BACKWARD UNIQUENESS AND THE EXISTENCE OF THE SPECTRAL LIMIT FOR SOME PARABOLIC SPDES

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ABSTRACT. The aim of this article is to study the asymptotic behaviour for large times of solutions to a certain class of stochastic partial differential equations of parabolic type. In particular, we will prove the backward uniqueness result and the existence of the spectral limit for abstract SPDEs and then show how these results can be applied to some concrete linear and nonlinear SPDEs. For example, we will consider linear parabolic SPDEs with gradient noise and stochastic NSEs with multiplicative noise. Our results generalize the results proved in [11] for deterministic PDEs.

CONTENTS
1. Introduction and formulation of the main results 1
2. Proof of the Theorems 1.2 and 1.5 on the backward uniqueness for SPDEs 5
3. Proof of Theorem 1.7 on the existence of a spectral limit 21
4. Applications and Examples 25
4.1. Backward Uniqueness for Linear SPDEs. 25
4.2. Backward Uniqueness for SPDEs with a quadratic nonlinearity 27
4.3. Existence of the spectral limit. 28
Appendix A. Some useful known results 28
References 30

1. INTRODUCTION AND FORMULATION OF THE MAIN RESULTS

The question of uniqueness of solutions to both deterministic and stochastic, both ordinary and partial, differential equations, is quite fairly well understood. There are plenty of positive results and there are some counterexample. This area is too vast to make any attempt of listed relevant papers. The question of backward uniqueness is equivalent to classical (i.e. forward) uniqueness in the case of ordinary differential equations. In the case of stochastic differential equations the backward uniqueness is closely related to the question of existence of a stochastic flow. In fact, the latter implies the former and the latter has been extensively studied since the pioneering works by Blagovescenski and Freidlin [1]. However, parabolic equations can only be solved forward and the backward uniqueness is completely unrelated to the forward uniqueness. To our knowledge, first results on backward uniqueness are due to Lees and Protter [14] and Mizohata [17]. This has been followed by a long series of papers, often using very different approaches, see e.g. Ghidaglia [11] and Escauriaza et al [8]. Primary applications of backward uniqueness is the study of long time behaviour of the solutions but there are also natural applications to control theory, see e.g. [16]. As in the case of PDEs also in the case of stochastic PDEs, the existence of a flow does not implies backward uniqueness. Furthermore, there are only few known examples of SPDEs which have flows, see e.g. [7], [3], [18] and references therein. Hence the question of backward uniqueness for SPDEs of parabolic type is even more important that the similar question...
for deterministic parabolic PDEs. Possible applications are paramount, let us just mention the most obvious: long time behaviour of solutions and control theory. The backward uniqueness we prove should be applicable to study regularity of the local spectral manifolds constructed in Flandoli-Schaumlöffel, c.f. Ruelle [23], Foias-Saut [10] and [2].

To the best of our knowledge our paper is the first one in which such an important question is being investigated. As mentioned in the abstract, in our paper we generalize the results proved in [11] for deterministic PDEs. One of the difficulties with extending the results from [11] to the stochastic case is that the standard Itô formula is not directly applicable to the case considered in this article. We use certain approximations to overcome this problem.

Let us now briefly present construction of the paper. At the end of this Introduction we will present notation, assumptions and the main results. In section 2 we state the proof of the Theorem about backward uniqueness of SPDEs. The argument is based on stochastic version of logarithmic convexity approach. The proof is separated in the series of Lemmas for convenience of the reader. Proof of the Lemma [2,3] can be omitted in the first reading. Section 3 contains proof of the Theorem about existence of spectral limit. The main difference of the proof comparing to backward uniqueness Theorem is that we use comparison Theorem for one dimensional diffusions to derive main a priori estimate. In section 4 we present examples of application of our result. It includes quite wide class of linear SPDEs and one example of application to 2D NSE with multiplicative noise of special form. In appendix we collect some auxiliary results applied in the proofs.

Assume that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ is a complete filtered probability space and $(w_t)_{t \geq 0}$ is an $\mathbb{R}^n$-valued $\mathbb{F}$-Wiener process, where $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$. We assume that

\begin{equation}
V \subset H \cong H' \subset V'
\end{equation}

is a Gelfand triple of Hilbert spaces. The norm in $V$, respectively $H$, will be denoted by $\| \cdot \|$, respectively by $| \cdot |$. The scalar product in $H$ (resp. $V$) will be denoted by $(\cdot, \cdot)_H$ (resp. $(\cdot, \cdot)_V$) and the duality pairing between $V'$ and $V$ will be denoted by $\langle \cdot, \cdot \rangle_{V', V}$. We will omit the indexes where no uncertainty appears. The Banach space of trace class operators in $H$ will be denoted $T_1(H)$.

We assume that $A(t), t \in [0, \infty)$ is a family of bounded linear operators from $V$ to $V'$ such that the sets $D(A(t)) =: D(A), t \in [0, \infty)$ are independent of time and $(B_k(t))_{k=1}^n, t \in [0, \infty)$ is a family of bounded linear operators both from $V$ to $H$ and from $H$ to $V'$. Let us set

\begin{equation}
\tilde{A}(t) = A(t) - \frac{1}{2} \sum_k B_k(t)^* B_k(t), t \in [0, T].
\end{equation}

We will assume that the sets $D(\tilde{A}(t)) =: D(\hat{A}), t \in [0, \infty)$ are independent of time. Here $\hat{A} : D(\hat{A}) \to H$ is selfadjoint strictly positive operator defined by

$$(\hat{A}u, v)_H := (u, v)_V, u, v \in V.$$ 

Then, see [15], p. 9-10, $V = D(\hat{A}^{1/2})$ and there exists an orthonormal basis $\{e_i\}_{i \geq 1} \subset D(\hat{A})$ of eigenvectors of $\hat{A}$ in $H$. For $N \in \mathbb{N}$ let $P_N : H \to H$ be the orthogonal projection onto the space $H_N = \text{lin}\{e_1, \ldots, e_N\}$ and let $Q_N = I - P_N$. We can notice that $P_N : V \mapsto V$ and $\|P_N\|_{\mathcal{L}(V, V)} \leq 1$. Denote $P_N' : V' \to V'$ adjoint operator to $P_N$ w.r.t. duality between $V$ and $V'$. Define a linear operator $\tilde{A}_N = P_N' \tilde{A} P_N$.

We assume that a map

$$F : [0, T] \times V \to V'$$

is such that it maps $[0, T] \times D(A)$ to $H$.

If $X$ is a separable Hilbert space then by $\mathcal{M}^p(0, T; X), p \geq 1$, we will understand the space of all progressively measurable stochastic processes $\xi : [0, T] \times \Omega \to X$, or rather their equivalence classes,
such that
\[ \mathbb{E} \int_0^T |\xi(s)|^p_X \, ds < \infty. \]

**Definition 1.1.** A progressively measurable \( H \)-valued stochastic process \( u(t), t \geq 0 \) is a solution of the problem

\[
\begin{aligned}
du(t) + (A(t)u(t) + F(t, u(t))) \, dt + \sum_{k=1}^n B_k(t)u(t) \, dw^k(t) &= 0, \quad t \geq 0 \\
u(0) &= u_0
\end{aligned}
\]

if and only if for each \( T > 0 \) \( u \in \mathcal{M}^2(0, T; V) \cap L^2(\Omega, C([0, T], H)) \) and for any \( t \in [0, \infty) \), the equality

\[
u(t) = u_0 - \int_0^t (A(s)u(s) \, ds + F(s, u(s))) \, ds - \sum_{k=1}^n \int_0^t B_k(s)u(s) \, dw(s)
\]
is satisfied \( \mathbb{P} \)-a.s..

The following assumptions will be used throughout the paper.

\(\textbf{(AC0)}\) The maps \( A : [0, T] \rightarrow \mathcal{L}(V, V') \) and \( B^k : [0, T] \rightarrow \mathcal{L}(V, H), k = 1, \ldots, n \) are strongly measurable and bounded, i.e. for each \( v \in V \), the functions \( [0, T] \ni t \mapsto A(t)v \in V' \) and \( [0, T] \ni t \mapsto B^k(t)v \in H, k = 1, \ldots, n \) are measurable and bounded.

\(\textbf{(AC1)}\) There exists a map \( \tilde{A}' : [0, T] \rightarrow \mathcal{L}(V, V') \) such that for all \( \phi \in V \) and \( \psi \in V' \), \( \langle \tilde{A}'(\cdot)\phi, \psi \rangle \in L^1(0, T) \) and, in the weak sense,

\[
\frac{d}{dt} \langle \tilde{A}'(\cdot)\phi, \psi \rangle = \langle \tilde{A}'(\cdot)\phi, \psi \rangle.
\]

\(\textbf{(AC2)}\) There exists constants \( \alpha > 0 \) and \( \lambda \in \mathbb{R} \) such that for all \( t \in [0, T] \),

\[
2\langle A(\cdot)u, u \rangle + \lambda |u|^2 \geq \alpha \|u\|^2 + \sum_{k=1}^n |B_k(\cdot)u|^2, \quad u \in V.
\]

\(\textbf{(AC3)}\) There exists a function \( \phi \in L^\infty(0, T) \) such that

\[
\sum_{k=1}^n |\langle u, B_k(\cdot)u \rangle| \leq \phi(\cdot)|u|^2, \quad \text{for all } t \in [0, T], \ u \in V.
\]

\(\textbf{(AC4)}\) There exist functions \( K_1 \in L^2(0, T) \) and \( K_2 \in L^1(0, T) \) such that \( K_2 \geq 0 \) and

\[
C(t) = \sum_{k=1}^n B_k(t)^* [\tilde{A}(t), B_k(t)] \leq K_1(t) I + K_2(t) \tilde{A}(t), \quad \text{for all } t \in [0, T],
\]

where \( I \) is the identity operator.

\(\textbf{(AC5)}\) There exists constants \( L_1, L_2 > 0 \) such that

\[
\sum_k |B_k(\cdot)x| \leq L_1 |A(\cdot)x| + L_2 |x|, \quad \text{for all } t \in [0, T], \ x \in D(A).
\]

\(\textbf{(AC6)}\) There exist constants \( \beta, \gamma > 0 \) such that for every \( t \in [0, T] \) we have

\[
|\langle A(\cdot)x, x \rangle| \leq \beta \|x\|^2 + \gamma |x|^2, \quad x \in V.
\]
Theorem 1.5. Assume that maps $A : [0, T] \to \mathcal{L}(V, V')$ and $B^k : [0, T] \to \mathcal{L}(V, H)$, $k = 1, \ldots, n$ satisfy the assumptions (AC0)-(AC6).

Assume that $u_0 \in H$ and that a process $u$ satisfies the following conditions. There exist constants $\delta_0 > 0$ and $\kappa > 2 + \frac{1}{\delta_0}$ such that

$$u \in L^{2 + \delta_0}(\Omega, C([0, T], V)), \quad u \in \mathcal{M}^2(0, T; D(\hat{A})).$$

There exists a progressively measurable process $\eta$ such that

$$\mathbb{E} \int_0^\kappa \left( \int_0^T n^2(s) \, ds \right) < \infty,$$

$$|F(t, u(t))| \leq n(t)\|u(t)\|, \text{ for a.a. } t \in [0, T].$$

Assume that $u$ is a solution of problem (1.3). If $u(T) = 0$ $\mathbb{P}$-a.s., then $u(t) = 0$, $\mathbb{P}$-a.s. for all $t \in [0, T]$.

Remark 1.3. It is well known that under some appropriate assumptions on $F$ the problem (1.3) has a unique solution see e.g. Theorem 1.4, p.140 in [20] for the case $F = 0$.

Remark 1.4. The Assumption (1.14) is satisfied if, e.g. $n \in L^2(0, T)$ is a deterministic function.

Theorem 1.5. Let us assume that maps $A : [0, T] \to \mathcal{L}(V, V')$ and $B^k : [0, T] \to \mathcal{L}(V, H)$, $k = 1, \ldots, n$ satisfy the assumptions (AC0)-(AC7). Furthermore, assumption (AC4) is satisfied with $K_1 = 0$.

Assume that $u_0 \in H$ and that a process $u$ satisfies the following conditions. There exist a constant $\delta_0 > 0$ such that

$$u \in L^{2 + \delta_0}(\Omega, C([0, T], V)), \quad u \in \mathcal{M}^2(0, T; D(\hat{A})).$$

There exists a progressively measurable process $\eta$ such that

$$\int_0^T n^2(s) \, ds < \infty \text{ a.a.},$$

$$|F(t, u(t))| \leq n(t)\|u(t)\|, \text{ for a.a. } t \in [0, T].$$

Assume that $u$ is a solution of problem (1.3). If $u(T) = 0$ $\mathbb{P}$-a.s., then $u(t) = 0$, $\mathbb{P}$-a.s. for all $t \in [0, T]$.

Corollary 1.6. Under the assumptions of Theorem 1.2 or Theorem 1.5 either $u(t) = 0$, $\mathbb{P}$-a.s. for all $t \in [0, T]$ or $|u(t)| > 0$, $\mathbb{P}$-a.s. for all $t \in [0, T]$.

We will use in the following Theorem the same notation as in Theorem 1.5.
**Theorem 1.7.** Let us assume that maps $A : [0, \infty) \to \mathcal{L}(V, V')$ and $B^k : [0, \infty) \to \mathcal{L}(V, H), k = 1, \ldots, n$ satisfy the assumptions (AC0)-(AC7) on any finite interval. Furthermore, we assume that the assumptions (AC1), (AC3), (AC4) and (AC7) are satisfied globally on $[0, \infty)$ and assumption (AC4) is satisfied with parameter $K_1 = 0$.

Suppose that $u$ is a unique solution of problem (1.3) with $u(0) \neq 0$. We also assume that there exist a progressively measurable process $n$ and a measurable set $\Omega' \subset \Omega$, $\mathbb{P}(\Omega') = 1$ such that for all $\omega \in \Omega'$, $n(\cdot, \omega) \in L^2(T_0, \infty)$ and

$$ (1.20) \quad |F(t, u(t))| \leq n(t)\|u(t)\|, \text{ for a.a. } t \in [T_0, \infty). $$

Assume that

$$ (1.21) \quad u \in \mathcal{M}_{loc}^2(0, \infty; D(\tilde{A})), $$

$$ (1.22) \quad \forall T > 0 \quad \mathbb{E} \sup_{0 \leq t \leq T} \|u(t)\|^2 + \delta_0 < \infty. $$

Then there exists a measurable map $\tilde{\Lambda} : \Omega \to \sigma(\tilde{A})$ such that $\mathbb{P}$-a.s.

$$ \lim_{t \to \infty} \frac{\langle \Lambda(u), u(t) \rangle}{\|u(t)\|^2} = \tilde{\Lambda}. $$

2. Proof of the Theorems 1.2 and 1.3 on the Backward Uniqueness for SPDEs

**Proof of Theorem 1.2.** We will argue by contradiction. Suppose that the assertion of the Theorem is not true. Then, because the process $u$ is adapted, we will be able to find $t_0 \in [0, T)$, an event $R \in \mathcal{F}_{t_0}$ and a constant $c > 0$ such that $\mathbb{P}(R) > 0$ and

$$ (2.1) \quad |u(t_0, \omega)| \geq c > 0, \omega \in R. $$

Without loss of generality we can assume that $\mathbb{P}(R) = 1$ and $t_0 = 0$. Otherwise, we can consider instead of measure $\mathbb{P}$ the conditional measure $\mathbb{P}_R := \frac{\mathbb{P}(\cdot \cap R)}{\mathbb{P}(R)}$.

