A Simpler Characterization of Sheffer Polynomials

A. Di Bucchianico∗
Department of Mathematics
University of Groningen
P. O. Box 800
9700 AV Groningen, Netherlands
A.Di.Bucchianico@math.rug.nl

D. Loeb†
LABRI
Université de Bordeaux I
33405 Talence, France
loeb@geocub.greco-prog.fr

Abstract

We characterize the Sheffer sequences by a single convolution identity

\[ F^{(y)} p_n(x) = \sum_{k=0}^{n} p_k(x) p_{n-k}(y) \]

where \( F^{(y)} \) is a shift-invariant operator. We then study a generalization of the notion of Sheffer sequences by removing the requirement that \( F^{(y)} \) be shift-invariant. All these solutions can then be interpreted as cocommutative coalgebras. We also show the connection with generalized translation operators as introduced by Delsarte. Finally, we apply the same convolution to symmetric functions where we find that the “Sheffer” sequences differ from ordinary full divided power sequences by only a constant factor.

1 Introduction

The basis of the Umbral Calculus (see [13] and [17]) is the convolution identity

\[ E^y p_n(x) = \sum_{k=0}^{n} p_k(x) p_{n-k}(y) \]  \hspace{1cm} (1)

where the shift operator \( E^y : K[x] \rightarrow K[x, y] \) is defined by \( E^y p(x) = p(x + y) \). A sequence of polynomials is a sequence \( (p_n(x))_{n=0}^{\infty} \) of polynomials such that \( \text{deg}(p_n(x)) = n \). It is said to be a divided powers sequence.

∗Author supported by NWO (Netherlands Organization for Scientific Research).
†Author supported by URA CNRS 1304.
if it obeys equation 1. The Umbral Calculus is the study of such sequences and their sister sequences of binomial type \((q_n(x))_{n=0}^{\infty}\) with \(q_n(x) = n! \ p_n(x)\) so called since they obey the “binomial” identity

\[ E^y q_n(x) = \sum_{k=0}^{n} \binom{n}{k} \ q_k(x) \ q_{n-k}(y). \]

Famous examples of sequences of binomial type include: the powers of \(x\), the lower factorials \(x(x-1) \cdots (x-n+1)\), the rising factorials \(x(x+1) \cdots (x+n-1)\), and the Abel polynomials \(x(x-na)^{n-1}\).

A related concept is that of the Sheffer sequences. A Sheffer sequence of polynomials \((s_n(x))_{n=0}^{\infty}\) has been traditionally defined algebraically by the identity

\[ E^y s_n(x) = \sum_{k=0}^{n} p_{n-k}(y) \ s_k(x) \]

where \((p_n(x))_{n=0}^{\infty}\) is itself a divided power sequence of polynomials. For example, the Bernoulli polynomials are Sheffer with respect to \((x^n/n!)_{n=0}^{\infty}\). Thus, \textit{a priori}, the Sheffer sequence is a less basic concept than that of divided power sequences—as far as convolution identities is concerned.

At the Franco-Québecois Workshop in May 1991, we asked what sort of “shiftless” Umbral Calculus would arise if the operator \(E^y\) was replaced by some other shift-invariant operator \(F^y : K[x] \rightarrow K[x, y]\).

\[ F^y p_n(x) = \sum_{k=0}^{n} p_k(x) \ p_{n-k}(y) \]

(We write \(F^y\) so as not to imply that \(F^y F^z\) is necessarily equivalent to \(F^{y+z}\)). In section \textsection 2 we will show that only Sheffer sequences obey equation \textsection 2. Thus, Sheffer sequences are a much more natural subject of study than is the special case of divided power sequences. We also show some connections with the theory of generalized translation operators and Cauchy problems as presented in \textsection 3.

In section \textsection 3, we seek parallel results for divided power sequences of symmetric functions. Surprisingly, up to a constant, the only “Sheffer” sequences of symmetric functions are the divided power sequences themselves.

We end sections \textsection 2 and \textsection 3 with applications to coalgebra theory. These results may be safely skipped by any non-specialist. Solutions to equation \textsection 2 are interpreted as cocommutative coalgebras, and classified according to their coalgebraic properties.

2 Polynomials

2.1 Notation

Let \(K\) be a field of characteristic zero, and let \(x, y\) be indeterminates. Consider a \(K\)-linear map \(\phi : K[x] \rightarrow K[x]\). Then \(\phi\) has a unique \(K[y]\)-linear extension \(\phi'\) to \(K[x, y]\). By an abuse of notation, we will denote \(\phi\)
and \( \phi' \) by the same symbol \( \phi \). Note that if \( \phi \) is a linear map on \( K[x] \) and \( \theta \) is a linear map on \( K[y] \), then \( \phi \) and \( \theta \) commute when considered as maps on \( K[x, y] \). Nevertheless, \( \phi \) and \( \theta \) do not necessarily commute with maps \( \psi : K[x] \to K[x, y] \) such as the shift operator.

