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Intersection numbers from higher-order partial differential equations

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ABSTRACT: We propose a new method for the evaluation of intersection numbers for twisted meromorphic $n$-forms, through Stokes’ theorem in $n$ dimensions. It is based on the solution of an $n$-th order partial differential equation and on the evaluation of multivariate residues. We also present an algebraic expression for the contribution from each multivariate residue. We illustrate our approach with a number of simple examples from mathematics and physics.

KEYWORDS: Differential and Algebraic Geometry, Scattering Amplitudes

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1 Introduction

The intersection number of differential $n$-forms [1–13] acts like an inner product in the de Rham twisted cohomology group, namely the quotient space of the closed forms modulo the exact forms, and therefore, it yields the decomposition of differential forms in a basis of forms, generating the vector space [14–17].

A wide class of integral functions, such as Aomoto-Gel’fand integrals, Euler-Mellin integrals, Gel’fand-Kapranov-Zelevinsky integrals, to name a few, which embed and generalise Feynman integrals, can be considered as the pairing of regulated integration domains, known as twisted cycles, and of regulated forms, known as twisted cocycles [18]. Within this definition, the integrand appears as the product of a multivalued function, called twist, and of a differential form. The twist carries information on the integral regularisation: for the case of dimensionally regulated Feynman integrals, the space-time dimensionality appears in the exponent of the twist. In this fashion, the algebraic properties of the integrals can be thought as coming more fundamentally from the algebraic properties of the corresponding cycles and cocycles. In particular, evaluation of intersection numbers for twisted differential forms becomes a crucial operation to derive linear and quadratic relations for integrals mentioned above, and to systematically derive differential and difference equations the latter obey [14–17, 19–24]. See [25–31], for recent reviews, and [24, 32] for applications to multi-loop calculus. Applications of intersection theory and cohomology to diagrammatic coaction have been presented in [33–36], and to other interesting physical contexts in [37–40].

The evaluation of the intersection numbers for twisted forms is based on the twisted version of Stokes’ theorem [41]. In particular, for the case of logarithmic $n$-forms, intersection numbers can be computed by applying the algorithm proposed in [2] or by means of the global residue theorem [8]. For generic meromorphic $n$-forms, the evaluation procedure can become computationally more demanding, and it can be performed by means of an iterative approach, as proposed in [16, 17, 42], elaborating on [10]. The iterative approach has been further refined in [43], by exploiting the invariance of the intersection numbers for forms belonging to the same cohomology classes. In [22, 23], this algorithm has been extended to account also for the relative cohomology cases [44], required when dealing with singularities of the integrand which are not regulated by the twist.

As an alternative to the evaluation procedure based on the Stokes’ theorem, intersection numbers can also be computed by solving the secondary equation built from the Pfaffian systems [9, 45, 46]. Within this algorithm, the determination of Pfaffian systems obeyed by the generators of the cohomology group is required, and efficient methods for their derivation have recently started to be proposed by means of Macaulay matrix in [46].

In this article, we propose a new algorithm for the computation of the intersection number of twisted $n$-forms, based on a novel way of applying Stokes’ theorem that requires the solution of a higher-order partial differential equation and application of the multivariate residues. The computational algorithm hereby proposed can be considered as a natural extension of the univariate case [41], and, just as the latter, its application requires the solution of a (partial) differential equation around each intersection point, that belongs to
the set of zeroes of the twist. In this work, we show that the solution of the differential equation can be found analytically by multiple Laurent series expansions, and that each residue admits a closed expression in terms of the Laurent coefficients of the two forms entering the pairing and of the twist.

The structure of the paper is as follows: in section 2 we discuss aspects of twisted cohomology theory and intersection numbers; we introduce a new method for computation of multivariate intersection numbers as multivariate residues using a higher-order partial differential equation, and discuss its solution locally around each intersection point. Section 3 contains application of our new approach to integrals and functions of interests for physics and mathematics. In section 4 we give a closed, algebraic expression for each residue, contributing to the multivariate intersection number. Section 5 contains our concluding remarks. The paper includes four appendices: appendix A contains the link of our new approach to Matsumoto’s algorithm, explicitly shown in the simple case of 2-forms; appendix B contains further details of the examples discussed in section 3; appendix C contains the derivation of the algebraic expression given in section 4.

2 Intersection numbers for twisted n-forms

2.1 Twisted cohomology

Let $B_i$, with $i = 1, \ldots , m$, be complex homogeneous polynomials in the homogeneous coordinates $Z = (Z_1, \ldots , Z_{n+1})$ of the complex projective space $\mathbb{C}P^n$. We introduce a manifold $X = \mathbb{C}P^n - \bigcup_{i=1}^m S_i$, where the hypersurfaces $S_i$ are identified by the equations:

$S_i := \{Z \mid B_i(Z) = 0\}$.  

(2.1)

In the following we work in the chart $Z_1 \neq 0$ with the local coordinates (see appendix B.1 for details)

$z = (z_1, \ldots , z_n) := (Z_2/Z_1, \ldots , Z_{n+1}/Z_1)$.  

(2.2)

We introduce the Aomoto-Gel’fand integrals, defined as twisted period integrals,

$\int_{\Gamma^{(n)}} u \varphi^{(n)} \equiv \int_{\Gamma^{(n)}} u \varphi^{(n)} d^nz$,  

(2.3)

where: $u$ is a multivalued function called the twist, which regulates the integral; $\Gamma^{(n)}$ is a regularised cycle called twisted or loaded cycle, i.e. an $n$-chain with empty boundary on $X$ (usually $\Gamma^{(n)}$ is denoted as $\Gamma^{(n)} \equiv \Gamma^{(n)} \otimes u$ to separate the integration domain $\Gamma^{(n)}$ and a specific choice of the branch of multivalued $u$ along it); $\varphi^{(n)}$ is a meromorphic differential $n$-form defined on $X$, called the twisted cocycle. In general $u$ is a multivalued function that “vanishes” on the integration boundary: $u(\partial \Gamma^{(n)}) = 0$. The latter property ensures that for any generic $(n - 1)$-form $\varphi^{(n-1)}$ the integral of the total differential is zero:

$0 = \int_{\Gamma^{(n)}} d(u \varphi^{(n-1)}) = \int_{\Gamma^{(n)}} u \nabla_\omega \varphi^{(n-1)}$,  

(2.4)
where we introduced the covariant derivative:

\[ \nabla_\omega := d + \omega \wedge \equiv u^{-1} \cdot d \cdot u, \quad \omega \equiv \sum_{i=1}^{n} \omega_i \, dz_i = d \log(u), \]  

(2.5)

with \( d = \sum_{i=1}^{n} dz_i \), where \( dz_i = \partial_{z_i} \) and \( \omega_i = u^{-1} \partial_{z_i} u \), using the shorthand notation \( \partial_{z_i} \equiv \partial/\partial z_i \). When dealing with individual integration variables, it might be convenient to consider the decomposition of the full covariant derivative:

\[ \nabla_\omega = \sum_{i=1}^{n} \nabla_\omega_i, \]  

(2.6)

with the partial covariant derivatives defined as:

\[ \nabla_\omega_i := \hat{\nabla}_\omega_i \, dz_i, \quad \text{and} \quad \hat{\nabla}_\omega_i = \partial_{z_i} + \hat{\omega}_i. \]  

(2.7)

Aomoto-Gel’fand integrals represent a wide class of special functions, such as Gauß hypergeometric functions, Lauricella functions, and their generalizations, Euler-type integrals, and Feynman integrals [14]. Integrals of this class are invariant under shifting of the differential \( n \)-form by a covariant derivative:

\[ \int_{\Gamma(n)} u \varphi = \int_{\Gamma(n)} u (\varphi + \nabla_\omega \varphi^{(n-1)}), \]  

(2.8)

Similar results are obtained also for the so called dual integrals, obtained from the integrals (2.3) by replacing \( u \to u^{-1} \) and \( \omega \to -\omega \) in definition (2.5).

In the case of Feynman integrals in the Baikov representation [47], the twist \( u \) admits the following factorized form:

\[ u = \prod_{i=1}^{m} B_i^{\gamma_i}, \]  

(2.9)

where the exponents \( \gamma_i \) are non-integer, and the factors \( B_i \) are polynomials build out of the kinematical and Baikov variables (corresponding to propagators). For this set of functions, analyticity, unitarity, and algebraic structure are related to the geometry captured by the Morse function \( h := \Re(\log(u)) \), see [48, 49].

The multivalued twist \( u \) carries information on the regularization: for dimensionally regulated Feynman integrals, it depends on the integration variables as well as on the external scales, such as the Mandelstam invariants and masses (all appearing in the polynomials \( B_i \)), and on the space-time dimensionality \( d \) (appearing in the \( \gamma_i \)). The topological information of integrals and dual integrals is contained in \( \omega \) that is a differential form with zeroes and poles,\(^1\) collected in the respective sets:

\[ Z_\omega = \{ \text{zeros of } \omega \}, \quad \text{and} \quad \mathbb{P}_\omega = \{ \text{poles of } \omega \}. \]  

(2.10)

\(^1\)We view \( \omega \) as a meromorphic form on the complex projective space \( \mathbb{C}P^n \), so it might also have singularities at “infinity”, see section 3 for some explicit examples and appendix B.1 for details on projective geometry.
The invariance of integrals and dual integrals under the transformation (2.8) can be used to expose the algebraic structure of Aomoto-Gel’fand integrals. Let us introduce two vector spaces of twisted cocycles: the \( H^\omega_n \) and dual twisted cohomology group: \( (H^\omega_n)^\vee := H^{n-\omega}_n \). These spaces are isomorphic, and their dimension:

\[
\nu = \dim(H^{\pm\omega}_n),
\]

can be determined by counting the number of critical points of \( B_i \), namely \( \nu = \dim(\mathbb{Z}_n) \) [49], or equivalently from the Euler characteristics \( \chi(P_{\omega}) \) of the projective variety generated by the poles of \( \omega \), as \( \nu = (-1)^n(n+1-\chi(P_{\omega})) \) [16], see also [50, 51], or by the Shape lemma [17].

We denote the elements of the twisted cohomology (2.11) as \( \langle \varphi_L \rangle \in H^\omega_n, \langle \varphi_R \rangle \in H^{-\omega}_n \), and use them to define the following natural twisted Poincaré pairings:

- **Integrals:**
  \[
  I = \int_{\Gamma(n)} u \varphi_L.
  \] (2.13)
- **Dual integrals:**
  \[
  I^\vee = \int_{\Gamma(n)} u^{-1} \varphi_R.
  \] (2.14)
- **Intersection numbers for twisted cocycles:**
  \[
  \langle \varphi_L \rangle \langle \varphi_R \rangle \equiv (2\pi i)^{-n} \int_X (u \varphi_{L,c}) \wedge (u^{-1} \varphi_R),
  \] (2.15)
  where \( \varphi_{L,c} \) is a compactly supported cocycle equivalent to \( \varphi_L \) [2] (see also section 2.3 below and appendix A).

### 2.2 Integrals and relations

Let us now briefly review some practical applications of the twisted cohomology theory to the study of Feynman integrals, focusing on the IBP decomposition method.

Consider the bases generating the cohomology groups introduced in eq. (2.11):

\[
\{\langle e_i \rangle\}_{i=1,\ldots,\nu} \in H^n_\omega \quad \text{and} \quad \{|h_i\rangle\}_{i=1,\ldots,\nu} \in H^{n-\omega}_\omega.
\]

These two bases can be used to express the identity operator in the cohomology space [14, 15] as follows:

\[
\mathbb{I}_c = \sum_{i,j=1}^\nu |h_i\rangle \left(C^{-1}\right)_{ij} \langle e_j |,
\] (2.16)

where we defined the metric matrix:

\[
C_{ij} := \langle e_i \mid h_j \rangle,
\] (2.17)

whose elements are intersection numbers of the twisted basis forms.
Linear relations for Aomoto-Gel’fand-Feynman integrals, the differential equations, and the finite difference equation they obey are consequences of the purely algebraic application of the identity operator (2.16), see also [14].

In the context of Feynman integral calculus, the decomposition of scattering amplitudes in terms of master integrals (MIs), as well as the equations obeyed by the latter, are derived by means of IBPs [52, 53] and of the Laporta method [54]. These relations emerge naturally from the algebraic properties of twisted cocycles.

Indeed, generic twisted cocycles can be projected onto the bases in the corresponding vector spaces as:

\[
\langle \phi_L | I_c \rangle = \sum_{i=1}^{\nu} c_i \langle e_i | \rangle, \quad \text{with} \quad c_i = \sum_{j=1}^{\nu} \langle \varphi_L | h_j \rangle \left( C^{-1} \right)_{ji}.
\] (2.18)

The latter formula is dubbed the master decomposition formula for twisted cocycles [14, 15]. It implies that the decomposition of any Aomoto-Gel’fand-Feynman integral as a linear combination of MIs is an algebraic operation (similarly to the decomposition/projection of any vector within a vector space), which can be carried out by computing intersection numbers of the twisted de Rham differential forms. Using the master decomposition formula, a Feynman integral \( I \) can be decomposed in terms the MIs \( J_i \) as:

\[
I = \sum_{i=1}^{\nu} c_i J_i.
\] (2.19)

with the decomposition coefficients \( c_i \) given by eq. (2.18).

