Green function for gradient perturbation of unimodal Lévy processes in the real line

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Abstract

We prove that the Green function of a generator of symmetric unimodal Lévy processes with the weak lower scaling order bigger than one and the Green function of its gradient perturbations are comparable for bounded $C^{1,1}$ subsets of the real line if the drift function is from an appropriate Kato class.

1 Introduction

Perturbations of Markovian generators are widely studied from many years. This theory may be considered from various points of view. Such perturbations appear, e.g., in local and non-local partial differential equations \[10, 11, 32, 33\], semigroup theory \[8, 29, 25, 4, 7\], stochastic processes \[24, 30, 30\], potential theory \[9, 12, 16\]. One of the natural question is: how this perturbation affects the solutions of the equations related to the unperturbed operator (e.g., the transition density of the semigroup, the Green function).

In this paper we are interested in the gradient perturbations and the potential theory of the perturbed operator. We briefly recall some results closely related to our research. Cranston and Zhao in \[15\] considered the operator $\Delta + b(x)\nabla$ in $\mathbb{R}^d$ for $d \geq 2$. They proved that the Green function and the harmonic measure of Lipschitz domains are comparable with those of $\Delta$ for the drift $b$ from the appropriate Kato class. In \[23\] and \[24\] Jakubowski studied the $\alpha$-stable Ornstein-Uhlenbeck process. He proved estimates for the first exit

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time from the ball and Harnack inequality for this process. In [9] Bogdan and Jakubowski
proved similar results as Cranson and Zhao for $\Delta^{\alpha/2} + b(x)\nabla$ in $C^{1,1}$ domains in $\mathbb{R}^d$, $d \geq 2$. In the recent paper [18] these results were generalizated to the case of pure-jump symmetric
unimodal Lévy processes possessing certain weak scaling properties. We note that in the
papers [9, 18] the case $d = 1$ was omitted. The aim of this paper is to fill this gap and
prove analogous results in one dimensional case.

We will denote by $\{X_t\}$ a pure-jump symmetric unimodal Lévy process on $\mathbb{R}$. That
is, a process with the symmetric density function $p_t(x)$ on $\mathbb{R} \setminus \{0\}$ which is non-increasing
on $\mathbb{R}_+$. The characteristic exponent of $\{X_t\}$ equals

$$\psi(x) = \int_{\mathbb{R}} (1 - \cos(xz)) \nu(dz), \quad x \in \mathbb{R}. $$

where $\nu$ is a Lévy measure, i.e., $\int_{\mathbb{R}} (1 \wedge |z|^2) \nu(dz) < \infty$. For general information on
unimodal processes, we refer the reader to [31, 34]. A primary example of the mentioned
class of processes is the symmetric $\alpha$-stable Lévy process having the fractional Laplacian
$\Delta^{\alpha/2}$ as a generator.

Let $$Lf(x) = \int_{\mathbb{R}} \left( f(x + z) - f(x) - 1_{|z| < 1}(z \cdot \nabla f(x)) \right) \nu(dz), \quad f \in C^2_b(\mathbb{R}), \quad (1.1)$$
be a generator of the process $X_t$. We will consider a non-empty bounded open $C^{1,1}$ set $D$
and the Green function $G_D(x, y)$ for $L$. Now, let $\tilde{G}_D(x, y)$ be a Green function for

$$\tilde{L} = L + b(x) \cdot \nabla,$$

where $b$ is a function from the Kato class $\mathcal{K}_1$ (see Section 2 for details). Our main result
is

**Theorem 1.1.** Let $b \in \mathcal{K}_1$, and let $D \subset \mathbb{R}$ be an union of finitely many open intervals with
positive distance between every two intervals. We assume that the characteristic exponent

$$\psi \in WLSC(\alpha, 0, \mathbb{C}) \cap WLSC(\alpha_1, 1, \mathbb{C}) \cap WUSC(\overline{\alpha}, 0, \mathbb{C}), \quad \text{where} \ \alpha_1 > 1,$$

Then, there exists a constant $C$ such that for $x, y \in D$,

$$C^{-1}G_D(x, y) \leq \tilde{G}_D(x, y) \leq CG_D(x, y). \quad (1.2)$$

Here WLSC and WUSC are the classes of functions satisfying a weak lower and weak
upper scaling condition, respectively (see Section 2 for definitions). Set $D$ should be
considered as an one-dimensional case of a bounded $C^{1,1}$ set, see Definition [2].

Generally, we follow the approach of [9] and [18], however there are some important
differences. Although the geometry of the set $D$ is much simpler than in higher dimensions,
it seems that the one dimensional case sometimes demands more delicate arguments. One of the main difficulties are the proper estimates for derivative of the Green function. As it was mentioned in [9], for $d = 1$, the available estimates

$$|\partial_x G_D(x, y)| \leq c G_D(x, y)/(\delta_D(x) \wedge |x - y|) \quad (1.3)$$

are not integrable near $y$. The estimates (1.3) hold, i.e. if $\nu'(r)/r$ is non-increasing (see [18, Lemma 3.2] and [28, Theorem 1.4]). To overcome this difficulty, we improve the estimates (1.3) near the pole in $y$, see Theorem 3.10. This result is new even for the fractional Laplacian. We emphasize here that we make no additional assumption on the monotonicity of $\nu'(r)/r$ as mentioned above. Like in the mentioned papers, our main tool is the perturbation formula. First, we use it to obtain estimates for sets $D$ with a small radius. Since the Green function $G_D(x, y)$ is bounded, we do not use the perturbation series as in [9] and [18]. Instead, we propose a simpler iteration argument.

We note also that one of our standing assumptions is $\alpha_1 > 1$. It may be understood that the rank of the operator $\mathcal{L}$ is larger than 1. Without this assumption the drift term may have the stronger effect than $\mathcal{L}$ on the behavior of the Green function of the $\tilde{\mathcal{L}}$. Any results concerning the cases $\alpha \leq 1$ would be interesting, however for $\alpha < 1$, Theorem 1.1 cannot hold in the form above (see the Introduction of [9] for more details).

The paper is organized as follows. In Section 2 we define the process $X$ and present its basic properties. In Section 3, we introduce the Green function of $X$, prove the estimates for its derivative and some 3G-like inequalities. In Section 4, we define the operator $\tilde{\mathcal{L}}$ and the Green function of the underlying Markov process. Lastly, in Section 5, we prove Theorem 1.1.

When we write $f(x) \approx C g(x)$, we mean that there is a number $0 < C < \infty$ independent of $x$, i.e. a constant, such that for every $x$, $C^{-1} f(x) \leq g(x) \leq C f(x)$. If the value of $C$ is not important we simply write $f(x) \approx g(x)$. The notation $C = C(a, b, \ldots, c)$ means that $C$ is a constant which depends only on $a, b, \ldots, c$.

We use a convention that numbered constants denoted by capital letters do not change throughout the paper. For a symmetric function $f : \mathbb{R} \to [0, \infty)$ we shall often write $f(r) = f(x)$ for any $x \in \mathbb{R}$ with $|x| = r$.

## 2 Preliminaries

In what follows, $\mathbb{R}$ denotes the Euclidean space of real numbers, $dy$ stands for the Lebesgue measure on $\mathbb{R}$. Without further mention we will only consider Borelian sets, measures and functions in $\mathbb{R}$. As usual, we write $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$. We let $B(x, r) = \{y \in \mathbb{R} : |x - y| < r\}$. For the arbitrary set $D \subset \mathbb{R}$, the distance to the complement of $D$, will be denoted by

$$\delta_x = \operatorname{dist}(x, D^c).$$
Definition 1. Let $\theta \in [0, \infty)$ and $\phi$ be a non-negative non-zero function on $(0, \infty)$. We say that $\phi$ satisfies the weak lower scaling condition (at infinity) if there are numbers $\alpha > 0$ and $c \in (0, 1)$ such that

$$\phi(\lambda \theta) \geq c \lambda^\alpha \phi(\theta) \quad \text{for} \quad \lambda \geq 1, \quad \theta > \theta.$$  \hspace{1cm} (2.1)

In short, we say that $\phi$ satisfies $\text{WLSC}(\alpha, \theta, c)$ and write $\phi \in \text{WLSC}(\alpha, \theta, c)$. If $\phi \in \text{WLSC}(\alpha, 0, c)$, then we say that $\phi$ satisfies the global weak lower scaling condition.

Similarly, we consider $\theta \in [0, \infty)$. The weak upper scaling condition holds if there are numbers $\alpha < 2$ and $C \in [1, \infty)$ such that

$$\phi(\lambda \theta) \leq C \lambda^\alpha \phi(\theta) \quad \text{for} \quad \lambda \geq 1, \quad \theta > \theta.$$  \hspace{1cm} (2.2)

In short, $\phi \in \text{WUSC}(\alpha, \theta, C)$. For global weak upper scaling we require $\theta = 0$ in (2.2).

