MIXING AND DECAY OF CORRELATIONS IN NON-UNIFORMLY EXPANDING MAPS

STEFANO LUZZATTO

ABSTRACT. I discuss recent results on decay of correlations for non-uniformly expanding maps. Throughout the discussion, I address the question of why different dynamical systems have different rates of decay of correlations and how this may reflect underlying geometrical characteristics of the system.

1. INTRODUCTION

One of the basic questions of the theory of Dynamical Systems is to describe the dynamics determined by the iterates of a map $f : M \to M$ on some space $M$. If $x \in M$, we say that the (forward) orbit of $x$ is the set $\theta(x) = \{f^i(x)\}_{i=0}^{\infty}$. It is well known that even for “simple” maps $f$, the topological and geometrical structure of orbits can be extremely complicated. However it is possible in some cases to obtain some remarkable results by focusing on the statistical properties of these orbits rather than their precise geometrical and topological structure. In this survey we shall always assume that $M$ is a smooth compact Riemannian manifold of dimension $d \geq 1$. For simplicity we shall call the Riemannian volume Lebesgue measure and denote it by $m$ and assume that it is normalized so that $m(M) = 1$. Let $f : M \to M$ be a piecewise $C^2$ map. For $x \in M$ we let $Df_x$ denote the derivative of $f$ at $x$ and define $\|Df_x\| = \max\{\|Df_x(v)\| : v \in T_xM, \|v\| = 1\}$. We are interested in formulating some expansion conditions on $f$. The condition $\|Df_x\| > 1$ implies that there is at least one vector which is expanded by $Df_x$. On the other hand, the condition $\|Df_x^{-1}\| < 1$, or equivalently $\|Df_x^{-1}\|^{-1} > 1$, means that all vectors are contracted by the inverse of $Df_x$ and thus that all vectors are expanded by $Df_x$. 

Date: Draft Version : September 2003.

2000 Mathematics Subject Classification. Primary: 37D25, 37A25.

Thanks to Gerhard Keller, Giulio Pianigiani, Mark Pollicott, Omri Sarig, and Benoit Saussol for useful comments on a preliminary version of this paper.
**Definition 1.** We say that $f$ is *expanding on average*, or *non-uniformly expanding* \(^1\) if there exists $\lambda > 0$ such that

\[
(*) \quad \liminf_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \| Df^{-1}(x) \|^{-1} > \lambda
\]

for almost every $x \in M$.

An equivalent, and perhaps more intuitive, formulation is that for almost every $x$ there exists a constant $C_x > 0$ such that

\[
\prod_{i=0}^{n-1} \| Df^{-1}(x) \|^{-1} \geq C_x e^{\lambda n} \quad \text{for every } n \geq 1.
\]

Thus every vector is expanded at a uniform exponential rate, although the constant $C_x$ which can in principle be arbitrarily small, indicates that arbitrarily large number of iterates may be needed before this exponential growth becomes apparent. The expansivity is reflected at the level of the manifold by an exponentially fast divergence of nearby orbits, which produces the well known phenomenon of *sensitive dependence on initial conditions*, in which small round-off errors increase very rapidly and orbits which start together appear very quickly to have completely independent dynamics. This is often interpreted in a somewhat negative sense as leading to *unpredictability* and *chaos*. However it turns out that the very same expansivity property can give rise to some remarkable probabilistic structures which imply that there is a well-defined statistical coherence in the dynamics of different orbits. More precisely, it is possible to show that in many cases there exists an *ergodic absolutely continuous invariant probability measure*. This implies that the asymptotic average distribution of orbits in space is the same for almost every initial condition $x$. Thus the sensitive dependence on initial conditions implies that we cannot control the exact location of a given point at a given time, but the probabilistic structure implies that we can know that typical points will spend certain average amounts of time in certain regions of the space.

The combination of the sensitive dependence on initial conditions and well defined average behaviour gives rise to the phenomenon of *mixing*, which is a formalization of the notion that the dynamics, albeit completely deterministic, behaves to a large extent like a stochastic process. Leaving the precise definitions until later, we mention that the *degree of stochasticity*

\(^1\)The *non-uniformly expanding* here should be interpreted in the sense of *not necessarily* uniformly expanding rather than strictly *not* uniformly expanding so that the uniformly expanding case to be defined below is a special case. The terminology is not optimal but unfortunately the term *expanding* is traditionally interpreted to mean *uniformly expanding*.
can be quantified through the notion of rate of mixing or rate of decay of correlations which, in some sense, measure the rate at which the deterministic system is approaching a stochastic process, or the speed at which memory is lost. It is known that a very wide range of rates can occur in different systems: from exponential to stretched exponential to polynomial to logarithmic. However it is not completely clear why different systems exhibit different rates of decay of correlations. The mathematics used to estimate the rate of decay of correlations for a particular system does not necessarily provide any intuitive justification for why a particular rate occurs. However some recent developments are beginning to give us some insight into the mechanisms involved, and point towards a subtle connection between the rate of decay of correlations and the finer geometrical structures associated to the dynamics. The purpose of these notes is to survey some recent results on rates of decay of correlations and to attempt, by bringing these results together, to formulate some opinion on the general question

*What aspect of a non-uniformly expanding map determines the rate of mixing?*

For completeness we review the basic definitions related to invariant measures and decay of correlations in Section 2. We then formulate the known results on decay for correlations for uniformly expanding maps, maps with indifferent fixed points, one-dimensional maps with critical points, and a class of two dimensional Viana maps. These are pretty much all the particular cases of maps for which results in this direction are known.  

This survey of known results suggests that one way of understanding the overall picture is the following. The exponential decay of correlations of uniformly expanding maps represents a kind of default rate, with slower rates occurring as a consequence of some intrinsic geometrical feature of a map which literally slows down the process of orbits distributing themselves over the entire space. In some of the simpler examples this slowing down phenomenon is perfectly apparent and it is even possible to tweak some parameters to get essentially any desired (subexponential) rate. We do not yet have a general theory which confirms this point of view rigorously, although we shall present some recent results in this direction, which establish a connection between the rate of decay of correlations and a certain measure of non-uniformity of the expansion, related to the speed at which the liminf in the definition of non-uniformly expanding maps is attained.

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2We will concentrate in this survey on non-uniformly expanding maps and not enter into any discussion of system which have contracting as well as expanding directions, such as partially or non-uniformly hyperbolic systems. The spirit of the discussion remains valid in those examples but there would be too much additional notation and different cases to consider and the essential points of the discussion would be overshadowed.
2. Basic notions

2.1. Invariant measures. The first key concept is that of an invariant probability measure.

Definition 2. A probability measure \( \mu \) on \( M \) is invariant under \( f : M \to M \) if

\[
\mu(f^{-1}(A)) = \mu(A)
\]

for every \( \mu \)-measurable set \( A \subset M \).

Notice that this condition is not equivalent to the condition \( \mu(f(A)) = \mu(A) \) unless \( A \) is invertible. In the general non-invertible case we define \( f^{-1}(A) = \{ x : f(x) \in A \} \). By a general result of Ergodic Theory, some mild conditions on the map \( f : M \to M \) ensure that there exists at least one invariant measure. However it is often the case, for example in the case of expanding maps which we are considering, that there are an infinite number (for example every periodic orbit admits an invariant probability measure concentrated on that orbit). In this case there is an issue about which of these measures is to be considered the most significant and what the relation is between them. To discuss this notion in more depth we need some more definitions.

