A Schur–Nevanlinna Type Algorithm for the Truncated Matricial Hausdorff Moment Problem

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Abstract
The main goal of this paper is to achieve a parametrization of the solution set of the truncated matricial Hausdorff moment problem in the non-degenerate and degenerate situations. We treat the even and the odd cases simultaneously. Our approach is based on Schur analysis methods. More precisely, we use two interrelated versions of Schur-type algorithms, namely an algebraic one and a function-theoretic one. The algebraic version, worked out in our former paper (Fritzsche et al.: Linear Algebra Appl 590:133–209. https://doi.org/10.1016/j.laa.2019.12.027, 2020), is an algorithm which is applied to finite or infinite sequences of complex matrices. The construction and discussion of the function-theoretic version is a central theme of this paper. This leads us to a complete description via Stieltjes transform of the solution set of the moment problem under consideration. Furthermore, we discuss special solutions in detail.

Keywords Truncated matricial Hausdorff moment problem · Schur–Nevanlinna type algorithm · Parametrization of the solution set via Stieltjes transform

Mathematics Subject Classification 44A60 · 47A57 · 30E05
1 Introduction

The main aim of this paper is to work out an algorithm of Schur–Nevanlinna type for functions, which leads via Stieltjes transforms to a full description of the solution set of the matricial Hausdorff moment problem in the general case. In order to realize this goal, we use our former investigations in [27–29] on matricial Hausdorff moment sequences, where such sequences were studied from the point of view of Schur analysis and an algebraic version of a corresponding Schur–Nevanlinna type algorithm was worked out. The synthesis of these two Schur–Nevanlinna type algorithms will provide us finally the desired result.

This strategy was already used by the authors to study the matricial Hamburger moment problem (see [23,32]) and the matricial Stieltjes moment problem (see [24,25,31]).

Certain aspects are more difficult with the matricial Hausdorff moment problem than with the matricial Hamburger or Stieltjes moment problem. This phenomenon could be already observed in the discussion of the so-called non-degenerate case of the matricial Hamburger problem (see [14,41]) and the matricial Stieltjes problem (see [15,16,18,19]) compared to the matricial Hausdorff problem (see [10,11]). One reason for this is the fact that the localization of the measure in a prescribed compact interval of the real axis requires to satisfy simultaneously more conditions. This implies that the possible moment sequences have a more complicated structure (see [27–29]).

Continuing the work done in [5,10,11] A. E. Choque-Rivero [6–9] investigated further aspects of the non-degenerate truncated matricial Hausdorff moment problem. As in [5,10,11] he distinguished between the case of an odd or even number of prescribed matricial moments. The approach used in [5,10,11] is based on V. P. Potapov’s method of Fundamental Matrix Inequalities. In the scalar case $q = 1$ the classical method used in [43] or [42, Ch. 4, § 7] is based on the application of orthogonal polynomials. A. E. Choque-Rivero [6] obtained a matrix generalization of the Krein–Nudelman representation of the resolvent matrix by using four families of orthogonal matrix polynomials on the interval $[\alpha, \beta]$. From the point of view of V. P. Potapov the Schur algorithm is interpreted as a multiplicative decomposition of a $J$-contractive matrix function into simplest elementary factors. Such multiplicative decompositions were constructed for the resolvent matrix for the moment problem under study by A. E. Choque-Rivero in [7,8].

An important feature of this paper is to achieve a simultaneous treatment of the even and odd truncated matricial Hausdorff moment problems in the general case. Our strategy is based on the application of Schur analysis methods.

This paper is organized as follows. In Sect. 2, we state some general facts on matricial power moment problems on Borel subsets of the real axis. In Sect. 3, we summarize essential insights about the structure of matricial $[\alpha, \beta]$-Hausdorff moment sequences, which were mostly obtained in our former papers [17–19]. A key observation is that we do not treat the original matricial moment problem, but an equivalent problem in the class $R_q(C \setminus [\alpha, \beta])$ of holomorphic matrix functions. In Sect. 4, we summarize some facts on the class $R_q(C \setminus [\alpha, \beta])$, which are needed in the sequel. In Sect. 5, we formulate a problem for functions of $R_q(C \setminus [\alpha, \beta])$ which is equivalent to the original matricial moment problem. This equivalence is caused by $[\alpha, \beta]$-Stieltjes Transform.
In [28], we constructed a Schur–Nevanlinna type algorithm for Non-negative Hermitian $q \times q$ measures on $[\alpha, \beta]$ by translating the Schur–Nevanlinna type algorithm for matricial $[\alpha, \beta]$-Hausdorff moment sequences into the language of measures. In Sect. 6, we translate now this algorithm via $[\alpha, \beta]$-Stieltjes Transform into the class $\mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$. On this way our main goal is to achieve a description of all solutions of the truncated matricial $[\alpha, \beta]$-Hausdorff moment problem via a linear fractional transformation of matrices. This requires to find the generating matrix function of this transformation and the corresponding domain. In the first step, we concentrate on that domain. Remember that in [10] we already studied the problem under consideration in the non-degenerate case, however by use of Potapov’s method of fundamental matrix inequalities. Doing this, we were led to a particular class $\mathcal{P}\mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$ of ordered pairs of $q \times q$ matrix-valued functions which are meromorphic in $\mathbb{C} \setminus [\alpha, \beta]$ (see [10, Def. 5.2]). In Sect. 7, we summarize some important facts about the class $\mathcal{P}\mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$ and present an example of a remarkable element of this class (see Example 7.15). The experiences from [10,11] teach us that it is necessary to introduce an equivalence relation within $\mathcal{P}\mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$. In order to take into account possible degeneracies of the moment problem, we have to single out an appropriate subclass of $\mathcal{P}\mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$, which is adapted to the prescribed matricial moments. Furthermore, we have to ensure that the construction of this subclass stands in harmony with the above mentioned equivalence relation in $\mathcal{P}\mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$. The just mentioned two themes are treated in Sect. 8. The goal of the following considerations is to prepare basic instruments for the version of our Schur–Nevanlinna type algorithm for functions. This algorithm should stand in correspondence with the Schur–Nevanlinna type algorithm for matricial $[\alpha, \beta]$-non-negative definite sequences, which was developed in [29]. A remarkable feature of this version is that this algorithm contains two elements of different nature. More precisely, the first step of the algorithm differs from the remaining steps. There occur (equivalence classes) of ordered pairs of matrix-valued functions in the first step, whereas the further steps require only matrix-valued functions. In Sects. 9 and 10, we study the corresponding transformations and its inverses for the first and remaining steps, resp. The transformations are defined by using Moore–Penrose inverses of matrices. It will turn out that under special conditions, which will be indeed satisfied in the case of interest for us, these transformations can be rewritten as usual linear fractional transformations of matrices the generating matrix-valued functions of which are quadratic $2q \times 2q$ matrix polynomials. Having a closer look at the considerations in Sects. 9 and 10 one can observe that the basic tools used there are of rather algebraic nature. In Sects. 11 and 12, we study the elementary steps of the forward and backward algorithm in more detail, resp. Namely, we turn our attention to the concrete classes of meromorphic matrix-valued functions occurring there. Moreover, we demonstrate that these elementary steps of the algorithm for functions are concordant with the elementary steps of the algebraic algorithm applied to the matricial moment sequences. In Sect. 13, we check (see Theorem 13.6) that the iteration of the elementary steps leads to a parametrization of the set of $[\alpha, \beta]$-Stieltjes Transforms of all solutions of the original matricial moment problem. In Sect. 14, we translate Theorem 13.6 into the language of linear fractional transformations of matrices and obtain our main results Theorems 14.2 and 14.5. In Sect. 15, via $[\alpha, \beta]$-Stieltjes Transform we determine all those solutions $\sigma$ of the moment problem associated with
a sequence \((s_j)_{j=0}^{m}\) for which the sequence \((s_j)_{j=0}^{m+1}\), where \(s_{m+1}\) is the \((m + 1)\)th power moment of \(\sigma\), is \([\alpha, \beta]\)-completely Degenerate. The main object of study in Sect. 16 is that solution of the moment problem associated with a sequence \((s_j)_{j=0}^{m}\) which corresponds to the central extension of \((s_j)_{j=0}^{m}\). We determine the position of the \([\alpha, \beta]\)-Stieltjes Transform of this solution within the general parametrization obtained in Theorem 14.2. In several appendices we summarize some needed facts on various topics such as particular aspects of matrix theory, Non-negative Hermitian measures and corresponding integration theory, Stieltjes of Non-negative Hermitian measures on the real line, ordered pairs of matrices corresponding to linear relations, linear fractional transformations of matrices, holomorphic matrix-valued functions.

## 2 Matricial Moment Problems on Borel Subsets of the Real Axis

In this section, we are going to formulate a certain class of matricial power moment problems. Before doing this, we have to introduce some terminology. We denote by \(\mathbb{Z}\) the set of all integers. Let \(\mathbb{N} := \{n \in \mathbb{Z}: n \geq 1\}\). Furthermore, we write \(\mathbb{R}\) for the set of all real numbers and \(\mathbb{C}\) for the set of all complex numbers. In the whole paper, \(p\) and \(q\) are arbitrarily fixed integers from \(\mathbb{N}\). We write \(\mathbb{C}^{p \times q}\) for the set of all complex \(p \times q\) matrices and \(\mathbb{C}^{p}\) is short for \(\mathbb{C}^{p \times 1}\). When using \(m, n, r, s, \ldots\) instead of \(p, q\) in this context, we always assume that these are integers from \(\mathbb{N}\). We write \(A^*\) for the conjugate transpose of a complex \(p \times q\) matrix \(A\). Denote by \(\mathbb{C}^{q \times q}_{\succeq}\) (resp. \(\mathbb{C}^{q \times q}_{>}\)) the set of all complex Non-negative (resp. positive) Hermitian \(q \times q\) matrices. If \(\mathcal{X}\) is a subset of \(\mathbb{C}^{q \times r}\) and if \(A \in \mathbb{C}^{p \times q}\), then let \(AX := \{AX: X \in \mathcal{X}\}\).

Let \((\mathcal{X}, \mathcal{A})\) be a measurable space. Each countably additive mapping whose domain is \(\mathcal{X}\) and whose values belong to \(\mathbb{C}^{q \times q}\) is called a Non-negative Hermitian \(q \times q\) measure on \((\mathcal{X}, \mathcal{A})\). For the integration theory with respect to Non-negative Hermitian measures, we refer to Kats [38] and Rosenberg [47]. For the convenience of the reader, a summary concerning this matter, sufficient for our purposes, is given in Appendix B.

Let \(\mathcal{B}_{\mathbb{R}}\) (resp. \(\mathcal{B}_{\mathbb{C}}\)) be the \(\sigma\)-algebra of all Borel subsets of \(\mathbb{R}\) (resp. \(\mathbb{C}\)). In the whole paper, \(\Omega\) stands for a non-empty set belonging to \(\mathcal{B}_{\mathbb{R}}\). Let \(\mathcal{B}_{\Omega}\) be the \(\sigma\)-algebra of all Borel subsets of \(\Omega\) and let \(\mathcal{M}_{q}^{\succeq}(\Omega)\) be the set of all Non-negative Hermitian \(q \times q\) measures on \((\Omega, \mathcal{B}_{\Omega})\). Observe that \(\mathcal{M}_{1}^{\succeq}(\Omega)\) coincides with the set of ordinary measures on \((\Omega, \mathcal{B}_{\Omega})\) with values in \([0, \infty)\).

Let \(\mathbb{N}_0 := \{m \in \mathbb{Z}: m \geq 0\}\). Throughout this paper, \(\kappa\) is either an integer from \(\mathbb{N}_0\) or \(\infty\). In the latter case, we have \(2\kappa := \infty\). Given \(\nu, \omega \in \mathbb{R} \cup \{-\infty, \infty\}\), we set \(\mathbb{Z}_{\nu, \omega} := \{k \in \mathbb{Z}: \nu \leq k \leq \omega\}\). Let \(\mathcal{M}_{q, \kappa}^{\succeq}(\Omega)\) be the set of all \(\mu \in \mathcal{M}_{q}^{\succeq}(\Omega)\) such that for each \(j \in \mathbb{Z}_{0, \kappa}\) the power function \(x \mapsto x^j\) defined on \(\Omega\) is integrable with respect to \(\mu\). If \(\mu \in \mathcal{M}_{q, \kappa}^{\succeq}(\Omega)\), then, for all \(j \in \mathbb{Z}_{0, \kappa}\), the matrix

\[
\left( s_j^{(\mu)} := \int_{\Omega} x^j \mu(dx) \right) \quad (2.1)
\]

is called (power) moment of \(\mu\) of order \(j\). Obviously, we have \(\mathcal{M}_{q}^{\succeq}(\Omega) = \mathcal{M}_{q, 0}^{\succeq}(\Omega) \subseteq \mathcal{M}_{q, \ell}(\Omega) \subseteq \mathcal{M}_{q, \ell+1}(\Omega) \subseteq \mathcal{M}_{q, \infty}(\Omega)\) for every choice of \(\ell \in \mathbb{N}_0\) and, furthermore,
s_0^\mu = \mu(\Omega) \text{ for all } \mu \in \mathcal{M}_q^\infty(\Omega). \text{ If } \Omega \text{ is bounded, then one can easily see that } \mathcal{M}_q^\infty(\Omega) = \mathcal{M}_{q,\infty}(\Omega). \text{ We now state the general form of the moment problem lying in the background of our considerations:}

**Problem 2.1** (MP[\Omega; (s_j)_{j=0}^\kappa, =]) Given a sequence \( (s_j)_{j=0}^\kappa \) of complex \( q \times q \) matrices, parametrize the set \( \mathcal{M}_{q,\kappa}(\Omega; (s_j)_{j=0}^\kappa, =) \) of all \( \sigma \in \mathcal{M}_{q,\kappa}(\Omega) \) satisfying \( s_j(\sigma) = s_j \) for all \( j \in \mathbb{Z}_{0,\kappa} \).

In the whole paper, let \( \alpha \) and \( \beta \) be two arbitrarily given real numbers satisfying \( \alpha < \beta \) and let \( \delta := \beta - \alpha \). In what follows, we mainly consider the case that \( \Omega \) is the compact interval \([\alpha, \beta]\) of the real axis \( \mathbb{R} \). As mentioned above, we have \( \mathcal{M}_q^\infty([\alpha, \beta]) = \mathcal{M}_q^\infty([\alpha, \beta]) \).

Since each solution of \( \text{MP}[[\alpha, \beta]; (s_j)_{j=0}^\kappa, =] \) generates in a natural way solutions of \( \text{MP}[[\alpha, \infty); (s_j)_{j=0}^\kappa, =], \text{MP}[(-\infty, \beta]; (s_j)_{j=0}^\kappa, =], \text{and} \text{MP}[\mathbb{R}; (s_j)_{j=0}^\kappa, =] \), we will also use results concerning the treatment of these moment problems.

### 3 Matricial [\alpha, \beta]-Hausdorff Moment Sequences

In this section, we recall a collection of results on the matricial Hausdorff moment problem and corresponding moment sequences of Non-negative Hermitian \( q \times q \) measures on the interval \([\alpha, \beta]\), which are mostly taken from [27–29]. To state a solvability criterion, we introduce the relevant class of sequences of complex matrices.

**Notation 3.1** Let \( (s_j)_{j=0}^\kappa \) be a sequence of complex \( p \times q \) matrices. Then let the block Hankel matrices \( H_n, K_n, \) and \( G_n \) be given by \( H_n := [s_{j+k}]_{j,k=0}^n \) for all \( n \in \mathbb{N}_0 \) with \( 2n \leq \kappa \), by \( K_n := [s_{j+k+1}]_{j,k=0}^n \) for all \( n \in \mathbb{N}_0 \) with \( 2n + 1 \leq \kappa \), and by \( G_n := [s_{j+k+2}]_{j,k=0}^n \) for all \( n \in \mathbb{N}_0 \) with \( 2n + 2 \leq \kappa \), resp.

To emphasize that a certain (block) matrix \( X \) is built from a sequence \( (s_j)_{j=0}^\kappa \), we sometimes write \( X^{(s)} \) for \( X \).

**Notation 3.2** Assume \( \kappa \geq 1 \) and let \( (s_j)_{j=0}^\kappa \) be a sequence of complex \( p \times q \) matrices. Then let the sequences \( (a_j)_{j=0}^{\kappa-1} \) and \( (b_j)_{j=0}^{\kappa-1} \) be given by

\[
a_j := -\alpha s_j + s_{j+1} \quad \text{and} \quad b_j := \beta s_j - s_{j+1},
\]

resp. Furthermore, if \( \kappa \geq 2 \), then let the sequence \( (c_j)_{j=0}^{\kappa-2} \) be given by

\[
c_j := -\alpha \beta s_j + (\alpha + \beta)s_{j+1} - s_{j+2}.
\]

For each matrix \( X_k = X_k^{(s)} \) built from the sequence \( (s_j)_{j=0}^\kappa \), we denote (if possible) by \( X_{\alpha,k,*} : X_k^{(a)} \), by \( X_{*,k,\beta} : X_k^{(b)} \), and by \( X_{\alpha,k,\beta} : X_k^{(c)} \) the corresponding matrix built from the sequences \( (a_j)_{j=0}^{\kappa-1}, (b_j)_{j=0}^{\kappa-1}, \) and \( (c_j)_{j=0}^{\kappa-2} \) instead of \( (s_j)_{j=0}^\kappa \), resp.
In view of Notation 3.1, we get in particular
\[ H_{\alpha,n,*} = -\alpha H_n + K_n \quad \text{and} \quad H_{\bullet,n,\beta} = \beta H_n - K_n \]
for all \( n \in \mathbb{N}_0 \) with \( 2n + 1 \leq \kappa \) and
\[ H_{\alpha,n,\beta} = -\alpha\beta H_n + (\alpha + \beta)K_n - G_n \]
for all \( n \in \mathbb{N}_0 \) with \( 2n + 2 \leq \kappa \). In the classical case \( \alpha = 0 \) and \( \beta = 1 \), we have furthermore \( a_j = s_{j+1} \) and \( b_j = s_j - s_{j+1} \) for all \( j \in \mathbb{Z}_{0,k-1} \) and \( c_j = s_{j+1} - s_{j+2} \) for all \( j \in \mathbb{Z}_{0,k-2} \).

**Remark 3.3** Let \((s_j)_{j=0}^{\infty}\) be a sequence of complex \( p \times q \) matrices. Then \( \delta s_j = a_j + b_j \) and \( \delta s_{j+1} = \beta a_j + \alpha b_j \) for all \( j \in \mathbb{Z}_{0,k-1} \). Furthermore, \( \delta s_{j+2} = \beta^2 a_j + \alpha^2 b_j - \delta c_j \) for all \( j \in \mathbb{Z}_{0,k-2} \).

**Definition 3.4** Let \( \mathcal{F}_{q,0,\alpha,\beta} \) (resp. \( \mathcal{F}_{q,0,\alpha,\beta}^< \)) be the set of all sequences \((s_j)_{j=0}^{\infty}\) of complex \( q \times q \) matrices, for which the block Hankel matrix \( H_0 \) is Non-negative (resp. positive) Hermitian, i.e., for which \( s_0 \in \mathbb{C}^{q \times q} \) (resp. \( s_0 \in \mathbb{C}_{\leq q}^{q \times q} \)) holds true. For each \( n \in \mathbb{N} \), denote by \( \mathcal{F}_{q,n,\alpha,\beta} \) (resp. \( \mathcal{F}_{q,n,\alpha,\beta}^< \)) the set of all sequences \((s_j)_{j=0}^{2n}\) of complex \( q \times q \) matrices, for which the block Hankel matrices \( H_n \) and \( H_{\alpha,n-1,\beta} \) are both Non-negative (resp. positive) Hermitian. For each \( n \in \mathbb{N}_0 \), denote by \( \mathcal{F}_{q,2n+1,\alpha,\beta} \) (resp. \( \mathcal{F}_{q,2n+1,\alpha,\beta}^< \)) the set of all sequences \((s_j)_{j=0}^{2n+1}\) of complex \( q \times q \) matrices for which the block Hankel matrices \( H_{\alpha,n,*} \) and \( H_{\bullet,n,\beta} \) are both Non-negative (resp. positive) Hermitian. Furthermore, denote by \( \mathcal{F}_{q,\infty,\alpha,\beta} \) (resp. \( \mathcal{F}_{q,\infty,\alpha,\beta}^< \)) the set of all sequences \((s_j)_{j=0}^{\infty}\) of complex \( q \times q \) matrices satisfying \((s_j)_{j=0}^{m} \in \mathcal{F}_{q,m,\alpha,\beta} \) (resp. \((s_j)_{j=0}^{m} \in \mathcal{F}_{q,m,\alpha,\beta}^< \)) for all \( m \in \mathbb{N}_0 \). The sequences belonging to \( \mathcal{F}_{q,\infty,\alpha,\beta} \) (resp. \( \mathcal{F}_{q,\infty,\alpha,\beta}^< \)) are said to be \([\alpha, \beta]\)-non-negative definite (resp. \([\alpha, \beta]\)-positive definite).

(Note that in [27], the sequences belonging to \( \mathcal{F}_{q,\infty,\alpha,\beta} \) were called \([\alpha, \beta]\)-Hausdorff non-negative definite.) A necessary and sufficient condition for the solvability of MP\([\alpha, \beta]; (s_j)_{j=0}^{\infty} = \# \) is the following:

**Theorem 3.5** (cf. [10, Thm. 1.3] and [11, Thm. 1.3]) Let \((s_j)_{j=0}^{\infty}\) be a sequence of complex \( q \times q \) matrices. Then \( \mathcal{M}_{q,*}[\alpha, \beta]; (s_j)_{j=0}^{\infty}, \# \neq \emptyset \) if and only if \((s_j)_{j=0}^{\infty} \in \mathcal{F}_{q,\infty,\alpha,\beta}^<\).

Since \( \Omega = [\alpha, \beta] \) is bounded, one can easily see that \( \mathcal{M}_{q,*}^{<}([\alpha, \beta]) = \mathcal{M}_{q,\infty}^{<}([\alpha, \beta]) \), i.e., the power moment \( s_j^{(\sigma)} \) defined by (2.1) exists for all \( j \in \mathbb{N}_0 \). If \( \sigma \in \mathcal{M}_{q}^{<}([\alpha, \beta]) \), then we call \( s_j^{(\sigma)} \) given by (2.1) the sequence of power moments associated with \( \sigma \).

Given the complete sequence of prescribed power moments \((s_j)_{j=0}^{\infty}\), the moment problem on the compact interval \( \Omega = [\alpha, \beta] \) differs from the moment problems on
the unbounded sets $\Omega = [\alpha, \infty)$ and $\Omega = \mathbb{R}$ in having necessarily a unique solution, assumed that a solution exists:

**Proposition 3.6** If $(s_j)_{j=0}^{\infty} \in \mathcal{F}_{q,\infty,\alpha,\beta}$, then the set $\mathcal{M}_{q,\infty}([\alpha, \beta])$ satisfies $\bigcup_{j \in \mathbb{Z}} \mathcal{F}_{q,\infty,\alpha,\beta}$ consists of exactly one element.

Proposition 3.6 is a well-known result, which can be proved, in view of Theorem 3.5, using the corresponding result in the scalar case $q = 1$ (see, [34] or [1, Thm. 2.6.4]).

We can summarize Proposition 3.6 and Theorem 3.5 for $\kappa = \infty$:

**Proposition 3.7** The mapping $\mathcal{E}[\alpha, \beta] : \mathcal{M}_{q}([\alpha, \beta]) \to \mathcal{F}_{q,\infty,\alpha,\beta}$ given by $\sigma \mapsto (s_j^{(\sigma)})_{j=0}^{\infty}$ is well defined and bijective.

For each $n \in \mathbb{N}_0$, denote by $\mathcal{H}_{q,2n}$ the set of all sequences $(s_j)_{j=0}^{2n}$ of complex $q \times q$ matrices, for which the corresponding block Hankel matrix $H_n$ is Non-negative Hermitian. Furthermore, for each $\ell \in \mathbb{N}_0 \cup \{\infty\}$ and each non-empty set $\mathcal{X}$, denote by $\mathcal{G}(\mathcal{X})$ the set of all sequences $(X_j)_{j=0}^{\ell}$ from $\mathcal{X}$. Obviously, $\mathcal{F}_{q,0,\alpha,\beta}$ coincides with the set of all sequences $(s_j)_{j=0}^{0}$ with $s \in \mathbb{C}^{q \times q}$. Furthermore, we have

$$\mathcal{F}_{q,2n,\alpha,\beta} = \{(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n} : (c_j)_{j=0}^{2n-1} \in \mathcal{H}_{q,2(n-1)}\} \quad (3.1)$$

for all $n \in \mathbb{N}$ and

$$\mathcal{F}_{q,2n+1,\alpha,\beta} = \{(s_j)_{j=0}^{2n+1} \in \mathcal{G}_{2n+1}((\mathbb{C}^{q \times q}) : \{(a_j)_{j=0}^{2n}, (b_j)_{j=0}^{2n}\} \subseteq \mathcal{H}_{q,2n}\} \quad (3.2)$$

for all $n \in \mathbb{N}_0$. Note that the following Propositions 3.8 and 3.9, which are proved in a purely algebraic way in [27], can also be obtained immediately from Theorem 3.5.

**Proposition 3.8** (cf. [27, Prop. 7.7(a)]) If $(s_j)_{j=0}^{\kappa} \in \mathcal{F}_{q,k,\alpha,\beta}$, then $(s_j)_{j=0}^{m} \in \mathcal{F}_{q,m,\alpha,\beta}$ for all $m \in \mathbb{Z}_{0,k}$.

In view of Proposition 3.8, the definition of the class $\mathcal{F}_{q,\infty,\alpha,\beta}$ seems to be natural.

**Proposition 3.9** ([27, Prop. 9.1]) Let $(s_j)_{j=0}^{\kappa} \in \mathcal{F}_{q,\kappa,\alpha,\beta}$. If $\kappa \geq 1$, then $(a_j)_{j=0}^{\kappa-1}, (b_j)_{j=0}^{\kappa-1} \subseteq \mathcal{F}_{q,\kappa-1,\alpha,\beta}$. If $\kappa \geq 2$, then furthermore $(c_j)_{j=0}^{\kappa-2} \subseteq \mathcal{F}_{q,\kappa-2,\alpha,\beta}$.

We write $\mathcal{R}(A) := \{Ax : x \in \mathbb{C}^q\}$ and $\mathcal{N}(A) := \{x \in \mathbb{C}^q : Ax = O_{p \times 1}\}$ for the column space and the null space of a complex $p \times q$ matrix $A$, resp.

**Notation 3.10** Let $\mathcal{D}_{p \times q,\kappa}$ be the set of all sequences $(s_j)_{j=0}^{\kappa}$ of complex $p \times q$ matrices satisfying $\bigcup_{j \in \mathbb{Z}} \mathcal{R}(s_j) \subseteq \mathcal{R}(s_0)$ and $\mathcal{N}(s_0) \subseteq \bigcap_{j \in \mathbb{Z}} \mathcal{N}(s_j)$.

**Remark 3.11** ([29, Prop. 7.11]) Let $(s_j)_{j=0}^{\kappa} \in \mathcal{F}_{q,\kappa,\alpha,\beta}$. Then $(s_j)_{j=0}^{\kappa} \in \mathcal{D}_{q,\kappa,\alpha,\beta}$. If $\kappa \geq 1$, furthermore $\{(a_j)_{j=0}^{\kappa-1}, (b_j)_{j=0}^{\kappa-1}\} \subseteq \mathcal{D}_{q,\kappa-1,\alpha,\beta}$. If $\kappa \geq 2$, moreover $\{(c_j)_{j=0}^{\kappa-2}\} \subseteq \mathcal{D}_{q,\kappa-2,\alpha,\beta}$. If $\kappa \geq 2$,
Lemma 3.12 ([28, Lem. 5.7]) Let $(s_j)^{j=0}_{j=\kappa} \in \mathcal{F}_{q,k,\alpha,\beta}$. Then $s_j \in C^{q \times q}$ for all $j \in \mathbb{Z}_{0,\kappa}$ and $s_{2k} \in C^{q \times q}$ for all $k \in \mathbb{N}_0$ with $2k \leq \kappa$. Furthermore, $\alpha s_{2k} \preceq s_{2k+1} \preceq \beta s_{2k}$ for all $k \in \mathbb{N}_0$ with $2k+1 \leq \kappa$.

Let $O_{p \times q}$ be the zero matrix from $C^{p \times q}$ and let $I_q := [\delta_{jk}]_{j,k=1}^q$ be the identity matrix from $C^{q \times q}$, where $\delta_{jk}$ is the Kronecker delta. Sometimes, if the size is clear from the context, we will omit the indices and write $O$ and $I$, resp. Taking into account Remark A.27, we obtain from Lemma 3.12:

Remark 3.13 If $(s_j)^{j=0}_{j=\kappa} \in \mathcal{F}_{q,k,\alpha,\beta}$, then $\mathcal{R}(a_{2k}) \cup \mathcal{R}(b_{2k}) \subseteq \mathcal{R}(s_{2k})$ and $\mathcal{N}(s_{2k}) \subseteq \mathcal{N}(a_{2k}) \cap \mathcal{N}(b_{2k})$ for all $k \in \mathbb{N}_0$ with $2k \leq \kappa - 1$.

Finite sequences from $\mathcal{F}_{q,m,\alpha,\beta}$ can always be extended to sequences from $\mathcal{F}_{q,\ell,\alpha,\beta}$ for all $\ell \in \mathbb{Z}_{m+1,\infty}$, which is due to the fact that a Non-negative Hermitian measure on the bounded set $[\alpha, \beta]$ possesses power moments of all non-negative orders. One of the main results in [27] states that the possible one-step extensions $s_{m+1} \in C^{q \times q}$ of a sequence $(s_j)^{j=0}_{j=m}$ to an $[\alpha, \beta]$-non-negative definite sequence $(s_j)^{j=m+1}_{j=0}$ fill out a matricial interval. In order to give an exact description of this interval, we are now going to introduce several matrices and recall their role in the corresponding extension problem for $[\alpha, \beta]$-non-negative definite sequences, studied in [27].

Given an arbitrary $n \in \mathbb{N}$ and arbitrary rectangular complex matrices $A_1, A_2, \ldots, A_n$, we write $\text{col} (A_j)^{j=1}_{j=n} := \text{col}(A_1, A_2, \ldots, A_n)$ (resp., $\text{row} (A_j)^{j=1}_{j=n} := [A_1, A_2, \ldots, A_n]$) for the block column (resp., block row) built from the matrices $A_1, A_2, \ldots, A_n$ if their numbers of columns (resp., rows) are all equal.

Notation 3.14 Let $(s_j)^{j=0}_{j=\kappa}$ be a sequence of complex $p \times q$ matrices. For all $\ell, m \in \mathbb{N}_0$ with $\ell \leq m \leq \kappa$, then let $y_{\ell,m} := \text{col} (s_j)^{j=\ell}_{j=m}$ and $z_{\ell,m} := \text{row} (s_j)^{j=\ell}_{j=m}$.

The block Hankel matrix $H_n$ admits the following block representations:

Remark 3.15 If $\kappa \geq 2$ and if $(s_j)^{j=0}_{j=\kappa}$ is a sequence of complex $p \times q$ matrices, then $H_n = \begin{bmatrix} H_{n-1} & y_{n,2n-1} \\ z_{n,2n-1} & s_{2n} \end{bmatrix}$ and $H_n = \begin{bmatrix} z_{1,n}^T \\ y_{1,n} \\ G_{n-1} \end{bmatrix}$ for all $n \in \mathbb{N}$ with $2n \leq \kappa$.

In this paper, the Moore–Penrose inverse of a complex matrix plays an important role. For each matrix $A \in C^{p \times q}$, there exists a unique matrix $X \in C^{q \times p}$, satisfying the four equations

$$AXA = A, \quad XAX = X, \quad (AX)^* = AX, \quad \text{and} \quad (XA)^* = XA$$

(see e.g. [13, Prop. 1.1.1]). This matrix $X$ is called the Moore–Penrose inverse of $A$ and is denoted by $A^\dagger$. Concerning the concept of Moore–Penrose inverse we refer to [46], [4, Ch. 1], and [2, Ch. 1] (see also [13, Sec. 1.1]).
If \(
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\) is the block representation of a complex \((p + q) \times (r + s)\) matrix \(M\) with \(p \times r\) block \(A\), then the matrix

\[M/A := D - CA^\dagger B\]  \hspace{1cm} (3.4)

is called the Schur complement of \(A\) in \(M\). Concerning a variety of applications of this concept in a lot of areas of mathematics, we refer to [54]. In the paper, various kinds of concrete Schur complements in block matrices will play an essential role. By virtue of Remark 3.15, we use in the sequel the following notation:

**Notation 3.16** If \((s_j)_{j=0}^\kappa\) is a sequence of complex \(p \times q\) matrices, then let \(L_0 := H_0\) and let \(L_n := H_n/H_{n-1}\) for all \(n \in \mathbb{N}\) with \(2n \leq \kappa\).

We write \(\text{rank } A\) for the rank of a complex matrix \(A\) and \(\det B\) for the determinant of a square complex matrix \(B\).

**Remark 3.17** (cf. [21, Lem. 3.5]) Let \(n \in \mathbb{N}_0\) and let \((s_j)_{j=0}^{2n} \in \mathcal{H}_q, 2n\). Then \(\text{rank } H_n = \sum_{k=0}^n \text{rank } L_k\) and \(\det H_n = \prod_{k=0}^n \det L_k\).

**Notation 3.18** Let \((s_j)_{j=0}^\kappa\) be a sequence of complex \(p \times q\) matrices. Then let \(\Theta_0 := O_{p \times q}\) and \(\Theta_n := \alpha \cdot s_{2n-1} H_{n-1}^{\dagger} y_{n,2n-1}\) for all \(n \in \mathbb{N}\) with \(2n - 1 \leq \kappa\).

**Definition 3.19** If \((s_j)_{j=0}^\kappa\) is a sequence of complex \(p \times q\) matrices, then (using Notation 3.2) the sequences \((a_j)_{j=0}^\kappa\) and \((b_j)_{j=0}^\kappa\) given by \(a_{2k} := \alpha s_{2k} + \Theta_{\alpha,k,\bullet}\) and \(b_{2k} := \beta s_{2k} - \Theta_{\bullet,k,\beta}\) for all \(k \in \mathbb{N}_0\) with \(2k \leq \kappa\) and by \(a_{2k+1} := \Theta_{k+1}\) and \(b_{2k+1} := -\alpha \beta s_{2k} + (\alpha + \beta) s_{2k+1} - \Theta_{\alpha,k,\beta}\) for all \(k \in \mathbb{N}_0\) with \(2k + 1 \leq \kappa\) are called the sequence of left matricial interval endpoints associated with \((s_j)_{j=0}^\kappa\) and the sequence of right matricial interval endpoints associated with \((s_j)_{j=0}^\kappa\), resp.

By virtue of Notation 3.18, we have in particular

\[a_0 = \alpha s_0, \quad b_0 = \beta s_0, \quad a_1 = s_1 s_0^{\dagger}, \quad \text{and} \quad b_1 = -\alpha \beta s_0 + (\alpha + \beta) s_1.\]  \hspace{1cm} (3.5)

Using Lemma 3.12 and Remark A.14, we easily obtain:

**Remark 3.20** If \((s_j)_{j=0}^\kappa \in \mathcal{F}_{q,\kappa,\alpha,\beta}\), then \(\{a_j, b_j\} \subseteq \mathbb{C}^{q \times q}_{\mathbb{H}}\) for all \(j \in \mathbb{Z}_{0,\kappa}\).

Observe that for arbitrarily given Hermitian \(q \times q\) matrices \(A\) and \(B\), the (closed) matricial interval

\[\llbracket A, B \rrbracket := \{X \in \mathbb{C}^{q \times q}_{\mathbb{H}} : A \preceq X \preceq B\}\]  \hspace{1cm} (3.6)

is non-empty if and only if \(A \preceq B\).

**Theorem 3.21** ([27, Thm. 11.2(a)]) If \(m \in \mathbb{N}_0\) and \((s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}\), then the matricial interval \(\llbracket a_m, b_m \rrbracket\) is non-empty and coincides with the set of all complex \(q \times q\) matrices \(s_{m+1}\) for which \((s_j)_{j=0}^{m+1}\) belongs to \(\mathcal{F}_{q,m+1,\alpha,\beta}\).
**Definition 3.22** If \((s_j)_j^{\kappa} = 0\) is a sequence of complex \(p \times q\) matrices, then we call \((d_j)_j^{\kappa} = 0\) given by \(d_j := b_j - a_j\) the sequence of \([\alpha, \beta]\)-interval lengths associated with \((s_j)_j^{\kappa} = 0\).

By virtue of (3.5), we have in particular
\[
d_0 = \delta s_0 \quad \text{and} \quad d_1 = -\alpha\beta s_0 + (\alpha + \beta)s_1 - s_1 s_0^\top s_1. \tag{3.7}
\]

**Remark 3.23** Let \((s_j)_j^{\kappa} = 0\) be a sequence of complex \(p \times q\) matrices with sequence of \([\alpha, \beta]\)-interval lengths \((d_j)_j^{\kappa} = 0\). For each \(k \in \mathbb{Z}_{0,\kappa}\), the matrix \(d_k\) is built from the matrices \(s_0, s_1, \ldots, s_k\). In particular, for each \(m \in \mathbb{Z}_{0,\kappa}\), the sequence of \([\alpha, \beta]\)-interval lengths associated with \((s_j)_j^{m} = 0\) coincides with \((d_j)_j^{m} = 0\).

**Remark 3.24** ([29, Rem. 7.26]) Suppose \(\kappa \geq 1\). If \((s_j)_j^{\kappa} = 0 \in \mathcal{D}_{p \times q, \kappa}\), then \(d_1 = a_0 s_0 b_0\) and \(d_1 = b_0 s_0 a_0\).

**Definition 3.25** Let \((s_j)_j^{\kappa} = 0\) be a sequence of complex \(p \times q\) matrices. Then the sequence \((\mathcal{A}_j)_j^{\kappa} = 0\) given by \(\mathcal{A}_0 := s_0\) and by \(\mathcal{A}_j := s_j - a_{j-1}\) is called the sequence of lower Schur complements associated with \((s_j)_j^{\kappa} = 0\) and \([\alpha, \beta]\). Furthermore, if \(\kappa \geq 1\), then the sequence \((\mathcal{B}_j)_j^{\kappa} = 0\) given by \(\mathcal{B}_j := b_{j-1} - s_j\) is called the sequence of upper Schur complements associated with \((s_j)_j^{\kappa} = 0\) and \([\alpha, \beta]\).

Because of (3.5), we have in particular
\[
\mathcal{A}_1 = a_0, \quad \mathcal{B}_1 = b_0, \quad \text{and} \quad \mathcal{B}_2 = c_0. \tag{3.8}
\]

**Remark 3.26** Let \((s_j)_j^{\kappa} = 0\) be a sequence of complex \(p \times q\) matrices. Then \(\mathcal{A}_{2n} = L_n\) for all \(n \in \mathbb{N}_0\) with \(2n \leq \kappa\) and \(\mathcal{A}_{2n+1} = L_{\alpha, n, \bullet}\) for all \(n \in \mathbb{N}_0\) with \(2n + 1 \leq \kappa\). In particular, if \(n \geq 1\), then \(\mathcal{A}_{2n}\) is the Schur complement of \(H_{n-1}\) in \(H_n\) and \(\mathcal{A}_{2n+1}\) is the Schur complement of \(H_{\alpha, n-1, \bullet}\) in \(H_{\alpha, n, \bullet}\). Furthermore, \(\mathcal{B}_{2n+1} = L_{\bullet, n, \beta}\) for all \(n \in \mathbb{N}_0\) with \(2n + 1 \leq \kappa\) and \(\mathcal{B}_{2n+2} = L_{\alpha, n, \beta}\) for all \(n \in \mathbb{N}_0\) with \(2n + 2 \leq \kappa\). In particular, if \(n \geq 1\), then \(\mathcal{B}_{2n+1}\) is the Schur complement of \(H_{\bullet, n-1, \beta}\) in \(H_{\bullet, n, \beta}\) and \(\mathcal{B}_{2n+2}\) is the Schur complement of \(H_{\alpha, n-1, \beta}\) in \(H_{\alpha, n, \beta}\).

If \(A\) and \(B\) are two complex \(p \times q\) matrices, then the matrix
\[
A \mp B := A(A + B)^\dagger B \tag{3.9}
\]
is called the parallel sum of \(A\) and \(B\).

**Proposition 3.27** ([27, Thm. 10.14]) If \((s_j)_j^{\kappa} = 0 \in \mathcal{F}_{q, \kappa, \alpha, \beta}\), then \(d_0 = \delta \mathcal{A}_0\) and furthermore \(d_k = \delta (\mathcal{A}_k \mp \mathcal{B}_k)\) and \(d_k = \delta (\mathcal{B}_k \mp \mathcal{A}_k)\) for all \(k \in \mathbb{Z}_{1,\kappa}\).

**Proposition 3.28** ([27, Prop. 10.15(a)]) If \((s_j)_j^{\kappa} = 0 \in \mathcal{F}_{q, \kappa, \alpha, \beta}^\circ\), then \(d_j \in \mathbb{C}^{q \times q}\) for all \(j \in \mathbb{Z}_{0,\kappa}\).
Proposition 3.29 ([27, Prop. 10.18]) Let \((s_j)^{k}_{j=0} \in \mathcal{F}_{q,k,\alpha,\beta}^\infty\). Then \(\mathcal{R}(\varnothing_0) = \mathcal{R}((\mathcal{A}_0))\) and \(\mathcal{N}(\varnothing_0) = \mathcal{N}((\mathcal{A}_0))\). Furthermore, \(\mathcal{R}(\varnothing_j) = \mathcal{R}(\mathcal{A}_j) \cap \mathcal{R}(\mathcal{B}_j)\) and \(\mathcal{N}(\varnothing_j) = \mathcal{N}(\mathcal{A}_j) \cap \mathcal{N}(\mathcal{B}_j)\) for all \(j \in \mathbb{Z}_{1,k}\), and \(\mathcal{R}(\varnothing_j) = \mathcal{R}(\mathcal{A}_{j+1}) + \mathcal{R}(\mathcal{B}_{j+1})\) and \(\mathcal{N}(\varnothing_j) = \mathcal{N}(\mathcal{A}_{j+1}) \cap \mathcal{N}(\mathcal{B}_{j+1})\) for all \(j \in \mathbb{Z}_{0,k-1}\).

The ranks of the matrices considered in Proposition 3.29 are connected by means of the well-known formula for the dimension of the sum of two arbitrary finite-dimensional linear subspaces:

**Remark 3.30** If \(\mathcal{U}_1\) and \(\mathcal{U}_2\) are finite-dimensional linear subspaces of some vector space, then \(\dim(\mathcal{U}_1 + \mathcal{U}_2) = \dim \mathcal{U}_1 + \dim \mathcal{U}_2 - \dim(\mathcal{U}_1 \cap \mathcal{U}_2)\).

**Corollary 3.31** Let \((s_j)^{k}_{j=0} \in \mathcal{F}_{q,k,\alpha,\beta}^\infty\). Then \(\text{rank } \varnothing_0 = \text{rank } \mathcal{A}_0\) and \(\text{rank } \varnothing_j = \text{rank } \mathcal{A}_j + \text{rank } \mathcal{B}_j\) for all \(j \in \mathbb{Z}_{1,k}\).

**Proof** From Proposition 3.29 we obtain \(\text{rank } \varnothing_0 = \text{rank } \mathcal{A}_0\) and, for all \(j \in \mathbb{Z}_{1,k}\), furthermore \(\text{rank } \varnothing_j = \dim(\mathcal{R}(\mathcal{A}_j) + \mathcal{R}(\mathcal{B}_j))\) and \(\text{rank } \varnothing_j = \dim(\mathcal{R}(\mathcal{A}_j) \cap \mathcal{R}(\mathcal{B}_j))\). The application of Remark 3.30 to the linear subspaces \(\mathcal{R}(\mathcal{A}_j)\) and \(\mathcal{R}(\mathcal{B}_j)\) of the finite-dimensional vector space \(\mathbb{C}^q\) yields then \(\text{rank } \varnothing_{j-1} = \text{rank } \mathcal{A}_j + \text{rank } \mathcal{B}_j - \text{rank } \varnothing_j\). □

Using Corollary 3.31, we are able to derive certain relations between the ranks of the matrices \(\varnothing_j\) and the ranks of the underlying block Hankel matrices:

**Lemma 3.32** Let \((s_j)^{k}_{j=0} \in \mathcal{F}_{q,k,\alpha,\beta}^\infty\). Then \(\text{rank } \varnothing_0 = \text{rank } H_0\). Furthermore, \(\sum_{j=0}^{2n+1} \text{rank } \varnothing_j = \text{rank } H_{\alpha,n,\bullet} + \text{rank } H_{\bullet,n,\beta}\) for all \(n \in \mathbb{N}_0\) with \(2n + 1 \leq \kappa\) and \(\sum_{j=0}^{2n} \text{rank } \varnothing_j = \text{rank } H_n + \text{rank } H_{\alpha,n-1,\beta}\) for all \(n \in \mathbb{N}\) with \(2n \leq \kappa\).

**Proof** Because of \(H_0 = \varnothing_0 = L_0 = \mathcal{A}_0\) and Corollary 3.31, we have \(\text{rank } \varnothing_0 = \text{rank } H_0\). Now consider an arbitrary \(n \in \mathbb{N}_0\) with \(2n + 1 \leq \kappa\). From Proposition 3.8 and (3.2) we see that the sequences \((a_j)^{2n}_{j=0}\) and \((b_j)^{2n}_{j=0}\) both belong to \(\mathcal{H}_{q,2,n}^\infty\). Thus, we apply Remark 3.17 to obtain \(\text{rank } H_{\alpha,n,\bullet} = \sum_{k=0}^{n} \text{rank } L_{\alpha,k,\bullet}\) and \(\text{rank } H_{\bullet,n,\beta} = \sum_{k=0}^{n} \text{rank } L_{\bullet,k,\beta}\). Using Corollary 3.31 and Remark 3.26, we get then

\[
\sum_{k=0}^{n} \text{rank } \mathcal{A}_{2k+1} + \sum_{k=0}^{n} \text{rank } \mathcal{B}_{2k+1} = \text{rank } H_{\alpha,n,\bullet} + \text{rank } H_{\bullet,n,\beta}.
\]

Now consider an arbitrary \(n \in \mathbb{N}\) with \(2n \leq \kappa\). From Proposition 3.8 and (3.1) we infer \((s_j)^{2n}_{j=0} \in \mathcal{H}_{q,2,n}^\infty\) and \((c_j)^{2(n-1)}_{j=0} \in \mathcal{H}_{q,2(n-1)}^\infty\). Thus, Remark 3.17 yields \(\text{rank } H_n = \sum_{k=0}^{n} \text{rank } L_k\) and \(\text{rank } H_{\alpha,n-1,\beta} = \sum_{k=0}^{n-1} \text{rank } L_{\alpha,k,\beta}\). Using Corollary 3.31 and
Remark 3.26, we get then

\[ \sum_{\ell=0}^{2n} \text{rank } \mathfrak{d}_\ell = \text{rank } \mathfrak{d}_0 + \sum_{m=1}^{n} (\text{rank } \mathfrak{d}_{2m-1} + \text{rank } \mathfrak{d}_{2m}) = \text{rank } \mathfrak{A}_0 + \sum_{m=1}^{n} (\text{rank } \mathfrak{A}_{2m} + \text{rank } \mathfrak{B}_{2m}) = \sum_{k=0}^{n} \text{rank } \mathfrak{A}_{2k} + \sum_{k=0}^{n-1} \text{rank } \mathfrak{B}_{2k+2} = \text{rank } H_n + \text{rank } H_{\alpha,n-1,\beta}. \]

\[ \square \]

**Proposition 3.33** ([27, Cor. 10.21]) Let \((s_j)_{j=0}^{K} \in \mathcal{F}_{q,k,\alpha,\beta}^{\infty}\) and assume \(\kappa \geq 1\). For all \(j \in \mathbb{Z}_{1,\kappa}\), then \(\mathfrak{d}_j = \delta \mathfrak{A}_j \delta_{j-1}^t \mathfrak{B}_j\) and \(\mathfrak{d}_j = \delta \mathfrak{B}_j \delta_{j-1}^t \mathfrak{A}_j\).

**Corollary 3.34** If \((s_j)_{j=0}^{K} \in \mathcal{F}_{q,k,\alpha,\beta}^{\infty}\), then \(\text{det } \mathfrak{d}_0 = \delta^q \text{det } \mathfrak{A}_0\) and, for all \(j \in \mathbb{Z}_{1,\kappa}\), furthermore

\[ \text{det } \mathfrak{d}_{j-1} \text{det } \mathfrak{d}_j = \delta^q \text{det } \mathfrak{A}_j \text{det } \mathfrak{B}_j. \quad (3.10) \]

**Proof** Because of Proposition 3.27 we have \(\text{det } \mathfrak{d}_0 = \delta^q \text{det } \mathfrak{A}_0\). Now assume \(\kappa \geq 1\) and let \(j \in \mathbb{Z}_{1,\kappa}\). First we consider the case that \(\text{det } \mathfrak{d}_{j-1} = 0\). From Proposition 3.29 we can infer \(\mathcal{N}(\mathfrak{d}_{j-1}) \subseteq \mathcal{N}(\mathfrak{A}_j)\). Consequently, \(\text{det } \mathfrak{A}_j = 0\) follows. Hence, (3.10) is fulfilled. Now we consider the case \(\text{det } \mathfrak{d}_{j-1} \neq 0\). In view of Remark A.13, then (3.10) is a consequence of Proposition 3.33.

\[ \square \]

**Lemma 3.35** Let \((s_j)_{j=0}^{K} \in \mathcal{F}_{q,k,\alpha,\beta}^{\infty}\). Then \(\text{det } \mathfrak{d}_0 = \delta^q \text{det } H_0\). Furthermore, \(\prod_{\ell=0}^{2n+1} \text{det } \mathfrak{d}_\ell = \delta^{(n+1)q} \text{det } (H_{\alpha,n,\bullet}) \text{det } (H_{\bullet,n,\beta})\) for all \(n \in \mathbb{N}_0\) with \(2n + 1 \leq \kappa\) and \(\prod_{\ell=0}^{2n} \text{det } \mathfrak{d}_\ell = \delta^{(n+1)q} \text{det } (H_n) \text{det } (H_{\alpha,n-1,\beta})\) for all \(n \in \mathbb{N}\) with \(2n \leq \kappa\).

**Proof** Because of \(H_0 = s_0 = L_0 = \mathfrak{A}_0\) and Corollary 3.34 we have \(\text{det } \mathfrak{d}_0 = \delta^q \text{det } H_0\). Now consider an arbitrary \(n \in \mathbb{N}_0\) with \(2n + 1 \leq \kappa\). With the same reasoning as in Lemma 3.32, we can infer from Remark 3.17 then \(\text{det } H_{\alpha,n,\bullet} = \prod_{k=0}^{n} \text{det } L_{\alpha,k,\bullet}\) and \(\text{det } H_{\bullet,n,\beta} = \prod_{k=0}^{n} \text{det } L_{\bullet,k,\beta}\). Using Corollary 3.34 and Remark 3.26, we get then

\[ \prod_{\ell=0}^{2n+1} \text{det } \mathfrak{d}_\ell = \prod_{k=0}^{n} (\text{det } \mathfrak{d}_{2k} \text{det } \mathfrak{d}_{2k+1}) = \prod_{k=0}^{n} (\delta^q \text{det } \mathfrak{A}_{2k+1} \text{det } \mathfrak{B}_{2k+1}) = \delta^{(n+1)q} \prod_{k=0}^{n} \text{det } \mathfrak{A}_{2k+1} \prod_{k=0}^{n} \text{det } \mathfrak{B}_{2k+1} = \delta^{(n+1)q} \text{det } H_{\alpha,n,\bullet} \text{det } H_{\bullet,n,\beta}. \]

Now consider an arbitrary \(n \in \mathbb{N}\) with \(2n \leq \kappa\). With the same reasoning as in Lemma 3.32, we can conclude from Remark 3.17 analogously \(\text{det } H_n = \prod_{k=0}^{n} \text{det } L_k\).
and $\det H_{\alpha,n-1,\beta} = \prod_{k=0}^{n-1} \det L_{\alpha,k,\beta}$. Using Corollary 3.34 and Remark 3.26, we obtain then

$$
\prod_{\ell=0}^{2n} \det \varnothing_{\ell} = \det \varnothing_{0} \prod_{m=1}^{n} (\det \varnothing_{2m-1} \det \varnothing_{2m})
= \delta^q \det \varnothing_{0} \prod_{m=1}^{n} (\delta^q \det \varnothing_{2m} \det \varnothing_{2m})
= \delta^{(n+1)q} \prod_{k=0}^{n-1} \det \varnothing_{2k} \prod_{k=0}^{n-1} \det \varnothing_{2k+2} = \delta^{(n+1)q} \det H_{n} \det H_{\alpha,n-1,\beta}. \quad \Box
$$

Now we state a consequence of Theorem 3.21.

**Corollary 3.36** (cf. [28, Cor. 5.25]) Let $m \in \mathbb{N}_0$, let $(s_j)_{j=0}^{m} \in \mathcal{F}_{q,m,\alpha,\beta}$, let $\lambda \in [0, 1]$, and let $s_{m+1} := a_m + \lambda \varnothing_m$. Then, the sequence $(s_j)_{j=0}^{m+1}$ belongs to $\mathcal{F}_{q,m+1,\alpha,\beta}$. Furthermore, $\varnothing_{m+1} = \lambda \varnothing_m$, $\varnothing_{m+1} = (1-\lambda)\varnothing_m$, and $\varnothing_{m+1} = \delta \lambda (1-\lambda) \varnothing_m$.

In [28, Def. 6.1], we subsumed the Schur complements mentioned in Remark 3.26 to a parameter sequence:

**Definition 3.37** Let $(s_j)_{j=0}^{k}$ be a sequence of complex $p \times q$ matrices. Let the sequence $(f_j)_{j=0}^{2k}$ be given by $f_0 := \varnothing_0$, by $f_{4k+1} := \varnothing_{2k+1}$ and $f_{4k+2} := \varnothing_{2k+1}$ for all $k \in \mathbb{N}_0$ with $2k + 1 \leq \kappa$, and by $f_{4k+3} := \varnothing_{2k+2}$ and $f_{4k+4} := \varnothing_{2k+2}$ for all $k \in \mathbb{N}_0$ with $2k + 2 \leq \kappa$. Then we call $(f_j)_{j=0}^{2k}$ the $\mathcal{F}_{\alpha,\beta}$-parameter sequence of $(s_j)_{j=0}^{n}$.

In view of (3.8) and (3.5), we have in particular

$$
f_0 = s_0, \quad f_1 = a_0 = s_1 - \alpha s_0, \quad \text{and} \quad f_2 = b_0 = \beta s_0 - s_1. \quad (3.11)
$$

The $[\alpha, \beta]$-non-negative definiteness as well as rank constellations among the Non-negative Hermitian block Hankel matrices $H_{\alpha, \alpha, \alpha, \alpha}$, $H_{\alpha, n, \alpha, \alpha}$, $H_{\alpha, n, \alpha, \alpha}$, and $H_{\alpha, n, \alpha, \alpha}$ can be characterized in terms of $\mathcal{F}_{\alpha,\beta}$-parameters (cf. [28, Propositions 6.13 and 6.14]).

**Remark 3.38** Let $(s_j)_{j=0}^{k}$ be a sequence of complex $p \times q$ matrices with $\mathcal{F}_{\alpha,\beta}$-parameter sequence $(f_j)_{j=0}^{2k}$. Then $f_0 = s_0$, furthermorn, for each $k \in \mathbb{Z}_{1,\kappa}$, the matrices $f_{2k-1}$ and $f_{2k}$ are built from the matrices $s_0, s_1, \ldots, s_k$. In particular, for each $m \in \mathbb{Z}_{0,\kappa}$, the $\mathcal{F}_{\alpha,\beta}$-parameter sequence of $(s_j)_{j=0}^{m}$ coincides with $(f_j)_{j=0}^{2m}$.

**Proposition 3.39** ([28, Prop. 6.14]) Let $(s_j)_{j=0}^{k}$ be a sequence of complex $q \times q$ matrices. Then $(s_j)_{j=0}^{k} \in \mathcal{F}_{q,k,\alpha,\beta}$ if and only if $f_j \in \mathbb{C}_{q \times q}$ for all $j \in \mathbb{Z}_{0,2k}$.

**Remark 3.40** ([28, Rem. 6.16]) Let $(s_j)_{j=0}^{k}$ be a sequence of complex $p \times q$ matrices. For all $k \in \mathbb{Z}_{1,\kappa}$, then $f_{2k-1} = \varnothing_{k-1} - f_{2k}$.
To single out all sequences \((f_j)_{j=0}^{2\kappa}\) of complex \(q \times q\) matrices which indeed occur as \(\mathcal{F}_{\alpha,\beta}\)-parameters of sequences \((s_j)_{j=0}^{\kappa} \in \mathcal{F}_{q,k,\alpha,\beta}^\kappa\), we introduced in [28, Notation 6.19] the following class:

**Notation 3.41** For each \(\eta \in [0, \infty)\), denote by \(C_{q,k,\eta}^\kappa\) the set of all sequences \((f_j)_{j=0}^{2\kappa}\) of non-negative Hermitian \(q \times q\) matrices satisfying, in the case \(\kappa \geq 1\), the equations \(\eta f_0 = f_1 + f_2\) and \(\eta(f_{2k-1} \ast f_{2k}) = f_{2k+1} + f_{2k+2}\) for all \(k \in \mathbb{Z}_{1,\kappa-1}\).

**Theorem 3.42** (cf. [28, Thm. 6.20]) The mapping \(\Gamma_{\alpha,\beta} : \mathcal{F}_{q,k,\alpha,\beta}^\kappa \to C_{q,k,\alpha,\beta}^\kappa\) given by \((s_j)_{j=0}^{\kappa} \mapsto (f_j)_{j=0}^{2\kappa}\) is well defined and bijective.

For each matrix \(A \in \mathbb{C}^{q \times q}\), there exists a uniquely determined matrix \(Q \in \mathbb{C}^{q \times q}\) with \(Q^2 = A\) called the non-negative Hermitian square root of \(A\). To uncover relations between the \(\mathcal{F}_{\alpha,\beta}\)-parameters \((f_j)_{j=0}^{2\kappa}\) and to obtain a parametrization of the set \(\mathcal{F}_{q,k,\alpha,\beta}^\kappa\), we introduced in [28, Def. 6.21 and Notation 6.28] another parameter sequence \((e_j)_{j=0}^{\kappa}\) and a corresponding class \(E_{q,k,\eta}^\kappa\) of sequences of complex matrices. (Observe that these constructions are well defined due to Proposition 3.28 and Remark A.25.)

**Definition 3.43** Let \((s_j)_{j=0}^{\kappa} \in \mathcal{F}_{q,k,\alpha,\beta}^\kappa\) with \(\mathcal{F}_{\alpha,\beta}\)-parameter sequence \((f_j)_{j=0}^{2\kappa}\) and sequence of \([\alpha, \beta]\)-interval lengths \((\delta_j)_{j=0}^{\kappa}\). Then we call \((e_j)_{j=0}^{\kappa}\) given by \(e_0 := f_0\) and by \(e_j := (\delta_j^{-1})^j f_j (\delta_j^{-1})^j\) for each \(j \in \mathbb{Z}_{1,\kappa}\) the \([\alpha, \beta]\)-interval parameter sequence of \((s_j)_{j=0}^{\kappa}\).

**Lemma 3.44** (cf. [28, Prop. 6.27]) If \((s_j)_{j=0}^{\kappa} \in \mathcal{F}_{q,k,\alpha,\beta}^\kappa\), then \(\tilde{f}_2 j = \delta_{j-1}^{1/2} e_j \delta_{j-1}^{1/2}\) for all \(j \in \mathbb{Z}_{1,\kappa}\).

With the Euclidean scalar product \(\langle \cdot, \cdot \rangle_E : \mathbb{C}^q \times \mathbb{C}^q \to \mathbb{C}\) given by \(\langle x, y \rangle_E := y^* x\), which is \(\mathbb{C}\)-linear in its first argument, the \(\mathbb{C}\)-vector space \(\mathbb{C}^q\) becomes a unitary space. Let \(\mathcal{U}\) be an arbitrary non-empty subset of \(\mathbb{C}^q\). The orthogonal complement \(\mathcal{U}^\perp := \{ v \in \mathbb{C}^q : \langle v, u \rangle_E = 0 \text{ for all } u \in \mathcal{U} \}\) of \(\mathcal{U}\) is a linear subspace of the unitary space \(\mathbb{C}^q\). If \(\mathcal{U}\) is a linear subspace itself, the unitary space \(\mathbb{C}^q\) is the orthogonal sum of \(\mathcal{U}\) and \(\mathcal{U}^\perp\). In this case, we write \(P_{\mathcal{U}}\) for the transformation matrix corresponding to the orthogonal projection onto \(\mathcal{U}\) with respect to the standard basis of \(\mathbb{C}^q\), i.e., \(P_{\mathcal{U}}\) is the uniquely determined matrix \(P \in \mathbb{C}^{q \times q}\) satisfying the three conditions \(P^2 = P\), \(P^* = P\), and \(\mathcal{R}(P) = \mathcal{U}\).

**Notation 3.45** For each \(\eta \in [0, \infty)\), let \(E_{q,k,\eta}^\kappa\) be the set of all sequences \((e_k)_{k=0}^{\kappa}\) from \(\mathbb{C}^{q \times q}\) which fulfill the following condition: If \(\kappa \geq 1\), then \(e_k \equiv P_{\mathcal{R}(d_{k-1})} (e_k) d_{k-1}^{1/2} d_k^{1/2}\) for all \(k \in \mathbb{Z}_{1,\kappa}\), where the sequence \((d_k)_{k=0}^{\kappa}\) is recursively given by \(d_0 := \eta e_0\) and

\[
d_k := \eta d_{k-1}^{1/2} e_k^{1/2} (P_{\mathcal{R}(d_{k-1})} - e_k) e_k^{1/2} d_{k-1}^{1/2}.
\]

Regarding Theorem 3.21, in the case \(q = 1\) (cf. [12, Sec. 1.3]), the (classical) canonical moments \(p_1, p_2, p_3, \ldots\) of a point in the moment space corresponding to a
probability measure $\mu$ on $[\alpha, \beta] = [0, 1]$ are given in our notation by

$$p_k = \frac{s_k - \alpha_{k-1}}{b_{k-1} - \alpha_{k-1}} = \frac{\mathcal{A}_k}{\mathcal{O}_{k-1}}, \quad k \in \mathbb{N},$$

where the sequence $(s_j)_{j=0}^{\infty}$ of power moments $s_j := \int_{[0,1]} x^j \mu(dx)$ associated with $\mu$ is $[0, 1]$-non-negative definite with $s_0 = 1$. Observe that the $[0, 1]$-interval parameters $(\varepsilon_j)_{j=0}^{\infty}$ of $(s_j)_{j=0}^{\infty}$ are connected to the canonical moments via

$$p_1 = 1 - \varepsilon_1, \quad p_2 = \varepsilon_2, \quad p_3 = 1 - \varepsilon_3, \quad p_4 = \varepsilon_4, \quad p_5 = 1 - \varepsilon_5, \quad \ldots \quad (3.12)$$

The quantities $q_k = 1 - p_k$ occur in the classical framework as well (see, e.g. [12, Sec. 1.3]). In the general case $q \in \mathbb{Q}$ we have the following:

**Theorem 3.46** ([28, Thm. 6.30]) The mapping $\Sigma_{\alpha, \beta} : \mathcal{F}_{\infty, q, \kappa, \alpha, \beta} \rightarrow \mathcal{E}_{\infty, q, \kappa, \delta}$ given by $(s_j)_{j=0}^{\infty} \mapsto (\varepsilon_j)_{j=0}^{\infty}$ is well defined and bijective.

**Proposition 3.47** (cf. [28, Prop. 6.32]) Let $(s_j)_{j=0}^{\infty} \in \mathcal{F}_{q, \kappa, \alpha, \beta}$ and $\varepsilon_j \in \| O_q \times q, P \mathcal{R}(\delta_{j-1}) \|$ for all $j \in \mathbb{Z}_{1, \kappa}$. Furthermore, $\delta_0 = \delta_\varepsilon_0$ and $\delta_j = \delta_{\varepsilon_{j-1}}^{1/2} \delta_{\varepsilon_j}^{1/2} (P \mathcal{R}(\delta_{j-1}) - \varepsilon_j) \delta_{\varepsilon_j}^{-1/2}$ for all $j \in \mathbb{Z}_{1, \kappa}$.

**Example 3.48** (cf. [29, Example 7.35]) Let $\lambda \in (0, 1)$, let $B \in \mathbb{C}^{q \times q}$, and let $P := \mathcal{P} \mathcal{R}(B)$. Then $(\varepsilon_j)_{j=0}^{\infty}$ given by $\varepsilon_0 := B$ and by $\varepsilon_j := \lambda P$ for all $j \in \mathbb{N}$ is the $[\alpha, \beta]$-interval parameter sequence of a sequence $(s_j)_{j=0}^{\infty} \in \mathcal{F}_{q, \infty, \alpha, \beta}$.

We continue by recalling the construction of a certain transformation for sequences of matrices. This transformation was introduced in [29] and constitutes the elementary step of a Schur type algorithm in the class of $[\alpha, \beta]$-non-negative definite sequences, i.e., Hausdorff moment sequences:

**Definition 3.49** Let $(s_j)_{j=0}^{\infty}$ be a sequence of complex $p \times q$ matrices. Further let $b_{-1} := -s_0$ and, in the case $\kappa \geq 1$, let $(b_j)_{j=0}^{\kappa-1}$ be given by Notation 3.2. Then we call the sequence $(b_j)_{j=0}^{\kappa}$ given by $b_j := b_{j-1}$ the $(-\infty, \beta]$-modification of $(s_j)_{j=0}^{\kappa}$.

In particular, if $\beta = 0$, then $(b_j)_{j=0}^{\kappa}$ coincides with the sequence $(-s_j)_{j=0}^{\kappa}$. For an arbitrary $\beta \in \mathbb{R}$, the sequence $(s_j)_{j=0}^{\kappa}$ is reconstructible from $(b_j)_{j=0}^{\kappa}$ as well.

Let $(s_j)_{j=0}^{\kappa}$ and $(t_j)_{j=0}^{\kappa}$ be sequences of complex $p \times q$ and $q \times r$ matrices, resp. As usual, the Cauchy product $(x_j)_{j=0}^{\kappa}$ of $(s_j)_{j=0}^{\kappa}$ and $(t_j)_{j=0}^{\kappa}$ is given by $x_j := \sum_{\ell=0}^{j} s_{\ell} t_{j-\ell}$. We call the sequence $(s_j)_{j=0}^{\kappa}$ defined by $s_0^\dagger := s_0$ and, for all $j \in \mathbb{Z}_{1, \kappa}$, recursively by $s_j^\dagger := -s_j^\dagger \sum_{\ell=0}^{j-1} s_{j-\ell} s_{\ell}^\dagger$ the reciprocal sequence associated to $(s_j)_{j=0}^{\kappa}$.
Remark 3.51 Let \((s_j)^{\kappa}_{j=0}\) be a sequence of complex \(p \times q\) matrices with reciprocal sequence \((r_j)^{\kappa}_{j=0}\). For each \(k \in \mathbb{Z}_{0,\kappa}\), then the matrix \(r_k\) is built from the matrices \(s_0, s_1, \ldots, s_k\). In particular, for each \(m \in \mathbb{Z}_{0,\kappa}\), the reciprocal sequence associated to \((s_j)^{m}_{j=0}\) coincides with \((r_j)^{m}_{j=0}\).

Using the Cauchy product and the reciprocal sequence associated to the \((-\infty, \beta]\)-modification of \((a_j)^{\kappa-1}_{j=0}\), we introduce now a transformation of sequences of complex matrices:

Definition 3.52 Suppose \(\kappa \geq 1\). Let \((s_j)^{\kappa}_{j=0}\) be a sequence of complex \(p \times q\) matrices. Denote by \((g_j)^{\kappa-1}_{j=0}\) the \((-\infty, \beta]\)-modification of \((a_j)^{\kappa-1}_{j=0}\) and by \((x_j)^{\kappa-1}_{j=0}\) the Cauchy product of \((b_j)^{\kappa-1}_{j=0}\) and \((g_j)^{\kappa-1}_{j=0}\). Then we call the sequence \((t_j)^{\kappa-1}_{j=0}\) given by \(t_j := -a_0 s_0 \delta x_j a_0\) the \(\mathcal{F}_{\alpha,\beta}\)-transform of \((s_j)^{\kappa}_{j=0}\).

Since, in the classical case that \(\alpha = 0\) and \(\beta = 1\), the sequence \((a_j)^{\kappa-1}_{j=0}\) coincides with the shifted sequence \((s_{j+1})^{\kappa-1}_{j=0}\), the \(\mathcal{F}_{0,1}\)-transform is given by \(t_j = -s_1 s_0^{\dagger} x_j s_1\) with the Cauchy product \((x_j)^{\kappa-1}_{j=0}\) of \((b_j)^{\kappa-1}_{j=0}\) and \((g_j)^{\kappa-1}_{j=0}\), where the sequence \((b_j)^{\kappa-1}_{j=0}\) is given by \(b_j = s_{j-1} - s_{j+1}\) and the sequence \((g_j)^{\kappa-1}_{j=0}\) is given by \(g_0 = -s_1\) and by \(g_j = s_j - s_{j+1}\) for \(j \in \mathbb{Z}_{1,\kappa-1}\).

Remark 3.53 Assume \(\kappa \geq 1\). Let \((s_j)^{\kappa}_{j=0}\) be a sequence of complex \(p \times q\) matrices with \(\mathcal{F}_{\alpha,\beta}\)-transform \((t_j)^{\kappa-1}_{j=0}\). Then one can see from Remark 3.51 that, for each \(m \in \mathbb{Z}_{0,\kappa}\), the \(\mathcal{F}_{\alpha,\beta}\)-transform of \((s_j)^{m}_{j=0}\) coincides with \((t_j)^{m-1}_{j=0}\).

Lemma 3.54 ([29, Lem. 8.29]) Suppose \(\kappa \geq 1\). Let \((s_j)^{\kappa}_{j=0}\) be a \(\mathcal{D}_{p \times q, \kappa}\). Denote by \((t_j)^{\kappa-1}_{j=0}\) the \(\mathcal{F}_{\alpha,\beta}\)-transform of \((s_j)^{\kappa}_{j=0}\). Then \(t_0 = \delta_1\), where \(\delta_1\) is given by Definition 3.22.

Using the parallel sum given via (3.9), the effect caused by \(\mathcal{F}_{\alpha,\beta}\)-transformation on the \(\mathcal{F}_{\alpha,\beta}\)-parameters can be completely described:

Corollary 3.55 ([29, Cor. 9.10]) Assume \(\kappa \geq 1\) and let \((s_j)^{\kappa}_{j=0} \in \mathcal{F}_{q, \kappa, \alpha, \beta}^{\geq}\) with \(\mathcal{F}_{\alpha,\beta}\)-transform \((t_j)^{\kappa-1}_{j=0}\). Denote by \((g_j)^{2(\kappa-1)}_{j=0}\) the \(\mathcal{F}_{\alpha,\beta}\)-parameter sequence of \((t_j)^{\kappa-1}_{j=0}\). Then \(g_0 = \delta(\delta_1 \oplus \delta_2)\) and \(g_j = \delta \delta_{j+2}\) for all \(j \in \mathbb{Z}_{1,2(\kappa-1)}\).

We are now going to iterate the \(\mathcal{F}_{\alpha,\beta}\)-transformation:

Definition 3.56 Let \((s_j)^{\kappa}_{j=0}\) be a sequence of complex \(p \times q\) matrices. Let the sequence \((s_j^{[0]})^{\kappa}_{j=0}\) be given by \(s_j^{[0]} := s_j\). If \(\kappa \geq 1\), then, for all \(k \in \mathbb{Z}_{0,\kappa}\), let the sequence \((s_j^{[k]})^{\kappa-k}_{j=0}\) be recursively defined to be the \(\mathcal{F}_{\alpha,\beta}\)-transform of the sequence \((s_j^{[k-1]})^{\kappa-(k-1)}_{j=0}\). For all \(k \in \mathbb{Z}_{0,\kappa}\), then we call the sequence \((s_j^{[k]})^{\kappa-k}_{j=0}\) the \(k\)-th \(\mathcal{F}_{\alpha,\beta}\)-transform of \((s_j)^{\kappa}_{j=0}\).
**Remark 3.57** Suppose $\kappa \geq 1$. Let $(s_j)_{j=0}^k$ be a sequence of complex $p \times q$ matrices. Then $(s_j)_{j=0}^{k-1}$ is exactly the $\mathcal{F}_{\alpha,\beta}$-transform of $(s_j)_{j=0}^k$ from Definition 3.56.

**Remark 3.58** Let $k \in \mathbb{Z}_{0,\kappa}$ and let $(s_j)_{j=0}^k$ be a sequence of complex $p \times q$ matrices with $k$-th $\mathcal{F}_{\alpha,\beta}$-transform $(u_j)_{j=0}^{k-1}$. In view of Remark 3.53, we see that, for each $\ell \in \mathbb{Z}_{0,\kappa-k}$, the matrix $u_{\ell}$ is built only from the matrices $s_0, s_1, \ldots, s_{\ell+k}$. In particular, for each $m \in \mathbb{Z}_{k,\kappa}$, the $k$-th $\mathcal{F}_{\alpha,\beta}$-transform of $(s_j)_{j=0}^m$ coincides with $(u_j)_{j=0}^{m-k}$.

**Proposition 3.59** ([29, Thm. 9.4]) If $(s_j)_{j=0}^k \in \mathcal{F}_{q,\kappa,\alpha,\beta}^\prec$, then $(s_j)_{j=0}^{k-k} \in \mathcal{F}_{q,\kappa-k,\alpha,\beta}$ for all $k \in \mathbb{Z}_{0,\kappa}$.

**Proposition 3.60** ([29, Prop. 9.8]) Let $(s_j)_{j=0}^k \in \mathcal{F}_{q,\kappa,\alpha,\beta}^\succ$, with $\mathcal{F}_{\alpha,\beta}$-parameter sequence $(f_j)_{j=0}^{2k}$. Then $f_0 = s_0$ and furthermore $f_{4k+1} = \delta^{-2k}a_0^{(2k)}$ and $f_{4k+2} = \delta^{-2k}b_0^{(2k)}$ for all $k \in \mathbb{N}_0$ with $2k + 1 \leq \kappa$ and $f_{4k+3} = \delta^{-(2k+1)}a_0^{(2k+1)}$ and $f_{4k+4} = \delta^{-(2k+1)}b_0^{(2k+1)}$ for all $k \in \mathbb{N}_0$ with $2k + 2 \leq \kappa$.

In the following result, we express the sequence of $[\alpha, \beta]$-interval lengths introduced in Definition 3.22 by the $\mathcal{F}_{\alpha,\beta}$-transforms of an $[\alpha, \beta]$-non-negative definite sequence:

**Corollary 3.61** ([29, Cor. 9.9]) If $(s_j)_{j=0}^k \in \mathcal{F}_{q,\kappa,\alpha,\beta}^\prec$, then $\delta_j = \delta^{-j-1}s_j^{(0)}$ for all $j \in \mathbb{Z}_{0,\kappa}$.

