WRONGKIAN-TYPE FORMULA FOR INHOMOGENEOUS $TQ$ EQUATIONS

Rafael I. Nepomechie

It is known that the transfer-matrix eigenvalues of the isotropic open Heisenberg quantum spin-1/2 chain with nondiagonal boundary magnetic fields satisfy a $TQ$ equation with an inhomogeneous term. We derive a discrete Wronskian-type formula relating a solution of this inhomogeneous $TQ$ equation to the corresponding solution of a dual inhomogeneous $TQ$ equation.

Keywords: Bethe ansatz, $TQ$ equation, discrete Wronskian, boundary integrability

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1. Introduction and summary of results

We consider the famous Baxter $TQ$ equation for the closed periodic $XXX$ spin chain of length $N$:

$$T(u)Q(u) = (u^+)^N Q^-(u) + (u^-)^N Q^+(u).$$ (1.1)

Here and hereafter, we use the brief notation $f^\pm(u) = f(u \pm i/2)$ and $f^{\pm\pm}(u) = f(u \pm i)$. It is known that for a given transfer-matrix eigenvalue $T(u)$, Eq. (1.1) can be regarded as a second-order finite-difference equation for $Q(u)$. The eigenvalue $T(u)$ is necessarily a polynomial in $u$ (of degree $N$) because the model is integrable. It is well known that Eq. (1.1) has two independent polynomial solutions [1]. One of them is a polynomial $Q(u)$ of degree $M \leq N/2$ of the form

$$Q(u) = \prod_{k=1}^{M} (u - u_k),$$ (1.2)

whose zeros $\{u_k\}$ are solutions of the Bethe equations that follow directly from (1.1)

$$\left(\frac{u_j + i/2}{u_j - i/2}\right)^N = \prod_{k=1}^{M} \frac{u_j - u_k + i}{u_j - u_k - i}, \quad j = 1, \ldots, M.$$ (1.3)

The other is a polynomial $P(u)$ of degree $N - M + 1 > N/2$ corresponding to Bethe roots “on the other side of the equator.” These two solutions are related by a discrete Wronskian (or Casoratian) formula

$$P^+(u)Q^-(u) - P^-(u)Q^+(u) \propto u^N,$$ (1.4)
where $\propto$ denotes equality up to a multiplicative constant. The existence of a second polynomial solution of the $TQ$ equation is equivalent to the admissibility of the Bethe roots [2], [3]. Using the Wronskian formula, we can succinctly reformulate the $Q$-system for this model [4] (which provides an efficient way to compute the admissible Bethe roots) in terms of $Q$ and $P$ [5], [6].

A generalization of Wronskian formula (1.4) for the open XXX spin chain with diagonal boundary fields was recently obtained [7]:

$$g(u)P^+(u)Q^-(u) - f(u)P^-(u)Q^+(u) \propto u^{2N+1},$$

(1.5)

where $Q(u)$ and $P(u)$ are the respective polynomial solutions of a $TQ$ and a dual $TQ$ equation (see Eqs. (2.7) and (2.12) below). Moreover, the functions $f(u)$ and $g(u)$ are given by (diagonal case)

$$f(u) = (u - i\alpha)(u + i\beta), \quad g(u) = f(-u) = (u + i\alpha)(u - i\beta),$$

(1.6)

where $\alpha$ and $\beta$ are boundary parameters. This result was used in [7] to formulate a $Q$-system for the model.

Our main result is a further generalization of the Wronskian formula to the case of non-diagonal boundary fields:

$$g(u)P^+(u)Q^-(u) - f(u)P^-(u)Q^+(u) = \mu(u)u^{2N+1},$$

(1.7)

where $Q(u)$ and $P(u)$ are the respective polynomial solutions of $TQ$ equation (2.7) and dual $TQ$ equation (2.12), $f(u)$ and $g(u)$ are now given by (2.9), and, most importantly, $\mu(u)$ is a polynomial that satisfies the remarkably simple relation

$$\mu^+(u) - \mu^-(u) = \gamma u(Q(u) - P(u)).$$

(1.8)

In other words, $\mu(u)$ is the discrete integral of $\gamma u(Q(u) - P(u))$. In the diagonal case, $\gamma = 0$; it then follows from (1.8) that $\mu(u) = \text{const}$, and (1.7) hence reduces to (1.5). The appearance of the nontrivial factor $\mu(u)$ in Wronskian-type formulas (1.7) and (1.8) is due to the presence of an inhomogeneous term in the $TQ$ equation for the model [9]–[11]. We expect that this Wronskian-type formula will be useful for formulating a $Q$-system for this model, but this remains a challenge.

In Sec. 2, we briefly review the considered model and its $TQ$ equations and then obtain a dual $TQ$ equation. In Sec. 3, we derive the Wronskian-type formulas (1.7) and (1.8).

