Abstract. We study the high-energy asymptotics of the steady velocity distributions for model systems of granular media in various regimes. The main results obtained are integral estimates of solutions of the hard-sphere Boltzmann equations, which imply that the velocity distribution functions $f(v)$ behave in a certain sense as $C \exp(-r|v|^s)$ for $|v|$ large. The values of $s$, which we call the orders of tails, range from $s = 1$ to $s = 2$, depending on the model of external forcing. The method we use is based on the moment inequalities and careful estimating of constants in the integral form of the Povzner-type inequalities.

Keywords: Boltzmann equation, inelastic interactions, granular media, moments, high-energy tails, Povzner-type inequalities.

Introduction

In this paper we address the problem of high-energy asymptotics for solutions of kinetic equations used for modeling dilute, rapid flows of granular media. Granular systems in such regimes are interesting from a physical point of view, since they show a variety of interesting and unexpected properties. They also appear in a growing number of industrial applications. Much of the interest to kinetic models in this context comes from the fact that such models provide a systematic way of derivation of hydrodynamic equations based on the principles of particle dynamics. They are also useful for numerical modeling of granular flows. We refer the reader to the review papers [10, 21, 22] for a general exposition of the subject.

In dilute flows, the binary collisions are often assumed to be the main mechanism of particle interactions. The effect of such collisions is modeled by collision terms of the Boltzmann or Enskog type. An important feature of collisions of granular particles is their inelastic character: in each collision a certain fraction of the kinetic energy is dissipated. This introduces some interesting features in the equations: in particular, the only functions on which the collision operator vanishes are delta functions corresponding to all particles at rest (or moving with the same velocity).

To obtain other nontrivial steady states in granular systems, as a general rule, a certain mechanism of the energy inflow is required. Experimentally this can be achieved, for example, by shaking a vessel with granular particles. In terms of equations, several simplified models have been proposed, in the space-homogeneous case, which include forcing terms of various types [34, 31, 15]. Examples of such terms are diffusion (in the velocity space) and Fokker-Planck operators which correspond to a physical model of a system of particles in a thermal bath.

Other important types of problems which lead to similar equations are related to self-similar solutions in the homogeneous cooling problem and the problem of shear flow [17, 11, 9]. In both cases the equations can be transformed, after an appropriate
change of variables, to space-homogeneous steady problems for the Boltzmann-type equations with force terms that correspond to the negative or anisotropic friction.

One of the interesting features of granular flows, which has been studied actively in the framework of kinetic theory, is the non-Maxwellian behavior of the steady velocity distributions. In fact, experimental data, theoretical predictions and numerical evidence suggest that typical velocity distributions in rapid granular flows have high-energy asymptotics (or “tails”) given by the “stretched exponentials” $\exp(-r|v|^s)$ with $s$ generally not equal to 2 (the classical, Maxwellian, case), or display power-like decay for $|v|$ large (see [19] and references therein).

The precise form of the asymptotics is determined by several factors, among which are the details of the interactions and the forcing models. In the present paper we study the model space-homogeneous system with hard-sphere collisions and four types of forcing terms. We distinguish between the cases of (i) diffusion (Gaussian heat bath), (ii) diffusion with friction (Fokker-Planck type terms), (iii) negative friction (obtained in a self-similar transformation in the homogeneous cooling problem, and (iv) anisotropic friction which appears in the shear flow transformation.

We obtain integral estimates for steady solutions using functionals of the form (1.13), which indicate that solutions have high-energy “tails” given by the “stretched exponentials” $\exp(-r|v|^s)$, with $s$ depending on the forcing terms. We obtain the values $s = \frac{3}{2}$ for the pure diffusion case, $s = 2$ for the diffusion-friction heat bath, $s = 1$ for the negative friction case and $s \geq 1$ in the case of the shear flow. Our method is based on representing functionals (1.13) in terms of symmetric moments, studying the infinite system of inequalities satisfied by these moments and using a sharp integral form of the so-called Povzner inequalities, similar to the one studied by one of the authors [4] in the case of the classical (elastic) space-homogeneous Boltzmann equation. We expect that the estimates obtained for the moment inequalities can be used for studying the time-dependent moments and the time-evolution of the tails, which should be an object of a separate study.

The problem of high-energy tails for solutions of the inelastic hard-sphere Boltzmann with diffusion model has been studied previously by several authors [17, 31, 16, 15] by the methods of formal asymptotic analysis (a formal argument becomes particularly simple if one discards the “gain” term in the kinetic equations). The general idea that appears in those papers is that for functions of the type $h(v) = C \exp(-r|v|^s)$, with $s \leq 2$, the “gain” term $Q^+(h, h)$ in the collision operator is a small perturbation of the loss term for $|v|$ large. Our approach based on studying the moments of the collision terms allows us to quantify this idea for the solutions of the original problem, without making apriori assumptions about their asymptotic behavior. A rigorous analysis of the problem with diffusive forcing has been performed in [19], where it was proved, in particular, that steady solutions are infinitely differentiable and decay faster than any polynomial for $|v|$ large. A lower bound of the type $C \exp(-r|v|^{3/2})$ was also established by using a comparison principle. The problem has also been studied numerically by a number of authors [28, 31, 29, 20].

Another series of related results was obtained for the so-called inelastic Maxwell models [5, 24, 26], which are approximate equations obtained by replacing the collision rate in the Boltzmann operator by a relative velocity independent mean value.
The equations for the Maxwell models simplify significantly by using the Fourier transform, and the equations for symmetric moments build a closed infinite recursive system \[5, 26\]. Using the Fourier transform methods, Bobylev and Cercignani \[6\] found solutions to the inelastic Maxwell model with a Gaussian heat bath which have high-energy tails \(\exp(-r|v|)\). In the case of self-similar scaling for the inelastic Maxwell models, solutions with power-like tails were found \[2, 14, 25\], and it was conjectured by Ernst and Brito \[16\] that such solutions determine the universal long-time asymptotics of the time-dependent solutions in the space-homogeneous cooling problem. This conjecture has recently been proved by Bobylev, Cercignani and Toscani \[7, 8\].

It is clear, however, that while Maxwell models may give reasonable approximations of the macroscopic quantities, the details of the velocity distributions can differ significantly from the hard-sphere case. In particular, this is true with respect of the high-energy asymptotics which depends crucially on the behavior of collision rate for large relative velocities, as can be easily seen from the formal asymptotic arguments of the type presented in \[15, 17, 31\].

On the other hand, there is an noticeable gap in the development of rigorous mathematical theory, between the Maxwell and hard sphere models. Therefore, the aim of this paper is to develop rigorous methods that would allow us to study solutions of the hard-sphere Boltzmann equation, with a particular attention to the high-energy asymptotics.

The paper is organized as follows. In Section \(1\) we present the problem and formulate the main results. One of the the most important technical aspects of our study is obtaining a precise integral form of the Povzner-type inequalities, which we study in Section \(2\). Section \(3\) is devoted to the moment inequalities specific to the hard-sphere case. We formulate the inequalities in terms of the normalized symmetric moments which appear as the coefficients of power series expansions of functionals \[1.13\]. We further study the dependence of the inequalities on the parameters to find the conditions under which the sequences of the normalized moments have geometric growth. Finally, Section \(4\) presents the proofs of the main theorems.

Most of our inequalities can be used in the time-dependent case, and therefore, we begin the next Section by considering the non-stationary Boltzmann equation.

1. Preliminaries and main results

We study kinetic models for space-homogeneous granular media, in which the one-particle distribution function \(f(v,t), v \in \mathbb{R}^3, t > 0\) is assumed to satisfy the following equation:

\[
\frac{\partial f}{\partial t} = Q(f,f) + G(f). \tag{1.1}
\]

Here \(Q(f,f)\) is the inelastic Boltzmann collision operator, expressing the effect of binary collisions of particles, and \(G(f)\) is a forcing term. We will consider three different examples of forcing. The first one is the pure diffusion thermal bath \[34, 31, 10\], in which case

\[
G_1(f) = \mu \Delta f, \tag{1.2}
\]
where \( \mu > 0 \) is a constant. The second example is the thermal bath with linear friction

\[
G_2(f) = \mu \Delta f + \lambda \text{div}(vf),
\]

(1.3)

where \( \lambda \) and \( \mu \) are positive constants.