Let us remark, that the metric matrix (2.17), in general, differs from the identity matrix. The Gram-Schmidt algorithm can be employed to build orthonormal bases from generic sets of independent elements, using the intersection numbers as scalar products. But more generally the coefficients appearing in the formulas (2.18), (2.19) are independent of the respective dual elements. Therefore, exploiting this freedom in choosing the corresponding dual bases may yield striking simplifications [15, 22, 23]. The decomposition formulas hold also in the case of the relative twisted de Rham cohomology, which allows for relaxation of the non-integer condition for the exponents \( \gamma_i \) that appear in eq. (2.9), see [22, 23, 44].

2.3 Partial differential equation

By elaborating on the method proposed in [2], we hereby propose to evaluate the intersection number for \( n \)-forms, using the multivariate Stokes’ theorem, yielding (see also appendix A):

\[
\langle \varphi_L^{(n)} | \varphi_R^{(n)} \rangle = (2\pi i)^{-n} \int_X (u \varphi_L^{(n)}) \wedge (u^{-1} \varphi_R^{(n)}) = \sum_{p \in P_\omega} \text{Res}_{z=p} (\psi \varphi_R^{(n)}),
\] (2.20)

where:

- \( \psi \) is a function (0-form), that obeys the following \( n \)-th order partial differential equation (nPDE):

\[
\frac{\partial^n}{\partial z_1 \partial z_2 \ldots \partial z_n} (u \psi) = u \varphi_L^{(n)}.
\] (2.21)
• \( p = (p_1, p_2, \ldots, p_n) \in \mathbb{P}_\omega \) is a pole of \( \omega \), i.e. an intersection point of singular hypersurface \( S_i \) defined in eq. (2.1), at finite location or at infinity.

• The residue symbol stands for:

\[
\text{Res}_{z=p}(f) = \text{Res}_{z=p_n} \cdots \text{Res}_{z=p_1} (f) = (2\pi i)^{-n} \int_{\mathcal{C}_1 \wedge \cdots \wedge \mathcal{C}_n} f \, dz_1 \wedge \cdots \wedge dz_n, \quad (2.22)
\]

where the integral goes over a product of small circles \( \mathcal{C}_i \), each encircling the corresponding pole \( z_i = p_i \) in the \( z_i \)-plane, see [55].

Representation (2.20) can be derived by rewriting the intersection number integral as a flux of a certain local form \( \eta \):

\[
\int_X (u \varphi^{(n)}_{L,c}) \wedge (u^{-1} \varphi^{(n)}_R) = \sum_{p \in \mathbb{P}_\omega} \int_{D_p} d z_1 \cdots d z_n \eta, \quad (2.23)
\]

Working term-by-term in the sum on the r.h.s., let us temporarily denote by \( (z_1, \ldots, z_n) \) the local coordinates centered at the intersection point \( p \). As the integration domain we take the polydisc \( \mathcal{D}_p = \{(z_1, \ldots, z_n) \mid |z_1|, \ldots, |z_n| \leq \epsilon \} \) and define:

\[
\eta := \bar{h}_1 \cdots \bar{h}_n \left( u \psi \right) \left( u^{-1} \varphi^{(n)}_{R} \right), \quad (2.24)
\]

where \( \bar{h}_i := 1 - h_i \) and \( h_i \) is the Heaviside step-function:

\[
h_i \equiv h(z_i) := \begin{cases} 
1 & \text{for } |z_i| < \epsilon, \\
0 & \text{otherwise,}
\end{cases} \quad (2.25)
\]

so that the differential \( dh_i \) is localized on the circle \( |z_i| = \epsilon \). The action of the partial derivatives in eq. (2.23) gives:

\[
d z_1 \cdots d z_n \eta = \left( \bar{h}_1 \cdots \bar{h}_n \left( u \nabla_{\omega_1} \cdots \nabla_{\omega_n} \psi \right) + \cdots + (-1)^n \left( u \psi \right) \right) \left( -1 \varphi^{(n)}_{R} \right) \wedge \left( u^{-1} \varphi^{(n)}_{R} \right). \quad (2.26)
\]

By requiring that the auxiliary 0-form \( \psi \) is the solution of the following \( n \)PDE:

\[
u \nabla_{\omega_1} \cdots \nabla_{\omega_n} \psi = u \varphi_{L}, \quad (2.27)
\]

we obtain:

\[
d z_1 \cdots d z_n \eta = \left( u \varphi_{L,c} \right) \wedge \left( u^{-1} \varphi_{R} \right), \quad (2.28)
\]

where the compactly supported \( n \)-form \( \varphi_{L,c} \) is defined as:

\[
\varphi_{L,c} := \bar{h}_1 \cdots \bar{h}_n \varphi_L + \cdots + (-1)^n \psi \bar{h}_1 \wedge \cdots \wedge \bar{h}_n \equiv \nabla_{\omega_1} \cdots \nabla_{\omega_n} \left( \bar{h}_1 \cdots \bar{h}_n \psi \right). \quad (2.29)
\]

The middle expression here is equivalent to the \( \varphi_{L,c} \) introduced by Matsumoto in [2] and, therefore, the integration of eq. (2.23) can be carried out via iterated residues. Indeed,
since $\varphi_R$ is a holomorphic $n$-form, in eq. (2.26) only the last term gives a non vanishing contribution:

$$
\int_X \left( u \varphi_L^{(n)} \right) \wedge \left( u^{-1} \varphi_R^{(n)} \right) = (-1)^n \sum_{p \in \mathcal{P}_\omega} \int_{D_p} (u \psi) \, d\bar{h}_1 \wedge \ldots \wedge d\bar{h}_n \wedge \left( u^{-1} \varphi_R^{(n)} \right)
$$

$$
= \sum_{p \in \mathcal{P}_\omega} \int_{\mathcal{C}_1 \cap \ldots \cap \mathcal{C}_n} \psi \varphi_R^{(n)}
$$

$$
= (2\pi i)^n \sum_{p \in \mathcal{P}_\omega} \text{Res}_{z=p}(\psi \varphi_R^{(n)}), \quad (2.30)
$$

where the product of small circles $\mathcal{C}_1 \cap \ldots \cap \mathcal{C}_n$ (i.e. an $n$-dimensional torus) is the distinguished boundary of the polydisc $D_p$. The last equation above\(^2\) reproduces the result shown in eq. (2.20). For more details we refer the interested reader to the discussion in appendix A.

Finally, let us once again highlight the crucial eq. (2.27) and write it as:

$$
\nabla_{\omega_1} \nabla_{\omega_2} \ldots \nabla_{\omega_n} \psi = \varphi_L^{(n)}. \quad (2.31)
$$

This $nPDE$, equivalent to eq. (2.21), is the natural extension of the equation $\nabla_{\omega_1} \psi = \varphi_L^{(1)}$ presented in [2] for the single variable case. Equation (2.31) constitutes the first main result of this communication, as it offers a new algorithm for the direct determination of the scalar function $\psi$, hence a simpler strategy for the evaluation of the intersection numbers between twisted $n$-forms.

### 2.4 Solution

The solution of eqs. (2.21), (2.31) can be formally written as:\(^3\)

$$
\psi = u^{-1} \int_{z_0}^z u \varphi_L^{(n)}. \quad (2.32)
$$

A crucial observation is in order. The *global* solution $\psi$ is, in general, a transcendental function. The evaluation of the intersection number in (2.20), though, because of the calculation of residues, requires the knowledge of $\psi$ only *locally* around each of the contributing pole, say $\psi_p = \psi|_{z \to p}$. To determine the *local* expression of $\psi$ around the point $p$, we propose two equivalent strategies:

1. **Solution by ansatz:**\(^4\)

$$
\psi_p \equiv \psi|_{z \to p} = \sum_{a=\alpha_{\text{min}}}^{\alpha_{\text{max}}} \psi_{p,a} (z-p)^a, \quad (2.33)
$$

\(^{\text{In the derivation, we used } \int_D d\bar{h} \wedge f(z) \, dz = \int_{\partial D} f(z) \, dz, \text{ with } \partial D := \partial D, \text{ to localize the integral on the boundary.}}\)

\(^{\text{In ref. [44], the solution } \psi \text{ for the twisted case with regulated pole is written by considering a modified integration contour, accounting for the contribution of monodromy. It can be shown that, around each (regulated) singular point, it is equivalent to the one considered here.}}\)

\(^{\text{Here and in the following, we adopt the multivariate exponent notation: } (z-p)^a = (z_1-p_1)^{a_1} \ldots (z_n-p_n)^{a_n}.}\)
whose coefficients $\psi_{p, a}$ can be determined by requiring the fulfillment of (2.31) around the pole $p$. The values of the $a_{\text{min}}$ and the $a_{\text{max}}$ depend on the Laurent expansion of $\varphi_L^{(n)}$ and $\varphi_R^{(n)}$ around $p$. The determination of the Laurent coefficients $\psi_{p, a}$ can be carried out by solving the triangular systems of linear equations that arises after inserting the ansatz in the multivariate differential equation.

2. Solution by series expansion:

$$\psi_p \equiv \psi \bigg|_{z \to p} = \left( u^{-1} \right) \bigg|_{z \to p} \int_p^z \left( u \varphi_L^{(n)} \right) \bigg|_{z \to p},$$

(2.34)

which can be directly obtained by a series expansion under the integral sign, and a subsequent multifold integration. Taking $p = 0$ without loss of generality, we observe that as $z \to 0$, the twist (2.9) admits a factorized expansion $u|_{z \to 0} = z^{\gamma'} \cdot \sum_{i \geq 0} u_i z^i$ with non-integer exponents $\gamma'$. A similar expansion holds for the cocycles: $\hat{\varphi}_L|_{z \to 0} = \sum_{i \geq \mu_L} \varphi_L, i z^i$ and $\hat{\varphi}_R|_{z \to 0} = \sum_{i \geq \mu_R} \varphi_R, i z^i$ with integer leading exponents $\mu_L, \mu_R \in \mathbb{Z}^n$. Using these expansions integration in eq. (2.34) can be done term by term via $\int dx x^a = z^{a+1}/(a+1)$.

Let us remark that the second strategy allows for a straightforward determination of the intersection numbers bypassing any linear system solving procedure, and it constitutes the second main result of this study.

2.5 Simplified formulas from rescaling

In the previous sections we saw how the intersection number was given by

$$\langle \varphi_L | \varphi_R \rangle = \sum_{p \in \mathbb{P}_\omega} \text{Res}_{z=p}(\psi \varphi_R), \quad \text{where formally} \quad \psi = u^{-1} \int u \varphi_L.$$  

(2.35)

It is not hard to see that the above expressions are invariant under the following simultaneous rescalings:

$$u \to u/q, \quad \varphi_L \to \varphi_L q, \quad \varphi_R \to \varphi_R/q, \quad \psi \to \psi q,$$

(2.36)

where $q$ may be any\footnote{To be precise, the function $q$ here has to be meromorphic and have all poles and zeroes regulated by $u$.} function of $z$. Since each individual residue of eq. (2.35) possesses this invariance, the rescaling may be done locally at each individual $p$. Such rescalings are of interest since certain specific choices for $q$ introduce simplifications. Two special choices of $q$ are of particular interest:

**Right rescaling.** This is defined as the choice $q = \hat{\varphi}_R$. It corresponds to the rescaling:

$$u \to u_R := u/\hat{\varphi}_R, \quad \varphi_L \to \phi := \varphi_L \hat{\varphi}_R, \quad \hat{\varphi}_R \to 1, \quad \psi \to f := \psi \hat{\varphi}_R,$$

(2.37)

In this case, the nPDE eq. (2.31) becomes

$$\hat{\nabla}_{\omega_R, 1} \cdots \hat{\nabla}_{\omega_R, n} f = \hat{\phi},$$

(2.38)
where $\omega_R := d\log(u_R)$, and the argument of the residues appearing in eq. (2.35) is directly given by $f$. Therefore, upon the right rescaling, it is sufficient to know the solution of the differential equation in (2.38) up to the simple pole term, because the higher order terms, from $O(1)$ on, would not contribute to the residue. This observation turns in a computational advantage, and, for this reason, eq. (2.38), which is a special form of eq. (2.31), will become in section 4 the starting point for deriving an algebraic expression of $\text{Res}(f)$.

**Left rescaling.** This form of rescaling is defined by the choice $q = 1/\hat{\varphi}_L$. It corresponds to

$$u \to u_L := u \hat{\varphi}_L, \quad \hat{\varphi}_L \to 1, \quad \varphi_R \to \phi := \hat{\varphi}_L \varphi_R, \quad \psi \to g := \psi / \hat{\varphi}_L, \quad (2.39)$$

Upon this rescaling eq. (2.31) becomes

$$\hat{\nabla} \omega_{L,1} \cdots \hat{\nabla} \omega_{L,n} g = 1, \quad (2.40)$$

and we also define $\omega_L := d\log(u_L)$, and where the argument of the residues appearing in eq. (2.35) becomes $g \hat{\varphi}_L \varphi_R$. This equation is conceptually simpler to solve than the case with a generic form on the r.h.s.

Finally let us remark, that another source of simplifications is the shift invariance of intersection numbers:

$$\langle \varphi_L | \varphi_R \rangle = \langle \varphi_L + \nabla_\omega \xi_L | \varphi_R + \nabla_{-\omega} \xi_R \rangle, \quad (2.41)$$

which is valid for generic $(n-1)$-forms $\xi_L$ and $\xi_R$, but discussion of this falls out of our scope. Now we move on to applications of the formalism that was introduced here.

