On Lagrangians of $r$-uniform hypergraphs

Yuejian Peng · Qingsong Tang · Cheng Zhao

Published online: 30 October 2013
© Springer Science+Business Media New York 2013

Abstract A remarkable connection between the order of a maximum clique and the Lagrangian of a graph was established by Motzkin and Straus in Can J Math 17:533–540 (1965). This connection and its extensions were successfully employed in optimization to provide heuristics for the maximum clique number in graphs. It has been also applied in spectral graph theory. Estimating the Lagrangians of hypergraphs has been successfully applied in the course of studying the Turán densities of several hypergraphs as well. It is useful in practice if Motzkin–Straus type results hold for hypergraphs. However, the obvious generalization of Motzkin and Straus’ result to hypergraphs is false. We attempt to explore the relationship between the Lagrangian of a hypergraph and the order of its maximum cliques for hypergraphs when the num-
ber of edges is in certain range. In this paper, we give some Motzkin–Straus type results for $r$-uniform hypergraphs. These results generalize and refine a result of Talbot in Comb Probab Comput 11:199–216 (2002) and a result in Peng and Zhao (Graphs Comb, 29:681–694, 2013).

**Keywords** Cliques of hypergraphs · Lagrangians of hypergraphs · Optimization

### 1 Introduction

In 1965, Motzkin and Straus (1965) established a continuous characterization of the clique number of a graph using the Lagrangian of a graph. Namely, the Lagrangian of a graph is the Lagrangian of its maximum clique which is determined by the order of a maximum clique. Applying this connection, they provided a new proof of classical Turán’s theorem (Turán 1941) on the Turán density of a complete graph. This connection has been also applied in spectral graph theory (Wilf 1986). Furthermore, the Motzkin–Straus result and its extension were successfully employed in optimization to provide heuristics for the maximum clique problem. The Motzkin–Straus theorem has been also generalized to vertex-weighted graphs (Gibbons et al. 1997) and edge-weighted graphs with applications to pattern recognition in image analysis (see Budinich 2003; Busygin 2006; Gibbons et al. 1997; Pavan and Pelillo 2003; Pardalos and Phillips 1990; Rota Buló et al. 2007). It is interesting to explore whether similar results hold for hypergraphs. The obvious generalization of Motzkin and Straus’ result to hypergraphs is false. In fact, there are many examples of hypergraphs that do not achieve their Lagrangian on any proper subhypergraph. In this paper, we provide evidences that the Lagrangian of an $r$-uniform hypergraph is related to the order of its maximum cliques under some conditions. Some definitions and notations are needed in order to state the questions and results precisely.

Let $\mathbb{N}$ be the set of all positive integers. Let $V$ be a set and $r \in \mathbb{N}$. Let $V^{(r)}$ denote the family of all $r$-subsets of $V$. An $r$-uniform graph or $r$-graph $G$ is a set $V(G)$ of vertices together with a set $E(G) \subseteq V(G)^{(r)}$ of edges. An edge $e = \{a_1, a_2, \ldots, a_r\}$ will be simply denoted by $a_1 a_2 \ldots a_r$. An $r$-graph $H$ is a subgraph of an $r$-graph $G$, denoted by $H \subseteq G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let $K_t^{(r)}$ denote the complete $r$-graph on $t$ vertices, that is the $r$-graph on $t$ vertices containing all possible edges. A complete $r$-graph on $t$ vertices is also called a clique with order $t$. For $n \in \mathbb{N}$, we denote the set $\{1, 2, 3, \ldots, n\}$ by $[n]$. Let $[n]^{(r)}$ represent the complete $r$-uniform graph on the vertex set $[n]$. When $r = 2$, an $r$-uniform graph is a simple graph. When $r \geq 3$, an $r$-graph is often called a hypergraph.

**Definition 1.1** Let $G$ be an $r$-uniform graph with vertex set $[n]$ and edge set $E(G)$. Let $S = \{\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n x_i = 1, x_i \geq 0 \text{ for } i = 1, 2, \ldots, n\}$. For $\vec{x} = (x_1, x_2, \ldots, x_n) \in S$, define

$$\lambda(G, \vec{x}) = \sum_{i_1 i_2 \ldots i_r \in E(G)} x_{i_1} x_{i_2} \ldots x_{i_r}.$$
The Lagrangian of $G$, denoted by $\lambda(G)$, is defined as

$$\lambda(G) = \max \{ \lambda(G, \bar{x}) : \bar{x} \in S \}.$$  

A vector $\bar{y} \in S$ is called an optimal weighting for $G$ if $\lambda(G, \bar{y}) = \lambda(G)$.

The following fact is easily implied by the definition of the Lagrangian.

**Fact 1.1** Let $G_1, G_2$ be $r$-uniform graphs and $G_1 \subseteq G_2$. Then $\lambda(G_1) \leq \lambda(G_2)$.

The following theorem by Motzkin and Straus in [Motzkin and Straus 1965] shows that the Lagrangian of a 2-graph is determined by the order of its maximum clique.

**Theorem 1.2** (Motzkin and Straus 1965) If $G$ is a 2-graph in which a largest clique has order $l$, then $\lambda(G) = \lambda(K^{(2)}_l) = \lambda([l]^{(2)}) = \frac{1}{2}(1 - \frac{1}{l})$.

As mentioned earlier, there are many examples of hypergraphs that do not achieve their Lagrangian on any proper subhypergraph and the obvious generalization of Motzkin and Straus’ result to hypergraphs is false. Sós and Straus attempted to generalize the Motzkin–Straus theorem to hypergraphs in [Sós and Straus 1982]. Recently, Rota Buló and Pelillo generalized the Motzkin and Straus’ result to $r$-graphs in some way using a continuous characterization of maximal cliques other than Lagrangians of hypergraphs in [Rota Buló and Pelillo 2008] and [Rota Buló and Pelillo 2009].

Lagrangians of hypergraphs has been proved to be a useful tool in hypergraph extremal problems. For example, Frankl and Rödl 1984 applied it in disproving Erdős’ long standing jumping constant conjecture. It has also been applied in finding Turán densities of hypergraphs in [Frankl and Füredi 1989], [Sidorenko 1987] and [Mubayi 2006].

