FUNCTIONAL INEQUALITIES FOR MODIFIED BESSEL FUNCTIONS

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Abstract. In this paper our aim is to show some mean value inequalities for the modified Bessel functions of the first and second kinds. Our proofs are based on some bounds for the logarithmic derivatives of these functions, which are in fact equivalent to the corresponding Turán type inequalities for these functions. As an application of the results concerning the modified Bessel function of the second kind we prove that the cumulative distribution function of the gamma-gamma distribution is log-concave. At the end of this paper several open problems are posed, which may be of interest for further research.

1. Introduction

Let us consider the probability density function \( \varphi : \mathbb{R} \to (0, \infty) \) and the reliability (or survival) function \( \overline{\varphi} : \mathbb{R} \to (0, 1) \) of the standard normal distribution, defined by

\[ \varphi(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \quad \text{and} \quad \overline{\varphi}(u) = \frac{1}{\sqrt{2\pi}} \int_u^\infty e^{-t^2/2} \, dt. \]

The function \( r : \mathbb{R} \to (0, \infty) \), defined by

\[ r(u) = \frac{\overline{\varphi}(u)}{\varphi(u)} = e^{u^2/2} \int_u^\infty e^{-t^2/2} \, dt, \]

is known in literature as Mills’ ratio [31] sect. 2.26] of the standard normal distribution, while its reciprocal \( 1/r \), defined by \( 1/r(u) = \varphi(u)/\overline{\varphi}(u) \), is the so-called failure (hazard) rate, which arises frequently in economics and engineering sciences. Recently, among other things, Baricz [12, Corollary 2.6] by using the Pinelis’ version of the monotone form of l’Hospital’s rule (see [35, 3, 4] for further details) proved the following result concerning the Mills ratio of the standard normal distribution:

Theorem A. If \( u_1, u_2 > u_0 \), where \( u_0 \approx 1.161527889 \ldots \) is the unique positive root of the transcendent equation \( u(u^2 + 2) \overline{\varphi}(u) = (u^2 + 1)\varphi(u) \), then the following chain of inequalities holds

\[ \frac{2r(u_1)r(u_2)}{r(u_1) + r(u_2)} \leq r \left( \frac{u_1 + u_2}{2} \right) \leq \sqrt{r(u_1)r(u_2)} \leq \sqrt{\overline{\varphi}(u_1)u_1} \leq \frac{r(u_1) + r(u_2)}{2} \leq r \left( \frac{2u_1u_2}{u_1 + u_2} \right). \]

Moreover, the first, second, third and fifth inequalities hold for all \( u_1, u_2 \) positive real numbers, while the fourth inequality is reversed if \( u_1, u_2 \in (0, u_0) \). In each of the above inequalities equality holds if and only if \( u_1 = u_2 \).

We note here that, since Mills’ ratio \( r \) is continuous, the second and third inequalities in (1) mean actually that under the aforementioned assumptions Mills’ ratio is log-convex and geometrically concave on the corresponding interval. More precisely, by definition a function \( f : [a, b] \subseteq \mathbb{R} \to (0, \infty) \) is log-convex if \( \ln f \) is convex, i.e. if for all \( u_1, u_2 \in [a, b] \) and \( \lambda \in [0, 1] \) we have

\[ f(\lambda u_1 + (1 - \lambda)u_2) \leq (f(u_1))^\lambda (f(u_2))^{1-\lambda}. \]

Similarly, a function \( g : [a, b] \subseteq (0, \infty) \to (0, \infty) \) is said to be geometrically (or multiplicatively) convex if \( g \) is convex with respect to the geometric mean, i.e. if for all \( u_1, u_2 \in [a, b] \) and \( \lambda \in [0, 1] \) we have

\[ g \left( u_1^{1/\lambda}u_2^{1-\lambda} \right) \leq (g(u_1))^{\lambda} (g(u_2))^{1-\lambda}. \]

We note that if \( f \) and \( g \) are differentiable then \( f \) is log-convex if and only if \( u \mapsto f'(u)/f(u) \) is increasing on \([a, b]\), while \( g \) is geometrically convex if and only if \( u \mapsto u g'(u)/g(u) \) is increasing on \([a, b]\). A similar
definition and characterization of differentiable log-concave and geometrically concave functions also holds.

Mean value inequalities similar to those presented above appear also in the recent literature explicitly or implicitly for other special functions, like the Euler gamma function and its logarithmic derivative (see for example the paper [2] and the references therein), the Gaussian and Kummer hypergeometric functions, generalized Bessel functions of the first kind, general power series (see the papers [5, 8, 9, 32], and the references therein), Bessel and modified Bessel functions of the first kind (see [13, 18, 32]).

In this paper, motivated by the above results, we are mainly interested in mean value functional inequalities concerning modified Bessel functions of the first and second kinds. The detailed content is as follows: in section 2 we present some preliminary results concerning some tight lower and upper bounds for the logarithmic derivative of the modified Bessel functions of the first and second kinds. These results will be applied in the sequel to obtain some interesting chain of inequalities for modified Bessel functions of the first and second kinds analogous to (1). To achieve our goal in section 2 we present some monotonicity properties of some functions which involve the modified Bessel functions of the first and second kinds. Section 3 is devoted to the study of the convexity with respect to Hölder (or power) means of modified Bessel functions of the first and second kinds. The results stated here complete and extend the results from section 2. As an application of our results stated in section 2, in section 4 we show that the cumulative distribution function of the three parameter gamma-gamma distribution is log-concave for arbitrary shape parameters. This result may be useful in problems of information theory and communications. Finally, in section 5 we present some interesting open problems, which may be of interest for further research.

2. Monotonicity properties of some functions involving modified Bessel functions

As usual, in what follows let us denote by $I_\nu$ and $K_\nu$ the modified Bessel functions of the first and second kinds of real order $\nu$ (see [11]), which are in fact the linearly independent particular solutions of the second order modified Bessel homogeneous linear differential equation [11, p. 77]

\[ u^2\nu''(u) + uu'(u) - (u^2 + \nu^2)v(u) = 0. \]

(2)

Recall that the modified Bessel function $I_\nu$ of the first kind has the series representation [11, p. 77]

\[ I_\nu(u) = \sum_{n \geq 0} \frac{(u/2)^{2n+\nu}}{n! (n + \nu + 1)}, \]

where $\nu \neq -1, -2, \ldots$ and $u \in \mathbb{R}$, while the modified Bessel function of the second kind $K_\nu$ (called sometimes as the MacDonald or Hankel function), is usually defined also as [11, p. 78]

\[ K_\nu(u) = \frac{\pi}{2} \frac{I_{-\nu}(u) - I_\nu(u)}{\sin \nu\pi}, \]

where the right-hand side of this equation is replaced by its limiting value if $\nu$ is an integer or zero. We note that for all $\nu$ natural and $u \in \mathbb{R}$ we have $I_\nu(u) = I_{-\nu}(u)$, and from the above series representation $I_\nu(u) > 0$ for all $\nu > -1$ and $u > 0$. Similarly, by using the familiar integral representation [11, p. 181]

\[ K_\nu(u) = \int_0^\infty e^{-u\cosh t} \cosh(\nu t) \, dt, \]

(3)

which holds for each $u > 0$ and $\nu \in \mathbb{R}$, one can see easily that $K_\nu(u) > 0$ for all $u > 0$ and $\nu \in \mathbb{R}$.

The following results provide some tight lower and upper bounds for the logarithmic derivatives of the modified Bessel functions of the first and second kinds $I_\nu$ and $K_\nu$ and will be used frequently in the sequel.

**Lemma B.** For all $u > 0$ and $\nu > 0$ the following inequalities hold

\[ \sqrt{\frac{\nu}{\nu + 1}} \frac{u}{u^2 + \nu^2} < \frac{uI_\nu''(u)}{I_\nu(u)} < \sqrt{u^2 + \nu^2}. \]

Moreover, the right-hand side of (4) holds true for all $\nu > -1$.

**Lemma C.** For all $u > 0$ and $\nu > 1$ the following inequalities hold

\[ -\sqrt{\frac{\nu}{\nu - 1}} \frac{u}{u^2 + \nu^2} < \frac{uK_\nu''(u)}{K_\nu(u)} < -\sqrt{u^2 + \nu^2}. \]
Moreover, the right-hand side of (5) holds true for all \( \nu \in \mathbb{R} \).

