Diffeomorphism group and conformal fields

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Abstract

Conformal fields are a new class of $\text{Vect}(N)$ modules which are more general than tensor fields. The corresponding diffeomorphism group action is constructed. Conformal fields are thus invariantly defined.

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1. Any meaningful object living in \( N \)-dimensional space must be invariantly defined, i.e. it must transform as a representation of \( Diff(N) \), the diffeomorphism group in \( N \) dimensions. \( Diff(N) \) representation theory therefore amounts to a classification of inequivalent meaningful objects, which appears to be of importance both in physics and geometry. In particular, the main concepts of differential geometry (tensor fields, connections, exterior derivatives) have a natural description in this language. This point of view has mainly been advocated by Russian mathematicians [1] – [5].

We have recently [6] [7] discovered a new class of modules of \( Vect(N) \), the algebra of vector fields in \( N \) dimensions, which is the Lie algebra of \( Diff(N) \). These modules, which were named conformal fields, are in a sense more natural than tensor fields. However, \( Vect(N) \) is not very interesting in its own right, but only as the infinitesimal version of \( Diff(N) \). Therefore, it is important to check that conformal fields can be exponented to \( Diff(N) \) representations. This is the subject of this letter.

It is known [1] – [5] that tensor fields can be described as those \( Diff(N) \) modules which are induced from the group \( GL(N) \) consisting of the linear transformations

\[
x^\mu \mapsto a^\mu_\nu x^\nu. \tag{1}
\]

However, the largest finite-dimensional subgroup of \( Diff(N) \) is the group of projective transformations,

\[
x^\mu \mapsto \frac{a^\mu_\nu x^\nu + a^\mu_0}{a_0^0 x^\nu + a_0^0}. \tag{2}
\]

This group is isomorphic to \( SL(N + 1) \) because two consecutive projective transformations correspond to multiplication of the matrices

\[
a^A_B \equiv \begin{pmatrix} a_0^0 & a_0^\nu \\ a^\mu_0 & a^\mu_\nu \end{pmatrix}. \tag{3}
\]

Since this matrix is only determined by (2) up to an over-all factor, we may put \( \det a = 1 \).

We thus have the following inclusions of groups

\[ GL(N) \subset SL(N + 1) \subset Diff(N). \tag{4} \]

A conformal field is a \( Diff(N) \) module which is induced from the projective group. Because of a theorem on induction in stages [9], this definition includes tensor fields: induce from an \( SL(N+1) \) module which in turn has been induced \( GL(N) \). However, if we start from an \( SL(N + 1) \) module which is not induced from \( GL(N) \), a new kind of invariantly defined object arises.

\(^1\)A better name would be “projective field”. The prefix “conformal” was chosen because we did not recognize the Lie algebra of the projective group, which is somewhat similar to the conformal algebra.
The above description of conformal fields is not very concrete. On the Lie algebra level, we can be much more explicit. \( \text{Vect}(N) \) reads
\[
[L_f, L_g] = L_{[f,g]},
\]
where \( L_f \) is the Lie derivative corresponding to the vector field \( f = f^\mu \partial_\mu \) (\( \partial_\mu \equiv \partial/\partial x^\mu \)). Tensor fields are defined by
\[
L_f = f^\mu \partial_\mu + \partial_\mu f^\nu T^\mu_\nu
\]
where \( T^\mu_\nu \) are the generators of \( \mathfrak{gl}(N) \), i.e.
\[
[T^\mu_\nu, T^\sigma_\tau] = \delta^\sigma_\tau T^\mu_\nu - \delta^\mu_\tau T^\sigma_\nu.
\]
By substituting different representations of \( \mathfrak{gl}(N) \) into (6), we obtain all kinds of tensor fields.

It should be noted that upper and lower indices have been switched, compared to our previous papers. This change in notation was performed in order to agree with other literature.

To describe conformal fields, it is useful to introduce the following notation. Let \( A = (0, \mu) \) be an \( N + 1 \)-component index, and let
\[
\begin{align*}
  f^A & = (0, f^\mu), & x^A & = (t, x^\mu), \\
  \partial_B & = (-t^{-1} x^\sigma \partial_\sigma, \partial_\nu), & k_B & = (t^{-1}, 0),
\end{align*}
\]
where \( t \) is a parameter. From (8) we deduce the following useful formulas.
\[
\begin{align*}
  [\partial_A, \partial_B] & = k_A \partial_B - k_B \partial_A, & [\partial_B, x^A] & = \delta^A_B - k_B x^A, \\
  x^A \partial_A & = k_A f^A = 0, & k_A x^A & = 1.
\end{align*}
\]
Now, the following expression satisfies \( \text{Vect}(N) \) [3].
\[
L_f = f^A \partial_A + \{ (\partial_A + k_A) f^B + cx^B \partial_A \partial_C f^C \} T^A_B,
\]
where \( c \) is another parameter and \( T^A_B \) are the generators of \( \mathfrak{sl}(N + 1) \), i.e.
\[
[T^A_B, T^C_D] = \delta^C_D T^A_B - \delta^A_B T^C_D, \quad T^A_A = 0.
\]

