Quantum and Randomised Algorithms for Non-linearity Estimation

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Non-linearity of a Boolean function indicates how far it is from any linear function. Despite there being several strong results about identifying a linear function and distinguishing one from a sufficiently non-linear function, we found a surprising lack of work on computing the non-linearity of a function. The non-linearity is related to the Walsh coefficient with the largest absolute value; however, the naive attempt of picking the maximum after constructing a Walsh spectrum requires $\Theta(2^n)$ queries to an $n$-bit function. We improve the scenario by designing highly efficient quantum and randomised algorithms to approximate the non-linearity allowing additive error, denoted $\lambda$, with query complexities that depend polynomially on $\lambda$. We prove lower bounds to show that these are not very far from the optimal ones. The number of queries made by our randomised algorithm is linear in $n$, already an exponential improvement, and the number of queries made by our quantum algorithm is surprisingly independent of $n$. Our randomised algorithm uses a Goldreich-Levin style of navigating all Walsh coefficients and our quantum algorithm uses a clever combination of Deutsch-Jozsa, amplitude amplification and amplitude estimation to improve upon the existing quantum versions of the Goldreich-Levin technique.

CCS Concepts: • Theory of computation → Quantum query complexity; • Security and privacy → Mathematical foundations of cryptography.

Additional Key Words and Phrases: Boolean function, Non-linearity, Quantum algorithm, Query complexity

1 INTRODUCTION

Boolean functions are an indispensable tool to design ciphers, codes and algorithms. Of particular interest are “simple” Boolean functions and the “hard” ones. The most common candidates are linear and bent functions, respectively, and the characterisation used for them is their non-linearity. Non-linearity of a function is defined as the smallest (Hamming) distance of that function to any affine function and is one of the conceptually simplest metric to evaluate a Boolean function.

Our subjects of investigation are $n$-bit input one-bit output Boolean functions. For any two such functions $f, g : \{0, 1\}^n \rightarrow \{0, 1\}$, define $\text{dist}(f, g)$ as the Hamming distance between the truth-tables of $f$ and $g$. For any $a \in \{0, 1\}^n$, define the function $\chi_a : \{0, 1\}^n \rightarrow \{0, 1\}$ as $\chi_a(x) = x \cdot a \pmod{2}$ in which $x \cdot a$ denotes the binary bitwise dot-product between $a$ and $x$. It turns out that these functions $\{\chi_a(\cdot)\}_{a \in \{0, 1\}^n}$ exactly represent the set of $n$-bit linear functions — these are the functions for which $g(x \oplus y) = g(x) \oplus g(y)$. Affine Boolean functions (those with algebraic degree 1) are generalisations of linear functions and satisfy $g(x \oplus y) = g(x) \oplus g(y) \oplus b$ for $b \in \{0, 1\}$; they are represented by functions $\chi_{a,b}(x) = (x \cdot a) \oplus b$. Of course, $\chi_a = \chi_{a,0}$.

The Walsh coefficient of a function $f$ at a point $a$ is defined as the average correlation of $f$ with $\chi_a$. We normalise it slightly differently to suit our approaches but that does not affect the main results of this work. We use the following definition.

$$\hat{f}(a) = \frac{1}{2^n} \sum_x (-1)^{f(x) \oplus \chi_a(x)} = \frac{1}{2^n} \sum_x (-1)^{f(x) \oplus x \cdot a}.$$  

Any Boolean function whose Walsh coefficients have the same absolute value is called a bent function; we know from Parseval’s inequality that these functions satisfy $\hat{f}(a) = \pm \frac{1}{\sqrt{2^n}}$ for all $a \in \{0, 1\}^n$. Such functions are considered to be the most “non-linear” ones. On the other hand,
linear functions satisfy $\hat{f}(a) = 1$ only when $f = x_a$, and $\hat{f}(a) = 0$ when $f$ is some different linear function. Observe that $\max_a |\hat{f}(a)|$ (which we denote $\hat{f}_{\text{max}}$) satisfies $\frac{1}{\sqrt{2^n}} \leq \hat{f}_{\text{max}} \leq 1$ — the first equality holds for Bent functions and the second equality holds for linear functions.

The (normalised) non-linearity of any $n$-bit Boolean function $f$ is defined as the minimum Hamming distance from $f$ to any affine Boolean function; mathematically,

$$\eta(f) = \min_{a,b} \frac{\text{dist}(f, x_{a,b})}{2^n}.$$ 

Using the above notation $0 \leq \eta(f) \leq \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2^{n/2}}$; however, this definition is computationally expensive to operate since one has to enumerate over all possible (exponentially many) linear functions and then compute distance between those functions and $f$ that also requires exponentially many function evaluations, giving a running time of $\Theta(2^n)$.

A seemingly simpler alternative arises from the Walsh-Hadamard (WH) transform of $f$. The WH transform generates a $2^n$-dimensional WH spectrum whose $a$-th coefficient is $\hat{f}(a)$. It is easy to see that $\hat{f}(a)$ can be expressed as $1 - 2^{\text{dist}(f, x'_{a})/2^n}$. That is, $\hat{f}(a) = \frac{1}{2}(1 - \hat{f}(a))$. Similarly, applying the same steps as above to the function $g(x) = f(x) \oplus 1$, it can be shown that $-\hat{f}(a) = 1 - 2^{\text{dist}(f, x'_{a})/2^n}$, i.e., $\text{dist}(f, x'_{a}) = \frac{1}{2}(1 + \hat{f}(a))$.

Combining the expressions above leads us to another expression for $\eta(f)$:

$$\eta(f) = \min_a \left\{ \frac{1}{2}(1 + \hat{f}(a)), \frac{1}{2}(1 - \hat{f}(a)) \right\} = \min_a \frac{1}{2} \left( 1 - |\hat{f}(a)| \right) = \frac{1}{2} - \frac{1}{2} \max_a |\hat{f}(a)| = \frac{1}{2} - \frac{1}{2} \hat{f}_{\text{max}}.$$ 

This expression can be computed using the $\Theta(n2^n)$ fast-Walsh-Hadamard transform algorithm and we get a $\Theta(n2^n)$ exact algorithm for computing $\eta(f)$. However, this algorithm has an additional overhead of $\Theta(2^n)$ space compared to the earlier deterministic approach. To the best of our knowledge, there are no better deterministic approaches known with asymptotically better time or space complexity for arbitrary Boolean functions. Therefore we focus on randomised, sampling-based, approaches. The subject of this paper is the non-linearity estimation problem where we want to estimate with high probability the non-linearity with $\lambda$ additive accuracy, for any given $\lambda$.

$\text{determine } 0 \leq a < b \leq 1 \text{ such that } b - a \leq \lambda \text{ and } \Pr[a < \eta(f) < b] \geq 1 - \delta.$ (1)

Recall that $\hat{f}(a)$ can also be expressed as the expected correlation of $f(x)$ and $x_a(x)$ for $x$ sampled uniformly at random. Suppose we denote this correlation as $\text{Corr}(f)$. We can use standard approaches for estimating $\mathbb{E}[\text{Corr}(f)]$; the number of samples of $x$ (and hence the running time complexity) to estimate $\mathbb{E}[\eta(f)]$ with additive accuracy $\lambda$ and error at most $\delta$ will be $O(\frac{1}{\delta} \log \frac{1}{\delta})$. There is no additional space overhead in these approaches. However, since $\eta(f)$ depends upon the Fourier coefficient with the largest absolute value, the time and query complexity still runs into $\Theta(2^n)$. In this paper we design algorithms with query complexities that are polynomial in $(n, \lambda, \log \frac{1}{\delta})$.

Non-linearity is an important property of Boolean functions and trying to compute or estimate it for a function, given either as a black-box or in some other representation, is a natural question in the realm of Boolean functions. However, beyond this academic curiosity lies the connection of non-linearity to other hardness measures of Boolean functions. In a recent paper, Boyar et al. [6] considered 5 common measures apart from nonlinearity (algebraic degree, annihilator immunity, algebraic thickness, normality, and multiplicative complexity) and obtained relationships among them; for example, they show that low multiplicative complexity implies low non-linearity and vice versa. Many of these measures, including non-linearity, are used to design cryptographic ciphers and hash-functions with interesting properties like collision-resistance [6] and propagation
characteristics [9]. Even though we leave out these interesting applications out of the scope of this paper, it would be worthwhile to understand the best use of a non-linearity estimation algorithm in cryptography [21].

The design of our quantum algorithm could be of independent interest. We were recently able to use the idea therein to formulate quantum algorithms for a few variants of the element distinctness problem [4].

1.1 Related work

To highlight the computational challenge of computing, or even estimating, the non-linearity of a Boolean function given as a black-box, recently Bera et al. [2] investigated this question in the context of the well-known BLR linearity testing algorithm [5]. The BLR test evaluates a Boolean function given in the form of a black-box, always accepting a linear function but sometimes accepting a non-linear function as well. They showed that the probability of false-positive in a BLR test is not monotonic with non-linearity, and hence, found it challenging to compute non-linearity by employing BLR.

However, if we want to estimate non-linearity, allowing some inaccuracy, then the above observation need not be a show-stopper. Indeed, using \( \rho \) to denote the probability that the BLR test accepts a function \( f \), it can be shown that \( \rho = \frac{1}{2} + \frac{1}{2} \sum_a \hat{f}^2(a) \leq \frac{1}{2} + \max_a \hat{f}(a) \). Suppose one runs the BLR test multiple times to get a close estimate \( \hat{\rho} \) of \( \rho \). Then it may be possible to estimate the lower bound \( \max_a \hat{f}(a) \geq 2\hat{\rho} - 1 \). However, the trouble is that \( 2\hat{\rho} - 1 \) can be positive or negative and, if negative, we fail to get any bound on \( \hat{f}_{\text{max}} \).

Hillery et al. proposed a property testing quantum algorithm that makes \( O\left( \frac{1}{\epsilon^2} \right) \) queries [14] and this was subsequently improved to \( O\left( \frac{1}{\epsilon} \right) \) queries [10]. But we faced hurdles when we tried to adapt these property testing algorithms that identify if a function \( f \) is linear (i.e., \( \eta(f) = 0 \)) or is \( \epsilon \)-far from linear (i.e., \( \eta(f) \geq \epsilon \)). Since we don’t consider promise problems ala property testing in this paper, so, if \( f \) is neither linear nor \( \epsilon \)-far (for any guessed \( \epsilon \)), the algorithm may erroneously return “linear” or “\( \epsilon \)-far” and we get no insights whatsoever.

Consider the simpler problem of computing \( x^* = \arg \max_x |\hat{f}^2(x)| \), and for simplicity, assume unique \( x^* \); estimating \( \eta(f) \) is easy with the knowledge of \( x^* \) (a randomised algorithm for this is given by Lemma 3.2 and a quantum algorithm is given by Lemma 4.1). It is known that the quantum circuit used in the Deutsch-Jozsa problem generates the state \( \sum_x \hat{f}(x) |x\rangle \) which, when observed, gives us a state \( |x\rangle \) sampled from the distribution \( \{ \Pr[x] = \hat{f}^2(x) \} \). Thus it is tempting to make multiple observations of independent runs of the Deutsch-Jozsa circuit and return the majority observation; the idea is that \( x^* \) has the largest probability in the entire spectrum, and so, may have the largest probability among the observed samples. This is the scenario of using the mode of a few \( i.i.d. \) samples as an estimator of the mode of a discrete distribution. However, Dutta et al. showed that if \( \min_{y \neq x^*} (\Pr[x^*] - \Pr[y]) \geq g(n) \), then the number of samples required is \( O\left( \frac{n}{g^2} \log \frac{1}{\delta} \right) \) [11, Theorem 4]. But this upper bound can be as large as exponential since we observed that \( g(n) \) could be \( O(1/2^{1.5n}) \) for \( n \)-bit functions \(^1\).

To the best of our knowledge, there exist very limited attempts towards this problem, even considering the classical computing framework. We are aware of an algorithm for non-linearity computation of a sparse Boolean function (sparsity is with respect to the truth table) [8], however, neither that approach provides any accuracy guarantees nor it is designed in the usual black-box query model — there the function is required to be given in its algebraic normal form. In a recent

\(^1\)Choose \( f \) with Hamming distance 1 from a bent function, say \( h \). It is straightforward to show that if \( \text{dist}(f, h) = 1 \), then \( \hat{f}(x) = h(x) \pm \frac{2}{\sqrt{n}} \); we obtain that the smallest gap between \( \hat{f}_{\text{max}}^2 \) and \( \hat{f}_{2\text{nd max}}^2 \) is \( \frac{8}{3n/2} \).
pre-print [15], its authors related $\hat{f}_{max}^2$ to the Gower’s $U_2$ norm of $f$ as $||f||_{U_2}^4 \leq \hat{f}_{max}^2$ and suggested that $||f||_{U_2}^4$ can be used to obtain a lower bound on $\hat{f}_{max}^2$ (which implies an upper-bound on non-linearity). There is a quantum circuit proposed by us in an earlier work [3] to estimate $||f||_{U_2}^4$ with additive accuracy $\lambda$ using $\tilde{O}(\frac{1}{\lambda^2})$ queries \footnote{The authors claimed a query complexity of $\tilde{O}(\frac{1}{\lambda})$ [15, Equation 28]. However, they used an incorrect form of Hoeffding’s inequality. Using the correct inequality gives a query complexity of $\tilde{O}(\frac{1}{\lambda^2})$ to obtain the estimate of the Gower’s $U_2$ norm.}. However, this approach does not control the accuracy of the estimate of $\hat{f}_{max}^2$ since there is no known theoretical upper bound on $\hat{f}_{max}^2 - ||f||_{U_2}^4$.

1.2 Overview of results

This paper resolves a few important questions in the light of the earlier discussions that non-linearity appears difficult without querying $f$ on exponentially-many inputs. The success of quantum query algorithms against Boolean functions motivated us to look into quantum algorithms. Can non-linearity be estimated with an additive constant inaccuracy using exponentially few queries to $f$? Or, even constant many queries? What is the minimum number of queries needed if accuracy is not a constant? What about classical randomised algorithms? After all, the BLR test is pretty effective.

Our techniques are primarily quantum in nature, but we also obtain results for randomised algorithms along the way. Here we are interested in query complexity, and so our algorithms require access to a unitary representation of a Boolean function $f$ (denoted $U_f$). $\lambda$ denotes the accuracy parameter and $\delta$ denotes the maximum allowed probability of error. Our results are summarised in Table 1.

Table 1. Our results on estimating $\eta(f)$ with additive error $\lambda$ (ignoring logarithmic factors). Note the independence of $n$ in the quantum algorithm complexity.

| Worst-case complexity | Algorithm | Query complexity | Number of qubits | Lower bound |
|-----------------------|-----------|------------------|------------------|-------------|
| Randomised algorithm  | $O(n/\lambda^6)$ [Theorem 5.2] | -                | $\Omega(1/\lambda)$ [Theorem 6.6] |
| Quantum algorithm     | $O(1/\lambda^3)$ [Theorem 5.1] | $O\left(\log \left(\frac{1}{\delta \lambda}\right) \cdot \left(n + \log \left(\frac{1}{\delta \lambda}\right)\right)\right)$ | $\Omega(1/\sqrt{\lambda})$ [Theorem 6.5] |

Our algorithms are most suitable for estimating non-linearity up to a constant or poly-logarithmic bits of precision (the smallest non-zero non-linearity requires roughly $n/2$ bits after the decimal point). It is easy to show that $|\hat{f}(x) - \hat{f}(y)| \geq \frac{2}{2^n}$ whenever $\hat{f}(x)$ and $\hat{f}(y)$ are distinct. Therefore, non-linearity can be exactly computed if we set $\lambda = 2/2^n$. Our lower bounds say that to compute non-linearity exactly, classically there is nothing better than querying $f$ at all the $2^n$ points; however, a $O(2^{n/2})$-query quantum algorithm probably exists.

