DEFORMATIONS OF PSEUDO-LAGRANGIAN SUBMANIFOLDS OF POISSON MANIFOLDS

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ABSTRACT. We consider Lagrangian-like submanifolds in certain even-dimensional ‘symplectic-like’ Poisson manifolds. We show that these have unobstructed deformations and that the deformations automatically preserve the Lagrangian-like property.

The study of holomorphic Lagrangian submanifolds of holomorphic symplectic manifolds and their deformation theory is well established (see e.g. [4] and references therein). Voisin [11] proved that pairs $(X, Y)$ consisting of a Kählerian symplectic manifold and a Lagrangian submanifold have unobstructed deformations, and under these deformations $Y$ stays Lagrangian. See also [6]. Recently, refining some results of Goto [3] and Hitchin [5], we studied in [8] certain even-dimensional Poisson manifolds called pseudo-symplectic, from the point of view that they are ‘like’ symplectic manifolds. A Poisson structure $\Pi \in H^0(X, \wedge^2 T_X)$ on a complex manifold $X$ of dimension $2n$ is said to be pseudo-symplectic if it is almost everywhere nondegenerate, hence degenerates along an anticanonical divisor $D = \left[ \Pi^n \right]$ called the Pfaffian of $\Pi$. We introduced a condition on $\Pi$ called normality, which is related to, and stronger than, normal crossing for $D$, but weaker than smoothness of $D$. Roughly speaking, the normality condition means that $(X, \Pi)$ is locally a product of Poisson manifolds of the form (smooth surface, smooth anticanonical divisor). We showed under this condition that $(X, \Pi)$ has, in a strong sense, unobstructed deformations.

Here we consider a analogue of Lagrangian submanifolds in the Poisson setting. An $n$-dimensional submanifold $Y$ in a $2n$-dimensional normal pseudo-symplectic manifold $(X, \Pi)$ is said to be pseudo-Lagrangian if

(i) $Y$ is transverse to the Pfaffian divisor $D$;
(ii) for all $y \in Y$, the conormal space $\tilde{N}_Y, y$ is an isotropic subspace for $\Pi$.

Note that condition (ii) implies that for all $y \in Y \setminus D$, i.e. all points $y \in Y$ where $\Pi$ is nondegenerate, $T_{Y,y}$ is a maximal isotropic subspace for the symplectic form $\Phi_y = \Pi_y^{-1}$. Condition (i) implies that the restriction $\tilde{D} = D \cap Y$ is a normal crossing divisor on $Y$.

Given $(X, \Pi)$ pseudo-symplectic Poisson and $Y \subset X$ pseudo-Lagrangian, one may consider deformations of the triple $(X, \Pi, Y)$ with or without the condition that $Y$ stay pseudo-Lagrangian.
In fact, we will prove that these deformation spaces are identical and smooth. Thus, our main purpose is to prove

**Theorem.** Let $Y$ be a pseudo-Lagrangian submanifold of a normal pseudo-symplectic Kählerian Poisson manifold $(X, \Pi)$. Then the triple $(X, \Pi, Y)$ has unobstructed deformations, and under these deformations $Y$ remains pseudo-Lagrangian.

This includes the special case where $Y$ is empty, which is the main result of [8], as well as the special case where $\Pi$ is symplectic (so $D$ is empty: Voisin’s theorem [11]). In fact, we will prove a more precise result (see Theorem 3 below).

From a different viewpoint, some results on deformations of submanifolds of Poisson manifolds were obtained by Baranovsky, Ginzburg et al. [1].

1. **The normal dg atom**

Let $Y$ be a pseudo-Lagrangian submanifold of a pseudo-symplectic manifold $(X, \Pi)$. Our purpose in this section is to describe a Lie-theoretic object (a dg Lie atom) which controls deformations of $Y$ in $X$ preserving the pseudo-Lagrangian property. Let $N$ denote the normal bundle of $Y$ and let $N = \bigoplus_{i=1}^{n} \wedge^i N$ be the exterior algebra on $N$. Our purpose in this section is to prove (compare [1]):

**Theorem 1.** Notations as above,

(i) $N$ admits the structure of differential graded Lie atom;

(ii) $N$ -deformations coincide with pseudo-Lagrangian deformations of $Y$ in $X$.

This result is not new. The existence of the differential on $N$ was certainly known to Baranovsky et al. [1], as was, in some form, the relationship of $N$ to pseudo-Lagrangian deformations.

**Proof of Theorem.** (i) Everything but the ‘differential’ assertion is more or less standard and valid independent of the Poisson structure. Thus, it is discussed at length in [2] and [10] that $N$ has the structure of Lie atom, deduced from viewing it as the mapping cone of the inclusion of Lie algebra sheaves

$$T_{X/Y} \to T_X$$

where $T_{X/Y}$ denotes the sheaf of vector fields on $X$ tangent to $Y$. This structure induces a graded Lie atom structure on $N$, deduced from the mapping cone of

$$T_{X/Y}T_X \to T_X$$

where $T_X$ is the Schouten graded Lie algebra and $T_{X/Y}T_X$ the exterior ideal generated by $T_{X/Y}$, which is easily seen to be a graded Lie subalgebra, though not a Lie ideal.

