GENERALIZED POINCARÉ SERIES FOR SU(2, 1)

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ABSTRACT. We define and study 'non-abelian' Poincaré series for the group $G = SU(2, 1)$, that is, Poincaré series attached to a Stone-Von Neumann representation of the unipotent subgroup $N$ of $G$. Such Poincaré series have in general exponential growth. In this study we use results on abelian and non-abelian Fourier term modules obtained in [2]. We compute the inner product of truncations of these series and of those associated to unitary characters of $N$, with square integrable automorphic forms in connection with their Fourier expansions. As a consequence, we obtain general completeness results that, in particular, generalize those valid for the classical holomorphic (and antiholomorphic) Poincaré series for $SL(2, \mathbb{R})$.

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In this paper we study Poincaré series on the Lie group SU(2, 1). Classically, in the study of automorphic forms one considers functions on a space $X$ that are invariant under the action of a group $\Gamma$ of transformations of $X$. The idea of a Poincaré series is to take as a germ a simple function $h$ on $X$ and to form the sum of transformed functions $x \mapsto \sum_{\gamma \in \Gamma} h(\gamma x)$. If this sum converges absolutely one has a $\Gamma$-invariant function on $X$. Poincaré [18] used this idea to construct what he called *fonctions thêtafuchsiennes*.

For the well-known cuspidal Poincaré series in the theory of holomorphic modular forms, the germ $h(z) = e^{2\pi inz}$, $n \in \mathbb{Z}_{\geq 1}$ on the upper half-plane is invariant under the transformation $z \mapsto z + 1$. With a suitable automorphy factor, it leads to series that converge absolutely, where the sum is over $\Gamma_1 \setminus \text{SL}_2(\mathbb{Z})$, and $\Gamma_1$ is generated by $\pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The resulting series may be identically zero. Hecke showed that the non-zero series of this type span the spaces of holomorphic cusp forms.
The same idea can be applied to the function
\[ h_s(x + iy) = e^{2\pi i n x} y^{1/2} I_{s-1/2}(2\pi|n|y) \]
on the upper half-plane, with \( I_{s-1/2} \) an exponentially increasing modified Bessel function. The function \( h_s \) is an eigenfunction of the hyperbolic Laplace operator. For \( \text{Re} \, s > 1 \), the sum \( \sum_{g \in \text{SL}_2(\mathbb{Z})} h_s(\gamma g) \) converges absolutely and defines a real-analytic Maass form with exponential growth. The resulting family of functions, called real-analytic Poincaré series, have a meromorphic continuation to the complex plane and the spaces of Maass cusp forms are spanned by residues of these families. (See Neunhöffer [15], Niebur [16].) This construction can be translated in a standard way yielding functions on the group \( G = \text{SL}(2, \mathbb{R}) \) that satisfy a transformation rule under the discrete subgroup \( \Gamma \).

Miatello and Wallach [14] extended the methods to automorphic forms on arbitrary Lie groups \( G \) of real rank one. For a discrete cofinite subgroup \( \Gamma \) of \( G \), they used a holomorphic family of germs \( h \) of Casimir eigenfunctions on the Lie group \( G \) such that \( h(ng) = \chi(n) h(g) \), where \( \chi \) is a character of a maximal unipotent subgroup \( N \) of \( G \) that is trivial on \( N \cap \Gamma \). Under suitable conditions on the parameters, this leads to Poincaré series on \( G \) with exponential growth. It is shown that there is a meromorphic continuation and a functional equation and the residues at singularities are square integrable automorphic forms. In many cases the corresponding spaces of cusp forms are spanned by residues of these Poincaré series (see [Theorem 3.2] and Corollary 3.4 in [14]). On the other hand, as it has been observed by several authors (for instance Gelbart-Piatetski-Shapiro ([6])) there exist many non-generic cusp forms, i.e. those having all their \( \chi \)-Fourier coefficients equal to zero, that cannot be detected by the series in [14]. These cusp forms correspond to representations of \( G \) that do not have Whittaker models (see [11],[20] for algebraic criteria satisfied by such representations).

The goal of the present paper is to refine the results in [14] in the case of the group \( \text{SU}(2, 1) \), by incorporating Poincaré series attached to irreducible, infinite dimensional representations of \( N \). For this purpose, we were led to study the so called Fourier term modules attached to the Stone-Von Neumann representation of \( N \), a task we carry out mainly in [2].

Here we define, initially for \( \text{Re} \, \nu > 2 \) in a way similar to [14], holomorphic families \( M_N(\xi, \nu) \) of exponentially increasing automorphic forms attached to an arbitrary irreducible, unitary representation of \( N \). We carry out its meromorphic continuation to \( \mathbb{C} \), proving that the poles are simple (except possibly at \( \nu = 0 \) and they occur at spectral parameters. We show that resolving these poles we obtain square integrable automorphic forms. In Section 6 we state our main results. We prove that what we call the main parts of these Poincaré series span the spaces of cusp forms (see Theorem 6.1). The main part is an automorphic form that in most cases is either a residue at a simple pole, or just a value at a point \( s \) of holomorphy of the series.

We consider the different cases of \( (\mathfrak{g}, K) \)-modules in Propositions 6.2 through 6.4. In particular we study the cases of holomorphic and antiholomorphic discrete series representations and the so called thin representations that come up in the
computation of the cohomology of the associated locally symmetric spaces. The resulting forms occur at integral parameters, mostly in the region of absolute convergence of the Poincaré series, and we show that their values or residues span the spaces of holomorphic cusp forms (see Proposition 6.4). A comparison with results in the classical case is made in § 6.4 and in Appendix D.

Some of our results are connected with results by Ishikawa [9], [10] (see [2, §16.2]). Similar methods as in this paper could be applied in the case of the more general group SU(n, 1), but we have preferred to stick to the case n = 2 to be able to give explicit results. We use computer aid to carry out some of calculations, specially in Proposition 7.1. We first work with discrete subgroups satisfying the conditions in [2], §5.1, and we further restrict the discussion to a class of discrete subgroups of SU(2, 1) having only one cusp to avoid notational complications that might obscure the methods. See §2.2.

2. Preliminaries. The group SU(2, 1)

We use the following realization of SU(2, 1)

\[ G := SU(2, 1) = \{ g \in SL_3(\mathbb{C}) : \; g^t I_{2,1} g = I_{2,1} \}, \]

with \( I_{2,1} = \text{diag}(1, 1, -1) \).

We fix subgroups \( K, A, \) and \( N \) in an Iwasawa decomposition \( G = NAK \). Here, the subgroup \( K \) consists of the matrices

\[ K = \left\{ \begin{pmatrix} u & 0 \\ 0 & \delta \end{pmatrix} : \; u \in U(2), \; \delta \det u = 1 \right\}, \]

and is a maximal compact subgroup of \( G \). There is an isomorphism \((U(1) \times SU(2))/[1, -1] \cong K \) via \((\eta, v) \mapsto \begin{pmatrix} \eta v & 0 \\ 0 & \eta^{-2} \end{pmatrix} \). The group

\[ A = \{ a(y) : \; y > 0 \}, \]

with \( a(y) = \begin{pmatrix} \frac{(y + y^{-1})/2}{2} & 0 & \frac{(y - y^{-1})/2}{2} \\ 0 & 1 & 0 \\ \frac{(y - y^{-1})/2}{2} & 0 & \frac{(y + y^{-1})/2}{2} \end{pmatrix} \)

is the identity component of a split torus.

The unipotent subgroup \( N = \{ n(b, r) : \; b \in \mathbb{C}, \; r \in \mathbb{R} \}, \) with

\[ n(b, r) = \begin{pmatrix} 1 + ir - |b|^2/2 & b & -ir + |b|^2/2 \\ -b & 1 & b \\ ir - |b|^2/2 & b & 1 - ir + |b|^2/2 \end{pmatrix}, \]

is isomorphic to the Heisenberg group on \( \mathbb{R}^2 \). The multiplication is given by

\[ n(b_1, r_1)n(b_2, r_2) = n(b_1 + b_2, r_1 + r_2 + \text{Im}(\overline{b_1}b_2)), \]

and its center is \( Z(N) = \{ n(0, r) : \; r \in \mathbb{R} \}. \)

The group \( K \) contains the subgroup \( M = \{ m(\eta) : \; |\eta| = 1 \}, \) with

\[ m(\eta) = \text{diag}(\eta, \eta^{-2}, \eta). \]

The group \( M \) commutes with \( A \), and \( AM \) normalizes \( N \). Namely,

\[ a(y)n(b, r)a(y)^{-1} = n(yb, y^2r), \quad m(\eta)n(b, r)m(\eta)^{-1} = n(\eta^3b, r). \]
Furthermore, $G$ has a Bruhat decomposition
\begin{equation}
G = NAM \sqcup NwAMN,
\end{equation}
with $w = \text{diag}(-1,-1,1)$ and the set $C = PwP = NwAMN$ is called the big cell.

In Appendix A.1 we discuss the Lie algebras of $G, N, A$ and $K$.

### 2.1. Upper half-plane model of the symmetric space.

The space
\begin{equation}
X = \{(z, u) \in \mathbb{C}^2 : \text{Im} z > |u|^2\}
\end{equation}
is a model of the symmetric space $G/K$. As a variety, $X$ is diffeomorphic to $NA$ with a left action
\begin{equation}
n(b, r) a(t) \cdot (z, u) = (t^2 z + 2tu + 2r + i|b|^2, tu + ib).
\end{equation}

We do not need to describe the action of $K$ explicitly. The point $(i, 0)$ is fixed by $K$.

The space $X$ inherits the complex structure of $\mathbb{C}^2$, which is preserved by the action of $G$. The boundary of $X$ consists of the points $(z, u)$ with $\text{Im} z = |u|^2$, plus a point at $\infty$ which is fixed by the subgroup $NAM$ of $G$. Thus, we have
\begin{equation}
\partial X = N \cdot (0, 0) \cup \{\infty\}.
\end{equation}

### 2.2. Discrete subgroups.

We will work with discrete subgroups $\Gamma$ of $G$ of finite covolume with one orbit of cusps, imposing one additional condition that makes the formulation of our results as simple as possible.

We can conjugate $\Gamma$ by an element $g \in G$ such that the unipotent subgroup $N$ intersects $\Gamma$ in a lattice $\Gamma_N$. We impose the additional condition that $g$ can be chosen in such a way that $\Gamma_N$ is a standard lattice $\Lambda_{σ}$, generated by $n(1, 0)$, $n(i, 0)$ and $n(0, 2/σ)$, with $σ \in \mathbb{Z}_{≥1}$. In this way we can apply results from [2].

In Appendix C in [2] we discuss two discrete subgroups satisfying these conditions. The first one is the group $Γ_0 = G \cap \text{SL}_2(\mathbb{Z}[i])$, which satisfies $Γ_0 \cap N = g_{∞} A_{4} g_{∞}$ with $g_{∞} = a(\sqrt{2}) m(e^{πi/4})$. So $g_{∞}^{-1} Γ_0 g_{∞}$ is a discrete subgroup satisfying all our conditions.

Francis and Lax [4] study a discrete subgroup in another realization of $SU(2, 1)$ of the form $G_{\text{FL}} = U_{\text{FL}}^{-1} GU_{\text{FL}}$ with
\begin{equation}
U_{\text{FL}} = \begin{pmatrix}
-\frac{i}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
0 & 1 & 0 \\
-\frac{i}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}
\end{pmatrix}.
\end{equation}

They give an explicit fundamental domain in $G/K$ for $G_{\text{FL}} \cap \text{SL}_3(\mathbb{Z}[i])$, from which it is clear that all cusps for this group are equivalent. Conjugating $G_{\text{FL}}$ to our realization $G$, we get the discrete subgroup $Γ_{\text{FL}} = G \cap U_{\text{FL}} \text{SL}_3(\mathbb{Z}[i]) U_{\text{FL}}^{-1}$. For this group one has $Γ_{\text{FL}} \cap N = m(e^{πi/4}) A_{4} m(e^{-πi/4})$. So $m(e^{-πi/4}) Γ_{\text{FL}} m(e^{πi/4})$ satisfies the conditions.

The requirements for $Γ$ simplify the book-keeping of Poincaré series and Fourier expansions. These choices are not essential for our results, but will simplify several expressions.
2.3. Irreducible representations of $K$. The irreducible representations of $K$ are obtained from pairs of irreducible representations of $U(1)$ and SU(2) having the same parity. See [21, (1), (2) on p. 185]. The irreducible representations $\tau_p$ of SU(2) are classified by their dimension $p + 1$ with $p \in \mathbb{Z}_{\geq 0}$. In this way we arrive at the representations $\tau^h_p$ of $K$, with $h \equiv p \mod 2$, $p \geq 0$ given by

$$\tau^h_p\left(\begin{array}{cc} \zeta^{1/2} & u \\ 0 & \zeta^{-1} \end{array}\right) = (\zeta^{1/2})^{-h} \tau_p(u).$$

The representation $\tau^h_p$ has $p + 1$ orthogonal realizations in $L^2(K)$, each with a basis of polynomial functions $^{h\Phi_{c,q}}$ on $K$. See the description in Appendix A.2.

2.4. Decomposition of $L^2(\Lambda_{c\sigma}\backslash N)$. We consider the quotient $\Lambda_{c\sigma}\backslash N$ for an arbitrary $\sigma \in \mathbb{Z}_{\geq 1}$.

We use $dn = dx dy dr$ for $n = n(x + iy, r)$ as the Haar measure on $N$, so $\Lambda_{c\sigma}\backslash N$ has volume $\frac{2}{\sigma}$.

2.4.1. Irreducible representations. The center of $N$ acts by a central character for any irreducible representation of $N$. If this central character is trivial, the representation corresponds to an irreducible representation of $N/Z(N) \cong \mathbb{R}^2$, hence to a character of $N$. Characters of $N$ occurring in $L^2(\Lambda_{c\sigma}\backslash N)$ are trivial on $\Lambda_{c\sigma}$, hence they are of the form

$$\chi_{\beta}: n(b, r) \mapsto e^{2\pi i \text{Im}(\beta b)},$$

with $\beta \in \mathbb{Z}[i]$. We note that characters of $N$ are trivial on the center $[n(0, r) : r \in \mathbb{R}]$.

If the center acts by $n(0, r) \mapsto e^{ir\lambda}$, then for each fixed $\lambda \in \mathbb{R}_{\neq 0}$, there is only one isomorphism class of irreducible representations of $N$ with this central character. It is unitary, and is called the Stone-von Neumann representation.

It can be realized on functions in $L^2(\mathbb{R})$ with a central character of the form $n(0, r) \mapsto e^{2\pi i \ell r}$, with $\ell \in \mathbb{R}_{\neq 0}$. The action of $n(x + iy, r) \in N$ is given by

$$\pi_{2\pi i\ell}(n(x + iy, r))\varphi(x) = e^{2\pi i\ell(r - 2\delta x - y)}\varphi(x + y).$$

For $\ell \in \mathbb{R}_{\neq 0}$, this defines a unitary representation on $L^2(\mathbb{R})$ for the standard inner product that leaves the Schwartz space invariant. It is called the Schrödinger representation.

2.4.2. Theta-functions. The Stone-Von Neumann representation can also be realized in $L^2(\Lambda_{c\sigma}\backslash N)$ by means of theta functions.

The central character equals $e^{2\pi i \ell r}$ on $n(0, r)$, and is trivial on $\Lambda_{c\sigma} \cap Z(N)$, hence $\ell \in \mathbb{Q}\mathbb{Z}$, with $\sigma \in \mathbb{Z}_{\geq 1}$ as usual.

Thus, for each $c \in \mathbb{Z}$ mod $2|\ell|$, we get a realization given by intertwining operators $\varphi \mapsto \Theta_{c,\ell}(\varphi)$ from the Schrödinger representation of $N$ in the space $\mathcal{S}(\mathbb{R})$ of Schwartz functions on $\mathbb{R}$, into the module $C^\infty(\Lambda_{c\sigma}\backslash N)$ with the action of $N$ by right translation. The theta functions are given by

$$\Theta_{c,\ell}(\varphi)(n(x + iy, r)) = \sum_{k \in \mathbb{Z}} \varphi\left(\frac{c}{2\ell} + k + y\right) e^{2\pi i\ell r} e^{-2\pi i\ell x(c/\ell + 2k + y)}.$$
In §A.3 we indicate how one can arrive at this intertwining operator, and how it leads to $2|\ell|$ orthogonal realizations of the Stone-von Neumann representation in $L^2(\Lambda_\sigma \backslash N)$.

We shall use a convenient orthogonal basis of $L^2(\mathbb{R})$. For $k \in \mathbb{Z}_{\geq 0}$ the Hermite polynomial $H_k$ is determined by the identity

$$e^{-\xi^2} H_k(\xi) = (-1)^k \partial^k_\xi e^{-\xi^2}.$$  

(2.16)

The normalized Hermite functions are given by

$$h_{\ell,k}(\xi) = 4|\ell|^{1/2} 2^{-k/2} (k!)^{-1/2} H_k(\sqrt{4\pi|\ell|}\xi)e^{-2|\ell|\xi^2}.$$  

(2.17)

In this way, for any $\ell \in \mathbb{R}$, $\ell \neq 0$, we obtain an orthogonal basis \{h_{\ell,k} : k \in \mathbb{Z}_{\geq 0}\} of $L^2(\mathbb{R})$.

Going over to theta-functions, for each $\ell \in \frac{\sigma}{2} \mathbb{Z}_{\neq 0}$, we get an orthogonal system

$$\big\{ \sqrt{\sigma/2} \Theta_{\ell,c}(h_{\ell,k}) : 0 \leq c \leq 2|\ell| - 1, \ k \geq 0 \big\}$$

of the subspace $L^2(\Lambda_\sigma \backslash N)_{\ell}$ determined by the central character $n(0,r) \mapsto e^{2\pi i r}$. On the other hand, an orthogonal basis of $L^2(\Lambda_\sigma \backslash N)_{0}$ is given by

$$\big\{ \sqrt{\sigma/2} \chi_\beta : \chi_\beta \in Z[i] \big\}.$$  

Thus

$$\mathcal{B} = \big\{ \sqrt{\sigma/2} \chi_\beta, \beta \in Z[i] \big\}$$

(2.19)

is an orthogonal basis of $L^2(\Lambda_\sigma \backslash N)$.

2.5. **Haar measures.** A Haar measure of $dg$ on $G$ can be chosen of the form $dg = a^{-2\rho} \ dn \ da \ dk$ with respect to the Iwasawa decomposition $G = NAK$, where $2\rho$ is the sum of the restricted roots of $\mathfrak{n}$, the Lie algebra of $N$. Thus $2\rho = 4\lambda$, where $\lambda$ is the simple restricted root. We take the coordinate $t \leftrightarrow a(t)$ on $A$, and the measure $da = \frac{dt}{t}$. The root $\lambda$ satisfies $a^4 = t$, thus $dg = r^{-5} \ dn \ dt \ dk$.

Furthermore, we normalize the Haar measure on $K$ so that $\int_K dk = 1$. In §2.4 we have already fixed the Haar measure $dn$ on $N$.

3. **Fourier expansions**

In this section we give an overview of the Fourier expansion of $\Lambda_\sigma$-invariant $K$-finite functions on $G = SU(2,1)$.

3.1. **Fourier term operators.** The basis $\mathcal{B}$ in (2.19) can be used to give a generalized Fourier expansion of elements $f \in C^0(\Lambda_\sigma \backslash G)_K$. We formulate it for a general $\sigma \in \mathbb{Z}_{\geq 1}$, though later we will use it only for $\sigma = 4$.

$$f(n g) = \sum_{b \in B} b(n) \int_{\Lambda_\sigma \backslash N} \overline{b(n_1)} f(n_1 g) \ dn_1$$

(3.1)

with $g \in G$ and $n \in N$. This Fourier expansion converges in the $L^2$-sense on $\Lambda_\sigma \backslash N$ for $f$ and for all its derivatives $uf$ with $u \in U(g)$.
For the terms with \( b = \sqrt{\sigma/2} \chi_\beta, \beta \in \mathbb{Z}[i], \) we define the Fourier term operator
\[
F_\beta f(g) = \frac{\sigma}{2} \int_{\mathbb{H}_r \backslash N} \chi_\beta(n_1)^{-1} f(n_1 g) \, dn_1. 
\]

The functions \( F_\beta f \) are \( \chi_\beta \)-equivariant under left translations by elements of \( N \). The abelian part of the Fourier expansion of \( f(g) \) is the sum
\[
\sum_{\beta \in \mathbb{Z}[i]} F_\beta f(g).
\]
The operator \( F_\beta \) is an intertwining operator for the action of \( g \) by right differentiation: \( F_\beta(X f) = X(F_\beta f) \) for \( X \in g \).

The action of \( g \) does not preserve the individual terms corresponding to the basis elements \( \sqrt{\sigma/2} \Theta_{\ell,c}(h_{\ell,m}). \) We group these basis elements together so that the action of \( g \) preserves the space of linear combinations of terms in each group. This leads to operators \( F_{\ell,c,d} \) with \( \ell \in \frac{\pi}{2} \mathbb{Z}_{\neq 0}, c \in \mathbb{Z} \mod 2\ell\mathbb{Z}, \) and \( d \equiv 1 + 2\mathbb{Z}. \) For \( n \in N, a \in A, k \in K: \)
\[
F_{\ell,c,d} f(nak) = \sum_{m,h,p,r,q} \Theta_{\ell,c}(h_{\ell,m}) \frac{h_{\Phi^{f}_{r,q}(k)}}{\|h_{\Phi^{f}_{r,q}}\|_K} \cdot \frac{\sigma}{2} \int_{n_1 \in \mathbb{H}_r \backslash N} \int_{k_1 \in K} \Theta_{\ell,c}(h_{\ell,m}(n') f(n_1 ak_1) \frac{h_{\Phi^{f}_{r,q}(k_1)}}{\|h_{\Phi^{f}_{r,q}}\|_K} \, dk_1 \, dn_1,
\]
where the sum runs over integers satisfying \( m \geq 0, h \equiv p \equiv r \equiv q \mod 2, |r| \leq p, |q| \leq p, \) and
\[
(6m + 3) \text{Sign}(\ell) + h - 3r = d.
\]
When dealing with these Fourier terms we often abbreviate \((\ell, c, d) = n \). We call the sum \( \sum_n F_{n} f \) the non-abelian part of the Fourier expansion.

See §B.1 for a further discussion.

3.2. Fourier term modules. As before, we define a Fourier term order \( N \) as a realization of an irreducible representation of \( N \) in \( C^{\infty}(\Lambda_r \backslash N) \); \( N \) is the index in the Fourier expansion of elements of \( C^{\infty}(\Lambda_r \backslash G)_K \); so \( F_N = F_\beta \) for \( N = N_\beta \) and \( F_N = F_{\ell,c,d} \) for \( N = N_{\ell,c,d} = N_n \). Since the Fourier term operators are intertwining operators of \((g, K)\)-modules, the spaces \( \mathcal{F}_N = F_N C^{\infty}(\Lambda_r \backslash G)_K \) are \((g, K)\)-modules. We call them Fourier term modules.

In the modules \( \mathcal{F}_N \) we have the submodules \( \mathcal{F}_N^\psi \) in which the center \( ZU(g) \) of the universal enveloping algebra of \( g \) acts according to the character \( \psi \) of \( ZU(g) \). These characters \( \psi \) are determined by their values on the generators \( C \) and \( \Delta_3 \). So \( f \in \mathcal{F}_N \) is in \( \mathcal{F}_N^\psi \) if \( C f = \lambda_2 f \) and \( \Delta_3 f = \lambda_3 f \) for fixed \( \lambda_2, \lambda_3 \in \mathbb{C} \). Later on we deal mainly with the Casimir element \( C \), and abbreviate \( \lambda_2 \) as \( \lambda \).

The characters of \( ZU(g) \) are parametrized by the spectral parameters \((\xi, \nu)\) where \( \xi \) is a character of \( M \) and \( \nu \in \mathbb{C} \) corresponds to the character \( a(t) \mapsto t^\nu \) of \( A \). A character \( \xi \) of \( M \) has the form \( m(\zeta) \mapsto \zeta^j \) with \( j \xi \in \mathbb{Z} \). Furthermore, if \((\nu_1, \xi)\) and \((\nu_2, \xi)\) in \( \mathbb{Z} \times \mathbb{C} \) are in the same orbit of the Weyl group, then they determine the same character of \( ZU(g) \).
The Weyl group of SU(2, 1) is isomorphic to the symmetric group on three elements. Its action on \( \mathbb{C} \times \mathbb{C} \) is generated by the transformation of \( \mathbb{C}^2 \) with matrices

\[
S_1 = \begin{pmatrix} -1/2 & 3/2 \\ 1/2 & -3/2 \end{pmatrix} \quad \text{and} \quad S_2 = \begin{pmatrix} -1/2 & -3/2 \\ 1/2 & 1/2 \end{pmatrix}.
\]

The Weyl group does not preserve \( \mathbb{Z} \times \mathbb{C} \subset \mathbb{C}^2 \). By \( O_W(\psi) \) we indicate the intersection of the Weyl group orbit corresponding to \( \psi \) with \( \mathbb{Z} \times \mathbb{C} \), and by \( O_W(\psi)^+ \) the subset \( \{ (j, \nu) \in O_W(\psi) : \text{Re} \, \nu \geq 0 \} \). The set of characters \( \psi \) of \( ZU(\mathfrak{g}) \) can be divided in the following way.

- **Generic parametrization:** All \( (j, \nu) \in O_W(\psi) \) satisfy \( \nu \notin j + 2\mathbb{Z} \), or \( O_W(\psi) = \{(0, 0)\} \).
- **Integral parametrization:** All \( (j, \nu) \in O_W(\psi) \) satisfy \( \nu \in j + 2\mathbb{Z} \) and \( (j, \nu) \neq (0, 0) \).

The study of the structure of the modules \( \mathcal{F}_N^\psi \) is the main object of [2]. Here we summarize the main results, with more details in the appendix. The main difference in the structure is between the case when \( N = N_0 \) and the other Fourier term orders.

Under generic parametrization the modules \( \mathcal{F}_N^\psi \) for are isomorphic for all choices of \( N \). The main difference in the structure occurs under integral parametrization, between \( N = N_0 \) and the other \( N \). The module \( \mathcal{F}_{N_0}^\psi \) still is the sum of a number of principal series modules, whereas for \( N \neq N_0 \) the structure is more complicated.

### 3.2.1. \( N \)-trivial Fourier term modules

The \( N \)-trivial Fourier term module \( \mathcal{F}_N^\psi \) contain the principal series modules \( H_{K}^{\xi, \nu} \) for which \( (j_\xi, \nu) \in O_W(\psi) \). They consist of the elements of \( C^\infty(N\backslash G)_K \) that transform on the left according to the character of \( AM, a(t)m(\xi) \mapsto \zeta^{j_\xi}t^{2\nu} \). For different spectral parameters \( (\xi, \nu) \) the corresponding principal series are disjoint. If \( O_W(\psi) \) does not contain elements of the form \( (\xi, 0) \), then

\[
\mathcal{F}_0^\psi = \bigoplus_{(j_\xi, \nu) \in O_W(\psi)} H_{K}^{\xi, \nu}.
\]

### 3.2.2. Submodules determined by boundary behavior

In the modules \( \mathcal{F}_N^\psi \) with \( N \neq N_0 \) there are submodules having the following characterization:

1. \( \mathcal{W}_N^\psi \) consists of the elements \( f \in \mathcal{F}_N^\psi \) with exponential decay

\[
f(na(t)k) = O(e^{-\varepsilon t}) \quad \text{as } t \uparrow \infty,
\]
uniform in \( n \) and \( k \) for some \( \varepsilon > 0 \). (In the non-abelian case \( N = N_{\ell, c, d} \) we can replace \( e^{-\varepsilon t} \) by \( e^{-\varepsilon t^2} \).)

2. \( \mathcal{M}_N^\psi \) consists of the linear combinations of elements \( f \in \mathcal{F}_N^\psi \) for which, for some \( (j_\xi, \nu) \in O_W(\psi)^+ \), the function

\[
t \mapsto f(na(t)k)t^{-2\nu}
\]
extends for each \( n \) and \( k \) to a holomorphic function of \( t \in \mathbb{C} \).
These spaces are \((g, K)\)-submodules by \([2, \text{Proposition 10.6}]\). In the particular case of \(\mathcal{N} = \mathcal{N}_0\) we obtain that \(W_0^\phi = \{0\}\), and that \(M_0^\phi\) contains all \(H_k^{\xi, \nu}\) with \((j_\xi, \nu) \in O_W(\psi)\) and \(\text{Re } \nu \geq 0\).

