Gravitational radiation from chiral string cusps

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We calculate the gravitational radiation from a cusp of the chiral cosmic strings as a function of the current on the string in the limit of small values of the current. The smoothing of the cosmic string cusp due to the presence of the superconducting current on the string leads to the different behavior of the gravitational radiation from the cusp as compared with the non-superconducting (ordinary) string. The difference between the gravitational energy radiation from the chiral cusp and the ordinary one is proportional to the value of the current on the string. It is shown that there is a high-frequency cutoff in the spectrum of the gravitational radiation from the chiral string. The frequency cutoff depends on the value of the string current. This effect is crucial for the detection of gravitational wave bursts from cosmic strings: the rather large current on the string would lower the amplitude of the incoming signal.

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I. INTRODUCTION

Cosmic strings are the linear topological defects that predicted by Grand Unified Theories (GUT). They may be naturally produced in the phase transitions in the early universe. A very remarkable feature of the ordinary (non-superconducting) cosmic string is the presence of the short-lived cusps, i.e. the regions of a string moving with the speed close to the speed of light (see for review e.g. [1, 2]). These cusps produce a highly beamed radiation that can be detected by the gravitational wave detectors [3, 4].

Witten in 1985 showed that cosmic strings can carry the superconducting current [5]. The presence of the current on the string leads to the new interesting features of the string motion: (i) the smoothing of the cusp and diminishing of their velocities in the case of small currents; (ii) the possibility for the existence of the stationary cosmic string configurations (vortons) [6, 7, 8] in the case of sufficiently large currents.

The equations of motions of the superconducting cosmic string are solvable analytically if (i) the current on the string is a chiral one and (ii) the radiation back reaction is neglected. The motion of the chiral cosmic string in the appropriate gauge is given as follows [9, 10, 11]:

\[ X^0 = t, \quad X(t, \sigma) = \frac{1}{2} \left[ X_-(\sigma_-) + X_+(\sigma_+) \right], \quad (1) \]

where \( t \) is the Minkowskian time, \( \sigma \) is a parameter along the string, \( X_-(\sigma_-) \) and \( X_+(\sigma_+) \) are vector functions of \( \sigma_- = \sigma - t \) and \( \sigma_+ = \sigma + t \) obeying the following constraints:

\[ X_-'^2(\sigma_-) = 1, \quad X_+'^2(\sigma_+) = \Delta^2(\sigma_+) \leq 1, \quad (2) \]

where \( \Delta(\sigma_+) \) determines the chiral current along the string: the more the value of the \( \Delta(\sigma_+) \) the less the current and vise versa. The smallness of chiral current means that \( \epsilon(\sigma_+) \equiv 1 - \Delta(\sigma_+) \ll 1. \)

Numerical computations of the electromagnetic and gravitational radiation from chiral cosmic strings as a function of current were done in [12]. The limiting case of radiation from cosmic string strings with a current close to the maximum value was considered in [13]. An opposite case for the electromagnetic radiation of strings with the small chiral currents considered by Blanco-Pillado and Ohnum [14]. Damour and Vilenkin [3, 4] considered the gravitational radiation from the cusp of the ordinary strings and argued that corresponding Gravitational Wave Bursts (GWB) can be detected by gravitational wave detectors LIGO, VIRGO and LISA. In this paper we find the gravitational radiation from the cusp of cosmic string in the limit of the small chiral current. We show that the chiral string cusp does not radiate the gravitational waves on the frequencies higher than some cutoff frequency \( \omega_{\text{cut}} \) which depends on the current value. Therefore the presence of the current on the string may impede the observability of the GWBs from the cosmic string cusps by gravitational wave detectors. We estimate the minimal chiral current leading to the lowering of amplitude of the gravitational wave on the given frequency from a single cusp and from the whole string network in the universe. We calculate also the dependence of full radiated gravitational energy from the cusp as a function of the small chiral current.

Throughout the paper we adopt the following conventions. We use the units \( \hbar = c = 1 \). The signature of the space-time is \((- , + , +, +)\). Greek indexes \( \mu, \nu, \ldots \) run through 0 to 3 and Latin labels designate the three spatial dimensions.

The paper is organized as follows: In Section II we find the dependence of waveforms for chiral string cusp on the value of the small current on the string and the angle of deviation of observer from the direction of the cusp.
motion. Then we discuss the influence of the presence of the chiral current on the amplitude of incoming GWBs from the whole universe. In Section III we make the detailed calculation of the gravitational energy radiation from the chiral cusp as a function of the small current. In Section IV we briefly discuss the obtained results.

II. WAVEFORMS FROM CHIRAL CUSPS

In the following we will consider the case of small current on the string (i.e. when \( \Delta \) in (2) is close to unity), otherwise cusps do not formed on the string at all. We also assume that the current is constant in the part of the string forming the cusp (it means that \( \Delta = \text{const} \)). Throughout this section we will follow the calculations of Damour and Vilenkin [3, 4], who considered the radiation from the ordinary (non-superconducting) string cusp.

