Derivation, calibration and verification of macroscopic model for urban traffic flow. Part 1

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Abstract

In this paper we present a second-order hydrodynamic traffic model that generalizes the existing second-order models of Payne-Whithem, Zhang and Aw-Rascle. In the proposed model, we introduce the pressure equation describing the dependence of “traffic pressure” on traffic density. The pressure equation is constructed for each road segment from the fundamental diagram that is estimated using measurements from traffic detectors. We show that properties of any phenomenological model are fully defined by the pressure equation. We verify the proposed model through simulations of the Interstate 580 freeway segment in California, USA, with traffic measurements from the Performance Measurement System (PeMS).

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1. Introduction

An excessive development of gas dynamics (generalized conservation laws, stable differencing schemes) has taken place in 1950s. At the same time, first macroscopic (hydrodynamical) models emerged, in which traffic flow was likened
to flow of “motivated” compressible liquid. In the Lighthill–Whitham–Richards (LWR) model (Lighthill and Whitham, 1955; Richards, 1956; Whitham, 1974) the traffic flow is described by law of conservation of vehicles. This model postulates the unique dependence of traffic flow on traffic density, which is called a fundamental diagram. In recent years, the class of macroscopic models has expanded significantly. Nowadays, the traffic flow is described using system of non-linear hyperbolic second-order PDEs (for traffic density and speed), in various formulations (Payne, 1971; Daganzo, 1995; Papageorgiou, 1998; Aw and Rascle, 2000; Zhang, 2002, 2003; Siebel and Mauser, 2006a,b), which do not assume unique dependence of speed on density.

In this paper we present new second-order hydrodynamical model that generalizes the existing second-order traffic models. In the existing second-order models, the traffic flow is described using system of non-linear hyperbolic second-order PDEs (for traffic density and speed) (Payne, 1971; Daganzo, 1995; Papageorgiou, 1998; Aw and Rascle, 2000; Zhang, 2002, 2003; Siebel and Mauser, 2006a,b), which account for the dependence of traffic flow (or speed) on its density in different ways. We introduce the pressure equation that unifies all second-order hydrodynamical models.

Furthermore, the form of the pressure equation for each roadway segment can be obtained empirically using data from traffic detectors. To obtain the pressure equation, we first determine the dependency of traffic flow on its density from detector data; and then, using the relation between pressure and flow, we obtain the pressure equation as a functional dependence between pressure and density.

We verify the proposed approach through numerical experiments using typical traffic detector data from the Performance Measurement System (PeMS) (California Department of Tranportation, 2012), on the I-580 freeway in California, USA. Simulations show different second-order models using the same pressure term produce the same result.

The rest of the paper is organized as follows. Section 2 has three parts: Subsection 2.1 describes the proposed second-order model and shows how the existing second-order models are special cases of the proposed one; Subsection 2.2
introduces the algorithm for constructing the pressure equation from the fundamental diagram; and Subsection 2.3 presents the numerical scheme for model integration. Section 3 discusses simulation results, which support the proposition that different second-order models with the same pressure term behave the same way. Finally, Section 4 concludes the paper.

2. Generalized model

2.1. System of second-order PDEs

The Lighthill-Whitham-Richards (LWR) model (Lighthill and Whitham, 1955; Richards, 1956) assumes unique relationship between traffic speed \( v(t,x) \) and density \( \rho(t,x) \): \( v(t,x) = V(\rho(t,x)) \), and the vehicle conservation law holds. Here, \( \rho(t,x) \) denotes the number of vehicles per unit length of roadway at moment \( t \), around point with coordinate \( x \); and \( v(t,x) \) denotes mean traffic speed at time \( t \) around point with coordinate \( x \). The function \( V(\rho) \) is non-increasing: \( V'(\rho) \leq 0 \). We denote the flow (i.e. the number of vehicles passing a reference point in unit time) as \( Q(\rho) = \rho \cdot V(\rho) \). The function \( Q(\rho) \) is usually referred to as the fundamental diagram (although sometimes this name is reserved for \( V(\rho) \)).

The vehicle conservation law in the LWR model is expressed in the differential form of continuity equation with zero right-hand side:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} = \frac{\partial (\rho V(\rho))}{\partial x} = \frac{\partial \rho}{\partial t} + \frac{\partial Q(\rho)}{\partial x} = 0. \tag{1}
\]

If we account for variations in vehicle number, for example due to lane changes or freeway entrances and exits, the Eq. (1) takes the form:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial Q(\rho)}{\partial x} = f_0, \tag{2}
\]

where \( f_0 \) is the number of vehicles entering (positive) or leaving (negative) in unit time.

The Eq. (2) by itself is not sufficient for the adequate description of all the phases of traffic (Daganzo, 1995). To correct this, we use the differential
transformation of the conservation law (Godunov and Romenskii, 2003). We multiply (2) by \( \partial Q / \partial \rho \):

\[
\frac{\partial Q}{\partial \rho} \left( \frac{\partial \rho}{\partial t} + \frac{\partial Q}{\partial x} \right) = \frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial \rho} \frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial \rho} f_0, \tag{3}
\]

and use following differential relations:

\[
\begin{align*}
\frac{\partial Q}{\partial t} &= \frac{\partial}{\partial t} \left( \rho v \right) = \rho \frac{\partial v}{\partial t} + v \frac{\partial \rho}{\partial t}, \\
\frac{\partial Q}{\partial x} &= \frac{\partial}{\partial x} \left( \rho v \right) = \rho \frac{\partial v}{\partial x} + v \frac{\partial \rho}{\partial x}.
\end{align*}
\]