The main theorems in \([2, \S2]\) give a further description of the modules \(\mathcal{F}_N^\phi\), with \(\mathcal{N} \neq \mathcal{N}_0\). We summarize the results.

### 3.2.3. Abelian Fourier term modules

Let \(\beta \in \mathbb{Z}[i], \beta \neq 0\). Then there is a direct sum decomposition

\[
\mathcal{F}_\beta^\phi = \mathcal{W}_\beta^\phi \oplus M_\beta^\phi.
\]

For each \((j_\xi, \nu) \in O_W(\psi)^+\) there are submodules \(\mathcal{W}_\beta^{\xi, \nu} \subset \mathcal{W}_\beta^\phi\) and \(M_\beta^{\xi, \nu} \subset M_\beta^\phi\) of the form

\[
\mathcal{W}_\beta^{\xi, \nu} = U(\beta) \omega_{0,0}(j_\xi, \nu), \quad M_\beta^{\xi, \nu} = U(\beta) \mu_{0,0}(j_\xi, \nu),
\]

where

\[
\omega_{\beta}^{0,0}(j_\xi, \nu)(na(t)k) = \chi_\beta(n) i^2 K_\nu(2\pi |\beta| t) 2^j \Phi_{0,0}^\beta(k)
\]

with the modified Bessel functions \(I_\nu\) (exponentially increasing) and \(K_\nu\) (exponentially decreasing).

Under generic parametrization the modules \(\mathcal{W}_\beta^{\xi, \nu}\) and \(M_\beta^{\xi, \nu}\) are isomorphic to the principal series module \(H_k^{\xi, \nu}\), and

\[
\mathcal{W}_\beta^\phi = \bigoplus_{(j_\xi, \nu) \in O_W(\psi)^+} \mathcal{W}_\beta^{\xi, \nu}, \quad M_\beta^\phi = \bigoplus_{(j_\xi, \nu) \in O_W(\psi)^+} M_\beta^{\xi, \nu}.
\]

Under integral parametrization the modules \(\mathcal{W}_\beta^{\xi, \nu}\) and \(M_\beta^{\xi, \nu}\) are reducible, isomorphic to each other, but for most \((j_\xi, \nu)\) not isomorphic to \(H_k^{\xi, \nu}\). We now have

\[
\mathcal{W}_\beta^\phi = \sum_{(j_\xi, \nu) \in O_W(\psi)^+} \mathcal{W}_\beta^{\xi, \nu}, \quad M_\beta^\phi = \sum_{(j_\xi, \nu) \in O_W(\psi)^+} M_\beta^{\xi, \nu}.
\]

If a \(K\)-type \(\tau_\beta^{j, \nu}\) occurs in modules \(\mathcal{W}_\beta^{\xi, \nu}\) and \(\mathcal{W}_\beta^{\xi, \nu}'\), then the (one-dimensional) subspaces \(\mathcal{W}_\beta^{\xi, \nu}_{\beta, h, p}\) and \(\mathcal{W}_\beta^{\xi, \nu}_{\beta, h, p}'\) with this \(K\)-type coincide, and similarly for \(M_\beta^{\xi, \nu}\).

### 3.2.4. Non-abelian Fourier term modules

In the non-abelian case several complications occur. We consider \(n = (\ell, c, d)\) with \(\ell \in \mathbb{Z}_{\ell \equiv 0 \text{ mod } 2}\), and \(d \in 1 + 2\mathbb{Z}\).

We need a further restriction on the elements of Weyl orbits. Set

\[
O_W(\psi)^n = \{(j, \nu) \in O_W(\psi) : \text{Re } \nu \geq 0, \text{ Sign } (\ell)(2j - d) + 3 \leq 0\}.
\]

In case the set \(O_W(\psi)^n\) is empty, then \(\mathcal{F}_n^\phi = \{0\}\). For each \((j_\xi, \nu)^n\) there are submodules \(\mathcal{W}_n^{\xi, \nu} \subset \mathcal{W}_n^\phi\) and \(M_n^{\xi, \nu} \subset M_n^\phi\). These modules contain the elements

\[
\omega_n^{0,0}(j_\xi, \nu)(na(t)k) = \Theta_{\ell, c}(h_{\ell, m_0}(n)) tW_{\ell, \nu/2}(2\pi |\ell| t^2) 2^j \Phi_{0,0}^\beta(k) \quad \text{for } \mathcal{W}_n^{\xi, \nu},
\]

\[
\mu_n^{0,0}(j_\xi, \nu)(na(t)k) = \Theta_{\ell, c}(h_{\ell, m_0}(n)) tM_{\ell, \nu/2}(2\pi |\ell| t^2) 2^j \Phi_{0,0}^\beta(k) \quad \text{for } \mathcal{M}_n^{\xi, \nu},
\]
with
\begin{equation}
(3.16) \quad m_0 = \frac{\text{Sign}(\ell)}{6}(d - 2j_\xi) - \frac{1}{2}, \quad \kappa = -m_0 - \frac{1}{2}(j_\xi \text{Sign}(\ell) + 1).
\end{equation}

We have that \(m_0 \in \mathbb{Z}_{\geq 0}\), and \(\kappa \in \frac{1}{2} \mathbb{Z}\). Here we use the Whittaker functions \(W_{\kappa,s}\) (exponentially decreasing) and \(M_{\kappa,s}\) (exponentially increasing for most combinations of \(\kappa\) and \(s\)).

Under generic parametrization, the modules \(W_{n,\xi,\nu}^\xi,\psi\) and \(M_{n,\xi,\nu}^\xi,\psi\) are generated respectively by \(\omega_{n,0}^\xi(j_\xi, \nu)\) and \(\mu_{n,0}^\xi(j_\xi, \nu)\). Both modules are irreducible and isomorphic to the principal series module \(H_{K}^\xi,\psi\). Moreover,
\begin{equation}
(3.17) \quad W_{n}^{\psi} = \bigoplus_{(j_\xi, \nu) \in \mathcal{O}_{W}(\psi)^{\sharp}_{n}} W_{n,\xi,\nu}^\xi, \quad M_{n}^{\psi} = \bigoplus_{(j_\xi, \nu) \in \mathcal{O}_{W}(\psi)^{\sharp}_{n}} M_{n,\xi,\nu}^\xi, \quad \psi \in \mathfrak{g}^*.
\end{equation}

Under integral parametrization both (3.18) and (3.17) break down. For some characters \(\psi\) the modules \(W_{n}^{\psi}\) and \(M_{n}^{\psi}\) have a non-empty intersection and do not span \(F_{n}^{\psi}\). The modules \(W_{n,\xi,\nu}^\xi,\psi\) and \(M_{n,\xi,\nu}^\xi,\psi\) are reducible, and in most cases non-isomorphic. Also, the elements in (3.17) need not generate the modules.

There are several properties that stay true under integral parametrization. In (3.17) we have to replace \(\bigoplus\) by \(\Sigma\). The \(K\)-type \(\tau_{h}^{\psi}\) occurs in \(W_{n}^{\psi}\) and in \(M_{n}^{\psi}\) with multiplicity one for \(|h - 2j_\xi| \leq p\). If \(\tau_{h}^{\psi}\) satisfies this condition for \((j_\xi, \nu)\) and \((j_{\xi'}, \nu')\) in \(\mathcal{O}_{W}(\psi)^{\sharp}_{n}\), then \(W_{n,\xi,\nu}^{\psi} = W_{n,\xi',\nu'}^{\psi}\) and \(M_{n,\xi,\nu}^{\psi} = M_{n,\xi',\nu'}^{\psi}\).

In §B.3 in the Appendix we give a further discussion of elements of Fourier term modules. Proposition B.1 gives, for \(N \neq N_{0}\), families \(\nu \mapsto \Omega_{N,h,p,q}(\xi, \nu)\) and \(\nu \mapsto \Omega_{N,h,p,q}(\xi, \psi)\) that we will use in the description of Fourier terms of automorphic forms and in the construction of Poincaré series.

4. Automorphic Forms

In this section we give the definitions of several modules of automorphic forms, and discuss the corresponding Fourier expansions. After that we turn to Poincaré series, which provide us with an explicit construction.

4.1. General facts. In §2.2 we have introduced the class of discrete subgroups \(\Gamma\) that we will use in this paper.

Definition 4.1. An automorphic form for \(\Gamma\) is a smooth function \(f\) on \(G = \text{SU}(2, 1)\) with the following properties:

a) \(f\) is \(\Gamma\)-invariant on the left and \(K\)-finite on the right: \(f \in C^\infty(\Gamma \backslash G)_{K}\),

b) \(f\) is an eigenfunction of the center of the enveloping algebra: \(uf = \psi(u)\) for all \(u \in ZU(g)\), for some character \(\psi\) of \(ZU(g)\),

c) \(f\) has polynomial growth:
\begin{equation}
(4.1) \quad f(na(t)k) = O(t^a) \quad \text{as } t \uparrow \infty \text{ for some } a \in \mathbb{R},
\end{equation}
uniform in \(n \in N\) and \(k \in K\).
Definition 4.2. We denote by $A(\psi)$ the $(\mathfrak{g}, K)$-module of automorphic forms with character $\psi$ of $ZU(q)$. By $A(\psi)_{h,p}$ we denote the subspace of $A(\psi)$ of $K$-type $\tau^h_{p}$, and by $A(\psi)_{h,p,q}$ the subspace of weight $q$ in that $K$-type.

The spaces $A(\psi)_{h,p}$ have finite dimension (see Theorem 1 of [8, p 8], for instance).

As in 2.5, we use the Haar measures $dn = dx dy dr$ on $N$, for $n = n(x + iy, r)$, $da(t) = t^{-1} dt$ on $A$, for $a = a(t)$, and the Haar measure $dk$ on $K$ that gives to $K$ volume 1. Then $dg = t^{-5} dn dt dk$ for $g = na(tk)$ is a Haar measure on $G$.

In $L^2(\Gamma \backslash G)$ we use the standard scalar product

$$ (f_1, f_2) = \int_{\Gamma \backslash G} f_1(g) \overline{f_2(g)} \, dg $$

and right translation gives a unitary representation of $G$ on $L^2(\Gamma \backslash G)$. We will be concerned mainly with the subspace $L^2(\Gamma \backslash G)_K$ of $K$-finite vectors. The integral in (4.2) can be computed by letting $g = nak$ run over $K$ and $na \in NA$ be such that $n = (i, 0)$ runs over a fundamental domain of $\Gamma \backslash X$. Then the norm in $L^2(\Gamma \backslash G)_K$ can be bounded by an integral over a Siegel set.

Definition 4.3. The submodule $A^{(2)}(\psi)$ of square integrable automorphic forms consists of the functions $f \in A(\psi)$ that represent an element of $L^2(\Gamma \backslash G)_K$.

The space $A^{(2)}(\psi) \subset L^2(\Gamma \backslash G)$ decomposes as the orthogonal direct sum of finitely many irreducible $(\mathfrak{g}, K)$-modules $V_i$ with a unitary structure, in which $ZU(q)$ acts via the character $\psi$. So there are finitely many isomorphism classes to which the $V_i$ belong. Furthermore, the sets of $K$-types occurring in different isomorphism classes are disjoint (see for instance [2, Proposition 12.2]). That means that each space $A^{(2)}(\psi)_{h,p}$ generates a $(\mathfrak{g}, K)$-module that is a finite sum of copies of one irreducible $(\mathfrak{g}, K)$-module with a unitary structure.

Fourier terms. There is an absolutely convergent Fourier expansion

$$ f(g) = \sum_{\mathcal{N}} F_{\mathcal{N}} f(g), $$

where the Fourier term order $\mathcal{N}$ runs over a system of realizations of irreducible representations of $N$ in $C^\infty(\Lambda_\mathcal{N} \backslash N)$. So $\mathcal{N}$ runs over the $\mathcal{N}_\beta$ with $\beta \in \mathbb{Z}[i]$ and the $\mathcal{N}_n$ with $n = (\ell, c, d)$, $\ell \in \frac{1}{2} \mathbb{Z}_{\geq 0}$, $c \mod 2 \ell$, and $d \in 1 + 2 \mathbb{Z}$.

In the case of automorphic forms, the Fourier term operators are intertwining operators of $(\mathfrak{g}, K)$-modules $F_{\mathcal{N}} : A(\psi) \to F^\psi_{\mathcal{N}}$. The property of polynomial growth is preserved by the Fourier term operators, and hence imposes restrictions on the Fourier terms. All elements in $F^\psi_{0}$ have polynomial growth. For $f \in A(\psi)$ the $\mathcal{N}$-trivial Fourier term $F_{0} f$ is, in general, a sum of contributions in the principal series modules $H^\psi_{\mathcal{N}}$ with $(\xi, \nu) \in \mathcal{O}_N(\psi)$, a set with at most six elements. However, on each space $A(\psi)_{h,p}$ for a fixed $K$-type, only two of these contributions can be non-zero. In case $\mathcal{O}_N(\psi)$ contains elements of the form $(j_\xi, 0)$ we may also have logarithmic contributions.
For all \( \mathcal{N} \neq \mathcal{N}_0 \) the property of polynomial growth forces that \( F_N f \in \mathcal{W}_N^\psi \). The multiplicity of each \( K \)-type in \( \mathcal{W}_N^\psi \) is at most 1. The operator \( F_N : A(\psi)_{h,p,q} \to \mathcal{W}_{N,h,p,q}^\psi \) lies in a one-dimensional space for each \( K \)-type and weight. (We use the notation given in Definition 4.2.) For each \( f \in A(\psi)_{h,p,q} \) there is a Fourier coefficient \( C_N(f) \) such that

\[
F_N f = c_N(f) \Omega_{N,h,p,q}(\xi, \nu),
\]

with \( \psi \) determined by \((\xi, \nu)\). See Proposition (B.1) for the family \( \Omega_{N,h,p,q} \).

The integral in (4.2) can be restricted to a fundamental domain contained in a Siegel set of the form

\[
\Xi(t_0) = \{ n(z, r) a(t) k : |\text{Re} z|, |\text{Im} z| \leq \frac{1}{2}, |r| \leq \frac{1}{2}, t \geq t_0, k \in K \}
\]

with \( t_0 > 0 \). For \( f \in A^2(\psi) \) the integral of \( |f|^2 \) over this Siegel set is finite, a property inherited by the Fourier terms. This implies that \( F_0 f \) can have only components in principal series modules \( H_K^{\psi,\nu} \) with \( \text{Re} \nu < 0 \). The square integrability imposes no further restriction on the other Fourier terms.

The form of the Fourier expansion can also be used to define other modules of automorphic forms.

**Definition 4.4.** The submodule \( A^0(\psi) \) of cusp forms consists of the functions \( f \in A(\psi) \) for which \( F_0 f = 0 \).

An automorphic form is rapidly decreasing if for all \( a \in \mathbb{R} \)

\[
f(n a(t) k) = O(t^{-a}) \quad \text{as } t \uparrow \infty,
\]

uniformly in \( n \) and \( k \).

Cusp forms are rapidly decreasing (see for instance [8, §4, Chap I], or Corollary to Lemma 3.4 on [13, p 46]).

All Poincaré series in this paper have exponential growth concentrated in a few Fourier terms. We define

**Definition 4.5.** Let \( \psi \) be a character of \( \text{ZU}(g) \). We say that a function \( f \in C^\infty(\Gamma \backslash G)_K \) that satisfies \( u f = \psi(u) f \) for all \( u \in \text{ZU}(g) \), has moderate exponential growth if there exists a finite set \( E \) of Fourier term orders containing \( \mathcal{N}_0 \) such that

\[
f - \sum_{N \in E} F_N f
\]

is in \( L^{2,\alpha}(\Lambda_r \backslash N A_T K)_K \) for some \( \alpha > 0, T > 0 \), where \( A_T = \{ a(t) : t \geq T \} \). We denote by \( A^1(\psi) \supset A(\psi) \) the space of automorphic forms of moderate exponential growth.

The dimensions of the spaces \( A^0(\psi)_{h,p} \) are finite, by the inclusion \( A^0(\psi) \subset A(\psi) \). The choice of the set \( E \) in Definition 4.5 depends on \( f \). The spaces \( A^1(\psi)_{h,p} \) can have infinite dimensions. For instance for \( \psi = \psi[\xi, \nu] \) described by generic parametrization and for one-dimensional \( K \)-types \( \tau^h \in \tau^0_0 \), the construction of Poincaré series will provide us with elements \( f = M_{\beta}(\psi, \nu) \in A^1(\psi)_{2,\beta,0} \) for which
\(F_{\beta}f\) has genuine exponential growth if \(\beta\) lies in one \(\mathbb{Z}[i]^*\)-orbit \(\{\beta_0, i\beta_0, -\beta_0, -i\beta_0\}\) in \(\mathbb{Z}[i] \setminus \{0\}\), and has exponential decay for all other \(\beta \in \mathbb{Z}[i] \setminus \{0\}\). See Proposition 4.8.

There is also the much larger module of functions \(\Lambda^u(\psi)\), satisfying only conditions a) and b) in Definition 4.4, the space of automorphic forms with unrestricted growth. However, for the purposes of this paper, the module \(\Lambda(\psi)\) is all we need.

4.2. Poincaré series. Given a \(K\)-finite function \(f\) on \(\Gamma_N \setminus G\), under some standard growth conditions one can form the Poincaré series

\[
P_f(g) = \sum_{\gamma \in \Gamma_N \setminus \Gamma} f(\gamma g).
\]

For instance, if we take \(f(0a(t)k) = t^{2+\nu} h^{\Phi_{r,q}}(k)\) with \(\Re \nu > 2\) and \(h - 3r = 2j_\xi\), then we have the Eisenstein series

\[
E(\xi, \nu)_{h,p,q}(g) = P_f(g).
\]

The convergence is absolute and uniform on compact subsets for \(\Re \nu > 2\), and \(E(\xi, \nu)_{h,p,q}\) depends holomorphically on \(\nu\) in this region. It is non-zero only if \(\xi(m(i)) = 1\) (see (2.5)), which is equivalent to \(j_\xi \in 4\mathbb{Z}\). The Eisenstein series \(E(\xi, \nu)_{h,p,q}\) are elements of \(A_{h,p,q}(\psi)\) for \(\Re \nu > 2\), and have a meromorphic continuation to \(\mathbb{C}\). We will apply the same approach to elements of \(M_{\xi,\nu}^E\) for \(N \neq N_0\).

Lemma 2.1 in [14] applied to the present situation (with \(\rho\) corresponding to \(\nu = 2\)) implies that, if \(f \in C(\Gamma_N \setminus G)_K\) satisfies

\[
f(0a(t)k) = O(a(t)^{2\rho+\varepsilon}) = O(t^{4+\varepsilon}) \quad \text{as } t \downarrow 0,
\]

for some \(\varepsilon > 0\), \(Pf\) converges absolutely and uniformly for \(n \in N, k \in K\). Moreover, this convergence is uniform when \(q\) and \(\nu\) vary over compact sets. This implies that if \(X \in g,\) and both \(f\) and \(Xf\) satisfy (4.10), then \(XPf\) is well-defined and equal to \(Pf\).

For \(f \in C^\infty(\Gamma_N \setminus G)_K\) satisfying condition (4.10) we have the decomposition

\[
Pf = P^\infty f + Pf',
\]

where \(P^\infty f\) is given by the finite sum over \(\Gamma_N \setminus \Gamma_P\), and \(Pf'\) is given by the remaining sum over \(\Gamma_N \setminus (\Gamma \cap NwAMN)\), with \(NwAMN\) the big cell in the Bruhat decomposition.

We will make use of the following lemma. We give a proof for completeness.

**Lemma 4.6.** For \(f \in C^\infty(\Gamma_N \setminus G)_K\) satisfying (4.10) one has

\[
Pf'(0a(t)k) = O(1) \quad \text{as } t \uparrow \infty
\]

uniformly in \(n \in N, k \in K\).

**Proof.** Let

\[
A_T = \{a(t) : t \geq T\}, \quad A_{<T} = A \setminus A_T.
\]
We have to prove that $P'f$ is bounded on a region $NA_T K$ for some sufficiently large $T > 1$. Below we will argue that if $T$ is sufficiently large, then

\begin{equation}
\gamma NA_T K \subset NA_{<1} K \quad \text{for all } \gamma \in \Gamma \cap NWAMN.
\end{equation}

So we need the bound in (4.10) only for $t < 1$, and can assume that $f$ is the positive function $f(na(t)k) = t^{2+\sigma}$ with $\sigma > 2$. So we consider a part of the Eisenstein series $E(1, \sigma)_{0,0,0}$.

We will also show below that there are a relatively compact neighborhood $\Omega$ of $1 \in G$ and a positive number $T_1$ such that for all $g \in NA_{<T_1}$

\begin{equation}
f(g) < 2f(g_1) \quad \text{for all } g_1 \in g\Omega.
\end{equation}

Now we apply the well-known approach of estimating a sum by an integral. For $p \in NA_T$

\[
\sum_{\gamma \in \Gamma_N \backslash (\Gamma \cap NWAMN)} f(p \gamma p) < \sum_{\gamma \in \Gamma_N \backslash (\Gamma \cap NWAMN)} \frac{1}{\text{vol} \Omega} \int_{g_1 \in p \Omega} 2f(\gamma g_1) \, dg_1
\]

\[
\leq \frac{2n}{\text{vol} \Omega} \int_{\Gamma_N \backslash NA_{<T_1} K} f(g_1) \, dg_1 \leq C_1 \int_{t=0}^{T_1} t^{2+\sigma} \frac{dt^5}{t} < \infty,
\]

uniform in $g$. The constant $C_1$ depends on the maximal number of fundamental domains for $\Gamma \backslash G$ intersecting a set $g\Omega$, on the volume of $\Omega$, on the volumes of $K$ and $\Gamma \backslash N$, and on $\sigma$. Since $f$ is left-invariant under $N$ we can replace the integral over $\gamma p\Omega$ by an integral over a $\Gamma_N$-equivalent set inside a fundamental domain for $\Gamma_N \backslash G$.

The bound is uniform for $g \in NA_T$. Since $f$ is right invariant under $K$ this gives the lemma.

We still have to establish (4.14) and (4.15). We use Lemma 2.1 in [2] which gives explicitly the transition from the big cell of the Bruhat decomposition to the Iwasawa decomposition. In particular,

\begin{equation}
N\text{wa}(t_1)mn(b, r)a(t)K = Na(t')K,
\end{equation}

\[
t' = \frac{t}{t[D]}, \quad D = 2ir + t^2 + |b|^2.
\]

We take $\Omega$ of the form

$\Omega = \{n(b, r)a(t)k \in NAK : |b| < \varepsilon, |r| < \varepsilon, k \in K, (1 - \varepsilon) < t < (1 + \varepsilon)\}$

with $\varepsilon > 0$ small. Elements of $n(a(t)) \Omega \in NA\Omega$ have the form

\[
n(b, r)a(t)n(b, r)k \in Na(tt_\varepsilon)K,
\]

with $tt_\varepsilon = t(1 + O(\varepsilon))$.

We apply this with

\[
\gamma = n_1\text{wa}(t_1)mn(b_1, r_1) \in NWAMN, \quad p = n(b_2, r_2)a(t_2)k_2 \in n(b, r)a(t)\Omega.
\]
we denote the character of $ZU$ is a well-defined element of $M$.

\begin{align*}
\gamma p &= n_1 \omega a(t_1) m(n(b_1, r_1) m(b_2, r_2) a(t_2) k_2 \\
& \in N \omega a(t_1) m(b_1 + b_2, r_1 + r_2 + \text{Im}(b_2)) a(t_2) K \\
& = N a(t') K, \quad \text{with } t' = \frac{t_2}{t_1 |2ir_3 + t_2^2 + |b_3|^2|}.
\end{align*}

Now we note that $t_2 = t(1 + O(\varepsilon))$, which implies that $t'$ can be made uniformly small by increasing $t$. This gives (4.14).

For (4.15) we take $p \in Na(t)$ and $q = n_\varepsilon a(t_\varepsilon) k_\varepsilon \in \Omega$. Then $pq \in Na(t_\varepsilon) K$, with $t_\varepsilon < t(1 + c\varepsilon)$ for some $c > 1$. It suffices to take $T_1 < (1 + c\varepsilon)^{-1}$.

**Automorphic Poincaré series.** We now apply $P$ to a meromorphic family of elements in a module $M_N^x$ with $\text{Re} \nu > 2$, to obtain an automorphic Poincaré series.

We work with a fixed Fourier term order $N = N_\beta, \beta \in \mathbb{Z}[i] \setminus \{0\}$ or $N = N_{\ell,c,d}$, with a character $\xi$ of $M$, and a variable spectral parameter $\nu$. All functions $f \in M_N^x$ satisfy (3.9), hence the Poincaré series $Pf$ are well defined and $f \mapsto Pf$ is an intertwining operator of $(g, K)$-modules $M_N^x \to \mathcal{C}^\infty(\Gamma \backslash G)_K$.

**Proposition 4.7.** Let $\xi$ be a character of $M \subset K$ and let $N$ be a Fourier term order that is not $N$-trivial. Then, there exist non-zero meromorphic families $M_N(\xi, \nu)$ of functions on $\Gamma N \backslash G$ such that:

a) For each $g \in G$, the function $\nu \mapsto M_N(\xi, \nu)(g)$ is meromorphic in $\mathbb{C}$ with first order singularities occurring possibly at points in $\mathbb{Z}_{< -1}$.

b) If $M_N(\xi, \nu)$ is holomorphic at $\nu_0$, then its value at $\nu = \nu_0$ is in $\mathcal{F}^\phi[\xi, \nu_0]$, and if furthermore $\text{Re} \nu_0 \geq 0$, then its value is in $M_N^{\xi, \nu_0}$. (By $\psi[\xi, \nu]$ we denote the character of $ZU(g)$ parametrized by $(\xi, \nu)$.)

c) If $M_N(\xi, \nu)$ has a singularity at $\nu_0 \in \mathbb{Z}_{< -1}$, then it is a simple pole and the residue at $\nu = \nu_0$ is an element of $\mathcal{F}^{\phi[\xi, \nu]}$.

Examples of such families are $\mu^{0,0}_\beta(j, \xi, \nu)$ in (3.11) and $\mu^{0,0}_n(j, \xi, \nu)$ in (3.15). See Appendix B, Proposition B.1 for more details.

**Remark.** In this definition we only prescribe the Fourier term order $N$ and the character $\xi$ of $M$. The holomorphy ensures that in a given family only finitely many $K$-types occur.

**Proposition 4.8.** Let $N \neq N_0$ be a Fourier term order, and let $\xi$ be a character of $M$. For each family $M_N$ as in Proposition 4.7, the automorphic Poincaré series

$$M_N(\xi, \nu) := PM_N(\xi, \nu)$$

is a well-defined element of $\mathcal{A}([\psi[\xi, \nu]])$ for each $\nu$ with $\text{Re} \nu > 2$. Here, by $\psi[\xi, \nu]$ we denote the character of $ZU(g)$ with spectral parameters $(\xi, \nu)$. Furthermore,

(i) The family $\nu \mapsto M_N(\xi, \nu)$ is holomorphic on the half-plane $\text{Re} \nu > 2$.

(ii) For each $X \in \mathfrak{g}$ the relation $XM_N(\xi, \nu) = PXM_N(\xi, \nu)$ holds.
(iii) The Fourier terms $F_N \cdot M_N(\xi, \nu)$ are holomorphic families of elements of $M^{\xi, \nu}$ on $\Re \nu > 2$. We have

a) Abelian cases. If $\mathcal{N} = N_\beta, \beta \in \mathbb{Z}[i] \setminus \{0\}$, then $F_N \cdot M_\beta(\xi, \nu) \in \mathcal{W}^{\xi, \nu}_{\mathcal{N}}$

for all $N' \notin \{N_{\mu \beta} : a \mod 4\} \cup \{0\}$.

b) Non-abelian cases. If $\mathcal{N} = N_{\ell, c, d}$, then $F_N \cdot M_{\ell, c, d}(\xi, \nu) \in \mathcal{W}^{\xi, \nu}_{\mathcal{N}'}$ for all $N' \notin \{N_{\ell', c, d'} : c' \mod 2\ell\} \cup \{0\}$.

Proof. It suffices to work with $\nu$ on a relatively compact open set in the half-plane $\Re \nu > 2$. The uniform convergence of the Poincaré series on compact sets in $(g, \nu)$ implies that holomorphy in $\nu$ is preserved and that $M_N(\xi, \nu) \mapsto M_N(\xi, \nu)$ is an intertwining operator $M^{\xi, \nu}_N \rightarrow C^\infty(\Gamma \backslash G)$. This gives parts (i) and (ii). The intertwining property implies that to have $M_N(\xi, \nu) \in \mathcal{A}^i(\psi, \xi, \nu)$ we need only prove parts a) and b) in (iii); see Definition 4.4.

