Symplectic Three-Algebra Unifying $\mathcal{N} = 5, 6$
Superconformal Chern-Simons-Matter
Theories

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Abstract

We define a 3-algebra with structure constants being symmetric in the first two indices. We also introduce an invariant anti-symmetric tensor into this 3-algebra and call it a symplectic 3-algebra. The general $\mathcal{N} = 5$ superconformal Chern-Simons-matter (CSM) theory with $SO(5)$ R-symmetry in three dimensions is constructed by using this algebraic structure. We demonstrate that the supersymmetry can be enhanced to $\mathcal{N} = 6$ if the symplectic 3-algebra and the fields are decomposed in a proper fashion. By specifying the 3-brackets, some presently known $\mathcal{N} = 5, 6$ superconformal theories are described in terms of this unified 3-algebraic framework. These include the $\mathcal{N} = 5, Sp(2N) \times O(M)$ CSM theory with $SO(5)$ R-symmetry , the $\mathcal{N} = 6, Sp(2N) \times U(1)$ CSM theory with $SU(4)$ R-symmetry, as well as the ABJM theory as a special case of $U(M) \times U(N)$ theory with $SU(4)$ R-symmetry.
1 Introduction

Recently, Chern-Simons-matter (CSM) theories with extended supersymmetries in three dimensions have attracted a lot of interests, because they are natural candidates of the dual gauge descriptions of M2 branes. About 20 years ago, generic Chern-Simons gauge theories (with or without matter) in 3D were demonstrated to be conformally invariant at the quantum level [1], [2], [3], [4], [5]. However, to describe M2 branes, one needs to further introduce (extended) supersymmetries into the CSM theories.

The $\mathcal{N} = 8$ CSM theory in $D = 3$ with $SO(4)$ gauge group was first constructed independently by Bagger and Lambert [9] and by Gustavsson [10] (BLG), in terms of the totally anti-symmetric Nambu 3-brackets [6, 7, 8]. The BLG model is known to be the dual gauge description of two M2 branes [14, 15, 16]. The Nambu 3-algebra equipped with a symmetric and positive-definite metric is essentially unique [17, 18]: It generates only an $SO(4)$ gauge symmetry. If, in place of the symmetric and positive-definite metric, one introduce a Lorentzian metric, then the so-called Lorentzian 3-algebra can be used to generate an arbitrary gauge group [19]. However, it was shown that the BLG model constructed from a Lorentzian 3-algebras is actually an $\mathcal{N} = 8$ super Yang-Mills theory [20, 21], not a CSM theory.

A little later, Aharony, Bergman, Jafferis and Maldacena (ABJM) have been able to construct an $\mathcal{N} = 6$ superconformal CSM theory with gauge group $U(N) \times U(N)$ and $SU(4)$ R-symmetry [22]. In their construction, the Nambu 3-brackets did not play any role. At level $k$, it has been argued that the ABJM theory describes the low energy limit of $N$ M2-branes probing a $C^4/Z_k$ singularity. As $k = 1, 2$, the supersymmetry is enhanced to $\mathcal{N} = 8$ [11, 12, 13]. In large-$N$ limit, the ABJM theory becomes the dual gauge theory of M theory on $AdS_4 \times S^7/Z_k$ [22]. Some further investigation of the ABJM theory can be found in Ref. [23, 25]. In Ref. [26, 27], it has been argued that one can also obtain the superconformal gauge theories with more or less supersymmetries by taking a conformal limit of $D = 3$ gauged supergravity theories. Using super Lie algebras to classify the gauge groups, Gaiotto and Witten (GW) have been able to construct a large class of $\mathcal{N} = 4$ CSM theories [28]. The GW theories are extended to include additional twisted hyper-multiplets [29, 30]. By generalizing Gaiotto and Witten’s construction, two new theories, $\mathcal{N} = 5$, $Sp(2M) \times O(N)$ and $\mathcal{N} = 6$, $Sp(2M) \times O(2)$ CSM theories, were constructed, and the ABJM theory was re-derived as a special case of $U(M) \times U(N)$ CSM theories [30]. The M theory and string theory
dualities of $\mathcal{N} = 5, Sp(2M) \times O(N)$ and $\mathcal{N} = 6, U(M) \times U(N)$ were studied in Ref. [31].

In an interesting paper, Bagger and Lambert (BL) have been able to construct the $\mathcal{N} = 6$ ABJM theory in a modified 3-algebra approach, in which the structure constants are antisymmetric only in the first two indices [32]. By introducing an anti-symmetric tensor into a 3-algebra (symplectic 3-algebra), the authors have constructed another class of $\mathcal{N} = 6$ CSM theories: the ones with gauge group $Sp(2M) \times O(2)$ [33]. Encouraged by the successes, it is natural to ask whether $\mathcal{N} = 5$ CSM theories can be constructed in terms of a 3-algebra or not. Furthermore, it is also natural to ask whether all $\mathcal{N} = 5, 6$ CSM theories can be constructed by a unified 3-algebraic framework. In this paper we will propose to solve these two problems.

In section 3, we define a symplectic 3-algebra in which the structure constants of the 3-brackets are symmetric in the first two indices. The general $\mathcal{N} = 5$ superconformal Chern-Simons-matter (CSM) theory with $SO(5)$ R-symmetry in three dimensions is constructed in terms of this symplectic 3-algebra. In section 3.2, we provide the $\mathcal{N} = 5, Sp(2N) \times O(M)$ CSM theory as an example by specifying the 3-brackets. In section 4, we demonstrate that the supersymmetry can be enhanced to $\mathcal{N} = 6$ by decomposing the symplectic 3-algebra and the fields properly, and the FI and the symmetry and reality properties of the structure constants of the $\mathcal{N} = 6$ 3-algebra can be derived from their $\mathcal{N} = 5$ counterparts. Therefore all $\mathcal{N} = 5, 6$ superconformal CSM theories are described by a unified (symplectic) 3-algebraic framework. By specifying the 3-brackets, the $\mathcal{N} = 6$, $Sp(2N) \times U(1)$ and $U(M) \times U(N)$ CSM are derived in section 4.2 and 4.3, respectively. Especially, the famous ABJM theory is obtained as a special case of $\mathcal{N} = 6, U(M) \times U(N)$ theory.

Note added: Very recently when we were working on the final version of our manuscript, a work [39] appeared, which contains some results overlapping partially with this paper.

2 Symplectic 3-algebras

It is known that one can construct $D = 3, \mathcal{N} = 6$ CSM theories by using a 3-algebra, in which the structure constants of the three-bracket are antisymmetric only in the first two indices [32]. This is a natural generalization of the Nambu 3-algebra whose structure constants are totally antisymmetric[6]. A further generalization would be to introduce a 3-algebra in which the struc-
ture constants are symmetric in the first two indices. In this section, we will define such a 3-algebra and then use it to construct $\mathcal{N} = 5$ CSM theories in next section.

By definition, a 3-algebra is a complex vector space equipped with a ternary, trilinear operation, called the 3-bracket. This operation from three vectors to one vector can be completely determined by its expressions in terms of a basis (or a set of generators) $T_a$ ($a = 1, 2, \cdots, K$):

$$[T_a, T_b; T_c] = f^d_{abc} T_d,$$  \hspace{1cm} (1)

where the set of complex numbers $f^d_{abc}$ are called the structure constants. Here we assume that

$$[T_a, T_b; T_c] = [T_b, T_a; T_c],$$  \hspace{1cm} (2)

i.e., the structure constants $f^d_{abc}$ are symmetric in the first two indices.

For a field $X$ valued in this 3-algebra, i.e., $X = X^c T_c$, we define the global transformation of the field as [9]:

$$\delta \tilde{\Lambda} X = \Lambda^{ab}[T_a, T_b; X],$$  \hspace{1cm} (3)

where the parameter $\Lambda^{ab}$ is independent of spacetime coordinate. (We will gauge this symmetry transformation in subsection 3.1). Because of Eq. (2), we require that $\Lambda^{ab}$ is symmetric in $ab$, i.e., $\Lambda^{ab} = \Lambda^{ba}$. Equation (3) is the natural generalization of $\delta \Lambda X = \Lambda^a[T_a, X]$ in an ordinary Lie 2-algebra. For an ordinary Lie 2-algebra, the Jacobi identity is equivalent to

$$\delta \Lambda (\{X, Y\}) = \{\delta \Lambda X, Y\} + \{X, \delta \Lambda Y\}. $$  \hspace{1cm} (4)

That is, $\delta \Lambda X = \Lambda^c[T_a, X]$ must act as a derivative. Analogously, one may require that Eq. (3) acts as a derivative [9]:

$$\delta \tilde{\Lambda} (\{X, Y; Z\}) = [\delta \tilde{\Lambda} X, Y; Z] + [X, \delta \tilde{\Lambda} Y; Z] + [X, Y; \delta \tilde{\Lambda} Z].$$  \hspace{1cm} (5)

Canceling $\Lambda^{ab}, X^c, Y^f$ and $Z^c$ from both sides, the above equation leads to the following fundamental identity (FI):

$$[T_a, T_b; [T_e, T_f; T_c]] = [[T_a, T_b; T_e], T_f; T_c] + [T_e, [T_a, T_b; T_f]; T_c] + [T_e, T_f; [T_a, T_b; T_c]].$$  \hspace{1cm} (6)

Later we will demonstrate that the FI is equivalent to the invariance of the structure constants: $\delta \tilde{\Lambda} f^d_{abc} = 0$ (see Eq. (11)) [32].
To define a symplectic 3-algebra, we introduce an anti-symmetric tensor \( \omega_{ab} \) and its inverse \( \omega^{ab} \) into the 3-algebra. The existence of the inverse of \( \omega_{ab} \) (\( \det \omega \neq 0 \)), and the Eq. \( \omega_{ab} = -\omega_{ba} \) imply that a 3-algebra index \( a \) must run from 1 to \( K = 2L \). The symplectic bilinear form is defined as follows:

\[
\omega(X, Y) = \omega_{ab} X^a Y^b. \tag{7}
\]

We require that the above bilinear form to be preserved under arbitrary global transformations, namely,

\[
\delta_\Lambda (\omega_{ab} X^a Y^b) = 0. \tag{8}
\]

This implies that the structure constants satisfy the condition:

\[
\omega_{de} f_{abc}^e = \omega_{ce} f_{abd}^e. \tag{9}
\]

Now the component form of Eq. (3) can be written as

\[
\delta_\Lambda X^a = \Lambda^{bc} f_{bcd}^a X^d \equiv \bar{\Lambda}^a d X^d.
\]

With the above definition of \( \bar{\Lambda}^a d \), Eq. (8) must be equivalent to

\[
\delta_\Lambda \omega_{ab} = -\bar{\Lambda}^c a \omega_{cb} - \bar{\Lambda}^c b \omega_{ac} = -\Lambda^{de} (f_{dea} \omega_{cb} + f_{deb} \omega_{ac}) = 0,
\]

where we used Eq. (9) in the last line. From point of view of ordinary Lie group, the (infinitesimal) matrices \( -\bar{\Lambda}^c a \) are in the Lie algebra of \( Sp(2L, \mathbb{C}) \), preserving the anti-symmetric tensor \( \omega_{ab} \) [33].

