Waves, damped wave and observation*

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Abstract  
This talk describes some applications of two kinds of observation estimate for the wave equation and for the damped wave equation in a bounded domain where the geometric control condition of C. Bardos, G. Lebeau and J. Rauch may failed.

1 The wave equation and observation

We consider the wave equation in the solution $u = u(x,t)$

$$
\left\{
\begin{array}{l}
\partial_t^2 u - \Delta u = 0 \quad \text{in } \Omega \times \mathbb{R}, \\
u = 0 \quad \text{on } \partial \Omega \times \mathbb{R}, \\
(u, \partial_t u)(\cdot,0) = (u_0, u_1),
\end{array}
\right.
$$

(1.1)

living in a bounded open set $\Omega$ in $\mathbb{R}^n$, $n \geq 1$, either convex or $C^2$ and connected, with boundary $\partial \Omega$. It is well-known that for any initial data $(u_0, u_1) \in H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega)$, the above problem is well-posed and have a unique strong solution.

Linked to exact controllability and strong stabilization for the wave equation (see [Li]), it appears the following observability problem which consists in proving the following estimate

$$
\|(u_0, u_1)\|_{H^1_0(\Omega) \times L^2(\Omega)}^2 \leq C \int_0^T \int_\omega |\partial_t u(x, t)|^2 \, dx \, dt
$$

*This work is supported by the NSF of China under grants 10525105 and 10771149. Part of this talk was done when the author visited Fudan University with a financial support from the "French-Chinese Summer Institute on Applied Mathematics" (September 1-21, 2008).
for some constant $C > 0$ independent on the initial data. Here, $T > 0$ and $\omega$ is a non-empty open subset in $\Omega$. Due to finite speed of propagation, the time $T$ have to be chosen large enough. Dealing with high frequency waves i.e., waves which propagates according the law of geometrical optics, the choice of $\omega$ can not be arbitrary. In other words, the existence of trapped rays (e.g, constructed with gaussian beams [see [Ra]]) implies the requirement of some kind of geometric condition on $(\omega, T)$ (see [BLR]) in order that the above observability estimate may hold.

Now, we can ask what kind of estimate we may hope in a geometry with trapped rays. Let us introduce the quantity

$$\Lambda = \frac{\| (u_0, u_1) \|_{H^2(\Omega) \times H^1_0(\Omega)}}{\| (u_0, u_1) \|_{H^1_0(\Omega) \times L^2(\Omega)}}$$

which can be seen as a measure of the frequency of the wave. In this paper, we present the two following inequalities

$$\| (u_0, u_1) \|^2_{H^1_0(\Omega) \times L^2(\Omega)} \leq e^{CA^{1/\beta}} \int_0^T \int_{\omega} |\partial_t u (x, t)|^2 \, dx \, dt \quad (1.2)$$

and

$$\| (u_0, u_1) \|^2_{H^1_0(\Omega) \times L^2(\Omega)} \leq C \int_0^\infty \int_{\omega} |\partial_t u (x, t)|^2 \, dx \, dt \quad (1.3)$$

where $\beta \in (0, 1)$, $\gamma > 0$. We will also give theirs applications to control theory.

The strategy to get estimate (1.2) is now well-known (see [Ro2],[LR]) and a sketch of the proof will be given in Appendix for completeness. More precisely, we have the following result.

**Theorem 1.1.** For any $\omega$ non-empty open subset in $\Omega$, for any $\beta \in (0, 1)$, there exist $C > 0$ and $T > 0$ such that for any solution $u$ of (1.1) with non-identically zero initial data $(u_0, u_1) \in H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega)$, the inequality (1.2) holds.

Now, we can ask whether is it possible to get another weight function of $\Lambda$ than the exponential one, and in particular a polynomial weight function with a geometry $(\Omega, \omega)$ with trapped rays. Here we present the following result.

**Theorem 1.2.** There exists a geometry $(\Omega, \omega)$ with trapped rays such that for any solution $u$ of (1.1) with non-identically zero initial data
(u_0, u_1) \in H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega), the inequality (1.3) holds for some \( C > 0 \) and \( \gamma > 0 \).

The proof of Theorem 1.2 is given in [Ph1]. With the help of Theorem 2.1 below, it can also be deduced from [LiR], [BuH].

2 The damped wave equation and our motivation

We consider the following damped wave equation in the solution \( w = w(x,t) \)

\[
\begin{cases}
\partial_t^2 w - \Delta w + 1_\omega \partial_t w = 0 & \text{in } \Omega \times (0, +\infty) , \\
w = 0 & \text{on } \partial \Omega \times (0, +\infty) ,
\end{cases}
\]

(2.1)

living in a bounded open set \( \Omega \) in \( \mathbb{R}^n \), \( n \geq 1 \), either convex or \( C^2 \) and connected, with boundary \( \partial \Omega \). Here \( \omega \) is a non-empty open subset in \( \Omega \) with trapped rays and \( 1_\omega \) denotes the characteristic function on \( \omega \). Further, for any \((w, \partial_t w)(\cdot,0) \in H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega)\), the above problem is well-posed for any \( t \geq 0 \) and have a unique strong solution.

Denote for any \( g \in C([0, +\infty); H^1_0(\Omega)) \cap C^1([0, +\infty); L^2(\Omega)) \),

\[
E(g,t) = \frac{1}{2} \int_{\Omega} \left( |\nabla g(x,t)|^2 + |\partial_t g(x,t)|^2 \right) dx .
\]

Then for any \( 0 \leq t_0 < t_1 \), the strong solution \( w \) satisfies the following formula

\[
E(w,t_1) - E(w,t_0) + \int_{t_0}^{t_1} \int_\omega |\partial_t w(x,t)|^2 dx dt = 0 .
\]

(2.2)

2.1 The polynomial decay rate

Our motivation for establishing estimate (1.3) comes from the following result.

**Theorem 2.1 .-** The following two assertions are equivalent. Let \( \delta > 0 \).
There exists $C > 0$ such that for any solution $w$ of (2.1) with the non-null initial data $(w, \partial_tw)(\cdot, 0) = (w_0, w_1) \in H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega)$, we have

$$
\| (w_0, w_1) \|^2_{H^1_0(\Omega) \times L^2(\Omega)} \leq C \int_0^\infty \left( \frac{E(\partial_tw, 0)}{E(w, 0)} \right)^{1/\delta} \int_\Omega |\partial_tw(x, t)|^2 \, dx \, dt .
$$

(ii) There exists $C > 0$ such that the solution $w$ of (2.1) with the initial data $(w, \partial_tw)(\cdot, 0) = (w_0, w_1) \in H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega)$ satisfies

$$
E(w, t) \leq C \| (w_0, w_1) \|^2_{H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega)} \quad \forall t > 0 .
$$

Remark. It is not difficult to see (e.g., [Ph2]) by a classical decomposition method, a translation in time and (2.2), that the inequality (1.3) with the exponent $\gamma$ for the wave equation implies the inequality of (i) in Theorem 2.1 with the exponent $\delta = 2\gamma/3$ for the damped wave equation. And conversely, the inequality of (i) in Theorem 2.1 with the exponent $\delta$ for the damped wave equation implies the inequality (1.3) with the exponent $\gamma = \delta/2$ for the wave equation.

Proof of Theorem 2.1.-

(ii) $\Rightarrow$ (i). Suppose that

$$
E(w, T) \leq C \| (w_0, w_1) \|^2_{H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega)} \quad \forall T > 0 .
$$

Therefore from (2.2)

$$
E(w, 0) \leq C \| (w_0, w_1) \|^2_{H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega)} + \int_0^T \int_\Omega |\partial_tw(x, t)|^2 \, dx \, dt .
$$

By choosing

$$
T = \left( 2C \frac{\| (w_0, w_1) \|^2_{H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega)}}{E(w, 0)} \right)^{1/\delta},
$$

we get the desired estimate

$$
E(w, 0) \leq 2 \int_0^T \left( \frac{\| (w_0, w_1) \|^2_{H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega)}}{E(w, 0)} \right)^{1/\delta} \int_\Omega |\partial_tw(x, t)|^2 \, dx \, dt .
$$

(i) $\Rightarrow$ (ii). Conversely, suppose the existence of a constant $c > 1$ such that the solution $w$ of (2.1) with the non-null initial data $(w, \partial_tw)(\cdot, 0) = (w_0, w_1) \in H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega)$ satisfies

$$
E(w, 0) \leq c \int_0^T \left( \frac{E(w, t) + E(\partial_tw, 0)}{E(w, 0)} \right)^{1/\delta} \int_\Omega |\partial_tw(x, t)|^2 \, dx \, dt .
$$
We obtain the following inequalities by a translation on the time variable and by using (2.2). ∀s ≥ 0

\[
\frac{E(w,s)}{E(w,0)+E(\partial_t w,0)} \leq c \int_s^{s+c} \left( \frac{E(w,x)}{E(w,0)+E(\partial_t w,0)} \right)^{1/\delta} dx dt 
\]

\[
\leq c \left( \frac{E(w,s)}{E(w,0)+E(\partial_t w,0)} - E \left( \frac{\frac{|\partial_t w(x,t)|^2}{E(w,0)+E(\partial_t w,0)}}{E(w,0)+E(\partial_t w,0)} \right)^{1/\delta} \right). 
\]

Denoting \( G(s) = \frac{E(w,s)}{E(w,0)+E(\partial_t w,0)} \), we deduce using the decreasing of \( G \) that

\[
G \left( s + c \left( \frac{1}{G(s)} \right)^{1/\delta} \right) \leq G(s) \leq c \left[ G(s) - G \left( s + c \left( \frac{1}{G(s)} \right)^{1/\delta} \right) \right]
\]

which gives

\[
G \left( s + c \left( \frac{1}{G(s)} \right)^{1/\delta} \right) \leq \frac{c}{1+c} G(s) . 
\]

Let \( c_1 = \left( \frac{1+c}{c} \right)^{1/\delta} - 1 > 0 \) and denoting \( d(s) = \left( \frac{c_1}{1+c} \right)^{1/\delta} \). We distinguish two cases.

