Estimating Ising Models from One Sample

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Abstract

Given one sample \( X \in \{\pm 1 \}^n \) from an Ising model \( \Pr[X = x] \propto \exp(x^T J x / 2) \), whose interaction matrix satisfies \( J := \sum_{i=1}^k \beta_i J_i \) for some known matrices \( J_i \) and some unknown parameters \( \beta_i \), we study whether \( J \) can be estimated to high accuracy. Assuming that each node of the Ising model has bounded total interaction with the other nodes, i.e. \( \|J\|_\infty \leq O(1) \), we provide a computationally efficient estimator \( \hat{J} \) with the high probability guarantee \( \|\hat{J} - J\|_F \leq \tilde{O}(\sqrt{k}) \), where \( \|J\|_F \) can be as high as \( \Omega(\sqrt{n}) \). Our guarantee is tight when the interaction strengths are sufficiently low, in particular when the Ising model satisfies Dobrushin’s condition. An example application of our result is in social networks, wherein nodes make binary choices, \( x_1, \ldots, x_n \), which may be influenced at varying strengths \( \beta_i \) by different networks \( J_i \) in which these nodes belong. By observing a single snapshot of the nodes’ behaviors the goal is to learn the combined correlation structure.

When \( k = 1 \) and a single parameter is to be inferred, we further show \( |\hat{\beta}_1 - \beta_1| \leq \tilde{O}(F(\beta_1 J_1)^{-1/2}) \), where \( F(\beta_1 J_1) \) is the log-partition function of the model. This was proved by Bhattacharya and Mukherjee [BM18] under additional assumptions and by Chatterjee [Cha07] assuming that \( F(\beta_1 J_1) \geq \Omega(n) \) is of maximal order. We generalize these results to any setting.

While our guarantees aim both high and low temperature regimes, our proof relies on sparsifying the correlation network by conditioning on subsets of the variables, such that the unconditioned variables satisfy Dobrushin’s condition, i.e. a high temperature condition which allows us to apply stronger concentration inequalities. We use this to prove concentration and anti-concentration properties of the Ising model, as well as re-derive inequalities that were previously shown using Chatterjee’s exchangeable pairs technique [Cha05]. We believe this sparsification result has applications beyond the scope of this paper as well.
1 Introduction

Loosely defined, statistical inference is the task of using multiple observations of some probabilistic phenomenon to estimate aspects of that phenomenon that are of interest and can be identified from the available data. An important setting that has seen much progress is that wherein the observations can be modeled as independent samples from a probability distribution that we are seeking to learn, in some way. However, rarely does this independence assumption hold. Observations collected on a spatial domain, temporal domain or social network are commonly intricately dependent. Rather than viewing it as a set of $n$ independent samples, in these settings it is more fitting to view the data as a single sample from an $n$-dimensional distribution. What can be inferred for that $n$-dimensional distribution from a single sample?

The ubiquity of data dependencies in the natural world, as well as in social, financial and networked environments has motivated the development of statistical inference frameworks, which both model and seek statistical guarantees under data dependencies. This pursuit has a long history in Statistical Physics, Probability Theory, Econometrics, and Machine Learning, and has recently been studied in the Theory of Computation—see e.g. [Bes74; Yu94; Cha07; KR+08; BCPR+09; MR09; Pes10; MR10; KM15; MS17; KR17; BM18; GM18; BN18; DDP19; DDDJ19; CVV19]. Our goal is to advance the algorithmic and probabilistic foundations of this important and emerging field.

We will focus our attention on the celebrated Ising model [Isi25], a canonical Markov Random Field with pairwise interactions, which samples a binary vector $x = (x_1, \ldots, x_n) \in \{\pm 1\}^n$, according to the measure:

$$
\Pr_{J^*}[x] = \exp \left( \frac{x^\top J^* x}{2} - F(J^*) - n \log 2 \right); \text{ where } F(J^*) = \log \left( 2^{-n} \sum_{x \in \{\pm 1\}^n} \exp(x^\top J^* x/2) \right). \quad (1)
$$

Here, $J^*$ is an $n \times n$ symmetric matrix with zero diagonal and $F$ is the so-called log-partition function. Notice that the term $J^*_{ij} x_i x_j$ in the exponent of the density encourages $x_i$ and $x_j$ to have equal or opposite signs depending on the sign and magnitude of $J^*_{ij}$, but this “local encouragement” can be overwritten by indirect interactions arising through paths between $i$ and $j$ in the graph defined by $J^*$. Whenever $i$ and $j$ are disconnected in this graph, $x_i$ and $x_j$ are independent.

The Ising model has found many applications in a range of disciplines, including Statistical Physics, Computer Vision, Computational Biology, and the Social Sciences [GG86; Ell93; Fel04; Cha05; DMR11; DDK17]. For example, it has been used to model binary behavior on a social network [Ell93; You06; MS10], wherein nodes choose between, say, whether to own an Android phone (+1) or an iPhone (-1), and $\sum_j J^*_{ij} x_i x_j$ captures the utility derived by node $i$ depending on her and the other nodes’ selection of phones. In this setting, it is easy to show that, when nodes update their strategies using logit best response dynamics, the stationary distribution of their joint behavior converges to (1).

Given its broad applicability, a long line of research has aimed at estimating Ising models from samples. For some exciting progress on this front in recent years, see e.g. [SW12; RWL+10; Bre15; VMLC16; KM17; HKM17; WSD19]. Importantly, these works assume access to multiple samples from the Ising model, and target estimating the interaction matrix $J^*$, under some conditions. Instead our focus in this work will be to estimate $J^*$ from a single sample. This is well-motivated in many applications of the model where we can realistically only collect a single sample from the
distribution. In our earlier example, we can take a snapshot of the network to see who is using an Android phone or an iPhone. If we take a snapshot of the network tomorrow, chances are that very little would have changed, and we certainly would not collect an independent sample from the model. Such lack of access to independent samples is ubiquitous in financial, meteorological, and geographical data, as well as social-network data [Man93, BDF09], and has been studied in topics as diverse as criminal activity (see e.g. [GSS96]), welfare participation (see e.g. [BLM00]), school achievement (see e.g. [Sac01]), participation in retirement plans [DS03], and obesity [CF13; TNP08].

Estimating the parameters of Ising models from a single sample has recently seen some important advances. As $J^*$ has $O(n^2)$ parameters, it is clear that some restriction must be placed on it for it to be estimable at all from a single observation. One well-studied assumption is that $J^* = \beta J$, where $J$ is a known matrix, and $\beta$ is an unknown scalar strength parameter, i.e. that the interaction strengths are all known up to a scalar multiple. In this setting, Chatterjee [Cha07] has shown that $\beta$ can be estimated to within $O(1/\sqrt{n})$ whenever the log-partition function of the model satisfies $F(J^*) \geq \Omega(n)$, and follow-up work has been able to generalize his result to broader settings [BM18, GM18, DDP19].

However, the afore-described assumption about $J^*$ is only a first step towards modeling more realistic scenarios. In practice, it might be an unreasonably strong assumption that the interaction strengths are known up to some scalar multiple. Indeed, nodes commonly belong to several networks which might contribute to their behavior at different degrees of importance. To capture such settings, we study the case that $J^*$ is an unknown linear combination of $k$ known community matrices, yielding the model

$$\Pr_{J^*}[x] = \exp \left( x^\top J^* x/2 - F(J^*) - n \log 2 \right); \quad \text{where } J^* = \sum_{i=1}^{k} \beta_i J_i. \tag{2}$$

Here, $\{J_i\}_{i=1}^{k}$ are community matrices, each symmetric with zeros on the diagonal, $\beta = \{\beta_i\}_{i=1}^{k}$ are scalar parameters signifying the varying importance of each community, and $F(J^*)$ is as defined in (1). Our goal is to estimate $J^*$ given $\{J_i\}_{i=1}^{k}$ and one sample $x \sim \Pr_{J^*}$, namely

**Single Sample Inference of an Ising Model with Multiple Parameters:** Given a collection of community matrices $\{J_i\}_{i=1}^{k}$ as well as a single sample $\{\pm 1\}^n \ni x$ from (2), compute an estimate $\hat{f}(x)$ to minimize $\|\hat{f}(x) - J^*\|_F$.

A few remarks are in order. First, model (2) adds a lot of expressive power to the single-parameter setting ($k = 1$) studied in earlier work. For example, we might have access to various social networks that individuals belong to (Facebook, LinkedIn, etc.) and expect that they all contribute to their behavior at different strengths. As another example, we might not know the interaction matrix, but might be comfortable postulating that it comes from some parametric family of matrices, and we might be seeking to estimate it. This is a common challenge in Spatial Econometrics [Ans01, LeS08, AF12, Ans13]. As a final example, we might be studying the temporal evolution of a network, such as the voting pattern within a district over a number of election cycles. A reasonable modeling assumption is that a person’s vote in any particular cycle correlates with the votes of his peers in that cycle and also his vote in the past cycle. However the strength of the spatial correlation (with peers) might be different from that of the temporal correlation (with past vote).
Our second remark is to point out that we have formulated our problem as that of estimating $J^*$ rather than estimating the parameters $\beta_1, \ldots, \beta_k$. This choice is necessary to make our problem statistically meaningful and feasible. Indeed, if the matrices $J_1, \ldots, J_k$ are (close to being) linearly dependent, then there is no hope in targeting an estimation of the parameters $\beta_1, \ldots, \beta_k$. Hence, we focus on learning the linear combination $\sum_i \beta_i J_i$ to high accuracy, where the error is measured according to the Frobenius norm, and this metric captures our learning capacity exactly in many cases, as we discuss below. In some applications, estimates of $\beta_1, \ldots, \beta_k$ can be directly obtained from an estimate of $\sum_i \beta_i J_i$. This is common when $J^*$ belongs to some parametric family of interaction matrices, or when the constituent $J_i$’s have structure, as we discuss below.

Finally, we need to assume a bound on the interaction strengths in $J^*$, otherwise we could observe nearly deterministic behavior, with different values of $J^*$ leading to indistinguishable distributions. We will make the common assumption that $\|J^*\|_\infty := \max_{i \in [n]} \sum_{j \neq i} |J^*_{ij}| \leq M$, where $M$ can be any constant. This models settings where no node has an overwhelming influence on the other nodes. It is important to note that a constant bound on $M$ is different from saying that the model satisfies the more stringent Dobrushin condition. Indeed, a constant upper bound on $M$ allows models that that do not satisfy Dobrushin’s condition and that are in low temperature.

**Section organization.** We present our main results in Section 1.1, starting with our theorem for general values of $k$, which generalizes prior work, and proceeding with our result for $k = 1$, which improves over prior work. We also present an impossibility result, establishing that our main result for $k > 1$ is tight. Additionally, we provide a detailed comparison of our work to the existing literature. In Section 1.2, we highlight some new techniques used in our proofs, which may have applications outside the scope of this paper, followed by a proof sketch of our main results. We postpone the full details of our paper to Sections 4—8.

### 1.1 Our Results

#### 1.1.1 Estimation of multiple parameters

We present our main result (the formal statement is Theorem 1):

**Informal Theorem 1.** There is an algorithm which, given a single sample $x$ from distribution \( (2) \), where $J_1, \ldots, J_k$ are known matrices with zero diagonals, and assuming that $J^* = \sum \beta_i J_i$ satisfies $\|J^*\|_\infty = O(1)$, runs in time $\text{poly}(n)$ and outputs $\hat{J}$ such that with probability $1 - \delta$,

\[
\|\hat{J} - J^*\|_F = O(\sqrt{k\log n} + \sqrt{\log(1/\delta)}).
\]

While one might have expected the error to go to zero as $n$ tends to $\infty$, notice here that $\sqrt{k}$ should be compared to the Frobenius norm of the matrix $J^*$, which can be as high as $\Omega(\sqrt{n})$. Our guarantee naturally worsens as $\|J^*\|_F$ decays, however, as we show in Informal Theorem 2 below,

\[\text{This is due to the flattening behavior of the sigmoid function when its argument is a large value.}\]

\[\text{2When this assumption is violated, it might become information-theoretically impossible to have a consistent estimator for } \beta.\]

\[\text{3Dobrushin’s uniqueness condition, discussed in Section 1.2.1 and formally in Appendix A, is a well-studied condition in Statistical Physics under which the correlations among variables are relatively weak. Ising models with } \|J^*\|_\infty < 1 \text{ satisfy this condition.}\]
it is optimal in many settings. Also, while it is generally information theoretically impossible to recover the parameters \( \beta^* = (\beta^*_j)_{ij} \), there are many settings where estimation of \( \beta^* \) can be obtained from the guarantees of Informal Theorem 1 and such examples are given in Section 2.

Next, we show that the error attained by Informal Theorem 1 is nearly optimal whenever the intersection of \( \text{span}(J_1, \ldots, J_k) \) with the set of matrices of bounded infinity norm contains a ball of large radius, in the sense presented below (the formal statement is Theorem 5):

**Informal Theorem 2.** Assume that there exist matrices \( J_1', \ldots, J_k' \in \text{span}(J_1, \ldots, J_k) \) that are (i) orthonormal with respect to the trace inner product, namely \( \text{trace}(J_i J_j') = 0 \) if \( i \neq j \) and \( \|J_i\|^2_F = \text{trace}(J_i J_i') = 1 \), and (ii) for all \( \sigma \in [-1,1]^k \), \( \sum_i \sigma_i J_i' \|_\infty \leq O(1) \). Then, any estimator \( \hat{J}(x) \) satisfies

\[
\max_{J^* \in \text{span}(J_1, \ldots, J_k) : \|J^*\|_\infty \leq 1} \mathbb{E}_{x \sim P_r}[\|\hat{J}(x) - J^*\|_F] \geq \Omega(\sqrt{k}).
\]

The bound, in fact, relies on the fact that \( \|J^*\|_\infty < 1 \), and may not hold when \( \|J^*\|_\infty > 1 \). In the proof, we argue that if \( J^* = c \sum_{i=1}^k \epsilon_i J_i' \), where \( \epsilon_i \in \{0,1\} \), one can estimate \( \epsilon_i \) correctly only with constant probability, hence \( \Omega(k) \) of the signs \( \epsilon_i \) are estimated incorrectly. Since the matrices \( J_i' \) are orthogonal, this translates to an \( \Omega(\sqrt{k}) \) Frobenius norm error. The formal proof is carried out using Assouad’s Lemma [Ass83], and includes an upper bound on the total variation distance between Ising models, proven using recent concentration results for polynomials from [AKPS+19].

### 1.1.2 Estimation of one parameter

While the focus of this paper is on multiple parameter estimation, our techniques can be used to remove some restrictive assumptions that were present in prior work studying the estimation of a single parameter, i.e. the setting \( k = 1 \). Recall that, in this case, \( J \) is a known matrix, we are given one sample from \( (1) \) with interaction matrix \( J^* = \beta^* J \), and the goal is to estimate \( \beta^* \). A direct application of Theorem 1 to this special case gives an estimation error of \( |\hat{\beta} - \beta^*| \leq O(\log n / \|J^*\|_F) \), which already extends a similar bound shown in [BM18] outside of the (generalized) Dobrushin condition regime.\(^4\)

Prior work in the single-parameter estimation problem has also focused on providing bounds in terms of the log-partition function \( F(\beta^* J) \) of the model, as defined in \( (1) \), which generally gives stronger bounds compared to those in terms of the Frobenius norm. We obtain the following (for a formal statement see Section 7):

**Informal Theorem 3.** Assume that \( \|J\|_\infty = 1 \), \( \beta^* = O(1) \) and \( \sqrt{F(\beta^* J)} \geq \Omega(\log \log n) \), where \( F(\cdot) \) is defined as in \( (1) \). There exists an estimator \( \hat{\beta} \) of \( \beta^* \) such that with probability \( 1 - \delta \) over \( x \sim P_{\beta^*} \),

\[
|\hat{\beta} - \beta^*| \leq O \left( \beta^* F(\beta^* J)^{-1/2} \log \log n + \log(1/\delta) \right).
\]

Compared to our bound, Chatterjee [Cha07] proves a rate of \( 1/\sqrt{n} \) when \( F(\beta^* J) \geq \Omega(n) \) and Bhattacharya and Mukherjee [BM18] prove a rate of \( \Omega(F(\beta^* J))^{-1/2} \), similarly to Informal Theorem 3. However, they use additional stability assumptions for the log-partition function, which typically do not hold at the vicinity of phase transitions, which are common in Ising models. For further discussion, see Section 7.\(^3\)

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\(^4\) Bhattacharya and Mukherjee [BM18] proved a rate of \( 1/\|J^*\|_F \) under the assumption that \( \|J\|_2 < 1 \). This is a slightly weaker assumption than \( \|J\|_\infty < 1 \), while still retaining most desirable properties, defined by [Hay06].
1.1.3 Quick Comparison to Prior Work

We have already discussed prior work on single-sample estimation in Ising models. While there have been several works, the topic remains largely underdeveloped. In particular, prior work has focused on the case where there is a single unknown parameter in the interaction matrix of the model. Moreover, estimation results are known under restrictive assumptions on the interaction matrix, the log-partition function, and other properties of the model. In light of these results, the goal of this paper is to gain a better understanding of single-sample estimation, while making it applicable to more complex settings. In particular, we target the simultaneous estimation of multiple interaction parameters in Ising models, while attempting to minimize the assumptions being made. These improvements necessitate a new proof approach, that further removes many of the conditions required by prior work even in the single parameter case. A more detailed exposition of prior work and a comparison to our results can be found in Section 1.3.

1.2 Overview of Techniques

We start by presenting the main techniques used in this paper in Section 1.2.1, proceed with a proof sketch in Section 1.2.2, compare the proof technique to prior work in Section 1.2.3 and discuss its possible future applications in Section 1.2.4.

1.2.1 Key Technical Insights and Vignettes

From Low-Temperature to High-Temperature (Dobrushin). While nodes of the Ising model can be complexly dependent, when the correlations are sufficiently weak, the model shares important similarities to product measures. A well-studied mathematical formulation of weak dependencies for general random vectors is Dobrushin’s uniqueness condition, defined formally in Section A. For Ising models, a sufficient condition implying Dobrushin’s is \( \|J^*\|_\infty = \alpha < 1 \), where \( \alpha \) is a constant; see e.g. [DS87; SZ92]. While Dobrushin’s condition implies multiple desirable properties (see e.g. [Cha05; Wei05]), we will specifically use the fact that functions of the Ising model concentrate well under this condition; see e.g. [Cha05; DDK17; GLP17; GSS+19; AKPS+19]. Unfortunately, the regimes we are considering in this paper may lie well outside Dobrushin’s condition, and the tools available to handle Ising models that do not satisfy Dobrushin’s condition are significantly weaker and restricted, and concentration does not hold in general.

In this work, we prove concentration inequalities for certain low-temperature Ising models via reductions to the Dobrushin regime: we show that we can condition on a subset of the variables, such that in the conditional distribution, the unconditioned variables are in high temperature. A basic example where we can see such behavior is when \( J \) is the incidence matrix of a bipartite graph, namely, there exists a set \( I \subseteq [n] \) such that \( J_{ij} = 0 \) whenever either \( i, j \in I \) or \( i, j \in [n] \setminus I \) (this model is also called a Boltzmann machine). If we condition on \( x_{-I} := x_{[n]\setminus I} \), then \( \{x_i : i \in I\} \) are conditionally independent. The following lemma generalizes this intuition. For the purposes of this lemma, we work with Ising models with external fields. Given an interaction matrix \( J^* \) and a vector \( h \) of external fields, we define the distribution over \( x \in \{\pm 1\}^n \) with density \( p_{J^*, h}(x) \propto \).
exp(\(x^T f^* x/2 + h^T x\)). Further, for a subset \(I \subseteq [n]\), define \(x_I\) as the vector of entries of \(x\) indexed by \(I\) and \(x_{-I}\) the one containing the remaining entries.

**Informal Lemma 1** (Conditioning Trick). Let \(p_{J^*,h}(x)\) be an Ising model with interaction matrix \(J^*\) satisfying \(\|J^*\|_{\infty} = M\) and any external field vector \(h\). Then there exist \(\ell = O(\log n)\) sets \(I_1, \ldots, I_\ell \subseteq [n]\) such that:

1. Each \(i \in [n]\) appears in exactly \(\ell' = \lceil \ell/(16M) \rceil\) different sets \(I_j\).
2. For all \(j \in [\ell]\), the conditional distribution of \(x_{I_j}\), conditioning on any setting of \(x_{-I_j}\), satisfies Dobrushin’s condition.

We apply this lemma repeatedly in our proof, as it allows us to tap into the flexibility of dealing with weakly dependent random variables. As a first application, given a vector \(a \in \mathbb{R}^n\), we obtain a lower bound on the variance of \(a^T x\). It is well known that if \(x\) is an i.i.d. vector of binary random variables, each with variance \(\nu\), then \(\text{Var}(a^T x) = \nu \|a\|^2\). Furthermore, if \(x\) has a low Dobrushin’s coefficient \(\alpha\), then the entries of \(x\) are nearly independent and we can also show that \(\text{Var}(a^T x) \geq \Omega(\|a\|^2\). We will use Informal Lemma 1 to show that a similar lower bound holds even beyond Dobrushin’s condition.

