Variants of Ahlfors’ lemma and properties of the logarithmic potentials

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Abstract

As a special class of conformal metrics with negative curvatures, SK-metrics play a crucial role in metric spaces. This paper concerns the variants of Ahlfors’ lemma in an SK-metric space and gives a higher order derivative formula for the logarithmic potential function, which can be applied for the estimates near the singularity of a conformal metric with negative curvatures.

1 Introduction

Let \( \mathbb{C} \) be the complex plane, \( \mathbb{D} \) be the open unit disk in \( \mathbb{C} \) and \( \mathbb{D}^* := \mathbb{D} \setminus \{0\} \) be the punctured unit disk. Let \( \mathcal{S} \) denote a Riemann surface. Let \( \mathcal{A} \) be the basic family consisting of homeomorphisms \( \sigma \) defined on the plane domains of \( \mathbb{C} \) into \( \mathcal{S} \) which defines the conformal structure of \( \mathcal{S} \). In our discussion we take the linear notation for a conformal metric \( ds = \lambda(z)|dz| \) and set the density function \( \lambda(z) \) to be positive on \( \mathcal{S} \). However, in Heins’ paper [4], he let the function \( \lambda(z) \) be non-negative for a conformal metric \( \lambda(z)|dz| \) on \( \mathcal{S} \), and then defined the SK-metrics.

Let \( \mathbb{P} \) denote the Riemann sphere \( \mathbb{C} \cup \{\infty\} \) with its canonical complex structure and a subdomain \( \tilde{\Omega} \subset \mathbb{P} \). For a point \( p \in \tilde{\Omega} \), let \( z \) be the local coordinates such that \( z(p) = 0 \). We say that a conformal metric \( \lambda(z)|dz| \) on the punctured domain \( \Omega := \tilde{\Omega} \setminus \{p\} \) has a singularity of order \( \alpha \leq 1 \) at the point \( p \), if, in local coordinates \( z \),

\[
\log \lambda(z) = \begin{cases} 
-\alpha \log |z| + O(1) & \text{if } \alpha < 1 \\
-\log |z| - \log \log(1/|z|) + O(1) & \text{if } \alpha = 1
\end{cases}
\]  

as \( z \to 0 \), with \( O \) being the Landau symbol throughout our study. Let \( M_u(r) := \sup_{|z|=r} u(z) \) for a real-valued function \( u(z) \) defined in a punctured neighborhood of \( z = 0 \) and call

\[
\alpha(u) := \lim_{r \to 0^+} \frac{M_u(r)}{\log(1/r)}
\]

the order of \( u(z) \) if this limit exists. For \( u(z) := \log \lambda(z) \), \( \alpha(u) \) in (1.1) and (1.2) are equivalent. We call the point \( p \) a conical singularity or corner of order \( \alpha \) if \( \alpha < 1 \) and a cusp if \( \alpha = 1 \). The generalized Gaussian curvature \( \kappa_\lambda(z) \) of the density function \( \lambda(z) \) is defined by

\[
\kappa_\lambda(z) = -\frac{1}{\lambda(z)^2} \liminf_{r \to 0} \frac{4}{\pi^2} \int_0^{2\pi} \log \lambda(z + re^{it})dt - \log \lambda(z).
\]  

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We say a conformal metric \( \lambda(z)|dz| \) on a domain \( \Omega \subseteq \mathbb{C} \) is regular, if its density \( \lambda(z) \) is of class \( C^2 \) in \( \Omega \). We will show that, if \( \lambda(z)|dz| \) is a regular conformal metric, then (1.3) is equivalent to

\[
\kappa_\lambda(z) = -\frac{\Delta \log \lambda(z)}{\lambda(z)^2},
\]

where \( \Delta \) denotes the Laplace operator. We will discuss details in Lemma [2,3]. It is well known that, if \( a < \kappa_\lambda(z) < b < 0 \), the metric \( \lambda(z)|dz| \) only has corners or cusps at isolated singularities, see [8]. If \( \kappa_\lambda(z) \geq 0 \) and the energy is finite, then only corners occur, see [5,10], also [7].

We have a fact that the Gaussian curvature is a conformal invariant. Let \( \lambda(z)|dz| \) be a conformal metric on a plane domain \( D \) and \( f : W \to D \) be a holomorphic mapping of a Riemann surface \( W \) into \( D \). Then the metric \( ds = f^*\lambda(w)|dw| \) on \( W \) induced by \( f \) from the original metric \( \lambda(z)|dz| \) is called the pullback of \( \lambda(z)|dz| \) and defined by

\[
ds = f^*\lambda(w)|dw| := \lambda(f(w))|f'(w)||dw|.
\]

It is evident that \( f^*\lambda(w)|dw| \) is a conformal metric on \( W \setminus \{ \text{critical points of } f \} \) with Gaussian curvature

\[
\kappa_{f^*\lambda}(w) = \kappa_\lambda(f(w)).
\]

Using this conformal invariance, we can only consider one Riemann surface with the conformal metric all over its conformal equivalence class.

