INVARIENTS OF COLORED LINKS AND A PROPERTY OF THE CLEBSCH-GORDAN COEFFICIENTS OF $U_q(g)$

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Abstract.

We show that multivariable colored link invariants are derived from the roots of unity representations of $U_q(g)$. We propose a property of the Clebsch-Gordan coefficients of $U_q(g)$, which is important for defining the invariants of colored links. For $U_q(sl_2)$ we explicitly prove the property, and then construct invariants of colored links and colored ribbon graphs, which generalize the multivariable Alexander polynomial.

1 Introduction

Recently a new family of invariants of colored oriented links and colored oriented ribbon graphs is introduced, which gives generalizations of the multivariable Alexander polynomial.\(^1\)\(^2\)\(^6\) The new invariants are related to the roots of unity representations of $U_q(sl_2)$ where $(X^\pm)^N = 0$ and $(K)^N = q^{2np}$ ($p \in \mathbb{C}$) and $q = \epsilon.5.6.7$ Here $\epsilon = \exp(\pi is/N)$, and the integers $N$ and $s$ are coprime. We call the representations nilpotent representations.

The invariants of colored links have a property that they vanish for disconnected links.\(^2\)\(^6\) Due to this property a proper regularization method is necessary for definition of the invariants.

In this paper we show an important property of the Clebsch-Gordan coefficients (CGC) of the nilpotent representations of $U_q(sl_2)$, which leads to the definition of the colored link invariants. We consider the nilpotent reps of $U_q(g)$ such that $(X^\pm)^N = 0$, $(K)^N = q^{2np}$ ($p \in \mathbb{C}$) and $q = \epsilon$. We give a conjecture that the property of CGC also holds for the nilpotent reps of $U_q(g)$ and we can define invariants of colored links for $U_q(g)$ in the same way as $U_q(sl_2)$.

2 CGC of the nilpotent representations

We introduce some symbols for a positive integer $n$ and a complex parameter $p$.

\[
[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q^! = \prod_{k=1}^{n}[k]_q, \quad [p; n]_q^! = \prod_{k=0}^{n-1}[p-k]_q. \quad (1)
\]
Proposition 3.1

The CGC of the nilpotent representations satisfy the irreducibility of the representations the sum $U_{\pi}$ for the nilpotent reps of $\pi$ the condition: $\max \{N_p \}$ where $N_p$ is 2-th primitive root of unity, while when $N$ is odd, $\epsilon$ may be $2N$-th or $N$-th primitive root of unity.

In Ref. 1 the explicit matrix representations of the colored braid group were introduced. It was shown that the representations are equivalent to the $R$ matrices of the nilpotent reps, which are derived from the universal $R$ matrix $R^{a_1,a_2} = \pi^{p_1} \otimes \pi^{p_2} (R)_{b_1,b_2}^{a_1,a_2}$. 5

We can show the fusion rule for the tensor product: $V^{(p_1)} \otimes V^{(p_2)} = \sum_{p_3} N_{p_1,p_2}^{p_3} V^{(p_3)}$, where $N_{p_1,p_2}^{p_3} = 1$ for $p_3 = p_1 + p_2 - n \ (0 \leq n \leq N - 1$ and $n \in \mathbb{Z})$; $N_{p_1,p_2}^{p_3} = 0$, otherwise. The CGC for $V^{(p_i)} \ (i = 1, 2, 3$) are given by 6

$C(p_1, p_2, p_3; z_1, z_2, z_3) = \delta(z_3, z_1 + z_2 - n)$

$\times \sqrt{[2p_1 + 2p_2 - 2n + 1]_e \left[ [n]_e [z_1]_e [z_2]_e [z_3]_e \right]}$

$\times \sum_{\nu} \frac{(-1)^{\nu} e^{-(\nu + p_1 + p_2 + p_3 + 1)} \times e^{(n-n^2)/2+(n-z_2)p_1+(n+z_1)p_2}}{[\nu]_e [n-\nu]_e [z_1-\nu]_e [z_2-n-\nu]_e}$

$\times \sqrt{\frac{[2p_1 - n; z_1-\nu]_e [2p_1 - z_1; n-\nu]_e [2p_2 - n; z_2 + \nu - n]_e [2p_2 - z_2; \nu]_e}{[2p_1 + 2p_2 - n + 1; z_1 + z_2 + 1]_e}}.$ \ (2)

Here $0 \leq z_i \leq N - 1$, for $i = 1, 2, 3$, and the sum over the integer $\nu$ in (2) is taken under the condition: $\max \{0, n - z_2\} \leq \nu \leq \min \{n, z_1\}$. The expression of the CGC was proved through the infinite dimensional representations. 6

3 A Property of CGC

We consider the following two sums;

$A_{p_1,p_2}^n = \sum_{z_1} [C(p_1, p_2, p_3; z_1, z_2, z_1 + z_2 - n)]^2 q^{2p(z_1)}$,

$B_{p_1,p_2}^n = \sum_{z_2} [C(p_1, p_2, p_3; z_1, z_2, z_1 + z_2 - n)]^2 q^{2p(z_2)}.$ \ (3)

For the nilpotent reps of $U_q(sl_2)$ we assume $p(z) = p - z + Np/2$ and $q = e$. Due to the irreducibility of the representations the sum $A_{p_1,p_2}^n \ (B_{p_1,p_2}^n)$ does not depend on $z_2 \ (z_1)$. We now introduce an important property of CGC.

