In this article, results from the previous paper (I) are applied to calculations of squeezed states for such well-known systems as the harmonic oscillator, free particle, linear potential, oscillator with a uniform driving force, and repulsive oscillator. For each example, expressions for the expectation values of position and momentum are derived in terms of the initial position and momentum, as well as in the \((\alpha, z)\)- and in the \((z, \alpha)\)-representations described in I. The dependence of the squeezed-state uncertainty products on the time and on the squeezing parameters are determined for each system.

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1 Introduction

In paper I, we discussed the general problem of the squeezed states for time-dependent systems in one spatial dimension described by the Schrödinger equation

\[ \mathcal{S}_1 \Psi(x, \tau) = 0, \]  

(1)

where the Schrödinger operator, \( \mathcal{S}_1 \), is

\[ \mathcal{S}_1 = \partial_{xx} + 2i\partial_\tau - 2V(x, \tau). \]  

(2)

The interaction, \( V(x, \tau) \), that we considered has the form

\[ V(x, \tau) = g^{(2)}(\tau)x^2 + g^{(1)}(\tau)x + g^{(0)}(\tau), \]  

(3)

where the coefficients, \( g^{(j)}(\tau) \), are differentiable, piecewise continuous, but otherwise arbitrary. The solution space of (1) was denoted by \( \mathcal{F}_{\mathcal{S}_1} \). We obtained the generalized squeezed states for this system and discussed their properties.

However, there are several common, well-known systems subsumed by the potential \( V(x, \tau) \) in (3). Let \( \mathbf{g} \) be the 3-tuple \( \mathbf{g} = (g^{(2)}(\tau), g^{(1)}(\tau), g^{(0)}(\tau)) \). Then, for example, when \( \mathbf{g} = (\frac{1}{2} \omega^2, 0, 0) \), we are dealing with the simple harmonic oscillator (HO). If \( \mathbf{g} = (0, 0, 0) \), then we have a free particle (FP). For the driven harmonic oscillator (DHO), we have \( \mathbf{g} = (\frac{1}{2} \omega^2, g(\tau), 0) \). Two other systems of interest are the linear potential (LP), where \( \mathbf{g} = (0, g(\tau), 0) \), and the repulsive oscillator (RO) for which \( \mathbf{g} = (-\frac{1}{2} \Omega^2, 0, 0) \). For both LP and DHO, we investigate the specific case \( g(\tau) = \kappa/2 \), where \( \kappa \) is a real constant. As is clear from I, all of these systems have isomorphic space-time symmetry algebras \( [1, 2, 3] \). Then, from our general results, we have algebraically calculated solution spaces for all these isomorphic systems and hence obtain their properties.

In Section 2, for convenience, we present a resumé of results from I. The time-dependent functions and symmetry operators for each of the systems mentioned above
are calculated in Section 3. Expectation values and uncertainty products for squeezed states for each of these examples are presented and discussed in Section 4. Finally, we summarize and comment on our results in Section 5.

2 Resumé of General Results

The generators of space-time symmetries were given in Eqs. (6) through (11) in I and will not be repeated here. The specific nature of the $\tau$-dependent solutions of the Schrödinger equation were determined in Section 3 of I. We called these solutions “number-operator states.” Some of the real and complex $\tau$-dependent are important here and we repeat their definitions below. Note that the complexified Lie algebra $\text{os}(1)$ is useful for computing expectation values for position and momentum operators. However, when dealing with specific examples, we have found it easier to compute expectation values in terms of real $\tau$-dependent functions.

The function $\xi$ of $\tau$,

$$\xi(\tau) = \frac{1}{\sqrt{2}}(\chi_1(\tau) + i\chi_2(\tau)), \quad (4)$$

and its complex conjugate, $\bar{\xi}$, are complex solutions of the second order differential equation

$$\ddot{a} + 2g_2(\tau)a = 0. \quad (5)$$

The $\tau$-dependent functions, $\chi_1$ and $\chi_2$, are real, linearly independent solutions of Eq. (5). The properties of these solutions are given in detail in I and in Refs. [1, 2, 4].

The remaining auxiliary $\tau$-dependent functions of interest in this paper are

$$C(\tau) = c(\tau) + C^o, \quad (6)$$

where $C^o$ is a complex integration constant and $c(\tau)$ is the definite integral

$$c(\tau) = \int_0^\tau d\rho \xi(\rho)g^{(1)}(\rho) \quad (7)$$

and

$$\phi_1(\tau) = \xi^2, \quad \phi_2(\tau) = \bar{\xi}^2, \quad \phi_3(\tau) = 2\xi\bar{\xi}, \quad (8)$$
The complex function $C$ can be written \[4\] in terms of real functions $C_1$ and $C_2$

$$C(\tau) = \frac{1}{\sqrt{2}} (C_1(\tau) + iC_2(\tau)),$$

where

$$C_\nu(\tau) = c_\nu(\tau) + C^\circ_\nu$$

and

$$c_\nu(\tau) = \int_0^\tau d\rho \chi_\nu(\rho)g^{(1)}(\rho),$$

for $\nu = 1, 2$. The complex integration constant $C^\circ$ and its complex conjugate are related to the real integration constants, $C^\circ_1$ and $C^\circ_2$ by

$$C^\circ = \frac{1}{\sqrt{2}} (C^\circ_1 + iC^\circ_2).$$

Also, we define the following initial values for the real and complex functions:

$$\xi^o = \xi(0), \quad \bar{\xi}^o = \bar{\xi}(0), \quad \phi^o_3 = \phi_3(0),$$

$$\chi^o_1 = \chi_1(0), \quad \chi^o_2 = \chi_2(0), \quad \hat{\phi}^o_3 = \hat{\phi}_3(0).$$

The coherent and squeezed state parameters are,

$$\alpha = |\alpha|e^{i\delta}, \quad z = re^{i\theta}, \quad r = |z|.$$ 

The expectation values of position and momentum in terms of complex functions and initial position and momentum are given by

$$\langle x(\tau) \rangle = i \left\{ [\bar{\xi}\xi^o - \xi\bar{\xi}^o]p_o - [\dot{\xi}\bar{\xi}^o - \dot{\bar{\xi}}\xi^o]x_o \right\}$$

$$+ i \left( \bar{\xi}\dot{c} - \xi\dot{\bar{c}} \right),$$

$$\langle p(\tau) \rangle = i \left\{ [\dot{\bar{\xi}}\xi^o - \ddot{\xi}\bar{\xi}^o]p_o + [\dot{\xi}\dot{\bar{\xi}}^o - \ddot{\xi}\dot{\xi}^o] \right\}$$

$$+ i \left( \dot{\bar{\xi}}\dot{c} - \dot{\xi}\dot{\bar{c}} \right),$$

where for the $(\alpha, z)$-representation

$$\alpha = i \left( p_o\xi^o - x_o\bar{\xi}^o \right) + iC^\circ,$$
and in the $(z, \alpha)$-representation,

$$|\alpha|[e^{i\delta} \cosh r - e^{i(\theta-\delta)} \sinh r] = i \left( p_o \xi_o - x_o \dot{\xi}_o \right) + i C_o. \quad (19)$$

Similarly for their complex conjugates.