Now, consider the natural isomorphism \( \pi : K[x] \to K[y] \). There is a unique map which we will denote \( T^y \phi \) such that \( T^y \phi \circ \pi = \pi \circ \phi \). Often a linear map \( K[x] \to K[x] \) will be denoted \( \phi_x \) with \( x \) as a subscript. In that case, \( T^y \phi_x \) will be denoted \( \phi_y \). For example, if \( D_x \) is the derivative with respect to \( x \), then \( D_y \) is the derivative with respect to \( y \). Similarly, if \( \epsilon_x \) is the evaluation map at \( x = 0 \), \( \epsilon_x p(x) = p(0) \), then \( \epsilon_y \) is the evaluation map at \( y = 0 \). Essentially, \( \phi_y \) behaves with respect to \( y \) in the same way as \( \phi_x \) does with respect to \( x \).

For \( n \) a nonnegative integer, let \( p_n(x) \) be a polynomial of degree \( n \) with coefficients in \( K \) and let \( F(y) \) be a \( K \)-linear operator \( K[x] \to K[x, y] \) which we propose as a possible solution to equation 3.

Note that we must assume that \( F(y) \) is linear in order to characterize such operators satisfying equation 3 since equation 3 specifies the value of \( F(y) \) only on a basis of \( K[x] \).

Finally, we note that in the above notation there are really two kinds of “shifts” \( E^y : p(x) \mapsto p(x+y) \). If \( y \) is taken as a constant, then \( E^y : K[x] \to K[x] \). Whereas, if \( y \) is taken as a variable, then \( E^y : K[x] \to K[x, y] \). However, commutation with one shift guarantees commutation with the other as the next lemma shows.

**Lemma 2.1** Let \( \theta \) be a linear operator on \( K[x] \) (and thus on \( K[x, y] \)). Then \( \theta E^c = E^c \theta \) for all \( c \in K \) if and only if \( \theta E^y = E^y \theta \).

**Proof:** (If) Trivial, evaluate at \( y = c \).

(Only if) For any polynomial \( p(x) \), the expressions \( E^y \theta p(x) \) and \( \theta E^y p(x) \) agree infinitely often when thought of as polynomials taking values in \( K[x] \). Thus, they must be the same polynomial, and \( \theta E^y = E^y \theta \). \( \square \)

### 2.2 Sheffer Theorem

In this section, we make the additional assumption that \( F(y) \) is shift-invariant for all \( y \).

**Theorem 2.1 (Sheffer Theorem)** Given \( F(y) \) and \( (p_n(x))_{n=0}^\infty \) as above, the following two statements are equivalent.

1. \( F(y) \) and \( (p_n(x))_{n=0}^\infty \) obey equation 3.
2. \( P_x = \epsilon_y \circ F(y) \) is an invertible shift-invariant operator on \( K[x] \), \( (p_n(x))_{n=0}^\infty \) is Sheffer relative to the divided power sequence \( (P_x^{-1} p_n(x))_{n=0}^\infty \), and \( F(y) = P_y E^y \).
2.2 Sheffer Theorem

Proof: (2 implies 1): Define \( q_n(x) = P_x^{-1} p_n(x) \) or equivalently \( q_n(y) = P_y^{-1} p_n(y) \). By hypothesis, \((q_n(x))_{n=0}^{\infty}\) is a divided power sequence. That is to say,

\[
E^y q_n(x) = \sum_{k=0}^{n} q_k(x) q_{n-k}(y).
\]

Now, operate on both sides of the identity with \( P_x P_y \). Keeping in mind that \( P_x E^y = E^y P_x \), we then get

\[
F(y) p_n(x) = P_y E^y P_x q_n(x) = \sum_{k=0}^{n} (P_x q_k(x))(P_y q_{n-k}(y)) = \sum_{k=0}^{n} p_k(x) p_{n-k}(y).
\]

(1 implies 2): By hypothesis, \( P_x \) is shift-invariant, but we must now show that \( P_x \) is invertible. Since \( p_0(0) \) is a nonzero constant, \( P_x p_n(x) \) is a polynomial of degree \( n \) for all \( n \). Hence, \( P_x \) is an invertible shift-invariant operator.

We may now let \( q_n(x) = P_x^{-1} p_n(x) \) and \( G(y) = P_y^{-1} F(y) \). It remains now to show that \( G(y) = E^y \).