**Proposition 3.62** ([29, Prop. 9.11]) Let $k \in \mathbb{Z}_{0,\kappa}$ and let $(s_j)_{j=0}^k \in \mathcal{F}_{q,\kappa,\alpha,\beta}^\prec$ with sequence of $[\alpha, \beta]$-interval lengths $(\delta_j)_{j=0}^k$ and $k$-th $\mathcal{F}_{\alpha,\beta}$-transform $(s_j)_{j=0}^{k-k}$. Then $(\delta^k\delta_{k+j})_{j=0}^{k-k}$ coincides with the sequence of $[\alpha, \beta]$-interval lengths associated with $(s_j)_{j=0}^{k-k}$.

**Theorem 3.63** ([29, Thm. 9.14]) Let $(s_j)_{j=0}^k \in \mathcal{F}_{q,\kappa,\alpha,\beta}^\prec$ with sequence of $[\alpha, \beta]$-interval lengths $(\delta_j)_{j=0}^k$ and $[\alpha, \beta]$-interval parameter sequence $(\epsilon_j)_{j=0}^k$. Let $k \in \mathbb{Z}_{0,\kappa}$. Then the $k$-th $\mathcal{F}_{\alpha,\beta}$-transform $(s_j)_{j=0}^{k-k}$ of $(s_j)_{j=0}^k$ belongs to $\mathcal{F}_{q,\kappa-k,\alpha,\beta}^\succ$ and the $[\alpha, \beta]$-interval parameter sequence $(p_j)_{j=0}^{k-k}$ of $(s_j)_{j=0}^{k-k}$ fulfills $p_0 = \delta^{k-1}\delta_k$ and $p_j = \epsilon_{k+j}$ for all $j \in \mathbb{Z}_{1,\kappa-k}$.

Now we look for a characterization of the fixed points of the $\mathcal{F}_{\alpha,\beta}$-transformation:

**Corollary 3.64** Suppose $\kappa \geq 1$. Let $(s_j)_{j=0}^k \in \mathcal{F}_{q,\kappa,\alpha,\beta}^\succ$ with $[\alpha, \beta]$-interval parameter sequence $(\epsilon_j)_{j=0}^k$ and $\mathcal{F}_{\alpha,\beta}$-transform $(t_j)_{j=0}^{k-1}$. Then $t_0 = s_0$ if and only if $\epsilon_1 - \epsilon_2 = \delta^{-2}\mathbb{P}\mathcal{R}(s_0)$.

**Proof** Proposition 3.47 yields $\delta_0 = \delta t_0$ and $\delta_1 = \delta^2\delta_0^{1/2}\epsilon_1^{1/2}(\mathbb{P}\mathcal{R}(s_0) - \epsilon_1)\epsilon_1^{1/2}\delta_0^{1/2}$. Because of Lemma 3.12, the matrix $s_0$ is Non-negative Hermitian. According to (3.7), we have $\delta_0 = \delta s_0$. In view of $\delta > 0$, then $\delta_0^{1/2} = \delta^{1/2}s_0^{1/2}$ as well as $\mathcal{R}(\delta_0) = \mathcal{R}(s_0)$ and $\mathcal{N}(\delta_0) = \mathcal{N}(s_0)$ follow. Consequently, we can conclude $\delta_1 = \ldots = \delta^{k-1}\delta_k = \ldots = \delta^{k-1}\delta_k = \ldots = \delta^{k-1}\delta_k$.
$\delta^2 s_0^{1/2} e_1^{1/2} (p_{R(s_0)} - e_1) e_1^{1/2} s_0^{1/2}$. Using Remark A.14, we can infer from Definition 3.43 furthermore $R(e_1) \subseteq R((0,0)^\dagger) = R((0,0)^\dagger) = R(0,0) = R(s_0)$ and, similarly, $N(s_0) \subseteq N(e_1)$. In particular, $R(e_1^{1/2}) = R(e_1) \subseteq R(s_0)$ follows, implying $e_1^{1/2} p_{R(s_0)} = e_1^{1/2}$. Hence, $\vartheta_1 = \delta^2 s_0^{1/2} (e_1 - e_1^2) s_0^{1/2}$. Therefore, $R(\vartheta_1) \subseteq R(s_0^{1/2})$ and $N(s_0^{1/2}) \subseteq N(\vartheta_1)$: 1. By virtue of Definition 3.43 and (3.11), we have $\vartheta_1 \subseteq R(s_0^{1/2})$, hence, it remains to show that $\delta^2 s_0^{1/2} (e_1 - e_1^2) s_0^{1/2} = s_0$ is equivalent to $e_1 - e_1^2 = \delta^{-2} p_{R(s_0)}$.

First assume that $\delta^2 s_0^{1/2} (e_1 - e_1^2) s_0^{1/2} = s_0$ is fulfilled. Taking into account Remark A.14, we see $R(e_1) \subseteq R(s_0^{1/2}) = R((s_0^{1/2})^\dagger) = R((s_0^{1/2})^\dagger)$ and similarly $N((s_0^{1/2})^\dagger) \subseteq N(e_1)$. Thus, we can infer from Remarks A.20, A.21, and A.14 that $\delta^2 (e_1 - e_1^2) = (s_0^{1/2})^\dagger s_0 (s_0^{1/2})^\dagger$. Because of $R(s_0) = R(s_0^{1/2}) = R((s_0^{1/2})^\dagger)$, the application of Remark A.18 yields $s_0 (s_0^{1/2})^\dagger = p_{R((s_0^{1/2})^\dagger)} = p_{R(s_0^{1/2})} = p_{R(s_0)}$. Consequently, $e_1 - e_1^2 = \delta^{-2} p_{R(s_0)}$ holds true. In view of $R(s_0^{1/2}) = R(s_0)$, we have $s_0^{1/2} p_{R(s_0)} = s_0^{1/2}$. Thus, we obtain $\delta^2 s_0^{1/2} (e_1 - e_1^2) s_0^{1/2} = s_0^{1/2} p_{R(s_0)} s_0^{1/2} = s_0$.

**Corollary 3.65** Let $(s_j)_{j=0}^{\infty} \in F_{q,\infty,\alpha,\beta}$ with $[\alpha, \beta]$-interval parameter sequence $(\epsilon_j)_{j=0}^{\infty}$ and $F_{a,\beta}$-transform $(t_j)_{j=0}^{\infty}$. Then the following statements are equivalent:

(i) $(t_j)_{j=0}^{\infty}$ coincides with $(s_j)_{j=0}^{\infty}$, i.e. the sequence $(s_j)_{j=0}^{\infty}$ is a fixed point of the $F_{a,\beta}$-transformation.

(ii) $\epsilon_1 - \epsilon_1^2 = \delta^{-2} p_{R(s_0)}$ and $\epsilon_j = \epsilon_1$ for all $j \in \mathbb{N}$.

**Proof** In view of Remark 3.57, we see from Theorem 3.63 that $(t_j)_{j=0}^{\infty}$ belongs to $F_{q,\infty,\alpha,\beta}$ and that the $[\alpha, \beta]$-interval parameter sequence $(p_j)_{j=0}^{\infty}$ of $(t_j)_{j=0}^{\infty}$ fulfills $p_0 = \vartheta_1$ and $p_j = \epsilon_j$ for all $j \in \mathbb{N}$. Because of Theorem 3.46, statement (i) holds if and only if $p_j = \epsilon_j$ for all $j \in \mathbb{N}$: Consequently, (i) is fulfilled if and only if $\vartheta_1 = \epsilon_0$ and $\epsilon_j = \epsilon_1$ for all $j \in \mathbb{N}$. Hence, it remains to show that $\vartheta_1 = \epsilon_0$ is equivalent to $\epsilon_1 - \epsilon_1^2 = \delta^{-2} p_{R(s_0)}$. In view of Remark 3.11, the application of Lemma 3.54 yields $t_0 = \vartheta_1$. By virtue of Definition 3.43 and (3.11), we have $\epsilon_0 = t_0 = s_0$. The application of Corollary 3.64 completes the proof.

Now we draw special attention to the scalar case $q = 1$.

**Example 3.66** Let $\Phi : F_{1,\infty,\alpha,\beta} \rightarrow F_{1,\infty,\alpha,\beta}$ be defined by $(s_j)_{j=0}^{\infty} \mapsto (t_j)_{j=0}^{\infty}$, where $(t_j)_{j=0}^{\infty}$ is the $F_{a,\beta}$-transform of $(s_j)_{j=0}^{\infty}$. Then:

(a) If $\delta < 2$, then $\Phi$ has one single fixed point $(s_0:j)_{j=0}^{\infty}$ given by $s_0:j = 0$.

(b) Suppose $\delta = 2$. For each $M \in [0, \infty)$ there exists a unique fixed point $(s_{M:j})_{j=0}^{\infty}$ of $\Phi$ with $s_{M:0} = M$. If $M = 0$, then $(s_{M:j})_{j=0}^{\infty}$ is given by $s_{M:j} = 0$. If $M > 0$, then $(s_{M:j})_{j=0}^{\infty}$ corresponds to the $[\alpha, \beta]$-interval parameter sequence $(\epsilon_{M:j})_{j=0}^{\infty}$ given by $\epsilon_{M:0} = M$ and by $\epsilon_{M:j} = \frac{1}{2}$ for $j \in \mathbb{N}$.
(c) Suppose \( \delta > 2 \). Then \( \Phi \) has exactly one fixed point \((s_0;j)_{j=0}^\infty \) with \( s_{0;0} = 0 \), namely \((s_0;j)_{j=0}^\infty \) given by \( s_{0;j} = 0 \). For each \( M \in (0, \infty) \), furthermore \( \Phi \) has exactly two fixed points \((s_{\pm M};j)_{j=0}^\infty \) with \( s_{M;0} = M \), namely \((s_{\pm M};j)_{j=0}^\infty \) corresponding to the \([\alpha, \beta]-\)interval parameter sequences \((e_{\pm M;j})_{j=0}^\infty \) given by \( e_{M;j} = M \) and by \( e_{M;j} = \frac{1}{2} + \frac{1}{2\sqrt{2}} \delta^2 - 4 \) for \( j \in \mathbb{N} \).

**Proof** (a) Assume \( \delta < 2 \). It is readily checked that the sequence \((s_0;j)_{j=0}^\infty \) defined by \( s_{0;j} := 0 \) belongs to \( \mathcal{F}_{1,\infty,\alpha,\beta} \) and that the sequence \((\varepsilon_{0;j})_{j=0}^\infty \) defined by \( \varepsilon_{0;j} := 0 \) is the \([\alpha, \beta]-\)interval parameter sequence of \((s_0;j)_{j=0}^\infty \). In view of \( \mathbb{P}_R(s_{0;0}) = 0 \), we can thus infer from Corollary 3.65 that \((s_0;j)_{j=0}^\infty \) is a fixed point of \( \Phi \).

Now consider an arbitrary fixed point \((s_j)_{j=0}^\infty \in \mathcal{F}_{1,\infty,\alpha,\beta} \) of \( \Phi \). If \( s_0 = 0 \), then from Remark 3.11 we can conclude that \( s_j = 0 \) for all \( j \in \mathbb{N} \), i.e., \((s_j)_{j=0}^\infty \) coincides with \((s_0;j)_{j=0}^\infty \). Consider now the case \( s_0 \neq 0 \). Then \( \mathbb{P}_R(s_0) = 1 \). Denote by \((\varepsilon_j)_{j=0}^\infty \) the \([\alpha, \beta]-\)interval parameter sequence of \((s_j)_{j=0}^\infty \). Because of Corollary 3.65, we have \( \varepsilon_1 - \varepsilon_2^2 = \delta - 2 \mathbb{P}_R(s_0) \). Consequently, \( 0 < \delta - 2 = \delta - 2 \mathbb{P}_R(s_0) = \varepsilon_1 - \varepsilon_2^2 = \frac{1}{4} - (\varepsilon_1 - \frac{1}{2})^2 \leq \frac{1}{4} \), contradicting \( \delta < 2 \). Thus, \((s_0;j)_{j=0}^\infty \) is the only fixed point of \( \Phi \).

(b) As above, we see that \((s_0;j)_{j=0}^\infty \in \mathcal{F}_{1,\infty,\alpha,\beta} \) is a fixed point of \( \Phi \) with \([\alpha, \beta]-\)interval parameter sequence \((\varepsilon_{0;j})_{j=0}^\infty \) given by \( \varepsilon_{0;j} = 0 \). Consider an arbitrary \( M \in (0, \infty) \) and let \((\varepsilon_{M;j})_{j=0}^\infty \) be defined by \( \varepsilon_{M;0} := M \) and by \( \varepsilon_{M;j} := \frac{1}{2} + \frac{1}{2\sqrt{2}} \delta^2 - 4 \) for all \( j \in \mathbb{N} \). In view of \( \delta > 2 \), we have \( \delta^2 - 4 > 0 \) and consequently \( 0 < \varepsilon_{M;j} < \frac{1}{2} < e_{M;j}^+ < 1 \) for all \( j \in \mathbb{N} \). Regarding \( \mathbb{P}_R(M) = 1 \), hence, due to Example 3.48, there exist sequences \((s_{M;j})_{j=0}^\infty \) and \((s_{M;j})_{j=0}^\infty \) belonging to \( \mathcal{F}_{1,\infty,\alpha,\beta} \) with \([\alpha, \beta]-\)interval
parameter sequence $(\epsilon^-_{M;j})_{j=0}^\infty$ and $(\epsilon^+_{M;j})_{j=0}^\infty$, resp. According to Theorem 3.46, the sequences $(s^-_{0;j})_{j=0}^\infty$, $(s^-_{M;j})_{j=0}^\infty$, and $(s^+_{M;j})_{j=0}^\infty$ are pairwise different. By virtue of Definition 3.43 and (3.11), we have $M = \epsilon^\pm_{M;0} = \delta^\pm_{M;0}$. In particular $P_R(s^\pm_{M;0}) = 1$.

Taking additionally into account $(\epsilon^\pm_{M;1} - \frac{1}{2})^2 = \frac{\delta^2 - 4}{4\delta^2} = \frac{1}{4} - \delta^{-2}$, hence $\epsilon^\pm_{M;1} - (\epsilon^\pm_{M;1} - \frac{1}{2})^2 = \delta^{-2}P_R(s^\pm_{M;0})$, follows. According to Corollary 3.65, then the sequences $(s^-_{M;j})_{j=0}^\infty$ and $(s^+_{M;j})_{j=0}^\infty$ are both fixed points of $\Phi$.

Now consider an arbitrary fixed point $(s_j)_{j=0}^\infty \in \mathcal{F}_{I,\infty,\alpha,\beta}^\infty$ of $\Phi$. Because of Lemma 3.12, we have $s_0 \in [0, \infty)$. If $s_0 = 0$, then, as above, the sequence $(s_j)_{j=0}^\infty$ coincides with $(s_0;j)_{j=0}^\infty$. Assume $s_0 > 0$. Then $P_R(s_0) = 1$. Denote by $(e_j)_{j=0}^\infty$ the $[\alpha, \beta]$-interval parameter sequence of $(s_j)_{j=0}^\infty$. Because of Corollary 3.65, we get $e_1 - e_1^2 = \delta^{-2}P_R(s_0)$ and $e_1 = e_2 = e_3 = \cdots$. Then $(e_1 - \frac{1}{2})^2 = e_1^2 - e_1 + \frac{1}{4} = -\delta^{-2}P_R(s_0) + \frac{1}{4} = -\frac{1}{\delta^{-2}} + \frac{1}{4} = \frac{\delta^2 - 4}{4\delta^2}$ follows, i.e. $e_1 = \frac{1}{2} - \frac{1}{2\delta}\sqrt{\delta^2 - 4}$ or $e_1 = \frac{1}{2} + \frac{1}{2\delta}\sqrt{\delta^2 - 4}$. By virtue of Definition 3.43 and (3.11), we have $e_0 = f_0 = s_0$.

Setting $M := s_0$, then $M \in (0, \infty)$ and $(e_j)_{j=0}^\infty$ coincides with $(e^\pm_{M;j})_{j=0}^\infty$ or with $(s^\pm_{M;j})_{j=0}^\infty$ defined above, implying that $(s_j)_{j=0}^\infty$ coincides with $(s^-_{M;j})_{j=0}^\infty$ or with $(s^+_{M;j})_{j=0}^\infty$, according to Theorem 3.46.

\begin{remark}
Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^\infty$. Then the following statements are equivalent:

(i) $(s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^\infty$.
(ii) $\mathcal{O}_j \subset \mathbb{C}^{q \times q}$ for all $j \in \mathbb{Z}_{0,m}$.
(iii) $\det \mathcal{O}_j \neq 0$ for all $j \in \mathbb{Z}_{0,m}$.
(iv) $\mathcal{O}_m \subset \mathbb{C}^{q \times q}$.
(v) $\det \mathcal{O}_m \neq 0$.

Indeed, in view of Proposition 3.28 we see that (ii) and (iii) resp. (iv) and (v) are equivalent. Furthermore, for each $j \in \mathbb{Z}_{0,m}$, from [27, Prop. 10.23] we obtain $\mathcal{O}_j \equiv (4/\delta)^{m-j} \mathcal{O}_m$. Thus, (iv) implies (ii). The equivalence of (iii) and (i) is a consequence of Lemma 3.35.

\begin{remark}
Let $(s_j)_{j=0}^\infty \in \mathcal{F}_{q,\infty,\alpha,\beta}^\infty$. In view of Remark 3.67, then the following statements are equivalent:

(i) $(s_j)_{j=0}^\infty \in \mathcal{F}_{q,\infty,\alpha,\beta}^\infty$.
(ii) $\mathcal{O}_j \subset \mathbb{C}^{q \times q}$ for all $j \in \mathbb{N}_0$.
(iii) $\det \mathcal{O}_j \neq 0$ for all $j \in \mathbb{N}_0$.

\end{remark}

4 The Class $\mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$

In this section, we introduce several classes of matrix-valued functions, holomorphic in the sense explained in Appendix F. We consider the following open half-planes in the complex plane: $\mathbb{H}_-(\alpha) := \{z \in \mathbb{C}: \text{Re}\,z < \alpha\}$, $\mathbb{H}_+(\beta) := \{z \in \mathbb{C}: \text{Re}\,z > \beta\}$,
\( \Pi_{\pm} := \{z \in \mathbb{C} : \text{Im} z < 0\} \), and \( \Pi_{+} := \{z \in \mathbb{C} : \text{Im} z > 0\} \). Furthermore, we write \( \text{Re} A := \frac{1}{2}(A + A^*) \) and \( \text{Im} A := \frac{1}{2i}(A - A^*) \) for the real part and the imaginary part of a complex square matrix \( A \), resp.

**Notation 4.1** Denote by \( \mathcal{R}_q(\Pi_+) \) the set of all matrix-valued functions \( F : \Pi_+ \to \mathbb{C}^{q \times q} \), which are holomorphic and satisfy \( \text{Im} F(z) \in \mathbb{C}^{q \times q}_R \) for all \( z \in \Pi_+ \).

By means of

\[
G(z) := \begin{cases} 
F(z), & \text{if } \text{Im} z > 0 \\
[F(\bar{z})]^*, & \text{if } \text{Im} z < 0
\end{cases}
\]

the matrix-valued functions \( F \) of the class \( \mathcal{R}_q(\Pi_+) \) can be extended to holomorphic matrix-valued functions \( G : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{q \times q} \), which satisfy \( \text{Im} G(z)/\text{Im} z \in \mathbb{C}^{q \times q}_R \) for all \( z \in \mathbb{C} \setminus \mathbb{R} \). In the scalar case \( q = 1 \), such a function is called a *R-function* in [39, p. 1]. The matrix-valued functions of the class \( \mathcal{R}_q(\Pi_+) \) are also called *Herglotz functions*, *Nevanlinna functions* or *Pick functions*. They admit a well-known integral representation, the scalar version of which can be found, e.g., in [39, Eq. (S1.1.1)].

Using the Euclidean norm \( \|x\|_E := \sqrt{x^*x} \) on \( \mathbb{C}^q \) corresponding to the Euclidean scalar product, we define the operator norm \( \|A\|_S := \max\{\|Au\|_E : u \in \mathbb{C}^q \text{ with } \|u\|_E = 1\} \) on \( \mathbb{C}^{p \times q} \) induced by the Euclidean norms on \( \mathbb{C}^q \) and \( \mathbb{C}^p \), which is also called the *spectral norm* on \( \mathbb{C}^{p \times q} \).

**Notation 4.2** Denote by \( \mathcal{R}_{0,q}(\Pi_+) \) the set of all \( F \in \mathcal{R}_q(\Pi_+) \) satisfying the growth condition \( \sup_{y \in [1, \infty)} y \|F(iy)\|_S < \infty \).

**Theorem 4.3** (cf. [10, Thm. 8.7])

(a) If \( F \in \mathcal{R}_{0,q}(\Pi_+) \), then there exists a unique \( \sigma \in \mathcal{M}_{q}^{\leq} (\mathbb{R}) \) such that

\[
F(z) = \int_{\mathbb{R}} \frac{1}{t - z} \sigma(\text{d}t)
\]

holds true for all \( z \in \Pi_+ \).

(b) If \( \sigma \in \mathcal{M}_{q}^{\geq} (\mathbb{R}) \), then \( F : \Pi_+ \to \mathbb{C}^{q \times q} \) defined via (4.1) belongs to \( \mathcal{R}_{0,q}(\Pi_+) \).

**Definition 4.4** Let \( F \in \mathcal{R}_{0,q}(\Pi_+) \). Then the unique measure \( \sigma \in \mathcal{M}_{q}^{\geq} (\mathbb{R}) \) such that (4.1) holds true for all \( z \in \Pi_+ \) is called the (matricial) spectral measure of \( F \) and is denoted by \( \sigma_F \).

In certain situations, an upper bound for \( y \|F(iy)\|_S \) can be obtained, using Lemma A.26:

**Lemma 4.5** (see e.g. [10, Lem. 8.9]) Let \( M \in \mathbb{C}^{q \times q} \) and let \( F : \Pi_+ \to \mathbb{C}^{q \times q} \) be a holomorphic matrix-valued function such that, for all \( z \in \Pi_+ \), the matrix

\[
\begin{bmatrix}
M & F(z) \\
[F(z)]^* & \frac{1}{\text{Im} z} \text{Im} F(z)
\end{bmatrix}
\]

is Non-negative Hermitian. Then \( F \in \mathcal{R}_{0,q}(\Pi_+) \) with \( \sup_{y \in (0, \infty)} y \|F(iy)\|_S \leq \|M\|_S \) and \( \sigma_F (\mathbb{R}) \preceq M \).
Now we introduce that class of holomorphic matrix-valued functions, which is relevant for the moment problem $MP[\{\alpha, \beta\}; (s_j)_j=0, \infty]$.

**Notation 4.6** Denote by $\mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$ the set of all matrix-valued functions $F: \mathbb{C} \setminus [\alpha, \beta] \to \mathbb{C}^{q \times q}$ which are holomorphic and satisfy the following conditions:

(I) $\text{Im} \ F(z) \in \mathbb{C}^{q \times q}$ for all $z \in \mathbb{R}_+$. 
(II) $F(x) \in \mathbb{C}^{q \times q}_\sigma$ for all $x \in (-\infty, \alpha)$ and $-F(x) \in \mathbb{C}^{q \times q}_\sigma$ for all $x \in (\beta, \infty)$.

Since such functions are holomorphic in $\mathbb{C} \setminus [\alpha, \beta]$ with Non-negative Hermitian imaginary part in $\mathbb{R}_+$, we can think of $\mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$ as a subclass of $\mathcal{R}_q(\mathbb{R}_+)$, by virtue of the identity theorem for holomorphic functions:

**Remark 4.7** By means of restricting matrix-valued functions of $\mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$ to $\mathbb{R}_+$, an injective mapping from $\mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$ into $\mathcal{R}_q(\mathbb{R}_+)$ is given.

Lemma 4.12 will show that the above mentioned restrictions even belong to $\mathcal{R}_{0,q}(\mathbb{R}_+)$. Since each function from $\mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$ is holomorphic in $\mathbb{C} \setminus [\alpha, \beta]$ with Hermitian values in $\mathbb{R}[\alpha, \beta]$, we can use the Schwarz reflection principle to obtain the following relation connecting its values on the open upper and lower half-plane:

**Remark 4.8** If $F \in \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$, then $[F(z)]^* = F(\bar{z})$ for all $z \in \mathbb{C} \setminus [\alpha, \beta]$.

The latter result can also be seen from the following integral representation:

**Theorem 4.9** (cf. [10, Thm. 1.1])

(a) If $F \in \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$, then there exists a unique $\tilde{\sigma} \in \mathcal{M}_q^\geq([\alpha, \beta])$ such that

$$F(z) = \int_{[\alpha, \beta]} \frac{1}{t-z} \tilde{\sigma}(dr) \quad (4.2)$$

holds true for all $z \in \mathbb{C} \setminus [\alpha, \beta]$.

(b) If $\tilde{\sigma} \in \mathcal{M}_q^\geq([\alpha, \beta])$, then $F: \mathbb{C} \setminus [\alpha, \beta] \to \mathbb{C}^{q \times q}$ defined via (4.2) belongs to $\mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$.

**Definition 4.10** Let $F \in \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$. In view of Theorem 4.9, let $\tilde{\sigma}$ be the uniquely determined measure from $\mathcal{M}_q^\geq([\alpha, \beta])$ such that (4.2) holds true for all $z \in \mathbb{C} \setminus [\alpha, \beta]$. Then $\tilde{\sigma}$ is called the $\mathcal{R}[\alpha, \beta]$-measure of $F$ and is denoted by $\tilde{\sigma}_F$.

As already mentioned in Sect. 3, the power moments of a measure belonging to $\mathcal{M}_q^\geq([\alpha, \beta])$ exist for each non-negative integer order:

**Remark 4.11** If $F \in \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$, then the $\mathcal{R}[\alpha, \beta]$-measure $\tilde{\sigma}_F$ of $F$ belongs to $\mathcal{M}_q^\geq([\alpha, \beta])$.

In view of Remark 4.7, we infer the following relation to the class $\mathcal{R}_{0,q}(\mathbb{R}_+)$:

**Lemma 4.12** Let $F \in \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$ with $\mathcal{R}[\alpha, \beta]$-measure $\tilde{\sigma}_F$ and denote by $f$ the restriction of $F$ onto $\mathbb{R}_+$. Then $f \in \mathcal{R}_{0,q}(\mathbb{R}_+)$ and the spectral measure $\sigma_f$ of $f$ fulfills $\sigma_f(\mathbb{R} \setminus [\alpha, \beta]) = O_{q \times q}$ and $\sigma_f(B) = \tilde{\sigma}_F(B)$ for all $B \in \mathfrak{B}[\alpha, \beta]$.
Proof. Use Theorems 4.9 and 4.3.

By virtue of Theorem 4.9, the following integral representations are readily checked:

\[ \text{Re } F(z) = \int_{[\alpha, \beta]} \frac{t - \text{Re } z}{|t - z|^2} \sigma_F(dt) \quad \text{and} \quad \text{Im } F(z) = \int_{[\alpha, \beta]} \frac{\text{Im } z}{|t - z|^2} \sigma_F(dt). \]

Now we state a useful characterization of the functions belonging to \( \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta]) \).

**Proposition 4.14** Let \( F : \mathbb{C} \setminus [\alpha, \beta] \to \mathbb{C}^{q \times q} \) be holomorphic. Then \( F \in \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta]) \) if and only if the following two conditions are fulfilled:

(I) \( \frac{1}{\text{Im } z} \text{Im } F(z) \in \mathbb{C}^{q \times q} \) for all \( z \in \mathbb{C} \setminus \mathbb{R} \).

(II) \( \text{Re } F(w) \in \mathbb{C}^{q \times q} \) for all \( w \in \mathbb{H}_-(\alpha) \) and \( -\text{Re } F(w) \in \mathbb{C}^{q \times q} \) for all \( w \in \mathbb{H}_+(\beta) \).

**Proof.** If \( F \in \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta]) \), then (I) and (II) are readily seen from Remark 4.13.

Conversely, suppose that (I) and (II) are fulfilled. Due to (I), we have \( \text{Im } F(z) \in \mathbb{C}^{q \times q} \) for all \( z \in \Pi_+ \). As in the proof of Lemma C.3, we can conclude from (I) that, for all \( x \in \mathbb{R} \setminus [\alpha, \beta] \), the equation \( \text{Im } F(x) = O_{q \times q} \) holds true, implying \( F(x) = \text{Re } F(x) \). Taking into account (II), we thus have \( F(x) \in \mathbb{C}^{q \times q} \) for all \( x \in (-\infty, \alpha) \) and \( -F(x) \in \mathbb{C}^{q \times q} \) for all \( x \in (\beta, \infty) \). Regarding Notation 4.6, hence \( F \in \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta]) \).

As an immediate consequence of the following result, we see that the column space \( \mathcal{R}(F(z)) \) and the null space \( \mathcal{N}(F(z)) \) of a matrix-valued function \( F \in \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta]) \) are both independent of the argument \( z \in \mathbb{C} \setminus [\alpha, \beta] \).

**Proposition 4.15** Let \( F \in \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta]) \). Then:

(a) \( \mathcal{R}(F(z)) = \mathcal{R}(\bar{\sigma}_F([\alpha, \beta])) \) and \( \mathcal{N}(F(z)) = \mathcal{N}(\bar{\sigma}_F([\alpha, \beta])) \) for all \( z \in \mathbb{C} \setminus [\alpha, \beta] \).

(b) \( \mathcal{R}(\text{Im } F(z)) = \mathcal{R}(\bar{\sigma}_F([\alpha, \beta])) \) and \( \mathcal{N}(\text{Im } F(z)) = \mathcal{N}(\bar{\sigma}_F([\alpha, \beta])) \) for all \( z \in \mathbb{C} \setminus \mathbb{R} \).

(c) \( \mathcal{R}(\text{Re } F(w)) = \mathcal{R}(\bar{\sigma}_F([\alpha, \beta])) \) and \( \mathcal{N}(\text{Re } F(w)) = \mathcal{N}(\bar{\sigma}_F([\alpha, \beta])) \) for all \( w \in \mathbb{H}_-(\alpha) \cup \mathbb{H}_+(\beta) \).

**Proof.** In view of Theorem 4.9, this follows from Lemma C.5 applied with \( \Omega = [\alpha, \beta] \).

We recall the definitions of two well-studied classes of matrices.

**Definition 4.16** Let \( A \) be a complex \( q \times q \) matrix. Then \( A \) is called EP matrix if \( \mathcal{R}(A^*) = \mathcal{R}(A) \). Furthermore, the matrix \( A \) is said to be almost definite if each \( x \in \mathbb{C}^q \) with \( x^*Ax = 0 \) necessarily fulfills \( Ax = O_{q \times 1} \). Denote by \( \mathbb{C}^{q \times q}_{\text{EP}} \) and \( \mathbb{C}^{q \times q}_{\text{AD}} \) the set of EP matrices and the set of almost definite matrices from \( \mathbb{C}^{q \times q} \), resp.

**Proposition 4.17** If \( F \in \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta]) \), then \( F(z) \in \mathbb{C}^{q \times q}_{\text{AD}} \) for all \( z \in \mathbb{C} \setminus [\alpha, \beta] \).
Proof In view of Theorem 4.9, this follows from Lemma C.6 applied with $\Omega = [\alpha, \beta]$. 

According to Remark A.31, we have $C_q^{q \times q} \subseteq \mathbb{C}_E^{q \times q}$. Hence, Proposition 4.17 implies that the values of a function $F \in \mathcal{R}_q(C \setminus [\alpha, \beta])$ fulfill $\mathcal{R}([F(z)]^*) = \mathcal{R}(F(z))$ for all $z \in \mathbb{C} \setminus [\alpha, \beta]$, a fact that also can be seen from Proposition 4.15 in combination with Remark 4.8.

By means of Theorem 4.9, a characterization of $\mathcal{R}_q(C \setminus [\alpha, \beta])$ in terms of the class $\mathcal{R}_q(\Pi_\perp)$ can be obtained:

Proposition 4.18 (cf. [10, Lem. 3.6]) Let $F : C \setminus [\alpha, \beta] \to C_q^{q \times q}$ be holomorphic and let the matrix-valued functions $g, h : \Pi_\perp \to C_q^{q \times q}$ be defined by $g(z) := (z - \alpha)F(z)$ and $h(z) := (\beta - z)F(z)$, resp. Then $F \in \mathcal{R}_q(C \setminus [\alpha, \beta])$ if and only if $g$ and $h$ both belong to $\mathcal{R}_q(\Pi_\perp)$.

In view of Remark 4.11, we can associate to a given function from the class $\mathcal{R}_q(C \setminus [\alpha, \beta])$ three auxiliary functions, which are intimately connected to the three sequences of complex matrices introduced in Notation 3.2 (cf. Remark 5.11):

Notation 4.19 Let $F \in \mathcal{R}_q(C \setminus [\alpha, \beta])$ with $\mathcal{R}[\alpha, \beta]$-measure $\tilde{\sigma}_F$. Then let the functions $F_a, F_b, F_c : C \setminus [\alpha, \beta] \to C_q^{q \times q}$ be defined by

$$F_a(z) := (z - \alpha)F(z) + \tilde{\sigma}_F([\alpha, \beta]), \quad F_b(z) := (\beta - z)F(z) - \tilde{\sigma}_F([\alpha, \beta]),$$

and

$$F_c(z) := (\beta - z)(z - \alpha)F(z) + (\alpha + \beta - z)\tilde{\sigma}_F([\alpha, \beta]) - \int_{[\alpha, \beta]} \tau_2 \tilde{\sigma}_F(\tau \, dt).$$

Proposition 4.20 Let $F \in \mathcal{R}_q(C \setminus [\alpha, \beta])$ with $\mathcal{R}[\alpha, \beta]$-measure $\tilde{\sigma}_F$. Then $F_a, F_b,$ and $F_c$ belong to $\mathcal{R}_q(C \setminus [\alpha, \beta])$ and their $\mathcal{R}[\alpha, \beta]$-measures $\tilde{\sigma}_a, \tilde{\sigma}_b,$ and $\tilde{\sigma}_c$ fulfill

$$\tilde{\sigma}_a(B) = \int_B (t - \alpha)\tilde{\sigma}_F(\tau \, dt), \quad \tilde{\sigma}_b(B) = \int_B (\beta - t)\tilde{\sigma}_F(\tau \, dt),$$

and

$$\tilde{\sigma}_c(B) = \int_B (\beta - t)(t - \alpha)\tilde{\sigma}_F(\tau \, dt)$$

for all $B \in \mathcal{B}[\alpha, \beta]$.

Proof Because of Remark 4.11, the integrals $\int_{[\alpha, \beta]} t\tilde{\sigma}_F(\tau \, dt)$ and $\int_{[\alpha, \beta]} t^2\tilde{\sigma}_F(\tau \, dt)$ exist. Since $t - \alpha > 0$ and $\beta - t > 0$ hold true for all $t \in [\alpha, \beta]$, we can thus conclude that $\int_B (t - \alpha)\tilde{\sigma}_F(\tau \, dt), \int_B (\beta - t)\tilde{\sigma}_F(\tau \, dt),$ and $\int_B (\beta - t)(t - \alpha)\tilde{\sigma}_F(\tau \, dt)$ are Non-negative Hermitian matrices for all $B \in \mathcal{B}[\alpha, \beta]$. Consequently, $\tilde{\sigma}_a, \tilde{\sigma}_b,$ and $\tilde{\sigma}_c$ belong to $\mathcal{M}_q^\infty([\alpha, \beta])$. 

Consider now an arbitrary \( z \in \mathbb{C} \setminus [\alpha, \beta] \). In view of Theorem 4.9, we have

\[
(z - \alpha) F(z) = \int_{[\alpha, \beta]} \frac{z - \alpha}{t - z} \tilde{\sigma}_F(dt) = \int_{[\alpha, \beta]} \frac{t - \alpha}{t - z} \tilde{\sigma}_F(dt) = \int_{[\alpha, \beta]} \frac{t - \alpha}{t - z} \tilde{\sigma}_F(dt) - \tilde{\sigma}_F([\alpha, \beta])
\]

and similarly \((\beta - z) F(z) = \int_{[\alpha, \beta]} (t - z)^{-1} (\beta - t) \tilde{\sigma}_F(dt) + \tilde{\sigma}_F([\alpha, \beta])\). Due to Theorem 4.9, then \( F_a \) and \( F_b \) belong to \( \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta]) \) having the asserted \( \mathcal{R}[\alpha, \beta] \)-measures. Using the representation above, we obtain furthermore

\[
(\beta - z)(z - \alpha) F(z) = \int_{[\alpha, \beta]} \frac{(\beta - z)(t - \alpha)}{t - z} \tilde{\sigma}_F(dt) - (\beta - z) \tilde{\sigma}_F([\alpha, \beta])
\]

Hence, \( F_c(z) = \int_{[\alpha, \beta]} \frac{1}{t - z} \tilde{\sigma}_F(dt) \). By virtue of Theorem 4.9, thus \( F_c \) belongs to \( \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta]) \) having the asserted \( \mathcal{R}[\alpha, \beta] \)-measure. \( \square \)

The combination of Proposition 4.20 with Remarks 4.13 and A.2 yields:

**Remark 4.21** Let \( F \in \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta]) \). For all \( z \in \mathbb{C} \setminus [\alpha, \beta] \), then

\[
\text{Im}[(z - \alpha) F(z)] = \text{Im} F_a(z) = \text{Im}(z) \int_{[\alpha, \beta]} \frac{t - \alpha}{|t - z|^2} \tilde{\sigma}_F(dt),
\]

\[
\text{Im}[(\beta - z) F(z)] = \text{Im} F_b(z) = \text{Im}(z) \int_{[\alpha, \beta]} \frac{\beta - t}{|t - z|^2} \tilde{\sigma}_F(dt),
\]

and

\[
\text{Im}[(\beta - z)(z - \alpha) F(z)] = \text{Im} F_c(z) + \text{Im}(z) \tilde{\sigma}_F([\alpha, \beta])
\]

\[
= \text{Im}(z)[\tilde{\sigma}_F([\alpha, \beta])] + \int_{[\alpha, \beta]} \frac{(\beta - t)(t - \alpha)}{|t - z|^2} \tilde{\sigma}_F(dt).
\]

Using Remark 4.21, the following result is readily checked:

**Remark 4.22** If \( F \in \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta]) \), then \( \frac{1}{\text{Im} z} \text{Im}[(\beta - z)(z - \alpha) F(z)] \gg \tilde{\sigma}_F([\alpha, \beta]) \gg O_{q \times q} \) for all \( z \in \mathbb{C} \setminus \mathbb{R} \).

**5 An Equivalent Problem in the Class \( \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta]) \)**

To describe the solution set of moment problems on the real axis, the transition to holomorphic functions by means of the Stieltjes transformation considered in detail...
in Appendix C has turned out to be very helpful. For the sake of a simpler description of the relation to \( R \)-functions treated in the previous section, as usual, we choose in the case \( \Omega = \mathbb{R} \) for the Stieltjes Transform \( S \) the domain \( \Pi_+ \) instead of \( \mathbb{C} \setminus \mathbb{R} \). As at the beginning of Appendix C, it can be shown that the following integral converges:

**Definition 5.1** Let \( \sigma \in \mathcal{M}_q^\infty (\mathbb{R}) \). Then we call the matrix-valued function \( S_{\sigma} : \Pi_+ \rightarrow \mathbb{C}^{q \times q} \) defined by

\[
S_{\sigma}(z) := \int_{\mathbb{R}} \frac{1}{t-z} \sigma (dt)
\]

the \( \mathbb{R} \)-Stieltjes transform of \( \sigma \).

From Theorem 4.3 and Definitions 4.4 and 5.1 we immediately see the well-known connection of the \( \mathbb{R} \)-Stieltjes transformation to the class \( R_{0,q}(\Pi_+) \):

**Proposition 5.2** (a) If \( F \in R_{0,q}(\Pi_+) \), then there exists a unique \( \sigma \in \mathcal{M}_q^\infty (\mathbb{R}) \) fulfilling \( F = S_{\sigma} \), namely \( \sigma = \sigma_F \).

(b) If \( \sigma \in \mathcal{M}_q^\infty (\mathbb{R}) \), then the \( \mathbb{R} \)-Stieltjes Transform \( S_{\sigma} \) of \( \sigma \) belongs to \( R_{0,q}(\Pi_+) \).

According to our interest in the matricial Hausdorff moment problem, we consider the integral transformation (C.1) for the particular case of Non-negative Hermitian measures \( \sigma \) belonging to \( \mathcal{M}_q^\infty ([\alpha, \beta]) \):

**Definition 5.3** Let \( \sigma \in \mathcal{M}_q^\infty ([\alpha, \beta]) \). Then we call the matrix-valued function \( \tilde{S}_{\sigma} : \mathbb{C} \setminus [\alpha, \beta] \rightarrow \mathbb{C}^{q \times q} \) defined by

\[
\tilde{S}_{\sigma}(z) := \int_{[\alpha, \beta]} \frac{1}{t-z} \sigma (dr) \quad (5.1)
\]

the \([\alpha, \beta]\)-Stieltjes transform of \( \sigma \).

The \([\alpha, \beta]\)-Stieltjes Transform of a Non-negative Hermitian measure from \( \mathcal{M}_q^\infty ([\alpha, \beta]) \) admits a power series representation at \( z_0 = \infty \) involving the corresponding moments:

**Proposition 5.4** ([5, Satz 1.2.16, p. 34]) Let \( \sigma \in \mathcal{M}_q^\infty ([\alpha, \beta]) \). Then the moments \( s_j^{(\sigma)} := \int_{[\alpha, \beta]} x^j \sigma (dx) \) exist for all \( j \in \mathbb{N}_0 \). For each \( z \in \mathbb{C} \) with \(|z| > \max\{|\alpha|, |\beta|\}\), furthermore \( z \in \mathbb{C} \setminus [\alpha, \beta] \) and

\[
\tilde{S}_{\sigma}(z) = - \sum_{j=0}^{\infty} z^{-(j+1)} s_j^{(\sigma)}.
\]

The following reformulation of Theorem 4.9 describes the relation between \([\alpha, \beta]\)-Stieltjes Transform \( \tilde{S}_{\sigma} \) and \( R[\alpha, \beta] \)-measure \( \tilde{\sigma}_F \):
Proposition 5.5 \(\text{The mapping } \Lambda_{[\alpha, \beta]} : \mathcal{M}_q^\leq ([\alpha, \beta]) \to \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta]) \text{ given by } \sigma \mapsto \check{S}_\sigma, \text{ where } \check{S}_\sigma \text{ is given by (5.1), is well defined and bijective. Its inverse } \Lambda_{[\alpha, \beta]}^{-1} : \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta]) \to \mathcal{M}_q^\leq ([\alpha, \beta]) \text{ is given by } F \mapsto \check{\sigma}_F, \text{ where } \check{\sigma}_F \text{ denotes the } \mathcal{R}([\alpha, \beta])\text{-measure of } F.\)

By virtue of Proposition 5.5, the moment problem \(\text{MP}[[\alpha, \beta]; (s_j)_{j=0}^\infty, \equiv] \) admits a reformulation as an equivalent problem for functions belonging to the class \(\mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])\):

**Problem 5.6** \((\text{FP}[[\alpha, \beta]; (s_j)_{j=0}^\kappa])\) Given a sequence \((s_j)_{j=0}^\kappa\) of complex \(q \times q\) matrices, parametrize the set \(\mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^\kappa]\) of all \(F \in \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])\) with \(\mathcal{R}[\alpha, \beta]\text{-measure } \check{\sigma}_F \text{ belonging to } \mathcal{M}_q^\leq [[\alpha, \beta]; (s_j)_{j=0}^\kappa, \equiv].\)

In particular, Problem FP\([\alpha, \beta]; (s_j)_{j=0}^\kappa\) has a solution if and only if the moment problem \(\text{MP}[[\alpha, \beta]; (s_j)_{j=0}^\kappa, \equiv] \) has a solution. From Theorem 3.5 we can therefore conclude:

**Proposition 5.7** \(\text{Let } (s_j)_{j=0}^\kappa \text{ be a sequence of complex } q \times q \text{ matrices. Then the set } \mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^\kappa] \text{ is non-empty if and only if the sequence } (s_j)_{j=0}^\kappa \text{ belongs to } \mathcal{F}_q^\leq,\alpha,\beta.\)

In view of Proposition 5.5, the solution set \(\mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^\kappa]\) of Problem FP\([\alpha, \beta]; (s_j)_{j=0}^\kappa\) can also be described in the following way:

**Remark 5.8** \(\mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^\kappa] = \{\check{S}_\sigma : \sigma \in \mathcal{M}_q^\leq [[\alpha, \beta]; (s_j)_{j=0}^\kappa, \equiv\}.\)

In combination with Propositions 3.6 and 5.4, we can infer from Remark 5.8 in particular:

**Proposition 5.9** \(\text{Let } (s_j)_{j=0}^\infty \in \mathcal{F}_q^\leq,\alpha,\beta. \text{ Then the set } \mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^\infty] \text{ consists of exactly one element } S. \text{ For all } z \in \mathbb{C} \text{ with } |z| > \max(|\alpha|, |\beta|), \text{ furthermore } z \in \mathbb{C} \setminus [\alpha, \beta] \text{ and }\)

\[
S(z) = -\sum_{j=0}^\infty z^{-(j+1)}s_j. \tag{5.2}
\]

In addition, we have the following result:

**Proposition 5.10** \(\text{Let } (s_j)_{j=0}^\infty \in \mathcal{F}_q^\leq,\alpha,\beta, \text{ let } F : \mathbb{C} \setminus [\alpha, \beta] \to \mathbb{C}^{q \times q} \text{ be holomorphic, and let } \rho \in \mathbb{R} \text{ with } \rho \geq \max(|\alpha|, |\beta|). \text{ Suppose that } F(z) = -\sum_{j=0}^\infty z^{-(j+1)}s_j \text{ holds true for all } z \in \mathbb{C} \text{ with } |z| > \rho. \text{ Then } F \in \mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^\infty].\)

**Proof** Due to Proposition 5.9, the set \(\mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^\infty]\) consists of exactly one element \(S\) and (5.2) holds true for all \(z \in \mathbb{C}\) with \(|z| > \rho\). In particular, \(F(z) = S(z)\) for all \(z \in \mathbb{C}\) with \(|z| > \rho\) follows. Consequently, the application of the identity theorem for holomorphic functions yields \(F = S\). Therefore, \(F\) belongs to \(\mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^\infty].\) \(\square\)
In view of Proposition 4.20, the sequences \((a_j)_{j=0}^{k-1}, (b_j)_{j=0}^{k-1}, \) and \((c_j)_{j=0}^{k-2}\) introduced in Notation 3.2 consist of the first power moments of the \(\mathcal{R}[\alpha, \beta]\)-measures \(\tilde{a}_0, \tilde{b}_0,\) and \(\tilde{c}_0\) of the matrix-valued functions \(F_a, F_b,\) and \(F_c\) built, according to Notation 4.19, from a given function \(F \in \mathcal{R}_q[[\alpha, \beta]]; (s_j)_{j=0}^{k}\):

\[ \text{Remark 5.11} \]

Let \((s_j)_{j=0}^{k} \in \mathcal{F}_{q,x,\alpha,\beta}^\geq \) and let \(F \in \mathcal{R}_q[[\alpha, \beta]]; (s_j)_{j=0}^{k}\). If \(k \geq 1\), then \(F_a \in \mathcal{R}_q[[\alpha, \beta]]; (a_j)_{j=0}^{k-1}\) and \(F_b \in \mathcal{R}_q[[\alpha, \beta]]; (b_j)_{j=0}^{k-1}\). If \(k \geq 2\), then \(F_c \in \mathcal{R}_q[[\alpha, \beta]; (c_j)_{j=0}^{k-2}\].

6 A Schur–Nevanlinna Type Algorithm in the Class \(\mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])\)

On the background of Proposition 3.7 and Theorem 3.46, we parametrized in [28, Sec. 8] the set \(\mathcal{M}_q^\geq ([\alpha, \beta])\) of Non-negative Hermitian \(q \times q\) measures on \([\alpha, \beta]\) and generalized several results from the scalar theory of canonical moments (cf. [12]) to the matrix case. To that end, we associated to such a measure the sequences built via Definitions 3.22 and 3.43 from its sequence of power moments:

**Definition 6.1** Let \(\sigma \in \mathcal{M}_q^\geq ([\alpha, \beta])\) with sequence of power moments \((s_j)_{j=0}^{\infty}\). Denote by \((e_j^\sigma)_{j=0}^{\infty}\) the \([\alpha, \beta]\)-interval parameter sequence of \((s_j)_{j=0}^{\infty}\) and by \((\sigma_j)_{j=0}^{\infty}\) the sequence of \([\alpha, \beta]\)-interval lengths associated with \((s_j)_{j=0}^{\infty}\). Then we call \((e_j^\sigma)_{j=0}^{\infty}\) the sequence of matricial canonical moments associated with \(\sigma\) and we say that \((\sigma_j)_{j=0}^{\infty}\) is the sequence of matricial interval lengths associated with \(\sigma\).

**Theorem 6.2** ([28, Thm. 8.2]) The mapping \(\Pi_{[\alpha, \beta]} : \mathcal{M}_q^\geq ([\alpha, \beta]) \rightarrow \mathcal{E}_{q,\infty,\delta}^\geq\) given by \(\sigma \mapsto (e_j^\sigma)_{j=0}^{\infty}\) is well defined and bijective.

On the basis of Proposition 5.5, a parametrization of the class \(\mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])\) immediately follows:

**Definition 6.3** Let \(F \in \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])\) with \(\mathcal{R}[\alpha, \beta]\)-measure \(\tilde{\sigma}_F\). Denote by \((e_j^F)_{j=0}^{\infty}\) the sequence of matricial canonical moments associated with \(\tilde{\sigma}_F\) and by \((\sigma_j^F)_{j=0}^{\infty}\) the sequence of matricial interval lengths associated with \(\tilde{\sigma}_F\). Then we call \((e_j^F)_{j=0}^{\infty}\) the sequence of \(\mathcal{R}[\alpha, \beta]\)-Schur parameters associated with \(F\) and we say that \((\sigma_j^F)_{j=0}^{\infty}\) is the \(\mathcal{R}[\alpha, \beta]\)-interval lengths associated with \(F\).

**Theorem 6.4** The mapping \(\Delta_{[\alpha, \beta]} : \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta]) \rightarrow \mathcal{E}_{q,\infty,\delta}^\geq\) given by \(F \mapsto (e_j^F)_{j=0}^{\infty}\) is well defined and bijective.

**Proof** Use Proposition 5.5 and Theorem 6.2.

By means of this one-to-one correspondence, results obtained in [28,29] on matricial canonical moments associated with Non-negative Hermitian measures
from $\mathcal{M}_q^\infty([\alpha, \beta])$ carry over to matrix-valued functions belonging to the class $\mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$ and their $\mathcal{R}[\alpha, \beta]$-Schur parameters.

By virtue of Propositions 3.6 and 3.59, the $\mathcal{F}_{\alpha,\beta}$-transformation (see Definitions 3.52 and 3.56) for $[\alpha, \beta]$-non-negative definite sequences of matrices gave rise to a transformation considered in [29, Def. 10.1] for Non-negative Hermitian measures from $\mathcal{M}_q^\infty([\alpha, \beta])$:

**Definition 6.5** Let $\sigma \in \mathcal{M}_q^\infty([\alpha, \beta])$ with sequence of power moments $(s_j)_{j=0}^\infty$ and let $k \in \mathbb{N}_0$. Denote by $(s_j^{(k)})_{j=0}^\infty$ the $k$-th $\mathcal{F}_{\alpha,\beta}$-transform of $(s_j)_{j=0}^\infty$ and by $\sigma^{(k)}$ the uniquely determined element in $\mathcal{M}_q^\infty([\alpha, \beta]; (s_j^{(k)})_{j=0}^\infty, =]$. Then we call $\sigma^{(k)}$ the $\mathcal{M}[\alpha, \beta]$-transform of $\sigma$.

**Remark 6.6** Let $\sigma \in \mathcal{M}_q^\infty([\alpha, \beta])$. According to Definitions 6.5 and 3.56, then $\sigma^{(0)} = \sigma$ and $\sigma^{(k)}$ is exactly the first $\mathcal{M}[\alpha, \beta]$-transform of $\sigma^{(k-1)}$ for each $k \in \mathbb{N}$.

**Remark 6.7** Let $(s_j)_{j=0}^\infty \in \mathcal{F}_{q,k,\alpha,\beta}$, let $\sigma \in \mathcal{M}_q^\infty([\alpha, \beta]; (s_j)_{j=0}^\infty, =]$, and let $k \in \mathbb{Z}_{0,k}$. Then, in view of Definition 6.5 and Remark 3.58, it is readily checked that $\sigma^{(k)}$ belongs to $\mathcal{M}_q^\infty([\alpha, \beta]; (s_j^{(k)})_{j=0}^\infty, =]$. In view of Proposition 5.5, we can define a corresponding transformation for functions belonging to the class $\mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$:

**Definition 6.8** Let $F \in \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$ with $\mathcal{R}[\alpha, \beta]$-measure $\sigma$ and let $k \in \mathbb{N}_0$. Denote by $\sigma^{(k)}$ the $k$-th $\mathcal{M}[\alpha, \beta]$-transform of $\sigma$ and by $F^{(k)}$ the $[\alpha, \beta]$-Stieltjes Transform of $\sigma^{(k)}$. Then we call $F^{(k)}$ the $k$-th $\mathcal{R}[\alpha, \beta]$-Schur transform of $F$.

**Remark 6.9** In the situation of Definition 6.8 we see from Proposition 5.5 that $F^{(k)}$ belongs to $\mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$ and that $\sigma^{(k)}$ is the $\mathcal{R}[\alpha, \beta]$-measure of $F^{(k)}$.

**Remark 6.10** Let $F \in \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$. Then, regarding Definition 6.8, Remark 6.6, and Proposition 5.5, it is readily checked that $F^{(0)} = F$ and that $F^{(k)}$ is exactly the first $\mathcal{R}[\alpha, \beta]$-Schur transform of $F^{(k-1)}$ for each $k \in \mathbb{N}$.

**Remark 6.11** Let $(s_j)_{j=0}^\infty \in \mathcal{F}_{q,k,\alpha,\beta}$, let $F \in \mathcal{R}_q([\alpha, \beta]; (s_j)_{j=0}^\infty)$, and let $k \in \mathbb{Z}_{0,k}$. Because of Remarks 6.7 and 6.9, then $F^{(k)} \in \mathcal{R}_q([\alpha, \beta]; (s_j^{(k)})_{j=0}^\infty)$.

One of the results in [29] states that the $\mathcal{M}[\alpha, \beta]$-transformation of a Non-negative Hermitian measure from $\mathcal{M}_q^\infty([\alpha, \beta])$ is essentially equivalent to left shifting its sequence of matricial canonical moments:

**Proposition 6.12** ([29, Prop. 10.4]) Let $k \in \mathbb{N}_0$ and let $\sigma \in \mathcal{M}_q^\infty([\alpha, \beta])$ with $k$-th $\mathcal{M}[\alpha, \beta]$-transform $\mu$. Then $\epsilon^{(\mu)}_0 = \delta^{k-1}\delta^{(\sigma)}_k$ and $\epsilon^{(\mu)}_j = \epsilon^{(\sigma)}_{k+j}$ for all $j \in \mathbb{N}$. Furthermore, $\delta^{(\mu)}_j = \delta^{k}\delta^{(\sigma)}_{k+j}$ for all $j \in \mathbb{N}_0$. In particular, $\mu([\alpha, \beta]) = \delta^{k-1}\delta^{(\sigma)}_k$.

The following analogous result for matrix-valued functions from the class $\mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$ justifies the notions $\mathcal{R}[\alpha, \beta]$-Schur parameters and $\mathcal{R}[\alpha, \beta]$-Schur transform chosen in Definitions 6.3 and 6.8, resp.
Proposition 6.13 Let $k \in \mathbb{N}_0$ and let $F \in \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$ with $k$-th $\mathcal{R}[\alpha, \beta]$-Schur transform $G$. Then $\delta_0^{(G)} = \delta^{k-1} \delta_k^{(F)}$ and $\delta_j^{(G)} = \delta_k^{(F)}$ for all $j \in \mathbb{N}$. Furthermore, $\delta_j^{(G)} = \delta_k^{(F)}$ for all $j \in \mathbb{N}_0$.

Proof In view of Definitions 6.3 and 6.8 and Remark 6.9, this is an immediate consequence of Proposition 6.12.

Let $\Omega \in \mathcal{B}_\mathbb{R}\setminus\{\emptyset\}$. A Non-negative Hermitian measure $\sigma \in \mathcal{M}_r^\leq(\Omega)$ is said to be molecular if there exists a finite subset $B$ of $\Omega$ satisfying $\sigma(\Omega \setminus B) = O_{q \times q}$. Obviously, this is equivalent to the existence of an $m \in \mathbb{N}$ and sequences $(\xi_\ell)_\ell$ and $(A_\ell)_\ell$ from $\Omega$ and $\mathbb{C}^{q \times q}$, resp., such that $\sigma = \sum_\ell \delta_{\xi_\ell} A_\ell$, where $\delta_{\xi_\ell}$ is the Dirac measure on $([\alpha, \beta], \mathcal{B}_{[\alpha, \beta]})$ with unit mass at $\xi_\ell$.

It was shown in [29, Prop. 10.5] that $\sigma \in \mathcal{M}_q^\leq([\alpha, \beta])$ is molecular if and only if for some $k \in \mathbb{N}_0$ its $k$-th $\mathcal{M}[\alpha, \beta]$-transform $\sigma^{(k)}$ coincides with the $q \times q$ zero measure on $([\alpha, \beta], \mathcal{B}_{[\alpha, \beta]})$. This leads to a characterization of rational matrix-valued functions from $\mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$ in terms of their $k$-th $\mathcal{R}[\alpha, \beta]$-Schur transforms:

Proposition 6.14 Let $F \in \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$. Then the following statements are equivalent:

(i) There exist complex $q \times q$ matrix polynomials $P$ and $Q$ such that $\det Q$ does not vanish identically and that $F$ coincides with the restriction of $P Q^{-1}$ onto $\mathbb{C} \setminus [\alpha, \beta]$.

(ii) There exists an integer $k \in \mathbb{N}_0$ such that $F^{(k)}$ coincides with the constant function on $\mathbb{C} \setminus [\alpha, \beta]$ with value $O_{q \times q}$.

Proof First observe that the $\mathcal{R}[\alpha, \beta]$-measure $\sigma := \delta_F$ given via Definition 4.10 belongs to $\mathcal{M}_q^\leq([\alpha, \beta])$. According to Lemma 4.12, the restriction $f$ of $F$ onto $\Pi_+$ belongs to $\mathcal{R}_{0,q}(\Pi_+)$. Furthermore, the spectral measure $\mu := \sigma_f$ of $f$ given via Definition 4.4 belongs to $\mathcal{M}_q^\leq(\mathbb{R})$ and, in view of Lemma 4.12, fulfills $\mu(\mathbb{R} \setminus [\alpha, \beta]) = O_{q \times q}$ and $\mu(B) = \sigma(B)$ for all $B \in \mathcal{B}_{[\alpha, \beta]}$. In particular, $\mu$ is molecular if and only if $\sigma$ is molecular. From [29, Prop. 10.5] we see that $\sigma$ is molecular if and only if, for some $k \in \mathbb{N}_0$, the $k$-th $\mathcal{M}[\alpha, \beta]$-transform $\sigma^{(k)}$ of $\sigma$ given via Definition 6.5 coincides with the $q \times q$ zero measure on $([\alpha, \beta], \mathcal{B}_{[\alpha, \beta]})$. For an arbitrary $k \in \mathbb{N}_0$, by virtue of Definitions 6.8 and 5.3 and Proposition 5.5, we infer that $\sigma^{(k)}$ is the $q \times q$ zero measure on $([\alpha, \beta], \mathcal{B}_{[\alpha, \beta]})$ if and only if $F^{(k)}$ is the constant function on $\mathbb{C} \setminus [\alpha, \beta]$ with value $O_{q \times q}$. Consequently, we have shown that (ii) is equivalent to the following statement:

(iii) $\mu$ is molecular.

(iii)$\Rightarrow$(i) Suppose that $\mu$ is molecular. From Proposition 5.5 we see that $F$ is exactly the $[\alpha, \beta]$-Stieltjes Transform $\mathcal{S}_\sigma$ of $\sigma$ given via Definition 5.3. If $\mu$ is the $q \times q$ zero
measure on \((\mathbb{R}, \mathcal{B}_\mathbb{R})\), then \(\sigma\) coincides with the \(q \times q\) zero measure on \(\{[\alpha, \beta], \mathcal{B}_{[\alpha, \beta]}\}\), hence \(\tilde{S}_\sigma\) is, by (5.1), the constant function on \(\mathbb{C} \setminus [\alpha, \beta]\) with value \(O_{q \times q}\), and, regarding \(F = \tilde{S}_\sigma\), thus (i) obviously holds true. Now assume that \(\mu\) is not the \(q \times q\) zero measure on \((\mathbb{R}, \mathcal{B}_\mathbb{R})\), i.e. \(\mu(\mathbb{R}) \neq O_{q \times q}\). Proposition 5.4 shows that the moments \(s_j := \int_{[\alpha, \beta]} x^j \sigma(dx)\) exist for all \(j \in \mathbb{N}_0\) and that we have, for each \(z \in \mathbb{C}\) with \(|z| > \max\{|\alpha|, |\beta|\}\), furthermore \(z \in \mathbb{C} \setminus [\alpha, \beta]\) and \(\tilde{S}_\sigma(z) = -\sum_{j=0}^{\infty} z^{-(j+1)} s_j\). For all \(j \in \mathbb{N}_0\), obviously \(s_j = \int_{\mathbb{R}} x^j \mu(dx)\). In view of (iii) and \(\mu(\mathbb{R}) \neq O_{q \times q}\), then [17, Rem. 4.6] shows that \((s_j)_{j=0}^{\infty}\) is, using the terminology of [17], a completely degenerate Hankel non-negative definite sequence of order \(n\) for some \(n \in \mathbb{N}\). Thus, from [17, Prop. 9.2 and Rem. 3.5] we obtain the existence of a constant \(\rho \in [0, \infty)\) and specific complex \(q \times q\) matrix polynomials \(a_n\) and \(b_n\) such that \(\det b_n\) does not vanish identically and \(\sum_{j=0}^{\infty} z^{-(j+1)} s_j = (a_n b_n^{-1})(z)\) holds true for all \(z \in \mathbb{C}\) with \(|z| > \rho\). Setting \(P := -a_n\) and \(Q := b_n\), then \(\det Q\) does not vanish identically and

\[
F(z) = \tilde{S}_\sigma(z) = -\sum_{j=0}^{\infty} z^{-(j+1)} s_j = -(a_n b_n^{-1})(z) = (PQ^{-1})(z)
\]

is valid for all \(z \in \mathbb{C}\) with \(|z| > \max\{|\alpha|, |\beta|, \rho\}\). Since \(F\) is holomorphic in \(\mathbb{C} \setminus [\alpha, \beta]\), then poles of the matrix-valued rational function \(PQ^{-1}\) can only occur in \([\alpha, \beta]\) and \(F\) coincides with the restriction of \(PQ^{-1}\) onto \(\mathbb{C} \setminus [\alpha, \beta]\). Consequently, (i) is valid.

\[
\square
\]

7 The Class \(\mathcal{P}\mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])\)

In the scalar case \(q = 1\), the set of all solutions of problem \(\mathcal{FP}([\alpha, \beta]; (s_j)_{j=0}^{2n+1})\) can be parametrized with functions of the class \(\mathcal{R}_1(\mathbb{C} \setminus [\alpha, \beta])\) augmented by the constant function with value \(\infty\) defined on \(\mathbb{C} \setminus [\alpha, \beta]\) (cf. [42, Thm. 7.2]). The corresponding approach for the matricial situation \(q \geq 1\) consists of extending the class \(\mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])\) of holomorphic matrix-valued functions according to Appendix D to some class of regular \(q \times q\) matrix pairs of meromorphic matrix-valued functions. Such a class was already considered in [10, Sec. 5]. As a first step, we extend the class \(\mathcal{R}_q(\Pi_+\), using the terminology from Appendix D and the end of Appendix F, without explaining these notations here. We only recall that a \(p \times q\) matrix pair \([P; Q]\) is said to be regular if it satisfies rank \(\begin{bmatrix} P & Q \end{bmatrix}\) = \(q\). Furthermore, we observe that the set \(\mathcal{P}(F)\) of poles of any meromorphic matrix-valued function \(F\) is discrete.

**Notation 7.1** Denote by \(\mathcal{P}\mathcal{R}_q(\Pi_+)\) the set of all ordered pairs \([P; Q]\) consisting of \(\mathbb{C}^{q \times q}\)-valued functions \(P\) and \(Q\) which are meromorphic in \(\Pi_+\), such that a discrete subset \(\mathcal{D}\) of \(\Pi_+\) exists, satisfying the following three conditions:

(I) \(\mathcal{P}(P) \cup \mathcal{P}(Q) \subseteq \mathcal{D}\).

(II) rank \(\begin{bmatrix} P(z) & Q(z) \end{bmatrix}\) = \(q\) for all \(z \in \Pi_+ \setminus \mathcal{D}\).

(III) \(\text{Im}([Q(z)]^* [P(z)]) \in \mathbb{C}^{q \times q}_{\infty}\) for all \(z \in \Pi_+ \setminus \mathcal{D}\).
Using a continuity argument, the following result is readily checked:

**Remark 7.2** If \( [P; Q] \in \mathcal{PR}_q(\Pi_+) \), then \( \text{Im}((Q(z))^{*}[P(z)]) \in \mathbb{C}^{q \times q} \) for all \( z \in \Pi_+ \setminus [\mathcal{P}(P) \cup \mathcal{P}(Q)] \).

Now we supplement Notation 7.1 in the following way:

**Notation 7.3** For each \( [P; Q] \in \mathcal{PR}_q(\Pi_+) \), denote by \( \mathcal{E}([P; Q]) \) the set of all \( z \in \Pi_+ \setminus [\mathcal{P}(P) \cup \mathcal{P}(Q)] \) satisfying \( \text{rank} \begin{bmatrix} P(z) \\ Q(z) \end{bmatrix} \neq q \).

Regarding Definition D.1, for each \( [P; Q] \in \mathcal{PR}_q(\Pi_+) \), we see that \( \mathcal{E}([P; Q]) \) is exactly the set of all points \( z \in \Pi_+ \) at which \( P \) and \( Q \) are both defined and the \( q \times q \) matrix pair \( [P(z); Q(z)] \) is not regular. In general, the linear subspace \( \mathcal{R}(\begin{bmatrix} P(z) \\ Q(z) \end{bmatrix}) \) depends on \( z \), whereas its dimension as well as the linear subspaces \( \mathcal{R}(Q(z)), \mathcal{R}(P(z)), (Q(z))(\mathcal{N}(P(z))), P(z)(\mathcal{N}(Q(z))) \), and the Non-negative integer \( \text{rk}([P(z); Q(z)]) \) given in (D.1) are essentially independent of \( z \):

**Proposition 7.4** Let \( [P; Q] \in \mathcal{PR}_q(\Pi_+) \). Then \( \mathcal{D} := \mathcal{P}(P) \cup \mathcal{P}(Q) \) is a discrete subset of \( \Pi_+ \) and \( \mathcal{E} := \mathcal{E}([P; Q]) \) is a discrete subset of \( \mathcal{G} := \Pi_+ \setminus \mathcal{P} \) admitting the representation \( \mathcal{E} = \{ z \in \mathcal{G} : \det(Q(z) - iP(z)) = 0 \} \). The set \( \mathcal{A} := \mathcal{P} \cup \mathcal{E} \) is the smallest discrete subset of \( \Pi_+ \) satisfying the conditions (I)–(III) in Notation 7.1. For all \( z \in \Pi_+ \setminus \mathcal{A} \), the \( q \times q \) matrix pair \( [P(z); Q(z)] \) is regular. For all \( z, w \in \Pi_+ \setminus \mathcal{A} \), furthermore

\[
\mathcal{R}(Q(z)) = \mathcal{R}(Q(w)), \quad \mathcal{R}(P(z)) = \mathcal{R}(P(w)),
\]

\[
Q(z)(\mathcal{N}(P(z))) = Q(w)(\mathcal{N}(P(w))), \quad P(z)(\mathcal{N}(Q(z))) = P(w)(\mathcal{N}(Q(w))),
\]

and \( \text{rk}([P(z); Q(z)]) = \text{rk}([P(w); Q(w)]) \) hold true.

**Proof** Observe that the matrix-valued functions \( P \) and \( Q \) are both meromorphic in \( \Pi_+ \). Hence, the sets \( \mathcal{P}(P) \) and \( \mathcal{P}(Q) \) of poles as well as their union \( \mathcal{P} \) are discrete subsets of \( \Pi_+ \). Consider an arbitrary discrete subset \( \mathcal{D} \) of \( \Pi_+ \), satisfying the conditions (I)–(III) in Notation 7.1. Such a subset exists by virtue of Notation 7.1. In view of Notation 7.1(II), and Notation 7.3, then the set \( \mathcal{E} \) is a subset of \( \mathcal{D} \) and hence discrete. In particular, \( \mathcal{E} \) is a discrete subset of \( \mathcal{G} \). Because of Notation 7.3 and Remark 7.2, the conditions (I)–(III) in Notation 7.1 are fulfilled where the set \( \mathcal{D} \) is substituted by \( \mathcal{A} \). Due to Notation 7.1(II), we have \( \mathcal{P} \subseteq \mathcal{D} \). Taking additionally into account \( \mathcal{E} \subseteq \mathcal{D} \), we see that the set \( \mathcal{A} \) is a subset of \( \mathcal{D} \) and thus a discrete subset of \( \Pi_+ \). Therefore, the set \( \mathcal{A} \) is the smallest discrete subset \( \mathcal{D} \) of \( \Pi_+ \) satisfying the conditions (I)–(III) in Notation 7.1. Obviously, the matrix-valued functions \( F := Q + iP \) and \( G := Q - iP \) are both holomorphic in \( \mathcal{G} \). From Lemma D.10 we infer that, for all \( z \in \mathcal{G} \) with \( \det G(z) \neq 0 \), the \( q \times q \) matrix pair \( [P(z); Q(z)] \) is regular, implying \( z \notin \mathcal{E} \). In view of Notation 7.3 and Definition D.1, we can conclude that, for all \( z \in \mathcal{G} \setminus \mathcal{E} \), the \( q \times q \) matrix pair \( [P(z); Q(z)] \) is regular and fulfills \( \text{Im}((Q(z))^{*}[P(z)]) \in \mathbb{C}^{q \times q} \), by virtue of Remark 7.2. Because of Lemma D.11, we conversely have then \( \det G(z) \neq 0 \) for all \( z \in \mathcal{G} \) with \( z \notin \mathcal{E} \). Consequently, \( \mathcal{E} = \{ z \in \mathcal{G} : \det G(z) = 0 \} \). Hence,
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Notation 7.5 (cf. [10, Def. 5.2]) Denote by \( \mathcal{PR}_q(\mathbb{C} \setminus [\alpha, \beta]) \) the set of all ordered pairs \([P; Q]\) consisting of \(\mathbb{C}^{q \times q}\)-valued functions \(P\) and \(Q\) which are meromorphic in \(\mathbb{C} \setminus [\alpha, \beta]\), for which a discrete subset \(D\) of \(\mathbb{C} \setminus [\alpha, \beta]\) exists, satisfying the following conditions:

(I) \( \mathcal{P}(P) \cup \mathcal{P}(Q) \subseteq D \).

(II) \( \text{rank} \begin{bmatrix} P(z) \\ Q(z) \end{bmatrix} = q \) for all \( z \in \mathbb{C} \setminus ([\alpha, \beta] \cup D) \).

(III) \( \frac{1}{\text{Im} z} \text{Im}((z - \alpha)[Q(z)]^*P(z)) \in \mathbb{C}^{q \times q} \) for all \( z \in \mathbb{C} \setminus (\mathbb{R} \cup D) \).

(IV) \( \frac{1}{\text{Im} z} \text{Im}((\beta - z)[Q(z)]^*P(z)) \in \mathbb{C}^{q \times q} \) for all \( z \in \mathbb{C} \setminus (\mathbb{R} \cup D) \).

Again using a continuity argument, the following result is readily checked:

Remark 7.6 If \([P; Q] \in \mathcal{PR}_q(\mathbb{C} \setminus [\alpha, \beta])\), then \( \frac{1}{\text{Im} z} \text{Im}((z - \alpha)[Q(z)]^*P(z)) \in \mathbb{C}^{q \times q} \) and \( \frac{1}{\text{Im} z} \text{Im}((\beta - z)[Q(z)]^*P(z)) \in \mathbb{C}^{q \times q} \) for all \( z \in \mathbb{C} \setminus [\mathbb{R} \cup \mathcal{P}(P) \cup \mathcal{P}(Q)] \).

As done for Notation 7.3 above, we analogously supplement Notation 7.5 in the following way:

Notation 7.7 For each \([P; Q] \in \mathcal{PR}_q(\mathbb{C} \setminus [\alpha, \beta])\) denote by \( \tilde{\mathcal{E}}([P; Q]) \) the set of all \( z \in \mathbb{C} \setminus ([\alpha, \beta] \cup \mathcal{P}(P) \cup \mathcal{P}(Q)) \) satisfying \( \text{rank} \begin{bmatrix} P(z) \\ Q(z) \end{bmatrix} \neq q \).

