The Parisi-Sourlas Mechanism in Yang-Mills Theory?

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Abstract

The Parisi-Sourlas mechanism is exhibited in pure Yang-Mills theory. Using the new scalar degrees of freedom derived from the non-linear gauge condition, we show that the non-perturbative sector of Yang-Mills theory is equivalent to a 4D O(1, 3) sigma model in a random field. We then show that the leading term of this equivalent theory is invariant under supersymmetry transformations where $x^2 + \theta \theta$ is unchanged. This leads to dimensional reduction proving the equivalence of the non-perturbative sector of Yang-Mills theory to a 2D O(1, 3) sigma model.

1 Introduction

There are now several new ideas on how to confine quarks inside hadrons in recent literature. The Seiberg-Witten paper [1] on the spontaneous breaking of N=2 supersymmetric Yang-Mills (SYM) to N=1 SYM is supposed to implement the dual type II superconductivity model of confinement proposed by Mandelstam and ’t Hooft [2]. Last year, Fadeev and Niemi [3] derived an effective field theory for knotlike solitons that is supposed to describe the infrared regime of pure Yang-Mills theory. In both approaches, confinement is essentially achieved through chromoelectric flux tubes, which provide the linear potential. These mechanisms break local Lorentz invariance because of the preferred direction defined by the flux tube.

There is a third approach to confinement, which is based on the non-linear generalization of the Coulomb gauge [4]. In a series of papers [5, 6, 7], this author argued the importance of generalizing the Coulomb gauge to the gauge condition

$$(\partial \cdot D)(\partial \cdot A) = 0,$$

because field configurations which satisfy $\partial \cdot A = f^a \neq 0$, and $(\partial \cdot D)f = 0$, are not gauge transformable to the Coulomb surface. In the non-linear regime (the linear sector being the Coulomb gauge) of the gauge condition (1), the Yang-Mills potential can be written as

$$A^a_\mu = \frac{1}{(1 + \vec{f} \cdot \vec{f})}(\delta^{ab} + \epsilon^{abc} f^c + f^a f^b(1 + \delta_{\mu} f^b + \epsilon_{\mu}).$$

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As discussed in reference [7], the effective dynamics of the new scalar field $f^a$ hints at non-perturbative effects. This was confirmed in reference [6] where the linear behavior of the instantaneous gluon propagator and the area law behavior of the Wilson loop were explicitly shown. The mechanism for these behaviors is not quantum mechanical but simply a statistical treatment of classical solutions of the action for pure $f$'s (all spherically symmetric $f^a(x)$ in 4D Euclidean). This confinement mechanism then maintains local Lorentz invariance unlike the mechanisms that make use of chromoelectric flux tubes.

Aside from not breaking Lorentz invariance, this mechanism relies on a straightforward, though non-trivial, difference between Abelian and non-Abelian theories. In an Abelian theory, gauge transformation is only a “translational” clue, but all orbits pass through the Coulomb surface. Furthermore, the Coulomb gauge describes the physical degrees of freedom, the transverse photon. But in a non-Abelian theory, gauge transformation is a combination of “translation” and “rotation” that depend on the gauge field being transformed. This fact leads to complications like gauge copying phenomena [6] and the existence of gauge fields that cannot be gauge transformed to the Coulomb gauge [6, 7]. Hence, the proposal to generalize the Coulomb gauge to the non-linear gauge defined by equation [1]. Note that in the Abelian limit or in the short distance limit (weak coupling), the non-linear gauge reduces to the Coulomb gauge ($f = 0$). But even before the coupling becomes very large, the fact that the non-linear term is proportional to $p^2$ ($p =$ momentum) while the linear term depends on $p^3$, the non-linear term would increasingly be more important in the long-distance regime. The non-zero $f$ terms must now be taken into account and their dynamics provide the confinement mechanism.

In this paper, we will prove a conjecture made in references [9] and [7] on the implementation of the Parisi-Sourlas mechanism [9] in pure Yang-Mills theory. The conjecture is based on two facts. First, the action for the pure $f$ term is clearly non-perturbative because of $(\frac{1}{p^2})$ and it is infinitely non-linear. Second, the action for $f$ is proportional to $(\partial f)^4$, thus hinting that it can be written as $(\frac{S^2}{M^2})^2$, where $S$ is an action with a usual kinetic term ($\sim (\partial f)^2$). If indeed we can write the action for the $f$ term as $(\frac{S^2}{M^2})^2$, then its dynamics is stochastic, i.e., driven by a random field. Clearly this hints of the possible realization of the Parisi-Sourlas mechanism.

The outline of this paper is as follows. In section 2, we give a detailed derivation of the path-integral in the non-linear gauge. Section 3 will prove the equivalence of the non-perturbative regime of Yang-Mills to an $O(1, 3)$ sigma model in random field. Section 4 presents the proof of dimensional reduction. The conclusion summarizes what the paper achieves.

2 The path-integral in the non-linear gauge

The Yang-Mills path-integral in the gauge defined by equation [3] is

$$PI = \int (dA^a_\mu) \delta(\partial \cdot D(\partial \cdot A)) det O e^{-SYM(A)},$$

(3)

where $O$ is the Fadeev-Popov operator

$$O^{ad} = (D \cdot \partial)^{ab}(\partial \cdot D)_{bd} - \epsilon^{abc}(\partial f^b) \cdot D^{cd}. \quad (4)$$

The important point about $O$ is that it is a non-singular operator even though $\partial \cdot D$ is singular [7]. The reason for this is that the gauge transform of a field configuration that satisfies $\partial \cdot A^a = f^a$, and $(\partial \cdot D)^{ab}f^b = 0$, remains on the surface $\partial \cdot A^a = f^a$ but no longer in the horizon. This can be seen by considering an infinitesimal parameter $A^a = \epsilon f^a$, resulting in an $A^a_\mu = A^a_\mu + \epsilon D^b_\mu f^b$, thus satisfying $\partial \cdot A^a = f^a$, but with $(\partial \cdot D^{ab}(A'))(\partial \cdot A^{b'}) = -\epsilon \epsilon^{abc}D^{cd}_\mu f^d\partial_\mu f^b \neq 0$. This should rectify the inconsistent claim in reference (7) that the field configurations in the non-linear sector of the non-linear gauge never leave the horizon yet having a non-singular $O$.