A notable feature of our algorithm is that, unlike the classical “fast Walsh-Hadamard transformation” approaches, our algorithms are iterative in nature requiring little additional space. Further, there is very little overhead in the running time on top of the queries to $U_f$. Hence the query complexity above directly translates to its time complexity as well, with poly($n$) overhead arising from the additional gates required to perform amplitude estimation, amplification and small sub-circuits.
1.3 Overview of techniques

We repeatedly estimate the probability \( p \) of an observation upon measuring the final state of a quantum circuit \( A \). Using quantum amplitude estimation [7] we can obtain an estimate \( \hat{p} \) such that \( \Pr[|p - \hat{p}| \geq \epsilon] \leq \delta \) for any \( \epsilon \leq \frac{1}{2} \) using a total of \( \Theta\left(\frac{1}{\epsilon^2} \log \frac{1}{\delta}\right) \) calls to \( A \) (details given as Corollary ?? in Appendix A). Note that a classical algorithm for the same task would require \( O\left(\frac{1}{\epsilon^2} \log \frac{1}{\delta}\right) \) calls to \( A \), or as we would often state, \( O\left(\frac{1}{\epsilon^2}\right) \) calls after ignoring log-factors.

We also use quantum amplitude amplification, in particular, its fixed point version [22] which gives us a quantum circuit that when measured at the end gives us a good state, if any such exists. If the probability of the original algorithm is lower bounded by \( p \), then the number of iterations, hence queries, can be upper bounded by \( O(\sqrt{1/p}) \). The fixed point version could have been replaced with other amplitude amplification variations that require only a lower bound on the success probability [7]; however, we prefer the fixed point version since it does not involve any intermediate measurement and can be used inside another amplitude amplification.

Both our randomised and quantum algorithms to estimate non-linearity with inaccuracy \( \lambda \) actually estimate \( \hat{f}^2_{\text{max}} \) with inaccuracy \( \Theta(\lambda^2) \). The latter is implemented as a binary search, named IntervalSearch, to find the largest threshold \( \tau \), among a discrete set of thresholds that depend on \( \lambda \), such that \( \hat{f}^2_{\text{max}} \geq \tau \). The difficulty lies in solving the BoundFMax decision problem that, given \( \tau \), decides if \( \hat{f}^2_{\text{max}} \geq \tau \) in sub-exponential time. There is a technical challenge in getting binary search to act since the algorithms use estimations to guide the search. The estimations have an additive error, and we have to be careful during the comparisons with the estimated values made by the binary search. Our technical contributions here are a randomised algorithm and a quantum algorithm for the BoundFMax problem.

The classical randomised approach, named CBoundFMax, searches among all \( f(x)^2 \) values; however, it uses the idea of the Goldreich-Levin algorithm [12] to restrict search among only a small subset of values. Value of any specific \( f(x)^2 \) is of course not readily available, but that can be easily estimated using a few \( f(x) \) values sampled randomly. The number of queries is linear in \( n \) and scales inversely with \( \tau^3 \). It is possible to convert this algorithm to a quantum one, but we would end up with a complexity that scales as \( \frac{n}{\tau^2} \) — indeed, that is the complexity of the quantum versions of the Goldreich-Levin algorithms that have been proposed so far [17, 19].

To understand how CBBoundFMax gets rid of the dependence on \( n \), it will be useful to understand CBoundFMax. Think of \( \epsilon \) to be something that is smaller than \( \tau \), say, \( \tau/2 \). CBoundFMax performs a level-order traversal of a binary tree built on all possible binary prefixes of length up to \( n \). At any particular node, say \( p \), the algorithm estimates \( PWC(p \lor 0) \) where \( PWC \) at any prefix \( q \) is defined as\( PWC(q) = \sum_{x \in \{0,1\}^n : x \lor q} f^2(q \lor x) \) where \( q \lor x \) denotes the concatenation of \( q \) with \( x \). Note that for \( PWC(p \lor 0) \), the summation is over all \( n \)-bit \( x \) with prefix \( p \lor 0 \) and \( PWC(p \lor 1) \) is defined similarly. It identifies those 1-bit extensions of \( p \) for which \( PWC(p \lor b) \geq \tau - \epsilon \) and adds them to a queue. Once all the nodes of a level is processed, the nodes in the queue are retrieved and processed in the manner described above. At the final level \( l = n \), \( PWC(p) = \hat{f}^2(p) \) for any \( n \)-bit prefix \( p \). If any prefix at the final level satisfy \( PWC(p) \geq \tau \), then the algorithm concludes that \( \hat{f}^2_{\text{max}} \geq \tau \).

It is immediate that the CBBoundFMax algorithm has three components that contribute to its complexity: a) the estimation of \( PWC(p) \) with additive accuracy \( \epsilon \) that takes \( O\left(\frac{n}{\epsilon^2(\tau-2\epsilon)}\right) \) queries, b) the total number of prefixes added to the queue at any particular level \( l \) which is \( O\left(\frac{1}{\epsilon^2(\tau-2\epsilon)}\right) \) by applying Parseval’s identity, and c) the outer loop for the level-order traversal which is \( O(n) \). Hence the total query complexity of CBBoundFMax is \( O\left(\frac{n}{\epsilon^2(\tau-2\epsilon)}\right) \) queries.

The CBBoundFMax algorithm can be improved if we reduce the complexity of any of the three components of CBBoundFMax and replace the classical loops and data structures with their quantum
equivalent ones. The most trivial way to improve the complexity is to replace the classical estimation with the quantum estimation. While classical estimation uses $O\left(\frac{1}{\epsilon^2}\right)$ queries, its quantum counterpart (implemented using the Deutsch-Jozsa circuit) uses $O\left(\frac{1}{\epsilon}\right)$ queries leading to the final query complexity $O\left(\frac{n}{\epsilon(\tau-2\epsilon)}\right)$ (this algorithm is explained in Appendix C.1). This is exactly what has been proposed earlier as the quantum version of Goldreich-Levin [17, 19] in which we set $\tau = \lambda^2$ and $\epsilon = \tau/2$ to get a list of all $x$ such that $f_{\max}^2(x) \geq \lambda^2$.

The above algorithm runs a classical subroutine around a quantum circuit (for estimating $PWC(p)$ on an eligible $p$) in each level. Its query complexity can be improved by using a single quantum circuit for the entire operations of a level: (i) Estimation of $PWC()$, followed by (ii) filtering based on comparison with $\tau$. This can be implemented by generating a superposition of all eligible $p$ in a level, say $\sum_p c(p) |p\rangle$ where $|c(p)|^2 \approx PWC(p)$, running amplitude estimation without the measurement to store $|c(p)|^2$ in some register and then comparing the value in this register to that of $\tau$ to mark some of the $|p\rangle$s in the superposition (see Appendix C.2 for the entire algorithm). The dependence of the query complexity on $n$ remains there, but it nevertheless improves to $\tilde{O}\left(\frac{n}{\epsilon \sqrt{\tau-2\epsilon}}\right)$.

To remove the dependence on $n$, we remove the classical level-order traversal altogether and replace the equally superposed initial state by an initial state in which each basis state has amplitude proportional to its Walsh coefficient. Using a clever combination of Deutsch-Jozsa, amplitude amplification and amplitude estimation, $Q$BoundFMax manages to achieve a complexity of $\tilde{O}\left(\frac{1}{\epsilon \sqrt{\tau-2\epsilon}}\right)$ queries. What is remarkable is that the final algorithm can be implemented as a single quantum circuit (see Figure 2) unlike many quantum algorithms which are essentially classical wrappers around amplitude amplification and amplitude estimation.

## 2 INTERVAL SEARCH FOR $f_{\max}^2$

In this section, we consider the problem of estimating an interval $J \subseteq (0, 1)$ of length $|J| \leq \epsilon$ such that $f_{\max}^2 \in J$ with high probability. We will use this as stepping stone for estimating $f_{\max}^2$ with any desired additive accuracy $\epsilon$. Let $k$ be the smallest integer such that $\frac{1}{2^k} \leq \epsilon/2$. For finding $J$, our IntervalSearch algorithm for the above problem divides the interval $[0, 1]$ into sub-intervals of length $\frac{1}{2^k}$ and then finds the right-most (i.e., towards 1) sub-interval that contains any non-zero Fourier coefficient-squared (i.e., $f^2(a)$ for any $a$). Clearly, $f_{\max}^2$ must belong to the same interval.

To implement the above strategy, we need to first solve the following problem that we call as BoundFMax: Given a function $f(x)$ as a blackbox, a threshold $\tau \in (0, 1)$, and accuracy $g$ (we will refer to this as the “gap”) perform the following with probability of error at most $\delta$.

- Return TRUE if $f_{\max}^2 \geq \tau$.
- Return FALSE if $f_{\max}^2 < \tau - 2g$.
- Return anything if $f_{\max}^2 \in [\tau - 2g, \tau]$.

Classical and quantum algorithms for the BoundFMax problem are explained in Sections 3 and 4, respectively. For now, assume that we have an homonymous algorithm for the problem.

The IntervalSearch algorithm is described in Algorithm 1. It uses binary search to find the rightmost interval (i.e., with highest boundary values) among all the $2^k$ sub-intervals that contain $f_{\max}^2$. It uses BoundFMax to decide whether to consider the “right-half” or “left-half” of the currently processing interval. To handle the third case of BoundFMax, we equipped IntervalSearch to search among slightly overlapping intervals.

It will be easier to understand IntervalSearch from the illustration given in Figure 1 in which we have assumed $\epsilon$ to be a power of $\frac{1}{2}$; in this case $\frac{1}{2^k} = \frac{\epsilon}{2}$ and the gap $g$ used for BoundFMax is $\frac{\epsilon}{16}$. It is easy to verify that the search interval $[\text{lower, upper}]$ before round 1 has length $1 - \frac{1}{2^k}$ and...
**Algorithm 1** Algorithm IntervalSearch to find out an ε-length interval containing $\hat{f}_{\text{max}}^2$

**Require:** accuracy $\epsilon$ and probability of error $\delta$

Set $k = \lceil \log_2 \frac{1}{\epsilon} \rceil + 1$ \quad \triangleright k is the smallest integer s.t. $\frac{1}{2^k} \leq \frac{\epsilon}{2}$; thus, $\frac{\epsilon}{4} < \frac{1}{2^k} \leq \frac{\epsilon}{2}$

Set gap $g = \frac{1}{2} \left( e - \frac{1}{2^k} \right)$ \quad \triangleright 8g + $\frac{1}{2^k} = \epsilon$ \Rightarrow $\frac{3}{2} \frac{\epsilon}{16} \geq g \geq \frac{\epsilon}{16}$

Set boundaries $\text{lower} = \frac{1}{2^n}$, $\text{upper} = 1$ and threshold $\tau = \frac{1}{2}$

for $i = 1 \ldots k$ do

if $\text{BoundFMax}(\tau, g, \frac{\epsilon}{2}) \rightarrow \text{TRUE}$ then

Update $\text{lower} = \tau - 2g$, $\text{upper} = \tau + \frac{1}{2^k}$; $\text{upper}$ is unchanged

else

Update $\text{upper} = \tau$, $\text{upper} = \tau - \frac{1}{2^k}$; $\text{lower}$ is unchanged

end if

end for

return $[\text{lower}, \text{upper}]$

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**Theorem 2.1.** Let $f$ be an $n$-bit Boolean function. Given an additive accuracy $\epsilon$ and probability of error $\delta$, there is a quantum algorithm of query complexity $O \left( \frac{1}{\epsilon^{9/2}} \right)$ and a classical algorithm of query complexity $O \left( \frac{n}{\epsilon^2} \right)$ that outputs an interval of length at most $\epsilon$ that, with probability at least $1 - \delta$, contains $\hat{f}_{\text{max}}^2$.

**Proof.** The Algorithm IntervalSearch makes $k$ invocations to BoundFMax. Let $c_b(\tau, g, \delta/k)$ be the number of calls that BoundFMax makes to the oracle to solve its problem with threshold $\tau$, accuracy $g$ and error $\delta/k$. Then, the total number of calls to $U_f$ is upper bounded by $\sum_{i=1}^{k} c_b(\tau_i, g, \delta/k)$ in which $\tau_i$ denotes the value of $\tau$ in the $i$-step of the binary-search.
Using the classical implementation of BoundFMax (Lemma 3.3), we have \( \sum_{i=1}^{k} c_b(\tau_i, g, \delta) = \frac{\delta}{g} \tilde{O}\left(\frac{n}{(e/16)^2}\right) \). The way IntervalSearch sets \( k \) and \( g \) ensures that \( \frac{e}{2} \leq 8g = \epsilon - \frac{1}{2k} < \frac{3e}{4} \). Furthermore, all \( \tau_i \geq \frac{1}{2e} > \frac{e}{4} \Rightarrow \tau_i - 2g > \frac{e}{16} \). This leads to the classical query complexity of

\[
\tilde{O}\left(\frac{n}{(e/16)^2}\right) = \tilde{O}\left(\frac{\delta}{k}\right) \quad \text{further hiding a } O(\log \frac{1}{\epsilon}) \text{ factor.}
\]

With the quantum implementation of BoundFMax (Lemma 4.2) we have \( \sum_{i=1}^{k} c_b(\tau_i, g, \delta) = \frac{\delta}{g} \tilde{O}\left(\frac{1}{(e/16)\sqrt{e}/16}\right) \). Therefore, we get the number of calls as \( \tilde{O}\left(\frac{1}{(e/16)^2}\right) \).

As for the error, there are \( k \) calls to BoundFMax that is allowed to return an incorrect answer with probability at most \( \frac{\delta}{k} \). Therefore, there is an overall probability of \( \delta \) that any of those calls return an incorrect answer. \( \square \)

3 CLASSICAL RANDOMISED ALGORITHM FOR BOUNDFMAX

Goldreich and Levin proposed a randomised algorithm for learning the “high” Walsh coefficients of a Boolean function [12]. Our randomised algorithm follows the presentation of this algorithm as a level-order traversal of a binary tree. We borrow from the book by O’Donnell [20] the definition \( W^{SU}[f] \), denoted here as PrefixWalshCoefficients (in short, PWC) of \( f \) at a prefix \( a \), and Proposition 3.40 as the forthcoming lemma.

**Definition 3.1 (PrefixWalshCoefficients).** For a prefix \( a \in \{0, 1\}^s \) such that \( 0 \leq s \leq n \), \( PWC(a) = \sum_{x \in \{0, 1\}^n} \hat{f}^2(x) \) where the summation is over all \( n \)-bit \( x \) with prefix \( a \).

It immediately follows that if \( a \) is \( n \)-bit, then \( PWC(a) = \hat{f}^2(a) \).

**Lemma 3.2 ([20]).** There is a \( O\left(\frac{1}{\epsilon} \log \frac{1}{\delta}\right) \)-query classical algorithm, denoted \( PWCE \), for estimating \( PWC(a) \) for a prefix \( a \) within \( \pm \epsilon \) and with probability at least \( 1 - \delta \). In particular, this algorithm can estimate \( \hat{f}^2(b) \) for an \( n \)-bit \( b \).

We include a proof of the lemma for completeness.

**Proof.** \( \hat{f}(b) \) can be expressed as \( \mathbb{E}_x(-1)^{f(x) \oplus x \cdot b} \); so it can be estimated by simply averaging \( (-1)^{f(x) \oplus x \cdot b} \) for some uniformly chosen random \( x \in \{0, 1\}^n \) — the number of queries required follow from Hoeffding’s bound for \( \pm 1 \) random variables.

Estimating \( PWC(a) \) requires expressing it too as the expectation of a \( \pm 1 \) random variable as shown below; here the length of \( a \) is denoted \( k \).

\[
PWC(a) = \sum_{b \in \{0, 1\}^{n-k}} \hat{f}(ab)^2 = \frac{1}{2^n} \sum_{b \in \{0, 1\}^{n-k}} \left( \sum_{x \in \{0, 1\}^n} (-1)^{f(x) \oplus x \cdot (ab)} \right)^2
\]

\[
= \frac{1}{2^n} \sum_{b \in \{0, 1\}^{n-k}} \left( \sum_{x, y \in \{0, 1\}^n} (-1)^{f(x) \oplus f(y) \oplus x \cdot (ab) \oplus y \cdot (ab)} \right)
\]

\[
= \frac{1}{2^n} \sum_{x_1, y_1 \in \{0, 1\}^k \quad b \in \{0, 1\}^{n-k}} \sum_{x_2, y_2 \in \{0, 1\}^{n-k}} (-1)^{f(x_1,x_2) \oplus f(y_1,y_2) \oplus (x_1,x_2) \cdot (ab) \oplus (y_1,y_2) \cdot (ab)} (\text{Split } x = x_1x_2, y = y_1y_2)
\]

\[
= \frac{1}{2^n} \sum_{x_1, y_1 \in \{0, 1\}^k \quad x_2, y_2 \in \{0, 1\}^{n-k}} (-1)^{f(x_1,x_2) \oplus f(y_1,y_2) \oplus x_1 \cdot a \oplus y_1 \cdot a} \sum_{b \in \{0, 1\}^{n-k}} (-1)^{x_2 \cdot y_2 \cdot b}
\]
\[
\begin{align*}
\Phi = \sum_{x_1, y_1 \in \{0, 1\}^k} \sum_{x_2 = y_2 \in \{0, 1\}^{n-k}} (-1)^{f(x_1, x_2) \oplus f(y_1, x_2) \oplus x_1 \cdot a \oplus y_1 \cdot a} \\
= \frac{1}{2^{n-k} 2^k} \sum_{x_1, y_1 \in \{0, 1\}^k} (-1)^{f(x_1, x_2) \oplus f(y_1, x_2) \oplus x_1 \cdot a \oplus y_1 \cdot a} \\
= \mathbb{E}_{x_1, y_1, x_2} \left[ (-1)^{f(x_1, x_2) \oplus f(y_1, x_2) \oplus x_1 \cdot a \oplus y_1 \cdot a} \right]
\end{align*}
\]

Lemma 3.3. Algorithm CBoundFMax solves the BoundFMax problem using \(\tilde{O}(\frac{n}{\epsilon^2 (\tau - 2\epsilon)})\) queries.

