Solving Sparse, Symmetric, Diagonally-Dominant Linear Systems in Time
$O(m^{1.31})$

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Abstract

We present a linear-system solver that, given an $n$-by-$n$ symmetric positive semi-definite, diagonally dominant matrix $A$ with $m$ non-zero entries and an $n$-vector $b$, produces a vector $\tilde{x}$ within relative distance $\epsilon$ of the solution to $Ax = b$ in time $O(m^{1.31} \log(n/\epsilon)^{O(1)})$, where $b$ is the log of the ratio of the largest to smallest non-zero entry of $A$. If the graph of $A$ has genus $m^{2\theta}$, or does not have a $K_m$ minor, then the exponent of $m$ can be improved to the minimum of $1 + 5\theta$ and $(9/8)(1 + \theta)$. The key contribution of our work is an extension of Vaidya’s techniques for constructing and analyzing combinatorial preconditioners.

1. Introduction

Sparse linear systems are ubiquitous in scientific computing and optimization. In this work, we develop fast algorithms for solving some of the best-behaved linear systems: those specified by symmetric, diagonally dominant matrices with positive diagonals. We call such matrices PSDDD as they are positive semi-definite and diagonally dominant. Such systems arise in the solution of certain elliptic differential equations via the finite element method, the modeling of resistive networks, and in the solution of certain network optimization problems [23, 19, 15, 25, 26].

While one is often taught to solve a linear system $Ax = b$ by computing $A^{-1}$ and then multiplying $A^{-1}$ by $b$, this approach is quite inefficient for sparse linear systems—the best known bound on the time required to compute $A^{-1}$ is $O(n^{2.373})$ [9] and the representation of $A^{-1}$ typically requires $\Omega(n^2)$ space. In contrast, if $A$ is symmetric and has $m$ non-zero entries, then one can use the Conjugate Gradient method, as a direct method, to solve for $A^{-1}b$ in $O(nnz)$ time and $O(n)$ space! Until Vaidya’s revolutionary introduction of combinatorial preconditioners [24], this was the best complexity bound for the solution of general PSDDD systems.

The two most popular families of methods for solving linear systems are the direct methods and the iterative methods. Direct methods, such as Gaussian elimination, perform arithmetic operations that produce $x$ treating the entries of $A$ and $b$ symbolically. As discussed in Section 1.4, direct methods can be used to quickly compute $x$ if the matrix $A$ has special topological structure.

Iterative methods, which are discussed in Section 1.5, compute successively better approximations to $x$. The Chebyshev and Conjugate Gradient methods take time proportional to $m \sqrt{\kappa_f(A)} \log(\kappa_f(A)/\epsilon)$ to produce approximations to $x$ with relative error $\epsilon$, where $\kappa_f(A)$ is the ratio of the largest to the smallest non-zero eigenvalue of $A$. These algorithms are improved by preconditioning—essentially solving $B^{-1}Ax = B^{-1}b$ for a precondiitioner $B$ that is carefully chosen so that $\kappa_f(A, B)$ is small and so that it is easy to solve linear systems in $B$. These systems in $B$ may be solved using direct methods, or by again applying iterative methods.

Vaidya [24] discovered that for PSDDD matrices $A$ one could use combinatorial techniques to construct matrices $B$ that provably satisfy both criteria. In his seminal work, Vaidya shows that when $B$ corresponds to a subgraph of the graph of $A$, one can bound $\kappa_f(A, B)$ by bounding the dilation and congestion of the best embedding of the graph of $A$ into the graph of $B$. By using preconditioners derived by adding a few edges to maximum spanning trees, Vaidya’s algorithm finds $\epsilon$-approximate solutions to PSDDD linear systems of maximum valence $d$ in time
When these systems have special structure, such as having a sparsity graph of bounded genus or avoiding certain minors, he obtains even faster algorithms. For example, his algorithm solves planar linear systems in time \( O((dn)^{1.75} \log(\kappa_f(A)/\epsilon)) \). This paper follows the outline established by Vaidya: our contributions are improvements in the techniques for bounding \( \kappa_f(A, B) \), a construction of better pre conditioners, a construction that depends upon average degree rather than maximum degree, and an analysis of the recursive application of our algorithm.

As Vaidya’s paper was never published, and his manuscript lacked many proofs, the task of formally working out his results fell to others. Much of its content appears in the thesis of his student, Anil Joshi [16]. Gremban, Miller and Zagh[13, 14] explain parts of Vaidya’s paper as well as extend Vaidya’s techniques. Among other results, they found ways of constructing pre conditioners by adding vertices to the graphs and using separator trees.

Much of the theory behind the application of Vaidya’s techniques to matrices with non-positive off-diagonals is developed in [3]. The machinery needed to apply Vaidya’s techniques directly to matrices with positive off-diagonal elements is developed in [5]. The present work builds upon an algebraic extension of the tools used to prove bounds on \( \kappa_f(A, B) \) by Boman and Hendrickson [6]. Boman and Hendrick son [7] have pointed out that by applying one of their bounds on support to the tree constructed by Alon, Karp, Peleg, and West [1] for the \( k \)-server problem, one obtains a spanning tree pre conditioner \( B \) with \( \kappa_f(A, B) = m \mathcal{O}(\sqrt{\log n \log \log n}) \). They thereby obtain a solver for PSDDD systems that produces \( \epsilon \)-approximate solutions in time \( m^{1.5+o(1)} \log(\kappa_f(A)/\epsilon) \). In their manuscript, they asked whether one could possibly augment this tree to obtain a better pre conditioner. We answer this question in the affirmative. An algorithm running in time \( O(mn^{1/2} \log^2(n)) \) has also recently been obtained by Maggs, et al. [18].

The present paper is the first to push past the \( O(n^{1.5}) \) barrier. It is interesting to observe that this is exactly the point at which one obtains sub-cubic time algorithms for solving dense PSDDD linear systems.

Reif [21] proved that by applying Vaidya’s techniques recursively, one can solve bounded-degree planar positive definite diagonally dominant linear systems to relative accuracy \( \epsilon \) in time \( O(m^{1+o(1)} \log(\kappa_f(A)/\epsilon)) \). We extend this result to general planar PSDDD linear systems.

Due to space limitations, some proofs of have been omitted, but will appear in the on-line and full versions of the paper.

### 1.1. Background and Notation

A symmetric matrix \( A \) is semi-positive definite if \( x^T A x \geq 0 \) for all vectors \( x \). This is equivalent to having all eigenvalues of \( A \) non-negative.

In most of the paper, we will focus on Laplacian matrices: symmetric matrices with non-negative diagonals and non-positive off-diagonals such that for all \( i \), \( \sum_j A_{ij} = 0 \). However, our results will apply to the more general family of positive semidefinite, diagonally dominant (PSDDD) matrices, where a matrix is diagonally dominant if \( |A_{ii}| \geq \sum_{j=1}^n |A_{ij}| \) for all \( i \). We remark that a symmetric matrix is PSDDD if and only if it is diagonally dominant and all of its diagonals are non-negative.

In this paper, we will restrict our attention to the solution of linear systems of the form \( Ax = b \) where \( A \) is a PSDDD matrix. When \( A \) is non-singular, that is when \( A^{-1} \) exists, there exists a unique solution \( x = A^{-1} b \) to the linear system. When \( A \) is singular and symmetric, for every \( b \in \text{Span}(A) \) there exists a unique \( x \in \text{Span}(A) \) such that \( A x = b \). If \( A \) is the Laplacian of a connected graph, then the null space of \( A \) is spanned by \( 1 \).