Let us introduce the scalar fields by inserting the following identity in equation [3]

$$I = \int (df^a)\delta(f^a - \partial \cdot A^a).$$

(5)

The two delta functionals (see equations [3] and [6]) imply that we can write

$$I^2_0 \Delta^a_\mu f^b = (A^a_\mu - \partial^a_\mu \frac{1}{\Box f^a}) - t^a_\mu,$$

(6)
where $t^a_\mu$ is a transverse vector field and $l_0$ is a length scale introduced for dimensional reasons. Equation (2) can be solved for $A^a_\mu$ in terms of $f^a$ and $t^b_\mu$:

$$A^a_\mu = \frac{1}{(1 + g^2 t_0^a \cdot \vec{f})} \left( \delta^{ab} + \epsilon^{abc} gl_0^a f^c + g^2 t_0^a f^b \right) \left( l_0^b \partial_\mu f^b + t^b_\mu \right).$$  \hspace{1cm} (7)

Shifting $t^a_\mu + \partial_\mu \frac{1}{g l_0^a} f^b \rightarrow t^a_\mu$ and rescaling $gl_0^a f^a \rightarrow f^a$, we find the expression given by equation (2). The new vector field $t^a_\mu$ and the original potential $A^a_\mu$ will now have divergence equal to $\frac{1}{g l_0^a} f^a$.

What equation (2) means is that we traded the 12 $A^a_\mu$’s satisfying the three constraints given by equation (1) to the 12 $t^a_\mu$’s and 3 $f^a$’s. The $t^a_\mu$’s satisfy $\partial_\mu t^a_\mu = \frac{1}{g l_0^a} f^a$ (3 conditions). The extra degrees of freedom are removed by the constraints

$$\rho^a = \frac{1}{(1 + \vec{f} \cdot \vec{f})^2} \left[ \epsilon^{abc} + \epsilon^{abc} \delta^{df} f^c + \epsilon^{abcd} f^{(a} f^{b)} + f^a f^b \delta^{abc} (1 + \vec{f} \cdot \vec{f}) \right]$$

$$- f^a (1 + \vec{f} \cdot \vec{f}) \delta^{abc} - f^c (1 + \vec{f} \cdot \vec{f}) \delta^{abc} \delta_\mu f^b \partial_\mu f^c = 0,$$  \hspace{1cm} (8)

which are to guarantee $\partial^\mu A^a_\mu = \frac{1}{g l_0^a} f^a$.

Let us incorporate $t^a_\mu$ in the path-integral by inserting the identity:

$$I = \int (dt^a_\mu) \delta (t^a_\mu - \tilde{t}^a_\mu),$$  \hspace{1cm} (9)

where $\tilde{t}^a_\mu$ is solved from equation (2). This means

$$\delta (A^a_\mu - \frac{1}{(1 + \vec{f} \cdot \vec{f})^2} (\delta^{ab} + \epsilon^{abc} f^c + f^a f^b (1 + \partial_\mu f^b + t^b_\mu)) = \frac{\delta (t^a_\mu - \tilde{t}^a_\mu)}{\det (\frac{\delta A^a_\mu}{\delta t^a_\mu})},$$  \hspace{1cm} (10)

where

$$\frac{\delta A^a_\mu}{\delta t^a_\mu (x')} = F^{ab} \delta_{\mu\nu} \delta^4 (x - x'),$$  \hspace{1cm} (11)

$$F^{ab} = \frac{1}{(1 + \vec{f} \cdot \vec{f})} (\delta^{ab} + \epsilon^{abc} f^c + f^a f^b).$$  \hspace{1cm} (12)

The path-integral now becomes

$$PI = \int (dt^a_\mu) (df^a) (dA^a_\mu) \delta (\partial \cdot D(\partial \cdot A)) \delta (\partial \cdot A^a - \frac{1}{g l_0^a} f^a)$$

$$\times \delta (A^a_\mu - F^{ab} (1 + \partial_\mu f^b + t^b_\mu)) \det (\mathcal{O}) \det \left( \frac{\delta A^a_\mu}{\delta t^a_\mu} \right) e^{-S_{YM}}(A).$$  \hspace{1cm} (13)

Before integrating out $A^a_\mu$, we can change the set of constraints from

$$\phi^a = (\partial \cdot D)^{ab} (\partial \cdot A^b) = 0,$$  \hspace{1cm} (14)

$$\chi^a = \partial \cdot A^a - \frac{1}{g l_0^a} f^a = 0,$$  \hspace{1cm} (15)

to

$$\phi^a = \rho^a,$$  \hspace{1cm} (16)

$$\chi^a = \partial \cdot t^a - \frac{1}{g l_0^a} f^a.$$  \hspace{1cm} (17)
This is implemented by

$$\delta(\phi^a)\delta(\chi^b) = \delta(\phi'^a)\delta(\chi'^b)J^{-1}(\phi, \chi)$$ \tag{18}$$

where

$$J(\phi, \chi) = \det \left( \begin{array}{cc} \delta^{ab} & \mathcal{F}^{ab} \\ \mathcal{F}^{ab} & \delta^{ab} \end{array} \right) \delta^4(x - x') \right)$$ \tag{19}$$

This Jacobian is 1, which can be shown by direct evaluation or by introducing fermionic coordinates and then doing the integrations. Substituting the new constraints, evaluating the determinant given by equations (11, 12), and integrating out $A^a_\mu$, we find

$$\mathcal{P}I = \int (dt^a)\delta(\partial \cdot t^a - \frac{1}{g^2} f^a) \delta(\rho^a) det^{-4}(1 + \vec{t} \cdot \vec{f}) det\mathcal{O} e^{-S_{YM}}, \tag{20}$$

where it is understood that equation (2) is substituted in $\mathcal{O}$ and $S_{YM}$.