Throughout the paper, $X_t$ will be the pure-jump symmetric unimodal Lévy process on $\mathbb{R}$. The Lévy measure $\nu$ of $X_t$ is symmetric and non-increasing, so it admits the density $\nu$, i.e., $\nu(dx) = \nu(|x|)dx$. Hence the characteristic exponent $\psi$ of $X_t$ is radial as well.

We assume that (see Theorem 1.1)

$$\psi \in \text{WLSC}(\alpha, 0, c) \cap \text{WUSC}(\alpha, 0, C),$$  \hspace{1cm} (2.3)

$$\psi \in \text{WLSC}(\alpha_1, 1, C_1), \quad \text{for some } \alpha_1 > 1.$$  \hspace{1cm} (2.4)

Following [31], we define

$$h(r) = \int_{\mathbb{R}} \left(1 \wedge \frac{|x|^2}{r^2}\right) \nu(|x|)dx, \quad r > 0.$$  

Let us notice that

$$h(\lambda r) \leq h(r) \leq \lambda^2 h(\lambda r), \quad \lambda > 1.$$  

Moreover, by [3] Lemma 1 and (6)]

$$2^{-1} \psi(1/r) \leq h(r) \leq C_1 \psi(1/r).$$

Here, we may choose $C_1 = \pi^2/2$ but it will be more convenient to write this constant as $C_1$. We define the function $V$ as follows,

$$V(0) = 0 \quad \text{and} \quad V(r) = 1/\sqrt{h(r)}, \quad r > 0.$$  

Since $h(r)$ is non-increasing, $V$ is non-decreasing. We have

$$V(r) \leq V(\lambda r) \leq \lambda V(r), \quad r \geq 0, \lambda > 1.$$  \hspace{1cm} (2.5)
By weak scaling properties of $\psi$ and the property $h(r) \approx \psi(1/r)$, we get

$$\left(\frac{c}{2C_1}\right)^{1/2} \lambda^{\alpha/2} \leq \frac{V(\lambda r)}{V(r)} \leq (2CC_1)^{1/2} \lambda^{\alpha/2}, \quad r > 0, \lambda > 1,$$

(2.6)

$$\frac{V(\eta r)}{V(r)} \leq \left(\frac{2C_1}{\varepsilon_1}\right)^{1/2} \eta^{\alpha/2}, \quad \eta < 1, r < 1.$$  

(2.7)

Therefore, $V \in WLSC(\alpha/2, 0, \sqrt{c/(2C_1)}) \cap WUSC(\alpha/2, 0, \sqrt{2CC_1})$.

**Remark 1.** The threshold $(0, 1)$ in scaling of $V$ in (2.7) may be replaced by any bounded interval at the expense of constant $\sqrt{2CC_1}/c$ (see [5, Section 3]), i.e., for any $R > 1$, there is a constant $c$ such that

$$\frac{V(\eta r)}{V(r)} \leq c\eta^{\alpha/2}, \quad \eta < 1, r < R.$$  

(2.8)

We define

$$M(r) = \frac{V^2(r)}{r^2}, \quad r > 0.$$  

We note that $M(\cdot)$ is decreasing and $\lim_{r \to 0} M(r) = \infty$. To simplify the notations how the constants depend on the parameters, we put

$$\sigma = (\overline{\alpha}, C, \alpha, c) \quad \text{and} \quad \overline{\sigma} = (\sigma, \overline{\alpha_1}, c_1).$$

Hence, e.g., writing $c = c(\sigma)$, we mean that $c$ depends on $\overline{\alpha}, C, \alpha, c$.

The global weak lower scaling condition (assumption (2.3)) implies $p_t(x)$ is jointly continuous on $(0, \infty) \times \mathbb{R}$ ($e^{-\psi} \in L^1(\mathbb{R})$) and (see [6, Lemma 1.5])

$$p_t(x) \overset{C}{\approx} \left[\frac{1}{\sqrt{2\pi t}}\right]^{-1} \land \frac{t}{V^2(|x|)}|x|, \quad t > 0, x \in \mathbb{R},$$  

(2.9)

$$\nu(x) \overset{C}{\approx} \frac{1}{V^2(|x|)|x|}, \quad x \neq 0,$$  

(2.10)

where $C = C(\sigma)$.

Let us denote

$$p(t, x, y) = p_t(y - x).$$

By [21, Theorem 1.1 (c)], we have

$$|\partial_x p(t, x, y)| \leq c \frac{1}{V^{-1}(|x|)} p(t, x, y), \quad t > 0, x, y \in \mathbb{R}.$$  

(2.11)
We consider a compensated potential kernel

\[ K(x) = \int_0^\infty (p_s(0) - p_s(x))ds, \quad x \in \mathbb{R}. \]

By symmetry and \cite{14} Theorem II.19, the monotone convergence theorem implies

\[ K(x) = \frac{1}{\pi} \int_0^\infty (1 - \cos(xs)) \frac{1}{\psi(s)} ds = \frac{1}{\pi x} \int_0^\infty (1 - \cos s\frac{1}{\psi(s/x)}) ds, \quad x \neq 0. \]

By \cite{20} Proposition 2.2, \( K \) is subadditive.

**Lemma 2.1.** For every \( R > 0 \) there exists a constant \( C_2 = C_2(\sigma, R \lor 1) \) such that

\[ |\partial_x K(x)| \leq C_2 M(|x| \land R), \quad x \in \mathbb{R}. \]

**Proof.** By symmetry we consider only \( x \geq 0 \). Let \( r = x \land R \). Since \( x \mapsto p_s(x) \) is nonincreasing on \((0, \infty)\), monotonicity of \( V(x) \), \((2.9)\) and \((2.11)\) imply

\[
0 \leq \partial_x K(x) = \partial_x \int_0^x \partial_p p_s(\rho) \, d\rho \, ds = \int_0^\infty |\partial_x p_s(x)| \, ds
\]

\[
\leq c \int_0^\infty \frac{1}{V^{-1}(\sqrt{s})^2} \wedge \frac{s}{V^2(x) x V^{-1}(\sqrt{s})} \, ds
\]

\[
\leq c \int_0^\infty \frac{1}{V^{-1}(\sqrt{s})^2} \wedge \frac{s}{V^2(r) r V^{-1}(\sqrt{s})} \, ds
\]

\[
= c \int_{V^2(r)}^\infty \frac{1}{V^{-1}(\sqrt{s})^2} \, ds + \frac{c}{V^2(r) r} \int_0^{V^2(r)} s \frac{s}{V^{-1}(\sqrt{s})} \, ds.
\]

By \cite{5} Remark 4, \( V^{-1} \in \text{WLSC}(\frac{2}{\pi}, 0, \overrightarrow{C^{-2/\pi}}) \), where \( 1 < \tau < 2 \), hence

\[
\int_{V^2(r)}^\infty \frac{1}{V^{-1}(\sqrt{s})^2} \, ds \leq \frac{c_1 V(r)^{4/\pi}}{r^2} \int_{V^2(r)}^\infty \frac{1}{s^{2/\pi}} \, ds = \frac{c_1}{1 - \frac{2}{\pi}} \frac{V^2(r)}{r^2},
\]

where \( c_1 = \overrightarrow{C^{4/\pi}} \). By explanation of \cite{5} Remark 4 and \((2.7)\), we have

\[
\frac{V^{-1}(\eta \ell)}{V^{-1}(t)} \geq c_\eta^2 \ell^{\frac{2}{\tau}}
\]

\[ (2.12) \]
for $0 < t < 1$, $\eta < 1$ and some constant $c = c(\alpha_1, c_1, C_1, V(1))$. This implies

$$
\int_0^{V^2(r)} \frac{s}{V^{-1}(\sqrt{s})} ds = \int_0^{V^2(r) \wedge 1} \frac{s}{V^{-1}(\sqrt{s})} ds + \int_{V^2(r) \wedge 1}^{V^2(r)} \frac{s}{V^{-1}(\sqrt{s})} ds
$$

$$
\leq \frac{c_2}{r} \int_0^{V^2(r) \wedge 1} \frac{V(r)^2/2}{s^{1/2\alpha - 1}} ds + c_3 \frac{V^4(r)}{2V^{-1}(1)} \leq c_4 (1 + R) \frac{V^4(r)}{r}.
$$

Hence,

$$
\frac{1}{V^2(r)r} \int_0^{V^2(r)} \frac{s}{V^{-1}(\sqrt{s})} ds \leq c_4 (1 + R) \frac{V^2(r)}{r^2}.
$$

By [20, Lemma 2.14 with $\alpha_1$], for $|x| \leq R$,

$$
K(x) \approx \frac{V^2(|x|)}{|x|}.
$$

(2.13)

Hence,

$$
|\partial K(x)| \leq M(|x| \wedge R) \approx \frac{K(|x| \wedge R)}{|x| \wedge R}.
$$

(2.14)

Analogously to $\alpha$-stable processes, we define the Kato class for gradient perturbations.

