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spaces we will be considering. Section 4 contains the main results. We begin in Section 4.1 with the setting

The Frechét manifold

Theorem 1.3.

smooth or of 'low-regularity' in the Sobolev sense. We further use these regularity properties to construct

denote the corresponding

Theorem 1.4.

ideal hydrodynamics. We record some examples of these here:

An immediate consequence of these theorems is that isochronal flows in these settings must either be

smooth or of 'low-regularity' in the Sobolev sense. We further use these regularity properties to construct

continuously Frechét differentiable exponential maps in the smooth setting.

Theorem 1.2. Let s > \frac{42}{7} and consider the group of axisymmetric diffeomorphisms of T^3 with respect to any of the Killing fields K = \partial_r, equipped with a right invariant L^2 metric. Given u_0 \in T_e \mathcal{A}_s^{\ast,0} let \gamma(t) denote the corresponding L^2 geodesic. If at time t = 1, \gamma passes through a point \eta \in \mathcal{A}_s^{\ast,1}, then we have u_0 \in T_e \mathcal{A}_s^{\ast,1} and consequently \gamma(t) evolves entirely in \mathcal{A}_s^{\ast,1}.

Theorem 1.3. The Frechét manifold \mathcal{D}_s(T^2) of smooth diffeomorphisms of T^2 equipped with the L^2 metric admits a well-defined exponential map which is a local C^1_d-diffeomorphism at the identity.

Theorem 1.4. The Frechét manifold \mathcal{A}_s^{\ast,0}(T^3) of smooth axisymmetric swirl-free diffeomorphisms of T^3 with respect to any of the Killing Fields K = \partial_r equipped with the L^2 metric admits a well-defined exponential map which is a local C^1_d-diffeomorphism at the identity.

Later sections will provide more precise statements of these results and notations.

Theorem 1.1 improves upon a similar result of Omori [27] which is infinitesimal in character. His method involves constructing a connection on the full diffeomorphism group which turns the volume-preserving diffeomorphisms into a totally geodesic submanifold. In the proof of Theorem 1.2 we make use of the recent work of Lichtenfelz, Misiołek and Preston [22] on the Euler equations in 3 dimensions with axisymmetric swirl-free initial data.

Throughout the arguments we will consider multiple configuration spaces: the full diffeomorphism group, the space of volumorphisms and its subspace of axisymmetric diffeomorphisms and the space of symplectomorphisms, all equipped with various right-invariant metrics. In each setting we will consider Sobolev diffeomorphisms of class H^s with varying requirements on the value of s and we may write \mathcal{D}^s(M) to refer to any of the aforementioned spaces. Similarly, we use \mathcal{G}(M) to refer to the corresponding smooth settings.

The paper is organised as follows: Sections 2 & 3 contain the necessary background information for the spaces we will be considering. Section 4 contains the main results. We begin in Section 4.1 with the setting of compressible fluids on T^n. The theorems presented here generalise to arbitrary dimensions the results from one and two dimensions contained in [10], [9] and [19] without any restrictions on whether or not the endpoints are conjugate, unlike the result contained in [8]. We then proceed to the Euler equations on T^2 in Section 4.2 where we also cover the cases of the Euler-\alpha and higher order Euler-Arnold equations.
mentioned above. This is followed by the axisymmetric results for 3D fluids in Section 4.3. The final case of the Symplectic Euler equations on $T^{2k}$ is covered in Section 4.4. In Section 5 we use our results from the Section 4 to construct continuously differentiable exponential maps in the Fréchet settings.

2. Manifold Structure of Sobolev Diffeomorphisms

Here we gather some basic facts about diffeomorphism groups. Further details concerning these spaces can be found in Ebin and Marsden [12], Arnold and Khesin [3], Ebin [11], Omori [28], Misiolek and Preston [26] and Inci, Kappeler and Topalov [17].

2.1. Sobolev Diffeomorphisms. Let $M$ be a compact, $n$-dimensional Riemannian manifold without boundary, with metric $g$ and volume form $\mu$. Let $g^\flat$ and $g^\sharp$ denote the usual “musical isomorphisms”. We define the Hodge Laplacian on $k$-forms by $\Delta = d\delta + \delta d$, where $d$ is the exterior derivative, $\delta = (-1)^{n(k+1)+1} \ast d \ast$ is its $L^2$ dual, and $\ast$ is the Hodge Star operator.

We define $H^s(TM)$ to be the space of vector fields on $M$ with $L^2$ derivatives up to order $s$. We equip $H^s(TM)$ with a $H^s$ inner product via:

$$\langle u, v \rangle_s = \sum_{k \leq s} \langle u, (1 + \Delta)^k v \rangle_{L^2}$$

In this paper we will concern ourselves with the case $s \in \mathbb{Z}_{\geq 0}$, but many of the constructions can be extended to the fractional case.

Recall that if $s > \frac{n}{2}$, then $H^s(M, M)$ is a Hilbert manifold modeled on $H^s(TM)$ equipped with $\langle \cdot, \cdot \rangle_s$. If we further require $s > \frac{n}{2} + 1$, then $\mathcal{D}^s(M)$ inherits a smooth submanifold structure as an open subset of $H^s(M, M)$. Its tangent space at the identity is $T_\epsilon \mathcal{D}^s(M) = H^s(TM)$ whose dual enjoys an $L^2$-orthogonal decomposition by the Hodge theorem:

$$H^s(T^*M) = \mathcal{H} \oplus d\delta H^{s+2}(T^*M) \oplus \delta dH^{s+2}(T^*M)$$

where $\mathcal{H}$ denotes the finite dimensional subspace of harmonic one-forms on $M$.

In what follows it will be convenient to define the projections:

$$\pi_0 : T_\epsilon \mathcal{D}^s \to d\delta H^{s+2} \oplus \delta dH^{s+2} \quad \text{and} \quad \pi_\mathcal{H} : T_\epsilon \mathcal{D}^s \to \mathcal{H}$$

2.2. Volume-Preserving Diffeomorphisms. The volumorphism group is defined in terms of the Riemannian volume form $\mathcal{D}^s_\mu(M) = \{ \eta \in \mathcal{D}^s(M) \mid \eta^* \mu = \mu \}$. It is a smooth Hilbert submanifold of $\mathcal{D}^s(M)$ and the tangent space at the identity consists of all divergence-free $H^s$ vector fields $T_\epsilon \mathcal{D}^s_\mu(M) = \{ u \in T_\epsilon \mathcal{D}^s(M) \mid \text{div } u = 0 \} = g^\sharp \{ \mathcal{H} \oplus \delta dH^{s+2}(T^*M) \}$, i.e the first and third summand of the Hodge Decomposition above.

2.3. Axisymmetric Diffeomorphisms. Let $M$ be a 3-dimensional manifold equipped with a smooth Killing field $K$. Following [22] we define a divergence-free vector field $u$ on $M$ to be axisymmetric if it commutes with the Killing field: $[K, u] = 0$. We denote the set of $H^s$ axisymmetric vector fields by $T_\epsilon \mathcal{A}_{\mathcal{K}}^s(M)$.

A volume-preserving diffeomorphism of $M$ is said to be axisymmetric if it commutes with the flow of the Killing field $K$. The set of all such $H^s$ volumorphisms, $\mathcal{A}_{\mathcal{K}}^s(M)$ is a topological group as well as a smooth totally geodesic Hilbert submanifold of $\mathcal{D}_\mu^s(M)$, cf. [22] Section 3].

Axisymmetric fluid flows are of great interest and their behaviour might be informally described as $2\frac{1}{2}$-dimensional fluids, cf. [35].

2.4. Symplectomorphisms. Let $M$ be a symplectic manifold of dimension $2k$ and let $\omega^\flat$ and $\omega^\sharp$ denote the standard “$\omega$-musical isomorphisms”. Analogously to volumorphisms, the symplectomorphism group $\mathcal{D}^s_\omega(M)$ is a closed Hilbert submanifold of $\mathcal{D}^s(M)$ consisting of those diffeomorphisms which preserve the symplectic form $\omega$ under pullback. The tangent space at the identity is $T_\epsilon \mathcal{D}^s_\omega(M) = \{ u \in T_\epsilon \mathcal{D}^s(M) \mid d\omega^\flat u = 0 \} = \omega^\sharp (\mathcal{H} \oplus d\delta H^{s+2}(T^*M))$, cf. [24].

For our purposes we will further require that $g$ and $\omega$ are compatible, that is, the map $J := g^\flat \omega^\flat : TM \to TM$ satisfies $J^2 = -Id$. In this case $J$ is said to give $M$ an almost complex structure, cf. [12, 11, 6, 29].
3. Lie Group Structure and Geodesics on Diffeomorphism Groups

Here we develop an infinite dimensional Lie group framework\(^1\) for \(\mathcal{D}^s(M)\) and its submanifolds of interest. We use this to present geodesic equations on diffeomorphism groups in a general formulation for a variety of metrics. The construction follows closely that of \[12\] and \[26\].