We attempt to explore the relationship between the Lagrangian of a hypergraph and the order of its maximum cliques for hypergraphs when the number of edges is in certain range though the obvious generalization of Motzkin and Straus’ result to hypergraphs is false. The following two conjectures are proposed in [Peng and Zhao 2013].

**Conjecture 1.3** (Peng-Zhao 2013) Let $l$ and $m$ be positive integers satisfying $(\binom{l-1}{r} - 1) \leq m \leq (\binom{l-1}{r} + \binom{l-2}{r-1})$. Let $G$ be an $r$-graph with $m$ edges and contain a clique of order $l - 1$. Then $\lambda(G) = \lambda([l-1]^{(r)})$.

The upper bound $(\binom{l-1}{r} + \binom{l-2}{r-1})$ in this conjecture is the best possible. For example, if $m = (\binom{l-1}{r} + \binom{l-2}{r-1} + 1$ then $\lambda(C_{r,m}) > \lambda([l-1]^{(r)})$, where $C_{r,m}$ is the $r$-graph on the vertex set $[l]$ and with the edge set $[l-1]^{(r)} \cup \{i_1 \ldots i_r \mid i_1 \ldots i_r \in [l-2]^{(r-1)} \cup \{1 \ldots (r-2)(l-1)l\}$. Take $\bar{x} = (x_1, \ldots, x_l) \in S$, where $x_1 = x_2 = \cdots = x_{l-2} = \frac{1}{l-1}$ and $x_{l-1} = x_l = \frac{1}{2(l-1)}$. Then $\lambda(C_{r,m}, \bar{x}) > \lambda([l-1]^{(r)})$.

**Conjecture 1.4** (Peng-Zhao 2013) Let $l$ and $m$ be positive integers satisfying $(\binom{l-1}{r} \leq m \leq (\binom{l-1}{r} + \binom{l-2}{r-1})$. Let $G$ be an $r$-graph with $m$ edges and contain no clique of order $l - 1$. Then $\lambda(G) < \lambda([l-1]^{(r)})$.

In [Peng and Zhao 2013], Conjecture 1.3 is proved for $r = 3$.  

$\copyright$ Springer
Theorem 1.5 (Peng-Zhao 2013) Let $l$ and $m$ be positive integers satisfying $(\binom{l-1}{3}) \leq m \leq \left(\binom{l-1}{3}\right) + \left(\binom{l-2}{2}\right)$. Let $G$ be a 3-graph with $m$ edges and $G$ contain a clique of order $l - 1$. Then $\lambda(G) = \lambda([l - 1]^{(3)})$.

In Peng et al. (Peng et al. x), an algorithm is proposed to check the validity of Conjecture 1.4 for 3-graphs and, as a demonstration, that algorithm confirms Conjecture 1.4 for some small $l$. For 3-graphs, the validity of Conjecture 1.4 for some small $l$ is verified in Peng and Zhao (2012) as well.

In Frankl and Füredi (1989), Frankl and Füredi applied the Lagrangians of related hypergraphs to estimate Turán densities of hypergraphs. They asked the following question: given $r \geq 3$ and $m \in \mathbb{N}$ how large can the Lagrangian of an $r$-graph with $m$ edges be? An answer to the above question would be quite useful in estimating Turán densities of hypergraphs.

The following definition is needed in order to state their conjecture on this problem. For distinct $A, B \in \mathbb{N}^{(r)}$, $A$ is less than $B$ in the colex ordering if $\max(A \triangle B) \in B$, where $A \triangle B = (A \setminus B) \cup (B \setminus A)$. For example, $246 < 156$ in $\mathbb{N}^{(3)}$ since $\max\{(2, 4, 6) \triangle \{1, 5, 6\}\} \in \{1, 5, 6\}$. In colex ordering, $123 < 124 < 134 < 234 < 125 < 135 < 235 < 145 < 245 < 345 < 126 < 136 < 236 < 136 < 246 < 346 < 125 < 256 < 356 < 456 < 127 < \cdots$. Note that the first $\binom{l}{r}$ $r$-tuples in the colex ordering of $\mathbb{N}^{(r)}$ are the edges of $[l]^{(r)}$. Let $C_{r,m}$ denote the $r$-graph with $m$ edges formed by taking the first $m$ elements in the colex ordering of $\mathbb{N}^{(r)}$.

Conjecture 1.6 (Frankl and Füredi 1989) The $r$-graph with $m$ edges formed by taking the first $m$ sets in the colex ordering of $\mathbb{N}^{(r)}$ has the largest Lagrangian of all $r$-graphs with $m$ edges. In particular, the $r$-graph with $\binom{l}{r}$ edges and the largest Lagrangian is $[l]^{(r)}$.

Theorem 1.2 implies that this conjecture is true when $r = 2$. For the case $r = 3$, Talbot in Talbot (2002) proved the following result.

Theorem 1.7 (Talbot 2002) Let $m$ and $l$ be positive integers satisfying

$$\left(\binom{l-1}{3}\right) \leq m \leq \left(\binom{l-1}{3}\right) + \left(\binom{l-2}{2}\right) - (l - 1).$$

Then Conjecture 1.6 is true for $r = 3$ and this value of $m$. Conjecture 1.6 is also true for $r = 3$ and $m = \binom{l}{3} - 1$ or $m = \binom{l}{3} - 2$.

For the case $r = 3$, Tang et al. (accepted) and proved the following.

Theorem 1.8 (Tang et al. accepted) Let $m$ and $t$ be integers. Then Conjecture 1.6 is true for $r = 3$ and $m = \binom{t}{3} - 3$, $m = \binom{t}{3} - 4$ or $m = \binom{t}{3} - 5$.

In He et al. (accepted), He, Peng, and Zhao verified Frankl and Füredi’s conjecture for $m \leq 50$ when $r = 3$.

The truth of Frankl and Füredi’s conjecture is not known in general for $r \geq 4$. Even in the case $r = 3$, it is still open.

The following result was given in Talbot (2002).
Lemma 1.9 Talbot (2002) For positive integers \( m, l, \) and \( r \) satisfying \( \binom{l-1}{r} \leq m \leq \binom{l-1}{r} + \binom{l-2}{r-1} \), we have \( \lambda(C_{r,m}) = \lambda([l-1]^{(r)}) \).