### 3 Applications

In this section we apply the $n$PDE method for computing intersection numbers of twisted 2-forms ($n = 2$ variables). Explicit results presented below agree with the iterative method of [16, 17, 42].

#### 3.1 Two-loop massless sunrise diagram

The first example is the massless sunrise diagram on the maximal cut, see figure 1. Using the Baikov parametrization, we denote the two irreducible scalar products as $z_1$ and $z_2$. The twist $u$ is built from the Baikov polynomial on the maximal cut. In the three coordinate charts $U_z$, $U_x$, and $U_y$ of the projective plane $\mathbb{C}P^2$ (see appendix B.1 for a brief review) it looks like this:

$$u = \begin{cases} 
  z_1^{r_1} z_2^{r_2} (z_1 + z_2 - s)^{r_3} & \text{in } U_z, \\
  x_1^{r_1} x_2^{r_2} (1 + x_1 - sx_2)^{r_3} & \text{in } U_x, \\
  y_1^{r_1} y_2^{r_2} (1 - sy_1 + y_2)^{r_3} & \text{in } U_y,
\end{cases} \quad (3.1)$$

- 9 -
where $\gamma_0 := \gamma_1 + \gamma_2 + \gamma_3$, and the polynomial factors (2.9) are $\{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3\} := \{z_1, z_2, z_1 + z_2 - s\}$. The corresponding singular hypersurfaces $\mathcal{S}_i$ defined in eq. (2.1) intersect at $3 + 2 + 1 = 6$ points in $\mathbb{C}P^2$ (see figure 2(a)):

$$
\mathbb{P}_\omega = \begin{cases} 
(z_1, z_2) \in \{(0,0), (s,0), (0,s)\} & \text{in } U_z, \\
(x_1, x_2) \in \{(0,0), (-1,0)\} & \text{in } U_x, \\
(y_1, y_2) = (0,0) & \text{in } U_y.
\end{cases} \tag{3.2}
$$

In this example we consider intersection numbers between generic $\varphi^{(2)}_L$ and $\varphi^{(2)}_R$ cocyles:

$$
\varphi^{(2)}_\bullet = z_1^{n_1} z_2^{n_2} (z_1 + z_2 - s)^{n_3} \, dz_1 \wedge dz_2, \tag{3.3}
$$

where $\bullet \in \{L, R\}$, and $n_i \in \mathbb{Z}$.

Our task is to compute the multivariate residues shown in the main formula (2.20) at each of the points (3.2). One way to do this is to find coordinate transformations that localize on these points $\mathbb{P}_\omega$, and allow for direct series expansion and solution of the nPDE (2.21) either using the algebraic formula (4.8) or the integral formula (2.32) (see also an alternative algebraic formula in appendix B.4). We collect such transformations in table 1 and refer the interested reader to appendix B.2 for further details.

**Example 1.** Consider the intersection $\langle \varphi_L | \varphi_R \rangle$ between:

$$
\hat{\varphi}_L = \frac{1}{z_1 z_2}, \quad \hat{\varphi}_R = \frac{1}{z_1 z_2}. \tag{3.4}
$$

Out of the six points collected in eq. (3.2) only two give non-trivial residues, yielding:

$$
\langle \varphi_L | \varphi_R \rangle = \text{Res}_{z=(0,0)}(\psi \varphi_R) + \text{Res}_{z=(0,s)}(\psi \varphi_R) = -\frac{1}{s^2} \frac{\gamma_3 (\gamma_1 + \gamma_3)}{\gamma_1 (1 - \gamma^2_1)} - \frac{1}{s^2} \frac{\gamma_3}{\gamma_1 (1 - \gamma^2_1)}. \tag{3.5}
$$

**Example 2.** We can also compute $\langle \varphi_L | \varphi_R \rangle$ of meromorphic forms containing the $\mathcal{B}_3$ factor:

$$
\hat{\varphi}_L = \frac{1}{z_1 z_2 (z_1 + z_2 - s)}, \quad \hat{\varphi}_R = \frac{1}{z_1 z_2 (z_1 + z_2 - s)^2}. \tag{3.6}
$$
(a) Sunrise diagram.  (b) Planar box diagram.  (c) $3F_2$

Figure 2. Singular hypersurfaces $S_i$ of the massless sunrise twist (3.1) on the left, massless 2-loop planar box twist (3.11) in the middle, and the $3F_2$ twist (3.19) on the right. The red sphere (with identified antipodal points) depicts the real slice of the projective plane $\mathbb{CP}^2$, whose equator is the line at infinity. The blue dots represent the intersection points collected in eqs. (3.2), (3.13), (3.20).

| Chart  | Point       | Coordinate transformation(s) |
|--------|-------------|------------------------------|
| $U_z$  | 1: (0, 0)   | $(z_1, z_2)$                 |
|        | 2: (s, 0)   | $\{(s + z_1 z_2, z_2), (s + z_1, z_1 z_2), (s + (z_1 - 1) z_2, z_2)\}$ |
|        | 3: (0, s)   | $\{(z_1 z_2, s + z_2), (z_1, s + z_1 z_2), ((z_1 - 1) z_2, s + z_2)\}$ |
| $U_x$  | 4: (0, 0)   | $\left(\frac{z_1 + z_2 - 1}{z_2}, \frac{1}{z_2}\right)$ |
|        | 5: (−1, 0)  | $\left\{(\frac{z_1 + z_2 - 1}{z_2}, \frac{1}{z_2}), (\frac{z_1 - 1}{z_1 z_2}, \frac{1}{z_1 z_2}), (\frac{z_1 z_2 - 1}{z_2}, \frac{1}{z_2})\right\}$ |
| $U_y$  | 6: (0, 0)   | $\left(\frac{1}{z_1}, \frac{z_2}{z_1}\right)$ |

Table 1. Intersection points $\mathbb{P}_i$ (middle) from the three coordinate charts of $\mathbb{CP}^2$ (left) that contribute to the intersection numbers between forms (3.3) with the massless sunrise twist (3.1). On the right we show the coordinate transformations used to compute the residues (2.20). We sum over all the contributions of each displayed transformation.

Out of the six points shown in eq. (3.2) the three points from the $U_z$ chart give non-zero residues:

\[
\begin{align*}
\text{Res}_{z=(0,0)}(\psi_\varphi R) &= -\frac{1}{s^3} \frac{1}{\gamma_1 \gamma_2}, \quad (3.7) \\
\text{Res}_{z=(s,0)}(\psi_\varphi R) &= 0 - \frac{1}{s^3} \frac{1 + \gamma_1 + 2 \gamma_2 + 2 \gamma_3}{\gamma_2 (\gamma_2 + \gamma_3)(1 + \gamma_2 + \gamma_3)} - \frac{1}{s^3} \left( \frac{1 - \gamma_1}{(1 + \gamma_3)(1 + \gamma_2 + \gamma_3)} + \frac{1 + \gamma_1}{\gamma_3 (\gamma_2 + \gamma_3)} \right), \quad (3.8) \\
\text{Res}_{z=(0,s)}(\psi_\varphi R) &= -\frac{1}{s^3} \frac{1 + 2 \gamma_1 + \gamma_2 + 2 \gamma_3}{\gamma_1 (\gamma_1 + \gamma_3)(1 + \gamma_1 + \gamma_3)} + 0 - \frac{1}{s^3} \frac{(1 + \gamma_1 + 2 \gamma_3)(1 + 2 \gamma_1 + \gamma_2 + 2 \gamma_3)}{\gamma_3 (\gamma_1 + \gamma_3)(1 + \gamma_1 + \gamma_3)(1 + \gamma_1 + \gamma_3)}, \quad (3.9)
\end{align*}
\]

where we separated contributions of the transformations shown in table 1 into individual
terms. By adding them up, the intersection number becomes:

\[
\langle \varphi_L | \varphi_R \rangle = \text{Res}_{z=(0,0)}(\psi \varphi_R) + \text{Res}_{z=(s,0)}(\psi \varphi_R) + \text{Res}_{z=(0,s)}(\psi \varphi_R)
\]

\[
= -\frac{1}{s^3} \frac{(\gamma_1 + \gamma_2 + \gamma_3)(1 + \gamma_1 + \gamma_2 + \gamma_3)}{\gamma_1 \gamma_2 \gamma_3 (1 + \gamma_3)}.
\] (3.10)

3.2 Two-loop planar box diagram

Now we turn to the massless two-loop planar box on the maximal cut in the Baikov representation, see figure 3, with the twist:

\[
u = \begin{cases}
\begin{aligned}
z_1^{\gamma_1} z_2^{\gamma_2} (st + s(z_1 + z_2) + z_1 z_2)^{\gamma_3} & \text{in } U_z, \\
x_1^{\gamma_1} x_2^{-\gamma_0 - \gamma_3} \left( x_1 + sx_2 + sx_1 x_2 + st x_2^2 \right)^{\gamma_3} & \text{in } U_x, \\
y_1^{-\gamma_0 - \gamma_3} y_2^{\gamma_2} (sy_1 + y_2 + sy_1 y_2 + st y_1^2)^{\gamma_3} & \text{in } U_y,
\end{aligned}
\end{cases}
\] (3.11)

where \(\gamma_0 := \gamma_1 + \gamma_2 + \gamma_3\), and the factors (2.9) are:

\[
\{B_1, B_2, B_3\} := \{z_1, z_2, st + s(z_1 + z_2) + z_1\}.
\] (3.12)

There are \(3 + 1 + 1 = 5\) intersection points in \(\mathbb{CP}^2\) of the singular hypersurfaces \(S_i\) (see figure 2(b)), their location is as follows:

\[
P_\omega = \begin{cases}
\begin{aligned}
(z_1, z_2) & \in \{(0,0), (-t,0), (0,-t)\} & \text{in } U_z, \\
(x_1, x_2) = (0,0) & \text{in } U_x, \\
y_1, y_2 = (0,0) & \text{in } U_y.
\end{aligned}
\end{cases}
\] (3.13)

Here we only consider intersection numbers between monomial \(\varphi_L^{(2)}\) and \(\varphi_R^{(2)}\) cocycles:

\[
\varphi^{(2)} = z_1^{n_1} z_2^{n_2} dz_1 \wedge dz_2,
\] (3.14)

with the same notation as in eq. (3.3).

Some of the intersection points (3.13) turn out to be degenerate: they have three singular hypersurfaces \(S_i\) passing through them. An example of this is the point \(x = (0,0)\) located on the line at infinity, as can be seen from eq. (3.11) and figure 2(b). One way to amend this issue is to employ the resolution of singularities procedure (closely related to the sector decomposition algorithm, see [56–58]). In appendix B.3 we give further details and collect the full list of coordinate transformations used for computation of intersection numbers in this setup.
Example 3. Let us consider the intersection number \( \langle \varphi_L \mid \varphi_R \rangle \) between the two logarithmic forms:

\[
\hat{\varphi}_L = \frac{1}{z_1 z_2}, \quad \hat{\varphi}_R = \frac{1}{z_1 z_2}.
\]

(3.15)

Only the origins of the three charts (3.13) contribute to this intersection number producing:

\[
\langle \varphi_L \mid \varphi_R \rangle = \text{Res}_{z=(0,0)}(\psi \varphi_R) + \text{Res}_{z=(0,0)}(\psi \varphi_R) + \text{Res}_{y=(0,0)}(\psi \varphi_R)
\]

\[
= \frac{1}{\gamma_1 \gamma_2} \left( -\frac{1}{\gamma_1 (\gamma_1 + \gamma_3)} + \frac{1}{(\gamma_2 + \gamma_3)(\gamma_0 + \gamma_3)} \right) + \left( \frac{1}{(\gamma_1 + \gamma_3)(\gamma_0 + \gamma_3)} + \frac{-1}{\gamma_2 (\gamma_1 + \gamma_3)} \right).
\]

(3.16)

Example 4. Similar to example 1, we compute the intersection between:

\[
\hat{\varphi}_L = \frac{1}{z_1^2 z_2}, \quad \hat{\varphi}_R = \frac{1}{z_1^2 z_2}.
\]

(3.17)

We find the three points from \( \mathbb{P}_\varphi \) contribute to this intersection number, resulting in:

\[
\langle \varphi_L \mid \varphi_R \rangle = \text{Res}_{z=(0,0)}(\psi \varphi_R) + \text{Res}_{z=-(0,0)}(\psi \varphi_R) + \text{Res}_{z=(0,0)}(\psi \varphi_R)
\]

\[
= -\frac{1}{\tilde{\gamma}^2 (1 - \tilde{\gamma}^2)} \gamma_3 (1 - \gamma_1 - \gamma_3) - \frac{1}{s^2 \tilde{t}^2 \gamma_1 (1 - \gamma_1^2)} \gamma_3 (1 - \gamma_1 - \gamma_3) + \frac{\gamma_3 (1 - \gamma_1 - \gamma_3)}{s^2 \tilde{t}^2 (\gamma_2 + \gamma_3)}.
\]

(3.18)

3.3 \( \mathcal{F}_2 \) hypergeometric function

Another setup with degenerate intersection points is the \( \mathcal{F}_2 \) hypergeometric function, whose twist looks like this:

\[
u = \begin{cases} 
\gamma_1 x_1 (z_1 - z_2) (z_1 - z_2)^7 (s - z_2)^7 (z_1 - z_2)^7 & \text{in } U_z, \\
(-1)^3 \gamma_1 (s_1 - z_1) x_1 (x_1 - x_2) (s_1 x_1 - 1)^7 (1 - x_1)^7 & \text{in } U_x, \\
(-1)^3 x_1 (1 - y_1)^3 (s y_1 - y_2)^7 (1 - y_2)^7 & \text{in } U_y,
\end{cases}
\]

(3.19)

where \( \gamma_0 := \gamma_1 + \ldots + \gamma_5 \). The five singular hypersurfaces \( \mathcal{S}_i \) intersect at the following \( 6 + 2 + 1 = 9 \) points (see figure 2(c)):

\[
\mathbb{P}_\varphi = \begin{cases} 
(z_1, z_2) \in \{(0,0), (0, s), (s, s), (1, 0), (1, s), (1, 1)\} & \text{in } U_z, \\
(x_1, x_2) \in \{(0,0), (1, 0)\} & \text{in } U_x, \\
(y_1, y_2) = (0,0) & \text{in } U_y.
\end{cases}
\]

(3.20)

We compute intersection numbers between generic \( \varphi_L^{(2)} \) and \( \varphi_R^{(2)} \) cocycles:

\[
\varphi^{(2)} = z_1^n z_2^m (1 - z_1)^{m_3} (s - z_2)^{m_4} (z_1 - z_2)^{m_5} dz_1 \wedge dz_2,
\]

(3.21)

with the same notation as in eq. (3.3). The corresponding coordinate transformations are reported in table 2.

\[\text{---} \quad 13 \quad \text{---}\]
Table 2. Intersection points $P_\omega$ (middle) from the three coordinate charts of $\mathbb{CP}^2$ (left) that contribute to the intersection numbers between forms (3.21) with the $3 F_2$ twist (3.19). On the right we show the coordinate transformations used to compute the residues (2.20). We sum over all the contributions of each displayed transformation.