The Fourier terms are obtained by finitely many integrals over compact sets, hence they depend holomorphically on $\nu$.

We have $M_N(\xi, \nu) = P^{\infty} M_N(\xi, \nu) + P' M_N(\xi, \nu)$ as in (4.11). Since $\Gamma_\mathcal{N} \setminus \Gamma_P$ is generated by $\Gamma_\mathcal{N} m(i)$ for the group that we have fixed in Subsection 2.2, there are four summands that contribute to $P^{\infty} M_N(\xi, \nu)$. In the abelian case, a computation based on (3.2) shows that $P^{\infty} M_\beta(\xi, \nu)$ is a linear combination of the functions $M_{\nu_\beta}(\xi, \nu)$, with $\nu = 0, 1, 2, 3$. These are among the excluded Fourier term orders in (iii) (a).

In the non-abelian case we proceed similarly on the basis of Proposition A.1 in Appendix A.

The component $P' M_N(\xi, \nu)$ is bounded, according to Lemma 4.6. The contribution of $P^{\infty} M_N(\xi, \nu)$ in general not bounded, but it contributes to only finitely many Fourier terms, with orders contained in the sets in (iii), a) and b). This implies that the Poincaré series $M_N(\xi, \nu)$ has at most moderate exponential growth.

In the next result we analyze the growth of the first sum in (4.11).

**Proposition 4.9.** Let $\mathcal{N} \neq N_0$. The finite sum

$$
(4.18) \quad P^{\infty} M_N(\xi, \nu)(g) = \sum_{\gamma \in \Gamma_\mathcal{N} \setminus \Gamma_P} M_N(\xi, \nu)(\gamma g)
$$

is holomorphic in $\nu \in \mathbb{C} \setminus \mathbb{Z}_{\leq -1}$, and satisfies

$$
(4.19) \quad F_N P^{\infty} M_N(\xi, \nu) = a M_N(\xi, \nu)
$$

for some $a \in \mathbb{C}$. The Poincaré family $\nu \mapsto M_N(\xi, \nu)$ is identically zero for $\nu \in \mathbb{C}$ if and only if $a = 0$.

Proof. In the proof of Proposition 4.8 we saw that the finite sum $P^{\infty} M_N(\xi, \nu)$ is a finite linear combination of $M_N(\xi, \nu)$. Let $\Delta$ be the subgroup of $\gamma \in \Gamma_P$ for which $M_N(\xi, \nu)(\gamma g)$ is a multiple of $M_N(\xi, \nu)(g)$. Then

$$
\sum_{\delta \in \Gamma_\mathcal{N} \setminus \Delta} M_N(\xi, \nu)(\delta g) = a M_N(\xi, \nu)(g)
$$
for some $a \in \mathbb{C}$. Thus, we have
\[
P^{\infty}M_N(\xi, \nu)(g) = a \sum_{\gamma \in \Delta \Gamma_p} M_N(\xi, \nu)(\gamma g).
\]
If $a \neq 0$ then all $\gamma \notin \Delta$ in this sum contribute to a Fourier term of order $N' \neq N$, and $F_N P^{\infty}M_N(\xi, \nu)$ is equal to $a M_N(\xi, \nu)(g)$. If $a = 0$, then
\[
M_N(\xi, \nu)(g) = \sum_{\gamma_1 \in \Gamma_p \Gamma} a \sum_{\gamma \in \Delta \Gamma_p} M_N(\xi, \nu)(\gamma \gamma_1 g) = 0. \quad \square
\]

**Growth.** If $M_N(\xi, \nu)$ is non-zero, then $P^{\infty}M_N(\xi, \nu)$ can have exponential growth. For instance if $M_N(\xi, \nu) = \mu_n^{0,0}(\xi, \nu)$, see (3.15), this follows from the fact that the $M$-Whittaker function has in general exponential growth. There are exceptions, since for special combinations of $\kappa$ and $s$, $M_{\kappa,s}$ and $W_{\kappa,s}$ are proportional.

### 5. Meromorphic continuation of Poincaré series

In this section we will prove the following result.

**Theorem 5.1.** Let $N \neq N_0$ be a Fourier term order, and let $\xi$ be a character of $M$. Then, for each family $\nu \mapsto M_N(\xi, \nu)$ as in Proposition 4.7, the family defined by $\nu \mapsto M_N(\xi, \nu) = PM_N(\xi, \nu)$ for $\Re \nu > 2$ has a meromorphic continuation to $\mathbb{C}$, and $M_N(\xi, \nu) = P_N(\xi, \nu)M_N(\xi, \nu) \in A^1(\psi|\xi, \nu))$, where defined.

(i) The singularities of $\nu \mapsto M_N(\xi, \nu) = PM_n(\xi, \nu)M_N(\xi, \nu)$ in the closed right half-plane occur at $\nu_0$ for which there exists a square integrable automorphic form in $A^2(\psi|\xi, \nu_0))$. The singularities at these points are simple poles, except for $\nu_0 = 0$, that may be a double pole.

(ii) For each $X \in \mathfrak{g}$ the relation $X M_N(\xi, \nu) = P_N(\xi, \nu)XM_N(\xi, \nu)$ holds as an identity of meromorphic families.

**Remarks.** (1) Here, for $\Re \nu > 2$, $P_N(\xi, \nu)$ is given by the operator $P$, while for other values of $\nu$ it is obtained by meromorphic continuation. From now on, we will just write $M_N(\xi, \nu) = P_N M_N(\xi, \nu)$ in all cases, for simplicity.

(2) Part ii) implies that $P_N(\xi, \nu)$ is an intertwining operator of $(\mathfrak{g}, K)$-modules for all $\nu \in \mathbb{C}$ where the Poincaré families have no singularity.

The proof of Theorem 5.1 follows the methods in [14, §2]. We shall give it in subsections 5.1–5.5. We carry out a reduction before we start on the actual proof. Proposition 4.7 provides us with a family $M_N(\xi, \nu)$ that may contain many $K$-types. As noted after Proposition B.1 in the appendix, such a function is obtained as a holomorphic linear combinations of families $M_{N;h,p,q}$ with $K$-type $\tau^h_p$ and weight $q$ in that $K$-type. Since the Poincaré series satisfies
\[
P\left(\sum_j c_j(\nu) M_{N;h_j,p_j,q}(\xi, \nu)\right) = \sum_j c_j(\nu)P M_{N;h_j,p_j,q}(\xi, \nu),
\]
it will thus suffice to prove the theorem for Poincaré series of the form $P M_{N;h,p,q}$. 
5.1. **Truncation.** We take a function \( \varphi \in C^\infty(0, \infty) \) that is equal to 1 on a neighborhood of 0, and equal to 0 on a neighborhood of \( \infty \). We use the same symbol for the extended function \( \varphi \in C^\infty(G) \) defined by \( \varphi(na(t)k) = \varphi(t) \).

We fix \( (h, p, q) \) satisfying \( h \equiv q \equiv p \mod 2 \), \( p \in \mathbb{Z}_{\geq 0} \), and consider a family \( \nu \mapsto M_{N,h,p,q}(\xi, \nu) \) as in Proposition B.1. We allow ourselves to drop the subscript \( h,p,q \) from the notation of the family, and from related families occurring later in the proof.

We write \( M_N(\xi, \nu) \) as
\[
M_N(\xi, \nu) = \varphi M_N(\xi, \nu) + (1 - \varphi) M_N(\xi, \nu). \tag{5.2}
\]

For \( \text{Re} \ \nu > 2 \) this leads to the decomposition
\[
M_N(\xi, \nu) = M_N(\xi, \nu) + \left( M_N(\xi, \nu) - \overline{M}_N(\xi, \nu) \right). \tag{5.3}
\]

The behavior of \( \varphi M_N(\xi, \nu)(na(t)k) \) as \( t \downarrow 0 \) is the same as that of \( M_N(\xi, \nu) \). The contribution of \( \varphi M_N(\xi, \nu)(na(t)k) \) as \( t \uparrow \infty \) is bounded (see Lemma 4.6). The finite sum \( \varphi M_N(\xi, \nu)(na(t)k) \) is zero for sufficiently large values of \( t \). Hence \( M_N(\xi, \nu) \) is bounded, and
\[
M_N(\xi, \nu) \in L^2(\Gamma \backslash G) \quad \text{for} \quad \text{Re} \ \nu > 2. \tag{5.4}
\]

The function \( ((1 - \varphi) M_N(\xi, \nu)) \) vanishes for all \( t \) near zero, uniformly in \( n \) and \( k \). Hence the sum \( M_N(\xi, \nu) - \overline{M}_N(\xi, \nu) \) is given, locally in \( g \), by a finite sum. So \( M_N(\xi, \nu) \) is meromorphic in \( \nu \in \mathbb{C} \), with at most first order singularities in \( \mathbb{Z}_{\leq -1} \). This difference has in general exponential growth as \( t \uparrow \infty \). In this way, the proof of the meromorphic continuation is reduced to the continuation of the truncated Poincaré series \( \overline{M}_N(\xi, \nu) \).

For each \( X \in \mathfrak{g} \)
\[
X \overline{M}_N(\xi, \nu)(g) = \sum_{\gamma \in \Gamma \backslash \Gamma} \varphi(\gamma g) X M_N(\xi, \nu)(\gamma g) + \sum_{\gamma \in \Gamma \backslash \Gamma} X \varphi(\gamma g) M_N(\xi, \nu)(\gamma g). \tag{5.5}
\]

The decay of \( X \varphi \) near zero is better than the decay of \( \varphi \), so both terms in (5.5) yield a bounded function (see also [14, Lemma 2.2]).

The Casimir operator \( C \) acts on \( M_N^{\xi, \nu} \) by multiplication by \( \lambda_2(\xi, \nu) = v^2 - 4 + \frac{1}{\xi} \ell^2_\xi \) (see (9.3) in [FItm]), which we now denote by \( \lambda(\xi, \nu) \). The quantity
\[
(C - \lambda(\xi, \nu))(\varphi M_N(\xi, \nu))(g)
\]

is non-zero only in the region where the cut-off function \( \varphi \) is not constant. So, the function
\[
\overline{M}_N(\xi, \nu)(g) := (C - \lambda(\xi, \nu)) \overline{M}_N(\xi, \nu)(g) \tag{5.6}
\]
has compact support in \( \Gamma \backslash G \). The argument in Lemma 2.3 in [14] gives the meromorphic continuation of \( \overline{M}_N(\xi, \nu)(g) \) to \( \nu \in \mathbb{C} \) with at most first order singularities in \( \mathbb{Z}_{\leq -1} \).
Since $M_N(\xi, \nu) = M_{N;h,p,q}$ we have
\begin{equation}
\tilde{M}_N(\xi, \nu) \in L^2(\Gamma \backslash \Gamma)_{h,p,q} \quad \text{for } \Re \nu > 2 ,
\end{equation}
\begin{equation}
M_N(\xi, \nu) = C_{\infty}^0(\Gamma \backslash \Gamma)_{h,p,q} \quad \nu \in \mathbb{C} \setminus \mathbb{Z}_{\leq -1} ,
\end{equation}
\begin{equation}
M_N(\xi, \nu) - \tilde{M}_N(\xi, \nu) \in C_{\infty}^0(\Gamma \backslash \Gamma)_{h,p,q} \quad \nu \in \mathbb{C} \setminus \mathbb{Z}_{\leq -1} .
\end{equation}

Here $L^2_{h,p,q}(\Gamma \backslash \Gamma)$ is the Hilbert space of $\Gamma$-invariant square integrable functions of $K$-type $\tau_p^h$ and weight $q$, with $p \geq 0$, $h \equiv p \equiv q \mod 2$, $|q| \leq p$. There is a direct sum decomposition
\begin{equation}
L^2_{h,p,q}(\Gamma \backslash \Gamma) = L^2_{h,p,q}^{\text{discr}}(\Gamma \backslash \Gamma) \oplus L^2_{h,p,q}^{\text{cont}}(\Gamma \backslash \Gamma) ,
\end{equation}
of subspaces in which the Casimir operator induces a self-adjoint operator having discrete (resp. continuous) spectrum on $L^2_{h,p,q}^{\text{discr}}(\Gamma \backslash \Gamma)$ (resp. on $L^2_{h,p,q}^{\text{cont}}(\Gamma \backslash \Gamma)$). The bounded function $\tilde{M}_N(\xi, \nu)$ is square integrable, and hence has a decomposition
\begin{equation}
\tilde{M}_N(\xi, \nu) = \tilde{M}_N^{\text{discr}}(\xi, \nu) + \tilde{M}_N^{\text{cont}}(\xi, \nu) \quad (\Re \nu > 2) .
\end{equation}
We shall consider the meromorphic continuation of these two components separately, with the help of the analogous decomposition of $M_N(\xi, \nu)$, valid at least for $\nu \in \mathbb{C} \setminus \mathbb{Z}_{\leq -1}$.

### 5.2. Contribution of the discrete spectrum

The space $L^2_{h,p,q}^{\text{discr}}(\Gamma \backslash \Gamma)$ has an orthonormal basis $\{f_l\}_{l \in \mathbb{N}}$ of eigenfunctions of $C$ with real eigenvalues $\{\mu_l\}$ with finite multiplicity, ordered increasingly, and chosen so that the $f_l$ are eigenfunctions of $\Delta_3$ as well.

We define for $\Re \nu > 2$
\begin{equation}
c_l(\nu) = \left( \tilde{M}_N(\xi, \nu), f_l \right) \quad \text{and} \quad d_l(\nu) = \left( \tilde{M}_N(\xi, \nu), f_l \right) .
\end{equation}

Since $\tilde{M}_N(\xi, \nu)$ is compactly supported and smooth, the series $\sum_l d_l(\nu) f_l$ converges absolutely and uniformly on any compact subset $\Omega$ of $\mathbb{C} \setminus \mathbb{Z}_{\leq -1}$. Furthermore, the coefficient $d_l$ inherits from $\tilde{M}_N(\xi, \nu)$ meromorphy in $\mathbb{C}$ with at most first order singularities in points of $\mathbb{Z}_{\leq -1}$, and we obtain from the fact that $\tilde{M}_N(\xi, \nu) = (C - \lambda(\xi, \nu))M_N(\xi, \nu)$, the relation
\begin{equation}
d_l(\nu) = (\mu_l - \lambda(\xi, \nu)) c_l(\nu) \quad \text{for } \Re \nu > 2 .
\end{equation}

So $c_l$ extends meromorphically to $\mathbb{C} \setminus \mathbb{Z}_{\leq -1}$, and if $\lambda(\xi, \nu) \neq \mu_l$ we have
\begin{equation}
\tilde{M}_N^{\text{discr}}(\xi, \nu) = \sum_l \frac{d_l(\nu)}{\mu_l - \lambda(\xi, \nu)} f_l .
\end{equation}

This series converges in the $L^2$-sense, uniformly for $\nu$ on compact sets $\Omega$ such that $\lambda(\xi, \nu) \neq \mu_l$ for all $\nu \in \Omega$ and all $l$ (see Theorem 2.5 in [14, p. 428]). In this way, $\tilde{M}_N^{\text{discr}}(\xi, \nu)$ has been extended to a meromorphic family in $\mathbb{C}$ with values in $L^2_{h,p,q}^{\text{discr}}(\Gamma \backslash \Gamma)$. Furthermore, we can repeatedly apply elements of $g$, and get that $u\tilde{M}_N^{\text{discr}}(\xi, \nu)$ is square integrable on $\Gamma \backslash \Gamma$ for all $u \in U(g)$. This implies that we have a family that is smooth in $(\nu, g)$ jointly, and holomorphic in $\nu$ as long as $\lambda(\xi, \nu) \neq \mu_l$.
for all \( l \). If there is a singularity at a point \( v_0 \), it is also a singularity of the sum of the terms in (5.12) for which \( \mu_l = \lambda(\xi, v_0) \).

5.3. **Contribution of the continuous spectrum.** We now consider the meromorphic continuation of \( \mathcal{M}_N^{\text{cont}}(\xi, \nu) \). In (4.9) we have introduced the Eisenstein series in their domain of absolute convergence \( \Re \nu > 2 \). For \( L^2_{\text{cont}}(\Gamma \setminus \mathbb{G})_{h,p,q} \) we need \( E(\xi_r, \nu)_{h,p,q} \) for the finitely many \( r \in \mathbb{Z} \) such that \( r \equiv p \mod 2 \), \( |r| \leq p \), and \( j_r = \frac{1}{2}(h - 3r) \in 4\mathbb{Z} \).

The Eisenstein series have a meromorphic continuation to \( \mathbb{C} \), and the possible singularities in the closed half-plane \( \Re \nu \geq 0 \) lie at a finite set of points in the interval \((0, 2] \). The residue at \( v_1 \in (0, 2] \) is a function in \( A^2(\psi[\xi_r, v_1]) \), and gives a contribution to the discrete spectrum.

The projection onto the continuous spectrum of a function \( f \in L^2(\Gamma \setminus \mathbb{G})_{h,p,q} \) takes the form

\[
f_{\text{cont}}(g) = \sum_r \frac{m_{h,p,q}}{2\pi i} \int_{\Re \lambda = 0} f_{\lambda}(\lambda) E(\xi_r, \lambda)_{h,p,q}(g) d\lambda,
\]

(5.13)

\[
f_{\lambda}(\lambda) = \int_{\Gamma \setminus \mathbb{G}} f(g) E(\xi_r, \lambda)_{h,p,q}(g) \, dg,
\]

for \( \Re \lambda = 0 \).

For \( \varepsilon \) sufficiently small, if \( 0 \leq \Re \lambda < \varepsilon \), the functions \( E(\xi_r, \lambda) = E(\xi_r, \lambda)_{h,p,q} \) do not have poles and are integrable on \( \Gamma \setminus \mathbb{G} \), and hence

\[
\varphi_{\lambda}(\lambda, \nu) = \int_{\Gamma \setminus \mathbb{G}} \mathcal{M}_N(\xi, \nu)(g) \overline{E(\xi_r, \lambda)(g)} \, dg
\]

(5.14)

\[
= \int_{\Gamma \setminus \mathbb{G}} \varphi(g) \mathcal{M}_N(\xi, \nu)(g) E(\xi_r, \lambda)(g) \, dg.
\]

is a holomorphic function of \((\lambda, \nu)\) for \( 0 \leq \Re \lambda < \varepsilon \) and \( \Re \nu > 2 \).

**Lemma 5.2.** The family \((\lambda, \nu) \mapsto \varphi_{\lambda}(\lambda, \nu)\) extends as a meromorphic function of \((\lambda, \nu)\) on \( \mathbb{C}^2 \) with singularities occurring only in the union of the following sets:

\[
\mathbb{C} \times (\mathbb{C} \setminus \mathbb{Z}_{\leq 1}), \quad \{(\lambda, \nu) : \lambda + \nu \text{ or } \lambda - \nu \in \mathbb{Z}_{\leq 0}\},
\]

\[
\{(\lambda, \nu) : \nu \in \mathbb{C}, \ E(\xi_r, \cdot) \text{ has a singularity at } \lambda\}.
\]

This is a preliminary continuation result, comparable to [14, Lemma 2.4].

**Proof.** The Eisenstein series \( E(\xi_r, \lambda)_{h,p,q} \) has a Fourier expansion in which the term of order \( N \) has the form \( C_N(\xi_r, \lambda) \Omega_{N,h,p,q}(\xi_r, \lambda) \), with the exponentially decreasing basis function given in Proposition B.1. The coefficient \( C_N(\xi_r, \lambda) \) is holomorphic where \( E(\xi_r, \lambda) \) is holomorphic.

We start with \((\lambda, \nu)\) for which (5.14) holds. The second integral in (5.14) is, up to a multiple, equal to

\[
C_N(\xi_r, \lambda) \int_{\Gamma \setminus \mathbb{G}} \varphi(g) \mathcal{M}_{N,h,p,q}(\xi, \nu)(g) \overline{\Omega_{N,h,p,q}(\xi_r, \lambda)} \, dg.
\]

(5.15)

The influence of the singularities of the Eisenstein series is given by the Fourier coefficient \( C_N(\xi_r, \cdot) \). The integral in (5.15) is a quantity depending on the basis
functions $M_N$ and $\Omega_N$ and the cut-off function $\varphi$. Only the Fourier coefficient depends on the group $\Gamma$.

For general $K$-types $t_h^\mu$ we do not have an explicit description of the function $M_{N,h,p,q}(\xi, \nu)$. However we know that for $na(t)k \in NAK$

$$M_{N,h,p,q}(\xi, \nu) = \sum_{r} w_r(n) t^{2+\nu} h_r^\mu(\xi, \nu; t) h_{\Phi_r^0}(k),$$

(5.16)

with $r$ satisfying the conditions $|r| \leq p$, $r \equiv p \mod 2$, and

$$h_r^\mu(\xi, \nu, t) = \sum_{m=0}^{\infty} \alpha_m(\xi, \nu)t^m$$

holomorphic in $(\nu, t) \in (\mathbb{C} \setminus \mathbb{Z}_{\leq -1}) \times \mathbb{C}$, with possible singularities at points $\nu = \nu_0$ with $\nu_0 \in \mathbb{Z}_{\leq -1}$. The functions $w_r$ are a character of $N$ or elements of an orthogonal system of theta functions on $N$. The $h_{\Phi_r^0}$ belong to an orthogonal system of polynomial functions on $K$. See Proposition B.1, and the discussion of $\nu$-regular behavior at 0 in [2, §10.2].

The function $\Omega_{N,h,p,q}(\xi, \nu)(na(t))$ does not have this form as $t \downarrow 0$. However, for $\lambda \notin \mathbb{Z}$ we can write it as a linear combination of $M_{N,h,p,q}(\xi, \nu)$ and $M_{N,h,p,q}(\xi, -\nu)$ with coefficients that are holomorphic in $\mathbb{C} \setminus \mathbb{Z}$. See [2, (10.21)]. In this way we find that the integral in (5.15) has the form

$$\int_{t=0}^{\infty} A_\pm(\lambda) \varphi(t) t^{\nu+k-1} \sum_{m,l\geq 0} \alpha_m(\xi, \nu) \alpha_r(\xi, \lambda) t^{m+l-1} dt,$$

(5.18)

where $A_+$ and $A_-$ are meromorphic functions which may have singularities only in $\mathbb{Z}$. The series in $m$ and $l$ converge absolutely on the compact support of $\varphi$.

For $\operatorname{Re} \nu - |\operatorname{Re} \lambda| > 0$ the integral converges and defines a holomorphic function of $(\lambda, \nu)$. We apply partial integration to the integral of individual terms to see that they have a meromorphic extension with singularities along lines $\nu \pm \lambda \in \mathbb{Z}_{\leq 0}$. Splitting off more and more terms, we get holomorphy of the remaining contributions on larger sets. Thus we have a meromorphic extension which is holomorphic outside the sets indicated in the lemma.

If $\lambda_0 \in \mathbb{Z}$ then there will be coinciding terms in (5.18). The singularities of $A_\pm(\lambda)$ at $\lambda_0$ are canceled, and the integrand in (5.18) extends holomorphically to $\lambda_0$. This reflects the fact that the family $\nu \mapsto \Omega_{N,h,p,q}(\xi, \nu)$ is holomorphic in $\mathbb{C}$. There arises a series with terms $t^{\nu+k-1}$ and $t^{\nu+k-N-1} \log t$, with $N \geq 0$. Partial integration now leads to singularities that may have second order.

In this way we arrive at a meromorphic extension of $\varphi_{\xi}$, with singularities along lines $\lambda - \nu = n$ and $\lambda + \nu = n$, with $n \in \mathbb{Z}_{\leq 0}$. The possibility that $M_{N,h,p,q}(\xi, \nu)$ has singularities at $\nu \in \mathbb{Z}_{\leq -1}$ may cause singularities of $\varphi_{\xi}$. The Fourier coefficient $C_N(\xi, \lambda)$ may also cause singularities. \hfill $\square$

The function $\tilde{M}_N(\xi, \nu)$ has a compact support determined by the cut-off function $\varphi$. It depends meromorphically on $\nu$ with singularities only in $\mathbb{Z}_{\leq -1}$. Hence

$$\tilde{\varphi}_{\xi}(\lambda, \nu) = \int_{\Gamma \backslash G} \tilde{M}_N(\xi, \nu)(g) \overline{E(\xi, \lambda)(g)} dg$$

(5.19)
is meromorphic, with singularities along $\nu = n, n \in \mathbb{Z}_{\leq -1}$ and along $\lambda = \lambda_0$ for points $\lambda_0$ at which the Eisenstein series has a singularity.

The functions $\tilde{M}_N(\xi, \nu)(g)$ and $\bar{M}_N(\xi, \nu)(g)$ and their derivatives by $X \in g$ are bounded. Differentiation under the integral sign gives

$$
\frac{\partial}{\partial \nu} \varphi_{j_i}(\lambda, \nu) = \left( M_N(\xi, \nu), (C - \lambda_2(\xi, \nu)) E(\xi, \lambda) \right)
= \left( \lambda_2(\xi, \lambda) - \lambda_2(\xi, \nu) \right) \left( \tilde{M}_N(\xi, \nu), E(\xi, \lambda) \right)
= \left( \lambda_2(\xi, \lambda) - \lambda_2(\xi, \nu) \right) \left( \lambda^2 - \nu^2 + \frac{1}{3}(j_r^2 - f^2) \right) \varphi_{j_i}(\lambda, \nu),
$$

valid for $\lambda \in \Omega$ and $\text{Re} \nu > 2$, and extended as a relation between meromorphic functions on $\mathbb{C}^2$, with use of Lemma 5.2. This relation describes the function $\varphi_{r_i}(\lambda, \nu)$ in terms of the better behaved function $\varphi_{r_i}(\lambda, \nu)$. The singularities of $\varphi_{j_i}$ come either from singularities of $\varphi_{r_i}$, or from the polynomial factor in (5.20). By repeating the differentiation in (5.20) we obtain quick decay of $\varphi_{j_i}(\nu, \lambda)$ for $\nu$ on vertical strips and for $\text{Re} \lambda$ in bounded sets. This implies the same result for $\varphi_{j_i}(\lambda, \nu)$ outside neighborhoods of its singularities.

We use the meromorphy of $\varphi_{j_i}$ and $\varphi_{r_i}$ to get the meromorphic continuation of the projections of $\tilde{M}_N(\xi, \nu)$ and $\bar{M}_N(\xi, \nu)$ to $L^2(\Gamma \backslash G)$.

$$
\tilde{M}_N(\xi, \nu)(g) = \sum_r \frac{m_{h,p,q}}{2\pi i} \int_{\text{Re} \lambda = 0} \varphi_{j_i}(\lambda, \nu) E(\xi, \lambda)_{h,p,q}(g) \, d\lambda,
$$

$$
\bar{M}_N(\xi, \nu)(g) = \sum_r \frac{m_{h,p,q}}{2\pi i} \int_{\text{Re} \lambda = 0} \varphi_{r_i}(\lambda, \nu) E(\xi, \lambda)_{h,p,q}(g) \, d\lambda.
$$

Initially, these relations hold for $\text{Re} \nu > 2$. The aim is to give a meromorphic extension. In the proof of Theorem 2.5 in [14] an analogous situation arose; see p 428–429.

In [14], the complex plane is divided into connected regions on which the integrals above describe a holomorphic function of $\nu$. Deformation of the path of integration causes changes of these regions. The resulting functions of $\nu$ are related by Cauchy’s residue theorem. Suitable choices of a complex square root of $\nu^2 + \frac{1}{4}(j_r^2 - f^2)$ lead to the desired meromorphic continuation. Below, we work out the proof of the meromorphic continuation of $\tilde{M}_N^\text{cont}(\xi, \nu)$ in an alternative way, in the present context and notation.