Throughout the various arguments in Section \[4\] we will deal with products of \(H^s\) Sobolev functions and compositions with elements of \(\mathcal{D}^s(M)\), so it is crucial that we have some control over the regularity of these objects. For this reason we recall the following results cf. \[1\], \[12\], \[17\].

**Lemma 3.1.** For any \(s > \frac{n}{2} + 1\), \(|m| \leq s\) and \(k \geq 0\) we have:

1. \(H^s(M, \mathbb{R}) \times H^m(M, \mathbb{R}) \rightarrow H^m(M, \mathbb{R})\); \((u, v) \mapsto uv\) is bounded.

2. \(H^{s+k}(M, \mathbb{R}^d) \times \mathcal{D}^s(M) \rightarrow H^s(M, \mathbb{R}^d)\); \((v, \phi) \mapsto v \circ \phi\) is \(C^k\)-smooth.

3. \(\mathcal{D}^{s+k}(M) \rightarrow \mathcal{D}^s(M)\); \(\phi \mapsto \phi^{-1}\) is \(C^k\)-smooth.

In particular this gives us that \(\mathcal{D}^s(M)\) is a topological group (one can readily show that \(\mathcal{D}^s(M)\), \(\omega\mathcal{D}^s(M)\) and \(\mathcal{D}_\omega^s(M)\) are all subgroups), where right translation \(R_\eta\) is smooth and left translation \(L_\eta\) is continuous (although not even Lipschitz continuous) in the \(H^s\) topology.

Again, using \(\mathcal{G}^s(M)\) to refer to any of the aforementioned manifolds, we denote the (almost) Lie algebra by \(\mathfrak{g}^s = T_e\mathcal{G}^s(M)\), we recall the group adjoint:

\[ Ad_\eta v = d_{\eta^{-1}}L_{d_\eta}R_{\eta^{-1}}v = (D\eta \cdot v) \circ \eta^{-1} \]

and the Lie algebra adjoint:

\[ ad_u v = -[u, v] \]

where \(\eta(t)\) is any curve in \(\mathcal{G}^s(M)\) with \(\eta(0) = e\) and \(\dot{\eta}(0) = u\). Observe that if \(u, v \in T_e\mathcal{G}^s\), their commutator is a priori only of Sobolev class \(H^{s-1}\).

**Remark 3.2.** The derivative loss coming from left translation and the Lie bracket are examples of why these diffeomorphism groups are not Lie Groups. However, the group structure they possess is sufficient for our purposes.

We now equip \(\mathcal{G}^s(M)\) with a (weak\(^2\)) right-invariant, Riemannian metric \(\langle \cdot, \cdot \rangle\). Note that the definitions of \(Ad_\eta\) and \(ad_u\) depend only on the group structure, and not on the choice of the metric. Hence, the geometry is, in some sense, encoded in the coadjoint operators defined as follows:

\[ (Ad_\eta^* u, v) = \langle u, Ad_\eta v \rangle \quad \text{and} \quad (ad_u^* v, w) = \langle v, ad_u w \rangle \]

for all \(u, v, w \in \mathfrak{g}^s\). Throughout this paper \(*\) will always refer to the adjoint of an operator with respect to the relevant metric and configuration space; which should be clear from the context.

We are specifically interested in the case where we equip \(\mathcal{G}^s(M)\) with a right-invariant \(H^r\) metric with \(r \geq 0\) defined at the identity by:

\[ (u, v)_{H^r} := \langle A^r u, v \rangle_{L^2} \quad u, v \in T_e\mathcal{G}^s \]

where \(A^r\) denotes an invertible elliptic pseudo-differential operator of order \(2r\); e.g. \(A^r = (1 + g^2 \Delta g^2)^r\), cf. Taylor \[24\]. We refer to such an \(A^r\) as an *inertia operator* and denote its inverse by \(A^{-r}\). We will always assume that \(A^r\) commutes with both \(d\) and \(\delta\) and that we have at least \(s > \frac{n}{2} + 2r + 1\) to guarantee a baseline level of control over the regularity of the vector fields involved in the later calculations. We will use \(*_r\) to denote the adjoint of an operator with respect to such a \(H^r\) metric.

\(^1\)It is important to note that, strictly speaking, these Hilbert manifolds of finite regularity mappings are not Lie Groups. However, they are topological groups and possess sufficient geometric structure for our purposes. See Remark \[32\] for more details.

\(^2\)We say a metric on \(\mathcal{G}^s\) is weak if it induces a weaker topology than the inherent \(H^s\) topology.
We define a geodesic on $(\mathcal{G}(M), \langle \cdot, \cdot \rangle)$ to be a critical path for the energy functional induced by the metric $\langle \cdot, \cdot \rangle$ and recall two important lemmas pertaining to geodesics on Lie groups with right-invariant metrics:

**Lemma 3.3** (cf. [26, Theorem 3.2]). If $\mathcal{G}$ is a Lie group equipped with a (possibly weak) right-invariant metric $\langle \cdot, \cdot \rangle$, then a curve $\sigma(t)$ is a geodesic if and only if the curve $u(t)$ in the Lie algebra given by the flow equation:

$$\dot{\sigma}(t) = d_e R_{\sigma(t)} u(t)$$

solves the Euler-Arnold equation:

$$\partial_t u(t) = - \text{ad}^*_{u(t)} u(t)$$

**Lemma 3.4** (cf. [26, Corollary 3.3]). If $\sigma(t)$ is a curve in $\mathcal{G}$ with $u(t) = d_{\sigma(t)} R_{\sigma(t)}^{-1} (\dot{\sigma}(t))$ satisfying (3.2) with initial conditions $\sigma(0) = e$ and $u(0) = u_0$, then we have the following conservation law:

$$\text{Ad}^*_{\sigma(t)} u(t) = u_0$$

and hence we can rewrite the flow equation as:

$$\dot{\sigma}(t) = d_e R_{\sigma(t)} \text{Ad}^*_{\sigma(t)} u_0 = d_e L_{\sigma(t)}^{-1} u_0$$

**Remark 3.5.** It is important to note at this point that Lemmas 3.3 & 3.4 do not apply seamlessly to our setting of $H^s$ Sobolev diffeomorphism groups. As mentioned earlier, $\mathcal{G}(M)$ and its subgroups of interest are not Lie groups, on account of left translation only being continuous, etc. However, in any of the settings we consider in this paper, analogues of (3.2) and (3.3) will hold, cf. Section 3 of [20].

Notable examples of Euler-Arnold equations (3.2) in the setting of diffeomorphism groups include the incompressible Euler equations in two and three dimensions, Burgers’ equation, the Hunter-Saxton equation, the Camassa-Holm equation, the $\mu$CH equation as well as the Euler-\(\alpha\) equations and the symplectic Euler equations. The associated Riemannian geometry of these equations as well as the existence of a $C^\infty$ exponential map and its properties has been studied extensively in the literature cf. [12], [28], [31], [13], [30], [20], [10], [9], [19], [11], [21] and many others.

### 4. Main Results

Let $\mathcal{G}$ be a group of diffeomorphisms of $M$ equipped with a (weak) Riemannian metric $\langle \cdot, \cdot \rangle$ and an associated exponential map where we assume that the results stated in Lemmas 3.3 & 3.4 hold. Our main focus is the following question: if a geodesic $\gamma(t)$ emanating from the identity in $\mathcal{G}$ at some later time $t_0 > 0$ passes through $\eta \in \mathcal{G}^{s+k}$, can it be shown that $\gamma(t)$ evolves entirely in $\mathcal{G}^{s+k}$? For our purposes, it will be sufficient to assume that both $t_0 = 1$ and $k = 1$. References for the various commutator estimates involved in the calculations include Kato-Ponce [20] and Taylor [34].

As mentioned in the introduction, previous results in this vein are due to Constantin and Kolev [10], [9] and Kappeler, Loubet and Topalov [19], who worked with compressible equations on the circle and the 2-torus respectively and Bruveris [8]. At the heart of our method lie the various conservation laws motivated by Lemma 3.4.

In this paper we will concern ourselves primarily with the flat case $M = \mathbb{T}^n$. However, many of the constructions can be extended to the setting of curved spaces by collecting any lower order terms arising in the various calculations due to derivatives of the components of the metric and its Christoffel symbols on $M$ into a single term which will be negligible for our purposes. It should be noted that we do not believe our results are sharp. We require varying conditions on the parameter $s$ due to the method of proof; all of which can be explained by Lemma 3.1.