**Example 5.** In continuation of example 3 we calculate the intersection of the logarithmic forms:

$$\hat{\varphi}_L = \frac{1}{z_1 z_2}, \quad \hat{\varphi}_R = \frac{1}{z_1 z_2}.$$  

(3.22)

Once again, only the origins of the charts contribute to this intersection number giving the total:

$$\left\langle \varphi_L | \varphi_R \right\rangle = \text{Res}_{z=(0,0)}(\psi \varphi_R) + \text{Res}_{x=(0,0)}(\psi \varphi_R) + \text{Res}_{y=(0,0)}(\psi \varphi_R)$$

(3.23)

$$= \left( \frac{1}{\gamma_1(\gamma_1 + \gamma_2 + \gamma_5)} + \frac{1}{\gamma_2(\gamma_1 + \gamma_2 + \gamma_5)} + 0 \right)$$

$$+ \left( \frac{1}{\gamma_1(\gamma_2 + \gamma_4 + \gamma_5)} + \frac{1}{\gamma_2(\gamma_2 + \gamma_4 + \gamma_5)} + 0 \right) + \left( \frac{1}{\gamma_1(\gamma_2 + \gamma_5)} + \frac{1}{\gamma_2(\gamma_1 + \gamma_2 + \gamma_3)} + 0 \right).$$

This concludes our collection of examples, for intersection numbers in $n = 2$ variables, involving Feynman integrals and other mathematical functions. Let us remark, that the presented method is also applicable to meromorphic $n$-forms in higher dimensions.

### 4 Algebraic solutions

In this section, we will introduce an algebraic solution for the contribution to the intersection number coming from each individual residue. The main formula of this section, eq. (4.8), is derived by solving the $n$PDE using an ansatz. We will start the section by introducing an essential combinatorial object, that of vector compositions.
4.1 Vector compositions

Vector compositions are a convenient tool for manipulating algebraic expressions involving multivariate monomials, in particular when dealing with product of multivariate series.

**Definition 1.** Vector compositions

\[ S = VC(\tau) \quad \text{with} \quad \tau = (\tau_1, \ldots, \tau_n) \quad (4.1) \]

are a function of \( n \) non-negative integers \( \tau_1 \) to \( \tau_n \). Its output \( S \) is a set of ordered lists \( \sigma \), each containing a number of \( n \)-vectors \( s \), with the property \( \sum_{s \in \sigma} s = \tau \) (\( s \) vectors contain non-negative integers; the zero-vector is excluded, corresponding to all \( s > 0 \)). Each \( s \) can be represented as a step in \( \mathbb{Z}^n \), therefore \( \sigma \), being a collection of steps, can be mapped to a path in \( \mathbb{Z}^n \), going from \( 0 \) to \( \tau \). Within this representation, \( S \) can be viewed as the set of all paths \( \mathbb{Z}^n \), connecting \( 0 \) to \( \tau \), in one or in multiple (ordered) steps, counted by the size of \( \sigma \), namely having \( |\sigma| = 1, 2, \ldots, m \), with \( m := \sum_{i=1}^n \tau_i \).

**Example 6 (\( n = 1 \) case).** In the univariate case, the vector \( \tau \) has just one entry, say \( \tau = (m) \) with \( m \in \mathbb{N} \), therefore the vector compositions reduce to standard compositions, i.e. \( VC(\tau) = \text{comp}(m) \). Compositions are a known operation from combinatorics which describe the number of (ordered) ways an integer can be expressed as a sum of positive integers. For illustration, let us choose \( \tau = (4) \)

\[ VC(\tau) = \text{comp}(4) = \{(4), (3, 1), (2, 2), (1, 3), (2, 1, 1), (1, 2, 1), (1, 1, 2), (1, 1, 1, 1)\} \quad (4.2) \]

**Example 7 (\( n = 2 \) case).** The two dimensional case, where the vector \( \tau \) has two components, is the first non-trivial case for applying vector composition. For illustration, let us consider \( \tau = (1, 2) \), yielding

\[ VC(\tau) = \left\{ \left( \begin{array}{c} 1 \\ 2 \end{array} \right) \right\}, \right. \]

\[ \left. \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \right\}, \left( \begin{array}{c} 0 \\ 1 \\ 1 \end{array} \right), \left( \begin{array}{c} 1 \\ 1 \\ 0 \end{array} \right) \right\}, \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right), \left( \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right), \left( \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right) \right\} \quad (4.3) \]

where the three rows correspond to entries with \(|\sigma|\) being 1, 2, and 3, respectively. Each element of \( VC(\tau) \) admits a representation in terms of lattice paths in \( \mathbb{Z}^2 \), as shown in figure 4.

More details on vector compositions can be found in e.g. [59]. In the current context, they turn out to be a useful tool to express the solution of the \( n \)PDE eq. (2.38).
4.2 Algebraic expression for residues

In section 2 we saw that multivariate intersection numbers can be expressed as a sum over contributions from a number of points, with each contribution expressed as a residue. In the “right rescaling” framework of section 2.5 the relation is given as

\[ \langle \varphi_L | \varphi_R \rangle = \sum_{p \in \mathbb{P}_\omega} \text{Res}_{z=p} (f) \tag{4.4} \]

where \( f \) is defined as the solution of eq. (2.38), namely

\[ \hat{\nabla} \omega_{R,1} \cdots \hat{\nabla} \omega_{R,n} f = \hat{\phi}, \quad \text{with} \quad \hat{\phi} = \hat{\varphi}_L \hat{\varphi}_R, \tag{4.5} \]

and where

\[ \hat{\nabla} \omega_{R,i} := (\partial_{z_i} + (\partial_{z_i} w)), \quad \text{with} \quad w := \log(u/\hat{\varphi}_R). \tag{4.6} \]

In section 2.4 it was discussed how to solve systems of the form of eq. (4.5) by making a series ansatz for \( f \) and solving it locally around each point, in what is essentially an analytic use of the “Frobenius method”. In this section we are going to pursue that direction a step further, to obtain an algebraic solution for each term in eq. (4.4). The first step is to write \( w \) and \( \hat{\phi} \) as series around each point, where the appropriate variable changes have been made in order to ensure that the point is located at \( z = 0 \) and in addition that the series expansions are well defined so no further variable changes (or blowups) are needed. With this we may write

\[ \hat{\phi} = \sum_{i=\mu_{\hat{\phi}}}^{\infty} \phi_i z^i \quad \text{and} \quad w = \log(u/\hat{\varphi}_R) = \sum_{i=0}^{\infty} w_i z^i + \sum_{j=1}^{n} v_j \log(z_j) \tag{4.7} \]
where \( z^i := z_1^{i_1} \cdots z_n^{i_n} \). In this notation \( \mu_\phi \) should be interpreted as the powers corresponding to the leading term in the expansion of \( \hat{\phi}_L \hat{\phi}_R \). The \( w \)-expansion begins at powers \( 0 \) (lower ones cannot appear when \( u \) is of the form \( \prod_i B_i^{\gamma_i} \), as in our case). Using an ansatz for \( f \) as a Laurent series expansion around each intersection point, the \( n \)PDE constraints the coefficients, so that the residue of \( f \) can be obtained in closed form as,

\[
\text{Res}(f) = \sum_{\tau=0}^{\mu_\phi - 2} \sum_{\sigma \in \text{VC}(\tau)} \left( \frac{(-1)^{|\sigma|}}{U(0,0)} \prod_{i=1}^{\sigma} \frac{U(\sigma_i, T_i)}{U(0, T_i)} \right) \phi_{\tau - 2}^{\mu_\phi - 2} \tag{4.8}
\]

where

\[
T_i := \tau - \sum_{j<i} \sigma_j \tag{4.9}
\]

and \( U \) is a function dependent only on the terms in the expansion of \( w \), which is conveniently defined in terms of an auxiliary function, \( R \), as

\[
U(\lambda, \eta) := R(\lambda, -\eta-1) \tag{4.10}
\]

whose general definition is reported in appendix C.1.2. Here, for illustration purposes, we showcase its expressions for a few small values of \( n \):

- \( n = 1 \) : \( R(\alpha, \beta) = (\beta + v) \delta_{\alpha,0} + \alpha w_\alpha \tag{4.11} \)
- \( n = 2 \) : \( R(\alpha, \beta) = (\beta_1 + v_1)(\beta_2 + v_2) \delta_{\alpha,0} + (\alpha_1 \alpha_2 + \alpha_1 (\beta_2 + v_2) + \alpha_2 (\beta_1 + v_1)) w_\alpha + \sum_{j=0}^{\alpha} (\alpha_1 - j_1) j_2 w_{\alpha-j} w_j \tag{4.12} \)
- \( n = 3 \) : \( R(\alpha, \beta) = (\beta_1 + v_1)(\beta_2 + v_2)(\beta_3 + v_3) \delta_{\alpha,0} + ((\alpha_1 + \beta_1 + v_1)(\alpha_2 + \beta_2 + v_2)(\alpha_3 + \beta_3 + v_3) - (\beta_1 + v_1)(\beta_2 + v_2)(\beta_3 + v_3)) w_\alpha + \sum_{j=0}^{\alpha} ((\alpha_1 - j_1) j_2 (j_3 + \beta_3 + v_3) + (\alpha_2 - j_2) j_3 (j_1 + \beta_1 + v_1)) + (\alpha_3 - j_3) j_1 (j_2 + \beta_2 + v_2) + \sum_{j=0}^{\alpha} \sum_{l=0}^{\alpha-j} (\alpha_1 - j_1 - l_1) j_2 l_3 w_{\alpha-j-l} w_j w_l. \tag{4.13} \)

For a general \( n \)-variate expression of \( R \), see appendix C.1.2.

The residue formula in eq. (4.8), using vector compositions, constitutes the third main result of this work. Further details on its derivation can be found in appendix C.

The arXiv version of this paper is supplemented with a \textsc{Mathematica} implementation of eq. (4.8), consisting of a package \texttt{algebraic_residue.m} and a notebook file \texttt{load_algebraic_residue.nb} that contains examples of its use.

### 4.3 Example

Here we show an application of the algebraic formula eq. (4.8).
Table 3. Values for $\hat{\Phi}$ and $-\mu_{\phi}$ at the twelve sub-intersection points for example 8. The points are defined in table 1, and the additional labels (a, b, c) refer to the adopted coordinate transformation.

| point | $\hat{\Phi}$ | $-\mu_{\phi} - 2$ |
|-------|--------------|------------------|
| 1     | $z_1^{-2}z_2^{-2}$ | {0, 0} |
| 2a    | $(s+z_1z_2)^{-2}$ | {-2, -2} |
| 2b    | $(s+z_1)^{-2}z_2^{-2}$ | {-2, 0} |
| 2c    | $(s+z_2(z_1 - 1))^{-2}$ | {-2, -2} |
| 3a    | $z_1^{-2}(s+z_2)^{-2}$ | {0, -2} |
| 3b    | $(s+z_1z_2)^{-2}$ | {-2, -2} |
| 3c    | $(z_1 - 1)^{-2}(z_2+s)^{-2}$ | {-2, -2} |
| 4     | $z_1^{-2}z_2^{-2}$ | {0, 0} |
| 5a    | $(z_1z_2 - 1)^{-2}$ | {-2, -2} |
| 5b    | $(z_1 - 1)^{-2}z_2^{-2}$ | {-2, 0} |
| 5c    | $((s+z_1)z_2 - 1)^{-2}$ | {-2, -2} |
| 6     | $z_1^{-2}z_2^{-2}$ | {0, 0} |

Example 8. Using the setup of section 3.1, let us compute $\langle \varphi_L | \varphi_R \rangle$ with

$$\hat{\varphi}_L = \frac{1}{z_1z_2}, \quad \hat{\varphi}_R = \frac{1}{z_1z_2}.$$ (4.14)

In the “right rescaling” scheme, the first step is to compute $\Phi = \varphi_L\hat{\varphi}_R$ at each of the points in table 1 and identify the leading term corresponding to the $\mu_{\phi}$-vector. The result is given in table 3.