There are two natural ways to formulate the problem of finding an approximate solution to a system \( Ax = b \). A vector \( \hat{x} \) has relative residual error \( \epsilon \) if \( \| A \hat{x} - b \| \leq \epsilon \| b \| \). We say that a solution \( \hat{x} \) is an \( \epsilon \)-approximate solution if it is at relative distance at most \( \epsilon \) from the actual solution—that is, if \( \| x - \hat{x} \| \leq \epsilon \| x \| \). One can relate these two notions of approximation by observing that relative distance of \( x \) to the solution and the relative residual error differ by a multiplicative factor of at most \( \kappa_f(A) \). We will focus our attention on the problem of finding \( \epsilon \)-approximate solutions.

The ratio \( \kappa_f(A) \) is the finite condition number of \( A \). The \( l_2 \) norm of a matrix, \( \|A\| \), is the maximum of \( \|A x\| / \|x\| \), and equals the largest eigenvalue of \( A \) if \( A \) is symmetric. For non-symmetric matrices, \( \lambda_{\min}(A) \) and \( \|A\| \) are typically different. We let \( |A| \) denote the number of non-zero entries in \( A \), and \( \min(A) \) and \( \max(A) \) denote the smallest and largest non-zero elements of \( A \) in absolute value, respectively.

The condition number plays a prominent role in the analysis of iterative linear system solvers. When \( A \) is PSD, it is known that, after \( \sqrt{\kappa_f(A)} \log(1/\epsilon) \) iterations, the Chebyshev iterative method and the Conjugate Gradient method produce solutions with relative residual error at most \( \epsilon \).
To obtain an $\varepsilon$-approximate solution, one need merely run $\log(\kappa_f(A))$ times as many iterations. If $A$ has $m$ non-zero entries, each of these iterations takes time $O(m)$. When applying the preconditioned versions of these algorithms to solve systems of the form $B^{-1}Ax = B^{-1}b$, the number of iterations required by these algorithms to produce an $\varepsilon$-accurate solution is bounded by $\sqrt{\kappa_f(A, B)} \log(\kappa_f(A) / \varepsilon)$ where

$$\kappa_f(A, B) = \left( \max_{x : x \neq 0} \frac{x^T Ax}{x^T Bx} \right) \left( \max_{x : x \neq 0} \frac{x^T Bx}{x^T Ax} \right),$$

for symmetric $A$ and $B$ with $\text{Span}(A) = \text{Span}(B)$. However, each iteration of these methods takes time $O(m)$ plus the time required to solve linear systems in $B$. In our initial algorithm, we will use direct methods to solve these systems, and so will not have to worry about approximate solutions. For the recursive application of our algorithms, we will use our algorithm again to solve these systems, and so will have to determine how well we need to approximate the solution. For this reason, we will analyze the Chebyshev iteration instead of the Conjugate Gradient, as it is easier to analyze the impact of approximation in the Chebyshev iterations. However, we expect that similar results could be obtained for the preconditioned Conjugate Gradient. For more information on these methods, we refer the reader to [12] or [8].

1.2. Laplacians and Weighted Graphs

All weighted graphs in this paper have positive weights. There is a natural isomorphism between weighted graphs and Laplacian matrices: given a weighted graph $G = (V, E, w)$, we can form the Laplacian matrix in which $A_{i,j} = -w(i, j)$ for $(i, j) \in E$, and with diagonals determined by the condition $A1 = 0$. Conversely, a weighted graph is naturally associated to each Laplacian matrix. Each vertex of the graph corresponds to both a row and column of the matrix, and we will often abuse notation by identifying this row/column pair with the associated vertex.

We note that if $G_1$ and $G_2$ are weighted graphs on the same vertex set with disjoint sets of edges, then the Laplacian of the union of $G_1$ and $G_2$ is the sum of their Laplacians.

1.3. Reductions

In most of this paper we just consider Laplacian matrices of connected graphs. This simplification is enabled by two reductions.

First, we note that it suffices to construct preconditioners for matrices satisfying $A_{i,i} = \sum_j [A_{i,j}]$, for all $i$. This follows from the observation in [3] that if $\hat{A} = A + D$, where $A$ satisfies the above condition, then $\kappa_f(\hat{A}, B + D) \leq \kappa_f(A, B)$. So, it suffices to find a preconditioner after subtracting off the maximal diagonal matrix that maintains positive diagonal dominance.

We then use an idea of Gremban [13] for handling positive off-diagonal entries. If $A$ is a symmetric matrix such that for all $i, A_{i,i} \geq \sum_j [A_{i,j}]$, then Gremban decomposes $A$ into $A = A_n + A_p$, where $D$ is the diagonal of $A$, $A_n$ is the matrix containing all negative off-diagonal entries of $A$, and $A_p$ contains all the positive off-diagonals. Gremban then considers the linear system

$$\begin{bmatrix} D + A_n & -A_p \\ -A_p & D + A_n \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix} = \begin{bmatrix} b \\ -b \end{bmatrix},$$

and observes that its solution will have $x' = -x$ and that $x$ will be the solution to $Ax = b$. Thus, by making this transformation, we can convert any $PSDDDD$ linear system into one with non-negative off diagonals. One can understand this transformation as making two copies of every vertex in the graph, and two copies of every edge. The edges corresponding to negative off-diagonals connect nodes in the same copy of the graph, while the others cross copies. To capture the resulting family of graphs, we define a weighted graph $G$ to be a Gremban cover if it has $2n$ vertices and

- for $i, j \leq n, (i, j) \in E$ if and only if $(i + n, j + n) \in E$, and $w(i, j) = w(i + n, j + n)$,
- for $i, j \leq n, (i, j + n) \in E$ if and only if $(i + n, j) \in E$, and $w(i, j + n) = w(i + n, j)$,
- the graph contains no edge of the form $(i, i + n)$.

When necessary, we will explain how to modify our arguments to handle Laplacians that are Gremban covers.

Finally, if $A$ is the Laplacian of an unconnected graph, then the blocks corresponding to the connected components may be solved independently.

1.4. Direct Methods

The standard direct method for solving symmetric linear systems is Cholesky factorization. Those unfamiliar with Cholesky factorization should think of it as Gaussian elimination in which one simultaneously eliminates on rows and columns so as to preserve symmetry. Given a permutation matrix $P$, Cholesky factorization produces a lower-triangular matrix $L$ such that $LL^T = PA^T$. Because one can use forward and back substitution to multiply vectors by $L^{-1}$ and $L^{-T}$ in time proportional to the number of non-zero entries in $L$, one can use the Cholesky factorization of $A$ to solve the system $Ax = b$ in time $O(|L|)$. 
Each pivot in the factorization comes from the diagonal of \( A \), and one should understand the permutation \( P \) as providing the order in which these pivots are chosen. Many heuristics exist for producing permutations \( P \) for which the number of non-zeros in \( L \) is small. If the graph of \( A \) is a tree, then a permutation \( P \) that orders the vertices of \( A \) from the leaves up will result in an \( L \) with at most \( 2n - 1 \) non-zero entries. In this work, we will use results concerning matrices whose sparsity graphs resemble trees with a few additional edges and whose graphs have small separators, which we now review.