Now we write the expectation values of position and momentum explicitly in terms of real $\tau$-dependent functions. These are more useful in constructing properties of squeezed states for specific systems. The expectation value for position is

$$\langle x(\tau) \rangle = (\chi_2 \chi_1^o + \chi_1 \chi_2^o) p_o - (\chi_1 \dot{\chi}_2^o - \chi_2 \dot{\chi}_1^o) x_o$$

$$+ \chi_1 c_2 - \chi_2 c_1, \quad (20)$$

and for momentum is

$$\langle p(\tau) \rangle = (\dot{\chi}_2 \chi_1^o + \dot{\chi}_1 \chi_2^o) p_o - (\dot{\chi}_1 \dot{\chi}_2^o - \dot{\chi}_2 \dot{\chi}_1^o) x_o$$

$$+ \dot{\chi}_1 c_2 - \dot{\chi}_2 c_1. \quad (21)$$

The coherent and squeezed state parameters can also be expressed in terms of the initial position and momentum and the initial values of the real $\tau$-dependent functions. For the $(\alpha, z)$-representation, the equations are

$$|\alpha|^2 = \frac{1}{2} \left( (\dot{\chi}_2^o x_o - \chi_2^o p_o - C_2^o)^2 + (\chi_1^o p_o - \dot{\chi}_1^o x_o + C_1^o)^2 \right),$$

$$\tan \delta = \frac{\chi_1^o p_o - \dot{\chi}_1^o x_o + C_1^o}{\chi_2^o x_o - \chi_2^o p_o - C_2^o}. \quad (22)$$

For the $(z, \alpha)$-representation, we have

$$|\alpha| \left[ \cos \delta \cosh r - \cos (\theta - \delta) \sinh r \right] = \sqrt{\frac{1}{2}} (\dot{\chi}_2^o x_o - \chi_2^o p_o - C_2^o),$$

$$|\alpha| \left[ \sin \delta \cosh r - \sin (\theta - \delta) \sinh r \right] = \sqrt{\frac{1}{2}} (\chi_1^o p_o - \dot{\chi}_1^o x_o + C_1^o). \quad (23)$$

Frequently, we wish to use the expressions for the expectation values of position and momentum in terms of the parameters $\alpha$ and $z$. We give both the $(\alpha, z)$- and $(z, \alpha)$-representations in terms of real $\tau$-dependent functions only. The complex expressions can be found in paper I.
The \((\alpha, z)\)-representation

Substituting Eqs.\,(4) and \,(9) into Eqs. \,(73) and \,(74) from paper I, we calculate real expressions for \(\langle x(\tau) \rangle_{(\alpha,z)}\) and \(\langle p(\tau) \rangle_{(\alpha,z)}\):

\[
\langle x(\tau) \rangle_{(\alpha,z)} = \sqrt{2} |\alpha| \left( \chi_1 \cos \delta + \chi_2 \sin \delta \right) + \chi_1 C_2 - \chi_2 C_1
\]

and

\[
\langle p(\tau) \rangle_{(\alpha,z)} = \sqrt{2} |\alpha| \left( \dot{\chi}_1 \cos \delta + \dot{\chi}_2 \sin \delta \right) + \dot{\chi}_1 C_2 - \dot{\chi}_2 C_1.
\]

The \((z, \alpha)\)-representation

Substituting Eqs. \,(4) and \,(9) into Eqs. \,(79) and \,(81) from paper I, we obtain the following expressions for \(\langle x(\tau) \rangle_{(z,\alpha)}\) and \(\langle p(\tau) \rangle_{(z,\alpha)}\):

\[
\langle x(\tau) \rangle_{(z,\alpha)} = \sqrt{2} |\alpha| \left\{ \chi_1 \left[ \cos \delta \cosh r - \cos (\theta - \delta) \sinh r \right] \\
+ \chi_2 \left[ \sin \delta \cosh r - \sin (\theta - \delta) \sinh r \right] \right\} \\
+ \chi_1 C_2 - \chi_2 C_1,
\]

and

\[
\langle p(\tau) \rangle_{(z,\alpha)} = \sqrt{2} |\alpha| \left\{ \dot{\chi}_1 \left[ \cos \delta \cosh r - \cos (\theta - \delta) \sinh r \right] \\
+ \dot{\chi}_2 \left[ \sin \delta \cosh r - \sin (\theta - \delta) \sinh r \right] \right\} \\
+ \dot{\chi}_1 C_2 - \dot{\chi}_2 C_1.
\]

The uncertainty product is representation independent and its expression in terms of the real \(\tau\)-dependent functions, \(\chi_1\) and \(\chi_2\), is given by Eq.\,(93) in paper I.

