Some topological problems on the configuration spaces of Artin and Coxeter groups

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Abstract

In the first part we review some topological and algebraic aspects in the theory of Artin and Coxeter groups, both in the finite and infinite case (but still, finitely generated). In the following parts, among other things, we compute the Schwartz genus of the covering associated to the orbit space for all affine Artin groups. We also give a partial computation of the cohomology of the braid group with non-abelian coefficients coming from geometric representations. We introduce an interesting class of ”sheaves over posets”, which we call ”weighted sheaves over posets”, and use them for explicit computations.

1 Introduction

This paper contains some results regarding some particular topological aspects in the theory of Artin and Coxeter groups. Among other things, we compute the Schwartz genus of the covering associated to the orbit space for all affine Artin groups. This result generalizes [DS00], [DPS04].

Our paper contains also a brief review of some results, concerning the topology of Artin and Coxeter groups, both in the finite and infinite case (but still, finitely generated) which are essential to our computations. Other reviews, even if considering some interesting aspects of the theory, are not very satisfactory about the topological underlying structure. Our review will still be very partial: the more than thirty years old literature on the subject would require a much longer paper. We concentrate essentially on a single line of research, which (in our opinion) gives the possibility to produce a neat picture of some basic topological situation underlying all the theory in very few pages. Such picture is based essentially on [Sal94] (and [DS96]) and [DS00] (both papers are based on the construction [Sal87]). Some of the several computations which use these constructions are cited below.

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For some literature containing many works related to Artin groups the reader can see the paper [L.P11] contained in this book. It should be added that almost all people who worked in the theory of Hyperplane Arrangements have given some contributions to the theory.

The new results that we present here concern some computations of:
- the cohomology of the braid groups with non-abelian coefficients, coming from geometric representations of the braid groups into the homology of an orientable surfaces (see part 3.1);
- the computation of the Schwartz genus of the covering associated to the orbit space of an affine Artin group (see part 3.2).

We skip the details of the computations of the first application, while we give details for the second one. We introduce here a particularly interesting class of “sheaves over posets”, which we call “weighted sheaves over posets”. We use them for some explicit computations for the top cohomology of the affine group of type $\tilde{A}_n$ (the whole cohomology was completely computed with different methods in [CMS08b], [CMS08a]). A natural spectral sequence is associated to such sheaves. We are going to exploit this general construction in future works.

Some of the ideas which we use here were partially explained in [Sal05], [Mor06]. The first of these papers essentially contains a talk given by the second author at a Conference in Tokyo, during the 60th-birthday fest in honor of F. Cohen.

2 General pictures

We will consider finitely generated Coxeter systems $(W,S)$ ($S$ finite), so

$$W = \langle s \in S \mid (ss')^{m(s,s')} = 1 \rangle$$

where $m(s,s') \in \mathbb{N} \cup \{\infty\}$, $m(s,s') = m(s',s)$, $m(s,s) = 1$ (for general reference see [Bou68], [Hum90]).

2.1 Case $W$ finite

The group $W$ can be realized as a group generated by reflections in $\mathbb{R}^n$ ($n = |S|$). Let $A$ be the reflection arrangement, i.e.

$$A = \{ H \subset \mathbb{R}^n \mid H \text{ is fixed by some reflection in } W \}.$$ 

Consider also the stratification into facets $\Phi := \{ F \}$ of $\mathbb{R}^n$ induced by $A$. The codimension-0 facets are called chambers. They are the connected components of the complement to the arrangement. All the chambers are simplicial cones, the group acting transitively over the set of all them. The Coxeter generator set $S$ corresponds to the set of reflections with respect to the walls of a fixed base-chamber $C_0$ (see fig. 1).

