Steiner Tree in $k$-star Caterpillar Convex Bipartite Graphs - A Dichotomy

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Abstract. The class of $k$-star caterpillar convex bipartite graphs generalizes the class of convex bipartite graphs. For a bipartite graph with partitions $X$ and $Y$, we associate a $k$-star caterpillar on $X$ such that for each vertex in $Y$, its neighborhood induces a tree. The $k$-star caterpillar on $X$ is imaginary and if the imaginary structure is a path (0-star caterpillar), then it is the class of convex bipartite graphs. The minimum Steiner tree problem (STREE) is defined as follows: given a connected graph $G = (V, E)$ and a subset of vertices $R \subseteq V(G)$, the objective is to find a minimum cardinality set $S \subseteq V(G)$ such that the set $R \cup S$ induces a connected subgraph. STREE is known to be NP-complete on general graphs as well as for special graph classes such as chordal graphs, bipartite graphs, and chordal bipartite graphs. The complexity of STREE in convex bipartite graphs, which is a popular subclass of chordal bipartite graphs, is open. In this paper, we introduce $k$-star caterpillar convex bipartite graphs, and show that STREE is NP-complete for 1-star caterpillar convex bipartite graphs and polynomial-time solvable for 0-star caterpillar convex bipartite graphs (also known as convex bipartite graphs). In [1], it is shown that STREE in chordal bipartite graphs is NP-complete. A close look at the reduction instances reveal that the instances are 3-star caterpillar convex bipartite graphs, and in this paper, we strengthen the result of [1].

Keywords: $k$-star caterpillar convex bipartite graphs, Steiner tree, chordal bipartite graphs, convex bipartite graphs.

1 Introduction

Many classical subset problems such as vertex cover, independent set and dominating set, have attracted the researchers in the field of theory and computing, examining the following aspects: (i) to know whether the problem is polynomial-time solvable or NP-complete on general graphs (ii) the status of the problem in well-known special graph classes such as chordal graphs, and bipartite graphs (iii) if NP-complete on general graphs, then investigate the instances generated out of the polynomial-time reduction in an attempt to identify easy vs hard instances. (iv) if NP-complete on general graphs, investigate the problem from a parameterized complexity perspective with a suitable parameter of interest.

The minimum Steiner tree problem (STREE) [2] is a classical subset problem. Given an unweighted connected graph $G$ and $R \subseteq V(G)$, the problem asks for a minimum cardinality set $S \subseteq V(G)$ such that the set $R \cup S$ induces a connected subgraph. Subsequently using traversals algorithm such as breadth first search or depth first search, one can obtain a tree on $R \cup S$, such a tree is known as the Steiner tree for the terminal set $R$. The sets $R$ and $S$ are known as the terminal set and the Steiner set, respectively, in the literature. Interestingly, STREE has applications in road construction [3], communication networks [1], computer networks and many more [5]. Two of the special cases of STREE are (i) $|R| = 2$; in this case, solving STREE is equivalent to solving the shortest path problem between the vertices in $R$ (ii) $|R| = |V(G)|$; solving this is equivalent to solving the minimum spanning tree problem assuming all edge weights are one.

On the complexity front, STREE is NP-complete on general, and bipartite graphs as there is a polynomial-time reduction from the Exact-3-Cover problem [4]. Further, it is NP-complete on bipartite graphs [6], split graphs [7], and chordal bipartite graphs [1]. For a computational problem known to be NP-complete on a

* This work is partially supported by DST-ECRA Project—ECR/2017/001442.
All graphs considered here are simple, undirected, connected, unweighted graphs. We follow the definitions and notation from [15,16]. For a graph $G$, let $V(G)$ denote the vertex set and $E(G)$ denote the edge set. The edge set $E(G) = \{u,v\}$ if $u$ is adjacent to $v$ in $G$. The open neighborhood of a vertex $v$ in $G$ is denoted as $N_G(v) = \{u \mid \{u,v\} \in E(G)\}$ and we denote the closed neighborhood of a vertex $v$ in $G$ as $N_G[v] = N_G(v) \cup \{v\}$. The degree of a vertex $v$ in $G$ is $d_G(v) = |N_G(v)|$. We denote by $\delta(G) = \min \{d_G(v) \mid v \in V(G)\}$. A vertex $v$ is said to be pendant, if $d_G(v) = 1$. For $V' \subseteq V(G)$, the graph induced on $V'$ is represented as $G[V']$. A bipartite graph is chordal bipartite, if every cycle of length strictly greater than four has a chord. A bipartite graph $G(X,Y)$ partitioned into $X$ and $Y$ is a convex bipartite graph, if there is an ordering of $X = (x_1,\ldots,x_m)$ such that for all $y \in Y$, $N_G(y)$ is consecutive with respect to the ordering of $X$, and $G$ is said to have convexity with respect to $X$. For $X = (x_1,\ldots,x_m)$, when we say $x_i < x_j$, we mean that $x_i$
appears before \( x_j \) in the ordering. Similarly, one can define convexity with respect to \( Y \). Convex bipartite graphs can also be interpreted as follows: there exists an imaginary path on \( X \) and for each \( y \in Y \), \( N_G(y) \) is an interval (subpath in the imaginary path) in \( X \). For every vertex \( y \in Y \), \( l(y) \) is the least vertex of \( X \) adjacent to \( y \) and \( r(y) \) is the greatest vertex of \( X \) adjacent to \( y \). We define \( T(x_i) \) and a vertex \( w(x_i) \in N(x_i) \) as follows: for \( i \geq 1 \), \( T(x_i) = \{ y \mid y \in N(x_i) \} \), and \( r(y) \) is the maximum, and \( w(x_i) \) is an arbitrary vertex of \( T(x_i) \). A \( k \)-star caterpillar, \( k \geq 1 \), is a tree \( T \) where \( V(T) = \{ x_1, \ldots, x_p \} \cup \{ x_{i1}, \ldots, x_{ik} \} \), \( 1 \leq i \leq p \) and \( E(T) = \{ \{ x_j, x_{i+1} \} \mid 1 \leq j \leq p-1 \} \cup \{ \{ x_j, x_{il} \} \mid 1 \leq j \leq p, 1 \leq l \leq k \} \). A 0-star caterpillar is a tree \( T \) where \( V(T) = \{ x_1, \ldots, x_p \} \) and \( E(T) = \{ \{ x_j, x_{i+1} \} \mid 1 \leq j \leq p-1 \} \). Equivalently, 0-star caterpillar is a path on \( p \) vertices.

3 A Polynomial-time Algorithm for STREE in Convex Bipartite graphs

We shall present our results by considering possible values for the terminal set; towards this, we partition the inputs into five sets. Throughout this paper, we assume convexity on \( X \). We shall next present the solution to STREE when \( R = X \).

3.1 STREE with \( R = X \)

We present a greedy algorithm (Algorithm 1) to solve this case. Note that if \( |X| = 1 \), then the Steiner set is empty. For \( |X| \geq 2 \), using convexity on \( X \), we identify the vertex \( y \) adjacent to \( x_1 \) such that \( r(y) \) is maximum and we continue from \( r(y) \). Interestingly, this greedy approach is indeed optimum, which we establish in this section through a classical cut-and-paste argument\(^{17}\). Let \( z_1 = x_1, z_{i+1} = r(w(z_i)), i \geq 1 \).

**Algorithm 1** *STREE with \( R = X \)*

1: Input: A connected convex bipartite graph \( G \) with \( R = X \).
2: Initialize \( i = 1, z_1 = x_1, z = r(w(z_1)) \)
3: Initialize Steiner set \( S = \{ w(z_1) \} \)
4: while \( z \neq x_m \) do
5: \( z_{i+1} = r(w(z_i)) \), and \( S = S \cup \{ w(z_{i+1}) \} \)
6: \( z = r(w(z_{i+1})) \), and \( i = i + 1 \)
7: end while

![Fig. 1: An illustration for the case \( R = X \)](image)

An illustration for \( R = X \) is given in Figure 1 and its trace for Algorithm 1 is given below. Note \( R = \{ x_1, x_2, x_3, x_4 \} \). As part of the initialization, we set \( i = 1, z_1 = x_1, z = r(w(z_1)) = x_3, S = \{ y_2 \} \).
During the first iteration, we see that \( z = x_3 \neq x_4 \) is true. Thus, \( z_2 = x_3 \) and \( S = \{y_2\} \cup \{y_4\} \). Also, \( z = x_4, \ i = 3 \). Hence the solution output by our algorithm is, \( S = \{y_2, y_4\} \). Note that \( \{y_1, y_3\} \) is also an optimal solution.

**Observation 1** For Algorithm [2], there exists \( k \) such that \( r(w(z_k)) = x_m \), and the Steiner set is \( S = \{w(z_1), \ldots, w(z_k)\} \). Thus, Algorithm [2] terminates.

**Theorem 2.** Let \( G \) be a convex bipartite graph. The set \( S \) of Steiner vertices of \( G \) obtained from Algorithm [2] is a minimum Steiner set.

