DIFFERENTIAL MODULAR FORMS ATTACHED TO NEWFORMS MOD \( p \)

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Abstract. In a previous paper \[7\] we attached to classical complex newforms \( f \) of weight 2 certain \( \delta_p \)-modular forms \( f^\sharp \) (in the sense of \[4, 6\]) of order 2 and weight 0; the forms \( f^\sharp \) can be viewed as “dual” to \( f \) and played a key role in some of the applications of the theory \[8\]. The aim of this paper is to provide a higher weight version of this “\( \sharp \)-duality” by attaching to classical newforms mod \( p \), \( \mathcal{f} \), of weight \( \kappa \) between 3 and \( p \), \( \delta_\pi \)-modular forms \( f^\sharp \) of order 2 and weight \( -\kappa' \), with \( \kappa' \) between 1 and \( p - 2 \).

1. Introduction and main result

In a series of papers beginning with \[3\] an arithmetic analogue of differential equations was developed in which derivatives of functions were replaced by Fermat quotient operators \( \delta_\pi \); the latter act on complete discrete valuation rings \( R_\pi \) of unequal characteristic and uniformizer \( \pi \), by the formula

\[
\delta_\pi x := \phi(x) - x^p \pi,
\]

where \( \phi : R_\pi \to R_\pi \) is a ring endomorphism whose reduction mod \( \pi \) is the \( p \)-power Frobenius. In particular spaces of \( \delta_\pi \)-modular forms were introduced and studied (cf. \[4, 6\] for \( \pi = p \) and \[10\] for arbitrary \( \pi \)).

A basic construction in \[7\] attaches to classical newforms \( f = \sum a_n q^n \) over \( \mathbb{C} \), of level \( N \) (i.e. on \( \Gamma_1(N) \)) and weight 2, some \( \delta_p \)-modular forms \( f^\sharp \) of level \( N \), order 2, and weight 0, with \( \delta_p \)-Fourier expansion nicely expressible in terms of \( f(\cdot^{-1})(q) := \sum a_n q^n \). The forms \( f^\sharp \) in \[7\] are arithmetic-differential objects that have no classical analogue but, rather, can be viewed as “dual” to the classical objects \( f \); by the way, the forms \( f^\sharp \) introduced in \[7\] played a key role in \[8\] where the theory of arithmetic differential equations was used to prove finiteness results for Heegner-like points.

The aim of this paper is to extend this “\( \sharp \)-duality” to the higher weight case. Specifically let \( N \) be an integer with \( (N, p) = 1, N > 4 \), let \( k \) be an algebraic closure of \( \mathbb{F}_p \), and let \( \mathcal{f} \) be a newform of level \( N \), over \( k \), of weight \( \kappa \), with \( 3 \leq \kappa \leq p \). Then, by a construction due to Serre \[14\], there exists a newform \( f \) of level \( N \) over \( \mathbb{C} \), of weight 2, and weight 0, with \( \delta_p \)-Fourier expansion nicely expressible in terms of \( f(\cdot^{-1})(q) := \sum a_n q^n \). The forms \( f^\sharp \) in \[7\] are arithmetic-differential objects that have no classical analogue but, rather, can be viewed as “dual” to the classical objects \( f \); by the way, the forms \( f^\sharp \) introduced in \[7\] played a key role in \[8\] where the theory of arithmetic differential equations was used to prove finiteness results for Heegner-like points.

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completion). On the other hand $\phi$ acts on $K_\pi[[q]]^\infty := \cup_n K_\pi[[q, \delta_\pi q, ..., \delta_\pi^n q]]$ by $\phi(q) = q^2 + \pi \delta_\pi q, \phi(\delta_\pi q) = (\delta_\pi q)^2 + \pi \delta_\pi^2 q$, etc. In particular $K_\pi[[q]]^\infty$ has a natural structure of left module over the polynomial ring $K_\pi[\phi]$. So in particular, for any polynomial $P(\phi) \in K_\pi[\phi]$ and any series $g(q) \in K_\pi[[q]]$ (e.g., for $g = f^{(-1)}(q)$) it makes sense to consider the series $P(\phi)g(q) \in K_\pi[[q]]^\infty$. The above concepts will be reviewed in detail in the body of the paper. We will then prove the following:

**Theorem 1.1.** For any newform $\mathcal{f}$ over $k$, of level $N$ and weight $\kappa$, with $3 \leq \kappa \leq p$, there exists a non-zero $\delta_\pi$-modular form $f^\sharp$ of level $N$, order 2, and weight $-\kappa'$, with $\kappa'$ a conjugate of $\kappa$, such that $f^\sharp$ has a $\delta_\pi$-Fourier expansion of the form

$$E(f^\sharp) = \sum_\sigma P_\sigma(\phi)(f^{(-1)}(q))^\sigma,$$

with $P_\sigma(\phi)$ polynomials in $K_\pi[\phi]$, $f$ a Serre lift of $\mathcal{f}$, and $\sigma$ running through the set of all embeddings of $K_f$ into $\mathbb{C}$.

The strategy of proof of our Theorem is as follows. The Serre lift $f$ being of weight 2 and level $Np$ one can use the method in [7] to attach to it an element $f^\sharp$ in the ring of $\delta_\pi$-modular forms of level $Np$ and weight 0. Using a construction from [10] one can interpret this $f^\sharp$ as a sum of $\delta_\pi$-modular forms of level $N$ and weights $0, -1, -2, ..., -p + 2$. Finally, an analysis of the action of the diamond operators shows that this sum of $\delta_\pi$-modular forms reduces to one term only; that term turns out to have weight $-\kappa'$ with $\kappa'$ conjugate to $\kappa$. Note that each given $\kappa$ has a priori several conjugates and the theorem does not tell which of these conjugates gives the weight of $f^\sharp$; be that as it may the weight $-\kappa'$ of $f^\sharp$ is always non-zero (in contrast to the weight of $f^\sharp$ in [7, 10] which is equal to 0). Finally note that, as in [7], $f^\sharp$ in our Theorem 1.1 is not unique.

The plan of the paper is as follows. In section 2 we review $\pi$-jet spaces and $\delta_\pi$-modular forms following [3, 10]. In section 3 we review (and complement) some (complex and arithmetic) facts about classical modular forms following [13, 10]. In section 4 we construct our forms $f^\sharp$. In section 5 we use some of the tools developed in the paper to construct a remarkable homomorphism. Indeed, we will recall, in that section, the definition of the ring $S_\pi^\infty$ of Igusa $\delta_\pi$-modular functions of level $N$ introduced in [9] and we will prove:

**Theorem 1.2.** There is a natural homomorphism from the ring of $\delta_\pi$-modular forms of level $Np$ and weight 0 to the ring $S_\pi^\infty \otimes_{R_\pi} R_\pi$. This homomorphism commutes with $\delta_\pi$ and raises orders by 1.

Cf. Theorem 5.1 for a more precise statement.

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2. Review of differential modular forms

In this section we review the basic definitions of $\pi$-jet spaces and $\delta_\pi$-modular forms introduced in [3, 4, 10]. Throughout this paper $p \geq 5$ is a fixed prime. For any
ring $A$ we denote the $p$-adic completion of $A$ by $\widehat{A}$. We denote by $R_p = \mathbb{Z}_p^u$ the $p$-adic completion of the maximum unramified extension of $\mathbb{Z}_p$. We set $K_p = R_p\mathbb{Q}$ (fraction field of $R_p$) and $k = R_p/pR_p$ (residue field of $R_p$); so $k$ is an algebraic closure of $\mathbb{F}_p$.