The left-hand side of (4) was proved for \( u > 0 \) and positive integer \( \nu \) by Phillips and Malin [34], and later by Baricz [14] for \( u > 0 \) and \( \nu > 0 \) real. The right-hand side of (4) appeared first in Gronwall’s paper [27] for \( u > 0 \) and \( \nu > 0 \) (motivated by a problem in wave mechanics), it was proved also by Phillips and Malin [34] for \( u > 0 \) and \( \nu \geq 1 \) integer, and recently by Baricz [14] for \( u > 0 \) and \( \nu \geq -1/2 \) real (motivated by a problem in biophysics; see [33]). For this inequality the case \( u > 0 \) and \( \nu > -1 \) real has been proved recently in [17].

The left-hand side of (5) was proved first by Phillips and Malin [34] for \( u > 0 \) and \( \nu > 1 \) positive integer, and was extended to the case \( u > 0 \) and \( \nu > 1 \) real recently by Baricz [14]. Finally, the right-hand side of (5) was proved first by Phillips and Malin [34] for \( u > 0 \) and \( \nu \geq 1 \) integer, and later extended to the case of \( u > 0 \) and \( \nu \) real arbitrary by Baricz [14].

It is worth mentioning that the inequalities (4) and (5), which have been proved recently also by Segura [37], are in fact equivalent to the Turán type inequalities for the modified Bessel functions of the first and second kinds. For further details the interested reader is referred to [14] [17] [19] [30] [37] and to the references therein.

Our first main result reads as follows.

**Theorem 1.** The following assertions are true:

(a) \( u \mapsto uI_{\nu}^2(u)/I_{\nu}^2(u) \) is strictly decreasing on \((0, \infty)\) for all \( \nu \geq 1 \);

(b) \( u \mapsto uI_{\nu}^2(u)/I_{\nu}(u) \) is strictly increasing on \((0, \infty)\) for all \( \nu > 1/2 \);

(c) \( u \mapsto \sqrt{u}I_{\nu}(u) \) is strictly log-concave on \((0, \infty)\) for all \( \nu \geq 1/2 \);

(d) \( u \mapsto u^2I_{\nu}^2(u)/I_{\nu}^2(u) \) is strictly decreasing on \((0, \infty)\) for all \( \nu \geq \nu_0 \), where \( \nu_0 \approx 1.373318506 \ldots \) is the positive root of the cubic equation \( 8\nu^2 - 9\nu^2 - 2\nu - 1 = 0 \).

In particular, for all \( u_1, u_2 > 0 \) and \( \nu \geq \nu_0 \) the following chain of inequalities holds

\[
\frac{2I_{\nu}(u_1)I_{\nu}(u_2)}{I_{\nu}(u_1) + I_{\nu}(u_2)} \leq I_{\nu}(\sqrt{u_1u_2}) \leq \sqrt{I_{\nu}(u_1)I_{\nu}(u_2)} \leq \sqrt{\frac{u_1 + u_2}{2\sqrt{u_1u_2}}} I_{\nu}(\frac{u_1 + u_2}{2}).
\]

Moreover, the second and third inequalities hold true for all \( \nu > 1 \), and the fourth inequality holds true for all \( \nu \geq 1/2 \). In each of the above inequalities equality holds if and only if \( u_1 = u_2 \).

We recall that part (b) of Theorem 1 was proved for \( \nu > 0 \) by Gronwall [27]. Notice also that recently Baricz [14] in order to prove the right-hand side of (4) proved implicitly part (b) of Theorem 1. For reader’s convenience we recall below that proof. Moreover, we give a somewhat different proof of this part, and two other completely different proofs.

We note that part (c) of Theorem 1 improves the result of Sun and Baricz [38], who proved that the function \( u \mapsto uI_{\nu}(u) \) is log-concave on \((0, \infty)\) for all \( \nu \geq 1/2 \). Recently, Baricz and Neuman [18] conjectured that the modified Bessel function \( I_{\nu} \) of the first kind is strictly log-concave on \((0, \infty)\) for all \( \nu > 0 \). As far as we know, this conjecture is still open and the much sharper result of this kind is of part (e) of Theorem 1.

**Proof of Theorem 1.** First we prove the monotonicity and log-concavity properties stated above.

(a) Recall that the modified Bessel function of the first kind \( I_{\nu} \) is a particular solution of the second-order differential equation (3) and thus

\[
I_{\nu}''(u) = (1 + \nu^2/u^2)I_{\nu}(u) - (1/u)I_{\nu}'(u).
\]

Using (7) and the left-hand side of (4), we obtain that for all \( u > 0 \) and \( \nu \geq 1 \)

\[
\frac{d}{du} \left[ \frac{uI_{\nu}'(u)}{I_{\nu}^2(u)} \right] = \left[ \frac{1}{uI_{\nu}(u)} \right] u^2 + \nu^2 - 2 \left[ \frac{uI_{\nu}'(u)}{I_{\nu}(u)} \right]^2 < \left[ \frac{1}{uI_{\nu}(u)} \right] \left[ -\nu^2 + \frac{1 - \nu}{1 + \nu} u^2 \right] \leq 0.
\]

(b) Consider the Turánian

\[
\Delta_{\nu}(u) = I_{\nu}^2(u) - I_{\nu-1}(u)I_{\nu+1}(u),
\]

which in view of the recurrence relations

\[
I_{\nu-1}(u) = (\nu/u)I_{\nu}(u) + I_{\nu}'(u)
\]

and

\[
I_{\nu+1}(u) = -(\nu/u)I_{\nu}(u) + I_{\nu}'(u),
\]

we obtain that

\[
\Delta_{\nu}(u) = (1 + \nu^2/u^2)I_{\nu}(u) - (1/u)I_{\nu}'(u)\frac{d}{du} \left[ \frac{uI_{\nu}'(u)}{I_{\nu}^2(u)} \right] \leq 0.
\]

In particular, for all \( u_1, u_2 > 0 \) and \( \nu \geq \nu_0 \) the following chain of inequalities holds

\[
\frac{2I_{\nu}(u_1)I_{\nu}(u_2)}{I_{\nu}(u_1) + I_{\nu}(u_2)} \leq I_{\nu}(\sqrt{u_1u_2}) \leq \sqrt{I_{\nu}(u_1)I_{\nu}(u_2)} \leq \sqrt{\frac{u_1 + u_2}{2\sqrt{u_1u_2}}} I_{\nu}(\frac{u_1 + u_2}{2}).
\]

Moreover, the second and third inequalities hold true for all \( \nu > 1 \), and the fourth inequality holds true for all \( \nu \geq 1/2 \). In each of the above inequalities equality holds if and only if \( u_1 = u_2 \).
can be rewritten as follows
\[ \Delta_{\nu}(u) = (1 + \nu^2/u^2)I_{\nu}^2(u) - [I_{\nu}(u)]^2. \]

Using (7) we get
\[ \Delta_{\nu}(u) = \frac{1}{u I_{\nu}(u)} \left[ \frac{u I_{\nu}(u)}{I_{\nu}(u)} \right]'. \]

It is known (see [10, 17]) that the Turán-type inequality \( \Delta_{\nu}(u) > 0 \) holds for all \( u > 0 \) and \( \nu > -1 \), and hence the required result follows. We may note incidentally that the result of this part actually follows also from the right-hand side of (3). More precisely, it is easy to see that the function \( u \mapsto u I_{\nu}(u)/I_{\nu}(u) \) satisfies the differential equation \( uu'(u) = u^2 + \nu^2 - v^2(u) \), and the right-hand side of (4) is clearly strictly increasing on \((0, \infty)\) for all \( \nu > -1 \). It is important to add here that in fact the right-hand side of (4) and the Turán-type inequality \( \Delta_{\nu}(u) > 0 \) are equivalent (see [14, 17]).

A third proof of this part can be obtained as follows. By using the infinite series representation of the modified Bessel function of the first kind we just need to show that the function
\[ u \mapsto \frac{u I_{\nu}(u)}{I_{\nu}(u)} = \sum_{n=0}^{\infty} \frac{(2n + \nu)(u/2)^{2n}}{n! \Gamma(n + 1)} \]
is strictly increasing on \((0, \infty)\) for all \( \nu > -1 \). To do this let us recall the following well-known result (see [20, 36]): Let us consider the power series \( f(u) = a_0 + a_1 u + \ldots + a_n u^n + \ldots \) and \( g(u) = b_0 + b_1 u + \ldots + b_n u^n + \ldots \), where for all \( n \geq 0 \) integer \( a_n \in \mathbb{R} \) and \( b_n > 0 \), and suppose that both converge on \((0, \infty)\). If the sequence \( \{a_n/b_n\}_{n \geq 0} \) is strictly increasing, then the function \( u \mapsto f(u)/g(u) \) is strictly increasing too on \((0, \infty)\). We note that we can see easily that the above result remains true in the case of even functions. Thus, to prove that \( u \mapsto u I_{\nu}(u)/I_{\nu}(u) \) is indeed strictly increasing it is enough to show that the sequence \( \{a_n/b_n\}_{n \geq 0} \), defined by \( a_n = 2n + \nu \) for all \( n \geq 0 \), is strictly increasing, which is certainly true.