By using the FI (6), one can prove that the structure constants are also preserved under the global symmetry transformations [32]:

\[
\delta_\Lambda f_{efc}^d = -\bar{\Lambda}^g e f_{gfc}^d - \bar{\Lambda}^g f f_{egc}^d - \bar{\Lambda}^g f e f_{fg}^d + \bar{\Lambda}^d g f_{efc}^g = \Lambda^{ab} (-f_{abc}^g f_{gfc}^d - f_{abf}^g f_{egc}^d - f_{abc}^g f_{efg}^d + f_{abc}^d f_{efc}^g) \tag{11}
\]

where we have used the FI (6) in the second line. In other words, Eq. (11) is equivalent to the FI (6). Thus we can use \( \omega_{ab} \) and \( f_{abc}^d \) to construct invariant Lagrangians, when the symmetry is gauged.
Later, when we gauge this global symmetry, we require that the gauge fields must be anti-hermitian, leading to a reality condition on the structure constants (See section (3.1)):

\[ f^*_{abc} d = -\omega^{ae} \omega^{bf} \omega^{cg} \omega^{dh} f_{efg} h. \]  

(12)

Since \( \Lambda^{ab} \) carries two symplectic 3-algebra indices, it obeys the following natural reality condition

\[ \Lambda^*_{ab} = \omega_{ac} \omega_{bd} \Lambda^{cd}. \]  

(13)

Since the 3-algebra is also a complex vector space, there is a hermitian bilinear form:

\[ h(X, Y) = X^a Y^a \]  

(14)

(with \( X^a \) the complex conjugate of \( X^a \)) which is positive-definite and will be used to construct the Lagrangian of matter fields in CSM theories. The hermitian bilinear form is also required to be preserved in the sense

\[ \delta \tilde{\Lambda} h(X, Y) = \delta \tilde{\Lambda} (X^a Y^a) = 0. \]  

(15)

As in Ref. [33], we will impose the reality conditions on the fields valued in the 3-algebra, so that respecting them will make the anti-symmetric tensor (7) and the hermitian bilinear form (14) compatible with each other. Namely the reality conditions essentially require that \( X^a \) transform in the same way as \( \omega_{ab} X^b \) under the above symmetry transformations. In fact, by using the reality conditions (12) and (13), it is easy to prove that

\[ \delta \tilde{\Lambda} X^a = \tilde{\Lambda}^a_{b} X^b = -\tilde{\Lambda}^b_{a} X^b. \]  

(16)

The last equality indicates that the matrix \( \tilde{\Lambda}^a_{b} \) is anti-hermitian. Comparing (16) with

\[ \delta \tilde{\Lambda} (\omega_{ab} X^b) = -\tilde{\Lambda}^b_{a} (\omega_{bc} X^c), \]  

(17)

we see that \( X^a \) indeed transform in the same way as \( \omega_{ab} X^b \). Therefore, it makes sense to denote \( X^a \) as \( \bar{X}_a \), i.e.,

\[ X^a = \bar{X}_a. \]  

(18)
Also, with (16), Eq. (15) is satisfied:

\[
\delta \tilde{\Lambda}(X^a Y^a) = (\delta \tilde{\Lambda} X^a) Y^a + X^a (\delta \tilde{\Lambda} Y^a) \\
= -\tilde{\Lambda}^b a X^b Y^a + X^a \tilde{\Lambda}^a b Y^b \\
= 0
\]  

(19)

By (18), the hermitian bilinear form (14) can be written in a manifest invariant form:

\[
X^a Y^a = \bar{X}_a Y^a = \bar{\bar{X}}_a \delta^a b Y^b,
\]  

(20)

and Eq. (19) is equivalent to the following equation:

\[
\delta \tilde{\Lambda} \delta^a b = \bar{\bar{\Lambda}}^a c \delta^c b - \tilde{\Lambda}^a b \delta^a c = 0.
\]  

(21)

In summary, the global transformations (10) preserve the hermitian bilinear form (14) and symplectic bilinear form (7) simultaneously. Or in other words,

\[
\delta \tilde{\Lambda} \omega_{ab} = 0 \quad \text{and} \quad \delta \tilde{\Lambda} \delta^a b = 0.
\]  

(22)

From point of view of ordinary Lie group, the symmetry group generated by the 3-algebra transformations (3) or (10) is the intersection of \(U(2L)\) and \(Sp(2L, \mathbb{C})\), which is \(Sp(2L)\).

We call the 3-algebra defined by the above Eq. (1), (2), (6), (7), (9), (14) and (12) a symplectic 3-algebra. \(^2\)

To construct \(N = 5\) CSM theories, the 3-bracket will be required to satisfy an additional constraint condition (see section 3.1): \(^3\)

\[
\omega([T_a, T_b; T_c], T_d) = 0
\]  

(23)

Or simply \(f_{(abc)}^e = 0\). Now Eq. (23) implies that \(\omega([T_a, T_b; T_c], T_d) = 0\) and \(\omega([T_a, T_b; T_c], T_d) = \omega([T_c, T_d; T_a], T_b)\). In summary, the structure constants have the following symmetry properties:

\[
\omega_{de} f_{abc}^e = \omega_{de} f_{bac}^e = \omega_{de} f_{abc}^e = \omega_{be} f_{cda}^e.
\]  

(24)

\(^1\)By our convention, the hermitian bilinear form is \(h(T_a, T_b) = \delta^a b\). In [32], it is denoted as \(h_{ab}\), which becomes \(\delta_{ab}\) in an orthonormal basis.

\(^2\)We gave the name ‘symplectic 3-algebra’ in our previous paper [33].

\(^3\)While we were writing this paper, the Ref. [34] appeared which contains a definition of 3-algebra similar to our definition of symplectic 3-algebra of this paper. See also [35]. Maybe there is a connection between our approach and theirs.
3 $D = 3, \mathcal{N} = 5$ CSM Theories

3.1 General $\mathcal{N} = 5$ CSM Theories

We first postulate that all matter fields are valued in the symplectic 3-algebra. We then assume the theory has an $SO(5) \cong Sp(4)$ R-symmetry. It is convenient to use the $Sp(4)$ indices for $R$-symmetry. We denote the eight complex scalar fields as $Z^a_A$, and their corresponding complex conjugate $\bar{Z}^A_a \equiv Z^*_a A$, where $A = 1, 2, 3, 4$ transforms in the 4-dimensional representation of $Sp(4)$, and $a$ is a 3-algebra index. Similarly, we denote the fermion fields and their complex conjugates as $\psi^a_A$ and $\bar{\psi}^A_a$, respectively. The gauge fields are defined as

$$\tilde{A}_\mu^c d \equiv A^a_{\mu} f_{abcd},$$  \hspace{1cm} (25)$$

where $\mu = 0, 1, 2$. Finally, we also impose the reality conditions on the fields:

$$Z^a_A = \omega^{AB} \omega_{ab} Z^b_B,$$

$$\psi^a_A = \omega^{AB} \omega_{ab} \psi^b_B,$$

$$\tilde{A}_\mu^c d = -\omega_{ca} \omega^{db} \tilde{A}^a_{\mu b},$$

$$A^a_{\mu} = \omega_{ae} \omega_{bf} A^e_{\mu f}. $$ \hspace{1cm} (26)$$

The last two equations of (26) and Eq. (25) require that the structure constants obey the reality condition:

$$f^*_{abc} d = -\omega^{ae} \omega^{bf} \omega_{dg} f_{efg} h. $$ \hspace{1cm} (27)$$

In terms of the symplectic 3-algebra, now we propose the following manifestly $Sp(4)$ covariant, $\mathcal{N} = 5$ SUSY transformations:

$$\delta Z^a_A = i \epsilon_A^B \psi^a_B,$$

$$\delta \psi^a_A = \gamma^\mu D_\mu Z^a_A \epsilon^B A + \frac{1}{3} f_{cde} a \omega^{BC} Z^b_B Z^c_C Z^d_D \epsilon^e A - \frac{2}{3} f_{cde} a \omega^{BD} Z^b_C Z^c_D Z^d_A \epsilon^e B - \frac{2}{3} f_{cde} a \omega^{BD} Z^b_C Z^c_D Z^d_A \epsilon^e B,$$

$$\delta \tilde{A}_\mu^c d = i \epsilon^A B \gamma^\mu \psi_B Z^a_A f_{abcd}. $$ \hspace{1cm} (28)$$

Here $\epsilon^{AB}$ is the antisymmetric supersymmetry parameter, satisfying

$$\epsilon^{AB} = -\epsilon^{BA},$$

$$\omega_{AB} \epsilon^{AB} = 0,$$

$$\epsilon_{AB} = \omega^{AC} \omega^{BD} \epsilon_{CD}. $$ \hspace{1cm} (29)$$
Namely, they transform as 5 of $Sp(4)$. The last equation of (29) is the reality condition on $\epsilon_{AB}$. The covariant derivatives are defined as

$$D_{\mu}Z^{A}_{d} = \partial_{\mu}Z^{A}_{d} - \tilde{A}_{\mu}^{c}Z^{A}_{c} \quad (30)$$

$$D_{\mu}Z^{d}_{A} = \partial_{\mu}Z^{d}_{A} + \tilde{A}_{\mu}^{c}Z^{c}_{A}. \quad (31)$$

Following BL’s strategy [32], we will derive the equations of motion by requiring that the supersymmetry transformations are closed on-shell. Let us first examine scalar supersymmetry transformation. By virtue of the identities in the appendix A, we find

$$[\delta_{1}, \delta_{2}]Z^{a}_{A} = v^{\mu}D_{\mu}Z^{a}_{A} - \frac{2}{3}f_{bdc}^{a}\Lambda^{cd}Z^{b}_{A} + \frac{2}{3}f_{cdb}^{a}\Lambda^{cd}Z^{b}_{A}, \quad (32)$$

where

$$v^{\mu} \equiv -\frac{i}{2}\bar{\epsilon}_{2BD}\epsilon^{1BD}, \quad (33)$$

$$\Lambda^{cd} \equiv -\frac{i}{2}Z_{D}^{a}Z_{C}^{d}(\epsilon^{CE}_{1}\epsilon^{2E}_{D} - \epsilon^{CE}_{2}\epsilon^{1E}_{D}) = \Lambda^{dc}, \quad (34)$$

and the $\epsilon$ bilinear is symmetric in $CD$. While the first term of Eq. (32) is the gauge covariant translation, we have to impose some conditions on the structure constants so that the remaining terms add up to be a gauge transformation. (We will read off the parameter of the gauge transformation by looking the closure of the algebra on the gauge fields.)