The Poisson structure $\Pi$ enters into the differential (on $T_X$, hence on $N$). To see that the differential of $T_X$, i.e. $[\cdot, \Pi]$, descends to $N$, suffices to show that the subalgebra $T_{X/Y}T_X$ is closed under $[\cdot, \Pi]$, and by elementary properties of the Schouten bracket it suffices to prove closedness
of $T_{X/Y}$. Indeed, let $v$ be a local vector field on $X$ tangent to $Y$ (i.e. preserving the ideal sheaf $I_Y$), and let $f_1, f_2$ be local functions in $I_Y$. Then by a standard formula of Lichnerowicz, we have

$$\langle df_1 \land df_2, [v, \Pi]\rangle = \pm v([f_1, f_2]) \pm \langle dv(f_1) \land df_2 - dv(f_2) \land df_1, \Pi\rangle.$$  

This vanishes on $Y$ by the Lagrangian condition, which shows that $[v, \Pi] \in T_{X/Y} T_X \subset \wedge^2 T_X$.

Assertion (ii) follows from the stronger result below.

We will denote the differential graded Lie algebra $T_{X/Y} T_X$ seen above by $T_X \{Y\}$. By a Poisson-Lagrange deformation of a triple $(X, \Pi, Y)$ as above we mean a triple $(\tilde{X}, \tilde{\Pi}, \tilde{Y})$ so that $(\tilde{X}, \tilde{\Pi})$ is a Poisson deformation of $(X, \Pi)$, $(\tilde{X}, \tilde{Y})$ is a deformation of $(X, Y)$, and $\tilde{Y}$ is pseudo-Lagrangian (isotropic) with respect to $\tilde{\Pi}$. Dropping the last condition leads to (plain) Poisson deformations of $(X, \Pi, Y)$.

**Theorem 2.** The deformation theory of $T_X \{Y\}$ coincides with the Poisson-Lagrange deformation theory of the triple $(X, \Pi, Y)$.

**Proof.** What’s being asserted is that given a local artinian algebra $R$, Poisson-Lagrange deformations of $(X, \Pi, Y)$ are in bijective correspondence with comultiplicative elements of the Jacobi-Bernoulli cohomology $\mathbb{H}^0(J(T_X \{Y\}, R)$. In proving this, we may assume the corresponding assertions for the dglas $T_{\tilde{X}}$ and $T_{X/Y}$ with $R$ coefficients, as well as for $T_X \{Y\}$ with coefficients in $R_1, \dim_{\mathbb{C}}(R_1) < \dim_{\mathbb{C}}(R)$, to be true.

Thus let $R_1 = R/(\eta)$ where $\eta$ is in the socle Ann$(m_R)$, and suppose given a deformation diagram

$$
\begin{array}{c}
\tilde{Y} \\
\downarrow
\end{array}
\to (\tilde{X}, \tilde{\Pi})

\begin{array}{c}
\Spec(R) \\
\uparrow
\end{array}
$$

(1)

so that $(\tilde{X}, \tilde{\Pi})$ is a Poisson deformation, $(\tilde{Y} \subset \tilde{X})$ is a flat deformation, and so that the pullback over $R_1$ is a Poisson-Lagrange deformation. The obstruction to $\tilde{Y}$ being pseudo-Lagrangian over $R$ is the Poisson bracket

$$\{\cdot, \cdot\} : \mathcal{I}_Y \times \mathcal{I}_Y \to \mathcal{O}_Y$$

and by our assumption that everything is ok over $R_1$ this factors through a pairing

$$\mathcal{I}_Y \times \mathcal{I}_Y \to \mathcal{O}_Y.$$  

(2)

Note that this obstruction is of a local nature, so in analyzing it we may choose compatible local coordinates on $X$ and $Y$ and assume that the deformation $\tilde{X}$ is trivial, i.e. $X \times \Spec(R)$, as is $\tilde{Y}$ abstractly. Then the deformation $\tilde{Y} \to \tilde{X}$ corresponds to a map

$$v : \mathcal{I}_Y \to m_R \otimes \mathcal{O}_Y, v \in H^0(N) \otimes m_R$$

$$\mathcal{I}_Y = \{f + v(f) : f \in \mathcal{I}_Y\}.$$  

Then in these terms the obstruction (2) is given by

$$(f_1, f_2) \mapsto \{v(f_1), f_2\} - \{v(f_2), f_1\} - v([f_1, f_2]).$$
(by our assumptions this is in \( \eta \mathcal{O}_Y \subset \mathfrak{m}_R \mathcal{O}_Y \)). On the other hand, in terms of the Poisson differential \([\cdot, \Pi]\), this is exactly \( \langle \nu, \Pi, df_1 \wedge df_2 \rangle \), QED.