In Sect. 3, we provide some evidences for Conjectures 1.3 and 1.4. In addition to several other results, we will prove the following result in Sect. 3.

Theorem 1.10 (a) Let \( m \) and \( l \) be positive integers satisfying \( \binom{l-1}{r} \leq m \leq \binom{l-1}{r} + \binom{l-2}{r-1} \). Let \( G \) be an \( r \)-graph on \( l \) vertices with \( m \) edges and contain a clique of order \( l-1 \). Then \( \lambda(G) = \lambda([l-1]^{(r)}) \).

(b) Let \( m \) and \( l \) be positive integers satisfying \( \binom{l-1}{3} \leq m \leq \binom{l-1}{2} + \binom{l-2}{1} - (l-2) \). Let \( G \) be a 3-graph with \( m \) edges and without containing a clique of order \( l-1 \). Then \( \lambda(G) < \lambda([l-1]^{(3)}) \).

When \( r = 3 \), Theorem 1.10 (a) and Lemma 2.5 imply Theorem 1.5. Theorem 1.10 (b) and Theorem 1.5 refine Theorem 1.7.

The results in this paper provide evidence for Conjectures 1.3 and 1.4 and extend some known results in the literature (Theorem 1.5 in Talbot 2002 and Theorem 1.7 in Peng and Zhao 2013). If Conjectures 1.3 and 1.4 are true, then the long standing conjecture of Frankl and Füredi-Conjecture 1.6 is true for this range of \( m \). They can be applied in estimating Lagrangians of some hypergraphs, for example, results involving estimating Lagrangians of several hypergraphs in Proposition 4.3 in Frankl and Füredi (1989) would be implied directly. The main results also provide solutions to the optimization problem of a class of homogeneous multilinear functions over the standard simplex of the Euclidean space. Theorem 1.10 also gives a connection between a continuous optimization problem and the maximum clique number of \( r \)-uniform hypergraphs when the number of edges is in certain range.

Some preliminary results will be stated in the following section.

2 Preliminary results

For an \( r \)-graph \( G = (V, E) \) on the vertex set \([n]\) and \( i \in V \), let \( E_i = \{A \in V^{(r-1)} : A \cup \{i\} \in E\} \) be the link of the vertex \( i \). Similarly, for a pair of vertices \( i, j \in V \), let \( E_{ij} = \{B \in V^{(r-2)} : B \cup \{i, j\} \in E\} \). Let \( E_i^c = \{A \in V^{(r-1)} : A \cup \{i\} \in V^{(r)} \setminus E\} \), and \( E_{ij}^c = \{B \in V^{(r-2)} : B \cup \{i, j\} \in V^{(r)} \setminus E\} \). Denote

\[
E_{i \setminus j} = E_i \cap E_j^c.
\]

Let us impose one additional condition on any optimal weighting \( \bar{x} = (x_1, x_2, \ldots, x_n) \) for an \( r \)-graph \( G \):

\[
|\{i : x_i > 0\}| \text{ is minimal, i.e. if } \bar{y} \in S \text{ satisfies } |\{i : y_i > 0\}| < |\{i : x_i > 0\}|,
\]

then \( \lambda(G, \bar{y}) < \lambda(G) \).

Note that \( \lambda(E_i, \bar{x}) \) corresponds to the partial derivative of \( \lambda(G, \bar{x}) \) with respect to \( x_i \). The following lemma gives some necessary conditions of an optimal weighting of \( \lambda(G) \).
Lemma 2.1 (Frankl and Rödl 1984) Let \( G = (V, E) \) be an \( r \)-graph on the vertex set \([n]\) and \( \vec{x} = (x_1, x_2, \ldots, x_n) \) be an optimal weighting for \( G \) with \( k \) (\( \leq n \)) non-zero weights satisfying condition (1). Then for every \( \{i, j\} \in [k]^2 \), (a) \( \lambda(E_i, \vec{x}) = \lambda(E_j, \vec{x}) = r \lambda(G) \), (b) there is an edge in \( E \) containing both \( i \) and \( j \).

Definition 2.1 An \( r \)-graph \( G = (V, E) \) on the vertex set \([n]\) is left-compressed if \( j_1 j_2 \ldots j_r \in E \) implies \( i_1 i_2 \ldots i_r \in E \) whenever \( i_k \leq j_k \), \( 1 \leq k \leq r \). Equivalently, an \( r \)-graph \( G = (V, E) \) on the vertex set \([n]\) is left-compressed if \( E_{j \setminus i} = \emptyset \) for any \( 1 \leq i < j \leq n \).

Remark 2.2 (a) In Lemma 2.1, part (a) implies that
\[
x_j \lambda(E_{ij}, \vec{x}) + \lambda(E_{i \setminus j}, \vec{x}) = x_i \lambda(E_{ij}, \vec{x}) + \lambda(E_{j \setminus i}, \vec{x}).
\]
In particular, if \( G \) is left compressed, then
\[
(x_i - x_j) \lambda(E_{ij}, \vec{x}) = \lambda(E_{i \setminus j}, \vec{x}) \tag{2}
\]
for any \( i, j \) satisfying \( 1 \leq i < j \leq k \) since \( E_{j \setminus i} = \emptyset \).

(b) By (2), if \( G \) is left-compressed, then an optimal weighting \( \vec{x} = (x_1, x_2, \ldots, x_n) \) for \( G \) must satisfy
\[
x_1 \geq x_2 \geq \ldots \geq x_n \geq 0. \tag{3}
\]

Denote
\[
\lambda^r_m = \max \{\lambda(G) : G \text{ is an } r - \text{ graph with } m \text{ edges}\}, \quad \lambda^r_{(m, l)} = \max \{\lambda(G) : G \text{ is an } r - \text{ graph with } m \text{ edges and contains a clique of order } l\}, \quad \text{and } \lambda^r_{(m, l)}^- = \max \{\lambda(G) : G \text{ is an } r - \text{ graph with } m \text{ edges and without a clique of order } l\}.
\]

The following two lemmas imply that we only need to consider left-compressed \( r \)-graphs when Conjecture 1.6 and Conjecture 1.3 are explored.

Lemma 2.3 Talbot (2002) There exists a left compressed \( r \)-graph \( G \) with \( m \) edges such that \( \lambda(G) = \lambda^r_m \).