**Proof.** Without loss of generality, we shall arrange the vertices in \( Y = (y_1, \ldots, y_n) \) such that \( S = (y_1, \ldots, y_k) \), are the vertices chosen by the Algorithm [2] in order. We use a binary vector \( A = (a_1, \ldots, a_n) \) to represent the output of our algorithm such that \( a_i = 1 \), if \( y_i \in S \), and \( a_i = 0 \), otherwise. It follows that \( a_i = 1, 1 \leq i \leq k \), \( a_j = 0, k + 1 \leq j \leq n \). Let \( S' \) denote an optimal Steiner set of \( G \). We use a binary vector \( B = (b_1, \ldots, b_n) \) to represent \( S' \) where \( b_i = 1 \), if \( y_i \in S' \), and \( b_i = 0 \), otherwise.

Since \( S' \) is optimal, \( |S'| \leq |S| \). Further, \( |B| = \sum_{i=1}^n b_i \leq \sum_{i=1}^n a_i = |A| \). To show that \( |S| = |S'| \), we need to prove that \( |S| \leq |S'| \), that is \( |A| \leq |B| \), we need to prove \( \sum_{i=1}^n a_i \leq \sum_{i=1}^n b_i \). We prove by mathematical induction on the number of indices \( d \) where \( A \) and \( B \) differ.

**Base case:** when \( d = 0 \), \( |A| = |B| \). Thus, \( |A| \leq |B| \).

**Induction Hypothesis:** Assume that for \( d \geq 1 \), if \( A \) and \( B \) differ in fewer than \( d \) positions, then \( |A| \leq |B| \).

**Induction Step:** Let the binary vectors \( A, B \) differ by \( d \geq 1 \) positions. Let \( j \) be the least index such that \( a_j \neq b_j \). Since \( S' \) is an optimal solution, it cannot be the case that \( a_j = 0 \) and \( b_j = 1 \). Therefore, \( a_j = 1 \) and \( b_j = 0 \). Further, \( j \leq k \). This implies that \( b_1 = 1 \) for \( 1 \leq i < j \), and \( b_j = 0 \). Recall that \( z_j = r(w(z_{j-1})) \), \( 2 \leq j \leq m \) and \( y_j = w(z_j) \). Observe that \( N(y_{j-1}) \cap N(y_j) \neq \emptyset \) and \( N(y_{j+1}) \cap N(y_j) \neq \emptyset \) as \( G \) is connected, and by our choice of \( y_j \), for each \( 1 \leq i \leq j-2, j+2 \leq i \leq k \), \( N(y_j) \cap N(y_i) = \emptyset \). Since \( S' \) is an optimal solution, there exists \( b_j = 1 \) with \( l > k \) such that \( \{z_j, y_l\} \in E(G) \). If \( \{z_j, y_l\} \notin E(G) \), then feasibility of the solution (connectedness) is lost. That is, the graph induced on \( S' \cup X \) has vertices \( z_j, z_{j+1} \) in different connected components. This contradicts the fact that \( S' \) is an optimal Steiner set. Therefore, \( \{z_j, y_l\} \in E(G) \).

Since our algorithm has chosen \( y_j \) over \( y_l \), it follows that \( r(y_j) \leq r(y_l) \), and \( N(\{y_1, \ldots, y_{j-1}, y_l\}) \subseteq N(\{y_1, \ldots, y_{j-1}, y_j\}) \). As part of our cut-and-paste argument, we modify the vector \( B \) to obtain a vector \( C = (c_1, \ldots, c_n) \) as follows: \( c_j = b_j, 1 \leq i \leq n, i \notin \{j, l\} \), \( c_j = 1, c_l = 0 \). It follows that the binary vectors \( C \) (modified \( B \)) and \( A \) differ in fewer than \( d \) positions and by the induction hypothesis, \( |A| \leq |C| \). Note that \( |C| = |B| \). Thus, \( |A| \leq |B| \). We continue this argument, if there is still a mismatch between \( A \) and \( C \), and stop this cut-and-paste argument, when \( d = 0 \). Thus, \( |A| = |B| \), and \( A \) is also an optimal solution. This completes the proof of the theorem.

**Remarks:** The proof is constructive in nature, and given an optimal solution, we can obtain another optimal solution by the constructive argument mentioned in the proof.

### 3.2 STREE with \( R \subset X \)

We shall now present a greedy algorithm (Algorithm [2]) for finding the Steiner tree in a convex bipartite graph with \( R \subset X \). When \( |X| \leq 2 \), the Steiner set is empty. As part of Algorithm [2] we shall consider \( |X| \geq 3 \). Consider \( R = \{z_1, \ldots, z_k\} \), recall that \( z_i \) appears before \( z_{i+1} \) in the ordering of \( X \). We start from \( z_1 \) and check whether the exploration can continue from \( r(w(z_1)) \) or \( z_j \), where \( z_j \) is the greatest indexed vertex in \( R \) adjacent to \( w(z_1) \). Let \( S_1 \) be the set of vertices chosen by algorithm for obtaining path from \( p = r(w(z)) \) until \( z_{j+1} \), and \( S_2 \) be the set of vertices chosen by algorithm for obtaining path from \( w(q) \) until \( z_{j+1} \), where \( q = z_j \). We choose the minimum output of these two subsolutions at each iteration. This greedy strategy is optimal.
which we establish in this section.

Algorithm 2 \textit{STREE with }$R \subset X$

1: \textbf{Input}: A connected convex bipartite graph $G$ with $R \subset X$.
2: Prune the vertices in $X$ less than $z = z_1$
3: Initialize Steiner set $S = \{w(z)\}$, and let $z_j$ be the greatest indexed vertex in $R$ adjacent to $w(z)$
4: \textbf{while} $j < k$ \textbf{do}
5: \hspace{1em} Initialize $p = r(w(z))$, $q = z_j$
6: \hspace{1em} if $p \neq q$ \textbf{then}
7: \hspace{2em} Initialize $S_1 = \{p\}$, $S_2 = \emptyset$
8: \hspace{1em} \textbf{else}
9: \hspace{2em} Initialize $S_1 = \emptyset$, $S_2 = \emptyset$
10: \hspace{1em} \textbf{end if}
11: \hspace{1em} \textbf{while} $\{z_{j+1}, w(p)\} \notin E(G)$ \textbf{do}
12: \hspace{2em} $S_1 = S_1 \cup \{w(p), r(w(p))\}$
13: \hspace{2em} $p = r(w(p))$
14: \hspace{1em} \textbf{end while}
15: \hspace{1em} \textbf{while} $\{z_{j+1}, w(q)\} \notin E(G)$ \textbf{do}
16: \hspace{2em} $S_2 = S_2 \cup \{w(q), r(w(q))\}$
17: \hspace{2em} $q = r(w(q))$
18: \hspace{1em} \textbf{end while}
19: \hspace{1em} \textbf{if} $|S_1| < |S_2|$ \textbf{then}
20: \hspace{2em} $S = S \cup S_1 \cup \{w(p)\}$; $z = p$
21: \hspace{2em} Update $z_j$ to be the greatest indexed vertex in $R$ adjacent to $w(p)$
22: \hspace{1em} \textbf{else}
23: \hspace{2em} $S = S \cup S_2 \cup \{w(q)\}$; $z = q$
24: \hspace{2em} Update $z_j$ to be the greatest indexed vertex in $R$ adjacent to $w(q)$
25: \hspace{1em} \textbf{end if}
26: \textbf{end while}
An illustration for $R \subset X$ is given in Figure 2 and its trace for Algorithm 2 is given below. The terminal vertices are $R = \{z_1 = x_1, z_2 = x_3, z_3 = x_4, z_4 = x_6, z_5 = x_8, z_6 = x_{11}, z_7 = x_{12}, z_8 = x_{13}\}$. Initially, $z = z_1 = x_1, S = \{y_1\}, k = 8$. In Iteration 1; we see that $2 < 8, p = r(w(z_1)) = x_3, q = z_2 = x_3, p = q$. Hence $S_1 = \emptyset, S_2 = \emptyset$. At Step 23, $S$ is updated to $S = \{y_1, y_4\}, z = x_3, z_j = x_4$. In Iteration 2; $3 \leq 8, p = r(w(z_2)) = x_5, q = x_4, p \neq q$. By Step 7, we get $S_1 = \{x_5\}, S_2 = \emptyset$. As per the first while loop; $\{x_6, y_3\} \in E(G)$, therefore the condition is false. In the second loop, $\{x_6, y_4\} \notin E(G), S_2$ is updated as $S_2 = \{y_3, x_5\}, q = x_5$. Further in the next iteration $\{x_6, y_3\} \in E(G)$, therefore the condition is false and the while loop terminates. We see that Step 19 is true, $S$ is updated to $S = \{y_1, y_4\} \cup \{x_5\} \cup \{y_5\}$. Further, $z = x_5, z_j = z_4 = x_6$. In Iteration 3; $4 \leq 8, p = x_6, q = x_6, p = q$. Hence by Step 9, $S_1 = \emptyset, S_2 = \emptyset$. Further in both while loops $\{x_8, y_7\} \in E(G)$, therefore conditions are false. At Step 23, $S$ is updated to $S = \{y_1, y_4, x_5, y_6\} \cup \{y_7\}, z = x_6, z_j = z_5 = x_8$. In Iteration 4; $5 \leq 8, p = x_10, q = x_8, p \neq q$ and $S_1 = \{x_10\}, S_2 = \emptyset$. Since $\{x_11, y_9\} \in E(G)$, the while loop condition fails at Step 11 and for the other while loop $\{x_11, y_7\} \notin E(G)$ is true, at Step 15. Inside the while loop $S_2$ is updated as $S_2 = \{y_7, x_10\}, q = x_10$. At Step 23, $S$ is updated to $S = \{y_1, y_4, x_5, y_6, y_7, x_10, y_9\}, z_j = z_7 = x_{12}, z = x_{10}$. In Iteration 5; $7 \leq 8, p = x_{12}, q = x_{12}$. We see that $p = q$, hence $S_1 = \emptyset, S_2 = \emptyset$. In while loops, since $\{x_{13}, y_{10}\} \in E(G)$, therefore conditions are false. At Step 23, $S$ is updated to $S = \{y_1, y_4, x_5, y_6, y_7, x_{10}, y_9, y_{10}\}, z = x_{12}, z_j = z_8 = x_{12}$. In the next iteration, $8 < 8$ is not true. Thus, Algorithm 2 outputs $S = \{y_1, y_4, x_5, y_6, y_7, x_{10}, y_9, y_{10}\}$.

