Gauge Fixing and Scattering Amplitudes in String Field Theory around Universal Solutions

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Abstract

We study a gauge fixed action of open string field theory expanded around the universal solution, which has been found as an analytic classical solution with one parameter, $a$. For $a > -1/2$, we are able to reproduce open string scattering amplitudes in the theory fixed in the Siegel gauge. At $a = -1/2$, all scattering amplitudes vanish and there is no open string excitation in the gauge fixed theory. These results support the conjecture that the universal solution can be regarded as a pure gauge or the tachyon vacuum solution.

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§1. Introduction

It has been conjectured that the tachyon vacuum solution\textsuperscript{1, 2, 3, 4} in open string field theory has been analytically constructed as the universal solution that is found in the universal subspace of string fields\textsuperscript{5}. This conjecture is strongly supported by the facts that the modified BRS charge in the theory expanded around the universal solution has vanishing cohomology in the Hilbert space of ghost number one\textsuperscript{6} and that it has been shown numerically that the non-perturbative vacuum vanishes in the theory\textsuperscript{7}. These facts imply that no open string excitation appears perturbatively around the solution and that from the outset the theory is on the tachyon vacuum.

There is an approach to describing the tachyon vacuum other than finding the tachyon vacuum solution, namely vacuum string field theory\textsuperscript{8, 9}. Though much progress has been made in vacuum string field theory\textsuperscript{10}, a kind of regularization is needed due to the existence of a pure ghost midpoint operator in the kinetic term\textsuperscript{9}. The kinetic operator of string field theory expanded around the universal solution is less singular than this pure ghost midpoint operator. Moreover, as Drukker has pointed out\textsuperscript{11}, the kinetic term constructed around the universal solution may become a pure ghost midpoint operator in a certain singular limit. It is thus evident that the theory expanded around the universal solution is much more general and less singular than vacuum string field theory.

In order to prove the equivalence of the tachyon vacuum and the universal solutions, we must show that the energy density of the universal solution cancels the D-brane tension exactly. In addition, we need to understand the existence of closed strings in the theory expanded around the universal solution. At present, it seems difficult to prove the exact cancellation due to technical problems\textsuperscript{7}, and therefore it is necessary to adopt other approaches to understand the relation between these solutions. To find closed strings, we should first study gauge fixing and scattering amplitudes in the theory.

The universal solution $\Psi_0(a)$ given in Ref.\textsuperscript{5} is believed to represent a pure gauge for $a > -1/2$ and the tachyon vacuum solution at $a = -1/2$. In Ref.\textsuperscript{11}, Drukker studies the theory expanded around this solution fixed in the Siegel gauge and suggests that original open string amplitudes can be reproduced around the solution for $a > -1/2$. In this paper, we investigate this gauge-fixed theory more precisely and show more explicitly the correspondence between the physical states and amplitudes of the original and expanded theories. Moreover, solving the cohomology under the Siegel gauge condition, we prove that amplitudes become zero in the theory expanded around the non-trivial solution at $a = -1/2$.

In §2, we solve the equation of motion in the Siegel gauge for $a > -1/2$ and we find a one-to-one correspondence between the spectra of the original and expanded theories. Next, we
derive the Feynman rule in the expanded theory. We find that this Feynman rule yields the same scattering amplitudes as in the original theory. In §3, we analyze the theory expanded around the solution at $a = -1/2$ and we find that the resulting scattering amplitudes become zero. We give some discussion in §4.

## §2. String field theory around pure gauge solutions

### 2.1. Pure gauge solutions

The equation of motion in cubic open string field theory is given by

$$Q_B \Psi + \Psi * \Psi = 0. \quad (2.1)$$

An analytic solution of the equation has been found in the form

$$\Psi_0 = Q_L (e^h - 1)I - C_L ((\partial h)^2 e^h) I, \quad (2.2)$$

where $I$ represents the identity string field, and the operators $Q_L$ and $C_L$ are defined as

$$Q_L(f) = \int_{C_{clt}} \frac{dw}{2\pi i} f(w) J_B(w), \quad C_L(f) = \int_{C_{clt}} \frac{dw}{2\pi i} f(w) c(w). \quad (2.3)$$

Here $J_B(w)$ and $c(w)$ are the BRS current and the ghost field, respectively. The function $h(w)$ in the solution satisfies $h(-1/w) = h(w)$ and $h(\pm i) = 0$. The solution can be expressed in terms of the matter Virasoro generators and the ghost and anti-ghost oscillators acting on the $SL(2, R)$ invariant vacuum, because it is constructed with the BRS current, the ghost field and the identity string field. Because the expression of the solution does not depend on any specific background, it is called the universal solution.

If we expand the string field $\Psi$ as

$$\Psi = \Psi_0 + \Phi, \quad (2.4)$$

the action for the fluctuation $\Phi$ becomes

$$S[\Phi] = -\frac{1}{g^2} \int \left( \frac{1}{2} \Phi * Q'_B \Phi + \frac{1}{3} \Phi * \Phi * \Phi \right), \quad (2.5)$$

where the modified BRS charge is given by

$$Q'_B = Q(e^h) - C((\partial h)^2 e^h). \quad (2.6)$$

The operators $Q(f)$ and $C(f)$ are defined as

$$Q(f) = \int_{C_{clt}} \frac{dw}{2\pi i} f(w) J_B(w), \quad C(f) = \int_{C_{clt}} \frac{dw}{2\pi i} f(w) c(w). \quad (2.7)$$
Let us consider the action expanded around the universal solution generated by the function

\[ h_a(w) = \log \left( 1 + \frac{a}{2} \left( w + \frac{1}{w} \right)^2 \right), \tag{2.8} \]

where \( a \) is a parameter larger than or equal to \(-1/2\). Substituting this function into (2.6), we find that the modified BRS charge can be expanded as

\[ Q'_B(a) = (1 + a)Q_B + \frac{a}{2}(Q_2 + Q_{-2}) + 4aZ(a) c_0 - 2aZ(a)^2(c_2 + c_{-2}) \]

\[ -2a(1 - Z(a)^2) \sum_{n=2}^{\infty} (-1)^n Z(a)^n-1(c_{2n} + c_{-2n}), \tag{2.9} \]

where we have expanded the BRS current and the ghost field as \( J_B(w) = \sum_n Q_n w^{-n-1} \) and \( c(w) = \sum_n c_n w^{-n+1} \). The function \( Z(a) \) is defined by

\[ Z(a) = (1 + a - \sqrt{1 + 2a})/a, \]

and it varies from \(-1\) to 1 for \( a \geq -1/2 \).

