Classical transport equation in non-commutative QED at high temperature

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We show that the high temperature behavior of non-commutative QED may be simply obtained from Boltzmann transport equations for classical particles. The transport equation for the charge neutral particle is shown to be characteristically different from that for the charged particle. These equations correctly generate, for arbitrary values of the non-commutative parameter $\theta$, the leading, gauge independent hard thermal loops, arising from the fermion and the gauge sectors. We briefly discuss the generating functional of hard thermal amplitudes.

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I. INTRODUCTION

The behavior of hot plasmas has been of considerable interest in the recent years \cite{1}. It is known, in particular, that in QCD the leading behavior of the $n$-point gluon functions at temperatures $T \gg p$, where $p$ represents a typical external momentum, is proportional to $T^2$ and these leading contributions to the amplitudes are all gauge independent \cite{2, 3}. In order to extract the leading order contributions to the amplitude leading to gauge invariant results for physical quantities, it is necessary to perform a resummation of hard thermal loops, which are defined by

$$p \ll k \sim T,$$

where $k$ denotes a characteristic internal loop momentum. Such a procedure, however, is quite technical and an alternate simpler method, based on classical transport equations, has been quite useful in deriving the gauge invariant effective action which incorporates all the effects of the hard thermal loops \cite{4, 5, 6, 7, 8, 9}. In such an approach, one pictures the constituents of the plasma as classical particles carrying color charge and interacting in a self-consistent manner. The main reason why such an approach works is that, for soft gauge fields, the occupation number per unit mode in a hot plasma is quite high due to the Bose-Einstein enhancement.

More recently, following from developments in string theory, there has been an increased interest in quantum field theories defined on a non-commutative manifold satisfying \cite{10, 11, 12, 13, 14, 15, 16, 17, 18}

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}$$

where $\theta^{\mu\nu}$ is assumed to be a constant anti-symmetric tensor with the dimensions of length squared. Furthermore, to avoid problems with unitarity, it is generally assumed that $\theta^{0\mu} = 0$ so that only the spatial coordinates are supposed to have non-commutativity. We will assume this in our entire discussion. The behavior of hot plasmas in such a non-commutative gauge theory is an interesting question. Let us note that, in non-commutative theories, because of the presence of a dimensional parameter, the hard thermal loop approximation allows for two extreme regions, namely,

$$\theta p T \gg 1, \quad \text{and} \quad \theta p T \ll 1$$

where perturbative calculations become simple. Here $\theta$ can be thought of as the magnitude of $\theta^{\mu\nu}$. In particular, the perturbative calculations \cite{19} show that in the region $\theta p T \gg 1$, all the oscillatory terms in the perturbative amplitudes are negligible and the leading behavior is proportional to $T^2$ as in “conventional” Yang-Mills theories (although $\theta$ dependent). However, in the other extreme limit, $\theta p T \ll 1$, the leading behavior is suppressed and is proportional to $T^2(\theta p T)^2$.

In an earlier paper \cite{20}, we had proposed a classical transport equation which reproduced the correct hard thermal loop amplitudes in the region $\theta p T \gg 1$. In this regime, of course, one is not expected to see the extended (dipole) nature of non-commutative quanta \cite{21, 22, 23} and a classical description based on the picture of a plasma consisting of charged constituents suffices. However, away from this extreme region, the extended nature of the non-commutative quanta will become relevant and will play an important role. It is important, therefore, to ask if the leading behavior of the amplitudes, at high temperature, can be described by a classical transport equation for arbitrary values of $\theta$. It is not obvious \textit{a priori} that such a description will work since it is known from the study of “conventional” QCD that the classical transport equation does not yield the correct contributions for the hard thermal loop amplitudes that are not proportional to $T^2$. In fact, such (sub-leading) contributions in “conventional” QCD are gauge dependent while
the transport equation is manifestly gauge covariant. Let us note that, in general, the perturbative hard thermal loop contributions can be written as $T^2 H(\theta p T)$ where $H$ is a given function of the dimensionless variable $\theta p T$. For $\theta p T \gg 1$, the leading behavior of $H(\theta p T) \sim 1$ while for $\theta p T \ll 1$, the leading behavior has the form $H(\theta p T) \sim (\theta p T)^2$. Although for arbitrary values of $\theta$, the hard thermal loop amplitudes cannot be evaluated in closed forms, nonetheless, it is easy to check from their integral representations that the leading terms in $H(\theta p T)$ are gauge independent. Therefore, it would seem that a classical description, based on a transport equation, may reproduce the leading terms in the hard thermal loop approximation for arbitrary $\theta$.