By the above reasoning,

\[
G(y) q_n(x) = \sum_{k=0}^{n} q_k(x) q_{n-k}(y).
\]

Now, apply \( \epsilon_y \) to both sides. Since \( \epsilon_y G(y) = \epsilon_y P_y^{-1} P_y E^y \) is the identity \( I_x \), we have

\[
q_n(x) = \sum_{k=0}^{n} q_k(x) q_{n-k}(0).
\]

In other words, \( q_n(0) = 0 \) for \( n > 0 \) and \( q_0(0) = 1 \).

Since \( F(y) \) and \( P_y \) are shift-invariant, \( G(y) \) is also shift-invariant. Thus, we can now apply [12, Theorem 5.3] which shows that \( G(y) = E^y \). That is to say, \((q_n(x))_{n=0}^{\infty}\) is a divided power sequence and \((p_n(x))_{n=0}^{\infty}\) is Sheffer. □

The above theorem yields interesting Sheffer sequences identities. We illustrate this with three examples: the Hermite polynomials, the Laguerre polynomials and the Bernoulli polynomials of the second kind. Let \((p_n(x))_{n=0}^{\infty}\) be a Sheffer sequence relative to \((q_n(x))_{n=0}^{\infty}\). By the First Expansion Theorem ([17, Theorem 2]), the operator \( P_y \) which maps \( q_n(y) \) to \( p_n(y) \) has expansion

\[
P_y = \sum_{k=0}^{\infty} p_n(0) Q_y^n
\]

where \( Q_y \) is the delta operator of \((q_n(y))_{n=0}^{\infty}\). Note that every shift-invariant operator can be represented as an integral operator (see [3]).
2.2.1 Hermite

Let \((H_n^\nu(x))_{n=0}^\infty\) be the sequence of Hermite polynomials of variance \(\nu\) where \(\nu\) is a real number (see [17, sect. 10]). Its generating function is

\[
\sum_{k=0}^\infty H_k^\nu(x) t^k = e^{x t - \nu t^2 / 2}.
\]

It follows that \(P_y = e^{-\nu D^2 / 2}\). If \(\nu < 0\), then

\[
e^{-\nu D^2 / 2} p(x) = \frac{1}{\sqrt{-2\pi \nu}} \int_{-\infty}^{\infty} e^{-u^2 / 2\nu} p(x + u) \, du,
\]

and

\[
\frac{1}{\sqrt{-2\pi \nu}} \int_{-\infty}^{\infty} e^{-u^2 / 2\nu} H_k^\nu(x + y + u) \, du = \sum_{k=0}^n H_k^\nu(x) H_{n-k}^\nu(y).
\]

2.2.2 Laguerre

Let \((L_\alpha^n(x))_{n=0}^\infty\) be the sequence of Laguerre polynomials of order \(\alpha\) where \(\alpha\) is a real number (see [17, sect. 11]). Its generating function is

\[
\sum_{k=0}^\infty L_k^\alpha(x) t^k = (1 - t)^{-\alpha - 1} e^{x + t / (\alpha + 1)}.
\]

Since \(Q = D/(D - I)\), it follows that \(P_y = (I - D)^{\alpha + 1}\). If \(\alpha < -1\), then

\[
(I - D)^{\alpha + 1} p(x) = \frac{1}{\Gamma(-\alpha - 1)} \int_0^\infty t^{-\alpha - 2} e^{-t} p(x + t) \, dt,
\]

and

\[
\frac{1}{\Gamma(-\alpha - 1)} \int_0^\infty t^{-\alpha - 2} e^{-t} L_n^\alpha(x + y + t) \, dt = \sum_{k=0}^n L_k^\alpha(x) L_{n-k}^\alpha(y).
\]

2.2.3 Bernoulli

Let \((b_n^\alpha(x))_{n=0}^\infty\) be the sequence of Bernoulli polynomials of the second kind. Its generating function is

\[
\sum_{k=0}^\infty b_k(x) t^k = \frac{1}{t \log(1 + t)} (1 + t)^x.
\]

In this case,

\[
P_y p(x) = \int_x^{x+1} p(u) \, du.
\]

Thus, we have

\[
\int_x^{x+1} b_n(y + u) \, du = \sum_{k=0}^n b_k(x) b_{n-k}(y).
\]
2.3 Generalized Sheffer

Let us now remove the condition that \( F(y) \) be shift-invariant which was so crucial to Theorem 2.1. Immediately, we have new solutions to equation 3. In fact, any sequence of polynomials \((p_n(x))_{n=0}^{\infty}\) (with \(\deg p_n(x) = n\)) gives rise to a unique operator of \( F(y) \) which verifies equation 3.