We see from the general expression (4.8) that a given point will only contribute if $-\mu_{\phi} - 2 \geq 0$. So we realize that there will only be contributions from three of the intersection points: 1, 4, and 6. In each case eq. (4.8) will contain one term only, of the form

$$\text{Res}(f) = \frac{\phi_{-2,-2}}{(v_1-1)(v_2-1)}.$$ (4.15)

We observe that $\phi_{-2,-2} = 1$ at each of the three points, while the values of the $v_i$ depend on the considered point, and are computed locally. The non-vanishing expressions of $\text{Res}(f)$, are:

- Point 1: $\frac{1}{\gamma_1\gamma_2}$
- Point 4: $\frac{-1}{\gamma_1(\gamma_1+\gamma_2+\gamma_3)}$
- Point 6: $\frac{-1}{\gamma_2(\gamma_1+\gamma_2+\gamma_3)}$,

therefore, the intersection number is obtained by adding them up, as:

$$\langle \varphi_R | \varphi_L \rangle = \frac{\gamma_3}{\gamma_1\gamma_2(\gamma_1+\gamma_2+\gamma_3)},$$ (4.17)

which is the correct expected results.

5 Conclusion

In this work we proposed a new computational method for the evaluation of intersection numbers for twisted meromorphic $n$-forms through Stokes’ theorem, which is based
on the solution of a $n$-th order partial differential equation ($n$PDE). Our finding can be summarised simply as:

$$
\langle \varphi_L | \varphi_R \rangle = \frac{1}{(2\pi i)^n} \int (u \varphi_{L,c}) \wedge (u^{-1} \varphi_R) = \frac{1}{(2\pi i)^n} \oint (u^{-1} \varphi_R) \int (u \varphi_L).
$$

(5.1)

The evaluation of the last integral is performed by multivariate Laurent series expansions and multivariate residues. The analytic properties of the intersection numbers and of the $n$PDE yielded the algebraic determination of the contributing residues, which we were able to cast in closed forms for an efficient evaluation.

The presented method requires the knowledge of the intersection points as input. In the case of a twist corresponding to a hyperplane arrangement and normal crossing at the intersection points, this information can be easily extracted from the integrand. Instead, for generic configurations, when more hypersurfaces pass through a given intersection point, the desingularization procedure constitutes a challenging, hence interesting, mathematical problem.

The formulation of the intersection number for twisted cohomology presented here applies to the case of regulated singularities. We are confident that it can be extended to the relative cohomology case, where the singularities of the integral are not regulated within the twist.

The new method presented here can be applied to derive linear relations, differential equations, difference equations, and quadratic relations for Feynman integrals as well as for a wider class of functions, such as Aomoto-Gel’fand integrals, Euler-Mellin integrals, and GKZ hypergeometric functions. Therefore the method can be useful for computational (quantum) field theory, and computational differential and algebraic topology.

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A Relation to Matsumoto’s algorithm

In this appendix we adopt the notation of [2] (see also [42] for a review) and explicitly derive the nPDE (2.31) in the n = 2 variable setting.

Let us consider an intersection number among two holomorphic 2-forms \( \varphi_L \) and \( \varphi_R \), and assume that any two hypersurfaces \( S_i \) and \( S_j \) defined in eq. (2.1) intersect at point(s), such that no other \( S_k \) with \( k \neq i, j \) passes through it. The main idea of [2] is to find an explicit 2-form \( \varphi_{L,c} \) with compact support in the same cohomology class of \( \varphi_L \). This 2-form appears in the definition of the intersection number integral (2.20):

\[
\langle \varphi_L \mid \varphi_R \rangle = (2\pi i)^{-2} \int_X \varphi_{L,c} \wedge \varphi_R .
\]  

(A.1)

We associate an intersection point \( p_{ij} := S_i \cap S_j \) with each pair of indices \( (i, j) \) with \( i < j \). For a singular hypersurface \( S_i \) we introduce two tubular neighborhoods \( V_i \) and \( U_i \) around it of radius \( \epsilon_1 \) and \( \epsilon_2 \) respectively, with \( \epsilon_1 < \epsilon_2 \), such that \( V_i \subset U_i \). Following [2], we then write an explicit formula for the compactly supported 2-form:

\[
\varphi_{L,c} = \varphi_L - \nabla_\omega \left( \sum_j h_j \left( \sum_{i<j} \psi_{p_{ij}} + \prod_{i<j} (1 - h_i) \psi^j_{p_{ij}} \right) \right),
\]  

(A.2)

where \( h_j \) is a smoothened version of the Heaviside step function: \( h_j = 1 \) in \( V_j \), \( h_j = 0 \) outside \( U_j \), and it smoothly interpolates between the two boundary values \( 0 \leq h_j \leq 1 \) in \( U_j \setminus V_j \). These properties of \( h_j \) imply that the two forms coincide \( \varphi_{L,c} = \varphi_L \) outside of the singular neighborhood \( \cup_j U_j \supset \cup_i S_i \).

Furthermore, it is not difficult to show that eq. (A.2) has compact support, i.e. the 2-form \( \varphi_{L,c} \) vanishes on the singular neighborhood \( \cup_i V_i \subset \cup_j U_j \), if we impose certain constraints on the auxiliary 0- and 1-forms appearing on the r.h.s. of eq. (A.2). For our purposes here it is enough to investigate these constraints locally\(^7\) around a given intersection point \( p_{ij} \) for some fixed \( i \) and \( j \), where we introduce local coordinates \((z_1, z_2)\), such that the hypersurfaces \( S_i \) and \( S_j \) are parametrized by \( z_1 = 0 \) and \( z_2 = 0 \) respectively. Then the auxiliary 1-forms \( \psi^i \) and \( \psi^j \), and the 0-form \( \psi \) (we will drop the subscripts in the following) from eq. (A.2) must satisfy the following differential equations [2]:

\[
\begin{align*}
\nabla_\omega \psi^i &= \varphi_L & \text{on } U_i := \{(z_1, z_2) \mid |z_1| \leq \epsilon_2 \}, \\
\nabla_\omega \psi^j &= \varphi_L & \text{on } U_j := \{(z_1, z_2) \mid |z_2| \leq \epsilon_2 \}, \\
\nabla_\omega \psi &= \psi^i - \psi^j & \text{on } U_i \cap U_j ,
\end{align*}
\]  

(A.3)

with the \( z \)-dependence shown explicitly:

\[
\begin{align*}
\psi^i &= \hat{\psi}^i(z_1, z_2) \, dz_2 , & \psi^j &= \hat{\psi}^j(z_1, z_2) \, dz_1 , & \psi &= \hat{\psi}(z_1, z_2) .
\end{align*}
\]  

(A.4)

Then on the polydisc \( U_i \cap U_j \) eq. (A.2) reduces to:

\[
\varphi_{L,c} = (1 - h_j)(1 - h_i) \varphi_L - (1 - h_j) \psi^i \wedge \varphi^j - (1 - h_i) \psi^j \wedge \varphi^i + \psi \, dh_i \wedge dh_j ,
\]  

(A.5)

---

\(^7\)See the original work [2] and the review in [42] for the full treatment.
which vanishes on the inner polydisc $V_i \cap V_j$. Similarly one can show that eq. (A.2) vanishes on the rest of the singular tube $V_j \cup \cup_{i \neq j} U_i$ for each $j$, which proves that $\varphi_{L,c}$ indeed has compact support.

In the integral (A.1) the compactly supported 2-form $\varphi_{L,c}$ is wedged against the holomorphic form $\varphi_R$, hence out of eq. (A.2) only the anti-holomorphic part will contribute to the intersection number. The only source of this is the last term in eq. (A.5), so that the intersection number integral localizes onto neighborhoods of intersection points $p_{ij}$ as:

$$\langle \varphi_L | \varphi_R \rangle = (2\pi i)^{-2} \sum_{p_{ij}} \int_{(U_i \setminus V_i) \cap (U_j \setminus V_j)} \psi \, dh_i \wedge dh_j \wedge \varphi_R, \quad \text{(A.6)}$$

where the integration domain is a difference of two polydiscs:

$$(U_i \setminus V_i) \cap (U_j \setminus V_j) = \{(z_1, z_2) | \epsilon_1 \leq |z_1|, |z_2| \leq \epsilon_2\} \equiv (U_i \cap U_j) \setminus (V_i \cap V_j) \quad \text{(A.7)}$$

Each integral in eq. (A.6) can be computed applying Stokes’ theorem one variable at time. Recalling that $h_i$ vanishes on $\partial U_i = \{z_1 | |z_1| = \epsilon_2\}$ (and similar for $h_j$), we obtain the following:

$$\langle \varphi_L | \varphi_R \rangle = (2\pi i)^{-2} \int_{\epsilon_1 \leq |z_1|, |z_2| \leq \epsilon_2} \psi \, dh_i \wedge dh_j \wedge \varphi_R = -(2\pi i)^{-1} \int_{|z_1| = \epsilon_1, \epsilon_1 \leq |z_2| \leq \epsilon_2} dh_j \wedge \varphi_R$$

$$= -(2\pi i)^{-1} \int_{\epsilon_1 \leq |z_2| \leq \epsilon_2} dh_j \wedge (\text{Res}_{z_1 = 0} \psi \varphi_R)$$

$$= (2\pi i)^{-1} \int_{|z_2| = \epsilon_1} \text{Res}_{z_1 = 0} \psi \varphi_R$$

$$= \text{Res}_{z_2 = 0} (\text{Res}_{z_1 = 0} \psi \varphi_R). \quad \text{(A.8)}$$

We would like to emphasise, that in ref. [2] the 0-form $\psi$ is determined via a multi-step procedure, in which the auxiliary 1-forms $\psi^i$, $\psi^j$ have to be determined first as solution to the system (A.3). In our work, instead, $\psi$ is determined at one go as the solution of a single higher-order differential equation:

$$\nabla_{\omega_1} \nabla_{\omega_2} \psi = \varphi_L, \quad \text{on } U_i \cap U_j. \quad \text{(A.9)}$$

In fact, considering $\nabla_\omega = \nabla_{\omega_1} + \nabla_{\omega_2}$, we have the following identity:

$$\nabla_{\omega_2} \psi = \psi^i. \quad \text{(A.10)}$$

Then the last equation of eq. (A.3) can be rearranged as:

$$0 = \nabla_\omega \left( \nabla_\omega \psi - (\psi^i - \psi^j) \right) = \nabla_\omega \left( \nabla_{\omega_1} \psi + \psi^j \right), \quad \text{(A.11)}$$

which indeed gives the nPDE eq. (2.31) in $n = 2$ variables: $\nabla_{\omega_1} \nabla_{\omega_2} \psi = \varphi_L.$
A.1 Examples of $\eta$

Here for illustrative purposes we collect the two examples of the $\eta$ introduced in eq. (2.24).

**Example 9 ($n = 1$ case).** Locally around a given intersection point the univariate version of eq. (2.24) is:

$$\eta = \tilde{h}_1 \left( u \psi \right) \left( u^{-1} \varphi_R \right). \quad (A.12)$$

The corresponding version of the intersection number integral (2.23) reads:

$$d_1 \eta = \left( \tilde{h}_1 \left( u \nabla_{\omega_1} \psi \right) - \left( u \psi \right) d\tilde{h}_1 \right) \wedge \left( u^{-1} \varphi_R \right)$$

$$\equiv (u\varphi_{L,c}) \wedge \left( u^{-1} \varphi_R \right), \quad (A.13)$$

where the compactly supported 1-form (2.29) becomes:

$$\varphi_{L,c} = \varphi_L - \nabla_{\omega_1} (h_1 \psi). \quad (A.15)$$

in agreement with [2].

**Example 10 ($n = 2$ case).** Specifying the formula (2.24) to 2-variate case gives:

$$\eta = \tilde{h}_1 \tilde{h}_2 \left( u \psi \right) \left( u^{-1} \varphi_R \right), \quad (A.16)$$

so that the integrand (2.23) turns into:

$$d_1 d_2 \eta = \left( \tilde{h}_1 \tilde{h}_2 \left( u \nabla_{\omega_1} \nabla_{\omega_2} \psi \right) - \tilde{h}_2 d\tilde{h}_1 \wedge \left( u \nabla_{\omega_2} \psi \right) - \tilde{h}_1 \left( u \nabla_{\omega_1} \psi \right) \wedge dh_2 + \left( u \psi \right) dh_1 \wedge dh_2 \right) \wedge \left( u^{-1} \varphi_R \right)$$

$$\equiv (u\varphi_{L,c}) \wedge \left( u^{-1} \varphi_R \right), \quad (A.18)$$

where the compactly supported 2-form $\varphi_{L,c}$ agrees with the definition (A.5).