3 Examples
3.1 Harmonic oscillator (HO)

For the harmonic oscillator \( g = (\frac{1}{2} \omega^2, 0, 0) \) and the differential equation (5) has the form

\[
\ddot{a} + \omega^2 a = 0.
\]  

(28)

Two real solutions are

\[
\chi_1 = \frac{1}{\sqrt{\omega}} \cos \omega \tau, \quad \chi_2 = \frac{1}{\sqrt{\omega}} \sin \omega \tau.
\]  

(29)

Using Eq. (4), the two complex solutions are

\[
\xi = \sqrt{\frac{1}{2\omega}} e^{i\omega \tau}, \quad \bar{\xi} = \sqrt{\frac{1}{2\omega}} e^{-i\omega \tau},
\]  

(30)

from which we obtain the three functions \( \phi_1, \phi_2, \) and \( \phi_3: \)

\[
\phi_1 = \frac{1}{2\omega} e^{2i\omega \tau}, \quad \phi_2 = \frac{1}{2\omega} e^{-2i\omega \tau}, \quad \phi_3 = \frac{1}{\omega}.
\]  

(31)

Therefore, the generators in Eqs. (6) to (11) in paper I have the form

\[
\mathcal{J}_- = \sqrt{\frac{1}{2\omega}} e^{i\omega \tau}(\partial_x + \omega x), \quad \mathcal{J}_+ = \sqrt{\frac{1}{2\omega}} e^{-i\omega \tau}(-\partial_x + \omega x),
\]  

(32)

\[
\mathcal{M}_- = \frac{1}{2\omega} e^{2i\omega \tau}(i\partial_\tau - \omega x \partial_x - \omega^2 x^2 - \frac{1}{4}\omega),
\]  

(33)

\[
\mathcal{M}_+ = \frac{1}{2\omega} e^{-2i\omega \tau}(i\partial_\tau + \omega x \partial_x - \omega^2 x^2 + \frac{1}{4}\omega),
\]  

(34)

\[
\mathcal{M}_3 = \frac{i}{\omega} \partial_\tau.
\]  

(35)

3.2 Free particle (FP)

In this specific case, \( g = (0, 0, 0) \) and Eq. (5) becomes

\[
\ddot{a} = 0,
\]  

(36)

which has real solutions,

\[
\chi_1 = 1, \quad \chi_2 = \tau.
\]  

(37)
The two complex solutions can be calculated directly using Eq. (4) and they are
\[ \xi = \sqrt{\frac{1}{2}} (1 + i\tau), \quad \bar{\xi} = \sqrt{\frac{1}{2}} (1 - i\tau). \] (38)

The remaining \( \tau \)-dependent functions can be obtained from Eqs. (8):
\[ \phi_1 = \frac{1}{2} (1 + i\tau)^2, \quad \phi_2 = \frac{1}{2} (1 - i\tau)^2, \quad \phi_3 = 1 + \tau^2. \] (39)

Therefore, the generators in Eqs. (6) to (11) of paper I can be written as
\[ J_- = \sqrt{\frac{1}{2}} \{ (1 + i\tau) \partial_x + x \}, \quad J_+ = \sqrt{\frac{1}{2}} \{ -(1 - i\tau) \partial_x + x \}, \] (40)
\[ M_- = \frac{i}{2} \{ (1 + i\tau)^2 \partial_x + i (1 + i\tau) x \partial_x + \frac{i}{2} x^2 + \frac{i}{2} (1 + i\tau) \}, \] (41)
\[ M_+ = \frac{i}{2} \{ (1 - i\tau)^2 \partial_x - i (1 - i\tau) x \partial_x + \frac{i}{2} x^2 - \frac{i}{2} (1 - i\tau) \}, \] (42)
\[ M_3 = \frac{i}{2} \{ (1 + \tau^2) \partial_x + \tau x \partial_x - \frac{i}{2} x^2 + \frac{1}{2} \tau \}. \] (43)

We have presented the generators in this and the previous example out of general interest. We will not show them in subsequent examples since they are usually long and not needed. In all cases described by the potential in Eq. (3), the calculations for which the generators are required can all be done in general. (See Section 3 of paper I.) All wave functions and expectation values can be obtained by computing the appropriate \( \tau \)-dependent functions and substituting them into expressions for the desired quantities. This is where the strength of this methodology lies.

### 3.3 Linear potential (LP)

Here we have \( g = (0, g(\tau), 0) \) and the differential equation is (36). The two real solutions are (37) and the two complex solutions are (38). Set \( g(0) = g_0 \).

The \( \tau \)-dependent function, \( C \) is defined in Eq. (10), where \( c(\tau) \) is
\[ c(\tau) = \sqrt{\frac{1}{2}} \int_0^\tau d\rho g(\rho)(1 + i\rho). \] (44)
The complex conjugate, $\bar{c}(\tau)$, can be obtained directly from Eq. (44). The two generators $J_-$ and $J_+$ can be written as

$$J_- = \sqrt{\frac{1}{2}} \{(1 + i\tau) \partial_x + x - iC\}, \quad J_+ = \sqrt{\frac{1}{2}} \{- (1 - i\tau) \partial_x + x + i\bar{C}\}. \quad (45)$$

The commutator of $J_-$ and $J_+$ is

$$[J_-, J_+] = I. \quad (46)$$

To determine the integration constants, $C^o$ and $\bar{C}^o$, calculate the operator $J_+J_- + \frac{1}{2}$ and find that

$$J_+J_- + \frac{1}{2} = \frac{1}{2} \{- (1 + \tau^2) \partial_{xx} + 2i\tau x \partial_x$$

$$+ \sqrt{2} \left[i(1 + i\tau)\bar{C} + i(1 - i\tau)C\right] \partial_x + x^2$$

$$+ i\sqrt{2}(\bar{C} - C)x + i\tau + C\bar{C}\}. \quad (47)$$

Taking the limit as $\tau \to 0$, we have

$$\lim_{\tau \to 0} \left\{ J_+J_- + \frac{1}{2} \right\} = \frac{1}{2} \{- \partial_{xx} + i\sqrt{2} \left(\bar{C}^o + C^o\right) \partial_x + x^2 + i\sqrt{2}(\bar{C}^o - C^o)x + C^o\bar{C}^o\}, \quad (48)$$

which is a Hamiltonian for a displaced oscillator if we choose $C^o$ to be pure imaginary and set

$$C^o = \frac{i}{\sqrt{2}} g_0. \quad (49)$$

Therefore, we find that

$$\lim_{\tau \to 0} \left\{ J_+J_- + \frac{1}{2} \right\} = \frac{1}{2} \{- \partial_{xx} + x^2 + i\sqrt{2}g_0x + \frac{1}{2}g_0^2\}. \quad (50)$$

Then, we see that

$$C = c(\tau) + \frac{i}{\sqrt{2}} g_0, \quad \bar{C} = \bar{c}(\tau) - \frac{i}{\sqrt{2}} g_0. \quad (51)$$

