Current correlations and quantum localization in 2D disordered systems with broken

time-reversal invariance

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We study long-range correlations of equilibrium current densities in a two-dimensional mesoscopic system with the time reversal invariance broken by a random or homogeneous magnetic field. Our result is universal, i.e. it does not depend on the type (random potential or random magnetic field) or correlation length of disorder. This contradicts recent $\sigma$-model calculations of Taras-Semchuk and Efetov (TS&E) for the current correlation function, as well as for the renormalization of the conductivity. We show explicitly that the new term in the $\sigma$-model derived by TS&E and claimed to lead to delocalization does not exist. The error in the derivation of TS&E is traced to an incorrect ultraviolet regularization procedure violating current conservation and gauge invariance.

The quantum coherence is well known to play a central role in the physics of mesoscopic systems. It induces, in particular, long-range spatial correlations of local densities of states and eigenfunction amplitudes. The long-range character of these correlations is due to the existence of massless modes, diffusons and Cooperons. This leads also to strong mesoscopic fluctuations of global quantities, such as the conductance or the inverse participation ratio.

In this paper, we present one more example of such long-range correlations induced by the phase coherence of a system. Specifically, we will calculate the correlation function of local equilibrium current densities of electrons subject to a (possibly smooth) random potential (RP) or random magnetic field (RMF). Non-zero local currents exist in a system if the time reversal invariance is broken by a magnetic field, either uniform or spatially fluctuating. We will demonstrate that the result is independent of the type of disorder and has a rather universal character.

This paper is to a large extent based on results obtained by two of us (A.D.M. and P.W.) in 1994 in collaboration with the late Arkadii Aronov. At that time our research was motivated by the paper $^1$, where it was claimed that the above current fluctuations lead to delocalization of electrons in random magnetic field, in contradiction to our finding in $^2$. Our results remained however unpublished. In fact, the unfortunate error in the calculation of Ref. $^1$ was quite obvious (these authors confused the correlation function of equilibrium currents with the current response function), and we did not see a necessity to devote an additional publication to it. Also, evaluation of the correlation function of local currents is quite straightforward and follows essentially the same lines as the calculation of a typical value of the global persistent current $^3$.

However, in the meantime, the result of Ref. $^1$ has been quoted in a number of publications as one of theoretical proposals concerning the problem of localization of two-dimensional (2D) fermions in a random magnetic field. Furthermore, the idea of $^1$ was recently revived, within a different line of reasoning, by Taras-Semchuk and Efetov (TS&E) $^4$. Despite the criticism of their work in the comment $^5$, TS&E have made public a rather extended paper $^6$ presenting in detail their calculations and conclusions. If true, their results, predicting a delocalizing one-loop contribution to the $\beta$-function governing the scaling of conductance for systems with broken time reversal invariance (unitary symmetry class), would change completely our understanding of localization properties of these systems. In particular, they would force one to reconsider the phase diagram of the quantum Hall effect. Most noteworthy, the findings of TS&E challenge the notion of universality for this class of systems: the new contribution to the $\beta$-function derived by them has a non-universal coefficient $\beta_{TS&E}$ (see below) depending on the type (RP or RMF) and the correlation length of the disorder. We thus believe that it is timely to present our results on current density correlations and our criticism of Refs. $^1$, $^4$, $^6$ without the length restrictions implied by the Comment format.

We consider the correlation function of local current densities $j_{\alpha}(r,E)$ at equilibrium in 2D (we set $\hbar = 1$ and omit the spin degree of freedom),

$$I_{\alpha\beta}(r - r', \omega) \equiv \langle j_{\alpha}(r, E + \omega) j_{\beta}(r', E) \rangle \quad (1)$$

$$j(r, E) = \frac{e}{2m} [i \nabla_r - i \nabla_{\tilde{r}} + \frac{2e}{c} \mathbf{A}(r)] \times \frac{1}{2\pi i} \left[ G_E^{R}(r, \tilde{r}) - G_E^{A}(r, \tilde{r}) \right]_{r=\tilde{r}}, \quad (2)$$

where $G_E^{R,A}$ are retarded and advanced Green’s functions, and $\langle \ldots \rangle$ denotes disorder (RP or RMF) averaging. In view of the current conservation, $\nabla j = 0$, the function $I_{\alpha\beta}$ is transverse in momentum space,

$$I_{\alpha\beta}(q, \omega) = e^2 F(q, \omega) \left( \delta_{\alpha\beta} - \frac{q_{\alpha}q_{\beta}}{q^2} \right). \quad (3)$$