If the parameter \( a \) is not equal to \(-1/2\), the action can be transformed into the action with the ordinary BRS charge through the string field redefinition

\[ \Phi' = e^{K(h_a)} \Phi, \tag{2.10} \]

where the operator \( K(f) \) is defined by using the ghost number current \( J_{gh} = cb \) as

\[ K(f) = \oint \frac{dw}{2\pi i} f(w) \left( J_{gh}(w) - \frac{3}{2} w^{-1} \right). \tag{2.11} \]

Substituting the function (2.8) into this definition, we obtain the mode expansion form of \( K(h_a) \) as

\[ K(h_a) = -\tilde{q}_0 \log(1 - Z(a))^2 - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} (q_{2n} + q_{-2n})Z(a)^n, \tag{2.12} \]

where \( \tilde{q}_0 \) and \( q_n \) are written in terms of the ghost oscillators as

\[ \tilde{q}_0 = \frac{1}{2} (c_0 b_0 - b_0 c_0) + \sum_{n=1}^{\infty} (c_{-n} b_n - b_{-n} c_n), \]

\[ q_n = \sum_{m=-\infty}^{\infty} c_{n-m} b_m \quad (n \neq 0). \tag{2.13} \]

Under the string field redefinition, the modified BRS charge transforms into the original one as follows

\[ Q_B \to Q'_B = e^{K(h_a)} Q_B(a) e^{-K(h_a)}. \tag{2.14} \]
Therefore we believe that the universal solution is pure gauge for \( a > -1/2 \), as discussed in Ref. [5]. Numerical analyses strongly support this belief.

Let us consider the BRS cohomology in the theory expanded around the universal solution for \( a > -1/2 \). As the cohomology of the Kato-Ogawa BRS charge, we know that any state \( |\psi\rangle \) satisfying \( Q_B |\psi\rangle = 0 \) can be written as

\[
|\psi\rangle = |P\rangle \otimes c_1 |0\rangle + |P'\rangle \otimes c_0 c_1 |0\rangle + Q_B |\phi\rangle,
\]

where \( |P\rangle \) and \( |P'\rangle \) are positive norm states in the matter sector, and if we consider flat backgrounds they are DDF states. The perturbative equation of motion for the fluctuation is given by \( Q'_B(a)\Phi = 0 \). Because the modified and original BRS charges are related by the similarity transformation, we can solve the equation of motion to obtain the solution

\[
|\Phi\rangle = |P\rangle \otimes e^{K(h_a)} c_1 |0\rangle + |P'\rangle \otimes e^{K(h_a)} c_0 c_1 |0\rangle + Q'_B(a) |\phi\rangle.
\]

In this solution, the non-trivial cohomology parts possess the same ghost number as the physical states in the original theory, and \( |P\rangle \) and \( |P'\rangle \) in the matter sector do not change. This result suggests that the physical spectrum is the same as that in the original theory and it is natural for the universal solution with \( a > -1/2 \) to correspond to the pure gauge.

### 2.2. Gauge fixing and the equation of motion

Though we solved the cohomology above, we must fix the gauge to determine the physical spectrum precisely. Here, we apply the Siegel gauge condition to the theory around the universal solution:

\[
b_0 \Phi = 0.
\]

Under the Siegel gauge condition, the equation of motion is given by

\[
L(a) \Phi = 0, \quad L(a) = \{Q'_B(a), b_0\}.
\]

From (2.9), we can write the operator \( L(a) \) in terms of an oscillator expression as

\[
L(a) = (1 + a) L_0 + \frac{a}{2} (L_2 + L_{-2}) + a(q_2 - q_{-2}) + 4aZ(a),
\]

where the operators \( L_n \) are the total Virasoro generators. We can also rewrite \( L(a) \) in terms of the twisted ghost Virasoro generators, \( L' = L + nq_n + \delta_{n,0} \), as

\[
L(a) = (1 + a) L'_0 + \frac{a}{2} (L'_2 + L'_{-2}) + 4aZ(a) - 1 - a.
\]
Because the twisted ghost conformal field theory has a central charge $c' = 24$, the three operators $L'_0 + 3$ and $L'_{\pm 2}$ form an $SL(2, R)$ algebra. Then, it is useful to rewrite $L(a)$ as

$$L(a) = 2(1 + a)l_0 + a(l_2 + l_{-2}) + 4aZ(a) - 4(1 + a), \quad (2.21)$$

where the operators $l_0$ and $l_{\pm 2}$ are defined by

$$l_0 = \frac{1}{2}(L'_0 + 3), \quad l_{\pm 2} = \frac{1}{2}L'_{\pm 2}, \quad (2.22)$$

and these form the algebra defined by the relations

$$[l_0, l_{\pm 2}] = \mp l_{\pm 2}, \quad [l_2, l_{-2}] = 2l_0. \quad (2.23)$$

In order to solve the gauge fixed equation of motion (2.18), we attempt to diagonalize the operator $L(a)$. Because this operator is expressed in terms of the $SL(2, R)$ generators as in (2.21), it can be diagonalized under $SL(2, R)$ transformations. If we restrict the $SL(2, R)$ group to a subgroup written by ‘normal ordered operators’, arbitrary elements in the subgroup can be represented as

$$U(s, t, u) = \exp(s l_{-2}) \exp(t l_0) \exp(u l_2), \quad (2.24)$$

where $s, t$ and $u$ are real parameters. Using the algebra defined by (2.23), we can calculate the $SL(2, R)$ transformation of the operator $L(a)$ as

$$U(s, t, u)L(a)U(s, t, u)^{-1} = \left\{ 2(1 + a + au) - 2s(au^2 + 2(1 + a)u + a)e^{-t} \right\} l_0$$
$$+ (au^2 + 2(1 + a)u + a)e^{-t} l_2$$
$$+ \left\{ a e^t - 2asu - 2(1 + a)s + s^2(au^2 + 2(1 + a)u + a)e^{-t} \right\} l_{-2}$$
$$+ 4aZ(a) - 4(1 + a). \quad (2.25)$$

If the coefficients of $l_{\pm 2}$ vanish, the parameters $s, t$ and $u$ satisfy

$$au^2 + 2(1 + a)u + a = 0, \quad (2.26)$$
$$a e^t - 2asu - 2(1 + a)s = 0. \quad (2.27)$$

From (2.26), we find that either $u = -Z(a)$ or $u = -1/Z(a)$. Substituting these values of $u$ into (2.27), we obtain

$$a e^t \mp 2\sqrt{1 + 2a} s = 0. \quad (2.28)$$

It should be noted that there is no solution of the equation (2.28) if $a = -1/2$ and $t$ is a finite real number. This fact implies that the operator $L(a)$ cannot be transformed into
a form linear in \( l_0 \) under regular \( SL(2,R) \) transformations. This difference between the cases \( a > -1/2 \) and \( a = -1/2 \) is natural, because the theory should have different physical spectrum in each case if our conjecture regarding the universal solution holds.