The paper is divided into two parts. Section 2 is devoted to a class of conformal metrics, SK-metrics. We begin with the definition and then discuss the maximum principle for these metrics. Section 3 is about the potential theory, which is a main tool we use to study the local term of a conformal metric near the singularity.

## 2 SK-metrics

For a topological space \( X \), a function \( u : X \to [-\infty, \infty) \) is called upper semi-continuous if the set \( \{ z \in X : u(z) < \alpha \} \) is open in \( X \) for each real number \( \alpha \). If \( \varphi \) is a uniformizer of \( \mathcal{S} \), i.e. \( \varphi \) is a univalent conformal map of a plane domain into \( \mathcal{S} \), then the conformal density function \( \lambda \) on \( \mathcal{S} \) can be extended to \( \mathcal{S} \cup \{ \varphi \} \) such that the extension is a conformal metric relative to \( \mathcal{S} \cup \{ \varphi \} \). We denote the image of \( \varphi \) with respect to this extension by \( \lambda_\varphi \) and call it the \( \varphi \)-scale of \( \lambda \). For two uniformizers \( \varphi \) and \( \psi \) of \( \mathcal{S} \), \( \varphi(a) = p, \psi(b) = p \), \( a, b \in \mathcal{S}, p \in \mathbb{C} \), if \( \lambda_\varphi \) and \( \lambda_\psi \) are upper semi-continuous at \( a \) and \( b \) respectively, we say \( \lambda \) is upper semi-continuous at \( p \). If \( \lambda \) is upper semi-continuous at every point of \( \mathcal{S} \), we say \( \lambda \) is upper semi-continuous on \( \mathcal{S} \).

Now we give the definition for SK-metrics. The concept of SK-metrics was given by Heins in [4], but its initial idea came from Ahlfors in [1] and [2]. Heins defined the SK-metrics by means of the Gaussian curvature and proved a more general maximum principle as a variant of Ahlfors’ lemma. According to Heins, the terminology of "SK-metric" is partly because its (Gaussian) "curvature is subordinate to \(-4\)”, see [4, p.3]. We call an upper semi-continuous metric \( \lambda(z)|dz| \) on \( \mathcal{S} \) an SK-metric if its Gaussian curvature is bounded above by \(-4\) at those points \( z \) in \( \mathcal{S} \) where \( z \) satisfies \( \lambda(z) > 0 \). A complete
metric with the negative constant Gaussian curvature is called the hyperbolic metric. The
hyperbolic metric on the unit disk $\mathbb{D}$ is defined by
\[
d\sigma = \lambda_{\mathbb{D}}(z)|dz| = \frac{|dz|}{1 - |z|^2}
\] (2.1)
with the constant Gaussian curvature $-4$ and it is an extremal SK-metric. For SK-metrics,
there is a generalization of the maximum principle mentioned by Ahlfors [2, Theorem A]
and Heins [4, Theorem 2.1], which claims that the hyperbolic metric is the unique maximal
SK-metric on $\mathbb{D}$.

**Theorem 2.1** [2] (Ahlfors’ lemma) Let $d\sigma$ be the hyperbolic metric on $\mathbb{D}$ given in (2.1)
and $ds$ be the metrics on $\mathbb{D}$ induce by an SK-metric on some Riemann surface
$W$. If the
function $f(z)$ is analytic in $\mathbb{D}$, then the inequality
\[
ds \leq d\sigma
\]
will hold throughout the circle.

The following result stated on the corresponding Riemann surface. It is a variant of
Theorem 2.1.

**Theorem 2.2** [4] Suppose that $W$ is a relatively compact domain of $S$ and that $\lambda(w)$ is
an SK-metric on $W$, $\mu(w)$ is a pullback on $W$ of $\lambda_{\mathbb{D}}(z)$ defined in (2.1). If for all $p \in \partial W$,
\[
\limsup_{w \to p} \frac{\lambda(w)}{\mu(w)} \leq 1,
\]
then throughout the boundary $\partial W$, it holds that
\[
\lambda(w) \leq \mu(w).
\]

Heins used condition (1.3) to define SK-metrics and he mentioned the equivalence
between (1.3) and (1.4) for SK-metrics in one word, see [4, (1.4)]. Here we present it in
detail.

**Lemma 2.3** Suppose $\Omega \in \mathbb{C}$ is a domain. If a function $u$ is of class $C^2(\Omega)$, then for $z \in \Omega$, we have
\[
\lim_{r \to 0} \frac{4}{r^2} \left( \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{it})dt - u(z) \right) = \Delta u(z).
\]