Regarding Definition D.1, we see that, for each \([P; Q] \in \mathcal{PR}_q(\mathbb{C} \setminus [\alpha, \beta])\), the set \( \tilde{\mathcal{E}}([P; Q]) \) is exactly the set of all points \( z \in \mathbb{C} \setminus [\alpha, \beta] \) at which \( P \) and \( Q \) are both defined and for which the \( q \times q \) matrix pair \([P(z); Q(z)]\) is not regular. The pairs belonging to \( \mathcal{PR}_q(\mathbb{C} \setminus [\alpha, \beta]) \) fulfill conditions analogous to those in Proposition 4.14 for matrix-valued functions belonging to \( \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta]) \):

Lemma 7.8 Let \([P; Q] \in \mathcal{PR}_q(\mathbb{C} \setminus [\alpha, \beta])\) and let \( \mathcal{P} := \mathcal{P}(P) \cup \mathcal{P}(Q) \). Then \( \frac{1}{\text{Im} z} \text{Im}([Q(z)]^*[P(z)]) \in \mathbb{C}^{q \times q} \) for all \( z \in \mathbb{C} \setminus [\mathbb{R} \cup \mathcal{P}] \) and \( \text{Re}([Q(w)]^*[P(w)]) \in \mathbb{C}^{q \times q} \) for all \( w \in [\mathbb{H}_-(\alpha)] \setminus \mathcal{P} \) and \( -\text{Re}([Q(w)]^*[P(w)]) \in \mathbb{C}^{q \times q} \) for all \( w \in [\mathbb{H}_+\beta)] \setminus \mathcal{P} \). Furthermore, \( [Q(x)]^*[P(x)] \in \mathbb{C}^{q \times q} \) for all \( x \in (-\infty, \alpha) \setminus \mathcal{P} \) and \( -[Q(x)]^*[P(x)] \in \mathbb{C}^{q \times q} \) for all \( x \in (\beta, \infty) \setminus \mathcal{P} \).
Consider an arbitrary \( z \in \mathbb{C}(\mathbb{R} \cup \mathcal{P}) \). We have

\[
\text{Im}((\beta - z)[Q(z)]^*[P(z)]) + \text{Im}((z - \alpha)[Q(z)]^*[P(z)]) = (\beta - \alpha) \text{Im}([Q(z)]^*[P(z)]).
\]

Regarding \( \alpha < \beta \), we obtain, by virtue of Remarks 7.6 and A.24, consequently

\[
\frac{1}{\text{Im}z} \text{Im}([Q(z)]^*[P(z)]) \in \mathbb{C}^{q \times q}. \text{ Let } A := [Q(z)]^*[P(z)]. \text{ Using Remarks A.2, A.24, and 7.6, we infer in the case } \text{Re} \ z < \alpha \text{ then }
\]

\[
\text{Re} \ A = \frac{1}{\text{Im}z} \text{Im}(zA) - \frac{\text{Re}z}{\text{Im}z} \text{Im} A \\
\geq \frac{1}{\text{Im}z} \text{Im}(zA) - \frac{\alpha}{\text{Im}z} \text{Im} A = \frac{1}{\text{Im}z} \text{Im}([z - \alpha]A) \supseteq O_{q \times q},
\]

i.e., \( \text{Re}([Q(z)]^*[P(z)]) \in \mathbb{C}^{q \times q} \). In the case \( \text{Re} \ z > \beta \), we can conclude analogously

\[
- \text{Re} \ A = \frac{\text{Re}z}{\text{Im}z} \text{Im}(A) - \frac{1}{\text{Im}z} \text{Im}(zA) \\
\geq \frac{\beta}{\text{Im}z} \text{Im}(A) - \frac{1}{\text{Im}z} \text{Im}(zA) = \frac{1}{\text{Im}z} \text{Im}([\beta - z]A) \supseteq O_{q \times q},
\]

i.e., \( -\text{Re}([Q(z)]^*[P(z)]) \in \mathbb{C}^{q \times q} \). Observe that the matrix-valued functions \( P \) and \( Q \) are both meromorphic in \( \mathbb{C} \setminus [\alpha, \beta] \). Since the set \( \mathcal{P} \) is the union of the poles of \( P \) and \( Q \), it is a discrete subset of \( \mathbb{C} \setminus [\alpha, \beta] \). Furthermore, \( P \) and \( Q \) are both holomorphic in \( \mathbb{C} \setminus ([\alpha, \beta] \cup \mathcal{P}) \). Consequently, a continuity argument shows that we have \( \text{Re}([Q(w)]^*[P(w)]) \in \mathbb{C}^{q \times q} \) for all \( w \in [\mathbb{H}_-(\alpha)] \setminus \mathcal{P} \) and \( -\text{Re}([Q(w)]^*[P(w)]) \in \mathbb{C}^{q \times q} \) for all \( w \in [\mathbb{H}_+(\beta)] \setminus \mathcal{P} \). Regarding the continuity of the function \( S: \mathbb{C} \setminus ([\alpha, \beta] \cup \mathcal{P}) \to \mathbb{C}^{q \times q} \) defined by \( S(z) := [Q(z)]^*[P(z)] \), we can conclude as in the proof of Lemma C.3 that \( \text{Im}([Q(x)]^*[P(x)]) = O_{q \times q} \) holds true for all \( x \in \mathbb{R} \setminus ([\alpha, \beta] \cup \mathcal{P}) \). Therefore, we get \( [Q(x)]^*[P(x)] = \text{Re}([Q(x)]^*[P(x)]) \) for all \( x \in \mathbb{R} \setminus ([\alpha, \beta] \cup \mathcal{P}) \). Taking into account the already shown inequalities, we can infer then \( [Q(x)]^*[P(x)] \in \mathbb{C}^{q \times q} \) for all \( x \in (-\infty, \alpha) \setminus \mathcal{P} \) and \( -[Q(x)]^*[P(x)] \in \mathbb{C}^{q \times q} \) for all \( x \in (\beta, \infty) \setminus \mathcal{P} \).

By virtue of Lemma 7.8, we can think of \( \mathcal{PR}_{q}(\mathbb{C} \setminus [\alpha, \beta]) \) as a subclass of \( \mathcal{PR}_{q}(\Pi_{+}) \) via restricting to the open upper half-plane \( \Pi_{+} \). Analogous as done for the class \( \mathcal{PR}_{q}(\Pi_{+}) \) in Proposition 7.4 above, we are now going to prove that for each pair \( [P; Q] \in \mathcal{PR}_{q}(\mathbb{C} \setminus [\alpha, \beta]) \) certain linear subspaces associated with the \( q \times q \) matrix pair \( [P(z); Q(z)] \) are essentially independent of \( z \). This is in accordance with Proposition 4.15. In the proof we will use Lemma 7.8 to reduce the situation to several open half-planes, in order to apply Proposition 7.4.

**Proposition 7.9** Let \( [P; Q] \in \mathcal{PR}_{q}(\mathbb{C} \setminus [\alpha, \beta]) \). Let \( \Pi_{1} := \Pi_{+}, \Pi_{2} := \mathbb{H}_{-}(\alpha), \Pi_{3} := \Pi_{-}, \) and \( \Pi_{4} := \mathbb{H}_{+}(\beta) \). Then \( \mathcal{P} := \mathcal{P}(\Pi_{1}) \cup \mathcal{P}(\Pi_{2}) \) is a discrete subset of \( \mathbb{C} \setminus [\alpha, \beta] \) and \( \mathcal{E} := \hat{\mathcal{E}}([P; Q]) \) is a discrete subset of \( \mathcal{G} := \mathbb{C} \setminus ([\alpha, \beta] \cup \mathcal{P}) \) with

\[
\text{Re} \ A = \frac{1}{\text{Im}z} \text{Im}(zA) - \frac{\text{Re}z}{\text{Im}z} \text{Im} A \\
\geq \frac{1}{\text{Im}z} \text{Im}(zA) - \frac{\alpha}{\text{Im}z} \text{Im} A = \frac{1}{\text{Im}z} \text{Im}([z - \alpha]A) \supseteq O_{q \times q},
\]
\( \Pi_k \cap \mathcal{E} = \{ z \in \Pi_k \setminus \mathcal{P} : \det (Q(z) - i^k P(z)) = 0 \} \) for each \( k \in \{1, 2, 3, 4\} \). The set \( \mathcal{A} := \mathcal{P} \cup \mathcal{E} \) is the smallest discrete subset \( \mathcal{D} \) of \( \mathbb{C} \setminus \{\alpha, \beta\} \) satisfying the conditions (I)--(IV) in Notation 7.5. For all \( z \in \mathbb{C} \setminus ((\alpha, \beta) \cup \mathcal{A}) \), the \( q \times q \) matrix pair \( [P(z); Q(z)] \) is regular. For every choice of \( z \) and \( w \) in \( \mathbb{C} \setminus ((\alpha, \beta) \cup \mathcal{A}) \), furthermore

\[
\begin{align*}
\mathcal{R}(Q(z)) &= \mathcal{R}(Q(w)), \\
\mathcal{R}(P(z)) &= \mathcal{R}(P(w)), \\
Q(z)(\mathcal{N}(P(z))) &= Q(w)(\mathcal{N}(P(w))), \\
P(z)(\mathcal{N}(Q(z))) &= P(w)(\mathcal{N}(Q(w))),
\end{align*}
\]

(7.1)

and \( \text{rk}([P(z); Q(z)]) = \text{rk}([P(w); Q(w)]) \) hold true.

**Proof** Since \( \mathcal{P} \) and \( \mathcal{Q} \) are matrix-valued functions meromorphic in \( \mathbb{C} \setminus \{\alpha, \beta\} \), the union \( \mathcal{P} \) of their poles is a discrete subset of \( \mathbb{C} \setminus \{\alpha, \beta\} \). Consider an arbitrary discrete subset \( \mathcal{D} \) of \( \mathbb{C} \setminus \{\alpha, \beta\} \) satisfying the conditions (I)--(IV) in Notation 7.5. Such a subset exists by virtue of Notation 7.5. In view of Notation 7.5(II) and Notation 7.7, then the set \( \mathcal{E} \) is a subset of \( \mathcal{D} \) and hence discrete. In particular, \( \mathcal{E} \) is a discrete subset of \( \mathcal{G} \). Because of Notation 7.7 and Remark 7.6, the conditions (I)--(IV) in Notation 7.5 are fulfilled with the set \( \mathcal{A} \) instead of \( \mathcal{D} \). Due to Notation 7.5(I), we have \( \mathcal{P} \subseteq \mathcal{D} \). Taking additionally into account \( \mathcal{E} \subseteq \mathcal{D} \), we see that the set \( \mathcal{A} \) is a subset of \( \mathcal{D} \) and thus a discrete subset of \( \mathbb{C} \setminus \{\alpha, \beta\} \). Therefore, the set \( \mathcal{A} \) is the smallest discrete subset \( \mathcal{D} \) of \( \mathbb{C} \setminus \{\alpha, \beta\} \) satisfying the conditions (I)--(IV) in Notation 7.5. For all \( z \in \mathbb{C} \setminus ((\alpha, \beta) \cup \mathcal{A}) \), we have furthermore \( z \in \mathcal{G} \) and \( z \notin \mathcal{E} \), implying rank \( \begin{bmatrix} P(z) \\ Q(z) \end{bmatrix} \) = \( q \) according to Notation 7.7, which, in view of Definition D.1, shows that the \( q \times q \) matrix pair \( [P(z); Q(z)] \) is regular. Let \( \phi_1(\omega) := \omega, \phi_2(\omega) := i\omega + \alpha, \phi_3(\omega) := -\omega, \) and \( \phi_4(\omega) := -i\omega + \beta \).

It is readily checked that, for each \( k \in \{1, 2, 3, 4\} \), the mapping \( \phi_k : \Pi_+ \to \Pi_k \) is bijective and that the union of their images \( \Pi_1, \Pi_2, \Pi_3, \Pi_4 \) is exactly the whole domain \( \mathbb{C} \setminus \{\alpha, \beta\} \).

Consider now an arbitrary \( k \in \{1, 2, 3, 4\} \). Since the inverse \( \psi_k := \phi_k^{-1} \) of \( \phi_k \) is an affine bijection from \( \Pi_k \) onto \( \Pi_+ \) and since the sets \( \mathcal{P} \) and \( \mathcal{E} \) are discrete, we can infer that \( \mathcal{D}_k := \psi_k(\Pi_k \cap \mathcal{A}) \) fulfills \( \mathcal{D}_k = \psi_k(\Pi_k \cap (\mathcal{P} \cup \mathcal{E})) = \psi_k(\Pi_k \cap \mathcal{P}) \cup \psi_k(\Pi_k \cap \mathcal{E}) \) and is a discrete subset of \( \Pi_+ \). Regarding Remark A.2 and Lemma 7.8, it is then readily checked that via

\[
P_k(\omega) := i^{k-1} P_0(\psi_k(\omega)) \quad \text{ and } \quad Q_k(\omega) := Q_0(\psi_k(\omega))
\]

(7.3)

matrix-valued functions \( P_k : \psi_k(\Pi_k \setminus \mathcal{P}(\omega)) \to \mathbb{C}^{q \times q} \) and \( Q_k : \psi_k(\Pi_k \setminus \mathcal{P}(\omega)) \to \mathbb{C}^{q \times q} \) are given, such that the pair \( [P_k; Q_k] \) consists of \( \mathbb{C}^{q \times q} \)-valued functions, which are meromorphic in \( \Pi_+ \), for which \( \mathcal{P}_k := \mathcal{P}(P_k) \cup \mathcal{P}(Q_k) \) fulfills \( \mathcal{P}_k = \psi_k(\Pi_k \cap \mathcal{P}(\omega)) \cup \psi_k(\Pi_k \cap \mathcal{P}(\omega)) = \psi_k(\Pi_k \cap \mathcal{P}) \subseteq \mathcal{D}_k \), and for which rank \( \begin{bmatrix} P_k(\omega) \\ Q_k(\omega) \end{bmatrix} \) = \( q \) and \( \text{Im}(Q_k(\omega))^* [P_k(\omega)] \in \mathbb{C}^{q \times q} \) hold true for all \( \omega \) in \( \psi_k(\Pi_k \setminus \mathcal{A}) = \psi_k(\Pi_k \setminus (\Pi_k \cap \mathcal{A})) = \Pi_+ \setminus \mathcal{D}_k \). Consequently, \( [P_k; Q_k] \in \mathcal{P}R_q(\Pi_+) \). In view of Proposition 7.4, then \( \mathcal{P}_k \) we see that is a discrete subset of \( \Pi_+ \) and \( \mathcal{E}_k := \mathcal{E}(P_k; Q_k) \) is a discrete
subset of $G_k := \Pi_+ \setminus \mathcal{P}_k$, admitting the representation

$$
\mathcal{E}_k = \{ \omega \in G_k : \det[Q_k(\omega) - iP_k(\omega)] = 0 \} = \{ \omega \in G_k : \det[Q(\phi_k(\omega)) - i^k P(\phi_k(\omega))] = 0 \}.
$$

Furthermore, $A_k := \mathcal{P}_k \cup \mathcal{E}_k$ is a discrete subset of $\Pi_+$ and the linear subspaces $\mathcal{R}(Q_k(\omega))$, $\mathcal{R}(P_k(\omega))$, $Q_k(\omega)(\mathcal{N}(P_k(\omega)))$, $P_k(\omega)(\mathcal{N}(Q_k(\omega)))$ and the difference of dimensions $\text{rk}([P_k(\omega); Q_k(\omega)])$ are independent of $\omega \in \Pi_+ \setminus A_k$. Since $\phi_k$ is an affine bijection from $\Pi_+$ onto $\Pi_k$, we can conclude that $Q_k := \Pi_k \cap \mathcal{P}$ fulfills $Q_k = \phi_k(\mathcal{P}_k)$ and is a discrete subset of $\Pi_k$ and that $F_k := \phi_k(\mathcal{E}_k)$ is a discrete subset of $H_k := \Pi_k \setminus Q_k$. We have $H_k = \phi_k(\Pi_+ \setminus \mathcal{P}) = \phi_k(\mathcal{G}_k)$ and

$$
\mathcal{F}_k = \{ \zeta \in H_k : \det[Q(\zeta) - i^k P(\zeta)] = 0 \}.
$$

Moreover, $B_k := Q_k \cup \mathcal{F}_k$ fulfills $B_k = \phi_k(\mathcal{P}_k) \cup \phi_k(\mathcal{E}_k) = \phi_k(\mathcal{P}_k \cup \mathcal{E}_k) = \phi_k(A_k)$ and is a discrete subset of $\Pi_k$. Thus, $\Pi_k \cap B_k = \phi_k(\Pi_+ \setminus A_k)$. Furthermore, $\mathcal{R}(Q(\zeta))$, $\mathcal{R}(P(\zeta))$, $Q(\zeta)(\mathcal{N}(P(\zeta)))$, $P(\zeta)(\mathcal{N}(Q(\zeta)))$, and $\text{rk}([P(\zeta); Q(\zeta)])$ are, in view of (7.3), independent of $\zeta \in \Pi_k \setminus B_k$, i.e., independent of $\zeta \in \phi_k(\Pi_+ \setminus A_k)$.

We are now going to verify $\Pi_k \cap \mathcal{E} = \mathcal{F}_k$. First consider an arbitrary $\zeta \in \Pi_k \cap \mathcal{E}$. Because of

$$
\mathcal{H}_k = \Pi_k \setminus Q_k = \Pi_k \setminus (\Pi_k \cap \mathcal{P}) = \Pi_k \setminus \mathcal{P}
$$

and Notation 7.7, we have $\zeta \in \mathcal{H}_k$ and $\text{rank}\left[\begin{array}{c} P(\zeta) \\ Q(\zeta) \end{array}\right] \neq q$. To the contrary, assume $\zeta \notin \mathcal{F}_k$. In view of (7.4), then $\det[Q(\zeta) - i^k P(\zeta)] \neq 0$. By virtue of Lemma D.10, hence the $q \times q$ matrix pair $[P(\zeta); Q(\zeta)]$ is regular, i.e., $\text{rank}\left[\begin{array}{c} P(\zeta) \\ Q(\zeta) \end{array}\right] = q$, according to Definition D.1. Since this is a contradiction, we necessarily have $\zeta \in \mathcal{F}_k$. Conversely, consider an arbitrary $\zeta \in \mathcal{F}_k$. Because of (7.4) and (7.5), then $\zeta \in \mathcal{H}_k = \Pi_k \setminus \mathcal{P} \subseteq \Omega \cup (\{\alpha, \beta\} \cup \mathcal{P})$. To the contrary, assume $\zeta \notin \mathcal{E}$. In view of Notation 7.7, then $\text{rank}\left[\begin{array}{c} P(\zeta) \\ Q(\zeta) \end{array}\right] = q$. According to Definition D.1, hence the $q \times q$ matrix pair $[P(\zeta); Q(\zeta)]$ is regular. Regarding (7.3) and Remark D.5, for $\omega := \psi(\zeta)$, we have $\omega \in \psi_k(\mathcal{H}_k) = G_k = \Pi_+ \setminus \mathcal{P}_k$ and

$$
\det([P_k(\omega)]^*[P_k(\omega)] + [Q_k(\omega)]^*[Q_k(\omega)]) = \det([P(\zeta)]^*[P(\zeta)] + [Q(\zeta)]^*[Q(\zeta)]) \neq 0.
$$

Consequently, due to Remark D.5, the $q \times q$ matrix pair $[P_k(\omega); Q_k(\omega)]$ is regular, i.e., $\text{rank}\left[\begin{array}{c} P_k(\omega) \\ Q_k(\omega) \end{array}\right] = q$, according to Definition D.1. By virtue of Notation 7.3, we have then $\omega \notin \mathcal{E}_k$, implying $\zeta \notin \mathcal{F}_k$. Since this is a contradiction, we see that $\zeta$ necessarily belongs to $\mathcal{E}$ and therefore to $\Pi_k \cap \mathcal{E}$. In view of (7.4) and (7.5), we obtain the equations $\Pi_k \cap \mathcal{E} = \mathcal{F}_k = \{ \zeta \in \Pi_k \setminus \mathcal{P} : \det[Q(\zeta) - i^k P(\zeta)] = 0 \}$. As already shown, for each $k \in \{1, 2, 3, 4\}$ the set $B_k$ is a discrete subset of $\Pi_k$ and the entities $\mathcal{R}(Q(\zeta))$, $\mathcal{R}(P(\zeta))$, $Q(\zeta)(\mathcal{N}(P(\zeta)))$, $P(\zeta)(\mathcal{N}(Q(\zeta)))$, and $\text{rk}([P(\zeta); Q(\zeta)])$
are independent of \( \zeta \in \Pi_k \setminus B_k \). In particular, \( \mathcal{B} := B_1 \cup \cdots \cup B_4 \) is a discrete subset of \( \Pi_1 \cup \cdots \cup \Pi_4 = \mathbb{C} \setminus [\alpha, \beta] \). Therefore, the sets \( (\Pi_1 \cap \Pi_2) \setminus \mathcal{B} \), \( (\Pi_2 \cap \Pi_3) \setminus \mathcal{B} \), and \( (\Pi_3 \cap \Pi_4) \setminus \mathcal{B} \) are non-empty. Consequently, we can infer that the linear subspaces \( \mathcal{R}(Q(\zeta)), \mathcal{R}(P(\zeta)), Q(\zeta)(\mathcal{N}(P(\zeta))), P(\zeta)(\mathcal{N}(Q(\zeta))) \), and the difference of dimensions \( \text{rk}([P(z); Q(z)]) \) are independent of \( \zeta \in \mathbb{C} \setminus ([\alpha, \beta] \cup \mathcal{B}) \). Because of

\[
P = \bigcup_{k=1}^{4}(\Pi_k \cap \mathcal{P}) = \bigcup_{k=1}^{4}Q_k \quad \text{and} \quad \mathcal{E} = \bigcup_{k=1}^{4}(\Pi_k \cap \mathcal{E}) = \bigcup_{k=1}^{4}F_k,
\]

we have furthermore \( \mathcal{A} = \mathcal{P} \cup \mathcal{E} = (Q_1 \cup F_1) \cup \cdots \cup (Q_4 \cup F_4) = B_1 \cup \cdots \cup B_4 = \mathcal{B} \). Thus, (7.1), (7.2), and \( \text{rk}([P(z); Q(z)]) = \text{rk}([P(w); Q(w)]) \) follow for all \( z, w \in \mathbb{C} \setminus ([\alpha, \beta] \cup \mathcal{A}) \). \( \Box \)

As is easily seen, the class \( \mathcal{PR}_q(\mathbb{C} \setminus [\alpha, \beta]) \) is closed under right multiplication by meromorphic matrix-valued functions \( R \) with not identically vanishing determinant:

**Remark 7.10** Let \( [P; Q] \in \mathcal{PR}_q(\mathbb{C} \setminus [\alpha, \beta]) \) and let \( R \) be a \( \mathbb{C}^q \times q \)-valued function meromorphic in \( \mathbb{C} \setminus [\alpha, \beta] \) such that \( \det R \) does not vanish identically in \( \mathbb{C} \setminus [\alpha, \beta] \). Then \( [PR; QR] \in \mathcal{PR}_q(\mathbb{C} \setminus [\alpha, \beta]) \).

In view of the remarks on meromorphic matrix-valued functions given at the end of Appendix F, it is readily checked that an equivalence relation on the set \( \mathcal{PR}_q(\mathbb{C} \setminus [\alpha, \beta]) \) is given. Regarding Remark D.8, this equivalence relation is in accordance with that considered in Appendix D for arbitrary \( p \times q \) matrix pairs.

**Definition 7.11** Two pairs \( [P; Q], [S; T] \in \mathcal{PR}_q(\mathbb{C} \setminus [\alpha, \beta]) \) are said to be equivalent if there exists a \( \mathbb{C}^q \times q \)-valued function \( R \) meromorphic in \( \mathbb{C} \setminus [\alpha, \beta] \) such that \( \det R \) does not vanish identically in \( \mathbb{C} \setminus [\alpha, \beta] \) which fulfills \( S = PR \) and \( T = QR \). In this case, we write \( [P; Q] \sim [S; T] \). Furthermore, denote by \( \langle [P; Q] \rangle \) the equivalence class of a pair \( [P; Q] \in \mathcal{PR}_q(\mathbb{C} \setminus [\alpha, \beta]) \) and by \( \langle Q \rangle := \{([S; T]) : [S; T] \in \mathcal{Q} \} \) the set of equivalence classes of pairs belonging to a subset \( \mathcal{Q} \) of \( \mathcal{PR}_q(\mathbb{C} \setminus [\alpha, \beta]) \).

Using Remark 4.8, the following remark can be easily concluded from Proposition 4.18:

**Remark 7.12** (cf. [10, Rem. 5.4]) Let \( F \in \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta]) \) and let the functions \( P, Q : \mathbb{C} \setminus [\alpha, \beta] \to \mathbb{C}^q \times q \) be defined by \( P(z) := F(z) \) and \( Q(z) := I_q \). Then the pair \( [P; Q] \) belongs to \( \mathcal{PR}_q(\mathbb{C} \setminus [\alpha, \beta]) \) and \( \det Q(z) \neq 0 \) holds true for all \( z \in \mathbb{C} \setminus [\alpha, \beta] \).

Conversely, we have:

**Lemma 7.13** (cf. [10, Prop. 5.7]) Let \( [P; Q] \in \mathcal{PR}_q(\mathbb{C} \setminus [\alpha, \beta]) \) be such that \( \det Q \) does not identically vanish in \( \mathbb{C} \setminus [\alpha, \beta] \). Then \( F := PQ^{-1} \) belongs to \( \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta]) \). Furthermore, the pair \( [S; T] \) consisting of the functions \( S, T : \mathbb{C} \setminus [\alpha, \beta] \to \mathbb{C}^q \times q \) defined by \( S(z) := F(z) \) and \( T(z) := I_q \) belongs to \( \mathcal{PR}_q(\mathbb{C} \setminus [\alpha, \beta]) \) and fulfills \( [P; Q] \sim [S; T] \) and \( \det T(z) \neq 0 \) for all \( z \in \mathbb{C} \setminus [\alpha, \beta] \).
Due to [10, Prop. 5.7], we have \( F \in \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta]) \). In view of Remark 7.12, we get then \([S; T] \in \mathcal{PR}_q(\mathbb{C} \setminus [\alpha, \beta])\) and \( \det T(z) \neq 0 \) for all \( z \in \mathbb{C} \setminus [\alpha, \beta] \). Furthermore, \( R := Q^{-1} \) is a \( \mathbb{C}^{q \times q} \)-valued function, which is meromorphic in \( \mathbb{C} \setminus [\alpha, \beta] \), satisfying \( S = PR \) and \( T = QR \). Since \( \det R = (\det Q)^{-1} \) does not identically vanish in \( \mathbb{C} \setminus [\alpha, \beta] \), thus \([P; Q] \sim [S; T]\) follows.

We end this section with an example of a simple family of pairs belonging to \( \mathcal{PR}_q(\mathbb{C} \setminus [\alpha, \beta]) \).

**Remark 7.14** Let \( z \in \mathbb{C} \), let \( x := z - \alpha \), and let \( y := \beta - z \). Involving \( \delta = \beta - \alpha \), it is readily checked that \( i(y\overline{x} - x\overline{y}) = 2\delta \Im(z) \) and \( |y|^2 x + |x|^2 y = \delta yx \).

Given two complex matrices \( A \) and \( B \), we will use the notation

\[
A \oplus B := \begin{bmatrix} A & O \\ O & B \end{bmatrix}.
\]

(7.6)

**Example 7.15** Let \( X, Y \in \mathbb{C}^{q \times q} \) satisfy \( \text{rank}\left[\begin{array}{c} X \\ Y \end{array}\right] = q \) and \( Y^*X \in \mathbb{C}^{q \times q} \). Let \( P, Q : \mathbb{C} \setminus [\alpha, \beta] \to \mathbb{C}^{q \times q} \) be defined by \( P(z) := X \) and \( Q(z) := Y \) and let \( g, h : \mathbb{C} \setminus [\alpha, \beta] \to \mathbb{C} \) be given by \( g(z) := z - \alpha \) and \( h(z) := \beta - z \), resp. Denote by \( \mathcal{I}_q \) and \( \mathcal{O}_q \) the constant \( q \times q \) matrix-valued functions defined on \( \mathbb{C} \setminus [\alpha, \beta] \) with values \( I_q \) and \( O_{q \times q} \), resp. Then:

(a) The pairs \([P; hQ]\) and \([-P; gQ]\) belong to \( \mathcal{PR}_q(\mathbb{C} \setminus [\alpha, \beta]) \).

(b) If \( X = O_{q \times q} \), then \([P; hQ]\) and \([-P; gQ]\) are equivalent to \([\mathcal{O}_q; \mathcal{I}_q]\).

(c) If \( Y = O_{q \times q} \), then \([P; hQ]\) and \([-P; gQ]\) are equivalent to \([\mathcal{I}_q; \mathcal{O}_q]\).

We verify that statements (a)–(c) are true:

(a) The functions \( P, hQ, -P, \) and \( gQ \) are holomorphic in \( \mathbb{C} \setminus [\alpha, \beta] \). Consider an arbitrary \( z \in \mathbb{C} \setminus [\alpha, \beta] \). Let \( x := z - \alpha \) and let \( y := \beta - z \). Observe that the matrices \( I_q \oplus (yI_q) \) and \( -I_q \oplus (xI_q) \) are invertible. Consequently, we get

\[
\text{rank}\left[\begin{array}{c} P(z) \\ h(z)Q(z) \end{array}\right] = \text{rank}\left[\begin{array}{c} X \\ yY \end{array}\right] = \text{rank}(I_q \oplus (yI_q)) = \text{rank}\left[\begin{array}{c} X \\ Y \end{array}\right] = q
\]

and

\[
\text{rank}\left[\begin{array}{c} -P(z) \\ g(z)Q(z) \end{array}\right] = \text{rank}\left[\begin{array}{c} -X \\ xY \end{array}\right] = \text{rank}((-I_q) \oplus (xI_q)) = \text{rank}\left[\begin{array}{c} -X \\ Y \end{array}\right] = q.
\]

From the first equation in Remark 7.14 we can conclude \( \Im(x\overline{y}) = \delta \Im z \) and \( \Im(y\overline{x}) = -\delta \Im z \). Taking additionally into account \( Y^*X \in \mathbb{C}^{q \times q} \), we thus obtain

\[
\Im((z - \alpha)[h(z)Q(z)]^*[P(z)]) = \Im(x\overline{y}Y^*X) = \Im(xy^*)Y^*X = \delta \Im(z)Y^*X,
\]

\[
\Im((\beta - z)[h(z)Q(z)]^*[P(z)]) = \Im(|y|^2Y^*X) = O_{q \times q},
\]

\[
\Im((z - \alpha)[g(z)Q(z)]^*[-P(z)]) = \Im(-|x|^2Y^*X) = O_{q \times q},
\]

and

\[
\Im((\beta - z)[g(z)Q(z)]^*[-P(z)]) = \Im(-y\overline{x}Y^*X)
\]

\[
= -\Im(y\overline{x})Y^*X = \delta \Im(z)Y^*X.
\]
Now assume in addition $z \notin \mathbb{R}$. Because of $\delta > 0$ and $Y^*X \in \mathbb{C}^{q \times q}$, we can infer

\[
\frac{1}{\text{Im} z} \text{Im}((z - \alpha)[h(z)Q(z)]^*P(z)) \in \mathbb{C}^{q \times q},
\]
\[
\frac{1}{\text{Im} z} \text{Im}((\beta - z)[h(z)Q(z)]^*P(z)) \in \mathbb{C}^{q \times q},
\]
\[
\frac{1}{\text{Im} z} \text{Im}((z - \alpha)[g(z)Q(z)]^*[-P(z)]) \in \mathbb{C}^{q \times q},
\]

and

\[
\frac{1}{\text{Im} z} \text{Im}((\beta - z)[g(z)Q(z)]^*[-P(z)]) \in \mathbb{C}^{q \times q},
\]

by virtue of Remark A.24. Hence, $[P; hQ]$ and $[-P; gQ]$ belong to $\mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$.

(b) Assume $X = O_{q \times q}$. Then $[P; hQ] = [\varnothing_q; hQ]$ and $[-P; gQ] = [\varnothing_q; gQ]$.

Since $\text{rank}\left[\begin{array}{c}
O_q \\
Y
\end{array}\right] = q$, we have $\det Y \neq 0$. Hence, $\det(hQ)$ and $\det(gQ)$ both do not vanish identically in $\mathbb{C} \setminus [\alpha, \beta]$. Because of

\[
\left[\begin{array}{c}
P(hQ)^{-1} \\
(hQ)(hQ)^{-1}
\end{array}\right] = \left[\begin{array}{c}
\varnothing_q(hQ)^{-1} \\
(hQ)(hQ)^{-1}
\end{array}\right] = \left[\begin{array}{c}
\varnothing_q \\
\varnothing_q
\end{array}\right],
\]

and

\[
\left[\begin{array}{c}
(-P)(gQ)^{-1} \\
(gQ)(gQ)^{-1}
\end{array}\right] = \left[\begin{array}{c}
\varnothing_q(gQ)^{-1} \\
(gQ)(gQ)^{-1}
\end{array}\right] = \left[\begin{array}{c}
\varnothing_q \\
\varnothing_q
\end{array}\right],
\]

the pairs $[P; hQ]$ and $[-P; gQ]$ are then both equivalent to $[\varnothing_q; \varnothing_q]$.

(c) Assume $Y = O_{q \times q}$. Then $[P; hQ] = [P; \varnothing_q]$ and $[-P; gQ] = [-P; \varnothing_q]$.

Since $\text{rank}\left[\begin{array}{c}
X \\
O_q
\end{array}\right] = q$, we have $\det X \neq 0$. Hence, $\det P$ and $\det(-P)$ both do not vanish identically in $\mathbb{C} \setminus [\alpha, \beta]$. Because of

\[
\left[\begin{array}{c}
P P^{-1} \\
(hQ)P^{-1}
\end{array}\right] = \left[\begin{array}{c}
P P^{-1} \\
\varnothing_q P^{-1}
\end{array}\right] = \left[\begin{array}{c}
\varnothing_q \\
\varnothing_q
\end{array}\right],
\]

and

\[
\left[\begin{array}{c}
(-P)(-P)^{-1} \\
(gQ)(-P)^{-1}
\end{array}\right] = \left[\begin{array}{c}
(-P)(-P)^{-1} \\
\varnothing_q(-P)^{-1}
\end{array}\right] = \left[\begin{array}{c}
\varnothing_q \\
\varnothing_q
\end{array}\right],
\]

the pairs $[P; hQ]$ and $[-P; gQ]$ are then both equivalent to $[\varnothing_q; \varnothing_q]$. 
8 The Class of Parameters

The pairs belonging to the subclass of \( \mathcal{PR}_q(\mathbb{C} \setminus [\alpha, \beta]) \) introduced below generate the equivalence classes, which will be used in Sect. 14 as parameters in the description of the set of all solutions to Problem FP\([\alpha, \beta]; \{s_j\}_{j=0}^m\].

**Notation 8.1** For each \( M \in \mathbb{C}^{q \times p} \), let \( \tilde{\mathcal{P}}[M] \) be the set of all pairs \( \{F; G\} \in \mathcal{PR}_q(\mathbb{C} \setminus [\alpha, \beta]) \) for which there exists a \( z_0 \in \mathbb{C}\setminus([\alpha, \beta] \cup \mathcal{P}(F) \cup \mathcal{P}(G) \cup \mathcal{E}([F; G])) \) such that \( \mathcal{R}(F(z_0)) \subseteq \mathcal{R}(M) \).

**Remark 8.2** If \( M \in \mathbb{C}^{q \times p} \) fulfills \( \text{rank } M = q \), then \( \tilde{\mathcal{P}}[M] = \mathcal{PR}_q(\mathbb{C} \setminus [\alpha, \beta]) \).

The class \( \tilde{\mathcal{P}}[M] \) can be characterized by an additional equation involving the transformation matrix \( \mathbb{P}_{\mathcal{R}(M)} \) corresponding to the orthogonal projection onto the column space \( \mathcal{R}(M) \):

**Lemma 8.3** Let \( M \in \mathbb{C}^{q \times p} \) and let \( \{F; G\} \in \mathcal{PR}_q(\mathbb{C} \setminus [\alpha, \beta]) \). Then \( \{F; G\} \in \tilde{\mathcal{P}}[M] \) if and only if \( \mathbb{P}_{\mathcal{R}(M)} F = F \). In this case, \( \mathcal{R}(F(z)) \subseteq \mathcal{R}(M) \) for all \( z \in \mathbb{C}\setminus([\alpha, \beta] \cup \mathcal{P}(F)) \).

**Proof** First observe that \( F \) is a matrix-valued function meromorphic in \( \mathbb{C} \setminus [\alpha, \beta] \) and that \( \mathcal{G} := \mathbb{C}\setminus([\alpha, \beta] \cup \mathcal{P}(F)) \) is exactly the set of points at which \( F \) is holomorphic.

Therefore, \( \mathbb{P}_{\mathcal{R}(M)} F \) is a matrix-valued function meromorphic in \( \mathbb{C} \setminus [\alpha, \beta] \) and \( \mathcal{G} \) is exactly the set of points at which \( \mathbb{P}_{\mathcal{R}(M)} F \) is holomorphic. In view of Proposition 7.9, the set \( \mathcal{A} := \mathcal{P}(F) \cup \mathcal{P}(G) \cup \mathcal{E}([F; G]) \) is a discrete subset of \( \mathbb{C}\setminus([\alpha, \beta] \cup \mathcal{A}) \).

Assume \( \{F; G\} \in \tilde{\mathcal{P}}[M] \). Then there exists some \( z_0 \in \mathbb{C}\setminus([\alpha, \beta] \cup \mathcal{A}) \) with \( \mathcal{R}(F(z_0)) \subseteq \mathcal{R}(M) \). As a subset of \( \mathcal{A} \), the set \( \mathcal{D} := \mathcal{P}(G) \cup \mathcal{E}([F; G]) \) is discrete and the function \( F \) which is holomorphic in \( \mathcal{G} \) fulfills, for all \( w \in \mathbb{C}\setminus[\alpha, \beta] \), \( \mathcal{R}(F(w)) = \mathcal{R}(F(z_0)) \subseteq \mathcal{R}(M) \), implying \( \mathbb{P}_{\mathcal{R}(M)} F(w) = F(w) \).

Conversely, assume \( \mathbb{P}_{\mathcal{R}(M)} F = F \). Since the set \( \mathcal{A} \) is discrete, there exists some \( z_0 \in \mathbb{C}\setminus([\alpha, \beta] \cup \mathcal{A}) \). In particular, \( z_0 \in \mathcal{G} \) and, consequently, \( \mathbb{P}_{\mathcal{R}(M)} F(z_0) = F(z_0) \).

Using Lemma 8.3, it is readily checked that the equivalence relation on the set \( \mathcal{PR}_q(\mathbb{C} \setminus [\alpha, \beta]) \) introduced in Definition 7.11 is compatible with the here considered subclass \( \tilde{\mathcal{P}}[M] \) in the following sense:

**Remark 8.4** Let \( M \in \mathbb{C}^{q \times p} \) and let \( \{F; G\} \in \tilde{\mathcal{P}}[M] \). Then \( \{\tilde{F}; \tilde{G}\} \in \tilde{\mathcal{P}}[M] \) for all \( \{F; G\} \in \{(F; G)\} \).

We can obtain a description of the set of equivalence classes \( \langle \tilde{\mathcal{P}}[M] \rangle \), depending on the rank \( r \) of the matrix \( M \) in terms of equivalence classes of pairs belonging to \( \mathcal{PR}_r(\mathbb{C} \setminus [\alpha, \beta]) \):

**Lemma 8.5** Let \( M \in \mathbb{C}^{q \times p} \) and let \( r := \text{rank } M \):
(a) If \( r = 0 \), then \( \langle \tilde{P} [M] \rangle = \{ \langle F_0; G_0 \rangle \} \) where \( F_0, G_0 : \mathbb{C} \setminus [\alpha, \beta] \to \mathbb{C}^{q \times q} \) are defined by \( F_0(z) := O_{q \times q} \) and \( G_0(z) := I_q \).

(b) Assume \( r \geq 1 \). Let \( u_1, u_2, \ldots, u_r \) be an arbitrary orthonormal basis of \( \mathcal{R}(M) \), let \( U := [u_1, u_2, \ldots, u_r] \), and let \( \Gamma_U : \{ P \mathcal{R}_r(\mathbb{C} \setminus [\alpha, \beta]) \} \to \{ \tilde{P} [M] \} \) be defined by \( \Gamma_U (\langle f; g \rangle) := \{ (UfU^*; UgU^* + \mathbb{P}_{[\mathcal{R}(M)]} ) \} \). Then \( \Gamma_U \) is well defined and bijective.

**Proof** First assume \( r = 0 \), i.e., \( M = O_{q \times p} \). Hence, \( \mathbb{P}_{\mathcal{R}(M)} = O_{q \times q} \). Consider now an arbitrary pair \( [F_1; G_1] \in \tilde{P} [M] \). By virtue of Lemma 8.3, then \( F_1 = \mathbb{P}_{\mathcal{R}(M)} F_1 = F_0 \) follows. Observe that \( [F_1; G_1] \) belongs to \( \mathcal{P} \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta]) \). In view of Notation 7.5, thus \( \text{rank} \left[ F_1(z_0) \right] = q \) for some \( z_0 \in \mathbb{C} \setminus ([\alpha, \beta] \cup \mathcal{P}(F_1) \cup \mathcal{P}(G_1)) \). Because of \( F_1(z_0) = F_0(z_0) = O_{q \times q} \), then necessarily \( \det G_1(z_0) \neq 0 \) holds true. In particular, det \( G_1 \) does not vanish identically in \( \mathbb{C} \setminus [\alpha, \beta] \). Consequently, the application of Lemma 7.13 to the pair \( [F_1; G_1] \) and regarding \( F_1 G_1^{-1} = F_0 G_1^{-1} = F_0 \), we obtain \( [F_1; G_1] \sim [F_0; G_0] \). Therefore, \( \{ \tilde{P} [M] \} \subseteq \{ \langle F_0; G_0 \rangle \} \) is verified. Using Remark 7.12 and Lemma 8.3, we can easily infer \( \langle F_0; G_0 \rangle \in \tilde{P} [M] \). Hence, \( \{ \tilde{P} [M] \} = \{ \langle F_0; G_0 \rangle \} \).

Now assume \( r \geq 1 \). We have \( U^* U = I_r \) and \( \mathcal{R}(U) = \mathcal{R}(M) \). Let \( N := \mathbb{P}_{[\mathcal{R}(M)]} \). Using Remarks A.11 and A.12, we immediately obtain the representations

\[
N = \mathbb{P}_{[\mathcal{R}(U)]} = I_q - \mathbb{P}_{\mathcal{R}(U)} = I_q - U U^*.
\]

(8.1)

In particular, we see

\[
N U = U - U U^* U = O_{q \times r} \quad \text{and} \quad U^* N = U^* - U U^* U^* = O_{r \times q}.
\]

(8.2)

We first consider an arbitrary pair \( [f; g] \in \mathcal{P} \mathcal{R}_r(\mathbb{C} \setminus [\alpha, \beta]) \) and show that \( \Gamma_U (\langle f; g \rangle) \) belongs to \( \{ \tilde{P} [M] \} \): According to Notation 7.5, the \( \mathbb{C}^{r \times r} \)-valued functions \( f \) and \( g \) are meromorphic in \( \mathbb{C} \setminus [\alpha, \beta] \) and there exists a discrete subset \( D \) of \( \mathbb{C} \setminus [\alpha, \beta] \) with \( \mathcal{P}(f) \cup \mathcal{P}(g) \subseteq D \) such that \( \text{rank} \left[ f(z) \right] = r \) for all \( z \in \mathbb{C} \setminus ([\alpha, \beta] \cup D) \) and furthermore \( \frac{1}{\text{Im} z} \text{Im}((z - \alpha)(g(z))^* [f(z)]) \in \mathbb{C}^{q \times q}_{<p} \) and \( \frac{1}{\text{Im} z} \text{Im}((\beta - z)(g(z))^* [f(z)]) \in \mathbb{C}^{q \times q}_{<p} \) for all \( z \in \mathbb{C} \setminus (\mathbb{R} \cup D) \) hold true. Obviously,

\[
F := U f U^* \quad \text{and} \quad G := U g U^* + \mathbb{P}_{[\mathcal{R}(M)]} \quad \text{(8.3)}
\]

are \( \mathbb{C}^{q \times q} \)-valued functions meromorphic in \( \mathbb{C} \setminus [\alpha, \beta] \) with \( \mathcal{P}(F) \subseteq \mathcal{P}(f) \) and \( \mathcal{P}(G) \subseteq \mathcal{P}(g) \). Thus, \( \mathcal{P}(F) \cup \mathcal{P}(G) \subseteq D \) follows. Consider an arbitrary \( z \in \mathbb{C} \setminus ([\alpha, \beta] \cup D) \). In view of Definition D.1, the \( r \times r \) matrix pair \( [f(z); g(z)] \) is regular. Furthermore, we have \( F(z) = U [f(z)] U^* \) and \( G(z) := U [g(z)] U^* + \mathbb{P}_{[\mathcal{R}(M)]} \). By virtue of Lemma D.12, then \( [F(z); G(z)] \) is a regular \( q \times q \) matrix pair fulfilling \( \mathcal{R}(F(z)) \subseteq \mathcal{R}(U) \) and \( [G(z)]^* [F(z)] = U ([g(z)]^* [f(z)]) U^* \). According to Definition D.1, in particular, \( \text{rank} \left[ F(z) \right] = q \). In the case \( z \notin \mathbb{R} \), using Remarks A.2 and A.25, we can infer \( \frac{1}{\text{Im} z} \text{Im} \left( (z - \alpha)(g(z))^*[f(z)] \right) = U \left[ \frac{1}{\text{Im} z} \text{Im}((z - \alpha)(g(z))^*[f(z)]) \right] U^* \in \mathbb{C}^{q \times q}_{<p} \).
and similarly $\frac{1}{\im z} \text{Im}((\beta - z)[G(z)]^* [F(z)]) \in \mathbb{C}^{q \times q}$. So $\mathcal{D}$ is a discrete subset of $\mathbb{C} \setminus [\alpha, \beta]$ such that the conditions (I)-(IV) in Notation 7.5 are fulfilled for $[P; Q] = [F; G]$. Consequently, $[F; G]$ belongs to $\mathcal{PR}_q(\mathbb{C} \setminus [\alpha, \beta])$. Because of rank $\begin{bmatrix} F(z) \\ G(z) \end{bmatrix} = q$, we see from Notation 7.7 that $z \not\in \mathcal{E}([F; G])$. Summarizing, we have $z \in \mathbb{C} \setminus ([\alpha, \beta] \cup \mathcal{P}(F) \cup \mathcal{P}(G) \cup \mathcal{E}([F; G]))$ and $\mathcal{R}(F(z)) \subseteq \mathcal{R}(U) = \mathcal{R}(M)$. Therefore, we obtain $[F; G] \in \tilde{\mathcal{P}}[M]$ and, regarding (8.3), thus $\Gamma_U(([f; g])) \in \tilde{\mathcal{P}}[M]$ follows.

Next we are going to show that $\Gamma_U(([f; g]))$ is independent of the choice of the particular representative $[f; g]$ of the equivalence class $([f; g]) \in \langle \mathcal{PR}_r(\mathbb{C} \setminus [\alpha, \beta]) \rangle$. To that end, consider two arbitrary pairs $[f_1; g_1]$ and $[f_2; g_2]$ from $\mathcal{PR}_r(\mathbb{C} \setminus [\alpha, \beta])$ satisfying $[f_1; g_1] \sim [f_2; g_2]$. For each $j \in \{1, 2\}$, let

$$F_j := U f_j U^* \quad \text{and} \quad G_j := U g_j U^* + N. \quad (8.4)$$

According to Definition 7.11, there is a $\mathbb{C}^r \times r$-valued function $\rho$ meromorphic in $\mathbb{C} \setminus [\alpha, \beta]$ such that $\det \rho$ does not vanish identically in $\mathbb{C} \setminus [\alpha, \beta]$, for which $f_2 = f_1 \rho$ and $g_2 = g_1 \rho$. Let $R := U \rho U^* + N$. Then $R$ is a $\mathbb{C}^{q \times q}$-valued function meromorphic in $\mathbb{C} \setminus [\alpha, \beta]$. Regarding $U^* U = I_r, N^2 = N,$ and (8.2), we get $F_1 R = U f_1 U^* \rho U^* + U f_1 U^* N = U f_1 \rho U^* = U f_2 U^* = F_2$ and

$$G_1 R = U g_1 U^* U \rho U^* + U g_1 U^* N + N U \rho U^* + N^2$$

$$= U g_1 \rho U^* + N^2 = U g_2 U^* + N = G_2.$$

Furthermore, there exists some $z_0 \in \mathbb{C} \setminus [\alpha, \beta]$ such that $\rho$ is holomorphic at $z_0$ with $\det \rho(z_0) \neq 0$. In addition, we are going to check now that $\det R(z_0) \neq 0$. For this reason, consider an arbitrary vector $v \in \mathcal{N}(R(z_0))$. Let $w := U^* v$. Then we have

$$U \rho(z_0) w + N v = [U \rho(z_0) U^* + N] v = R(z_0) v = O_{q \times 1}, \quad (8.5)$$

implying $U^* U \rho(z_0) w + U^* N v = O_{r \times 1}$. In view of $U^* U = I_r$ and (8.2), thus $\rho(z_0) w = O_{r \times 1}$ follows. Because of $\det \rho(z_0) \neq 0$, then necessarily $w = O_{r \times 1}$ holds true. Substituting this into (8.5), we get $N v = O_{q \times 1}$. Regarding (8.1), hence $v = U U^* v = U w = O_{q \times 1}$. Therefore, the linear subspace $\mathcal{N}(R(z_0))$ is trivial, implying $\det R(z_0) \neq 0$. In particular, det $R$ does not vanish identically in $\mathbb{C} \setminus [\alpha, \beta]$. According to Definition 7.11, then $[F_1; G_1] \sim [F_2; G_2]$. Consequently, $\Gamma_U(([f_1; g_1])) = \Gamma_U(([f_2; g_2]))$. Thus, the mapping $\Gamma_U$ is well defined. We are now going to show that the mapping $\Gamma_U$ is injective. To that end, consider two arbitrary pairs $[f_1; g_1]$ and $[f_2; g_2]$ from $\mathcal{PR}_r(\mathbb{C} \setminus [\alpha, \beta])$ satisfying $\Gamma_U(([f_1; g_1])) = \Gamma_U(([f_2; g_2]))$. Let $[F_1; G_1]$ and $[F_2; G_2]$ be given via (8.4). Then $[F_1; G_1] \sim [F_2; G_2]$. Hence, according to Definition 7.11, there exists a $\mathbb{C}^{q \times q}$-valued function $R$ meromorphic in $\mathbb{C} \setminus [\alpha, \beta]$ such that $\det R$ does not vanish identically in $\mathbb{C} \setminus [\alpha, \beta]$, fulfilling $F_2 = F_1 R$ and $G_2 = G_1 R$. Let $\rho := U^* R U$. Then $\rho$ is a $\mathbb{C}^r \times r$-valued function meromorphic in $\mathbb{C} \setminus [\alpha, \beta]$. Regarding $U^* U = I_r$, (8.4), and (8.2),
we have \( f_1 \rho = U^* U f_1 U^* R U = U^* F_1 R U = U^* F_2 U = U^* U f_2 U^* U = f_2 \)
and
\[
\begin{align*}
g_1 \rho &= U^* U g_1 U^* R U = U^* (G_1 - N) R U = U^* G_1 R U \\
&= U^* G_2 U = U^* U g_2 U^* U = g_2.
\end{align*}
\]

Furthermore, there exists some \( z_0 \in \mathbb{C} \setminus [\alpha, \beta] \) such that \( R \) is holomorphic at \( z_0 \) with \( \det R(z_0) \neq 0 \). In addition, we now prove that \( \det (z_0) \neq 0 \). To do this, we consider an arbitrary vector \( w \in \mathcal{N}(\rho(z_0)) \). Let \( v := U w \). Then \( U^* R(z_0) v = U^* R(z_0) U w = \rho(z_0) w = O_{r \times 1} \). Because of \((8.4)\) and \((8.2)\), we have \( N G_j = N^2 = N \) for each \( j \in \{1, 2\} \). In view of \( G_2 = G_1 R \) and \((8.1)\), we hence get \( N = N R = R - U U^* R \). From \((8.2)\) we infer \( N v = N U w = O_{q \times 1} \). Taking additionally into account \( U^* R(z_0) v = O_{q \times 1} \), we can then conclude \( R(z_0) v = O_{q \times 1} \). Since \( \det R(z_0) \neq 0 \) holds true, necessarily \( v = O_{q \times 1} \) follows. Regarding \( U^* U = I_r \), we thus obtain \( w = U^* v = O_{r \times 1} \). Therefore, the linear subspace \( \mathcal{N}(\rho(z_0)) \) is trivial, implying \( \det (z_0) \neq 0 \). In particular, \( \rho \) does not vanish identically in \( \mathbb{C} \setminus [\alpha, \beta] \). According to Definition 7.11, consequently \( \{f_1; g_2\} \sim \{f_2; g_2\} \), i.e., \( \{f_1; g_1\} = \{f_2; g_2\} \).

We finish the proof by showing that the mapping \( \rho : U \rightarrow \text{ } \) is surjective. To that end, consider an arbitrary pair \([F_1; G_1]\) from \( \mathcal{P}[M] \). Then \([F_1; G_1] \in \mathcal{P} \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta]) \). According to Notation 7.5, the \( \mathbb{C}^{q \times d} \)-valued functions \( F_1 \) and \( G_1 \) are meromorphic in \( \mathbb{C} \setminus [\alpha, \beta] \). Consequently, \( B := G_1 - i F_1 \) is a \( \mathbb{C}^{q \times q} \)-valued function meromorphic in \( \mathbb{C} \setminus [\alpha, \beta] \). Due to Proposition 7.9, the set \( A := \mathcal{P}(F_1) \cup \mathcal{P}(G_1) \cup \mathcal{E}([F_1; G_1]) \) is a discrete subset of \( \mathbb{C} \setminus [\alpha, \beta] \). Consider an arbitrary \( w \in \Pi^+ \setminus A \). In view of Proposition 7.9, the \( q \times q \) matrix pair \([F_1(w); G_1(w)]\) is regular. Using Lemma 7.8 and Remark A.24, we get \( \text{Im}([G(w)]^*[F(w)]) \in \mathbb{C}^{q \times q} \). We see from Lemma 8.3 moreover \( \mathcal{R}(F_1(w)) \subseteq \mathcal{R}(M) \). Therefore, Proposition D.13 applies to the \( q \times q \) matrix pair \([F_1(w); G_1(w)]\) and we get \( \det B(w) \neq 0 \). In particular, \( B \) does not vanish identically in \( \mathbb{C} \setminus [\alpha, \beta] \). Hence, \( R := B^{-1} \) is a \( \mathbb{C}^{q \times q} \)-valued function meromorphic in \( \mathbb{C} \setminus [\alpha, \beta] \). Consequently,
\[
f := U^* F_1 R U \quad \text{and} \quad g := U^* G_1 R U
\]
are \( \mathbb{C}^{r \times r} \)-valued functions meromorphic in \( \mathbb{C} \setminus [\alpha, \beta] \). In addition, \( f \) and \( g \) are both holomorphic at \( w \) with \( f(w) = U^*[F_1(w)][B(w)]^{-1} U \) and \( g(w) = U^*[G_1(w)][B(w)]^{-1} U \). Thus,
\[
F_2 := U f U^* \quad \text{and} \quad G_2 := U g U^* + N
\]
are \( \mathbb{C}^{q \times q} \)-valued functions, which are meromorphic in \( \mathbb{C} \setminus [\alpha, \beta] \) and holomorphic at \( w \) with \( F_2(w) = U^*[f(w)] U^* \) and \( G_2(w) = U^*[g(w)] U^* + \mathbb{P}_{[R(M)]}^{-1} \). Due to Proposition D.13, the \( q \times q \) matrix pair \([F_2(w); G_2(w)]\) is regular and satisfies \( F_2(w) = [F_1(w)][B(w)]^{-1} \) and \( G_2(w) = [G_1(w)][B(w)]^{-1} \). Let \( \mathcal{H}(F_1), \mathcal{H}(G_1), \mathcal{H}(R), \mathcal{H}(F_2), \) and \( \mathcal{H}(G_2) \) be the sets of complex numbers at which \( F_1, G_1, R, F_2, \) and \( G_2 \) are holomorphic, resp. Taking into account the arbitrary choice of \( w \in \Pi^+ \setminus A \), we can infer that \( \Pi^+ \setminus A \) is a subset of each of the sets \( \mathcal{H}(F_1), \mathcal{H}(G_1), \mathcal{H}(R), \mathcal{H}(F_2), \) and \( \mathcal{H}(G_2) \). Since \( \mathcal{A} \) is discrete, the set \( \Pi^+ \setminus A \) has in particular an accumulation point in \( \mathcal{H}(F_1) \cap \mathcal{H}(G_1) \cap \mathcal{H}(R) \cap \mathcal{H}(F_2) \cap \mathcal{H}(G_2) \). Using the identity theorem for
holomorphic functions, we thus can conclude
\[ \begin{align*}
F_2 &= F_1 R \\
G_2 &= G_1 R.
\end{align*} \tag{8.8}
\]

Observe that \( \det R = (\det B)^{-1} \) does not vanish identically in \( \mathbb{C} \setminus [\alpha, \beta] \). Consequently, Remark 7.10 yields \([F_2; G_2] \in \mathcal{PR}_q(\mathbb{C} \setminus [\alpha, \beta]) \) and \([F_1; G_1] \sim [F_2; G_2] \). We are now going to show that \([f; g] \) belongs to \( \mathcal{PR}_r(\mathbb{C} \setminus [\alpha, \beta]) \): Since \( \det B \) does not vanish identically in \( \mathbb{C} \setminus [\alpha, \beta] \), we obtain from the identity theorem for holomorphic functions that \( \mathcal{N} := \{ \xi \in \mathbb{C} \setminus ([\alpha, \beta] \cup \mathcal{P}(\det B)) : \det B(\xi) = 0 \} \) is a discrete subset of \( \mathbb{C} \setminus [\alpha, \beta] \). As already mentioned, \( \mathcal{A} \) is a discrete subset of \( \mathbb{C} \setminus [\alpha, \beta] \). Therefore, \( \mathcal{D} := \mathcal{A} \cup \mathcal{P}(\mathbb{R}) \cup \mathcal{P}(\mathbb{B}) \cup \mathcal{N} \) is a discrete subset of \( \mathbb{C} \setminus [\alpha, \beta] \) as well. In view of (8.6), we have \( \mathcal{P}(f) \subseteq \mathcal{P}(F_1) \cup \mathcal{P}(R) \) and \( \mathcal{P}(g) \subseteq \mathcal{P}(G_1) \cup \mathcal{P}(R) \). Hence, \( \mathcal{P}(f) \cup \mathcal{P}(g) \subseteq \mathcal{D} \) follows. Consider an arbitrary \( z \in \mathbb{C} \setminus ([\alpha, \beta] \cup \mathcal{D}) \). Let \( P := U[f(z)]U^* \) and \( Q := U[g(z)]U^* + \mathbb{R}([\mathcal{P}(U)])^2 \). By virtue of (8.1), (8.7), and (8.8), we immediately see that \( P = F_2(z) = [F_1(z)][B(z)]^{-1} \) and \( Q = G_2(z) = [G_1(z)][B(z)]^{-1} \). Observe that, due to Proposition 7.9, the \( q \times q \) matrix pair \([F_1(z); G_1(z)] \) is regular. Because of Remark D.6, thus the \( q \times q \) matrix pair \([P; Q] \) is regular and \( Q^*P = [B(z)]^{-1}([G_1(z)]^*[F_1(z)])[B(z)]^{-1} \) holds true. Regarding \( U^*U = I_r \), we can now apply Lemma D.12 to the \( r \times r \) matrix pair \([f(z); g(z)] \) to see that \([f(z); g(z)] \) is regular and that \( Q^*P = U([g(z)]^*[f(z)])U^* \) is fulfilled. In particular, we obtain \( \text{rank}\begin{bmatrix} f(z) \\ g(z) \end{bmatrix} = r \), according to Definition D.1, and furthermore
\[
[g(z)]^*[f(z)] = U^*[g(z)]^*[f(z)]U^*U = U^*Q^*PU
\]
\[
= U^*[B(z)]^{-1}([G_1(z)]^*[F_1(z)])[B(z)]^{-1}U
\]
\[
= ([B(z)]^{-1}U)^*[G_1(z)]^*[F_1(z)][B(z)]^{-1}U.
\]

In the case \( z \notin \mathbb{R} \), due to Proposition 7.9, we have \( \frac{1}{\text{Im}z} \text{Im}((\beta - z)[G_1(z)]^*[F_1(z)]) \in \mathbb{C}^{q \times q} \) and \( \frac{1}{\text{Im}z} \text{Im}((\beta - z)[G_1(z)]^*[F_1(z)]) \in \mathbb{C}^{q \times q} \), implying, by virtue of Remarks A.2 and A.25, then \( \frac{1}{\text{Im}z} \text{Im}((\beta - z)[G_1(z)]^*[F_1(z)]) \in \mathbb{C}^{q \times q} \) and, similarly, \( \frac{1}{\text{Im}z} \text{Im}((\beta - z)[G_1(z)]^*[F_1(z)]) \in \mathbb{C}^{q \times q} \). According to Notation 7.5, hence \([f; g] \) belongs to \( \mathcal{PR}_r(\mathbb{C} \setminus [\alpha, \beta]) \). Applying \( \Gamma_U \) to the equivalence class of \([f; g] \), we get with (8.7) and \([F_1; G_1] \sim [F_2; G_2] \) then \( \Gamma_U([f; g]) = ([F_2; G_2] - [F_1; G_1]) \). \( \square \)

**Example 8.6** Let \( M \in \mathbb{C}^{q \times p} \) and let \( F, G : \mathbb{C} \setminus [\alpha, \beta] \to \mathbb{C}^{q \times q} \) be defined by \( F(z) := O_{q \times q} \) and \( G(z) := I_q \). Then \([F; G] \in \mathcal{P}[M] \).

**Remark 8.7** Let \( M \in \mathbb{C}^{q \times q}_H \) and let \( f, g : \mathbb{C} \setminus [\alpha, \beta] \to \mathbb{C} \) be both holomorphic and not identically vanishing. Let \( U \) be a linear subspace of \( \mathcal{R}(M) \) and let \( P := \mathbb{P}U \). Let \( F, G : \mathbb{C} \setminus [\alpha, \beta] \to \mathbb{C}^{q \times q} \) be defined by \( F(z) := f(z)MPM \) and \( G(z) := g(z)(I_q - M^\dagger PM) \). Then \([F; G] \) belongs to \( \mathcal{P}[M] \) and
\[
\mathcal{R}(G(z)) = \mathcal{N}(M^\dagger PM),
\]
\[
G(z)(\mathcal{N}(F(z))) = \mathcal{N}(PM),
\]
\[
\mathcal{R}(F(z)) = \mathcal{R}(MP),
\]
\[
F(z)(\mathcal{N}(G(z))) = \mathcal{R}(MP),
\]
and, in view of (D.1), moreover $\text{rk}([F(z); G(z)]) = 0$ hold true for all $z \in \mathbb{C} \setminus [\alpha, \beta]$ with $f(z) \neq 0$ and $g(z) \neq 0$.

Indeed, since $f$ and $g$ are both holomorphic and not identically vanishing, the matrix-valued functions $F$ and $G$ are holomorphic and in particular meromorphic in $\mathbb{C} \setminus [\alpha, \beta]$ and $\mathcal{D} := \{z \in \mathbb{C} \setminus [\alpha, \beta] : f(z) = 0 \text{ or } g(z) = 0\}$ is a discrete subset of $\mathbb{C} \setminus [\alpha, \beta]$. Observe that $M^* = M$ implies $(M^*)^* = M^\dagger$ and $M^\dagger M = MM^\dagger$, by virtue of Remarks A.14 and A.18. Consider an arbitrary $z \in \mathbb{C} \setminus ([\alpha, \beta] \cup \mathcal{D})$. Let $v \in \mathcal{N}\left(\begin{bmatrix} F(z) \\ G(z) \end{bmatrix}\right)$, i.e., $f(z)MPMv = O_{q \times 1}$ and $g(z)(I_q - M^\dagger PM)v = O_{q \times 1}$. Since $f(z) \neq 0$ and $g(z) \neq 0$, hence $MPMv = O_{q \times 1}$ and $v = M^\dagger PMv$. Thus, taking additionally into account (3.3), we can conclude $v = M^\dagger PMv = M^\dagger MM^\dagger PMv = M^\dagger M^\dagger MPMv = O_{q \times 1}$. This shows $\text{rank}\left[\begin{bmatrix} F(z) \\ G(z) \end{bmatrix}\right] = q$. Observe that $\mathcal{U} \subseteq \mathcal{R}(M)$ implies $MM^\dagger P = P$, by virtue of Remarks A.11 and A.20. Taking additionally into account $P^* = P = P^2$, we can furthermore conclude

$$(I_q - M^\dagger PM)^*(MPM) = (I_q - MM^\dagger M^\dagger)MPM = MPM - MMPM^\dagger MPM$$

$$= MPM - MPMM^\dagger PM = MPM - MP^2M = O_{q \times q}.$$ 

Consequently, $[G(z)]^*[F(z)] = \frac{1}{i\pi(z)}f(z)(I_q - M^\dagger PM)^*(MPM) = O_{q \times q}$. If $z \notin \mathbb{R}$, thus the matrices $\frac{1}{i\pi(z)} \text{Im}((z - \alpha)[G(z)]^*[F(z)])$ and $\frac{1}{i\pi(z)} \text{Im}((z - \beta)[G(z)]^*[F(z)])$ are both Non-negative Hermitian. Hence, $[F; G]$ belongs to $\mathcal{P}\mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$. Since obviously $\mathcal{P}\mathcal{R}_q(M)F = F$, then $[F; G] \in \mathcal{P}\mathcal{R}_q(M)$ follows by virtue of Lemma 8.3. Because of $(M^\dagger PM)^2 = M^\dagger PMPM^\dagger PM = M^\dagger MPMPM^\dagger PM = M^\dagger MPM$, we have $\mathcal{R}(I_q - M^\dagger PM) = \mathcal{N}(M^\dagger PM)$ and $\mathcal{N}(I_q - M^\dagger PM) = \mathcal{R}(M^\dagger PM)$. In view of $f(z) \neq 0$ and $g(z) \neq 0$, we get then

$$\mathcal{R}(G(z)) = \mathcal{R}(I_q - M^\dagger PM) = \mathcal{N}(M^\dagger PM)$$

and $\mathcal{N}(G(z)) = \mathcal{N}(I_q - M^\dagger PM) = \mathcal{R}(M^\dagger PM)$ as well as

$$\mathcal{R}(F(z)) = \mathcal{R}(MPM) = \mathcal{R}(MPM^*M^*) = \mathcal{R}(MP)$$

and $\mathcal{N}(F(z)) = \mathcal{N}(MPM) = \mathcal{N}(M^*P^*M) = \mathcal{N}(PM)$. Taking again into account $f(z) \neq 0$ and $g(z) \neq 0$, consequently

$$G(z)(\mathcal{N}(F(z))) = g(z)(I_q - M^\dagger PM)(\mathcal{N}(PM)) = g(z)\mathcal{N}(PM) = \mathcal{N}(PM)$$

and

$$F(z)(\mathcal{N}(G(z))) = f(z)MPM(\mathcal{R}(M^\dagger PM)) = \mathcal{R}(f(z)MPMPM^\dagger PM)$$

$$= \mathcal{R}(f(z)MP^2M) = \mathcal{R}(MPM^*M^*) = \mathcal{R}(MP)$$

follow. Regarding (D.1), in particular $\text{rk}([F(z); G(z)]) = 0$. 

9 The $\mathcal{F}_{\alpha,\beta}(M)$-transformation and Its Inverse

Our next considerations are aimed at preparing the foundations for the desired function-theoretic Schur–Nevanlinna type algorithm. This algorithm consists of two different instances, because the first step differs from the remaining ones. In this section, we treat the algebraic formalism for the first step. Doing this, we take into account as well the forward as the backward form of the algorithm.

We are now going to introduce a transformation of matrix-valued functions, which is intimately connected with the $\mathcal{F}_{\alpha,\beta}$-transformation for sequences of complex matrices (see Definition 3.52).

In this section, for an arbitrarily given complex matrix $E$, we write

$$P_E := P_{\mathcal{R}(E)}$$ and $$Q_E := P_{\mathcal{N}(E)}$$

for the transformation matrix corresponding to the orthogonal projection onto $\mathcal{R}(E)$ and $\mathcal{N}(E)$, resp. In view of Remarks A.11, A.10, and A.18, we have

$$\mathcal{R}(P_E) = \mathcal{R}(E), \quad \mathcal{N}(P_E) = \mathcal{N}(E^*), \quad \mathcal{R}(Q_E) = \mathcal{N}(E), \quad \mathcal{N}(Q_E) = \mathcal{R}(E^*),$$

$$P_E^2 = P_E, \quad P_E^* = P_E, \quad Q_E^2 = Q_E, \quad Q_E^* = Q_E,$$ (9.1)

and

$$P_E = EE^\dagger, \quad P_E^* = E^\dagger E, \quad Q_E = I_q - E^\dagger E, \quad Q_E^* = I_p - EE^\dagger.$$ (9.2)

In the sequel, we will also use these identities without explicitly mentioning. Furthermore, we consider here a complex matrix $M$, which in the context of the matricial Hausdorff moment problem will later be the Non-negative Hermitian matrix $s_0$ taken from a sequence $(s_j)_{j=0}^\kappa$ belonging to $\mathcal{F}_{q,k,\kappa,\alpha,\beta}$.

**Definition 9.1** Let $\mathcal{G}$ be a non-empty subset of $\mathbb{C}$, let $F: \mathcal{G} \to \mathbb{C}^{p\times q}$ be a matrix-valued function, and let $M$ be a complex $p \times q$ matrix. Then the pair $[G_1; G_2]$ built with the functions $G_1$ and $G_2$ defined on $\mathcal{G}$ by

$$G_1(z) := (\beta - z)F(z) - M$$

and

$$G_2(z) := (\beta - z)[(z - \alpha)M^\dagger F(z) + P_M^*] + \delta Q_M$$

is called the $\mathcal{F}_{\alpha,\beta}(M)$-transformed pair of $F$.

In connection with the $\mathcal{F}_{\alpha,\beta}(M)$-transformation, we consider the following quadratic $(p + q) \times (p + q)$ matrix polynomial:
Notation 9.2 Let $M \in \mathbb{C}^{p \times q}$. Then let $	ilde{W}_M : \mathbb{C} \to \mathbb{C}^{(p+q) \times (p+q)}$ be defined by

$$
\tilde{W}_M(z) := \begin{bmatrix}
-(\beta - z) P_M & M \\
-(\beta - z) (z - \alpha) M^\dagger & -(\beta - z) P_M^* - \delta Q_M
\end{bmatrix}.
$$

In what follows, we will use the notation given via (7.6) to calculate, in view of Remark A.36, certain forms involving the signature matrix $\tilde{J}_q$ given by (A.1). For an arbitrarily given $z \in \mathbb{C}$, we will furthermore write abbreviatory $x := z - \alpha$ and $y := \beta - z$. Obviously, we have

$$y + x = \beta - \alpha = \delta \quad \text{and} \quad \alpha y + \beta x = \alpha \beta - \alpha z + \beta z - \beta \alpha = (\beta - \alpha) z = \delta z$$

as well as

$$\alpha x + \beta y = \alpha z - \alpha^2 + \beta^2 - \beta z = (\beta + \alpha) (\beta - \alpha) - (\beta - \alpha) z = \delta (\beta + \alpha - z).$$

Lemma 9.3 Let $M \in \mathbb{C}^{q \times q}_H$. Let $z \in \mathbb{C}$, let $x := z - \alpha$, and let $y := \beta - z$.

(a) Let $W_0 := \tilde{W}_M(z)$. Then

$$W_0^* \tilde{J}_q W_0 = \begin{bmatrix}
-2 |y|^2 \text{Im}(z) M^\dagger & i (\overline{yx} + |y|^2) P_M \\
-i (yx + |y|^2) P_M & -2 \text{Im}(z) M
\end{bmatrix}. \quad (9.5)$$

(b) Let $W_1 := [(x I_q) \oplus I_q] W_0$ and let $W_2 := [(y I_q) \oplus I_q] W_0$. Then

$$W_1^* \tilde{J}_q W_1 = \delta \begin{bmatrix}
O_{q \times q} & i \overline{yx} P_M \\
-i y x P_M & -2 \text{Im}(z) M
\end{bmatrix}$$

$$= \begin{bmatrix}
O_{q \times q} & i (|y|^2 \overline{x} + |x|^2 \overline{y}) P_M \\
-i (|y|^2 x + |x|^2 y) P_M & -2 \delta \text{Im}(z) M
\end{bmatrix} \quad (9.6)$$

and

$$W_2^* \tilde{J}_q W_2 = |y|^2 \begin{bmatrix}
-2 \text{Im}(z) M^\dagger & iP_M \\
-i P_M & O_{q \times q}
\end{bmatrix}. \quad (9.7)$$

(c) Let $W_3 := [(yx I_q) \oplus I_q] W_0$. Then

$$W_3^* \tilde{J}_q W_3 = |y|^2 \begin{bmatrix}
-2 |x|^2 \text{Im}(z) M^\dagger & i (\overline{xy} + |x|^2) P_M \\
-i (yx + |x|^2) P_M & -2 \text{Im}(z) M
\end{bmatrix}. \quad (9.8)$$
In view of (9.1), (9.2), and Remarks A.14 and 7.14, the proof of Lemma 9.3 is straightforward. We omit the details.

In addition, we now rewrite the right-hand sides of the equations (9.5)–(9.8), using the signature matrix $\tilde{J}_q$.

**Proposition 9.4** Let $M \in \mathbb{C}^{q \times q}$. Let $z \in \mathbb{C}$, let $x := z - \alpha$, and let $y := \beta - z$.

(a) Let $W_0 := \ddot{W}_M(z)$. Then

$$W_0^* \tilde{J}_q W_0 = [(yx P_M) \op I_q]^* \tilde{J}_q [(yx P_M) \op I_q] + |y|^2 [(P_M \op I_q)^* \tilde{J}_q (P_M \op I_q) - 2 \Im(z)(M^\dagger \op O_{q \times q})] - 2 \Im(z)(O_{q \times q \op M}).$$

(b) Let $W_1 := [(x I_q) \op I_q] W_0$ and let $W_2 := [(y I_q) \op I_q] W_0$. Then

$$W_1^* \tilde{J}_q W_1 = \delta \left( [(yx P_M) \op I_q]^* \tilde{J}_q [(yx P_M) \op I_q] - 2 \Im(z)(O_{q \times q \op M}) \right) = |y|^2 [(x P_M) \op I_q]^* \tilde{J}_q [(x P_M) \op I_q] + |x|^2 [(y P_M) \op I_q]^* \tilde{J}_q [(y P_M) \op I_q] - 2 \delta \Im(z)(O_{q \times q \op M})$$

and

$$W_2^* \tilde{J}_q W_2 = \delta |y|^2 [(P_M \op I_q)^* \tilde{J}_q (P_M \op I_q) - 2 \Im(z)(M^\dagger \op O_{q \times q})].$$

(c) Let $W_3 := [(yx I_q) \op I_q] W_0$. Then

$$W_3^* \tilde{J}_q W_3 = |y|^2 \left\{ [(yx P_M) \op I_q]^* \tilde{J}_q [(yx P_M) \op I_q] + |x|^2 [(P_M \op I_q)^* \tilde{J}_q (P_M \op I_q) - 2 \Im(z)(M^\dagger \op O_{q \times q})] - 2 \Im(z)(O_{q \times q \op M}) \right\}.$$

**Proof** Taking into account $P_M^* = P_M$ and (A.1), the asserted identities immediately follow from Lemma 9.3. \(\Box\)

It will be clear from Lemmata 9.13 and 9.14 below that the following transformation for pairs of meromorphic matrix-valued functions is, under certain conditions, essentially the inversion of the $F_{\alpha,\beta}(M)$-transformation. To define this inverse transformation, we use the terminology given at the end of Appendix F.

**Definition 9.5** Let $\mathcal{G}$ be a domain. Let $G_1$ be a $\mathbb{C}^{p \times q}$-valued function meromorphic in $\mathcal{G}$ and let $G_2$ be a $\mathbb{C}^{q \times q}$-valued function meromorphic in $\mathcal{G}$. Let $M \in \mathbb{C}^{p \times q}$, let the functions $g, h : \mathcal{G} \to \mathbb{C}$ be defined by

$$g(z) := z - \alpha \quad \text{and} \quad h(z) := \beta - z,$$

(9.9)
resp., and let
\[ F_1 := h P_M G_1 + MG_2 \quad \text{and} \quad F_2 := -h g M^\dagger G_1 + h G_2. \quad (9.10) \]

Suppose that \( \det F_2 \) does not identically vanish in \( G \). Then we call the \( \mathbb{C}^{p \times q} \)-valued function \( F := F_1 F_2^{-1} \), which is meromorphic in \( G \), the inverse \( \mathcal{F}_{\alpha,\beta}(M) \)-transform of \([G_1; G_2]\).