Finally, a fourth proof is as follows. By using the Weierstrassian factorization
\[ I_{\nu}(u) = \frac{u^\nu}{2 \Gamma(\nu + 1)} \prod_{n \geq 1} \left( 1 + \frac{u^2}{j_{\nu,n}^2} \right), \]
where \( \nu > -1 \) and \( j_{\nu,n} \) is the \( n \)-th positive zero of the Bessel function \( J_{\nu} \) of the first kind, we obtain that
\[ \frac{d}{du} \left[ \frac{u I_{\nu}(u)}{I_{\nu}(u)} \right] = \frac{d}{du} \left[ \nu + 2 \sum_{n \geq 1} \frac{u^2}{u^2 + j_{\nu,n}^2} \right] = 4 \sum_{n \geq 1} \frac{u j_{\nu,n}^2}{(u^2 + j_{\nu,n}^2)^2} > 0 \]
for all \( u > 0 \) and \( \nu > -1 \). We note that this proof reveals that the function \( u \mapsto u I_{\nu}(u)/I_{\nu}(u) \) is in fact strictly decreasing on \((-\infty, 0)\) for all \( \nu > -1 \). This is in agreement with the fact that the function \( u \mapsto u I_{\nu}(u)/I_{\nu}(u) \) is even, as we can see in the above series representations.

(c) Owing to Duff [24] it is known that the function \( u \mapsto \sqrt{u} K_{\nu}(u) \) is strictly completely monotonic, and consequently (see [12, p. 167]) strictly log-convex on \((0, \infty)\) for each \( |\nu| \geq 1/2 \). On the other hand, due to Hartman [28] the function \( u \mapsto u I_{\nu}(u)K_{\nu}(u) \) is concave, and consequently log-concave on \((0, \infty)\) for all \( \nu > 1/2 \). Since \( u \mapsto 2u I_{1/2}(u)K_{1/2}(u) = 1 - e^{-2u} \) is concave on \((0, \infty)\), we conclude that in fact the function \( u \mapsto u I_{\nu}(u)K_{\nu}(u) \) is concave, and hence log-concave on \((0, \infty)\) for all \( \nu \geq 1/2 \). Now, combining these results, in view of the fact that the product of log-concave functions is log-concave, the required result follows.

(d) Using (4) and (7) we obtain that
\[ \frac{d}{du} \left[ \frac{u^2 I_{\nu}(u)}{I_{\nu}(u)} \right] = \frac{1}{I_{\nu}(u)} \left[ u^2 + \nu^2 + \frac{u I_{\nu}(u)}{I_{\nu}(u)} - 2 \left( \frac{u I_{\nu}(u)}{I_{\nu}(u)} \right)^2 \right] < \left[ u^2 + \nu^2 + \sqrt{u^2 + \nu^2} - 2 \left( \frac{u^2 \nu}{\nu + 1} + \nu^2 \right) \right] \]
for all \( u > 0 \) and \( \nu > 0 \). Observe that the last expression is nonpositive if and only if we have
\[ \left( \frac{\nu - 1}{\nu + 1} \right)^2 u^4 + \left( 2\nu^2 \frac{\nu - 1}{\nu + 1} - 1 \right) u^2 + \nu^2 (\nu^2 - 1) \geq 0. \]
A computation shows that this is satisfied if
\[ \left[ 2\nu^2 \frac{\nu - 1}{\nu + 1} - 1 \right]^2 - 4 \left( 2\nu^2 \frac{\nu - 1}{\nu + 1} - 1 \right) \nu^2 (\nu^2 - 1) = - \frac{8\nu^3 - 9\nu^2 - 2\nu - 1}{(\nu + 1)^2} \leq 0. \]
Now, since $\nu \geq \nu_0$ we have $8\nu^3 - 9\nu^2 - 2\nu - 1 \geq 0$ and thus the proof of part (d) is complete.

It should be mentioned here that part (a) of this theorem for $\nu \geq \nu_0$ actually is an immediate consequence of this part. More precisely, the proof of part (a) of this theorem can be simplified significantly as follows: in view of part (d) of this theorem, the function

$$u \mapsto \frac{uK'_\nu(u)}{K^2_\nu(u)} = \frac{1}{u} \cdot \frac{u^2K'_\nu(u)}{K^2_\nu(u)}$$

is strictly decreasing as a product of two positive and strictly decreasing functions.

Now, let us focus on the chain of inequalities (8). To prove this we use Corollary 2.5 from [5]. More precisely, the first inequality in (8) follows from part (d) of this theorem, while the second inequality in (8) is an immediate consequence of the fact that $I_\nu$ is a strictly increasing function on $(0, \infty)$ for all $\nu > -1$. The third inequality in (8) means actually the strict geometrical convexity of $I_\nu$ and is equivalent to part (b) of this theorem; the fourth inequality is equivalent to part (c) of this theorem.

Finally, observe that part (a) of this theorem is equivalent to the inequality

$$\frac{2I_\nu(u_1)I_\nu(u_2)}{I_\nu(u_1) + I_\nu(u_2)} \leq I_\nu\left(\sqrt{u_1u_2}\right),$$

which holds for all $u_1, u_2 > 0$ and $\nu \geq 1$. Moreover, in this inequality equality holds if and only if $u_1 = u_2$.

The following result is a companion of Theorem 1 for modified Bessel functions of the second kind. We note that part (b) of the following theorem is well-known (see for example [29, 35, 36]), and part (c) was proved by Baricz [17]. For part (b) we give here a different proof, while for part (c) we recall the proof from [17] and we present a simple alternative proof.

**Theorem 2.** The following assertions are true:

(a) $u \mapsto K'_\nu(u)/K^2_\nu(u)$ is strictly decreasing on $(0, \infty)$ for all $|\nu| \geq 1$;
(b) $u \mapsto K'_\nu(u)/K_\nu(u)$ is strictly increasing on $(0, \infty)$ for all $\nu \in \mathbb{R}$;
(c) $u \mapsto uK'_\nu(u)/K_\nu(u)$ is strictly decreasing on $(0, \infty)$ for all $\nu \in \mathbb{R}$;
(d) $u \mapsto uK'_\nu(u)$ is strictly increasing on $(0, \infty)$ for all $\nu \in \mathbb{R}$;
(e) $u \mapsto u^2K'_\nu(u)$ is strictly increasing on $(0, \infty)$ for all $|\nu| \geq 5/4$;
(f) $u \mapsto u^2K''_\nu(u)$ is strictly increasing on $(2, \infty)$ for all $\nu \in \mathbb{R}$.

In particular, for all $u_1, u_2 > 0$ and $|\nu| \geq 1$ the following chain of inequalities holds

$$2K_\nu(u_1)K_\nu(u_2) \leq K_\nu\left(\frac{u_1 + u_2}{2}\right) \leq K_\nu(u_1)K_\nu(u_2) \leq K_\nu\left(\sqrt{u_1u_2}\right) \leq \frac{K_\nu(u_1) + K_\nu(u_2)}{2}. \quad (8)$$

Moreover, the second, third and fourth inequalities hold true for all $\nu \in \mathbb{R}$. In addition, for $|\nu| \geq 5/4$ and $u_1, u_2 > 0$ the fourth inequality can be improved as

$$K_\nu\left(\frac{2u_1u_2}{u_1 + u_2}\right) \leq \frac{K_\nu(u_1) + K_\nu(u_2)}{2}. \quad (9)$$

This inequality holds true for all $u_1, u_2 > 2$ and $\nu \in \mathbb{R}$. In each of the above inequalities equality hold if and only if $u_1 = u_2$.

**Proof.** First we prove the monotonicity properties for modified Bessel functions of the second kind.

(a) Recall that the modified Bessel function of the second kind $K_\nu$ is a particular solution of the second-order differential equation (2), and this in turn implies that

$$K''_\nu(u) = (1 + u^2/\nu^2)K_\nu(u) - (1/u)K'_\nu(u). \quad (10)$$

Consequently, by using two times the right-hand side of (5), for all $u > 0$ and $\nu \geq 1$ we have

$$\frac{d}{du} \left[ \frac{K'_\nu(u)}{K^2_\nu(u)} \right] = \left[ \frac{1}{u^2K_\nu(u)} \right] \left[u^2 + \nu^2 \frac{uK'_\nu(u)}{K_\nu(u)} - 2 \left[ \frac{uK'_\nu(u)}{K_\nu(u)} \right]^2 \right] < - \left[ \frac{1}{u^2K_\nu(u)} \right] \left[ \frac{uK'_\nu(u)}{K_\nu(u)} \right] \left[ \frac{uK'_\nu(u)}{K_\nu(u)} + 1 \right] \leq 0.$$

On the other hand the function $\nu \mapsto K'_\nu(u)$ is even, and thus from the above result we obtain that indeed the function $u \mapsto K'_\nu(u)/K^2_\nu(u)$ is strictly decreasing on $(0, \infty)$ for all $|\nu| \geq 1$. 