Proposition 3.1 The CGC of the nilpotent representations satisfy

$A_{p_1,p_2}^n / B_{p_1,p_2}^n = g(p_1, p_2) = f(p_1)/f(p_2)$, independent of $n \ (p_3 = p_1 + p_2 - n). \ (4)$
Definition 3.4

Proposition 3.3

3.1 we have the following.

\[ \text{CGC. In the same way we can define the multivariable invariants of colored ribbon graphs.} \]

\[ \text{The second eq. in (5) (for } R_{p_1p_2}^n \text{) can be shown by exchanging } p_1 \text{ with } p_2, \text{ and by setting } \epsilon \rightarrow \epsilon^{-1}. \]

\[ \text{It is easy to see that the property (4) of CGC also holds for the finite dimensional (spin) representations of } U_q(sl_2) \text{ with } q \text{ generic, where we have } g(j_1, j_2) = [2j_1]/[2j_2]. \]

\[ \text{We can define invariants of colored links using the proposition 3.1. Let } \hat{T}, \text{ we denote by } \hat{T} \text{ the link obtained by closing the open strings of } T. \text{ We note the following proposition.} \]

Proposition 3.2 Let } T_1 \text{ and } T_2 \text{ be two } (1, 1)-\text{tangles. If } \hat{T}_1 \text{ is isotopic to } \hat{T}_2 \text{ as a link in } S^3 \text{ by an isotopy which carries the closing component of } \hat{T}_1 \text{ to that of } \hat{T}_2. \text{ Then } T_1 \text{ is isotopic to } T_2 \text{ as a } (1, 1)-\text{tangle.} \]

Let us introduce the functor } \phi(\cdot) \text{ for the tangle diagrams.} \[ \text{We denote by } \phi(T, \alpha)_a^b \text{ the value } \phi \text{ for the tangle with variables } a \text{ and } b \text{ on the closing component (or edge). It is easy to show that } \phi(T, \alpha)_a^b = \lambda \delta_{ab}. \]

We put } L = \hat{T} \text{ and } s \text{ is the color of the closing component (or edge) of } \hat{T}. \text{ For a colored link } (L, \alpha) \text{ and a color } s \text{ of closing component (or edge), we define } \Phi \text{ by } \Phi(L, s, \alpha) = \lambda \text{ where } L, T, s \text{ are as above and } \phi(T, \alpha)_a^b = \lambda \delta_{ab}. \text{ We can show that } \Phi \text{ is well-defined, i.e. } \Phi(L, s, \alpha) \text{ does not depend on a choice of } T. \]

Proposition 3.3 \[ \text{For a link } L \text{ and its color } \alpha = (p_1, \cdots, p_n), \text{ we have} \]

\[ \Phi(L, s, \alpha)([p_s; N - 1]_\epsilon) = \Phi(L, s', \alpha)([p_{s'}; N - 1]_\epsilon). \]

\[ \text{Thus we have constructed the multivariable invariants from the property (4) of CGC. In the same way we can define the multivariable invariants of colored ribbon graphs.} \]
We now consider CGC of $U_q(g)$, where $g$ is a simple Lie algebra.

\[ A_{\Lambda_1 \Lambda_2}^{\Lambda_3} = \sum_{\vec{z}_1} [C(\Lambda_1, \Lambda_2, \Lambda_n; \vec{z}_1, \vec{z}_2, \vec{z}_1 + \vec{n} - \vec{n})] q^{2\rho(\vec{z}_1)}, \]
\[ B_{\Lambda_1 \Lambda_2}^{\Lambda_3} = \sum_{\vec{z}_2} [C(\Lambda_1, \Lambda_2, \Lambda_3; \vec{z}_1, \vec{z}_2, \vec{z}_1 + \vec{z}_2 - \vec{n})] q^{2\rho(\vec{z}_2)}. \] (9)

We assume that $\rho$ is given by "half the sum of positive roots". Finally, we propose the following conjecture.

**Conjecture 3.5**

1. The CGC of the nilpotent representations $\Lambda_i$ of $U_q(g)$ satisfy

\[ A_{\Lambda_1 \Lambda_2}^{\Lambda_3} / B_{\Lambda_1 \Lambda_2}^{\Lambda_3} = g(\Lambda_1, \Lambda_2), \text{ independent of } \Lambda_3. \] (10)

2. With a proper normalization of the quantum trace ($\rho(\vec{z}) \rightarrow \rho(\vec{z}) + \text{constant}$), we can set $g(\Lambda_1, \Lambda_2) = f(\Lambda_1)/f(\Lambda_2)$.

3. $f(\Lambda)$ is equivalent to the tangle invariant for the Hopf link.

If the conjecture is true, we can construct multivariable invariants of colored links and colored ribbon graphs from $U_q(g)$ in the same way as $U_q(sl_2)$.

It is easy to show that the property (10) holds for CGC of finite dimensional representations of $U_q(g)$ with $q$ generic.

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