Let us denote \( X_{+}^{0} = \sigma_{+}, X_{-}^{0} = -\sigma_{-} \). Then the solutions of equations of motion of the chiral string (11) can be rewritten in the following form:

\[
X^{\mu} = \frac{1}{2} \left[ X_{-}^{\mu}(\sigma_{-}) + X_{+}^{\mu}(\sigma_{+}) \right],
\]

(3)

where \( X_{-}^{\mu}(\sigma_{-}) \) and \( X_{+}^{\mu}(\sigma_{+}) \) are the three-dimensional components of vector functions \( X_{-}(\sigma_{-}) \) and \( X_{+}(\sigma_{+}) \) correspondingly. The energy-momentum tensor of the chiral cosmic string is given by the following expression:

\[
T^{\mu\nu} = \mu \int d\sigma \, (\dot{X}^{\mu} \dot{X}^{\nu} - X^{\mu} \dot{X}^{\nu}) \delta^{(4)}(x - X(t, \sigma)),
\]

(4)

where \( \mu \) is the energy of the string per unit length. Inserting (11) into general expression for Fourier-transform of the energy-momentum tensor:

\[
T^{\mu\nu}(\omega, k) = \frac{1}{2\pi} \int dt \, d\sigma \, e^{i(\omega t - \mathbf{k} \cdot \mathbf{x})} T^{\mu\nu}(t, x)
\]

(5)

and using \( d\sigma = (1/2) d\sigma_{+} d\sigma_{-} \) we obtain the Fourier-transform of the energy-momentum tensor for chiral cosmic string:

\[
T^{\mu\nu}(\omega, k) = -\frac{\mu}{4\pi} I^{(\mu \nu)},
\]

(6)

where \( I^{(\mu \nu)} = (1/2)(I^{(\mu \nu)} + I^{(\nu \mu)}) \) and

\[
I^{\mu} = \int d\sigma_{-} X_{-}^{\mu} e^{-i\mathbf{k} \cdot \mathbf{x}_{-}} Y^{\mu} = \int d\sigma_{+} X_{+}^{\mu} e^{-i\mathbf{k} \cdot \mathbf{x}_{+}},
\]

(7)

where \( k = (\omega, \mathbf{k}) \). The small perturbations of metric \( h_{\mu\nu} \) for any relativistic source in the linearized approximation is given by [10]:

\[
\bar{h}^{\mu\nu}(t, \mathbf{n}) = \frac{4G}{r} \int d\omega \, e^{-i\omega(t-r)} T^{\mu\nu}(\omega, \mathbf{k}),
\]

(8)

where \( r \) is a distance from the source, \( \mathbf{n} = \mathbf{x}/r \) and \( \bar{h}_{\mu\nu} \equiv h_{\mu\nu} - (1/2) g_{\mu\nu} h_{\lambda\lambda} \). For description of the gravitational radiation from cusps it is more convenient to use waveforms \( \kappa^{\mu\nu}(t-r, \mathbf{n}) = \bar{h}^{\mu\nu}(t, \mathbf{n})/r \) instead of \( h^{\mu\nu} \):

\[
\kappa^{\mu\nu}(t-r, \mathbf{n}) = 4G \int d\omega \, e^{-i\omega(t-r)} T^{\mu\nu}(\omega, \mathbf{k}).
\]

(9)

The Fourier-transform of the waveform:

\[
\kappa^{\mu\nu}(\omega, \mathbf{n}) = \int dt \, \kappa^{\mu\nu}(t) e^{i\omega t}.
\]

(10)

Inserting (9) into (10) and using the expression for the Fourier-transform of the energy-momentum tensor (6) we find:

\[
\kappa^{\mu\nu}(\omega, \mathbf{n}) = -2G\mu I^{(\mu \nu)}.
\]

(11)

For the energy-momentum tensor the following relations are valid: \( k^{\mu} T^{\mu\nu}(\omega, \mathbf{k}) = 0 \) (16), therefore using (11) and (10) one can see that the 00 and 0j components of \( \kappa^{\mu\nu} \) can be expressed through the purely spatial components of \( \kappa^{ij} \). By this reason we will be interested in the following only in the \( \kappa^{ij} \).

Now let us consider the radiation from the relativistic cusp. We take the following coordinates of the cusp: \( \sigma_{+}^2 = 0 \) and \( X_{+}^0 = 0 \). It is convenient to define the cusp as the point on the two-dimensional world-sheet of the string where \( X_{-} \) and \( X_{+} \) are anti-parallel [14]:

\[
\mathbf{n}^{(c)} = -\frac{X_{-}}{|X_{-}|} = \frac{X_{+}}{|X_{+}|},
\]

(12)

where \( X_{-} \) and \( X_{+} \) in (12) is taken in at \( \sigma_{\pm} = 0 \) and \( \mathbf{n}^{(c)} \) is the direction of the cusp motion (i.e. the concentrating of the gravitational radiation). The position of the string near the cusp can be expressed as follows:

\[
\begin{align*}
X_{-}(\sigma_{-}) &= -\mathbf{n}^{(c)} \sigma_{-} + \frac{1}{2} X_{-}^{\mu} \sigma_{-}^2 + \frac{1}{6} X_{-}^{\mu} \sigma_{-}^3, \\
X_{+}(\sigma_{+}) &= \Delta \mathbf{n}^{(c)} \sigma_{+} + \frac{1}{2} X_{+}^{\mu} \sigma_{+}^2 + \frac{1}{6} X_{+}^{\mu} \sigma_{+}^3,
\end{align*}
\]