If \( c_1 s \leq c \left( \frac{1}{G(s)} \right)^{1/\delta} \), then \( G(s) \leq \left( \frac{1}{c_1 s} \right)^{1/\delta} \) and

\[
G \left( (1+c_1) s \right) \leq d(s) . 
\]

If \( c_1 s > c \left( \frac{1}{G(s)} \right)^{1/\delta} \), then \( s + c \left( \frac{1}{G(s)} \right)^{1/\delta} < (1+c_1) s \) and the decreasing of \( G \) gives \( G \left( (1+c_1) s \right) \leq G \left( s + c \left( \frac{1}{G(s)} \right)^{1/\delta} \right) \) and then

\[
G \left( (1+c_1) s \right) \leq \frac{c}{1+c} G(s) . 
\]

Consequently, we have that ∀s > 0, ∀n ∈ N, n ≥ 1,

\[
G \left( (1+c_1) s \right) \leq \max \left\{ d(s), \frac{c}{1+c} d \left( \frac{s}{1+c_1} \right)^{1/\delta}, \cdots, \right. 
\]

\[
\left. \cdot \frac{c}{1+c} \left( \frac{s}{1+c_1} \right)^{1/\delta}, \cdots, \left( \frac{c}{1+c} \right)^{n+1} G \left( \frac{s}{1+c_1} \right)^{1/\delta} \right\}. 
\]

Now, remark that with our choice of \( c_1 \), we get

\[
\frac{c}{1+c} d \left( \frac{s}{1+c_1} \right) = d(s) \quad \forall s > 0 . 
\]
Thus, we deduce that \( \forall n \geq 1 \)
\[
G \left( (1 + c_1) s \right) \leq \max \left\{ d(s), \left( \frac{s}{1 + c_1} \right)^{n+1} G \left( \frac{s}{1 + c_1} \right) \right\}
\]
\[
\leq \max \left\{ d(s), \left( \frac{s}{1 + c_1} \right)^{n+1} \right\} \quad \text{because } G \leq 1 ,
\]
and conclude that \( \forall s > 0 \)
\[
\frac{E(w, s)}{E(w, 0) + E(\partial_t w, 0)} = G(s) \leq d \left( \frac{s}{1 + c_1} \right) = \left( \frac{c(1 + c_1)}{c_1} \right)^{\frac{s}{1 + c_1}} .
\]
This completes the proof.

### 2.2 The approximate controllability

The goal of this section consists in giving an application of estimate (1.2).

For any \( \omega \) non-empty open subset in \( \Omega \), for any \( \beta \in (0, 1) \), let \( T > 0 \) be given in Theorem 1.1.

Let \( (v_0, v_1, v_{0d}, v_{1d}) \in (H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega))^2 \) and \( u \) the solution of (1.1) with initial data \( (u, \partial_t u) (\cdot, 0) = (v_0, v_1) \).

For any integer \( N > 0 \), let us introduce
\[
I_N(x, t) = -1_{\omega} \sum_{\ell=0}^{N} \left[ \partial_t w^{(2\ell+1)}(x, t) + \partial_t w^{(2\ell)}(x, T - t) \right] , \tag{2.3}
\]
where \( w^{(0)} \in C([0, T]; H^2(\Omega) \cap H^1_0(\Omega)) \) is the solution of the damped wave equation (2.1) with initial data
\[
\left( w^{(0)}, \partial_t w^{(0)} \right) (\cdot, 0) = (v_{0d}, -v_{1d}) - (u, -\partial_t u) (\cdot, T) \text{ in } \Omega ,
\]
and for \( j \geq 0 \), \( w^{(j+1)} \in C([0, T]; H^2(\Omega) \cap H^1_0(\Omega)) \) is the solution of the damped wave equation (2.1) with initial data
\[
\left( w^{(j+1)}, \partial_t w^{(j+1)} \right) (\cdot, 0) = \left( -w^{(j)}, \partial_t w^{(j)} \right) (\cdot, T) \text{ in } \Omega .
\]
Introduce
\[
M = \sup_{j \geq 0} \left\| w^{(j)} (\cdot, 0), \partial_t w^{(j)} (\cdot, 0) \right\|_{H^2(\Omega) \times H^1_0(\Omega)}^2 .
\]
Our main result is as follows.

**Theorem 2.2.** Suppose that $M < +\infty$. Then there exists $C > 0$ such that for all $N > 0$, the control function $f_N$ given by (2.3), drives the system

$$
\begin{cases}
\partial_t^2 v - \Delta v = 1_{\omega \times (0,T)}f_N & \text{in } \Omega \times (0,T), \\
v = 0 & \text{on } \partial \Omega \times (0,T), \\
(v, \partial_t v) (\cdot, 0) = (v_0, v_1) & \text{in } \Omega,
\end{cases}
$$

to the desired data $(v_{0d}, v_{1d})$ approximately at time $T$ i.e.,

$$
\|v (\cdot, T) - v_{0d}, \partial_t v (\cdot, T) - v_{1d}\|_{H^1_0 (\Omega) \times L^2(\Omega)}^2 \leq \frac{C}{[\ln (1 + 2N)]^{2\beta} M},
$$

and satisfies

$$
\|f_N\|_{L^\infty(0,T; L^2(\Omega))} \leq C (N + 1) \|(v_0, v_1, v_{0d}, v_{1d})\|_{(H^1_0 (\Omega) \times L^2(\Omega))^2}.
$$

**Remark.** For any $\varepsilon > 0$, we can choose $N$ such that

$$
\frac{C}{[\ln (1 + 2N)]^{2\beta} M} \simeq \varepsilon^2 \quad \text{and} \quad (2N + 1) \simeq \varepsilon (\frac{\sqrt{M}}{2})^{1/\beta},
$$

in order that

$$
\|v (\cdot, T) - v_{0d}, \partial_t v (\cdot, T) - v_{1d}\|_{H^1_0 (\Omega) \times L^2(\Omega)} \leq \varepsilon,
$$

and

$$
\|f\|_{L^\infty(0,T; L^2(\Omega))} \leq \varepsilon \left(\left(\frac{\sqrt{\varepsilon}}{\sqrt{M}}\right)^{1/\beta}\right) \|(v_0, v_1, v_{0d}, v_{1d})\|_{(H^1_0 (\Omega) \times L^2(\Omega))^2}.
$$

In [Zu], a method was proposed to construct an approximate control. It consists of minimizing a functional depending on the parameter $\varepsilon$. However, no estimate of the cost is given. On the other hand, estimate of the form (1.2) was originally established by [Ro2] to give the cost (see also [Le]). Here, we present a new way to construct an approximate control by superposing different waves. Given a cost to not overcome, we construct a solution which will be closed in the above sense to the desired state. It takes ideas from [Ru] and [BF] like an iterative time reversal construction.
2.2.1 Proof

Consider the solution
\[ V(\cdot, t) = \sum_{\ell=0}^{N} \left[ w^{(2\ell+1)}(\cdot, t) + w^{(2\ell)}(\cdot, T-t) \right]. \]

We deduce that for \( t \in (0, T) \)
\[
\begin{cases}
\partial_t^2 V(\cdot, t) - \Delta V(\cdot, t) = -1_{\omega} \sum_{\ell=0}^{N} \left[ \partial_t w^{(2\ell+1)}(\cdot, t) + \partial_t w^{(2\ell)}(\cdot, T-t) \right], \\
V = 0 \quad \text{on } \partial\Omega \times (0, T), \\
(V, \partial_t V)(\cdot, 0) = 0 \quad \text{in } \Omega.
\end{cases}
\]

Now, from the definition of \( w^{(0)} \), the property of \((w^{(j+1)}, \partial_t w^{(j+1)}) (\cdot, 0)\) and a change of variable, we obtain that
\[
(V, \partial_t V)(\cdot, T) = (w^{(0)}, -\partial_t w^{(0)}) (\cdot, 0) + (w^{(2N+1)}, \partial_t w^{(2N+1)}) (\cdot, T)
\]
\[
= (v_{0d}, v_{1d}) - (u, \partial_t u) (\cdot, T) + (w^{(2N+1)}, \partial_t w^{(2N+1)}) (\cdot, T).
\]

Finally, the solution \( v = V + u \) satisfies
\[
\begin{cases}
\partial_t^2 v - \Delta v = 1_{\omega \times (0, T)} f_N \quad \text{in } \Omega \times (0, T), \\
v = 0 \quad \text{on } \partial\Omega \times (0, T), \\
(v, \partial_t v)(\cdot, 0) = (v_0, v_1) \quad \text{in } \Omega, \\
(v, \partial_t v)(\cdot, T) = (v_{0d}, v_{1d}) + (w^{(2N+1)}, \partial_t w^{(2N+1)}) (\cdot, T) \quad \text{in } \Omega.
\end{cases}
\]

Clearly,
\[
\|v(\cdot, T) - v_{0d}, \partial_t v(\cdot, T) - v_{1d}\|_{H^1_0(\Omega) \times L^2(\Omega)}^2 = 2E \left( w^{(2N+1)}, T \right).
\]

It remains to estimate \( E \left( w^{(2N+1)}, T \right) \). We claim that
\[
\exists C > 0 \quad \forall N \geq 1 \quad E \left( w^{(2N+1)}, T \right) \leq \frac{C}{\ln(1 + 2N)}^2 M.
\]

Indeed, from Theorem 1.1, we can easily see by a classical decomposition method that there exist \( C > 0 \) and \( T > 0 \) such that for any \( j \geq 0 \),
\[
\|w^{(j+1)}(\cdot, 0), \partial_t w^{(j+1)}(\cdot, 0)\|_{H^1_0(\Omega) \times L^2(\Omega)}^2 \leq C \exp \left( C \left( \|w^{(j+1)}(\cdot, 0), \partial_t w^{(j+1)}(\cdot, 0)\|_{H^1(\Omega) \times H^1(\Omega)} \| w^{(j+1)}(\cdot, 0), \partial_t w^{(j+1)}(\cdot, 0)\|_{H^1_0(\Omega) \times L^2(\Omega)} \right)^{1/\beta} \right) \int_0^T \int_\omega |\partial_t w^{(j+1)}(x, t)|^2 dx dt.
\]
Since
\[ E\left(w^{(j+1)}, 0\right) = E\left(w^{(j)}, T\right) \quad \forall j \geq 0, \]
we deduce from (2.2) that for any \( j \geq 0 \)

\[ E\left(w^{(j+1)}, 0\right) \leq C \exp\left(C \frac{M}{\pi 2N}\right)^{1/(2\beta)} \left[ E\left(w^{(j)}, T\right) - E\left(w^{(j+1)}, T\right) \right]. \]

Let
\[ d_j = E\left(w^{(j+1)}, T\right). \]

By using the decreasing property of the sequence \( d_j \), that is \( d_j \leq d_{j-1} \), we obtain that for any integer \( 0 \leq j \leq 2N \)
\[ d_j \leq C e^{\left(C \frac{M}{\pi 2N}\right)^{1/(2\beta)} [d_{j-1} - d_j]}. \]

By summing over \([0, 2N]\), we deduce that
\[ (2N + 1) d_{2N} \leq C e^{\left(C \frac{M}{\pi 2N}\right)^{1/(2\beta)} [d_{-1} - d_{2N}]} . \]

Finally, using the fact that \( d_{-1} \leq M \), it follows that
\[ d_{2N} \leq \frac{C}{\ln \left(1 + 2N\right)^{2\beta}} M . \]

This completes the proof of our claim.