\( \square \)

2. Unobstructed deformations

We will keep the notations of the previous section. Thus, \((X, \Pi)\) is a pseudo-symplectic Kählerian Poisson manifold with (normal-crossing) Pfaffian divisor \(D\), and \(Y\) is a pseudo-Lagrangian submanifold. We denote by \(\text{Def}_{\text{loc. trivial}}(X, D, Y)\) the space of deformation of the triple \((X, D, Y)\) where \(D\) deforms locally trivially. This space corresponds to the dgla \(T_X(Y)(\log D)\).

**Theorem 3.** (i) The triple \((X, \Pi, Y)\) has unobstructed deformations and these deformations are Poisson-Lagrange and induce locally trivial deformations on \(D\).

(ii) The deformation space \(\text{Def}_{\text{loc. trivial}}(X, D, Y)\) is unobstructed.

(iii) There is a deformation space of quadruples \((X, \Pi, D, Y)\) that maps smoothly to \(\text{Def}(X, \Pi, Y)\) and to \(\text{Def}_{\text{loc. trivial}}(X, D, Y)\).

As in [8], we deduce

**Corollary 4.** Given a deformation \((\tilde{X}, \tilde{Y})\) of \((X, Y)\), the Poisson structure \(\Pi\) extends to \((\tilde{X}, \tilde{Y})\) iff \(D\) extends locally trivially to \((\tilde{X}, \tilde{Y})\).

**Proof of Theorem.** Let \(\tilde{D}\) be the restriction of the Pfaffian divisor \(D\) on \(Y\). By our hypotheses, both \(D\) and \(\tilde{D}\) have normal crossings. Henceforth, we will denote by \(\Omega\) various de Rham complexes in strictly positive degrees (i.e. omitting the zeroth term \(\Omega^0 = O\)). Denote by \(\Omega_X^i(Y)\) the kernel of the pullback map \(\Omega_X^1 \rightarrow \Omega_Y^1\). Then it is not hard to check that \(\Omega_X^i(Y)\) has the structure of a differential graded Lie algebra so that the inclusion into the Lie-Poisson algebra \(\Omega_X^i\) is a Lie subalgebra. This turns the cokernel \(\Omega_Y^i\) into a differential graded Lie atom. Likewise, for the log differentials \(\Omega_X^i(Y)(\log D)\), a subalgebra of \(\Omega_X^i(\log D)\) with cokernel atom \(\Omega_Y^i(\log \tilde{D})\). Now recall the homomorphism \(\wedge^i \Pi^\#\) already used in [8]. It yields a map of short exact sequences

\[
\begin{align*}
0 & \rightarrow \Omega_X^i(Y)(\log D) \rightarrow \Omega_X^i(\log D) \rightarrow \Omega_Y^i(\log D) \rightarrow 0 \\
0 & \rightarrow T_X^i(Y)(\log D) \rightarrow T_X^i(\log D) \rightarrow N \rightarrow 0
\end{align*}
\]

The first two vertical maps are dgla homomorphisms, hence the right vertical arrow is a Lie atom homomorphism. In any event, a local computation in [8] shows that the middle vertical arrow is bijective, and the same computation also shows that the left vertical arrow is bijective.

Now we can argue as in [8]: Deligne’s \(E_1\) degeneration theorem implies \(E_1\)-degeneration for \(\Omega_X^i(\log D)\) and \(\Omega_Y^i(\log \tilde{D})\), hence for \(\Omega_X^i(Y)(\log D)\). Consequently, the bracket pairing induces the trivial pairing on cohomology for the algebra \(T_X^i(Y)(\log D)\), hence this algebra has unobstructed deformations. Then we see as in [8] that the inclusion \(T_X^i(Y)(\log D) \rightarrow T_X^i(Y)\) is a direct summand projection, so that \((X, \Pi, Y)\) has unobstructed Poisson-Lagrange deformations. Finally, the fact that the map induced by the differential \(H^0(N) \rightarrow H^0(\wedge^2 N)\) is trivial shows that Poisson-Lagrange deformations coincide with Poisson deformations.
Finally, assertions (ii) and (iii) follow as in [8] from, respectively, the vanishing of the bracket-induced map on $H(\Omega_X^1(Y)(\log D))$, and from Deligne’s result on $E_1$ degeneration for $\Omega_X^1(Y)(\log D)$ [2], which implies surjectivity on cohomology of the edge map

$$\Omega_X^1(Y)(\log D) \to \Omega_X^1(Y)(\log D)$$

\[ \square \]

Remark 5. Christian Lehn [7] has generalized the Voisin theorem to normal-crossing subvarieties $Y$. The analogous statement in the Poisson setting is open.

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