If \( B \) is the Laplacian matrix of a weighted graph \((V, E, w)\), and one eliminates a vertex \( a \) of degree 1, then the remaining matrix has the form

\[
\begin{bmatrix}
1 & 0 \\
0 & A_1
\end{bmatrix}
\]

where \( A_1 \) is the Laplacian of the graph in which \( a \) and its attached edge have been removed. Similarly, if a vertex \( a \) of degree 2 is eliminated, then the remaining matrix is the Laplacian of the graph in which the vertex \( a \) and its adjacent edges have been removed, and an edge with weight \( 1/(1/w_1 + 1/w_2) \) is added between the two neighbors of \( a \), where \( w_1 \) and \( w_2 \) are the weights of the edges connecting \( a \) to its neighbors.

Given a graph \( G \) with edge set \( E = R \cup D \), where the edges in \( R \) form a tree, we will perform a partial Cholesky factorization of \( G \) in which we successively eliminate all the degree 1 and 2 vertices that are not endpoint of edges in \( D \). We introduce the algorithm \texttt{trim} to define the order in which the vertices should be eliminated, and we call the \texttt{trim order} the order in which \texttt{trim} deletes vertices.

**Algorithm:** \texttt{trim}(\( V, R, D \))

1. While \( G \) contains a vertex of degree one that is not an endpoint of an edge in \( D \), remove that vertex and its adjacent edge.
2. While \( G \) contains a vertex of degree two that is not an endpoint of an edge in \( D \), remove that vertex and its adjacent edges, and add an edge between its two neighbors.

**Proposition 1.1.** The output of \texttt{trim} is a graph with at most \( 4|D| \) vertices and \( 5|D| \) edges.

**Remark 1.2.** If \((V, R)\) and \((V, D)\) are Gremban covers, then we can implement \texttt{trim} so that the output graph is also a Gremban cover. Moreover, the genus and maximum size clique minor of the output graph do not increase.

After performing partial Cholesky factorization of the vertices in the trim order, one obtains a factorization of the form

\[
B = LCL^T, \text{ where } C = \begin{bmatrix} I & 0 \\ 0 & A_1 \end{bmatrix},
\]

\( L \) is lower triangular, and the left column and right columns in the above representations correspond to the eliminated and remaining vertices respectively. Moreover, \(|L| \leq 2n - 1 \), and this Cholesky factorization may be performed in time \( O(n + |D|) \).

The following Lemma may be proved by induction.

**Lemma 1.3.** Let \( B \) be a Laplacian matrix and let \( L \) and \( A_1 \) be the matrices arising from the partial Cholesky factorization of \( B \) according to the trim order. Let \( U \) be the set of eliminated vertices, and let \( W \) be the set of remaining vertices. For each pair of vertices \((a, b)\) in \( W \) joined by a simple path containing only vertices of \( U \), let \( B_{(a,b)} \) be the Laplacian of the graph containing just one edge between \( a \) and \( b \) of weight \( 1/(\sum \frac{1}{w_i}) \), where the \( w_i \) are the weights on the path between \( a \) and \( b \). Then,

\( a \) the matrix \( A \) is the sum of the Laplacian of the induced graph on \( W \) and the sum all the Laplacians \( B_{(a,b)} \).

\( b \) \( 1L = 1W \), where \( 1 \) is the all-ones vector, and \( 1W \) is the characteristic vector of \( W \), and

\( c \) the matrix \( LL^T \) is the sum of the Laplacian of the graph containing all edges deleted by \texttt{trim} and the diagonal matrix that is \( 1 \) for vertices in \( W \) and \( 0 \) elsewhere, minus the sum of all the Laplacians \( B_{(a,b)} \).

**Corollary 1.4.** For \( B \) and \( A_1 \) as in Lemma 1.3, \( \max(A_1) \leq n \max(B) \) and \( \min(A_1) \geq \min(B)/n \).

Other topological structures may be exploited to produce elimination orderings that result in sparse \( L \). In particular, Lipton, Rose and Tarjan [17] prove that if a graph can be dissected by a family of small separators, then \( L \) can be made sparse. The precise definition and theorem follow.

**Definition 1.5.** A subset of vertices \( C \) of a graph \( G = (V,E) \) with \( n \) vertices is an \( f(n) \)-separator if \( |C| \leq f(n) \), and the vertices of \( V - C \) can be partitioned into two sets \( U \) and \( W \) such that there are no edges from \( U \) to \( W \), and \( |U|, |W| \leq 2n/3 \).

**Definition 1.6.** Let \( f() \) be a positive function. A graph \( G = (V,E) \) with \( n \) vertices has a family of \( f() \)-separators if for every \( s \leq n \), every subgraph \( G_s \subseteq G \) with \( s \) vertices has a \( f(s) \)-separator.

**Theorem 1.7 (Nested Dissection: Lipton-Rose-Tarjan).** Let \( A \) be an \( n \) by \( n \) symmetric PSD matrix, \( \alpha > 0 \) be a constant, and \( h(n) \) be a positive function of \( n \). Let \( f(x) = \frac{x}{x + \alpha} \) for \( x \geq 0 \). Let \( x \) be a positive integer, and let \( X \) be a subset of vertices, then the \( \ell \)-neighborhood of \( X \) is defined as \( X^\ell \). If \( X \) is a separator, then \( |X^{\ell + 1}| \leq 2 \cdot |X^\ell| \).
h(n)\alpha^n. If G(A) has a family of f()-separator, then the Nested Dissection Algorithm of Lipton, Rose and Tarjan can, in O \((n + (h(n)\alpha^n)^3)\) time, factor A into A = LL^T so that L has at most O \((h(n)\alpha^n)^2 \log n\) non-zeros.

To apply this theorem, we note that many families of graphs are known to have families of small separators. Gilbert, Hutchinson, and Tarjan [10] show that all graphs of n vertices with genus bounded by g have a family of O(\sqrt{gn})-separators, and Plotkin, Rao and Smith [20] show that any graph that excludes K_5 as minor has a family of O(\sqrt{n \log n})-separators.

1.5. Iterative Methods

Iterative methods such as Chebyshev iteration and Conjugate Gradient solve systems such as \(Ax = b\) by successively multiplying vectors by the matrix A, and then taking linear combinations of vectors that have been produced so far. The preconditioned versions of these iterative methods take as input another matrix B, called the preconditioner, and also perform the operation of solving linear systems in B. In this paper, we will restrict our attention to the preconditioned Chebyshev method as it is easier to understand the effect of imprecision in the solution of the systems in B on the method’s output. In the non-recursive version of our algorithms, we will exploit the standard analysis of Chebyshev iteration (see [8]), adapted to our situation:

**Theorem 1.8 (Preconditioned Chebyshev).** Let A and B be Laplacian matrices, let b be a vector, and let x satisfy Ax = b. At each iteration, the preconditioned Chebyshev method multiplies one vector by A, solves one linear system in B, and performs a constant number of vector additions. At the kth iteration, the algorithm maintains a solution \(\tilde{x}\) satisfying

\[
\|\tilde{x} - x\| \leq e^{-k/\sqrt{\kappa_f(A,B)}} \kappa_f(A) \kappa_f(B) \|x\|.
\]

In the non-recursive versions of our algorithms, we will precompute the Cholesky factorization of the preconditioners B, and use these to solve the linear systems encountered by preconditioned Chebyshev method. In the recursive versions, we will perform a partial Cholesky factorization of B, into a matrix of the form \(L[f, 0; 0, A_1]L^T\), construct a preconditioner for A_1, and again use the preconditioned Chebyshev method to solve the systems in A_1.