On the other hand, the computation of the bound of \( f_N \) is immediate. Therefore, we check that for some \( C > 0 \) and \( T > 0 \),
\[ \|f_N\|_{L^\infty(0, T; L^2(\Omega))} \leq C (N + 1) \|\langle v_0, v_1, v_{0d}, v_{1d}\rangle\|_{(H^1_0(\Omega) \times L^2(\Omega))^2}, \]
\[ \|v(\cdot, T) - v_{0d}, \partial_t v(\cdot, T) - v_{1d}\|_{H^1_0(\Omega) \times L^2(\Omega)}^2 \leq \frac{C}{\ln \left(1 + 2N\right)^{2\beta}} M , \]
for any \( \beta \in (0, 1) \) and any integer \( N > 0 \). This completes the proof of our Theorem.
2.2.2 Numerical experiments

Here, we perform numerical experiments to investigate the practical applicability of the approach proposed to construct an approximate control. For simplicity, we consider a square domain $\Omega = (0, 1) \times (0, 1)$, $\omega = (0, 1/5) \times (0, 1)$. The time of controllability is given by $T = 4$.

For convenience we recall some well-known formulas. Denote by $\{e_j\}_{j \geq 1}$ the Hilbert basis in $L^2(\Omega)$ formed by the eigenfunctions of the operator $-\Delta$ with eigenvalues $\{\lambda_j\}_{j \geq 1}$, such that $\|e_j\|_{L^2(\Omega)} = 1$ and $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$, i.e.,

\[
\begin{cases}
\lambda_j = \pi^2 (k_j^2 + \ell_j^2), & k_j, \ell_j \in \mathbb{N}^*, \\
e_j(x_1, x_2) = 2\sin(\pi k_j x_1) \sin(\pi \ell_j x_2).
\end{cases}
\]

The solution of

\[
\begin{cases}
\partial_t^2 v - \Delta v = f & \text{in } \Omega \times (0, T), \\
v = 0 & \text{on } \partial\Omega \times (0, T), \\
(v, \partial_t v)(\cdot, 0) = (v_0, v_1) & \text{in } \Omega,
\end{cases}
\]

where $f$ is in the form

\[ f(x_1, x_2) = -1_\omega \sum_{j \geq 1} f_j(t) e_j(x_1, x_2), \]

is given by the formula

\[
v(x_1, x_2, t) = \lim_{G \to +\infty} \sum_{j=1}^{G} \left\{ a_0^j \cos(t \sqrt{\lambda_j}) + a_1^j \frac{1}{\sqrt{\lambda_j}} \sin(t \sqrt{\lambda_j}) \\
+ \frac{1}{\sqrt{\lambda_j}} \int_0^t \sin \left( (t - s) \sqrt{\lambda_j} \right) R_j(s) \, ds \right\} e_j(x_1, x_2),
\]

where

\[
\begin{cases}
v_0(x_1, x_2) = \lim_{G \to +\infty} \sum_{j=1}^{G} a_0^j e_j(x_1, x_2), & \sum_j \lambda_j |a_0^j|^2 < +\infty, \\
v_1(x_1, x_2) = \lim_{G \to +\infty} \sum_{j=1}^{G} a_1^j e_j(x_1, x_2), & \sum_j |a_1^j|^2 < +\infty, \\
R_j(t) = -\lim_{G \to +\infty} \sum_{i=1}^{G} (\int_\omega e_i e_j dx_1 dx_2) f_i(t).
\end{cases}
\]

Here, $G$ will be the number of Galerkin mode. The numerical results are shown below. The approximate solution of the damped wave equation is established via a system of ODE solved by MATLAB.
Example 1: low frequency  The initial condition and desired target are specifically as follows. \((v_0, v_1) = (0, 0)\) and \((v_{0d}, v_{1d}) = (e_1 + e_2, e_1)\). We take the number of Galerkin mode \(G = 100\) and the number of iterations in the time reversal construction \(N = 30\).

Below, we plot the graph of the desired initial data \(v_{0d}\) and the controlled solution \(v(\cdot, t = T = 4)\).

Below, we plot the graph of the energy of the controlled solution and the cost of the control function.

Example 2: high frequency  The initial condition and desired target are specifically as follows. \((v_{0d}, v_{1d}) = (0, 0)\) and with \((k_0, a_0, b_0, k_1, a_1, b_1) = \)
(200, 1/2, 10000), for \((x_1, x_2) \in (0, 1) \times (0, 1)\),

\[
\begin{align*}
v_0 (x_1, x_2) &= \sum_{j=1}^{G} \left( \int_0^1 \int_0^1 g_0 (x_1, x_2) e_j (x_1, x_2) dx_1 dx_2 \right) e_j (x_1, x_2), \\
v_1 (x_1, x_2) &= \sum_{j=1}^{G} \left( \int_0^1 \int_0^1 g_1 (x_1, x_2) e_j (x_1, x_2) dx_1 dx_2 \right) e_j (x_1, x_2), \\
g_0 (x_1, x_2) &= e^{-\frac{k_0 a_0}{2} (x_1-x_0_1)^2} e^{-\frac{k_0 b_0}{2} (x_2-x_0_2)^2} \cos \left( \frac{k_0 (x_2-x_0_2)}{2} \right), \\
g_1 (x_1, x_2) &= e^{-\frac{k_0 a_0}{2} (x_1-x_0_1)^2} e^{-\frac{k_0 b_0}{2} (x_2-x_0_2)^2} \\
&\quad \times \left[ k_0 b_0 (x_2-x_0_2) \cos \left( \frac{k_0 (x_2-x_0_2)}{2} \right) \\
&\quad \quad + \frac{k_0}{2} + a_0 \sin \left( \frac{k_0 (x_2-x_0_2)}{2} \right) \\
&\quad \quad - k_0 a_0^2 (x_1-x_0_1)^2 \sin \left( \frac{k_0 (x_2-x_0_2)}{2} \right) \right].
\end{align*}
\]

Notice that we have chosen as initial data the \(G\)-first projections on the basis \(\{e_j\}_{j \geq 1}\) of a gaussian beam \(g (x_1, x_2, t)\) such that \(g (\cdot, t = 0) = g_0\), \(\partial_t g (\cdot, t = 0) = g_1\) and which propagates on the direction \((0, 1)\).

We take the number of Galerkin mode \(G = 1000\) and the number of iterations in the time reversal construction \(N = 100\).

Below, we plot the graph of the energy of the controlled solution and the cost of the control function.

\begin{itemize}
\item[3 Conclusion]
\end{itemize}

In this talk, we have considered the wave equation in a bounded domain (eventually convex). Two kinds of inequality are described when occurs trapped rays. Applications to control theory are given. First, we link such kind of estimate with the damped wave equation and its decay
rate. Next, we describe the design of an approximate control function by an iterative time reversal method. We also provide a numerical simulation in a square domain. I’m grateful to Prof. Jean-Pierre Puel, the “French-Chinese Summer Institute on Applied Mathematics” and Fudan University for the kind invitation and the support of my visit.

4 Appendix

In this appendix, we recall most of the material from the works of I. Kukavica [Ku2] and L. Escauriaza [E] for the elliptic equation and from the works of G. Lebeau and L. Robbiano [LR] for the wave equation.

In the original paper dealing with doubling property and frequency function, N. Garofalo and F.H. Lin [GaL] study the monotonicity property of the following quantity

\[ \frac{r \int_{B_{y_0, r}} |\nabla v(y)|^2 \, dy}{\int_{\partial B_{y_0, r}} |v(y)|^2 \, d\sigma(y)}. \]

However, it seems more natural in our context to consider the monotonicity properties of the frequency function (see [Ze]) defined by

\[ \frac{\int_{B_{y_0, r}} |\nabla v(y)|^2 \left( r^2 - |y|^2 \right) \, dy}{\int_{B_{y_0, r}} |v(y)|^2 \, dy}. \]

4.1 Monotonicity formula

Following the ideas of I. Kukavica ([Ku2], [Ku], [KN], see also [E], [AE]), one obtains the following three lemmas. Detailed proofs are given in [Ph3].

