A finite-temperature periodic structure in (super)string theory

A.A. Bytsenko
Department of Theoretical Physics, State Technical University,
St Petersburg 195251, Russia

E. Elizalde
Department E.C.M., Faculty of Physics, University of Barcelona,
Diagonal 647, 08028 Barcelona, Spain

S.D. Odintsov
Department of Physics, Faculty of Science, Hiroshima University,
Higashi-Hiroshima 724, Japan

S. Zerbini
Department of Physics, University of Trento, 38050 Povo, Italy
I.N.F.N., Gruppo Collegato di Trento

Abstract

Using a Laurent series representation for the (super)string one-loop free energy, an explicit form for the analytic continuation of the Laurent series beyond the critical (Hagedorn) temperature is obtained. As an additional result, a periodic structure is found in (super)string thermodynamics. A brief physical discussion about the origin and meaning of such structure is carried out.

1E-mail address: eli @ ebubecm1.bitnet
2On sabbatical leave from Tomsk Pedagogical Institute, 634041 Tomsk, Russia.
1 Introduction

There is a number of motivations for the study of the behaviour of extended objects —here we will mainly concentrate on strings— at non-zero temperature $T = \beta^{-1}$ (for a short list of references see [1-6]). Perhaps one of the main reasons for these investigations is connected with the thermodynamics of the early universe (see [7] and references therein) as well as with the attempts to use exented objects for the description of the high temperature limit of the confining phase of large-N SU(N) Yang-Mills theory [8,9].

It is well known since the early days of dual string models, that an essential ingredient of string theory at non-zero temperature is the so-called Hagedorn temperature. The appearance of the Hagedorn temperature is a consequence of the fact that the asymptotic form of the state level density has an exponential dependence of the mass. A naive argument leads to the conclusion that above such a temperature the free energy diverges. According to the popular viewpoint, the Hagedorn temperature is the critical temperature for a phase transition to some new phase (probably associated with topological strings [10]).

There are different representations (in particular, those connected with different ensembles) for the string free energy. One of these representations [6] —which is very useful for formal manipulations— gives a modular-invariant expression for the free energy. However, this and all the other well known representations [5] are integral ones, in which the Hagedorn temperature appears as the convergence condition in the ultraviolet limit of a certain integral. In order to discuss the high- or the low-temperature limits in such representations one must expand the integral in terms of a corresponding series. Thus, a specific series expansion appears in string theory at non-zero temperature.

In the present note, making use of the so-called Laurent series representation for the one-loop open superstring free energy introduced in refs. [11,12] and of the analytic continuation of such series, we discuss the possible appearance of a periodic thermodynamic structure valid for any value of $\beta$. Some consideration about the open bosonic string are also reported.
2 One-loop (super)string free energy at finite temperature

It is well-known that the one-loop free energy for the bosonic \((b)\) or fermionic \((f)\) degree of freedom in \(d\)-dimensional space is given by

\[
F_{b,f} = \pm \frac{1}{\beta} \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \log \left(1 \mp e^{-\beta \omega_k}\right)
\]

(1)

where \(\omega_k = \sqrt{k^2 + m^2}\) and \(m\) is the mass for the corresponding degree of freedom. Let us recall the Mellin-Barnes representation for the one-loop free energy discussed in field theory in ref. \[13\] (see also \[14,15\]). Integrating eq. (1) by parts, we obtain

\[
F_{b,f} = -\frac{(4\pi)^{(1-d)/2}}{(d-1)\Gamma((d-1)/2)} \int dk^2 \frac{k^{d-1}}{\omega_k (e^{\beta \omega_k} \mp 1)}.
\]

(2)

For the factor in the integrand, \((e^{\beta \omega_k} \mp 1)^{-1}\), we shall use the Mellin transform in the following form \[13\]

\[
\frac{1}{e^{ax} \mp 1} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \zeta^{(\mp)}(s) \Gamma(s)(ax)^{-s}, \quad ax > 0,
\]

(3)

where \(\text{Re } s = c, c > 1\), for bosons and \(c > 0\) for fermions, \(\zeta^{(-)}(s) = \zeta(s)\) is the Riemann-Hurwitz zeta function and \(\zeta^{(+)}(s) = (1 - 2^{1-s})\zeta(s) = \sum_{n=1}^{\infty} (-1)^{n-1} n^{-s}\), \(\text{Re } s > 0\).