**Definition 2.** We say that a function $b: \mathbb{R} \to \mathbb{R}$ belongs to the Kato class $\mathcal{K}_1$ if

$$
\lim_{r \to 0^+} \sup_{x \in \mathbb{R}} \int_{B(x, r)} \frac{K(|x - z|)}{|x - z|} |b(z)| dz = 0.
$$

We note that $L^\infty(\mathbb{R}) \subset \mathcal{K}_1$. Since $\frac{K(r)}{r} \approx \frac{V^2(r)}{r^2}$ for small $r > 0$, in this paper we will use the condition (2.15) in the form

$$
\lim_{r \to 0^+} \sup_{x \in \mathbb{R}} \int_{B(x, r)} \frac{V^2(|x - z|)}{|x - z|^2} |b(z)| dz = 0.
$$

(2.15)

We consider the time-homogeneous transition probabilities

$$
P_t(x, A) = \int_A p(t, x, y) dy, \quad t > 0, x \in \mathbb{R}, A \subset \mathbb{R}.
$$
By Kolmogorov’s and Dinkin-Kinney’s theorems the transition probability $P_t$ define in the usual way Markov probability measure \{\mathbb{P}^x, x \in \mathbb{R}\} on the space $\Omega$ of the right-continuous and left-limited functions $\omega : [0, \infty) \rightarrow \mathbb{R}$. We let $\mathbb{E}^x$ be the corresponding expectations. We will denote by $X = \{X_t\}_{t \geq 0}$ the canonical process on $\Omega$, $X_t(\omega) = \omega(t)$. Hence,

$$\mathbb{P}(X_t \in B) = \int_B p(t, x, y)dy.$$ 

For any open set $D$, we define the first exit time of the process $X_t$ from $D$,

$$\tau_D = \inf\{t > 0 : X_t \notin D\}.$$ 

Now, by the usual Hunt’s formula, we define the transition density of the process killed when leaving $D$ (see [2], [14], [6]):

$$p_D(t, x, y) = p(t, x, y) - \mathbb{E}^x[\tau_D < t; p(t - \tau_D, X_{\tau_D}, y)], \quad t > 0, x, y \in \mathbb{R}.$$ 

We briefly recall some well known properties of $p_D$ (see [6]). The function $p_D$ satisfies the Chapman-Kolmogorov equations

$$\int_{\mathbb{R}} p_D(s, x, z)p_D(t, z, y)dz = p_D(s + t, x, y), \quad s, t > 0, x, y \in \mathbb{R}.$$ 

Furthermore, $p_D$ is jointly continuous when $t \neq 0$, and we have

$$0 \leq p_D(t, x, y) = p_D(t, y, x) \leq p(t, x, y). \quad (2.16)$$ 

In particular,

$$\int_{\mathbb{R}} p_D(t, x, y)dy \leq 1. \quad (2.17)$$ 

If $D$ is a $C^{1,1}$ domain (see definition in Section 3), by Blumenthal’s 0-1 law, symmetry of $p_t$, we have $\mathbb{P}^x(\tau_D = 0) = 1$ for every $x \in D^c$. In particular, $p_D(t, x, y) = 0$ if $x \in D^c$ or $y \in D^c$.

### 3 Green function of $L$

We define the Green function of $X_t$ for $D$,

$$G_D(x, y) = \int_0^\infty p_D(t, x, y)dt, \quad x, y \in \mathbb{R} \quad (3.1)$$

and the Green operator

$$G_D \phi(x) = \int_{\mathbb{R}} G_D(x, y)\phi(y)dy, \quad x \in \mathbb{R}. \quad (3.2)$$

From now on, every time we will mention the Green function, it should be understand as a Green function of $D$, and then $G = G_D$. 

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Definition 3. We say that a function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is a \( \mathcal{L} \)-harmonic (or simply harmonic) function on an open bounded set \( D \subset \mathbb{R} \) if for any open \( F \subset D \) and \( x \in F \)
\[
f(x) = E^x f(X_{\tau_F}).
\]
We say that a function \( f \) is a regular \( \mathcal{L} \)-harmonic (or simply regular harmonic) function on an open bounded set \( D \subset \mathbb{R} \) if for every \( x \in D \)
\[
f(x) = E^x f(X_{\tau_D}).
\]

Note that for fixed \( x \in D \) the function \( G(x, \cdot) \) is harmonic on \( D \setminus \{x\} \) and regularly harmonic on \( D \setminus B(x, \varepsilon) \), where \( B(x, \varepsilon) \subset D \). By [35, Theorem 1.1], we know that the function \( K(x) \) is harmonic on \( \{0\}^c \).

Definition 4. We call a set \( D \subset \mathbb{R} \) a \( C^{1,1} \) class set at scale \( r > 0 \) if it is an union of open intervals of length at least \( r \) and distanced one from another at least \( r \). The number \( r_0 = \sup \{r : D \text{ is at scale } r\} \) is called a localization radius.

Definition 4 corresponds with the definition of multidimensional \( C^{1,1} \) set with localization radius \( r_0 \). In what follows, we assume that \( D \) is a \( C^{1,1} \) set with \( \text{diam} D < \infty \) and localization radius \( r_0 = r_0(D) \).

Some constants will depend on the ratio \( \text{diam} D / r_0 \) called the distortion of the set \( D \).

Lemma 3.1. There exists a constant \( C_3 = C_3(\sigma, \text{diam}(D)/r_0, 1 \vee \text{diam}(D)) \) such that
\[
G(x, y) \approx V(\delta_x) V(\delta_y) \left( \frac{1}{\sqrt{\delta_x \delta_y}} \wedge \frac{1}{|x - y|} \right), \quad x, y \in D. \tag{3.3}
\]

Proof. Note that (see [6, Proposition 4.4 and Theorem 4.5]),
\[
p_D(t, x, y) \approx e^{-2\gamma(D)t} \left( \frac{V(\delta_x)}{\sqrt{t/2} \wedge V(r_0)} \wedge 1 \right) \left( \frac{V(\delta_y)}{\sqrt{t/2} \wedge V(r_0)} \wedge 1 \right) p(t \wedge V^2(r_0), x, y),
\]
where \( \frac{1}{8}(\text{diam}(D)/r_0)^2 \leq \gamma(D)V^2(r_0) \leq c(\text{diam}(D)/r_0)^{1/2} \). Now, integrating them against time, we get
\[
G(x, y) \approx (V(\delta_x) \wedge V(r_0)) (V(\delta_y) \wedge V(r_0)) p(V^2(r_0), x, y)
+ \int_0^{V^2(r_0)} \left( \frac{V(\delta_x)}{\sqrt{t}} \wedge 1 \right) \left( \frac{V(\delta_y)}{\sqrt{t}} \wedge 1 \right) p(t, x, y) dt,
\]

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where the comparability constant $c_1$ depends on the scaling characteristics in (2.6) and (2.7) and a distortion of $D$. Now, by the same calculation as in the proof of \[13, \text{Theorem 7.3 (iii) and Corollary 7.4}\], we obtain

$$G(x, y) \approx V(\delta_x) V(\delta_y) \left( \frac{1}{V^{-1}\left(\sqrt{V(\delta_x) V(\delta_y)}\right)} \wedge \frac{1}{|x - y|}\right),$$

where the comparability constant $c_2$ depends on the scaling characteristics in (2.6) and (2.7), a distortion of $D$ and $1 \lor \text{diam}(D)$.

Let us consider $x, y \in D$ such that

$$V^2(|x - y|) < V(\delta_x) V(\delta_y),$$

(3.4)

Without a loss of generality we may and do assume $\delta_x \leq \delta_y$. Then,

$$V^2(|x - y|) < V(\delta_x) V(\delta_x + (\delta_y - \delta_x)) \leq V(\delta_x) V(\delta_x + |x - y|) \leq V(\delta_x) [V(\delta_x) + V(|x - y|)],$$

which implies $V(|x - y|) \leq 2V(\delta_x)$. By monotonicity and subadditivity of $V$ we obtain that

$$V(\delta_x) \leq V(\delta_y) \leq V(|x - y|) + V(\delta_x) \leq 3V(\delta_x).$$

As a consequence of (2.6), we obtain

$$\delta_y \leq (18C_1/\alpha)^{1/(2\alpha)} \delta_x.$$  

(3.5)

Again, by (2.6), we get

$$V^{-1}\left(\sqrt{V(\delta_x) V(\delta_y)}\right) \approx \sqrt{\delta_x \delta_y},$$

(3.6)

where $c_3 = c_3(\alpha, \gamma)$. Now, let

$$V^2(|x - y|) \geq V(\delta_x) V(\delta_y).$$

We only need to show that $2|x - y|^2 \geq \delta_x \delta_y$. Without the loss of generality we can and do assume $\delta_x < \delta_y$. By monotonicity of $V$, $|x - y| \geq \delta_x$. The case $\delta_y \leq |x - y|$ is obvious. For $\delta_x \leq |x - y| < \delta_y$, we have $\delta_y \leq |x - y| + \delta_x \leq 2|x - y|$, which completes the proof. \qed

### 3.1 Estimates of the Poisson kernel

If $D$ is $C^{1,1}$, it is known that the harmonic measure of $D$ has a density and we call it the Poisson kernel. By the Ikeda-Watanabe formula \[22\] it is equal to

$$P_D(x, z) = \int_D G(x, y) \nu(z - y) dy, \quad x \in D, \ z \in \overline{D}^c.$$  

(3.7)
Lemma 3.2. Let \( R > 0 \) and \( B = B(0, R) \). Then

\[
P_B(x, z) \leq \frac{C_1}{V(\delta_x) V(\delta_z) |x-z|} \left( \frac{V(R)}{V(\delta_z)} \wedge 1 \right), \quad x \in B, \ z \in B^c,
\]

where \( C_4 = C_4(\alpha_1, 1 \vee R) > 0 \).