The third example relates to self-similar solutions of equation (1.1) for \( G(f) = 0 \). We denote

\[
f(v, t) = \frac{1}{v_0(t)} \tilde{f}(\tilde{v}(v, t), \tilde{t}(t)), \quad \tilde{v} = \frac{v}{v_0(t)},
\]

where

\[
v_0(t) = (a + \kappa t)^{-1}, \quad \tilde{t}(t) = \frac{1}{\kappa} \ln(1 + \frac{\kappa}{a} t), \quad a, \kappa > 0.
\]

Then, the equation for \( \tilde{f}(\tilde{v}, \tilde{t}) \) coincides (after omitting the tildes) with equation (1.1), where

\[
G_3(f) = -\kappa \text{div}(vf), \quad \kappa > 0.
\]

(1.4)

Finally, the last type of forcing is given by the term appearing in the shear flow transformation (see, for example, [11, 9])

\[
G_4(f) = -\kappa v_1 \frac{\partial f}{\partial v_2},
\]

(1.5)

where \( \kappa \) is a positive constant.

We assume the granular particles to be perfectly smooth hard spheres performing inelastic collisions characterized by a single parameter: the coefficient of normal restitution \( 0 < e < 1 \). To define the collision operator we write

\[
Q(f, f) = Q^+(f, f) - Q^-(f, f),
\]

(1.6)

where the “loss” term \( Q^-(f, f) \) is

\[
Q^-(f, f) = f(f * |v|),
\]

(1.7)

and the “gain” term \( Q^+(f, f) \) is most easily defined through its weak form:

\[
\int_{\mathbb{R}^3} Q^+(f, f)(v) \psi(v) dv = \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(v) f(w) |u| \int_{S^2} \psi(v') d\sigma dw dv,
\]

(1.8)

where \( u = v - w \) is the relative velocity of two particles about to collide, and \( v' \) is the velocity after the collision. The collision transformation that puts \( v \) and \( w \) into correspondence with the post-collisional velocities \( v' \) and \( w' \) can be expressed as follows:

\[
v' = v + \frac{\beta}{2} (|u|\sigma - u),
\]

\[
w' = w - \frac{\beta}{2} (|u|\sigma - u),
\]

(1.9)

where we set \( \beta = \frac{1 + e}{2} \), and \( 0 < e < 1 \) is the restitution coefficient. Notice that we always have \( \frac{1}{2} < \beta < 1 \).

Combining (1.7) and (1.8) and using the symmetry that allows us to exchange \( v \) with \( w \) in the integrals we obtain the following symmetrized weak form

\[
\int_{\mathbb{R}^3} Q(f, f)(v) \psi(v) dv = \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(v) f(w) |u| A_{\beta}[\psi](v, w) dw dv,
\]

(1.10)
where

\[
A_\beta[\psi](v, w) = \frac{1}{4\pi} \int_{S^2} (\psi(v') + \psi(w') - \psi(v) - \psi(w)) \, d\sigma.
\] (1.11)

The weak form (1.8) will be sufficient for the purposes of our study. The usual strong form [10, 21, 22] can be obtained from (1.8) by taking \( \psi(v) = \delta(v - v_0) \) (see also [19]).

We will assume that the solutions are normalized as follows

\[
\int_{\mathbb{R}^3} f(v, t) \, dv = 1, \quad \int_{\mathbb{R}^3} f(v, t) v_i \, dv = 0, \quad i = 1, 2, 3.
\] (1.12)

Since the expected behavior of solutions for \(|v|\) large is \( C \exp(-r |v|^s) \), we introduce the following functionals:

\[
F_{r,s}(f) = \int_{\mathbb{R}^3} f(v) \exp(r |v|^s) \, dv,
\] (1.13)

and study the values of \( s \) and \( r \) for which these functionals are finite. This motivates the following definition.

**Definition.** We say that the function \( f \) has an exponential tail of order \( s > 0 \), if the following supremum

\[
r^*_s = \sup \{ r > 0 \mid F_{r,s}(f) < +\infty \}
\] (1.14)

is positive and finite.

In the case \( s = 2 \) the value of \((r^*_s)^{-1}\) is known as the tail temperature of \( f \) [4]. It is easy to see that the number \( s \) in the above definition is determined uniquely. Indeed, if for certain \( s > 0 \),

\[
0 < r^*_s < +\infty,
\]

then we have \( r^*_{s'} = +\infty \), for every \( s' < s \), and also \( r^*_{s'} = 0 \), for every \( s' > s \).

Another useful representation of the functionals (1.13) is obtained by using the symmetric moments of the distribution function. Setting

\[
m_p = \int_{\mathbb{R}^3} f(v)|v|^{2p} \, dv, \quad p \geq 0,
\] (1.15)

and expanding the exponential function in (1.13) into the Taylor series we obtain (formally)

\[
F_{r,s}(f) = \int_{\mathbb{R}^3} f(v) \left( \sum_{k=0}^{\infty} \frac{r^k}{k!} |v|^s \right) \, dv = \sum_{k=0}^{\infty} \frac{m_{sk}}{k!} r^k.
\] (1.16)

Then the value \( r^*_s \) from (1.14) can be interpreted as the radius of convergence of the series (1.16), and the order of the tail \( s \) is therefore the unique value for which the series has a positive and finite radius of convergence.

We can now formulate the main results of this study. Our first result concerns the steady states of equation (1.1) corresponding to the first three types of forcing.

**Theorem 1.** Let \( f_i(v), i = 1, 2, 3, \) be nonnegative steady solutions of the equations (1.1), with the forcing terms (1.2), (1.3) and (1.4), respectively, and assume that \( f_i(v) \) have finite moments of all orders. Then \( f_i(v) \) have exponential tails of orders \( 3/2, 2 \) and 1, respectively.
For the shear flow model (1.5), we obtain the following weaker result.

**Theorem 2.** Let \( f_s(v) \) be a nonnegative steady solution of the shear flow model (1.1), (1.5) that has finite moments of all orders. Then the supremum \( r_1^s \), defined in (1.14), is finite, and therefore, \( s \geq 1 \).

**Remark.** The assumption of finiteness of moments of all orders is obviously required for the functionals (1.13) to be finite and for the expansions (1.16) to make sense. However, the moment inequalities we establish below also imply the following apriori estimates for all cases of the solutions: Suppose that a moment \( m_{p_0} \) of any order \( p_0 > 1 \) is finite. Then, in fact, all moments are finite and the solutions have exponential tails of the corresponding order. This observation is important, since it excludes the possibility of power-like decay for solutions of the considered equations, as soon as solutions have finite mass and finite moment of any order higher than kinetic energy.

The approach that we take in order to establish the above results is based on the moment method, in the form developed by one of the authors [4], for the classical space-homogeneous Boltzmann equation. We study the moment equations obtained by integrating (1.1) against \(|v|^2p\):

\[
\frac{\partial m_p}{\partial t} = Q_p + G_p
\]  

\[(1.17)\]

(in the steady case the time-derivative term drops out), where

\[
Q_p = \int_{\mathbb{R}^3} Q(f, f) |v|^{2p} \, dv \quad \text{and} \quad G_p = \int_{\mathbb{R}^3} G(f) |v|^{2p} \, dv. 
\]  

\[(1.18)\]

To investigate the summability of the series (1.16) we look for estimates of the sequence of moments \((m_p)\), with \( p = \frac{s+1}{k} \), \( k = 0, 1, 2, \ldots \), and study the dependence of the estimates on \( s \). We will be interested in the situation when the series has a finite and positive radius of convergence, which means that the sequence of the coefficients satisfies

\[ c q^k \leq \frac{m_{s+1}}{k!} \leq C Q^k, \quad k = 0, 1, 2, \ldots, \]

for certain constants \( q \) and \( Q > 0 \).