We tentatively assume that the third term of Eq. (32) is proportional to the gauge transformation. So the second term of Eq. (32) should be also proportional to the gauge transformation. This leads us to impose an additional constraint condition on the structure constants:

$$\frac{1}{2}(f_{bdc}^{a} + f_{bcd}^{a}) = \frac{\alpha}{2}f_{cdb}^{a}, \quad (35)$$

where $\alpha$ is a constant, to be determined later. Now the second and third term of Eq. (32) can be combined as

$$\frac{1}{3}(-\alpha + 2)f_{cdb}^{a}\Lambda^{cd}Z^{b}_{A}, \quad (36)$$

which should be the gauge transformation.
Let us now look at the gauge fields:

$$[\delta_1, \delta_2] \tilde{A}_\mu^a = \nu^\nu \tilde{F}_{\mu\nu}^a - (D_\mu \Lambda) f_{cdb}^a$$

$$+ \nu^\nu [\tilde{F}_{\mu\nu}^a - \varepsilon_{\mu\nu\lambda} (Z_A^\lambda D^d Z^\lambda_{A} - \frac{i}{2} \bar{\psi}^{Bc} \gamma^\lambda \psi_B^d) f_{cdb}^a]$$

$$+ \mathcal{O}(Z^4), \quad (37)$$

where the last term $\mathcal{O}(Z^4)$ is fourth order in the scalar fields $Z$. We recognize the second term of the first line as a gauge transformation

$$-(D_\mu \Lambda) f_{cdb}^a = -D_\mu (\Lambda f_{cdb}^a) \quad (38)$$

by a parameter $\tilde{\Lambda}^a_b = \Lambda^d f_{cdb}^a$, since the FI (6) or (11) implies that $D_\mu f_{cdb}^a = 0$ [32]. In accordance with the parameter, now (36) must satisfy the following equation:

$$\frac{1}{3} (\alpha + 2) f_{cdb}^a \Lambda f_{cdb}^b = \Lambda f_{cdb}^a Z^b_A. \quad (39)$$

This equation can be solved by setting $\alpha = -1$. Or in other words, Eq. (39) can be solved if Eq. (35) can be written as

$$f_{(bcd)}^a = 0, \quad (40)$$

which is equivalent to Eq. (23). Now Eq. (32) becomes

$$[\delta_1, \delta_2] Z^a_A = \nu^\nu D_\mu Z^a_A + \tilde{\Lambda}^a_b Z^b_A, \quad (41)$$

as expected.

Following Gustavsson’s approach [10], one can demonstrate that the FI (6) admits an explicit solution in terms of a tensor product: $f_{abc}^d = k_{mn} r_{ab} T_{cd}^m$ where $k_{mn}$ is the Killing-Cartan metric of $Sp(2L)$, and $r_{ab}^m = \omega_{ac} T^m_{bc}$ [28]. The matrix $T^m_{bc}$ is in the fundamental representation of $Sp(2L)$, and $\omega_{ac}$ is the $Sp(2L)$-invariant anti-symmetric tensor. Now Eq. (40) implies that $k_{mn} r_{(ab)}^m r^d_{cd} = 0$, which is first derived by GW [28]. In the GW theories, it is the key requirement for enhancing the $\mathcal{N} = 1$ supersymmetry to the $\mathcal{N} = 4$ supersymmetry.

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4 According to our convention, if $\delta \tilde{A}^a_A = \tilde{\Lambda}^a_b Z^b_A$, we must set $\delta \tilde{A}^a_A = -D_\mu \tilde{\Lambda}^a_b$ so that $\delta \tilde{A} (D_\mu Z^a_A) = \tilde{\Lambda}^a_b (D_\mu Z^b_A)$. 

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By using the FI (6) and the symmetry conditions (24), one can prove that the last term of Eq. (37) vanishes:

$$\mathcal{O}(Z^4) = 0.$$  \hspace{1cm} (42)

So the second line of Eq. (37) must be the equations of motion for the gauge fields:

$$\tilde{F}_{\mu\nu}^{\ a \ b} - \varepsilon_{\mu\nu\lambda}(Z_A^c D^\lambda Z^{Ad} - \frac{i}{2} \bar{\psi}_C \gamma^\lambda \gamma^d \psi_B^a) f_{cdb}^a = 0.$$  \hspace{1cm} (43)

Now only the first line of Eq. (37) remains:

$$[\delta_1, \delta_2] \tilde{A}_\mu^a_b = v^\nu \tilde{F}_{\mu\nu}^{\ a \ b} - D_\mu \tilde{\Lambda}_b^a,$$  \hspace{1cm} (44)

which is the desired result.

Finally we turn to the fermion supersymmetry transformation:

$$[\delta_1, \delta_2] \psi_A^a = v^\mu D_\mu \psi_A^a + \tilde{\Lambda}_b^a \psi_A^b$$

$$+ \frac{i}{2} (\epsilon_1^{BC} \epsilon_{2BA} - \epsilon_2^{BC} \epsilon_{1BA}) E_A^a$$

$$- \frac{1}{2} v_\nu \gamma^\nu E_A^a,$$  \hspace{1cm} (45)

where

$$E_A^a = \gamma^\mu D_\mu \psi_A^a - f_{cdb}^a Z_B^c Z_B^e \psi_A^d + 2f_{cde}^a Z_B^c Z_B^d \psi_A^B.$$  \hspace{1cm} (46)

Hence the equations of motion for fermionic fields are $E_A^a = 0$. The scalar equations of motion can be derived by taking the super-variation of the fermionic equations of motion:

$$\delta E_A^a = 0.$$  \hspace{1cm} (47)

After Fierz transformation, we obtain two independent parts, containing $\gamma^\mu \epsilon_{BC}$ and $\epsilon_{BC}$, respectively. The part containing $\gamma^\mu \epsilon_{BC}$ merely implies the equations of motion for the gauge fields, so we will not write it down here. The part containing $\epsilon_{BC}$ reads

$$\left( \delta_A^{[C} F_{B]}^a + G_A^{BCa} \right) \epsilon_{BC} = 0,$$  \hspace{1cm} (48)

where

$$F_{Ba} \equiv -D^2 Z^B_a + i f_{cdb}^a Z^C_b \bar{\psi}_C^d \psi_B^d + \frac{1}{3} f_{efd}^a f_{gcb}^a Z^C_b Z^C_c Z^d_D Z^d_E Z^f_Z^f,$$  \hspace{1cm} (49)
and

\[ G_A^{B C a} \epsilon_{B C} \equiv \left[ i f_{c d b}^a \left( \frac{3}{2} Z^{B d} \bar{\psi}^C \psi^b_A + Z^c A \bar{\psi}^C \psi^{B d} \right) + i f_{d e b}^a Z^{B b} \bar{\psi}^C \psi^d_A \right. \]

\[ \left. + \frac{2}{3} (f_{e f d}^a f_{g c b}^a + f_{c e b}^a f_{f d a}^a + 2 f_{e b d}^a f_{f c e}^a) Z^b D Z^{B c} Z^{C d} Z^{B e} Z^f \right] \epsilon_{B C}. \]

Since the parameters \( \epsilon_{B C} \) are traceless, in the sense that \( \omega^{B C} \epsilon_{B C} = \epsilon_{B B} = 0 \), Eq. (48) must be equivalent to the following traceless equation:

\[ \delta_{\omega}^{[C} (F^{B] a} - \frac{1}{4} \omega^{B C} F^a + G_A^{B C a} - \frac{1}{4} \omega^{B C} \omega_{D E} G_A^{E D a} = 0. \]

Contracting on \( AC \) gives the scalar equations of motion:

\[ F^{B a} + \frac{4}{5} G_A^{B A a} - \frac{1}{5} G^{B A a} = 0. \]

After some simplification we obtain

\[ 0 = -D^2 Z_a^B - i f_{a b c} (Z_d^B \bar{\psi}^C \psi^b_c - 2 Z^{C d} \bar{\psi}^b_c \psi^B_d) \]

\[ -\frac{1}{5} (f_{a b c}^g f_{d e f}^g + f_{a b d}^g f_{c e f}^g + 3 f_{a b c}^g f_{d g e}^f - 3 f_{a b e}^g f_{d g f}^e) Z^b_A Z^C Z^B C \]

All the equations of motion can be derived as the Euler-Lagrangian equations from the following action:

\[ \mathcal{L} = \frac{1}{2} \left( -D_\mu \bar{Z}_a^A D^2 Z_a^A + i \bar{\psi}_a^A D_\mu \gamma^\mu \psi_a^A \right) \]

\[ -\frac{i}{2} \omega^{A B} \omega^{C D} \bar{\psi}_d f_{a b c}^e (Z_a^B \bar{\psi}_C \psi^d_d - 2 Z^d_A \bar{\psi}_C \psi^B_d) \]

\[ + \frac{1}{2} \epsilon^{\mu \nu \lambda} (\omega_{d e f} f_{a b c}^e A_\mu \partial_\nu A_\lambda^f + \frac{2}{3} \omega_{d h} f_{a b c}^e A_\mu A_\nu A_\lambda^f) \]

\[ - \frac{1}{60} (2 f_{a b c}^g f_{d e f}^g - 9 f_{d c a}^g f_{g b e}^f + 2 f_{a b e}^g f_{d g f}^e) Z^f_A Z^A Z^B Z^{C d} Z^C Z_e. \]

With the reality conditions (12) and the first equation of (26), one can recast the potential term into the following form:

\[ V = \frac{2}{15} (\Upsilon_{A B C}^d)^* \Upsilon_{A B C}^d, \]

where

\[ \Upsilon_{A B C}^d \equiv f_{a b c}^d (Z_a^A Z_b^B Z_c^C + \frac{1}{4} \omega_{B C} Z_A^A Z_B^B Z_C^C), \]
Therefore the potential term is actually positive definite. Also it is not difficult to verify that the Lagrangian (54) has manifest $\mathcal{N} = 5$ supersymmetry with $Sp(4)$ R-symmetry; namely it is indeed invariant (up to some boundary terms) under the supersymmetry transformations (28). It is easy to check that the above Lagrangian is a scale invariant, local field theory, provided that the structure constants are dimensionless. This implies that the theory is classically conformal invariant. We expect that after quantization it is conformally invariant at the quantum level.

In the same manner as in our previous paper [33], if we specify the 3-brackets properly, certain Lie algebra of the gauge groups can be generated by the FI (6) of the 3-algebra. In the next subsection, we will provide the $\mathcal{N} = 5, Sp(2N) \times O(M)$ CSM theory as an example.