In this paper, we describe classical transport equations, for QED, which precisely reproduce the leading hard thermal loop contributions for arbitrary values of $\theta$. We show that the transport equation for the charge neutral particle is characteristically different from that for the charged particle. In section II, we give the results for the leading hard thermal loop amplitudes obtained from perturbation calculations in an integral form, both for the fermion as well as the photon and the ghost loops. In section III, we describe the classical transport equation for a non-commutative charged particle, which reproduces all the hard thermal loop amplitudes for the fermion loop for arbitrary $\theta$. In section IV, we describe the transport equation for a charge neutral non-commutative particle which reproduces the hard thermal loop amplitudes for the photon and the ghost loops. In section V, we derive from the transport equation a manifestly covariantly conserved form of the current, which is related to the generating functional of hard thermal loops.

II. RESULTS FROM PERTURBATION THEORY

Let us consider the Lagrangian density for non-commutative QED of the form

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} \star F^{\mu\nu} + \bar{\psi} \star (i\gamma^\mu D_\mu - m) \psi$$

where

$$D_\mu \psi = \partial_\mu \psi - ie [A_\mu, \psi]_{MB}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie [A_\mu, A_\nu]_{MB}$$

Here the Moyal bracket is defined to be

$$[A, B]_{MB} = A \star B - B \star A$$

where the Grönewold-Moyal star product has the form

$$A(x) \star B(x) = e^{\frac{i}{2} \theta^\mu_\nu \partial_\mu (x) \partial_\nu (x)} A(x + \zeta) B(x + \eta) \bigg|_{\zeta = 0 = \eta}$$

We can add, to this Lagrangian density, gauge fixing and ghost terms which, in a covariant gauge, have the form

$$\mathcal{L}_{gf} + \mathcal{L}_{ghost} = -\frac{1}{2\xi} (\partial_\mu A^\mu) \star (\partial_\nu A^\nu) + \partial^\mu \bar{c} \star (\partial_\mu c - ie [A_\mu, c]_{MB})$$

with $\xi$ representing the gauge fixing parameter.

Our method of calculation employs an analytic continuation of the imaginary-time formalism [1]. Using this approach, we relate the Green’s functions to forward scattering amplitudes of on-shell thermal particles, a technique that has been previously applied in the $SU(N)$ gauge theory as well as in gravity [24, 25]. The one-loop Feynman graphs which contribute to the photon self-energy are shown in figure [2], while the one-loop diagrams associated with the three-point function are represented in figure [3].
FIG. 2: One-loop diagrams which contribute to the three photon function in the non-commutative QED. Wavy, dashed and solid lines denote respectively photons, ghosts and electrons. All the external momenta are incoming.

Using the Feynman rules given in the appendix, the leading order contributions to the photon amplitudes in the hard thermal loop approximation, arising from the fermion loop, are obtained to be

\[
\Pi_{\mu\nu}^{(\text{fermion})} = \frac{8\ e^2}{(2\pi)^3} \int \frac{d^3k}{|k|} \left[ \frac{1}{e^{i\frac{\bar{k}}{T}} + 1} \right] \left| G_{\mu\nu}(k;\ p) \right|_{k_0=|\bar{k}|} \tag{9}
\]

\[
\Gamma_{\mu\nu\lambda}^{(\text{fermion})} = \frac{8\ e^3}{(2\pi)^3} \sin \left( \frac{p_1 \times p_2}{2} \right) \int \frac{d^3k}{|k|} \left[ \frac{1}{e^{i\frac{\bar{k}}{T}} + 1} \right] \left| L_{\mu\nu\lambda}^{(\text{fermion})}(k; p_1, p_2, p_3) \right|_{k_0=|\bar{k}|}, \tag{10}
\]

where the Lorentz structures are defined as

\[
G_{\mu\nu}(k;\ p) = \eta_{\mu\nu} - \frac{k_\mu p_\nu + k_\nu p_\mu}{(k \cdot p)} + \frac{p^2 k_\mu k_\nu}{(k \cdot p)^2} \tag{11}
\]

\[
L_{\mu\nu\lambda}^{(\text{fermion})}(k; p_1, p_2, p_3) = \left\{ \frac{p_1^2 k_\mu k_\nu k_\lambda}{(k \cdot p_1)^2 (k \cdot p_3)} + \frac{k_\mu k_\nu p_3 k_\lambda}{(k \cdot p_1)(k \cdot p_3)} + \frac{k_\lambda}{(k \cdot p_3)} \eta_{\mu\nu} - (\mu, p_1) \leftrightarrow (\lambda, p_3) \right\} + \text{cyclic perm of } (\mu, p_1), (\nu, p_2), (\lambda, p_3) \tag{12}
\]

