AN INVARIANT SUPERTRACE FOR THE CATEGORY OF REPRESENTATIONS OF LIE SUPERALGEBRAS

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ABSTRACT. In this paper we give a re-normalization of the supertrace on the category of representations of Lie superalgebras of type I, by a kind of modified superdimension. The genuine superdimensions and supertraces are generically zero. However, these modified superdimensions are non-zero and lead to a kind of supertrace which is non-trivial and invariant. As an application we show that this new supertrace gives rise to a non-zero bilinear form on a space of invariant tensors of a Lie superalgebra of type I. The results of this paper are completely classical results in the theory of Lie superalgebras but surprisingly we can not prove them without using quantum algebra and low-dimensional topology.

INTRODUCTION

The theory of quantum groups and classical representation theory of Lie algebras has been widely and productively used in low-dimensional topology. There are fewer examples of low-dimensional topology or quantum groups being used to produce results in the classical theory of Lie algebras. Good examples of such work include the theory of crystal bases (see [7]) and the use of the Kontsevich integral to give a new proof of the multiplicativity of the Duflo-Kirillov map $S(\mathfrak{g}) \rightarrow U(\mathfrak{g})$ for metrized Lie (super-)algebras $\mathfrak{g}$ (see [11]). In this paper we use low-dimensional topology and quantum groups to define a non-trivial kind of supertrace on the category of representations of a Lie superalgebra of type I. It should be noted that the genuine supertrace is generically zero on such a category (see Proposition 2.2).

In [3, 4], the authors give a re-normalization of the Reshetikhin-Turaev quantum invariants, by modified quantum dimensions. In the case of simple Lie algebras these modified quantum dimensions are proportional to the genuine quantum dimensions. For Lie superalgebras of type I the genuine quantum dimensions are generically zero but the modified quantum dimensions are non-zero and lead to non-trivial link
invariants. In this case the modified quantum dimension of a quantized module is given by an explicit formula which is determined by the underlying Lie superalgebra module. In this paper we take the classical limit of the modified quantum dimension to obtain a modified superdimension. Then we use this modified superdimension to re-normalize the supertrace and define a non-trivial bilinear form on a space of invariant tensor.

Our proof that the modified supertrace is well defined and has the desired properties is as follows. We first formulate the desired statements at the level of the Lie superalgebra. Then we “deform” these statements to the quantum level and use low-dimensional topology to prove these “deformed” statements. Taking the classical limit we recover the original statements. To make this proof precise we use the Etingof-Kazhdan theory of quantization.

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1. Preliminaries

In this section we review background material that will be used in the following sections.

A super-space is a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$ over $\mathbb{C}$. We denote the parity of an homogeneous element $x \in V$ by $\bar{x} \in \mathbb{Z}_2$. We say $x$ is even (odd) if $x \in V_0$ (resp. $x \in V_1$). In the Appendix we recall some basic features and conventions concerning the category of super-spaces.

A Lie superalgebra is a super-space $g = g_0 \oplus g_1$ with a super-bracket $[,] : g \otimes g \to g$ that preserves the $\mathbb{Z}_2$-grading, is super-antisymmetric ($[x, y] = -(-1)^{\bar{x}\bar{y}}[y, x]$), and satisfies the super-Jacobi identity (see [5]). Throughout, all modules will be $\mathbb{Z}_2$-graded modules (module structures which preserve the $\mathbb{Z}_2$-grading, see [5]).

1.1. Lie superalgebras of type I. In this subsection we recall notations and properties related to Lie superalgebras of type I.

Let $g = g_0 \oplus g_1$ be a Lie superalgebra of type I, i.e. $g$ is equal to $\mathfrak{sl}(m|n)$ or $\mathfrak{osp}(2|2n)$. We will assume that $m \neq n$. Let $b$ be the distinguished Borel sub-superalgebra of $g$. Then $b$ can be written as the direct sum of a Cartan sub-superalgebra $\mathfrak{h}$ and a positive nilpotent sub-superalgebra $n_+$. Moreover, $g$ admits a decomposition $g = n_- \oplus \mathfrak{h} \oplus n_+$. Let $W$ be the Weyl group of the even part $g_0$ of $g$. 
Let $\Delta^+_{\mathfrak{t}}$ (resp. $\Delta^+_{\mathfrak{t}}$) be the even (resp. odd) positive roots. Let $\rho_{\mathfrak{t}}$ (resp. $\rho_{\mathfrak{t}}$) denote the half sum of all the even (resp. odd) positive roots. Set $\rho = \rho_{\mathfrak{t}} - \rho_{\mathfrak{t}}$. A positive root is called simple if it cannot be decomposed into a sum of two positive roots.

A Cartan matrix associated to a Lie superalgebra is a pair consisting of a $r \times r$ matrix $A = (a_{ij})$ and a set $\tau \subset \{1, \ldots, r\}$ determining the parity of the generators. Let $(A, \tau)$ be the Cartan matrix arising from $\mathfrak{g}$ and the distinguished Borel sub-superalgebra $\mathfrak{b}$. Here the set $\tau = \{s\}$ consists of only one element because of our choice of Borel sub-algebra $\mathfrak{b}$. (See the appendix.)

By Proposition 1.5 of [6] there exists $e_i \in \mathfrak{n}^+$, $f_i \in \mathfrak{n}^-$ and $h_i \in \mathfrak{h}$ for $i = 1, \ldots, r$ such that the Lie superalgebra $\mathfrak{g}$ is generated by $e_i, f_i, h_i$ where

$$[e_i, f_j] = \delta_{ij}h_i, \quad [h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij}e_j, \quad [h_i, f_j] = -a_{ij}f_j.$$ 

Note that these generators also satisfy the Serre relations and higher order Serre type relations (see [9]).

There are $d_1, \ldots, d_r$ in $\{\pm 1, \pm 2\}$ such that the matrix $(d_ia_{ij})$ is symmetric. Let $\langle , \rangle$ be the symmetric non-degenerate form on $\mathfrak{h}$ determined by $\langle h_i, h_j \rangle = d_j^{-1}a_{ij}$. This form gives an identification of $\mathfrak{h}$ and $\mathfrak{h}^*$. Moreover, the form $\langle , \rangle$ induces a $W$-invariant bilinear form on $\mathfrak{h}^*$, which we will also denote by $\langle , \rangle$.

1.2. The category $\mathfrak{g}$-Mod. Modules over Lie superalgebras of type I are different in nature than modules over semi-simple Lie algebras. For example, each Lie superalgebra of type I has one parameter families of modules. Any module in such a family has superdimension zero and so the supertrace of an endomorphism of such a module is zero.

Let $\mathfrak{g}$-Mod be the category of finite dimensional $\mathfrak{g}$-modules (see Appendix). We will now describe this category in more detail. If $U$ and $V$ are two $\mathfrak{g}$-modules we denote by $\text{Hom}_\mathfrak{g}(U, V)$ the super-space of $\mathfrak{g}$-module morphisms. The super-space $\text{Hom}_\mathfrak{g}(U, V)$ should not be confused with $\text{Hom}_C(U, V)$ (where $U$ and $V$ are viewed as super-spaces) which is naturally equipped with a $\mathfrak{g}$-module structure.

Let $\lambda \in \mathfrak{h}^*$ be a linear functional on $\mathfrak{h}$. Kac [5] defined a $\mathfrak{g}$ irreducible highest weight module $V(\lambda)$ of weight $\lambda$ with a highest weight vector $v_0$ having the property that $h.v_0 = \lambda(h)v_0$ for all $h \in \mathfrak{h}$ and $\mathfrak{n}_+v_0 = 0$. Let $a_i = \lambda(h_i)$. In [5] Kac showed that $V(\lambda)$ is finite-dimensional if and only if $a_i \in \mathbb{N}$ for $i \neq s$. Therefore, $a_s$ can be an arbitrary complex number. Irreducible finite-dimensional $\mathfrak{g}$-modules are divided into two classes: typical and atypical.
There are many equivalent definition for a weight module to be typical (see [6]). Here we say that \( V(\lambda) \) is typical if it splits in any finite-dimensional \( g \)-module (i.e. if it is a submodule or a factor-module of a finite dimensional \( g \)-module then it is a direct summand). By Theorem 1 of [6] this is equivalent to requiring that

\[
< \lambda + \rho, \alpha > \neq 0
\]

for all \( \alpha \in \Delta^+ \). If \( V(\lambda) \) is (a)typical we will say the weight \( \lambda \) is (a)typical.