### 3.2 $\mathcal{N} = 5, Sp(2N) \times O(M)$ CSM theory

To generate a direct product gauge group, such as $Sp(2N) \times O(M)$, we first split one 3-algebra index into two indices: $a \to k\hat{k}$. As a result, a 3-algebra valued field becomes $Z^A_a \to Z^{(k\hat{k})}_A$. We also decompose the antisymmetric tensor as $\omega_{ab} \to \omega_{k\hat{l}}\delta_{kl}$, where $\omega_{k\hat{l}}$ is anti-symmetric, and require $Z^{(k\hat{k})}_A$ to be valued in the bi-fundamental representation of $Sp(2N) \times O(M)$. (Here $k,l = 1, \cdots, M$ are the $O(M)$ indices while $\hat{k}, \hat{l} = 1, \cdots, 2N$ the $Sp(2N)$ indices.) With this decomposition of $\omega_{ab}$, we can rewrite the reality condition (26) as

$$Z^{(k\hat{k})}_A = \omega^B\omega_{k\hat{l}}\delta_{kl}Z^B_{(l\hat{l})},$$

and similar conditions for the fermion and gauge fields. Consequently, the hermitian bilinear form of two fields

$$\omega^{AB}\omega_{ab}Z^b_BZ^a_A = Z^a_A Z^a_A = \bar{Z}^A\bar{Z}^A,$$

can be rewritten in a trace form:

$$Z^{(k\hat{k})}_A \bar{Z}^{(k\hat{k})}_A = \text{Tr}(Z^{A\dagger}Z_A)$$

We then specify the 3-brackets as follows:

$$[T_{kk}, T_{li}; T_{m\hat{m}}] = k(\delta_{k\hat{m}}\omega_{l\hat{m}}T_{ml} + \delta_{kl}\omega_{l\hat{m}}T_{mk} - \delta_{km}\omega_{k\hat{m}}T_{lm} + \delta_{lm}\omega_{k\hat{m}}T_{k\hat{m}}).$$
The overall coefficient $k$ on the right-hand side is assumed to be a real constant. It is straightforward to verify that the 3-brackets satisfy the FI (6) and the constraints (23). The corresponding structure constants are

$$ f_{kk,lt,nn} = -k[(\delta_{km}\delta^m_l - \delta^m_k\delta_{lm})\omega_{kl}\delta^m_n - \delta_{kl}\delta^m_n(\delta_{km}\delta^0_l + \delta^0_m\omega_{lm})]. \quad (61) $$

It is not hard to check that the structure constants have the symmetry properties (24), and satisfy the reality condition (27). We observe that the structure constants are the same as the components of an embedding tensor in Ref. [27]. This is not merely an accident, and we will explore their relations in a coming paper. With this choice of structure constants, the gauge fields (25) become: (We re-scale $A_{\mu}^{ab}$ by $\frac{1}{k}$ in eq. (25).)

$$ \hat{A}_{\mu}^{mn}\hat{n}\hat{n} = A_{\mu}^{kl}f_{kk,lt,nn}^{mn}. $$

It is easy to see that $\hat{A}_{\mu}^{mn}\hat{n}$ is the $Sp(2N)$ part of the gauge potential, because it can be written as $A_{\mu}^{kl}(t_{kl})^{mn}\hat{n}$, where $(t_{kl})^{mn}\hat{n}$ is the fundamental representation of the ordinary Lie algebra $Sp(2N)$. Similarly, we can identify $A_{\mu}^{mn}$ as the $O(M)$ part of the gauge potential. As we explained in our previous paper [33], the Lie algebra of the gauge group $Sp(2N) \times O(M)$ is actually generated by the FI (6) after we specify the structure constants by Eq. (61).

We would like to derive the $\mathcal{N} = 5$, $Sp(2N) \times O(M)$ Lagrangian and the corresponding supersymmetry transformation law in the 3-algebraic framework. With the notation (59), the kinetic terms for matter fields in the Lagrangian (54) read

$$ -\frac{1}{2}\text{Tr}(D_{\mu}Z^{\dagger A}D^{\mu}Z - i\bar{\psi}^{\dagger A}D_{\mu}\gamma^{\mu}\psi_A). \quad (63) $$

With the choice of the structure constants (61), we learn that

$$ \omega_{de}f_{abc}e^{X^aY^bZ^cW^d} = -k\text{Tr}(XY^{\dagger}ZW^{\dagger} + YX^{\dagger}ZW^{\dagger} - ZX^{\dagger}YW^{\dagger} - ZY^{\dagger}XW^{\dagger}). $$

Hence the Yukawa terms in the Lagrangian (54) become

$$ ik\varepsilon^{ABCD}\text{Tr}(Z_A\bar{\psi}^BZ_C\psi_D) $$

$$ -\frac{k}{2}\text{Tr}(\bar{\psi}^{\dagger A}Z_BZ^{\dagger B}\psi^A - \bar{\psi}^{\dagger A}Z_BZ^{\dagger B}\psi^A - 2\bar{\psi}^{\dagger A}Z_BZ^{\dagger A}\psi^B + 2\bar{\psi}^{\dagger A}Z^{\dagger B}Z_A\psi^B), \quad (65) $$
where we have used the following $S_4(4)$ identity:

$$\varepsilon^{ABCD} = -\omega^{AB}\omega^{CD} + \omega^{AC}\omega^{BD} - \omega^{AD}\omega^{BC}. \quad (66)$$

Substituting the definition of the gauge fields (62) into the ‘twisted’ Chern-Simons term in the Lagrangian (54) gives the conventional Chern-Simons term

$$\frac{1}{4k} \epsilon^\mu\nu\lambda \text{Tr}(\hat{A}_\mu \partial_\nu \hat{A}_\lambda + \frac{2}{3} A_\mu A_\nu A_\lambda - \frac{2}{3} A_\mu A_\nu A_\lambda). \quad (67)$$

Finally we want to calculate the potential terms in the Lagrangian (54). By using $\omega_{de} f_{abce} = \omega_{ce} f_{abde}$, they can be re-written as

$$\frac{1}{60} \text{Tr}(2[Z^A, Z_B; Z^C, Z_A] - 9[Z^B, Z_C; Z^A][Z^C, Z_B; Z_A] \text{Tr}(Z^B Z^D Z^E Z^F). \quad (68)$$

The last two terms can be combined together:

$$-\frac{4k^2}{15} (\omega_{AF}\omega_{BE}\omega_{CD} - 2\omega_{AF}\omega_{BC}\omega_{DE} + 2\omega_{AC}\omega_{BF}\omega_{DE} - \omega_{AE}\omega_{BF}\omega_{CD} + \omega_{AC}\omega_{BE}\omega_{DF} - \omega_{AE}\omega_{BC}\omega_{DF}) \text{Tr}(Z^B Z^D Z^A Z^C Z^E Z^F). \quad (69)$$

The first term becomes

$$\frac{k^2}{30} (2\omega_{AD}\omega_{BE}\omega_{CF} + 4\omega_{AB}\omega_{CF}\omega_{DE} - 2\omega_{AE}\omega_{BF}\omega_{CF} + \omega_{AD}\omega_{BC}\omega_{EF} - 2\omega_{AB}\omega_{CD}\omega_{EF} - \omega_{AC}\omega_{BD}\omega_{EF} + \omega_{AD}\omega_{BF}\omega_{CE} + 2\omega_{AB}\omega_{CE}\omega_{DF} - \omega_{AF}\omega_{BD}\omega_{CE}) \text{Tr}(Z^B Z^D Z^A Z^C Z^E Z^F). \quad (70)$$

Clearly, they can be simplified further. Taking account of the cyclic property of the trace, there are only four possible potential terms:

$$(c_1 \omega_{AD}\omega_{BE}\omega_{CF} + c_2 \omega_{BD}\omega_{DE}\omega_{AF} + c_3 \omega_{AD}\omega_{CE}\omega_{BF} + c_4 \omega_{CD}\omega_{AE}\omega_{BF}) \times \text{Tr}(Z^A Z^D Z^B Z^E Z^C Z^F), \quad (71)$$

where $c_1, \cdots, c_4$ are constants. After some work, we reach the final expression for the potential:

$$\frac{k^2}{6} \text{Tr}(-6Z_A Z^A Z_B Z^B Z_C Z^C + 4Z_A Z^C Z_B Z^D Z^A Z_C Z^B + Z_A Z^A Z_B Z^B Z_C Z^C + Z_A Z^B Z_B Z^C Z_C Z^A). \quad (72)$$
In deriving this potential, we have used another $Sp(4)$ identity [30]:

\[ \varepsilon_{GABC} \varepsilon^{GDEF} = 3! \delta_{[A}^{D} \delta_{B}^{E} \delta_{C]}^{F} \]

\[ = 3(-\delta_{[A}^{D} \omega_{EF}^{BC}] + \delta_{[A}^{E} \omega_{DF}^{BC} - \delta_{[A}^{F} \omega_{DE}^{BC})]. \]

In summary, with the choice of the structure constants (61), the Lagrangian (54) is given by

\[ \mathcal{L} = -\frac{1}{2} \text{Tr}(D_{\mu} Z^A \nabla^{\mu} Z_A - i\bar{\psi}^A D_{\mu} \gamma^\mu \psi_A) + i k \varepsilon^{ABCD} \text{Tr}(Z_A \bar{\psi}_B Z_C \psi_D) \]

\[-i \frac{k}{2} \varepsilon^{\mu\nu\lambda} \text{Tr}(\hat{A}_\mu \partial_{\nu} \hat{A}_\lambda + \frac{2}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda - A_\mu \partial_\nu A_\lambda - \frac{2}{3} A_\mu A_\nu A_\lambda) \]

\[+ \frac{k^2}{6} \text{Tr}(-6Z_A \nabla^A Z_B \nabla^B Z_C Z^\dagger \psi_A Z_B Z^\dagger A + 4Z_A \nabla^A Z_B Z^\dagger C Z^\dagger B \]

\[+ Z_A \nabla^A Z_B Z^\dagger B Z_C Z^\dagger C + Z_A \nabla^B Z_B Z^\dagger C Z^\dagger A), \]

(74)

Substituting the structure constants (61) into (28), the SUSY transformation law reads

\[ \delta Z_A = i \bar{\epsilon}_A^B \psi_B \]

\[ \delta \psi_A = \gamma^\mu D_\mu Z_B \epsilon_A^B - i k \epsilon^C_A (Z_B \nabla^B Z_C + Z_B Z^\dagger C Z^\dagger B) \]

\[+ \frac{4k}{3} \epsilon^C_B (Z_C Z^\dagger B Z_A + Z_C Z^\dagger A Z^\dagger B) \]

\[ \delta A_\mu = i k \epsilon^A B \gamma_\mu (Z_A \psi^\dagger_B + \psi_B Z^\dagger_A) \]

\[ \delta \hat{A}_\mu = -i k \epsilon^A B \gamma_\mu (\psi_B Z_A + Z^\dagger_A \psi_B). \]

(75)

The $\mathcal{N} = 5$, $Sp(2N) \times O(M)$ Lagrangian (74) and the supersymmetry transformation law (75) are in agreement with those given in ref. [30], which were derived in terms of ordinary Lie algebra. This theory has been conjectured to be the dual gauge theory of M2 branes probing a $\mathbf{C}^4/\hat{D}_k$ singularity, where $\hat{D}_k$ is the binary dihedral group [30, 31].