**Proof.** Since $u$ is of $C^2$, using Taylor’s expansion of $u(z)$ at $z_0 \in \Omega$,
\[
u(z_0 + z) = u(z_0) + u_x(z_0) \cdot x + u_y(z_0) \cdot y + \frac{1}{2} u_{xx}(z_0) \cdot x^2 + \frac{1}{2} u_{yy}(z_0) \cdot y^2 + u_{xy}(z_0) \cdot xy + \varepsilon(z),
\]
where $\varepsilon(z) \to 0$ as $z \to 0$ and $z = x + yi$, we have
\[
\frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{it})dt - u(z_0) = \frac{r^2}{4} (u_{xx}(z_0) + u_{yy}(z_0)) + \frac{1}{2\pi} \int_0^{2\pi} \varepsilon(z)dt.
\]
As $r = |z| \to 0$,
\[
\frac{1}{r^2} \int_0^{2\pi} \varepsilon(z)dt = \int_0^{2\pi} \frac{\varepsilon(z)}{r^2}dt = 0,
\]
then
\[ \lim_{r \to 0} \frac{4}{r^2} \left( \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{it}) dt - u(z) \right) = u_{xx}(z_0) + u_{yy}(z_0) = \Delta u(z_0) \]
as required. \( \square \)

The following theorem offers us a simple way to construct a new SK-metric on a plane domain by means of the maximum principle. This is called the "gluing lemma".

**Theorem 2.4** [7] Let \( \lambda(z)|dz| \) be an SK-metric on a domain \( G \subset \mathbb{C} \) and let \( \mu(z)|dz| \) be an SK-metric on a subdomain \( U \subset G \) such that the "gluing condition"

\[ \limsup_{U \ni z \to \xi} \mu(z) \leq \lambda(\xi) \]
holds for all \( \xi \in \partial U \cap G \). Then \( \sigma(z)|dz| \) defined by

\[ \sigma(z) := \begin{cases} \max\{\lambda(z), \mu(z)\} & \text{for } z \in U, \\ \lambda(z) & \text{for } z \in G \setminus U \end{cases} \]
is an SK-metric on \( G \).

We end this section with the discussion on SK-metrics in the punctured domain. On the punctured unit disk \( \mathbb{D}^* \), the hyperbolic metric is expressed by

\[ \lambda_{\mathbb{D}^*}(z)|dz| = \frac{|dz|}{2|z|\log(1/|z|)} \tag{2.2} \]
with the constant curvature \(-4\). From the definition (1.1) of the singularity and its order, we know that the metric (2.2) has order 1 at the origin. In any punctured disk, Kraus, Roth and Sugawa gave the expression of the hyperbolic metric which has a singularity at the origin of order \( \alpha < 1 \) in [7] without any detailed discussion. Now we give a complete presentation of the proof.

**Theorem 2.5** [7] For \( R > 0 \), let \( D_R^* := \{ z : 0 < |z| < R \} \) and

\[ \lambda_{\alpha, R}(z) := \begin{cases} \frac{1 - \alpha}{2|z|\sinh \left( (1 - \alpha)\log \frac{R}{|z|} \right)} & \text{if } \alpha < 1, \\ \frac{1}{2|z|\log \frac{R}{|z|}} & \text{if } \alpha = 1 \end{cases} \]
for \( z \in D_R^* \), then for an arbitrary SK-metric \( \sigma(z) \) on \( D_R^* \) which has a singularity at \( z = 0 \) of order \( \alpha \), we have \( \sigma(z) \leq \lambda_{\alpha, R}(z) \).

**Proof.** We consider the case \( \alpha < 1 \). First, choose an arbitrary \( 0 < R_0 < R \), consider \( \lambda_{\beta, R_0}(z) \) on \( 0 < |z| < R_0 \) for \( \alpha < \beta < 1 \), and let \( u(z) := \log \sigma(z) \), \( v(z) := \log \lambda_{\beta, R_0}(z) \), \( E := \{ z : 0 < |z| < R_0, u(z) > v(z) \} \).

Now we have the assertion that \( 0 \notin \overline{E} \). Since \( \sigma(z)|dz| \) and \( \lambda_{\beta, R_0}(z)|dz| \) both have a singularity at \( z = 0 \) with order \( \alpha \), \( \beta \) respectively, then

\[ v(z) = -\beta \log |z| + O(1), \quad u(z) = -\alpha \log |z| + O(1), \]
so \( u - v = (\beta - \alpha) \log |z| + O(1) \). Since \( u - v \to -\infty \) as \( z \to 0 \), then on a sufficiently small neighborhood of \( z = 0 \), \( u - v < 0 \) holds, thus \( 0 \notin \overline{E} \).
Similarly, we have $\partial E \cap \{z : 0 < |z| < R_0\} = \emptyset$, because $\nu \to +\infty$ as $|z| \to R_0$, and $u$ is bounded in $\{z : 0 < |z| < R_0\}$.