Let $H_C := H + iH \subset \mathbb{C}^n$ be the complexification of the hyperplane $H$, and set

$$Y := M(A) := \mathbb{C}^n \setminus \bigcup_{H \in A} H_C$$

be the complement to the complexified arrangement. The group $W$ acts freely over $Y$ so we can form the orbit space

$$Y/W := Y/W$$
which is a complex manifold, actually an affine manifold, by classical results.

Define

\[ G_W := \pi_1(Y_W) \]

as the Artin group of type \( W \) (somebody calls it the Artin-Brieskorn group, somebody else calls it the Artin-Tits group: for brevity, we just take the intersection of them) (see [Bri71, Del72, L.P11]). A presentation of \( G_W \) is obtained by that of \( W \) by removing the relations \( s^2 = 1 \)

\[ G_W = \langle g_s, s \in S \mid g_s g'_s g_s \cdots = g'_s g_s g'_s \cdots \rangle \]  

(same number \( m(s, s') \) of factors on each side).

From Deligne’s theorem ([Del72]) the orbit space \( Y_W \) is a space of type \( k(\pi, 1) \), so we have

\[ H^*(G_W; L) = H^*(Y_W; L) \]

where \( L \) is any \( G_W \)-module and \( L \) is the corresponding local system over \( Y_W \).

We also recall ([Sal94, Sal87]):

**Theorem 1** The orbit space \( Y_W \) contracts over a cell complex \( X_W \) obtained from a convex polyhedron \( Q \) by explicit identifications over its faces.

More precisely, one takes one point \( x_0 \in C_0 \), and set

\[ Q := \text{convex hull } [W.x_0] \]

a polyhedron obtained by the convex hull of the orbit of one point \( x_0 \) in the base-chamber.
For example, in the case of the braid arrangement one obtains the so called permutohedron.

One verifies the following facts.

The \( k \)-faces of \( Q \) are also polyhedra, each of them corresponding to a coset of a parabolic subgroup \( W_{\Gamma} \), where \( \Gamma \subset S \) is a \( k \)-subset of \( S \). The correspondence

\[
\{ \text{faces of } Q \} \leftrightarrow \{ w.W_{\Gamma}, \ \Gamma \subset S \}
\]

is obtained by taking the polyhedron given by the convex-hull of the orbit \( W_{\Gamma}.x_0 \) and translating it by \( w \).

One has also (Bou68, Hum90):

**Proposition 2.1** Inside each coset \( w.W_{\Gamma} \) there exists an unique element of minimal length.

Here the length is the minimal number of letters (coming from \( S \)) in a reduced expression.

For every face \( e \) of \( Q \), which corresponds to a coset \( w.W_{\Gamma} \), let \( \beta(e) \in w.W_{\Gamma} \) be the element of minimal length. Notice that \( W \) permutes faces of the same dimension. Then each pair of faces \( e, e' \) belonging to the same orbit is identified by using the homeomorphism \( \beta(e)\beta(e')^{-1} \).

We give in fig. 2, 3 the example of the group \( W \) of type \( A_2 \), so \( G_W \) is the braid group in three strands. The orbit space turns out to have the homotopy type of an hexagon whose edges are identified according to the given arrows.

### 2.2 Case \( W \) infinite

When \( W \) is infinite (but still \( S \) is finite) the theory is analogue with the following changes (see Bou68, Vin71). One can still realize \( W \) as a group of (non-orthogonal) reflections in \( \mathbb{R}^n \), \( (n = |S|) \) starting from a base chamber \( C_0 \). In [Bou68], the standard first octant is considered, (so \( C_0 = \{ x_i > 0, \ i = 1, \ldots, n \} \)) but one can start from a more general open cone with vertex \( 0 \) (see [Vin71]). Again, \( S \) corresponds to the set of reflections with respect to the walls of \( C_0 \).

We recall here the main points which we need.

Let \( U := W.C_0 \) be the orbit of the closure of the base chamber. \( U \) is called the Tits cone of the Coxeter system.