4 **$D = 3, \mathcal{N} = 6$ CSM Theories from 3-algebras**

In Ref. [30], the $\mathcal{N} = 6$ theories are derived from the $\mathcal{N} = 5$ theories by enhancing the R-symmetry from $Sp(4)$ to $SU(4)$. In this section we
will implement the same idea in the context of 3-algebras. We will call the symplectic 3-algebras presented in this paper and in Ref. [32], respectively, to construct the $\mathcal{N} = 5, \mathcal{N} = 6$ theories as the “$\mathcal{N} = 5, \mathcal{N} = 6$ 3-algebra”, respectively. We will see that the symplectic 3-algebra provides framework unifying the $\mathcal{N} = 5$ and $\mathcal{N} = 6$ CSM models.

4.1 General $\mathcal{N}=6$ CSM Theories

The enhancement of R-symmetry from $Sp(4)$ to $SU(4)$ in ref. [30] is based on the following observation: The reality condition (26) implies that the complex conjugate of a matter field can be obtained by a similarity transformation. Therefore the matter fields actually furnish a pseudo-real presentation of the gauge group. If we decompose this pseudo-real representation into a complex representation and its conjugate representation, then the $Sp(4)$ R-symmetry will be enhanced to $SU(4)$, and the global $\mathcal{N} = 5$ SUSY will get enhanced to $\mathcal{N} = 6$.

In this section, we will show that this enhancement can be implemented exclusively in the framework of symplectic 3-algebra, which thus provides a unified framework for both $\mathcal{N} = 5$ and $\mathcal{N} = 6$ theories. Since in our approach the ordinary Lie algebra of the gauge groups is generated by the FI and the 3-brackets, the challenge we face is to derive the $\mathcal{N} = 6$ 3-algebra from the 3-algebra proposed in this paper.

Following ref. [30], we first decompose an $\mathcal{N} = 5$ scalar field as a direct sum of an $\mathcal{N} = 6$ scalar field and its complex conjugate (See eq. (80)):

\[
(Z^a_A)_{\mathcal{N}=5} \rightarrow Z^{aa}_A = \bar{Z}^a_A \chi_1^a + \omega_{AB} Z^B_a \chi_2^a = \bar{Z}^a_A \delta_1^{\alpha} + \omega_{AB} Z^B_a \delta_2^{\alpha},
\]

where the right hand side of the arrow contains $\mathcal{N} = 6$ fields. Here the index $a$ of the left hand side of the arrow runs from 1 to $2L$, while the index $a$ of the right hand side of the arrow runs from 1 to $L$. And $\chi_1^a$ and $\chi_2^a$ are “spin up” and “spin down” spinor, respectively; i.e.,

\[
\chi_1^a = \begin{pmatrix}
1 \\
0
\end{pmatrix}, \quad \chi_2^a = \begin{pmatrix}
0 \\
1
\end{pmatrix}.
\]

To make the $\mathcal{N} = 5$ SUSY transformation law (28) consistent with that of $\mathcal{N} = 6$ (see below the first two equations of eq. (95)), we have to decompose the $\mathcal{N} = 5$ fermion fields as follows:

\[
(\psi^a_A)_{\mathcal{N}=5} \rightarrow \psi^{aa}_A = \omega_{AB} \psi^{Ba}_A \delta_1^{\alpha} - \psi_{Aa} \delta_2^{\alpha},
\]
where the right hand side contains $N = 6$ fermion fields. We further decompose the anti-symmetric tensor $\omega_{ab}$ and its inverse as

$$
\omega_{ab} \rightarrow \omega_{\alpha a, b\beta} = \delta^a_{\alpha} \delta_{2\alpha} \delta_1 \delta_{1\beta} - \delta^a_{\beta} \delta_{2\beta} \delta_1 \delta_{1\alpha},
$$

$$
\omega^{ab} \rightarrow \omega^{a\alpha, b\beta} = \delta^a_{\alpha} \delta_{2\alpha} \delta_1 \delta_{1\beta} - \delta^a_{\beta} \delta_{2\beta} \delta_1 \delta_{1\alpha}.
$$

(79)

Then the reality condition (26) reads

$$
Z^{*A}_a = \bar{Z}^a_A, \quad \psi^* A a = \psi A a,
$$

(80)
in agreement with those for $N = 6$ theories. This justifies the above decomposition (79) of the anti-symmetric tensor of the $N = 5$ 3-algebra to derive the $N = 6$ 3-algebra.

To be compatible with the decomposition of scalar and fermion fields, one has to decompose the gauge fields as

$$
(\tilde{A}_\mu^a)^{N=5} \rightarrow (\tilde{A}_\mu^a)^{a\alpha, b\beta} = \tilde{A}_\mu^a \delta_{1\alpha} \delta_{1\beta} - \tilde{A}_\mu^b \delta_{2\alpha} \delta_{2\beta},
$$

(81)

where the right hand side is a direct sum of $N = 6$ gauge fields and their complex conjugates. Since our gauge fields $(\tilde{A}_\mu^c)^{N=5}$ are defined in terms of the structure constants of a 3-algebra, i.e.,

$$
(\tilde{A}_\mu^c)^{N=5} = (A_\mu^{ab})^{f_{abc}^c}_{N=5},
$$

(82)

(see also Eq. (25)), we have to decompose its structure constants properly to result in the desired decomposition Eq. (81). We find that Eq. (81) indeed follows from the decomposition of the structure constants given by

$$
(\omega_{\mu\nu} f_{abc}^c)^{N=5} \rightarrow \omega_{\delta\varepsilon, en} f_{aa, b\beta, c\gamma}^{en} = f_{ac}^{ad} \delta_{2\alpha} \delta_{1\beta} \delta_{1\delta} + f_{cb}^{ad} \delta_{2\alpha} \delta_{1\beta} \delta_{1\varepsilon} + f_{bc}^{ad} \delta_{2\alpha} \delta_{1\gamma} \delta_{1\delta} + f_{bd}^{ad} \delta_{2\alpha} \delta_{1\gamma} \delta_{1\varepsilon},
$$

(83)

combined with the decomposition of $(A_\mu^{ab})^{N=5}$ given by

$$
(A_\mu^{ab})^{N=5} \rightarrow A_\mu^{a\alpha, b\beta} = -\frac{1}{2}(A_\mu^a b \delta_{1\alpha} \delta_{2\beta} + A_\mu^b a \delta_{2\alpha} \delta_{1\beta}).
$$

(84)

With these decompositions, the $N = 6$ gauge fields become: (see the right side of Eq. (81))

$$
\tilde{A}_\mu^c d = A_\mu^b a f^{ca}_{bd}.
$$

(85)
Later we will identify the above $f^{cd}_{ab}$ in the right side of eq. (83) as the structure constants of the $\mathcal{N} = 6$ 3-algebra. With eq. (79) and (83), the reality condition of the structure constants (27) reduces to

$$f^{*ab}_{cd} = f^{cd}_{ab},$$

as desired for the $\mathcal{N} = 6$ 3-algebra \[32, 33\].

Since we decompose a matter field as a direct sum of a $\mathcal{N} = 6$ matter field and its complex conjugate, it is necessary to decompose a generator of the 3-algebra as a direct sum of a generator of a 3-algebra and its complex conjugate. This can be accomplished by setting

$$(T_a)_{\mathcal{N}=5} \rightarrow T_{a\alpha} = \bar{t}_a \delta_{1\alpha} - t^a \delta_{2\alpha},$$

where $t^a$ is a generator of the 3-algebra, and $\bar{t}_a$ its complex conjugate.

The hermitian bilinear form of two $\mathcal{N} = 5$ fields will be (for instance):

$$(Z_{1A}^a Z_{2A}^a)_{\mathcal{N}=5} = (\omega_{ab} \omega_{1B}^{\dot{a}} Z_{1B}^b Z_{2A}^a)_{\mathcal{N}=5} \rightarrow \omega_{ab, \dot{a}b} \omega_{1B}^{\dot{a}} Z_{1B}^b Z_{2A}^a = Z_{2A}^a Z_{1A}^a + Z_{1A}^a Z_{2A}^a.$$  \[88\]

Namely, it becomes a sum of the hermitian bilinear form of two $\mathcal{N} = 6$ fields and its complex conjugate. Generally speaking, the hermitian bilinear form of two arbitrary $\mathcal{N} = 6$ 3-algebra valued fields will become

$$h(X, Y) = X^a_{\alpha} Y_{\alpha} = \bar{X}^a Y_a.$$  \[89\]

The reality condition (86) and Eq. (83) imply that the $\mathcal{N} = 5$ 3-bracket (1) can be decomposed as a direct sum of $\mathcal{N} = 6$ brackets and their complex conjugates as follows:

$$[T_a, T_b; T_c]_{\mathcal{N}=5} \rightarrow [T_{a\alpha}, T_{b\beta}; T_{c\gamma}]$$

$$= [t^a, t^c; \bar{t}_b] \delta_{2\alpha} \delta_{1\beta} \delta_{2\gamma} + [t^a, t^c; \bar{t}_b]^{*} \delta_{1\alpha} \delta_{2\beta} \delta_{1\gamma}$$

$$+ [t^b, t^c; \bar{t}_a] \delta_{1\alpha} \delta_{2\beta} \delta_{2\gamma} + [t^b, t^c; \bar{t}_a]^{*} \delta_{2\alpha} \delta_{1\beta} \delta_{1\gamma}.$$  \[90\]

Here the 3-brackets

$$[t^a, t^c; \bar{t}_b] = f^{ac}_{bd} t^d.$$  \[91\]

are those for the $\mathcal{N} = 6$ 3-algebra. Such 3-brackets were first proposed by Bagger and Lambert \[32\] for a $\mathcal{N} = 6$ CSM theory. An unusual feature of
the 3-brackets is that it involves complex conjugate for the third generator. Our above decomposition from the $N = 5$ 3-algebra reveals clearly the origin of the need for complex conjugation of the third generator.

Later we will see that the structure constants defined in Eq. (91) are indeed anti-symmetric in the first two indices. (See Eq. (93).) With Eq. (90), the fundamental identity (6) reduces to

$$f^{eg}_d g^{ae}_b f^{bc}_d = 0,$$

as desired. Also the constraint condition (23) on the structure constants and the symmetry properties (24) of the structure constants reduce to

$$f^{ab}_{cd} = -f^{ba}_{cd} = f^{ba}_{dc}.$$

One easily recognizes that eqs. (89), (91), (92), (86), and (93) are those defining the $N = 6$ 3-algebra used in ref. [32]. (The relation between the $N = 6$ 3-algebra and super Lie algebra was discussed in Ref. [36].)