The inequality (9) is equivalent to part concavity of $K$ we have

and strict log-convexity of (a) Corollary 2.5 from [5]. The first inequality in (8) follows from part

for all $u > 0$ and $\nu \in \mathbb{R}$. However, in view of (3), for all $n \in \{0, 1, 2, \ldots\}$, $u > 0$ and $\nu \in \mathbb{R}$, we easily have

\[ (-1)^n K^{(n)}_\nu(u) = \int_0^\infty (\cosh t)^n e^{-u \cosh t} \cosh(\nu t) \, dt > 0, \]

i.e. the function $u \mapsto K_\nu(u)$ is strictly completely monotonic. Now, since each strictly completely monotonic function is strictly log-convex, we obtain that $u \mapsto K_\nu(u)$ is strictly increasing on $(0, \infty)$ for all $\nu \in \mathbb{R}$.

(c) Consider the Turánian

\[ \Delta_\nu(u) = K_\nu^2(u) - K_{\nu - 1}(u)K_{\nu + 1}(u). \]

Using the recurrence relations

\[ K_{\nu - 1}(u) = (\nu/u) K_\nu(u) - K_\nu'(u) \]

and

\[ K_{\nu + 1}(u) = (\nu/u) K_\nu(u) - K_\nu'(u) \]

we have

\[ \Delta_\nu(u) = (1 + \nu^2/u^2)K_\nu^2(u) - [K_\nu'(u)]^2. \]

Combining this with (10), we obtain

\[ \Delta_\nu(u) = \frac{1}{u} K^2_\nu(u) \left[ \frac{uK_\nu'(u)}{K_\nu(u)} \right]' . \]

But, the function $\nu \mapsto K_\nu(u)$ is strictly log-convex on $\mathbb{R}$ for each fixed $u > 0$ (see [16]), which implies that for all $\nu \in \mathbb{R}$ and $u > 0$ the Turán-type inequality $\Delta_\nu(u) < 0$ holds. This shows that the function $u \mapsto uK_\nu'(u)/K_\nu(u)$ is strictly decreasing on $(0, \infty)$ for all $\nu \in \mathbb{R}$. Another proof for this part can be obtained as follows. First observe that the function $u \mapsto uK_\nu'(u)/K_\nu(u)$ satisfies the differential equation

\[ u^2 K''_\nu(u) - uK_\nu''(u) - [K_\nu(u)]^2 = 0 \]

for all $u > 0$ and $\nu > 1$. The right-hand side of the above inequality is positive if and only if the expression

\[ Q_\nu(u) = u^4 + [2\nu^2 - \nu/(\nu - 1)]u^2 + \nu^2(\nu^2 - 1) \]

is positive. It is easy to see that the discriminant of the equation $Q_\nu(\sqrt{u}) = 0$ is $(5 - 4\nu)/\nu^2/(\nu - 1)^2$ and this is negative if and only if $\nu \geq 5/4$. Finally, since the function $\nu \mapsto K_\nu(u)$ is even, the proof is complete.

(f) In view of (3) we obtain that

\[ u^2 K_\nu'(u) = -u^2 \int_0^\infty e^{-u \cosh t}(\cosh t)(\cosh(\nu t)) \, dt \]

and thus

\[ [u^2 k_\nu'(u)]' = u \int_0^\infty (u \cosh t - 2) e^{-u \cosh t}(\cosh t)(\cosh(\nu t)) \, dt > 0 \]

for all $u > 2$ and $\nu \in \mathbb{R}$.

Now, let us focus on the inequalities (8) and (9). As in the proof of the chain of inequalities (3), we use Corollary 2.5 from [3]. The first inequality in (8) follows from part (a), the second inequality is just the strict log-convexity of $K_\nu$ proved in part (b), while the third inequality is equivalent to the geometrical concavity of $K_\nu$ proved in part (c). The fourth inequality is equivalent to part (d) of this theorem, while the inequality (9) is equivalent to part (e).
3. Convexity of modified Bessel functions with respect to power means

In this section we are going to complement and extend the results of the above section. To this aim we study the convexity of modified Bessel functions of the first and second kinds with respect to Hölder means. For reader’s convenience we recall here first some basics.

Let \( \varphi : [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a strictly monotonic continuous function. The function \( M_\varphi : [a, b]^2 \rightarrow [a, b] \), defined by

\[
M_\varphi(u_1, u_2) = \varphi^{-1} \left( \frac{\varphi(u_1) + \varphi(u_2)}{2} \right)
\]

is called the quasi-arithmetic mean (or Kolmogorov mean) associated to \( \varphi \), while the function \( \varphi \) is called a generating function (or a Kolmogorov-Nagumo function) of the quasi-arithmetic mean \( M_\varphi \). A function \( f : [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{R} \) is said to be convex with respect to the mean \( M_\varphi \) (or \( M_\varphi \)-convex) if for all \( u_1, u_2 \in [a, b] \) and all \( \lambda \in [0, 1] \) the inequality

\[
f(M_\varphi^\lambda(u_1, u_2)) \leq \lambda f(u_1) + (1 - \lambda) f(u_2)
\]

holds, where \( M_\varphi^\lambda(u_1, u_2) = \varphi^{-1}(\lambda \varphi(u_1) + (1 - \lambda) \varphi(u_2)) \) is the weighted version of \( M_\varphi \). It can be proved easily (see for example [22]) that \( f \) is convex with respect to \( M_\varphi \) if and only if \( \varphi \circ f \circ \varphi^{-1} \) is convex in the usual sense on \( \varphi([a, b]) \). Now, for any two quasi-arithmetic means \( M_\varphi \) and \( M_\psi \) (with Kolmogorov-Nagumo functions \( \varphi \) and \( \psi \) defined on intervals \( [a, b] \) and \( [c, d] \)), a function \( f : [a, b] \rightarrow [c, d] \) is called \((M_\varphi, M_\psi)\)-convex if it satisfies

\[
f(M_\varphi^\lambda(u_1, u_2)) \leq M_\psi^\lambda(f(u_1), f(u_2))
\]

for all \( u_1, u_2 \in [a, b] \) and \( \lambda \in [0, 1] \), where \( M_\psi^\lambda(u_1, u_2) = \psi^{-1}(\lambda \psi(u_1) + (1 - \lambda) \psi(u_2)) \). In this case, if \( f \) is \((M_\varphi, M_\psi)\)-concave. Due to Aczél [1] it is known from a long time ago that if \( \psi \) is increasing then the function \( f \) is \((M_\varphi, M_\psi)\)-convex if and only if the function \( \psi \circ f \circ \varphi^{-1} \) is convex in the usual sense on \( \varphi([a, b]) \). This is because, if \( \psi \) is increasing and we denote with \( s \) and \( t \) the values \( \varphi(u_1) \) and \( \varphi(u_2) \), then by definition \( f \) is \((M_\varphi, M_\psi)\)-convex if and only if

\[
\psi \left( f \left( \varphi^{-1}(\lambda s + (1 - \lambda)t) \right) \right) \leq \lambda \psi \left( f \left( \varphi^{-1}(s) \right) \right) + (1 - \lambda) \psi \left( f \left( \varphi^{-1}(t) \right) \right)
\]

holds for all \( s, t \in \varphi([a, b]) \) and \( \lambda \in [0, 1] \). See also [22] for more details.

Now, if \( \psi \) is decreasing, then clearly the above inequality is reversed, and this in turn implies that the function \( f \) is \((M_\varphi, M_\psi)\)-convex if and only if the function \( \psi \circ f \circ \varphi^{-1} \) is concave in the usual sense on \( \varphi([a, b]) \). Moreover, a similar characterization of \((M_\varphi, M_\psi)\)-concave functions is also valid, depending on the monotonicity of the function \( \psi \).