### 2. The model and its $TQ$ equations

We consider the isotropic (XXX) open Heisenberg quantum spin-1/2 chain of length $N$ with boundary magnetic fields, whose Hamiltonian is given by

$$H = \sum_{k=1}^{N-1} \vec{\sigma}_k \cdot \vec{\sigma}_{k+1} - \frac{\xi}{\beta} \sigma_1^x - \frac{1}{\beta} \sigma_1^z + \frac{1}{\alpha} \sigma_N^z,$$

(2.1)

where $\vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$ are the usual Pauli matrices and $\alpha$, $\beta$, and $\xi$ are arbitrary real parameters. This model is not $U(1)$-invariant in the nondiagonal case $\xi \neq 0$.

To construct the corresponding transfer matrix, we use the R-matrix (solution of the Yang–Baxter equation) given by the $4 \times 4$ matrix

$$R(u) = \left( u - \frac{i}{2} \right) I + i\mathbb{P}$$

(2.2)

$\mathbb{P}$.

1After completing this work, we learned of a similar result for the closed XXX spin chain with a nondiagonal twist (see Theorem 4.10 in [8].
(where $\mathbb{P}$ is the permutation matrix and $I$ is the identity matrix) and the K-matrices (solutions of boundary Yang–Baxter equations) given by the $2 \times 2$ matrices [12], [13]

$$K^R(u) = \begin{pmatrix} i(\alpha - 1/2) + u & 0 \\ 0 & i(\alpha + 1/2) - u \end{pmatrix},$$

$$K^L(u) = \begin{pmatrix} i(\beta - 1/2) - u & -\xi(u + i/2) \\ -\xi(u + i/2) & i(\beta + 1/2) + u \end{pmatrix},$$

(2.3)

which depend on the boundary parameters $\alpha$, $\beta$, and $\xi$.

The transfer matrix $T(u) = T(u; \alpha, \beta, \xi)$ is given by [14]

$$T(u) = \text{tr}_0 K^L_0(u)M_0(u)K^R_0(u)\tilde{M}_0(u),$$

(2.4)

where $M$ and $\tilde{M}$ are monodromy matrices given by

$$M_0(u) = R_{01}(u)R_{02}(u) \cdots R_{0N}(u), \quad \tilde{M}_0(u) = R_{0N}(u) \cdots R_{02}(u)R_{01}(u).$$

(2.5)

The transfer matrix is constructed to satisfy the fundamental commutativity condition

$$[T(u), T(v)] = 0$$

(2.6)

and the relation $T(-u) = T(u)$. Hamiltonian (2.1) is proportional to $(dT(u)/du)_{u=i/2}$ up to an additive constant.

The eigenvalues $T(u)$ of the transfer matrix $T(u)$ are polynomials in $u$ (as a consequence of (2.6)) and satisfy the $TQ$ equation [9]–[11]

$$-uT(u)Q(u) = g^-(u)(u^+)^{2N+1}Q^--(u) + f^+(u)(u^-)^{2N+1}Q^+- (u - u^+)^{2N+1},$$

(2.7)

where $Q(u)$ is an even polynomial of degree $2N$

$$Q(u) = \prod_{k=1}^{N} (u - u_k)(u + u_k),$$

(2.8)

the functions $f(u)$ and $g(u)$ are given by

$$f(u) = (u - i\alpha)(u\sqrt{1 + \xi^2} + i\beta), \quad g(u) = f(-u) = (u + i\alpha)(u\sqrt{1 + \xi^2} - i\beta),$$

(2.9)

and $\gamma$ is defined by

$$\gamma = -2(1 - \sqrt{1 + \xi^2}).$$

(2.10)

We note the presence of an inhomogeneous term (proportional to $\gamma$) in $TQ$ equation (2.7). In the diagonal case $\xi = 0$, from (2.10), we see that $\gamma = 0$ and the inhomogeneous term hence disappears. In this case, the functions $f(u)$ and $g(u)$ in (2.9) reduce to (1.6).

The transfer matrix transforms under charge conjugation by reflection (negation) of all the boundary parameters:

$$C T(u; \alpha, \beta, \xi) C = T(u; -\alpha, -\beta, -\xi), \quad C = (\sigma^x)^{\otimes N}. $$

(2.11)
We thus obtain a dual $TQ$ equation from (2.7) as in [7] by changing $Q(u) \rightarrow P(u)$ and reflecting the boundary parameters $\alpha \rightarrow -\alpha$, $\beta \rightarrow -\beta$, and $\xi \rightarrow -\xi$, which implies that $f(u)$ and $g(u)$ become interchanged,

$$- uT(u)P(u) = f^-(u)(u^+)^{2N+1}P^--(u) + g^+(u)(u^-)^{2N+1}P^{++}(u) - \gamma u(u^-u^+)^{2N+1}, \quad (2.12)$$

where $P(u)$ is also an even polynomial of degree $2N$,

$$P(u) = \prod_{k=1}^{N} (u - \tilde{u}_k)(u + \tilde{u}_k), \quad (2.13)$$

whose zeros can be regarded as dual Bethe roots. We emphasize that the same eigenvalue $T(u)$ appears in both (2.7) and (2.12).