Once $g(\tau)$ is known, the remaining $\tau$-dependent functions can be found and the symmetry generators constructed from Eqs. (6) to (11) in paper I.
For example, suppose that we choose

\[ g(\tau) = \frac{\kappa}{2}, \quad (52) \]

where \( \kappa \) is a real constant. Then, \( g_o = \kappa/2 \), and we have for the complex function \( C \)

\[ C(\tau) = \frac{\kappa}{4\sqrt{2}} \left[ 2\tau + i(2 + \tau^2) \right]. \quad (53) \]

We will need the real functions \( C_1 \) and \( C_2 \), as well as the values of the real integration constants \( C^o_1 \) and \( C^o_2 \). They can be evaluated with the help of Eqs. (5) through (12). For arbitrary \( g(\tau) \), we have

\[ C_1(\tau) = c_1(\tau) = \int_0^\tau d\rho g(\rho), \quad (54) \]
\[ C_2(\tau) = c_2(\tau) + C_2^o, \quad (55) \]

where

\[ c_2(\tau) = \int_0^\tau d\rho g(\rho)\rho, \quad (56) \]

and

\[ C_1^o = 0, \quad C_2^o = g_o. \quad (57) \]

When \( g(\tau) \) is a constant, \( \kappa/2 \), then

\[ C_1(\tau) = c_1(\tau) = \frac{\kappa}{2}\tau, \quad (58) \]
\[ C_2(\tau) = c_2(\tau) + C_2^o, \quad (59) \]

where

\[ c_2(\tau) = \frac{\kappa}{4}\tau^2, \quad (60) \]

and

\[ C_2^o = \frac{\kappa}{2}. \quad (61) \]
3.4 Driven Harmonic Oscillator (DHO)

For this system, we have $g = (\frac{1}{2}\omega^2, g(\tau), 0)$, where $g(\tau)$ is a real function of $\tau$ and $g(0) = g_o$. The differential equation for $\chi_1$ and $\chi_2$ is (28). The real solutions are given in (29) and the complex solutions in (30). The function, $C$, can be written as in Eq. (6) with

$$c(\tau) = \sqrt{\frac{1}{2\omega}} \int_0^\tau d\rho g(\rho)e^{i\omega\rho}. \quad (62)$$

From Eq. (62) and its complex conjugate all the remaining $\tau$-dependent functions can be found, if needed.

The two operators $J_{\pm}$ are

$$J_- = \sqrt{\frac{1}{2\omega}} e^{i\omega\tau} \left\{ \partial_x + \omega x + i\sqrt{2\omega}e^{-i\omega\tau}C \right\}, \quad (63)$$
$$J_+ = -\sqrt{\frac{1}{2\omega}} e^{-i\omega\tau} \left\{ \partial_x - \omega x + i\sqrt{2\omega}e^{i\omega\tau}C \right\}. \quad (64)$$

Repeating the procedure of the previous subsection, computing the operator $J_+J_- + \frac{1}{2}$, and taking the limit $\tau \rightarrow 0$, we obtain the integration constant

$$C^o = -\frac{i}{\omega\sqrt{2\omega}}g_o, \quad (65)$$

and its complex conjugate.

For the specific case when

$$g(\tau) = g_o = \frac{\kappa}{2}, \quad (66)$$

$\kappa$ a real number, we have

$$C = -\frac{i\kappa}{(2\omega)^{3/2}}e^{i\omega\tau}. \quad (67)$$

The real functions $C_1$ and $C_2$ and have the form Eq. (10) where

$$c_1(\tau) = \int_0^\tau d\rho g(\rho) \cos \omega\rho, \quad c_2(\tau) = \int_0^\tau d\rho g(\rho) \sin \omega\rho, \quad (68)$$

with real integration constants

$$C_1^o = 0, \quad C_2^o = -\frac{1}{\omega^{3/2}}g_o. \quad (69)$$
When \( g(\tau) = \kappa/2 \), Eqs. (68) become

\[
c_1(\tau) = \frac{\kappa}{2}\omega^{3/2} \sin \omega \tau, \quad c_2(\tau) = -\frac{\kappa}{2}\omega^{3/2} (1 - \cos \omega \tau),
\]

with real integration constants

\[
C_1^0 = 0, \quad C_2^0 = -\frac{1}{2}\omega^{3/2} \kappa.
\]

### 3.5 Repulsive oscillator (RO)

In this final example we have \( g = (-\frac{1}{2} \Omega^2, 0, 0) \). Eq. (4) becomes

\[
\ddot{a} - \Omega^2 a = 0.
\]

This differential equation has two real solutions:

\[
\chi_1 = \frac{1}{\sqrt{\Omega}} \cosh \Omega \tau, \quad \chi_2 = \frac{1}{\sqrt{\Omega}} \sinh \Omega \tau.
\]

Using Eq. (4), the two complex solutions are

\[
\xi = \frac{1}{\sqrt{2}\Omega} \left[ \cosh \Omega \tau + i \sinh \Omega \tau \right],
\]

\[
\bar{\xi} = \frac{1}{\sqrt{2}\Omega} \left[ \cosh \Omega \tau - i \sinh \Omega \tau \right].
\]

For each of model systems discussed above, their Schrödinger algebras are isomorphic, \((\mathcal{S}A)_1^c = su(1, 1) \circ w^c_1\). An explicit operator basis for each system can be obtained by substituting the appropriate functions for each example into Eqs. (6) to (11) of paper I. As examples, see Eqs. (32) to (35) for the harmonic oscillator and Eqs. (40) to (43) for the free particle.

In each case, the subalgebra of operators \( \{M_3, J_\pm, I\} \) forms a basis for an oscillator algebra \( os(1) \). Therefore, each of the examples above will have a complete set of number-operator states. Only, for HO will these states be energy eigenstates since, in that case, \( M_3 \) is proportional to the energy operator. For the remaining examples, there may be no simple interpretation for the operator \( M_3 \).
4 Squeezed States for Specific Examples

Explicit calculation of coherent states for HO, FP, DHO, LP, and RO have been calculated elsewhere [3]. These results are contained as specific examples, $z = 0$, of our results here.