B Additional details for the applications

In this appendix we provide further details for the examples presented in section 3.

B.1 Projective plane

Here we review some of the basic features of the complex projective space $\mathbb{C}P^n$, which is the ambient space of the integration domain $X$ introduced in eq. (2.15). To support the examples discussed in section 3, we focus on the $n = 2$ dimensional case known as the complex projective plane $\mathbb{C}P^2$.

Consider a triple of complex variables $(Z_1, Z_2, Z_3) \in \mathbb{C}^3 \setminus \{0, 0, 0\}$. The projective plane $\mathbb{C}P^2$ consists of equivalence classes of these triples under a uniform rescaling: $(Z_1, Z_2, Z_3) \equiv (\lambda Z_1, \lambda Z_2, \lambda Z_3)$ for $\lambda \in \mathbb{C}^\ast$. In other words, $\mathbb{C}P^2$ is the space of complex lines in $\mathbb{C}^3$ passing through the origin.
the series expansion around the new origin $z_1 = z_2 = 0$ is ill-defined. For example, expanding $1/(z_1 + z_2)$ in the following two ways we get different results:

$$1/(z_1 + z_2) = \begin{cases} 
    z_2^{-1} - z_1 z_2^{-2} + \ldots & \text{first in } z_1, \text{ then in } z_2, \\
    z_1^{-1} - z_1^{-2} z_2 + \ldots & \text{first in } z_2, \text{ then in } z_1.
\end{cases} \quad (B.3)$$
To derive the transformation rule for $(B)$ is located at the intersection of the two singular hypersurfaces with the twist (3.11).

Here we provide further details for section 3.2, in particular in table 5 we collect the coordinate transformations needed for evaluation of intersection numbers between forms (3.14) with the massless 2-loop planar box twist (3.11). On the right we show the coordinate transformations used to compute the residues (2.20). Some transformations are parametrized by $N \in \mathbb{Z}_{\geq 0}$. We sum over all the contributions of each displayed transformation.

Table 5. Intersection points (middle) from the three coordinate charts of $\mathbb{CP}^2$ (left) that contribute to the intersection numbers between forms (3.3) with the massless 2-loop planar box twist (3.11). On the right we show the coordinate transformations used to compute the residues (2.20).

| Chart | Point | Coordinate transformation(s) |
|-------|-------|-----------------------------|
| $U_z$ | $(0, 0)$ | $(z_1, z_2)$ |
|       | $(-t, 0)$ | $(-t + (s + t) \sum_{i=1}^{N}(-1/z_2)^i + z_1 (z_1 z_2)^N, z_1 z_2)$ |
|       | $(0, -t)$ | $(z_1 z_2, -t + (s - t) \sum_{i=1}^{N}(-1/z_2)^i + (z_1 z_2)^N z_2)$ |
| $U_x$ | $(0, 0)$ | $\left\{ \left( \frac{1}{z_2}, \frac{1}{z_1}, \frac{1}{z_2} \right), \left( \frac{1}{z_2}, \frac{1}{z_1}, \frac{1}{s} \right), \left( -s + (s - t) \sum_{i=1}^{N}(-1/z_2)^i + z_1 (z_1 z_2)^N, \frac{1}{z_1 z_2} \right) \right\}$ |
| $U_y$ | $(0, 0)$ | $\left\{ \left( \frac{1}{z_1}, \frac{1}{z_2}, \frac{1}{z_1 z_2} \right), \left( \frac{1}{z_1}, \frac{1}{z_2}, \frac{1}{z_2} \right), \left( 1 + z_2, \frac{1}{z_1 z_2}, 1 + z_2 \right) \right\}$ |

To fix this we further change $(z_1, z_2) \mapsto (z_1, z_1 z_2)$, so that the twist (B.2) becomes regular:

$$u = z_1^{2s + 3} z_2^2 (s + z_1)^{\gamma_1} (1 + z_2)^{\gamma_3},$$  \hspace{1cm} (B.4)

meaning that all the polynomial factors now have constant terms. Series expansion of eq. (B.4) no longer depends on the order of operations and we may proceed with evaluation of the multivariate residue in eq. (2.20) as discussed in section 2 and section 3.1. Putting together the transformations that led us to the twist (B.4) gives:

$$(z_1, z_2) \mapsto (s + z_1, z_1 z_2),$$  \hspace{1cm} (B.5)

where we recognize one of the elements listed on the second row of table 1.

In general, coordinate transformations such as (B.5) are also known as resolutions of singularities of algebraic varieties. They can be computed algorithmically with the help of, for example, pySecDec [56, 57].

B.3 Two-loop planar box diagram

Here we provide further details for section 3.2, in particular in table 5 we collect the coordinate transformations needed for evaluation of intersection numbers between forms (3.14) with the twist (3.11).

Consider the point $z = (-t, 0)$ appearing on the second row of table 5. This point is located at the intersection of the two singular hypersurfaces $\mathcal{S}_2 \cap \mathcal{S}_3$, i.e. satisfies $\mathcal{B}_2 = \mathcal{B}_3 = 0$ (see eq. (3.12)). We look for a pair of local coordinates that roughly looks like $(z_1, z_2) \mapsto (B_3(z_1, z_2), B_2(z_1, z_2))$. Since $B_2(z_1, z_2) := z_2$ we may keep $z_2$ as one of our new coordinates. To derive the transformation rule for $z_1$ solve:

$$B_3(z_1, z_2) := st + s(z_1 + z_2) + z_1 z_2 = 0 \implies z_1 = -t + (s - t) \frac{-z_2/s}{1 + z_2/s},$$  \hspace{1cm} (B.6)
and expand this solution near \( z_2 = 0 \) producing:

\[
  z_1 = -t + (s - t) \left( -z_2/s + (-z_2/s)^2 + \ldots + (-z_2/s)^N + \ldots \right).
\]

(B.7)

Truncating this expansion up to the \( N \)th order and an additional condition \( (z_1, z_2) \mapsto (z_1, z_1 z_2) \) reproduces (up to the \( z_1 (z_1 z_2)^N \) term) the coordinate transformation shown on the second row of table 5.

In these new coordinates, the twist becomes:

\[
  u = z_1^{2(N+1)} z_2^{N+1}(s - (s - N z_2 + z_1 z_2)^N z_1 - (s - t)s^{-N} z_2 + z_1 z_2)^{\gamma_3},
\]

(B.8)

where each polynomial factor now has a constant term, so that the series expansion around \( z_1 = z_2 = 0 \) is well-defined. A generic monomial 2-form \((3.14)\) transforms as follows:

\[
  \varphi_2 := z_1^{n_1} z_2^{n_2} d z_1 \wedge d z_2 \mapsto z_1^{N+1+n_2} z_2^{N+n_2} (-t + \ldots + z_1 (z_1 z_2)^N n_1) d z_1 \wedge d z_2.
\]

(B.9)

The pole order of the form shifts by \( N \) in both variables, which implies that for large enough \( N \) the non-vanishing condition \((B.19)\) will no longer be satisfied and the corresponding transformation rule will not contribute to the intersection number. Hence we conclude, that even though the table 5 collects an infinite number of coordinate transformations, only a finite number of them contributes to a given intersection number.

### B.4 Series expansion of the integral formula

Here we derive the expression of a given residue appearing in the main formula \((2.20)\) that utilises the series expansion method presented in section 2.4. Throughout this section we assume that the pole of the residue is located at the origin \( z = 0 \).

Using the multi-index notation introduced in section 2.4, we write the \( z \)-expanded cocycles as:

\[
  \hat{\varphi}_L = \sum_{\mu \geq \mu_L} \varphi_{L, \mu} z^\mu, \quad \hat{\varphi}_R = \sum_{\mu \geq \mu_R} \varphi_{R, \mu} z^\mu.
\]

(B.10)

Examples of the domains of summation appearing here are shown in figure 5(a). In the following we will also refer to such domains of \( z \)-expansions as supports. Expansion of the twist \( u \) takes the following form:

\[
  u(z) = u_0 z^7 \left( 1 + \sum_{i, 1 \geq 1} u_i z^i \right) \equiv u_0 z^7 (1 + \tilde{u}(z)), \quad \tilde{u}(z) := \sum_{i, 1 \geq 1} u_i z^i,
\]

(B.11)

where \( u_0 \) is a \( z \)-independent constant, the vector \( \gamma \) is a linear combination of exponents appearing in \((2.9)\), \( \tilde{u} \) collects terms of the expansion that have at least one power of \( z \) inside, and we denote:

\[
  i \cdot 1 = i_1 + \ldots + i_n.
\]

(B.12)

We show the supports of \( u \) and \( \tilde{u} \) in figure 6(a) and figure 6(b) respectively. In the following we implicitly assume an additional condition \( i \geq j \) whenever we write \((i - j) \cdot 1 \geq m \) for some vector shift \( j \).

\footnote{Recall that the support of a Laurent series \( f = \sum_{i \in \mathbb{Z}^n} f_i z^i \) is the set \( \text{supp}(f) := \{ i \in \mathbb{Z}^n \mid f_i \neq 0 \} \).}
Figure 5. Supports of the $\hat{\phi}_L$ and $\hat{\phi}_R$ expansions (B.10) in (a), and the corresponding summation domains for eqs. (B.15), (B.18) in ((b), (c), (d)).

\[ \{ i \in \mathbb{Z}^n \mid i \cdot 1 \geq 0 \} \quad \{ i \in \mathbb{Z}^n \mid i \cdot 1 \geq 1 \} \quad \{ i \in \mathbb{Z}^n \mid i \cdot 1 \geq 2 \} \quad \{ i \in \mathbb{Z}^n \mid i \cdot 1 \leq -1 \} \]

(a) supp($\hat{\phi}_L$) (b) $\rho_0$ (c) $\rho_1$ (d) $\rho_2$

Figure 6. Supports of the expansions (B.11) in ((a), (b), (c)), where we implicitly assume an extra $i \geq 0$ condition. In ((d)) we illustrate the notation of eq. (B.18) (an additional $i \leq 0$ here is implied as well).

According to the integral formula (2.32), our goal is to compute the following residue:

$$\rho := \text{Res}_{z=0} \left[ \varphi_R u^{-1} \int z \varphi_L u \right].$$

(B.13)

The key idea is to expand the inverse of the twist $u(z)^{-1}$ in terms of $\tilde{u}(z)$ using the geometric series:

$$\rho = \text{Res}_{z=0} \left[ \varphi_R \int \varphi_L \right] + \sum_{m=1}^{(-\mu_L-\mu_R-2)\cdot 1} (-1)^m \text{Res}_{z=0} \left[ \varphi_R \cdot \tilde{u}^{m-1} \cdot \left( \tilde{u} \int \varphi_L - \int \tilde{u} \varphi_L \right) \right].$$

(B.14)

where in $\rho_m$ we gather the residues with $m$ insertions of $\tilde{u}(z)$. The summation bounds in the second term follows from the fact that the $z$-expansion of $\tilde{u}(z)^m$ has terms with at

---

For brevity we omit the $z^{-\gamma}$ and $z^{\gamma}$ factors around the integral sign that come from the twist (B.11), but they can easily be restored via: $\int \varphi \rightarrow z^{-\gamma} \int z^{\gamma} \varphi$ (with some abuse of notation).
least $m$ powers of $z$ inside, as reflected in figure 6(b) and figure 6(c) for $m = 1$ and $m = 2$ respectively.

The residue $\rho_0$ in eq. (B.14) is straightforward to evaluate:

$$\rho_0 = \sum_{i=\mu_L+1}^{-\mu_R-1} \hat{\phi}_{R,-i-1} \hat{\phi}_{L,i-1} \frac{1}{i + \gamma},$$

(B.15)

where the fraction stands for $1/(i + \gamma) := 1/\prod_j(i_j + \gamma_j)$, and the $n$-dimensional summation domain is shown in figure 5(b).

To compute $\rho_m$ in eq. (B.14) we separately expand its three factors, multiply them, and take the residue. The first factor is given in eq. (B.10). The second factor is a power of a series, so we may write its $i^{th}$ non-zero coefficient $\tilde{u}(z)^m = \sum_i (\tilde{u}^m)_i z^i$ using the Kronecker symbol $\delta_{i,j}$, as:

$$\tilde{u}(z)^m = \sum_{j_1 \geq 1} \cdots \sum_{j_m \geq 1} \delta_{(\sum_j j_k),i} u_{j_1} \cdots u_{j_m}, \quad \text{for } i \cdot 1 \geq m \text{ and } i \geq 0.$$  

One can come up with various evaluation strategies for this multiple sum using, for example, the nested parametrization as in eq. (C.32), or the (precomputed) multivariate Bell polynomials [60]. For the $i^{th}$ non-zero coefficient of the last factor of $\rho_m$ we get

$$\left(\tilde{u} \int \varphi_L - \int \tilde{u} \varphi_L\right)_i = \sum_{j \geq i - \mu_L - 1} u_j \hat{\varphi}_{L,i-j-1} \left(\frac{1}{j - \gamma} - \frac{1}{i + \gamma}\right), \quad \text{for } (i - \mu_L - 1) \cdot 1 \geq 1$$

and $i \geq \mu_L + 1$. Integrals here are done via the same technique as discussed in section 2.4.