Proof. By (3.7), Lemma 3.1 and (2.10), there is \( c_1 = c_1(\alpha, 1 \vee R) \) such that

\[
P_B(x, z) = \int_B G_B(x, y) \nu(|y-z|) \, dz
\]

\[
\approx \int_B V(\delta_x) V(\delta_y) \left( \frac{1}{\sqrt{\delta_x \delta_y}} \wedge \frac{1}{|x-y|} \right) \frac{dy}{V^2(|z-y|)|z-y|}. 
\]

By Remark 1, we obtain inequality (2.7) for \( r < 3 \) with constant \( c_2 = c_2(\alpha_1, \alpha_1, 1 \vee R) \). Hence, for \( |z| < 2R \), we have

\[
(2CC_1)^{-1/2} \left( \frac{\delta_z}{|z-y|} \right)^{\pi/2} \leq \frac{V(\delta_z)}{V(|z-y|)} \leq c_2 \left( \frac{\delta_z}{|z-y|} \right)^{\alpha_1/2},
\]

\[
(2CC_1)^{-1/2} \left( \frac{\delta_y}{|z-y|} \right)^{\pi/2} \leq \frac{V(\delta_y)}{V(|z-y|)} \leq c_2 \left( \frac{\delta_y}{|z-y|} \right)^{\alpha_1/2}.
\]

These imply

\[
P_B(x, z) \leq c_1 c_2 \int_B \frac{V(\delta_x)}{V(\delta_z)} \left( \frac{1}{\sqrt{\delta_x \delta_y}} \wedge \frac{1}{|x-y|} \right) \frac{(\delta_y \delta_z)^{\alpha_1/2}}{|z-y|^{1+\alpha_1}} \, dy
\]

\[
\approx \int_B V(\delta_x) V(\delta_z) \left( \frac{\delta_z}{\delta_x} \right)^{\alpha_1/2} G_B^{S_{\alpha_1}S}(x, y) \frac{dy}{|z-y|^{1+\alpha_1}}
\]

\[
\approx V(\delta_x) V(\delta_z) \left( \frac{\delta_z}{\delta_x} \right)^{\alpha_1/2} P_B^{S_{\alpha_1}S}(x, z).
\]

Here, \( c_3 = c_3(c_1, c_2, \alpha_1) \) and \( S_{\alpha_1}S \) refers to the symmetric \( \alpha \)-stable process with index of stability \( \alpha_1 \). Similarly, we obtain

\[
P_B(x, z) \geq c_4 \frac{V(\delta_x)}{V(\delta_z)} \left( \frac{\delta_z}{\delta_x} \right)^{\pi/2} P_B^{S_{\alpha_1}S}(x, z),
\]

where \( c_4 = c_4(c_1, \alpha_1, C) \). By formula for \( P_B^{S_{\alpha_1}S}(x, z) \) \cite{1} Theorem A1, we get the assertion of the lemma for \( |z| < 2R \).

If \( |z| \geq 2R \), by (2.5) and \cite{19} Proposition 3.5, we get

\[
P_B(x, z) \approx \nu(|z|) E^x T_B \approx V(\delta_x) V(R) \nu(|z|),
\]

which implies the claim of the lemma.

\( \square \)
Proposition 3.3. There exists a constant $C_5 = C_5(c, \text{diam}(D)/r_0, 1 \vee \text{diam}(D))$ such that

$$P_D(x, z) \overset{c_5}{=} \frac{V(\delta_z)}{V(\delta_z)|x - z|} \left( \frac{V(\text{diam}(D))}{V(\delta_z)} \wedge 1 \right), \quad x \in D, z \in D^c.$$  

Proof. Let $x \in D, z \in D^c$. By Lemma 3.2 we consider only the case when $D$ is a sum of at least two open intervals. Let $B$ be an open interval such that $x \in B$ and $\tilde{D} = D \setminus B$ is open. By the Ikeda-Watanabe formula

$$P_D(x, z) = \int_B G(x, y)\nu(|y - z|)dy + \int_{\tilde{D}} G(x, y)\nu(|y - z|)dy =: I + II.$$  

Lemma 3.2 implies

$$G(x, y) \overset{c_1}{=} G_B(x, y), \quad x, y \in B,$$

for $c_1 = c_1(C_2)$. Hence, by Lemma 3.2

$$I \overset{c_2}{=} \int_B G_B(x, y)\nu(|y - z|)dy \overset{c_2}{=} \frac{V(\delta_z)}{V(\text{diam}(B))}\frac{V(\text{diam}(B))}{|x - z|} \left( \frac{V(\text{diam}(B))}{V(\delta_z)} \wedge 1 \right).$$  

where $c_2 = c_2(C_2, C_3)$. If $\text{dist}(z, B) = \delta_z$, the lower bound follows by (3.9). Suppose $\text{dist}(z, B) < \delta_z$ and let $B$ be a connected component of $\tilde{D}$ such that $\text{dist}(z, \tilde{B}) = \delta_z$. Therefore, by Lemma 3.1

$$II \geq C_2 \int_B \frac{V(\delta_z)V(\delta_y)}{	ext{diam}(D)} \nu(|y - z|)dy \geq \frac{C_2r_0}{2\text{diam}(D)} \frac{V(\delta_z)}{|x - z|} \int_B (V(\delta_y)\nu(|y - z|)dy.$$  

Now, (2.10) and (2.5) imply

$$\int_B V(\delta_y)\nu(|y - z|)dy \geq \frac{c_4}{V^2(2\delta_z)} \int_0^{\delta_z \wedge r_0/2} V(s)ds \geq \frac{c_4}{V^2(2\delta_z)} \int_{(\delta_z \wedge r_0/2)}^{\delta_z \wedge r_0/2} V(s)ds \geq \frac{c_4 (\delta_z \wedge r_0/2)}{4V^2(2\delta_z)}.$$

Hence, we obtain the lower bound in this case.

Next, we will prove the upper bound for the second integral. Let $\lambda = \delta_z \wedge \text{diam}(D)$ and $D_1 = \tilde{D} \cap \{ y : \delta_y \leq \lambda \}$ and $D_2 = \tilde{D} \cap \{ y : \delta_y > \lambda \}$. By weak scaling conditions, we obtain

$$II \overset{c_5}{=} \int_{D_2} \frac{V(\delta_z)V(\delta_y)}{\text{diam}(D)} dy \frac{dy}{V^2(\text{diam}(D)/|y - z|)|y - z|} \leq \frac{V(\delta_z)}{r_0} \int_D \frac{V(\delta_y)dy}{V^2(|y - z|)|y - z|} \leq \frac{V(\delta_z)}{r_0} \left( \frac{|D_1|V(\lambda)}{V^2(\delta_z)} + \int_{D_2} \frac{dy}{V(\delta_y)V(\delta_y)} \right) \leq \frac{V(\delta_z)}{r_0} \left( \frac{2\text{diam}(D)}{V^2(\delta_z)} \frac{\lambda V(\lambda)}{r_0} + c_6 \frac{1}{V(\lambda)} \right) \leq \frac{V(\delta_z)}{V(\delta_z)|x - z|} \left( \frac{V(\text{diam}(D))}{V(\delta_z)} \wedge 1 \right),$$

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where \( c_5 = c_5(C_2, C) \) and \( c_6 \) depends only on the scaling characteristics. This completes the proof. \( \square \)

### 3.2 Estimates of \( \partial_x G(x, y) \)

Below, we will prove various estimates of \( \partial_x G(x, y) \) according to the range of variables \( x \) and \( y \). We summarize these results in Theorem 3.10. First, we will need the following auxiliary lemma.

