Gödel-type Solutions in Einstein-Maxwell-Scalar Field Theories

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We show that one-parameter family of Gödel-type metrics are exact solutions in a wide class of Einstein-Maxwell-scalar field theories. In these solutions, the gauge field has a 1-form expectation value parallel to the timelike contact form that represents the rotation of the Gödel universe.

Subject Index B02, E01

1. Introduction
The cosmological metric found by Gödel, so-called Gödel universe [1], is one of the most intriguing exact solutions to the Einstein equations. Striking properties of the metric are the homogeneous geometry in space and time, the cosmological rotation, and the appearance of closed timelike curves (CTCs) through each points. For a long period after the pioneering work by Gödel, much attention is paid on many Gödel-type solutions with a cosmological constant and fluids in general relativity [2–10] (see also references therein).

The Gödel-type spacetime is a direct product of a 3-dimensional space with a Lorentzian metric and a 1-dimensional space. The 3-dimensional spacetimes make a one-parameter family of squashed 3-dimensional anti-de Sitter spaces (AdS$^3$) [11]. Such a 3-dimensional spacetime is a contact space with a timelike contact form that describes the rotation of the spacetime [12]. This structure is easily extended to higher-dimensional theories, then, recently, the Gödel-type metrics in higher dimensions are studied in the context of supersymmetric theories [13–15].

We show, in this paper, that the one-parameter family of Gödel-type metrics are exact solutions to the Einstein equations couples to purely field theories that consist of a U(1)-gauge field, a real scalar field, and a complex scalar field with a wide class of potentials. The interesting property of the solutions is that the gauge field has a 1-form expectation value parallel to the timelike contact form. Owing to the non-vanishing 1-form expectation value, the complex scalar field has an expectation value that is shifted off the potential minimum in these solutions. These solutions provide new non-trivial symmetry-breaking states in simple models, then it suggests that a variety of gauge field theories admit the solutions with the Gödel-type metrics. As is seen later, for example, a field theory admits two Gödel-type
solutions: one is with CTCs, and the other is without CTC. It would open the possibility of phase transition from the former state to the latter state.

2. Basic Model

We consider the action

\[
S = \int \sqrt{-g} d^4x \left( \frac{1}{16\pi G} R - g^{\mu\nu} (\nabla_\mu \Phi)^* (\nabla_\nu \Phi) - V(\Phi^* \Phi) - \frac{1}{2} g^{\mu\nu} (\partial_\mu \Psi)(\partial_\nu \Psi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right),
\]

(1)

where \( R \) is the scalar curvature with respect to a metric \( g_{\mu\nu} \), \( \Phi \) is a complex scalar field, \( \Psi \) is a real scalar field, and \( F_{\mu\nu} \) is the field strength of a U(1) gauge field \( A_\nu \). The complex scalar field has a potential \( V(\Phi^* \Phi) \), and couples with the gauge field through \( \nabla_\nu \Phi := (\partial_\nu - ieA_\nu)\Phi \).

By variation of the action (1), we derive field equations:

\[
\frac{1}{\sqrt{-g}} \nabla_\mu (\sqrt{-g} g^{\mu\nu} \nabla_\nu \Phi) - \frac{\partial V}{\partial \Phi^*} = 0,
\]

(2)

\[
\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Psi) = 0,
\]

(3)

\[
\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} F^{\mu\nu}) = J^\nu,
\]

(4)

\[
R_{\mu\nu} = 8\pi G \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\alpha_{\alpha} \right),
\]

(5)

where \( J_\mu \) is the electric current defined by

\[
J_\mu := ie \left( \Phi^* \nabla_\mu \Phi - \Phi (\nabla_\mu \Phi)^* \right),
\]

(6)

and the energy-momentum tensor \( T_{\mu\nu} \) is given by

\[
T_{\mu\nu} = 2(\nabla_\mu \Phi)^* (\nabla_\nu \Phi) - g_{\mu\nu} \left( g^{\alpha\beta} (\nabla_\alpha \Phi)^* (\nabla_\beta \Phi) + V(\Phi^* \Phi) \right) + (\partial_\mu \Psi)(\partial_\nu \Psi) - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} (\partial_\alpha \Psi)(\partial_\beta \Psi) + F_{\mu\alpha} F^{\alpha\nu} - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}.
\]