2. Support Theory

The essence of support theory is the realization that one can bound \(\lambda_f(A, B)\) by constructing an embedding of A into B. We define a weighted embedding of A into B to be a function \(\pi\) that maps each edge e of A into a weighted simple path in B linking the endpoints of A. Formally, \(\pi : E_A \times E_B \to \mathbb{R}^+\) is a weighted embedding if for all \(e \in A\), \(\{f \in B : \pi(e, f) > 0\}\) is a simple path connecting from one endpoint of e to the other. We let \(\text{path}_\pi(e)\) denote this set of edges in this path in B. For \(e \in A\), we define \(w_d(e) = \sum_{f \in \text{path}_\pi(e)} \frac{a_{e,f}}{\pi(e, f)}\) and the weighted congestion of an edge \(f \in B\) under \(\pi\) to be \(w_c_\pi(f) = \sum_{e : f \in \text{path}_\pi(e)} w_d(e) \pi(e, f)\).

Our analysis of our preconditioners relies on the following extension of the support graph theory.

**Theorem 2.1 (Support Theorem).** Let A be the Laplacian matrix of a weighted graph G and B be the Laplacian matrix of a subgraph S of G. Let \(\pi\) be a weighted embedding of G into S. Then

\[
\kappa_f(A, B) \leq \max_{f \in S} w_c_\pi(f).
\]

To understand this statement, the reader should first consider the case in which all the weights \(a_{e,f}\) and \(\pi(e, f)\) are 1. In this case, the Support Theorem says that \(\kappa_f(A, B)\) is at most the maximum over edges \(f\) of the sum of the lengths of the paths through \(f\). This improves upon the upper bound on \(\kappa_f(A, B)\) stated by Vaidya and proved in Bern et al. of the maximum congestion times the maximum dilation, and it improves upon the bound proved by Boman and Hendrickson which was the sum of the dilations. This statement also extends the previous theories by using fractions of edges in B to route edges in A. That said, our proof of the Support Theorem owes a lot to the machinery developed by Boman and Hendrickson and our \(\pi\) is analogous to their matrix \(M\).

We first recall the definition of the support of A in B, denoted \(\sigma(A, B)\):

\[
\sigma(A, B) = \min \{\tau : \forall t \geq \tau, \ tB \geq A\}.
\]

Gremban proved that one can use support to characterize \(\lambda_f\):

**Lemma 2.2.** If \(\text{Null}(A) = \text{Null}(B)\), then

\[
\lambda_f(A, B) = \sigma(A, B)\sigma(B, A).
\]

Vaidya observed

**Lemma 2.3.** If S is a subgraph of the weighted graph G, A is the Laplacian of G and B is the Laplacian of S, then \(\sigma(B, A) \leq 1\).

Our proof of the Support Theorem will use the Splitting Lemma of Bern et al. and the Rank-One Support Lemma of Boman-Hendrickson:
Lemma 2.4 (Splitting Lemma). Let \( A = A_1 + A_2 + \cdots + A_k \) and let \( B = B_1 + B_2 + \cdots + B_k \). Then,

\[
\sigma(A, B) \leq \max_i \sigma(A_i, B_i).
\]

For an edge \( e \in A \) and a weighted embedding \( \pi \) of \( A \) into \( B \), we let \( A_i \) denote the Laplacian of the graph containing only the weighted edge \( e \) and \( B_e \) denote the Laplacian of the graph containing the edges \( f \in \text{path}_\pi(e) \) with weights \( a_f \pi(e, f) \). We have:

Lemma 2.5 (Weighted Dilation). For an edge \( e \in A \),

\[
\sigma(A, B, e) = wd_\pi(e).
\]

Proof. Follows from Boman and Hendrickson’s Rank-One Support Lemma.

Proof of Theorem 2.1. Lemma 2.5 implies

\[
\sigma(A_e, B_e) = 1.
\]

We then have

\[\begin{align*}
\sigma(A, \max_{f \in B} wc_\pi(f) B) & \leq \sigma(A, \sum_{f \in B} wc_\pi(f) A_f) \\
& = \sigma(A, \sum_{e \in A} wd_\pi(e) B_e) \\
& \leq \max_{e \in A} \sigma(A_e, B_e) \\
& \leq 1,
\end{align*}\]

where the second-to-last inequality follows from the Splitting Lemma.

3. The Preconditioner

In this section, we construct and analyze our preconditioner.

Theorem 3.1. Let \( A \) be a Laplacian and \( G = (V, E, w) \) its corresponding weighted graph. Let \( G \) have \( n \) vertices and \( m \) edges. For any positive integer \( t \leq n \), the algorithm precondition, described below, runs in \( O(m \log m) \) time and outputs a spanning tree \( R \subseteq E \) of \( G \) and a set of edges \( D \subseteq E \) such that

1. if \( B \) is the Laplacian corresponding to \( R \cup D \), then \( \sigma_f(A, B) \leq 2^{O(\sqrt{\log n \log \log \log n})} \), and
2. \( |D| \leq O(t^2 \log n / \log \log n) \).

Moreover, if \( G \) has genus \( s^2 \) or has no \( K_s \) minor, then

1. \( |D| \leq O(t s \log s \log n / \log \log n) \).

and if \( G \) is the Gremban cover of such a graph, then the same bound holds and we can ensure that \( D \) is a Gremban cover as well.

Proof. Everything except the statement concerning Gremban covers follows immediately from Theorem 2.1 and Lemmas 3.7, 3.8, and 3.13.

In the case that \( G \) is Gremban cover, we apply the algorithm precondition to the graph that it covers, but keeping all weights positive. We then set \( R \) and \( D \) to both images of each edge output by the algorithm. Thus, the size of the set \( D \) is at most twice what it would otherwise be.

For our purposes, the critical difference between these two graphs is that a cycle in the covered graph corresponds in the Gremban cover to either two disjoint cycles or a double-traversal of that cycle. Altering the arguments to compensate for this change increases the bound of Lemma 3.10 by at most a factor of 3, and the bound of Lemma 3.13 by at most 9.

The spanning tree \( R \) is built using an algorithm of Alon, Karp, Peleg, and West [1]. The edges in the set \( D \) are constructed by using other information generated by this algorithm. In particular, the AKPW algorithm builds its spanning tree by first building a spanning forest, then building a spanning forest over that forest, and so on. Our algorithm works by decomposing the trees in these forests, and then adding a representative edge between each set of vertices in the decomposed trees.

Throughout this section, we assume without loss of generality that the maximum weight of an edge is 1.

3.1. The Alon-Karp-Peleg-West Tree

We build our preconditioners by adding edges to the spanning trees constructed by Alon, Karp, Peleg, and West [1]. In this subsection, we review their algorithm, state the properties we require of the trees it produces, and introduce the notation we need to define and analyze our preconditioner.

The AKPW algorithm is run with the parameters \( x = 2\sqrt{\log n \log \log n} \) and \( \rho = \left[ \frac{\log n}{\log \log n} \right] \), and the parameters \( \mu = 9\rho \log n \) and \( y = x \mu \) are used in its analysis.

We assume, without loss of generality, that the maximum weight edge in \( E \) has weight 1. The AKPW algorithm begins by partitioning the edge set \( E \) by weight as follows:

\[ E_i = \{ e \in E : 1/y^i < w(e) \leq 1/y^{i-1} \} \cdot \]

For each edge \( e \in E \), let \( \text{class}(e) \) be the index such that \( e \in E_{\text{class}(e)} \).
The AKPW algorithm iteratively applies a modification of an algorithm of Awerbuch [2], which we call cluster, whose relevant properties are summarized in the following lemma.