**Lemma 3.4.** Let \( x \in D \). There is a constant \( C_6 = C_6(\sigma, \text{diam}(D)/r_0, 1 \lor \text{diam}(D)) \) such that

\[
\int_{\mathbb{R}} \frac{M(|x-z|)}{V(\delta_x)} \, dz \leq C_6 \frac{V(\delta_x)}{\delta_x}.
\]

**Proof.** Let \( B_1 = B(x, \delta_x/2) \). By (2.8), we have

\[
\int_{B_1} \frac{M(|x-z|)}{V(\delta_x)} \, dz \leq 2 \int_{B_1} \frac{V^2(|x-z|)}{|x-z|^2 V(\delta_x)} \, dz \leq c_1 \int_{B_1} \frac{V^2(\delta_x)|x-z|^{\alpha} - 2}{\delta_x^2 V(\delta_x)} \, dz \leq c_2 \frac{V(\delta_x)}{\delta_x}.
\]

Note that for \( z \not\in B \), we have \( \delta_x \leq 3|x-z| \) and \( \delta_x \leq 2|x-z| \). Hence, by (2.6),

\[
\int_{B_1 \cap \{\delta_x < \delta \}} \frac{M(|x-z|)}{V(\delta_x)} \, dz \leq c_3 \int_{\{\delta_x < \delta \}} \delta_x^2 \, dz \leq c_4 \int_{\{\delta_x < \delta \}} \frac{V(\delta_x)\delta_x^{7/2 - 2}}{\delta_x^{1/2}} \, dz \leq c_5 \frac{V(\delta_x)}{\delta_x},
\]

\[
\int_{B_1 \cap \{\delta_x > \delta \}} \frac{M(|x-z|)}{V(\delta_x)} \, dz \leq c_6 \int_{\{\delta_x < \delta \}} \frac{V^2(\delta_x)}{\delta_x^2 V(\delta_x)} \, dz \leq c_7 \int_{\{\delta_x < \delta \}} \frac{V^2(\delta_x)}{\delta_x^2 V(\delta_x)} \, dz \leq c_8 \frac{V(\delta_x)}{\delta_x}.
\]

\( \square \)

**Proposition 3.5.** There is a constant \( C_7 = C_7(\sigma, \text{diam}(D)/r_0, 1 \lor \text{diam}(D)) \) such that

\[
|\partial_x G(x, y)| \leq C_7 \left( M(|x-y|) + \frac{G(x, y)}{\delta_x} \mathbb{1}_{|x-y| > \delta_x} \right)
\]

**Proof.** Since \( X_t \) is translation invariant, we may and do assume that \( 0 \not\in D \). Let \( x, y \in D \) and \( x \neq y \). It is known (see [20, Lemma 2.3])

\[
G_{\{0\}^c}(x, y) = K(x) + K(y) - K(y - x).
\]

Hence, by symmetry,

\[
G(x, y) = G_{\{0\}^c}(x, y) - \mathbb{E}^x G_{\{0\}^c}(x, X_{\tau_D}) = K(y) - K(x - y) - \mathbb{E}^y K(X_{\tau_D}) + \mathbb{E}^y K(x - X_{\tau_D}).
\]
By Lemma 2.1 and the dominated convergence theorem,

$$\partial_x G(x, y) = \mathbb{E}^y \partial_x K(x - X_{\tau_D}) - \partial_x K(x - y).$$

Again, by Lemma 2.1 and (2.14), for $|x - z| \geq |x - y|/2$, we have

$$|\partial_x K(x - z)| \leq c_1 M(|x - z| \wedge \text{diam}(D)) \leq c_2 M(|x - y|).$$

This implies

$$|\partial_x G(x, y)| \leq c_3 M(|x - y|) + \mathbb{E}^y |\partial K(x - X_{\tau_D})| \leq c_4 M(|x - y|) + \mathbb{E}^y \left[ |\partial K(x - X_{\tau_D})|, |x - X_{\tau_D}| \leq \frac{|x - y|}{2} \right].$$

It remains to estimate

$$I := \mathbb{E}^y \left[ |\partial K(x - X_{\tau_D})|, |x - X_{\tau_D}| \leq \frac{|x - y|}{2} \right]. \quad (3.11)$$

If $\delta_x \geq \frac{|x - y|}{2}$, $I = 0$. So let $\delta_x < \frac{|x - y|}{2}$. Note that if $|x - z| \leq |x - y|/2$, then $|y - z| \geq |x - y|/2$, and in consequence, by Proposition 3.3,

$$P_D(y, z) \lesssim \frac{V(\delta_y)}{V(\delta_z)} \cdot \frac{1}{|y - z|} \lesssim \frac{V(\delta_y)}{V(\delta_z)} \cdot \frac{2}{|x - y|}.$$

By Lemma 3.4,

$$I \leq \int_{D \cap B(x, \frac{|x - y|}{2})} M(|x - z|) P_D(y, z)dz \leq c_5 \frac{V(\delta_y)}{|x - y|} \int_{\mathbb{R}} M(|x - z|) \frac{dz}{V(\delta_z)} \leq c_6 \frac{V(\delta_y)}{|x - y|} \frac{V(\delta_x)}{\delta_x}.$$  

Since $\delta_x \leq \frac{|x - y|}{2}$, we have $\delta_y \leq \frac{3}{2} |x - y|$ and

$$\frac{V(\delta_y)}{|x - y|} \frac{V(\delta_x)}{\delta_x} \lesssim \frac{G(x, y)}{\delta_x}.$$

Hence,

$$|\partial_x G(x, y)| \lesssim M(|x - y|) + \frac{G(x, y)}{\delta_x} 1_{|x - y| > \delta_x},$$

which ends the proof.

By Lemma 2.1 and Proposition 3.3, we get a weaker but also useful estimate.
Corollary 3.6. There is a constant $C_8 = C_8(\sigma, \text{diam}(D) \lor 1)$ such that for any open $D \neq \mathbb{R}$

$$|\partial_x G(x, y)| \leq C_8 M(\delta_x \land |x - y|). \tag{3.12}$$

Lemma 3.7. If $f \in \mathcal{K}_1$, then

$$\partial_y \int_D G(y, z) f(z) \, dz = \int_D \partial_y G(y, z) f(z) \, dz, \quad y \in D. \tag{3.13}$$

Proof. Let $0 < h < \delta_y / 2$. Then,

$$\frac{G(y + h, z) - G(y, z)}{h} = \frac{1}{h} \left| \int_0^1 \partial_s G(y + sh, z) \, ds \right|$$

$$= \left| \int_0^1 \partial_s G(y + sh, z) \, ds \right| \leq C_8 \int_0^1 (M(\delta_y + sh \land |y + sh - z|)) \, ds$$

$$\leq C_8 \int_0^1 (M(\delta_y / 2) + M(|y + sh - z|)) \, ds.$$

Since $f \in \mathcal{K}_1$ and the integrand is uniformly in $h$ integrable on and $(0, 1) \times D$, which ends the proof.

Proposition 3.8. Let $x \in D$, $0 < \varepsilon < \delta_x$, $B = B(x, \varepsilon)$ and $A = B^c \cap D$. Then,

$$\partial_x G(x, y) = \int_B \partial_x G(x, z) P_A(y, z) \, dz.$$

Proof. Fix $x \in D$. Then, $G(x, \cdot)$ is regular harmonic on $A = D \cap [x - \varepsilon, x + \varepsilon]^c$ for every $0 < \varepsilon < \delta_x$. This means

$$G(x, y) = \mathbb{E}^x G(x, X_{\tau_A}) = \int_B G(x, z) P_A(y, z) \, dz, \quad y \in A.$$

Let us fix $y \in A$. For $z \in B$, we define $P_1(y, z) = P_A(y, z) 1_{B(x, \varepsilon/2)}(z)$ and $P_2(y, z) = P_A(y, z) - P_1(y, z)$. Since $P_1$ is bounded, we have $P_1 \in \mathcal{K}_1$ and by Lemma 3.7

$$\partial_x \int_B G(x, z) P_1(y, z) \, dz = \int_B \partial_x G(x, z) P_1(y, z) \, dz.$$

Since $\partial_x G(x, z)$ is finite on the support of $P_2(y, \cdot)$, by the mean value theorem and the dominated convergence theorem, we get
\[
\lim_{h \to 0} \frac{\int_B G(x+h, z) P_2(y, z) dz - \int_B G(x, z) P_2(y, z) dz}{h} = \lim_{h \to 0} \int_B \frac{G(x+h, z) - G(x, z)}{h} P_2(y, z) dz = \int_B \partial_x G(x, z) P_2(y, z) dz. 
\]

These imply
\[
\partial_x \int_B G(x, z) P_B(y, z) dz = \partial_x \int_B G(x, z) P_1(y, z) dz + \partial_x \int_B G(x, z) P_2(y, z) dz 
\]
\[
= \int_B \partial_x G(x, z) P_1(y, z) dz + \int_B \partial_x G(x, z) P_2(y, z) dz 
\]
\[
= \int_B \partial_x G(x, z) P_A(y, z) dz, 
\]

which completes the proof. \(\square\)

**Lemma 3.9.** Let \(x, y \in D\) and \(\delta_x < 2|x - y|\). Then, there exists a constant \(C_9 = C_9(\sigma, \text{diam}(D)/r_0, 1 \vee \text{diam}(D))\) such that

\[
|\partial_x G(x, y)| \leq C_9 \frac{G(x, y)}{\delta_x}. 
\]

**Proof.** Let \(B \subset D\) be any interval such that \(\overline{B} \subset D\) and put \(A = B^c \cap D\). For any \(x \in B\) and \(y \in D\) such that \(x \neq y\), by Propositions 3.3 and 3.5 and harmonicity of \(G\),

\[
|\partial_x G(x, y)| = |\int_B \partial_x G(x, z) P_A(y, z) dz| 
\]
\[
\leq C_7 \int_B \left( M(|x - z|) + \frac{G(x, z)}{\delta_x} \mathbb{1}_{|x - z| > \delta_x} \right) P_A(y, z) dz 
\]
\[
\leq C_7 \int_B M(|x - z|) P_A(y, z) dz + C_7 \frac{G(x, y)}{\delta_x}. 
\]