(7)

We assume the total spacetime \( \mathcal{M}^4 \) is a direct product of a 3-dimensional spacetime \( \mathcal{N}^3 \) and \( \mathbb{R}^1 \), where the metric takes the form

\[
ds_{\mathcal{M}}^2 = ds_{\mathcal{N}}^2 + d\chi^2.
\]

(8)

The metric on \( \mathcal{N}^3 \) is assumed to be

\[
ds_{\mathcal{N}}^2 = -(dt + f(r)d\theta)^2 + dr^2 + h^2(r)d\theta^2,
\]

(9)

where the coordinate \( \theta \) is an angular coordinate with the period \( 2\pi \). The functions \( f(r) \) and \( h(r) \) should be determined later.
We introduce the 1-form basis,
\[ \sigma^0 = dt + f(r)d\theta, \quad \sigma^1 = dr, \quad \sigma^2 = h d\theta, \quad \sigma^3 = d\chi, \] (10)
so as to rewrite the metric (8) as
\[ ds^2 = -(\sigma^0)^2 + (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2. \] (11)
We note that
\[ d\sigma^0 = f' h \sigma^1 \wedge \sigma^2, \] (12)
where the prime denotes the derivative with respect to \( r \). Therefore, if \( f' \neq 0 \),
\[ \sigma^0 \wedge d\sigma^0 \neq 0 \] (13)
holds. The 1-form \( \sigma^0 \) satisfying (13) is called a contact form on the 3-dimensional Lorentzian space \( \mathcal{N}^3 \).
Furthermore, if we assume \( f' = 2\omega h \), (\( \omega : \text{const.} \)) (14),
it is known that \( \mathcal{N}^3 \) is a quasi-Sasaki space with a Lorentzian metric[16, 17]. The Ricci curvature tensor with respect to the metric (11) reduces to the simple form:
\[ R_{ab} = 2\omega^2 \sigma^0_a \sigma^0_b + \left( 2\omega^2 \frac{h''}{h} \right) \left( \sigma^1_a \sigma^1_b + \sigma^2_a \sigma^2_b \right). \] (15)

3. Equations of scalar fields and Maxwell field
We assume the scalar fields and the gauge field as
\[ \Phi = v, \quad \Psi = \Psi(\chi), \quad A = B \sigma^0, \] (16)
where \( v \) and \( B \) are assumed to be non-vanishing real positive constants. The equation (2) of \( \Phi \) reduces to
\[ \frac{1}{2} \frac{dV(v)}{dv} = e^2 B^2 v, \] (17)
where \( V(v) \) is defined by replacement \( \Phi^* \Phi \) by \( v^2 \) in \( V(\Phi^* \Phi) \). Owing to the non-vanishing gauge field, the expectation value \( v \) of \( \Phi \) is shifted off the stationary points of the potential \( V(v) \).
The equation (3) of Ψ reduces to
\[ \partial^2 \chi \Psi(\chi) = 0, \]  
then by using a real constant \( k \), we have
\[ \Psi = k \chi + \text{const.} \]  
The Maxwell equation (4) in the differential form is written as
\[ \star (d \star F) = -J, \]  
where \( \star \) denotes the Hodge dual operator. The current (6) carried by Φ reduces to
\[ J = 2e^2v^2A. \]  
Then, (20) becomes
\[ \star (d \star F) = -2e^2v^2A. \]  
Owing to the scalar field Φ has the expectation value \( v \), the gauge field acquires the mass \( 2e^2v^2 \), then we have the Proca equation (22). On the other hand, from the assumptions (16) that the gauge 1-form \( A \) is parallel to the contact form \( \sigma_0 \), the field strength \( F \) is
\[ F = dA = 2\omega B \sigma_1 \wedge \sigma_2, \]  
then the left hand side of (20) becomes
\[ \star (d \star F) = -4\omega^2A. \]  
Therefore, (22) requires
\[ \omega^2 = \frac{1}{2} e^2v^2. \]  

