)

Define for each \(F_T \in C(\hat{G}_{S,\Gamma}, C_0(\hat{A}_T))\) the norm
\[
\|F_T\|_u := \sup \left\{\|\Phi_M \times U^N_L(F_T)\|\right\}
\]
where the supremum is taken over all covariant Hilbert space representations \((\Phi_M, U^N_L)\) of the \(C^*-\text{dynamical system} \ (\hat{G}_{S,\Gamma}, C_0(\hat{A}_T), \alpha^N_L)\).

Then \(\|\|\_u\) is a norm on \(C(\hat{G}_{S,\Gamma}, C_0(\hat{A}_T))\), which is called the universal norm. The universal norm is dominated by the \(\|\|_1\)-norm, and the completion of \(C(\hat{G}_{S,\Gamma}, C_0(\hat{A}_T))\) with respect to \(\|\|_u\) is a \(C^*-\text{algebra called} \ G^*_r(\hat{G}_{S,\Gamma}, C_0(\hat{A}_T))\) holonomy-flux cross-product \(\text{C*-algebra} \ C_0(\hat{A}_T)\) of \(\hat{G}_{S,\Gamma}\) for a finite orientation preserved graph system associated to a graph \(\Gamma\) and a set \(S\) of surfaces. Shortly this algebra is denoted by \(C_0(\hat{A}_T) \times_{\alpha^N_L} \hat{G}_{S,\Gamma}\).

Notice that, for a surface \(S\) having the same surface intersection property for a finite orientation preserved graph system associated to \(\Gamma\), the \(L^2(\hat{A}_T, \mu_{\Gamma})\)-norm of an element \(F_T \in C_0(\hat{A}_T) \times_{\alpha^N_L} \hat{G}_{S,\Gamma}\) is given by
\[
\|\pi^{I,\Gamma}_{E(S)}(F_T)\psi_T\|_2 = \int_{\hat{A}_T} \int_{\hat{G}_{S,\Gamma}} \mathrm{d}\mu_{S,G}(\rho_{S,G}(\Gamma)) \mathrm{d}\mu_{\Gamma} (\beta_{\Gamma}(\Gamma))
\]
\[
|f_T(\rho_{S,G}(\Gamma); \beta_{\Gamma}(\Gamma))\psi_T | L(\rho_{S,G}(\Gamma))(\beta_{\Gamma}(\Gamma)), \ldots, L(\rho_{S,G}(\Gamma))(\beta_{\Gamma}(\Gamma))|)^2
\]
whenever \(\rho_{S,G}(\Gamma) = \rho_{S,G}(\Gamma) = g_S \in \hat{G}_{S,\Gamma}\) for \(i \neq j\) and \(1 \leq i, j \leq N\).

The general holonomy-flux cross-product algebra \(C_0(\hat{A}_T) \times_{\alpha} \hat{G}_{S,\Gamma}\) for an action \(\alpha \in \text{Act}(\hat{G}_{S,\Gamma}, C_0(\hat{A}_T))\) is in the case of a locally compact group \(G\) a non-commutative and non-unital \(C^*-\text{algebra.}\)

Refer to the definitions of restricted graph-diffeomorphisms presented in [7] Definition 6.2.10] and consider the non-standard identification of the configuration space \(\hat{A}_T\) with \(G^{\Gamma}\).
Proposition 4.6. The state $\omega_E^F(\bar{S})$ on $C_0(\bar{A}_F)\times_{\alpha^F_E} \bar{Z}_{S,\Gamma}$ associated to the GNS-representation $(\mathcal{H}_\Gamma, \pi_{\ell}(\mathcal{E}^F_S))$ is not surface-orientation-preserving graph-diffeomorphism invariant, but it is a surface-preserving graph-diffeomorphism invariant state.

Notice that,

$$\zeta_\sigma \circ \alpha(\rho_{S,\Gamma}(\Gamma)) \neq \alpha(\rho_{S,\Gamma}(\Gamma_\sigma)) \circ \zeta_\sigma$$

holds for every bisection $\sigma \in \mathcal{B}(\mathcal{P}_E^F)$ and $\rho_{S,\Gamma} \in G_{S,\Gamma}$. Therefore, it is necessary to restrict the holonomy-flux cross-product $C^*$-algebra to $C_0(\bar{A}_F)\times_{\alpha^F_E} \bar{Z}_{S,\Gamma}$.

**Proof**: Let $(\varphi_F, \Phi_F)$ be a graph-diffeomorphism on $\mathcal{P}_F$ over $V_F$, which is surface orientation preserving. Then investigate the following computation:

$$\omega_E^F(\theta_{(\varphi_F, \Phi_F)}(F_G)) = \int_{A_F} \int_{G_{S,\Gamma}} d\mu_F(\theta_{(\Phi_F(\gamma_1)),...,\Phi_F(\gamma_N)} \cdot \mu_{\bar{S},\Lambda}(\rho_{\varphi_F(S_i)}(\Phi_F(\gamma_1)),...,\rho_{\varphi_F(S_N)}(\Phi_F(\gamma_N))))$$

$$\left| F_G(\rho_{S,\Gamma}(\gamma_1),...,\rho_{S,\Gamma}(\gamma_N)) \right|^2$$

$$= \int_{A_F} \int_{G_{S,\Gamma}} d\mu_F(\theta_{(\Phi_F(\gamma_1)),...,\Phi_F(\gamma_N)}) \cdot \mu_{\bar{S},\Lambda}(\rho_{S_i}(\gamma_1),...,\rho_{S_N}(\gamma_N)) \cdot \left| F_G(\rho_{S,\Gamma}(\gamma_1),...,\rho_{S,\Gamma}(\gamma_N)) \right|^2$$

$$= \omega_E^F(F_G)$$

whenever $\varphi_F(S_i) = \bar{S}_i, \bar{S}_i \in \bar{S}$ for all $1 \leq i \leq N$ and $\Gamma_\sigma = (\Phi_F(\gamma_1),...,\Phi_F(\gamma_N))$. Clearly for $\varphi_F(S_i) = S_i$ the invariance property is easy to deduce.

The different possibilities of orientation of surfaces and the graphs allow to define a bulk of automorphic actions and $C^*$-dynamical systems for the holonomy algebra $C_0(\bar{A}_F)$. Therefore, speak about different surface configurations with respect to graphs and define many different holonomy-flux cross-product $C^*$-algebras. For example there are the following holonomy-flux cross-product $C^*$-algebras constructable: $C_0(\bar{A}_F)\times_{\alpha^F_E} \bar{G}_{S,\Gamma}$, $C_0(\bar{A}_F)\times_{\alpha^F_E} \bar{G}_{S,\Gamma}$, $C_0(\bar{A}_F)\times_{\alpha^F_E} \bar{G}_{S,\Gamma}$, $C_0(\bar{A}_F)\times_{\alpha^F_E} \bar{G}_{S,\Gamma}$, and $C_0(\bar{A}_F)\times_{\alpha^F_E} \bar{G}_{S,\Gamma}$ whenever $1 \leq M \leq N$ and for a set $\{\bar{S}_i\}$ of suitable surface sets.

If the tensor $C^*$-algebra $C_0(\bar{A}_F)\otimes C_0(\bar{A}_F)$ is used, then the $C^*$-algebra $C_0(\bar{A}_F)\times_{\alpha^F_E} \bar{G}_{S,\Gamma}\otimes C_0(\bar{A}_F)\times_{\alpha^F_E} \bar{G}_{S,\Gamma}$ with respect to the minimal $C^*$-norm is constructed.

Observe that, a generalised Stone - von Neumann theorem presented in [26, Theorem 4.24] is not achievable, since the objects $\bar{G}_{S,\Gamma}$ and $\bar{A}_F$ are not identified in general. It is necessary to distinguish between the two objects, since the holonomies are independent whereas the fluxes are dependent on a surface or surface set. Nevertheless, if it is assumed that $\bar{G}_{S,\Gamma}$ is identified with $G^M$ and $\bar{A}_F$ is identified with $G^N$, then the holonomy-flux cross-product algebra is identified with $C_0(G^N)\times_{\alpha^M_E} G^M$. But a generalised Stone - von Neumann theorem is only available for $M$ equal to $N$. This is the result of theorem [33] and theorem [310].

Hence only for $M = N$ the $C^*$-algebra $C_0(G^N)\times_{\alpha^F_E} G^N$ is isomorphic to $K(L^2(G^N, \mu_N))$. Notice that, the state $\omega_E^F$ is now given in this particular case by

$$\omega_E^F(F_G) = \int_{G^N} \int_{G^N} d\mu_N(h) d\mu_N(h) |F_G(h, h)|^2$$

for $F_G \in C_0(G^N)\times_{\alpha^F_E} G^N$ and does not depend on the surfaces anymore. If $\bar{G}_{S,\Gamma}$ for example is identified with $G^{N-1}$, then a problem occurs. The Morita equivalent $C^*$-algebra to $C_0(G^N)\times_{\alpha^F_E} G^M$ where $M < N$ is not of the form $C^*(G^K)$ for a suitable $K$ where $1 \leq K \leq N$. The author does not know any Morita equivalent $C^*$-algebra to the $C^*$-algebra $C_0(G^N)\times_{\alpha^F_E} G^M$ where $M < N$. 

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Pedersen [15 section 7.7] has presented a generalisation of regular representations of cross-products. His results are adapted to the case of a set $\tilde{S}$ of surfaces with simple surface intersection property for a finite orientation preserved graph system associated to $\Gamma$ in the following paragraphs.

Set $\mathcal{H}^\Gamma_{E(S)} := L^2(G_{S,\Gamma}, \mathcal{H}_\Gamma)$, where this Hilbert space is identified with $L^2(G_{S,\Gamma}) \otimes \mathcal{H}_\Gamma$. In the following investigation the elements $f_\Gamma$ are understood as elements of $\mathcal{K}(G_{S,\Gamma}, C_0(\mathcal{A}_\Gamma))$.

First observe that, $\Psi^\Gamma_{E(S)}(\rho_{S,\Gamma}(\Gamma))$ is an element of $\mathcal{H}^\Gamma_{E(S)}$, if there is a map

$$\rho_{S,\Gamma}(\Gamma) \mapsto \Psi^\Gamma_{E(S)}(f_\Gamma, \rho_{S,\Gamma}(\Gamma))$$

such that $\Psi^\Gamma_{E(S)}(\rho_{S,\Gamma}(\Gamma)) \in \mathcal{H}_\Gamma$.