Since we are interested in long-range current correlations, we will study the low-momentum, low-frequency behavior of $I_{\alpha\beta}$. A finite value of $F(q, \omega)$ in the low-$q$ limit would imply dipole-type long-range correlations in coordinate space,
We will make use of the representation of the correlation function $I_{\alpha\beta}$ in terms of a functional derivative of a generating functional $F\{A, A'\}$ with respect to vector-potential source fields $A(r), A'(r)$,

$$I_{\alpha\beta}(r - r') = e^2 \frac{\delta}{\delta A_{\alpha}(r)} \frac{\delta}{\delta A'_{\beta}(r')} F\{A, A'\} \bigg|_{A = A' = 0}, \quad (5)$$

where

$$F\{A, A'\} = \left[ \left( \text{Tr} \ln G_{E}^R\{A\} - \text{Tr} \ln G_{E}^A\{A\} \right) \right. \times \left[ \left( \text{Tr} \ln G_{E + \omega}^R\{A'\} - \text{Tr} \ln G_{E + \omega}^A\{A'\} \right) \right]. \quad (6)$$

It is seen from Eq. (3) that the transverse character of $I_{\alpha\beta}$ is intimately related to gauge invariance of the generating functional $F\{A, A'\}$.

We start from a diagramatic analysis in the framework of a conventional impurity diagram technique. The diagrams for $F$ are obtained by connecting two closed electron loops by impurity lines; the diagrams for $I_{\alpha\beta}$ are then generated by inserting two current vertices (one in each loop) in all possible ways.

Let us first consider the case of a random scalar potential with a correlation function $\langle \bar{U}(r)\bar{U}(r') \rangle = W(r - r')$, its correlation length $d$ is assumed to be finite, though it can be arbitrary as compared to the wave length $\lambda_F$. In particular, if $d \gg \lambda_F$ (smooth RP), the scattering is of a small-angle nature, implying that the transport relaxation time is much longer than the single-particle one, $\tau_{tr} \gg \tau$. It is clear that in any finite order $n$ of the perturbative expansion ($n$ is a number of impurity lines connecting the fermion loops), the current correlation function $I^{(n)}_{\alpha\beta}$ is finite-ranged, i.e., it has a finite correlation length, beyond which the correlations decay exponentially in view of the finite-range character of $W(r - r')$.

This means that in momentum space $I^{(n)}_{\alpha\beta}(q)$ should have no singularity as $q \to 0$, implying that $F(q) \propto q^2$. Indeed, the sum of all the diagrams of the $n$-th order can be presented in the form

$$I^{(n)}_{\alpha\beta}(q) = \frac{1}{n!} \int (dq_1) . . . (dq_{n-1}) W(q_1) . . . W(q_n) \times T_{\alpha}(q_1, . . . , q_n) T_{\beta}(-q_1, . . . , -q_n), \quad (7)$$

where $(dq) = d^2q/(2\pi)^2$, and $T_{\alpha}(q_1, . . . , q_n)$ denotes a vertex part to which $n$ impurity lines with momenta $q_1, . . . , q_n$ satisfying $\sum_i q_i = q$ are attached in all possible ways. Because of its vector character and the symmetry with respect to permutation of $q_i$, the block $T_{\alpha}$ should be proportional to $q$: the current conservation implies its transverse character $T_{\alpha} \propto \bar{q}_{\alpha}$, where $\bar{q}_\alpha = \epsilon_{\alpha\beta\gamma} q_\gamma$ ($\epsilon_{\alpha\beta}$ is the antisymmetric tensor). When substituted in Eq. (7), this yields again

$$F_{\text{RP}}(q) \propto q^2, \quad q \to 0. \quad (8)$$

We now turn to the RMF case. The situation now is somewhat less trivial, since although the RMF correlation function $W_B(r - r') = \langle B(r)B(r') \rangle$ is assumed to be of a finite range, the correlation function of the corresponding vector potential $A(r)$ is long-ranged, which is reflected by a singularity at $q = 0$ in momentum space,

$$W_{\alpha\beta}(q) = \frac{W_B(q)}{q^2} \left( \delta_{\alpha\beta} - \frac{q_\alpha q_\beta}{q^2} \right). \quad (9)$$

Let us, however, analyze the RMF analog of Eq. (7),

$$I^{(n)}_{\alpha\beta}(q) = \frac{1}{n!} \int (dq_1) . . . (dq_{n-1}) \times \frac{\delta}{\delta A_{\alpha}(q_1)} \cdots \frac{\delta}{\delta A_{\alpha}(q_n)} \frac{1}{ T_{\alpha}(q_1, . . . , q_n) T_{\beta}(-q_1, . . . , -q_n).} \quad (10)$$