Similarly, the leading order contributions to the photon amplitudes coming from the photon and the ghost loops are determined to be

\[
\Pi_{\mu\nu}^{(\text{gauge})} = \frac{4\ e^2}{(2\pi)^3} \int \frac{d^3k}{|k|} \left[ \frac{1}{e^{i\frac{\bar{k}}{T}} - 1} \right] (1 - \cos(p \times k)) \left| G_{\mu\nu}(k;\ p) \right|_{k_0=|\bar{k}|} \tag{13}
\]
\[ \Gamma_{\mu\nu\lambda}^{(gauge)} = -\frac{4i e^3}{(2\pi)^3} \sin \left( \frac{p_1 \times p_2}{2} \right) \int \frac{d^3k}{|k|} \frac{1}{e^{\frac{\eta}{k}} - 1} \left\{ [1 - \cos(p_3 \times k)] L_{\mu\nu\lambda}^{(gauge)}(k; p_1, p_2, p_3) - [1 - \cos(p_2 \times k)] L_{\mu\lambda\nu}^{(gauge)}(k; p_1, p_3, p_2) - [1 - \cos(p_1 \times k)] L_{\lambda\nu\mu}^{(gauge)}(k; p_3, p_2, p_1) \right\} \Big|_{k_0 = |\vec{k}|}, \]

where

\[ L_{\mu\nu\lambda}^{(gauge)}(k; p_1, p_2, p_3) = k_\mu k_\nu k_\lambda \left( \frac{p_1^2}{(k \cdot p_1)^2(k \cdot p_3)} + \frac{p_2^2}{(k \cdot p_2)^2(k \cdot p_3)} - \frac{p_3^2}{(k \cdot p_3)^2(k \cdot p_1)} \right) + \frac{1}{(k \cdot p_2)(k \cdot p_3)} \left[ k_\nu k_\lambda (p_2 - p_3)_\mu + k_\mu (k_\lambda p_2 - k_\nu p_3)_\lambda \right] + \frac{1}{(k \cdot p_1)(k \cdot p_3)} k_\mu (k_\lambda p_2 - k_\nu p_3)_\nu + 2 \frac{k_\mu}{(k \cdot p_1)} \eta_{\nu\lambda} - (\mu, p_1) \leftrightarrow (\nu, p_2). \] (15)

In computing these results, the gauge parameter \( \xi \) in Eq. (8) has been kept arbitrary, but the dependence on \( \xi \) cancels out in the final result in the leading order terms. The above hard thermal loop amplitudes are gauge independent and satisfy simple Ward identities. For example, from the structure in (11), one can easily verify the transversality of the photon self-energy

\[ p^\mu \Pi_{\mu\nu}(p) = 0, \]

for both \( \Pi_{\mu\nu}^{(fermion)} \) and \( \Pi_{\mu\nu}^{(gauge)} \). Similarly, the identity relating the two and the three point functions

\[ p_3^\lambda \Gamma_{\mu\nu\lambda}(p_1, p_2, p_3) = 2i e \sin \left( \frac{p_1 \times p_2}{2} \right) \left[ \Pi_{\mu\nu}(p_1) - \Pi_{\nu\mu}(p_2) \right]. \]

can also be seen to hold separately for the contributions from the fermion and the gauge sectors. This suggests the possibility that these leading order hard thermal loop amplitudes may be obtained from gauge covariant, classical transport equations.