4. Einstein’s equations
The total energy-momentum tensor (7) in the 1-form basis (10) is given by
\[ T_{ab} = T_{ab}^\Phi + T_{ab}^\Psi + T_{ab}^F, \]  
where
\[ T_{ab}^\Phi = (e^2B^2v^2 + V(v)) \sigma_a^0\sigma_b^0 + (e^2B^2v^2 - V(v)) (\sigma_a^1\sigma_b^1 + \sigma_a^2\sigma_b^2 + \sigma_a^3\sigma_b^3), \]  
\[ T_{ab}^\Psi = \frac{k^2}{2} (\sigma_a^0\sigma_b^0 - \sigma_a^1\sigma_b^1 - \sigma_a^2\sigma_b^2 + \sigma_a^3\sigma_b^3), \]  
\[ T_{ab}^F = 2\omega^2B^2(\sigma_a^0\sigma_b^0 + \sigma_a^1\sigma_b^1 + \sigma_a^2\sigma_b^2 - \sigma_a^3\sigma_b^3). \]  

From (15) and (26)-(28), the Einstein equation (5) reduces to
\[ 2\omega^2\sigma_a^0\sigma_b^0 + \left( 2\omega^2 - \frac{k^2}{2} \right) (\sigma_a^1\sigma_b^1 + \sigma_a^2\sigma_b^2) \]
\[ = (2e^2B^2v^2 - V(v) + 2\omega^2B^2) \sigma_a^0\sigma_b^0 + (V(v) + 2\omega^2B^2) (\sigma_a^1\sigma_b^1 + \sigma_a^2\sigma_b^2) \]
\[ + (V(v) + k^2 - 2\omega^2B^2) \sigma_a^3\sigma_b^3, \]  
(29)
where we use the unit $8\pi G = 1$. Namely, we have a set of coupled equations:

$$
2\omega^2 = 2e^2B^2v^2 - V(v) + 2\omega^2B^2,
$$

(30)

$$
2\omega^2 - \frac{h''}{h} = V(v) + 2\omega^2B^2,
$$

(31)

$$
0 = V(v) + k^2 - 2\omega^2B^2.
$$

(32)

From (30), (32) with (25), for non-vanishing $v$, we have

$$
B^2 = \frac{1}{2} - \frac{k^2}{2e^2v^2} - \frac{1}{3e^2v^2}(e^2v^2 + V(v)),
$$

(33)

$$
k^2 = \frac{1}{3}(e^2v^2 - 2V(v)),
$$

(34)

and (17) becomes

$$
3v \frac{dV(v)}{dv} - 2e^2v^2 - 2V(v) = 0.
$$

(35)

For a given potential $V(\Phi^*\Phi)$, let $v_0$ be a root of the algebraic equation (35). Inserting $v = v_0$ into (33) and (34), we express $B^2$ and $k^2$ by $v_0$.

Using (25), (33) and (34) we rewrite (31) as

$$
h'' = 2k^2 h.
$$

(36)

For regularity, $h$ should behave $h \to r$ as $r \to 0$, then we have

$$
h = \frac{1}{\sqrt{2k}} \sinh \sqrt{2kr}, \quad f = \frac{2\omega}{k^2} \sinh^2 \frac{kr}{\sqrt{2}}.
$$

(37)

Therefore, we obtain Gödel-type metrics:

$$
ds^2 = - \left( dt + \frac{2\omega}{k^2} \sinh^2 \frac{kr}{\sqrt{2}} d\theta \right)^2 + dr^2 + \frac{1}{2k^2} \sinh^2 \sqrt{2kr} d\theta^2 + d\chi^2,
$$

(38)

where the parameters $\omega$ and $k$ are given by (25) and (34) for $v = v_0$. The metric (38) is already discussed by Rebouças and Tiomno [7].

From the explicit form of the metric (38), we see that if $2\omega^2/k^2 > 1$ closed timelike curves (CTCs) appear in the region $r_c < r$, where the critical radius $r_c$ is given by

$$
\sinh^2 \frac{kr_c}{\sqrt{2}} = \left( \frac{2\omega^2}{k^2} - 1 \right)^{-1}.
$$

(39)

A circle of a radius $r = const.$ $> r_c$ on a $t = const.$ surface is a CTC. The radius of CTC becomes larger and diverges as $k^2$ approaches to $2\omega^2$.