**Lemma A**. Let \( D \subset \mathbb{R}^{N+1}, N \geq 1 \), be a connected bounded open set such that \( \overline{B_{y_0, R_0}} \subset D \) with \( y_0 \in D \) and \( R_0 > 0 \). If \( v = v(y) \in H^2(D) \) is a solution of \( \Delta_y v = 0 \) in \( D \), then

\[ \Phi (r) = \frac{\int_{B_{y_0, r}} |\nabla v(y)|^2 \left( r^2 - |y - y_0|^2 \right) \, dy}{\int_{B_{y_0, r}} |v(y)|^2 \, dy}. \]
is non-decreasing on $0 < r < R_o$, and
\[
\frac{d}{dr} \ln \int_{B_{y_o,r}} |v(y)|^2 \, dy = \frac{1}{r} (N + 1 + \Phi(r)) .
\]

**Lemma B** - Let $D \subset \mathbb{R}^{N+1}$, $N \geq 1$, be a connected bounded open set such that $B_{y_o,R_o} \subset D$ with $y_o \in D$ and $R_o > 0$. Let $r_1$, $r_2$, $r_3$ be three real numbers such that $0 < r_1 < r_2 < r_3 < R_o$. If $v = v(y) \in H^2(D)$ is a solution of $\Delta_y v = 0$ in $D$, then
\[
\int_{B_{y_o,r_2}} |v(y)|^2 \, dy \leq \left( \int_{B_{y_o,r_1}} |v(y)|^2 \, dy \right)^\alpha \left( \int_{B_{y_o,r_3}} |v(y)|^2 \, dy \right)^{1-\alpha},
\]
where $\alpha = \frac{1}{\ln \frac{r_2}{r_1}} \left( \frac{1}{\ln \frac{r_3}{r_2}} + \frac{1}{\ln \frac{r_1}{r_2}} \right)^{-1} \in (0,1)$.

The above two results are still available when we are closed to a part $\Gamma$ of the boundary $\partial \Omega$ under the homogeneous Dirichlet boundary condition on $\Gamma$, as follows.

**Lemma C** - Let $D \subset \mathbb{R}^{N+1}$, $N \geq 1$, be a connected bounded open set with boundary $\partial D$. Let $\Gamma$ be a non-empty Lipschitz open subset of $\partial D$. Let $r_o$, $r_1$, $r_2$, $r_3$, $R_o$ be five real numbers such that $0 < r_1 < r_o < r_2 < r_3 < R_o$. Suppose that $y_o \in D$ satisfies the following three conditions:
\begin{enumerate}
  \item $B_{y_o,r} \cap D$ is star-shaped with respect to $y_o$ $\forall r \in (0, R_o)$,
  \item $B_{y_o,r} \subset D$ $\forall r \in (0, r_o)$,
  \item $B_{y_o,r} \cap \partial D \subset \Gamma$ $\forall r \in [r_o, R_o)$.
\end{enumerate}
If $v = v(y) \in H^2(D)$ is a solution of $\Delta_y v = 0$ in $D$ and $v = 0$ on $\Gamma$, then
\[
\int_{B_{y_o,r_2} \cap D} |v(y)|^2 \, dy \leq \left( \int_{B_{y_o,r_1}} |v(y)|^2 \, dy \right)^\alpha \left( \int_{B_{y_o,r_3} \cap D} |v(y)|^2 \, dy \right)^{1-\alpha},
\]
where $\alpha = \frac{1}{\ln \frac{r_2}{r_1}} \left( \frac{1}{\ln \frac{r_3}{r_2}} + \frac{1}{\ln \frac{r_1}{r_2}} \right)^{-1} \in (0,1)$.

### 4.1.1 Proof of Lemma B

Let
\[
H(r) = \int_{B_{y_o,r}} |v(y)|^2 \, dy .
\]
By applying Lemma A, we know that
\[
\frac{d}{dr} \ln H (r) = \frac{1}{r} (N + 1 + \Phi (r)) .
\]
Next, from the monotonicity property of \(\Phi\), one deduces the following two inequalities
\[
\ln \left( \frac{H (r_2)}{H (r_1)} \right) = \int_{r_1}^{r_2} \frac{N + 1 + \Phi (r)}{r} \, dr \\
\leq (N + 1 + \Phi (r_2)) \ln \frac{r_2}{r_1} ,
\]
\[
\ln \left( \frac{H (r_3)}{H (r_2)} \right) = \int_{r_2}^{r_3} \frac{N + 1 + \Phi (r)}{r} \, dr \\
\geq (N + 1 + \Phi (r_2)) \ln \frac{r_3}{r_2} .
\]
Consequently,
\[
\frac{\ln \left( \frac{H (r_2)}{H (r_1)} \right)}{\ln \frac{r_2}{r_1}} \leq (N + 1 + \Phi (r_2)) \leq \frac{\ln \left( \frac{H (r_3)}{H (r_2)} \right)}{\ln \frac{r_3}{r_2}} ,
\]
and therefore the desired estimate holds
\[
H (r_2) \leq (H (r_1))^\alpha (H (r_3))^{1-\alpha} ,
\]
where \(\alpha = \frac{1}{\ln \frac{r_2}{r_1}} \left( \frac{1}{\ln \frac{r_1}{r_3}} + \frac{1}{\ln \frac{r_2}{r_3}} \right)^{-1} .\)

4.1.2 Proof of Lemma A

We introduce the following two functions \(H\) and \(D\) for \(0 < r < R_o\) :\n\[
H (r) = \int_{B_{yo,r}} |v (y)|^2 \, dy ,
D (r) = \int_{B_{yo,r}} |\nabla v (y)|^2 \left( r^2 - |y - yo|^2 \right) \, dy .
\]
First, the derivative of \(H (r) = \int_0^r \int_{S_N} |v (ps + yo)|^2 \rho^N \, d\rho \, d\sigma (s)\) is given by \(H' (r) = \int_{\partial B_{yo,r}} |v (y)|^2 \, d\sigma (y)\). Next, recall the Green formula
\[
\int_{\partial B_{yo,r}} |v|^2 \partial_\nu G (y) - \int_{\partial B_{yo,r}} \partial_\nu \left( |v|^2 \right) G (y) \\
= \int_{B_{yo,r}} |v|^2 \Delta G (y) - \int_{B_{yo,r}} \Delta \left( |v|^2 \right) G (y) .
\]
We apply it with $G(y) = r^2 - |y - y_o|^2$ where $G|_{\partial B_{y_o,r}} = 0$, $\partial_\nu G|_{\partial B_{y_o,r}} = -2r$, and $\Delta G = -2(N + 1)$. It gives

$$H'(r) = \frac{1}{r} \int_{B_{y_o,r}} (N + 1)|v|^2 \, dy + \frac{1}{2r} \int_{\partial B_{y_o,r}} \Delta \left(|v|^2\right) \left(r^2 - |y - y_o|^2\right) \, dy$$

$$= \frac{N+1}{r} H(r) + \frac{1}{r} \int_{B_{y_o,r}} \text{div} (v \nabla v) \left(r^2 - |y - y_o|^2\right) \, dy$$

Consequently, when $\Delta_y v = 0$,

$$H'(r) = \frac{N+1}{r} H(r) + \frac{1}{r} D(r) \text{,} \quad (A.1)$$

that is $\frac{H'(r)}{H(r)} = \frac{N+1}{r} + \frac{1}{r} \frac{D(r)}{H(r)}$ the second equality in Lemma A.

Now, we compute the derivative of $D(r)$.

$$D'(r) = \frac{d}{dr} \left( r \int_{S_N} |(\nabla v)_{\rho s + y_o}|^2 \rho^N d\rho d\sigma \right)$$

$$- \int_{S_N} r^2 |(\nabla v)_{\rho s + y_o}|^2 r^N d\sigma$$

$$= 2r \int_{S_N} |(\nabla v)_{\rho s + y_o}|^2 \rho^N d\rho d\sigma$$

On the other hand, we have by integrations by parts that

$$2r \int_{B_{y_o,r}} |\nabla v|^2 \, dy = \frac{N+1}{r} D(r) + \frac{4}{r} \int_{B_{y_o,r}} |(y - y_o) \cdot \nabla v|^2 \, dy$$

$$- \frac{1}{r} \int_{B_{y_o,r}} \nabla v \cdot (y - y_o) \Delta v \left(r^2 - |y - y_o|^2\right) \, dy$$

Therefore,

$$= 2r \int_{B_{y_o,r}} |\nabla v|^2 \left(r^2 - |y - y_o|^2\right) \, dy$$

$$- 4 \int_{B_{y_o,r}} |(y - y_o) \cdot \nabla v|^2 \, dy$$

$$+ 2 \int_{B_{y_o,r}} (y - y_o) \cdot \nabla v \Delta v \left(r^2 - |y - y_o|^2\right) \, dy$$

and this is the desired estimate (A.3).

Consequently, from (A.2) and (A.3), we obtain, when $\Delta_y v = 0$, the following formula

$$D'(r) = \frac{N+1}{r} D(r) + \frac{4}{r} \int_{B_{y_o,r}} |(y - y_o) \cdot \nabla v|^2 \, dy \text{.} \quad (A.4)$$
The computation of the derivative of $\Phi (r) = \frac{D(r)}{H(r)}$ gives

$$
\Phi'(r) = \frac{1}{H^2(r)} [D'(r) H(r) - D(r) H'(r)],
$$

which implies using (A.1) and (A.4) that

$$
H^2(r) \Phi'(r) = \frac{1}{r} \left( 4 \int_{B_{y_0,r}} |(y - y_0) \cdot \nabla v|^2 dy H(r) - D^2(r) \right) \geq 0,
$$

indeed, thanks to an integration by parts and using Cauchy-Schwarz inequality, we have

$$
D^2(r) = 4 \left( \int_{B_{y_0,r}} v \nabla \cdot (y - y_0) dy \right)^2 \leq 4 \left( \int_{B_{y_0,r}} |(y - y_0) \cdot \nabla v|^2 dy \right) \left( \int_{B_{y_0,r}} |v|^2 dy \right) \leq \frac{4}{H(r)} \left( \int_{B_{y_0,r}} |(y - y_0) \cdot \nabla v|^2 dy \right) H(r).
$$

Therefore, we have proved the desired monotonicity for $\Phi$ and this completes the proof of Lemma A.