Suppose that there exists a constant $c > 0$ and a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that $\mathbb{E}_\mathbb{Q}|u(t)|^2 \geq c$, for all $t \in [0, T]$.

Then, by taking $t = T$, we infer that $\mathbb{E}_\mathbb{Q}|u(T)|^2 > 0$ what is a clear contradiction with the assumption that $u(T) = 0$, $\mathbb{P}$-a.s..

Now we shall prove that such a measure exists.

For this let us fix $\delta \geq 0$, $k = 1, \ldots, n$ and let us define progressively measurable processes

$$ \rho^\delta_k(s) = \frac{\langle u(s), B_k u(s) \rangle}{\|u(s)\|^2 + \delta}, \quad s \in [0, T]. $$

Because of the assumption (1.7), $|\rho^\delta_k| \leq \phi$ and since $\phi \in L^\infty(0, T) \subset L^2(0, T)$ we infer that $\rho^\delta_k \in \mathcal{M}(0, T; \mathbb{R})$. Therefore, a process

$$ \sum_{k=1}^n \int_0^t \rho^\delta_k(s) \, dw^k_s, \quad t \in [0, T], $$

is square integrable $\mathbb{R}$-valued martingale which satisfies condition 5.7 from [12] Theorem 5.3 p.142]. Hence the process $M_\delta = (M_\delta(t))_{t \in [0, T]}$ defined by

$$ (2.2) \quad M_\delta(t) = \exp(-2 \sum_k \int_0^t \rho^\delta_k(s) \, dw^k(s) - 2 \int_0^t \sum_k |\rho^\delta_k(s)|^2 \, ds), \quad t \in [0, T], $$

\footnote{Here and below by $\mathbb{E}_\mathbb{Q}$ we will denote the mathematical expectation w.r.t. the measure $\mathbb{Q}$.}
satisfies $\mathbb{E}M_\delta(t) = 1$ for all $t \geq 0$ and

$$dM_\delta(t) = -2M_\delta(t)\sum_k \frac{\langle u, B_k u \rangle}{|u|^2 + \delta} dw^k(t), \quad t \in [0, T].$$

Therefore, $M_\delta$ is a continuous square integrable martingale. The above allows us to define a probability measure $Q^\delta$ by

$$\frac{dQ^\delta}{dp} = M_\delta(T).$$

Next let us fix $\varepsilon > 0$ and define a process $\psi^\varepsilon$ by

$$\psi^\varepsilon(t) = -\frac{1}{2}M_\varepsilon(t) \log (|u(t)|^2 + \varepsilon), \quad t \in [0, T].$$

Now we will prove the following result.

**Lemma 2.1.** The process $\psi^\varepsilon$ defined in (2.4) is an Itô process.

**Proof of Lemma 2.1.** First, by invoking Theorem 1.2 from Pardoux [20], we will show that $\log (|u(t)|^2 + \varepsilon)$ is an Itô process. For this we need to check that the assumptions of that result for the function $R : H \ni x \mapsto \log(|x|^2 + \varepsilon) \in \mathbb{R}$ and the process $u$ are satisfied. Obviously the function $R$ is of $C^2$-class and since

$$R'(x)h = \frac{2\langle x, h \rangle}{|x|^2 + \varepsilon}, \quad x, h \in H,$$

$$R''(x)(h_1, h_2) = \frac{2}{|x|^2 + \varepsilon} (\langle h_2, h_1 \rangle - \frac{2\langle x, h_2 \rangle \langle x, h_1 \rangle}{|x|^2 + \varepsilon}), \quad x, h_1, h_2 \in H,$$

the 1st and 2nd derivatives of $R$ are bounded and hence the assumptions (i) and (ii) of [20, Theorem 1.2] are satisfied. Since the embedding $V \hookrightarrow H$ is continuous, we infer that the assumptions (iv) and (v) of [20, Theorem 1.2] are satisfied as well. Moreover, for any $Q \in \mathcal{T}_1(H)$, we have

$$Tr(Q \circ R''(x)) = \frac{2}{|x|^2 + \varepsilon} (TrQ - \frac{2\langle Qx, x \rangle}{|x|^2 + \varepsilon}), \quad x \in H$$

and hence the map $H \ni x \mapsto Tr(Q \circ R''(x)) \in \mathbb{R}$ is continuous. Thus also the condition (iii) in [20, Theorem 1.4] is satisfied. Therefore, $\log (|u(t)|^2 + \varepsilon), \; t \geq 0$, is an Itô process and

$$-d\frac{1}{2} \log (|u(s)|^2 + \varepsilon)) = \sum_{k=1}^n \frac{\langle u, B_k u \rangle}{|u|^2 + \varepsilon} dw^k(s)$$

$$+ \left( \sum_{k=1}^n \frac{\langle u, B_k u \rangle^2}{(|u|^2 + \varepsilon)^2} + \frac{\langle (A - \frac{1}{2} \sum_{k=1}^n B_k^* B_k)u + F(s, u), u \rangle}{(|u|^2 + \varepsilon)^2} \right) ds.$$ 

(2.5)

Secondly, since $M_\varepsilon(t), \; t \geq 0$, is a continuous square integrable martingale satisfying equality (2.3), the process $\psi^\varepsilon(t) = -\frac{1}{2}M_\varepsilon(t) \log (|u(t)|^2 + \varepsilon), \; t \in [0, T]$, is an Itô process and

$$d\psi^\varepsilon(t) = -\frac{1}{2} \left( \log (|u|^2 + \varepsilon) dM_\varepsilon(t) + M_\varepsilon d\log (|u|^2 + \varepsilon) \right)$$

$$+ d\langle M_\varepsilon, \log (|u|^2 + \varepsilon) \rangle_t.$$

(2.6)

This completes the proof the Lemma.

\footnote{Note that the measure $Q^\delta$ is also well defined by this formula.}
Let us define functions \( \bar{\Lambda}_\varepsilon \) and \( \bar{\Lambda}_\varepsilon^F \) by

\[
\bar{\Lambda}_\varepsilon(u) = \frac{\langle (A - \frac{1}{2} \sum_{k=1}^n B_k^* B_k)u, u \rangle}{|u|^2 + \varepsilon} + \sum_{k=1}^n \frac{\langle u, B_k u \rangle^2}{(|u|^2 + \varepsilon)^2}, \quad u \in V,
\]

\[
\bar{\Lambda}_\varepsilon^F(t, u) = \bar{\Lambda}_\varepsilon(u) + \frac{\langle F(t, u), u \rangle}{|u|^2 + \varepsilon}, \quad t \in [0, T], \quad u \in V, \varepsilon \geq 0.
\]

We will omit index \( \varepsilon \) if \( \varepsilon = 0 \).

Combining equalities (2.6) and (2.8) we infer that

\[
du(s) = \frac{1}{2} V, \varepsilon (s, u(s)) ds + \sum_{k=1}^n \frac{\langle u, B_k u \rangle}{|u|^2 + \varepsilon} dw^k(s)
\]

It follows from assumption (1.7) that the integrands in the stochastic integrals in (2.9) belong to \( \mathcal{M}^2(0, T; \mathbb{R}) \). Therefore, we can apply mathematical expectation to (2.9) and consequently, by Assumption (1.7) we infer that for all \( t \in [0, T] \),

\[
\frac{1}{2} \mathbb{E} M_\varepsilon(0) \log (|u(0)|^2 + \varepsilon) - \frac{1}{2} \mathbb{E} M_\varepsilon(t) \log (|u(t)|^2 + \varepsilon) \leq C_1 + C_2 \int_0^t \mathbb{E} M_\varepsilon(s) \bar{\Lambda}_\varepsilon^F(s, u(s)) ds.
\]

Suppose now that the following result is true.

**Lemma 2.2.** In the above framework we have

\[
\sup_{\varepsilon > 0} \int_0^T \mathbb{E} M_\varepsilon(s) \bar{\Lambda}_\varepsilon^F(s, u(s)) ds = \sup_{\varepsilon > 0} \int_0^T \mathbb{E} \bar{\Lambda}_\varepsilon^F(s, u(s)) ds < \infty.
\]

Then, in conjunction with (2.10) and (A.1) we have that

\[
\mathbb{E}_\varepsilon \log (|u(t)|^2 + \varepsilon) \geq \mathbb{E}_\varepsilon \log (|u(0)|^2 + \varepsilon) - K = \mathbb{E} \log (|u(0)|^2 + \varepsilon) - K, \quad t \in [0, T],
\]

where \( K = 2C_1 + 2C_2 \sup_{\varepsilon > 0} \int_0^T \mathbb{E}_\varepsilon \bar{\Lambda}_\varepsilon^F(s, u(s)) ds \). Hence by the Jensen inequality

\[
\mathbb{E}_\varepsilon (|u(t)|^2 + \varepsilon) = \mathbb{E}_\varepsilon e^{\log (|u(t)|^2 + \varepsilon)} \geq e^{\mathbb{E}_\varepsilon \log (|u(t)|^2 + \varepsilon)} \geq e^{\mathbb{E} \log (|u(0)|^2 + \varepsilon) - K}.
\]

Therefore, by the Fatou Lemma and (2.1), we infer that

\[
\liminf_{\varepsilon \to 0} e^{\mathbb{E} \log (|u(0)|^2 + \varepsilon) - K} \geq e^{\mathbb{E} \log (|u(0)|^2) - K} > 0.
\]

Combining (2.11) and (A.1) we get

\[
\varepsilon + \mathbb{E} [M_\varepsilon(t) |u(t)|^2] = \mathbb{E}_\varepsilon (|u(t)|^2 + \varepsilon) \geq e^{\mathbb{E} \log (|u(0)|^2) - K} > 0
\]

Choose \( \varepsilon = \frac{1}{2} e^{\mathbb{E} \log (|u(0)|^2) - K} \). Since \( u \) is an \( \mathbb{F} \)-adapted process we get by (A.1) and (2.13)

\[
\mathbb{E}_\varepsilon |u(t)|^2 = \mathbb{E} [M_\varepsilon(t) |u(t)|^2] \geq \frac{1}{2} e^{\mathbb{E} \log (|u(0)|^2) - K} > 0,
\]

what contradicts our assumption that \( u(t) = 0 \) and the Theorem follows. \( \square \)
Hence, it only remains to prove Lemma 2.2. The proof of Lemma 2.2 will be preceded by the following auxiliary results:

**Lemma 2.3.** *In the above framework there exists set \( \Omega' \subset \Omega, \mathbb{P}(\Omega') = 1 \) and sequence \( \{N_l\}_{l=1}^{\infty} \) such that for all \( \omega \in \Omega', t \in [0, T] \), as \( l \to \infty \), the following holds*

\[
\int_0^t M_\varepsilon(s) \frac{|(\tilde{A}_t - \tilde{A})u|^2}{|u|^2 + \varepsilon} \, ds \to 0,
\]

\[
\int_0^t M_\varepsilon(s) \left| \sum_{k=1}^n Q_k B_k u \right|^2 \, ds \to 0,
\]

\[
\frac{< (\tilde{A}_{N_l} - \tilde{A})u(t), u(t) >}{|u(t)|^2 + \varepsilon} \to 0,
\]

\[
\int_0^t M_\varepsilon(s) \left| \sum_{k=1}^n \frac{< (\tilde{A}_{N_l} - \tilde{A})u(s), B_k u(s) >}{|u(s)|^2 + \varepsilon} \right| \, du^k(s) \to 0.
\]

**Proof of Lemma 2.3.** It is enough to prove that for each of convergences (2.15)-(2.18) we can find \( \Omega'_i \subset \Omega, i = 1, \ldots, 4 \) of measure 1 such that the corresponding convergence holds. Denote \( v_N = P_N v, v \in H \). Then, for \( t \in [0, T] \), the following inequalities holds

\[
\int_0^t M_\varepsilon(s) \frac{|(\tilde{A}_N - \tilde{A})u|^2}{|u|^2 + \varepsilon} \, ds \leq \int_0^T M_\varepsilon(s) \frac{|(\tilde{A}_N - \tilde{A})u|^2}{|u|^2 + \varepsilon} \, ds \leq \frac{\sup_{s \in [0, T]} M_\varepsilon(s)}{\varepsilon} \int_0^T |(\tilde{A}_N - \tilde{A})u|^2 \, ds,
\]

and

\[
\int_0^T |(\tilde{A}_N - \tilde{A})u|^2 \, ds \leq \left( \int_0^T |P_N \tilde{A}(P_N u - u)|^2 \, ds + \int_0^T |(P_N^* - 1)\tilde{A}u|^2 \, ds \right) \leq \left( |P_N u - u|_{L^2(0, T; D(\tilde{A}))}^2 + |Q_N \tilde{A}u|^2_{L^2(0, T; H)} \right).
\]

Since \( M_\varepsilon \) is a square integrable martingale, then by Doob inequality there exist a set \( \Omega'_1 \subset \Omega \) of measure 1 such that \( \sup_{s \in [0, T]} M_\varepsilon(s, \omega) < \infty, \omega \in \Omega' \). Moreover by Assumption (1.13) there exist \( \Omega_2 \subset \Omega \) of measure 1 such that

\[
u(\cdot, \omega) \in L^2(0, T; D(\tilde{A})), \omega \in \Omega_2.
\]

Therefore, if \( \omega \in \Omega'_1 = \Omega_1 \cap \Omega_2 \), then by the Lebesgue Dominated Convergence Theorem,

\[
2 \sup_{s \in [0, T]} \frac{M_\varepsilon(s, \omega)}{\varepsilon} \left( |P_N u(\omega) - u(\omega)|_{L^2(0, T; D(\tilde{A}))}^2 + |Q_N \tilde{A}u(\omega)|_{L^2(0, T; H)}^2 \right) \to 0, N \to \infty.
\]

Consequently, by inequalities (2.19) and (2.20) we infer that

\[
\int_0^t M_\varepsilon(s) \frac{|(\tilde{A}_N - \tilde{A})u|^2}{|u|^2 + \varepsilon} \, ds \to 0, N \to \infty, \omega \in \Omega_1 \cap \Omega_2, t \in [0, T].
\]

This concludes the proof of (2.15).
Similarly to (2.19) we get

\[
\int_0^T M_\varepsilon(s) \frac{k}{|u|^2 + \varepsilon} ds \leq \sup_{s \in [0,T]} M_\varepsilon(s, \omega) \frac{T}{\varepsilon} \sum_{k=1}^n \|Q_N B_k u\|^2 ds.
\]

If \( \omega \in \Omega_2 \) then by assumptions (1.9) and (2.21) that \( B_k u(\omega) \in L^2(0, T; V) \), \( k = 1, \ldots, n \). Hence, by the Lebesgue Dominated Convergence Theorem for \( \omega \in \Omega_1 \cap \Omega_2 \),

\[
\sup_{s \in [0,T]} M_\varepsilon(s, \omega) \frac{T}{\varepsilon} \sum_{k=1}^n \|Q_N B_k u\|^2 ds \to 0, \text{ as } N \to \infty.
\]

Therefore, by combining (2.24) and (2.25), we infer that

\[
\int_0^T M_\varepsilon(s) \frac{k}{|u|^2 + \varepsilon} ds \to 0, N \to \infty, \omega \in \Omega_1 \cap \Omega_2, t \in [0, T].
\]

This proves (2.16) with \( \Omega'_2 = \Omega'_1 \).