Substituting Eq. (76), (78), (83), and (84) into the $N = 5$ Lagrangian (54) and the SUSY transformation law (28), and using the $Sp(4)$ identity (66) and (73), we reproduce the $N = 6$ Lagrangian

$$\mathcal{L} = -D_{\mu} Z^{a}_{A} D^{\mu} Z^{a}_{A} - i \bar{\psi}^{Aa}_{\mu} \gamma^{\mu} D_{\mu} \psi^{Aa}$$

$$- i f^{ab}_{cd} \bar{\psi}^{Ad}_{\mu} \gamma^{\mu} D_{\mu} \psi^{Bc}_{\mu} + 2 i f^{ab}_{cd} \bar{\psi}^{Ad}_{\mu} \gamma^{\mu} D_{\mu} \psi^{Bc}_{\mu}$$

$$- \frac{i}{2} \varepsilon_{ABCD} f^{ab}_{cd} \bar{\psi}^{Ac}_{\mu} \psi^{Bd}_{\mu} Z_{a}^{A} Z_{b}^{B} - \frac{i}{2} \varepsilon_{ABCD} f^{cd}_{ab} \bar{\psi}^{Ac}_{\mu} \psi^{Bd}_{\mu} \bar{Z}_{a}^{A} \bar{Z}_{b}^{B}$$

$$+ \frac{1}{2} \varepsilon^{\mu \lambda \rho} f^{ab}_{cd} A_{\mu}^{\epsilon} \bar{\psi}^{Ad}_{\mu} \psi^{Bc}_{\mu} + \frac{2}{3} \varepsilon f^{acbg}_{def} f^{bg}_{ac} f^{ab}_{cd}$$

$$- \frac{1}{2} \varepsilon f^{cd}_{ab} f^{fe}_{bg} \bar{Z}_{a}^{A} Z_{b}^{A} Z_{c}^{B} Z_{d}^{B} Z_{e}^{C} Z_{f}^{C}$$

and the $N = 6$ SUSY transformation law reads

$$\delta Z^{a}_{A} = -i \varepsilon^{AB} \psi_{Bd}$$

$$\delta \bar{Z}^{a}_{A} = -i \bar{\varepsilon}_{AB} \psi^{Bd}$$

$$\delta \psi_{Bd} = \gamma^{\mu} D_{\mu} Z^{a}_{A} \epsilon_{AB} + f^{ab}_{cd} Z^{a}_{A} Z^{b}_{b} \bar{Z}^{c}_{c} \epsilon_{AB} + f^{ab}_{cd} Z^{a}_{A} Z^{B}_{B} \bar{Z}^{c}_{c} \epsilon_{CD}$$

$$\delta \bar{\psi}^{Bd} = \gamma^{\mu} D_{\mu} \bar{Z}^{a}_{A} \epsilon^{AB} + f^{cd}_{ab} \bar{Z}^{a}_{C} \bar{Z}^{B}_{A} Z^{C}_{c} \epsilon^{AB} + f^{cd}_{ab} \bar{Z}^{a}_{C} \bar{Z}^{B}_{D} Z^{C}_{c} \epsilon^{CD}$$

$$\delta A^{a}_{b} = -i \varepsilon_{AB} Z^{a}_{A} \gamma_{\mu} \bar{\psi}^{Bb}_{\mu} f^{ca}_{bd} + \frac{i}{2} \varepsilon \gamma_{\mu} \bar{Z}^{a}_{A} \psi^{Bb}_{\mu} f^{ca}_{bd}.$$

Here the SUSY transformation parameters $\epsilon_{AB}$ satisfy

$$\epsilon_{AB} = \epsilon_{BA}$$

$$\epsilon^{*} = \epsilon_{AB} = \frac{1}{2} \varepsilon^{ABCD} \epsilon_{CD}$$
Now the parameters $\epsilon_{AB}$ transform as the 6 of $SU(4)$. It is in this sense that the global $\mathcal{N} = 5$ SUSY gets enhanced to $\mathcal{N} = 6$. The Lagrangian (94) and the transformation law (95) are the same as the ones obtained in the 3-algebra approach for $\mathcal{N} = 6$ theories in ref. [32].

The $\mathcal{N} = 6$ superconformal CSM theories in three dimensions can be classified by super Lie algebras [28, 30, 38] or by using group theory [37]. Two primary types are allowed: with gauge group $U(M) \times U(N)$ and $Sp(2N) \times U(1)$, respectively. In the next two subsections we will drive these two theories by specifying the structure constants of the $\mathcal{N} = 6$ 3-algebra.

4.2 $\mathcal{N} = 6$, $Sp(2N) \times U(1)$

The Lagrangian and SUSY transformation law for this theory were first constructed in ref. [33], starting from a formalism for the symplectic 3-algebra, involving an anti-symmetric tensor, that is different from the 3-algebra formalism of Bagger and Lambert [32]. Here we would like to present the theory completely in the framework of ref. [32]. We first specify the structure constants as

$$\omega_{a-e, f+} = -k[(\omega_{ab} \omega_{cd} + \omega_{ac} \omega_{bd})h_{+-}h_{+-} + (\omega_{ad} \epsilon_{+-})(\omega_{bc} \epsilon_{+-})],$$

(98)

where $k$ is a real constant, $\omega^{ab}$ an antisymmetric bilinear form ($a, b = 1, 2, \cdots, 2N$), $h_{+-} = h_{-+} = 1$ and $\epsilon_{+-} = -\epsilon_{-+} = ih_{+-}$. Here $a, b$ are the $Sp(2N)$ indices while $+,-$ the $SO(2)$ indices. And $\omega_{a-e, +} \equiv \omega_{ae} h_{+-}$ is the gauge invariant antisymmetric tensor. Since $\omega_{a-e, +}$ is non-singular, Eq. (98) is equivalent to the following equation:

$$f^{a+b}_{c+d} = k(\omega^{ab} \omega_{cd} + \delta^a_d \delta^b_c - \delta^a_c \delta^b_d)\delta^{+-} + \delta^{+-}.$$  

(99)

Suppressing the $SO(2)$ indices gives

$$f^{ab}_{cd} = k(\omega^{ab} \omega_{cd} + \delta^a_d \delta^b_c - \delta^a_c \delta^b_d).$$

(100)

It is not too difficult to check that the structure constants satisfy the FI (92) and the reality condition (86), and also have the desired symmetry properties.

\[5\text{In the Lagrangian (94) of section 4.1, the index } a \text{ runs from 1 to } L. \text{ In this subsection, we split it into two indices: } a \rightarrow a\pm, \text{ and set } L = 4N. \text{ We hope this will not cause any confusion.}\]
We see that after suppressing the $SO(2)$ indices, the structure constants are the same as the components of an embedding tensor in Ref. [27].

In fact, in accordance with Eqs. (85) and (100), the gauge fields can be decomposed into two parts:

$$\tilde{A}_\mu^{cd} = A_\mu^b f^{ca}_{\quad bd}$$

$$\equiv - (A_\mu^c A^d_c + A_\mu^d A^c_d) + (A_\mu^a A^a_d) \delta^c_d$$

$$\equiv B_\mu^{cd} + A_\mu^c \delta^d_c.$$ 

It is natural to identify $A_\mu$ as the $U(1)$ part of the gauge potential, and $B_\mu^{cd}$ as the $Sp(2N)$ part. The reason is that we can recast $B_\mu^{cd}$ as $A_{ab}^{\mu} (t_{ab})^{cd}$, where $(t_{ab})^{cd}$ is in the fundamental representation of the Lie algebra of $Sp(2N)$.

We substitute the structure constants (100) into (95). We then obtain the $N=6$ (on-shell) SUSY transformation law in the theory (see Appendix B.1). The equations of motion can be derived from the Lagrangian obtained by substituting eqs. (100) into the Lagrangian (94) and replacing $A_\mu^a b$ by $\frac{1}{k} A_\mu^b a$ (see Appendix B.1). The SUSY transformation law (116) and the Lagrangian (115) are indeed in agreement with the $N=6, Sp(2M) \times U(1)$ superconformal CSM theory derived from the symplectic 3-algebra in ref. [33], or from the ordinary Lie algebra in ref. [30].

4.3 $\mathcal{N} = 6$, $U(M) \times U(N)$

The Lagrangian this theory has been constructed in ref. [32]. For this paper to be self-contained, it is worth to present the Lagrangian and SUSY transformation law of $D = 3, \mathcal{N} = 6$, $U(M) \times U(N)$ theory in this subsection.

To generate a direct gauge group such as $U(M) \times U(N)$, we split up a lower 3-algebra index $a$ into two indices: $a \rightarrow n\hat{n}$, where $n = 1, ..., M$ is a fundamental index of $U(M)$, $\hat{n} = 1, ..., N$ an anti-fundamental index of $U(N)$. With this decomposition, the hermitian inner product (89) can be written as a trace:

$$X^a Y_a \rightarrow X^a_{n\hat{n}} Y_{n\hat{n}} = X^a_{\hat{n}n} Y_{\hat{n}n} \equiv \text{Tr}(X^\dagger Y),$$

where the superscript “t” stands for the usual transpose. On the other hand, according to the definition (89), the hermitian inner product can be also written as: $X^a Y_a \equiv \bar{X}^a Y_a$, which leads us to decompose an upper index $a$ as $a \rightarrow \hat{n}n$. Thus the hermitian inner product can be written as

$$X^a Y_a \equiv \bar{X}^a Y_a \rightarrow \bar{X}^{\hat{n}n} Y_{n\hat{n}} \equiv \text{Tr}(\bar{X} \bar{Y}) = \text{Tr}(X^\dagger Y).$$

22
We then specify the 3-bracket (91) to be

$$[\hat{t}^{kk}, \hat{t}^{ll}; \bar{t}^{m\hat{m}}] = k (\delta^{k}_{\hat{m}} \delta^{l}_{m} \hat{t}^{kl} - \delta^{l}_{\hat{m}} \delta^{k}_{m} \hat{t}^{kl}). \quad (104)$$

The structure constants can be easily read off as

$$f^{kk, ll}_{m\hat{m}, n\hat{n}} = k (\delta^{k}_{\hat{m}} \delta^{l}_{n} \delta^{k}_{\hat{n}} \delta^{l}_{m} - \delta^{l}_{\hat{m}} \delta^{k}_{n} \delta^{k}_{\hat{n}} \delta^{l}_{m}). \quad (105)$$

It is straightforward to check that the structure constants $f^{kk, ll}_{m\hat{m}, n\hat{n}}$ satisfy the FI (92) and the reality conditions (86), and has the symmetry properties (93). The structure constants are first discovered by BL [32] (though they did not write down Eq. (105) explicitly), and they are also the same as the components of an embedding tensor in Ref. [27].