In the case \( a > -1/2 \), we can set \( s = -u \) in order to diagonalize \( L(a) \). In this case, the operator \( L(a) \) can be transformed into \( L_0 \) as

\[
U'(a)L(a)U'(a)^{-1} = \sqrt{1 + 2a} L_0,
\]

where \( U'(a) \) is given by

\[
U'(a) = \exp \left\{ \frac{1}{2} Z(a) L'_{-2} \right\} \exp \left\{ \frac{1}{2} (L_0 + 3) \log(1 - Z(a)^2) \right\} \exp \left\{ -\frac{1}{2} Z(a) L'_2 \right\} \\
= \exp \left\{ -\frac{1}{4} \left( L'_2 - L'_{-2} \right) \log \left( \frac{1 + Z(a)}{1 - Z(a)} \right) \right\}.
\]

Therefore, we obtain the solution \( \Phi = U'(a)^{-1} \Phi_0 \) of the gauge fixed equation of motion \((2.18)\), where \( \Phi_0 \) satisfies \( L_0 \Phi_0 = 0 \).

For later convenience, we present here another procedure for the diagonalization of \( L(a) \) making use of conformal field theory. In the twisted ghost conformal field theory, the operator \( U'(a) \) induces a conformal mapping represented by the function

\[
f_a(w) = \left( \frac{w^2 + Z(a)}{Z(a) w^2 + 1} \right)^{\frac{1}{2}};
\]

that is, if \( \phi(w) \) is a primary field of dimension \( h \), the transformation of \( \phi(w) \) by \( U'(a) \) is given by

\[
U'(a)\phi(w)U'(a)^{-1} = \left( \frac{df_a(w)}{dw} \right)^h \phi(f_a(w)).
\]

For \( 0 < Z(a) < 1 \) (\( a > 0 \)), this conformal mapping is depicted in Fig. 1. In the \( z \) plane, there is a branch cut connecting \( -\sqrt{Z(a)} \) and \( \sqrt{Z(a)} \). This branch cut corresponds to a line segment on the imaginary axis between \( -i\sqrt{Z(a)} \) and \( i\sqrt{Z(a)} \) in the \( w \) plane. There is another branch cut from \( \pm 1/\sqrt{Z(a)} \) to infinity in the \( z \) plane. For \( -1 < Z(a) < 0 \) (\( -1/2 < a < 0 \)), the mapping is represented by the figure that is obtained by rotating Fig. 1 clockwise \( 90^\circ \).

From \((2.20)\), we find that the operator \( L(a) \) can be expressed in terms of the twisted energy momentum tensor \( T'(w) = \sum L'_n w^{-n-2} \) as

\[
L(a) = \oint \frac{dw}{2\pi i} w e^{h_a(w)} T'(w) + 4Z(a) - 1 - a,
\]

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where the integration contour is a unit circle. Generally, the kinetic operator can be written using the twisted energy momentum tensor in the theory around universal solutions in the Siegel gauge. Then, the transformation of $L(a)$ by $U'(a)$ is given by

$$U'(a) L(a) U'(a)^{-1} = \int \frac{dw}{2\pi i} w e^{h_a(w)} U'(a) T'(w) U'(a)^{-1} + 4Z(a) - 1 - a. \quad (2.34)$$

Under the conformal mapping $z = f_a(w)$, the twisted energy momentum tensor is transformed to

$$U'(a) T'(w) U'(a)^{-1} = \left( \frac{df_a(w)}{dw} \right)^2 T'(f_a(w)) + \frac{c'}{12} S(f_a, w), \quad (2.35)$$

where the central charge $c'$ is 24 and $S(f, w)$ denotes the Schwartzian derivative:

$$S(f, w) = \frac{\partial^3 f(w)}{\partial f(w)} - \frac{3}{2} \left( \frac{\partial^2 f(w)}{\partial f(w)} \right)^2. \quad (2.36)$$

Using (2.8) and (2.31), we see that

$$e^{h_a(w)} = \frac{1}{(1 - Z(a))^2} w^{-2} (w^2 + Z(a))(Z(a)w^2 + 1), \quad (2.37)$$

$$S(f_a, w) = -\frac{3Z(a)}{2} w^{-2} \frac{(1 + 2Z(a)w^2 + w^4)(Z(a) + 2w^2 + Z(a)w^4)}{(w^2 + Z(a))^2(Z(a)w^2 + a)^2}, \quad (2.38)$$

and therefore

$$w e^{h_a(w)} f'_a(w) = \frac{1 + Z(a)}{1 - Z(a)} f_a(w), \quad (2.39)$$

$$w e^{h_a(w)} S(f_a, w) = -\frac{3Z(a)}{2(1 - Z(a))^2} \times \frac{(1 + 2Z(a)w^2 + w^4)(Z(a) + 2w^2 + Z(a)w^4)}{w^3 (w^2 + Z(a))(Z(a)w^2 + a)}. \quad (2.40)$$

Fig. 1. Conformal mapping of the $w$ plane into the $z$ plane under the mapping $z = f_a(w)$ in the case $0 < Z(a) < 1 (a > 0)$.
Substituting (2.34) into (2.35) and then using (2.39) and (2.40), we find

\[
U'(a)L(a)U'(a)^{-1} = \frac{1 + Z(a)}{1 - Z(a)} \int \frac{dw}{2\pi i} f_a(w) \frac{df_a(w)}{dw} T'(f_a(w))
\]

\[
- \frac{3Z(a)}{(1 - Z(a))^2} \int \frac{dw}{2\pi i} (1 + 2Z(a)w^2 + w^4)(Z(a) + 2w^2 + Z(a)w^4)
\]

\[
+ 4Z(a) - 1 - a.
\]  

(2.41)

In the first term on the right-hand side of this expression, we can change the variable of integration to \( z = f_a(w) \), since the integration contour does not cross the branch cut in the \( z \) plane, as seen in Fig. 1. Then, the first term becomes

\[
\frac{1 + Z(a)}{1 - Z(a)} \int \frac{dz}{2\pi i} z T'(z) = \sqrt{1 + 2a} L'_0.
\]  