Observation 3 Let $S$ be any optimal Steiner set. For each Steiner vertex $y \in Y$, there exists at most two Steiner vertices adjacent to $y$ in $S$.

Lemma 1. In Algorithm 2, for each iteration, the difference between $|S_1|$ and $|S_2|$ is at most one.

Proof. Let $z$ be the vertex under consideration in $R$ and $z_j$ is the greatest indexed vertex in $R$ adjacent to $w(z), p = r(w(z)), q = z_j$.

Case 1: $p = q$. Steps 11-18 of Algorithm 2 computes $S_1$ and $S_2$. Since $p = q$, it implies that $|S_1| = |S_2|$.

Case 2: $p \neq q$. Let the path starting from $p$ to $z_{j+1}$ be $P_1$ and, the path starting from $q$ to $z_{j+1}$ be $P_2$. Let $P_1 = w(u_1, u_2, \ldots, u_s, w(u_s), z_{j+1})$ and $P_2 = w(v_1, v_2, \ldots, v_t, v_{t+1} = z_{j+1})$. Observe that Steps 11-14 of Algorithm 2 constructs $P_1$ and updates $S_1; S_1 = V(P_1) \setminus \{w(u_s), z_{j+1}\}$. Similarly, Steps 15-18 of Algorithm 2 constructs $P_2$ and updates $S_2; S_2 = V(P_2) \setminus \{z_{j}, w(v_t), v_{t+1}\}$. Since $G$ is convex on $X$, for $1 \leq k \leq s, 1 \leq t \leq t$, $u_k \geq v_t$ and $\{v_{t+1}, w(u_k)\} \in E(G)$.

Case: For some $i \geq 2, u_i = v_i$. In this case, we observe that $s = t$. Further, $|S_1| = 2s - 1, |S_2| = 2t - 2$. Therefore, $|S_1| - |S_2| = 1$.

Case: For all $i \geq 2, u_i > v_i$. We observe that $s = t - 1$. Therefore, $|S_2| - |S_1| = 1$.

By the definition of $p$ and $q$, the case $u_i < v_i$ cannot happen. From above two cases we see that $|S_1|$ and $|S_2|$ can differ by at most one.

Theorem 4. Let $G$ be a convex bipartite graph. The set $S$ of Steiner vertices of $G$ obtained from Algorithm 2 is a minimum Steiner set.

Proof. Without loss of generality, we shall order the vertices in $G$ as $\sigma = [v_1, \ldots, v_t], t = |V(G)|$ in a way that $S = [v_1, \ldots, v_t]$ are the vertices chosen by Algorithm 2 in order. Note that the ordering $\sigma$ is with respect to the ordering of vertices chosen by the algorithm and not in accordance with the convex ordering of $X$. We use a binary vector $A = (a_1, \ldots, a_t)$ to represent the output of our algorithm such that $a_i = 1$, if $v_i \in S$, and $a_i = 0$, otherwise. It follows that $a_i = 1, 1 \leq i \leq t, a_i = 0, l + 1 \leq j \leq t$. Let $S'$ denote an optimal Steiner set of $G$. We use a binary vector $B = (b_1, \ldots, b_t)$ to represent $S'$ where $b_i = 1$, if $v_i \in S'$, and $b_i = 0$, otherwise. Since $S'$ is optimal, $|S'| \leq |S|$. Further, $|B| = \sum_{i=1}^{t} b_i \leq \sum_{i=1}^{t} a_i = |A|$. To show that $|S'| = |S|$, we need to show that $|S| \leq |S'|$, that is $|A| \leq |B|$. To show that $|A| \leq |B|$, we need to prove $\sum_{i=1}^{t} a_i \leq t b_i$. We prove by strong mathematical induction on the number of indices $d$ where $A$ and $B$ differ.

Base case: when $d = 0, |A| = |B|$. Thus, $|A| \leq |B|$.
**Induction Hypothesis:** Assume that for \( d \geq 1 \), if \( A \) and \( B \) differ in less than \( d \) positions, then \( |A| \leq |B| \).

**Induction Step:** Let the binary vectors \( A, B \) differ by \( d \geq 1 \) positions. Let \( j \) be the least index such that \( a_j \neq b_j \). Note that \( j \leq l \), otherwise \( S' \) is not optimal. This implies that \( b_i = 1, 1 \leq i < j \), and \( b_j = 0 \). We consider the following cases to complete our proof.

**Case 1:** \( v_{j-1} \in X \). Since \( S' \) is an optimal solution, there exists \( b_k = 1, k > l \) such that \( \{v_{j-1}, v_k\} \in E(G) \). Note that \( v_j \in Y \) and \( v_j = w(v_{j-1}) \). Similar to the proof of the previous theorem, we modify \( B \) to obtain a vector \( C \) as follows: \( C = (c_1, \ldots, c_t), \ c_i = b_i, 1 \leq i \leq t, \ i \neq \{j, k\}, \ c_j = 1, \ c_k = 0 \). Note that \( |C| = |B| \). It follows that the binary vectors \( C \) and \( A \) differ in less than \( d \) positions and by the induction hypothesis, \( |A| \leq |C| \).

**Case 2:** \( v_{j-1} \in Y \). We have the following subcases.

**Case 2.1:** \( N(v_{j-1}) \cap R = \emptyset \). Observe that there exists \( b_k = 1, k > l \) such that \( \{v_{j-1}, x_k\} \in E(G) \). Note that \( v_j = r(v_{j-1}) \). In this case, an optimal solution with the corresponding vector \( C \) is obtained from \( B \) by changing the values of \( b_j, b_k \) as \( b_j = 1, b_k = 0 \). It follows that the binary vectors \( C, A \) differ in less than \( d \) positions and by the induction hypothesis, \( |A| \leq |C| \). Note that \( |C| = |B| \). Thus, \( |A| \leq |B| \).

**Case 2.2:** \( N(v_{j-1}) \cap R \neq \emptyset \). Let \( z_0 \) be the greatest indexed vertex in \( N(v_{j-1}) \cap R \). If \( N(z_0) \cap N(z_{k+1}) \neq \emptyset \), then note that \( v_j \in Y \). Observe that there exists \( b_r = 1, r > l \) such that \( \{z_0, v_r\} \in E(G) \). Let \( v_j = w(z_0) \). In this case, an optimal solution with the corresponding vertex \( C \) is obtained from \( B \) by changing the values of \( b_j, b_r \) as \( b_j = 1, b_r = 0 \). Note that \( |C| = |B| \). It follows that the binary vectors \( C, A \) differ in less than \( d \) positions and by the induction hypothesis, \( |A| \leq |C| \). Thus, \( |A| \leq |B| \).

If \( N(z_0) \cap N(z_{k+1}) = \emptyset \), note that \( v_j \in X \) or \( v_j \in Y \). Let \( v_r, r \leq l \) be the least vertex in \( S \) adjacent to \( z_{k+1} \). Note that steps 11-18 of Algorithm 2 construct two paths \( P_1 \) and \( P_2 \), and choose the minimum out of these two paths. Let \( P_1 = (v_j = r(v_{j-1}), w(v_j), \ldots, v_r, z_{k+1}) \), and \( P_2 = (z_0, v_j = w(z_0), \ldots, v_r, z_{k+1}) \).

If \( v_j \in Y \), then algorithm chooses \( P_2 \) as \( P_2 \) is shortest. In this case, the number of vertices included in \( S_B \) by \( P_2 \) is \( r - j + 1 \) and all these vertices appear after \( v_{j-1} \) with respect to \( \sigma \). Let \( Q \) be the vertices chosen by the optimal algorithm to connect \( z_0 \) and \( z_{k+1} \). Since \( P_2 \) is the shortest path and \( Q \) is part of optimal solution, cardinality of \( Q \) is \( r - j + 1 \). We now bring our cut-and-paste argument and update \( S' \) as \( S' = (S' \setminus Q) \cup S_2 \).