Among the quasi-arithmetic means the Hölder means (or power means) are of special interest. They are associated to the generating function \( \varphi_p : (0, \infty) \rightarrow \mathbb{R} \), defined by

\[
\varphi_p(u) = \begin{cases} 
  u^p, & \text{if } p \neq 0 \\
  \ln u, & \text{if } p = 0,
\end{cases}
\]

and have the following form

\[
M_\varphi^\lambda(u_1, u_2) = \begin{cases} 
  [(1 - \lambda)u_1^p + \lambda u_2^p]^{1/p}, & \text{if } p \neq 0 \\
  u_1^{1/\lambda} u_2^{1-\lambda}, & \text{if } p = 0.
\end{cases}
\]

Now, let \( p \) and \( q \) be two arbitrary real numbers. Using the above definitions of generalized convexities we say that a function \( f : [a, b] \subseteq (0, \infty) \rightarrow (0, \infty) \) is \((M_\varphi, M_\psi)\)-convex, or simply \((p, q)\)-convex, if the inequality

\[
f(M_\varphi^\lambda(u_1, u_2)) \leq M_\psi^\lambda(f(u_1), f(u_2))
\]

is valid for all \( p, q \in \mathbb{R} \), \( u_1, u_2 \in [a, b] \) and \( \lambda \in [0, 1] \). If the above inequality is reversed, then we say that the function \( f \) is \((M_\varphi, M_\psi)\)-concave, or simply \((p, q)\)-concave. Observe that the \((1, 1)\)-convexity is the usual convexity, the \((1, 0)\)-convexity is exactly the log-convexity, while the \((0, 0)\)-convexity corresponds to the case of the geometrical convexity. We note that motivated by the works [5] and [7], recently Baricz [10] considered the \((p, p)\)-convexity of the zero-balanced Gaussian hypergeometric functions and general power series. The \((p, q)\)-convexity of zero-balanced Gaussian hypergeometric functions was considered recently by Zhang et al. [43].
The following result gives a characterization of differentiable \((p, q)\)-convex functions and will be applied in the sequel in the study of the convexity of modified Bessel functions of the first and second kinds with respect to power means. For a proof see [15].

**Lemma D.** Let \(p, q \in \mathbb{R}\) and let \(f : [a, b] \subseteq (0, \infty) \rightarrow (0, \infty)\) be a differentiable function. The function \(f\) is (strictly) \((p, q)\)-convex \(((p, q)\)-concave) if and only if \(u \mapsto u^{1-p}f'(u)[f(u)]^{q-1}\) is (strictly) increasing (decreasing) on \([a, b]\).

The next result completes and extends parts (a), (b) and (d) of Theorem 1. Notice that if we choose in part (b) of Theorem 3 the values \(p = 0\) and \(q = -1\), then we reobtain part (a) of Theorem 1. Similarly, choosing \(p = q = 0\) in part (a) of Theorem 3 we obtain the strict geometrical convexity stated in part (b) of Theorem 1. Finally, by taking \(p = q = -1\) in part (b) of Theorem 3 we obtain the monotonicity result stated in part (d) of Theorem 1.

**Theorem 3.** Let \(p, q \in \mathbb{R}\) and let \(\nu > -1\). Then the following assertions are true:

(a) if \(p \leq 0\) and \(q \geq 0\), then \(I_{\nu}\) is strictly \((p, q)\)-convex on \((0, \infty)\);

(b) if \(p \leq 0\) and \(q < 0\), then \(I_{\nu}\) is strictly \((p, q)\)-concave on \((0, \infty)\) provided if \(\nu \geq -1/q\) and \(4q(q-1)\nu^3 - (p^2 - 4(q-1))\nu^2 - 2p^2 \nu - p^2 \geq 0\);

(c) if \(p \geq 0\) and \(q \leq -1\), then \(I_{\nu}\) is strictly \((p, q)\)-concave on \((0, \infty)\) provided if \(\nu \geq 1\);

(d) if \(p \geq 0\) and \(q > 0\), then \(I_{\nu}\) is strictly \((p, q)\)-convex on \((0, \infty)\) provided if \(\nu \geq p/q\);

(e) if \(p \leq 1\) and \(q \geq 1\), then \(I_{\nu}\) is strictly \((p, q)\)-convex on \((0, \infty)\).

**Proof.** For convenience first we introduce the following notation

\[
\lambda_{p, q, \nu}(u) = \frac{d}{du} \left[ \frac{u^{1-p}I_{\nu}'(u)}{I_{\nu}^{-q}(u)} \right] = \frac{I_q(u)}{u^{p+1}} \left[ u^2 + \nu^2 - p \left( \frac{uI_{\nu}'(u)}{I_{\nu}(u)} \right) - (1-q) \left( \frac{uI_{\nu}'(u)}{I_{\nu}(u)} \right)^2 \right].
\]

We note that in view of Lemma 4 the \((p, q)\)-convexity \(((p, q)\)-concavity) of \(I_{\nu}\) depends only on the sign of the expression \(\lambda_{p, q, \nu}(u)\).

(a) This follows easily from the fact that if \(\nu > -1\), \(p \leq 0\) and \(q \geq 0\), then \(\lambda_{p, q, \nu}(u) > 0\) for all \(u > 0\). More precisely, from the right-hand side of (4) we have

\[
\lambda_{p, q, \nu}(u) > \frac{I_q(u)}{u^{p+1}} \left[ -p \left( \frac{uI_{\nu}'(u)}{I_{\nu}(u)} \right) + q \left( \frac{uI_{\nu}'(u)}{I_{\nu}(u)} \right)^2 \right] \geq 0
\]

for all \(\nu > -1\), \(p \leq 0\), \(q \geq 0\) and \(u > 0\). It should be mentioned here that this part follows actually from part (b) of Theorem 1. Namely, the function \(u \mapsto u^{1-p}I_{\nu}'(u) [I_{\nu}(u)]^{q-1}\) is strictly increasing on \((0, \infty)\) for all \(p \leq 0\), \(q \geq 0\) and \(\nu > -1\) as a product of the strictly increasing functions \(u \mapsto uI_{\nu}'(u)/I_{\nu}(u)\) and \(u \mapsto u^{1-p}I_q(u)\). Now, since for \(p = q = 0\) this part reduces to part (b) of Theorem 1 the above remark reveals that in fact part (b) of Theorem 1 and part (a) of Theorem 3 are equivalent.

(b) First assume that \(p < 0\) and \(q < 0\). Then by using (4) we obtain that

\[
\lambda_{p, q, \nu}(u) < \frac{I_q(u)}{u^{p+1}} \left[ u^2 + \nu^2 - p\sqrt{u^2 + \nu^2} - (1-q) \left( \frac{\nu}{\nu + 1} u^2 + \nu^2 \right) \right]
\]

and this is nonpositive if

\[
p^2(u^2 + \nu^2) \leq \left( q\nu^2 + \frac{q\nu + 1}{\nu + 1} u^2 \right)^2, \quad \text{i.e.} \quad 0 \leq Q_{\nu}(u^2),
\]

where \(Q_{\nu}(u) = au^2 + bu + c\) with \(\nu \geq -1/q\),

\[
a = \left( \frac{q\nu + 1}{\nu + 1} \right)^2, \quad b = 2q\nu^2 \frac{q\nu + 1}{\nu + 1} - p^2, \quad c = \nu^2(q^2\nu^2 - p^2).
\]

This gives a necessary condition to be \(b^2 - 4ac \leq 0\). A computation shows that the condition \(b^2 - 4ac \leq 0\) is equivalent to the inequality

\[
4q(q-1)\nu^3 - (p^2 - 4(q-1))\nu^2 - 2p^2 \nu - p^2 \geq 0.
\]

Now, assume that \(p = 0\) and \(q < 0\). Then from the left-hand side of (4) we have

\[
\lambda_{0, q, \nu}(u) = \frac{I_q(u)}{u} \left[ u^2 + \nu^2 - (1-q) \left( \frac{uI_{\nu}'(u)}{I_{\nu}(u)} \right)^2 \right] < \frac{I_q(u)}{u} \left( \frac{q\nu + 1}{\nu + 1} u^2 + q\nu^2 \right) < 0
\]
for all \( \nu > -1/q, q < 0 \) and \( u > 0 \), as we requested.

(c) This follows directly from part (a) of Theorem 1. More precisely, it is easy to see that the function \( u \mapsto u^{-p}I'_\nu(u)/I_\nu(u) \) is strictly decreasing on \((0, \infty)\) for all \( \nu \geq 1 \) as a product of the strictly decreasing function \( u \mapsto u^{-p}I'_\nu(u)/I_\nu(u) \) and the decreasing function \( u \mapsto u^{-p}I_{\nu+1}(u) \). Since part (c) of Theorem 3 reduces to part (a) of Theorem 1 when \( p = 0 \) and \( q = -1 \), the above proof reveals that in fact part (c) of Theorem 3 is equivalent to part (a) of Theorem 1.