2.2. Ergodicity and absolutely continuous measures. The relevance of invariant probability measures is that they capture the statistical features of the dynamics in a sense to be defined below. First we need one more definition.

Definition 3. Let \( \mu \) be an \( f \)-invariant measure. We say that \( \mu \) is ergodic if there do not exist measurable sets \( A, A^c = M \setminus A \) with \( f^{-1}(A) = A \) (and thus also \( f^{-1}(A^c) = A^c \)) and \( \mu(A) \in (0, 1) \) (and thus also \( \mu(A^c) = \mu(M \setminus A) \in (0, 1) \)).

This means that if \( \mu \) is not ergodic, there are (at least) two components of the ambient space \( M \) which never interact; thus the dynamical properties of \( f \) on \( A \) completely independent of the dynamics \( A^c \). Any measure can be decomposed into a number of ergodic components and, by a particular case of what is perhaps the most remarkable result of the Theory of Dynamical Systems, there is a strong statistical coherence in the dynamics of typical points associated to each ergodic component.

Theorem (Birkhoff). Let \( \mu \) be an ergodic invariant measure for \( f \). Then, for any measurable set \( A \subset M \) and \( \mu \) almost every \( x \in M \), we have

\[
\frac{\# \{ 1 \leq j \leq n : f^j(x) \in A \}}{n} \to \mu(A).
\]
Here \( \# \{1 \leq j \leq n : f^j(x) \in A \} \) denotes the cardinality of the set of indices \( j \) for which \( f^j(x) \in A \). The statement contains two parts. The first is the convergence of the ratio on the left to some limit, a fact which is in itself quite remarkable. If we interpret \( f^j(x) \in A \) as meaning that the event \( A \) occurs at time \( j \) for the initial condition \( x \) then this convergence means that there is a statistical pattern to the occurrence of such an event: as the number of iterates increases, the proportion of times for which the event occurs stabilises. The second part of the statement is that this limit is precisely the \( \mu \)-measure of \( A \), sometimes stated in the form time averages equal space averages, where “space” is measured according to the invariant measure \( \mu \).

Of course, if the measure \( \mu \) is singular with respect to the reference measure \( m \) on \( M \) given by the Riemannian volume, then this result only concerns a set of zero volume and leaves us in the dark about the dynamics of all other points. Things are better if the measure is absolutely continuous with respect to the Riemannian volume.

**Definition 4.** The measure \( \mu \) is absolutely continuous with respect to \( m \) if \( m(A) = 0 \) always implies \( \mu(A) = 0 \).

In particular, if \( \mu \) is absolutely continuous with respect to \( m \), any set which has positive measure for \( \mu \) must also have positive measure for \( m \). Therefore, in this case, Birkhoff’s Theorem immediately implies that there exists at least a positive volume set of points which are typical with respect to \( \mu \).

### 2.3. Mixing and decay of correlations.

An important first step in the analysis of non-uniformly expanding maps in general, and in specific examples, is to show that they admit an ergodic absolutely continuous invariant probability measure. The next is to study the mixing properties of the map with respect to that measure [46, 49, 55, 87, 88].

**Definition 5.** An invariant probability measure \( \mu \) is mixing if

\[
|\mu(A \cap f^{-n}(B)) - \mu(A)\mu(B)| \to 0
\]

as \( n \to \infty \), for all measurable sets \( A, B \subseteq M \).

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3In many examples of area-contracting diffeomorphisms any invariant measure is necessarily supported on some attractor which has zero volume, and thus any invariant measure is necessarily singular with respect to the Riemannian volume. In many such cases there is a very sophisticated theory of stable manifolds which allows us to show that typical points in \( M \) actually converge not just to the attractor as a whole but to typical points in the attractor, and are therefore typical with respect to the invariant measure; a conclusion which does not follow immediately from Birkhoff’s Theorem.
One way to interpret this condition is to notice that it is equivalent to the condition
\[ \left| \frac{\mu(A \cap f^{-n}(B))}{\mu(B)} - \mu(A) \right| \to 0 \]
as \( n \to \infty \), for all measurable sets \( A, B \subseteq M \), with \( \mu(B) \neq 0 \). Then one can think of \( f^{-n}(B) \) as a “redistribution of mass”, notice that \( \mu(f^{-n}(B)) = \mu(B) \) by the invariance of the measure. Then the mixing condition says that for large \( n \) the proportion of \( f^{-n}(B) \) which intersects \( A \) is just proportional to the measure of \( A \). In other words \( f^{-n}(B) \) is spreading itself uniformly with respect to the measure \( \mu \). A more probabilistic point of view is to think of \( \mu(A \cap f^{-n}(B)) = \mu(B) \) as the conditional probability of having \( x \in A \) given the fact that \( f^n(x) \in B \), i.e. the probability that the occurrence of the event \( B \) today is a consequence of the occurrence of the event \( A \) exactly \( n \) steps in the past. The mixing condition then says that this probability converges precisely to the probability of \( A \). Thus, asymptotically, there is no causal relation between the two events.

The mixing condition can be written in an integral form as
\[ \left| \int \chi_{A \cap f^{-n}(B)} d\mu - \int \chi_A d\mu \int \chi_B d\mu \right| \to 0 \]
or even
\[ \left| \int \chi_A (\chi_B \circ f^n) d\mu - \int \chi_A d\mu \int \chi_B d\mu \right| \to 0 \]
This last formulation now admits a natural generalization by replacing the characteristic functions with arbitrary measurable functions.

**Definition 6.** For real valued measurable functions \( \varphi, \psi : M \to \mathbb{R} \) we define the correlation function
\[ C_n(\varphi, \psi) = \left| \int \psi(\varphi \circ f^n) d\mu - \int \psi d\mu \int \varphi d\mu \right| \]
In this context, the functions \( \varphi \) and \( \psi \) are often called observables. Assuming that the measure \( \mu \) is mixing, a natural question is whether \( C_n(\varphi, \psi) \) tends to 0 for general observables and, if so, whether there is any particular speed at which the correlation function decays, i.e. does there exist a sequence \( \{ \gamma_n \} \) with \( \gamma_n \to 0 \) as \( n \to \infty \) (for example \( \gamma_n = e^{-\alpha n} \) or \( \gamma_n = n^{-\alpha} \)), depending only on the map \( f \) and on the class of admissible observables, such that for any two admissible \( \varphi, \psi \) there exists a constant \( C_{\varphi, \psi} \) such that
\[ C_n(\varphi, \psi) \leq C_{\varphi, \psi} \gamma_n \quad \forall n. \]
It turns out that the answer is negative if the class of admissible observables is too large, e.g. \( L^\infty(\mu) \) which in particular contains the characteristic functions. Indeed, it is a classical result that for any given sequence \( \{ \gamma_n \} \)
it is possible to choose measurable sets $A, B$ such that the correlation function $C_n(\chi_A, \chi_B)$ decays slower than the rate determined by the sequence $\{\gamma_n\}$. However, if the class of admissible observables is restricted to functions with some regularity, e.g. continuous functions with some conditions on the modulus of continuity such as Hölder continuous functions, it turns out that for a large and significant class of dynamical systems it is indeed possible to speak of a particular rate of mixing or rate of decay of correlations. Unless explicitly mentioned below, we shall always assume that we are dealing with Hölder continuous observables.