To the inverse \( \mathcal{F}_{\alpha,\beta}(M) \)-transform we can associate the following matrix polynomial:

**Notation 9.6** Let \( M \in \mathbb{C}^{p \times q} \). Then let \( \tilde{V}_M : \mathbb{C} \to \mathbb{C}^{(p+q) \times (p+q)} \) be defined by
\[
\tilde{V}_M(z) := \begin{bmatrix}
(\beta - z) P_M & M \\
-(\beta - z)(z - \alpha) M^\dagger & (\beta - z) I_q
\end{bmatrix}.
\]

**Remark 9.7** Let \( M \in \mathbb{C}^{q \times q} \) and let \( z \in \mathbb{C} \). Then:

(a) If \( M = O_q \), then \( \tilde{V}_M(z) = \begin{bmatrix} O_{q \times q} & O_{q \times q} \\
O_{q \times q} & (\beta - z) I_q
\end{bmatrix} \).
(b) If \( M \) is invertible, then \( \tilde{V}_M(z) = \begin{bmatrix} (\beta - z) I_q \\
-(\beta - z)(z - \alpha) M^{-1} & (\beta - z) I_q
\end{bmatrix} \).

Regarding \( y + x = \delta \) and \( Q_{\cdot \cdot} = I_q - P_M^* \), it is readily checked that the matrix polynomials \( \tilde{V}_M \) and \( \tilde{W}_M \) are connected in the following way by the signature matrix \( \tilde{j}_{pq} \) given in (A.1):

**Remark 9.8** If \( z \in \mathbb{C} \), then \( [\tilde{V}_M(z)] \tilde{j}_{pq} = -j_{pq} [\tilde{W}_M(z) + (z - \alpha)(O_{p \times p} \oplus Q_M)] \).

Consequently, a result analogous to Proposition 9.4 follows:

**Proposition 9.9** Let \( M \in \mathbb{C}_H^{q \times q} \). Let \( z \in \mathbb{C} \), let \( x := z - \alpha \), and let \( y := \beta - z \).

(a) Let \( V_0 := \tilde{V}_M(z) \). Then
\[
V_0^* \tilde{j}_q V_0 = [(y x P_M) \oplus I_q]^* \tilde{j}_q [(y x P_M) \oplus I_q] + |y|^2 [(P_M \oplus I_q)^* \tilde{j}_q (P_M \oplus I_q) + 2 \text{Im}(z)(M^\dagger \oplus O_{q \times q})] + 2 \text{Im}(z)(O_{q \times q} \oplus M).
\]

(b) Let \( V_1 := [(x I_q) \oplus I_q] V_0 \) and let \( V_2 := [(y I_q) \oplus I_q] V_0 \). Then
\[
V_1^* \tilde{j}_q V_1 = \delta [(y x P_M) \oplus I_q]^* \tilde{j}_q [(y x P_M) \oplus I_q] + 2 \text{Im}(z)(O_{q \times q} \oplus M)) = |y|^2 [(x P_M) \oplus I_q]^* \tilde{j}_q [(x P_M) \oplus I_q] + |x|^2 [(y P_M) \oplus I_q]^* \tilde{j}_q [(y P_M) \oplus I_q] + 2 \delta \text{Im}(z)(O_{q \times q} \oplus M)
\]
and
\[
V_2^* \tilde{j}_q V_2 = \delta |y|^2 [(P_M \oplus I_q)^* \tilde{j}_q (P_M \oplus I_q) + 2 \text{Im}(z)(M^\dagger \oplus O_{q \times q})].
\]
(c) Let \( V_3 := [(yxI_q) \oplus I_q]V_0 \). Then

\[
V_3^* \tilde{J}_q V_3 = |y|^2 \left\{ [(yxP_M) \oplus I_q]^* \tilde{J}_q [(yxP_M) \oplus I_q] \right. \\
+ |x|^2 \left[ (P_M \oplus I_q)^* \tilde{J}_q (P_M \oplus I_q) + 2 \text{Im}(z)(M^\dagger \oplus O_q \times q) \right] \\
+ 2 \text{Im}(z)(O_q \times q \oplus M) \right\}.
\]

**Proof** Consider an arbitrary \( \ell \in \{0, 1, 2, 3\} \). Using the notation given in Proposition 9.4, we obtain, by virtue of Remarks A.37 and 9.8, then \( V_\ell j_{qq} = -j_{qq} [W_\ell + x(O_q \times q \oplus Q_M)] \) and \( \tilde{J}_q j_{qq} = -j_{qq} \tilde{J}_q \). Taking additionally into account \( j_{QQ}^2 = I_{2q} \) and \( j_{QQ} = j_{qq} \) and setting \( U_\ell := W_\ell + x(O_q \times q \oplus Q_M) \), we can conclude hence

\[
V_\ell^* \tilde{J}_q V_\ell = (-j_{qq} U_\ell j_{qq})^* \tilde{J}_q (-j_{qq} U_\ell j_{qq}) \\
= j_{qq} U_\ell^* (j_{qq} \tilde{J}_q j_{qq}) U_\ell j_{qq} = -j_{qq} (U_\ell^* \tilde{J}_q U_\ell) j_{qq}.
\]

Because of \( Q_M^* = Q_M \) and \( M^* = M \), we have \( Q_M M = (M Q_M)^* = O_q \times q \). Consequently, \( Q_M P_M = Q_M M M^\dagger = O_q \times q \) follows. In view of Notation 9.2, thus we obtain

\[
(Q_M \oplus O_q \times q) W_0 = (Q_M \oplus O_q \times q) \begin{bmatrix} -y P_M & M^* \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -y Q_M P_M & Q_M M^\dagger \\ O_q \times q & O_q \times q \end{bmatrix} = O_{2q \times 2q}.
\]

In particular, \( (Q_M \oplus O_q \times q) W_\ell = O_{2q \times 2q} \). Using \( \tilde{J}_q^* = \tilde{J}_q \) and Remark A.37, we get then

\[
U_\ell^* \tilde{J}_q U_\ell \\
= W_\ell^* \tilde{J}_q W_\ell + 2 \text{Re}[x(O_q \times q \oplus Q_M) \tilde{J}_q W_\ell] + x(O_q \times q \oplus Q_M) \tilde{J}_q (O_q \times q \oplus Q_M) \\
= W_\ell^* \tilde{J}_q W_\ell + 2 \text{Re}[x \tilde{J}_q (Q_M \oplus O_q \times q) W_\ell] + |x|^2 \tilde{J}_q Q_M (O_q \times q \oplus Q_M) \]

implying \( V_\ell^* \tilde{J}_q V_\ell = -j_{qq} (W_\ell^* \tilde{J}_q W_\ell) j_{qq} \). Remark A.37 yields \( j_{qq} (R \oplus S) j_{qq} = R \oplus S \) and

\[
j_{qq} (R \oplus S)^* \tilde{J}_q (R \oplus S) j_{qq} = (R \oplus S)^* (j_{qq} \tilde{J}_q j_{qq}) (R \oplus S) = -(R \oplus S)^* \tilde{J}_q (R \oplus S)
\]

for all \( R, S \in \mathbb{C}^{q \times q} \). The asserted identities can now be deduced from Proposition 9.4.

We are now going to consider the composition of the transformations introduced in Definitions 9.1 and 9.5.

**Lemma 9.10** Let \( M \in \mathbb{C}^{p \times q} \) and let \( z \in \mathbb{C} \). Then

\[
[\hat{V}_M(z)][\hat{W}_M(z)] = -(\beta - z)\delta(P_M \oplus I_q) = [\hat{W}_M(z)][\hat{V}_M(z)].
\]
Proof Let $x := z - \alpha$ and let $y := \beta - z$. We have then

$$
[\tilde{V}_M(z)][\tilde{W}_M(z)] = \begin{bmatrix}
yP_M & M \\
yxM^\dagger yI_q & -yP_M - \delta Q_M
\end{bmatrix}
\begin{bmatrix}
yP_M & M \\
yxM^\dagger & -yxM^\dagger - yP_M - \delta Q_M
\end{bmatrix}
= \begin{bmatrix}
y^2P_M^2 - yxMM^\dagger & yPM_M - yMP_M^* - \delta MQ_M \\
y^2xM^\dagger PM - y^2xM^\dagger & -yxM^\dagger M - y^2P_M^* - y\delta Q_M
\end{bmatrix}
$$

and

$$
[\tilde{W}_M(z)][\tilde{V}_M(z)] = \begin{bmatrix}
yP_M & M \\
yxM^\dagger & -yxM^\dagger - yP_M - \delta Q_M
\end{bmatrix}
\begin{bmatrix}
yP_M & M \\
yxM^\dagger yI_q & -yxM^\dagger
\end{bmatrix}
= \begin{bmatrix}
y^2P_M^2 - yxMM^\dagger & -yPM_M + yM \\
y^2xM^\dagger PM + y^2xPM^*_M + yx\delta Q_M^*M^\dagger & -yxM^\dagger M - y^2P_M^* - y\delta Q_M
\end{bmatrix}.
$$

Consequently, in view of (9.1), (9.2), and $y + x = \delta$, the assertion follows. \( \square \)

For a given Non-negative Hermitian matrix $M$, the condition in Definition 9.5 is satisfied for pairs belonging to the subclass $\tilde{\mathcal{P}}[M]$ of $\mathcal{P}\mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$, introduced in Sect. 8. Hence, for suchlike pairs the corresponding inverse $\mathcal{F}_{\alpha,\beta}(M)$-transform exists and can be written as a linear fractional transformation, as considered in Appendix E:

**Proposition 9.11** Let $M \in \mathbb{C}_{\geq}^{q \times q}$ and let $[G_1; G_2] \in \tilde{\mathcal{P}}[M]$. In view of the functions

$$
g, h : \mathbb{C} \setminus [\alpha, \beta] \to \mathbb{C}
$$

defined by (9.9), let $F_1$ and $F_2$ be given via (9.10) as matrix-valued functions meromorphic in $\mathbb{C} \setminus [\alpha, \beta]$. Then $det F_2$ does not identically vanish in $\mathbb{C} \setminus [\alpha, \beta]$. Furthermore, $det F_2(z) \neq 0$ and $F(z) = [F_1(z)]^{-1}F_2(z)$ for all $z \in \mathbb{C} \setminus ([\alpha, \beta] \cup \mathcal{P}(G_1) \cup \mathcal{P}(G_2) \cup \mathcal{E}([G_1; G_2]))$, where $F$ denotes the inverse $\mathcal{F}_{\alpha,\beta}(M)$-transform of $[G_1; G_2]$.

**Proof** According to Notation 8.1, the pair $[G_1; G_2]$ belongs to $\mathcal{P}\mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$. Hence, $G_1$ and $G_2$ are $\mathbb{C}_{\geq}^{q \times q}$-valued functions, which are meromorphic in $\mathbb{C} \setminus [\alpha, \beta]$. Furthermore, by virtue of Proposition 7.9, the set $\mathcal{A} := \mathcal{P}(G_1) \cup \mathcal{P}(G_2) \cup \mathcal{E}([G_1; G_2])$ is a discrete subset of $\mathbb{C} \setminus [\alpha, \beta]$. Consequently, $\mathbb{C} \setminus ([\alpha, \beta] \cup \mathcal{A}) \neq \emptyset$. Consider an arbitrary $z \in \mathbb{C} \setminus ([\alpha, \beta] \cup \mathcal{A})$. Then $G_1$ and $G_2$ are both holomorphic in $z$. Thus $F_1$ and $F_2$ are both holomorphic in $z$ as well. Consider an arbitrary $v \in \mathcal{N}(F_2(z))$. Regarding Remark A.16, we are going to show in a first step that

$$
\|R[G_1(z)]v\|_E = 0 \quad (9.11)
$$

holds true, where $R := \sqrt{M^\dagger}$. Because of $z \neq \beta$, we have, according to (9.9) and (9.10), the equation

$$
(z - \alpha)M^\dagger[G_1(z)v = [G_2(z)]v. \quad (9.12)
$$
In view of Remark A.14, hence
\[ v^*[G_2(z)]^*[G_1(z)]v = (\overline{z} - \alpha)v^*[G_1(z)]^*M^\dagger[G_1(z)]v = (\overline{z} - \alpha)\|R[G_1(z)]v\|_E^2. \] 
(9.13)

In the case \( z \in \mathbb{C}\setminus\mathbb{R} \), we see from Lemma 7.8 and Remark A.2 then that
\[ 0 \leq v^*(\frac{1}{\text{Im} z} \text{Im}([G_2(z)]^*[G_1(z)]))v = \frac{1}{\text{Im} z} \text{Im}(v^*[G_2(z)]^*[G_1(z)]v) \]
\[ = \frac{1}{\text{Im} z} \text{Im}((\overline{z} - \alpha)\|R[G_1(z)]v\|_E^2) = -\|R[G_1(z)]v\|_E^2 \leq 0, \]
implying (9.11). If \( z \in (-\infty, \alpha) \), then \( \overline{z} = z < \alpha \) and we obtain, by virtue of Lemma 7.8 and (9.13), thus
\[ 0 \leq v^*[G_2(z)]^*[G_1(z)]v = (\overline{z} - \alpha)\|R[G_1(z)]v\|_E^2 \leq 0, \]
implying again (9.11). In the case \( z \in (\beta, \infty) \), we have \( \overline{z} = z > \beta > \alpha \) and, because of Lemma 7.8 and (9.13), similarly
\[ 0 \leq v^*(-[G_2(z)]^*[G_1(z)])v = -v^*[G_2(z)]^*[G_1(z)]v \]
\[ = (\alpha - \overline{z})\|R[G_1(z)]v\|_E^2 \leq 0, \]
i.e., (9.11). Thus, (9.11) is verified. Consequently, we can infer
\[ P_M[G_1(z)]v = MM^\dagger[G_1(z)]v = MR^2[G_1(z)]v = O_{q \times 1}. \]

In view of Lemma 8.3, we have furthermore \( P_MG_1 = G_1 \). Hence, \( [G_1(z)]v = O_{q \times 1} \) follows. Because of (9.12), this implies \([G_2(z)]v = O_{q \times 1}\). Observe that, due to Proposition 7.9, the \( q \times q \) matrix pair \([G_1(z); G_2(z)]\) is regular. According to Remark D.5, thus necessarily \( v = O_{q \times 1} \) holds true. Therefore, the linear subspace \( \mathcal{N}(F_2(z)) \) is trivial, implying \( \det F_2(z) \neq 0 \). In particular, \( \det F_2 \) does not identically vanish in \( \mathbb{C} \setminus [\alpha, \beta] \) and \( F(z) = [F_1(z)][F_2(z)]^{-1} \).

For any Non-negative Hermitian matrix \( M \), the inverse \( \mathcal{F}_{\alpha, \beta}(M) \)-transformation induces, according to Definition 7.11 and Remark 8.4, a well-defined transformation for equivalence classes from \( (\mathcal{P}[M]) \):

**Corollary 9.12** Let \( M \in \mathbb{C}^{n \times q}_{\mathbb{R}} \) and let the pairs \([G_1; G_2], [\tilde{G}_1; \tilde{G}_2] \in \mathcal{P}[M] \) be equivalent. Then the inverse \( \mathcal{F}_{\alpha, \beta}(M) \)-transform \( F \) of \([G_1; G_2]\) coincides with the inverse \( \mathcal{F}_{\alpha, \beta}(M) \)-transform \( \tilde{F} \) of \([\tilde{G}_1; \tilde{G}_2]\).

**Proof** Using the functions \( g, h: \mathbb{C} \setminus [\alpha, \beta] \to \mathbb{C} \) given via (9.9), we define by (9.10) two \( \mathbb{C}^{q \times q} \)-valued functions \( F_1 \) and \( F_2 \) meromorphic in \( \mathbb{C} \setminus [\alpha, \beta] \). Because of Proposition 9.11, then \( \det F_2 \) does not vanish identically in \( \mathbb{C} \setminus [\alpha, \beta] \). By virtue of Definition 9.5, we thus have \( F = F_1F_2^{-1} \). Furthermore, due to Definition 7.11,
there exists a $\mathbb{C}^{q \times q}$-valued function $R$ meromorphic in $\mathbb{C} \setminus [\alpha, \beta]$ such that det $R$ does not vanish identically in $\mathbb{C} \setminus [\alpha, \beta]$ satisfying $\tilde{G}_1 = G_1 R$ and $\tilde{G}_2 = G_2 R$. Using again the functions $g$ and $h$, we define according to (9.10) by $\tilde{F}_1 := h P_M \tilde{G}_1 + MG_2 \tilde{G}_2$ and $\tilde{F}_2 := -hgM^1 \tilde{G}_1 + h\tilde{G}_2$ two $\mathbb{C}^{q \times q}$-valued functions $\tilde{F}_1$ and $\tilde{F}_2$ meromorphic in $\mathbb{C} \setminus [\alpha, \beta]$. Then $\tilde{F}_1 = F_1 R$ and $\tilde{F}_2 = F_2 R$. The application of Proposition 9.11 to the pair $[\tilde{G}_1; \tilde{G}_2]$ yields furthermore that det $\tilde{F}_2$ does not vanish identically in $\mathbb{C} \setminus [\alpha, \beta]$. By virtue of Definition 9.5, hence $\tilde{F}_2 = \tilde{F}_1 \tilde{F}_2^{-1}$. Consequently, $\tilde{F} = (F_1 R)(F_2 R)^{-1} = F$. 

\[\Box\]

Furthermore, after transition to equivalence classes as mentioned above, the $\mathcal{F}_{\alpha, \beta}(M)$-transformation turns out to be inverse to the inverse $\mathcal{F}_{\alpha, \beta}(M)$-transformation:

**Lemma 9.13** Let $M \in \mathbb{C}^{q \times q}$ and let $[G_1; G_2] \in \tilde{\mathcal{P}}[M]$ with inverse $\mathcal{F}_{\alpha, \beta}(M)$-transform $F$. Then $[G_1; G_2]$ is equivalent to the $\mathcal{F}_{\alpha, \beta}(M)$-transformed pair of $F$.

**Proof** Using the functions $g$, $h : \mathbb{C} \setminus [\alpha, \beta] \rightarrow \mathbb{C}$ given via (9.9), we define by (9.10) two $\mathbb{C}^{q \times q}$-valued functions $F_1$ and $F_2$ meromorphic in $\mathbb{C} \setminus [\alpha, \beta]$. Then $P_M F_1 = F_1$. Denote by $[H_1; H_2]$ the $\mathcal{F}_{\alpha, \beta}(M)$-transformed pair of $F$ and by $W$ the restriction of the holomorphic $\mathbb{C}^{2q \times 2q}$-valued function $\tilde{W}_M$ onto $\mathbb{C} \setminus [\alpha, \beta]$. In view of Definition 9.1 and Notation 9.2, then $\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} = -W \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$. Due to Proposition 9.11, the function det $F_2$ does not vanish identically in $\mathbb{C} \setminus [\alpha, \beta]$. According to Definition 9.5, we thus have $F = F_1 F_2^{-1}$. Denote by $V$ the restriction of the holomorphic $\mathbb{C}^{2q \times 2q}$-valued function $\tilde{V}_M$ onto $\mathbb{C} \setminus [\alpha, \beta]$. Regarding Notation 9.6, then $\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = V \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$. Taken all together, we get

$$
\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} = -W \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \tilde{F}_2^{-1} = -W \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \tilde{F}_2^{-1} = -WV \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \tilde{F}_2^{-1} = -WV \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \tilde{F}_2^{-1}.
$$

From Lemma 8.3 we see $P_M G_1 = G_1$. Using Lemma 9.10, we thus can infer $H_1 = h \delta P_M G_1 F_2^{-1} = h \delta G_1 F_2^{-1}$ and $H_2 = h \delta G_2 F_2^{-1}$. Observe that $R := \delta h \tilde{F}_2^{-1}$ is a $\mathbb{C}^{q \times q}$-valued function, which is meromorphic in $\mathbb{C} \setminus [\alpha, \beta]$ satisfying $H_1 = G_1 R$ and $H_2 = G_2 R$. Furthermore, because of $\delta \neq 0$, the function det $R$ does not vanish identically in $\mathbb{C} \setminus [\alpha, \beta]$. According to Definition 7.11, consequently $[G_1; G_2] \sim [H_1; H_2]$. 

\[\Box\]

Conversely, we have:

**Lemma 9.14** Let $M \in \mathbb{C}^{q \times q}$ and let $F : \mathbb{C} \setminus [\alpha, \beta] \rightarrow \mathbb{C}^{q \times q}$ be a matrix-valued function with $\mathcal{F}_{\alpha, \beta}(M)$-transformed pair $[G_1; G_2]$ such that $P_M F = F$ and $[G_1; G_2] \in \tilde{\mathcal{P}}[M]$ hold true. Then the inverse $\mathcal{F}_{\alpha, \beta}(M)$-transform of $[G_1; G_2]$ coincides with $F$.

**Proof** Since $[G_1; G_2]$ belongs to $\tilde{\mathcal{P}}[M]$, we see that $G_1$ and $G_2$ are $\mathbb{C}^{q \times q}$-valued functions, which are meromorphic in $\mathbb{C} \setminus [\alpha, \beta]$. Using the functions $g$, $h : \mathbb{C} \setminus [\alpha, \beta] \rightarrow \mathbb{C}$ given via (9.9), we can thus define by (9.10) two $\mathbb{C}^{q \times q}$-valued functions $F_1$ and
$F_2$, which then are meromorphic in $\mathbb{C} \setminus [\alpha, \beta]$ as well. Denote by $V$ the restriction of the holomorphic $\mathbb{C}^{2q \times 2q}$-valued function $\tilde{V}_M$ onto $\mathbb{C} \setminus [\alpha, \beta]$. From Notation 9.6 we see $\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = V \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$. Denote by $W$ the restriction of the holomorphic $\mathbb{C}^{2q \times 2q}$-valued function $\tilde{W}_M$ onto $\mathbb{C} \setminus [\alpha, \beta]$. Regarding Definition 9.1, Notation 9.2, and $P_M F = F$, we have furthermore $W \begin{bmatrix} F \\ Iq \end{bmatrix} = - \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$. Taken all together, we obtain $\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = - VW \begin{bmatrix} F \\ Iq \end{bmatrix}$. In view of Lemma 9.10, thus $F_1 = h \delta P_M F = h \delta F$ and $F_2 = h \delta Iq$ follow. Taking into account $[G_1; G_2] \in \mathcal{P}[M]$, we see from Proposition 9.11 that $\det F_2$ does not vanish identically in $\mathbb{C} \setminus [\alpha, \beta]$. Denote by $H$ the inverse $F_{\alpha, \beta}(M)$-transform of $[G_1; G_2]$. According to Definition 9.5, then $H = F_1 F_2^{-1}$. Consequently, $H = F$. \hfill \square

10 The $F_{\alpha, \beta}(A, M)$-transformation and Its Inverse

In this section, we continue the preceding considerations concerning the construction of the function-theoretic version of the Schur–Nevanlinna type algorithm. We prepare the algebraic formalism for the remaining steps after the first one.

In what follows, we consider two complex $q \times q$ matrices $A$ and $M$, which, in the context of the matricial Hausdorff moment problem, will later be the Non-negative Hermitian matrices $a_0$ and $s_0$ for a given sequence $(s_j)^{\kappa}_{j=0} \in \mathcal{F}_{q, \kappa, \alpha, \beta}$. In this reading, the matrices

$$B := \delta M - A$$

and

$$N := A + \alpha M$$

(10.1)

correspond to $b_0$ and $s_1$, resp., and we have

$$A = -\alpha M + N$$

and

$$B = \beta M - N,$$

(10.2)

according to Notation 3.2. Consider an arbitrarily given $z \in \mathbb{C}$ and let $x := z - \alpha$ and $y := \beta - z$. Taking additionally into account (9.3) and (9.4), we then infer

$$yA - xB = (y + x)N - (\alpha y + \beta x)M = \delta(N - zM)$$

(10.3)

and

$$xA - yB = (x + y)N - (\alpha x + \beta y)M = \delta[N - (\beta + \alpha - z)M].$$

(10.4)

**Definition 10.1** Let $G$ be a non-empty subset of $\mathbb{C}$, let $F : G \to \mathbb{C}^{p \times q}$ be a matrix-valued function, and let $A$ and $M$ be two complex $p \times q$ matrices. Then $G : G \to \mathbb{C}^{p \times q}$ defined by

$$G(z) := AM^\dagger[(\beta - z)F(z) - M][(\beta - z)[(z - \alpha)F(z) + M)]^\dagger A$$

is called the $F_{\alpha, \beta}(A, M)$-transform of $F$. 

In connection with the $F_{\alpha,\beta}(A, M)$-transformation, we consider the following complex $(p+q) \times (p+q)$ matrix polynomial:

**Notation 10.2** Let $A$ and $M$ be two complex $p \times q$ matrices. Then let $\tilde{W}_{A,M}: \mathbb{C} \rightarrow \mathbb{C}^{(p+q) \times (p+q)}$ be defined by

$$\tilde{W}_{A,M}(z) := \begin{bmatrix} -(\beta - z)AM^\dagger & A \\ -(\beta - z)(z - \alpha)A^\dagger & -(\beta - z)A^\dagger M - QA \end{bmatrix}. $$

Under certain conditions, we can write the $F_{\alpha,\beta}(A, M)$-transform as a linear fractional transformation with the generating matrix-valued function $\tilde{W}_{A,M}$.

**Lemma 10.3** Let $A, M \in \mathbb{C}^{p \times q}$ be such that $N(M) \subseteq N(A)$. Let $F: \mathbb{C} \setminus [\alpha, \beta] \rightarrow \mathbb{C}^{p \times q}$ be a matrix-valued function with $F_{\alpha,\beta}(A, M)$-transform $G$ and let $G_1, G_2: \mathbb{C} \setminus [\alpha, \beta] \rightarrow \mathbb{C}^{q \times q}$ be defined by

$$G_1(w) := -(\beta - w)AM^\dagger F(w) + A \quad (10.5)$$

and

$$G_2(w) := -(\beta - w)(w - \alpha)A^\dagger F(w) - (\beta - w)A^\dagger M - QA. \quad (10.6)$$

Let $z \in \mathbb{C} \setminus [\alpha, \beta]$ be such that $\mathcal{R}(F(z)) \subseteq \mathcal{R}(M)$ and $\mathcal{N}(M) \subseteq \mathcal{N}(F(z))$ as well as $\mathcal{R}((z - \alpha)F(z) + M) = \mathcal{R}(A)$ and $\mathcal{N}((z - \alpha)F(z) + M) = \mathcal{N}(A)$ are fulfilled. Then $\det G_2(z) \neq 0$ and $G(z) = [G_1(z)][G_2(z)]^{-1}$.

**Proof** Let $X := (z - \alpha)F(z) + M$ and let $Y := (\beta - z)F(z) - M$. From Remark A.21 we get $AM^\dagger M = A$. Remark A.18 shows that $I_q - A^\dagger A = QA$. Setting $y := \beta - z$ and $Z := yX$, we obtain then

$$-AM^\dagger Y = -yAM^\dagger F(z) + AM^\dagger M = -yAM^\dagger F(z) + A = G_1(z) \quad (10.7)$$

and

$$-A^\dagger Z - (I_q - A^\dagger A) = -yA^\dagger X - QA = G_2(z). \quad (10.8)$$

Using Remark A.22, we get $XM^\dagger Y = YM^\dagger X$. In view of Remark A.18, furthermore $X^\dagger X = A^\dagger A$ holds true. Thus, we can infer

$$AM^\dagger Y = AA^\dagger AM^\dagger Y = AX^\dagger XM^\dagger Y = AX^\dagger YM^\dagger X$$

and, therefore,

$$AM^\dagger YA^\dagger A = AX^\dagger YM^\dagger XA^\dagger A = AX^\dagger YM^\dagger XX^\dagger X = AX^\dagger YM^\dagger X = AM^\dagger Y.$$
Regarding $y \neq 0$, we have $\mathcal{R}(Z) = \mathcal{R}(A)$ and $\mathcal{N}(Z) = \mathcal{N}(A)$. By virtue of Remark A.18, hence $ZZ^\dagger = AA^\dagger$ and $Z^\dagger Z = A^\dagger A$. Consequently,

$$-[Z^\dagger A - (I_q - A^\dagger A)](-A^\dagger Z - (I_q - A^\dagger A))$$

$$= Z^\dagger AA^\dagger Z + Z^\dagger A(I_q - A^\dagger A) + (I_q - A^\dagger A)A^\dagger Z + (I_q - A^\dagger A)^2$$

$$= Z^\dagger ZZ^\dagger Z + I_q - A^\dagger A = Z^\dagger Z + I_q - Z^\dagger Z = I_q.$$

Hence, $\det[-A^\dagger Z - (I_q - A^\dagger A)] \neq 0$ and $[-A^\dagger Z - (I_q - A^\dagger A)]^{-1} = -Z^\dagger A - (I_q - A^\dagger A)$. Thus,

$$-AM^\dagger Y[-A^\dagger Z - (I_q - A^\dagger A)]^{-1} = AM^\dagger Y[Z^\dagger A + (I_q - A^\dagger A)]$$

$$= AM^\dagger YZ^\dagger A + AM^\dagger Y(I_q - A^\dagger A) = AM^\dagger YZ^\dagger A = G(z).$$

In view of (10.7) and (10.8), the proof is complete. □

From Lemmata 10.12 and 10.13 we will see that the following transformation for matrix-valued functions is in a generic situation indeed the inversion of the $\mathcal{F}_{\alpha,\beta}(A, M)$-transformation. Against this background we introduce the following notation:

**Definition 10.4** Let $\mathcal{G}$ be a non-empty subset of $\mathbb{C}$, let $G: \mathcal{G} \rightarrow \mathbb{C}^{p \times q}$ be a matrix-valued function, and let $A$ and $M$ be two complex $p \times q$ matrices. Let $B := \delta M - A$ and let $F: \mathcal{G} \rightarrow \mathbb{C}^{p \times q}$ be defined by

$$F(z) := -[(\beta - z)MA^\dagger G(z) + A + MQ_A M^\dagger B]$$

$$\times(\beta - z)[(z - \alpha)A^\dagger G(z) - M^\dagger A] + (z - \alpha)Q_A M^\dagger B)^\dagger.$$

Then we call the matrix-valued function $F$ the inverse $\mathcal{G}_{\alpha,\beta}(A, M)$-transform of $G$.

**Lemma 10.5** Let $A \in \mathbb{C}^{q \times q}_{\mathbb{H}}$ and let $M \in \mathbb{C}^{q \times q}$ with $\mathcal{R}(A) \subseteq \mathcal{R}(M)$. Let $G \in \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$ be such that $\mathcal{R}(G(z_0)) \subseteq \mathcal{R}(A)$ holds true for some $z_0 \in \mathbb{C} \setminus [\alpha, \beta]$. Let $B := \delta M - A$ and let $E_1, E_2: \mathbb{C} \setminus [\alpha, \beta] \rightarrow \mathbb{C}^{q \times q}$ be defined by

$$E_1(w) := (\beta - w)MA^\dagger G(w) + A + MQ_A M^\dagger B \quad (10.9)$$

and

$$E_2(w) := -(\beta - w)(w - \alpha)A^\dagger G(w) + (\beta - w)M^\dagger A - (w - \alpha)Q_A M^\dagger B. \quad (10.10)$$

For all $z \in \mathbb{C} \setminus [\alpha, \beta]$, then $\mathcal{R}(E_1(z)) \subseteq \mathcal{R}(M)$, $\mathcal{N}(M) \subseteq \mathcal{N}(E_1(z))$, $\mathcal{R}(E_2(z)) = \mathcal{R}(M)$, and $\mathcal{N}(E_2(z)) = \mathcal{N}(M)$. Furthermore, the inverse $\mathcal{F}_{\alpha,\beta}(A, M)$-transform of $G$ is holomorphic in $\mathbb{C} \setminus [\alpha, \beta]$. 
**Proof** Consider an arbitrary \( z \in \mathbb{C} \setminus [\alpha, \beta] \). Regarding Remark A.6, we have \( \mathcal{R}(E_1(z)) \subseteq \mathcal{R}(MA^\dagger G(z)) + \mathcal{R}(A) + \mathcal{R}(MQ_A M^\dagger B) \subseteq \mathcal{R}(M) \). From Proposition 4.15 we infer \( \mathcal{R}(G(\overline{z})) = \mathcal{R}(G(z_0)) \subseteq \mathcal{R}(A) \subseteq \mathcal{R}(M) \). Consequently, \([\mathcal{R}(M)]^\perp \subseteq [\mathcal{R}(A)]^\perp \subseteq [\mathcal{R}(G(\overline{z})))^\perp \). In view of Remarks A.10 and 4.8, then

\[
\mathcal{N}(M) \subseteq \mathcal{N}(A) \subseteq \mathcal{N}([G(\overline{z}])]^\ast) = \mathcal{N}(G(z)) \quad (10.11)
\]

follows. Regarding \( \mathcal{N}(M) \subseteq \mathcal{N}(A) \) and Remark A.6, we can conclude \( \mathcal{N}(M) \subseteq \mathcal{N}((z - \alpha)A^\dagger [G(z)]v + (z - \alpha)Q_A M^\dagger Bv) \). In view of Proposition 4.15, we have \( \mathcal{R}(A^\dagger) = \mathcal{R}(A) \subseteq \mathcal{R}(M) = \mathcal{R}(M^\dagger) \). Therefore, we obtain \( \mathcal{R}(Q_A M^\dagger) \subseteq \mathcal{R}(M^\dagger) \) and, regarding additionally Remark A.6, we can conclude \( \mathcal{R}(Q_A M^\dagger) \subseteq \mathcal{R}(M^\dagger) \). Consequently, we obtain, by virtue of Remark A.14, that \( (\beta - z)MA^\dagger Av = (\beta - z)(z - \alpha)A^\dagger [G(z)]v + (z - \alpha)Q_A M^\dagger Bv \). (10.13)

In view of Proposition 4.15, we have \( \mathcal{R}(G(z)) = \mathcal{R}(G(z_0)) \subseteq \mathcal{R}(A) \). According to Remark A.20, the set \( AA^\dagger G(z) = G(z) \). Regarding \( z \neq \beta \), we can multiply Eq. (10.13) from the left by \((\beta - z)^{-1}A\) to obtain then \( AM^\dagger Av = (z - \alpha)[G(z)]v \). Left multiplication of this identity by \((z - \alpha)v^*\) yields

\[
|z - \alpha|^2 v^*[G(z)]v = (\overline{z} - \alpha)v^* AM^\dagger Av = (\overline{z} - \alpha)\|Rv\|^2_E. \quad (10.14)
\]

In the case \( z \in \mathbb{C} \setminus \mathbb{R} \), we can infer from Proposition 4.14 and Remark A.2 that

\[
0 \leq |z - \alpha|^2 v^*[\frac{1}{\text{Im} z}] \text{Im} G(z)]v = \frac{1}{\text{Im} z} \text{Im}(|z - \alpha|^2 v^*[G(z)]v) = \frac{1}{\text{Im} z} \text{Im}[(\overline{z} - \alpha)\|Rv\|^2_E] \geq \|Rv\|^2_E \leq 0,
\]

implying (10.12). If \( z \in (-\infty, \alpha) \), then \( \overline{z} = z < \alpha \) and we obtain, by virtue of Notation 4.6 and (10.14), thus \( 0 \leq |z - \alpha|^2 v^*[G(z)]v = (\overline{z} - \alpha)\|Rv\|^2_E \leq 0 \), implying again (10.12). In the case \( z \in (\beta, \infty) \), we have \( \overline{z} = z > \beta > \alpha \) and, because of Notation 4.6 and (10.14), similarly

\[
0 \leq |z - \alpha|^2 v^*[-G(z)]v = -|z - \alpha|^2 v^*[G(z)]v = (\alpha - \overline{z})\|Rv\|^2_E \leq 0,
\]

i.e. (10.12). Hence, (10.12) is verified. Consequently, using Remark A.20, we can infer \( Av = MM^\dagger Av = M\sqrt{M^\dagger}Rv = O_{q \times 1} \). Regarding (10.11), we thus obtain
\[ [G(z)]v = O_{q \times 1}. \] Because of (10.11) and Remark A.21, we have \( AM^\dagger M = A \). In view of \( Q_A = I_q - A^\dagger A \), in particular \( Q_A M^\dagger M = M^\dagger M - A^\dagger A \) holds true. Taking into account \( Av = O_{q \times 1} \) and \([G(z)]v = O_{q \times 1} \), we see from (10.13) then

\[ O_{q \times 1} = (z - \alpha) Q_A M^\dagger Bv = (z - \alpha) Q_A M^\dagger (\delta M)v = (z - \alpha) \delta M^\dagger Mv. \]

Left multiplication of the latter by \( M \) yields \( (z - \alpha) \delta Mv = O_{q \times 1} \). Since \( z \neq \alpha \) and \( \delta > 0 \) hold true, necessarily \( Mv = O_{q \times 1} \) follows. Hence, \( \mathcal{N}(E_2(z)) \subseteq \mathcal{N}(M) \). Because of (10.11) and \( \mathcal{N}(M) \subseteq \mathcal{N}(B) \), we obtain, by virtue of Remark A.6, on the other hand \( \mathcal{N}(M) \subseteq \mathcal{N}(A^\dagger G(z)) \cap \mathcal{N}(M^\dagger A) \cap \mathcal{N}(Q_A M^\dagger B) \subseteq \mathcal{N}(E_2(z)) \).

Consequently, \( \mathcal{N}(E_2(z)) = \mathcal{N}(M) \) is verified. Using Remark A.3, we can, in view of \( \mathcal{R}(E_2(z)) \subseteq \mathcal{R}(M) \), then easily conclude \( \mathcal{R}(E_2(z)) = \mathcal{R}(M) \). Observe that the matrix-valued function \( G \) is holomorphic. Thus, \( E_1 \) and \( E_2 \) are holomorphic in \( C \setminus [\alpha, \beta] \) as well. Let \( D_2 : C \setminus [\alpha, \beta] \rightarrow C^{q \times q} \) be defined by \( D_2(w) := [E_2(w)]^\dagger \). As already shown, the linear subspaces \( \mathcal{R}(E_2(w)) \) and \( \mathcal{N}(E_2(w)) \) do not depend on the point \( w \in C \setminus [\alpha, \beta] \). Due to Proposition F.3, thus the matrix-valued function \( D_2 \) is holomorphic. Denote by \( F \) the inverse \( F_{\alpha, \beta}(A, M) \)-transform of \( G \). In view of Remarks A.15 and A.13, we have \( E_1 D_2 = F \). Using Remark F.2, we can conclude then that the matrix-valued function \( F \) is holomorphic in \( C \setminus [\alpha, \beta] \).

The following complex \((p + q) \times (p + q)\) matrix polynomial is intimately connected to the inverse \( F_{\alpha, \beta}(A, M) \)-transform:

**Notation 10.6** Let \( A \) and \( M \) be two complex \( p \times q \) matrices and let \( B := \delta M - A \). Then let \( \tilde{V}_{A,M} : C \rightarrow C^{(p+q) \times (p+q)} \) be defined by

\[
\tilde{V}_{A,M}(z) := \begin{bmatrix}
(\beta - z)MA^\dagger & A + MQ_AM^\dagger B \\
-(\beta - z)(z - \alpha)A^\dagger & (\beta - z)(\delta Q_M + M^\dagger A) - (z - \alpha)Q_AM^\dagger B
\end{bmatrix}.
\]

**Remark 10.7** Let \( A, M \in C^{q \times q} \) and let \( z \in C \). Let \( B := \delta M - A \). Then:

(a) If \( A = O_{q \times q} \), then \( B = \delta M \) and \( \tilde{V}_{A,M}(z) = \delta \begin{bmatrix}
O_{q \times q} & M \\
O_{q \times q} & (\beta - z)I_q - \delta M^\dagger M
\end{bmatrix}. \)

(b) If \( B = O_{q \times q} \), then \( A = \delta M \) and \( \tilde{V}_{A,M}(z) = \begin{bmatrix}
\delta^{-1}(\beta - z)MM^\dagger & \delta M \\
-\delta^{-1}(\beta - z)(z - \alpha)M^\dagger & \delta(\beta - z)I_q
\end{bmatrix}. \)

(c) If all the matrices \( M, A, B \) are invertible, then \( \tilde{V}_{A,M}(z) = \begin{bmatrix}
(\beta - z)MA^{-1} & A \\
-(\beta - z)(z - \alpha)A^{-1} & (\beta - z)M^{-1}A
\end{bmatrix}. \)

Under certain conditions, we can write the inverse \( F_{\alpha, \beta}(A, M) \)-transform as a linear fractional transformation with the generating matrix-valued function \( \tilde{V}_{A,M} \).

**Lemma 10.8** Let \( A \in C^{q \times q} \) and let \( M \in C^{q \times q} \) with \( \mathcal{R}(A) \subseteq \mathcal{R}(M) \). Let \( G \in \mathcal{R}_q(C \setminus [\alpha, \beta]) \) with inverse \( F_{\alpha, \beta}(A, M) \)-transform \( F \) be such that \( \mathcal{R}(G(z_0)) \subseteq \mathcal{R}(A) \) holds true for some \( z_0 \in C \setminus [\alpha, \beta] \). Let \( B := \delta M - A \) and let \( F_1, F_2 : C \setminus [\alpha, \beta] \rightarrow C^{q \times q} \) be defined by

\[
F_1(w) := (\beta - w)MA^\dagger G(w) + A + MQ_AM^\dagger B \tag{10.15}
\]
and
\[ F_2(w) := -(\beta - w)(w - \alpha)A^\dagger G(w) + (\beta - w)(\delta Q_M + M^\dagger A) - (w - \alpha)Q_A M^\dagger B. \]

(10.16)

For all \( z \in \mathbb{C} \setminus [\alpha, \beta] \), then \( \det F_2(z) \neq 0 \) and \( F(z) = [F_1(z)][F_2(z)]^{-1} \).

**Proof** Let \( E_1, E_2 : \mathbb{C} \setminus [\alpha, \beta] \to \mathbb{C}^{q \times q} \) be defined by (10.9) and (10.10). Consider an arbitrary \( z \in \mathbb{C} \setminus [\alpha, \beta] \). We have \( F_1(z) = E_1(z) \) and \( F_2(z) = E_2(z) + (\beta - z)\delta Q_M \). From Lemma 10.5 we get \( \mathcal{R}(E_2(z)) = \mathcal{R}(M) \) and \( \mathcal{N}(E_2(z)) = \mathcal{N}(M) \). Remark A.10 yields then \( \mathcal{R}(E_2(z)) = \mathcal{R}(M^*) = \mathcal{R}([E_2(z)]^*) \). In view of \( z \neq \beta \) and \( \delta > 0 \), we thus can apply Lemma A.19 with \( \eta := (\beta - z)\delta \) to see that the matrix \( E_2(z) + \eta Q_M = F_2(z) \) is invertible and that \([E_2(z)]^\dagger = [F_2(z)]^{-1} - \eta^{-1}Q_M \) holds true. By virtue of Lemma 10.5, we have \( \mathcal{N}(M) \subseteq \mathcal{N}(E_1(z)) \). Consequently, we obtain

\[ [F_1(z)][F_2(z)]^{-1} = [E_1(z)][E_2(z)]^\dagger + \eta^{-1}Q_M = [E_1(z)][E_2(z)]^\dagger = F(z). \quad \square \]

**Lemma 10.9** Let \( A \in \mathbb{C}^{q \times q} \) and let \( M \in \mathbb{C}^{q \times q} \) with inverse \( F_{\alpha,\beta}(A, M) \)-transform \( F \). Suppose that \( \mathcal{R}(G(z_0)) \subseteq \mathcal{R}(A) \) holds true for some \( z_0 \in \mathbb{C} \setminus [\alpha, \beta] \). For all \( z \in \mathbb{C} \setminus [\alpha, \beta] \), then \( \mathcal{R}(F(z)) \subseteq \mathcal{R}(M) \) and \( \mathcal{N}(M) \subseteq \mathcal{N}(F(z)) \) and furthermore \( \mathcal{R}((z - \alpha)F(z) + M) = \mathcal{R}(A) \) and \( \mathcal{N}((z - \alpha)F(z) + M) = \mathcal{N}(A) \).

**Proof** Consider an arbitrary \( z \in \mathbb{C} \setminus [\alpha, \beta] \). Let \( E_1, E_2 : \mathbb{C} \setminus [\alpha, \beta] \to \mathbb{C}^{q \times q} \) be defined by (10.9) and (10.10). According to Definition 10.4, then \( F(z) = [E_1(z)][E_2(z)]^\dagger \). Lemma 10.5 yields furthermore \( \mathcal{R}(E_1(z)) \subseteq \mathcal{R}(M) \) and \( \mathcal{R}(E_2(z)) = \mathcal{R}(M) \). By virtue of Remarks A.14 and A.10, we can infer from the last identity \( \mathcal{N}([E_2(z)]^\dagger) = \mathcal{N}(M^*) = \mathcal{N}(M) \). Consequently, we obtain \( \mathcal{R}(F(z)) \subseteq \mathcal{R}(M) \) and \( \mathcal{N}(M) \subseteq \mathcal{N}(F(z)) \). Taking additionally into account Remark A.14, we can conclude \( M[E_2(z)][E_2(z)]^\dagger = M \). Remark A.20 yields \( MM^\dagger A = A \). Let \( x := z - \alpha \), let \( y := \beta - z \), and let \( X := xF(z) + M \). Taken all together, we get

\[ X = [xE_1(z) + ME_2(z)][E_2(z)]^\dagger = (xA + yMM^\dagger A)[E_2(z)]^\dagger = \delta A[E_2(z)]^\dagger. \]

Analogous to the corresponding considerations in the proof of Lemma 10.8, we can show that the matrix \( R := E_2(z) + Q_M \) is invertible and that \([E_2(z)]^\dagger = R^{-1} - Q_M \) holds true. As in the proof of Lemma 10.5, we can obtain (10.11). Thus, \( X = \delta A(R^{-1} - Q_M) = \delta AR^{-1} \) follows. Regarding \( \delta > 0 \), we see from Remark A.8 hence \( \mathcal{R}(X) = \mathcal{R}(A) \) and \( \mathcal{N}(X) = R\mathcal{N}(A) \). Let \( B := \delta M - A \). In view of (10.10) and (10.11), each \( v \in \mathcal{N}(A) \) satisfies \( Rv = -xQM^\dagger Bv + Q_Mv \) and thus \( ARv = O_{q \times 1} \). Consequently, \( \mathcal{N}(X) \subseteq \mathcal{N}(A) \) is verified. Taking additionally into account \( \mathcal{R}(X) = \mathcal{R}(A) \), we infer by virtue of Remark A.3 then easily \( \mathcal{N}(X) = \mathcal{N}(A) \). \quad \square

Now we are going to study the composition of the two transformations introduced in Definitions 10.1 and 10.4. Doing this, we will take into account that, in view of Lemmata 10.3 and 10.8, these transformations can be written under certain conditions as linear fractional transformations of matrices with generating matrix-valued functions \( \tilde{W}_{A,M} \) and \( \tilde{V}_{A,M} \), resp.
Lemma 10.10 Let $A, M \in \mathbb{C}^{p \times q}$ with $\mathcal{R}(A) \subseteq \mathcal{R}(M)$ and $\mathcal{N}(M) \subseteq \mathcal{N}(A)$, let $B := \delta M - A$, and let $N := A + \alpha M$. Let $z \in \mathbb{C}$, let $x := z - \alpha$, and let $y := \beta - z$. Then

$$[\tilde{W}_{A,M}(z)][\tilde{V}_{A,M}(z)] = -\delta([y P_A] \oplus [y P_A^* - \beta Q_M + Q_A(z I_q - M^\dagger N)])$$

and

$$y P_A^* - \beta Q_M + Q_A(z I_q - M^\dagger N) = (M^\dagger B + P_A^* M^\dagger A) - y Q_A - x P_A^*.$$  

(10.17)

Proof For the block representation $[\tilde{W}_{A,M}(z)][\tilde{V}_{A,M}(z)] = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$ with $p \times p$ block $X_{11}$, we have

$$X_{11} = -y M^\dagger A (y M A^\dagger) + A(-y x A^\dagger),$$
$$X_{12} = -y M^\dagger A + M Q A M^\dagger B + A[y(\delta Q_M + M^\dagger A) - x Q_A M^\dagger B],$$
$$X_{21} = -y x A^\dagger (y M A^\dagger) - (y A^\dagger M - Q_A)(-y x A^\dagger),$$

and

$$X_{22} = -y x A^\dagger (A + M Q A M^\dagger B) - (y A^\dagger M - Q_A)[y(\delta Q_M + M^\dagger A) - x Q_A M^\dagger B].$$

The application of Remarks A.20 and A.21 yields $M M^\dagger A = A$ and $A M^\dagger M = A$. Regarding (9.1), (9.2), and $y + x = \delta$, we obtain then

$$X_{11} = -y^2 M^\dagger A M^\dagger A^\dagger - y x A A^\dagger = -y(y A A^\dagger + x A A^\dagger) = -y \delta P_A,$$
$$X_{12} = -y A A^\dagger M + M Q A M^\dagger B + y A M^\dagger A = -y A Q A M^\dagger B = O_{p \times q},$$
$$X_{21} = -y^2 x A^\dagger M A^\dagger + y^2 x A^\dagger M A^\dagger + y x Q_A A^\dagger = y x(I_q - A^\dagger A) A^\dagger = O_{q \times p},$$

and

$$X_{22} = -y x A^\dagger A - y x A^\dagger M Q_A M^\dagger B - y^2 A^\dagger M M^\dagger A + y x A^\dagger M Q_A M^\dagger B + y^2 Q_A Q M + y Q_A M^\dagger A - x Q_A M^\dagger B$$
$$= -y x A^\dagger A - y^2 A^\dagger A + y \delta Q_M + y Q_A M^\dagger A - x Q_A M^\dagger B$$
$$= -y \delta P_A^* + (\beta - z) \delta Q_M + y Q_A M^\dagger A - x Q_A M^\dagger B.$$  

(10.18)

In view of (10.3), we have

$$y M^\dagger A - x M^\dagger B = M^\dagger (y A - x B) = \delta M^\dagger (N - z M).$$
By virtue of \( Q_M = I_q - M^\dagger M \), we thus get
\[
yM^\dagger A - xM^\dagger B - z\delta Q_M = \delta(M^\dagger N - zM^\dagger M - zI_q + zM^\dagger M) = \delta(M^\dagger N - zI_q).
\]
\[10.19\]

From (10.18) we can infer then
\[
X_{22} + y\delta P_{A^*} - \beta\delta Q_M = -z\delta Q_M + yQA^\dagger A - xQAM^\dagger B\\
= yQA^\dagger A - xQAM^\dagger B - z\delta Q_AQ_M = \delta QA(M^\dagger N - zI_q),
\]
i.e., \( X_{22} = -\delta[yP_{A^*} - \beta Q_M + QA(zI_q - M^\dagger N)] \). Because of (9.2), we have
\[
M^\dagger B = M^\dagger (\delta - A) = \delta M^\dagger M - M^\dagger A = \delta P_{M^*} - M^\dagger A.
\]
\[
P_{A^*} - Q_M = A^\dagger A - (I_q - M^\dagger M) = M^\dagger M - (I_q - A^\dagger A) = P_{M^*} - Q_A,
\]
and, taking additionally into account \( AM^\dagger M = A \), furthermore
\[
QAP_{M^*} = (I_q - A^\dagger A)M^\dagger M = M^\dagger M - A^\dagger A = P_{M^*} - P_{A^*}.
\]

From (10.18) we thus can conclude
\[
X_{22} = -y[\delta(P_{A^*} - Q_M) - QA^\dagger A] - xQA(\delta P_{M^*} - M^\dagger A)\\
= -y[\delta(P_{M^*} - Q_A) - QA^\dagger A] - x[\delta(P_{M^*} - P_{A^*}) - QA^\dagger A]\\
= -y(\delta P_{M^*} - Q_A^\dagger A) + y\delta Q_A + x\delta P_{A^*}\\
= -\delta(P_{M^*} - M^\dagger A + A^\dagger AM^\dagger A) + \delta(yQ_A + xP_{A^*})\\
= -\delta[(M^\dagger B + P_{A^*}M^\dagger A) - yQ_A - xP_{A^*}].
\]

Comparing the two representations of \( X_{22} \), we can infer then (10.17).

The next result is concerned with the matrix polynomial from (10.17):

**Lemma 10.11** Let \( A, M \in \mathbb{C}^{p \times q} \) with \( \mathcal{N}(M) \subseteq \mathcal{N}(A) \) and let \( z \in \mathbb{C} \setminus [\alpha, \beta] \). Let \( N := A + \alpha M \), let \( y := \beta - z \), and let \( H := yP_{A^*} - \beta Q_M + QA(zI_q - M^\dagger N) \). Then \( \det H \neq 0 \) and \( A^\dagger H^{-1} = y^{-1}A \).

**Proof** Consider an arbitrary \( v \in \mathcal{N}(H) \). We have then
\[
\beta Q_Mv - yA^\dagger Av = (\beta Q_M - yP_{A^*})v\\
= QA(zI_q - M^\dagger N)v = (I_q - A^\dagger A)(zI_q - M^\dagger N)v.
\]

Regarding \( z \neq \beta \), left multiplication of the latter identity by \(-y^{-1}A\) yields \( Av = O_{p \times 1} \). Consequently, \( Nv = \alpha Mv \). Taking into account Remark A.21, thus \( AM^\dagger Nv =...\)
\[ \alpha AM^\dagger Mv = \alpha Av = O_{p \times 1}. \] From (10.20) we then infer
\[
\beta Q_M v = (I_q - A^\dagger A)(zI_q - M^\dagger N)v = (zI_q - M^\dagger N)v = zv - \alpha M^\dagger Mv.
\]
(10.21)

Left multiplying this by \( M \), we get \( O_{p \times 1} = (z - \alpha)Mv. \) Since \( z \neq \alpha \), then necessarily \( Mv = O_{p \times 1} \). Substituting this into (10.21) and regarding \( Q_M = I_q - M^\dagger M \), we obtain \( \beta v = zv \). Because of \( z \neq \beta \), hence \( v = O_{p \times 1} \) follows. Consequently, the linear subspace \( \mathcal{N}(H) \) is trivial, implying \( \det H \neq 0 \). By virtue of \( AH = yA\mathcal{P}_A^* = yA \) and \( z \neq \beta \), we thus have \( AH^{-1} = y^{-1}A \).

In generic situations, the \( \mathcal{F}_{\alpha, \beta}(A, M) \)-transformation turns out to be inverse to the inverse \( \mathcal{F}_{\alpha, \beta}(A, M) \)-transformation:

**Lemma 10.12** Let \( A \in \mathbb{C}_H^{q \times q} \) and let \( M \in \mathbb{C}_H^{q \times q} \) with \( \mathcal{R}(A) \subseteq \mathcal{R}(M) \). Let \( G \in \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta]) \) with inverse \( \mathcal{F}_{\alpha, \beta}(A, M) \)-transform \( F \). Suppose that \( \mathcal{R}(G(z_0)) \subseteq \mathcal{R}(A) \) holds true for some \( z_0 \in \mathbb{C} \setminus [\alpha, \beta] \). Then \( G \) is exactly the \( \mathcal{F}_{\alpha, \beta}(A, M) \)-transform of \( F \).

**Proof** Consider an arbitrary \( z \in \mathbb{C} \setminus [\alpha, \beta] \). Let \( F_1, F_2: \mathbb{C} \setminus [\alpha, \beta] \to \mathbb{C}_H^{q \times q} \) be defined by (10.15) and (10.16). Then
\[
\begin{bmatrix}
F_1(z) \\
F_2(z)
\end{bmatrix} = \begin{bmatrix}
\mathcal{V}_{A,M}(z) \\
\mathcal{G}(z)
\end{bmatrix}.
\]
Due to Lemma 10.8, furthermore \( \det F_2(z) \neq 0 \) and \( F(z) = [F_1(z)][F_2(z)]^{-1} \). Denote by \( H \) the \( \mathcal{F}_{\alpha, \beta}(A, M) \)-transform of \( F \) and let \( H_1, H_2: \mathbb{C} \setminus [\alpha, \beta] \to \mathbb{C}_H^{q \times q} \) be defined by
\[
H_1(w) := - (\beta - w)AM^\dagger F(w) + A
\]
and
\[
H_2(w) := - (\beta - w)(w - \alpha)A^\dagger F(w) - (\beta - w)A^\dagger M - QA.
\]
Using Remark A.10, we infer \( \mathcal{N}(M) \subseteq \mathcal{N}(A) \). In view of Lemma 10.9, we thus can apply Lemma 10.3 to \( F \) to obtain \( \det H_2(z) \neq 0 \) and \( H(z) = [H_1(z)][H_2(z)]^{-1} \). Since by construction
\[
\begin{bmatrix}
H_1(z) \\
H_2(z)
\end{bmatrix} = \begin{bmatrix}
\mathcal{W}_{A,M}(z) \\
\mathcal{I}_q
\end{bmatrix} \begin{bmatrix}
F_1(z) \\
F_2(z)
\end{bmatrix}^{-1}
\]
holds true, we have
\[
\begin{bmatrix}
H_1(z) \\
H_2(z)
\end{bmatrix} = \begin{bmatrix}
\mathcal{W}_{A,M}(z) \\
\mathcal{I}_q
\end{bmatrix} \begin{bmatrix}
\mathcal{V}_{A,M}(z) \\
\mathcal{G}(z)
\end{bmatrix}^{-1} \begin{bmatrix}
F_1(z) \\
F_2(z)
\end{bmatrix}^{-1}.
\]
Let \( x := z - \alpha \), let \( y := \beta - z \), and let \( N := A + \alpha M \). Taking into account Lemma 10.10, then
\[
H_1(z) = - \delta y P_A G(z) [F_2(z)]^{-1},
\]
\[
H_2(z) = - \delta [y P_A^* - \beta Q_M + Q_A (zI_q - M^\dagger N)] [F_2(z)]^{-1}.
\]
follow by comparing both sides of the latter identity. According to Proposition 4.15, we get \( \mathcal{R}(G(z)) = \mathcal{R}(G(z_0)) \subseteq \mathcal{R}(A) \). Thus, \( P_A G(z) = G(z) \). Using again Proposition 4.15, we can conclude \( \mathcal{R}(G(\overline{z})) = \mathcal{R}(G(z_0)) \subseteq \mathcal{R}(A) \), implying \( [\mathcal{R}(A)]^\perp \subseteq [\mathcal{R}(G(\overline{z}))]^\perp \). In view of Remarks A.10 and 4.8, we then obtain \( \mathcal{N}(A) \subseteq \mathcal{N}(G(z)) \). Due to Remark A.21, therefore \( [G(z)]^\dagger A = G(z) \) holds true. Regarding \( \det H_2(z) \neq 0 \) and Lemma 10.11, we have furthermore

\[
-\delta A[F_2(z)]^{-1}[H_2(z)]^{-1} = A(-\delta^{-1}[H_2(z)][F_2(z)])^{-1} = y^{-1}A.
\]

Consequently,

\[
 H(z) = [H_1(z)][H_2(z)]^{-1} = -\delta y[G(z)]^\dagger A[F_2(z)]^{-1}[H_2(z)]^{-1} = [G(z)]^\dagger A = G(z).
\]

Conversely, we have:

**Lemma 10.13** Let \( A \in \mathbb{C}^{q \times q}_H \) and let \( M \in \mathbb{C}^{q \times q}_\geq \) with \( \mathcal{R}(A) \subseteq \mathcal{R}(M) \). Let \( F: \mathbb{C} \setminus [\alpha, \beta] \to \mathbb{C}^{q \times q} \) with \( F_{\alpha, \beta}(A, M) \)-transform \( G \) and denote by \( H \) the inverse \( F_{\alpha, \beta}(A, M) \)-transform of \( G \). Let \( z \in \mathbb{C} \setminus [\alpha, \beta] \) be such that \( \mathcal{R}(F(z)) \subseteq \mathcal{R}(M) \) and \( \mathcal{N}(M) \subseteq \mathcal{N}(F(z)) \) as well as \( \mathcal{R}((z - \alpha)F(z) + M) = \mathcal{R}(A) \) and \( \mathcal{N}((z - \alpha)F(z) + M) = \mathcal{N}(A) \) are fulfilled. Suppose that \( G \) belongs to \( \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta]) \) and that \( \mathcal{R}(G(z_0)) \subseteq \mathcal{R}(A) \) holds true for some \( z_0 \in \mathbb{C} \setminus [\alpha, \beta] \). Then \( H(z) = F(z) \).

**Proof** Because of Remark A.10, we have \( \mathcal{N}(M) \subseteq \mathcal{N}(A) \), implying \( AM^\dagger M = A \), by virtue of Remark A.21. Let \( G_1, G_2: \mathbb{C} \setminus [\alpha, \beta] \to \mathbb{C}^{q \times q} \) be defined by (10.5) and (10.6). The application of Lemma 10.3 yields then \( \det G_2(z) \neq 0 \) and \( G(z) = [G_1(z)][G_2(z)]^{-1} \). Let \( x := z - \alpha \) and let \( y := \beta - z \). Setting \( X := xF(z) + M \) and \( Y := yF(z) - M \), we get

\[
 G_1(z) = -yAM^\dagger F(z) + A = -yAM^\dagger F(z) + AM^\dagger M = -AM^\dagger Y
\]

and

\[
 G_2(z) = -yxA^\dagger F(z) - yA^\dagger M - QA = -yA^\dagger X - QA.
\]

Taking into account the assumptions, we have \( XQA = O_{q \times q} \) and, in view of Remarks A.20 and A.21, furthermore \( MM^\dagger A = A, AA^\dagger X = X, \) and \( XA^\dagger A = X \). Let \( B := \delta M - A \) and let \( E_1, E_2: \mathbb{C} \setminus [\alpha, \beta] \to \mathbb{C}^{q \times q} \) be defined by (10.9) and
(10.10). Since \( A Q_A = O_{q \times q} \) holds obviously true, we can infer then

\[
F(z)E_2(z) - E_1(z) = F(z)[-yxA^\dagger G(z) + yM^\dagger A - xQ_AM^\dagger B] - [yMA^\dagger G(z) + A + MQ_AM^\dagger B] \\
= -yx[F(z)]A^\dagger G(z) + y[F(z)]M^\dagger A - x[F(z)]Q_AM^\dagger B \\
= -yMA^\dagger G(z) - MM^\dagger A - MQ_AM^\dagger B \\
= -yXA^\dagger G(z) + YM^\dagger A - XQ_AM^\dagger B \\
= -yXA^\dagger [G_1(z)][G_2(z)]^{-1} + YM^\dagger A[G_2(z)][G_2(z)]^{-1} \\
= [-yXA^\dagger (-AM^\dagger Y) + YM^\dagger A(-yA^\dagger X - QA)][G_2(z)]^{-1} \\
= y(XM^\dagger Y - YM^\dagger X)[G_2(z)]^{-1}.
\]

Regarding Remark A.22, we see \( XM^\dagger Y = YM^\dagger X \). Consequently, \( F(z)E_2(z) = E_1(z) \) follows. Observe that \( H(z) = [E_1(z)][E_2(z)]^\dagger \). Due to Lemma 10.5, we have \( \mathcal{R}(E_2(z)) = \mathcal{R}(M) \). Using Remarks A.14 and A.10, we thus can conclude \( \mathcal{N}((E_2(z))^\dagger) = \mathcal{N}(M^\ast) = \mathcal{N}(M) \). In view of \( \mathcal{N}(M) \subseteq \mathcal{N}(F(z)) \), hence \( \mathcal{N}((E_2(z))^\dagger) \subseteq \mathcal{N}(F(z)) \). Because of Remarks A.21 and A.14, then \( F(z)[E_2(z)](E_2(z))^\dagger = F(z) \) holds true. Consequently, we obtain

\[
H(z) = [E_1(z)][E_2(z)]^\dagger = F(z)[E_2(z)](E_2(z))^\dagger = F(z). \quad \square
\]

In the particular completely degenerate situation \( B = O_{p \times q} \) we have \( A = \delta M \), according to (10.1). Because of (9.2), then the matrix polynomials \( \tilde{V}_{A,M} \) and \( \tilde{W}_{A,M} \) from Notations 10.6 and 10.2 essentially coincide with \( \tilde{V}_M \) and \( \tilde{W}_M \) introduced in Notations 9.6 and 9.2, resp.: 

**Remark 10.14** If \( M \in \mathbb{C}^{p \times q} \), then the equations \( \tilde{V}_{\delta M,M} = \tilde{V}_M[(\delta^{-1}I_p) \oplus (\delta I_q)] \) and \( \tilde{W}_{\delta M,M} = [(\delta^{-1}I_p) \oplus (\delta I_q)]\tilde{W}_M \) hold true.

In addition to the matrices \( A \) and \( M \) and the matrices \( B \) and \( N \) built from them via (10.1), we now consider the matrix \( D := AM^\dagger B \), which, in view of Remark 3.24, corresponds to \( \partial_1 \). Because of (10.2), we have

\[
D = (-\alpha M + N)M^\dagger (\beta M - N) = -\alpha \beta M + \alpha P_MN + \beta NP_M^\ast - NM^\dagger N
\]

in analogy to the second equation in (3.7). Taking into account (10.1) and (3.9), we get furthermore

\[
D = A[\frac{1}{\delta}(A + B)]^\dagger B = \delta[A(A + B)^\dagger B] = \delta(A \mp B). \quad (10.22)
\]
Notation 10.15 Let $A, M \in \mathbb{C}^{p \times q}$ and let $B := \delta M - A$ and $D := AM^\dagger B$. Then let $\tilde{U}_{A,M} : \mathbb{C} \to \mathbb{C}^{(p+q) \times (p+q)}$ be defined by

$$\tilde{U}_{A,M}(z) := \begin{bmatrix} M[(\beta - z)PA^*M^\dagger B + (z - \alpha)QA^*M^\dagger A]D^\dagger + B & \beta - z)(\delta Q_M + M^\dagger A) \\ -(\beta - z)(z - \alpha)M^\dagger AD^\dagger & \end{bmatrix}. $$

Remark 10.16 Let $A, M \in \mathbb{C}^{q \times q}$ and let $z \in \mathbb{C}$. Let $B := \delta M - A$ and $D := AM^\dagger B$. Then:

(a) If $D = O_{q \times q}$, then $\tilde{U}_{A,M}(z) = \begin{bmatrix} O_{q \times q} & B \\ O_{q \times q} & (\beta - z)(\delta Q_M + M^\dagger A) \end{bmatrix}.$

(b) If $A = O_{q \times q}$, then $B = \delta M, D = O_{q \times q}$, and $\tilde{U}_{A,M}(z) = \delta \begin{bmatrix} O_{q \times q} & M \\ O_{q \times q} & (\beta - z)Q_M \end{bmatrix}.$

(c) If $B = O_{q \times q}$, then $A = \delta M, D = O_{q \times q}$ and $\tilde{U}_{A,M}(z) = \delta \begin{bmatrix} O_{q \times q} & O_{q \times q} \\ O_{q \times q} & (\beta - z)I_q \end{bmatrix}.$

(d) If the matrices $M, A, B, D$ are invertible and $AM^{-1}B = BM^{-1}A$, then $\tilde{U}_{A,M}(z) = \begin{bmatrix} (\beta - z)M^\dagger A^{-1} & B \\ -(\beta - z)(z - \alpha)B^{-1}(\beta - z)M^{-1}A \end{bmatrix}.$

Lemma 10.17 Let $A, M \in \mathbb{C}^{q \times q}$ with $\mathcal{R}(A) \subseteq \mathcal{R}(M)$ and let $z \in \mathbb{C}$. Let $B := \delta M - A$, $D := AM^\dagger B$, and $N := A + \alpha M$ and let $x := z - \alpha$ and $y := \beta - z$. Then

$$[\tilde{V}_{A,M}(z)][\tilde{V}_D(z)] = [\tilde{V}_M(z)][\tilde{U}_{A,M}(z)]. \quad (10.23)$$

Proof Because of Remark A.6, we have $\mathcal{R}(B) \subseteq \mathcal{R}(M)$ and $\mathcal{R}(N) \subseteq \mathcal{R}(M)$. Consequently, Remark A.20 shows $MM^\dagger A = A, MM^\dagger B = B$, and $MM^\dagger N = N$. In view of $Q_A = I_q - A^\dagger A$, then

$$MQ_A^\dagger N = MM^\dagger N - MA^\dagger AM^\dagger N = N - MA^\dagger AM^\dagger N \quad (10.24)$$

and, by virtue of

$$Q_A^\dagger B = M^\dagger B - A^\dagger AM^\dagger B = M^\dagger B - A^\dagger D. \quad (10.25)$$

furthermore

$$MQ_A^\dagger B = MM^\dagger B - MA^\dagger D = B - MA^\dagger D \quad (10.26)$$

follow. Taking into account $A + B = \delta M$ and $\delta > 0$, we can infer $\mathcal{R}(A) \subseteq \mathcal{R}(A + B)$. Since $A$ and $M$ are Hermitian, Remark A.24 shows that $B$ is Hermitian as well. Using
[44, Thm. 2.2(b)], we can thus conclude \((A \Leftrightarrow B)^* = A \Leftrightarrow B\). In view of (10.22), then \(D^* = D\) follows. Furthermore, we have \(\mathcal{R}(D) \subseteq \mathcal{R}(A) \subseteq \mathcal{R}(M)\). Using Remark A.14, we obtain then \(\mathcal{R}(D^\dagger) \subseteq \mathcal{R}(A) \subseteq \mathcal{R}(M^\dagger)\). From Remarks A.20 and A.14, we can thus conclude

\[
M^\dagger M D^\dagger = D^\dagger, \quad M^\dagger M A^\dagger = A^\dagger, \quad \text{and} \quad A^\dagger A D^\dagger = D^\dagger. \tag{10.27}
\]

In particular,

\[
A^\dagger A M^\dagger M D^\dagger = A^\dagger A D^\dagger = D^\dagger = M^\dagger M D^\dagger
\]

follows. In view of (10.2), we infer then

\[
A^\dagger D D^\dagger = A^\dagger A M^\dagger B D^\dagger = A^\dagger A M^\dagger (\beta M - N) D^\dagger = (\beta M^\dagger M - A^\dagger A M^\dagger N) D^\dagger. \tag{10.28}
\]

Taking into account \(P_D = DD^\dagger\) and Notations 10.6 and 9.6, the computation of the \(q \times q\) matrices in the block representation \([\tilde{V}_A, M(z)] [\tilde{V}_D(z)] = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}\) yields

\[
X_{11} = y^2 MA^\dagger P_D - yx(A + MQ_A M^\dagger B) D^\dagger
= y(y MA^\dagger D - x A - x MQ_A M^\dagger B) D^\dagger,
\]

\[
X_{12} = y MA^\dagger D + y(A + MQ_A M^\dagger B) = y(MA^\dagger D + A + MQ_A M^\dagger B),
\]

\[
X_{21} = -y^2 x A^\dagger P_D - yx[y(\delta Q_M + M^\dagger A) - x Q_A M^\dagger B] D^\dagger
= -yx(y A^\dagger D + y\delta Q_M + y M^\dagger A - x Q_A M^\dagger B) D^\dagger,
\]

and

\[
X_{22} = -yx A^\dagger D + y[y(\delta Q_M + M^\dagger A) - x Q_A M^\dagger B]
= -yx(A^\dagger D - y\delta Q_M - y M^\dagger A + x Q_A M^\dagger B).
\]

By virtue of (10.26), (10.28), and (10.24), we obtain

\[
X_{11} = y(y MA^\dagger D - x A - x B + x MA^\dagger D) D^\dagger = y(\delta MA^\dagger D - x \delta M) D^\dagger
= -y \delta(x M - MA^\dagger D) D^\dagger
= -y \delta[(z - \alpha)M - M(\beta M^\dagger M - A^\dagger A M^\dagger N)] D^\dagger
= -y \delta[(z - \alpha - \beta) M + M A^\dagger A M^\dagger N] D^\dagger
= -y \delta[(z - \alpha - \beta) M + N - MQ_A M^\dagger N] D^\dagger.
\]

Because of (10.26), we have

\[
X_{12} = y(MA^\dagger D + A + B - MA^\dagger D) = y\delta M = -y \delta(-M).
\]
Using (10.3), we get (10.19) by the same reasoning as in the proof of Lemma 10.10. The combination of (10.25), (10.19), (10.28), and (9.2) yields

\[ X_{21} = -yx[yA^\dagger D + (\beta - z)\delta Q_M + yM^\dagger A - xM^\dagger B + xA^\dagger D]D^\dagger \]
\[ = -yx[\delta A^\dagger D + \beta\delta Q_M + yM^\dagger A - xM^\dagger B - z\delta Q_M]D^\dagger \]
\[ = -yx[\delta A^\dagger D + \beta\delta Q_M + \delta(M^\dagger N - zI_q)]D^\dagger \]
\[ = -yx\delta[\beta M^\dagger M - A^\dagger AM^\dagger N + \beta(I_q - M^\dagger M) + M^\dagger N - zI_q]D^\dagger \]
\[ = -yx\delta(y I_q + QAM^\dagger N)D^\dagger . \]

Taking into account (10.25) and (10.19), we get furthermore

\[ X_{22} = -y[xA^\dagger D - (\beta - z)\delta Q_M - yM^\dagger A + xM^\dagger B - xA^\dagger D] \]
\[ = -y(z\delta Q_M - yM^\dagger A + xM^\dagger B - \beta\delta Q_M) = -y\delta(z I_q - M^\dagger N - \beta Q_M). \]

Hence, the first equation in (10.23) is verified.

Because of (10.4) and (10.27), we have

\[ MA^\dagger AM^\dagger(yB - xA)D^\dagger = \delta MA^\dagger AM^\dagger[(\beta + \alpha - z)M - N]D^\dagger \]
\[ = \delta[(\beta + \alpha - z)M - MA^\dagger AM^\dagger N]D^\dagger. \] (10.29)

Taking into account \( P_M = MM^\dagger \) and Notations 9.6 and 10.15, the computation of the \( q \times q \) matrices in the block representation \([\hat{V}_M(z)][\hat{U}_{A,M}(z)] = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \) yields

\[ Y_{11} = yP_MM(yPA^\dagger M^\dagger B + xQAM^\dagger A)D^\dagger - yxMMA^\dagger AD^\dagger \]
\[ = y(yMP_A^\dagger M^\dagger B + xMQA^\dagger A - xMM^\dagger A)D^\dagger, \]
\[ Y_{12} = yP_MB + yM(\delta Q_M + M^\dagger A) \]
\[ = yP_M(B + A) = yP_M(\delta M) = y\delta M = -y\delta(-M), \]
\[ Y_{21} = -yxM^\dagger M(yPA^\dagger M^\dagger B + xQAM^\dagger A)D^\dagger - y^2xM^\dagger AD^\dagger \]
\[ = -yxM^\dagger(yMP_A^\dagger M^\dagger B + xMQA^\dagger A + yA)D^\dagger, \]

and

\[ Y_{22} = -yxM^\dagger B + y^2(\delta Q_M + M^\dagger A) = -yxM^\dagger B - (\beta - z)\delta Q_M - yM^\dagger A. \]

In view of \( PA^\dagger = A^\dagger A \) and \( QA = I_q - A^\dagger A \), we can infer from (10.29) and (10.24) then

\[ Y_{11} = y(AM^\dagger AM^\dagger B + xMM^\dagger A - xMA^\dagger AM^\dagger A - xMM^\dagger A)D^\dagger \]
\[ = yMA^\dagger AM^\dagger(yB - xA)D^\dagger = -y\delta[(z - \alpha - \beta)M + MA^\dagger AM^\dagger N]D^\dagger \]
\[ = -y\delta[(z - \alpha - \beta)M + N - MQAM^\dagger N]D^\dagger. \]
Because of $MM^\dagger A = A$ and the identities (10.29) and (10.27), we have furthermore

\[
Y_{21} = -yxM^\dagger (yMA^\dagger AM^\dagger B + xMM^\dagger A - xMA^\dagger AM^\dagger A + yA)D^\dagger \\
= -yxM^\dagger [MA^\dagger AM^\dagger (yB - xA)D^\dagger + xAD^\dagger + yAD^\dagger] \\
= -yxM^\dagger (\delta[(\beta + \alpha - z)M - MA^\dagger AM^\dagger N]D^\dagger + \delta AD^\dagger) \\
= -yx\delta[(y + \alpha)M^\dagger MD^\dagger - M^\dagger MA^\dagger AM^\dagger ND^\dagger + M^\dagger AD^\dagger] \\
= -yx\delta(yD^\dagger + \alpha M^\dagger MD^\dagger - A^\dagger AM^\dagger ND^\dagger + M^\dagger AD^\dagger) \\
= -yx\delta[yI_q + M^\dagger (\alpha M + A) - A^\dagger AM^\dagger N]D^\dagger \\
= -yx\delta(yI_q + Q_AM^\dagger N)D^\dagger. 
\]

From (10.19) moreover $Y_{22} = -y\delta(zI_q - M^\dagger N - \beta Q_M)$ follows. By virtue of $Y_{12} = -y\delta(-M)$, thus the second equation in (10.23) is verified. \(\square\)

## 11 On the Elementary Steps of the Forward Algorithm

This section is aimed to work out the elementary step of the forward algorithm by applying the transformations studied in the previous section.