### 4.1.3 Proof of Lemma C

Under the assumption $B_{y_0,r} \cap \partial D \subset \Gamma$ for any $r \in [r_o, R_o)$, we extend $v$ by zero in $B_{y_0, R_o} \setminus D$ and denote by $\bar{v}$ its extension. Since $v = 0$ on $\Gamma$, we have

$$
\left\{ \begin{array}{ll}
\bar{v} = v & \text{in } B_{y_0,R_o}, \\
\bar{v} = 0 & \text{on } B_{y_0,R_o} \cap \partial D, \\
\nabla \bar{v} = \nabla v & \text{in } B_{y_0,R_o}.
\end{array} \right.
$$

Now, we denote $\Omega_r = B_{y_0,r} \cap D$, when $0 < r < R_o$. In particular, $\Omega_r = B_{y_0,r}$, when $0 < r < r_o$. We introduce the following three functions:

$$
H(r) = \int_{\Omega_r} |v(y)|^2 dy,
$$

$$
D(r) = \int_{\Omega_r} |\nabla v(y)|^2 \left( r^2 - |y - y_0|^2 \right) dy,
$$

and

$$
\Phi(r) = \frac{D(r)}{H(r)} \geq 0.
$$

Our goal is to show that $\Phi$ is a non-decreasing function. Indeed, we will prove that the following equality holds

$$
\frac{d}{dr} \ln H(r) = (N + 1) \frac{d}{dr} \ln r + \frac{1}{r} \Phi(r). \quad (C.1)
$$
Therefore, from the monotonicity of $\Phi$, we will deduce (in a similar way than in the proof of Lemma A) that
\[
\frac{\ln \left( \frac{H(r_2)}{H(r_1)} \right)}{\ln \frac{1}{r_1}} \leq (N + 1) + \Phi (r_2) \leq \frac{\ln \left( \frac{H(r_1)}{H(r_2)} \right)}{\ln \frac{1}{r_2}},
\]
and this will imply the desired estimate
\[
\int_{\Omega_{r_2}} |v(y)|^2 \, dy \leq \left( \int_{B_{y_o,r_1}} |v(y)|^2 \, dy \right)^\alpha \left( \int_{\Omega_{r_3}} |v(y)|^2 \, dy \right)^{1-\alpha},
\]
where $\alpha = \frac{1}{\ln r_2} \left( \frac{1}{\ln r_1} + \frac{1}{\ln r_2} \right)^{-1}$.

First, we compute the derivative of $H(r) = \int_{B_{y_o,r}} |\nabla v(y)|^2 \, dy$.
\[
H'(r) = \int_{S_N} |\nabla v(rS + y_o)|^2 r^N \, d\sigma (s)
= \frac{1}{r} \int_{S_N} |\nabla v(rS + y_o)|^2 rS \cdot sr^N \, d\sigma (s)
= \frac{1}{r} \int_{B_{y_o,r}} \text{div} \left( |\nabla v(y)|^2 (y - y_o) \right) \, dy
= \frac{1}{r} \int_{B_{y_o,r}} (N + 1) |\nabla v(y)|^2 + \nabla |\nabla v(y)|^2 \cdot (y - y_o) \, dy
= \frac{N + 1}{r} H(r) + \frac{1}{r} \int_{\Omega} v(y) \nabla v(y) \cdot (y - y_o) \, dy.
\]
\[(C.2)\]

Next, when $\Delta_y v = 0$ in $D$ and $v|_{\Gamma} = 0$, we remark that
\[
D(r) = 2 \int_{\Omega} v(y) \nabla v(y) \cdot (y - y_o) \, dy,
\]
\[(C.3)\]

indeed,
\[
\int_{\Omega} |\nabla v|^2 \left( r^2 - |y - y_o|^2 \right) \, dy
= \int_{\Omega} \text{div} \left[ v \nabla v \left( r^2 - |y - y_o|^2 \right) \right] \, dy - \int_{\Omega} v \text{div} \left[ v \nabla v \left( r^2 - |y - y_o|^2 \right) \right] \, dy
= - \int_{\Omega} v \Delta v \left( r^2 - |y - y_o|^2 \right) \, dy - \int_{\Omega} v \nabla v \cdot \nabla \left( r^2 - |y - y_o|^2 \right) \, dy
\]
because on $\partial B_{y_o,r}$, $r = |y - y_o|$ and $v|_{\Gamma} = 0$
\[
= 2 \int_{\Omega} v \nabla v \cdot (y - y_o) \, dy \quad \text{because } \Delta_y v = 0 \text{ in } D.
\]

Consequently, from (C.2) and (C.3), we obtain
\[
H'(r) = \frac{N + 1}{r} H(r) + \frac{1}{r} D(r),
\]
\[(C.4)\]
and this is (C.1).
On another hand, the derivative of $D(r)$ is

$$D'(r) = 2r \int_{S^r} \left| (\nabla \mathbf{u})_{p_s+y_o} \right|^2 \rho^N d\rho d\sigma(s)$$

$$= 2r \int_{\Omega_2} |\nabla v(y)|^2 dy .$$  \hfill (C.5)

Here, when $\Delta_y v = 0$ in $D$ and $v|_{\Gamma} = 0$, we will remark that

$$2r \int_{\Omega_2} |\nabla v(y)|^2 dy = \frac{N+1}{2} D(r) + \frac{1}{r} \int_{B_{\rho r}} |y - y_o| \cdot \nabla v(y)|^2 dy$$

$$+ \frac{1}{r} \int_{\Gamma \cap B_{\rho r}} \left| \partial_y v \right|^2 \left( r^2 - |y - y_o|^2 \right) (y - y_o) \cdot \nu d\sigma(y)$$

indeed,

$$\left( N + 1 \right) \int_{\Omega_2} |\nabla v|^2 \left( r^2 - |y - y_o|^2 \right) dy$$

$$= \int_{\Omega_2} \text{div} \left( |\nabla v|^2 \right) \left( r^2 - |y - y_o|^2 \right) (y - y_o) dy$$

$$- \int_{\Omega_2} \nabla \left( |\nabla v|^2 \right) \left( r^2 - |y - y_o|^2 \right) \cdot (y - y_o) dy$$

$$= \int_{\Gamma \cap B_{\rho y_o}} |\nabla v|^2 \left( r^2 - |y - y_o|^2 \right) (y - y_o) \cdot \nu d\sigma(y)$$

$$- \int_{\Omega_2} \partial_y \left( |\nabla v|^2 \right) \left( r^2 - |y - y_o|^2 \right) (y - y_o) dy$$

$$= \int_{\Gamma \cap B_{\rho y_o}} |\nabla v|^2 \left( r^2 - |y - y_o|^2 \right) (y - y_o) \cdot \nu d\sigma(y)$$

indeed,

and

$$\int_{\Omega_2} \partial_y v \partial_{y, y_j} v \left( r^2 - |y - y_o|^2 \right) (y - y_o) dy$$

$$= - \int_{\Omega_2} \partial_y \left( \partial_{y, y_j} v \partial_y v \left( r^2 - |y - y_o|^2 \right) \right) dy$$

$$+ \int_{\Omega_2} \partial_y \left( (y - y_o) \partial_y v \partial_y v \left( r^2 - |y - y_o|^2 \right) \right) dy$$

$$+ \int_{\Omega_2} \left( (y - y_o) \partial_y^2 v \partial_y v \left( r^2 - |y - y_o|^2 \right) \right) dy$$

$$= - \int_{\Gamma \cap B_{\rho y_o}} \nu \cdot \left( (y - y_o) \partial_y v \partial_y v \left( r^2 - |y - y_o|^2 \right) \right) d\sigma(y)$$

$$+ \int_{\Omega_2} |\nabla v|^2 \left( r^2 - |y - y_o|^2 \right) dy$$

$$+ 0 \quad \text{because} \quad \Delta_y v = 0 \quad \text{in} \quad D$$

$$= \int_{\Gamma \cap B_{\rho y_o}} |\nabla v|^2 \left( r^2 - |y - y_o|^2 \right) (y - y_o) \cdot \nu d\sigma(y)$$

Therefore, when $\Delta_y v = 0$ in $D$, we have

$$\left( N + 1 \right) \int_{\Omega_2} |\nabla v|^2 \left( r^2 - |y - y_o|^2 \right) dy$$

$$= \int_{\Gamma \cap B_{\rho y_o}} |\nabla v|^2 \left( r^2 - |y - y_o|^2 \right) (y - y_o) \cdot \nu d\sigma(y)$$

$$- 2 \int_{\Gamma \cap B_{\rho y_o}} \partial_y v \nu \cdot \left( (y - y_o) \partial_y v \left( r^2 - |y - y_o|^2 \right) \right) d\sigma(y)$$

$$+ 2r^2 \int_{\Omega_2} |\nabla v|^2 dy - 4 \int_{\Omega_2} |(y - y_o) \cdot \nabla v|^2 dy .$$
By using the fact that \( v|_\Gamma = 0 \), we get \( \nabla v = (\nabla v \cdot \nu) \nu \) on \( \Gamma \) and we deduce that

\[
(N + 1) \int_{\Omega} |\nabla v|^2 \left( r^2 - |y - y_o|^2 \right) dy \\
= - \int_{\Gamma \cap B_{y_o,r}} |\partial_y v|^2 \left( r^2 - |y - y_o|^2 \right) (y - y_o) \cdot \nu d\sigma (y) \\
+ 2r^2 \int_{\Omega} |\nabla v|^2 dy - 4\int_{\Omega} |(y - y_o) \cdot \nabla v|^2 dy ,
\]

and this is (C.6).

Consequently, from (C.5) and (C.6), when \( \Delta_y v = 0 \) in \( D \) and \( v|_\Gamma = 0 \), we have

\[
D'(r) = \frac{N+1}{r} D(r) + \frac{4}{r} \int_{\Omega} |(y - y_o) \cdot \nabla v (y)|^2 dy \\
+ \frac{4}{r} \int_{\Gamma \cap B_{y_o,r}} |\partial_y v|^2 \left( r^2 - |y - y_o|^2 \right) (y - y_o) \cdot \nu d\sigma (y) .
\]

The computation of the derivative of \( \Phi (r) = \frac{D(r)}{H^2(r)} \) gives

\[
\Phi' (r) = \frac{1}{H^2(r)} \left[ D'(r) H(r) - D(r) H'(r) \right] ,
\]

which implies from (C.4) and (C.7), that

\[
H^2(r) \Phi' (r) = \frac{1}{r} \left( 4 \int_{\Omega} |(y - y_o) \cdot \nabla v (y)|^2 dy H(r) - D^2(r) \right) \\
\quad + \frac{H(r)}{r} \int_{\Gamma \cap B_{y_o,r}} |\partial_y v|^2 \left( r^2 - |y - y_o|^2 \right) (y - y_o) \cdot \nu d\sigma (y)
\]

Thanks to (C.3) and Cauchy-Schwarz inequality, we obtain that

\[
0 \leq 4 \int_{\Omega} |(y - y_o) \cdot \nabla v (y)|^2 dy H(r) - D^2(r) .
\]

The inequality \( 0 \leq (y - y_o) \cdot \nu \) on \( \Gamma \) holds when \( B_{y_o,r} \cap D \) is star-shaped with respect to \( y_o \) for any \( r \in (0, R_o) \). Therefore, we get the desired monotonicity for \( \Phi \) which completes the proof of Lemma C.