3. Uniformly Expanding Maps

3.1. The smooth case. The definition of (non-uniformly) expanding includes as a special case the classical uniformly expanding case.

**Definition 7.** We say that $f$ is uniformly expanding if there exist constants $C, \lambda > 0$ such that for all $x \in M$, all $v \in T_x M$, and all $n \geq 0$, we have

$$\|Df^n(v)\| \geq Ce^{\lambda n}\|v\|.$$

In this case, as in most of the cases which will be discussed below, the proof of the existence of an absolutely continuous invariant measure historically predates the estimates for the rate of decay of correlations, and can usually be obtained through significantly simpler arguments.

**Theorem 1** ([10, 41, 60, 62, 82, 86, 89, 106]). Let $f : M \to M$ be $C^2$ uniformly expanding. Then there exists a unique absolutely continuous mixing invariant probability measure $\mu$.

**Theorem 2** ([17, 83, 90, 96]). Let $f : M \to M$ be $C^2$ uniformly expanding. Then the correlation function decays exponentially fast.

Several of the arguments used in the proofs of these results rely on the fact that smooth uniformly expanding systems have a geometrical Markov structure, that is there exists a partition of $M$ modulo sets of measure 0 (finite in this particular case), such that $f$ is $C^2$ on each partition element and maps each element to the whole manifold, or to some suitably large union of other partition elements. The results then generalize quite naturally to the more general Markov case, even if the partition is countable, as long as some mild technical conditions are satisfied [20, 74, 75, 108, 109].

3.2. The piecewise smooth case. The general (non-Markov) piecewise expanding case is significantly more complicated and even the existence of an absolutely continuous invariant measure is no longer guaranteed [27, 43, 63, 99]. The main problems lie in the fact that the images of the discontinuity
set can be very badly distributed and cause havoc with any kind of structure. In the Markov case this does not happen because the set of discontinuities gets mapped to itself by definition. Also the possibility of components being translated in different directions can destroy on a global level the local expansiveness given by the derivative. Moreover, where results exist for rates of decay of correlations, they do not always apply to the case of Hölder continuous observables, as technical reasons sometimes require that different functions spaces be considered which are more compatible with the discontinuous nature of the maps. We shall not explicitly comment on the particular classes of observables considered in each case.

In the one-dimensional case these problems are somewhat more controllable and relatively simple conditions guaranteeing the existence of an ergodic invariant probability measure can be formulated even in the case of a countable number of domains of smoothness of the map. These essentially require that the size of the image of all domains on which the map is $C^2$ strictly positive and that certain conditions on the second derivative are satisfied \cite{2,18,19,63}. In the higher dimensional case, the situation is considerably more complicated and there are a variety of possible conditions which can be assumed on the discontinuities. The conditions of \cite{63} were generalized to the two-dimensional context in \cite{56} and then to arbitrary dimensions in \cite{26,42,100}. There are also several other papers which prove similar results under various conditions, we mention \cite{3,28,30,32,33,95}. In \cite{29,37} it is shown that conditions sufficient for the existence of a measure are generic in a certain sense within the class of piecewise expanding maps.

Exponential decay of correlations has also been proved for non-Markov piecewise smooth maps, although again the techniques have had to be considerably generalized. In terms of setting up the basic arguments and techniques, a similar role to that played by \cite{63} for the existence of absolutely continuous invariant measures might be attributed to \cite{47,57,92} for the problem of decay of correlations in the one-dimensional context. More recently, alternative approaches have been proposed and implemented in \cite{64,108}. The approach of \cite{108} has proved particularly suitable for handling some higher dimensional cases such as \cite{31} in which assumptions on the discontinuity set are formulated in terms of the topological pressure of this set and exponential decay of correlations is proved, and \cite{6,8} in which the assumptions on the discontinuity set are formulated as geometrical non-degeneracy assumptions, which are essentially conditions on the first and second derivatives of the map near the discontinuity, and dynamical assumptions on the rate of recurrence of typical points to the discontinuities. We remark however that the estimates in \cite{6,8} are sub-exponential (and sub-optimal) and clearly weaker than the exponential decay of correlations.
obtained in [31], perhaps because they are carried out in a general framework which allows the possible presence of critical points (this will be discussed in more detail in Section 7). It would be interesting to know how the assumptions of [6, 8] and those of [31] are related.

4. Almost uniformly expanding maps

4.1. Neutral fixed points. Perhaps the simplest way to weaken the uniform expansivity condition is to consider a one-dimensional map which is expanding, i.e. \(|f(x)| > 1\), everywhere except at some fixed point \(p\) at which \(f'(p) = 1\). The fixed point \(p\) is still repelling, but nearby points remain close to \(p\) much longer than they would if the derivative were > 1. On the other hand the dynamics away from the fixed point is uniformly expanding and orbits tend to distribute themselves over the whole interval quite quickly. Thus the overall effect is that orbits tend to spend a long time trapped in a neighbourhood of the fixed point with relatively short bursts of chaotic activity outside this neighbourhood. This is characteristic of the phenomenon of intermittency which appears to be common in many natural phenomena and is an important feature for example in the theory of Self-Organized Criticalities [11, 54]. Indeed, this was one of the motivations which led to the study of maps with this kind of characteristics, see [73].

More formally, we suppose that there exists a partition \(P\) of \([0, 1]\) into a finite number of subintervals and that \(f\) is \(C^2\) in the interior of each partition element with a \(C^1\) extension to the boundaries and that the derivative is strictly greater than 1 everywhere except at a fixed point \(p\) (which for simplicity we can assume lies at the origin) where \(f'(p) = 1\). For the moment we assume also a strong Markov property: each partition element is mapped bijectively to the whole interval. First of all we want to focus on the consequences of the presence of the neutral (or indifferent) fixed point \(p\). For definiteness, let us suppose that on a small neighbourhood of 0 the map takes the form

\[ f(x) \approx x + x^2 \phi(x) \]

where \(\approx\) means that the terms on the two sides of the expression as well as their first and second order derivatives converge as \(x \to 0\). We assume moreover that \(\phi\) is \(C^\infty\) for \(x \neq 0\); the precise form of \(\phi\) determines the precise degree of neutrality of the fixed point, and in particular affects the second derivative \(f''\). It turns out that it plays a crucial role in determining the mixing properties and even the very existence of an absolutely continuous invariant measure.

4.2. Loss of mixing. The following result shows that the situation is drastically different from the uniformly expanding case.
Theorem 3. \[84\] If \( f \) is \( C^2 \) at the neutral fixed point, then \( f \) does not admit any absolutely continuous invariant probability measures.

This case occurs if, for example, \( \phi(x) \equiv 1 \). It is interesting to note that \( f \) has the same topological behaviour as a uniformly expanding map, typical orbits continue to wander densely on the whole interval, but the proportion of time which they spend in various regions tends to concentrate on the fixed point, so that, asymptotically, typical orbits spend all their time near 0. It turns out that in this situation there exists an infinite (\( \sigma \)-finite) absolutely continuous invariant measure which gives finite mass to any measurable set not containing the fixed point 0 and infinite mass of any neighbourhood of \( p \) \[97\].