From Assumption (1.12) by the Lebesgue Dominated Convergence Theorem and Dini’s Theorem we infer that

\[
\mathbb{E} \sup_{s \in [0,T]} \|u_m(s) - u(s)\|_V^{1+\delta_0} \to 0, m \to \infty
\]

Therefore, by Chebyshev inequality, \( \sup_{s \in [0,T]} \|u_m(s) - u(s)\|_V^{1+\delta_0} \) converges to 0 as \( m \to \infty \) in probability. Consequently, there exist sequence \( \{m_k\}_{k=1}^\infty \) and set \( \Omega_3 \subset \Omega, \mathbb{P}(\Omega_3) = 1 \) such that

\[
\sup_{s \in [0,T]} \|u_{m_k}(s, \omega) - u(s, \omega)\|_V \to 0, k \to \infty, \omega \in \Omega_3.
\]

Now we shall prove that (2.17) holds with the subsequence \( \{m_l\}_{l=1}^\infty \) defined above and \( \omega \in \Omega_3 \). The following sequence of inequalities holds for \( v \in V, t \in [0, T] \)

\[
\frac{|<\tilde{A}_m(t) - \tilde{A}(t)>v, v>}{|v|^2 + \varepsilon} \leq \frac{1}{\varepsilon} \left( |<\tilde{A}(t)v_m, v_m>-<\tilde{A}(t)v, v>| + |<\tilde{A}(t)v_m, v>-<\tilde{A}(t)v, v>| \right)
\]

\[
\leq \frac{1}{\varepsilon} \left( |<\tilde{A}(t)v_m, v_m>-<\tilde{A}(t)v_m, v>+|<\tilde{A}(t)v_m, v>-<\tilde{A}(t)v, v>| \right)
\]

\[
\leq \frac{1}{\varepsilon} \left( |<\tilde{A}(t)v, v_m-v>+|<\tilde{A}(t)(v_m-v), v>| \right)
\]

\[
\leq \frac{\tilde{A}(t)\mathcal{L}(V, V')}{\varepsilon} (||v_m||_V ||v_m-v||_V+||v||_V ||v_m-v||_V) \leq \frac{2\tilde{A}(t)\mathcal{L}(V, V')}{\varepsilon} ||v||_V ||v_m-v||_V
\]

Therefore, by (2.28), Assumption (1.12) and Assumption (AC0) we infer that

\[
\lim_{l \to \infty} \sup_{s \in [0,T]} \frac{|<\tilde{A}_m(t) - \tilde{A}(t)>u(s), u(s)> - |<\tilde{A}(t)u(s), u(s)>|}{|u(s)|^2 + \varepsilon} = 0, \omega \in \Omega_3.
\]

This proves (2.17).

It remains to prove convergence in (2.18). By Assumption (1.13) and the Lebesgue Dominated Convergence Theorem there exist a subsequence \( \{M_l\}_{l=1}^\infty \) of the sequence \( \{m_k\}_{k=1}^\infty \) such that

\[
|u_{M_l} - u|_{\mathcal{M}^2(0,T; D(\tilde{A}))} + |P_{M_l} \tilde{A}(\cdot)u(\cdot) - \tilde{A}(\cdot)u(\cdot)|_{\mathcal{M}^2(0,T; H)} \leq \frac{1}{l}
\]
Let us observe that \( \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_3} = 1 \). Therefore, by Hölder and Burkholder inequalities, see e.g. Corollary iv.4.2 in [22], we have

\[
\begin{align*}
\mathbb{E} \left[ \sup_{t \in [0,T]} \int_0^t M_\varepsilon(s) \sum_{k=1}^n \frac{< (\tilde{A}_M - \tilde{A})u(s), B_ku(s) >}{|u(s)|_H^2 + \varepsilon} \, du^k(s) \right] \\
\leq C \mathbb{E} \left( \int_0^T M_\varepsilon^2(s) \sum_{k=1}^n \frac{|< (\tilde{A}_M - \tilde{A})u(s), B_ku(s) >|^2}{(|u(s)|_H^2 + \varepsilon)^2} \, ds \right)^{1/2} \\
\leq \frac{C}{\varepsilon} \mathbb{E} \left[ \sup_{s \in [0,T]} \left( M_\varepsilon(s) \sum_{k=1}^n |B_ku(s)|^2 \right)^{1/p_1} \left( \int_0^T |(\tilde{A}_M - \tilde{A})u(s)|^2 \, ds \right)^{1/2} \right] \\
\leq \frac{C}{\varepsilon} \left( \mathbb{E} \sup_{s \in [0,T]} M_\varepsilon^{p_1}(s) \right)^{1/p_1} \left( \mathbb{E} \sup_{s \in [0,T]} \left( \sum_{k=1}^n |B_ku(s)|^2(s) \right)^{p_2/2} \right)^{1/p_2} \\
\times \left( \mathbb{E} \left( \int_0^T |(\tilde{A}_M - \tilde{A})u|^2(s) \, ds \right)^{p_3/2} \right)^{1/p_3} \\
= \frac{C}{\varepsilon} \left( \mathbb{E} \sup_{s \in [0,T]} M_\varepsilon^{p_1}(s) \right)^{1/p_1} \left( \mathbb{E} \sup_{s \in [0,T]} \left( \sum_{k=1}^n |B_ku(s)|^2(s) \right)^{(2+\delta_0)/2} \right)^{1/p_2} \\
\times \left( \mathbb{E} \int_0^T |(\tilde{A}_M - \tilde{A})u|^2(s) \, ds \right)^{1/2} \\
\end{align*}
\]

(2.32)

Next we will show that the RHS of (2.32) is finite. Notice that the first factor on the RHS of (2.32) is finite by the Doob inequality and Assumption (1.7). Furthermore, the second term is finite by assumptions (1.6), (1.10) and (1.12). Now we will find the upper bound for the last term in the product \( W \). We have

\[
\begin{align*}
\int_0^T \mathbb{E}|(\tilde{A}_M - \tilde{A})u|^2(s) \, ds \\
\leq C \left( \int_0^T \mathbb{E}|P_{M^*}^* \tilde{A}P_{M^*} u - P_{M^*}^* \tilde{A}u|^2(s) \, ds + \int_0^T \mathbb{E}|P_{M^*}^* \tilde{A}u - \tilde{A}u|^2(s) \, ds \right) \\
\leq C \left( \int_0^T \mathbb{E}|\tilde{A}P_{M^*} u - \tilde{A}u|^2(s) \, ds + \int_0^T \mathbb{E}|Q_{M^*} \tilde{A}u|^2(s) \, ds \right) \\
\end{align*}
\]

(2.33)

\[
= C \left( |u_{M_t} - u|_{M^2(0,T;D(\tilde{A}))}^2 + |Q_{M_t} \tilde{A}u|_{M^2(0,T;H)}^2 \right) \leq \frac{C}{T^2}
\]
where last inequality follows from assumption (2.31). Combining (2.32) and (2.33) we infer that

\[ \mathbb{E} \sup_{t \in [0,T]} \left| \int_0^t M_\varepsilon(s) \sum_{k=1}^n \frac{(\hat{A}_{M_k} - \hat{A}) u(s), B_k u(s)}{|u(s)|^2_H + \varepsilon} \, dw_k(s) \right| \leq C \frac{1}{t^2}. \tag{2.34} \]

By Borel-Cantelli Lemma and Doob inequality we infer from inequality (2.34) that there exist \( \Omega_4, \mathbb{P}(\Omega_4) = 1 \) such that

\[ \int_0^t M_\varepsilon(s) \sum_{k=1}^n \frac{(\hat{A}_{M_k} - \hat{A}) u(s), B_k u(s)}{|u(s)|^2_H + \varepsilon} \, dw_k(s) \to 0, \quad l \to \infty, \quad \omega \in \Omega_4, \quad t \in [0, T]. \tag{2.35} \]

Put \( \Omega' = \bigcap_{i=1}^4 \Omega_i \). Combining convergence results (2.23), (2.26), (2.30) and (2.35) we prove the Lemma with sequence \( \{M_i\}_{i=1}^\infty \) and space \( \Omega' \).

**Proof of Lemma 2.2** We have by the assumptions (1.6), (1.14), (1.15) the following chain of inequalities

\[
\int_0^t \mathbb{E}_{Q^\varepsilon} \tilde{A}_\varepsilon(s, u(s)) \, ds \leq \int_0^t \mathbb{E}_{Q^\varepsilon} \tilde{A}_\varepsilon(u(s)) \, ds + \mathbb{E}_{Q^\varepsilon} \int_0^t \frac{n(s)|u||u|}{|u|^2 + \varepsilon} \, ds \\
\leq \int_0^t \mathbb{E}_{Q^\varepsilon} \tilde{A}_\varepsilon(u(s)) \, ds + (\mathbb{E}_{Q^\varepsilon} \int_0^t n^2(s) \, ds)^{1/2} (\mathbb{E}_{Q^\varepsilon} \int_0^t \frac{|u|^2||u||^2}{(|u|^2 + \varepsilon)^2} \, ds)^{1/2} \\
\leq \int_0^t \mathbb{E}_{Q^\varepsilon} \tilde{A}_\varepsilon(u(s)) \, ds + C(\mathbb{E}_{Q^\varepsilon} \int_0^t \frac{|u|^2}{|u|^2 + \varepsilon} \, ds)^{1/2} \\
\leq \int_0^t \mathbb{E}_{Q^\varepsilon} \tilde{A}_\varepsilon(u(s)) \, ds + C(\int_0^t \mathbb{E}_{Q^\varepsilon} \tilde{A}_\varepsilon(u(s)) - \lambda ds)^{1/2}.
\]

Therefore, it is enough to estimate from above the term \( \mathbb{E}_{Q^\varepsilon} \tilde{A}_\varepsilon(u(s)) \). Because of the assumption (1.7) we have only to consider the following function \( \tilde{A}_\varepsilon(u(t)), \ t \geq 0, \) where

\[
\tilde{A}_\varepsilon(u) = \frac{(A - \frac{1}{2} \sum_{k=1}^n B_k^* B_k) u, u}{|u|^2 + \varepsilon} = \frac{\langle \hat{A} u, u \rangle}{|u|^2 + \varepsilon}, \quad u \in V.
\]

We will prove that

\[ \sup_t \mathbb{E}_{Q^\varepsilon} \tilde{A}_\varepsilon(u(t)) < \infty. \tag{2.36} \]

Since we cannot directly apply the Itô formula to the function \( \tilde{A}_\varepsilon \) we will consider finite dimensional its approximations.

For this aim define function \( \tilde{A}_\varepsilon^N(u) := \frac{\langle \hat{A}_N u, u \rangle}{|u|^2 + \varepsilon}, \ u \in H \). Then, by the Lemma [A.10] applied with \( C = \hat{A}_N, \tilde{A}_\varepsilon^N \) is of \( C^2 \) class and it has bounded 1st and 2nd derivatives. By the Itô formula and Lemma
The above equality (2.38) can be rewritten as
\[
\begin{align*}
\langle \tilde{A} N u, du \rangle & \quad \rightarrow \quad \langle \tilde{A} N u, u \rangle \quad \rightarrow \quad \langle \tilde{A} N B_k u, B_k u \rangle \quad \rightarrow \quad \langle \tilde{A} N u, B_k u \rangle \quad \rightarrow \quad \langle \tilde{A} N u, u \rangle \langle \tilde{A} N u, B_k u \rangle^2 \quad \rightarrow \quad \langle \tilde{A} N u, B_k u \rangle^2 \quad \rightarrow \quad \langle \tilde{A} N u, u \rangle \langle \tilde{A} N u, B_k u \rangle^2.
\end{align*}
\]

Because \( u \) is a solution of problem (1.3) we have, still with \( u = u(t) \),
\[
\begin{align*}
\langle \tilde{A} N u, u \rangle & \quad \rightarrow \quad \langle \tilde{A} N B_k u, B_k u \rangle \quad \rightarrow \quad \langle \tilde{A} N u, u \rangle \langle \tilde{A} N u, B_k u \rangle^2 \quad \rightarrow \quad \langle \tilde{A} N u, B_k u \rangle^2 \quad \rightarrow \quad \langle \tilde{A} N u, u \rangle \langle \tilde{A} N u, B_k u \rangle^2.
\end{align*}
\]
Therefore,
\[
\begin{align*}
\langle \tilde{A} N u, u \rangle & \quad \rightarrow \quad \langle \tilde{A} N B_k u, B_k u \rangle \quad \rightarrow \quad \langle \tilde{A} N u, u \rangle \langle \tilde{A} N u, B_k u \rangle^2 \quad \rightarrow \quad \langle \tilde{A} N u, B_k u \rangle^2 \quad \rightarrow \quad \langle \tilde{A} N u, u \rangle \langle \tilde{A} N u, B_k u \rangle^2.
\end{align*}
\]

