On a relation of overconvergence and $F$-analyticity on $p$-adic Galois representations of a $p$-adic field $F$

Megumi Takata

September 17, 2019

Abstract

Let $p$ be a prime number. There are properties called “overconvergence” and “$F$-analyticity” for $p$-adic Galois representations of a $p$-adic field $F$. By Berger’s work, it is known that $F$-analyticity is stricter than overconvergence. In this article, we show that, in many cases, an overconvergent Galois representation is $F$-analytic up to a twist by a character. This result emphasizes the necessity of the theory of $(\varphi, \Gamma)$-modules over the multivariable Robba ring, by which we expect to study all $p$-adic Galois representations.

1 Introduction

In the $p$-adic local Langlands program for $GL_2(\mathbb{Q}_p)$, which has finally been established in [Col10] and [CDP14], an important result in the early stage is a theorem due to Cherbonnier and Colmez [CC98]. It says that all of the $p$-adic Galois representations of a $p$-adic field are overconvergent with respect to the cyclotomic $\mathbb{Z}_p$-extension. It enables us to study every $p$-adic Galois representation of a $p$-adic field in the terms of modules over the univariable Robba ring, which is the ring of (not necessarily bounded) functions on $p$-adic annuli. One of the benefits appears in Berger’s work [Ber02], which relates the following three notions: 1) $(\varphi, \Gamma)$-modules, 2) $p$-adic differential equations, and 3) filtered $(\varphi, N)$-modules.

Thus, if we want to generalize the $p$-adic local Langlands program from $GL_2(\mathbb{Q}_p)$ to $GL_2(F)$ for an arbitrary $p$-adic field $F$, a similar question occurs: is any $p$-adic Galois representation of $F$ overconvergent if we consider a general Lubin-Tate extension instead of the cyclotomic one? For this, Fourquaux and

*Kyushu Sangyo University, 3-1 Matsukadai 2-chome, Higashi-ku, Fukuoka 813-8503 Japan.
E-mail address: m.takata@ip.kyusan-u.ac.jp
2010 Mathematics Subject Classification. 11S20, 11S31.
Xie [FX13] show a negative answer. That is, contrary to the cyclotomic case, there exist (an infinite number of) $p$-adic Galois representations which is not overconvergent with respect to a Lubin-Tate extension. Then we face a next question: how strict is overconvergence in a general Lubin-Tate setting? Berger [Ber16] has proved that, an another property called $F$-analyticity is a sufficient condition for overconvergence. In addition, he has proved the following:

**Theorem 1.1** ([Ber13, Cor. 4.3]). *Any absolutely irreducible and overconvergent $p$-adic Galois representations of $F$ are $F$-analytic up to twist by a character.*

We remark that Berger and Fourquaux [BF17, Th. 1.3.1, Cor. 1.3.2] describe an arbitrary overconvergent representation by using an $F$-analytic one and one which factors through $\Gamma$, where $\Gamma$ is the Galois group of the considering Lubin-Tate extension.

The main theorem of this article is a generalization of Theorem 1.1. To state it in detail, we need to introduce some notations. We fix an algebraic closure $\overline{F}$ of $F$ and write $G_{F}=\text{Gal}(\overline{F}/F)$. Let $L \subset \overline{F}$ denote a finite extension of $F$. Let $F_{LT}^{\varpi}$ be the Lubin-Tate extension of $F$ with respect to a fixed uniformizer $\varpi \in F$. We write $\Gamma = \text{Gal}(F_{LT}^{\varpi}/F)$. Let $q$ be the order of the residue field of $F$. Let $R$ be the Robba ring with coefficients in $L$. It has an action of $\Gamma$ and $\varphi_q$, where $\varphi_q$ is a lift of the $q$-th power map. We have a fully faithful functor $R \otimes_{\varphi_q} \Gamma^d$ from the category of $L$-representations of $G_{F}$ to that of the $(\varphi_q, \Gamma)$-modules over $R$ (for more precise descriptions of $R$, the notion of $(\varphi_q, \Gamma)$-modules, and $R \otimes_{\varphi_q} \Gamma^d$, see §2).

**Theorem 1.2.** Let $V$ be an overconvergent $L$-representation of $G_{F}$. Suppose that $D = R \otimes_{\varphi_q} \Gamma^d(V)$ has a filtration

$$0 = D_0 \subset D_1 \subset \cdots \subset D_r = D$$

of $(\varphi_q, \Gamma)$-modules over $R$. We put $\Delta_i = D_i/D_{i-1}$. We assume that the following conditions hold.

(a). For any $1 \leq i \leq r$, we have $\text{End}_{(\varphi_q, \Gamma)\text{-mod}/R}(\Delta_i) = L$.

(b). For any $1 \leq i < j \leq r$, we have $\text{Hom}_{(\varphi_q, \Gamma)\text{-mod}/R}(\Delta_j, \Delta_i) = 0$.

(c). For any $1 < i \leq r$, the short exact sequence $0 \rightarrow D_{i-1} \rightarrow D_i \rightarrow \Delta_i \rightarrow 0$ does not split.

Then there exist a finite extension $L'$ of $L$ and a character $\delta: G_{F}^{\times} \rightarrow (L')^{\times}$ such that $V \otimes_{L} L'(\delta)$ is $F$-analytic.

**Remark 1.3.** (i). The case $r = 1$ of Theorem 1.2 says that, if an overconvergent $L$-representation $V$ of $G_{F}$ satisfies $\text{End}(V) = L$, then $V$ becomes $F$-analytic after twisting by a character. It is still a generalization of Theorem 1.1.
(ii). Of course, there are overconvergent representations which cannot be $F$-analytic after twisting by any character: for example, the direct sum of $F$-analytic one and not $F$-analytic one. However, a large part of overconvergent representations seem to satisfy the conditions of Theorem 1.2 for example, most of trianguline representations. While, $F$-analyticity is very strict since, by definition, it demands the Hodge-Tate weights with respect to any $\tau \in \text{Hom}(F, \overline{F}) \setminus \{\text{id}_F\}$ to be zero. Hence this result predicts that the size of the class of overconvergent representations is thought to be very small in the whole. Therefore we seem to need the theory of $(\varphi, \Gamma)$-modules over the multivariable Robba ring ([Ber13], [Ber16], [BF17]), by which we expect to study all $p$-adic Galois representations.