Finally the sum representation of the $\rho_m$ residue reads:

$$\rho_m = \sum_{(i-\mu_L-1) \cdot 1 \geq m} \hat{\phi}_{R,-i-1} \sum_{j \geq i - \mu_R - 1} \sum_{j \geq i - \mu_L - 1} \tilde{(\tilde{u}^m)}_j \left(\tilde{u} \int \varphi_L - \int \tilde{u} \varphi_L\right)_{i-j},$$

(B.18)

where examples of the outer sum’s domain are shown in figures 5(c) and 5(d) (see also figure 6(d) for an illustration of the $\leq$ notation in the inner sum). Alongside the non-vanishing condition:

$$-\mu_L - \mu_R - 2 \geq 0,$$

(B.19)

we obtain the algebraic expression for a given residue:

$$\text{Res}(\psi \varphi_R) = \sum_{m=0}^{(-\mu_L-\mu_R-2) \cdot 1} (-1)^m \rho_m$$

(B.20)

with $\rho_m$ defined by eqs. (B.15) and (B.18), (B.16), (B.17).

C Algebraic solution: proof

The purpose of this appendix is to show the validity of the algebraic solution given by eq. (4.8) and discuss some features not mentioned in the main text.
C.1 Deriving the algebraic expression

We will consider a more general case than in section 4, and only at the end specify to the “right rescaling” framework. Our starting point will be eqs. (2.20) and (2.31):

\[ \langle \varphi_L | \varphi_R \rangle = \sum_{p \in P} \text{Res}_{z=p}(\psi \varphi_R) \]  

where \( \psi \) is defined through

\[ \hat{\nabla}_{\omega_1} \cdots \hat{\nabla}_{\omega_n} \psi = \hat{\varphi}_L \quad \text{where} \quad \hat{\nabla}_{\omega_i} := \partial_{z_i} + (\partial_{z_i} \log(u)) . \]  

We want to approach the differential equation locally as a series expansions. So in the following we will consider a point corresponding to one of the terms of eq. (C.1) with the assumption that the proper variable changes has been made so \( p = 0 \) and in addition that the series expansions are well defined so no further blowups are needed as also discussed in section 4.

Let us write the objects involved\(^{10}\) as such expansions:

\[ \hat{\varphi}_L = \sum_{i=\mu_L}^{\infty} \hat{\varphi}_{L,i} z^i \quad \hat{\varphi}_R = \sum_{i=\mu_R}^{\infty} \hat{\varphi}_{R,i} z^i \quad \psi = \sum_{i=\mu_\psi}^{\infty} \psi_i z^i \]  

\[ w := \log(u) = \sum_{i=\mu_w}^{\infty} w_i z^i + \sum_{j=1}^{n} v_j \log(z_j) . \]  

The exponent of the leading powers \( \mu_L, \mu_R, \mu_w, \)\(^{11}\) are given by the physical problem, while \( \mu_\psi \) is not since the \( \psi \)-expression is an ansatz so \( \mu_\psi \) is just assumed to chosen to be small enough that all manipulations done in the following are valid.

With these expansions we may easily write the residue we are looking for as

\[ \text{Res}_{z=0}(\psi \varphi_R) = \sum_{i=\mu_\psi}^{-\mu_R-1} \psi_i \hat{\varphi}_{R,-i-1} . \]  

The challenging part is solving eq. (C.2) to identify the expressions for the \( \psi_i \).

C.1.1 The case \( n = 2 \)

**Solving the nPDE.** We first discuss the case \( n = 2 \), and later we extend it to arbitrary \( n \)-forms. Solving eq. (C.2) corresponds to solving \( \Delta = 0 \) with

\[ \Delta := \hat{\nabla}_{\omega_1} \hat{\nabla}_{\omega_2} \psi - \hat{\varphi}_L . \]  

\(^{10}\)One might consider why \( w \) has these log terms. Clearly such terms will appear if \( u \) has a factor of \( z \gamma \), and in addition the requirement is that the derivatives of \( w \) are Laurent polynomials, not that \( w \) itself is. This is what rules out terms of the form \( z_i \log(z_j) \) as such terms do not have all derivatives being Laurent. This is also why the branch choice of the logarithms is irrelevant, since potential factors of \( 2\pi i \) will vanish when taking the derivatives.

\(^{11}\)\( \mu_w \) will be 0 for all problems where \( u = \prod B^\gamma \) as for the Feynman integrals in Baikov representation discussed in this paper. Yet we keep it general in this appendix since the theory is valid also for other functional forms for \( u \). For instance a \( u \) containing a factor of \( e^{i/2} \) will give \( \mu_w = -1 \).
This corresponds to

\[ \Delta = \partial_{1} \partial_{2} \psi + (\partial_{1} \partial_{2} w) \psi + (\partial_{1} w)(\partial_{2} \psi) + (\partial_{2} w)(\partial_{1} \psi) + (\partial_{1} w)(\partial_{2} w) \psi - \hat{\varphi}_{L} \] (C.7)

where we have used the obvious notation \( \partial_{i} := \partial_{z_{i}} \). Inserting the expansions we get

\[ \Delta = \sum_{i=\mu_{\psi}}^{\infty} (i_{1}i_{2} \psi_{i} - \hat{\varphi}_{L, i-1}) z_{1}^{i_{1}-1} z_{2}^{i_{2}-1} \]

\[ + \sum_{i=\mu_{\psi}}^{\infty} \sum_{j=\mu_{w}}^{\infty} (j_{1}j_{2} w_{j} + i_{2}(j_{1} w_{j} + \delta_{j_{1}, 0} v_{1}) + i_{1}(j_{2} w_{j} + \delta_{j_{2}, 0} v_{2})) \psi_{i} z_{1}^{i_{1}+j_{1}-1} z_{2}^{i_{2}+j_{2}-1} \]

\[ + \sum_{i=\mu_{\psi}}^{\infty} \sum_{j=\mu_{w}}^{\infty} \sum_{k=\mu_{w}}^{\infty} (j_{1} w_{j} + \delta_{j_{1}, 0} v_{1})(k_{2} w_{k} + \delta_{k_{2}, 0} v_{2}) \psi_{i} z_{1}^{i_{1}+j_{1}+k_{1}-1} z_{2}^{i_{2}+j_{2}+k_{2}-1}. \] (C.8)

Changing the summation variables we may rewrite this as

\[ \Delta = \sum_{s=\mu_{\psi}+n\mu_{w}}^{\infty} \Delta_{s} z^{s-1} \] (C.9)

where\(^{12}\)

\[ \Delta_{s} := \sum_{t=\mu_{\psi}}^{s-n\mu_{w}} R(s-t, t) \psi_{t} - \hat{\varphi}_{L, s-1} \] (C.10)

with

\[ R(\alpha, \beta) = (\beta_{1}+v_{1})(\beta_{2}+v_{2})\delta_{\alpha, 0} + (\alpha_{1}\alpha_{2} + \alpha_{1}(\beta_{2}+v_{2}) + \alpha_{2}(\beta_{1}+v_{1})) w_{\alpha} \]

\[ + \sum_{j=\mu_{w}}^{\infty} (\alpha_{1} j_{1} j_{2} w_{\alpha} - j_{1} w_{j}) \] (C.11)

which reduces to the expression given in eq. (4.12) in the case where \( \mu_{w} = 0 \). In general this function \( R \) may be interpreted as \( R(\alpha, \beta) \) being the coefficient in \( \Delta \) of \( \psi_{\beta} z^{\alpha+\beta-1} \).

Having \( \Delta = 0 \) in general requires that each term in the \( z \)-expansion of eq. (C.9) is zero independently. This means that our task now is to solve \( \Delta_{s} = 0 \) for each value of \( s \).

This can be done. If we define

\[ \tilde{\mu}_{\psi} := \mu_{L} - n\mu_{w} + 1 \] (C.12)

we find the recursive solution

\[ \psi_{q} = 0 \] if \( q \not\geq \tilde{\mu}_{\psi} \)

\[ \psi_{q} = \left( \hat{\varphi}_{L, q+n\mu_{w}-1} - \sum_{r: \mu_{\psi} \leq r < q} Q(q, r) \psi_{r} \right) / Q(q, q) \] if \( q \geq \tilde{\mu}_{\psi} \) (C.13)

\(^{12}\)In eqs. (C.9) and (C.10) we have reinserted \( n \) reflecting the way the corresponding equations will look in the \( n \)-variate case.
where
\[ Q(q, r) := R(q-r+n\mu_w, r) \] (C.14)
and where the exact definitions of \( \geq \) and other binary operators applied to the index vectors are given in section C.4. The quantity \( \mu_\psi \) may be interpreted as the “correct” value for \( \mu_\psi \), or at least as a value for which \( \mu_\psi \neq \mu_\psi \) would make it impossible for the above derivation to go through.

The recursive solution given by eq. (C.13) can be useful on its own to find \( \psi \)-solutions to use as arguments of the residue function using for instance eq. (C.5), but let us continue the derivation in order to find a closed expression.

**Evaluation of the residue.** The next step is to re-express eq. (C.13) in a form in which only the coefficients of the individual \( \hat{\varphi}_{L,i} \) are recursive. This can be done as
\[ \psi_q = \sum_{\kappa=\mu_\psi}^{q} \hat{\varphi}_{L,\kappa} \Psi(q, q-\kappa) \] (C.15)
with
\[ \Psi(q, \tau) = \begin{cases} 0 & \text{if } \tau \not\geq 0 \\ 1/Q(q, q) & \text{if } \tau = 0 \\ -\sum_{i, 0 < i < \tau} Q(q-i, q-\tau) \Psi(q, i) / Q(q-r, q-r) & \text{if } \tau > 0 \end{cases} \] (C.16)
or alternatively
\[ \Psi(q, \tau) = \begin{cases} 0 & \text{if } \tau \not\geq 0 \\ 1/W(q, 0) & \text{if } \tau = 0 \\ -\sum_{i, 0 < i < \tau} W(q-\tau-i, 0) \Psi(q, i) / W(q-\tau, 0) & \text{if } \tau > 0 \end{cases} \] (C.17)
where \( W(\eta, \lambda) := Q(\eta+\lambda, \eta) \) or correspondingly
\[ W(\eta, \lambda) = R(\lambda+n\mu_w, \eta). \] (C.18)

Eq. (C.17) is a recursive expression that only involves the coefficients of the expansion of \( w \). The recursion can be solved with the result
\[ \Psi(q, \tau) = \sum_{\sigma \in \text{VC}(\tau)} (-1)^{\sigma_{i}} \prod_{i=1}^{\sigma_{i}} W_1(q-\tau+\sum_{j<i} \sigma_j, 0) / W(q-\tau, 0) \] (C.19)
where \( \text{VC} \) refers to the vector compositions defined and discussed in section 4.1. By combining eqs. (C.15) and (C.5), and changing the summation variables we get
\[ \text{Res}(\psi \hat{\varphi}_R) = \sum_{\tau=0}^{n\mu_w-\mu_L-\mu_R-2} \sum_{h=\mu_\psi}^{(-\mu_R-\tau-1} \Psi(h+\tau, \tau) \hat{\varphi}_{L, h+n\mu_w-1} \hat{\varphi}_{R, -h-\tau-1} \] (C.20)
and finally we may insert the expression for \( \Psi \) from eq. (C.19) and we get the most general form of the algebraic expression:

\[
\text{Res}(\psi \varphi_R) = \sum_{\tau=0}^{n \mu_w - n \mu_L - n \mu_R - 2} \sum_{\sigma \in \text{VC}(\tau)} \sum_{h=n \mu_L - n \mu_w - 1}^{n \mu_R - \tau - 1} Y(h, \tau, \sigma) \hat{\varphi}_{L, h+n \mu_w - 1} \hat{\varphi}_{R, -h - \tau - 1}
\] (C.21)

where

\[
Y(h, \tau, \sigma) := \frac{(-1)^{|\sigma|}}{W(h+\tau, 0)} \prod_{i=1}^{|\sigma|} \frac{W(h + \sum_{j<i} \sigma_j, \sigma_i)}{W(h + \sum_{j<i} \sigma_j, 0)}.
\] (C.22)

We notice that \( \tau \) is not really needed as an argument to \( Y \) since \( \tau = \sum_j \sigma_j \).

### C.1.2 Beyond \( n = 2 \)

Even though the derivation in the previous section was done in the case of \( n = 2 \), it will go through in the same way for different values. In particular the final result eq. (C.21) will look the same for all values of \( n \). The main exceptions are eqs. (C.6), (C.7), (C.8), (C.11). Looking briefly at some other cases, for \( n = 1 \) we have

\[
\Delta = \hat{\nabla}_w \psi - \hat{\varphi}_L = (\partial_2 \psi) + (\partial_2 w) \psi - \hat{\varphi}_L
\] (C.23)

which will correspond to

\[
R(\alpha, \beta) = (\beta + w) \delta_{\alpha, 0} + \alpha w_\alpha
\] (C.24)

as in eq. (4.11). Likewise for \( n = 3 \) we have

\[
\Delta = \hat{\nabla}_w \hat{\nabla}_w \hat{\nabla}_w \psi - \hat{\varphi}_L
\] (C.25)

\[
= (\partial_1 w)(\partial_2 \psi) + (\partial_2 w)(\partial_1 \psi) + (\partial_3 w)(\partial_1 \psi) + (\partial_1 w)(\partial_3 \psi) + (\partial_3 w)(\partial_2 \psi) + (\partial_2 w)(\partial_3 \psi) + (\partial_2 w)(\partial_3 \psi) + (\partial_1 w)(\partial_2 \psi)
\]

which after a similar derivation yields

\[
R(\alpha, \beta) = (\beta_1 + v_1)(\beta_2 + v_2)(\beta_3 + v_3) \delta_{\alpha, 0}
\]

\[
\quad + \sum_{j=\mu_w}^{\alpha-\mu_w} (\alpha_1 - j_1)(j_2 + \beta_3 + v_3) + (\alpha_2 - j_2)(j_1 + \beta_1 + v_1) + (\alpha_3 - j_3)(j_2 + \beta_2 + v_2) \) w_{\alpha-j} w_j
\]

\[
\quad + \sum_{j=\mu_w}^{\alpha-2\mu_w} \sum_{l=\mu_w}^{\alpha-j-\mu_w} (\alpha_1 - j_1 - l_1)(j_2 l_3 w_{\alpha-j-l} w_l)
\] (C.27)

which reduces to eq. (4.13) in the case of \( \mu_w = 0 \).