The above equality (2.38) can be rewritten as
\[
\begin{align*}
\langle \tilde{A} N u, u \rangle & \quad \rightarrow \quad \langle \tilde{A} N B_k u, B_k u \rangle \quad \rightarrow \quad \langle \tilde{A} N u, u \rangle \langle \tilde{A} N u, B_k u \rangle^2 \quad \rightarrow \quad \langle \tilde{A} N u, B_k u \rangle^2 \quad \rightarrow \quad \langle \tilde{A} N u, u \rangle \langle \tilde{A} N u, B_k u \rangle^2.
\end{align*}
\]
The drift term on the right hand side of (2.39) can be written as:

\[
(2.40) \quad M_\varepsilon \left( 2\tilde{\Lambda}^N - \frac{\tilde{\Lambda}^N u}{|u|^2 + \varepsilon} \langle \tilde{A}u, u \rangle - 2\tilde{\Lambda}^N \frac{\langle F(t, u), u \rangle}{|u|^2 + \varepsilon} - 2\tilde{\Lambda}^N \frac{\langle \tilde{A}N u, F(t, u) \rangle}{|u|^2 + \varepsilon} + \langle C_N u, u \rangle \right)
\]

\[
= M_\varepsilon \left( -2\frac{\tilde{\Lambda}^N u}{|u|^2 + \varepsilon} - 2\tilde{\Lambda}^N \frac{\langle \tilde{A}N - \tilde{\Lambda}^N, u \rangle}{|u|^2 + \varepsilon} + \langle C_N u, u \rangle \right),
\]

where \( C_N = \sum_{k=1}^n B_k^* [\tilde{A}, B_k] + \tilde{A}'_N (\cdot) \). The first term on the right hand side (2.40) can be rewritten as:

\[
-2M_\varepsilon \frac{\tilde{\Lambda}^N u}{|u|^2 + \varepsilon} = M_\varepsilon \left( -2\frac{\tilde{\Lambda}^N u}{|u|^2 + \varepsilon} - 2\frac{\langle \tilde{A}N - \tilde{\Lambda}^N, u \rangle}{|u|^2 + \varepsilon} \right)
\]

Therefore we have

\[
M_\varepsilon(t)\tilde{\Lambda}^N(u(t)) + 2 \int_0^t M_\varepsilon(s) \frac{\langle \tilde{A}N - \tilde{\Lambda}^N, u \rangle}{|u|^2 + \varepsilon} ds + 2\varepsilon \int_0^t M_\varepsilon(s) \frac{\tilde{\Lambda}^N u}{|u|^2 + \varepsilon} ds
\]

\[
= M_\varepsilon(0)\tilde{\Lambda}^N(u(0)) - 2 \int_0^t M_\varepsilon(s) \frac{\langle \tilde{A}N - \tilde{\Lambda}^N, u \rangle}{|u|^2 + \varepsilon} ds - 2 \int_0^t M_\varepsilon(s) \frac{\langle \tilde{A}N - \tilde{\Lambda}^N, F(s, u) \rangle}{|u|^2 + \varepsilon} ds
\]

\[
+ \int_0^t M_\varepsilon(s) \frac{\langle C_N u, u \rangle}{|u|^2 + \varepsilon} ds - \int_0^t M_\varepsilon(s) \sum_{k=1}^n 2\frac{\tilde{\Lambda}^N u, B_k u}{|u|^2 + \varepsilon} dw^k(s) = (i) + (ii) + (iii) + (iv) + (v).
\]

Next we shall deal with estimating the term (ii) above. By the Young inequality we have, with \( \varepsilon_1 < \frac{1}{2} \),

\[
\int_0^t M_\varepsilon(s) \frac{\langle \tilde{A}N - \tilde{\Lambda}^N, u \rangle}{|u|^2 + \varepsilon} ds \leq \varepsilon_1 \int_0^t M_\varepsilon(s) \frac{\tilde{\Lambda}^N u}{|u|^2 + \varepsilon} ds + \frac{C}{\varepsilon_1} \int_0^t M_\varepsilon(s) \frac{\langle \tilde{A}N - \tilde{\Lambda}^N, u \rangle}{|u|^2 + \varepsilon} ds.
\]
A similar method can be applied to the term (iii). As a result, we get

\[
M_\varepsilon(t) \tilde{\Lambda}_N^\varepsilon(u(t)) + \int_0^t M_\varepsilon(s) \frac{|\tilde{\Lambda}_N - \tilde{\Lambda}_N^\varepsilon u|^2}{|u|^2 + \varepsilon} ds + 2 \varepsilon \int_0^t M_\varepsilon(s) \frac{(\Lambda_N^\varepsilon)^2}{|u|^2 + \varepsilon} ds
\]

\[
\leq M_\varepsilon(0) \tilde{\Lambda}_N^\varepsilon(0) + \frac{C}{\varepsilon} \int_0^t M_\varepsilon(s) \frac{|(\tilde{A} - \tilde{\Lambda}_N)u|^2}{|u|^2 + \varepsilon} ds + \frac{C}{\varepsilon^2} \int_0^t M_\varepsilon(s) \frac{|F(s, u)|^2}{|u|^2 + \varepsilon} ds
\]

(2.41)

\[
+ \int_0^t M_\varepsilon(s) \frac{(C_N u, u)}{|u|^2 + \varepsilon} ds - \int_0^t M_\varepsilon(s) \sum_{k=1}^n \frac{2(\tilde{\Lambda}_N u, B_k u)}{|u|^2 + \varepsilon} dw^k(s).
\]

Let us estimate the term (iv). By the assumption AC4 and the definition of operator C_N we have the following chain of inequalities:

\[
|\langle C_N x, x \rangle| \leq |\langle C x, x \rangle| + |\tilde{\Lambda}_N(\cdot)\|_{L(V^\prime,V)}\|x\|^2 + \sum_k (\tilde{\Lambda}_N - \tilde{\Lambda})B_k x, B_k x)\]

\[
+ |\langle (\tilde{\Lambda}_N - \tilde{\Lambda}) x, \sum_k B_k^2 B_k x \rangle| \leq K_1(\cdot)|x|^2 + (K_2(\cdot) + |\tilde{\Lambda}'(\cdot)\|_{L(V^\prime,V')}\langle \tilde{\Lambda} x, x \rangle|
\]

\[
+ \sum_k \langle \tilde{\Lambda} Q_N B_k x, Q_N B_k x \rangle + |(\tilde{\Lambda}_N - \tilde{\Lambda})x|_{D(A)} \leq K_1(\cdot)|x|^2
\]

\[
+ (K_2(\cdot) + |\tilde{\Lambda}'(\cdot)\|_{L(V^\prime,V')}\langle \tilde{\Lambda} x, x \rangle) + |(\tilde{\Lambda}_N - \tilde{\Lambda})x|_{D(A)} + \sum_k \|Q_N B_k x\|^2.
\]

where assumption (1.8) has been used in second inequality and assumption (1.10) in last inequality. Therefore

(2.42)

\[
\int_0^t M_\varepsilon(s) \frac{\langle C_N u, u \rangle}{|u|^2 + \varepsilon} ds \leq \int_0^t K_1(s) M_\varepsilon(s) ds + \int_0^t (K_2(s) + |\tilde{\Lambda}'(s)|_{L(V^\prime,V')} M_\varepsilon(s) \tilde{\Lambda}_N^\varepsilon(u(s)) ds
\]

\[
+ \left( \int_0^t M_\varepsilon(s) \frac{|(\tilde{\Lambda}_N - \tilde{\Lambda})u|^2}{|u|^2 + \varepsilon} ds \right)^{1/2} \left( \int_0^t M_\varepsilon(s) \frac{|\tilde{\Lambda} u|^2}{|u|^2 + \varepsilon} ds \right)^{1/2} + \int_0^t M_\varepsilon(s) \frac{\sum_k \|Q_N B_k u\|^2}{|u|^2 + \varepsilon} ds.
\]

We will denote

\[
K_3(\varepsilon, N) = \int_0^t M_\varepsilon(s) \frac{|(\tilde{\Lambda}_N - \tilde{\Lambda})u|^2}{|u|^2 + \varepsilon} ds,
\]

\[
K_4(\varepsilon, N) = \int_0^t \sum_k \frac{\|Q_N B_k u\|^2}{|u|^2 + \varepsilon} ds,
\]

\[
K_5(\varepsilon) = \int_0^t M_\varepsilon(s) \frac{|\tilde{\Lambda} u|^2}{|u|^2 + \varepsilon} ds,
\]

\[
K_6(s) = |\tilde{\Lambda}'(s)|_{L(V^\prime,V')}.
\]
Combining (2.41), (2.42) with the assumption (1.15) we get

\begin{equation}
M_\varepsilon(t)\tilde{\Lambda}_\varepsilon (u(t)) + \int_0^t M_\varepsilon(s) \frac{\tilde{A}_N - \tilde{\Lambda}_\varepsilon u}{|u|^2 + \varepsilon} \, ds + 2\varepsilon \int_0^t M_\varepsilon(s) \frac{(\Lambda_\varepsilon N)^2}{|u|^2 + \varepsilon} \, ds
\leq M_\varepsilon(0)\tilde{\Lambda}_\varepsilon (0) + \int_0^t K_1(s)M_\varepsilon(s) \, ds + \frac{C}{\varepsilon} K_3(\varepsilon, N) + (K_3(\varepsilon, N)K_5(\varepsilon))^{1/2} + K_4(\varepsilon, N)
\end{equation}

\begin{equation}
+ \frac{C}{\varepsilon_2} \int_0^t (n^2(s) + K_2(s))M_\varepsilon(s)\Lambda_\varepsilon N(u(s)) \, ds - \int_0^t M_\varepsilon(s) \sum_{k=1}^n \frac{2\langle \tilde{A}_N u, B_k u \rangle}{|u|^2 + \varepsilon} \, dw^k(s)
\end{equation}

By assumption (1.13) we can find \( \Omega_5 \subset \Omega, \mathbb{P}(\Omega_5) = 1 \) such that

\begin{equation}
K_5(\varepsilon) < \infty, u \in L^2(0, T; D(\tilde{A})), \omega \in \Omega_5.
\end{equation}

By Lemma 2.3 we can find \( \Omega' \subset \Omega, \mathbb{P}(\Omega') = 1 \) and sequence \( \{M_\varepsilon\}_{\varepsilon=1}^\infty \) such that \( \forall \omega \in \Omega', t \in [0, T] \)

\begin{equation}
limit_{\varepsilon \to 0} K_3(\varepsilon, M_\varepsilon(t)) = 0, \quad \lim_{\varepsilon \to 0} K_4(\varepsilon, M_\varepsilon(t)) = 0,
\end{equation}

\begin{equation}
\lim_{\varepsilon \to 0} \int_0^t M_\varepsilon(s) \sum_{k=1}^n \frac{2\langle \tilde{A}_M u, B_k u \rangle}{|u|^2 + \varepsilon} \, dw^k(s) = \int_0^t M_\varepsilon(s) \sum_{k=1}^n \frac{2\langle \tilde{A}_u u, B_k u \rangle}{|u|^2 + \varepsilon} \, dw^k(s).
\end{equation}

Therefore, applying (2.43) with subsequence \( \{M_\varepsilon\}_{\varepsilon=1}^\infty \) and tending \( t \to \infty \) in equality (2.43) we infer that

\begin{equation}
M_\varepsilon(t)\tilde{\Lambda}_\varepsilon (t) \leq M_\varepsilon(0)\tilde{\Lambda}_\varepsilon (0) + \int_0^t K_1(s)M_\varepsilon(s) \, ds + \frac{C}{\varepsilon_2} \int_0^t (n^2(s) + K_2(s))M_\varepsilon(s)\Lambda_\varepsilon N(u(s)) \, ds - \int_0^t M_\varepsilon(s) \sum_{k=1}^n \frac{2\langle \tilde{A}_u u, B_k u \rangle}{|u|^2 + \varepsilon} \, dw^k(s), \omega \in \Omega_5 \cap \Omega'.
\end{equation}

Let \( X_\varepsilon = (X_\varepsilon(t))_{t \geq 0} \)—solution of the following SDE

\begin{equation}
X_\varepsilon(t) = M_\varepsilon(0)\tilde{\Lambda}_\varepsilon (0) + \int_0^t K_1(s)M_\varepsilon(s) \, ds + \frac{C}{\varepsilon_2} \int_0^t (n^2(s) + K_2(s) + K_6(s))X_\varepsilon(s) \, ds
\end{equation}

\begin{equation}
- \sum_{k=1}^n \int_0^t M_\varepsilon(s) \frac{2\langle \tilde{A}_u u, B_k u \rangle}{|u|^2 + \varepsilon} \, dw^k(s)
\end{equation}

Then, we have that a.a.

\begin{equation}
M_\varepsilon(t)\tilde{\Lambda}_\varepsilon (t) + \int_0^t M_\varepsilon(s) \frac{(|\tilde{A} - \tilde{\Lambda}_\varepsilon| u)^2}{|u|^2 + \varepsilon} \, ds \leq X_\varepsilon(t).
\end{equation}
Indeed, it is enough to subtract from the inequality (2.46) the identity (2.47) and then use the Gronwall Lemma. On the other hand equation (2.47) can be solved explicitly. It’s unique solution is given the following formula explicitly and we have

\[(2.49) \quad X_\varepsilon(t) = M_\varepsilon(0) \tilde{A}_\varepsilon(0)e^0 \int_0^t (n^2(s) + K_2(s) + K_6(s)) \, ds \]
\[+ \int_0^t e^s \int (n^2(\tau) + K_2(\tau) + K_6(\tau)) \, d\tau \quad K_1(s) M_\varepsilon(s) \, ds \]
\[- \sum_{k=1}^n \int_0^t e^s \int (n^2(\tau) + K_2(\tau) + K_6(\tau)) \, d\tau \quad M_\varepsilon(s) 2\langle \tilde{A}_\varepsilon B_k, u \rangle \frac{2}{|u|^2 + \varepsilon} \, dw^k(s). \]

Denote

\[(2.50) \quad L_\varepsilon(t) = \sum_{k=1}^n \int_0^t e^s \int (n^2(\tau) + K_2(\tau) + K_6(\tau)) \, d\tau \quad M_\varepsilon(s) 2\langle \tilde{A}_\varepsilon B_k, u \rangle \frac{2}{|u|^2 + \varepsilon} \, dw^k(s), \quad t \geq 0. \]

Let us show that this definition is correct. It is enough to show that a.a.

\[\sum_{k=1}^n \int_0^t e^s \int (n^2(\tau) + K_2(\tau) + K_6(\tau)) \, d\tau \quad M_\varepsilon(s) 2\langle \tilde{A}_\varepsilon B_k, u \rangle \frac{2}{|u|^2 + \varepsilon} \, ds < \infty. \]

We have

\[\sum_{k=1}^n \int_0^t e^s \int (n^2(\tau) + K_2(\tau) + K_6(\tau)) \, d\tau \quad (M_\varepsilon(s))^2 \frac{2}{|u|^2 + \varepsilon} \, ds \leq e^0 \int_0^t e^s \int (n^2(\tau) + K_2(\tau) + K_6(\tau)) \, d\tau \quad \times \sup_{s \leq t} |M_\varepsilon(s)|^2 \frac{1}{\varepsilon^2} \sup_{s \leq t} \sum_{k=1}^n |B_k u(s)|^2 \int_0^t |\tilde{A}(s)|^2 \, ds \leq C e^0 \int_0^t (|u^2| + |u|^2) \int_0^t |\tilde{A}(s)|^2 \, ds, \]

and the result follows from assumptions (1.13) and (1.12).