In §2, we recall some definitions and theorems on the Lubin-Tate $(\varphi_q, \Gamma)$-modules. In particular, we define overconvergence and $F$-analyticity here. In §3, after introducing several lemmas, we prove Theorem 3.4. Then we obtain Theorem 1.2 as a corollary.

Acknowledgments
The author would like to thank Kentaro Nakamura for many useful comments, in particular, for suggesting him to generalize the original version of the main theorem.

2 Preliminary on the Lubin-Tate $(\varphi_q, \Gamma)$-modules

In this section, we recall some notion on the Lubin-Tate $(\varphi_q, \Gamma)$-modules.

Let $F$ be a finite extension of $\mathbb{Q}_p$, $\mathcal{O}_F$ the ring of integers of $F$, and $q$ the order of the residue field of $\mathcal{O}_F$. We fix an algebraic closure $\overline{F}$ of $F$. We write $G_F = \text{Gal}(\overline{F}/F)$. We fix a uniformizer $\varpi$ of $F$ and a power series $f_\varpi(T) \in \mathcal{O}_F[[T]]$ such that $f_\varpi(T) \equiv \varpi T$ modulo degree 2 and $f_\varpi(T) \equiv T^q$ modulo $\varpi$. Then there exists a unique formal group law $G_\varpi(X, Y) \in \mathcal{O}_F[[X, Y]]$ such that $f_\varpi \in \text{End}(G_\varpi)$, which we call the Lubin-Tate formal group associated to $\varpi$. It has a natural formal $\mathcal{O}_F$-module structure $\cdot : \mathcal{O}_F \to \text{End}(G_\varpi)$ such that, for any $a \in \mathcal{O}_F$, the first term of $[a](T)$ is $aT$ and $[\varpi](T) = f_\varpi(T)$. For any $a \in \mathcal{O}_F$, we put $G_\varpi[a] = \{ \alpha \in \overline{F} \mid [a](\alpha) = 0 \}$. We denote by $F_\varpi^{LT}$ the extension of $F$ in $\overline{F}$ obtained by adding all elements of $G_\varpi[\varpi^n]$ for all integers $n \geq 1$. We call $F_\varpi^{LT}$ the Lubin-Tate extension of $F$ associated to $\varpi$. We define the Tate module $T G_\varpi$ of $G_\varpi$ by $T G_\varpi = \lim_{\leftarrow n} G_\varpi[\varpi^n]$. This is a free $\mathcal{O}_F$-module of rank 1 on which $G_\varpi$ naturally acts. It induces a character $\chi_{LT} : \mathcal{G}_F \to \mathcal{O}_F^\times$, which we call the Lubin-Tate character associated to $\varpi$. We write $\mathcal{H}_F = \text{Gal}(\overline{F}/F_\varpi^{LT})$, which is the kernel of $\chi_{LT}$. For any integer $n \geq 1$, we put

$$
\Gamma = \text{Gal}(F_\varpi^{LT}/F) = \mathcal{G}_F/\mathcal{H}_F \cong \mathcal{O}_F^\times \quad \text{and}
$$

$$
\Gamma_n = \text{Gal}(F_\varpi^{LT}/F(G_\varpi[\varpi^n])).
$$
Let $L \subset \overline{F}$ be a finite extension of $F$. We put

$$\mathcal{O}_E = \mathcal{O}_{E_L} = \left\{ \sum_{n \in \mathbb{Z}} a_n T^n \middle| a_n \in \mathcal{O}_L \text{ for any } n \in \mathbb{Z}, \ a_n \to 0 \ (n \to -\infty) \right\},$$

which is a discrete valuation ring such that a uniformizer $\varpi_L$ of $L$ generates the maximal ideal. We put $\mathcal{E} = \mathcal{E}_L = \text{Frac}(\mathcal{O}_E)$. In this article, we consider the weak topology. For any $n \in \mathbb{Z}$, we define a topology on $\varpi^*_n \mathcal{O}_E$ with $\{\varpi^*_n \mathcal{O}_E + T^j \varpi^*_n \mathcal{O}_E[T] \}_{i,j}$ as a fundamental system of neighborhoods of 0. We define a topology on $\mathcal{E} = \cup_{n \in \mathbb{Z}} \varpi^*_n \mathcal{O}_E$ by the inductive limit topology.

An $L$-linear $(\varphi_q, \Gamma)$-action on $\mathcal{O}_E$ or on $\mathcal{E}$ is defined such that, for any $\gamma \in \Gamma$, we have

$$\varphi_q(T) = f_{\varpi}(T), \quad \gamma.T = [\chi_{\text{LT}}(\gamma)](T).$$

A free $\mathcal{O}_E$-module $D_0$ of finite rank is called a $(\varphi_q, \Gamma)$-module over $\mathcal{O}_E$ if:

1. $\Gamma$ acts continuously and semilinearly on $D_0$ and
2. a $\varphi_q$-semilinear map $\Phi: D_0 \to D_0$ is equipped such that
   - the action of $\Gamma$ and $\Phi$ commute, and
   - the $\mathcal{O}_E$-linear homomorphism $\mathcal{O}_E \otimes_{\varphi_q, \mathcal{O}_E} D_0 \to D_0$ induced by $\Phi$ is an isomorphism.

We abuse notation and use the same symbol $\varphi_q$ to denote $\Phi$. The notion of $(\varphi_q, \Gamma)$-modules over $\mathcal{E}$ is defined in a similar way as above. We call a $(\varphi_q, \Gamma)$-module $D$ over $\mathcal{E}$ étale if there exists a $(\varphi_q, \Gamma)$-module $D_0$ over $\mathcal{O}_E$ such that $D \simeq \mathcal{E} \otimes_{\mathcal{O}_E} D_0$ as $(\varphi_q, \Gamma)$-modules over $\mathcal{E}$.