(13)

where the derivatives of \( X_{+}^{\mu} \) is taken at \( \sigma_{\pm} = 0 \). Further, using the conditions (2) one can find:

\[
\begin{align*}
\mathbf{n}^{(c)} X_{-}'' &= 0, \\
\mathbf{n}^{(c)} X_{+}'' &= (X_{-}^\nu)^2, \\
\mathbf{n}^{(c)} X_{+}' &= -\frac{(X_{-}^\nu)^2}{\Delta}.
\end{align*}
\]

(14)

Damour and Vilenkin [3, 4] found for the ordinary string cusp the qualitative dependence of waveform from the \( \theta \) angle between \( \mathbf{n}^{(c)} \) and direction of emission \( \mathbf{n} \). It was obtained that the waveform can be approximated by the power dependence from the frequency \( \kappa \propto 1/\omega^{4/3} \) for \( \omega \lesssim \omega_{\text{cut}}(\theta) \propto 1/\theta^3 \). Note that the dependence of \( \kappa \) on \( \omega \) as \( \kappa \propto 1/\omega^{4/3} \) differs from the result in [3, 4] because our definition of \( \kappa \) (16) does not contain additional \( \omega \). It was found that for large frequencies \( \omega \gtrsim \omega_{\text{cut}} \) the waveforms are exponentially suppressed therefore one can set \( \kappa \approx 0 \) for \( \omega \gtrsim \omega_{\text{cut}} \).

Our aim now is to find the dependence of \( \kappa^{ij} \) not only on the angle \( \theta \) but also on the string current. Following
the analysis made in [3, 4] we insert [13] into [7] and use the conditions [14]. We find that function \( Y \) defined in [7] can be rewritten as follows (we omit 4-dimensional indexes \( \mu \) as they are not significant for such an analysis):

\[
Y \propto \int du (\varepsilon + u) e^{-i\phi_u},
\]

where

\[
\phi_+ \equiv \frac{1}{2} k X_+ \sim u^3 + \varepsilon u^2 + \frac{\theta^2 + 2\epsilon}{\theta^2} \varepsilon^2 u.
\]

Here we made the replacement \( u = [\omega (X'_u)^2]^{1/3} \sigma_+ \) and value \( \varepsilon = \theta (\omega / |X'_u|)^{1/3} \) was introduced. The main difference from the case of the ordinary string is the presence of the additional multiplier \((\theta^2 + 2\varepsilon)/\theta^2\) in [10] for \( \varepsilon^2 u \). One can see that in the case of sufficiently small angles \( \theta \ll \epsilon \) the behavior of \( \phi_+ \) is different from that for ordinary strings. Now the condition determining the properties of integral [15] (i.e. when this integral become exponentially small) is

\[
\sqrt{\theta^2 + 2\epsilon / \theta^2} \varepsilon \approx 1
\]

instead of \( \varepsilon \approx 1 \) for ordinary string cusp [3, 4]. This leads to the different expression for the cutoff frequency:

\[
\omega_{\text{cut}}(\theta, \epsilon) \propto 1/(\theta^2 + \epsilon)^{3/2},
\]

where \( \epsilon = 1 - \Delta \ll 1 \). Therefore, the cutoff frequency \( \omega_{\text{cut}} \) for small angles \( \theta \ll \epsilon \) does not depend on the \( \theta \) and proportional to \( \epsilon^{-3/2} \). For other values of \( \theta \) we can take that the radiation from chiral string cusp is equal to that of the ordinary string cusp, i.e. \( \kappa(\epsilon, \theta) \simeq \kappa(0, \theta) \). Thus we see that in the case of chiral string cusp the radiation is very small for frequencies \( \omega \gtrsim \omega_{\text{cut}} \), where

\[
\omega_{\text{cut}} = \frac{1}{Lg^{3/2}},
\]

even when the observer is placed in the direction of the cusp.

Let us now consider the problem of the detection of gravitational bursts from chiral string cusps. Damour and Vilenkin argued [3, 4] that the gravitational bursts from ordinary string cusps can be detected by the gravitational wave detectors. They find that it is possible for LIGO, VIRGO and especially for LISA to detect the wave bursts from ordinary strings. How does change the inferences about the detection of GWB in the case of the presence of the chiral current on the string? Let us suppose for simplicity that all strings in the universe carry the same small but non-zero current (that is \( \epsilon = \text{const} \ll 1 \)). The main difference of the radiation of chiral cusp from the radiation of ordinary cusp is the existence of cutoff frequency [15], such that signal from cusp with frequencies \( \omega \gtrsim \omega_{\text{cut}} \) is highly suppressed. In the simplified model of string network evolution adopted by Damour and Vilenkin [3, 4] the typical length of the closed strings is given:

\[
L \sim \Gamma G\mu t,
\]

where the dimensionless coefficient \( \Gamma \sim 100 \) determines the gravitational radiation from a string (\( \mathcal{E} = \Gamma G\mu^2 \)) and \( t \) is the cosmological time. It is assumed that signals from cusps reaches the Earth from different distances that corresponds to different times \( t \). It is more convenient to work with redshifts \( z \) rather than with times \( t \). The cosmological time \( t \) is connected with the redshift \( z \) in a different way in dependence on what epoch we consider: radiation or matter dominated. Let \( \omega_{\text{eq}} \) be the redshift of equal matter and radiation densities then we have:

\[
t = \begin{cases} 
t_0 (1 + z)^{-3/2}, & z \lesssim \omega_{\text{eq}}; \\
t_0 (1 + \omega_{\text{eq}})^{1/2} (1 + z)^{-2}, & z \gtrsim \omega_{\text{eq}},
\end{cases}
\]

where \( t_0 \approx 10^{18} \) s is the present age of the universe.

To find the cutoff frequency of incoming signal \( \omega_{\text{cut}} \) one should insert successively the time \( t \) into [16] into [19]. And finally we multiply the resulting expression by the additional factor \((1 + z)\) which takes into account the gravitational signal redshift. As a result we find:

\[
\omega_{\text{cut}} = \begin{cases} 
(1 + z)^{1/2} \left[ G\mu t_0 \frac{1}{3/2} \right]^{-1}, & z \lesssim \omega_{\text{eq}}; \\
(1 + z) \left[ G\mu t_0 (1 + \omega_{\text{eq}})^{1/2} \right]^{-1}, & z \gtrsim \omega_{\text{eq}}.
\end{cases}
\]

We see that, in any way, matter or radiation dominated era is considered, the typical length of loops decreases with \( z \) and consequently the cutoff frequency increases with redshift.

The analysis given by Damour and Vilenkin [3, 4] leads to the following result: for a given frequency \( \omega \) and for a given string parameter \( \mu \) (or, strictly speaking, for a given parameter \( \Gamma G\mu \)) the amplitude of incoming signals \( h \) from cusps in the universe decrease with \( z \), but the number of such signals per unit time \( N(\mu, \omega) \) increase with \( z \). That is we should find the compromise between the rate of signals and their amplitude. For a given \( N \) one can find the minimal amplitude of incoming signals \( h(\mu, N, \omega) \) and corresponding maximal redshift \( z_m(\mu, N, \omega) \) (in fact, because \( dN(\omega) / dz \) increase with \( z \) we can think for simplicity that all signals come from \( z_m \) and have the same amplitude corresponding to \( z_m \)).

How does the existence of the cutoff frequency [21] modify these arguments? The answer depends on the correlation of given frequency \( \omega \) and the cutoff frequency \( \omega_{\text{cut}} \) with \( z = z_m(\mu, N, \omega) \). If \( \omega_{\text{cut}} \lesssim \omega \) then the result does not change, i.e. the amplitude of incoming signal from chiral string cusp is the same as in the case of ordinary string. In the opposite case \( \omega_{\text{cut}} \gtrsim \omega \) the strings with redshift \( z_m(\mu, N, \omega) \) does not radiate the gravitational waves on frequency \( \omega \). However the dependence of \( \omega_{\text{cut}} \) on \( z \) allows us to shift to a higher \( z \) until the relation \( \omega_{\text{cut}} = \omega \) is satisfied. This corresponds to a some new (higher) redshift \( z_m \) and consequently to a new (smaller) amplitude of the incoming signal \( h(\mu, N, \omega_{\text{cut}}) \).

This means that current may effectively lower the amplitude of GWBs from cosmic string cusps in the whole universe.
Let us estimate the minimal current on the string that may affect on the amplitude $z$. There are three regimes of behavior of $z_m$ in dependence of function $y(\mu, \bar{N}, \omega)$:

$$y(\mu, \bar{N}, \omega) = 10^{-2}(\bar{N}/c)^{5/3} (\Gamma G\mu)^{8/3} \omega^{2/3},$$  \hspace{1cm} (22)

here $c$ is the average number of cusps per loop period (usually it is taken that $c \sim 0.1$). Maximal redshift $z_m(y)$ can be expressed as follows:

$$z_m(y) = \begin{cases} \frac{y^{1/3}}{10^{-2}(\bar{N}/c)/(\Gamma G\mu^2 \omega^{3})^{2/11}}, & y \lesssim 1; \\ \frac{y^{6/11}}{10^{-2}(\bar{N}/c)/(\Gamma G\mu^2 \omega^{3})^{3/11}}, & y \gtrsim y_{eq}, \end{cases}$$  \hspace{1cm} (23)

where $y_{eq} = \frac{11}{\sqrt{6}}$. Inserting (23) in (21) we can find from the equation $\omega_{cut} = \omega$ the critical value $\epsilon_{cut}$ when the current becomes to lower the incoming amplitude of signal from the cusp. We obtain:

$$\epsilon_{cut} = \left\{ \begin{array}{ll} \Gamma G\mu \omega^{2/3}, & y \lesssim 1; \\ 10^{-2}(\bar{N}/c)/(\Gamma G\mu^2 \omega^{3})^{2/11}, & y \gtrsim 1. \end{array} \right.$$  \hspace{1cm} (24)