Substituting them into (3):

\[
v \frac{\partial \rho}{\partial t} + \rho \frac{\partial v}{\partial t} + \frac{\partial Q}{\partial \rho} \frac{\partial v}{\partial x} + \frac{\partial Q}{\partial \rho} \rho \frac{\partial v}{\partial x} = \frac{\partial Q}{\partial \rho} f_0 \Rightarrow \\
\rho \left( \frac{\partial v}{\partial t} + \frac{\partial Q}{\partial \rho} \frac{\partial \rho}{\partial x} \right) + v \left( \frac{\partial \rho}{\partial t} + \frac{\partial Q}{\partial \rho} \frac{\partial v}{\partial x} \right) = \rho \left( \frac{\partial v}{\partial t} + \frac{\partial Q}{\partial \rho} \frac{\partial v}{\partial x} \right) + v f_0 = \frac{\partial Q}{\partial \rho} f_0,
\]

we get:

\[
\frac{\partial v}{\partial t} + \frac{\partial Q}{\partial \rho} \frac{\partial v}{\partial x} = \left( \frac{\partial Q}{\partial \rho} - v \right) \frac{f_0}{\rho}.
\]

Thus, we constructed second-order macroscopic model, which does not assume unique dependency between traffic speed and density, as was in the LWR:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial Q}{\partial x} &= \frac{\partial \rho}{\partial t} + \left( \frac{\partial Q}{\partial \rho} \right) \frac{\partial v}{\partial x} = f_0, \\
\frac{\partial v}{\partial t} + \left( \frac{\partial Q}{\partial \rho} \right) \frac{\partial v}{\partial x} &= \left( \frac{\partial Q}{\partial \rho} - v \right) \frac{f_0}{\rho}. \tag{4}
\end{align*}
\]

The right-hand side \( f_0 \) of the first equation in (4) accounts for the number of vehicles (positive) or leaving (negative) in unit time. Looking at the right-hand side of the second equation in (4), we see the impact of these vehicles on traffic speed.

The number of left and right boundary conditions depends on the signs of the eigenvalues of system (4): \( \lambda_{1,2} = \frac{\partial Q(\rho)}{\partial \rho} \). At a freeway entrance, their number is either two, if \( \frac{\partial Q(\rho)}{\partial \rho} > 0 \), or zero, if \( \frac{\partial Q(\rho)}{\partial \rho} \leq 0 \). The opposite is true for a freeway exit: zero boundary conditions, if \( \frac{\partial Q(\rho)}{\partial \rho} \geq 0 \), and two, if \( \frac{\partial Q(\rho)}{\partial \rho} < 0 \). Therefore, we can use time-dependent traffic flow \( Q(t) \) and speed \( v(t) \) as boundary conditions.
Besides boundary conditions, we set the following initial conditions:

$$\rho(x, 0) = \rho_0(x), \quad v(x, 0) = v_0(x).$$

Existing macroscopic second-order models describe traffic in various formulations (Payne, 1971; Daganzo, 1995; Papageorgiou, 1998; Aw and Rascle, 2000; Zhang, 2002, 2003; Siebel and Mauser, 2006a,b). These models differ in the way they describe the dependency of flow (or speed) on density. Now, we will discuss the most popular second-order models and show how they turn out to be the special cases of our proposed model (4).

We start with the Payne–Whitham model (Payne, 1971; Whitham, 1974) that has zero right-hand side:

$$\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial Q}{\partial x} &= 0, \\
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{c_0^2}{\rho} \frac{\partial \rho}{\partial x} &= 0, \\
\end{align*}$$

(5)

and generalize it for the arbitrary form of the pressure equation — the dependency of the pressure on the density,

$$P(\rho) = \int_0^\rho c^2(\tilde{\rho}) d\tilde{\rho}. \quad (6)$$

With $P(\rho)$ defined in (6), model (4) can be rewritten as:

$$\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial Q}{\partial x} &= 0, \\
\frac{\partial}{\partial t} \left( \rho v \right) + \frac{\partial}{\partial x} \left( \rho v^2 + P(\rho) \right) &= 0, \\
\end{align*}$$

(7)

which generalizes (5).

System (7) could be rewritten in the divergent form:

$$\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial Q}{\partial x} &= 0, \\
\frac{\partial (\rho v)}{\partial t} + \frac{\partial (\rho v^2 + P(\rho))}{\partial x} &= 0, \\
\end{align*}$$

(8)

Now we need to find a way to express pressure $P(\rho) = \int_0^\rho c^2(\tilde{\rho}) d\tilde{\rho}$, or disturbance propagation speed $c(\rho) = \pm \sqrt{\frac{\partial P(\rho)}{\partial \rho}}$, through calculated variables,
speed \( v \) and density \( \rho \). First, we express partial derivative \( \frac{\partial P}{\partial \rho} \):

\[
\frac{\partial P}{\partial \rho} = c^2(\rho) = \frac{\partial P}{\partial x} \frac{\partial x}{\partial \rho} = P_x \frac{\rho_x}{\rho} = \left( \rho_x = -\frac{\rho t}{Q_\rho} \right) = -\frac{Q_\rho}{\rho} \frac{\partial P}{\rho_t} \frac{\partial x}{\partial \rho}.
\] (9)

Then, we express \( \frac{\partial P}{\partial \rho} \):

\[
-\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial t} + \frac{\partial (Qv)}{\partial x} = \frac{\partial Q}{\partial t} + \frac{\partial (Qv)}{\partial Q} \frac{\partial Q}{\partial x} = \left( \frac{\partial Q}{\partial \rho} = -\frac{\rho t}{Q_\rho} \right) = \frac{\partial Q}{\partial t} - \frac{\partial (Qv)}{\partial x} \frac{\partial \rho}{\partial t}
\]

and substitute it into Eq. (9):

\[
c^2(\rho) = \frac{\partial P}{\partial \rho} = \left( \frac{\partial P}{\partial x} \frac{\partial x}{\partial \rho} \right) = \frac{\partial Q}{\partial \rho} \left( \frac{\partial Q}{\partial \rho} - \frac{\partial (Qv)}{\partial x} \frac{\partial \rho}{\partial t} \right) = \left( \frac{\partial Q}{\partial \rho} \right)^2 - \frac{\partial (Qv)}{\partial \rho}.
\]