Therefore, it remains to estimate the integral

\[
\int_B M(|x - z|) P_A(y, z) dz. \tag{3.14} 
\]

Let \(B = B(x, \delta_x/4)\). By the assumption \(y \notin B\), \(\text{dist}(y, B) \geq \delta_x/4\) and \(|y - z| \approx |x - y|\) for \(z \in B\). Denote \(\delta_x^A = \text{dist}(x, \partial A)\). Note that \(\delta_x^A \approx \delta_x\) and \(\delta_y^A \approx \delta_y\). By Proposition 3.3 and Lemmas 3.4 3.1, we get

\[
\int_B M(|x - z|) P_A(y, z) dz \leq C_4 \int_B M(|x - z|) \frac{V(\delta^A_y)}{V(\delta^A_x) |y - z|} dz 
\]
\[
\leq c_1 \frac{V(\delta_y)}{|x - y|} \int_B M(|x - z|) \frac{V(\delta^A_y)}{V(\delta^A_x)} dz \leq c_2 \frac{V(\delta_y) V(\delta^A_x)}{|x - y| \delta_x} \leq c_3 \frac{V(\delta_y) V(\delta_x)}{|x - y| \delta_x} \leq c_4 \frac{G(x, y)}{\delta_x}. 
\]
Since constants $c_1 - c_4$ depend on $D$ only via constants $C_2, C_4$ and $C_6$, the proof is completed.

**Theorem 3.10.** There is a constant $C_{10} = C_{10}(\sigma, \text{diam}(D)/r_0, 1 \lor \text{diam}(D))$ such that

$$|\partial_x G(x, y)| \leq C_{10} \frac{G(x, y) \wedge K(|x|)}{|x-y| \wedge \delta_x}, \quad x, y \in D.$$  

**Proof.** Due to Corollary [3.6](#) and Lemmas [3.9](#) and [2.13](#), it remains to prove existing of a constant $c$ such that

$$G(x, y) \geq cM(|x-y|),$$  

when $|x-y| \leq \delta_x/2$. But in this case $\delta_x \approx \delta_y$ and therefore, by Lemma [3.1](#),

$$G(x, y) \approx V^2(\delta_x).$$

Since $\alpha_1 > 1$, by (2.7), we obtain that $s \mapsto V^2(s)/s$ is almost increasing (bounded from below by an increasing function). Hence, we get the claim.

We end this section we the proof of the uniform intergability of $\partial_z G(z, y)$.

**Lemma 3.11.** The function $\partial_z G(z, y)$ is uniformly in $y$ integrable against $|b(z)|dz$.

**Proof.** It is enough to show that

$$\lim_{N \to \infty} \sup_{y \in \mathbb{R}} \int_{|\partial_z G(z, y)| > N} |\partial_z G(z, y)||b(z)|dz = 0.$$  

Let $N > 0$ and $r_N = \inf\{r > 0 : M(r) \leq N/C_8\} \land r_0$. Note that $\lim_{r \to 0} M(r) = \infty$, hence $r_N \to 0$ as $N \to \infty$. Fix $y \in \mathbb{R}$ and take $N$ such that $r_N \leq r_0$. By (3.12), \{ $z : |\partial_z G(z, y)| > N$\} $\subset$ \{ $z : M(\delta_z) > N/C_8$\} $\cup$ \{ $z : M(|z-y|) > N/C_8$\} $\subset$ \{ $z : \delta_z < r_N$\} $\cup$ \{ $z : |y-z| < r_N$\}. We may assume that the set $D$ is an union of $k$ distinctive intervals. By Proposition [3.5](#) and monotonicity of $M(\cdot)$, we have

$$\int_{|\partial_z G(z, y)| > N} |\partial_z G(z, y)||b(z)|dz \leq C_8 \left( \int_{\delta_z < r_N} M(\delta_z)|b(z)|dz + \int_{|z-y| < r_N} M(|z-y|)|b(z)|dz \right) \leq (2k + 1)C_8K_{r_N},$$

where

$$K_r = \sup_{y \in \mathbb{R}} \int_{B(y, r)} M(|y-z|)|b(z)|dz.$$  

By (2.15), $\lim_{N \to \infty} K_{r_N} = 0$, which completes the proof.
3.3 3G inequalities

Now, we apply the estimates of the Green function and its derivative to obtain the following 3G-type inequalities.

**Proposition 3.12.** There is a constant $C_{11} = C_{11} (\sigma, 1 \lor \text{diam}(D))$ such that

$$\frac{G(x, z) G(z, y)}{G(x, y)} \leq C_{11} V(\delta_z) \left( \frac{G(x, z)}{V(\delta_x)} \lor \frac{G(z, y)}{V(\delta_y)} \right).$$

**Proof.** For $x, y \in D$, we define

$$G(x, y) = \frac{G(x, y)}{V(\delta_x) V(\delta_y)}.$$

It suffices to prove that for any $x, y, z \in D$, we have

$$G(x, z) \land G(z, y) \leq c G(x, y).$$

By Lemma 3.1,

$$C_{3}^{-1} (G(x, z) \land G(z, y)) \leq \frac{1}{(\delta_x \delta_z)^{1/2}} \land \frac{1}{|x - z|} \land \frac{1}{(\delta_x \delta_y)^{1/2}} \land \frac{1}{|z - y|}$$

$$= \frac{1}{(\delta_x \delta_y)^{1/2}} \left( \frac{\delta_x \land \delta_y}{\delta_z} \right)^{1/2} \land \frac{1}{|x - z| \lor |z - y|}.$$

If $\frac{\delta_x \land \delta_y}{\delta_z} \leq 2$, Lemma 3.1 imply

$$G(x, z) \land G(z, y) \leq 2C_3 \left( \frac{1}{(\delta_x \delta_y)^{1/2}} \land \frac{1}{|x - y|} \right) \leq 2C_3^2 G(x, y).$$

If $\frac{\delta_x \land \delta_y}{\delta_z} \geq 2$, then $\sqrt{\delta_x \delta_y} \leq \delta_x \lor \delta_y \leq 2 (|x - z| \lor |y - z|)$ and in consequence

$$G(x, z) \land G(z, y) \leq \frac{C_3}{|x - z| \lor |y - z|} \leq 2C_3^2 G(x, y).$$

Lemma 3.13. There is a constant $C_{12} = C_{12} (\sigma, \text{diam}(D)/r_0, 1 \lor \text{diam}(D))$ such that for any $x, y, z \in D$, we have

$$\frac{G(x, z) \partial_z G(z, y)}{G(x, y)} \leq C_{12} M(\delta_z \land |y - z|).$$
Proof. Note that \( \delta_z^2 \leq 4(\delta_z^2 + |x-z|^2) \), hence,

\[
\frac{V(\delta_z)G(x,z)}{V(\delta_x)} \approx V(\delta_z)^2 \left( \frac{1}{(\delta_z^2)^{1/2}} \land \frac{1}{|x-z|} \right) \leq 2 \frac{V(\delta_z)^2}{\delta_z}.
\]

(3.16)

By Proposition 3.12 and (3.16),

\[
\frac{G(x,z)G(z,y)}{G(x,y)} \leq C_{11} \left( \frac{V(\delta_z)G(x,z)}{V(\delta_x)} \lor \frac{V(\delta_z)G(z,y)}{V(\delta_y)} \right) \leq c_1 \frac{V(\delta_z)^2}{\delta_z},
\]

(3.17)

where \( c_1 = 2C_3C_{11} \). For \( \delta_z < 2|y-z| \), by Lemma 3.9 and (3.17), we get

\[
\frac{G(x,z)|\partial_z G(z,y)|}{G(x,y)} \leq c_2 \frac{V(\delta_z)G(z,y)}{\delta_z} \leq c_2 M(\delta_z),
\]

where \( c_2 = c_1 C_9 \). Now, let \( \delta_z \geq 2|y-z| \). Note that \( \delta_z \approx \delta_y \) and in consequence \( G(z,y) \approx V^2(\delta_z) \delta_z \). Hence, by (3.12) and (3.17), we have

\[
\frac{G(x,z)|\partial_z G(z,y)|}{G(x,y)} \leq c_3 \frac{V(\delta_z)^2}{G(z,y)\delta_z} M(\delta_z) \leq c_4 M(\delta_z),
\]

where \( c_3 = c_1 C_8 \) and \( c_4 = c_3 \left( \frac{2C_1}{\xi} \right) \sqrt{\frac{3}{2}} \). Now, by (2.5), the assertion of the lemma holds.