**Informal Lemma 2** (Anti-Concentration). Suppose that \(x\) is sampled from an Ising model whose interaction matrix satisfies \(\|J^*\|_{\infty} = O(1)\) and whose external field vector satisfies \(\|h\|_{\infty} = O(1)\). Then, for all \(a \in \mathbb{R}^n\),

\[
\text{Var}(a^T x) \geq \Omega(\|a\|^2).
\]

To prove this lemma, consider the sets \(I_1, \ldots, I_\ell\) from Informal Lemma 1. First, we claim that there exists \(j \in [\ell]\) such that \(\|a_{I_j}\|^2 \geq \Omega(\|a\|^2)\). Indeed, by linearity of expectation, if we draw \(j \in [\ell]\) uniformly at random then,

\[
\mathbb{E}_j[\|a_{I_j}\|^2] = \mathbb{E} \left[ \sum_{i=1}^n 1(i \in I_j) a_i^2 \right] = \sum_{i=1}^n \frac{\ell'}{\ell} a_i^2 \geq \frac{\ell'}{\ell} \|a\|^2 \geq \Omega(\|a\|^2).
\]

Hence, there exists a set \(I_j\) that achieves this expectation, namely, \(\|a_{I_j}\|^2 \geq \Omega(\|a\|^2)\). Now using that, conditioning on \(x_{-I_j}, x_{I_j}\) has a low Dobrushin coefficient, as implied by Informal Lemma 1, we can bound \(\text{Var}[a^T x | x_{-I_j}] \geq \Omega(\|a_{I_j}\|^2)\) as discussed above, using weak dependence. Since conditioning reduces the variance on expectation, we conclude that

\[
\text{Var}(a^T x) \geq \mathbb{E}_{x_{-I_j}}[\text{Var}[a^T x | x_{-I_j}]] \geq \Omega(\|a_{I_j}\|^2) \geq \Omega(\|a\|^2).
\]

**Measure Concentration for Non-Polynomials.** There are multiple recent works studying the concentration of polynomial functions of the Ising model [DDK17; GLP17; GSS+19; AKPS+19]. Here, we would like to bound the tails of general functions, in terms of their polynomial Taylor approximations. By a simple modification to the proof of [AKPS+19], we can derive the following:

**Informal Theorem 4.** Let \(f : \{0,1\}^n \to \mathbb{R}\) be an arbitrary function and \(X\) be sampled from an Ising model which satisfies Dobrushin’s condition. Then

\[
\Pr[|f(x) - \mathbb{E} f(x)| > t] \leq \exp \left( -c \min \left( \frac{t^2}{\|\mathbb{E}_x Df(x)\|^2 + \max_x \|Hf(x)\|^2}, \frac{t}{\max_x \|Hf(x)\|^2} \right) \right).
\]
Here $D_if(x) = (f(x_{i+}) - f(x_{i-}))/2$ is the discrete derivative, where $x_{i+}$ and $x_{i-}$ are obtained from $x$ by replacing the value of $x_i$ with 1 and $-1$, respectively. The vector of discrete derivatives is denoted by $Df$ and $Hf$ is the $n \times n$ matrix of second discrete derivatives.

Theorem 4 can be trivially extended to derive bounds based on higher order Taylor expansion, extending [AKPS+19, Theorem 2.2] for multi-linear polynomials.

1.2.2 Proof Sketch of our Upper Bound

Using the tools from Section 1.2.1, we present a sketch of the proof of our main results. We start by looking at the setting $k = 1$. In this case, estimating the interaction matrix $\beta^* J$ with error $\tilde{O}(\sqrt{k}) = \tilde{O}(1)$ in Frobenius norm is equivalent to estimating $\beta^*$ to within additive error $\tilde{O}(1)/\|J\|_F$ in absolute value. A standard approach is maximum likelihood estimation (MLE), which outputs the maximizer $\hat{\beta}$ of the probability of the given sample $x$, namely, $\hat{\beta} := \arg\max_\beta \Pr_{\beta J}[x]$.

For Ising models, the MLE requires computing the partition function which is computationally hard to approximate [SS+14]. A recourse is to compute the maximum pseudo-likelihood estimator (MPLE) of [Bes74; Bes75] instead. One typically minimizes the negative log pseudo-likelihood,

$$\phi(x; \beta) := -\sum_{i=1}^n \log \Pr_{\beta J}[x_i \mid x_{-i}],$$

where $\Pr_{\beta J}[x_i \mid x_{-i}]$ is the probability of $\Pr_{\beta J}$ to draw $x_i$ conditioned on the remaining entries of $x$, denoted $x_{-i}$. This is a convex function in the parameters $\beta$ which can be optimized using an appropriate first-order optimization technique to find an optimum $\hat{\beta}$.

Consistency of this estimator can then be proven by bounding

$$\frac{|\phi'(x; \beta^*)|}{\max_{\beta \in [\beta^*, \hat{\beta}]} \phi''(x; \beta)}.$$ (4)

Indeed, it is easy to see that

$$|\hat{\beta} - \beta^*| \max_{\beta \in [\beta^*, \hat{\beta}]} \phi''(x; \beta) \leq |\phi'(x; \hat{\beta}) - \phi'(x; \beta^*)| = |\phi'(x; \beta^*)|,$$

where we used that $\hat{\beta}$ is an optimizer of $\phi(x; \beta)$. Hence, if the above ratio is small enough, we get our desired result.

We now turn to the specific challenges encountered when trying to bound this ratio. The derivative $\phi'(x; \beta^*)$ takes the form

$$\phi'(x; \beta^*) = \sum_{i=1}^n \phi_i'(x; \beta^*) \quad \text{;} \quad \phi_i'(x; \beta^*) := -\frac{d}{d\beta} \log \Pr_{\beta J}[x_i \mid x_{-i}]|_{\beta = \beta^*}.$$ (5)

We notice that $\mathbb{E}[\phi_i'(x; \beta^*) \mid x_{-i}] = 0$, hence it suffices to show concentration of the derivative around its mean to obtain a good upper bound. However, tail bounds on the gradient from prior work do not lead us to the optimal bound on the ratio (4) in our setting. Instead, we use Lemma 1 to select a number of subsets $I_1, \ldots, I_i$ of $[\tilde{n}]$, such that conditioned on $x_{-I_i}, x_{I_i}$ satisfies
Dobrushin’s condition. The lemma also guarantees that each \( i \in [n] \) belongs to \( \ell' \) different subsets \( I_j \) where \( \ell' \) is a constant fraction of \( \ell \), which means we can write

\[
|\phi'(x; \beta^*)| = \left| \sum_{i=1}^{n} \phi'_i(x; \beta^*) \right| \leq \frac{\ell}{\ell'} \sum_{j=1}^{\ell} \sum_{i \in I_j} \phi'_i(x; \beta^*) \leq \frac{\ell}{\ell'} \max_{j \in [\ell']} \sum_{i \in I_j} \phi'_i(x; \beta^*) \leq O \left( \max_{j \in [\ell']} \left| \sum_{i \in I_j} \phi'_i(x; \beta^*) \right| \right).
\]

(5)

Hence, it suffices to bound each one of the terms that appear in the max. In fact, since each term \( \sum_{i \in I_j} \phi'_i(x) \) has zero mean conditioned on \( x_{-I_j} \), it suffices to show that it concentrates around its mean conditioned on \( x_{-I_j} \). Given that conditioning on \( x_{-I_j} \), \( x_{I_j} \) satisfies Dobrushin’s condition, we can use the concentration inequality from Informal Theorem 4 to derive that

\[
\left| \sum_{i \in I_j} \phi'_i(x; \beta^*) \right| \leq O \left( \| \mathbb{E} [Jx \mid x_{-I_j}] \|_2 + \| J \|_F \right),
\]

with high probability. Applying (5) and union bounding over \( j \in [\ell] \), we deduce that with high probability,

\[
|\phi'(x; \beta^*)| \leq O \left( \max_{j \in [\ell]} \| \mathbb{E} [Jx \mid x_{-I_j}] \|_2 + \| J \|_F \right).
\]

(6)

We now show a lower bound on \( \phi''(x; \beta) \). Some simple calculations show that

\[
\phi''(x; \beta) \geq \| Jx \|_2^2.
\]

(7)

We then proceed by showing that: (a) its expectation is lower bounded appropriately; and (b) it concentrates around its expectation. Note that (7) reduces (a) to showing an expectation bound for a sum of squares of linear functions. This can also be phrased as a variance bound for linear functions of the Ising model, which is exactly the type of result that Informal Lemma 2 provides. Using it, we manage to prove that the expectation of the second derivative conditioned on \( x_{-I_j} \) is at least

\[
\mathbb{E} [\phi''(x; \beta) \mid x_{-I_j}] \geq \Omega \left( \| \mathbb{E} [Jx \mid x_{-I_j}] \|_2^2 + \| J \|_F^2 \right).
\]

(8)

By concentration of polynomials under Dobrushin’s condition [AKPS+19], we will show that \( \phi''(x; \beta) \) is at least the right hand side of (8) with high probability, and taking a union bound over \( j \in [\ell] \), we derive that

\[
\phi''(x; \beta) \geq \tilde{\Omega} \left( \max_{j \in [\ell]} \| \mathbb{E} [Jx \mid x_{-I_j}] \|_2^2 + \| J \|_F^2 \right).
\]

(9)

By (4), (6) and (9), we derive that \( |\hat{\beta} - \beta^*| \leq O(1/\| J \|_F) \) as required.

When we have \( k > 1 \), we assume that \( J^* \) lies in a \( k \) dimensional linear subspace of \( \mathbb{R}^{n \times n} \). A natural way of extending our approach to this setting is by using directional derivatives, which quantify how much \( \phi \) changes along a specific direction of the subspace. Similar to the approach when \( k = 1 \), we would like to show that in each direction the ratio of the first and second derivatives is small, with high probability. However, there are infinitely many directions to consider. Thus, the strategy of just bounding a single direction with high probability and taking the union bound does not suffice. Instead, we will construct a finite subset \( \mathcal{U} \) of these directions
such that every direction is $\epsilon$ close to one in $\mathcal{U}$ ($\mathcal{U}$ forms an $\epsilon$-net). By a union bound over $\mathcal{U}$ we prove that with high probability, the ratio between the derivatives is bounded for all the directions in $\mathcal{U}$. Since the derivatives are Lipschitz as a function of the direction, this suffices to argue that the ratio is bounded for all directions. We note that union bounding over $|\mathcal{U}| = (1/\epsilon)^k$ events increases the error by a multiplicative factor of $\sqrt{k \log(1/\epsilon)}$. This completes the proof for $k > 1$.

1.2.3 Comparison of the Proof Technique to Prior Work

Our approach uses the conditioning technique (Informal Lemma 1) to obtain sharper bounds compared to prior techniques. To explain why this conditioning is crucial, we note that prior work \cite{DDP19, BM18, Cha07, GM18} aims at identifying values $L, \mu$ that depend only on $n$ and on other problem parameters, such that $|\varphi'(x; \beta^*)| \leq L$ and $\varphi''(x; \beta) \geq \mu$, and then they derive the rate $L/\mu$. Here, instead, we bound the ratio between the derivatives in a joint manner, obtaining both bounds as a function of the conditioned entries of $x$. Indeed, it is not clear whether an approach that targets separate absolute bounds for the first and second derivatives can yield optimal estimation bounds without the assumptions made in the statements of prior work, since functions of general Ising models tend not to concentrate, and this issue is resolved if we ensure Dobrushin by conditioning.

The techniques for bounding the first and second derivatives are completely different than those appearing in prior work: there, $\varphi'(x; \beta^*)$ is bounded using Chatterjee’s exchangeable pair technique \cite{Cha05}, and $\varphi''(x; \beta)$ is bounded by specifically tailored arguments. Here, we use strong concentration results under Dobrushin.

Lastly, \cite{DDP19, GM18} address estimation of Ising models with external fields, where they target estimation of one interaction parameter $\beta$ and one or several external field parameters. (For further details of what they are estimating see Section 1.3). They follow a similar approach as \cite{Cha05, BM18}. In particular, they lower-bound the Hessian of the negative log-pseudolikelihood to prove it is strongly convex, and they separately upper-bound the norm of its gradient at the true parameters. Lower bounding the Hessian boils down to showing anti-concentration for a scalar quantity in their setting. In contrast, we have a more complex setting with multiple interaction parameters. This makes a union bound over $\exp(k)$ many directional derivatives along with a single-dimensional comparison of first and second derivatives on each direction more appropriate. This requires deriving high probability guarantees in each direction, while prior work was only concerned with constant probability bounds.

1.2.4 Possible Applications of our Techniques

Dobrushin’s condition can be defined for general distributions rather than just Ising models (see Appendix A), and our conditioning trick (Informal Lemma 1) can be trivially extended. Further, we believe that our techniques can be used more generally to prove convergence of the MPLE for general multi-dimensional distributions. For instance, it will be possible to derive a bound on the derivative in a similar way: each summand in the decomposition (5) is zero mean conditioned on $x_{-I}$ regardless of the distribution of $x^I$, so it remains to show that it conditionally concentrates, which might be significantly easier than arguing about unconditional concentration.

\footnote{This holds under standard regularity assumptions.}
Lastly, note that by conditioning we can re-prove concentration inequalities outside of Dobrushin, that were previously proven using the exchangeable pairs technique [Cha05], while also allowing more flexibility in combining with results that hold under Dobrushin.

1.3 Detailed Comparison to Prior Work

The first to study estimation of $\beta$ from one sample is Chatterjee [Cha07], who considered the case $k = 1$ and proved a rate of $O(1/\sqrt{n})$ when the partition function is of maximal order, $F(\beta^* J) \geq \Omega(n)$, and when additionally $\|\beta^* J\|_2 = O(1)$. Bhattacharya and Mukherjee [BM18] extend this result, proving a rate of $F(\beta^* J)^{-1/2}$ under additional assumptions: one is a Frobenius norm upper bound on $J$ and an important one is that $F((\beta^* + \delta) J)/F((\beta^* - \delta) J)$ remains constant for some constant $\delta > 0$. This last condition makes their bound non-applicable in the vicinity of certain phase transitions, where the behavior of the partition function abruptly changes. Notice that many Ising models undergo such phase transitions at specific values of $\beta$. For example, in their application for learning Erdos Renyi random graphs and regular graphs, guarantees were given only when $\beta^*$ is bounded away from 1, an assumption removed here. As another application of their main statement, they show a rate of $\|J\|_F^{-1}$ under the assumption that $J$ has only positive entries and $\|\beta^* J\|_2 < 1$, which is a generalization of Dobrushin’s condition discussed in Section A.

Ghosal and Mukherjee [GM18] were the first to study estimation of more than one parameter from one sample, in the Ising model with external fields $\theta$, where the density is $\propto \exp(\beta x^T J x/2 + \theta \cdot \sum_i x_i)$. They proved consistency assuming that some quantity $T_n(x)$ that quantifies the strength of correlations is sufficiently large and showed that it suffices to assume that $\|J\|_F \geq \Omega(\sqrt{n})$. Extending that work, Daskalakis et al. [DDP19] studied logistic regression from dependent samples: given as input multiple observations $(z_i, x_i)_{i=1}^n \in (\mathbb{R}^d \times \{-1, 1\})^n$, and assuming that conditioning on the $z_i$’s the $x_i$’s were sampled from a density that is $\propto \exp(\beta x^T J x/2 + \sum_i (\theta^T z_i) x_i)$, they showed consistent estimators for $\beta$ and $\theta$, assuming $\|J\|_F \geq \Omega(\sqrt{n})$ and under some additional assumptions about the design matrix defined by the $z_i$’s. In particular, Daskalakis et al. [DDP19] estimate a single interaction parameter $\beta_i$ along with a coefficient vector $\theta$ that projects the features $z_i$ of each observation to an external field $\theta^T \cdot z_i$ for that observation.

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2 Estimating the Parameters

In this section, we show how Theorem 1 can be used to derive rates on the estimated parameters. Notice that the ratio $\lambda := \inf_{\beta \neq 0} \|\sum_i \beta_i J_i\|_F / \|\beta\|_2$, which corresponds to the minimal singular value of the operator $\beta \mapsto \sum_i \beta_i J_i$ between the Hilbert spaces $(\mathbb{R}^d, \|\cdot\|_2)$ and $(\mathcal{M}_{n \times n}(\mathbb{R}), \|\cdot\|_F)$,
captures the following guarantee in estimating $\beta$: $\|\beta^* - \hat{\beta}\|_2 \leq \|\hat{f} - f^*\|_F / \lambda$. It is therefore desirable that $\lambda$ be large.

A simple setting where this holds is when the $J_i$’s are orthogonal with respect to the trace inner product, namely, trace$(J_i^T J_j) = 0$ for $i \neq j$. In this case, $\lambda = \min_i \|J_i\|_F$, and if all matrices have $\|J_i\|_F \geq \Omega(\sqrt{n})$ then the rate is $\|\hat{\beta} - \beta^*\| \leq \sqrt{k \log n / n}$. Note that orthogonality holds when the matrices have a disjoint support. For an example, assume that each $x_t$ represents a person which resides in one of $\ell$ countries and $k = \ell + \binom{\ell}{2}$: each $J_i$ either represents the connections within a specific country or connections between people across two specific countries. The matrices $J_i$ have disjoint support, hence they are orthogonal. For another example of disjoint supports, assume that we are taking snapshots of a network $(V, E)$ across $T$ different time steps, where $n = |V|T$ and $x_{v,t}$ corresponds to the snapshot of node $v \in V$ at time $t \in [T]$. We have two interaction matrices, $J_1$ which corresponds to temporal interactions: $(J_1)_{(v,t),(u,s)} = 1(u = v \wedge t - s = 1)$ and $J_2$ which corresponds to spatial interactions: $(J_2)_{(v,t),(u,s)} = 1((u, v) \in E \wedge t = s)$. These matrices clearly have disjoint supports.

For an example where estimation of $\beta$ is possible without orthogonality, suppose that each $J_i$ is the incidence matrix of a graph $(|n|, E_i)$, and define $\lambda_i = |E_i \setminus \bigcup_{j \in [i]} E_j|$, namely, $\lambda_i$ is the number of connections unique to network $i$. Then, Theorem 1 guarantees

$$\sum_{i} \lambda_i (\hat{\beta}_i - \beta_i^*)^2 \leq \sqrt{k \log n}; \quad \text{and} \quad \|\hat{\beta} - \beta^*\|_2 \leq \sqrt{k \log n / \min_i \lambda_i},$$

Hence, we can accurately learn the parameters of networks with many unique connections.

## 3 Preliminaries

This section establishes the notational conventions used and presents some background definitions used throughout the paper. We start with the notational conventions.

### Standard notations and definitions.

- **Sets of indices**: denote by $[n] = \{1, \ldots, n\}$. Given $I \subseteq [n]$, let $-I := [n] \setminus I$, and given $i \in [n]$ let $-i = -\{i\} = [n] \setminus \{i\}$.

- **Indexed vectors and matrices**: Given a vector $a = (a_1, \ldots, a_n)$ and a subset $I \subseteq [n]$, let $a_I$ denote the $|I|$ coordinate vector $\{a_i : i \in I\}$. Similarly, for a matrix $A$ of dimension $n \times m$, $I \subseteq [n]$ and $I' \subseteq [m]$, let $A_{I,I'}$ denote the corresponding submatrix. Similarly, let $A_I := A_{I,n}$ and $A_{I} = A_{[n]I}$, and let $A_i := A_{\{i\}}$.

- **Standard mathematical sets**: Let $S^{k-1}$ denote the $k$-dimensional unit sphere, $\{x \in \mathbb{R}^k : \|x\|_2 = 1\}$. And given $m, n > 0$ integers, let $M_{n \times m}(\mathbb{R}) := M_{n \times m}$ denote the space of real matrices of dimension $n \times m$.

- **Ising model distributions**: Given a symmetric matrix $J$ with zeros on the diagonal, let $\Pr_J$ denote the Ising model with interaction matrix $J$, defined as in (1). We say that a random variable $x$ with interaction matrix $J$ and external field $h$ is $(M, \gamma)$-bounded, if $\|J\|_\infty \leq M$.

\footnote{Notice that $\|J\|_F$ is always upper bounded by $O(\sqrt{n})$ if $\|J\|_\infty = O(1)$.

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Theorem 1. Let $\var(x_i|x_{-i}) \geq \gamma$ with probability 1. In other words, if for all $x$ and all $i$, $\Pr[x_i = 1|x_{-i}](1 - \Pr[x_i = 1|x_{-i}]) \geq \gamma$.

- Absolute constants: We let the notations $C, C', C_1, \ldots$ denote constants that depend only on $M$ and $\gamma$, and are bounded whenever $M$ is bounded from above and $\gamma$ from below (unless the dependence on $\gamma$ and $M$ is stated explicitly).

- Conditional variance: given two random variables $X$ and $Y$, define $\Var[X|Y] := \mathbb{E}_X[(X - \mathbb{E}[X|Y])^2|Y]$. Since the conditional expectation $\mathbb{E}[X|Y]$ is a random variable which is a function of $Y$, so is $\Var[X|Y]$.

Matrix norms. Given a real matrix $A$ of dimension $m \times n$, let $\|A\|_2^2 = \sum_{ij} A^2_{ij}$ denote the Frobenius norm, let

$$\|A\|_2 = \max_{u \in \mathbb{R}^n \setminus \{0\}} \frac{\|Au\|_2}{\|u\|_2}$$

and let

$$\|A\|_\infty = \max_{u \in \mathbb{R}^n \setminus \{0\}} \frac{\|Au\|_\infty}{\|u\|_\infty} = \max_{i=1,\ldots,m} \sum_{j=1}^n |A_{ij}|.$$ 

The following inequalities are known: $\|A\|_2 \leq \|A\|_F$ and $\|A\|_2 \leq \|A\|_\infty$.

Dobrushin’s condition. Next we define a variant of Dobrushin’s uniqueness condition (high-temperature condition) for Ising models that we use. The more general form of the condition is presented in Section A.

Definition 1 (Dobrushin’s condition). Given an Ising model $x$ with interaction matrix $J$, we say that it satisfies Dobrushin’s condition if $\|J\|_\infty < 1$, where $\alpha := \|J\|_\infty$ is called the Dobrushin’s coefficient.

Optimization over a vector space $\mathcal{V}$. Notice that replacing the interaction matrices $J_1, \ldots, J_k$ with other matrices $J'_1, \ldots, J'_k$ that span the same linear subspace of $M_{n \times n}(\mathbb{R})$ does not change the studied problem. Hence, we will forget about $J_1, \ldots, J_k$ and replace them with their span $\mathcal{V}$.

4 Analyzing the Maximum Pseudo-Likelihood Estimator

This section contains the proof of our main result which is the following theorem.