The specific $\tau$-dependent functions that we need were developed in Section 3. In each case, we express the expectation values for position and momentum in three ways: in terms of the initial position and momentum [Eqs. (20) and (21)], in the $(\alpha, z)$-representation [Eqs. (24) and (25)], and in the $(z, \alpha)$-representation [Eqs. (26) and (27)]. However, the parameters $|\alpha|$, $\delta$, $|z|$, and $\theta$ are defined in terms of $x_o$ and $p_o$ differently in the $(\alpha, z)$- and $(z, \alpha)$-representations, Eqs. (22) and (23), respectively. Finally, for each system we give the uncertainty product, which is independent of representation.

4.1 Harmonic oscillator (HO)

Combining Eqs. (20) and (21) with the real functions (29) for the oscillator, we find that the expectation values for position and momentum in terms of $x_o$ and $p_o$, are

$$\langle x(\tau) \rangle = \frac{1}{\omega} (p_o \sin \omega \tau + \omega x_o \cos \omega \tau),$$  
$$\langle p(\tau) \rangle = p_o \cos \omega \tau - \omega x_o \sin \omega \tau. \quad (75)$$

In the $(\alpha, z)$-representation, using Eqs. (24) and (25), we obtain

$$\langle x(\tau) \rangle_{(\alpha,z)} = \sqrt{\frac{2}{\omega}} |\alpha| \cos (\omega \tau - \delta),$$  
$$\langle p(\tau) \rangle_{(\alpha,z)} = -\sqrt{2 \omega} |\alpha| \sin (\omega \tau - \delta), \quad (77)$$

where $\alpha$ and $z$ are defined in Eq. (15). In addition, according to Eq. (22), we have

$$|\alpha|^2 = \frac{1}{2\omega} (p_o^2 + \omega^2 x_o^2), \quad \delta = \tan^{-1} \left( \frac{p_o}{\omega x_o} \right). \quad (79)$$

Note that in this representation, these expectation values are independent of the squeeze parameters, $|z|$ and $\theta$. This is because both coherent and squeezed states
follow the classical motion \( x_c(\tau) \) and \( p_c(\tau) \). The differences are that the squeezed state is a Gaussian wave packet whose width oscillates with time and it has a time-dependent uncertainty product, as we come to below.

With Eqs. (26) and (27), we find that, for the \((z,\alpha)\)-representation,

\[
\langle x(\tau) \rangle_{(z,\alpha)} = 2|\alpha| \{ \cos (\omega \tau - \delta) \cosh r \\
- \cos (\omega \tau + \delta - \theta) \sinh r \},
\]

\[
\langle p(\tau) \rangle_{(z,\alpha)} = -2\omega |\alpha| \{ \sin (\omega \tau - \delta) \cosh r \\
- \sin (\omega \tau + \delta - \theta) \sinh r \},
\]

with the connection (23) to the initial position and momentum

\[
|\alpha| \left[ \cos \delta \cosh r - \cos (\theta - \delta) \sinh r \right] = \sqrt{\frac{\omega}{2}} x_0,
\]

\[
|\alpha| \left[ \sin \delta \cosh r - \sin (\theta - \delta) \sinh r \right] = \sqrt{\frac{1}{2\omega}} p_0.
\]

From Eq. (82), we find the identity

\[
\frac{1}{2\omega} (p_0^2 + \omega^2 x_0^2) = |\alpha|^2 \{ \cosh 2r - \cos (\theta - 2\delta) \sinh 2r \}.
\]

The uncertainty in position and momentum are

\[
(\Delta x)^2 = \frac{1}{2\omega} \{ \cosh 2r + \cos (2\omega \tau - \theta) \sinh 2r \}
\]

and

\[
(\Delta p)^2 = \frac{\omega}{2} \{ \cosh 2r + \cos (2\omega \tau - \theta) \sinh 2r \},
\]

respectively. Therefore, the uncertainty product is

\[
(\Delta x)^2(\Delta p)^2 = \frac{1}{4} \left[ 1 + \sin^2 (2\omega \tau - \theta) \sinh^2 2r \right]
\]

\[
= \frac{1}{4} \left[ 1 + \sin^2 \left( \frac{2\omega \tau - \theta}{4} \right) \left( s^2 - \frac{1}{s^2} \right)^2 \right],
\]

where [6, 7]

\[
s = \exp r
\]
is the “squeeze parameter.” The uncertainty relation (86) is identical to Eq. (30) in Ref. [5] after a suitable choice for the phase $\theta$.

For $z$ real and positive, the squeezed-state wave function is

$$\psi_{ss}(x) = \left[\pi s^2\right]^{-1/4} \exp \left[-\frac{(x - x_o)^2}{2s^2} + ip_o x\right], \quad (88)$$

with $\hbar/m\omega = 1$. For a more general expression for the squeezed state wave function, see Ref. [5].

4.2 Free particle (FP)

Substituting the real functions (37) into Eqs. (20) and (21), we get the expectation values for position and momentum in terms of $x_o$ and $p_o$:

$$\langle x(\tau) \rangle = x_o + p_o \tau, \quad (89)$$
$$\langle p(\tau) \rangle = p_o. \quad (90)$$

In the $(\alpha, z)$-representation, we obtain

$$\langle x(\tau) \rangle_{(\alpha,z)} = \sqrt{2}|\alpha|[\cos \delta + \tau \sin \delta], \quad (91)$$
$$\langle p(\tau) \rangle_{(\alpha,z)} = \sqrt{2}|\alpha| \sin \delta. \quad (92)$$

where

$$\sqrt{2}|\alpha| \cos \delta = x_o, \quad \sqrt{2}|\alpha| \sin \delta = p_o, \quad (93)$$

and

$$|\alpha|^2 = \frac{1}{2}(p_o^2 + x_o^2), \quad \delta = \tan^{-1}\left(\frac{p_o}{x_o}\right). \quad (94)$$

For the $(z, \alpha)$-representation, we find that

$$\langle x(\tau) \rangle_{(z,\alpha)} = \sqrt{2}|\alpha|\{\cos \delta \cosh r - \cos (\theta - \delta) \sinh r \quad + \tau[\sin \delta \cosh r - \sin (\theta - \delta) \sinh r]\}, \quad (95)$$
$$\langle p(\tau) \rangle_{(z,\alpha)} = \sqrt{2}|\alpha|\{\sin \delta \cosh r - \sin (\theta - \delta) \sinh r\}, \quad (96)$$
where
\[
\sqrt{2} |\alpha| \cos \delta \cosh r - \cos (\theta - \delta) \sinh r = x_o,
\]
\[
\sqrt{2} |\alpha| \sin \delta \cosh r - \sin (\theta - \delta) \sinh r = p_o, \tag{97}
\]
and we have the relationship
\[
\frac{1}{2} (x_o^2 + p_o^2) = |\alpha|^2 \left[ \cosh 2r - \cos (\theta - 2\delta) \sinh 2r \right]. \tag{98}
\]