Now let us show that the 3-bracket (104) is indeed equivalent to Bagger and Lambert’s 3-bracket [32]. Writing $X = X^{kk} t^{kk}$, and $\bar{Z} = \bar{Z}^{m\hat{m}} \bar{t}^{m\hat{m}}$, by Eq. (104), one can get

$$[X, Y; \bar{Z}] = k (X \bar{Z} Y - Y \bar{Z} X) n\hat{n} t^{n\hat{n}}. \quad (106)$$

The right hand side is the ordinary matrix multiplication. It is exactly the same as Eq. (53) of Ref. [32]. In accordance with eq. (105), the gauge fields can be decomposed as

$$\hat{A}_{\mu}^{kk} n\hat{n} = A_{\mu}^{\hat{m}\hat{m}} f^{kk, ll}_{m\hat{m}, n\hat{n}}$$

$$= A_{\mu}^{kl} t^{kl} \delta^{k}_{n} A_{\mu}^{lk} n\hat{n} \delta^{k}_{\hat{n}}$$

$$\equiv \hat{A}_{\mu}^{k\hat{n}} n\hat{n} + A_{\mu}^{k\hat{n}} n\hat{n} \delta^{k}_{\hat{n}}. \quad (107)$$

So the 3-bracket (106) and the FI (92) generate a $U(M) \times U(N)$ gauge group [32], with $\hat{A}_{\mu}^{k\hat{n}}$ the $U(M)$ part and $A_{\mu}^{k\hat{n}}$ the $U(N)$ part of the gauge potential.

The supersymmetry transformation law and the Lagrangian in this theory can be obtained by substituting the expression (105) of the structure constants into eqs. (95) and (94), and replacing $A_{\mu}^{ba} b_{a}$ by $\hat{A}_{\mu}^{b\hat{a}} a$. To make the paper self-contained, we include the results in Appendix B.2. The SUSY transformation law (120) and the Lagrangian (117) are in agreement with the $D = 3, \mathcal{N} = 6$ $U(M) \times U(N)$ CSM theory, which has been derived from the ordinary Lie algebra approach in ref. [30] and from the 3-algebra approach in ref. [32].

This theory is conjectured to be the dual gauge theory of M2 branes a $\mathbb{C}^{4}/\mathbb{Z}_{k}$ singularity. If $M = N$, this theory becomes the well-known ABJM model [22, 23, 25].
5 Conclusions

In this paper, we first introduce an anti-symmetric tensor $\omega_{ab}$ into a 3-algebra, with structure constants of the 3-bracket being symmetric in the first two indices. We call it a symplectic 3-algebra. We then construct the general $\mathcal{N} = 5$ superconformal CSM theory with $Sp(4)$ R-symmetry in three dimensions in terms of this symplectic 3-algebra. All matter fields take values in this symplectic 3-algebra. The gauge symmetry is generated by the 3-bracket and FI. By specifying the 3-brackets, we provide the $\mathcal{N} = 5, Sp(2N) \times O(M)$ CSM theory as an example of our theory. It would be nice to see if CSM theories with other gauge groups (for example, $G_2 \times SU(2)$ [27]) for multiple M2 branes can be generated in a similar way with the 3-algebra approach. Also it would be interesting to generalize the symplectic 3-algebra theory, so that it can describe CSM quiver gauge theories.

We have succeeded in enhancing the $\mathcal{N} = 5$ supersymmetry to $\mathcal{N} = 6$ by decomposing the symplectic 3-algebra and the fields properly. At the same time, we also demonstrate that the FI and the symmetry and reality properties of the structure constants of the $\mathcal{N} = 6$ 3-algebra can be derived from the $\mathcal{N} = 5$ counterparts. Hence some of $\mathcal{N} = 5,6$ superconformal CSM theories are described by a unified symplectic 3-algebraic framework. It would be nice to investigate the relation between these two kinds of 3-algebras further. By specifying the 3-brackets, the $\mathcal{N} = 6, Sp(2N) \times U(1)$ and $U(M) \times U(N)$ CSM, including the ABJM theory, are derived. We also compare the approach used in our previous paper [33] with that of this paper. The same theory $(Sp(2N) \times U(1))$ are derived by starting from different 3-algebra formalisms.

It would be nice to construct the $\mathcal{N} \leq 4$ superconformal CSM theories [39, 40] for multiple M2 branes in terms of 3-algebras.

6 Acknowledgement

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A Conventions and Useful Identities

In $1 + 2$ dimensions, the gamma matrices are defined as $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$. For the metric we use the $(-,+,+,+)$ convention. The gamma matrices can be defined as the Pauli matrices: $\gamma_\mu = (i\sigma_2, \sigma_1, \sigma_3)$, satisfying the important identity

$$\gamma_\mu \gamma_\nu = \eta_{\mu\nu} + \epsilon_{\mu\nu\lambda} \gamma^\lambda.$$  \hfill (108)

We also define $\epsilon^{\mu\nu\lambda} = -\epsilon_{\mu\nu\lambda}$. So $\epsilon_{\mu\nu\lambda} \epsilon^{\rho\nu\lambda} = -2\delta^\rho_\mu$. The Fierz transformation is

$$(\bar{\lambda} \chi) \psi = -\frac{1}{2}(\bar{\lambda} \psi) \chi - \frac{1}{2}(\bar{\lambda} \gamma_\nu \psi) \gamma^\nu \chi.$$ \hfill (109)

Some useful $Sp(4)$ identities are

$$\tilde{\epsilon}^1_{AC} \epsilon_{2C} - \tilde{\epsilon}^2_{AC} \epsilon_{1C} = \tilde{\epsilon}^{BC} \epsilon_{2C} A - \tilde{\epsilon}^{BC} \epsilon_{1C} A$$ \hfill (110)

$$\frac{1}{2} \tilde{\epsilon}^1_{CD} \gamma_\nu \epsilon_{2CD} \delta^A_B = -\tilde{\epsilon}^1_{AC} \gamma_\nu \epsilon_{2BC} - \tilde{\epsilon}^2_{AC} \gamma_\nu \epsilon_{1BC}$$ \hfill (111)

$$2\tilde{\epsilon}^1_{AC} \epsilon_{2BD} - 2\tilde{\epsilon}^2_{AC} \epsilon_{1BD} = \tilde{\epsilon}^{CE} \epsilon_{2DE} \delta^A_B - \tilde{\epsilon}^{CE} \epsilon_{1DE} \delta^A_B$$
$$- \tilde{\epsilon}^{AE} \epsilon_{2DE} \delta^C_B + \tilde{\epsilon}^{AE} \epsilon_{1DE} \delta^C_B$$
$$+ \tilde{\epsilon}^{AE} \epsilon_{2BE} \delta^C_D - \tilde{\epsilon}^{AE} \epsilon_{1BE} \delta^C_D$$
$$- \tilde{\epsilon}^{AE} \epsilon_{2BE} \delta^A_D + \tilde{\epsilon}^{AE} \epsilon_{1BE} \delta^A_D$$ \hfill (112)

$$\frac{1}{2} \epsilon^{ABCD} \tilde{\epsilon}^1_{EF} \gamma_\mu \epsilon_{2EF} = \tilde{\epsilon}_{1AB} \gamma_\mu \epsilon_{2CD} - \tilde{\epsilon}_{2AB} \gamma_\mu \epsilon_{1CD}$$
$$+ \tilde{\epsilon}_{1AD} \gamma_\mu \epsilon_{2BC} - \tilde{\epsilon}_{2AD} \gamma_\mu \epsilon_{1BC}$$
$$- \tilde{\epsilon}_{1BD} \gamma_\mu \epsilon_{2AC} + \tilde{\epsilon}_{2BD} \gamma_\mu \epsilon_{1AC}$$ \hfill (113)

$$\epsilon^{ABCD} = -\omega^{AB} \omega^{CD} + \omega^{AC} \omega^{BD} - \omega^{AD} \omega^{BC}.$$ \hfill (114)

The $Sp(4)$ indices can lowered and raised by the anti-symmetric tensor $\omega_{AB}$ and its inverse $\omega^{AB}$.

B SUSY Transformation Law and Lagrangian in $D = 3, \mathcal{N} = 6$ CSM Theories

For this paper to be self contained, below we give the explicit form of the SUSY transformation law and the Lagrangian for the $D = 3, \mathcal{N} = 6$ CSM theories with $SU(4)$ R-symmetry. For the notations, see the corresponding subsections in the text.
B.1 \( Sp(2N) \times U(1) \) CSM Theory

The Lagrangian of the theory is given by

\[
\mathcal{L} = -D_{\mu} \tilde{Z}_A^a D^a Z^A_{\mu} - i\tilde{\psi}^A \gamma^\mu D_{\mu} \psi^A - 2iD_{\mu} \tilde{\psi}_B^a \psi^A_{\mu} Z^A_{\mu} - \tilde{\psi}^A \gamma^\mu D_{\mu} \psi^A + \mathcal{L}_{CS}^{(1)}
\]

The SUSY transformation laws are given by

\[
\begin{align}
\delta Z^A_{\mu} &= -ie^{AB} \partial_{\mu} A^A + \frac{1}{8} e^{\mu \nu \lambda} Tr(B_{\mu} \partial_{\nu} B_{\lambda} + \frac{2}{3} B_{\mu} B_{\nu} B_{\lambda}) \\
\delta \tilde{Z}^a_{\mu} &= -i\bar{\epsilon}_{AB} \psi^{B\dagger}_{\mu} \\
\delta \psi^A_{\mu} &= \epsilon^A_{AB} \psi^{B\dagger}_{\mu} \\
\delta \psi^B_{\mu} &= \epsilon^{AB} \psi^{B\dagger}_{\mu} \\
\delta A_{\mu} &= -i\epsilon^{AB} \gamma_{\mu} \psi^{B\dagger}_{A} \\
\delta B_{\mu}^a &= i\epsilon^{AB} \gamma_{\mu} \omega^{ca} Z^A_{\omega} \psi^{B\dagger}_{a} - i\epsilon^{AB} \gamma_{\mu} \bar{Z}^a_{BD} \psi^{B\dagger}_{A} Z^C_{\omega} \psi^{B_{\omega}} \\
\end{align}
\]

B.2 \( U(M) \times U(N) \) CSM Theory

The Lagrangian of the theory is given by

\[
\mathcal{L} = -Tr(D_{\mu} \bar{Z}_A^a D^a Z^A_{\mu}) - iTr(\bar{\psi}^A \gamma^\mu D_{\mu} \psi^A) - V + \mathcal{L}_{CS}^{(2)}
\]

\[
\begin{align}
\delta \bar{Z}^a_{\mu} &= -i\epsilon_{AB} \gamma_{\mu} \omega^{ca} Z^A_{\omega} \psi^{B\dagger}_{a} \\
\delta \bar{\psi}^A_{\mu} &= -i\epsilon^{AB} \gamma_{\mu} \psi^{B\dagger}_{A} \\
\delta \psi^A_{\mu} &= -i\epsilon^{AB} \gamma_{\mu} \bar{Z}^a_{BD} \psi^{B\dagger}_{A} Z^C_{\omega} \psi^{B_{\omega}} \\
\delta Z^A_{\mu} &= -i\epsilon^{AB} \gamma_{\mu} \bar{Z}^a_{BD} \psi^{B\dagger}_{A} Z^C_{\omega} \psi^{B_{\omega}} + i\epsilon^{AB} \gamma_{\mu} \bar{Z}^a_{BD} \psi^{B\dagger}_{A} Z^C_{\omega} \psi^{B_{\omega}} \\
\end{align}
\]
The Lagrangian (117) is the same obtained by BL [32], except for that we re-scale the gauge fields by a factor $\frac{1}{k}$. The potential term is

$$V = 2k^2 \text{Tr}(\bar{Z}_A Z^A \bar{Z}_B Z^B Z^C \bar{Z}_C) - \frac{4k^2}{3} \text{Tr}(Z^A \bar{Z}_B Z^C \bar{Z}_A Z^B \bar{Z}_C)$$

$$- \frac{k^2}{3} \text{Tr}(Z^A \bar{Z}_A Z^B \bar{Z}_B Z^C \bar{Z}_C + \bar{Z}_A Z^A \bar{Z}_B Z^B \bar{Z}_C Z^C).$$