We begin the groundwork for the main results. We aim to establish an explicit relationship between the regularity of the end configuration $\eta$ and the regularity of the initial velocity $u_0$. To do this we will make use of (3.3), suitably modified to each case considered below.
Lemma 4.1. Let \( s > \frac{n}{2} + 3 \) and let \( \gamma(t) \) be a smooth curve in \( \mathscr{P}^s(T^n) \) with \( v(t) = d_{\gamma(t)}R_{\gamma(t)^{-1}}(\dot{\gamma}(t)), \gamma(0) = e \) and \( \gamma(1) = \eta \). We have the following identity

\[
\Delta \eta = D\eta \int_0^1 D\gamma(t)^{-1}(P_\lambda v) \circ \gamma(t) \, dt + G
\]

where \( G = G(\gamma) \) is of class \( H^{s-1} \), \( \Delta \eta \) is defined by considering \( \eta : T^n \rightarrow T^n \) in coordinates and applying \( \Delta \) to each component and, for any \( \lambda \in \mathbb{R} \), \( P_\lambda \) is a differential operator acting component-wise given by:

\[
P_\lambda(t) = \lambda - \sum_{i,j=1}^n p_{ij}(t) \partial_i \partial_j \quad \text{with} \quad p_{ij}(t) = (D\gamma D\gamma^\top)_{ij} \circ \gamma^{-1}(t)
\]

with \( D\gamma^\top \) denoting the pointwise adjoint of \( D\gamma \).

Proof. Applying the Laplacian \( \Delta \) to the tangent vector to the curve \( \gamma(t) \), we get:

\[
\Delta \left( \frac{d\gamma}{dt} \right) = \Delta(v \circ \gamma) = (Dv \circ \gamma)\Delta \gamma - \lambda v \circ \gamma + (P_\lambda v \circ \gamma)
\]

Rearranging, we get:

\[
\begin{cases}
\frac{d}{dt}(\Delta \gamma) - (Dv \circ \gamma)(\Delta \gamma) = (P_\lambda v) \circ \gamma - \lambda v \circ \gamma \\
\Delta \gamma(0) = 0
\end{cases}
\]

On the other hand, differentiating the flow equation \( d\gamma/dt = v \circ \gamma \) in the spatial variables gives:

\[
\begin{cases}
\frac{d}{dt}(D\gamma) - Dv \circ \gamma D\gamma = 0 \\
D\gamma(0) = \text{Id}
\end{cases}
\]

Using (4.4) and the Duhamel formula, we can now rewrite (4.3) as an integral equation in the form:

\[
\Delta \gamma(t) = D\gamma(t) \int_0^t D\gamma(\tau)^{-1}(P_\lambda v) \circ \gamma(\tau) \, d\tau + G(t)
\]

where

\[
G(t) = -\lambda D\gamma(t) \int_0^t D\gamma(\tau)^{-1}(v \circ \gamma)(\tau) \, d\tau
\]

is a curve in \( H^{s-1}(T^n, \mathbb{R}^n) \). Evaluating at \( t = 1 \) and denoting \( G(1) \) by \( G \) we arrive at (4.1). 

Suppose now that \( \gamma(t) \) is a geodesic of the metric \( \langle \cdot, \cdot \rangle \) in \( \mathscr{P}^s \) with \( \gamma(0) = e \) and \( \dot{\gamma}(0) = u_0 \in T_e \mathscr{P}^s \). Suppose further that at time \( t_0 = 1 \), \( \gamma \) passes through \( \eta \). Since \( \text{Ad}_{\gamma(t)}^{-1} = D\gamma(t)^{-1}R_{\gamma(t)} \), using the conservation law (4.3) and Lemma 4.1 we may rewrite (4.1) as:

\[
\Delta \eta = D\eta \int_0^1 \text{Ad}_{\gamma(t)}^{-1} P_\lambda(t) \left( \text{Ad}_{\gamma(t)}^{-1} \right)^* u_0 \, dt + G
\]

where \( \star \) denotes the metric adjoint as before.

We can see from (4.5) that \( P_\lambda \) will play a central role in establishing a relationship between the regularity of \( \eta \) and our initial data \( u_0 \). To this end we will establish that it defines a norm equivalent to the \( H^1 \) norm for sufficiently large \( \lambda > 0 \).

Lemma 4.2. There exists a \( \lambda > 0 \) such that, for any \( t \in [0,1] \) and \( v \in H^1 \), the operator \( P_\lambda \) satisfies the estimate:

\[
\langle P_\lambda(t)v, v \rangle_{L^2} \simeq \| v \|_{H^1}^2
\]

with the constants depending on the curve \( \gamma : [0,1] \rightarrow \mathscr{P}^s(T^n) \).
Proof. We first derive an estimate for the coefficients $p_{ij}$. For any $w \in \mathbb{R}^n$, $t \in [0,1]$ and $1 \leq i, j \leq n$, we have

\[
\sum_{i,j=1}^{n} p_{ij}(t)w_i w_j = w^T [D\gamma(t)] [D\gamma(t)^T] \circ \gamma^{-1}(t) w
\]

\[
= |D\gamma(t)^T \circ \gamma^{-1} w|^2.
\]

As $s > \frac{n}{2} + 3$, it follows from a compactness argument and the fact that $D\gamma \circ \gamma^{-1}$ is a linear isomorphism of $\mathbb{R}^n$ for all $t \in [0,1]$ that:

\[
(4.6) \quad \sum_{i,j=1}^{n} p_{ij}(t)w_i w_j \simeq |w|^2
\]

Integrating by parts and using estimate (4.6), we have:

\[
(P_\lambda(t)v, v)_{L^2} = \lambda \langle v, v \rangle_{L^2} - \sum_{i,j} \langle p_{ij}(t)\partial_i \partial_j v, v \rangle_{L^2}
\]

\[
= \lambda \|v\|_{L^2}^2 + \frac{1}{2} \int_{\mathbb{T}^n} \sum_{i,j} \partial_i p_{ij}(t)\partial_j |v|^2 \, dx + \int_{\mathbb{T}^n} \sum_{i,j} p_{ij}(t)\partial_j v \partial_i v \, dx
\]

\[
= \lambda \|v\|_{L^2}^2 - \frac{1}{2} \sum_{i,j} \langle \partial_i \partial_j p_{ij}(t)v, v \rangle_{L^2} + \int_{\mathbb{T}^n} \sum_{i,j} p_{ij}(t)\partial_j v \partial_i v \, dx
\]

\[
\geq \lambda \|v\|_{L^2}^2 - \frac{1}{2} \sum_{i,j} \|\partial_i \partial_j p_{ij}(t)v\|_{L^2} \|v\|_{L^2} + \int_{\mathbb{T}^n} \sum_i |\partial_i v_1|^2 |\partial_i v_2|^2 \, dx
\]

where the penultimate and last inequalities follow from the uniform boundedness in $t \in [0,1]$ of the coefficients $\partial_i \partial_j p_{ij}$ and taking $\lambda$ sufficiently large.

On the other hand using (4.6) again we have

\[
(P_\lambda(t)v, v)_{L^2} \lesssim \lambda \|v\|_{L^2}^2 + \frac{1}{2} \sum_{i,j} \|\partial_i \partial_j p_{ij}(t)v\|_{L^2} \|v\|_{L^2} + \int_{\mathbb{T}^n} \sum_i |\partial_i v_1|^2 \|\partial_i v_2\|^2 \, dx
\]

\[
\lesssim \|v\|_{H^1}^2.
\]

\[\blacksquare\]

4.1. Diffeomorphisms of the Flat $n$-Torus $\mathbb{T}^n$. We are now ready to present our first theorem, which generalises to $n$ dimensions the corresponding results proved in [10], [9] and [19]. Our method follows along the lines of the aforementioned. The main difference is the explicit use of the coadjoint operators and the conservation law (3.3) which shortens the argument.

**Theorem 4.3.** Let $r \geq 1$ be an integer, $s > \frac{n}{2} + 2r + 5$ and consider the space of orientation preserving diffeomorphisms of $\mathbb{T}^n$ equipped with a right-invariant $H^r$ metric (3.1). Given $u_0 \in T_s(\mathcal{D}^s)$ let $\gamma(t)$ denote the corresponding $H^r$ geodesic. If at time $t = 1$, $\gamma$ passes through a point $\eta \in \mathcal{D}^{s+1}$, then we have $u_0 \in T_s(\mathcal{D}^{s+1})$ and consequently $\gamma(t)$ evolves entirely in $\mathcal{D}^{s+1}$.

Recall from [26], Section 3, that the group coadjoint on $\mathcal{D}^s(\mathbb{T}^n)$ equipped with a $H^r$ metric has the form:

\[
\text{Ad}_{\eta}^{\ast} v = A^{-r} (\det(D\eta)D\eta^{\top}(A^r v \circ \eta))
\]

Hence we have the following version of (3.3):

\[
(4.7) \quad u(t) = (\text{Ad}_{\gamma(t)}^{-1})^r u_0 = A^{-r} R_{\gamma(t)}^{-1} (\det(D\gamma(t))D\gamma(t)^{\top})^{-1} A^r u_0
\]
Remark 4.4. Later in our argument we will require that, for each $t \in [0,1]$, $(\text{Ad}_{\gamma(t)}^{-1})^* : T_c \mathcal{D}^{s+1} \rightarrow T_c \mathcal{D}^{s+1}$ be a bounded invertible linear operator, so it is at this point that we can explicitly see the necessity of $r \geq 1$. A simple derivative count shows that $A'$ must be at least of order 2 to prevent a loss of derivatives coming from the multiplication by the coefficients of the matrix $D\gamma(t)$ which are apriori only of class $H^{s-1}$ (recall that only the endpoints of $\gamma(t)$ will be assumed to be of class $H^{s+1}$).