Let us fix the rate of observable GWBs $\bar{N}/c \sim 1$, the frequency $\omega \sim 10^2$ (preferable for detecting GWB on LIGO/VIRGO [3, 4]) and find $\epsilon_{cut}$ as a function of $G\mu$. From (24) we obtain:

$$\epsilon_{cut} \sim \left\{ \begin{array}{ll} 10^{-14}(G\mu)^{-2/3}, & G\mu \lesssim 10^{-10}; \\ 10^{-10}(G\mu)^{-2/11}, & G\mu \gtrsim 10^{-10}. \end{array} \right.$$  \hspace{1cm} (25)

For example for the string with $G\mu \sim 10^{-12}$ we have $\epsilon_{cut} \sim 10^{-6}$; if we take $G\mu \sim 10^{-10}$ (corresponding to $z_m \sim 1$) then $\epsilon_{cut} \sim 10^{-8}$; for usually accepted $G\mu \sim 10^{-8}$ for GUT string we find $\epsilon_{cut} \sim 10^{-9}$. It may seems that such small values of $\epsilon_{cut}$ corresponds to small currents, but in fact the physical current is expressed through $\epsilon$ as follows (for $\epsilon \ll 1$):

$$j = \sqrt{2\mu \sqrt{\epsilon}},$$  \hspace{1cm} (26)

what corresponds to $j_{cut} \sim 10^{10}$ GeV for $G\mu \sim 10^{-10}$ and $j_{cut} \sim 10^{11}$ GeV for $G\mu \sim 10^{-6}$. Note, that current on the string, generated due to the oscillation of the string loop in the magnetic field in the universe, is in fact much smaller, $j_B \sim 10^7$ GeV [13].

### III. CUSP GRAVITATIONAL RADIATION

In this section we calculate the full gravitational radiation of the energy from the single cusp and obtain the dependence of radiation from the current along the string. The correct behavior of radiated gravitational energy as function of $\epsilon$ can be obtained from the following qualitative consideration. For the radiated gravitational energy we have

$$E(\epsilon) \propto \int_0^\theta \int_0^\infty \omega^2 d\omega \kappa^2(\epsilon),$$  \hspace{1cm} (27)

therefore the difference between the radiated energy from the ordinary and the superconducting string cusps is proportional to:

$$\delta E \propto \int_0^\theta \int_0^\infty \omega^2 d\omega \left[ \kappa^2(0) - \kappa^2(\epsilon) \right] \propto \sqrt{\epsilon}. $$  \hspace{1cm} (28)

(In the last estimation the result for cutoff frequency (18) was used). That is we may conclude that the gravitational radiation from the cusp of the superconducting chiral string depends on the current as follows:

$$E(\epsilon) = E(0) - G\mu^2 B \sqrt{\epsilon},$$  \hspace{1cm} (29)

where $B$ is a numerical constant depending on the string configuration ($G\mu^2$ was introduced in (28) for dimension reasons). In the following we make the detailed calculation of the radiation from the cusp that confirms of our estimation (29) and gives the exact value of $B$ for any given configuration of the string.

The gravitational radiation from any relativistic source is given [12]

$$\frac{dE}{d\Omega} = 2G \int_0^\infty \omega^2 d\omega P_{ij} P_{lm} \times$$

$$\times \left[ T_{ij}^r(\omega, k)T_{jm}(\omega, k) - \frac{1}{2} T_{ij}^r(\omega, k)T_{lm}(\omega, k) \right],$$

where $P_{ij} = \delta_{ij} - n_in_j$ is the projection operator on the plane orthogonal to $n$ and $T_{jm}(\omega, k)$ is given by (30). The expression (30) can be greatly simplified if we rewrite it in 'co-rotating' basis $(\mathbf{n}, \mathbf{v}, \mathbf{w})$ where $\mathbf{v}$ and $\mathbf{w}$ are the unit vectors, perpendicular each other and to the vector $\mathbf{n}$ [17]. In a new basis the radiated energy is

$$\frac{dE}{d\Omega} = 2G \int_0^\infty \omega^2 d\omega \times$$

$$\times \left[ \tau_{pq}(\omega, k)\tau_{pq}(\omega, k) - \frac{1}{2} \tau_{pq}(\omega, k)\tau_{pq}(\omega, k) \right],$$

where $\tau_{pq}(\omega, k)$ are the Fourier-transforms of an energy-momentum tensor in the co-rotating basis $(\mathbf{n}, \mathbf{v}, \mathbf{w})$. Only indexes $p, q$ with values 2 and 3 appear in the equation (31). The Fourier-transforms $\tau_{pq}(\omega, k)$ can be expressed in the following way

$$\tau_{pq}(\omega, k) = -\frac{\mu}{4\pi} f(p, \hat{\mathbf{Y}} \cdot \mathbf{q}),$$