2. **Povzner-type inequalities for inelastic collisions**

In this section we establish an important technical result that will allow us to control the contribution of the “gain” operator in the moment equations (1.17). We consider radially symmetric test functions \( \psi(v) = \Psi(|v|^2) \). The weak form (1.18) of the “gain” operator can then be written as

\[
\int_{\mathbb{R}^3} Q^+(f, f)(v) \Psi(|v|^2) \, dv = \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(v) f(w) |u| A^+_\beta[\Psi](v, w) \, dw \, dv, 
\]

where

\[
A^+_\beta[\Psi](v, w) = \frac{1}{4\pi} \int_{S^2} \left( \Psi(|v'|^2) + \Psi(|w'|^2) \right) d\sigma. 
\]  

\[(2.1)\]

A series of results [30, 13, 23, 12, 33, 4, 27] obtained in the case of the classical Boltzmann equation develops the general idea that for convex functions \( \Psi \) the expression (2.1) is in a certain sense “smaller” than the corresponding contribution of
the “loss” term, which is given by
\[ A^{-}[\Psi](v, w) = \Psi(|v|^2) + \Psi(|w|^2). \]
This type of results is generally known as Povzner-type inequalities. An approach for obtaining such inequalities in the case of inelastic collisions has recently been developed in [19]. However, for the purposes of the present study we need a better control of the constants in the inequalities than those provided by the results of [19]. We will therefore establish a sharper version of the Povzner-type inequality for inelastic collisions, using the ideas of [4]. The key point, as in [4], is to look for estimates of the integral quantity (2.1), rather than for pointwise estimates of the integrand.

Lemma 1. For every \( \beta \in \left[ \frac{1}{2}, 1 \right] \) there exists a function \( \bar{g}_\beta(\mu) \), on \( \mu \in (-1, 1) \), which we define explicitly in the course of the proof, such that \( \bar{g}_\beta(\mu) \) is nonnegative, continuous, even, nondecreasing for \( \mu \in [0, 1] \), satisfies
\[ 2 \int_0^1 \bar{g}_\beta(2z - 1) z \, dz = 1, \]
and
\[ \bar{g}_\beta(\mu) \leq 1 + \left( \frac{1}{\beta} - 1 \right)^2, \]
and for every smooth function \( \Psi(x) \), defined for \( x > 0 \), nondecreasing and convex,
\[ A^+_\beta[\Psi] \leq 2 \int_0^1 \bar{g}_\beta(2z - 1) \Psi \left( z \left( |v|^2 + |v_s|^2 \right) \right) \, dz. \]

Remarks. 1) The above inequality is a generalization of inequalities (12), (16) from [4] to the inelastic case, under the extra assumption of \( \Psi \) being nondecreasing. Indeed, from the conditions on \( \bar{g}_\beta(\mu) \) it is easy to see that in the elastic case \( \beta = 1 \) we must have \( \bar{g}_1(\mu) = 1 \). Then the inequality of the lemma reduces to
\[ A^+_\beta[\Psi] \leq 2 \int_0^1 \Psi \left( z \left( |v|^2 + |v_s|^2 \right) \right) \, dz, \]
which is equivalent to the form used in [4]. 2) The smoothness assumption on \( \Psi \) can be dispensed with relatively easily, by elaborating some of the arguments we use in the proof. On the other hand, in the most important examples \( \Psi(x) = x^p \) with \( p > 1 \), which will be used in the moment estimates, the needed smoothness is readily available.

Proof. In the proof we use repeatedly the following argument: suppose that \( \psi(v) \), \( v \in \mathbb{R}^3 \) is a convex function. Then, for almost every \( a \in \mathbb{R}^3 \), and for every \( b \in \mathbb{R}^3 \),
\[ \psi(a + tb) + \psi(a - tb) \quad (2.2) \]
is a nondecreasing function of \( t > 0 \). (To see this differentiate \( (2.2) \) in \( t \) and notice that a convex function has almost everywhere a nondecreasing directional derivative in every direction.)

To apply this argument we notice that since \( \Psi(x) \) is convex and nondecreasing, then also \( \psi(v) = \Psi(|v|^2) \) is convex as a function of \( v \in \mathbb{R}^3 \). In order to introduce
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the symmetric structure (2.2) into the integrand of (2.1) we then rewrite \( v' \) and \( w' \) as

\[
v' = U + \frac{u'}{2}, \quad w' = U - \frac{u'}{2},
\]

where \( U = \frac{u + w}{2} \) is the velocity of the center of mass, and \( u' \) is the relative velocity after the collision. We then have, according to (1.9),

\[
u' = (1 - \beta)u + \beta|u|\sigma,
\]

(2.3)

where \( u \) is the relative velocity before the collision. Further, we pass to the spherical coordinates \((\rho, \omega)\) in \( u' \) by setting

\[
u' = \rho |u|\omega, \quad \rho \in \mathbb{R}, \quad \omega \in S^2.
\]

Denoting by \( \nu \) the unit vector in the direction of \( u \) and using (2.3), we can write

\[
\rho \omega = (1 - \beta)\nu + \beta\sigma.
\]

(2.4)

We then perform the change of variables from \( \sigma \) to \( \omega \) in the integral (2.1). To do so, notice that for every test function \( \varphi(k), k \in \mathbb{R}^3 \), we can formally extend the integration from \( S^2 \) to \( \mathbb{R}^3 \) by writing

\[
\int_{S^2} \varphi(\sigma) d\sigma = \int_{\mathbb{R}^3} \delta \left( \frac{|k|^2 - 1}{2} \right) \varphi(k) dk,
\]

(2.5)

where \( \delta \) is the one-dimensional \( \delta \)-function.

Changing variables from \( k \) to

\[
k' = (1 - \beta)\nu + \beta k, \quad k' = \rho \omega
\]

and then passing to the spherical coordinates \((\rho, \omega)\) we can rewrite the integral (2.5) as

\[
\frac{1}{\beta^3} \int_{S^2} \int_0^\infty \rho^2 \delta \left( \frac{\rho \omega - (1 - \beta)\nu^2 - \beta^2}{2\beta^2} \right) \varphi \left( \frac{\rho \omega - (1 - \beta)\nu}{\beta} \right) d\rho d\omega
\]

\[
= \frac{1}{\beta} \int_{S^2} \int_0^\infty \rho^2 \delta \left( \frac{(\rho - a)^2 - (a^2 + b^2)}{2} \right) \varphi \left( \frac{\rho \omega - (1 - \beta)\nu}{\beta} \right) d\rho d\omega,
\]

(2.6)

where

\[
a = (1 - \beta)(\nu \cdot \omega) \quad \text{and} \quad b^2 = 2\beta - 1.
\]

The radial integration in (2.6) can be performed explicitly, since

\[
\int_0^\infty \rho^2 \delta \left( \frac{(\rho - a)^2 - (a^2 + b^2)}{2} \right) d\rho = \int_a^\infty \frac{(\rho + a)^2}{\rho} \delta \left( \frac{\rho^2 - (a^2 + b^2)}{2} \right) \rho d\rho
\]

\[
= \frac{(a + \sqrt{a^2 + b^2})^2}{\sqrt{a^2 + b^2}}.
\]

After the radial integration, \( \rho \) in (2.6) will be expressed through \( \mu = (\nu \cdot \omega) \) according to

\[
\rho = \lambda(\mu) = (1 - \beta)\mu + \sqrt{(1 - \beta)^2 \mu^2 + 2\beta - 1}.
\]

(2.7)

(The last equation is nothing but the condition \(|k|^2 = 1\) expressed in the new variables.) Thus, we obtain the formula

\[
\int_{S^2} \varphi(\sigma) d\sigma = \int_{S^2} \varphi \left( \frac{\rho \omega - (1 - \beta)\nu}{\beta} \right) g_\beta((\nu \cdot \omega)) d\omega,
\]

(2.8)
where
\[
g_{\beta}(\mu) = \frac{1}{\beta} \left( \frac{a + \sqrt{a^2 + b^2}}{\sqrt{a^2 + b^2}} \right)^2 = \frac{\lambda^2(\mu)}{\beta(\lambda(\mu) - (1 - \beta)\mu)}. \tag{2.9}
\]

Applying identity (2.8) to (2.1) we get
\[
A_{\beta}^+ = \frac{1}{4\pi} \int_{S^2} g_{\beta}(\nu \cdot \omega) \left\{ \Psi \left( \left| U + \lambda \frac{|u|}{2} \omega \right|^2 \right) + \Psi \left( \left| U - \lambda \frac{|u|}{2} \omega \right|^2 \right) \right\} d\omega. \tag{2.10}
\]

Our next goal is to simplify (2.10) to get a convenient upper bound. First, due to the convexity of \( \Psi(| \cdot |^2) \), the expression in braces, considered as a function of \( \lambda > 0 \), is monotonically increasing. Using (2.7) it is easy to see that
\[
0 < 2\beta - 1 \leq \lambda(\mu) \leq 1,
\]
for all \( \mu \in [-1, 1] \). Thus, estimating \( \lambda \) by one and setting
\[
E = 2U^2 + \frac{|u|^2}{2} = |v|^2 + |w|^2, \tag{2.11}
\]
we find:
\[
A_{\beta}^+ = \frac{1}{4\pi} \int_{S^2} g_{\beta}(\nu \cdot \omega) \left\{ \Psi \left( E \left( \text{1} + \frac{|\nu \cdot \omega|}{2} \right) \right) + \Psi \left( E \left( \text{1} - \frac{|\nu \cdot \omega|}{2} \right) \right) \right\} d\omega, \tag{2.12}
\]
where
\[
\xi = \frac{2|U||u|}{E} (m \cdot \omega), \tag{2.13}
\]
and \( m \) is the unit vector in the direction of \( U \).

