Thermodynamics of phantom energy in the presence of a Reissner-Nordström black hole

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Abstract

In this paper, we study the validity of the generalized second law (GSL) in phantom dominated universe in the presence of a Reissner-Nordström (RN) black hole. Our study is independent of the origin of the phantom like behavior of the considered universe. We also discuss the GSL in the neighborhood of transition from quintessence to phantom regime. We show that for a constant equation of state parameter, the GSL may be satisfied provided that the temperature is proportional to de Sitter temperature. It is shown that in models with (only) a transition from quintessence to phantom regime the generalized second law does not hold in the transition epoch. Next we show that if the phantom energy has a chemical potential, then the GSL will hold if the mass of black hole is above from a critical value.

Keywords: Phantom energy; Reissner-Nordström black hole; Generalized second law of thermodynamics.

I. INTRODUCTION

The discovery that current era of the universe is in an accelerating expansion phase obtained from many cosmological observations, such as the type Ia supernova (SN Ia), Wilkinson Microwave Anisotropy Probe (WMAP), the Sloan Digital Sky Survey (SDSS) etc., is one of the most outstanding achievements in modern cosmology. This unusual phenomenon provides an intense interest to the scientific community to understand the nature of the hidden force responsible for this accelerating expansion. It is assumed that this acceleration is mainly due to the presence of unusual stuff dubbed "dark energy", which possesses positive energy density $\rho > 0$ and negative pressure $p < 0$ which induce repulsive gravity. Except for negative pressure, we do not know the other components and properties of this mysterious form of energy. Despite the strong observational evidence for the existence of dark energy, we have no idea about how the dark energy evolves.

To explain the evolutionary behavior of the dark energy, various models have been proposed. In all these models, the dark energy is characterized by the equation of state $\omega = p/\rho$ ($p$ and $\rho$ are the pressure and energy density of the dark energy respectively) which facilitates us to understand the nature of dark energy that accelerates the universe.

An unknown force (dark energy) which explains the accelerated behavior of the universe is usually represented by a cosmological constant which is nothing but a vacuum energy. However, to explain the cosmic expansion, one requires the value of $\Lambda$ to be of the order of $10^{-120}$, which can not be explained by current particle physics. This is commonly known as the cosmological constant problem.

In the study of the present accelerated expansion of the universe driven by phantom energy, one may face a crucial situation in which the phantom energy density and the scale factor blows up in a finite time called Big Rip. This Big Rip may be avoided by introducing the effect of gravitational back-reactions which may end the phantom dominated regime. So in this view, we can introduce horizons for the accelerating universe and associate entropy and temperature to them. In this way one can make thermodynamical interpretation of a system comprising of phantom dark energy in the form of perfect fluid and the cosmological horizon.

Babichev et al. have shown that black holes lose mass by accreting phantom energy and finally disappear completely. As a result their areas will go down along with their entropies. By keeping this picture into mind, we proceed to investigate whether the GSL of thermodynamics holds in this scenario. For the sake of interest one may expect that if the parameters assigned to the universe are supposed to be the same as that of the ordinary thermodynamics parameters related to the physical system, then the thermodynamics laws must hold true by considering the global picture (i.e. by considering universe as an object).

One can take the present universe as one thermody-
namical entity. Gibbons and Hawking \[19\] firstly investigated the thermodynamical properties for the de Sitter spacetime, the event horizon and the apparent horizon of the universe coincide and so there is only one cosmological horizon. It was shown that the cosmological horizon area can be interpreted as an entropy (measure of some one ignorance about information of the regions behind it) and thermal radiations coming from the cosmological horizon. The thermodynamical study of the universe has been extended to the quasi-de Sitter space in \[13, 20, 21\]. If the apparent horizon and the event horizon of the universe are different, it has been shown that first law and second law of thermodynamics hold true for the apparent horizon. On the other hand these laws break down in the case of event horizon \[21\].

In this paper, we explore the thermodynamical behavior of a universe containing a RN black hole and phantom energy \((\omega < -1)\). The universe is considered as a closed thermodynamic system with a boundary of future event horizon. Assuming the temperature of the phantom energy and the cosmic horizon is the same, we check the validity of GSL whether the total entropy of all components of the system is an increasing function of time. The phantom energy interacts with the black hole which leads the black hole mass to decrease. This, henceforth, violates the ordinary second law of black hole thermodynamics. We determined the condition under which the GSL holds. We then compute the GSL near the time of phantom transition and discuss its implications. Next we study GSL by taking phantom energy having a chemical potential, which leads to a certain critical value of black hole mass above which the GSL will hold. Finally we conclude the paper.