Similarly, if \( v_j \in X \), then the algorithm chooses \( P_1 \) as \( P_1 \) is the shortest between \( P_1 \) and \( P_2 \). In this case the number of vertices included in \( S_1 \) by \( P_1 \) is \( r - j + 1 \). Let \( Q \) be the vertices chosen by the optimal algorithm to connect \( z_0 \) and \( z_{k+1} \). Since \( P_1 \) is the shortest path and \( Q \) is part of the optimal solution, the cardinality of \( Q \) is \( r - j + 1 \). We now bring our cut-and-paste argument and update \( S' \) as \( S' = (S' \setminus Q) \cup S_1 \). Let \( C \) (modified \( B \)) be the corresponding binary vector of \( S' \). Note that \( |C| = |B| \). Thus, the binary vectors \( C, A \) differ in less than \( d \) positions and by the induction hypothesis, \( |A| \leq |C| \). This completes the case analysis. We conclude \( |A| = |B| \) and \( A \) is also an optimal solution. This completes the proof of Theorem 3. \( \square \)

### 3.3 STREE when \( R = Y \)

We shall present a greedy algorithm (Algorithm 3) to output a minimum Steiner tree when \( R = Y \). Note that if \( |Y| = 1 \), then the Steiner set is empty. Therefore we work with \( |Y| \geq 2 \). By definition, for each \( y_i \in Y \), \( N(y_i) = \{x_p, x_{p+1}, \ldots, x_q\} \) is an interval. Further, \( l(y_i) = x_p \) and \( r(y_i) = x_q \). For all \( y_i \in Y \), let \( [l_i, r_i] \) represent the interval such that \( l_i = p \) and \( r_i = q \). We arrange the vertices of \( Y \) as \( (y_1, y_2, \ldots, y_n) \) such that for all \( i, j, 1 \leq i < j \leq n, r_i \leq r_j \). We use \( y_i \) to represent the vertex \( y_i \in Y \) as well as the interval corresponding to \( y_i \).
Algorithm 3  \textit{STREE} when $R = Y$

1: \textbf{Input}: A connected convex bipartite graph $G$ with $R = Y$.
2: All intervals are unmarked initially, let $|Y| = n$, Steiner set $S = \{}$
3: \textbf{for} $i = 1, \ i \leq n, \ i = i+1$ \textbf{do}
4: \hspace{1em} \text{if } y_i \text{ is unmarked} \text{ then}
5: \hspace{2em} $S = S \cup \{r_i\}$
6: \hspace{1em} Mark all intervals $y_j$ such that $r_i \in N(y_j)$
7: \hspace{1em} \textbf{else}
8: \hspace{2em} \text{if } r_i = x_m \text{ then}
9: \hspace{3em} \text{Continue}
10: \hspace{2em} \text{else}
11: \hspace{3em} \text{if there exists a marked } y_j \text{ such that } r_i+1 \in N(y_j) \text{ then}
12: \hspace{4em} \text{Continue}
13: \hspace{3em} \text{else}
14: \hspace{4em} $S = S \cup \{r_i\}$
15: \hspace{3em} Mark all intervals $y_j$ such that $r_i \in N(y_j)$
16: \hspace{2em} \textbf{end if}
17: \hspace{1em} \textbf{end if}
18: \hspace{1em} \textbf{end if}
19: \textbf{end for}

\begin{figure}[h]
    \centering
    \includegraphics[width=\textwidth]{algorithm3.png}
    \caption{An illustration for $R = Y$}
\end{figure}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
Iteration number & Condition checking and marking status & Update on S & Update on marking \\
\hline
1 & $1 \leq 5, y_1$ is unmarked & $S = S \cup \{x_2\}$ & Mark $y_1, y_2$ \\
\hline
2 & $2 \leq 5, y_2$ is marked & $r_1 = x_4 \neq x_8, \exists y_j$ such that $\{x_6, y_j\} \in E(G), S = S \cup \{x_4\}$ & Mark $y_3, y_4$ \\
\hline
3 & $3 \leq 5, y_3$ is marked & $r_1 = x_6 \neq x_8, \exists y_j$ such that $\{y_4, x_7\} \in E(G)$ & - \\
\hline
4 & $4 \leq 5, y_4$ is marked & $r_1 = x_7 \neq x_8, \exists y_j$ such that $\{x_8, y_j\} \in E(G), S = S \cup \{x_7\}$ & Mark $y_5$ \\
\hline
5 & $5 \leq 5, y_5$ is marked & $r_1 = x_8 = x_8$ & - \\
\hline
6 & $6 \leq 5$ & & Thus, $S = \{x_2, x_4, x_7\}$ \\
\hline
\end{tabular}
\caption{Trace of Algorithm 3}
\end{table}
An illustration and its interval representation is given in Figure 3, and its trace of Algorithm 3 is given in Table 1. Let \( Z = \{ z_1, z_2, \ldots, z_p \} \subseteq X \) be the vertices selected by our algorithm and \( Z' = \{ z'_1, z'_2, \ldots, z'_q \} \subseteq X \) be the vertices selected by any optimum algorithm. Note that \( z_1 \leq z_2 \leq \ldots \leq z_p \). Further we arrange \( Z' \) such that \( z'_1 \leq z'_2 \leq \ldots \leq z'_q \). For the set \( \{ z_1, z_2, \ldots, z_i \} \), \( N(\{ z_1, z_2, \ldots, z_i \}) = \bigcup_{j=1}^{i} N(z_j) \).

**Theorem 5.** For all indices \( i \leq q \), the following statements are true:
1. \( z'_i \leq z_i \)
2. \( N(\{ z_1, z_2, \ldots, z_i \}) \supseteq N(\{ z'_1, z'_2, \ldots, z'_i \}) \)

**Proof.** By mathematical induction on \( i \).

**Base Case:** Since \( z'_i \leq z'_j \), \( j > 1 \), we have \( \{ y_1, z'_1 \} \in E(G) \). Since our algorithm has chosen \( z_1 \), \( \{ y_1, z_1 \} \in E(G) \). Therefore, \( z'_i \leq z_i \). The ordering of \( Y \) and the convexity of \( X \) imply that \( N(z_1) \supseteq N(z'_1) \).

**Induction Hypothesis:** Assume for \( i \geq 2 \), \( z'_{i-1} \leq z_{i-1} \) and \( N(\{ z_1, z_2, \ldots, z_{i-1} \}) \supseteq N(\{ z'_1, z'_2, \ldots, z'_{i-1} \}) \) are true.

**Induction Step:** We have to prove that when \( i \geq 2 \), \( z'_i \leq z_i \) and \( N(\{ z_1, z_2, \ldots, z_i \}) \supseteq N(\{ z'_1, z'_2, \ldots, z'_i \}) \).

By the induction hypothesis, we know that up to \( i-1 \), \( z'_{i-1} \leq z_{i-1} \) and \( N(\{ z_1, z_2, \ldots, z_{i-1} \}) \supseteq N(\{ z'_1, z'_2, \ldots, z'_{i-1} \}) \). By Steps 5 and 14 of Algorithm 3, it is clear that our algorithm always includes \( z_i = r(y) \) of an interval \( y \), hence \( z'_i \leq z_i \).

Assume on the contrary, \( N(\{ z_1, z_2, \ldots, z_i \}) \not\supseteq N(\{ z'_1, z'_2, \ldots, z'_i \}) \). Then, there exists an interval \( y \) such that \( y \in N(z'_j) \) and \( y \notin N(z_i) \). It is clear that \( r(y) < z_i \) (Recall that \( r(y) \) appears before \( z_i \) in the ordering). Since for each interval \( w \) our algorithm includes some \( x \in N(w) \) in the solution, it must be the case that \( y \in N(z_j) \) for some \( j, 1 \leq j \leq i-1 \) (as illustrated in Figure 4). This implies that \( y \in N(\{ z_1, z_2, \ldots, z_{i-1} \}) \), which is a contradiction.

**Fig. 4: Interval representation of G**

Therefore, \( N(\{ z_1, z_2, \ldots, z_i \}) \supseteq N(\{ z'_1, z'_2, \ldots, z'_i \}) \). Hence the proof.