**Theorem 2.2 (Generalized Sheffer Theorem)** Let \((p_n(x))_{n=0}^{\infty}\) be any sequence of polynomials such that \(\deg p_n(x) = n\). The relation

\[
Qp_n(x) = \begin{cases} 
p_{n-1}(x) & \text{if } n > 0, \\
0 & \text{if } n = 0
\end{cases}
\]

defines a unique linear operator \(Q\). Furthermore, the relations

\[
q_n(0) = \delta_{n0},
\]

and

\[
Qq_n(x) = \begin{cases} 
q_{n-1}(x) & \text{if } n > 0, \\
0 & \text{if } n = 0
\end{cases}
\]

define a unique sequence of polynomials \((q_n(x))_{n=0}^{\infty}\) which in the philosophy of [14] would be called a divided power sequence relative to or basic for \(Q\). The relation

\[
P_xq_n(x) = p_n(x)
\]

defines a K-linear operator \(P_x\). (Incidentally, \(P_x\) is \(Q\)-invariant and invertible.) The only solution \(F(y)\) to equation 3 is \(P_yG^{(y)}\) where \(G^{(y)}\) is given by the convergent sum

\[
G^{(y)} = \sum_{n=0}^{\infty} q_n(y) Q^n.
\]

Proof: Let us first check that all the objects mentioned above are well-defined. Since \((p_n(x))_{n=0}^{\infty}\) is a basis for \(K[x]\), \(Q\) is well defined and it lowers the degree of any polynomial by one. Thus, \(Q^{-1}\) is well defined up to a constant. Since the constant term of \(q_n(x)\) is given, \(q_n(x)\) is well defined. By induction, \(Q^n\) lowers the degree of any polynomial by \(n\); thus, the sum giving \(G^{(y)}\) is in fact convergent. \(P_x\) is of course well defined and invertible since \((q_n(x))_{n=0}^{\infty}\) and \((p_n(x))_{n=0}^{\infty}\) are both sequences of polynomials. Thus, \(F^{(y)}\) is well defined.

\(P_x\) is \(Q\)-invariant because

\[
P_xQp_n(x) = P_xp_{n-1}(x) = Qq_{n-1}(x) = QP_xq_n(x).
\]

Again uniqueness of solution is automatic, so it will suffice to verify that \(F^{(y)}\) is in fact a solution. Now, as in [12, Lemma 5.2], we have

\[
G^{(y)}q_n(x) = \sum_{k=0}^{\infty} q_k(y) Q^k q_n(x) = \sum_{k=0}^{n} q_k(y) q_{n-k}(x)
\]

which given the \(Q\)-invariancy of \(P_x\) can be transformed into equation 3 by applying \(P_xP_y\) to both sides, and exchanging \(x\) and \(y\).

Two explicit examples that illustrate Theorem 2.2 are:
Proposition 2.1 Let \((q_n(x))_{n=0}^{\infty}\), \(Q\), and \(G^{(y)}\) be as in Theorem 2.2. Then we have

\[
\lim_{y \to 0} \frac{G^{(y)} - G^0}{y} = \sum_{k=0}^{\infty} (Dq_k)(0) Q^k.
\]
2.3 Generalized Sheffer

Proof: Apply the left hand side to the basis \((q_n(x))_{n=0}^{\infty}\). Then equation 3 yields

\[
\lim_{y \to 0} \frac{G^y - G^0}{y} q_n(x) = \lim_{y \to 0} \sum_{k=1}^{n} q_{n-k}(x) q_k(y)/y
\]

Now, it follows from \(q_k(0) = \delta_{0k}\) that

\[
\lim_{y \to 0} \frac{G^y - G^0}{y} q_n(x) = \sum_{k=1}^{n} (Dq_k)(0) q_{n-k}(x) = \left( \sum_{k=0}^{\infty} (Dq_k)(0) Q^k \right) q_n(x). \]

In particular, it follows from the First Expansion Theorem \[17, Theorem 2\] that the right hand side sums to \(D\) if \(Q\) is a delta operator with basic set \((q_n(x))_{n=0}^{\infty}\).

In \[8\], Levitan also gives a systematic exposition of the relation between generalized translation operators and Cauchy problems (i.e., partial differential equations with initial data). In our case, we have the following Cauchy problem (cf. \[2, Theorem 5\]):

**Proposition 2.2 (Cauchy problem)** Let \((q_n(x))_{x=0}^{\infty}\), \(Q\), and \(G(y)\) be as in Theorem 2.2, then for all polynomials \(p(x)\) we have \(u(x,y) = G(y)p(x)\) as a solution of the following Cauchy problem

\[
Q_x u = Q_y u \\
u(x,0) = p(x)
\]

Proof: First, note that \(u(x,0) = p(x)\) because \(G(0) = I\). Since \((q_n(x))_{n=0}^{\infty}\) is a basis, it suffices to show that \(Q_x G(y) q_n(x) = Q_y G(y) q_n(x)\). This follows directly from equation 3. \(\Box\)

If \(Q = D\) in Proposition 2.2, then we can easily compute \(u\) as follows: Define new variables \(\xi = x\) and \(\eta = x + y\). Since \(D_x = D_\xi + D_\eta\) and \(D_y = D_\eta\), the differential equation transforms into \(D_\xi u = 0\) with solution \(u(x,y) = f(\eta) = f(x+y)\). Now, set \(y = 0\) which yields \(p = f\). Hence, \(u(x,y) = p(x+y)\) as expected.