4.3. Non-uniform expansivity. The situation changes if we relax the condition that \( f \) be \( C^2 \) at \( p \) and allow the second derivative \( f''(x) \) to diverge to infinity as \( x \to p \). This means that the derivative increases quickly near \( p \) and thus nearby points are repelled at a faster rate. Although this is apparently a very subtle change, it makes all the difference. With some mild conditions on the rate of divergence of \( f'' \) near \( p \) it is possible to recover the existence of an absolutely continuous probability measure \( \mu \) \[51\] \[84\]. Typical points still spend a large proportion of time near \( p \) but they now also spend a positive proportion of time in the remaining part of the space. In particular they are non-uniformly expanding: by a simple application of Birkhoff’s Ergodic Theorem to the function \( \log |f'(x)| \), we have that, for \( \mu \)-almost every \( x \),

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log |f'(f^i(x))| \to \int \log |f'| d\mu > 0.
\]

The fact that \( \int \log |f'| d\mu > 0 \) follows from the simple observation that \( \mu \) is absolutely continuous and finite, and that \( \log |f'| > 0 \) except at the neutral fixed point.

4.4. Polynomial decay of correlations. If \( \phi \) is of the form

\[ \phi(x) = x^{-\alpha} \quad \text{for some } \alpha \in (0, 1), \]

then we have the following

Theorem 4. \[50\] \[52\] \[61\] \[94\] \[109\] \( f \) admits a mixing (in particular ergodic) invariant probability measures with decay of correlations

\[
C_n = O(n^{1-\frac{1}{\alpha}}).
\]

Thus, the existence of an absolutely continuous invariant measure as in the uniformly expanding case has been recovered, but the exponential rate of decay of correlation has not. Thinking of the rate of decay of correlation
as related to the mixing process, and in particular to the speed at which the mass gets redistributed by the dynamics, we can think of the indifferent fixed point as having the effect of *slowing down* this process by trapping nearby points for disproportionately long time. In fact we have a *slower rate of decay* if $\alpha$ is larger, which corresponds to the second derivative $f''$ diverging more slowly as the fixed point is approached. Geometrically, the order of the tangency between the graph of $f$ and the diagonal increases with $\alpha$, thus the fixed point is becoming *less repelling*, points tend to remain trapped for longer, and the rate of decay of correlations is slower.

We remark that the estimate in [94] shows that the rate stated is optimal by obtaining lower bounds as well as upper bounds. Most of the results stated in this survey are upper bounds. A brief discussion of this issue is given at the end of the paper.

### 4.5. Logarithmic decay of correlations

This intuitive explanation for the connection between the order of the fixed point and the rate of decay of correlations is supported by recent work [48] in which the situation is taken to an extreme by considering very general functions $\phi$ satisfying a *slowly varying condition* [11]. This gives a range of possible rates of decay of correlation, including logarithmic, intermediate logarithmic and intermediate polynomial. As an example, if we let $\phi$ be of the form:

$$\phi(1/x) = \log x \log^{(2)} x \ldots \log^{(r-1)} x (\log^{(r)} x)^{1+\alpha}$$

for some $r \geq 1$, $\alpha \in (-1, \infty)$ where $\log^{(r)} = \log \log \ldots \log$ repeated $r$ times, we get the following result:

**Theorem 5.** [48] $f$ admits a mixing absolutely continuous invariant probability measure with decay of correlations $$C_n = O(\log^{(r)} n)^{-\alpha}.$$  

We remark that the methods of [109] for the proof of Theorem 4 apply to a considerably larger class of maps than those explicitly defined here. In particular we can consider maps with any arbitrary finite number of neutral fixed or periodic orbits and, most importantly, the Markov condition can be relaxed by adding some mild additional conditions on the expansivity (i.e. assuming that $f' \geq \mu > 2$ on the partition elements which do not contain the neutral fixed point). The arguments of [48] are based on generalizations of the methods of [109] and are therefore very likely to extend to give logarithmic decay of correlations in these additional cases as well. Recently there have been some generalizations of the results above to higher-dimensional situations, see [50, 85].
5. ONE-DIMENSIONAL MAPS WITH CRITICAL POINTS

5.1. Unimodal and multimodal maps. We now consider another class of systems which can also exhibit various rates of decay of correlations, but where the mechanism for producing these different rates is significantly more subtle. We consider the class of $C^3$ maps $f : [0, 1] \to [0, 1]$ with some finite number of non-flat critical points. We say that $c \in [0, 1]$ is a critical point if $f'(c) = 0$; the critical point is non-flat if there exists an $0 < \ell < \infty$ called the order of the critical point, such that $|f(x)| \approx |x - c|^{\ell}$ for $x$ near $c$; $f$ is unimodal if it has only one critical point, and multimodal if it has more than one. We shall allow $f$ to have an arbitrary finite number of critical points but assume that they all have the same order. We shall also assume a standard technical condition called negative Schwarzian derivative which is a kind of convexity assumption on the derivative of $f$. Although the results we shall describe apply in significant generality, they are already interesting and highly non-trivial in the case of maps belonging to the well known logistic family

$$f_a(x) = ax(1 - x)$$

which has a unique critical point of order $\ell = 2$ (and satisfies the negative Schwarzian derivative condition).

5.2. Non-uniform expansion. As mentioned above, the questions concerning the existence and ergodicity of an absolutely continuous invariant probability measure are of a somewhat different nature from those which concern the statistical properties of such measures. A first important observation is that the expanding properties of the maps, which play an important role in the analysis of the two classes of examples mentioned above, are not at all obvious here. Around the critical point there is a region in which the derivative is arbitrarily small, and close to the boundaries of the interval $[0, 1]$ there are regions in which the derivative is relatively large. The non-uniform expansivity condition is still possible in principle but depends on typical orbits spending on average more time in the expanding region than in the contracting region near the critical points. The first observation that this actually does happen goes back to Ulam and von Neumann [103] in which they show that the so called top quadratic map, $f(x) = 4x(1 - x)$ (they actually use a different but equivalent representation of the quadratic family), is $C^1$ conjugate to a uniformly expanding map. This allows them to show explicitly that there exists an absolutely continuous invariant probability measure and to show that the map is therefore non-uniformly expanding in the sense of our definition.
The approach of Ulam and von Neumann however relies on some special characteristics of the particular map they consider. It can be generalized to work for some relatively small (albeit infinite) set of parameter values, although this generalization depends on results from the theory of one-dimensional dynamics which were not available until the late 70’s and the beginning of the 80’s, many decades after the work of [103]. This theory has spawned a huge amount of research and it falls well beyond the scope of this survey to enter into the technical details of the differences between one result and another. We really just mention the basic principle of what has been understood, which is that a substantial amount of information about the overall expansivity properties of the map is contained in the expansivity properties of the orbits of the critical points, i.e. in the behaviour of the sequence $D_n(c) = |(f^n)'(f(c))|$. In the unimodal case, there are several papers proving the existence of absolutely continuous invariant probability measures under weaker and weaker assumptions: finite critical orbit [91], non-recurrent critical point [77] (both of these conditions imply exponential growth of $D_n$), exponential growth [36, 80], a summability condition [81], and more recently the remarkable paper [25] which supersedes all previous results by showing that an absolutely continuous invariant probability measure exists under the simple condition that $D_n \to \infty$ without any further assumption about the rate of growth of this derivative. The multimodal case has proved significantly harder and was first addressed in the following

**Theorem 6.** [23] If $f$ satisfies

$$
\sum_n D_n^{-1/(2\ell-1)} < \infty
$$

for each critical point $c$, then there exists an $f$-invariant probability measure $\mu$ absolutely continuous with respect to Lebesgue measure.