By use of (25) and (34) the existence condition of CTC becomes

$$
V(v_0) > -e^2v_0^2.
$$

(40)

Positivity of $B^2$ and $k^2$ in (33) and (34) requires

$$
\frac{1}{2} e^2v_0^2 > V(v_0) > -e^2v_0^2,
$$

(41)

where the upper bound corresponds to $k \to 0$ and the lower to $B \to 0$. Then, if $B \neq 0$, CTCs appear generally. In the special case $B = 0$, from (30) and (32) we have $k^2 = 2\omega^2$, and (38) reduces to a metric of $AdS^3 \times \mathbb{R}$ in the form

$$
ds^2 = \frac{1}{2k^2} \left( - \left( d\tau + 2 \sinh^2 \frac{\rho}{2} d\theta \right)^2 + d\rho^2 + \sinh^2 \rho \ d\theta^2 \right) + d\chi^2,
$$

(42)

where $\rho := \sqrt{2kr}$ and $\tau := \sqrt{2kt}$. In this case, there is no CTC.
5. Examples

Here, we present two examples of the potential $V(\Phi^*\Phi)$. The first case is a massive free complex scalar field $\Phi$ with a negative vacuum energy. The potential is given by $V(\Phi^*\Phi) = m^2\Phi^*\Phi - V_0$, where $m$ and $V_0$ are positive constants. Solving (35), we have

$$v_0^2 = \frac{V_0}{e^2 - 2m^2},$$

(43)

where $e^2 > 2m^2$ should be assumed. From (33) and (34), we have

$$B^2 = \frac{m^2}{e^2}, \quad k^2 = \frac{1}{3}V_0,$$

(44)

and the metric given by (38).

In the special case $e^2 = 2m^2$ and $V_0 = 0$, (35) allows an arbitrary value of $v_0$, and we have

$$B^2 = \frac{1}{2}, \quad k^2 = 0,$$

(45)

and the metric becomes the one found by Som and Raychaudhri [2] in the form

$$ds^2 = -(dt + \omega r^2 d\theta)^2 + dr^2 + r^2 d\theta^2 + d\chi^2,$$

(46)

where $\omega^2 = e^2v_0^2/2$ can take an arbitrary value.

Next, we consider the potential of the form $V(\Phi) = \lambda^2 (\Phi^*\Phi - \eta^2)^2 - e^2\eta^2$, where $\eta$ is a constant. Solving (35), we have two cases:

(i) $v_0^2 = \eta^2,$ \quad (ii) $v_0^2 = \frac{1}{5\lambda}(2e^2 - \lambda\eta^2).$

(47)

In the case (i), we obtain

$$B^2 = 0, \quad k^2 = e^2\eta^2,$$

(48)

and the metric of $\text{AdS}^3 \times \mathbb{R}^1$ given by (42).

In the case (ii), we have

$$B^2 = \frac{2}{5e^2}(e^2 - 3\lambda\eta^2), \quad k^2 = \frac{1}{25\lambda}(2e^4 + 23e^2\lambda\eta^2 - 12\lambda^2\eta^4).$$

(49)

If $e^2 = 3\lambda\eta^2$, the case reduces to the case (i), and if $e^2 > 3\lambda\eta^2$, the metric takes the form of (38).

In this example, we see that there are two non-trivial solutions in a theory with the potential $V$. One is a Gödel-type with a non-vanishing 1-form expectation value of the gauge field, and the other is $\text{AdS}^3 \times \mathbb{R}^1$ with the vanishing gauge field. The former has CTCs, while the latter does not.

6. Summary

We have shown that the one-parameter family of Gödel-type metrics are exact solutions to the coupled system that consist of a U(1)-gauge field, a real scalar field, a complex scalar field with a wide class of potentials, and Einstein gravity. The solutions that have an expectation value of the complex scalar field and a 1-form expectation value of the gauge field. We have presented a field theory that allowed two Gödel-type solutions: metric with CTCs and metric without CTC. It seems interesting to study the possibility of transition between two symmetry-breaking states described by these solutions.
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