Taking \( \left( \frac{\partial Q}{\partial \rho} \right)^2 \) and \( \frac{\partial (Qv)}{\partial \rho} \) from the known relations,

\[
\begin{align*}
\left( \frac{\partial Q}{\partial \rho} \right)^2 &= \left( \frac{\partial (Qv)}{\partial \rho} \right)^2 = \left( v + \rho \frac{\partial v}{\partial \rho} \right)^2 = v^2 + 2\rho v \frac{\partial v}{\partial \rho} + \rho^2 \left( \frac{\partial v}{\partial \rho} \right)^2, \\
\end{align*}
\]

we obtain the final equation for \( \frac{\partial P}{\partial \rho} = c^2(\rho) \):

\[
c^2(\rho) = \frac{\partial P}{\partial \rho} = \left( \frac{\partial Q}{\partial \rho} \right)^2 - \frac{\partial (Qv)}{\partial \rho} = v^2 + 2\rho v \frac{\partial v}{\partial \rho} + \rho^2 \left( \frac{\partial v}{\partial \rho} \right)^2 - v^2 - 2\rho v \frac{\partial v}{\partial \rho} = \rho^2 \left( \frac{\partial v}{\partial \rho} \right)^2.
\] (10)

Using (10) in the second equation of (7):

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial Q}{\partial x} &= 0, \\
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + c^2(\rho) \frac{\partial \rho}{\partial x} &= \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \rho \frac{\partial v}{\partial \rho} \frac{\partial \rho}{\partial x} = \frac{\partial v}{\partial t} + \left( v + \rho \frac{\partial v}{\partial \rho} \right) \frac{\partial v}{\partial x} = 0.
\end{align*}
\] (11)

Substituting \( \frac{\partial Q}{\partial \rho} = v + \rho \frac{\partial v}{\partial \rho} \) into (11), we arrive at:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial Q}{\partial x} &= 0, \\
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + c^2(\rho) \frac{\partial \rho}{\partial x} &= \frac{\partial v}{\partial t} + \left( v + \rho \frac{\partial v}{\partial \rho} \right) \frac{\partial v}{\partial x} = \frac{\partial v}{\partial t} + \left( \frac{\partial Q}{\partial \rho} \right) \frac{\partial v}{\partial x} = 0.
\end{align*}
\] (12)
which is our model (4) with zero right-hand side.

**Remark.** It should be noted that (Zhang, 2002) was the first to propose the expression for disturbance propagation speed of the form: $c(\rho) = \rho \frac{\partial V(\rho)}{\partial \rho}$. This differs from (11) in that instead of the observed speed $v$, it used equilibrium speed $V(\rho)$ and the momentum equation $\frac{\partial v}{\partial t} + (v + c(\rho)) \frac{\partial v}{\partial x} = 0$. The expression for the disturbance propagation speed $c(\rho) = \rho \frac{\partial V(\rho)}{\partial \rho}$ in (Zhang, 2002) was derived from the car-following model (Helbing, 2001), while we obtained our expression $c(\rho) = \pm \sqrt{\frac{\partial P(\rho)}{\partial \rho}} = \pm \sqrt{\rho^2 \left(\frac{\partial v}{\partial \rho}\right)^2} = \pm \rho \frac{\partial v(\rho)}{\partial \rho}$ from (8), with no additional restrictions on the form of $v(\rho)$.

Now, let’s look at other second-order macroscopic models. The work (Zhang, 2002) proposes model defined by following system:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial Q}{\partial x} &= 0, \\
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + c(\rho) \frac{\partial v}{\partial x} &= 0,
\end{align*}
\] (13)

where the disturbance propagation speed is expressed through the derivative of the equilibrium speed: $c(\rho) = \rho \frac{\partial V(\rho)}{\partial \rho}$.

Note that the system (13) is almost identical to the non-divergent form of PDE system proposed in (Aw and Rascle, 2000). The difference is in notation: in (Aw and Rascle, 2000) smooth function $p(\rho)$ is used instead of the equilibrium speed $V(\rho)$:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial Q}{\partial x} &= 0, \\
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \beta(\rho,v) \left( V(\rho) - v \right) &= 0.
\end{align*}
\] (14)

From the physical standpoint, the use of the equilibrium speed $V(\rho)$ in (14) instead of an arbitrary smooth increasing function $p(\rho) \sim \rho^\gamma, \gamma > 0$ is more justified (see (Zhang, 2002)): function $V(\rho)$ cannot be smooth in points of transition between different traffic phases.

The system similar to (13), but with additional relaxation term $\beta(\rho,v) (V(\rho) - v)$
In the right-hand side, was used in (Siebel and Mauser, 2006a,b):

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial Q}{\partial x} &= 0, \\
\frac{\partial v}{\partial t} + \left( v + \rho \frac{\partial V(\rho)}{\partial \rho} \right) \frac{\partial v}{\partial x} &= \beta(\rho, v) (V(\rho) - v),
\end{align*}
\]

(15)

where the equilibrium speed is expressed as 

\[ V(\rho) = \frac{V_{\text{max}}}{\left(1 - \left(\frac{\rho}{\rho_{\text{max}}}\right)^{n_1}\right)^{n_2}} \]

in (Siebel and Mauser, 2006a); and as 

\[ V(\rho) = \frac{V_{\text{max}}}{\left(1 - \exp\left(\frac{-\lambda}{V_{\text{max}}}(\frac{1}{\rho} - \frac{1}{\rho_{\text{max}}})\right)\right)} \]

in (Siebel and Mauser, 2006b).