The assumptions have since been weakened to the summability condition $\sum D_n^{-1/\ell} < \infty$ and to allow the possibility of critical points of different orders [24].

It is worth mentioning that, unlike the examples of the previous sections, the conditions on the growth of the derivative along the critical orbit which define this class of examples, are generally not directly verifiable since they

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4The fact that the derivative is calculated in the critical value $f(c)$ rather than the critical point $c$ often causes confusion to people not used to working with maps with critical points. It is sufficient to observe however that by the chain rule we have $(f^n)'(x) = f'(x)f'(f(x)) \ldots f'(f^{n-1}(x))$ and therefore this would always be equal to zero if we choose $x = c$ since $f'(c) = 0$ by definition. Calculating the derivatives at $f(c)$ on the other hand gives an accurate reflection of the behaviour of the critical orbit.
involve the full forward orbit of the critical point. Moreover, these conditions are extremely unstable: arbitrarily small changes in the parameter can destroy the delicate balance between the number of iterates spent in the contracting region and in the expanding regions which cause the orbit to exhibit derivative growth on average.\textsuperscript{5} Non-trivial arguments are thus required to show that the appropriate growth conditions are satisfied for a significant set of parameter values. It turns out that in generic one-parameter families there exist nowhere dense sets of positive Lebesgue measure\textsuperscript{6} for which the corresponding maps admit an absolutely continuous invariant probability measure \cite{16,53,66,76,93,98,101,102}. There are also some generalizations to families of piecewise smooth maps with critical points \cite{67,68}.

5.3. Basic strategy. These results rely in a crucial way on the (one could say miraculous) property of such one-dimensional maps that, as long as there are no periodic attractors, the dynamics outside any fixed neighbourhood of the critical set is uniformly expanding \cite{71,72}, and on the remarkable observation that if the derivative along the critical orbit is bounded away from 0 there can be no attracting periodic orbits (as an immediate consequence of the fact that any attracting periodic orbit must have a critical point in its basin of attraction, see \cite{76}). In general this expansivity will degenerate as the size of the neighbourhood tends to zero, but the crucial fact remains that we can divide the interval into some arbitrarily small neighbourhood \(\Delta\) and its complement \([0,1] \setminus \Delta\); since the dynamics is uniformly expanding on \([0,1] \setminus \Delta\) it is necessary, and to some extent sufficient, to control the dynamics and recurrence of points in \(\Delta\). Orbits that pass through \(\Delta\) may lose a lot of the expansion which they had previously accumulated outside \(\Delta\), but if they do not fall in \(\Delta\) too often, and if, when they do, they do not fall too close to the critical point, it may happen that this loss is not sufficient to destroy the expansivity completely. One possible strategy for controlling the extent of this loss of expansion is to take advantage of the fact that points in \(\Delta\) are mapped extremely close to a critical value and thus inevitably shadow the orbit of the critical value for a certain amount of time. The closer the point is to the critical point \(c\), the longer the shadowing time. During this time, the point has essentially the same behaviour as the critical value and in particular has the same pattern of derivative growth. Thus if we know that the critical value has some expansivity properties and we can show that our original orbit shadows it for long enough, we can conclude

\textsuperscript{5}Strictly speaking this has only been proved in certain unimodal cases \cite{45,59,70}, but it is widely believed to be a very general fact.

\textsuperscript{6}Thus the non-uniform expansion property in the context of one-dimensional maps with critical points occurs for a set of parameters which is topologically small but measure-theoretically large.
that by the end of this shadowing period it will have regained the expansion it lost by having an iterate in $\Delta$.

The details of this argument rely on a balance between the rate of growth of the derivative along the critical orbits and the number of iterates for which a point $x \in \Delta$ shadows this orbit which in turn depend on how close $x$ is to the critical point which in turn determines how much expansion is lost at the first iterate. If the derivative is growing very fast, e.g. exponentially, then $x$ tends to get pushed away much faster, and the shadowing period is relatively short. However, since the derivative is growing very fast, this length of time is sufficient to recover the loss of derivative incurred at the return in $\Delta$. If the derivative along the critical orbit is growing more slowly, e.g. at a polynomial rate, then nearby points tend to get pushed away more slowly and the shadowing lasts significantly longer. This is good from the point of view of recovering expansion as more time is needed due to the very same slow derivative growth along the critical orbit. As far as the existence of an absolutely continuous invariant measure is concerned, the estimates do indeed work, and we can show that even for a relatively slow rate of growth of the derivative along the critical orbits, such as that given by the summability condition in the Theorem, the shadowing of the critical orbit is sufficient to compensate the small derivative at returns to $\Delta$ for almost all orbits, giving rise to a map which satisfies the non-uniform expansivity condition [23].

5.4. Rate of mixing. This approach becomes particularly interesting when addressing the question of the decay of correlations. It turns out that the existence of critical points is much less of an issue than the particular rate of growth of the derivative along the critical orbits, and that the conceptual picture is much more similar to the case of expanding maps with indifferent fixed points discussed above, than would appear at first sight. Indeed, we can think of the case in which the rate of growth of $D_n$ is subexponential as a situation in which the critical orbit is in some sense neutral or indifferent and in this respect very similar to the neutral or indifferent fixed point. The consequences of this are also very similar. Points which land close to the critical point tend to remain close to it's orbit for a particularly long time. Thus even though the orbit is not a fixed or even a periodic point, and may even be dense, it makes sense to still think of nearby points as being trapped by it for a certain length of time which depends on the particular rate of growth of the derivative. During this time small intervals are not distributing themselves over the whole space as uniformly as they should and thus the mixing process is delayed and the rate of decay of correlations is correspondingly slower. The same argument works also in the case in which the rate of growth of the derivatives $D_n$ is exponential. In this case the critical
orbits behave analogously to hyperbolic repelling orbits and nearby points are pushed away exponentially fast. Thus there is no significant loss in the rate of mixing and the decay of correlations is exponential in this case.