**Lemma 11.1** Let $(s_j)^0_{j=0} \in C_{\alpha,\beta}^{q,0}$ and let $F \in \mathcal{R}_q[[\alpha, \beta]; (s_j)^0_{j=0}]$. Then the $\mathcal{F}_{\alpha,\beta}(s_0)$-transformed pair of $F$ belongs to $\mathcal{P}[s_0]$.

**Proof** Denote by $[G_1; G_2]$ the $\mathcal{F}_{\alpha,\beta}(s_0)$-transformed pair of $F$. Obviously, $\mathcal{D} := \emptyset$ is a discrete subset of $C \setminus [\alpha, \beta]$. Observe that $F$ is holomorphic. In view of Definition 9.1, then $G_1$ and $G_2$ are holomorphic as well. In particular, $G_1$ and $G_2$ are $C^{q \times q}$-valued functions, which are meromorphic in $C \setminus [\alpha, \beta]$ with $\mathcal{P}(G_1) \cup \mathcal{P}(G_2) \subseteq \mathcal{D}$. Consequently, condition (I) in Notation 7.5 is fulfilled with the set $\mathcal{D}$ for the pair $[P; Q] = [G_1; G_2]$. Consider an arbitrary $z \in C \setminus [\alpha, \beta]$. By assumption, the $\mathcal{R}[\alpha, \beta]$-measure $\tilde{\sigma}_F$ of $F$ belongs to $\mathcal{M}_q^{\infty}[[\alpha, \beta]; (s_j)^0_{j=0}]$, i.e., $\tilde{\sigma}_F([\alpha, \beta]) = s_0$. Taking additionally into account Proposition 4.15, hence $\mathcal{R}(F(z)) = \mathcal{R}(s_0)$ follows. Thus, $\mathbb{P}_{\mathcal{R}(s_0)} F(z) = F(z)$. Regarding Notation 9.2, we can infer then

\[
\begin{bmatrix}
G_1(z) \\
G_2(z)
\end{bmatrix} = -[\tilde{W}_{s_0}(z)] 
\begin{bmatrix}
F(z) \\
I_q
\end{bmatrix}.
\]

Using Lemma 10.9, we obtain

\[
[\tilde{V}_{s_0}(z)] 
\begin{bmatrix}
G_1(z) \\
G_2(z)
\end{bmatrix} = -[\tilde{V}_{s_0}(z)][\tilde{W}_{s_0}(z)] 
\begin{bmatrix}
F(z) \\
I_q
\end{bmatrix} \\
= (\beta - z)\delta 
\begin{bmatrix}
\mathbb{P}_{\mathcal{R}(s_0)} F(z) \\
I_q
\end{bmatrix} = (\beta - z)\delta 
\begin{bmatrix}
F(z) \\
I_q
\end{bmatrix}.
\]

In view of $z \neq \beta$ and $\delta > 0$, we can conclude $q \geq \text{rank} \begin{bmatrix} G_1(z) \\ G_2(z) \end{bmatrix} \geq \text{rank} \begin{bmatrix} F(z) \\ I_q \end{bmatrix} = q$, implying

\[
\begin{bmatrix}
G_1(z) \\
G_2(z)
\end{bmatrix} = q. \text{ Let } x := z - \alpha \text{ and let } y := \beta - z. \text{ Furthermore, let}
\]
$W_0 := \tilde{W}_{s_0}(z)$, let $W_1 := [(xI_q) \oplus I_q]W_0$, and let $W_2 := [(yI_q) \oplus I_q]W_0$. As already mentioned above, we have $-W_0\begin{bmatrix} F(z) \\ I_q \end{bmatrix} = \begin{bmatrix} G_1(z) \\ G_2(z) \end{bmatrix}$. Consequently,

$$-W_1\begin{bmatrix} F(z) \\ I_q \end{bmatrix} = \begin{bmatrix} xG_1(z) \\ G_2(z) \end{bmatrix} \quad \text{and} \quad -W_2\begin{bmatrix} F(z) \\ I_q \end{bmatrix} = \begin{bmatrix} yG_1(z) \\ G_2(z) \end{bmatrix}.$$

Taking additionally into account $P_{\mathcal{R}(s_0)}F(z) = F(z)$, the application of Proposition 9.4 yields

$$\begin{bmatrix} xG_1(z) \\ G_2(z) \end{bmatrix}^* \tilde{j}_q \begin{bmatrix} xG_1(z) \\ G_2(z) \end{bmatrix} = \delta\begin{bmatrix} yxF(z) \\ I_q \end{bmatrix}^* \tilde{j}_q \begin{bmatrix} yxF(z) \\ I_q \end{bmatrix} - 2 \text{Im}(z)s_0$$

and

$$\begin{bmatrix} yG_1(z) \\ G_2(z) \end{bmatrix}^* \tilde{j}_q \begin{bmatrix} yG_1(z) \\ G_2(z) \end{bmatrix} = 2|y|^2\begin{bmatrix} F(z) \\ I_q \end{bmatrix}^* \tilde{j}_q \begin{bmatrix} F(z) \\ I_q \end{bmatrix} - 2 \text{Im}(z)[F(z)]^*s_0^\dagger[F(z)].$$

Because of Remark A.36, we have

$$\begin{bmatrix} xG_1(z) \\ G_2(z) \end{bmatrix}^* \tilde{j}_q \begin{bmatrix} xG_1(z) \\ G_2(z) \end{bmatrix} = 2 \text{Im}([G_2(z)]^*[xG_1(z)])$$

$$= 2 \text{Im}((z - \alpha) [G_2(z)]^*[G_1(z)]),$$

$$\begin{bmatrix} yG_1(z) \\ G_2(z) \end{bmatrix}^* \tilde{j}_q \begin{bmatrix} yG_1(z) \\ G_2(z) \end{bmatrix} = 2 \text{Im}([G_2(z)]^*[yG_1(z)])$$

$$= 2 \text{Im}((\beta - z) [G_2(z)]^*[G_1(z)]),$$

and, furthermore,

$$\begin{bmatrix} yxF(z) \\ I_q \end{bmatrix}^* \tilde{j}_q \begin{bmatrix} yxF(z) \\ I_q \end{bmatrix} = 2 \text{Im}[yxF(z)] \quad \text{and} \quad \begin{bmatrix} F(z) \\ I_q \end{bmatrix}^* \tilde{j}_q \begin{bmatrix} F(z) \\ I_q \end{bmatrix} = 2 \text{Im}[F(z)].$$

Now assume in addition $z \notin \mathbb{R}$. Taken all together, we get then

$$\frac{1}{\text{Im} z} \text{Im}((z - \alpha) [G_2(z)]^*[G_1(z)]) = \delta\left(\frac{1}{\text{Im} z} \text{Im}[yxF(z)] - s_0\right) \quad (11.1)$$

and

$$\frac{1}{\text{Im} z} \text{Im}((\beta - z) [G_2(z)]^*[G_1(z)]) = \delta|y|^2\left(\frac{1}{\text{Im} z} \text{Im}[F(z)] - [F(z)]^*s_0^\dagger[F(z)]\right). \quad (11.2)$$
Remark 4.22 provides us $\frac{1}{\text{Im } z} \text{ Im}[yx F(z)] \succ s_0$. Taking additionally into account $\delta > 0$ and Remark A.24, we conclude from (11.1) then $\frac{1}{\text{Im } z} \text{ Im}((z - \alpha)[G_2(z)]^*[G_1(z)]) \in \mathbb{C}_\succ^{q \times q}$. Moreover, in view of Theorem 4.9, we apply Lemma C.7 to $F$ and obtain $[F(\bar{z})]s_0^\dagger [F(\bar{z})]^* \preceq (\text{Im } \bar{z})^{-1} \text{ Im}[F(\bar{z})]$. Because of Remark 4.8, we have $(\text{Im } \bar{z})^{-1} \text{ Im}[F(\bar{z})] - [F(\bar{z})]s_0^\dagger [F(\bar{z})]^* = \frac{1}{\text{Im } z} \text{ Im}[F(z)] - [F(z)]^*s_0^\dagger [F(z)]$. Taking additionally into account $\delta > 0$ and Remark A.24, we infer from (11.2) then similarly $\frac{1}{\text{Im } z} \text{ Im}((\beta - z)(G_2(z)]^*[G_1(z)]) \in \mathbb{C}_\succ^{q \times q}$. In view of the choice of $z$ in $\mathbb{C} \setminus [\alpha, \beta] = \mathbb{C} \setminus ((\alpha, \beta) \cup D)$, we have thus shown that the conditions (II)–(IV) in Notation 7.5 are fulfilled with the set $D$ for the pair $[P; Q] = [G_1; G_2]$. Consequently, $[G_1; G_2] \in \mathcal{P}_R(a \setminus [\alpha, \beta])$. Furthermore, $\mathbb{P}_{R[s_0]} F = F$ verified. According to Definition 9.1, then $\mathbb{P}_{R[s_0]} G_1 = G_1$. By virtue of Lemma 8.3, hence $[G_1; G_2] \in \mathbb{P}[s_0]$ follows.

\begin{lemma}
Assume $\kappa \geq 1$. Let $(s_j)_{j=0}^{\kappa-1} \in \mathcal{F}_{q,\kappa,\alpha,\beta}$ with $\mathcal{F}_{\alpha,\beta}$-transform $(t_j)_{j=0}^{\kappa-1}$ and let $F \in \mathcal{R}_q([\alpha, \beta]; (s_j)_{j=0}^{\kappa-1})$. Further, let $a_0$ be given via Notation 3.2. Then the $\mathcal{F}_{\alpha,\beta}(a_0, s_0)$-transform of $F$ belongs to $\mathcal{R}_q([\alpha, \beta]; (t_j)_{j=0}^{\kappa-1})$.
\end{lemma}

\begin{proof}
We first consider the case $\kappa = \infty$. Let $\rho := \max|\alpha|, |\beta|$ and let $\mathcal{C}_\rho := \{z \in \mathbb{C} : |z| > \rho \}$. Obviously, $\mathcal{C}_\rho \subseteq \mathbb{C} \setminus [\alpha, \beta]$ and $0 \notin \mathcal{C}_\rho$. According to Proposition 3.9, the sequences $(a_j)_{j=0}^{\infty}$ and $(b_j)_{j=0}^{\infty}$ introduced in Notation 3.2 both belong to $\mathcal{F}_{\alpha,\beta}^{\infty}$. Because of Remark 5.11, the matrix-valued functions $F_a$ and $F_b$ given in Notation 4.19 fulfill $F_a \in \mathcal{R}_q([\alpha, \beta]; (a_j)_{j=0}^{\infty})$ and $F_b \in \mathcal{R}_q([\alpha, \beta]; (b_j)_{j=0}^{\infty})$. Consequently, $F_b$ and $F_a$ are holomorphic in $\mathbb{C} \setminus [\alpha, \beta]$ and Proposition 5.9 yields, for all $z \in \mathcal{C}_\rho$, the series expansions

$$F_b(z) = - \sum_{j=0}^{\infty} z^{-(j+1)} b_j \quad \text{and} \quad F_a(z) = - \sum_{j=0}^{\infty} z^{-(j+1)} a_j.$$

In accordance with Definition 3.49, denote by $(g_j)_{j=0}^{\infty}$ the $(-\infty, \beta]$-modification of $(a_j)_{j=0}^{\infty}$. Then $g_0 = -a_0$ and, in view of Notation 3.2, furthermore $g_j = b a_{j-1} - a_j$ for all $j \in \mathbb{N}$. The matrix-valued functions $Y, Z : \mathbb{C} \setminus [\alpha, \beta] \to \mathbb{C}_\succ^{q \times q}$ defined by $Y(z) := -z F_b(z)$ and $Z(z) := -(\beta - z) F_a(z)$, resp., are both holomorphic in $\mathbb{C} \setminus [\alpha, \beta]$ with series expansions

$$Y(z) = \sum_{n=0}^{\infty} z^{-n} b_n \quad \text{and} \quad Z(z) = \sum_{n=0}^{\infty} z^{-n} g_n$$

for all $z \in \mathcal{C}_\rho$. Let $R : \mathbb{C} \setminus [\alpha, \beta] \to \mathbb{C}_\succ^{q \times q}$ be defined by $R(z) := [Z(z)]^\dagger$. Observe that the $\mathcal{R}_{\alpha, \beta}$-measure $\bar{\alpha}_a$ of $F_a$ fulfills $\bar{\alpha}_a([\alpha, \beta]) = a_0$. Using Proposition 4.15, we obtain, for all $z \in \mathbb{C} \setminus [\alpha, \beta]$, thus $\mathcal{R}(Z(z)) = \mathcal{R}(F_a(z)) = \mathcal{R}(a_0) = \mathcal{R}(g_0)$ and, analogously, $\mathcal{N}(Z(z)) = \mathcal{N}(g_0)$. Therefore, we see from Proposition F.3 that the matrix-valued function $R$ is holomorphic in $\mathbb{C} \setminus [\alpha, \beta]$. Denote by $(r_j)_{j=0}^{\infty}$ the reciprocal sequence associated to $(g_j)_{j=0}^{\infty}$. The application of Lemma F.7 yields the
series expansion
\[ R(z) = \sum_{n=0}^{\infty} z^{-n} r_n \]
for all \( z \in \mathbb{C}_\rho \). Denote by \((x_j)_{j=0}^{\infty}\) the Cauchy product of \((b_j)_{j=0}^{\infty}\) and \((r_j)_{j=0}^{\infty}\). From Lemma F.6 we see then that the function \( X := Y R \) is holomorphic in \( \mathbb{C} \setminus [\alpha, \beta] \) with series expansion
\[ X(z) = \sum_{n=0}^{\infty} z^{-n} x_n \]
for all \( z \in \mathbb{C}_\rho \). Observe that the \( \mathcal{R}[\alpha, \beta] \)-measure \( \check{\sigma}_F \) of \( F \) fulfills \( \check{\sigma}_F([\alpha, \beta]) = s_0 \). Denote by \( G \) the \( \mathcal{F}_{\alpha, \beta}(a_0, s_0) \)-transform of \( F \). Regarding Notation 4.19 and Definition 10.1, we obtain
\[
G(z) = a_0 s_0^\dagger [F_b(z)][(\beta - z) F_a(z)]^\dagger a_0 = a_0 s_0^\dagger [F_b(z)][-Z(z)]^\dagger a_0 = -a_0 s_0^\dagger [F_b(z)][R(z)] a_0
\]
for all \( z \in \mathbb{C} \setminus [\alpha, \beta] \). In particular, by virtue of Remarks F.2 and F.1, thus \( G \) is holomorphic in \( \mathbb{C} \setminus [\alpha, \beta] \). Let \( H : \mathbb{C} \setminus [\alpha, \beta] \to \mathbb{C}^{q \times q} \) be defined by \( H(z) := zG(z) \). We have
\[
H(z) = -za_0 s_0^\dagger [F_b(z)][R(z)] a_0 = a_0 s_0^\dagger [F_b(z)][X(z)] a_0 = a_0 s_0^\dagger [Y(z)][R(z)] a_0
\]
for all \( z \in \mathbb{C} \setminus [\alpha, \beta] \). From Remark F.5 we see then that the matrix-valued function \( H \) is holomorphic in \( \mathbb{C} \setminus [\alpha, \beta] \) with series expansion
\[
H(z) = \sum_{n=0}^{\infty} z^{-n} (a_0 s_0^\dagger x_n a_0)
\]
for all \( z \in \mathbb{C}_\rho \). In view of Definition 3.52, consequently
\[
G(z) = \frac{1}{z} H(z) = -\sum_{j=0}^{\infty} z^{-(j+1)} t_j
\]
for all \( z \in \mathbb{C}_\rho \) follows. Due to Proposition 3.59, the sequence \((t_j)_{j=0}^{\infty}\) belongs to \( \mathcal{F}_{q, \infty, \alpha, \beta} \). Since \( G \) is holomorphic in \( \mathbb{C} \setminus [\alpha, \beta] \), the application of Proposition 5.10 thus yields \( G \in \mathcal{R}_q([\alpha, \beta]; (t_j)_{j=0}^{\infty}) \), completing the proof in the case \( \kappa = \infty \).

Now we consider the case \( \kappa < \infty \). Then \( m := \kappa \) belongs to \( \mathbb{N} \). Regarding Remark 4.11, denote by \( \hat{s}_j := \int_{[\alpha, \beta]} \xi^j \check{\sigma}_F(d\xi) \) for all \( j \in \mathbb{N}_0 \) the power moments of the \( \mathcal{R}[\alpha, \beta] \)-measure \( \check{\sigma}_F \) of \( F \). Then \( \check{\sigma}_F \in \mathcal{M}_{q, \infty}([\alpha, \beta]; (\hat{s}_j)_{j=0}^{\infty}, =) \),
i.e., $F \in \mathcal{R}_q([\alpha, \beta]; (\hat{s}_j)_{j=0}^\infty)$. In particular, by virtue of Proposition 5.7, we therefore have $(\hat{s}_j)_{j=0}^\infty \in \mathcal{F}_{q, \alpha, \beta}^\infty$. Denote by $(\hat{t}_j)_{j=0}^\infty$ the $\mathcal{F}_{\alpha, \beta}$-transform of $(\hat{s}_j)_{j=0}^\infty$. Let $\hat{a}_0 := -\alpha s_0 + s_1$ and denote by $G$ the $\mathcal{F}_{\alpha, \beta}((\hat{a}_0, \hat{s}_0))$-transform of $F$. Since the assertion is already proved for $k = \infty$, we see that $G$ belongs to $\mathcal{R}_q([\alpha, \beta]; (\hat{t}_j)_{j=0}^\infty)$. Observe that by assumption $m \geq 1$ and $s_j = \hat{s}_j$ for all $j \in \mathbb{Z}_0, m$ hold true. Hence, we have $\hat{a}_0 = a_0$ and, because of Remark 3.53, furthermore $\hat{t}_j = t_j$ for all $j \in \mathbb{Z}_0, m-1$. Consequently, $G$ is exactly the $\mathcal{F}_{\alpha, \beta}((a_0, s_0))$-transform of $F$ and belongs to $\mathcal{R}_q([\alpha, \beta]; (t_j)_{j=0}^{m-1})$. □

12 On the Elementary Steps of the Backward Algorithm

This section can be considered as the analogue of the preceding one for the backward algorithm. More precisely, we will work out the elementary step of the backward algorithm by applying the transformation studied in Sect. 10.

Lemma 12.1 Let $(s_j)_{j=0}^0 \in \mathcal{F}_{q, 0, \alpha, \beta}$ and let $[G_1; G_2] \in \hat{\mathcal{P}}[s_0]$. Then the inverse $\mathcal{F}_{\alpha, \beta}(s_0)$-transform of $[G_1; G_2]$ belongs to $\mathcal{R}_q([\alpha, \beta]; (s_j)_{j=0}^0)$.

Proof According to Notation 8.1, we have $[G_1; G_2] \in \mathcal{P}\mathcal{R}_q(\mathbb{C}\setminus[\alpha, \beta])$. In particular, $G_1$ and $G_2$ are $\mathbb{C}^{q \times q}$-valued functions, which are meromorphic in $\mathbb{C}\setminus[\alpha, \beta]$. Using the functions $g, h : \mathbb{C}\setminus[\alpha, \beta] \to \mathbb{C}$ given via (9.9), we define by (9.10) two $\mathbb{C}^{q \times q}$-valued functions $F_1$ and $F_2$ meromorphic in $\mathbb{C}\setminus[\alpha, \beta]$. From Remark 3.12 we get $s_0 \in \mathbb{C}^{q \times q}$. By virtue of Proposition 9.11, thus $F_2$ does not vanish identically in $\mathbb{C}\setminus[\alpha, \beta]$ and the inverse $\mathcal{F}_{\alpha, \beta}(s_0)$-transform $F$ of $[G_1; G_2]$ admits the representation $F = F_1F_2^{-1}$, according to Definition 9.5.

In a first step, we are now going to show that the pair $[F_1; F_2]$ belongs to $\mathcal{P}\mathcal{R}_q(\mathbb{C}\setminus[\alpha, \beta])$. Due to Proposition 7.9, the set $\mathcal{A} := \mathcal{P}(G_1) \cup \mathcal{P}(G_2) \cup \hat{\mathcal{E}}(G_1; G_2)$ is a discrete subset of $\mathbb{C}\setminus[\alpha, \beta]$, satisfying the conditions (I)–(IV) in Notation 7.5 for the pair $[P; Q] = [G_1; G_2]$. In view of (9.10), we have $\mathcal{P}(F_1) \subseteq \mathcal{P}(G_1) \cup \hat{\mathcal{E}}(G_1; G_2)$ and $\mathcal{P}(F_2) \subseteq \mathcal{P}(G_1) \cup \mathcal{P}(G_2)$. Consequently, $\mathcal{P}(F_1) \cup \mathcal{P}(F_2) \subseteq \mathcal{A}$, i.e., condition (I) in Notation 7.5 is fulfilled with the set $\mathcal{D} = \mathcal{A}$ for the pair $[P; Q] = [F_1; F_2]$. Consider an arbitrary $z \in \mathbb{C}\setminus([\alpha, \beta] \cup \mathcal{A})$. Regarding $s_0 \in \mathbb{C}^{q \times q}$, Proposition 9.11 yields \[ \det F_2(z) \neq 0 \] and

\[ F(z) = [F_1(z)][F_2(z)]^{-1}. \]  

(12.1)

In particular, rank \( \begin{bmatrix} F_1(z) \\ F_2(z) \end{bmatrix} = q \). Let $x := z - \alpha$ and let $y := \beta - z$. Furthermore, let $V_0 := \tilde{V}_s(z)$, let $V_1 := [(x I_q) \oplus I_q]V_0$, and let $V_2 := [(y I_q) \oplus I_q]V_0$. According to Notation 9.6, we have $V_0 \begin{bmatrix} G_1(z) \\ G_2(z) \end{bmatrix} = \begin{bmatrix} F_1(z) \\ F_2(z) \end{bmatrix}$. Hence,

\[ V_1 \begin{bmatrix} G_1(z) \\ G_2(z) \end{bmatrix} = \begin{bmatrix} xF_1(z) \\ F_2(z) \end{bmatrix} \quad \text{and} \quad V_2 \begin{bmatrix} G_1(z) \\ G_2(z) \end{bmatrix} = \begin{bmatrix} yF_1(z) \\ F_2(z) \end{bmatrix}. \]
From Lemma 8.3 we see $\mathbb{P}_{R(0)} G_1 = G_1$. Using Proposition 9.9, we thus can infer
\[
\begin{bmatrix}
   xF_1(z) \\
   F_2(z)
\end{bmatrix}^* \tilde{j}_q \begin{bmatrix}
   xF_1(z) \\
   F_2(z)
\end{bmatrix} = |y|^2 \begin{bmatrix}
   xG_1(z) \\
   G_2(z)
\end{bmatrix}^* \tilde{j}_q \begin{bmatrix}
   xG_1(z) \\
   G_2(z)
\end{bmatrix} + |x|^2 \begin{bmatrix}
   yG_1(z) \\
   G_2(z)
\end{bmatrix}^* \tilde{j}_q \begin{bmatrix}
   yG_1(z) \\
   G_2(z)
\end{bmatrix}
+ 2\delta \text{Im}(z)[G_2(z)]^* s_0[G_2(z)]
\]
and
\[
\begin{bmatrix}
   yF_1(z) \\
   F_2(z)
\end{bmatrix}^* \tilde{j}_q \begin{bmatrix}
   yF_1(z) \\
   F_2(z)
\end{bmatrix} = \delta |y|^2 \begin{bmatrix}
   G_1(z) \\
   G_2(z)
\end{bmatrix}^* \tilde{j}_q \begin{bmatrix}
   G_1(z) \\
   G_2(z)
\end{bmatrix} + 2 \text{Im}(z)[G_1(z)]^* s_0[G_1(z)]
\]
Because of Remark A.36, we see that
\[
\begin{bmatrix}
   \xi F_1(z) \\
   F_2(z)
\end{bmatrix}^* \tilde{j}_q \begin{bmatrix}
   \xi F_1(z) \\
   F_2(z)
\end{bmatrix} = 2 \text{Im}([F_2(z)]^*[\xi F_1(z)]) = 2 \text{Im}(\xi [F_2(z)]^*[F_1(z)])
\]
and
\[
\begin{bmatrix}
   \xi G_1(z) \\
   G_2(z)
\end{bmatrix}^* \tilde{j}_q \begin{bmatrix}
   \xi G_1(z) \\
   G_2(z)
\end{bmatrix} = 2 \text{Im}([G_2(z)]^*[\xi G_1(z)]) = 2 \text{Im}(\xi [G_2(z)]^*[G_1(z)])
\]
hold true for each $\xi \in \{x, y\}$ and that
\[
\begin{bmatrix}
   G_1(z) \\
   G_2(z)
\end{bmatrix}^* \tilde{j}_q \begin{bmatrix}
   G_1(z) \\
   G_2(z)
\end{bmatrix} = 2 \text{Im}([G_2(z)]^*[G_1(z)])
\]
is valid. Now assume in addition $z \notin \mathbb{R}$. Taken all together, we obtain then
\[
\frac{1}{\text{Im} z} \text{Im}(x[F_2(z)]^*[F_1(z)])
= |y|^2 \left[ \frac{1}{\text{Im} z} \text{Im}(x[G_2(z)]^*[G_1(z)]) \right] + |x|^2 \left[ \frac{1}{\text{Im} z} \text{Im}(y[G_2(z)]^*[G_1(z)]) \right]
+ \delta [G_2(z)]^* s_0[2G_2(z)]
\text{ (12.2)}
\]
and
\[
\frac{1}{\text{Im} z} \text{Im}(y[F_2(z)]^*[F_1(z)])
= \delta |y|^2 \left[ \frac{1}{\text{Im} z} \text{Im}([G_2(z)]^*[G_1(z)]) \right] + [G_1(z)]^* s_0^\dagger[G_1(z)].
\text{ (12.3)}
\]
Since the conditions (III) and (IV) in Notation 7.5 are satisfied with the set $\mathcal{D} = \mathcal{A}$ for the pair $[P; Q] = [G_1; G_2]$, we have

$$\frac{1}{\text{Im} z} \text{Im}(x[G_2(z)]^*[G_1(z)]) \in \mathbb{C}^{q \times q} \quad \text{and} \quad \frac{1}{\text{Im} z} \text{Im}(y[G_2(z)]^*[G_1(z)]) \in \mathbb{C}^{q \times q}. \quad (12.4)$$

Taking into account (12.4), $\delta > 0$, and $s_0 \in \mathbb{C}^{q \times q}$, we use Remarks A.24 and A.25 to infer from (12.2) that $\frac{1}{\text{Im} z} \text{Im}(x[F_2(z)]^*[F_1(z)]) \in \mathbb{C}^{q \times q}$. By virtue of Lemma 7.8, the matrix $\frac{1}{\text{Im} z} \text{Im}([G_2(z)]^*[G_1(z)])$ is Non-negative Hermitian. Because of Remark A.16, we have $s_0^\dagger \in \mathbb{C}^{q \times q}$. Regarding additionally $\delta > 0$, from Remarks A.24 and A.25 (12.3) we conclude similarly $\frac{1}{\text{Im} z} \text{Im}(y[F_2(z)]^*[F_1(z)]) \in \mathbb{C}^{q \times q}$. In view of the choice of $z \in \mathbb{C} \setminus ([\alpha, \beta] \cup \mathcal{A})$, we thus have verified that conditions (II)–(IV) in Notation 7.5 are fulfilled with the set $\mathcal{D} = \mathcal{A}$ for the pair $[P; Q] = [F_1; F_2]$. Consequently, $[F_1; F_2] \in \mathcal{P}\mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$. Therefore, the application of Lemma 7.13 to the pair $[P; Q] = [F_1; F_2]$ yields

$$F = F_1F_2^{-1} \in \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta]). \quad (12.5)$$

In a second step, we are now going to show that $\tilde{\sigma}_F([\alpha, \beta]) = s_0$. First we verify

$$\tilde{\sigma}_F([\alpha, \beta]) \succ s_0. \quad (12.6)$$

Let $W_0 := \tilde{W}_{s_0}(z)$. Because of Lemma 9.10 and $\mathbb{P}_{\mathcal{R}(s_0)} G_1 = G_1$, we have

$$W_0 \begin{bmatrix} F_1(z) \\ F_2(z) \end{bmatrix} = W_0 V_0 \begin{bmatrix} G_1(z) \\ G_2(z) \end{bmatrix} = -y\delta \begin{bmatrix} \mathbb{P}_{\mathcal{R}(s_0)} G_1(z) \\ G_2(z) \end{bmatrix} = -y\delta \begin{bmatrix} G_1(z) \\ G_2(z) \end{bmatrix}. \quad (12.7)$$

Setting $W_1 := [(yI_q) \oplus I_q]W_0$, hence $W_1 \begin{bmatrix} F_1(z) \\ F_2(z) \end{bmatrix} = -y\delta \begin{bmatrix} xG_1(z) \\ G_2(z) \end{bmatrix}$ follows. In view of (9.10), we have $\mathbb{P}_{\mathcal{R}(s_0)} F_1 = F_1$. Thus, using Proposition 9.4, we can conclude then

$$|y|^2\delta^2 \begin{bmatrix} xG_1(z) \\ G_2(z) \end{bmatrix}^* \tilde{J}_q \begin{bmatrix} xG_1(z) \\ G_2(z) \end{bmatrix} = \delta \begin{bmatrix} yxF_1(z) \\ F_2(z) \end{bmatrix}^* \tilde{J}_q \begin{bmatrix} yxF_1(z) \\ F_2(z) \end{bmatrix} - 2 \text{Im}(z)[F_2(z)]^*s_0[F_2(z)].$$
Remark A.36 yields \[
\begin{bmatrix}
xG_1(z) \\
G_2(z)
\end{bmatrix}^* \tilde{J}_q \begin{bmatrix}
xG_1(z) \\
G_2(z)
\end{bmatrix} = 2 \text{Im}(x[G_2(z)]^*[G_1(z)]). 
\]
Consequently,

\[
|y|^2 \delta \left( \frac{1}{\text{Im} z} \text{Im}(x[G_2(z)]^*[G_1(z)]) \right) = \frac{1}{2 \text{Im} z} \begin{bmatrix}
xyF_1(z) \\
F_2(z)
\end{bmatrix}^* \tilde{J}_q \begin{bmatrix}
xyF_1(z) \\
F_2(z)
\end{bmatrix} - [F_2(z)]^* s_0 [F_2(z)].
\]

Taking into account \( \delta > 0 \) and (12.4), we see from Remark A.24 that the matrix on the left-hand side of the last equation is Non-negative Hermitian. Since \( \tilde{J}_q \) and \( s_0 \) are Hermitian matrices, we can then use Remarks A.24 and A.25 to conclude

\[
[F_2(z)]^* s_0 [F_2(z)] \preceq \frac{1}{2 \text{Im} z} \begin{bmatrix}
xyF_1(z) \\
F_2(z)
\end{bmatrix}^* \tilde{J}_q \begin{bmatrix}
xyF_1(z) \\
F_2(z)
\end{bmatrix}.
\]

In view of (12.1) and Remarks A.25 and A.36, hence

\[
s_0 \preceq \frac{1}{2 \text{Im} z} \begin{bmatrix}
xyF(z) \\
I_q
\end{bmatrix}^* \tilde{J}_q \begin{bmatrix}
xyF(z) \\
I_q
\end{bmatrix} = \frac{1}{\text{Im} z} \text{Im}[xyF(z)]
\]

\[
= \frac{1}{\text{Im} z} \text{Im}[(\beta - z)(z - \alpha)F(z)]
\]

follows. Since \( A \) is a discrete set, we in particular infer

\[
\lim_{\eta \to \infty} \frac{1}{\eta} \text{Im}[(\beta - i\eta)(i\eta - \alpha)F(i\eta)] \succ s_0. 
\]  

(12.8)

For all \( \eta > 0 \), we have \( \eta^{-2}(\beta - i\eta)(i\eta - \alpha) = (\beta \eta^{-1} - i)(i - \alpha \eta^{-1}) \) and therefore

\[
\lim_{\eta \to \infty} \eta^{-2}(\beta - i\eta)(i\eta - \alpha) = 1. 
\]

Regarding (12.5) and Theorem 4.9, we can apply Lemma C.4 to \( F \) and obtain \( \lim_{\eta \to \infty} i\eta F(i\eta) = -\tilde{\sigma}_F([\alpha, \beta]) \). Consequently,

\[
-\tilde{\sigma}_F([\alpha, \beta]) = \left[ \lim_{\eta \to \infty} \frac{1}{\eta^2} (\beta - i\eta)(i\eta - \alpha) \right] \lim_{\eta \to \infty} i\eta F(i\eta)
\]

\[
= \lim_{\eta \to \infty} \frac{i}{\eta} [(\beta - i\eta)(i\eta - \alpha)F(i\eta)].
\]

Because of \( \tilde{\sigma}_F([\alpha, \beta]) \in \mathbb{C}_H^{q \times q} \), we obtain from Remark A.2 then

\[
\tilde{\sigma}_F([\alpha, \beta]) = -\text{Re} \left( \lim_{\eta \to \infty} \frac{i}{\eta} [(\beta - i\eta)(i\eta - \alpha)F(i\eta)] \right)
\]

\[
= \lim_{\eta \to \infty} \frac{1}{\eta} \text{Im}[(\beta - i\eta)(i\eta - \alpha)F(i\eta)].
\]

In combination with (12.8), this implies (12.6).
Conversely, we now verify $\sigma_F([\alpha, \beta]) \ll s_0$. Because of (12.5) and Proposition 4.15, we have $R(F(z)) = R(\tilde{\sigma}_F([\alpha, \beta]))$. Using Remark A.8 and (12.1), we infer $R(F(z)) = R(F_1(z))$. In view of $P_{R(s_0)}F_1 = F_1$, we have $R(F_1(z)) \subseteq R(s_0)$. Consequently, $R(\tilde{\sigma}_F([\alpha, \beta])) \subseteq R(s_0)$. Let $W_2 := \{(y I_q) \oplus I_q\} W_0$. In view of (12.7), then $W_2 [F_1(z)] = -y \delta \left[ yG_1(z) G_2(z) \right]$. Taking additionally into account $P_{R(s_0)}F_1 = F_1$, from Proposition 9.4 we get

$$|y|^2 \delta^2 \left[ yG_1(z) G_2(z) \right] \tilde{J}_q \left[ yG_1(z) G_2(z) \right] = \delta |y|^2 \left( F_1(z) F_2(z) \right) \tilde{J}_q \left[ F_1(z) F_2(z) \right] - 2 \text{Im}(z)[F_1(z)]^* s_0^t [F_1(z)].$$

**Remark A.36** yields

$$\left[ yG_1(z) G_2(z) \right] \tilde{J}_q \left[ yG_1(z) G_2(z) \right] = 2 \text{Im}(y)[G_2(z)]^*[G_1(z)].$$

Consequently,

$$\frac{1}{\text{Im} z} \text{Im}(y)[G_2(z)]^*[G_1(z)] = \frac{1}{2 \text{Im} z} \left[ F_1(z) F_2(z) \right] \tilde{J}_q \left[ F_1(z) F_2(z) \right] - [F_1(z)]^* s_0^t [F_1(z)].$$

Regarding $\delta > 0$ and (12.4), we see from Remark A.24 that the matrix on the left-hand side of the last equation is Non-negative Hermitian. Since $s_0$ is Hermitian, **Remark A.14** shows that $s_0^t$ is Hermitian as well. Taking additionally into account that $\tilde{J}_q$ is Hermitian, we can use Remarks A.24 and A.25 to conclude

$$[F_1(z)]^* s_0^t [F_1(z)] \ll \frac{1}{2 \text{Im} z} \left[ F_1(z) F_2(z) \right] \tilde{J}_q \left[ F_1(z) F_2(z) \right].$$

Because of (12.1), the application of Remarks A.25 and A.36 thus yields

$$[F(z)]^* s_0^t [F(z)] \ll \frac{1}{2 \text{Im} z} \left[ F(z) I_q \right] \tilde{J}_q \left[ F(z) I_q \right] = \frac{1}{\text{Im} z} \text{Im} F(z).$$

Regarding that the set $A$ is discrete and that, according to (12.5), the function $F$ is holomorphic, we can apply a continuity argument to show that $[F(\zeta)]^* s_0^t [F(\zeta)] \ll (\text{Im} \zeta)^{-1} \text{Im} F(\zeta)$ holds true for all $\zeta \in \Pi_+$. In view of (12.5) and Theorem 4.9, we can apply Lemma C.8 to $F$. Taking additionally into account account $s_0 \in C_+^{q \times q}$ and $R(\tilde{\sigma}_F([\alpha, \beta])) \subseteq R(s_0)$, then $\tilde{\sigma}_F([\alpha, \beta]) \ll s_0$ follows by Lemma C.8. In combination with (12.6), this implies $\tilde{\sigma}_F([\alpha, \beta]) = s_0$. Because of (12.5), thus $F$ belongs to $R(q[[\alpha, \beta]]; s_j^0)_{j=0}^q$.  

**Lemma 12.2** Assume $\kappa \geq 1$. Let $(s_j)^{\kappa}_{j=0} \in F_{q, \kappa, \alpha, \beta}$ with $F_{\alpha, \beta}$-transform $(t_j)^{\kappa-1}_{j=0}$ and let $G \in R_q[[\alpha, \beta]; (t_j)^{\kappa-1}_{j=0}]$. Then the inverse $F_{\alpha, \beta}(a_0, s_0)$-transform of $G$ belongs to $R_q[[\alpha, \beta]; (s_j)^{\kappa}_{j=0}]$.  

\[\Box\]
**Proof** Because of Proposition 3.59, we have \((t_j)_{j=0}^{\kappa-1} \in \mathcal{F}_{q,\kappa-1,\alpha,\beta}^\infty\). First we consider the case \(\kappa = \infty\). By virtue of Proposition 5.9, the set \(\mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^{\infty}]\) consists of exactly one element, say \(F\). According to Lemma 11.2, the \(\mathcal{F}_{\alpha,\beta}(a_0, s_0)\)-transform of \(F\) belongs to \(\mathcal{R}_q[[\alpha, \beta]; (t_j)_{j=0}^{\infty}]\). Since, due to Proposition 5.9, the set \(\mathcal{R}_q[[\alpha, \beta]; (t_j)_{j=0}^{\infty}]\) consists of exactly one element, we conclude that \(G\) coincides with the \(\mathcal{F}_{\alpha,\beta}(a_0, s_0)\)-transform of \(F\). Using Remark 3.12, we easily infer \(a_0 \in \mathbb{C}_H^{q \times q}\) and \(s_0 \in \mathbb{C}_H^{q \times q}\). Due to Remark 3.13, we have \(\mathcal{R}(a_0) \subseteq \mathcal{R}(s_0)\). Consider now an arbitrary \(z \in \mathbb{C} \setminus [\alpha, \beta]\). Observe that the \(\mathcal{R}(\alpha, \beta)\)-measure \(\delta_F\) of \(F\) fulfills \(\delta_F((\alpha, \beta)) = s_0\). Taking into account Proposition 4.15, we obtain thus \(\mathcal{R}(F(z)) = \mathcal{R}(s_0)\) and \(\mathcal{N}(F(z)) = \mathcal{N}(s_0)\). Because of Remark 5.11, the function \(F_a\) given in Notation 4.19 belongs to \(\mathcal{R}_q[[\alpha, \beta]; (a_j)_{j=0}^{\infty}]\). Hence, we analogously get \(\mathcal{R}(F_a(z)) = \mathcal{R}(a_0)\) and \(\mathcal{N}(F_a(z)) = \mathcal{N}(a_0)\). In a similar way, we can conclude \(\mathcal{R}(G(z)) = \mathcal{R}(t_0)\). Regarding Definition 3.52, we see furthermore \(\mathcal{R}(t_0) \subseteq \mathcal{R}(a_0)\). Consequently, \(\mathcal{R}(G(z)) \subseteq \mathcal{R}(a_0)\). Thus, we can apply Lemma 10.13 to the function \(F\) and its \(\mathcal{F}_{\alpha,\beta}(a_0, s_0)\)-transform \(G\) and obtain with the inverse \(\mathcal{F}_{\alpha,\beta}(a_0, s_0)\)-transform \(H\) of \(G\) then \(H(z) = F(z)\). Hence, \(H = F\), implying \(H \in \mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^{\infty}]\).

Now we consider the case \(\kappa < \infty\). Then \(m := \kappa \in \mathbb{N}\). Regarding Remark 4.11, denote by \(\hat{\iota}_j := \int[\alpha, \beta] \xi^j \delta_G(d\xi)\) for all \(j \in \mathbb{N}_0\) the power moments of the \(\mathcal{R}(\alpha, \beta)\)-measure \(\delta_G\) of \(G\). Then \(\delta_G \in \mathcal{M}_{q,\infty}([\alpha, \beta]; (\hat{\iota}_j)_{j=0}^{\infty}, =]\), i.e., \(G \in \mathcal{R}_q[[\alpha, \beta]; (\hat{\iota}_j)_{j=0}^{\infty}]\). By virtue of Proposition 5.7, we have in particular \((\hat{\iota}_j)_{j=0}^{\infty} \in \mathcal{F}_{q,\infty,\alpha,\beta}^\infty\). We are now going to construct the \(\mathcal{F}_{\alpha,\beta}\)-parameter sequence of a sequence from \(\mathcal{F}_{q,\infty,\alpha,\beta}\) with \(\mathcal{F}_{\alpha,\beta}\)-transform \((\hat{\iota}_j)_{j=0}^{\infty}\). Due to Theorem 3.42, the \(\mathcal{F}_{\alpha,\beta}\)-parameter sequence \((\hat{\delta}_j)_{j=0}^{\infty}\) of \((\hat{\iota}_j)_{j=0}^{\infty}\) belongs to the class \(\mathcal{C}_{q,\infty,\delta}\) introduced in Notation 3.41. Since \(G\) belongs to \(\mathcal{R}_q[[\alpha, \beta]; (t_j)_{j=0}^{m-1}]\), we have \(\hat{\iota}_j = t_j\) for all \(j \in \mathbb{N}_0, m-1\). Denote by \((\hat{g}_j)_{j=0}^{2(m-1)}\) the \(\mathcal{F}_{\alpha,\beta}\)-parameter sequence of \((\hat{\iota}_j)_{j=0}^{m-1}\). Because of Remark 3.38, then \(\hat{g}_j = g_j\) for all \(j \in \mathbb{N}_0, 2(m-1)\). Denote by \((f_j)_{j=0}^{2m}\) the \(\mathcal{F}_{\alpha,\beta}\)-parameter sequence of \((s_j)_{j=0}^{m}\). Regarding \(\delta > 0\), let the sequence \((\hat{f}_j)_{j=0}^{\infty}\) be given by

\[
\hat{f}_j := \begin{cases} 
 f_j, & \text{if } j \leq 2m \\
 \delta^{-1}\hat{g}_{j-2}, & \text{if } j > 2m + 1
\end{cases}
\]

Theorem 3.42 yields \((f_j)_{j=0}^{2m} \in \mathcal{C}_{q,m,\delta}^\infty\). Taking additionally into account \(\delta > 0\) and \(m \geq 1\), we can conclude then that the sequence \((\hat{f}_j)_{j=0}^{\infty}\) is a sequence of Non-negative Hermitian matrices fulfilling the relations \(\delta \hat{f}_0 = \delta f_0 = f_1 + f_2 = \hat{f}_1 + \hat{f}_2\) and

\[
\delta(\hat{f}_{2k-1} \oplus \hat{f}_{2k}) = \delta(f_{2k-1} \oplus f_{2k}) = f_{2k+1} + f_{2k+2} = \hat{f}_{2k+1} + \hat{f}_{2k+2}
\]
for all \( k \in \mathbb{N} \) with \( k \leq m - 1 \), and, regarding Remark A.17, furthermore

\[
\delta(\hat{f}_{2k-1} \equiv \hat{f}_{2k}) = \delta[(\delta^{-1} \hat{g}_{2k-3}) \equiv (\delta^{-1} \hat{g}_{2k-2})]
\]

\[
= \hat{g}_{2k-3} \equiv \hat{g}_{2k-2} = \delta^{-1}(\hat{g}_{2k-1} + \hat{g}_{2k}) = \hat{f}_{2k+1} + \hat{f}_{2k+2}
\]

for all \( k \in \mathbb{N} \) with \( k \geq m + 1 \). In view of Corollary 3.55, in addition \( g_0 = \delta(\hat{f}_1 \equiv \hat{f}_2) \) and \( g_j = \delta f_{j+2} \) hold true for all \( j \in \mathbb{Z}_{1,2(m-1)} \). In the case \( m = 1 \), we therefore have

\[
\delta(\hat{f}_{2m-1} \equiv \hat{f}_{2m}) = \delta(\hat{f}_1 \equiv \hat{f}_2) = g_0 = \hat{g}_0 = \delta^{-1}(\hat{g}_1 + \hat{g}_2) = \delta^{-1}(\hat{g}_{2m-1} + \hat{g}_{2m}),
\]

whereas, because of Remark A.17, in the case \( m \geq 2 \) then

\[
\delta(\hat{f}_{2m-1} \equiv \hat{f}_{2m}) = (\delta \hat{f}_{2m-1}) \equiv (\delta \hat{f}_{2m})
\]

\[
= g_{2m-3} \equiv g_{2m-2} \equiv g_{2m-3} \equiv g_{2m-2} = \delta^{-1}(g_{2m-1} + \hat{g}_{2m})
\]

follows. Consequently, \( \delta(\hat{f}_{2m-1} \equiv \hat{f}_{2m}) = \hat{f}_{2m+1} + \hat{f}_{2m+2} \). Hence, the sequence \( (\hat{f}_j)_{j=0}^{\infty} \) belongs to \( C_{q, \infty, \beta}^{\infty} \). According to Theorem 3.42, then there exists a sequence \( (\hat{s}_j)_{j=0}^{\infty} \) from \( F_{q, \infty, \alpha, \beta}^{\infty} \) with \( F_{\alpha, \beta}^{\infty} \)-parameter sequence \( (\hat{f}_j)_{j=0}^{\infty} \). Denote by \( (\hat{\delta}_j)_{j=0}^{\infty} \) the \( F_{\alpha, \beta}^{\infty} \)-transform of \( (\hat{s}_j)_{j=0}^{\infty} \) and by \( (\hat{g}_j)_{j=0}^{\infty} \) the \( F_{\alpha, \beta}^{\infty} \)-parameter sequence of \( (\hat{\delta}_j)_{j=0}^{\infty} \). Using Corollary 3.55, we infer

\[
\hat{g}_0 = \delta(\hat{f}_1 \equiv \hat{f}_2) = \delta(\hat{f}_1 \equiv \hat{f}_2) = g_0 = \hat{g}_0 \quad \text{and} \quad \hat{g}_j = \delta \hat{f}_{j+2} = \delta \hat{f}_{j+2} = \hat{g}_j = \hat{g}_j
\]

for all \( j \in \mathbb{N} \) with \( j \leq 2m - 2 \) and, furthermore \( \hat{g}_j = \delta \hat{f}_{j+2} = \hat{g}_j \) for all \( j \in \mathbb{N} \) with \( j \geq 2m - 1 \). Consequently, the \( F_{\alpha, \beta}^{\infty} \)-parameter sequence \( (\hat{g}_j)_{j=0}^{\infty} \) of \( (\hat{f}_j)_{j=0}^{\infty} \) coincides with the \( F_{\alpha, \beta}^{\infty} \)-parameter sequence \( (\hat{g}_j)_{j=0}^{\infty} \) of \( (\hat{f}_j)_{j=0}^{\infty} \). Since Proposition 3.59 yields \( (\hat{f}_j)_{j=0}^{\infty} \in F_{q, \infty, \alpha, \beta}^{\infty} \), we can conclude from Theorem 3.42 that the sequences \( (\hat{f}_j)_{j=0}^{\infty} \) and \( (\hat{f}_j)_{j=0}^{\infty} \) coincide. In particular, \( (\hat{f}_j)_{j=0}^{\infty} \) is the \( F_{\alpha, \beta}^{\infty} \)-transform of \( (\hat{s}_j)_{j=0}^{\infty} \). The application of the already for \( \kappa = \infty \) proved assertion of the present lemma yields with \( \hat{a}_0 := -\alpha \hat{s}_0 + \hat{s}_1 \) for the inverse \( F_{\alpha, \beta}(\hat{a}_0, \hat{s}_0) \)-transform \( H \) of \( G \) thus \( H \in \mathcal{R}_q[[\alpha, \beta]]; (\hat{s}_j)_{j=0}^{\infty} \). In view of Proposition 3.8 and Remark 3.38, the sequence \( (\hat{s}_j)_{j=0}^{m} \) belongs to \( F_{q, m, \alpha, \beta}^{\infty} \) and its \( F_{\alpha, \beta}^{\infty} \)-parameter sequence is exactly \( (\hat{f}_j)_{j=0}^{2m} \). Consequently, the \( F_{\alpha, \beta}^{\infty} \)-parameter sequences of \( (\hat{s}_j)_{j=0}^{m} \) and \( (s_j)_{j=0}^{m} \) coincide as well. By virtue of Theorem 3.42, then the sequences \( (\hat{s}_j)_{j=0}^{m} \) and \( (s_j)_{j=0}^{m} \) coincide. Hence, \( H \) is exactly the inverse \( F_{\alpha, \beta}(\hat{a}_0, \hat{s}_0) \)-transform of \( G \) and belongs to \( \mathcal{R}_q[[\alpha, \beta]]; (s_j)_{j=0}^{m} \). \( \square \)

### 13 Parametrization of the Set of All Solutions

We are now going to iterate the \( F_{\alpha, \beta}(M) \)-transformation introduced in Definition 9.1 with the \( F_{\alpha, \beta}(A, M) \)-transformation introduced in Definition 10.1. To that end, we use the \( k \)-th \( F_{\alpha, \beta} \)-transform \( (s_j^{(k)})_{j=0}^{\kappa-k} \) of a sequence \( (s_j)_{j=0}^{\kappa} \) constructed in Definition 3.56.
and, in addition, the sequence \((a^{[k]}_j)_{j=0}^{\kappa-k}\) given by \(a^{[k]}_j := -\alpha s^{[k]}_j + s^{[k]}_{j+1}\), i.e., the sequence built from \((s^{[k]}_j)_{j=0}^{\kappa-k}\) according to Notation 3.2:

**Definition 13.1** Let \(G\) be a non-empty subset of \(\mathbb{C}\), let \(F: G \to \mathbb{C}^{p \times q}\) be a matrix-valued function, and let \((s_j)_{j=0}^\kappa\) be a sequence of complex \(p \times q\) matrices. Let 

\[\mathcal{G}_0(F, (s_j)_{j=0}^\kappa) := F.\]

Recursively, for all \(k \in \mathbb{Z}_{\geq 1}\), denote by \(\mathcal{G}_k(F, (s_j)_{j=0}^\kappa)\) the \(\mathcal{F}_{\alpha, \beta}(\alpha^{[k-1]}_0, \beta^{[k-1]}_0)\)-transform of \(\mathcal{G}_{k-1}(F, (s_j)_{j=0}^\kappa)\). In view of Definition 9.1, for all \(k \in \mathbb{Z}_{\geq 0}\), denote by \(\mathcal{P}_k(F, (s_j)_{j=0}^\kappa)\) the \(\mathcal{F}_{\alpha, \beta}(s^{[k]}_0)\)-transformed pair of \(\mathcal{G}_k(F, (s_j)_{j=0}^\kappa)\). Then, for all \(m \in \mathbb{Z}_{\geq 0}\), we call \(\mathcal{P}_m(F, (s_j)_{j=0}^\kappa)\) the \(m\)-th \(\mathcal{F}_{\alpha, \beta}\)-transformed pair of \(F\) with respect to \((s_j)_{j=0}^\kappa\) and we call \(\mathcal{G}_m(F, (s_j)_{j=0}^\kappa)\) the \(m\)-th \(\mathcal{F}_{\alpha, \beta}\)-transform of \(F\) with respect to \((s_j)_{j=0}^\kappa\).

**Remark 13.2** The pair \(\mathcal{P}_0(F, (s_j)_{j=0}^\kappa)\) is exactly the \(\mathcal{F}_{\alpha, \beta}(s_0)\)-transformed pair of \(F\). If \(\kappa \geq 1\), then \(\mathcal{G}_1(F, (s_j)_{j=0}^\kappa)\) is exactly the \(\mathcal{F}_{\alpha, \beta}(a_0, s_0)\)-transform of \(F\).

Regarding Proposition 9.11, we will use the following mappings:

**Notation 13.3** For each matrix \(M \in \mathbb{C}^{q \times q}\), let the mapping \(\tilde{F}_M\) be defined on the class \(\tilde{\mathcal{P}}[M]\) by \(\tilde{F}_M([G_1; G_2]) := F\), where \(F\) is the inverse \(\mathcal{F}_{\alpha, \beta}(M)\)-transform of \([G_1; G_2]\). Furthermore, given two matrices \(A, M \in \mathbb{C}^{p \times q}\), let the mapping \(\tilde{F}_{A, M}\) be defined on the set of all matrix-valued functions \(G: \mathbb{C} \setminus [\alpha, \beta] \to \mathbb{C}^{p \times q}\) by \(\tilde{F}_{A, M}(G) := F\), where \(F\) is the inverse \(\mathcal{F}_{\alpha, \beta}(A, M)\)-transform of \(G\).

**Proposition 13.4** Let \((s_j)_{j=0}^\kappa \in \mathcal{F}_{\alpha, \beta, \gamma}^{\geq 0}\). Then \(\psi: \langle \tilde{\mathcal{P}}[s_0] \rangle \to \mathcal{R}_q([\alpha, \beta]; (s_j)_{j=0}^\kappa)\) defined by \(\psi([G_1; G_2]) := \tilde{F}_s([G_1; G_2])\) is a bijection with inverse \(\psi^{-1}\) given by \(\psi^{-1}(F) = \langle \mathcal{P}_{\tilde{G}_0}(F, (s_j)_{j=0}^\kappa) \rangle\).

**Proof** Consider arbitrary \([G_1; G_2] \in \tilde{\mathcal{P}}[s_0]\) and \(F \in \mathcal{R}_q([\alpha, \beta]; (s_j)_{j=0}^\kappa)\). Due to Remark 3.12, we have \(s_0 \in \mathbb{C}^{q \times q}\). According to Corollary 9.12, then \(\psi((G_1; G_2))\) is independent of the concrete representative of the equivalence class \((G_1; G_2)\).

Because of Lemma 12.1, furthermore \(\tilde{F}_s([G_1; G_2])\) belongs to \(\mathcal{R}_q([\alpha, \beta]; (s_j)_{j=0}^\kappa)\), the mapping \(\psi\) is well defined. Regarding Remark 13.2, we obtain from Lemma 11.1 moreover \(\mathcal{P}_{\tilde{G}_0}(F, (s_j)_{j=0}^\kappa) \in \tilde{\mathcal{P}}[s_0]\). Consequently, the mapping \(\chi : \mathcal{R}_q([\alpha, \beta]; (s_j)_{j=0}^\kappa) \to \langle \mathcal{P}[s_0] \rangle\) defined by \(\chi(F) := \langle \mathcal{P}_{\tilde{G}_0}(F, (s_j)_{j=0}^\kappa) \rangle\) is well defined as well. Using Lemma 9.13 and Remark 13.2, we conclude \(\chi \circ \psi) = \mathcal{R}_q(F) = \mathcal{R}(s_0)\) for all \(z \in \mathbb{C} \setminus [\alpha, \beta]\). Therefore, \(\mathbb{P}_F = F\). Taking additionally into account Remark 13.2 and \(\mathcal{P}_{\tilde{G}_0}(F, (s_j)_{j=0}^\kappa) \in \tilde{\mathcal{P}}[s_0]\), Lemma 9.14 then yields \(\psi = \psi^{-1}\) with inverse \(\psi^{-1}\).

**Proposition 13.5** Let \(m \in \mathbb{N}\) and let \((s_j)_{j=0}^m \in \mathcal{F}_{q, m, \alpha, \beta}^{\geq 0}\) with \(\mathcal{F}_{\alpha, \beta}\)-transform \((t_j)_{j=0}^{m-1}\) Then \(\psi: \mathcal{R}_q([\alpha, \beta]; (t_j)_{j=0}^{m-1}) \to \mathcal{R}_q([\alpha, \beta]; (s_j)_{j=0}^m)\) defined by \(\psi(G) := \tilde{F}_{a_0, s_0}(G)\) is a bijection with inverse \(\psi^{-1}\) given by \(\psi^{-1}(F) = \tilde{G}_1(F, (s_j)_{j=0}^m)\).
Proof In view of Lemma 12.2, the mapping \( \psi \) is well defined. According to Remark 13.2, we see from Lemma 11.2 that the mapping \( \chi : R_q[[\alpha, \beta]; (s_j)^{m-1}] \rightarrow R_q[[\alpha, \beta]; (t_j)^{m-1}] \) given by \( \chi(F) := \tilde{G}_1(F, (s_j)^{m-1}) \) is also well defined. Using Remark 3.12, we easily infer \( a_0 \in C_{H}^{q \times q} \) and \( s_0 \in C_{H}^{q \times q} \). Because of Remark 3.13, we have \( R(a_0) \subseteq R(s_0) \). Consider now an arbitrary \( G \in R_q[[\alpha, \beta]; (t_j)^{m-1}] \), then \( G \in R_q(C \setminus [\alpha, \beta]) \). Taking into account Remark 5.8, we conclude from Lemma C.5 further \( \mathcal{R}(G(z)) = \mathcal{R}(t_0) \) for all \( z \in C \setminus [\alpha, \beta] \). Regarding Definition 3.52, we then obtain \( \mathcal{R}(G(z)) \subseteq R(a_0) \) for all \( z \in C \setminus [\alpha, \beta] \). By construction, \( \psi(G) \) is the inverse \( \mathcal{F}_{\alpha, \beta}(a_0, s_0) \)-transform of \( G \). Denote by \( H \) the \( \mathcal{F}_{\alpha, \beta}(a_0, s_0) \)-transform of \( \psi(G) \). In view of Remark 13.2, then \( H = \chi(\psi(G)) \). Using Lemma 10.12, hence \( H = G \) follows. Consequently, \( (\chi \circ \psi)(G) = G \). Now we consider an arbitrary \( F \in R_q[[\alpha, \beta]; (s_j)^{m-1}] \). By virtue of Lemma 11.2, the \( \mathcal{F}_{\alpha, \beta}(a_0, s_0) \)-transform \( \chi(F) \) of \( F \) then belongs to \( R_q[[\alpha, \beta]; (t_j)^{m-1}] \). As above, we thus have \( \chi(F) \in R_q(C \setminus [\alpha, \beta]) \) and \( \mathcal{R}(\chi(F)(z)) \subseteq R(q_0) \) for all \( z \in C \setminus [\alpha, \beta] \). Observe that the \( \mathcal{R}[\alpha, \beta] \)-measure \( \bar{\mathcal{R}}(F) \) satisfies \( \bar{\mathcal{R}}(\chi(F)) = s_0 \). Using Proposition 4.15, we get \( R(F(z)) = R(s_0) \) and \( N(F(z)) = N(s_0) \) for all \( z \in C \setminus [\alpha, \beta] \). Because of Remark 5.11, the function \( F_a \) given in Notation 4.19 belongs to \( R_q[[\alpha, \beta]; (a_j)^{m-1}] \). Therefore, we obtain analogously \( R(F_a(z)) = R(a_0) \) and \( N(F_a(z)) = N(a_0) \) for all \( z \in C \setminus [\alpha, \beta] \). Hence, we can apply Lemma 10.13 to the function \( F \) and its \( \mathcal{F}_{\alpha, \beta}(a_0, s_0) \)-transform \( \chi(F) \) and obtain with the inverse \( \mathcal{F}_{\alpha, \beta}(a_0, s_0) \)-transform \( H = \psi(\chi(F)) \) of \( \chi(F) \) then \( H(z) = F(z) \) for all \( z \in C \setminus [\alpha, \beta] \). Thus, \( (\psi \circ \chi)(F) = F \). Consequently, \( \psi \) is bijective with inverse \( \chi \).

The combination of Propositions 13.4 and 13.5 now yields a first parametrization of the solution set of the matricial Hausdorff moment problem \( MP[[\alpha, \beta]; (s_j)^{m}] = \) \( \emptyset \), where, however, the set of parameters still depends on the given data.

**Theorem 13.6** Let \( m \in \mathbb{N} \) and let \( (s_j)^{m-1} \in F_{q,m,\alpha,\beta} \). Let \( \psi_m : (\tilde{P})_{0}^{[m]} \rightarrow R_q[[\alpha, \beta]; (s_j^{[m-1]})] \) be defined by \( \psi_m((G_1; G_2)) := \tilde{F}_{0}^{[m]}((G_1; G_2)) \). In the case \( m \geq 1 \) let, for all \( k \in \mathbb{Z}_{0,m-1} \), furthermore \( \psi_k : R_q[[\alpha, \beta]; (s_j^{[m-1]})] \rightarrow R_q[[\alpha, \beta]; (s_j^{[k-1]})] \) be given by \( \psi_k(G) := \tilde{F}_{0}^{[k]}(G) \). Then \( \psi_m : (\tilde{P})_{0}^{[m]} \rightarrow R_q[[\alpha, \beta]; (s_j^{[m-1]})] \) is a bijection with inverse \( \psi_m^{-1} \) given by \( \psi_m^{-1}(F) = (\tilde{P}G_m(F, (s_j)^{m-1}_j)) \).

**Proof** Because of Proposition 3.59, we have \( (s_j^{[k-1]})_j \in F_{q,m-k,\alpha,\beta} \) for all \( k \in \mathbb{Z}_{0,m} \). According to Proposition 13.4, then \( \psi_m \) is a bijection with inverse \( \psi_m^{-1} \) given by \( \psi_m^{-1}(F) = (\tilde{P}G_m(F, (s_j^{[m-1]})_j)) \). Regarding Definition 3.56, we infer in the case \( m \geq 1 \) for all \( k \in \mathbb{Z}_{0,m-1} \) from Proposition 13.5 that \( \psi_k \) is a bijection with inverse \( \psi_k^{-1} \) given by \( \psi_k^{-1}(F) = \tilde{G}_1(F, (s_j^{[m-1]})_j) \). In view of Remark 13.2, we see, for all \( F \in R_q[[\alpha, \beta]; (s_j^{[m-1]})] \), that \( \psi_m^{-1}(F) \) is exactly the equivalence class of the \( \mathcal{F}_{\alpha, \beta}(0, (s_j)^{m-1}) \)-transformed pair of \( F \). In the case \( m \geq 1 \), for all \( k \in \mathbb{Z}_{0,m-1} \) and all \( F \in R_q[[\alpha, \beta]; (s_j^{[k-1]})] \), furthermore \( \psi_k^{-1}(F) \) coincides with the equivalence class
of the $\mathcal{F}_{\alpha,\beta}(a_{0}^{(k)}, s_{0}^{(k)})$-transform of $F$. Regarding Definition 13.1, we obtain in the case $m \geq 1$, for all $F \in \mathcal{R}_{q}(\alpha, \beta) \{ (s_{j})_{j=0}^{m} \}$, the equation $(\psi_{m}^{-1} \circ \cdots \circ \psi_{0}^{-1})(F) = \tilde{G}_{m}(F, (s_{j})_{j=0}^{m})$ and hence $(\psi_{m}^{-1} \circ \cdots \circ \psi_{0}^{-1})(F) = \langle \mathbf{P} \tilde{G}_{m}(F, (s_{j})_{j=0}^{m}) \rangle$, implying that $\Psi_{m}$ is a bijection with inverse $\Psi_{m}^{-1}$ given by $\Psi_{m}^{-1}(F) = \langle \mathbf{P} \tilde{G}_{m}(F, (s_{j})_{j=0}^{m}) \rangle$.

We end this section by mentioning a relation between the $k$-th $\mathcal{F}_{\alpha,\beta}$-transform given in Definition 13.1 of a matrix-valued function with respect to a sequence of matrices and the $k$-th $\mathcal{R}[\alpha, \beta]$-Schur transform introduced in Definition 6.8. We start with the case $k = 1$:

**Lemma 13.7** Suppose $\kappa \geq 1$. Let $(s_{j})_{j=0}^{\kappa} \in \mathcal{F}_{q, \kappa, \alpha, \beta}^{\infty}$ and let $F \in \mathcal{R}_{q}(\alpha, \beta) \{ (s_{j})_{j=0}^{\kappa} \}$ with $\mathcal{F}_{\alpha,\beta}(a_{0}, s_{0})$-transform $G$ and first $\mathcal{R}[\alpha, \beta]$-Schur transform $F^{[1]}$. Then $G = F^{[1]}$.

**Proof** By assumption we have $F \in \mathcal{R}_{q}(\mathbb{C} \setminus [\alpha, \beta])$ with $\mathcal{R}[\alpha, \beta]$-measure $\sigma := \tilde{\sigma}_{F}$ belonging to $\mathcal{M}_{q, \kappa}^{\infty}(\alpha, \beta) \{ (s_{j})_{j=0}^{\kappa} \}$. Observe that $\sigma \in \mathcal{M}_{q, \infty}^{\infty}(\alpha, \beta)$ according to Remark 4.11. Setting $s_{j} := \int_{[\alpha, \beta]} x^{j} \sigma(dx)$ for all $j \in \mathbb{Z}_{\kappa+1, \infty}$, we have then $\sigma \in \mathcal{M}_{q, \infty}^{\infty}(\alpha, \beta) \{ (s_{j})_{j=0}^{\infty} \}$ and hence $F \in \mathcal{R}_{q}(\alpha, \beta) \{ (s_{j})_{j=0}^{\infty} \}$. In particular, Theorem 3.5 shows $(s_{j})_{j=0}^{\infty} \in \mathcal{F}_{q, \infty, \alpha, \beta}^{\infty}$. Regarding $\kappa \geq 1$, the matrix $a_{0} = -\alpha s_{0} + s_{1}$ is not affected by the above extension of the sequence $(s_{j})_{j=0}^{\kappa}$. Thus, we can apply Lemma 11.2 to obtain $G \in \mathcal{R}_{q}(\alpha, \beta) \{ (t_{j})_{j=0}^{\infty} \}$, where $(t_{j})_{j=0}^{\infty}$ is the $\mathcal{F}_{\alpha,\beta}$-transform of $(s_{j})_{j=0}^{\infty}$. Consequently, we have $G \in \mathcal{R}_{q}(\mathbb{C} \setminus [\alpha, \beta])$ with $\mathcal{R}[\alpha, \beta]$-measure $\nu := \tilde{\sigma}_{G}$ belonging to $\mathcal{M}_{q, \infty}^{\infty}(\alpha, \beta) \{ (t_{j})_{j=0}^{\infty} \}$. By virtue of Proposition 5.5, in particular $G = \tilde{S}_{\nu}$. According to Definition 6.5, the first $\mathcal{M}[\alpha, \beta]$-transform $\mu := \sigma_{[1]}$ of $\sigma$ belongs to $\mathcal{M}_{q, \kappa, \infty}^{\infty}(\alpha, \beta) \{ (t_{j})_{j=0}^{\kappa} \}$ as well. Due to Proposition 3.6 and Theorem 3.5, the set $\mathcal{M}_{q, \kappa}^{\infty}(\alpha, \beta) \{ (t_{j})_{j=0}^{\kappa} \}$ consists of at most one element. Hence, $\nu = \mu$ follows. Since $F^{[1]} = \tilde{S}_{\mu}$ by Definition 6.8, we infer then $G = \tilde{S}_{\nu} = \tilde{S}_{\mu} = F^{[1]}$. □

**Proposition 13.8** Let $(s_{j})_{j=0}^{\kappa} \in \mathcal{F}_{q, \kappa, \alpha, \beta}^{\infty}$, let $k \in \mathbb{Z}_{0, \kappa}$, and let $F \in \mathcal{R}_{q}(\alpha, \beta) \{ (s_{j})_{j=0}^{\kappa} \}$ with $k$-th $\mathcal{F}_{\alpha,\beta}$-transform $\tilde{G}_{k}(F, (s_{j})_{j=0}^{\kappa})$ of $F$ with respect to $(s_{j})_{j=0}^{\kappa}$ and $k$-th $\mathcal{R}[\alpha, \beta]$-Schur transform $F^{[k]}$ of $F$. Then $\tilde{G}_{k}(F, (s_{j})_{j=0}^{\kappa}) = F^{[k]}$.

**Proof** We use mathematical induction. According to Definition 13.1, we have $\tilde{G}_{0}(F, (s_{j})_{j=0}^{\kappa}) = F$, whereas Remark 6.10 shows $F^{[0]} = F$. Hence, the assertion holds true for $k = 0$. Now assume that $\kappa \geq 1$ and that $\tilde{G}_{k-1}(F, (s_{j})_{j=0}^{\kappa}) = F^{[k-1]}$ is valid for some $k \in \mathbb{Z}_{1, \kappa}$. Setting $T := F^{[k-1]}$ and $t_{j} := s_{j}^{(k-1)}$ for all $j \in \mathbb{Z}_{0, \kappa - (k-1)}$, then $\tilde{G}_{k}(F, (s_{j})_{j=0}^{\kappa})$ is, by Definition 13.1, exactly the $\mathcal{F}_{\alpha,\beta}(-\alpha t_{0} + t_{1}, t_{0})$-transform of $T$. Furthermore, Remark 6.11 yields $T \in \mathcal{R}_{q}(\alpha, \beta) \{ (t_{j})_{j=0}^{k-(k-1)} \}$. From Proposition 3.59 we infer $(t_{j})_{j=0}^{k-(k-1)} \in \mathcal{F}_{q, \kappa - (k-1), \alpha, \beta}^{\infty}$. Taking additionally into account $\kappa - (k - 1) \geq 1$, we can apply Lemma 13.7 to the sequence $(t_{j})_{j=0}^{k-(k-1)}$ and the function $T$ to see that the $\mathcal{F}_{\alpha,\beta}(-\alpha t_{0} + t_{1}, t_{0})$-transform of $T$ coincides with the
first $\mathcal{R}[\alpha, \beta]$-Schur transform of $T$, i.e., $\hat{G}_k(F, (s_j)_{j=0}^\infty) = T^{[1]}$. Since Remark 6.10 provides $F^{[k]} = T^{[1]}$, the proof is complete. 