Then consider the curve $|z| = R_0$. It is clear that $v(z)$ and $u(z)$ satisfy

$$\lim_{r \to 0} \frac{4}{r^2} \left( \frac{1}{2\pi} \int_0^{2\pi} v(z + re^{it}) dt - v(z) \right) = e^{2v},$$

and

$$\liminf_{r \to 0} \frac{4}{r^2} \left( \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{it}) dt - u(z) \right) \geq e^{2u}$$

by Lemma 2.3. Thus

$$\liminf_{r \to 0} \frac{4}{r^2} \left( \frac{1}{2\pi} \int_0^{2\pi} \left( u(z + re^{it}) - v(z + re^{it}) \right) dt - (u(z) - v(z)) \right) \geq e^{2u} - e^{2v},$$

which is positive on $E$. By definition of limit inferior, we have for $z \in E$

$$\frac{1}{2\pi} \int_0^{2\pi} \left( u(z + re^{it}) - v(z + re^{it}) \right) dt - (u(z) - v(z)) > 0,$$

therefore,

$$u(z) - v(z) \leq \frac{1}{2\pi} \int_0^{2\pi} \left( u(z + re^{it}) - v(z + re^{it}) \right) dt.$$

Now we recall the definition of subharmonic functions. Let $\Omega$ be an open subset of $\mathbb{C}$. A function $u : \Omega \to (-\infty, \infty)$ is called subharmonic if $u$ is upper semi-continuous and satisfies the local sub-mean inequality, i.e. given $z \in \Omega$, there exists $\rho > 0$ such that

$$u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{it}) dt$$

(2.3)

for $0 \leq r < \rho$. If we adopt the definition as above, then $u - v$ is subharmonic on $E$, hence $u - v$ has no maximum in $E$ and $u - v$ approaches its least upper bound on a sequence tending to $\partial E$. A contradiction. So $E = \emptyset$.

Finally, letting $R_0 \to R$ and $\beta \to \alpha$ gives the maximality of $\lambda_{\alpha,R}(z)$ for $\alpha < 1$. According to Kraus, Roth and Sugawa, for the case $\alpha = 1$ this expression has to be interpreted in the limit sense $\alpha \nearrow 1$ to obtain $\lambda_{1,R}(z)$, i.e.

$$\lambda_{1,R}(z) = \lim_{\alpha \nearrow 1} \lambda_{\alpha,R}(z) = \frac{1}{2|z| \log \frac{R}{|z|}}$$

\[\Box\]

**Remark.** The righthand side of (2.3) is called the circumferential mean of function $u$. Heins used it to describe the curvature in the definition of SK-metrics in [4] with $\rho = 1$ and $z = 0$. 

5
3 Potential theory

Generally speaking, the SK-metric is defined by the fact that its Gaussian curvature no greater than some negative constant. So the maximum principle for the SK-metric is common and useful. After a combination with PDEs, the asymptotic behavior of a metric has something in-between with the local properties of the solution to the corresponding PDE. We can consider the curvature equation

$$\Delta u = -\kappa(z) e^{2u}, \quad (3.1)$$

where $\kappa(z)$ is known, then the definition of SK-metrics is fruitful in the case that the curvature function $\kappa(z)$ is strictly negative and Hölder continuous in $\mathbb{D}^*$, see [6], also [9] for details. For an SK-metric $\lambda(z)$ on $\mathbb{D}^*$, regarding $u := \log \lambda$ as a solution to the equation $(3.1)$, the global properties of $u$ have been well known by means of the study on equation $(3.1)$. However, near the singularity $z = 0$, the local properties are still not explicit. We can employ a way related to partial differential equations to investigate the asymptotic behavior of $u$ near the origin. Potential theory is a powerful tool in our case. In this section, we present a formula of the higher order derivatives for the logarithmic potential, and give an asymptotic estimate for $u$ near the origin without any proof as an application of potential theory. We only refer to the logarithmic potential.

We identify $\mathbb{C}$ with $\mathbb{R}^2$, and write $z = x_1 + ix_2$, $\zeta = y_1 + iy_2$. Set $0 < r \leq 1$ and denote $D_R := \{ z \in \mathbb{C} : |z| < R \}$, $D_R^* := D_R \setminus \{0\}$ for a positive number $R$. For a bounded, integrable function $f(z)$ defined on a domain $\Omega \subseteq \mathbb{C}$, the integral

$$\frac{1}{2\pi} \int_\Omega \log |z - \zeta| f(\zeta) d\sigma_\zeta$$

is called the logarithmic potential of $f$, where $d\sigma_\zeta$ is the area element. The Hölder spaces $C^{n,\nu} (D_R)$ are defined as the subspaces of $C^n (D_R)$ consisting of functions whose $n$-th order partial derivatives are locally Hölder continuous with exponent $\nu$ in $D_R$, $0 < \nu \leq 1$. Then the following proposition for the first and the second order derivatives of the logarithmic potential is valid.

**Proposition 3.1** [3, 6] Let $f : D_r \rightarrow \mathbb{R}$ be a locally bounded, integrable function in $D_r$ and $\omega$ be the logarithmic potential of $f$. Then $\omega \in C^1 (D_r)$ and for any $z = x_1 + ix_2 \in D_r$,

$$\frac{\partial}{\partial x_j} \omega(z) = \frac{1}{2\pi} \int_{D_r} \frac{\partial}{\partial x_j} \log |z - \zeta| f(\zeta) d\sigma_\zeta$$

for $j \in \{1, 2\}$.

