A time domain approach for the exponential stability of a linearized compressible flow-structure PDE system

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This work is motivated by a longstanding interest in the long time behavior of flow-structure interaction (FSI) PDE dynamics. We consider a linearized compressible flow-structure interaction (FSI) PDE model with a view of analyzing the stability properties of both the compressible flow and plate solution components. In our earlier work, we gave an answer in the affirmative to question of uniform stability for finite energy solutions of said compressible flow-structure system, by means of a “frequency domain” approach. However, the frequency domain method of proof in that work is not “robust” (insofar as we can see), when one wishes to study longtime behavior of solutions of compressible flow-structure PDE models, which track the appearance of the ambient state onto the boundary interface. Nor is a frequency domain approach in this earlier work availing when one wishes to consider the dynamics, in long time, of solutions to physically relevant nonlinear versions of the compressible flow-structure PDE system under present consideration (e.g., the Navier–Stokes nonlinearity in the PDE flow component or a nonlinearity of Berger/Von Karman type in the plate equation). Accordingly, in the present work, we operate in the time domain by way of obtaining the necessary energy estimates, which culminate in an alternative proof for the uniform stability of finite energy compressible flow-structure solutions. Since there is a need to avoid steady states in our stability analysis, as a prerequisite result, we also show here that zero is an eigenvalue for the generators of flow-structure systems, whether the material derivative term be absent or present. Moreover, we provide a clean characterization of the (one dimensional) zero eigenspace, with or without material derivative, under an appropriate assumption on the underlying ambient vector field.

KEYWORDS
flow-structure interaction, compressible flows, uniform stability

MSC CLASSIFICATION
35M30; 35Q35; 76N10

1 | INTRODUCTION

The linearized compressible flow-structure PDE model which we will consider arises in the context of the design of various engineering systems and the study of gas dynamics. This coupled system describes the interaction between a plate and a given compressible gas flow. In contrast to incompressible fluid flows, wherein the fluid density is assumed...
to be a constant, compressible gas flow models will contain an additional fluid density variable and involve other state spaces. For further details, the reader is referred to previous studies.\textsuperscript{1-4} The presence of the density (pressure) equations in compressible cases will tend to make the analysis quite different than that for incompressible flows; in particular, one must deal with the extra density (pressure) solution variable. Moreover, and intrinsic to the problem under consideration, linearization of the compressible Navier–Stokes equation occurs around a rest state that contains an arbitrary ambient vector field \( \mathbf{U} \). Qualitative properties of this model—that is, wellposedness and long-term analysis in the sense of global attractors (in the presence of the von Karman plate nonlinearity) were firstly analyzed in Chueshov,\textsuperscript{4} in the case \( \mathbf{U} = 0 \).

However, the case \( \mathbf{U} \neq 0 \) was recognized to be challenging, since the presence of a nonzero ambient field introduces problematic terms such as \( \mathbf{U} \cdot \nabla p \), (where \( p \) is the pressure variable), a term which is “unbounded” with respect to the underlying finite energy space of wellposedness.

With respect to compressible flow-structure PDE systems with underlying nonzero ambient terms, a positive answer to the wellposedness question was given in Avalos and Geredeli,\textsuperscript{1} again in the case \( \mathbf{U} \neq 0 \). With a view of handling the aforesaid troublesome term \( \mathbf{U} \cdot \nabla p \), a frequency domain approach is invoked in Avalos and Geredeli.\textsuperscript{1} By way of appropriately estimating \( \mathbf{U} \cdot \nabla p \) in Avalos and Geredeli,\textsuperscript{1} as a static (and not time-dependent term), the frequency domain approach allows for an appropriate decomposition of static Stokes flow, and an eventual invocation of the nonsmooth domain version of the Agmon–Douglas–Nirenberg Theorem; see p. 75 of Dauge.\textsuperscript{5} In addition to dealing with unbounded term \( \mathbf{U} \cdot \nabla p \), a large part of the work in Avalos and Geredeli\textsuperscript{1} is devoted to a spectral analysis of the compressible flow-structure generator, by way of ultimately invoking the well-known resolvent criteria for exponential decay in Huang\textsuperscript{6} and Prüss.\textsuperscript{7}

Although the methodology set forth in Avalos and Geredeli\textsuperscript{1} is effective in establishing exponential stability for solutions of the compressible flow-structure PDE models (2)–(4) below, with \( \kappa = 0 \), it’s use seems limited when dealing with (2)–(4) when the physically relevant material derivative term is present (i.e., \( \kappa = 1 \); see Avalos et al.\textsuperscript{3} for the modeling aspects of this problem; also Dowell\textsuperscript{8}). In particular, since the presence of the material derivative term \( \mathbf{U} \cdot \nabla w \) on the boundary interface constitutes an unbounded perturbation of the compressible flow-structure PDE system, a necessary spectral analysis for \( \kappa = 1 \), analogous to that in Avalos and Geredeli,\textsuperscript{1} is problematic. In addition, the critical frequency domain estimates which were obtained in Avalos and Geredeli\textsuperscript{1} for the linear problem do not lend themselves readily to adaptation so as to handle nonlinear versions of (2)–(4), versions in which the Navier–Stokes or von Karman plate nonlinearities are present. Consequently, the problem of analyzing long time behavior for nonlinear compressible flow-structure PDE systems—particularly in the sense of global attractors—must be undertaken in the “time domain” rather than the “frequency domain.”

Accordingly, the principal contribution of the present manuscript is to give an alternative proof for the exponential stability of the solutions via a certain multiplier method in the time domain. In particular, our main (gradient type) multiplier is based upon the solution of a certain Neumann problem, a solution which is sufficiently smooth, even considering the unavoidable boundary interface singularities; see Dauge\textsuperscript{9} and Jerison and Kenig.\textsuperscript{10} We should also note that the application of this multiplier is practicable and convenient, due to the characterization (compatibility condition) of the stabilizable finite energy space \( \text{Null}(\mathcal{A}) \), where \( \mathcal{A} \) is the semigroup generator of the system.

Though there are some wellposedness and long time behavior results for nonlinear “uncoupled” compressible flows,\textsuperscript{11-18} besides being of intrinsic interest in its own right, this manuscript will also serve as a blue print for our forthcoming work which will address the existence of compact attractors for nonlinear flow-structure PDE interactions in which the material derivative appears in the normal component boundary condition of the compressible flow variable (i.e., \( \kappa = 1 \)). This material derivative term on the boundary interface represents an unbounded perturbation of the compressible flow-structure semigroup generator, in addition to the presence of a nonlocal nonlinearity (von Karman or Berger). With this future work in mind, in this manuscript, we also analyze the Null space of the generator of the system, in the presence or absence of the material derivative on the interaction surface \( \Omega \). Again, our main reason for doing this is to avoid finite energy initial data which give rise to steady states. In the course of proof, which partly involves a necessary assumption on the ambient field \( \mathbf{U} \), we show that, despite the unboundedness presented by the material derivative perturbation, the Null Space of the “material derivative” generator \( \mathcal{A}_1 \), (\( \kappa = 1 \)) will coincide with Null Space of the “material derivative free” generator \( \mathcal{A}_0 \), (\( \kappa = 0 \)). It is also worth mentioning that there will be nontrivial steady states for the system once the long time behavior–attractor–work is under consideration. This will make the attractor nontrivial and require quite similar (smallness) assumptions on the size of the ambient flow as imposed in this manuscript.