**Theorem 6.** For all \( k \leq p \), the graph induced on \( N(\{ z_1, z_2, \ldots, z_k \}) \) is connected.

**Proof.** By mathematical induction on \( k \).

**Base Case:** For \( i = 1 \), by definition \( G[N[z_1]] \) is connected.

**Induction Hypothesis:** Assume that for \( i \geq 1 \), \( G[N(\{ z_1, z_2, \ldots, z_i \})] \) is connected.

**Induction Step:** We have to prove that when \( i \geq 1 \), \( G[N(\{ z_1, z_2, \ldots, z_{i+1} \})] \) is connected. By our induction hypothesis, we know that \( G[N(\{ z_1, z_2, \ldots, z_i \})] \) is connected. A vertex \( z_{i+1} \) can be added to \( S \) for two reasons:

**Case 1:** There exists \( y \) such that \( r(y) = z_{i+1} \) and \( y \) is unmarked. As per Step 6 of our algorithm, \( z_{i+1} \) is included in the solution and \( y \) is labelled as marked. Since \( G \) is connected, there exists a marked interval \( w \in Y \) adjacent to \( z_{i+1} \) such that \( z_{i+1} = r(w) \) or \( z_{i+1} < r(w) \). Therefore, graph induced on \( N(\{ z_1, z_2, \ldots, z_i \}) \cup \{ z_{i+1}, w, y \} \) is connected. (Inclusion of \( z_{i+1} \) as per the illustration in Figure 5)
Case 2: There exists \( y \) such that \( r(y) = z_{i+1} \) and \( y \) is marked. It must be the case that there exists an unmarked \( w \in Y \) which is adjacent to \( z_{i+1} \) and not adjacent to \( z_i \). To ensure connectedness between \( G[N[\{z_1, z_2, \ldots, z_i\}]] \) and \( w \), our algorithm chooses \( z_{i+1} \). Since \( y \) is ending at \( z_{i+1} \), then \( y \) is adjacent to one of \( z_1, z_2, \ldots, z_i \). Therefore, graph induced on \( N[\{z_1, z_2, \ldots, z_i\}] \cup \{z_{i+1}, w, y\} \) is connected. (Inclusion of \( z_{i+1} \) as per the illustration in Figure 6)

Therefore, by both Case 1 and Case 2, \( G[N[\{z_1, z_2, \ldots, z_{i+1}\}]] \) is connected. \( \square \)

**Theorem 7.** Algorithm 3 outputs a minimum Steiner set, that is \( p = q \).

**Proof.** By Theorem 5, we know that if \( i = q \), then \( N(\{z_1, z_2, \ldots, z_q\}) \supseteq N(\{z'_1, z'_2, \ldots, z'_q\}) \). By Theorem 6, \( N[\{z_1, z_2, \ldots, z_q\}] \) is connected. Hence \( p = q \).

**Time complexity analysis:** For vertices in \( Y \), we can maintain an additional data structure so that for each \( y \in Y \), \( l(y) \) and \( r(y) \) can be computed in linear time. Further, using this data structure and adjacency list of the underlying graph Algorithms 1, 2, and 3 can be implemented in \( O(m + n) \), linear in the input size.

### 3.4 STREE when \( R \subset Y \)

We shall present a dynamic programming based solution for the case \( R \subset Y \). Let \( \sigma = (y_1, y_2, \ldots, y_n) \) be the ordering of vertices in \( Y \) satisfying the following conditions; for all \( i, j, 1 \leq i \leq j \leq n \), \( y_i \) appears before \( y_j \) in \( \sigma \), if (i) \( l_i < l_j \), or (ii) \( l_i = l_j \) and \( r_i \geq r_j \).

We denote by \( \sigma(y_i) < \sigma(y_j) \), if \( y_i \) appears before \( y_j \) in \( \sigma \). Similar to Section 3.3, in this section we work with the underlying interval representation of \( G \). Recall that for \( y_i \in Y \), \( N(y_i) = \{x_p, x_{p+1}, \ldots, x_q\}, l_i = p \) and \( r_i = q \).
For \( z \in Y \), \( N(z) = \{x_p, x_{p+1}, \ldots, x_q\} \) such that \( l(z) = x_p \) and \( r(z) = x_q \), we denote by \( l_z = p \) and \( r_z = q \). Let \( R = \{z_1, z_2, \ldots, z_k\} \subset Y \) such that \( \sigma(z_1) < \sigma(z_2) < \ldots < \sigma(z_k) \). For \( z_k \in R \), \( u = l_k \) and similarly \( x_i = r(z_k) \) and \( v = r_k \). Let \( x_r = l(z_1) \) and \( W = \{w_1, w_2, \ldots, w_t\} = \{x_r, \ldots, x_m\} \), \( t = m - r + 1 \).

Note that \( x_1, \ldots, x_{r-1} \) is not considered for our discussion, since no \( z \in R \) is adjacent to \( x_1, \ldots, x_{r-1} \). Therefore, we work with \( W \) and \( Y \). Further, for \( y \in Y \), we remove the edges \( \{y, x_i\} \in E(G), 1 \leq i \leq r - 1 \).

Let \( S \) be the set of Steiner vertices required to connect \( R \) in \( G \).

We classify \( I = [G, R = \{z_1, z_2, \ldots, z_k\}] \) into four equivalence classes which are defined as follows:

\[
E_1 = \{I \mid \exists y_c \text{ such that } y_c \in N(w_{u-1}) \text{ and } r_c \geq r_k, \text{ and } \exists y_d \text{ such that } y_c \neq y_d, y_d \in N(w_{u-1}) \text{ and } l_k \leq r_d < r_k\}.
\]

\[
E_2 = \{I \mid \exists y_d \text{ such that } y_d \in N(w_{u-1}) \text{ and } l_k \leq r_d < r_k, \text{ and } \exists y_c \text{ such that } y_c \neq y_d, y_c \in N(w_{u-1}) \text{ and } r_c \geq r_k\}.
\]

\[
E_3 = \{I \mid \exists y_c \text{ such that } y_c \in N(w_{u-1}) \text{ and } r_c \geq r_k, \text{ and } \exists y_d \text{ such that } y_d \neq y_c, y_d \in N(w_{u-1}) \text{ and } l_k \leq r_d < r_k\}.
\]

\[
E_4 = \{I \mid l_k = 1\}.
\]

Informally, \( E_1 \) considers all those inputs such that in the underlying interval representation there exists an interval \( y_c \) which overlaps with \( z_k \), adjacent to \( l_k - 1 \) and it ends on or after \( r_k \), further, there does not exist an interval \( y_d \) which overlaps with \( z_k \), adjacent to \( l_k - 1 \) and it ends before \( r_k \).

Similarly, \( E_2 \) considers all those inputs such that in the underlying interval representation there exists an interval \( y_d \) which overlaps with \( z_k \), adjacent to \( l_k - 1 \) and it ends before \( r_k \), further, there does not exist an interval \( y_c \) which overlaps with \( z_k \), adjacent to \( l_k - 1 \) and it ends on or after \( r_k \).

Likewise, \( E_3 \) considers all those inputs such that in the underlying interval representation there exists an interval \( y_d \) which overlaps with \( z_k \) and it ends before \( r_k \), and there exists an interval \( y_c \) which overlaps with \( z_k \) and it ends on or after \( r_k \).

In \( E_4 \), we consider all intervals such that \( l_k = 1 \). This means each \( z_i \in R \) is adjacent to \( x_1 \).

Note that, \( E_1, E_2, E_3, \) and \( E_4 \) clearly partitions the set of all inputs.

We define an indicator function \( b(y) \) for each \( y \in Y \) such that:

\[
b(y) = \begin{cases} 1 & \text{if } y \in Y \setminus R \\ 0 & \text{if } y \in R \end{cases}
\]

Note that \( b(z_1) = b(z_2) = \ldots = b(z_k) = 0 \).

**Optimal Substructure Property:** We now show that an optimal solution to the Steiner tree problem for the case \( R \subset Y \) lies within its optimal solutions to subproblems. Let \( T \) be an optimal Steiner tree containing \( R \). Clearly, each \( z \in R \) appears as a leaf in \( T \). Let \( w \) be a parent of \( z_k \). If we root the tree at \( w \), then both left and right subtrees of \( w \) must be optimal. Note that the optimal right subtree contains each \( z_i \), \( 1 \leq i \leq k - 1 \) as a leaf. Further, \( w \) is in \( X \) and \( w \) is adjacent to \( z_k \) and \( y \in Y \). Note that \( y \) is \( z_i \), \( 1 \leq i \leq k - 1 \) or \( y \in Y \setminus R \). Moreover, there are many candidates for \( y \) whose corresponding intervals overlap with \( z_k \). Clearly, if all choices of \( y \) are considered, then we are sure of obtaining an optimal \( y \) using which \( z_k \) is connected with the rest of vertices in \( R \).

Using our optimal substructure, we define a function \( F \) which computes the minimum number of Steiner vertices required to connect \( z_k \) with the rest of \( R \). If \( z_k \) overlaps with some \( z_i \), \( 1 \leq i \leq k - 1 \), then to obtain an optimal solution to the problem we include the appropriate \( x \in N_G(z_k) \cap N_G(z_i) \) and the optimal solutions obtained from the subproblems. If \( z_k \) has no overlap with any \( z_i \), \( 1 \leq i \leq k - 1 \), then \( z_k \) overlaps with \( y \in Y \setminus R \) and there may be many such \( y \). To obtain an optimal solution to the problem we include the appropriate \( y \) and \( x \in N_G(z_k) \cap N_G(y) \) and the optimal solutions obtained from the subproblems. We now present our recursive solution to compute \( F \).

We define a function \( F[u, v] \) which denotes the number of Steiner vertices required in \( G \) to connect \( z_k \in R \) with \( z \in R, 1 \leq i \leq k - 1 \).