Proof of Theorem 4.4. The proof consists of three stages. We begin by combining (4.5) with some commutator estimates to acquire an expression of the form $\Delta \eta = D\eta M u_0 + \tilde{G}$, where $\tilde{G}$ is of class $H^{s-1}$. We then establish that, for $v \in H^s$, $Mv$ being of class $H^{s-1}$ implies that $v$ is of class $H^{s+1}$. We finish by concluding that, as $Mu_0 = D\eta^{-1}(\Delta \eta - \tilde{G}) \in H^{s-1}$, we have that $u_0 \in H^{s+1}$. Hence, by the results for the initial value problem, we have that $\gamma(t)$ evolves entirely in $H^{s+1}$ and the proof will be complete. Recall (4.5):

$$\Delta \eta = D\eta \int_0^1 \text{Ad}_{\gamma(t)}^{-1} P_{\lambda}(t) (\text{Ad}_{\gamma(t)}^{-1})^* u_0 \ dt + G$$

where again $G$ is of class $H^{s-1}$. Introducing a commutator term $P_{\lambda} = A^{-\bar{\gamma}} \text{Ad}_{\lambda}(t) A^{\bar{\gamma}} + A^{-\bar{\gamma}} \text{Ad}_{\lambda}(t) P_{\lambda}(t)$ we have

$$\Delta \eta = D\eta \int_0^1 \text{Ad}_{\gamma(t)}^{-1} P_{\lambda}(t) A^{\bar{\gamma}} (\text{Ad}_{\gamma(t)}^{-1})^* u_0 \ dt + D\eta \int_0^1 \text{Ad}_{\gamma(t)}^{-1} A^{\bar{\gamma}} (\text{Ad}_{\gamma(t)}^{-1})^* u_0 \ dt + G$$

Notice now that $[A^{\bar{\gamma}}, P_{\lambda}(t)] = [p^{ij}, A_{\bar{\gamma}}]\partial_i \partial_j$ and, as $p^{ij}$ are of class $H^{s-1}$, we have that $[A^{\bar{\gamma}}, P_{\lambda}(t)] : H^r \rightarrow H^{s-r-1}$ by the following Kato-Ponce [20] type commutator estimate, cf. [31]:

$$(4.8) \quad \|A^{\bar{\gamma}}(fg) - fA^{\bar{\gamma}}g\|_{H^{s-r-1}} \lesssim \|\nabla f\| \|g\|_{H^{s-1}} + \|f\|_{H^{s-1}} \|g\|_{\infty}$$

Hence, the term $D\eta \int_0^1 \text{Ad}_{\gamma(t)}^{-1} A^{\bar{\gamma}} (\text{Ad}_{\gamma(t)}^{-1})^* u_0 \ dt$ belongs to $H^{s-1}$ and we rewrite:

$$\Delta \eta = D\eta \int_0^1 \text{Ad}_{\gamma(t)}^{-1} A^{\bar{\gamma}} P_{\lambda}(t) A^{\bar{\gamma}} (\text{Ad}_{\gamma(t)}^{-1})^* u_0 \ dt + \tilde{G}$$

where $\tilde{G} := D\eta \int_0^1 \text{Ad}_{\gamma(t)}^{-1} A^{\bar{\gamma}} (\text{Ad}_{\gamma(t)}^{-1})^* u_0 \ dt + G$ is of class $H^{s-1}$.

Next, we claim that the operator defined by:

$$(4.9) \quad Mv = \int_0^1 M_t v \ dt := \int_0^1 \text{Ad}_{\gamma(t)}^{-1} A^{\bar{\gamma}} P_{\lambda}(t) A^{\bar{\gamma}} (\text{Ad}_{\gamma(t)}^{-1})^* v \ dt$$

is a linear isomorphism from $H^{r+1}$ to $H^{s-1}$. We can see that, for $r + 1 \leq k \leq s + 1$, $M : H^k \rightarrow H^{k-2}$ is a bounded linear operator, as the integrand is a composition of bounded linear operators. Consider the case $k = r + 1$, and define the bilinear form:

$$\Lambda : H^{r+1} \times H^{r+1} \rightarrow \mathbb{R} \quad \Lambda(v, w) = \langle Mv, w \rangle_{H^r}$$

It follows from the boundedness of $M$ that $\Lambda$ is bounded. Furthermore, by Lemma 4.2 we have:

$$\Lambda(v, v) = \int_0^1 \langle A^{\bar{\gamma}} P_{\lambda}(t) A^{\bar{\gamma}} (\text{Ad}_{\gamma(t)}^{-1})^* v, (\text{Ad}_{\gamma(t)}^{-1})^* v \rangle_{H^r} \ dt$$

$$= \int_0^1 \langle P_{\lambda}(t) A^{\bar{\gamma}} (\text{Ad}_{\gamma(t)}^{-1})^* v, A^{\bar{\gamma}} (\text{Ad}_{\gamma(t)}^{-1})^* v \rangle_{L^2} \ dt$$

$$\gtrsim \int_0^1 \|A^{\bar{\gamma}} (\text{Ad}_{\gamma(t)}^{-1})^* v\|_{H^1}^2 \ dt$$

$$\gtrsim \|v\|_{H^{r+1}}^2$$
Hence, by the Lax-Milgram theorem, we have that $M : H^{r+1} \to H^{r-1}$ is a linear isomorphism. We now proceed by induction. Assume $M : H^k \to H^{k-2}$ is a linear isomorphism for some $r + 1 \leq k \leq s$, and let $g \in H^{k-1}$. By the induction hypothesis, there exists $f \in H^k$ such that $Mf = g$. We claim that $f$ is in fact of class $H^{k+1}$. For $j = 1, \ldots, n$ consider:

$$\partial_j f = \partial_j M^{-1} g = M^{-1} \partial_j g + [\partial_j, M^{-1}] g = M^{-1} \partial_j g + M^{-1} [M, \partial_j] M^{-1} g = M^{-1} \partial_j g + M^{-1} [M, \partial_j] f.$$ 

By assumption, the first term on the RHS is in $H^k$. As for the commutator term, from \textbf{(4.9)} is sufficient to show that for $j = 1, \ldots, n$ and $r + 1 \leq k \leq s$, $[M, \partial_j] : H^k \to H^{k-2}$. This will give us that $[M, \partial_j] : H^k \to H^{k-2}$, and hence $\partial_j f \in H^k$.

First, for an operator $B$, define $C_\gamma(B) = R_\gamma BR_\gamma^{-1}$, and its inverse $\tilde{C}_\gamma(B) = R_\gamma^{-1}BR_\gamma$. Then, using \textbf{(1.7)}, the fact that $[A^r, \partial_j] = 0$ and some standard commutator algebra, we have:

$$[M_t, \partial_j] = [D_\gamma R_\gamma A^{-\tilde{z}} P_\lambda A^{-\tilde{z}} R_\gamma^{-1} (\det (D_\gamma) D_\gamma^{-\top})^{-1} A^r, \partial_j]$$

$$= [D_\gamma C_\gamma (A^{-\tilde{z}} P_\lambda A^{-\tilde{z}}) (\det (D_\gamma) D_\gamma^{-\top})^{-1}, \partial_j] A^r$$

$$= [D_\gamma, \partial_j] C_\gamma (A^{-\tilde{z}} P_\lambda A^{-\tilde{z}}) (\det (D_\gamma) D_\gamma^{-\top})^{-1} A^r$$

$$+ D_\gamma C_\gamma (A^{-\tilde{z}} P_\lambda A^{-\tilde{z}}) [(\det (D_\gamma) D_\gamma^{-\top})^{-1}, \partial_j] A^r$$

$$+ D_\gamma [C_\gamma (A^{-\tilde{z}} P_\lambda A^{-\tilde{z}}), \partial_j] (\det (D_\gamma) D_\gamma^{-\top})^{-1} A^r.$$ 