In Section 2 we construct a trace on the “ideal” generated by typical modules. With this in mind let us recall some properties of these modules. The space of typical weights is dense in the space of weights corresponding to finite-dimensional modules. In particular, if \( a_i \in \mathbb{N} \) for \( 1 \leq i \leq r \) and \( i \neq s \) then there are only finitely many atypical weights with \( a_i = \lambda(h_i) \). Furthermore, if \( \lambda \) is atypical then \( a_s = \lambda(h_s) \in \mathbb{Z} \). Thus, the name typical is fitting.

For any object \( V \) of \( g\text{-Mod} \) whose \( \mathbb{Z}_2 \) grading is given by \( V = V_0 \oplus V_1 \) let \( \text{sdim}(V) = \dim(V_0) - \dim(V_1) \) be the superdimension of \( V \). From Proposition 2.10 of [6] we have that if \( V \) is a typical \( g \)-module then \( \text{sdim}(V) = 0 \). This vanishing can make other mathematical objects trivial. For example, the supertrace on endomorphisms of a typical module and quantum invariants of links arising from Lie superalgebras (see Proposition 2.2 and [3], resp.).

Fix a typical module \( V_0 \). Let \( \mathcal{I}_{V_0} \) be the set of objects \( V \) of \( g\text{-Mod} \) such that there exists an object \( W \) of \( g\text{-Mod} \) and even \( g \)-linear morphisms \( \alpha : V \to V_0 \otimes W \) and \( \beta : V_0 \otimes W \to V \) with \( \beta \circ \alpha = \text{Id}_V \).

Proposition 1.1.

1. The definition of \( \mathcal{I}_{V_0} \) does not depend on the choice of \( V_0 \), i.e. \( \mathcal{I}_{V_0} = \mathcal{I}_{V_1} \) for any two typical modules \( V_0 \) and \( V_1 \).

2. The set \( \mathcal{I}_{V_0} \) is an ideal in the sense that for any \( V, V' \in \mathcal{I}_{V_0} \) and \( W \in g\text{-Mod} \) we have \( V \otimes W \in \mathcal{I}_{V_0} \) and \( V \oplus V' \in \mathcal{I}_{V_0} \).

We define \( \mathcal{I} \) to be the set \( \mathcal{I}_V \) where \( V \) is any typical module, which is well define by the proposition.

Proof of Proposition 1.1. We will prove the first statement, the second follows easily from the definition of \( \mathcal{I}_{V_0} \). First, we have \( W \in \mathcal{I}_V \) if and only if \( \mathcal{I}_W \subset \mathcal{I}_V \). We will use this fact in the remainder of the proof.

As mentioned above irreducible finite dimensional \( g \)-modules are in one to one correspondence with \( \mathbb{N}^{r-1} \times \mathbb{C} \). We will denote \( V_\alpha \) as the module corresponding to \( (\bar{c}, \alpha) \in \mathbb{N}^{r-1} \times \mathbb{C} \). Let \( V_\alpha^0 \) and \( V_\alpha^1 \) be typical modules. From the character formula for typical modules we know that
$V^c_\beta$ is a submodule of $V^0_\alpha \otimes V^c_\beta_{-\alpha}$. Since typical modules always split we have $V^c_\beta \in \mathcal{I}_{V^c_\beta}$ and so $\mathcal{I}_{V^c_\beta} \subset \mathcal{I}_{V^c_\beta}$. 

On the other, from the discussion in the previous paragraph we have $\text{Hom}_g(V^0_\alpha \otimes V^c_\beta_{-\alpha}, V^c_\beta) \neq 0$, implying $\text{Hom}_g(V^0_\alpha, V^c_\beta \otimes (V^c_{\beta-\alpha})^*) \neq 0$. Therefore, as $V^0_\alpha$ is typical, $V^0_\alpha \in \mathcal{I}_{V^c_\beta}$ and so $\mathcal{I}_{V^c_\beta} \subset \mathcal{I}_{V^c_\beta}$. 

2. A Trace

In this section we define a non-zero supertrace on $\text{End}_g(V)$ for $V \in \mathcal{I}$. First, let us prove that the usual supertrace on $\text{End}_g(V)$ is zero.

Let $V$ be a super-space and let $\{v_i\}$ be a basis of $V$ with homogeneous vectors. Let $\{v_i^*\}$ be the dual basis of $V^*$. We have that $\overline{v}_i^* = \overline{v}_i = \overline{v}_i^*$. Define the supertrace on $\text{End}_\mathbb{C}(V)$ to be the function $\text{str}_V : \text{End}_\mathbb{C}(V) \to \mathbb{C}$ given by $f \mapsto \sum_i (-1)^i \overline{v}_i^*(f(v_i))$. Then str has the property that if $f \in \text{Hom}_\mathbb{C}(V, W)$ and $g \in \text{Hom}_\mathbb{C}(W, V)$ then $\text{str}_W(f \circ g) = (-1)^{\overline{f}} \text{str}_V(g \circ f)$.

Let us define the partial supertrace that is a generalization of the supertrace. For this, we first define the evaluation and coevaluation morphisms $ev_V : V \otimes V^* \to \mathbb{C}$ and $coev_V : \mathbb{C} \to V \otimes V^*$ given by $v \otimes f \mapsto (-1)^{\overline{f}} f(v)$ and $1 \mapsto \sum_i v_i^* \otimes v_i$, respectively.

Definition 2.1. Let $U$ and $V$ be super-spaces and $f \in \text{End}_\mathbb{C}(U \otimes V)$. Then we call the partial supertrace of $f$ the endomorphism

$$\text{ptr}(f) = (\text{Id}_U \otimes ev_V) \circ (f \otimes \text{Id}_{V^*}) \circ (\text{Id}_V \otimes \text{coev}_V) \in \text{End}_\mathbb{C}(U).$$

For $f$ as in Definition 2.1 we have $\text{str}_{U \otimes V}(f) = \text{str}_V(\text{ptr}(f))$. In addition, if $f \in \text{End}_g(U \otimes V)$ then $\text{ptr}(f) \in \text{End}_g(U)$.

Let $V$ be an element of $\mathcal{I} = \mathcal{I}_{V^c_\beta}$ and $f \in \text{End}_g(V)$. Choose morphisms $\alpha : V_0 \otimes W \to V$ and $\beta : V \to V_0 \otimes W$ such that $\alpha \circ \beta = \text{Id}_V$. Then ptr$(\beta \circ \alpha)$ is an invariant map of $V_0$ and so ptr$(\beta \circ \alpha) = \mathfrak{c} \text{Id}_{V_0}$ for some $\mathfrak{c} \in \mathbb{C}$. We define the bracket of the triple $(f, \alpha, \beta)$ to be $<f; \alpha, \beta>$.

Proposition 2.2. Let $V \in \mathcal{I}$ and $f \in \text{End}_g(V)$ then $\text{str}_V(f) = 0$.

Proof. Using the notation above, we have $\text{str}_V(f) = \text{str}_V(f \circ \alpha \circ \beta) = \text{str}_{V_0 \otimes W}(\beta \circ f \circ \alpha) = \text{str}_{V_0}(\text{ptr}(\beta \circ f \circ \alpha))$. But ptr$(\beta \circ f \circ \alpha) = <f; \alpha, \beta > \text{Id}_{V_0}$ so

$$\text{str}_V(f) = \text{str}_{V_0}(<f; \alpha, \beta > \text{Id}_{V_0}) = <f; \alpha, \beta > \text{sdim}(V_0) = 0$$

as the superdimension of $V_0$ is zero. 