In what follows, we provide the PDE description of the interaction model under the study. Let the flow domain \( \partial \Omega \) be a bounded subset of \( \mathbb{R}^3 \), with boundary \( \partial \Omega \). Moreover, \( \partial \Omega = \bar{S} \cup \bar{\Omega} \), with \( S \cap \Omega = \emptyset \) and with (structure) domain \( \Omega \subset \mathbb{R}^3 \)
being a flat portion of \( \partial \Omega \). In particular, assume that \( \partial \Omega \) has the following specific configuration:

\[
\Omega \subset \{ x = (x_1, x_2, 0) \} \quad \text{and} \quad S \subset \{ x = (x_1, x_2, x_3) : x_3 \leq 0 \},
\]

(1) and \( n(x) \) denotes the unit outward normal vector to \( \partial \Omega \) where \( n|_{\Omega} = [0, 0, 1] \). We also impose additional following geometrical conditions on \( \Omega \):

**Condition 1** The flow domain \( \Omega \) should be a curvilinear polyhedral domain—that is, \( \Omega \) has a finite set of smooth edges and corners; see Dauge\(^9\)—which satisfies the following assumptions:

1. Each corner of the boundary \( \partial \Omega \)—if any—is diffeomorphic to a convex cone,
2. Each point on an edge of the boundary \( \partial \Omega \) is diffeomorphic to a wedge with opening \( < \pi \).

**Remark 1.** The point of making such assumptions on the geometry of domain \( \Omega \) is that they will allow for our application of elliptic results for solutions of second-order boundary value problems on corner domains; see Dauge\(^9\). In particular, these results will be invoked in our time dependent multiplier method by way of proving our uniform stability result.

The usual, familiar geometries can be considered as in Figure 1.

The coupled PDE system which we will consider is the result of a linearization which is undertaken in Chueshov\(^4\) and Avalos et al.:\(^2\) Within the three-dimensional geometry \( \Omega \), the compressible Navier–Stokes equations are present, assuming the flow which they describe to be barotropic. This system is linearized with respect to some reference rest state of the form \( \{ p_*, U, \varrho_* \} \): The pressure and density components \( p_*, \varrho_* \) are scalars, and the arbitrary ambient field \( U : \partial \Omega \to \mathbb{R}^3 \)

\[
U(x_1, x_2, x_3) = [U_1(x_1, x_2, x_3), U_2(x_1, x_2, x_3), U_3(x_1, x_2, x_3)]
\]

is given.

In Chueshov\(^4\) and Avalos et al:\(^2\), we already see that noncritical lower-order terms are deleted, and the aforesaid pressure and density reference constants are set equal to unity. Thus, we are presented with the following system of equations, in solution variables \( u(x_1, x_2, x_3, t) \) (flow velocity), \( p(x_1, x_2, x_3, t) \) (pressure), \( w(x_1, x_2, t) \) (elastic plate displacement), and \( w_t(x_1, x_2, t) \) (elastic plate velocity):

\[
\begin{align*}
\rho \left( u \cdot \nabla u + \frac{1}{2} \nabla u + \frac{1}{2} \nabla \rho \right) &= 0 \quad \text{in} \quad \Omega \times (0, \infty), \\
\rho \left( u \cdot \nabla u - \nabla p + \eta \Delta u \right) &= 0 \quad \text{in} \quad \Omega \times (0, \infty), \\
\rho \frac{\partial u}{\partial t} &= \rho_0 \quad \text{on} \quad \partial \Omega \times (0, \infty), \\
\rho \frac{\partial u}{\partial t} &= \rho_0 \quad \text{on} \quad S \times (0, \infty), \\
u \cdot n &= w_t + \kappa u \cdot \nabla w \quad \text{on} \quad \Omega \times (0, \infty), \\
\Delta^2 w + \frac{2}{3} \partial^2_{x_3} (u_3) + \frac{1}{2} \nabla \rho - \rho &= 0 \quad \text{on} \quad \Omega \times (0, \infty) \\
\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial t} \quad \text{on} \quad \partial \Omega \times (0, \infty), \\
\rho \left( p(0), u(0), w(0), w_t(0) \right) &= \left( p_0, u_0, w_0, w_t(0) \right).
\end{align*}
\]

(2) (3) (4)

(where above, \( \nabla w = [w_{x_1}, w_{x_2}, 0] \); see Avalos et al.). We note that the flow linearization is taken with respect to some generally inhomogeneous compressible Navier–Stokes system; thus, \( U \) does not need to be divergence free generally; there
are also initially forcing terms in the pressure and flow equations (and energy level terms) which we have neglected, since they do not effect the current analysis. However, we retained the energy level term div(U)p, for reasons of “invariance”; see Remark 6 below. It is also implicitly used in the proof of stability result Theorem 2—in relation (47). This flow-structure system is a generalization of that considered by the late Igor Chueshov in with therein fixed vector field U = 0. In contrast, the PDE systems (2)–(4) depend upon a generally nonzero, fixed, ambient vector field U about which the linearization takes place. The quantity η > 0 represents a drag force of the domain on the viscous flow. In addition, the quantity τ in (2) is in the space TH^{1/2}(∂Ω) of tangential vector fields of Sobolev index 1/2; that is,

\[ \tau \in TH^{1/2}(\partial \Omega) = \{ \mathbf{v} \in H^{1/2}_T(\partial \Omega) : \mathbf{v}|_{\partial \Omega} \cdot \mathbf{n} = 0 \text{ on } \partial \Omega \}. \]

(See, e.g., p.846 of Buffa et al.) In addition, we take ambient field U to be in the space

\[ \mathbf{V}_0 = \{ \mathbf{v} \in H^1(\Omega) : \mathbf{v}|_{\partial \Omega} \cdot \mathbf{n} = 0 \text{ on } \partial \Omega \}. \]

(This vanishing of the boundary for ambient fields is a standard assumption in compressible flow literature; see previous studies.)

Moreover, the stress and strain tensors in the flow PDE component of (2)–(4) are defined respectively as

\[ \sigma(\mathbf{\mu}) = 2\nu\mathbf{e}(\mathbf{\mu}) + \lambda I_3 \cdot \mathbf{e}(\mathbf{\mu}) I_3 \quad : \quad \mathbf{e}_{ij}(\mathbf{\mu}) = \frac{1}{2} \left( \frac{\partial \mathbf{\mu}_j}{\partial x_i} + \frac{\partial \mathbf{\mu}_i}{\partial x_j} \right), \quad 1 \leq i, j \leq 3, \]

where Lamé Coefficients \( \lambda \geq 0 \) and \( \nu > 0 \). The associated finite energy space will be

\[ H = L^2(\Omega) \times L^2(\Omega) \times H^1_0(\Omega) \times L^2(\Omega), \]

which is a Hilbert space, topologized by the following standard inner product:

\[ \langle \mathbf{y}_1, \mathbf{y}_2 \rangle_H = (p_1, p_2)_{L^2(\Omega)} + (\mathbf{u}_1, \mathbf{u}_2)_{L^2(\Omega)} + (\Delta w_1, \Delta w_2)_{L^2(\Omega)} + (v_1, v_2)_{L^2(\Omega)}, \]

for any \( \mathbf{y}_i = (p_i, \mathbf{u}_i, w_i, v_i) \in H, \ i = 1, 2. \)

**Remark 2.** As we noted in Avalos et al., the flow PDE boundary conditions which are in (2) are the so-called impermeability-slip conditions (no flow passes through the boundary), and these conditions will necessarily involve the plate's deflections \( w \) on \( \Omega \subset \partial \Omega \).

**Remark 3.** In many uncoupled or coupled gas flow PDE dynamics, the underlying ambient flow profile is quite specific. Namely, \( U = [0, 0, U_0] \), where \( U_0 > 0 \); see Chueshov et al. In the present work, along with handling this canonical case, flow-structure PDE systems of the types (2)–(4) with more arbitrary underlying ambient flow field may be considered; see, for example, the ambient fields in previous studies.