Lemma 2.4 (Peng and Zhao 2013) Let \( m \) and \( l \) be positive integers satisfying \( m \leq \binom{l}{r} - 1 \). Then there exists a left compressed \( r \)-graph \( G \) containing the clique \([l - 1]^{(r)}\) with \( m \) edges such that \( \lambda(G) = \lambda^r_{(m, l - 1)} \).

When Conjectures 1.3 and 1.4 were discussed for \( r = 3 \) in Peng and Zhao (2013) and Peng et al. (accepted), the following results were proved.

Lemma 2.5 (Peng and Zhao 2013) Let \( m \) and \( l \) be positive integers satisfying \( \binom{l - 1}{3} \leq m \leq \binom{l - 1}{3} + \binom{l - 2}{2} \). Then there exists a left compressed 3-graph \( G \) on the vertex set \([l]\) with \( m \) edges and containing the clique \([l - 1]^{(3)}\) such that \( \lambda(G) = \lambda^3_{(m, l - 1)} \).
Lemma 2.6 Peng et al. (accepted) Let \( m \) and \( l \) be positive integers satisfying \( \binom{l-1}{3} \leq m \leq \binom{l-1}{3} + \binom{l-2}{2} \). Then there exists a left compressed 3-graph \( G \) on the vertex set \([l]\) with \( m \) edges and without containing the clique \([l-1]^{(3)}\) such that \( \lambda(G) = \lambda_{3_{m,l-1}} \).

3 Evidence for Conjectures 1.3 and 1.4

Frank and Füredi Frankl and Füredi (1989) originally asked how large the Lagrangian of an \( r \)-graph with \( l \) vertices and \( m \) edges can be, where \( m \leq \binom{l}{r} \). For a given \( r \)-graph with \( l \) vertices and \( m \) edges, let

\[
\lambda(l, r, m) = \max \{ \lambda(G) : G = (V, E) \text{ is an r-graph, } |V| = l, |E| = m \}.
\]

In Talbot (2002), the following result is proved, which is the evidence for Conjecture 1.3 for \( r \)-graphs \( G \) on exactly \( l \) vertices.

Theorem 3.1 (Talbot 2002) For any \( r \geq 4 \) there exists constants \( \gamma_r \) and \( \kappa_0(r) \) such that if \( m \) satisfies

\[
\binom{l-1}{r} \leq m \leq \binom{l-2}{r} - \gamma_r (l-1)^{r-2},
\]

with \( l \geq \kappa_0(r) \), then \( \lambda(l, r, m) = \lambda(C_{r,m}) = \lambda([l-1]^{(r)}) \).

In Tang et al. (accepted), we proved:

Theorem 3.2 (Tang et al. accepted) Let \( m \) and \( l \) be positive integers satisfying \( \binom{l}{r} - 4 \leq m \leq \binom{l}{r} - 1 \). Then the \( r \)-graph with \( m \) edges formed by taking the first \( m \) sets in the colex ordering of \( \mathbb{N}^{(r)} \) has the largest Lagrangian of all \( r \)-graphs with \( m \) edges and \( l \) vertices.

Next, we point out a useful lemma.

Lemma 3.3 Let \( G \) be a left-compressed \( r \)-graph on the vertex set \([l]\) containing the clique \([l-1]^{(r)}\). Let \( \tilde{x} = (x_1, x_2, \ldots, x_l) \) be an optimal weighting for \( G \). Then

\[
x_1 \leq x_{l-1} + x_l \leq 2x_{l-1}.
\]

Proof Note that \( x_{l-1} > 0 \). Otherwise \( \lambda(G, \tilde{x}) \leq \lambda([l-2]^{(r)}) \) contradicting to that \( \tilde{x} \) is an optimal weighting for \( G \). Since \( G \) is left compressed, applying Remark 2.2(a) by taking \( i = 1 \), \( j = l-1 \), we get

\[
x_1 = x_{l-1} + \frac{\lambda(E_1 \setminus (l-1), \tilde{x})}{\lambda(E_1(l-1), \tilde{x})}.
\]

Since \( G \) contains the clique \([l-1]^{(r)}\), then any \( (r-1) \)-tuple in \( E_1 \setminus (l-1) \) must contain \( l \) but not \( 1 \) or \( l-1 \). Therefore
We will show that there exists a set of edges $F$ such that
\[
\lambda(E_1 \setminus (l-1), \bar{x}) \leq \sum_{2 \leq i_1 < i_2 < \cdots < i_{r-2} \leq l-2} x_{i_1} x_{i_2} \cdots x_{i_{r-2}} x_l. \tag{6}
\]

Since $G$ contains the clique $[l-1](r)$, then every $(r-2)$-tuple in $[2, 3, \ldots, l-2](r-2)$ belongs to $E_1(l-1)$, then
\[
\lambda(E_1(l-1), \bar{x}) \geq \sum_{2 \leq i_1 < i_2 < \cdots < i_{r-2} \leq l-2} x_{i_1} x_{i_2} \cdots x_{i_{r-2}}. \tag{7}
\]

Combining inequalities (6) and (7), we get
\[
\frac{\lambda(E_1(l-1), \bar{x})}{\lambda(E_1(l-1), \bar{x})} \leq x_l. \tag{8}
\]

Applying inequality (8) to (5), we get (4).

Note that the only left-compressed $r$-graph on the vertex set $[r + 1]$ is $C_{r,r+1}$. So we assume that an $r$-graph has at least $r + 2$ vertices in this paper.

Next we give some results refining Theorem 1.7 when $r = 3$.

**Theorem 3.4** Let $r \geq 3$ and $l \geq r + 2$ be positive integers. Let $G$ be a left-compressed $r$-graph on the vertex set $[l]$ satisfying $|[l-2](r-1)\setminus E_l| \geq 2^{r-3} |E(l-1)_l|$.

(a) If $G$ contains $[l-1](r)$, then $\lambda(G) = \lambda([l-1](r))$.

(b) If $G$ does not contain $[l-1](r)$, then $\lambda(G) < \lambda([l-1](r))$.