The Yang-Mills field strength in terms of $f^a$ and $t^a_\mu$ has the following form

$$F^a_{\mu\nu} = \frac{1}{g} Z^a_{\mu\nu} + L^a_{\mu\nu} + gQ^a_{\mu\nu}(f; t), \tag{21}$$

where $Z^a_{\mu\nu}$, $L^a_{\mu\nu}$, and $gQ^a_{\mu\nu}(f; t)$ are zeroth-order linear and quadratic in $t^a_\mu$, respectively. The action becomes

$$S_{YM} = \int \left[ \frac{1}{g^2} Z^a_{\mu\nu}(f)Z^a_{\nu\mu}(f) + \frac{2}{g} Z^a_{\mu\nu}(f)L^a_{\mu\nu}(f; t) \right. \right.$$  

$$+ \left. \{2Z^a_{\mu\nu}(f)Q^a_{\mu\nu}(f; t) + L^a_{\mu\nu}(f; t)L^a_{\mu\nu}(f; t) \} \right]$$  

\hspace{1cm} + 2gL^a_{\mu\nu}(f; t)Q^a_{\mu\nu}(f; t) + g^2Q^a_{\mu\nu}(f; t)Q^a_{\mu\nu}(f; t) \right]. \tag{22}$$

### 3 Equivalence to a non-linear sigma model

Consider the pure $f$ path-integral

$$\mathcal{P}I(f) = \int (df^a) det^{-4}(1 + \vec{f} \cdot \vec{f}) det\mathcal{O} e^{-\frac{4}{g^2} \int d^4z^2}, \tag{23}$$

where $\mathcal{O}$ is the operator given by equation (3) with $t^a_\mu = 0$. To get the minus one power of $\det(1 + \vec{f} \cdot \vec{f})$, we changed the $(1 + \vec{f} \cdot \vec{f})$ factor of $\rho^a$ from minus two to plus one (see equation (8)). Why this change was made will be clarified later. In isolating the path-integral for pure $f$ term (as given in equation (23)) from the full path-integral, we are essentially claiming that the pure $f$ dynamics is dominant and the remaining $(f^a; t^a_\mu)$ constrained dynamics can be treated as corrections.

We will now show that this path-integral is equal to

$$\mathcal{P}I(f) = \int (df^a) \det \left( \frac{\delta^2 S}{\delta f^a \delta f^b} \right) \exp \left[ -\frac{1}{2} \int d^4x \left( \frac{\delta S}{\delta t^a} \right)^2 \right], \tag{24}$$

where

$$S = \frac{1}{2g} \int d^4x \eta^{ab} \partial_\mu f^a \partial_\mu f^b, \tag{25}$$

$$\eta^{ab} = -\delta_{ab} + \frac{f^a f^b}{(1 + \vec{f} \cdot \vec{f})}. \tag{26}$$

Equations (22, 23) give the action for an $O(1, 3)$ sigma model in the non-linear form. Equation (24) says that the non-perturbative regime of Yang-Mills theory is equivalent to an $O(1, 3)$ sigma model in a random field. The rest of this section is devoted to the proof of this claim.
The pure $f^a$ field strength can be written as

$$Z_{\mu \nu}^a = \sum_{abc} \partial_\mu f^b \partial_\nu f^c, \quad (27)$$

$$\sum_{abc} = \frac{1}{(1 + \vec{f} \cdot \vec{f})^2} \left\{ (1 + 2 \vec{f} \cdot \vec{f}) \epsilon_{abc} + 2 \delta^{ab} f^c - 2 \delta^{ac} f^b 
+ 3 \epsilon_{abd} f^d f^e - 3 \epsilon_{acd} f^b f^d + \epsilon_{bcd} f^a f^d \right\}. \quad (28)$$

Equations (27,28) give

$$\frac{1}{4} Z^2 = \frac{1}{4} \left( \frac{1}{1 + \vec{f} \cdot \vec{f}} \right)^2 \left\{ (1 + \vec{f} \cdot \vec{f}) \right( (\partial_\mu f^a \partial_\nu f^a)^2 - (\partial_\mu f^a \partial_\nu f^a)^2 \right) 
+ 6(1 + 2 \vec{f} \cdot \vec{f}) \left( (\partial_\mu f^a)^2 (f^b \partial_\nu f^b)^2 - (\partial_\mu f^a \partial_\nu f^b)^2 \right) \right\}. \quad (29)$$

It is interesting to note that if we impose $\vec{f} \cdot \vec{f} = 1$, equation (29) gives the interaction term of the Fadeev-Niemi action, which supposedly describes knotted strings [3]. This observation shows that indeed the $f^a$ dynamics describes the non-perturbative regime of Yang-Mills theory.