Let us notice that formula (2.50) can be rewritten as follows

\[(2.52) \quad X_\varepsilon(t) = M_\varepsilon(0) \tilde{A}_\varepsilon(0)e^0 \int (n^2(s) + K_2(s) + K_6(s)) \, ds \]
\[+ \int_0^t e^s \int (n^2(\tau) + K_2(\tau) + K_6(\tau)) \, d\tau \quad K_1(s) M_\varepsilon(s) \, ds - L_\varepsilon(t), \quad t \geq 0. \]

By the definition (2.50), the process \(L_\varepsilon\) is a local martingale. We will show that in our assumptions it is martingale. By proposition A.6 it is enough to show that there exists \(\delta > 0, K < \infty\)

\[\mathbb{E}|L_\varepsilon|^1 + \delta \leq K, \quad \tau \in [0, T].\]
Let \( p_i, i = 1, 2, 3, 4 \) be real numbers such that \( \frac{4}{\sum_{i=1}^{4} \frac{1}{p_i}} = 1, p_i > 1, i = 1, 2, 3, 4 \). By Burkholder and Hölder inequalities we infer that for \( t \geq 0 \),

\[
\mathbb{E}|L_\varepsilon(t)|^{1+\delta} \leq C \mathbb{E} \left( \int_{0}^{t} e^{\int_{0}^{s} (n^2 + K_2 + K_0) du} \left( M_\varepsilon(s) \right)^2 \sum_{k=1}^{n} \left| \tilde{A}u_k \right|^2 \left| B_k u \right|^2 ds \right)^{(1+\delta)/2}
\]

\[
\leq C \varepsilon^{1+\delta} \mathbb{E} \left[ e^{\int_{0}^{t} (n^2 + K_2 + K_0) du} \sup_{s \in [0, t]} \left( M_\varepsilon(s) \right)^{(1+\delta)} \right] 
\]

\[
\times \sup_{s \in [0, t]} \sum_{k=1}^{n} \left| B_k u(s) \right|^{(1+\delta)} \left( \int_{0}^{t} \left| \tilde{A}u \right|^2 ds \right)^{(1+\delta)/2} \right] 
\]

\[
\leq C(\varepsilon) \left( \mathbb{E} e^{\int_{0}^{t} (n^2 + K_2 + K_0) du} \right)^{\frac{1}{p_1}} \left( \mathbb{E} \sup_{s \in [0, t]} \left( M_\varepsilon(s) \right)^{(1+\delta)p_2} \right)^{\frac{1}{p_2}} 
\]

\[
\times \left( \mathbb{E} \sup_{s \in [0, t]} \sum_{k=1}^{n} \left| B_k u(s) \right|^{2+\delta_0} \right)^{\frac{1}{p_3}} \left( \mathbb{E} \int_{0}^{t} \left| \tilde{A}u \right|^2 ds \right)^{\frac{1}{p_4}}. 
\]  

(2.53)

Choose \( \delta \in \left( 0, \frac{\delta_0 \kappa - (4+2\delta_0)}{4+2\delta_0 + \kappa(4+\delta_0)} \right) \). With a special choice of exponents \( p_i, i = 1, \ldots, 4 \):

\[
p_1 = \frac{\kappa}{1+\delta}, p_3 = \frac{2+\delta_0}{1+\delta}, p_2 = \frac{2}{1+\delta}, p_4 = 1/ \left( 1 - \left( \frac{1}{p_1} + \frac{1}{p_3} + \frac{1}{p_4} \right) \right),
\]

we obtain

\[
S = C(\varepsilon) \left( \mathbb{E} e^{\int_{0}^{t} (n^2 + K_2 + K_0) du} \right)^{\frac{1}{p_1}} \left( \mathbb{E} \sup_{s \in [0, t]} \left( M_\varepsilon(s) \right)^{(1+\delta)p_2} \right)^{\frac{1}{p_2}} 
\]

\[
\times \left( \mathbb{E} \sup_{s \in [0, t]} \sum_{k=1}^{n} \left| B_k u(s) \right|^{2+\delta_0} \right)^{\frac{1}{p_3}} \left( \mathbb{E} \int_{0}^{t} \left| \tilde{A}u \right|^2 ds \right)^{\frac{1}{p_4}}. 
\]  

(2.53)

Notice that by Assumption (1.14) the first factor in the product (2.53) is finite. Furthermore, by Doob inequality and Assumption (1.7) the second factor is finite. The third factor is finite by Assumptions (1.6), (1.10) and (1.12). The last factor in the product (2.53) is finite by Assumption (1.13). Thus, \( S < \infty \) and hence, the process \( L_\varepsilon \) is martingale. In particular, \( \mathbb{E}L_\varepsilon(t) = 0 \) for all \( t \geq 0 \).

It follows from (2.52), (2.48) and the Hölder inequality that

\[
\mathbb{E} \tilde{M}_\varepsilon(t) \tilde{A}_\varepsilon(t) \leq \mathbb{E} X_\varepsilon(t) \leq C(\mathbb{E} \tilde{M}_\varepsilon(0)^{1+\delta_0} + t \|K_1\|_{L^2(0,T)}) \mathbb{E} e^{\kappa(\delta_0) \int_{0}^{t} n^2(s) ds} 
\]

\[
\leq C \left( \frac{\mathbb{E} \left\| u(0) \right\|^{1+\delta_0} + t \|K_1\|_{L^2(0,T)} }{C^{1+\delta_0}} \right) \mathbb{E} e^{\kappa(\delta_0) \int_{0}^{t} n^2(s) ds}. 
\]  

(2.54)

Therefore, we get the estimate (2.36) from (2.54) and (1.12). Hence the proof of Lemma 2.2 is complete.

\[\square\]

**Proof of Theorem 1.2**  As mentioned earlier completion of the proof of Lemma 2.2 also completes the proof of the Theorem.

\[\square\]
Proof of Theorem 1.5. We will argue by contradiction. Suppose that the assertion of the Theorem is not true. Then, because the process \( u \) is adapted, we will be able to find \( t_0 \in [0, T) \), an event \( R \in \mathcal{F}_{t_0} \) and a constant \( c > 0 \) such that \( \mathbb{P}(R) > 0 \) and

\[
|u(t_0, \omega)| \geq c > 0, \ \omega \in R.
\]

Without loss of generality we can assume that \( \mathbb{P}(R) = 1 \) and \( t_0 = 0 \). Otherwise, we can consider instead of measure \( \mathbb{P} \) the conditional measure \( \mathbb{P}_R := \frac{\mathbb{P}(\cdot \cap R)}{\mathbb{P}(R)} \).

Suppose that there exists a constant \( c > 0 \) such that

\[
|u(t)|^2 \geq c, \text{ for all } t \in [0, T].
\]

Then, by taking \( t = T \), we infer that \( |u(T)|^2 > 0 \) what is a clear contradiction with the assumption that \( u(T) = 0 \), \( \mathbb{P} \)-a.s..

Now we shall prove that such a constant exists. Let \( \phi_r : \mathbb{R} \to \mathbb{R}, r > 0 \) a mollifying function such that \( \phi_r \in C_0^\infty (\mathbb{R}) \) and

\[
\phi_r(x) = \begin{cases} 
1, & \text{if } |x| \geq r \\
0 \leq \phi_r(x) \leq 1, & \text{if } r/2 < |x| < r \\
0, & \text{if } |x| \geq r/2
\end{cases}
\]

Next let us fix \( r > 0 \) and define a process \( \psi^r \) by

\[
(2.56) \quad \psi^r(t) = -\frac{1}{2} \phi_r(u(t)) \log |u(t)|^2, \ t \in [0, T].
\]

Now for any \( r \geq 0 \) we define stopping time \( \tau_r \) as follows

\[
(2.57) \quad \tau_r(\omega) = \inf\{t \in [0, T] : |u(t, \omega)| \leq r\}, \ \omega \in \Omega.
\]

Note that \( \tau_r \) is well defined since \( u(T) = 0 \) a.s. and \( \tau_r \leq \tau_\delta \leq \tau_0 \leq T \) if \( 0 \leq \delta \leq r \).

As in the Theorem 1.2 we have the following result.

**Lemma 2.4.** For every \( r > 0 \) the process \( \psi^r \) defined in (2.56) is an Itô process and

\[
(2.58) \quad \psi^r(t \wedge \tau_r) = \psi^r(0) + \int_0^{t \wedge \tau_r} \sum_{k=1}^n \frac{\langle u, B_k u \rangle}{|u|^2} \, dw^k(s) + \int_0^{t \wedge \tau_r} \left( \sum_{k=1}^n \frac{\langle u, B_k u \rangle^2}{|u|^4} + \frac{\langle (A - \frac{1}{2} \sum_{k=1}^n B^*_k B_k) u + F(s, u), u \rangle}{|u|^4} \right) \, ds, \ t \geq 0.
\]

Combining equality (2.58) and definition (2.8) we infer that

\[
(2.59) \quad \psi^r(t \wedge \tau_r) = \psi^r(0) + \int_0^{t \wedge \tau_r} \Lambda F(s, u(s)) \, ds + \int_0^{t \wedge \tau_r} \sum_{k=1}^n \frac{\langle u, B_k u \rangle}{|u|^2} \, dw^k(s)
\]

Suppose for the time being, that the following result is true.

**Lemma 2.5.** In the above framework we have, \( \mathbb{P} \)-a.s.

\[
\int_0^{\tau_0} \Lambda F(s, u(s)) \, ds < \infty.
\]
Then it follows from Assumption (AC3) that
\[
\int_0^T \sum_{k=1}^n \frac{\langle u, B_k u \rangle}{|u|^2 + \varepsilon} \, dw^k(s) \text{ exists a.e.}
\]
and
\[
(2.60) \quad \mathbb{E} \left[ \int_0^T \sum_{k=1}^n \frac{\langle u, B_k u \rangle}{|u|^2 + \varepsilon} \, dw^k(s) \right]^2 \leq \int_0^T \mathbb{E} \sum_{k=1}^n \frac{|\langle u, B_k u \rangle|^2}{|u|^4} \, ds < \infty.
\]
Therefore we infer that
\[
\log |u(t \wedge \tau_r)|^2 \geq \log |u(0)|^2 - K, \quad t \in [0, T],
\]
where
\[
K = \int_0^{\tau_0} \tilde{\Lambda}^F(s, u(s)) \, ds + \int_0^T \sum_{k=1}^n \frac{\langle u, B_k u \rangle}{|u|^2} \, dw^k(s).
\]
Hence
\[
(2.61) \quad |u(t \wedge \tau_r)|^2 = e^{\log (|u(t \wedge \tau_r)|^2)} \geq e^{\log |u(0)|^2 - K} = e^{-K} |u(0)|^2.
\]

Tend $r$ to 0. Therefore,
\[
(2.62) \quad |u(\tau_0)|^2 \geq \frac{1}{2} e^{-K} |u(0)|^2 > 0,
\]
what contradicts our assumption that $u(\tau_0) = 0$ and the Theorem follows. \(\square\)

Hence, it only remains to prove Lemma 2.5.

**Proof of Lemma 2.5** We have by the assumptions (AC2), (1.18), (1.19) the following chain of inequalities
\[
\int_0^t \tilde{\Lambda}^F(s, u(s)) \, ds \leq \int_0^t \tilde{\Lambda}(u(s)) \, ds + \int_0^t n(s)|u||u| \left| \frac{1}{|u|^2} \right| ds
\]
\[
\leq \int_0^t \tilde{\Lambda}(u(s)) \, ds + (\int_0^t n^2(s) \, ds)^{1/2} (\int_0^t \left| \frac{|u|^2}{|u|^4} \right| ds)^{1/2}
\]
\[
\leq \int_0^t \tilde{\Lambda}(u(s)) \, ds + C(\int_0^t \left| \frac{|u|^2}{|u|^4} \right| ds)^{1/2}
\]
\[
\leq \int_0^t \tilde{\Lambda}(u(s)) \, ds + C(\int_0^t \tilde{\Lambda}(u(s)) - \lambda ds)^{1/2}.
\]
Therefore, it is enough to estimate from above the term $\int_0^{\tau_0} \tilde{\Lambda}(s, u(s)) \, ds$. Because of the assumption (1.7) we have only to consider the following function $\int_0^{\tau_0} \tilde{\Lambda}(u(s)) \, ds$, where
\[
\tilde{\Lambda}(u) = \frac{\langle (A - \frac{1}{2} \sum_{k=1}^n B_k^* B_k) u, u \rangle}{|u|^2} = \frac{\langle \tilde{A} u, u \rangle}{|u|^2}, \quad u \in V.
\]

We will prove that
\[
(2.63) \quad \sup_{t \in [0, \tau_0]} \tilde{\Lambda}(u(t)) < \infty \text{ a.s.}
\]
Fix $\varepsilon > 0$. By the same argument as in the proof of the Theorem \[12\], we get inequality

\[
M_\varepsilon(t)\tilde{\Lambda}_\varepsilon(t) + \int_\tau^t M_\varepsilon(s)\frac{|\tilde{A} - \tilde{\Lambda}_\varepsilon|u|^2}{|u|^2 + \varepsilon} ds + 2\varepsilon \int_\tau^t M_\varepsilon(s)\frac{(\tilde{\Lambda}_\varepsilon)^2}{|u|^2 + \varepsilon} ds
\]

\[
\leq M_\varepsilon(0)\tilde{\Lambda}_\varepsilon(0) + \int_\tau^t K_1(s)M_\varepsilon(s) ds + \frac{C}{\varepsilon^2} \int_\tau^t (n^2(s) + K_2(s)
\]

\[
+ K_6(s)M_\varepsilon(s)\tilde{\Lambda}_\varepsilon(s) ds - \int_\tau^t M_\varepsilon(s)\sum_{k=1}^n 2(\tilde{A}u, B_ku) |u|^2 + \varepsilon dw^k(s), t \geq \tau.
\]

Notice that $K_1(s) = 0, s \in [0, T]$ by assumptions of the Theorem. Consequently, for $t \geq \tau$,

\[
M_\varepsilon(t)\tilde{\Lambda}_\varepsilon(t) + \int_\tau^t M_\varepsilon(s)\frac{|\tilde{A} - \tilde{\Lambda}_\varepsilon|u|^2}{|u|^2 + \varepsilon} ds \leq (M_\varepsilon(\tau)\tilde{\Lambda}_\varepsilon(\tau))e^{\int_\tau^T (n^2(s) + K_2(s) + K_6(s)) ds} - L_\varepsilon(t).
\]

where,

\[
L_\varepsilon(t) = \int_\tau^t e^s \frac{\int (n^2(s) + K_2(s) + K_6(s)) ds}{M_\varepsilon(s)} \sum_{k=1}^n 2(\tilde{A}u, B_ku) |u|^2 + \varepsilon dw^k(s), t \geq \tau.
\]

Let us denote, for $t \geq 0$,

\[
S_\varepsilon(t) = -\int_0^t (n^2(s) + K_2(s) + K_6(s)) ds M_\varepsilon(t)\tilde{\Lambda}_\varepsilon(t),
\]

\[
N_\varepsilon(t) = e^{-\int_0^t (n^2(s) + K_2(s) + K_6(s)) ds} \int_\tau^t M_\varepsilon(s)\frac{|\tilde{A} - \tilde{\Lambda}_\varepsilon|u|^2}{|u|^2 + \varepsilon} ds.
\]