We need a ramified version of the above (which is slightly more general than the one encountered in § 10). Indeed let $F/\mathbb{Q}$ be a normal finite extension with ring of integers $O_F$ and let $\varphi \subset O_F$ be a prime dividing $p$. There exists a (not necessarily unique) ring automorphism $\phi = \phi_\varphi$ of $O_F$ preserving $\varphi$ and inducing the $p$-power Frobenius on $O_F/\varphi$. Let $\hat{O}_\varphi$ be the completion of the localization of $O$ at $\varphi$ and $\pi \in O_F$ a uniformizer of $O_\varphi$. Then $\phi : O_F \to O_F$ induces an automorphism of $O_\varphi$ whose reduction mod $\pi$ is the $p$-power Frobenius on $O_\varphi/(\pi)$. For any discrete valuation ring $O'$ which is a finite unramified extension of $O_\varphi$, $O'$ is étale over $O_\varphi$ so $\phi : O_\varphi \to O_\varphi$ lifts uniquely to a ring automorphism $\phi : O' \to O'$ whose reduction mod $\pi$ is the $p$-power Frobenius. Taking limit and completion we get an automorphism $\phi$ of the ring $\widehat{O}_\varphi^w$, completion of the maximum unramified extension of $O_\varphi$, whose reduction mod $\pi$ is the $p$-power Frobenius on the residue field. Let $K_\pi = K_p(\pi)$ and let $R_\pi$ be the valuation ring of $K_\pi$. Since $R_\pi \subset \widehat{O}_\varphi^w$ is an unramified extension of complete discrete valuation rings with the same residue field it follows that $R_\pi = \widehat{O}_\varphi^w$ so $\phi$ is an automorphism of $R_\pi$. We let $e$ be the ramification index of $R_\pi \subset R_\pi$. Note that the resulting embedding $\rho : O_F \to R_\pi$ depends on $F$ and $\varphi$ only.

Consider the polynomial

$$C_\pi(X, Y) := \pi^{-1}(X^p + Y^p - (X + Y)^p) \in \mathbb{Z}[p/\pi][X, Y].$$

A $\pi$-derivation from an $\mathbb{Z}[p/\pi]$-algebra $A$ into an $A$-algebra $B$ is a map $\delta_\pi : A \to B$ such that $\delta_\pi(1) = 0$ and

$$\delta_\pi(x + y) = \delta_\pi x + \delta_\pi y + C_\pi(x, y)$$

$$\delta_\pi(xy) = x^p \cdot \delta_\pi y + y^p \cdot \delta_\pi x + \pi \cdot \delta_\pi x \cdot \delta_\pi y,$$

for all $x, y \in A$. For a $\pi$-derivation we denote by $\phi : A \to B$ the map $\phi(x) = x^p + \pi \delta_\pi x$; then $\phi$ is a ring homomorphism. A $\delta_\pi$-prolongation sequence is a sequence $S^* = (S^n)_{n \geq 0}$ of $\mathbb{Z}[p/\pi]$-algebras $S^n$, $n \geq 0$, together with $\mathbb{Z}[p/\pi]$-algebra homomorphisms $\varphi : S^n \to S^{n+1}$ and $\pi$-derivations $\delta_\pi : S^n \to S^{n+1}$ (where $S^{n+1}$ is viewed as an $S^n$-algebra via $\varphi$) such that $\delta_\pi \circ \varphi = \varphi \circ \delta_\pi$ on $S^n$ for all $n$. A morphism of $\delta_\pi$-prolongation sequences, $u^* : S^* \to S^*$ is a sequence $u^n : S^n \to S^n$ of $\mathbb{Z}[p/\pi]$-algebra homomorphisms such that $\delta_\pi \circ u^n = u^{n+1} \circ \delta_\pi$ and $\varphi \circ u^n = u^{n+1} \circ \varphi$. Let $W$ be the ring of polynomials $\mathbb{Z}[\phi]$ in the indeterminate $\phi$. For $w = \sum_{i=0}^{r} a_i \phi^i \in W$ (respectively for $w$ with $a_i \geq 0$), $S^*$ a $\delta_\pi$-prolongation sequence, and $x \in (S^0)^\times$ (respectively $x \in S^0$) we can consider the element $x^w := \prod_{i=0}^{r} (\varphi^{-1} \phi^i(x))^{a_i} \in (S^r)^\times$ (respectively $x^w \in S^r$). On the other hand we may consider the $\pi$-derivation $\delta_\pi : R_\pi \to R_\pi$ given by $\delta_\pi x = (\phi(x) - x^p)/\pi$. One can consider the $\delta_\pi$-prolongation sequence $R_\pi^*$ where $R_\pi^r = R_\pi$ for all $n$. By a $\delta_\pi$-prolongation sequence over $R_\pi$ we understand a prolongation sequence $S^*$ equipped with a morphism $R_\pi^* \to S^*$. From now on all our $\delta_\pi$-prolongation sequences are assumed to be over $R_\pi$. For any affine $R_\pi$-scheme of finite type $Y = Spec A$ there exists a (unique) $\delta_\pi$-prolongation sequence, $A^* = (A^n)_{n \geq 0}$, with $A^0 = A$ such that for any $A^*$-prolongation sequence $B^*$ and any $R_\pi$-algebra homomorphism $u : A \to B^0$ there exists a unique morphism of $\delta_\pi$-prolongation sequences $u^* : A^* \to B^*$ with $u^0 = u$. We define the $\pi$-jet spaces
\( J^n_\pi(Y) \) of \( Y \) as the formal schemes \( J^n_\pi(Y) := Spf \hat{A}^n \). This construction globalizes to the case when \( Y \) is not necessarily affine (such that the construction commutes, in the obvious sense, with open immersions). A \( \delta_\pi \)-function on \( Y \) (of order \( n \)) is an element \( f \in \mathcal{O}(J^n_\pi(Y)) \), equivalently a morphism \( f : J^n_\pi(Y) \to \hat{A}^1 \) over \( R_\pi \); such a map induces a map \( f_* : Y(R_\pi) \to R_\pi \). (Indeed any \( R_\pi \)-point of \( Y \) canonically lifts, by the universality property, to an \( R_\pi \)-point of \( J^n_\pi(Y) \) which can then be mapped by \( f_* \) to an \( R_\pi \)-point of the affine line.) If \( Y \) is smooth \( f_* \) uniquely determines \( f \); we then write \( f_* = f \) and we also refer to \( f_* \) as a \( \delta_\pi \)-function. If \( Y = G \) is a smooth group scheme over \( R_\pi \) then a \( \delta_\pi \)-character of \( G \) is a homomorphism \( f : J^n_\pi(G) \to \hat{G} \); such an \( f \) induces a group homomorphism \( f = f_* : G(R_\pi) \to R_\pi \), still referred to as a \( \delta_\pi \)-character. Note that the \( \pi \)-jet spaces \( J^n_\pi(Y) \) only depend on the \( \pi \)-adic completion of \( Y \) and not on \( Y \) so one can introduce \( \pi \)-jet spaces \( J^n_\pi(Y) \) attached to formal \( \pi \)-adic schemes \( Y \) over \( R_\pi \) which are locally \( \pi \)-adic completions of schemes of finite type over \( R_\pi \).

Next we review differential modular forms (\( \delta \)-modular forms and \( \delta_\pi \)-modular forms) following [4] [10]. Recall the modular curve \( X_1(N) \) over \( \mathbb{Z}_p \) with \( (N, p) = 1 \), \( N > 4 \); it is a smooth curve and it carries a line bundle \( \mathcal{O} \). This construction globalizes to the case when \( \mathbb{Z}_p \) is replaced by \( \mathbb{Q}_p \). A \( \mathbb{Q}_p \)-element \( \mathbb{Q}_p \) induces a map \( f \) which we continue to denote by \( \mathbb{Q}_p \). We then write \( \mathbb{Q}_p \) and we also refer to \( \mathbb{Q}_p \) as a \( \delta_\pi \)-function. If \( Y = G \) is a smooth group scheme over \( R_\pi \) then a \( \delta_\pi \)-character of \( G \) is a homomorphism \( f : J^n_\pi(G) \to \hat{G} \); such an \( f \) induces a group homomorphism \( f = f_* : G(R_\pi) \to R_\pi \), still referred to as a \( \delta_\pi \)-character. Note that the \( \pi \)-jet spaces \( J^n_\pi(Y) \) only depend on the \( \pi \)-adic completion of \( Y \) and not on \( Y \) so one can introduce \( \pi \)-jet spaces \( J^n_\pi(Y) \) attached to formal \( \pi \)-adic schemes \( Y \) over \( R_\pi \) which are locally \( \pi \)-adic completions of schemes of finite type over \( R_\pi \).