We denote by $\text{Rep}_L(\mathcal{G}_F)$ the category of continuous finite dimensional $L$-representations of $\mathcal{G}_F$ and by $\text{Mod}_{\varphi_q, \Gamma}^{\text{ét}}$ the category of $(\varphi_q, \Gamma)$-modules over $\mathcal{E}$. Let $\text{Mod}_{\varphi_q, \Gamma}^{\text{ét}}$ denote the full subcategory of $\text{Mod}_{\varphi_q, \Gamma}$ consisting of étale objects. Fontaine have found an equivalence of categories of $\text{Rep}_L(\mathcal{G}_F)$ and $\text{Mod}_{\varphi_q, \Gamma}^{\text{ét}}$ in the cyclotomic case. In the general Lubin-Tate cases, it is given by Kisin and Ren. To construct it, we use a big field $B$, which contains $\mathcal{E}_F$ and has a $(\varphi_q, \mathcal{G}_F)$-action. Moreover, we have $B^{\varphi_q} = \mathcal{E}_F$. For a precise definition of $B$, the reader may refer to [KR09] or [FX13 1B]. Note that, in [KR09] (resp. [FX13]), this is denoted by $E^{\text{ur}}$ (resp. $B$).

**Theorem 2.1** ([Fon91 Th. 3.4.3, Rem. 3.4.4], [KR09 Th. 1.6]). We have a functor

$$D: \text{Rep}_L(\mathcal{G}_F) \to \text{Mod}_{\varphi_q, \Gamma}^{\text{ét}}: V \mapsto (V \otimes_F B)^{\varphi_q} = 1,$$

which gives an equivalence of categories. Its quasi-inverse functor is given by $D \mapsto (D \otimes_{\mathcal{E}_F} B)^{\varphi_q} = 1$.

Now we define a subfield $\mathcal{E}^\dagger$ of $\mathcal{E}$ and a ring $R$ containing $\mathcal{E}^\dagger$. Let $v_p$ denote the $p$-adic additive valuation on $\overline{F}$ normalized as $v_p(p) = 1$. For any formal
series \( f(T) = \sum_{n \in \mathbb{Z}} a_n T^n \) with coefficients in \( L \) and any real number \( r \geq 0 \), we define \( v^{(r)}(f) = \inf_n v_p(a_n) + nr \), which may be \( \pm \infty \). For any \( 0 \leq s \leq r \), we put \( v^{[s,r]}(f) = \inf_{s \leq t \leq r} v^{(r)}(f) \). We define

\[
\mathcal{E}^{[s,r]} = \mathcal{E}^{[s,r]}_L = \left\{ f(T) = \sum_{n \in \mathbb{Z}} a_n T^n \left| a_n \in L \text{ for any } n \in \mathbb{Z}, v^{[s,r]}(f) \neq -\infty \right. \right\},
\]

\[
\mathcal{E}^{[0,r]} = \mathcal{E}^{[0,r]}_L = \bigcap_{0 < s \leq r} \mathcal{E}^{[s,r]},
\]

\[
\mathcal{R} = \mathcal{R}_L = \bigcup_{r > 0} \mathcal{E}^{[0,r]},
\]

\[
\mathcal{E}^{(0,r]} = \mathcal{E}^{(0,r]}_L = \mathcal{E}^{[0,r]} \cap \mathcal{E}, \text{ and}
\]

\[
\mathcal{E}^\dagger = \mathcal{E}^\dagger_L = \bigcup_{r > 0} \mathcal{E}^{[0,r]}.
\]

They have ring structure and \( \Gamma \) acts on them such that \( \gamma \cdot f = [\chi_{LT}(\gamma)](T) \) for any \( \gamma \in \Gamma \). Moreover, if \( r \) is sufficiently small, then we have a ring endomorphism \( \varphi_q : \mathcal{E}^{[s,r]} \to \mathcal{E}^{[s,r]}_{\Gamma} \) such that \( \varphi_q(T) = f_{\mathbb{Z}}(T) \). Hence the rings \( \mathcal{E}^\dagger \) and \( \mathcal{R} \) are endowed with a \((\varphi_q, \Gamma)\)-action. We call \( \mathcal{R} \) the Robba ring and \( \mathcal{E}^\dagger \) the bounded Robba ring. We have inclusions \( \mathcal{E} \supset \mathcal{E}^\dagger \subset \mathcal{R} \) as rings with a \((\varphi_q, \Gamma)\)-action.

The ring \( \mathcal{E}^{[s,r]} \) is equipped with the topology defined by the valuation \( v^{[s,r]} \). Then \( \mathcal{E}^{[s,r]} \) is complete. We endow \( \mathcal{E}^{[0,r]} \) with the projective limit topology, and \( \mathcal{R} \) with the inductive limit topology. On \( \mathcal{E}^{(0,r]} \), we consider the topology defined by the valuation \( v^{(r)} \). We endow \( \mathcal{E}^\dagger \) with the inductive limit topology.

The notion of \((\varphi_q, \Gamma)\)-modules over \( \mathcal{E}^\dagger \) or \( \mathcal{R} \) is defined in a similar way as those over \( \mathcal{O}_E \) or \( \mathcal{E} \). A \((\varphi_q, \Gamma)\)-module \( D^\dagger \) over \( \mathcal{E}^\dagger \) is called étale if \( D^\dagger \otimes_{\mathcal{E}^\dagger} \mathcal{E} \) is étale. A \((\varphi_q, \Gamma)\)-modules \( D \) over \( \mathcal{R} \) is called étale or of slope \( 0 \) if there exists an étale \((\varphi_q, \Gamma)\)-module \( D^\dagger \) over \( \mathcal{E}^\dagger \) such that \( D \simeq D^\dagger \otimes_{\mathcal{E}^\dagger} \mathcal{R}^\dagger \). Let \( \text{Mod}^{\varphi_q, \Gamma}_{\mathcal{E}^\dagger} \) (resp. \( \text{Mod}^{\varphi_q, \Gamma}_{\mathcal{R}^\dagger} \)) denote the category of \((\varphi_q, \Gamma)\)-modules over \( \mathcal{E}^\dagger \) (resp. \( \mathcal{R}^\dagger \)). We denote by \( \text{Mod}^{\varphi_q, \Gamma, \text{ét}}_{\mathcal{E}^\dagger} \) (resp. \( \text{Mod}^{\varphi_q, \Gamma, \text{ét}}_{\mathcal{R}^\dagger} \)) the full subcategory of \( \text{Mod}^{\varphi_q, \Gamma}_{\mathcal{E}^\dagger} \) (resp. \( \text{Mod}^{\varphi_q, \Gamma}_{\mathcal{R}^\dagger} \)) consisting of étale objects.