Then recall the $C^*$-algebra dynamical system $(G_{S,\Gamma}, C_0(\mathcal{A}_\Gamma), \alpha^\Gamma_L)$ and the covariant pair $(\Phi_M, U^\Gamma_L)$ of this $C^*$-dynamical system. There is a morphism $\Phi^M_L$ of the $C^*$-algebras, which maps from $C_0(\mathcal{A}_\Gamma)$ to $\mathcal{K}(\mathcal{H}^\Gamma_{E(S)})$, and a representation $U^\Gamma_L$ of the group $G_{S,\Gamma}$ on the $C^*$-algebra $\mathcal{K}(\mathcal{H}^\Gamma_{E(S)})$. Both objects are defined by

$$(\Phi^M_L(f_\Gamma)\Psi^\Gamma_{E(S)})(\rho_{S,\Gamma}(\Gamma)) = \Phi_M(\alpha^\Gamma_L(\rho_{S,\Gamma}(\Gamma))(f_\Gamma))\Psi^\Gamma_{E(S)}(\rho_{S,\Gamma}(\Gamma))$$

for $\Psi^\Gamma_{E(S)} \in \mathcal{H}^\Gamma_{E(S)}$, $f_\Gamma \in C_0(\mathcal{A}_\Gamma)$ and

$$(U^\Gamma_L(\hat{\rho}_{S,\Gamma}(\Gamma)))\Psi^\Gamma_{E(S)}(\rho_{S,\Gamma}(\Gamma)) := \Psi^\Gamma_{E(S)}(L(\hat{\rho}_{S,\Gamma}(\Gamma)))(\rho_{S,\Gamma}(\Gamma))$$

for $U^\Gamma_L \in \text{Rep}(G_{S,\Gamma}, \mathcal{K}(\mathcal{H}^\Gamma_{E(S)}))$, $\rho_{S,\Gamma}, \hat{\rho}_{S,\Gamma}, \tilde{\rho}_{S,\Gamma} \in G_{S,\Gamma}$. Then $(\Phi^M_L, U^\Gamma_L)$ defines a covariant representation of $(G_{S,\Gamma}, C_0(\mathcal{A}_\Gamma), \alpha^\Gamma_L)$ in $\mathcal{K}(\mathcal{H}^\Gamma_{E(S)})$.

**Definition 4.7.** Let $\tilde{S}$ be a set of surfaces with simple surface intersection property for a finite orientation preserved graph system associated to a graph $\Gamma$.

The left regular representation of the holonomy-flux cross-product $C^*$-algebra

$C_0(\mathcal{A}_\Gamma) \rtimes_{\alpha^\Gamma_L} G_{S,\Gamma}$ induced by $(\Phi_M, \mathcal{H}_\Gamma)$ is the representation $\pi^\Gamma_{E(S)}$ on $L^2(G_{S,\Gamma}, \mathcal{H}_\Gamma)$, which is expressed by

$$(\pi^\Gamma_{E(S)}(f_\Gamma))\Psi^\Gamma_{E(S)}(\rho_{S,\Gamma}(\Gamma)) = \left(\left((\Phi^M_L \times U^\Gamma_L)(f_\Gamma)\right)\Psi^\Gamma_{E(S)}(\rho_{S,\Gamma}(\Gamma))\right)$$

for $f_\Gamma(\hat{\rho}_{S,\Gamma}(\Gamma)) \in C_0(\mathcal{A}_\Gamma)$, $\rho_{S,\Gamma}, \hat{\rho}_{S,\Gamma}, \tilde{\rho}_{S,\Gamma} \in G_{S,\Gamma}$ and $\Psi^\Gamma_{E(S)} \in \mathcal{H}^\Gamma_{E(S)}$. The representation $\pi^\Gamma_{E(S)}$ is also denoted by $\Phi^M_L \times U^\Gamma_L$.

Then recall a general $C^*$-algebra dynamical system given by $(G_{S,\Gamma}, C_0(\mathcal{A}_\Gamma), \alpha)$. There is a morphism $\Phi^M_{E(S)}$ from the $C^*$-algebra $C_0(\mathcal{A}_\Gamma)$ to $\mathcal{K}(\mathcal{H}^\Gamma_{E(S)})$ and a representation $U$ of the group $G_{S,\Gamma}$ on the $C^*$-algebra $\mathcal{K}(\mathcal{H}^\Gamma_{E(S)})$. They are defined by

$$(\Phi^M_{E(S)}(f_\Gamma))\Psi^\Gamma_{E(S)}(\rho_{S,\Gamma}(\Gamma)) := \Phi_M(\alpha(\rho_{S,\Gamma}(\Gamma))(f_\Gamma))\Psi^\Gamma_{E(S)}(\rho_{S,\Gamma}(\Gamma))$$

for $\Psi^\Gamma_{E(S)} \in \mathcal{H}^\Gamma_{E(S)}$, $f_\Gamma \in C_0(\mathcal{A}_\Gamma)$ and $\rho_{S,\Gamma}, \rho_{S,\Gamma} \in G_{S,\Gamma}$. Consequently, a general regular representation of the holonomy-flux cross-product is given by

$$\pi^\Gamma_{E(S)}(f_\Gamma)\Psi^\Gamma_{E(S)} = (\Phi^M_{E(S)} \times U)(f_\Gamma)\Psi^\Gamma_{E(S)}$$

whenever $U \in \text{Rep}(G_{S,\Gamma}, \mathcal{K}(\mathcal{H}^\Gamma_{E(S)}))$ and $f_\Gamma \in C_0(\mathcal{A}_\Gamma)$.

Until now a unification of the different holonomy-flux cross-product $C^*$-algebras for certain surface sets has been not presented. The following algebra plays an important role.
Definition 4.8. Let $\tilde{S}$ be a set of surfaces with simple surface intersection property for a finite orientation preserved graph system associated to a graph $\Gamma$.

The multiplier algebra of the holonomy-flux cross-product $C^*$-algebra $C_0(\mathcal{A}_\Gamma) \rtimes_{\alpha_L} \hat{G}_{S,\Gamma}$ is given by all linear operators

$$M : C_0(\mathcal{A}_\Gamma) \rtimes_{\alpha_L} \hat{G}_{S,\Gamma} \to C_0(\mathcal{A}_\Gamma) \rtimes_{\alpha_L} \hat{G}_{S,\Gamma}$$

such that for any $F_\Gamma \in C_0(\mathcal{A}_\Gamma) \rtimes_{\alpha_L} \hat{G}_{S,\Gamma}$ there exists a $\hat{F}_\Gamma \in C_0(\mathcal{A}_\Gamma) \rtimes_{\alpha_L} \hat{G}_{S,\Gamma}$ such that for all $F_\Gamma \in C_0(\mathcal{A}_\Gamma) \rtimes_{\alpha_L} \hat{G}_{S,\Gamma}$ it is true that

$$\hat{F}_\Gamma^* M(F_\Gamma) = \left\langle \hat{F}_\Gamma, M(F_\Gamma) \right\rangle_{C_0(\mathcal{A}_\Gamma) \rtimes_{\alpha_L} \hat{G}_{S,\Gamma}} = \left\langle \hat{F}_\Gamma, F_\Gamma \right\rangle_{C_0(\mathcal{A}_\Gamma) \rtimes_{\alpha_L} \hat{G}_{S,\Gamma}} = \hat{F}_\Gamma^* F_\Gamma$$

In particular, the multiplier algebra of the reduced holonomy-flux group $C^*$-algebra $C_0^r(\hat{G}_{S,\Gamma}, C_0(\mathcal{A}_\Gamma))$ consists of such linear maps $M$ such that for any $F_\Gamma \in C_0(\mathcal{A}_\Gamma)$ there exists a $\hat{F}_\Gamma \in C_0(\mathcal{A}_\Gamma)$ such that for all $F_\Gamma \in C_0(\mathcal{A}_\Gamma)$ it is true that

$$\left\langle (p \Gamma S(\hat{F}_\Gamma(\hat{\rho}_{S,\Gamma}(\Gamma)))) \Psi^\Gamma_{E(S)}(\pi^\Gamma S_L(\pi^\Gamma S_L(M(\hat{\rho}_{S,\Gamma}(\Gamma)))) \Phi^\Gamma_{E(S)}) \right\rangle_{H^\Gamma_{E(S)}} = \left\langle \pi^\Gamma S_L(\hat{F}_\Gamma(\hat{\rho}_{S,\Gamma}(\Gamma))) \Psi^\Gamma_{E(S)}(\pi^\Gamma S_L(F_\Gamma(\hat{\rho}_{S,\Gamma}(\Gamma)))) \Phi^\Gamma_{E(S)}) \right\rangle_{H^\Gamma_{E(S)}} \tag{26}$$
yields whenever $\Psi^\Gamma_{E(S)} \in H^\Gamma_{E(S)}$.

Example 4.1: In [3] Definition 3.19, [7] Definition 6.1.19] the following map $I$ has been introduced. The map $I : C_0(\mathcal{A}_\Gamma) \to C_0(\mathcal{A}_\Gamma^{-1})$ is given by

$$I : f_\Gamma \mapsto f_{\Gamma^{-1}}, \text{ where } (I \circ f_\Gamma)(\gamma_1, \ldots, \gamma_N) := f_{\Gamma^{-1}}(\gamma_1^{-1}, \ldots, \gamma_N^{-1})$$
such that $I^2 = \text{id}$, where id is the identical automorphism on $C_0(\mathcal{A}_\Gamma)$.

Consider a suitable set $\tilde{S}$ of surfaces that is contained in the set $S$ and let $M \leq N$. Note that, if $M < N$, then there is a set of paths $\Gamma'' := \Gamma \setminus \Gamma'$ such that each path of this set does not intersect a surface in $\tilde{S}$. Each path in $\Gamma''$ intersects only one surface in $\tilde{S}$ at the source vertex of this path. Then $G_{S,\Gamma} \leq \Gamma''$ is a subgroup of $G_{S,\Gamma}$ and is embedded by $G_{S,\Gamma} := G_{S,\Gamma} \times \{ e_G \} \times \ldots \times \{ e_G \}$ in $\hat{G}_{S,\Gamma}$. Denote the set of surfaces, which has the simple surface intersection property for the finite orientation preserved graph system $\mathcal{P}_\Gamma$, which is contained in $S$, and which is not contained in $\tilde{S}$, by $\tilde{R}$. Note that $G_{\tilde{R},\Gamma''} \leq \Gamma''$ is a subgroup of $G_{\tilde{R},\Gamma}$ and is embedded by $G_{\tilde{R},\Gamma} := G_{\tilde{R},\Gamma''} \times \{ e_G \} \times \ldots \times \{ e_G \}$ in $\hat{G}_{\tilde{R},\Gamma}$. Let $\tilde{R}$ be a set of surfaces, which has the same surface intersection property for a path $\gamma$ in a graph, which is contained in the finite orientation preserved graph system $\mathcal{P}_\Gamma$.

**Situation 1:**

Then there is a $C^*$-dynamical system in $\mathcal{K}(H^G_{E(S),\Gamma})$, which is given by $(\hat{G}_{S,\Gamma^{-1}}, C_0(\mathcal{A}_\Gamma^{-1}), \alpha^M_{\Gamma^{-1}})$. Let $(\Phi^\_M, U^\_M)_{\Gamma^{-1}}$ be a covariant pair associated to the $C^*$-dynamical system.