In view of Lemma 13.7, a more explicit description of the Schur–Nevanlinna type algorithm for the class $\mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$ considered in Sect. 6 can be given by means of matricial linear fractional transformations. For the sake of simplicity, we illustrate this for the scalar case $q = 1$, where this amounts to a scalar linear fractional transformation or a continued fraction expansion of functions belonging to $\mathcal{R}_1(\mathbb{C} \setminus [\alpha, \beta])$. These considerations are along the lines of the classical results by Schur [50, 51] for the class $S_{1 \times 1}(\mathbb{D})$ of holomorphic functions mapping the open unit disc $\mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \}$ into the closed unit disc $\overline{\mathbb{D}}$ (cf. Notation F.8) and by Nevanlinna [45] for the class $\mathcal{R}_{0, 1}(\Pi_+)$ introduced in Notation 4.2:

Let $f \in \mathcal{R}_1(\mathbb{C} \setminus [\alpha, \beta])$ with $\mathcal{R}[\alpha, \beta]$-measure $\tilde{\sigma}_f$. Then $f \in \mathcal{R}_1([\alpha, \beta]; (s_j)_{j=0}^\infty)$ with the sequence $(s_j)_{j=0}^\infty$ of power moments $s_j := \int_{[\alpha, \beta]} x^j \tilde{\sigma}_f(dx)$ associated with $\tilde{\sigma}_f$ belonging to $\mathcal{F}_{1, \infty, [\alpha, \beta]}$, by virtue of Proposition 3.7. We see from Lemma 13.7 that the first $\mathcal{R}[\alpha, \beta]$-Schur transform $f^{[1]}$ of $f$ is exactly the $\mathcal{F}_{\alpha, \beta}(a_0, s_0)$-transform of $f$. Consequently, Lemma 11.2 yields $f^{[1]} \in \mathcal{R}_1([\alpha, \beta]; (s_j)_{j=0}^\infty)$ with the $\mathcal{F}_{\alpha, \beta}$-transform $(s_j^{[1]})_{j=0}^\infty$ of $(s_j)_{j=0}^\infty$ belonging to $\mathcal{F}_{1, \infty, [\alpha, \beta]}$, by virtue of Proposition 3.59. Hence, we can conclude from Proposition 5.9 that the expansions

$$f(z) = -\frac{s_0}{z} - \frac{s_1}{z^2} - \frac{s_2}{z^3} - \cdots$$

and

$$f^{[1]}(z) = -\frac{s_0^{[1]}}{z} - \frac{s_1^{[1]}}{z^2} - \frac{s_2^{[1]}}{z^3} - \cdots$$

are valid for all $z \in \mathbb{C}$ with $|z| > \max(|\alpha|, |\beta|)$. Recall that $a_0 = -\alpha s_0 + s_1$ and $b_0 = \beta s_0 - s_1$, according to Notation 3.2. Lemma 3.12 yields $s_0 \geq 0$ and $a_0 \geq 0$. In what follows, we assume $s_0 > 0$. Because of (3.7) and Corollary 3.61, we have then

$$\vartheta_0 = \delta s_0 \quad \text{and} \quad s_0^{[1]} = \vartheta_1 = -\alpha \beta s_0 + (\alpha + \beta) s_1 = \frac{s_0^2}{s_0} = \frac{a_0 b_0}{s_0}.$$ 

In view of $\delta = \beta - \alpha > 0$, thus $\vartheta_0 > 0$. Regarding Remark A.13 and $a_0 + b_0 = \delta s_0$, we obtain, by virtue of Definition 3.43 and (3.11), hence

$$\epsilon_0 = f_0 = s_0 \quad \text{and} \quad \epsilon_1 = \frac{f_2}{\vartheta_0} = \frac{b_0}{\delta s_0} = \frac{\delta s_0 - a_0}{\delta s_0} = 1 - \frac{a_0}{\delta s_0}.$$ 

Taken all together, we can infer by direct calculation $s_0 = \epsilon_0$, $a_0 = \delta \epsilon_0 (1 - \epsilon_1)$, $b_0 = \delta \epsilon_0 \epsilon_1$, and $s_0^{[1]} = \delta^2 \epsilon_0 \epsilon_1 (1 - \epsilon_1)$. In view of $s_0 = \tilde{\sigma}_f([\alpha, \beta])$, we can conclude from Remark 5.11 that the function $f_{\alpha} : \mathbb{C} \setminus [\alpha, \beta] \to \mathbb{C}$ given, according to Notation 4.19, by $f_{\alpha}(z) := (z - \alpha)f(z) + s_0$ belongs to $\mathcal{R}_1(\mathbb{C} \setminus [\alpha, \beta])$ with $\mathcal{R}[\alpha, \beta]$-measure $\tilde{\sigma}_\alpha$ fulfilling $\tilde{\sigma}_{\alpha}(\{\alpha, \beta\}) = a_0$. If $a_0 = 0$, then Proposition 4.15 yields $f_{\alpha}(z) = 0$, i.e., $f(z) = (\alpha - z)^{-1}s_0$ for all $z \in \mathbb{C} \setminus [\alpha, \beta]$, implying $\tilde{\sigma}_f = s_0 \delta_\alpha$, by virtue of Proposition 5.5, where $\delta_\alpha$ is the Dirac measure on $[\alpha, \beta]$, $\mathcal{B}_{[\alpha, \beta]}$ with unit mass at $\alpha$. Now assume $a_0 > 0$. Proposition 4.15 yields, for all $z \in \mathbb{C} \setminus [\alpha, \beta]$,
then \( f_a(z) \neq 0 \). In view of Definition 10.1 and Remark A.13, we thus can infer the representation

\[
f^{[1]}(z) = \frac{a_0^2/s_0}{\beta - z} \cdot \frac{(\beta - z)f(z) - s_0}{(\alpha - z)f(z) + s_0} = \frac{\delta^2 \epsilon_0(1 - \epsilon_1)^2}{\beta - z} \cdot \frac{(\beta - z)f(z) - \epsilon_0}{(\alpha - z)f(z) + \epsilon_0}
\]

(13.1)

for all \( z \in \mathbb{C} \setminus [\alpha, \beta] \). As seen above, we have \( f^{[1]} \in R_q(\mathbb{C} \setminus [\alpha, \beta]) \). Taking additionally into account \( f_a(z) \neq 0 \) for all \( z \in \mathbb{C} \setminus [\alpha, \beta] \) and the assumptions \( s_0 > 0 \) and \( a_0 > 0 \), the conditions of Lemma 10.13 are fulfilled. So its application shows that \( f \) coincides with the inverse \( F_{\alpha, \beta}(a_0, s_0) \)-transform of \( f^{[1]} \). Observe that \( Q_{s_0} = 0 \) and \( Q_{a_0} = 0 \) by (9.2) and the assumptions \( s_0 > 0 \) and \( a_0 > 0 \). From Lemma 10.8 we can conclude for all \( z \in \mathbb{C} \setminus [\alpha, \beta] \) then \( a_0^2/s_0 - (z - \alpha) f^{[1]}(z) \neq 0 \) and the representation

\[
f(z) = \frac{s_0}{\beta - z} \cdot \frac{a_0^2/s_0 + (\beta - z)f^{[1]}(z)}{(z - \alpha)f^{[1]}(z)} = \frac{\epsilon_0}{\beta - z} \cdot \frac{\delta^2 \epsilon_0(1 - \epsilon_1)^2 + (\beta - z)f^{[1]}(z)}{\delta^2 \epsilon_0(1 - \epsilon_1)^2 - (z - \alpha)f^{[1]}(z)},
\]

which also follows by direct calculation from (13.1). Regarding \((\beta - z)^{-1} + (z - \alpha)^{-1} = \delta(\beta - z)^{-1}(z - \alpha)^{-1}\), we can rewrite this, for all \( z \in \mathbb{C} \setminus [\alpha, \beta] \), as

\[
f(z) = \frac{s_0}{z - \alpha} \cdot \frac{a_0^2/s_0 + f^{[1]}(z)}{(z - \alpha)} = \frac{-s_0}{z - \alpha} \cdot \frac{\delta a_0^2/s_0}{(\beta - z)(z - \alpha)} + \frac{f^{[1]}(z) - a_0^2/s_0}{z - \alpha} = \frac{s_0}{(\alpha - z)} \cdot \frac{\delta a_0^2}{(\beta - z)(z - \alpha)} - \frac{f^{[1]}(z)}{z - \alpha}
\]

\[
= \frac{s_0}{(\alpha - z)} \cdot \frac{a_0^2/s_0 - f^{[1]}(z)}{z - \alpha} + (\alpha - z) \frac{\delta a_0^2}{(\beta - z)(z - \alpha)} - \frac{f^{[1]}(z)}{z - \alpha} = \frac{s_0}{\alpha - z} + \frac{\delta a_0^2}{(\beta - z)(z - \alpha)} - \frac{f^{[1]}(z)}{z - \alpha},
\]

giving rise to a continued fraction expansion of functions \( f \in R_1(\mathbb{C} \setminus [\alpha, \beta]) \).

### 14 Description via Linear Fractional Transformation

We are now going to write the parametrization \( \Psi_m \) from Theorem 13.6 as a matricial linear fractional transformation, as considered in Appendix E. The generating matrix-valued function of this transformation is a composition of certain instances of the matrix polynomials introduced in Notations 9.6 and 10.6:

**Notation 14.1** Let \((s_j)_{j=0}^k\) be a sequence of complex \( p \times q \) matrices and let \( m \in \mathbb{Z}_{0, \kappa} \).

Then let \( \bar{V}_m := V_0 V_1 \cdots V_m \), where \( V_k := \bar{V}_{a_0/s_0}^{[k]} \) for all \( k \in \mathbb{Z}_{0, m-1} \) and \( V_m := \bar{V}_{a_0/s_0}^{[m]} \).
Regarding Notations 9.6 and 10.6, we see that $\mathcal{W}_m$ is a complex $(p + q) \times (p + q)$ matrix polynomial with $\deg \mathcal{W}_m \leq 2(m + 1)$. As a main result of the present paper, we now obtain a description of the set of all solutions to Problem FP\([\alpha, \beta]; (s_j)^m_{j=0}\) via a matricial linear fractional transformation generated by that matrix polynomial:

**Theorem 14.2** Let $m \in \mathbb{N}_0$ and let $(s_j)^m_{j=0} \in \mathcal{F}^{\geq}_{q,m,\alpha,\beta}$. Denote by $\begin{bmatrix} \tilde{w}_m & \tilde{z}_m \\ \tilde{h}_m & \tilde{\zeta}_m \end{bmatrix}$ the $q \times q$ block representation of the restriction of $\mathcal{W}_m$ onto $\mathbb{C} \setminus [\alpha, \beta]$:

(a) Let $\Gamma \in \langle \mathcal{P}[s_0^m] \rangle$ and let $[G_1; G_2] \in \Gamma$. Then $\det(\tilde{h}_m G_1 + \tilde{\zeta}_m G_2)$ does not vanish identically in $\mathbb{C} \setminus [\alpha, \beta]$ and the function

$$F = (\tilde{w}_m G_1 + \tilde{z}_m G_2)(\tilde{h}_m G_1 + \tilde{\zeta}_m G_2)^{-1} \quad (14.1)$$

belongs to $\mathcal{R}_q[[\alpha, \beta]; (s_j)^m_{j=0}]$.

(b) For each $F \in \mathcal{R}_q[[\alpha, \beta]; (s_j)^m_{j=0}]$, there exists a unique equivalence class $\Gamma \in \langle \mathcal{P}[s_0^m] \rangle$ such that (14.1) holds true for all $[G_1; G_2] \in \Gamma$, namely the equivalence class $\langle \mathcal{P}\mathcal{E}_m(F, (s_j)^m_{j=0}) \rangle$ of the $m$-th $\mathcal{F}_{\alpha,\beta}$-transformed pair $\mathcal{P}\mathcal{E}_m(F, (s_j)^m_{j=0})$ of $F$ with respect to $(s_j)^m_{j=0}$.

**Proof** Let the mappings $\psi_0, \psi_1, \ldots, \psi_m$ be defined as in Theorem 13.6. Let $V_m := \tilde{V}_0^m$ and, in the case $m \geq 1$, let $V_k := \tilde{V}_0^{[k]} \tilde{s}_0^{[k]}$ for all $k \in \mathbb{Z}_{0,m-1}$. For all $k \in \mathbb{Z}_{0,m}$, let $\begin{bmatrix} \tilde{a}_k & \tilde{b}_k \\ \tilde{c}_k & \tilde{d}_k \end{bmatrix}$ be the $q \times q$ block representation of the restriction of $V_k$ onto $\mathbb{C} \setminus [\alpha, \beta]$. Because of Proposition 3.59, we have $(s_j)^{m-k}_{j=0} \in \mathcal{F}^{\geq}_{q,m-k,\alpha,\beta}$ for all $k \in \mathbb{Z}_{0,m}$. Using Remarks 3.12 and 3.13, we thus easily obtain $\tilde{a}_0^{[k]} \in \mathbb{C}^{q \times q}$ and $\mathcal{R}(\tilde{a}_0^{[k]}) \subseteq \mathcal{R}(s_0^{[k]})$ for all $k \in \mathbb{Z}_{0,m-1}$ and, furthermore, $s_0^{[k]} \in \mathbb{C}^{q \times q}$ for all $k \in \mathbb{Z}_{0,m}$.

Consider now an arbitrary pair $[G_1; G_2] \in \mathcal{P}[s_0^m]$. According to Notation 8.1, then $[G_1; G_2]$ belongs to $\mathcal{P}\mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$. In particular, $G_1$ and $G_2$ are $\mathbb{C}^{q \times q}$-valued functions meromorphic in $\mathbb{C} \setminus [\alpha, \beta]$. Because of Proposition 7.9, the set $A := \mathcal{P}(G_1) \cup \mathcal{P}(G_2) \cup \mathcal{E}(G_1; G_2)$ is a discrete subset of $\mathbb{C} \setminus [\alpha, \beta]$. Hence, $\mathbb{C} \setminus ([\alpha, \beta] \cup A) \neq \emptyset$. Let $H_m := \psi_m([G_1; G_2])$. In view of Definition 9.5 and Notations 9.6 and 13.3, we conclude from Proposition 9.11 then $\det[\tilde{c}_m(z)G_1(z) + \tilde{d}_m(z)G_2(z)] \neq 0$ and

$$[\tilde{a}_m(z)G_1(z) + \tilde{b}_m(z)G_2(z)][\tilde{c}_m(z)G_1(z) + \tilde{d}_m(z)G_2(z)]^{-1} = [\psi_m([G_1; G_2])](z) = H_m(z)$$

for all $z \in \mathbb{C} \setminus ([\alpha, \beta] \cup A)$. In the case $m \geq 1$, let $H_k := \psi_k(H_{k+1})$ for all $k \in \mathbb{Z}_{0,m-1}$. By virtue of Theorem 13.6, we have $H_m \in \mathcal{R}_q[[\alpha, \beta]; (s_j)^m_{j=0}]$ and, in the case $m \geq 1$, moreover $H_k \in \mathcal{R}_q[[\alpha, \beta]; (s_j)^{m-k}_{j=0}]$ for all $k \in \mathbb{Z}_{0,m-1}$.

In the case $m \geq 1$, we now consider an arbitrary $\ell \in \mathbb{Z}_{0,m-1}$. Then $H_{\ell+1} \in \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$ and the $\mathcal{R}([\alpha, \beta])$-measure $\tilde{\sigma}_{\ell+1}$ of $H_{\ell+1}$ fulfills $\tilde{\sigma}_{\ell+1}([\alpha, \beta]) = s_0^{[\ell+1]}$. Using Proposition 4.15, we obtain in particular $\mathcal{R}(H_{\ell+1}(z)) = \mathcal{R}(s_0^{[\ell+1]})$ for all
Regarding Definitions 3.56 and 3.52, we infer furthermore $R(0^{(l+1)}_{q}) \subseteq R(a_{0}^{(l)}).$ Consequently, $R(H_{l+1}(z)) \subseteq R(a_{0}^{(l)})$ follows for all $z \in \mathbb{C} \setminus \{\alpha, \beta\}.$

Thus, in the case $m \geq 1,$ Lemma 10.8 yields, in view of Notations 10.6 and 13.3, for all $k \in \mathbb{Z}_{0,m-1},$ then $\det(\bar{c}_{k}(z)H_{k+1}(z) + \bar{d}_{k}(z)) \neq 0$ and

$$[\bar{a}_{k}(z)H_{k+1}(z) + \bar{b}_{k}(z)][\bar{c}_{k}(z)H_{k+1}(z) + \bar{d}_{k}(z)]^{-1} = [\psi_{k}(H_{k+1})](z) = H_{k}(z)$$

for all $z \in \mathbb{C} \setminus \{\alpha, \beta\}.$ By virtue of $\hat{\omega}_{m} = V_{0}V_{1} \cdots V_{m},$ we can conclude from Proposition E.2 hence $\det(\hat{\eta}_{m}(z)G_{1}(z) + \tilde{\eta}_{m}(z)G_{2}(z)) \neq 0$ and

$$H_{0}(z) = [(\psi_{0} \circ \psi_{1} \circ \cdots \circ \psi_{m})(\{G_{1}; G_{2}\})](z) = [\hat{\omega}_{m}(z)G_{1}(z) + \tilde{\eta}_{m}(z)G_{1}(z)][\hat{\eta}_{m}(z)G_{1}(z) + \tilde{\eta}_{m}(z)G_{2}(z)]^{-1}$$

for all $z \in \mathbb{C} \setminus \{\alpha, \beta\}.$ In particular, $\det(\hat{\eta}_{m}G_{1} + \tilde{\eta}_{m}G_{2})$ does not vanish identically in $\mathbb{C} \setminus \{\alpha, \beta\}.$ Consequently, $(\hat{\omega}_{m}G_{1} + \tilde{\eta}_{m}G_{2})(\hat{\eta}_{m}G_{1} + \tilde{\eta}_{m}G_{2})^{-1}$ is a $\mathbb{C}^{q \times q}$-valued function meromorphic in $\mathbb{C} \setminus \{\alpha, \beta\}.$ By virtue of $H_{0} \in R_{q}([\alpha, \beta]; (s_{j})^{m}_{j=0}),$ the matrix-valued function $H_{0}$ is holomorphic in $\mathbb{C} \setminus \{\alpha, \beta\}.$ Since the set $A$ is discrete, we can conclude from the identity theorem for holomorphic functions then

$$(\psi_{0} \circ \psi_{1} \circ \cdots \circ \psi_{m})(\{G_{1}; G_{2}\}) = H_{0} = (\hat{\omega}_{m}G_{1} + \tilde{\eta}_{m}G_{2})(\hat{\eta}_{m}G_{1} + \tilde{\eta}_{m}G_{2})^{-1}.$$ 

The application of Theorem 13.6 completes the proof. □

**Remark 14.3** Let $m \in \mathbb{N}_{0}$ and let $(s_{j})^{m}_{j=0} \in \mathcal{F}_{q,m,\alpha,\beta}^{\geq}$. In view of $\delta > 0$ and Corollary 3.61, we have $\delta^{m-1}d_{m} = s_{0}^{(m)},$ and consequently $R(\partial_{m}) = R(s_{0}^{(m)})$ and $\mathcal{N}(\partial_{m}) = \mathcal{N}(s_{0}^{(m)}).$ According to Notation 8.1, hence $\hat{\mathcal{P}}[\partial_{m}] = \hat{\mathcal{P}}[s_{0}^{(m)}].$

Now we study separately three cases depending on the rank $r$ of the matrix $\partial_{m} = \delta^{-(m-1)}s_{0}^{(m)}$ determining the amount of determinacy of the truncated moment problem in question. We distinguish between the case $r = q,$ the case $1 \leq r < q - 1,$ and the case $r = 0.$

First we consider the so-called non-degenerate case $r = q.$ In view of Remark 3.67, this is exactly the case of $(s_{j})^{m}_{j=0} \in \mathcal{F}_{q,m,\alpha,\beta}^{>}$ for which Problem $FP([\alpha, \beta]; (s_{j})^{m}_{j=0})$ was already considered in [10,11], applying Potapov’s method of fundamental matrix inequalities separately for even and odd Non-negative integers $m,$ resp. The generating matrix-valued functions of the linear fractional transformation obtained there are matrix polynomials. For the scalar case $q = 1$ we refer to Krein/Nudelman [42, Ch. IV, § 7], where the generating matrix function of the linear fractional transformation is built from orthogonal polynomials of first kind and second kind.

**Theorem 14.4** Let $m \in \mathbb{N}_{0}$ and let $(s_{j})^{m}_{j=0} \in \mathcal{F}_{q,m,\alpha,\beta}^{>}$ be such that the matrix $\partial_{m}$ is non-singular. Then all statements of Theorem 14.2 are valid with the class $\mathcal{P}_{\mathcal{R}_{q}}(\mathbb{C} \setminus \{\alpha, \beta\})$ instead of $\hat{\mathcal{P}}[s_{0}^{(m)}].$

**Proof** Use Remarks 14.3 and 8.2 and Theorem 14.2. □
Now we turn our attention to the degenerate, but not completely degenerate case $1 \leq r \leq q - 1$:

**Theorem 14.5** Assume $q \geq 2$. Let $m \in \mathbb{N}_0$ and let $(s_j)^m_{j=0} \in \mathcal{F}_{q,m,\alpha,\beta}^\circ$. Let $r := \text{rank } \mathcal{D}_m$ and assume $1 \leq r \leq q - 1$. Denote by $\left[ \begin{array}{cc} \tilde{m} & \tilde{m} \\ \tilde{m} & \tilde{m} \end{array} \right]$ the $q \times q$ block representation of the restriction of $\tilde{\mathcal{D}}_m$ onto $\mathbb{C} \setminus [\alpha, \beta]$. Let $u_1, u_2, \ldots, u_q$ be an orthonormal basis of $\mathbb{C}^q$ with $\{u_1, u_2, \ldots, u_r\} \subseteq \mathcal{R}(\mathcal{D}_m)$ and let $W := [u_1, u_2, \ldots, u_q]$. Then:

(a) Let $[g_1; g_2] \in \mathcal{P}(\mathcal{R}_r(\mathbb{C} \setminus [\alpha, \beta])$. Then $\det(\tilde{m}_m W(g_1 \oplus O_{(q-r) \times (q-r)}) + \tilde{m}_m W(g_2 \oplus I_{q-r})$ does not vanish identically in $\mathbb{C} \setminus [\alpha, \beta]$.

(b) For each $[g_1; g_2] \in \mathcal{P}(\mathcal{R}_r(\mathbb{C} \setminus [\alpha, \beta])$ let $S_{W,[g_1; g_2]} : \mathbb{C} \setminus [\alpha, \beta] \to \mathbb{C}^{q \times q}$ be defined by

$$
S_{W,[g_1; g_2]} := [\tilde{m}_m W(g_1 \oplus O_{(q-r) \times (q-r)}) + \tilde{m}_m W(g_2 \oplus I_{q-r})]
\times [\tilde{m}_m W(g_1 \oplus O_{(q-r) \times (q-r)}) + \tilde{m}_m W(g_2 \oplus I_{q-r})]^{-1}.
$$

Then $\Sigma_W : (\mathcal{P}(\mathcal{R}_r(\mathbb{C} \setminus [\alpha, \beta]) \to \mathcal{R}_q([\alpha, \beta]; (s_j)^m_{j=0})$ defined by $\Sigma_W([g_1; g_2]) := S_{W,[g_1; g_2]}$ is well defined and bijective.

**Proof** Obviously, $W$ is a unitary $q \times q$ matrix and $U := [u_1, u_2, \ldots, u_r]$ is the left $q \times r$ block of $W$. In view of $\dim \mathcal{R}(\mathcal{D}_m) = r$, we furthermore see that $u_1, u_2, \ldots, u_r$ is an orthonormal basis of $\mathcal{R}(\mathcal{D}_m)$. Due to Lemma 8.5, the mapping $\Gamma_U : (\mathcal{P}(\mathcal{R}_r(\mathbb{C} \setminus [\alpha, \beta])) \to (\beta \mathcal{D}_m)$ defined by $\Gamma_U([f_1; f_2]) := \langle Uf_1 U^*, Uf_2 U^* + \mathbb{P}_{[\mathcal{R}(\mathcal{D}_m)]^\perp} \rangle$ is thus well defined and bijective. By virtue of Remark 14.3, we have $\beta \mathcal{D}_m = \beta \mathcal{D}_m^{[m]}$. Consider now an arbitrary pair $[g_1; g_2] \in \mathcal{P}(\mathcal{R}_r(\mathbb{C} \setminus [\alpha, \beta])$. Then $g_1$ and $g_2$ are $\mathbb{C}^r \times r$-valued functions, which are meromorphic in $\mathbb{C} \setminus [\alpha, \beta]$. Observe that $\tilde{w}_m, \tilde{x}_m, \tilde{y}_m$, and $\tilde{z}_m$, as restrictions of $q \times q$ matrix polynomials, are $\mathbb{C}^q \times q$-valued functions, which are holomorphic in $\mathbb{C} \setminus [\alpha, \beta]$. Consequently, we can easily conclude that

$$
X_{U,[g_1; g_2]} := \tilde{w}_m U g_1 U^* + \tilde{x}_m (U g_2 U^* + \mathbb{P}_{[\mathcal{R}(\mathcal{D}_m)]^\perp})
$$

and

$$
Y_{U,[g_1; g_2]} := \tilde{y}_m U g_1 U^* + \tilde{z}_m (U g_2 U^* + \mathbb{P}_{[\mathcal{R}(\mathcal{D}_m)]^\perp})
$$

are $\mathbb{C}^q \times q$-valued functions which are meromorphic in $\mathbb{C} \setminus [\alpha, \beta]$ with $\mathcal{P}(X_{U,[g_1; g_2]} \subseteq \mathcal{P}(g_1) \cup \mathcal{P}(g_2)$ and $\mathcal{P}(Y_{U,[g_1; g_2]} \subseteq \mathcal{P}(g_1) \cup \mathcal{P}(g_2)$. Therefore, $\det Y_{U,[g_1; g_2]}$ is a complex-valued function which is meromorphic in $\mathbb{C} \setminus [\alpha, \beta]$ with $\mathcal{P}(\det Y_{U,[g_1; g_2]} \subseteq \mathcal{P}(g_1) \cup \mathcal{P}(g_2)$. Similarly,

$$
T_{W,[g_1; g_2]} := \tilde{m}_m W(g_1 \oplus O_{(q-r) \times (q-r)}) + \tilde{m}_m W(g_2 \oplus I_{q-r})
$$

and

$$
R_{W,[g_1; g_2]} := \tilde{m}_m W(g_1 \oplus O_{(q-r) \times (q-r)}) + \tilde{m}_m W(g_2 \oplus I_{q-r})
$$
are $\mathbb{C}^{q \times q}$-valued functions which are meromorphic in $\mathbb{C} \setminus [\alpha, \beta]$ with $\mathcal{P}(T_{W, [g_1; g_2]}) \subseteq \mathcal{P}(g_1) \cup \mathcal{P}(g_2)$ and $\mathcal{P}(R_{W, [g_1; g_2]}) \subseteq \mathcal{P}(g_1) \cup \mathcal{P}(g_2)$, and det $R_{W, [g_1; g_2]}$ is a complex-valued function which is meromorphic in $\mathbb{C} \setminus [\alpha, \beta]$ with $\mathcal{P}(\det R_{W, [g_1; g_2]}) \subseteq \mathcal{P}(g_1) \cup \mathcal{P}(g_2)$. In view of $[g_1; g_2] \in \mathcal{P}R_r(\mathbb{C} \setminus [\alpha, \beta])$, we have $\Gamma_U([g_1; g_2]) \subseteq \langle \tilde{P}[s_0^{[m]}] \rangle$. Due to Theorem 14.2(a), hence det $Y_{U, [g_1; g_2]}$ does not vanish identically in $\mathbb{C} \setminus [\alpha, \beta]$. Regarding the identity theorem for holomorphic functions, then $\mathcal{N} := \{ \xi \in \mathbb{C} \setminus [\alpha, \beta] : \det Y_{U, [g_1; g_2]}(\xi) = 0\}$ is a discrete subset of $\mathbb{C} \setminus [\alpha, \beta]$. Consequently, $\mathcal{D} := \mathcal{P}(g_1) \cup \mathcal{P}(g_2) \cup \mathcal{N}$ is a discrete subset of $\mathbb{C} \setminus [\alpha, \beta]$, which fulfills $\mathcal{P}(X_{U, [g_1; g_2]}) \cup \mathcal{P}(Y_{U, [g_1; g_2]}) \cup \mathcal{P}(\det Y_{U, [g_1; g_2]}) \subseteq \mathcal{D}$ and $\mathcal{P}(T_{W, [g_1; g_2]}) \cup \mathcal{P}(R_{W, [g_1; g_2]}) \cup \mathcal{P}(\det R_{W, [g_1; g_2]}) \subseteq \mathcal{D}$. In particular, $\mathbb{C} \setminus ([\alpha, \beta] \cup \mathcal{D})$ is non-empty.

Consider now an arbitrary $z \in \mathbb{C} \setminus ([\alpha, \beta] \cup \mathcal{D})$. Then det $Y_{U, [g_1; g_2]}(z) \neq 0$ and, furthermore,

$$X_{U, [g_1; g_2]}(z) = \tilde{\nu}_m(z)[Ug_1(z)U^*] + \tilde{\nu}_m(z)[Ug_2(z)U^* + \mathbb{P}_{[R(\alpha_m)]^1}],$$

$$Y_{U, [g_1; g_2]}(z) = \tilde{\nu}_m(z)[Ug_1(z)U^*] + \tilde{\nu}_m(z)[Ug_2(z)U^* + \mathbb{P}_{[R(\alpha_m)]^1}],$$

$$T_{W, [g_1; g_2]}(z) = \tilde{\nu}_m(z)W[g_1(z) \oplus O_{(q-r) \times (q-r)}] + \tilde{\nu}_m(z)W[g_2(z) \oplus I_{q-r}],$$

and

$$R_{W, [g_1; g_2]}(z) = \tilde{\nu}_m(z)W[g_1(z) \oplus O_{(q-r) \times (q-r)}] + \tilde{\nu}_m(z)W[g_2(z) \oplus I_{q-r}].$$

Consequently, using Lemma E.3(b), we obtain det $R_{W, [g_1; g_2]}(z) \neq 0$ and

$$[X_{U, [g_1; g_2]}(z)][Y_{U, [g_1; g_2]}(z)]^{-1} = [T_{W, [g_1; g_2]}(z)][R_{W, [g_1; g_2]}(z)]^{-1}.$$

In particular, det $R_{W, [g_1; g_2]}$ does not vanish identically in $\mathbb{C} \setminus [\alpha, \beta]$, showing (a).

Therefore, $S_{W, [g_1; g_2]} = T_{W, [g_1; g_2]}R_{W, [g_1; g_2]}^{-1}$ is a $\mathbb{C}^{q \times q}$-valued function which is meromorphic in $\mathbb{C} \setminus [\alpha, \beta]$ and holomorphic at $z$ with

$$S_{W, [g_1; g_2]}(z) = [X_{U, [g_1; g_2]}(z)][Y_{U, [g_1; g_2]}(z)]^{-1}$$

$$= (\tilde{\nu}_m(z)[Ug_1(z)U^*] + \tilde{\nu}_m(z)[Ug_2(z)U^* + \mathbb{P}_{[R(\alpha_m)]^1}])$$

$$\times (\tilde{\nu}_m(z)[Ug_1(z)U^*] + \tilde{\nu}_m(z)[Ug_2(z)U^* + \mathbb{P}_{[R(\alpha_m)]^1}])^{-1}. \quad (14.2)$$

Since (14.2) holds true for all $z \in \mathbb{C} \setminus ([\alpha, \beta] \cup \mathcal{D})$ and the set $\mathcal{D}$ is discrete, we can conclude from the identity theorem for holomorphic functions that

$$S_{W, [g_1; g_2]} = [\tilde{\nu}_m(Ug_1U^*) + \tilde{\nu}_m(Ug_2U^* + \mathbb{P}_{[R(\alpha_m)]^1})]$$

$$\times [\tilde{\nu}_m(Ug_1U^*) + \tilde{\nu}_m(Ug_2U^* + \mathbb{P}_{[R(\alpha_m)]^1})]^{-1}. \quad (14.2)$$

Regarding Theorem 14.2(a), for each pair $[P, Q] \in \tilde{P}[\alpha_m] = \tilde{P}[s_0^{[m]}]$, we can consider the function $F_{[P, Q]} := (\tilde{\nu}_mP + \tilde{\nu}_mQ)(\tilde{\nu}_mP + \tilde{\nu}_mQ)^{-1}$. In view of Definition 7.11, it is readily checked that, given two arbitrary pairs $[P_1; Q_1], [P_2; Q_2] \in \tilde{P}[\alpha_m]$, the equivalence $[P_2; Q_2] \sim [P_2; Q_2]$ implies $F_{[P_1; Q_1]} = F_{[P_2; Q_2]}$. According to
Theorem 14.2, thus the mapping  \( \Pi : \langle \dot{\mathcal{P}}[\mathcal{D}_m] \rangle \to \mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^m] \) defined by \( \Pi((\{P; Q\})) := F_{\{P; Q\}} \) is well defined and bijective. By virtue of that we have already shown, we get

\[
\Sigma_W((g_1; g_2)) = S_{W,\{g_1; g_2\}} = F_{Ug_1U^*; Ug_2U^* + P_{[\mathcal{R}(\mathcal{D}_m)]}} = \Pi((\{Ug_1U^*; Ug_2U^* + P_{[\mathcal{R}(\mathcal{D}_m)]}\})) = \Pi(F_{\{g_1; g_2\}}).
\]

Since this holds true for all pairs \([g_1; g_2] \in \mathcal{P} \mathcal{R}_r(\mathbb{C}\setminus[\alpha, \beta])\), we thus verified that \( \Sigma_W = \Pi \circ \Gamma_U \) is well defined and bijective, i.e., (b) holds true.

Now we treat the completely degenerate case \( r = 0 \). We will see in particular that in this situation the solution is unique.

**Theorem 14.6** Let \( m \in \mathbb{N}_0 \) and let \((s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^*\) with sequence of \([\alpha, \beta]\)-interval lengths \((\delta_j)_{j=0}^m\). Then \( \mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^m] \) consists of exactly one element if and only if \( \delta_m = O_{\mathcal{Q} \times q} \). In this case, \( \mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^m] = \{\tilde{\mathbb{I}}_m \tilde{\mathbb{I}}_m^{-1}\}, \) where \([\tilde{\mathbb{I}}_m \tilde{\mathbb{I}}_m^{-1}]\) denotes the \( q \times q \) block representation of the restriction of \( \tilde{\mathcal{D}}_m \) onto \( \mathbb{C}\setminus[\alpha, \beta] \).

**Proof** Assume \( \delta_m \neq O_{\mathcal{Q} \times q} \). For each \( \ell \in \{0, 1\} \) let \((s_{\ell,j})_{j=0}^{m+1}\) be given by

\[
s_{\ell,j} := \begin{cases} 
  s_j, & \text{if } 0 \leq j \leq m, \\
  \alpha_m + \ell \delta_m, & \text{if } j = m + 1.
\end{cases}
\]

Due to Corollary 3.36, then \(((s_{0,j})_{j=0}^{m+1}, (s_{1,j})_{j=0}^{m+1}) \subseteq \mathcal{F}_{q,m+1,\alpha,\beta}^*\). By virtue of Proposition 5.7, thus \( \mathcal{R}_q[[\alpha, \beta]; (s_{\ell,j})_{j=0}^{m+1}] \) is non-empty for each \( \ell \in \{0, 1\} \). Consequently, we can choose, for each \( \ell \in \{0, 1\} \), a function \( F_{\ell} \in \mathcal{R}_q[[\alpha, \beta]; (s_{\ell,j})_{j=0}^{m+1}] \). Observe that, for each \( \ell \in \{0, 1\} \), the \( \mathcal{R}[\alpha, \beta] \)-measure \( \sigma_{\ell} \) of \( F_{\ell} \) belongs to \( \mathcal{M}_{q,m+1}[\alpha, \beta]; (s_{\ell,j})_{j=0}^{m+1} \), implying \( \int_{[\alpha, \beta]} \chi_{x=1}^m \sigma_{\ell}(dx) = s_{\ell,m+1} \). Because of \( s_{1, m+1} - s_{0,m+1} = \delta_m \neq O_{\mathcal{Q} \times q} \), we have \( \sigma_0 \neq \sigma_1 \) and hence \( F_0 \neq F_1 \). Since the functions \( F_0 \) and \( F_1 \) both belong to \( \mathcal{R}_q[[\alpha, \beta]; (s_{j})_{j=0}^{m}] \), therefore this set consists of at least two elements. Thus, we can reversely conclude that if the set \( \mathcal{R}_q[[\alpha, \beta]; (s_{j})_{j=0}^{m}] \) consists of exactly one element, then necessarily \( \delta_m = O_{\mathcal{Q} \times q} \) follows.

Assume \( \delta_m = O_{\mathcal{Q} \times q} \). By virtue of Lemma 8.5, then \( \langle \dot{\mathcal{P}}[\mathcal{D}_m] \rangle \) consists of exactly one element, namely the equivalence class \( ([G_1; G_2]) \) of the pair \([G_1; G_2]\) built from the functions \( G_1, G_2 : \mathbb{C}\setminus[\alpha, \beta] \to \mathbb{C}^{q \times q} \) defined by \( G_1(z) := O_{\mathcal{Q} \times q} \) and \( G_2(z) := I_q \).

Due to Remark 14.3, we have \( \dot{\mathcal{P}}[\mathcal{D}_m] = \dot{\mathcal{P}}[\mathcal{D}_0^m] \). Consequently,

\[
\langle \dot{\mathcal{P}}[\mathcal{D}_0^m] \rangle = \langle \dot{\mathcal{P}}[\mathcal{D}_m] \rangle = \{([G_1; G_2])\}
\]

follows. Because of Proposition 5.7, the set \( \mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^m] \) is non-empty. Consider an arbitrary \( F \in \mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^m] \). Taking into account (14.3), we can infer from Theorem 14.2 then

\[
F = (\tilde{\mathbb{I}}_m G_1 + \tilde{\mathbb{I}}_m G_2)(\tilde{\mathbb{I}}_m G_1 + \tilde{\mathbb{I}}_m G_2)^{-1} = \tilde{x}_m \tilde{x}_m^{-1}.
\]
implying \( R_q([\alpha, \beta]; (s_j)^m) = \{ \bar{m}_3 \bar{m}_4 \} \). In particular, \( R_q([\alpha, \beta]; (s_j)^m) \) consists of exactly one element.

In view of Propositions 5.5 and 3.29 and Remark 3.26, we obtain from Theorem 14.6 immediately the following two results:

**Corollary 14.7** Let \( n \in \mathbb{N} \) and let \((s_j)^{2n+1} = \in \mathcal{F}_{q, 2n+1, \alpha, \beta} \). Then 
\( M_{q, 2n+1}^\{[\alpha, \beta]; (s_j)^{2n+1} = \} \) consists of exactly one element if and only if \( R(L_{\alpha, n, \bullet}) \cap R(L_{\bullet, n, \beta}) = \{ O_q \times 1 \} \).

**Corollary 14.8** Let \( n \in \mathbb{N} \) and let \((s_j)^{2n} = \in \mathcal{F}_{q, 2n, \alpha, \beta} \). Then 
\( M_{q, 2n}^\{[\alpha, \beta]; (s_j)^{2n} = \} \) consists of exactly one element if and only if \( R(L_n) \cap R(L_{\alpha, n-1, \beta}) = \{ O_q \times 1 \} \).

We continue with a necessary condition for unique solvability of Problem MPI([\alpha, \beta]; (s_j)^m) of \( m \in \mathbb{N} \) and let \((s_j)^m_\bullet \) be a sequence of complex \( q \times q \) matrices such that 
\( M_{q, m}^\{[\alpha, \beta]; (s_j)^m_\bullet = \} \) consists of exactly one element. Then \( \mathfrak{A}_m + \text{rank} \mathfrak{B}_m \leq q \).

**Proof** Due to Theorem 3.5, we have \((s_j)^m_\bullet = \in \mathcal{F}_{q, m, \alpha, \beta} \). In view of Proposition 5.5, we obtain from Theorem 14.6 thus \( \mathfrak{d}_m = O_q \times q \), i.e. \( \text{rank} \mathfrak{d}_m = 0 \). Since \( \text{rank} \mathfrak{d}_m \leq q \) holds true, we can infer by virtue of Corollary 3.31 then \( \text{rank} \mathfrak{A}_m + \text{rank} \mathfrak{B}_m \leq q \).

We are now going to factorize \( \mathcal{D}_m \) in a way alternative to Notation 14.1. In a first step, we derive by virtue of Lemma 10.17 a connection between \( \mathcal{D}_m \) and \( \mathcal{D}_{m-1} \), which against the background of Theorem 14.2 correlates the solution sets: 
\( R_q([\alpha, \beta]; (s_j)^m) \subseteq R_q([\alpha, \beta]; (s_j)^{m-1}) \). To that end, we make use of the sequences \((a_j^{[k]})_{j=0}^{k-1} \) and \((b_j^{[k]})_{j=0}^{k-1} \) given by \( a_j^{[k]} := -\alpha s_j^{[k]} + s_j^{[k]+1} \) and \( b_j^{[k]} := \beta s_j^{[k]} - s_j^{[k]+1} \), resp., i.e., the sequences built from the \( k \)-th \( \mathcal{F}_{\alpha, \beta} \)-transform \((s_j^{[k]})_{j=0}^{k-1} \) according to Notation 3.2:

**Lemma 14.10** Suppose \( \kappa \geq 1 \). Let \((s_j^{[\kappa]})_{j=0}^{\kappa} = \in \mathcal{F}_{q, \kappa, \alpha, \beta} \). Then 
\( \mathcal{D}_m = \mathcal{D}_{m-1} \tilde{U} a_0^{[m-1]} \alpha_0^{[m-1]} \) for all \( m \in \mathbb{Z}_{1, \kappa} \).

**Proof** Consider an arbitrary \( m \in \mathbb{Z}_{1, \kappa} \). In view of Notation 14.1 it is sufficient to verify the identity \( \tilde{V}_{A, M} \tilde{V}_D = \tilde{V}_M \tilde{U}_{A, M} \) with \( A := a_0^{[\kappa-1]}, M := s_0^{[\kappa-1]} \), and \( D := s_0^{[m]} \).

Due to Proposition 3.59, the sequence \((s_j^{[m-1]})_{j=0}^{\kappa-m+1} \) belongs to \( \mathcal{F}_{q, \kappa-m+1, \alpha, \beta} \). Using Remarks 3.12 and 3.13, we can hence infer \( A, M \in \mathcal{C}_H^{q \times q} \) and \( R(A) \subseteq R(M) \). Let \( B := \delta M - A \). Because of Remark 3.3, then \( B = \delta s_0^{[m-1]} - a_0^{[m-1]} = b_0^{[m-1]} \).

According to Definitions 3.22 and 3.52, denote by \((a_j^{[m-1]})_{j=0}^{\kappa-m+1} \) the sequence of \([\alpha, \beta]-\text{interval lengths} \) associated with \((s_j^{[m-1]})_{j=0}^{\kappa-m+1} \) and by \((t_j)_{j=0}^{\kappa-m+1} \) the \( \mathcal{F}_{\alpha, \beta} \)-transform of \((s_j^{[m-1]})_{j=0}^{\kappa-m+1} \), resp. Remark 3.11 yields \((a_j^{[m-1]})_{j=0}^{\kappa-m+1} \in \mathcal{D}_{q \times q, \kappa-m+1} \). By virtue of Remark 3.24 and Lemma 3.54, we can hence infer \( A M^\dagger B = a_0^{[m-1]} = \).
Proposition 14.11 Let \((s_j)_{j=0}^m \in \mathcal{F}_{q,k,\alpha,\beta}\) and let \(m \in \mathbb{Z}_{0,k}\). Then \(\hat{\mathcal{D}}_m = U_0U_1 \cdots U_m\), where \(U_\ell := \hat{U}_{a_\ell}^{(\epsilon - 1),s_\ell}^{(\ell - 1)}\) for all \(\ell \in \mathbb{Z}_{1,m}\) and \(U_0 := \hat{V}_{s_0}\).

Lemma 14.12 Suppose \(\kappa \geq 1\). Let \((s_j)_{j=0}^{m} \in \mathcal{F}_{q,k,\alpha,\beta}\) with \(\mathcal{F}_{\alpha,\beta}\)-parameter sequence \((f_j)_{j=0}^{m}\) and sequence of \([\alpha, \beta]\)-interval lengths \((d_j)_{j=0}^{m}\). Let \(k \in \mathbb{Z}_{0,k-1}\) and let \(z \in \mathbb{C}\).

Then \(\hat{U}_{a_0^{(k)},s_0^{(k)}}(z) = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}\), where

\[
\begin{align*}
U_{11} &= \delta_k[(\beta - z)f_{2k+1}f_{2k+1}\delta_k^{+}f_{2k+2} + (z - \alpha)(I_q - f_{2k+1}^{+}f_{2k+1})\delta_k^{+}f_{2k+1}], \\
U_{12} &= \delta_k^{+}f_{2k+2}, \\
U_{21} &= -[(\beta - z)(z - \alpha)\delta^{-1}k\delta_k^{+}f_{2k+1}\delta_k^{+}], \\
U_{22} &= (\beta - z)\delta[(I_q - \delta_k^{+}f_{2k+1}) + \delta_k^{+}f_{2k+1}].
\end{align*}
\]

Proof Let \(A := a_0^{(k)}\), let \(M := s_0^{(k)}\), and let \(B := \delta M - A\). Because of Remark 3.3, then \(B = \delta s_0^{(k)} - a_0^{(k)} = b_0^{(k)}\). Corollary 3.61 shows \(M = \delta^{k-1}a_k\). Furthermore, Proposition 3.60 yields \(A = a_0^{(k)} = \delta^{k}f_{2k+1}\) and \(B = \delta^{k}f_{2k+2}\). Let \(D := AM^\dagger B\) denote by \((d_j)_{j=0}^{m}\) the sequence of \([\alpha, \beta]\)-interval lengths associated with \((s_j)_{j=0}^{m}\).

Due to Proposition 3.59 we have \((s_j)_{j=0}^{m} \in \mathcal{F}_{q,k-1,\alpha,\beta}\). Because of Remark 3.11, in particular \((s_j)_{j=0}^{m} \in \mathcal{D}_{q \times q,k-1}\). Consequently, from Remark 3.24 we conclude \(\delta_k^{(k)} = D\). Proposition 3.62, then implies \(D = \delta^{k}d_{k+1}\). Using Remarks A.15 and A.13 and taking into account \(\delta > 0\), Notation 10.15, and (9.2), the assertion follows.

15 On the Sets \(\mathcal{R}_q([\alpha, \beta]; (s_j)_{j=0}^{m+1})\) in the Case of \([\alpha, \beta]\)-completely Degenerate Extensions of a Sequence \((s_j)_{j=0}^{m} \in \mathcal{F}_{q,m,\alpha,\beta}\)

In this section, we study \([\alpha, \beta]\)-completely Degenerate extensions of a sequence \((s_j)_{j=0}^{m} \in \mathcal{F}_{q,m,\alpha,\beta}\). First we recall the notion of \([\alpha, \beta]\)-completely Degenerate sequences belonging to \(\mathcal{F}_{q,m,\alpha,\beta}\) and a characterization of this class of sequences.

Definition 15.1 ([27, Def. 10.24]) Let \(\ell \in \mathbb{N}_0\) and let \((s_j)_{j=0}^{\ell} \in \mathcal{F}_{q,\ell,\alpha,\beta}\) with sequence of \([\alpha, \beta]\)-interval lengths \((d_j)_{j=0}^{\ell}\) given in Definition 3.22. Then \((s_j)_{j=0}^{\ell}\) is called \([\alpha, \beta]\)-completely degenerate if \(d_\ell = O_{q \times q}\). We denote by \(\mathcal{F}_{q,\ell,\alpha,\beta}\) the set of all sequences \((s_j)_{j=0}^{\ell} \in \mathcal{F}_{q,\ell,\alpha,\beta}\) which are \([\alpha, \beta]\)-completely Degenerate.
Proposition 15.2 (cf. [28, Prop. 6.38]) Let \( \ell \in \mathbb{N} \) and let \((s_j)_{j=0}^\ell \in \mathcal{F}_{q,\ell,\alpha,\beta}^\infty \) with \([\alpha, \beta]\)-interval parameter sequence \((\epsilon_j)_{j=0}^\ell \) given in Definition 3.43. Then \((s_j)_{j=0}^\ell \) is \([\alpha, \beta]\)-completely Degenerate if and only if \( \epsilon_\ell^2 = \epsilon_\ell \).

Observe that in the situation of Proposition 15.2, due to \( \epsilon_\ell \gtrless O_{q \times q} \), we have \( \epsilon_\ell^* = \epsilon_\ell \) and thus the condition \( \epsilon_\ell^2 = \epsilon_\ell \) is equivalent to \( \epsilon_\ell \) being a transformation matrix corresponding to an orthogonal projection, i.e., \( \epsilon_\ell = \mathbb{P}_{R(\epsilon_\ell)} \).

Against the background of Proposition 15.2, we are looking now for a description of the set

\[
\{s_{m+1} \in \mathbb{C}^{q \times q} : (s_j)_{j=0}^{m+1} \in \mathcal{F}_{q,m+1,\alpha,\beta} \}.
\]

We will show that this set stands in a bijective correspondence to the set of all linear subspaces of \( R(\delta_m) \).

**Notation 15.3** Let \( m \in \mathbb{N}_0 \), let \((s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^\infty \) with sequence of \([\alpha, \beta]\)-interval lengths \((\delta_j)_{j=0}^m \), and let \( \mathcal{U} \) be a linear subspace of \( R(\delta_m) \). Then let \( s_m, \mathcal{U} := b_m - \delta_m^{1/2} \mathbb{P}_U \delta_m^{1/2} \) if \( m \) is even, and \( s_m, \mathcal{U} := a_m + \delta_m^{1/2} \mathbb{P}_U \delta_m^{1/2} \) if \( m \) is odd.

**Example 15.4** Let \( m \in \mathbb{N}_0 \) and let \((s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^\infty \) with sequence of \([\alpha, \beta]\)-interval lengths \((\delta_j)_{j=0}^m \). Let \( \mathcal{U}_0 := \{O_{q \times q}\} \) and let \( \mathcal{U}_1 := R(\delta_m) \). Then \( s_m, \mathcal{U}_0 = b_m \) and \( s_m, \mathcal{U}_1 = a_m \) if \( m \) is even, and \( s_m, \mathcal{U}_0 = a_m \) and \( s_m, \mathcal{U}_1 = b_m \) if \( m \) is odd.

Indeed, we have \( \mathbb{P}_U \mathcal{U}_0 = O_{q \times q} \) and, in view of \( R(\delta_m^{1/2}) = R(\delta_m) \), furthermore \( \mathbb{P}_U \mathcal{U}_1 = \mathbb{P}_{R(\delta_m^{1/2})} \). Consequently, \( \delta_m^{1/2} \mathbb{P}_U \mathcal{U}_0 \delta_m^{1/2} = O_{q \times q} \) and \( \delta_m^{1/2} \mathbb{P}_U \mathcal{U}_1 \delta_m^{1/2} = \delta_m \). Since, according to Definition 3.22, we have \( \delta_m = b_m - a_m \), the assertions follow by virtue of Notation 15.3.

**Proposition 15.5** Let \( m \in \mathbb{N}_0 \) and let \((s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^\infty \). Then:

(a) Let \( \mathcal{U} \) be a linear subspace of \( R(\delta_m) \) and let \( s_{m+1} := s_m, \mathcal{U} \). Then \((s_j)_{j=0}^{m+1} \in \mathcal{F}_{q,m+1,\alpha,\beta}^\infty \). Furthermore,

\[
f_{2m+1} = \delta_m^{1/2}(I_q - \mathbb{P}_U) \delta_m^{1/2}, \quad f_{2m+2} = \delta_m^{1/2} \mathbb{P}_U \delta_m^{1/2}, \quad \text{and} \quad \epsilon_{m+1} = \mathbb{P}_U.
\]

(b) Let \( s_{m+1} \in \mathbb{C}^{q \times q} \) be such that \((s_j)_{j=0}^{m+1} \in \mathcal{F}_{q,m+1,\alpha,\beta}^\infty \). Then there exists a linear subspace \( \mathcal{U} \) of \( R(\delta_m) \) such that \( s_{m+1} = s_m, \mathcal{U} \), namely \( \mathcal{U} = R(\epsilon_{m+1}) \).

(c) Let \( \mathcal{U} \) and \( \mathcal{V} \) be linear subspaces of \( R(\delta_m) \). Then \( \mathcal{U} = \mathcal{V} \) if and only if \( s_m, \mathcal{U} = s_m, \mathcal{V} \).

**Proof** First observe that \( \delta_m \) belongs to \( \mathbb{C}^{q \times q} \), due to Proposition 3.28. Let \( D := \delta_m^{1/2} \).

(a) By virtue of Definitions 3.37 and 3.25 and Notation 15.3, we have

\[
f_{2m+2} = f_{4n+4} = \mathfrak{A}_{2n+2} = \mathfrak{B}_{2n+1} = b_m - s_{m+1} = D \mathbb{P}_U D
\]

in the case \( m = 2n \) for some \( n \in \mathbb{N}_0 \), and

\[
f_{2m+2} = f_{4n+4} = \mathfrak{A}_{2n+2} = \mathfrak{B}_{2n+1} = s_{m+1} - a_m = D \mathbb{P}_U D
\]
in the case \( m = 2n + 1 \) for some \( n \in \mathbb{N}_0 \). Using Remark 3.40, we can infer then
\[
\hat{f}_{2m+1} = \vartheta_m - \hat{f}_{2m+2} = DD - D\mathbb{P}_\mathcal{U}D = D(I_q - \mathbb{P}_\mathcal{U})D.
\]
Because of \( O_{q \times q} \preceq \mathbb{P}_\mathcal{U} \preceq I_q \) and Remark A.25, consequently the matrices \( \hat{f}_{2m+2} \) and \( \hat{f}_{2m+1} \) are both Non-negative Hermitian. From Proposition 3.39, we can conclude now \( (s_j)_{j=0}^{m+1} \in \mathcal{F}_{q,m+1,\alpha,\beta}^\preceq \). According to Definition 3.43, then \( \varepsilon_{m+1} = D^\dagger \hat{f}_{2m+2}D^\dagger = D^\dagger D\mathbb{P}_\mathcal{U}DD^\dagger \).

Because of \( \mathcal{R}(\vartheta_m) = \mathcal{R}(D) \), we have \( \mathcal{U} \subseteq \mathcal{R}(D) \). Remarks A.11 and A.20 then yield \( DD^\dagger \mathbb{P}_\mathcal{U} = \mathbb{P}_\mathcal{U} \). Furthermore \( D^\ast \mathbb{P}_\mathcal{U} = D \) implies \( D^\dagger D = DD^\dagger \), by virtue of Remark A.18. Hence, \( D^\dagger D\mathbb{P}_\mathcal{U} = \mathbb{P}_\mathcal{U} \) follows. Taking account \( \mathbb{P}_\mathcal{U}^\ast = \mathbb{P}_\mathcal{U} \) and (3.3), we furthermore obtain \( \mathbb{P}_\mathcal{U} = (DD^\dagger \mathbb{P}_\mathcal{U})^\ast = \mathbb{P}_\mathcal{U}DD^\dagger \). Consequently, we conclude \( \varepsilon_{m+1} = D^\dagger D\mathbb{P}_\mathcal{U}DD^\dagger = \mathbb{P}_\mathcal{U} \). In view of \( \mathbb{P}_\mathcal{U}^2 = \mathbb{P}_\mathcal{U} \), the application of Proposition 15.2 provides then \( (s_j)_{j=0}^{m+1} \in \mathcal{F}_{q,m+1,\alpha,\beta}^{\preceq,cd} \).

(b) According to Proposition 15.2, we have \( \varepsilon_{m+1}^2 = \varepsilon_{m+1} \). From Proposition 3.47 and (3.6), we get \( \varepsilon_{m+1}^2 = \varepsilon_{m+1} \) and \( O_{q \times q} \preceq \varepsilon_{m+1} \preceq \mathbb{P}_\mathcal{R}(\vartheta_m) \). Consequently, \( \varepsilon_{m+1} = \mathbb{P}_\mathcal{U} \) with \( \mathcal{U} := \mathcal{R}(\varepsilon_{m+1}) \). Furthermore, \( \mathbb{P}_\mathcal{U} \preceq \mathbb{P}_\mathcal{R}(\vartheta_m) \), implying \( \mathcal{U} \subseteq \mathcal{R}(\vartheta_m) \).

Lemma 3.44 yields \( \hat{f}_{2m+2} = D\varepsilon_{m+1}D \). By virtue of Definitions 3.37 and 3.25, we have then
\[
D\mathbb{P}_\mathcal{U}D = \hat{f}_{2m+2} = \hat{f}_{4n+2} = \mathcal{B}_{2n+1} = \mathcal{B}_{m+1} = b_m - s_{m+1}
\]
in the case \( m = 2n \) for some \( n \in \mathbb{N}_0 \), and
\[
D\mathbb{P}_\mathcal{U}D = \hat{f}_{2m+2} = \hat{f}_{4n+2} = \mathcal{A}_{2n+2} = \mathcal{A}_{m+1} = s_{m+1} - a_m
\]
in the case \( m = 2n + 1 \) for some \( n \in \mathbb{N}_0 \). In view of Notation 15.3, hence \( s_{m+1} = s_{m,\mathcal{U}} \) follows.

(c) Obviously, \( \mathcal{U} = \mathcal{V} \) implies \( s_{m,\mathcal{U}} = s_{m,\mathcal{V}} \), according to Notation 15.3. Conversely, suppose \( s_{m,\mathcal{U}} = s_{m,\mathcal{V}} \). From Notation 15.3, then \( D\mathbb{P}_\mathcal{U}D = D\mathbb{P}_\mathcal{V}D \) follows. By the same reasoning as in the proof of part (a), we can infer \( D^\dagger D\mathbb{P}_\mathcal{U}DD^\dagger = \mathbb{P}_\mathcal{U} \) and
\[
D^\dagger D\mathbb{P}_\mathcal{V}DD^\dagger = \mathbb{P}_\mathcal{V}.
\]
Consequently, \( \mathbb{P}_\mathcal{U} = \mathbb{P}_\mathcal{V} \), implying \( \mathcal{U} = \mathcal{V} \).

\[\square\]

**Notation 15.6** Let \( m \in \mathbb{N}_0 \), let \( (s_j)_{j=0}^{m} \in \mathcal{F}_{q,m,\alpha,\beta}^{\preceq} \) with sequence of \([\alpha, \beta]-\)interval lengths \((\vartheta_j)_{j=0}^{m}\), and let \( \mathcal{U} \) be a linear subspace of \( \mathcal{R}(\vartheta_m) \). Then let \( X_{m,\mathcal{U}} \), \( Y_{m,\mathcal{U}} : \mathbb{C} \setminus [\alpha, \beta] \rightarrow \mathbb{C}^{q \times q} \) be defined by
\[
X_{m,\mathcal{U}}(z) := \delta^{m-1} \mathbb{P}_\mathcal{U} \delta_m^{1/2}, \quad Y_{m,\mathcal{U}}(z) := (\beta - z)[I_q - (\delta_m^{1/2})^\dagger \mathbb{P}_\mathcal{U} \delta_m^{1/2}].
\]

**Example 15.7** Let \( m \in \mathbb{N}_0 \) and let \( (s_j)_{j=0}^{m} \in \mathcal{F}_{q,m,\alpha,\beta}^{\preceq} \) with sequence of \([\alpha, \beta]-\)interval lengths \((\vartheta_j)_{j=0}^{m}\). Let \( \mathcal{U}_0 := \{O_{q \times 1}\} \) and let \( \mathcal{U}_i := \mathcal{R}(\vartheta_m) \) for all \( z \in \mathbb{C} \setminus [\alpha, \beta] \), then
\[
X_{m,\mathcal{U}_0}(z) = O_{q \times q} \quad \text{and} \quad Y_{m,\mathcal{U}_0}(z) := (\beta - z)I_q.
\]

Indeed, as in the proof of Example 15.4, we obtain
\[
\delta_m^{1/2} \mathbb{P}_{\mathcal{U}_0} \delta_m^{1/2} = O_{q \times q} \quad \text{and} \quad \delta_m^{1/2} \mathbb{P}_{\mathcal{U}_0} \delta_m^{1/2} = \delta_m.
\]
Taking into account \( \delta_m^{1/2} \mathcal{U}_0^{1/2} = (\delta_m^{1/2})^\dagger \), then \( (\delta_m^{1/2})^\dagger \mathbb{P}_{\mathcal{U}_0} \delta_m^{1/2} = O_{q \times q} \) and \( (\delta_m^{1/2})^\dagger \mathbb{P}_{\mathcal{U}_0} \delta_m^{1/2} = \delta_m \). Hence, \( I_q - (\delta_m^{1/2})^\dagger \mathbb{P}_{\mathcal{U}_0} \delta_m^{1/2} = I_q \) and, in
view of Remark A.18, furthermore $I_q-\left(\partial_m^{1/2}\right)^{\top}\mathbb{P}_U\partial_m^{1/2} = \mathbb{P}_N(\partial_m)$. Now, the assertions follow by virtue of Notation 15.6.

We now determine the subspaces studied in Proposition 7.9 associated with the particular pairs $[X_{m,U}; Y_{m,U}]$. It turns out that these pairs are degenerate in the sense that the value of the Non-negative integer $\text{rk}([X_{m,U}(z); Y_{m,U}(z)])$ given in (D.1), which can be seen as an analogue of the matrix rank for pairs of matrices (cf. Remark D.3), is zero:

**Remark 15.8** Let $m \in \mathbb{N}_0$, let $(s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^\varphi$ with sequence of $[\alpha, \beta]$-interval lengths $(\delta_j)_{j=0}^m$ and, let $U$ be a linear subspace of $\mathcal{R}(\partial_m)$. In view of Notations 15.6 and 8.1 and $\mathcal{R}(\partial_m) = \mathcal{R}(\partial_m^{1/2})$ as well as Remark 8.7, then $[X_{m,U}; Y_{m,U}] \in \tilde{\mathcal{P}}[\partial_m]$ and

$$\mathcal{R}(Y_{m,U}(z)) = \mathcal{N}(\partial_m^{1/2}), \quad \mathcal{R}(X_{m,U}(z)) = \mathcal{N}(\partial_m^{1/2}, \mathbb{P}_U),$$

$$\mathcal{R}(Y_{m,U}(z)) = \mathcal{N}(\partial_m^{1/2}), \quad \mathcal{R}(X_{m,U}(z)) = \mathcal{N}(\partial_m^{1/2}, \mathbb{P}_U),$$

and $\text{rk}([X_{m,U}(z); Y_{m,U}(z)]) = 0$ hold true for $z \in \mathbb{C} \setminus [\alpha, \beta]$.

**Proof** In view of Remarks 15.8 and 14.3, this is a consequence of Theorem 14.2(a). \(\Box\)

Lemma 15.9 shows that the following notation is correct.

**Notation 15.10** Let $m \in \mathbb{N}_0$, let $(s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^\varphi$ with sequence of $[\alpha, \beta]$-interval lengths $(\delta_j)_{j=0}^m$ and, let $U$ be a linear subspace of $\mathcal{R}(\partial_m)$. Denote by $\begin{bmatrix} \tilde{\omega}_m & \tilde{\eta}_m & \tilde{\gamma}_m \end{bmatrix}$ the $q \times q$ block representation of the restriction of $\tilde{\mathcal{U}}_m$ onto $\mathbb{C} \setminus [\alpha, \beta]$. Then the function $\det(\tilde{\omega}_m X_{m,U} + \tilde{\eta}_m Y_{m,U})$ does not vanish identically.

**Proof** In view of Remarks 15.8 and 14.3 and Notation 15.10, this is a consequence of Theorem 14.2(a). \(\Box\)

Given a sequence $(s_j)_{j=0}^{m+1} \in \mathcal{F}_{q,m+1,\alpha,\beta}^\varphi$, now we look for the $q \times q$ matrix polynomials in the four $q \times q$ blocks of the $2q \times 2q$ matrix polynomial $\tilde{\mathcal{U}}_{m+1}$ defined in Notation 14.1.
Lemma 15.12 Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{m+1} \in \mathcal{F}_{q,m+1}^{\gamma} \alpha, \beta$ with $\mathcal{F}_{\alpha, \beta}$-parameter sequence $(f_j)_{j=0}^{2m+2}$ and sequence of $[\alpha, \beta]$-interval lengths $(\varnothing_j)_{j=0}^{m+1}$. Denote by 
\[
\begin{bmatrix}
w_m & \varphi_m \\
\eta_m & \beta_m
\end{bmatrix}
\] and 
\[
\begin{bmatrix}
w_{m+1} & \varphi_{m+1} \\
\eta_{m+1} & \beta_{m+1}
\end{bmatrix}
\] the $q \times q$ block representations of $\hat{\mathcal{H}}_m$ and $\hat{\mathcal{H}}_{m+1}$, resp. Setting $P := \mathcal{F}_{2m+1}^{\gamma} \alpha, \beta$, for all $z \in \mathbb{C}$, then
\[
\begin{align*}
w_{m+1}(z) &= \left\{ w_m(z)\delta_m[(\beta - z)P\varphi_m^{\gamma} f_{2m+2} + (z - \alpha)(I_q - P)\delta_m^{\gamma} f_{2m+1}] \\
&\quad - (\beta - z)(z - \alpha)^{-m+1} \varphi_m(z)\delta_m^{\gamma} f_{2m+1} \right\} \delta_{m+1}^{\gamma}, \\
\varphi_{m+1}(z) &= \delta(\delta^{-m+1} w_m(z)\varphi_{2m+2} + (\beta - z)\varphi_m(z)(I_q - \delta_m^{\gamma} \varphi_m + \delta_m^{\gamma} \varphi_{2m+1})), \\
\eta_{m+1}(z) &= \left\{ \eta_m(z)\delta_m[(\beta - z)P\varphi_m^{\gamma} f_{2m+2} + (z - \alpha)(I_q - P)\delta_m^{\gamma} f_{2m+1}] \\
&\quad - (\beta - z)(z - \alpha)^{-m+1} \eta_m(z)\delta_m^{\gamma} f_{2m+1} \right\} \delta_{m+1}^{\gamma},
\end{align*}
\]
and
\[
\beta_{m+1}(z) = \delta(\delta^{-m+1} \eta_m(z)\beta_{2m+2} + (\beta - z)\beta_m(z)(I_q - \delta_m^{\gamma} \eta_m + \delta_m^{\gamma} \beta_{2m+1})).
\]

Proof From Lemma 14.10 we obtain $\hat{\mathcal{H}}_{m+1} = \hat{\mathcal{H}}_m \hat{\mathcal{U}}_{\varnothing_j}^{(m)} \alpha, \beta$. Using the $q \times q$ block representations of $\hat{\mathcal{H}}_{m+1}$ and $\hat{\mathcal{H}}_m$ as well as Lemma 14.12, a straightforward calculation completes the proof. \hfill $\Box$

Now we specify Lemma 15.12 for the case of a sequence $(s_j)_{j=0}^{m+1} \in \mathcal{F}_{q,m+1}^{\gamma, \alpha, \beta}$.

Lemma 15.13 Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{m+1} \in \mathcal{F}_{q,m+1}^{\gamma, \alpha, \beta}$ with $\mathcal{F}_{\alpha, \beta}$-parameter sequence $(f_j)_{j=0}^{2m+2}$ and sequence of $[\alpha, \beta]$-interval lengths $(\varnothing_j)_{j=0}^{m+1}$. Denote by 
\[
\begin{bmatrix}
w_m & \varphi_m \\
\eta_m & \beta_m
\end{bmatrix}
\] and 
\[
\begin{bmatrix}
w_{m+1} & \varphi_{m+1} \\
\eta_{m+1} & \beta_{m+1}
\end{bmatrix}
\] the $q \times q$ block representations of $\hat{\mathcal{H}}_m$ and $\hat{\mathcal{H}}_{m+1}$, resp. For all $z \in \mathbb{C}$, then $w_{m+1}(z) = O_{q \times q}$ and $\varphi_{m+1}(z) = O_{q \times q}$ as well as
\[
\begin{align*}
\varphi_{m+1}(z) &= \delta(\delta^{-m+1} w_m(z)\varphi_{2m+2} + (\beta - z)\varphi_m(z)(I_q - \delta_m^{\gamma} \varphi_m + \delta_m^{\gamma} \varphi_{2m+1})), \\
\eta_{m+1}(z) &= \delta(\delta^{-m+1} \eta_m(z)\beta_{2m+2} + (\beta - z)\beta_m(z)(I_q - \delta_m^{\gamma} \eta_m + \delta_m^{\gamma} \beta_{2m+1})).
\end{align*}
\]

Proof According to Definition 15.1, we have $(s_j)_{j=0}^{m+1} \in \mathcal{F}_{q,m+1}^{\gamma, \alpha, \beta}$ and $\varnothing_{m+1} = O_{q \times q}$. The application of Lemma 15.12 completes the proof. \hfill $\Box$

Let $(s_j)_{j=0}^{m} \in \mathcal{F}_{q,m}^{\gamma, \alpha, \beta}$ with sequence of $[\alpha, \beta]$-interval lengths $(\varnothing_j)_{j=0}^{m}$ and let $U$ be a linear subspace of $R(\varnothing_m)$. Using Notation 15.3 to define $s_{m+1} := s_{m,U}$, we determine now the set $R_q[(\alpha, \beta); (s_j)_{j=0}^{m+1}]$. 


Proposition 15.14 Let \( m \in \mathbb{N}_0 \) and let \( (s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^{\infty} \) with sequence of \([\alpha, \beta]\)-interval lengths \((\delta_j)_{j=0}^m\). Let \( \mathcal{U} \) be a linear subspace of \( \mathcal{R}(\mathcal{O}_m) \) and let \( s_{m+1} := s_{m,\mathcal{U}} \). Then \( \mathcal{R}_q[(\alpha, \beta); (s_j)_{j=0}^{m+1}] = \{S_{m,\mathcal{U}}\} \), where \( S_{m,\mathcal{U}} \) is given via Notation 15.10.

Proof Denote by \((f_j)_{j=0}^{2m+2}\) the \( \mathcal{F}_{\alpha,\beta} \)-parameter sequence of \((s_j)_{j=0}^{m+1}\). Because of Proposition 15.15(a), we have \((s_j)_{j=0}^{m+1} \in \mathcal{F}_{q,m,\alpha,\beta}^{\infty,cd} \) and furthermore \( f_{2m+1} = \delta_{m/2}(I_q - P_{\mathcal{U}})\mathcal{O}_m^{1/2} \) and \( f_{2m+2} = \mathcal{O}_m^{1/2}P_{\mathcal{U}}\mathcal{O}_m^{1/2} \). In particular, \( \mathcal{O}_m^t f_{2m+1} = \mathcal{O}_m^t \mathcal{O}_m^{1/2} - \mathcal{O}_m^t \mathcal{O}_m^{1/2}P_{\mathcal{U}}\mathcal{O}_m^{1/2} \) and, in view of \( \mathcal{O}_m^t \mathcal{O}_m^{1/2} = (\mathcal{O}_m^{1/2})^t \), consequently \((I_q - \mathcal{O}_m^{1/2} + \mathcal{O}_m^t f_{2m+1} = I_q - (\mathcal{O}_m^{1/2})^t P_{\mathcal{U}}\mathcal{O}_m^{1/2} \). By virtue of Notation 15.6, for all \( z \in \mathbb{C} \setminus [\alpha, \beta] \), hence \( \delta_j f_{2m+2} = X_{m,\mathcal{U}}(z) \) and \( \mathcal{O}_m (\beta - z)[(I_q - \mathcal{O}_m^{1/2} + \mathcal{O}_m^t f_{2m+1} = Y_{m,\mathcal{U}}(z) \). Denote by \( \begin{bmatrix} w_m & \tilde{w}_m \\ \tilde{w}_{m+1} & \tilde{w}_{m+1} \end{bmatrix} \) and \( \begin{bmatrix} \tilde{y}_m & \tilde{y}_{m+1} \\ \tilde{y}_{m+1} & \tilde{y}_{m+1} \end{bmatrix} \) the \( q \times q \) block representations of the restrictions of \( \tilde{\mathcal{O}}_m \) and \( \tilde{\mathcal{O}}_{m+1} \), resp., onto \( \mathbb{C} \setminus [\alpha, \beta] \). From Lemma 15.13 we can infer then \( \tilde{y}_{m+1}(z) = \delta_j (\tilde{w}_m(z)X_{m,\mathcal{U}}(z) + \tilde{w}_m(z)Y_{m,\mathcal{U}}(z)) \) and \( \tilde{y}_{m+1}(z) = \delta_j (\tilde{y}_m(z)X_{m,\mathcal{U}}(z) + \tilde{y}_m(z)Y_{m,\mathcal{U}}(z)) \) for all \( z \in \mathbb{C} \setminus [\alpha, \beta] \). In view of \( \delta > 0 \), Lemma 15.9, and Notation 15.10, we see that \( \det \tilde{y}_{m+1} \) does not identically vanish and that \( S_{m,\mathcal{U}} = \tilde{y}_{m+1} \tilde{y}_{m+1}^{-1} \). By virtue of Definition 15.1, we can apply Theorem 14.6 to complete the proof. \( \Box \)

Now we recall some facts and notions from [27].

Proposition 15.15 ([27, Prop. 11.4]) Let \( m \in \mathbb{N}_0 \), let \( (s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^{\infty} \), and let \( s_{m+1} \in \{a_m, b_m\} \). Then \((s_j)_{j=0}^{m+1} \in \mathcal{F}_{q,m,\alpha,\beta}^{\infty,cd} \).

Definition 15.16 (cf. [27, Def. 11.5]) Let \( m \in \mathbb{N}_0 \) and let \((s_j)_{j=0}^m) \in \mathcal{F}_{q,m,\alpha,\beta}^{\infty,cd} \). Let the sequence \((s_j)_{j=0}^\infty \) be recursively defined by \( s_j := a_{j-1} \) (resp. \( s_j := b_{j-1} \)). Then \((s_j)_{j=0}^\infty \) is called the lower (resp. upper) \([\alpha, \beta]\)-completely Degenerate sequence associated with \((s_j)_{j=0}^m \).

Proposition 15.17 (cf. [27, Prop. 211]) Let \( m \in \mathbb{N}_0 \) and let \((s_j)_{j=0}^m) \in \mathcal{F}_{q,m,\alpha,\beta}^{\infty,cd} \). Denote by \((s_j)_{j=0}^\infty \) and \((\tilde{s}_j)_{j=0}^\infty \) the lower and upper \([\alpha, \beta]\)-completely Degenerate sequence associated with \((s_j)_{j=0}^m \) resp. Then the set \( \mathcal{M}_{q,\infty}^iap([\alpha, \beta]; (s_j)_{j=0}^\infty, =) \) contains exactly one element \( s_m \) and the set \( \mathcal{M}_{q,\infty}^iap([\alpha, \beta]; (\tilde{s}_j)_{j=0}^\infty, =) \) contains exactly one element \( \tilde{s}_m \).

Definition 15.18 ([27, Def. 12.4]) Let \( m \in \mathbb{N}_0 \) and let \( (s_j)_{j=0}^m) \in \mathcal{F}_{q,m,\alpha,\beta}^{\infty,cd} \). Then the Non-negative Hermitian \( q \times q \) measure \( s_m \) (resp. \( \tilde{s}_m \)) is called the lower (resp. upper) CD-measure associated with \((s_j)_{j=0}^m \) and \([\alpha, \beta]\).

Now we are interested in the \([\alpha, \beta]\)-Stieltjes Transforms of \( s_m \) and \( \tilde{s}_m \), resp.

Definition 15.19 Let \( m \in \mathbb{N}_0 \) and let \((s_j)_{j=0}^m) \in \mathcal{F}_{q,m,\alpha,\beta}^{\infty,cd} \). Denote by \( s_m^\alpha \) (resp. \( \tilde{s}_m^\alpha \)) the lower (resp. upper) CD-measure associated with \((s_j)_{j=0}^m \) and \([\alpha, \beta]\). Let \( S_m \) be the \([\alpha, \beta]\)-Stieltjes Transform of \( s_m \) and let \( \tilde{S}_m \) be the \([\alpha, \beta]\)-Stieltjes Transform of \( \tilde{s}_m \). Then we call \( S_m \) (resp. \( \tilde{S}_m \)) the lower (resp. upper) \( R[\alpha, \beta]\)-function associated with \((s_j)_{j=0}^m \).
In Theorem 14.2, we obtained a complete description of the set $R_q[[\alpha, \beta]; (s_j)_j^{m=0}]$ of $[[\alpha, \beta]]$-Stieltjes Transforms of measures belonging to $M^{\hat{\infty}}[[\alpha, \beta]; (s_j)_j^{m=0}, =]$. Now we are interested in the position of $S_m$ and $\overline{S}_m$ in the set $R_q[[\alpha, \beta]; (s_j)_j^{m=0}, =]$. In particular, we determine the equivalence classes $\Gamma_m \in \langle \hat{P}[s_0^{[m]}] \rangle$ and $\overline{\Gamma}_m \in \langle \hat{P}[s_0^{[m]}] \rangle$, which correspond to $S_m$ and $\overline{S}_m$, resp., according to Theorem 14.2(b). It can be expected that these equivalence classes possess certain extremal properties within the set $\langle \hat{P}[s_0^{[m]}] \rangle$. The preceding considerations lead us quickly to explicit expressions for $S_m$ and $\overline{S}_m$.

**Proposition 15.20** Let $m \in \mathbb{N}_0$ and let $(s_j)_j^{m=0} \in X_q,m,\alpha,\beta$ with sequence of $[[\alpha, \beta]]$-interval lengths $(s_j)_j^{m=0} = 0$. Let $U_0 := \{O_q \times 1\}$ and let $U_1 := R_m$. Then $S_m = S_{m,U_0}$ and $\overline{S}_m = S_{m,U_0}$ if $m$ is even, and $S_m = S_{m,U_0}$ and $\overline{S}_m = S_{m,U_1}$ if $m$ is odd. In particular, $\Gamma_m = \{(X_{m,U_0}, Y_{m,U_0})\}$ and $\overline{\Gamma}_m = \{(X_{m,U_0}, Y_{m,U_0})\}$ if $m$ is even, and $\Gamma_m = \{(X_{m,U_0}, Y_{m,U_1})\}$ and $\overline{\Gamma}_m = \{(X_{m,U_1}, Y_{m,U_1})\}$ if $m$ is odd.

**Proof** Denote by $(s_j)_j^{m=0}$ and $(\overline{s}_j)_j^{m=0}$ the lower and upper $[[\alpha, \beta]]$-completely Degenerate sequence associated with $(s_j)_j^{m=0}$, resp. From Proposition 15.17 we can conclude $\sigma_m \in M_q,m+1[[\alpha, \beta]; (s_j)_j^{m=0}, =]$ and $\overline{\sigma}_m \in M_q,m+1[[\alpha, \beta]; (s_j)_j^{m=0}, =]$. Remark 5.8 then shows $S_m \in R_q[[\alpha, \beta]; (s_j)_j^{m+1}, =]$ and $\overline{S}_m \in R_q[[\alpha, \beta]; (s_j)_j^{m+1}, =]$.