**Proof** (a) If $G$ contains $[l-1](r)$, then clearly $\lambda(G) \geq \lambda([l-1](r))$. We show that $\lambda(G) = \lambda([l-1](r))$ as well. Let $\bar{x} = (x_1, x_2, \ldots, x_l)$ be an optimal weighting for $G$. Since $G$ is left-compressed, by Remark 2.2(a), $x_1 \geq x_2 \geq \cdots \geq x_l \geq 0$. If $x_l = 0$, then the conclusion holds obviously, so we assume that $x_l > 0$.

Consider a new weighting for $G$, $\bar{z} = (z_1, z_2, \ldots, z_l)$ given by $z_i = x_i$ for $i \neq l-1, l$, $z_{l-1} = 0$ and $z_l = x_{l-1} + x_l$. By Lemma 2.1(a), $\lambda(E_{l-1}, \bar{x}) = \lambda(E_l, \bar{x})$, so
\[
\lambda(G, \bar{z}) - \lambda(G, \bar{x}) = x_{l-1} (\lambda(E_l, \bar{x}) - x_l \lambda(E_{l-1} \setminus l, \bar{x})) - x_{l-1} (\lambda(E_{l-1} \setminus l, \bar{x}) - x_l \lambda(E_{l-1} \setminus l, \bar{x})) - x_{l-1} x_l (\lambda(E_{l-1} \setminus l, \bar{x}))
= x_{l-1} (\lambda(E_l, \bar{x}) - \lambda(E_{l-1} \setminus l, \bar{x})) - x_{l-1} x_l (\lambda(E_{l-1} \setminus l, \bar{x})) + x_{l-1} \lambda(E_{l-1} \setminus l, \bar{x})
= -x_{l-1} x_l (\lambda(E_{l-1} \setminus l, \bar{x})) + x_{l-1} \lambda(E_{l-1} \setminus l, \bar{x}). \tag{9}
\]

We will show that there exists a set of edges $F \subset [1, \ldots, l-2, l](r) \setminus E$ satisfying
\[
\lambda(F, \bar{z}) \geq x_{l-1}^2 \lambda(E_{l-1} \setminus l, \bar{x}). \tag{10}
\]

Then using (9) and (10), the $r$-graph $G^* = ([l], E^*)$, where $E^* = E \cup F$, satisfies $\lambda(G^*, \bar{z}) \geq \lambda(G)$. Since $\bar{z}$ has only $l-1$ positive weights, then $\lambda(G^*, \bar{z}) \leq \lambda([l-1](r))$, and consequently, $\lambda(G) \leq \lambda([l-1](r))$. 

\(\square\) Springer
We now construct the set of edges $F$. Let $D = [l - 2]^{(r - 1)} \setminus E_l$. Then by the assumption, $|D| \geq 2^{r - 3} |E_{(l-1)}|!$ and $\lambda(D, \bar{x}) \geq 2^{r - 3} |E_{(l-1)}|! (x_{l-1})^{r-1}$.

Let $F$ consist of those edges in $\{1, ..., l - 2, l\}^{(r)} \setminus E$ containing the vertex $l$. Then

$$\lambda(F, \bar{z}) = (x_l - 1 + x_l) \lambda(D, \bar{x})$$

$$\geq x_l 2^{r-3} |E_{(l-1)}|! (x_{l-1})^{r-1}$$

$$= x_l 2^{r-3} |E_{(l-1)}|! x_l (2^{r-3} (x_{l-1})^{r-3})$$

$$\geq x_l 2^{r-3} |E_{(l-1)}|! x_1^{r-2} \text{ by (4)}$$

$$\geq x_l 2^{r-3} \sum_{i_1 i_2 \ldots i_{r-2} \in E_{(l-1)}!} x_i x_j \ldots x_{i_{r-2}}$$

$$= x_l 2^{r-3} \lambda(E_{(l-1)}|, \bar{x}). \quad (11)$$

This proves part (a).

(b) Let $\bar{x} = (x_1, x_2, \ldots, x_l)$ be an optimal weighting for $G$. Since $G$ is left-compressed, by Remark 2.2(a), $x_1 \geq x_2 \geq \cdots \geq x_l \geq 0$. If $x_{l-1} = 0$, since $G$ does not contain the clique $[l - 1]^{(r)}$, then the conclusion holds obviously, so we assume that $x_{l-1} > 0$. We add all edges in $[l - 1]^{r} - E(G)$ to $G$ and get a new $r$-graph $H$. Observe that the new $r$-graph $H$ is still left-compressed, satisfies $||l - 2|^{(r - 1)} \setminus E_l| \geq 2^{r - 3} |E_{(l-1)}|!$ and contains the clique $[l - 1]^{(r)}$. So by part (a), $\lambda(H) = \lambda([l - 1]^{(r)})$. On the other hand,

$$\lambda(G) = \lambda(G, \bar{x}) < \lambda(H, \bar{x}) \leq \lambda(H).$$

Therefore, $\lambda(G) < \lambda([l - 1]^{(r)})$. This proves part (b). \qed

We are now ready to prove Theorem 1.10.

**Proof of Theorem 1.10** (a) Let $m$ and $l$ be positive integers satisfying $\binom{l-1}{r} \leq m \leq \binom{l-1}{r} + \binom{l-2}{r-1} - (2^{r-3} - 1) \binom{l-2}{r-2} - 1$. Let $G$ be an $r$-graph on $l$ vertices with $m$ edges and a clique of order $l - 1$ such that $\lambda(G) = \lambda^{(r)}(m, l - 1)$. Applying Lemma 2.4, we can assume that $G$ is left-compressed and contains the clique $[l - 1]^{(r)}$. By Theorem 3.4, it is sufficient to show that $||l - 2|^{(r - 1)} \setminus E_l| \geq 2^{r - 3} |E_{(l-1)}|!$. If not, then $||l - 2|^{(r - 1)} \setminus E_l| < 2^{r - 3} |E_{(l-1)}|!$. Since $G$ contains the clique $[l - 1]^{(r)}$, then

$$m = \binom{l - 1}{r} + \binom{l - 2}{r - 1} - ||l - 2|^{(r - 1)} \setminus E_l| + |E_{(l-1)}|!$$

$$> \binom{l - 1}{r} + \binom{l - 2}{r - 1} - (2^{r - 3} - 1) |E_{(l-1)}|!$$

$$\geq \binom{l - 1}{r} + \binom{l - 2}{r - 1} - (2^{r - 3} - 1) \left( \binom{l - 2}{r - 2} - 1 \right)$$