**Theorem 7.** [23] Let \( f \) satisfy

\[
\sum_n D_n^{-1/(2\ell-1)} < \infty
\]

for each critical point \( c \), and let \( \mu \) be an absolutely continuous invariant probability measure with support \( \text{supp}(\mu) \). If \( f \) is not renormalizable on \( \text{supp}(\mu) \), then \( (\text{supp}(\mu), \mu, f) \) is mixing with the following rates:

**Polynomial case:** If there exists \( C > 0, \tau > 2\ell - 1 \) such that

\[
D_n(c) \geq Cn^\tau,
\]

for all \( c \in C \) and \( n \geq 1 \), then, for any \( \tilde{\tau} < \frac{\tau-1}{\ell-1} - 1 \), we have

\[
C_n = \Theta(n^{-\tilde{\tau}})
\]

**Exponential case:** If there exist \( C, \beta > 0 \) such that

\[
D_n(c) \geq Ce^{\beta n}
\]

for all \( c \in C \) and \( n \geq 1 \), then there exist \( \tilde{\beta} > 0 \) such that

\[
C_n = \theta e^{-\tilde{\beta}n}.
\]

Thus the existence of the critical point, which certainly plays a fundamental role in determining many characteristics of the dynamics of these maps, can be considered a bit of a red herring as far as the rate of decay of correlations in concerned, i.e. it does not seem to be a crucial ingredient in and of itself. It is more useful to keep in mind the property of uniform expansion outside some small neighbourhood \( \Delta \) of the critical points, and to think of \( \Delta \) in a similar way to the neighbourhood of the indifferent fixed point in the previous class of examples. Due to the uniform expansion outside \( \Delta \), points behave in a stochastic-like away there and thus tend to fall in \( \Delta \) with some definite frequency. Once they fall in \( \Delta \) they remain *trapped* for a certain amount of time, not in \( \Delta \) itself, but in a small neighbourhood of some finite number of iterates of the corresponding critical orbit. If the critical orbit is neutral, this slows down the mixing process, but if it is exponential, this number of iterates is so small that it hardly affects the mixing process.

We remark that exponential decay of correlations in the unimodal case has been proved in [15, 107] assuming exponential derivative growth and a bounded recurrence condition along the critical orbit, and in [53] assuming only the exponential growth condition. The extension of this case to the
multimodal context, and the subexponential estimates, are proved for the first time in [23].

6. VIANA MAPS

Viana maps were introduced in [104] as an example of a class of systems which are strictly not uniformly expanding but for which the non-uniform expansivity condition is satisfied and, most remarkably, is persistent under small $C^3$ perturbations, which is not the case for any of the examples discussed above. These maps are defined as skew-products on a two dimensional cylinder of the form $f: \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{R}$

$$f(\theta, x) = (\kappa \theta, x^2 + a + \varepsilon \sin 2\pi \theta)$$

where $\varepsilon$ is assumed sufficiently small and $a$ is chosen so that the one-dimensional quadratic map $x \mapsto x^2 + a$ for which the critical point lands after a finite number of iterates onto a hyperbolic repelling periodic orbit (and thus is a good parameter value and satisfies the non-uniform expansivity conditions as mentioned above). The map $\kappa \theta$ is taken modulo $2\pi$, and the constant $\kappa$ is a positive integer which was required to be $\geq 16$ in [104] although it was later shown in [34] that any integer $\geq 2$ will work. The $\sin$ function in the skew product can also be replaced by more general Morse functions.

Viana proved that such class of skew-products are non-uniformly expanding, by showing directly that Lebesgue almost every point satisfies the non-uniform expansivity condition. It was then proved in [3, 9] that $f$ is topologically mixing and has a unique ergodic invariant measure which is absolutely continuous with respect to the two-dimensional Lebesgue measure. As regards the rate of decay of correlation however, the situation is more subtle than in the classes of examples mentioned above. There is no particular finite set of orbits which to play the role of either the indifferent fixed point or the critical points. There is a whole curve of critical points and different points on this curve have different behaviour which is very difficult to control. Thus the characteristics of the map which determine the rate of decay of correlation are less easy to pinpoint in a geometrical sense. Nevertheless, some geometrical structure can be obtained and used to obtain some estimates for the rate of decay of correlations. First estimates were obtained in [6–8] as a Corollary of a general theory of decay of correlations for non-uniformly expanding maps which will be discussed in the next section.

Theorem 8. [8] Viana maps have super-polynomial decay of correlations: for any $\gamma > 0$ we have

$$C_n = O(n^{-\gamma})$$
Sharper results have since been obtained in [14] by concentrating on Viana maps and taking advantage of some additional information which is not available in the general abstract setting, and also adapting some ideas from previous related work [13].

**Theorem 9.** [14] For all small enough \( \varepsilon > 0 \) there exists a constant \( C_\varepsilon > 0 \) such that Viana maps have stretched exponential decay of correlations:

\[
C_n = O(e^{\sqrt{n}/C_\varepsilon})
\]

Both results rely on some estimates from [104] and the results of Theorem 9 are probably the best one can do on the basis of those estimates. However it is quite possible that a fresh approach to Viana maps, or possibly even just an improved set of estimates based on the original arguments of Viana, could lead to estimates giving a faster rate of decay of correlations.

7. **General theory of non-uniformly expanding maps**

So far we have addressed the question of the rate of decay of correlations in several examples of non-uniformly expanding maps. In each case, the arguments used to obtain the results rely on particular features of the system. Here we want to discuss some general abstract theory of non-uniformly expanding maps. It turns out that it is possible to formulate a quantitative measure of the degree of non-uniformity of the system, which contains information about how close or how far the system is to being uniformly expanding. The following results show that the rate of decay of correlations is closely related to this measure of the non-uniformity of the expansion. They have been announced in [6] and are given in full details in [7] for the one-dimensional case, and in [8] in the case of maps in manifolds of arbitrary dimension. We start with the case in which \( f \) is a \( C^2 \) local diffeomorphism since the conceptual picture is more straightforward. We then show how they can be extended to map with critical points and even discontinuities and/or points with infinite derivative.

7.1. **Non-uniformly expanding local diffeomorphisms.** Let \( f : M \to M \) be a \( C^2 \) local diffeomorphism of the compact manifold \( M \) of dimension \( d \geq 1 \). We suppose that \( f \) satisfies the non-uniform expansivity condition given above: there exists a constant \( \lambda > 0 \) such that

\[
\liminf_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \left\| Df_{f^i(x)}^{-1} \right\|^{-1} > \lambda
\]

for almost every \( x \in M \). For simplicity we suppose also that \( f \) is topologically transitive, i.e. there exists a point \( x \) whose orbit is dense in \( M \). Since we have no geometrical information whatsoever about \( f \) we want to show
that the statistical properties such as the rate of decay of correlations somehow depends on abstract information related to the non-uniform expansivity condition only. Thus we make the following

**Definition 8.** For \( x \in M \), we define the expansion time function

\[
E(x) = \min \left\{ N : \frac{1}{n} \sum_{i=0}^{n-1} \log \| Df^{-1}_{f^i(x)} \|^{-1} \geq \frac{\lambda}{2} \quad \forall n \geq N \right\}.
\]

By condition \((\ast)\) this function is defined and finite almost everywhere. It measures the amount of time one has to wait before the uniform exponential growth of the derivative kicks in. If \( E(x) \) was uniformly bounded, we would essentially be in the uniformly expanding case. In general it will take on arbitrarily large values and not be defined everywhere. If \( E(x) \) is large only on a small set of points, then it makes sense to think of the map as being not very non-uniform, whereas, if it is large on a large set of points it is in some sense, very non-uniform. To formalize this notion, we define the set

\[
\Gamma_n = \{ x \in M : E(x) > n \}
\]

and formulate our assumptions about the degree of non-uniformity of \( f \) in terms of the rate of decay of the measure of \( \Gamma_n \).