Then observe that $\alpha^M_{\Gamma^{-1}} = I \circ \alpha^M_{\Gamma} \circ I^{-1}$ and $U^M_{\Gamma^{-1}} = I \circ U^M_{\Gamma} \circ I^{-1}$ hold. Then $(\hat{G}_{S,\Gamma}, C_0(\mathcal{A}_\Gamma^{-1}), I \circ \alpha^M_{\Gamma^{-1}} \circ I^{-1})$ is a $C^*$-dynamical system in $\mathcal{K}(H^G_{E(S),\Gamma})$. Respectively, $(\hat{G}_{S,\Gamma}, C_0(\mathcal{A}_\Gamma), \alpha^M_{\Gamma})$ is a $C^*$-dynamical system in $\mathcal{K}(H^G_{E(S),\Gamma})$.

Note that, if $\tilde{S}$ is equal to $S$, then $\tilde{S}$ has the simple surface intersection property for the finite orientation preserved graph system $\mathcal{P}_\Gamma$, and $M = N$. Then $(\hat{G}_{S,\Gamma}, C_0(\mathcal{A}_\Gamma^{-1}), I \circ \alpha^N_{\Gamma} \circ I^{-1})$ and $(\hat{G}_{S,\Gamma}, C_0(\mathcal{A}_\Gamma), \alpha^N_{\Gamma})$ are two $C^*$-dynamical systems in $\mathcal{K}(H^G_{E(S),\Gamma})$.

**Situation 2:**

Furthermore, there is a $C^*$-dynamical system in $\mathcal{K}(H^G_{E(S)})$ given by $(\hat{G}_{R,\Gamma}, C_0(\mathcal{A}_\Gamma), \alpha^N_{\Gamma})$ for $K$ suitable.

**Situation 3:**

There is a $C^*$-dynamical system in $\mathcal{K}(H^G_{E(S)})$ given by $(\hat{G}_{R,\Gamma'}, C_0(\mathcal{A}_\Gamma'), \alpha^N_{\Gamma'})$.
Situation 4:
Finally, there is $C^*$-dynamical system in $\mathcal{K}(\mathcal{H}_F^G(S))$ given by $(\tilde{G}_{S_{\Gamma}^{-1}} \times G, C_0(A), (I^{-1} \circ M_R \circ I) \circ \alpha_{K_L}^K)$. Note that, $(I^{-1} \circ M_R \circ I) \circ \alpha_{K_L}^K = (I^{-1} \circ M_R \circ I)$. Reformulate $(\tilde{G}_{S_{\Gamma}^{-1}} \times G, C_0(A), (I^{-1} \circ M_R \circ I) \circ \alpha_{K_L}^K)$. For each $C^*$-dynamical system given above there is a cross-product $C^*$-algebra.

In the following proposition the situation 1 is studied.

Proposition 4.9. Let $\hat{T} := \{T_1, ..., T_N\}$ be a set of surfaces with simple surface intersection property for the orientation preserved graph system $\mathcal{P}_F^S$. Let $\hat{S} := \{S_1, ..., S_M\}$ be a set of surfaces that is contained in $\hat{T}$ an such that $M \leq N$.

The unitaries $U_M^R(\rho_{S_{\Gamma}^{-1}}(\Gamma^{-1}))$, whenever $\rho_{S_{\Gamma}^{-1}}(\Gamma^{-1}) \in \tilde{G}_{S_{\Gamma}^{-1}}$, are elements of the multiplier algebra of the $C^*$-algebra $C_0(\tilde{A}) \times_{\alpha_{K_L}^K} \tilde{G}_{S_{\Gamma}}$. Moreover, the elements of the holonomy-flux cross-product algebra $C_0(\tilde{A}) \times_{\alpha_{K_L}^K} \tilde{G}_{S_{\Gamma}}$, are multipliers of the $C^*$-algebra $C_0(\tilde{A}) \times_{\alpha_{K_L}^K} \tilde{G}_{S_{\Gamma}}$.

Proof. Choose the two surface sets $\hat{S}$ and $\hat{T}$ and a graph $\Gamma$ such that $(C_0(\tilde{A}), \tilde{G}_{S_{\Gamma}^{-1}}, I^{-1} \circ M_R \circ I)$ and $(C_0(\tilde{A}), \tilde{G}_{T_{\Gamma}^{-1}}, I^{-1} \circ M_R \circ I)$ are two $C^*$-dynamical systems. Then notice that

$$(F_T \ast \tilde{F}_1)(\hat{\rho}_{S_{\Gamma}^{-1}}(\Gamma^{-1})) = \int_{\tilde{G}_{S_{\Gamma}^{-1}}} d\mu_{S_{\Gamma}^{-1}}(\hat{\rho}_{S_{\Gamma}^{-1}}(\Gamma^{-1})) F_T(\hat{\rho}_{S_{\Gamma}^{-1}}(\Gamma^{-1})) \left(I^{-1} \circ M_R(\rho_{S_{\Gamma}^{-1}}(\Gamma^{-1})) \right)$$

holds whenever $F_T \in L^1(\tilde{G}_{S_{\Gamma}^{-1}}, C_0(\tilde{A}), I^{-1} \circ M_R \circ I)$ and $\tilde{F}_1 \in L^1(\tilde{G}_{S_{\Gamma}^{-1}}, C_0(\tilde{A}), I^{-1} \circ M_R \circ I)$. Furthermore recognize that,

$$F_T(\hat{\rho}_{S_{\Gamma}^{-1}}(\Gamma^{-1})) = (I^{-1} \circ M_R(\hat{\rho}_{S_{\Gamma}^{-1}}(\Gamma^{-1})) \left(I^{-1} \circ M_R(\hat{\rho}_{S_{\Gamma}^{-1}}(\Gamma^{-1})) \right)$$

is true.

Notice that,

$$\hat{\alpha}_N^R(\hat{\rho}_{S_{\Gamma}^{-1}}(\Gamma^{-1})) = (I \circ \alpha_{K_L}^K(\hat{\rho}_{S_{\Gamma}^{-1}}(\Gamma^{-1})))$$

and

$$\int_{\mathcal{A}_T} d\mu_T(\hat{\rho}_{S_{\Gamma}^{-1}}(\Gamma^{-1})) \circ \alpha_{K_L}^K(\hat{\rho}_{S_{\Gamma}^{-1}}(\Gamma^{-1})) \circ I(\hat{T}_T)(\hat{T}_T) = \int_{\mathcal{A}_T} d\mu_T(\hat{T}_T)(\hat{T}_T)$$

yields whenever $T_T \in C_0(\tilde{T})$ and $\tilde{\rho}_{S_{T_T}}^N \in G_{T_{\Gamma}}$. Clearly, there is a representation $\pi_{I, R}^M$ of $L^1(\tilde{G}_{S_{\Gamma}^{-1}}, C_0(\tilde{A}), I^{-1} \circ M_R \circ I)$ on $\mathcal{H}_T$, which is given by

$$\pi_{I, R}^M(\hat{T}_T) := \int_{\tilde{G}_{S_{\Gamma}^{-1}}} d\mu_{S_{\Gamma}^{-1}}(\hat{\rho}_{S_{\Gamma}^{-1}}(\Gamma^{-1})) \phi_M(\hat{T}_T(\hat{\rho}_{S_{\Gamma}^{-1}}(\Gamma^{-1}))) \circ I(\hat{T}_T(\hat{\rho}_{S_{\Gamma}^{-1}}(\Gamma^{-1}))) \circ I(\hat{T}_T(\hat{\rho}_{S_{\Gamma}^{-1}}(\Gamma^{-1})))$$

where $\hat{\rho}_{S_{\Gamma}^{-1}}(\Gamma^{-1}) \in \tilde{G}_{S_{\Gamma}^{-1}}$, $\hat{T}_T \in L^1(\tilde{G}_{S_{\Gamma}^{-1}}, C_0(\tilde{A}), I^{-1} \circ M_R \circ I)$ and $\psi_T \in \mathcal{H}_T$. Then derive that there is an isomorphism $T$ from $L^1(\tilde{G}_{S_{\Gamma}^{-1}}, C_0(\tilde{A}), I^{-1} \circ M_R \circ I)$ to $L^1(\tilde{G}_{S_{\Gamma}^{-1}}, C_0(\tilde{A}), I^{-1} \circ M_R \circ I)$. 

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Then the Hilbert space $L^2(G_{S,Γ}, μ_{S,Γ})$ is embedded into $L^2(G_{T,Γ}, μ_{T,Γ})$. The left regular representation of $C_0(\mathcal{A}_T) \rtimes_{\alpha^N_R} G_{S,Γ-1}$ on $L^2(G_{T,Γ}, μ_{T,Γ}) \otimes \mathcal{H}_Γ$ is given by

$$(\pi^{-1}_{Γ}S(F_Γ)\Psi_{E(T)}^{Γ}(ρ_{T,Γ}(Γ))) = \left( (\Phi^{M}_{R} \times (I^{-1} \circ U^{N}_{R} \circ I)(F_Γ))\Psi_{E(T)}^{Γ}(ρ_{T,Γ}(Γ)) \right) \cdot \int_{G_{S,Γ-1}} dμ_{S,Γ-1}(\hat{ρ}_{S,Γ-1}(Γ^{-1}))$$

for $F_Γ(\hat{ρ}_{S,Γ-1}(Γ^{-1})) \in C_0(\mathcal{A}_T)$, $ρ_{T,Γ} \in G_{T,Γ}$, $\hat{ρ}_{S,Γ-1}, \hat{ρ}_{S,Γ-1} \in G_{S,Γ-1}$, and $Ψ_{E(S)}^{Γ} \in \mathcal{H}_{E(S)}^{Γ}$.

Set $U^{N}_{L}(\hat{ρ}_{T,Γ}(Γ)) := \hat{Ψ}_{E(T)}^{Γ}(ρ_{T,Γ}(Γ))$ and $(I^{-1} \circ U^{N}_{R}(\hat{ρ}_{S,Γ-1}(Γ^{-1})) \circ I)\hat{Ψ}_{E(T)}^{Γ}(ρ_{T,Γ}(Γ)) := \hat{Ψ}_{E(T)}^{Γ}(ρ_{T,Γ}(Γ))$. Then the unitaries $I^{-1} \circ U^{N}_{R}(\hat{ρ}_{S,Γ-1}(Γ^{-1})) \circ I$, whenever $\hat{ρ}_{S,Γ-1} \in G_{S,Γ-1}$, are multipliers. This is verified by the following computation:

$$(\pi_{L}^{Γ,T}((F_Γ))\Psi_{E(T)}^{Γ}(ρ_{T,Γ}(Γ)), (\pi_{L}^{Γ,T}(M_U(F_Γ)))\Psi_{E(T)}^{Γ}(ρ_{T,Γ}(Γ))) \cdot \int_{G_{S,Γ-1}} \int_{G_{T,Γ}} dμ_{S,Γ-1}(\hat{ρ}_{S,Γ-1}(Γ^{-1})) dμ_{T,Γ}(\hat{ρ}_{T}(Γ))$$

holds whenever $(I^{-1} \circ U^{N}_{R}(\hat{ρ}_{S,Γ-1}(Γ^{-1})) \circ I)\hat{F}_{Γ} := \hat{F}_{Γ}$.