The field equation of the non-linear sigma-model is:

$$\frac{\delta S}{\delta f^a} = \eta^{ab} \Box^2 f^b - \frac{f^a}{(1 + \vec{f} \cdot \vec{f})} \eta^{bc} \partial_\mu f^b \partial_\mu f^c. \quad (30)$$

If we identify $f^a$ of above with the $f^a$ of Yang-Mills in the non-linear gauge, then we should have

$$\Box^2 f^a = \frac{1}{1 + \vec{f} \cdot \vec{f}} \left[ \partial_\mu f^a \partial_\mu f^b \partial_\mu f^b - f^a \partial_\mu f^b \partial_\mu f^b + \epsilon_{abc} f^c \partial_\mu f^b \partial_\mu f^d \partial_\mu f^d \right] 
+ \text{linear term in } t^a_\mu \quad (31)$$

Equation (31) comes from the non-linear gauge condition. Since we are considering pure $f$ dynamics, we will put $t^a_\mu = 0$. Clearly we find that

$$\frac{1}{4} \int d^4 x Z^2 = \frac{1}{2} \int d^4 x \left( \frac{\delta S}{\delta f^a} \right)^2.$$

However, we can add a zero, a surface term, to the Yang-Mills action such that

$$\frac{1}{4} \int d^4 x Z^2 + \int d^4 x \partial_\mu H_\mu = \frac{1}{2} \int d^4 x \left( \frac{\delta S}{\delta f^a} \right)^2 \quad (32)$$

Inspection shows that

$$\partial_\mu H_\mu = \alpha (\partial_\mu f^a \partial_\mu f^a)^2 + \beta (\partial_\mu f^a \partial_\nu f^a)^2 
+ \gamma (\partial_\mu f^a)^2 (f^b \partial_\nu f^b)^2 + \delta (\partial_\mu f^a \partial_\nu f^b \partial_\nu f^b)^2 
+ \rho \left( (f^a \partial_\mu f^a)^2 \right)^2, \quad (33)$$

and equating corresponding terms in (32) yields

$$\alpha = \frac{2(\vec{f} \cdot \vec{f})^3 + 7(\vec{f} \cdot \vec{f})^2 - 6(\vec{f} \cdot \vec{f}) - 1}{4(1 + \vec{f} \cdot \vec{f})^4},$$

$$\beta = \frac{1}{4(1 + \vec{f} \cdot \vec{f})^2}.$$
\[ \gamma = \frac{(\vec{f} \cdot \vec{f})^3 - 11(\vec{f} \cdot \vec{f}) - 7}{2(1 + \vec{f} \cdot \vec{f})^4}, \]
\[ \delta = \frac{(\vec{f} \cdot \vec{f})^2 + 8(\vec{f} \cdot \vec{f}) + 4}{2(1 + \vec{f} \cdot \vec{f})^4}, \]
\[ \rho = \frac{-[\vec{f} \cdot \vec{f})^2 + 3(\vec{f} \cdot \vec{f}) + 1]}{2(1 + \vec{f} \cdot \vec{f})^4}. \]

To prove that the added term is indeed zero, we note that in 4D Euclidean space,

\[ H_\mu(x) = \partial^x_\mu \int d^4y \frac{1}{(x-y)^2} [\partial \cdot H]_y, \]

giving

\[ \int d^4x \partial_\mu H_\mu = \int_{x \to \infty} \left\{ \int d^4y \frac{(x-y)_\mu}{(x-y)^3} [\partial \cdot H]_y \right\} n_\mu dS. \quad (34) \]

Since we will only be considering \( L^2 \) fields, i.e., \( \int d^4x A^a_\mu A^a_\mu = \text{finite} \), then \( A^a_\mu \sim \frac{1}{x^{2+\varepsilon}} \) as the 4D Euclidean radius \( x \to \infty \). From equation (34), i.e., \( x_\mu - y_\mu = y'_\mu \), we find

\[ \int d^4x (\partial \cdot H) = \int_{x \to \infty} \left\{ \int d^4y'y'_\mu y'_\beta \left[ (\partial \cdot H)_x - \partial_\alpha(\partial \cdot H)_x y'_\alpha \right. \\
+ \left. \frac{1}{2} \partial_\alpha \partial_\beta (\partial \cdot H)_x y'_\alpha y'_\beta \cdots \right] \right\} n_\mu dS. \quad (35) \]

We only have to look at the even number of integrands in \( y' \) because all the odd vanish. From the expression of \( (\partial \cdot H) \), we see that

\[ (\partial \cdot H) \sim \left\{ \frac{1}{x^{16+\varepsilon}}, \frac{1}{x^{14+\varepsilon}}, \frac{1}{x^{12+\varepsilon}}, \frac{1}{x^{10+\varepsilon}} \right\}_{x \to \infty}. \]

Because of these behaviors, even though each \( y' \) integration diverges as \( y' \to \infty \), the \( H \) factor that goes with it goes to zero faster as \( x \to \infty \) resulting in a zero for each term. Thus,

\[ \int d^4x \partial_\mu H_\mu = \int_{x \to \infty} H_\mu n_\mu dS = 0. \quad (36) \]

To complete the proof of the equivalence to an \( O(1,3) \) sigma model in random field, we need to establish the following determinant relation

\[ \det^{-1}(1 + \vec{f} \cdot \vec{f}) \det \bar{O} = \det \left( \frac{\delta^2 S}{\delta f^a \delta f^b} \right). \quad (37) \]

The proportionality constant may be infinite but it should be field \( f \) independent. At first glance, equation (37) does not seem to make sense because \( \bar{O} \) is a fourth-order differential operator while \( \frac{\delta^2 S}{\delta f^a \delta f^b} \) is only second-order. However, as we will show in the following arguments equation (37) is valid.

