Melting and viscous dissipation effect on upper-convected Maxwell and Williamson nanofluid

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Abstract
This article mainly addresses the influence of the viscous dissipation, melting, and chemical reaction on Williamson and Maxwell nanofluids over a stretching sheet embedded in porous media. The system of partial differential equations which is obtained by conservation principles, is transformed by means of an appropriate similarity transformation into a system of ordinary differential equations. The numerical results are obtained by employing the Keller box method. The impacts of different germane parameters on velocity profiles, thermal and concentration fields, Nusselt number, skin friction coefficient, and Sherwood number are selected by means of graphical and tabular representations. Our numerical solution detects that the dimensionless melting parameter highly affects the velocity boundary layer of a Williamson nanofluid when compared with an upper-convected Maxwell nanofluid. Moreover, the velocity, temperature, and concentration distributions decrease for both fluids when the permeability parameter increases. Furthermore, the temperature distribution increases with an increase of the Eckert number.

KEYWORDS
chemical reaction, melting heat transfer, porous media, upper-convected Maxwell fluid, viscous dissipation, Williamson fluid

1 | INTRODUCTION

An engineered fluid that can be used in modern high technological area with huge heat diffusivity capacity is called nanofluid. For the first time such fluid is experimental studied by Choi and Eastman.1 The experimental result reveals the pumping powers during heat exchange were significantly reduced when nanofluid is involved in the processes. Das,2 Pak and Cho,3 Xuan and Li4 experimentally discovered that convective fluid thermal conductivity can be elevated up 10% to 50% with addition of small amount of volumetric fraction of nanoparticles. The study further indicates that in common fluids, to improve the heat transfer by a factor of 2, the pumping power were increased by a factor of 10. The examinations of nanofluids were encompassing wide area of fluid dynamic studies. Accordingly, Ramesh and Gireesha5 have computed the impacts of nanofluid past a Riga plate with nonlinear radiative heat transfer. Moreover, the influences of energy generation of nanoliquid past a disk were studied by Mahanthesh et al.6 They reckoned out that the Nusselt number is boosted with an addition of nanoparticle. Furthermore, the influences of velocity slip on MHD flow...
of Eyring-Powell nanofluid past isothermal sphere were deliberated by Swarnalathamma. The thermal performance of single-walled carbon nanotube nanofluid under turbulent flow conditions was investigated by Fadodun et al. Their results indicate that convective heat transfer (average Nusselt number) increases by 7.48% while the pressure drop and pumping power increase by 119% and 199%, respectively. Mathematical modeling of non-Newtonian fluid with chemical aspects via numerical technique was investigated by Hayat et al. Ijaz Khan et al. examined new thermodynamics of entropy generation minimization in the presence of nonlinear thermal radiation and nanomaterial. Raju and Ojjela examined the effects of the induced magnetic field motion on mixed convective Jeffrey nanofluid flow through a porous channel due to thermophoresis and Brownian. They found that when the values of viscoplastic Eyring-Powell fluid parameter increase the flow, thermal, and concentration boundary layer thickness decelerates. The Brownian motion and thermophoresis aspects in nonlinear flow of micropolar nanoliquid overstretching surface have been investigated by Hayat et al.

A substantial amount of experimental and theoretical work has been voted for to determine the role of natural convection in the kinetics of heat transfer escorted with melting or solidification effect. The study of heat transfer escorted by melting has received much attention because of its important applications in areas such as liquid polymer extrusion, frozen ground thawing, permafrost melting, casting, and welding processes as well as phase change material, hot eutrophication, polymers, ceramics, geothermal energy recovery, silicon wafer process, thermal insulation, and so on. The dynamics of melting processes under different circumstances were described by Roberts, Tien and Yen, Epstein and Cho, and Kairi and Murthy. The result reveals that melting enhances heat and mass transfer. Moreover, different researchers such as, Krishnamurthy et al., Kumar and Gireesha, Khan et al., Kairi and Ram Reddy, and Babu and Narayana have discussed the influences of melting on non-Newtonian fluids like Williamson and Burgers fluid exposed to different physical conditions like porosity, radiation, and so on. Still further, simultaneous effects of melting heat transfer and inclined magnetic field flow of tangent hyperbolic fluid over a nonlinear stretching surface with homogeneous-heterogeneous reactions was studied by Quyuum et al.