**Lemma 3.2 (Colored Awerbuch).** There exists an algorithm with template

\[ F = \text{cluster}(G, x, E_1, \ldots, E_k), \]

where \( G = (V, E) \) is a graph, \( x \) is a number, \( E_1, \ldots, E_k \) are disjoint subsets of \( E \), and \( F \) is a spanning forest of \( V \), such that

1. each forest of \( F \) has depth at most \( 3mk \log n \),
2. for each \( 1 \leq i \leq k \), the number of edges in class \( E_i \) between vertices in the same tree of \( F \) is at most \( x \) times the number of edges in class \( E_i \) between vertices in distinct trees of \( F \), and
3. \( \text{cluster} \) runs in time \( O(\sum_i |E_i|) \).

The other part of the AKPW algorithm is a subroutine with template

\[ G' = \text{contract}(G, F), \]

that takes as input a graph \( G \) and a spanning forest \( F \) of \( G \), and outputs the multigraph \( G' \) obtained by contracting the vertices of each tree in \( F \) to a single vertex. This contraction removes all resulting self-loops (which result from edges between vertices in the same tree), but keeps an image of each edge between distinct trees of \( F \). The classes, weights, and names of the edges are preserved, so that each edge in \( G' \) can be mapped back to a unique pre-image in \( G \).

We can now state the AKPW algorithm:

**Algorithm:** \( R = \text{AKPW}(G) \)

1. Set \( j = 1 \) and \( G^{(j)} = G \).
2. While \( G^{(j)} \) has more than one vertex
   - (a) Set \( R^j = \text{cluster}(G^{(j)}, x, E_{j-\rho+1}, \ldots, E_j) \).
   - (b) Set \( G^{(j+1)} = \text{contract}(G^{(j)}, R^j) \).
   - (c) Set \( j = j + 1 \).
3. Set \( R = \bigcup_j R^j \).

The tree output by the AKPW algorithm is the union of the pre-images of the edges in forests \( R^j \). Our preconditioner will include these edges, and another set of edges \( D \) constructed using the forests \( F^j \).

To facilitate the description and analysis of our algorithm, we define

\[ F^j \] to be the forest on \( V \) formed from the union of the pre-images of edges in \( R^j \cup \cdots \cup R^{j-1} \).

\[ T^j \] to be the tree of \( F^j \) containing vertex \( v \).

\[ E^j_i = \{(u, v) \in E_i : T^j_u \neq T^j_v \} \]

\[ H^j_i = E^j_i - E^{j+1}_i, \]

\[ H^j = \bigcup_i H^j_i. \]

We observe that \( F^{j+1} \) is comprised of edges from \( E_1, \ldots, E_j \), and that each edge in \( H^j \) has both endpoints in the same tree of \( F^{j+1} \).

Alon, et. al. prove:

**Lemma 3.3 (AKPW Lemma 5.4).** The algorithm AKPW terminates. Moreover, for every \( i \leq j \),

\[ |E^j_i| \leq |E^{(j-1)}_i| /x \leq |E_i| /x^{j-1}. \]

We remark that \( x^\rho > |E| \), so for \( i \leq j - \rho \), \( E^j_i = \emptyset \). The following lemma follows from the proof of Lemma 5.5 of [1] and the observation that \( y^\rho \geq |E| \).

**Lemma 3.4.** For each simple path \( P \) in \( F^{j+1} \) and for each \( l \), \( |P \cap E_i| \leq \min(y^{j+1}, y^\rho) \).

### 3.2. Tree Decomposition

Our preconditioner will construct the edge set \( D \) by decomposing the trees in the forests produced by the AKPW algorithm, and adding edges between the resulting sub-trees. In this section, we define the properties the decomposition must satisfy and describe the decomposition algorithm.

**Definition 3.5.** For a tree \( T \) and a set of edges \( H \) between the vertices of \( T \), we define an \( H \)-decomposition of \( T \) to be a pair \((W, \sigma)\) where \( W \) is a collection of subsets of the vertices of \( T \) and \( \sigma \) is a map from \( H \) into sets or pairs of sets in \( W \) satisfying

1. for each set \( W \in W \), the graph induced by \( T \) on \( W \) is connected,
2. for each edge in \( T \) there is exactly one set \( W \in W \) containing that edge, and
3. for each edge in \( e \in H \), if \( |\sigma(e)| = 1 \), then both endpoints of \( e \) lie in \( \sigma(e) \); otherwise, one endpoint of \( e \) lies in one set in \( \sigma(e) \), and the other endpoint lies in the other.

We note that there can be sets \( W \in W \) containing just one vertex of \( T \).

For a weighted set of edges \( H \) and an \( H \)-decomposition \((W, \sigma)\), we define the \( H \)-weight of a set \( W \in W \) by

\[ w_H(W) \overset{\text{def}}{=} \sum_{e \in H, W \in \sigma(e)} w(e) \]

We also define \( w_{tot}(H) \overset{\text{def}}{=} \sum_{e \in H} w(e) \).
Our preconditioner will use an algorithm for computing small \(H\)-decompositions in which each set \(W \in \mathcal{W}\) with \(|W| > 1\) has bounded \(H\)-weight.

**Lemma 3.6 (Tree Decomposition).** There exists an algorithm with template

\[
(W, \sigma) = \text{decompose}(T, H, \phi)
\]
that runs in time \(O(|H| + |T|)\) and outputs an \(H\)-decomposition \((W, \sigma)\) satisfying

1. for all \(W \in \mathcal{W}\) such that \(|W| > 1\), \(w_H(W) \leq \phi\), and
2. \(|W| \leq 4w_{1,\phi}(H)/\phi\).

**Proof.** We let \(T(v)\) denote the set of vertices in the subtree rooted at \(v\), and for a set of vertices \(W\), let \(H(W) = \{e \in H : e \cap H \neq \emptyset\}\). We then define \(\hat{\sigma}(v) \defeq H(T(v))\).

Let \(v_0\) denote the root of the tree. Our algorithm will proceed as if it were computing \(\hat{\sigma}(v_0)\) via a depth-first traversal of the tree, except that whenever it encounters a subtree of weight more than \(\phi/2\), it will place nodes from that subtree into a set in \(\mathcal{W}\) and remove them from the tree. There are three different cases which determine how the nodes are placed into the set and how \(\sigma\) is constructed.

If, when processing a node \(v\), the algorithm has traversed a subset of the children of \(v\), \(\{v_1, \ldots, v_k\}\), such that \(\hat{\sigma}(v_1) + \cdots + \hat{\sigma}(v_k) \geq \phi/2\), then a set \(W\) is created, all the nodes in \(\{v\} \cup_{i=1}^k T(v_i)\) are placed in \(W\), and those nodes in \(\cup_{i=1}^k T(v_i)\) are deleted from the tree. If a node \(v\) is encountered such that \(\phi/2 \leq H(T(v)) \leq \phi\), then a set \(W\) is created, the nodes in \(T(v)\) are placed in \(W\), and those nodes in \(W\) are deleted from the tree. In either case, for each node \(e \in H(W)\) we set \(\sigma(e) = \sigma(e) \cup \{W\}\).

If a node \(v\) is encountered which is not handled by either of the preceeding cases and for which \(\hat{\sigma}(v) > \phi\), then two sets \(W_1 = T(v)\) and \(W_2 = \{v\}\) are created, and those nodes in \(T(v)\) are deleted from the tree. For each edge \(e \in H(v)\), \(W_2\) is added to \(\sigma(e)\) and for each edge \(e \in H(T(v) - \{v\})\), \(W_1\) is added to \(\sigma(e)\).

When the algorithm finally returns from examining the root, all the remaining nodes are placed in a final set, and this set is added to \(\sigma(e)\) for each edge \(e \in H\) with endpoints in this set. The algorithm maintains the invariant that whenever it returns from examining a node \(v\), it has either deleted \(v\), or removed enough vertices below \(v\) so that \(\hat{\sigma}(v) < \phi/2\).