**Proof.** The algorithm traverses in level-order a binary tree on all strings of lengths up to \(n\) to find some \(x\) such that \(\hat{f}^2(x) \geq \tau\); children of a node \(a \in \{0, 1\}^s\) are denoted \(a^0\) and \(a^1\).

It uses the observation that if \(PWC(a) = \sum_{b \in \{0, 1\}^s} \hat{f}^2(a \oplus b)\) is less than \(\tau - \epsilon\), then there cannot be any \(x\) with prefix \(a\) for which \(\hat{f}^2(x) \geq \tau - \epsilon\) and hence the subtree under \(a\) need not be further explored. However there may be an inaccuracy in estimating \(PWC(a)\) which we handle in the two cases below.

**Case — CBoundFMax returns TRUE;** This happens only when the algorithm finds some \(n\)-bit \(cp\) for which the \(PWC\) estimate \(e\) satisfies \(e \geq \tau - \epsilon\). From Lemma 3.2, \(e\) satisfies \(PWC(cp) - \epsilon \leq e \leq PWC(cp) + \epsilon\) with probability at least \(1 - \delta'\). Furthermore, \(PWC(cp) = \hat{f}^2(cp)\). Combining all these results, we see that the following holds with high probability.

\[
\tau \leq e + \epsilon \leq (PWC(cp) + \epsilon) + \epsilon = \hat{f}^2(cp) + 2\epsilon \implies \hat{f}_{max}^2 \geq \tau - 2\epsilon
\]

**Case — CBoundFMax returns FALSE;** For this case, assume that on the contrary \(\hat{f}_{max}^2 \geq \tau\), i.e., there is some \(n\)-bit \(x\) for which \(\hat{f}^2(x) \geq \tau\). Therefore, for all prefixes \(p\) of \(x\), \(PWC(p) \geq \hat{f}^2(x) \geq \tau\). From Lemma 3.2, with probability at least \(1 - \delta'\), \(e\) satisfies \(PWC(p) - \epsilon \leq e \leq PWC(p) + \epsilon\). For a moment assume that the estimator of \(PWC\) makes no error; thus, \(e \geq \tau - \epsilon\) and when that holds, \(p\) is added to \(Q\) and eventually retrieved and processed. Since the above fact holds for all prefixes of \(x\), so, all

---

**Algorithm 2 Algorithm CBoundFMax**

**Require:** threshold \(\tau \in (0, 1)\), confidence \(\epsilon \in (0, \tau)\), error \(\delta \in (0, 1)\)

Initialise a FIFO list \(Q = \{\epsilon\}\), where \(\epsilon\) denotes the empty string

**while** \(Q\) is not empty **do**

- remove prefix \(p\) from \(Q\)
- for suffix \(s\) from \(\{0, 1\}\) **do**
  - childprefix \(cp = p^s\)
  - obtain estimate \(e \leftarrow PWCE(cp, \epsilon, (\frac{\tau - \epsilon}{2n}) \delta)\) > With accuracy \(\epsilon\) and error \(\delta' = \frac{\tau - \epsilon}{2n} \delta\)
  - if \(e \geq \tau - \epsilon\) **then**
    - If \(\text{len}(cp) < n\), add \(cp\) to \(Q\)
  - Else (i.e., \(\text{len}(cp) = n\)), **return** TRUE
- **end if**
- **end for**
- **end while**

**return FALSE**

**LEMMA 3.3.** Algorithm CBoundFMax solves the BoundFMax problem using \(\tilde{O}(\frac{n}{\epsilon^2 (\tau - 2\epsilon)})\) queries.
of them will be stored in Q and retrieved which means that x will also be added to Q and retrieved and processed. When \( p = x, e \geq PWC(x) - e = f_2^*(x) - e \geq \tau - e \). Thus, CBoundFMax should be returning \text{TRUE} when that happens — this leads to a contradiction. Therefore, when CBoundFMax returns \text{FALSE} it must be true that, with high probability, \( f_2^{\max} < \tau \).

Let PWCE be the classical estimator for PWC. The total number of calls to PWCE will be at most \( 2 \times (\frac{n}{\tau - 2\epsilon}) \) — there are two suffixes to try for each prefix and due to Parseval’s identity, for every length \( s \in \{1, \ldots, n\} \), there are at most \( \frac{1}{\tau - 2\epsilon} \) prefixes \( \{p_1, p_2, \ldots\} \) of length \( s \) such that \( PWCE(p, e, \delta') \geq \tau - e \) where \( \delta' \) is some error. This is because of the fact that for any prefix \( p \), \( PWCE(p) < \tau - 2\epsilon \implies PWCE(p, e, \delta) < \tau - e \). So we have \( \text{PWCE}(p, e, \delta') \geq \tau - e \implies PWC \geq \tau - 2\epsilon \). Since at any level the total number of prefixes \( p \) such that \( PWC(p) \geq \tau - 2\epsilon \) is at most \( \frac{1}{\tau - 2\epsilon} \), we have that the total number of prefixes \( p \) such that \( PWC(p, e, \delta') \geq \tau - e \) is at most \( \frac{1}{\tau - 2\epsilon} \). The query-complexity is obtained by combining the number of calls to PWCE with Lemma 3.2.

The algorithm works in a flawless manner as described if all the PWCE calls are within their promised accuracy with no error. Since PWCE is called with error parameter \( \delta' = (\frac{\tau - 2\epsilon}{2n}) \delta \) therefore the probability of CBoundFMax facing any error is at most \( \delta \). □

4 QUANTUM ALGORITHM FOR BOUNDFMAX

In the previous section we obtain a classical algorithm to solve the BoundFMax problem using \( O(\frac{n}{\epsilon^{(\tau - 2\epsilon)}}) \) queries. Note that the estimation of PWC in Algorithm 2 is done classically. A naive approach to reduce the complexity of the algorithm is to replace the classical estimation by a quantum algorithm for PWC estimation.

This quantum algorithm simply executes the Deutsch-Jozsa circuit and measure the first s qubits in the standard basis. The probability of observing \( |a\rangle \) is \( \sum_{b \in \{0,1\}^s} f_2^*(ab) = PWC(a) \). It directly follows that \( PWC(a) \) can be estimated using amplitude estimation. The number of calls to \( U_f \) that is required to achieve additive accuracy \( \epsilon \) and probability of error \( \delta \) is \( O(\frac{1}{\epsilon} \log \frac{1}{\delta}) \).

**Lemma 4.1.** There is a \( O(\frac{1}{\epsilon} \log \frac{1}{\delta}) \)-query quantum algorithm for estimating \( PWC(a) \) for a prefix a within \( \pm \epsilon \) and with probability at least \( 1 - \delta \). The algorithm computes \( \hat{f}_2(b) \) for an n-bit b.

Using the quantum estimation subroutine for PWC estimation inside Algorithm 2 gives us a simple quantum algorithm for the BoundFMax problem. The number of queries can easily be shown to be \( \hat{O}(\frac{n}{\epsilon^{(\tau - 2\epsilon)}}) \) which is already better compared to the classical algorithm discussed in Lemma 3.3. We now explain how to remove the dependency on \( n \) and improve the dependency on \( (\tau - 2\epsilon) \).

Our quantum algorithm QBoundFMax is described in Algorithm 3 and a quantum circuit for the same is illustrated in Figure 2. It is to be noted that \( R_l \) is used to represent the \( i^{th} \) register used in the quantum circuit corresponding to the algorithm. We use DJ to represent the circuit for Deutsch-Jozsa: \( H^{\otimes n} \cdot U_f \cdot H^{\otimes n} \) and we use a few smaller circuits listed below.

\( \text{EQ:} \) Checking for equality of two n-bit strings, \( \text{EQ} \) maps 2n-qubit basis states \( |x, y\rangle \) to \( (-1)^{xy} |x, y\rangle \) if \( x = y \) and does nothing otherwise.

\( \text{HDQ:} \) When the target qubit is \( |0^q\rangle \), and with a \( q \)-bit string \( y \) in the control register, \( \text{HD} \) computes the absolute difference of \( y_{int} \) from \( 2^{q-1} \) and outputs it as a string where \( y_{int} \) is the integer corresponding to the string \( y \). It can be represented as \( \text{HDQ}(y) |b\rangle = |b \oplus \tilde{y}\rangle |y\rangle \) where \( y, b \in \{0, 1\}^q \) and \( \tilde{y} \) is the bit string corresponding to the integer \( [2^{q-1} - y_{int}] \). Even though the operator \( \text{HD} \) requires two registers, the second register will always be in the state \( |0^q\rangle \) and shall be reused by uncomputing (using HD\(^T\)) after the CMP gate. Hence, we have not explicitly mentioned it in Algorithm 3 and Figure 2. For all practical purposes, this operator can be treated as the mapping \( |y\rangle \mapsto |\tilde{y}\rangle \).
Algorithm 3 Algorithm QBoundFMax

**Require:** Threshold $\tau$, accuracy $\epsilon$ and error $\delta$.

1. Set $r' = \tau - \frac{\epsilon}{8}$, $q = \lceil \log(\frac{1}{\delta}) \rceil + 4$ and $l = q + 3$.
2. Set $\tau_1 = \frac{2^l}{\pi} \sin^{-1}(\sqrt{r'})$
3. on $21 \ln \left( \frac{1}{\delta^{r'}} \right)$ many independent copies do ✓ index $i$ ranges from 1 to $21 \ln \left( \frac{1}{\delta^{r'}} \right)$
   4. Initialize 5 registers $R_f^i, R_o^i, R_b^i, R_m^i, R_n^i$ as $|0^n\rangle |0^n\rangle |0^l\rangle |\tau_1\rangle |0\rangle$. The $4^{th}$ register is on $l$ qubits.
5. **Stage 1:** Apply $DJ = H^\otimes n \cdot U_f \cdot H^\otimes n$ on $R_f^i$.
6. **Stage 2:** Apply quantum amplitude estimation sans measurement (AmpEst) on $DJ$ with $R_m^i$ as the input register, $R_b^i$ as the precision register and $R_f^i$ used to determine the “good state”.
   AmpEst is called with error at most $1 - \frac{8}{\pi^2}$ and additive accuracy $\frac{1}{2\pi}$.
7. **Stage 3:** Use $HD_1$ on $R_b^i$ and $R_n^i$ individually.
8. Use $CMP$ on $R_b^i$ and $R_n^i = |\tau_1\rangle$ as input registers and $R_f^i$ as output register.
9. Use $HD_2$ on $R_b^i$ and $R_n^i$ individually again.
10. end loop
11. **Stage 4:** Initialize two new registers $R_f, R_o$ as $\sum_{x\in\{0,1\}^n} \hat{f}(x) |x\rangle |0\rangle$.
12. For each basis state $|x\rangle$ in $R_f$, for $i = 1 \ldots 21 \ln \left( \frac{1}{\delta^{r'}} \right)$ compute the majority of the basis states of each $R_n^i$, register conditioned on the $R_b^i$ to be in $|x\rangle$, and store the result in $R_o$.
13. **Stage 5:** Apply Fixed Point Amplitude Amplification (AA) $\frac{1}{\sqrt{\epsilon - 2\delta}} \log \left( \frac{\sqrt{2}}{\sqrt{\delta}} \right) \times \text{times on } R_f$ and measure $R_o$ as $m$.
14. if $m = |1\rangle$ then
15. return TRUE
16. else
17. return FALSE
18. end if

**CMP:** $\text{CMP}$ is defined as $\text{CMP} |y_1\rangle |y_2\rangle |b\rangle = |y_1\rangle |y_2\rangle |b \oplus (y_1 \leq y_2)\rangle$ where $y_1, y_2 \in \{0,1\}^n$ and $b \in \{0,1\}$ and it simply checks if the integer corresponding to the basis state in the first register is at most that in the second register.

**Cond-MAJ:** Given $k$ copies of the form $|x_i^k\rangle = |x_i\rangle |c_i\rangle$ where $x_i \in \{0,1\}^n$ and $c_i \in \{0,1\}$ and an answer register $|b\rangle$, $\text{Cond-MAJ}_z$ flips $|b\rangle$ if $|x_i^k\rangle = |z\rangle |1\rangle$ for at least $k/2$ many indices $i$.

The EQ circuit is trivial to implement, but the other three are slightly non-trivial. We have discussed their implementation details in Appendix D.

**Lemma 4.2.** The Algorithm QBoundFMax makes $O\left( \frac{1}{\epsilon \sqrt{r - 2\delta}} \log \left( \frac{1}{\delta} \right) \log \left( \frac{1}{\sqrt{\delta}} \right) \right) = \tilde{O}\left( \frac{1}{\epsilon \sqrt{r - 2\delta}} \right)$ calls to the oracle and with error at most $\delta$ behaves as follows:

- (1) if $\text{QBoundFMax}$ returns TRUE then $\hat{f}_{\text{max}} \geq \tau - 2\epsilon$
- (2) if $\text{QBoundFMax}$ returns FALSE then $\hat{f}_{\text{max}} < \tau$.

when given an $n$–bit Boolean function $f$ as an oracle, a threshold $\tau$, accuracy parameter $\epsilon$ and error parameter $\delta$ such that $\tau > 2\epsilon$.

Alternatively, we get that (a) if $\hat{f}_{\text{max}} \geq \tau$ then the algorithm returns TRUE and (b) if $\hat{f}_{\text{max}} < \tau - 2\epsilon$ then it returns FALSE.

**Proof.** Before we prove the correctness of the algorithm, we introduce a few propositions that will help us in the proof.
Proposition 4.1. For any two angles \( \theta_1, \theta_2 \in [0, \pi] \), \( \sin \theta_1 \leq \sin \theta_2 \iff \frac{\pi}{2} - \theta_1 \geq \frac{\pi}{2} - \theta_2 \).

Proof. The proof uses trigonometric identities and transformations.

\[
\sin \theta_1 \leq \sin \theta_2 \iff \cos \left( \frac{\pi}{2} - \theta_1 \right) \leq \cos \left( \frac{\pi}{2} - \theta_2 \right) \equiv \cos \left( \frac{\pi}{2} - \theta_1 \right) \leq \cos \left( \frac{\pi}{2} - \theta_2 \right) .
\]

Now, since \( \theta_1 \in [0, \pi] \), we have \( \left( \frac{\pi}{2} - \theta_1 \right) \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \). This in turn implies \( \frac{\pi}{2} - \theta_1 \in \left[ 0, \frac{\pi}{2} \right] \). The proposition now follows from the fact that \( \cos \theta \) is decreasing in the range \( \theta \in \left[ 0, \frac{\pi}{2} \right] \). □

Proposition 4.2. For any two \( q \)-bit integers \( u, v \),

\[
2^{q-1} - v \leq 2^{q-1} - u \iff \sin^2 \left( \frac{\pi}{2} \frac{v}{2^q} \right) \geq \sin^2 \left( \frac{\pi}{2} \frac{u}{2^q} \right) .
\]

Proof. If \( z \) is a \( q \)-bit integer then \( \frac{\pi}{2} \frac{z}{2^q} \in [0, \pi] \). Therefore, \( \sin^2 \left( \frac{\pi}{2} \frac{v}{2^q} \right) \leq \sin^2 \left( \frac{\pi}{2} \frac{u}{2^q} \right) \) is equivalent to \( \sin \left( \frac{\pi}{2} \frac{v}{2^q} \right) \leq \sin \left( \frac{\pi}{2} \frac{u}{2^q} \right) \). Using Proposition 4.1, this is equivalent to \( \left| \frac{\pi}{2} - \frac{v}{2^q} \right| \geq \left| \frac{\pi}{2} - \frac{u}{2^q} \right| \) which is same as \( 2^{q-1} - v \geq 2^{q-1} - u \). □

Proposition 4.3. \( \tau \) and \( \tau_1 \) satisfy \( 0 \leq \tau - \epsilon - \sin^2 \left( \frac{\tau_1}{2^q} \right) \leq \frac{2\pi}{2^q} \).