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3. Review of classical modular forms

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3.1. Complex modular forms [12]. We denote by \( M_m(\Gamma_1(M), \mathbb{C}) \) the space of classical complex modular forms of weight \( m \) on \( \Gamma_1(M) \) and by \( T_m(n) \) and \( (n), \) \( n \geq 1, \) the Hecke operators and the diamond operators acting on this space. The diamond operators belong to the ring generated by the Hecke operators. An eigenform is an element \( f \in M_m(\Gamma_1(M), \mathbb{C}) \) which is an eigenvector for all \( T_m(n) \) (and hence also for all \( (n) \)). Let \( S_m(\Gamma_1(M), \mathbb{C}) \) be the space of cusp forms. An eigenform has \( a_1 \neq 0. \) If a cuspidal eigenform \( f = \sum a_nq^n \) is normalized by \( a_1 = 1 \) then...
For each $n \geq 1$ and there exists a Dirichlet character $\epsilon \bmod M$ such that $\langle n \rangle f = \epsilon(n)f$ for $n \geq 1$. Let $f$ be a normalized cuspidal eigenform as above; one lets $O_f \subset \mathbb{C}$ denote the ring generated by all $a_n$ (it is finitely generated) and one lets $K_f$ be the fraction field of $O_f$. The Dirichlet character $\epsilon$ attached to $f$ takes then values in $O_f$. A newform is a cuspidal normalized eigenform such that the system \{a_n\} \not\subset [M] does not occur for a cuspidal normalized eigenform form of the same weight and smaller level dividing $M$. Let $X_1(M)$ be the complete complex modular curve for $\Gamma_1(M)$; as an algebraic curve it is defined over $\mathbb{Q}$ hence its Jacobian $J_1(M)$ is also defined over $\mathbb{Q}$. The diamond operators act on $X_1(M)$ and are defined over $\mathbb{Q}$ and hence induce endomorphisms of $J_1(M)$ defined over $\mathbb{Q}$. We recall the following basic:

**Theorem 3.1. (Eichler-Shimura)** Let $f \in S_2(\Gamma_1(M), \mathbb{C})$ be a newform with character $\epsilon$. Then there exists an abelian variety $A_{f/\mathbb{Q}}$ over $\mathbb{Q}$, a ring homomorphism

$$
\iota : O_f \to \text{End}(A_{f/\mathbb{Q}}/\mathbb{Q}),
$$

and a dominant homomorphism

$$
\pi : J_1(M)_{\mathbb{Q}} \to A_{f/\mathbb{Q}},
$$

defined over $\mathbb{Q}$, such that for any prime $l$ we have commutative diagrams

$$
\begin{array}{cccc}
J_1(M)_{\mathbb{Q}} & \xrightarrow{T(l)^*} & J_1(M)_{\mathbb{Q}} & \xrightarrow{(l)^*} \\
\downarrow \pi & & \downarrow \pi & \\
A_{f/\mathbb{Q}} & \xrightarrow{\iota(a_l)} & A_{f/\mathbb{Q}} & \xrightarrow{\pi(\iota(l))} A_{f/\mathbb{Q}}
\end{array}
$$

Moreover the image of the pull-back map

$$
\pi^* : H^0(A_{f/\mathbb{C}}, \Omega) \to H^0(J_1(N)_{\mathbb{C}}, \Omega) \simeq S_2(\Gamma_1(N), \mathbb{C})
$$

is the $\mathbb{C}$-linear span of $\{f^* \mid \sigma : K_f \to \mathbb{C}\}$.

Here $T(l)^*, (l)^* \in \text{End}(J_1(M)_{\mathbb{Q}}/\mathbb{Q})$ are the naturally induced Hecke and diamond operators.

**3.2. Model of $X_1(Np)$.** For modular curves and forms we use the standard notation from [14]. Let $N > 4$, $(N, p) = 1$. Recall that the modular curve $X_1(Np)$ over $\mathbb{C}$ has a model (still denoted by $X_1(Np)$ in what follows) over $\mathbb{Z}[1/N, \zeta_p]$ considered in [14], p. 470. Recall some of the main properties of $X_1(Np)$. First $X_1(Np)$ is a regular scheme proper and flat of relative dimension 1 over $\mathbb{Z}[1/N, \zeta_p]$ (where $\zeta_p$ is a primitive $p$-th root of unity) and is smooth over $\mathbb{Z}[1/Np, \zeta_p]$. Also the special fiber of $X_1(Np)$ at $p$ is a union of two smooth projective curves $I$ and $I'$ crossing transversally at a finite set $\Sigma$ of points. Furthermore $I$ is isomorphic to the Igusa curve $I_1(N)$ in [14], p. 160, so $I$ is the smooth compactification of the curve classifying triples $(E, \alpha, \beta)$ with $E$ an elliptic curve over a scheme of characteristic $p$, and $\alpha : \mu_N \to E$, $\beta : \mu_p \to E$ are embeddings (of group schemes). Similarly $I'$ is the smooth compactification of the curve classifying triples $(E, \alpha, b)$ with $E$ an elliptic curve over a scheme of characteristic $p$, and $\alpha : \mu_N \to E$, $b : \mathbb{Z}/p\mathbb{Z} \to E$ are embeddings. Finally $\Sigma$ corresponds to the supersingular locus on the corresponding curves.

Let now $F_0 = \mathbb{Q}(\zeta_p)$ and $\pi_0 = 1 - \zeta_p$ and consider the embedding of $\mathbb{Z}[\zeta_N, \zeta_p, 1/N]$ into $R_{\pi_0}$ (hence of $\mathbb{Z}[\zeta_N, 1/N]$ into $R_p$). Denote by $X_1(Np)_{R_{\pi_0}}$ the base change of
parameterizes pairs \((E, \alpha)\) consisting of elliptic curves \(E\) with an embedding \(\alpha : \mu_N \to E\). The morphism \(X_1(Np) \to X_1(N)\) over \(\mathbb{C}\) induces a morphism

\[ e : X_1(Np)_{R_{\pi_0}} \setminus \Sigma \to X_1(N)_{R_{\pi_0}} \setminus (ss) \]

over \(R_{\pi_0}\), where \((ss)\) is the supersingular locus in the closed fiber of \(X_1(N)_{R_{\pi_0}}\). (See [10] for details.) Let \(X \subset X_1(N)_{R_{\pi_0}}\) be an affine open set, \(X_{R_{\pi_0}} := X \otimes_{R_{\pi_0}} X_1(N)_{R_{\pi_0}} \setminus (ss)\) its base change to \(R_{\pi_0}\), and \(X_1 := e^{-1}(X_{R_{\pi_0}})\). Denote by \(\mathcal{X}_{R_{\pi_0}}\) the \(\pi_0\)-adic completion of \(X_{R_{\pi_0}}\). Also note that the \(\pi_0\)-adic completion of \(X_1\) has two connected components; let \(\mathcal{X}_1\) be the component whose reduction mod \(\pi_0\) is contained in \(I \setminus \Sigma\). We get a morphism \(e : \mathcal{X}_1 \to \mathcal{X}_{R_{\pi_0}}\).