Another way to solve this Cauchy problem is to proceed as Heaviside did in the previous century: Fix \(x\) and treat \(D_x\) as a formal constant. Then the Cauchy problem becomes an ordinary differential equation whose solution is readily seen to be \(u(x,y) = e^{yD_x} p(x)\) which equals \(p(x+y)\) by the First Expansion Theorem \[17, Theorem 2\].

If \(Q = cD + xD^2\) (cf. the second example below Theorem 2.2 where \(c = 1\)) and \(c \geq \frac{1}{2}\), then it follows from \[8, Theorem 2.4.2.6\] that

\[
u(x,y) = \frac{1}{2\pi} \frac{1}{2\pi B(c - \frac{1}{2}, \frac{1}{2})} \int_0^{2\pi} p(x+y - 2\sqrt{xy} \cos \phi) (\sin^2 \phi)^{c-1} \, d\phi
\]
where $B$ denotes the beta function.

The relation between Cauchy problems and generalized translation operators is due to Delsarte (see [5], for recent developments see [11] and references therein). Delsarte mainly considered the Hankel translation, which is associated with the Sturm-Liouville operator

$$
\Delta_x = \frac{d^2}{dx^2} + \frac{2\nu}{x} \frac{d}{dx}.
$$

A closed form for the Hankel translation is given by (see e.g. [4, p. 4])

$$
G(y)p(x) = \frac{\Gamma(\nu + 1/2)}{\Gamma(\nu)\Gamma(1/2)} \int_0^\pi p\left[\left\{x^2 + y^2 - 2yx \cos \theta\right\}^{1/2}\right](\sin \theta)^{2\nu-1} d\theta.
$$

An Umbral Calculus based on the Hankel translation operator is presented in [4]. This Umbral Calculus is related to Bessel functions.

### 2.4 Coalgebra

The above can be profitably recast in the terminology of coalgebras (see [14] for the relation between Umbral Calculus and coalgebras). A coalgebra is a vector space $V$ equipped with a comultiplication $\Delta : V \to V \otimes V$ and a counitary map $\epsilon : V \to K$. These maps must be coassociative

\[(I \otimes \Delta) \circ \Delta = (\Delta \otimes I) \circ \Delta\]  \hspace{1cm} (6)

and obey the counitary property

\[(\epsilon \otimes I) \circ \Delta = I = (I \otimes \epsilon) \circ \Delta.\]  \hspace{1cm} (7)

Now, $K[x, y]$ is isomorphic to the tensor product $K[x] \otimes K[y]$, so any $F = F(y)$ (satisfying equation 3) would be a potential candidate for a comultiplication map. Equation 4 is automatically satisfied:

\[ (F \otimes I) \circ Fp_n(x) = \sum_{i+j+k=n} p_i(x) \otimes p_j(x) \otimes p_k(x) \]

\[ = (I \otimes F) \circ Fp_n(x).\]

Moreover, $F$ is automatically cocommutative since $\sum_{k=0}^n p_k(x) p_{n-k}(y)$ is symmetric in $x$ and $y$.

By equation 6,

\[ p_n(x) = (\epsilon \otimes I)Fp_n(x) \]

\[ = (\epsilon \otimes I) \sum_{k=0}^n p_k(x) \otimes p_{n-k}(x) \]

\[ = \sum_{k=0}^n (\epsilon p_k(x)) p_{n-k}(x).\]

Since $\{p_n(x) : n \in \mathbb{N}\}$ is a basis, we have $\epsilon p_k(x) = \delta_{k0}$. For example, $\epsilon$ is the “evaluation at zero” operator if, as in [2], $p_n(0) = \delta_{n0}$.

We have thus proven the following proposition.
Proposition 2.3  All $F(y)$ satisfying equation 3 define distinct (yet isomorphic) cohomogeneous cocommutative coalgebras. Conversely, any cohomogeneous coalgebra isomorphic to $(K[x], E^y)$ yields a solution to equation 3.