(d) Recall that part (b) of Theorem 1 states that \( I_\nu \) is strictly geometrically convex on \((0, \infty)\) for all \( \nu > -1 \), i.e., the function \( u \mapsto u^{-p}I'_\nu(u)/I_\nu(u) \) is strictly increasing on \((0, \infty)\) for all \( \nu > -1 \). To prove that \( I_\nu \) is strictly \((p, q)\)-convex on \((0, \infty)\) for all \( p \geq 0, q > 0 \) and \( \nu \geq p/q \) in what follows we show that the function \( u \mapsto u^{-p}I'_\nu(u)/I_{\nu+1}(u) \) is strictly increasing as a product of the strictly increasing functions \( u \mapsto u^{-p}I'_\nu(u)/I_\nu(u) \) and \( u \mapsto u^{-p}I_{\nu+1}(u) \). On the other hand, observe that since \( u \mapsto u^{-p}I'_\nu(u)/I_\nu(u) \) is strictly increasing on \((0, \infty)\), we obtain that

\[
u I'_\nu(u)/I_\nu(u) > \nu
\]

for all \( \nu > -1 \) and \( u > 0 \) (actually for \( \nu > 0 \) this inequality follows directly from the left-hand side of (4)). Here we used that if \( u \) tends to zero then \( u^{-p}I'_\nu(u)/I_\nu(u) \) tends to \( \nu \), which can be verified from (4) or from

\[
u I'_\nu(u)/I_\nu(u) = \nu + 2 \sum_{n \geq 1} \frac{u^2}{u^2 + j_{\nu,n}^2}.
\]

The above inequality implies that

\[
\frac{d}{du} \left[ \frac{I_\nu(u)}{u^p} \right] = \frac{I_\nu(u)}{u^{p+1}} \left[ -p + q \frac{uI'_\nu(u)}{I_\nu(u)} \right] > \frac{I_\nu(u)}{u^{p+1}}(-p + q\nu) \geq 0,
\]

and with this the proof of this part is complete.

(e) This follows from the fact that \( I_\nu \) is strictly increasing and convex on \((0, \infty)\) for all \( \nu > -1 \). Namely, the function \( u \mapsto u^{-p}I'_\nu(u)/I_{\nu+1}(u) \) is strictly increasing as a product of the strictly increasing function \( u \mapsto I'_\nu(u) \) and the increasing functions \( u \mapsto u^{-p} \) and \( u \mapsto I_{\nu+1}(u) \).

Now, we are going to present the analogous result of Theorem 3 for modified Bessel functions of the second kind. We note that part (c) of Theorem 4 (when \( p = 1 \) and \( q = -1 \)) reduces to part (a) of Theorem 2 part (e) of Theorem 1 (when \( p = 1 \) and \( q = 0 \)) becomes part (b) of Theorem 2 part (b) of Theorem 1 (when \( p = q = 0 \)) reduces to part (c) of Theorem 2 and part (d) of Theorem 1 (when \( p = 0 \) and \( q = 1 \)) becomes part (d) of Theorem 2. Finally, observe that if we choose \( p = -1 \) and \( q = 1 \) in part (a) of Theorem 1 then we obtain part (e) of Theorem 2.

**Theorem 4.** Let \( p, q \in \mathbb{R} \) and let \( \nu \in \mathbb{R} \). Then the following assertions are true:

(a) if \( p \leq 0 \) and \( q \geq 1 \), then \( K_\nu \) is strictly \((p, q)\)-convex on \((0, \infty)\) provided if \( \nu > 1 \) and

\[4(1-q)p^2\nu^2 + 4(q-2)p^2\nu + p^2(p^2 + 4) \leq 0;\]

(b) if \( p \leq 0 \) and \( q \leq 0 \), then \( K_\nu \) is strictly \((p, q)\)-concave on \((0, \infty)\);

(c) if \( p \geq 0 \) and \( q < 0 \), then \( K_\nu \) is strictly \((p, q)\)-concave on \((0, \infty)\) provided if \( |\nu| \geq -p/q;\)

(d) if \( p \geq 0 \) and \( q \geq 1 \), then \( K_\nu \) is strictly \((p, q)\)-concave on \((0, \infty)\);

(e) if \( p \geq 1 \) and \( q \geq 0 \), then \( K_\nu \) is strictly \((p, q)\)-concave on \((0, \infty)\).

**Proof.** For convenience first we introduce the following notation

\[
\mu_{p, q, \nu}(u) = \frac{d}{du} \left[ \frac{u^{1-p}K'_\nu(u)}{K_{\nu-1}(u)} \right] = \frac{K'_\nu(u)}{K_{\nu-1}(u)} \left[ u^2 + \nu^2 - p \left[ \frac{uK'_\nu(u)}{K_\nu(u)} \right] - (1-q) \left[ \frac{uK'_\nu(u)}{K_\nu(u)} \right]^2 \right].
\]

Observe that in view of Lemma 3 the \((p, q)\)-convexity \((p, q)\)-concavity of \( K_\nu \), depends only on the sign of the expression \( \mu_{p, q, \nu}(u) \).

(a) Notice that for all \( \nu \in \mathbb{R} \) fixed when \( u \) tends to zero \( uK'_\nu(u)/K_\nu(u) \) tends to \(-\nu\). This can be verified for example from the integral representation (3). On the other hand, in view of part (c) of Theorem 2 the function \( u \mapsto uK'_\nu(u)/K_\nu(u) \) is strictly decreasing on \((0, \infty)\) for all \( \nu \in \mathbb{R} \), and this in turn implies that for all \( \nu \in \mathbb{R} \) and \( u > 0 \) the inequality

\[uK'_\nu(u)/K_\nu(u) < -\nu\]

(12)
holds. We note that actually this follows also from the right-hand side of [5]. Now, by using [12] and the left-hand side of [5] we obtain that
\[
\mu_{p,q,v}(u) > \frac{K_\nu^2(u)}{u^{p+1}} \left[ u^2 + \nu^2 + p\sqrt{\frac{\nu}{\nu-1}}u^2 + \nu^2 + (q-1)\nu^2 \right]
\]
and the right hand side of the last inequality is nonnegative if and only if
\[
Q_{\nu}(u) = u^4 + \left( 2q\nu^2 - \frac{\nu}{\nu-1}p^2 \right)u^2 + \nu^2 (q^2\nu^2 - p^2) \geq 0.
\]
Now, under assumptions the discriminant of the quadratic equation \(Q_{\nu}(\sqrt{u}) = 0\), i.e.
\[
\frac{\nu^2}{(\nu-1)^2} \left[ 4(1-q)p^2\nu^2 + 4(q-2)p^2\nu + p^2(p^2 + 4) \right]
\]
is negative and with this the proof of this part is complete.

(b) This follows from the fact that if \(\nu \in \mathbb{R}\) and \(p,q \leq 0\), then \(\mu_{p,q,v}(u) < 0\) for all \(u > 0\). Namely, from the right-hand side of [5] we have
\[
\mu_{p,q,v}(u) < \frac{K_\nu^2(u)}{u^{p+1}} \left[ -p \left( \frac{uK'_\nu(u)}{K_\nu(u)} \right) + q \left( \frac{uK'_\nu(u)}{K_\nu(u)} \right)^2 \right] \leq 0
\]
for all \(\nu \in \mathbb{R}\), \(p,q \leq 0\) and \(u > 0\). Here we used that \(K_\nu\) is strictly decreasing on \((0,\infty)\) for all \(\nu \in \mathbb{R}\).

We note here that this part follows actually from part (c) of Theorem 2. Namely, the function \(u \mapsto u^{1-p}K_\nu^q(u) [K_\nu(u)]^{\nu-1}\) is strictly decreasing on \((0,\infty)\) for all \(p,q \leq 0\) and \(\nu \in \mathbb{R}\) as a product of the strictly decreasing and negative function \(u \mapsto uK'_\nu(u)/K_\nu(u)\) and the strictly increasing and positive function \(u \mapsto u^{-p}K_\nu^q(u)\). Now, since for \(p = q = 0\) this part reduces to part (c) of Theorem 2, the above remark suggests that in fact part (c) of Theorem 2 is equivalent to part (b) of Theorem 4.