### 4.2 Quantitative unique continuation property for the Laplacian

Let \( D \subset \mathbb{R}^{N+1}, N \geq 1 \), be a connected bounded open set with boundary \( \partial D \). Let \( \Gamma \) be a non-empty Lipschitz open part of \( \partial D \). We consider the
Laplacian in $D$, with a homogeneous Dirichlet boundary condition on $\Gamma \subset \partial \Omega$:

\[
\begin{aligned}
\Delta_y v &= 0 \quad \text{in } D, \\
v &= 0 \quad \text{on } \Gamma, \\
v &= v(y) \in H^2(D).
\end{aligned}
\]

The goal of this section is to describe interpolation inequalities associated to solutions $v$ of (D.1).

**Theorem D .-** Let $\omega$ be a non-empty open subset of $D$. Then, for any $D_1 \subset D$ such that $\partial D_1 \cap \partial D \subseteq \Gamma$ and $\overline{D_1} \setminus (\Gamma \cap \partial D_1) \subset D$, there exist $C > 0$ and $\mu \in (0, 1)$ such that for any $v$ solution of (D.1), we have

\[
\int_{D_1} |v(y)|^2 \, dy \leq C \left( \int_{\omega} |v(y)|^2 \, dy \right)^{\mu} \left( \int_{D} |v(y)|^2 \, dy \right)^{1-\mu}.
\]

Or in an equivalent way by a minimization technique,

**Theorem D' .-** Let $\omega$ be a non-empty open subset of $D$. Then, for any $D_1 \subset D$ such that $\partial D_1 \cap \partial D \subseteq \Gamma$ and $\overline{D_1} \setminus (\Gamma \cap \partial D_1) \subset D$, there exist $C > 0$ and $\mu \in (0, 1)$ such that for any $v$ solution of (D.1), we have

\[
\int_{D_1} |v(y)|^2 \, dy \leq C \left( \frac{1}{\varepsilon} \right)^{\frac{1-\mu}{2\mu}} \int_{\omega} |v(y)|^2 \, dy + \varepsilon \int_{D} |v(y)|^2 \, dy \quad \forall \varepsilon > 0.
\]

**Proof of Theorem D .-** We divide the proof into two steps.

**Step 1 .-** We apply Lemma B, and use a standard argument (see e.g., [Ro]) which consists to construct a sequence of balls chained along a curve. More precisely, we claim that for any non-empty compact sets in $D$, $K_1$ and $K_2$, such that $\text{meas}(K_1) > 0$, there exists $\mu \in (0, 1)$ such that for any $v = v(y) \in H^2(D)$, solution of $\Delta_y v = 0$ in $D$, we have

\[
\int_{K_2} |v(y)|^2 \, dy \leq \left( \int_{K_1} |v(y)|^2 \, dy \right)^{\mu} \left( \int_{D} |v(y)|^2 \, dy \right)^{1-\mu}.
\]

**Step 2 .-** We apply Lemma C, and choose $y_o$ in a neighborhood of the part $\Gamma$ such that the conditions $i$, $ii$, $iii$, hold. Next, by an adequate partition of $D$, we deduce from (D.2) that for any $D_1 \subset D$ such that $\partial D_1 \cap \partial D \subseteq \Gamma$ and $\overline{D_1} \setminus (\Gamma \cap \partial D_1) \subset D$, there exist $C > 0$ and $\mu \in (0, 1)$
such that for any \( v = v(y) \in H^2(D) \) such that \( \Delta_y v = 0 \) on \( D \) and \( v = 0 \) on \( \Gamma \), we have
\[
\int_{D_1} |v(y)|^2 \, dy \leq C \left( \int_{\omega} |v(y)|^2 \, dy \right)^\mu \left( \int_{D} |v(y)|^2 \, dy \right)^{1-\mu}.
\]
This completes the proof.

### 4.3 Quantitative unique continuation property for the elliptic operator \( \partial_t^2 + \Delta \)

In this section, we present the following result.

**Theorem E .** Let \( \Omega \) be a bounded open set in \( \mathbb{R}^n \), \( n \geq 1 \), either convex or \( C^2 \) and connected. We choose \( T_2 > T_1 \) and \( \delta \in (0, (T_2 - T_1) / 2) \). Let \( f \in L^2(\Omega \times (T_1, T_2)) \). We consider the elliptic operator of second order in \( \Omega \times (T_1, T_2) \) with a homogeneous Dirichlet boundary condition on \( \partial \Omega \times (T_1, T_2) \),
\[
\begin{aligned}
\partial_t^2 w + \Delta w &= f \quad \text{in } \Omega \times (T_1, T_2), \\
w &= 0 \quad \text{on } \partial \Omega \times (T_1, T_2), \\
w &= w(x, t) \in H^2(\Omega \times (T_1, T_2)).
\end{aligned}
\]

Then, for any \( \varphi \in \mathcal{C}_0^\infty(\Omega \times (T_1, T_2)) \), \( \varphi \neq 0 \), there exist \( C > 0 \) and \( \mu \in (0, 1) \) such that for any \( w \) solution of (E.1), we have
\[
\int_{T_1 - \delta}^{T_1 + \delta} \int_{\Omega} |w(x, t)|^2 \, dx \, dt \leq C \left( \int_{T_1 - \delta}^{T_1} \int_{\Omega} |w(x, t)|^2 \, dx \, dt \right)^{1-\mu} \left( \int_{T_1 - \delta}^{T_1} \int_{\Omega} |w(x, t)|^2 \, dx \, dt + \int_{T_1 - \delta}^{T_1} \int_{\Omega} |\varphi w(x, t)|^2 \, dx \, dt \right)^\mu.
\]

**Proof .** First, by a difference quotient technique and a standard extension at \( \Omega \times \{T_1, T_2\} \), we check the existence of a solution \( u \in H^2(\Omega \times (T_1, T_2)) \) solving
\[
\begin{aligned}
\partial_t^2 u + \Delta u &= f \quad \text{in } \Omega \times (T_1, T_2), \\
u &= 0 \quad \text{on } \partial \Omega \times (T_1, T_2) \cup \Omega \times \{T_1, T_2\},
\end{aligned}
\]
such that
\[
\|u\|_{H^2(\Omega \times (T_1, T_2))} \leq c \|f\|_{L^2(\Omega \times (T_1, T_2))},
\]
for some \( c > 0 \) only depending on \( (\Omega, T_1, T_2) \). Next, we apply Theorem D with \( D = \Omega \times (T_1, T_2), \Omega \times (T_1 + \delta, T_2 - \delta) \subset D_1, y = (x, t), \Delta_y = \partial_t^2 + \Delta, \) and \( v = w - u \).
4.4 Application to the wave equation

From the idea of L. Robbiano [Ro2] which consists to use an interpolation inequality of Hölder type for the elliptic operator $\partial_t^2 + \Delta$ and the Fourier-Bros-Iagolnitzer transform introduced by G. Lebeau and L. Robbiano [LR], we obtain the following estimate of logarithmic type.

**Theorem F.** Let $\Omega$ be a bounded open set in $\mathbb{R}^n$, $n \geq 1$, either convex or $C^2$ and connected. Let $\omega$ be a non-empty open subset in $\Omega$. Then, for any $\beta \in (0,1)$ and $k \in \mathbb{N}^*$, there exist $C > 0$ and $T > 0$ such that for any solution $u$ of

$$\begin{cases} 
\partial_t^2 u - \Delta u = 0 & \text{in } \Omega \times (0,T) , \\
u = 0 & \text{on } \partial \Omega \times (0,T) , \\
(u, \partial_t u) (\cdot, 0) = (u_0, u_1) ,
\end{cases}$$

with non-identically zero initial data $(u_0, u_1) \in D(A^{k-1})$, we have

$$\| (u_0, u_1) \|_{D(A^{k-1})} \leq C e \left( C \| (u_0, u_1) \|_{L^2(\Omega) \times L^2(\Omega)} \right)^{1/(2k)} \| u \|_{L^2(\omega \times (0,T))} .$$

**Proof.** First, recall that with a standard energy method, we have that

$$\forall t \in \mathbb{R} \quad \| (u_0, u_1) \|_{H^1(\Omega) \times L^2(\Omega)}^2 = \int_\Omega \left( |\partial_t u (x,t)|^2 + |\nabla u (x,t)|^2 \right) \, dx ,$$

and there exists a constant $c > 0$ such that for all $T \geq 1$,

$$T \| (u_0, u_1) \|_{L^2(\Omega) \times H^{-1}(\Omega)}^2 \leq c \int_0^T \int_\Omega |u (x,t)|^2 \, dx . \tag{F.1}$$

Next, let $\beta \in (0,1)$, $k \in \mathbb{N}^*$, and choose $N \in \mathbb{N}^*$ such that $0 < \beta + \frac{1}{2N} < 1$ and $2N > k$. For any $\lambda \geq 1$, the function $F_\lambda (z) = \frac{1}{2} \int_\mathbb{R} e^{iz} e^{-\left( \frac{1}{\lambda^2} \right)^{2N}} \, dz$ is holomorphic in $\mathbb{C}$, and there exists four positive constants $C_0, c_0, c_1$ and $c_2$ (independent on $\lambda$) such that

$$\begin{cases} 
\forall z \in \mathbb{C} \quad |F_\lambda (z)| \leq C_0 \lambda^{-c_0 |\text{Im} z|^{1/\gamma}} , \\
|\text{Im} z| \leq c_2 |\text{Re} z| \Rightarrow |F_\lambda (z)| \leq C_0 \lambda^{-c_1 |\text{Re} z|^{1/\gamma}} ,
\end{cases} \tag{F.3}$$

(see [LR]).