Notice that the closure of the chamber \( C_0 \) is endowed with a natural stratification into facets (which are still relatively open cones with vertex \( 0 \)). When \( C_0 \) is the standard positive octant, each facet is given by imposing some coordinates equal to 0, and the remaining coordinates positive.

Each reflection in \( W \) is conjugated to a reflection with respect to a wall of \( C_0 \). So, the arrangement \( A \) of reflection hyperplanes is just the orbit of the walls of \( C_0 \). Each connected component of the complement inside \( U \) of the arrangement (again called a chamber) is of the shape \( w.C_0 \) for a unique \( w \in W \). Of course \( A \) is not locally finite (e.g. \( 0 \) is contained in all the hyperplanes). The orbits of the facets of \( C_0 \) give a “stratification” of \( U \) into relatively open cells, also called facets (in general, \( U \) is neither open nor closed in \( \mathbb{R}^n \)).

Recall also (see Bou68, Vin71):

1. \( U \) is a convex cone in \( \mathbb{R}^n \) with vertex \( 0 \).

2. \( U = \mathbb{R}^n \) iff \( W \) is finite
Figure 2: for each 1-cell $e$ we indicate the corresponding coset $\beta(e) \cdot W_F$.

3. The stabilizer of a facet $F$ in $U$ is the subgroup $W_F$ generated by all the reflections with respect to hyperplanes (in $A$) containing $F$. So, in general $W_F$ is not finite.

4. $U^0 := \text{int}(U)$ is open in $\mathbb{R}^n$ and a (relatively open) facet $F \subset \mathcal{T}_0$ is contained in $U^0$ iff the stabilizer $W_F$ is finite.

By property 4 the arrangement is locally finite in the interior part $U^0$. So we take in this case

$$Y := \left[ U^0 + i\mathbb{R}^n \right] \setminus \bigcup_{H \in A} H_C$$

which corresponds to complexifying only the interior part of the Tits cone. The group $W$ acts (as before) diagonally onto $Y$, and one shows easily (exactly as in the finite case) that the action is free. Therefore, one has an orbit space

$$Y_W := Y/W$$

which is still a manifold, and a regular covering

$$Y \to Y_W$$

with group $W$. 5.
We still define $G_W := \pi_1(Y_W)$ as the Artin group of type $W$. We will see in a moment that for $G_W$ one has a presentation similar to (2). In fact, we have very similar constructions.

Take $x_0 \in C_0$ and let $Q$ be the finite CW-complex constructed as follows. For all subsets $\Gamma \subset S$ such that $W_\Gamma$ is finite, construct a $|\Gamma|$-cell $Q_\Gamma$ in $U^0$ as the convex hull of the $W_\Gamma$ orbit of $x_0$. Each $Q_\Gamma$ is a finite convex polyhedron which contains the point $x_0$.

Let $X_{W_\Gamma}$ be obtained from $Q_\Gamma$ by identifications on its faces defined as in the finite case (relative to the finite group $W_\Gamma$). Define

$$Q := \bigcup Q_\Gamma$$  \hspace{1cm} (3)

(a finite union of convex polyhedra) where the union is taken on all the above $\Gamma$ for which $W_\Gamma$ is finite. Define also

$$X_W := \bigcup X_{W_\Gamma}.$$  \hspace{1cm} (4)

**Remark 2.2** The definition of $X_W$ makes sense because of the following easy fact:

For any common cell $e \subset Q_\Gamma \cap Q_{\Gamma'}$, the minimal element $\beta(e)$ is the same when computed in $W_\Gamma$ and in $W_{\Gamma'}$.

Moreover, we have the following generalization of the finite case (see [Sa94])

**Theorem 2** The CW-complex $X_W$ is deformation retract of the orbit space $Y_W$.

**Proof.** Notice that $X_W$ is a finite complex in all cases.

First, there exists a regular CW-complex $X \subset Y$ which is deformation retract of $Y$, and $X$ is constructed as in [Sa87]. That paper already worked for the affine cases; however, the same procedure works in general because we reduce to the locally finite case around faces with finite stabilizer.