Now the diamond operators \(\langle n \rangle\) on \(X_1(Np)_{\mathbb{Q}}\) give rise to diamond operators \(\langle n \rangle_N\) (for \(n \in (\mathbb{Z}/N\mathbb{Z})^\times\)) and \(\langle d \rangle_p\) (for \(d \in (\mathbb{Z}/p\mathbb{Z})^\times\)) on \(X_1(Np)_{\mathbb{Q}}\) corresponding to the Chinese remainder theorem spitting \((\mathbb{Z}/Np\mathbb{Z})^\times \simeq (\mathbb{Z}/N\mathbb{Z})^\times \times (\mathbb{Z}/p\mathbb{Z})^\times\). Recall from [13] that \(G := (\mathbb{Z}/p\mathbb{Z})^\times\) acts via \(\langle \_ \rangle_p\) on the covering \(X_1(Np) \to X_1(N)\) over \(\mathbb{Z}[1/N, \zeta_p]\); this action preserves the Igusa curve \(I\) and induces on \(I\) the usual diamond operators. So \(G\) acts on the covering \(e : \mathcal{X}_1 \to \mathcal{X}_{R_{\pi_0}}\). Also, as a matter of notation, if \(\epsilon : (\mathbb{Z}/Np\mathbb{Z})^\times \to \mathbb{C}\) is a Dirichlet character we denote by \(\epsilon_N : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}\) and \(\epsilon_p : (\mathbb{Z}/p\mathbb{Z})^\times \to \mathbb{C}\) the induced Dirichlet characters.

### 3.3. Model of \(J_1(Np)\)

In the discussion below we continue to assume that \(\pi_0 = 1 - \zeta_p\) and we continue to consider the embedding embedding \(\mathbb{Z}[\zeta_N, \zeta_p, 1/N] \to R_{\pi_0}\). Let \(A_{K_{\pi_0}} = J_1(Np)_{K_{\pi_0}}\) be the Jacobian of \(X_1(Np)_{K_{\pi_0}}\) over \(K_{\pi_0}\). Let \(A_{R_{\pi_0}}\) be the Néron model of \(A_{K_{\pi_0}}\) over \(R_{\pi_0}\); cf. [2], p. 12. By the Néron property the Abel-Jacobi map \(X_1(Np)_{R_{\pi_0}} \to J_1(Np)_{R_{\pi_0}}\) that sends \(\infty\) into 0 can be extended to a morphism

\[ X_1(Np)_{R_{\pi_0}} \setminus \Sigma \to A_{R_{\pi_0}} \]

and hence induces a morphism

\[ \mathcal{X}_1 \to \mathcal{A} := (A_{R_{\pi_0}})^\ast \]

where \(A_{R_{\pi_0}}^0\) is the connected component of \(A_{R_{\pi_0}}\). Note that, since \(\pi_0 = 1 - \zeta_p\), one has by [2], pp. 246 and 267, that \(A_{R_{\pi_0}}^0\) coincides with \(Pic^0_{X_1(Np)_{R_{\pi_0}}}\) and is a semi-abelian scheme whose special fiber modulo its maximum torus is a product of the Jacobians of \(I\) and \(I'\) respectively.

### 3.4. Pulling back to arbitrary \(R_\pi\)

Consider an arbitrary number field \(F\) as in section 2, containing \(F_0 = \mathbb{Q}(\zeta_p)\), and let \(\pi\) be as in section 2. We will allow ourselves to enlarge \(F\) and change \(\pi\) repeatedly in what follows. Denote by

\[ (3.1) \quad X_{R_\pi}, X_1, \mathcal{X}_1, A_{R_\pi}, A_{R_\pi}^0, \mathcal{A} \]

the objects obtained from the corresponding objects over \(R_{\pi_0}\) by base change via \(R_{\pi_0} \subset R_\pi\). Note that the latter extension of rings can be ramified so \(A_{R_\pi}\) is not necessarily the Néron model of \(A_{K_\pi}\); but all the schemes (respectively formal schemes) above continue to be smooth over \(R_\pi\) and \(A_{R_\pi}^0\) is semi-abelian, equal to \(Pic^0_{X_1(Np)/R_\pi}\).
3.5. Schemes attached to a fixed complex newform. Let \( f \in S_2(\Gamma_1(Np), \mathbb{C}) \) be a newform and assume we are given an embedding \( \rho : \mathcal{O}_f \to R_\pi \) with \( \pi \) as above. Let \( A_{\pi} \to A_f/K_\pi \) be the induced morphism of abelian varieties over \( K_\pi \) obtained by base change from the morphism of abelian varieties over \( \mathbb{Q} \), \( J_1(Np)_{\mathbb{Q}} \to A_f/\mathbb{Q} \). Let \( \hat{A}_f/R_\pi \) be the Néron model of \( A_f/K_\pi \), let \( A^0_f/R_\pi \) be the connected component of the latter and let \( A^0_f/R_\pi \to \hat{A}_f/R_\pi \) be the induced morphism. By the theory of Raynaud extensions [13], pp. 33-34, there exists a group scheme \( C \) over \( R_\pi \) for all primes \( \pi \) such that \( (A^0_f/R_\pi)^c \simeq \hat{C} \). So we have an induced homomorphism \( A \to (A^0_f/R_\pi)^c \to \hat{B} \), hence an induced morphism \( X_1 \to \hat{B} \), hence an induced morphism \( J^0_n(X_1) \to J^0_n(B) \). Furthermore note that by the Néron property over \( R_\pi_0 \), plus base change from \( R_\pi_0 \) to \( R_\pi \), one has induced endomorphisms \( T(l)_* \) of \( A_{R_\pi_0}, A^0_{R_\pi_0}, \) and \( A \) over \( R_\pi \) for all primes \( l \). Similarly, by the Néron property over \( R_\pi \) we have a \( \mathbb{O}_f \to \text{End}(A) \) acts on \( A \) and \( A^0 \). Clearly this action is compatible with the Hecke action on \( A^0_\pi \). Similarly, for \( d \in (\mathbb{Z}/p\mathbb{Z})^\times \), the diamond operators act on \( X_1 \) and \( A \). By the functoriality of the Raynaud extension (cf. [13], pp. 33-34) we have an induced action of \( \mathcal{O}_f \) on \( C \) and hence on \( B \); so we have a ring homomorphism \( \iota : \mathcal{O}_f \to \text{End}(B/R_\pi) \).

Assume first \( B \neq 0 \). Then, for any prime \( l \), and any \( d \in (\mathbb{Z}/p\mathbb{Z})^\times \), we have commutative diagrams

\[
\begin{array}{ccc}
A & \xrightarrow{T(l)^*} & A \\
\downarrow & & \downarrow \\
\hat{B} & \xrightarrow{\iota(a_l)} & \hat{B}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
X_1 & \xrightarrow{(d)_\pi} & X_1 \\
\downarrow & & \downarrow \\
A & \xrightarrow{(d)_\pi} & A \\
\downarrow & & \downarrow \\
\hat{B} & \xrightarrow{\iota(\epsilon p(d))} & \hat{B}
\end{array}
\]