Corollary 2.1  Suppose $F(y)$ together with $(p_n(x))_{n=0}^\infty$ obeys equation 3, and $F(y)$ together with $(p'_n(x))_{n=0}^\infty$ also obeys equation 3. Then the two maps $\epsilon$ and $\epsilon'$ defined by

$$
\epsilon p_n(x) = \delta_{n0} \\
\epsilon' p'_n(x) = \delta_{n0}
$$

are identical.

Corollary 2.2  Suppose $F(y)$ together with $(p_n(x))_{n=0}^\infty$ obeys equation 3, and the resulting coalgebra is in fact a bialgebra with respect to the usual multiplication of polynomials. Then $F(y)$ is the map $E^y-c$ for some constant $c$. Thus, $(p_n(x))_{n=0}^\infty$ is a Sheffer sequence. The counitary map $\epsilon$ is evaluation at $x = c$. These bialgebras are then Hopf algebras when equipped with the antipode $\omega : (y + c)^n \mapsto (-1)^n(y + c)^n$.

Proof: If $F(y)$ is an algebra map, then $F(y)$ is the substitution for $x$ of some polynomial $r(x, y)$. By degree considerations in equation 3, $r(x, y)$ must be of degree one. Moreover, since $F(y)$ is cocommutative, $r(x, y)$ must be symmetric in $x$ and $y$. Thus, $r(x, y) = a(x + y) - c$. Consideration of the leading coefficients in equation 3 indicates that $a$ must be zero. Thus, $F(y) = E^y-c$. The remaining results are easily verified.

3 Symmetric Functions

3.1 Introduction

In 3, the notion (and combinatorial interpretation) of divided power sequences is extended to the domain of symmetric functions. A linear divided powers sequence of symmetric functions $(p_n(x_1, x_2, \ldots))_{n=0}^\infty$ is a sequence of homogeneous symmetric functions—one of each degree—obeying the following convolution identity

$$
E^y p_n(x_1, x_2, \ldots) = \sum_{k=0}^n p_k(x_1, x_2, \ldots) p_{n-k}(y, 0, 0, \ldots)
$$

where the symmetric shift $E^y$ is defined by the rule

$$
E^y q(x_1, x_2, \ldots) = q(y, x_1, x_2, \ldots).
$$

Well-known examples of linear divided power sequences of symmetric functions include the elementary $e_n(x_1, x_2, \ldots)$ and complete $h_n(x_1, x_2, \ldots)$ symmetric functions.
Suppose we now generalize to
\[ F^y p_n(x_1, x_2, \ldots) = \sum_{k=0}^{n} p_k(x_1, x_2, \ldots) p_{n-k}(y, 0, 0, \ldots) \]

where \( p_n(x_1, x_2, \ldots) \) is a sequence of homogeneous symmetric functions—one for each degree—and \( F^y \) is a linear operator. In this case, there is not much to say about \( F^y \). It is not defined on a basis, so there are not enough constraints to characterize it completely.

Clearly, we are considering the wrong generalization of polynomial sequences. We must turn to the subject of \([10]\), full sequences of symmetric functions, since it is those sequences which serve as a useful basis for the space of symmetric functions.

### 3.2 Notation

A partition \( \lambda \) is an eventually zero, decreasing sequence of natural numbers \( \lambda_1 \geq \lambda_2 \geq \cdots = 0 \). Its conjugate, denoted \( \lambda' \), is defined by the rule
\[ \lambda'_i = |\{j : \lambda_j \geq i\}|. \]

We will compare partitions and/or vectors in two different ways.

- First, they can be compared coordinate wise: \( \alpha \leq \beta \) if and only if \( \alpha_i \leq \beta_i \) for all \( i \).
- Second, they can be compared using the reverse lexicographical order. That is to say, they are ordered as if they were words written in Hebrew or Arabic (from right to left). \( \alpha \ll \beta \) if and only if there is an \( i \) such that \( \alpha_i < \beta_i \) and \( \alpha_j = \beta_j \) for all \( j > i \).

Let \( \mathcal{P} \) be the set of all partitions and \( \mathcal{P}_n \) be the set of all partitions summing to \( n \). Clearly, only \( \ll \) is a total ordering of \( \mathcal{P} \). In fact, \( \ll \) is a strengthening of the \( < \) relation which itself is so weak as to be equality when restricted to \( \mathcal{P}_n \).

The monomial symmetric functions \( m_\lambda(x_1, x_2, \ldots) \) for \( \lambda \in \mathcal{P}_n \) form a basis for the vector space of homogeneous symmetric functions of degree \( n \). In fact, \( (m_\lambda(x_1, x_2, \ldots))_{\lambda \in \mathcal{P}} \) will be our canonical example of a full sequence (just as \( (x^n)_{n=0}^\infty \) is the typical sequence of polynomials).