The construction of $X$ can be chosen invariantly with respect to the action of $W$, which permutes cells of the same dimension.
The action on $X$ being free, we look at the orbit space $X/W$. By remark 2.2 this reduces to finite cases.

(A similar proof can be obtained by generalizing "combinatorial stratifications" to this situation, see [BZ92]).

Below we give a picture for the case $\tilde{A}_2$ (fig 4).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{Q is the union of three hexagons. $X_W$ is obtained by gluing 1-cells with same type of arrow.}
\end{figure}

**Conjecture 1** The space $Y$ is a $k(\pi, 1)$ space.

Of course, if one of the spaces $Y$, $Y_W$, $X$, $X_W$ is a $k(\pi, 1)$, so are all the other spaces. This conjecture is known, besides the finite case, for some affine groups: $\tilde{A}_n$, $\tilde{C}_n$ ([Oko79]; in case $\tilde{A}_n$ a different proof is given in [Cha95]); case $\tilde{B}_n$ was solved in ([CMS10]). For other known cases see [Hen85], [CD95].

As an immediate corollary to Theorem 2 we re-find in a very short way a presentation of the fundamental group (see [vdL83]).
Theorem 3 A presentation for the Artin group $G_W$ is similar to 2 where we have to consider only pairs $s$, $s'$ such that $m(s,s')$ is finite.

Proof. We have just to look at the 2-skeleton of $X_W$, which is given as follows. For each pair $s$, $s'$ such that $m(s,s')$ is finite, one has a $2m(s,s')$-gon with two orbits of edges, which glue in a similar way as fig. 3. This gives the relation
$$g_s g_{s'} g_s \ldots = g_{s'} g_s g_{s'} \ldots (m(s,s') \text{ factors}).$$

2.3 Algebraic complexes for Artin groups

We refer here mainly to [Sal94], [DS96].

We consider the algebraic complex related to the cell structure of $X_W$. It is given in the following way.

Let $Z[GW]$ be the group algebra of $GW$. Let $(C_*, \partial_*)$ be the algebraic complex of free $Z[GW]$-modules such that in degree $k$ it is free with basis $e_J$ corresponding to subsets $J \subset S$ such that $W_J$ is finite:
$$C_k := \bigoplus_{|J| = k} Z[GW] e_J$$

Let
$$\partial(e_J) := \sum_{I \subset J, |I| = k-1} [I : J] T^J_I . e_I$$

where $[I : J]$ is the incidence number (= 0, 1 or $-1$) of the cells in $X_W$ and
$$T^J_I := \sum_{\beta \in W^I_J} (-1)^{\ell(\beta)} g_\beta$$

where

1. $W^I_J := \{ \beta \in W_J : \ell(\beta s) > \ell(\beta), \forall s \in W_I \}$ is the set of elements of minimal length for the cosets $W^I_J$ (prop. 2.1);

2. if $\beta \in W^I_J$ and $\beta = s_{i_1} \ldots s_{i_k}$ is a reduced expression then $\ell(\beta) = k$;

3. if $\beta$ is as in 2 then $g_\beta := g_{s_{i_1}} \ldots g_{s_{i_k}}$. One shows that this map
$$\psi : W \to GW$$

is a well-defined section (not a homomorphism) of the standard surjection $GW \to W$.

Remark 2.3 The orbit space $X_W$ is a $k(\pi,1)$ iff $(G_*, \partial_*)$ is acyclic (in this case the augmentation gives a resolution of $Z$ into free $Z[GW]$-modules).
In papers [Sal94] [DS96] we used the knowledge of the fact that $X_W$ is a $k(\pi,1)$ in the finite case to deduce that $X_W$ is acyclic. One could try the converse: prove algebraically that the algebraic complex is exact and conclude that $X_W$ is a $k(\pi,1)$. It is interesting to consider the following abelian representation ([Sal87], [SS97], [DSS97]).