On the other hand assume \( B = 0 \); then \( C = T \). So \( \mathcal{O}_f \) acts on \( T \) via a homomorphism \( \iota^\perp : \mathcal{O}_f \to \text{End}(T/R_\pi) \). Enlarging \( R_\pi \) again and replacing the schemes (3.1) by their pull-backs to this new \( R_\pi \) we may (and will!) assume \( T_{K_\pi} \) is split over \( K_\pi \). So the identity component of its locally (!) finite type Néron model has the form \( D = \mathbb{G}^*_m/R_\pi \) and, by the universality property of the Néron model, we get a homomorphism \( T \to D \) over \( R_\pi \). By the Néron property over \( R_\pi \) we have a homomorphism \( \iota^\perp : \mathcal{O}_f \to \text{End}(D/R_\pi) \) and corresponding diagrams:

\[
\begin{array}{ccc}
A & \xrightarrow{T(l)^*} & A \\
\downarrow & & \downarrow \\
\hat{D} & \xrightarrow{\iota^\perp(a_l)} & \hat{D}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
X_1 & \xrightarrow{(d)_\pi} & X_1 \\
\downarrow & & \downarrow \\
A & \xrightarrow{(d)_\pi} & A \\
\downarrow & & \downarrow \\
\hat{D} & \xrightarrow{\iota^\perp(\epsilon p(d))} & \hat{D}
\end{array}
\]

3.6. Lift of Igusa curve to characteristic zero [10]. Recall that \( L \) denotes the line bundle on \( X_1(N)_{\mathbb{R}} \) such that the sections of the powers of \( L \) identify with the modular forms of various weights on \( \Gamma_1(N) \); cf. [14] p. 450. Let \( E_{p-1} \in H^0(X_1(N)_{\mathbb{R}}, L^{p-1}) \) be the normalized Eisenstein form of weight \( p-1 \) and let \((ss)\) be the supersingular locus on \( X_1(N)_{\mathbb{R}} \), defined as the locus of \( E_{p-1} \). Here \( E_{p-1} \) is normalized by the condition that its Fourier expansion has constant term 1. Take
an open covering

\[(3.4) \quad X_1(N) = \bigcup X_1(N)_i\]

such that \(L\) is trivial on each \(X_1(N)_i\) and let \(x_i\) be a basis of \(L\) on \(X_1(N)_i\). Then \(E_p^{-1} = \varphi_i x_i^{p-1}\) where \(\varphi_i \in \mathcal{O}(X_1(N)_i)\). Set \(x_i = u_{ij} x_j, \ u_{ij} \in \mathcal{O}^\times(X_1(N)_{ij}), \ X_1(N)_{ij} = X_1(N)_i \cap X_1(N)_j\). Set \(z_i = x_i^{-1}\) (in \([10]\) this \(z_i\) was denoted by \(t_i\)) and consider the \(R_\pi\)-scheme \(X_1(N)_{\Pi}\) obtained by gluing the schemes

\[X_1(N)_{\Pi} := \text{Spec} \ \mathcal{O}(X_1(N)_{i,R_\pi})[z_i]/(z_i^{p-1} - \varphi_i)\]

via \(z_i = u_{ij}^{-1} z_j\) where

\[X_1(N)_{i,R_\pi} := X_1(N)_i \otimes_{R_\pi} R_\pi.\]

In the discussion below denote by an upper bar the functor \(\otimes k\). Then \(\overline{X_1(N)_{\Pi}}\) is clearly birationally isomorphic (hence isomorphic) to the Igusa curve \(I\) (cf. [14], pp. 460, 461). Now let \(X \subset X_1(N)\) over \(R_\pi\) be an affine open set disjoint from the supersingular locus. Consider the open covering

\[(3.5) \quad X = \bigcup X_i\]

where \(X_i = X_1(N)_i \cap X\). Consider the \(R_\pi\)-scheme \(X_{\Pi} \subset X_1(N)_{\Pi}\) obtained by gluing the schemes \(X_{\Pi} := \text{Spec} \ \mathcal{O}(X_{1,R_\pi})[z_i]/(z_i^{p-1} - \varphi_i)\) via \(z_i = u_{ij}^{-1} z_j\) (where \(X_{1,R_\pi} := X_i \otimes_{R_\pi} R_\pi\)). Note that \(z_i^{p-1} - \varphi_i\) are monic polynomials whose derivatives are invertible in \(\mathcal{O}(X_{1,R_\pi})[z_i]/(z_i^{p-1} - \varphi_i)\) so \(X_{\Pi}\) is an \(\text{étale}\) cover of \(X_{1,R_\pi}\) and is the pull back of the latter in \(X_1(N)_{\Pi}\). Note that the scheme \(X_{\Pi} = X_{\Pi} \otimes k\) is isomorphic to \(X_{\Pi} = X_{\Pi} \otimes k\). Recall from [10] that the isomorphism \(X_{\Pi} \simeq X_{\Pi}\) lifts uniquely to an isomorphism \((X_{\Pi})^* \simeq X_{\Pi}\). This follows immediately by applying the standard Lemma [3.2] below to \(S := \mathcal{O}(X_i), X_i = \hat{X}_i, T = \mathcal{O}(X_{\Pi}), X_{\Pi} = \mathcal{O}^c(X_i)\).

\[\square\]

**Lemma 3.2.** Let \(S \rightarrow T\) be a homomorphism of flat \(\pi\)-adically complete \(R_\pi\)-algebras, let \(f \in S[z]\) be a monic polynomial and assume we have a homomorphism \(\overline{\mathcal{O}}[z]/(f) \rightarrow \overline{T}\) such that \(df/dz\) is invertible in \(\overline{S}[z]/(f)\). Then \(\overline{\mathcal{O}}\) lifts uniquely to a homomorphism \(\sigma : S[z]/(f) \rightarrow T\). If \(\overline{\mathcal{O}}\) is an isomorphism then so is \(\sigma\).

**Proof.** One defines \(\sigma\) by sending the class \(\xi \in S[z]/(f)\) of \(z\) into the unique solution \(\tau \in T\) (which exists by Hensel) of the equation \(f = 0\) with \(\overline{\mathcal{O}}[\overline{\xi}] = \overline{\mathcal{O}}\), where the upper bar means class mod \(\pi\).

\[\square\]

### 3.7. Serre lifts

Let \(M\) be any positive integer. In what follows a classical modular form over a ring \(B\), of weight \(\kappa\), on \(\Gamma_1(M)\) will be understood in the sense of [11] as a rule that attaches to any \(B\)-algebra \(C\) and any triple consisting of an elliptic curve \(E/C\), an embedding \(\mu_{M,C} \rightarrow E[M]\), an embedding \(M_\kappa(\Gamma_1(M), B)\) the \(B\)-module of all these forms and by \(S_\kappa(\Gamma_1(M), B)\) the module of corresponding cusp forms. For the concepts of Hecke operators, diamond operators, and newforms used below we refer to [14]. Let \(\Theta_p : (\mathbb{Z}/p\mathbb{Z})^* \rightarrow \mathbb{Z}_p^\times \subset R_\pi^\times\) be the Teichmüller character. We have the following “reduction to weight 2” result [14], p. 478:

**Proposition 3.3.** Let \((N, p) = 1, N > 4\) and let \(f \in S_\kappa(\Gamma_1(N), k)\) be a newform, where \(3 \leq \kappa \leq p\). Then there exists a newform \(f' \in S_2(\Gamma_1(Np), \mathbb{C})\) with character \(\epsilon\) and there exists an embedding \(\rho : \mathcal{O}_f \rightarrow R_\pi\) such that:
1) The image of \( f \) via \( \mathcal{O}_f \xrightarrow{\Delta} R_\pi \to k \) is \( \overline{T} \);

2) \( \rho \circ \epsilon_p = \Theta_p^{g-2} \).

In the above \( \epsilon_p \) is the “\( p \)-part” in the decomposition \( \epsilon = \epsilon_p \epsilon_N \). We shall refer to the pair \((f, \rho)\), or simply to \( f \), as a Serre lift of \( \overline{T} \). Recall from [3] that Serre lifts are not unique.

### 4. Construction of differential modular forms

#### 4.1. \( \delta_\pi \)-characters.

Let \( B \) be an abelian scheme over \( R_\pi \). Recall that by a \( \delta_\pi \)-character of \( B \), of order \( n \), we understand a homomorphism \( J^n_\pi(B) \to \widehat{\mathbb{G}}_m \).