To see that the algorithm produces at most \(4w_{1,\phi}/\phi\) sets, we note that each edge in \(H\) can contribute its weight to at most two sets, and that every time the algorithm forms sets, it either forms one set with weight at least \(\phi/2\) or two sets with total weight at least \(\phi\). \(\square\)

### 3.3. Constructing the Preconditioner

We can now describe our algorithm for constructing the preconditioner. We will defer a discussion of how to efficiently implement the algorithm to Lemma 3.8.

The algorithm will make use of the parameter

\[
\theta^{(j)} \defeq \left\{ \begin{array}{ll}
 x^{j-1} & \text{if } j \leq \rho \\
 x^\rho y^{j-\rho-1} & \text{otherwise}
\end{array} \right.
\]

**Algorithm:** \((T, D) = \text{Precondition}(G)\)

1. Run \(T = \text{AKPW}(G)\). Set \(h\) to the number of iterations taken by \(\text{AKPW}\), and record \(R^1, \ldots, R^h\) and \(H^1, \ldots, H^h\).

2. For \(j = 1\) to \(h\)
   
   (a) let \(\{T_1, \ldots, T_k\}\) be the set of trees in \(F^{j+1}\).
   
   (b) for \(i = 1\) to \(k\)

   i. let \(H\) be the subset of edges in \(H_j\) with endpoints in \(T_i\)
   
   ii. Set \((\{W_1, \ldots, W_l\}, \sigma)\) to

   \[
   \text{decompose}(T_i, H, |E| / \theta^{(j)})
   \]
   
   iii. for each \(\mu \leq \nu \leq l\), let \(\sigma_{\mu, \nu}\) be the maximum weight edge in \(H\) between \(W_\mu\) and \(W_\nu\), and add \(\sigma_{\mu, \nu}\) to \(D\).

**Lemma 3.7.** Let \(D\) be the set of edges produced by precondition. Then,

\[
|D| \leq 8\rho^2 t^2 = O(t^2 \log n / \log \log n).
\]

Moreover, if \(G\) has no \(K_s\) minor, then \(|D| = O(ts \log s \log n / \log \log n)|\).

**Proof.** Let \(c_j\) be the total number of sets produced by applying \(\text{decompose}\) to the trees in \(F^{j+1}\). We first bound \(\sum_j c_j\). We have

\[
\sum_j c_j \leq \sum_j 4\theta^{(j)} w_{1,\phi}(H^j)/|E| = (4t/|E|) \sum_j \theta^{(j)} \sum_i w_i,\phi(H^j_i).
\]

To bound this sum, we set

\[
h^j_i = \left\{ \begin{array}{ll}
 0 & \text{if } j < i,
 4t |H^j_i| & \text{if } i = j, \text{ and}
 4t |H^j_i| & \text{if } j > i.
\end{array} \right.
\]
We observe that Lemma 3.3 implies $h^j_i \leq |E_i|/x^{j-i}$, and $h^j_i = 0$ for $j \geq i + \rho$. As $\theta^{(j)}$ is increasing, we have
\[
\sum_j \sum_i \theta^{(j)} w_{i,d}(H^j_i) \leq \sum_j \sum_{i+j-i+\rho+1} \theta^{(j)} h_i^j / y^{j-i-1}
= \sum_i \sum_{j=i}^{i+j-i+\rho+1} \theta^{(j)} h_i^j / y^{j-i}
\leq \sum_i \sum_{j=i}^{i+j-i+\rho+1} \theta^{(j)} / (x^{j-i} y^{j-i+1})
\leq \sum_i |E_i| \rho
\leq |E| \rho,
\]
as $\theta^{(j)} \leq x^{j-i} y^{j-i-1}$ for $j \leq i + \rho - 1$. Thus, $\sum_j c_j \leq 4pt$, and, because we add at most one edge between each pair of these sets, we have $|D| \leq 8pt^2 s^2$.

As observed by Vaidya, a result of Mader [4] implies that if a graph does not have a complete graph on $s$ vertices as a minor, then the average degree of every minor of $G$ is $O(s \log s)$. Hence, the number of edges added to $D$ at iteration $j$ is at most $c_j s \log s$, and so
\[
|D| \leq \sum_j c_j s \log s \leq 8pt s \log s.
\]

Finally, a graph of genus $s^2$ does not have a $K_{s(s)}$ minor. \hfill \Box

Using the dynamic trees data structure of Sleator and Tarjan [22], we prove:

**Lemma 3.8.** If $G$ is a graph with $n$ vertices and $m$ edges, then the output of precondition can be produced in $O(m \log m)$ time.

### 3.4. Analyzing the Preconditioner

We will use weighted embeddings of edges into paths in $R \cup D$ to bound the quality of our preconditioners. The weights will be determined by a function $\tau(j,l)$, which we now define to be
\[
\tau(j,l) = \begin{cases} 
1 & j - l < \rho \\
\frac{(j-l-\rho+1)^2}{y^{j-l+1} x^{\rho+1}} & \text{Otherwise.}
\end{cases}
\]

For each edge $e \in H^j_i$ and each edge $f \in \text{path}_\pi(e)$, we will set $\pi(e,f) = \tau(j,\text{class}(f))$. We will construct $\pi$ so as to guarantee $\text{class}(e) < \text{class}(f) + \rho$.

It remains to define the paths over which edges are embedded. For an edge $e = (u,v)$ in $H^j_i$, if $e \in R \cup D$ then we set $\text{path}_\pi(e) = e$ and $\pi(e,e) = 1$. Otherwise, we let $T$ be the tree in $F_{j+1}^i$ containing the endpoints of $e$ and let $\sigma$ be the function output by decompose on input $T$. If $|\sigma(e)| = 1$, then we let $\text{path}_\pi(e)$ be the simple path in $T$ connecting the endpoints of $e$. Otherwise, we let $\{W_\nu, W_\mu\} = \sigma(e)$ and let $a_{\nu,\mu}$ be the edge added between $W_\nu$ and $W_\mu$. We then let $\text{path}_\pi(e)$ be the concatenation of the simple path in $T$ from $u$ to $a_{\nu,\mu}$, the edge $a_{\nu,\mu}$, and the simple path in $T$ from $a_{\nu,\mu}$ to $v$.

The two properties that we require of $\pi$ are encapsulated in the following lemma.

**Lemma 3.9.** (a) For all $j \geq 1$, $\sum_{i=1}^{j} \frac{y^i \min(y^{i-t+1}, y^j)}{\tau(j,l)} \leq y^{j+1}(\rho+2)$, and
(b) For all $l \geq 1$, $\sum_{j \geq l} \tau(j,l) \leq (\rho+1)$.

We now derive the upper bound we need on the maximum weighted congestion of the embedding $\pi$.