where the 'modified' functions $\hat{I}_p$ and $\hat{Y}_q$ are determined as follows:

$$\hat{I}_2 \equiv \hat{I}(\mathbf{n}, \mathbf{v}), \quad \hat{I}_3 \equiv \hat{I}(\mathbf{n}, \mathbf{w}),$$

$$\hat{Y}_2 \equiv \hat{Y}(\mathbf{n}, \mathbf{v}), \quad \hat{Y}_3 \equiv \hat{Y}(\mathbf{n}, \mathbf{w}),$$

with

$$\hat{I}(\mathbf{n}, \mathbf{z}) = \int_{-\infty}^\infty d\sigma_-(X'_-z)e^{-\frac{\mu}{2}X'_-},$$

$$\hat{Y}(\mathbf{n}, \mathbf{z}) = \int_{-\infty}^\infty d\sigma_+(X'_+z)e^{-\frac{\mu}{2}X'_+}.$$
We extended the integration on $\sigma_\pm$ in (3) from $-\infty$ to $\infty$ as the radiation mostly come from the vicinity of the cusp. Inserting (32) in (31) we find:

$$\frac{dE}{d\Omega} = \frac{G \mu^2}{8 \pi^2} \int_0^\infty \omega^2 d\omega K,$$

where

$$K = |\hat{I}_2 \hat{Y}_2|^2 + |\hat{I}_3 \hat{Y}_3|^2 + \frac{1}{2} |\hat{I}_2 \hat{Y}_3 + \hat{I}_3 \hat{Y}_2|^2 - \frac{1}{2} |\hat{I}_2 \hat{Y}_2 + \hat{I}_3 \hat{Y}_3|^2. \quad (36)$$

For the definiteness let us choose the following co-rotating basis:

$$\mathbf{n} = (\cos \theta, -\sin \phi \sin \theta, \cos \phi \sin \theta),$$

$$\mathbf{v} = (-\sin \theta, -\sin \phi \cos \theta, \cos \phi \cos \theta),$$

$$\mathbf{w} = (0, \cos \phi, \sin \phi), \quad (37)$$

and $\mathbf{n}^{(c)} = (1, 0, 0)$. The difference between vectors $\mathbf{n}$ and $\mathbf{n}^{(c)}$ is $\delta$ so that

$$\mathbf{n} = \mathbf{n}^{(c)} + \delta \approx \mathbf{n}^{(c)} + \theta \vec{\psi}, \quad (38)$$

where $\vec{\psi} = (-\theta/2, -\sin \phi, \cos \phi)$. In this basis we have $\mathbf{w} \mathbf{n}^{(c)} = 0$ therefore from (34) we find that $\text{Re} \hat{I}_3 = \text{Re} \hat{Y}_3 = 0$. Thus the expression for $K$ (30) can be rewritten in a following way:

$$K = \frac{1}{2} \left( |\hat{I}_2|^2 + |\hat{I}_3|^2 \right) \left( |\hat{Y}_2|^2 + |\hat{Y}_3|^2 \right) + 2 \text{Re} \hat{I}_2 \text{Im} \hat{I}_3 \text{Re} \hat{Y}_2 \text{Im} \hat{Y}_3. \quad (39)$$

Let us denote further $\mathbf{X}_\pm'' = \hat{\rho}_\pm / L$, where $L$ is the invariant length of the string. Using the relations (13), (14) and (38) after some algebra one can find the expressions for phases $\varphi_\pm = (\omega/2)n \cdot \mathbf{X}_\pm$ (where the notation $n^{(c)} = (1, \mathbf{n}^{(c)})$ was introduced) entering the formulas for functions $\hat{I}_p$ and $\hat{Y}_q$ (31):

$$\varphi_- = \frac{\omega}{4} \left[ \theta^2 \sigma_- + \frac{\theta}{L} (\vec{\psi} \hat{\rho}_-) \sigma_+ + \frac{1}{3} \frac{(\hat{\rho}_-)^2}{L^2} \sigma_+^3 \right], \quad (40)$$

$$\varphi_+ = -\frac{\omega}{4} \left[ (\theta^2 + 2\epsilon) \sigma_+ - \frac{\theta}{L} (\vec{\psi} \hat{\rho}_+) \sigma_+^2 + \frac{1}{3 \Delta} \frac{(\hat{\rho}_-)^2}{L^2} \sigma_+^3 \right].$$