**Proposition 4.1.** Consider an abelian scheme \( B \) over \( R_\pi \) of relative dimension \( g \). Then the \( R_\pi \)-module of \( \delta_\pi \)-characters of \( B \) of order \( 2 \) has rank \( \geq g \).

In [3] this was proved under the assumption that the ramification index of \( R_\pi \) satisfies \( e \leq p - 2 \); the method there does not seem to extend to arbitrary \( e \). The argument below, for general \( e \), follows a method in [5].

**Proof.** Let \( \mathcal{F} \in R_\pi[[T_1, T_2]]^g \) be the formal group law of \( B \) and \( l(T) \in K_\pi[[T]]^g \) its logarithm; here \( T, T_1, T_2 \) are \( g \)-tuples of variables. For any \( n \geq 1 \) let

\[
L^n_\pi = \frac{1}{\pi} l(\phi^n(T))|_{T=0} \in K_\pi[[T, \delta_\pi T, \ldots, \delta^n_\pi T]]
\]

where \( \phi: K_\pi[[T, \delta_\pi T, \ldots]] \to K_\pi[[T, \delta_\pi T, \ldots]] \),

\[
\phi(T) = T^p + \pi \delta_\pi T, \quad \phi(\delta_\pi T) = (\delta_\pi T)^p + p \delta^2_\pi T, \ldots
\]

We claim that there is an integer \( \nu \geq 1 \) such that

\[
\pi^\nu L^n_\pi \in R_\pi[[T]][\delta_\pi T, \ldots, \delta^n_\pi T];
\]

and, by the way, if in addition \( e \leq p - 1 \) then one can take \( \nu = 0 \). Indeed by [15], p. 64, we have

\[
l(T) = \sum_{|\alpha| \geq 1} \frac{A_\alpha}{|\alpha|} T^\alpha
\]

for some \( A_\alpha \in R_\pi \). Hence setting \( G^n_\pi = \frac{1}{\pi} \phi^n(T)|_{T=0} \) we have

\[
L^n_\pi = \frac{1}{\pi} \sum_{|\alpha| \geq 1} \phi^n(A_\alpha) \frac{\pi^{\nu-1}}{|\alpha|} (G^n_\pi)^\alpha
\]

Now if \( v_\pi \) is the \( \pi \)-adic valuation we have

\[
v_\pi \left( \frac{\pi^{\nu-1}}{|\alpha|} \right) \geq |\alpha| - 1 - e \cdot \log_p |\alpha|
\]

So the above goes to \( \infty \) for \( |\alpha| \to \infty \); moreover the above is \( \geq 0 \) if \( e \leq p - 1 \). This proves the claim. Let \( N^2 \) be the kernel of \( J^2_\pi(B) \to \hat{B} \). Exactly as in [5] the \( g \)-tuples \( \pi^\nu L^n_\pi \) and \( \pi^\nu L^n_\pi \) define 2\( g \) homomorphisms \( N^2 \to \widehat{\mathbb{G}}_\alpha \) that are \( R_\pi \)-linearly independent. So \( \text{Hom}(N^2, \widehat{\mathbb{G}}_\alpha) \) has rank \( \geq 2g \). We are done by considering the standard exact sequence (cf. [3]):

\[
0 = H^0(B, \mathcal{O}) \to \text{Hom}(J^2_\pi(B), \widehat{\mathbb{G}}_\alpha) \to \text{Hom}(N^2, \widehat{\mathbb{G}}_\alpha) \to H^1(B, \mathcal{O}) \simeq R_\pi^g.
\]

\( \square \)
Remark 4.2. Recall from [3] that $G_m = \text{Spec} \mathbb{Z}[x, x^{-1}]$ has a remarkable $\delta_\pi$-character, i.e. a homomorphism $\psi : J^1_\pi(G_m) \to \widehat{G}_a$ defined by

$$\psi = \pi^m \sum_{n \geq 1} (-1)^{n-1} \frac{\pi^n}{n} \left( \frac{\delta_\pi x}{xp} \right)^n$$

for an appropriate $m \in \mathbb{Z}$. So if $D = G_m^\psi$ then there is a canonical system $\psi_1, ..., \psi_\nu : J^1_\pi(D) \to \widehat{G}_a$ of $\delta_\pi$-characters obtained by composing $\psi$ above with the canonical projections onto the factors.

4.2. Semi-invariant $\delta_\pi$-characters. Assume the notation of Section 3.5 and let $\rho : \mathbb{K}_f(\zeta_N, \zeta_p) \to \mathbb{K}_\pi$ be an embedding. (One can always choose $\pi$ such that such a $\rho$ exists.) So $\rho(\mathcal{O}_f[1/N, \zeta_N, \zeta_p]) \subset R_\pi$. Then, exactly as in [7], Proposition 4.5 and using the same argument as on the top of page 992, loc. cit., we get:

Lemma 4.3. Assume $B \neq 0$ and let $\mathcal{T} = \iota(\mathcal{O}_f) \subset \text{End}(B/R_\pi)$. Then there exists a non-zero $\delta_\pi$-character $\psi \in \mathcal{O}(J^2_\pi(B))$ and a ring homomorphism $\chi : \mathcal{T} \to R_\pi$ such that $\psi \circ \tau = \chi(\tau) \cdot \psi$ for all $\tau \in \mathcal{T}$.

Similarly we get:

Lemma 4.4. Assume $B = 0$ (hence $D$ is defined and non-zero). Let $\mathcal{T} = \iota^\dagger(\mathcal{O}_f) \subset \text{End}(D/R_\pi)$. Then there exists a non-zero $\delta_\pi$-character $\psi \in \mathcal{O}(J^2_\pi(D))$ and a ring homomorphism $\chi : \mathcal{T} \to R_\pi$ such that $\psi \circ \tau = \chi(\tau) \cdot \psi$ for all $\tau \in \mathcal{T}$.

4.3. $\delta_\pi$-modular forms from $X_l$. Recall from [10] the following:

Proposition 4.5. There is a natural isomorphism

$$\mathcal{O}(J^1_\pi(X_l)) \simeq \bigoplus_{\kappa=0}^{p-2} M^\kappa_\pi(-\kappa).$$

Via this isomorphism $M^\kappa_\pi(-\kappa)$ is identified with the subspace of all $\varphi \in \mathcal{O}(J^1_\pi(X_l))$ on which the diamond operators $(d)_p$ with $d \in (\mathbb{Z}/p\mathbb{Z})^\times$ act by the character $d \mapsto \Theta_p(d)^{-\kappa}$ (i.e. $\langle d \rangle_p \varphi = \Theta_p(d)^{-\kappa} \varphi$).

The ring $\mathcal{O}(J^0_\pi(X_l))$ can be referred to as the ring of $\delta_\pi$-modular forms of level $Np$, order $n$, and weight $0$.

Let now $f$ be as in section 3.5 and let $\rho : \mathcal{O}_f[\zeta_N] \to R_\pi$ be an embedding. Assume the notation in section 3.5 and let $\epsilon = \epsilon_N \epsilon_p$ be the character of $f$.

Assume first $B \neq 0$.