Let $R := A[q, q^{-1}]$ be the ring of Laurent polynomials over a ring $A$. One can represent $G_W$ by

$$g_s \mapsto \text{[multiplication by $-q$]}, \forall s \in S$$

(\in Aut(R)). This representation, which coincides with the determinant of the Burau representation, has a very interesting meaning in the finite case: the cohomology of $G_W$ with coefficient in this representation equals the trivial cohomology of the Milnor fibre of the associated discriminant bundle. In other words, the orbit space has a Milnor fibration over $S^1$, with Milnor fibre $F$, and the above twisted cohomology of $G_W$ gives the trivial cohomology of $F$ as a module over $R$, the $(-q)$-multiplication corresponding to the monodromy of the bundle.

The tensor product $C_* \otimes R$ has boundary

$$\partial (e_J) = \sum_{I \subset J} [I : J] \frac{W_J(q)}{W_I(q)} e_I$$

where

$$W_J(q) := \sum_{w \in W_J} q^{l(w)}$$

is the Poincaré series of the group $W_J$ (here, a polynomial since the stabilizers are finite). The denominator $W_I(q)$ divides the numerator $W_J(q)$ so the quotient is still a polynomial.

**Example.** Case $A_n$. We have Dynkin graph

**Example.** Case $A_n$. We have Dynkin graph

$$\circ \quad \cdots \quad \circ$$

$$1 \quad 2 \quad \ldots \quad n-1 \quad n$$

Let

$$[k] := \frac{q^k - 1}{q - 1}; \quad [k]! := \prod_{i=1}^{k} [i]; \quad \left[ \begin{array}{c} k \\ h \end{array} \right] := \frac{[k]!}{[h]![k-h]!}$$

1Actually this is known in the finite case: an algebraic complex equivalent to the given one was found in [Squ94], who proved algebraically that the complex is exact in the finite case. This paper appeared slightly later than [Sal94], but actually it was known (it seems, only to the author, who never published something similar) much before because it was essentially the content of his unpublished PhD thesis discussed in 1980. Paper [Squ94] was written posthumous from his colleagues, who published the main part of his thesis with some changes (and adding some remarks and theorems which are not contained in the thesis. This whole thing seems very mysterious: it is not clear why nothing was published before ’94 about Squier’s thesis work, as Squier continued to work and did other publications in the 80’s. Also, somebody would probably like to know who are the actual authors of the paper appeared in ’94.)
For $J \subseteq S \cong \{1, \ldots, n\}$ one has

$$W_J(q) = \prod_{i=1}^{m} \left( |\Gamma_i(J)| + 1 \right)!$$

where $\Gamma_1(J), \Gamma_2(J), \ldots$ are the connected components of the subgraph of $A_n$ generated by $J$ (fig 5).

Figure 5: here $W_J(q) = [3][4][2]!$

So, all coefficients are of the shape $\begin{bmatrix} k \\ h \end{bmatrix}$ ($k, h$ depending on $J, I$).

For some computations using these methods for the finite case:
for $H^*(G_W, Q[q, q^{-1}])$ \cite{DPSS01}, \cite{Fre88} (case $A_n$) \cite{DPSS99} (all other cases);
for the top cohomology of $H^*(G_W, Z[q, q^{-1}])$ in all cases \cite{DSS97};
for the cohomology $H^*(G_W, Z[q, q^{-1}])$ for all exceptional cases see \cite{CS04};
the same for case $A_n$: see \cite{Cal06};
In the affine cases: see \cite{CMS08a} (cohomology in case $\tilde{A}_n$), \cite{CMS08a} ($k(\pi, 1)$ problem and cohomology for $\tilde{B}_n$).

2.4 CW-complexes for Coxeter groups

We refer here essentially to \cite{DS00}.