As indicated in different literatures viscous dissipation is the unidirectional processes by which the work done by a fluid on adjacent layers due to the action of shear forces are transformed into heat. It has significant impact on heat transfer; especially for high-velocity flows, fluids with a moderate Prandtl number, highly viscous flows at moderate velocities and moderate velocities with small wall-to-fluid temperature difference. Accordingly, Pop and Yasin et al. remarked that energy dissipation and Joule heating in fluid dynamic has a great significance for the design of energy-conversion systems and energy-efficient circuits. Furthermore, the impact of Joule heating, viscous dissipation, and heat generation of fluid like Oldroyd B and Casson nanofluid has been discussed by Kumar et al., Reddy et al., Ajayi et al., and Yohannes and Shankar. A modified homogeneous-heterogeneous reaction for MHD stagnation flow with viscous dissipation and Joule heating was investigated by Khan et al. Still further, Hayat et al. examined by entropy generation in magnetohydrodynamic radiative flow due to rotating disk in presence of viscous dissipation and Joule heating.

In most industries nowadays the importance of non-Newtonian fluids dominates the Newtonian fluids. The rheological possessions of non-Newtonian fluids cannot be elucidated by the classical Naiver-Stokes equations. Also no single model is sufficient to describe non-Newtonian fluids characteristics. To overcome this difficulty several models have come into being. The rheological models that were proposed were Williamson, Cross, Ellis, power law, Carreau fluid model, and so on. Typical of a non-Newtonian fluid model with shave retreating property is Williamson fluid model and was first projected by Williamson. The examination of Williamson fluid has been carried out by researchers such as, Dapra and Scarpi, Reddy et al., Sreelakshmi et al., Talha et al., Immaculate et al., Esvaramoorthi et al., and AL-Qaisy and Abdulhadi past different physical geometry such as stretching surface, vertical porous plate, and asymmetric channels. Furthermore, magnetohydrodynamic bio convective flow of Williamson nanofluid containing gyrotactic microorganisms subjected to thermal radiation and Newtonian conditions was studied by Zaman and Gul. Still further, Hayata et al. studied mathematical modeling of non-Newtonian fluid with chemical aspects. From their simulation it can be seen that increasing values of $S_{C}$ enhance the concentration field but opposite trend is seen for higher values of $K_{1}$ and $K_{2}$.

A Maxwell fluids model can foresee the stress lessening and become the most popular model. This model excludes the complicated effects of shear-dependent viscosity and thus enables us to focus merely on the effects of fluid's elasticity on the characteristics of its boundary layer. Typically, many researchers have been studied non-Newtonian upper-convected Maxwell fluid flows. Accordingly, the slip effect on non-Newtonian upper-convected Maxwell and micropolar fluid flow over a stretching sheet were studied by Vijayalakshmi et al. Furthermore, the study of the influence of cross diffusion on Casson and Maxwell fluid flows past a stretching surface was examined by Kumaran et al. In a porous medium
mass and heat transfer of a non-Newtonian Maxwell nanofluid above a stretching surface with variable thickness was investigated by Elbashbeshy et al.\textsuperscript{43} With variable thermophysical possessions mass and heat transfer of upper-convected Maxwell fluid flow properties over a horizontal melting surface were studied by Adegbie et al.\textsuperscript{44} The result indicated when thermal conductivity parameter increases the temperature of the fluid increases. Hayat et al\textsuperscript{45} investigated the effect of Cattaneo-Christov heat flux stagnation point flow with homogeneous-heterogeneous reactions. Khana et al\textsuperscript{46} studied the influence of chemically reaction on the flow of Maxwell liquid with variable ticked surface. The heat, mass, and motile microorganisms transfer rates in the convective stretched flow of Maxwell fluid consisting of nanoparticles and gyrotactic microorganisms were studied by Khan et al.\textsuperscript{47} From their result it is observed that thermal, concentration, and motile microorganisms' density distributions, respectively.

In view of all the above mentioned studies, it is decided that impacts of melting and viscous dissipation on boundary flow of upper-convected Maxwell and Williamson nanofluids in porous medium with chemical reaction is not inspected yet. Thus, to fill this gap, the main object of this article is to explore the effect of melting heat transfer and viscous dissipation on boundary flow of upper-convected Maxwell and Williamson nanofluids in porous medium with chemical reaction. For the information our work is innovative in terms of the proposed fluids and incorporated parameters in the boundary layer flow. By employing appropriate similarity transformations the nonlinear coupled dimensionless equations from the governing equations are achieved. Finally, the resulting dimensionless equations are solved computationally by instigating the Keller Box method via MATLAB software. We scrutinize the influence of dimensionless melting, magnetic field, thermal radiation, chemical reaction, Brownian motion, thermophoresis, permeability parameter, Prandtl number, Eckert number, and Lewis number on Williamson and Maxwell nanofluids that are derived from the governing equations. From the result we observed that the dimensionless melting parameter affects highly the velocity boundary layer of Williamson nanofluid when compared with upper-convected Maxwell nanofluid.