Going through the derivation for various values of \( n \) allows us to realize the pattern and find the general expression for \( R \) which is given in the following.
For $R$ in the $n$-variate case, we may write
\[ R(\alpha, \beta) = \sum_{k=0}^{n} R_{n,k} \]  
(C.28)

where $R_{n,k}$ contains the terms with $k$ $w$-factors and correspondingly $k-1$ sums. We find it convenient to use distinct expressions for $R_{n,k}$ for $k = 0$, $k = 1$, and $k \geq 2$. Defining
\[ \lambda_i := \beta_i + v_i \]  
(C.29)

we have
\[ R_{n,0} = \left( \prod_{i=1}^{n} \lambda_i \right) \delta_{\alpha,0} \]  
(C.30)
\[ R_{n,1} = \left( \left( \prod_{i=1}^{n} (\lambda_i + \alpha_i) \right) - \left( \prod_{i=1}^{n} \lambda_i \right) \right) w_{\alpha} \]  
(C.31)
\[ R_{n,k} = \sum_{i_2 = \mu_w}^{\alpha-(k-1)\mu_w} \cdots \sum_{i_k = \mu_w}^{\alpha-\sum_{q=2}^{k-1} i_q - (k-1)\mu_w} \cdots \sum_{i_k = \mu_w}^{\alpha-\sum_{q=2}^{k} i_q - \mu_w} S_{n,k} \left( \alpha-\sum_{q=2}^{k} i_q, i_2, \ldots, i_k, \lambda \right) w_{\alpha} w_{i_1} \cdots w_{i_k} \]  
(C.32)

where we have defined
\[ S_{n,k}(i_1, i_2, \ldots, i_k, \lambda) := \sum_{s \in \text{KSN}(n,k)} \left( \prod_{l \in \text{CS}(n,s)} \lambda_l \right) \prod_{y=1}^{k} \prod_{j \in s_y} i_{y,j}. \]  
(C.33)

Here we have $\text{KSN}(n,k)$ being defined as the set of all sets with size $k$ of (non-empty and non-overlapping) subsets of the set of integers from 1 to $n$. An example being
\[ \text{KSN}(3,2) = \left\{ \{\{1\}, \{2\}\}, \{\{1\}, \{3\}\}, \{\{2\}, \{3\}\}, \{\{1\}, \{2, 3\}\}, \{\{2\}, \{1, 3\}\}, \{\{3\}, \{1, 2\}\} \right\} \]  
(C.34)
where it should be possible to see the structure reflected in the single-sum term of eq. (C.27).

The other set-function $\text{CS}(n,s)$ gives the counter-set to one of these sets, that is
\[ \text{CS}(n,s) := \{1, \ldots, n\} \big/ \bigcup_{i} s_i \]  
(C.35)

with an example being
\[ \text{CS}(3, \{\{1\}, \{3\}\}) = \{2\}. \]  
(C.36)

We should point out that it is possible to write eq. (C.32) nicer and more symmetrically at the cost of introducing an extra sum and a delta function, that is\(^{14}\)
\[ R_{n,k} = \sum_{i_1 = \mu_w}^{\infty} \cdots \sum_{i_k = \mu_w}^{\infty} \delta\left( \sum_{j} i_j \right) \alpha S_{n,k}(i_1, \ldots, i_k, \lambda) w_{i_1} \cdots w_{i_k} \]  
(C.37)

where evaluating the $i_1$-sum would bring back eq. (C.32).

\(^{13}\)We could in principle get the $k = 1$ expression as a special case of eq. (C.32).

\(^{14}\)For practical implementations one can use finite upper limits and replace $\infty \rightarrow \alpha - (k-1)\mu_w$. 

\[ \text{– 32 –} \]
C.2 Algebraic expression in the rescaling frameworks

In section 2.5 it was discussed how to simplify the computation of the intersection number by rescaling the quantities on which it depends, locally at each intersection point. Two different cases were presented, right rescaling and left rescaling, which we will discuss below.

**Right rescaling.** Right rescaling is defined by the rescalings and renamings given in eq. (2.37)

\[ \hat{\phi}_L \rightarrow \hat{\phi} = \hat{\phi}_L \hat{\phi}_R, \quad \hat{\phi}_R \rightarrow 1, \quad u \rightarrow u_R = u/\hat{\phi}_R, \quad \psi \rightarrow f. \]  

(C.38)

Eq. (C.21) may then be used for these rescaled variables, with \( \hat{\phi}_L, i, \hat{\phi}_R, i, w_i, \) and \( v_j \) interpreted as the coefficients in the expansions of the rescaled variables \( \hat{\phi}, 1, \) and \( \log(u_R) \), and \( \mu_L, \mu_R, \mu_w \) as the corresponding powers of the leading coefficients. But the expansion of 1 is of course trivial, corresponding to \( \hat{\phi}_R, i = \delta_{i,0} \) and \( \mu_R = 0 \). This allows us to do the \( h \)-sum of eq. (C.21) giving

\[ \text{Res}(f) = \sum_{\tau=0}^{n|\mu_w|-\mu_L-2} \sum_{\sigma \in \mathcal{VC}(\tau)} Y(-\tau-1, \tau, \sigma) \hat{\phi}_{n\mu_w-\tau-2} \]  

and inserting the expression for \( Y \) the residue becomes

\[ \text{Res}(f) = \sum_{\tau=0}^{n|\mu_w|-\mu_L-2} \sum_{\sigma \in \mathcal{VC}(\tau)} (-1)^{|\sigma|} |\sigma| U(\sigma, \tau - \sum_{j<i} \sigma_j) \prod_{i=1}^{\tau} U(0, \tau - \sum_{j<i} \sigma_j) \hat{\phi}_{n\mu_w-\tau-2} \]  

(C.40)

where

\[ U(\alpha, \beta) = W(-\beta-1, \alpha) = R(\alpha+n\mu_w, -\beta-1). \]  

(C.41)

Inserting \( \mu_w = 0 \), eq. (C.40) reduces to the algebraic expression discussed in the main text in eq. (4.8).

**Left rescaling.** One may go through the same steps for the left rescaling framework, which is defined in terms of the renamings and rescalings defined in eq. (2.39)

\[ \hat{\phi}_L \rightarrow 1, \quad \hat{\phi}_R \rightarrow \hat{\phi} = \hat{\phi}_L \hat{\phi}_R, \quad u \rightarrow u_L = u\hat{\phi}_L, \quad \psi \rightarrow g. \]  

(C.42)

Inserting this in eq. (C.21), we get

\[ \text{Res}(g\hat{\phi}_L\hat{\phi}_R) = \sum_{\tau=0}^{n|\mu_w|-\mu_L-2} \sum_{\sigma \in \mathcal{VC}(\tau)} (-1)^{|\sigma|} |\sigma| \hat{U}(\sigma, \sum_{j<i} \sigma_j) \prod_{i=1}^{\tau} \hat{U}(0, \sum_{j<i} \sigma_j) \hat{\phi}_{n\mu_w-\tau-2} \]  

(C.43)
with

\[ \tilde{U}(\alpha, \beta) = W(\beta+1-n\mu_w, \alpha) = R(\alpha+n\mu_w, \beta+1-n\mu_w). \]  
(C.44)

Eq. (C.43) is the algebraic expression in the left rescaling framework.

C.3 Additional details

Expansion maxima. In eqs. (C.3), (C.4) we wrote the expansions of \( \hat{\phi}_L \), \( \hat{\phi}_R \), and \( w \) as formal expansions from the corresponding \( \mu \) up to \( \infty \). Yet we do not need infinitely many terms in the series, there is a maximal value above which the terms do not contribute to the result of eq. (C.21). Considering eq. (C.21) as well as the general \( n \) expression for \( R \) given in and below eq. (C.28) we may deduce what these maximal values are. Using the notation \( M \) in parallel with that used for the minimum values \( \mu \) we may write an expansion only containing the terms that are actually needed:

\[ \hat{\phi}_L = \sum_{i=\mu_L}^{M_L} \hat{\phi}_{L,i} z^i \quad \hat{\phi}_R = \sum_{i=\mu_R}^{M_R} \hat{\phi}_{R,i} z^i \quad w = \sum_{i=\mu_w}^{M_w} w_i z^i + \sum_{j=1}^{n} v_j \log(z_j) \]  
(C.45)

where we have

\[ M_L = n\mu_w-\mu_R-2, \quad M_R = n\mu_w-\mu_L-2, \quad M_w = (n+1)\mu_w-\mu_L-\mu_R-2. \]  
(C.46)

This fact is useful for implementation purposes, since it tells which coefficients to extract before applying the algebraic expression. Applying one of the rescaled versions of the algebraic expression does not change these limits if expressed in terms of the original \( \varphi_L \) and \( \varphi_R \).

The univariate case. In the univariate case there are two simplifications taking place which deserve discussion. The first is that the vector compositions VC reduce to standard compositions as discussed in section 4.1. The second is that only one \( \omega \) is present which makes it desirable to express the result in terms of expansions of that object, replacing the \( w \)-expansion of eq. (C.4) with its separately treated log-terms. That is

\[ \omega = \sum_{i=\mu_w}^{\infty} \omega_i z^i \]  
(C.47)

with the relation between the two expansions being \( \omega_i = (i+1)w_{i+1} \) except for \( \omega_{-1} = v_1 \). We also have \( \mu_w = \mu_w - 1 \). Inserting this in eq. (C.21) we get

\[ \text{Res}(\psi \varphi_R) = \sum_{\tau=0}^{\mu_w-\mu_\phi-\mu_\xi-1} \sum_{\sigma \in \text{comp}(\tau)} \sum_{h=\mu_\phi-\mu_\omega}^{\mu_\xi-\tau-1} Y(h, \tau, \sigma) \hat{\phi}_{L,h+\mu_\omega} \hat{\phi}_{R,-h-\tau-1} \]  
(C.48)
\[ Y(h, \tau, \sigma) = \frac{(-1)^{|\sigma|}}{W(h+\tau, 0)^{|\sigma|}} \prod_{i=1} W \left( h + \sum_{j<i} \sigma_j, \sigma_i \right) \]  

(C.49)

as in eq. (C.22) but with

\[ W(\eta, \lambda) = \eta \delta_{\lambda+\mu, -1} + \omega_{\lambda+\mu}. \]  

(C.50)

**Further discussion.** It is worthwhile to look at the special case in which everything (e.g. \( \varphi_L, \varphi_R, \) and all \( \omega_i \)) has simple poles only. This corresponds to \( \mu_L = \mu_R = -1 \) and \( \mu_w = 0 \). Inserting this in eq. (C.21) we get that each of the sums contain one term only, leaving

\[ \text{Res}(\psi \varphi_R) = \frac{\hat{\varphi}_{L, -1} \hat{\varphi}_{R, -1}}{\prod_{i=1}^n v_i}. \]  

(C.51)

Combining this with eq. (2.20) we get an expression for the intersection number similar to an expression from refs. [2, 8].

Additionally it is worth noticing that that the intersection number is *linear* in both \( \varphi_L \) and \( \varphi_R \) which may be seen from the integral definition in eq. (2.15). That property is easy to see in the algebraic expression as it is given by eq. (C.21). But in the rescaling framework linearity becomes hidden. For instance looking at the solution in the right-rescaling framework as it is given by eq. (4.8), linearity in \( \varphi_L \) is obvious. Linearity in \( \varphi_R \), however, becomes obscured as the \( \varphi_{R, i} \)-terms mix with the \( \omega_i \)-factors, which enter the expressions in a very non-linear fashion.

### C.4 Comparing index vectors

In this appendix and in section 4 we used the notation for the \( n \)-variate formula eq. (4.8) where the sums and products are parameterized by index vectors written in bold, of length \( n \) and containing integers. Such vectors can be compared using the so-called *product order*, let us briefly review how it is done. If we let \( a \) and \( b \) be two such index vectors of length \( n \), the relations we use in this paper are defined as in table 6, and some are illustrated in figure 7. We see that \(< \) and \( > \) are not equivalent, for instance it is true that \( (1, 2) < (2, 0) \) while it is false that \( (1, 2) > (2, 0) \). We notice that the index vectors form a *partially ordered set* under the \( \leq \) operator.
Figure 7. Illustration of some of the binary operators defined in table 6. The shaded areas cover the integer values for which the illustrated relation is true.

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