In general, in a full sequence \( (p_\lambda(x_1, x_2, \ldots))_{\lambda \in \mathcal{P}} \), the symmetric functions \( p_\lambda(x_1, x_2, \ldots) \) must be homogeneous of degree \( n \) (for \( \lambda \in \mathcal{P}_n \)). Moreover, they must have expansions in terms of the monomial symmetric functions whose index follows \( \lambda' \) in reverse lexicographical order
\[
p_\lambda(x_1, x_2, \ldots) = \sum_{\mu \geq \lambda'} b_{\lambda\mu} m_\mu(x_1, x_2, \ldots)
\]
where \( b_{\lambda\lambda} \) is never zero.
A full sequence is thus a basis for the space of symmetric functions.

Even though \( p_\lambda(x_1, x_2, \ldots) \) is only defined for \( \lambda \) a partition, it will be convenient to extend its definition to all vectors of integers with finite support. If \( \alpha_i \) is always nonnegative, then there is a unique partition \( \lambda \) which is a permutation of \( \alpha \). We then write

\[
p_\alpha(x_1, x_2, \ldots) = p_\lambda(x_1, x_2, \ldots).
\]

On the other hand, if \( \alpha_i < 0 \) for some \( i \), we write

\[
p_\alpha(x_1, x_2, \ldots) = 0.
\]

Finally, we must define a few linear operators; the multivariate symmetric derivative \( D_\lambda \) is most simply defined by

\[
D_\lambda m_\mu(x_1, x_2, \ldots) = m_\mu - \lambda
\]

while the augmentation \( \epsilon \) is defined by

\[
\epsilon p(x_1, x_2, \ldots) = p(0, 0, \ldots).
\]

Note that \( E^a = \sum_{n=0}^{\infty} a^n D_{(n)} \). A linear operator \( \theta \) is said to be shift-invariant is \( E^a \theta = \theta E^a \). In that case, we have the following convergent expansion of \( \theta \) in terms of \( D_\lambda \):

\[
\theta = \sum_\lambda \epsilon(\theta m_\lambda(x_1, x_2, \ldots)) \ D_\lambda.
\]

Now, we can define the object of interest; a full divided powers sequence is a full sequence of symmetric functions \((p_\lambda(x_1, x_2, \ldots))_{\lambda \in \mathcal{P}} \) which obeys the convolution identity

\[
E^y p_\lambda(x_1, x_2, \ldots) = \sum_\alpha p_\alpha(x_1, x_2, \ldots) \ p_{\lambda-\alpha}(y, 0, 0, \ldots)
\]

where the sum is over all integer vectors \( \alpha \) with finite support.

### 3.3 Sheffer Theorem

What linear operators \( F^y \) and full sequences \( p_\lambda(x_1, x_2, \ldots) \) obey

\[
F^y p_\lambda(x_1, x_2, \ldots) = \sum_\alpha p_\alpha(x_1, x_2, \ldots) \ p_{\lambda-\alpha}(y, 0, 0, \ldots)?
\]

**Theorem 3.1 (Sheffer Theorem)** Given that \( F^y \) is a shift-invariant operator obeying equation \( \Box \), then \( F^y = c E^y \). In other words, \((p_\lambda(x_1, x_2, \ldots))_{\lambda \in \mathcal{P}} \) is up to constant factor equal to a full divided power sequence of symmetric functions.
Proof: First, consider $F^0$.

$$F^0 p_\lambda(x_1, x_2, \ldots) = \sum_\alpha p_\alpha(x_1, x_2, \ldots) p_{\lambda - \alpha}(0, 0, 0, \ldots).$$

However, for $\alpha \neq (0)$, $p_\alpha(0, 0, 0, \ldots)$ is zero while $p_{(0)}(0, 0, 0, \ldots) = c \neq 0$.

Without loss of generality, we can assume that $c = 1$. Otherwise, replace $F^y$ with $\frac{1}{c} F^y$ and $p_\lambda(x_1, x_2, \ldots)$ with $\frac{1}{c} p_\lambda(x_1, x_2, \ldots)$. It remains then to show that $F^y = E^y$.

Since $F^y$ and $E^y$ are both shift-invariant, so is their difference which we can then expand in the form

$$F^y - E^y = \sum_\lambda c_\lambda D_\lambda.$$ 

We will show by induction on $\lambda$ (ordered reverse lexicographically) that $c_\lambda = 0$ and thus $F^y = E^y$. The base case $\lambda = (0)$ has already been dispensed with.