**Lemma 3.10.** For each $j$ and each simple path $P$ in $F_{j+1}^i$,
\[
\sum_{f \in P} \frac{1}{w(f) \tau(j, \text{class}(f))} \leq (\rho+2) y^{j+1}
\]

**Proof.**
\[
\sum_{f \in P} \frac{1}{w(f) \tau(j, \text{class}(f))} \leq \sum_{i=1}^{j} \sum_{f \in P \cap E_i} \frac{1}{w(f) \tau(j,l)}
\leq \sum_{i=1}^{j} \min_{l \leq j} \frac{y^l \min(y^{j-l+1}, y^j)}{\tau(j,l)}
\leq \frac{1}{\tau(j,l)}
\leq y^{j+1}(\rho+2)
\]
where the third-to-last inequality follows from Lemma 3.4, the second-to-last inequality follows from $f \in E_i$, and the last inequality follows from Lemma 3.9 (a). \hfill \Box

**Lemma 3.11.** For each edge $e \in E$,
\[
wd_\pi(e) \leq (2\rho + 5) y^{j+1} w(e)
\]

**Proof.** Let $e \in H^j_i$, let $T$ be the forest in $F_{j+1}^i$ containing the endpoints of $e$, and let $(W, \sigma)$ be the output of decompose on input $T$. If $|\sigma(e)| = 1$, the $e$ is routed over the simple path in $T$ connecting its endpoints, so we can apply Lemma 3.10 to show
\[
wd_\pi(e) \leq (\rho+2) y^{j+1} w(e).
\]

Otherwise, let $\sigma(e) = \{W_\nu, W_\mu\}$, and observe that $\text{path}_\pi(e)$ contains two simple paths in $T$ and the edge
Applying Lemma 3.10 to each of these paths and recalling class \( \text{class}(a_{v,\mu}) \leq j \), which implies \( w(a_{v,\mu}) \geq 1/y^j \), we obtain

\[
\text{wd}(e) \leq 2(\rho^j + 2)g^j w(e) + y^j w(e) \leq (2\rho + 5)g^j w(e).
\]

Proof. Let \( T \) be the tree in \( F_{j+1} \) containing the endpoints of \( f \), and let \( (W,\sigma) \) be the output of \text{decompose} on input \( T \). There are two cases two consider: \( f \) can either be an edge of \( T \), or \( f \) can be one of the edges \( a_{v,\mu} \). If \( f \) is an edge of \( T \), let \( W \) be the set of \( W \) containing its endpoints. Otherwise, if \( f \) is one of the edges \( a_{v,\mu} \), let \( W \) be the larger of the sets \( W_v \) or \( W_\mu \). If \( |W_\mu| = |W_v| = 1 \), then the only edge having \( f \) in its path is \( f \) itself, in which case the lemma is trivial. So, we may assume \( |W| > 1 \). In either case, each edge \( e \) for which \( f \in \text{path}(e) \) must have \( W \in \sigma(e) \). Thus,

\[
\sum_{e \in H : f \in \text{path}(e)} \text{wd}(e) \leq \sum_{e \in H : W \in \sigma(e)} \text{wd}(e) \\
\leq \sum_{e \in H : W \in \sigma(e)} w(e)(2\rho + 5)g^j \\
= w_H(W)(2\rho + 5)g^j \\
\leq (2\rho + 5)g^j |E|/t.
\]

\[\square\]

Lemma 3.13. Let \( R, D \) and \( \pi \) be constructed as above. Let \( S = R \cup D \). Then,

\[
\max_{f \in S} \text{wc}(f) = \frac{m}{l}2^o(\sqrt{\log n / \log \log n}).
\]

Proof. For any edge \( f \in S \), we let \( l = \text{class}(f) \) and compute

\[
\text{wc}(f) = \sum_{e \in E : f \in \text{path}(e)} \pi(e, f) \\
= \sum_{j} \sum_{e \in H : f \in \text{path}(e)} \tau(j, l) \\
\leq \sum_{j} \tau(j, l)(2\rho + 5)\mu^j g^j |E|/t \\
\leq (\rho + 1)(2\rho + 5)\mu^j g^j |E|/t, \\
= 2^o(\sqrt{\log n / \log \log n}) |E|/t.
\]

where the second-to-last inequality follows from Lemma 3.12, the last inequality follows from Lemma 3.9 (b), and the last equality follows from \( \mu^j = 2^o(\sqrt{\log n / \log \log n}) \). \[\square\]

4. One-Shot Algorithms

Our first algorithm constructs a preconditioner \( B \) for the matrix \( A \), performs a partial Cholesky factorization of \( B \) by eliminating the vertices in trim order to obtain \( B = L[J, 0, 0, A_1]L^T \), performs a further Cholesky factorization of \( A_1 \) into \( L_1L_1^T \), and applies the preconditioned Chebyshev algorithm. In each iteration of the preconditioned Chebyshev algorithm, we solve the linear systems in \( B \) by back-substitution through the Cholesky factorizations.

Theorem 4.1 (One-Shot). Let \( A \) be an \( n \)-by-\( n \) PSD matrix with \( m \) non-zero entries. Using a single application of our preconditioner, one can solve the system \( Ax = b \) to relative accuracy \( \epsilon \) in time \( O(m^{18/13 + o(1)} \log(k(A)/\epsilon)) \). Moreover, if the sparsity graph of \( A \) does not contain a minor isomorphic to the complete graph on \( m^2 \) vertices, or if it has genus at most \( m^{2\theta} \), for \( \theta < 1/3 \), then the exponent of \( m \) can be reduced to \( 1.125(1 + \theta) + o(1) \).

Proof. The time taken by the algorithm is the sum of the time required to compute the preconditioner, perform the partial Cholesky factorization of \( B \), pre-process \( A_1 \), and solve each linear system \( Bx = y \) by eliminating the vertices in trim order. In each case, we will set \( t = m^\gamma \) for some constant \( \gamma \), and note that the number of iterations will be \( m^{1-\gamma}/2+o(1) \), and that the matrix \( A_1 \) will depend on \( m^\gamma \).

If we do not assume that \( A \) has special topological structure, then \( A_1 \) is a matrix on \( m^{2\gamma+o(1)} \) vertices. If we solve systems in \( A_1 \) by Cholesky factorization, then it will take time \( O(m^{m/\gamma+o(1)}) \) to perform the factorization and time \( O(m^{m/\gamma+o(1)}) \) to solve each system. So, the total time will be \( m^{(1-\gamma)/2+4\gamma+o(1)} + m^{m/\gamma+o(1)} \). Setting \( \gamma = 3/13 \), we obtain the first result.

If the graph has genus \( \theta \) or does not have a \( K_{m^2} \), minor, or is the Gremban cover of such a graph, then can apply part (2’’) of Theorem 3.1. Thus, \( A_1 \) is a matrix on \( m^{m^2+o(1)} \) vertices. In the Gremban cover case, the preconditioner is a Gremban cover, and so the partial Cholesky factorization can ensure that \( A_1 \) is a Gremban cover as well. As the Gremban cover of a graph has a similar family of separators to the graph it covers, in either case we can apply the algorithm of Lipton, Rose and Tarjan to obtain the Cholesky factorization of \( A_1 \). By Theorem 1.7, with \( \alpha = 1/2 + \theta \), the time required to perform the factorization will be \( O(m + m^{2(3\theta+3/2)+o(1)}) \), and the time required to
solve the system will be
\[ O \left( m^{1 - \gamma} 2 (m + m^{(2^\theta + 1)} + o(1)) \right) = O \left( m^{1 - \gamma} 2^{1 + o(1)} \right), \]
provided \( \gamma(2\theta + 1) \leq 1 \). We will obtain the desired result by setting \( \gamma = (3 - 9\theta)/4 \).

5. Recursive Algorithms

We now show how to apply our algorithm recursively to improve upon the running time of the algorithm presented in Theorem 4.1.