**Theorem 2.2** ([FX13] Prop. 1.6]). The functor \( D^\dagger \mapsto \mathcal{R} \otimes_{\mathcal{E}^\dagger} D^\dagger \) gives an equivalence of categories of \( \text{Mod}^{\varphi_q, \Gamma, \text{ét}}_{\mathcal{E}^\dagger} \) and \( \text{Mod}^{\varphi_q, \Gamma, \text{ét}}_{\mathcal{R}^\dagger} \).

We have a subfield \( B^\dagger \) of \( B \) as in [FX13] 1B, which contains \( \mathcal{E}^\dagger_F \) and has a \((\varphi_q, \mathcal{G}_F)\)-action. Moreover, we have \( (B^\dagger)^{\mathcal{G}_F} = \mathcal{E}^\dagger_F \). For any object \( V \) of \( \text{Rep}_L^\text{oc} (\mathcal{G}_F) \), we put \( D^\dagger(V) = (V \otimes_{\mathcal{G}_F} B^\dagger)^{\mathcal{G}_F} \). We have a natural inclusion \( D^\dagger(V) \subset D(V) \).

**Definition 2.3.** An object \( V \) of \( \text{Rep}_L^\text{oc} (\mathcal{G}_F) \) is overconvergent if \( D^\dagger(V) \) generates \( D(V) \) as an \( \mathcal{E} \)-vector space. We denote by \( \text{Rep}_L^\text{oc}^\dagger (\mathcal{G}_F) \) the subcategory of \( \text{Rep}_L^\text{oc} (\mathcal{G}_F) \) consisting of overconvergent objects.
Theorem 2.4 ([FX13 Prop. 1.5]). The operation $D^!$ gives an equivalence of categories of $\text{Rep}^\text{oc}_L(G_F)$ and $\text{Mod}_{\xi}^{\omega,\text{ét}}$. Moreover, the diagram

$$
\begin{array}{ccc}
\text{Rep}^\text{oc}_L(G_F) & \xrightarrow{D^!} & \text{Mod}_{\xi}^{\omega,\text{ét}} \\
\downarrow & & \downarrow \\
\text{Rep}_L(G_F) & \xrightarrow{D} & \text{Mod}_{\xi}^{\omega,\text{ét}}
\end{array}
$$

commutes up to canonical isomorphisms.

Remark 2.5. Cherbonnier and Colmez [CC98] have shown that, in the cyclotomic case, we have $\text{Rep}^\text{oc}_L(G_F) = \text{Rep}_L(G_F)$, i.e. all of the $L$-representations of $G_F$ are overconvergent with respect to the cyclotomic extension. Even in the general Lubin-Tate case, all of the 1-dimensional $L$-representations of $G_F$ are overconvergent [FX13 Remark 1.8]. However, there exist $L$-representations of $G_F$ which are not overconvergent, as shown by Fourquaux and Xie [FX13 Theorem 0.6].

Now, let us recall another property of $L$-representations of $G_F$ so-called $F$-analyticity. We write $\mathbb{C}_p$ for the $p$-adic completion of $\mathbb{T}$.

Definition 2.6. An object $V$ of $\text{Rep}_L(G_F)$ is called $F$-analytic if, for any $\tau \in \text{Hom}_{\mathbb{Q}_p\text{-alg}}(F, \mathbb{T}) \setminus \{\text{id}_F\}$, the $\mathbb{C}_p$-representation $\mathbb{C}_p \otimes \tau, F V$ is trivial.

For $F$-analytic representations, Berger shows the following:

Theorem 2.7 ([Ber10 Thm. C]). Any $F$-analytic $L$-representation of $G_F$ is overconvergent.

There is also a notion of $F$-analyticity for $(\varphi_q, \Gamma)$-modules over $\mathcal{R}$. To define it, we introduce an action of a Lie algebra. Let $\text{Lie} \Gamma$ denote the Lie algebra associated to the $p$-adic Lie group $\Gamma$. Note that there is an isomorphism $\text{Lie} \Gamma \xrightarrow{\sim} \mathcal{O}_F$ induced by $\chi_{LT}$. Let $D$ be any object in $\text{Mod}_{\xi}^{\omega,\text{ét}}$. We will define an action of $\text{Lie} \Gamma$ on $D$. Take any $x \in D$. If $\beta \in \text{Lie} \Gamma$ is sufficiently close to 0, then we can define $\gamma = \exp \beta \in \Gamma$. Now we recall the following:

Lemma 2.8 ([FX13 Lemma 1.7]). For any $r > 0$ and $0 < s \leq r$, there exists $n = n(s, t)$ such that, for any $\gamma' \in \Gamma_n$ and $f \in \mathcal{E}^{[s, r]}$, we have $v^{[s, r]}((\gamma' - 1)f) \geq v^{[s, r]}(f) + 2$.

Note that, in the original version, the right hand side is $v^{[s, r]}(f) + 1$. However, by the proof, we can replace 1 with any other positive real number, for example 2.

We fix a basis $e_1, \ldots, e_d$ of $D$. For any $r > 0$, we put $D^{[0, r]} = \oplus_{i=1}^d \mathcal{E}^{[0, r]} e_i$. It depends on the choice of $e_1, \ldots, e_d$. However, for another choice $e'_1, \ldots, e'_d$, there exists $0 < r' < r$ such that $\oplus_{i=1}^d \mathcal{E}^{[0, r']} e_i = \oplus_{i=1}^d \mathcal{E}^{[0, r']} e'_i$. We put $D^{[s, r]} = \mathcal{E}^{[s, r]} \otimes_{\mathcal{E}^{[0, r]}} D^{[0, r]}$. 