The rest of this article is structured as follows. The second section describes statement of the problem and mathematical formulation. Solution of methodology and scheme of methodology are provided under Section 3. Section 4 tells us that result and discussion of the article. Conclusion is provided under Section 5.

2 | MATHEMATICAL FORMULATION

The main theme in this article is based on a time-independent flow of an upper-convected Maxwell and Williamson nanofluid with chemical reaction and viscous dissipation effect past a stretching surface embedded in porous medium. The study considered uniform stretching velocity \( u_w \) and the uniform free stream velocity \( u_e \) in the similar direction as revealed in Figure 1. Above the stretching sheet the concentration and the ambient concentration, respectively, represented by \( C_w \) and \( C_\infty \). The melting surface of wall subjected to fixed temperature \( T_m \) and the free stream condition \( T_\infty \) with \( T_\infty > T_m \) is considered. In addition, thermal radiation influence is taken into account. The Williamson fluid model essential equations are specified as follows:

\[
S = -p I + \tau, \tag{1}
\]

\[
\tau = \left[ \mu_\infty + \frac{\mu_0 - \mu_\infty}{1 - \Gamma \dot{\gamma}} \right] A_1, \tag{2}
\]

where \( p \) is pressure, \( I \) is identity vector, \( S \) is the Cauchy stress tensor, \( \tau \) is extra stress tensor, \( \mu_0 \) and \( \mu_\infty \) are the limiting viscosities at zero and at infinite shear rate, \( \Gamma > 0 \) is the time constant, \( A_1 \) is the first Rivlin-Ericksen tensor and \( \dot{\gamma} \) is defined as follows

\[
\dot{\gamma} = \sqrt{\frac{\pi}{2}} \gamma = \text{trace}(A_1^2), \tag{3}
\]

where \( \gamma \) is the second invariant strain tensor. Here, we have only considered the case for which \( \mu_\infty = 0 \) and \( \Gamma \dot{\gamma} < 1 \). Then we obtain.

\[
\tau = \left[ \frac{\mu_0}{1 - \Gamma \dot{\gamma}} \right] A_1 \text{ or } \tau = \mu_0(1 - \Gamma \dot{\gamma})A_1. \tag{4}
\]
Under these assumptions the governing equations can be described in Cartesian system as listed below.

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (5) \]

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} + \sqrt{2\nu \Gamma} \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial x \partial y} - \xi \left( u^2 \frac{\partial^2 u}{\partial x^2} + v^2 \frac{\partial^2 u}{\partial y^2} + 2uv \frac{\partial^2 u}{\partial x \partial y} \right) + U_e \frac{\partial u_e}{\partial x} + \frac{\sigma B_0^2}{\rho_f} \left( \xi \frac{\partial u}{\partial y} + u \right) - \frac{\nu}{k'} u, \quad (6) \]

\[ u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha_m \frac{\partial^2 T}{\partial y^2} + \tau \left\{ D_t \left( \frac{\partial C}{\partial y} \frac{\partial T}{\partial y} \right) + \frac{D_r}{T_\infty} \left( \frac{\partial T}{\partial y} \right)^2 \right\} - \frac{1}{(\rho c_p)_f} \frac{\partial q_r}{\partial y} + \frac{\nu}{\rho_f} \left( \frac{\partial u}{\partial y} \right)^2, \quad (7) \]

\[ u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D_B \frac{\partial^2 C}{\partial y^2} + \frac{D_r}{T_\infty} \frac{\partial ^2 T}{\partial y^2} - K_r (C - C_\infty). \quad (8) \]

The appropriate boundary conditions are:

\[ u = u_w = ax, \quad T = T_m, \quad C = C_w \text{ at } y = 0, \quad (9) \]

\[ u \to 0, \quad T \to T_\infty, \quad C \to C_\infty \text{ as } y \to \infty, \quad \text{and} \quad k \left( \frac{\partial u}{\partial y} \right)_{y=0} = \rho (\varphi + c_s (T_m - T_\infty)) v(x, 0), \quad (10) \]

where \( u \) and \( v \) are the velocity components along the \( x \) and \( y \) directions, respectively, \( \rho_f \) the density of the base fluid, \( \alpha_m = \frac{1}{\rho c_f} \) the thermal diffusivity, \( \xi \) is the relaxation time parameter of the fluid, \( B_0 \) is the strength of the magnetic field, \( \nu \) is the kinematic viscosity of the fluid, \( k \) is the thermal conductivity of the fluid, \( D_t \) is the thermophoretic diffusion coefficient, \( D_r \) is the Brownian diffusion coefficient, \( \tau = \frac{(\rho c_v)}{(\rho c_f)} \) is the ratio between the effective heat capacity of the nanoparticle material and heat capacity of the fluid, \( c \) is the volumetric volume expansion coefficient and \( \rho \) is the density of the particles, \( k' \) is the permeability of porous medium, \( \varphi \) is the latent heat of the fluid and \( c_s \) is the heat capacity of the solid surface, \( c_f \) is the heat capacity of the fluid, \( \sigma \) is the electrical conductivity. We can write for the radiation using Rosseland approximation