Now we will address an important question regarding the eigenvalues of the second-order macroscopic models. The PDE systems (4),(7),(8),(13),(15) are of hyperbolic type. By introducing vectors 

\[ W = \{\rho, v\}^T, \quad U = \{\rho, \rho v\}^T, \quad F = \{\rho v, \rho v^2 + P(\rho)\}^T, \]

we can write the equations (4),(7),(8),(13),(15) with zero right-hand side in vector form:

\[
\frac{\partial W}{\partial t} + A_0 \frac{\partial W}{\partial x} = 0,
\]

(16)

where \( A_0 \) is a Jacobian for the non-divergent form of these equations:

\[
A_0 = \begin{pmatrix}
 v & \rho \\
 \frac{c^2(\rho)}{\rho} & v
\end{pmatrix} = \begin{pmatrix}
 v & \rho \\
 0 & v + \rho \frac{\partial V(\rho)}{\partial \rho}
\end{pmatrix} = \begin{pmatrix}
 \frac{\partial Q(\rho)}{\partial \rho} & 0 \\
 0 & \frac{\partial Q(\rho)}{\partial \rho}
\end{pmatrix}.
\]

(17)

Or, with Jacobian \( A = \partial F/\partial U \) for the divergent form (8):

\[
\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = \frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x} = 0.
\]

(18)

The eigenvalues of Jacobian \( A_0 \), or Jacobian \( A = \partial F/\partial U = U_W A_0 U_W^{-1} \), vary depending on the equation form used: \( \lambda_{1,2} = v \pm c(\rho) \) or \( \lambda_1 = v + \rho \frac{\partial V(\rho)}{\partial \rho} \leq \lambda_2 = v \).

If we express disturbance propagation speed \( c(\rho) \) through flow density and speed:

\[ c^2(\rho) = \rho^2 \left(\frac{\partial v}{\partial \rho}\right)^2, \]

both eigenvalues are reduced to one: \( \lambda_{1,2} = \frac{\partial Q(\rho)}{\partial \rho} = v + \rho \frac{\partial v(\rho)}{\partial \rho}. \)

This resolves the issue of anisotropic properties of second-order equations that existed due to one of the eigenvalues being larger than traffic speed.

In (Zhang, 2003) a theorem defining anisotropic or isotropic properties of traffic flow from its fundamental diagram \( Q(\rho) \) was proved. Anisotropy is defined
from the solution of the Riemann problem with arbitrary initial conditions: the Jacobian eigenvalues must not exceed traffic speed at any point. This means that vehicles cannot influence the ones in front of them.

If there is only one eigenvalue defining the disturbance propagation, \( \lambda(\rho) = \frac{\partial Q(\rho)}{\partial \rho} \), the flow will be anisotropic when \( \lambda(\rho) = \frac{\partial Q}{\partial \rho} \leq v(\rho) = \frac{Q}{\rho} \), and isotropic when \( \lambda(\rho) = \frac{\partial Q}{\partial \rho} > v(\rho) \). Thus, if we consider the fundamental diagram for multi-lane road, shown in Fig. 1, the anisotropy will not be preserved in flows where \( \alpha_0 = \frac{Q}{\rho} < \alpha_1 = \frac{\partial Q}{\partial \rho} < 90^\circ \), as shown. This example agrees well with research by Zhang, since (Zhang, 2003) shows, that anisotropy of traffic flow may be violated for multi-lane road.

![Figure 1: An example of fundamental diagram, generated for the Third Ring Road of Moscow, Russia.](image)

Finally, we will show how PDE system in the divergent form, proposed in (Aw and Rascle, 2000):

\[
\begin{align*}
\rho_t + (\rho v)_x &= 0, \\
(\rho (v + p(\rho)))_t + (\rho v (v + p(\rho)))_x &= 0,
\end{align*}
\]

(19)
can be derived from (8) using $P(\rho) = \int_{0}^{\rho} c^2(\tilde{\rho}) d\tilde{\rho} = \int_{0}^{\rho} \left( \tilde{\rho} \frac{\partial p(\tilde{\rho})}{\partial \rho} \right)^2 d\tilde{\rho}$.

Conservation equations are identical, so we will focus on the momentum equations:

\begin{align*}
\frac{\partial (\rho v)}{\partial t} + \frac{\partial (\rho v^2 + P(\rho))}{\partial x} &= 0 \\
\frac{\partial (\rho v)}{\partial t} + \frac{\partial (\rho^2 v)}{\partial x} &= \rho \frac{\partial p(\rho)}{\partial \rho} \frac{\partial \rho}{\partial x} = 0 \\
\frac{\partial (\rho v)}{\partial t} + \frac{\partial (\rho v^2)}{\partial x} &= \rho \frac{\partial p(\rho)}{\partial \rho} \frac{\partial v}{\partial x} + \rho^2 \frac{\partial ^2 \rho}{\partial x^2} = 0.
\end{align*}

From the conservation equation of (8), using $\frac{\partial Q(\rho)}{\partial \rho} = v - \rho \frac{\partial p(\rho)}{\partial \rho}$, we obtain:

\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} \left( \rho \frac{\partial v}{\partial x} \right) &= 0 \\
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} \left( \rho \frac{\partial v}{\partial x} \right) &= 0 \\
\rho \frac{\partial ^2 v}{\partial x^2} &= \rho \left( \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} \right).
\end{align*}

By substituting $\rho^2 \left( \frac{\partial p}{\partial \rho} \frac{\partial v}{\partial x} \right) = \rho \left( \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} \right)$ into the momentum equation of (19):

\begin{align*}
\frac{\partial (\rho v)}{\partial t} + \frac{\partial (\rho v^2)}{\partial x} + \rho \left( \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} \right) &= 0 \\
\frac{\partial (\rho v)}{\partial t} + \frac{\partial (\rho v^2)}{\partial x} + \rho \left( \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} \right) &= 0 \\
\rho \left( v + p(\rho) \right)_{x} + \rho v \left( v + p(\rho) \right)_{x} &= 0.
\end{align*}

So, we arrived at the momentum equation of (19), QED.