since $|E_{(l-1)}|! \leq \binom{l-2}{r-2} - 1$. (If $|E_{(l-1)}|! = \binom{l-2}{r-2}$, then $E = [l]^{(r)}$ since $G$ is left-compressed and $m = \binom{l}{r}$ which is a contradiction). This proves part (a) of Theorem 1.10.
(b) Let $G$ be a 3-graph with $m$ edges without containing a clique of order $l - 1$ such that $\lambda(G) = \lambda_{(m,l-1)}^3$. Then by Lemma 2.6, we can assume that $G$ is left-compressed with vertex set $[l]$. Let $\bar{x} = (x_1, x_2, \ldots, x_l)$ be an optimal weighting of $G$ satisfying $x_1 \geq x_2 \geq \cdots \geq x_l$. If $x_l = 0$, then $\lambda(G) < \lambda([l - 1]^{(3)})$ and the conclusion follows. So we assume that $x_l > 0$. Now we will use the following result which is proved in Peng et al. (accepted)

**Lemma 3.5** (see Peng et al. (accepted)) Let $m$ and $l$ be positive integers satisfying $(l^{-1}) \leq m \leq (l^{-1}) + (l^{-2}) - (l - 2)$. Let $G$ be a left-compressed 3-graph on the vertex set $[l]$ with $m$ edges and without containing a clique of order $l - 1$ such that $\lambda(G) = \lambda_{(m,l-1)}^3$. Let $\bar{x}$ be an optimal weighting for $G$ with $l$ positive weights. Then $\lambda(G) < \lambda([l-1]^{(3)})$ or

$$|[l-1]^{(3)} \setminus E| \leq l - 2.$$ 

Let $H$ be obtained by adding all triples in $[l-1]^{(3)} \setminus E(G)$ to $G$. By Lemma 3.5, there are at most $l - 2$ such triples. Therefore, $H$ is a 3-graph with at most $(l^{-1}) + (l^{-2})$ edges and containing $[l-1]^{(3)}$. By Theorem 1.5, $\lambda(H) = \lambda([l-1]^{(3)})$. Since each $x_l > 0$ and $G$ does not contain the clique $[l-1]^{(3)}$, then

$$\lambda(G) = \lambda(G, \bar{x}) < \lambda(H, \bar{x}) \leq \lambda(H) = \lambda([l-1]^{(3)})$$

which proves part (b) of Theorem 1.10. \hfill \Box

**Theorem 3.6** Let $l$ be a positive integer. Let $G$ be a left-compressed $r$-graph on the vertex set $[l]$ such that there is a one-to-one function $f$ from $E([l-1])$ to $[l - 2]^{(r-1)} \setminus E_l$ satisfying the condition: for $i_1 i_2 \cdots i_{r-2} \in E([l-1])$, $f(i_1 i_2 \cdots i_{r-2}) = j_1 j_2 \cdots j_{r-1}$ satisfies $j_k \leq i_{k+1}$ for all $1 \leq k \leq r - 3$, where $i_1 \leq i_2 \leq \cdots \leq i_{r-2}$ and $j_1 \leq j_2 \leq \cdots \leq j_{r-1}$.

(a) If $G$ contains $[l-1]^{(r)}$, then $\lambda(G) = \lambda([l-1]^{(r)})$.

(b) If $G$ does not contain $[l-1]^{(r)}$, then $\lambda(G) < \lambda([l-1]^{(r)})$.

**Proof** (a) Let $G$ contain the clique $[l-1]^{(r)}$.

Let $\bar{x} = (x_1, x_2, \ldots, x_l)$ be an optimal weighting for $G$. Now we proceed to show that $\lambda(G) \leq \lambda([l-1]^{(r)})$.

Consider a new weighting for $G$, $\bar{z} = (z_1, z_2, \ldots, z_l)$ given by $z_i = x_i$ for $i \neq l - 1, l$, $z_{l-1} = 0$ and $z_l = x_{l-1} + x_l$. By Lemma 2.1(a), $\lambda(E_{l-1}, \bar{x}) = \lambda(E_l, \bar{x})$, so

$$\lambda(G, \bar{z}) - \lambda(G, \bar{x}) = x_{l-1} \lambda(E_l, \bar{x}) - x_{l-1} \lambda(E_{l-1}, \bar{x})$$

$$= -x_{l-1} \lambda(E_{l-1}, \bar{x}) - x_l \lambda(E_{l-1}, \bar{x}) - x_{l-1} x_l \lambda(E_{l-1}, \bar{x})$$

$$= x_{l-1} \lambda(E_l, \bar{x}) - \lambda(E_{l-1}, \bar{x}) - x_{l-1}^2 \lambda(E_{l-1}, \bar{x})$$

$$= -x_{l-1}^2 \lambda(E_{l-1}, \bar{x}). \quad (12)$$

We will show that there exists a set of edges $F \subset \{1, \ldots, l - 2, l\}^{(r)} \setminus E$ satisfying

$$\lambda(F, \bar{z}) \geq x_{l-1}^2 \lambda(E_{l-1}, \bar{x}). \quad (13)$$

\hfill \Box
Then using (12) and (13), the r-graph \( G^* = ([k], E^*) \), where \( E^* = E \cup F \), satisfies \( \lambda(G^*, \vec{z}) \geq \lambda(G) \). Since \( \vec{z} \) has only \( l - 1 \) positive weights, then \( \lambda(G^*, \vec{z}) \leq \lambda([l - 1]^r) \), and consequently,

\[
\lambda(G) \leq \lambda([l - 1]^r).
\]

Let \( F \) consist of all \( j_1 j_2 \cdots j_{r-1} l \), where \( j_1 j_2 \cdots j_{r-1} \in f(E([l-1]) \). Then

\[
\lambda(F, \vec{z}) = (x_{j-1} + x_l) \sum_{j_1 j_2 \cdots j_{r-1} \in f(E([l-1])} x_{j_1} \cdots x_{j_{r-1}}.
\]

Recall that \( f \) is a one-to-one function and for each element \( i_1 i_2 \cdots i_{r-2} \) of \( E([l-1]) \), \( f(i_1 i_2 \cdots i_{r-2}) = j_1 j_2 \cdots j_{r-1} \) satisfies \( j_k \leq i_{k+1} \) for all \( 1 \leq k \leq r - 3 \). Combining with Lemma 3.3, we get

\[
\lambda(F, \vec{z}) \geq x_{j-1}^2 \sum_{i_1 i_2 \cdots i_{r-2} \in E([l-1])} x_{i_1} x_{i_2} \cdots x_{i_{r-2}} \\
\geq x_{j-1}^2 \sum_{i_1 i_2 \cdots i_{r-2} \in E([l-1])} x_{i_1} x_{i_2} \cdots x_{i_{r-2}} \\
= x_{j-1}^2 \lambda(E([l-1]) \vec{x}).
\]

(14)

This proves (a).