As in the RP case, the block $T$ represents a sum of all possible diagrams with one current vertex and $n$ disorder lines attached. This includes, in addition to the diagrams appearing in the RP case, diagrams having vertices originating from the $A^2$ term in the Hamiltonian, $\bar{H} = (-i\nabla - eA(r)/c)^2/2m$ (i.e., with two RMF lines joining the electron line at the same point, or with a RMF line starting at the external current vertex). Such diagrams, while being of minor importance for the diffusion contribution considered below (they are suppressed by the factor $(k_Fl_{tr})^{-1}$, where $k_F$ is the Fermi momentum and $l_{tr} = v_F\tau_{tr}$), are crucially important in the perturbation theory [3]. It is not difficult to see that all the diagrams for the vertex part $T$ in Eq. (10) can be generated by variation of a current with respect to the vector potential

$$T_{\alpha_1 . . . \alpha_n}(q_1, . . . , q_n) = \frac{\delta}{\delta A_{\alpha_1}(q_1)} . . . \frac{\delta}{\delta A_{\alpha_n}(q_n)} \langle j_\alpha(A + \tilde{A}) \rangle \bigg|_{\tilde{A} = 0.} \quad (11)$$

In view of the gauge invariance, the vertex part in the limit $q \to 0$ must have the form

$$T_{\alpha_1 . . . \alpha_n}(q_1, . . . , q_n) \propto \bar{q}_\alpha(q_1)\alpha_1 . . . \bar{q}_\alpha(q_n)\alpha_n, \quad (12)$$

so that all the singularities in the vector potential correlators $W_{\alpha_\beta}(q_i)$ are canceled by the vertex parts, yielding

$$F_{\text{RMF}}(q) \propto q^2, \quad q \to 0. \quad (13)$$

Therefore, in any order $n$ the perturbative contribution to the current correlator is of finite range in the RMF case as well, despite the singular nature of the vector potential correlator [3].

As usual in mesoscopic physics, long-range correlations are determined by a diffusion contribution. We assume Cooperons to be suppressed by the random or homogeneous magnetic field; in the latter case the magnetic field
is assumed to be classically weak ($\omega, \tau_{tr} \ll 1$). The corresponding diagrams are shown in Fig. 1. The sum of these two diagrams satisfies the current conservation requirement, since they can be obtained from Eq. (3) with the generating functional $F$ shown in Fig. 2.

\[ I_{\alpha\beta}^{b} = \frac{e^{2}}{2\pi^{2}} Re \int (d\bf{k})(d\bf{k}')(d\Omega) \]
\[ \times V_{\alpha}(\bf{k}, \Omega, q)[V_{\beta}(\bf{k}', -\Omega, -q)]^{*} \]
\[ \times D_{\bf{k}\bf{k}'}(\bf{Q} + \bf{q}, \omega), \]

(16)

where $V_{\alpha}(\bf{k}, \Omega, q)$ are the triangle vertex parts,

\[ V_{\alpha}(\bf{k}, \Omega, q) = v_{\alpha}G^{R}(\bf{k} + \bf{q})G^{A}(\bf{k} - \bf{Q})G^{R}(\bf{k}). \]

(17)

Substituting (14) in (16), we find the following two relevant contributions. If one takes into account the singular parts of both diffusons, then the Green functions can be expanded in small momenta $\bf{q}$ and $\bf{Q}$, which yields for $q \gg l_{tr}$

\[ I_{\alpha\beta}^{b1} = -\frac{e^{2}}{2\pi^{2}} \frac{D^{2}}{2} Re \int_{0}^{\frac{1}{2}\tau_{tr}} (\Omega) \frac{(Q\alpha + q\alpha)(2Q\beta + q\beta)}{DQ^{2} - i\omega} \frac{DQ^{2} - i\omega}{D(Q + Q)^{2} - i\omega} \]
\[ = \frac{e^{2}}{2\pi^{2}} \left[ \delta_{\alpha\beta} \ln(q\tau_{tr}) - \frac{q\alpha q\beta}{q^{2}} \ln(qL_{\omega}) \right]. \]

(18)

Another contribution

\[ I_{\alpha\beta}^{b2} = \frac{e^{2}}{2\pi^{2}} \frac{\tau_{tr}}{\tau_{tr}} \ln \frac{L_{\omega}}{l_{tr}}, \]

(19)

arises when the regular part of one of the two diffusons is taken into account.