6
Lemma 2.11. Since we can prove it in the same way as [Ber02, Lem. 5.2], we omit it.

Proof. Since we can prove it in the same way as [Ber02, Lem. 5.2], we omit it.

We choose an integer $m \geq 0$ such that $\gamma^m \in \Gamma_{n(s, t)}$. By Corollary 2.9 the series

$$
(\log \gamma)_{s,r}(x) = \frac{1}{p^m} \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\gamma^m - 1)^i}{i} x
$$

converges in $D^{[s,r]}$, and it is independent of the choice of $m$. We can show that, if $s' < s$, then the image of $(\log \gamma)_{s',r}(x)$ via the natural map $D^{[s',r]} \to D^{[s,r]}$ coincides with $(\log \gamma)_{s,r}(x)$. Thus we obtain a projective system $((\log \gamma)_{s,r})_{0 < s \leq r}$, which gives an element $(\log \gamma)x \in D^{[0,r]}$. We put $d\Gamma x = (\log \gamma)x$. For a general $\beta \in \text{Lie } \Gamma$, we choose a sufficiently large $n \in \mathbb{Z}$ and put $d\Gamma x = p^{-n}d\Gamma_{p^n}x$, which is independent of the choice of $n$. As a result, we have an action $d\Gamma_\bullet : \text{Lie } \Gamma \to \text{End}_L(D)$.

For an element $\beta \in \text{Lie } \Gamma$ sufficiently close to 1, $\nabla_{\beta} \mid_D$ denotes the operator $(\log \chi_{L\Gamma}(\exp \beta))^{-1}d\Gamma_{\beta}$ on $D$.

Definition 2.10. An object $D$ of $\text{Mod_\nu_\text{\Gamma}}^R$ is called $F$-analytic if the operator $\nabla_{\beta} \mid_D$ is independent of the choice of $\beta \in \text{Lie } \Gamma$. If so, we often denote $\nabla_{\beta} \mid_D$ by $\nabla_\beta \mid_D$ or $\nabla$ if no confusion occurs.

We have a formula to calculate $\log \gamma$ as follows:

Lemma 2.11. For any $x \in D$ and an element $\gamma \in \Gamma$ sufficiently close to 1, we have

$$
(\log \gamma)x = \lim_{n \to \infty} \gamma^n x - x.
$$

Proof. It suffices to show the formula for $D^{[s,r]}$. We fix a basis $e_1, \ldots, e_d$ of $D$. For any $y = \sum_{i=1}^{d} y_i e_i \in D^{[s,r]} = \oplus_{i=1}^{d} \mathcal{L}^{[s,r]}e_i$, we define $v^{[s,r]}(y) = \inf_{1 \leq i \leq d} v^{[s,r]}(y_i)$. Let $L$ be an $L$-linear operator on $D^{[s,r]}$. We define a valuation of $\mathcal{L}$ by

$$
v^{[s,r]}(\mathcal{L}) = \inf_{y \in D^{[s,r]}} (v^{[s,r]}(Ly) - v^{[s,r]}(y)).
$$

If $v^{[s,r]}(\mathcal{L}) > (p - 1)^{-1}$, then the operator

$$
\exp \mathcal{L} = \sum_{i=0}^{\infty} \frac{\mathcal{L}^i}{i!}
$$

on $D^{[s,r]}$ is well-defined.

By corollary 2.9 if an element $\gamma \in \Gamma$ is sufficiently close to 1, then the operator $\log \gamma$ on $D^{[s,r]}$ is well-defined and $v^{[s,r]}(\log \gamma) \geq 2$. Thus we can define $\exp(\log \gamma)$. Moreover, we have

$$
\gamma x = [\exp(\log \gamma)](x).
$$
Hence, for any integer $n \geq 1$, we have
\[
\gamma^{p^n}x = [\exp(p^n \log \gamma)](x) = \sum_{i=0}^{\infty} \frac{(p^n \log \gamma)^i}{i!} x, \quad \text{and}
\]
\[
\gamma^{p^n}x - x = \frac{p^n}{p^n} \sum_{i=0}^{\infty} \frac{p^{ni}(\log \gamma)^{i+2}}{(i+2)!} x.
\] (1)

Note that the second term of the right-hand side of (1) is well-defined since $p^{[s,r]}(p^{ni}(\log \gamma)^{i+2}x/(i+2)!)$ tends to 0 as $i \to \infty$. As $n$ goes to $\infty$ in (1), we have the required equality.

Berger shows that $F$-analyticity of an $L$-representation of $G_F$ is equivalent to that of the corresponding $(\phi_q, \Gamma)$-module.

**Theorem 2.12** ([Ber16, Thm. D]). Let $V$ be any object of $\text{Rep}^\text{oc}_L(G_F)$. Then $V$ is $F$-analytic if and only if $R \otimes_{E^1} D(V)$ is $F$-analytic.

### 3 Proof of the main theorem

In this section, we introduce several lemmas first. Next, we prove Theorem 3.4, which is a $(\phi_q, \Gamma)$-module version of the main theorem. Then we obtain Theorem 1.2 as a corollary.

**Lemma 3.1.** Let $\Delta$ and $D$ be $(\phi_q, \Gamma)$-modules over $R$. Suppose that $D$ is $F$-analytic and
\[
\text{End}(\phi_q, \Gamma)_{\text{mod}/R}(\Delta) = L.
\] If $\text{Ext}(\Delta, D) \neq 0$, then $\Delta$ is $F$-analytic.