## 2 | FUNCTIONAL SETTING OF THE PROBLEM

Throughout, for a given domain \( D \), the norm of corresponding space \( L^2(D) \) will be denoted as \( ||\cdot||_D \) (or simply \( ||\cdot|| \) when the context is clear). Inner products in \( L^2(\Omega) \) or \( L^2(\partial \Omega) \) will be denoted by \( \langle \cdot, \cdot \rangle_\partial \), whereas inner products \( L^2(\partial \Omega) \) will be written as \( \langle \cdot, \cdot \rangle_{\partial \Omega} \). We will also denote pertinent duality pairings as \( \langle \cdot, \cdot \rangle_{XX'} \) for a given Hilbert space \( X \). The space \( H^s(\Omega) \) will denote the Sobolev space of order \( s \), defined on a domain \( D; H^s_0(\Omega) \) will denote the closure of \( C^\infty(\Omega) \) in the \( H^s(D) \)-norm \( ||\cdot||_{H^s(D)} \). We make use of the standard notation for the boundary trace of functions defined on \( \partial \), which are sufficiently smooth; for example, for a scalar function \( \Phi \in H^s(\Omega), \frac{1}{2} < s < \frac{3}{2}, \gamma(\Phi) = \Phi|_{\partial \Omega} \), which is a well-defined and surjective mapping on this range of \( s \), owing to the Sobolev Trace Theorem on Lipschitz domains (see, e.g., Nečas or Theorem 3.38 of McLean).

With respect to the above setting, the PDE system given in (2)–(4) can be written as an ODE in Hilbert space \( H \). That is, if \( \Phi(t) = [p, u, w, w] \in C([0, T]; H) \) solves the problems (2)–(4), then, for the respective cases \( \kappa = 0 \) or \( \kappa = 1 \), there is a
modeling operator \( A_\kappa : D(A_\kappa) \subset H \to H \) such that \( \Phi(\cdot) \) satisfies

\[
\begin{align*}
\frac{d}{dt} \Phi(t) &= (A_\kappa + B_0)\Phi(t), \\
\Phi(0) &= \Phi_0.
\end{align*}
\]  

(9)

Here \( A_\kappa : D(A_\kappa) \subset H \to H \) is defined as follows:

\[
A_\kappa = \begin{bmatrix}
-U \cdot V(\cdot) & -\text{div}(\cdot) & 0 & 0 \\
V(\cdot) & -\text{div}(\cdot) - \eta I - U \cdot V(\cdot) & 0 & 0 \\
0 & 0 & 0 & I \\
\| \cdot \|^2_{\Omega} - [2\nu \partial_3(u_0) + \lambda \text{div}(\cdot)] & -\Delta^2 & 0 & 0
\end{bmatrix};
\]  

(10)

\[
D(A_\kappa) = \{ (p_0, u_0, w_1, w_2) \in L^2(\Omega) \times H^1(\Omega) \times H^2(\Omega) \times H^2(\Omega) : \text{properties (A.i) -- (A.v) hold} \},
\]

where

(A.i) \( U \cdot V p_0 \in L^2(\Omega) \),

(A.ii) \( \text{div}(u_0) - \nu p_0 \in L^2(\Omega) \) (Consequently, we infer the boundary trace regularity

\[
[\sigma(u_0)n - p_0n]_{\partial \Omega} \in H^{-\frac{1}{2}}(\partial \Omega),
\]

(A.iii) \( -\Delta^2 w_1 + [2\nu \partial_3(u_0) + \lambda \text{div}(u_0)]_{\Omega} + p_0_{\Omega} \in L^2(\Omega) \),

(A.iv) \( (\sigma(u_0)n - p_0n) \perp TH^{1/2}(\partial \Omega) \). That is,

\[
\langle \sigma(u_0)n - p_0n, \tau \rangle_{H^{-\frac{1}{2}}(\partial \Omega) \times H^{\frac{1}{2}}(\partial \Omega)} = 0 \text{ for every } \tau \in TH^{1/2}(\partial \Omega)
\]

(and so \( \sigma(u_0)n - p_0n, \tau = 0 \) in the sense of distributions; see Remark 3.1 of Avalos et al.2.)

(A.v) The flow velocity component \( u_0 = f_0 + \tilde{f}_0 \), where \( f_0 \in V_0 \) and \( \tilde{f}_0 \in H^1(\Omega) \) satisfies

\[
\tilde{f}_0|_{\partial \Omega} = \begin{cases}
0 & \text{on } S \\
(w_2 + \kappa U \cdot \nu w_1) n & \text{on } \Omega
\end{cases}
\]

(11)

(and so \( f_0|_{\partial \Omega} \in TH^{1/2}(\partial \Omega) \).

Moreover, \( B_0 \in \mathcal{L}(H) \) is given by

\[
B_0 \begin{bmatrix}
p_0 \\
u_0 \\
w_0 \\
w_1
\end{bmatrix} = \begin{bmatrix}
-\text{div}(U)p_0 \\
0 \\
0 \\
w_1
\end{bmatrix}.
\]  

(12)

In Avalos et al.2, it was shown that solutions to the compressible flow-structure PDE systems (2)–(4), again for \( \kappa = 0 \) or \( \kappa = 1 \), with initial data in said finite energy space \( H \), can be associated with a strongly continuous semigroup \( \{e^{\lambda t}\}_{t \geq 0} \subset \mathcal{L}(H) \) which yields the following wellposedness result:

**Theorem 1.** (See Theorem 3.1 of Avalos et al.2, and Theorem 5.1 of Avalos et al.3.) Assume that ambient vector field \( U \in V_0 \cap H^1(\Omega) \) (when \( \kappa = 0 \) and \( \kappa = 1 \)). Additionally, in the case \( \kappa = 1 \), let \( U|_{\Omega} \in C^2(\bar{\Omega}) \).

(i) (Wellposedness) The flow-structure operator \( A_\kappa : D(A_\kappa) \subset H \to H \) generates a \( C_0 \)-semigroup on \( H \). Accordingly, the solution of (2)–(4) — with initial data \( [p_0, u_0, w_0, w_1] \in H \) — may be given by

\[
\begin{bmatrix}
p(t) \\
u(t) \\
w(t) \\
w_1(t)
\end{bmatrix} = e^{(A_\kappa + B_0)t} \begin{bmatrix}
p_0 \\
u_0 \\
w_0 \\
w_1
\end{bmatrix} \in C([0, \infty); H].
\]  

(12)

(ii) (Energy relation) We define

\[
E(t) = \frac{1}{2} \left[ \|p(t)\|_{\Omega}^2 + \|u(t)\|_{\Omega}^2 + \|\Delta w(t)\|_{\Omega}^2 + \|w_1(t)\|_{\Omega}^2 \right],
\]  

(13)

1 The existence of an \( H^1(\Omega) \)-function \( \tilde{f}_0 \) with such a boundary trace on Lipschitz domain \( \Omega \) is assured; see, for example, Theorem 3.33 of McLean.31
where \([p(t), u(t), w(t), w_i(t)]\) is the solution of compressible flow-structure systems (2)–(4), given explicitly by (12). Then for all \(0 \leq s < t\), we have the relation

\[
E(t) + \int_s^t \left[ \sigma(u(r)), \varepsilon(u(r))_\sigma + \eta \|u(r)\|_\sigma^2 \right] \, dr = E(s) - \frac{1}{2} \int_s^t \int_\sigma \text{div}(U) \, \left[ |p(r)|^2 + |u(r)|^2 \right] \, d\sigma \, dr
\]

\[+ \kappa \int_s^t \left[ 2v \partial_x \cdot (u_2) + \lambda \text{div}(u) - p \right]_{\Omega} \, U \cdot \nabla w \, \Omega \, dr.\]  

(14)

From the expression (14), it is seen that the generators \(A_\kappa, A_\kappa + B_0 : D(A_\kappa) \subset H \rightarrow H\) do not dissipate the energy of the systems (2)–(4). Nonetheless, in Avalos and Geredeli, we establish that solutions of (2)–(4) decay uniformly, with respect to initial data which is \(H\)-orthogonal to the one dimensional null space of \(A_0\) (see Theorem 2 [ii] therein).