(2.42)

If \( 1 > Z(a) > 0 \), the integral in the second term in (2.41) can be reduced to the summation of the residues at \( w = 0 \) and \( \pm i \sqrt{Z(a)} \), which individually are

\[
\int_{C_0} = \frac{1 + Z(a)^2}{Z(a)}, \quad \int_{C_{i\sqrt{Z(a)}}} = \int_{C_{-i\sqrt{Z(a)}}} = -1 + \frac{Z(a)^2}{2Z(a)}.
\]  

(2.43)

If \( -1 < Z(a) < 0 \), the residues at \( \pm i \sqrt{Z(a)} \) are replaced by those at \( \pm \sqrt{-Z(a)} \), but their values are unchanged. Finally, substituting (2.42) and (2.43) into (2.34), we can derive the same result as obtained from operator expressions:

\[
U'(a)L(a)U'(a)^{-1} = \sqrt{1 + 2a} L'_0 - \frac{3Z(a)}{(1 - Z(a))^2} 2Z(a) + 4Z(a) - 1 - a
\]

\[
= \sqrt{1 + 2a} L_0.
\]  

(2.44)

Note that this result cannot be applied to the case \( a = -1/2 \), because the operator \( U'(a) \) becomes singular at \( a = -1/2 \) as seen in (2.30).

2.3. Physical subspace

First, we show that the modified BRS charge is transformed into the original one through the similarity transformation with \( U'(a) \):

\[
U'(a)Q'_B(a)U'(a)^{-1} = \sqrt{1 + 2a} Q_B.
\]  

(2.45)

The commutation relations of \( L_m \) and \( q_m \) with \( Q_n \) are given by

\[
[L_m, Q_n] = -nQ_{m+n},
\]

\[
[q_m, Q_n] = Q_{m+n} - 2mn c_{m+n}.
\]  

(2.46)
Then, we obtain the commutation relation of $L'_m$ with $Q_n$ as

$$[L'_m, Q_n] = (m - n)Q_{m+n} - 2m^2n c_{m+n}. \quad (2.47)$$

The first term on the left-hand side implies that $Q_n$ is transformed as oscillators of a dimension two field in the twisted ghost theory. Due to the second term, $Q_n$ is not an oscillator of a primary field. This is a natural result of the fact that the BRS current can be written $J_B(w) = wc'(w)T_X(w) + \cdots$, where $c'(w)$ and $T_X(w)$ denote the ghost field of dimension zero and the matter energy momentum tensor of dimension two in the twisted theory, and the dots stand for terms containing no matter oscillators.

From the above, we see that, because $Q'_B(a)$ and $L(a)$ have similar oscillator expressions,

$$Q'_B(a) = (1 + a)Q_B + \frac{a}{2}(Q_2 + Q_{-2}) + \cdots, \quad (2.48)$$

$$L(a) = (1 + a)L_0 + \frac{a}{2}(L'_2 + L'_{-2}) + \cdots, \quad (2.49)$$

and the operators $L'_n$ are oscillators of a dimension 2 field, we can obtain the transformation of $Q'_B(a)$ by analogy to the previous result for $L(a)$ as

$$U'(a) Q'_B(a) U'(a)^{-1} = \sqrt{1 + 2a} Q_B + \cdots, \quad (2.50)$$

where the dots represent pure ghost contributions. Because the right-hand side of (2.50) is nilpotent, the pure ghost contributions are found to be zero. This follows from the fact that an operator of the form $Q(f) + C(g)$ must be $Q(e^f) - C((\partial f)^2 e^f)$ or $C(f)$ if it is nilpotent. Therefore, we have shown that the transformation law given in (2.45) holds.

Here, we should discuss the cohomology without the Siegel gauge condition. Using the equation (2.45), we show that the state $|\Phi\rangle$ with the condition $Q'_B(a) |\Phi\rangle = 0$ can be written as

$$|\Phi\rangle = U'(a)^{-1}(|P\rangle \otimes c_1 |0\rangle) + U'(a)^{-1}(|P'\rangle \otimes c_0 c_1 |0\rangle) + Q'_B(a) |\phi\rangle. \quad (2.51)$$

However, this expression is different from that for the state given in (2.16), though these states are imposed by the same condition. We can show with a natural line of reasoning that the apparent difference between these states reduces to merely a BRS-exact state.

In order to see this fact, we prove that

$$U'_f = \text{const} \times U_f \times \exp K \left( \log \frac{w \partial f(w)}{f(w)} \right), \quad (2.52)$$

where $U'_f$ and $U_f$ are the operators that implement the conformal transformation $z = f(w)$ in the twisted and untwisted ghost conformal field theory, respectively.
In the untwisted theory, the ghost and anti-ghost fields are expanded in a unit disc as

\[ c(w) = \sum_{n=-\infty}^{\infty} c_n w^{-n+1}, \quad b(w) = \sum_{n=-\infty}^{\infty} b_n w^{-n-2}, \]  

(2.53)

and in the twisted theory they are expanded as

\[ c'(w) = \sum_{n=-\infty}^{\infty} c_n w^{-n}, \quad b'(w) = \sum_{n=-\infty}^{\infty} b_n w^{-n-1}. \]  

(2.54)

Then, the ghost and anti-ghost fields in both theories can be connected by the relations

\[ c'(w) = w^{-1} c(w), \quad b'(w) = w b(w). \]  

(2.55)

For these ghost fields, the operators \( U_f \) and \( U'_f \) induce the transformations

\[ U_f c(w) U_f^{-1} = (\partial f(w))^{-1} c(f(w)), \]  

(2.56)

\[ U_f b(w) U_f^{-1} = (\partial f(w))^2 b(f(w)), \]  

(2.57)

\[ U'_f c'(w) U'_f^{-1} = (\partial f(w))^{0} c'(f(w)), \]  

(2.58)

\[ U'_f b'(w) U'_f^{-1} = (\partial f(w))^{1} b'(f(w)). \]  

(2.59)

Combining (2.55) and (2.56), we can find the transformation of \( c'(w) \) through \( U_f \):

\[ U_f c'(w) U_f^{-1} = w^{-1} U_f c(w) U_f^{-1} = w^{-1} (\partial f(w))^{-1} c(f(w)) = \frac{f(w)}{w \partial f(w)} c'(f(w)). \]  

(2.60)

Similarly, from (2.55) and (2.56), the transformation of \( b(w) \) is given by

\[ U_f b'(w) U_f^{-1} = \frac{w (\partial f(w))^2}{f(w)} b'(f(w)). \]  