The general integral. We consider functions $I^+_r(\Lambda, \nu)$ on the region $\{ (\Lambda, \nu) \in \mathbb{C}^2 : \text{Re} \Lambda > 0 \}$, and $I^-_r(\Lambda, \nu)$ on $\{ (\Lambda, \nu) \in \mathbb{C}^2 : \text{Re} \Lambda < 0 \}$, both given by the integral

$$
\frac{1}{2\pi i} \int_{\text{Re} \Lambda = 0} \frac{\varphi_{j_i}(\lambda, \nu)}{\lambda^2 - \Lambda^2} E(\xi, \Lambda)_{h,p,q}(g) \, d\lambda.
$$

If $\Lambda^2 = \nu^2 + \frac{1}{4}(j_r^2 - f^2)$, then $I^+_r(\Lambda, \nu)$ corresponds to the integral in the term of order $r$ in the expression for $\tilde{M}_N^\text{cont}(\xi, \nu)(g)$. We first consider $\Lambda$ and $\nu$ as two independent variables. We drop $g$ from the notation in the further computations.
The functions $I^\pm$ are meromorphic, with singularities at most along (complex) lines of the form \( \{ (\Lambda, n) : \pm \text{Re} \Lambda > 0, \ n \in \mathbb{Z}_{\leq -1} \} \). We have the relation $I^+_r(\Lambda, \nu) = I^-_r(-\Lambda, \nu)$. The singularities of the integrand in (5.21) as a function of $\lambda$ do not occur on the path of integration.

We take $a > 0$, and consider two deformations of the path of integration, as depicted in Figure 1. We take $\varepsilon > 0$ sufficiently small such that both deformed paths are contained in $\Omega$. Denote by $R_{a,\varepsilon}$ the open rectangular region bounded by the joined paths $C_{a,\varepsilon}^\pm$.

Deforming the path $\text{Re} \lambda = 0$ to $C_{a,\varepsilon}^-$ we get an extension of $I^+_r(\Lambda, \nu)$ with $\Lambda$ in the region on the right of $C_{a,\varepsilon}^-$. Similarly, we have an extension of $I^-_r(\Lambda, \nu)$ with $\Lambda$ in the region on the left of $C_{a,\varepsilon}^+$.

For $\Lambda \neq 0 \in R_{a,\varepsilon}$, the difference of these two extensions is given by

\[
I^-_r(\Lambda, \nu) - I^+_r(\Lambda, \nu) = \frac{1}{2\pi i} \left( \int_{C_{a,\varepsilon}^+} - \int_{C_{a,\varepsilon}^-} \right) \tilde{\phi}_j(\lambda, \nu) \frac{\lambda^2 - \Lambda^2}{A^2 - \Lambda^2} E(jr, \lambda) d\lambda \]

\[
= \frac{1}{2\pi i} \int_{\partial R_{a,\varepsilon}} \tilde{\phi}_j(\lambda, \nu) \frac{\lambda^2 - \Lambda^2}{A^2 - \Lambda^2} E(jr, \lambda) d\lambda
\]

\[
= \sum_{\pm} \text{Res}_{\lambda = \pm \Lambda} \tilde{\phi}_j(\lambda, \nu) E(jr, \lambda) = \sum_{\pm} \frac{\pm 1}{2\Lambda} \tilde{\phi}_j(\pm \Lambda, \nu) E(jr, \pm \Lambda).
\]

The right hand side is meromorphic on $\mathbb{C}^2$. The singularities are along lines $\mathbb{C} \times \{ n \}$ with $n \in \mathbb{Z}_{\leq -1}$, and along lines $\{ \lambda_0 \} \times \mathbb{C}$ with $\pm \lambda_0$ or $\pm \lambda_0$ a singularity of the Eisenstein series. Such points $\lambda_0$ do not occur on the paths $C_{a,\varepsilon}^\pm$ or in the rectangle $R_{a,\varepsilon}$. In the sum $\sum_{\pm}$ the singularities at $\Lambda = 0$ cancel each other. The formula

\[
I^+_r(\Lambda, \nu) = I^-_r(\Lambda, \nu) + \sum_{\pm} \frac{\pm 1}{2\Lambda} \tilde{\phi}_j(\pm \Lambda, \nu) E(jr, \pm \Lambda),
\]
valid at first for $\Lambda \in R_{\mu, \nu}$, gives the meromorphic extension for $(\Lambda, \nu)$ on $\mathbb{C}^2$ minus the subsets $i[a, \infty) \times \mathbb{C}$ and $-i[a, \infty) \times \mathbb{C}$. Since $a > 0$ can be chosen arbitrarily, the function $(\Lambda, \nu) \mapsto I^r_\nu(\Lambda, \nu)$ is meromorphic on $\mathbb{C}^2$. On the region $\Re \Lambda \geq 0$ it can have singularities only along lines $\nu = n, n \in \mathbb{Z}_{\leq -1}$. For $\Re \Lambda < 0$ there may be singularities along lines $\{\lambda_0\} \times \mathbb{C}$ as well.

**Meromorphic continuation for $|j_r| = |j|$.** To obtain the meromorphic continuation as a function of $\nu$ of the contributions to $M_{\nu}^\text{cont}(\xi, \nu)$, we restrict the functions $I^r_\nu(\Lambda, \nu)$ to the curve of equation $\Lambda^2 = \nu^2 + \delta$, with $\delta = \frac{1}{4}(j^2 - \bar{j}^2)$. In the case $|j_r| = |j|$ we can do that by taking $\Lambda = \nu$. The singular set of $I^r_\nu(\Lambda, \nu)$ is contained in a union of lines $\mathbb{C} \times \{n\}, n \in \mathbb{Z}_{\leq -1}$, and lines $\{\lambda_0\} \times \mathbb{C}$ with $\lambda_0$ determined by the singularities of the Eisenstein series. This singular set intersects the line $\Lambda = \nu$ discretely. Hence the restriction $\nu \mapsto I^r_\nu(\nu, \nu)$ gives a meromorphic function of $\nu$ with singularities at values related to singularities of Eisenstein series and at negative integers. The extension satisfies the following identity of meromorphic functions on $\mathbb{C}$.

\[
(5.24) \quad I^r_\nu(\nu, \nu) = I^r_\nu(-\nu, \nu) + \sum_{\pm} \frac{\pm 1}{2\nu} \hat{\varphi}_{jr}(\pm \nu, \nu) E(j_r, \pm \nu).
\]

We note that the singularities of $\hat{\varphi}_{jr}(\Lambda, \nu)$ occur along lines of the form $\mathbb{C} \times \{\lambda_0\}$ and $\{n\} \times \mathbb{C}$, although $\varphi_{jr}(\Lambda, \nu)$ can have singularities along the line $\pm \Lambda = \nu$. See Lemma 5.2.

**Meromorphic continuation for $|j_r| \neq |j|$.** Now we are concerned with the restriction of $I^r_\nu(\Lambda, \nu)$ to a quadratic curve $C_r$ with equation $\Lambda^2 = \nu^2 + \delta$ with $\delta \neq 0$.

The meromorphic function $(\Lambda, \nu) \mapsto \frac{1}{2}I^r_\nu(\Lambda, \nu) + \frac{1}{2}I^r_\nu(\Lambda, \nu)$ on $\mathbb{C}^2$ is even in $\Lambda$. Hence it is of the form $(\Lambda, \nu) \mapsto \tilde{I}_\nu(\Lambda^2, \nu)$ for some meromorphic function $\tilde{I}_\nu$ on $\mathbb{C}^2$. This function inherits from $I^r_\nu$ and $I^r_{\bar{r}}$ the property that its singularities occur only along lines $\mathbb{C} \times \{n\}$ and $\{\lambda_0\} \times \mathbb{C}$. Hence $(\Lambda, \nu) \mapsto \tilde{I}_\nu(\Lambda^2, \nu)$ has a meromorphic restriction to $C_r$. Relation (5.22) implies the following

\[
\tilde{I}_\nu(\Lambda^2, \nu) = I^r_\nu(\Lambda, \nu) + \sum_{\pm} \frac{\pm 1}{2\nu} \hat{\varphi}_{jr}(\pm \Lambda, \nu) E(j_r, \pm \Lambda).
\]

The set of singularities of $\varphi_{jr}$ is a union of lines described in Lemma 5.2. This set intersects $C_r$ in isolated points. So the restriction of $\varphi_{jr}$ to $C_r$ is a meromorphic function, and by (5.20), the restriction of $\hat{\varphi}_{jr}(\Lambda, \nu) = (\Lambda^2 - \nu^2 - \delta) \varphi_{jr}(\Lambda, \nu)$ to $C_r$ is zero. Hence we get the equality $\tilde{I}_\nu(\Lambda^2, \nu) = I^r_\nu(\Lambda, \nu)$ on $C_r$. The meromorphic function $\nu \mapsto \tilde{I}_\nu(\nu^2 + \delta, \nu)$ gives the desired meromorphic continuation.

**Regularity.** In each of the three cases, the meromorphically continued quantity is given by integrals and terms with a factor that is quickly decreasing for $\lambda$ on vertical strips. So all contributions to the meromorphic continuation of $M_{\nu}^\text{cont}(\xi, \nu)(g)$ are locally $C^\infty$ in $(\nu, g)$ and holomorphic in $\nu$ at all points $(\nu, g)$ with $\Re \nu \geq 0$.

**Remark.** We note that identity (5.22) shows that in the cases $|j_r| = |j|$ there is no pole of the continuation of $I^0_\nu(\nu)$ at $\nu = 0$, and hence $M_{\nu}^\text{cont}(\xi, \nu)$ is holomorphic
at \( \nu = 0 \). Actually, in the same way one can see that \( M^\cont_N(\xi, \nu) \) is holomorphic at \( \nu = 0 \) for the abelian Poincaré series studied in [14] in the case of arbitrary semisimple Lie groups of real rank one.

5.4. Conclusion. The considerations in the previous subsections show that the meromorphic continuation can be given in the form

\[
M_N(\xi, \nu) = M^\disc_N(\xi, \nu) + M^\cont_N(\xi, \nu) + P(1 - \varphi)M_N(\xi, \nu).
\]

For open sets \( \Omega \subset \mathbb{C} \) in which no singularities occur, \( M_N(\xi, \nu) \in C^\infty(\Omega \times G) \), and it depends holomorphically on \( \nu \in \Omega \). At a singularity at \( \nu = \nu_0 \) with \( \text{Re} \nu_0 \geq 0 \) the function \( M_N(\xi, \nu) \) is the sum of an explicit finite sum, describing the singularity, plus an element of \( C^\infty(\Omega \times G) \) that is holomorphic in \( \nu \) on a neighborhood \( \Omega \) of \( \nu_0 \).

The order of singularities in the closed right half-plane is 1, or possibly 2 at \( \nu = \nu_0 \). The Laurent series at a singularity at \( \nu = \nu_0 \) with \( \text{Re} \nu_0 \geq 0 \), starts with a term given by a square integrable automorphic form for which the eigenvalue of the Casimir operator is equal to \( \lambda_2(\xi, \nu_0) \).

It might happen that the sum \( \sum_{\gamma \in \Gamma_N \setminus \Gamma} M_N(\xi, \nu; \gamma g) \) is identically zero. In this case, \( PM_N(\xi, \nu) \) is zero, and Theorem 5.1 is trivially true for \( M_N(\xi, \nu) \).

5.5. Main part and order. To be able to complete the proof of Theorem 5.1, we now introduce a useful concept.

Definition 5.3. Assume the family \( \nu \mapsto PM_N(\xi, \nu) \) is not identically zero. We let the order at the point \( \nu_0 \) of the Poincaré family, denoted \( \text{po}_N(\nu_0) = \text{po}(M_N; \nu_0) \), be the integer such that the limit

\[
P_{\nu_0}(M_N) = \lim_{\nu \to \nu_0} (\nu - \nu_0)^{\text{po}_N(\nu_0)} M_N(\xi, \nu)
\]

exists and is non-zero. We call the limit function, \( P_N(\nu_0) \), the main part of \( M_N(\xi, \cdot) \) at \( \nu_0 \). If the Poincaré family is identically zero we put \( \text{po}_N(\nu_0) = -\infty \) and \( P_N(\nu_0) = 0 \) for all \( \nu_0 \in \mathbb{C} \).

Thus, if the order is a positive integer, then \( PM_N(\xi, \nu) \) has a singularity at \( \nu = \nu_0 \) and it equals 1, the main part is the residue at \( \nu_0 \). If the order is non-positive, the main part at \( \nu_0 \) is the value \( M_N(\xi, \nu_0) \) (for \( m = 0 \)) or the value of a derivative at \( \nu_0 \) (for \( m \geq 1 \)).

Proposition 5.4. Let \( \text{Re} \nu \geq 0 \).

i) The main part \( P_{\nu_0}(M_N) \) is an automorphic form with moderate exponential growth lying in \( \mathcal{A}(\psi[\xi, \nu]) \).

ii) If \( M_N(\xi, \nu) \) has a singularity at \( \nu = \nu_0 \), then the main part is a square integrable automorphic form in \( \mathcal{A}^{(2)}(\psi[\xi, \nu]) \).

Proof: On the region \( \text{Re} \nu > 0 \) we have \( uM_N(\xi, \nu) = \psi[\xi, \nu](u) M_N(\xi, \nu) \) for all \( u \in ZU(\mathfrak{g}) \). The pointwise holomorphy implies that this relation persists in the complement of all singularities. The same reasoning applies to the function \( \nu \mapsto (\nu - \nu_0)^{\text{po}_N(\nu_0)} M_N(\xi, \nu) \) on a neighborhood of \( \nu_0 \), and extends to the main part by holomorphy.
The Fourier terms $F_N \cdot M_N(\xi, \nu)$ depend holomorphically on $\nu$, since they are defined by integration over compact sets. For almost all Fourier term orders $N'$ we have for $\Re \nu > 2$

$$F_N \cdot M_N(\xi, \nu) \in W_{N', b, p, q}^{\xi, \nu}.$$ 

Since $\nu \mapsto W_{N', b, p, q}^{\xi, \nu}$ is a holomorphic family of one-dimensional subspaces of $C^\infty(\Gamma_N \backslash G)_{b, p, q}$ this relation is preserved by holomorphy. Thus, we obtain part i).

If $M_N(\xi, \nu)$ has a singularity at $\nu = \nu_0$, then by (5.12), its main part at $\nu_0$ is a linear combination of basis functions $f_i \in L^2, \text{discr}(\Gamma \backslash G)_{h, p, q}$ which lie in $A^\infty(\psi)$ for a character of $ZU(q)$ that satisfies $\psi_f(C) = \lambda_2(\xi, \nu_0)$ and $\psi_f(\Delta_3) = \lambda_3(\xi, \nu_0)$. By i), the only $f_i$ that can occur satisfy $\psi_i = \psi[\xi, \nu_0]$. This gives ii).

Proposition 5.4 implies the assertion in Theorem 5.1 that $M_N(\xi, \nu)$ is a family of automorphic forms with moderate exponential growth. It also proves part (i) of the theorem, and gives the additional information that the main part is in $A^\infty(\psi[\xi, \nu_0])$.

Having shown that the family is locally in $C^\infty(\Omega \times G)$, it follows that the intertwining property in part ii) of the theorem stays valid under meromorphic continuation. This completes the proof of Theorem 5.1.

6. Statements of completeness results

For cuspidal Maass wave forms on $\Gamma \backslash SL_2(\mathbb{R})$ all square integrable automorphic forms arise from Poincaré series. The spaces of holomorphic cusp forms are spanned by absolutely convergent Poincaré series. Spaces of real analytic Maass forms are obtained by resolving meromorphically continued Poincaré series. Theorem 3.2 in [14] gives a generalization of the latter results to Lie groups of real rank one under restrictions on the isomorphism class of the $(\mathfrak{g}, K)$-module generated by the square integrable automorphic form.

For $SU(2, 1)$ not all square integrable automorphic forms are values or residues of Poincaré series. However, we will see that in all cases when this does not hold, a non-zero square integrable automorphic form can be detected by using the family of scalar products $\nu \mapsto (f, M_N(\xi, \nu))_{\Gamma \backslash G}$ with a suitable family of truncated Poincaré series. As a preparation for a more precise formulation, we start with a discussion on $(\mathfrak{g}, K)$-modules generated by square integrable automorphic forms.

6.1. Preparation. As in Definition 5.3, given a non-zero meromorphic family $\nu \mapsto F(\nu)$ of functions in $C^\\infty(G)$ and $\nu_0 \in \mathbb{C}$, the main part of $F$ at $\nu_0$ is defined as the non-zero function

$$(6.1) \quad \lim_{\nu \to \nu_0} (\nu - \nu_0)^m F(\nu) \in C^\\infty(G),$$

where $m$ is the order of $F$ at $\nu_0$, i.e. the unique integer such that the limit exists and is non-zero. For the family that is identically zero we define the order to be $-\infty$ and the main part to be zero.

We will work with Poincaré families for Fourier term orders $N \neq N_0$. We allow $\nu \mapsto P^\\infty M_N(\xi, \nu)$ (and hence $\nu \mapsto PM_N(\xi, \nu)$) to be identically zero.

As before, for $\nu_0$ with $\Re \nu_0 \geq 0$, we denote the main part at $\nu_0$ of Poincaré families $M_N(\xi, \nu)$ by $P_N(\xi, \nu_0)$. For truncated Poincaré families $\tilde{M}_N(\xi, \nu)$, it will
be denoted by $\tilde{P}_N(\xi, \nu_0)$. We denote by $\text{po}_N(\xi, \nu_0)$ the order of $\nu \mapsto \text{M}_N(\xi, \nu)$ at $\nu_0$. If $\text{po}_N(\nu_0) \leq 0$, the order of the truncated family may depend on the choice of the cut-off function.

We note that for singularities at $\nu = \nu_0$ we have $P_N(\xi, \nu_0) = \tilde{P}_N(\xi, \nu_0)$, independently of the cut-off function used for truncation. This follows from the fact that $\text{M}_N(\xi, \nu) - \tilde{\text{M}}_N(\xi, \nu)$ is holomorphic at $\nu \in \mathbb{C} \setminus \mathbb{Z}_{\leq -1}$.

6.1.1. Modules of square integrable automorphic forms. Let $\psi$ be a character of $ZU(g)$. There are only finitely many isomorphism classes of $(g, K)$-modules in which $ZU(g)$ acts according to a given character. The $K$-types in these different isomorphism classes are disjoint (see for instance [2, Proposition 12.2]). Hence $A^{(2)}(\psi)$ is the direct sum of finitely many irreducible $(g, K)$-modules, and for a given $K$-type $\tau^h_0$ the space $A^{(2)}(\psi)_{h, p}$ generates a finite sum of irreducible $(g, K)$-modules with isomorphism class determined by $\psi$ and $\tau^h_0$.

We leave the one-dimensional $(g, K)$-module out of our consideration. It consists of constant functions, which occur as residues of Eisenstein series. We proceed with an infinite-dimensional irreducible $(g, K)$-module $V$ contained in $A^{(2)}(\psi)$. Since it does not consist of constant functions, some of the Fourier term operators

$$F_N : V \rightarrow \mathcal{W}_N(\psi)$$

with $N \neq N_0$ must be non-zero.

We will need a number of quantities, determined by the isomorphism class of $V$:

- A specific choice of spectral parameters for the character $\psi$ of $ZU(g)$: a character $\xi$ of $M$ and a complex number $v_0$ in the closed right half plane.
- The minimal $K$-type $\tau^h_0$ of $V$.

Table 1 gives a list of relevant isomorphism classes and the corresponding values of the 4-tuple $(\xi, v_0, h_0, p_0)$.

We fix such a 4-tuple and consider square integrable automorphic forms $f \in A^{(2)}(\psi[\xi, v_0])_{h_0, p_0, q_0}$. Each non-zero function $f$ in this space generates a $(g, K)$-module $V_f \subset A^{(2)}(\psi[\xi, v_0])$. These modules $V_f$ are all submodules of $L^2, \text{disc} \{ \Gamma \setminus G \}$ in the isomorphism class specified by $(\xi, v_0, h_0, p_0)$.

Given the family of basis functions $\nu \mapsto \Omega_N(\xi, \nu) = \Omega_{N; h_0, p_0, q_0}(\xi, \nu)$ in Proposition B.1, we define the Fourier coefficient $c_N(f)$ by

$$F_N f = c_N(f) \Omega_N(\xi, \nu_0).$$

6.2. General completeness result. We can now state the relation between the non-vanishing of Fourier term operators and Poincaré series, valid in the context indicated above.

**Theorem 6.1.** Let $V$ be an infinite-dimensional irreducible $(g, K)$-module of square integrable automorphic forms, with isomorphism type determined by the quadruple $(\xi, v_0, h_0, p_0)$, as indicated in Table 1. For each Fourier term order $N \neq N_0$ we use the Poincaré family $\text{M}_N(\xi, \nu) = \text{P}_N \text{M}_{N; h_0, p_0, q_0}(\xi, \nu)$. Let $f \in V_{h_0, p_0, q_0}$ be non-zero. Then the following statements are equivalent:

i) The Fourier coefficient $c_N(f)$ is non-zero.
| Type                        | Parameters                                                                 |
|-----------------------------|-----------------------------------------------------------------------------|
| unitary principal series    | \(H(j, \nu_0)\) \(\nu_0 \in i[0, \infty)\)                                 |
|                            | if \(\nu_0 = 0\), then \(j \in \{0\} \cup 1 + 2\mathbb{Z}\) \(h_0 = 2j, p_0 = 0\) |
| complementary series        | \(H(j, \nu_0)\) \(0 < \nu_0 < 2, j = 0\) or \(0 < \nu_0 < 1, j \in 1 + 2\mathbb{Z}\) |
|                            | \(h_0 = 2j, p_0 = 0\)                                                    |

| Discrete series type        |                                                                                     |
|----------------------------|------------------------------------------------------------------------------------|
| large d.s.t.                | \(H_+(j, \nu_0)\) \(\nu_0 \equiv j \mod 2, \nu_0 \geq |j|, \nu_0 \geq 1\) \(h_0 = -j, p_0 = \nu_0\) |
| holomorphic d.s.t.          | \(IF(j, \nu_0)\) \(\nu_0 \equiv j \mod 2, 0 \leq \nu_0 \leq j - 2\) \(h_0 = 2j, p_0 = 0\) |
| antiholomorphic d.s.t.      | \(FI(j, \nu_0)\) \(\nu_0 \equiv j \mod 2, 0 \leq \nu_0 \leq -j - 2\) \(h_0 = 2j, p_0 = 0\) |

| Thin representations (non-tempered) |                                                                                     |
|-------------------------------------|------------------------------------------------------------------------------------|
| \(T_1^+\) \(k \in \mathbb{Z}_{>0}\) | \(IF(1, -1)\) \(j = 1, \nu_0 = 1\) \(h_0 = 2, p_0 = 0\)                          |
| \(T_k^+\) \(k \in \mathbb{Z}_{>0}\) | \(IF_+(j, -1)\) \(\nu_0 = 1, j = 3 + 2k\) \(h_0 = k + 3, p_0 = k + 1\)         |
| \(T_{-1}^-\) \(k \in \mathbb{Z}_{>0}\) | \(IF(-1, -1)\) \(j = 1, \nu_0 = 1\) \(h_0 = -2, p_0 = 0\)                     |
| \(T_k^-\) \(k \in \mathbb{Z}_{>0}\) | \(IF_+(j, -1)\) \(\nu_0 = 1, j = -3 - 2k\) \(h_0 = -k - 3, p_0 = k + 1\)      |

Table 1. List of isomorphism classes of infinite-dimensional irreducible \((g, \mathcal{K})\)-modules with a unitary structure, from [2, Theorem 16.1]. We use the notation for the isomorphism types given in [2, §12.2].

We give spectral parameters \((j, \nu_0) = (j_\xi, \nu_0)\). The Weyl group orbit of \((j, \nu_0)\) determines a character \(\psi = \psi(j, \nu_0)\) of \(ZU(g)\). We give also the \(K\)-type \(\tau_{p_0}^{j_0}\) with minimal dimension \(p_0 + 1\) occurring in the module (and generating it).

ii) The scalar product \((f, \check{P}_N(\xi, \nu_0))_{\Gamma \setminus G} \neq 0\) for some cut-off function \(\varphi\). Here \(\check{P}_N(\xi, \nu_0)\) is the main part at \(\nu_0\) of the truncated Poincaré family \(\nu \mapsto P_N(\varphi M_N(\xi, \nu))\).

In ii), in many cases depending on \(V\), one has that \(P_N(\xi, \nu_0) = \check{P}_N(\xi, \nu_0)\), independently of the cut-off function.

Remarks. (1) See Proposition B.1 for the family \(M_N(\xi, \nu) = M_N(h_0, p_0, \nu_0(\xi, \nu))\). For the completeness result we use Poincaré families with minimal \(K\)-type \(\tau_{p_0}^{j_0}\) of \(V\), and maximal weight \(p_0\) in that \(K\)-type. We will often omit the index \((h_0, p_0, p_0)\) from the notation.
(2) The spaces \( V_{h_0,p_0,p_0} \) and \( \mathcal{W}_{\mathcal{N},h_0,p_0,p_0}^{(2)} \) with minimal \( K \)-type \( T^{h_0}_{p_0} \) and weight \( p_0 \) have dimension one, so the Fourier term operator \( F_\mathcal{N} \) is determined by the constant \( c_\mathcal{N}(f) \).

(3) Since \( V \) is non-zero and is not the trivial module, there are Fourier term orders \( \mathcal{N} \) for which statement i) holds. This implies that the scalar products in statement ii) detect all non-zero elements in \( A^{(2)}(\psi)^{h_0,p_0,p_0} \).

The proof of Theorem 6.1 will be completed in Subsection 7.4.

6.3. Singularities and zeros of Poincaré families. The completeness result Theorem 6.1 puts the emphasis on square integrable automorphic forms. Now we put the emphasis on Poincaré families and their singularities and zeros at a point \( v_0 \).

We handle not all isomorphism classes in Table 1 together, but consider the various cases separately. Given \( (\xi,v_0,h_0,p_0) \), we consider the behavior of the associated meromorphic family \( \nu \mapsto \mathcal{M}_\mathcal{N}(\xi,v) = \mathcal{M}_\mathcal{N}(\xi) \) at \( v = v_0 \).

The proofs of the following propositions are in §7.6.

Proposition 6.2. Let \( (\xi,v_0,h_0,p_0) \) be the parameters of a representation in the irreducible unitary principal series, the complementary series, or the thin representations \( T_{-1}^+ \) or \( T_{-1}^- \). In particular \( h_0 = 2j_\xi \) and \( p_0 = 0 \). Let \( \mathcal{N} \) be a generic abelian or non-abelian Fourier term order. If \( v_0 \neq 0 \) (resp. \( v_0 = 0 \)), then either the Poincaré family \( \nu \mapsto \mathcal{M}_\mathcal{N}(\xi,v) \) is holomorphic at \( v = v_0 \) or it has a singularity of order one (resp. 2).

a) The family \( \nu \mapsto \mathcal{M}_\mathcal{N}(\xi,v) \) has a singularity at \( v = v_0 \) if and only if there are \( f \in A^{(2)}(\psi)_{2j_\xi,0,0} \) with \( c_\mathcal{N}(f) \neq 0 \).

b) If the family is holomorphic at \( v = v_0 \) and is not identically zero, then \( \mathcal{M}_\mathcal{N}(\xi,v_0) \) in \( A^{(2)}(\psi)_{2j_\xi,0,0} \) is not square integrable.

Proposition 6.2 a) couples the presence of a singularity to the presence of square integrable automorphic forms for which the specific Fourier coefficient \( c_\mathcal{N} \) does not vanish. If there is no singularity at \( v_0 \), there may exist non-zero square integrable automorphic forms in \( A^{(2)}(\psi)_{2j_\xi,0,0} \), but \( c_\mathcal{N} \) vanishes for all of them.

Proposition 6.3. Let \( (\xi,v_0,h_0,p_0) \) be parameters of a representation of large discrete series type or of a thin representation \( T_k^\pm \) with \( k \in \mathbb{Z}_{\geq 0} \). In particular \( h_0 = -j_\xi \) and \( p_0 = v_0 \in \mathbb{Z}_{\geq 1} \) for the large discrete series and \( h_0 = \pm(k+3), \) \( p_0 = k+1 \) for \( T_k^\pm \).