Substituting (3) into (2), we get

\[
F_{b,f} = -\frac{(4\pi)^{(1-d)/2}}{(d-1)\Gamma((d-1)/2)} \int dk^2 \frac{k^{d-1}}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \zeta^{(\mp)}(s) \Gamma(s) \beta^{-s} \omega_k^{-1-s}.
\]

(4)

Performing the integration over \(k\) with the help of the Euler beta function \(B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)\) (notice that, owing to absolute convergence, the order of integration over \(k\) and \(s\) can be interchanged), we obtain

\[
\int dk^2 k^{d-1} \omega_k^{-1-s} = (m^2)^{(d-s)/2} B\left(\frac{d+1}{2}, \frac{s-d}{2}\right) = (m^2)^{(d-s)/2} \frac{\Gamma\left(\frac{d+1}{2}\right) \Gamma\left(\frac{s-d}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)}, \quad \text{Re } s > d.
\]

(5)

Finally, one gets \[13\]

\[
F_{b,f} = -2^{-d} \pi^{(1-d)/2} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \Gamma\left(\frac{s}{2}\right) \zeta^{(\mp)}(s) \beta^{-s} \Gamma\left(\frac{s-d}{2}\right) (m^2)^{(d-s)/2}, \quad c > d.
\]

(6)
For (super)string theory one can write the representation (6) in the form

\[
F_{\text{bosonic string}} = -2^{d-1} \pi^{-d/2} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \zeta(s)(\beta/2)^{-s} \Gamma\left(\frac{s-d}{2}\right) \Gamma\left(\frac{s-d}{2}\right) \times \text{Tr} (M^2)^{(d-s)/2}, \quad d = 26, \tag{7}
\]

\[
F_{\text{superstring}} = -2^{-d} \pi^{-d/2} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \zeta(s)(1 - 2^{-s})(\beta/2)^{-s} \Gamma\left(\frac{s-d}{2}\right) \Gamma\left(\frac{s-d}{2}\right) \times \text{STr} (M^2)^{(d-s)/2}, \quad d = 10. \tag{8}
\]

Here \(M^2\) is the mass operator, the symbol \(\text{STr}\) means the trace over fermion and boson fields. For closed strings the constraint should be introduced via the usual identity [2].

For the bosonic strings the mass operator contains both infrared (due to the presence of the tachyon in the spectrum) and ultraviolet divergences, while for superstrings it contains only ultraviolet divergences. Hence, the consideration of superstrings is much simpler from a technical point of view.

In the following we shall consider open superstrings. For the open superstring (without gauge group) the spectrum is given by (see for example [16])

\[
M^2 = 2 \sum_{i=1}^{d-2} \sum_{n=1}^{\infty} n \left(N_{ni}^b + N_{ni}^f\right), \quad d = 10. \tag{9}
\]

The quantity \(\text{STr}(M^2)^{(d-s)/2}\) which appears in equation (8), requires a regularization because a naive definition of it leads to a formal divergent expression, namely at \(d = 10\) we can write

\[
\text{STr} (M^2)^{5-\frac{s}{2}} = \frac{1}{\Gamma\left(\frac{s}{2} - 5\right)} \int_0^\infty dt \, t^{\frac{s}{2} - 6} \text{STr} e^{-tM^2}. \tag{10}
\]

As a result we need the heat-kernel expansion for \(\text{STr} e^{-tM^2}\).

It is known that

\[
\text{STr} e^{-tM^2} = 8 \prod_{n=1}^{\infty} \left(\frac{1 - e^{-2tn}}{1 + e^{-2tn}}\right)^{-8} = 8 \left[\theta_4 (0|e^{-2t})\right]^{-8}, \tag{11}
\]

the presence of the factor 8 in eq. (11) is due to the degeneracy of the ground state. Recall that the Jacobi’s elliptic theta function \(\theta_4 (0|e^{-t})\) when \(t \to 0\) has the asymptotic form

\[
\theta_4 (0|e^{-t}) = \sqrt{\frac{\pi}{t}} \theta_2 (0|e^{-\pi^2/t}) = \sqrt{\frac{\pi}{t}} \sum_{n=-\infty}^{+\infty} \exp \left[-\frac{\pi^2}{t} \left( n - \frac{1}{2} \right)^2\right] = 2 \sqrt{\frac{\pi}{t}} e^{-\pi^2/(4t)} \left( 1 + e^{-2\pi^2/t} + e^{-6\pi^2/t} + \cdots\right), \tag{12}
\]
hence
\[ \left[ \theta_4 \left( 0 | e^{-t} \right) \right]^{-8} \bigg|_{t \to 0} = \frac{t^4}{2^8 \pi^4} e^{2\pi^2/t} - \frac{t^4}{2^5 \pi^4} + O \left( e^{-2\pi^2/t} \right). \] (13)