**Proof.** This lemma is a generalization of [FX13] Theorem 5.20. Suppose that $\Delta$ is not $F$-analytic. Then there exist $\beta, \beta' \in \text{Lie } \Gamma$ such that $\nabla_\beta|_\Delta \neq \nabla_{\beta'}|_\Delta$. We put $\nabla'|_\Delta = \nabla_\beta|_\Delta - \nabla_{\beta'}|_\Delta$. Since $\nabla_\beta|_\Delta$ and $\nabla_{\beta'}|_\Delta$ are both $R$-derivations and the trivial $(\phi_q, \Gamma)$-module $R$ is $F$-analytic, the operator $\nabla'|_\Delta$ is $R$-linear. Moreover, it is stable under $(\phi_q, \Gamma)$-action. Hence we have $\nabla'|_\Delta \in \text{End}(\phi_q, \Gamma)_{\text{mod}/R}(\Delta)$, and by assumption, there exists $c \in L^\times$ such that $\nabla'|_\Delta = c \cdot \text{id}_\Delta$. Now we choose any extension $0 \to D \to \tilde{D} \to \Delta \to 0$, and consider the operator $\nabla'|_{\tilde{D}} = \nabla_\beta|_{\tilde{D}} - \nabla_{\beta'}|_{\tilde{D}}$, which is also $R$-linear and $(\phi_q, \Gamma)$-stable. Since $D$ is $F$-analytic, we have $\tilde{D} \subset \ker(\nabla'|_{\tilde{D}})$ and $\nabla'|_{\tilde{D}}$ induces an $R$-linear and $(\phi_q, \Gamma)$-stable homomorphism $\Delta \to \tilde{D}$, which is a section of $\tilde{D} \to \Delta$. Therefore we have $\text{End}(\Delta, D) = 0$ as conclusion.

\[\Box\]
Let $\Delta$ and $D$ be $(\varphi_q, \Gamma)$-modules over $\mathcal{R}$. Then $\text{Hom}_{R\text{-mod}}(\Delta, D)$ has a canonical $(\varphi_q, \Gamma)$-module structure, and we have a natural isomorphism

$$\text{Ext}(R, \text{Hom}_{R\text{-mod}}(\Delta, D)) \xrightarrow{\sim} \text{Ext}(\Delta, D). \quad (2)$$

Here, we describe the isomorphism (2) explicitly. Take any extension

$$0 \to \text{Hom}_{R\text{-mod}}(\Delta, D) \to \tilde{\mathcal{H}} \to \mathcal{R} \to 0$$

and choose a lift $\tilde{h} \in \mathcal{H}$ of $1 \in \mathcal{R}$. Then both $(\varphi_q - 1)\tilde{h}$ and $(\gamma - 1)\tilde{h}$ are in $\text{Hom}_{R\text{-mod}}(\Delta, D)$. We put $\tilde{D} = D \oplus \Delta$ as an $\mathcal{R}$-module, and define a $(\varphi_q, \Gamma)$-action on $\tilde{D}$ as follows: on $D \subset \tilde{D}$, we use the original $(\varphi_q, \Gamma)$-action. For any $x \in \Delta$ and $\gamma \in \Gamma$, we define

$$\varphi_q|_{\tilde{D}}(x) = \varphi_q|_{\Delta}(x) + [(\varphi_q - 1)\tilde{h}](x),$$

$$\gamma|_{\tilde{D}}(x) = \gamma|_{\Delta}(x) + [(\gamma - 1)\tilde{h}](x).$$

Here, for two $\mathcal{R}$-modules $V_1 \subset V_2$, an element $x \in V_1$ and an operator $T$ which acts on both $V_1$ and $V_2$, the notation $T|_{V_1}(x)$ means the image of $x$ by $T$ regarding $x$ as an element in $V_1$. Since the isomorphism class $[\tilde{D}]$ of the $(\varphi_q, \Gamma)$-module $\tilde{D}$ is independent of the choice of $\tilde{\mathcal{H}}$ and $\tilde{h}$, we have a map

$$\text{Ext}(R, \text{Hom}_{R\text{-mod}}(\Delta, D)) \to \text{Ext}(\Delta, D): [\tilde{\mathcal{H}}] \mapsto [\tilde{D}],$$

which is the isomorphism (2).

Now we present the inverse map of (2). Take any extension $0 \to D \to \tilde{D} \to \Delta \to 0$ and choose a section $s: \Delta \to \tilde{D}$ as an $\mathcal{R}$-module. Note that $s$ is an element of the $(\varphi_q, \Gamma)$-module $\text{Hom}_{R\text{-mod}}(\Delta, \tilde{D})$. Then we have

$$(\varphi_q - 1)s, (\gamma - 1)s \in \text{Hom}_{R\text{-mod}}(\Delta, D)$$

for any $\gamma \in \Gamma$. We put $\tilde{\mathcal{H}} = \text{Hom}_{R\text{-mod}}(\Delta, D) \oplus \mathcal{R}\tilde{h}$ as an $\mathcal{R}$-module, where $\mathcal{R}\tilde{h}$ is a free $\mathcal{R}$-module of rank 1 of which $\tilde{h}$ gives a basis. We define a $(\varphi_q, \Gamma)$-action on $\tilde{\mathcal{H}}$ as follows: on $\text{Hom}_{R\text{-mod}}(\Delta, D) \subset \tilde{\mathcal{H}}$, we use the original $(\varphi_q, \Gamma)$-action. We define

$$\varphi_q\tilde{h} = \tilde{h} + (\varphi_q - 1)s,$$

$$\gamma\tilde{h} = \tilde{h} + (\gamma - 1)s \quad \text{for any } \gamma \in \Gamma.$$

Since the isomorphism class $[\tilde{\mathcal{H}}]$ is independent of the choice of $\tilde{\mathcal{H}}$ and $s$, we obtain a map

$$\text{Ext}(\Delta, D) \to \text{Ext}(R, \text{Hom}_{R\text{-mod}}(\Delta, D)): [\tilde{D}] \mapsto [\tilde{\mathcal{H}}],$$

which gives the inverse of (2).

For $F$-analytic $(\varphi_q, \Gamma)$-modules $D_1$ and $D_2$ over $\mathcal{R}$, $\text{Ext}_{\text{an}}(D_1, D_2)$ denotes the $L$-subspace of $\text{Ext}(D_1, D_2)$ consisting of $F$-analytic extensions.
Lemma 3.2. Let $\Delta$ and $D$ be $F$-analytic $(\varphi_q, \Gamma)$-modules over $R$. Then the image of $\text{Ext}_{\text{an}}(R, \text{Hom}_{R\text{-mod}}(\Delta, D))$ by the isomorphism \([\cdot]\) is $\text{Ext}_{\text{an}}(\Delta, D)$.