Now we formulate our result as the following theorem:

**Theorem 2.1.** Any macroscopic second-order hyperbolic PDE system for description of traffic flow can be formulated in the following diagonal form:

\begin{equation}
\begin{cases}
\frac{\partial \rho}{\partial t} + \left( \frac{\partial Q}{\partial \rho} \right) \frac{\partial \rho}{\partial x} = f_0, \\
\frac{\partial v}{\partial t} + \left( \frac{\partial Q}{\partial v} \right) \frac{\partial v}{\partial x} = \left( \frac{\partial Q}{\partial p} \right) - v \left( \frac{\partial Q}{\partial \rho} \right),
\end{cases}
\end{equation}

with the use of pressure equation (6): $P(\rho) = \int_{0}^{\rho} c^2(\tilde{\rho}) d\tilde{\rho} = \int_{0}^{\rho} \left( \tilde{\rho} \frac{\partial p(\tilde{\rho})}{\partial \rho} \right)^2 d\tilde{\rho}$.

The term $f_1$ in the right-hand side of (21) plays the role of a relaxation term if necessary.
One can define functions $Q(\rho)$ or $V(\rho)$ according to his or her preferences. We propose an empirical method using traffic detector data for calibration of the fundamental diagram for each segment of road network. This algorithm is described in the next section. The use of relaxation term $\frac{1}{\tau} (V(\rho) - v)$ will be discussed in the second part of this paper.

2.2. Empirical pressure equation

In the second part of this paper we will show how measurements from freeway detectors and vehicles’ GPS tracks could be used to automatically obtain fundamental diagrams $Q(\rho)$ and $V(\rho) = Q(\rho)/\rho$ for respective road segments. In this part, we will briefly cover main stages of this algorithm.

According to the three-phase traffic state theory (Kerner, 2009), we distinguish three phases of traffic on freeways, and determine functional dependencies $Q(\rho)$ for each phase, and then stitch them together in the following transition points: $\rho_1$ — transition from free flow to synchronized flow, and $\rho_2$ — transition from synchronized flow to wide moving jam.

1. Free flow: $Q(\rho) = \alpha_2 \rho^2 + \alpha_1 \rho$, $0 \leq \rho < \rho_1$;
2. Synchronized flow: $Q(\rho) = \beta_2 \rho^2 + \beta_1 \rho + \beta_0$, $\rho_1 \leq \rho < \rho_2$;
3. Wide moving jam: $Q(\rho) = c_\ast (\rho_{\text{max}} - \rho)$, $\rho_2 \leq \rho \leq \rho_{\text{max}}$.

The coefficients for $Q(\rho)$ are determined from key points of the fundamental diagram (this subject will be covered in detail in Part 2). Then, knowing $Q(\rho)$, we can find the equilibrium speed $V(\rho) = \frac{Q(\rho)}{\rho}$, eigenvalues $\lambda_{1,2}(\rho) = \frac{\partial Q(\rho)}{\partial \rho}$ and disturbance propagation speed $c(\rho) = \rho \frac{\partial V(\rho)}{\partial \rho}$ as functions of traffic density.

$$
\begin{cases}
    V(\rho) = \alpha_2 \rho + \alpha_1 \\
    \frac{\partial Q(\rho)}{\partial \rho} = 2\alpha_2 \rho + \alpha_1, \ 0 \leq \rho < \rho_1; \\
    c(\rho) = \alpha_2 \rho
\end{cases}
$$
2. Synchronized flow:

\[
\begin{align*}
V(\rho) &= \beta_2 \rho + \beta_1 + \frac{\beta_0}{\rho} \\
\frac{\partial Q(\rho)}{\partial \rho} &= 2 \beta_2 \rho + \beta_1, \quad \rho_1 \leq \rho < \rho_2; \\
c(\rho) &= \beta_2 \rho - \frac{\beta_0}{\rho} \\
V(\rho) &= c_* \left( \frac{\rho_{\text{max}}}{\rho} - 1 \right)
\end{align*}
\]

3. Wide moving jam:

\[
\begin{align*}
\frac{\partial Q(\rho)}{\partial \rho} &= -c_*, \quad \rho_2 \leq \rho \leq \rho_{\text{max}}; \\
c(\rho) &= -\frac{c_* \rho_{\text{max}}}{\rho}
\end{align*}
\]

The pressure equations are obtained using the familiar expression: 

\[
P(\rho) = \alpha_2 \rho^3, \quad 0 \leq \rho < \rho_1;
\]

2. Synchronized flow:

\[
P(\rho) = \frac{\beta_2^2}{3} (\rho^3 - \rho_1^3) + 2 \beta_0 \beta_2 (\rho_1 - \rho) + \beta_0^2 \left( \frac{1}{\rho_1} - \frac{1}{\rho} \right) + \frac{\alpha_2^2 \rho_1^3}{3}, \quad \rho_1 \leq \rho < \rho_2;
\]

3. Wide moving jam:

\[
P(\rho) = c_*^2 \rho_{\text{max}}^2 \left( \frac{1}{\rho_2} - \frac{1}{\rho} \right) + \beta_0^2 \left( \frac{1}{\rho_2^2} - \frac{1}{\rho^2} \right) + \frac{\beta_2^2}{3} (\rho_2^3 - \rho_1^3) + \\
+2 \beta_0 \beta_2 (\rho_1 - \rho_2) + \frac{\alpha_2^2 \rho_2^3}{3}, \quad \rho_2 \leq \rho \leq \rho_{\text{max}}.
\]

To verify proposed methodology for unification of second-order models through using the generalized form of empirical pressure equation, we used archival data from the PeMS database (California Department of Tranportation, 2012). Fundamental diagrams for two segments of I-580 freeway (California, USA), based on PeMS data from January 2012 to December 2012, are shown in Fig. 2.