If $(W, S)$ is a finite Coxeter system, which is realized as a reflection group in $\mathbb{R}^n$, with reflection arrangement $\mathcal{A}$, consider the subspace arrangement in $\mathbb{R}^{nd} \cong (\mathbb{R}^n)^d$ given by

$$\mathcal{A}^{(d)} := \{ H^{(d)} \}$$

where $H^{(d)}$ is the codimensional-$d$ subspace given by “$d$-complexification” of the hyperplane $H \in \mathcal{A}$:

$$H^{(d)} := \{ (X_1, \ldots, X_d) : X_i \in \mathbb{R}^n, X_i \in H \}.$$ 

For $d = 2$ we have the standard complexification of the hyperplanes. Let

$$Y^{(d)} := \mathbb{R}^{nd} \setminus \bigcup_{H \in \mathcal{A}} H^{(d)}.$$
As before, the group $W$ acts freely on $Y^{(d)}$ and we consider the orbit space

$$Y_W^{(d)} := Y^{(d)}/W.$$ 

Recall:

**Theorem 4** The space

$$Y_W^{(\infty)} := \left[ \lim_{d \to \infty} Y^{(d)} \right]/W = \left[ \lim_{d \to \infty} Y_W^{(d)} \right]$$

is a space of type $k(W, 1)$.

In case $(W, S)$ is infinite (still, $S$ finite) one has to substitute $\mathbb{R}^n \times \cdots \times \mathbb{R}^n$ ($d$ factors) with the space $Y^{(d)}$ becomes

$$Y^{(d)} := U_W^{(d)} \setminus \bigcup_{H \in A} H^{(d)}.$$ (11) 

**Theorem 5** For finitely generated $W$, the same conclusion as in theorem 4 holds, by taking definition (11) for $Y^{(d)}$.

So, different from the case of Artin groups, we always get a $k(W, 1)$ space here.

Recall also the construction of a CW-complex which generalizes that given for Artin groups.

**Theorem 6** When $W$ is finite, the space $Y_W^{(d)}$ contracts over a CW-complex $X_W^{(d)}$ such that

$$\{ k\text{-cells of } X_W^{(d)} \} \leftrightarrow \{ \text{flags } \Gamma := (\Gamma_1 \supset \cdots \supset \Gamma_d) : \Gamma_1 \subset S, \sum_{i=1}^{d} |\Gamma_i| = k \}$$

Passing to the limit, $Y_W^{(\infty)} = k(W, 1)$ contracts over a CW-complex $X_W^{(\infty)}$ such that

$$\{ k\text{-cells of } X_W^{(\infty)} \} \leftrightarrow \{ \text{flags } \Gamma := (\Gamma_1 \supset \Gamma_2 \supset \cdots) : \Gamma_1 \subset S, \sum_{i \geq 1} |\Gamma_i| = k \}$$

Notice that $X_W^{(\infty)}$ does not have finite dimension but the number of $k$-cells is finite, given by $\binom{n+k-1}{k}$.

The case $W$ infinite has to be modified as in case of Artin groups, by considering flags composed with subsets $\Gamma \subset S$ such that $W_{\Gamma}$ is finite. In the limit, the theorem is

**Theorem 7** For $S$ finite, the space $Y_W^{(\infty)} = k(W, 1)$ contracts over a CW-complex $X_W^{(\infty)}$ such that the set of $k$-cells of $X_W^{(\infty)}$ corresponds to

$$\{ \text{flags } \Gamma := (\Gamma_1 \supset \Gamma_2 \supset \cdots) : \Gamma_1 \subset S, W_{\Gamma_1} \text{ finite, } \sum_{i \geq 1} |\Gamma_i| = k \}$$

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2.5 Algebraic complexes for Coxeter groups

Consider the algebraic complex \((C^d_*, \partial)\) of free \(\mathbb{Z}[W]\)-modules, where
\[
C^d_k := \bigoplus_{\Gamma : \sum_{i=1}^{d} |\Gamma_i| = k \atop |W_{\Gamma_1}| < \infty} \mathbb{Z}[W]e(\Gamma)
\]
The generators of \(C_*\) are in one to one correspondence with the cells of \(X^{(\infty)}_W\).