### III. TRANSPORT EQUATION FOR CHARGED PARTICLES

The basic idea behind the transport equation approach is to picture the thermal particles, moving in an internal loop, as classical particles in equilibrium in the hot plasma whose dynamics is governed by the classical transport equation. The transport equation can, of course, be derived in a straightforward manner once we know the dynamical equations for a particle in the background of an electromagnetic field. Let us assume that the equations of motion for a particle is given by

\[ m \frac{dx^\mu}{d\tau} = k^\mu \]  
\[ m \frac{dk^\mu}{d\tau} = eX^\mu \] (18)  
(19)

where \( \tau \) denotes the proper time of the particle and \( X^\mu \) represents the force it feels in the presence of a background electromagnetic field. The explicit form of \( X^\mu \) will, of course, be different depending on whether a particle is charged or neutral and we will discuss these two cases separately. In general, the form of \( X^\mu \) must be such that \( k^2 = k^\mu k_\mu \) is a constant and that the time evolution for \( k^\mu \) transforms covariantly under a gauge transformation. For the present, let us note that, given the equations for the dynamics of the particle, the transport equation can be derived in a straightforward manner as follows.

Let us define the current associated with the particle as

\[ j_\mu(x) = e \sum \int dK k_\mu f(x, k) \]

(20)
where the sum is over helicities (One must also allow for a sum over particle species in the case where there are more particle types.). \( f(x, p) \) represents the distribution function for the particle and the integration measure is defined to be

\[
dK = \frac{d^4k}{(2\pi)^3} \, 2\theta(k_0) \, \delta(k^2 - m^2)
\]

so as to guarantee that the particle has positive energy and is on-shell. This is, of course, a natural generalization of the conventional definition of the current to a non-commutative theory, which will reduce naturally to the usual current in the commutative limit. However, in the non-commutative theory, the current transforms in the adjoint representation of the \( U(1) \) group and correspondingly, the distribution function must also transform covariantly under a \( U(1) \) gauge transformation. The current is covariantly conserved,

\[
D_\mu j^\mu = \partial_\mu j^\mu - ie [A_\mu, j^\mu]_{\text{MB}} = 0
\]

as a consequence of the fact that it belongs to the adjoint representation of the gauge group.

Given the equations for the particle and the covariant nature of the distribution function, the transport equation has the general form

\[
D_\tau f(x, k) = C
\]

where \( D_\tau \) is the covariant derivative along the trajectory and \( C \), in general, can be thought of as a collision term. Using the equations of motion for the particle, we can rewrite this as

\[
k^\mu D_\mu f(x, k) + eX^\mu \star \frac{\partial f}{\partial k^\mu} = mC
\]

It is a simple matter to check from the definition of the current in (20) and the transport equation in (24) that current conservation, (22), will hold provided we identify

\[
C = -e \frac{\partial X^\mu}{\partial k^\mu} \star f(x, k)
\]

so that we can write the transport equation, in general, as

\[
k^\mu D_\mu f(x, k) + \frac{\partial(eX^\mu \star f(x, k))}{\partial k^\mu} = 0
\]

We note here that, for a “conventional” charged point particle, we have

\[
X^\mu_{\text{(conventional)}} = F^\mu\nu k_\nu
\]

so that

\[
C_{\text{(conventional)}} = 0
\]

Therefore, in such a case, one can naturally talk of a “collisionless” plasma. However, we note that, in principle, if \( X^\mu \) is a more complicated function of \( k^\mu \), then, a basic “collision” term is inevitable in discussing a hot plasma. This will become quite clear when we discuss the transport equation for a charge neutral particle in the next section.

The particles, in a non-commutative theory, have dipole characteristics. Therefore, when we talk of a charged particle in such a theory, it is natural to generalize the form of the force to include a dipole term, namely,

\[
X^\mu_{\text{(charged)}} = (1 + \sin k \times \iota D) F^\mu\nu k_\nu
\]

where we have introduced, for simplicity, the standard notation (within the context of non-commutative theories) that

\[
A \times B = \theta^{\mu\nu} A_\mu B_\nu
\]

and \( D_\mu \) is the covariant derivative introduced earlier in the adjoint representation. The first term, on the right hand side of (29), represents the standard Lorentz force for a point charged particle. The second term, when the external momentum is small (or the variations of the external fields are slow), can be thought of as a dipole interaction, which
would be a natural generalization considering that non-commutative particles have dipole characteristics. Furthermore, this term vanishes when $\theta^{\mu\nu} \to 0$ so that the force reduces naturally to the conventional one for a charged point particle. If such an additional term is present in the equation of motion \( \Box \), it is clear that we will have a non-trivial contribution to the transport equation coming from \( C \). However, a posteriori, it turns out that such a dipole interaction term is not manifest in the leading perturbative amplitudes at high temperatures. Namely, the presence of such a dipole interaction term is incompatible with the required symmetries of the two and three point photon amplitudes in the leading order. This is also otherwise clear, namely, we know that fermion loops (charged loops) only give planar (but $\theta$ dependent) contributions.