We further symmetrize (2.12) by using the change of variables \( \omega \mapsto -\omega \). Since the expression in braces is invariant under this transformation, we can replace the function \( g_{\beta}(\mu) \) in (2.12) by its symmetrized version
\[
\bar{g}_{\beta}(\mu) = \frac{1}{2} (g_{\beta}(\mu) + g_{\beta}(-\mu)). \tag{2.14}
\]

It is now a somewhat tedious but straightforward computation to check that \( \bar{g}_{\beta}(\mu) \) has the properties listed in the formulation of the lemma. Noticing that
\[
\frac{2|U||u|}{E} \leq 1
\]
and using the convexity argument again, this time for the function \( \Psi(E(\frac{1+\xi}{2})) \), we can replace \( \xi \) in (2.12) by \( (m \cdot \omega) \).

Next, we see that, for \( |U| \) and \( |u| \) fixed, the integral (2.12) has the structure
\[
\int_{S^2} \varphi_1(\nu \cdot \omega) \varphi_2(m \cdot \omega) d\omega,
\]
where \( \varphi_1 \) and \( \varphi_2 \) are nonnegative, even and monotonically increasing on \([0, 1]\). It is easy to show that the maximum value of the integral is attained when the vectors \( \nu \) and \( m \) are parallel. The integral in (2.12) is then bounded by
\[
\frac{1}{4\pi} \int_{S^2} \bar{g}_{\beta}(\nu \cdot \omega) \left\{ \Psi \left( E \left( \frac{1 + (\nu \cdot \omega)}{2} \right) \right) + \Psi \left( E \left( \frac{1 - (\nu \cdot \omega)}{2} \right) \right) \right\} d\omega
\]
\[
= \frac{1}{2} \int_{-1}^{1} \bar{g}_{\beta}(\mu) \left\{ \Psi \left( E \left( \frac{1 + \mu}{2} \right) \right) + \Psi \left( E \left( \frac{1 - \mu}{2} \right) \right) \right\} d\mu. \tag{2.15}
\]
Using that $\bar{g}_\beta$ is an even function of $\mu$ and performing the change of variables $z = (1 + \mu)/2$, we arrive at the conclusion of the lemma. □

In the case when the function $\Psi(x)$ is a power function of $x$, the bounds of the lemma take a more explicit form, and we obtain the following important corollary.

**Corollary 3.** Let $\psi(v) = |v|^{2p}$, where $p > 1$. Then $A_\beta[\psi]$, given by (1.11), satisfies the inequality

$$A_\beta(|v|^{2p}) \leq \gamma_p(|v|^2 + |w|^2)^p - |v|^{2p} - |w|^{2p},$$

where the constant $\gamma_p$, defined by (2.16), is strictly decreasing for $p \geq 1$ and satisfies $\gamma_p < \min\{1, \frac{4}{p+1}\}$.

**Proof.** Taking $\Psi(x) = x^p$, we can write

$$A_\beta(|v|^{2p}) \leq A_\beta^+[\Psi] - |v|^{2p} - |w|^{2p}.$$

Using Lemma 1 with $\Psi(x) = x^p$ we get the bound

$$A_\beta^+[\Psi] \leq \gamma_p(|v|^2 + |w|^2)^p,$$

where

$$\gamma_p = 2 \int_0^1 \bar{g}_\beta(2z - 1) z^p dz. \tag{2.16}$$

By Lemma 1, $\gamma_1 = 1$, and so, since $z^p < z$ for all $z \in (0, 1)$, we have

$$\gamma_p < 1,$$

for all $p > 1$. On the other hand, estimating $\bar{g}_\beta(\mu)$ by its maximum

$$\bar{g}_\beta(1) = 1 + \left(\frac{1}{\beta} - 1\right)^2 \leq 2,$$

we get

$$\gamma_p \leq 4 \int_0^1 z^p dz = \frac{4}{p+1}. \tag{2.17}$$

This completes the proof. □

**Remark.** The expression (2.16) for the constant $\gamma_p$ simplifies in the cases $\beta = 1$ (elastic interactions), when

$$\gamma_p = \frac{2}{p+1},$$

and in the case $\beta = 1/2$ (“sticky” particles), when

$$\gamma_p = \frac{p^{2p} + 1}{2^{p-2}(p+1)(p+2)}.$$

In the general case the integrand of (2.16) is too complicated to yield an answer in closed form. However, the bound (2.17) is quite useful for $p > 3$ and shows the correct “inverse first power” decay for $p$ large.
3. Moment inequalities

The estimate of Corollary 3 is a crucial step to obtaining the moment inequalities in the form characteristic for the Boltzmann equation with “hard interactions” [12, 4]. The basic estimate takes a particularly simple form when \( p \) is an integer, since then the binomial formula can be used to obtain the inequality

\[
A_\beta[v|v|^{2p}] + (1 - \gamma_p)(|v|^{2p} + |w|^{2p}) \leq \gamma_p \sum_{k=1}^{p-1} \binom{p}{k} |v|^{2k} |w|^{2(p-k)}
\]

(cf. [4, p. 1189]). In the case of non-integer \( p \) a similar result can be obtained, after we establish the following estimates of the binomial expansion.

Lemma 2. Assume that \( p > 1 \), and let \( k_p \) denote the integer part of \( \frac{p+1}{2} \). Then for all \( x, y > 0 \) the following inequalities hold

\[
\sum_{k=1}^{k_p-1} \binom{p}{k} (x^k y^{p-k} + x^{p-k} y^k) \leq (x + y)^p - x^p - y^p
\]

(3.1)

Remarks. 1) The binomial coefficients for non-integer \( p \) are defined as

\[
\binom{p}{k} = \frac{p(p-1)\ldots(p-k+1)}{k!}, \quad k \geq 1; \quad \binom{p}{0} = 1.
\]

2) In the case when \( p \) is an odd integer the first of the inequalities becomes an equality which then coincides with the binomial formula for \((x + y)^p\).

Proof. The proof will be achieved by induction on \( n = k_p = 1, 2, 3 \ldots \) In the case \( k_p = 0 \) the following inequality is satisfied for \(-1 < p \leq 1\):

\[(x + y)^p - x^p - y^p \leq 0.\]

Next, for \( n = 1 \) and \( 1 < p \leq 3 \), using the above inequality and the identity

\[0 \leq (x + y)^p - x^p - y^p = \int_0^x \int_0^y p(p-1)(t+\tau)^{p-2} d\tau dt,\]

we obtain

\[(x + y)^p - x^p - y^p \leq \int_0^x \int_0^y p(p-1)(t^{p-2} + \tau^{p-2}) d\tau dt = p (xy^{p-1} + x^{p-1}y),\]

which provides the base for the induction.