The uncertainty in position is given by
\[
(\Delta x)^2 = \frac{1}{2} \left[ 1 + \tau^2 \right] \cosh 2r
- \frac{1}{2} \left[ 1 - \tau^2 \right] \cos \theta + 2\tau \sin \theta \right] \sinh 2r, \tag{99}
\]
and the uncertainty in momentum is
\[
(\Delta p)^2 = \frac{1}{2} \left[ \cosh 2r + \cos \theta \sinh 2r \right]. \tag{100}
\]

Therefore, the uncertainty product is
\[
(\Delta x)^2 (\Delta p)^2 = \frac{1}{4} \left[ 1 + \tau^2 + \left[ \tau^2 \cos \theta - \tau \sin \theta \right] \sinh 4r \right]
+ \frac{1}{2} + \frac{3}{2} \tau^2 - \frac{1}{2} \left[ 1 - \tau^2 \right] \cos 2\theta - \tau \sin 2\theta \right] \sinh^2 2r. \tag{101}
\]

### 4.3 Linear potential (LP)

When the interaction is linear, we use the time-dependent functions computed in Section 3.3. Calculating the expectation values of position and momentum in terms of the initial position and momentum, we find
\[
\langle x(\tau) \rangle = x_o + p_o \tau + \int_0^\tau d\rho g(\rho) \rho - \tau \int_0^\tau d\rho g(\rho), \tag{102}
\]
\[
\langle p(\tau) \rangle = p_o - \int_0^\tau d\rho g(\rho). \tag{103}
\]

For the particular case of \( g(\tau) = \kappa/2 \), a constant,
\[
\langle x(\tau) \rangle = x_o + p_o \tau - \frac{\kappa}{4} \tau^2 \tag{104}
\]
\[
\langle p(\tau) \rangle = p_o - \frac{\kappa}{2} \tau. \tag{105}
\]
To connect to the \((\alpha, z)\)-representation, we would have

\[
|\alpha|^2 = \frac{1}{2} \left[ (x_0 - g_o)^2 + p_o^2 \right],
\]

\[
\delta = \tan^{-1} \left( \frac{p_o}{x_0 - g_o} \right).
\]

(106)

For the \((z, \alpha)\)-representation, Eqs. (23) become

\[
|\alpha| \cos \delta \cosh r - \cos (\theta - \delta) \sinh r = \sqrt{\frac{1}{2}} (x_0 - g_o),
\]

\[
|\alpha| \sin \delta \cosh r - \sin (\theta - \delta) \sinh r = \sqrt{\frac{1}{2}} p_o.
\]

(107)

We can combine these two equations into one to yield

\[
|\alpha| \cosh 2r - \cos (\theta - 2\delta) \sinh 2r = \frac{1}{2} \left( p_o^2 + x_0^2 - 2g_o x_0 + g_o^2 \right).
\]

(108)

To obtain the corresponding equations for constant \(g(\tau)\), replace \(g_o\) by \(\kappa/2\) in each of the above equations.

For the \((\alpha, z)\)-representation, the corresponding expectation values for \(x\) and \(p\) are

\[
\langle x(\tau) \rangle_{(\alpha,z)} = \sqrt{2} |\alpha| \cos \delta + \tau \sin \delta + \int_0^\tau d\rho g(\rho) \rho + g_0 - \tau \int_0^\tau d\rho g(\rho),
\]

\[
\langle p(\tau) \rangle_{(\alpha,z)} = \sqrt{2} |\alpha| \sin \delta - \int_0^\tau d\rho g(\rho).
\]

(109)

(110)

When \(g(\tau) = \kappa/2\), we find from the previous two equations that

\[
\langle x(\tau) \rangle_{(\alpha,z)} = \sqrt{2} |\alpha| \cos \delta + \tau \sin \delta - \frac{\kappa}{4} \tau^2 + \frac{\kappa}{2},
\]

\[
\langle p(\tau) \rangle_{(\alpha,z)} = \sqrt{2} |\alpha| \sin \delta - \frac{\kappa}{2} \tau.
\]

(111)

(112)

For the \((z, \alpha)\)-representation, we see that

\[
\langle x(\tau) \rangle_{(z,\alpha)} = \sqrt{2} |\alpha| \cos \delta \cosh r - \cos (\theta - \delta) \sinh r
\]

\[
+ \tau \sin \delta \cosh r - \sin (\theta - \delta) \sinh r) + \int_0^\tau d\rho g(\rho) \rho + g_0 - \tau \int_0^\tau d\rho g(\rho),
\]

\[
\langle p(\tau) \rangle_{(z,\alpha)} = \sqrt{2} |\alpha| \sin \delta \cosh r - \sin (\theta - \delta) \sinh r \int_0^\tau d\rho g(\rho),
\]

(113)

(114)
When $g(\tau) = \kappa/2$,

$$\langle x(\tau) \rangle_{(z,\alpha)} = \sqrt{2}|\alpha| \left\{ \cos \delta \cosh r - \cos (\theta - \delta) \sinh r 
+ \tau \left[ \sin \delta \cosh r - \sin (\theta - \delta) \sinh r \right] \right\}$$

$$- \frac{\kappa}{4} r^2 + \frac{\kappa}{2}, \quad (115)$$

$$\langle p(\tau) \rangle_{(z,\alpha)} = \sqrt{2}|\alpha| \left\{ \sin \delta \cosh r - \sin (\theta - \delta) \sinh r \right\} - \frac{\kappa}{2} \tau. \quad (116)$$

The uncertainties in position and momentum are given by Eq. (99) and (100), respectively. Therefore, the uncertainty product (101) is still valid for a system with a linear interaction since it is independent of $g(\tau)$.

### 4.4 Driven Harmonic Oscillator (HDO)

Referring to the results of Section 3.4 and Eqs. (20) and (21), the expectation values for position and momentum are presented below.