It is important to note that we can write

\[ \frac{\delta^2 S}{\delta f^a \delta f^b} = -\eta(\Box + R(f) + M_\mu(f)\partial_\mu), \quad (38) \]

where

\[ M_\mu^{ab} = -2 f^a \partial_\mu f^b + \frac{2 f^a f^b f^c \partial_\mu f^c}{1 + \vec{f} \cdot \vec{f}}, \quad (39) \]
Equations (43,44,45) enable us to write ✷(37). Second, it leaves ∇^{2} without a field-dependent coefficient, which in turn enables us to carry out the following identification:

\[ \text{det} \eta = \text{det} \det (1 + f^a \cdot f^a) \]

Equation (41) implies that the proportionality factor in equation (37) is \( a^{2} \text{ + } M \cdot \partial + R \). It is hinted by the fact that for \( f^a = \text{constant} \), \( \hat{O} = (\nabla^{2} + M \cdot \partial + R) \).

Now we prove that equation (41) is true for any \( f^a \). Neglecting the \( t \) dependent term in equation (31), we find \( \partial_{\mu} \hat{A}_{\mu} = 0 \). This does not mean, however, that we are expanding about a field configuration on the Coulomb surface. Equation (4) clearly shows that this is not a case of background decomposition (where \( A = \hat{A} + a \)) because \( t_{\mu} \) is linked in a very nonlinear manner to \( f^a \). Besides, we are not considering a particular field configuration \( f^a \). Thus, if anything, the vanishing of the divergence of \( \hat{A}_{\mu} \) is purely a coincidence, but a welcome one because it leads to a significant amplification of \( O \).

Equations (4) and (2), with \( t_{\mu} = 0 \), give

\[ O = (\nabla^{2} + V_{\mu} \nabla^{2} \partial_{\mu} + W_{\mu \nu} \partial_{\mu} \partial_{\nu} + T_{\mu} \partial_{\mu}), \]

where

\[ V_{\mu} = -2 \epsilon^{abc} \hat{A}_{\mu}, \]

\[ W_{\mu \nu} = -\delta^{ab} \hat{A}_{\mu} \hat{A}_{c} + 2 \epsilon^{abc} \partial_{\mu} \hat{A}_{c}, \]

\[ T_{\mu} = -\delta^{ab} \hat{A}_{c} \partial_{\mu} \hat{A}_{c} + \partial_{\mu} \hat{A}_{a} \hat{A}_{b} - \epsilon^{abc} \nabla^{2} \hat{A}_{c}. \]

Equations (43,44,45) enable us to write

\[ \hat{O} = (\nabla^{2} + N \cdot \partial)^2, \]

where

\[ N_{\mu} = -\epsilon^{abc} \hat{A}_{c}. \]

Since the determinant is invariant under a similarity transformation,

\[ \text{det} (\nabla^{2} + N \cdot \partial) = \text{det} (J^{-1}(\nabla^{2} + N \cdot \partial)J) = \text{det} (\nabla^{2} + Y \cdot \partial + P), \]

where

\[ Y_{\mu} = 2J^{-1} \partial_{\mu} J + J^{-1} N_{\mu} J, \]

\[ P = J^{-1} \nabla^{2} J + J^{-1} N_{\mu} \partial_{\mu} J. \]

Using equations (43,49,50), equation (41) is true if we can find a \( J \) such that

\[ \text{tr ln}(J + M \cdot \partial + R) = \text{tr ln} \left( J + \frac{2}{\nabla^{2}} (Y \cdot \partial + P) \right) \]

\[ + \frac{1}{\nabla^{2}} (Y \cdot \partial + P) \text{ tr ln} \left( J + \frac{1}{\nabla^{2}} (Y \cdot \partial + P) \right), \]
where the trace is over space-time and isospin. Using the expansion for \( \ln(1 + x) \), we find that equation (51) is equivalent to a set of infinite, global equations,

\[
\text{tr} \left\{ \left[ \frac{1}{\sqrt{2}} (Y \cdot \partial + P) \right]^n \right\} = \frac{1}{2} \text{tr} \left\{ \left[ \frac{1}{\sqrt{2}} (M \cdot \partial + R) \right]^n \right\}, \tag{52}
\]

with \( n = 1, 2, \ldots, \infty \), for the nine elements of \( J \) for each space-time point. Equation (52) in diagram form can be written as in Fig 1 where the insertions on the left are \( M \cdot \partial + P \) while those on the right are \( M \cdot \partial + R \). Because of the \((1 + \vec{f} \cdot \vec{f})\) factors in the denominators of the insertions, each term is essentially non-perturbative.

Can \( J \) be determined by the infinite number of global equations given by equation (52)? Naively, the answer is yes because we have infinite equations for the infinite unknowns (9 components at each space-time point). We can visualize the problem better if we discretize space-time (lattice formulation). The set of equations given by (52) becomes a set of non-linear algebraic equations for the values of \( J^{ab}(x_i) \).

By choosing a suitable ansatz for \( J \), we can always have the number of equations equal to the number of unknowns. It should be noted that both sides have divergencies arising from the same point limit of the Green function of \( \Box^2 \). This must be handled carefully using a suitable regulator. In the continuum, what can be done is to solve for \( J \) for \( n = 1 \) and then verify the relationship for \( n = 2, 3, \ldots \). For \( n=1 \), equation (52) can be solved by the local equations

\[
Y^{aa}_\mu = \frac{1}{2} M^{aa}_\mu, \tag{53}
\]
\[
R^{aa} = P^{aa}. \tag{54}
\]

Using equations (53,54), and the fact that \( N^{aa}_\mu = 0 \), we find

\[
J^{-1ab}_\mu \partial^a J^{ba} = \frac{1}{4} M^{aa}_\mu, \tag{55}
\]
\[
J^{-1ab}_\mu \Box^2 J^{ba} + J^{-1ab}_\mu N^{bc}_\mu \partial^a J^{ca} = \frac{1}{2} R^{aa}. \tag{56}
\]

There are many possible solutions to equations (55,56). Assume the simplest ansatz

\[
J^{ab} = \alpha(f) \delta^{ab} + \beta(f) f^a f^b, \tag{57}
\]

With \( f = (\vec{f} \cdot \vec{f})^{1/2} \) the inverse exists under a very broad condition, i.e., \( \alpha^2(\alpha + \beta \vec{f} \cdot \vec{f}) \neq 0 \). The inverse is given by \( J^{-1ab} = \left( \frac{1}{\alpha} \right) \delta^{ab} - \left( \frac{1}{\alpha} \right) \frac{\beta \vec{f} \cdot \vec{f}}{\alpha + \beta \vec{f} \cdot \vec{f}} f^a f^b \).