Theorem 1. Let $\mathcal{V}$ denote a subspace of $M_{n \times n}(\mathbb{R})$ of dimension $k \geq 1$ containing symmetric matrices with zero diagonals, fix $M > 0$, let $\mathcal{J} := \{J \in \mathcal{V}: \|J\|_\infty \leq M\}$ and fix $\delta > 0$. There exists an algorithm that runs in time $\text{poly}(n)$, that given one sample $x \sim \Pr[\mathcal{V}]$, where $J^* \in \mathcal{J}$, outputs $\hat{J} = \hat{J}(x)$, such that with probability $1 - \delta$:

$$\|J^* - \hat{J}\|_F \leq C(M) \sqrt{k \log n + \log(1/\delta)},$$

where $C(M) > 0$ is bounded when $M$ is bounded.

Theorem [1] guarantees that we can find a matrix $\hat{J}$ that is $\tilde{O}(\sqrt{k})$ close to the true interaction matrix $J^*$ in Frobenius norm. Notice that both matrices can have a Frobenius norm as high as $\sqrt{n}$, which makes this a non-trivial guarantee. Section 4.1 contains an overview of the proof, while the main lemmas are presented in the following sections.
4.1 Overview of the proof

The algorithm used to estimate $f^*$ will be the maximum pseudo-likelihood estimator (MPLE), as mentioned in Section 3:

$$\arg \max_{J \in \mathcal{J}} \log \text{PPL}(J; x) := \arg \max_{J \in \mathcal{J}} \prod_{t \in [n]} \Pr[x_i | x_{-i}].$$  \hspace{1cm} (11)

In fact, the above maximization problem is concave and we will be able to find an approximate solution using first-order methods. We will address the question of how to do this efficiently in Section 4.4. For convenience in calculations, the function we will actually optimize is the negative log pseudo-likelihood:

$$\varphi(J) := -\log \text{PPL}(J; x) = \sum_{i=1}^n (\log \cosh(J x) - x_i J x + \log 2).$$  \hspace{1cm} (12)

A standard approach for showing consistency of the MPLE is by showing high probability upper bounds on the first derivatives of $\varphi$ combined with a high probability lower bound on the smallest eigenvalue of the Hessian. To formalize this intuition in our setting, we begin by reviewing the definition of differentiation with respect to a matrix.

**Definition 2.** If $f : M_{n \times n}(\mathbb{R}) \to \mathbb{R}$ is a twice continuously differentiable function and $A \in M_{n \times n}(\mathbb{R}) \setminus \{0\}$, we define for all $J \in M_{n \times n}(\mathbb{R})$

$$\frac{\partial f(J)}{\partial A} = \lim_{t \to 0} \frac{f(J + tA) - f(J)}{t} = \frac{d f(J + tA)}{dt} \bigg|_{t=0}.$$

In the case of the MPLE, some simple calculations show that

$$\frac{\partial \varphi(J)}{\partial A} = \frac{1}{2} \sum_{i=1}^n (A_i x) (\tanh(J x) - x_i); \quad \frac{\partial^2 \varphi(J)}{\partial A^2} = \frac{1}{2} \sum_{i=1}^n (A_i x)^2 \text{sech}^2(J x).$$  \hspace{1cm} (13)

If we had the usual partial derivatives for a function in $\mathbb{R}^n$, we would try to bound the gradient and Hessian of this function. In our case, a similar strategy is to bound the first and second derivatives of $\varphi$ with respect to matrices $A \in M_{n \times n}(\mathbb{R})$ of unit norm, namely, those residing in $\mathcal{A} := \{A \in M_{n \times n}(\mathbb{R}): ||A||_F = 1\}$, as defined in Section 3. Specifically, we would like a high probability upper bound on the derivative of $\varphi$ at the true matrix $J^*$ and a lower bound on the second derivative of $\varphi$ at any point. This is exactly the content of the following Lemma, which is of central importance to the proof.

**Lemma 1.** Let $x$ be an ising model with interaction matrix $J^* \in \mathcal{J}$ and zero external field. Fix $\xi \geq 1$ and $t \in (0, \sqrt{n}]$. Define $\mathcal{A}_\xi := \{A \in \mathcal{A}: ||A||_2 \leq 1/\xi\}$. Then, with probability at least $1 - (Cn)^k \exp(-c \min(t^2, \xi^2))$, for all $A \in \mathcal{A}_\xi$ and all $J \in \mathcal{J}$, the following holds:

$$\frac{\partial^2 \varphi(J)}{\partial^2 A} \geq \epsilon; \quad \text{and} \quad \left| \frac{\partial \varphi(J^*)}{\partial A} \right| \leq C t \min_{J \in \mathcal{J}} \frac{\partial^2 \varphi(J)}{\partial A^2}. \hspace{1cm} (14)$$

We will break Lemma 1 into two separate Lemmas involving the first and second derivative. Notice that the high probability bounds of Lemma 1 hold for any $A \in \mathcal{A}_\xi$, where the appropriate
value for $\xi$ will be specified later. Note that we consider only $A_\xi$ instead of the whole set $A$, since the derivatives with respect to $A \in A$ concentrate better when the spectral norm of $A$ is lower. Another characteristic of Lemma 1 is that it does not give a uniform bound for the derivative, but instead it depends on the strong convexity. Since we only need the ratio of the two derivatives to be small, this does not present any difficulties in the proof. However, one cannot help but wonder if we could bound the derivative by a constant quantity. We provide some more insight regarding this phenomenon at the end of this section.

We will now show that a bound on these ‘good’ directions suffices to obtain the result we need. Using Lemma 1, we can prove that if we were to execute the optimization algorithm that returns an approximate solution $\hat{J}$ to (11), then $\hat{J}$ would be close to the true interaction matrix $J^*$ with high probability.

**Lemma 2.** Fix $\delta > 0$. With probability at least $1 - \delta$, for any $J \in J$ that satisfies $\varphi(J) \leq \varphi(J^*) + k$, we have that

$$\|J - J^*\|_F \leq C \left( \sqrt{k \log n} + \sqrt{\log(1/\delta)} \right).$$

We defer the proof to Subsection 4.5. It uses the standard technique of bounding the error in terms of the ratio of the derivative with the strong convexity, taking into account the approximation error of the optimization procedure.

Furthermore, we need to show that an approximate solution to (11) can be found in polynomial time. This is achieved via gradient descent, together with a regularizer that ensures that the algorithm returns a solution with small infinity norm. We note that the running time is not optimized.

**Lemma 3.** There exists an algorithm that outputs $\hat{J}$ satisfying $\hat{J} \in V$ and $\|\hat{J}\|_\infty \leq 2M$, such that

$$\varphi(\hat{J}) \leq \varphi(J^*) + \epsilon$$

and runs in time $O(k^2 n^6 M^2 / \epsilon^2)$.

The regularizer should be large enough to exclude solutions with large infinity norm, but also small enough to allow solutions with infinity norm up to $M$. The full proof is given in Section 4.4.

Lemma 2 together with Lemma 3 conclude the proof of Theorem 1. Of course, Lemma 2 uses Lemma 1. Therefore, we proceed with a discussion of the main results required for proving Lemma 1. First, we present a simple but powerful lemma that converts Ising models $P_{\mathcal{J}}$ with $1 < \|J\|_\infty \leq C$ to models satisfying Dobrushin’s condition, namely $\|J\|_\infty < 1$ by conditioning on a set of nodes. While in the first setting standard concentration inequalities are not guaranteed to hold and fundamental quantities of the distribution such as the partition function are generally hard to approximate, the second setting resembles the i.i.d. scenario. While this reduction is simple and has many limitations, it suffices to show that the learning rate obtained in the constant influence regime is at least as good as the optimal rate achievable under Dobrushin’s condition. This optimality is made precise in Section 8.

**Lemma 4.** Let $x = (x_1, \ldots, x_n)$ be an $(M, \gamma)$-Ising model, and fix $\eta \in (0, M]$. Then, there exist subsets $I_1, \ldots, I_\ell \subseteq [n]$ with $\ell \leq CM^2 \log n / \eta^2$ such that:
1. For all $i \in [n],$

$$|\{j \in [\ell] : i \in I_j\}| = \left\lceil \frac{\eta \ell}{8M} \right\rceil .$$

2. For all $j \in [\ell]$ and any value of $x_{-I_j}$, the conditional distribution of $x_{I_j}$ conditioned on $x_{-I_j}$ is an $(\eta, \gamma)$-Ising model.

Furthermore, for any non-negative vector $\theta \in \mathbb{R}^n$ there exists $j \in [\ell]$ such that

$$\sum_{i \in I_j} \theta_i \geq \frac{\eta}{8M} \sum_{i=1}^n \theta_i.$$ 

The proof of Lemma 4 is presented in Section 5. It is a simple application of the probabilistic method and will be used many times throughout this work.

We now present the two lemmas that directly imply Lemma 1. The concentration bounds are stated in terms of sets $I_j$ that are selected as in Lemma 4 with $\eta = 1/2$. By conditioning on $x_{-I_j}$ for $j \in [\ell]$, we will be able to derive the following guarantees:

**Lemma 5.** Fix $\xi \geq 1$, $t \in [1, \sqrt{n}]$. Let $I_1, \ldots, I_\ell$ be the subsets obtained by Lemma 4 for $\eta = 1/2$. Then, with probability at least $1 - \log n (Cn)^{k-1} e^{-\min(t, \eta \xi)}$ over $x$, for all $A \in \mathcal{A}_\xi$,

$$\frac{\partial \varphi(J^*)}{\partial A} \leq C t \left( 1 + \max_{j \in [\ell]} \left\| \mathbb{E}[Ax | x_{-I_j}] \right\|_2 \right). \quad (15)$$

**Lemma 6.** Fix $\xi > 0$. Let $I_1, \ldots, I_\ell$ be the subsets obtained by Lemma 4 for $\eta = 1/2$. Then, with probability at least $1 - \log n (Cn)^{k-1} e^{-\xi^2}$, for any $A \in \mathcal{A}_\xi$ and any $J \in \mathcal{J}$:

$$\frac{\partial^2 \varphi(J)}{\partial A^2} \geq C \left( 1 + \max_{j \in [\ell]} \left\| \mathbb{E}[Ax | x_{-I_j}] \right\|_2^2 \right). \quad (16)$$

It is clear that these two lemmas combined directly imply Lemma 1. Notice that the derivative is upper bounded by a value which is not constant, but it is rather bounded in terms of conditional expectations with respect to $x_{-I_j}$. Similarly, the second derivative is lower bounded in terms of the same quantity. Since the rate is a function of the ratio between the first and the second derivative, such a non-constant bound suffices to derive Theorem 1. Furthermore, one might not, in general, replace these bounds with constant bounds, since the term $\max_{j \in [\ell]} \|\mathbb{E}[Ax | x_{-I_j}]\|_2$ might not concentrate outside of Dobrushin’s condition, and, $\partial \varphi(J^*) / \partial A$ and $\partial^2 \varphi(J) / \partial A^2$ will fluctuate along with it. Next, we present the proofs of Lemmas 5 and 6.

**Section organization.** Section 4.2 contains the proof of Lemma 5 that provides an upper bound for the derivative of the PMLE. In Section 4.3 we present the bound on the second derivative of the PMLE. In Section 4.4 we describe the algorithm used and prove Lemma 3 regarding its performance. Lastly, in Section 4.5 give a proof of Lemma 2 that connects the bounds on the derivatives with the estimation rate. Notice that Lemma 4 will be presented in Section 5.
4.2 Bounding the Derivative of Log Pseudo-Likelihood

The central goal of this section is to prove Lemma 5. The general form of the derivative is a sum of tanh functions. The first thing one notices is that its mean is 0. Hence, it is enough to prove a strong enough concentration bound for this quantity. There are many challenges with this approach, which we will now present.

First, most concentration bounds that are known hold for Ising models that satisfy Dobrushin’s condition. However, our model may or may not be in this state. Thus, we utilize Lemma 4 to find a small number of sets $I_j \subseteq [n]$, such that each $i \in [n]$ belongs to a small number of these sets. This allows us to write the derivative sum as a sum over all the sets $I_j$, thus focusing on the behavior of the function in each set. Lemma 4 also guarantees that if we condition on $x_{-I_j}$, the resulting Ising model will satisfy Dobrushin’s condition. Hence, the problem reduces to bounding the terms of the sum belonging to $I_j$ when we condition on $x_{-I_j}$.

This brings us to the next challenge, which involves the concentration bound. Most of the existing results on concentration inequalities for the Ising model focus on the case where the function is a multilinear polynomial. Since this is not the case for the derivative, we introduce a simple modification to the already existing techniques so that we obtain the desired concentration in our case. This will give us a bound that depends on the conditional expectation of a quadratic form conditioned on $x_{-I_j}$. This might seem insufficient at first, since we are not getting a uniform bound for the derivative, but rather one that depends on the values of $x_{-I_j}$. However, an analogous lower bound is proven for the strong convexity in Lemma 6. Hence, the two bounds match in all instantiations of $x_{-I_j}$, allowing us to complete the proof for the concentration of the derivative in a single direction $A \in A_\xi$.

Finally, we need this bound to hold for all directions $A \in A_\xi$ uniformly. The way we chose $A_\xi$ allows us to obtain a bound for each direction that holds with probability proportional to $n^{-2\xi}$. Hence, by taking a union bound over a set of directions of cardinality $n^{\xi}$ we still obtain a high probability concentration bound for all of them. Since we would like this bound to hold for all directions in $A_\xi$, we should choose this set so that it covers all the possible directions in this set. A natural candidate for such a set is an $\epsilon$-net of $A_\xi$. Intuitively, it is a set of points such that every element of $A_\xi$ is close to at least one of them. By taking a union bound over this set, we ensure that the concentration bound will hold for all directions, thus completing the proof.

4.2.1 Proof of Lemma 5

First, we decompose the derivative according to terms corresponding to the sets $I_1, \ldots, I_\ell$ as specified by Lemma 4 for $\eta = 1/2$. Recall that each element $i \in [n]$ appears in exactly $\ell' = \lceil \eta \ell / 8 \rceil$ sets:

$$\frac{\partial \phi(I^*)}{\partial A} = \frac{1}{2} \sum_{i \in [n]} A_i x_i (x_i - \tanh(J_i^* x)) \leq \frac{1}{\ell'} \sum_{j \in [\ell]} \sum_{i \in I_j} A_i x_i (x_i - \tanh(J_i^* x)) := \frac{1}{\ell'} \sum_{j \in [\ell]} |\psi_j(x; A)|. \quad (17)$$

We will bound the terms $\{\psi_j(x; A) : j \in [\ell]\}$ separately, and show that each term concentrates around zero. In order to do so, we will show that conditioned on any value for $x_{-I_j}$, $\psi_j(x; A)$ concentrates around zero, while the radius of concentration can depend on the specific value of $x_{-I_j}$. First, we would like to claim that this term is conditionally zero mean:
Claim 1. For any \( j \in [\ell] \), \( \mathbb{E} \left[ \psi_j(x; A) \Big| x_{-I_j} \right] = 0 \).

Proof. First, fix \( i \in I_j \), and notice that since \( A \) and \( J^* \) have zeros on the diagonal, both \( A_i x \) and \( J_i x \) are constant conditioned on \( x_{-i} \), hence
\[
\mathbb{E}[A_i x(x_i - \tanh(J_i x))]_{x_{-i}} = A_i x(\mathbb{E}[x_i|x_{-i}] - \tanh(J^*_i x)) = 0,
\]
where the last equality follows from definition of the Ising model. Next, notice that
\[
\mathbb{E} \left[ \psi_j(x; A) \Big| x_{-I_j} \right] = \sum_{i \in I_j} \mathbb{E} \left[ A_i x(x_i - \tanh(J^*_i x)) \Big| x_{-I_j} \right] = \sum_{i \in I_j} \mathbb{E}_{x_i} \left[ A_i x(x_i - \tanh(J^*_i x)) \mid x_{-i} \right] = 0.
\]

\( \square \)

Next, we will bound the radius of concentration of \( \psi_j(x; A) \) conditioned on \( x_{-I_j} \), and show that it is roughly bounded by \( O \left( 1 + \left\| \mathbb{E} \left[ A x \mid x_{-I_j} \right] \right\|_2 \right) \). Additionally, we derive concentration inequalities for the above term. In order to achieve this task, we utilize Lemma 4, which states that \( x_{I_j} \) is conditionally Dobrushin, conditioned on \( x_{-I_j} \). Hence, we can utilize concentration inequalities for Dobrushin random variables, and achieve the following bound:

**Lemma 7.** For any symmetric matrix \( A \) with zeros on the diagonal, we have
\[
\Pr \left[ |\psi_j(x; A)| \geq t \mid x_{-I_j} \right] \leq \exp \left( -c \min \left( \frac{t^2}{\left\| \mathbb{E} \left[ A x \mid x_{-I_j} \right] \right\|^2_2 + \|A\|_F^2, \|A\|_2^2} \right) \right).
\]

Most existing concentration bounds for the Ising model focus on multilinear polynomials, which is obviously not the form \( \psi_j \) has. Hence, in order to prove this bound, we modified slightly the existing proof of Theorem 2 to present a concentration inequality for general functions of ising models satisfying Dobrushin’s condition, presented in Section 6. The concentration radius we get depend on bounds of the first and second discrete derivatives of the function. The full proof of Lemma 7 can be found in Section B.1.

Combining Lemma 7 with (17), one could already derive a concentration bound on \( \text{partialvarphi}(J^*)/\partial A \):

**Lemma 8.** For any \( A \) and \( t > 0 \),
\[
\Pr \left[ \left\| \frac{\partial \varphi_j(J^*)}{\partial A} \right\| \geq t \left( \|A\|_F + \max_{j} \left\| \mathbb{E}[Ax|x_{-I_j}] \right\|_2 \right) \right] \leq C \log n \exp \left( -c \min \left( t^2, \frac{t \|A\|_F}{\|A\|_2} \right) \right).
\]

Proof. For any \( j \in [\ell] \), Lemma 7 implies that
\[
\Pr \left[ |\psi_j(x; A)| \geq t \left( \|A\|_F + \left\| \mathbb{E}[Ax|x_{-I_j}] \right\|_2 \right) \right] \leq \exp \left( -c \min \left( t^2, \frac{t \|A\|_F}{\|A\|_2} \right) \right).
\]

By a union bound over \( j \in [\ell] \), with probability at least \( 1 - C \log n \exp(-c \min(t^2, t\|A\|_F/\|A\|_2)) \),
\[
|\psi_j(x; A)| \leq t(\|A\|_F + \left\| \mathbb{E}[Ax|x_{-I_j}] \right\|_2).
\]
holds for all $j \in [\ell]$. By (17), whenever this holds, we also have

$$\left| \frac{\partial \varphi(f^*)}{\partial A} \right| \leq \frac{1}{\ell'} \sum_{j \in [\ell]} |\psi_j(x; A)| \leq \ell \ell' t \left( \|A\|_F + \max_{j \in [\ell]} \|\mathbb{E}[Ax|x_{-j}]\|_2 \right).$$

Since $\ell/\ell'$ is at most a constant, the proof is complete.

Lemma 8 is certainly a very desirable result for one direction $A$. However, we would like to prove a bound that holds for all directions $A \in A_\delta^c$ at the same time. To prove that, we will utilize the following simple and well-known, but powerful technique. We begin with a definition:

**Definition 3.** Given a metric space $(\mathcal{U}, d)$ and $\epsilon > 0$ we say that a set $\mathcal{N} \subseteq \mathcal{U}$ is an $\epsilon$-net for $\mathcal{U}$ if for all $u \in \mathcal{U}$ there exists $v \in \mathcal{N}$ such that $d(u, v) \leq \epsilon$.

Intuitively, if we take the union of balls with radius $r$ centered by the elements of the $\epsilon$-net, it covers the whole space. Hence, to bound the maximal deviation of $\psi_j(x; A)$ within $A$ we select a finite $\epsilon$-net of $(\mathcal{A}_\delta^c, \| \cdot \|_F)$ with respect to the distance $d(A, B) = \|A - B\|_F$. By a union bound, we show that with high probability, for all $A \in \mathcal{N}_\delta^c$, $|\psi_j(x; A)|$ is small. At this point, note that $\mathcal{A}_\delta^c$ was constructed to contain only elements with low spectral norm since they admit a better concentration than elements with large spectral norm. Hence, we have a factor of $e^{-t^2}$ in the probability bound. Therefore, the union bound over exponential directions does not increase the probability by a lot. After we apply it, since $\psi_j(x; A)$ is Lipschitz in $A$, it is easily implied that with high probability all $\{\psi_j(x; A) : A \in A_\delta^c\}$ are small at the same time. We formalize the above intuition in the next lemma:

**Lemma 9.** Let $\{Z_A\}_{A \in A_\delta^c}$ denote a collection of random variables. Assume that $|Z_A - Z_B| \leq r$ whenever $\|A - B\|_F < \epsilon$, and assume that for all $A \in A_\delta^c$,

$$\Pr[Z_A > 0] \leq \delta,$$

where $r, \epsilon, \delta > 0$. Then,

$$\Pr[\exists A \in A_\delta^c : Z_A > r] \leq \delta (3/\epsilon)^{k-1}.$$  

**Proof.** First, we claim that there exists an $\epsilon$-net $\mathcal{N}$ of $(\mathcal{A}_\delta^c, \| \cdot \|_F)$ of size at most $(C/\epsilon)^{k-1}$. We utilize the duality between packing and covering, starting with the following definition: given a metric space $(\mathcal{U}, d)$, a subset $\mathcal{M} \subseteq \mathcal{U}$ is called an $\epsilon$-separated set if all distinct $u, v \in \mathcal{U}$ satisfy $|u - v| \geq \epsilon$. It holds that an $\epsilon$-separated set of maximal size is also an $\epsilon$-net (otherwise, one could add more elements to the set while keeping it $\epsilon$-separated). Hence, to show that there exists an $\epsilon$-net of small size it suffices to bound the size of any $\epsilon$-separated set.