For \( x,y \in D \), we define

\[
\kappa(x,y) = \int_D \left| b(z) \frac{G(x,z)\partial_z G(z,y)}{G(x,y)} \right| dz,
\]

(3.18)

Lemma 3.14. Let \( \lambda < \infty, R < 1 \). There is a constant \( C_{13} = C_{13}(\sigma, b, \lambda, R) \) such that if \( \text{diam}(D)/r_0(D) \leq \lambda \) and \( \text{diam}(D) \leq R \), then

\[
\kappa(x,y) \leq C_{13}, \quad x,y \in D.
\]

(3.19)

Furthermore, \( C_{13} \to 0 \) as \( R \to 0 \).

Proof. Since \( b \in K_1 \), (3.19) follows by Lemma 3.13.

\[
\square
\]

4 Green function of \( \tilde{L} \)

Following [8] and [26] we recursively define, for \( t > 0 \) and \( x,y \in \mathbb{R} \),

\[
p_0(t,x,y) = p(t,x,y),
\]

\[
p_n(t,x,y) = \int_0^t \int_\mathbb{R} p_{n-1}(t-s,x,z)b(z)\partial_z p(s,z,y) \, dz \, ds, \quad n \geq 1,
\]
and we let
\[ \tilde{p}(t, x, y) = \sum_{n=0}^{\infty} p_n(t, x, y). \]  

(4.1)

By [26, Theorem 1.1] the series converges absolutely, \( \tilde{p} \) is a continuous probability transition density function, and
\[ c_T^{-1} p(t, x, y) \leq \tilde{p}(t, x, y) \leq c_T p(t, x, y), \quad x, y \in \mathbb{R}, \ 0 < t < T, \]  

(4.2)

where \( c_T \to 1 \) if \( T \to 0 \), see [8, Theorem 2].

By Chapman-Kolmogorov equation, there is \( C_{14} > 0 \) such that
\[ C_{14}^{-1-t} p(t, x - y) \leq \tilde{p}(t, x, y) \leq C_{14}^{t+1} p(t, x - y), \ t > 0, \ x, y \in \mathbb{R}. \]  

(4.3)

We let \( \tilde{P}, \tilde{E} \) be the Markov distributions and expectations defined by transition density \( \tilde{p} \) on the canonical path space. By Hunt formula,
\[ \tilde{p}_D(t, x, y) = \tilde{p}(t, x, y) - \tilde{E}^x [\tau_D < t; \tilde{p}(t - \tau_D, X_{\tau_D}, y)]. \]  

(4.4)

Except symmetry, \( \tilde{p}_D \) has analogous properties as \( p_D \), i.e. the Chapman-Kolmogorov equation holds
\[ \int_{\mathbb{R}^4} \tilde{p}_D(s, x, z) \tilde{p}_D(t, z, y) dz = \tilde{p}_D(s + t, x, y), \ s, t > 0, \ x, y \in \mathbb{R}, \]

\[ 0 \leq \tilde{p}_D(t, x, y) \leq \tilde{p}(t, x, y) \] and \( \tilde{p}_D \) is jointly continuous on \( (0, \infty) \times D \times D \).

We denote by \( \tilde{G}_D(x, y) \) the Green function of \( \tilde{L} = \mathcal{L} + b \partial \) on \( D \),
\[ \tilde{G}_D(x, y) = \int_0^\infty \tilde{p}_D(t, x, y) dt. \]  

(4.5)

As for \( G \), from now on, every time we will mention the Green function \( \tilde{G} \), it should be understand as a Green function of \( \tilde{L} \) on \( D \), and then \( \tilde{G} = \tilde{G}_D \).

By Blumenthal’s 0-1 law and (4.3), \( \tilde{p}_D(t, x, y) = 0 \) and \( \tilde{G}(x, y) = 0 \) if \( x \in D^c \) or \( y \in D^c \).

By (4.2), we have
\[ \lim_{t \to 0} \frac{\tilde{p}(t, x, y)}{t} = \lim_{t \to 0} \frac{p(t, x, y)}{t} = \nu(y - x). \]

Thus the intensity of jumps of the canonical process \( X_t \) under \( \tilde{P}^x \) is the same as under \( P^x \). Accordingly, we obtain the following description.

**Lemma 4.1.** The \( \tilde{P}^x \)-distribution of \( (\tau_D, X_{\tau_D}) \) on \( (0, \infty) \times (D)^c \) has density
\[ \int_D \tilde{p}_D(u, x, y) \nu(z - y) dy, \quad u > 0, \ \delta_z > 0. \]  

(4.6)
We define the Poisson kernel of $D$ for $\tilde{L}$,

$$\tilde{P}_D(x, y) = \int_D \tilde{G}(x, z) \nu(|y - z|) \, dz, \quad x \in D, \ y \in D^c. \quad (4.7)$$

By (4.5), (4.7) and (4.6) we have

$$\tilde{P}_x(X_{\tau_D} \in A) = \int_A \int_D \tilde{G}(x, z) \nu(|y - z|) \, dz \, dy = \int_A \tilde{P}_D(x, y) \, dy, \quad (4.8)$$

if $A \subset (\bar{D})^c$. For the case of $A \subset \partial D$, we refer the reader to Lemma 5.8.

**Lemma 4.2.** $\tilde{G}(x, y)$ is continuous and

$$\tilde{G}(x, y) \leq C_{15}, \quad x, y \in \mathbb{R},$$

where $C_{15} = C_{15}(\sigma, b, \text{diam}(D))$.

**Proof.** In the same way as in [9, Lemma 7] we get that there are constants $c$ and $C$ such that

$$\tilde{p}_D(t, x, y) \leq Ce^{-ct}, \quad t > 1, \ x, y \in \mathbb{R}. \quad (4.9)$$

By (4.5), (4.2) and (4.9) we obtain

$$\tilde{G}(x, y) \leq \int_0^1 C_{14} \rho(t, x, y) \, dt + \int_1^\infty Ce^{-ct} \, dt \leq \int_0^1 \rho(t, 0, 0) \, dt + \int_1^\infty Ce^{-ct} \, dt \leq c_1 + C/c,$$

where $c_1$ is finite bound for $\int_0^1 \rho(t, 0, 0) \, dt$. We put $C_{15} = c_1 + C/c$. By (4.5), continuity of $\tilde{p}_D$ and the dominated convergence theorem, $\tilde{G}(x, y)$ is continuous. \hfill $\square$

By Lemmas 4.2 and 3.11 for every $x \in D$, the function

$$f_x(y) := \tilde{G}(x, y) - G(x, y) - \int_D \tilde{G}(x, z) b(z) \partial_z G(z, y) \, dz$$

is well defined, integrable and bounded on $\mathbb{R}$. Hence, following [18, Theorem 3.1], we obtain the following perturbation formula (for the proof see [18]).

**Lemma 4.3.** Let $x, y \in \mathbb{R}$. We have

$$\tilde{G}(x, y) = G(x, y) + \int_D \tilde{G}(x, z) b(z) \partial_z G(z, y) \, dz. \quad (4.10)$$
5 Proof of Theorem 1.1

First, we will prove the comparability of \( G \) and \( \tilde{G} \) for small sets \( D \) from the \( C^{1,1} \) class. For this purpose we could consider the perturbed series for \( \tilde{G} \) as it was presented in [18]. We could define by induction the functions \( G_n \) and show the convergence and estimates of the series

\[
\tilde{G}(x, y) = \sum_{n=0}^{\infty} G_n(x, y).
\]

However, since \( \tilde{G} \) is bounded, we present a simpler proof of the following lemma (compare [18, Lemma 3.11]).

**Lemma 5.1.** Let \( b \in K_1 \) and \( \lambda > 0 \). There is \( \varepsilon = \varepsilon(\sigma, b, \lambda) > 0 \) such that if \( \text{diam}(D)/r_0(D) \leq \lambda \) and \( \text{diam}(D) \leq \varepsilon \), then

\[
\frac{1}{2} G(x, y) \leq \tilde{G}(x, y) \leq \frac{3}{2} G(x, y), \quad x, y \in \mathbb{R}. \quad (5.1)
\]

**Proof.** By Lemma 3.14 there exists \( \varepsilon_1 > 0 \) such that if \( \text{diam}(D) < \varepsilon_1 \), then

\[
\int_D G(x, z)|\partial_z G(z, y)b(z)|dz \leq C_{13}G(x, y), \quad (5.2)
\]

and \( C_{13} < \frac{1}{3} \). Let \( 0 < \eta < 1 \). By Lemma 3.11 there exists \( \varepsilon_2 > 0 \) such that if \( \text{diam}(D) < \varepsilon_2 \), then

\[
\sup_{y \in \mathbb{R}} \int_D |\partial_z G(z, y)b(z)|dz \leq \eta.
\]

We put \( \varepsilon = \min(\varepsilon_1, \varepsilon_2) \) and \( \text{diam}(D) \leq \varepsilon \). By Lemma 4.3

\[
\tilde{G}(x, y) \leq G(x, y) + \int_D \tilde{G}(x, z)|b(z)|\partial_z G(z, y)|dz
\]

\[
\leq G(x, y) + C_{13}\eta. \quad (5.3)
\]

By putting the estimates of \( \tilde{G} \) from (5.4) into (5.3) and applying (5.2), we get

\[
\tilde{G}(x, y) \leq G(x, y) + \int_D (G(x, y) + C_{13}\eta)|b(z)|\partial_z G(z, y)|dz
\]

\[
\leq G(x, y)(1 + C_{13}) + C_{13}\eta^2. \quad (5.4)
\]