Combining Eqs. (15), (18) and (19), we finally get

\[ I_{\alpha\beta} = \frac{e^{2}}{2\pi^{2}} \ln(qL_{\omega}) \left( \delta_{\alpha\beta} - \frac{q\alpha q\beta}{q^{2}} \right), \]

(20)

which has the required transverse form (3) with

\[ F(q, \omega) = \frac{1}{2\pi^{2}} \ln(qL_{\omega}), \]

(21)

\[ L_{\omega} \ll q \ll l_{tr}^{-1}. \]

Note, that neither the single-particle nor the transport scattering time enter the final result, which is universal and does not depend on the character of the disorder.

Our result (20) differs from that of TS&E (Ref. [6]) in two crucial aspects. First of all, in the formula by TS&E the factor $F(q, \omega) \sim \ln(qL_{\omega})$ is replaced by a large constant $\beta_{TSE} \gg 1$. Specifically, for the RMF case TS&E find

\[ \beta_{TSE} \sim \frac{\tau_{tr}}{\tau}, \]

(22)

while for a smooth RP their consideration yields

\[ \beta_{TSE} \sim \left( \frac{\tau_{tr}}{\tau} \right)^{1/2}. \]

(23)

Secondly, while our contribution comes from the IR region $Q \ll l_{tr}^{-1}$, their term is claimed to be determined by the ultraviolet (UV) region $Q \gg l_{tr}^{-1}$. We will now show that the result of TS&E is wrong.
In fact, already the above considerations imply that the findings of TS&E are incorrect. Indeed, we have demonstrated that \( F(q, \omega) \propto q^2 \) as \( q \to 0 \). In other words, Eq. (2) implies \( F(q = 0, \omega) = 0 \), since a gauge-invariant generating functional \( F \) cannot depend on a change of the vector potential by a constant. Therefore the TS&E result \( F(q = 0, \omega) = \beta_{\text{TSE}} > 0 \) cannot be correct.

To make closer contact with the work of TS&E and, in particular, to show explicitly their mistake, we will now rederive Eq. (24) in the framework of the ballistic \( \sigma \)-model approach \([1,2]\). The \( \sigma \)-model can be derived by averaging over a smooth RP or a RMF following \([3,4]\), as explained in detail in \([1]\). Performing a standard calculation and treating the \( \sigma \)-model correlator perturbatively, one gets the leading contribution in the form

\[
I_{\alpha\beta}(r - r', \omega) = \frac{\epsilon^2}{2\pi^2} \epsilon_F \int \frac{dn}{(2\pi)^2} \times n_0 g(r, n; r', n'; \omega)\delta(r - r') \delta(n - n'),
\]

where \( g(r, n; r', n'; \omega) \) is a propagator describing the motion from the point \( (r', n') \) of the phase space to the point \( (r, n) \), with the unit vector \( n \) staying for the velocity direction of a particle on the Fermi surface. The propagator \( g \) obeys the Liouville–Boltzmann equation

\[
\left( L - i\omega \right) g(r, n; r', n'; \omega) + \int \frac{dn'}{2\pi} w(n, n') g(r, n''; r', n''; \omega) - g(r, n; r', n'; \omega)] = \delta(r - r') \delta(n - n'),
\]

where \( L = v_F n \nabla_r \) and \( w(n, n') \) is the scattering cross-section (\( \phi \) is the angle between \( n \) and \( n' \)),

\[
w(n, n') = \begin{cases} 2\pi v_F W(2\pi F \sin \frac{\phi}{2}) \text{ RP,} \\ 2\pi v_F \cot^2 \frac{\phi}{2} \left( \frac{2\pi m}{e \hbar c} \right) W_B(2\pi F \sin \frac{\phi}{2}) \text{ RMF.} \end{cases}
\]

Note that \( (24) \) is identical to the Altshuler–Shklovskii-type contribution to the local density of states correlation function \([12]\), up to the velocity factors \( v_F n_\alpha, v_F n_\beta \).

Equation (24) has the same form as the formula (37) of TS&E, Ref. [1]. In fact, there is a slight difference between the two formulas: while our function \( g \) is a full propagator of the Liouville–Boltzmann equation, the function \( \Gamma(r, n; r', n') \) of TS&E is obtained by projecting this equation onto the space of non-zero harmonics in the velocity space. As a consequence, our propagator has a diffusive behavior at large distances (see below), while \( \Gamma \) of TS&E does not. This is because we treat the \( \sigma \)-model field fully perturbatively, while TS&E do this with respect to the non-zero harmonics only. Therefore some of the contributions to our Eq. (24) are shifted by them into other terms not included in Eq. (37) of Ref. [1].