So we can define the regularized supertrace in the following way
\[
\text{STr} \left( M^2 \right)^{5 - \frac{s}{2}} = \frac{2^{8 - s/2}}{\Gamma \left( \frac{s}{2} - 5 \right)} \left\{ \int_0^\infty dt \frac{t^{s/2 - 6}}{\pi^4} \left[ \left[ \theta_4 \left( 0 | e^{-t} \right) \right]^{-8} - \frac{t^4}{2^8 \pi^4} \left( e^{2\pi^2/t} - 8 \right) \right] \right. \\
+ \frac{1}{2^8 \pi^4} \int_0^\infty dt \left( e^{2\pi^2/t} - 8 \right) \right\}. \] (14)

Since the regularization of the integral \( \int_0^\infty dx x^\lambda \) as the analytical function of \( \lambda \) gives \( \int_0^\infty dx x^\lambda = 0 \), it turns out that the last integral in eq. (14) is equal to zero. Moreover, we have
\[
\int_0^\infty dt t^{s/2 - 2} e^{2\pi^2/t} = \int_0^\infty dt \left( -\frac{s+1}{2} \right)^{s-1} e^{2\pi^2/t} = \left( -2\pi^2 \right)^{s/2 - 1} \Gamma \left( 1 - \frac{s}{2} \right), \] (15)

and therefore
\[
\text{STr} \left( M^2 \right)^{5 - \frac{s}{2}} = \frac{2^{1-s/2} \pi^{-6}}{\Gamma \left( \frac{s}{2} - 5 \right)} \left[ \pi^s \Gamma \left( 1 - \frac{s}{2} \right) \text{Re} \left( -1 \right) \pi^{-1} + 2\pi^{3/2} G(s, \mu) \right], \] (16)

where
\[
G(s, \mu) = 2^{1-s/2} \pi^{(s-1)/2} \int_0^\mu dt \frac{t^{s/2 - 6}}{2^8 \pi^4} \left[ \left[ \frac{1}{2} \theta_4 \left( 0 | e^{-t} \right) \right]^{-8} - t^4 \left( e^{2\pi^2/t} - 8 \right) \right]. \] (17)

In eq. (17) the infrared cutoff parameter \( \mu \) has been introduced. Such a regularization is necessary for \( s \geq 4 \). On the next stage of our calculations this regularization will be removed \( (\mu \to \infty) \).

For the one-loop free energy we get
\[
F_{\text{superstring}} = -(2\pi)^{-11} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \left[ \varphi(s) + \psi(s) \right], \] (18)

where
\[
\varphi(s) = (1 - 2^{-s}) \left[ \text{Re} \left( -1 \right) \pi^{-1} \right] \frac{\pi}{\sin \frac{s\pi}{2}} \zeta(s) \left( \frac{\beta}{2\pi} \right)^{-s}, \] (19)
\[
\psi(s) = (1 - 2^{-s}) (\pi)^{3/2-s} G(s, \mu) \Gamma(s/2) \zeta(s) \left( \frac{\beta}{2\pi} \right)^{-s}. \] (20)