With this, we conclude that the correct transport equation for a charged particle, in the presence of (non-commutative) background electromagnetic fields, has the form

$$k^\mu D_\mu f(x,k) + \frac{\partial(e k_\nu F^{\mu\nu} \ast f(x,k))}{\partial k^\mu} = 0 \quad (30)$$

where, for hard thermal loop contributions, we have effectively

$$X^\mu_{\text{eff}} = F^{\mu\nu} k_\nu \quad (31)$$

Equation (30) is manifestly gauge covariant and, although its form is very similar to the transport equation for “conventional” QED, the star products bring in the necessary non-commutative dependence into the amplitudes.

Let us recall that the distribution function is determined from the transport equation iteratively in order to obtain the current (20), which yields the amplitudes. Thus, writing

$$f(x,k) = f^{(0)} + e f^{(1)} + e^2 f^{(2)} + \cdots \quad (32)$$

and substituting this into the transport equation (30), we obtain, to the lowest order,

$$f^{(0)} \sim n_B(|k^0|) \quad \text{or} \quad n_F(|k^0|) \quad (33)$$

which are the conventional equilibrium distribution functions for bosons and fermions, depending on the nature of the charged particle in the loop. In the present case, where we are considering the contributions from the fermion loop, we choose $f^{(0)} \sim n_F(|k^0|)$. With this, we see that, to the lowest order in $e$, the current (20) vanishes (actually, the time component of the current to the lowest order is a constant which can be set to zero by a simple redefinition). Substituting the lowest order solution, $f^{(0)}$, into the transport equation (30), leads to

$$f^{(1)} = \frac{1}{k \cdot \partial} \frac{\partial}{\partial k_\nu} \left( (k \cdot \partial A_\nu - \partial_\nu k \cdot A) f^{(0)} \right) \quad (34)$$

Substituting this into the definition of the current, \( \Box \), we obtain the leading contribution of the fermions to the photon self-energy as

$$\Pi^{\text{(fermion)}}_{\mu\nu}(p) = \frac{\delta j_\mu(p)}{\delta A_\nu(-p)} \bigg|_{A_\nu=0} = -\frac{8e^2}{(2\pi)^3} \int \frac{d^3k}{|k|} n_F(|\vec{k}|) G_{\mu\nu}(k;p). \quad (35)$$

Here, and in the following, the on-shell condition $k_0 = |\vec{k}|$ is to be always understood. It is clear that this agrees completely with (9).

With the form of $f^{(1)}$ determined, we can iterate the transport equation (30) to determine the distribution function in the next order which has the form

$$f^{(2)} = \frac{i}{k \cdot \partial} \left( [k \cdot A, f^{(1)}]_{\text{MB}} - [k \cdot A, A_\nu]_{\text{MB}} \frac{\partial f^{(0)}}{\partial k_\nu} + \cdots \right) \quad (36)$$

where “$\cdots$” represent terms that do not contribute at the leading order to the current (and, therefore, to the three point amplitude). Substituting this into the current, \( \Box \), we obtain the leading contribution of the fermions to the three point photon amplitude as $\langle p_1 + p_2 + p_3 = 0 \rangle$

$$\Gamma^{(\text{fermion})}_{\mu\nu\lambda}(p_1,p_2,p_3) = \frac{\delta^2 j_\mu(p_1)}{\delta A_\nu(p_2) \delta A_\lambda(p_3)} \bigg|_{A_\nu=0} = \frac{16i e^3}{(2\pi)^3} \sin \left( \frac{p_1 \times p_2}{2} \right) \int \frac{d^3k}{|k|} n_F(|\vec{k}|) \frac{1}{k \cdot p_3} \left[ G_{\mu\nu}(k;p_2) k_\lambda + \frac{k_\mu k_\nu}{k \cdot p_2} \left( \frac{p_2 \cdot p_3}{k \cdot p_3} k_\lambda - p_{2\lambda} \right) - (p_1 \leftrightarrow p_2) \right]. \quad (37)$$

This can be checked to be completely equivalent to the result from perturbative calculations in (11).
The case of the transport equation for a charge neutral particle is even more interesting and challenging. A “conventional” charge neutral point particle, of course, does not feel any electromagnetic force. However, the particles in a non-commutative theory have an extended structure (because of the non-commutativity of coordinates). Consequently, even a charge neutral particle in a non-commutative theory can have a dipole structure, as is generally believed, and such a particle can feel a dipole force in the presence of background electromagnetic fields. On the other hand, such a dipole force, as we have argued in the last section, is not manifest in the leading order amplitudes calculated from perturbation theory. Therefore, one has to think more carefully with the only guidance coming from the explicit results of perturbation theory. In principle, since the dipole force does not work, one may try a properly covariantized quadrupole form of the interaction in (19) for a charge neutral particle in a non-commutative theory.