The main intent of the present work is to (i) discern the null space for \(A_1 : D(A_1) \subset H \rightarrow H\), under appropriate assumptions, and (ii) give an alternative proof of uniform stability for the (material derivative-free) semigroup \(e^{\lambda t A_0}\) in the case \(\kappa = 1\). Also an explicit characterization for the corresponding zero eigenspace was given there, again in the case \(\kappa = 0\). However, in the presence of the unbounded—material derivative—term \(U \cdot \nabla w\), it is not at all clear a priori that the case \(\kappa = 1\) should give rise to the same zero eigenspace. In this section, we give a positive answer to this question: Indeed, zero is also an eigenvalue for the generator \(A_1\) whose null space is identical with that for \(A_0\). (As we said, this spectral information will be needed in our future work on longtime behavior properties of nonlinear compressible flow-structure PDE dynamics, with material derivative term in place.) Before giving this result, let us recall the following lemma (see Lemma 10 of Avalos and Geredeli):

**Lemma 1.** Let \(U \in V_0 \cap H^3(\Omega)\) and \(\|\text{div}(U)\|_\infty \leq C_0^*(\Psi)\) (sufficiently small), where \(C_0^*(\Psi)\) is a positive constant with \(\Psi = \Psi(U) \equiv \|U\|_{H^3(\Omega)} + 1\). Then one has the following:

The subspace \(\text{Null}(A_0) \subset H\) of the flow-structure generator \(A_0 : D(A_0) \subset H \rightarrow H\) is one dimensional. In particular, \(\text{Null}(A_0)\) is given explicitly as

\[
\text{Null}(A_0) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ A^{-1}(1) \\ 0 \end{bmatrix} \right\},
\]

(15)

where \(A : L^2(\Omega) \rightarrow L^2(\Omega)\) is the elliptic operator

\[
\hat{A} \sigma = \Delta^2 \sigma, \quad \text{with} \quad D(\hat{A}) = \{ w \in H^4(\Omega) \cap H_0^2(\Omega) : \Delta^2 w \in L^2(\Omega) \}.
\]

(16)

Now, we give the same result for the generator \(A_1\) but we note that if we wish for the generator \(A_1 : D(A_1) \subset H \rightarrow H\) of the flow-structure PDE system to have the same one dimensional null space in (15), as for \(A_0\), despite the additional (unbounded) material derivative term, then: from the domain criterion (A.v) and (15), one must necessarily have, for \(\kappa = 1\), the relation

\[
U \cdot \nabla A^{-1}(1) = 0 \quad \text{on} \quad \Omega
\]

(17)

where the biharmonic operator \(\hat{A}\) is as in (16). That is,

\[
[U_1(x_1, x_2, 0), U_2(x_1, x_2, 0) : \frac{\partial}{\partial x_1} A^{-1}(1), \frac{\partial}{\partial x_2} A^{-1}(1)] = 0.
\]
Lemma 2. (a) Suppose that the ambient field \( U \) satisfies (17), and moreover \( \| \text{div}(U) \|_{\infty} \leq C_0^*(\Psi) \) (sufficiently small), where \( C_0^*(\Psi) \) is a positive constant with \( \Psi = \Psi(U) \equiv \| U \|_{H^1(\Omega)} + 1 \). Then, as in Lemma 1, the subspace \( \text{Null}(A_1) \subset H \) of the (when the material derivative is taken into account) flow-structure generator \( A_1 \) is given by

\[
\text{Null}(A_1) = \text{Null}(A_0) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ A^{-1}(1) \end{bmatrix} \right\}
\]

(18)

(b) The orthogonal complement of \( \text{Null}(A_\kappa) \) (\( \kappa = 0 \) or \( \kappa = 1 \)) admits of the characterization

\[
[\text{Null}(A_\kappa)]^\perp = \{ [p_0, u_0, w_1, w_2] \in H : \int_\Omega p_0 d\Omega + \int_\Omega w_1 d\Omega = 0 \}.
\]

(19)

Remark 4. Suppose the curvilinear polyhedron \( \Omega \) is strictly an “edge domain” – i.e., the geometry has no corners – such that each point on an edge of \( \partial \Omega \) is diffeomorphic to a wedge with opening \( <\pi/2 \). Then an example of an ambient field \( U \in V_0 \cap H^3(\Omega) \) which meets the compatibility condition (17) can be constructed as follows: Let boundary data \( g \) satisfy

\[
\begin{bmatrix}
-\frac{\partial}{\partial x_3} A^{-1}(1) \\
\frac{\partial}{\partial x_3} A^{-1}(1) \\
0
\end{bmatrix} \in H^4(\Omega) \text{ on } \Omega
\]

\[
g = \begin{cases}
0 \text{ on } \partial \Omega.
\end{cases}
\]

Therewith, \( g \in H^3(\partial \Omega) \); see, for example, Theorem 3.33, p. 95, of McLean.31 Here, we are implicitly using the fact that \( \left. \frac{\partial}{\partial x_3} A^{-1}(1) \right|_{\partial \Omega} = 0 \). Subsequently, we can invoke the well-established elliptic regularity results on edge domains—see, for example, Dauge19—for Neumann problem, with boundary data \( g \), so as to secure a \( U \) which meets the requirements of Lemma 2, after an appropriate scaling.

Proof. We focus here on the touchier case \( \kappa = 1 \), in as much as the details for \( \kappa = 0 \) are subsumed in the material derivative case. (See also Avalos and Geredeli3)

Suppose \( \Phi = [p_0, u_0, w_1, w_2] \in D(A_1) \) is a solution of

\[
A_1 \Phi = 0.
\]

(20)

where \( A_1 : D(A_1) \subset H \to H \) is as given in (10). (Without loss of generality, we take \( \Phi \) to be real-valued.) With respect to the pressure component, we invoke the \( L^2 \)-decomposition

\[
p_0 = q_0 + c_0,
\]

(21)

where

\[
q_0 \text{ satisfies } \int_\Omega q_0 d\Omega = 0, \text{ and } c_0 = \text{constant}.
\]

(22)

Therewith, in PDE terms, the abstract relation then becomes

\[
\begin{cases}
-U \cdot \nabla q_0 - \text{div} u_0 = 0 \text{ in } \Omega \\
-\nabla q_0 + \text{div} \sigma(u_0) - \eta u_0 - U \cdot \nabla u_0 = 0 \text{ in } \Omega \\
w_2 = 0 \text{ on } \Omega \\
-\Delta^2 w_1 - [2\nu \partial_{x_3}(u_0)_3 + \lambda \text{div}(u_0) - p_0]_\Omega = 0 \text{ on } \Omega.
\end{cases}
\]

(23)
\[
\begin{align*}
\sigma(u_0) - p_0 n & = 0 \text{ on } \partial \Omega \\
\mathbf{u}_0 \cdot n & = 0 \text{ on } \mathcal{S} \\
\mathbf{u}_0 \cdot n & = \mathbf{U} \cdot \nabla w_1 \text{ on } \Omega \\
w_1 & = \frac{\partial w_1}{\partial n} = 0 \text{ on } \partial \Omega.
\end{align*}
\]

We have immediately then,

\[w_2 = 0.\]  \hfill (25)