II. RN BLACK HOLE IN THE PHANTOM DOMINATED FRIEDMANN-LEMAITRE-ROBERTSON-WALKER UNIVERSE AND GSL

The line element of spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric with scale factor \(a(t)\) is given by

\[
ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2). \tag{1}
\]

The Hubble parameter is defined by \(H = \dot{a}/a\), where the over-dot denotes derivative with respect to the comoving time \(t\). Here we assume the cosmology with the equation of state \(p = \omega \rho\). For \(\omega < -1\) we have the phantom fluid and for the accelerated expansion of the universe i.e. \(\ddot{a} > 0, \omega < -1/3\).

The future event horizon, \(R_h\), is defined by

\[
R_h(t) = a(t) \int^\infty_t \frac{d\tau}{a(\tau)}. \tag{2}
\]

Eq. (2) corresponds to the distance that light will travel from the present time till far in the future. In a phantom energy dominated universe, the universe lasts for a finite time \(t_\ast\). At the Big Rip singularity i.e. at time \(t = t_\ast\), \(\infty\) must be replaced with \(t_\ast\) in the integration. Hence the future event horizon is a finite distance.

For a system in quintessence i.e. \(-1 < \omega < -1/3\) \((\dot{H} < 0)\), the future event horizon satisfy \(\dot{R}_h \geq 0\) and for a phantom dominated universe i.e. \(\omega < -1\) \((\dot{H} > 0)\), \(\dot{R}_h \leq 0\). If the phantom ends to quintessence phase, one may have \(\dot{R}_h \geq 0\) even in the phantom dominated regime.

An entropy for the future event horizon is given by

\[
S_h = \pi R_h^2. \tag{3}
\]

The total entropy of the universe, \(S\) can be obtained as a sum of the entropy inside the horizon, \(S_{in}\), and \(S_h\) \[9\]

\[
S = S_{in} + S_h. \tag{4}
\]

Here it is assumed that the perfect fluid is in thermal equilibrium with the future event horizon, as required for FLRW universe model. When the future event horizon is de Sitter horizon i.e. when the spacetime (1) is de Sitter, then the temperature can be considered as \(T = H/2\pi\). When \(R_h \neq H^{-1}\) i.e. for a non-de Sitter spacetime, it is assumed that the future event horizon temperature is proportional to the de Sitter temperature \[10, 11\]

\[
T = \frac{bH}{2\pi}, \tag{5}
\]

where \(b\) is a constant.

In the presence of dark energy and dark matter, the RN black hole is introduced inside the future event horizon. Further, it is assumed that the mass of the black hole, \(M\), is small such that the FLRW model is unaltered. With the use of \(\rho = 3H^2/8\pi\), this condition becomes

\[
MH \ll \frac{R_h^3H^3}{2}. \tag{6}
\]

In terms of black hole entropy \(S_{bh}\) and entropy of the perfect fluid \(S_{d}\) the entropy inside the horizon may be given as

\[
S_{in} = S_{bh} + S_{d}. \tag{7}
\]

The rate of change of mass of the RN black hole (due to phantom energy accretion) is \[18\]

\[
\dot{M} = 4\pi A_1M^2(\rho + p) = -A_1M^2\dot{H}. \tag{8}
\]

where \(r_h\) is the horizon of the black hole and \(A_1\) is a constant. In terms of Hubble parameter, \(H\), the mass of the black hole may be given as

\[
M = \frac{1}{B + A_1H}, \tag{9}
\]

where \(B\) is a constant.
shown that in the case of transition from quintessence to phantom phase the order of the first nonzero derivative of \(H\) in the phantom regime is the magnitude of mass. The black hole entropy is \(S_h = 4\pi M^2\), thereby

\[ S_{bh} = -32\pi A_1 M^3 \dot{H}. \]  

The first law of black hole thermodynamics relates the entropy of the phantom fluid to the energy and pressure

\[ TdS_d = dE + pdV = (\rho + p)dV + V dp, \]

where \(V = 4\pi R_h^3/3\) is the inside volume and \(E = \rho V\) is the total energy inside the future event horizon. Eq. (11) yields

\[ T \dot{S}_d = \dot{H} R_h^2, \]

To satisfy the GSL we must have

\[ \dot{S}_f + \dot{S}_{bh} + \dot{S}_h \geq 0. \]  

This gives

\[ \dot{H} \left( \frac{R_h^2}{H} - 2 A_1 b M^2 \left( M - 2 \sqrt{M^2 - Q^2} - \frac{2M^2}{\sqrt{M^2 - Q^2}} \right) + b R_h \dot{R}_h \geq 0. \]  

The above inequality is satisfied if the quantity inside the square brackets is positive. In other words

\[ R_h^2 \geq 2 A_1 b H M^2 \left( M - 2 \sqrt{M^2 - Q^2} - \frac{2M^2}{\sqrt{M^2 - Q^2}} \right). \]  

Note that in the above analysis, we assumed that the black hole is non-extremal i.e. the black hole contains small electric charge compared to the corresponding magnitude of mass.