If, in addition, $f$ is locally Hölder continuous with exponent $\nu \leq 1$, then $\omega \in C^2 (D_r)$, $\Delta \omega = f$ in $D_r$ and for $z \in D_r$,

$$\frac{\partial^2}{\partial x_i \partial x_j} \omega(z) = \frac{1}{2\pi} \int_{D_R} \frac{\partial^2}{\partial x_i \partial x_j} \log |z - \zeta| (f(\zeta) - f(z)) d\sigma_\zeta$$

$$- \frac{1}{2\pi} f(z) \int_{\partial D_R} \frac{\partial}{\partial x_j} \log |z - \zeta| N_1(\zeta) d\zeta,$$

where $N(\zeta) = (N_1(\zeta), N_2(\zeta))$ is the unit outward normal at the point $\zeta \in \partial D_R$, $R > r$ such that the divergence theorem holds on $D_R$ and $f$ is extended to vanish outside of $D_r$. 


There is a similar proposition for higher order derivatives of the logarithmic potential.

Define a multi-index \( \mathbf{j} = (j_1, j_2) \), \( |\mathbf{j}| = j_1 + j_2 \), \( j_1, j_2 = 0, 1, 2, \ldots \). For \( z = x_1 + ix_2 \), denote

\[
\frac{\partial}{\partial x_1} = \partial_1, \quad \frac{\partial}{\partial x_2} = \partial_2, \quad \partial^j = \partial_1^{j_1} \partial_2^{j_2} \text{ and } \frac{\partial^j}{\partial \zeta} = \frac{\partial^{j_1}}{\partial y_1} \frac{\partial^{j_2}}{\partial y_2}.
\]

Let \( e_\tau = (0, 1) \) or \( (1, 0) \) for \( \tau = 1, 2, \ldots \). Then \( \mathbf{j} \) can be expressed in the form \( e_1 + e_2 + \cdots + e_n \). We define two vectors \( \theta_\tau := e_1 + \cdots + e_\tau, \phi_\tau := e_{\tau+1} + \cdots + e_n \) for \( \tau = 1, \ldots, n-1 \) where \( \phi_{n-1} = (0, 0) \), so \( \mathbf{j} \) has a decomposition as \( \mathbf{j} = \theta_\tau + e_{\tau+1} + \phi_\tau \). Write \( \zeta = y_1 + iy_2 \) and set

\[
P_n[f](z, \zeta) := \begin{cases} 
\sum_{|\mathbf{i}|\leq n} \frac{(\zeta - z)^{\mathbf{i}}}{\mathbf{i}!} f(z) & \text{if } n \geq 1 \\
 f(z) & \text{if } n = 0,
\end{cases}
\]

where \( \mathbf{i} \) is a multi-index, \( \mathbf{i} = (i_1, i_2) \), \( (\zeta - z)^{\mathbf{i}} = (y_1 - x_1)^{i_1}(y_2 - x_2)^{i_2} \), \( \mathbf{i}! = i_1! i_2! \). We have the following recurrent formula for \( P_n[f](z, \zeta) \).

**Lemma 3.2** For \( P_n[f](z, \zeta) \) defined as above, then

\[
\frac{\partial^e}{\partial \zeta} P_n[f](z, \zeta) = P_{n-1}[\partial^e f](z, \zeta)
\]

holds for \( e = (0, 1) \) or \( (1, 0) \).

**Proof.** We take the case \( e = (1, 0) \) as an example, when \( e = (0, 1) \) it is similar. Let \( n \geq 1 \). Then

\[
\begin{align*}
\frac{\partial}{\partial y_1} P_n[f](z, \zeta) &= \frac{\partial}{\partial y_1} \sum_{i_1+i_2 \leq n, 0 \leq i_1 \leq n} \frac{(y_1 - x_1)^{i_1}(y_2 - x_2)^{i_2}}{i_1! i_2!} \partial_1^{i_1} \partial_2^{i_2} f(z) \\
&= \sum_{i_1+i_2 \leq n} \frac{(y_1 - x_1)^{i_1-1}(y_2 - x_2)^{i_2}}{(i_1-1)! i_2!} \partial_1^{i_1-1} \partial_2^{i_2} f(z) \\
&= \sum_{i_1+i_2 \leq n-1} \frac{(y_1 - x_1)^{i_1-1}(y_2 - x_2)^{i_2}}{(i_1-1)! i_2!} \partial_1^{i_1-1} \partial_2^{i_2} \partial_1 f(z) \\
&= \sum_{i_1+i_2 \leq n-1} \frac{(y_1 - x_1)^{i_1}(y_2 - x_2)^{i_2}}{i_1! i_2!} \partial_1^{i_1} \partial_2^{i_2} \partial_1 f(z) \\
&= \sum_{|\mathbf{i}| \leq n-1} \frac{(\zeta - z)^{\mathbf{i}}}{\mathbf{i}!} \partial_1^e f(z) \\
&= P_{n-1}[\partial_1 f](z, \zeta).
\end{align*}
\]

Using the multi-index notation, the Taylor expansion of \( f \) can be written in short.
Theorem 3.3  [cf. 1] If \( f(\zeta) \) is analytic in a domain \( \Omega \in \mathbb{C} \), containing the point \( z \), it is possible to write
\[
f(\zeta) = \sum_{t=0}^{n} \frac{f^{(t)}(z)}{t!} (\zeta - z) + R_{n+1}(z, \zeta),
\]
where \( R_{n+1}(z, \zeta) \) is the error term and \( R_{n+1}(z, \zeta) = f_{n+1}(z)(\zeta - z)^{n+1} \) with \( f_{n+1}(z) \) analytic in \( \Omega \). This expression is equivalent to
\[
f(\zeta) = P_n[z](z, \zeta) + R_{n+1}(z, \zeta), \tag{3.2}
\]
with \( R_{n+1}(z, \zeta) \) as above.