(b) Let \( \vec{x} = (x_1, x_2, \ldots, x_l) \) be an optimal weighting for \( G \). Since \( G \) is left-compressed, by Remark 2.2(a), \( x_1 \geq x_2 \geq \cdots \geq x_l \geq 0 \). If \( x_{l-1} = 0 \), since \( G \) does not contain the clique \([l - 1]^r\), then the conclusion holds obviously, so we assume that \( x_{l-1} > 0 \). We add all edges in \([l - 1]^r - E(G)\) to \( G \) and get a new r-graph \( H \). Observe that the new r-graph \( H \) is still left-compressed, contains the clique \([l - 1]^r\), and still satisfies the condition that there is a one-to-one function \( f \) from \( E([l-1]) \) to \([l - 2]^r \setminus E_l \) such that for \( i_1 i_2 \cdots i_{r-2} \in E([l-1])\), \( f(i_1 i_2 \cdots i_{r-2}) = j_1 j_2 \cdots j_{r-1} \) satisfies \( j_k \leq i_{k+1} \) for all \( 1 \leq k \leq r - 3 \). So by part (a), \( \lambda(H) = \lambda([l - 1]^r) \). On the other hand,

\[
\lambda(G) = \lambda(G, \vec{x}) < \lambda(H, \vec{x}) \leq \lambda(H).
\]

Therefore, \( \lambda(G) < \lambda([l - 1]^r) \). This proves part (b). \( \square \)

**Remark 3.7** When \( r = 3 \), Theorem 3.6(a) implies Theorem 1.5 and Theorem 3.6(b) implies Theorem 1.10(b).

**Proof** Let \( G \) be a 3-graph with \( m \) edges containing a clique of order \( l - 1 \), where \( \binom{l - 1}{3} \leq m \leq \binom{l - 1}{3} + \binom{2}{2} \). By Lemma 2.5, we can assume that \( G \) is left-compressed and on the vertex set \([l]\). If \( |E([l-1])| > |[l - 2]^2 \setminus E_l| \), then

\[
m \geq \binom{l - 1}{3} + \binom{l - 2}{2} - |[l - 2]^2 \setminus E_l| + |E([l-1])| > \binom{l - 1}{3} + \binom{l - 2}{2}
\]

\( \square \) Springer
which is a contradiction. So \(|E_{(l-1)l}| \leq |(l-2)\backslash E_i|\). Therefore, there is a one-to-one function \(f\) from \(E_{(l-1)l}\) to \(|(l-2)\backslash E_i|\) and \(f\) automatically satisfies the condition in Theorem 3.6. Applying Theorem 3.6, we have \(\lambda(G) = \lambda([l-1]^{(3)})\).

Applying Lemma 3.5, we can similarly show that Theorem 3.6(b) implies Theorem 1.10(b).

In our results below, note that it does not matter how many vertices we are allowed to use.

**Theorem 3.8** Let \(G\) be an \(r\)-graph containing a clique of order \(l - 1\) with \(m\) edges. If \(m \leq \binom{l-1}{r} + 2(l - r)\), then \(\lambda(G) = \lambda([l-1]^{(r)})\).

**Proof** Let \(G\) be an \(r\)-graph containing a clique of order \(l - 1\) with \(m\) edges such that \(\lambda(G) = \lambda_{(m,l-1)}^{(r)}\). Clearly \(\lambda(G) \geq \lambda([l-1]^{(r)})\). Next we show that \(\lambda(G) \leq \lambda([l-1]^{(r)})\). Since \(\lambda_{(m,l-1)}^{(r)}\) does not decrease as \(m\) increases, it is sufficient to show the case that \(m = \binom{l-1}{r} + 2(l - r)\). Based on Lemma 2.4, we may assume that \(G\) is left compressed and the optimal weighting \(\bar{x} = (x_1, x_2, \ldots, x_n)\) of \(G\) satisfying \(x_i \geq x_j\) when \(i < j\). Note that \(x_{l+1} = 0\). Otherwise, then by Lemma 2.1, \(G\) contains edge \(12\ldots(r-2)\) for all \(i\), where \(r - 1 \leq i \leq l\) and \(12\ldots(r-2)\) for all \(j\), where \(r - 1 \leq j \leq l - 1\). Then \(m \geq \binom{l-1}{r} + 2(l - r) + 3\) which is a contradiction. If \(x_l = 0\), then \(\lambda(G) \leq \lambda([l-1]^{(r)})\) and we are done. If \(x_l > 0\), then by Lemma 2.1, \(G\) contains edge \(12\ldots(r-2)(l-1)\), it should contain all edges \(12\ldots(r-2)il\) for all \(i\), where \(r - 1 \leq i \leq l - 1\). Note that \(E_{(l-1)l} = \{12\ldots(r-2)\}\). Otherwise, \(12\ldots(r-3)(r-1)(l-1)il \in E\) and \(G\) contains all edges \(12\ldots(r-3)(r-1)il\) for all \(i\), where \(r \leq i \leq l - 1\). So \(m \geq \binom{l-1}{r} + 2(l - r) + 1\) which is a contradiction. So we can assume that \(G\) is on the vertex set \([l]\) and \(E_{(l-1)l} = \{12\ldots(r-2)\}\).