**Theorem 10.** \([6-8]\) Let \( f : M \to M \) be a transitive \( C^2 \) local diffeomorphism satisfying condition \((\ast)\) and suppose that there exists \( \gamma > 1 \) such that

\[
m(\Gamma_n) = O(n^{-\gamma}).
\]

Then there exists an absolutely continuous, \( f \)-invariant, probability measure \( \mu \). Some finite power of \( f \) is mixing with respect to \( \mu \) and the correlation function \( C_n \) for Hölder continuous observable on \( M \) satisfies

\[
C_n = O(n^{-(\gamma+1)}).
\]

We remark that the existence and ergodicity of the measure \( \mu \) was proved in \([5]\) assuming only condition \((\ast)\). The arguments in \([6-8]\) give an alternative proof under the additional assumption on the rate of decay of \( m(\Gamma_n) \). We also remark that the choice of \( \lambda/2 \) in the definition of the expansion time function \( E(x) \) is fairly arbitrary and does not affect the asymptotic rate estimates. Any positive number smaller than \( \lambda \) would yield the same results.

### 7.2. Critical points and discontinuities.

The arguments used in the proof of Theorem \([10]\) apply to maps which fail to be local diffeomorphism on some zero measure set \( C \) satisfying some simple non-degeneracy conditions. Points in \( C \) maybe the the higher dimensional analogue of critical points (i.e. points at which derivative is degenerate), or may be points of discontinuities for \( f \), or points at which the map is not differentiable.
and for which the derivative blows up to infinity. Remarkably, all these cases are dealt with, as “problematic” points, in the same way and need to satisfy the same conditions which are just the natural generalization of the non-degeneracy (or non-flatness) condition for critical points of one-dimensional maps.

**Definition 9.** The critical set \( C \subset M \) is non-degenerate if \( m(C) = 0 \) and there is a constant \( \beta > 0 \) such that for every \( x \in M \setminus C \) we have \( \text{dist}(x, C)^\beta \lesssim \|Df_x v\|/\|v\| \lesssim \text{dist}(x, C)^{-\beta} \) for all \( v \in T_x M \), and the functions \( \log \det Df \) and \( \log \|Df^{-1}\| \) are locally Lipschitz with Lipschitz constant \( \lesssim \text{dist}(x, C)^{-\beta} \).

These are geometrical conditions which have nothing to do with the dynamics. We also need to assume some dynamical conditions concerning the rate of recurrence of typical points near the critical set. We let \( d_\delta(x, C) \) denote the \( \delta \)-truncated distance from \( x \) to \( C \) defined as \( d_\delta(x, C) = \text{dist}(x, C) \) if \( d(x, C) \leq \delta \) and \( d_\delta(x, C) = 1 \) otherwise.

**Definition 10.** We say that \( f \) satisfies the property of subexponential recurrence to the critical set if for any \( \epsilon > 0 \) there exists \( \delta > 0 \) such that for Lebesgue almost every \( x \in M \)

\[
\limsup_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} - \log \text{dist}_\delta(f^j(x), C) \leq \epsilon.
\]

Again, we want to differentiate between different degrees of recurrence in a similar way to the way we differentiated between different degrees of non-uniformity of the expansion.

**Definition 11.** For \( x \in M \), we define the recurrence time function

\[
\mathcal{R}(x) = \min \left\{ N \geq 1 : \frac{1}{n} \sum_{j=0}^{n-1} - \log \text{dist}_\delta(f^j(x), C) \leq 2\epsilon, \forall n \geq N \right\}
\]

Then, for a map satisfying both conditions (\( * \)) and (\( ** \)) we let

\[
\Gamma_n = \{ x : E(x) > n \text{ or } \mathcal{R}(x) > n \}
\]

**Theorem 11.** \([6–8]\) Let \( f : M \to M \) be a transitive \( C^2 \) local diffeomorphism outside a non-degenerate critical set \( C \), satisfying conditions (\( * \)) and (\( ** \)). Suppose that there exists \( \gamma > 0 \) such that

\[
|\Gamma_n| = O(n^{-\gamma}).
\]

Then there exists an absolutely continuous, \( f \)-invariant, probability measure \( \mu \). Some finite power of \( f \) is mixing with respect to \( \mu \) and for any Hölder continuous function \( \varphi, \psi \) on \( M \) we have

\[
C_n = O(n^{-\gamma+1}).
\]
We remark that although condition (**) might appear to be a very technical condition, it is actually quite natural and in fact almost necessary. Indeed, suppose that an absolutely continuous invariant measure $\mu$ did exist for $f$. Then, a simple application of Birkhoff’s Ergodic theorem implies that condition (**) is equivalent to the integrability condition

$$\int_M |\log \text{dist}_S(x, S)| d\mu < \infty$$

which is simply saying that the invariant measure does not give too much weight to a neighbourhood of the discontinuity set.

8. Concluding remarks

8.1. What causes slow decay of correlations? The results on maps with indifferent fixed points and those on one dimensional maps with critical points suggest that slow decay of correlation is caused, literally, by a slowing down of the dynamics due to some indifferent fixed or periodic point, or even some indifferent non-periodic orbit. In these cases, the responsible orbits can be identified exactly and the mechanism through which they slow down the mixing process can be described quite explicitly and even quantified in terms of the degree of “neutrality” of the orbit. In higher dimensional cases where it is impossible to identify specific orbits with neutral behaviour it is much more difficult to obtain optimal estimates. It is not known if Viana maps have any neutral orbits and the current estimates for decay of correlations are based on estimates which may not be optimal. The results of [6–8] described in the last section develop a connection between the rate of decay of correlations and the rate at which orbits start to exhibit exponential growth of the derivative. However this theory still does resolve the issue of what geometrical or other characteristics of the system will cause this rate to be of a certain kind rather than another. In view of the above discussion, it seems reasonable to conjecture in the first instance that the difference between exponential and subexponential behaviour lies in the absence or presence of a neutral orbit of some kind.

To formalize this notion we recall a few standard notions of Ergodic Theory. We suppose that $f : M \rightarrow M$ is a $C^2$ map of the compact manifold $M$ of dimension $d \geq 1$, and we let $\mathcal{M}_{\text{inv}}$ denote the space of all probability invariant measures $\mu$ on $M$ which satisfy the integrability condition

$$\int_M \log \|Df_x\| d\mu < \infty.$$
such that the decomposition is invariant by the derivative and such that for all \( j = 1, \ldots, k \) and for all non zero vectors \( v^{(j)} \in E^j_x \) we have

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \| Df^n_x(v^{(j)}) \| = \lambda_j.
\]

The constants \( \lambda_1, \ldots, \lambda_k \) are called the \textit{Lyapunov exponents} associated to the measure \( \mu \). The set of Lyapunov exponents associated to \( \mu \) is sometimes called the \textit{Lyapunov spectrum} of \( \mu \) and denoted by \( \text{Lyap}(\mu) \). The standard terminology is to call \( \mu \) hyperbolic if it has no associated zero Lyapunov exponents.

\textbf{Definition 12.} \( \mu \in \mathcal{M}_{\text{inv}} \) is expanding if every Lyapunov exponent associated to \( \mu \) is \( > 0 \).