The function \( F[u, v] \) for \( I \) in \( E_1 \) or \( E_2 \) or \( E_3 \) or \( E_4 \) is defined as follows: \( F[u, v] = \min_{z} f[u, v], \) for each \( z \in Y \) such that \( u = l_z \) and \( v = r_z \), where \( f[u, v] \) is defined as follows:
Case 1: $I \in E_1$. Then, $\exists y_c$ such that $y_c \in N(w_{u-1})$ and $r_c \geq r_k$, and $\exists y_d$ such that $y_c \neq y_d$, $y_d \in N(w_{u-1})$ and $l_k \leq r_d < r_k$.

$$f[u, v] = 1 + \min_{y_c} F[p, q], \; 1 \leq p \leq u - 1, \; v \leq q \leq t$$

Case 2: $I \in E_2$. Then, $\exists y_d$ such that $y_d \in N(w_{u-1})$ and $l_k \leq r_d < r_k$, and $\exists y_c$ such that $y_c \neq y_d$, $y_c \in N(w_{u-1})$ and $r_c \geq r_k$.

$$f[u, v] = 1 + b(z_k) + \min_{y_d} F[p, s], \; 1 \leq p \leq u - 1, \; u \leq s \leq v - 1$$

Case 3: $I \in E_3$. Then, $\exists y_c$ such that $y_c \in N(w_{u-1})$ and $r_c \geq r_k$, and $\exists y_d$ such that $y_d \neq y_c$, $y_d \in N(w_{u-1})$ and $l_k \leq r_d < r_k$.

$$f[u, v] = \min\{1 + \min_{y_c} F[p, q], 1 + b(z_k) + \min_{y_d} F[p, s]\}, \; 1 \leq p \leq u - 1, \; v \leq q \leq t, \; u \leq s \leq v - 1$$

Case 4: $I \in E_4$

$F[u, v] = 1$, since $l_k = 1$, for each $z_i \in R$, $l_i = 1$.

Note that, when the input comes from equivalence class $E_i$, there may be many identical intervals of type $z$ such that $l_z = u$ and $r_z = v$. Further, we compute $f[u, v]$ for each interval $z$, and $F[u, v]$ is precisely the minimum among $f[u, v]$.

We observe that $F[u, v]$ depends on $F[p, q]$ or $F[p, s]$. The above definition has overlapping subproblems which we shall exploit and present a solution using dynamic programming paradigm. Towards this end we now define a recursive solution using which we populate the dynamic programming table in a bottom-up.

Recursive solution:

**Base case:**
For $z \in Y$, $l_z = 1$, $j = r_z$, we define $F[1, j] = \min_z f[1, j], \; 1 \leq j \leq t$, the value of $f[1, j]$ is

- $f[1, j] = 1$, if $z \in Y \setminus R$.
- $f[1, j] = 0$, if $z \in R$.
- $f[1, j] = \infty$, if no such $z$ exists.

For $2 \leq i \leq j \leq t$, for each $z \in Y$, $i = l_z$ and $j = r_z$, $F[i, j] = \min_z f[i, j]$, where $f[i, j]$ is

Case 1: $\exists y_c$ such that $y_c \in N(w_{i-1})$ and $r_c \geq j$, and $\exists y_d$ such that $y_d \neq y_c$, $y_d \in N(w_{i-1})$ and $i \leq r_d < j$

$$f[i, j] = 1 + \min_{y_c} F[p, q], \; 1 \leq p \leq i - 1, \; j \leq q \leq t$$

Case 2: $\exists y_d$ such that $y_d \in N(w_{i-1})$ and $i \leq r_d < j$, and $\exists y_c$ such that $y_c \neq y_d$, $y_c \in N(w_{i-1})$ and $r_c \geq j - 1$

$$f[i, j] = 1 + b(z_k) + \min_{y_d} F[p, s], \; 1 \leq p \leq i - 1, \; i \leq s \leq j - 1$$

Case 3: $\exists y_c$ such that $y_c \in N(w_{i-1})$ and $r_c \geq j$, and $\exists y_d$ such that $y_d \neq y_c$, $y_d \in N(w_{i-1})$ and $i \leq r_d < j$

$$f[i, j] = \min\{1 + \min_{y_c} F[p, q], 1 + b(z_k) + \min_{y_d} F[p, s]\}, \; 1 \leq p \leq i - 1, \; j \leq q \leq t, \; i \leq s \leq j - 1$$

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The function $F[i, j] = \infty$, if no such $z$ exist.

**Computation of $F[i, j]$:** We know that for each interval $z \in Y$, the corresponding function $f[i, j]$ is computed. We compute $f[i, j]$ as per the ordering $\sigma$. That is, for two intervals $y_a$ and $y_b$ such that $\sigma(y_a) < \sigma(y_b)$, then $F[l_a, r_a]$ is computed first followed by $F[l_b, r_b]$. We compute $F[i, j]$ for each interval $y_a \in Y$ such that $i = l_a$ and $j = r_a$. The value of $F$ depends on the case (the above three cases) in which $y_a$ falls in. Thus, we consider the following three cases and describe how $F$ is computed in each of them.

**Case 1:** Note that in this case, we consider all interval $y \in Y$ such that $y$ overlaps with $y_a$, and $l_y < l_a$ and $r_y \geq r_a$. As per $\sigma$, for each $y$, we compute $f[y, r_y]$. Since $G$ is connected, $y_a$ is connected with some $y$. We examine each $y$ and choose $y$ for which $f[y, r_y]$ is minimum. Let $y_{min} = y \in Y$ is such that $f[y, r_y]$ is minimum. Clearly, some $x \in N(y) \cap N(y_{min})$ is in the solution to connect $y_a$ and $y_{min}$. As part of our approach, we include $x_{i_a} \in N(y_a) \cap N(y_{min})$ in our solution. Thus, we obtain $f[l_a, r_a] = 1 + \min_y F[l_y, r_y]$. The ‘$1$’ in the expression indicates the inclusion of $x_{i_a}$ in the solution, further, it is connected with a $y$ vertex as indicated by the recursive solution $F[l_y, r_y]$ in the expression. In this case, we do not include $y_a$ in the solution. Finally, we consider all $y_a$ and for each we compute $f[l_a, r_a]$, the minimum over all $f[l_a, r_a]$ is precisely $F[l_a, r_a]$. An illustration is given in Figure 7.

![Fig. 7: (a) An instance of G for Case 1, (b) An instance of G for Case 2, (c) An instance of G for Case 3](image)

**Case 2:** Note that in this case, we consider all interval $y' \in Y$ such that $y'$ overlaps with $y_a$, and $l_y < l_a$ and $r_y < r_a$. The description for computation of $f[l_y, r_y]$ is same as Case 1 and the only change is that $f[l_y, r_y]$ includes $x_{i_a}$ and $y_a$ in the solution. Thus, we obtain $f[l_y, r_y] = 1 + b(y_a) + \min_y F[l_y, r_y]$. The value of $b(y_a)$ is ‘$0$’ if $y_a \in R$, and ‘$1$’, otherwise. An illustration is given in Figure 8.

**Case 3:** This case is a blend of Case 1 and 2. With $y_a$ being the reference interval, we find two intervals $y$ and $y'$ in $Y$ such that $y$ satisfies Case 1 and $y'$ satisfies Case 2. Accordingly, we compute $F$ for $y$ and $y'$ and take the minimum of the two. An illustration is given in Figure 7.

**Case 4:** Since $\forall z \in R$, $l(z) = 1$, including $w_1$ will connect all the vertices in $R$. Hence $F[i, j] = 1$.

**Overlapping subproblems in $F$:**

Consider the subproblems $F[p, q]$, $F[a, b]$, $F[c, d]$ such that $p < a$, $p < c$ and $q \geq a$, $q \geq c$. Since we compute $F[i, j]$ in bottom-up and $p < a$, $p < c$, $F[p, q]$ is computed before $F[a, b]$ and $F[c, d]$. We observe that $F[p, q]$ is a subproblem in $F[a, b]$ and $F[c, d]$. We compute $F[p, q]$ once and reuse the solution when it is referred again. Therefore each subproblem is computed exactly once.

**Computing the optimal Steiner set using $F$:**

Using $F[u, v]$, we construct the solution set $S$ in a bottom up starting from minimum $f[u, v]$. If suppose, $f[u, v]$ is updated due to $f[p, q]$ of $F[p, q]$, then include the vertex $w_u$, $w_p$ in $S$, and also include the corresponding $y$ vertex in $S$. We continue this process until we reach either some $f[1, j]$, $1 \leq j \leq n$ or $f[p, q]$ such that $p = l_z$. If there exist $z_i$ such that $N(z_i) \cap S = \emptyset$ and there does not exist $z_j$ such that $\sigma(z_i) < \sigma(z_j)$, $N(z_i) \cap N(z_j) \neq \emptyset$, then include $l(z_i) \in R$ (An illustration for inclusion of $z_i$ in $S$ is in Figure 8). The vertices

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S \ R are the desired Steiner vertices of G for the terminal set R.

An illustration for inclusion of \( z_i \) in \( S \), for \( z_3, z_4, z_5 \) we include \( l(z_4), l(z_5) \) in \( S \).

Fig. 8: An illustration for inclusion of \( z_i \) in \( S \).