If the Poincaré family \( \nu \mapsto \mathcal{M}_\mathcal{N}(\xi,v) \) is not identically zero, then it is holomorphic at \( v = v_0 \), with a value at \( v_0 \) that is not square integrable. Furthermore, in this case there are cut-off functions \( \varphi \) such that for any \( f \in A^{(2)}(\psi)_{h_0,p_0,p_0} \)

\[
c_\mathcal{N}(f) \neq 0 \iff \left( \mathcal{P}_\mathcal{N}(\varphi \mathcal{M}_\mathcal{N}(\xi,v_0)), f \right) \neq 0.
\]

Proposition 6.4. Let \( (\xi,v_0,h_0,p_0) \) be parameters of a representation of holomorphic or antiholomorphic discrete series type. In particular, \( h_0 = 2j_\xi \), \( p_0 = 0 \). Let \( \mathcal{N} = \mathcal{N}_n \) be a non-abelian Fourier term order.

i) If \( v_0 \geq 1 \), the Poincaré family \( \nu \mapsto \mathcal{M}_n(\xi,v) \) is holomorphic at \( v_0 \), and its value \( \mathcal{M}_n(\xi,v_0) \) is in \( A^{(2)}(\psi)_{2j_\xi,0,0} \).
a) The value $M_n(\xi, \nu_0)$ is non-zero if and only if there are square integrable automorphic forms $f \in A^2(\psi_{2\xi,0})$ for which $c_N(f) \neq 0$.

b) If $M_n(\xi, \nu_0) = 0$ and the family is not identically zero, then the family has a first order zero at $\nu = \nu_0$ and $\partial_\nu M_n(\xi, \nu)|_{\nu=\nu_0}$ is not square integrable.

ii) If $\nu_0 = 0$, then $M_n(\xi, \nu)$ can have at most a first order singularity at $\nu_0$.

Furthermore:

a) The family $\nu \mapsto M_n(\xi, \nu)$ has a first order pole at $\nu = 0$ if and only if there are $f \in A^2(\psi_{2\xi,0})$ with $c_N(f) \neq 0$.

b) If the family is holomorphic at $\nu = 0$ and not identically zero, then $M_n(\xi, 0) = 0$, and $\partial_\nu M_n(\xi, \nu)|_{\nu=0} \in A'(\psi_{2\xi,0}) \setminus A^2(\psi_{2\xi,0})$.

**Remark.** Part i), with $\nu_0 \in \mathbb{Z}_{\geq 1}$, refers to discrete series representations. In part ii) we deal with limits of discrete series, occurring as direct summands of reducible principal series representations. The proofs of the propositions will be given in Section 7 (see Subsection 7.6).

### 6.4. Comparison.

In [14] Poincaré series attached to a character $\chi_\beta$ of $\Lambda_\gamma \setminus N$ were studied for $G$ semisimple Lie group of real rank one. It was proved that the meromorphically continued family $M_\beta(\xi, \nu)$ has at most simple poles located at spectral points $\nu_0$, except for $\nu_0 = 0$ where it may have a double pole. In particular, if $\nu_0 \neq 0$, under suitable conditions on the parameters, for any square integrable automorphic form $f$ one has the relation

$$\left(f, \text{Res}_{\nu=\nu_0} M_\beta(\xi, \nu)\right) \overset{\cdot}{=} c_\beta(f),$$

where $\cdot$ indicates equality up to a non-zero constant. That is, the residues of the Poincaré families $M_\beta(\xi, \cdot)$ detect all cusp forms having a non-zero abelian Fourier coefficient. There is a similar result for $\nu_0 = 0$. However, these results do not give information on non-abelian Fourier terms, and in particular, on non-generic automorphic forms, i.e. those having all abelian coefficients $c_\beta(f)$ equal to zero. Thus, for the group $SU(2, 1)$, they do not include cusp forms generating a $(g, K)$-module lying in the holomorphic (or antiholomorphic) discrete series or in the thin representations $T^\pm_k$, (with $k \geq -1$), which do not admit abelian Whittaker models.

In Theorem 6.1 we prove a version of (6.4) that involves both, abelian and non-abelian Fourier terms, showing that the main parts of the corresponding Poincaré series (in some cases their truncated main parts) do detect the whole space of cusp forms.

If we compare with the classical results for $SL(2, \mathbb{R})$ ([17], [15],[16],[19]), we see that there are similarities and some differences.

In the case of the unitary principal series and the complementary series the results in Proposition 6.2 are similar to the results for Maass forms of weight zero.

In the case of holomorphic and antiholomorphic series, for $\nu_0 \in \mathbb{Z}_{\geq 1}$ the absolutely convergent Poincaré series span the corresponding spaces of square integrable automorphic forms. This extends to $\nu_0 = 2$ and $\nu_0 = 1$ by working with the values of the analytically continued Poincaré series. In the case when $\nu_0 = 0$, we deal with limits of discrete series, occurring as direct summands of reducible principal series representations. The proofs of the propositions will be given in Section 7 (see Subsection 7.6).
we deal with limits of holomorphic or antiholomorphic discrete series, and, for the detection, we need the residues at the first order singularities of the non-abelian Poincaré series. All this is similar to what happens with holomorphic automorphic forms on the upper half-plane. For weights \( k \in \mathbb{Z}_{\geq 2} \) the values of Poincaré series span the spaces of integrable automorphic forms, with absolute convergence for \( k \geq 3 \) and conditional convergence for \( k = 2 \). Weight 1 corresponds to \( \nu_0 = 0 \). Then we need residues of Poincaré families to span the spaces of square integrable automorphic forms. In Appendix D we give a more detailed discussion following [1].

There are also differences between \( \text{SL}_2(\mathbb{R}) \) and \( \text{SU}(2,1) \). Maximal unipotent subgroups of \( \text{SL}_2(\mathbb{R}) \) are abelian, and the Fourier expansions are simpler than for \( \text{SU}(2,1) \). Since \( \text{SL}_2(\mathbb{R}) \) has a non-trivial center, we need to work with automorphic forms with a character of \( \Gamma \) when we consider odd weights.

The cases of the large discrete series and of the representations \( T_{\pm}^k \) do not have analogues for \( \text{SL}(2,\mathbb{R}) \). For \( \text{SL}_2(\mathbb{R}) \) the square integrable automorphic forms are always obtained from values or by resolving singularities of Poincaré series. We do not need statements like in Proposition 6.3, where we had to use scalar products of automorphic forms with truncated Poincaré families.

7. Proofs of completeness results

7.1. Scalar product formulas and Wronskian order. The results to prove depend on spectral parameters \((\xi, \nu_0)\) and on a \( K \)-type \( \tau_{\mathbb{Q}_k} \) that together determine an irreducible class of irreducible \((g, K)\)-modules in Table 1. We first state and prove some facts that are valid more generally. Proposition 7.1 is valid for an arbitrary \( k \)-type \( \tau_{\mathbb{Q}_k} \) and a weight \( q \) in that \( K \)-type. In Subsection 7.3 we assume that the \( K \)-type \( \tau_{\mathbb{Q}_k} \) occurs in \( L^2, \text{discr}(\Gamma \setminus G) \). For a square integrable automorphic form \( f \) and a Fourier term order \( \mathcal{N} \neq \mathcal{N}_0 \), the Fourier coefficient \( c_{\mathcal{N}}(f) \) is defined as in (6.3) by the equation

\[
F_{\mathcal{N}} f = c_{\mathcal{N}}(f) \Omega_{\mathcal{N}}(\xi, \nu_0).
\]

The first main point in this section is to obtain the relation (7.18), which expresses the scalar product \( (\mathbf{M}_N(j, \nu), f)_{\Gamma \setminus G} \) as the product of \( c_{\mathcal{N}}(f) \) and a quantity that is a meromorphic function of \( \nu \), not depending on \( f \). After that, we specialize to \((h, p, q) = (h_0, p_0, p_0)\) and proceed to prove the results in Section 6. We shall make use of Theorem 7.8, concerning Wronskian orders. Comparison of the orders with the Wronskian orders will lead to a proof of Theorem 6.1.

7.2. Sesquilinear forms. The relations we will obtain in Proposition 7.1 below are valid in more generality than we need it. In this subsection the \( K \)-type \( \tau_{\mathbb{Q}_k} \) is general.
Given $F_1, F_2 \in L^2(\Lambda, \sigma | G_K)$ we have

$$(F_1, F_2)_{\Gamma_N} = \int_{n \in \Lambda, \sigma | N} \int_{t=0}^{\infty} \int_{k \in K} F_1(na(t)k) \overline{F_2(na(t)k)} \, dn \frac{dt}{t^5} \, dk$$

(7.2)

$$= \int_{t=0}^{\infty} \{F_1, F_2\}(t) \frac{dt}{t^5},$$

where we use the sesquilinear form

$$(F_1, F_2) \mapsto \{F_1, F_2\}(t) = \int_{\Lambda, \sigma | N} \int_{K} F_1(na(t)k) \overline{F_2(na(t)k)} \, dn \, dk$$

(7.3)

on $C^\infty(\Lambda, \sigma | G_K)$ with values in $C^\infty(0, \infty)$. The integration is over compact sets, so the sesquilinear map is well-defined.

We shall also use the Maass-Selberg form

$\text{MS}(F_1, F_2)(t) = \{F_1, CF_2\}(t) - \{CF_1, F_2\}(t).$

(7.4)

Functions in $C^\infty(\Lambda, \sigma | G)_{h, p, q}$ of weight $q$ in the $K$-type $\tau^h_p$ that transform on the left according to the same representation of $N$ have an expansion

$$f(na(t)k) = \sum_r w_r(n) f_r(t) \Phi_{r, q}^p(k),$$

(7.5)

where $r$ runs over $|r| \leq p$, $r \equiv p \text{ mod } 2$. If $N = N_{\beta}$, then all $w_r$ are equal to the character $\chi_{\beta}$ of $N$ in (2.13). If $N = N_{\ell, c, d}$ then the $w_r$ are theta-functions $\Theta_{\ell, c}(h_{\ell, m})$ as in §2.4.2. They are mutually orthogonal in $L^2(\Lambda, \sigma | N)$ and have norm $\sqrt{2/\sigma}$. The $\Phi_{r, q}^p$ are elements of an orthogonal system in $L^2(K)$, with norms satisfying relation (A.10).

We compute

$$\{F_1, F_2\}(t) = \int_{n \in \Lambda, \sigma | N} \int_{k \in K} \sum_r w_r(n) f_{1, r}(t) \Phi_{r, q}^p(k) \cdot \sum_r w_r(n) f_{2, r}(t) \overline{\Phi_{r, q}^p(k)} \, dn \, dk$$

Using the properties of the functions $w_r$ and $\Phi_{r, q}^p$, we arrive at

$$\{F_1, F_2\}(t) = \frac{2}{\sigma} \| \Phi_{p, q}^p \|_K^2 \sum_r \left(\frac{p}{p+r}\right) f_{1, r}(t) \overline{f_{2, r}(t)}.$$

(7.6)

We apply this formula to get an expression for the Maass-Selberg form.

**Proposition 7.1.** Let $F_1, F_2 \in C^\infty(\Lambda, \sigma | G)_{h, p, q}$ have expansions as in (7.5). Consider the generalized Wronskian for $h_1, h_2 \in C^\infty(0, \infty)$ and $n \in \mathbb{Z}_{\geq 1}$, given by

$$\text{Wr}_n(h_1, h_2)(t) = h_1(t) h_2^{(n)}(t) - h_1^{(n)}(t) h_2(t).$$

(7.7)
Then
\[
\text{MS}(F_1, F_2)(t) = \frac{2}{\sigma} \left\| h\Phi_{p,q}^p \right\|^2_K \sum_{r=p(2), |r| \leq p} \left( \frac{p}{p+r} \right)^2 \frac{d}{dt} \left( t^3 \text{W}(F_1, F_2)(t) \right),
\]
(7.8)
with
\[
\text{W}(F_1, F_2)(t) = \sum_{r=p(2), |r| \leq p} \left( \frac{p}{p+r} \right) \text{Wr}(f_{1,r}, f_{2,r})(t).
\]
(7.9)
The form \( \text{Wr} = \text{Wr}_1 \) is the usual Wronskian.

**Proof.** The action of the Casimir element \( C \) on \( F_1 \) and \( F_2 \) can be made explicit in the components \( f_{j,r} \) with the Mathematica routines discussed in [2, §9.2]. This gives the first relation in (7.8). The second equality in (7.8) becomes clear in a direct check. □

### 7.3. Scalar product formulas

Our goal in this subsection is to prove two useful relations. We will work with the basis families \( \nu \mapsto \mathcal{M}_\nu(\xi) = \mathcal{M}_\nu(\mathcal{M}, \nu) \), and the Poincaré family \( \nu \mapsto \mathbf{P}(\xi) \mathcal{M}_\nu(\mathcal{M}, \nu) = \mathcal{M}_\nu(\xi) \). Given a square integrable automorphic form \( f \), the Fourier coefficients \( c_N(f) \) are defined by the identity
\[
\mathbf{F}_N f = c_N(f) \Omega_N(\xi, \nu_0) \quad (N \neq N_0).
\]
See Proposition B.1 for the basis function \( \Omega_N \).

**Lemma 7.2.** For \( \text{Re} \nu \) sufficiently large
\[
\mathbf{M}_N(\mathcal{M}, \nu) f |_{\mathcal{M}, G} = c_N(f) \int_0^\infty \varphi(t) \left\{ \mathcal{M}_N(\xi, \nu)(t) \Omega_N(\xi, \nu_0)(t) \right\} \frac{dt}{t^5}.
\]
(7.11)

**Proof.** The automorphic form \( f \) has a decomposition into Iwasawa coordinates
\[
f(nak) = \sum_r f_r(na) \ h\Phi_{r,q}^p(k),
\]
(7.12)
with components \( f_r \in C^\infty(\mathbb{A}_\nu \backslash \mathbb{N}A) \). The sum is over \( r \equiv p \) mod 2, \( |r| \leq p \).

The function \( \Omega_N(\xi, \nu) \) has also a description in components:
\[
\Omega_N(\xi, \nu)(nak) = \sum_r w_r(n) \ h\Phi_{r,q}^p(k),
\]
(7.13)
where the functions \( h\Phi_{r,q}^p(\nu, \cdot) \) are linear combinations of \( K \)-Bessel functions or \( W \)-Whittaker functions with exponential decay as \( t \to \infty \), that satisfy an estimate \( O(t^{-\alpha_t}) \) as \( t \to 0 \) for some \( \alpha_t > 0 \). See (B.4).

In the abelian case the Fourier term in [2, (5.5)] is
\[
\mathbf{F}_\beta f(na(t)k) = \frac{2}{\sigma} \int_{\nu' \in \mathbb{A}_\nu \backslash \mathbb{N}} \chi_{\beta}(\nu') \sum_r f_r(n'na(t)) \ h\Phi_{r,q}^p(k) d\nu'
\]
Using (7.10) and the notation in (B.3), we see that the left-hand side is equal to
\[ c_\beta(f) \Omega_\beta(\xi, v_0)(na(t)k) = c_\beta(f) \sum_h \chi_\beta(n) h_\nu^\omega(v_0, t) h_\nu^\rho(k). \]
This gives
\[ c_\beta(f) h_\nu^\mu(v_0, t) = \frac{2}{\sigma} \int_{n' \in \Lambda_r \setminus N} \chi_\beta(n') f_r(na(t)) \, dn'. \]
In the non-abelian case we use [2, (8.21)], with \( n = (\ell, c, d) \), \( w_r = \Theta_{\ell, c}(h_{\ell, m(h, r)}) \), and \( m(h, r) \) as in (B.8).
\[ F_n f(na(t)k) = \sum_r w_r(n) \frac{h_\nu^\rho(k)}{\| h_\nu^\rho \|^2} 2 \int_{n' \in \Lambda_r \setminus N} \int_{k' \in K} w_r(n') \cdot \sum_{r'} f_r(n'a(t)) h_\nu^\rho(k') h_\nu^\rho(k') \, dk' \, dn'. \]
\[ = \frac{2}{\sigma} \sum_r w_r(n) h_\nu^\rho(k) \int_{n' \in \Lambda_r \setminus N} w_r(n') f_r(na(t)) \, dn'. \]
We compare this with
\[ F_n f = c_n(f) \Omega_n(\xi, v_0), \]
to obtain
\[ c_n(f) h_\nu^\mu(v_0, t) = \frac{2}{\sigma} \int_{n' \in \Lambda_r \setminus N} w_r(n) f_r(na(t)) \, dn', \]
which is similar to the expression in the abelian case.
For the function \( M_N(\xi, \nu) \) as well, we use the notation in (B.3):
\[ M_N(\xi, \nu)(na(t)k) = \sum_r w_r(n) h_\nu^\mu(v, t) h_\nu^\rho(k), \]
with \( h_\nu^\mu \) expressed in \( I \)-Bessel functions or \( M \)-Whittaker functions, and with \( w_r \) as above. We have
\[ h_\nu^\mu(v, t) = O(t^{2+\nu}). \]
Now we take \( \text{Re } \nu > 2 \) to have convergence of the Poincaré series, and \( \text{Re } \nu > \alpha_r \) in the exponent of the estimate for \( h_\nu^\mu(v_0, t) \). We take apart the absolutely convergent Poincaré series and with the use of (7.6) we arrive at the scalar product formula in the lemma. Indeed
\[ (\tilde{M}_N(\xi, \nu), f)_{\Gamma|G} = \int_{\Gamma|G} \sum_{g \in \tilde{G}} \varphi(\gamma g)M_N(\xi, \nu)(\gamma g) f(g) \, dg \]
\[ = \int_{\tilde{G}} \varphi(g) M_N(\xi, \nu)(g) f(g) \, dg, \]
the derivatives. Enlarging Re \( \sigma \) have (C.2) and (C.7). Using the contiguous relations in Appendix C we can handle expressions in modified Bessel functions or Whittaker functions for which we have (7.18) and (7.19). The left-hand side depends meromorphically on \( \nu \). The components \( h_i^\nu(v, t) \) and \( h_i^\nu(v_0, t) \) and their derivatives, with expressions in modified Bessel functions or Whittaker functions for which we have (C.2) and (C.7). Using the contiguous relations in Appendix C we can handle the derivatives. Enlarging Re \( \nu \) sufficiently, we arrive at (7.18) in the proposition.

The left-hand side depends meromorphically on \( \nu \). The integral in the right-hand side is over a compact set in \((0, \infty)\) and we thus get its meromorphic continuation with singularities only in \( \mathbb{Z}_{\leq -1} \), due to the Wronskian functions.

**Proposition 7.3.** Let \( N \neq N_0 \), and let \( f \in \mathcal{A}^2(\xi, v_0) \). Then we have

\[
\left( \mathcal{M}_N(\xi, v), f \right)_{\Gamma \backslash G} = \sum_{r} \frac{1}{\nu^2 - \nu_0^2} \int_{\tau} \varphi(t) \left( \mathcal{W}(\mathcal{M}_N(\xi, v), \Omega_N(\xi, v_0))(t) - \mathcal{W}(\mathcal{M}_N(\xi, v), \Omega_N(\xi, v_0))(t) \right) dt,
\]

as an identity of meromorphic functions in \( \nu \in \mathbb{C} \). The singularities of the function \( \nu \mapsto \mathcal{W}(\mathcal{M}_N(\nu), \Omega_N(\nu)) \) are contained in the set \( \mathbb{Z}_{\leq -1} \).

**Proof.** Since \( \mathcal{M}_N(\xi, v) \) and \( \Omega_N(\xi, v_0) \) are eigenfunctions of the Casimir element \( C \),

we have

\[
\text{MS}(\mathcal{M}_N(\xi, v), \Omega_N(\xi, v_0))(t) = (\nu^2 - \nu_0^2) \left\{ \mathcal{M}_N(\xi, v), \Omega_N(\xi, v_0) \right\}(t).
\]

Applying Lemma 7.2 and (7.8) we get for Re \( \nu \) sufficiently large

\[
\left( \mathcal{M}_N(\xi, v), f \right)_{\Gamma \backslash G} = \sum_{r} \frac{1}{\nu^2 - \nu_0^2} \int_{\tau} \varphi(t) \left( \mathcal{W}(\mathcal{M}_N(\xi, v), \Omega_N(\xi, v_0))(t) - \mathcal{W}(\mathcal{M}_N(\xi, v), \Omega_N(\xi, v_0))(t) \right) dt
\]

To apply partial integration, the Wronskian function has to have sufficient decay at zero. This is based on the components \( h_i^\nu(v, t) \) and \( h_i^\nu(v_0, t) \) and their derivatives, with expressions in modified Bessel functions or Whittaker functions for which we have (C.2) and (C.7). Using the contiguous relations in Appendix C we can handle the derivatives. Enlarging Re \( \nu \) sufficiently, we arrive at (7.18) in the proposition.

The left-hand side depends meromorphically on \( \nu \). The integral in the right-hand side is over a compact set in \((0, \infty)\) and we thus get its meromorphic continuation with singularities only in \( \mathbb{Z}_{\leq -1} \), due to the Wronskian functions.

**Proof of Theorem 6.1.** The results up till here hold for any square integrable automorphic form \( f \in \mathcal{A}^2(\xi, v_0) \). The isomorphism class of the irreducible module \( V_f = U(g) f \) is determined by the spectral parameters \( (\xi, v_0) \) and the \( K \)-type. To use Proposition 7.3 we have to study the Wronskian function \( \mathcal{W}(\mathcal{M}_N(\xi, v), \Omega_N(\xi, v_0)) \). We now proceed with the minimal \( K \)-type \( \tau_{\rho_0}^k \) in the irreducible module under consideration in order to use the explicit formulas available for that \( K \)-type.
Let $V$ be an irreducible submodule of $A^2(\psi[\xi, v_0])$, with isomorphism class characterized by $(\xi, v_0, h_0, p_0)$ as indicated in Table 1. This submodule is of the form $V_f$ for any non-zero $f \in A^2(\psi[\xi, v_0])_{h_0, p_0, p_0}$.

For $\mathcal{N} \neq \mathcal{N}_0$ we put

$$I_{\mathcal{N}}(\varphi; v, v_0) = \int_{T=0}^{\infty} \varphi'(t) W(M_N(\xi, \nu), \Omega_{\mathcal{N}}(\xi, v_0))(t) \frac{dt}{t}. \tag{7.20}$$

The parameters $\xi, h_0$ and $p_0$ are not visible in the notation.

At points $\nu = v_0$ with $\text{Re} \ v_0 \geq 0$ the function $\nu \mapsto I_{\mathcal{N}}(\varphi; v, v_0)$ is holomorphic. So the order at $\nu = v_0$ of the function

$$\nu \mapsto \frac{1}{v^2 - v_0^2} I_{\mathcal{N}}(\varphi; v, v_0) \tag{7.21}$$

is at most 1 if $v_0 \neq 0$, and at most 2 if $v_0 = 0$. The cut-off function $\varphi$ may have an influence on this order.

**Definition 7.4.** The Wronskian order, denoted by $\omega_{\mathcal{N}}(v_0)$, is the maximum of the orders of the functions (7.21) as $\varphi$ varies over all cut-off functions. A cut-off function $\varphi$ is said to be good if the order of the function in (7.21) coincides with the Wronskian order.

**Lemma 7.5.** For each $T > 0$ there is a good cut-off function $\varphi_T$ for $\omega_{\mathcal{N}}(v_0)$ such that $\varphi_T = 1$ on $[0, T]$, $\varphi_T(t) \geq 0$ for $T \leq t \leq T + 1$ and $\varphi_T = 0$ on $[T + 1, \infty)$.

**Proof.** The function $(\nu, t) \mapsto W(M_N(\xi, \nu), \Omega_{\mathcal{N}}(\xi, v_0))(t)$ is of the form $(\nu, t) \mapsto (\nu - v_0)^u A(\nu, t)$, with $u \in \mathbb{Z}$ and $A \in C^0(U \times (0, \infty))$ a real-analytic function which is holomorphic in $\nu$ on a neighborhood $U$ of $v_0$ in $\mathbb{C}$, and such that $t \mapsto A(\nu_0, t)$ is not identically zero. Then the order of the function in (7.21) is at most $1 - u$ if $v_0 \neq 0$ and $2 - u$ if $v_0 = 0$.

For each $T > 0$ there exists $\alpha \in C^0(0, \infty)$ with $\text{Supp} \alpha \subset [T, T + 1]$, $\alpha \geq 0$, such that $\int_{T}^{\infty} \alpha(t) A(\nu_0, t) \frac{dt}{T} \neq 0$ and $\int_{T}^{T+1} \alpha(t) dt = 1$. Then $\varphi_T(t) = \int_{T}^{T+1} \alpha(x) dx$ is a cut-off function for which the order of the function in (7.21) is equal to $1 - u$ if $v_0 \neq 0$ and $2 - u$ if $v_0 = 0$. So $\varphi_T$ is a good cut-off function. \hfill \Box

Each non-zero square integrable automorphic form $f \in A^2(\psi)_{h_0, p_0, p_0}$ generates a $(\mathfrak{g}, K)$-module $V_f$ in $A^2$ in the isomorphism class specified by $(\xi, v_0, h_0, p_0)$. We now consider Poincaré families $M_{\mathcal{N}}(\xi, \nu)_{h_0, p_0, p_0}$ for a Fourier term order $\mathcal{N} \neq \mathcal{N}_0$.

**Lemma 7.6.** Let $\varphi$ be a good cut-off function. Let $\psi = \psi[\xi, v_0]$ and let $f$ be a non-zero square integrable automorphic form in $A^2(\psi)_{h_0, p_0, p_0}$.

Then the Fourier coefficient $c_{\mathcal{N}}(f)$ is non-zero if and only if the scalar product $(\hat{P}_{\mathcal{N}}(v_0), f)_{\Gamma \backslash G}$ is non-zero. Furthermore, if for some $f \in A^2(\psi)_{h_0, p_0, p_0}$ these equivalent conditions hold, then $p_{\mathcal{N}}(v_0) = \omega_{\mathcal{N}}(v_0)$.

In particular, if $M_{\mathcal{N}}(\xi, \nu)$ is not identically zero and the main part $P_{\mathcal{N}}(v_0)$ is square integrable then $c_{\mathcal{N}}(P_{\mathcal{N}}(v_0)) \neq 0$. If $v_0 = 0$ and $p_{\mathcal{N}}(0) = 2$ (resp.1) then $\omega_{\mathcal{N}}(0) = 2$ (resp.1).
Proof: Since $\tilde{M}(\xi, \nu) = P_N(\varphi M_N(\xi, \nu))$ is a meromorphic family, the scalar product in the lemma is also meromorphic in $\nu$. We use Proposition 7.3 to see that

$$\langle \tilde{M}_N(\xi, \nu), f \rangle_{\Gamma \setminus G} \doteq \frac{c_N(f)}{\nu^2 - \nu_0^2} I_N(\varphi; \nu, \nu_0).$$

(7.22)

If $c_N(f) = 0$, then the scalar product vanishes identically in $\nu$, and hence

$$\langle \tilde{P}_N(\nu_0), f \rangle_{\Gamma \setminus G} = 0.$$

If $c_N(f) \neq 0$, then

$$\langle \tilde{M}_N(\xi, \nu), f \rangle_{\Gamma \setminus G} \sim (\nu - \nu_0)^{-w_0(\nu_0)}$$

and hence

$$\langle \tilde{P}_N(\nu_0), f \rangle_{\Gamma \setminus G} \sim (\nu - \nu_0)^{\nu_0 - w_0(\nu_0)}$$

as $\nu \to \nu_0$.

We see from (7.23) that $p_o(\nu_0) < w_0(\nu_0)$ is not possible, and if $p_o(\nu_0) > w_0(\nu_0)$ then $p_o(\nu_0) = 0$ and $\tilde{P}_N(\nu_0) = P_N(\nu_0)$ is square integrable and nonzero. Now by taking $f = P_N(\nu_0)$ we have that

$$\langle P_N(\nu_0), P_N(\nu_0) \rangle_{\Gamma \setminus G} \sim \frac{c_N(P_N(\nu_0))}{(\nu - \nu_0)^{\nu_0 - w_0(\nu_0)}},$$

(7.24)

a contradiction, since the r.h.s tends to 0 as $\nu \to \nu_0$.

Therefore we must have that $p_o(\nu_0) = w_0(\nu_0)$, hence $\langle \tilde{P}_N(\nu_0), f \rangle_{\Gamma \setminus G} \sim 1$, and $\langle \tilde{P}_N(\nu_0), f \rangle_{\Gamma \setminus G}$ is non-zero, as asserted.

Suppose that $P_N(\nu_0)$ is square integrable. By the first assertion, applied with $f = P(\nu_0)$, the coefficient $c_N(P_N(\nu_0)) \neq 0$ if and only if $\langle \tilde{P}(\nu_0), P(\nu_0) \rangle \neq 0$ for all good cut-off functions.

If $p_o(\nu_0) > 0$, then $\tilde{P}_N(\nu_0) = P_N(\nu_0)$, and the statement follows.

If $p_o(\nu_0) = 0$, then $\langle \tilde{P}_N(\nu_0), P_N(\nu_0) \rangle = \langle \tilde{M}_N(\xi, \nu_0), M_N(\xi, \nu_0) \rangle$ approximates $\|M_N(\xi, \nu_0)\|^2 \neq 0$, if we take good cut-off functions $\varphi_T$ as in Lemma 7.5 and let $T \uparrow \infty$.