Getting back to the space $(\mathcal{A}_\delta^c, \| \cdot \|_F)$, notice that $(M_{n \times n}(\mathbb{R}), \| \cdot \|_F)$ is isometric to $(\mathbb{R}^{n^2}, \| \cdot \|_2)$, hence any $k$ dimensional linear subspace of $(M_{n \times n}(\mathbb{R}), \| \cdot \|_F)$ is isometric to $(\mathbb{R}^k, \| \cdot \|_2)$, hence $(\mathcal{A}_\delta, \| \cdot \|_F)$ is isomorphic to $(\mathcal{S}^{k-1}, \| \cdot \|_2)$, recalling that $\mathcal{A}$ is the intersection of a $k$-dimensional subspace of $M_{n \times n}(\mathbb{R})$ with the set of all matrices with unit Frobenius norm. By a volume argument, any $\epsilon$-separated set with respect to $(\mathcal{S}^{k-1}, \| \cdot \|_2)$ is of size at most $(3/\epsilon)^{k-1}$, hence, any $\epsilon$-separated set with respect to $(\mathcal{A}_\delta, \| \cdot \|_F)$ is of this size. In particular, since $A_\delta^c \subseteq A$, we derive that any $\epsilon$-separated set for $A_\delta^c$ is of size at most $(3/\epsilon)^{k-1}$, hence, by the duality discussed above, there exists an $\epsilon$-net $\mathcal{N}$ of size $(3/\epsilon)^{k-1}$.
To conclude the proof, by a union bound over \( \mathcal{N} \),

\[
\Pr[\forall A \in \mathcal{N}, \ Z_A \leq 0] \geq 1 - \delta(3/\epsilon)^{k-1}.
\] (18)

Whenever (18) holds, we have that for all \( A \in \mathcal{A} \) there exists \( B \in \mathcal{N} \) such that \( \|A - B\|_F \leq \epsilon \), hence by the assumption of this lemma,

\[
Z_A = Z_B + (Z_A - Z_B) \leq 0 + r = r,
\]

which concludes the proof. \( \square \)

**Remark 1.** Using the technique of chaining one can possibly improve the guarantee of Lemma 9 hence deriving better bounds that may remove the \( \sqrt{\log n} \) term in the rate obtained in Theorem 7. This may, however, require to modify some of the concentration inequalities in this paper.

We will apply Lemma 9 together with Lemma 8 to show that with high probability, all \( A \in \mathcal{A} \) satisfies:

\[
\left| \frac{\partial \varphi(J^*)}{\partial A} \right| < t \left( \|A\|_F + \max_j \|\mathbb{E}[Ax_j]\|_2 \right) + \epsilon.
\] (19)

Indeed, recall that (19) holds for any specific \( A \in \mathcal{A} \) with high probability by Lemma 8 and we can generalize for all \( A \in \mathcal{A} \) by applying Lemma 9 with

\[
Z_A = \left| \frac{\partial \varphi(J^*)}{\partial A} \right| - t \left( \|A\|_F + \max_j \|\mathbb{E}[Ax_j]\|_2 \right).
\]

The worst case Lipschitzness of \( Z_A \) as a function of \( A \) is \( O(n) \), which follows from simple calculations involving matrix norms. This will cost us a factor of \( O(n^k) \) in the union bound. This suffices to conclude the proof:

**Proof of Lemma 5** For \( A \in \mathcal{A} \), define

\[
Z_A = \left| \frac{\partial \varphi(J^*)}{\partial A} \right| - t \left( 1 + \max_j \|\mathbb{E}[Ax_j]\|_2 \right).
\]

To apply Lemma 9, we have to show that each \( Z_A \) is non-positive with large probability, and additionally, that \( Z_A \) is Lipschitz as a function of \( A \) for all \( x \). First, by applying Lemma 8 we derive that

\[
\Pr[|Z_A| > 0] \leq C \log n \exp(-c \min(t^2, t_\epsilon)) \nu,
\]

using the fact that \( \|A\|_2 \leq 1/\xi \) and \( \|A\|_F = 1 \). Next, we bound

\[
|Z_A - Z_B| \leq \left| \frac{\partial \varphi(J^*)}{\partial A} - \frac{\partial \varphi(J^*)}{\partial B} \right| + t \max_{j \in [t]} \|\mathbb{E}[Ax_j]\|_2 - \max_{j \in [t]} \|\mathbb{E}[Bx_j]\|_2
\]

\[
\leq \left| \frac{\partial \varphi(J^*)}{\partial A} - \frac{\partial \varphi(J^*)}{\partial B} \right| + t \max_{j \in [t]} \|\mathbb{E}[Ax_j]\|_2 - \|\mathbb{E}[Bx_j]\|_2
\]

\[
\leq \left| \frac{\partial \varphi(J^*)}{\partial A} - \frac{\partial \varphi(J^*)}{\partial B} \right| + t \max_{j \in [t]} \|\mathbb{E}[(A - B)x_j]\|_2,
\] (20)
using the triangle inequality. We bound the two terms at the right hand side of (20). First, 
\[
\left| \frac{\partial \varphi(J^*)}{\partial A} - \frac{\partial \varphi(J^*)}{\partial B} \right| \leq \sum_{i \in [n]} |(A_i - B_i)x(x_i - \tanh(J^*_i x))| \leq 2 \sum_{i \in [n]} |(A_i - B_i)x| \leq 2 \| (A - B)x \|_1 \\
\leq 2 \sqrt{n} \| (A - B)x \|_2 \leq 2 \sqrt{n} \| (A - B) \|_2 \|x\|_2 \leq 2n \| A - B \|_F.
\]
Secondly, for any \( j \in [\ell] \), 
\[
t \left\| E[Ax - Bx]_{x_\ell} \right\|_2 \leq t E[\| Ax - Bx \|_2]_{x_\ell} \leq t \max_{x \in \{-1,1\}^n} \| Ax - Bx \|_2 \\
\leq t \max_{x \in \{-1,1\}^n} \| A - B \|_2 \|x\|_2 \leq t \sqrt{n} \| A - B \|_F \leq n \| A - B \|_F,
\]
using \( \|x\|_2 = \sqrt{n} \), \( \| \cdot \|_2 \leq \| \cdot \|_F \) and the assumption in this lemma that \( t \leq \sqrt{n} \). Hence 
\[
|Z_A - Z_B| \leq 3n \| A - B \|_F.
\]
Applying Lemma \ref{lem:anticoncentration} with 
\[
\delta = \exp(-c \min(t^2, t^2 \xi)) ; \quad \epsilon = 1/(3n) ; \quad r = 1,
\]
we obtain that 
\[
\Pr \left[ \max_{A \in \mathcal{A}_i} Z_A > 1 \right] \leq C \log n(9n)^{k-1} e^{-c \min(t^2, t^2 \xi)}.
\]
This concludes the proof.

\[\square\]

### 4.3 Strong Convexity of Log Pseudo-Likelihood

The central goal of this section is to prove Lemma \ref{lem:strong-convexity}. We break it into two parts. Section 4.3.1 deals with the overall proof of the lemma, while Section 4.3.2 contains an auxiliary lemma utilized in the proof. The approach here is quite similar to the one for the derivative. However, the specific tools differ, since this is an anticoncentration result. The argument begins by lower bounding the second derivative by a quadratic form depending only on the direction \( A \) of the derivative, namely \( \| Ax \|_2^2 \). Then, the problem reduces to showing anticoncentration for this quadratic form. This will be accomplished by establishing two claims.

First, we show that the mean of this form is bounded away from 0 by a quantity that depends on the Frobenius norm of \( Ax \), conditional on the values \( x_{-j} \). This would trivially be true if the spins were independent. In the case of the Ising model, we use Lemma \ref{lem:good-set} to find a subset of the nodes that has a small Dobrushin constant. This means that these nodes will be weakly correlated conditional on the rest, hence close to independent. This translates to the covariance matrix being diagonally dominant, from which the claim follows.

Second, we need to show that \( \| Ax \|_2^2 \) concentrates around its mean in a radius that is of the same or less order that the mean, conditional on \( x_{-j} \). This concentration will be easier than the one obtained for the derivative, since we are now dealing with a quadratic function, to which known concentration results apply \cite{AKPS19}. Therefore, we have shown that the second derivative in a particular direction \( A \) is sufficiently large.

Finally, we need to show that the preceding anticoncentration holds for all directions \( A \in \mathcal{A}_g \) at the same time. This will require a very similar argument to the one used for the derivative. Thus, we consider an \( \epsilon \)-net that covers \( \mathcal{A}_g \) and take a union bound over all the elements in this set.
4.3.1 Proof of Lemma 6

The first step will be to lower bound the second derivative by a quadratic form. This way, the task of proving strong convexity becomes significantly easier. For all $J \in \mathcal{J}$ and $A \in \mathcal{A}$, we obtain that

$$
\frac{\partial^2 \varphi(J)}{\partial A^2} = \sum_{i=1}^{n} (A_i x)^2 \text{sech}^2(J x) \geq \sum_{i=1}^{n} (A_i x)^2 \text{sech}^2(\|J x\|_\infty)
$$

$$
= \|A x\|^2_2 \text{sech}^2(\|J x\|_\infty) \geq \|A x\|^2_2 \text{sech}^2(\|J\|_\infty \|x\|_\infty) \tag{21}
$$

$$
\geq \|A x\|^2_2 \text{sech}^2(M), \tag{22}
$$

where we used the facts that $\text{sech} x \geq \text{sech} y$ whenever $|x| \leq |y|$ and $\|J\|_\infty \leq M$ for all $J \in \mathcal{J}$. Notice that the bound we get only depends on the direction $A$ of the derivative and not on the point $J$ where we are calculating it. Hence, any bounds we manage to prove on this quantity will hold for all $J \in \mathcal{J}$. Our goal is to show that with high probability $\|A x\|^2_2$ is sufficiently large for all directions $A \in \mathcal{A}_z$.

The general strategy to showing this anticoncentration property will be to bound it’s expectation away from 0 and then show that the function concentrates well around that value. We now focus on the first goal. Since $\|A x\|_2^2$ is a sum of squares of linear functions, we proceed with a lower bound on the second moment of any linear function. What we need essentially is a variance lower bound. Therefore, we have the following lemma.

**Lemma 10.** Let $y$ be an $(M, \gamma)$-Ising model. Then, for any vector $a \in \mathbb{R}^m$,

$$
\text{Var}(a^\top x) \geq \frac{c \gamma^2 \|a\|^2_2}{M}.
$$

The proof is presented in Subsection 4.3.2. Here now give a short outline of it.

**Proof sketch.** First, let’s examine what would happen if all the coordinates of $x$ were independent, each coordinate having a variance of at least $\gamma$. Then,

$$
\text{Var}(a^\top x) = \sum_i \text{Var}(a, x_i) = \sum_i a_i^2 \text{Var}(x_i) \geq \gamma \|a\|^2_2.
$$

We would like to apply the same logic in our situation. Specifically, we would like the coordinates of $x$ to be as close to independent as possible. Therefore, we will use Lemma 2 to find a set of coordinates $I$ such that $x_I$ is an $(a, \gamma)$-Ising model conditioned on any value for $x_{-I}$, where $a = c \gamma$. In this regime, the interactions between the nodes are weak, resulting in the covariance matrix of $x_I$ conditioned on $x_{-I}$ being diagonally dominant. This allows us to derive the same variance bound as if $x_I$ were independent:

$$
\text{Var} [a^\top x \mid x_{-I}] = \text{Var} [a_i^\top x_I \mid x_{-I}] \geq c \gamma \|a_i\|^2_2.
$$

Lemma 4 guarantees that we can select the set $I$ such that $\|a_i\|^2_2 \geq a / (8M) \|a\|^2_2$, by substituting $\theta_i$ with $a_i^2$ in the lemma. Hence,

$$
\text{Var} [a^\top x \mid x_{-I}] \geq c \gamma \|a_i\|^2_2 \geq \frac{c' \gamma a \|a\|^2_2}{M} \geq \frac{c'' \gamma^2 \|a\|^2_2}{M}.
$$

Finally, since conditioning can only decrease the conditional variance on expectation, the same bound holds for $\text{Var} [a^\top x]$.
Now, we would like to apply Lemma 10 to obtain a lower bound for the expectation of $\|Ax\|_2^2$. A simple application of the Lemma shows that:

$$
\mathbb{E}[\|Ax\|_2^2] = \sum_i \mathbb{E}[(A;x)_i^2] \geq \sum_i (\text{Var}[(A;x)_i] + \mathbb{E}[A;x_i^2]) \\
\geq \sum_i \Omega(\|A_i\|_2^2) + \|\mathbb{E}[Ax]\|_2^2 = \Omega(\|A\|_F^2) + \|\mathbb{E}[Ax]\|_2^2.
$$

(23)

(24)

Recall, however, that our goal in Lemma 6 is to lower bound the second derivative of $\phi$ in terms of the conditional expectation of $Ax$, conditioned on $x_{-j}$, for $j \in [\ell]$. The following lemma presents a variant of (23) when we condition on $x_{-j}$. Its proof is an application of Lemma 4 along with some simple calculations with conditional expectations.

**Lemma 11.** Fix $A \in A$. For any $I_j$ and any $x_{-j}$, it holds that

$$
\mathbb{E}[\|Ax\|_2^2 \mid x_{-j}] \geq c\gamma^2 \|A_{-j}\|_F^2 + \mathbb{E}[Ax \mid x_{-j}]^2.
$$

Furthermore, there exists some $j \in [\ell]$ such that

$$
\mathbb{E}[\|Ax\|_2^2 \mid x_{-j}] \geq c\gamma^2 / M
$$

with probability 1.

**Proof.** Since conditioned on $x_{-j}$, $x_j$ is a $(1/2, M)$ Ising model, we derive by Lemma 10 that

$$
\mathbb{E}[\|Ax\|_2^2 \mid x_{-j}] = \sum_{i=1}^n \mathbb{E}[(A;x)_i^2 \mid x_{-j}] = \sum_{i=1}^n \left( \mathbb{E}[A_i|x_{-j}]^2 + \text{Var}[A_i|x_{-j}] \right) \\
= \left\| \mathbb{E}[Ax \mid x_{-j}] \right\|_2^2 + \sum_{i=1}^n \text{Var}[A_{i,j}] \geq \left\| \mathbb{E}[Ax \mid x_{-j}] \right\|_2^2 + \sum_{i=1}^n c\gamma^2 \|A_{i,j}\|_2^2 \\
\geq \left\| \mathbb{E}[Ax \mid x_{-j}] \right\|_2^2 + c\gamma^2 \|A_{-j}\|_F^2.
$$

To prove the second part of the lemma, it suffices to show that there exists $j \in [\ell]$ such that $\|A_{-j}\|_F^2 \geq c' \|A\|_F^2 / M = c' / M$. Recall that the sets $\{I_j\}_{j \in [\ell]}$ were obtained from Lemma 4 with $\eta = 1/2$. By the last part of this lemma, if we substitute $\theta_i = \|A_{i,j}\|_2^2$, we derive that there exists a $j \in [\ell]$ such that

$$
\|A_{i,j}\|_F^2 = \sum_{i \in I_j} \theta_i \geq \frac{c}{M} \sum_{i \in [n]} \theta_i = \frac{c\|A\|_F^2}{M} = \frac{c}{M},
$$

recalling that $\|A\|_F = 1$ for all $A \in A$. □

According to Lemma 11 if we can show that $\|Ax\|_2^2$ is concentrated at a radius of

$$
O \left( \|A_{i,j}\|_F^2 + \mathbb{E}[Ax \mid x_{-j}]^2 \right)
$$

around its mean, conditional on $x_{-j}$, then anticoncentration follows. The next Lemma contributes to the proof in this direction.
Lemma 12. Fix symmetric matrix $A$ with zeros on the diagonal and $j \in [t]$. Then, for any fixed value of $x_{-I_j}$ and any $t > 0$

$$\Pr \left[ \|Ax\|_2^2 < E \left[ \|Ax\|_2^2 \mid x_{-I_j} \right] - t \mid x_{-I_j} \right] \leq \exp \left( -\frac{c}{\|A\|_2^2} \min \left( \frac{t^2}{\|A\|_F^2 + E \left[ \|Ax\|_2 \mid x_{-I_j} \right] } , t \right) \right).$$

The proof is given in Section 5.2. Note that by Lemma 4 we know that $x_{I_j}$ is conditionally Dobrushin, conditioned on $x_{-I_j}$. The reason why Lemma 12 does not directly follow from the concentration inequality for quadratic forms (Theorem 2) is that the matrix $A^\top A$ does not necessarily have zeros on the diagonal. Hence, the proof consists of a simple argument that reduces to the concentration of a polynomial of the form $x^\top Ax$ where $A$ has zeros on the diagonal.

By combining Lemmas 11 and 12, we are now able to prove that $\|Ax\|_2^2$ is lower bounded with high probability, conditioned on $x_{-I_j}$. This is accomplished by the following Lemma.

We can apply Lemma 12 to obtain a lower bound on $\|Ax\|_2^2$:

Lemma 13. Let $A$ be a symmetric matrix with zeros on the diagonal. Then, for any $t > 0$:

$$\Pr \left[ \|Ax\|_2^2 < c\|A\|_2^2 + \max_j \|E[Ax \mid x_{-I_j}]\|_2^2 - t \left( \|A\|_F + \max_j \|E[Ax \mid x_{-I_j}]\|_2 \right) \right]$$

$$\leq C \log n \exp \left( -\frac{c \min (t^2, t\|A\|_F) }{\|A\|_2^2} \right).$$

Proof. Let $E_j$ denote the event that

$$\|Ax\|_2^2 > E[\|Ax\|_2^2 \mid x_{-I_j}] - t \left( \|A\|_F + \|E[Ax \mid x_{-I_j}]\|_2 \right). \quad (25)$$

By Lemma 12

$$\Pr[E_j] = E_{x_{-I_j}} \left[ \Pr[E_j \mid x_{-I_j}] \right] \geq 1 - \exp \left( -c \min (t^2, t\|A\|_F) / \|A\|_2^2 \right).$$

By a union bound, $\Pr \left[ \bigcap_j E_j \right] \geq 1 - C \log n \exp \left( -c \min (t^2, t\|A\|_F) / \|A\|_2^2 \right)$.

From now onward, assume that $\bigcap_j E_j$ holds. Lemma 11 implies that

$$E \left[ \|Ax\|_2^2 \mid x_{-I_j} \right] \geq c\|A_{I_j}\|_2^2 + \|E[Ax \mid x_{-I_j}]\|_2^2$$

and additionally, that there exists some $j$ such that $\|A_{I_j}\|_2^2 \geq c\|A\|_F^2$. We derive that

$$\|Ax\|_2^2 > c\|A_{I_j}\|_2^2 + \|E[Ax \mid x_{-I_j}]\|_2^2 - t \left( \|A\|_F + \|E[Ax \mid x_{-I_j}]\|_2 \right) \quad (26)$$

holds for all $j$. Let $j_1 = \max_j \|E[Ax \mid x_{-I_j}]\|_2$ and $j_2 = \max_j \|A_{I_j}\|_F$. Then, substituting $j_1$ in (26), we derive that

$$\|Ax\|_2^2 > \max_j \|E[Ax \mid x_{-I_j}]\|_2^2 - t \left( \|A\|_F + \max_j \|E[Ax \mid x_{-I_j}]\|_2 \right).$$
By substituting \( j_2 \) in (26) we derive that
\[
\|Ax\|^2 > c'\|A\|^2_F - t \left( \|A\|_F + \max_j \|\mathbb{E}[Ax | x_{-j}]\|_2 \right).
\]

Taking an average of the above two inequalities, we derive that
\[
\|Ax\|^2 > c'\|A\|^2_F/2 + \max_j \|\mathbb{E}[Ax | x_{-j}]\|^2/2 - t \left( \|A\|_F + \max_j \|\mathbb{E}[Ax | x_{-j}]\|_2 \right).
\]

This concludes the proof.

Next, recall that we want a lower bound on \( \|Ax\|^2 \) that holds for \( A \in \mathcal{A}_\xi \) at the same time. We will use Lemma 9 to conclude the proof:

**Proof of Lemma 6** First of all, applying Lemma 13 with \( t \) being a sufficiently small constant, we derive that
\[
\Pr \left[ \|Ax\|^2 < c + c \max_j \|\mathbb{E}[Ax | x_{-j}]\|^2 \right] \leq C \log n \exp(-c'\xi^2),
\]

(27)

since \( \|A\|_F = 1 \) and \( \|A\|_2 \leq 1/\xi \) for all \( A \in \mathcal{A}_\xi \). We will use Lemma 9 with
\[
Z_A = c + c \max_j \|\mathbb{E}[Ax | x_{-j}]\|^2 - \|Ax\|^2.
\]

To apply this lemma, we have to prove both that each \( Z_A \) is non-positive with high probability, and that \( |Z_A - Z_B| \) is small whenever \( \|A - B\|_F \) is small. Firstly, (27) implies that \( \Pr[Z_A > 0] \leq C \log n \exp(-c'\xi^2) \). Secondly, we bound \( |Z_A - Z_B| \):

\[
|Z_A - Z_B| \leq \|Ax\|^2 - \|Bx\|^2 + c \max_j \|\mathbb{E}[Ax | x_{-j}]\|^2 - \max_j \|\mathbb{E}[Bx | x_{-j}]\|^2
\]

(29)

\[
\leq \|Ax\|^2 - \|Bx\|^2 + c \max_j \|\mathbb{E}[Ax | x_{-j}]\|^2 - \|E[Bx | x_{-j}]\|^2.
\]

(30)

We will bound the two terms in the right hand side of (29) in a similar way. First,
\[
\|Ax\|^2 - \|Bx\|^2 = \|Ax\|^2 - \|Bx\|^2 \cdot \|Ax\|^2 + \|Bx\|^2 \leq \|Ax - Bx\|^2 (\|A\|^2 \|x\|^2 + \|B\|^2 \|x\|^2) \leq \|A - B\|^2 \|x\|^2 (\|A\|^2 + \|B\|^2) \leq 2n\|A - B\|^2_F,
\]

using the fact that \( \|u\| - \|v\| \leq \|u - v\| \) by the triangle inequality for any norm, that \( \|A\|_2 \leq \|A\|_F = 1 \) for any \( A \in \mathcal{A}_\xi \), that \( \|x\|^2 = n \) and the fact that \( \|\cdot\|_2 \leq \|\cdot\|_F \). Similarly, we bound the second term in the right hand side of (29): for all \( j \in [\ell] \),
\[
\|\mathbb{E}[Ax | x_{-j}]\|^2 - \|\mathbb{E}[Bx | x_{-j}]\|^2
\]
\[
= \|\mathbb{E}[Ax | x_{-j}]\|^2 - \|\mathbb{E}[Bx | x_{-j}]\|^2 \cdot \|\mathbb{E}[Ax | x_{-j}]\|^2 + \|\mathbb{E}[Bx | x_{-j}]\|^2
\]
\[
\leq \|\mathbb{E}[(A - B)x | x_{-j}]\|^2 (\|A\| + \|B\|) \|x\|^2
\]
\[
\leq \mathbb{E} \left[ \| (A - B)x \|^2 | x_{-j} \right] \cdot 2\sqrt{n}
\]
\[
\leq 2n\|A - B\|^2_F.
\]
by Jensen’s inequality. We use Lemma 9 with
\[ \delta = C \log n \exp \left( -c' \xi^2 \right) ; \quad \epsilon = c''/n ; \quad r = c/2, \]
where \( c \) is the constant (28), and derive that with probability at least \((Cn)^{k-1} \delta = \log n (Cn)^{k-1} \exp \left( -c' \xi^2 \right)\)

it holds that \( Z_A \leq r = c/2 \) for all \( A \in A_\xi \). Assuming that this holds, then (22) implies that
\[
\frac{\partial^2 \varphi(J)}{\partial A^2} \geq c''' \|Ax\|^2 \geq c''' c/2 + c''' c \max_j \|E[Ax | x_{-j}]\|^2
\]
holds for all \( A \in A_\xi \) and \( J \in \mathcal{J} \), as required. \( \square \)

4.3.2 Proof of Lemma 10

We begin by proving Lemma 10. The idea is that if the total interaction strength of a node with all of its neighbors is small compared to the variance of each node, then the behavior is similar to that of independent samples. To prove that, we employ the following Lemma, the proof of which can be found in [DDDJ19, Lemma 4.9], and is also a direct corollary of the earlier paper [BN+19, Theorem 3.1].