The proof of Proposition 4.3 is given in Appendix E.

We analyze the algorithm in stages. Consider one of the 21 \( \ln \left( \frac{1}{\delta \tau^2} \right) \) independent copies. Then the state of that copy after stage-1 can be given as

\[
|\psi_1\rangle = \sum_{x \in \{0,1\}} \hat{f}(x) |x\rangle |0^n\rangle |0\rangle |\tau_1\rangle |0\rangle .
\]

In stage-2, on applying amplitude estimation with \( R_i^1 \) as the input register, \( R_i^2 \) as the precision register and \( R_i^1 \) as the register indicating the "good state" whose amplitude should be amplified, we obtain the state of the system as,

\[
|\psi_2\rangle = \sum_{x \in \{0,1\}^n} \hat{f}(x) |x\rangle |\phi\rangle \left( \beta_{x,3} |a_x\rangle + \beta_{x,3} |E_x\rangle \right) |\tau_1\rangle |0\rangle .
\]
Here $|\phi\rangle$ itself is $D_{\beta} |0^n\rangle$ [the state on which amplitude estimation happens], and $|a_x\rangle$ is a normalized state of the form $|a_x\rangle = y_+ |a_{x^+}\rangle + y_- |a_{x^-}\rangle$ that on measurement outputs $a \in \{a_{x^+}, a_{x^-}\}$ which is an $l$-bit string that behaves as $\sin^2 \left( \frac{a x}{2l} \right) - \hat{f}(x) \leq \frac{1}{2\epsilon}$. We denote the normalized amplitude of $|a_x\rangle$ $\beta_{x,s}$ (s indicates “success”). We denote the set $\{a_{x^+}, a_{x^-}\}$ by $S_{a_x}$. The state $|E_x\rangle$ is the normalized error state defined as $|E_x\rangle = \sum_{a \in S_{a_x}} Y_{x,a} |a\rangle$. Notice that since the the amplitude amplification routine is called with error at most $1 - \frac{8}{\pi^2}$, we have

$$|\beta_{x,s}|^2 \leq 1 - \frac{8}{\pi^2}; \text{ furthermore, } |\beta_{x,s}|^2 + |\beta_{x,s'}|^2 = 1.$$  \hfill (2)

In stage-3, the action of $HD_1$ on any $l$-bit computational basis state $|y\rangle$ can be given as

$$HD_1 |y\rangle = \left( 2^{(l-1)} - y \right).$$

Meanwhile, the action of $CMP$ on a basis state of the form $|y\rangle |z\rangle |0\rangle$ can be given as,

$$CMP |y\rangle |z\rangle |0\rangle \rightarrow |y\rangle |z\rangle |y \leq z\rangle.$$

Combining these with Proposition 4.2, the action of stage-3 on a system with the basis state $|y\rangle |z\rangle |0\rangle$ can be given as

$$(HD_1^\perp \otimes HD_1^\perp \otimes I) \text{ CMP } (HD_1 \otimes HD_1 \otimes I) |y\rangle |z\rangle |0\rangle \rightarrow |y\rangle |z\rangle |\mathbb{I} \{y, z\}\rangle$$

where $\mathbb{I} \{y, z\}$ is the indicator function that takes on value 1 if $\sin^2 \left( \frac{uy}{\pi^2} \right) \geq \sin^2 \left( \frac{uz}{\pi^2} \right)$ and 0 else. Let $\tilde{a} = \sin^2 \left( \frac{uy}{\pi^2} \right)$. We use the notation $\tilde{y}$ for the expression $\sin^2 \left( \frac{uy}{\pi^2} \right)$. Thus, we can define $\mathbb{I} \{y, z\}$ as 1 if $\tilde{y} \geq \tilde{z}$ and 0 otherwise.

Using the above notations, we can give the transformation of $|\psi_3\rangle$ through stage-3 as

$$|\psi_3\rangle = \sum_{x \in \{0,1\}^n} \hat{f}(x) |x\rangle |\phi\rangle \left( \beta_{x,s} y_+ |a_{x^+}\rangle + \beta_{x,s} y_- |a_{x^-}\rangle + \beta_{x,s} |E_x\rangle \right) |\tau_1\rangle |0\rangle$$

$$\xrightarrow{\text{Stage-3}} \sum_{x \in \{0,1\}^n} \hat{f}(x) |x\rangle |\phi\rangle \left[ \beta_{x,s} \left( y_+ |a_{x^+}\rangle |\tau_1\rangle + \beta_{x,s} y_- |a_{x^-}\rangle |\tau_1\rangle + \beta_{x,s} |E_x\rangle |\mathbb{I} \{a_{x^+}, \tau_1\}\rangle \right) 

+ \beta_{x,s} \sum_{a \in S_{a_x}} Y_{x,a} |a\rangle |\tau_1\rangle |\mathbb{I} \{a, \tau_1\}\rangle \right] 

= \sum_{x \in \{0,1\}^n} \hat{f}(x) |x\rangle |\phi\rangle \left[ \beta_{x,s} \left( y_+ |a_{x^+}\rangle |\tau_1\rangle + \beta_{x,s} y_- |a_{x^-}\rangle |\tau_1\rangle + \beta_{x,s} |E_x\rangle |\mathbb{I} \{a_{x^+}, \tau_1\}\rangle \right) 

+ \beta_{x,s} \sum_{a \in S_{a_x}} Y_{x,a} |a\rangle |\tau_1\rangle |\mathbb{I} \{a, \tau_1\}\rangle \right] + \sum_{a \in S_{a_x}} Y_{x,a} |a\rangle |\tau_1\rangle |0\rangle \right)$$

$$= |\psi_3\rangle$$

Now, we analyse the value of the indicator function $\mathbb{I} \{a_x, \tau_1\}$ for $a \in S_{a_x}$ under different scenarios. Recall that $\mathbb{I} \{a, \tau_1\} = 1$ iff $\tilde{a} \geq \tilde{\tau}_1$.

**Scenario (i):** Consider the scenario where $x$ is such that $\hat{f}(x) \leq \tau - 2\epsilon$. Then, for any $a \in S_{a_x}$, we have $\tilde{a} = \sin^2 \left( \frac{ax}{2l} \right) \leq \left[ \hat{f}(x) - \frac{1}{l} \right] \leq \left[ \frac{1}{\pi^2} \right]$. This gives $\tilde{a} \leq \tau - 2\epsilon + \frac{1}{\pi^2}$. Since, $q \geq \log \left( \frac{1}{\epsilon} \right) + 4$, we get $\frac{1}{\pi^2} \leq \frac{\epsilon}{16}$ and hence $\tilde{a} \leq \tau - \epsilon - \frac{15}{16}\epsilon$. Using the inequality of Proposition 4.3, we have
\[ \alpha \leq \sin^2 \left( \frac{\pi \alpha}{12} \right) + \frac{2\pi}{16} - \frac{15\pi}{16} \leq \sin^2 \left( \frac{\pi \alpha}{12} \right) + \frac{2\pi}{16} - \frac{15\pi}{16} < \sin^2 \left( \frac{\pi \alpha}{12} \right) = \tau \] where the second inequality comes from the fact that \( \frac{1}{2\pi} \leq \frac{\alpha}{16} \). Hence, for any \( a \in S_{a_e} \), we have \( \mathbb{I} \{ a, \tau \} = 0 \).

**Scenario (ii):** Next, consider the scenario where \( x \) is such that \( \hat{f}(x) \geq \tau \). Again for any \( a \in S_{a_e} \), we have \( \alpha \in \left[ \hat{f}(x) - \frac{1}{12}, \hat{f}(x) + \frac{1}{12} \right] \). Accordingly, \( \alpha \geq \hat{f}(x) - \frac{1}{12} \). Since, \( \hat{f}(x) \geq \tau \) and \( \frac{1}{12} \leq \frac{\alpha}{16} \), we have \( \alpha \geq \tau - \frac{\pi}{16} \geq \tau - \epsilon = \tau' \). Now, we have set \( \tau_1 = \left[ \frac{\pi}{\alpha} \sin^{-1}(\sqrt{\tau}) \right] \leq \frac{\pi}{\alpha} \sin^{-1}(\sqrt{\tau'}) \), so we get \( \sin^2 \left( \frac{\pi \alpha}{12} \right) \leq \tau' \). Hence, we have \( \sin^2 \left( \frac{\pi \alpha}{12} \right) \leq \tau' \leq \alpha = \sin^2 \left( \frac{\pi \alpha}{12} \right) \), i.e, \( \tau_1 \leq \alpha \). This implies that for any \( a \in S_{a_e} \), we get \( \mathbb{I} \{ a, \tau_1 \} = 1 \).

For an analysis of stages 4 and 5, consider the following two cases.

**Case (i):** For all \( x \in \{0,1\}^n \), \( \hat{f}(x) < \tau - 2\epsilon \).

Then, the state after stage-3 can be written as

\[
|\psi_3'\rangle = \sum_{x \in \{0,1\}^n} \hat{f}(x) |x\rangle |\phi\rangle \left[ \beta_{x,s} |a_x\rangle |\tau_1\rangle |0\rangle + \beta_{x,s} \left( \sum_{a \in S_{a_e}} \sum_{\alpha < \tau_1} y_{x,a} |a\rangle |\tau_1\rangle |0\rangle + \sum_{a \in S_{a_e}} \sum_{\alpha \geq \tau_1} y_{x,a} |a\rangle |\tau_1\rangle |1\rangle \right) \right] \]

The gates in stage-4 operate only on the registers \( \{R^i_1, R^i_2\}_i \) (along with \( R_{f_0} \) and \( R_{f_1} \)). Hence, we can ignore the \( R^i_2 \), \( R^i_3 \) and \( R^i_4 \) registers, and rewrite the residual state of \( |\psi_3'\rangle \) as

\[
|x'\rangle = \sum_{x \in \{0,1\}^n} \hat{f}(x) |x\rangle (\eta_{x,0} |0\rangle + \eta_{x,1} |1\rangle) \]

where \( |\eta_{x,0}\rangle^2 \geq |\beta_{x,s}\rangle^2 \). Since \( |\beta_{x,s}\rangle^2 + |\beta_{x,s}\rangle^2 = 1, |\eta_{x,1}\rangle^2 \leq |\beta_{x,s}\rangle^2 \).

Now, in stage-4, for every basis state \( \{ |x\rangle : x \in \{0,1\}^n \} \) of \( R_{f_1} \), we perform a conditional majority of the 21 \( \ln \left( \frac{1}{\delta^2 \tau^2} \right) \) copies of \( R^i_4 \) conditioned on \( R^i_4 \) being \( |x\rangle \), and we store the result in \( R_{f_0} \). Suppose \( k \) denotes the number of independent copies. To compute the probability that \( R_{f_0} \) is \( |1\rangle \) when \( R_{f_1} \) is in \( |x\rangle \), observe that this event happens when at least \( k/2 \) of the \( \{R^i_1, R^i_2\}_i \) pairs of registers are in the state \( |1\rangle \). For any \( i \), the probability that \( R^i_1 R^i_2 \) of \( |x'\rangle \) is in the state \( |x\rangle |1\rangle \) is \( |\hat{f}(x)|^2 |\beta_{x,s}|^2 \) which is less than \( |\hat{f}(x)|^2 |\beta_{x,s}|^2 \leq |\beta_{x,s}|^2 \) that can be upper bounded by \( 1 - \frac{2}{\pi^2} \) (using Equation 2).

Then, using Chernoff bound\(^3\) it is straightforward to see the following relation for each \( x \in \{0,1\}^n \):

\[
\Pr \left[ R_{f_0} = |1\rangle \left| R_{f_1} = |x\rangle \right. \right] \leq \delta^2 \tau^2.
\]

Since, in this case, the above relation holds true for all \( x \in \{0,1\}^n \), we have that \( \Pr \left[ R_{f_0} = |1\rangle \right] \leq \delta^2 \tau^2 \).

In stage-5, we perform amplitude amplification on \( R_{f_0} \) with \( |1\rangle \) being the “good” state using the fixed-point amplitude amplification algorithm (FPAA) \([22]\). \( \frac{1}{\sqrt{\lambda}} \log \left( \frac{1}{\delta} \right) \) calls to the oracle are

\(^3\)Given a coin whose probability of head is \( \rho > \frac{1}{2} \), Chernoff’s bound says that the probability that tail is observed in at least \( \frac{n}{4} \) trials out of \( n \) Bernoulli trials is upper bounded by \( \exp \left( -\frac{n}{2\rho} (\rho - \frac{1}{2})^2 \right) \). In our case, \( 1 \geq \rho > \frac{4}{\pi^2} \), so \( n \geq \frac{1}{\pi^2} \frac{2}{(\frac{4}{\pi^2} - \frac{1}{2})^2} \ln \frac{1}{\delta} \) is sufficient for error at most \( b \).
necessary and sufficient for FPAA to output a good state with probability at least $1 - \delta$ where $
abla$ is the probability of the good state prior to amplification.\textsuperscript{4} Thus, the number of iterations required to amplify the probability of a state from some probability that is at most $\delta^2 \tau^2$ to $\delta$ is $\Omega\left(\frac{1}{\delta^2 \tau^2} \log \frac{2}{\sqrt{\delta}}\right)$ which is the number of amplifications that the algorithm performs.

Hence, the error will be (much) larger than $\delta$, or in other words, the probability of obtaining $R_{fo}$ as $|0\rangle$ on measurement after stage-5 is at least $1 - \delta$.

**Case (ii):** Now, let there exist some $z \in \{0, 1\}^n$ such that $\hat{f}(z) \geq \tau$.

Let $G$ be the set of all such $z$, i.e., $G = \{z \in \{0, 1\}^n : \hat{f}(z) \geq \tau\}$. Then the state after stage-3 can be written as

$$|\psi_3\rangle = \sum_{x \in \{0, 1\}^n} \hat{f}(x) |x\rangle |\phi\rangle \left[ \beta_{x,s} |a_x\rangle |\tau_1\rangle |1\{a_x, \tau_1\}\right]$$

$$+ \beta_{x,3} \left[ \sum_{a \notin S_{ax} \atop a < \tau_1} y_{x,a} |a\rangle |\tau_1\rangle |0\rangle + \sum_{a \notin S_{ax} \atop a \geq \tau_1} y_{x,a} |a\rangle |\tau_1\rangle |1\rangle \right]$$

$$= \sum_{x \in G} \hat{f}(x) |x\rangle |\phi\rangle \left[ \beta_{x,s} |a_x\rangle |\tau_1\rangle |1\rangle \right]$$

$$+ \beta_{x,3} \left[ \sum_{a \notin S_{ax} \atop a < \tau_1} y_{x,a} |a\rangle |\tau_1\rangle |0\rangle + \sum_{a \notin S_{ax} \atop a \geq \tau_1} y_{x,a} |a\rangle |\tau_1\rangle |1\rangle \right]$$

$$+ \sum_{x \notin G} \hat{f}(x) |x\rangle |\phi\rangle \left[ \beta_{x,s} |a_x\rangle |\tau_1\rangle |0\rangle \right]$$

$$+ \beta_{x,3} \left[ \sum_{a \notin S_{ax} \atop a < \tau_1} y_{x,a} |a\rangle |\tau_1\rangle |0\rangle + \sum_{a \notin S_{ax} \atop a \geq \tau_1} y_{x,a} |a\rangle |\tau_1\rangle |1\rangle \right]$$

As before, if we trace out the registers $R_{i_1}^l, R_{i_2}^l$ and $R_{i_4}^l$ in $|\psi_3\rangle$, the residual state can be expressed as

$$|\chi^l\rangle = \sum_{x \in G} \hat{f}(x) |x\rangle \left( \eta_{x,1} |1\rangle + \eta_{x,0} |0\rangle \right) + \sum_{x \notin G} \hat{f}(x) |x\rangle \left( \eta_{x,0} |0\rangle + \eta_{x,1} |1\rangle \right)$$

where $|\eta_{x,1}|^2 \geq |\beta_{x,s}|^2$ and $|\eta_{x,0}|^2 \leq |\beta_{x,s}|^2$ for $x \in G$, and $|\eta_{x,0}|^2 \geq |\beta_{x,s}|^2$ and $|\eta_{x,1}|^2 \leq |\beta_{x,s}|^2$ for $x \notin G$.