The expression of the boundary is the following:
\[
\partial e(\Gamma) = \sum_{1 \leq i \leq d} \sum_{\substack{\tau \in \Gamma_i \\setminus \{\}\,\text{s.t.} \, j \leq \tau \,\text{for some} \, j \in \Gamma_i}} (-1)^0(\Gamma, i, \tau, \beta) \beta e(\Gamma')
\]
where
\[
\Gamma' = (\Gamma_1 \supset \ldots \supset \Gamma_{i-1} \supset \Gamma_i \setminus \{\} \supset \beta^{-1}\Gamma_{i+1} \supset \ldots \supset \beta^{-1}\Gamma_d \beta)
\]
and \((-1)^0(\Gamma, i, \tau, \beta)\) is an incidence index. To get a precise expression for \(\alpha(\Gamma, i, \tau, \beta)\), fix a linear order on \(S\) and let
\[
\mu(\Gamma_i, \tau) := |\{j \in \Gamma: j \leq \tau\}|
\]
\[
\sigma(\beta, \Gamma_j) := |\{(a, b) \in \Gamma_j \times \Gamma_j: a < b \text{ and } \beta(a) > \beta(b)\}|
\]
in other words, \(\mu(\Gamma_i, \tau)\) is the number of reflections in \(\Gamma_i\) less or equal to \(\tau\) and \(\sigma(\beta, \Gamma_j)\) is the number of inversions operated by \(\beta\) on \(\Gamma_j\). Then we define:
\[
\alpha(\Gamma, i, \tau, \beta) = i(\ell(\beta) + \sum_{j=1}^{i-1} |\Gamma_j| + \mu(\Gamma_i, \tau) + \sum_{j=i+1}^{d} \sigma(\beta, \Gamma_j))
\]
where \(\ell\) is the length function in the Coxeter group.

Let now \(C_* := \lim_{d \to \infty} C^d_*\). The flags in \(C_k\) are the (infinite) sequences
\[
\Gamma = (\Gamma_1 \supset \Gamma_2 \supset \ldots)
\]
such that \(\sum_{i \geq 1} |\Gamma_i| = k\), so they still have a finite number of nonempty \(\Gamma_i\).

**Theorem 8** For any finitely generated \(W\), the algebraic complex \((C_*, \partial_*)\) gives a free resolution of the trivial \(\mathbb{Z}[W]\)-module \(\mathbb{Z}\).

The proof follows straightforward from the remark that the limit space \(Y^{(\infty)}_W\), so \(X^{(\infty)}_W\), is a space of type \(k(\pi, 1)\).

3 Applications

...... continue in the paper cited in the footnote in the first page .................

References

[Bou68] N. Bourbaki, *Groupes et algebres de Lie*, vol. Chapters IV-VI, Hermann, 1968.
[Bri71] E. Brieskorn, *Die Fundamentalgruppe des Raumes der regulären Orbits einer endlichen komplexen Spiegelungsgruppe*, Invent. Math. **12** (1971), 57–61.

[BZ92] A. Bjorner and G. Ziegler, *Combinatorial stratifications of complex arrangements*, Jour. Amer. Math. Soc. **5** (1992), 105–149.

[Cal06] F. Callegaro, *The homology of the Milnor fiber for classical braid groups*, Alg. Geom. Top. **6** (2006), 1903–1923.

[CD95] R. Charney and M.W. Davis, *The $k(\pi,1)$-problem for hyperplane complements associated to infinite reflection groups*, J. of AMS **8** (1995), 597–627.

[Cha95] R. Charney, *Geodesic automation and growth functions for Artin groups of finite type*, Math. Ann. **301** (1995), 307–324.