Multiplying inequality (2.65) by $e^{-\int_0^t (n^2(s) + K_2(s) + K_6(s)) ds}$ we infer that

\[
S_\varepsilon(t) + N_\varepsilon(t) \leq S_\varepsilon(\tau) - 2 \int_\tau^t e^{-\int_0^s (n^2(r) + K_2(r) + K_6(r)) dr} M_\varepsilon(s) \sum_{k=1}^n 2(\tilde{A}u, B_ku) |u|^2 + \varepsilon dw^k(s).
\]

Therefore, by the definition of $S_\varepsilon$, we infer that

\[
S_\varepsilon(t) + N_\varepsilon(t) \leq S_\varepsilon(\tau) - 2 \int_\tau^t S_\varepsilon(s) \sum_{k=1}^n \frac{(\tilde{A}u, B_ku)}{|(\tilde{A}u, u)|} dw^k(s).
\]

Since $N_\varepsilon \geq 0$, by the Comparison Theorem for the one dimensional diffusions, see e.g. Theorem 1.1 p. 352 in [12], we have for $t \geq \tau$, $P$-a.s.

\[
S_\varepsilon(t) \leq S_\varepsilon(\tau)e^{-2 \int_\tau^t \sum_{k=1}^n \frac{(\tilde{A}u, B_ku)}{|(\tilde{A}u, u)|} dw^k(s) - 2 \int_\tau^t \sum_{k=1}^n \frac{|(\tilde{A}u, B_ku)|^2}{|(\tilde{A}u, u)|} ds}, t \in [0, T].
\]

Hence

\[
\tilde{\Lambda}_\varepsilon(t) \leq \tilde{\Lambda}_\varepsilon(\tau) \frac{M_\varepsilon(t)}{M_\varepsilon(\tau)} e^{\frac{1}{M_\varepsilon(t)} \int (n^2(r) + K_2(r) + K_6(r)) dr} e^{-2 \int_\tau^t \sum_{k=1}^n \frac{(\tilde{A}u, B_ku)}{|(\tilde{A}u, u)|} dw^k(s) - 2 \int_\tau^t \sum_{k=1}^n \frac{|(\tilde{A}u, B_ku)|^2}{|(\tilde{A}u, u)|^2} ds}, t \in [0, T].
\]
Put $\tau = 0$ in the equality (2.70). Hence,
\begin{equation}
\liminf_{\varepsilon \to 0} \bar{\Lambda}_\varepsilon(t) \leq e^r \int_0^t (n^2(r) + K_2(r) + K_0(r)) dr \left\{ -2 \int_0^\tau \frac{\sum_{k=1}^n \langle \hat{A} \hat{u}, B_k u \rangle}{|\hat{u}|^2 + \varepsilon} ds \right\} \Lambda(0) \frac{\tau_0 - \tau}{M_\varepsilon(t)}, t \in [0, T].
\end{equation}

Notice that
\begin{equation}
\bar{\Lambda}(t) = \liminf_{\varepsilon \to 0} \bar{\Lambda}_\varepsilon(t), t \in [0, \tau_0).
\end{equation}
Thus
\begin{equation}
\bar{\Lambda}(t) \leq e^r \int_0^t (n^2(r) + K_2(r) + K_0(r)) dr \left\{ -2 \int_0^\tau \frac{\sum_{k=1}^n \langle \hat{A} \hat{u}, B_k u \rangle}{|\hat{u}|^2 + \varepsilon} ds \right\} \bar{\Lambda}(0) \frac{\tau_0 - \tau}{M_\varepsilon(t)}, t \in [0, \tau_0).
\end{equation}

It follows from assumptions (AC3), (AC7) that RHS is uniformly bounded w.r.t. $t \in [0, T]$ a.s. for any $\varepsilon > 0$. Hence, the result follows.

\begin{flushright}
\Box
\end{flushright}

3. PROOF OF THEOREM 1.7 ON THE EXISTENCE OF A SPECTRAL LIMIT

Proof of Theorem 1.7 Without loss of generality we can suppose that $T_0 = 0$ and $\lambda = 0$ in the assumption (1.6). Otherwise, we can replace $\hat{A}$ by $A + \lambda I$ and $F$ by $F - \lambda I$.

Let us begin the proof with an observation that $|u(t)| > 0$ for all $t > 0$. Indeed, otherwise by the Theorem 1.2 we would have that $u(0)$ is identically 0 what would contradict one of our assumptions. Hence the process
\begin{equation}
\bar{\Lambda}(u(t)) = \frac{\langle \hat{A} u(t), u(t) \rangle}{|u(t)|^2}, t \geq 0
\end{equation}
is well defined. Let us also note that $\bar{\Lambda}^\varepsilon$ converges pointwise and monotonously to $\bar{\Lambda}$.

Step 1: Proof of the existence of the limit $\lim_{t \to \infty} \bar{\Lambda}(t)$.

Let us fix $\tau \geq 0$. By the same argument as in the proof of the Theorem 1.2 we get inequality (2.46) i.e. $\mathbb{P}$-a.s.
\begin{align}
M_\varepsilon(t) \bar{\Lambda}_\varepsilon(t) + \int_\tau^t M_\varepsilon(s) (|\hat{A} - \bar{\Lambda}_\varepsilon| u^2) ds &+ 2 \varepsilon \int_\tau^t M_\varepsilon(s) \frac{(\bar{\Lambda}_\varepsilon)^2}{|u|^2 + \varepsilon} ds \\
&\leq M_\varepsilon(\tau) \bar{\Lambda}_\varepsilon(\tau) + \int_\tau^t K_1(s) M_\varepsilon(s) ds + C \varepsilon \int_\tau^t (n^2(s) + K_2(s)) ds \\
&+ K_0(s) M_\varepsilon(s) \bar{\Lambda}_\varepsilon(s) ds - \int_\tau^t M_\varepsilon(s) \sum_{k=1}^n \frac{2(\hat{A} u, B_k u)}{|u|^2 + \varepsilon} dw^k(s), t \geq \tau.
\end{align}

By the assumptions of Theorem 1.7 $K_1(s) = 0$, $s \in [0, \infty)$. Consequently, for $t \geq \tau$,
\begin{align}
M_\varepsilon(t) \bar{\Lambda}_\varepsilon(t) + \int_\tau^t M_\varepsilon(s) (|\hat{A} - \bar{\Lambda}_\varepsilon| u^2) ds &+ 2 \varepsilon \int_\tau^t M_\varepsilon(s) \frac{(\bar{\Lambda}_\varepsilon)^2}{|u|^2 + \varepsilon} ds \\
&\leq M_\varepsilon(\tau) \bar{\Lambda}_\varepsilon(\tau) + C \varepsilon \int_\tau^t (n^2(s) + K_2(s) + K_0(s)) M_\varepsilon(s) \bar{\Lambda}_\varepsilon(s) ds - \int_\tau^t M_\varepsilon(s) \sum_{k=1}^n \frac{2(\hat{A} u, B_k u)}{|u|^2 + \varepsilon} dw^k(s).
\end{align}
Thus, for \( t \geq \tau \),

\[
M_\varepsilon(t)\tilde{\Lambda}_\varepsilon(t) + \int_\tau^t M_\varepsilon(s) \left( \frac{|\tilde{A} - \tilde{\Lambda}_\varepsilon|u|^2}{|u|^2 + \varepsilon} \right) ds \leq M_\varepsilon(\tau)\tilde{\Lambda}_\varepsilon(\tau)e^{\int_\tau^t (\tilde{n}^2(s) + K^2 + K_0(s)) \, ds} - L_\varepsilon(\tau(t)).
\]

where,

\[
L_\varepsilon(\tau(t)) = \int_\tau^t e^{-\int_s^t (\tilde{n}^2(r) + K^2 + K_0(r)) \, dr} M_\varepsilon(s) \sum_{k=1}^n \frac{2(\tilde{A}_k, B_k u)}{|u|^2 + \varepsilon} \, dw^k(s), \quad t \geq \tau.
\]

Let us denote, for \( t \geq 0 \),

\[
S_\varepsilon(t) = e^{-\int_0^t (\tilde{n}^2(r) + K^2 + K_0(r)) \, dr} M_\varepsilon(t)\tilde{\Lambda}_\varepsilon(t),
\]

\[
N_\varepsilon(t) = e^{-\int_0^t (\tilde{n}^2(r) + K^2 + K_0(r)) \, dr} \int_\tau^t M_\varepsilon(s) \left( \frac{|\tilde{A} - \tilde{\Lambda}_\varepsilon|u|^2}{|u|^2 + \varepsilon} \right) ds.
\]

Multiplying inequality (3.3) by \(-\int_0^t (\tilde{n}^2(r) + K^2 + K_0(r)) \, dr\) we infer that

\[
S_\varepsilon(t) + N_\varepsilon(t) \leq S_\varepsilon(\tau) - 2\int_\tau^t e^{-\int_s^t (\tilde{n}^2(r) + K^2 + K_0(r)) \, dr} M_\varepsilon(s) \sum_{k=1}^n \frac{2(\tilde{A}_k, B_k u)}{|u|^2 + \varepsilon} \, dw^k(s).
\]

Therefore, by the definition of \( S_\varepsilon \), we infer that

\[
S_\varepsilon(t) + N_\varepsilon(t) \leq S_\varepsilon(\tau) - 2\int_\tau^t S_\varepsilon(s) \sum_{k=1}^n \frac{(\tilde{A}_k, B_k u)}{|(\tilde{A}_k, u)|} \, dw^k(s).
\]

Since \( N_\varepsilon \geq 0 \), by the comparison principle for the one dimensional diffusions, see e.g. Theorem 1.1 p. 352 in [12], we have for \( t \geq \tau, \, \mathbb{P}\text{-a.s.} \).

\[
S_\varepsilon(t) \leq S_\varepsilon(\tau) e^{-2\int_\tau^t \sum_{k=1}^n \frac{(\tilde{A}_k, B_k u)}{|(\tilde{A}_k, u)|} \, dw^k(s) - 2\int_\tau^t \sum_{k=1}^n \frac{|(\tilde{A}_k, B_k u)|}{|(\tilde{A}_k, u)|^2} \, ds}.
\]

Let us observe that inequality (3.7) makes sense when \( \varepsilon = 0 \). Indeed, as already mentioned before, \(|u(t)| > 0\) for all \( t > 0 \) \( \mathbb{P}\text{-a.s.}\). Suppose that we can show that there exist sequence \( \{\varepsilon_l\}_{l=1}^\infty, \, \varepsilon_l \to 0 \) such that \( S_{\varepsilon_l}(s) \to S_0(s), \, s \in [0, t] \) \( \mathbb{P}\text{-a.s.}\). Then we will be able to conclude that inequality (3.7) holds with \( \varepsilon = 0 \).

Thus it is easy to show that \( \xi^\varepsilon_t = \sum_{k=1}^n \int_0^t \frac{\langle B_k u(s), u(s) \rangle}{|u(s)|^2 + \varepsilon} \, dw^k(s) \) converges to \( \xi_t = \sum_{k=1}^n \int_0^t \frac{\langle B_k u(s), u(s) \rangle}{|u(s)|^2 + \varepsilon} \, dw^k(s) \) as \( \varepsilon \to 0 \) in probability uniformly on any finite interval \( t \in [0, m], \, m \in \mathbb{N} \).

Indeed, in this case there exist subsequence \( \{\varepsilon_{l_1}\}_{l=1}^\infty, \, \varepsilon_{l_1} \to 0 \) such that \( \xi_{\varepsilon_{l_1}} \to \xi \) as \( l \to \infty \) uniformly on any finite interval with probability 1 and, therefore, \( M_{\varepsilon_{l_1}} \to M_0 \) \( \mathbb{P}\text{-a.s.}\). Convergence \( S_{\varepsilon_{l_1}} \to S_0 \) uniformly on any finite interval with probability 1 as \( l \to \infty \) follow.

We will show convergence of martingales \( \xi^\varepsilon \) to \( \xi \) in \( \mathbb{L}^2(\Omega \times [0, m]) \) for any \( m \in \mathbb{N} \). Convergence in probability follows from Doob’s inequality. Notice that \( \sum_{k=1}^n \frac{\langle B_k u(t), u(t) \rangle}{|u(t)|^2 + \varepsilon}, \, t \in [0, \infty) \) converges a.s.
to \( \sum_{k=1}^{n} \frac{<B_k u(t), u(t)>}{|u(t)|^2}, t \in [0, \infty) \). Indeed, \(|u(t)| > 0\) for all \( t \geq 0 \) \( \mathbb{P}\)-a.s. Furthermore, by Assumption (AC3)

\[
\sup_t \left| \sum_{k=1}^{n} \left( \frac{<B_k u(t), u(t)>}{|u(t)|^2 + \varepsilon} - \frac{<B_k u(t), u(t)>}{|u(t)|^2} \right) \right| = \\
\sup_t \left| \sum_{k=1}^{n} \frac{<B_k u(t), u(t)>}{|u(t)|^2 + \varepsilon} \right| |\frac{\varepsilon}{|u(t)|^2 + \varepsilon}| \leq |\phi(\cdot)|_{L^\infty} < \infty.
\]

Hence, by the Lebesgue Dominated Convergence Theorem, \( \sum_{k=1}^{n} \frac{<B_k u(t), u(t)>}{|u(t)|^2 + \varepsilon} \) \( \rightarrow \varepsilon \) \( \mathbb{P}\)-a.s. Indeed, in view of the assumption (1.11), we infer that \( \varepsilon \rightarrow 0 \) in \( L^2(\Omega \times [0, m]) \) for any \( m \in \mathbb{N} \). Thus we have shown that inequality (3.7) holds with \( \varepsilon = 0 \), i.e. that for \( t \geq \tau \), \( \mathbb{P}\)-a.s.

\[
S(t) \leq S(\tau) e^{-2 \int_{\tau}^{t} \sum_{k=1}^{n} \frac{(\Delta u, B_k u)}{|(A u, u)|^2} \, ds} \leq e^{-2 \int_{\tau}^{t} \sup_{k} \frac{|(\Delta u, B_k u)|^2}{|A u, u|^2} \, ds},
\]

where \( S(t) = S_{0}(t), N(t) = N_{0}(t), t \geq 0 \).