(c) By using [12] and the right-hand side of [5] we have for all \(u > 0\), \(p \geq 0\), \(q < 0\) and \(\nu \geq -p/q\)
\[
\mu_{p,q,v}(u) < \frac{K_\nu^2(u)}{u^{p+1}} \left[ -p \left( \frac{uK'_\nu(u)}{K_\nu(u)} \right) + q \left( \frac{uK'_\nu(u)}{K_\nu(u)} \right)^2 \right] < 0,
\]
\[
\leq -(p + q\nu) \frac{K_\nu^2(u)}{u^{p+1}} \left( \frac{uK'_\nu(u)}{K_\nu(u)} \right) \leq 0.
\]
\[
\text{(d) Since } p \geq 0 \text{ and } q \geq 1, \text{ the function } u \mapsto u^{-p}K_\nu^q(u) \text{ is decreasing on } (0,\infty) \text{ for all } \nu \in \mathbb{R}. \text{ Now, by using part (d) of Theorem 2 we conclude that } u \mapsto u^{1-p}K_\nu^q(u) [K_\nu(u)]^{\nu-1} \text{ is strictly increasing as a product of the strictly increasing and negative function } u \mapsto uK'_\nu(u) \text{ and the decreasing and positive function } u \mapsto u^{-p}K_\nu^q(u) \text{. Observe that since for } p = 0 \text{ and } q = 1 \text{ this part reduces to part (d) of Theorem 2 in fact they are equivalent. Finally, we note that the proof of this part can be obtained also simply from the fact that under assumptions } \mu_{p,q,v}(u) > 0.\]
\[
\text{(e) The proof of this part is very similar to the proof of part (d) above. Under assumptions the function } u \mapsto u^{1-p}K_\nu^q(u) \text{ is decreasing. Consequently, by using part (b) of Theorem 2, the function } u \mapsto u^{1-p}K_\nu^q(u) [K_\nu(u)]^{\nu-1} \text{ is strictly increasing as a product of the strictly increasing and negative function } u \mapsto K'_\nu(u)/K_\nu(u) \text{ and the decreasing and positive function } u \mapsto u^{-p}K_\nu^q(u) \text{. Observe that since for } p = 1 \text{ and } q = 0 \text{ this part reduces to part (b) of Theorem 2 in fact they are equivalent.}\]

4. Application to the log-concavity of the gamma-gamma distribution

The probability density function \(f_{a,b,\alpha} : (0,\infty) \to (0,\infty)\) of the three parameter gamma-gamma random variable is defined by (see [21])
\[
f_{a,b,\alpha}(u) = \frac{2(ab)^{\frac{a+1}{\alpha}} u^{\frac{a+1}{\alpha} - 1}}{\Gamma(a)\Gamma(b)\alpha^{\frac{a+1}{\alpha}}} K_{a-b} \left( 2 \sqrt{\frac{ab}{\alpha}} u \right),
\]
where \(a,b > 0\) are the distribution shaping parameters, \(K_\nu\) stands for the modified Bessel function of the second kind, and \(\alpha > 0\) is the mean of the gamma-gamma random variable. The gamma-gamma distribution is produced from the product of two independent gamma random variables and has been widely used in a variety of applications, for example in modeling various types of land and sea radar clutters, in modeling the effects of the combined fading and shadowing phenomena, encountered in
the mobile communications channels. Of particular interest is the application of the gamma-gamma distribution in optical wireless systems, where transmission of optical signals through the atmosphere is involved. For more details see [21] [23].

Now, consider the functions \( f_{a,b,\alpha} : (0, \infty) \to (0, \infty) \) and \( F_{a,b,\alpha} : (0, \infty) \to (0, 1) \) defined by

\[
\tilde{f}_{a,b,\alpha}(u) = f_{a,b,\alpha}(u) \left( \frac{au^2}{4ab} \right) = \frac{2^{1-(a+b)}(ab)u^{a+b-2}}{a\Gamma(a)\Gamma(b)} K_{a-b}(u)
\]

and

\[
F_{a,b,\alpha}(u) = \int_0^u f_{a,b,\alpha}(t) \, dt = \frac{1}{\Gamma(a)\Gamma(b)} \cdot G_{1,1,3}^2 \left[ \frac{ab}{\alpha} u \left( \begin{array}{c} 1 \\ a, b, 0 \end{array} \right) \right],
\]

where \( G_{1,1,3}^2 \) is a Meijer \( G \)-function [26] eq. 9.301. Here \( \tilde{f}_{a,b,\alpha} \) is just a transformation of the probability density function \( f_{a,b,\alpha} \), while \( F_{a,b,\alpha} \) is the cumulative distribution function of the gamma-gamma distribution.

In probability theory usually the cumulative distribution functions does not have closed-form, and thus sometimes it is quite difficult to study their properties directly. In statistics, economics and industrial engineering frequently appears some problems which are related to the study of log-concavity (log-convexity) of some univariate distributions. An interesting unified exposition of related results on the log-concavity and log-convexity of many distributions, including applications in economics, were communicated by Bagnoli and Bergstrom [7]. Some of their main results were reconsidered by András and Baricz [6] by using the monotone form of l’Hospital’s rule. Moreover, by using the idea from [6], recently, Baricz [15] showed, among others, that if a probability density function is geometrically concave then the corresponding cumulative distribution function will be also geometrically concave. In this section we use this result to prove that the cumulative distribution function \( F_{a,b,\alpha} \) is strictly log-concave on \((0, \infty)\) for all \(a, b, \alpha > 0\). This result may be useful in problems of information theory and communications.

**Theorem 5.** Let \(a, b, \alpha > 0\). Then the following assertions are true:

(a) \( u \mapsto uf_{a,b,\alpha}'(u) / f_{a,b,\alpha}(u) \) is strictly decreasing on \((0, \infty)\);
(b) \( u \mapsto uf_{a,b,\alpha}'(u) / F_{a,b,\alpha}(u) \) is strictly decreasing on \((0, \infty)\);
(c) \( u \mapsto F_{a,b,\alpha}'(u) / F_{a,b,\alpha}(u) \) is strictly decreasing on \((0, \infty)\);
(d) \( u \mapsto F_{a,b,\alpha}'(u) / F_{a,b,\alpha}(u) \) is strictly decreasing on \((0, \infty)\).

**Proof.** (a) From part (c) of Theorem 2 we have that the function

\[
\frac{uf_{a,b,\alpha}'(u)}{f_{a,b,\alpha}(u)} = a + b - 2 \frac{uK_{a-b}^'(u)}{K_{a-b}(u)}
\]

is strictly decreasing on \((0, \infty)\) for all \(a, b, \alpha > 0\).

(b) Observe that part (a) of this theorem actually means that the function \(\tilde{f}_{a,b,\alpha}\) is strictly geometrically concave, i.e. for all \(a, b, \alpha > 0\), \(\lambda \in (0, 1)\) and \(u_1, u_2 > 0\), \(u_1 \neq u_2\) we have

\[
\tilde{f}_{a,b,\alpha}(u_1^{1-\lambda}u_2^{\lambda}) > \left[ \tilde{f}_{a,b,\alpha}(u_1) \right]^{1-\lambda} \left[ \tilde{f}_{a,b,\alpha}(u_2) \right]^{\lambda}.
\]

Now, changing in the above inequality \(u_i\) with \(2\sqrt{abu_i}/\alpha\), where \(i \in \{1, 2\}\), we obtain

\[
f_{a,b,\alpha}(u_1^{1-\lambda}u_2^{\lambda}) > \left[ f_{a,b,\alpha}(u_1) \right]^{1-\lambda} \left[ f_{a,b,\alpha}(u_2) \right]^{\lambda}
\]

for all \(a, b, \alpha > 0\), \(\lambda \in (0, 1)\) and \(u_1, u_2 > 0\), \(u_1 \neq u_2\). This means that the function \(f_{a,b,\alpha}\) is strictly geometrically concave and hence the function \(u \mapsto uf_{a,b,\alpha}'(u) / f_{a,b,\alpha}(u)\) is strictly decreasing on \((0, \infty)\).

(c) This follows from part (b) of this theorem. Namely, it is known (see [15]) that if the probability density function is strictly geometrically concave, then the corresponding cumulative distribution function is also strictly geometrically concave.

(d) Part (c) of this theorem states that the cumulative distribution function \(F_{a,b,\alpha}\) is strictly geometrically concave. Now, by using the fact that \(F_{a,b,\alpha}\), as a distribution function, is increasing, for all \(a, b, \alpha > 0\), \(\lambda \in (0, 1)\) and \(u_1, u_2 > 0\), \(u_1 \neq u_2\) we have

\[
F_{a,b,\alpha}(\lambda u_1 + (1 - \lambda)u_2) > F_{a,b,\alpha}(u_1^{1-\lambda}u_2^{\lambda}) > \left[ F_{a,b,\alpha}(u_1) \right]^{1-\lambda} \left[ F_{a,b,\alpha}(u_2) \right]^{\lambda},
\]

that is, \(F_{a,b,\alpha}\) is strictly log-concave on \((0, \infty)\). \(\Box\)
5. Open Problems

In this section our aim is to complement the results from the previous sections and to present certain open problems, which may be of interest for further research.