While being crucial for a proper account of the diffusive contribution, this difference is irrelevant for the analysis of the contribution of the ballistic region, with characteristic momenta of the propagators satisfying \( Q \gg l^{-1}_B \), which is claimed by TS&E to be responsible for the results of Ref. [1].

In the hydrodynamic limit \( (Q \ll 1, \omega \tau_{tr} \ll 1) \) the propagator \( g(Q; n, n'; \omega) \) has the conventional diffusive behavior (in view of the vector nature of the factors \( n_\alpha, n_\beta \) in \( (24) \) we have to take into account the leading correction in \( Q \tau_{tr} \)),

\[
g(Q; n, n'; \omega) = \frac{(1 - i\omega \nu Q)(1 - i\omega \nu Q')}{DQ^2 - i\omega}.
\]

Substituting \( (27) \) in \( (24) \), we arrive again at Eq. (20).

Let us now demonstrate how the statement \( F(q = 0, \omega) = 0 \) is obtained within this approach. This can be most clearly done if one introduces the path integral representation for the propagator

\[
g(r, n; r', n'; \omega) = \int_0^\infty dT e^{i\omega T} g(r, n; r', n'; T);
\]

\[
g(r, n; r', n'; T) = \int_{x(0) = (r, n)}^{x(T) = (r, n)} \mathcal{D}r(t) e^{iS[r(t)]},
\]

where \( \dot{r}(t) = v_F n(t) \) and \( z(t) = \{r(t), n(t)\} \). Here \( S \) is the action corresponding to the dynamics determined by Eq. (25). Using (28), it is easy to show that

\[
\int dr' g(r, n; r', n'; \omega) n_\beta g(r', n'; \tilde{r}, \tilde{n}; \omega)
\]

\[
= \int_0^\infty dT e^{i\omega T} \int_{x(0) = (r, n)}^{x(T) = (r, n)} \mathcal{D}r(t) \int_0^T dt n_\beta(t) e^{iS[r(t)]}
\]

\[
= g(r, n; \tilde{r}, \tilde{n}; \omega) \frac{(r - \tilde{r})_\beta}{v_F}.
\]

Substituting this (with \( r = \tilde{r} \)) in Eq. (24) integrated over \( r' \), we immediately get \( I_{\alpha\beta}(q = 0, \omega) = 0 \), in agreement with our earlier calculation. This can be traced back to the fact that an integral of velocity vector over a closed loop is equal to zero, see Fig. 3a. Note that one should exclude from Eq. (25) a contribution of a straight trajectory of zero length in the limit \( \tilde{r} \to r \) (see a detailed discussion below). In fact, this proof of \( F(q = 0, \omega) = 0 \) can be cast in the form fully analogous to the above diagrammatic derivation based on Eq. (3). Indeed, Eq. (24) can be rewritten in the form (3) with the generating functional given by an integral over closed paths in the phase space,

\[
\mathcal{F}\{A, A'\} = \int_0^\infty \frac{dT}{T} e^{i\omega T} \int_{x(0) = x(T)} \mathcal{D}r(t)
\]

\[
\times \exp \left( iS[r(t)] - i\frac{e}{c} \int dt \dot{r}(t) [A(r(t)) + A'(r(t))] \right).
\]

The trajectories \( r(t) \) in Eqs. (23), (30) are classical counterparts of the diffuson ladders in Fig. 1. Since the integral in (30) goes over closed loops, the functional \( \mathcal{F} \)
is gauge-invariant (the second term in the action yields the flux through the loop), which immediately leads to $F(q = 0, \omega) = 0$ as explained above.

![Diagram](image)

**FIG. 3.** (a) The paths contributing to the current correlation function $I_{\alpha \beta}$, Eq. (24); (b) paths giving rise to the spurious contribution to $I_{\alpha \beta}$ found by TS&E.