**Remark.** If \( f \in C^{n, \nu}(\Omega) \) with \( 0 < \nu \leq 1 \), \( n \geq 1 \), and the Hölder continuity is a local property, then the error term \( R_{n+1}(z, \zeta) \) satisfies
\[
R_{n+1}(z, \zeta) = O(|z - \zeta|^{\nu + n}). \tag{3.3}
\]

On the basis of Lemma 3.2, we can present the analogue of Proposition 3.1 as follows.

**Proposition 3.4** Let \( r < 1 \), \( f : D_r \to \mathbb{R} \), \( f \in C^{n-2, \nu}(D_r) \) with \( 0 < \nu \leq 1 \), \( n \geq 3 \) and \( \omega \) be the logarithmic potential of \( f \). Then \( \omega(z) \in C^{n}(D_r) \) and for a multi-index \( j \), \( |j| = n \),
\[
\partial^j \omega(z) = \frac{1}{2\pi} \int_{D_R} \partial^j \log |z - \zeta| \cdot (f(\zeta) - P_{n-2}[f](z, \zeta)) d\sigma_{\zeta}
- \frac{1}{2\pi} \sum_{\tau=1}^{n-1} \int_{\partial D_R} \partial^\tau \log |z - \zeta| \cdot P_{\tau-1} [\partial^\tau f](z, \zeta) \cdot \langle N(\zeta), e_{\tau+1} \rangle |d\zeta|, \tag{3.4}
\]
where \( N(\zeta) = (N_1(\zeta), N_2(\zeta)) \) is the unit outward normal at the point \( \zeta \in \partial D_R \), \( \langle \cdot, \cdot \rangle \) is the inner product, \( R > r \) such that the divergence theorem holds on \( D_R \) and the function \( f \) is extended to vanish outside of \( D_r \).

We need the following Divergence Theorem for the proof. For a point \( z = (x_1, x_2) \), a vector field \( w(z) = (w_1(z), w_2(z)) \) and a function \( u(z) \), denote
\[
\text{div } w = \frac{\partial w_1}{\partial x_1} + \frac{\partial w_2}{\partial x_2} = \text{divergence of } w,

D u = (\partial_1 u, \partial_2 u) = \text{gradient of } u,
\]	hen \( \Delta u = \text{div } Du. \)

**Theorem 3.5**  [cf. 3]  (Divergence Theorem) Let \( \Omega \) be a bounded domain with \( C^1 \) boundary \( \partial \Omega \), for any vector field \( w \) in \( C^0(\Omega) \cap C^1(\Omega) \), we have
\[
\iint_{\Omega} \text{div } w d\sigma_z = \int_{\partial \Omega} \langle N(z), w \rangle d|z|, \tag{3.5}
\]
where \( \langle \cdot, \cdot \rangle \) is the inner product.
In (3.5) we select $w(z) = v(z) Du(z)$, then
\[
\int\int_{\Omega} Du Dv \, d\sigma_z + \int\int_{\Omega} v \, Du \, d\sigma_z = \int_{\partial\Omega} v \langle Du, N(z) \rangle \, d|z|. \tag{3.6}
\]
Since we only need one $\partial_m$ for $m = 1, 2$, we can fix the other component $x_{3-m}$ in (3.6) and relabel $u$, we obtain the following Green’s (first) identity:
\[
\int\int_{\Omega} u \, \partial_m v \, d\sigma_z + \int\int_{\Omega} v \, \partial_m u \, d\sigma_z = \int_{\partial\Omega} uv \, N_m(z) \, d|z|. \tag{3.7}
\]

**Proof of Proposition 3.4** Let
\[
u_j(z) = \frac{1}{2\pi} \int_{D_R^j} \partial^j \log |z - \zeta| \cdot (f(\zeta) - P_{n-2}[f](z, \zeta)) \, d\sigma_\zeta
- \frac{1}{2\pi} \sum_{\tau=1}^{n-1} \int_{\partial D_R} \partial^\theta \log |z - \zeta| \cdot P_{\tau-1}[\partial^\phi f](z, \zeta) \cdot \langle N(\zeta), e_{\tau+1} \rangle \, d\zeta. \tag{3.8}
\]
Note that
\[
|\partial^j \log |z - \zeta|| = \frac{n!}{|z - \zeta|^n}, \tag{3.9}
\]
for $n = |j|$, and $\log |z - \zeta|$ is harmonic for $\zeta \neq z$, then by the local Hölder continuity of $f$ in $D_r$, the function $u_j(z)$ is well defined.