By induction,

\[
\tilde{G}(x, y) \leq G(x, y)(1 + C_{13} + \cdots + C_{13}^{m-1}) + C_{13}\eta^m. \quad (5.5)
\]

Now, taking \( n \to \infty \), for every \( x, y \in D \), we obtain

\[
\tilde{G}(x, y) \leq G(x, y)\frac{1}{1 - C_{13}}. \quad (5.6)
\]
Since $C_{13} < \frac{1}{3}$, by Lemma 4.3, (5.6) and (5.2), we get
\[
\tilde{G}(x, y) \geq G(x, y) - \frac{1}{1 - C_{13}} \int_D G(x, z) |b(z) \partial_z G(z, y)| dz \geq G(x, y) \left( 1 - \frac{C_{13}}{1 - C_{13}} \right).
\]

We note that the comparison constants in the proof above will improve to 1 if $\text{diam}(D) \to 0$ and the distortion of $D$ is bounded. By (4.8),

\[
\tilde{P}^x(X_{\tau_D} \in A) \approx \tilde{P}(X_{\tau_D} \in A), \quad x \in D, \quad A \subset (D)^c,
\]

where $C_{16} = C_{16}(\sigma, b, \lambda, \text{diam}(D))$ and $\text{diam}(D) < \varepsilon$ from Lemma 5.1.

Following [9, Proof of Lemma 14], we obtain that the boundary of our general $C^{1,1}$ open set $D$ is not hit at the first exit, i.e.

\[
\tilde{P}^x(X_{\tau_D} \in \partial D) = 0, \quad x \in D.
\]

Hence, in the context of Lemma 5.1 the $\tilde{P}^x$ distribution of $X_{\tau_D}$ is absolutely continuous with respect to the Lebesgue measure, and has density function

\[
\tilde{P}_D(x, y) \approx P_D(x, y), \quad y \in D^c,
\]

provided $x \in D$. This follows from (4.8) and (5.8).

The definition of $\tilde{L}$-harmonicity is analogous to that of $L$-harmonicity

**Definition 5.** We say that a function $f : \mathbb{R} \to \mathbb{R}$ is $\tilde{L}$-harmonic on an open bounded set $D \subset \mathbb{R}$, if for any open $F \subset \overline{F} \subset D$ and $x \in F$

\[
f(x) = \tilde{E}^x f(X_{\tau_F}).
\]

We say that a function $f$ is regular $\tilde{L}$-harmonic on an open bounded set $D \subset \mathbb{R}$, if for every $x \in D$

\[
f(x) = \tilde{E}^x f(X_{\tau_D}).
\]

Following [9] and [18], we get the following Harnack inequality.

**Lemma 5.2** (Harnack inequality for $\tilde{L}$). Let $x, y \in \mathbb{R}$, $0 < s < 1$ and $k \in \mathbb{N}$ satisfy $|x - y| \leq 2^k s$. Let $u$ be nonnegative in $\mathbb{R}$ and $L$-harmonic in $B(x, s) \cup B(y, s)$. There is $C_{17} = C_{17}(\overline{\alpha}, \overline{C}, b)$ such that

\[
C_{17}^{-2^{-k(1+\overline{\alpha})}} u(x) \leq u(y) \leq C_{17}2^{k(1+\overline{\alpha})} u(x).
\]

We obtain a boundary Harnack principle for $L$ and general $C^{1,1}$ sets $D$. See proof of [18, Lemma 4.3]
Lemma 5.3 (BHP). Let $z \in \partial D$, $0 < r \leq r_0(D)$, and $0 < q < 1$. If $\tilde{u}, \tilde{v}$ are nonnegative in $\mathbb{R}$, regular $\tilde{\mathcal{L}}$-harmonic in $D \cap B(z, r)$, vanish on $D^c \cap B(z, r)$ and satisfy $\tilde{u}(x_0) = \tilde{v}(x_0)$ for some $x_0 \in D \cap B(z, qr)$ then

$$C_{18}^{-1} \tilde{v}(x) \leq \tilde{u}(x) \leq C_{18} \tilde{v}(x), \quad x \in D \cap B(z, qr),$$

(5.11)

with $C_{18} = C_{18}(\sigma, b, q, r_0(D))$.

Now, we have all the tools necessary to prove the main result of our paper. Since in the proof we follow the idea from [9], we only give its basic steps (for details see [9, Proof of Theorem 1]).

Proof of Theorem 1.1. By (4.10), we have the estimate

$$\tilde{G}(x, y) \leq G(x, y) + \int_D |\partial_z G(z, y)| |b(z)| dz, \quad x, y \in D.$$  

(5.12)

We consider $\eta < 1$, say $\eta = 1/2$. By Lemma 3.11 there is a constant $r > 0$ so small that

$$\int_{D^r} |\partial_z G(z, y)b(z)| dz < \eta, \quad y \in D,$$

(5.13)

and

$$\int_{D^r} G(x, z) |\partial_z G(z, y)| |b(z)| dz < \eta, \quad y \in D,$$

(5.14)

Where $D^r = \{z \in D : \delta_z \leq r\}$. We denote

$$\rho = [\varepsilon \wedge r_0(D) \wedge r]/16,$$

with $\varepsilon = \varepsilon(\psi, b, \lambda, \text{diam}(D))$ of Lemma 5.1.

To prove (1.2) we will consider $x$ and $y$ in a partitions of $D \times D$.

First, we consider $y$ far from the boundary of $D$, say $\delta_y \geq \rho/4$.

- For $|x - y| \leq \rho/8$, $G(x, y) \approx G_B(x, y) \approx \tilde{G}_B(x, y) \approx \tilde{G}_D(x, y)$ (we use Lemmas 3.1, 5.1, 4.2).

- If $\rho/8 < \delta_x$ we use Harnack inequalities for $\mathcal{L}$ and $\tilde{\mathcal{L}}$.

- For $\delta_x < \rho/8$ we use Boundary Harnack principles (see Lemma 5.3, 27. Theorem 2.18).

Next, suppose that $\delta_D(y) \leq \rho/4$. Here, the difficulty lies in the fact $\tilde{G}$ is non-symmetric.

In the proof of lower bounds we consider two cases: $x$ close to $y$ and $x$ far away from $y$.  

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In the case $|x - y| \leq \rho$, we locally approximate $D$ by the small $C^{1,1}$ set $F$ such that $\delta_D(x) = \delta_F(x)$ and $\delta_D(x) = \delta_F(x)$ (see [9, Lemma 1]). Then $\tilde{G}(x, y) \geq \tilde{G}_F(x, y) \approx G_F(x, y) \approx G_D(x, y)$ (see Lemma 3.1).

For $|x - y| > \rho$ and $\delta_D(x) \geq \rho/4$ we use Harnack inequalities. For $\delta_D(x) \leq \rho/4$ we use boundary Harnack principles.

In the next step, we prove the upper bound in (1.2) for $\delta_D(x) \geq \rho/4$. We have already proved that for $y \in D \setminus D^r$,

$$c_1^{-1}G(x, y) \leq \tilde{G}(x, y) \leq c_1G(x, y).$$

By (3.18), Lemma 3.14 Lemma 4.2 (5.12) and (5.13),

$$\tilde{G}(x, y) \leq AG(x, y) + \int_{D^r} \tilde{G}(x, z) |\partial_z G(z, y)b(z)| dz, \quad (5.15)$$

$$\leq AG(x, y) + B(x), \quad (5.16)$$

where $A = 1 + c_1C_3$ and $B(x) = \eta C_7$. Now, plugging (5.16) into (5.15), and using (5.18), (5.14) and induction, we get for $n = 0, 1, \ldots$,

$$\tilde{G}_D(x, y) \leq A(1 + \eta + \cdots + \eta^n)G_D(x, y) + \eta^n B(x). \quad (5.17)$$

In consequence,

$$\tilde{G}_D(x, y) \leq \frac{A}{1 - \eta}G_D(x, y). \quad (5.18)$$

Finally, we prove the upper bound in (1.2) when $\delta_x < \rho/4$.

- If $|x - y| > \rho$, we use boundary Harnack principles.

- For $|x - y| \leq \rho$, consider the same set $F$ as above. We have

$$\tilde{G}_D(x, y) = \tilde{G}_F(x, y) + \int_{D \setminus F} \tilde{P}_F(x, z) \tilde{G}_D(z, y) dz.$$

By Lemma 5.1 and (5.9), $\tilde{G}_F(x, y) \approx G_F(x, y)$ and $\tilde{P}_F(x, z) \approx P_F(x, z)$. We already know that for $|z - y| > \rho$, $\tilde{G}_D(z, y) \approx G(z, y)$. Thus,

$$\tilde{G}_D(x, y) \approx G_F(x, y) + \int_{D \setminus F} P_F(x, z)G_D(z, y) dz = G_D(x, y).$$

The proof of Theorem 1.1 is complete.
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