Equation (57) leads to a first-order, non-linear, ordinary differential equation for \( \alpha \) and \( \beta \) by equating coefficients of \( f^a \partial^a f^a \), which is the only possible structure of \( M^{aa}_\mu \) and \( Y^{aa}_\mu \). Equation (54), on the other
hand, leads to two equations – a second-order, non-linear ordinary differential equation by equating coefficients of $f^a \partial_{\mu} f^a f^b \partial_{\nu} f^b$ and a first-order, non-linear ordinary differential equation by equating coefficients of $\partial_{\mu} f^a \partial_{\nu} f^a$. Seemingly, we have an overspecified problem. Fortunately, the two first-order equations have the same structure and they only result in the following linear relationship between $\alpha$ and $\beta$

$$\alpha(f) = K(f) \beta(f), \quad (58)$$

where

$$K = -f^2 a(f) \pm \sqrt{a^2(f) f^4 + 8 f^2 (f^4 + 3 f^2 + 2) a(f) \over 2 a(f)}, \quad (59)$$

$$a(f) = 2 f^4 + 5 f^2 - 5. \quad (60)$$

We already see from equation (59) the multiplicity of solutions even for the simplest ansatz for $f$.

Substituting (58) in the second-order equation, the problem simplifies tremendously, the differential equation becomes linear given by

$$\frac{d^2 \beta}{df^2} + B(f) \beta + C(f) \beta = 0, \quad (61)$$

where

$$B(f) = 4 \frac{f^2}{3 K + f^2} + \frac{2 K'}{K} - 2 \frac{f^2 K'}{3 K(K + f^2)} - \frac{1}{f(1 + f^2)} \quad (62)$$

$$C(f) = \frac{1}{3} K''(3 K + f^2) - \frac{K'(3 K + f^2)}{K(K + f^2)}$$

$$+ \frac{(2 K - f^2 - 2) f^2}{3 K(1 + f^2)(K + f^2)} - \frac{(1 - f^2) f^2}{3(1 + f^2)}. \quad (63)$$

Equations (51), (62), (63) are too complicated to have a closed form solution. However, a power series solution can be given because there are no poles in $f$. Remember that in the non-linear regime $f \neq 0$ and that $f$ is always real. This verifies the existence of a $J$ that at least satisfies the one insertion condition of the trace $\ln$. Note that even at the one insertion level, the “equality” of the determinant is already non-perturbative.

Finally, we answer the question, “Why not prove equation (41) directly using $N_{\mu}$?” The answer is simple, $\sum_{a} a = 0$ thus we cannot have an insertion per insertion comparison of the $tr \ ln$. The single insertion trace of $Y \cdot \theta + P$ is contained in the higher insertion traces of $N \cdot \theta$ and comparison is difficult to make (but the terms are there but with different coefficients).

### 4 Proof of dimensional reduction

The path-integral given by (24) can be written as

$$PI(f; w; \bar{\psi}^a; \psi^a) = \int (df^a)(dw^a)(d\bar{\psi}^a)(d\psi^a)e^{-A}, \quad (64)$$

where

$$A = \int d^4 x \left\{ -{1 \over 2} w^2 + w_a {\delta S \over \delta f^a} + \bar{\psi} \frac{\delta^2 S}{\delta f^a \delta f^b} \psi^b \right\}. \quad (65)$$

Because of the metric $\eta^{ab}$, this action could not be derived from the supersymmetric version of $S$, i.e.,

$$A \neq \int d^4 \bar{\theta} d\theta {1 \over 2} \eta_{ab}(\Phi) (\partial_{\mu} \Phi^a \partial_{\nu} \Phi^b + \partial_{\theta} \Phi^a \partial_{\theta} \Phi^b), \quad (66)$$

where

$$\Phi^a = f^a + \bar{\psi} \psi^a + \bar{\psi}^a \theta + \bar{\theta} \theta w^a. \quad (67)$$
Also because of \( \eta \), the supersymmetrized \( S(\Phi) \) is not invariant under the following transformation that leaves \( x^2 + \overline{\theta} \theta \) invariant,

\[
\begin{align*}
x_\mu &\rightarrow x'_\mu = x_\mu + \varepsilon_\mu \overline{\theta} \theta + \varepsilon_\mu \overline{\sigma} \rho, \\
\theta &\rightarrow \theta' = \theta - 2 \rho \varepsilon \cdot x, \\
\overline{\theta} &\rightarrow \overline{\theta}' = \overline{\theta} - 2 \overline{\rho} \varepsilon \cdot x.
\end{align*}
\]

In equations (68, 69, 70), \( \varepsilon_\mu, \rho, \overline{\sigma} \) are infinitesimal coordinate and grassman parameters. Dimensional reduction could not be proven using the non-linear action of \( O(1,3) \).