For data shown in Fig. 2, we obtained the following parameters for fundamental diagrams:
Figure 2: Fundamental diagrams for two segments of I-580 freeway (California, USA), based on PeMS data. Left: detector #1 (PeMS Vehicle Detector Station 402423), right: detector #2 (PeMS Vehicle Detector Station 402425).

- Detector #1: $Q_1 = \max [Q(\rho)] = 2.10 \text{ veh./s}, \rho_1 = 0.084 \text{ veh./m}, \alpha_1 = 49.6, \alpha_2 = -293.2; Q_2 = \max \left( \frac{Q}{Q_{\text{max}}} \right) + \left( \frac{\rho}{\rho_{\text{max}}} \right) = 1.84 \text{ veh./s}, \rho_2 = 0.141 \text{ veh./m}, \beta_0 = 2.5, \beta_1 = -4.9, \beta_2 = 1.6; c_* = 4.20 \text{ m/s}, \rho_{\text{max}} = 0.58 \text{ veh./m};$

- Detector #2: $Q_1 = \max [Q(\rho)] = 2.11 \text{ veh./s}, \rho_1 = 0.076 \text{ veh./m}, \alpha_1 = 50.2, \alpha_2 = -295.7; Q_2 = \max \left( \frac{Q}{Q_{\text{max}}} \right) + \left( \frac{\rho}{\rho_{\text{max}}} \right) = 1.49 \text{ veh./s}, \rho_2 = 0.165 \text{ veh./m}, \beta_0 = 2.30, \beta_1 = -0.4, \beta_2 = -27.6; c_* = 3.58 \text{ m/s}, \rho_{\text{max}} = 0.58 \text{ veh./m};$

Fig. 3 shows the values of the equilibrium speed $V(\rho) = \frac{Q(\rho)}{\rho}$, eigenvalues $\lambda_{1,2}(\rho) = \frac{\partial Q(\rho)}{\partial \rho}$ and disturbance propagation speed $c(\rho) = \rho \frac{\partial V(\rho)}{\partial \rho}$ as a function of density. Pressure as a function of density, obtained empirically for two segments of I-580, is shown in Fig. 4.

In Fig. 3 we can see that for all values of density ($0 \leq \rho \leq \rho_{\text{max}}$) holds $\lambda_{1,2} = \frac{\partial Q(\rho)}{\partial \rho} < v(\rho)$. According to the theorem from (Zhang, 2003), it means that the flow on both segments of I-580 is always anisotropic.

2.3. Computational method

The PDE system (4) is of hyperbolic type, and a variety of finite-difference methods for solving such systems exist. By introducing vectors $W = \{\rho, v\}^T$,
Figure 3: Equilibrium speed (solid green line), eigenvalues (dotted red line), and disturbance propagation speed (dashed blue line) as a functions of density for two segments of I-580. Left: detector #1, right: detector #2.

Figure 4: Pressure as a function of density for two segments of I-580. Left: detector #1, right: detector #2.

\[ f = \left\{ f_0, \left( \frac{\partial Q}{\partial \rho} - v \right) \frac{f_0}{\rho} \right\}^T, \]  
the system (4) can be written in the vector form:

\[ \frac{\partial W}{\partial t} + A_0 \frac{\partial W}{\partial x} = f(W, x, t) \]  \hspace{1cm} (22)

with the diagonal Jacobian:

\[ A_0 = \begin{pmatrix} \frac{\partial Q(\rho)}{\partial \rho} & 0 \\ 0 & \frac{\partial Q(\rho)}{\partial \rho} \end{pmatrix} \]  \hspace{1cm} (23)

The eigenvalues of Jacobian \( A_0 \), \( \lambda_{1,2} = \frac{\partial Q(\rho)}{\partial \rho} \), are always real.

When choosing a numerical method, we must take into account that the solution for the model equations at every road segment is defined by the change of
computed values in its boundary points. In this case, we can choose conservative first-order grid-characteristic method (Magomedov and Kholodov, 1969). For system (4), since equations for density and speed are linearly independent, the algorithm for numerical solution (Magomedov and Kholodov, 1969) can be split into two independent stages:

\[ \rho_{m+1}^n = \rho_m^n - \frac{\Delta t}{2\Delta x} \left( \left| \frac{\partial Q_{m+1/2}^n}{\partial \rho} \right| + \left| \frac{\partial Q_{m-1/2}^n}{\partial \rho} \right| \right) (Q_m^n - Q_{m-1}^n) + \]
\[ + \left( \left| \frac{\partial Q_{m+1/2}^n}{\partial v} \right| - \left| \frac{\partial Q_{m-1/2}^n}{\partial v} \right| \right) (Q_m^n - Q_{m-1}^n) \]

\[ v_{m+1}^n = v_m^n - \frac{\Delta t}{2\Delta x} \left( \left| \frac{\partial Q_{m+1/2}^n}{\partial \rho} \right| + \left| \frac{\partial Q_{m-1/2}^n}{\partial \rho} \right| \right) (v_m^n - v_{m-1}^n) + \]
\[ + \Delta t \left( \frac{\partial Q_{m+1/2}^n}{\partial \rho} - \frac{\partial Q_{m-1/2}^n}{\partial \rho} \right) \frac{f_0(\rho_m^n)}{\rho_m^{n+1/2}} + f_0(\rho_m^n) \]  

In (24), \( m = 1, \ldots, M \) is the grid node index along \( x-axis, n = 1, \ldots, N \) is the grid node index along \( t-axis, \Delta t \) and \( \Delta x \) are numerical integration steps. In the first stage, we use \( \rho_m^n = \frac{\rho_{m+1}^n + \rho_{m-1}^n}{2} \). In the second stage, we use \( \rho_{m+1/2}^n = \frac{\rho_{m+1}^n + \rho_{m}^n}{2} \) and \( \rho_{m-1/2}^n = \frac{\rho_{m}^n + \rho_{m-1}^n}{2} \), because we have already found \( \rho_{m+1}^n \) during the first stage.