Secondly, we multiply the pressure PDE component in (23) and (24) by \(p_0\) and the fluid PDE component by \(\mathbf{u}_0\). Subsequent integrations and integrations by parts, and a consideration of domain criteria (A.iv), (A.v) and (25), yield then

\[
\begin{align*}
\langle \sigma(u_0), e(u_0) \rangle_{\mathcal{S}} & \geq \frac{1}{2} (\text{div}(u_0)_\mathcal{S} - p_0)_{\mathcal{S}} + \frac{1}{2} (\text{div}(u_0)_{\mathcal{S}} - u_0)_{\mathcal{S}} \\
- \langle \nabla u_0, u_0 \rangle_{\mathcal{S}} & \geq \left\{ [2 v \partial_3(u_0)_3 + \lambda \text{div}(u_0) - q_0]_{\Omega}, u_3 \right\}_{\Omega} \\
+ \frac{1}{2} (\text{div}(u_0)_{\mathcal{S}} - u_0)_{\mathcal{S}}.
\end{align*}
\]

(For the last term on RHS, we are also using the fact that \(n = [0, 0, 1]\) on \(\Omega\).) Therewith, combining the decomposition (21) with Green’s formula and the fact that \(\mathbf{U} \in \mathbf{V}_0\), we then obtain

\[
\begin{align*}
\langle \sigma(u_0), e(u_0) \rangle_{\mathcal{S}} & \geq (\text{div}(u_0), c_0) \\
- \langle \nabla u_0, u_0 \rangle_{\mathcal{S}} & \geq \left\{ [2 v \partial_3(u_0)_3 + \lambda \text{div}(u_0) - q_0]_{\Omega}, u_3 \right\}_{\Omega} \\
+ \frac{1}{2} (\text{div}(u_0)_{\mathcal{S}} - u_0)_{\mathcal{S}}.
\end{align*}
\]

To estimate the first term on right hand side of (27), we multiply the pressure equation in (23) by constant component \(c_0\) of (21) and integrate over \(\mathcal{S}\). This gives,

\[- (\mathbf{U} \cdot \nabla q_0, c_0) = (\text{div}(u_0), c_0).\]

Subsequently, we integrate by parts both sides of this relation, while bearing in mind the domain criterion (A.v) and (25) (and the fact that \(\mathbf{U} \in \mathbf{V}_0\), so as to have

\[(\text{div}(u_0)_{\mathcal{S}} - u_0)_{\mathcal{S}} = (\text{div}(c_0, \mathbf{U} \cdot \nabla w_1)_{\mathcal{S}}.
\]

Applying this relation to (27) gives now

\[
\begin{align*}
\langle \sigma(u_0), e(u_0) \rangle_{\mathcal{S}} & \geq \frac{1}{2} (\text{div}(u_0)_{\mathcal{S}} - u_0)_{\mathcal{S}} \\
& \geq \left\{ [2 v \partial_3(u_0)_3 + \lambda \text{div}(u_0) - q_0]_{\Omega}, u_3 \right\}_{\Omega} \\
& \quad + \frac{1}{2} (\text{div}(u_0)_{\mathcal{S}} - u_0)_{\mathcal{S}}.
\end{align*}
\]

and subsequently invoking the domain criterion (A.v) (for \(\kappa = 1\)) and the elastic equation on \(\Omega\), we have now

\[
\begin{align*}
\langle \sigma(u_0), e(u_0) \rangle_{\mathcal{S}} & \geq \frac{1}{2} (\text{div}(u_0)_{\mathcal{S}} - u_0)_{\mathcal{S}} \\
& \geq \left\{ [2 v \partial_3(u_0)_3 + \lambda \text{div}(u_0) - q_0]_{\Omega}, U \cdot \nabla \left( A^{-1} [2 v \partial_3(u_0)_3 + \lambda \text{div}(u_0) - q_0]_{\Omega} \right) \right\}_{\Omega} \\
& \quad + \frac{1}{2} (\text{div}(u_0)_{\mathcal{S}} - u_0)_{\mathcal{S}}.
\end{align*}
\]

and subsequently invoking (17) yields then

\[
\begin{align*}
\langle \sigma(u_0), e(u_0) \rangle_{\mathcal{S}} & \geq - \left\{ [2 v \partial_3(u_0)_3 + \lambda \text{div}(u_0) - q_0]_{\Omega}, U \cdot \nabla \left( A^{-1} [2 v \partial_3(u_0)_3 + \lambda \text{div}(u_0) - q_0]_{\Omega} \right) \right\}_{\Omega} \\
& \quad + \frac{1}{2} (\text{div}(u_0)_{\mathcal{S}} - u_0)_{\mathcal{S}}.
\end{align*}
\]

(28)
To make use of this relation, we appeal to the estimate provided by Theorem 2.4 or Remark 2.5 of Temam, for solution pair \((u_0, q_0)\) of (Stokes system) (23). Namely, we have

\[
\| q_0 \|_\Omega^2 + (\sigma(u_0), e(u_0))_\Omega + \eta \| u_0 \|_\Omega^2 \lesssim C \left( \| U \cdot \nabla u_0 \|_\Omega^2 + \| \nabla \cdot u_0 \|_\Omega^2 + \| u_0 \|_{H^2(\partial \Omega)}^2 \right)
\]

(We are also implicitly using Korn’s Inequality.) Estimating right hand side by Korn’s Inequality and the Sobolev Trace Theorem gives now

\[
\| q_0 \|_\Omega^2 + (\sigma(u_0), e(u_0))_\Omega + \eta \| u_0 \|_\Omega^2 \lesssim C \| U \|_{H^2(\partial \Omega)} \| \nabla (U \cdot \nabla u_0) - q_0 \|_\Omega + C \| U \|_{H^2(\partial \Omega)} \| \nabla \cdot u_0 \|_\Omega^2 - C \eta \| u_0 \|_\Omega^2.
\]

(29)

where the term \(\Psi(U)\) is as given in the statement of Lemma 2, and positive constant \(C_1\) is independent of \(U\). (We are implicitly using here the Sobolev Imbedding Theorem with respect to field \(U \in H^3(\Omega)\). Applying (29) to the right hand side of (28) gives then,

\[
(\sigma(u_0), e(u_0))_\Omega + \eta \| u_0 \|_\Omega^2 \lesssim \left[ \left( \| 2v \partial_3 (u_0)_3 + \lambda \nabla \cdot u_0 - q_0 \|_\Omega \right)^2 + C_2(\Psi(U)) \| \nabla (U \cdot \nabla u_0) - q_0 \|_\Omega \right] \| \sigma(u_0), e(u_0) \|_\Omega + \eta \| u_0 \|_\Omega^2.
\]

whence we obtain

\[
(\sigma(u_0), e(u_0))_\Omega + \eta \| u_0 \|_\Omega^2 \lesssim C \| U \|_{H^2(\partial \Omega)} \| \nabla (U \cdot \nabla u_0) - q_0 \|_\Omega + C_2(\Psi(U)) \| \nabla \cdot u_0 \|_\Omega^2 - C \eta \| u_0 \|_\Omega^2.
\]

(30)

In turn, an integration by parts, the estimates (29) and (30) (and a rescaling of parameter \(\epsilon > 0\)) give

\[
\| \sigma(u_0) \cdot n - q_0 n \|_{H^1(\partial \Omega)}^2 \lesssim C \left( \| q_0 \|_\Omega^2 + (\sigma(u_0), e(u_0))_\Omega + \eta \| u_0 \|_\Omega^2 + \| U \cdot \nabla u_0 \|_\Omega^2 \right).
\]

(31)