2. We now present the analogs of (3) and (10) at the group level. Under the diffeomorphism \( x \to y(x) \), a \( Diff(N) \) representation \( \phi(x) \) transforms as \( \phi(x) \to \phi'(x) \). We say that this representation is field-like, if
\[
\phi'(y) = U(y, x) \phi(x),
\]
where \( \phi \) is a vector and \( U \) is a matrix in some finite-dimensional vector space. Under two consecutive diffeomorphisms, \( x \to y \to z \), the field transforms as \( \phi \to \phi' \to \phi'' \), where
\[
\phi''(z) = U(z, y) \phi'(y) = U(z, y) U(y, x) \phi(x).
\]
Comparing with (12), we see that the matrices $U(y, x)$ must fulfil
\[
U(z, y)U(y, x) = U(z, x), \quad U(x, x) = 1.
\] (14)

Clearly, the dual field, which transforms as
\[
\phi'(y) = \phi(x)U(x, y),
\] (15)

is also a $Diff(N)$ representation. Tensor products of (12) and (15) can also be considered.

The classical field-like objects are tensor fields,
\[
U^\mu_\nu(y, x) = \frac{\partial y^\mu}{\partial x^\nu},
\] (16)

because
\[
\frac{\partial z^\mu}{\partial y^\nu} \frac{\partial y^\nu}{\partial x^\sigma} = \frac{\partial z^\mu}{\partial x^\sigma}, \quad \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu.
\] (17)

Eq. (14) also admits a one-dimensional representation, which corresponds to densities with weight $\lambda$. Let
\[
\left| \frac{\partial y}{\partial x} \right| = \text{det} \left( \frac{\partial y}{\partial x} \right)
\] (18)

be the Jacobian of the mapping $x \to y$. Then
\[
U(y, x) = \left| \frac{\partial y}{\partial x} \right|^{-\lambda}
\] (19)

satisfies the representation conditions (14).

Our main result is the following field-like representation of $Diff(N)$.

**Theorem 1**: Let $c$ be a parameter, and let $x^A$, $\partial/\partial x^B$ and $k_B$ be defined as in (8). Then the following $(N + 1) \times (N + 1)$-dimensional matrices satisfy (14).
\[
U^A_B(y, x) = \left\{ \frac{\partial}{\partial x^B} + k_B + c \frac{\partial}{\partial x^B} h(y, x) \right\} y^A,
\] (20)

where
\[
h(y, x) = \log \left| \frac{\partial y}{\partial x} \right| = \text{tr} \log \left( \frac{\partial y}{\partial x} \right).
\] (21)

More explicitly, the matrix reads
\[
U^A_B(y, x) = \begin{pmatrix}
U^0_0 & U^0_\nu \\
U_0^\mu & U_\nu^\nu
\end{pmatrix} = \left( t^{-1} \left( y^\mu - x^\sigma \partial_\sigma y^\mu - cy^{\mu 0} \partial_0 h \right) \partial_\nu y^\mu + cy^{\mu 0} \partial_\nu h \right).
\] (22)

Note that $|\partial y/\partial x|$ is the determinant of the $N \times N$-dimensional matrix $\partial y^\mu/\partial x^\nu$; because $y^0 \equiv t$, the determinant of $\partial y^A/\partial x^B$ vanishes. The corresponding fields
\( \phi^A(x), \phi^B(x), \) and tensor products thereof, are called \textit{conformal fields}. Of course, we can also consider tensor products of conformal fields with tensor fields or densities.