The case $p_o(\nu_0) < 0$ is impossible, by (7.23) and the relation $w_0(\nu_0) \geq 0$ (Theorem 7.8).

The last assertions follow by similar arguments, by taking $f = P_N(\nu_0)$ and again using (7.23).

Proof of Theorem 6.1. Let $V$ be an irreducible $(\mathfrak{g}, K)$-module generated by a square integrable automorphic form $f$. Then $F_N f = c_N(f) \Omega_N(\xi, \nu_0)$, and the vanishing of the Fourier term operator is determined by the vanishing of $c_N(f)$. Thus the theorem follows from Lemma 7.6. □

7.5. Singularities and zeros of Poincaré families. The general completeness result in Theorem 6.1 tells us about the relation between vanishing of Fourier terms of square integrable automorphic forms and its detection by truncated Poincaré series. The results in §6.3 involve the Poincaré families themselves, their singularities and zeros. We point out that the proof does not use any of the results in the previous section, since it only considers Wronskians and not Poincaré series.
Proposition 7.7. Let $N$, $j = j_0$ and $\nu_0$ be as above. Let $\psi = \psi(j, \nu_0)$.

Then, the order $\nu_0 N$ at $\nu_0$ of the Poincaré family $\nu \mapsto M_N(\xi, \nu)$ can only have values in $\{0, 1, 2, -1, -\infty\}$ and the main part $P_N(\nu_0)$ at $\nu_0$ of the family is an automorphic form in $A(\psi)_{h_0, p_0, p_0}$.

i) If $\nu_0 N > 0$, then $P_N(\nu_0) = \tilde{P}_N(\nu_0)$ for each choice of the cut-off function $\varphi$, and $P_N(\nu_0)$ is a square integrable automorphic form in the space $A^2(\psi)_{h_0, p_0, p_0}$.

ii) If $\nu_0 N = 0$, then $P_N(\nu_0) = A^2(\psi)$ if and only if the $V_f$ are of holomorphic- or antiholomorphic discrete series type.

iii) If $\nu_0 N = -1$, then the $V_f$ are of holomorphic or antiholomorphic discrete series type, and $P_N(\nu_0)$ is not a square integrable automorphic form.

iv) If $\nu_0 N = -\infty$, then $P_N(\nu_0) = 0$.

Proof. Most of these statements have been established in earlier sections. Theorem 5.1 (meromorphic continuation) and Proposition 5.4 imply that $P_N(\nu_0) \in A(\psi)$, and $P_N(\nu_0) \in A^2(\psi)$ if $\nu_0 N > 0$. Also, we have that $\nu_0 N \leq 1$ if $\nu_0 \neq 0$, and $\nu_0 N(0) \leq 2$. The fact that $P_N(\nu_0) = \tilde{P}_N(\nu_0)$ if $\nu_0 N > 0$ was discussed in §6.1. Part iv) of the proposition is a direct consequence of the definitions. This leaves us to prove ii) and iii) and the fact that Poincaré orders in $\mathbb{Z}_{\xi-2}$ do not occur.

We suppose that $M_N(\xi, \nu)$ is holomorphic at $\nu = \nu_0$. We use the decomposition (4.11). In the region of absolute convergence, the term $P' M_N(\xi, \nu)$ has polynomial decay, so its Fourier term of order $N$ is a multiple of $\Omega_N(\xi, \nu)$. Hence the Fourier term of order $N$ has the following form, first in the region of absolute convergence, and then extended by meromorphic continuation:

$$F_N M_N(\xi, \nu) = F_N M_N^0(\xi, \nu) + C_{N, N}(\xi, \nu) \Omega_N(\xi, \nu),$$

$$M_N^0(\xi, \nu)(\gamma g) = \sum_{\gamma \in \Gamma_N \backslash \Gamma} M_N(\xi, \nu)(\gamma g).$$

The function $\nu \mapsto C_{N, N}(\xi, \nu)$ is holomorphic on a neighborhood of $\nu_0$.

If $M_N^0(\xi, \nu)$ is identically zero, then we are in the case $\nu_0 N = -\infty$. Otherwise, the term $M_N^0$ is a linear combination of non-zero elements in a finite number of one-dimensional spaces $M_N(\xi, \nu)$, among which $N' = N$ occurs with a non-zero factor. With Proposition 4.8 this leads to

$$(7.26) \quad F_N M_N(\xi, \nu) = \nu M_N(\xi, \nu) + C_{N, N}(\xi, \nu) \Omega_N(\xi, \nu),$$

with a non-zero factor $\nu$. If $M_N(\xi, \nu_0)$ and $\Omega_N(\xi, \nu_0)$ are linearly independent, then the total Fourier term is non-zero, and $\nu_0 N = 0$. Moreover, the function $t \mapsto M_N(\xi, \nu_0)(a(t))$ has exponential growth, and $P_N(\nu_0) = M_N(\xi, \nu_0)$ is not square integrable. Linear dependence of the two functions occurs only in the non-abelian case, when $V$ is of holomorphic or antiholomorphic discrete series type. See §B.3.1.

Thus, to complete the proof we have to take a closer look at the case when $V$ is of holomorphic or of antiholomorphic discrete series type. Then, $M_N(\xi, \nu_0)$ and
The Fourier term takes the form
\[ F_n M_n(\xi, \nu) = a(\nu) Y_{\xi}(j_\xi, \nu) + b(\nu) \Omega_n(j_\xi, \nu), \]
with holomorphic functions \( a(\nu) \) and \( b(\nu) \) in a neighborhood of \( \nu_0 \), with \( a(\nu_0) = 0 \). The function \( a(\nu) \) is essentially the factor in front of \( V_{\kappa, s} \) in (C.6). This implies that the zero of \( a(\nu) \) at \( \nu_0 \) has order 1.

Clearly, \( p_0(\nu_0) = 0 \) implies that \( b(\nu_0) \neq 0 \). If \( M_n(\xi, \nu_0) = 0 \), then \( b(\nu_0) = 0 \) as well. The derivative \( \partial_\nu M_n(\xi, \nu) \big|_{\nu=\nu_0} \) is an element of \( \mathcal{A}(\psi)_{h_0, p_0, p_0} \) (possibly zero). It has Fourier term of order \( n \) of the form
\[ a'(\nu_0) \omega_n^{0,0}(j_\xi, \nu_0) + b'(\nu_0) \omega_n^{0,0}(j_\xi, \nu_0). \]
Since \( a'(\nu_0) \neq 0 \), the limit is non-zero, and \( p_0(\nu_0) = -1 \), the main part \( P_n(\xi, \nu_0) \) is the derivative \( \partial_\nu M_n(\xi, \nu) \big|_{\nu=\nu_0} \), and \( F_n P_n(\xi, \nu_0) \) has an exponentially increasing Fourier term; see (C.7).

7.6. **Proofs of the results in §6.3.** For this task we will make use of the values of the Wronskian order for each type of irreducible \((g, K)\)-module, which are given in Theorem 7.8. The proof of the theorem is very technical and we postpone it to Section 8.

**Theorem 7.8.** Table 2 specifies the values of \( w_0 N(\nu_0) \) for each type of irreducible \((g, K)\)-module. When two values are given, the value between brackets applies to \( \nu_0 = 0 \). The empty spaces do not occur in the Fourier expansion.

| \( N \) | u.principal s. | compl.s. | large d.s. | holo-aholo d.s. | \( T^0_{-1} \) | \( T^0_k \), \( k \geq 0 \) |
|-------|-------------|---------|----------|----------------|------------|----------------|
| gen.abelian | 1 (2) | 1 | 0 | 0 (1) | 1 | 0 |
| non-abelian | 1 (2) | 1 | 0 | | | |

Table 2. Values of the Wronskian order \( w_0 N(\nu_0) \)

We now turn to the proofs of the results in §6.3.

**Proof of Proposition 6.2.** We have \( h_0 = 2 j_\xi \) and \( p_0 = 0 \) in the cases of reducible unitary principal series, complementary series and thin representations \( T^0_{-1} \). The Wronskian order is positive. By Table 2, \( w_0 N(\nu_0) = 1 \) if \( \nu_0 \neq 0 \) and \( w_0 N(0) = 2 \). We consider Fourier term orders \( N \neq N_0 \).

Proposition 7.7 i) implies that if \( \nu \mapsto M_N(\xi, \nu) \) has a singularity at \( \nu_0 \), then \( P_N(\nu_0) \in A^2(\psi)_{2 j_\xi, 0, 0}. \) If \( \nu_0 \neq 0 \), the second statement in Lemma 7.6 implies that \( c_N(P_N(\nu_0)) \neq 0 \) and \( p_0 N(\nu_0) = w_0 N(\nu_0) \).

If there is a singularity at \( \nu_0 = 0 \) we have to rule out the possibility that \( p_0 N(0) = 1 \). From (7.22) with \( f = P_N(0) \) we get the relation
\[ (M_N(\xi, \nu), P_N(0))_{\Gamma, G} = c_N(P_N(0)) \frac{I_N(\psi; \nu, \nu_0)}{\nu^2}. \]
If \( c_N(P_N(0)) \neq 0 \), the right-hand side has order two by Lemma 7.6 and Theorem 7.8. So, the assumption that \( po_N(0) = 1 \) implies that \( c_N(P_N(0)) \) vanishes, and that

\[
0 = \lim_{\nu \to 0} (\nu M_N(\xi, \nu), P_N(0))_{\Gamma \backslash G} = \| P_N(0) \|^2.
\]

This implies that \( P_N(0) = 0 \), a contradiction. Thus \( po_N(0) = 2 \).

Conversely, suppose that there exists a square integrable automorphic form \( f \in \mathcal{A}^2(\psi)_{2; k, 0, 0} \) satisfying \( c_N(f) \neq 0 \). For such an \( f \) we have by Lemma 7.6

\[
(\tilde{M}_N(\xi, \nu), f)_{\Gamma \backslash G} \sim (\nu - \nu_0)^{-wo(N(\nu_0))} (\nu \to \nu_0),
\]

and, since \( po_N(\nu_0) = wo(\nu_0) \),

\[
((\nu - \nu_0)^{po_N(\nu_0)} M_N(\xi, \nu), f)_{\Gamma \backslash G} \sim 1 \quad (\nu \to \nu_0).
\]

Since \( wo_N(\nu_0) > 0 \), this implies that \( M_N(\xi, \nu) \) has a singularity at \( \nu = \nu_0 \).

In case \( M_N(\xi, \nu) \) is holomorphic at \( \nu_0 \), we use Proposition 4.9. We have the Fourier term

\[
F_N M_N(\xi, \nu_0) = a M_N(\xi, \nu) + C_{N, N}(\xi, \nu_0) \Omega_N(\xi, \nu_0),
\]

with \( a \neq 0 \).

The second term on the right is square integrable and the first term has exponential growth for the isomorphism classes under consideration. This proves assertion b) in the proposition and completes the proof.

Proof of Proposition 6.3. For the large discrete series type and the thin representations \( T^k_k \) with \( k \geq 0 \), the Wronskian order is \( wo_N(\nu_0) = 0 \), according to Table 2. A singularity of the Poincaré family at \( \nu = \nu_0 \) would contradict the fact that \( po_N(\nu_0) \leq wo_N(\nu_0) \), as shown in Lemma 7.6. Further, one shows that \( M_N(\xi, \nu_0) \) is not square integrable by arguing as in Proposition 6.2.

The last assertion in the proposition is a direct consequence of Lemma 7.6. 

Proof of Proposition 6.4. For the holomorphic and antiholomorphic discrete series types we need to consider only non-abelian Fourier term orders \( N = N_0 \) with \( n = (\ell, c, d) \). The Wronskian order is \( wo_0(\nu_0) = 0 \) for \( \nu_0 \neq 0 \), and \( wo_0(0) = 1 \).

Case \( \nu_0 \neq 0 \). Parts ii)–iv) of Proposition 7.7 show that the Poincaré family is holomorphic at \( \nu = \nu_0 \). If it has a non-zero value it is square integrable, if it has a zero of order 1 then the derivative at \( \nu = \nu_0 \) is not square integrable. The last possibility is that the Poincaré family is identically zero.

If there are \( f \in \mathcal{A}^2(\psi)_{2, k, 0, 0} \) with \( F_n f \neq 0 \), then we have, by Proposition 7.3 and (C.11), for each cut-off function \( \varphi \):

\[
(\tilde{M}_n(\xi, \nu), f)_{\Gamma \backslash G} \sim \frac{c_n(f)}{\nu^2 - \nu_0^2} \frac{\Gamma(1 + \nu)}{\Gamma(1 + \nu - \kappa_0)} \cdot (-1),
\]

as \( \nu \to \nu_0 \), using that \( \int_{\nu_0}^{\infty} \varphi'(t) dt = -1 \).

Since \( \frac{1}{\nu^2} - \kappa_0 \in \mathbb{Z} \leq 0 \) the gamma factor in the denominator has a pole at \( \nu_0 \), canceling the zero in \( \nu^2 - \nu_0^2 \). This implies that the limit as \( \nu \to \nu_0 \) of the left-hand side \( (\tilde{M}_n(\xi, \nu_0), f) \) does not depend on \( \varphi \). Letting \( \varphi = 1 \) on a growing interval we
have that $\tilde{M}_n(\xi, \nu_0) \to M_n(\xi, \nu_0)$ in the $L^2$-sense, since $M_n(\xi, \nu_0) = \Omega_n(\xi, \nu_0)$ has quick decay. Thus we get $(M_n(\xi, \nu_0))_{\Gamma\backslash G} = c_n(f) \neq 0$, and hence $p_0(\nu_0) = 0$.

Conversely, if $p_0(\nu_0) = 0$, by taking $f = M_n(\xi, \nu_0)$ and arguing as before we see that $c_n(f) \neq 0$. In this way we have obtained the assertions in part i) of the proposition.

Case $\nu_0 = 0$. Now $w_0(0) = 1$, hence the singularity of $M_n(\xi, \nu)$ at $\nu = 0$ can be of order at most 1, by Lemma 7.6. At a singularity the main part $P_n(0)$ is square integrable, and $P_n(0) = P_n(0)$ for each cut-off function $\varphi$. So if $c_n(f) \neq 0$ for some $f \in \mathcal{A}^2(\psi)_{\nu, 0}$, then $p_0(0, 0)$ must be equal to one.

Assume now $M_n(\xi, \nu)$ is holomorphic at $\nu = 0$, and not identically zero. Then $M_n(\xi, 0) \in \mathcal{A}^2(\psi)_{\nu, 0}$, since $M_n(\xi, 0) = \Omega_n(\xi, 0)$. Proposition 7.3, together with the fact that $w_0(0) = 1$, implies that

$$(\tilde{M}_n(\xi, \nu), M_n(\xi, 0))_{\Gamma\backslash G} = \frac{c_n(M_n(\xi, 0))}{\nu(c_0 + O(\nu))}$$

as $\nu \to 0$, with $c_0 \neq 0$. The left-hand side is holomorphic in a neighborhood of $\nu = 0$. Hence $c_n(M_n(\xi, 0)) = 0$. This gives

$$(\tilde{M}_n(\xi, 0), M_n(\xi, 0))_{\Gamma\backslash G} = 0.$$ 

On the other hand, $\tilde{M}_n(\xi, 0) \to M_n(\xi, 0)$ in $L^2(\Gamma\backslash G)$ as $\varphi \uparrow 1$. Hence we have $M_n(\xi, 0) = 0$. Then Proposition 7.7 leaves only the possibility that $p_0(0) = -1$, and $P_n(0)$ not square integrable. This completes the proof. 

8. Wronskian orders for irreducible $(g, K)$-modules.

Our last and final task in the paper will be to prove Theorem 7.8. We first formulate and prove a general result on the Wronskian order and next we carry out a separate scrutiny for each type of irreducible $(g, K)$-module in Table 1.

**Proposition 8.1.** Let $\nu \mapsto M_N(\xi, \nu)$ and $\Omega_N(\xi, \nu)$ be as above, with $\nu_0 \in \mathbb{R} \cup (0, \infty)$. Then

i) If $W(M_N(\xi, \nu), \Omega_N(\xi, \nu))(t) \neq 0$ for some $t > 0$, then $w_0(\nu_0) = 0$ and $w_0(\nu_0) = 1$ otherwise.

ii) Let $\nu_0 \neq 0$. If $W(M_N(\xi, \nu), \Omega_N(\xi, \nu))(t)$ is identically zero and, for some $t > 0$, $[M_N(\xi, \nu), \Omega_N(\xi, \nu)](t) \neq 0$, then $w_0(\nu_0) = 0$.

iii) Let $\nu_0 = 0$, and suppose that $W(M_N(\xi, \nu), \Omega_N(\xi, \nu))$ is identically zero. Then the derivative $\partial_\nu W(M_N(\xi, \nu), \Omega_N(\xi, \nu))(t)$ has the form $a^3$ for some $a \in \mathbb{C}$.

a) If $a \neq 0$, then $w_0(0) = 1$.

b) If $a = 0$, and $[M_N(\xi, 0), \Omega_N(\xi, 0)](t) \neq 0$ for some $t > 0$, then $w_0(0) = 0$.

**Proof.** We first recall some notation. If two quantities $a$ and $b$ differ by a non-zero factor, we write $a = b$. Furthermore, by $a \sim b$ we indicate that the ratio between $a$ and $b$ tends to a non-zero value.

We get into the proof of the lemma. The integral $I_N(\varphi; \nu, \nu_0)$ in (7.20) is a holomorphic function of $\nu$ in a neighborhood of $\nu_0$. If $W(M_N(\xi, \nu), \Omega_N(\xi, \nu)) \neq 0$, we
can find $\alpha \in C^\infty_c[0, \infty)$ such that $\nu \mapsto \int_0^\infty \alpha(t) \mathbf{W}(M_N(\xi, \nu), \Omega_N(\xi, \nu_0)) t^{-3} \, dt \neq 0$. This implies that there is a smooth function $\varphi$, such that $\varphi = 1$ on $[0, T]$ and $\varphi = 0$ on $[T_1, \infty)$, with $0 < T < T_1$, and such that $I_N(\varphi; \nu, \nu_0)$ is non-zero. This gives part i).

We now consider ii). Since $\mathbf{W}(M_N(\xi, \nu_0), \Omega_N(\xi, \nu_0))$ is identically zero, then $I_N(\varphi; \nu, \nu_0)$ has a zero at $\nu = \nu_0$. To establish the order of this zero we denote by $M_N$ the derivative $\left. \frac{dM_N(\xi, \nu)}{d\nu} \right|_{\nu = \nu_0}$. Differentiation with respect to $\nu$ commutes with the differentiations involved in the definition of the Maass-Selberg form in (7.8), hence we get from (7.19)

$$\text{MS}(M_N, \Omega_N(\xi, \nu_0)) = \left. \frac{d}{d\nu} \right|_{\nu = \nu_0} \text{MS}(M_N(\xi, \nu), \Omega_N(\xi, \nu_0)) = -2\nu_0 \{ M_N(\xi, \nu_0), \Omega_N(\xi, \nu_0) \}.$$ 

Now, by Proposition 7.1 we have

$$\partial_t (r^{-3} \mathbf{W}(M_N(\xi, \nu), \Omega_N(\xi, \nu_0)))(t) = c_1 r^{-5} \text{MS}(M_N(\xi, \nu), \Omega_N(\xi, \nu_0))(t)$$

for some $c_1 \neq 0$. Differentiations with respect to $t$ and $\nu$ commute. Thus, by (8.1)

$$\partial_t (r^{-3} \mathbf{W}(M_N, \Omega_N(\xi, \nu_0))(t)) = \partial_t (r^{-3} \mathbf{W}(M_N(\xi, \nu), \Omega_N(\xi, \nu_0))(t)) \big|_{\nu = \nu_0}$$

(8.2)

If $\nu_0 \neq 0$, then the assumption that $\{ M_N(\xi, \nu_0), \Omega_N(\xi, \nu_0) \}$ is non-zero implies that $t \mapsto r^{-3} \mathbf{W}(M_N, \Omega_N(\xi, \nu_0))(t)$ is a real-analytic non-zero function. Hence, by the assumption, as a function of $\nu$

$$\int_0^\infty \varphi'(t) \mathbf{W}(M_N(\xi, \nu), \Omega_N(\xi, \nu_0))(t) \frac{dt}{t^3} \sim (\nu - \nu_0) \int_0^\infty \varphi'(t) \mathbf{W}(M_N, \Omega_N(\xi, \nu_0))(t) \frac{dt}{t^3} = (\nu - \nu_0),$$

by (8.2), for some choice of cut-off function $\varphi$. This gives part ii).

Now let $\nu_0 = 0$. Put $\tilde{M}_N = \left. \frac{d^2}{d\nu^2} M_N(\xi, \nu) \right|_{\nu = 0}$. We start with the holomorphic family $

\nu \mapsto \mathbf{W}(M_N(\xi, \nu), \Omega(\xi, 0))(t).$ On a neighborhood of $\nu = 0$ it has an expansion

$$\mathbf{W}(M_N(\xi, \nu), \Omega_N(\xi, 0))(t) = \mathbf{W}(M_N(\xi, 0), \Omega_N(\xi, 0))(t) + \nu \mathbf{W}(M_N, \Omega_N(\xi, 0))(t) + \nu^2 \mathbf{W}(M_N, \Omega_N(\xi, 0))(t) + \mathcal{O}(\nu^3).$$

Since we are in part iii) of the lemma, this expansion starts with a zero term. This brings us to

$$t^5 \partial_t r^{-3} \mathbf{W}(M_N(\xi, \nu), \Omega_N(\xi, 0))(t) = \nu t^5 \partial_t r^{-3} \mathbf{W}(M_N, \Omega_N(\xi, 0))(t) + \nu^2 t^5 \partial_t r^{-3} \mathbf{W}(M_N, \Omega_N(\xi, 0))(t) + \mathcal{O}(\nu^3).$$

(8.4)

On the other hand, by (8.1)

$$t^5 \partial_t r^{-3} \mathbf{W}(M_N(\xi, \nu), \Omega_N(\xi, 0))(t) = \left. \frac{d}{d\nu} \right|_{\nu = 0} \text{MS}(M_N(\xi, \nu), \Omega_N(\xi, 0))(t)$$

$$= -\nu^2 \{ M_N(\xi, \nu), \Omega_N(\xi, 0) \}(t)$$

(8.5)

$$= -\nu^2 \{ M_N(\xi, 0), \Omega_N(\xi, 0) \}(t) + \mathcal{O}(\nu^3).$$

(8.5)
Comparison of the expansions in (8.4) and (8.5) gives
\[ \partial_t t^{-3} W(M_N, \Omega_N(\xi, 0))(t) = 0, \]
(8.6)
\[ \partial_t t^{-3} W(\tilde{M}_N, \Omega_N(\xi, 0))(t) = t^{-5}[M_N(\xi, 0), \Omega_N(\xi, 0)](t). \]
The first relation gives \( W(M_N(\xi, 0), \Omega_N(\xi, 0))(t) = at^3 \) for some \( a \in \mathbb{C}. \)

If \( a \neq 0 \), we get from (8.3)
\[ \int_0^\infty \varphi'(t) W(M_N(\xi, \nu), \Omega_N(\xi, 0))(t) \frac{dt}{t^3} = av + O(\xi, \nu^2). \]
This implies that \( \text{wo}_N(\xi, 0) = 1 \), and gives part iii) a).

Suppose now that \( a = 0 \). From the second line in (8.6) we see that
\[ W(\tilde{M}_N, \Omega_N(\xi, 0))(t) = t^3 F_N(t), \]
where \( F_N \) is a primitive of \( t \rightarrow t^{-5}[M_N(\xi, 0), \Omega_N(\xi, 0)](t). \) The assumptions in iii) b) imply that \( F_N \) is a non-zero real analytic function on \( (0, \infty) \).

Then, by (8.3) we get
\[ \int_{t=0}^\infty \varphi'(t) W(M_N(\xi, \nu), \Omega_N(\xi, 0))(t) \frac{dt}{t^3} \]
\[ \begin{align*} &\equiv \nu^2 \int_{t=0}^\infty \varphi'(t) t^3 F_N(t) dt + O(\nu^3). \end{align*} \]
The assumptions in iii) b) imply that we can choose the cut-off function in such a way that the integral on the right is non-zero. This gives part iii) b). \( \square \)

8.1. **Proof of Theorem 7.8.** Our next and final task will be to check the values of \( \text{wo}_N(\nu_0) \) in Table 2 by applying Proposition 8.1. We have to handle the various isomorphism classes in Table 1 for the generic abelian cases \( \mathcal{N} = \mathcal{N}_0, \beta \neq 0, \) and for the non-abelian cases \( \mathcal{N} = \mathcal{N}_f. \) Some cases will require quite long computations and references to [2]. Namely, we will consider:

- **Unitary principal series and complementary series** in §8.1.1 and §8.1.2
- **Holomorphic and antiholomorphic discrete series types** mostly in §8.1.2.

Some computations for the case \( \nu_0 = 0 \) (limits of discrete series) in §8.1.4.
- **Large discrete series type** in §8.1.5.
- **Thin representations:** \( T_{+k}^\pm \) in §8.1.2, \( T_k^\pm \) with \( k \geq 0 \) in §8.1.5.

8.1.1. **One-dimensional minimal \( K \)-types, generic abelian cases.** Let \( p_0 = 0. \) Then \( h_0 = 2_j, \) and
\[ M_{\beta; 2_j; 0, 0}(\xi, \nu)(na(t)k) = \chi_\beta(n) t^2 I_v(2\pi|\beta|t)^{2/j} \Phi_{0, 0}^0(k), \]
\[ \Omega_{\beta; 2_j; 0, 0}(\xi, \nu_0)(na(t)k) = \chi_\beta(n) t^2 K_{\nu_0}(2\pi|\beta|t)^{2/j} \Phi_{0, 0}^0(k), \]
which leads with (C.11) to
\[ W(M_{\beta}(\xi, \nu), \Omega_{\beta}(\xi, \nu))(t) = -t^3. \]
We use that \( \nu_0 \in (0, \infty) \cup \iota \mathbb{R} \), and that \( K_\nu \) is even in \( \nu \), to get rid of the complex conjugation.
Proposition 8.1 i) gives \( \omega_0(v_0) = 1 \) if \( v_0 \neq 0 \) and \( \omega_0(0) = 2 \). This concerns the unitary principal series and the complementary series.

8.1.2. One-dimensional minimal K-types, non-abelian cases. Again \( p_0 = 0, h_0 = 2 j \xi \). \( \Omega_n(\xi, v_0) = \omega_n^{0,0}(j \xi, \nu) \) and \( M_n(\xi, \nu) = \mu_n^{0,0}(\xi, \nu) \) in (3.15). From (C.11) we obtain

\[
W(M_\beta(\xi, v_0), \Omega_\beta(\xi, v_0))(t) = -4\pi |\ell| \Gamma(v_0 + 1) t^2 / \Gamma(\frac{|\nu+1|}{2} - \kappa).
\]

Since \( \Re v_0 \geq 0 \), zeros may be due only to the gamma factor in the denominator. In (3.16) we see that \( \kappa = -m_0 - \frac{1}{2} - \frac{1}{2} j \xi \text{Sign} (\ell) \), with a non-negative integer \( m_0 \). In the cases under consideration we use

\[
\frac{v_0 + 1}{2} - \kappa = m_0 + 1 + \frac{1}{2}(v_0 + j \xi \text{Sign}(\ell)).
\]

The present assumption concerns the following isomorphism classes:

- Isomorphism class \( \Pi(j, v_0) \), unitary principal series and complementary series, with \( v_0 \neq j \xi \mod 2 \) or \( (j \xi, v_0) = (0, 0) \). The quantity \( W(v_0) \) is non-zero, and we have \( \omega_0(v_0) = 1 \) if \( v_0 \neq 0 \), and \( \omega_0(0) = 2 \).
- Isomorphism class \( \IF(j, v_0) \), holomorphic discrete series type, and isomorphism class \( \FI(j, v_0) \), antiholomorphic discrete series type, with \( v_0 \equiv j \xi \mod 2, 0 \leq v_0 \leq |j \xi| - 2 \).

Table 23 in [2] shows that it occurs in \( \omega_n^{[j \xi, v_0]} \) under the conditions

\[
\begin{align*}
& \text{for } \IF(j, v_0) : \quad \ell < 0, \quad j \xi > 0, \quad 0 \leq m_0 < \frac{1}{2}(j \xi - v_0), \\
& \text{for } \FI(j, v_0) : \quad \ell > 0, \quad j \xi < 0, \quad 0 \leq m_0 < \frac{1}{2}(j \xi + v_0). 
\end{align*}
\]

This gives in both cases \( \frac{v_0 + 1}{2} - \kappa \in \mathbb{Z}_{\leq 0} \), and vanishing of the Wronskian function at \( v = v_0 \). We cannot apply Proposition 8.1, i).