The current, to this order, can now be obtained and leads to the photon two point function

$$f^{(1)} = \frac{2}{k \cdot \partial} \partial_{\mu} \left( (1 - \cos k \times i \partial)(k \cdot A_{\mu} - \partial_{\mu} k \cdot A) f^{(0)}(k) \right)$$

The current, to this order, can now be obtained and leads to the photon two point function

$$\Pi^{(gauge)}_{\mu\nu}(p) = \frac{\delta j_{\mu}(-p)}{\delta A^{\nu}(p)} \bigg|_{A_{\mu} = 0} = -\frac{4 e^{2}}{(2\pi)^{3}} \int \frac{d^{3}k}{|k|} n_{B}(|k|) G_{\mu\nu}(k;p)(1 - \cos p \times k)$$

This, in fact, agrees completely with eq. (13) and is purely transverse.

We can now use $f^{(1)}$ in the transport equation (20) to determine the distribution function at the next order. With a little bit of work, it is easy to see that

$$f^{(2)} = \frac{1}{k \cdot \partial} \left( \frac{X^{\mu}_{(\text{charge neutral})} \ast f}{e} \right)_{AB}$$

Here, the restriction “quadratic” refers to quadratic terms in the $A_{\mu}$ fields (or linear in $e$). Let us note that $X^{\mu}_{(\text{charge neutral})}$ involves trigonometric functions of non-commuting operators and the expansion of these functions, even to linear order in $A_{\mu}$ (the trigonometric functions are multiplied by a field strength), is highly nontrivial in the coordinate space. Namely, one must use the Baker-Campbell-Hausdorff formula for such an expansion which involves an infinite series and a priori, it is not clear if a closed form expression necessary for the computations (and comparison with the perturbative results) can be obtained. The solution lies in the fact that these operators take a
much simpler form in the momentum space. Namely, substituting this into (20), the form of the current to second order is determined to have a closed form expression in the momentum space given by

\[ j^{(2)}_\mu(p_1) = 8ie^3 \int dK d^4p \sin \frac{p_1 \times p}{2} \frac{k_\mu}{k \cdot p_1} \left[ \frac{k \cdot A(-(p_1 + p))}{k \cdot p} \theta((1 - \cos k \times (p_1 + p))H_\sigma(p, k)f^{(0)}) \right. \\
- \frac{\partial}{\partial k_\sigma} \left( (1 - \cos(k \times p_1))k \cdot A(-(p_1 + p))A_\sigma(p)f^{(0)} \right) \\
+ \left. \frac{\partial}{\partial k_\sigma} \left( \frac{k \cdot A(-(p_1 + p))}{k \cdot (p_1 + p)}(\cos k \times p_1 - \cos k \times p)H_\sigma(p, k)f^{(0)} \right) + \cdots \right] \] (44)

Here, we have defined, for simplicity,

\[ H_\sigma(p, k) = p_\sigma k \cdot A(p) - k \cdot pA_\sigma(p) \] (45)

and the “…” represent terms that do not contribute at the leading order.

It is now easy to obtain the three point amplitude from the second order current \((p_1 + p_2 + p_3 = 0)\)

\[ \Gamma^{(gauge)}_{\mu \nu \lambda}(p_1, p_2, p_3) = \frac{\delta^2 j_\mu(p_1)}{\delta A^\nu(p_2) \delta A^\lambda(p_3)} \bigg|_{A_\sigma = 0} \]

\[ = \frac{8ie^3}{(2\pi)^3} \sin \left( \frac{p_1 \times p_2}{2} \right) \int \frac{d^3k}{|k|} n_B(|k|) \left\{ [1 - \cos(k \times p_1)] k_\mu G_{\mu \nu}(k; p_1) \right. \\
+ [1 - \cos(k \times p_3)] [k_\mu G_{\nu \lambda}(k; p_3) + k_\nu G_{\mu \sigma}(k; p_1) G_{\sigma \lambda}^*(k; p_3)] \\
+ [\cos(k \times p_1) - \cos(k \times p_3)] \left. \frac{k_\mu}{k \cdot p_3} \frac{k_\nu}{k \cdot p_2} G_{\mu \nu}(k; p_1) G_{\sigma \lambda}^*(k; p_3) - (p_2 \leftrightarrow p_3; \nu \leftrightarrow \lambda) \right\} \] (46)