3. The Wronskian-type formula

We now consider the relation between $Q(u)$ (a solution of $TQ$ equation (2.7) for some transfer-matrix eigenvalue $T(u)$) and the corresponding $P(u)$ (a solution of dual $TQ$ equation (2.12) for the same transfer-matrix eigenvalue $T(u)$). For this, we use the ansatz

$$g(u)P^+(u)Q^-(u) - f(u)P^-(u)Q^+(u) = \mu(u)u^{2N+1}, \quad (3.1)$$

where $f(u)$ and $g(u)$ are given by (2.9) and the function $\mu(u)$ is yet to be determined. This ansatz is motivated by result (1.5) in the diagonal case.

We next define $R(u)$, following the results in [1], as

$$R = \frac{u^{2N+1}}{Q^+Q^-} = \frac{1}{\mu} \left( g \frac{P^+}{Q^+} - f \frac{P^-}{Q^-} \right), \quad (3.2)$$

where the second equality follows from ansatz (3.1). Dividing both sides of $TQ$ equation (2.7) by $QQ^{++}Q^{--}$, we obtain

$$- \frac{uT}{Q^{++}Q^{--}} = f^+R^- + g^-R^+ - \gamma uQR^+R^- \quad (3.3)$$

Substituting $R$ in this equation using the second equality in (3.2) and then multiplying both sides of the resulting equation by $\mu^+\mu^-Q^{++}Q^{--}$, we obtain

$$-uT\mu^+\mu^- = f^+g \frac{PQ^{++}Q^{--}}{Q}(\mu^+ - \mu^- + \gamma uP) - f^+f^-P^--Q^{++}(\mu^+ + \gamma uP) +$$

$$+ g^+g^-P^{++}Q^{--}(\mu^- - \gamma uP) + f^-g^+P^--P^{++}Q(\gamma u). \quad (3.4)$$

Similarly, we define $S(u)$ as

$$S = \frac{u^{2N+1}}{P^+P^-} = \frac{1}{\mu} \left( g \frac{Q^-}{P^-} - f \frac{Q^+}{P^+} \right), \quad (3.5)$$

and divide both sides of dual $TQ$ equation (2.12) by $PP^{++}P^{--}$. We thus obtain

$$-\frac{uT}{P^{++}P^{--}} = f^+S^+ + g^+S^- - \gamma uPS^+S^- \quad (3.6)$$
Substituting the expression for $S$ in this equation using the second equality in (3.5) and then multiplying both sides by $\mu^+\mu^-P^{++}P^{--}$, we obtain

$$-uT\mu^+\mu^- = f^+g^-PQ^{++}Q^{--}(\gamma u) - f^+f^-P^{--}Q^{++}(\mu^- + \gamma uQ) +$$

$$+ g^+g^-P^{++}Q^{--}(\mu^- - \gamma uQ) - f^+g^+\frac{P^{--}P^{++}Q}{P}(\mu^- - \mu^- - \gamma uQ).$$

(3.7)

Equating the right-hand sides of (3.4) and (3.7), we obtain the constraint

$$[\mu^+ - \mu^- + \gamma u(P - Q)]\frac{1}{QP}(gP^+Q^- - fP^-Q^+)^+ (gP^+Q^- - fP^-Q^+)^- = 0,$$

(3.8)

which is obviously satisfied if we set

$$\mu^+ - \mu^- = \gamma u(Q - P)$$

(3.9)

as required by (1.8). For given polynomials $Q(u)$ and $P(u)$, Eq. (3.9) can be solved for a polynomial function $\mu(u)$ up to an arbitrary additive constant.

Result (3.9) can in fact be obtained more directly. Let

$$\mu(u) = \frac{1}{u^{2N+1}}(g(u)P^+(u)Q^-(u) - f(u)P^-(u)Q^+(u)),$$

(3.10)

which is equivalent to (1.7). We multiply $TQ$ equation (2.7) by $P(u)/(u^2-u)^{2N+1}$, multiply dual $TQ$ equation (2.12) by $Q(u)/(u^2-u)^{2N+1}$, and subtract the second equation from the first. The obtained result written in terms of $\mu$ given by (3.10) exactly coincides with (3.9).

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