Pseudo code to compute \( F \):

```
Algorithm 4 Computing \( F \)
1: Input: A connected convex bipartite graph \( G \) with \( R \subset Y \).
2: oldFvalue = \( \infty \), newFvalue = \( \infty \)
3: for each \( z \in Y \) such that \( l_z = 1 \) and \( r_z = j \) do
4:     if \( z \in Y \setminus R \) then
5:         \( F[1, j] = 1 \)
6:     end if
7:     if \( z \in R \) then
8:         \( F[1, j] = 0 \)
9:     end if
10: end for
11: for \( j = 1, j \leq m, j = j + 1 \) do
12:     if there does not exists a vertex such that \( l_s = 1 \) and \( r_s = j \) then
13:         \( F[1, j] = \infty \)
14:     end if
15: end for
16: for \( i = 2, i \leq t, i = i + 1 \) do
17:     for each \( z \in Y \) such that \( l(z) = x_i \) do
18:         let \( j = r(y) \)
19:         if \( \exists y_c \) such that \( y_c \in N(w_i - 1) \) and \( r_c \geq j \), and \( \exists y_d \) such that \( y_d \neq y_c, y_d \in N(w_i - 1) \) and \( i \leq r_d < j \) then
20:             \( f[i, j] = \min \{ 1 + \min_{y_c} F[p, q], 1 + b(z_k) + \min_{y_d} F[p, s] \}, 1 \leq p \leq i - 1, j \leq q \leq t, i \leq s \leq j - 1 \)
21:         else if \( \exists y_d \) such that \( y_d \in N(w_i - 1) \) and \( i \leq r_d < j \) then
22:             \( f[i, j] = 1 + b(z_k) + \min_{y_d} F[p, s], 1 \leq p \leq i - 1, i \leq s \leq j - 1 \)
23:         else
24:             \( f[i, j] = 1 + \min_{y_c} F[p, q], 1 \leq p \leq i - 1, j \leq q \leq t \)
25:         end if
26:         newFvalue = \( \min \{ f[i, j], oldFvalue \} \)
27:     oldFvalue = newFvalue
28: end for
29: \( F[i, j] = newFvalue \)
30: end for
```

Time complexity of the function \( F \):
As the range of \( i (j) \) is \( 1 \) to \( m \) and for each \( z \in Y \), we compute the function \( F \), the number of subproblems \( F \)
created by our dynamic programming is at most $O(n^2)$. Further, the number of updates on $F$ is $O(m^2n)$. Thus, Steiner tree when $R \subset Y$ runs in $O(m^2n)$, polynomial in the input size.

**Theorem 8.** For a convex bipartite graph $G$ and a terminal set $R \subset V(G)$, the Steiner set output by our algorithm is an optimal Steiner set.

**Proof.** Let $R = \{z_1, \ldots, z_k\}$. With $z_k$ as the reference interval, we first identify the equivalent class in which $G$ falls into. Further, we compute $F[i, j]$ in a specific order so that solutions to subsubproblems are made available to the subproblems and further to the actual problem. Thus, the optimal solution to $F[u, v]$ is obtained by considering all optimal subproblems. Therefore, the set output by our algorithm is an optimal Steiner set. □

We now trace our algorithm for the example given in Figure 9.

**Base case: $F[1, 2] = 0$.**

At $i = 2$, there exist two intervals $y_2, y_6$ which starts at $x_2$. The $F$ values computed are:

- For $y_2$, $F[2, 3] = 1 + b(z) + \min (F[1, 2], F[1, 3]) = 1 + 1 + 0 = 2$
- For $y_6$, $F[2, 7] = 1 + b(z) + \min (F[1, 2], F[1, 3], F[1, 4], F[1, 5], F[1, 6], F[1, 7]) = 1 + 1 + 0 = 2$

At $i = 3$, there exists an interval $y_3$ which starts at $x_3$. The function $F$ is computed for $y_3$ is $F[3, 4] = 1 + \min (F[1, 3], F[1, 4], F[1, 5], F[1, 6], F[1, 7], F[2, 3], F[2, 4], F[2, 5], F[2, 6], F[2, 7]) = 1 + 2 = 3$

At $i = 4$, there exists an interval $y_4$ which starts at $x_4$. The function $F$ computed for $y_4$ is $F[4, 5] = 1 + \min (F[1, 4], F[1, 5], F[1, 6], F[1, 7], F[2, 4], F[2, 5], F[2, 6], F[2, 7], F[3, 4], F[3, 5], F[3, 6], F[3, 7]) = 1 + 2 = 3$

At $i = 5$, there exists an interval $y_5$ which starts at $x_5$. The function $F$ computed for $y_5$ is $F[5, 6] = 1 + \min (F[2, 5], F[2, 6], F[2, 7], F[3, 5], F[3, 6], F[3, 7], F[4, 5], F[4, 6], F[4, 7]) = 1 + 2 = 3$

**Constructing an optimal solution:** For this input instance $F[u, v]$ is $F[5, 6]$. The ‘$1’ in the expression $F[5, 6]$ indicates the inclusion of $x_5$ in $S$, $S = x_5$. Since the value of $F[5, 6]$ is updated due to $F[2, 7]$, we next consider $F[2, 7]$. Now in the expression $F[2, 7]$, ‘$1’ indicates the inclusion of $x_2$ in $S$, and $b(z) = 1$, which refers to the inclusion of $y_6$ in $S$, $S = \{x_5, x_2, y_6\}$. On the similar line $F[2, 7]$ is updated due to $F[1, 2]$. Thus include $x_2$ in $S$. Finally, there exists an interval $z_2$ such that $N(z_2) \cap S = \emptyset$, hence include $x_3$ in $S$, $S = \{x_5, x_2, y_6, x_3\}$. Therefore, the Steiner vertices of $G$ is $\{x_2, x_3, y_6, x_5\}$.

### 3.5 STREE when $R \cap X \neq \emptyset$ and $R \cap Y \neq \emptyset$

Let $R = \{z_1, \ldots, z_l\}$ such that $R \cap X = \{z_1, \ldots, z_k\}$, $1 \leq k < l$, and $R \cap Y = \{z_{k+1}, \ldots, z_l\}$. To describe the solution for this case, we transform the graph $G(X, Y)$ to $G^*(X^*, Y^*)$ such that $X^* = X, Y^* = Y \cup W$, $W = \{w_i \mid z_i \in R \cap X, 1 \leq i \leq k\}$ and $E(G^*) = E(G) \cup \{(w_i, z_i) \mid w_i \in Y^*, z_i \in R \cap X, 1 \leq i \leq k\}$. Note that each $w_i$ is a pendant vertex in $G^*$. Observe that the convex ordering of $X^*$ is same as $X$, and for each $y \in Y^*$, $N_{G^*}(y)$ is consecutive with respect to the ordering of $X^*$. Therefore, $G^*$ is a convex bipartite graph. Moreover,
this transformation is a solution preserving transformation. That is, using the transformed graph \( G^* \), we obtain a solution to STREE in \( G \). In particular \((G, R)\) is mapped to \((G^*, R^*)\) such that \( R^* = (R \cap Y) \cup W \).

Clearly, \( R^* \subset Y^* \). Using the dynamic programming presented in Section 3.4 we solve \((G^*, R^*)\), and let \( S^* \) be the solution to \( G^* \). Note that since each \( w_i \) is pendant and \( w_i \in R^* \). Hence no \( w_i \) is in \( S^* \). Therefore \( S^* \) is also a solution in \( G \).

**Remarks:** To solve STREE in convex bipartite graphs, it is enough to consider the case STREE when \( G \)

We define the Vertex Cover problem (VC)

\[ \text{Instance: A graph } G, \text{ a terminal set } R \subseteq V(G), \text{ a non-negative integer } k. \]

\[ \text{Question: Does there exist a vertex cover } S \subseteq V(G) \text{ such that } G[R \cup S] \text{ is connected and } |S| \leq k? \]

**The Vertex Cover problem (VC)**

\[ \text{Instance: A graph } G, \text{ a non-negative integer } k. \]

\[ \text{Question: Does there exist a vertex cover } S \subseteq V(G) \text{ such that for each edge } e = \{u, v\} \in E(G), u \in S \text{ or } v \in S \text{ and } |S| \leq k? \]

**Theorem 9.** STREE is NP-complete on 1-star caterpillar convex bipartite graphs.

**Proof.** STREE is in NP: Given an input instance \((G, R, k)\) of STREE and a certificate set \( S \subseteq V(G) \), whether \( S \) is a Steiner set of cardinality at most \( k \) can be verified in polynomial time by using standard graph traversal algorithms [17].