### A. GSL Near Transition Time

The astrophysical data favors the transition of the dark energy state parameter from the sub-negative to super-negative values around \(-1\) [2]. Following this observation, we check the validity of the GSL near the transition time \(t_0\) \((w = -1)\) in the presence of the RN black hole. In the phantom regime \(H > 0\) and in the quintessence regime we have \(H < 0\), therefore if the Hubble parameter has a Taylor series at transition time, which is taken to be at \(t_0 = 0\), \(H(0) = 0\) and this is done by applying the Taylor series expansion of the Hubble parameter about \(t_0 = 0\), we obtain

\[ H = h_0 + h_1 t^a, \quad h_0 = H_0, \quad h_1 = \frac{1}{a!} \frac{d^a}{dt^a} H, \]

where \(h_0 = H(t = 0)\) and \(a\), a positive even integer number, is the order of the first nonzero derivative of \(H\) at \(t = 0\). \(h_1 = H^{(a)}/a!\) and \(H^{(a)} = d^a H/dt^a\). In the case of transition from quintessence to phantom phase we must have \(h_1 > 0\). Using \(\dot{R}_h = H R_h - 1\), it can be shown that \(\dot{R}(t)\) has the following expansions:

\[ R_h(t) = R_h(0) + (h_0 R_h(0) - 1) t + O(t^2), \]

for \(\dot{R}_h(0) \neq 0\), and

\[ R_h(t) = R_h(0) \left( 1 + \frac{h_1}{a + 1} t^{a+1} \right) + O(t^{a+2}), \]

for \(\dot{R}_h(0) = 0\), at \(t = 0\). Near the transition time, Eq. (6) reduces to \(h_0^2 R_h^2(0) \gg 2M(0)\).

The condition of validity of GSL near the transition time, \(t = 0\), for \(\dot{R}_h(0) = 0\), can be investigated by inserting \(H = h_0 + h_1 t^a\) and [18] into (13):

\[ a_1 h_1 \left[ \left( \frac{R_h(0)}{h_0} \right)^2 - 2h_1 M(0) \left( M(0) - 2\sqrt{M(0)^2 - Q^2} - \frac{2M(0)^2}{\sqrt{M(0)^2 - Q^2}} \right) \right] t^{a-1} + O(t^a) \geq 0. \]

Note that \((a - 1)\) is an odd integer, therefore if the quantity in the square bracket \(\geq 0\), GSL is not respected in quintessence (phantom) phase before (after) the transition. Indeed the black hole mass \(M(0)\), gives the possibility that GSL becomes respected in the quintessence era before the transition.

### B. Phantom Energy With Chemical Potential and GSL

We now proceed to study the GSL by assuming the dark energy having chemical potential. Thermodynamical studies reveal several interesting features of phantom energy: the temperature of the phantom fluid without chemical potential is positive definite but its co-moving entropy is negative. More recently, the thermodynamic and statistical properties of phantom fluids were reexamined by considering the existence of a non-zero chemical potential \(\mu\). In this case, it was found that the entropy condition, \(S \geq 0\), implies that the possible values of \(\omega\) are heavily dependent on the value, as well as on the sign of the chemical potential [22]. In terms of the present day quantities (appearing below with subscript 0), the energy density of a dark energy fluid can be written as [23]

\[ \rho = \rho_0 \left( \frac{T}{T_0} \right)^{\frac{1+w}{1-w}}, \]

whereas its entropy (including a chemical potential) reads [24]

\[ S = \left( \frac{(1+w)\rho_0 - \mu_0 n_0}{T_0} \right) \left( \frac{T}{T_0} \right)^{\frac{1}{1+w}} V, \]

It thus follows that the total entropy of the system consisting of a charged black hole plus a dark energy fluid reads

\[ S = \pi (M + \sqrt{M^2 - Q^2})^2 + \left( \frac{(1+w)\rho_0 - \mu_0 n_0}{T_0} \right) \left( \frac{\rho}{\rho_0} \right)^{\frac{1}{1+w}} V, \]

for \(\dot{R}_h(0) \neq 0\), and

\[ R_h(t) = R_h(0) \left( 1 + \frac{h_1}{a + 1} t^{a+1} \right) + O(t^{a+2}), \]

for \(\dot{R}_h(0) = 0\), at \(t = 0\). Near the transition time, Eq. (6) reduces to \(h_0^2 R_h^2(0) \gg 2M(0)\).