Finally each element of the $C^*$-algebra $C_0(\mathcal{A}_T) \rtimes_{\alpha^N_R} G_{S,Γ-1}$ defines a linear map $M$ from $C_0(\mathcal{A}_T) \rtimes_{\alpha^N_R} G_{S,Γ-1}$.
\(G_{T,\Gamma}\) to \(C_0(\hat{A}_T) \times_{\alpha_L^\infty} G_{T,\Gamma}\) by

\[
((\pi_{\hat{F},T}^E(M(\hat{F}_T)))\Psi_{E(\hat{T})}(\rho_{T,\Gamma}(\Gamma))) = ((\pi_{\hat{F},T}^E(M(\hat{F}_T)))\Psi_{E(\hat{T})}(\rho_{T,\Gamma}(\Gamma)))
\]

\[
\Phi_M \left( F_T(\hat{\rho}_{S,\Gamma}(\Gamma^{-1}))(\alpha_{\hat{M}}^M(\hat{\rho}_{S,\Gamma}(\Gamma^{-1})) \circ I \circ \alpha_{\hat{N}}^N(\rho_{T,\Gamma}(\Gamma)))(\hat{F}_T)(\hat{\rho}_{S,\Gamma}(\Gamma^{-1}) \hat{\rho}_{T,\Gamma}(\Gamma)) \right)
U_{\hat{N}}^N(\hat{\rho}_{T,\Gamma}(\Gamma))\Psi_{E(\hat{T})}(\rho_{T,\Gamma}(\Gamma))
\]

for \(F_T(\hat{\rho}_{S,\Gamma}(\Gamma)), \hat{F}_T(\hat{\rho}_{T,\Gamma}(\Gamma)) \in C_0(\hat{A}_T), \hat{\rho}_{S,\Gamma}, \hat{\rho}_{T,\Gamma}, \rho_{T,\Gamma} \in G_{T,\Gamma}, \rho_{T,\Gamma}(\Gamma) := (\hat{\rho}_{T,\Gamma}(\Gamma), e_G, \ldots, e_G) \in G_{T,\Gamma}\) and \(\Psi_{E(\hat{T})} \in H_{E(\hat{T})}\). Clearly, if the set \(\hat{S}\) is replaced by a set \(\hat{R}^{-1}\), which is contained in \(\hat{T}\), then \(\hat{\rho}_{R^{-1},\Gamma}(\Gamma^{-1}) = \rho_{R,\Gamma}(\Gamma) \in G_{R,\Gamma}\) and \(\alpha_{\hat{N}}^N(\rho_{R,\Gamma}(\Gamma)) = \alpha_{\hat{N}}^N(\rho_{R,\Gamma}(\Gamma)) \in Aut(C_0(\hat{A}_T))\) yield.

Set \((I^{-1} \circ U_{\hat{N}}^N(\rho_{S,\Gamma}(\Gamma^{-1})) \circ I \circ \alpha_{\hat{N}}^N(\rho_{T,\Gamma}(\Gamma))\) \(\Psi_{E(\hat{T})}(\rho_{T,\Gamma}(\Gamma)) := \hat{\Psi}_{E(\hat{T})}(\rho_{T,\Gamma}(\Gamma))\). Then \(M\) is a multiplier since the following derivation is true:

\[
\left(\left((\pi_{\hat{F},T}^E(M(\hat{F}_T)))\Psi_{E(\hat{T})}(\rho_{T,\Gamma}(\Gamma)), ((\pi_{\hat{F},T}^E(M(\hat{F}_T)))\Psi_{E(\hat{T})}(\rho_{T,\Gamma}(\Gamma))\right)_{H_{E(\hat{T})}}
\]

\[
= \int_{G_{S,\Gamma}} \int_{G_{T,\Gamma}} d\mu_{S,\Gamma^{-1}}(\hat{\rho}_{S,\Gamma}(\Gamma^{-1})) \, d\mu_{T,\Gamma}(\rho_{T,\Gamma}(\Gamma))
\]

\[
\left(\Phi_M \left( \alpha_{\hat{N}}^N(\rho_{T,\Gamma}(\Gamma)) \right)(\hat{F}_T)(\hat{\rho}_{T,\Gamma}(\Gamma)) \right) U_{\hat{N}}^N(\hat{\rho}_{T,\Gamma}(\Gamma))\Psi_{E(\hat{T})}(\rho_{T,\Gamma}(\Gamma)),
\]

\[
\Phi_M \left( F_T(\hat{\rho}_{S,\Gamma}(\Gamma^{-1}))(\alpha_{\hat{M}}^M(\hat{\rho}_{S,\Gamma}(\Gamma^{-1})) \circ I \circ \alpha_{\hat{N}}^N(\rho_{T,\Gamma}(\Gamma)))(\hat{F}_T)(\hat{\rho}_{S,\Gamma}(\Gamma^{-1}) \hat{\rho}_{T,\Gamma}(\Gamma)) \right)
U_{\hat{N}}^N(\hat{\rho}_{T,\Gamma}(\Gamma))\Psi_{E(\hat{T})}(\rho_{T,\Gamma}(\Gamma))
\]

\[
= \int_{G_{S,\Gamma}} \int_{G_{T,\Gamma}} d\mu_{S,\Gamma^{-1}}(\hat{\rho}_{S,\Gamma}(\Gamma^{-1})) \, d\mu_{T,\Gamma}(\rho_{T,\Gamma}(\Gamma))
\]

\[
\left(\Phi_M \left( \alpha_{\hat{N}}^N(\rho_{T,\Gamma}(\Gamma)) \right)(\hat{F}_T)(\hat{\rho}_{T,\Gamma}(\Gamma)) \right) \hat{\Psi}_{E(\hat{T})}(\rho_{T,\Gamma}(\Gamma)),
\]

\[
\Phi_M \left( F_T(\hat{\rho}_{S,\Gamma}(\Gamma^{-1}))(\alpha_{\hat{M}}^M(\hat{\rho}_{S,\Gamma}(\Gamma^{-1})) \circ I \circ \alpha_{\hat{N}}^N(\rho_{T,\Gamma}(\Gamma)))(\hat{F}_T)(\hat{\rho}_{S,\Gamma}(\Gamma^{-1}) \hat{\rho}_{T,\Gamma}(\Gamma)) \right)
\hat{\Psi}_{E(\hat{T})}(\rho_{T,\Gamma}(\Gamma))
\]

\[
= \int_{G_{S,\Gamma}} \int_{G_{T,\Gamma}} d\mu_{S,\Gamma}(\hat{\rho}_{S,\Gamma}(\Gamma)) \, d\mu_{T,\Gamma}(\rho_{T,\Gamma}(\Gamma))
\]

\[
\left(\Phi_M \left( \alpha_{\hat{N}}^N(\rho_{T,\Gamma}(\Gamma)) \right)(\hat{F}_T)(\hat{\rho}_{T,\Gamma}(\Gamma)) \right) U_{\hat{N}}^N(\hat{\rho}_{T,\Gamma}(\Gamma))\Psi_{E(\hat{T})}(\rho_{T,\Gamma}(\Gamma)),
\]

\[
\Phi_M \left( F_T(\hat{\rho}_{S,\Gamma}(\Gamma^{-1}))(\alpha_{\hat{M}}^M(\hat{\rho}_{S,\Gamma}(\Gamma^{-1})) \circ I \circ \alpha_{\hat{N}}^N(\rho_{T,\Gamma}(\Gamma)))(\hat{F}_T)(\hat{\rho}_{S,\Gamma}(\Gamma^{-1}) \hat{\rho}_{T,\Gamma}(\Gamma)) \right)
U_{\hat{N}}^N(\hat{\rho}_{T,\Gamma}(\Gamma))\Psi_{E(\hat{T})}(\rho_{T,\Gamma}(\Gamma))
\]

\[
= \int_{G_{S,\Gamma}} \int_{G_{T,\Gamma}} d\mu_{S,\Gamma^{-1}}(\hat{\rho}_{S,\Gamma}(\Gamma^{-1})) \, d\mu_{T,\Gamma}(\rho_{T,\Gamma}(\Gamma))
\]

\[
\left(\Phi_M \left( I^{-1} \circ \alpha_{\hat{M}}^M(\hat{\rho}_{S,\Gamma}(\Gamma^{-1})) \circ I \right) \left(\hat{F}_T(\hat{\rho}_{S,\Gamma}(\Gamma^{-1})) \right) \right)
U_{\hat{N}}^N(\hat{\rho}_{T,\Gamma}(\Gamma))\Psi_{E(\hat{T})}(\rho_{T,\Gamma}(\Gamma)),
\]

\[
\Phi_M \left( \alpha_{\hat{N}}^N(\rho_{T,\Gamma}(\Gamma)) \right)(\hat{F}_T)(\hat{\rho}_{T,\Gamma}(\Gamma)) \right) \hat{\Psi}_{E(\hat{T})}(\rho_{T,\Gamma}(\Gamma))
\]

\[
= \left(\left((\pi_{\hat{F},T}^E(M(\hat{F}_T)))\Psi_{E(\hat{T})}(\rho_{T,\Gamma}(\Gamma)), \pi_{\hat{F},T}^E(\hat{F}_T)\Psi_{E(\hat{T})}(\rho_{T,\Gamma}(\Gamma))\right)_{H_{E(\hat{T})}}
\]

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Notice that, the same arguments are used for a surface set $\tilde{T} := \{T_1, ..., T_N\}$, which has the simple surface intersection property for the orientation preserved graph system $\mathcal{P}_k^R$ and where $\tilde{R}^{-1} := \{R_1^{-1}, ..., R_N^{-1}\}$ is a set of surfaces that has the simple surface intersection property for the orientation preserved graph system $\mathcal{P}_k^{R^{-1}}$. Indeed it can be shown that for all situations of example 4 except situation 2 similar results can be obtained. The situation 2 is not needed in the next theorem and hence is briefly discussed in the following remark.

**Remark 4.10.** In situation 2 the sets $\tilde{R}$ and $\tilde{S}$ are disjoint. Let $\tilde{T}_2$ and $\tilde{T}_3$ be two disjoint surface sets such that the holonomy-flux cross-product algebras are given by $C_0(\tilde{A}_T) \times_{\alpha_N} \tilde{G}_{T_3,\Gamma}$ and $C_0(\tilde{A}_T) \times_{\alpha_N} \tilde{G}_{T_5,\Gamma}$.