The meromorphic function \( \varphi(s) \) has first order poles at \( s = 1 \) and \( s = 2k, \ k = 0, 1, 2, \ldots \). The pole of \( \varphi \) for \( s = 1 \) is also of first order but its residue is imaginary. The meromorphic function \( \psi(s) \) has first order poles at \( s = 1 \). One can see that the regularization cutoff parameter is removed automatically.
As a result, we obtain
\[
F_{\text{superstring}} = 2(2\pi)^{-11} \left[ \sum_{k=1}^{\infty} (1 - 2^{-2k}) \zeta(2k)x^{2k} - \frac{\pi x}{4} G(1, \infty) \right] + F_R(x),
\] (21)
where \(x = \beta_c / \beta\), \(\beta_c = 2\pi\) and \(F_R(x)\) is the contribution coming from the contour integral along the arc of radius \(R\). If \(|x| < 1\), then the value of the contour integral on the right half-plane is vanishing when \(R \to \infty\). Therefore the series converges when \(\beta > \beta_c = 2\pi\), \(\beta_c\) being the Hagedorn temperature [1-4]. The sum of the series can be explicitly evaluated and the result is
\[
\sum_{k=1}^{\infty} (1 - 2^{-2k}) \zeta(2k)x^{2k} = \frac{\pi x}{4} \tan\left(\frac{\pi x}{2}\right), \quad |x| < 1.
\] (22)
As a consequence, the free energy is given by
\[
F_{\text{superstring}} = \frac{\pi x}{2(2\pi)^{11}} \left[ \tan\left(\frac{\pi x}{2}\right) - G(1, \infty) \right].
\] (23)

Now, observing that
\[
\sum_{k=1}^{\infty} \zeta(2k)y^{2k} = \frac{1}{2} - \frac{\pi y}{2} \cot(\pi y), \quad |y| < 1,
\] (24)

it is easy to show that the free energy for the open bosonic string has the form
\[
F_{\text{bosonic string}} = \frac{1}{2^{23}\pi^{16}} \left[ -y \cot(\pi y) - \frac{2y}{\beta_c} D(1, \mu) + \pi^{-1} D(0, \mu) \right].
\] (25)

Here \(y = \beta_c / \beta\) and the related Hagedorn temperature is \(\beta_c = \sqrt{8\pi}\),
\[
D(s, \mu) = 2\pi \int_0^\mu dt t^{s/2-14} \left[ \eta(it)^{-24} - t^{12} e^{2\pi it} \right],
\] (26)
and \(\eta(\tau) = e^{i\tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi in\tau})\) is Dedekind’s eta function. However, in this case, the infrared cutoff parameter \(\mu\) cannot be removed (there is a tachion in the spectrum).

The Laurent series have been obtained in each case for for \(|x| < 1\) and \(|y| < 1\), respectively, namely for \(\beta > \beta_c\). However the right hand sides of the above formulas (22) and (24) may also be understood as analytic continuations of these series to arguments \(|x| > 1\) and \(|y| > 1\), respectively (i.e. to \(\beta < \beta_c\)). As a consequence, we realize that we have obtained a periodic structure for the one-loop free energy of the corresponding (super)strings.
3 Conclusions

We finish the paper with some remarks. The results we have obtained here are based on the Mellin-Barnes representation for the one-loop free energy of the critical (super)strings. Such a novel representation for the lowest order in string perturbation theory has allowed us to obtain explicit thermodynamic expressions in term of a Laurent series. The critical temperature arises in this formalism as the convergence condition (namely, the radius of convergence) of these series. Furthermore, an explicit analytic continuation of the free energy to temperatures beyond the critical Hagedorn temperature (i.e., to $\beta \leq \beta_c$) has been constructed. As a result, there is now the possibility to have the free energy which corresponds to new string phases.

Actually, it is somewhat surprising to see that there exists such a finite temperature periodic structure in the behaviour of (super)string thermodynamical quantities. However, we should remember that a similar asymptotic behaviour was obtained in ref. [1]. So there is a possibility for a new interpretation of the critical temperature and of the transitions associated with this thermodinamical structure.

In the case of the closed bosonic string —which has not been considered here— these transitions may well be connected with the dual $\beta$-symmetry which is present in the function $\text{Tr} \exp(-tM^2)$ and hidden in the final form of the free energy.

The typical widths of the periodic sectors depend on the Regge slope parameter $\alpha$. The widths of the sectors grow together with the parameter $\alpha$, and in the limit $\alpha \to \infty$ (string tension goes to zero), the thermodynamic system can be associated with an ideal gas of free quantum fields present in the normal modes of the string [2,17]. Of course the precise interpretation of “domain”-like structure in quantum (super)string theory obtained in this paper is somehow delicate and many questions remain unanswered.

Finally it should be noticed that higher orders of string perturbation theory do not modify the critical temperature, at least for the bosonic string [18]. In consequence, there exists the possibility that the periodic structure found above might also be present when dealing with a Riemann surface of arbitrary genus.