STREE is NP-hard: It is known from [17] that VC on general graphs is NP-complete and this can be reduced in polynomial time to STREE in 1-star caterpillar convex bipartite graphs using the following reduction algorithm. We map an instance \((G, k)\) of VC on general graphs to the corresponding instance \((G^*, R, k')\) of STREE as follows: \( V(G^*) = V_1 \cup V_2 \cup V_3 \),

\[ V_1 = \{x_i \mid v_i \in V(G)\}, \]

\[ V_2 = \{y_{i_1}, y_{i_2} \mid e_i \in E(G)\}, \]

\[ V_3 = \{z_{i_1}, z_{i_2} \mid e_i \in E(G)\}. \]

We shall now describe the edges of \( G^* \),

\[ E(G^*) = E_1 \cup E_2, \]

\[ E_1 = \{(y_{i_1}, x_k), (y_{i_1}, x_l), (y_{i_2}, x_k), (y_{i_2}, x_l), (v_k, v_l) \in E(G), x_k, x_l \in V_1, y_{i_1}, y_{i_2} \in V_2, 1 \leq i \leq m, 1 \leq k, l \leq n\} \]

\[ E_2 = \{(x, z_{i_1}), (x, z_{i_2}) \mid x \in V_1, z_{i_1}, z_{i_2} \in V_2, 1 \leq i \leq m\}. \]

We define \( X^* = V_2 \cup V_3, Y^* = V_1 \), and imaginary 1-star caterpillar \( T \) on \( X^* \) is defined with \( V_3 \) as the backbone and \( V_2 \) as the pendant vertex set. That is, \( V(T) = X^* \) and \( E(T) = \{(y_{i_1}, y_{i_2}), (y_{i_2}, y_{21}), (y_{21}, y_{22}), \ldots, (y_{m_1}, y_{m_2})\} \cup \{(y_{i_1}, z_{i_1}), (y_{i_2}, z_{i_2}) \mid 1 \leq i \leq m\}. \)

An example is illustrated in Figure 10, the vertex cover instance \( G(V, E) \) with \( k = 2 \) is mapped to STREE instance of 1-star caterpillar convex bipartite graph \( G^*(V^*, E^*) \) with \( R = \{y_{11}, y_{12}, y_{21}, y_{22}, y_{31}, y_{32}, z_{11}\}, k' = 2 \).
Claim. $G^*$ is a 1-star caterpillar convex bipartite graph.

Proof. By construction, $T$ is a 1-star caterpillar on $X^*$. Each $x_i \in V_1$ (or $x_i \in Y^*$) is adjacent to all vertices in $V_3$ and also to each element in some subset $Y' \subseteq V_2$. Therefore, for each $x_i$, $N(x_i)$ is a subtree in $T$. Hence $G^*$ is a 1-star caterpillar convex bipartite graph.

Claim. $(G, k)$ has a vertex cover with at most $k$ vertices if and only if $(G^*, R = \{y_{i1}, y_{i2} \mid 1 \leq i \leq m\} \cup \{z_{11}\}, k' = k)$ has a Steiner tree of size at most $k'$.

Proof. (Only if) Let $V' = \{v_i \mid 1 \leq i \leq k\}$ is a vertex cover of size $k$ in $G$. Then we construct the Steiner set $S$ of $G^*$ for $R = \{y_{i1}, y_{i2} \mid 1 \leq i \leq m\} \cup \{z_{11}\}$ as follows $S = \{x_i \mid 1 \leq i \leq k, v_i \in V', x_i \in V(G^*)\}$. Indeed, for any edge $e_i = \{v_k, v_l\} \in E(G)$, $v_k$ or $v_l$ in $V'$. Then by our construction, we know that $y_{i1}$ and $y_{i2}$ are adjacent to $x_1$ and $x_1$, and $x_k$ or $x_i$ is in $S$. So each vertex in $\{y_{i1}, y_{i2} \mid 1 \leq i \leq m\}$ is adjacent to at least one vertex in $S$. Further, by our construction, each vertex in $V_1$ is adjacent to each vertex in $V_3$. Hence $S \cup R$ induces a connected subgraph in $G^*$.

(If) For $R$ in $G^*$, let $S = \{x_i \mid 1 \leq i \leq k'\}$ is a Steiner set of $G^*$ of size $k'$. Then, we construct the vertex cover $V'$ of size $k$ in $G$ as follows: $V' = \{v_i \mid x_i \in S, v_i \in V(G), 1 \leq i \leq k'\}$. We now claim that $V'$ is a vertex cover in $G$. Suppose that there is an edge $e_i = \{v_k, v_l\} \in E(G)$ for which neither $v_k$ nor $v_l$ is in $V'$. This implies that neither $x_k$ nor $x_i$ is in $S$. Since $R$ contains $y_{i1}, y_{i2}$, it follows that $N(y_{i1}) \cap S = \emptyset$ and $N(y_{i2}) \cap S = \emptyset$. Further, $S$ not a Steiner set. A contradiction. Thus $V'$ is a vertex cover of size $k$ in $G$.

Therefore, we conclude STREE on 1-star caterpillar convex bipartite graphs is NP-complete.

Corollary 1. STREE is NP-complete on $k$-star caterpillar convex bipartite graphs, $k \geq 1$. Further, STREE is NP-complete on tree convex bipartite graphs.

Proof. Since the class of 1-star caterpillar convex bipartite graphs is a special case of $k$-star caterpillar convex bipartite graphs, $k \geq 1$, and the fact $k$-star caterpillar convex bipartite graphs are a subclass of tree convex bipartite graphs, this result follows from Theorem [9].

Remark: In [1], it is shown that STREE in chordal bipartite graphs is NP-complete, and it is important to highlight that it is a 3-star caterpillar convex bipartite graph. Hence STREE is NP-complete for 3-star caterpillar convex bipartite graph.

In Theorem [9] we strengthen the result of [1] by establishing the NP-complete result for 1-star caterpillar convex bipartite graphs.

We shall next present two applications of our result. We use STREE in convex bipartite graphs as a framework and solve (a) STREE in intervals graphs, and (b) Domination in convex bipartite graphs. To the best of our knowledge STREE in interval graphs is open, and the study of domination in convex bipartite graphs is already reported in [8].
5 An Application: STREE in Interval graphs

It is known from [7] that STREE on chordal graphs is NP-complete. The class of interval graphs is a popular subclass of chordal graphs on which STREE is open. In this paper, we present a polynomial-time algorithm for STREE in interval graphs using STREE in convex bipartite graphs as a black box. In particular, we invoke STREE in convex bipartite graphs with \( R = Y \) algorithm to solve STREE in interval graphs. It is important to highlight that Domination on chordal bipartite graphs is NP-complete [1]. A micro-level set problem on 1-star caterpillar convex bipartite graphs is NP-complete. Thus we obtain a dichotomy for the convex bipartite graphs in \( O \).

6 Another Application: Domination in convex bipartite graphs

It is known from [8] that the minimum domination set problem in convex bipartite graphs is polynomial-time solvable. In this section, we propose an approach which uses STREE in convex bipartite graphs as a black box, and this approach is different from the one reported in [8]. Further, we obtain a solution to the domination in convex bipartite graphs in \( O(nm) \) time. Our reduction algorithm takes an instance of domination problem in convex bipartite graphs and maps to the corresponding instance of STREE in convex bipartite graphs. For \( G(X, Y) \) of domination problem, we invoke (i) STREE on \( G \) with \( R = X \), and (ii) STREE on \( G \) with \( R = Y \). Let \( D_1 \) and \( D_2 \) be the minimum set of Steiner vertices output by the algorithm when invoked on \((G, R = X)\), and \((G, R = Y)\), respectively.

**Theorem 10.** \( D = D_1 \cup D_2 \) is a minimum dominating set.

**Proof.** Suppose that \( D \) is not a minimum dominating set, then there exists a minimum dominating set \( D' \) such that \(|D'| < |D|\). This implies that \(|D' \cap Y| < |D_1|\) or \(|D' \cap X| < |D_2|\). Further, \( D' \cap Y \) is a Steiner set for the case \( R = X \) and \( D' \cap X \) is a Steiner set for the case \( R = Y \). It contradicts the fact that \( D_1 \) and \( D_2 \) are minimum Steiner sets. Therefore, \( D \) is a minimum dominating set.

Remark: It is shown in [19] that the dominating set problem on comb-convex bipartite graphs is NP-complete. Since comb-convex bipartite graphs are precisely 1-star caterpillar convex bipartite graphs, the dominating set problem on 1-star caterpillar convex bipartite graphs is NP-complete. Thus we obtain a dichotomy for the domination in \( k \)-star caterpillar convex bipartite graphs similar to STREE.

It is important to highlight that Domination on chordal bipartite graphs is NP-complete [1]. A micro-level analysis of the reduction instances shows that the instances are a variant of 3-star caterpillar convex bipartite graph: exactly one of the 3-stars is such that one branch is \( P_1 \) (path of length one) and the other two branches
are \(P_2\) (path of length two). In this paper, we strengthen the result of [1] and show that on 1-star caterpillar convex graphs, the Domination is NP-complete.

**Conclusions and Directions for Further Research:** In this paper, we present an interesting dichotomy: we show that STREE on 0-star caterpillar convex bipartite graphs (convex bipartite graphs) are polynomial-time solvable, whereas STREE on 1-star caterpillar convex bipartite graphs is NP-complete. Further we show that STREE in interval graphs and Domination in convex bipartite graphs are polynomial-time solvable by using the STREE algorithm for convex bipartite graphs. Our greedy strategies and dynamic programming based solution exploits the structure of convex bipartite graphs which can be used in the study of other combinatorial problems such as Steiner path, variants of dominating set, variants of Hamiltonicity. Also, P vs NPC boundary investigation for other combinatorial problems in generalization of convex bipartite graphs would be an interesting direction to explore with.

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