Assuming now that the inequalities (3.1) are true for \( 2n-1 < p \leq 2n+1 \), we write

\[(x + y)^{p+2} - x^{p+2} - y^{p+2} = \int_0^x \int_0^y (p+2)(p+1)(t + \tau)^p d\tau dt.\]
By induction hypothesis, the right-hand side of (3.2) is bounded from below by
\[
\int_0^x \int_0^y (p+2)(p+1)(t^p + \tau^p) \, d\tau \, dt \\
+ \int_0^x \int_0^y (p+2)(p+1) \sum_{k=1}^{k_p-1} \binom{p}{k} (t^{k_p-k} + t^{p-k+1}) \, d\tau \, dt,
\]
and from above by a similar expression with \(k_p - 1\) replaced by \(k_p\). Performing the integration, using the identity
\[
\frac{(p+2)(p+1)}{(k+1)(p-k+1)} \binom{p}{k} = \binom{p+2}{k+1},
\]
and noticing that \(k_p + 1 = k_{p+2}\), we obtain the lower bound for (3.2) in the form
\[
(p+2)(x^{p+2} + y^{p+2}) + \sum_{k=1}^{k_{p+2}-1} \binom{p+2}{k} (x^{k+1}y^{p+1-k} + x^{p+1-k}y^{k+1}) = \sum_{k=1}^{k_{p+2}-1} \binom{p+2}{k} (x^{k+2}y^{p+2-k} + x^{p+2-k}y^{k}),
\]
and the upper bound with \(k_{p+2} - 1\) replaced by \(k_{p+2}\). This completes the induction argument. \(\square\)

We now establish the following bounds for the moments of the collision term \(Q_p\) defined in (1.18) in terms of moments \(m_p\) of the distribution function.

**Lemma 3.** For every \(p > 1\),
\[-m_{p+\frac{1}{2}} \leq Q_p \leq -(1 - \gamma_p) m_{p+\frac{1}{2}} + \gamma_p S_p,\]
where
\[S_p = \sum_{k=1}^{k_p} \binom{p}{k} (m_{k+\frac{1}{2}} m_{p-k} + m_k m_{p-k+\frac{1}{2}}) \quad (3.3)\]
and \(\gamma_p\) is the constant from Corollary 3.

**Proof.** Multiplying the inequality of Corollary 3 by \(f(v)f(w) |v - w|\) and integrating with respect to \(v\) and \(w\), we obtain
\[
Q_p \leq \frac{\gamma_p}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(v)f(w) |v - w| \left( (|v|^2 + |w|^2)^p - |v|^{2p} - |w|^{2p} \right) \, dv \, dw \\
- (1 - \gamma_p) \int_{\mathbb{R}^3} f(v) |v|^{2p} \int_{\mathbb{R}^3} f(w) |v - w| \, dv \, dw. \quad (3.4)
\]
The inner integral in the last term can be estimated as
\[
\int_{\mathbb{R}^3} f(w) |v - w| \, dw \geq |v|.
\]
The last inequality follows by Jensen’s inequality, since \(f\) is normalized to have unit mass and zero mean, and the function \(|v - w|\) is convex in \(w\) for every \(v\) fixed. Thus, the last integral term in (3.4) is estimated below by
\[
\int_{\mathbb{R}^3} f(v) |v|^{2p+1} \, dv = m_{p+1/2}.
\]
In the first integral term in (3.4), we use the inequality \(|v - w| \leq |v| + |w|\) and the upper estimate of Lemma 2 to get
\[
|v - w| \left(\left(\frac{1}{2} + |w|^{2p} - |w|^{2p} - |w|^{2p}\right)\right)
\leq \sum_{k=1}^{k_p} \left(\frac{1}{k} \left(|v|^{2k+1/2}|w|^{2p} + |v|^{2k+1/2}|w|^{2p}\right)\right)^{1/2}.
\]  
(3.5)
Substituting the estimate (3.5) into (3.4) and performing the integration we obtain the upper bound of the Lemma.

For the lower bound we use the splitting (1.6), neglecting the nonnegative \(Q^+\) term and estimating the moments of \(Q^-\) in the same way as we did for the second integral term in (3.4). This completes the proof.

We next apply the bounds for the moments of the collision terms obtained in Lemma 3 to the steady moment equations
\[
Q_p + G_p = 0, \quad (3.6)
\]
obtained from (1.17). Under suitable conditions on smoothness and decay for \(|v|\) large of the solutions \(f(v)\), the moments of the forcing terms are calculated as follows.

In the case of pure diffusion (1.2) we have
\[
G_p = \int_{\mathbb{R}^3} f(v) \mu \Delta |v|^{2p} dv = 2\mu p \left(2p + 1\right) m_{p-1}. \tag{3.7}
\]
In the case of diffusion with friction (1.3), we obtain
\[
G_p = \int_{\mathbb{R}^3} f(v) \left(\mu \Delta |v|^{2p} - \lambda v \cdot \nabla |v|^{2p}\right) dv = -2\lambda p m_p + 2\mu p \left(2p + 1\right) m_{p-1}. \tag{3.8}
\]
Setting \(\mu = 0\) and \(\lambda = -\kappa\) in the above identity, we obtain the case of the self-similar solutions: (1.4)
\[
G_p = 2\kappa p m_p. \tag{3.9}
\]
Finally, for the shear flow term (1.5), we obtain the inequality
\[
G_p = 2\kappa p \int_{\mathbb{R}^3} f(v) v_1 v_2 |v|^{2p-2} dv \leq 2\kappa p m_p. \tag{3.10}
\]

Hence, combining the bounds of Lemma 3 with the above identities, we find for every \(p > 1\), in the first three cases of forcing the double inequalities
\[
G_p \leq m_{p+\frac{1}{2}} \leq \frac{1}{1 - \gamma_p} \left(G_p + \gamma_p S_p\right), \tag{3.11}
\]
and in the case of the shear flow the one-sided inequality
\[
m_{p+\frac{1}{2}} \leq \frac{1}{1 - \gamma_p} \left(2\kappa p m_p + \gamma_p S_p\right), \tag{3.12}
\]
where \(G_p\) are given by (3.7), (3.8) and (3.9), and \(S_p\) is given by (3.3).

Notice that since the terms \(G_p\) and \(S_p\) depend on the moments \(m_k\) of order at most \(p\) \((p - \frac{1}{2} \text{ in the case of } S_p\)\), inequalities (3.11) and (3.12) can be “solved” recursively. More precisely, assuming some properties of the moments of lower order
we can use the recursive inequalities to obtain information about the behavior of the moments \( m_p \), for \( p \) large.

In order to study the summability of the series (3.14), it is convenient to formulate the moment inequalities in terms of the normalized moments

\[
z_p = \frac{m_p}{\Gamma(ap + b)}, \quad p \geq 0,
\]

(3.13)

where \( a \) and \( b \) are constants to be determined. Indeed, the coefficients of the power series (1.16) represent a particular case of (3.13), with \( p = \frac{s^k}{2} \), \( a = \frac{2}{s} \) and \( b = 1 \). We will therefore study the conditions on \( a \) and \( b \) under which the sequences of normalized moments \( z_p = z_{\frac{s^k}{2}} \) have geometric (exponential) bounds.

We will first look for estimates of the term \( S_p \) in the moment inequalities (3.11) and (3.12), expressed in terms of the normalized moments \( z_p \). We recall the definition of the Beta function

\[
B(x, y) = \int_0^1 s^{x-1}(1-s)^{y-1} \, ds = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}
\]

(3.14)

which will be used in the proof of next lemma.

**Lemma 4.** Let \( m_p = z_p \Gamma(ap + b) \) with \( a \geq 1 \) and \( b > 0 \). Then for every \( p > 1 \),

\[
S_p \leq A \Gamma(ap + \frac{a}{2} + b) Z_p,
\]

where

\[
Z_p = \max_{1 \leq k \leq k_p} \{ z_{k+1/2} z_{p-k}, z_k z_{p-k+1/2} \}
\]

(3.15)

and \( A = A(a,b) \) is a constant independent of \( p \).