Proof. Take any extension $0 \to \text{Hom}_{R\text{-mod}}(\Delta, D) \to \tilde{H} \to R \to 0$, and suppose that, by the isomorphism \([\cdot]\), the class $[\tilde{H}]$ maps to $[\tilde{D}] \in \text{Ext}(\Delta, D)$. First, we assume that $\tilde{H}$ is $F$-analytic. Then we must show that $\Delta$ as a $\Gamma$-submodule of $\tilde{D}$ is $F$-analytic. We choose a lift $\tilde{h}$ of $1 \in R$ in $\tilde{H}$ as above. Take any $\gamma \in \Gamma$ sufficiently close to $1$. By Lemma 2.11, the element $(\log \gamma)\tilde{h}$ is indeed in $\text{Hom}_{R\text{-mod}}(\Delta, D)$. By using Lemma 2.11 again, for any $x \in \Delta$, we have

$$(\log \gamma)\tilde{h}(x) = (\log \gamma)(\Delta)(x) + \left[\left(\log \gamma\right)(\tilde{h})(x)\right]$$

and

$$\frac{\log \gamma}{\log \chi_{LT}(\gamma)}\tilde{h}(x) = \nabla \Delta(x) + \left[\nabla \tilde{h}(x)\right].$$

Therefore the operator $\log \gamma/(\log \chi_{LT}(\gamma))$ on $\tilde{D}$ is independent of the choice of $\gamma$ and $\tilde{D}$ is $F$-analytic.

Conversely, we will show that the $F$-analyticity of $\tilde{D}$ yields that of $\tilde{H}$. Take any section $s: \Delta \to \tilde{D}$ of the projection $\tilde{D} \to \Delta$ as an $R$-module. Since both $\Delta$ and $D$ are $F$-analytic, the $(\varphi_q, \Gamma)$-module $\text{Hom}_{R\text{-mod}}(\Delta, D)$ is $F$-analytic. Thus $\nabla$ on $\text{Hom}_{R\text{-mod}}(\Delta, D)$ is well-defined. By Lemma 2.11, we have

$$(\log \gamma)\tilde{h} = (\log \gamma)s \quad \text{and} \quad \frac{\log \gamma}{\log \chi_{LT}(\gamma)}\tilde{h} = \nabla s.$$ 

Therefore $|\log \gamma/(\log \chi_{LT}(\gamma))|\tilde{h}$ is independent of the choice of $\gamma$ and $\tilde{H}$ is $F$-analytic. \qed

By Lemma 3.2 and \cite[Cor. 4.4]{FX13}, we have the following:

Corollary 3.3. Let $\Delta$ and $D$ be $F$-analytic $(\varphi_q, \Gamma)$-modules over $R$. Then $\text{Ext}_{\text{an}}(\Delta, D)$ is of codimension

$$[F: Q_p] \dim L \text{Hom}_{(\varphi_q, \Gamma)-\text{mod}/R}(\Delta, D)$$

in $\text{Ext}(\Delta, D)$. In particular, if $\text{Hom}_{(\varphi_q, \Gamma)-\text{mod}/R}(\Delta, D) = 0$, then we have

$$\text{Ext}_{\text{an}}(\Delta, D) = \text{Ext}(\Delta, D).$$

Now we state and prove the main theorem. For a $(\varphi_q, \Gamma)$-module $D$ over $R$, a finite extension $L' \subset F$ of $L$ and a character $\delta: F^\times \to (L')^\times$, we denote by $D(\delta)$ the $(\varphi_q, \Gamma)$-module over $R_{L'}$ whose underlying $R_{L'}$-module is $L' \otimes_L D$ and whose $(\varphi_q, \Gamma)$-action is defined by

$$\varphi_q|_{D(\delta)}(x) = \delta(\varphi)(\text{id}_{L'} \otimes \varphi_q|D)(x) \quad \text{and} \quad \gamma|_{D(\delta)}(x) = \delta(\chi_{LT}(\gamma))(\text{id}_{L'} \otimes \gamma_q|D)(x)$$

for any $x \in D(\delta)$ and any $\gamma \in \Gamma$. 

10
Theorem 3.4. Let $D$ be a $(\varphi_q, \Gamma)$-module over $\mathcal{R}$. Suppose that $D$ has a filtration

$$0 = D_0 \subset D_1 \subset \cdots \subset D_r = D$$

of $(\varphi_q, \Gamma)$-modules over $\mathcal{R}$. We put $\Delta_i = D_i/D_{i-1}$. We assume that the following conditions hold.

(a). For any $1 \leq i \leq r$, we have $\text{End}_{(\varphi_q, \Gamma)-\text{mod}/\mathcal{R}}(\Delta_i) = L$.

(b). For any $1 \leq i < j \leq r$, we have $\text{Hom}_{(\varphi_q, \Gamma)-\text{mod}/\mathcal{R}}(\Delta_j, \Delta_i) = 0$.

(c). For any $1 \leq i \leq r$, the short exact sequence $0 \rightarrow D_{i-1} \rightarrow D_i \rightarrow \Delta_i \rightarrow 0$ does not split.

Then there exist a finite extension $L'$ of $L$ and a character $\delta : F^\times \rightarrow (L')^\times$ such that, for any $0 \leq i \leq r$, the $(\varphi_q, \Gamma)$-module $D_i(\delta)$ is $F$-analytic.

Proof. We prove it by induction on $r$. First, we prove the case $r = 1$. Let $D$ be a $(\varphi_q, \Gamma)$-module over $\mathcal{R}$ such that $\text{End}_{(\varphi_q, \Gamma)-\text{mod}/\mathcal{R}}(D) = L$. We choose a $\mathbb{Z}$-basis $\beta_1, \ldots, \beta_n$ of $\text{Lie} \Gamma$. For any $1 \leq i \leq n$, we have $\nabla_{\beta_i} - \nabla_{\beta_i} \in \text{End}_{(\varphi_q, \Gamma)-\text{mod}/\mathcal{R}}(D)$. Thus there exist $c_1 = 0, c_2, \ldots, c_n \in L$ such that $\nabla_{\beta_i} - \nabla_{\beta_i} = c_i \cdot \text{id}_D$ for any $1 \leq i \leq n$.