Then we can evaluate the integrals in (31) using the formulas (in the analogy with analysis carried out in (14)):

$$\int_0^\infty dx \cos \left[ \frac{3}{2} \xi \left( x + \frac{x^3}{3} \right) \right] = \frac{1}{\sqrt{3}} K_{1/3}(\xi),$$

$$\int_0^\infty dx \sin \left[ \frac{3}{2} \xi \left( x + \frac{x^3}{3} \right) \right] = \frac{1}{\sqrt{3}} K_{2/3}(\xi), \quad (41)$$

where $K_{1/3}(\xi)$ and $K_{2/3}(\xi)$ are the modified Bessel functions of order 1/3 and 2/3 correspondingly. Denoting

$$\xi_- = \frac{1}{6} \omega \frac{L}{\hat{\rho}_-} \theta^3 \mathcal{Y}_3, \quad \chi_- = \frac{L}{\hat{\rho}_-} \theta \mathcal{Y}_-, \quad \xi_+ = \frac{1}{6} \omega \frac{L}{\hat{\rho}_+} (\theta^2 \mathcal{Y}_+^2 + 2\epsilon)^{3/2}, \quad \chi_+ = \frac{L}{\hat{\rho}_+} (\theta^2 \mathcal{Y}_+^2 + 2\epsilon)^{1/2},$$

we finally find:

$$\mathcal{Y}_\pm = \sqrt{\frac{\hat{\rho}_-^2 - (\hat{\rho}_- \vec{\psi})^2}{\hat{\rho}_+}},$$

after some calculations one can obtain from (34):

$$\hat{I}_2 = \frac{2L}{\sqrt{3} \hat{\rho}_-} \theta^2 \mathcal{Y} \mathcal{K}_{1/3}(\xi_-)$$

$$- \frac{2L}{\sqrt{3} \hat{\rho}_-} \theta^2 \mathcal{Y}_- K_{2/3}(\xi_-),$$

$$\hat{I}_3 = -i \frac{2L}{\sqrt{3} \rho_-} \theta^2 \mathcal{Y}_2 K_{2/3}(\xi_-),$$

$$\hat{Y}_2 = \frac{2L}{\sqrt{3} \hat{\rho}_+} \theta \sqrt{\theta^2 \mathcal{Y}_+^2 + \epsilon K_{1/3}(\xi_+)} \quad (42)$$

$$- i \frac{2L}{\sqrt{3} \hat{\rho}_+} \theta \sqrt{\theta^2 \mathcal{Y}_+^2 + \epsilon} K_{2/3}(\xi_+),$$

$$\hat{Y}_3 = -i \frac{2L}{\sqrt{3} \hat{\rho}_+} \theta \sqrt{\theta^2 \mathcal{Y}_+^2 + \epsilon} K_{2/3}(\xi_+).$$

The expressions for $\hat{I}_p$ and $\hat{Y}_q$ in the case of the ordinary string can be found from (12) by setting $\epsilon = 0$. All we have to do is to insert the result (12) in (39) and calculate the radiated power using (36). Expressions (12) was found under the assumption that $\theta \ll 1$. In the case of the electromagnetic radiation Blanco-Pillado and Olum (14) extended the obtained results for small $\theta$ to any values of $\theta$, because the electromagnetic radiation is highly beam and integral for radiated power over $\theta$ converges vary fast. This procedure can not be applied for the gravitational radiation because the radiated gravitational power (24) does not converges for large $\theta$. Nevertheless we can find the difference between the radiated powers of ordinary and chiral cosmic string cusps in the case $\epsilon \ll 1$ using the fact that the difference of powers arises for the $\theta \lesssim \epsilon$ where our calculations are valid. In other words in our analysis we are able to calculate the value $B$ in formula (25) (which will be verified below) but we can not find the value $E(0)$, i.e. the radiated power for the ordinary string cusp.

Without the loss of the generality we can choose $\hat{\rho}_- = \rho_- (0, 0, 1)$ and $\hat{\rho}_+ = \rho_+ (0, -\sin \phi_0, \cos \phi_0)$ where $\phi_0$ is the angle between $\hat{\rho}_-$ and $\hat{\rho}_+$. Then we have $\mathcal{Y}_- = |\sin \phi|$ and $\mathcal{Y}_+ = |\sin (\phi - \phi_0)|$. Substituting (17) into (16), then obtained expression into (33) and making the successive change of variables:

$$\omega = \frac{6}{\mathcal{Y}_3^3 \mathcal{Y}_-^3 L} \rho_-^3, \quad \theta = \theta' \sqrt{\epsilon}, \quad (43)$$

Fig. 1: The coefficient $B$ in (47) as a function of an angle $\phi_0$ which parameterizes the geometry of the string loop (49).

\[ E(\epsilon) = E(0) - \sqrt{\frac{\epsilon}{\pi^2}} \frac{\rho_+^{1/3}}{\rho_-^{4/3}} G \mu^2 L \int_0^{2\pi} \frac{d\phi}{\gamma^2} \int_0^\infty d\theta' \left[ G(a) - G(a_0) \right], \]

where

\[ G(a) = \int_0^\infty dz \, z^2 a^{2/3} \left\{ \frac{1}{2} \left[ K_{1/3}^2(z) + Y_a^2 K_{2/3}^2(z) \right] \left[ K_{1/3}^2(az) + Y_a^2 a^{2/3} \left( \frac{\rho_+}{\rho_-} \right)^{2/3} Y_a^2 K_{2/3}^2(az) \right] \right. \]
\[ + 2 Y_a^2 (\bar{\rho}_- \bar{\psi}) (\bar{\rho}_+ \bar{\psi}) a^{1/3} \left( \frac{\rho_+}{\rho_-} \right)^{1/3} K_{1/3}^2(az) K_{2/3}^2(az) K_{1/3}^2(az) K_{2/3}^2(az) \}, \]

with

\[ a = \rho_- \left[ \sin^2(\phi - \phi_0) + 2 \sin^2(\phi) \right]^{3/2}, \]
\[ a_0 = \rho_- \left[ \sin(\phi - \phi_0) \right]^3. \]