Now we can employ induction. Since Proposition 3.4 has been obtained already, and $j$ has the decomposition $j = \theta_{n-1} + e_n$, we may assume that the formula (3.4) is true for $\theta_{n-1}$. Fix a function $\eta(t) \in C^{n-1}(\mathbb{R})$ such that $0 \leq \eta \leq 1$, $0 \leq \eta^{(n-1)} \leq 2$, $\eta(t) = 0$ for $t \leq 1$, $\eta(t) = 1$ for $t \geq 2$, and set
\[
\eta_\varepsilon := \eta\left(\frac{|z - \zeta|}{\varepsilon}\right), \quad L := \frac{1}{2\pi} \log |z - \zeta|.
\]
Note that $\eta_\varepsilon$ and $L$ are both skew symmetric with respect to $x_1$ and $y_1$, $x_2$ and $y_2$. Then
\[
\partial^e L \eta_\varepsilon = -\frac{\partial^e}{\partial \zeta} L \eta_\varepsilon \tag{3.10}
\]
for $e = (0,1)$ or $(1,0)$.

For $\varepsilon > 0$, define the function
\[
u_j(z, \varepsilon) := \int\int_{D_r} \partial^j \eta_\varepsilon \cdot f(\zeta) \, d\sigma_\zeta.
\]
We obtain $\nu_{\theta_{n-1}}(z, \varepsilon) \in C^{n-1}(D_r)$ for a fixed $\varepsilon$ by induction.

From (3.9) we know that, $\zeta = z$ is a singularity of $\log |z - \zeta|$ when $|j| \geq 3$. To overwhelm the blow-up behavior near the singularity, we need the Taylor expansion (3.2). To prevent a singularity from appearing on the boundary $\partial \Omega$, we have to enlarge the domain $D_r$ of the integral (3.4) into a larger domain $D_R$ where the divergence theorem holds. Thus for sufficiently small $\varepsilon$,
\[
\partial^e_n
\nu_j(z, \varepsilon) = \int\int_{D_r} \partial^e_n (\partial^\theta_{n-1} L \eta_\varepsilon) \cdot f(\zeta) \, d\sigma_\zeta
= \int\int_{D_R} \partial^e_n (\partial^\theta_{n-1} L \eta_\varepsilon) \cdot (f(\zeta) - P_{n-2}[f]) \, d\sigma_\zeta + \int\int_{D_R} \partial^e_n (\partial^\theta_{n-1} L \eta_\varepsilon) \cdot P_{n-2}[f] \, d\sigma_\zeta.
\]
Combining the skew symmetry (3.10), Green’s identity (3.7) and Theorem 3.2 for sufficiently small \( \varepsilon \), we have

\[
\int_{D_R} \partial^{\nu_n}(\partial^{\nu_n-1}L\eta_{\varepsilon}) \cdot P_{n-2}[f] d\sigma_\zeta
\]

\[
= - \int_{\partial D_R} \frac{\partial^{\nu_n}}{\partial \zeta} \partial^{\nu_n}(\partial^{\nu_n-1}L\eta_{\varepsilon}) \cdot P_{n-2}[f] d\sigma_\zeta
\]

\[
= - \int_{\partial D_R} \partial^{\nu_n-1}L \cdot P_{n-2}[f] \langle N(\zeta), e_n \rangle d\zeta + \int_{D_R} \partial^{\nu_n-1}L\eta_{\varepsilon} \cdot P_{n-3}[\partial^{\nu_n}f] d\sigma_\zeta
\]

\[
= \ldots
\]

\[
= - \int_{\partial D_R} \partial^{\nu_n-1}L \cdot P_{n-2}[f] \langle N(\zeta), e_n \rangle d\zeta - \ldots - \int_{\partial D_R} \partial^{\nu_2}L \cdot P_1[\partial^{\nu_2}f] \langle N(\zeta), e_3 \rangle d\zeta
\]

\[
+ \int_{D_R} \partial^{\nu_3}L\eta_{\varepsilon} \cdot P_0[\partial^{\nu_3}f] d\sigma_\zeta
\]

\[
= - \int_{\partial D_R} \partial^{\nu_n-1}L \cdot P_{n-2}[f] \langle N(\zeta), e_n \rangle d\zeta - \ldots - \int_{\partial D_R} \partial^{\nu_2}L \cdot P_1[\partial^{\nu_2}f] \langle N(\zeta), e_3 \rangle d\zeta
\]

\[
- \int_{\partial D_R} \partial^{\nu_1}L \cdot P_0[\partial^{\nu_2}f] \langle N(\zeta), e_2 \rangle d\zeta
\]

\[
= - \sum_{\tau=1}^{n-1} \int_{\partial D_R} \partial^{\nu_\tau}L \cdot P_{\tau-1}[\partial^{\nu_\tau}f] \langle N(\zeta), e_{\tau+1} \rangle d\zeta.
\]