$\square$
Definition 2.3. Let \( \mathbf{d} : \{ \text{typical modules} \} \to \mathbb{C} \) be the function defined by

\[
\mathbf{d}(V(\lambda)) = \prod_{\alpha \in \Delta^+} \frac{\lambda + \rho, \alpha >}{\rho, \alpha >} \prod_{\alpha \in \Delta^+} \frac{\lambda + \rho, \alpha >}.
\]

Note that Equation (1) implies that \( \mathbf{d} \) is well defined. As an example, for \( \mathbf{g} = \mathfrak{sl}(n|1) \) with \( n \geq 2 \), and for \( \Lambda = (0, \ldots, 0|a) \) with \( a \notin \{0, -1, \ldots, 1 - n\} \), we have \( \mathbf{d}(V(\lambda)) = \prod_{i=0}^{n-1} 1/(a + i) \).

Theorem 1. Let \( V \in I \) and \( f \in \text{End}_g(V) \). Choose a typical module \( V_0 \), \( \alpha \in \text{Hom}_g(V_0 \otimes W, V) \) and \( \beta \in \text{Hom}_g(V, V_0 \otimes W) \) such that \( \alpha \circ \beta = \text{Id}_V \).

Then

\[ \text{str}'(f) = \mathbf{d}(V_0) < f; \alpha; \beta > \]

depends only on \( f \), i.e. does not depend on the choice of \( V_0 \), \( \alpha \) or \( \beta \).

Furthermore, \( \text{str}' \) is a trace in the following sense: for any \( V, V' \in I \) and any \( g \)-module \( U \),

1. \( \text{str}' : \text{End}_g(V) \to \mathbb{C} \) is linear.
2. \( \text{str}'(f \circ g) = (-1)^{|\mathfrak{g} T|} \text{str}'(g \circ f) \) for any \( f \in \text{Hom}_g(V, V') \) and \( g \in \text{Hom}_g(V', V) \),
3. \( \text{str}'_{V \otimes U}(f \otimes g) = \text{str}'(f) \text{str}_U(g) \) for any \( f \in \text{End}_g(V) \) and any \( g \in \text{End}_g(U) \), in particular \( \text{str}'(f \otimes g) = \text{str}(g) = 0 \) if \( U \in I \).
4. \( \text{str}'_{V \otimes U}(f) = \text{str}'(\text{ptr}(f)) \) for any \( f \in \text{End}_g(V \otimes U) \).

The proof of Theorem 1 will be given in Section 4. Let us now make a few comments about this theorem. First, remark that property (4) implies property (3). Next, property (4) implies a kind of invariance for \( \text{str}' \). Let us make this statement more precise.

Let \( U, U' \) be \( g \)-modules and \( V, V' \) be in \( I \). The following spaces of morphisms are canonically isomorphic:

\[
\text{Hom}_g(\text{Hom}_C(U', V'), \text{Hom}_C(U, V)) \cong \text{Hom}_g(U \otimes V', V \otimes U')
\]

\[
\cong \text{Hom}_g(V' \otimes U, U' \otimes V) \cong \text{Hom}_g(\text{Hom}_C(V, U), \text{Hom}_C(V', U'))
\]

Let \( \Psi \in \text{Hom}_g(\text{Hom}_C(U', V'), \text{Hom}_C(U, V)) \) and respectively \( h, h^\#, \Psi^\# \) be the corresponding morphisms in the three other spaces. We have \( h^\# = \tau \circ h \circ \tau \) where \( \tau \) is the super permutation (see Appendix). Also, if \( f \in \text{Hom}_C(U', V') \) and \( g \in \text{Hom}_C(V, U) \) then \( \Psi(f) = \text{ptr}(h \circ (\text{Id}_V \otimes f)) \) and \( \Psi^\#(g) = \text{ptr}(h^\# \circ (\text{Id}_V \otimes g)) \) (here we use a generalization of the partial trace \( \text{ptr} : \text{Hom}(A \otimes C, B \otimes C) \to \text{Hom}(A, B) \)). Thus, applying property (4), we get that

\[
\text{str}'(\Psi(f) \circ g) = (-1)^{|\mathfrak{g} T|} \text{str}'(f \circ \Psi^\#(g))
\]
Indeed,

\[
\text{str}' (\Psi(f) \circ g) = \text{str}' (pt(h \circ (\text{Id}_U \otimes f)) \circ g) \\
= (-1)^{\frac{d}{2}} \text{str}' (pt(h \circ (g \otimes f))) \\
= (-1)^{\frac{d}{2}} \text{str}' (h \circ (g \otimes f)) \\
= \text{str}' (h^\# \circ (f \otimes g)) \\
= \text{str}' (pt(h^\# \circ (f \otimes g))) \\
= (-1)^{\frac{d}{2}} \text{str}' (pt(h^\# \circ (\text{Id}_{V'} \otimes g)) \circ f) \\
= (-1)^{\frac{d}{2}} \text{str}' (f \circ \Psi^\# (g)) .
\]

The results of this section can be stated in the language of symmetric monoidal category with duality or more generally ribbon categories. We will not make this formalism precise, however we will end this section by giving the following graphs which we hope will shed light on the above results. For more details on ribbon categories see [8].

Here we will represent morphisms with ribbon graphs, which are read from bottom to top. The tensor product of two morphisms is represented by setting the two corresponding graphs next to each other. For example, if \( f : V \rightarrow V' \) and \( g : U \rightarrow U' \) are even morphism of \( g\)-Mod then we represent \( f \) and \( f \otimes g \) by:

\[
\begin{array}{c}
\begin{array}{ccc}
V' & \downarrow f & \downarrow V \\
V & \downarrow & \downarrow \\
\end{array}
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\begin{array}{ccc}
V' & \downarrow f & \downarrow V' \\
V & \downarrow & \downarrow \\
U & \downarrow g & \downarrow U' \\
U' & \downarrow & \downarrow \\
\end{array}
\end{array}
= \begin{array}{c}
\begin{array}{ccc}
V' & \downarrow f \otimes g & \downarrow V' \\
V & \downarrow & \downarrow \\
U & \downarrow & \downarrow \\
U' & \downarrow & \downarrow \\
\end{array}
\end{array}
\]

respectively. Let the graphs \( \begin{array}{c}
\begin{array}{ccc}
V & \downarrow & \downarrow \\
V & \downarrow & \downarrow \\
\end{array}
\end{array} \) and \( \begin{array}{c}
\begin{array}{ccc}
V & \downarrow & \downarrow \\
V & \downarrow & \downarrow \\
\end{array}
\end{array} \) represent the evaluation and coevaluation morphisms \( \text{ev}_V : V \otimes V^* \rightarrow \mathbb{C} \) and \( \text{coev}_V : \mathbb{C} \rightarrow V \otimes V^* \) given by \( v \otimes f \mapsto (-1)^{|f|} f(v) \) and \( 1 \mapsto \sum_i v_i \otimes v_i^* \), respectively.