(118)

The Chern-Simons term reads

$$\mathcal{L}_{CS} = \frac{1}{2k} \varepsilon^{\mu\nu\lambda} \text{Tr} \left( \hat{A}_\mu \partial_\nu \hat{A}_\lambda + \frac{2}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda - A_\mu \partial_\nu A_\lambda - \frac{2}{3} A_\mu A_\nu A_\lambda \right).$$

(119)

The $\mathcal{N}=6$ SUSY transformation laws, which are closed on-shell with the equations of motion derivable from the above lagrangian (117), are given by

$$\delta Z^A = -i \bar{\epsilon}^{AB} \psi_B$$

$$\delta \bar{Z}_A = -i \bar{\epsilon}_{AB} \bar{\psi}^B$$

$$\delta \psi_B = \gamma^\mu D_\mu Z^A \epsilon_{AB} + k(Z^C \bar{Z}_C Z^A - Z^A \bar{Z}_C Z^C) \epsilon_{AB} + 2k Z^C \bar{Z}_B Z^D \epsilon_{CD}$$

$$\delta \bar{\psi}^B = \gamma^\mu D_\mu Z^A \epsilon^{AB} + k(Z^A Z^C \bar{Z}_C - Z^C Z^C Z_\lambda) \epsilon^{AB} + 2k Z^D \bar{Z}_B \bar{Z}_C \epsilon^{CD}$$

$$\delta \hat{A}_\mu = -ik \bar{\epsilon}_{AB} \gamma_\mu \bar{\psi}^B Z^A + ik \bar{\epsilon}^{AB} \gamma_\mu Z_\lambda \psi_B$$

$$\delta A_\mu = ik \bar{\epsilon}_{AB} \gamma_\mu Z^A \bar{\psi}^B - ik \bar{\epsilon}^{AB} \gamma_\mu \psi_B \bar{Z}_A.$$  

(120)

References

[1] W. Chen, G.W. Semenoff and Y.S. Wu, “Scale and Conformal Invariance in Chern-Simons-Matter Field Theory”, Phys. Rev. D44, 1625 (1991).

[2] W. Chen, G. Semenoff and Y.S. Wu, “Probing Topological Features in Perturbative Chern-Simons Gauge Theory”, Mod. Phys. Lett. A5, 1833 (1990).

[3] W. Chen, G.W. Semenoff and Y.S. Wu, “Two-Loop Analysis of Chern-Simons-Matter Theory”, Phys. Rev. D46, 5521 (1992); arXiv:hep-th/9209005.

[4] O.M. Del Cima, D.H.T. Franco, J.A. Helayel-Neto and O. Piguet, “An algebraic proof on the finiteness of Yang-Mills-Chern-Simons theory in D=3”, Lett. Math. Phys. 47, 265 (1999); arXiv:math-ph/9904030.
[5] Nikolas Akerblom, Christian Saemann, Martin Wolf, “Marginal Deformations and 3-Algebra Structures”, arXiv:0906.1705 [hep-th].

[6] Y. Nambu, Generalized Hamiltonian mechanics, Phys. Rev. D7 (1973), 2405-2412.

[7] H. Awata, M. Li, D. Minic, T. Yoneya, “On the quantization of nambu brackets,” arXiv:hep-th/9906248.

[8] Giulio Bonelli, Alessandro Tanzini, Maxim Zabzine, “Topological branes, p-algebras and generalized Nahm equations ;” Phys.Lett.B 672: 390-395, 2009, arXiv:0807.5113 [hep-th].

[9] J. Bagger and N. Lambert, “Modeling multiple M2’s,” Phys. Rev. D 75, 045020 (2007), arXiv:hep-th/0611108; J. Bagger and N. Lambert, “Gauge Symmetry and Supersymmetry of Multiple M2-Branes,” Phys. Rev. D 77, 065008 (2008), arXiv:0711.0955 [hep-th]; J. Bagger and N. Lambert, “Comments On Multiple M2-branes,” JHEP 0802, 105 (2008), arXiv:0712.3738 [hep-th].

[10] A. Gustavsson, “Algebraic structures on parallel M2-branes,” arXiv:0709.1260 [hep-th]; “Selfdual strings and loop space Nahm equations,” JHEP 0804, 083 (2008), arXiv:0802.3456 [hep-th].

[11] Marcus K. Benna, Igor R. Klebanov, Thomas Klose, “Charges of Monopole Operators in Chern-Simons Yang-Mills Theory,” arXiv:0906.3008 [hep-th].

[12] Andreas Gustavsson, Soo-Jong Rey, “Enhanced N=8 Supersymmetry of ABJM Theory on R(8) and R(8)/Z(2),” arXiv:0906.3568 [hep-th].

[13] O-Kab Kwon, Phillial Oh, Jongsu Sohn, “Notes on Supersymmetry Enhancement of ABJM Theory,” arXiv:0906.4333 [hep-th].

[14] J. Distler, S. Mukhi, C. Papageorgakis and M. Van Raamsdonk, “M2-branes on M-folds,” JHEP 0805, 038 (2008) arXiv:0804.1256 [hep-th].

[15] N. Lambert and D. Tong, “Membranes on an Orbifold,” arXiv:0804.1114 [hep-th].

[16] Chethan Krishnan, Carlo Maccaferri, “Membranes on Calibrations,” JHEP 0807, 005 (2008) arXiv:0805.3125 [hep-th].
[17] Jerome P. Gauntlett, Jan B. Gutowski, “Constraining Maximally Supersymmetric Membrane Actions,” arXiv:0804.3078 [hep-th].

[18] G. Papadopoulos “M2-branes, 3-Lie Algebras and Plucker relations ,” arXiv:0804.2662 [hep-th].

[19] J. Gomis, G. Milanesi and J. G. Russo, “Bagger-Lambert Theory for General Lie Algebras,” arXiv:0805.1012 [hep-th]; S. Benvenuti, D. Rodriguez-Gomez, E. Tonni and H. Verlinde, “N=8 superconformal gauge theories and M2 branes,” arXiv:0805.1087 [hep-th]; P. M. Ho, Y. Imamura and Y. Matsuo, “M2 to D2 revisited,” arXiv:0805.1202 [hep-th];

[20] M. A. Bandres, A. E. Lipstein and J. H. Schwarz, “Ghost-Free Supersymmetric Action for Multiple M2-Branes,” arXiv:0806.0054 [hep-th];

[21] J. Gomis, D. Rodriguez-Gomez, M. Van Raamsdonk and H. Verlinde, “Supersymmetric Yang-Mills Theory From Lorentzian Three-Algebras,” arXiv:0806.0738 [hep-th].

[22] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, “N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals,” arXiv:0806.1218 [hep-th].

[23] M. Benna, I. Klebanov, T. Klose and M. Smedback, “Superconformal Chern-Simons Theories and AdS4/CFT3 Correspondence,” arXiv:0806.1519 [hep-th].

[24] Andreas Gustavsson, Soo-Jong Rey, “Enhanced N=8 Supersymmetry of ABJM Theory on R(8) and R(8)/Z(2),” arXiv:0906.3568 [hep-th].

[25] M. Bandres, A. Lipstein and J. Schwarz “Studies of the ABJM Theory in Formulation with Manifest SU(4) R-Symmetry,” arXiv:0807.0880 [hep-th].

[26] E. A. Bergshoeff, M. de Roo and O. Hohm, “Multiple M2-branes and the Embedding Tensor,” arXiv:0804.2201 [hep-th]; E. A. Bergshoeff, M. de Roo, O. Hohm and D. Roest, “Multiple Membranes from Gauged Supergravity,” arXiv:0806.2584 [hep-th].
[27] E. A. Bergshoeff, O. Hohm, D. Roest, H. Samtleben and E. Sezgin, “The Superconformal Gaugings in Three Dimensions,” arXiv:0807.2841 [hep-th].

[28] D. Gaiotto and E. Witten, “Janus Configurations, Chern-Simons Couplings, And The Theta-Angle in N=4 Super Yang-Mills Theory,” arXiv:0804.2907 [hep-th].

[29] K. Hosomichi, K. M. Lee, S. Lee, S. Lee and J. Park, “N=4 Superconformal Chern-Simons Theories with Hyper and Twisted Hyper Multiplets,” arXiv:0805.3662 [hep-th].

[30] K. Hosomichi, K. M. Lee, S. Lee, S. Lee and J. Park, “N=5,6 Superconformal Chern-Simons Theories and M2-branes on Orbifolds,” arXiv:0806.4977 [hep-th].

[31] O. Aharony, O. Bergman and D. L. Jafferis, “Fractional M2-branes,” arXiv:0807.4924 [hep-th].

[32] J. Bagger and N. Lambert, “Three-Algebras and N=6 Chern-Simons Gauge Theories,” arXiv:0807.0163 [hep-th].

[33] Fa-Min Chen, Yong-Shi Wu, “Symplectic Three-Algebra and \( \mathcal{N} = 6,\, \text{Sp}(2N) \times U(1) \) Superconformal Chern-Simons-Matter Theory ,” arXiv:0902.3454[hep-th].

[34] Jos Figueroa-O’Farrill, “Simplicity in the Faulkner construction,” arXiv:0905.4900 [hep-th].

[35] Paul de Medeiros, Jos Figueroa-O’Farrill, Elena Mndez-Escobar, Patricia Ritter, “On the Lie-algebraic origin of metric 3-algebras,” arXiv:0809.1086 [hep-th].

[36] Jakob Palmkvist, “Three-algebras, triple systems and 3-graded Lie superalgebras,” arXiv:0905.2468 [hep-th].

[37] M. Schnabl and Y. Tachikawa, “Classification of \( \mathcal{N} = 6 \) superconformal theories of ABJM type,” arXiv:0807.1102 [hep-th].

[38] V. G. Kac, “Lie Superalgebras,” Adv. Math. 26 (1977) 8.
[39] Paul de Medeiros, Jos Figueroa-O’Farrill, Elena Mndez-Escobar, “Superpotentials for superconformal Chern-Simons theories from representation theory,” arXiv:0908.2125 [hep-th].

[40] S. Cherkis and C. Saemann, “Multiple M2-branes and generalized 3-Lie algebras,” Phys. Rev. D 78: 066019 (2008), arXiv:0807.0808 [hep-th].