We see that obtained result confirm our estimation (29) with:

\[ B = \frac{48 \rho_+^{1/3}}{\pi^2 \rho_-^{4/3}} L \int_0^{2\pi} \frac{d\phi}{\gamma^2} \int_0^\infty d\theta' \left[ G(a) - G(a_0) \right]. \]

The dependence of radiated energy (29) on $\epsilon$ can be expressed in terms of the physical current $j$, using (26):

\[ E(j) = E(0) - \frac{G \mu^2 B}{\sqrt{2}\mu j} \]

On the Fig. 1 the calculated values of $B/L$ as a function of $\phi_0$ is shown for the following particular loop:

\[ a = \frac{L}{2\pi} \left( -\sin \frac{2\pi \sigma_-}{L}, 0, -\cos \frac{2\pi \sigma_-}{L} \right). \]

We see that obtained result confirm our estimation (29) with:

\[ b = \frac{L}{2\pi} \times \left[ \sin \frac{2\pi \sigma_+}{L}, \sin \phi_0 \cos \frac{2\pi \sigma_+}{L}, -\cos \phi_0 \cos \frac{2\pi \sigma_+}{L} \right]. \]

Note that the function $B$ is not reflection symmetric with respect to the point $\phi_0 = \pi/2$. This is because of a presence of the term proportional to $\cos \phi \cos(\phi - \phi_0)$ in (45).

IV. CONCLUSION

The gravitational radiation from the chiral string cusp is considered analytically in the limit of the small current, $\epsilon = 1 - \Delta \ll 1$, where $\Delta$ is from (2). We showed that if the back-reaction of charge carriers is taken into account then the chiral string cusp (and consequently the whole string) does not radiate the gravitational energy at frequencies $\omega > \omega_{\text{cut}}$. The cutoff frequency $\omega_{\text{cut}}$ depends on the current and is defined by (18). If the current on the string is sufficiently large then this cutoff may lead to an effective lowering of the amplitude of GWBs. This would
affect the possible detection of GWBs coming from the whole universe by the interferometric gravitational detectors LIGO, VIGRO and especially LISA. The critical value of the chiral current (when the smoothing of the cusp becomes to influence the detectability) depends according to (24) on the observation frequency $\omega$, assumed rate of GWBs $\dot{N}$ and the mass parameter of the string $\mu$. For example, for GUT string with $G\mu \sim 10^{-6}$ the critical current $j = \sqrt{2 \mu / \epsilon_{\text{cut}}} \sim 10^{11}$ GeV, which corresponds to $\epsilon_{\text{cut}} \sim 10^{-9}$.

The analytical dependence of the total gravitational energy radiation from the cusp as a function of the small current was calculated. Namely it was found that the radiation decreases with the current on the string as follows: $E(\epsilon) = E(0) - G\mu^2B\sqrt{\epsilon}$. For the one-parametric family of string configurations we found the numerical values of $B \sim 50 \div 400$.

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[1] E.P.S. Shellard and A. Vilenkin, *Cosmic Strings and other Topological Defects* (Cambridge University Press, Cambridge, England, 1994).
[2] M.B. Hindmarsh and T.W.B. Kibble, Rep. Prog. Phys. **58**, 477 (1995).
[3] T. Damour and A. Vilenkin, Phys. Rev. Lett. **85**, 3761 (2000).
[4] T. Damour and A. Vilenkin, Phys. Rev. D **64**, 064008 (2001).
[5] E. Witten, Nucl. Phys. **B249**, 557 (1985).
[6] R.L. Davis and E.P.S. Shellard, Phys. Lett. B **209**, 485 (1988).
[7] D. Haws, M. Hindmarsh and N. Turok, Phys. Lett. B **209**, 255 (1988).
[8] E. Copeland, D. Haws, M. Hindmarsh and N. Turok, Nucl. Phys. **B306**, 908 (1988).
[9] B. Carter and P. Peter, Phys. Lett. B **466**, 41 (1999).
[10] A.C. Davis, T.W.B. Kibble, M. Pickles and D.A. Steer, Phys. Rev. D **62**, 083516 (2000).
[11] J.J. Blanco-Pillado, Ken D. Olum and A. Vilenkin, Phys. Rev. D **63**, 103513 (2001).
[12] E. Babichev and V. Dokuchaev, Nucl. Phys. **B645**, 134 (2002).
[13] E. Babichev and V. Dokuchaev, Phys. Rev. D **66**, 025007 (2002).
[14] J.J. Blanco-Pillado and K.D. Olum, Nucl. Phys. **B599**, 435 (2001).
[15] V. Berezinsky, B. Hnatyk and A. Vilenkin, Phys. Rev. D **64**, 043004 (2001).
[16] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1974).
[17] R. Durrer, Nucl. Phys. **B328**, 238 (1989).