First consider the case $m = 2n$ with some $n \in \mathbb{N}_0$. Because of Example 15.4, we have then $s_{m,U_0} = b_m$ and $s_{m,U_1} = a_m$. In view of Definition 15.16, we can thus apply Proposition 15.14 to the sequences $(s_j)_j^{m+1} = 0$ and $(\overline{s}_j)_j^{m+1} = 0$, resp., to obtain $R_q[[\alpha, \beta]; (s_j)_j^{m+1}, =] = \{S_{m,U_0}\}$ and $R_q[[\alpha, \beta]; (s_j)_j^{m+1}, =] = \{S_{m,U_1}\}$. Thus, $S_m = S_{m,U_0}$ and $\overline{S}_m = S_{m,U_0}$ follow. Regarding Lemma 15.11, Remarks 15.8 and 14.3, and Notation 15.10, we can infer from Theorem 14.2(b) that $\Gamma_m = \{(X_{m,U_0}, Y_{m,U_0})\}$ and $\overline{\Gamma}_m = \{(X_{m,U_1}, Y_{m,U_1})\}$.

Now consider the case $m = 2n + 1$ with some $n \in \mathbb{N}_0$. Because of Example 15.4, we have then $s_{m,U_0} = a_m$ and $s_{m,U_1} = b_m$. In view of Definition 15.16, we can thus apply Proposition 15.14 to the sequences $(s_j)_j^{m+1} = 0$ and $(\overline{s}_j)_j^{m+1} = 0$, resp., to obtain $R_q[[\alpha, \beta]; (s_j)_j^{m+1}, =] = \{S_{m,U_0}\}$ and $R_q[[\alpha, \beta]; (s_j)_j^{m+1}, =] = \{S_{m,U_1}\}$. Thus, $S_m = S_{m,U_0}$ and $\overline{S}_m = S_{m,U_1}$ follow. Regarding Lemma 15.11, Remarks 15.8 and 14.3, and Notation 15.10, we can infer from Theorem 14.2(b) then $\Gamma_m = \{(X_{m,U_0}, Y_{m,U_0})\}$ and $\overline{\Gamma}_m = \{(X_{m,U_1}, Y_{m,U_1})\}$. \hfill \Box

Finally, we want to indicate the announced extremal properties of the equivalence classes $\Gamma_m$ and $\overline{\Gamma}_m$ from $\langle \hat{P}[s_0^{[m]}] \rangle$ which correspond to $S_m$ and $\overline{S}_m$ according to Theorem 14.2(b).

**Remark 15.21** If we look back to Proposition 15.20, Example 15.7, and Remark 15.8 and if we consider the corresponding pairs $\{X_{m,U_0}, Y_{m,U_0}\}$ and $\{X_{m,U_1}, Y_{m,U_1}\}$ belonging to $\langle \hat{P}[s_0^{[m]}] \rangle$, then it should be mentioned that these pairs have in addition to $\text{rk}(\{X_{m,U_0}, Y_{m,U_0}\}) = \text{rk}(\{X_{m,U_1}, Y_{m,U_1}\}) = 0$ further extremal rank properties. Indeed, the function $X_{m,U_1}$ satisfies $\text{rank} X_{m,U_1} (z) = \text{rank} \hat{d}_m$ for all $z \in \mathbb{C} \setminus [\alpha, \beta]$, which is the maximal possible rank of a $q \times q$ matrix-valued function $X$ fulfilling
At the beginning of this section we state the necessary background information. Recall that the sequences \((a_j)_{j=0}^\infty\) and \((b_j)_{j=0}^\infty\) were introduced in Definition 3.19.

Definition 16.1 (cf. [27, Def. 10.11]) If \((s_j)_{j=0}^\infty\) is a sequence of complex \(p \times q\) matrices, then we call \((m_j)_{j=0}^\infty\) given by \(m_j := \frac{1}{2}(a_j + b_j)\) the sequence of \([\alpha, \beta]\)-interval mid points associated with \((s_j)_{j=0}^\infty\).

Definition 16.2 (cf. [27, Def. 10.33]) Let \((s_j)_{j=0}^\infty\) be a sequence of complex \(p \times q\) matrices with sequence of \([\alpha, \beta]\)-interval mid points \((m_j)_{j=0}^\infty\). Assume \(k \geq 1\) and let \(k \in \mathbb{Z}_{1,k}\). Then \((s_j)_{j=0}^\infty\) is said to be \([\alpha, \beta]\)-central of order \(k\) if \(s_j = m_{j-1}\) for all \(j \in \mathbb{Z}_{k,k}\).

Definition 16.3 ([27, Def. 11.9]) Let \(m \in \mathbb{N}_0\) and let \((s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^\geq\). Let the sequence \((s_j)_{j=m+1}^\infty\) be recursively defined by \(s_j := m_{j-1}\), where \(m_{j-1}\) is given by Definition 16.1. Then \((s_j)_{j=0}^\infty\) is called the \([\alpha, \beta]\)-central sequence associated with \((s_j)_{j=0}^m\).

Proposition 16.4 ([27, Prop. 11.10]) Let \(m \in \mathbb{N}_0\) and let \((s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^\geq\). Then the \([\alpha, \beta]\)-central sequence associated with \((s_j)_{j=0}^m\) is \([\alpha, \beta]\)-non-negative definite and \([\alpha, \beta]\)-central of order \(m + 1\).

Proposition 16.5 Let \(m \in \mathbb{N}_0\) and let \((s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^\geq\). Denote by \((\hat{s}_j)_{j=0}^\infty\) the \([\alpha, \beta]\)-central sequence associated with \((s_j)_{j=0}^m\). Then the set \(\mathcal{M}_{q,\infty}([\alpha, \beta]; (\hat{s}_j)_{j=0}^\infty, =)\) contains exactly one element \(\hat{\sigma}_m\).

Proof Combine Propositions 16.4 and 3.6.

Proposition 16.5 leads us to the following notion.

Definition 16.6 Let \(m \in \mathbb{N}_0\) and let \((s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^\geq\). Then the Non-negative Hermitian \(q \times q\) measure \(\hat{\sigma}_m\) mentioned in Proposition 16.5 is called the \([\alpha, \beta]\)-central measure associated with \((s_j)_{j=0}^m\).

Definition 16.7 Let \(m \in \mathbb{N}_0\) and let \((s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^\geq\). Denote by \(\hat{\sigma}_m\) the \([\alpha, \beta]\)-central measure associated with \((s_j)_{j=0}^m\). Then the \([\alpha, \beta]\)-Stieltjes Transform \(\hat{S}_m\) of \(\hat{\sigma}_m\) is call the \([\alpha, \beta]\)-central function associated with \((s_j)_{j=0}^m\).
Our next goal is now to determine the position of the \([\alpha, \beta]-\text{Stieltjes Transform} of \tilde{\sigma}_m\) within the parametrization of \(R_q([\alpha, \beta]; (s_j)_j^{m=0})\) obtained in Theorem 14.2. In order to realize this plan, we continue our investigations in [29, Sec. 10] where we studied a Schur type transformation for matrix measures on \([\alpha, \beta]\) which transforms the concrete matrix measure under consideration in accordance with the Schur type algorithm considered in Definition 3.56, which has to be applied to the corresponding moment sequence.

**Definition 16.8** ([29, Def. 10.6]) Let \(\sigma \in M^\infty_\alpha([\alpha, \beta])\) with sequence of power moments \((s^\sigma_j)_j^{\infty=0}\) and let \(k \in \mathbb{N}\). Then \(\sigma\) is called central of order \(k\) if \((s^\sigma_j)_j^{\infty=0}\) is \([\alpha, \beta]\)-central of order \(k\).

Against the background of centrality of measures on \([\alpha, \beta]\), we consider now the scalar case, in particular the following object discussed in [29, Sec. 10].

**Notation 16.9** Let \(a, b \in \mathbb{R}\) with \(a < b\) and let \(v_{[a,b]} : \mathcal{B}_{[a,b]} \to [0, \infty)\) be the arcsine distribution on \([a, b]\) given by \(v_{[a,b]}(B) := \int_B h \, d\lambda\), where \(\lambda : \mathcal{B}_{[a,b]} \to [0, \infty)\) is the Lebesgue measure on \([a, b]\) and \(h : [a, b] \to [0, \infty)\) is defined by \(h(x) := 0\) if \(x \in (a, b)\) and by \(h(x) := \frac{\pi}{2\sqrt{(x-a)(b-x)}}\) if \(x \in \{a, b\}\).

Now we turn our attention to the ordinary and canonical moments of \(v_{[a,b]}\).

**Example 16.10** Let \(a, b \in \mathbb{R}\) with \(a < b\). Then \(v_{[a,b]} \in M^\infty_1([a, b])\). Denote by \((s_j)_j^{\infty=0}\) the sequence of power moments associated with \(v_{[a,b]}\) and by \((\epsilon_j)_j^{\infty=0}\) the sequence of (matricial) canonical moments associated with \(v_{[a,b]}\) via Definition 6.1.

Then \(s_j = \sum_{k=0}^j \binom{j}{k} 2^{-2k} (b-a)^k a^{j-k}\) for all \(j \in \mathbb{N}_0\). In particular, \(v_{[a,b]}([a, b]) = 1\) and \(\int_{[a,b]} v_{[a,b]}(dr) = (a + b)/2\). Furthermore, \(\epsilon_0 = 1\) and \(\epsilon_j = 1/2\) for all \(j \in \mathbb{N}\).

Indeed, the measure \(\mu := v_{[0,1]}\) is a probability measure on \([0, 1]\) with moments \(\int_{[0,1]} x^k \mu(dx) = \binom{2k}{k} 2^{-2k}\) for all \(k \in \mathbb{N}_0\) (see, e.g. [36, formula (25.1)]) and (classical) canonical moments \(p_k = 1/2\) for all \(k \in \mathbb{N}\) (see, e.g. [12, Example 1.3.6]). By virtue of (3.12), then the sequence of matricial canonical moments associated with \(\mu\) via Definition 6.1 fulfills \(\epsilon_0^{(\mu)} = 1\) and \(\epsilon_j^{(\mu)} = 1/2\) for all \(j \in \mathbb{N}\). With \(d := b - a\) let \(T : [0, 1] \to [a, b]\) be defined by \(T(x) = dx + a\). Then it is readily checked that \(v_{[a,b]}\) is the image measure of \(\mu\) under \(T\). Consequently, we can infer \(v_{[a,b]} \in M^\infty_1([a, b])\) and

\[
\int_{[a,b]} t^j v_{[a,b]}(dr) = \int_{[0,1]} [T(x)]^j \mu(dx) = \int_{[0,1]} \sum_{k=0}^j \binom{j}{k} d^k x^k a^{j-k} \mu(dx)
\]

\[
= \sum_{k=0}^j \binom{j}{k} d^k \int_{[0,1]} x^k \mu(dx) a^{j-k} = \sum_{k=0}^j \binom{j}{k} \binom{2k}{k} 2^{-2k} d^k a^{j-k}
\]

for all \(j \in \mathbb{N}_0\). Furthermore, the sequence of matricial canonical moments associated with \(v_{[a,b]}\) coincides, according to [28, Prop. 8.12], with \((\epsilon_j^{(\mu)})_j^{\infty=0}\).

We reformulate now Example 3.66(b) in the language of measures.
Proposition 16.11 Suppose $\delta = 2$. Denote by $\mu$ the first $\mathcal{M}[\alpha, \beta]$-transform of $v_{[\alpha, \beta]}$. Then $\mu = v_{[\alpha, \beta]}$, i.e., the measure $v_{[\alpha, \beta]}$ is a fixed point of the $\mathcal{M}[\alpha, \beta]$-transformation. In particular, the measure $v_{[-1, 1]}$ is a fixed point of the $\mathcal{M}[-1, 1]$-transformation.

Proof Regarding Example 16.10, denote by $(s_j)_{j=0}^{\infty}$ the sequence of power moments associated with $v_{[\alpha, \beta]}$. According to Proposition 3.7, then $(s_j)_{j=0}^{\infty} \in \mathcal{F}_{1, \infty, \alpha, \beta}$. Denote by $(t_j)_{j=0}^{\infty}$ the $\mathcal{F}_{\alpha, \beta}$-transform of $(s_j)_{j=0}^{\infty}$ and by $(\epsilon_j)_{j=0}^{\infty}$ the $[\alpha, \beta]$-interval parameter sequence of $(s_j)_{j=0}^{\infty}$ given in Definitions 3.52 and 3.43, resp. Taking into account Remark 3.57 and Definition 6.5, we infer that $(t_j)_{j=0}^{\infty}$ is the sequence of power moments associated with $\mu$. By virtue of Definition 6.1, we see that $(\epsilon_j)_{j=0}^{\infty}$ is the sequence of matricial canonical moments associated with $v_{[\alpha, \beta]}$. From Example 16.10 we thus obtain $\epsilon_0 = 1$ and $\epsilon_j = 1/2$ for all $j \in \mathbb{N}$. Using Example 3.66(b), we can conclude then that $(t_j)_{j=0}^{\infty}$ coincides with $(s_j)_{j=0}^{\infty}$. The application of Proposition 3.7 hence yields $\mu = v_{[\alpha, \beta]}$. 

The following result indicates that the notion of $[\alpha, \beta]$-centrality of order $k$ of matrix measures is intimately connected via Stieltjes Transform with the scalar probability measure $v_{[\alpha, \beta]}$ introduced in Notation 16.9. More precisely, this property is characterized by the fact that the $(k-1)$-th $\mathcal{M}[\alpha, \beta]$-transform of the matrix measure under consideration is a $q$-dimensional inflation of $v_{[\alpha, \beta]}$, where the corresponding matrix coefficient is a multiple of the $(k-1)$-th matricial interval length.

Theorem 16.12 (cf. [29, Thm. 10.9]) Let $\sigma \in \mathcal{M}_{q}^{\infty}([\alpha, \beta])$ and let $k \in \mathbb{N}$. Denote by $(\sigma^{(\sigma)})_{j=0}^{\infty}$ the sequence of matricial interval lengths associated with $\sigma$ given in Definition 6.1 and by $\sigma^{[k-1]}$ the $(k-1)$-th $\mathcal{M}[\alpha, \beta]$-transform of $\sigma$ given via Definition 6.5. Let $M := \delta^{k-2}Q_{k-1}^{(\sigma)}$ and let $\mu : \mathcal{B}_{[\alpha, \beta]} \to \mathbb{C}^{q \times q}$ be defined by $\mu(B) := [v_{[\alpha, \beta]}(B)]M$. Then $\sigma$ is central of order $k$ if and only if $\sigma^{[k-1]} = \mu$.

A closer look at the proof of Theorem 16.12 given in [29] shows that one of the central points of it is [29, Example 10.8], where we took from [12, Example 1.3.6] the observation that the sequence $(p_k)_{k=1}^{\infty}$ of canonical moments of $v_{[0, 1]}$ is the constant sequence with value $1/2$. This result originates in Karlin/Shaqlay [36, Sec. 25]. For an updated presentation, we refer also to Karlin/Studden [37, Ch. 4, § 4]. The essential method used by Karlin and Shaqlay is a careful study of the geometry of Chebyshev polynomials.

Example 16.13 Let $a, b \in \mathbb{R}$ with $a < b$. Then $v_{[a, b]}$ is central of order 1.

Indeed, let $\sigma := v_{[a, b]}$ and let $M := (b-a)^{-1}Q_{0}^{(\sigma)}$. By virtue of Example 16.10, we see $\sigma \in \mathcal{M}_{1}^{\infty}([a, b])$ and $\sigma([a, b]) = 1$. According to Remark 6.6, we have $\sigma^{[0]} = \sigma$. Proposition 6.12 yields $\sigma^{[0]}([a, b]) = (b-a)^{-1}Q_{0}^{(\sigma)}$. Consequently, we can infer $M = 1$ and hence $\sigma^{[0]}(B) = [v_{[a, b]}(B)]M$ for all $B \in \mathcal{B}_{[a, b]}$ follows. Applying Theorem 16.12 shows that $v_{[a, b]}$ is central of order 1.

Now we turn our attention via $[\alpha, \beta]$-Stieltjes Transform to functions belonging to $\mathcal{R}_{q}(\mathbb{C} \setminus [\alpha, \beta])$. 
Definition 16.14 Let \( F \in \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta]) \) with \( \mathcal{R}[\alpha, \beta] \)-measure \( \sigma \) and let \( k \in \mathbb{N} \). We call \( F \) central of order \( k \) if \( \sigma \) is central of order \( k \).

Notation 16.15 Let \( a, b \in \mathbb{R} \) with \( a < b \). Then denote by \( g_{[a, b]} \) the \([a, b]\)-Stieltjes Transform of \( \nu_{[a, b]} \).

The following observation is an easy consequence of the construction of the objects under consideration.

Remark 16.16 Let \( M \in \mathbb{C}^{q \times q} \) and let \( \nu \in \mathcal{M}_q^\mathbb{C}([\alpha, \beta]) \) with \([\alpha, \beta]\)-Stieltjes Transform \( f \). Then \( \mu : \mathcal{B}_{[\alpha, \beta]} \to \mathbb{C}^{q \times q} \) defined by \( \mu(B) := [\nu(B)]M \) belongs to \( \mathcal{M}_q^\mathbb{C}([\alpha, \beta]) \) and \( G : \mathbb{C} \setminus [\alpha, \beta] \to \mathbb{C}^{q \times q} \) defined by \( G(z) := f(z)M \) coincides with the \([\alpha, \beta]\)-Stieltjes Transform of \( \mu \).

Theorem 16.17 Let \( F \in \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta]) \) and let \( k \in \mathbb{N} \). Denote by \( (\mathcal{V}_j^{[F]})_{j=0}^\infty \) the sequence of \( \mathcal{R}[\alpha, \beta] \)-interval lengths associated with \( F \) given via Definition 6.3 and by \( F^{(k-1)} \) the \((k-1)\)-th \( \mathcal{R}[\alpha, \beta] \)-Schur transform of \( F \) given in Definition 6.8. Let \( N := \delta_k - 2 \delta_{k-1} \) and let \( G : \mathbb{C} \setminus [\alpha, \beta] \to \mathbb{C}^{q \times q} \) be defined by \( G(z) := g_{[\alpha, \beta]}(z)N \). Then \( F \) is central of order \( k \) if and only if \( F^{(k-1)} = G \).

Proof Let \( \sigma \) be the \( \mathcal{R}[\alpha, \beta] \)-measure of \( F \). Denote by \( (\mathcal{V}_j^{(\sigma)})_{j=0}^\infty \) the sequence of matricial interval lengths associated with \( \sigma \) and by \( \sigma^{[k-1]} \) the \((k-1)\)-th \( \mathcal{M}[\alpha, \beta] \)-transform of \( \sigma \). Let \( M := \delta_k - 2 \delta_{k-1} \) and let \( \mu : \mathcal{B}_{[\alpha, \beta]} \to \mathbb{C}^{q \times q} \) be defined by \( \mu(B) := [\nu_{[\alpha, \beta]}(B)]M \). According to Definitions 6.3 and 6.1, we have \( \delta_{k-1}^{(\sigma)} = \delta_{k-1}^{(\sigma)} \), implying \( N = M \). From Proposition 6.12 we can infer \( M = \sigma^{[k-1]}([\alpha, \beta]) \). In particular, \( M \in \mathbb{C}^{q \times q} \). Taking additionally into account Example 16.10 and Notation 16.15, the application of Remark 16.16 now shows that \( \mu \) belongs to \( \mathcal{M}_q^\mathbb{C}([\alpha, \beta]) \) and that \( G \) coincides with the \([\alpha, \beta]\)-Stieltjes Transform of \( \mu \). Consequently, in view of Definition 6.8, we can conclude from Proposition 5.5 that \( \sigma^{[k-1]} = \mu \) if and only if \( F^{(k-1)} = G \). By virtue of Definition 16.14, the application of Theorem 16.12 completes the proof. \( \square \)

Theorems 16.12 and 16.17 contain further results, which indicate the importance of the arcsine distribution introduced in Notation 16.9. For other topics in which the arcsine distribution plays a significant role, we refer to sum rules for Jacobi matrices, which are compact perturbations of the free Jacobi matrix associated with the arcsine distribution (see Killip/Simon [40] and Simon [52]) and free probability and random matrices (see Hiai/Petz [35]).

Now we are going to determine the position of the \([\alpha, \beta]\)-central function associated with a sequence \( (s_j)_{j=0}^m \in \mathcal{F}_q^{[m, \alpha, \beta]} \) within the description of \( \mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^m] \) given in Theorem 14.2.
Notation 16.18 Let \( m \in \mathbb{N}_0 \) and let \((s_j)^m_{j=0} \in \mathcal{F}_{q,m,\alpha,\beta}^\prec\) with sequence of \([\alpha, \beta]\)-interval lengths \((\delta_j)^m_{j=0}\). Then let \( \hat{G}_m, \hat{X}_m, \hat{Y}_m : \mathbb{C} \setminus [\alpha, \beta] \to \mathbb{C}^{q \times q} \) be defined by \( \hat{G}_m(z) := \delta_{m-1}^p g_{[\alpha, \beta]}(z) \delta_m, \hat{X}_m(z) := \delta_{m-1}^p [(\beta - z)g_{[\alpha, \beta]}(z) - 1] \delta_m, \) and \( \hat{Y}_m(z) := (\beta - z) [(z - \alpha)g_{[\alpha, \beta]}(z) + 1] \mathbb{P}_R(\delta_m) + \delta \mathbb{P}_N(\delta_m) \).

Proposition 16.19 Let \( m \in \mathbb{N}_0 \), let \((s_j)^m_{j=0} \in \mathcal{F}_{q,m,\alpha,\beta}^\prec\) with sequence of \([\alpha, \beta]\)-interval lengths \((\delta_j)^m_{j=0}\). Then:

(a) \( \hat{G}_m \in \mathcal{R}_q([\alpha, \beta]) \) and \( \hat{X}_m : \mathcal{R}_q([\alpha, \beta]) \to \mathbb{C}^{q \times q} \) is the \( \mathcal{F}_{\alpha, \beta}(s_0^m) \)-transformed pair of \( \hat{G}_m \).

(b) \( [\hat{X}_m; \hat{Y}_m] \in \mathcal{P}[s_0^m] \) and the inverse \( \mathcal{F}_{\alpha, \beta}(s_0^m) \)-transform of \([\hat{X}_m; \hat{Y}_m]\) coincides with \( \hat{G}_m \).

Proof Setting \( M := \delta_{m-1}^p \delta_m \), we have \( \hat{G}_m = g_{[\alpha, \beta]}M \), according to Notation 16.18. From \( \delta > 0 \) and Proposition 3.28, we can infer \( M \in \mathbb{C}^{q \times q} \). Remark 14.3 furthermore yields \( s_0^m = M \). Denote by \([G_1; G_2]\) the \( \mathcal{F}_{\alpha, \beta}(M) \)-transformed pair of \( \hat{G}_m \).

(a) Taking into account Example 16.10 and Notation 16.15, the application of Remark 16.16 shows that \( \mu : \mathcal{B}_{[\alpha, \beta]} \to \mathbb{C}^{q \times q} \), defined by \( \mu(B) := [v_{[\alpha, \beta]}(B)]M \) belong to \( \mathcal{M}_q^\prec([\alpha, \beta]) \) and that \( \hat{G}_m \) coincides with the \([\alpha, \beta]\)-Stieltjes Transform of \( \mu \). In view of Example 16.10, we have \( \mu([\alpha, \beta]) = M \). Consequently, \( \mu \in \mathcal{M}_q^\prec,0([\alpha, \beta]; (s_j^m)^0_{j=0}, =) \) follows. Remark 5.8 then shows \( \hat{G}_m \in \mathcal{R}_{q,0}([\alpha, \beta]; (s_j^m)^0_{j=0}, =) \). Taking into account \( M^* = M \), we can infer from Remark 14.3 moreover \( \mathbb{P}_{R}(M^*) = \mathbb{P}_{R}(\delta_m) \) and \( \mathbb{P}_{N}(M) = \mathbb{P}_{N}(\delta_m) \). Using Remark A.18, in particular \( M^* M = \mathbb{P}_{R}(\delta_m) \) follows. By virtue of Definition 9.1 and Notation 16.18 we have, for all \( z \in \mathbb{C} \setminus [\alpha, \beta] \), then

\[
G_1(z) = (\beta - z) \hat{G}_m(z) - M = (\beta - z)g_{[\alpha, \beta]}(z)M - M = [(\beta - z)g_{[\alpha, \beta]}(z) - 1]M = \hat{X}_m(z)
\]

and

\[
G_2(z) = (\beta - z)((z - \alpha)M^* \hat{G}_m(z) + \mathbb{P}_{R}(M^*)) + \delta \mathbb{P}_{N}(M)
\]

\[
= (\beta - z)((z - \alpha)g_{[\alpha, \beta]}(z)M^* \hat{G}_m(z) + \mathbb{P}_{R}(M^*)) + \delta \mathbb{P}_{N}(M)
\]

\[
= (\beta - z)((z - \alpha)g_{[\alpha, \beta]}(z)\mathbb{P}_{R}(\delta_m) + \mathbb{P}_{R}(\delta_m)) + \delta \mathbb{P}_{N}(\delta_m)
\]

\[
= (\beta - z)((z - \alpha)g_{[\alpha, \beta]}(z) + 1)\mathbb{P}_{R}(\delta_m) + \delta \mathbb{P}_{N}(\delta_m) = \hat{Y}_m(z).
\]

(b) By virtue of Proposition 3.59, we have \((s_j^m)^0_{j=0} \in \mathcal{F}_{q,0,\alpha,\beta}^\prec\). Taking additionally into account part (a), Lemma 11.1 yields \([\hat{X}_m; \hat{Y}_m] \in \mathcal{P}[s_0^m] \). Obviously, \( \mathbb{P}_{R}(M) \hat{G}_m = g_{[\alpha, \beta]} \mathbb{P}_{R}(M) \hat{G}_m = \hat{G}_m \). Because of \( M \in \mathbb{C}^{q \times q} \) and part (a), the application of Lemma 9.14 shows that the inverse \( \mathcal{F}_{\alpha, \beta}(M) \)-transform of \([G_1; G_2]\) coincides with \( \hat{G}_m \). \( \square \)
Proposition 16.20 Let $m \in \mathbb{N}_0$, let $(s_j)^m_{j=0} \in \mathcal{F}^\prec_{q,m,\alpha,\beta}$ with sequence of $[\alpha, \beta]$-interval lengths $(\delta_j)^m_{j=0}$, and let $F \in \mathcal{R}_q[[\alpha, \beta]; (s_j)^m_{j=0}]$. Denote by $\tilde{G}_m(F, (s_j)^m_{j=0})$ the $m$-th $\mathcal{F}_{\alpha,\beta}$-transform of $F$ with respect to $(s_j)^m_{j=0}$ and by $\mathcal{P}\tilde{G}_m(F, (s_j)^m_{j=0})$ the $m$-th $\mathcal{F}_{\alpha,\beta}$-transformed pair of $F$ with respect to $(s_j)^m_{j=0}$ given in Definition 13.1. Regarding Notation 16.18 and Definition 7.11, then the following statements are equivalent:

(i) $F$ is central of order $m + 1$.
(ii) $\tilde{G}_m(F, (s_j)^m_{j=0}) = \hat{G}_m$.
(iii) $\mathcal{P}\tilde{G}_m(F, (s_j)^m_{j=0}) \sim [\hat{X}_m; \hat{Y}_m]$.

Proof Denote by $F^{[m]}$ the $m$-th $\mathcal{R}_{[\alpha, \beta]}$-Schur transform of $F$. Due to Proposition 13.8, we have $\tilde{G}_m(F, (s_j)^m_{j=0}) = F^{[m]}$.

(i)$\Leftrightarrow$(ii) Denote by $\sigma$ the $\mathcal{R}_{[\alpha, \beta]}$-measure of $F$ and by $(\sigma(j))^{\infty}_{j=0}$ and $(\delta(j))^{\infty}_{j=0}$ the sequence of power moments and the sequence of matricial interval lengths associated with $\sigma$, resp. Then $\sigma \in \mathcal{M}_{q,m}[[\alpha, \beta]; (s_j)^m_{j=0}, =]$ and hence $s(j) = s_j$ for all $j \in \mathbb{Z}_{0,m}$. By virtue of Remark 3.23, consequently $\delta(j) = \delta_m$. Denote by $(\delta(F))^{\infty}_{j=0}$ the sequence of $\mathcal{R}_{[\alpha, \beta]}$-interval lengths associated with $F$. Taking additionally into account Definitions 6.3 and 6.1, then $\delta_m = \delta_m = \delta_m$ follows. According to Notation 16.18, thus $\hat{G}_m = \hat{g}_{[\alpha, \beta]}(m, \alpha, \beta)$. Now, in view of $\mathcal{P}\tilde{G}_m(F, (s_j)^m_{j=0}) = F^{[m]}$, the application of Theorem 16.17 yields the equivalence of (i) and (ii).

(ii)$\Rightarrow$(iii) From Proposition 16.19(a) we see that $[\hat{X}_m; \hat{Y}_m]$ is the $\mathcal{F}_{\alpha,\beta}(s_j)^m_{j=0}$-transformed pair of $\hat{G}_m$. In view of (ii) and Definition 13.1, then $\mathcal{P}\tilde{G}_m(F, (s_j)^m_{j=0}) = [\hat{X}_m; \hat{Y}_m]$ follows. In particular, (iii) holds true.

(iii)$\Rightarrow$(ii) Setting $M := s_0^{[m]}$, the combination of Proposition 3.59 and Lemma 3.12 yields $M \in \mathbb{Z}_q^{q \times q}$. From Proposition 16.19(b) we see $[\hat{X}_m; \hat{Y}_m] \in \hat{\mathcal{P}}[M]$ and that $\hat{G}_m$ is the inverse $\mathcal{F}_{\alpha,\beta}(M)$-transform of $[\hat{X}_m; \hat{Y}_m]$. Because of (iii) and Remark 8.4, in particular $\mathcal{P}\tilde{G}_m(F, (s_j)^m_{j=0}) \in \hat{\mathcal{P}}[M]$ follows. Denote by $G$ the inverse $\mathcal{F}_{\alpha,\beta}(M)$-transform of $\mathcal{P}\tilde{G}_m(F, (s_j)^m_{j=0})$. Taking into account (iii), we can infer from Corollary 9.12 that $G = \hat{G}_m$. Observe that $\mathcal{P}\tilde{G}_m(F, (s_j)^m_{j=0})$ is the $\mathcal{F}_{\alpha,\beta}(M)$-transformed pair of $\hat{G}_m(F, (s_j)^m_{j=0})$, according to Definition 13.1. Remark 6.11 yields $F^{[m]} \in \mathcal{R}_q[[\alpha, \beta]; (s_j^{[m]}_{j=0})^{0}_{j=0}]$. Consequently, the $\mathcal{R}_{[\alpha, \beta]}$-measure $\mu$ of $F^{[m]}$ belongs to $\mathcal{M}_{q,0}[[\alpha, \beta]; (s_j^{[m]}_{j=0})^{0}_{j=0}, =]$, i.e., $\mu([\alpha, \beta]) = M$. Taking additionally into account Theorem 4.15(a), hence $\mathcal{R}(F^{[m]}(z)) = \mathcal{R}(M)$ for all $z \in \mathbb{C}\setminus[\alpha, \beta]$ follows. In view of $\hat{G}_m(F, (s_j)^m_{j=0}) = F^{[m]}$, hence $\mathcal{P}\mathcal{R}(M)\hat{G}_m(F, (s_j)^m_{j=0}) = \hat{G}_m(F, (s_j)^m_{j=0})$. Thus, we can apply Lemma 9.14 to obtain $G = \hat{G}_m(F, (s_j)^m_{j=0})$. Therefore, (ii) holds true.

Now we are able to give a representation according to Theorem 14.2 of the $[\alpha, \beta]$-central function associated with a sequence $(s_j)^m_{j=0} \in \mathcal{F}^\prec_{q,m,\alpha,\beta}$.

Proposition 16.21 Let $m \in \mathbb{N}_0$ and let $(s_j)^m_{j=0} \in \mathcal{F}^\prec_{q,m,\alpha,\beta}$. Denote by $\begin{bmatrix} \tilde{m}_m & \tilde{y}_m \\ \tilde{h}_m & \tilde{s}_m \end{bmatrix}$ the $q \times q$ block representation of the restriction of $\hat{\tilde{G}}_m$ onto $\mathbb{C}\setminus[\alpha, \beta]$. Then $\det(\tilde{h}_m \tilde{x}_m + \tilde{m}_m \tilde{y}_m \tilde{s}_m)$
\( \tilde{s}_m \tilde{Y}_m \) does not vanish identically in \( \mathbb{C} \setminus [\alpha, \beta] \) and \( \tilde{s}_m = (\tilde{w}_m \tilde{X}_m + \tilde{z}_m \tilde{Y}_m)(\tilde{\eta}_m \tilde{X}_m + \tilde{\tilde{z}}_m \tilde{Y}_m)^{-1}. \)

**Proof** Remark 16.19(b) shows \( \tilde{X}_m; \tilde{Y}_m \in \tilde{\mathcal{P}}[s_0^{(m)}] \). Consequently, we can apply Theorem 14.2(a) to see that \( \det(\tilde{\eta}_m \tilde{X}_m + \tilde{s}_m \tilde{Y}_m) \) does not vanish identically in \( \mathbb{C} \setminus [\alpha, \beta] \) and that the matrix-valued function \( F := (\tilde{w}_m \tilde{X}_m + \tilde{z}_m \tilde{Y}_m)(\tilde{\eta}_m \tilde{X}_m + \tilde{\tilde{z}}_m \tilde{Y}_m)^{-1} \) belongs to \( \mathcal{R}_q[[\alpha, \beta]]; (s_j)^m_{j=0} \). Theorem 14.2(b) then yields \( \tilde{X}_m; \tilde{Y}_m \in \mathcal{PG}_m(F; (s_j)^m_{j=0}) \).

From Proposition 16.20 we can thus conclude that \( F \) is central of order \( m + 1 \). Denote by \( \sigma \) the \( \mathcal{R}[[\alpha, \beta]] \)-measure of \( F \). Then \( \sigma \in \mathcal{M}^\infty_{q, \infty}[[\alpha, \beta]]; (s_j)^\infty_{j=0}, = \) follows. According to Definition 16.14, furthermore \( \sigma \) is central of order \( m + 1 \). In view of Definition 16.8, this means that the sequence of power moments \( (s_j^{(\sigma)})^\infty_{j=0} \) associated with \( \sigma \) is \([\alpha, \beta]\)-central of order \( m + 1 \). Since \( s_j^{(\sigma)} = s_j \) for all \( j \in \mathbb{Z}_{0,m} \), thus Definitions 16.2 and 16.3 show that \( (s_j^{(\sigma)})^\infty_{j=0} \) coincides with the \([\alpha, \beta]\)-central sequence \( (s_j)^\infty_{j=0} \) associated with \( (s_j)^m_{j=0} \). Consequently, \( \sigma \in \mathcal{M}^\infty_{q, \infty}[[\alpha, \beta]]; (s_j)^\infty_{j=0}, = \) follows. Proposition 16.5 and Definition 16.6 then yield \( \sigma = \tilde{s}_m \). From Proposition 5.5 we can furthermore conclude that \( F \) is the \([\alpha, \beta]\)-Stieltjes Transform of \( \sigma \). In view of Definition 16.7, the proof is complete. \( \square \)

In our following considerations, we concentrate on the case of a sequence \( (s_j)^m_{j=0} \in \mathcal{F}_{q,m,\alpha,\beta}^\infty \). Before doing that we state some elementary preparations in the scalar case, which are of own interest.

**Lemma 16.22** Let \( a, b \in \mathbb{R} \) with \( a < b \) and let \( z \in \mathbb{C}\setminus[a, b] \). Then there exists a unique \( w \in \mathbb{C} \) satisfying \( w^2 = (z - a) (z - b) \) and \( |w - z + c| < d \), where \( c := (a + b)/2 \) and \( d := (b - a)/2 \).

**Proof** First observe that \( (z - a)(z - b) = (z - c)^2 - d^2 \) and that there exists either a single one or two different solutions \( w \in \mathbb{C} \) satisfying \( w^2 = (z - a)(z - b) \). We choose a particular solution \( w_0 \). If \( w_0 = 0 \), then \( z = a \) or \( z = b \), contradicting \( z \notin [a, b] \). Thus, we have \( w_0 \neq -w_0 \) and hence \( w_1 := w_0 \) and \( w_2 := -w_0 \) are the only solutions of the equation \( w^2 = (z - a)(z - b) \). Consequently, \( t_1 := w_1 - z + c \) and \( t_2 := w_2 - z + c \) fulfill \( t_1 \neq t_2 \) and \( t_1, t_2 \leq z - c \). Moreover, \( (\pm w_0)^2 = (z - c)^2 - d^2 \). Therefore, \( t_1 \) and \( t_2 \) are the two solutions of the equation \( t^2 + 2(z-c)t + d^2 = 0 \). Hence, \( t_1 + t_2 = -2(z-c) \) and \( t_1 t_2 = d^2 \). In particular, \( |t_1| \cdot |t_2| = d^2 \). We are now going to show \( |t_1| \neq |t_2| \). Assume to the contrary \( |t_1| = |t_2| \). In view of \( t_1 t_2 = d^2 \) and \( d > 0 \), then \( t_2 = \overline{t_1} \) and \( |t_1| = d \) follow. Using \( t_1 + t_2 = -2(z-c) \), we can thus infer \( z = c - \text{Re} \ t_1 \in \mathbb{R} \) and furthermore \( |z - c| = |\text{Re} \ t_1| \leq |t_1| \). Taking additionally into account \( |t_1| = d \), then \( -d \leq z - c \leq d \) follows, contradicting \( z \notin [a, b] \). Thus, we have shown \( |t_1| \neq |t_2| \). Since \( |t_1| \cdot |t_2| = d^2 \), then either \( |t_1| < d \) and \( |t_2| > d \) or \( |t_1| > d \) and \( |t_2| < d \). Consequently, exactly one of the two solutions of the equation \( w^2 = (z - a)(z - b) \) fulfills \( |w - z + c| < d \). \( \square \)

**Lemma 16.23**

(a) The function \( g_{[-1,1]} \) belongs to \( \mathcal{R}_1(\mathbb{C}\setminus[-1,1]) \) with \( \mathcal{R}_{[-1,1]} \)-measure \( \nu_{[-1,1]} \) and is central of order 1.

(b) Let \( z \in \mathbb{C}\setminus[-1,1] \). Then \( g_{[-1,1]}(z) \neq 0 \) and \( w_z := -1/g_{[-1,1]}(z) \) is the unique complex number \( w \) satisfying \( w^2 = z^2 - 1 \) and \( |w - z| < 1 \).
Proof (a) In view of Example 16.10 and Notation 16.15, we can infer from Proposition 5.5 that \( g_{[-1,1]} \) belongs to \( \mathcal{R}_1(\mathbb{C}\setminus[\alpha,\beta]) \) and that \( v_{[-1,1]} \) is the \( \mathcal{R}[1] \)-measure of \( g_{[-1,1]} \). According to Definition 16.14 and Example 16.13, thus \( g_{[-1,1]} \) is central of order 1.

(b) Because of part (a), we can apply Proposition 4.15(a) to obtain \( \mathcal{N}(g_{[-1,1]}(z)) = \mathcal{N}(v_{[-1,1]}([(-1,1])) \). From Example 16.10 we know \( v_{[-1,1]}([(-1,1)] = 1 \). Hence, \( g_{[-1,1]}(z) \neq 0 \) follows. According to [12, p. 125, especially (4.5.4)], the remaining assertion of part (b) holds true. (Observe that in [12], for probability measures \( \mu \) on \([-1,1]\), the integral \( S(z, \mu) = \int_{-1}^{1} (z-x)^{-1} d\mu(x) = -\tilde{S}_{\mu}(z) \) is considered.) \( \square \)

Proposition 16.24 (a) The function \( g_{[\alpha,\beta]} \) belongs to \( \mathcal{R}_1(\mathbb{C}\setminus[\alpha,\beta]) \) with \( \mathcal{R}[\alpha,\beta] \)-measure \( v_{[\alpha,\beta]} \) and is central of order 1.

(b) Let \( z \in \mathbb{C}\setminus[\alpha,\beta] \). Then \( g_{[\alpha,\beta]}(z) \neq 0 \) and \( w_z := -1/g_{[\alpha,\beta]}(z) \) is the unique complex number \( w \) satisfying \( w^2 = (z-\alpha)(z-\beta) \) and \( |w-z+(\alpha+\beta)/2| < (\beta-\alpha)/2 \).

Proof By the same reasoning as in the proof of Lemma 16.23, we can conclude that \( g_{[\alpha,\beta]} \) belongs to \( \mathcal{R}_1(\mathbb{C}\setminus[\alpha,\beta]) \) and is central of order 1, that \( v_{[\alpha,\beta]} \) is the \( \mathcal{R}[\alpha,\beta] \)-measure of \( g_{[\alpha,\beta]} \), and that \( g_{[\alpha,\beta]}(z) \neq 0 \). Let \( c := (\alpha+\beta)/2 \) and let \( d := (\beta-\alpha)/2 \). Let \( T: [-1,1] \to [\alpha,\beta] \) be defined by \( T(x) = dx + c \). Then it is readily checked that \( v_{[\alpha,\beta]} \) is the image measure of \( v_{[-1,1]} \) under \( T \). Observe that \( d > 0 \) and that \( z \notin [\alpha,\beta] \) implies \( \xi \notin [-1,1] \) for \( \xi := (z-c)/d \). By virtue of Definition 16.15 and Definition 5.3, we thus can infer

\[
g_{[\alpha,\beta]}(z) = \int_{[\alpha,\beta]} \frac{1}{t-z} v_{[\alpha,\beta]}(dt) = \int_{[-1,1]} \frac{1}{T(x)-z} v_{[-1,1]}(dx) = \int_{[-1,1]} \frac{1}{dx-z+c} v_{[-1,1]}(dx) = \frac{1}{d} \int_{[-1,1]} \frac{1}{x-(z-c)/d} v_{[-1,1]}(dx) = \frac{1}{d} g_{[-1,1]}(\xi).
\]

In view of Lemma 16.23(b), consequently \( w_z = d\omega_{\xi} \), where \( \omega_{\xi} \) is the unique complex number \( \omega \) satisfying \( \omega^2 = \xi^2 -1 \) and \( |\omega-\xi| < 1 \). Hence, \( w_z^2 = d^2(\xi^2 - 1) = (z-c)^2 - d^2 = (z-\alpha)(z-\beta) \) and \( |w_z - z + (\alpha+\beta)/2| = |d\omega_{\xi} - z + c| = d|\omega_{\xi} - \xi| < d = (\beta-\alpha)/2 \). By virtue of Lemma 16.22, the proof is complete. \( \square \)

Lemma 16.25 Suppose \( \delta = 2 \). Denote by \( f \) the first \( \mathcal{R}[\alpha,\beta] \)-Schur transform of \( g_{[\alpha,\beta]} \). Then \( f = g_{[\alpha,\beta]} \gray{\text{i.e.}, the function } g_{[\alpha,\beta]} \gray{\text{is a fixed point of the } \mathcal{R}[\alpha,\beta] \gray{\text{-Schur transformation.}} \gray{\text{In particular, the function } g_{[-1,1]} \gray{\text{is a fixed point of the } \mathcal{R}[1,1] \gray{\text{-Schur transformation.}}}}\)

Proof In view of Example 16.10 and Notation 16.15, we can infer from Proposition 5.5 that \( g_{[\alpha,\beta]} \) belongs to \( \mathcal{R}_1(\mathbb{C}\setminus[\alpha,\beta]) \) and that \( v_{[\alpha,\beta]} \) is the \( \mathcal{R}[\alpha,\beta] \)-measure of \( g_{[\alpha,\beta]} \). Denote by \( \mu \) the first \( \mathcal{M}[\alpha,\beta] \)-transform of \( v_{[\alpha,\beta]} \). According to Definition 6.8, then \( f \) is the \( [\alpha,\beta] \)-Stieltjes Transform of \( \mu \). Since Proposition 16.11 yields \( \mu = v_{[\alpha,\beta]} \), hence \( f \) is the \( [\alpha,\beta] \)-Stieltjes Transform of \( v_{[\alpha,\beta]} \). Regarding Notation 16.15, the proof is complete. \( \square \)
Proposition 16.26  Let \( m \in \mathbb{N}_0 \) and let \( (s_j)_{j=0}^m \in \mathcal{F}^\to_{q,m,a,b} \) with sequence of \([a, b]\)-interval lengths \((\delta_j)_{j=0}^m\). Let \( G_1, G_2 : \mathbb{C} \setminus [a, b] \rightarrow \mathbb{C}^{q \times q} \) be defined by \( G_1(z) := \delta^{m-1} g_{[a,b]}(z) \delta_m \) and \( G_2(z) := I_q \). For all \( z \in \mathbb{C} \setminus [a, b] \), then

\[
\hat{X}_m(z) = \delta^{m-1}[(\beta - z)g_{[a,b]}(z) - 1] \delta_m
\]

and

\[
\hat{Y}_m(z) = (\beta - z)[(z - \alpha)g_{[a,b]}(z) + 1] I_q.
\]

Regarding Definition 7.11, furthermore \([G_1; G_2] \in \mathcal{PR}_q(\mathbb{C} \setminus [a, b])\) and \([\hat{X}_m; \hat{Y}_m] \sim [G_1; G_2] \).

**Proof**  Remark 3.67 yields \( \det \delta_m \neq 0 \). Hence, \( \mathcal{R}(\delta_m) = \mathbb{C}^q \) and \( \mathcal{N}(\delta_m) = \{O_q \times 1\} \), which imply \( \mathcal{PR}(\delta_m) = I_q \) and \( \mathcal{PN}(\delta_m) = O_q \times q \). For all \( z \in \mathbb{C} \setminus [a, b] \), now the asserted representations of \( \hat{X}_m(z) \) and \( \hat{Y}_m(z) \) follow immediately from Notation 16.18. By virtue of Proposition 16.19(b) and Notation 8.1, we get \( [\hat{X}_m; \hat{Y}_m] \in \mathcal{PR}_q(\mathbb{C} \setminus [a, b]) \). Let \( f : \mathbb{C} \setminus [a, b] \rightarrow \mathbb{C} \) be defined by \( f(z) := \gamma - z)g_{[a,b]}(z) + 1 \). From Proposition 16.24(a) we see that \( g_{[a,b]} \) belongs to \( \mathcal{K}_q(\mathbb{C} \setminus [a, b]) \) and that \( v_{[a,b]} \) is the \( \mathcal{R}[a,b] \)-measure of \( g_{[a,b]} \). Example 16.10 furthermore shows \( v_{[a,b]}([a,b]) = 1 \) and \( \int_{[a,b]} f(t) v_{[a,b]}(dt) = (\alpha + \beta)/2 \). Taking into account Notation 4.19, we can thus infer from Proposition 4.20 that \( f \) belongs to \( \mathcal{K}_1(\mathbb{C} \setminus [a, b]) \) and that the \( \mathcal{R}[a,b] \)-measure \( \sigma \) of \( f \) fulfills \( \sigma([a,b]) = \int_{[a,b]} f(t) v_{[a,b]}(dt) = (\alpha + \beta)/2 - \alpha = (\beta - \alpha)/2 \neq 0 \). Consider now an arbitrary \( z \in \mathbb{C} \setminus [a, b] \). Using Proposition 4.15(a) we obtain then \( \mathcal{N}(f(z)) = \mathcal{N}(\sigma([a,b])) = \{0\} \) and hence \( f(z) \neq 0 \). Regarding \( \hat{Y}_m(z) = (\beta - z)f(z) I_q \), then \( \hat{Y}_m(z) \neq 0 \) follows. From Proposition 16.24(b) we see \( g_{[a,b]}(z) \neq 0 \) and that \( w_z := -1/g_{[a,b]}(z) \) satisfies \( w_z^2 = (z - \alpha)(z - \beta) \). Hence, \([\beta - z) + w_z] w_z = (\beta - z)(z - \alpha) \) and, in view of \( w_z \neq 0 \), therefore \( \beta - z) + w_z = (\beta - z) [1 - (z - \alpha) / w_z] = (\beta - z) f(z) \). Multiplication by \( g_{[a,b]}(z) \) yields \( (\beta - z) g_{[a,b]}(z) - 1 = (\beta - z) f(z) g_{[a,b]}(z) \). Consequently, we get the identity \( (\beta - z) g_{[a,b]}(z) - 1)/(\beta - z) f(z) = g_{[a,b]}(z) \), from which we can conclude \([\hat{X}_m(z)][\hat{Y}_m(z)]^{-1} = G_1(z) \). The application of Lemma 7.13 completes the proof. \( \Box \)

Proposition 16.27  Let \( m \in \mathbb{N}_0 \) and let \( (s_j)_{j=0}^m \in \mathcal{F}^\to_{q,m,a,b} \). Let \[
\left[
\delta_m \tilde{Y}_m \tilde{Y}_m
\right]
\]
be the \( q \times q \) block representation of the restriction of \( \delta^m \tilde{Y}_m \) onto \( \mathbb{C} \setminus [a, b] \). Then

\[
\det(\delta^{m-1} g_{[a,b]} \delta_m \delta_m + 3_m) \neq 0
\]
does not vanish identically in \( \mathbb{C} \setminus [a, b] \) and \( \tilde{Y}_m = (\delta^{m-1} g_{[a,b]} \delta_m \delta_m + 3_m) (\delta^{m-1} g_{[a,b]} \delta_m \delta_m + 3_m)^{-1} \).

**Proof**  From Proposition 16.21 we see that \( \det(\delta_m \hat{X}_m + 3_m \hat{Y}_m) \) does not vanish identically in \( \mathbb{C} \setminus [a, b] \) and that \( \hat{S}_m = (\delta_m \hat{X}_m + 3_m \hat{Y}_m) (\delta_m \hat{X}_m + 3_m \hat{Y}_m)^{-1} \). Remark 16.19(b) shows \([\hat{X}_m; \hat{Y}_m] \in \mathcal{PR}_{[0]^m} \). Let \( G_1, G_2 : \mathbb{C} \setminus [a, b] \rightarrow \mathbb{C}^{q \times q} \) be defined by \( G_1(z) := \delta^{m-1} g_{[a,b]}(z) \delta_m \) and \( G_2(z) := I_q \). According to Proposition 16.26, then
If \( Z = U \), then \( \tilde{X}_m; \tilde{Y}_m \sim [G_1; G_2] \). The application of Definition 7.11 completes the proof.

\[ \text{Data Availability Statement} \quad \text{Data sharing not applicable to this article as no datasets were generated or analysed during the current study.} \]

**A Some Facts from Matrix Theory**

This appendix contains a summary of results from matrix theory, which are used in this paper. What concerns results on the Moore–Penrose inverse \( \tilde{A}^+ \) of a complex matrix \( A \), we refer, e.g., to [13, Sec. 1].

**Remark A.1** Let \( m, n \in \mathbb{N} \) and let \( A_1, A_2, \ldots, A_m \) and \( B_1, B_2, \ldots, B_n \) be complex \( p \times q \) matrices. If \( M \in \mathbb{C}^{q \times p} \) is such that \( A_j M B_k = B_k M A_j \) holds true for all \( j \in \mathbb{Z}_{1,m} \) and all \( k \in \mathbb{Z}_{1,n} \), then \( (\sum_{j=1}^{m} \eta_j A_j) M (\sum_{k=1}^{n} \theta_k B_k) = (\sum_{k=1}^{n} \theta_k B_k) M (\sum_{j=1}^{m} \eta_j A_j) \) for all complex numbers \( \eta_1, \eta_2, \ldots, \eta_m \) and \( \theta_1, \theta_2, \ldots, \theta_n \).

**Remark A.2** (a) If \( Z \in \mathbb{C}^{q \times q} \) and \( \eta \in \mathbb{C} \), then \( \text{Re}(\eta Z) = \text{Re}(\eta) \text{Re}(Z) - \text{Im}(\eta) \text{Im}(Z) \) and \( \text{Im}(\eta Z) = \text{Re}(\eta) \text{Im}(Z) + \text{Im}(\eta) \text{Re}(Z) \).

(b) If \( Z \in \mathbb{C}^{q \times q} \) and \( X \in \mathbb{C}^{q \times p} \), then \( \text{Re}(X^* ZX) = X^* \text{Re}(Z) X \) and \( \text{Im}(X^* ZX) = X^* \text{Im}(Z) X \).

**Remark A.3** If \( A \in \mathbb{C}^{p \times q} \), then \( \dim \mathcal{R}(A) + \dim \mathcal{N}(A) = q \).

**Remark A.4** If \( A \in \mathbb{C}^{p \times q} \), then \( \mathcal{R}(A) = \mathcal{R}(AA^*) \) and \( \mathcal{N}(A) = \mathcal{N}(A^* A) \).

**Remark A.5** Let \( A \in \mathbb{C}^{p \times r} \), let \( B \in \mathbb{C}^{p \times s} \), and let \( C \in \mathbb{C}^{q \times r} \). In view of Remark A.4, then \( \mathcal{R}(A) + \mathcal{R}(B) = \mathcal{R}([A, B]) = \mathcal{R}(AA^* + BB^*) \) and \( \mathcal{N}(A) \cap \mathcal{N}(C) = \mathcal{N} \left( \begin{bmatrix} A \\ C \end{bmatrix} \right) = \mathcal{N}(A^* A + C^* C) \).

**Remark A.6** Let \( n \in \mathbb{N} \) and let \( A_1, A_2, \ldots, A_n \in \mathbb{C}^{p \times q} \). For all \( \eta_1, \eta_2, \ldots, \eta_n \in \mathbb{C} \), then \( \mathcal{R}(\sum_{j=1}^{n} \eta_j A_j) \subseteq \sum_{j=1}^{n} \mathcal{R}(A_j) \) and \( \mathcal{N}(\sum_{j=1}^{n} \eta_j A_j) \subseteq \mathcal{N}(\sum_{j=1}^{n} \eta_j A_j) \).

**Remark A.7** If \( A \in \mathbb{C}^{p \times q} \) has rank \( q \) and \( B \in \mathbb{C}^{q \times s} \) has rank \( s \), then \( \text{rank}(AB) = s \).

**Remark A.8** Let \( L \in \mathbb{C}^{p \times p} \) and \( R \in \mathbb{C}^{q \times q} \) be both invertible. Let \( A \in \mathbb{C}^{p \times q} \) and let \( X := LAR \). Then \( \mathcal{R}(X) = L \mathcal{R}(A) \) and \( \mathcal{N}(X) = R^{-1} \mathcal{N}(A) \).

We think that the following result is well-known. However, we did not succeed in finding a reference.

**Lemma A.9** Let \( A \in \mathbb{C}^{p \times q} \) and let \( R \in \mathbb{C}^{q \times r} \). Then \( \mathcal{R}(AR) = \mathcal{R}(A) \) if and only if \( \mathcal{N}(A) + \mathcal{R}(R) = \mathbb{C}^q \).

**Proof** Observe that \( U := \mathcal{N}(A) + \mathcal{R}(R) \) is a linear subspace of the \( \mathbb{C} \)-vector space \( \mathbb{C}^q \). Let \( \phi: U \to \mathbb{C}^p \) be defined by \( \phi(x) := Ax \). Then \( \phi \) is linear with \( \ker \phi = U \cap \mathcal{N}(A) = \mathcal{N}(A) \) and \( \phi(U) = A \mathcal{R}(A) = \mathcal{R}(AR) \). Regarding \( \dim \ker \phi + \dim \phi(U) = \dim U \), then \( \dim U = \dim \mathcal{N}(A) + \dim \mathcal{R}(AR) = q - \text{rank} A + \text{rank}(AR) \) follows. Hence, \( U = \mathbb{C}^q \) if and only if \( \text{rank}(AR) = \text{rank} A \). Because of \( \mathcal{R}(AR) \subseteq \mathcal{R}(A) \), the latter is equivalent to \( \mathcal{R}(AR) = \mathcal{R}(A) \).
Remark A.10 If $A \in \mathbb{C}^{p \times q}$, then $\mathcal{R}(A^*) = [\mathcal{N}(A)]^\perp$ and $\mathcal{N}(A^*) = [\mathcal{R}(A)]^\perp$.

We write $\mathbb{P}_U$ for the transformation matrix corresponding to the orthogonal projection onto a linear subspace $\mathcal{U}$ of the unitary space $\mathbb{C}^p$ with respect to the standard basis.

Remark A.11 Let $\mathcal{U}$ be a linear subspace of $\mathbb{C}^p$. Then $\mathbb{P}_U$ is the unique complex $p \times p$ matrix satisfying $\mathbb{P}_U^2 = \mathbb{P}_U$, $\mathbb{P}_U^* = \mathbb{P}_U$, and $\mathcal{R}(\mathbb{P}_U) = \mathcal{U}$. Furthermore, $\mathcal{N}(\mathbb{P}_U) = \mathcal{U}^\perp$ and $\mathbb{P}_U + \mathbb{P}_U^\perp = I_p$.

Remark A.12 If $\mathcal{U}$ is a linear subspace of the unitary space $\mathbb{C}^p$ with dimension $d := \dim \mathcal{U} \geq 1$ and some orthonormal basis $u_1, u_2, \ldots, u_d$, then $\mathbb{P}_U = UU^*$, where $U := [u_1, u_2, \ldots, u_d]$.

Remark A.13 If $A \in \mathbb{C}^{q \times q}$ fulfills $\det A \neq 0$, then $A^\dagger = A^{-1}$.

Remark A.14 If $A \in \mathbb{C}^{p \times q}$, then $(A^\dagger)^\dagger = A$, $(A^*)^\dagger = (A^\dagger)^*$, $\mathcal{R}(A^\dagger) = \mathcal{R}(A^*)$, and $\mathcal{N}(A^\dagger) = \mathcal{N}(A^*)$.

Remark A.15 Let $\eta \in \mathbb{C}$ and let $A \in \mathbb{C}^{p \times q}$, then $(\eta A)^\dagger = \eta^* A^\dagger$.

Remark A.16 If $A \in \mathbb{C}_{\geq}^{q \times q}$, then $A^\dagger \in \mathbb{C}_{\geq}^{q \times q}$ and $(A^\dagger)^{1/2} = (A^{1/2})^\dagger$.

Regarding (3.9), we easily obtain with Remark A.15:

Remark A.17 Let $\eta \in \mathbb{C}$ and let $A, B \in \mathbb{C}^{p \times q}$. Then $(\eta A) \notin (\eta B) = \eta(A \notin B)$.

Remark A.18 If $A \in \mathbb{C}^{p \times q}$, then $\mathbb{P}_{\mathcal{R}(A)} = AA^\dagger$, $\mathbb{P}_{\mathcal{N}(A)} = I_q - A^\dagger A$, $\mathbb{P}_{\mathcal{R}(A^*)} = A^\dagger A$, and $\mathbb{P}_{\mathcal{N}(A^*)} = I_p - AA^\dagger$.

Lemma A.19 Let $A \in \mathbb{C}^{p \times q}$ and let $B \in \mathbb{C}^{q \times q}$ be such that $\mathcal{R}(B) \subseteq \mathcal{R}(A^*) \subseteq \mathcal{R}(B^*)$. Let $\eta \in \mathbb{C}\setminus\{0\}$. Then the matrix $B + \eta \mathbb{P}_{\mathcal{N}(A)}$ is invertible and $B^\dagger = (B + \eta \mathbb{P}_{\mathcal{N}(A)})^{-1} - \eta^{-1} \mathbb{P}_{\mathcal{N}(A)}$.

Proof First observe that $\mathcal{R}(B) = \mathcal{R}(A^*) = \mathcal{R}(B^*)$ follows from the assumption, since rank$(B^*) = \text{rank } B$ holds true. In view of Remark A.18, we thus obtain $BB^\dagger = A^\dagger A = B^\dagger B$ and $\mathbb{P}_{\mathcal{N}(A)} = I_q - A^\dagger A$. Taking additionally into account Remark A.11, we infer then

$$
(B + \eta \mathbb{P}_{\mathcal{N}(A)})(B^\dagger + \eta^{-1} \mathbb{P}_{\mathcal{N}(A)}) = BB^\dagger + \eta^{-1} B \mathbb{P}_{\mathcal{N}(A)}B^\dagger + \mathbb{P}_{\mathcal{N}(A)}^2
$$

$$
= A^\dagger A + \eta^{-1} B(I_q - A^\dagger A) + \eta(I_q - A^\dagger A)B^\dagger + \mathbb{P}_{\mathcal{N}(A)}
$$

$$
= I_q + \eta^{-1} B(I_q - B^\dagger B) + \eta(I_q - B^\dagger B)B^\dagger = I_q.
$$

Remark A.20 Let $A \in \mathbb{C}^{p \times q}$ and let $B \in \mathbb{C}^{p \times m}$. Then $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ if and only if $AA^\dagger B = B$.

Remark A.21 Let $A \in \mathbb{C}^{p \times q}$ and let $C \in \mathbb{C}^{n \times q}$. Then $\mathcal{N}(A) \subseteq \mathcal{N}(C)$ if and only if $CA^\dagger A = C$. 

The combination of Remarks A.20, A.21, and A.1 yields:

**Remark A.22** Let \( A \in \mathbb{C}^{p \times q} \) and \( M \in \mathbb{C}^{q \times p} \) be such that \( \mathcal{R}(A) \subseteq \mathcal{R}(M) \) and \( \mathcal{N}(M) \subseteq \mathcal{N}(A) \). For all \( \eta_1, \eta_2, \theta_1, \theta_2 \in \mathbb{C} \), then \((\eta_1 A + \eta_2 M) M^\dagger (\theta_1 A + \theta_2 M) = (\theta_1 A + \theta_2 M) M^\dagger (\eta_1 A + \eta_2 M)\).

Regarding Remark A.20, we can easily conclude from Lemma A.9:

**Remark A.23** Let \( A \in \mathbb{C}^{p \times q} \) and let \( B \in \mathbb{C}^{p \times r} \). Then \( \mathcal{R}(B) = \mathcal{R}(A) \) if and only if there exists a complex \( q \times r \) matrix \( R \) fulfilling \( \mathcal{N}(A) + \mathcal{R}(R) = \mathbb{C}^q \) and \( B = AR \).

**Remark A.24** The set \( \mathbb{C}^{q \times q}_H \) is an \( \mathbb{R} \)-vector space and \( \mathbb{C}^{q \times q}_\succ \) is a convex cone in \( \mathbb{C}^{q \times q}_H \).

**Remark A.25** Let \( A \in \mathbb{C}^{q \times q} \) and let \( X \in \mathbb{C}^{q \times p} \). Then \( X^*AX \in \mathbb{C}^{p \times p} \). If \( A \in \mathbb{C}^{q \times q}_\succ \), then \( X^*AX \in \mathbb{C}^{p \times p}_\succ \).

We now state a well-known characterization of Non-negative Hermitian block matrices in terms of the Schur complement (3.4):

**Lemma A.26** (cf. [13, Lem. 1.1.9 and 1.1.10]) Let \( \left[ \begin{array}{cc} A & B \\ \text{sps} & D \end{array} \right] \) be the block representation of a complex \((p + q) \times (p + q)\) matrix \( M \) with \( p \times p \) block \( A \). Then \( M \) is Non-negative Hermitian if and only if \( A \) and \( M/A := D - CA^\dagger B \) are both Non-negative Hermitian and furthermore \( \mathcal{R}(B) \subseteq \mathcal{R}(A) \) and \( C = B^* \) are fulfilled. In this case, \( \|B\|_2^2 \leq \|A\|_s \|D\|_s \).

**Lemma A.27** (cf. [28, Lem. A.13]) Let \( A, B \in \mathbb{C}^{q \times q}_H \) with \( O_{q \times q} \lesssim A \lesssim B \). Then \( \mathcal{R}(A) \subseteq \mathcal{R}(B) \) and \( \mathcal{N}(B) \subseteq \mathcal{N}(A) \). Furthermore, \( O_{q \times q} \preceq \mathbb{P}_{\mathcal{R}(A)} B^\dagger \mathbb{P}_{\mathcal{R}(A)} \preceq A^\dagger \).

We continue with some observations on the classes of EP matrices and almost definite matrices given in Definition 4.16. From Remark A.6 we easily see:

**Remark A.28** If \( A \in \mathbb{C}^{q \times q}_{\text{EP}} \), then \( \mathcal{R}(\text{Re} A) \subseteq \mathcal{R}(A) \) and \( \mathcal{R}(\text{Im} A) \subseteq \mathcal{R}(A) \).

**Remark A.29** If \( A \in \mathbb{C}^{q \times q}_{\text{AD}} \), then \( \eta A \in \mathbb{C}^{q \times q}_{\text{AD}} \) for all \( \eta \in \mathbb{C} \).

**Remark A.30** If \( A \in \mathbb{C}^{q \times q}_{\text{AD}} \), then \( \mathcal{N}(A^*) = \mathcal{N}(A) \) and \( \mathcal{R}(A^*) = \mathcal{R}(A) \).

**Remark A.31** Taking into account Remark A.30, one can easily check that \( \{ M \in \mathbb{C}^{q \times q} : \eta M \in \mathbb{C}^{q \times q}_{\succ} \} \subseteq \mathbb{C}^{q \times q}_{\text{AD}} \subseteq \mathbb{C}^{q \times q}_{\text{EP}} \) for all \( \eta \in \mathbb{C} \).

**Lemma A.32** Let \( A \in \mathbb{C}^{q \times q} \) satisfy \( \text{Im} A \in \mathbb{C}^{q \times q}_{\text{EP}} \). Then \( A \in \mathbb{C}^{q \times q}_{\text{EP}} \).

**Proof** Let \( x \in \mathcal{N}(A) \). Then \( x^* \text{Im}(A)x = 0 \). Since, by virtue of Remark A.31, we have \( \text{Im} A \in \mathbb{C}^{q \times q}_{\text{AD}} \), then \( \text{Im}(A)x = 0 \) follows. Consequently, \( A^*x = Ax = 0 \). Hence, \( \mathcal{N}(A) \subseteq \mathcal{N}(A^*) \), implying \( \mathcal{R}(A) = \mathcal{R}(A^*) \), i.e., \( A \in \mathbb{C}^{q \times q}_{\text{EP}} \).

**Lemma A.33** Let \( A \in \mathbb{C}^{q \times q} \) satisfy \( \text{Im} A \in \mathbb{C}^{q \times q}_{\succ} \) and \( \text{rank}(\text{Im} A) = \text{rank} A \). Then \( A \in \mathbb{C}^{q \times q}_{\text{AD}} \) and \( \mathcal{R}(\text{Im} A) = \mathcal{R}(A) \).
**Proof** Lemma A.32 yields \( \mathcal{R}(A^*) = \mathcal{R}(A) \). In view of Remark A.28 and rank(\( \text{Im} \, A \)) = rank \( A \), we get \( \mathcal{R}(\text{Im} \, A) = \mathcal{R}(A) \). Consider an arbitrary \( x \in \mathbb{C}^q \) with \( x^*Ax = 0 \). Then \( x^*A^*x = x^*Ax = 0 \). Consequently, \( x^* \text{Im} \, (A)x = 0 \). Since Remark A.31 yields \( \text{Im} \, A \in \mathbb{C}^{q \times q} \), we have \( \text{Im}(A)x = O_{q \times 1} \). Because of \( \mathcal{N}(\text{Im} \, A) = [\mathcal{R}((\text{Im} \, A)^*)] = [\mathcal{R}(\text{Im} \, A)] = [\mathcal{R}(A)] = [\mathcal{R}(A^*)] = \mathcal{N}(A) \), we get \( Ax = O_{q \times 1} \) and, thus, \( A \in \mathbb{C}^{q \times q} \).

A complex \( p \times q \) matrix \( K \) is said to be **contractive** if \( \| K \|_S \leq 1 \).

**Remark A.34** Let \( K \in \mathbb{C}^{p \times q} \). Then the matrix \( K \) is contractive if and only if \( I_q - K^*K \) is Non-negative Hermitian.

**Remark A.35** Let \( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \) be the block representation of a contractive complex \((p + q) \times (p + q)\) matrix \( K \) with \( p \times p \) block \( A \). Suppose that \( A \) or \( D \) is unitary. In view of Remark A.34 then one can easily see that \( B = O_{p \times q} \) and \( C = O_{q \times p} \).

A complex square matrix \( J \) is called a **signature matrix** if it satisfies \( J^* = J \) and \( J^2 = I \). In this paper, we focus on the particular signature matrices

\[
\tilde{J}_q := \begin{bmatrix} O_{q \times q} & iI_q \\ -iI_q & O_{q \times q} \end{bmatrix} \quad \text{and} \quad j_{pq} := \begin{bmatrix} -I_p & O_{p \times q} \\ O_{q \times p} & I_q \end{bmatrix}.
\tag{A.1}
\]

What concerns several aspects of the so-called \( J \)-Theory for arbitrary signature matrices \( J \), we refer to [13, §§1.3–1.4, p. 26–44].

**Remark A.36** If \( P, Q \in \mathbb{C}^{q \times p} \), then

\[
\begin{bmatrix} P \\ Q \end{bmatrix}^* \tilde{J}_q \begin{bmatrix} P \\ Q \end{bmatrix} = 2 \text{Im}(Q^*P) \quad \text{and} \quad \begin{bmatrix} P \\ Q \end{bmatrix}^* j_{pq} \begin{bmatrix} P \\ Q \end{bmatrix} = Q^*Q - P^*P.
\]

**Remark A.37** Let \( A \in \mathbb{C}^{p \times p} \) and let \( D \in \mathbb{C}^{q \times q} \). Then

\[
\begin{bmatrix} A & O_{p \times q} \\ O_{q \times p} & D \end{bmatrix} j_{pq} = \begin{bmatrix} -A & O_{p \times q} \\ O_{q \times p} & D \end{bmatrix} \tilde{J}_q = \begin{bmatrix} -A & O_{p \times q} \\ O_{q \times p} & D \end{bmatrix} \begin{bmatrix} A & O_{q \times p} \\ O_{q \times p} & D \end{bmatrix}.
\]

Furthermore, if \( p = q \), then

\[
\begin{bmatrix} A & O_{q \times q} \\ O_{q \times q} & D \end{bmatrix} j_{pq} = \begin{bmatrix} -A & O_{q \times q} \\ -iD & O_{q \times q} \end{bmatrix} \tilde{J}_q = \begin{bmatrix} -A & O_{q \times q} \\ -iD & O_{q \times q} \end{bmatrix} \begin{bmatrix} A & O_{q \times q} \\ O_{q \times q} & D \end{bmatrix}.
\]

In particular, \( \tilde{J}_q j_{qq} = -j_{qq} \tilde{J}_q \).

**B Integration with Respect to Non-negative Hermitian Measures**

Consider a measurable space \((\mathcal{X}, \mathcal{F})\) consisting of a non-empty set \(\mathcal{X}\) and a \(\sigma\)-algebra \(\mathcal{F}\) on \(\mathcal{X}\). A mapping \(\mu : \mathcal{X} \rightarrow \mathbb{C}_+^{q \times q}\) is called **non-negative Hermitian** \(q \times q\) measure on \((\mathcal{X}, \mathcal{F})\) if it is \(\sigma\)-additive, i.e., \(\mu(\bigcup_{n=1}^{\infty} X_n) = \sum_{n=1}^{\infty} \mu(X_n)\) holds true for all sequences \((X_n)_{n=1}^{\infty}\) of pairwise disjoint sets from \(\mathcal{X}\). We are using the integration theory of pairs of matrix-valued functions with respect to Non-negative Hermitian measures, developed independently by Kats [38] and Rosenberg [47–49] (cf. [13, Sec. 2.2]):

First consider an arbitrary ordinary measure \(\nu\) on a measurable space \((\mathcal{X}, \mathcal{F})\). A measurable function \(F : \mathcal{X} \rightarrow \mathbb{C}^{p \times q}\) is said to be **integrable with respect to \(\nu\)** if
Let $F = \{f_{jk}\}_{j=1,...,p}$ belong to $[\mathcal{L}^1(\nu)]^{p \times q}$, i.e., all entries $f_{jk}$ belong to the class $L^1(\nu)$ of functions $f : \mathcal{X} \to \mathbb{C}$, which are integrable with respect to $\nu$. In this case, let $\int_X F d\nu := \{\int_X f_{jk} d\nu\}_{j=1,...,p}$ for all $X \in \mathcal{X}$.

Throughout this part of the appendix, let an arbitrary Non-negative Hermitian $q \times q$ measure $\mu = [\mu_{jk}]_{j,k=1}^q$ be given. Then all the entries $\mu_{jk}$ are complex-valued measures and all the diagonal entries $\mu_{jj}$ are ordinary measures with values in $[0, \infty)$. Consequently, the trace measure $\tau := \text{tr} \mu = \sum_{j=1}^q \mu_{jj}$ of $\mu$ is an ordinary measure with values in $[0, \infty)$. For each $X \in \mathcal{X}$, from $\tau(X) = 0$ necessarily $\mu(X) = O_{q \times q}$ follows. Consequently, there exist $\tau$-a.e. uniquely determined Radon–Nikodym derivatives $d\mu_{jk}/d\tau$. The $\tau$-a.e. uniquely determined measurable mapping $\mu'_{jk} := [d\mu_{jk}/d\tau]_{j,k=1}^q$ is called trace derivative of $\mu$ and satisfies $\mu(X) = \int_X \mu'_{jk} d\tau$ for all $X \in \mathcal{X}$ and $O_{q \times q} \leq \mu'(x) \leq L_q$ for $\tau$-a.e. $x \in \mathcal{X}$.

A measurable function $f : \mathcal{X} \to \mathbb{C}$ is said to be integrable with respect to $\mu$ if $\int_{\mathcal{X}} |f| d\nu_{jk} < \infty$ holds true for all $j, k \in \mathbb{Z}_{1,q}$, where $\nu_{jk}$ denotes the variation of the complex measure $\mu_{jk}$. In this case, let $\int_X f d\mu := [\int_X f_{jk} d\mu_{jk}]_{j,k=1}^q$ for all $X \in \mathcal{X}$. Denote by $\mathcal{L}(\mu)$ the set of all such functions $f$, which are integrable with respect to $\mu$ in this sense.

**Remark B.1** Let $u \in \mathbb{C}^q$ and let $v := u^* \mu u$. Then $v$ is a bounded measure on $(\mathcal{X}, \mathcal{X})$, which is absolutely continuous with respect to $\tau$. For all $f \in \mathcal{L}(\mu)$, furthermore $\int_{\mathcal{X}} |f| d\nu < \infty$ and $\int_{\mathcal{X}} f d\nu = u^* (\int_{\mathcal{X}} f d\mu) u$.

**Remark B.2** The mapping defined on the $\mathbb{C}$-vector space $\mathcal{L}(\mu)$ by $f \mapsto \int_{\mathcal{X}} f d\mu$ is $\mathbb{C}$-linear.

**Remark B.3** If $f \in \mathcal{L}(\mu)$, then $\overline{f} \in \mathcal{L}(\mu)$ and $\int_{\mathcal{X}} \overline{f} d\mu = (\int_{\mathcal{X}} f d\mu)^*$.

An ordered pair $(\Phi, \Psi)$ consisting of measurable functions $\Phi : \mathcal{X} \to \mathbb{C}^{p \times q}$ and $\Psi : \mathcal{X} \to \mathbb{C}^{r \times q}$ is said to be left-integrable with respect to $\mu$ if the matrix-valued function $\Phi_{jk}^*, \Psi^* \in [L^1(\tau)]^{p \times r}$ in this case, let $\int_X \Phi_{jk} d\mu_{jk}^* := \int_X \Phi_{jk}^* d\tau$ for all $X \in \mathcal{X}$. In particular, denote by $\mathcal{L}^2_{p \times q}(\mu)$ the set of all measurable functions $\Phi : \mathcal{X} \to \mathbb{C}^{p \times q}$ for which the pair $(\Phi, \Psi)$ is left-integrable with respect to $\mu$.