Why do TS&E fail to observe that $F(q = 0, \omega)$ is identically zero and find instead the results (22), (23) for this quantity? The reason is as follows. TS&E evaluate Eq. (24) by writing it in momentum space and imposing an UV cut-off $Q_0 \gg l_r^{-1}$ which is set $Q_0 = (v_F \tau)^{-1}$ in the end of the calculation. It is easy to see that within this procedure Eq. (24) is replaced by

$$I_{\alpha \beta}^{\text{TSE}}(r - r', \omega) = \frac{e^{2}v_F^2}{4\pi^2} \int d\mathbf{r} \Delta(r - \mathbf{r}, Q_0) J_{\alpha \beta}(r, r', \mathbf{r}; \omega),$$

where

$$J_{\alpha \beta}(r, r', \mathbf{r}; \omega) = \int \frac{d\mathbf{n} d\mathbf{n}'}{(2\pi)^2} n_\alpha n_\beta \left< \mathbf{n} \mathbf{n}' | \omega \right> \omega \delta \left( \mathbf{n} - \mathbf{r}' \right) \delta \left( \mathbf{n}' - \mathbf{r} \right),$$

and $\Delta(\rho, Q_0)$ is a broadened $\delta$-function,

$$\Delta(\rho, Q_0) = \int_{|Q| < Q_0} (dQ)e^{iQ\rho} = \frac{Q_0 J_1(Q_0 \rho)}{2\pi \rho},$$

with $J_1(x)$ the Bessel function. The crucial difference between Eq. (31) and the exact formula (24) is that the former is determined by non-closed paths with $|\mathbf{r} - \mathbf{r}'| \sim Q_0^{-1}$. In particular, there is a contribution to (31) from a very short, almost straight path of a length $\sim |\mathbf{r} - \mathbf{r}'|$, with the point $\mathbf{r}'$ lying between the points $\mathbf{r}$ and $\mathbf{r}$ (see Fig. 3). It is precisely this spurious contribution that determines the result of TS&E. To demonstrate this, we simply calculate this contribution. According to (31), evaluation of the $q \to 0$ limit of $I_{\alpha \beta}^{\text{TSE}}(q, \omega)$ involves integration of $J_{\alpha \beta}$ over $\mathbf{r}'$. Using (24), we easily find that the contribution of the almost-direct trajectories to $\int d\mathbf{r}' J_{\alpha \beta}^{\text{TSE}}$ is equal to

$$\int d\mathbf{r}' J_{\alpha \beta}(r, r', \mathbf{r}; \omega) = \int \frac{dn_\alpha n_\beta}{2\pi \rho v_F} \delta(\rho - \rho_0),$$

where $\rho = r - r'$. If one completely neglects the disorder, $g$ becomes the free propagator (since we discuss a ballistic contribution, the frequency $\omega$ is fully irrelevant assuming $\omega < \tau^{-1}$, and one can simply set $\omega = 0$),

$$g_0(r, n; r', n') = \frac{1}{2\pi v_F |r - r'|} \delta \left( \mathbf{n} - \mathbf{n}' \right),$$

(35)

substitution of which into (34) leads to a divergency of the type $\delta(0)$. To see how this divergency is smeared by the disorder scattering, we rewrite (34) in the form

$$\int d\mathbf{r}' J_{\alpha \beta}(r, r', \mathbf{r}; \omega) = \frac{\rho_\alpha \rho_\beta}{v_F^2 \rho^2} P(0, \rho/v_F),$$

where $P(\delta \phi, t)$ is the distribution of the variation $\delta \phi$ of the velocity angle in time $t$. In the case of a RP the velocity angle diffuses with the diffusion constant $\tau^{-1}$,

$$P_{\text{RP}}(\delta \phi, t) = \left( \frac{\tau^{-1}}{4\pi t} \right)^{1/2} \exp \left[ -\frac{\tau^{-1}}{4\pi} (\delta \phi)^2 \right],$$

while in a RMF one has a Cauchy distribution

$$P_{\text{RMF}}(\delta \phi, t) = \frac{1}{\pi} \left( \frac{2t/\tau^{-1}}{2t/\tau^{-1} + (\delta \phi)^2} \right).$$

Substituting (37), (38) in (36) we arrive at

$$\int d\mathbf{r}' J_{\alpha \beta}(r, r', \mathbf{r}; \omega) = \frac{1}{v_F^2} \rho_\alpha \rho_\beta \times \left\{ \begin{array}{ll} \left( l_r/4\pi \rho \right)^{1/2}, & \text{RP} \\ \left( l_r/2\pi \rho \right)^{1/2}, & \text{RMF} \end{array} \right.,$$

Finally, substituting (33) in (31), we get

$$I_{\alpha \beta}^{\text{TSE}}(q = 0, \omega) = \frac{\rho_\alpha \rho_\beta}{2\pi^3} \times \left\{ \begin{array}{ll} \frac{\gamma(Q_0 l_r)}{4\Gamma(\frac{3}{2})}, & \text{RP} \\ \frac{Q_0 l_r}{4}, & \text{RMF} \end{array} \right.,$$

with a numerical coefficient $\gamma = \sqrt{\pi/2}[\Gamma(\frac{3}{2})/4\Gamma(\frac{3}{2})]$, reproducing the results (22), (23) of TS&E.