Next, for every $x \in \{0, 1\}^n$, conditioned on all the $R_i^l$ registers and $R_{f1}$ being in state $|x\rangle$, we perform a conditional majority over all the $R_{i_2}^l$ registers and store the output in $R_{fo}$. Then, using Chernoff bounds as in case(i), we get that for any $x \in G$,

$$Pr\left[R_{fo} = |1\rangle \left| R_{f1} = |x\rangle \right. \right] \geq 1 - \delta^2 \tau^2 \geq 1 - \delta,$$

and for any $x \notin G$ we have,

$$Pr\left[R_{fo} = |1\rangle \left| R_{f1} = |x\rangle \right. \right] \leq \delta^2 \tau^2 < \delta.$$

\textsuperscript{4}We can replace FPAA by vanilla amplitude amplification where success probability is unknown but a lower-bound is known. A similar analysis will follow.
Therefore, the overall probability of obtaining $|1\rangle$ in $R_{fo}$ after stage-4 can be expressed as

$$\Pr \left[ R_{fo} = |1\rangle \right] \geq \sum_{x \in \mathbb{C}} |\hat{f}(x)|^2 \cdot (1 - \delta) \geq \tau(1 - \delta) \geq \tau / 2$$

under the reasonable assumption that the target error probability $\delta < \frac{1}{2}$.

Thus on amplifying the amplitude of the state $|1\rangle$ in $R_{fo}$ for $\frac{1}{\sqrt{\tau - 2\delta}} \log \left( \frac{2}{\sqrt{\delta}} \right)$ iterations using fixed point amplitude amplification, we observe $|1\rangle$ in $R_{fo}$ with probability at least $1 - \delta$ as required.

Now, we evaluate the query complexity of the algorithm. It is straightforward to observe that the number of calls made by amplitude estimation in $QBoundFMax$ is $O(2^k) = O(\frac{1}{\epsilon})$. The amplitude estimation subroutine is implemented on $21 \ln \left( \frac{1}{\delta^2 \tau} \right) = O\left( \log \left( \frac{1}{\delta \tau} \right) \right)$ many independent copies. Hence the query complexity at the end of stage-2 is $O\left( \frac{1}{\epsilon} \log \left( \frac{1}{\delta \tau} \right) \right)$. In the last stage, amplitude amplification is performed $O\left( \frac{1}{\sqrt{\tau - 2\delta}} \log \left( \frac{1}{\sqrt{\delta}} \right) \right)$ iterations. Hence, the algorithm $QBoundFMax$ makes $O\left( \frac{1}{\epsilon} \log \left( \frac{1}{\delta \tau} \right) \right) \cdot \hat{O}\left( \frac{1}{\epsilon \sqrt{\tau - 2\delta}} \right)$ queries to the oracle in total.

\[\square\]

4.1 Fine tuning of interval search

Earlier we saw how to obtain a small interval $J$ that contains $\hat{f}_{max}^2$ with high probability. However, there may be a requirement to fine-tune this estimation.

Suppose $f$ is linear, i.e., $\eta(f) = 0$. For such a function $\hat{f}_{max}^2 = 1$; however, due to the nature of Algorithm 1 we will get the interval $[1 - \epsilon, 1]$ for $\hat{f}_{max}^2$, and hence an interval for $\eta(f)$. We feel that a non-linearity estimation algorithm should be able to clearly identify a linear function instead of presenting approximate values close to 0.

Consider the other extreme of Bent functions with the largest non-linearity; these would have $\hat{f}_{max}^2 = \frac{1}{2\pi}$. However, our Algorithm 1 will, most-likely, return the interval $\approx [\frac{1}{2\pi}, \epsilon]$. We wonder if it is possible to obtain an even tighter interval.

\[
\begin{align*}
|0^{\otimes n}\rangle & \xrightarrow{H^{\otimes n}} |U_f^{\otimes n}\rangle & \xrightarrow{H^{\otimes n}} |f^{\otimes n}\rangle \\
|0^{\otimes n}\rangle & \xrightarrow{H^{\otimes n}} |U_f^{\otimes n}\rangle & \xrightarrow{H^{\otimes n}} |f^{\otimes n}\rangle \\
|0\rangle & \xrightarrow{U_f} |f\rangle
\end{align*}
\]

Fig. 3. Circuit for estimating $\sum_x \hat{f}^4(x)$

For handling these extreme values of $\hat{f}_{max}^2$ consider the quantum circuit illustrated in Figure 3. Three registers $R_1$, $R_2$ and $R_3$ are initialized to $|0^{\otimes n}\rangle$, $|0^{\otimes n}\rangle$ and $|0\rangle$, respectively. Then it applies the Deutsch-Jozsa circuit independently on $R_1$ and $R_2$, to obtain the state $\sum_x \sum_y f^4(x) |x\rangle |y\rangle |\hat{f}(y)\rangle |\hat{f}(x)\rangle |\hat{f}(y)\rangle |\hat{f}(x)\rangle |0\rangle$. Finally, the circuit flips the state of $R_3$ if $x = y$ to obtain the state $\sum_x \sum_{y \neq x} f^4(x) |x\rangle |x\rangle |1\rangle + \sum_{x,y,x \neq y} f^4(x) |x\rangle |y\rangle |y\rangle |0\rangle$. Now $R_3$ is measured in the standard basis. The probability of observing $|1\rangle$ in $R_3$ is

$$p = \sum_x \hat{f}^4 \leq \hat{f}_{max}^2 \sum_x \hat{f}^2 = \hat{f}_{max}^2$$

(Using Parseval’s equality).
At this point amplitude estimation can be used to estimate this probability with any desired additive error $\epsilon'$, and with low error probability denoted by $\delta'$. Denoting this estimate by $p^*$, with probability at least $1 - \delta'$ it satisfies $p^* - \epsilon' \leq p \leq p^* + \epsilon'$, which implies that $\hat{\epsilon}_{\max} \geq p^* - \epsilon'$. This is used to get a tighter lower bound for $\hat{\epsilon}_{\max}$. The value of $\delta'$ is set to the same $\delta$ that is used in IntervalSearch. A tight estimate for $\hat{\epsilon}_{\max}$ is computed based on the interval $J$ obtained from IntervalSearch.

If $J$ is the rightmost interval, then we want to primarily detect if $\hat{\epsilon}_{\max} = 1$. We will use $\epsilon' = \epsilon$. Note that amplitude estimation will return 1 in this case without fail. Thus, if $p^* = 1$, then the interval $[1, 1]$ is returned. Otherwise, the left boundary of $J$ can be tightened to $p^* - \epsilon$, provided it is larger than the original left boundary of $J$. Clearly, we get an interval of length at most $\epsilon$. The number of calls to the circuit, and hence to $U_f$, will be $O(\frac{1}{\epsilon} \log \frac{1}{\delta})$.

If $J$ is the leftmost interval, then we want to primarily get a lower bound on $\hat{\epsilon}_{\max}$ larger than $\frac{1}{3\pi}$. For this reason, we will use $\epsilon' = \epsilon^{3/2}$ and employ the improved amplitude estimation proposed by Montanaro [18]. Observe that the output probability of the circuit is $p = \sum_x \hat{j}^4(x)$; so, its variance can be computed as

$$p(1 - p) \leq \hat{\epsilon}_{\max} (1 - \sum_x \hat{j}^4(x)) \leq \hat{\epsilon}_{\max} \leq \epsilon,$$

where the last inequality is implied by the fact that IntervalSearch returned $J$ as the last interval whose right boundary ($\frac{1}{3\pi}$) is less than $\epsilon$. Montanaro showed that if variance of an output is bounded, then a better estimation method exists for additive errors; in our case, the number of calls to the circuit, and so also to $U_f$ will be $O(\frac{\sqrt{\epsilon}}{\epsilon^{3/2}} \log \frac{1}{\delta}) = O(\frac{1}{\epsilon} \log \frac{1}{\delta})$. The estimation procedure will return an estimate $p^*$. If $p^* \leq \epsilon^{3/2}$ then we will can safely use $\frac{1}{2\pi}$ as the left boundary of $J$. Otherwise, we can increase the left boundary of $J$ to $p^* - \epsilon^{3/2}$. In both cases, the length of $J$ remains at most $\epsilon$.

To summarise, we saw above how to make $O(\frac{1}{\epsilon} \log \frac{1}{\delta})$ queries to $U_f$ and return an estimate for $\hat{\epsilon}_{\max}$ that is at most $\epsilon$ away from the actual value. Moreover, if $f$ is linear, it correctly identifies that $\hat{\epsilon}_{\max} = 1$.

## 5 NON-LINEARITY ESTIMATION

Now we combine the earlier results to present our algorithm for estimating $\eta(f)$.

**Theorem 5.1.** Given an oracle $U_f$ to an $n$-bit function $f$, an additive accuracy $\lambda$ and an error $\delta$, there is a quantum algorithm that outputs an interval $I$ of length at most $\lambda$ such that the normalised nonlinearity of the function $f$, denoted by $\eta(f)$, belongs to $I$ with probability at least $1 - \delta$. The algorithm makes $\tilde{O}\left(\frac{1}{\lambda^2}\right)$ queries to $U_f$. If $\eta(f) = 0$ then the algorithm always outputs the correct non-linearity.

The algorithm simply calls IntervalSearch (along with fine-tuning described in Section 4.1) using accuracy $\epsilon = 2d$ (for some $d$ to be decided below) and using the error probability $\delta/2$. The combined probability of error, from IntervalSearch and fine-tuning, remains bounded by $\delta$. Let $J = [c - d, c + d]$ be the interval returned by IntervalSearch. If $J = [1, 1]$ then the algorithm shall output $J$, else $I = \left(\frac{1}{2}(1 - \sqrt{c}) - \frac{\sqrt{c}}{2}, \frac{1}{2}(1 - \sqrt{c}) + \frac{\sqrt{c}}{2}\right)$. We prove the correctness of this algorithm below.

**Proof.** $\eta(f) = 0$ is equivalent to $\hat{\epsilon}_{\max} = 1$ and in that case, IntervalSearch after fine-tuning returns $[1, 1]$ without fail. Thus, this case is correctly handled.

For non-linear functions, suppose IntervalSearch returns $J = [c - d, c + d]$. IntervalSearch guarantees that $\hat{\epsilon}_{\max} \in J$. Then, $\hat{\epsilon}_{\max} \in J' = \left[\sqrt{c - d}, \sqrt{c + d}\right]$ and so, $\eta(f) \in I' = \left(\frac{1}{2}(1 - \sqrt{c - d}, \sqrt{c + d}\right).$
\[\sqrt{c+d}, \frac{1}{2}(1-\sqrt{c-d})].\] From the fact that \(\sqrt{a} - \sqrt{b} \leq \sqrt{a-b}\) and \(\sqrt{a} + \sqrt{b} \geq \sqrt{a+b}\), we can say that \(I'\) is contained in \(I = \left(\frac{1}{2}(1-\sqrt{c} - \sqrt{d}), \frac{1}{2}(1-\sqrt{c} + \sqrt{d})\right) = \left(\frac{1}{2}(1-\sqrt{c}) - \frac{\sqrt{a}}{2}, \frac{1}{2}(1-\sqrt{c}) + \frac{\sqrt{a}}{2}\right)\).

The length of \(I\) is \(\sqrt{a}\) that we want to be \(\lambda\). So, we set \(d = \lambda^2\). From Theorem 2.1, we know that the complexity of \(\text{IntervalSearch}\) when using \(\text{QBoundFMax}\) is \(\tilde{O}\left(\frac{1}{\varepsilon^2}\right)\). Here, we have \(e = 2d = 2\lambda^2\). Hence, the number of \(U_f\) queries in \(\text{IntervalSearch}\) while using \(\text{QBoundFMax}\) is upper bounded by \(\tilde{O}\left(\frac{1}{\lambda^2}\right)\). \(\square\)

**Theorem 5.2.** There is a classical algorithm that makes \(\tilde{O}(\frac{n}{\lambda^2})\) queries to \(U_f\) and outputs an interval \(I\) of length at most \(\lambda\) such that \(\eta(f)\) belongs to \(I\) with constant probability.

The algorithm and the proof are exactly as in Theorem 5.1 but using \(\text{QBoundFMax}\) and without the fine-tuning subroutine.

## 6 LOWER BOUNDS ON NON-LINEARITY

Now we present our query complexity lower bounds for a non-linearity separation problem. We employ the well-known quantum adversary method proposed by Ambainis [1] which is stated in the form of Theorem 6.1.

**Theorem 6.1.** Let \(F\) be a \(p\)-bit Boolean function and \(X\) and \(Y\) be two sets of inputs such that \(F(x) \neq F(y)\) for any \(x \in X\) and \(y \in Y\). Let \(R \subseteq X \times Y\) be a relation such that

1. for every \(x \in X\), \(\exists\) at least \(m\) different \(y \in Y\) such that \((x,y) \in R\).
2. for every \(y \in Y\), \(\exists\) at least \(m'\) different \(x \in X\) such that \((x,y) \in R\).
3. for every \(x \in X\) and \(i \in \{1, \ldots, p\}\), \(\exists\) at most \(l\) different \(y \in Y\) such that \(x_i \neq y_i\) and \((x,y) \in R\).
4. for every \(y \in Y\) and \(i \in \{1, \ldots, p\}\), \(\exists\) at most \(l'\) different \(x \in X\) such that \(x_i \neq y_i\) and \((x,y) \in R\).

Then any quantum algorithm uses \(\Omega\left(\sqrt{\frac{m-m'}{1}}\right)\) queries to compute \(F\) on \(X \cup Y\).

Consider the following \(\hat{f}_{\text{max}}\) decision problem: Given a Boolean function \(f\) with a promise that \(\hat{f}_{\text{max}} = 1\) or \(\hat{f}_{\text{max}} = 1 - 4\lambda\), decide the case of \(f\). Using the notations of Theorem 6.1, consider a Boolean function \(F : \{0,1\}^m \rightarrow \{0,1\}\) that takes as input a truth-table of an \(n\)-bit Boolean function encoded as a \(2^n\)-bit binary string and outputs 1 iff the encoded function, say \(f()\), satisfies \(\hat{f}_{\text{max}} = 1\). The \(\hat{f}_{\text{max}}\) decision problem is to compute \(F()\) on \(X \cup Y\) where \(X\) contains some \(n\)-bit functions with \(\hat{f}_{\text{max}} = 1\) and \(Y\) contains some \(n\)-bit functions with \(\hat{f}_{\text{max}} = 1 - 4\lambda\). Note that any algorithm that solves non-linearity estimation with \(\lambda\) accuracy also computes \(\hat{f}_{\text{max}}\) with \(2\lambda\) accuracy and, thus, can solve the above decision problem. So, a lower bound on the \(\hat{f}_{\text{max}}\) decision problem naturally is a lower bound on the problem of non-linearity estimation.

For the case of \(\lambda > \frac{1}{4}\), we have a constant time quantum algorithm for non-linearity estimation and so the lower bound is \(\Omega(1)\). So suppose that \(\lambda \leq \frac{1}{4}\). Consider the following sets \(X = \{0^n\}\) and \(Y = \{y \in \{0,1\}^n : wt(y) = 2\lambda \cdot 2^n\}\) where \(wt()\) denotes the Hamming weight. Notice that deciding if an \(n\)-bit Boolean function \(g\) belongs to \(X\) or \(Y\) given that \(g\) belongs to one of them conforms to the generalized Grover search problem. We present the lower bound of this problem below.

**Lemma 6.2.** Any quantum algorithm uses \(\Omega\left(\frac{1}{\sqrt{\lambda}}\right)\) queries to the oracle of the given function \(g\) to decide if \(g \in X\) or \(g \in Y\) given the promise that \(g\) belongs to one of them.