Let $\lambda \in P_n$ ($n > 0$), and suppose that $c_\mu = 0$ for $\mu \ll \lambda \in P_n$ and for $\mu \in P_m$ with $m < n$. We must show that $c_\lambda = 0$. By induction,

$$(F^y - E^y)p_\lambda(x_1, x_2, \ldots) = c_\lambda b_{\lambda' \lambda}$$

where the $b$ sequence is defined by equation 8. However, the right hand side is equal to

$$p_{\lambda'}(x_1, x_2, \ldots) - p_{\lambda'}(y, x_1, x_2, \ldots) + \sum_{\alpha \neq (0)} p_{\lambda' - \alpha}(x_1, x_2, \ldots) p_\alpha(y, 0, 0, \ldots)$$

which is homogeneous of degree $n$ in the variables $x_1, x_2, \ldots$, and $y$. Thus, the right hand side has no constant term. Therefore, the constant $c_\lambda b_{\lambda' \lambda}$ must be zero. However, $b_{\nu' \nu}$ is never zero, so we must have $c_\lambda = 0$. \(\square\)

Open Problem: What happens if we no longer assume that $F^y$ is shift-invariant? Do we get an analog of Proposition 2.2?

3.4 Coalgebra

As seen in [9], all operators of the form $F^y$ obeying 8 serve as the comultiplication of a (strongly) homogeneous cocommutative Hopf algebra over the symmetric functions, and conversely. For the symmetric shift operator, for example, the augmentation $\epsilon$ is the counitary map, and the antipode is the classical involution of symmetric functions

$$\omega h_n(x_1, x_2, \ldots) = (-1)^n \epsilon_n(x_1, x_2, \ldots).$$


References

[1] A. Di Bucchianico, *Representations of Sheffer polynomials*, submitted to J. Math. Anal. Appl.

[2] V. M. Bukhtabber and A. N. Kholodov, *Groups of formal diffeomorphisms of the superline, generating functions for sequences of polynomials, and functional equations*, Izv. Akad. Nauk SSSR 53 (1989), 944-970 (English transl. in Math. USSR Izvestiya 35 (1990), 277-305; MR 91h:58014).

[3] V. M. Bukhtabber and A. N. Kholodov, *Boas-Buck structures on sequences of polynomials*, Funct. Anal. Appl. 23 (4) (1990), 266-276 (MR 91d:26017).

[4] F. M. Cholewinski, The Finite Calculus associated with Bessel Functions, Contemporary Mathematics 75, American Mathematical Society, 1988.

[5] J. Delsarte, *Sur une extension de la formule de Taylor*, J. Math. Pur. Appl. 17 (1938), 213-231.

[6] P. Feinsilver and R. Schott, *Algebraic structures and operator calculus*, to appear: Kluwer, The Netherlands.

[7] S. G. Kurbanov and V. M. Maksimov, *Mutual expansions of differential operators and divided difference operators*, Dokl. Akad. Nauk UzSSR 4 (1986), 8-9 (MR 87k:05021).

[8] B. M. Levitan, *Generalized Translation Operators*, Israel Program Sci. Translations, Jerusalem, 1964.

[9] D. Loeb, *Sequences of symmetric functions of binomial type*, Stud. Appl. Math. 83 (1990), 1-30.

[10] D. Loeb, *Sequences of symmetric functions of binomial type II: Full Sequences*, In progress.

[11] C. Markett, *A new proof of Watson’s product formula for Laguerre polynomials via a Cauchy problem associated with a singular differential operator*, SIAM J. Math. Anal. 17 (1986), 1010-1032.

[12] G. Markowsky, *Differential operators and the theory of binomial enumeration*, J. Math. Anal. Appl. 63 (1978), 145-155.

[13] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1979.

[14] R. Morris (ed.), *Umbral Calculus and Hopf Algebras*, Contemporary Mathematics 6, American Mathematical Society, 1982.

[15] R. Mullin and G.-C. Rota, *On the Foundations of Combinatorial Theory: III. Theory of Binomial Enumeration*, in: B. Harris (ed.), *Graph Theory and Its Applications*, Academic Press, 1970, 167-213.

[16] E. D. Rainville, *Special Functions*, MacMillan, New York, 1960.

[17] G.-C. Rota, D. Kahaner and A. Odlyzko, *On the Foundations of Combinatorial Theory: VIII. Finite Operator Calculus*, J. Math. Anal. Appl. 42 (1973), 684-760.

[18] I. M. Sheffer, *Some properties of polynomial sets of type zero*, Duke Math. J. 5 (1939), 590-622 (MR 1, 15).
[19] O. V. Viskov, *Operator characterization of generalized Appell polynomials*, Dokl. Akad. Nauk SSR 225 (1975), 749-752 (English transl. in Soviet Math. Dokl. 16 (1975), 1521-1524).