Let \( g : V \rightarrow V \) be an even invariant morphism of a \( g \)-module \( V \) and let \( G \) be a ribbon graph representing \( g \) (as in Equation (3)). If \( V \) is simple then the morphism \( g \) is a scalar times the identity, which we denote by \( < g > = < G >. \)

The elements \( \text{str}_V(g) \) and \( \text{str}'_V(g) \) can be represented by

\[
\text{str}_V(g) = \left( \begin{array}{c}
\begin{array}{c}
V \\
\end{array}
\end{array} \right) ,
\]

\[
\text{str}'_V(g) = \left( \begin{array}{c}
\begin{array}{c}
V \\
\end{array}
\end{array} \right) .
\]
where we require $V \in \mathcal{I}$ in (4). When $V$ is simple the supertrace can be rewritten as

$$\text{str}_V(g) = \left\langle V \circlearrowright \right\rangle \left\langle \begin{array}{c} V \\ g \\ v \end{array} \right\rangle = \text{sdim}(V) \left\langle \begin{array}{c} V \\ g \\ v \end{array} \right\rangle$$

where $\text{sdim}(V) = 0$ if $V$ is typical. Also, when $V$ is a typical module the $\text{str}'$ becomes

$$\text{str}'_V(g) = d(V) \left\langle \begin{array}{c} V \\ g \\ v \end{array} \right\rangle$$

Thus, the function $d$ can be thought of as a nonzero replacement of the usual superdimension. Moreover, $d$ can be thought of as the classical analogue of the modified quantum dimensions defined in \[4\].

If $f : V \to V'$ is an even invariant morphism let $f^* : (V')^* \to V^*$ be the “super-transpose” of $f$ defined in the Appendix. We can represent $f^*$ by

$$f^* = \begin{array}{c} V' \\ f \\ V \\ \sigma \end{array}$$

We will use the “super-transpose” in the next section.

3. INVARIANT TENSORS

In this section we define a non-trivial bilinear form on a space of invariant tensors of $\mathfrak{g}$. The standard bilinear form on $\mathfrak{g}$ is zero on this space of tensors.

Let $V$ be an object of $\mathfrak{g}$-$\text{Mod}$ and let $T(V) = \bigoplus_i T(V)_i$ be the tensor algebra of $V$, where $T(V)_i$ is the space $V^\otimes i$. Let $T(V)^\theta$ be the invariant tensors of $T(V)$.

**Lemma 3.1.** All invariant tensors of $T(\mathfrak{g})$ are even.
Proof. We will prove the lemma for \( \mathfrak{g} = \mathfrak{sl}(m|n) \), the prove for \( \mathfrak{osp}(2|2n) \) is similar. We can identify \( \mathfrak{sl}(m|n) \) with the Lie superalgebra of supertrace zero \( (m+n) \times (m+n) \) matrices. This standard representation is obtained by sending \( e_i \) to the elementary matrix \( E_{i,i+1} \), \( f_i \) to \( E_{i+1,i} \), \( h_i \) to \( E_{i,i} - E_{i+1,i+1} \) if \( i \neq m \) and \( h_m \) to \( E_{m,m} + E_{m+1,m+1} \). The Cartan subalgebra \( \mathfrak{h} \) with basis \( (h_i) \) is contained in the space of diagonal matrices \( X \). The space \( X^* \) has a canonical basis \( (\epsilon_1, \ldots, \epsilon_{m+n}) \) which is dual to the basis formed by the matrices \( E_{i,i} \). Set \( \delta_i = \epsilon_{i+m} \), then \( \mathfrak{h}^* \) is the kernel of the supertrace \( \text{str} = \sum \epsilon_i - \sum \delta_j \). Therefore, \( \mathfrak{h}^* \) is the quotient of \( X^* \) by the supertrace.

Let \( \Lambda \subset \mathfrak{h} \) be the root lattice generated by the positive roots. Let \( f : \Lambda \rightarrow \mathbb{Z} \) be the linear function determined by \( \epsilon_i \mapsto n \) and \( \delta_j \mapsto m \) (note that \( \text{str} \mapsto 0 \)). By definition the simple positive even roots \( \epsilon_i - \epsilon_j \) and \( \delta_i - \delta_j \) map to zero and the simple positive odd roots \( \epsilon_i - \delta_j \) map to \(- (m-n) \). Therefore, the image of \( f \) is \((m-n)\mathbb{Z} \) and \( f \) induces a linear map \( \tilde{f} : \Lambda \rightarrow \mathbb{Z}/2\mathbb{Z} \) given by \( \alpha \mapsto \frac{f(\alpha)}{m-n} \) modulo 2. The map \( \tilde{f} \) in turn induces a map on the weight vectors of \( T(\mathfrak{g}) \) (which we also denote by \( \tilde{f} \)) that satisfies \( \tilde{f}(x \otimes y) = \tilde{f}(x) + \tilde{f}(y) \) for \( x, y \in T(\mathfrak{g}) \). Note that \( \tilde{f} \) gives the parity of a weight vector of \( T(\mathfrak{g}) \).

Let \( t \) be an element of \( T(\mathfrak{g})_k \) with weight \( a_1 \epsilon_1 + \cdots + a_m \epsilon_m + b_1 \delta_1 + \cdots + b_n \delta_n \). If \( t \) is in \( (T(\mathfrak{g})_k)^\theta \) then the Cartan subalgebra acts by zero and so the weight of \( t \) is zero, i.e. \( a_i = b_j = 0 \) for all \( i \) and \( j \). But from above we have that parity of \( t \) is equal to \( f(t) = \frac{n \sum a_i + m \sum b_i}{m-n} \) modulo 2, which is zero if \( t \) is in \( (T(\mathfrak{g})_k)^\theta \). Thus, all the invariant tensors of \( T(\mathfrak{g}) \) are even. \( \square \)

From Propositions 2.5.3 and 2.5.5 of \([3]\) there exists a unique (up to constant factor) non-degenerate supersymmetric invariant even bilinear form \( \langle , \rangle \) on \( \mathfrak{g} \). Let \( b : \mathfrak{g} \rightarrow \mathfrak{g}^* \) be the isomorphism given by the assignment \( x \mapsto (x, \cdot) \).

We extend this bilinear form to \( T(\mathfrak{g}) \) by

\[
(x_1 x_2 \ldots x_k, x'_1 x'_2 \ldots x'_l) = \delta_{kl} \prod_{i=1}^{k} (-1)^i \sigma_i \tau_i(x_i, x'_i)
\]

where \( x_i, x'_j \in \mathfrak{g} \). Since \( \langle , \rangle \) is non-degenerate on \( \mathfrak{g} \) we have that this extension is a non-degenerate bilinear form on \( T(\mathfrak{g}) \). Moreover, since \( \langle , \rangle \) is supersymmetric on \( \mathfrak{g} \) and \( \langle x, x' \rangle = 0 \) for all \( x, x' \in \mathfrak{g} \) such that \( \tau \neq \tau' \) we have that the extension is supersymmetric on \( T(\mathfrak{g}) \).

For \( t \in (T(\mathfrak{g})_N)^\theta \simeq \text{Hom}_\mathbb{C}(T(\mathfrak{g})_N, \mathbb{C}) \) we have \( t^* \in \text{Hom}_\mathbb{C}(T(\mathfrak{g}^*)_N, \mathbb{C}) \), where \( * \) is the “super-transpose” defined in the Appendix. Using this
notation the bilinear form is given by \((t, t') = \langle t^* \circ b^\otimes_N \circ t' \rangle\). Here and after, if \(g \in \text{End}_C(\mathbb{C})\) then we will denote \(\langle g \rangle\) as the scalar \(g(1)\).

Recall the definition of the coevaluation morphism \(\text{coev}_V\) given in Section 2.

**Definition 3.2.** For \(N \in \mathbb{N}\) define

\[
\mathcal{IT}_N = \{ f(\text{coev}_V(1)) : f \in \text{Hom}_g(V \otimes V^*, g^\otimes n) \text{ for some } V \in \mathcal{I} \}
\]

and \(\mathcal{IT} = \oplus_N \mathcal{IT}_N\).