Let $\psi : J^2_\pi(B) \to \widehat{G}_a = \widehat{\mathbb{A}}^1$ and $\chi$ be as in Lemma 4.3. So for $d \in (\mathbb{Z}/(p\mathbb{Z})^\times$ we have

$$\psi \circ \iota(\epsilon_p(l)) = \chi(l(\epsilon_p(d))) \cdot \psi.$$

Then if $\beta : X_l \to \widehat{B}$ is the vertical morphism in equation 3.2 and if $\beta^2 : J^2_\pi(X_l) \to J^2_\pi(B) = J^2_\pi(B)$ is the induced morphism set

$$f^\sharp = \psi \circ \beta^2 : J^2_\pi(X_l) \to \widehat{\mathbb{A}}^1$$

We get

$$\langle d \rangle_p f^\sharp = \psi \circ \beta^2 \cdot \langle d \rangle_p = \psi \circ \iota(\epsilon_p(d)) \circ \beta^2 = \chi(l(\epsilon_p(d))) \cdot \psi \circ \beta^2 = \chi(l(\epsilon_p(d))) \cdot f^\sharp.$$
Set $\rho':=\chi \circ \pi: \mathcal{O}_f \to R_\pi$. Then $f^\sharp$ has weight $-\kappa'$ for some integer $0 \leq \kappa' \leq p-2$ if and only if

$$\rho' \circ \epsilon_p = \Theta_p^{-\kappa'}.$$ 

If in addition $f$ and $\rho$ are as in Proposition 3.3 so in particular $\overline{f}$ has weight $\kappa$, then

$$\rho \circ \epsilon_p = \Theta_p^{\kappa-2}.$$ 

Let $g \in (\mathbb{Z}/p\mathbb{Z})^\times$ be a generator; so $\Theta_p(g) \in \mathbb{Z}_p^\times$ is a primitive $(p-1)$-th root of unity in $\mathbb{Q}_p$ and $\epsilon_p(g)$ is a (not necessarily primitive) $(p-1)$-th root of unity in $\mathbb{C}$. Then $\rho, \rho'$ induce embeddings $r, r': \mathbb{Q}(\epsilon_p(g)) \to \mathbb{Q}(\Theta_p(g))$ hence $r(\epsilon_p(g)) = \Theta_p(g)^{\kappa-2}$. Now since $\mathbb{Q}(\Theta_p(g))/\mathbb{Q}$ is Galois there is an automorphism $\gamma$ of this extension such that $\gamma \circ r = r'$; by the structure of the Galois group we must have $\gamma(\Theta_p(g)) = \Theta_p(g)^c$ for some integer $c$ coprime to $p-1$. We get

$$\rho'(\epsilon_p(g)) = r'(\epsilon_p(g))) = \gamma(r(\epsilon_p(g)) = \Theta_p(g)^c(\kappa-2)$$

hence $\rho' \circ \epsilon_p = \Theta_p^{c(\kappa-2)} = \Theta_p^{\kappa'}$, where $\kappa'$ is the unique integer between 0 and $p-2$ such that $\kappa' \equiv c(2 - \kappa) \mod p-1$. Then we have proved the following:

**Proposition 4.6.** Let $\overline{f} \in S_\kappa(\Gamma_1(N), k)$ be a newform as in Proposition 3.3 and let $(f, \rho)$ be a Serre lift (cf. loc.cit.). Assume $B \neq 0$ and let $f^\sharp$ be as in (4.1). Then $f^\sharp \in M^2_{n}(\kappa')$ where $\kappa'$ is a conjugate of $\kappa$.

Similarly if $B = 0$ (hence $D$ is defined and non-zero) we let $\psi: J^1_n(D) \to \widehat{G}_n = \widehat{\mathbb{A}}^1$ and $\chi$ be as in Lemma 4.4. Then if $\beta: \mathcal{X}_1 \to \widehat{D}$ is the vertical morphism in equation (4.3) and if $\beta_1: J^1_n(\mathcal{X}_1) \to J^2_n(\widehat{D}) = J^2_n(D)$ is the induced morphism we set

$$f^\sharp = \psi \circ \beta_1: J^1_n(\mathcal{X}_1) \to \widehat{\mathbb{A}}^1$$

and we get:

**Proposition 4.7.** Let $\overline{f} \in S_\kappa(\Gamma_1(N), k)$ be a newform as in Proposition 3.3 and let $(f, \rho)$ be a Serre lift (cf. loc.cit.). Assume $B = 0$ and let $f^\sharp$ be as in (4.2). Then $f^\sharp \in M^1_n(\kappa')$ where $\kappa'$ is a conjugate of $\kappa$.

The proof of Theorem 1.1 will be concluded if we prove the following Proposition, where we assume that the embedding $\rho: \mathcal{O}_f \to R_\pi$ is extended (which is always possible by changing $\pi$) to an embedding of the Galois closure of $K_f$ into $K_\pi$.

**Proposition 4.8.** Let $f^\sharp$ be as in (4.1) or (4.2). Then $f^\sharp$ is non-zero and its $\delta_\pi$-Fourier expansion has the form

$$E(f^\sharp) = \sum_{\sigma} P_\sigma(\phi)(f^{(1)}(q)) \sigma,$$

with $P_\sigma(\phi)$ polynomials in $K_\pi[\phi]$ and $\sigma$ running through the set of all embeddings of $K_f$ into $\mathbb{C}$.

**Proof.** We have $f^\sharp \neq 0$ because $\psi \neq 0$ and the images of $\beta: \mathcal{X}_1 \to \widehat{B}$ and $\beta: \mathcal{X}_1 \to \widehat{D}$ generate $\widehat{B}$ and $\widehat{D}$ respectively. The shape of $E(f^\sharp)$ follows directly from the way $f^\sharp$ was defined, exactly as in (7), proof of Theorem 6.3. For convenience we sketch here the argument. Assume $B \neq 0$; the case $B = 0$ is similar (with $D$ playing the role of $B$). Let $g$ be the dimension of $B$, let $\omega_1, ..., \omega_g$ be a basis of global 1-forms on $B$ and consider the formal logarithm $L = L(T_1, ..., T_g)$ of $B$. 
with components \( L_1, \ldots, L_g \) (which are series in the variables \( T_1, \ldots, T_g \)) such that \( \omega_i = dL_i \). One can write

\[
\beta^* \omega_i = \sum_{\sigma} c_{i\sigma} \sum_n a_n^\sigma q^{n-1} dq,
\]

with \( c_{i\sigma} \in R_\pi \). On the other hand, if \( \beta^*(T_i) = \varphi_i(q) \in R_\pi[[q]] \) and \( \varphi = (\varphi_1, \ldots, \varphi_g) \), we have

\[
\beta^* \omega_i = \beta^* \left( \sum_j \frac{dL_i}{dT_j} dT_j \right) = \sum_j \frac{dL_i}{dT_j}(\varphi(q)) \frac{d\varphi_j}{dq} dq = \frac{d}{dq}(L_i(\varphi(q))) dq.
\]

One then gets

\[
L_i(\varphi(q)) = \sum_{\sigma} c_{i\sigma} \sum_n a_n^\sigma q^n = \sum_{\sigma} c_{i\sigma}(f^{(-1)}(q))^\sigma.
\]

On the other hand one has

\[
\psi = \sum_i Q_i(\phi)L_i
\]

for some polynomials \( Q_i(\phi) \in K_\pi[\phi] \). We get

\[
f^t = \beta^* \psi = \sum_i Q_i(\phi) \left( \sum_{\sigma} c_{i\sigma}(f^{(-1)}(q))^\sigma \right) = \sum_{\sigma} \left( \sum_i c_{i\sigma} Q_i(\phi) \right) (f^{(-1)}(q))^\sigma,
\]

and we are done by setting \( P_\sigma(\phi) = \sum_i c_{i\sigma} Q_i(\phi) \).

5. Link with Igusa differential modular functions

The aim of this section is to construct a natural (and somewhat unexpected) ring homomorphism from the ring \( O(J^p_\pi(X)) \) of \( \delta_\pi \)-modular forms of level \( N \), order \( n \), and weight 0 to the ring \( S_{\pi}^{n+1} \otimes_{R_\pi} R_\pi \) where \( S_{\pi}^{n+1} \) is the ring of Igusa \( \delta_\pi \)-modular forms of level \( N \) \cite{1}. We start by reviewing the rings \( S_{\pi}^0 \).