**Lemma 14.** Let \( x \) be an \((M, \gamma)\) Ising model with \( M < 1 \). Fix \( i \in [n] \) and let \( P_{x_{-i}|x_i=1} \) denote the conditioned distribution over \( x_{-i} \) conditioned on \( x_i = 1 \) and \( P_{x_{-i}|x_i=-1} \) denote the conditioned distribution conditioned on \( x_i = -1 \). Then,
\[
W_1(P_{x_{-i}|x_i=1}, P_{x_{-i}|x_i=-1}) \leq \frac{2M}{1-M},
\]
where \( W_1 \) is the \( \ell_1 \)-Wasserstein distance, namely \( W_1(P, Q) = \min_\pi \mathbb{E}_{(x,y) \sim \pi}[\|x - y\|_1] \). The minimum is taken over all distributions \( \pi \) over \([−1, 1]^n \times [−1, 1]^n \), such that the marginals are \( P \) and \( Q \), respectively.

This lemma essentially tells us that if \( M \) is small enough, then changing the spin of a single node is unlikely to influence the remaining ones. Using this fact, we can proceed as in the proof sketch of Section 4.3.1 to prove the claim. To prove it in the general case, we employ once more the conditioning trick to reduce the Dobrushin constant of the model. Specifically, Lemma 4 guarantees that we can find a subset \( I \) of nodes that are conditionally very weakly dependent, while still maintaining a constant fraction of the total variance of the linear function. The details are given below.

**Proof of Lemma 10** Fix \( a \in \mathbb{R}^d \). We first assume that \( M \leq \gamma/4 \), and then we prove it for all \( M \). We start by arguing that for all \( i \in [n] \),
\[
\sum_{j \in [n] \setminus \{i\}} |\text{Cov}(x_i, x_j)| \leq \frac{M}{1-M}.
\]
The strategy will be to bound the Wasserstein distance between two Ising models conditional on different values of a single node. Using a tensorization argument, we have that if \( X = (X_1, \ldots, X_n), Y = (Y_1, \ldots, Y_n) \) are random vectors over the same domain, then \( W_1(X, Y) \geq \)
\[\sum_{i \in [n]} W_1(X_i, Y_i).\] Applying this on the distributions \(P_{X_+|X_i=1}\) and \(P_{X_-|X_i=-1}\), we derive by Lemma 14 that
\[
\sum_{j \in [n]\setminus\{i\}} |E[X_j|X_i=1] - E[X_j|X_i=-1]| = \sum_{j \in [n]\setminus\{i\}} W_1(P_{X_j|X_i=1}, P_{X_j|X_i=-1}) \\
\leq W_1(P_{X_+|X_i=1}, P_{X_-|X_i=-1}) \leq \frac{2M}{1-M'}
\]
where we also used the easily established property that for any two random variables \(U, V\) supported in a set of size 2, \(W_1(U, V) = |E[U] - E[V]|\). We will use the above bound to get a bound on \(\text{Cov}(x_i, x_j)\). Indeed,
\[
\text{Cov}(x_i, x_j) = E[(x_i - E[x_i])x_j] = E_x[(x_i - E[x_i])E_x[x_j|x_i]] \\
= \Pr[x_i = 1](1 - E[x_i])E_x[x_j|x_i = 1] + \Pr[x_i = -1](1 - E[x_i])E_x[x_j|x_i = -1] \\
= \frac{1}{2}(1 + E[x_i])(1 - E[x_i])E_x[x_j|x_i = 1] + \frac{1}{2}(1 - E[x_i])(1 - E[x_i])E_x[x_j|x_i = -1] \\
= \frac{1}{2}(1 - E[x_i])^2(E_x[x_j|x_i = 1] - E[x_j|x_i = -1]).
\]
Combining the above inequalities, we obtain that
\[
\sum_{j \in [n]\setminus\{i\}} |\text{Cov}(x_i, x_j)| \leq \frac{\sum_{j \in [n]\setminus\{i\}} |E[x_j|x_i = 1] - E[x_j|x_i = -1]|}{2} \leq \frac{M}{1-M'}.
\]
To conclude the proof for the setting where \(M \leq \gamma/4\):
\[
\text{Var}(a^\top x) = E[a^\top (x - E x)(x - E x)^\top a] = \sum_{i,j} a_i a_j \text{Cov}(x_i, x_j) \geq \sum_{i=1}^n a_i^2 \text{Var}(x_i) - \sum_{i \neq j} |a_i a_j \text{Cov}(x_i, x_j)| \\
\geq \sum_{i=1}^n a_i^2 \text{Var}(x_i) - \sum_{i \neq j} \frac{(a_i^2 + a_j^2)|\text{Cov}(x_i, x_j)|}{2} = \sum_{i=1}^n a_i^2 \left(\text{Var}(x_i) - \sum_{j \neq i} |\text{Cov}(x_i, x_j)|\right) \\
\geq \sum_{i=1}^n a_i^2 \left(\gamma - \frac{\sqrt{M}}{1-M}\right) \geq \|a\|^2 \frac{\gamma}{2},
\]
where we used the fact that \(M \leq \gamma/4 < 1/2\), which implies that \(M/(1-M) \leq 2M \leq \gamma/2\).

In order to prove the Lemma without the above assumption, we again use the conditioning trick. Specifically, we find a subset of the nodes that satisfies Dobrushin’s condition with a small enough \(M\) and then apply our result. To make this formal, notice that by Lemma 11 there exists a subset \(I\) of nodes such that conditioned on any \(x_{-i}, x_i\) is a \((\gamma/4, \gamma)\)-Ising model. Moreover, it is guaranteed that
\[
\sum_{i \in I} a_i^2 \geq \frac{c' \gamma}{M} \sum_{i \in [n]} a_i^2.
\]
Hence, if we apply the previous result when conditioning on \(x_{-i}\) we get
\[
\text{Var}[a^\top x|x_{-i}] = \text{Var}[a^\top x_i|x_{-i}] \geq \frac{\gamma \|a_i\|^2}{2} \geq \frac{c' \gamma^2 \|a\|^2}{M}.
\]
Since conditioning decreases the variance in expectation, we have that
\[
\text{Var}[a^\top x] \geq \mathbb{E}_{x \leftarrow [\text{Var}[a^\top x| x \leftarrow] \geq \frac{c\gamma^2\|a\|^2}{M},
\]
as required.

4.4 Algorithm for Maximizing Pseudo-Likelihood

Next, we show that there exists a polynomial time algorithm for the MPLE. In the rest of this section, we prove Lemma 3. The central problem is maximizing the pseudo-likelihood with the constraint that the matrix we output has low infinity norm. To achieve this, we use a regularizer of the form \( \lambda \max(0, \|A_\beta\|_\infty - M) \). This ensures that the optimal solution that the algorithm outputs will have small infinity norm. We should also argue that the output of the regularized procedure is close to the minimizer of \( \phi \). To do this, we need to calculate bounds for \( \phi \) and its derivative. Since the regularized part is not differentiable, we have to use subgradient optimization methods.

Proof of Lemma 3

The algorithm essentially solves the constrained optimization problem described in (11). To design the optimization algorithm, it will be more convenient to work with a specific base of matrices that span \( \mathcal{V} \). For this reason, let \( \{A_1, \ldots, A_k\} \) be a fixed orthonormal basis of \( \mathcal{V} \) with respect to the inner product induced by the Frobenius norm. Then, each \( J \in \mathcal{V} \) can be uniquely written in the form
\[
J = \sum_{i=1}^k \beta_i A_i
\]
We use the notation \( \beta = (\beta_1, \ldots, \beta_k) \) and \( A_\beta = \sum_{i=1}^k \beta_i A_i \). Thus, minimizing \( \varphi(J) \) subject to \( J \in \mathcal{J} \) is the same as minimizing
\[
\psi(\beta) = \varphi(A_\beta)
\]
subject to the constraint \( \|A_\beta\|_\infty \leq M \). Thus, in order to solve (11), we can equivalently solve
\[
\arg \min_{\beta : \|A_\beta\|_\infty \leq M} \psi(\beta)
\]
(31)

One way to solve this is to use constrained optimization. However, we would like to avoid the complicated projection procedure associated with this strategy. We note that projecting to the set of matrices with low infinity norm does not directly translate to a similar procedure in the space of \( f_i \), since \( A_i \) might have large infinity norm. Instead, we will solve the following regularized optimization problem:
\[
\arg \min_{\beta} \left( \psi(\beta) + \lambda \max(0, \|A_\beta\|_\infty - M) \right)
\]
Set \( h(\beta) = \psi(\beta) + \lambda \max(0, \|A_\beta\|_\infty - M) \), which is clearly a convex function. We will describe a way to pick \( \lambda \) shortly after. Denote by \( \beta_1 \) the solution of this optimization problem and by \( \beta^* \) the one corresponding to \( J^* \). Also, suppose the algorithm outputs a point \( \hat{\beta} \) for accuracy \( \epsilon \). The details are given in Algorithm 4.4. We will prove that \( \|A_{\beta_1}\|_\infty, \|A_{\beta^*}\|_\infty \leq 3M \). First, for \( \beta = 0 \) we have \( \|A_0\|_\infty \leq M \) and by equation (12) we get \( \psi(0) = n \log 2 \).
Algorithm 1 Optimization Procedure

**Input:**
- Basis of matrices $J_1, \ldots, J_k$
- Accuracy $\epsilon$
- Sample $(x_1, \ldots, x_n)$
- Infinity norm bound $M$

**Output:**
- Vector $\hat{\beta}$

1. $(A_1, \ldots, A_k) := \text{GRAM-SCHMIDT}(J_1, \ldots, J_k)$ // Find orthonormal basis
2. for $i := 1, \ldots, k$ do
3. $\beta^0_i := 0$ // Initializations
4. end for
5. $T = \frac{M^2 n^k}{\epsilon^2}$ // Number of steps of gradient descent
6. $\eta := \sqrt{\frac{M}{n k T}}$ // stepsize
7. for $t := 1, \ldots, T$ do
8. $U := \beta^t_1 A_1 + \cdots + \beta^t_k A_k$ // The interaction matrix at time $t$
9. for $i := 1, \ldots, k$ do
10. $S_i := 0$
11. end for
12. for $j := 1, \ldots, n$ do
13. $S_i := S_i + ((A_i)x) (\tanh(U_j x) - x_j)$ // Compute the derivative
14. end for
15. if $\|U\|_{\infty} \geq M$ then
16. $\text{MAX} := -\infty$ // If regularized part is nonzero, also need subgradient
17. $\text{ARGMAX} := 1$ // First find max row sum
18. for $l := 1, \ldots, n$ do
19. $\text{TEMP} := 0$
20. for $v := 1, \ldots, n$ do
21. $\text{TEMP} := \text{TEMP} + |U_{lv}|$
22. end for
23. if $\text{TEMP} > \text{MAX}$ then
24. $\text{MAX} := \text{TEMP}$
25. $\text{ARGMAX} := l$
26. end if
27. end for
28. for $i := 1, \ldots, k$ do
29. for $v := 1, \ldots, n$ do
30. $S_i := S_i + 5n \cdot \text{sgn}(\beta^t_i) \cdot \|(A_i)_{\text{MAX},v}\|$ // compute subgradient for max row
31. end for
32. end for
33. end if
34. for $i := 1, \ldots, k$ do
35. $\beta^{t+1}_i = \beta^t_i - \eta S_i$ // Iteration of subgradient descent
36. end for
37. end for
38. Output $\hat{\beta} := \frac{1}{T} \sum_{t=1}^{T} \beta^t$
We set $\lambda = 5n$. We now examine what the value of the function is $\beta$ such that $\|A_\beta\|_\infty \geq 3M$. If we manage to show that it is greater than the value at 0, then it is clear that the minimizer will not lie in this set. Using the inequality $\cosh(x) \geq \exp(-|x|)$ and equation 12 we have:

$$\psi(\beta) = \sum_{i=1}^{n} (\log \cosh((A_\beta)x_i) - x_i(A_\beta)x + \log 2) \geq \sum_{i=1}^{n} (-|A_\beta|x_i - x_i(A_\beta)x + \log 2)$$

Using the triangle inequality and the fact that $x$ is a $\{-1, 1\}^n$ vector, we have:

$$\sum_{i=1}^{n} x_i(A_\beta)x = x^\top A_\beta x \leq \|A_\beta\|_2 \|x\|_2^2 = n\|A_\beta\|_2 \leq n\|A_\beta\|_\infty$$

Overall, we get:

$$\psi(\beta) \geq -n\|A_\beta\|_\infty - n\|A_\beta\|_\infty + n\log 2 = -2n\|A_\beta\|_\infty + n\log 2$$

So, the value of the optimized function at $\beta$ is $h(\beta) = \psi(\beta) + 5n(\|A_\beta\|_\infty - M) \geq -2n\|A_\beta\|_\infty + n\log 2 + 3n\|A_\beta\|_\infty = n\log 2 = h(0) + n\|A_\beta\|_\infty$

Hence, we conclude that $\|A_\beta\|_\infty \leq 3M$. Moreover, we have

$$h(\hat{\beta}) - h(\beta_1) \leq \epsilon \leq Mn \implies h(\hat{\beta}) \leq Mn + n\log 2 \implies \|A_\hat{\beta}\|_\infty \leq 3M$$

As for the guarantee of the algorithm, we notice that:

$$\psi(\hat{\beta}) \leq h(\hat{\beta}) \leq h(\beta_1) + \epsilon \leq h(\beta^*) + \epsilon = \psi(\beta^*) + \epsilon$$

since $\|A_\beta\|_\infty \leq M$. Thus, if we denote by $\hat{\beta} = A_\hat{\beta}$, we conclude that:

$$\phi(\hat{\beta}) \leq \phi(\beta^*) + \epsilon$$

which is what we wanted to prove. It remains now to argue about the computational complexity of the procedure. The algorithm we will use is subgradient descent, which performs reasonably well when the subgradient is upper bounded. The following Theorem can be found in [Bub+15].

**Lemma 15 ([Bub+15]).** Suppose $f$ is a convex function with minimum $x^*$ and $g \in \partial f$ is a subgradient such that $\|g\| \leq L$. Suppose we are running the following iterative procedure:

$$x_{t+1} = x_t - \eta g(x_t)$$

with $\|x_1 - x^*\| \leq R$. If we choose $\eta = R/(L\sqrt{t})$ then

$$f \left( \frac{1}{t} \sum_{i=1}^{t} f(x_i) \right) - f(x^*) \leq \frac{RL}{\sqrt{t}}$$

29
Hence, if we want accuracy $\epsilon$, we need to run the algorithm for $t = O(R^2L^2/\epsilon^2)$ iterations. We now argue that both $R$ and $L$ are polynomially bounded in our case. To bound a subgradient of $h$, we begin by bounding each partial derivative of $\psi$, since for this function it is also a subgradient. First of all, it is easy to see that $\partial \psi(\beta)/\partial \beta_i \leq 2\sqrt{n}$. Indeed, by equation $\ref{eq:13}$ we get:

$$
\left| \frac{\psi(\beta)}{\partial \beta_i} \right| = \left| \sum_{k=1}^{n} ((A_i)_k x) (\tanh((A_\beta)_k x) - x_k) \right| \leq \sum_{k=1}^{n} |(A_i)_k x| \left| \tanh((A_\beta)_k x) - x_k \right|
$$

$$
\leq \sum_{k=1}^{n} |(A_i)_k x| \left( \left| \tanh((A_\beta)_k x) \right| + 1 \right) \leq 2 \sum_{k=1}^{n} |(A_i)_k x| 
$$

$$
\leq 2 \sum_{k=1}^{n} \sum_{i=1}^{n} |(A_i)_k| \leq 2\sqrt{n} \|A_i\|_F = 2\sqrt{n}
$$

In the preceding calculation, we used the Cauchy Schwarz inequality along with the fact that $A_i$ belongs to the orthonormal basis. Now, we turn our attention to the non-smooth part of $h$. We use the following general fact about subgradients.

**Lemma 16** (folklore). Suppose $f_1, f_2 : \mathbb{R}^n \mapsto \mathbb{R}$ are differentiable convex functions. Then, $h = \max(f_1, f_2)$ has a subgradient of the form

$$
g(x) = \begin{cases} 
\nabla f_1(x), & \text{if } f_1(x) \geq f_2(x) \\
\nabla f_2(x), & \text{otherwise}
\end{cases}
$$

This means that to bound the subgradient of $\lambda \max(0, \|A_\beta\|_\infty - M)$ we just need to bound the gradient of each function in the max. The function 0 obviously has gradient 0. We have

$$
\|A_\beta\|_\infty = \max_{u \in [n]} \sum_{i=1}^{n} |(A_\beta)_{uv}|
$$

Hence, to bound the subgradient of this function, we focus on a fixed row $u$. Set

$$
L(\beta) = \sum_{i=1}^{n} |(A_\beta)_{uv}| = \sum_{i=1}^{n} \sum_{k=1}^{n} (A_i)_{uv}
$$

. Using the fact that a subgradient for $|x|$ is $\text{sgn}(x)$, we obtain that a subgradient of $L$ w.r.t. variable $\beta_i$ is $\text{sgn}(\beta_i) \sum_{i=1}^{n} |(A_i)_{uv}|$, which is clearly bounded in norm by $\sum_{i=1}^{n} |(A_i)_{uv}| \leq \sqrt{n}$. Hence, the subgradient w.r.t. $\beta_i$ of the function $\lambda \max(0, \|A_\beta\|_\infty - M)$ is bounded in absolute value by $\lambda \sqrt{n} = 5n \sqrt{n}$. This means that the subgradient $g_i$ of $h$ is $O(n \sqrt{n})$ in absolute value. It follows that $\|g\|_2 \leq n \sqrt{k} \sqrt{n}$. Finally, by choosing $x_1 = 0$ we have

$$
\|x_1 - \beta^*\|_2 = \|A_\beta - A_\beta^*\|_F \leq \sqrt{n} \|A_\beta - A_\beta^*\|_\infty \leq M \sqrt{n}
$$

Hence, after $t = O(n^4 M^2 k/\epsilon^2)$ rounds we achieve error of at most $\epsilon$. The only part of the algorithm we haven’t analysed is the Gram-Schmidt computation. It is well known that for a subspace of $\mathbb{R}^k$ of dimension $l$, the Gram-Schmidt procedure takes $O(ml^2)$. For $m = k, l = n^2$ we see that this is less than the complexity for the optimization part.

The last issue we should address to get the time complexity of the algorithm is the calculation of the subgradient. The gradient of $\psi$ amounts to the calculation of $A_\beta x$, which takes $O(n^2k)$
time(A_i x can be precomputed for all i). The calculation of the subgradient for the nonsmooth part is easy once we determine which row has the maximum absolute sum for the particular value of β. This also takes O(kn^2) time, since we have to calculate the sum of the matrices in each row. Once we do that, a precomputation of the row sums of each matrix can give us the value of β. Let us derive that a precomputation of the row sums of each matrix can give us the subgradient in O(1) time. Overall, computation of the gradient takes O(kn^2) time, which means that our algorithm runs in O(k^2n^6M^2/ε^2) time.

4.5 Proof of Lemma 2

Proof of Lemma 2: First, notice that \|J - J^*\|_F \leq \sqrt{n}\|J - J^*\|_2 \leq \sqrt{n}\|J - J\|_\infty \leq C\sqrt{n}. Hence, it suffices for the proof to assume that \sqrt{k\log n + \log(1/δ)} = O(\sqrt{n}).