- Thin representations \( T^+_{\ell} \) (type \( \IF(1, -1) \)) and \( T^-_{\ell} \) (type \( \IF(-1, -1) \)), with \( v_0 = 1, j \xi = \pm 1 \). It occurs in \( \omega_n^{[[1, 1]]} \) under the condition \( m_0 = 0, \neq \ell > 0 \). We obtain a non-zero Wronskian function, and can apply part i) in Proposition 8.1 to conclude that \( \omega_0(1) = 1 \).

We proceed with the holomorphic and antiholomorphic series types. Since \( W(M_\beta(\xi, v_0), \Omega_\beta(\xi, v_0)) = 0 \), the components \( t M_{\kappa, v_0/2}(2\pi|\ell|^2) \) and \( W'_{\kappa, v_0/2}(2\pi|\ell|^2) \) are proportional,

\[
[M_{\lambda}(v_0), \Omega_{\lambda}(v_0)](t) \equiv t^2 W_{v_0, v_0/2}(2\pi|\ell|^2)^2
\]

is non-zero. This gives \( \omega_0(v_0) = 0 \) if \( v_0 \neq 0 \), by Proposition 8.1, ii). See §8.1.4 for the remaining case \( v_0 = 0 \).

8.1.3. Higher-dimensional minimal K-types, generic abelian case. Here only the large discrete series type \( \Pi^*(j, v_0) \) has to be considered, with the basis functions as in (B.6) and (B.7).
Hence we have that
\[ w_{\nu}(8.17) \]
\[ \text{iii). We use the following proportionality} \]
\[ \text{8.1.4. Limits of holomorphic and antiholomorphic discrete series.} \]
\[ \partial \]
\[ \text{value} \]
\[ \nu \]
\[ \phi \]
\[ \text{morphism classes} \]
\[ \text{IF} \]
\[ \text{For} \]
\[ \beta \]
\[ \text{m} \]
\[ \text{K} \]
\[ \text{In a higher-dimensional minimal} \]
\[ \text{K-type} \]
\[ t \]
\[ \text{runs through the range of integers from} \]
\[ \frac{-j_\xi+\nu_0}{2} \]
\[ \text{to} \]
\[ \frac{\nu_0-j_\xi}{2} \].
\[ \text{Since} \]
\[ \nu_0 \geq j_\xi, \text{the value} \]
\[ m = 0 \]
\[ \text{occurs in this range. With} \]
\[ \text{(C.3)} \]
\[ \text{we have the following} \]
\[ \text{asymptotic behavior as} \]
\[ t \downarrow 0 \]
\[ \text{(8.16)} \]
\[ \text{In a higher-dimensional minimal} \]
\[ K\text{-type} \]
\[ \frac{t_{00}}{p_0} \]
\[ \text{we have} \]
\[ h_{00} = -j_\xi \]
\[ \text{and} \]
\[ p_0 = \nu_0. \]
\[ \text{So} \]
\[ m := \frac{h_{00}+p_0}{2} - n \]
\[ \text{runs through the range of integers from} \]
\[ \frac{-j_\xi+\nu_0}{2} \]
\[ \text{to} \]
\[ \frac{\nu_0-j_\xi}{2} \].
\[ \text{Since} \]
\[ \nu_0 \geq j_\xi, \text{the value} \]
\[ m = 0 \]
\[ \text{occurs in this range. With} \]
\[ \text{(C.3)} \]
\[ \text{we have the following} \]
\[ \text{asymptotic behavior as} \]
\[ t \downarrow 0 \]
\[ \text{(8.16)} \]
\[ \text{For} \]
\[ t \]
\[ \text{near zero, the (non-zero) term with} \]
\[ m = 0 \]
\[ \text{is larger than the other terms.} \]
\[ \text{Hence we have that} \]
\[ \text{W}_{\nu_0}(\nu_0) = 0 \]
\[ \text{by ii) in Proposition 8.1.} \]
8.1.4. Limits of holomorphic and antiholomorphic discrete series. For the isomorphism classes \text{IF}(j_\xi, 0) \text{ and } \text{FI}(j_\xi, 0) \text{ the Wronskian function is zero if } \nu \text{ takes the value } \nu_0 = 0. \text{ To apply Proposition 8.1 iii) we need to compute the derivative} \]
\[ \partial_{\nu} W(M_n(\xi, \nu), \Omega_n(\xi, 0))(t) \big|_{\nu=0}. \]
\[ \text{In this situation we have} \]
\[ \frac{1+\nu}{2} - \kappa = m_0 + \frac{\nu+j_\xi}{2} \]
\[ \text{with} \]
\[ \epsilon = \text{Sign}(\ell), -\epsilon j_\xi \in 2\mathbb{Z}_{\geq 1}, \]
\[ \text{and} \]
\[ 0 \leq m_0 < -\frac{\epsilon}{2} j_\xi. \text{ See [2, Table 23]. This implies that} \]
\[ \nu \mapsto \frac{1+\nu}{2} - \kappa \]
\[ \text{has values} \]
\[ \epsilon = \nu_0 = 0 \].
\[ \text{The zero of the Wronskian in (C.11) at} \]
\[ \nu = 0 \]
\[ \text{implies that} \]
\[ \mu_n(\xi, 0) \]
\[ \text{and} \]
\[ \Omega_n(\xi, 0) \]
\[ \text{are proportional. Hence, the holomorphic function} \]
\[ \nu \mapsto \Gamma\left(\frac{1+\nu}{2} - \kappa\right) \]
\[ W(M_n(\xi, \nu), \Omega_n(\xi, 0))(t) \]
\[ \text{has a zero at} \]
\[ \nu = 0 \]
\[ \text{and we apply Proposition 8.1 iii).} \]
\[ \text{We use the following proportionality} \]
\[ \text{(8.17)} \]
\[ \partial W(M_n(\xi, \nu), \Omega_n(\xi, 0))(t) \big|_{\nu=0} \approx \lim_{\nu \to 0} \Gamma\left(\frac{1+\nu}{2} - \kappa\right) W(M_n(\xi, \nu), \Omega_n(\xi, 0)). \]
\[ \text{For} \]
\[ \nu \]
\[ \text{near} \]
\[ 0 \]
\[ \text{and the abbreviation} \]
\[ 2\pi |\ell| = \alpha, \text{we have} \]
\[ \Gamma\left(\frac{1+\nu}{2} - \kappa\right) W(M_n(\xi, \nu), \Omega_n(\xi, 0))(t) = \Gamma\left(\frac{1+\nu}{2} - \kappa\right) W(t M_{\nu/2}(\alpha^2), t W_{\kappa,0}(\alpha^2)), \right. \]
\[ = \Gamma\left(\frac{1+\nu}{2} - \kappa\right) 2\alpha^2 \left( M_{\nu/2}(\alpha^2) W_{\nu,0}(\alpha^2) - M_{\nu/2}(\alpha^2) W_{\kappa,0}(\alpha^2) \right) \]
\[ \left. = \Gamma\left(\frac{1+\nu}{2} - \kappa\right) 2\alpha^2 \left( - \frac{1}{\alpha^2} M_{\nu/2}(\alpha^2) W_{\kappa+1,0}(\alpha^2) - \frac{2\kappa + \nu}{2\alpha^2} M_{\nu/2}(\alpha^2) W_{\kappa,0}(\alpha^2) \right) \right. \]
where we have used the contiguous relations in [2, (A.18)].

Next we determine the asymptotic behavior as \( t \to \infty \), using the formulas in (C.7). The main term in the expansion of the first summand is

\[
-2t \Gamma(1 + \nu) (\alpha t^2)^{-k+k+1} = -4\pi t \Gamma(1 + \nu) t^3 .
\]

The second term is of size \( O(r^{-1}) \). Hence the value of \( a \) in Proposition 8.1 iii) equals \( a = -4\pi t \), thus \( \text{wo}_n(0) = 1 \).

### 8.1.5. Higher-dimensional minimal K-types, non-abelian cases.

We still have to consider the large discrete series type and the thin representations \( T^+_k \) with \( k \in \mathbb{Z}_{\geq 0} \) in the non-abelian cases. The minimal \( K \)-type \( \tau \) has dimension \( p_0 + 1 > 1 \), for which we have the explicit descriptions (B.9) and (B.10) of the basis functions \( \Omega_n \) and \( M_n \). These expressions depend on \( \kappa \) and \( s \) in (B.8). We collect in Table 3 information from Tables 15, 23, and equation (16.6) in [2], and from Table 1 above. In (B.9) and (B.10) the summation variable runs over \( r \equiv p_0 \mod 2 \), \( |r| \leq p_0 \),

| Parameters | \( T^+_{\ell} \) | \( T^-_{\ell} \) |
|-----------|------------------|------------------|
| type \( v_0 \) | \( H_\pm (j_\kappa, v_0) \) | \( IF_+(j_\kappa, -1) \) | \( IF_+(j_\kappa, -1) \) |
| \( j_\kappa \) | \( j_\kappa \equiv v_0 \mod 2 \) | \( 2k + 3 \) | \( -(2k + 3) \) |
| \( h_0 \) | \( -j_\kappa \) | \( k + 3 \) | \( -(k + 3) \) |
| \( p_0 \) | \( v_0 \) | \( k + 1 \) | \( k + 1 \) |
| \( \varepsilon = \text{Sign}(\ell) \) | \( \varepsilon \in \{1, -1\} \) | \( -1 \) | \( 1 \) |
| \( m_0 \) | \( \geq \frac{v_0 - |j_\kappa|}{2} \) | \( k + 1 \) | \( k + 1 \) |
| \( m(h_0, r) \) | \( m_0 + \frac{\varepsilon}{2} (r + j_\kappa) \) | \( \frac{k + 1 - r}{2} \) | \( \frac{k + 1 + r}{2} \) |
| \( \kappa(r) \) | \( -m_0 - \frac{\varepsilon}{2} (r + j_\kappa) - \frac{1}{2} \) | \( \frac{k + 1 - r}{2} \) | \( \frac{k + 1 + r}{2} \) |
| \( s(r) = s(h_0, r) \) | \( -\frac{1}{2} (j_\kappa + r) \) | \( \frac{k + 3 - r}{4} \) | \( -\frac{1}{4} \) |

Table 3. Parameters for the cases of large discrete series type and thin representations.

and \( m(h_0, r) \geq 0 \). For the large discrete series type this gives no restriction, since \( m(h_0, r) \geq \frac{p_0 + \varepsilon r}{2} = \frac{p_0 + \varepsilon r}{2} \geq 0 \). For the thin representations the same holds: \( m(h, r) = \frac{k + 1 - r}{2} = \frac{p_0 + \varepsilon r}{2} \geq 0 \).

First we consider the Wronskian function at \( \nu = v_0 \).

\[
W(M_n(\xi, \nu_0), \Omega_n(\xi, \nu_0)) = \sum_r \left( \frac{p_0}{2} \right)^2 (-e^{\pi i (\theta_0 + \kappa) - \kappa}) \frac{\Gamma(\frac{1}{2} + |s(r)| - \kappa)}{\sqrt{m(h_0, r)!} (2\pi |\ell|^2)!} \cdot (-1)^{m(h_0, r)} \sqrt{m(h_0, r)} \Wr(t M_{\kappa(r), \ell}(r), 2\pi |\ell|^2) \). 
\]

For all terms in the sum we have \( \frac{1}{2} + |s(r)| - \kappa(r) \geq 1 + m_0 \geq 1 \).

The factor \( -e^{\pi i (\theta_0 + \kappa) - \kappa} \) is invariant under \( r \mapsto r + 2 \), and can be taken outside the sum as a non-zero constant. The factor \( (-1)^{m(h_0, r)} \) can be replaced by
\((-1)^{(p_0+r)/2}\) up to a non-zero constant. With (C.11) we arrive at

\[
W(M_n(\xi, v_0), \Omega_n(\xi, v_0)) = t^2 p_0 \sum_r \left( \frac{p_0}{p_0 + r} \right) (-1)^{(p_0+r)/2} \frac{\Gamma\left(\frac{1}{2} + |s(r)| - \kappa(r)\right)}{\Gamma(2|s(r)| + 1) r^3} \frac{\Gamma\left(\frac{1}{2} + |s(r)| - \kappa(r)\right)}{(2|s(r)|)!}
\]

\[
= t^2 p_0 \sum_{n=0}^{p_0} \left( \frac{p_0}{n} \right) (-1)^n r^3 = 0 .
\]

This brings us to part ii) of Proposition 8.1.

With similar simplifications we obtain

\[
|(M_n(\xi, v_0), \Omega_n(\xi, v_0))|/(t) = \sum_r \left( \frac{p_0}{p_0 + r} \right) t^{2p_0+2} (-1)^{(p_0+r)/2} \frac{\Gamma\left(\frac{1}{2} + |s(r)| - \kappa(r)\right)}{\Gamma(2|s(r)|)!} M_{\kappa(r),|s(r)|}(2\pi|\ell|^2) W_{\kappa(r),|s(r)|}(2\pi|\ell|^2).
\]

(8.18)

For the behavior as \( t \downarrow 0 \) of the separate terms we use (C.7) and (C.8). If \( s(r) \neq 0 \) we find

\[
\sim t^{2p_0+2} \left( \frac{p_0}{p_0 + r} \right) (-1)^{(p_0+r)/2} \frac{\Gamma\left(\frac{1}{2} + |s(r)| - \kappa(r)\right)}{(2|s(r)|)!} \frac{\Gamma\left(\frac{1}{2} + |s(r)| - \kappa(r)\right)}{2\pi|\ell|^2}.
\]

(8.19)

For \( s(r) = 0 \) we have

\[
t^{2p_0+2} \left( \frac{p_0}{p_0 + r} \right) (-1)^{(p_0+r)/2} \frac{\Gamma\left(\frac{1}{2} - \kappa(r)\right)}{\Gamma\left(\frac{1}{2} - \kappa(r)\right)} 2\pi|\ell|^2 \log 2\pi|\ell|^2.
\]

(8.20)

If \( s(r) = 0 \) occurs in the sum, then the corresponding term has the largest size near \( t = 0 \), and by ii) we thus get \( w_0(v_0) = 0 \). We next verify that this is the case.

We have \(-p_0 \leq r \leq p_0\). For the large discrete series type

\[
s(-p_0) = \frac{1}{4}(-j_\ell + p_0) = \frac{1}{4}(-j_\ell + v_0) \geq 0, \quad s(p_0) = \frac{1}{4}(-j_\ell - v_0) \leq 0 .
\]

Since \( h_0 \equiv p_0 \equiv r \mod 2 \), the values of \( 2s(r) \) are integral. Thus the value \( s(r) = 0 \) occurs, and we get \( w_0(v_0) = 0 \) in this case.

In the case of \( T_k^\pm \) we have

\[
\pm s(r) = \frac{k + 3 + r}{4} \geq \frac{k + 3 - p_0}{4} = \frac{1}{2} .
\]

So \( s(r) \) does not have the value 0 in the sum over \( r \), and all terms in the sum have an asymptotic expansion with a main term that is a non-zero constant times \( t^{2p_0+4} \).
We have to determine whether the following quantity is zero or non-zero.

\[ 2\pi |\ell|^2 |p_\ell|^4 \sum_r \left( \frac{p_0}{p_\ell + r} \right) (-1)^{(p_\ell + r)/2} \frac{2}{k + 3 \mp r}. \]

With the substitution \( r = \mp 2n \pm p_0 \) we get, up to a non-zero factor,

\[
\sum_{n=0}^{p_0} \binom{p_0}{n} (-1)^n \frac{1}{n + 1} = \sum_{n=0}^{p_0} \frac{p_0!}{n! (p_0 - n)! (n + 1)}
\]

\[
= \sum_{n=0}^{p_0 + 1} \binom{p_0 + 1}{n + 1} (-1)^n \frac{1}{p_0 + 1} - 1 \frac{(-1)^{p_0 + 1}}{p_0 + 1} = \frac{(1 - 1)^{p_0 + 1}}{p_0 + 1} + \frac{(-1)^{p_0}}{p_0 + 1} \neq 0.
\]

So the value of \( \{ (M_n(\xi, v_0), \Omega_n(\xi, v_0)) \} (t) \) is non-zero for \( t \) sufficiently close to 0. Therefore, this implies that \( w_0(0) = 0 \) for the thin representations \( T_k^\pm \), with \( k \geq 0 \), as well.

In this way we have taken care of the last case in the list of isomorphism classes, thus the proof is now complete.
Appendix

In this appendix we give further explanations and computations, mostly based on [2]. In this way we avoid in the main text frequent references to that paper.

APPENDIX A. LIE ALGEBRA AND SUBGROUPS OF SU(2, 1)

A.1. Subalgebras and Casimir element. We let $\mathfrak{g}$ be the real Lie algebra of $G = SU(2, 1)$, and let $\mathfrak{g}_c$ be the complex Lie algebra $\mathbb{C} \otimes \mathfrak{g}$. For Lie algebras of subgroups we use a similar notation.

We fix a basis of $\mathfrak{g}$ by indicating bases for the summands in $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{t}$:

(A.1) $\mathfrak{n} = \mathbb{R}X_0 \oplus \mathbb{R}X_1 \oplus \mathbb{R}X_2$, 
\[ e^{X_0} = n(0, t/2), \quad e^{X_1} = n(t, 0), \quad e^{X_2} = n(it, 0); \]

(A.2) $\mathfrak{a} = \mathbb{R}H_r$, $e^{H_r} = a(e')$;

(A.3) $\mathfrak{t} = \mathbb{R}H_t \oplus \mathbb{R}W_0 \oplus \mathbb{R}W_1 \oplus \mathbb{R}W_2$,
\[ e^{H_t} = m(e^t), \quad e^{W_0} = \text{diag}[e^t, e^{-it}, 1], \]
\[ e^{W_1} = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad e^{W_2} = \begin{pmatrix} \cos t & i \sin t & 0 \\ i \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]

The Lie algebra of $M$ is $\mathfrak{m} = \mathbb{R}H_t$. The Lie algebra $\mathfrak{g}$ is a real form of $\mathfrak{s}\mathfrak{l}_3$, of type $A_2$. A Cartan algebra contained in $\mathfrak{t}_c$ is $\mathbb{C}C_1 \oplus \mathbb{C}W_0$, where $C_1 = 3W_0 - 2H_t$. It has root spaces spanned by

\begin{align*}
\text{in } \mathfrak{t}_c: & \quad Z_{12} = W_1 - iW_2, \quad Z_{21} = W_1 + iW_2; \\
\text{not in } \mathfrak{t}_c: & \quad Z_{13} = \frac{1}{2}H_t + i\left(X_0 - \frac{1}{4}W_0 - \frac{1}{4}C_1\right), \\
& \quad Z_{31} = \frac{1}{2}H_t - i\left(X_0 - \frac{1}{4}W_0 - \frac{1}{4}C_1\right), \\
& \quad Z_{23} = \frac{1}{2}\left(X_1 - W_1 + iX_2 - iW_2\right), \\
& \quad Z_{32} = \frac{1}{2}\left(X_1 - W_1 - iX_2 + iW_2\right)
\end{align*}

(A.4)

with corresponding roots
\[ \alpha_{12}(C_1) = 0 \quad \alpha_{13}(C_1) = 3i \quad \alpha_{23}(C_1) = 3i \]
\[ \alpha_{12}(W_0) = 2i \quad \alpha_{13}(W_0) = i \quad \alpha_{23}(W_0) = -i \]
\[ \alpha_{ij} = -\alpha_{ji} \]

The center of the enveloping algebra can be generated by two independent elements. One of them is the Casimir operator given by $\sum_j X_j X_j^*$, where $\{X_j\}$ is a basis of $\mathfrak{g}_c$ and $\{X_j^*\}$ the dual basis with respect to a suitable multiple of the Killing form.

For the two bases discussed above this leads to

(A.5) $C = H_t^2 - 4H_4 - \frac{1}{3}H_r^2 + 4X_0H_t - 8X_0W_0 + 4X_0^2 - 2X_1W_1$
\[ + X_1^2 - 2X_2W_2 + X_2^2 \]
\( A.3. \) \( \Theta \) functions. We refer to [2, §3] for more details.

Schrödinger representation.

\( A.2. \) Realization of irreducible representations of \( e \) (resp. left) translation by \( (A.8) \)

functions \( h \) \( (A.7) \)

\( \Lambda \) the standard generators of \( h \)

The norms of \( \delta \) the delta-distribution

relation to a relation between Schwartz functions and tempered distributions. Let

\[ \| \Phi \|_K^2 = \frac{p!}{(p+2)!} \| \Phi \|_K^2. \]

The Lie algebra \( \mathfrak{t} \) acts by right differentiation:

\begin{align*}
\mathfrak{c}_i^h \Phi_{r,q}^p &= -ih \Phi_{r,q}^p, \\
W_0^h \Phi_{r,q}^p &= -iq \Phi_{r,q}^p, \\
(W_1 \pm iW_2)^h \Phi_{r,q}^p = (q \mp p) \Phi_{r,q+2}^p.
\end{align*}

We refer to [2, §3] for more details.

A.3. \( \Theta \) functions. We indicate how \( \Theta \) functions in \((2.15)\) arise from the Schrödinger representation.

Under the bilinear form \( [\varphi_1, \varphi_2]_{\mathcal{K}} = \int_{\mathbb{R}} \varphi_1(\xi) \varphi_2(\xi) \, d\xi \), the Schrödinger representations \( \pi_{2\pi} \) and \( \pi_{-2\pi} \) in \((2.14)\) are dual to each other. One can extend this relation to a relation between Schwartz functions and tempered distributions. Let the delta-distribution \( \delta \) be defined by \( \delta(\varphi) = \varphi(0) \) for \( \varphi \in \mathcal{S}(\mathbb{R}) \). Let \( \sigma \in \mathbb{Z}_{\geq 1} \). For the standard generators of \( \Lambda_{\sigma} \) we have:

\begin{align*}
\pi_{-2\pi}(n(0, 2/\sigma)) \delta_\sigma &= e^{-\pi i n/\sigma} \delta_\sigma, \\
\pi_{-2\pi}(n(1, 0)) \delta_\sigma &= e^{\pi i \sigma} \delta_\sigma, \\
\pi_{-2\pi}(n(i, 0)) \delta_\sigma &= \delta_{\sigma-1}.
\end{align*}
This implies that if we take $\ell \in \frac{\sigma}{2} \mathbb{Z}$, $c \in \mathbb{Z} \mod 2|\ell|$, we have invariant distributions

$$\mu_{\ell,c} = \sum_{k \in \mathbb{Z}} \delta_{k+c/2\ell}.$$  

(A.13)

Each such $\pi_{-2\pi \ell}(\Lambda_\sigma)$-invariant tempered distribution $\mu = \mu_{\ell,c}$ on $\mathbb{R}$ gives rise to a theta-function by the formula

$$\Theta_{\mu}(\varphi)(n) = [\pi_{2\pi \ell}(n)\varphi, \mu]_\mathbb{R} \quad (\varphi \in \mathcal{S}(\mathbb{R})),  

(A.14)$$

yielding left-$\Lambda_\sigma$-invariant functions on $N$. In this way we have the operator $\varphi \mapsto \Theta_{\ell,c}(\varphi)$ given by

$$\Theta_{\ell,c}(\varphi)(n(x + iy, r)) = \sum_{k \in \mathbb{Z}} [\varphi, \pi_{-2\pi \ell}(n(-x - iy, -r)) \delta_{c/2\ell+k}]_\mathbb{R},$$  

which expands to (2.15). It is an intertwining operator:

$$\Theta_{\ell,c}(\varphi)(nn_1) = \Theta_{\ell,c}(\pi_{2\pi \ell}(n_1)\varphi)(n) \quad (n,n_1 \in N),  

X \Theta_{\ell,c}(\varphi) = \Theta_{\ell,c}(d\pi_{2\pi \ell}(X)\varphi) \quad (X \in \mathfrak{a}).$$  

(A.16)

Furthermore, for $\ell, \ell' \in \frac{\sigma}{2} \mathbb{Z}_{\geq 0}, c, c' \in \mathbb{Z}$, $\varphi, \varphi' \in \mathcal{S}(\mathbb{R})$, one has the orthogonality relations

$$\Theta_{\ell,c}(\varphi), \Theta_{\ell',c'}(\psi)_{\Lambda_{\sigma}\setminus N} = \begin{cases} 
\frac{2}{\sigma} (\varphi, \psi)_\mathbb{R} & \text{if } \ell' = \ell, \ c' \equiv c \mod 2|\ell|,
0 & \text{otherwise}.  
\end{cases}$$  

(A.17)

So for each couple $(\ell, c)$ of integers with $\ell \in \frac{\sigma}{2} \mathbb{Z}_{\geq 0}$, $0 \leq c \leq 2|\ell| - 1$, the linear map $\varphi \mapsto \sqrt{\sigma/2} \Theta_{\ell,c}(\varphi)$ from $\mathcal{S}(\mathbb{R})$ to $C^\infty_\ell(\Lambda_\sigma \setminus N)$, extends to a unitary map $L^2(\mathbb{R}) \to L^2(\Lambda_\sigma \setminus N)$. The images for different choices of $(\ell, c)$ are orthogonal to each other.

**Further transformation properties.** The distribution $\mu_{\ell,c}$ in (A.13) satisfies

$$\pi_{-2\pi \ell}(n(1/2\ell, 0))\mu_{\ell,c} = e^{\pi i c/\ell} \mu_{\ell,c},$$  

(A.18)

$$\pi_{-2\pi \ell}(n(i/2\ell, 0))\mu_{\ell,c} = \mu_{\ell,c-1}.$$  

This implies

$$\Theta_{\ell,c}(\varphi)(n(1/2\ell, 0)n) = e^{\pi i c/\ell} \Theta_{\ell,c}(\varphi)(n),$$  

(A.19)

$$\Theta_{\ell,c}(\varphi)(n(i/2\ell, 0)n) = \Theta_{\ell,c+1}(\varphi)(n),$$

which go beyond the left invariance under $\Lambda_\sigma$.

The group $\Gamma$ that we have fixed in §2.2 contains the element $m(i)$, which normalizes the unipotent group $N$, and also the lattice $\Gamma \cap N$. Indeed, we have that $m(-i)n(b, r)m(i) = n(ib, r)$ and furthermore (see [2, Proposition 4.2]):

**Proposition A.1.** Let $m \in \mathbb{Z}_{\geq 0}$, $\sigma \in \mathbb{Z}_{\geq 1}$, and $\ell \in \frac{\sigma}{2} \mathbb{Z}_{\geq 0}$. The automorphism $n(b, r) \mapsto n(ib, r)$ of $N$ induces a linear bijection in the $2|\ell|$-dimensional space of
theta functions with basis \( \{ \Theta_{\ell,c}(h_{\ell,m}) : 0 \leq c < 2|\ell| \} \), where \( \Theta_{\ell,c}(h_{\ell,m}) \) is the linear transformation determined by

\[
\Theta_{\ell,c}(h_{\ell,m})(nib, r) = \frac{(-i \text{Sign}(\ell))^{m} 2^{(|\ell|-1)/2}}{\sqrt{2|\ell|}} \sum_{c'=0}^{2|\ell|-1} e^{\pi ic'/\ell} \Theta_{\ell,c'}(h_{\ell,m}(n(b, r))).
\]

**Appendix B. Fourier term modules**

**B.1. Fourier term operators.** The fact that for given pairs \((\ell, c)\) the operator \(\varphi \mapsto \Theta_{\ell,c}(\varphi)\) is an intertwining operator makes it fairly natural to consider the Fourier terms

\[
F_{\ell,c}f(na) = \frac{\sigma}{2} \sum_{m \geq 0} \Theta_{\ell,c}(h_{\ell,m})(n) \int_{\Lambda r \setminus N} \Theta_{\ell,c}(h_{\ell,m})(n_{1}) f(n_{1}ak) \, dn_{1},
\]

with \(n \in N, a \in A, k \in K\). Using the orthogonal system \(\{^h\Phi_{p, r}^{\nu}\}\) in \(L^2(K)\) we get for \(F_{\ell,c}f\) an expansion as in (3.4), without the restriction (3.5). One can work out the action of \(U(g)\) on the individual terms explicitly; see \[2, \S 7\]. Then it turns out that the action leaves the quantity in the left hand side of (3.5) constant, with an odd value. This implies that the operators \(F_{\ell,c,d}\) are indeed intertwining operators for the action of \(g\).