\textbf{Proof of Theorem 1}: We begin by noting that

\[
\frac{\partial}{\partial x^B} = \frac{\partial y^C}{\partial x^B} \frac{\partial}{\partial y^C}.
\]

Namely, the LHS equals

\[
(- \frac{x^\mu}{t}, 1) \frac{\partial}{\partial x^\mu} = (- \frac{x^\mu}{t}, 1) \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial}{\partial y^\nu},
\]

whereas the RHS is

\[
(- \frac{y^\nu}{t}, 1) \left( \begin{array}{cc}
0 & 0 \\
\frac{x^\mu y^\nu}{t} & \frac{\partial y^\nu}{\partial x^\mu}
\end{array} \right) \frac{\partial}{\partial y^\nu}.
\]

The theorem asserts that

\[
U^A_B(z, x) = \frac{\partial z^A}{\partial x^B} + k_B z^A + c \frac{\partial}{\partial x^B} h(z, x) z^A
\]

and

\[
U^A_C(z, y) U^C_B(y, x)
= \left( \frac{\partial}{\partial y^C} + k_C + c \frac{\partial}{\partial y^C} h(z, y) \right) z^A \left( \frac{\partial}{\partial x^B} + k_B + c \frac{\partial}{\partial x^B} h(y, x) \right) y^C
\]

\[
= \frac{\partial y^C}{\partial x^B} \frac{\partial z^A}{\partial y^C} + k_B z^A + c \frac{\partial y^C}{\partial x^B} \frac{\partial}{\partial y^C} h(z, y) z^A + z^A \frac{\partial}{\partial x^B} h(y, x)
\]

are equal. Using

\[
k_C \frac{\partial y^C}{\partial x^B} = y^C \frac{\partial}{\partial y^C} = 0, \quad k_C y^C = 1,
\]

and (23) we see that the terms independent of \( c \) agree. The terms linear in \( c \) yield

\[
\frac{\partial}{\partial x^B} h(z, x) = \frac{\partial}{\partial x^B} h(z, y) + \frac{\partial}{\partial x^B} h(y, x),
\]

i.e.

\[
h(z, x) = h(z, y) + h(y, x).
\]

That this functional equation is solved by

\[
h(y, x) = \log \left| \frac{\partial y}{\partial x} \right|
\]

follows from

\[
\frac{\partial z}{\partial y} \left| \frac{\partial y}{\partial x} \right| = \left| \frac{\partial z}{\partial y} \right|.
\]
Finally, we note that $U^A_B(x, x) = \delta^A_B$, which completes the proof. \hfill \blacksquare

Because the matrix (20) satisfies (14), so does its determinant. However, this does not give us any new representation, because $\det U$ equals the Jacobian (18). To see this, we note that (20) can be written as

$$U^A_B = \left( \frac{\partial}{\partial x^B} + k_B \right) y^C \left( \delta^A_C + c M^A_C \right), \quad (33)$$

where

$$M^A_B = y^A \left( \frac{\partial}{\partial y^B} + k_B \right) x^C \frac{\partial h}{\partial x^C}. \quad (34)$$

Now,

$$\log \det(1 + cM) = \text{tr} \log(1 + cM) = \sum_{n=1}^{\infty} \frac{(-c)^n}{n} \text{tr} M^n = 0, \quad (35)$$

because

$$\text{tr} M^n \propto y^A \frac{\partial}{\partial y^A} = 0. \quad (36)$$

We thus conclude that $\det U$ is independent of $c$. When $c = 0$,

$$U^A_B = \begin{pmatrix} 1 & 0 \\ \vdots & \partial y^\mu \partial x^\nu \end{pmatrix} \quad (37)$$

has a block triangular form, and its determinant is hence

$$\det U = \det \frac{\partial y}{\partial x} = \left| \frac{\partial y}{\partial x} \right|, \quad (38)$$

as claimed.

3. Eq. (10) is the infinitesimal form of a conformal field. Consider the infinitesimal diffeomorphism

$$y^A = x^A + f^A. \quad (39)$$

To first order in $f$,

$$\partial_B y^A \approx \delta^A_B - k_B x^A + \partial_B f^A, \quad (40)$$

$$h(y, x) \approx \text{tr} \log \left( 1 + \frac{\partial f}{\partial x} \right) \approx \text{tr} \left( \frac{\partial f}{\partial x} \right) = \partial_\sigma f^\sigma, \quad (41)$$

and

$$U^A_B(y, x) \approx \delta^A_B + (\partial_B + k_B) f^A + cx^A \partial_B \partial_\sigma f^\sigma. \quad (42)$$

Thus,

$$\phi^A(y) \approx \phi^A(x) - f^\mu \partial_\mu \phi^A(x) + U^A_B(y, x) \phi^A(x) \quad (43)$$

from which we deduce that

$$L_f \phi^A(x) = f^\mu \partial_\mu \phi^A(x) - \left[ (\partial_B + k_B) f^A + cx^A \partial_B \partial_\sigma f^\sigma \right] \phi^B(x). \quad (44)$$
This is recognized as (10) in the vector case, with
\[ T^A_B \phi^C = -\delta^C_B \phi^A. \] (45)