Recall our basic setting in section 2. With \( X \times X_1(N) \) disjoint from the super-singular locus, and with \( L \) and \( V \rightarrow X \) as in section 2 set:

\[
S^n = O(J^p_\pi(X)), \quad M^n = O(J^p_\pi(V)), \quad S^\infty = \lim \rightarrow S^n, \quad M^\infty = \lim \rightarrow M^n;
\]

the elements of \( M^n \) are the \( \delta_\pi \)-modular functions of order \( n \). For \( w \in W \) recall that we let \( M^n(w) \) be the the \( R_\pi \)-module of all \( f \in M^n \) such that \( \lambda \ast f = \lambda^w f \) for \( \lambda \in R_\pi^\times \), where \( \ast \) is the natural action of the multiplicative group; the elements of \( M^n(w) \) are the \( \delta_\pi \)-modular forms of weight \( w \). On the other hand recall the natural \( \delta_\pi \)-expansion map

\[
E : M^n \rightarrow R_p((q))^n := R_p((q))[q', \ldots, q^{(n)}]^-;
\]

its restriction to each \( M^n(w) \) is injective. Let \( S_{\pi}^0 \subset R_p((q))^n \) be the image of \( M^n \) via \( E \) and let

\[
S^\infty_{\pi} = \lim \rightarrow S^0_{\pi} \subset R_p((q))^\infty := \lim \rightarrow R_p((q))^n.
\]

The ring \( S^\infty_{\pi} \) can be referred to as the ring of Igusa \( \delta_\pi \)-modular functions of level \( N \) (or the ring of the \( \delta_\pi \)-Igusa curve of level \( N \) \cite{11}). Recall from \cite{11} and \cite{16}, p. 269, that there is a (necessarily unique) \( f^0 \in M^1(\phi - 1) \) such that \( E(f^0) = 1 \). Moreover we have a congruence

\[
(5.1) \quad f^0 \equiv E_{p-1} \mod p \text{ in } M^1.
\]
Theorem 5.1. There exists a sequence of \(S^0\)-algebra homomorphisms,
\[
O(J^n_\pi(\mathcal{X}_i)) \to S^{n+1}_{\pi} \otimes_{R_\pi} R_\pi,
\]
compatible with each other as \(n\) varies, and commuting with \(\delta_\pi\).

As we shall see these homomorphisms are “natural”. Also if we consider the limit
\[
O(J^\infty_\pi(\mathcal{X}_i)) := \lim_{n \to \infty} O(J^n_\pi(\mathcal{X}_i))
\]
our Theorem 5.1 yields an \(S^0\)-algebra homomorphism, commuting with \(\delta_\pi\),
\[
(5.2) \quad O(J^\infty_\pi(\mathcal{X}_i)) \to S^\infty_{\pi} \otimes_{R_\pi} R_\pi.
\]

Proof of Theorem 5.1. We begin by constructing an \(S^0\)-algebra homomorphism
\[
(5.3) \quad O(\mathcal{X}_i) \to S^1_\pi \otimes_{R_\pi} R_\pi.
\]
Indeed consider a cover as in Equation 3.5 and, using the notation preceding that equation, set \(f^0 = \Phi_i x_i^{\phi_\pi - 1}\) where \(\Phi_i \in S^1_\pi := O(J^0_\pi(\mathcal{X}_i))\). Let
\[
M^1_i = S^1_\pi[x_i, x_i^{-1}, \delta_\pi x_i^\pi] = S^1_\pi[z_i, z_i^{-1}, \delta_\pi z_i]^{\pi}
\]
Then
\[
M^1_i/(f^0 - 1) \cong S^1_\pi[z_i, z_i^{-1}, \delta_\pi z_i]/(z_i^{\phi_\pi - 1} - \Phi_i).
\]
Since the latter is Noetherian and \(p\)-adically complete, hence \(\pi\)-adically complete, and since
\[
(M^1_i/(f^0 - 1)) \otimes_{R_\pi} R_\pi
\]
is finite over it, it follows that \((M^1_i/(f^0 - 1)) \otimes_{R_\pi} R_\pi\) is \(p\)-adically complete, hence \(\pi\)-adically complete. Recall that, setting \(S_i = O(\mathcal{X}_i)\), we have
\[
O(\mathcal{X}_i) = \frac{(S_i \otimes_{R_\pi} R_\pi)[z_i]}{(z_i^{\phi_\pi - 1} - \varphi_i)}, \quad (M^1_i/(f^0 - 1)) \otimes_{R_\pi} R_\pi = \frac{S^1_\pi[z_i, z_i^{-1}, \delta_\pi z_i]}{(z_i^{\phi_\pi - 1} - \Phi_i)}.
\]
Hence
\[
O(\mathcal{X}_i) = \frac{\overline{S}_i[z_i]}{(z_i^{\phi_\pi - 1} - \varphi_i)}, \quad (M^1_i/(f^0 - 1)) \otimes_{R_\pi} R_\pi = \frac{S^1_\pi[z_i, z_i^{-1}, \delta_\pi z_i]}{(z_i^{\phi_\pi - 1} - \Phi_i)}.
\]
Since, by Equation 5.1, \(\overline{\Phi}_i = \varphi_i\) in \(\overline{S}_i\) and \(\overline{S}_i \subset S^1_\pi\) we get natural inclusions, compatible with varying \(i\):
\[
O(\mathcal{X}_i) \subset (M^1_i/(f^0 - 1)) \otimes_{R_\pi} R_\pi
\]
and hence, by Lemma 5.2, unique liftings of the above to homomorphisms
\[
O(\mathcal{X}_i) \to (M^1_i/(f^0 - 1)) \otimes_{R_\pi} R_\pi.
\]
By uniqueness these liftings glue together to give a homomorphism
\[
O(\mathcal{X}_i) \to (M^1_i/(f^0 - 1)) \otimes_{R_\pi} R_\pi.
\]
Composing with the natural map
\[
(M^1_i/(f^0 - 1)) \otimes_{R_\pi} R_\pi \to S^0_{\pi} \otimes_{R_\pi} R_\pi
\]
we get the desired homomorphism 5.3.

Now by the universality property of \((O(J^n_\pi(\mathcal{X}_i)))_{n \geq 0}\) and by the fact that
\[
(S^{n+1}_{\pi} \otimes_{R_\pi} R_\pi)_{n \geq 0}
\]
is a $\delta_n$-prolongation sequence we get that the homomorphism 5.3 induces natural compatible ring homomorphisms commuting with $\delta_n$ as in the statement of the theorem. □

Remark 5.2. It would be interesting to see if the homomorphism 5.2 is injective. On the other hand we claim that its reduction mod $\pi$,

\[(5.4) \quad \mathcal{O}(J^n_\infty(A_i)) \to \mathcal{S}^\infty_\infty,\]

is not injective. Indeed, since $X_\infty$ is étale over $X$ it follows from [3], Proposition 1.4, that

\[\mathcal{O}(J^n_\infty(A_i)) = \mathcal{O}(A_i)[\delta_\pi z_i, \ldots, \delta^n_\pi z_i],\]

hence

\[\mathcal{O}(J^n_\infty(A_i)) = \mathcal{O}(A_i)[\delta_\pi z_i, \ldots, \delta^n_\pi z_i].\]

On the other hand by [9], Introduction, we have

\[\mathcal{S}^\infty_\infty \otimes R_p \mathcal{R}_\pi = \mathcal{S}^\infty_\infty = \mathcal{S}^\infty_\infty[z_i, \delta_p z_i, \delta^2_p z_i, \ldots, (f^\theta - 1, \delta_p(f^\theta - 1), \delta^2_p(f^\theta - 1), \ldots).\]

So the map 5.4 sends $\delta^n_\pi z_i$ into 0 for all $1 \leq n \leq p - 2$; in particular the map 5.4 is not injective.

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