Let \(t \in \mathcal{IT}_N\) and \(t' \in (T(g)_N)^0\). We will now show that \((t, t')\) can be written in terms of the supertrace. We regard \(t, t'\) as elements of \(\text{Hom}_g(C, g^\otimes n)\). As \(t = f(\text{coev}_V)\) for some \(f \in \text{Hom}_g(V \otimes V^*, g^\otimes n)\) where \(V \in \mathcal{I}\), we have \(t^* = \text{coev}_V^* \circ f^*\) and

\[
(t, t') = \langle \text{coev}_V^* \circ f^* \circ b^\otimes_N \circ t' \rangle.
\]

The morphism \(f^* \circ b^\otimes_N \circ t' \in \text{Hom}_g(C, V^* \otimes V) \simeq \text{Hom}_g(C, V \otimes V^*)\) can be identified with a \(g\)-linear endomorphism of \(V\) which we denote by \([f^* \circ b^\otimes_N \circ t']\). Thus, we have \((t, t') = \text{str}_V([f^* \circ b^\otimes_N \circ t'])\) which is zero by Proposition 2.2. The above discussion can be summarized in the following lemma.

**Lemma 3.3.** If \(t \in \mathcal{IT}_N\) and \(t' \in (T(g)_N)^0\) then \((t, t') = \text{str}_V([f^* \circ b^\otimes_N \circ t'])\) which is zero.

**Proposition 3.4.** The sets \(\mathcal{IT}_N\) are vector spaces. Moreover, \(\mathcal{IT} = \oplus_N \mathcal{IT}_N\) is a two sided ideal of \(T(g)^0\) which is in the kernel of the restriction of \((., .)\) to the space of invariant tensor \(T(g)^0\).

**Proof.** We will first show that \(\mathcal{IT}_N\) is a vector space. Let \(t_1, t_2 \in \mathcal{IT}_N\) and \(\lambda \in \mathbb{C}\). Then \(t_i = f_i(\text{coev}_V(1))\) for some \(f_i\) and \(V_i\). Set \(V = V_1 \oplus V_2\). Let \(f : V \otimes V^* \rightarrow g^\otimes N\) be the invariant map given by

\[
f((v_1 \oplus v_2) \otimes (\varphi_1 \oplus \varphi_2)) = f_1(v_1 \otimes \varphi_1) + \lambda f_2(v_2 \otimes \varphi_2).
\]

Then \(f(\text{coev}_V(1)) = t_1 + \lambda t_2\). Thus, \(\mathcal{IT}_N\) is a vector space.

Now we will show that \(\mathcal{IT}\) is an ideal. Let \(t' \in (g^\otimes M)^0\) and let \(t_1\) be as above. Let \(g : V_1 \otimes V_1^* \rightarrow g^\otimes (M+N)\) be the invariant map given by

\[
g(v_1 \otimes \varphi_1) = t' \otimes f_1(v_1 \otimes \varphi_1).
\]

Then \(g(\text{coev}_{V_1}(1)) = t' \otimes t_1\) and so \(t' \otimes t_1 \in \mathcal{IT}_{M+N}\).

The last statement of the proposition follows from Lemma 3.3. \(\square\)

Next we define a bilinear form on \(\mathcal{IT}\). The following definition is motivated by Lemma 3.3 and justified by Theorem 2.
Definition 3.5. For $t_1 \in \mathcal{IT}_N$ and $t_2 \in \mathcal{IT}_M$ with $t_i = f_i(\text{coev}_{V_i})$, define
\[(t_1, t_2)' = \delta_{M,N} \text{str}_V'( [f_1^* \circ b^{\otimes N} \circ t_2])\]

We can represent $[f_1^* \circ b^{\otimes N} \circ t_2]$ by the following picture where $M = N = 3$ for simplicity:

\[
\begin{array}{c}
V_2 \\
\vdots \\
V_1 \\
\end{array}
\begin{array}{c}
f_2 \\
\vdots \\
f_1 \\
\end{array}
\]

It is tempting to think that the above construction could work for $t_1 \in \mathcal{IT}$ and any $t_2 \in T(\mathfrak{g})$ but this is false because there are examples of $t_2 \in T(\mathfrak{g})$ for which the above scalar depends not only of $t_1$ but also of $f_1$.

To simplify notation we will identify $\mathfrak{g}$ and $\mathfrak{g}^*$ using the isomorphism $b$ but will no longer write $b$.

**Theorem 2.** $(\cdot, \cdot)'$ is a well define symmetric bilinear form on $\mathcal{IT}$ satisfying $(G(t_1), t_2)' = (t_1, G^*(t_2))'$ for any $t_1 \in \mathcal{IT}_M, t_2 \in \mathcal{IT}_N$ and $G \in \text{Hom}_\mathfrak{g}(T(\mathfrak{g})_M, T(\mathfrak{g})_N)$. In particular, the symmetric group $S_N$ acts orthogonally on $\mathcal{IT}_N$.

**Proof.** Let $t_1$ and $t_2$ be elements of $\mathcal{IT}_N$ with $t_i = f_i(\text{coev}_{V_i})$. We need to show that the definition of $(t_1, t_2)'$ is independent of $f_1, f_2, V_1$, and $V_2$.

Using the canonical isomorphism given in Equation (11), we can identify $\text{Hom}_\mathfrak{g}(V_2 \otimes V_2^*, V_1 \otimes V_1^*) \cong \text{Hom}_\mathfrak{g}(\mathbb{C}, V_1 \otimes V_1^*) \cong \text{End}_\mathfrak{g}(V_1 \otimes V_2)$. Therefore, below we will consider $f_1^* \circ f_2$ as an element of $\text{End}_\mathfrak{g}(V_1 \otimes V_2)$. Notice that for fixed $t_1 = f_1(\text{coev}_{V_1})$ the map $\mathcal{IT}_N \to \mathbb{C}$ given by
\[t \mapsto \text{str}_{V_1}'(f_1^* \circ t)\]
is well defined and linear. Then from Theorem 1 (4) we have
\[
\begin{align*}
\text{str}_{V_1}'(f_1^* \circ t_1) &= \text{str}_{V_1 \otimes V_2}'(f_1^* \circ f_2) \\
&= \text{str}_{V_1 \otimes V_2}'(f_2^* \circ f_1) \\
&= \text{str}_{V_2}'(f_2^* \circ t_1),
\end{align*}
\]
which does not depend on $f_1$ or $V_1$. Thus, $(\cdot, \cdot)'$ is a well defined symmetric bilinear form.

For the last statement of the theorem we have $(G(t_1), t_2)' = \text{str}_{V_1 \otimes V_2}'(f_1^* \circ G^* \circ f_2) = (t_1, G^*(t_2))'$. □
4. Proof of Theorem 1

The proof of Theorem 1 uses quantized Lie superalgebras and low-dimensional topology. In particular, we have the following general plan: (1) start with the desired statement at the level of $\mathfrak{g}$-Mod, (2) translate these statements to the quantum level, (3) use properties of invariants of ribbon graphs to prove these statements and (4) take the classical limit to obtain the proof of the original statements. With this in mind we will begin this section by recalling some properties about the Drinfeld-Jimbo type quantization of $\mathfrak{g}$.

Let $h$ be an indeterminate and set $q = e^{h/2}$. We use the notation $q^z = e^{zh/2}$ for $z \in \mathbb{C}$. Let $U_{h}^{DJ}(\mathfrak{g})$ be the Drinfeld-Jimbo type quantization of $\mathfrak{g}$ defined in [9]. The quantization $U_{h}^{DJ}(\mathfrak{g})$ is a braided $\mathbb{C}[[h]]$-Hopf superalgebra given by generators and relations. As we will explain now $U_{h}^{DJ}(\mathfrak{g})$ is related to a quasi-Hopf superalgebra.

For each Lie algebra Drinfeld defined a quasi-Hopf quantized universal enveloping algebra:

$$(U(\mathfrak{g})[[h]], \Delta_0, \epsilon_0, \Phi_{KZ}).$$

The morphisms $\Delta_0$ and $\epsilon_0$ are the standard coproduct and counit of $U(\mathfrak{g})[[h]]$. The element $\Phi_{KZ}$ is the KZ-associator. Let $A_{\mathfrak{g}}$ be the analogous topologically free quasi-Hopf superalgebra (for more details see [2]).