(a) In the $(x_o, p_o)$-representation:

$$\langle x(\tau) \rangle = \frac{1}{\omega} (p_o \sin \omega \tau + \omega x_o \cos \omega \tau)$$

$$+ \frac{1}{\omega} \left\{ \cos \omega \tau \int_0^\tau d\rho g(\rho) \sin \omega \tau - \sin \omega \tau \int_0^\tau d\rho g(\rho) \cos \omega \tau \right\}, \quad (117)$$

$$\langle p(\tau) \rangle = p_o \cos \omega \tau - \omega x_o \sin \omega \tau$$

$$- \sin \omega \tau \int_0^\tau d\rho g(\rho) \sin \omega \tau - \cos \omega \tau \int_0^\tau d\rho g(\rho) \cos \omega \tau, \quad (118)$$

where, for the $(\alpha, z)$-representation

$$|\alpha|^2 = \frac{1}{2\omega} \left( p_o^2 + \omega^2 x_o^2 + g_o x_o + \frac{1}{\omega^2} g_o^2 \right),$$

$$\delta = \tan^{-1} \frac{\omega p_o}{\omega x_o + g_o}. \quad (119)$$

For the $(z, \alpha)$-representation, the connecting formulas are

$$|\alpha| \left[ \cos \delta \cosh r - \cos (\theta - \delta) \sinh r \right] = \sqrt{\frac{r}{2}} \left( \sqrt{\frac{\omega}{\omega^2 + g_o}} \right),$$

$$|\alpha| \left[ \sin \delta \cosh r - \sin (\theta - \delta) \sinh r \right] = \sqrt{\frac{r}{2\omega}} p_o. \quad (120)$$
(b) In the \((\alpha,z)\)-representation:

\[
\langle x(\tau) \rangle_{(\alpha,z)} = \sqrt{2} |\alpha| \cos (\omega \tau - \delta) + (\chi_1 C_2 - \chi_2 C_1),
\]

\(\quad (121)\)

\[
\langle p(\tau) \rangle_{(\alpha,z)} = -\sqrt{2} \omega |\alpha| \sin (\omega \tau - \delta) + (\dot{\chi}_1 C_2 - \dot{\chi}_2 C_1),
\]

\(\quad (122)\)

(c) In the \((z,\alpha)\)-representation:

\[
\langle x(\tau) \rangle_{(z,\alpha)} = 2 |\alpha| \{ \cos (\omega \tau - \delta) \cosh r
\]

\[
- \cos (\omega \tau + \delta - \theta) \sinh r \} + (\chi_1 C_2 - \chi_2 C_1),
\]

\(\quad (123)\)

\[
\langle p(\tau) \rangle_{(z,\alpha)} = -2 \omega |\alpha| \{ \sin (\omega \tau - \delta) \cosh r
\]

\[
- \sin (\omega \tau + \delta - \theta) \sinh r \} + \dot{\chi}_1 C_2 - \dot{\chi}_2 C_1.
\]

\(\quad (124)\)

In each of (b) and (c) above, we have

\[
\chi_1 C_2 - \chi_2 C_2 = \frac{1}{2} \{ \cos \omega \tau \int_{\tau_0}^{\tau} d\rho g(\rho) \sin \omega \rho
\]

\[
- \sin \omega \tau \int_{\tau_0}^{\tau} d\rho g(\rho) \cos \omega \rho \},
\]

\(\quad (125)\)

and

\[
\dot{\chi}_1 C_2 - \dot{\chi}_2 C_2 = -\sin \omega \tau \int_{\tau_0}^{\tau} d\rho g(\rho) \sin \omega \rho
\]

\[
- \cos \omega \tau \int_{\tau_0}^{\tau} ds g(\rho) \cos \omega \rho.
\]

\(\quad (126)\)

When \(g(\tau) = \kappa/2\), explicit expressions of expectation values for \(x\) and \(p\) can be obtained. From Eqs. \((117)\) and \((118)\), we find

\[
\langle x(\tau) \rangle = \frac{1}{\omega} (p_o \sin \omega \tau + \omega x_o \cos \omega \tau) + \frac{\kappa}{2 \omega^2} (\cos \omega \tau - 1),
\]

\(\quad (127)\)

\[
\langle p(\tau) \rangle = p_o \cos \omega \tau - \omega x_o \sin \omega \tau - \frac{\kappa}{2 \omega} \sin \omega \tau.
\]

\(\quad (128)\)

For the \((\alpha, z)\)-representation, applying Eqs. \((121)\) and \((122)\), we find that

\[
\langle x(\tau) \rangle_{(\alpha,z)} = \sqrt{2} |\alpha| \cos (\omega \tau - \delta) - \frac{\kappa}{2 \omega \tau},
\]

\(\quad (129)\)

\[
\langle p(\tau) \rangle_{(\alpha,z)} = -\sqrt{2} \omega |\alpha| \sin (\omega \tau - \delta).
\]

\(\quad (130)\)
Making use of Eqs. (123) and (124), we obtain for the \((z, \alpha)\)-representation,

\[
\langle x \rangle = \sqrt{\frac{2}{\omega}} \{ \cos (\omega \tau - \delta) \cosh r - \sin (\omega \tau + \delta - \theta) \sinh r \} - \frac{\kappa}{2 \omega^2}, \tag{131}
\]

\[
\langle p \rangle = -\sqrt{2 \omega} |\alpha| \{ \sin (\omega \tau - \delta) \cosh r - \sin (\omega \tau + \delta - \theta) \sinh r \}. \tag{132}
\]

The connecting formulas for this case can be obtained from Eqs. (119) and (120) by substituting \(g_o = k/2\).

Expressions for the uncertainties in position and momentum, (84) and (85), respectively, derived for HO remain valid here. As a consequence, the uncertainty product (86) holds for DHO.