Acknowledgments

We would like to thank G. Cognola, K. Kirsten and L. Vanzo for discussions. A.A.B. is grateful to the Faculty of Science of the University of Trento for hospitality. E.E. thanks DGICYT (Spain) for financial support through research project PB90-0022. S.D.O. wishes to thank the Particle Group at Hiroshima University for kind hospitality.
References

[1] K. Huang and S. Weinberg, *Phys. Rev. Lett.* 25 (1970) 895; R. Hagedorn, *Nuovo Cim. Suppl.* 3 (1965) 147; S. Fubini and G. Veneziano, *Nuovo Cim.* A64 (1969) 1640.

[2] R. Rohm, *Nucl. Phys.* B237 (1984) 553; K. Kikkawa and M. Yamasaki, *Phys. Lett.* B149 (1984) 357; M. Gleiser and J.G. Taylor, *Phys. Lett.* B164 (1985) 36; M. Bowick and L.C.R. Wijewardhana, *Phys. Rev. Lett.* 54 (1985) 2485; M. McGuigan, *Phys. Rev.* D38 (1988) 552; E. Alvarez and M. Osorio, *Phys. Rev.* D36 (1987) 1175; S.D. Odintsov, *Europhys. Lett.* 8 (1989) 207; N. Deo, S. Jain and C.-I. Tan, *Phys. Rev.* D40 (1989) 2626.

[3] J.J. Atick and E. Witten, *Nucl. Phys.* B310 (1988) 291.

[4] I. Antoniadis, J. Ellis and D.V. Nanopoulos, *Phys. Lett.* B199 (1987) 402; Y. Aharonov, F. Englert and J. Orloff, *Phys. Lett.* B199 (1987) 366; I. Antoniadis and C. Kounnas, *Phys. Lett.* B261 (1991) 369; F. Englert and J. Orloff, *Nucl. Phys.* B334 (1990) 472.

[5] S.D. Odintsov, *Rivista Nuovo Cim.* 15 (1992) 1.

[6] N. Sakai and I. Senda, *Progr. Theor. Phys.* 75 (1986) 692; J. Polchinski, *Commun. Math. Phys.* 104 (1987) 539; S.D. Odintsov, *Phys. Lett.* B252 (1990) 573; K.H. O’Brien and C.-I. Tan, *Phys. Rev.* D36 (1987) 1184.

[7] R. Brandenberger and C. Vafa, *Nucl. Phys.* B316 (1989) 391; E. Alvarez, *Nucl. Phys.* B269 (1986) 596.

[8] J. Polchinski, *Phys. Rev. Lett.* 68 (1992) 1267.

[9] M.B. Green, *Phys. Lett.* B282 (1992) 380.

[10] E. Witten, *Commun. Math. Phys.* 117 (1988) 353.

[11] A.A. Bytsenko, E. Elizalde, S.D. Odintsov and S. Zerbini, Laurent series representation for the open superstring free energy, preprint HUPD-92-09, UB-ECM-PF 92/21, *Phys. Lett. B*, to appear.
[12] A.A. Bytsenko, E. Elizalde, S.D. Odintsov and S. Zerbini, A novel representation for the free energy in string theory at non-zero temperature, preprint HUPD-92-12, UB-ECM-PF 92/25, *Nucl. Phys. B*, to appear.

[13] A.A. Bytsenko, L. Vanzo and S. Zerbini, *Mod. Phys. Lett.* **A7** (1992) 2669; ibid. *Phys. Lett.* **B291** (1992) 26.

[14] B. Allen, *Phys. Rev.* **D33** (1986) 3640.

[15] E. Elizalde and A. Romeo, *Phys. Rev.* **D40** (1989) 436; ibid. *Int. J. Mod. Phys.* **A7** (1992) 365.

[16] M.B. Green, J.H. Schwarz and E. Witten, *Superstring Theory*, Cambridge Univ. Press, Cambridge, 1987.

[17] A. Strumia and G. Venturi, *Lett. Nuovo Cim.* **13** (1975) 337.

[18] P. Murphy and S. Sen, *Phys. Lett.* **B233** (1989) 322; Y. Leblanc , *Phys. Rev.* **D39** (1989) 3731; E. Alvarez and T. Ortin, *Phys. Lett.* **B241** (1990) 215.