Combining (30) and (31) gives now

\[
\| U \|_{H^2(\partial \Omega)} \| \nabla (U \cdot \nabla u_0) - q_0 \|_\Omega + C \| \nabla \cdot u_0 \|_\Omega^2 - C \eta \| u_0 \|_\Omega^2 \lesssim C_2(\Psi(U)) \left( \| U \|_{H^2(\partial \Omega)} \| \nabla (U \cdot \nabla u_0) - q_0 \|_\Omega + C_2(\Psi(U)) \| \nabla \cdot u_0 \|_\Omega^2 \right).
\]

(32)

If we now specify

\[
\max \left\{ \| \nabla \cdot U \|_\Omega, \| U \|_{H^2(\partial \Omega)} \right\} < \frac{1}{C_2(\Psi(U))} \equiv C_3(\Psi(U))
\]

(33)

where again \(\Psi(U) \equiv \| U \|_{H^2(\partial \Omega)} + 1\), we then deduce that

\[
u_0 = 0 \text{ and } q_0 = 0.
\]

(34)

(We are also implicitly using Korn’s Inequality). Consequently, via (21), we obtain

\[
p_0 = c_0, \text{ where } c_0 \text{ is constant.}
\]

(35)

Returning to the elastic equation on \(\Omega\) in (23), we then have

\[
w_1 = \hat{A}^{-1}(c_0).
\]

(36)

where again \(\hat{A}^{-1}: L^2(\Omega) \rightarrow L^2(\Omega)\) is the operator defined in (16). The relations (25) and (34)–(36) give the conclusion of (a) of Lemma 2 for \(\kappa = 1\). Given the definition of the \(H\)-inner product as well as the definition in (16) of \(\hat{A}: L^2(\Omega) \rightarrow L^2(\Omega)\), the relation in (b) of Lemma 2 is immediate.
**4 | EXPONENTIAL STABILITY—A TIME DOMAIN APPROACH**

This section is devoted to giving an alternative proof for the exponential stability of the solutions to material derivative-free system (2)–(4) (i.e., the case $\kappa = 0$). Unlike the frequency domain approach followed in Avalos and Geredeli\(^4\) to obtain this stability result, we give the proof of Theorem 2 in time domain, an approach which involves a gradient type multiplier by way of obtaining necessary energy estimates. As we mentioned before, the alternative proof which we provide here can in principle be used in any long term analysis for corresponding nonlinear systems. In fact, the time domain approach which we develop here will be our key departure point in the forthcoming paper related to compact global attractors of the systems (2)–(4), in the presence of the material derivative, as well as given structural (von Karman) nonlinearity.

With the notation adopted above, we give the following exponential decay result for all initial data taken from $[\text{Null}(A)]^\perp$, which is the complement of one dimensional subspace of $H$ (see Lemma 2b):

**Theorem 2.** Let the ambient vector field $U \in V_0 \cap H^3(\Omega)$ and the geometrical assumptions in Condition 1 hold. Additionally, assume that $\| \text{div}(U) \|_\infty$ is sufficiently small. Then the (material derivative-free) $C_0$-semigroup $\{ e^{tA_0 + B_0} \} \in \mathcal{L} ([\text{Null}(A_0)]^\perp)$ decays exponentially. In particular, there exist constants $\Re > 0$ and $\Omega > 0$ such that the solution to the flow-structure PDE systems (2)–(4), with initial data $\Phi_0 = [p_0, u_0, w_0 w_1] \in [\text{Null}(A_0)]^\perp$, obeys the uniform decay rate

$$\| e^{tA_0 + B_0} \Phi_0 \|_H \leq \Re e^{-\Omega t} \| \Phi_0 \|_H, \text{ for all } t > 0.$$  \(37\)

**Remark 5.** In our time domain approach, the geometrical assumptions on the polyhedral flow domain $\Omega$ are necessary, since in the course of proof of Theorem 8, we appeal to higher regularity results for the Neumann problem on domains with corners (see Jerison and Kenig\(^10\) and Dauge\(^19\) for further details.) They are analogous to the geometrical assumptions made in Avalos and Geredeli\(^1\) in which the frequency domain approach requires the invocation of higher regularity results for static Stokes flow on corner domains; see Dauge\(^5\).

**Remark 6.** We note that for $\kappa = 0$, we indeed have the invariance of $(A_0 + B_0)$ and its associated $C_0$-semigroup $\{ e^{tA_0 + B_0} \} \in \mathcal{L} ([\text{Null}(A_0)]^\perp)$ (this invariance being needed for Theorem 2): From the expression for the flow-structure adjoint given in (56) of Avalos and Geredeli\(^1\) and the null space characterization given in the present Lemma 2 (a), we have that

$$\text{Null}([A_0 + B_0]^*) = \text{Null}(A_0).$$

Thus for $\Phi_0 = [p_0, u_0, w_0 w_1] \in D(A_0) \cap [\text{Null}(A_0)]^\perp$ and zero eigenfunction $\Phi_e$ as given in (15), we have for every $t \geq 0$,

$$(e^{tA_0 + B_0} \Phi_0, \Phi_e)_H = (\Phi_0, e^{tA_0 + B_0}^* \Phi_e)_H = (\Phi_0, \Phi_e)_H = 0.$$

**Proof.** The proof is based on the application of a multiplier which is intrinsic to the compressible flow-structure PDE system under consideration. In order to deal with the lack of $H^1$-regularity of the pressure variable $p$—which is partly manifested in the unbounded term $U \cdot \nabla p$—this special multiplier exploits the compatibility condition given in (19) for any data of $[\text{Null}(A_0)]^\perp$ so as to ultimately enable us to obtain the necessary observability estimate for the energy of the system.

We will consider the following system (in the case $\kappa = 0$ in (2)–(4)) and initial data $\Phi(0) = [p(0), u(0), w(0), w_1(0)] = [p_0, u_0, w_0, w_1] = \Phi_0 \in [\text{Null}(A_0)]^\perp$:

\[
\begin{align*}
  p_t + U \cdot \nabla p + \text{div}U + \text{div}(U)p &= 0 \text{ in } \Omega \times (0, \infty) \\
  u_t + U \cdot \nabla u - \text{div}\sigma(u) + \eta u + \nabla p &= 0 \text{ in } \Omega \times (0, \infty) \\
  (\sigma(u)n - \eta n) \cdot \tau &= 0 \text{ on } \partial \Omega \times (0, \infty) \text{ for all } \tau \in TH^{1/2}(\partial \Omega) \\
  u \cdot n &= 0 \text{ on } S \times (0, \infty) \\
  u \cdot n &= w_1 \text{ on } \Omega \times (0, \infty).
\end{align*}
\]

\[
\begin{align*}
  w_{tt} + \Delta^2 w + [2\nu \partial_3(u) + \lambda \text{div}(u) - p]_{\Omega} &= 0 \text{ on } \Omega \times (0, \infty) \\
  w &= 0 \text{ on } \partial \Omega \times (0, \infty). \\
  [p(0), u(0), w(0), w_1(0)] &= [p_0, u_0, w_0, w_1] \in [\text{Null}(A_0)]^\perp.
\end{align*}
\]
From Theorem 1(ii), we have the following energy relation:

\[
E(t) + \int_s^t \left[ (\sigma(u(\tau)), e(u(\tau))_\sigma + \eta \|u(\tau)\|_2^2 \right] d\tau = E(s) - \frac{1}{2} \int_s^t \int_\sigma \text{div}(U) \left[ |p(\tau)|^2 + |u(\tau)|^2 \right] d\sigma d\tau,
\]

where \(E(t)\) is as given in (13).