The system (4) should also be integrated at boundary points. For example, at the end of the freeway segment, \( (t^{n+1}, x_M) \) and when \( \frac{\partial Q(\rho_{M-1/2})}{\partial \rho} > 0 \), we get:

\[ \rho_{M+1}^n = \rho_M^n - \frac{\Delta t}{\Delta x} (Q_M^n - Q_{M-1}^n) \]
\[ v_{M+1}^n = v_M^n - \frac{\Delta t}{\Delta x} \left( \frac{\partial Q(\rho_{M+1/2})}{\partial \rho} + \frac{\partial Q(\rho_{M-1/2})}{\partial \rho} \right) (v_M^n - v_{M-1}^n) \]  

If \( \frac{\partial Q(\rho_{M-1/2})}{\partial \rho} < 0 \), the values \( (\rho, v)^{n+1}_M \) should be computed from the known time series \( \{\rho, v\}^{n+1}_M = \{\rho(t^{n+1}, x_M), v(t^{n+1}, x_M)\} \). In the case when these time series are not known, transparent boundary conditions could be used: \( (\partial \rho/\partial x) = 0, (\partial v/\partial x) = 0 \), and \( \{\rho, v\}^{n+1}_M = \{\rho, v\}^{n+1}_{M-1} \). We use the same algorithm at the beginning of the freeway segment, but for different sign of
\[
\frac{\partial Q(\rho_{3/2}^n)}{\partial \rho} < 0:
\]

I. \[\rho_1^{n+1} = \rho_1^n - \frac{\Delta t}{\Delta x} (Q_2^n - Q_1^n)\]

II. \[v_1^{n+1} = v_1^n - \frac{\Delta t}{\Delta x} \left( \frac{\partial Q(\rho_{3/2}^{n+1/2})}{\partial \rho} + \left| \frac{\partial Q(\rho_{3/2}^{n+1/2})}{\partial \rho} \right| \right) (v_2^n - v_1^n)\]

If \(\frac{\partial Q(\rho_{3/2}^n)}{\partial \rho} > 0\), then the boundary conditions are: \(\{\rho, v\}_1^{n+1} = \{\rho(t^{n+1}, x_1), v(t^{n+1}, x_1)\}\).

If we express our system in vector form with the non-diagonal Jacobian (16) or (18), the numerical algorithm gets more complicated. For system (16), (18), the whole family of differential schemes could be expressed in the following form:

\[
\begin{cases}
W_{m+1}^{n+1} = W_m^n - \frac{\Delta t}{\Delta x} \left( G_{m+1/2}^{n+1} - G_{m-1/2}^{n+1} \right), \\
U_{m+1}^{n+1} = U_m^n - \frac{\Delta t}{\Delta x} \left( F_{m+1/2}^{n+1} - F_{m-1/2}^{n+1} \right).
\end{cases}
\]

The choice of interpolation for \(G_{m+1/2}^{n+1}\) and \(F_{m+1/2}^{n+1}\) in (27) is crucial for obtaining the scheme with the desired properties. When choosing grid-characteristic method of the first order of approximation (Magomedov and Kholodov, 1969), expressions \(G_{m+1/2}^{n+1}\) and \(F_{m+1/2}^{n+1}\) are as follows:

\[
\begin{cases}
G_{m+1/2}^{n+1} = \frac{1}{2} \left( A_0 \right)_{m+1/2} (W_m^n + W_{m+1}^n) + \frac{1}{2} \left( \Omega_0^{-1} |\Lambda| \Omega_0 \right)_{m+1/2} (W_m^n - W_{m+1}^n), \\
F_{m+1/2}^{n+1} = \frac{1}{2} \left( F_m^n + F_{m+1}^n \right) + \frac{1}{2} \left( \Omega^{-1} |\Lambda| \Omega \right)_{m+1/2} (U_m^n - U_{m+1}^n).
\end{cases}
\]

or these could be more complex expressions, which allow building of higher-order integration scheme on the given stencil. In (28), \(\Lambda\) is a diagonal matrix of eigenvalues of \(A_0\) or \(A = \partial F/\partial U = U \cdot A_0 U^{-1}\) for systems (16), (18), and, respectively, \(\Omega_0\) and \(\Omega = \Omega_0 U^{-1}\) are matrices of their left eigenvectors. The variable values in intermediate nodes \(m \pm 1/2\) in (28) can be computed using simple linear interpolation without loss of accuracy.

The characteristic form of compatibility equations, equivalent to systems (16),
(18), along characteristics \( dx = \lambda_i dt \), \( i = 1,2 \), can be expressed as:

\[
\begin{align*}
\omega_{0,i} \cdot \left( \frac{\partial W}{\partial t} + \lambda_i \frac{\partial W}{\partial x} \right) &= 0, \\
\omega_i \cdot \left( \frac{\partial U}{\partial t} + \lambda_i \frac{\partial U}{\partial x} \right) &= 0,
\end{align*}
\]

(29)

Each of the equations (29) is, in essence, an ordinary differential equation along the characteristic \( dx = \lambda_i dt \). Sometimes we need to numerically integrate the compatibility conditions (29) at boundary points along characteristics entering the integration area. For example, at the beginning of the freeway segment at boundary point \( (t^{n+1}, x_1) \) with negative eigenvalue \( (\lambda_i)_{3/2}^n < 0 \) we get:

\[
\begin{align*}
(\omega_{0,i})_{3/2}^n \cdot \left( \frac{W_{M}^{n+1} - W_{M}^n}{\Delta t} + (\lambda_i)_{3/2}^n \frac{W_{M}^n - W_{M-1}^n}{\Delta x} \right) &= 0, \\
(\omega_i)_{3/2}^n \cdot \left( \frac{U_{M}^{n+1} - U_{M}^n}{\Delta t} + (\lambda_i)_{3/2}^n \frac{U_{M}^n - U_{M-1}^n}{\Delta x} \right) &= 0.
\end{align*}
\]