(2.61)

We can realize the same transformations of (2.60) and (2.61) using \( U_f e^{K(g)} \). For an arbitrary function \( g(w) \), the operator \( K(g) \) generates the transformations\[8,11,14\]

\[ e^{K(g)} c'(w) e^{-K(g)} = e^{g(w)} c'(w), \]  

(2.62)

\[ e^{K(g)} b'(w) e^{-K(g)} = e^{-g(w)} b'(w). \]  

(2.63)

Combining (2.60), (2.61), (2.62) and (2.63), we find

\[ U_f e^{K(g)} c'(w) e^{-K(g)} U_f^{-1} = e^{g(w)} \left( \frac{f(w)}{w \partial f(w)} \right) c'(f(w)), \]  

(2.64)

\[ U_f e^{K(g)} b'(w) e^{-K(g)} U_f^{-1} = e^{-g(w)} \left( \frac{w \partial f(w)^2}{f(w)} \right) b'(f(w)). \]  

(2.65)
Comparing (2.58) and (2.59) with (2.64) and (2.65), we find that these transformation laws are equivalent if \( g(w) \) is given by
\[
g(w) = \log \left( \frac{w \partial f(w)}{f(w)} \right). \tag{2.66}
\]

The ghost part of \( U_f' \) is uniquely determined by the transformation laws (2.58) and (2.59), up to a multiplicative constant. The matter parts of \( U_f \) and \( U_f' \) have a same form, because they are uncharged by the twist operation. Therefore the equation (2.52) is proved. In Appendix A, we generalize this equation to the case of general background charge.

We now apply the formula (2.52) to the mapping \( f(w) = f_a(w) \) given by (2.31). We can calculate the function in the operator \( K \) as
\[
\log \left( \frac{w \partial f_a(w)}{f_a(w)} \right) = \log \left\{ \frac{(1 - Z(a))^2}{(w^2 + Z(a))(Z(a)w^2 + 1)} \right\} - h_a(w) + \log \frac{1 + Z(a)}{1 - Z(a)} \tag{2.67}
\]
Therefore, we can represent the operator \( U'(a) \) in terms of the untwisted Virasoro generators as
\[
U'(a) = U(a) \exp \left( \tilde{q}_0 \log \frac{1 + Z(a)}{1 - Z(a)} \right), \tag{2.68}
\]
where the operator \( U(a) \) is defined as
\[
U(a) = \exp \left\{ -\frac{1}{4} (L_2 - L_{-2}) \log \frac{1 + Z(a)}{1 - Z(a)} \right\} = \exp \left\{ \frac{1}{2} Z(a)L_{-2} \right\} \exp \left\{ \frac{1}{2} L_0 \log(1 - Z(a)^2) \right\} \exp \left\{ -\frac{1}{2} Z(a)L_2 \right\}. \tag{2.69}
\]
Here, the multiplicative constant is 1 because \( U'(a) \) is a unitary operator. Since the untwisted Virasoro generators commute with the BRS charge, it is easily seen from (2.68) that (2.14) and (2.45) hold simultaneously.

Using (2.68), we can express the operator \( U'(a)^{-1} \) as
\[
U'(a)^{-1} = e^{K(h_a)}U(a)^{-1} \exp \left( -\tilde{q}_0 \log \frac{1 + Z(a)}{1 - Z(a)} \right) = e^{K(h_a)} \times \sum_{n=0}^{\infty} \frac{1}{n!} (L_2 - L_{-2})^n \theta^n \times e^{-4\tilde{q}_0 \theta} \left( \theta = \frac{1}{4} \log \frac{1 + Z(a)}{1 - Z(a)} \right) \tag{2.69}
\]
\(^*\) The hermiticity property is given by \( (q_n)^\dagger = -q_{-n} \ (n \neq 0), \ (\tilde{q}_0)^\dagger = -\tilde{q}_0 \) and \( (L_n)^\dagger = L_{-n} \).
\[ = e^{K(h_a)} e^{-4\tilde{q}_0 \theta} + e^{K(h_a)} \times \left\{ Q_B \sum_{n=1}^{\infty} \frac{1}{n!} (b_2 - b_{-2})(L_2 - L_{-2})^{n-1} \theta^n \right\} \times e^{-4\tilde{q}_0 \theta} \]  

(2.70)

Then we can rewrite the first term of (2.51) as

\[ U'(a)^{-1} (|P \rangle \otimes c_1 |0\rangle) = e^{2\theta} |P \rangle \otimes e^{K(h_a)} c_1 |0\rangle \]

+ \[ e^{K(h_a)} Q_B \sum_{n=1}^{\infty} \frac{1}{n!} (b_2 - b_{-2})(L_2 - L_{-2})^{n-1} \theta^n (e^{2\theta} |P \rangle \otimes c_1 |0\rangle) \]

= \[ e^{2\theta} |P \rangle \otimes e^{K(h_a)} c_1 |0\rangle + Q'_B(a) |\phi\rangle, \]  

(2.71)

where \( |\phi\rangle \) is given by

\[ |\phi\rangle = e^{K(h_a)} \sum_{n=1}^{\infty} \frac{1}{n!} (b_2 - b_{-2})(L_2 - L_{-2})^{n-1} \theta^n (e^{2\theta} |P \rangle \otimes c_1 |0\rangle) \].  

(2.72)

Similarly, the second term of (2.51) can be written

\[ U'(a)^{-1} (|P' \rangle \otimes c_0 c_1 |0\rangle) = e^{-2\theta} |P' \rangle \otimes e^{K(h_a)} c_0 c_1 |0\rangle + Q'_B(a) |\phi'\rangle. \]  

(2.73)

Hence, from (2.71) and (2.73), it is shown that the difference between the cohomologies (2.16) and (2.51) can be expressed as a BRS-exact state, as asserted above.

Now that the cohomology of \( Q'_B(a) \) has been established, we can obtain the physical subspace specified by the conditions

\[ Q'_B(a) |\Phi\rangle = 0, \quad b_0 |\Phi\rangle = 0. \]  

(2.74)

Because, under the same similarity transformation, \( Q'_B(a) \) and \( L(a) \) are transformed into \( Q_B \) and \( L_0 \), the state \( |\Phi\rangle \) satisfying (2.71) can be written

\[ |\Phi\rangle = U'(a)^{-1} \left( |P \rangle \otimes c_1 |0\rangle + \tilde{Q}_B |\phi\rangle \right), \]  

(2.75)

where \( \tilde{Q}_B \) denotes the terms of \( Q_B \) that do not contain the ghost and anti-ghost zero modes, \( c_0 \) and \( b_0 \). Here, we have used the cohomology for the Kato-Ogawa BRS charge (2.15) and the commutation relations \([L'_n, b_0] = 0\). Hence, there is a one-to-one correspondence between the spectra of the original theory and those of the theory around the universal solution for \( a > -1/2 \). They are connected through the similarity transformation with \( U'(a) \).