### 4.5 Repulsive oscillator (RO)

Referring to Section 3.5, for the time-dependent functions for the repulsive oscillator, we find the expectation values in the \((x_o, p_o)\)-representation are

\[
\langle x(\tau) \rangle = \frac{1}{\Omega} [p_o \sinh \Omega \tau + \Omega x_o \cosh \Omega \tau], \tag{133}
\]

\[
\langle p(\tau) \rangle = p_o \cosh \Omega \tau + \Omega x_o \sinh \Omega \tau. \tag{134}
\]

For the \((\alpha, z)\)-representation, we obtain

\[
\langle x(\tau) \rangle_{(\alpha, z)} = \sqrt{\frac{2}{\Omega}} |\alpha| [\cosh \Omega \tau \cos \delta + \sinh \Omega \tau \sin \delta], \tag{135}
\]

\[
\langle p(\tau) \rangle_{(\alpha, z)} = \sqrt{2\Omega} |\alpha| [\sinh \Omega \tau \cos \delta + \cosh \Omega \tau \sin \delta], \tag{136}
\]

where we have

\[
x_o = \sqrt{\frac{2}{\Omega}} |\alpha| \cos \delta, \quad p_o = \sqrt{2\Omega} |\alpha| \sin \delta, \tag{137}
\]

and

\[
|\alpha|^2 = \frac{1}{2\Omega} \left(p_o^2 + \Omega^2 x_o^2 \right). \tag{138}
\]

In the \((z, \alpha)\)-representation, we see that

\[
\langle x(\tau) \rangle_{(z, \alpha)} = \sqrt{\frac{2}{\Omega}} |\alpha| \{ \cos \delta \cosh r \}
\]
\[
- \cos (\theta - \delta) \sinh r \cosh \Omega \tau \\
+ [\sin \delta \cosh r - \sin (\theta - \delta) \sinh r] \sinh \Omega \tau, \\
\langle p(\tau) \rangle_{(z, \alpha)} = \sqrt{2\Omega} |\alpha| \{ [\cos \delta \cosh r \\
- \cos (\theta - \delta) \sinh r] \sinh \Omega \tau \\
+ [\sin \delta \cosh r - \sin (\theta - \delta) \cosh \Omega \tau] \}, \\
\]
where we have
\[
x_o = \sqrt{\frac{2}{\Omega}} |\alpha| [\cos \delta \cosh |z| - \cos (\theta - \delta) \sinh |z|], \\
p_o = \sqrt{2\Omega} |\alpha| [\sin \delta \cosh |z| - \sin (\theta - \delta) \sinh |z|]. \\
\]
In addition, we have the identity
\[
\frac{1}{2\Omega} (p_o^2 + \Omega^2 x_o^2) = |\alpha|^2 [\cosh 2r - \cos (\theta - \delta) \sinh 2r]. \\
\]

We obtain the uncertainty product directly from Eq. (93) of paper I. We have
\[
(\Delta x)^2(\Delta p)^2 = \frac{1}{4} \left( 1 + \sinh^2 2\Omega \tau \right) - \frac{1}{4} \sinh^2 2\Omega \tau \sin \theta \sinh 4r, \\
+ \frac{1}{8} \left\{ 1 + 3 \sinh^2 2\Omega \tau + \cosh^2 2\Omega \tau \cos 2\theta \right\} \sinh^2 2r. \\
\]
Initially, the Gaussian wave packet describing this state satisfies the minimum uncertainty condition, but spreads out over time.

5 Discussion

All quantum systems described by a Schrödinger equation \((\Box)\) with potential \((V)\) have isomorphic symmetry algebras, designated by \((\mathcal{SA})_c^1\) or its oscillator subalgebra os(1) = \{\(\mathcal{M}_3, J_\pm, I\}\). This isomorphism means that for each such system it is possible to construct a complete set of eigenstates of the operator \(\mathcal{M}_3\) and the Casimir operator, \(C\), of os(1). These states form a representation space for os(1). They are
also eigenstates of the number operator, \( J_+ J_- \), constructed from the ladder operators, \( J_\pm \), of \( \text{os}(1) \). This is a consequence of the relationship between the operators \( M_3 \) and \( K_3 \) (Eq. (28) of paper I). Only for HO do these states correspond to energy eigenstates.

In Sec. 3 of paper I, we showed that the extremal state is a Gaussian function. For all of the systems discussed in this paper, the limit as \( \tau \to 0 \) of \( K_3 \) is an time-independent oscillator Hamiltonian (HO, FP, RO) or a time-independent driven oscillator Hamiltonian (LP, DHO). (See Sec. 3.3 and 3.4 of this paper.) Therefore, each system has effectively been transformed by the \( \mathcal{R} \)-separable coordinates, \((\zeta, \eta)\) (Sec. 3, paper I), into a time-independent oscillator or a driven oscillator.

In the general treatment of paper I, we computed squeezed-state wave functions for both the \((\alpha, z)\)- and \((z, \alpha)\)-representations. [See Eqs. (63) and (64) of I.] Each of them were written as expansions in eigenstates of the number operator. We may think of these expansions as representing transformed Gaussian functions [8]. Expectation values for position and momentum, uncertainties in position and momentum, and their uncertainty product were derived.

In the \((\alpha, z)\)-representation, according to Eqs. (24) and (25), \( \langle x \rangle \) and \( \langle p \rangle \) depend only on \( \alpha \) and not on \( z \). Since \( \alpha \) is fixed by the initial position and momentum, \( z \) is free to vary. [See Eq. (22).] However, in the \((z, \alpha)\)-representation, according to Eqs. (26) and (27), the expectation values depend upon all four parameters \(|\alpha|, \delta, r, \) and \( \theta \). From Eq. (23), we can determine any two of these in terms of the initial position and initial momentum. A third way of expressing the expectation values of \( x \) and \( p \) is in terms of initial position and momentum. [See Eqs. (20) and (21).] This way is independent of the complex parameters \( \alpha \) and \( z \), and therefore is identical for both representations. Only then are the relationships between the four parameters, \( \alpha, \delta, r, \) and \( \theta \), and the initial position and momentum sensitive to which representation we are using.
The expectation values of $x$ and $p$ satisfy the classical equations of motion and describe classical trajectories in phase space: for HO the trajectory is an ellipse; for FP it is a straight line; for LP it is a parabola; for DHO it is a displaced ellipse, and for RO it is a hyperbola.

The uncertainty products do not depend on $\alpha$, but do depend on $z$. Also, they are independent of representation. For HO and DHO the time-dependence of the uncertainty product is linked to the squeeze parameter, $z$. If $z = 0$, then the uncertainty product is minimized. When $z \neq 0$, there is an oscillation in the uncertainty product, subject to $(\Delta x)^2(\Delta p)^2 \geq \frac{1}{4}$. For the other three systems, the uncertainty product increases with time, starting from a state of minimum uncertainty. The Gaussian wave packet for these three systems will eventually dissipate.

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