Denote

\[
\vartheta_{\tau}(t) = e^{-2 \int_{\tau}^{t} \sum_{k=1}^{n} \frac{(\Delta u, B_k u)}{|(A u, u)|^2} \, ds} \leq e^{-2 \int_{\tau}^{t} \sup_{k} \frac{|(\Delta u, B_k u)|^2}{|A u, u|^2} \, ds}, \quad t \geq \tau.
\]

Let us note that a process \( (\vartheta_{\tau}(t))_{t \geq \tau} \) is a local martingale. Moreover it is a uniformly integrable martingale. Indeed, in view of the assumption (1.11), we infer that

\[
\sup_{t \geq \tau} \mathbb{E}[\vartheta_{\tau}(t)^{1+\delta}] \leq \mathbb{E} e^{2(\delta^2 + \delta) \int_{\tau}^{t} \sup_{k} \frac{|(\Delta u, B_k u)|^2}{|A u, u|^2} \, ds} < \infty.
\]

Hence, by the Doob Martingale Convergence Theorem the following limit exists \( \mathbb{P}\)-a.s. (and in \( L^1(\mathbb{P}) \))

\[
\vartheta_{\tau}(\infty) := \lim_{t \to \infty} \vartheta_{\tau}(t),
\]

and \( \mathbb{E}\vartheta_{\tau}(\infty) = 1 < \infty \). Therefore, \( \vartheta_{\tau}(\infty) < \infty \) \( \mathbb{P}\)-a.s.. Furthermore, \( \vartheta_{\tau}(\infty) > 0 \) \( \mathbb{P}\)-a.s.. Indeed, by Fatou Lemma

\[
\mathbb{E} [\vartheta_{\tau}(\infty)^{-1}] \leq \lim_{t \to \infty} \mathbb{E} [\vartheta_{\tau}(t)^{-1}] \leq \mathbb{E} e^{4 \int_{\tau}^{\infty} \sup_{k} |C_k^2(s)|^2 \, ds} < \infty.
\]

Consequently, by Lemma the we infer that

\[
\lim_{\tau \to \infty} \vartheta_{\tau}(\infty) = 1, \mathbb{P}\text{-a.s.}
\]

Thus,

\[
\limsup_{t \to \infty} S(t) \leq S(\tau) \vartheta_{\tau}(\infty).
\]

Since the above holds for any \( \tau > 0 \), we infer that \( \mathbb{P} - a.s. \)

\[
\limsup_{t \to \infty} S(t) \leq \liminf_{\tau \to \infty} S(\tau) \vartheta_{\tau}(\infty) = \liminf_{\tau \to \infty} S(\tau).
\]

We infer that the following limit exists \( \mathbb{P} - a.s. \)

\[
\lim_{t \to \infty} S(t) = \lim_{t \to \infty} e^{-\int_{0}^{t} (\Delta u + K_2 u + K_6 u) \, d\tau} M_0(t) \Lambda(t).
\]
Moreover, since \( n \in L^2(0, \infty), K_2 \in L^1(0, \infty) \) and \( K_6 \in L^1(0, \infty) \), the following limit exists \( \mathbb{P} - \text{a.s.} \)

\[
\lim_{t \to \infty} e^{-\int_0^t (n^2(\tau)+K_2(\tau)+K_6(\tau))d\tau} = e^{-\int_0^\infty (n^2(\tau)+K_2(\tau)+K_6(\tau))d\tau} > 0.
\]

Furthermore, from Assumption (AC3) it follows that \((M_0(t))_{t \geq 0}\) is uniformly integrable exponential martingale and therefore "as above" exists \( \mathbb{P} - \text{a.s.} \)

\[
M_0(\infty) := \lim_{t \to \infty} M_0(t) \neq 0.
\]

Combining (3.10), (3.11) and (3.12) we infer that the following limit exists \( \mathbb{P}\text{-a.s.} \)

\[
\tilde{M}(\infty) := \lim_{t \to \infty} \tilde{M}(t).
\]

\[
\square
\]

**Step 2: Proof that** \( \tilde{M}(\infty) \in \sigma(\tilde{A}) \).

We will need an estimate for \( N_{\varepsilon}(t), t \geq \tau \). Denote

\[
R_{\tau}(t) = 2 \int \sum_{k=1}^n \frac{\langle \tilde{A}u(s), B_k u(s) \rangle}{\| \langle \tilde{A}u(s), u(s) \rangle \|} dw^k(s), t \geq \tau.
\]

Let us observe that the process \( R_{\tau}\) martingale in the formula (3.6). The formula (3.6) can be rewritten as follows.

\[
S_{\varepsilon}(t) \leq S_{\varepsilon}(\tau) - N_{\varepsilon}(t) - \int_{\tau}^t S_{\varepsilon}(s)dR_{\tau}(s).
\]

Denote

\[
\Psi_{\tau}(t) = e^{R_{\tau}(t)+\frac{1}{2}(R_{\tau})(t)}, t \geq \tau.
\]

By the Comparison Theorem for the one dimensional diffusions, see e.g. Theorem 1.1 p. 352 in [12], we have for \( t \geq \tau \),

\[
S_{\varepsilon}(t) \leq \frac{1}{\Psi_{\tau}(t)}\left( S_{\varepsilon}(\tau) - \int_{\tau}^t \Psi_{\tau}(s)dN_{\varepsilon}(s) \right) = -N_{\varepsilon}(t) + \frac{S_{\varepsilon}(\tau)}{\Psi_{\tau}(t)} + \int_{\tau}^t N_{\varepsilon}(s)\Psi_{\tau}(s)d\Psi_{\tau}(s)
\]

\[
(3.15) = -N_{\varepsilon}(t) + \frac{S_{\varepsilon}(\tau)}{\Psi_{\tau}(t)} + \int_{\tau}^t N_{\varepsilon}(s)\Psi_{\tau}(s)dR_{\tau}(s) + \int_{\tau}^t N_{\varepsilon}(s)\Psi_{\tau}(s)d(R_{\tau})(s).
\]

Also by the assumption (1.6) with \( \lambda = 0 \) we have that \( S_{\varepsilon}(t) \geq 0, t \geq \tau \). Thus, we infer from (3.15) that

\[
N_{\varepsilon}(t) \leq \frac{S_{\varepsilon}(\tau)}{\Psi_{\tau}(t)} + \int_{\tau}^t N_{\varepsilon}(s)\Psi_{\tau}(s)dR_{\tau}(s) + \int_{\tau}^t N_{\varepsilon}(s)\Psi_{\tau}(s)d(R_{\tau})(s).
\]

(3.16) Denote \( K_{\varepsilon}(t) = N_{\varepsilon}(t)\Psi_{\tau}(t), t \geq \tau \). Then we can rewrite (3.16) as follows

\[
K_{\varepsilon}(t) \leq S_{\varepsilon}(\tau) + \int_{\tau}^t K_{\varepsilon}(s)dR_{\tau}(s) + \int_{\tau}^t K_{\varepsilon}(s)d(R_{\tau})(s), t \geq \tau.
\]

Applying the Comparison Theorem we have that

\[
K_{\varepsilon}(t) \leq S_{\varepsilon}(\tau)\Psi_{\tau}(t), t \geq \tau.
\]
Consequently, we have an estimate for $N^\varepsilon$:

$$N_\varepsilon(t) \leq S_\varepsilon(\tau), \quad t \geq \tau.$$  

We infer from (3.19) that $\mathbb{P}$ a.s.

$$\lim_{t \to \infty} N_\varepsilon(t) \leq S_\varepsilon(\tau).$$

i.e. $\mathbb{P}$ a.s.

$$-\int_0^{(n^2(\tau) + K_2(\tau) + K_6(\tau))d\tau} e^{\frac{1}{2}M(s)} \frac{(\bar{A} - \bar{\Lambda}_\varepsilon(s))u(t)^2}{|u(t)|^2 + \varepsilon} ds \leq e^{-\int_0^{(n^2(\tau) + K_2(\tau) + K_6(\tau))d\tau} \frac{1}{2} \bar{\Lambda}_\varepsilon(\tau)} \int_\tau^\infty Mg(s)|\bar{A} - \bar{\Lambda}(s)|u(s)^2 ds < \infty.$$  

As we have already mentioned $|u(t)| > 0$, $t > 0$. Hence the right hand side of inequality (3.21) is uniformly bounded w.r.t. $\varepsilon$. Therefore, by Fatou Lemma $\mathbb{P}$ a.s.

$$\int_\tau^\infty Mg(s)|\bar{A} - \bar{\Lambda}(s)|u(s)^2 ds < \infty.$$  

Let $\psi : \mathbb{R} \ni t \mapsto \frac{u(t)}{|u(t)|} \in H$. It follows from (3.22) that there exists sequence $\{t_j\}_{j=1}^\infty : t_j \to \infty$ as $j \to \infty$ and $(\bar{A} - \bar{\Lambda}(t_j))\psi(t_j) \to 0$ in $H$. Therefore, by (3.13) $h_j = (\bar{A} - \bar{\Lambda}(\infty))\psi(t_j) \to 0$ in $V'$ as $j \to \infty$. If $\bar{\Lambda}(\infty) \notin \sigma(\bar{A})$ then $(\bar{A} - \bar{\Lambda}(\infty))^{-1} \in \mathcal{L}(V', V)$. Since $h_j \to 0$ in $V'$ we have $\psi(t_j) = (\bar{A} - \bar{\Lambda}(\infty))^{-1}h_j \to 0$ in $H$. This is contradiction with the fact that $|\psi(t)| = 1$. This completes the argument in Step 2.

The proof of Theorem 1.7 is now complete.

4. Applications and Examples

Now we will show how to apply Theorems 1.2 and 1.7 to certain linear and nonlinear SPDEs.

4.1. Backward Uniqueness for Linear SPDEs. We will consider following equation:

$$du + (Au + F(t)u) dt + \sum_{k=1}^n B_k u dw^k_t = f dt + \sum_{k=1}^n g_k dw^k_t,$$

where $f \in \mathcal{M}^2(0, T; V)$, $g \in \mathcal{M}^2(0, T; D(\bar{A}))$ and the operators $A, B_k, k = 1, \ldots, n$ satisfy the same assumptions of Theorem 1.2. We will suppose that $F \in L^2(0, T; \mathcal{L}(V, H))$. Then, we notice that assumption (1.15) is satisfied with $n = |F(\cdot)|_{\mathcal{L}(V, H)}$. Applying Theorem 1.2 we have the following result:

**Theorem 4.1.** Let $u_1, u_2$ be two solutions of (4.1), such that for some $\delta_0 > 0$

$$u_1, u_2 \in \mathcal{M}_a(0, T; D(\bar{A})) \cap L^{2+\delta_0}(\Omega, C([0, T]; V)).$$  

Then if $u_1(T) = u_2(T), \mathbb{P}$-a.s., $u_1(t) = u_2(t), \quad t \in [0, T] \mathbb{P}$-a.s.

**Proof.** Denote $u = u_1 - u_2$. Applying Theorem 1.2 to the process $u$ we immediately get the result.

**Example 4.2.** Assume $a^{i,j} \in C^{1,1}_{t,x}([0, T] \times \mathbb{R}^n), i, j = 1, \ldots, n$ and $b_k, c_i : [0, T] \times \mathbb{R}^n \to \mathbb{R}, k = 1, \ldots, n, l = 1, \ldots, m$ are measurable functions. Denote $\bar{A}_0(\cdot) = (a^{i,j}(\cdot))_{i,j=1}^n$. Assume that the following inequalities are satisfied

$$C_1 I \leq \bar{A}_0 \leq C_2 I,$$

for some $0 < C_1 \leq C_2 < \infty$,
\[ \int_{0}^{T} |\partial_t \tilde{A}_0| ds \leq C_3 I, \text{ for some } C_3 > 0 \]

\[ \sup_{x \in \mathbb{R}^n} \sum_{k} |b_k(t, \cdot)| + |c(t, \cdot)| \in L^2(0, T). \]

Then the equation satisfied:

\[ c_l \in L^\infty([0, T] \times \mathbb{R}^n), \nabla c_l \in L^1(0, T; L^\infty(\mathbb{R}^n)), \triangle c_l \in L^2(0, T; L^\infty(\mathbb{R}^n)). \]

Then the equation

\[ du = \left( \sum_{i,j} \frac{\partial}{\partial x_i} \left( a^{ij} \frac{\partial u}{\partial x_j} \right) + \sum_{k} b_k(t, \cdot) \frac{\partial u}{\partial x_k} + c(t, \cdot) u + f \right) dt + \sum_{l=1}^{m} \left( c_l(t, \cdot) u + g_l \right) \circ dw^l_t, \]

where stochastic integral is in Stratonovich sense, satisfies conditions of the Theorem 4.1. Indeed, we have in this case that

\[ \tilde{A} = -\sum_{i,j} \frac{\partial}{\partial x_i} \left( a^{ij} \frac{\partial}{\partial x_j} \right), \quad F(t) = -\sum_{k} b_k(t, \cdot) \frac{\partial}{\partial x_k} - c(t, \cdot), \quad B_l = c_l, \]

\[ H = L^2(\mathbb{R}^n), V = H_0^{1,2}(\mathbb{R}^n). \]

**Example 4.3.** Assume \( b_k, c, \sigma_k \in C_{t,x}^{0,1}([0, T] \times \mathbb{R}^n), k = 1, \ldots, n \) and following inequalities are satisfied:

\[ \sup_{x \in \mathbb{R}^n} \sum_{k} |\nabla \sigma_k(t, \cdot)|^2 \in L^\infty(0, T), \]

\[ \sup_{x \in \mathbb{R}^n} \sum_{k} |b_k(t, \cdot)| + |c(t, \cdot)| \in L^2(0, T). \]

Then the equation

\[ du = \left( \Delta u + \sum_{k} b_k(t, \cdot) \frac{\partial u}{\partial x_k} + c(t, \cdot) u + f \right) dt + \sum_{k} \left( \sigma_k(t, \cdot) \frac{\partial u}{\partial x_k} + g_k \right) \circ dw^k_t, \]

where stochastic integral is in Stratonovich sense, satisfies conditions of the Theorem 4.1. Indeed, we have in this case that

\[ \tilde{A} = -\Delta, F(t) = -\sum_{k} b_k(t, \cdot) \frac{\partial}{\partial x_k} - c(t, \cdot), B_k = \sigma_k(t, \cdot) \frac{\partial}{\partial x_k}, \]

\[ H = L^2(\mathbb{R}^n), V = H_0^{1,2}(\mathbb{R}^n). \]

We need to check only assumption (1.8). Other conditions are trivial. We have

\[ \sum_{k} ([\tilde{A}, B_k]u, B_k u) = \sum_{k} \int_{\mathbb{R}^n} \sigma_k \frac{\partial u}{\partial x_k} \left( \sigma_k \frac{\partial \Delta u}{\partial x_k} - \Delta \left( \sigma_k \frac{\partial u}{\partial x_k} \right) \right) dx \]

\[ = \sum_{k} \int_{\mathbb{R}^n} |\nabla \sigma_k|^2 \frac{\partial u}{\partial x_k}^2 dx \leq \sup_{x \in \mathbb{R}^n} \sum_{k} |\nabla \sigma_k(t, \cdot)|^2 \|u\|^2. \]

The existence of a regular solution has been established in [20].