2.4. Scattering amplitudes

Here, we consider scattering amplitudes in the theory around the universal solution. The general amplitude is calculated by evaluating an expression of the form

\[ A = \left( \prod \langle V | \right) \left( \prod \frac{b_0}{L(a)} \right) \left( \prod |R\rangle \right) \left( \prod |\text{external}' \rangle \right), \]  

(2.76)
where $\langle V |, |R \rangle$ and $|\text{external}\rangle'$ are vertices, reflectors and external states, respectively. Also, $b_0/L(a)$ is the propagator in the theory expanded around the universal solution. From (2.29) and $[L'_n, b_0] = 0$, we can rewrite the propagator as

$$\frac{b_0}{L(a)} = \frac{1}{\sqrt{1 + 2a}} \times U'(a)^{-1} b_0 L'_0 U'(a). \quad (2.77)$$

From (2.73), the external states can be written as similarity transformations of the external states in the original theory:

$$|\text{external}\rangle' = U'(a)^{-1} |\text{external}\rangle. \quad (2.78)$$

It can be easily seen that on the reflector, the operator $K'_n = L'_n - (-1)^n L'_{-n}$ satisfies

$$12 \langle R | (K'_n^{(1)} + K'_n^{(2)}) = 0, \quad (2.79)$$

and then we find

$$12 \langle R | \prod_{r=1}^{2} (U'(a)^{(r)})^{-1} = 12 \langle R |. \quad (2.80)$$

Using (2.77), (2.78) and (2.80), we can rewrite the amplitude into the form

$$\mathcal{A} = \left( \prod \langle V' \rangle \right) \left( \prod \frac{1}{\sqrt{1 + 2a}} \frac{b_0}{L_0} \right) \left( \prod |R\rangle \right) \left( \prod |\text{external}\rangle \right). \quad (2.81)$$

The difference between this and the original amplitude is in the normalization factor of the propagator and the change of the vertex $123 \langle V \rangle$ to

$$123 \langle V' \rangle = 123 \langle V \prod_{r=1}^{3} U'(a)^{(r)}^{-1}. \quad (2.82)$$

The operator $K_n = L_n - (-1)^n L_{-n}$ is conserved on the original vertex. In addition, we find that the operator $K(h_a)$ also is conserved on the original vertex. Using these conservation laws and the expression (2.68) of $U'(a)$, we find that the modified vertex can be rewritten as

$$123 \langle V' \rangle = 123 \langle V \prod_{r=1}^{3} \exp \left( -\log \sqrt{1 + 2a} \tilde{q}_0^{(r)} \right). \quad (2.83)$$

Under the $SL(2, R)$ normal ordering, the zero mode $q_0$ of the ghost number current yields an anomalous contribution on the vertex.

$$123 \langle V \sum_{r=1}^{3} q_0^{(r)} = 3 \times 123 \langle V \rangle. \quad (2.84)$$
Since $\tilde{q}_0 = q_0 - 3/2$, we find
\[ 123 \langle V' \rangle = \left( \sqrt{1 + 2a} \right)^{\frac{3}{2}} \times 123 \langle V \rangle. \hspace{1cm} (2.85) \]

Combining the above results, we find that the perturbative amplitude in the theory around the solution takes the form
\[ \mathcal{A} = \left( \prod \left( \sqrt{1 + 2a} \right)^{\frac{3}{2}} \langle V \rangle \right) \left( \prod \frac{1}{\sqrt{1 + 2a}} \frac{b_0}{L_0} \right) \left( \prod |R \rangle \right) \left( \prod |\text{external} \rangle \right). \hspace{1cm} (2.86) \]

The normalization factors of the propagators and the vertices cancel and the remaining factors can be absorbed into the normalization of the external states as $\left( \sqrt{1 + 2a} \right)^{1/2} \times |\text{external} \rangle$. Finally, it is found that the amplitude becomes equal to the corresponding amplitude in the original theory. Hence, we conclude that the theory for $a > -1/2$ describes the same physics as the theory with the original BRS charge.

\section*{§3. String field theory around non-trivial solutions}

\subsection*{3.1. Non-trivial solutions}

In this section we consider the theory expanded around the universal solution which is obtained by setting the parameter $a$ to $-1/2$. At $a = -1/2$, the function (2.8) is given by
\[ h(w) = \log \left( -\frac{1}{4} \left( \frac{w - 1}{w} \right)^2 \right), \hspace{1cm} (3.1) \]
and the modified BRS charge (2.9) becomes
\[ \bar{Q}_B \equiv Q'_B(-1/2) \]
\[ = \frac{1}{2} Q_0 - \frac{1}{4}(Q_2 + Q_{-2}) + 2c_0 + c_2 + c_{-2}. \hspace{1cm} (3.2) \]

Through a similarity transformation, this modified BRS charge can be transformed into a form that contains specific level oscillators,
\[ e^{K(\rho)} \bar{Q}_B e^{-K(\rho)} = -\frac{1}{4} Q_2 + c_2, \hspace{1cm} (3.3) \]
where $\rho(w)$ and $K(\rho)$ are given by
\[ \rho(w) = -2 \log(1 - w^{-2}) = 2 \sum_{n=1}^{\infty} \frac{1}{n} w^{-2n}, \hspace{1cm} (3.4) \]
\[ K(\rho) = 2 \sum_{n=1}^{\infty} \frac{1}{n} q_{-2n}. \hspace{1cm} (3.5) \]
Here we introduce an operator $O^{(k)}$ to a certain operator $O$ that is defined by replacing the ghost oscillator modes $c_n$ and $b_n$ in $O$ by $c_n^{(k)} = c_{n+k}$ and $b_n^{(k)} = b_{n-k}$ without changing their order. With this definition, we can rewrite the equation (3.3) as

$$Q_B = -\frac{1}{4} e^{-K(\rho)} Q_B^{(2)} e^{K(\rho)}.$$  \hspace{0.5cm} (3.6)

For this bc-shift transformation, it is important that the original algebra of the operator $O$ is realized to the operator $O^{(k)}$. This follows from the anti-commutation relations \{ $c_m^{(k)}, b_n^{(k)}$ \} = $\delta_{m+n,0}$. Consequently, through the relation (3.6), we can determine the cohomology of $\bar{Q}_B$ by referring to the original cohomology of $Q_B$. We find that any state $|\psi\rangle$ satisfying $\bar{Q}_B |\psi\rangle = 0$ can be written

$$|\psi\rangle = |P\rangle \otimes e^{-K(\rho)} b_{-2} |0\rangle + |P'\rangle \otimes e^{-K(\rho)} |0\rangle + \bar{Q}_B |\phi\rangle.$$  \hspace{0.5cm} (3.7)

Here, the states $b_{-2} |0\rangle$ and $|0\rangle$ correspond to $c_1 |0\rangle$ and $c_0 c_1 |0\rangle$ in the original cohomology (2.15), respectively.