We have thus demonstrated that the findings of TS&E result from an incorrect regularization procedure. Note that their term (40) is not transverse with respect to $q$ and thus violates the current conservation. One could anticipate this without any calculations: a non-zero $q \to 0$ limit of $I_{\alpha \beta}$ in combination with current conservation would imply, according to (8), existence of a singular term $\sim q_\alpha q_\beta/q^2$ of a long-range nature (4) in coordinate space. However, TS&E deal with short scales $\ll l_r$ only and thus have no chance to obtain such a term. This
violation of current conservation remains, however, hidden in \( I_{\alpha\beta} \), where only the trace \( I^\text{TSE}_{\alpha\xi}(q = 0, \omega = 0) \) is calculated, while the transverse structure of \( I^\text{TSE}_{\alpha\beta} \) is then “restored” by hand.

The violation of current conservation is a direct consequence of the violation of gauge invariance of the generating functional \( F \) by the regularization of TS&E. Indeed, this gauge invariance is guaranteed by the fact that only those closed loops contribute to \( I_{\alpha\beta} \) while TS&E allow for non-closed short paths which produce their spurious term. It is clear from our discussion what would be a proper UV regularization procedure respecting the gauge invariance constraint. One should discard the contribution of those relevant trajectories \( \rho \rightarrow 0 \) is removed, \( Q \rightarrow \infty \). Such “zero-length trajectories” are well known to appear in the quasiclassical representation of the Green’s function \( G_E(r, R) \) at \( r \rightarrow R \). It is also well known that the contribution of such a direct trajectory of length zero should be split off from the remaining sum over periodic orbits in the trace formula for \( G_E(r, R) \); the former yields the average (Weyl) contribution to the density of states, while the latter describes fluctuations around it \( \omega \). In the present context of current correlations, the contribution of the zero-length trajectory is zero since \( \langle \rho \rangle \) is described by a diagram with no impurity lines at all. The rest is represented by a sum over closed classical paths (Eq. (24) and Fig. 3a) with the direct zero-length trajectory excluded. This is in full analogy with the semiclassical consideration of the spectral correlator \( \langle \rho(E)\rho(E+\omega) \rangle \) \( (24) \), where the zero length trajectory yields the disconnected part \( \langle \rho \rangle^2 \), while the sum over periodic orbits reproduces the Altshuler–Shklovskii diffuson contribution. Note that the procedure used by TS&E would produce an additional spurious contribution of very short, almost straight paths in this case as well.

We now turn to the closely related analysis (already presented in a brief form in the Comment \( [3] \) of the new term in the \( \sigma \)-model obtained by TS&E in Refs. \( [1] \)). According to these authors, this term leads to delocalization at the one loop level, so that the corresponding diagrams should contain only one true (singular at low momenta and frequency) diffuson. One can show, however, that, very generally, such a contribution to conductivity does not diverge. For white-noise RP this was shown in Ref. \( [4] \). We will now prove that the statement holds also in the case of RMF or smooth RP with time reversal invariance broken. Indeed, the sum of all one-diffuson diagrams with \( n \) impurity lines crossing the diffuson and having their starting (end) point anywhere on the left (right) block has the form

\[
\Delta\sigma^{(n)}_{xx}(\omega) = \frac{e^2}{2\pi} \int (dq) \frac{M_{xx}^{(n)}(q, \omega)}{2\pi \sigma^2 (Dq^2 - i\omega)},
\]

where

\[
M_{xx}^{(n)}(q, \omega) = \frac{1}{n!} \int (dq_1) \ldots (dq_{n-1}) W(q_1) \ldots W(q_n)
\times S^{L}_{q_1}(q_1, \ldots, q_n) S^{R}_{-q_1}(q_1, \ldots, -q_n), \quad \text{RP}
\]

\[
M_{xx}^{(n)}(q, \omega) = \frac{1}{n!} \int (dq_1) \ldots (dq_{n-1}) W(q_1) \ldots W(q_n)
\times W_{\alpha_1\beta_1}(q_1) \ldots W_{\alpha_n\beta_n}(q_n)
\times S^{L}_{q_1}(q_1, \ldots, q_n)
\times S^{R}_{-q_1}(q_1, \ldots, -q_n), \quad \text{RMF}.
\]