Let \( \xi = C\sqrt{k\log n + \log(1/δ)} \) and \( t = C'\sqrt{k\log n + \log(1/δ)} \), for some sufficiently large C and C', as discussed above, we can assume that \( t \leq \sqrt{n} \), as in the requirement of Lemma 1. Let \( \Delta = \hat{f} - J^* \) and set \( \hat{A} = \Delta/\|\Delta\|_F \). We begin by distinguishing between two cases. If \( \hat{A} \notin A_\xi \), then by definition we should have

\[
\frac{\|\Delta\|_F}{\|\Delta\|_\infty} \leq \frac{\|\Delta\|_F}{\|\Delta\|_2} \leq \xi \implies \|\Delta\|_F \leq C\sqrt{k\log n + \log(1/δ)}\|\Delta\|_\infty.
\]

By assumption, we have \( \|\hat{f}\|_F \leq M \) and \( \|J^*\|_F \leq M \). We conclude that in this case

\[
\|\Delta\|_F \leq 2MC\sqrt{k\log n + \log(1/δ)}.
\]

Now suppose \( \hat{A} \in A_\xi \). Define the function \( h : [0, \|\Delta\|_F] \mapsto \mathbb{R} \) such that

\[
h(s) = \varphi(J^* + sA).
\]

By definition we have \( h'(0) = \partial \varphi(J^*)/\partial A \), and also notice that \( h''(s) = \partial^2 \varphi(J^* + sA)/\partial A^2 \) for all \( s \in [0, \|\Delta\|_F] \). Those first and second derivatives are bounded in Lemma 1 and we will use them to bound \( \|\hat{f} - J^*\|_F \).

Next, define the quadratic polynomial \( p : \mathbb{R} \to \mathbb{R} \) by

\[
p(s) = h(0) + h'(0)s + \frac{s^2}{2} + h(0) - \frac{h'(0)^2}{2\mu} \tag{32}
\]

where \( \mu = \inf_{s \in [0, \|\Delta\|_F]} h''(s) \). Notice that \( p \) is is tangent to \( h \) at 0 and since \( h \) is \( \mu \)-strongly convex, hence it follows that \( h \geq p \) in \([0, \|\Delta\|_F] \). For all \( s \in \mathbb{R} \), we use (32) to extract \( |s - 0| = |s| \):

\[
|s + h'(0)/\mu| = \sqrt{\frac{2(p(s) - h(0))}{\mu} + \frac{h'(0)^2}{\mu^2}} \leq \sqrt{\frac{2}{\mu} \max(p(s) - h(0), 0) + \frac{h'(0)^2}{\mu^2}},
\]

by using \( \sqrt{a + b} \leq \sqrt{a} + \sqrt{b} \) for \( a, b \geq 0 \) and the fact that \( s \mapsto \sqrt{s} \) is monotonic non-decreasing. We derive that

\[
|s| \leq 2 \left| \frac{h'(0)}{\mu} \right| + \sqrt{\frac{2}{\mu} \max(p(s) - h(0), 0)}. \tag{33}
\]

We will bound this quantity, substituting \( s = \|\Delta\|_F \). First, by Lemma 1, \( \frac{h'(0)}{\mu} = \frac{\partial \varphi(J^*)/\partial A}{\partial^2 \varphi(J^*)/\partial A^2} \leq tC \leq C\sqrt{k\log n + \log(1/δ)}, \)
where \( J = J^* + sA \) for some \( s \in [0, \|\Delta\|_F] \). Secondly, using the fact that \( p \) is tangent to \( h \) at 0 and otherwise \( p \leq h \), we derive that
\[
p(\|\Delta\|_F) - p(0) \leq h(\|\Delta\|_F) - h(0) = \varphi(\tilde{J}) - \varphi(J^*) \leq k,
\]
where the last inequality follows from the assumption of this lemma. We derive that
\[
\sqrt{\frac{2}{\mu} \max (p(s) - h(0), 0)} \leq C\sqrt{k}.
\]
By (33) we derive that \( \|\Delta\|_F \leq C\sqrt{k\log n + \log(1/\delta)} \) as required. \( \square \)

5 The Conditioning Trick

The main purpose of this section is to prove the following Lemma, which is used multiple times in the proof of Theorem 1.

**Lemma 4.** Let \( x = (x_1, \ldots, x_n) \) be an \((M, \gamma)\)-Ising model, and fix \( \eta \in (0, M] \). Then, there exist subsets \( I_1, \ldots, I_\ell \subseteq [n] \) with \( \ell \leq C\eta M^2 \log n / \eta^2 \) such that:

1. For all \( i \in [n] \),
   \[
   |\{ j \in [\ell] : i \in I_j \}| = \left\lceil \frac{\eta \ell}{8M} \right\rceil
   \]
2. For all \( j \in [\ell] \) and any value of \( x_{-I_j} \), the conditional distribution of \( x_{I_j} \) conditioned on \( x_{-I_j} \) is an \((\eta, \gamma)\)-Ising model.

Furthermore, for any non-negative vector \( \theta \in \mathbb{R}^n \) there exists \( j \in [\ell] \) such that
\[
\sum_{i \in I_j} \theta_i \geq \frac{\eta}{8M} \sum_{i=1}^n \theta_i.
\]

Intuitively, this lemma says that we can select a small number of subsets, such that each element \( i \in [n] \) is contained in at least a constant fraction of these sets. At the same time, we want the variables in these subsets to be weakly dependent when we condition on the rest. The proof relies on the following technical lemma.

**Lemma 17.** Let \( A \) be a matrix with zero on the diagonal, and \( \|A\|_\infty \leq 1 \). Let \( 0 < \eta < 1 \). Then, there exist subsets \( I_1, \ldots, I_\ell \subseteq [n] \), with \( \ell \leq C\log n / \eta^2 \) such that:

1. For all \( i \in [n] \),
   \[
   |\{ j \in [\ell] : i \in I_j \}| = \left\lceil \frac{\eta \ell}{8} \right\rceil
   \] (34)
2. For all \( j \in [\ell] \) and all \( i \in I_j \),
   \[
   \sum_{k \in I_j} |A_{ik}| \leq \eta
   \] (35)
This Lemma lies in the heart of the proof. It essentially ensures that we can select subsets of nodes that conditionally satisfy Dobrushin’s condition with an arbitrarily small constant. It also shows that these subsets can cover all the nodes. Thus, it allows breaking up a sum over all the nodes to a sum over all these subsets with only a logarithmic error factor. The proof is an application of the probabilistic method, using a simple, but strong technique found in [AS04].

Proof of Lemma 17. The proof uses the probabilistic method. To prove that there exists an object with a specific property, we find a way to select the object at random so that with positive probability the property is satisfied. In our case, the object is the collection of subsets \( I_1, \ldots, I_\ell \). The properties are 34 and 35.

We first define a way to select these subsets at random. This is done with the following sampling procedure:

Algorithm 2 Sampling Procedure

\[
\text{for } j := 1, \ldots, \ell \text{ do} \\
\text{Draw a random set } I'_j \text{ of coordinates (a subset of the set of all coordinates } [n]), \text{ where each } \ i \in [n] \text{ is selected to be in } I'_j \text{ independently with probability } \eta. \\
\text{Set } I_j \text{ to be the set of all coordinates } i \in I'_j \text{ such that } \sum_{k \in I'_j} |A_{ik}| \leq \eta. \\
\text{end for} \\
\text{Output } (I_1, \ldots, I_\ell)
\]

Notice that each subset is selected independently of the others. We are now going to prove that the output of this sampling procedure satisfies 34 and 35 with positive probability. By the nature of the procedure, 35 is automatically satisfied. It remains to check that 34 holds. First, for a fixed \( i \in [n] \) and \( j \in [\ell] \), we calculate the probability that \( i \in I_j \).

\[
\Pr [i \in I_j] = \Pr [i \in I'_j] \Pr [i \in I_j | i \in I'_j] = \Pr [i \in I'_j] \Pr \left( \sum_{k \in I'_j} |A_{ik}| \leq \eta \mid i \in I_j \right)
\]

\[
= \Pr [i \in I'_j] \Pr \left( \sum_{k \in I'_j} |A_{ik}| \leq \eta \right) = \frac{\eta}{2} \Pr \left( \sum_{k \in I'_j} |A_{ik}| \leq \eta \right).
\]

By linearity of expectation we have:

\[
\mathbb{E} \left[ \sum_{k \in I'_j} |A_{ik}| \right] = \frac{\eta}{2} \sum_{k \in [n]} |A_{ik}| \leq \frac{\eta}{2}
\]

Thus, by applying Markov’s inequality we get:

\[
\Pr \left[ \sum_{k \in I'_j} |A_{ik}| \geq \eta \right] \leq \frac{1}{2},
\]

which yields

\[
\Pr [i \in I_j] \geq \frac{\eta}{4}.
\]
Now, fix an \( i \in [n] \) and define
\[
S_i := |\{j \in [\ell] : i \in I_j\}|
\]
Notice that the events \( \{i \in I_j\}_{j=1}^\ell \) are independent from each other. Hence, we can write
\[
S_i = \sum_{j=1}^\ell 1(i \in I_j)
\]
which means that \( S_i \) is a sum of independent Bernoulli random variables. By the preceding calculation, we get:
\[
\mathbb{E}[S_i] = \sum_{j=1}^\ell \mathbb{E}[1(i \in I_j)] = \sum_{j=1}^\ell \mathbb{P}[i \in I_j] \geq \frac{\eta \ell}{4}.
\]
Now, using Hoeffding’s inequality and setting \( \ell = 32 \log 4 \log n / \eta^2 \) we get:
\[
\Pr \left[ S_i \leq \frac{\eta \ell}{8} \right] \leq \Pr \left[ |S_i - \mathbb{E}[S_i]| \geq \frac{\eta \ell}{8} \right] \leq 2 \exp \left( -2 \frac{\left( \frac{\eta \ell}{8} \right)^2}{\ell} \right) \\
\leq 2 \exp \left( -\frac{\eta^2 \ell}{32} \right) \leq \frac{1}{2n}.
\]
By a simple union bound we get
\[
\Pr \left[ \exists i \in [n] : S_i \leq \frac{\eta \ell}{8} \right] \leq n \frac{1}{2n} = \frac{1}{2}
\]
That means that with positive probability we have \( S_i \geq \eta \ell / 8 \). Hence, there exists a collection of subsets \( I_1, \ldots, I_\ell \) having this property. Notice that we do not have Equation \( 34 \) but an inequality instead. However, for each \( i \) such that \( S_i > \eta \ell / 8 \), we can remove \( i \) from some sets so that the number of sets it appears is exactly \( \lceil \eta \ell / 8 \rceil \). Clearly, by removing elements from a set, inequality \( 35 \) still holds.

**Proof of Lemma 4** Let \( J \) be the interaction matrix of the Ising model distribution of \( X \) and let \( h \) denote the external field. By the hypothesis, we have that \( \|J\|_\infty / M \leq 1 \). Hence, by applying Lemma \( 17 \) for the matrix \( \|J\|_\infty / M \leq 1 \) with \( \eta' = \eta / M \), we obtain that there is a collection of subsets \( I_1, \ldots, I_\ell \) such that
- For all \( i \in [n] \),
  \[
  |\{j \in [\ell] : i \in I_j\}| = \left\lceil \frac{\eta \ell}{8M} \right\rceil
  \]
- For all \( j \in [\ell] \) and all \( i \in I_j \)
  \[
  \sum_{k \in I_j} \frac{|A_{ik}|}{M} \leq \frac{\eta}{M} \implies \sum_{k \in I_j} |A_{ik}| \leq \eta
  \] (36)
We now argue that if inequality \[36\] holds, then the conditional distribution of \(X_{-l_j}\) conditioned on \(X_{-l_j} = x_{-l_j}\) is an \((\eta, \gamma)\)-Ising model. To show that, it suffices to argue that \(\Pr[X_{l_j} = y|X_{-l_j} = x_{-l_j}] \propto \exp(y^\top J'y + h^\top y)\), for some interaction matrix \(J'\) and external field \(h'\). Hence, we calculate the ratio of conditional probabilities for two configurations \(y\) and \(y'\):

\[
\frac{\Pr[X_{l_j} = y|X_{-l_j} = x_{-l_j}]}{\Pr[X_{l_j} = y'|X_{-l_j} = x_{-l_j}]} = \frac{\exp\left(\sum_{u \in I, v \in I_j} J_{uv} y_u y_v + \sum_{u \in I, v \notin I_j} J_{uv} y_u x_v + h^\top y\right)}{\exp\left(\sum_{u \in I, v \in I_j} J_{uv} y'_u y'_v + \sum_{u \in I, v \notin I_j} J_{uv} y'_u x_v + h^\top y'\right)}
\]

By the preceding equality, it is clear that the conditional distribution of \(X_{l_j}\) conditioned on \(X_{-l_j} = x_{-l_j}\) is an Ising model with interaction matrix \(J' = \{J_{uv}\}_{u,v \in I_j}\) and external field \(h'_i = h_i + \sum_{v \notin I_j} J_{uv}x_v\). This proves the claim.

To conclude with the last part of the lemma, fix \(a \in \mathbb{R}^n\), and drawing \(j \in [\ell]\) uniformly at random, one obtains

\[
\mathbb{E}_j[\sum_{i \in I_j} a_i] = \mathbb{E}_j[\sum_{i = 1}^n a_i 1(i \in I_j)] = \frac{\eta \ell / (8M)}{\ell} \sum_{i = 1}^n a_i \geq \frac{\eta}{8M} \sum_{i = 1}^n a_i.
\]

In particular, there exists some \(j \in [\ell]\) which achieves a value of at least this expectation. \(\square\)

## 6 Concentration Inequalities for General Functions

One of the main goals of the proof is to show that the derivative of the log pseudo-likelihood concentrates around its mean value. To do this, we rely on suitably conditioning on some of the spins, which guarantees that the conditional distribution on the remaining spins satisfies Dobrushin’s condition. However, the task of showing concentration of the derivative in this regime remains. We begin with an overview of the results regarding concentration of functions on the boolean hypercube when Dobrushin’s condition is satisfied. We then state and prove a modification of these results that serves our purposes well.

Many concentration results in the weak dependence regime concern functions that are multi-linear polynomials. Specifically, in [AKPS+19] they prove general concentration results for arbitrary multilinear polynomials of weakly dependent random variables. For polynomials of degree 2, they show the following:

**Theorem 2 ([AKPS+19])**. Let \(A\) be a symmetric matrix with zeros in the diagonal and \(X\) be an \((\alpha, \gamma)\)-bounded random variable. Let \(p(x) = x^\top Ax\). Then, for any \(t > 0\),

\[
\Pr[|p(X) - \mathbb{E}p(X)| > t] \leq \exp\left(-c \min\left(\frac{t^2}{\|A\|_F^2 + \|\mathbb{E}Ax\|_2^2}, \frac{t}{\|A\|_2}\right)\right).
\]

where the constant \(c\) only depends on \(\alpha, \gamma\) and not on the entries of \(A\).

This inequality is tight up to constants. A nice property of this result is that it explicitly connects the radius of concentration with the Frobenius norm of the matrix.

Unfortunately, the derivative of the PMLE is not a polynomial. Hence, we cannot directly apply Theorem 2. Instead, we will modify it’s proof so that it holds for arbitrary functions over
the hypercube. The original proof relied on the fact that the Hessian matrix of a second degree polynomial is constant. Despite this not being true in our case, we will follow the same strategy and prove concentration using the second order Taylor approximation of the function. First, we need to define these quantities precisely. In the following, for a vector $x$ and an index $i$, we denote by $x_{i+}$ the vector obtained from $x$ by replacing that $i$th coordinate with 1 and by $x_{i-}$ the one that is obtained by replacing this coordinate with $-1$.

**Definition 4.** For an arbitrary function $f : \{0, 1\}^n \mapsto \mathbb{R}$, we define the discrete derivative of the function as

$$D_i f(x) := \frac{f(x_{i+}) - f(x_{i-})}{2}$$

Let $D f(x)$ denote the $n$-coordinate vector of discrete derivatives. Similarly, the function $H : \{0, 1\}^n \mapsto \mathbb{R}^{n \times n}$ defined as

$$H_{ij}(x) = D_i(D_j f(x))$$

is called the discrete Hessian of $f$.

As we will see, our concentration bound will depend on the discrete derivative and Hessian of a function. However, in some cases it is more convenient to provide bounds for these quantities rather than explicitly calculate them. Therefore, we can replace these two quantities by the ones defined below.

**Definition 5.** Let $f : \{0, 1\}^n \mapsto \mathbb{R}$ be an arbitrary function. A function $\tilde{D} : \{0, 1\}^{n_1} \mapsto \mathbb{R}^{n}$ is called a pseudo discrete derivative for $f$ if

$$\|\tilde{D}(x)\|_2 \geq \|D f(x)\|_2$$

for all $x \in \{0, 1\}^n$. Additionally, we say that a function $\tilde{H} : \{-1, 1\}^n \to \mathbb{R}^{n_1 \times n_2}$ is a pseudo discrete Hessian with respect to the pseudo discrete derivative $\tilde{D}$ if for all $u \in \mathbb{R}^{n_1}, x \in \mathbb{R}^n$,

$$\|u^\top \tilde{H}(x)\|_2 \geq \|D(u^\top \tilde{D}(x))\|_2.$$ 

We are now ready to state the modification of Theorem 2.

**Theorem 3.** Let $f : \{0, 1\}^n \mapsto \mathbb{R}$ be an arbitrary function and $X$ an $(\alpha, \gamma)$-bounded random variable. Then

$$\text{Pr}[|f(x) - \mathbb{E} f(x)| > t] \leq \exp \left( -c \min \left( \frac{t^2}{\|\mathbb{E} \tilde{D}(x)\|_2^2 + \max_x \|\tilde{H}(x)\|_F^2}, \frac{t}{\max_x \|\tilde{H}(x)\|_2} \right) \right).$$

The proof of Theorem 3 follows the structure of Theorem 2 with only slight modifications. Hence, we are now going to describe the machinery used in the proof of Theorem 2.

The main ingredient of the proof is a discrete logarithmic Sobolev inequality, first proven in [Mar03]. First, we need a definition.

**Definition 6.** For any function $f : \{0, 1\}^n \mapsto \mathbb{R}$ and $i \in [n]$, we define

$$\delta_i f(x) = \frac{1}{2} \left( \mathbb{E} \left( (f(X) - f(X_1, \ldots, X_{i-1}, \bar{X}_i, X_{i+1}, \ldots, X_n))^2 | X = x \right) \right)^{1/2}$$

where the random variable $X = (X_1, \ldots, X_n)$ has distribution $\mu$ and the conditional distribution of $\bar{X}_i$ given that $X = x$ is $\mu(\cdot | x_i)$, which is the conditional distribution of $X_i$ given that $X_j = x_j$ for $j \neq i$. We denote by $\delta f(x)$ the vector in $\mathbb{R}^n$ whose $i$-th coordinate is $\delta_i f(x)$.
This quantity intuitively captures how much function $f$ changes on average when we resample one of its input variables independently. By the classical theory of Concentration Inequalities for i.i.d. random variables, we know that this average Lipschitzness directly affects the concentration properties of the function. This is indeed the case here, as the next lemma shows.

**Lemma 18** ([Mar03; GSS+19]). Let $f : \{0, 1\}^n \mapsto \mathbb{R}$ be an arbitrary function on the hypercube. Suppose $X$ is an $(\alpha, \gamma)$-bounded random variable. Then for any $p \geq 2$ it holds

$$\|f(X) - \mathbb{E}f(X)\|_p := \left(\mathbb{E} \left(\|f(X) - \mathbb{E}f(X)\|_p^p\right)\right)^{1/p} \leq \sqrt{2Cp} \left(\mathbb{E} \left(\|f(X)\|_2^p\right)\right)^{1/p}$$

where $C$ is a function of $\alpha, \gamma$ which is bounded when $\alpha$ is bounded from 1 and $\gamma$ is bounded from zero.

Lemma 18 essentially tells us that in order to bound the $p$-th moment of the function we wish to show concentration for, it is enough to control the $p$-th moment of $\|d_1 f(X)\|_2$. In the proof of Theorem 3, the authors bound this moment by the corresponding moment of a multilinear form of gaussian random variables. In doing so, they exploit the fact that $d_1 f(X)$ can be very conveniently bounded when $f$ is a multilinear polynomial. We will follow exactly the same technique, while relying on the discrete derivative of $f$ instead to bound $d_1 f(X)$.