The first and second term in the sum mapping to the correct space follow from the fact that, for $f \in H^{s-1}$, we have $[f, \partial_j] : H^{k-2} \to H^{k-2}$ and $H^{k-2r} \to H^{k-2r}$, which is clear. All that remains is to show that $[C_\gamma (A^{-\tilde{z}} P_\lambda A^{-\tilde{z}}), \partial_j] : H^{k-2r} \to H^{k-2}$, which is equivalent to showing $[A^{-\tilde{z}} P_\lambda A^{-\tilde{z}}, \tilde{C}_\gamma(\partial_j)] = [A^{-\tilde{z}} P_\lambda A^{-\tilde{z}}, \partial_j \gamma \circ \gamma^{-1} \partial_j] : H^{k-2r} \to H^{k-2}$. Let $f_j = \partial_j \gamma \circ \gamma^{-1} \in H^{s-3}$. Now we have:

$$[A^{-\tilde{z}} P_\lambda A^{-\tilde{z}}, f_j \partial_j] = A^{-\tilde{z}} [f_j, A^\tilde{z}] \partial_j A^{-\tilde{z}} P_\lambda A^{-\tilde{z}} + A^{-\tilde{z}} [P_\lambda, f_j \partial_j] A^{-\tilde{z}} + A^{-\tilde{z}} P_\lambda A^{-\tilde{z}} [f_j, A^\tilde{z}] \partial_j A^{-\tilde{z}}$$

and the result follows by observing that:

- $[P_\lambda, f_j \partial_j]$ is a second order differential operator with coefficients in $H^{s-3}$
- $[f_j, A^\tilde{z}] : H^{k-3} \to H^{k-r-2}$
- $[f_j, A^\tilde{z}] : H^{k-r-1} \to H^{k-2r}$

where the latter two follow by commutator estimates analogous to \textbf{(4.8)}. Hence $\partial_j f \in H^k$, which implies that $f \in H^{k+1}$, and by the induction argument $M : H^{s+1} \to H^{s-1}$ is a linear isomorphism.

Finally, we have:

$$\Delta \eta = D\eta M(u_0) + \tilde{G},$$

which implies

$$u_0 = M^{-1}(D\eta^{-1}(\Delta \eta - \tilde{G})) \in H^{s+1}$$

and the proof of Theorem \textbf{4.3} is complete. \hfill \Box

4.2. Volume-Preserving Diffeomorphisms of the Flat 2-Torus $T^2$. The main result presented in this section is as follows:

**Theorem 1.1** Let $s > 6$ and consider the group of volume-preserving diffeomorphisms of $T^2$ equipped with a right-invariant $L^2$ metric. Given $u_0 \in T_c \mathcal{G}_\mu^s$ let $\gamma(t)$ denote the corresponding $L^2$ geodesic. If at time $t = 1$, $\gamma$ passes through a point $\eta \in \mathcal{G}_\mu^{s+1}$, then we have $u_0 \in T_c \mathcal{G}_\mu^{s+1}$ and consequently $\gamma(t)$ evolves entirely in $\mathcal{G}_\mu^{s+1}$. 
For the proof we make use of the conservation of vorticity in 2D, which takes the form:

\[(4.10)\quad \text{rot}(u) \circ \gamma = \text{rot}(u_0),\]

where \(u\) is the Eulerian velocity of the flow \(\gamma\), cf. (3.3). Applying the symplectic gradient \(\nabla^\perp := (-\partial_2, \partial_1)\) to both sides of (4.10) we obtain the following conservation law.

\[(4.11)\quad \Delta u = \text{Ad}_\gamma(\Delta u_0)\]

Now, recall by the Hodge Decomposition Theorem (2.1) that we may decompose the tangent space \(T_c\mathcal{D}_\mu^s\) into an \(L^2\)-orthogonal sum:

\[T_c\mathcal{D}_\mu^s = T_c\mathcal{D}_\mu^s,\text{ex} \oplus \mathcal{H}\]

where \(T_c\mathcal{D}_\mu^s,\text{ex} = g^*(d\delta H^{s+2}(T^*\mathbb{T}^2))\) is known as the space of exact volume-preserving vector fields and again \(\mathcal{H}\) denotes the space of harmonic vector fields. Hence, we can rewrite \(u_0 = \nabla^\perp f_0 + h_0\), where \(f_0 \in H_0^{s+1}(\mathbb{T}^2, \mathbb{R}) := \{g \in H^{s+1}(\mathbb{T}^2, \mathbb{R}) \mid \tilde{g}(0) = 0\}\) and \(h_0 \in \mathcal{H}\). So we may reformulate (4.11) as:

**Lemma 4.5.** For \(u_0 = \nabla^\perp f_0 + h, \gamma\) and \(u\) as above, we have:

\[(4.12)\quad u(t) = \nabla^\perp (\Delta^{-1}R^{-1}_{\gamma(t)} \Delta f_0) + h(t)\]

where \(h(t)\) evolves in \(\mathcal{H}\) and \(\Delta^{-1} : H_0^{s+1}(\mathbb{T}^2, \mathbb{R}) \to H_0^{s+1}(\mathbb{T}^2, \mathbb{R}),\) defined in frequency space by:

\[\Delta^{-1} f(\xi) := \begin{cases} 0 & \text{if } \xi = (0,0) \\ \hat{f}(\xi) & \text{if } \xi \in \mathbb{Z}^2 \setminus (0,0) \end{cases}\]

is a linear isomorphism.

**Proof.** Using (4.11) we have:

\[
\Delta u = \text{Ad}_\gamma(\Delta u_0) \\
= \text{Ad}_\gamma(\Delta(\nabla^\perp f_0 + h)) \\
= \text{Ad}_\gamma(\nabla^\perp \Delta f_0) \\
= \nabla^\perp (R^{-1}_{\gamma(t)} \Delta f_0)
\]

which, as \(\Delta\) acts component-wise on vector fields, immediately yields (4.12).

Formula (4.12) will be more convenient for our purposes than (4.10).

**Proof of Theorem 1.1.** The proof consists of three stages. We begin by combining (4.15) and (4.12), along with some commutator estimates to acquire an expression of the form \(\Delta \eta = D\eta M \nabla^\perp \Delta f_0 + G\), where \(G\) is of class \(H^{s-1}\). We then establish that, for \(v \in H_0^{s-2}\), \(Mv\) being of class \(H^{s-1}\) implies that \(v\) is of class \(H_0^{s-1}\). We finish by concluding that, as \(M \nabla^\perp \Delta f_0 = D\eta^{-1}(\Delta \eta - G)\) is of class \(H^{s-1}\), we have that \(f_0 \in H^{s+2}\), and hence \(u_0 \in H^{s+1}\). Then, by the results for the initial value problem, we have that \(\gamma(t)\) evolves entirely in \(H^{s+1}\) and the proof will be complete. Combining (4.15) and (4.12):

\[
\Delta \eta = D\eta \int_0^1 D\gamma(t)^{-1} R_{\gamma(t)} P_\lambda(t) \nabla^\perp (\Delta^{-1} R_{\gamma(t)}^{-1} \Delta f_0) \, dt + D\eta \int_0^1 D\gamma(t)^{-1} R_{\gamma(t)} P_\lambda(t) h(t) \, dt + G
\]

where again \(G\) is of class \(H^{s-1}\). Some straightforward calculus now yields:

\[
\Delta \eta = D\eta \int_0^1 D\gamma(t)^{-1} R_{\gamma(t)} P_\lambda(t) \Delta^{-1} R_{\gamma(t)}^{-1} (D\gamma(t) \nabla^\perp \Delta f_0) \, dt + D\eta \int_0^1 D\gamma(t)^{-1} R_{\gamma(t)} P_\lambda(t) h(t) \, dt + G
\]

\[= D\eta \int_0^1 R_{\gamma(t)} \Gamma^{-1}(t) P_\lambda(t) \Delta^{-1} \Gamma(t) R_{\gamma(t)}^{-1} \nabla^\perp \Delta f_0 \, dt + G\]
where $G$ has absorbed the term $D\eta \int_0^1 D\gamma(t)^{-1} R_{\gamma(t)} P_\lambda(t) h(t) \, dt$ and is still of class $H^{s-1}$ and we denote $\Gamma := D\gamma \circ \gamma^{-1}$ for notational simplicity. Next, we introduce commutator terms in order to achieve a more advantageous symmetry. We recall the Hodge projections:

$$
\pi_0 : T_e \mathcal{D}^\sigma \to T_e \mathcal{D}^{\sigma, e} \oplus \nabla H^{s+1} \quad \text{and} \quad \pi_H : T_e \mathcal{D}^\sigma \to \mathcal{H},
$$

and rewrite:

$$
\Delta \eta = D\eta \int_0^1 R_{\gamma(t)} \Delta^{-1/2} \pi_0 P_\lambda(t) \Delta^{-1/2} R_{\gamma(t)}^{-1} (\nabla^+ \Delta f_0) \, dt + D\eta \int_0^1 R_{\gamma(t)} \Omega(t) R_{\gamma(t)}^{-1} (\nabla^+ \Delta f_0) \, dt + G
$$

where

$$
\Omega(t) = \Gamma^{-1}(t) P_\lambda(t) \Delta^{-1}(t) - \Delta^{-1/2} \pi_0 P_\lambda(t) \Delta^{-1/2}.
$$