Then the elements of these algebras are represented on two different Hilbert spaces $\mathcal{H}^{E}_{\tilde{T}_2} := L^2(\hat{\tilde{G}}_{T_3,\Gamma}, \mu_{T_5,\Gamma}) \otimes \mathcal{H}_\Gamma$ and $\mathcal{H}^{E}_{\tilde{T}_3} := L^2(\hat{\tilde{G}}_{T_3,\Gamma}, \mu_{T_5,\Gamma}) \otimes \mathcal{H}_\Gamma$. Set $\mathcal{H}_{E}(\tilde{T}_i) := L^2(\hat{\tilde{G}}_{T_3,\Gamma}, \mu_{T_5,\Gamma})$ for $i = 2, 3$. Hence there are two representations $\pi_{E}(\tilde{T}_2)$ and $\pi_{E}(\tilde{T}_3)$ such that $\pi_{E}(\tilde{T}_2) \otimes \pi_{E}(\tilde{T}_3)$ is a representation on $\mathcal{H}_{E}(\tilde{T}_2) \otimes \mathcal{H}_{E}(\tilde{T}_3)$.

The holonomy-flux cross-product $C^*$-algebra $C_0(\tilde{A}_T) \times_{\alpha_N} \tilde{G}_{T_3,\Gamma}$ is represented on $\mathcal{H}^{E}_{\tilde{T}_2}$ by

$$\langle \pi_{\tilde{T}_2}(F) \Psi_{E}(\tilde{T}_2), \pi_{\tilde{T}_2}(M(F)) \Phi_{E}(\tilde{T}_2) \rangle_{\mathcal{H}_{E}(\tilde{T}_2) \otimes \mathcal{H}_\Gamma}$$

whenever $\pi_{\tilde{T}_2}(F) \in C_0(\tilde{A}_T) \times_{\alpha_N} \tilde{G}_{T_3,\Gamma}$ and $F \in C_0(\tilde{A}_T)$ and the computation

$$\langle \pi_{\tilde{T}_2}(F) \Psi_{E}(\tilde{T}_2), \pi_{\tilde{T}_2}(M(F)) \Phi_{E}(\tilde{T}_2) \rangle_{\mathcal{H}_{E}(\tilde{T}_2) \otimes \mathcal{H}_\Gamma} = \langle \int_{\tilde{G}_{T_3,\Gamma}} d \mu_{T_3,\Gamma}(\tilde{G}_{T_3,\Gamma}) \Phi_{M} \left( \left( \omega_N \rho_{T_3,\Gamma}(\Gamma) \right) \Phi_{E}(\tilde{T}_2) \right) \Psi_{E}(\tilde{T}_2) \rangle_{\mathcal{H}_{E}(\tilde{T}_2) \otimes \mathcal{H}_\Gamma}$$

Then show that, the holonomy-flux cross-product $C^*$-algebra $C_0(\tilde{A}_T) \times_{\alpha_N} \tilde{G}_{T_3,\Gamma}$ is a subset of the multiplier algebra $M(C_0(\tilde{A}_T) \times_{\alpha_N} \tilde{G}_{T_3,\Gamma})$. The multiplier $M$ is assumed to be the map $C_0(\tilde{A}_T) \times_{\alpha_N} \tilde{G}_{T_3,\Gamma} \ni F \mapsto \tilde{F} * F \in C_0(\tilde{A}_T) \times_{\alpha_N} \tilde{G}_{T_3,\Gamma}$ for a $\tilde{F} \in C_0(\tilde{A}_T) \times_{\alpha_N} \tilde{G}_{T_3,\Gamma}$. But since $L(\rho_{T_3,\Gamma}(\Gamma)^{-1})(\tilde{G}_{T_3,\Gamma})$ is not well-defined, the convolution

$$\langle \tilde{F} * F \rangle \left( \tilde{\rho}_{T_3,\Gamma}(\Gamma) \right) = \int_{\tilde{G}_{T_3,\Gamma}} d \mu_{T_3,\Gamma}(\tilde{G}_{T_3,\Gamma}) \tilde{F} \left( \tilde{\rho}_{T_3,\Gamma}(\Gamma) \right) (\omega_N \rho_{T_3,\Gamma}(\Gamma)) F \left( \tilde{\rho}_{T_3,\Gamma}(\Gamma) \right)$$

is not well-defined, too. Consequently, it has to be assumed that either $\tilde{G}_{T_3,\Gamma}$ is embedded into $\tilde{G}_{T_3,\Gamma}$ as a subgroup or the other way around. Clearly the situation 4 is of this form.
Remark 4.11. Let $S$ contains only the surface $S$ and let $\bar{S}$ be a surface set with the same surface intersection property for a path $\gamma$. Then $U_{\bar{R}}^{1}(\rho_{\bar{S},\gamma}(\gamma))$ is contained in the multiplier algebra of $C_{0}(\bar{A}_{\bar{T}}) \times_{\alpha_{\bar{T}}} \bar{G}_{\bar{S},\gamma}$. This follows by showing that, the map
\[
C_{0}(\bar{A}_{\bar{T}}) \times_{\alpha_{\bar{T}}} \bar{G}_{\bar{S},\gamma} \ni \pi_{L}^{\gamma}S(F_{\gamma}) \mapsto U_{\bar{R}}^{1}(\rho_{\bar{S},\gamma}(\gamma))\pi_{L}^{\gamma}S(F_{\gamma}) \in C_{0}(\bar{A}_{\bar{T}}) \times_{\alpha_{\bar{T}}} \bar{G}_{\bar{S},\gamma}
\]
defines a multiplier map. Furthermore it can be shown that,
\[
C_{0}(\bar{A}_{\bar{T}}) \times_{\alpha_{\bar{T}}} \bar{G}_{\bar{S},\gamma} \ni \pi_{L}^{\gamma}S(F_{\gamma}) \mapsto \pi_{L}^{\gamma}S(F_{\gamma} \ast \bar{F}_{\bar{T}}) \in C_{0}(\bar{A}_{\bar{T}}) \times_{\alpha_{\bar{T}}} \bar{G}_{\bar{S},\gamma}
\]
defines a multiplier map for each function $\bar{F}_{\bar{T}} \in C_{0}(\bar{A}_{\bar{T}}) \times_{\alpha_{\bar{T}}} \bar{G}_{\bar{S},\gamma}$.

Theorem 4.12. Let $\bar{S}$ be a set of surfaces with simple surface intersection property for a finite orientation preserved graph system associated to a graph $\Gamma$. Let $\{\bar{S}_{i}\}$ be a set of surfaces such that each surface set $\bar{S}_{i}$ is suitable for a finite (orientation preserved) graph system associated to a graph $\Gamma$.

Then the following statements are true:

(i) The algebra $C_{0}(\bar{A}_{\bar{T}})$, the group $\bar{G}_{\bar{S}_{i},\bar{T}}$ and the group $\bar{G}_{\bar{S},\bar{G}}$ are not contained in $C_{0}(\bar{A}_{\bar{T}}) \times_{\alpha_{\bar{T}}} \bar{G}_{\bar{S},\bar{G}}$.

(ii) The analytic holonomy algebra $C_{0}(\bar{A}_{\bar{T}})$ and the unitaries $U_{\bar{R}}^{M}(\rho_{\bar{S},\bar{G}}(\Gamma))$, whenever $\rho_{\bar{S},\bar{G}}(\Gamma) \in \bar{G}_{\bar{S},\bar{G}}$ where $1 \leq M \leq N$, are elements of the multiplier algebra of the $C^{*}$-algebra $C_{0}(\bar{A}_{\bar{T}}) \times_{\alpha_{\bar{T}}} \bar{G}_{\bar{S},\bar{G}}$.

(iii) The unitaries $U_{\bar{R}}^{M}(\rho_{\bar{S},\bar{G}}(\Gamma))$, $U_{\bar{R}}^{M}(\rho_{\bar{S},\bar{G}}(\Gamma))$, $U_{\bar{R}}^{M}(\rho_{\bar{S},\bar{G}}(\Gamma))$, $U_{\bar{R}}^{M}(\rho_{\bar{S},\bar{G}}(\Gamma))$ and so on, whenever $\rho_{\bar{S},\bar{G}}(\Gamma) \in \bar{G}_{\bar{S},\bar{G}}$ where $1 \leq M \leq N$ and all $i$, are elements of the multiplier algebra of the $C^{*}$-algebra $C_{0}(\bar{A}_{\bar{T}}) \times_{\alpha_{\bar{T}}} \bar{G}_{\bar{S},\bar{G}}$.

(iv) The elements of the holonomy-flux cross-product algebra $C_{0}(\bar{A}_{\bar{T}}) \times_{\alpha_{\bar{T}}} \bar{G}_{\bar{S},\bar{G}}$ are multipliers of the $C^{*}$-algebra $C_{0}(\bar{A}_{\bar{T}}) \times_{\alpha_{\bar{T}}} \bar{G}_{\bar{S},\bar{G}}$.

(v) Moreover, all elements of $C_{0}(\bar{A}_{\bar{T}}) \times_{\alpha_{\bar{T}}} \bar{G}_{\bar{S},\bar{G}}$, $C_{0}(\bar{A}_{\bar{T}}) \times_{\alpha_{\bar{T}}} \bar{G}_{\bar{S},\bar{G}}$, $C_{0}(\bar{A}_{\bar{T}}) \times_{\alpha_{\bar{T}}} \bar{G}_{\bar{S},\bar{G}}$, $C_{0}(\bar{A}_{\bar{T}}) \times_{\alpha_{\bar{T}}} \bar{G}_{\bar{S},\bar{G}}$, $C_{0}(\bar{A}_{\bar{T}}) \times_{\alpha_{\bar{T}}} \bar{G}_{\bar{S},\bar{G}}$, $C_{0}(\bar{A}_{\bar{T}}) \times_{\alpha_{\bar{T}}} \bar{G}_{\bar{S},\bar{G}}$ for $1 \leq M \leq N$ are contained in the multiplier algebra the $C^{*}$-algebra $C_{0}(\bar{A}_{\bar{T}}) \times_{\alpha_{\bar{T}}} \bar{G}_{\bar{S},\bar{G}}$.

Proof: The proof is similar to proposition [10] and remarks [4.10] and [4.11].

In [10, 7, Section 8.2] the Lie algebra-valued quantum flux operators $E_{\bar{S}}(\Gamma)$ for different surfaces $S$ are considered. Similarly, they are not contained in $C_{0}(\bar{A}_{\bar{T}}) \times_{\alpha_{\bar{T}}} \bar{G}_{\bar{S},\bar{G}}$ or $C_{0}(\bar{A}_{\bar{T}}) \times_{\alpha_{\bar{T}}} \bar{G}_{\bar{S},\bar{G}}$, but they are affiliated in the sense of Woronowicz [27].

Remark 4.13. If the action of the flux group $\bar{G}_{\bar{S},\bar{G}}$ on $C_{0}(\bar{A}_{\bar{T}})$ is assumed to be the identity, then $C_{0}(\bar{A}_{\bar{T}}) \times_{\alpha_{\bar{T}}} \bar{G}_{\bar{S},\bar{G}}$ is equivalent to $C_{0}(\bar{A}_{\bar{T}}) \otimes_{\text{max}} C^{*}(\bar{G}_{\bar{S},\bar{G}})$ where $\otimes_{\text{max}}$ denotes the maximal $C^{*}$-tensor product.
Definition 4.14. Let $\Gamma_\infty$ be the inductive limit of a family of graphs $\{\Gamma_i\}$ such that each graph $\Gamma_i$ of the family has the same intersection surface property for the set $\tilde{S}$ (or the set $\hat{S}$) of surfaces. Set $|\Gamma_i| = N_i$. Then $P_{\Gamma_\infty}$ is the inductive limit of an inductive family $\{P_{\Gamma_i}\}$ of finite orientation preserved graph systems.