4. As any representation of \( Diff(N) \), a conformal field yields a representation of every subgroup by restriction. The group of projective transformations (2) consists of the diffeomorphisms of the form
\[ y^A = \Delta^{-1} a_B^A x^B, \quad \Delta = k_A a_B^A x^B, \] (46)
where we have put \( t = 1 \). To calculate the form of the matrix (20), we need
\[ \partial_B y^A = \Delta^{-1} (a^A_B - \Delta^{-1} a_C^D x^D k_D a_B^C). \] (47)

In particular,
\[ \partial_\nu y^\mu = \Delta^{-1} \delta_\nu^\mu (\delta_\sigma^\nu - M_\sigma^\nu), \] (48)
where
\[ M_\mu^\nu = \Delta^{-1} a_0^\nu (x^\mu + (a^{-1})_\tau^\mu a_0^\tau). \] (49)
\((a^{-1}) \) is the inverse of \((a^\mu_\nu)\). We note that
\[ \partial_\nu \Delta = a_0^\nu \] (50)
and
\[ \partial_\nu M_\mu^\nu = \Delta^{-1} a_0^\mu (\delta_\sigma^\nu - M_\sigma^\nu). \] (51)

Hence,
\[ h(y, x) = \log \det \left| \frac{\partial y}{\partial x} \right| = -N \log \Delta + \log \det a + \text{tr} \log(1 - M), \] (52)
and
\[ \partial_\nu h(y, x) = -N \Delta^{-1} a_0^\nu + \text{tr} \partial_\nu \log(1 - M) = -(N + 1) \Delta^{-1} a_0^\nu. \] (53)

Collecting all terms, we find that
\[ U_A^B(y, x) = \Delta^{-1} \left\{ a_B^A + (1 + c(N + 1)) \Delta^{-1} a_D^A x^D (k_B - \Delta^{-1} k_C a_B^C) \right\}. \] (54)

Actually, (54) is the product of two factors, each of which satisfies (14). In an obvious matrix notation, two consecutive projective transformations take the form
\[ y = \frac{ax}{k^T ax}, \quad z = \frac{by}{k^T by}. \] (55)

The proof that
\[ \Delta^{-1}(z, y) \Delta^{-1}(y, x) = \Delta^{-1}(z, x), \] (56)
reads
\[ (k^T by)(k^T ax) = k^T \frac{ax}{k^T ax} (k^T ax) = k^T ba x. \] (57)
That the other factor in (54) also satisfies (14) can be verified in a similar fashion. In particular, we have the following infinite-dimensional $SL(N + 1)$ representation for $c = -1/(N + 1)$.

$$\phi'^A(y) = a^A_B \phi^B(x),$$

(58)

where $y$ is related to $x$ by (55). We believe that the corresponding representations for $c \neq -1/(N + 1)$ are new.

5. In this letter we have only dealt with the local properties of the diffeomorphism group. The name $Diff(N)$ then makes sense, because any manifold is locally diffeomorphic to a subset of $\mathbb{R}^N$. It would be interesting to determine for which $N$-dimensional manifolds $X$ the group action be extended to $Diff X$, the diffeomorphism group on $X$. It is clear that the construction goes through for the group $Diff_{Loc} X$ consisting of maps with local support, reducing to the identity map outside some fixed region. Evidently,

$$Diff_{Loc} X \subset Diff X,$$

(59)

so we may consider the $Diff X$ representation induced from a $Diff_{Loc} X$ conformal field. However, it is not clear to us if any subtleties arise since both groups in (59) are infinite dimensional. Indeed, (20) is not well defined on the torus, because it explicitly involves the coordinates and not just their derivatives.

In [7] we constructed first-order differential operators (exterior derivatives), which commutes with the action of $Vect(N)$. Preliminary calculations show that the corresponding result holds on the group level.

A point worth noting is that a conformal field depends on two parameters: $c$, which appears explicitly in (20), and $t$ which enters through (8). We may thus denote it by $\phi(x, t; c)$. As this notation suggests, the parameter $t$ in many ways behave as an extra dimension (“time”), in addition to the $N$ (“space”) dimensions labelled by $x$. Moreover, conformal fields are useful in the construction of $Vect(N)$ lowest-weight modules, which may be relevant to quantum gravity [8].

We conclude by observing that invariantly defined field-like objects (12) are scarce; previously, only tensor fields were known (spinor fields are more complicated, because their definition requires additional structure in the form of a metric or a vielbein). Presumably, no further field-like objects exist, because $SL(N + 1)$ is the largest finite-dimensional subgroup from which a $Diff(N)$ module can be induced. Therefore, we believe that conformal fields have an important role to play in physics and geometry.
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