Let $U_{h}^{DJ}(\mathfrak{g})$-$\text{Mod}_{fr}$ ($A_{\mathfrak{g}}$-$\text{Mod}_{fr}$) be the tensor category of topologically free $U_{h}^{DJ}(\mathfrak{g})$-modules (resp. $A_{\mathfrak{g}}$-modules) of finite rank, i.e. $U_{h}^{DJ}(\mathfrak{g})$-modules (resp. $A_{\mathfrak{g}}$-modules) of the form $V[[h]]$ where $V$ is a finite dimensional $\mathfrak{g}$-module. We say a module $V[[h]]$ in $U_{h}^{DJ}(\mathfrak{g})$-$\text{Mod}_{fr}$ is typical if $V$ is a typical $\mathfrak{g}$-module.

In [2] the first author proves that there exists a functor $G : A_{\mathfrak{g}}$-$\text{Mod}_{fr} \rightarrow U_{h}^{DJ}(\mathfrak{g})$-$\text{Mod}_{fr}$ which is an equivalence of tensor categories. There is a natural tensor functor $G' : \mathfrak{g}$-$\text{Mod} \rightarrow A_{\mathfrak{g}}$-$\text{Mod}_{fr}$ given by $V \mapsto V[[h]]$ and $f \mapsto G'(f)$ where the action of $\mathfrak{g}$ on $V$ extends to an action of $U(\mathfrak{g})[[h]]$ on $V[[h]]$ be linearity and $G'(f)(\sum v_i h^i) = \sum f(v_i) h^i$. We have the following commutative diagram of functors

$$
\begin{array}{ccc}
A_{\mathfrak{g}}$-$\text{Mod}_{fr} & \xrightarrow{G} & U_{h}^{DJ}(\mathfrak{g})$-$\text{Mod}_{fr} \\
G' \downarrow & & \downarrow \text{classical limit} \\
\mathfrak{g}$-$\text{Mod} & \xrightarrow{\text{classical limit}} &
\end{array}
$$

(5)

where the down left arrow is the classical limit given by taking the limit as $h$ goes to zero. For any object $V$ and morphism $g$ of $\mathfrak{g}$-$\text{Mod}$ let us denote $G \circ G'(V)$ and $G \circ G'(g)$ by $\tilde{V}$ and $\tilde{g}$, respectively. Here the
functor $G \circ G'$ composed with the classical limit is the identity functor, i.e. $V \equiv \widetilde{V} \mod h$ and $g \equiv \widetilde{g} \mod h$.

In [3] the authors define an invariant of framed colored links. Let us now recall the basic construction and some properties of this invariant. Here we say that a link or more generally a tangle is colored if each of its components are assigned an object of $U^D_J(g)$-$\text{Mod}_{fr}$.

Let $F$ be the usual Reshetikhin-Turaev functor from the category of framed colored tangles to the category of $U^D_J(g)$-$\text{Mod}_{fr}$. In [3] a function from the set of typical $U^D_J(g)$-module to the ring $C[[h]][h^{-1}]$ is defined. As remarked in [3] this function can be multiplied by $h^{[-\Delta_t]}$ to obtain a function which takes values in $C[[h]]$. Let us denote this function by $d_h$.

**Lemma 4.1.** We have

$$d_h(\widetilde{V}(\lambda)) = h^{[\Delta_t]} \prod_{\alpha \in \Delta_+^\pm} \frac{q^{<\lambda+\rho,\alpha> - q^{-<\lambda+\rho,\alpha>}}}{q^{<\rho,\alpha> - q^{-<\rho,\alpha>}}} \prod_{\alpha \in \Delta_+^\pm} (q^{<\lambda+\rho,\alpha> - q^{-<\lambda+\rho,\alpha>}}).$$

In particular, $d(V(\lambda))$ is equal to $d_h(\widetilde{V}(\lambda)) \mod h$.

**Proof.** The proof follows from the formulas for $h^{[-\Delta_t]}d_h$ given in the Appendix of [3] and from the definition of $d$. □

Suppose $L$ is a framed colored link such that by cutting some component of $L$ one obtains a framed colored (1,1)-tangle $T_{V(\lambda)}$ such that the open string is colored by the deformed typical module $\widetilde{V}(\lambda)$ of highest weight $\lambda$. Then we have $F(T_{V(\lambda)}) = x. \text{Id}_{\widetilde{V}(\lambda)}$, for some $x$ in $C[[h]]$. Set $\langle T_{V(\lambda)} \rangle = x$. In [3] it is shown that the assignment

$$L \mapsto d_h(\widetilde{V}(\lambda)) \langle T_{V(\lambda)} \rangle$$

is a well defined colored framed link invariant denoted by $F'$. In particular, $F'(L)$ is independent of $V(\lambda)$, $T_{V(\lambda)}$ and where $L$ is cut.

An even morphism $f : V_1 \otimes ... \otimes V_n \rightarrow W_1 \otimes ... \otimes W_m$ in the category $U^D_J(g)$-$\text{Mod}_{fr}$ can be represented by the following box and arrows:

$$\begin{array}{c}
\text{W}_1 \\
\vdots \\
\text{W}_m
\end{array} \quad \begin{array}{c}
\text{V}_1 \\
\vdots \\
\text{V}_n
\end{array} \quad \begin{array}{c}
f \\
\text{f}
\end{array}$$

Such a box is called a coupon, which we denote by $C_{V_1,\ldots,V_n}^{W_1,\ldots,W_m}(f)$. Here we will say a ribbon graph is a framed tangle with coupons and colors
coming from the category $U^\text{DJ}_h(\mathfrak{g})$-$\text{Mod}_{fr}$. In \cite{4} it is shown that the construction of $F'$ can be extended to ribbon graphs having at least one component colored by a typical $U^\text{DJ}_h(\mathfrak{g})$-module.

The invariant $F'$ can also be extended to ribbon graphs having at least one component colored by a deformed module in $\mathcal{I}$ (see \cite{4}). We will now describe this extension in the following situation. Let $C$ ($C'$) be a $(1,1)$-tangle (resp. $(2,2)$-tangle) ribbon graph such that the input(s) and output(s) are equal. Let $L_C$ be the closed ribbon graph obtained from closing the coupon $C$. Let $T_{C'}$ be the $(1,1)$-tangle ribbon graph obtained from closing right most component. The ribbon graphs $L_C$ and $T_{C'}$ can be represented by the following pictures

$$L_C = \begin{array}{c}
\text{[Diagram]}
\end{array} \quad T_{C'} = \begin{array}{c}
\text{[Diagram]}
\end{array}$$

These pictures represent respectively the trace and the partial trace of the morphisms in the coupon.

Let $V \in \mathcal{I}$ and let $\alpha : V_0 \times W \to V$ and $\beta : V \to V_0 \otimes W$ be morphisms in $\mathfrak{g}$-$\text{Mod}$ such that $\alpha \circ \beta = \text{Id}_V$. Let $f \in \text{End}_{\mathfrak{g}}(V)^\pi$ and let $T(f; \alpha; \beta)$ be the $(1,1)$-tangle ribbon graph $T_{C_V^\beta \otimes \overline{W}(\beta) \circ C_V^\alpha(f) \circ C_V^\beta \otimes \overline{W}(\alpha)}$. That is

$$T(f; \alpha; \beta) = \begin{array}{c}
\text{[Diagram]}
\end{array}$$

Then we define

$$F'(L_{C_V^\beta(f)}) = d_h(\overline{V}_0) < T(f; \alpha; \beta) >.$$  \hfill (6)

In \cite{3,4} it is shown that $F'$ is well defined. Now we are ready to prove the main theorem of the paper.