**Remark B.4** If $\Phi \in \mathcal{L}^2_{p \times q}(\mu)$ and $\Psi \in \mathcal{L}^2_{r \times q}(\mu)$, then $(\Phi, \Psi)$ is left-integrable with respect to $\mu$.

**Remark B.5** If $\Phi, \Psi \in \mathcal{L}^2_{p \times q}(\mu)$, then $\int_X \Phi d\mu \Psi^* = (\int_X \Psi d\mu \Phi^*)^*$ and $\int_X \Phi d\mu \Phi^* \in \mathbb{C}^{r \times p}$ for all $X \in \mathcal{X}$.

**Lemma B.6** Let $\Phi \in \mathcal{L}^2_{p \times q}(\mu)$ and let $\Psi \in \mathcal{L}^2_{r \times q}(\mu)$. For all $X \in \mathcal{X}$, then

$$\mathcal{R}(\int_X \Phi d\mu \Psi^*) \subseteq \mathcal{R}(\int_X \Phi d\mu \Phi^*), \quad \mathcal{N}(\int_X \Phi d\mu \Phi^*) \subseteq \mathcal{N}(\int_X \Psi d\mu \Phi^*),$$

and

$$\left(\int_X \Psi d\mu \Phi^*\right) \left(\int_X \Phi d\mu \Phi^*\right) = \int_X \Psi d\mu \Psi^*.$$
**Proof** Consider an arbitrary \( X \in \mathcal{X} \). First observe that \( \Theta := \begin{bmatrix} \Phi \\ \Psi \end{bmatrix} \) is a measurable function satisfying

\[
\Theta \mu_\tau' \Theta^* = \begin{bmatrix} \Phi \\ \Psi \end{bmatrix} \mu_\tau' [\Phi^*, \Psi^*] = \begin{bmatrix} \Phi \mu_\tau' \Phi^* & \Phi \mu_\tau' \Psi^* \\
\Psi \mu_\tau' \Phi^* & \Psi \mu_\tau' \Psi^* \end{bmatrix}.
\]

By assumption, we have \( \Phi \mu_\tau' \Phi^* \in [L^1(\tau)]^{p \times p} \) and \( \Psi \mu_\tau' \Psi^* \in [L^1(\tau)]^{r \times r} \). Due to Remark B.4, the pairs \((\Phi, \Psi)\) and \((\Psi, \Phi)\) are both left-integrable with respect to \( \mu \), i.e., \( \Phi \mu_\tau \Psi^* \in [L^1(\tau)]^{p \times r} \) and \( \Psi \mu_\tau \Phi^* \in [L^1(\tau)]^{r \times p} \) are true. Thus, we infer \( \Theta \mu_\tau \Theta \in [L^1(\tau)]^{(p+r) \times (p+r)} \), i.e., \( \Theta \in L_{(p+r) \times q}^2(\mu) \). Setting \( A := \int_X \Phi \text{d} \mu \Phi^* \), \( B := \int_X \Phi \text{d} \mu \Psi^* \), \( C := \int_X \Psi \text{d} \mu \Phi^* \), and \( D := \int_X \Psi \text{d} \mu \Psi^* \), we get

\[
\begin{bmatrix} A & B \\
C & D \end{bmatrix} = \int_X \begin{bmatrix} \Phi \mu_\tau' \Phi^* & \Phi \mu_\tau' \Psi^* \\
\Psi \mu_\tau' \Phi^* & \Psi \mu_\tau' \Psi^* \end{bmatrix} \text{d} \tau = \int_X \Theta \mu_\tau' \Theta^* \text{d} \tau = \int_X \Theta \text{d} \mu \Theta^*. 
\]

Because of Remark B.5, the matrix \( \int_X \Theta \text{d} \mu \Theta^* \) is Non-negative Hermitian. Using Lemma A.26, we can conclude then \( \mathcal{R}(B) \subseteq \mathcal{R}(A) \), \( C = B^* \), and that the matrices \( A, D \), and \( E/A = D = CA^\dagger B \) are Non-negative Hermitian. By virtue of Remark A.14, thus \( D \) and \( CA^\dagger B \) are Hermitian matrices, which satisfy \( CA^\dagger B \preceq D \). Taking into account Remark A.10, we can furthermore infer \( \mathcal{N}(A) \subseteq \mathcal{N}(C) \). \( \Box \)

**Remark B.7** Let \( f : \mathcal{X} \rightarrow \mathbb{C} \) be a measurable function and let \( \Phi, \Psi : \mathcal{X} \rightarrow \mathbb{C}^{q \times q} \) be defined by \( \Phi(x) := f(x)I_q \) and \( \Psi(x) := I_q \). Then \( f \in \mathcal{L}(\mu) \) if and only if \((\Phi, \Psi)\) is left-integrable with respect to \( \mu \). In this case, \( f \mu_\tau' \in [L^1(\tau)]^{q \times q} \) and \( \int_X f \text{d} \mu = \int_X (f \mu_\tau') \text{d} \tau = \int_X \Phi \text{d} \mu \Phi^* \) for all \( X \in \mathcal{X} \), where \( \mu_\tau' \) denotes the trace derivative of \( \mu \).

Using Remark B.7 and Lemma B.6, the following result can be easily verified:

**Lemma B.8** (cf. [22, Lem. B.2(b)]) If \( f \in \mathcal{L}(\mu) \), then \( \mathcal{R}(\int_X f \text{d} \mu) \subseteq \mathcal{R}(\mu(\mathcal{X})) \) and \( \mathcal{N}(\mu(\mathcal{X})) \subseteq \mathcal{N}(\int_X f \text{d} \mu) \).

**Lemma B.9** Let \( g : \mathcal{X} \rightarrow \mathbb{R} \) satisfy \( g \in \mathcal{L}(\mu) \) and \( \mu(\{g \leq 0\}) = O_{q \times q} \). Then \( \mathcal{R}(\int_X g \text{d} \mu) = \mathcal{R}(\mu(\mathcal{X})) \) and \( \mathcal{N}(\int_X g \text{d} \mu) = \mathcal{N}(\mu(\mathcal{X})) \).

**Proof** Consider an arbitrary \( u \in \mathcal{N}(\int_X g \text{d} \mu) \). In view of Remark B.1, then \( \nu := u^* \mu u \) is a bounded measure with \( \int_X g \text{d} \nu = u^* (\int_X g \text{d} \mu) u = 0 \) and \( \nu(\{g \leq 0\}) = 0 \). Thus, \( \nu(\mathcal{X}) = 0 \). Therefore, \( u^* \mu(\mathcal{X}) u = 0 \). Since, by virtue of \( \mu(\mathcal{X}) \in \mathbb{C}^{q \times q}_\mathbb{R} \) and Remark A.31, we have \( \mu(\mathcal{X}) \in \mathbb{C}^{q \times q}_\mathbb{R} \), we conclude then \( u \in \mathcal{N}(\mu(\mathcal{X})) \). So we have \( \mathcal{N}(\int_X g \text{d} \mu) \subseteq \mathcal{N}(\mu(\mathcal{X})) \). Observe that, due to Lemma B.8, furthermore \( \mathcal{N}(\mu(\mathcal{X})) \subseteq \mathcal{N}(\int_X g \text{d} \mu) \) holds true. Hence, \( \mathcal{N}(\int_X g \text{d} \mu) = \mathcal{N}(\mu(\mathcal{X})) \). Using Remarks B.3 and A.10, we can then easily infer \( \mathcal{R}(\int_X g \text{d} \mu) = \mathcal{R}(\mu(\mathcal{X})) \). \( \Box \)

We will particularly apply the following result on integrable functions \( f \) satisfying \( \text{Re} f > 0 \) or \( \text{Im} f > 0 \).

Lemma B.10  Let $f \in \mathcal{L}(\mu)$, let $\eta, \theta \in \mathbb{R}$, and let $g := \eta \Re f + \theta \Im f$. Suppose that $\mu([g \leq 0]) = O_{q \times q}$. Then $g \in \mathcal{L}(\mu)$ with $\mathcal{R}(\int_{\mathcal{X}} f \, d\mu) = \mathcal{R}(\int_{\mathcal{X}} g \, d\mu)$.

\begin{align}
\mathcal{R}(\int_{\mathcal{X}} g \, d\mu) &= \mathcal{R}(\mu(\mathcal{X})), \\
\mathcal{N}(\int_{\mathcal{X}} g \, d\mu) &= \mathcal{N}(\mu(\mathcal{X})), \\
\mathcal{R}(\int_{\mathcal{X}} f \, d\mu) &= \mathcal{R}(\mu(\mathcal{X})), \\
\mathcal{N}(\int_{\mathcal{X}} f \, d\mu) &= \mathcal{N}(\mu(\mathcal{X})).
\end{align}

\textbf{Proof} Because of Remarks B.3 and B.2, the real-valued function $g$ belongs to $\mathcal{L}(\mu)$. Hence, Lemma B.9 yields (B.1). Let $h \in \{f, f^\ast\}$. By virtue of Remark B.3, we have $h \in \mathcal{L}(\mu)$. Consider an arbitrary $u \in \mathcal{N}(\int_{\mathcal{X}} h \, d\mu)$. In view of Remark B.1, then $v := u^* \mu u$ is a bounded measure and $\int_{\mathcal{X}} h \, d\nu = u^*(\int_{\mathcal{X}} h \, d\mu)u = 0$. Remarks B.3 and B.2 provide us $\int_{\mathcal{X}} \Re h \, d\nu = 0$ and $\int_{\mathcal{X}} \Im h \, d\nu = 0$. Regarding $\Re h = \Re f$ as well as $\Im h = \Im f$ if $f = f^\ast$ and $\Im h = -\Im f$ if $h = f^\ast$, we thus obtain $\int_{\mathcal{X}} \Re f \, d\nu = 0$ and $\int_{\mathcal{X}} \Im f \, d\nu = 0$. Remark B.2 implies $\int_{\mathcal{X}} g \, d\nu = 0$. The assumption $\nu([g \leq 0]) = 0$ yields then $\nu(\mathcal{X}) = 0$. So we have $u^* \mu(\mathcal{X})u = 0$. Since, by virtue of $\mu(\mathcal{X}) \in \mathbb{C}^{q \times q}$ and Remark A.31, we have $u \in \mathcal{N}(\mu(\mathcal{X}))$.

Observe that, due to Lemma B.8, furthermore $\mathcal{N}(\mu(\mathcal{X})) \subseteq \mathcal{N}(\int_{\mathcal{X}} h \, d\mu)$ holds true. Consequently, $\mathcal{N}(\int_{\mathcal{X}} h \, d\mu) = \mathcal{N}(\mu(\mathcal{X}))$. Using Remarks B.3 and A.10, we easily conclude $h \in \mathcal{L}(\mu)$ and $\mathcal{R}(\int_{\mathcal{X}} h \, d\mu) = \mathcal{R}(\int_{\mathcal{X}} h \, d\mu^\ast) = \mathcal{R}(\mu(\mathcal{X}))$. Choosing the appropriate $h$, we thus obtain (B.2) and $\mathcal{R}(\int_{\mathcal{X}} f \, d\mu^\ast) = \mathcal{R}(\mu(\mathcal{X})) = \mathcal{R}(\int_{\mathcal{X}} f \, d\mu)$.

We end this section with a matricial version of Lebesgue’s dominated convergence theorem:

\textbf{Proposition B.11} ([26, Prop. A.6]) Let $f, f_1, f_2, \ldots : \mathcal{X} \to \mathbb{C}$ be measurable functions and let $g \in \mathcal{L}(\mu)$ be such that $\lim_{n \to \infty} f_n(x) = f(x)$ for $\tau$-almost all $x \in \mathcal{X}$ and $|f_n(x)| \leq |g(x)|$ for all $n \in \mathbb{N}$ and $\tau$-almost all $x \in \mathcal{X}$ hold true. Then the functions $f, f_1, f_2, \ldots$ belong to $\mathcal{L}(\mu)$ and $\lim_{n \to \infty} \int_{\mathcal{X}} f_n \, d\mu = \int_{\mathcal{X}} f \, d\mu$.

\section{The Stieltjes Transform of Non-negative Hermitian Measures}

In this section, we consider a non-empty closed subset $\Omega$ of $\mathbb{R}$ and a Non-negative Hermitian $q \times q$ measure $\sigma$ on $(\Omega, \mathcal{B}(\Omega))$. So $\Omega$ is also a closed subset of $\mathbb{C}$, whereas $\mathcal{G} := \mathbb{C} \setminus \Omega$ is an open subset of $\mathbb{C}$. Observe that, for each $t \in \Omega$, the function $h_t : \mathcal{G} \to \mathbb{C}$ defined by $h_t(z) := 1/(t - z)$ is holomorphic. Consider an arbitrary $z \in \mathcal{G}$ and let $d_z := \inf_{x \in \Omega} |x - z|$. Then $d_z > 0$ and $1/|t - z| \leq 1/d_z$ for all $t \in \Omega$. Consequently, the function $g_z : \Omega \to \mathbb{C}$ defined by $g_z(t) := 1/(t - z)$ belongs to $\mathcal{L}(\sigma)$. For each closed disk $K \subseteq \mathcal{G}$, we have with $d_K := \inf_{(x, w) \in \Omega \times K} |x - w|$, furthermore $d_K > 0$ and $1/|t - z| \leq 1/d_K$. By means of that, one can check that the matrix-valued function $\hat{S}_\sigma : \mathcal{G} \to \mathbb{C}^{q \times q}$ defined by

$$
\hat{S}_\sigma(z) := \int_{\Omega} \frac{1}{t - z} \sigma(dt)
$$

\[\text{(C.1)}\]
is holomorphic (see, e.g. [20, Satz 5.8, p. 147, Kapitel IV]). The mapping $\sigma \mapsto \hat{\Sigma}_\sigma$ is called Stieltjes transformation. Accordingly, the function $\hat{\Sigma}_\sigma$ itself is called Stieltjes transform of $\sigma$. For $\Omega = \mathbb{R}$, the restriction of $\hat{\Sigma}_\sigma$ onto $\Pi_+$ is exactly the $\mathbb{R}$-Stieltjes Transform $\Sigma_\sigma$ of $\sigma$ introduced in Definition 5.1. Thus, we obtain:

**Lemma C.1** Denote by $F$ the restriction of $\hat{\Sigma}_\sigma$ onto $\Pi_+$. Then $F \in \mathcal{R}_{0,q}(\Pi_+)$ and the spectral measure $\sigma_F$ of $F$ fulfills $\sigma_F(\mathbb{R} \setminus \Omega) = O_{q \times q}$ and $\sigma_F(B) = \sigma(B)$ for all $B \in \mathcal{B}_{\Omega}$.

**Proof** Let $\chi : \mathcal{B}_{\mathbb{R}} \to \mathbb{C}^{q \times q}$ be defined by $\chi(B) := \sigma(B \cap \Omega)$. Then $\chi$ is a Non-negative Hermitian $q \times q$ measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ satisfying $F(z) = \int_{\mathbb{R}} (t-z)^{-1} \chi(dt)$ for all $z \in \Pi_+$. By virtue of Proposition 5.2, then $F \in \mathcal{R}_{0,q}(\Pi_+)$ and $\chi = \sigma_F$ follow. $\Box$

For $\Omega = [\alpha, \beta]$ the function $\hat{\Sigma}_\sigma$ coincides with the $[\alpha, \beta]$-Stieltjes Transform $\hat{\Sigma}_\sigma$ of $\sigma$ introduced in Definition 5.3. By virtue of (C.1), we easily see:

**Remark C.2** For all $z \in \mathbb{C} \setminus \Omega$, we have

$$\text{Re}\ \hat{\Sigma}_\sigma(z) = \int_{\Omega} \frac{t - \text{Re} z}{|t-z|^2} \sigma(dt) \quad \text{and} \quad \text{Im}\ \hat{\Sigma}_\sigma(z) = \int_{\Omega} \frac{\text{Im} z}{|t-z|^2} \sigma(dt).$$

**Lemma C.3** For all $z \in \mathbb{C} \setminus \mathbb{R}$, the matrix $\frac{1}{\text{Im} z} \text{Im}\ \hat{\Sigma}_\sigma(z)$ is Non-negative Hermitian. Furthermore, $\text{Im}\ \hat{\Sigma}_\sigma(x) = O_{q \times q}$ for all $x \in \mathbb{R} \setminus \Omega$. Moreover, for all $w \in \mathbb{C}$ with $\text{Re} w < \inf \Omega$, the matrix $\text{Re}\ \hat{\Sigma}_\sigma(w)$ is Non-negative Hermitian and, for all $w \in \mathbb{C}$ with $\text{Re} w > \sup \Omega$, the matrix $-\text{Re}\ \hat{\Sigma}_\sigma(w)$ is Non-negative Hermitian.

**Proof** Except for $\text{Im}\ \hat{\Sigma}_\sigma(x) = O_{q \times q}$ for all $x \in \mathbb{R} \setminus \Omega$, the assertions are a consequence of Remark C.2. Consider now an arbitrary $x \in \mathbb{R} \setminus \Omega$. The matrix-valued function $\hat{\Sigma}_\sigma$ is holomorphic. Hence, the two sequences $(\pm \text{Im} \hat{\Sigma}_\sigma(x \pm i/n))_{n=1}^\infty$ converge to $\pm \text{Im} \hat{\Sigma}_\sigma(x)$, resp. As already mentioned, we have $\frac{1}{\text{Im} z} \text{Im}\ \hat{\Sigma}_\sigma(z) \in \mathbb{C}_n^{q \times q}$ for all $z \in \mathbb{C} \setminus \mathbb{R}$. In particular, the sequences $(\pm \text{Im} \hat{\Sigma}_\sigma(x \pm i/n))_{n=1}^\infty$ both consist of Non-negative Hermitian matrices. Consequently, we obtain $\pm \text{Im} \hat{\Sigma}_\sigma(x) \in \mathbb{C}_n^{q \times q}$ for their limits, implying $\text{Im}\ \hat{\Sigma}_\sigma(x) = O_{q \times q}$. $\Box$

Using Proposition B.11, one can easily deduce the following representations for $\sigma(\Omega)$ via limits along the imaginary axis (cf. [26, Lem. A.8(c)]):

**Lemma C.4** The following equations hold true:

$$\lim_{y \to \infty} y \text{Re}\ \hat{\Sigma}_\sigma(iy) = O_{q \times q}, \quad \lim_{y \to \infty} y \text{Im}\ \hat{\Sigma}_\sigma(iy) = \sigma(\Omega), \quad \text{and} \quad \lim_{y \to \infty} iy\hat{\Sigma}_\sigma(iy) = -\sigma(\Omega).$$

The following lemma in particular shows that the matrix-valued function $\hat{\Sigma}_\sigma$ has constant column and null space on $\mathbb{C} \setminus [\inf \Omega, \sup \Omega]$. 
Lemma C.5 (cf. [26, Lem. A.8(b)]) For all \( z \in \mathbb{C} \setminus [\inf \Omega, \sup \Omega] \), the equations
\[
\mathcal{R}(\hat{S}_\sigma(z)) = \mathcal{R}(\sigma(\Omega)) \quad \text{and} \quad \mathcal{N}(\hat{S}_\sigma(z)) = \mathcal{N}(\sigma(\Omega)) \tag{C.2}
\]
hold true. Furthermore,
\[
\mathcal{R}(\im \hat{S}_\sigma(z)) = \mathcal{R}(\sigma(\Omega)) \quad \text{and} \quad \mathcal{N}(\im \hat{S}_\sigma(z)) = \mathcal{N}(\sigma(\Omega)) \tag{C.3}
\]
are valid for all \( z \in \mathbb{C} \setminus \mathbb{R} \) and
\[
\mathcal{R}(\re \hat{S}_\sigma(w)) = \mathcal{R}(\sigma(\Omega)) \quad \text{and} \quad \mathcal{N}(\re \hat{S}_\sigma(w)) = \mathcal{N}(\sigma(\Omega)) \tag{C.4}
\]
are fulfilled for all \( w \in \mathbb{C} \) with \( \re w < \inf \Omega \) or \( \re w > \sup \Omega \).

Proof For each \( \zeta \in \mathbb{C} \setminus \Omega \), let \( g_\zeta : \Omega \to \mathbb{C} \) be defined by \( g_\zeta(t) := 1/(t - \zeta) \). As already mentioned at the beginning of this section, we have \( g_\zeta \in \mathcal{L}(\sigma) \) for each \( \zeta \in \mathbb{C} \setminus \Omega \). Because of Remarks B.3 and B.2, then the functions \( \re g_\zeta \) and \( \im g_\zeta \) both belong to \( \mathcal{L}(\sigma) \) and, in view of \( \hat{S}_\sigma(\zeta) = \int_\Omega g_\zeta \, d\sigma \), fulfill \( \re \hat{S}_\sigma(\zeta) = \int_\Omega \re g_\zeta \, d\sigma \) and \( \im \hat{S}_\sigma(\zeta) = \int_\Omega \im g_\zeta \, d\sigma \) for each \( \zeta \in \mathbb{C} \setminus \Omega \). Furthermore, we have \( \re g_\zeta(t) = (t - \re \zeta)/|t - \zeta|^2 \) and \( \im g_\zeta(t) = \im \zeta/|t - \zeta|^2 \) for each \( \zeta \in \mathbb{C} \setminus \Omega \). Consider now an arbitrary \( z \in \mathbb{C} \setminus \mathbb{R} \). In the case \( \im z > 0 \), we have \( \im g_z > 0 \) and thus the application of Lemma B.10 with \( f = g_z, \eta = 0, \) and \( \theta = 1 \) yields then (C.3) and (C.2). If \( \im z < 0 \), then \( -\im g_z > 0 \) and we can infer (C.3) and (C.2) in a similar way with \( \theta = -1 \). Now let \( w \in \mathbb{C} \) with \( \re w < \inf \Omega \) or \( \re w > \sup \Omega \). In the case \( \re w < \inf \Omega \), we have \( \re g_w > 0 \) and thus the application of Lemma B.10 with \( f = g_w, \eta = 1, \) and \( \theta = 0 \) yields then (C.4) and (C.2). If \( \re w > \sup \Omega \), then \( -\re g_w > 0 \) and we can infer (C.4) and (C.2) in a similar way with \( \eta = -1 \). \( \square \)

Recall that a matrix \( A \in \mathbb{C}^{q \times q} \) is called \textit{EP matrix} if \( \mathcal{R}(A^*) = \mathcal{R}(A) \) and that \( A \) is said to be \textit{almost definite} if each \( x \in \mathbb{C}^q \) with \( x^*Ax = 0 \) necessarily fulfills \( Ax = O_{q \times 1} \). The corresponding classes are denoted by \( \mathbb{C}_\text{EP}^{q \times q} \) and \( \mathbb{C}_\text{AD}^{q \times q} \) (cf. Definition 4.16).

In view of (C.1), we obviously have \( [\hat{S}_\sigma(z)]^* = \hat{S}_\sigma(\overline{z}) \) for all \( z \in \mathbb{C} \setminus \Omega \). Consequently, from Lemma C.5 we can infer \( \mathcal{R}([\hat{S}_\sigma(z)]^*) = \mathcal{R}(\hat{S}_\sigma(z)) \), i.e., \( \hat{S}_\sigma(z) \in \mathbb{C}_\text{EP}^{q \times q} \) for all \( z \in \mathbb{C} \setminus [\inf \Omega, \sup \Omega] \). Regarding Remark A.31, the values of \( \hat{S}_\sigma \) satisfy a stronger condition:

Lemma C.6 For all \( z \in \mathbb{C} \setminus [\inf \Omega, \sup \Omega] \), we have \( \hat{S}_\sigma(z) \in \mathbb{C}_\text{AD}^{q \times q} \).

Proof First consider an arbitrary \( z \in \mathbb{C} \setminus \mathbb{R} \). Let \( \eta := \frac{1}{\im z} \). Lemma C.3 yields \( \im(\eta[\hat{S}_\sigma(z)]) \in \mathbb{C}_\geq^{q \times q} \). From Lemma C.5 we can infer rank \( \im \hat{S}_\sigma(z) = \rank \hat{S}_\sigma(z) \), implying rank \( \im(\eta[\hat{S}_\sigma(z)]) = \rank(\eta[\hat{S}_\sigma(z)]) \). Thus, we can apply Lemma A.33 to obtain \( \eta[\hat{S}_\sigma(z)] \in \mathbb{C}^{q \times q}_\text{AD} \). By virtue of Remark A.29, then \( \hat{S}_\sigma(z) \in \mathbb{C}_\text{AD}^{q \times q} \) follows. Now consider an arbitrary \( x \in \mathbb{R} \setminus [\inf \Omega, \sup \Omega] \). Assume \( x > \sup \Omega \). From Lemma C.3 we can infer then \( -\hat{S}_\sigma(x) = -\re \hat{S}_\sigma(x) \in \mathbb{C}^{q \times q}_\geq \). Because of Remark A.31, hence \( -\hat{S}_\sigma(x) \in \mathbb{C}_\text{AD}^{q \times q} \). By virtue of Remark A.29, thus \( \hat{S}_\sigma(x) \in \mathbb{C}_\text{AD}^{q \times q} \). If \( x < \inf \Omega \),
then Lemma C.3 yields \( \hat{S}_\sigma(x) = \text{Re} \hat{S}_\sigma(x) \in \mathbb{C}^{q \times q} \), implying \( \hat{S}_\sigma(x) \in \mathbb{C}^{q \times q} \) by Remark A.31.

**Lemma C.7** For all \( z \in \mathbb{C} \setminus \mathbb{R} \), the following matrix inequalities hold true:

\[
[\hat{S}_\sigma(z)]^*[\frac{1}{\text{Im} z} \text{Im} \hat{S}_\sigma(z)]^t [\hat{S}_\sigma(z)] \preceq \sigma(\Omega) \text{ and } [\hat{S}_\sigma(z)]^t [\hat{S}_\sigma(z)]^* \preceq \frac{1}{\text{Im} z} \text{Im} \hat{S}_\sigma(z).
\]

**Proof** Consider an arbitrary \( z \in \mathbb{C} \setminus \mathbb{R} \). Let \( g_z : \Omega \to \mathbb{C} \) be defined by \( g_z(t) := 1/(t-z) \) and let \( \Lambda, \Xi : \Omega \to \mathbb{C}^{q \times q} \) be defined by \( \Lambda(t) := |g_z(t)|^2 I_q \) and \( \Xi(t) := I_q \). As already mentioned at the beginning of this section, we have \( g_z \in \mathcal{L}(\sigma) \). Because of \( \text{Im} g_z/\text{Im} z = |g_z|^2 \), we can conclude with Remarks B.2 and B.3 then \(|g_z|^2 \in \mathcal{L}(\sigma)\).

According to Remark B.7, hence the pair \((\Lambda, \Xi)\) is left-integrable with respect to \( \sigma \). Consequently, the matrix-valued function \( \Lambda \sigma' \Xi^* \) belongs to \([\mathcal{L}^1(\tau)]^q \times q\), where \( \tau \) denotes the trace measure of \( \sigma \) and \( \sigma'_t \) is the trace derivative of \( \sigma \).

Let \( \Theta : \Omega \to \mathbb{C}^{q \times q} \) be defined by \( \Theta(t) := g_z(t)I_q \). Then \( \Theta \) is measurable and fulfills \( \Lambda \sigma'_t \Xi^* = \Theta \sigma'_t \Theta^* \).

Thus, the pair \((\Theta, \Theta)\) is left-integrable with respect to \( \sigma \), i.e., \( \Theta \in \mathcal{L}^2_{q \times q} (\sigma) \). By virtue of Remarks B.7 and B.2, we have

\[
\int_\Omega \Theta d\sigma \Theta^* = \int_\Omega \Theta \sigma'_t \Theta^* d\tau = \int_\Omega \Lambda \sigma'_t \Xi^* d\tau = \int_\Omega \Lambda d\sigma \Xi^* = \int_\Omega |g_z|^2 d\sigma = \frac{1}{\text{Im} z} \text{Im} \hat{S}_\sigma(z).
\]

Furthermore, \( \Xi \) belongs to \( \mathcal{L}^2_{q \times q} (\sigma) \) and fulfills

\[
\int_\Omega \Xi d\sigma \Xi^* = \int_\Omega \Xi \sigma'_t \Xi^* d\tau = \int_\Omega \sigma'_t d\tau = \sigma(\Omega).
\]

In view of Remark B.4, then the pairs \((\Theta, \Xi)\) and \((\Xi, \Theta)\) are both left-integrable with respect to \( \sigma \). Using Remarks B.7 and B.5, we then can infer

\[
\int_\Omega \Theta d\sigma \Xi^* = \int_\Omega g_z d\sigma = \hat{S}_\sigma(z) \quad \text{and} \quad \int_\Omega \Xi d\sigma \Theta^* = (\int_\Omega \Theta d\sigma \Xi^*)^* = [\hat{S}_\sigma(z)]^*.
\]

The proof is completed by applying Lemma B.6 once with \( \Phi = \Theta \) and \( \Psi = \Xi \) and a second time with \( \Phi = \Xi \) and \( \Psi = \Theta \). \( \square \)

In combination with Lemma C.7, the following result reveals a certain minimality of the Non-negative Hermitian matrix \( \sigma(\Omega) \) with respect to the Löwner partial order:

**Lemma C.8** Let \( A \in \mathbb{C}^{q \times q} \) be such that \( \mathcal{R}(\sigma(\Omega)) \subseteq \mathcal{R}(A) \) and \( [\hat{S}_\sigma(z)]^* A^\dagger [\hat{S}_\sigma(z)] \preceq \frac{1}{\text{Im} z} \text{Im} \hat{S}_\sigma(z) \) for all \( z \in \Pi_+ \). Then \( \sigma(\Omega) \preceq A \).

**Proof** Denote by \( F \) the restriction of \( \hat{S}_\sigma \) onto \( \Pi_+ \). Because of Lemma C.1, then \( F \in \mathcal{R}_0 q (\Pi_+) \) and \( \sigma_F(\mathbb{R}) = \sigma(\Omega) \). Taking additionally into account Lemma C.5 and the assumptions, we get, for all \( z \in \Pi_+ \), then \( \mathcal{R}(F(z)) \subseteq \mathcal{R}(A) \) and furthermore \( \frac{1}{\text{Im} z} \text{Im} F(z) = [F(z)]^* A^\dagger [F(z)] \in \mathbb{C}^{q \times q} \). By virtue of Lemma A.26, hence the matrix...
\[
\begin{bmatrix}
A & F(z) \\
[F(z)]^* \frac{1}{1 - \frac{1}{z}} \Im F(z)
\end{bmatrix}
\]
is Non-negative Hermitian for all \( z \in \mathbb{H}_+ \). The application of Lemma 4.5 thus yields \( \sigma_F(\mathbb{R}) \preceq A \), implying the assertion. \( \square \)

### D Particular Pairs of Matrices

Regular pairs of matrices considered in this section implement the extension of the set of complex matrices by corresponding points at infinity analogous to the transition from the affine to the projective space. In this sense, they can be thought of as homogeneous coordinates.

**Definition D.1** An ordered pair \([P; Q]\) of complex \( p \times q \) matrices \( P \) and \( Q \) is called \( p \times q \) matrix pair. Such a pair is said to be regular if \( \text{rank} \left[ \begin{bmatrix} P \\ Q \end{bmatrix} \right] = q \) and proper if \( \text{rank} \ Q = q \).

Each \( p \times q \) matrix pair \([P; Q]\) generates a linear relation \( R := \{ (Qv, Pv) : v \in \mathbb{C}^q \} \) in the \( \mathbb{C} \)-vector space \( \mathbb{C}^p \). In accordance with that, we associate to each \( p \times q \) matrix pair \([P; Q]\) the linear subspaces \( \mathcal{R}(\begin{bmatrix} P \\ Q \end{bmatrix}), \mathcal{R}(Q), \mathcal{R}(P), Q(\mathcal{N}(P)), \) and \( P(\mathcal{N}(Q)) \). Obviously, we have \( Q(\mathcal{N}(P)) \subseteq \mathcal{R}(Q) \) and \( P(\mathcal{N}(Q)) \subseteq \mathcal{R}(P) \). Consequently, we get for

\[
\text{rank}([P; Q]) := \dim \mathcal{R}(P) - \dim(P(\mathcal{N}(Q))) \tag{D.1}
\]

the inequalities \( 0 \leq \text{rank}([P; Q]) \leq q \).

**Lemma D.2** Let \([P; Q]\) be a \( p \times q \) matrix pair. Then

\[
\dim \mathcal{R}(Q) + \dim(P(\mathcal{N}(Q))) = \dim \left( \begin{bmatrix} P \\ Q \end{bmatrix} \right) = \dim \mathcal{R}(P) + \dim(Q(\mathcal{N}(P))).
\]

**Proof** Let \( U := \mathcal{N}(P) \) and \( V := \mathcal{N}(Q) \). The mappings \( \phi : U \to \mathbb{C}^p \) and \( \psi : V \to \mathbb{C}^p \) defined by \( \phi(u) := Qu \) and \( \psi(v) := Pv \), resp., are \( \mathbb{C} \)-linear with \( \ker \phi = U \cap V, \phi(U) = Q(U), \) ker \( \psi = V \cap U, \) and \( \psi(V) = P(V) \). Regarding \( \dim \ker \phi + \dim \phi(U) = \dim U \) and \( \dim \ker \psi + \dim \psi(V) = \dim V \), then \( \dim U = \dim(U \cap V) + \dim(Q(\mathcal{N}(P))) \) and \( \dim V = \dim(U \cap V) + \dim(P(\mathcal{N}(Q))) \) follow. The application of Remark A.3 yields \( \dim \mathcal{R}(P) + \dim U = q \) and \( \dim \mathcal{R}(Q) + \dim V = q \). By virtue of Remark A.5, we see from Remark A.3 that \( \dim \mathcal{R}(\begin{bmatrix} P \\ Q \end{bmatrix}) + \dim(U \cap V) = q \). Taking all together, we obtain

\[
\dim \mathcal{R}(Q) + \dim(P(\mathcal{N}(Q))) = q - \dim(U \cap V) = \dim \left( \begin{bmatrix} P \\ Q \end{bmatrix} \right)
\]

and

\[
\dim \mathcal{R}(P) + \dim(Q(\mathcal{N}(P))) = q - \dim(U \cap V) = \dim \left( \begin{bmatrix} P \\ Q \end{bmatrix} \right). \quad \square
\]
A generalization of Remark A.3 for \( p \times q \) matrix pairs immediately follows from Lemma D.2:

**Remark D.3** The equation \( \text{rk}([P; Q]) + \dim(Q(\mathcal{N}(P))) = \dim \mathcal{R}(Q) \) holds true for each \( p \times q \) matrix pair \([P; Q]\).

**Lemma D.4** Let \([P; Q]\) be a \( p \times q \) matrix pair with \( \text{rk}([P; Q]) = q \). Then \( \text{rank} P = q \), \( \text{rank} Q = q \), \( Q(\mathcal{N}(P)) = \{O_{p \times 1}\} \), and \( P(\mathcal{N}(Q)) = \{O_{p \times 1}\} \). In particular, the pair \([P; Q]\) is proper.

**Proof** By assumption and (D.1), we have

\[
q = \text{rk}([P; Q]) = \dim \mathcal{R}(P) - \dim(P(\mathcal{N}(Q))) \leq \dim \mathcal{R}(P) = \text{rank} P \leq q.
\]

Consequently, \( P(\mathcal{N}(Q)) = \{O_{p \times 1}\} \) and \( \text{rank} P = q \). Thus, \( \mathcal{N}(P) = \{O_{q \times 1}\} \) and hence \( Q(\mathcal{N}(P)) = \{O_{p \times 1}\} \). We infer from \( P(\mathcal{N}(Q)) = \{O_{p \times 1}\} \) furthermore \( \mathcal{N}(Q) \subseteq \mathcal{N}(P) \), implying \( \mathcal{N}(Q) = \{O_{q \times 1}\} \). Therefore, \( \text{rank} Q = q \), i.e., \([P; Q]\) is proper. \( \square \)

Each proper \( p \times q \) matrix pair \([P; Q]\) is necessarily regular with \( P(\mathcal{N}(Q)) = \{O_{p \times 1}\} \) and \( \text{rk}([P; Q]) = \dim \mathcal{R}(P) \). Furthermore, using Remarks A.3, A.4, and A.5, the following result is readily checked:

**Remark D.5** Let \([P; Q]\) be a \( p \times q \) matrix pair. Then \([P; Q]\) is proper if and only if \( \det(Q^*Q) \neq 0 \). Furthermore, the following statements are equivalent:

(i) \([P; Q]\) is regular.

(ii) \( \det(P^*P + Q^*Q) \neq 0 \).

(iii) \( \mathcal{N}(P) \cap \mathcal{N}(Q) = \{O_{q \times 1}\} \).

Using Remark A.7, we obtain furthermore:

**Remark D.6** Let \([P; Q]\) be a \( p \times q \) matrix pair and let \( V \in \mathbb{C}^{q \times s} \). Let \( \phi := PV \) and let \( \psi := QV \). Then \([\phi; \psi]\) is a \( p \times s \) matrix pair fulfilling \( \psi^*\phi = V^*(Q^*P)V \). If \( \text{rank} V = s \) and \([P; Q]\) is regular (resp., proper), then \([\phi; \psi]\) is regular (resp., proper).

It is readily checked that by the following definition an equivalence relation on the set of \( p \times q \) matrix pairs is given:

**Definition D.7** Two \( p \times q \) matrix pairs \([P; Q]\) and \([S; T]\) are said to be equivalent if \( \mathcal{R}(\begin{bmatrix} P & Q \end{bmatrix}) = \mathcal{R}(\begin{bmatrix} S & T \end{bmatrix}) \). In this case, we write \([P; Q] \equiv [S; T] \). Furthermore, denote by \([([P; Q])]\) the corresponding equivalence class of a \( p \times q \) matrix pair \([P; Q]\).

**Remark D.8** Remarks A.23 and D.5 show that two regular \( p \times q \) matrix pairs \([P; Q]\) and \([S; T]\) are equivalent if and only if there is an \( R \in \mathbb{C}^{q \times q} \) with \( \det R \neq 0 \) fulfilling \( S = PR \) and \( T = QR \).

**Remark D.9** Each proper \( q \times q \) matrix pair \([P; Q]\) satisfies \( \det Q \neq 0 \) and \([P; Q] \equiv [PQ^{-1}, I_q] \).
Consequently, the set of equivalence classes of proper $q \times q$ matrix pairs can be identified with the set of complex $q \times q$ matrices by means of $[P; Q] \mapsto A := PQ^{-1}$, where $\mathcal{R}(Q) = \mathbb{C}^q$, $\mathcal{R}(P) = \mathcal{R}(A)$, $Q(\mathcal{N}(P)) = \mathcal{N}(A)$, and $P(\mathcal{N}(Q)) = \{O_{q \times 1}\}$.

In the remaining part of this section, we are concerned with reducing certain $q \times q$ matrix pairs $[P; Q]$, which satisfy a condition of the form $\mathcal{R}(P) \subseteq \mathcal{R}(M)$ with a given complex $q \times p$ matrix $M$ of rank $r \geq 1$, to $r \times r$ matrix pairs $[\phi; \psi]$ without loosing any information:

**Lemma D.10** Let $\theta \in \mathbb{C}$ with $|\theta| = 1$ and let $[P; Q]$ be a $q \times q$ matrix pair. Let $A_\theta := Q + \theta P$ and let $B_\theta := Q - \theta P$. Suppose that $\det B_\theta \neq 0$ and let $K_\theta := A_\theta B_\theta^{-1}$. Then

\[
\begin{align*}
\mathcal{R}(P) &= \mathcal{R}(I_q - K_\theta), & \mathcal{N}(P) &= B_\theta^{-1}(\mathcal{N}(I_q - K_\theta)), & \text{(D.2)} \\
\mathcal{R}(Q) &= \mathcal{R}(I_q + K_\theta), & \mathcal{N}(Q) &= B_\theta^{-1}(\mathcal{N}(I_q + K_\theta)), & \text{(D.3)} \\
Q(\mathcal{N}(P)) &= \mathcal{N}(I_q - K_\theta), & \text{and } P(\mathcal{N}(Q)) &= \mathcal{N}(I_q + K_\theta). & \text{(D.4)}
\end{align*}
\]

Furthermore, $\text{rk}([P; Q]) = \text{rank}(I_q - K_\theta) + \text{rank}(I_q + K_\theta) - q$ and $[P; Q]$ is regular.

**Proof** We have $K_\theta B_\theta = A_\theta$ and hence $(I_q + K_\theta)B_\theta = B_\theta + A_\theta = 2Q$ and $(I_q - K_\theta)B_\theta = B_\theta - A_\theta = -2\theta P$. Thus, we can easily infer (D.3) and (D.2), using Remark A.8. From (D.2) and (D.3) we obtain

\[
Q(\mathcal{N}(P)) = (Q - \theta P)(\mathcal{N}(P)) = B_\theta B_\theta^{-1}(\mathcal{N}(I_q - K_\theta)) = \mathcal{N}(I_q - K_\theta)
\]

and

\[
P(\mathcal{N}(Q)) = (Q - \theta P)(\mathcal{N}(Q)) = B_\theta B_\theta^{-1}(\mathcal{N}(I_q + K_\theta)) = \mathcal{N}(I_q + K_\theta),
\]

i.e., (D.4). Due to Remark A.3, we have

\[
\dim \mathcal{R}(I_q + K_\theta) + \dim \mathcal{N}(I_q + K_\theta) = q, \quad \dim \mathcal{R}(I_q - K_\theta) + \dim \mathcal{N}(I_q - K_\theta) = q.
\]

(D.5)

Taking into account (D.1), (D.2), and (D.4), we conclude from the first equation in (D.5) then

\[
\text{rk}([P; Q]) = \dim \mathcal{R}(I_q - K_\theta) - \dim \mathcal{N}(I_q + K_\theta) = \text{rank}(I_q - K_\theta) + \text{rank}(I_q + K_\theta) - q.
\]

Lemma D.2 yields $\dim \mathcal{R}(\begin{bmatrix} P & Q \end{bmatrix}) = \dim \mathcal{R}(P) + \dim Q(\mathcal{N}(P))$. Because of (D.2), (D.4), and the second equation in (D.5), we infer $\text{rk}(\begin{bmatrix} P & Q \end{bmatrix}) = q$, i.e., $[P; Q]$ is regular.

$\square$
We think that the following result is well-known. However, we did not succeed in finding an available reference.

**Lemma D.11** (cf. [53, Lem. 1.6]) Let $[P; Q]$ be a regular $q \times q$ matrix pair satisfying $\text{Im}(Q^*P) \subset \mathbb{C}_{\geq}^{q\times q}$. Let $A := Q + iP$ and let $B := Q - iP$. Then $\det B \neq 0$ and the matrix $K := AB^{-1}$ satisfies $\|K\|_S \leq 1$.

**Proof** We have

$$A^*A = (Q^* - iP^*)(Q + iP) = Q^*Q + i(Q^*P - P^*Q) + P^*P = Q^*Q + P^*P - 2\text{Im}(Q^*P)$$

and

$$B^*B = (Q^* + iP^*)(Q - iP) = Q^*Q - i(Q^*P - P^*Q) + P^*P = Q^*Q + P^*P + 2\text{Im}(Q^*P).$$

In view of Remarks A.24 and A.25, in particular $B^*B \succ Q^*Q + P^*P \succ O_{q\times q}$ follows. Using Remark A.4 and Lemma A.27, we infer then $\mathcal{N}(B) = \mathcal{N}(B^*B) \subseteq \mathcal{N}(Q^*Q + P^*P)$. From Remark D.5 we see furthermore $\det(P^*P + Q^*Q) \neq 0$. Consequently, $\mathcal{N}(B) = \{0\}$, implying $\det B \neq 0$. Regarding $KB = A$, we have moreover $B^*(I_q - K^*K)B = B^*B - A^*A = 4\text{Im}(Q^*P)$. Taking into account Remarks A.24 and A.25, we can conclude then $I_q - K^*K = 4B^{-*}\text{Im}(Q^*P)B^{-1} \succ O_{q\times q}$. Thus, the application of Remark A.34 yields $\|K\|_S \leq 1$. □

**Lemma D.12** Assume $r \leq q$. Let $U \in \mathbb{C}^{q\times r}$ with $U^*U = I_r$ and let $[\phi; \psi]$ be an $r \times r$ matrix pair. Let $P := U\phi U^*$ and let $Q := U\psi U^* + P_{[\mathcal{R}(U)]^\perp}$. Then $[P; Q]$ is a $q \times q$ matrix pair with $\mathcal{R}(P) \subseteq \mathcal{R}(U)$ fulfilling $\det(P^*P + Q^*Q) = \det(\phi^*\phi + \psi^*\psi)$ and $Q^*P = U(\psi^*\phi)U^*$. In particular, $[P; Q]$ is regular if and only if $[\phi; \psi]$ is regular.

**Proof** Observe that $\mathcal{R}(P) = \mathcal{R}(U\phi U^*) \subseteq \mathcal{R}(U)$. Furthermore, we have $P^*P = U\phi^*U^*U\phi U^* = U\phi^*\phi U^*$. Let $N := P_{[\mathcal{R}(U)]^\perp}$. By virtue of Remark A.11, then $\mathcal{N}(N) = \mathcal{R}(U)$, implying $NU = O$. Consequently, we infer

$$Q^*Q = U\psi^*U^*U\psi U^* + U\psi^*(NU)^* + NU\psi U^* + N^*N = U\psi^*\psi U^* + N$$

and

$$Q^*P = U\psi^*U^*U\phi U^* + N^*U\phi U^* = U\psi^*\phi U^* + NU\phi U^* = U\psi^*\phi U^*.$$

We are now going to show that

$$\det(P^*P + Q^*Q) = \det(\phi^*\phi + \psi^*\psi) \quad \text{(D.6)}$$

holds true. Observe that, because of Remark D.5, the asserted equivalence immediately follows from (D.6). Using the already shown identities, we get

$$P^*P + Q^*Q = U(\phi^*\phi + \psi^*\psi)U^* + N. \quad \text{(D.7)}$$
First assume \( r = q \). Then the matrix \( U \) is unitary, implying \( N = O \). Thus, (D.6) is a consequence of (D.7).

Now we consider the case \( r < q \). Then there exists some \( V \in \mathbb{C}^{q \times (q-r)} \) such that \( W := [U, V] \) is a unitary \( q \times q \) matrix. In particular, we get

\[
W^*W = \begin{bmatrix} U^*U & U^*V \\ V^*U & V^*V \end{bmatrix} = \begin{bmatrix} I_r & O_{r \times (q-r)} \\ O_{(q-r) \times r} & I_{q-r} \end{bmatrix} \quad \text{and} \quad WW^* = UU^* + VV^* = I_q.
\]

(D.8)

Because of \( U^*U = I_r \) and Remark A.12, we have \( UU^* = \mathbb{P}_R(U) \). In view of Remark A.11, we obtain from the last equation in (D.8) thus \( VV^* = I_q - UU^* = N \). Taking additionally into account (D.7) and (D.8), we hence can infer

\[
W^*(P^*P + Q^*Q)W = \begin{bmatrix} U^* \\ V^* \end{bmatrix} [U(\phi^*\phi + \psi^*\psi)U^* + VV^*][U, V] = \begin{bmatrix} \phi^*\phi + \psi^*\psi & O_{r \times (q-r)} \\ O_{(q-r) \times r} & I_{q-r} \end{bmatrix}.
\]

In particular, (D.6) holds true. \( \square \)

**Proposition D.13** (cf. [3, Lem. 4.3]) Let \( M \in \mathbb{C}^{p \times p} \) with rank \( r \geq 1 \), let \( u_1, u_2, \ldots, u_r \) be an orthonormal basis of \( \mathcal{R}(M) \), and let \( U := [u_1, u_2, \ldots, u_r] \). Let \( [P; Q] \) be a regular \( q \times q \) matrix pair fulfilling \( \text{Im}(Q^*P) \in \mathbb{C}^{r \times q} \) and \( \mathcal{R}(P) \subseteq \mathcal{R}(M) \) and let \( B := Q - iP. \) Then \( \det B \neq 0 \). Let \( \phi := U^*PB^{-1}U \) and let \( \psi := U^*QB^{-1}U \). Then \( [\phi; \psi] \) is a regular \( r \times r \) matrix pair satisfying \( \psi^*\phi = (B^{-1}U)^*(Q^*P)(B^{-1}U) \). Let \( S := U\phi U^* \) and let \( T := U\psi U^* + \mathbb{P}_{[\mathcal{R}(M)]} \). Then \([S; T] \) is a regular \( q \times q \) matrix pair satisfying \( \det(S^*S + T^*T) = \det(\phi^*\phi + \psi^*\psi) \) and \( T^*S = B^{-1}Q^*PB^{-1} \). Furthermore, \([P; Q] \cong [S; T] \) with \( S = PB^{-1} \) and \( T = QB^{-1} \).

**Proof** We only consider here the case \( r < q \). Then there exists some \( V \in \mathbb{C}^{q \times (q-r)} \) such that \( W := [U, V] \) is a unitary \( q \times q \) matrix. In particular, we get (D.8). Let \( A := Q + iP. \) From Lemma D.11 we see \( \det B \neq 0 \) and that \( K := AB^{-1} \) satisfies \( \|K\|_S \leq 1 \). Consequently the matrix \( L := W^*KW \) then satisfies \( \|L\|_S \leq 1 \) as well. Furthermore, we have

\[
L = \begin{bmatrix} U^* \\ V^* \end{bmatrix} K[U, V] = \begin{bmatrix} U^*KU & U^*KV \\ V^*KU & V^*KV \end{bmatrix}.
\]

Obviously, \( B - A = -2iP \) and \( B + A = 2Q \) hold true. Consequently, we obtain

\[
\frac{1}{2}(I_q - K) = PB^{-1} \quad \text{and} \quad \frac{1}{2}(I_q + K) = QB^{-1}.
\]

(D.9)

Remark A.12 yields \( UU^* = \mathbb{P}_R(M) \). Hence, \( UU^*P = P \) follows. With (D.9) and (D.8), we can thus conclude

\[
V^*(I_q - K) = -2iV^*PB^{-1} = -2iV^*UU^*PB^{-1} = O_{(q-r) \times q},
\]
implying $V^*K = V^*$. Regarding (D.8), we infer for the lower blocks of $L$ then $V^*KU = V^*U = O_{q-r}$ and $V^*KV = V^*V = I_{q-r}$. In particular, the lower right block $V^*KV$ of $L$ is unitary. Consequently, the application of Remark A.35 to $L$ yields the block representation

$$ L = \begin{bmatrix} U^*KU & O_{r \times (q-r)} \\ O_{(q-r) \times r} & I_{q-r} \end{bmatrix}. \quad (D.10) $$

First we verify the assertions for the pair $[S; T]$: Using (D.8) and (D.10), we obtain

$$ I_q = UU^* + VV^* = UU^*UU^* + VV^*, \quad K = WLW^* = [U, V]L \begin{bmatrix} U^* \\ V^* \end{bmatrix} = UU^*KUU^* + VV^* $$

and, consequently, $I_q - K = UU^*(I_q - K)UU^*$ as well as $I_q + K = UU^*(I_q + K)UU^* + 2VV^*$. Because of (D.9), then

$$ PB^{-1} = UU^*PB^{-1}U^* = U\Phi U^* \quad \text{and} \quad QB^{-1} = UU^*QB^{-1}U^* + VV^* = U\Phi U^* + VV^* $$

follow. In view of (D.8) and $UU^* = \mathbb{P}_{\mathcal{R}(M)}$, we infer from Remark A.11 furthermore $VV^* = I_q - UU^* = \mathbb{P}_{\mathcal{R}(M)^\perp}$. Thus, we get $PB^{-1} = S$ and $QB^{-1} = T$. By virtue of Remark D.6, hence $[S; T]$ is a regular $q \times q$ matrix pair fulfilling $T^*S = B^{-*}(Q^*P)B^{-1}$. In addition, Remark D.8 yields $[P; Q] \cong [S; T]$. It remains to show the assertions involving the pair $[\phi; \psi]$: Regarding $U^*U = I_r$ and $\mathcal{R}(U) = \mathcal{R}(M)$, we can apply Lemma D.12 to the $r \times r$ matrix pair $[\phi; \psi]$ to obtain $\det(S^*S+T^*T) = \det(\phi^*\phi + \psi^*\psi)$ and to see that $[\phi; \psi]$ is regular and that $T^*S = U(\psi^*\phi)U^*$ holds true. From the last equation we can infer then

$$ \psi^*\phi = U^*U\psi^*\phi^*U^* = U^*T^*SU = U^*Q(QB^{-1})^*(PB^{-1})U = (B^{-1}U)^*(Q^*P)(B^{-1}U). \quad \square $$

### E Linear Fractional Transformations of Matrices

In this appendix, we consider a matricial generalization of the transformation $z \mapsto \frac{az+b}{cz+d}$ of the extended complex plane. We thereby follow [13, Sec. 1.6], while restricting ourselves to the version $Z \mapsto (AZ + B)(CZ + D)^{-1}$ with denominator on the right side. Let $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be the block representation of a complex $(p+q) \times (p+q)$ matrix $M$ with $p \times p$ block $A$. If the set

$$ Q_{C,D} := \{Z \in \mathbb{C}^{p \times q} : \det(CZ + D) \neq 0\} \quad (\text{resp., } P Q_{C,D} := \{(P, Q) \in \mathbb{C}^{p \times q} \times \mathbb{C}^{q \times q} : \det(CP + DQ) \neq 0\}) $$

is non-empty, then let the linear fractional transformation $\Phi_M^{(p,q)} : Q_{C,D} \rightarrow \mathbb{C}^{p \times q}$ (resp., $\Psi_M^{(p,q)} : P Q_{C,D} \rightarrow \mathbb{C}^{p \times q}$) be defined by

$$ \Phi_M^{(p,q)}(Z) := (AZ + B)(CZ + D)^{-1} \quad (\text{resp., } \Psi_M^{(p,q)}([P; Q]) := (AP + BQ)(CP + DQ)^{-1}) $$
In this context, the block matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is called the generating matrix of the linear fractional transformation. For each matrix $Z \in Q_{C,D}$, we obviously have $(Z, I_q) \in P Q_{C,D}$ and $\Phi_M^{(p,q)}(Z) = \Psi_M^{(p,q)}((Z, I_q))$. We first characterize the case, that the corresponding domain is non-empty:

**Lemma E.1** ([31, Lem. D.2]) The following statements are equivalent:

(i) $Q_{C,D} \neq \emptyset$.

(ii) $P Q_{C,D} \neq \emptyset$.

(iii) rank$[C, D] = q$.

The composition of two linear fractional transformations is again a linear fractional transformation with generating matrix $M$ emerging from ordinary matrix multiplication:

**Proposition E.2** (cf. [31, Propositions D.3 and D.4]) Let $\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}$ and $\begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}$ be the block representations of two given complex $(p + q) \times (p + q)$ matrices $M_1$ and $M_2$ with $p \times p$ block $A_1$ and $A_2$, resp. Let $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be the block representation of the product $M := M_2 M_1$ with $p \times p$ block $A$.

(a) Suppose that the set $Q := \{ Z \in Q_{C_1,D_1} : \Phi_M^{(p,q)}(Z) \in Q_{C_2,D_2} \}$ is non-empty. Then $Q \subseteq Q_{C,D}$ and $\Phi_M^{(p,q)}(Z) = \Phi_{M_2}^{(p,q)}(\Phi_{M_1}^{(p,q)}(Z))$ for all $Z \in Q$.

(b) Suppose that the set $P Q := \{ [P; Q] \in P Q_{C_1,D_1} : \Psi_M^{(p,q)}([P; Q]) \in Q_{C_2,D_2} \}$ is non-empty. Then $P Q \subseteq P Q_{C,D}$ and $\Psi_M^{(p,q)}([P; Q]) = \Phi_{M_2}^{(p,q)}(\Psi_{M_1}^{(p,q)}([P; Q]))$ for all $[P; Q] \in P Q$.

In connection with the particular embedding of $r \times r$ matrix pairs into the class of $q \times q$ matrix pairs for $r \leq q$ considered in Lemma D.12, the following auxiliary result is of interest:

**Lemma E.3** Suppose $q \geq 2$ and let $r \in \mathbb{Z}_{1,q-1}$. Let $[U, V]$ be the block representation of a unitary $q \times q$ matrix $W$ with $q \times r$ block $U$. Let $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be the block representation of a complex $2q \times 2q$ matrix $M$ with $q \times q$ block $A$ and let $N := \begin{bmatrix} AW & BW \\ CW & DW \end{bmatrix}$.

(a) Let $f \in \mathbb{C}^{r \times r}$ and let $F := f \oplus O_{(q-r) \times (q-r)}$. Then $U f U^* \in Q_{C,D}$ if and only if $F \in Q_{C W, D W}$. In this case, $\Phi_N^{(q,q)}(U f U^*) = \Phi_N^{(q,q)}(F)$.

(b) Let $f, g \in \mathbb{C}^{r \times r}$, let $F := f \oplus O_{(q-r) \times (q-r)}$, and let $G := g \oplus I_{q-r}$. Then $(U f U^*, U g U^* + P_{[R(U)]^\perp}) \in P Q_{C,D}$ if and only if $(F, G) \in P Q_{C W, D W}$. In this case, $\Psi_N^{(q,q)}((U f U^*, U g U^* + P_{[R(U)]^\perp})) = \Psi_N^{(q,q)}([F; G])$.

**Proof** (a) Because of $W F W^* = U f U^*$ and $W^{-1} = W^*$, we have $[(A W)F + (B W)]W^{-1} = A(W F W^*) + B = A(U f U^*) + B$ and similarly $[(C W)F + (D W)]W^{-1} = C(U f U^*) + D$. Consequently, (a) follows.

(b) As in the proof of Lemma D.12, we have (D.8) and we can conclude $V V^* = P_{[R(U)]^\perp}$. Beside $W F W^* = U f U^*$, we get $W G W^* = U g U^* + V V^* = U g U^* +$...
\[ \mathbb{P}_{[R(U)]^\perp}. \text{ The equation } [(AW)F + (BW)G]W^{-1} = A(WFW^*) + B(WGW^*) = A(UFU^*) + B(UGU^* + \mathbb{P}_{[R(U)]^\perp}) \] then follows from \( W^{-1} = W^* \). Similarly, we obtain moreover \([(CW)F + (DW)G]W^{-1} = C(UFU^*) + D(UGU^* + \mathbb{P}_{[R(U)]^\perp})\). Consequently, (b) follows.

**F Holomorphic Matrix-valued Functions**

Let \( \mathcal{G} \) be a domain, i.e., an open, non-empty, and connected subset of \( \mathbb{C} \). A matrix-valued function \( F: \mathcal{G} \to \mathbb{C}^{p \times q} \) is said to be holomorphic if all entries \( f_{jk}: \mathcal{G} \to \mathbb{C} \) of \( F = [f_{jk}]_{j=1,\ldots,p} \) are holomorphic functions. In this case, the matrix-valued function \( F \) admits, for each \( z_0 \in \mathcal{G} \), a unique power series representation \( F(z) = \sum_{n=0}^{\infty} (z - z_0)^n A_n \). The corresponding disk of convergence coincides with the largest open disk with center \( z_0 \) lying entirely in \( \mathcal{G} \). The coefficients \( A_n = [a_{jk,n}]_{j=1,\ldots,p} \) are given by the Taylor series \( f_j(z) = \sum_{n=0}^{\infty} a_{jk,n}(z - z_0)^n \) at \( z_0 \). Setting \( F^{(n)} \) with the \( n \)-th derivatives \( f_j^{(n)} \) of the infinitely differentiable functions \( f_j \), we have \( A_n = \frac{1}{n!} F^{(n)}(z_0) \). Basic results on holomorphic functions can be generalized to the matrix case considered here in an appropriate way:

**Remark F.1** Let \( F: \mathcal{G} \to \mathbb{C}^{p \times q} \) be holomorphic, let \( U \in \mathbb{C}^{r \times p} \), and let \( V \in \mathbb{C}^{q \times s} \). Then \( H := UFV \) is holomorphic with \( H^{(n)} = UF^{(n)}V \) for all \( n \in \mathbb{N}_0 \).

The Cauchy product for sequences of matrices determines the coefficients of the product of two matrix-valued power series:

**Remark F.2** Let \( F: \mathcal{G} \to \mathbb{C}^{p \times q} \) and \( G: \mathcal{G} \to \mathbb{C}^{q \times r} \) be two holomorphic functions. Let \( z \in \mathcal{G} \) and let the sequences \( (A_n)_{n=0}^{\infty} \) and \( (B_n)_{n=0}^{\infty} \) be given by \( A_n := \frac{1}{n!} F^{(n)}(z) \) and \( B_n := \frac{1}{n!} G^{(n)}(z) \), resp. Then \( H := FG \) is holomorphic and the sequence \( (C_n)_{n=0}^{\infty} \) given by \( C_n := \frac{1}{n!} H^{(n)}(z) \) coincides with the Cauchy product of \( (A_n)_{n=0}^{\infty} \) and \( (B_n)_{n=0}^{\infty} \).

If, in the case \( p = q \), the values \( F(z) \) of the holomorphic matrix-valued function \( F \) are invertible matrices for all \( z \in \mathcal{G} \), then the function \( G: \mathcal{G} \to \mathbb{C}^{q \times q} \) defined by \( G(z) := [F(z)]^{-1} \) is holomorphic as well. Now suppose that \( F \) satisfies only the weaker condition of having constant column space \( \mathcal{R}(F(z)) \) and constant null space \( \mathcal{N}(F(z)) \), independent of the argument \( z \in \mathcal{G} \). Then, even in the case \( p \neq q \), the function \( G: \mathcal{G} \to \mathbb{C}^{q \times p} \) defined by \( G(z) := [F(z)]^\dagger \) turns out to be holomorphic. Furthermore, the sequences of Taylor coefficients of \( G \) and \( F \) both belong to the class introduced in Notation 3.10 and are mutually reciprocal in the sense of Definition 3.50. The following is a specification of a result due to Campbell and Meyer [4, Thm. 10.5.4]:

**Proposition F.3** Let \( F: \mathcal{G} \to \mathbb{C}^{p \times q} \) be holomorphic. Then the following statements are equivalent:

(i) The function \( G: \mathcal{G} \to \mathbb{C}^{q \times p} \) defined by \( G(z) := [F(z)]^\dagger \) is holomorphic.

(ii) \( \mathcal{R}(F(z)) = \mathcal{R}(F(w)) \) and \( \mathcal{N}(F(z)) = \mathcal{N}(F(w)) \) for all \( z, w \in \mathcal{G} \).

(iii) \( (\frac{1}{n!} F^{(n)}(z))_{n=0}^{\infty} \in \mathcal{D}_{p \times q, \infty} \) for all \( z \in \mathcal{G} \).
If (i) is fulfilled and \( z_0 \in \mathcal{G} \), then \( \left( \frac{1}{n!} G^{(n)}(z_0) \right)_{n=0}^{\infty} \) is exactly the reciprocal sequence associated to \( \left( \frac{1}{n!} F^{(n)}(z_0) \right)_{n=0}^{\infty} \).

**Proof** The equivalence of (i) and (ii) is an immediate consequence of [33, Prop. 8.4].

Let (i) be fulfilled. Consider an arbitrary \( z_0 \in \mathcal{G} \). Because of [33, Thm. 8.9 and 4.21], the sequence \( \left( \frac{1}{n!} F^{(n)}(z_0) \right)_{n=0}^{\infty} \) belongs to \( 
abla_{p,q,\infty} \) and \( \left( \frac{1}{n!} G^{(n)}(z_0) \right)_{n=0}^{\infty} \) is exactly the reciprocal sequence associated to \( \left( \frac{1}{n!} F^{(n)}(z_0) \right)_{n=0}^{\infty} \). In particular, (iii) holds true. Conversely, suppose that (iii) is fulfilled. From [33, Thm. 8.9] we can then infer that the function \( G \) is holomorphic in all points \( z \in \mathcal{G} \). Consequently, (i) holds true. \( \square \)

Next, we give analogous results for power series expansions at \( z_0 = \infty \). To that end, let \( \rho \in (0, \infty) \) and suppose that the improper open annulus \( C_\rho := \{ z \in \mathbb{C} : |z| > \rho \} \) is entirely contained in \( \mathcal{G} \). Furthermore, let a holomorphic matrix-valued function \( F : \mathcal{G} \to \mathbb{C}^{p \times q} \) be given, admitting the series representation

\[
F(z) = \sum_{n=0}^{\infty} z^{-n} C_n \tag{D.1}
\]

for all \( z \in C_\rho \) with certain complex \( p \times q \) matrices \( C_0, C_1, C_2, \ldots \). This is the matricial version of a special case of the general situation of a given complex-valued function \( f \) which is holomorphic in an annulus \( A := \{ z \in \mathbb{C} : r < |z-c| < R \} \) centered at \( c \in \mathbb{C} \) with radii \( 0 \leq r < R \leq \infty \). As is well known, such a function \( f \) has a Laurent series \( f(z) = \sum_{\ell=-\infty}^{\infty} a_\ell (z-c)^\ell \) at the point \( c \) converging on \( A \) with uniquely determined coefficients \( a_\ell \in \mathbb{C} \). In the particular situation of interest considered here, we have \( c = 0, R = \infty \), and \( a_\ell = 0 \) for all \( \ell \in \mathbb{N} \). This case can be easily reduced to the ordinary power series expansion of holomorphic functions, discussed at the beginning of this section. By means of the substitution \( z \mapsto w := 1/z \), we can proceed to a holomorphic function \( \Phi \) defined on the open disk \( B_{1/\rho} := \{ w \in \mathbb{C} : |w| < 1/\rho \} \) with Taylor series \( \Phi(w) = \sum_{n=0}^{\infty} w^n C_n \) at the point \( w_0 = 0 \):

**Lemma F.4** Let \( F : \mathcal{G} \to \mathbb{C}^{p \times q} \) be holomorphic, admitting for all \( z \in C_\rho \) the series representation (D.1) with certain complex \( p \times q \) matrices \( C_0, C_1, C_2, \ldots \). Then \( \lim_{\xi \to 0} F(1/\xi) = C_0 \) and the matrix-valued function \( \Phi : B_{1/\rho} \to \mathbb{C}^{p \times q} \) defined by \( \Phi(w) := F(1/w) \) for \( w \neq 0 \) and by \( \Phi(0) := \lim_{\xi \to 0} F(1/\xi) \) is holomorphic with \( \frac{1}{n!} \Phi^{(n)}(0) = C_n \) for all \( n \in \mathbb{N}_0 \).

We continue with the analogue of Remark F.1 for power series expansion at \( z_0 = \infty \):

**Remark F.5** Let \( F : \mathcal{G} \to \mathbb{C}^{p \times q} \) be holomorphic, admitting the series representation (D.1) for all \( z \in C_\rho \) with certain complex \( p \times q \) matrices \( C_0, C_1, C_2, \ldots \). Let \( U \in \mathbb{C}^{r \times p} \) and let \( V \in \mathbb{C}^{q \times s} \). Then \( H := UV \) is holomorphic and \( H(z) = \sum_{n=0}^{\infty} z^{-n} (UC_n V) \) for all \( z \in C_\rho \).

Likewise, Remark F.2 can be modified in a well-known matter for power series expansion at \( z_0 = \infty \):
Lemma F.6 Let \( F: G \to \mathbb{C}^{p \times q} \) and \( G: G \to \mathbb{C}^{q \times r} \) be holomorphic functions, admitting the series representations \( F(z) = \sum_{n=0}^{\infty} z^{-n} C_n \) and \( G(z) = \sum_{n=0}^{\infty} z^{-n} D_n \) for all \( z \in \mathcal{C}_\rho \) with certain complex \( p \times q \) matrices \( C_0, C_1, C_2, \ldots \) and certain complex \( q \times r \) matrices \( D_0, D_1, D_2, \ldots \), resp. Let \( H := FG \) and denote by \((E_n)_{n=0}^{\infty}\) the Cauchy product of \((C_n)_{n=0}^{\infty}\) and \((D_n)_{n=0}^{\infty}\). Then \( H \) is holomorphic and \( H(z) = \sum_{n=0}^{\infty} z^{-n} E_n \) for all \( z \in \mathcal{C}_\rho \).

Using Proposition F.3, we are able to expand the function \( z \mapsto [F(z)]^\dagger \) under certain conditions at \( z_0 = \infty \) into a series with coefficients given, according to Definition 3.50, by the reciprocal sequence associated to the sequence \((C_n)_{n=0}^{\infty}\) from (D.1):

**Lemma F.7** Let \( F: G \to \mathbb{C}^{p \times q} \) be holomorphic and let \((C_n)_{n=0}^{\infty}\) be a sequence of complex \( p \times q \) matrices such that \((D.1)\) and furthermore \( \mathcal{R}(F(z)) = \mathcal{R}(C_0) \) and \( \mathcal{N}(F(z)) = \mathcal{N}(C_0) \) hold true for all \( z \in \mathcal{C}_\rho \). Let \( G: \mathcal{C}_\rho \to \mathbb{C}^{q \times p} \) be defined by \( G(z) := [F(z)]^\dagger \) and denote by \((D_n)_{n=0}^{\infty}\) the reciprocal sequence associated to \((C_n)_{n=0}^{\infty}\). Then \( G \) is holomorphic and \( G(z) = \sum_{n=0}^{\infty} z^{-n} D_n \) for all \( z \in \mathcal{C}_\rho \).

**Proof** According to Lemma F.4, we proceed to a holomorphic function \( \Phi: B_{1/\rho} \to \mathbb{C}^{p \times q} \), which satisfies \( \frac{1}{n!} \Phi^{(n)}(0) = C_n \) for all \( n \in \mathbb{N}_0 \). Consider an arbitrary \( w \in B_{1/\rho} \).

If \( w = 0 \), then \( \Phi(w) = C_0 \). In the case \( w \neq 0 \), we see that \( z := 1/w \) belongs to \( \mathcal{C}_\rho \) and that \( \Phi(w) = F(z) \). Consequently, \( \mathcal{R}(\Phi(w)) = \mathcal{R}(C_0) \) and \( \mathcal{N}(\Phi(w)) = \mathcal{N}(C_0) \) for all \( w \in B_{1/\rho} \). In particular, \( \mathcal{R}(\Phi(w)) \) and \( \mathcal{N}(\Phi(w)) \) are independent of \( w \in B_{1/\rho} \).

Let \( \Psi: B_{1/\rho} \to \mathbb{C}^{q \times p} \) be defined by \( \Psi(w) := [\Phi(w)]^\dagger \). From Proposition F.3 we see then that \( \Psi \) is holomorphic and that \((1/n!) \Psi^{(n)}(0))_{n=0}^{\infty} \) is exactly the reciprocal sequence associated to \((1/n!) \Phi^{(n)}(0))_{n=0}^{\infty} \). Hence, we have \( \frac{1}{n!} \Psi^{(n)}(0) = D_n \) for all \( n \in \mathbb{N}_0 \) and thus \( \Psi(w) = \sum_{n=0}^{\infty} w^n D_n \) for all \( w \in B_{1/\rho} \). Consider an arbitrary \( z \in \mathcal{C}_\rho \). Then \( w := 1/z \) belongs to \( B_{1/\rho} \setminus \{0\} \) and we have \( \Psi(w) = [F(1/w)]^\dagger \), implying

\[
G(z) = [F(z)]^\dagger = \Psi(w) = \sum_{n=0}^{\infty} w^n D_n = \sum_{n=0}^{\infty} z^{-n} D_n.
\]

In the remaining part of this section, let \( G \) be again an arbitrary domain. Next, we consider the matricial generalization of a special class of holomorphic functions, which is well studied, especially in the generic case of \( G \) being the open unit disk:

**Notation F.8** Denote by \( S_{p \times q}(G) \) the set of all functions \( S: G \to \mathbb{C}^{p \times q} \), which are holomorphic in \( G \) and satisfy \( \|S(z)\|_S \leq 1 \) for all \( z \in G \).

The matrix-valued functions belonging to \( S_{p \times q}(G) \) are called **Schur functions** (in \( G \)).

**Lemma F.9** (cf. [30, Lem. 3.9]) Let \( S \in S_{p \times q}(G) \) and let \( U, V \in \mathbb{C}^{p \times q} \) with \( UU^* = I_p \) and \( V^*V = I_q \). For all \( z, w \in G \), then \( \mathcal{R}(U + S(z)) = \mathcal{R}(U + S(w)) \) and \( \mathcal{N}(V + S(z)) = \mathcal{N}(V + S(w)) \).

We end this section with some remarks concerning meromorphic matrix-valued functions. A subset \( D \) of \( G \) is said to be **discrete in \( G \)** if \( G \) does not contain any
accumulation point of \( \mathcal{D} \). So, according to the identity theorem for holomorphic functions, two holomorphic functions \( F, G : \mathcal{G} \to \mathbb{C}^{p \times q} \) coincide if and only if the set \( \{ z \in \mathcal{G} : F(z) = G(z) \} \) is not discrete in \( \mathcal{G} \). Speaking in the following of a discrete subset of \( \mathcal{G} \), we always mean a subset of \( \mathcal{G} \), which is discrete in \( \mathcal{G} \). For such a discrete subset \( \mathcal{D} \) of \( \mathcal{G} \), the set \( \mathcal{G} \setminus \mathcal{D} \) is a domain.

A complex-valued function \( f \) is said to be meromorphic in \( \mathcal{G} \) if there exists a discrete subset \( \mathcal{P}(f) \) of \( \mathcal{G} \) such that \( f \) is a holomorphic function defined on the domain \( \mathcal{G} \setminus \mathcal{P}(f) \), which in each point from \( \mathcal{P}(f) \) has a pole (of positive order). In particular, each holomorphic function \( f : \mathcal{G} \to \mathbb{C} \) is meromorphic in \( \mathcal{G} \) with \( \mathcal{P}(f) = \emptyset \). We call a \( \mathbb{C}^{p \times q} \)-valued function \( F \) meromorphic in \( \mathcal{G} \) if all entries \( f_{jk} \) of \( F = [f_{jk}]_{j=1,...,p \atop k=1,...,q} \) are complex-valued functions meromorphic in \( \mathcal{G} \). In this case, the union \( \mathcal{P}(F) := \bigcup_{j=1}^{p} \bigcup_{k=1}^{q} \mathcal{P}(f_{jk}) \) of the sets of poles of all entries \( f_{jk} \) is a discrete subset of \( \mathcal{G} \). In particular, each holomorphic function \( F : \mathcal{G} \to \mathbb{C}^{p \times q} \) is meromorphic in \( \mathcal{G} \) with \( \mathcal{P}(F) = \emptyset \). Since \( \mathcal{G} \) is assumed to be connected, the set of complex-valued functions meromorphic in \( \mathcal{G} \) has the algebraic structure of a field. Using the arithmetic of this field, the usual operations from matrix algebra can be formally carried over to matrix-valued functions, which are meromorphic in \( \mathcal{G} \). Thus, corresponding sums and products of such matrix-valued functions are again meromorphic in \( \mathcal{G} \). Furthermore, it is readily checked that the determinant \( \det F \) of a \( \mathbb{C}^{q \times q} \)-valued function \( F \) meromorphic in \( \mathcal{G} \) is a complex-valued function, which is meromorphic in \( \mathcal{G} \). If \( \det F \) does not identically vanish, then the mapping \( F^{-1} \) given by formal matrix inversion of \( F \) (seen as a matrix with entries in the field of complex-valued functions meromorphic in \( \mathcal{G} \)) is again a \( \mathbb{C}^{q \times q} \)-valued function, which is meromorphic in \( \mathcal{G} \) with not identically vanishing determinant.

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