The holonomy-flux cross-product $C^*$-algebra $\mathfrak{A} \rtimes_{\alpha_E} \mathcal{G}_S$ (of a special surface configuration $\bar{S}$) is an inductive limit $C^*$-algebra $\varprojlim_{\mathcal{P}_{\Gamma_i} \in \mathcal{P}} C(\mathcal{A}_{\Gamma_i}) \rtimes_{\alpha_E} \mathcal{G}_{S,\Gamma_i}$ of the inductive family of $C^*$-algebras given by

$$\{(C(\mathcal{A}_{\Gamma_i}) \rtimes_{\alpha_E} \mathcal{G}_{S,\Gamma_i}, \beta_{\Gamma_i,\Gamma_j}) : \beta_{\Gamma_i,\Gamma_j} : \ast \text{- homomorphisms s.t. } \beta_{\Gamma_i,\Gamma_j} = \beta_{\Gamma_i,\Gamma_k} \circ \beta_{\Gamma_k,\Gamma_j} \text{ for } \Gamma_i \leq \Gamma_k \leq \Gamma_j\}$$

completed in the norm (where elements of norm 0 are divided out)

$$\|F\| := \inf_{\mathcal{P}_{\Gamma_i} \in \mathcal{P}} \|\beta_{\Gamma_i,\Gamma_j}(F_{\Gamma_i})\|_{\Gamma_j} \text{ for } F_{\Gamma_i} \in \mathfrak{A}_{\Gamma_i} \rtimes_{\alpha_E} \mathcal{G}_{S,\Gamma_i}$$

with $\|F_{\Gamma_i}\|_{\Gamma_i} := \sup_{\mathcal{P}_{\Gamma_i}} \|\pi_E(F_{\Gamma_i})\|_2$ where the supremum is taken over all non-degenerate $L^1$-norm decreasing $*$-representations of $L^1(\bar{G}_{S,\Gamma_i}, C(\mathcal{A}_{\Gamma_i}))$.

Proposition 4.15. Let $\Gamma_\infty$ be the inductive limit of a family of graphs $\{\Gamma_i\}$ such that each graph $\Gamma_i$ of the family has the same intersection surface property for the set $\tilde{S}$ (or the set $\hat{S}$) of surfaces and such that there is only a finite number of intersections of $\tilde{S}$ and all graphs in $\Gamma_\infty$. Set $|\Gamma_i| = N_i$. Then $P_{\Gamma_\infty}$ is the inductive limit of an inductive family $\{P_{\Gamma_i}\}$ of finite orientation preserved graph systems. Denote the center of the inductive limit group $\bar{G}_S$ by $Z_S$.

The state $\omega_{E(\bar{S})}$ on $\mathfrak{A} \rtimes_{\alpha_E} \mathcal{Z}_S$ associated to the GNS-representation $(\mathcal{H}_\Gamma, \pi_{E(S)}^{\Gamma}, \Omega^{\Gamma}_{E(S)})$ is not surface-orientation preserving graph-diffeomorphism invariant, but it is a surface preserving graph-diffeomorphism invariant state.

Proof: This is deduced similarly to proposition 4.6

Theorem 4.16. The multiplier algebra $M(\mathfrak{A} \rtimes_{\alpha_E} \mathcal{G}_S)$ of the holonomy-flux cross-product $C^*$-algebra $\mathfrak{A} \rtimes_{\alpha_E} \mathcal{G}_S$ contains all elements of the holonomy-flux cross-product $C^*$-algebra of any suitable surface set $\tilde{S}$ in $\mathcal{S}$.

Proof: This is derived by using theorem 4.12

5 The holonomy-flux-graph-diffeomorphism cross-product $C^*$-algebra

In this section the holonomy-flux cross-product $C^*$-algebra is enlarged further such that the new $C^*$-algebra contains in a suitable sense the finite graph-diffeomorphisms. Hence, this algebra contains some constraints of the theory of quantum gravity. This is one further step to the aim of the project $AQV$. Notice that, the construction in this section is restricted to surface-preserving graph-diffeomorphisms, but the development can be generalised to surface-orientation-preserving graph-diffeomorphisms. The latter are necessary for the interplay with the quantum flux operators.

Recall the $C^*$-dynamical system $(\mathfrak{B}(\mathfrak{P}_\Gamma), C_0(\mathcal{A}_\Gamma), \zeta)$ defined in [9, Section 3.2], [7, Proposition 6.2]. Similarly to the construction of the Banach $\ast$-algebra $L^1(\bar{G}_{S,\Gamma}, C_0(\mathcal{A}_\Gamma), \zeta)$ in subsection 4.1 the Banach $\ast$-algebra $L^1(\mathfrak{B}_{S,surf}(\mathfrak{P}_\Gamma), C_0(\mathcal{A}_\Gamma), \zeta)$ is developed in the next paragraphs.

Due to the fact that, the number of subgraphs of $\Gamma$ generated by the edges of $\Gamma$ is finite, there exists a finite set $\mathfrak{B}_{S,\Gamma}(\mathfrak{P}_\Gamma)$ of bisections, such that each of bisection is a map from the set $V_\Gamma$ to a distinct subgraph of $\Gamma$ such that all elements of $\mathfrak{P}_\Gamma$ are construed from the finite set $\mathfrak{B}_{S,\Gamma}(\mathfrak{P}_\Gamma)$. Call such a set of bisections a generating system of bisections for a graph $\Gamma$. 

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The function $F_{\Gamma, 3}$ is contained in $l^1(\mathfrak{B}_{S, \text{surf}}^F(\mathcal{P}_T), C_0(\mathcal{A}_T), \zeta)$ if $F_{\Gamma, 3}$ satisfies

$$\|F_{\Gamma, 3}\|_1 := \sum_{l=1, \ldots, k_T} \|F_{\Gamma, 3}(h_{\Gamma}(\Gamma'_\sigma))\|_2 < \infty$$

Then the product of two elements $F_{\Gamma, 3}, K_{\Gamma, 3} \in l^1(\mathfrak{B}_{S, \text{surf}}^F(\mathcal{P}_T), C_0(\mathcal{A}_T), \zeta)$ is defined by

$$(F_{\Gamma, 3} * K_{\Gamma, 3})(h_{\Gamma}(\Gamma'_\sigma)) = \sum_{\sigma, \delta, \beta \in \mathfrak{B}_{S, \text{surf}}^F(\mathcal{P}_T)} F_{\Gamma, 3}(h_{\Gamma}(\Gamma'_\beta)) K_{\Gamma, 3}(h_{\Gamma}(\Gamma'_\delta))$$

and the involution is

$$F_{\Gamma, 3}(h_{\Gamma}(\Gamma'_\sigma)) := \overline{F_{\Gamma, 3}(h_{\Gamma}(\Gamma'_{\sigma}^{-1}))}$$

There is a *-representation $\pi_{\Gamma, 3}^+$ of $l^1(\mathfrak{B}_{S, \text{surf}}^F(\mathcal{P}_T), C_0(\mathcal{A}_T), \zeta)$ on $l^2(\mathfrak{B}_{S, \text{surf}}^F(\mathcal{P}_T), C_0(\mathcal{A}_T), \zeta)$ given by

$$\pi_{\Gamma, 3}^+(F_{\Gamma, 3}) = \sum_{\sigma \in \mathfrak{B}_{S, \text{surf}}^F(\mathcal{P}_T)} F_{\Gamma, 3}(h_{\Gamma}(\Gamma'_\sigma)) U(h_{\Gamma}(\Gamma'_\sigma))$$

where $U(h_{\Gamma}(\Gamma'_\sigma)) = \delta_\sigma$ and $\delta_\sigma(h_{\Gamma}(\Gamma'_\sigma)) := \delta(h_{\Gamma}(\Gamma'_\sigma))$.

**Lemma 5.1.** Let $\mathfrak{B}_{S, \text{surf}}^F(\mathcal{P}_T) := \{\sigma \in \mathfrak{B}(\mathcal{P}_T)\}^{1 \leq i \leq k}$ be a subset of $\mathfrak{B}(\mathcal{P}_T)$ that forms a generating system of bisections for the graph $\Gamma$.

The integrated *-representation $\pi_{\Gamma, 3}^+$ of $l^1(\mathfrak{B}_{S, \text{surf}}^F(\mathcal{P}_T), C_0(\mathcal{A}_T), \zeta)$ is non-degenerate.

**Proof:** This follows from the fact that, $\pi_{\Gamma, 3}^+(F_{\Gamma, 3}(h_{\Gamma}(\Gamma'))\delta_{ad}(h_{\Gamma}(\Gamma')) = F_{\Gamma, 3}(h_{\Gamma}(\Gamma'))$.

Since $\mathfrak{B}_{S, \text{surf}}^F(\mathcal{P}_T)$ is finite-dimensional and discrete, the reduced holonomy-graph-diffeomorphism group $C^\ast$-algebra coincides with the holonomy-graph-diffeomorphism cross-product $C^\ast$-algebra $C_0(\mathcal{A}_T) \rtimes \mathfrak{B}_{S, \text{surf}}^F(\mathcal{P}_T)$.

But this algebra does not contain any flux variables. Hence recall that in proposition [7, Proposition 6.2.15], it was shown that, the triple $(\mathfrak{B}(\mathcal{P}_T), W(G_{S, \Gamma}), \zeta)$ of a surface preserving group $\mathfrak{B}(\mathcal{P}_T)$ of bisections, a $C^\ast$-algebra $W(G_{S, \Gamma})$ associated to a suitable set $S$ of surfaces and a graph $\Gamma$ is a $C^\ast$-dynamical system in $\mathcal{L}(\mathcal{H}_T)$.

The pair $(\Phi, V)$, which consists of a morphism $\Phi \in \text{Mor}(W(G_{S, \Gamma}), \mathcal{L}(\mathcal{H}_T))$ and a unitary representation $V$ of $\mathfrak{B}(\mathcal{P}_T)$ on $\mathcal{L}(\mathcal{H}_T)$, i.e. $V \in \text{Rep}(\mathfrak{B}(\mathcal{P}_T), \mathcal{L}(\mathcal{H}_T))$ such that

$$\Phi(\zeta_\sigma(W)) = V(\sigma)\Phi(W)V^\ast(\sigma)$$

is a covariant representation of $(\mathfrak{B}(\mathcal{P}_T), W(G_{S, \Gamma}), \zeta)$ in $\mathcal{L}(\mathcal{H}_T)$.