With the help of some simple algebraic identities, it may be verified that this expression is in complete agreement with the diagrammatic result (14). This is indeed a nontrivial check that the transport equation for the charge neutral particle is, in fact, correct.

V. SUMMARY AND DISCUSSIONS

In this paper, we have given transport equations, for charged as well as charge neutral particles, that reproduce exactly the leading two and the three point photon amplitudes in non-commutative QED, in the hard thermal loop approximation, for arbitrary value of \( \theta \). Since the leading amplitudes satisfy tree level Ward identities and the transport equation is manifestly gauge covariant, it follows that these are the true equations which will generate all the amplitudes in the leading hard thermal loop approximation, both for the charged as well as the charge neutral loops.

The force for the charge neutral particle, \((38)\), is completely new and is worth discussing a little since it adds to the physical picture of particles in non-commutative theories. We note that, in the limit of small external momenta (or small variations in the background), the leading order term in the force \( X^\mu_{\text{charge neutral}} \) is that of a quadrupole (in fact, in the configuration of a pair of dipoles aligned back to back). This is quite interesting in the sense that while the conventional picture of a charge neutral particle in a non-commutative theory is that of a dipole, we see that the dipole interaction is not manifest in the hard thermal loop approximation. Instead, the leading effect comes from a quadrupole interaction. Furthermore, a local quadrupole interaction (properly covariantized) is not sufficient; rather, the true equations need a further modification by a nonlocal interaction.

Let us also note from the covariant form of the transport equation, \((20)\), that we can write

\[ f = -\frac{e}{k \cdot D} \frac{\partial (X^\mu * f)}{\partial k^\mu} \] (47)

which provides an integral equation representation for the distribution function. Substituting this into the definition of the current, \((20)\), we obtain

\[ j_\mu = -2e^2 \int dK \frac{k_\mu}{k \cdot D} \frac{\partial (X^\nu * f)}{\partial k^\nu} \] (48)

We note that this form of the current has several interesting features. It transforms covariantly and is manifestly covariantly conserved. Furthermore, even though equation \((47)\) can only be solved iteratively leading to an iterative
solution for the current \( (48) \), experience shows that the leading contributions to the current, in the hard thermal loop approximation, can be obtained from \( (48) \) by simply replacing \( f \rightarrow f(0) \) on the right hand side. With this, the leading form of the current becomes

\[
j_\mu = 2e^2 \int dK \left( \eta_{\mu\nu} - k_\mu \frac{1}{k \cdot D} D_\nu \right) \frac{1}{k \cdot D} X^\nu f(0)
\]

Substituting the form of \( X_\mu^{(\text{charge neutral})} \) given by Eq. \((53)\) this takes the form

\[
j_\mu^{(\text{charge neutral})} = \frac{4e^2}{(2\pi)^2} \int \frac{d^3k}{|k|} n_B(|\vec{k}|) \left( \eta_{\mu\nu} - k_\mu \frac{1}{k \cdot D} D_\nu \right) \frac{1}{k \cdot D} \left\{ 1 - \cos k \times \left( iD + e \left[ \frac{1}{k \cdot D} F_\mu \right] \right) \right\} F_\nu,
\]

with \( F \) defined in Eq. \((39)\). We note that in the regime \( k \times i\partial \gg 1 \), where the cosine oscillates rapidly and becomes negligible, this reduces to our earlier result \( (20) \) (which shows only planar contributions). In the commutative limit, \( \theta^{\mu\nu} \rightarrow 0 \), the current vanishes as should be the case for a charge neutral particle in a commutative theory. Furthermore, it is worth noting here that, for a charged particle, the current in the commutative limit, reduces to that in “conventional” QED.