**Proof of Theorem 3.** As mentioned earlier, the proof follows the same strategy as [AKPS+19], with a small adjustment. Our general strategy will be to bound the $p$-th moment of $f(X) - \mathbb{E}f(X)$ by the $p$-th moment of a gaussian multilinear form. By Jensen’s Inequality, we have:

$$\mathbb{E} \left(\|d_1 f(X)\|_2^p\right) = \mathbb{E} \left[\left(\frac{1}{2} \left(\mathbb{E} \left(\left|f(X) - f(X_1, \ldots, X_{i-1}, \tilde{X}_i, X_{i+1}, \ldots, X_n)\right|^2 \mid X = x\right)\right)\right)^{p/2}\right]$$

$$\leq \mathbb{E} \left[\left(\frac{1}{2} \left(\sum_{i=1}^{n} \mathbb{E} \left(\left|f(X) - f(X_1, \ldots, X_{i-1}, \tilde{X}_i, X_{i+1}, \ldots, X_n)\right|^2 \mid X = x\right)\right)\right)^{p/2}\right]$$

We now note that for each $i$, either $X_i$ and $\tilde{X}_i$ will be the same or they will have opposite signs. This means that in any case

$$\left|f(X) - f(X_1, \ldots, X_{i-1}, \tilde{X}_i, X_{i+1}, \ldots, X_n)\right| \leq |D_1 f(X)|$$

This means that

$$\mathbb{E} \left(\|d_1 f(X)\|_2^p\right) \leq \mathbb{E} \left(\frac{1}{2} |D_1 f(X)|^p\right)$$

Combining this with the log Sobolev inequality of Lemma 18 we get:

$$\|f(X) - \mathbb{E}f(X)\|_p \leq \sqrt{2Cp} \left(\mathbb{E} \|D_1 f(X)\|_2^p\right)^{1/p} \leq \sqrt{2Cp} \left(\mathbb{E} \|D(X)\|_2^p\right)^{1/p}$$

(37)

Now suppose $g \sim \mathcal{N}(0, I_n)$ is an $n$-dimensional gaussian vector independent of $X$. For a fixed $X$, the random variable $\frac{D(X)^\top g}{\|D(X)\|_2}$ is a single dimensional gaussian $\mathcal{N}(0, 1)$. Hence, by elementary properties of the gaussian distribution, there exists a constant $M > 0$ independent of $p$ such that

$$\left(\mathbb{E}_g \left(\frac{D(X)^\top g}{\|D(X)\|_2}\right)^p\right)^{1/p} \geq \frac{1}{M} \sqrt{p}$$

37
The symbol $\mathbb{E}_X$ means that we are integrating only with respect to the random variable $g$. This implies that for a fixed $X$

$$\sqrt{\mathbb{P}}\|\bar{D}(X)\|_2 \leq M \left( \mathbb{E}_X \left( \bar{D}(X)^\top g \right)^p \right)^{1/p}$$

Combining with Equation 37, we get that for all functions $f$ and pseudo derivatives $\bar{D}$:

$$\|f(X) - \mathbb{E}f(X)\|_p \leq K \left( \mathbb{E}_{X,g} \left( \bar{D}(X)^\top g \right)^p \right)^{1/p}$$

(38)

where $K = M\sqrt{C}$. Inequality 38 is a first step in proving the bound on the $p$-th moment. We want to use this inequality to make the second derivative appear on the right hand side. To do this, we "fix" the value of $g$. By Minkowski’s inequality we have:

$$(\mathbb{E}_{X,g} \left( \bar{D}(X)^\top g \right)^p)^{1/p} = \|\bar{D}(X)^\top g\|_p \leq \|\bar{D}(X)^\top g - \mathbb{E}_X \bar{D}(X)^\top g\|_p + \|\mathbb{E}_X \bar{D}(X)^\top g\|_p$$

First, we bound the second term. By linearity of expectation, we have:

$$\|\mathbb{E}_X \bar{D}(X)^\top g\|_p = \left( \mathbb{E}_g \left( (\mathbb{E}_X \bar{D}(X))^{\top} g \right)^p \right)^{1/p}$$

Now, the variable $(\mathbb{E}_X \bar{D}(X))^{\top} g$ is clearly a single dimensional gaussian, which means that it’s $p$-th moment is bounded as:

$$\|\mathbb{E}_X \bar{D}(X)^\top g\|_p \leq M \sqrt{\mathbb{P}}\|\mathbb{E}_X \bar{D}(X)\|_2$$

(39)

and that concludes the bound for the second term. For the first term, fix $g$ and define the function $h : \{0,1\}^n \mapsto \mathbb{R}$ as $h(x) = \bar{D}(x)^\top g - \mathbb{E}_X \bar{D}(X)^\top g$. Now, we apply inequality 37 to the function $h$, which gives us

$$(\mathbb{E}_X \left( \bar{D}(X)^\top g - \mathbb{E}_X \bar{D}(X)^\top g \right)^p)^{1/p} \leq \sqrt{C_p} \left( \mathbb{E}_X \|D(\bar{D}(X)^\top g)\|_2^p \right)^{1/p}$$

$$\leq \sqrt{C_p} \left( \mathbb{E}_X \|g^\top \bar{H}(X)\|_2^p \right)^{1/p}$$

$$\leq \sqrt{CM} \left( \mathbb{E}_{X,g'} \|g^\top \bar{H}(X)g'\|_p^p \right)^{1/p}$$

We conclude that

$$\|\bar{D}(X)^\top g - \mathbb{E}_X \bar{D}(X)^\top g\|_p = \left( \mathbb{E}_{X,g} \left( \bar{D}(X)^\top g - \mathbb{E}_X \bar{D}(X)^\top g \right)^p \right)^{1/p}$$

$$\leq K \left( \mathbb{E}_{X,g,g'} \|g^\top \bar{H}(X)g'\|_p^p \right)^{1/p}$$

Notice now that we have to bound the $p$-th norm of a quadratic form $g^\top \bar{H}(X)g'$ where $g,g'$ are independent gaussian vectors. That is precisely what the Hanson-Wright inequality gives us.
Note that the matrix \( \hat{H} \) need not have zeroes in the diagonal, as mentioned in \([RV+13; Lat+06]\). For a fixed \( X \) there is a constant \( C' \) such that:

\[
\left( \mathbb{E}_{g,g'} \left( g^\top \hat{H}(X) g' \right) \right)^{1/p} \leq C' \left( \sqrt{p} \max_{x \in \{0,1\}^n} \|\hat{H}(x)\|_F + p \max_{x \in \{0,1\}^n} \|\hat{H}(x)\|_2 \right)
\]

This gives us the final inequality:

\[
\|D(X)^\top g - \mathbb{E}_X D(X)^\top g\|_p \leq C' \left( \sqrt{p} \max_{x \in \{0,1\}^n} \|\hat{H}(x)\|_F + p \max_{x \in \{0,1\}^n} \|\hat{H}(x)\|_2 \right)
\]

Putting together inequalities 39 and 40 we conclude that there exists a universal constant \( C'' \) such that

\[
\|f(X) - \mathbb{E}f(X)\|_p \leq C'' \left( \sqrt{p} \|\mathbb{E}_X D(X)\|_2 + \sqrt{p} \max_{x \in \{0,1\}^n} \|\hat{H}(x)\|_F + p \max_{x \in \{0,1\}^n} \|\hat{H}(x)\|_2 \right)
\]

To conclude the proof, we notice that by Markov’s inequality and the preceding result:

\[
\Pr \left[ |f(X) - \mathbb{E}f(X)| > eC'' \left( \sqrt{p} \|\mathbb{E}_X D(X)\|_2 + \sqrt{p} \max_{x \in \{0,1\}^n} \|\hat{H}(x)\|_F + p \max_{x \in \{0,1\}^n} \|\hat{H}(x)\|_2 \right) \right] \leq e^{-p}
\]

We now set

\[
p = \min \left( \frac{t^2}{\|\mathbb{E}_X \hat{D}(X)\|_2^2 + \max_{x \in \{0,1\}^n} \|\hat{H}(x)\|_F^2}, \frac{t}{\max_{x \in \{0,1\}^n} \|\hat{H}(x)\|_2} \right)
\]

and the result follows.

\[\square\]

7 Improved Bound for Estimating a Single Parameter

In this section, we prove a formal statement of Theorem 3.

**Theorem 4.** Assume that \( \|f\|_\infty = 1 \) and \( |\beta^*| \leq M \) for any parameter \( M \). Let \( \delta > 0 \) and further assume that

\[
\frac{\sqrt{F(|\beta^*|)}}{\log \log n + \log(1/\delta)} \geq C(M).
\]

Then, there exists an efficient estimator \( \hat{\beta} \), such that with probability \( 1 - \delta \) satisfies

\[
|\hat{\beta} - \beta^*| \leq C(M)\beta^*F(\beta^*)^{-1/2}(\log \log n + \log(1/\delta)).
\]

Similarly to the proof of Theorem 1, our estimator is the maximum pseudo likelihood. Define the negative log pseudo-likelihood as

\[
\varphi(\beta) = - \sum_{i=1}^n \log \Pr_{\beta}[x_i | x_{-i}].
\]

---

8The standard result in \([RV+13; Lat+06]\) applies to square matrices. Since \( \hat{H} \) might not be necessarily a square matrix, we can just make it square by adding zeroes to the missing dimensions. The quadratic form doesn’t change and the matrix norms \( \| \cdot \|_F, \| \cdot \|_2 \) remain the same.
Then, the following holds:

\[ \phi'(\beta) := \frac{d\phi(\beta)}{d\beta} = \frac{1}{2} \sum_{i=1}^{n} (J_i x)(\tanh(J_i \beta x) - x_i) \]

(41)

and further,

\[ \phi''(\beta) := \frac{d^2\phi(\beta)}{d\beta^2} = \frac{1}{2} \sum_{i=1}^{n} (J_i x)^2 \text{sech}^2(\beta J_i x) \geq c \|J x\|^2_2, \]

(42)

where the last inequality holds uniformly for all \(|\beta| \leq M\), since \(\text{sech}^2\) is lower bounded whenever its argument is bounded from above. Similarly to the arguments in Lemma 2 and Lemma 3 it suffices to bound the ratio between the first and second derivatives of \(\phi\) and we obtain that

\[ |\beta^* - \hat{\beta}| \leq \frac{\min_{|\beta| \leq M} |\phi'(|\beta^*|)|}{\phi''(\beta^*)} \leq C \frac{\min_{|\beta| \leq M} |\phi'(|\beta^*|)|}{\|J x\|^2_2}. \]

(43)

We prove a lower bound on \(\|J x\|^2_2\) based on arguments from [Cha07; BM18], and use lemmas from the proof of Theorem 1 to show that the derivative is bounded in terms of \(\|J x\|_2\).

We begin with the second part. For this purpose, we provide a restatement of the lemmas from the proof of Theorem 1 that we will use here. First, the sub-sampling lemma, that shows how to reduce the correlations in the Ising model by subsampling:

**Lemma 4.** Let \(x = (x_1, \ldots, x_n)\) be an \((M, \gamma)\)-Ising model, and fix \(\eta \in (0, M]\). Then, there exist subsets \(I_1, \ldots, I_\ell \subseteq [n]\) with \(\ell \leq C M^2 \log n / \eta^2\) such that:

1. For all \(i \in [n]\),

\[ |\{j \in [\ell] : i \in I_j\}| = \left\lceil \frac{\eta \ell}{8M} \right\rceil. \]

2. For all \(j \in [\ell]\) and any value of \(x_{-I_j}\), the conditional distribution of \(x_{I_j}\) conditioned on \(x_{-I_j}\) is an \((\eta, \gamma)\)-Ising model.

Furthermore, for any non-negative vector \(\theta \in \mathbb{R}^n\) there exists \(j \in [\ell]\) such that

\[ \sum_{i \in I_j} \theta_i \geq \frac{\eta}{8M} \sum_{i=1}^{n} \theta_i. \]

We create the sets \(I_1, \ldots, I_\ell\) from this lemma with \(\eta = 1/2\). The following two lemmas would provide an upper bound on \(\phi'\) and a lower bound on \(\|J x\|_2\), respectively, both in terms of the same quantity, which is a function of the sets \(I_1, \ldots, I_\ell\):

**Lemma 19** (A special case of Lemma 8). For any \(t > 0\),

\[ \Pr \left[ |\phi'(\beta^*)| > t \left( \|J\|_F + \max_j \|E[J x|x_{-I_j}]\|_2 \right) \right] \leq C \log n \exp \left( -c \min \left( t^2, \frac{t \|J\|_F}{\|J\|_2} \right) \right). \]
Lemma 20 (A special case of Lemma 13). For any $t > 0$:  
\[
\Pr \left[ \|Jx\|_2^2 < c \|J\|_F^2 + c \max_j \|E[Jx | x_{-j}]\|_2^2 - t \left( \|J\|_F + \max_j \|E[Jx | x_{-j}]\|_2 \right) \right] 
\leq C \log n \exp \left( -c \frac{\min(t^2, t \|J\|_F)}{\|J\|_F^2} \right).
\]

Lemmas 19 and 20 follow from Lemma 8 and Lemma 13 by substituting $J^*$ with $\beta^* J$ and $A$ with $J$, and noting that the quantity $\frac{2\phi'(\beta^*)}{\beta^*}$ in Section 4 corresponds to $\phi'(\beta^*)$ in this section. As a direct corollary, we can bound $\phi'$ with respect to $\|Jx\|$:  

Lemma 21. For any $t \geq 1$, with probability at least $1 - \log ne^{-ct}$,  
\[
|2\phi'(\beta^*)| = \left| x^\top Jx - \sum_{i=1}^n J_ix \tanh(\beta^* Jx) \right| \leq Ct\|Jx\|_2 + Ct.
\]

Proof: The first equality follows from definition, hence we will prove the inequality. From Lemma 19, it holds that for any $t > 0$,  
\[
\Pr \left[ |\phi'(\beta^*)| > t \left( \|J\|_F + \max_j \|E[Jx | x_{-j}]\|_2 \right) \right] 
\leq C \log n \exp \left( -c \min(t^2, t \|J\|_F) \|J\|_F^2 \right).
\]  

(44)  

(45)  

since $\|J\|_F \leq \|J\|_2$ for all $J$ and since $t \geq 1$. Second of all, from Lemma 20, for any $t > 0$, we have that  
\[
\Pr \left[ \|Jx\|_2^2 < c \|J\|_F^2 + c \max_j \|E[Jx | x_{-j}]\|_2^2 - t \left( \|J\|_F + \max_j \|E[Jx | x_{-j}]\|_2 \right) \right] 
\leq C \log n \exp \left( -c \frac{\min(t^2, t \|J\|_F)}{\|J\|_F^2} \right).
\]  

(48)  

(49)  

(50)  

(51)  

using the fact that $\|J\|_F \geq \|J\|_2$ and that $\|J\|_2$ is at most some constant.

Next, with probability at least $1 - 2 \log ne^{-c \min(t^2)}$, both the events that their probabilities are estimated in (44) and (48) hold, and assume that they do hold till the rest of the proof. Let $\zeta = \|J\|_F + \max_j \|E[Jx | x_{-j}]\|_2$ and we derive that $|\phi'(\beta^*)| \leq t\zeta$ and that $\|Jx\|_2^2 \geq c\zeta^2 - t\zeta$. Hence,  
\[
|\phi'(\beta^*)| \leq Ct\|Jx\|_2 + Ct.
\]  

(52)  

Indeed, if $\zeta$ is at most a constant times $t$ then (52) follows and otherwise, $\|Jx\| \geq c\zeta^2 / 2$ while $|\phi'(\beta^*)| \leq \zeta t$ and (52) follows as well. This concludes the proof. \qed
Next, we lower bound $\| J x \|_2^2$ in terms of the log partition function $F(\beta^* J)$, based on arguments from [Cha07; BM18],

$$\| J x \|_2^2 = \sum_{i=1}^{n} (J_i x)^2 \geq \sum_{i=1}^{n} J_i x \tanh(\beta^* J_i x)/\beta^*, \quad (53)$$

since for all $y \in \mathbb{R}$, $y$ and $\tanh(y)$ have the same sign and additionally, $|\tanh(y)| \leq |y|$. From Lemma 21, the right hand side of (53) can be approximated by $x^\top J x/\beta^*$. Further, the term $x^\top J x$ can be lower bounded by the log partition function:

**Lemma 22.** With probability at least $1 - e^{-F(\beta^*)/2}$, $x^\top J x \geq F(\beta^*)/\beta^*$.

**Proof.** Recall that

$$\Pr_{\beta^*} [ x ] = 2^{-n} e^{\beta^* x^\top J x/2 - F(\beta^*)}$$

hence,

$$\mathbb{E}_{\beta^*} \left[ e^{-\beta^* x^\top J x/2} \right] = \sum_x e^{-\beta^* x^\top J x/2} \Pr_{\beta^*} [ x ] = e^{-F(\beta^*)}.$$

Therefore, by Markov’s inequality,

$$\Pr_{\beta^*} \left[ \beta^* x^\top J x < F(\beta^*) \right] = \Pr_{\beta^*} \left[ e^{-\beta^* x^\top J x/2} > e^{-F(\beta^*)/2} \right] \leq \mathbb{E}_{\beta^*} [ e^{-\beta^* x^\top J x/2} ] / e^{-F(\beta^*)/2} = e^{-F(\beta^*)/2}.$$

By the above lemmas, we derive the following:

**Lemma 23.** For any $1 \leq t \leq c \sqrt{F(\beta^*)}$, with probability at least $e^{-ct}$,

$$\frac{q'(\beta^*)}{\| J x \|_2^2} \leq Ct\beta^*/\sqrt{F(\beta^*)}.$$

**Proof.** We derive from (53), Lemma 21 and Lemma 23 and from $t \leq c \sqrt{F(\beta^*)}$, that with probability at least $\log ne^{-ct}$,

$$\| J x \|_2^2 \geq \frac{1}{\beta^*} \sum_{i=1}^{n} J_i x \tanh(\beta^* J_i x) \geq \frac{x^\top J x - Ct \| J x \|_2 - Ct}{\beta^*} \geq \frac{F(\beta^*) - Ct \beta^*}{(\beta^*)^2} - \frac{Ct \| J x \|_2}{\beta^*} \geq \frac{F(\beta^*)/2}{(\beta^*)^2} - \frac{Ct \| J x \|_2}{\beta^*}, \quad (54)$$

where the last inequality follows from the above assumption that $1 \leq t \leq c \sqrt{F(\beta^*)}$ for a sufficiently small $c > 0$ and the $|\beta|$ is bounded by a constant. Assume that (54) holds from now onward. We derive that

$$(\| J x \|_2^2)^2 + \| J x \|_2 \beta^* C t - F(\beta^*)/2 \geq 0.$$}

By solving the quadratic inequality for $\| J x \|_2 \beta^*$, we derive that

$$\| J x \|_2 \beta^* \geq \frac{-Ct + \sqrt{C^2t^2 + 2F(\beta^*)}}{2} \geq \sqrt{F(\beta^*)/2} - Ct.$$
Applying the bound $t \leq c \sqrt{F(J\beta^*)}$, we derive that
\[ \|Jx\|_2 \geq \sqrt{F(J\beta^*)}/(2\beta^*). \] (55)

Further, from Lemma 21 with probability at least $\log n e^{-ct}$,
\[ \phi'(\beta^*) \leq Ct\|Jx\|_2 + Ct, \] (56)
and assume that this holds for the rest of the proof. By (55) and (56),
\[ \frac{\phi'(\beta^*)}{\|Jx\|_2} \leq C t\|Jx\|_2 + t \leq \frac{2\beta^* Ct}{\sqrt{F(J\beta^*)}} + \frac{4(\beta^*)^2 Ct}{F(J\beta^*)} \leq \frac{Ct\beta^*}{\sqrt{F(J\beta^*)}}, \] (57)
by the assumption of this lemma that $\sqrt{F(J\beta^*)}$ is at least a constant and since $\beta^*$ is bounded by a constant. \qed

By (43) and by Lemma 23, Theorem 4 follows, by taking $t = \log \log n + \log(1/\delta)$.

8 The Lower Bound on $\|\hat{J} - J^*\|_F$

The estimation error of Theorem 1 is (nearly)-optimal up to constants when $\|J^*\|_\infty < 1$ (more generally Dobrushin’s condition). We show this using Assouad’s Lemma [Ass83] in the form presented in [Yu97]. We begin by bounding the total variation distance between two Ising models in terms of the Frobenius norm distance between their interaction matrices. First, recall that the $\chi^2$ divergence is defined as follows:

**Definition 7 ($\chi^2$-divergence).** Given two discrete distributions $P$ and $Q$, the $\chi^2$-divergence between them is
\[ \chi^2(Q, P) = \mathbb{E}_{x \sim P} \left[ \left( \frac{Q(x)}{P(x)} - 1 \right)^2 \right]. \]

We have the following relation between the total variation distance $TV$ and the $\chi^2$ divergence.

**Lemma 24.** For any two distributions $P$ and $Q$,
\[ TV(Q, P) \leq \sqrt{\frac{\chi^2(Q, P)}{2}}. \]

Additionally, we use the following lemma regarding subexponential distributions:

**Lemma 25 ([Ver18], Proposition 2.7.1).** Let $X$ be a zero mean random variable, and suppose that there exists $K > 0$ such that for all $t > 0$,
\[ \Pr[|X| > t] \leq 2 \exp(-t/K). \] (58)

Then, for any $\lambda$ that satisfies $|\lambda| \leq 1/(CK)$, it holds that
\[ \mathbb{E}[\exp(\lambda x)] \leq \exp(C^2K^2\lambda^2), \] (59)
where $C > 0$ is a universal constant.

43
In fact, both (58) and (59) are equivalent, and a random variable that satisfies any of these is called subexponential. We are ready to bound the distance between two Ising models:

**Lemma 26.** Consider two Ising models $P, Q$ with interaction matrices $J, J + A$ respectively such that $\|J\|_\infty < 1$. Moreover, assume that $\|A\|_F$ is bounded by a sufficiently small constant $c < 1$. Then,

$$TV(Q, P) \leq \sqrt{\chi^2(Q, P)/2} \leq c\|A\|_F,$$

where $c$ and $c'$ are bounded constants whenever $1 - \|J\|_\infty$ is bounded away from zero.

**Proof.** We will bound the $\chi^2$ divergence between $Q$ and $P$ and the bound on the total variation will follow from Lemma [24]. The $\chi^2$ divergence will be bounded in terms of the moment generating function of some polynomial $g(x)$ where $x \sim P$. And the MGF of $g(x)$ will be bounded using Theorem 2 on the concentration of polynomials, combined with Lemma 25.