**Proof.** Substituting (3.13) in the expression (3.3) for \( S_p \) we get

\[
S_p = \sum_{k=1}^{k_p} \left( \begin{array}{c} p \\ k \end{array} \right) \left( \Gamma(ak + \frac{a}{2} + b) \Gamma(a(p-k) + b) z_{k+1/2} z_{p-k} \right. \\
+ \left. \Gamma(ak + b) \Gamma(a(p-k) + \frac{a}{2} + b) z_k z_{p-k+1/2} \right).
\]

(3.16)

Using (3.14), we can rewrite (3.16) as

\[
\Gamma(ap + \frac{a}{2} + b) \sum_{k=1}^{k_p} \left( \begin{array}{c} p \\ k \end{array} \right) \left( B(ak + \frac{a}{2} + b, a(p-k) + b) z_{k+1/2} z_{p-k} \right. \\
+ \left. B(ak + b, a(p-k) + \frac{a}{2} + b) z_k z_{p-k+1/2} \right).
\]

(3.17)

Next, we estimate the products \( z_{k+1/2} z_{p-k} \) and \( z_k z_{p-k+1/2} \) by their maximum \( Z_p \), obtaining the following bound for the sum in (3.17)

\[
Z_p \sum_{k=1}^{k_p} \left( \begin{array}{c} p \\ k \end{array} \right) \left( B(ak + \frac{a}{2} + b, a(p-k) + b) + B(ak + b, a(p-k) + \frac{a}{2} + b) \right)
\]

(3.18)

\[
= Z_p \int_0^1 s^{\frac{a}{2}+b-1}(1-s)^{b-1} \sum_{k=1}^{k_p} \left( \begin{array}{c} p \\ k \end{array} \right) \left\{ s^a + k(1-s)^a(p-k) + s^a(p-k)(1-s)^a \right\} \, ds.
\]
Since the expression in braces depends monotonically on $a$, we estimate it from above by setting $a = 1$. Further, using the lower bound of Lemma 2 the right-hand side of (3.18) is bounded above by

$$Z_p \int_0^1 \left\{ s^{\frac{p}{2}+b-1}(1-s)^{b-1}(1-s^p -(1-s)^p) + \chi_p \left( \frac{p}{k_p} \right) s^{\frac{p}{2}+b-1}(1-s)^{b-1} \left( s^{k_p}(1-s)^{p-k_p} + s^{p-k_p}(1-s)^{k_p} \right) \right\} ds,$$

(3.19)

where $\chi_p = 0$ if $p$ is an odd integer, and 1, otherwise. Neglecting the negative terms in $1-s^p -(1-s)^p$ and using the definition of the Beta function again, we obtain the bound

$$B \left( \frac{a}{2} + b \right) + \chi_p \left( \frac{p}{k_p} \right) \left( B \left( k_p + \frac{a}{2} + b, p-k_p + b \right) + B \left( k_p + b, p-k_p + \frac{a}{2} + b \right) \right).$$

(3.20)

The first term of (3.20) is a constant independent on $p$; to estimate the second term we recall the following asymptotic formula for the Gamma functions [1]:

$$\lim_{p \to \infty} \frac{\Gamma(p+r)}{\Gamma(p+s)} p^{s-r} = 1,$$

(3.21)

for all $r, s > 0$. Therefore, taking the first Beta function in the second term of (3.20) for definiteness, we obtain

$$\left( \frac{p}{k_p} \right) B \left( k_p + \frac{a}{2} + b, p-k_p + b \right) = \frac{\Gamma(p+1) \Gamma(k_p + \frac{a}{2} + b) \Gamma(p-k_p + b)}{\Gamma(p + \frac{a}{2} + 2b) \Gamma(k_p + 1) \Gamma(p-k_p + 1)} \leq C p^{1-\frac{a}{2}} k_p^{\frac{a}{2}+b-1} (p-k_p)^{b-1}.$$

A similar inequality can be obtained for the other Beta function term. It is clear now that the second term in (3.20) is $O(p^{-1})$ for $p \to \infty$, and since it also is locally bounded for $p \geq 0$, it is bounded uniformly in $p$. Denoting now by $A = A(a,b)$ the uniform bound of (3.20) we obtain the conclusion of the lemma.

\[ \square \]

**Remark.** A more careful analysis of the expression (3.18) would allow us to obtain a sharper upper bound

$$C p^{-a} Z_p$$

(3.22)

for that expression, at least for $1 \leq a \leq 2$. Thus, the factor $\Gamma \left( ap + \frac{a}{2} + b \right)$ in the estimate of the lemma could be improved to $\Gamma \left( ap - \frac{a}{2} + b \right)$. However, the result in the present formulation will be sufficient to obtain the necessary bounds for the moments, so we will not pursue the improved estimates based on the bound (3.22).

We next obtain the simplified inequalities for the normalized moments (3.13). Substituting (3.13) in the inequalities (3.11) and using the estimate of Lemma 4 we obtain in the case of pure diffusion (3.7)

$$2 \mu \frac{\Gamma \left( ap - a + b \right)}{\Gamma \left( ap + \frac{a}{2} + b \right)} p(2p+1) z_{p-1} \leq z_{p+1} \leq \frac{2 \mu}{1 - \gamma_p} \frac{\Gamma \left( ap - a + b \right)}{\Gamma \left( ap + \frac{a}{2} + b \right)} p(2p+1) z_{p-1} + \frac{\gamma_p A}{1 - \gamma_p} \frac{\Gamma \left( ap + \frac{a}{2} + 2b \right)}{\Gamma \left( ap + \frac{a}{2} + b \right)} Z_p,$$

(3.23)
for all $p \geq 1$. In the case of diffusion with friction (3.24), the terms

$$-2\lambda \frac{\Gamma(ap+b)}{\Gamma(ap+\frac{a}{2}+b)} p z_p$$

and

$$-\frac{2\lambda}{1-\gamma_p} \frac{\Gamma(ap+b)}{\Gamma(ap+\frac{a}{2}+b)} p z_p$$

will be added to the left and the right-hand sides of (3.23), respectively. For the shear flow case (3.10) we obtain

$$z_{p+\frac{1}{2}} \leq \frac{2\kappa}{1-\gamma_p} \frac{\Gamma(ap+b)}{\Gamma(ap+\frac{a}{2}+b)} p z_p + \gamma_p A \frac{\Gamma(ap+\frac{a}{2}+2b)}{1-\gamma_p} \frac{\Gamma(ap+\frac{a}{2}+b)}{\Gamma(ap+\frac{a}{2}+b)} z_p.$$  (3.25)

Using Corollary 3, for every $\varepsilon > 0$ and for all $p > 1 + \varepsilon$, the constants involving $\gamma_p$ can be estimated as follows:

$$1 \leq \frac{1}{1-\gamma_p} \leq \frac{1}{1-\gamma_{1+\varepsilon}} = K_{\varepsilon}$$  (3.26)

and

$$\gamma_p \frac{1}{1-\gamma_p} \leq \frac{4K_{\varepsilon}}{p+1}.$$  (3.27)

Further, using the identities

$$z \Gamma(z) = \Gamma(z+1) \quad \text{and} \quad z (z+1) \Gamma(z) = \Gamma(z+2)$$

and estimating

$$0 < c_3 \leq \frac{2p (2p+1)}{(ap-a+b)(ap+1-a+b)} \leq C_3,$$

and

$$ap + \frac{a}{2} + 2b - 1 \leq C_4 \frac{p+1}{4},$$

we can reduce the inequalities (3.23) to

$$c_3 \mu \frac{\Gamma(ap-a+b+2)}{\Gamma(ap+\frac{a}{2}+b)} z_{p-1} \leq z_{p+\frac{1}{2}} \leq C_3 K_{\varepsilon} \mu \frac{\Gamma(ap-a+b+2)}{\Gamma(ap+\frac{a}{2}+b)} z_{p-1}$$

$$+ C_4 K_{\varepsilon} \frac{\Gamma(ap+\frac{a}{2}+2b-1)}{\Gamma(ap+\frac{a}{2}+b)} z_p.$$  (3.28)

For the additional terms (3.24) appearing in the equation with friction, we use the inequalities

$$c_5 \leq \frac{2p}{ap+b} \leq C_5$$

to estimate them as

$$-C_5 K_{\varepsilon} \lambda \frac{\Gamma(ap+b+1)}{\Gamma(ap+\frac{a}{2}+b)} z_p$$

and

$$-c_5 \lambda \frac{\Gamma(ap+b+1)}{\Gamma(ap+\frac{a}{2}+b)} z_p.$$  (3.29)