**Lemma 5.2.** Let $S$ be a set of surfaces with same surface intersection property for $\Gamma$. Furthermore, let $\mathfrak{B}_{S, \text{surf}}^F(\mathcal{P}_T) := \{\sigma \in \mathfrak{B}(\mathcal{P}_T)\}^{1 \leq i \leq k}$ be a subset of $\mathfrak{B}(\mathcal{P}_T)$ that forms a generating system of bisections for the graph $\Gamma$.

Then the triple $(\mathfrak{B}_{S, \text{surf}}^F(\mathcal{P}_T), C_0(\mathcal{A}_T) \rtimes \mathfrak{B}_{S, \text{surf}}^F(\mathcal{P}_T), \zeta)$ is a $C^\ast$-dynamical system in $\mathcal{L}(\mathcal{H}_T)$.

**Proof:** Set $\Gamma = \{\gamma_1, \ldots, \gamma_N\}, \Gamma_\sigma = \{\gamma_1 \circ \sigma(v_1), \ldots, \gamma_N \circ \sigma(v_N)\}$.

Let $F_{\Gamma} : C_c(\mathbb{Z}_{S, \Gamma}) \to C_0(\mathcal{A}_T)$ and denote the image of $F_{\Gamma}(\rho_{S_1}(\gamma_1), \ldots, \rho_{S_N}(\gamma_1))$ by $F_{\Gamma}(\rho_{S_1}(\gamma_1), \ldots, \rho_{S_N}(\gamma_1); h_{\Gamma}(\gamma_1), \ldots, h_{\Gamma}(\gamma_N))$. Notice that,

$$(\zeta_{F_{\Gamma}}(\rho_{S_1}(\gamma_1), \ldots, \rho_{S_N}(\gamma_1)) = F_{\Gamma}(\rho_{S_1}(\gamma_1), \ldots, \rho_{S_N}(\gamma_1); h_{\Gamma_\sigma(\gamma_1 \circ \sigma(v_1)), \ldots, h_{\Gamma_\sigma(\gamma_N \circ \sigma(v_N))})$$

holds. Clearly this defines a point-norm continuous automorphic action.
Proposition 5.3. Let $\hat{S}$ be a set of surfaces with same surface intersection property for $\Gamma$. Furthermore, let $\mathfrak{B}^\Gamma_{\text{surf}}(\mathcal{P}_T) := \{\sigma_i \in \mathfrak{B}(\mathcal{P}_T)\}_{1 \leq i \leq k}$ be a subset of $\mathfrak{B}(\mathcal{P}_T)$ that forms a generating system of bisections for the graph $\Gamma$.

The pair $(\pi^\Gamma_{E(\hat{S})}^E, V)$ is a covariant pair of the $C^*$-dynamical system $(\mathfrak{B}^\Gamma_{\text{surf}}(\mathcal{P}_T \Sigma), C_0(\hat{A}_T) \rtimes \mathbb{Z}_{\Gamma, \Gamma}, \zeta)$ in $\mathcal{L}(\mathcal{H})$.

Proof: Take the $\pi^\Gamma_{E(\hat{S})}^E$-representation of $C_0(\hat{A}_T) \rtimes \mathbb{Z}_{\Gamma, \Gamma}$ on $\mathcal{H}_T$ and $V$ a regular representation of $\mathfrak{B}^\Gamma_{\text{surf}}(\mathcal{P}_T \Sigma)$ on $\mathcal{H}_T$ to observe that,

$$\pi^\Gamma_{E(\hat{S})}^E(\zeta^E(F_T))\Omega^E_{E(\hat{S})} = \int_{Z_{\hat{S}, T}} d\mu_{\hat{S}, T}(\rho_{S_1}(\gamma_1), ..., \rho_{S_N}(\gamma_N))\zeta^E(F_T)(\rho_{S_1}(\gamma_1), ..., \rho_{S_N}(\gamma_N))\Omega^E_{E(\hat{S})} = \int_{Z_{\hat{S}, T}} d\mu_{\hat{S}, T}(\rho_{S_1}(\gamma_1 \circ \sigma(v_1)), ..., \rho_{S_N}(\gamma_N \circ \sigma(v_N)))$$

$$\Gamma_T(\rho_{S_1}(\gamma_1 \circ \sigma(v_1)), ..., \rho_{S_N}(\gamma_N \circ \sigma(v_N)))\Omega^E_{E(\hat{S})} = \int_{Z_{\hat{S}, T}} d\mu_{\hat{S}, T}(\rho_{S_1}(\gamma_1 \circ \sigma(v_1)), ..., \rho_{S_N}(\gamma_N \circ \sigma(v_N))))\Gamma_T(\rho_{S_1}(\gamma_1 \circ \sigma(v_1)), ..., \rho_{S_N}(\gamma_N \circ \sigma(v_N))V_{\sigma}^\Gamma_{E(\hat{S})} = V_{\sigma}^\Gamma_{E(\hat{S})}(F_T)V_{\sigma}^\Gamma_{E(\hat{S})}\Omega^E_{E(\hat{S})}$$

yields if $v_i = t(\gamma_i)$ for $i = 1, ..., N$. Consequently, $(\pi^\Gamma_{E(\hat{S})}^E, V)$ is a covariant representation.

In proposition 4.6 it has been shown that, the state $\omega_{E(\hat{S})}$ of $C_0(\hat{A}_T) \rtimes \mathbb{Z}_{\Gamma, \Gamma}$ is graph-diffeomorphism invariant in general. There is a finite surface-orientation-preserving graph-diffeomorphism and hence a $\mathfrak{B}^\Gamma_{\text{surf}}(\mathcal{P}_T)$-invariant state of $C_0(\hat{A}_T) \rtimes \mathbb{Z}_{\Gamma, \Gamma}$ on $\mathcal{H}_T$ given by

$$\omega^E_{E(\hat{S})}(\zeta(F_{T, \hat{S}})) = \Omega^E_{E(\hat{S})}V_{\sigma}^\Gamma_{E(\hat{S})}(F_{T, \hat{S}})V_{\sigma}^\Gamma_{E(\hat{S})}\Omega^E_{E(\hat{S})} = \omega^E_{E(\hat{S})}(F_{T, \hat{S}})$$

whenever $\sigma \in \mathfrak{B}^\Gamma_{\text{surf}}(\mathcal{P}_T)$ and $F_{T, \hat{S}} \in C_0(\hat{A}_T) \rtimes \mathbb{Z}_{\Gamma, \Gamma}$.

Proposition 5.4. Let $\hat{S}$ be a set of surfaces with same surface intersection property for $\Gamma$. Furthermore, let $\mathfrak{B}^\Gamma_{\text{surf}}(\mathcal{P}_T) := \{\sigma_i \in \mathfrak{B}(\mathcal{P}_T)\}_{1 \leq i \leq k}$ be a subset of $\mathfrak{B}(\mathcal{P}_T)$ that forms a generating system of bisections for the graph $\Gamma$.

The space $l^1(\mathfrak{B}^\Gamma_{\text{surf}}(\mathcal{P}_T), C_0(\hat{A}_T) \rtimes \mathbb{Z}_{\Gamma, \Gamma}, \zeta)$ is defined by all functions $F_{T, \hat{S}} : \mathfrak{B}^\Gamma_{\text{surf}}(\mathcal{P}_T) \to C_0(\hat{A}_T) \rtimes \mathbb{Z}_{\Gamma, \Gamma}$ for which

$$\|F_{T, \hat{S}}\|_1 = \sum_{\sigma \in \mathfrak{B}^\Gamma_{\text{surf}}(\mathcal{P}_T)} \|F_{T, \hat{S}}(\sigma(t(\gamma_1)), ..., \sigma(t(\gamma_N)))\|_2 < \infty$$

is true.

The convolution *-algebra $l^1(\mathfrak{B}^\Gamma_{\text{surf}}(\mathcal{P}_T), C_0(\hat{A}_T) \rtimes \mathbb{Z}_{\Gamma, \Gamma}, \zeta)$ is presented by the multiplication

$$(G_{T, \hat{S}} * F_{T, \hat{S}})(\sigma(t(\gamma_1)), ..., \sigma(t(\gamma_N))) = \sum_{\sigma, \sigma' \in \mathfrak{B}^\Gamma_{\text{surf}}(\mathcal{P}_T)} G_{T, \hat{S}}(\sigma(t(\gamma_1)), ..., \sigma(t(\gamma_N)))\zeta_{\sigma}(F_{T, \hat{S}}((\sigma^{-1} * \sigma')(t(\gamma_1)), ..., (\sigma^{-1} * \sigma')(t(\gamma_N))))$$
and the involution

$$F_{r,s}(\sigma(t(\gamma_1)), \ldots, \sigma(t(\gamma_\gamma))) = \zeta_\sigma(F_{r,s}(\sigma^{-1}(t(\gamma_1)), \ldots, \sigma^{-1}(t(\gamma_N)))^*)$$

where the involution $^*$ of $l^1(B^F_{S,\text{surf}}(\mathcal{P}_T), C_0(\mathcal{A}_T) \rtimes_\alpha \bar{Z}_{S,\Gamma}, \zeta)$ is inherited from the involution $^*$ of $C_0(\mathcal{A}_T) \rtimes_\alpha \bar{Z}_{S,\Gamma}$

$$F_{r,s}(\sigma(t(\gamma_1)), \ldots, \sigma(t(\gamma_N))) = \alpha(\rho_s(\Gamma))(F_{r,s}^+(\sigma(t(\gamma_1)), \ldots, \sigma(t(\gamma_N)); \rho_s(\gamma_1)^{-1}, \ldots, \rho_s(\gamma_N)^{-1})$$

and

$$F_{r,s}^+(\sigma(t(\gamma_1)), \ldots, \sigma(t(\gamma_N)); \rho_s(\gamma_1)^{-1}, \ldots, \rho_s(\gamma_N)^{-1}) = F_{r,s}(\sigma(t(\gamma_1)), \ldots, \sigma(t(\gamma_N)); \rho_s(\gamma_1)^{-1}, \ldots, \rho_s(\gamma_N)^{-1})$$

where the map

$$(\sigma(t(\gamma_1)), \ldots, \sigma(t(\gamma_N))) \mapsto F_{r,s}(\sigma(t(\gamma_1)), \ldots, \sigma(t(\gamma_N)))$$

defines an element in $l^1(B^F_{S,\text{surf}}(\mathcal{P}_T), C_0(\mathcal{A}_T) \rtimes_\alpha \bar{Z}_{S,\Gamma}, \zeta)$, the map

$$(\sigma(t(\gamma_1)), \ldots, \sigma(t(\gamma_N)); \rho_s(\gamma_1)^{-1}, \ldots, \rho_s(\gamma_N)^{-1}) \mapsto F_{r,s}(\sigma(t(\gamma_1)), \ldots, \sigma(t(\gamma_N)); \rho_s(\gamma_1)^{-1}, \ldots, \rho_s(\gamma_N)^{-1})$$

defines an element in $C_0(\mathcal{A}_T) \rtimes_\alpha \bar{Z}_{S,\Gamma}$ and, finally, the map

$$(\sigma(t(\gamma_1)), \ldots, \sigma(t(\gamma_N)); \rho_s(\gamma_1)^{-1}, \ldots, \rho_s(\gamma_N)^{-1}; \hfr(\gamma_1), \ldots, \hfr(\gamma_N))$$

$$\mapsto F_{r,s}(\sigma(t(\gamma_1)), \ldots, \sigma(t(\gamma_N)); \rho_s(\gamma_1)^{-1}, \ldots, \rho_s(\gamma_N)^{-1}; \hfr(\gamma_1), \ldots, \hfr(\gamma_N))$$

defines an element in $C_0(\mathcal{A}_T)$.