Consider a new weighting for \(G\), \(\tilde{z} = (z_1, z_2, \ldots, z_l)\) given by \(z_i = x_i\) for \(i \neq l - 1, l\), \(z_{l-1} = 0\) and \(z_l = x_{l-1} + x_l\). By Lemma 2.1(a), \(\lambda(E_{l-1}, \tilde{x}) = \lambda(E_l, \tilde{x})\), so

\[
\lambda(G, \tilde{z}) - \lambda(G, \tilde{x}) = x_{l-1}(\lambda(E_l, \tilde{x}) - \lambda(E_{l-1}, \tilde{x})) - x_{l-1}^2 \lambda(E_{(l-1)l}, \tilde{x})
\]

\[
= -x_{l-1}^2 \lambda(E_{(l-1)l}, \tilde{x})
\]

\[
= -x_{l-1}^2 x_1 x_2 \cdots x_{r-2}. \tag{15}
\]

Let \(F\) consist of those edges in \(\{1, \ldots, l - 2, l\}^{(r)}\) \(\backslash E\) containing the vertex \(l\). Then clearly

\[
\lambda(F, \tilde{z}) \geq (x_{l-1} + x_l)x_2 \cdots x_{r-2}x_{l-1}^2 \tag{16}
\]

since \(23\ldots(r-2)(r-1)(l-2)l\) is in \(F\). Otherwise, \(23\ldots(r-2)(r-1)(l-2)l \in E\). Since \(G\) is left compressed, then \(23\ldots(r-2)(r-1)il \in E\) for all \(r \leq i \leq l - 2\), \(12\ldots(r-2)jl \in E\) for all \(r - 1 \leq j \leq l - 2\) and \(13\ldots(r-1)kl \in E\) for all \(r \leq k \leq l - 2\). Recall that \(12\ldots(r-2)il \in E\) for all \(i\), where \(r - 1 \leq i \leq l - 1\). Then \(m \geq \binom{l-1}{r} + 3(l - r) - 1 \geq \binom{l-1}{r} + 2(l - r) + 1\) which is a contradiction.
By Lemma 3.3, we have $x_1 \leq x_{l-1} + x_l$. Applying this to (16), we get

$$\lambda(F, \vec{z}) \geq x_{l-1}^2 x_1 x_2 \cdots x_{r-2}. \quad (17)$$

Then using (15) and (17), the $r$-graph $G^* = ([l], E^*)$, where $E^* = E \cup F$, satisfies $\lambda(G^*, \vec{z}) \geq \lambda(G)$. Since $\vec{z}$ has only $l - 1$ positive weights, then $\lambda(G^*, \vec{z}) \leq \lambda([l - 1]^{(r)})$, and consequently,

$$\lambda(G) \leq \lambda([l - 1]^{(r)}).$$

This completes the proof. $\Box$

4 Concluding remarks

As we have seen in Sect. 3, Lemmas 2.5, 2.6, and Theorem 3.4 for the case $r = 3$ refine Theorem 1.7. If one can have some results similar to Lemmas 2.5 and 2.6 for general $r$, then one can get results similar to Theorem 1.7 for general $r$.

We also remark that in some applications, estimating $\lambda(l, r, m)$ is sufficient. So Theorems 1.10 and 3.1 might still be applicable in some situations though Conjectures 1.3 and 1.4 cannot be verified in general at this moment.

Acknowledgments We thank an anonymous referee and the editor for helpful and insightful comments.

References

Budinich M (2003) Exact bounds on the order of the maximum clique of a graph. Discrete Appl Math 127:535–543

Busygin S (2006) A new trust region technique for the maximum weight clique problem. Discrete Appl Math 304(4):2080–2096

Frankl P, Füredi Z (1989) Extremal problems whose solutions are the blow-ups of the small Witt-designs. J Comb Theory (A) 52:129–147

Frankl P, Rödl V (1984) Hypergraphs do not jump. Combinatorica 4:149–159

Gibbons LE, Hearn DW, Pardalos PM, Ramana MV (1997) Continuous characterizations of the maximum clique problem. Math Oper Res 22:754–768

He G, Peng Y, Zhao C, On finding Lagrangians of 3-uniform hypergraphs, Ars Combinatoria (accepted)

Motzkin TS, Straus EG (1965) Maxima for graphs and a new proof of a theorem of Turán. Can J Math 17:533–540

Mubayi D (2006) A hypergraph extension of Turan’s theorem. J Comb Theory B 96:122–134

Pavan M, Pelillo M (2003) Generalizing the motzkin-stras theorem to edge-weighted graphs, with applications to image segmentation. Lect Notes Comput Sci 2683:485–500

Pardalos PM, Phillips AT (1990) A global optimization approach for solving the maximum clique problem. Int J Comput Math 33:209–216

Peng Y, Zhao C (2013) A Motzkin–Straus type result for 3-uniform hypergraphs. Graphs Comb 29:681–694

Peng Y, Zhao C (2012) On Lagrangians of hypergraphs and cliques. Recent Adv Comput Sci Inf Eng 125:7–12

Peng Y, Zhu HG, Zheng Y, Zhao C On cliques and Lagrangians of 3-uniform hypergraphs. http://arxiv.org/abs/1211.6508

Rota Bulò S, Pelillo M (2008) A continuous characterization of maximal cliques in k-uniform hypergraphs. Learn Intellig Optim 5313:220–233
Rota Buló S, Pelillo M (2009) A generalization of the Motzkin–Straus theorem to hypergraphs. Optim Lett 3(2):287–295
Rota Buló S, Torsello A, Pelillo M (2007) A continuous-based approach for partial clique enumeration. Graph-Based Represent Patt Recogn 4538:61–70
Sidorenko AF (1987) Solution of a problem of Bollobás on 4-graphs. Mat Zametki 41:433–455
Sós V, Straus EG (1982) Extremal of functions on graphs with applications to graphs and hypergraphs. J Comb Theory B 63:189–207
Talbot J (2002) Lagrangians of hypergraphs. Comb Probab Comput 11:199–216
Tang QS, Peng YJ, Zhang XD, Zhao C Some results on Lagrangians of hypergraphs. Discrete Appl Math (accepted)
Tang QS, Peng YJ, Zhang XD, Zhao C On Frankl and Füredi’s conjecture for 3-uniform hypergraphs. http://arxiv.org/abs/1211.7056
Turán P (1941) On an extremal problem in graph theory. Mat Fiz Lapok 48:436–452 (in Hungarian)
Wilf HS (1986) Spectral bounds for the clique and independence number of graphs. J Comb Theory B 40:113–117