**Proof.** Consider the sets \(X\) and \(Y\). Let the relation \(R = X \times Y\). Also let \(\lambda = 2\lambda \cdot 2^n\). Then for every \(x \in X\), \(\exists\) exactly \(\left(\frac{2^n}{\lambda}\right)\) different \(y \in Y\) such that \((x,y) \in R\) and for every \(y \in Y\), \(\exists\) exactly \(\left(\frac{2^n}{\lambda}\right)\) different \(x \in X\) such that \((x,y) \in R\). Therefore, the length of \(R\) is \(\Omega(2^n)\). From the fact that \(\sqrt{a} - \sqrt{b} \leq \sqrt{a-b}\) and \(\sqrt{a} + \sqrt{b} \geq \sqrt{a+b}\), we can say that \(I'\) is contained in \(I = \left(\frac{1}{2}(1-\sqrt{c} - \sqrt{d}), \frac{1}{2}(1-\sqrt{c} + \sqrt{d})\right) = \left(\frac{1}{2}(1-\sqrt{c}) - \frac{\sqrt{a}}{2}, \frac{1}{2}(1-\sqrt{c}) + \frac{\sqrt{a}}{2}\right)\).

The length of \(I\) is \(\sqrt{a}\) that we want to be \(\lambda\). So, we set \(d = \lambda^2\). From Theorem 2.1, we know that the complexity of \(\text{IntervalSearch}\) when using \(\text{QBoundFMax}\) is \(\tilde{O}\left(\frac{1}{\varepsilon^2}\right)\). Here, we have \(e = 2d = 2\lambda^2\). Hence, the number of \(U_f\) queries in \(\text{IntervalSearch}\) while using \(\text{QBoundFMax}\) is upper bounded by \(\tilde{O}\left(\frac{1}{\lambda^2}\right)\). \(\square\)
exactly \(1 \in X\) such that \((x, y) \in R\). So we have \(m = \binom{2^n}{\ell}\) and \(m' = 1\). Now, for any \(x \in X\) and \(i \in \{1, 2, \cdots, 2^n\}\), fix \(y_i = 1\). Naturally, \(x_i \neq y_i\). There are exactly \(\binom{2^n-1}{\ell-1}\) different \(y \in Y\) such that \(y_i = 1\) for any \(i\). Hence, we have \(l = \binom{2^n-1}{\ell-1}\). On the other side it is trivial that \(l' = 1\) since there is only a single element in \(X\). Hence, from Theorem 6.1, we obtain our lower bound as \(\Omega\left(\frac{1}{\sqrt{\lambda}}\right)\) queries.

Note: Since the problem in Lemma 6.2 is a version of the unstructured search problem, the lower bound in Lemma 6.2 is also a direct consequence of the optimality of the generalized Grover’s search [13].

**Theorem 6.3.** Any quantum algorithm uses \(\Omega\left(\frac{1}{\sqrt{\lambda}}\right)\) queries to the oracle of the given function \(f\) to decide if \(\hat{f}_{\text{max}} = 1\) or \(\hat{f}_{\text{max}} = 1 - 4\lambda\) given the promise that it is either of the cases.

**Proof.** Consider the set \(X\) and \(Y\) as defined earlier. We know that for any \(a \in \{0, 1\}^n\), \(\hat{f}(a)\) can be written as \(\frac{1}{\sqrt{2^n}}(|\{x : f(x) = a \cdot x\}| - |\{x : f(x) \neq a \cdot x\}|)\). Now, since \(\lambda \leq \frac{1}{8}\), we have \(\hat{f}_y(0^n) = 1 - 4\lambda \geq \frac{1}{2}\) for any \(y \in Y\) where \(y\) represents the truth table of \(f_y\). Next, as for any \(a \neq 0^n\), \(f_y(x) = a \cdot x\) for at most \(\frac{1}{2} + 2\lambda\) many \(x\)'s. Hence, we have \(\hat{f}_y(a) \leq 4\lambda \leq \frac{1}{2}\) for any \(y \in Y\). Hence for any \(y \in Y\), \((\hat{f}_y)_\text{max}\) occurs at \(0^n\) and \((\hat{f}_y)_\text{max} = 1 - 4\lambda\). Now, for \(x \in X\), we have \(\hat{f}(z) = 0\) for all \(z \in \{0, 1\}^n\). Hence, \(\hat{f}_x(0^n) = 1\) and \(\hat{f}_x(a) = 0\) for any \(a \neq 0^n\). So for \(x \in X\), \((\hat{f}_x)_\text{max}\) occurs at \(0^n\) and \((\hat{f}_x)_\text{max} = 1\).

Thus, the problem of deciding if a given function \(f\) belongs to \(X\) or \(Y\) can be reduced to deciding if for the function \(\hat{f}\), \(\hat{f}_{\text{max}} = 1\) or \(\hat{f}_{\text{max}} = 1 - 4\lambda\). So, using Lemma 6.2, we obtain that any quantum algorithm uses \(\Omega\left(\frac{1}{\sqrt{\lambda}}\right)\) queries to the oracle of the given function \(f\) to decide if \(\hat{f}_{\text{max}} = 1\) or \(\hat{f}_{\text{max}} = 1 - 4\lambda\) given the promise that it is either of the cases. \(\square\)

Note that the lower bound in Theorem 6.3 indicates the difficulty in separating functions that are linear and \(4\lambda\)-close to linear. We also obtain the same lower bound separating functions that are bent and are \(4\lambda\)-close to bent. Recall that a function \(f\) is called a bent function if \(|\hat{f}(a)| = \frac{1}{\sqrt{2^n}}\) for all \(a \in \{0, 1\}^n\). Therefore, for such a function, if \(\hat{f}(0^n) = \frac{1}{\sqrt{2^n}}\), then \(\text{wt}(x_f) = \frac{1}{2}\left(2^n - \sqrt{2^n}\right)\) where \(x_f\) represents the truth table of \(f\).

Characterising functions that are \(4\lambda\)-close to bent requires a little bit of work, and the following proposition does the heavy-lifting.

**Proposition 6.1.** Let \(f\) be some \(n\)-bit function such that \(\text{wt}(x_f) \geq k\) and a Boolean function \(g\) be obtained from \(f\) by flipping \(k\) bits in \(x_f\) from 1 to 0. Then, \(\hat{g}(0^n) = \hat{f}(0^n) + k \frac{2}{2^n}\), and for any \(a \neq 0^n\), \(\hat{g}(a) = \hat{f}(a) + \sum_{i=1}^{k} \frac{2}{2^n}\).

**Proof.** We will actually prove the proposition for \(k = 1\), i.e., \(g\) is obtained from \(f\) by flipping the value at some point, say \(y\); so \(f(y) = 1\) was flipped to \(g(y) = 0\). The general case can be applied by repeatedly applying this case, in turn, to the \(g\) that is obtained.

For any \(a \in \{0, 1\}^n\), define \(S_a^0(f)\) as the number of elements in the set \(\{x : f(x) = a \cdot x\}\) and \(S_a^1(f)\) as the number of elements in the set \(\{x : f(x) \neq a \cdot x\}\). When \(a = 0^n\), \(S_a^0(f) = \text{wt}(f)\) and \(S_a^0(f) = 2^n - \text{wt}(f)\). In general, \(\hat{f}(a) = \frac{1}{\sqrt{2^n}} (S_a^0(f) - S_a^1(f))\).

Now, \(g\) is obtained by flipping the value of \(f(y) = 1\) to \(g(y) = 0\), i.e., \(\text{wt}(x_g) = \text{wt}(x_f) - 1\). First, consider the case of \(a = 0^n\). For this \(a\), \(S_a^1(g) = S_a^1(f) - 1, S_a^0(g) = S_a^1(f) + 1\), and \(\hat{g}(0^n) = \hat{f}(0^n) + \frac{2}{2^n}\). This proves the first claim of the proposition.
Now we prove the case of general $a \neq 0^n$. There are two possibilities here. On one hand, if $a$ is such that $f(y) = a \cdot y$, then $S^0_a(g) = S^0_a(f) - 1$ and $S^1_a(g) = S^1_a(f) + 1$, and thus, $\hat{g}(a) = \hat{f}(a) - \frac{2a}{\lambda}$. On the other hand, if $a$ is such that $f(y) = 1 \oplus a \cdot y$, then a similar argument shows that $\hat{g}(a) = \hat{f}(a) + \frac{2a}{\lambda}$. Combining both the cases we get that $\hat{g}(a) = \hat{f}(a) \pm \frac{2a}{\lambda}$ as required. □

If we choose bent functions such that $\hat{f}(0^n) = \frac{1}{\sqrt{2^n}}$, we always have $\hat{f}(0^n) \geq \hat{f}(a)$ for any $a$. Moreover, if $g$ is obtained by flipping some bits of $x_f$ from 1 to 0, then from the above proposition we get that $\hat{g}(0^n) = \hat{f}(0^n) + \frac{\sqrt{2}}{2^n} \geq \hat{f}(0^n) + \sum_{i=1}^{k} \pm \frac{2}{2^n} \geq \hat{f}(a) + \sum_{i=1}^{k} \pm \frac{2}{2^n} = \hat{g}(a)$ for any $a$. That is, $\hat{g}_{\text{max}} = \hat{g}(0^n)$. Clearly, if exactly $2\lambda \cdot 2^n$ 1s are flipped to 0, then $\hat{g}_{\text{max}}$ occurs at $0^n$ and $\hat{g}(0^n) = \frac{1}{\sqrt{2^n}} + 4\lambda$.

**Theorem 6.4.** Any quantum algorithm uses $\Omega\left(\frac{1}{\sqrt{\lambda}}\right)$ queries to the oracle of the given function $f$ to decide if $\hat{f}_{\text{max}} = \frac{1}{\sqrt{2^n}}$ or $\hat{f}_{\text{max}} = \frac{1}{\sqrt{2^n}} + 4\lambda$ given the promise that it is either of the cases.

**Proof.** Let $x$ be the truth table string of some bent function $f_x$ such that $\hat{f}_x(0^n) = \frac{1}{\sqrt{2^n}}$. Then we know $wt(x) = \frac{1}{2}(2^n - \sqrt{2^n}) = k$. Let $X = \{x\}$ and $\tilde{\lambda} = 2 \cdot \lambda \cdot 2^n$. Now let $Y$ be the set that contains all strings $y$ which can be obtained by flipping exactly $\tilde{\lambda}$ number of 1s to 0. Then for any $y \in Y$, $wt(y) = \frac{1}{2}(2^n - \sqrt{2^n}) - \tilde{\lambda}$. Let $R = X \times Y$. Now, for $x \in X$, we can see that there are exactly $\binom{k}{\tilde{\lambda}}$ number of $y \in Y$ such that $(x, y) \in R$ and so $m = \binom{k}{\tilde{\lambda}}$. Similarly, we can see that for any $i \in \{1, \ldots, 2^n\}$ there are at most $l = \binom{k-1}{\lambda-1}$ $y \in Y$ such that $x_i \neq y_i$ and $(x, y) \in R$. Since the set $X$ contains only a single element, we have $m' = 1$ and $l' = 1$. So we have $\frac{m-m'}{l-l'} = \frac{k-1}{\lambda} = \frac{2^{n-n/2\lambda}}{2\lambda} \geq \frac{2^n/2}{2\lambda} = \frac{1}{8\lambda}$. Thus we have the lower bound as $\Omega\left(\frac{1}{\sqrt{\lambda}}\right)$.

These results lead to our required quantum lower bound.

**Theorem 6.5.** Any quantum algorithm uses $\Omega\left(\frac{1}{\sqrt{\lambda}}\right)$ queries to estimate the non-linearity of a given function with $\lambda$ accuracy.

Next we show the classical randomised complexity of the non-linearity estimation. The adversary method has been extended to randomised algorithms as well which is what we use. Using a theorem given by Laplante et al. [16] in which we use the values obtained in the proof of Theorem 6.5, we obtain the following result. (Complete proof is given in Appendix B.)

**Theorem 6.6.** Any classical randomised algorithm uses $\Omega\left(\frac{1}{\lambda}\right)$ queries to estimate the non-linearity of a given function with $\lambda$ accuracy.

7 **CONCLUSION**

In this paper we looked at the problem of estimating, within additive accuracy $\lambda$, the non-linearity of a Boolean function given to us as a black-box. We devised a quantum strategy that makes $O\left(\frac{1}{\lambda^2}\right)$ queries to the black-box in the worst-case and is independent of the size of the function. In contrast, our classical randomised algorithm makes linear in $n$ many queries apart from depending on $\lambda$ as $\frac{1}{\sqrt{\lambda}}$. We proved a lower bound of $\Omega\left(\frac{1}{\lambda^2}\right)$ on the quantum query complexity and a lower bound of $\Omega\left(\frac{1}{\lambda}\right)$ on the classical randomised query complexity of the non-linearity estimation problem. We conclude with a conjecture that the quantum query complexity of non-linearity estimation is $\Theta\left(\frac{1}{\lambda}\right)$ — that will allow us to construct a sub-exponential query quantum algorithm for exactly computing $\eta(f)$ using only a few additional qubits.
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A AMPLITUDE ESTIMATION AND AMPLIFICATION

In this section, we provide details on the quantum amplitude estimation and amplitude amplification subroutines that are used as part of our algorithms.

A.1 Amplitude Estimation

The amplitude estimation problem is the problem of estimating the probability $p$ of an observation upon measuring the final state of a quantum circuit $A$. Let $k$ and $m$ be some parameters that we shall fix later. A quantum amplitude estimation algorithm (say, named as $AmpEst$) was proposed by Brassard et al. [7] that acts on two registers of $m$ and $n$ qubits, makes $2^m$ calls to controlled-$A$ and outputs a $\hat{p} \in [0, 1]$ that is a good approximation of $p$ in the following sense.

**Theorem A.1.** The $AmpEst$ algorithm returns an estimate $\hat{p}$ that has a confidence interval $|p - \hat{p}| \leq 2\pi k \frac{\sqrt{p(1-p)}}{2^m} + \pi^2 \frac{k^2}{2^{2m}}$ with probability at least $\frac{8}{\pi^2}$ if $k = 1$ and with probability at least $1 - \frac{1}{2(k-1)}$ if $k \geq 2$. If $p = 0$ or $1$ then $\hat{p} = p$ with certainty.

The $AmpEst$ algorithm can be used to estimate $p$ with desired accuracy and error. The following corollary is obtained directly from the above theorem.

**Corollary A.2.** The amplitude estimation algorithm returns an estimate $\hat{p}$ that has a confidence interval $|p - \hat{p}| \leq \frac{1}{2q}$ with probability at least $\frac{8}{\pi^2}$ using $q + 3$ qubits and $2^{q+3} - 1$ queries. If $p = 0$ or $1$ then $\hat{p} = p$ with certainty.

**Proof.** Set $k = 1$ in Theorem A.1. Since $p \leq 1$, we get $\sqrt{p(1-p)} \leq \frac{1}{2}$. Then we have

$$2\pi k \frac{\sqrt{p(1-p)}}{2^m} + \pi^2 \frac{k^2}{2^{2m}} \leq 2\pi \frac{1}{2} \frac{1}{2^m} + \pi^2 \frac{1}{2^m} \leq \frac{\pi^2}{2^m} \leq \frac{2\pi}{2^m} \leq \frac{8}{\pi^2}.$$

The last inequality follows from the fact that $\frac{\pi^2}{2^m} < 1$ (which is true when $m \geq 2$). Now, set $m = q + 3$ to prove the corollary.

Now, let $p_\alpha$ be the probability of obtaining the basis state $|\alpha\rangle$ on measuring the state $|\psi\rangle$. The amplitude estimation circuit referred to above uses an oracle, denoted $O_\alpha$ to mark the "good state" $|\alpha\rangle$, and involves measuring the output of the $AmpEst$ circuit in the standard basis; actually, it suffices to only measure the first register. We can summarise the behaviour of the $AmpEst$ circuit (without the final measurement) in the following lemma.