Actually, the paper of Parisi and Sourlas hints that we should use instead

\[
S_\sigma = \int d^4x \left\{ \frac{1}{2} \partial_\mu \sigma \partial_\mu \sigma - \frac{1}{2} \partial_\mu f^a \partial_\mu f^a + \lambda (\sigma^2 - \overline{f} \cdot f - 1) \right\}.
\]

The equation of motion for the multiplier \( \lambda(x) \) implements the equivalence of \( S_\sigma \) to \( S \). We will show that the path-integral given by equation (24) is equal to

\[
PI(f, \sigma, \lambda) = \int (df^a)(d\sigma)(d\lambda) \delta \left( \frac{\delta S_\sigma}{\delta \lambda} \right) \delta \left( \frac{\delta S_\sigma}{\delta \sigma} \right) \times det \left( \frac{\delta^2 S_\sigma}{\delta \phi_i \delta \phi_j} \right) \exp \left\{ -\frac{1}{2} \int d^4x \left( \frac{\delta S_\sigma}{\delta f^a} \right)^2 \right\},
\]

where \( \phi_i = (f^a, \sigma, \lambda) \). The field derivatives are

\[
\begin{align*}
\frac{\delta S_\sigma}{\delta f^a} &= \Box^2 f^a - 2 \lambda f^a, \\
\frac{\delta S_\sigma}{\delta \sigma} &= -\Box^2 \sigma + 2 \lambda \sigma, \\
\frac{\delta S_\sigma}{\delta \lambda} &= \sigma^2 - (1 + \overline{f} \cdot f).
\end{align*}
\]

It is straightforward to show that

\[
\frac{\delta S_\sigma}{\delta f^a} \bigg|_{\delta S_\sigma = \delta S = 0} = \frac{\delta S}{\delta f^a}.
\]

We will now give a “physicist’s” proof of

\[
\begin{align*}
\delta^2 S_\sigma \bigg|_{\delta S_\sigma = \delta S = 0} &\approx det \left( \frac{\delta^2 S}{\delta f^a \delta f^b} \right),
\end{align*}
\]

where the proportionality factor is anything as long as it is field-independent. Note that the matrix at the LHS of (77) is an “infinite 5 \times 5” matrix while the RHS is an “infinite 3 \times 3” matrix. However, direct inspection shows both determinants have leading terms \( \Box^2 \), thus equation (77) is not really surprising. Equation (76) can be written as

\[
\frac{\delta S}{\delta f^b} = \left( \frac{\delta S_\sigma}{\delta \phi_i} \frac{\delta \phi_i}{\delta f^b} \right) \bigg|_{\delta S_\sigma = \delta S = 0}.
\]

Differentiating again with \( f^a \), we get

\[
\frac{\delta^2 S}{\delta f^a \delta f^b} = \left[ \frac{\delta \phi_i}{\delta f^a} \left( \frac{\delta^2 S_\sigma}{\delta \phi_i \delta \phi_j} \right) \frac{\delta \phi_j}{\delta f^b} \right] \bigg|_{\delta S_\sigma = \delta S = 0}.
\]

The first matrix of the RHS of this equation is “3 \times 5 (infinite)”, the second is “5 \times 5”, while the last is “5 \times 3” for consistency with the LHS which is “3 \times 3”. Taking the determinant of both sides and
using the cyclic property of the determinant, we need to evaluate the determinant of the “$5 \times 5$ infinite” matrix

$$
\begin{bmatrix}
1 & 0 & 0 & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} \\
0 & 1 & 0 & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} \\
\frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} \\
\frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} \\
\frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} \\
0 & 1 & 0 & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} \\
\frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} \\
\frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} \\
\frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} \\
\end{bmatrix}
$$

\begin{equation}
= \begin{bmatrix}
1 & 0 & 0 & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} \\
0 & 1 & 0 & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} \\
0 & 0 & 1 & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} \\
\frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} \\
\frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} \\
0 & 1 & 0 & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} \\
0 & 0 & 1 & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} \\
\frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} \\
\frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} & \frac{\delta \lambda}{\delta f} \\
\end{bmatrix}
\end{equation}

Above we made use of $\sigma = \left(1 + \frac{\delta}{\delta f} \cdot \frac{\delta}{\delta f}\right)^{1/2}$, and $\lambda = \frac{1}{2} \left(\frac{\delta S}{\delta \sigma}\right)^{2} = \frac{1}{2} \frac{\delta \lambda}{\delta \sigma} \partial_{\sigma} f \partial_{\sigma} f$. The important point is that by a series of manipulations, the fourth row or the fifth row can be made zero. Thus,

\begin{equation}
det\left(\frac{\delta^{2} S}{\delta f^{a} \delta f^{b}}\right) = “0 \cdot \infty” \quad \text{det}\left(\frac{\delta^{2} S_{\sigma}}{\delta \phi_{i} \delta \phi_{j}}\right) \bigg|_{\frac{\delta S_{\sigma}}{\delta \phi_{i}} = \frac{\delta S_{\sigma}}{\delta \phi_{j}} = 0}.
\end{equation}

The $\infty$ comes from the $\delta^{4}(x - x')$.

Effectively, we have established that the dynamics of $f$ with action $S$ in a random field is equivalent to a constrained stochastic dynamics of $f$, $\sigma$, $\lambda$ with action $S_{\sigma}$. That is true follows from the fact that we can always add $\frac{1}{2} \left(\frac{\delta S_{\sigma}}{\delta \sigma}\right)^{2}$ and $\frac{1}{2} \left(\frac{\delta S_{\lambda}}{\delta \lambda}\right)^{2}$ in the exponentials of (72).