By the classic “Pazy-Datko” result (see Theorem 4.1, p. 116, of Pazy\(^{33}\); also Datko\(^{34}\) and Pazy\(^{35}\)), it is enough to establish the following integral estimate, for some \(C^* > 0\) that is independent of \(t\):

\[
\int_0^\infty E(t) dt \leq C^* E(0).
\]

At this point, we note that since \(\Phi_0 = [p_0, u_0, w_0, w_1] \in [\text{Null}(A_0)]^1\), then as pointed out in Remark 6, \(e^{A_0 t} \Phi_0 \in C([0, T]; [\text{Null}(A)]^1)\) (see Proposition 13 of Avalos and Geredeli\(^{1}\). Now, with the objective estimate (42) in mind, we consider the elliptic map which was originally invoked in Chueshov\(^4\) for the case \(U = 0\). Namely, let \(\Psi = \Psi(f, g) \in H^1(\partial\Omega)\) solve the following BVP for data \(f \in L^2(\Omega)\) and \(g \in L^2(\Omega)\):

\[
\begin{cases}
-\Delta \Psi = f & \text{in } \Omega \\
\sigma \frac{\partial \Psi}{\partial n} = 0 & \text{on } S \\
\frac{\partial \Psi}{\partial n} = g & \text{on } \Omega,
\end{cases}
\]

with \([f, g]\) satisfying the compatibility condition

\[
\int_\partial f d\partial\Omega + \int_\Omega g d\Omega = 0.
\]

By known elliptic regularity results for the Neumann problem on Lipschitz domains—see, for example, Jerison and Kenig\(^{10}\),—we have

\[
\|\Psi(f, g)\|_{H^1(\partial\Omega)} \leq C \left[ \|f\|_{\partial\Omega} + \|g\|_{\Omega} \right].
\]

In order to get the estimate (42), we invoke a multiplier born of \(\Psi(\cdot, \cdot)\):

**Step I:** With respect to the fluid PDE component in (38), we multiply both sides of this equation by \(\nabla \Psi(p(t), w(t))\), where \(\Psi(p(t), w(t))\) satisfies (43) with \(f = p\) and \(g = w\). Here, we note that since \(\Phi_0 = [p_0, u_0, w_0, w_1] \in [\text{Null}(A)]^1\), then \([p(t), w(t)]\) also satisfy said compatibility condition for BVP (43). An integration in time and space then gives

\[
\int_0^T \left( u_t + \nabla p + U \cdot \nabla u - \text{div}\sigma(u) + \eta u, \nabla \Psi(p, w) \right)_\partial \tau = 0.
\]

We now look at each term in this relation separately:

**I.(i):** We start with the first term of last relation, and we have

\[
\int_0^T \left( u_t, \nabla \Psi(p, w) \right)_\partial \tau = \left( u, \nabla \Psi(p, w) \right)_\partial \tau - \int_0^T \left( u, \nabla \Psi(p_t, w_t) \right)_\partial \tau.
\]
Now, let us focus on the second term of RHS of (46): Using the pressure equation in (38)–(40), we get

\[
- \int_0^T (u, \nabla \Psi(p_t, w_t)) \, d\tau = \int_0^T (u, \nabla \Psi(U \cdot \nabla p + \text{div}(U)p, 0)) \, d\tau + \int_0^T (u, \nabla \Psi(\text{div}(u), w_t)) \, d\tau.
\]

(47)

Now, for the first term on RHS of (47), we invoke the Leray (or Helmholtz) Projector

\[ P \in \mathcal{L}(L^2(\Omega), L^2(\Omega) \cap \text{Null(div)}) \]

; subsequently,

\[ u = Pu + (I - P)u \]

satisfies

\[
\begin{aligned}
\text{div}(Pu) &= 0 \text{ in } \Omega; \quad Pu \cdot n = 0 \text{ on } \partial \Omega \\
(I - P)u &= \nabla q(u), \text{ for } q \in H^1(\Omega), \int_{\partial \Omega} qd\sigma = 0.
\end{aligned}
\]

(48)

(See, e.g., Temam \(32\) [Theorem 1.4, p. 11]) Therewith, we have

\[
\int_0^T (u, \nabla \Psi(U \cdot \nabla p + \text{div}(U)p, 0)) \, d\tau = \int_0^T (Pu + \nabla q(u), \nabla \Psi(U \cdot \nabla p + \text{div}(U)p, 0)) \, d\tau
\]

\[
= - \int_0^T (\text{div}Pu, \Psi(U \cdot \nabla p + \text{div}(U)p, 0)) \, d\tau
\]

\[
+ \int_0^T \langle Pu \cdot n, \Psi(U \cdot \nabla p + \text{div}(U)p, 0) \rangle_{\partial \Omega} \, d\tau
\]

\[
+ \int_0^T \langle q(u), \nabla \Psi(U \cdot \nabla p + \text{div}(U)p, 0) \cdot n \rangle_{\partial \Omega} \, d\tau
\]

\[
- \int_0^T (q(u), \text{div}\nabla \Psi(U \cdot \nabla p + \text{div}(U)p, 0)) \, d\tau,
\]

and so after considering (43) and (48), we have

\[
\int_0^T (u, \nabla \Psi(U \cdot \nabla p + \text{div}(U)p, 0)) \, d\tau = \int_0^T (q(u), U \cdot \nabla p + \text{div}(U)p) \, d\tau.
\]

(50)

Now, to deal with RHS of (50), we apply Green's Formula:

\[
\int_0^T (q(u), U \cdot \nabla p) \, d\tau = \int_0^T \int_{\partial \Omega} (U \cdot n)(q(u)p) \, d\partial \Omega \, d\tau
\]

\[
- \int_0^T \langle (\text{div}(U)q(u), p) \rangle_{\partial \Omega} \, d\tau - \int_0^T \langle (U \cdot \nabla q(u), p) \rangle_{\partial \Omega} \, d\tau
\]

\[
= - \int_0^T \langle (\text{div}(U)q(u), p) \rangle_{\partial \Omega} \, d\tau - \int_0^T \langle (U \cdot (I - P)u), p \rangle_{\partial \Omega} \, d\tau.
\]

(51)
Applying (47), (50), and (51) to RHS of (46) now gives

\[ \int_0^T (u_t, \nabla \Psi(p, w))_\sigma d\tau = \int_0^T (u, \nabla \Psi(p, w))_\sigma d\tau \]

(52)

Using the estimate for \( \Psi(\cdot, \cdot) \) in (44), we will then have from (52)

\[ \left| \int_0^T (u, \nabla \Psi(p, w))_\sigma d\tau \right| = \mathcal{O} \left( [E(T) + E(0)] + \Psi(U) \int_0^T \left( (\sigma(u(\tau)), \epsilon(u(\tau)))_\sigma + \eta \|u(\tau)\|_{\sigma}^2 \right)^{\frac{1}{2}} E(\tau)^{\frac{1}{2}} d\tau \right), \]

(53)

where we also implicitly use Korn's Inequality. (Here again, \( \Psi(U) \equiv \left[ \|U\|_{C(\sigma)} + 1 \right] \). Now let us continue with the second term on LHS of (45).