**Remark 4.4.** Instead of the Laplacian one can consider a second order time dependent operator \( A(t) = \sum_{ij} a^{ij}(t) \frac{\partial^2}{\partial x_i \partial x_j} \) where matrix \( a = (a^{ij}) : [0, T] \to \mathbb{R}^n \) is uniformly (w.r.t. \( t \)) positively definite.
Example 4.5. Let $H$ be the real separable Hilbert space, $A : D(A) \subset H \to H$ be a strictly negative linear operator on $H$; $V = D((-A)^{1/2})$ be the Hilbert space endowed with the natural norm. Identifying $H$ with its dual we can write $V \subset H \subset V'$. Let also $B : V \times V \to V'$ be a bilinear continuous operator and $b_k \in \mathbb{R}$, $k = 1, \ldots, n$ are given. Assume that $B$ satisfies

\begin{equation}
|B(u, v)| \leq C|u|^{1/2}|Au|^{1/2}|v|, \ u \in D(A), \ v \in V
\end{equation}

and

\begin{equation}
|B(u)| \leq C|u||v|_{D(A)}, \ u \in H, \ v \in D(A).
\end{equation}

Then equation

\begin{equation}
 du = (Au + B(u, u_{stat}(t)) + B(u_{stat}(t), u))dt + \sum_{k=1}^{n} b_k u \circ dw_t^k,
\end{equation}

where $u_{stat} \in L^2(0, T; D(A)) \ \mathbb{P}$-a.s. satisfies conditions of the Theorem 1.2. Indeed, it is enough to put

\[
\tilde{A} = A, \ F(t) = B(u_{stat}(t), \cdot) + B(\cdot, u_{stat}(t)), \ B_k = b_k.
\]

In particular, in this scheme falls linearisation around solution $u_{stat}$ of two dimensional stochastic Navier-Stokes equation with multiplicative noise (see [6]).

4.2. Backward Uniqueness for SPDEs with a quadratic nonlinearity. Assume that the linear operators $A, B_k, k = 1, \ldots, n$ satisfy the same assumptions as in the Theorem 1.5 $B \in \mathcal{L}(V \times V, V')$, $R \in \mathcal{L}(V, H)$ and

\begin{equation}
|B(u, v)| + |B(v, u)| \leq K||u|||Av|, \ u, v \in D(\tilde{A}).
\end{equation}

Assume that $f \in L^2(0, T; H)$ and $g_k \in \mathcal{M}^2(0, T; H), k = 1, \ldots, n$. Consider the following problem.

\begin{equation}
 du + (Au + B(u, u) + R(u)) dt + \sum_{k=1}^{n} B_k u \circ dw_t^k = f dt + \sum_{k=1}^{n} g_k dw_t^k,
\end{equation}

Applying Theorem 1.5, we have the following result.

Theorem 4.6. Suppose that $u_1, u_2$ are two solutions of (4.6), such that for some $\delta_0 > 0$ and $i = 1, 2$,

\begin{equation}
 u_i \in \mathcal{M}^2(0, T; D(\tilde{A})) \cap L^{2+\delta_0}([0, T]; V)), \ i = 1, 2,
\end{equation}

If $u_1(T) = u_2(T), \ \mathbb{P}$-a.s., then for all $t \in [0, T], u_1(t) = u_2(t), \ \mathbb{P}$-a.s..

Proof of Theorem 4.6. We denote $u = u_1 - u_2$. Then $u \in \mathcal{M}^2(0, T; D(\tilde{A})) \cap L^{2+\delta_0}([0, T]; V))$ and $u$ is a solution to

\begin{equation}
 du + (Au + B(u_1, u) + B(u_2, u) + R(u)) dt + \sum_{k=1}^{n} B_k u \circ dw_t^k = 0.
\end{equation}

By the assumption (4.5) it follows that

\[
|B(u_1, u) + B(u_2, u) + R(u)| \leq ||R||_{\mathcal{L}(V, H)} + K(||Au_1| + |Au_2||)||u||
\]

\[
\leq C||R||_{\mathcal{L}(V, H)} + K(||\tilde{A}u_1| + |\tilde{A}u_2||)||u||.
\]

Therefore, by (4.7) we have that $n = ||R||_{\mathcal{L}(V, H)} + K(||\tilde{A}u_1| + |\tilde{A}u_2||)$ satisfy assumption (1.19) and Theorem 1.5 applies to $u$. 

The stochastic Navier-Stokes equations with multiplicative noise fit into the framework described above.
Example 4.7. Let $d = 2$ or $d = 3$, $H = \{ f \in L^2(\mathbb{T}^d, \mathbb{R}^d) : \text{div } f = 0 \}$, $V = W^{1,2}(\mathbb{T}^d, \mathbb{R}^d) \cap H$, $\nu > 0$, $A$ is a Stokes operator, $P : L^2(\mathbb{T}^d, \mathbb{R}^d) \to H$ is a projection on divergence free vector fields, $B_k \in L^\infty([0, T] \times \mathbb{T}^d, \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d))$, $k = 1, \ldots, n$, $\{ W^k_t \}_{k=1}^n$ is a sequence of independent one dimensional Wiener processes on $(\Omega, \mathcal{F}, \{ \mathcal{F}_t \}_{t \geq 0}, \mathbb{P})$. Assume also that

$$\sum_{k=1}^n |B_k|_{L^\infty([0, T] \times \mathbb{T}^d, \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d))} + |\nabla B_k|_{L^\infty([0, T] \times \mathbb{T}^d, \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d))} + |\Delta B_k|_{L^\infty([0, T] \times \mathbb{T}^d, \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d))} < \infty.$$ 

Let also $f \in L^2(0, T; H)$ and $g_k \in \mathcal{M}^2(0, T; H)$, $k = 1, \ldots, n$. Assume that $u_1, u_2 \in \mathcal{M}^2(0, T; D(A)) \cap L^{2+\delta_0}(\Omega, C([0, T]; V))$ are two solutions of equation:

$$(4.9) \quad du(t) + P((u(t) \nabla) u(t)) dt = (\nu Au(t) + f(t)) dt + \sum_{k=1}^n (PB_k(t) u(t) + g_k(t)) \circ dW^k_t, \quad t \in [0, T],$$

where stochastic integral is in a Stratonovich sense. Then the assumptions of Theorem 4.6 are satisfied and backward uniqueness holds i.e. if $u_1(T) = u_2(T)$, $\mathbb{P}$-a.s., then for all $t \in [0, T]$, $u_1(t) = u_2(t)$, $\mathbb{P}$-a.s.

4.3. Existence of the spectral limit.

Example 4.8. As it is usual, we denote by $\mathbb{T}^n$ the $n$-dimensional torus. We put $L = L^2(\mathbb{T}^n, \mathbb{R}^n)$ and $V = W^{1,2}(\mathbb{T}^n, \mathbb{R}^n)$. Assume that $u \in \mathcal{M}^2(0, T; W^{2,2}(\mathbb{T}^n, \mathbb{R}^n)) \cap L^{2+\delta_0}(\Omega, C([0, T]; V))$ is a unique solution of equation:

$$du^k = \left( \sum_{i,j=1}^n \frac{\partial}{\partial x_i} (a^{ij}(\cdot) \frac{\partial u^k}{\partial x_j}) + \sum_{l=1}^n b_l(t, \cdot) \frac{\partial u^k}{\partial x_l} + \sum_{l=1}^n c^{kl}(t, \cdot) u^l \right) dt + \sum_{m,l=1}^n h^{ml}_k(t) u^l \circ dW^m_t$$

$$u(0) = u_0 \in V, \quad k = 1, \ldots, n.$$ 

Here we assume that matrix $\tilde{A} = (a^{ij}(x))_{i,j=1}^n, x \in \mathbb{T}^n$ is strictly positive definite, $a^{ij} \in L^\infty(\mathbb{T}^n)$, $i, j = 1, \ldots, n, h^{ml}_k \in L^\infty([0, \infty) \times \mathbb{T}^n)$, $k, m, l = 1, \ldots, n$,

$$\int_0^\infty (|h(s)|_{L^\infty(\mathbb{T}^n)}^2 + |\nabla h(s)|_{L^\infty(\mathbb{T}^n)}^2 + |\Delta h(s)|_{L^\infty(\mathbb{T}^n)}^2) ds < \infty,$$

$$\int_0^\infty (|b(s)|_{L^\infty(\mathbb{T}^n)} + |c(s)|_{L^\infty(\mathbb{T}^n)}) ds < \infty.$$ 

Then assumptions of the Theorem 1.7 are satisfied and the spectral limit exists.

APPENDIX A. SOME USEFUL KNOWN RESULTS

We present here, for convenience of readers, some standard definitions, lemmas and theorems used in the article. We follow here book [19], appendix C and references therein.

Definition A.1. A family $(f_j)_{j \in J}$ of measurable functions $f_j : \Omega \to \mathbb{R}$ is called uniformly integrable if

$$\lim_{M \to \infty} \left( \sup_{j \in J} \int_{|f_j| > M} |f_j| d\mathbb{P} \right) = 0.$$
Definition A.2. An increasing and convex function $\psi : [0, \infty) \to [0, \infty)$ is called a uniform integrability test function if and only if $\psi$ is and
\[
\lim_{x \to \infty} \frac{\psi(x)}{x} = \infty.
\]

Example A.3. $\psi(x) = x^p, p > 1, x \geq 0$.

Theorem A.4. The family $(f_j)_{j \in J}$ of measurable functions $f_j : \Omega \to \mathbb{R}$ is uniformly integrable if and only if there is a uniform integrability test function $\psi$ such that
\[
\sup_{j \in J} \int \psi(|f_j|) \, d\mathbb{P} < \infty.
\]

Theorem A.5. Suppose $\{f_k\}_{k \geq 1}$ is a family of measurable functions $f_k : \Omega \to \mathbb{R}$ such that
\[
\lim_{k \to \infty} f_k(\omega) = f(\omega), \text{ for a.a. } \omega.
\]

Then the following are equivalent:
1. The family $\{f_k\}_{k \geq 1}$ is uniformly integrable.
2. $f \in L^1(\mathbb{P})$ and $f_k$ converges to $f$ in $L^1(\mathbb{P})$.

Now we will give two applications of the notion of uniform integrability:

Proposition A.6 (Ex. 7.12, a), p.132 of [19]). Suppose that $\{Z_t\}_{t \in [0, \infty)}$ is a local martingale such that for some $T > 0$ the family
\[
\{Z(\tau) : \tau \leq T, \tau \text{ is a stopping time}\}
\]
is uniformly integrable. Then $\{Z_t\}_{t \in [0, T]}$ is a martingale.

Theorem A.7 (Doob’s martingale convergence Theorem). Let $\{Z_t\}_{t \geq 0}$ be a right continuous super-martingale. Then the following are equivalent:
1. $\{Z_t\}_{t \geq 0}$ is uniformly integrable.
2. There exist $Z \in L^1(\mathbb{P})$ such that $Z_t \to Z$ $\mathbb{P}$-a.e. and $Z_t \to Z$ in $L^1(\mathbb{P})$.

Proposition A.8. If $t \in [0, T]$ and $H \in L^1(\Omega, \mathcal{F}_t, \mathbb{P})$, then
\[
\mathbb{E}_{\mathbb{Q}^t} H = \mathbb{E}M(\mathbb{F}_T | H) = \mathbb{E} | \mathbb{E}(M_\epsilon(T)H | \mathcal{F}_t) | = \mathbb{E}H \mathbb{E}(M_\epsilon(T) | \mathcal{F}_t) = \mathbb{E}M_\epsilon(t)H.
\]

Lemma A.9. Assume that function $\theta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfies properties
i. $\theta(s, t) = \theta(s, u)\theta(u, t)$, for any $s, u, t \in \mathbb{R}^+$.
ii. $\theta(s, s) = 1$ for any $s \in \mathbb{R}^+$.
iii. There exist finite limit $\theta(s) = \lim_{t \to \infty} \theta(s, t), s \in \mathbb{R}^+$.
iv. There exist $u_0 \in \mathbb{R}^+$ such that $\theta(u_0) > 0$.

Then $\lim_{s \to \infty} \theta(s) = 1$.

Proof. By property (i) applied with $s = \tilde{s}, u = \tilde{t}, t = \tilde{s}$ and property (ii) we infer that
\[
\theta(\tilde{s}, \tilde{t})\theta(\tilde{t}, \tilde{s}) = \theta(\tilde{s}, \tilde{s}) = 1, \tilde{s}, \tilde{t} \in \mathbb{R}^+.
\]

Therefore $\theta(s, t) > 0$ for all $s, t \in \mathbb{R}^+$ and
\[
\theta(s, t) = \theta(t, s)^{-1}, t, s \in \mathbb{R}^+.
\]

By properties (i) and (iii) we deduce that
\[
\theta(s) = \lim_{t \to \infty} \theta(s, u)\theta(u, t) = \theta(s, u)\theta(u), s, u \in \mathbb{R}^+.
\]

Therefore, by equality (A.4) with $u = u_0$ and property (iv) we infer that
\[
\theta(s) = \theta(s, u_0)\theta(u_0) > 0, s \in \mathbb{R}^+.
\]
Combining equalities (A.3) and (A.4) we get
\begin{equation}
(A.6)
\theta(s) = \frac{\theta(u)}{\theta(u, s)}, \quad u, s \in \mathbb{R}^+.
\end{equation}
Hence, by identity (A.5) we infer that
\begin{equation}
\lim_{s \to \infty} \theta(s) = \lim_{s \to \infty} \frac{\theta(u)}{\theta(u, s)} = \frac{\theta(u)}{\theta(u)} = 1.
\end{equation}

**Lemma A.10.** Let $H$ be Hilbert space, $C \in \mathcal{L}(H, H)$, $\varepsilon > 0$, $F : H \to \mathbb{R}$ is defined by
\[ F(x) = \frac{<Cx,x>}{|x|^2 + \varepsilon}, \quad x \in H. \]
Then $F$ is of $C^2$-class, the 1st and 2nd derivatives of $F$ are continuous bounded functions and they are given by the following formulas:
\begin{equation}
(A.7)
F'(x)h_1 = \frac{2<(Cx,h_1)>}{|x|^2 + \varepsilon} - \frac{2<(Cx,x)<h_1,h_1>}{(|x|^2 + \varepsilon)^2},
\end{equation}
\begin{equation}
(A.8)
F''(x)(h_1,h_2) = 2\frac{<(Ch_1,h_2)>}{|x|^2 + \varepsilon} - 4\frac{(Cx,h_1)<h_2,h_2>}{(|x|^2 + \varepsilon)^2} - 4\frac{(Cx,h_2)<h_1,h_1>}{(|x|^2 + \varepsilon)^2} - 2\frac{(Cx,x)<h_2,h_1>}{(|x|^2 + \varepsilon)^2} + 8\frac{(Cx,x)<h_1,h_1><h_2,h_2>}{(|x|^2 + \varepsilon)^3}.
\end{equation}

**Proof of Lemma A.10** Proof is omitted.

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