We claim the integral involving $\Omega(t)$ is of class $H^{s-1}$ and hence can be absorbed into $G$. Observe that $\Gamma(t)$ and its inverse have entries in $H^{s-1}$. We suppress $t$ and continue

$$
\Omega = \Gamma^{-1}[P_\lambda \Delta^{-1}, \Gamma] + \Delta^{-1/2} \pi_0 \Delta^{-1} P_\lambda \Delta^{-1} - \Delta^{-1/2} \pi_0 P_\lambda \Delta^{-1/2} + \pi_H P_\lambda \Delta^{-1}
$$

As $\pi_0 = \Delta^{-1/2} \pi_0 \Delta^{1/2}$, we have:

$$
\Omega = \Gamma^{-1}[P_\lambda \Delta^{-1}, \Gamma] + \Delta^{-1/2} \pi_0 \Delta^{-1} P_\lambda \Delta^{-1} - \Delta^{-1/2} \pi_0 P_\lambda \Delta^{-1/2} + \pi_H P_\lambda \Delta^{-1}
$$

Examining the terms separately we have:

$$
[P_\lambda \Delta^{-1}, \Gamma] = \lambda[\Delta^{-1}, \Gamma] - [p^{ij} \partial_i \partial_j \Delta^{-1}, \Gamma]
$$

$$
= \lambda[\Delta^{-1}, \Gamma] - [p^{ij} \Delta^{-1} \partial_i \partial_j, \Gamma]
$$

$$
= \lambda[\Delta^{-1}, \Gamma] - p^{ij} \Delta^{-1} \partial_i \partial_j \Gamma + \Gamma p^{ij} \Delta^{-1} \partial_i \partial_j
$$

Commuting $p^{ij}$ with $\Gamma$ and introducing commutator terms, we have:

$$
[P_\lambda \Delta^{-1}, \Gamma] = \lambda[\Delta^{-1}, \Gamma] - p^{ij} \Delta^{-1} \partial_i \partial_j \Gamma + p^{ij} \Gamma \Delta^{-1} \partial_i \partial_j
$$

$$
= \lambda[\Delta^{-1}, \Gamma] - p^{ij} \Delta^{-1} \partial_i \partial_j \Gamma + p^{ij} \Delta^{-1} \partial_i \partial_j - p^{ij} \Delta^{-1} \partial_i \partial_j + p^{ij} \Gamma \Delta^{-1} \partial_i \partial_j
$$

$$
= \lambda[\Delta^{-1}, \Gamma] - p^{ij} \Delta^{-1} \partial_i \partial_j \Gamma - p^{ij} \Delta^{-1} \partial_i \partial_j \Gamma + p^{ij} \Delta^{-1} \partial_i \partial_j
$$

We can see by direct calculation of $[\Gamma, \Delta]$ and $[\partial_i \partial_j, \Gamma]$ that these terms map $H^{s-2}$ to $H^{s-3}$. As for the term $\Delta^{-1/2} \pi_0 [\Delta^{1/2}, P_\lambda] \Delta^{-1}$ we have

$$
[\Delta^{1/2}, P_\lambda] = [\Delta^{1/2}, p^{ij}] \partial_i \partial_j
$$

and, as $p^{ij}$ are of class $H^{s-1}$, the fact that this maps $H^s$ to $H^{s-2}$ is a consequence of the following commutator estimate, cf. [31]:

$$
(4.13) \quad \left\| \Delta^{1/2}(fg) - f \Delta^{1/2} g \right\|_{H^{s-2}} \lesssim \| \nabla f \|_\infty \| g \|_{H^{s-1}} + \| f \|_{H^{s-1}} \| g \|_\infty
$$

Lastly, observe that the term $\pi_H P_\lambda \Delta^{-1}$ maps $H^{s-2} \to C^\infty$. Hence we have

$$
(4.14) \quad \Delta \eta = D\eta \int_0^1 R_{\gamma(t)} \Delta^{-1/2} \pi_0 P_\lambda(t) \Delta^{-1/2} R_{\gamma(t)}^{-1} (\nabla^+ \Delta f_0) \, dt + \bar{G}
$$

where now $\bar{G} = G + D\eta \int_0^1 R_{\gamma(t)} \Omega(t) R_{\gamma(t)}^{-1} (\nabla^+ \Delta f_0) \, dt$ evolves in $H^{s+1}$. 

...
Our goal is now to show that, for $0 \leq \lambda \leq 4.2$ and the $L^2$-orthogonality in the Hodge decomposition (2.1) that:

$$\|Mv\|_{L^2} \geq \|\{M, \partial_j\}v\|_{L^2}$$

From which it follows that if $Mv \in L^2$, then $v \in L^2$. Now assume the lemma holds for some $k$ with $0 \leq k \leq s - 2$ and $Mv \in H^{k+1}$. By the inductive hypothesis, $v \in H^k_0$. Furthermore, for $j = 1, 2$ we have:

$$M\partial_j v = \partial_j Mv + [M, \partial_j]v$$

By assumption, $\partial_j Mv \in H^k$. As for the latter term, we show that for $0 \leq k \leq s - 2$, we have $[M, \partial_j]H^k_0 \rightarrow H^k_0$, and hence $[M, \partial_j] : H^k_0 \rightarrow H^k_0$. Expanding yields:

$$[M, \partial_j] = C_j \left( [\Delta^{-1/2} \pi_0 P_\lambda \Delta^{-1/2}, \bar{C}_j(\partial_j)] \right)$$

where again $C_j(B) := R_\gamma BR_\gamma^{-1}$ and $\bar{C}_j(B) := R_\gamma^{-1} BR_\gamma$. Examining the term inside the conjugation above, we have:

$$\left[ \Delta^{-1/2} \pi_0 P_\lambda \Delta^{-1/2}, \bar{C}_j(\partial_j) \right] = \Delta^{-1/2} \pi_0 P_\lambda \Delta^{-1/2} \bar{C}_j(\partial_j) - \bar{C}_j(\partial_j) \Delta^{-1/2} \pi_0 P_\lambda \Delta^{-1/2}$$

where $C_j(\partial_j) = R_\gamma BR_\gamma^{-1}$ and $\bar{C}_j(\partial_j) := R_\gamma^{-1} BR_\gamma$. Examining the term inside the conjugation above, we have:

$$\left[ \Delta^{-1/2} \pi_0 P_\lambda \Delta^{-1/2}, \bar{C}_j(\partial_j) \right] = \Delta^{-1/2} \pi_0 P_\lambda \Delta^{-1/2} \bar{C}_j(\partial_j) - \bar{C}_j(\partial_j) \Delta^{-1/2} \pi_0 P_\lambda \Delta^{-1/2}$$

Notice that the term

$$\left[ \bar{C}_j(\partial_j), \Delta^{1/2} \right] = \left[ \partial_j \gamma^{\dagger} \partial_i, \Delta^{1/2} \right] = \left[ \partial_j \gamma^{\dagger}, \Delta^{1/2} \right] \partial_i$$

maps $H^{k+1}_0 \rightarrow H^k_0$ for any $0 \leq k \leq s - 2$. Furthermore we have:

$$\left[ \Delta^{-1/2} \pi_0 P_\lambda, \bar{C}_j(\partial_j) \right] = \Delta^{-1/2} \pi_0 P_\lambda \bar{C}_j(\partial_j) - \bar{C}_j(\partial_j) \Delta^{-1/2} \pi_0 P_\lambda$$

The commutator term $[P_\lambda, \bar{C}_j(\partial_j)]$ can be explicitly calculated to be a second order operator with $H^{s-3}$ coefficients. As for $[\Delta^{-1/2} \pi_0, \bar{C}_j(\partial_j)]P_\lambda$, we compute:

$$\left[ \Delta^{-1/2} \pi_0, \bar{C}_j(\partial_j) \right] = \Delta^{-1/2} \pi_0 \bar{C}_j(\partial_j) - \bar{C}_j(\partial_j) \Delta^{-1/2} \pi_0$$

$$= \Delta^{-1/2} \pi_0 \bar{C}_j(\partial_j)(\pi_0 + \pi_H) - \bar{C}_j(\partial_j) \Delta^{-1/2} \pi_0$$
To proceed we introduce the identity on $H^k_0$ as $\Delta^{1/2} \Delta^{-1/2}$ into the term involving the projection $\pi_0$. So we have:

$$[\Delta^{-1/2} \pi_0, \tilde{C}_\gamma(\partial_j)] = \Delta^{-1/2} \pi_0 \tilde{C}_\gamma(\partial_j) \Delta^{1/2} \Delta^{-1/2} \pi_0 \pi_0 - \tilde{C}_\gamma(\partial_j) \Delta^{-1/2} \pi_0 + \Delta^{-1/2} \pi_0 \tilde{C}_\gamma(\partial_j) \pi_0$$

$$= \Delta^{-1/2} [\tilde{C}_\gamma(\partial_j), \Delta^{1/2}] \Delta^{-1/2} \pi_0 + \Delta^{-1/2} \pi_0 \tilde{C}_\gamma(\partial_j) \pi_0$$

As before $[\tilde{C}_\gamma(\partial_j), \Delta^{1/2}] : H^{k+1}_0 \to H^k_0$, and we have $[M, \partial_j] : H^k_0 \to H^k_0$ for $0 \leq k \leq s - 2$. Hence we have shown that $M\partial_j v = \partial_j M v + [M, \partial_j] v \in H^k$. So by the inductive hypothesis we have $\partial_j v \in H^k_0$ which gives us $v \in H^{k+1}_0$. Therefore, we have that, for $0 \leq k \leq s - 1$, if $Mv \in H^k$, then $v \in H^k_0$.