I.(ii): We have via Green's Formula and using (43),

\[ \int_0^T (\nabla p, \nabla \Psi(p, w))_\sigma d\tau = \int_0^T \langle p n, \nabla \Psi(p, w) \rangle_{\sigma \sigma} d\tau - \int_0^T \langle p, \Delta \Psi(p, w) \rangle_{\sigma} d\tau \]

\[ = \int_0^T \|p\|_{\sigma}^2 d\tau + \int_0^T \langle p n, \nabla \Psi(p, w) \rangle_{\sigma \sigma} d\tau. \]

(54)

I.(iii): To proceed with the fourth term on LHS of (45), we will appeal to the elliptic regularity results for solutions of second order BVP on corner domains, which are established in Dauge\(^{9,25}\) (see also Dauge\(^{19}\)); it is at this point where our geometrical assumptions in Condition 1 come into play. Using these assumptions, we have the following higher regularity:

\[ \|\Psi(p, w)\|_{H^1(\sigma)} \leq C \left[ \|p\|_{\sigma} + \|w_{ex}\|_{H^{1/2}(\sigma \sigma)} \right] \]

\[ \leq C[\|p\|_\sigma + \|w\|_{H^1(\Omega)}]. \]

(55)

where

\[ w_{ex}(x) = \begin{cases} 0, & x \in S \\ w(\tilde{x}), & x \in \Omega. \end{cases} \]

(For the second inequality in 55, we are invoking Theorem 3.33, pp. 95 of McLean\(^{31}\).) Therewith, for the fourth term on LHS of (45), we have

\[ -\int_0^T (\text{div} \sigma(u), \nabla \Psi(p, w))_\sigma d\tau = \int_0^T (\sigma(u), \epsilon(\nabla \Psi(p, w)))_\sigma d\tau - \int_0^T (\sigma(u) \cdot n, \nabla \Psi(p, w))_{\sigma \sigma} d\tau. \]

And so applying (55) then gives

\[ -\int_0^T (\text{div} \sigma(u), \nabla \Psi(p, w))_\sigma d\tau \]

\[ = \mathcal{O} \left( \int_0^T \left( (\sigma(u(\tau)), \epsilon(u(\tau)))_\sigma + \eta \|u(\tau)\|_{\sigma}^2 \right)^{\frac{1}{2}} E(\tau)^{\frac{1}{2}} d\tau \right) - \int_0^T \langle \sigma(u) \cdot n, \nabla \Psi(p, w) \rangle_{\sigma \sigma} d\tau. \]

(56)
Lastly, by means of the regularity result in (44), we have for the remaining terms in (45),

$$\int_0^T (U \cdot \nabla u + \eta u, \nabla \Psi(p, w))_\Omega$$

$$= \mathcal{O} \left( \Psi(U) \int_0^T \left[ (\sigma(u(\tau)), c(u(\tau)))_\sigma + \eta \|u(\tau)\|_\Omega^2 \right] \frac{1}{2} E(\tau)^{\frac{3}{2}} d\tau \right). \quad (57)$$

where again we implicitly use Korn’s inequality. Now, if we combine (45), (53), (54), (56), and (57) and keep in mind that $[\sigma(u)n - p n]_\sigma \cdot \tau = 0$, as well as the BC in (43) then we obtain

$$\int_0^T \|p\|_\Omega^2 d\tau - \int_0^T \left[ [2
\nu \partial_{x_i}(u)_3 + \lambda \text{div}(u) - p]_\Omega, w \right]_\Omega d\tau$$

$$= \mathcal{O} \left( [E(T) + E(0)] + \Psi(U) \int_0^T \left[ (\sigma(u(\tau)), c(u(\tau)))_\sigma + \eta \|u(\tau)\|_\Omega^2 \right] \frac{1}{2} E(\tau)^{\frac{3}{2}} d\tau \right). \quad (58)$$

**Step II:** To continue with the energy estimates, in this step, we apply the multiplier $w$ to the plate equation in (39), integrate in time and space to have

$$\int_0^T (w_{tt} + \Delta^2 w + \left[ 2 \nu \partial_{x_i}(u)_3 + \lambda \text{div}(u) - p \right]_\Omega, w)_\Omega d\tau = 0.$$

An integration by parts then gives

$$\int_0^T \|\Delta w\|_\Omega^2 d\tau + \int_0^T \left[ [2 \nu \partial_{x_i}(u)_3 + \lambda \text{div}(u) - p]_\Omega, w \right]_\Omega d\tau$$

$$= \mathcal{O} \left( \int_0^T \|w_t\|_\Omega^2 d\tau + E(T) + E(0) \right). \quad (59)$$

Using the domain criterion (A.v) (for $\kappa = 0$) and the Sobolev Trace Theorem, we also have

$$\int_0^T \|w_t\|_\Omega^2 d\tau = \int_0^T \|u_3\|_\Omega^2 d\tau = \mathcal{O} \left( \int_0^T \left[ (\sigma(u(\tau)), c(u(\tau)))_\sigma + \eta \|u(\tau)\|_\Omega^2 \right] d\tau \right). \quad (60)$$

Now, adding the relations (58), (59), and (60), we have now

$$\int_0^T E(\tau) d\tau \leq \mathcal{O} \left( [E(T) + E(0)] + \Psi(U) \int_0^T \left[ (\sigma(u(\tau)), c(u(\tau)))_\sigma + \eta \|u(\tau)\|_\Omega^2 \right] \frac{1}{2} E(\tau)^{\frac{3}{2}} d\tau \right.$$

$$+ \int_0^T \left[ (\sigma(u(\tau)), c(u(\tau)))_\sigma + \eta \|u(\tau)\|_\Omega^2 \right] d\tau \right).$$
whence we obtain after using Young Inequality

\[
(1 - \varepsilon) \int_0^T E(\tau) d\tau \leq C_0 [E(T) + E(0)] + C_1 |\Psi(U)|^2 \int_0^T [(\sigma(u(\tau)), e(u(\tau)))_\Omega + \eta \|u(\tau)\|_{\Omega}^2] d\tau,
\]

where \( C \) is a constant independent of \( T > 0 \). To conclude the proof of Theorem 2, we invoke the energy relation (41) and get

\[
(1 - \varepsilon) \int_0^T E(\tau) d\tau \leq 2C_0 E(T) + C_1 |\Psi(U)|^2 \int_0^T [(\sigma(u(\tau)), e(u(\tau)))_\Omega + \eta \|u(\tau)\|_{\Omega}^2] d\tau
\]

\[
+ C_2 \left| \int_0^T \int_{\partial\Omega} \text{div}(U)[|p(\tau)|^2 + |u(\tau)|^2] d\theta d\tau \right|.
\]

Using the energy relation (41) one more time, we have

\[
2C_0 E(T) + C_1 |\Psi(U)|^2 \int_0^T [(\sigma(u(\tau)), e(u(\tau)))_\Omega + \eta \|u(\tau)\|_{\Omega}^2] d\tau
\]

\[
\leq C_2 |\Psi(U)|^2 \left( E(0) + \frac{1}{2} \left| \int_0^T \int_{\partial\Omega} \text{div}(U)[|p(\tau)|^2 + |u(\tau)|^2] d\theta d\tau \right| \right).
\]

Applying this to (61), we then obtain

\[
(1 - \varepsilon) \int_0^T E(\tau) d\tau \leq C_3 |\Psi(U)|^2 E(0) + C_4 |\Psi(U)|^2 \|\text{div}(U)\|_{\infty} \int_0^T E(\tau) d\tau.
\]

This gives the estimate (42), for

\[
\|\text{div}(U)\|_{\infty} < \frac{1}{C_4 |\Psi(U)|^2},
\]

and with subsequent \( C^* = \frac{2C_3 |\Psi(U)|^2}{1 - 2\varepsilon} \). This completes the proof of Theorem 2.

Remark 7. In many applications, the geometrical setting with also nonzero in flow/out flow boundary conditions is really interesting from the engineering and physical points of view both on the theoretical level and on the level of numerical analysis. In principle, one can modify the details of the proof above to handle in flow/out flow boundary conditions on the hard walls of \( S \)—instead of the homogeneous impermeability BC’s—and with ambient flow profile \( U \) being a constant vector field with the “right signs” (so as to handle certain boundary terms in the associated PDE estimates).

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CONFLICT OF INTEREST

The author would like to state that this work does not have any conflicts of interest.

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