(30)

If \( (\lambda_i)_{3/2}^n > 0 \), we use boundary conditions \( \{ \rho, v \}_{1}^{n+1} = \{ \rho \left( t^{n+1}, x_1 \right), v \left( t^{n+1}, x_1 \right) \} \)

At the end of the freeway segment, we do the same for the opposite sign of \( (\lambda_i)_{M-1/2}^n > 0 \):

\[
\begin{align*}
(\omega_{0,i})_{M-1/2}^n \cdot \left( \frac{W_{M}^{n+1} - W_{M}^n}{\Delta t} + (\lambda_i)_{M-1/2}^n \frac{W_{M}^n - W_{M-1}^n}{\Delta x} \right) &= 0, \\
(\omega_i)_{M-1/2}^n \cdot \left( \frac{U_{M}^{n+1} - U_{M}^n}{\Delta t} + (\lambda_i)_{M-1/2}^n \frac{U_{M}^n - U_{M-1}^n}{\Delta x} \right) &= 0.
\end{align*}
\]

(31)

With negative \( (\lambda_i)_{M-1/2}^n < 0 \), values \( \{ \rho, v \}_{M}^{n+1} \) should be determined from known time series \( \{ \rho, v \}_{M}^{n+1} = \{ \rho \left( t^{n+1}, x_M \right), v \left( t^{n+1}, x_M \right) \} \), and, if these time series are unknown, we use transparent boundary conditions: \( (\partial \rho / \partial x) = 0, \ (\partial v / \partial x) = 0 \), and \( \{ \rho, v \}_{M}^{n+1} = \{ \rho, v \}_{M-1}^{n+1} \).

3. Numerical results

To verify the proposed empirical pressure equation, we carried out numerical experiments using traffic detector data for segment of I-580 freeway in California, USA, obtained from PeMS (California Department of Transportation, 2012). We used two loop detectors, denoted #1 and #2, separated by ~ 1 km segment.
of 4-lane freeway with no entrances or exits, as shown in Fig. 5. Data (traffic flow and speed, with 30-second temporal resolution) from detector #1 were used as the left boundary condition, and data from downstream detector #2 were used for verification of modeling results. We used transparent right boundary condition \((\partial Q/\partial x) = 0, (\partial v/\partial x) = 0\). We simulated a 24-hour interval for a single weekday for all four lanes of I-580. The results are shown in Fig. 6 for flows and in Fig. 7 for speeds. The results obtained with different models are shown in different subplots: top left — proposed anisotropic second-order model (4); top right — generalized Payne–Whitham in divergent form with zero right-hand side (8); bottom left — model system (13) proposed in (Zhang, 2002); bottom right — divergent system (19) proposed in (Aw and Rascle, 2000). All hydrodynamical models studied (4),(8),(13),(19) were using the same empirical pressure equation (as presented in section 2.2).

Figure 5: Traffic detectors #1 and #2 chosen on the segment of the I-580 freeway in California, USA. Data from detector #1 were used as a left boundary condition for modeling, and data from downstream detector #2 were used for verification of modeling results.

The modeling shows complete matching of results, obtained using different second-order macroscopic models, when using the same pressure equation. This confirms that second-order hydrodynamical models can be generalized through the use of the pressure term \(P(\rho)\).

4. Conclusion

In this paper we explored the unification of the second-order hydrodynamic traffic models in various formulations (Payne, 1971; Daganzo, 1995; Papageorgiou, 1998; Aw and Rascle, 2000; Zhang, 2002, 2003; Siebel and Mauser, 2006a,b).
using algorithm for building pressure equation $P(\rho)$ adequate to empirical data from traffic detectors.

Existing macroscopic models describe traffic with a non-linear system of the second-order hyperbolic equations (for density and speed), which differ in the way they account for dependency between traffic flow (or speed) and density. We have shown that all these second-order hydrodynamical models can be generalized through the use of pressure equation (6).

We have also shown that the form of the pressure equation for each segment of road network can be obtained empirically using the data from traffic detectors. First we derive the equation for $Q(\rho)$ or $V(\rho)$ using traffic measurements. Then, knowing relation between $P(\rho)$ and $Q(\rho)$, we obtain the pressure equation.

To verify proposed methodology, we conducted numerical experiments by modeling a segment of I-580 freeway in California, USA, using data from PeMS (California Department of Transportation, 2012). The simulations show complete agreement of results obtained using different macroscopic models, when using the same empirical pressure equation. This suggests that the properties of every phenomenological model are defined by the form of its pressure equation, that in its turn is determined from the observed fundamental diagram $Q(\rho)$. 

Figure 6: The comparison of calculated flow (solid green line), aggregated over four lanes of I-580, and observed values from detector \#2 (dashed blue line). The results obtained with different models are shown in different subplots: top left — proposed anisotropic second-order model (4); top right — generalized Payne–Whitham in divergent form with zero right-hand side (8); bottom left — model system (13) proposed in (Zhang, 2002); bottom right — divergent system (19) proposed in (Aw and Rascle, 2000).
Figure 7: The comparison of calculated traffic speed (solid green line), aggregated over four lanes of I-580, and observed values from detector #2 (dashed blue line). The results obtained with different models are shown in different subplots: top left — proposed anisotropic second-order model (4); top right — generalized Payne–Whitham in divergent form with zero right-hand side (8); bottom left — model system (13) proposed in (Zhang, 2002); bottom right — divergent system (19) proposed in (Aw and Rascle, 2000).

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