Recall that Neuman [32] proved that the modified Bessel function $I_\nu$ is strictly log-convex on $(0, \infty)$ for all $\nu \in (-1/2, 0]$. Since $I_{-1/2}(u) = \sqrt{\pi/(2u)} \cosh u$, we conclude that in fact $I_\nu$ is strictly log-convex on $(0, \infty)$ for all $\nu \in [-1/2, 0]$. Thus, for all $\nu \in [-1/2, 0]$ and $u_1, u_2 > 0$ the third inequality in (6) can be improved as follows

$$I_{\nu}(\sqrt{u_1u_2}) \leq I_{\nu}\left(\frac{u_1 + u_2}{2}\right) \leq \sqrt{I_{\nu}(u_1)I_{\nu}(u_2)}.$$ 

Moreover, this implies that the function $I_{\nu}$ is strictly $(p, q)$-convex on $(0, \infty)$ for all $\nu \in [-1/2, 0]$, $p \leq 1$ and $q \geq 0$. This can be verified by writing the function $u \mapsto u^{1-p}I_\nu'(u)/I_\nu^2(u)$ as a product of the functions $u \mapsto I_\nu'(u)/I_\nu(u)$ and $u \mapsto u^{1-p}I_\nu^2(u)$.

Concerning Theorem 1 we have the following open problem.

**Question 1.** What can we say about the monotonicity of the functions $u \mapsto uI_\nu'(u)/I_\nu^2(u)$ and $u \mapsto u^2I_\nu'(u)/I_\nu^2(u)$ for $|\nu| < 1$ and $\nu \in (-1, 0)$, respectively? Is it true that $u \mapsto \sqrt{u}I_\nu(u)$ is strictly log-concave on $(0, \infty)$ for all $\nu \geq 0$?

Now, concerning Theorems 2, 3 and 4 we may ask the following.

**Question 2.** What can we say about the monotonicity of $u \mapsto K_\nu'(u)/K_\nu^2(u)$ when $|\nu| < 1$?

**Question 3.** What can we say about the $(p, q)$-convexity (concavity) of $I_\nu$ when $p \geq 0$, $q \in (-1, 0)$? Moreover, the conditions for $\nu$ in parts (b), (c) and (d) of Theorem 1 can be relaxed?

**Question 4.** What can we say about the $(p, q)$-convexity (concavity) of $K_\nu$ when $p \leq 1$, $q \in (0, 1)$? Moreover, the conditions for $\nu$ in parts (a) and (c) of Theorem 5 can be relaxed?

It is well-known that the function $\nu \mapsto K_\nu(u)$ is strictly log-convex on $\mathbb{R}$ for all $u > 0$ fixed (see [16]). On the other hand $\nu \mapsto K_\nu(u)$ is strictly increasing on $(0, \infty)$ for all $u > 0$ fixed. Clearly these imply that the function $\nu \mapsto K_\nu(u)$ is strictly $(p, q)$-convex on $(0, \infty)$ for all $p \leq 1$ and $q \geq 0$, and all fixed $u > 0$. This suggest the following.

**Question 5.** What can we say about the $(p, q)$-convexity (concavity) of the function $\nu \mapsto K_\nu(u)$ on $(0, \infty)$ when $p$ and $q$ are arbitrary real numbers?

Similarly, the function $\nu \mapsto I_\nu(u)$ is strictly log-concave on $(-1, \infty)$ for all $u > 0$ fixed (see [16]). On the other hand $\nu \mapsto I_\nu(u)$ is strictly decreasing on $(-1, \infty)$ for all $u > 0$ fixed. Clearly these imply that the function $\nu \mapsto I_\nu(u)$ is strictly $(p, q)$-concave on $(0, \infty)$ for all $p \geq 1$ and $q \geq 0$, and all fixed $u > 0$. Thus, it is natural to ask the following.

**Question 6.** What can we say about the $(p, q)$-convexity (concavity) of the function $\nu \mapsto I_\nu(u)$ on $(0, \infty)$ when $p$ and $q$ are arbitrary real numbers? And what about the $(p, q)$-convexity (concavity) of $\nu \mapsto I_\nu(u)$ on $(-1, \infty)$?

Due to Laforgia [29] it is known that $K_\nu'(u)/K_\nu(u) \leq -\nu/u - 1$ for all $u > 0$ and $\nu \in (0, 1/2)$. First observe that the above inequality is valid for all $\nu \in [0, 1/2]$. Since $K_0'(u) = -K_1(u)$ for $\nu = 0$ the above inequality is equivalent to $K_1(u) > K_0(u)$, which is clearly true, since the function $\nu \mapsto K_\nu(u)$ is strictly increasing on $(0, \infty)$ for all $u > 0$ fixed. Now, since $K_{1/2}(u) = \sqrt{\pi/(2u)}e^{-u}$ we obtain that in Laforgia’s inequality for $\nu = 1/2$ we have equality and since $\nu \mapsto K_\nu(u)$ is even, we deduce that $K_\nu'(u)/K_\nu(u) \leq -\nu/u - 1$ holds true for all $u > 0$ and $|\nu| \leq 1/2$, with equality for $\nu = 1/2$.

By using this result we obtain that

$$\left[\frac{u^2K_\nu''(u)}{K_\nu(u)}\right] - \frac{\nu^2K_\nu'(u)}{K_\nu(u)} + \frac{u^2K_\nu''(u)}{K_\nu(u)} = \left[\frac{uK_\nu'(u)}{K_\nu(u)} + u^2 + \nu^2\right] \leq u^2 - u + \nu^2 - \nu < 0$$

for all $u \in (0, 1)$ and $|\nu| \leq 1/2$. This implies that the function $u \mapsto u^2K_\nu'(u)$ is strictly decreasing on $(0, 1)$ for all $|\nu| \leq 1/2$, i.e. the modified Bessel function of the second kind $K_\nu$ is strictly $(-1, 1)$-concave on $(0, 1)$ for all $|\nu| \leq 1/2$. This completes parts (e) and (f) of Theorem 2.

Taking into account the above discussion we may ask the following.

**Question 7.** Is it true that $u \mapsto u^2K_\nu'(u)$ is strictly decreasing on $(0, 2)$ for all $|\nu| \leq 1/2$?
In reliability analysis it has been found very useful to classify life distributions (i.e. distributions of which cumulative distribution function satisfies $F(u) = 0$ for $u \leq 0$) according to the monotonicity properties of the failure rate. By definition a life distribution (with probability density function $f$ and survival or reliability function $F$) has the increasing failure rate (IFR) property if the function $u \mapsto f(u)/F(u)$ is increasing on $(0, \infty)$. Since by definition $F(u) = 1 - F(u)$ for all $u > 0$, clearly we have $F(u) = -f(u)$ for all $u > 0$. Thus, a life distribution is IFR if and only if $u \mapsto -F(u)/F(u)$ is increasing on $(0, \infty)$, i.e. the reliability function $F$ is log-concave. It is well-known that if a probability density function is log-concave then this implies that the corresponding cumulative distribution function and the complementary cumulative distribution function (or survival function) have the same property (for more details see [6][7][15]). Another class of life distributions is the NBU, which has been shown to be fundamental in the study of replacement policies. By definition a life distribution satisfies the new-is-better-than-used (NBU) property if $u \mapsto \log F(u)$ is sub-additive, i.e.

$$F(u_1 + u_2) \leq F(u_1)F(u_2)$$

for all $u_1, u_2 > 0$. The corresponding concept of a new-is-worse-than-used (NWU) distribution is defined by reversing the above inequality. The NBU property may be interpreted as stating that the chance $F(u_1)$ that a new unit will survive to age $u_1$ is greater than the chance $F(u_1 + u_2)/F(u_2)$ that an unfailed unit of age $u_2$ will survive an additional time $u_1$. It can be shown easily that if a life distribution is IFR then it is NBU (see for example [11]), but the inverse implication in general does not hold. Since the most important life distribution satisfies the NBU property it is natural to ask the following.

**Question 8.** Is it true that the gamma-gamma distribution satisfies the NBU property?

To answer this question it would be enough to prove that the probability density function $f_{a,b,\alpha}$ is log-concave, and for this in view of part (b) of Theorem 5 it is quite enough to show that $f_{a,b,\alpha}$ is increasing. Similarly, observe that for the log-concavity of $f_{a,b,\alpha}$ we just need to show that $\tilde{f}_{a,b,\alpha}$ is increasing and log-concave. However, by part (a) of Theorem 5 if $\tilde{f}_{a,b,\alpha}$ is increasing, then it is log-concave. Thus, to prove that the gamma-gamma distribution is NBU we need to show that either $f_{a,b,\alpha}$ or $\tilde{f}_{a,b,\alpha}$ is increasing.

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