In gauge unfixed string field theory, all component fields of the string field correspond to states of ghost number one. Therefore, the resulting cohomology (3.7) implies that all on-shell modes are reduced to gauge degrees of freedom in the gauge unfixed theory.

3.2. No open string theorem

First, we demonstrate the formula

$$U_f^{(k)} = \text{const} \times U'_f \times \exp K \left( \log \frac{f(w)^{k+1}}{w^{k+1} \partial f(w)} \right).$$  \hspace{0.5cm} (3.8)

Because the bc-shift transformation preserves the forms of commutation relations, the operators $c^{(k)}(w)$ and $b^{(k)}(w)$ are transformed as primary fields of dimension $-1$ and 2 under the similarity transformation with $U_f^{(k)}$:

$$U_f^{(k)} c^{(k)}(w) U_f^{(k)-1} = (\partial f(w))^{-1} c^{(k)}(w),$$  \hspace{0.5cm} (3.9)

$$U_f^{(k)} b^{(k)}(w) U_f^{(k)-1} = (\partial f(w))^2 b^{(k)}(w).$$  \hspace{0.5cm} (3.10)

The operators $c^{(k)}(w)$ and $b^{(k)}(w)$ can be written in terms of the twisted operators as

$$c^{(k)}(w) = \sum_{n=-\infty}^{\infty} c_{n+k} w^{-n+1} = w^{k+1} c'(w),$$  \hspace{0.5cm} (3.11)

$$b^{(k)}(w) = \sum_{n=-\infty}^{\infty} b_{n-k} w^{-n-2} = w^{-k-1} b'(w).$$  \hspace{0.5cm} (3.12)

Note that $O^{(k)}$ here does not represent the operator $O$ of the $k$-th string.
Then, from these relations and the equations (2.56), (2.57), (2.62) and (2.63), it follows that
\[
U_f' e^K(g) c^{(k)}(w) e^{-K(g)} U_f'^{-1} = w^{k+1} e^{g(w)} (f(w))^{-k-1} c^{(k)}(f(w)), \tag{3.13}
\]
\[
U_f' e^K(g) b^{(k)}(w) e^{-K(g)} U_f'^{-1} = w^{-k-1} e^{-g(w)} \partial f(w) (f(w))^{k+1} b^{(k)}(f(w)). \tag{3.14}
\]

If the function \(g(w)\) is given by
\[
g(w) = \log \frac{f(w)^{k+1}}{w^{k+1} \partial f(w)}, \tag{3.15}
\]
these transformation laws coincide with the similarity transformation with \(U_f'(k)\). Because the \(bc\)-shift transformation does not affect matter oscillators, the operators \(U_f'(k)\) and \(U_f e^K(g)\), in which \(g(w)\) is given by (3.15), are equal up to a multiplicative constant. Hence, the formula (3.8) is proved.

We now consider the case \(f(w) = w/\sqrt{1 - w^2}\). For this function, \(U_f\) is given by
\[
U_f = \exp \left( \frac{1}{2} L_2 \right). \tag{3.16}
\]

For \(k = -2\), the function \(g(w)\) is found to be
\[
g(w) = 2 \log(1 - w^2). \tag{3.17}
\]

Then, applying the formula (3.8) to this case, we obtain the equation
\[
U_f'^{-2} = U_f' \times \exp \left( \sum_{n=1}^{\infty} \frac{2}{n} q_{2n} \right), \tag{3.18}
\]
where the multiplicative constant can be shown to be 1 by expanding the both sides and using the relations \(L'_n = L_n + n q_n + \delta_{n,0}\) and \(L^{(k)}_n = L_n + k q_n + (k^2 - 3k) \delta_{n,0}/2\). Noting that \((L^{(k)}_n)^\dagger = L^{(-k)}_{-n}\) \((n \neq 0)\), we obtain the following relation by taking the Hermitian conjugate of (3.18):
\[
U_f' = e^{-K(\rho)} U_F^{(2)}. \tag{3.19}
\]

The operator \(K(\rho)\) here is given by (3.5), and \(U_F\) is defined as
\[
U_F = (U_f')\dagger = \exp \left( \frac{1}{2} L_{-2} \right). \tag{3.20}
\]

The function corresponding to \(U_F\) is given by \(F(w) = \sqrt{w^2 + 1}\).

Since \(Q_B^{(2)}\) and \(L_n^{(2)}\) commute, the operator \(Q_B^{(2)}\) is transformed into \(\bar{Q}_B\) under the similarity transformation with \(U_F'\):
\[
- \frac{1}{4} U_F' Q_B^{(2)} U_F'^{-1} = - \frac{1}{4} e^{-K(\rho)} U_F^{(2)} Q_B^{(2)} (U_F'^{-1})^{-1} e^{K(\rho)} = \bar{Q}_B. \tag{3.21}
\]
where we have used the equation (3.6). Consequently, from a similar derivation of (3.7), it follows that if a state $|\psi\rangle$ satisfies $\bar{Q}_B |\psi\rangle = 0$, this state can be written

$$|\psi\rangle = U_F' (|P\rangle \otimes b_{-2} |0\rangle) + U_F' (|P'\rangle \otimes |0\rangle) + \bar{Q}_B |\phi\rangle. \quad (3.22)$$

As in the case $a > -1/2$, the difference between the cohomologies (3.7) and (3.22) turns out to be merely a BRS-exact state, as can be shown by using the relation (3.19).