Finally, for the self-similar solution case we obtain the inequalities

$$c_5 \kappa \frac{\Gamma(ap+b+1)}{\Gamma(ap+\frac{a}{2}+b)} z_p \leq z_{p+\frac{1}{2}} \leq C_5 K_{\varepsilon} \kappa \frac{\Gamma(ap+b+1)}{\Gamma(ap+\frac{a}{2}+b)} z_p$$

$$+ C_4 K_{\varepsilon} \frac{\Gamma(ap+\frac{a}{2}+2b-1)}{\Gamma(ap+\frac{a}{2}+b)} z_p,$$  (3.30)

the last of which is also true in the shear flow case.
We now study the inequalities (3.28)–(3.30) for the values of $a = \frac{2}{3}$ corresponding to the proposed orders of tails of the solutions. In the case of pure diffusion we take $a = \frac{4}{3}$ and the inequalities (3.28) take the form

$$c_3 \mu z_{p-1} \leq z_{p+\frac{1}{2}} \leq C_3 K_\varepsilon \mu z_{p-1} + C_4 K_\varepsilon \frac{\Gamma\left(\frac{4}{3} p + \frac{2}{3} + 2b - 1\right)}{\Gamma\left(\frac{4}{3} p + \frac{2}{3} + b\right)} Z_p,$$

(3.31)

for $p > 1 + \varepsilon$. We notice that if $b < 1$, the asymptotic formula (3.21) allows us to control the factor in front of the $Z_p$ term in (3.31) in the following way:

$$C_4 K_\varepsilon \frac{\Gamma\left(\frac{4}{3} p + \frac{2}{3} + 2b - 1\right)}{\Gamma\left(\frac{4}{3} p + \frac{2}{3} + b\right)} \leq \frac{1}{2}, \quad \text{for} \quad p \geq p_1,$$

(3.32)

if we take $p_1$ sufficiently large. Inequality (3.31) then becomes

$$c_3 \mu z_{p-1} \leq z_{p+\frac{1}{2}} \leq C_3 K_\varepsilon \mu z_{p-1} + \frac{1}{2} Z_p, \quad \text{for} \quad p \geq p_1.$$

(3.33)

In the case of diffusion with friction the choice $a = 1$ gives us the inequalities

$$-C_5 K_\varepsilon \lambda z_p + c_3 \mu z_{p-1} \leq \frac{\Gamma(p + \frac{1}{2} + b)}{\Gamma(p + 1 + b)} z_{p+\frac{1}{2}} \leq -c_5 \lambda z_p$$

$$+ C_3 K_\varepsilon \mu z_{p-1} + C_4 K_\varepsilon \frac{\Gamma(p - \frac{1}{2} + 2b)}{\Gamma(p + b + 1)} Z_p.$$

(3.34)

Taking now $b < \frac{3}{2}$ and choosing $p_1$ large enough, we obtain using (3.21),

$$C_4 \frac{\Gamma(p - \frac{1}{2} + 2b)}{\Gamma(p + b + 1)} \leq \frac{c_5 \lambda}{2} \quad \text{and} \quad \frac{\Gamma(p + \frac{1}{2} + b)}{\Gamma(p + 1 + b)} \leq 1, \quad \text{for} \quad p \geq p_1.$$

(3.35)

We can then use (3.31) to obtain the following simple inequalities

$$C_5 K_\varepsilon \lambda z_p \geq c_3 \mu z_{p-1} - z_{p+\frac{1}{2}}$$

(3.36)

and

$$c_5 \lambda z_p \leq C_3 \mu z_{p-1} + \frac{1}{2} c_5 \lambda Z_p,$$

(3.37)

for all $p \geq p_1$.

Finally, in the case of self-similar solutions we take $a = 2$, and (3.30) becomes

$$c_5 \kappa z_p \leq z_{p+\frac{1}{2}} \leq C_5 K_\varepsilon \kappa z_p + C_4 K_\varepsilon \frac{\Gamma(2p + 2b)}{\Gamma(2p + b + 1)} Z_p.$$

(3.38)

We then take $b < 1$ and choose $p_1$ large enough to obtain

$$C_4 K_\varepsilon \frac{\Gamma(2p + 2b)}{\Gamma(2p + b + 1)} \leq \frac{1}{2}, \quad \text{for} \quad p \geq p_1.$$

(3.39)

Inequality (3.38) then simplifies to

$$c_5 \kappa z_p \leq z_{p+\frac{1}{2}} \leq C_5 K_\varepsilon \kappa z_p + \frac{1}{2} Z_p, \quad \text{for} \quad p \geq p_1.$$

(3.40)

The second of these inequalities is also satisfied in the shear flow case.

We have now obtained inequalities for the normalized moments (3.13) in the form which is simple enough to be analyzed and which, as we will see below, will allow us to prove the results about the tail behavior stated in Section 1. Abstracting
now from the precise meaning of the terms in inequalities (3.33), (3.36), (3.37) and (3.40) we can say that they express the balance between the “loss terms” (moments of order $p + \frac{1}{2}$), “gain terms” (terms involving $Z_p$), diffusion (moments of order $p - 1$) and friction or the force terms in the shear flow (moments of order $p$). We notice also that inequalities in the form (3.28), (3.29) and (3.30) together with the asymptotic formula (3.21) can be used to actually derive the values of $a$ for which the series (1.16) has finite and positive radius of convergence. For the sake of simplicity, since these values are already known from the formal arguments, we will not perform these computations here.

4. Proofs of Theorems 1 and 2

Proof of Theorem 1. We will establish the following statement that will imply the conclusion of the Theorem (see also the Remark that follows Theorem 2). We show that for every $p_0 > 1$ there are positive constants $c, q, \alpha, \beta, \gamma, \delta, \xi, \eta, \zeta, \theta$, depending on $m_0$ and $m_1$ only, and $C, Q, \rho, \sigma, \tau, \upsilon, \phi, \chi, \psi, \omega$, depending on $m_0, p_0$ and $m_0$, only, such that

$$cq^k \leq \frac{m_\alpha k}{k!} \leq CQ^k,$$

(4.1)

for all $k \geq s$, where $s = \frac{2}{3}$ in the case of the pure diffusion, $s = 2$ in the case of diffusion with friction, and $s = 1$ for the self-similar solutions (1.4). Equivalently, we can set $a = \frac{4}{3}, \alpha = 1$ and $\beta = 2$ in the respective cases and look for the estimates

$$cq^p \leq z_p \leq CQ^p,$$

(4.2)

for all $p \geq 1$, with $z_p$ defined as in (3.13). (The constants in (4.2) have to be modified to match those in (4.1)).

Notice that it would be sufficient to prove (4.2) for a certain value of $b > 0$ in the definition of $z_p$ (3.13). Indeed, since

$$C_1 p^{b_1 - b_2} \leq \frac{\Gamma(ap + b_1)}{\Gamma(ap + b_2)} \leq C_2 p^{b_1 - b_2},$$

changing the value of $b$ in (3.13) essentially results in the multiplication of $z_p$ by the factor $C p^{b_1 - b_2}$, which can be compensated for by adjusting the constants in (4.2). We fix the value of $b < 1$ so that inequalities (3.11), (3.32) and (3.34) are available for $p$ sufficiently large.

The proof of the inequalities (4.2) will be accomplished in two steps. The first one will be to show that (4.2) holds on the initial interval, $1 \leq p \leq p_1$, where $p_1$ (dependent on $p_0$ and $b$) is chosen so that inequalities (3.32) and (3.33) hold with $\varepsilon = \frac{1 - p_0}{2}$.

Step 1: Initial interval. We notice that for $1 \leq p \leq p_1$, the Gamma function is bounded both from above and from below:

$$0 < c_0 \leq \frac{1}{\Gamma(ap + b)} \leq C_0,$$

(4.3)

where for $a > 0$ and $b > 0$ the constants $c_0$ and $C_0$ depend only on $a, b$ and $p_0$. Thus, on the initial interval it suffices to estimate $m_\alpha$ instead of $z_p$ in (4.2).
To obtain the desired estimate, we first use Jensen’s inequality to derive for every \(0 < p' < p < p''\) the inequalities
\[
(m_{p'}^{1/p'})^p \leq m_p \leq (m_{p''}^{1/p''})^p.
\] (4.4)
Taking \(p' = 1\) and \(p'' = p_0\) we obtain the bounds
\[
c q^p \leq m_p \leq C Q^p
\] (4.5)
for \(1 \leq p \leq p_0\), with \(c = C = 1\), \(q = m_1\) and \(Q = Q_0 = \max\{1, m_{p_0}^{1/p_0}\}\).