The recursive algorithm is quite straightforward: it first constructs the top-level preconditioner \( B_1 \) for matrix \( A_0 = A \). It then eliminates to vertices of \( B_1 \) in the trim order to obtain the partial Cholesky factorization \( B_1 = L_1 C_1 L_1^T \), where \( C_1 = [I, 0; 0, A_1] \). When an iteration of the preconditioned Chebyshev algorithm needs to solve a linear system in \( B_1 \), we use forward- and backward-substitution to solve the systems in \( L_1 \) and \( L_1^T \), but recursively apply our algorithm to solve the linear system in \( A_1 \).

We will use a recursion of depth \( r \), a constant to be determined later. We let \( A_0 = A \) denote the initial matrix. We let \( B_{i+1} \) denote the preconditioner for \( A_i, L_i C_i L_i^T \) be the partial Cholesky factorization of \( B_i \) in trim order, and \( C_i = [I, 0; 0, A_i] \). To analyze the algorithm, we must determine the relative error \( \epsilon_i \) to which we will solve the systems in \( A_i \). The bound we apply is derived from the following lemma, which we derive from a result of Golub and Overton [11].

**Lemma 5.1 (Preconditioned Inexact Chebyshev Method).** Let \( A \) and \( B \) be Laplacian matrices satisfying \( \sigma(B, A) \geq 1 \). Let \( x \) be the solution to \( Ax = b \). If, in each iteration of the preconditioned Chebyshev Method, a vector \( z_k \) is returned satisfying
\[ B z_k = r_k + q_k, \text{ where } ||q_k|| \leq \delta ||r_k||, \]
where \( \delta \leq \left( 128 \sqrt{\kappa_f(B)} \sigma(A,B) \right)^{-1} \), then the \( k \)-th iterate, \( x_k \), output by the algorithm will satisfy
\[ ||x - x_k|| \leq 6 \cdot 2^{-k/\sqrt{\kappa_f(A,B)}} \sqrt{\kappa_f(A)} \sqrt{\kappa_f(B)} ||x||. \]

Our main theorem is:

**Theorem 5.2 (Iterative).** Let \( A \) be an \( n \)-by-\( n \) PSDD matrix with \( m \) non-zero entries. Using the recursive algorithm, one can solve the system \( Ax = b \) to relative accuracy \( \epsilon \) in time
\[ O \left( m^{1.31 + o(1)} \left( \log(\epsilon^{-1}) \log(\frac{\max(A)}{\min(A)}) \right)^{O(1)} \right). \]

Moreover, if the graph of \( A \) does not contain a minor isomorphic to the complete graph on \( m^\theta \) vertices, or has genus at most \( m^{2\theta} \), or is the Gremban cover of such a graph, then the exponent of \( m \) can be reduced to \( 1 + 3\theta + o(1) \).

We note that if \( G(A) \) is planar, then the algorithm take time nearly linear in \( m \). We believe that we can improve the term \( \min(\gamma) \) to \( \kappa_f(A) \).

The following lemma is needed to bound \( \kappa_f(A_i) \) and \( \kappa_f(B_i) \).

**Proposition 5.3 (Fiedler).** Suppose \( A \) is a \( n \) by \( n \) Laplacian matrix. Then the smallest non-zero eigenvalue of \( A \) is at least \( \min(A)/n^2 \) and the largest eigenvalue of \( A \) is at most \( \max(A) \) Thus, \( \kappa_f(A) \leq n^3 \max(A)/\min(A) \).

The following two lemmas allow us to bound the accuracy of the solutions to systems in \( B_i \) in terms of the accuracy of the solutions to the corresponding systems in \( A_i \).

**Lemma 5.4.** Let \( B \) be a Laplacian matrix and let \( L C L^T \) be the partial Cholesky factorization obtained by eliminating vertices of \( B \) in the trim order. Then
\[ \kappa(L) \leq \sqrt{8(n + 1)n^4 \max(1, \max(A)) \min(A)}. \]

**Lemma 5.5.** Let \( B \) be a Laplacian matrix and let \( L C L^T \) be the partial Cholesky factorization obtained by eliminating vertices of \( B \) in the trim order. For any \( c \in \text{Span}(B) \), let \( s \) be the solution to \( D s = L^{-1} c \) and let \( \tilde{s} \) satisfy \( ||s - \tilde{s}|| \leq \epsilon ||s|| \). Let \( \tilde{y} \) be the solution to \( L^T \tilde{y} = \tilde{s} \). Then
\[ ||c - B \tilde{y}|| \leq \epsilon \kappa(L) ||c||. \]

**Proof of Theorem 5.2.** Without loss of generality, we assume \( \max(A) = 1 \). For \( A_0, \ldots, A_r, B_1, \ldots, B_r, \) and \( L_1, \ldots, L_r \) as defined above, we can apply Proposition 5.3, Corollary 1.4, and Lemma 5.4 to show:
- \( \max(B_{i+1}) \leq \max(A_i) \leq \max(A)/n^i \),
- \( \min(B_{i+1}) \geq \min(A_i) \geq \max(A)/n^i \),
- \( \kappa(B_i) \leq n^{2i+1}(\max(A)/\min(A)) \),
- \( \kappa(A_i) \leq n^{2i+3}(\max(A)/\min(A)) \),
- \( \kappa(L_i) \leq n^{i}(\max(A)/\min(A)) \).

In the recursive algorithm we will solve systems in \( A_i \), for \( i \geq 1 \), to accuracy
\[ \epsilon_i = \left( 128n^{4i+7}(\max(A)/\min(A))^2 \right)^{-1}. \]
By Lemma 5.5 and the above bounds, we then obtain solutions to the systems in \( B_i \) to sufficient accuracy to apply Lemma 5.1.
Let $m_i$ be the number of edges of $A_k$. When constructing the preconditioner, we set $t_i = (m_i)^\gamma$, for a $\gamma$ to be chosen later.

Thus, by Theorem 3.1 and Proposition 1.1, $m_i \leq m_i^{(2\gamma)}$, and $\kappa(A_i, B_{i+1}) = m_i^{(2\gamma)(1-\gamma)+o(1)}$.

We now prove by induction that the running time of the algorithm obtained from a depth $r$ recursion is

$$O\left( m_i^{\beta_r + o(1)} \left( r \log(n_{\max}(A)/n_{\min}(A)) \right)^r \right),$$

where

$$\beta_r \overset{\text{def}}{=} \frac{1 - \gamma}{2} \sum_{i=1}^{r} (2\gamma)^{r-i-1} + 2(2\gamma)^r,$$

and $\gamma \overset{\text{def}}{=} (3 - \sqrt{5})/2$. In the limit, $\beta_r$ approaches $\beta_\infty = (3 + \sqrt{5})/4$ from above. The base case, $r = 1$, follows from Theorem 4.1.

The preprocessing time is negligible as the partial Cholesky factorizations used to produce the $C_i$ take linear time, and the full Cholesky factorization is only performed on $A^r$.

Thus, the running time is bounded by the iterations. The induction follows by observing that the iteration time is $m_i^{1+o(1)} \log (\kappa(A_i)/\kappa(B_i)/\epsilon_r)$, which proves the inductive hypothesis because $m_i^{\beta_r - 1} > m$. As 1.31 > $\beta_\infty$, there exists an $r$ for which $\beta_r < 1.31$.

When the graph of $A$ does not contain a $K_m$ minor or has genus at most $m^{2\theta}$, we apply a similar analysis. In this case, we have $m_i \leq m_{i-1}^{1+o(1)}$. Otherwise, our proof is similar, except that we set $\gamma = (3 - \theta - \sqrt{1 + 6\theta + \theta^2})/2$, and obtain $\beta_\infty = (3 + \theta + \sqrt{1 + 6\theta + \theta^2})/4$, and note that $\beta_\infty \leq 1 + 5\theta$.

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