For $a = -1/2$, the operator $L(a)$ becomes

$$\tilde{L} = L(-1/2) = \frac{1}{2} L_0' - \frac{1}{4} (L_2' + L_{-2}') + \frac{3}{2}. \quad (3.23)$$

Though the operator $\tilde{L}$ cannot be transformed into $L_0$ under the similarity transformation with $U(s,t,u)$, as seen in the previous section, it can be transformed into $L_2'$ under a transformation with $U_F'$:

$$\tilde{L} = -\frac{1}{4} U_F' L_2' U_F'^{-1}. \quad (3.24)$$

Finally, we consider scattering amplitudes in the theory expanded around the non-trivial solution. Because $b_0$ commutes with $L'_n$, the cohomology in the Siegel gauge is given by

$$U_F' (|P\rangle \otimes b_{-2} |0\rangle) + U_F' (|P'\rangle \otimes |0\rangle). \quad (3.25)$$

These states have ghost number $-1$ or $0$. If we calculate an amplitude with these as external states, it becomes zero because its total ghost number differs from that needed to realize non-zero amplitudes. Hence, we conclude that open strings do not appear perturbatively in the theory expanded around the universal solution at $a = -1/2$

§4. Discussion

In this paper, we have studied open string field theory expanded around the universal solution under the Siegel gauge condition. In the theory for $a > -1/2$, we derived physical states and the Feynman rule, and then we showed that open string amplitudes in the original theory can be reproduced. In the theory with $a = -1/2$, we proved no open string theorem, which states that no open string exists around the non-trivial universal solution, because perturbative amplitudes become zero there. These results provide further evidence for the conjecture that universal solutions correspond to a pure gauge or the tachyon vacuum. Though the function in the solution is restricted to the form given in (2.8), it is possible to extend it to other functions, for example to the function given in Ref. [6] in a straightforward way.
For $a > -1/2$, through a string field redefinition, we can understand the coincidence of the amplitudes of the original and expanded theories. The action of the gauge unfixed theory expanded around the solution can be transformed into the original action under the similarity transformation with the operator $K(h_a)$. Similarly, as found in §2.4, if we carry out a string field redefinition using the operator $U'(a)$ and change the normalization of the string field, we can transform the action in the Siegel gauge to the original action fixed in the same gauge. Consequently, the amplitudes in the two theories are identical. Here, this scale transformation of the string field provides the factor absorbed into external states of the amplitudes.

It is plausible that the universal solution for $a = -1/2$ is the tachyon vacuum solution. If this is the case, we should be able to observe closed strings, at least as on-shell states, in the theory expanded around this solution. This is an open problem, which is important for proving the equivalence of these solutions and Sen’s conjectures. As Drukker pointed out, it should be possible to reproduce closed string scattering amplitudes with the Feynman rule around the non-trivial solution.

**Acknowledgements**

We would like to thank Hiroshi Itoyama for discussions and encouragement.
Appendix A

Generalization of (2.52)

In the twisted ghost conformal field theory with a general background charge,\textsuperscript{9,21} the energy momentum tensor is given by

\[ T'(w) = T(w) + (\lambda - 2) \partial J_{gh}(w), \]  

(A.1)

where \( \lambda \) denotes a real number. The theory with \( \lambda = 1 \) corresponds to the case investigated in the main text. The mode expansions of the ghost and anti-ghost fields in the twisted theory are given by

\[ c'(w) = \sum_{n=-\infty}^{\infty} c_n w^{-n+1+\lambda}, \quad b'(w) = \sum_{n=-\infty}^{\infty} b_n w^{-n-\lambda}. \]  

(A.2)

Thus, the twisted and untwisted ghost fields have the relations

\[ c'(w) = w^{\lambda-2} c(w), \quad b'(w) = w^{-(\lambda-2)} b(w). \]  

(A.3)

The twisted Virasoro generators can be written in terms of the untwisted operators as

\[ L'_n = L_n + (2 - \lambda) q_n + a_n \delta_{n,0}, \]  

(A.4)

where \( a_n = -(\lambda - 2)(\lambda + 1)/2 \) if \( \lambda = 0, \pm 1, \pm 2, \ldots \).

If the operators \( U_f \) and \( U'_f \) implement the conformal mapping \( z = f(w) \) in the untwisted and twisted theories, respectively, the primary fields \( \phi(w) \) of dimension \( h \) and \( \phi'(w) \) of dimension \( h' \) in each theory satisfy

\[ U_f \phi(w) U_f^{-1} = (\partial f(w))^h \phi(f(w)), \quad U'_f \phi'(w) U'_f^{-1} = (\partial f(w))^{h'} \phi'(f(w)). \]  

(A.5)

The ghost operators \( c(w), b(w), c'(w) \) and \( b'(w) \) have dimensions \( h = -1, h = 2, h' = 1 - \lambda \) and \( h' = \lambda \), respectively. Using (A.3) and (A.5), we find that

\[ U_f e^{K(g)} c'(w) e^{-K(g)} U_f^{-1} = w^{\lambda-2} U_f e^{K(g)} c(w) e^{-K(g)} U_f^{-1} \]
\[ = w^{\lambda-2} e^{g(w)} U_f c(w) U_f^{-1} \]
\[ = w^{\lambda-2} e^{g(w)} (\partial f(w))^{-1} c(f(w)) \]
\[ = w^{\lambda-2} e^{g(w)} (\partial f(w))^{-1} (f(w))^{2-\lambda} c'(f(w)). \]  

(A.6)

Similarly, we obtain the following equation for the anti-ghost:

\[ U_f e^{K(g)} b'(w) e^{-K(g)} U_f^{-1} = w^{-\lambda+2} e^{-g(w)} (\partial f(w))^2 (f(w))^{\lambda-2} b'(f(w)). \]  

(A.7)
If these transformation laws coincide with the conformal mapping \( z = f(w) \) in the twisted theory, the function \( g(w) \) satisfies

\[
\begin{align*}
    w^{\lambda - 2} e^{g(w)} (\partial f(w))^{-1} (f(w))^{2-\lambda} &= (\partial f(w))^{1-\lambda}, \\
    w^{-\lambda + 2} e^{-g(w)} (\partial f(w))^{2} (f(w))^{\lambda - 2} &= (\partial f(w))^{\lambda}.
\end{align*}
\]  

(A.8)

In this case, \( g(w) \) is given by

\[
g(w) = \log \left( \frac{w \partial f(w)}{f(w)} \right)^{2-\lambda}.
\]  

(A.9)

Because the operator \( U'_f \) that induces the conformal mapping \( z = f(w) \) can be uniquely determined up to a multiplicative constant, we obtain the formula

\[
U'_f = \text{const} \times U_f \times \exp K \left( \log \left( \frac{w \partial f(w)}{f(w)} \right)^{2-\lambda} \right).
\]  

(A.10)

If we set \( \lambda = 1 \), this formula reduces to the equation (2.52).
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