The condition of validity of GSL near the transition time, \(t = 0\), for \(\dot{R}_h(0) = 0\), can be investigated by inserting \(H = h_0 + h_1 t^a\) and [18] into (13):

\[ a_1 h_1 \left[ \left( \frac{R_h(0)}{h_0} \right)^2 - 2h_1 M(0) \left( M(0) - 2\sqrt{M(0)^2 - Q^2} - \frac{2M(0)^2}{\sqrt{M(0)^2 - Q^2}} \right) \right] t^{a-1} + O(t^a) \geq 0. \]

Note that \((a - 1)\) is an odd integer, therefore if the quantity in the square bracket \(\geq 0\), GSL is not respected in quintessence (phantom) phase before (after) the transition. Indeed the black hole mass \(M(0)\), gives the possibility that GSL becomes respected in the quintessence era before the transition.
where the first term represents the black hole entropy and the second term is the phantom fluid entropy inside a co-moving volume $V$, written in terms of the energy density. Now, due to the accretion process, in an arbitrarily short time interval, the black hole mass varies by $\Delta M$ and the phantom field energy varies by $\Delta \rho$. Therefore, the total entropy variation within the cavity takes the form

$$\Delta S = 2\Delta M \left[ \pi (M + \sqrt{M^2 - Q^2}) \left( 1 + \frac{2M}{\sqrt{M^2 - Q^2}} \right) \right]$$

$$+ \frac{1}{1+w} \left[ \frac{(1+w)\rho_0 - \mu_0 \eta_0}{T_0} \left( \frac{\rho}{\rho_0} \right) \frac{\pi}{\sqrt{\pi}} V \Delta \rho. \right]$$

(23)

For a phantom fluid modeled by a scalar field, only the kinetic term contributes to the accretion, so that the energy conservation inside the cavity implies [25]

$$\Delta M = -\frac{1}{2}(1+w) V \Delta \rho. \quad (24)$$

Now, by inserting Eq. (24) into Eq. (23), we obtain an expression for the total entropy variation of the black hole plus dark energy

$$\Delta S = 2\Delta M \left[ \pi (M + \sqrt{M^2 - Q^2}) \left( 1 + \frac{2M}{\sqrt{M^2 - Q^2}} \right) \right]$$

$$- \frac{1}{(1+w)^2} \left[ \frac{(1+w)\rho_0 - \mu_0 \eta_0}{T_0} \left( \frac{\rho}{\rho_0} \right) \frac{\pi}{\sqrt{\pi}} \right] \geq 0.$$ 

In the limit when the electric charge is negligibly small $M \gg Q$, we have

$$M \left( \frac{6 + \frac{Q^2}{2M^2}}{2} \right) \geq M_{\text{crit}} = \frac{1}{\pi (1+w)^2} \left( \frac{(1+w)\rho_0 - \mu_0 \eta_0}{T_0} \right) \times \left( \frac{\rho}{\rho_0} \right) \frac{\pi}{\sqrt{\pi}}.$$ 

(25)

Expression (25) shows that GSL will hold if the black hole mass is above a certain critical mass $M_{\text{crit}}$. In a special case, if mass of the black hole is also very small and equal in magnitude to the charge then the expression (24), yields

$$M \geq M_{\text{crit}} = \frac{2}{13\pi (1+w)^2} \left( \frac{(1+w)\rho_0 - \mu_0 \eta_0}{\rho_0 T_0} \right) \left( \frac{\rho}{\rho_0} \right) \frac{\pi}{\sqrt{\pi}}.$$ 

(26)

Notice that the critical mass is negative when $w < -1$ and the inevitable conclusion is that the process is physically forbidden. In the point of view of [26], such a result should be physically expected since phantom fluids with negative temperature cannot exist in nature. On the other hand, we see that for negative values of $\mu_0$ there exists a positive critical mass above which the black hole can accrete the phantom fluid. If we adopt the results from Babichev et al. [18], where $\Delta \rho < 0$, the condition to the mass is $M < M_{\text{crit}}$. Therefore, only black holes with mass below the critical mass can accrete phantom energy.

III. CONCLUSION

In this work we have discussed the accretion of phantom fluids with negative chemical potential by RN black hole. As we have seen, there is a positive critical mass in order to enable the phantom accretion. As physically expected, the GSL of thermodynamics determines the thermodynamic viability of the whole process and the amount of dark energy accretion. Phantom fluids with zero chemical potential are not consistent because they require either a negative entropy (which is microscopically unacceptable) or a negative temperature (which needs a bounded spectrum which has not been justified from any scalar field model).

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