The space $l^1(B^F_{S,\text{surf}}(\mathcal{P}_T), C_0(\mathcal{A}_T) \rtimes_\alpha \bar{Z}_{S,\Gamma}, \zeta)$ is a well-defined Banach $^*$-algebra.

**Definition 5.5.** Let $\mathcal{S}$ be a set of surfaces with same surface intersection property for $\Gamma$. Furthermore, let $B^F_{S,\text{surf}}(\mathcal{P}_T) := \{\sigma_1 \in \mathcal{B}(\mathcal{P}_T)\}_{1 \leq i \leq k}$ be a subset of $\mathcal{B}(\mathcal{P}_T)$ that forms a generating system of bisections for the graph $\Gamma$.

Let $(\pi^{I,G}_{E(\mathcal{S})}, V)$ be a covariant representation of $(B^F_{S,\text{surf}}(\mathcal{P}_T), C_0(\mathcal{A}_T) \rtimes_\alpha \bar{Z}_{S,\Gamma}, \zeta)$ in $L(H_\Gamma)$.

Define the **integrated holonomy-flux-graph-diffeomorphism representation of**

$$l^1(B^F_{S,\text{surf}}(\mathcal{P}_T), C_0(\mathcal{A}_T) \rtimes_\alpha \bar{Z}_{S,\Gamma}, \zeta)$$

by

$$\pi_{I,B}(F_{r,s}(\sigma(t(\gamma_1)), \ldots, \sigma(t(\gamma_N)))) = \sum_{\sigma \in B^F_{S,\text{surf}}(\mathcal{P}_T)} \pi^{I,G}_{E(\mathcal{S})}(F_{r,s}(\sigma(t(\gamma_1)), \ldots, \sigma(t(\gamma_N)))) V_\sigma$$

$$= \sum_{\delta_i \in \mathcal{P}_T, \delta_i(\gamma_i)} \pi^{I,G}_{E(\mathcal{S})}(F_{r,s}(\delta_1, \ldots, \delta_N)) V(\delta_1, \ldots, \delta_N)$$

such that the sum is over all paths $\delta_i$, which start at $t(\gamma_i)$ and $\delta_i \in \mathcal{P}_T$.

**Definition 5.6.** Let $\mathcal{S}$ be a set of surfaces with same surface intersection property for $\Gamma$. Furthermore, let $B^F_{S,\text{surf}}(\mathcal{P}_T) := \{\sigma_1 \in \mathcal{B}(\mathcal{P}_T)\}_{1 \leq i \leq k}$ be a subset of $\mathcal{B}(\mathcal{P}_T)$ that forms a generating system of bisections for the graph $\Gamma$.

The **reduced holonomy-flux-graph-diffeomorphism group** $C^*$-algebra $C^*(B^F_{S,\text{surf}}(\mathcal{P}_T), C_0(\mathcal{A}_T) \rtimes_\alpha \bar{Z}_{S,\Gamma})$ of a graph $\Gamma$ and a set of surfaces $\mathcal{S}$ is defined as the closure of $l^1(B^F_{S,\text{surf}}(\mathcal{P}_T), C_0(\mathcal{A}_T) \rtimes_\alpha \bar{Z}_{S,\Gamma}, \zeta)$ in the norm $\|F_{r,s}\|_2 := \|\pi_{I,B}(F_{r,s})\|_2$.

**Proposition 5.7.** Let $\mathcal{S}$ be a set of surfaces with same surface intersection property for $\Gamma$. Furthermore, let $B^F_{S,\text{surf}}(\mathcal{P}_T) := \{\sigma_1 \in \mathcal{B}(\mathcal{P}_T)\}_{1 \leq i \leq k}$ be a subset of $\mathcal{B}(\mathcal{P}_T)$ that forms a generating system of bisections for the graph $\Gamma$. 

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Suppose that \((\mathfrak{W}_\Sigma(P_T), C_0(\bar{A}_T) \rtimes_\alpha \bar{Z}_{S,T}, \zeta)\) in \(\mathcal{L}(\mathcal{H})\) is a C*-dynamical system and that for each 
\(F_T \in l^1(\mathfrak{W}_\Sigma(P_T), C_0(\bar{A}_T) \rtimes_\alpha \bar{Z}_{S,T}, \zeta)\) define

\[
\|F_T, S\| := \sup \left\{ \|(\pi \otimes V)(F_T, S)\| : (\pi, V) \text{ is a covariant representation of } (\mathfrak{W}_\Sigma(P_T), C_0(\bar{A}_T) \rtimes_\alpha \bar{Z}_{S,T}, \zeta) \right\}
\]

Then \(\|\cdot\|\) is a norm on \(l^1(\mathfrak{W}_\Sigma(P_T), C_0(\bar{A}_T) \rtimes_\alpha \bar{Z}_{S,T}, \zeta)\) called the universal norm. The universal norm is dominated by the \(\|\cdot\|_1\)-norm, and the completion of \(l^1(\mathfrak{W}_\Sigma(P_T), C_0(\bar{A}_T) \rtimes_\alpha \bar{Z}_{S,T}, \zeta)\) with respect to \(\|\cdot\|\) is a C*-algebra. This C*-algebra is called the holonomy-flux-graph-diffeomorphism cross-product C*-algebra \((C_0(\bar{A}_T) \rtimes_\alpha \bar{Z}_{S,T}) \rtimes_\zeta \mathfrak{W}_\Sigma(P_T)\) associated to a graph \(\Gamma\) and a set \(\mathcal{S}\) of surfaces.

In \cite{9} Proposition 3.32 or \cite{7} Proposition 6.2.2, it has been argued that, there are several C*-dynamical systems available for the analytic holonomy C*-algebra and the group of bisections. This is used to define a bunch of holonomy-flux-graph-diffeomorphism cross-product C*-algebras, which are constructed from C*-dynamical systems. These cross-product C*-algebras are exterior equivalent, too. Clearly there is a multiplier algebra of the holonomy-flux-graph-diffeomorphism cross-product algebra associated to a graph and a set of surfaces is derivable. The author of this article suggests that it can be proven that, the different holonomy-flux-graph-diffeomorphism cross-product C*-algebras are contained in this multiplier algebra by using similar arguments used in the proof of theorem \cite{11,12}. The construction of the inductive limit C*-algebra of a family of C*-algebras defined above is not mathematically understood very well until now. A detailed study of these objects is a further project.

## 6 Comparison table

The Weyl C*-algebra for surfaces and the holonomy-flux cross-product C*-algebra associated to a certain surface set are constructed from functions depending on holonomies along paths of a graph, and the strongly continuous unitary representation of the quantum flux group for surfaces. In contrast to the Weyl algebra, where the group-valued quantum flux operators are implemented as unitary operators, the elements of the holonomy-flux cross-product C*-algebra are operator-valued functions depending on group-valued quantum flux variables for surfaces. In both cases these operators are represented on Hilbert spaces.

| Ingredients                  | Weyl C*-algebra for surfaces                                                                 | holonomy-flux cross-product C*-algebra                              |
|------------------------------|---------------------------------------------------------------------------------------------|---------------------------------------------------------------------|
| Set of fin. set of surfaces \(\mathcal{S}\) |                                                                                              | set of fin. set of surfaces \(\mathcal{S}\)                        |
| \(G\) locally compact group |                                                                                              | \(G\) locally compact group                                        |
| \(\bar{A}_T\) graph systems |                                                                                              | fin. orientation-preserved graph systems \(\bar{A}_T\)             |
| Natural or non-standard identif. |                                                                                              | natural or non-standard identification of \(\bar{A}_T\)            |
| A set of independent paths in \(\mathcal{P}_T\) |                                                                                              | a set of independent paths in \(\mathcal{P}_T\)                  |
| Hilbert space                | \(\mathcal{H}_T := L^2(\bar{A}_T, d\mu_T)\) or \(\mathcal{H}_\infty := L^2(\bar{A}, d\mu_{\infty})\) | \(\mathcal{H}_T := L^2(\bar{A}_T, d\mu_T)\) or \(\mathcal{H}_\infty := L^2(\bar{A}, d\mu_{\infty})\) |
| \(\Phi_M \in \text{Rep}(C_0(\bar{A}_T), \mathcal{L}(\mathcal{H}_T))\) | left regular representation of the flux group \(\Phi_M \in \text{Rep}(C_0(\bar{A}_T), \mathcal{L}(\mathcal{H}_T))\) | Weyl-integrated holonomy-flux repres. of the holonomy-flux cross-product C*-alg. \(\pi^\Gamma_{E(\mathcal{S})} \in \text{Rep}(L^1(\mathcal{G}_{S,T}, C_0(\bar{A}_T)), \mathcal{K}(\mathcal{H}_T))\) |

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| *-algebra | Weyl algebra generated by \( C_0(\mathcal{A}_\Gamma) \) and \( \{ U \in \text{Rep}(\tilde{G}_S, K(Hr)) \} \) |
| completion w.r.t. | \( L^2(\tilde{A}_\Gamma, \mu_\Gamma) \)-norm |
| \( C^* \)-algebra | \( \text{Weyl}(\tilde{S}, \Gamma) \) |
| induct. limit \( C^* \)-alg. state | unique and pure state \( \tilde{\omega}_M \) on \( \text{Weyl}_L(\tilde{S}) \) s.t. \( \tilde{\omega}_M \circ \zeta(\Phi, \varphi) = \tilde{\omega}_M \) \( \forall (\Phi, \varphi) \) certain diffeomorphism |
| | \( L^1(\tilde{G}_S, C_0(\mathcal{A}_\Gamma)) \) universal-norm |
| | \( C_0(\mathcal{A}_\Gamma) \rtimes_\alpha \tilde{G}_S \) multiplier algebra of \( C_0(\mathcal{A}_\Gamma) \rtimes_\alpha \tilde{G}_S \) |
| | \( C(\tilde{A}) \rtimes_\alpha \tilde{G}_S \) \( (G \text{ compact}) \) |
| | state \( \omega_{E(\tilde{S})} \) on \( C(\tilde{A}) \rtimes_\alpha \tilde{Z}_S \) s.t. \( \omega_{E(\tilde{S})} \circ \zeta(\Phi, \varphi) = \omega_{E(\tilde{S})} \) \( \forall (\Phi, \varphi) \) certain diffeomorphism, which preserve the surfaces in \( \tilde{S} \) |

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