If we think of the current as arising from an effective action as,

\[
j_\mu[A] = \frac{\delta \Gamma[A]}{\delta A_\mu}
\]

one can, in principle, functionally integrate \((50)\) to obtain the effective action which generates the hard thermal loops. However, except for the planar regime \((20)\), this is very difficult to carry out, in general.

The covariant conservation of the current \((22)\), ensures that the effective action is gauge invariant. This follows because, under a gauge transformation generated by the infinitesimal gauge parameter \( \omega(x) \),

\[
\frac{\delta \Gamma[A]}{\delta \omega(x)} = \int d^4y \frac{\delta A_\mu(y)}{\delta \omega(x)} \frac{\delta \Gamma[A]}{\delta A_\mu(y)} = -D_\mu j^\mu(x) = 0.
\]

One may take advantage of the fact that the effective action is gauge invariant, and try to evaluate it in a light-like axial gauge in which \( k \cdot A_\mu(k) = 0 \), with \( k_0 = |\vec{k}| \). In this gauge, it is possible to integrate \((50)\), to obtain an effective action of the form

\[
\Gamma^{(\text{charge neutral})}[A] = -\frac{2e^2}{(2\pi)^2} \int \frac{d^4k}{|k|} n_B(|\vec{k}|) \int d^4x A^{(k)}_\mu(x) \left[ 1 - \cos(k \times i\partial) \right] A^{(k)}_\mu(x).
\]

Although this expression has a deceptively simple structure, in reality, the \( k \)-integration is nontrivial due to the complicated \( k \)-dependence implicit in the potential \( A^{(k)}_\mu(x) \). This question clearly deserves further study.

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APPENDIX A: FEYNMAN RULES

The Feynman rules following from the Lagrangian given by Eqs. \((3)\) and \((8)\) take the forms

\[
\begin{align*}
\begin{array}{c}
\text{\( \mu \)}
\end{array} & \quad \begin{array}{c}
p \end{array} & : & \quad \begin{array}{c}
i \\
p - m + i\epsilon
\end{array} = iS(p) \\
\begin{array}{c}
\text{\( \nu \)}
\end{array} & \quad \begin{array}{c}
p \end{array} & : & \quad \begin{array}{c}
i \\
p^2 + i\epsilon
\end{array} = iD(p)
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
p \end{array} & : & \quad \begin{array}{c}
i \\
p^2 + i\epsilon
\end{array} = iD(p)
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{\( \mu \)}
\end{array} & \quad \begin{array}{c}
p \end{array} & \quad \begin{array}{c}
\nu \end{array} & : & \quad \begin{array}{c}
i \\
p^2 + i\epsilon
\end{array} = iD(p)
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
p \end{array} & : & \quad \begin{array}{c}
i \\
p^2 + i\epsilon
\end{array} = iD(p)
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
p \end{array} & : & \quad \begin{array}{c}
i \\
p^2 + i\epsilon
\end{array} = iD(p)
\end{align*}
\]
\[ i e \gamma^\mu \frac{1}{2} \hat{p}_i \times \hat{p}_f \]

\[ -2 e \sin \left( \frac{p_1 \times p_2}{2} \right) [ (p_1 - p_2)^\lambda \eta^{\mu\nu} + (p_2 - p_3)^\mu \eta^{\nu\lambda} + (p_3 - p_1)^\nu \eta^{\lambda\mu} ] \]

\[ -2 e \sin \left( \frac{p_2 \times p_3}{2} \right) p_f^\mu \]

\[ -4 i e^2 \left[ \sin \left( \frac{p_1 \times p_2}{2} \right) \sin \left( \frac{p_3 \times p_4}{2} \right) (\eta^{\mu\lambda} \eta^{\rho\sigma} - \eta^{\mu\rho} \eta^{\lambda\sigma}) \right. \]

\[ + \sin \left( \frac{p_1 \times p_3}{2} \right) \sin \left( \frac{p_4 \times p_2}{2} \right) (\eta^{\mu\rho} \eta^{\lambda\nu} - \eta^{\mu\nu} \eta^{\lambda\rho}) \]

\[ + \sin \left( \frac{p_1 \times p_4}{2} \right) \sin \left( \frac{p_2 \times p_3}{2} \right) (\eta^{\mu\nu} \eta^{\lambda\rho} - \eta^{\mu\lambda} \eta^{\nu\rho}) \right], \]

\[(A2)\]

where all momenta are incoming and the Dirac delta functions representing the conservation of momenta are understood.

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