Now, let $s, \ell_0 \in \mathbb{R}$, we introduce the following Fourier-Bros-Iagolnitzer transformation in [LR]:

$$W_{\ell_0, \lambda} (x, s) = \int_{\mathbb{R}} F_\lambda (\ell_0 + is - \ell) \Phi (\ell) u (x, \ell) \, d\ell , \tag{F.4}$$
where $\Phi \in C_0^\infty (\mathbb{R})$. As $u$ is solution of the wave equation, $W_{t, \lambda}$ satisfies:

$$
\begin{aligned}
&\left\{ \begin{array}{l}
\frac{\partial^2}{\partial t^2} W_{t, \lambda}(x, s) + \Delta W_{t, \lambda}(x, s) \\
= \int_{\mathbb{R}} -F_{\lambda}(x - t + is - t) \left[ \Phi''(t) u(x, \ell) + 2\Phi'(t) \partial_t u(x, \ell) \right] d\ell, \\
W_{t, \lambda}(x, s) = 0 \quad \text{for } x \in \partial \Omega, \\
W_{t, \lambda}(x, 0) = (F_{\lambda} \ast \Phi(u, \cdot))(t_0) \quad \text{for } x \in \Omega.
\end{array} \right.
\end{aligned}
$$

(F.5)

In another hand, we also have for any $T > 0$,

$$
\|\Phi u(x, \cdot)\|_{L^2((\frac{1}{T} - 1, \frac{1}{T} + 1))} \leq \|\Phi u(x, \cdot) - F_{\lambda} \ast \Phi u(x, \cdot)\|_{L^2((\frac{1}{T} - 1, \frac{1}{T} + 1))}
+ \|F_{\lambda} \ast \Phi u(x, \cdot)\|_{L^2((\frac{1}{T} - 1, \frac{1}{T} + 1))}
\leq \|\Phi u(x, \cdot) - F_{\lambda} \ast \Phi u(x, \cdot)\|_{L^2(\mathbb{R})}
+ \left( \int_{t \in (\frac{1}{T} - 1, \frac{1}{T} + 1)} |W_{t, \lambda}(x, 0)|^2 dt \right)^{1/2}.
$$

(F.6)

Denoting $\mathcal{F}(f)$ the Fourier transform of $f$, by using Parseval equality and $\mathcal{F}(F_{\lambda})(\tau) = e^{-\left(\frac{\tau}{T}\right)^2}$, one obtain

$$
\|\Phi u(x, \cdot) - F_{\lambda} \ast \Phi u(x, \cdot)\|_{L^2(\mathbb{R})}
= \frac{1}{\sqrt{2\pi}} \|\mathcal{F}(\Phi u(x, \cdot) - F_{\lambda} \ast \Phi u(x, \cdot))\|_{L^2(\mathbb{R})}
= \frac{1}{\sqrt{2\pi}} \left( \int_{\mathbb{R}} \left|1 - e^{-\left(\frac{\tau}{T}\right)^2}\right| \mathcal{F}(\Phi u(x, \cdot))(\tau)^2 d\tau \right)^{1/2}
\leq C \left( \int_{\mathbb{R}} \left|\left(\frac{\tau}{T}\right)^k \mathcal{F}(\Phi u(x, \cdot))(\tau)^2 \right| d\tau \right)^{1/2}
\leq C \frac{1}{\sqrt{T}} \left( \int_{\mathbb{R}} \left|\mathcal{F}(\partial_\tau^k (\Phi u(x, \cdot)))(\tau)^2 \right| d\tau \right)^{1/2}
\leq C \frac{1}{\sqrt{T}} \|\partial_\tau^k (\Phi u(x, \cdot))\|_{L^2(\mathbb{R})}.
$$

(F.7)

Therefore, from (F.6) and (F.7), one gets

$$
\|\Phi u(x, \cdot)\|_{L^2((\frac{1}{T} - 1, \frac{1}{T} + 1))}
\leq C \frac{1}{\sqrt{T}} \|\partial_\tau^k (\Phi u(x, \cdot))\|_{L^2(\mathbb{R})} + \left( \int_{t \in (\frac{1}{T} - 1, \frac{1}{T} + 1)} |W_{t, \lambda}(x, 0)|^2 dt \right)^{1/2}.
$$

(F.8)

Now, recall that from the Cauchy’s theorem we have:

**Proposition 1.** Let $f$ be a holomorphic function in a domain $D \subseteq \mathbb{C}$. Let $a, b > 0$, $z \in \mathbb{C}$. We suppose that

$$
D_o = \{(x, y) \in \mathbb{R}^2 \simeq \mathbb{C} \mid |x - \text{Re} z| \leq a, \ |y - \text{Im} z| \leq b\} \subseteq D,
$$

then

$$
f(z) = \frac{1}{\pi ab} \int \int_{\left|\frac{x - \text{Re} z}{a}\right|^2 + \left|\frac{y - \text{Im} z}{b}\right|^2 \leq 1} f(x + iy) d\alpha d\beta.
$$
Choosing \( z = t \in \left( \frac{T}{2} - 1, \frac{T}{2} + 1 \right) \subset \mathbb{R} \) and \( x + iy = \ell_o + is \), we deduce that

\[
|W_{t,\lambda}(x, 0)| \leq \frac{1}{n!} \int_{|s| \leq b} \int_{|\lambda| \leq a} |W_{t,\lambda}(x, 0)| \, df_o \, ds.
\]

Then, we have

\[
\begin{align*}
|W_{t,\lambda}(x, 0)| &\leq \frac{1}{n!} \int_{|s| \leq b} \int_{|\lambda| \leq a} |W_{t,\lambda}(x, 0)| \, ds \, df_o \\
&\leq \frac{2}{\pi n!} \left( \int_{|s| \leq b} \int_{|\lambda| \leq a} |W_{t,\lambda}(x, 0)| \, ds \, df_o \right)^{1/2},
\end{align*}
\]

and with \( a = 2b = 1 \),

\[
\begin{align*}
\int_{\Omega} \int_{|s| \leq b} \int_{|\lambda| \leq a} &|W_{t,\lambda}(x, 0)|^2 \, dt \\
&\leq \int_{\Omega} \int_{|s| \leq b} \int_{|\lambda| \leq a} \left| W_{t,\lambda}(x, 0) \right|^2 \, ds \, df_o \, dt
\end{align*}
\]

Consequently, from (F.8), (F.10) and integrating over \( \Omega \), we get the existence of \( C > 0 \) such that

\[
\begin{align*}
\int_{\Omega} \int_{|s| \leq b} \int_{|\lambda| \leq a} &|W_{t,\lambda}(x, 0)|^2 \, dt \\
&\leq C(1 + \int_{\Omega} \int_{|s| \leq b} \int_{|\lambda| \leq a} \left| W_{t,\lambda}(x, 0) \right|^2 \, ds \, df_o \, dt
\end{align*}
\]

Now recall the following quantification result for unique continuation of elliptic equation with Dirichlet boundary condition (Theorem E applied to \( T_1 = -1, T_2 = 1, \delta = 1/2, \varphi \in C_0^\infty (\omega \times (-1, 1)) \):

**Proposition 2** - Let \( \Omega \) be a bounded open set in \( \mathbb{R}^n \), \( n \geq 1 \), either convex or \( C^2 \) and connected. Let \( \omega \) be a non-empty open subset in \( \Omega \). Let \( f = f(x, s) \in L^2(\Omega \times (-1, 1)) \). Then there exists \( \tilde{c} > 0 \) such that for all \( w = w(x, s) \in H^2(\Omega \times (-1, 1)) \) solution of

\[
\begin{align*}
\partial_s^2 w + \Delta w &= f & \text{in } \Omega \times (-1, 1), \\
w &= 0 & \text{on } \partial \Omega \times (-1, 1),
\end{align*}
\]

for all \( \varepsilon > 0 \), we have:

\[
\begin{align*}
\int_{|s| \leq 1/2} \int_{|\lambda| \leq 1} |w(x, s)|^2 \, dx \, ds &\leq \tilde{c} \varepsilon^{1/2} \left( \int_{|s| \leq 1} \int_{\Omega} |w(x, s)|^2 \, dx \, ds + \int_{|s| \leq 1} \int_{\Omega} |f(x, s)|^2 \, dx \, ds \right) \\
&\quad + e^{-4\alpha/\varepsilon} \int_{|s| \leq 1} \int_{\Omega} |w(x, s)|^2 \, dx \, ds.
\end{align*}
\]
Applying to $W_{t_0, \lambda}$, from (F.5) we deduce that for all $\varepsilon > 0$,

$$
\int_{|s| \leq 1/2} \int_\Omega |W_{t_0, \lambda}(x, s)|^2 \, dx \, ds \\
\leq e^{-2\varepsilon^2/\varepsilon} \int_{|s| \leq 1} \int_\Omega |W_{t_0, \lambda}(x, s)|^2 \, dx \, ds \\
+ \varepsilon e^{2/\varepsilon} \int_{|s| \leq 1} \int_\omega |W_{t_0, \lambda}(x, s)|^2 \, dx \, ds \\
+ \varepsilon e^{2/\varepsilon} \int_{|s| \leq 1} \int_\Omega \left| \int_\mathbb{R} - F_\lambda(t_0 + is - \ell) \right| \, dt \|u\|^2 \, dx \, ds \\
\leq e^{-2\varepsilon^2/\varepsilon} \int_{|s| \leq 1} \int_\Omega |W_{t_0, \lambda}(x, s)|^2 \, dx \, ds \\
+ \varepsilon e^{2/\varepsilon} \int_{|s| \leq 1} \int_\omega |W_{t_0, \lambda}(x, s)|^2 \, dx \, ds \\
+ \varepsilon e^{2/\varepsilon} \int_{|s| \leq 1} \int_\Omega \left| \int_\mathbb{R} - F_\lambda(t_0 + is - \ell) \right| \, dt \|u\|^2 \, dx \, ds .
$$