**Lemma A.3.** Given an oracle $O_x$ that marks $|x\rangle$ in some state $|\psi\rangle$, $AmpEst$ on an input state $|\psi\rangle |0^m\rangle$ generates the following state.

$$AmpEst |\psi\rangle |0^m\rangle \rightarrow \beta_{x,s} |\psi\rangle |\hat{p}_x\rangle + \beta_{x,s} |\psi\rangle |E_x\rangle$$

where $|\beta_{x,s}\rangle^2$, the probability of obtaining the good estimate, is at least $\frac{8}{\pi^2}$, and $|\hat{p}_x\rangle$ is an $m$-qubit normalized state of the form $|\hat{p}_x\rangle = \gamma_+ |\hat{p}_{x,+}\rangle + \gamma_- |\hat{p}_{x,-}\rangle$ such that for $p \in \{\hat{p}_{x,+}, \hat{p}_{x,-}\} = S_{p_x}$ (say), $\sin^2(\pi \frac{p}{2^m})$ approximates $p_x$ up to $m - 3$ bits of accuracy. Further, $|E_x\rangle$ is an $m$-qubit error state (normalized) such that any basis state in $|E_x\rangle$ corresponds to a bad estimate, i.e., we can express it as $|E_x\rangle = \sum_{t \in \{0,1\}^m} \gamma_{t,x} |t\rangle$ in which $|\sin^2(\pi \frac{p}{2^m}) - p_x| > \frac{1}{2^m}$ for any $t \notin S_{p_x}$.

In an alternate setting where the oracle $O_x$ is not provided, $AmpEst$ can still be performed if the basis state $|\alpha\rangle$ is provided — one can construct a quantum circuit, say $EQ$, that takes as input $|\phi\rangle |x\rangle$ and marks the state $|x\rangle$ of the superposition state $|\phi\rangle$ as described in section 4. We name this extended-$AmpEst$ circuit as $EQAmpEst$ which implements the following operation.
EQAmpEst (|x⟩ |ψ⟩ |0^m⟩) → |x⟩ (β_x,s |ψ⟩ |^s x⟩ + β_x,π |ψ⟩ |E_x⟩)

where the notations are as defined above and the quantum circuit EQ is used wherever the oracle O_x was used in the previous setting. In such a scenario, since EQAmpEst is a quantum circuit, we could replace the state |x⟩ by a superposition ∑x α_x |x⟩. We then obtain the following.

**Corollary A.4.** Given an EQ circuit, the EQAmpEst on an input state ∑x α_x |x⟩ |ψ⟩ |0^m⟩ outputs a final state of the form

EQAmpEst ( ∑x α_x |x⟩ |ψ⟩ |0^m⟩ ) → ∑x α_x β_x,s |x⟩ |ψ⟩ |^s x⟩ + ∑x α_x β_x,π |x⟩ |ψ⟩ |E_x⟩.

Notice that on measuring the first and the third registers of the output, with probability |α_x β_x,s|^2 ≥ \(\frac{8}{π^2} |α_x|^2\) we would obtain as measurement outcome a pair |a⟩ |x⟩ where \(\sin^2(\pi \frac{m}{π^2}) = \hat{p} \) is within \(\pm \frac{1}{2m}\) of the probability \(p_x\) of observing the basis state |x⟩ when the state |ψ⟩ is measured. Observe in this setting that the subroutine essentially estimates the amplitude of all the basis states |x⟩. However, with a single measurement we can obtain the information of at most one of the estimates.

### A.2 Amplitude amplification

The amplitude amplification algorithm (AA) is a generalization of the novel Grover’s algorithm. Given an n-qubit algorithm A that outputs the state |ϕ⟩ = ∑k α_k |k⟩ on |0^n⟩ and a set of basis states G = { |a⟩ } of interest, the goal of the amplitude amplification algorithm is to amplify the amplitude α_a corresponding to the basis state |a⟩ for all |a⟩ ∈ G such that the probability that the final measurement output belongs to G is close to 1. In the most general setting, one is given access to the set G via an oracle \(O_G\) that marks all the states |a⟩ ∈ G in any given state |ϕ⟩; i.e., \(O_G\) acts as

\[O_G \sum_k α_k |k⟩ |0⟩ → \sum_{a ∉ G} α_a |a⟩ |0⟩ + \sum_{a ∈ G} α_a |a⟩ |1⟩.\]

Now, for any G, any state |ϕ⟩ = ∑k α_k |k⟩ can be written as

\[|ϕ⟩ = ∑_k α_k |k⟩ = \sin(θ) |v⟩ + \cos(θ) |\overline{v}⟩\]

where \(\sin(θ) = √{∑_{a ∈ G} |α_a|^2}, |v⟩ = \frac{∑_{a ∉ G} α_a |a⟩}{√{∑_{a ∉ G} |α_a|^2}}\) and \(|\overline{v}⟩ = \frac{∑_{a ∈ G} α_a |a⟩}{√{∑_{a ∈ G} |α_a|^2}}\). Notice that the states |v⟩ and \(|\overline{v}⟩\) are normalized and are orthogonal to each other. The action of the amplitude amplification algorithm can then be given as

\[AA\left( ∑_k α_k |k⟩ |0⟩ \right) = AA\left( \sin(θ) |v⟩ + \cos(θ) |\overline{v}⟩ \right) |0⟩ → \sqrt{1 - \beta} |v⟩ |1⟩ + \sqrt{β} |\overline{v}⟩ |0⟩\]

where \(β\) satisfies |β| < \(δ\) and \(δ\) is the desired error probability. This implies that on measuring the final state of AA, the measurement outcome |a⟩ belongs to G with probability \(|1 - β|\) which is at least \(1 - δ\).

### B LOWER BOUND FOR RANDOMISED NON-LINEARITY TESTING

Consider the following theorem from [16].

**Theorem B.1 (Theorem 2 [16]).** Let S and S’ be two sets and let f : S → S’ be a function. Consider a weight scheme where

1. Every pair (x, y) ∈ S × S is assigned a non-negative weight \(w(x, y)\) such that \(w(x, y) = 0\) whenever \(f(x) = f(y)\).
(2) Every tuple \((x, y, i)\) is assigned a non-negative weight \(w'(x, y, i)\) such that \(w'(x, y, i) = 0\) whenever \(f(x) = f(y)\) or \(x_i = y_i\).

Define \(w(x) = \sum_y w(x, y)\) and \(v(x, i) = \sum_y w'(x, y, i)\) for all \(x\) and for all \(i\). If \(w'(x, y, i)w'(y, x, i) \geq w^2(x, y)\) for all \(x, y, i\) such that \(x_i \neq y_i\), then

\[
Q_2(f) = \Omega \left( \min_{x, y, i} \sqrt{\frac{w(x)w(y)}{v(x, i)v(y, i)}} \right).
\]

Moreover, if \(w'(x, y, i)w'(y, x, i) \geq w(x, y)\) for all \(x, y, i\) such that \(x_i \neq y_i\), then

\[
R(f) = \Omega \left( \max_{x, y, i} \min_{x, y, i} \frac{w(x)w(y)}{v(x, i)v(y, i)} \right).
\]

where \(Q_2(f)\) is the bounded error quantum query complexity and \(R(f)\) is the classical randomised query complexity of \(f\).

Define a weight scheme such that

\[
w(x, y) = \begin{cases} 0, & \text{if } f(x) = f(y) \\ 1, & \text{else} \end{cases} \quad ; \quad w'(x, y, i) = \begin{cases} 0, & \text{if } f(x) = f(y) \text{ or } x_i = y_i \\ 1, & \text{else} \end{cases}
\]

Note that in this weight scheme it is always true that \(w'(x, y, i)w'(y, x, i) \geq w^2(x, y)\) and \(w'(x, y, i)w'(y, x, i) \geq w(x, y)\) for all \(x, y, i\) such that \(x_i \neq y_i\). Let \(X\) be a non-empty subset of the set of strings \(x\) such that \(f(x) = 0\) and \(Y\) be a non-empty subset of the set of strings \(y\) such that \(f(y) = 1\) where \(f\) is some decision function which takes inputs of size \(n\). Let \([n] = \{1, \cdots, n\}\). Also let \(S = X \cup Y\) and \(R = X \times Y\). Define \(m_x\) (resp. \(m_y\)) for any \(x \in X\) (resp. \(y \in Y\)) as the number of \(y \in Y\) (resp. \(x \in X\)) such that \((x, y) \in R\). Similarly, for any \(x \in X\) (resp. \(y \in Y\)) and \(i \in [n]\) define \(l_{(x, i)}\) (resp. \(l_{(y, i)}\)) as the number of \(y \in Y\) (resp. \(x \in X\)) such that \((x, y) \in R\) and \(x_i \neq y_i\). From the definition of the weights from Theorem B.1, we can see that \(w(x) = m_x\), \(w(y) = m_y\), \(v(x, i) = l_{(x, i)}\) and \(v(y, i) = l_{(y, i)}\) for any \(x \in X\), \(y \in Y\) and \(i \in [n]\). Adapting the definition from Theorem 6.1, let \(m = \min_x \{m_x\}\), \(m' = \min_y \{m_y\}\), \(l = \max_{x, i} \{l_{(x, i)}\}\) and \(l' = \max_{y, i} \{l_{(y, i)}\}\). Using these definitions, we can explicitly see the equivalence of the forms of \(Q_2(f)\) in Theorem 6.1 and Theorem B.1 as

\[
Q_2(f) = \Omega \left( \min_{x, y, i} \sqrt{\frac{w(x)w(y)}{v(x, i)v(y, i)}} \right) = \Omega \left( \sqrt{\frac{m \cdot m'}{l \cdot l'}} \right).
\]

Since \(X\) and \(Y\) are non-empty and \(R = X \times Y\), we know for any \(y, m_y \geq 1\). Next, since for any \(x \in X\) and \(y \in Y\), we have \(f(x) \neq f(y)\), it is obvious that \(x_i \neq y_i\). So for any \(y \in Y\) there is at least one index \(i \in [n]\) such that \(x_i \neq y_i\) and hence \(l_{(y, i)} \geq 1\). Now, see in the proofs of Theorem 6.3 and Theorem 6.4 that \(m' = l' = 1\). Hence we have \(w(y) = m_y = 1\) and \(v(y, i) = l_{(y, i)} = 1\) for any \(y \in Y\) and \(i \in [n]\). From the proofs of Theorem 6.3 and Theorem 6.4 we also have \(\frac{m}{l} = \min_{x, i} (\frac{m_x}{l_{(x, i)}}) \geq 1\).

Hence \(\frac{m_x}{l_{(x, i)}} \geq 1\) for any \(x \in X\) and \(i \in [n]\) and so max \(\left\{ \frac{w(x)}{v(x, i)}, \frac{w(y)}{v(y, i)} \right\} = \frac{w(x)}{v(x, i)}\). This implies that

\[
R(f) = \Omega \left( \min_{x, y, i} \frac{w(x)}{v(x, i)} \right) = \Omega \left( \frac{m}{l} \right).
\]

leading us the following theorem.
**Theorem B.2.** Any classical randomised algorithm uses $\Omega(\frac{1}{\lambda})$ queries to estimate the non-linearity of a given function with $\lambda$ accuracy.

### C: Alternative Quantum Approaches for BoundFMax

Recall that there are three components that contribute to the complexity of $\text{CBoundFMax}$. The innermost component is the classical estimation with complexity $\tilde{O}(\frac{1}{\epsilon^2 \log \frac{1}{\delta}})$. This is nested inside the loop over all suffixes of any particular length which contributes $O(\frac{1}{\tau - 2\epsilon})$. This in turn is nested inside an outer loop over all levels of the binary tree which gives $O(n)$ to the complexity. Improving the speed of any of the three components will result in an overall speedup. In the algorithms that follow, we tried to replace the existing classical component with a quantum component which has a speedup over its classical counterpart.

#### C.1 CBoundFMax–QPWC

The most naive way to introduce “quantumness” in $\text{CBoundFMax}$ is by replacing the classical estimation of $\text{PWC}$ in Algorithm 2 by a quantum estimation of $\text{PWC}$. The new algorithm which we call $\text{CBoundFMax–QPWC}$ is given as Algorithm 4.

**Algorithm 4 Algorithm CBoundFMax–QPWC**

**Require:** threshold $\tau \in (0, 1)$, confidence $\epsilon \in (0, \tau)$, error $\delta \in (0, 1)$

Initialise a FIFO list $Q = \{\epsilon\}$, where $\epsilon$ denotes the empty string

**while** $Q$ is not empty **do**

remove prefix $p$ from $Q$

**for** suffix $s$ from $\{0, 1\}$ **do**

childprefix $cp = p \overline{s}$

Estimate $e = \sum_x$ with prefix $cp \hat{f}(x)^2$ with accuracy $\epsilon$ and error $(\frac{\tau - 2\epsilon}{2n})\delta$. \(\triangleright\) Lemma 4.1

if $e \geq \tau - \epsilon$ **then**

If $\text{len}(cp) < n$, add $cp$ to $Q$

Else (i.e., $\text{len}(cp) = n$), **return** TRUE

**end if**

**end for**

**end while**

**return** FALSE

The correctness of the algorithm clearly follows from the correctness of Algorithm 2. Now see that the complexity of the quantum estimation can be obtained as $O(\frac{1}{\epsilon^2 \log(\frac{1}{\delta})})$ from Theorem 4.1. The complexity of the classical outer loop is $O(\frac{n}{\epsilon(\tau - 2\epsilon)})$ as derived in Section 3. Hence, the query complexity of $\text{CBoundFMax–QPWC}$ turns out as $O(\frac{n}{\epsilon(\tau - 2\epsilon) \log(\frac{1}{\delta})})$ queries, a slight improvement over the $\text{CBoundFMax}$.

Using $\text{CBoundFMax–QPWC}$ to solve the $\text{BoundFMax}$ problem in Algorithm 1, we obtain a hybrid algorithm of query complexity $O(\frac{1}{\epsilon^2 \log(\frac{1}{\delta})})$ that outputs an interval of length at most $\epsilon$ that contains $\hat{f}_{\text{max}}^2$ with probability at least $1 - \delta$, given $\epsilon$ and $\delta$.

#### C.2 QCBoundFMax

Now we describe another optimisation that changes one more classical operation to a quantum one. Recall that in the level order traversal of the binary tree, at any particular level, there can be at most $\frac{\tau}{\tau - 2\epsilon}$ many prefixes such that the estimate of the $\text{PWC}$ at those points are greater than the threshold. Since the estimation of $\text{PWC}$ at the next level is performed over all points with these
prefixes, the number of estimations due to these prefixes is of the order $O(\frac{1}{\tau^3})$. In this version of the algorithm, which we call QCBoundFMax, we replace the classical level order traversal with a quantum level order traversal and use superposition and amplitude amplification to our benefit.

Our quantum algorithm QCBoundFMax is described in Algorithm 5. We use $R_i$ to represent the $i^{th}$ register used in the quantum circuit corresponding to the algorithm. In the algorithm, we also use a few smaller circuits as described in Section 4.