\textbf{Proof of Theorem 1} Let $V_1$ be a typical $\mathfrak{g}$-module. Then we have $\mathcal{I} = \mathcal{I}_{V_0} = \mathcal{I}_{V_1}$. Choose $\alpha_i : V_i \times W_i \to V$ and $\beta_i : V \to V_i \otimes W$ such that $\alpha_i \circ \beta_i = \text{Id}_{V_i}$, for $i = 0, 1$. If $f \in \text{End}_{\mathfrak{g}}(V)^\pi$ then $< f; \alpha_0; \beta_0 > = < f; \alpha_1; \beta_1 > = 0$ as $\text{ptr}(\beta \circ f \circ \alpha) = < f; \alpha; \beta > \text{Id}_{V_0}$ and $\beta \circ f \circ \alpha$ is odd. Therefore, we can assume that $f \in \text{End}_{\mathfrak{g}}(V)^\pi$ (i.e. $f$ is a morphism in the symmetric monoidal category $\mathfrak{g}$-$\text{Mod}_{\pi}$ defined in the Appendix). We will show that

$$d(V_0) < f; \alpha_0; \beta_0 > = d(V_1) < f; \alpha_1; \beta_1 >.$$  \hfill (6)
By definition of the ribbon category $U_h^{DJ}(\mathfrak{g})$-Mod, we have $\langle f; \alpha_i; \beta_i \rangle$ is equal to $\langle T(f; \alpha_i; \beta_i) \rangle \mod h$, for $i = 0, 1$. Combining this with Lemma 4.1 we have that $d(V_i) < \langle f; \alpha_i; \beta_i \rangle$ is equal to $d_h(V_i) < \langle T(f; \alpha_i; \beta_i) \rangle \mod h$, for $i = 0, 1$. Finally, from [4] we have that the extension of $F'$ to ribbon graphs is well defined. In particular, we have $d_h(V_0) < \langle T(f; \alpha_0; \beta_0) \rangle = d_h(V_1) < \langle T(f; \alpha_1; \beta_1) \rangle$. Thus, Equation (6) holds and $\text{str}^t_v(f)$ only depends on $f$.

Now we prove the remaining statements of the theorem. The function $\text{str}^t_v$ is linear because $F(C\bar{V}'(a\bar{f} + b\bar{g})) = aF(C\bar{V}'(\bar{f})) + bF(C\bar{V}'(\bar{g}))$ for $f, g \in \text{End}_\mathfrak{g}(V)\pi$ and $a, b \in \mathbb{C}$. Number (3) follows from the property that $F'(L \sqcup L') = F'(L)F'(L')$ for any two links $L$ and $L'$ (see [4]). The proof of Number (4) follows from the behavior of $F'$ with respect to cabling (see [4]).

To prove Number (2) we need to be careful because coupons must be labeled by even morphisms, but the morphisms in the statement of (2) can be odd. If $V$ is an object of $\mathfrak{g}$-Mod then denote $V^-$ as the $\mathfrak{g}$-module obtained from $V$ by taking the opposite parity. Then $V$ and $V^-$ are isomorphic by an odd isomorphism $\sigma_V : V \to V^-$, which changes the parity.

**Lemma 4.2.** Let $\gamma \in \text{End}_{U_h^{DJ}(\mathfrak{g})}(\bar{W} \otimes \bar{V})\pi$ and set $\eta = (\text{Id} \otimes \bar{\sigma}_V)\gamma(\text{Id} \otimes \bar{\sigma}_V)$. Then

$$F\left(T_{C\bar{W} \otimes \bar{V}}(\gamma)\right) = -F\left(T_{C\bar{W} \otimes \bar{V}^-}(\eta)\right).$$

**Proof.** Let $\{w_i\}_{i=1}^q$ and $\{v_j\}_{j=1}^p$ be bases of the $\mathfrak{g}$-modules $V$ and $W$, respectively. Then $\{v_j\}_{j=1}^p$, $\{\sigma_V(v_j)\}_{j=1}^p$ and $\{w_i\}_{i=1}^q$ are bases for the $U_h^{DJ}(\mathfrak{g})$-modules $\bar{V}$, $\bar{V}^-$ and $\bar{W}$, respectively.

Let $\gamma_{ij}^{kl}$ be the elements of $\mathbb{C}[[h]]$ defined by

$$\gamma(w_i \otimes v_j) = \sum_{k=1}^q \sum_{l=1}^p \gamma_{ij}^{kl} w_k \otimes v_l.$$

A direct calculation shows:

$$F\left(T_{C\bar{W} \otimes \bar{V}}(\gamma)\right)(w_i) = \sum_{k=1}^q \sum_{l=1}^p (-1)^{\overline{ij} + \overline{kl}} \gamma_{ij}^{kl} w_k,$$ (7)

$$F\left(T_{C\bar{W} \otimes \bar{V}^-}(\eta)\right)(w_i) = \sum_{k=1}^q \sum_{l=1}^p (-1)^{\overline{ij} + \overline{kl}} (-1)^{\overline{ij} + \overline{kl}} \delta_{ij} \gamma_{ij}^{kl} w_k,$$ (8)

where $\delta_{ij}(-1)^{\overline{ij} + \overline{kl}} = (-1)^{\overline{ij} + \overline{kl}}$ and $\overline{w}_i = \overline{w}_k$ since $\eta$ is an even morphism. Therefore, the right sides of (7) and (8) are the negative of each other and the lemma follows. \qed
Lemma 4.3. For $V \in \mathcal{I}$ and $f \in \text{End}_g(V)_\mathcal{P}$ we have
\[ F'(L_{C_V^\psi}(\bar{f})) = -F'(L_{C_{V-}^\psi(\bar{\sigma}_0 \bar{f} \bar{g})}). \]

Proof. Let $\alpha \in \text{End}_g(V_0 \otimes W, W)_\mathcal{P}$ and $\beta \in \text{End}_g(V, V_0 \otimes W)_\mathcal{P}$ such that $\text{Id}_V = \alpha \circ \beta$. Then for $\alpha^- = (\text{Id}_{V_0} \otimes \sigma_W) \circ \alpha \circ \sigma_V \in \text{End}_g(V_0 \otimes W^-, V^-)_\mathcal{P}$ and $\beta^- = \sigma_V \circ \beta \circ (\text{Id}_{V_0} \otimes \sigma_W) \in \text{End}_g(V^-, V_0 \otimes W^-)_\mathcal{P}$, we have $\text{Id}_{V^-} = \alpha^- \circ \beta^-$. Now, we also denote $\bar{f}^- = \bar{\sigma}_V \circ \bar{f} \circ \bar{\sigma}_V^{-1} \in \text{End}_g(V^-)_\mathcal{P}$ and it is convenient to give the following pictorial proof.

\[ \begin{array}{ccc}
F'(f) & = & F'(\alpha) \\
& = & F'(\beta) \\
& = & F'(\bar{f})
\end{array} \]

where the fourth equality comes from Lemma 4.2.

Now we are ready to prove Number (2). Let $f : V \to V'$ and $g : V' \to V$ be morphisms of $g$-Mod such that $f \circ g$ is even. If $f$ and $g$ are both even then Number (2) follows from the fact that the closure of $C_{V'}^\psi(f) \circ C_V^\psi(g)$ is isotopic to closure of $C_{V'}^\psi(g) \circ C_V^\psi(f)$. If $f$ and $g$ are both odd then Number (2) follows from the following lemma.