Here \( S^{L} (S^{R}) \) denotes the left (right) vertex part which has one external current vertex and to which \( n \) impurity lines (momenta \( q_i \)) as well as one diffuson (momentum \( q = \sum q_i \)) are attached. Using the same arguments as presented above for the vertex part \( T \) of the current correlator, one finds \( S^{L} \times S^{L} \propto q \) for RP and \( S^{L} \times q \ll q \) for RMF, so that the correction \( [4] \) to conductivity is non-divergent as \( \omega \rightarrow 0 \), independently of the type of disorder and presence or absence of the time-reversal symmetry,

\[
\Delta\sigma^{(n)}_{xx}(\omega) \propto \int (dq) \frac{q^2}{(Dq^2 - i\omega)} \propto \text{const} + |\omega|.
\]

In fact, the absence of a divergent correction is intimately related to the above analysis of the \( q \rightarrow 0 \) form of the current correlator. Indeed, summing up diagrams with a diffuson vertex inserted in all possible ways and using

\[
G^R_E G^L_E = i\tau \left[ G^R_E - G^A_E \right]
\]

one finds

\[
e^2 M^{(n)}(q = 0, \omega) = (2\pi \sigma)^2 I^{(n)}_{xx}(q = 0, \omega),
\]

so that the prefactor in front of the logarithmic divergence \( \int (dq) / \omega(Dq^2 - i\omega) \) in \( [4] \) is equal to \( I^{(n)}_{xx}(q = 0, \omega) \) (as also found by TS&E). Since, however, \( I^{(n)}_{xx}(q = 0, \omega) = 0 \), this just amounts to saying that the divergent one-diffuson contribution does not exist, neither for the RP nor for the RMF problem. As to the error in the \( \sigma \)-model calculation of conductivity by TS&E in \( [3] \), it is identically the same as in their calculation of the current correlator (discussed above in great detail), in view of the relation \( [3] \) between the two quantities.

As is well known, divergent contributions to the conductivity appear if one allows for more than one diffuson. In particular, the counterpart of the one- and
two–diffusion diagrams for the current correlator (Fig. 1) are the conventional two-loop diagrams for conductivity with two and three diffusons, yielding the usual weak-localization correction in the unitary ensemble,

$$\frac{\Delta g}{g} = -\frac{1}{2\pi^2 g^2} \ln \frac{L_\omega}{l_{tr}},$$

(45)

where $g = \sigma_{xx}/(e^2/h)$.

Finally, let us return to the one-diffusion corrections to conductivity (11). Though non-divergent, they are still of considerable interest, because of their non-analytic ($\propto |\omega|$) character at $\omega \rightarrow 0$. Such contributions correspond to $1/t^2$ long-time tails in the velocity correlation function and are induced by return processes neglected in the Boltzmann transport theory. They were discovered long ago in the context of the classical Lorentz gas and were recently studied for a 2D electron gas subject to a smooth RP or RMF. It was found that, in contrast to the universal weak localization corrections, the prefactor and the sign of the $|\omega|$ term is non-universal (i.e. depends on the type of disorder).

In conclusion, we have studied long-range correlations of local current densities $j(r, E)$ in a 2D mesoscopic system with time reversal invariance broken by a homogeneous or random magnetic field. We have considered two types of disorder: a (possibly smooth) random potential and a random magnetic field. We have demonstrated that within any given order of the perturbation theory the range of correlations remains finite even in the RMF case, despite the IR-singular nature of the vector potential correlator. The long-range correlations are determined by the diagrams with diffusons, Fig. 1, yielding the result (20). The current correlation function is found to be universal, i.e. it does not depend on the type or correlation length of disorder.

Our results demonstrate that recent findings of TS&E on the current correlations and on the conductivity renormalization leading to delocalization in 2D electron systems with a smooth RP or a RMF (as well as a similar statement of an earlier paper [1]) are fundamentally wrong. To make this point as clear as possible, we have demonstrated how our results are reproduced within the ballistic $\sigma$–model approach. We have further shown that the spurious contribution found by TS&E results from an incorrect regularization procedure violating current conservation and gauge invariance. This term does not arise if one uses diagrammatic perturbation theory or a properly regularized ballistic $\sigma$–model. Therefore, the new term in the diffusive $\sigma$–model which was obtained in [11,12] and which was claimed there to lead to delocalization, does not exist. In particular, the RMF problem belongs to the conventional unitary symmetry class with the leading quantum correction being of two-loop order and of localizing nature as found earlier [11,12].

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