[CMS08a] F. Callegaro, D. Moroni, and M. Salvetti, *Cohomology of affine Artin groups and applications*, Trans. Amer. Mat. Soc. **360** (2008), 4169–4188.

[CMS08b] _____, *Cohomology of Artin groups of type $\tilde{A}_n, \tilde{B}_n$ and applications*, Geom. & Top. Mon. **13** (2008), 85–104.

[CMS10] _____, *The $K(\pi,1)$ problem for the affine Artin group of type $\tilde{B}_n$ and its cohomology*, Jour. Eur. Math. Soc. **12** (2010), 1–22.

[CMS09] F. Callegaro and M. Salvetti, *Integral cohomology of the Milnor fibre of the discriminant bundle associated with a finite Coxeter group*, C. R. Acad. Sci. Paris, Ser. I **339** (2004), 573–578.

[Del72] P. Deligne, *Les immeubles des groupes de tresses généralisés*, Inventiones math. **17** (1972), 273–302.

[DPS01] C. De Concini, C. Procesi, and M. Salvetti, *Arithmetic properties of the cohomology of braid groups*, Topology **40** (2001), no. 4, 739–751.

[DPS04] _____, *On the equation of degree 6*, Comm. Math. Helv. **79** (2004), 605–617.

[DS99] C. De Concini, C. Procesi, M. Salvetti, and F. Stumbo, *Arithmetic properties of the cohomology of Artin groups*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **28** (1999), no. 4, 695–717.

[DS96] C. De Concini and M. Salvetti, *Cohomology of Artin groups*, Math. Res. Lett. **3** (1996), 293–297.

[DS00] _____, *Cohomology of Artin groups and Coxeter groups*, Math. Res. Lett. **7** (2000), 213–232.

[DSS97] C. De Concini, M. Salvetti, and F. Stumbo, *The top-cohomology of Artin groups with coefficients in rank-1 local systems over $\mathbb{Z}$*, Topology and its Applications **78** (1997), 5–20.

[Fre88] E. V. Frenkel, *Cohomology of the commutator subgroup of the braids group*, Func. Anal. Appl. **22** (1988), no. 3, 248–250.
[Hen85] H. Hendriks, Hyperplane complements of large type, Invent. Math. 79 (1985), 375–381.

[Hum90] J.E. Humphreys, Reflection groups and Coxeter groups, Cambridge University Press, 1990.

[L.P11] L. Paris, Basic question on Artin-Tits groups, these Proceedings (2011).

[Mor06] D. Moroni, Finite and infinite type artin groups: Topological aspects and cohomological computations, PhD thesis (2006).

[Oko79] C. Okonek, Das $k(\pi,1)$-problem für die affinen wurzelsysteme vom typ $A_n, C_n$, Mathematische Zeitschrift 168 (1979), 143–148.

[Sal87] M. Salvetti, Topology of the complement of real hyperplanes in $\mathbb{C}^n$, Invent. Math. 88 (1987), no. 3, 603–618.

[Sal94] , The homotopy type of Artin groups, Math. Res. Lett. 1 (1994), 567–577.

[Sal05] , On the Cohomology and Topology of Artin and Coxeter groups, Pubblicazioni Dipartimento di Matematica L.Tonelli, Pisa (2005).

[Squ94] C.C. Squier, The homological algebra of Artin groups, Math. Scand. 75 (1994), no. 1, 5–43.

[SS97] M. Salvetti and F. Stumbo, Artin groups associated to infinite Coxeter groups, Discrete Mathematics 163 (1997), 129–138.

[vdL83] H. van der Lek, The homotopy type of complex hyperplane complements, Ph.D. thesis, University of Nijmegan, 1983.

[Vin71] E.B. Vinberg, Discrete linear groups generated by reflections, Math. USSR Izvestija 5 (1971), no. 5, 1083–1119.