Finally, since (4.10) says

$$M(\nabla^\perp \Delta f_0) \in H^{s-1}$$

we have

$$\nabla^\perp \Delta f_0 \in H^{s-1}$$

which implies that

$$\nabla^\perp f_0 \in H^{s+1}$$

which finally gives us that $u_0 \in H^{s+1}$ and the proof is complete.

**Corollary 4.6.** For $s > 0$, if $\gamma(t)$ is an $L^2$ geodesic on $D_\mu^s(T^2)$ with, for some $T > 0$, $\gamma(0) = \gamma(T) = e$, then $\gamma(t)$ evolves entirely in $D_\mu^s(T^2)$. In other words, any isochronal Euler flow on $T^2$ must either be smooth or of some ‘low-regularity’ in a $H^s$ sense.

4.2.1. **Higher Order Euler-Arnold Equations.** In this section we equip the group of volumorphisms with a right-invariant $H^r$ metric (3.1) induced by the interia operator $A^r$, for integer $r \geq 1$, then (3.2) becomes:

$$\begin{align*}
\partial_t A^r u + \nabla_u A^r u + (\nabla u)^T A^r u &= -\nabla p \\
(\text{div}(u)) &= 0
\end{align*}$$

The main result in this section is as follows:

**Theorem 4.7.** Let $r \geq 1$ be an integer, $s > 2r + 6$ and consider the group of volume-preserving diffeomorphisms of $T^2$ equipped with a right-invariant $H^r$ metric (3.1). Given $u_0 \in T_e D_\mu^s$ let $\gamma(t)$ denote the corresponding $H^r$ geodesic. If at time $t = 1$, $\gamma$ passes through a point $\eta \in D_\mu^{s+1}$, then we have $u_0 \in T_e D_\mu^{s+1}$ and consequently $\gamma(t)$ evolves entirely in $D_\mu^{s+1}$.

Taking the curl of both sides of (4.17), we acquire a “conservation of vorticity”-type equation in this context.

$$\text{rot}(A^r u) \circ \gamma = \text{rot}(A^r u_0)$$

This allows us to derive an analogous conservation law to (4.12).

**Lemma 4.8.** For $u_0 = \nabla^\perp f_0 + h$, $\gamma$ and $u$ as above, we have:

$$u(t) = \nabla^\perp (\Delta^{-1} A^{-r} R^{-1}_{\gamma(t)} A^r \Delta f_0) + h(t)$$

where $h(t)$ evolves in $\mathcal{H}$.

**Proof.** Again, as in the $L^2$ setting, applying the symplectic gradient $\nabla^\perp := (-\partial_2, \partial_1)$ to both sides of (4.18) gives us:

$$A^r \Delta u = \text{Ad}_\gamma A^r u_0$$

Which immediately yields (4.19).

**Proof of Theorem 4.7** Using Lemma 4.1 and (4.19), this follows from a completely analogous argument as in Theorem 4.1.

**Remark 4.9.** If we define $A = 1 - \alpha^2 \Delta$, the above theorem covers the case of the Euler-$\alpha$ equations studied in [16] and [30].
4.3. Swirl-Free Axisymmetric Diffeomorphisms of the Flat 3-Torus $\mathbb{T}^3$. In this section we consider the group of axisymmetric diffeomorphisms of the flat periodic box $\mathbb{T}^3$, equipped with any of the Killing fields $K = \frac{\partial}{\partial s}$ and a certain subclass of swirl-free initial data.

Given an axisymmetric vector field $u$, we define its swirl to be the function $g(v, K)$ on $\mathbb{T}^3$. A vector field is called swirl-free if this function identically vanishes. In our argument for this section, the swirl-free condition will deliver a conservation law which will play the same role that conservation of vorticity did in the previous section. It is shown in [22] that the swirl of an axisymmetric velocity field is transported by its flow. More precisely, if $u_0 \in T_e \mathcal{A}_\mu^s$ and $\gamma(t)$ is the corresponding geodesic in $\mathcal{A}_\mu^s$ then $g(u, K) \circ \gamma(t) = g(u_0, K)$ as long as it is defined. We denote the space of swirl-free axisymmetric vector fields by $T_e \mathcal{A}_\mu^s,0$. The proof of the following lemma can be found in [22].

**Lemma 4.10.** Let $K$ be any of the Killing fields $\partial_1$. Then:

1. If $v \in T_e \mathcal{A}_{\mu+1}^s$, then curl $v = \phi K$, where $\phi$ is a function of class $H^s$.
2. If $u_0 \in T_e \mathcal{A}_{\mu,0}^s$ and $u(t)$ is the corresponding solution of the Euler equations (1.1), then, by the above, we can write curl $u_0 = \phi_0 K$ and curl $u(t, x) = \phi(t, x)K(x)$. The function $\phi$ is transported along the flow lines: $\phi(t, \gamma(t)) = \phi_0(x)$.
3. If $u_0 \in T_e \mathcal{A}_{\mu,0}^s$ then the corresponding solution $u(t)$ of the Euler equations (1.1) can be extended globally in time.
4. If $u_0$ and $u(t)$ are as above, and $\gamma(t) \in \mathcal{A}_{\mu,0}^s$ is the flow of $u(t)$, then $K$ is preserved by the adjoint: $Ad_{\gamma(t)}K = K$.
5. For $v \in T_p \mathbb{T}^3$ with $g(v, K(p)) = 0$, we have $g(D\gamma(t)v, K(\gamma(t, p))) = 0$.

We now proceed to the main result for this section:

**Theorem 1.2.** Let $s > \frac{3}{4}$ and consider the group of axisymmetric diffeomorphisms of $\mathbb{T}^3$ with respect to any of the Killing fields $K = \partial_3$, equipped with a right invariant $L^2$ metric. Given $u_0 \in T_e \mathcal{A}_{\mu,0}^s$ let $\gamma(t)$ denote the corresponding $L^2$ geodesic. If at time $t = 1$, $\gamma$ passes through a point $\eta \in \mathcal{A}_{\mu+1}^s$, then we have $u_0 \in T_e \mathcal{A}_{\mu,0}^s$ and consequently $\gamma(t)$ evolves entirely in $\mathcal{A}_{\mu+1}^s$.

Without loss of generality, we will assume, for the duration of the section, that $K = \partial_3$. So, if we assume a vector field is axisymmetric and swirl-free, this is now equivalent to saying $v = v_1 \partial_1 + v_2 \partial_2$, where $\partial_3 v_1 = \partial_3 v_2 = 0$. Proceeding as before we establish a conservation law:

**Lemma 4.11.** Let $u(t)$ be the Eulerian velocity of the flow $\gamma(t)$. Then we have:

$$\Delta u(t) = Ad_{\gamma(t)} \Delta u_0.$$  

**Proof.** As $u(t)$ evolves in $T_e \mathcal{A}_{\mu,0}^s$, we may write $u(t) = u_1(t) \partial_1 + u_2(t) \partial_2$, where $\partial_3 u_1(t) = \partial_3 u_2(t) = 0$. Hence by taking the curl we have:

$$\text{curl } u(t) = ( - \partial_2 u_1(t) + \partial_1 u_2(t)) \partial_3.$$

Comparing this with Lemma 4.10, we see $\phi = - \partial_2 u_1(t) + \partial_1 u_2(t)$ and thus, as $\phi$ is preserved along the flow lines ($\phi \circ \gamma = \phi_0$), we have as in 2D:

$$\partial_2 u_1(t) + \partial_1 u_2(t) \circ \gamma = - \partial_2 u_0,1(t) + \partial_1 u_0,2(t).$$

Since

$$\Delta u_0 = \Delta u_{0,1} \partial_1 + \Delta u_{0,2} \partial_2,$$

$$\Delta u(t) = \Delta u_1(t) \partial_1 + \Delta u_2(t) \partial_2,$$

and

$$D\gamma(t) = \begin{bmatrix} \partial_1 \gamma_1 & \partial_2 \gamma_1 & 0 \\ \partial_1 \gamma_2 & \partial_2 \gamma_2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

the lemma follows in an identical fashion to the computation for (4.11). \(\square\)

We proceed with the proof of Theorem 1.2.
Proof of Theorem 4.12. Using (4.1) and (4.20), this follows from a completely analogous argument as in Theorem 1.1.