The splitting \(F_{\ell,c} = \sum_{d=1(2)} F_{\ell,c,d}\) can be understood from the action of the double cover of \(M\) on the elements \(F_{\ell,c}\) in \(C^{\infty}(\Lambda r \setminus G)_{K}\). See the discussion of the metaplectic action in \[2, \S 8.2\].

The convergence of the total Fourier expansion

\[
f = \sum_{\beta} F_{\beta} f + \sum_{n} F_{n} f
\]

is discussed in \[2, Proposition 5.2\].

**B.2. N-trivial Fourier term modules.** The principal series module \(H_{K}^{\xi, \nu}\) contains the \(K\)-types \(\tau_{p}^{h}\) satisfying \(|h - 2j| \leq p\), with multiplicity one. The subspace \(H_{K, h, p, q}^{\xi, \nu}\) of \(K\)-type \(\tau_{p}^{h}\) and weight \(q\) is spanned by the function

\[
^{h}\psi_{r,q}^{p}(\nu) : na(t)k \mapsto t^{2+j} \Phi_{r,q}^{p}(k),
\]

with \(r = \frac{h-2j}{2}\). Here, by the Iwasawa decomposition, if \(g \in G\), we write \(g = na(t)k\) with \(n \in N\) \(a(t) \in A\) and \(k \in K\). See \[2, \S 10.1\].

If \(\nu \not\equiv j_{k} \mod 2\), then the principal series module \(H_{K}^{\xi, \nu}\) is an irreducible \((g, K)\)-module. All isomorphism classes of irreducible \((g, K)\)-modules are represented by a submodule in some principal series representation (see \[3\]).

If the Weyl orbit contains elements of the form \((j, 0)\), then there are logarithmic submodules of \(F_{0}^{\varphi[\nu]}\), as discussed in \[2, Propositions 10.4 and 12.3\].
B.3. Families of Fourier terms. The families \( \nu \mapsto \omega_{N,j}^{0,0}(\xi,\nu) \) in (3.11) (for \( N = N_{0}, \beta \neq 0 \)) and (3.15) (for \( N = N_{0} \)) are holomorphic and even in \( \mathbb{C} \). Furthermore, the function \( (\nu, g) \mapsto \omega_{N,j}^{0,0}(\xi, \nu; g) \) is in \( \mathcal{C}^\infty(\mathbb{C} \times G) \). The families \( \nu \mapsto \mu_{N,j}^{0,0}(\xi, \nu) \) have similar properties, except that in the non-abelian case \( N = N_{0} \) there may be first order singularities at points of \( \mathbb{Z}_{\leq -1} \), as one can see in the expansion in [2, (A.9)].

The values of these families at points \( \nu \) in the closed right half-plane \( \Re \nu \geq 0 \) are elements of \( \mathcal{W}_{N}^{\xi,\nu} \) or \( \mathcal{M}_{N}^{\xi,\nu} \), respectively. For the values and residues in the left half-plane we are content to note that they are in \( \mathcal{F}_{N}^{\psi[\xi,\nu]} \), where \( \psi[\xi, \nu] \) is the character of \( ZU(g) \) parametrized by \( (\xi, \nu) \).

If we multiply these families by a holomorphic function of \( \nu \) we get families with the same properties. The special choice of \( \omega_{N}^{0,0} \) and \( \mu_{N}^{0,0} \) is not intrinsic, but governed by the traditional choice of basis elements of differential equations for the modified Bessel functions and the Whittaker functions.

Differentiation by an element of \( g \) preserves these properties. So if \( u \in U(g) \) then \( u \omega_{N,j}^{0,0}(\xi, \nu) \) and \( u \mu_{N,j}^{0,0}(\xi, \nu) \) are again holomorphic (or meromorphic). There are choices of \( u \in U(g) \) that send \( \mathcal{F}_{N,0,0,0}^{\psi} \) to \( \mathcal{F}_{N,h,p,q}^{\psi} \) for any \( K \)-type \( \tau_{h}^{p} \) and weight \( q \) in that \( K \)-type.

In the case \( q = p \) this is done in [2] by repeated application of the shift operators; see Proposition 6.1, and the application to Fourier term modules in Tables 9 and 11. Other weights in a given \( K \)-type are reached by powers of \( Z_{1,2} \) \( t \). This leads to the families \( \omega_{N,j}^{a,b} \) and \( \mu_{N,j}^{a,b} \) in [2, (10.17)] with \( a, b \in \mathbb{Z}_{\geq 0} \). Under generic parametrization this is used to define \( \mathcal{W}_{N}^{\xi,\nu} \) and \( \mathcal{M}_{N}^{\xi,\nu} \). In the abelian case, this also works under integral parametrization. See [2, (10.18), (13.8)]. In the non-abelian case the families \( \omega_{N,j}^{a,b}(\xi, \nu) \) and \( \mu_{N,j}^{a,b}(\xi, \nu) \) may have zeros in the closed right half-plane, which have to be divided out. See Proposition 14.2 and Definition 14.8 in [2].

In this way we arrive at the following result:

**Proposition B.1.** Let \( \xi \) be a character of \( M \) and let \( N \neq N_{0} \) a Fourier term order. For each \( K \)-type \( \tau_{h}^{p} \) satisfying \( |h - 2j| \leq 3p \), and each weight \( q \equiv p \mod 2 \), \( |q| \leq p \) there are an even holomorphic family \( \Omega_{N,h,p,q}(\xi, \nu) \) in \( \mathbb{C} \) and a meromorphic family \( \mathcal{M}_{N,h,p,q}(\xi, \nu) \) in \( \mathcal{C} \), with at most first order singularities in points of \( \mathbb{Z}_{\leq -1} \), with the following properties:

a) The values and residues of these families at \( \nu \) are in \( \mathcal{F}_{N}^{\psi[\xi,\nu]} \).

b) For \( \Re \nu \geq 0 \) the value \( \Omega_{N,h,p,q}(\xi, \nu_{0}) \) spans the one-dimensional space \( \mathcal{W}_{N,h,p,q}^{\xi,\nu_{0}} \).

c) For \( \Re \nu \geq 0 \) the value \( \mathcal{M}_{N,h,p,q}(\xi, \nu_{0}) \) spans the one-dimensional space \( \mathcal{M}_{N,h,p,q}^{\xi,\nu_{0}} \).

Any (finite) non-zero linear combination \( \sum j c_{j}(\nu) \mathcal{M}_{N,h,j,p,q}(\xi, \nu) \) with holomorphic functions \( c_{j} \) gives a function as required in Proposition 4.7.
In all cases, we have a decomposition in Iwasawa coordinates of the form

$$\Omega_{N;h,p,q}(\xi, \nu)(na(t)k) = \sum_r w_r(n) h^\nu_r(v, t)^h \Phi^\nu_{r,q}(k),$$

(B.3)

$$M_{N;h,p,q}(\xi, \nu)(na(t)k) = \sum_r w_r(n) h^\nu_r(v, t)^h \Phi^\nu_{r,q}(k).$$

The sum is over \( r \equiv p \mod 2, |r| \leq p \). In the non-abelian case there is an additional restriction on \( r \); here we just take the components \( h^\nu_r(v, t) \) and \( h^\nu_r(v, t) \) to be zero if the additional condition is not satisfied. By \( w_r \) we mean \( \chi_\beta \) in (2.13) if \( N = N_\beta \), with \( \beta \neq 0 \). In the non-abelian case \( N = N_{e,d} \) and we have \( w_r = \Theta_{t,e}(h_{\ell,m(h,r)}) \) with \( m(h, r) \in \mathbb{Z}_{\geq 0} \) given in (B.8) below.

The component \( h^\nu_r(v, t) \) is a linear combination of positive powers of \( t \) times factors \( K_{v+n}(2\pi|\beta|t) \) if \( N = N_\beta \) or factors \( W_{k+n,\nu/2}(2\pi|\ell|t^2) \) if \( N = N_{e,c} \). In both cases \( n \in \mathbb{Z}_{\geq 0} \). The component \( h^\nu_r(v, t) \) has a similar structure, with Bessel functions \( I_{v+n}(2\pi|\beta|t) \) and Whittaker functions \( W_{k+n,\nu/2}(2\pi|\ell|t^2) \). The description of these components is far from unique. With the contiguous relations mentioned in §C we can rewrite the components in many ways. From (C.2) and (C.7), and from the properties of \( M^\nu_N \), we obtain estimates valid at least for \( \text{Re } \nu \geq 0 \), of the form

$$\Omega_{N;h,p,q}(\xi, \nu)(na(t)k) = O(e^{-\alpha t}) \quad \text{as } t \uparrow \infty, \text{ for some } \alpha > 0,$$

(B.4)

$$M_{N;h,p,q}(\xi, \nu)(na(t)k) = O(t^{2-\alpha}) \quad \text{as } t \downarrow 0, \text{ for some } \alpha > 0,$$

Drawbacks. (1) The families \( \omega^{0,0}_N(j, \nu) \) and \( \mu^{0,0}_N(j, \nu) \) generating \( \mathcal{W}^{\xi,v}_N \) and \( \mathcal{M}^{\xi,v}_N \) are nicely explicit. The action of the shift operators can be carried out explicitly, at least with help of a computer. In the \( K \)-type \( t^\nu_p \) and weight \( q \) the elements have the form

$$\sum_{r \equiv p \mod 2, |r| \leq p} \chi_\beta(n) F_i(t)^h \Phi^\nu_{r,q}(k),$$

$$\sum_{r \equiv p \mod 2, |r| \leq p} \Theta_{t,e}(h_{\ell,m(r)}(n)) F_i(t)^h \Phi^\nu_{r,q}(k),$$

where the \( F_i(t) \) are linear combinations of modified Bessel functions or Whittaker functions multiplied by powers of \( t \). After a few applications of shift operators these linear combinations become in general very complicated.

(2) Parts b) and c) of the proposition seem to give a basis of \( \mathcal{F}^{\xi,v}_{N;h,p,q} \). This fails in some cases in the non-abelian case, for values of \( \nu \in \mathbb{Z}_{\geq 0} \). (This is due to the fact that the Whittaker functions \( M_{e,s} \) and \( W_{s,s} \) may be proportional; see [2, (A.13)].)

B.3.1. An additional family. Proposition 14.2, ii) in [2] tells us that the sole instances in which \( \mathcal{M}^{\xi,v}_N \) and \( \mathcal{W}^{\xi,v}_N \) have a non-zero intersection occur in those non-abelian cases in which \( \mathcal{W}^{\xi,v}_N \) contains a submodule in the holomorphic or in the antiholomorphic discrete series. Then the intersection \( \mathcal{W}^{\xi,v}_n \cap \mathcal{M}^{\xi,v}_n \) is equal to that submodule.
Proposition 16.6 in [2] gives a third family \( \nu \mapsto T_{n,h,p,q}(j, \nu) \) in the non-abelian case. It is based on the function \( V_{\kappa,s} \) in (C.5), and is holomorphic and even in \( \mathbb{C} \) such that for \( \text{Re } \nu_0 \geq 0 \)

\[
\mathcal{F}_{n,h,p,q}^{\nu} = \bigoplus_{(j, \nu_0) \in \Omega_{W}(\phi_{h}^n)} \left( \mathbb{C} \Omega_{n,h,p,q}(j_\xi, \nu_0) \oplus \mathbb{C} T_{n,h,p,q}(j_\xi, \nu_0) \right).
\]

B.3.2. **Explicit cases.** There are only a few cases where we know explicit and rather simple expressions of the functions in Proposition B.1. In the one-dimensional \( K \)-type with \( (h, p, q) = (2j, 0, 0) \) we have the explicit expressions in (3.11) and (3.15).

There are some higher dimensional cases as well. We mention the cases that we will need later on. They are based on Propositions 13.2, 14.4 and 14.15 in [2]. (Ishikawa gives similar expressions in [9, Theorem 5.3.1].) This concerns \( K \)-types \( \tau^K_h \) that have minimal dimension among the \( K \)-types occurring in a submodule of \( \mathcal{F}^{\psi}_{N} \). Then \( p \geq 1 \), and \( \psi = \psi[-h, p] \). We take \((\xi, \nu) \in \Omega_{W}(\psi)^{+}\) such that \( j_\xi = -h \) and \( \nu = p \).

If, in the generic abelian cases, \( \tau^K_h \) is minimal in \( \mathcal{W}_{\beta}^h \), then

\[
\Omega_{\beta,h,p,q}^{\nu}(-h, p)(na(t)k) = \sum_{r \in \{2\}, |r| \leq p} \chi_{\beta}(n) \left( \frac{ij^{|r|}}{|r|} \right)^{(r+p)/2} t^{2+p} K_{h-r}/2(2\pi|\beta| t)^h \Phi_{r,q}^p(k).
\]

By \( \doteq \) we denote equality up to a non-zero factor.

If \( \tau^K_h \) is minimal in \( \mathcal{M}_{\beta}^h, \beta \neq 0 \), then

\[
\mathcal{M}_{\beta,h,p,q}^{\nu}(-h, p)(na(t)k) = \sum_{r \in \{2\}, |r| \leq p} \chi_{\beta}(n) \left( \frac{-ij^{|r|}}{|r|} \right)^{(r+p)/2} t^{2+p} I_{h-r}/2(2\pi|\beta| t)^h \Phi_{r,q}^p(k).
\]

In the non-abelian cases \( N = N_{\ell,c,d} \) we use the quantities

\[
m(h, r) = \frac{1}{2} \text{Sign}(\ell)(r - r_0(h)), \quad r_0(h) = \frac{1}{3}(h - d) + \text{Sign}(\ell),
\]

\[
\kappa(r) = -m(h, r) - \text{Sign}(\ell)s(h, r) - \frac{1}{2} s(r) = s(h, r) = \frac{1}{4}(h - r).
\]

The summation runs over \( r \equiv p \mod 2 \) such that \( |r| \leq p \) and \( m(h, r) \geq 0 \).

If \( \tau^K_h \) is minimal in \( \mathcal{W}_{n}^h \), then

\[
\Omega_{n,h,p,q}^{\nu}(-h, p)(na(t)k) \doteq \sum_{r} \Theta_{\ell,c}(h_{\ell,m(h, r)})(n) \cdot t^{\kappa(r)} \sqrt{m(h, r)!} t^{1+p} W_{\kappa(r), \nu}(2\pi|\ell|^2 t)^h \Phi_{r,q}^{p}(k).
\]
If \( \tau^h_p \) is minimal in \( \mathcal{M}^h \), then
\[
\mathcal{M}_{n,h,p,q}(-h, p)(n(\tau)k) \\
= \sum_r \Theta_{\ell} e(h_{\ell,m(h,r)}(n)) c^M(r) t^{1+p} M_{k(r),s(r)}(2\pi |t|^2)^h \Phi^h_{r,q}(k),
\]
(B.10)
\[
c^M(r) = -e^{\pi i(m(h,r) - s(r))} \frac{\Gamma(\frac{1}{2} + |s(r)|) - \kappa(r))}{\sqrt{m(h, r)!} (2|s(r)|)!}.
\]

We stress that these formulas give an explicit expression only for the value at \( \nu = p \) of the holomorphic families \( \nu \mapsto \Omega_{N; h, p, q}(-h, \nu) \) and \( \nu \mapsto \mathcal{M}_{N; h, p, q}(-h, \nu) \).

**Appendix C. Special Functions and Wronskians**

The \( N \)-non-trivial Fourier term functions have expressions in terms of modified Bessel functions and Whittaker functions. Appendix A of [2] gives many results and references for these special functions. Here we mention facts that we use here.

**C.1. Modified Bessel functions.** The modified Bessel differential equation
\[
s^2 j''(x) + x j'(x) - (s^2 + \nu^2) j(x) = 0
\]
has the standard solutions \( I_\nu \) and \( K_\nu \); [2, (A.2), (A.4)]. Their boundary behavior is
\[
I_\nu(x) \sim (2\pi x)^{-1/2} e^x \quad \text{as } x \uparrow \infty,
\]
\[
I_\nu(x) \sim \Gamma(\nu) (s/2)^\nu \quad \text{as } x \downarrow 0;
\]
\[
K_\nu(x) \sim (\pi/2x)^{1/2} e^{-x} \quad \text{as } x \uparrow \infty,
\]
\[
K_\nu(x) = \frac{\Gamma(\nu) (s/2)^\nu}{\pi x} \quad \text{as } x \downarrow 0.
\]

For \( n \in \mathbb{Z} \) we need the following more precise information, as \( x \downarrow 0 \).
\[
I_n(x) = I_{n}(x) = (s/2)^{|n|/|n|!} + O(x^{|n|+1}),
\]
\[
K_n(x) = K_{n}(x)
\]
\[
= \frac{(|n| - 1)!}{2} (s/2)^{-|n|} + O(x^{-|n|+2} \log x) \quad \text{if } n \neq 0,
\]
\[
= -\log(s/2) + O(x^2 \log x) \quad \text{if } n = 0.
\]

The contiguous relations in [2, (A.7)] allow to express derivatives of \( I_\nu \) and \( K_\nu \) as linear combinations of \( I_{\nu\pm 1} \) or of \( K_{\nu\pm 1} \), respectively.

**C.2. Whittaker functions.** The Whittaker differential equation
\[
f''(x) = \left( \frac{1}{4} - \frac{\kappa}{x} + \frac{s^2 - 1/4}{x^2} \right) f(x) = 0
\]
has standard solutions \( M_{\kappa,s} \) and \( \text{W}_{\kappa,s} \). Since these two solutions are proportional for some combinations, we also use an ad hoc solution \( V_{\kappa,s} \), in [2, (A.12)]. It is the holomorphic continuation in \( s \) of
\[
V_{\kappa,s}(x) = \frac{\pi i}{\sin \pi s} \sum_{\pm} \frac{\pm e^{-\pi i s}}{s} \Gamma(\frac{1}{2} + 2 + s) \Gamma(1 \pm 2s) M_{\kappa,\pm s}(x).
\]
This implies the relation
\[
M_{\kappa,s}(\tau) = e^{\pi i\tau} \Gamma(1 + 2s) \left( \frac{-i e^{-\pi i\tau}}{\Gamma(1/2 + s + \kappa)} W_{\kappa,s}(\tau) - \frac{1}{\Gamma(1/2 + s - \kappa)} V_{\kappa,s}(\tau) \right).
\]  
(C.6)

We note the boundary behavior:
\[
W_{\kappa,s}(x) \sim x^\kappa e^{-x^{1/2}} \quad \text{as } x \uparrow \infty,
\]
\[
= O(x^{1/2-s}) \quad \text{as } x \downarrow 0 \text{ and } s \not\in \mathbb{Z} ;
\]
\[
M_{\kappa,s}(x) = \frac{\Gamma(1 + 2s) \Gamma(\frac{1}{2} - \kappa + s)}{\Gamma(\frac{1}{2} + |\kappa| - s)} x^{-\kappa} e^{x^{1/2}} \left( 1 + O(1/x) \right) \quad \text{as } x \uparrow \infty, \text{ for } \Re s \geq 0 ,
\]
\[
\sim x^{s+1/2} \quad \text{as } x \downarrow 0 \text{ for } s \not\in \mathbb{Z}_{\leq -1};
\]
\[
V_{\kappa,s}(x) \sim -e^{-\pi i\kappa} x^{-\kappa} e^{x^{1/2}} \quad \text{as } x \uparrow \infty .
\]

The functions \(W_{\kappa,s}\) and \(V_{\kappa,s}\) are even in the parameter \(s\). If \(\frac{1}{2} + s - \kappa \not\in \mathbb{Z}_{\geq 0}\), we have as \(x \downarrow 0\)
\[
W_{\kappa,s}(x) \sim \begin{cases} 
\frac{\Gamma(2|s|)}{\Gamma(\frac{1}{2} + |\kappa| - s)} x^{1/2 - |s|} & \text{if } s > 0 , \\
-\frac{1}{\Gamma(\frac{1}{2} - s)} x^{1/2} \log x & \text{if } s = 0 . 
\end{cases}
\]  
(C.8)

There are the specializations
\[
M_{\kappa,\kappa-1/2}(x) = W_{\kappa,\kappa-1/2}(x) = x^\kappa e^{-x^{1/2}} ,
\]
\[
M_{\kappa,-\kappa-1/2}(x) = -e^{-\pi i\kappa} V_{\kappa,\kappa+1/2}(x) = \tau^{-\kappa} e^{\kappa^{1/2}} .
\]  
(C.9)

In \(\text{[2, (A.18)-(A.21)]}\) we quote many contiguous relations, allowing us to express derivatives of Whittaker functions in terms of Whittaker functions with neighboring values of the parameters.

C.3. \textbf{Wronskians.} For two expressions \(X\) and \(Y\) we denote by \(\text{Wr}(X, Y)_t\) the quantity \(X \frac{dY}{dt} - \frac{dX}{dt} Y\). The skew-symmetry implies that \(\text{Wr}(X, Y)_t = 0\).

For solutions \(X\) and \(Y\) of the differential equation (C.1) one can easily check that \(\text{Wr}(X, Y)_t\) is a multiple of \(\frac{1}{X}\), and for solutions of (C.4) the Wronskian is a constant.

The asymptotic behavior near zero gives
\[
\text{Wr}(I_\nu(x), I_{-\nu}(x))_x = -\frac{2 \sin \pi \nu}{\pi x} , \quad \text{Wr}(M_{\kappa,s}(x), M_{\kappa,-s}(x))_x = -2x .
\]  
(C.10)

Using the expression \(\text{[2, (A.4)]}\) for \(K_\nu\) in terms of \(I_\nu\) and \(I_{-\nu}\) we get
\[
\text{Wr}(I_\nu(x), K_\nu(x))_x = -x^{-1} ,
\]
and similarly with \(\text{[2, (A.11)]}\)
\[
\text{Wr}(M_{\kappa,s}(x), W_{\kappa,s}(x))_x = \frac{\Gamma(1 + 2s)}{\Gamma(\frac{1}{2} - \kappa + s)} .
\]
This leads to the relations that we use in the proof of Theorem 7.8:
\[
\text{Wr}(t^2 I_n(2\pi|\beta|t), t^2 K_{\nu}(2\pi|\beta|t)) = -t^3, \\
\text{Wr}(t M_{k,v/2}(2\pi|\ell|^t), t W_{k,v/2}(2\pi|\ell|^t)) = -4\pi t \Gamma(\nu + 1) \frac{\Gamma\left(\frac{\nu+1}{2} - \kappa\right)}{\Gamma\left(\frac{\nu+1}{2}\right)} t^3.
\]

**Appendix D. Poincaré series for \(SL_2(\mathbb{R})\)**

Here we give some more information related to the discussion in Subsection 6.4. We work with functions on the group \(SL_2(\mathbb{R})\), like we do for \(SU(2, 1)\). We use Iwasawa coordinates \((x, y, \theta) \mapsto n(x)a(y)k(\theta)\), with \(n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\), \(x \in \mathbb{R}; a(y) = \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}\), \(y > 0\); and \(k(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}\). \(\theta \in \mathbb{R}\) mod \(2\pi\). A function \(g\) on \(SL_2(\mathbb{R})\) has weight \(k \in \mathbb{Z}\) if \(f(gk(\theta)) = f(g)e^{ik\theta}\), so the weight coincides with the \(K\)-type. The Casimir operator \(C\) is given by

\[
C = -y^2 \partial_x^2 - y^2 \partial_y^2 + y \partial_x \partial_y.
\]

For automorphic forms of odd weight we need a character on the discrete subgroup of the same parity as the weight.

Automorphic forms on \(\Gamma\backslash SL_2(\mathbb{R})\) for the character \(\chi\) are defined as in Definition 4.1, except that only in a) we require \(f(\gamma g) = \chi(\gamma)f(g)\) for all \(\gamma \in \Gamma\). The ring \(ZU(sl_2)\) is the polynomial ring in the Casimir operator. We write the eigenvalue of the Casimir operator as \(\frac{1}{2} - \nu^2\).

For simplicity we choose the discrete subgroup \(\Gamma = SL_2(\mathbb{Z})\) which has twelve characters: \(\chi_n, n \in \frac{1}{12}\mathbb{Z}\) mod \(\mathbb{Z}\), determined by

\[
\chi_n(n(1)) = e^{2\pi in}, \quad \chi_n(k(-\pi/2)) = e^{-6\pi in}.
\]

The character \(\chi_n\) is suitable for weight \(k\) if \(12n \equiv k \mod 2\).

**Basis functions** for the spaces of Fourier terms of order \(n \neq 0\) are

\[
\Omega_k(n, \nu; n(x)a(y)k(\theta)) = e^{2\pi inx} W_{ek/2, \nu}(4\pi|n|y) e^{ik\theta},
\]

\[
M_k(n, \nu; n(x)a(y)k(\theta)) = e^{2\pi inx} M_{ek/2, \nu}(4\pi|n|y) e^{ik\theta},
\]

with \(\varepsilon = \text{Sign} (n)\). These functions have weight \(k\) and eigenvalue \(\frac{1}{4} - \nu^2\) under the Casimir operator.

The **Poincaré series**

\[
P_k(n, \nu; g) = \sum_{\gamma \in \Gamma \backslash \Gamma} \chi(\gamma)^{-1} M_k(n, \nu; \gamma g)
\]

converges absolutely for \(\text{Re} \nu > \frac{1}{2}\) and gives a non-trivial family \(\nu \mapsto P_k(n, \nu)\) of real-analytic automorphic forms on \(\Gamma\backslash SL_2(\mathbb{R})\) for the character \(\chi\), of weight \(k\), and with moderate exponential growth. (Only the Fourier term of order \(n\) has exponential growth for general values of \(\nu\).)

The family has a meromorphic extension to \(\nu \in \mathbb{C}\) and all its main parts are automorphic forms with moderate exponential growth. This can be shown by spectral theory, as in [14].
Several results in Section 6.3 are similar to results for $\Gamma\backslash SL_2(\mathbb{R})$. We quote some statements from [1, Proposition 11.3.9].

For even weights $k$, the Poincaré series $P_k(n, \nu)$ has a first order singularity at $\nu = \nu_0 \in i\mathbb{R} \setminus \{0\} \cup (0, \frac{1}{2})$ if and only if there is a square integrable automorphic form $\varphi$ of weight $k$ with character $\chi_n$ and eigenvalue $\frac{1}{4} - \nu_0^2$ that has a non-zero Fourier term of order $n$. Such an automorphic form $\varphi$ generates a submodule of $L^2,\text{discr}(\Gamma \backslash G, \chi)_K$ in the unitary principal series or in the complementary series. For $\nu_0 = 0$ the same equivalence holds, but now with a second order singularity.

For odd weights, a similar statement holds, with $\nu = \nu_0 \in i\mathbb{R} \setminus \{0\}$ (no complementary series). These results are similar to those in Proposition 6.2 above. For $SL_2(\mathbb{R})$ there are no irreducible modules analogous to the thin representations.

The discrete series type representations of $SL_2(\mathbb{R})$ are analogous to the holomorphic and antiholomorphic discrete series type modules for $SU(2, 1)$. Part i) of Proposition 6.4 is analogous to the fact that spaces of holomorphic cusp forms of weights at least three on the upper half plane are spanned by values of Poincaré series. These results are similar to those in Proposition 6.2 above. For $SL_2(\mathbb{R})$ it is known that the space of holomorphic square integrable automorphic forms has dimension 1 for the character $\chi_{1/12}$, and is zero for other characters. For character $\chi_{1/12}$ the space is spanned by the square of the Dedekind eta function. The corresponding function on $SL_2(\mathbb{R})$ is

$$\varphi_+((n(x)a(y)k(\theta))) = y^{1/2} \eta(x + iy)^2 e^{i\theta}.$$  

We have also

$$\varphi_-((n(x)a(y)k(\theta))) = y^{1/2} \overline{\eta(x + iy)^2} e^{-i\theta}$$  

with character $\chi_{-1/12}$ and weight $-1$. Here, the element $\varphi_+$ is a lowest weight vector in a submodule of a principal series representation, and $\varphi_-$ is a highest weight vector in the complementary module.

From an initial part of the Fourier expansion

$$\eta(z)^2 = q^{1/12} \left( 1 - 2q - q^2 + 2q^3 + q^4 + 2q^5 - q^6 - 2q^8 - 2q^9 + \cdots \right), \quad q = e^{2\pi iz},$$

we can conclude that $P_{\pm 1}(n, \nu)$ has a first order singularity at $\nu = 0$ for $n = \pm (\frac{1}{12} + m)$ with $0 \leq m \leq 6$ and $m = 8, 9$, and that it is holomorphic at $\nu = 0$ for $m = 7$.

For the other five characters suitable for odd weights, the Poincaré families are holomorphic at $\nu = 0$.

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