Next we complete the proof of dimensional reduction. First, we will exponentiate $\delta \left(\frac{\delta S_{\sigma}}{\delta \phi_{i}}\right)$ by using

\begin{equation}
\int (d\varphi) \delta(\varphi - \varphi_{0}) F(\varphi) = \int (d\varphi) \exp\{-\frac{1}{2} \int dx(\varphi - \varphi_{0})^{2}\} F(\varphi)
+ C_{2} \frac{\delta^{2} F(\varphi)}{\delta \varphi^{2}} \bigg|_{\varphi_{0}} + \cdots
\end{equation}

It may seem that the use of the above approximation unnecessarily complicates things because we could have exponentiated $\delta \left(\frac{\delta S_{\sigma}}{\delta \phi_{i}}\right)$ by

\begin{equation}
lim_{\varepsilon \to 0} \exp\left\{-\frac{1}{2} \int d^{4}x \left(\frac{\delta S_{\sigma}}{\delta \phi_{i}}\right)^{2}\right\}.
\end{equation}

Unfortunately, the presence of $\varepsilon$ invalidates $SU(3)$ except for $\varepsilon = 1$.

Second, we will exponentiate $\delta \left(\frac{\delta S_{\lambda}}{\delta \lambda}\right)$ by introducing an auxiliary field $w$ by

$$
\delta \left(\frac{\delta S_{\lambda}}{\delta \lambda}\right) = \int (dw) e^{-\int d^{4}x \frac{\delta S_{\lambda}}{\delta \lambda}}.
$$

This was not done for $\frac{\delta S_{\sigma}}{\delta \sigma}$ because $\sigma$ has a kinetic term (and $\lambda$ does not), which requires the presence of a $\frac{1}{2} w_{\sigma}^{2}$ (will arise from $\partial_{\sigma} \Phi_{i} \partial_{\sigma} \Phi_{i}$).

Taking everything into account, we have

\begin{align*}
PI(23) &= PI(24) \\
&= PI(22)
\end{align*}

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\[ \approx \int (df^a)(d\sigma)(d\lambda) \det \left( \frac{\delta^2 S_\sigma}{\delta \phi_i \delta \phi_j} \right) \]
\[ \times \exp \left\{ -\int d^4x \left[ + \frac{1}{2} \left( \frac{\delta S_\sigma}{\delta f^a} \right)^2 + \frac{1}{2} \left( \frac{\delta S_\sigma}{\delta \sigma} \right)^2 + w_\lambda \frac{\delta S_\lambda}{\delta \lambda} \right] \right\} \],

where the numbers in the PI refer to the equation numbers in this paper. Note that the last equation only involves an approximation, because we neglected the other terms in equation (79). Since the kinetic term of \( f^a \) has the wrong sign, we momentarily shift \( f^a \to i f^a \), effectively rotating the \( O(1,3) \) symmetry to \( O(4) \). Finally, we introduce the fermion fields for the determinant and the auxiliary fields \( w_a \) and \( w_\sigma \) to complete the squares of the first two terms of the exponential to get

\[ \text{PI}(72) \approx \int (d\phi_i)(dw_i)(d\psi_i)(d\psi_j) \exp\{-A_{SS}\}, \]

where \( \phi_i = (f^a, \sigma, \lambda); \ w_i = (w^a, w_\sigma, w_\lambda); \ \psi_i = (\psi^a, \psi_\sigma, \psi_\lambda); \) and

\[ A_{SS} = \int d^4x \left\{ -\frac{1}{2} \frac{\delta^2 S_\sigma}{\delta f^a} + w_a \frac{\delta S_\sigma}{\delta f^a} + w_\sigma \frac{\delta S_\sigma}{\delta \sigma} \right\} \]

Equation (82) can be derived from the supersymmetric version of \( S_\sigma \), i.e.,

\[ A_{SS} = S_\sigma(\Phi) = \int d^4x d\theta d\bar{\theta} \left\{ \frac{1}{2} \partial_\mu \Phi_\sigma \partial_\mu \bar{\Phi}_\sigma + \frac{1}{2} \partial_\theta \Phi_\sigma \partial_\theta \bar{\Phi}_\sigma \right\} \]

\[ + \frac{1}{2} \partial_\mu \Phi^a \partial_\mu \Phi^a + \frac{1}{2} \partial_\theta \Phi^a \partial_\theta \Phi^a + \Phi_\lambda (\Phi^2 + \Phi^a \Phi^a - 1) \]

where \( \Phi^a \) is given by equation (67) and

\[ \Phi_\sigma = \sigma + \bar{\theta} \psi_\sigma + \bar{\psi}_\sigma \theta + \bar{\theta} \bar{\psi}_\sigma, \]

\[ \Phi_\lambda = \lambda + \bar{\theta} \psi_\lambda + \bar{\psi}_\lambda \theta + \bar{\theta} \bar{\psi}_\lambda. \]

Equation (83) explicitly shows invariance under equations (68, 69, 70) and dimensional reduction follows. Then we rotate back \( f^a \to i f^a \), yielding again the \( O(1,3) \) sigma model in the linear form but in dimension reduced by two.

This completes the proof that the non-perturbative regime of Yang-Mills theory in 4D is equivalent to a non-linear \( O(1,3) \) sigma model in 2D.

5 Conclusion

We have exhibited the Parisi-Sourlas mechanism in Yang-Mills theory. Since the starting point of the proof is the scalar field derived from the non-linear gauge condition, this paper proves further that the Coulomb gauge is not an appropriate gauge fixing in non-Abelian theories. Important field configurations will be missed in the path-integral as shown in reference [4], where the linear potential was derived, and in this paper, where the Parisi-Sourlas mechanism was exhibited.

It is also important to point out that the derivation of the linear potential is not quantum mechanical but merely a statistical treatment of a class of classical configurations, i.e., all spherically symmetric \( f^a(x) \) in 4D Euclidean space. But as this paper showed, taking into account the full dynamics, including quantum effects, results in dimensional reduction and equivalence to an \( O(1,3) \) sigma model in 2D.
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