Step 1: Pure diffusion. We take \(\varepsilon = \frac{p_0 - 1}{2}\), use the bounds (3.26) and (3.27) in the moment inequalities (3.11), (3.7) and estimate
\[
S_p \leq 2^{p+1} M_p, \quad \text{where} \quad M_p = \max_{1 \leq k \leq k_p} \{m_k m_{p-k+\frac{1}{2}} m_{k+\frac{1}{2}} m_{p-k}\}. \quad (4.6)
\]
We then obtain, for all \(p > 1 + \varepsilon\), the inequalities
\[
2 \mu (2p + 1) m_{p-1} \leq m_{p+\frac{1}{2}} \leq 2 K_\varepsilon \mu p (2p + 1) m_{p-1} + K_\varepsilon 2^{p+1} M_p. \quad (4.7)
\]
Now we see that using (4.7) we can extend the bounds (3.3) (by augmenting the constants \(q\) and \(Q\) if necessary) to the interval \(\frac{3}{2} + \varepsilon \leq p \leq p_0 + \frac{1}{2}\). Using the interpolation inequality (4.4) we can then extend the bounds (3.3) to all intermediate values \(p_0 < p < p_0 + \frac{1}{2}\).

Further, by iterating inequalities (4.7) we can cover the interval \(p_0 \leq p \leq p_1\) by a fixed number of subintervals of length at most \(\frac{1}{2}\), so that finally inequalities (4.6), with the constants depending on \(m_0\), \(m_1\), \(p_0\) and \(m_{p_0}\) only, will be extended to the whole interval \(1 \leq p \leq p_1\).

Step 1: Diffusion with friction. We argue as in the previous case and obtain using (3.11), (3.8) the following upper bounds for \(m_{p+\frac{1}{2}}\):
\[
m_{p+\frac{1}{2}} \leq -2 K_\varepsilon \lambda p m_p + 2 K_\varepsilon \mu p (2p + 1) m_{p-1} + K_\varepsilon 2^{p+1} M_p, \quad (4.8)
\]
for all \(p > 1 + \varepsilon\). Neglecting the non-positive friction term on the right-hand side yields the same upper bounds as in the pure diffusion case. On the other hand, the lower bound can be written in the form
\[
2 \lambda p m_p + m_{p+\frac{1}{2}} \geq 2 \mu p (2p + 1) m_{p-1}, \quad (4.9)
\]
which implies that for every \(p > 1\) one of the following inequalities is true:
\[
2 \lambda p m_p \geq \mu p (2p + 1) m_{p-1} \quad \text{or} \quad m_{p+\frac{1}{2}} \geq \mu p (2p + 1) m_{p-1}
\]
Combining the two inequalities and using the interpolation inequality (4.4) in the second of the cases we obtain
\[
m_p \geq \min \left\{ \frac{2}{\lambda} (p + \frac{1}{2}) m_{p-1}, (\mu p (2p + 1) m_{p-1})^{\frac{2p}{2p+1}} \right\}.
\]
This allows us to extend the lower bound (1.5) iteratively to the interval \(1 \leq p \leq p_1\).

Step 1: Self-similar solutions. Using the moment inequalities (3.11), (3.9) and arguing as above we obtain
\[
2 \kappa p m_p \leq m_{p+\frac{1}{2}} \leq 2 K_\varepsilon \kappa p m_p + K_\varepsilon 2^{p+1} M_p, \quad (4.10)
\]
for all \(p \geq 1 + \varepsilon\). Using these bounds we extend (1.3) to the interval \(1 \leq p \leq p_1\) by the same iterative argument as in the pure diffusion case.
We now pass to Step 2 and use inequalities (3.33), (3.36) and (3.37) to extend bounds (1.2) to all \( p \geq 1 \) by an induction argument. The base of the induction is established by virtue of the bounds (1.5) and (1.3) on the interval \( 1 \leq p \leq p_1 \). We further verify the induction step separately in each of the three cases.

**Step 2: Pure diffusion.** Our aim is to find the constants \( q \) and \( Q \) in such a way that for every \( n = 1, 2, 3 \ldots \), the inequalities (4.2) for \( 1 \leq p \leq p_1 + \frac{n-1}{2} \) imply the same inequalities for \( p_1 + \frac{n-1}{2} \leq p \leq p_1 + \frac{n}{2} \). Thus, assuming (4.2) for \( 1 \leq p \leq p_1 + \frac{n-1}{2} \) we use (3.33) to find

\[
c_3 \mu c q^{p-1} \leq z_{p+\frac{1}{2}} \leq C \left( C_3 K p_0 \mu Q^{-\frac{1}{2}} + \frac{1}{2} \right) Q^{p+\frac{1}{2}}.
\]

Taking \( q \leq (c_3 \mu)^{\frac{1}{2}} \) and \( Q \geq (2C_3 K p_0 \mu)^{\frac{1}{2}} \) we obtain the inequality

\[
c q^{p+\frac{1}{2}} \leq z_{p+\frac{1}{2}} \leq C Q^{p+\frac{1}{2}},
\]

from which it follows that (4.2) is true for \( p_1 + \frac{n-1}{2} \leq p \leq p_1 + \frac{n}{2} \).

**Step 2: Diffusion with friction.** The upper bound case can be treated similarly to the previous one, with the difference that inequalities (3.37) will allow us to increase \( p \) by one in each step, instead of one half, as in the pure diffusion case. Thus, for every \( n = 1, 2, 3 \ldots \), we assume (4.2) for \( 1 \leq p \leq p_1 + n - 1 \) and obtain using (3.37),

\[
z_{p+1} \leq \left( \frac{C \mu}{C_5 \lambda} Q^{-1} + \frac{1}{2} \right) C Q^{p+1}.
\]

We now take \( Q \geq \frac{2C \mu}{C_5 \lambda} \) to obtain the inequalities

\[
z_{p+1} \leq C Q^{p+1},
\]

which imply the upper bound (4.2) for \( p_1 + n - 1 \leq p \leq p_1 + n \).

For the lower bound we see that assuming (4.2) to be true for \( 1 \leq p \leq p_1 + \frac{n-1}{2} \), the inequalities (3.36) imply that at least one of the inequalities

\[
K_\epsilon C_5 \lambda z_p \geq \frac{1}{2} c_3 \mu c q^{p-1} \quad \text{or} \quad z_{p+\frac{1}{2}} \geq \frac{1}{2} c_3 \mu c q^{p-1}
\]

is true. By choosing \( q < \min \left\{ \left( \frac{1}{2} c_3 \mu \right)^{\frac{1}{2}}, \frac{c_3 \mu}{2K_\epsilon C_5 \lambda} \right\} \) we obtain (4.2) for \( p_1 + \frac{n-1}{2} \leq p \leq p_1 + \frac{n}{2} \).

**Step 2: Self-similar solutions.** We use inequalities (3.40) and argue as in the pure diffusion case, assuming for every \( n = 1, 2, 3 \ldots \) that (4.2) holds for \( 1 \leq p \leq p_1 + \frac{n-1}{2} \). We then find

\[
c_5 \kappa c q^{p} \leq z_{p+\frac{1}{2}} \leq \left( C_5 K_\epsilon \kappa Q^{-\frac{1}{2}} + \frac{1}{2} \right) C Q^{p+\frac{1}{2}}.
\]

Therefore, taking \( q < (c_5 \kappa)^2 \) and \( Q > (2C_5 K_\epsilon \kappa)^2 \) we obtain (4.2) for \( p_1 + \frac{n-1}{2} \leq p \leq p_1 + \frac{n}{2} \).

We can now complete the proof of Theorem 1 by an induction argument. \( \square \)

The above proof contains the proof of Theorem 2 as a special case: indeed the inequalities for the normalized moments in the shear flow case coincide with the upper inequalities for the case of self-similar solutions. The result of Theorem 2 is weaker than in the latter case, since we were not able to obtain suitable lower bounds for the moments in the shear flow problem.
Concluding Remarks

The estimates for the normalized moments that we established certainly deserve more attention that we were able to give them in the framework of the steady solutions to the kinetic equations. In fact, we hope to return to the problem of the time-evolution of the tails by the moment method in a separate paper. Another promising direction of study stems from the use of the integral bounds together with maximum principles for kinetic equations, in the form suggested by C. Villani [32]. There are indications that such methods may yield more precise asymptotics (in particular, pointwise upper bounds) for some cases of kinetic equations [18].

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