Remark 4.12. Further work in this direction will include the study of curved manifolds, possibly with boundary. As mentioned previously, many of our constructions extend to the setting of curved spaces by collecting any lower order terms arising in the various calculations due to derivatives of the components of the metric and its Christoffel symbols on \( M \) into a single term which is negligible for our purposes. As an initial case, the author has established the result for swirl-free initial data for certain Killing fields on the round 3-sphere \( \mathbb{S}^3 \).

The strategy for manifolds with boundary will have to be modified however. For example, the estimates involved need to be reproved and the presence of boundary terms when integrating by parts would have to be accounted for. Preliminary calculations in the case of axisymmetric flows on the vertically periodic cylinder \( D_2 \times S^1 \) suggest that this can be done; for instance the “inverted” version of (3.3) obtained in this setting matches that of the version involved in Theorem 1.2 up to negligible lower order terms.

4.4. Symplectomorphisms of the Flat Torus \( T^{2k} \). In this section we consider the group of symplectomorphisms of the torus \( T^{2k} \), equipped with the standard symplectic form \( \omega \). The main result presented in this section is

**Theorem 4.13.** Let \( s > k + 5 \) and consider the group of symplectomorphisms of \( T^{2k} \) equipped with a right-invariant \( L^2 \) metric. Given \( u_0 \in T_e D_s \) let \( \gamma(t) \) denote the corresponding \( L^2 \) geodesic. If at time \( t = 1 \), \( \gamma \) passes through a point \( \eta \in D^{s+1}_\omega \) then we have \( u_0 \in T_e D^{s+1}_\omega \) and consequently \( \gamma(t) \) evolves entirely in \( D^{s+1}_\omega \).

As before we begin with a consequence of (3.3) and establish a new conservation law from it. The following can be found in [11].

\[
(4.22) \quad \gamma^* \delta \omega^b u = \delta \omega^b u_0
\]

Using this we obtain:

**Proposition 4.14.**

\[
(4.23) \quad \Delta u = \text{Ad}_\eta \Delta u_0
\]

**Proof.** Applying \( g^* d \) to both sides of (4.22):

\[
g^* \gamma^* (\Delta \omega^b u) = g^* \Delta \omega^b u_0 \]

\[
\Rightarrow D\gamma^* \gamma^* \omega^b (\Delta u \circ \gamma) = g^* \omega^b \Delta u_0
\]

\[
\Rightarrow D\gamma^* \gamma^* J(\Delta u \circ \gamma) = J \Delta u_0
\]

\[
\Rightarrow \Delta u = R_0 \gamma^{-1} J^{-1} (D\eta^*)^{-1} J \Delta u_0
\]

\[
= R_0^{-1} D\eta \Delta u_0
\]

\[
= \text{Ad}_\eta \Delta u_0
\]

Where, in the penultimate line, we have used the fact that \( D\eta \) is a symplectic matrix.

**Proof of Theorem 4.13.** This follows in a completely analogous fashion to the proof of Theorem 1.1 using the above conservation law (4.23).

5. The Frechét Setting

As mentioned in the introduction, to the best of the author’s knowledge, the earliest investigations into the kind of regularity property considered in this paper are due to Constantin & Kolev [10], [9] and Kappeler, Loubet & Topalov [18], [19]. In each instance, the authors used this property to construct exponential maps...
Lemma 5.1. Let $U$ map on a neighbourhood of $0$ differentiable $u$ theorem at each point

Theorem 5.2. Hence, given by $u$ at definitions. for Frechét spaces admitting Hilbert approximations cf. [14] & [19, Theorem A.5]. First we recall some basic settings we have considered. As in the above, we make use of a Nash-Moser-type inverse function theorem for Frechét spaces admitting Hilbert approximations cf. [14] & [19, Theorem A.5].

Then $\exp := \exp$ is a bounded linear bijection in the $C$ which is a $C^{1}_F$-diffeomorphism. We obtain analogous results here for the Frechét category. We obtain analogous results here for the $C^{1}_F$-diffeomorphisms’ in the Frechét category. We obtain analogous results here for the settings we have considered in Section 4, we can construct an exponential map on a neighbourhood of $0 \in T_e\mathcal{G}$ which is a $C^{1}_F$ diffeomorphism onto its image.

---

3It is important to note that this notion of continuously differentiable is weaker than the standard definition, cf. [14] Page 73.
Remark 5.3. We will prove Theorem 5.2 for the case of $\mathcal{D}_\mu(T^2)$ equipped with the $L^2$ metric. We note that this case is not a new result, cf. [27], however we have used a different method of proof. The analogous results in the other settings considered in Section 6 follow from an analogous argument, making use of the literature pertaining to Fredholmness of exponential maps on groups of diffeomorphisms cf. [6], [7], [4], [22] and [26]. We will use the notation defined in Lemma 5.1.

Proof. Let $s_0 > 6$. From [12], we have that $\mathcal{D}_\mu^s(T^2)$ equipped with the $L^2$ metric admits a well-defined exponential map which is a local $C^\infty$ diffeomorphism at the identity $\exp^s : U^{s_0} \to V^{s_0}$. We may shrink $U^{s_0}$ if necessary so that $V^{s_0}$ lies in the image of a chart map for $\mathcal{D}_\mu^{s_0}(T^2)$.

We know from [12] Theorem 12.1 that, for all $k \in \mathbb{Z}_{>0}$, $\exp^{s_0}(U^k) \subseteq V^k$. Furthermore, uniqueness and smooth dependence of Lagrangian solutions on initial data in each $H^k$ topology gives us that $\exp^s := \exp^{s_0}|_{U^k} : U^k \to V^k$ is a well-defined $C^\infty$ injection. Theorem 1.1 now guarantees that, for all $k \in \mathbb{Z}_{>0}$, $\exp^s : U^k \to V^k$ is in fact a $C^\infty$ bijection.

Next, from [13], we know that $\exp^{s_0}$ is a non-linear Fredholm map of index zero. Hence, we can further restrict $U^{s_0}$ if necessary to guarantee that, for all $u \in U$, $d_u \exp^{s_0} : T_{\eta} \mathcal{D}_\mu^{s_0}(T^2) \to T_{\exp^{s_0}(u)} \mathcal{D}_\mu^{s_0}(T^2)$ is a linear isomorphism. Furthermore, defining $\eta := \exp^{s_0}(u) \in \mathcal{D}_\mu(T^2)$ we in fact have:

$$d_u \exp^{s_0} = D\eta(\Omega_u - \Gamma_u)$$

where, for any $k \in \mathbb{Z}_{>0}$, $D\eta : T_{\eta} \mathcal{D}_\mu^{s_0}(T^2) \to T_{\eta} \mathcal{D}_\mu^{s_0}(T^2)$ and $\Omega_u : T_{\eta} \mathcal{D}_\mu^{s_0}(T^2) \to T_{\eta} \mathcal{D}_\mu^{s_0}(T^2)$ are linear isomorphisms and $\Gamma_u : T_{\eta} \mathcal{D}_\mu^{s_0}(T^2) \to T_{\eta} \mathcal{D}_\mu^{s_0}(T^2)$ is a compact operator. So, if for $w \in T_{\eta} \mathcal{D}_\mu^{s_0}(T^2)$,

$$d_u \exp^{s_0}(w) \in T_{\eta} \mathcal{D}_\mu^{s_0+k+1}(T^2),$$

we have that $w = \Omega_u^{-1}(D\eta^{-1}(d_u \exp^{s_0}(w)) + \Gamma_u(w)) \in T_{\eta} \mathcal{D}_\mu^{s_0+k+1}(T^2)$.

Hence, $d_u \exp^{s_0}(T_{\eta} \mathcal{D}_\mu^{s_0}(T^2) \setminus T_{\eta} \mathcal{D}_\mu^{s_0+k+1}(T^2)) \subseteq T_{\eta} \mathcal{D}_\mu^{s_0+k+1}(T^2) \setminus T_{\eta} \mathcal{D}_\mu^{s_0+k+1}(T^2)$ and we may apply Lemma 5.1.

Remark 5.4. It is important to note that Theorem 5.2 does not follow immediately from the work of Ebin and Marsden in [12]. While they define an exponential map for each Sobolev index $s > \frac{n}{2} + 1$, $\exp^s : \tilde{U}^s \to \tilde{V}^s$ and, indeed, their Theorem 12.1 ensures that each $\exp^s$ will map smooth initial data to a geodesic in $\mathcal{D}_\mu(T^2)$, they do so by applying the inverse function theorem in separately for each index. Hence, there is no apriori relationship between $\tilde{U}^s$ which guarantees that their intersection is not a single point, cf. [27] page 87.

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