Consequently, from (F.11) and (F.12), there exists a constant $C > 0$, such that for all $\varepsilon > 0$,

$$
\int_\Omega \int_{t \in \left(\frac{T}{2} - 1, \frac{T}{2} + 1\right]} |\Phi(t)u(x, t)|^2 \, dt \, dx \\
\leq C e^{-4\varepsilon^2/\varepsilon} \int_\Omega \int_{t \in \left(\frac{T}{2} - 2, \frac{T}{2} + 2\right]} \left( \int_{|s| \leq 1} \int_\Omega |W_{t_0, \lambda}(x, s)|^2 \, dx \, ds \right) \, dt \, dx \\
+ 4C e^{-4\varepsilon^2/\varepsilon} \int_{t \in \left(\frac{T}{2} - 2, \frac{T}{2} + 2\right]} \left( \int_{|s| \leq 1} \int_\Omega |W_{t_0, \lambda}(x, s)|^2 \, dx \, ds \right) \, dt \, dx \\
+ 4C e^{-4\varepsilon^2/\varepsilon} \int_{t \in \left(\frac{T}{2} - 2, \frac{T}{2} + 2\right]} \left( \int_{|s| \leq 1} \int_\Omega \left| \int_\mathbb{R} - F_\lambda(t_0 + is - \ell) \right| \, dt \right) \|u\|^2 \, dx \, ds .
$$

Let define $\Phi \in C^\infty_0(\mathbb{R})$ more precisely now: we choose $\Phi \in C^\infty_0([0, T])$, $0 \leq \Phi \leq 1$, $\Phi \equiv 1$ on $\left(\frac{T}{2}, \frac{T}{4}\right)$. Furthermore, let $K = \left[0, \frac{T}{4}\right] \cup \left[\frac{T}{4}, T\right]$ such that $\text{supp}(\Phi') = K$ and $\text{supp}(\Phi'') \subset K$.

Let $K_0 = \left[\frac{T}{4}, \frac{T}{2}\right]$. In particular, $\text{dist}(K, K_0) = \frac{T}{4}$. Let define $T > 0$ more precisely now: we choose $T > 16 \max(1, 1/c_2)$ in order that $\left(\frac{T}{2} - 2, \frac{T}{2} + 2\right) \subset K_0$ and $\text{dist}(K, K_0) \geq \frac{T}{c_2}$.

Now, we will choose $\ell_0 \in \left(\frac{T}{2} - 2, \frac{T}{2} + 2\right) \subset K_0$ and $s \in [-1, 1]$. Consequently, for any $\ell \in K$, $|\ell_0 - \ell| \geq \frac{T}{c_2} \geq \frac{1}{c_2} |s|$ and it will imply from the second line of (F.3) that

$$
\forall \ell \in K \quad |F_\lambda(\ell_0 + is - \ell)| \leq A \lambda^{\gamma} e^{-c_1 \lambda^{\frac{2}{1+\gamma}}} .
$$

Till the end of the proof, $C$ and respectively $C_T$ will denote a generic positive constant independent of $\varepsilon$ and $\lambda$ but dependent on $\Omega$ and respectively $(\Omega, T)$, whose value may change from line to line.

The first term in the right hand side of (F.13) becomes, using (F.1),

$$
\frac{1}{\lambda^{2k}} \int_\Omega \int_\mathbb{R} \left| \frac{\partial_{\ell}^k (\Phi(t)u(x, t))}{\lambda^{2k}} \right|^2 \, dt \, dx \leq C_T \frac{1}{\lambda^{2k}} \|(u_0, u_1)\|_{D(A^{k-1})}^2 .
$$

(F.15)
The second term in the right hand side of (F.13) becomes, using the first line of (F.3),
\[
\begin{align*}
e^{-4\varepsilon} & \int_{\Omega} \int_{|s| \leq 1} \int_{\Omega} |W_{o,\lambda}(x,s)|^2 \, dx \, ds \, d\ell_o \\
\leq & \left( C_{\alpha} \lambda^{1.5} e^{2\lambda_0} \right)^2 C_{\alpha} \int_{\Omega} \int_{|s| \leq 1} \int_{\Omega} \left| \int_0^T |u(x,\ell)| \, d\ell \right|^2 \, dx \, ds \\
\leq & C_T\lambda^{2.5} e^{2\lambda_0} e^{-4\alpha_0/\varepsilon} \| (u_0, u_1) \|^2_{H^1(\Omega) \times L^2(\Omega)} .
\end{align*}
\]
(16)

The third term in the right hand side of (F.13) becomes, using the first line of (F.3),
\[
\begin{align*}
e^{C_1/\varepsilon} & \int_{|s| \leq 1} \int_{\Omega} |W_{o,\lambda}(x,s)|^2 \, dx \, ds \, d\ell_o \\
\leq & \left( C_{\alpha} \lambda^{1.5} e^{2\lambda_0} \right)^2 C_{\alpha} \int_{|s| \leq 1} \int_{\Omega} \left| \int_0^T |u(x,\ell)| \, d\ell \right|^2 \, dx \, ds \\
\leq & C_T\lambda^{2.5} e^{2\lambda_0} e^{C_1/\varepsilon} \int_0^T \| u(x,t) \|^2 \, dt \, dx .
\end{align*}
\]
(17)

The fourth term in the right hand side of (F.13) becomes, using (F.14) and the choice of \( \Phi \),
\[
\begin{align*}
e^{\gamma/\varepsilon} & \int_{|s| \leq 1} \int_{\Omega} \int_{|s| \leq 1} \left[ \Phi' \left( \int_{\Omega} \int_R - F_\lambda(\ell_0 + is - \ell) \right) \\
\quad + 2\Phi' \left( \int_{\Omega} \int_R \partial_t u(x,\ell) \right) \right] \, d\ell \, dx \, ds \\
\leq & C \left( A\lambda^{1.5} e^{-c_1 \lambda(1/\gamma)} \right)^2 e^{\gamma/\varepsilon} \left( \int_{\Omega} \left| \left( \partial_u u(x,\ell) \right) \right| \right) \left| \partial_u u(x,\ell) \right| \right] \, d\ell \, dx \\
\leq & C_T\lambda^{2.5} e^{-2c_1 \lambda(1/\gamma)} e^{\gamma/\varepsilon} \| (u_0, u_1) \|^2_{H^1(\Omega) \times L^2(\Omega)} .
\end{align*}
\]
(18)

We finally obtain from (F.15), (F.16), (F.17), (F.18) and (F.13), that
\[
\begin{align*}
\int_0^T & \int_{|s| \leq 1} \int_{\Omega} |\Phi(t) u(x,t)|^2 \, dt \, dx \\
\leq & C_T \frac{1}{\varepsilon} \| (u_0, u_1) \|_{H^1(\Omega) \times L^2(\Omega)}^2 \\
& + C_T\lambda^{2.5} e^{2\lambda_0} e^{-4\alpha_0/\varepsilon} \| (u_0, u_1) \|^2_{H^1(\Omega) \times L^2(\Omega)} \\
& + C_{\lambda} e^{2\lambda_0} e^{C_1/\varepsilon} \int_0^T \| u(x,t) \|^2 \, dt \, dx \\
& + C_{\lambda} e^{-2c_1 \lambda(1/\gamma)} e^{\gamma/\varepsilon} \| (u_0, u_1) \|^2_{H^1(\Omega) \times L^2(\Omega)} .
\end{align*}
\]
(19)

We begin to choose \( \lambda = \frac{1}{2} \) in order that
\[
\begin{align*}
\int_0^T & \int_{|s| \leq 1} \int_{\Omega} |\Phi(t) u(x,t)|^2 \, dt \, dx \\
\leq & e^{2\beta_0 C_T \| (u_0, u_1) \|_{H^1(\Omega) \times L^2(\Omega)}} \\
& + e^{-2c_0/\varepsilon} C_T \| (u_0, u_1) \|^2_{H^1(\Omega) \times L^2(\Omega)} \\
& + C_{\lambda} e^{2\lambda_0} e^{C_1/\varepsilon} \int_0^T \| u(x,t) \|^2 \, dt \, dx \\
& + C_{\lambda} \exp \left( \left( -2c_1 \left( \frac{1}{\gamma} + \frac{\varepsilon}{2} \right) \right) \right) \| (u_0, u_1) \|^2_{H^1(\Omega) \times L^2(\Omega)} .
\end{align*}
\]
(20)
We finally need to choose $T > 16 \max \left(1, \frac{1}{c_2} \right)$ large enough such that 
\[-2c_1 \left(\frac{T}{\varepsilon}\right)^{1/\gamma} + \tilde{c} \right) \leq -1 \text{ that is } 8 \left(\frac{1+c}{2c_1}\right)^{\gamma} \leq T,\]
to deduce the existence of $C > 0$ such that for any $\varepsilon \in (0, 1)$,
\[
\int_I \int_{\Omega} \left| u(x, t) \right|^2 dtdx \leq \int_I \int_{\Omega} \left| \Phi(t) u(x, t) \right|^2 dtdx
\leq C\varepsilon^{2\theta k} \left\| (u_0, u_1) \right\|^2_{D(A^{k-1})}
+ Ce^{C/\varepsilon} \int_0^T \int_{\Omega} \left| u(x, t) \right|^2 dtdx .
\]

(F.21)

Now we conclude from (F.2), that there exist a constant $c > 0$ and a time $T > 0$ large enough such that for all $\varepsilon > 0$ we have
\[
\left\| (u_0, u_1) \right\|^2_{L^2(\Omega) \times H^{-1}(\Omega)}
\leq e^{c/\varepsilon} \int_0^T \int_{\Omega} \left| u(x, t) \right|^2 dtdx + \varepsilon^{2\theta k} \left\| (u_0, u_1) \right\|^2_{D(A^{k-1})} .
\]

(F.22)

Finally, we choose
\[
\varepsilon = \left( \frac{\left\| (u_0, u_1) \right\|^2_{L^2(\Omega) \times H^{-1}(\Omega)}}{\left\| (u_0, u_1) \right\|^2_{D(A^{k-1})}} \right)^{1/(2\theta k)} .
\]

Theorem 1.1 is deduced by applying Theorem F to $\partial_t u$.

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