Therefore

\[
\partial^{\nu_n} v_j(z, \varepsilon) = \int_{D_R} \partial^{\nu_n}(\partial^{\nu_n-1}L\eta_{\varepsilon}) \cdot (f(\zeta) - P_{n-2}[f]) d\sigma_\zeta
\]

\[
- \sum_{\tau=1}^{n-1} \int_{\partial D_R} \partial^{\nu_\tau}L \cdot P_{\tau-1}[\partial^{\nu_\tau}f] \langle N(\zeta), e_{\tau+1} \rangle d\zeta.
\] (3.11)

Now we compare (3.8) and (3.11). By the local Hölder continuity of \( f \), Theorem 3.3 and the estimate (3.3), there exist constants \( M_1 \) and \( M_2 \) such that

\[
|u_j(z) - \partial^{\nu_n} v_j(z, \varepsilon)|
\]

\[
= \left| \int_{|\zeta - z| \leq 2\varepsilon} (\partial^j L - \partial^j L\eta_{\varepsilon}) R_{n-1}(z, \zeta) d\sigma_\zeta \right|
\]

\[
\leq M_1 \int_{|\zeta - z| \leq 2\varepsilon} \left( \frac{n!}{|\zeta - z|^n} + \frac{2(n-1)!}{\varepsilon|\zeta - z|^{n-1}} \right) |z - \zeta|^\nu + n - 2 d\sigma_\zeta
\]

\[
= M_1 \int_{|\zeta - z| \leq 2\varepsilon} \left( \frac{n!}{|\zeta - z|^2} + \frac{2(n-1)!}{\varepsilon|\zeta - z|} \right) |z - \zeta|^\nu d\sigma_\zeta
\]

\[
\leq M_2 2(2\varepsilon)^\nu.
\]

The last inequality comes from Lemma 4.2 in 8. Hence \( \partial^{\nu_n} v_j(z, \zeta) \) converges to \( u_j(z) \) uniformly on any compact subset of \( D_r \) as \( \varepsilon \to 0 \). It is easy to see \( v_j(z, \varepsilon) \) converges uniformly to \( \partial^{\nu_n-1}\omega \) in the disk \( D_r \), then \( \omega \in C^\nu(D_r) \) and \( u_j(z) = \partial^j\omega(z) \). The proof is complete. \( \square \)

We list two results on a class of conformal metrics with negative curvatures as an application of potential theory. No proof is involved here. For more details, see [6, 7], also [11].
Theorem 3.6 [6] Let $\kappa : \mathbb{D} \to \mathbb{R}$ be a locally Hölder continuous function with $\kappa(0) < 0$. If $u : \mathbb{D}^* \to \mathbb{R}$ is a $C^2$-solution to $\Delta u = -\kappa(z)e^{2u}$ in $\mathbb{D}^*$, then $u$ has the order $\alpha \in (-\infty, 1]$ and

$$
\begin{align*}
  u(z) &= -\alpha \log |z| + v(z), \quad \text{if } \alpha < 1, \\
  u(z) &= -\log |z| - \log \log(1/|z|) + w(z), \quad \text{if } \alpha = 1,
\end{align*}
$$

where the remainder functions $v(z)$ and $w(z)$ are continuous in $\mathbb{D}$. Moreover, the second partial derivatives satisfy the following,

$$
\begin{align*}
  v_{zz}(z), v_{zz}(z) \text{ and } v_{zz}(z) \text{ are continuous at } z = 0 & \quad \text{if } \alpha \leq 0; \\
  v_{zz}(z), v_{zz}(z), v_{zz}(z) = O(|z|^{-2\alpha}) & \quad \text{if } 0 < \alpha < 1, \\
  w_{zz}(z), w_{zz}(z), w_{zz}(z) = O(|z|^{-2} \log^{-2}(1/|z|)) & \quad \text{if } \alpha = 1,
\end{align*}
$$

when $z$ tends to $z = 0$.

Theorem 3.7 [11] Let $\kappa : \mathbb{D} \to \mathbb{R}$ satisfy $\kappa(0) < 0$, $\kappa(z) \in C^{n-2+\nu}(\mathbb{D}^*)$ for an integer $n \geq 3$, $0 < \nu \leq 1$ and let $u : \mathbb{D}^* \to \mathbb{R}$ be a $C^n$-solution to $\Delta u = -\kappa(z)e^{2u}$ in $\mathbb{D}^*$. Then $u(z)$ has a singularity at the origin of the order $0 < \alpha \leq 1$, and for $n_1$, $n_2 \geq 1$, $n_1 + n_2 = n$, near the origin, $v(z)$, $w(z)$ as in Theorem 3.6 satisfy

$$
\begin{align*}
  \partial^n v(z), \bar{\partial}^n v(z), \bar{\partial}^{n_1} \partial^{n_2} v(z) = O(|z|^{2-2\alpha-n}), \\
  \partial^n w(z), \bar{\partial}^n w(z) = O(|z|^{-n} \log^{-2}(1/|z|)), \\
  \bar{\partial}^{n_1} \partial^{n_2} w(z) = O(|z|^{-n} \log^{-3}(1/|z|))
\end{align*}
$$

where

$$
\partial^n = \frac{\partial^n}{\partial z^n}, \quad \bar{\partial}^n = \frac{\partial^n}{\partial \bar{z}^n}
$$

for a positive natural number $n$.

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