Lemma 4.4. If $f$ and $g$ are both odd then
\[ F'(L_{C_V^\psi(f \circ g)}) = -F'(L_{C_{V'}^\psi(g \circ f)}). \]

Proof. From Lemma 4.3 we have
\[ F'(L_{C_V^\psi(f \circ g)}) = -F'(L_{C_{V-}^\psi(\bar{\sigma}_0 \bar{f} \bar{g} \bar{\sigma})}). \tag{9} \]

Now since $\bar{\sigma} \circ \bar{f}$ and $\bar{g} \circ \bar{\sigma}$ are even we have the right side of Equation (9) is equal to
\[ -F'(L_{C_{V-}^\psi(\bar{\sigma} \circ \bar{f} \circ \bar{g})}) = -F'(L_{C_{V'}^\psi(g \circ \sigma f \circ \sigma g)}). \]

\[ \begin{array}{ccc}
\alpha & \beta \\
f & \bar{f}^- \\
\bar{g} \bar{\sigma} & \bar{\sigma} \bar{g}
\end{array} \]
Thus we have proved the lemma. □

This finishes the proof of Number (2) and the theorem. □

APPENDIX

The theory of super-spaces follows the rule “whenever you permute two odd elements in an expression, put a − sign”. With this in mind, many concepts of linear algebra have super analogs. These analogs have new and different properties which are relevant to this paper. Let us discuss some of these differences.

In all the following, elements of super-spaces are generally assumed to be homogeneous and thus their parity is well defined. The definitions must be generalized by linearity for non homogeneous elements.

The category $SV$ of super-spaces. The category $SV$ of super-spaces is a category whose objects are super-spaces. The morphisms in $SV$ between two object $U$ and $V$ denoted by $\text{Hom}_C(U, V)$ is the super-space of linear maps with the parity given by:

\[
\begin{align*}
\text{Hom}_C(U, V)_\Pi &= \text{Hom}_C(U_\Pi, V_\Pi) \oplus \text{Hom}_C(U_\uppi, V_\uppi) \\
\text{Hom}_C(U, V)_\uppi &= \text{Hom}_C(U_\Pi, V_\pi) \oplus \text{Hom}_C(U_\pi, V_\uppi).
\end{align*}
\]

This category is “super-monoidal” with the super version of the operator $\otimes$ (let us denote $\otimes$ the usual tensor product in the category Vect).

For two objects $U$, $V$ of $SV$ their tensor product is the vector space $U \otimes V$ with the $\mathbb{Z}_2$-grading given by

\[
\begin{align*}
(U \otimes V)_\Pi &= U_\Pi \otimes V_\Pi \oplus U_\pi \otimes V_\uppi \\
(U \otimes V)_\uppi &= U_\Pi \otimes V_\pi \oplus U_\pi \otimes V_\uppi.
\end{align*}
\]

and for morphisms $f \in \text{Hom}_C(U, U')$ and $g \in \text{Hom}_C(V, V')$, $f \otimes g$ is given by

\[
f \otimes g = \begin{cases} 
& f \otimes g \text{ on } U_\Pi \otimes V_\Pi \\
& (-1)^q f \otimes g \text{ on } U_\pi \otimes V_\uppi
\end{cases}
\]

So that $(f \otimes g)(x \otimes y) = (-1)^q f(x) \otimes g(y)$. This tensor product realizes an isomorphism:

\[
\text{Hom}_C(U, U') \otimes \text{Hom}_C(V, V') \simeq \text{Hom}_C(U \otimes V, U' \otimes V').
\] (10)

Let $SV_{\Pi}$ be the subcategory of $SV$ with the same objects but only even morphisms (i.e. $\text{Hom}_{SV_{\Pi}}(U, V) = \text{Hom}_C(U, V)_\Pi$). The tensor product $\otimes$ restricted to $SV_{\Pi}$ is the usual bifunctor of Vect with an appropriate grading on objects. Moreover, $SV_{\Pi}$ is a symmetric monoidal category with symmetry isomorphisms $\tau_{U,V} : U \otimes V \simeq V \otimes U$ given by the super permutation $\tau_{U,V}(u \otimes v) = (-1)^{m_n} v \otimes u$. The category $SV$ is not a symmetric monoidal category because in general there
are morphisms $f$ and $g$ with the property that $(\text{Id} \otimes g) \circ (f \otimes \text{Id}) \neq (f \otimes \text{Id}) \circ (\text{Id} \otimes g)$.

For a super-space $U$, the ‘super-dual’ $U^*$ is defined to be the super-space $\text{Hom}_\mathbb{C}(U, \mathbb{C})$. The tensor product gives the following canonical isomorphism

$$U^* \otimes V^* = \text{Hom}_\mathbb{C}(U, \mathbb{C}) \otimes \text{Hom}_\mathbb{C}(V, \mathbb{C}) \simeq \text{Hom}_\mathbb{C}(U \otimes V, \mathbb{C} \otimes \mathbb{C}) = (U \otimes V)^*.$$  

If $f \in \text{Hom}_\mathbb{C}(U, V)$, the “super-transpose” of $f$ is the linear map $f^* \in \text{Hom}_\mathbb{C}(V^*, U^*)$ given by

$$f^*(\phi) = (-1)^{|\phi|} f \circ \phi$$

for $\phi \in V^*$. Then, if $f, g$ are composable morphisms of $SV$, we have

$$(f \circ g)^* = (-1)^{|g|} g^* \circ f^*.$$  

By convention the dual is a left dual:

- (left duality) $\text{ev}_V \in \text{Hom}_\mathbb{C}(V^* \otimes V, \mathbb{C})$ is simply the contraction

$$< \phi, x > = \phi(x).$$

- (right duality) $\text{ev}_V' \in \text{Hom}_\mathbb{C}(V \otimes V^*, \mathbb{C})$ is given by

$$< x, \phi > = (-1)^{|\phi|} \phi(x).$$

This defines a canonical isomorphism $V \rightarrow V^{**}$ when $V$ is finite dimensional. Again here, when restricted to $SV_\mathfrak{g}$ the * became a functor, i.e. the usual contravariant duality functor with some grading information.

The category $\mathfrak{g}$-Mod of $\mathfrak{g}$-modules. The universal enveloping algebra $U\mathfrak{g}$ of $\mathfrak{g}$ is a Hopf super-algebra, i.e. $U\mathfrak{g}$ is a Hopf algebra object in $SV_\mathfrak{g}$. Let $\mathfrak{g}$-Mod be the category where objects are finite dimensional super-space $V$ with a structure of $\mathfrak{g}$-module (i.e. $U\mathfrak{g}$-modules). The morphisms of $\mathfrak{g}$-Mod are the morphisms $f$ of $SV$ that are (“super”) $\mathfrak{g}$-linear:

$$\forall x \in \mathfrak{g}, \forall v \in V, f(x.v) = (-1)^{|\mathfrak{g}|} x.f(v).$$

The structure of Hopf super-algebra on $U\mathfrak{g}$ gives the tensor product of two $\mathfrak{g}$-modules a natural structure of $\mathfrak{g}$-modules and the tensor product of two $\mathfrak{g}$-linear morphisms is $\mathfrak{g}$-linear. Similarly, if $V$ is an object of $\mathfrak{g}$-Mod then the super-space $V^*$ is a $\mathfrak{g}$-module whose action is induced from the antipodal map of $U\mathfrak{g}$. $\text{Hom}_\mathfrak{g}(U, V)$ is canonically isomorphic to the super-space of invariant elements of $V \otimes U^*$ and so

$$\text{Hom}_\mathfrak{g}(U, V) \cong \text{Hom}_\mathfrak{g}(\mathbb{C}, V \otimes U^*)$$  

Let $\mathfrak{g}$-Mod$_{\mathfrak{g}}$ be the category whose objects are the objects of $\mathfrak{g}$-Mod and whose morphisms are morphisms of $SV_\mathfrak{g}$ which are $\mathfrak{g}$-linear. Then as above $\mathfrak{g}$-Mod$_{\mathfrak{g}}$ becomes a symmetric monoidal category with duality. Note that in general $\mathfrak{g}$-Mod is not such a category. This is the reason
we require that the morphisms $\alpha$ and $\beta$ in the definition of $\mathcal{I}$ (see Proposition 1.1) are in $\mathfrak{g} \text{-} \text{Mod}_{\Pi}$. In other words, the proof of Theorem 1 requires that we work in the category $\mathfrak{g} \text{-} \text{Mod}_{\Pi}$.

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