Define $g(x) = x^\top Ax/2 - \mathbb{E}_{x \sim P}[x^\top Ax/2]$ and notice that $\mathbb{E}_{x \sim P}[g(x)] = 0$. Further, define $w = \mathbb{E}_{x \sim P}[\exp(g(x))]$. We will present $\chi^2(Q, P)$ in terms of $g(x)$ and $w$. First, notice that for any $x \in \{\pm 1\}^n$,

$$Q(x)/P(x) = e^{x^\top (J + A)x/2} \sum_x e^{x^\top (J + A)x/2} = e^{x^\top Ax/2} \left( \sum_x e^{x^\top (J + A)x/2} \right)^{-1} = e^{x^\top Ax/2} \left( \sum_x e^{x^\top (J + A)x/2} \right)^{-1} = \exp(x^\top Ax/2) / \mathbb{E}_{x \sim P}[\exp(x^\top Ax/2)] = \exp(g(x)) / w. \quad (60)$$

From (60) we can present $\chi^2(Q, P)$ as follows:

$$\chi^2(Q, P) = \mathbb{E}_{x \sim P} \left[ \left( \frac{Q(x)}{P(x)} - 1 \right)^2 \right] = \mathbb{E}_{x \sim P} \left[ \left( \frac{\exp(g(x))}{w} - 1 \right)^2 \right] = \mathbb{E}_{x \sim P} \left[ \frac{(\exp(g(x)) - w)^2}{w^2} \right]. \quad (63)$$

Moreover from Jensen’s inequality we have

$$w = \frac{\mathbb{E}_{x \sim P}[\exp(x^\top Ax/2)]}{\exp(\mathbb{E}_{x \sim P}[x^\top Ax/2])} \geq 1, \quad (64)$$

which implies that

$$\chi^2(Q, P) \leq \mathbb{E}_{x \sim P} \left[ (\exp(g(x)) - w)^2 \right] = \mathbb{E}_{x \sim P} \left[ \exp(2g(x)) - \exp(g(x))w + w^2 \right] = \mathbb{E}_{x \sim P} \left[ \exp(2g(x)) - w^2 \right] \leq \mathbb{E}_{x \sim P} \left[ \exp(2g(x)) - 1 \right]. \quad (65)$$

In order to bound $\exp(2g(x))$ we will use Lemma 25 and in order to apply that lemma, we will bound the tail of $g(x)$. Since $P$ satisfies Dobrushin’s condition, from Theorem 2 on the concentration of polynomials (taken from [AKPS+19]):

$$\Pr_{x \sim P}[|g(x)| > t] \leq 2 \exp \left( -c \min \left( \frac{t^2}{\|A\|^2_F + \|EAx\|^2_F}, \frac{t}{\|A\|^2_F} \right) \right) \quad (66)$$

\begin{align*}
&\leq 2 \exp \left( -c \min \left( \frac{t^2}{\|A\|^2_F + \|EAx\|^2_F}, \frac{t}{\|A\|^2_F} \right) \right), \quad (67)
\end{align*}
using the fact that $x \sim P$ is zero mean, and that $\| \cdot \|_2 \leq \| \cdot \|_F$. We further derive that $\Pr[|g(x)| > t] \leq 2 \exp(-c't/\|A\|_F)$, for some other constant $c' < -\ln 2$, since for $t/\|A\|_F < 1$, the right hand side of (67) is greater than 1 and trivially upper bounds $\Pr[|g(x)| > t]$. Using Lemma 25 with $K = \|A\|_F/c'$, we derive that

$$
\mathbb{E}_{x \sim P}[\exp(\theta g(x))] \leq \exp(C^2\theta^2\|A\|_F^2)
$$

(68)

for all $\theta \leq c_1/\|A\|_F$ where $c_1, C > 0$ are some constants. Since we assumed that $\|A\|_F$ is bounded by a sufficiently small constant, we can assume that $2 \leq c_1/\|A\|_F$, and in particular, derive that

$$
\mathbb{E}_{x \sim P}[\exp(2g(x))] \leq \exp(4C^2\|A\|_F^2) \leq 1 + 8C^2\|A\|_F^2,
$$

where the last inequality follows from the fact that $e^x \leq 1 + 2x$ for $x \leq 1$, and we can assume that $4C^2\|A\|_F^2 \leq 1$. In combination with (65), the proof follows. 

Next we state a simple variant of Assouad’s Lemma in the form we apply it. It is a popular tool for showing statistical minimax lower bounds on the error achieved by any estimator.

**Lemma 27** ([Yu97], Lemma 2). Let $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$, where $\Theta = \{-1, 1\}^k$. Let $\hat{\theta}(x)$ be an estimator for $\theta$ based on $x$ drawn from $P \in \mathcal{P}$. Assume that for any $\theta, \theta' \in \Theta$ that differ on one coordinate, $TV(P_\theta, P_\theta') \leq 1/2$. Then,

$$
\max_{\tau \in \Theta} \mathbb{E}_{x \sim P_\theta}[\|\hat{\theta}(x) - \tau\|_1] \geq k/4.
$$

Using Lemma 27, we show that the $\tilde{O}(\sqrt{k})$ error achieved by Theorem 1 is (nearly) tight.

**Theorem 5.** Let $\mathcal{V}$ be a $k$ dimensional vector space, such that there exist $J_1, \ldots, J_k \in \mathcal{V}$ that are (i) orthonormal with respect to the trace inner product, namely $\text{trace}(J_i^T J_j) = 0$ if $i \neq j$ and $\|J_i\|_F^2 = \text{trace}(J_i^T J_i) = 1$, and (ii) for all $\tilde{\sigma} \in [-1, 1]^k$, $\|\sum_i \sigma_i J_i\|_\infty \leq C$ for some constant $C > 0$. Then, any estimator $\hat{f}(x)$ satisfies

$$
\max_{f \in \mathcal{V} : \|f\|_\infty \leq 1} \mathbb{E}_{x \sim P_\theta}[\|\hat{f}(x) - f\|_F] \geq c'\sqrt{k}.
$$

**Proof.** We first present an overview of our approach. Given the vector space $\mathcal{V}$, let $J_1, \ldots, J_k$ denote the matrices guaranteed from this lemma. Let $A_i = c J_i$ where $c > 0$ is a sufficiently small constant to be defined later and define for all $\theta \in \{-1, 1\}^k$: $A_\theta = \sum_{i=1}^k \theta_i A_i$, and $P_\theta = \Pr_{A_\theta}$, namely, the Ising model with interaction matrix $A_\theta$. Notice that if $c$ is sufficiently small then the following holds:

- For all $\theta \in \{-1, 1\}^k$, $\|A_\theta\|_\infty \leq 1/2$, but the assumption of this theorem.
- For any $\theta, \theta' \in \{-1, 1\}^k$ that differ on one coordinate, it holds that $TV(P_\theta, P_\theta') \leq 1/2$, by Lemma 26.

We define $c$ such that these two requirements hold, and derive from Lemma 27 that any estimator $\hat{\theta}(x)$ satisfies

$$
\max_{\tau \in \Theta} \mathbb{E}_{x \sim P_\theta}[\|\hat{\theta}(x) - \tau\|_1] \geq k/4.
$$
Since the $\ell_2$ distance and the $\ell_1$ distance over $\mathbb{R}^k$ differ by a multiplicative factor of at most $\sqrt{k}$, we derive that

$$\max_{\tau \in \Theta} \mathbb{E}_{x \sim P_{\tau}} \left[ \| \hat{\theta}(x) - \tau \|_2 \right] \geq \sqrt{k}/4.$$ 

As a consequence, any estimator $\hat{A}$ for $A_{\tau} := \sum_{i=1}^k \tau_i A_i$ has an expected error of $\Omega(\sqrt{k})$ in Frobenius norm. Indeed, given $\hat{A}$ let $\hat{\theta}$ denote the vector that satisfies $\sum_i \hat{\theta}_i A_i = \hat{A}$. Then, since $A_1, \ldots, A_j$ are orthogonal with respect to the trace inner product and satisfy $\|A_i\|_F = c$ for the constant $c$ defined above, we derive that $\|A_{\tau} - A_{\tau'}\|_F = c \| \tau - \tau' \|_2$ for all $\tau, \tau' \in \mathbb{R}^k$. Hence, any estimator $\hat{A}$ satisfies

$$\max_{\tau \in \Theta} \mathbb{E}_{x \sim P_{\tau}} \left[ \| \hat{A}(x) - A_{\tau} \|_F \right] = c \max_{\tau \in \Theta} \mathbb{E}_{x \sim P_{\tau}} \left[ \| \hat{\theta}(x) - \tau \|_2 \right] \geq c\sqrt{k}/4.$$ 

This concludes the proof.

\[ \square \]

## A Dobrushin’s Uniqueness Condition

Here we present Dobrushin’s uniqueness condition in its full generality. First we define the influence of a node $j$ on a node $i$.

**Definition 8 (Influence in Graphical Models).** Let $\pi$ be a probability distribution over some set of variables $V$. Let $B_j$ denote the set of state pairs $(X, Y)$ which differ only in their value at variable $j$. Then the influence of node $j$ on node $i$ is defined as

$$I(j, i) = \max_{(X, Y) \in B_j} \left\| \pi_i(\cdot|X^{-i}) - \pi_i(\cdot|Y^{-i}) \right\|_{TV}$$

Now, we are ready to state Dobrushin’s condition.

**Definition 9 (Dobrushin’s Uniqueness Condition).** Consider a distribution $\pi$ defined on a set of variables $V$. Let

$$\alpha = \max_{i \in V} \sum_{j \in V} I(j, i)$$

$\pi$ is said to satisfy Dobrushin’s uniqueness condition if $\alpha < 1$.

Notice that $\|J^*\|_\infty < 1$ implies Dobrushin’s condition and a proof can found in [Cha05]. A generalization of this condition was given by [Hay06] who defined the generalized Dobrushin’s condition as $\|I\|_2 < 1$, where $I = I(i, j)$ is the influence matrix. This condition, while being weaker, retains most desirable properties of the original Dobrushin’s condition.

## B Technical Proofs

We begin with the proof of Lemma 7 and then move to the proof of Lemma 12.
B.1 Proof of Lemma \([7]\)

In this section, we prove the following lemma:

**Lemma 7.** For any symmetric matrix \(A\) with zeros on the diagonal, we have

\[
\Pr \left[ |\psi_j(x; A)| \geq t \bigg| x_{-I_j} \right] \leq \exp \left( -c \min \left( \frac{t^2}{\|E Ax + b\|_2^2 + \|A\|_F^2}, \frac{t}{\|A\|_2} \right) \right),
\]

where \(A = A_{I_j} ; \ b' = A_{I_j-1} x_{-I_j} ; \ J' = J'_{I_j} \) and \(h' = J'_{I_j} x_{-I_j} \) and \(x' = x_I\).

Notice that \(A', b', J', h'\) are fixed conditioned on \(x_{-I_j}\) and \(x'\) is distributed as the conditional distribution of \(x_I\) conditioned on \(x_{-I_j}\). Furthermore, \(x'\) is a \((1/2, \gamma)\)-Ising model, with interaction matrix \(J'\) and external field \(h'\), conditioned on \(x_{-I_j}\). Hence, the following lemma will imply that the right hand size of (69) concentrates conditioned on \(x_{-I_j}\).

**Lemma 28.** Let \(x\) be a \((1/2, \gamma)\) Ising model over \(\{-1, 1\}^m\) with interaction matrix \(J\) and external field \(h\). Let \(A\) be a symmetric real matrix of dimension \(m \times m\) with zeros on the diagonal, let \(b \in \mathbb{R}^m\) be a vector and let

\[
f(x) = \sum_{i \in [m]} (A_i x + b_i)(x - \tanh(J_i x + h))\]

Then, for any \(t > 0\),

\[
\Pr[|f(x)| \geq t] \leq \exp \left( -c \min \left( \frac{t^2}{\|EAx + b\|_2^2 + \|A\|_F^2}, \frac{t}{\|A\|_2} \right) \right),
\]

where \(c > 0\) is lower bounded by a constant constant whenever \(\|A\|_\infty, \|b\|_\infty, \|A'\|_\infty\) and \(\|b'\|_\infty\) are bounded from above by a constant.

First we derive Lemma \([7]\) based on Lemma \([28\). Substituting \(A = A', b = b', J = J'\) and \(h = h'\) and \(x = x'\), we derive that,

\[
\Pr \left[ \psi_j(x; A) \geq t \bigg| x_{-I_j} \right] \leq \exp \left( -c \min \left( \frac{t^2}{\|EA'x + b'\|_2^2 + \|A'\|_F^2}, \frac{t}{\|A'\|_2} \right) \right)
= \exp \left( -c \min \left( \frac{t^2}{\|EA_I x\|_2^2 + \|A_{I_j}\|_F^2}, \frac{t}{\|A_{I_j}\|_2} \right) \right)
\leq \exp \left( -c \min \left( \frac{t^2}{\|EA_I x\|_2^2 + \|A\|_2}, \frac{t}{\|A\|_2} \right) \right),
\]

using the fact that \(\|A_{I_j}\|_2 \leq \|A\|_2\) and that \(\|A_{I_j}\|_F \leq \|A\|_F = 1\) for all \(A \in \mathcal{A}\). This concludes the proof of Lemma \([7]\). Lastly, we prove Lemma \([28]\).
Proof of Lemma 28. By abuse of notation, define \( \text{tanh} : \mathbb{R}^m \to \mathbb{R}^m \) by

\[
\text{tanh}(y_1, \ldots, y_m) = (\text{tanh}(y_1), \ldots, \text{tanh}(y_m)).
\]

Decompose \( f(x) \) as

\[
f(x) = g(x)^\top h(x),
\]

where \( g(x) = Ax + b \) and \( h(x) = x - \text{tanh}(Jx + h) \).

We start by bounding \( \sum_i (D_i f(x))^2 \). Decompose

\[
2D_i f(x) = f(x_i^+) - f(x_i^-) = \langle g(x_i^+) - g(x_i^-), h(x_i^+) - h(x_i^-) \rangle + \langle h(x_i^+) - h(x_i^-), g(x_i^+) - g(x_i^-) \rangle
\]

By Cauchy Schwartz, for any \( a, b, c, d \in \mathbb{R} \), we have that \( (a + b + c + d)^2 \leq 4(a^2 + b^2 + c^2 + d^2) \). Hence,

\[
\sum_{i=1}^m (D_i f(x))^2 \leq \sum_{i=1}^m ((g(x_i^+) - g(x_i^-))^\top (h(x_i^+) - h(x_i^-)))^2 + \sum_{i=1}^m (g(x_i^+) - g(x_i^-))^\top (h(x_i^+) - h(x_i^-))^2
\]

\[
+ \sum_{i=1}^m ((g(x_i^+) - g(x_i^-))^\top (h(x_i^-) - h(x)))^2 + \sum_{i=1}^m ((g(x_i^+) - g(x_i^-))^\top h(x))^2. \quad (70)
\]

We will bound the terms in the right hand side of (70) one after the other.

Term 4:

\[
\langle g(x_i^+) - g(x_i^-), h(x) \rangle = \langle x_i^+ - x_i^-, A(x - \text{tanh}(Jx + h)) \rangle = 2e_i^\top A(x - \text{tanh}(Jx + h)).
\]

Summing over all \( i \), we get

\[
\sum_i (\langle g(x_i^+) - g(x_i^-), h(x) \rangle)^2 = 4\|A(x - \text{tanh}(Jx + h))\|_2^2.
\]

Term 3: using the fact that \( \text{tanh} \) is 1-Lipschitz and Cauchy Schwartz,

\[
|\langle g(x_i^+) - g(x_i^-), (h(x_i^+) - h(x_i^-)) \rangle| = |2e_i^\top A(x_i^+ - \text{tanh}(Jx_i^+ + h) - x - \text{tanh}(Jx + h))|
\]

\[
\leq 2\|e_i^\top A\|_2 \|x_i^+ - x\|_2 + \|x - \text{tanh}(Jx_i^+ + h) - \text{tanh}(Jx + h)\|_2
\]

\[
\leq 2\|e_i^\top A\|_2 (2 + \|Jx_i^+ + h - Jx + h\|_2)
\]

\[
\leq 2\|e_i^\top A\|_2 (2 + \|J\|_2 \|x_i^+ - x\|_2)
\]

\[
\leq C\|e_i^\top A\|_2.
\]

Summing over all \( i \), we get

\[
\sum_i |\langle g(x_i^+) - g(x_i^-), h(x_i^-) - h(x) \rangle|^2 \leq C \sum_i \|e_i^\top A\|_2^2 = C\|A\|_2^2.
\]
We will bound both terms from the right hand side of (73). Starting with the first term, for any fixed $i \in [m]$ we have

$$\bar{D}_i(x_{i+}) - \bar{D}_i(x_{i-}) = 2A_{ij}.$$
Hence,
\[ \sum_{j=1}^{m} \left( \xi_{i}(\tilde{D}_{i}(x_{j+}) - \tilde{D}_{i}(x_{j-})) \right)^{2} = 4 \sum_{j} \left( \sum_{i} \xi_{i}A_{ij} \right)^{2} = 4\|\tilde{\xi}_{1\ldots m}A\|_{2}^{2}, \]
where \( \tilde{\xi}_{1\ldots m} = (\tilde{\xi}_{1}, \ldots, \tilde{\xi}_{m}) \). For the second term of \((73)\), we have:
\[ \sum_{j=1}^{m} \left( \sum_{i=n+1}^{2m} \xi_{i}(\tilde{D}_{i}(x_{j+}) - \tilde{D}_{i}(x_{j-})) \right)^{2} = \|\tilde{\xi}_{n+1\ldots 2m}A \left( \sum_{i=n}^{m} x_{i+} - x_{i-} - \tanh(A'x_{i+} + h) + \tanh(A'x_{i-} + h) \right) \|_{2}^{2} = \|\tilde{\xi}_{n+1\ldots 2m}AW\|_{2}^{2}, \]
for the matrix \( W \) that we defined in the calculations of the discrete derivative, in \((71)\). By \((72)\) we get that
\[ \|\tilde{\xi}_{n+1\ldots 2m}AW\|_{2}^{2} \leq \|\tilde{\xi}_{n+1\ldots 2m}A\|_{2}^{2}\|W\|_{2}^{2} \leq C\|\tilde{\xi}_{n+1\ldots 2m}A\|_{2}^{2}. \]
By \((73)\) and the bounds on both terms in its right hand side, we derive that
\[ \sum_{j\in[m]} D_{j}(\tilde{\xi}^{\top} \tilde{D}(x))^{2} \leq \|\tilde{\xi}^{\top} \tilde{H}\|_{2}^{2}, \]
where \( \tilde{H} = C(A|A|0) \) is the matrix of dimension \((2m + 1) \times m\) obtained from stacking two copies of \( A \) one on top of each other on top of one row of zeros at the bottom, all multiplied by a sufficiently large constant \( C \). Hence, we can define the pseudo Hessian as the constant function \( \tilde{H}(x) = \tilde{H} \).
Lastly, we would like to apply Theorem 3 applying it with the pseudo discrete derivative and Hessian defined above. We would just have to calculate:
\[ \|E_{x}[\tilde{D}(x)]\|_{2}^{2} = C\sum_{i=1}^{m} (E_{x}[A_{i}^{\top} x + b_{i}])^{2} + C\sum_{i=1}^{m} (E_{x}[A_{i}^{\top} (x - \tanh(Jx + h))])^{2} + C\|A\|_{2}^{2}. \]
The first summand equals \( \|E[Ax + b]\|_{2}^{2} \), while the second equals zero, from the same argument as in Claim 1. This implies that \( \|E[\tilde{D}(x)]\|_{2}^{2} \leq C\|E[Ax + b]\|_{2}^{2} + C\|A\|_{2}^{2} \). Next, we bound the terms corresponding to the pseudo Hessian: we have that \( \|\tilde{H}\|_{2} \leq C\|A\|_{2} \), and \( \|\tilde{H}\|_{F}^{2} \leq C\|A\|_{2}^{2} \). Plugging these in Theorem 5 concludes the proof. \( \square \)

**B.2 Proof of Lemma 12**

Now, we move on to the proof of Lemma 12. The function that we wish to show concentration about is a second degree polynomial, hence Theorem 2 applies. However, this Theorem requires the matrix to have 0 in the diagonal, which is not necessarily the case for \( A^{\top}A \). Hence, we need to modify the matrix so that it is zero-diagonal, obtain the concentration bound for the modified matrix and then translate the result in terms of the original matrix. This is done in the following proof.

**Proof of Lemma 12** Denote \( p(x) = \|Ax\|_{2}^{2} = x^{\top}A^{\top}Ax \). Let \( E \) be obtained from \( A^{\top}A \) by zeroing all elements of the diagonal and denote \( \tilde{p}(x) = x^{\top}Ex \). Note that \( \tilde{p}(x) - p(x) \) is a constant as a
function of $x$, since $x_i^2 = 1$ for all $i$. This means that it suffices to bound the deviation of $\tilde{p}(x)$. By Lemma 4, conditioning on $x_{-I}$, yields an Ising model with Dobrushin constant $1/2$. Hence, we can apply Theorem 2 and get that for any $t \geq 0$,

$$\Pr \left[ \left| \tilde{p}(x) - \mathbb{E} \left( \tilde{p}(x) \mid x_{-I} \right) \right| > t \mid x_{-I} \right] \leq \exp \left( -c \min \left( \frac{t^2}{\|E\|_F^2 + \|\mathbb{E}[E x_{-I}]\|_2^2} , \frac{t}{\|E\|_2} \right) \right). \quad \text{(74)}$$

Now, we show how $E$ can be replaced with $A^\top A$ in (74). First:

$$\|\mathbb{E}[E x_{-I}]\|_2 \leq \|\mathbb{E}[A^\top A x_{-I}]\|_2 + \|\mathbb{E}[(E - A^\top A)x_{-I}]\|_2,$$

which, using the inequality $(a + b)^2 \leq 2a^2 + 2b^2$, implies that

$$\|E\|_F^2 + \|\mathbb{E}[E x_{-I}]\|_2^2 \leq 2\|E\|_F^2 + 2\|\mathbb{E}[A^\top A x_{-I}]\|_2^2 + 2\|\mathbb{E}[(E - A^\top A)x_{-I}]\|_2^2$$

$$= 2\|A^\top A\|_F^2 + 2\|\mathbb{E}[A^\top A x_{-I}]\|_2^2.$$

In the last equality, we used the fact that $\|\mathbb{E}[(E - A^\top A)x_{-I}]\|_2^2$ is just the sum of the squares of the diagonal entries of $A^\top A$, which means that together with $\|E\|_F^2$ they add up to $\|A\|_F^2$. Next, notice that $\|E\|_2 \leq \|A^\top A\|_2$. To prove this, we note that for all $x \in \mathbb{R}^n$,

$$x^\top E x = x^\top A^\top A x - \sum_{i \in [n]} (A^\top A)_{ii} \leq x^\top A^\top A x.$$

Putting all of this together, we obtain that the right hand side of (74) is bounded by

$$\exp \left( -c' \min \left( \frac{t^2}{\|A^\top A\|_F^2 + \|\mathbb{E}[A^\top A x_{-I}]\|_2^2} , \frac{t}{\|A^\top A\|_2} \right) \right).$$

Finally, we want to make this bound depend on $A$ rather than $A^\top A$. First,

$$\|A^\top A\|_F^2 \leq \|A\|_2^2 \|A\|_2^2$$

using the well known inequality $\|AB\|_F \leq \|A\|_2 \|B\|_F$.

Next, we have:

$$\|\mathbb{E}[A^\top A x_{-I}]\|_2^2 = \|A^\top \mathbb{E}[A x_{-I}]\|_2^2 \leq \|A\|_2^2 \|\mathbb{E}[A x_{-I}]\|_2^2 \quad \text{(75)}$$

Lastly,

$$\|A^\top A\|_2 = \|A\|_2^2.$$

This concludes the proof. \qed
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