### Algorithm 5 Algorithm QCBoundFMax

**Require:** Threshold $\tau$, accuracy $\varepsilon$ and error $\delta$.

1. Set $\tau' = \tau - \varepsilon$ and $q = \lceil \log(\frac{1}{\tau'}) \rceil + 3$
2. Set $\tau_1 = \lceil \frac{\pi}{2^n} \sin^{-1}(\sqrt{2}) \rceil$ if $\sin^{-1}(\sqrt{2}) \leq \frac{\pi}{2}$ and $\tau_1 = \lceil \frac{\pi}{2^n} \sin^{-1}(\sqrt{2}) \rceil$ if $\sin^{-1}(\sqrt{2}) > \frac{\pi}{2}$.
3. Initialise the circuit as $|0^n\rangle |0^n\rangle |0^n\rangle |\tau_1\rangle |0^\alpha\rangle$.
4. Apply $H^{\otimes n}$ on $R_1$.
5. For $i$ in $\{1, 2, ..., n\}$ do
6. Apply quantum amplitude estimation ($AmpEst$) on $DJ$ with $i$ qubits of $R_2$ as the input register, $R_3$ as the precision register and $R_1$ is used to determine the “good state”. $AmpEst$ is called with error at most $\delta/2n$ and additive accuracy $\frac{1}{2} \cdot \frac{1}{2^q}$.
7. Use $HD_q$ on $R_3$ and $R_4$ separately.
8. Use $CMP$ on $R_3$ and $R_4$ as input registers and $R_5$ as output register.
9. Apply Fixed Point Amplitude Amplification (FPAA) $\frac{1}{\sqrt{\tau' - 2\varepsilon}}$ times on the $i^{th}$ qubit of $R_5$ with error at most $\delta/2n$ and measure $i^{th}$ qubit of $R_5$ as $m_i$.
10. If $m_i = |0\rangle$ then
11. Return FALSE
12. Else if $m_i = |1\rangle$ and $i = n$ then
13. Return TRUE
14. End if
15. Reset $R_3$ to $|0^i\rangle$ and $R_4$ to $|\tau_1\rangle$ by applying $HD_q^+$ individually on them.
16. End for

One can see that in QCBoundFMax the selection of desired prefixes (i.e., any prefix $p$ such that $PWC(p) \geq \tau$) for the next round is done quite differently from what is used in CBoundFMax-QPWC. While in CBoundFMax-QPWC the desired prefixes for the round $i$ are selected by estimating $PWC$ of each of the prefixes selected at round $i - 1$ one at a time and then comparing them with the threshold, in QCBoundFMax we use a clever combination of amplitude estimation without measurement, amplitude amplification and two constant time subroutines $HD_q$ and $CMP$. The estimation and comparison using $AmpEst$, $HD_q$ and $CMP$ is exactly as explained in Section 4 except that here the estimate is of the amplitude of the prefixes $p$ that were selected in round $i$. Also of difference is that the EQ circuit in the amplitude estimation module here works such that in the $i^{th}$ round, EQ maps the $2n$-qubit basis states $|x, y\rangle$ to $(-1)^j |x, y\rangle$ if $x_j = y_j$ for all $j \in \{0, 1, \cdots, i - 1\}$ where $x_j$ denotes the $j^{th}$ bit of the string $x$ and similarly for $y_j$. Post estimation and comparison, the $i^{th}$ output bit of any $i$-bit prefix $p$ contains $|1\rangle$ if $PWC(p)$ is greater than the threshold and $|0\rangle$ otherwise. We next use amplitude amplification to amplify only those states whose $i^{th}$ output qubit is $|1\rangle$. Since at any level there can be at most $\frac{1}{\sqrt{\tau' - 2\varepsilon}}$ many prefixes which are greater than the threshold, we use FPAA $O(\frac{1}{\sqrt{\tau' - 2\varepsilon}})$ many times to amplify the desired states. This process is repeated in loop over all possible lengths of prefixes. Hence the total query complexity of QCBoundFMax is
\[ O(n) \times O\left(\frac{1}{\sqrt{\tau - 2\epsilon}}\right) \times O(1) \times O\left(\frac{1}{\epsilon} \right) = O\left(\frac{n}{\epsilon \sqrt{\tau - 2\epsilon}}\right) \] queries. Notice that this is an \( O\left(\frac{1}{\sqrt{\tau - 2\epsilon}}\right) \) speedup over CBoundFMax-QPWC.

It easily follows that the use of QCBBoundFMax to solve the BoundFMax problem in Algorithm 1 gives us a quantum algorithm that given additive accuracy \( \epsilon \) and error probability \( \delta \) outputs an interval of length at most \( \epsilon \) that contains \( \hat{f}_\text{max}^2 \) with probability \( 1 - \delta \) using \( O\left(\frac{n^3}{\epsilon^{3/2} \log \frac{1}{\delta}}\right) \) queries to the oracle of the function.

Finally, observe that by cleverly setting the initial state and ignoring the first \( n - 1 \) rounds in QCBBoundFMax algorithm we obtain Algorithm 3 with a better query complexity.

### D. SUBROUTINES USED IN QCBBoundFMax ALGORITHM

In this section we present the algorithms of the \( \text{HD}_q \), CMP and Cond-MAJ\(_y \) subroutines used in Algorithm 3. The string representations of the integers are indexed in the reverse order, i.e., the bit-representation of an integer \( z \) is represented as \( z = z_0 z_1 z_2 \ldots \) where \( z_0 \) denotes the most significant bit of \( z \). We have used the following quantum operations in those algorithms.

- \( C\text{X}(a,b) \) : Controlled-NOT gate where \( X \) gate is applied on qubit \( b \) if the control qubit \( a \) is in the state \( |1\rangle \).
- \( \overline{C}\text{X}(a,b) \) : Controlled-NOT gate where \( X \) gate is applied on qubit \( b \) if the control qubit \( a \) is in the state \( |0\rangle \).
- \( C\overline{C}^i\text{X}(a,b_1,b_2,\ldots,b_i,c) \) : Multi-controlled NOT gate where \( X \) gate is applied on qubit \( c \) if the first control qubit \( a \) is in the state \( |1\rangle \) and the next \( i \) control qubits \( b_1,b_2,\ldots,b_i \) are in the state \( |0\rangle \).
- \( \text{SWAP}(A,B) \) : Swap gate that swaps register \( A \) with register \( B \) given that the registers \( A \) and \( B \) are of the same size.

#### D.1 Quantum circuit for \( \text{HD}_q \) operation

The \( \text{HD}_q \) operation was defined as \( |y\rangle |b\rangle \mapsto |b \oplus \tilde{y}\rangle |y\rangle \) where \( y, b \in \{0, 1\}^q \) and \( \tilde{y} \) is the bit string corresponding to the integer \( 2^{q-1} - y_{int} \). An algorithm to implement this operator is described in Algorithm 6. We also show the quantum circuit corresponding to \( \text{HD}_q \) in Figure 4. Note that the register \( a \) is an ancilla register.

The algorithm behind \( \text{HD}_q \) is based on bitwise manipulations. We use \( y \) to denote the input integer \( (0 \leq y < 2^q) \), available in the first register. We will focus on obtaining \( ||y - 2^{q-1}|| \) in a register. The
Algorithm 6 Subroutine HDₜₚ

1: Initialise R₁R₂R₃ = |y⟩|0ⁿ⟩|0⟩.
2: for i in {0, 1, 2, · · · , n − 1} do
3:   Apply CX(R₁[i], R₂[i]).
4: end for
5: Apply CX(R₁[0], R₃).
6: Apply CX(R₃, R₂[0]).
7: Initialise an empty array S.
8: for i in {0, 1, 2, · · · , n − 1} do
9:   Append R₂[n − i − 1] to S.
10: for k in {1, n − i − 2} do
11:   Apply ³⁻¹CX(R₃, S[0], S[1], · · · , S[i], R₂[k]).
12: end for
13: end for
14: Apply ³₀X(R₁[0], R₁[1], · · · , R₁[n − 1], R₂[0]).
15: Apply CX(R₁[0], R₃).
16: Apply SWAP(R₁, R₂).

simple case is y ≥ 2⁵⁻¹ when we need to simply set the most significant bit of y to 0.⁵ For the other case, i.e., y < 2⁵⁻¹, we have to subtract y (available in binary) from 2⁵⁻¹ (1 followed by q − 1 zeroes). The algorithm implements this by first finding out the rightmost bit of y that is set to 1, denoted t; all bits to the right of t are 0 and do not contribute to the difference. The t-th bit of the difference is set to 1. And all the bits to the left of t are flipped (they are subtracted from 1 created by the cascading borrowing effect when the t-bit of the difference was calculated)⁶. And of course, if all the bits of y are 0, then we just flip the first bit of the answer.

D.2 Quantum circuit for CMP operation

Next, we present the algorithm for the subroutine CMP in Algorithm 7. Figure 5 contains the circuit for the CMP subroutine with x as the first input and y as the second input. It is to be noted that the register a in the circuit is an ancilla register. The state of the register a before and after the subroutine is applied is |0ⁿ⟩. The out register contains the result of the comparison in the form |x ≤ y⟩.

The algorithm applies the standard method of comparing two bit-strings. It compares the k-th bit of x and y, starting from the most significant bit and moving right, until it finds some k such that xₖ ≠ yₖ. For this position, the answer bit is flipped (i.e., we claim that x > y) if xₖ = 1, yₖ = 0; otherwise we claim that x ≤ y. We also start the answer qubit in |1⟩, so if x ≤ y, then the answer qubit remains in the state |1⟩.

---
⁵For instance if y = 111, then ÿ = 100 − 111 = 011.
⁶For instance, if we have y = 0110, then we need 10000 − 00110. The first bit from the right that is set to one is the second least significant bit. Then all bits between the second least significant bit and the most significant bit is flipped. So we have 10000 − 00110 = 01010.
Algorithm 7 Subroutine CMP

1. Initialise $R_1 R_2 R_3 R_4 = |x\rangle |y\rangle |0^n\rangle |0\rangle$.
2. Apply $X(R_4)$.
3. Initialise an empty array $A$.
4. for $i$ in $\{0, 1, \ldots, n-1\}$ do
   5. Apply $C^{n+1} X(R_1[i], R_2[i], A[0], A[1], \ldots, A[i-1], R_3[n-1])$.
   6. if $i \neq n-1$ then
      7. Apply $CX(R_1[i], R_2[i])$.
      8. Apply $CX(R_2[i], R_3[i])$.
      9. Apply $CX(R_1[i], R_2[i])$.
   10. end if
   11. Append $R_3[i]$ to array $A$.
4. end for
12. Apply $CX(R_3[n-1], R_4)$.
14. Apply the conjugate transpose of all the operations from line 4 to line 12.

D.3 Quantum circuit for Cond-MAJ$_y$ operation

Let $X_1 \ldots X_k$ be Bernoulli random variables with success probability $p > 1/2$. Let $Maj$ denote their majority value (that appears more than $k/2$ times). A common result that is often used, and can be proved easily using Chernoff’s bound \footnote{$\Pr[\sum X_i > \frac{k}{2}] \geq 1 - \exp(-\frac{1}{2p}n(p - \frac{1}{2})^2)$}, is that $Maj$ has a success probability at least $1 - \delta$, for any given $\delta$, if we choose $k \geq \frac{2p}{(p-1/2)^2} \ln \frac{1}{\delta}$. We require a quantum formulation of the same.

Suppose we have $k$ copies of the quantum state $|\phi\rangle = \sum_x \alpha_x |x\rangle \left( |\psi_{0,x}\rangle |0\rangle + |\psi_{1,x}\rangle |1\rangle \right)$ in which we define "success" as observing $|1\rangle$ in the last register (without loss of generality) of $|\phi\rangle$ given that the first register is $|y\rangle$. Let $p = \| |\psi_{1,y}\rangle \|^2$ denote the probability of success and let $p > 1/2$. Now, define the conditional majority Cond-MAJ$_y$ as the operator $U_y$ that performs the MAJ operator on the third registers of each copy of $|\phi\rangle$ if the corresponding first register in that copy is $|y\rangle$ and stores the output in the second register of an answer state if the first register of the answer state is $|y\rangle$.
Suppose we apply $U_y$ on $|\phi\rangle \otimes \sum_x \hat{f}(x) |x\rangle |0\rangle$ and we measure the answer register after applying $U_y$. Then it is easy to show, essentially using the same analysis as above, that

$$U_y |\phi\rangle \otimes |0\rangle = \alpha_y |y\rangle \left( |\Gamma_{0,y}\rangle |0\rangle + |\Gamma_{1,y}\rangle |1\rangle \right) + \gamma$$

in which $\|\Gamma_{1,y}\| ^2 \geq 1 - \delta$ and $\gamma = \sum_{x \neq y} \alpha_x |x\rangle |\phi\rangle \otimes |0\rangle$.

The Cond-MAJ$_y$ operator can be implemented without additional queries and with poly$(k)$ gates and log$(k)$ qubits. The algorithm for the implementation is presented in Algorithm 8.

**Algorithm 8 Subroutine Cond-MAJ$_y$**

1. Let $t = \lceil \log(k) \rceil$ and $l = \lfloor k/2 \rfloor$.
2. Define $R_1 = R_{x_1}R_{c_1} = |x_i\rangle |c_i\rangle$ and $n = |y|$.
3. Initialise $R_1 \cdots R_l R'_1 \cdots R'_t R''_1 R''_t$ $= \left( \bigotimes_{i=1}^{k} |x_i\rangle |c_i\rangle \right) |0^{\otimes t}\rangle |l-1\rangle |0\rangle$.
4. for $i$ in $\{1, \ldots, k\}$ do
5. for $j$ in $\{0, \ldots, n-1\}$ do
6. Apply $X(R_{x_i}[j])$ if $j^{th}$ bit of $y$ equals 1.
7. end for
8. for $m$ in $\{1, \ldots, t\}$ do
9. Apply $C^0 \cdots C^{t+1-m}X(R_{x_1}, R_{c_1}, R'_t, R'_t, R''_1, \cdots, R''_t)$.
10. end for
11. end for
12. Apply CMP$(R', R'', R_3)$ and apply $X(R_3)$.
13. Apply the conjugate transpose of all the operations from line 4 to line 11.

### E PROOF OF PROPOSITION 4.3

**Proposition 4.3.** $\tau$ and $\tau_1$ satisfy $0 \leq \tau - \epsilon - \sin^2 \left( \frac{\tau_1}{2q} \right) \leq \frac{2\tau}{2q}$.

**Proof.** We defined $\tau' = \tau - \epsilon$ and $\tau_1 = \lfloor \frac{2q}{\pi} \cdot \sin^{-1}(\sqrt{\tau'}) \rfloor$. Then,

$$\tau_1 \leq \frac{2q}{\pi} \cdot \sin^{-1}(\sqrt{\tau'})$$

$$\Rightarrow \sin \left( \frac{\pi}{2q} \tau_1 \right) \leq \sqrt{\tau'}$$

$$\Rightarrow \sin^2 \left( \frac{\pi}{2q} \tau_1 \right) = p_{\tau_1} \leq \tau'$$

$$\Rightarrow \tau' - p_{\tau_1} \geq 0$$

On the other hand, we have,

$$\tau_1 > \lfloor (2q/\pi) \cdot \sin^{-1}(\sqrt{\tau'}) \rfloor - 1$$

$$\Rightarrow \sin \left( \frac{\pi}{2q} \tau_1 \right) > \sin \left( \sin^{-1}(\sqrt{\tau'}) - \frac{\pi}{2q} \right)$$

$$= \sqrt{\tau'} \cos \left( \frac{\pi}{2q} \right) - \cos \left( \sin^{-1}(\sqrt{\tau'}) \right) \sin \left( \frac{\pi}{2q} \right)$$

$$= \sqrt{\tau'} \cos \left( \frac{\pi}{2q} \right) - (\sqrt{1 - \tau'}) \sin \left( \frac{\pi}{2q} \right) \quad (\text{Uses } \cos(\sin^{-1}(x)) = 1 - x^2)$$

$$\Rightarrow p_{\tau_1} = \sin^2 \left( \frac{\pi \tau_1}{2q} \right)$$
\[ \geq \tau' \cos^2\left(\frac{\pi}{2q}\right) + (1 - \tau') \sin^2\left(\frac{\pi}{2q}\right) - 2\sqrt{\tau'} \sqrt{1 - \tau'} \cos\left(\frac{\pi}{2q}\right) \sin\left(\frac{\pi}{2q}\right) \]

\[ \geq \tau' \cos^2\left(\frac{\pi}{2q}\right) - (1 - \tau') \sin^2\left(\frac{\pi}{2q}\right) - 2\sqrt{\tau'} \sqrt{1 - \tau'} \cos\left(\frac{\pi}{2q}\right) \sin\left(\frac{\pi}{2q}\right) \]

\[ = \tau' - \sin^2\left(\frac{\pi}{2q}\right) - \sqrt{\tau'} \sqrt{1 - \tau'} \sin\left(\frac{2\pi}{2q}\right) \]

Now, notice that,

\[ \sin^2\left(\frac{\pi}{2q}\right) + \sqrt{\tau'} \sqrt{1 - \tau'} \sin\left(\frac{2\pi}{2q}\right) \]

\[ \leq \sin^2\left(\frac{\pi}{2q}\right) + \frac{1}{2} \sin\left(\frac{2\pi}{2q}\right) \quad \text{(Uses } x(1 - x) \leq 1/4) \]

\[ = \sin^2\left(\frac{\pi}{2q}\right) + \sin\left(\frac{\pi}{2q}\right) \cos\left(\frac{\pi}{2q}\right) \]

\[ = \sin\left(\frac{\pi}{2q}\right) \left(\sin\left(\frac{\pi}{2q}\right) + \cos\left(\frac{\pi}{2q}\right)\right) \]

\[ \leq \frac{2\pi}{2q} \]

The last inequality is due to the fact that \(\sin \theta \leq \theta\), \(\sin(\theta) \leq 1\) and \(\cos(\theta) \leq 1\). This gives us the following:

\[ p_{\tau_1} \geq \tau' - \sin^2\left(\frac{\pi}{2q}\right) - \sqrt{\tau'} \sqrt{1 - \tau'} \sin\left(\frac{2\pi}{2q}\right) \]

\[ > \tau' - \frac{2\pi}{2q} \]

So we have

\[ 0 \leq \tau' - p_{\tau_1} \leq \frac{2\pi}{2q} \]

\[ \square \]