Now we choose a $\mathbb{Z}$-basis $\gamma_1, \ldots, \gamma_n$ of the free part of $\Gamma$ such that $\gamma_i^{m_i} = \exp(p^{m_i} \beta_i)$ for some integers $m_i \geq 0$. We fix a finite extension $L'$ of $L$ and a character $\delta : F^\times \rightarrow (L')^\times$. By Lemma 2.11, for any $x \in D(\delta)$, we can compute

$$\nabla_{\beta_i}(\delta(x)) = (\text{id}_{L'} \otimes \nabla_{\beta_i}(D))(x) + \frac{\log \delta(\chi_{\text{LT}}(\gamma_i))}{\log \chi_{\text{LT}}(\gamma_i)} x.$$ 

Thus, if we find a character $\delta$ such that the equality

$$c_i + \frac{\log \delta(\chi_{\text{LT}}(\gamma_1))}{\log \chi_{\text{LT}}(\gamma_1)} - \frac{\log \delta(\chi_{\text{LT}}(\gamma_i))}{\log \chi_{\text{LT}}(\gamma_i)} = 0$$

holds for each $1 \leq i \leq n$, then $D(\delta)$ is $F$-analytic. Actually, we can do it: on the torsion part of $F^\times$ and on $\chi_{\text{LT}}(\gamma_1)$, put $\delta = 1$. On $\delta(\chi_{\text{LT}}(\gamma_2)), \ldots, \delta(\chi_{\text{LT}}(\gamma_n))$, put

$$\delta(\chi_{\text{LT}}(\gamma_i)) = \exp(c_i \log \chi_{\text{LT}}(\gamma_i)),$$

where $c_i$ is a sufficiently large integer such that the right-hand side is well-defined. Finally, define $\delta(\chi_{\text{LT}}(\gamma_i))$ as an $e_i$-th root of $\delta(\chi_{\text{LT}}(\gamma_i))$ (hence we must extend $L$ in general).

Next we suppose that the statement for $r - 1$ holds. By extending scalar and twisting by a character, we may assume that, for any $1 \leq i \leq r - 1$, $D_i$ is $F$-analytic. Then we must show that $D_r$ is $F$-analytic. By the assumption (a), (c) and Lemma 3.1, $\Delta_r$ is $F$-analytic. Now we apply the functor $\text{Hom}_{(\varphi_q, \Gamma)-\text{mod}/\mathcal{R}}(\Delta_r, \bullet)$ to the short exact sequences

$$0 \rightarrow \Delta_1 \rightarrow D_2 \rightarrow \Delta_2 \rightarrow 0,$$

$$0 \rightarrow D_2 \rightarrow D_3 \rightarrow \Delta_2 \rightarrow 0,$$

$$\vdots$$

$$0 \rightarrow D_{r-1} \rightarrow D_r \rightarrow \Delta_r \rightarrow 0.$$
Then, by the assumption (b), we inductively obtain \( \text{Hom}_{(\varphi, \Gamma)\text{-mod}/R}(\Delta_r, D_i) = 0 \) for any \( 1 \leq i \leq r - 1 \). By Corollary 3.3 we have \( \text{Ext}_{\text{an}}(\Delta_r, D_{r-1}) = \text{Ext}(\Delta_r, D_{r-1}) \). Therefore \( D_r \) is \( F \)-analytic and we have the conclusion.

This theorem, together with Theorem 2.12, implies the following:

**Corollary 3.5** (Theorem 1.2). Let \( V \) be an overconvergent \( L \)-representation of \( \mathcal{G}_F \) such that \( R \otimes_{\mathcal{F}} D^1(V) \) satisfies the assumptions of Theorem 3.4. Then there exist a finite extension \( L' \) of \( L \) and a character \( \delta: \mathcal{G}_F \to (L')^\times \) such that \( V \otimes_{L} L'(\delta) \) is \( F \)-analytic.

**References**

[Ber02] L. Berger, *Représentations p-adiques et équations différentielles*, Invent. Math. **148** (2002), 219–284.

[Ber13] , *Multivariable Lubin-Tate \((\varphi, \Gamma)\)-modules and filtered \( \varphi \)-modules*, Math. Res. Lett. **20** (2013), no. 3, 409–428.

[Ber16] , *Multivariable \((\varphi, \Gamma)\)-modules and locally analytic vectors*, Duke Math. J. **165** (2016), no. 18, 3567–3595.

[BF17] L. Berger and L. Fourquaux, *Iwasawa theory and \( F \)-analytic Lubin-Tate \((\varphi, \Gamma)\)-modules*, Documenta Math. **22** (2017), 999–1030.

[CC98] F. Cherbonnier and P. Colmez, *Représentations p-adiques surconvergentes*, Invent. Math. **133** (1998), 581–611.

[CDP14] P. Colmez, G. Dospinescu, and V. Paškūnas, *The \( p \)-adic local Langlands correspondence for \( GL_2(\mathbb{Q}_p) \)*, Cambridge J. Math. **2** (2014), no. 1, 1–47.

[Col10] P. Colmez, *Représentations de \( GL_2(\mathbb{Q}_p) \) et \((\varphi, \Gamma)\)-modules*, Astérisque **330** (2010), 281–509.

[Fon91] J. M. Fontaine, *Représentations p-adiques des corps locaux*, The Grothendieck Festschrift Volume II, Prog. Math., vol. 87, Birkhauser, 1991, pp. 249–309.

[FX13] L. Fourquaux and B. Xie, *Triangulable \( \mathcal{O}_F \)-analytic \((\varphi, \Gamma)\)-modules of rank 2*, Algebra and Number Theory **7** (2013), no. 10, 2545–2592.

[KR09] M. Kisin and W. Ren, *Galois representations and Lubin-Tate groups*, Documenta Math. **14** (2009), 441–461.