Notice that if \( f \) satisfies the non-uniform expansivity condition (*) defined above, then the corresponding absolutely continuous invariant measure \( \mu \) (which exists by [5]) is expanding in the sense of definition 12. Conversely, if \( \mu \) is an expanding absolutely continuous invariant probability measure, then it clearly satisfies condition (*). Thus these notions characterize the class of (non-uniformly) expanding maps. Notice however that in general there are infinitely many invariant measures, for example the Dirac measures on fixed and periodic points. Even if the absolutely continuous invariant measure is expanding, there may be other singular measures which are not, i.e. have a zero Lyapunov exponent. To distinguish between these two cases we formulate the following

\textbf{Definition 13.} We say that \( f \) is totally expanding if all measures in \( \mathcal{M}_{\text{inv}} \) are uniformly expanding in the sense that \( \exists \lambda > 0 \) such that every Lyapunov exponent of every measure \( \mu \in \mathcal{M}_{\text{inv}} \) is \( \geq \lambda \).

A totally expanding map is in some sense the strongest possible version of a non-uniformly expanding map without necessarily being uniformly expanding: a kind of “uniformly” non-uniformly expanding! Uniformly expanding maps are clearly totally expanding. Remarkably, if \( f \) is a \( C^1 \) local diffeomorphisms then the two notions are equivalent: completely expanding implies uniformly expanding [4, 35]. However general non-uniformly expanding maps are certainly not necessarily totally expanding. The examples of maps with indifferent fixed points of Section 4 are not totally expanding, as the Dirac measure on the indifferent fixed point is not expanding. In the context of one-dimensional smooth maps with critical points it is known that in the unimodal case exponential growth of the derivative along the critical orbit (the Collet-Eckmann condition) implies uniform hyperbolicity on periodic orbits [78] which in turn implies total expansivity [22]. Conversely total expansivity clearly implies uniform hyperbolicity on
periodic orbits which implies exponential growth of the derivative along the critical orbit \cite{79}. Moreover \cite{79} closes a chain of implications which finally imply that in the unimodal case, total hyperbolicity is equivalent to exponential decay of correlations. This suggests the following

**Conjecture.** Suppose \( f : M \to M \) is non-uniformly expanding with absolutely continuous invariant measure \( \mu \). Then \( f \) has exponential decay of correlations if and only if it is totally expanding.

The idea is that any invariant measure with a zero Lyapunov exponent causes a slowing down effect analogous to that caused by an indifferent fixed point or an indifferent critical orbit. This conjecture is supported by known examples as described above, but at the moment it is not clear how it could be tackled in general. Even the implication in just one of the directions would interesting and support the intuitive picture of the cause of slow decay of correlation given in the discussion.

We remark that the assumption of non-uniform expansion is crucial here. There are several examples of systems which have exponential decay of correlations but clearly have invariant measures with zero Lyapunov exponents, e.g. partially hyperbolic maps or maps obtained as time-1 maps of certain flows \cite{38–40}. These examples however are not non-uniformly expanding, and are generally partially hyperbolic which means that there are two continuous subbundles such that the derivative restricted to one subbundle has very good expanding properties or contracting properties and the other subbundle has the zero Lyapunov exponents. For reasons which are not at all clear, this might be better from the point of view of decay of correlations than a situation in which all the Lyapunov exponents of the absolutely continuous measure are positive but there is some embedded singular measure with zero Lyapunov exponent slowing down the mixing process. Certainly there is still a lot to be understood on this topic.

### 8.2. Arguments and techniques.

The existence of absolutely continuous invariant measures can, in many cases, be proved by fairly direct geometric arguments. Estimates on the rate of decay of correlations, on the other hand usually require considerably more sophisticated arguments. In the case of expanding maps, these generally involve an abstract functional-analytic or probabilistic framework in which particular geometric characteristics of the system (expansion, smoothness) are used as ingredients.

The pioneering work of Sinai, Ruelle and Bowen on uniformly hyperbolic systems (with uniformly expanding systems essentially as a special case), introduced the basic idea of approaching the problem via the Perron-Frobenius operator. This is an operator on a suitable function space of appropriate densities with the property that any fixed point for the operator is
the density of an invariant absolutely continuous measure. Thus, various functional-analytic techniques can be brought to bear on the problem of the existence of such a fixed point and on the speed of convergence of arbitrary densities to the fixed points which turns out to be closely related to the rates of decay of correlations. In particular, a \textit{spectral gap} in the spectrum of the operator implies exponential decay of correlations. Despite the great success of such an approach in dealing with various classes of systems, it has the intrinsic limitation of producing results which are necessarily exponential: if there is no spectral gap then one cannot deduce any other rate for the decay. Another functional-analytic method has been introduced in \cite{65} to deal with maps with discontinuities. This still involves a direct study of the Perron-Frobenius operator but using the so-called \textit{Birkhoff metric} and the notion of \textit{invariant cones}. This method appears to have more in-built flexibility and in certain cases allows better estimates for the actual constants and exponents involved in decay of correlations. Moreover it has been adapted in \cite{75} to deal with systems with subexponential decay.

More recently, a quite different approach has been introduced in \cite{21,108,109} which relies on a probabilistic \textit{coupling} argument. As in the functional-analytic approach, it depends ultimately on the fact that the rate of decay correlations is related to the rate at which arbitrary absolutely continuous measures with densities satisfying some regularity conditions converge to the invariant measure under the dynamics. However, in this case the conclusions do not ultimately depend on a spectral estimate, but on more direct geometric estimates. This approach has proved to be very flexible and far reaching in its scope and underlies several recent results such as those in \cite{6,7,23} as well as those in the pioneering papers \cite{108,109}. It has also proved successful in generalizing known results to observables which satisfy significantly weaker summability conditions on the modulus of continuity of the derivative, rather than the usual H"older continuity conditions \cite{69}. It is beyond the scope of this note to enter into a more detailed discussion of the various techniques. See the excellent and comprehensives texts \cite{12,105} for detailed discussions of the functional-analytic methods in particular, and \cite{109} for the coupling method.

8.3. \textbf{Lower bounds for decay of correlations}. Finally we make the important observation that all the estimates given above have been upper bounds for the decay of correlations. An important question for a full understanding of the phenomenon of decay of correlations is that of whether these are actually lower bounds as well. It turns out that lower bounds require some sophisticated arguments which are not always directly related to the arguments used to prove upper bounds. There are not many results in this
direction although there have been some important recent developments in [44, 94].

8.4. **Hyperbolicity.** There is also extensive research work on higher dimensional systems which exhibit some degree of hyperbolicity, a combination of expansion and contraction in different directions. As well as the natural analogues of the uniformly and non-uniformly expanding systems, uniformly and non-uniformly hyperbolic systems, there are other categories such as partially hyperbolic or projectively hyperbolic. These definitions all try to distinguish different possible kinds of hyperbolicity both in terms of how the expansion and contraction estimates behave within certain contracting and expanding subbundles of the tangent space, as well as how these subbundles are related to each other. Again we refer to [12, 105] for more details and additional references.

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**DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE, LONDON**

*E-mail address:* stefano.luzzatto@imperial.ac.uk

*URL:* http://www.ma.ic.ac.uk/~luzzatto