\[ q_r = -\frac{4 \sigma^* \partial T^4}{3k^* \partial y}, \quad (11) \]

where \( \sigma^* \) is the Stefan-Boltzman constant, \( k' \) is the absorption constant. Assuming the temperature difference within the flow such that \( T^4 \) may be expanded in a Taylor series about \( T_\infty \) and neglecting higher orders we get \( T^4 = 4TT^3_\infty - 3T^4_\infty \), Hence

\[ \frac{\partial q_r}{\partial y} = -\frac{16 \sigma^* T^3_\infty \partial^2 T}{3k^* \partial y^2}. \quad (12) \]
Introducing similarity transformations

\[ \psi = \sqrt{c} f(\eta), \quad \theta(\eta) = \frac{(T - T_\infty)}{(T_w - T_\infty)}, \quad \phi(\eta) = \frac{(C - C_\infty)}{(C_w - C_\infty)}, \quad \eta = \sqrt{\frac{c}{v}} y. \]  
(13)

We choose the stream function \( \psi(x, y) \) such that

\[ \frac{\partial \psi}{\partial y} = u \quad \text{and} \quad \frac{\partial \psi}{\partial x} = v. \]  
(14)

By applying the similarity transformation in Equation (13) and Equations (6) to (8) are transformed into the nondimensional ordinary differential equation form as follows:

\[ f''' + (\beta M + 1) f'' + \beta(2ff'f'' - f^2f''') + \gamma f''f''' -(M + d) f' = 0, \]  
(15)

\[ \left(1 + \frac{4}{3} R \right) \theta'' + Prf \theta' + N b f \theta' + N t f \theta'^2 + E c f'^2 = 0, \]  
(16)

\[ \phi'' + L e f \phi' + \frac{N t}{N b} \theta'' - K L e f \phi = 0. \]  
(17)

With corresponding boundary conditions.

\[ f'(\eta) = 1, Q \theta'(\eta) + Pr f(\eta) = 0, \theta(\eta) = 0, \phi(\eta) = 0, \text{at} \; \eta = 0, \]  
(18)

\[ f'(\eta) \rightarrow 0, \theta(\eta) \rightarrow 1, \phi(\eta) \rightarrow 1, \text{as} \; \eta \rightarrow \infty. \]  
(19)

where \( f' \) is dimensionless velocity, \( \theta \) is dimensionless temperature, \( \phi \) is dimensionless concentration, and \( \eta \) is the similarity variable. The prime denotes differentiation with respect to \( \eta \). The overall governing parameters are defined as;

\( \gamma = \sqrt{\frac{3\nu}{\nu}} x T \) is non-Newtonian Williamson parameter, \( d = \frac{\nu}{k} \) is permeability parameter, \( \beta = \xi a \) is Deborah number, \( M = \frac{\sigma B^2}{\nu} \) is magnetic field parameter, \( R = \frac{4\nu T^3}{kk} \) is thermal radiation parameter, \( k = a_m (\rho c) \) is thermal conductivity, \( Pr = \frac{\nu}{a_m} \) is Prandtl number, \( N b = \frac{c D W}{\nu} \) is Brownian motion parameter, \( N t = \frac{c D W (T_w - T_m)}{T_m} \) is thermophoresis parameter, \( Le = \frac{\nu}{D_k} \) is Lewis number, \( K = \frac{K_a}{a} \) is chemical reaction parameter, \( Ec = \frac{c_k (T_w - T_m)}{c_p (T_w - T_m)} \) is Eckert number, \( Q = \frac{c_k (T_w - T_m)}{c_p (T_w - T_m)} \) is dimensionless melting parameter.

The skin friction \( C_f \), local Nusselt number \( N u_s \), and the Sherwood number \( S h_x \) are the important physical quantities of interest in this problem which are defined as

\[ C_f = \frac{\tau_w}{\rho u_w^2}, \quad N u_s = \frac{x q_w}{k (T_w - T_m)}, \quad S h_x = \frac{x q_m}{D_B (C_w - C_m)}. \]  
(20)

Here, \( \tau_w = -\mu (1 + \beta) \left(1 + \frac{R}{\sqrt[3]{2}} \frac{\partial u}{\partial y}\right) \frac{\partial u}{\partial y} \) at \( y = 0 \) is the surface shear stress, \( q_w = -k \left(\frac{\partial T}{\partial y}\right) \) at \( y = 0 \) is the surface heat flux, and \( q_m = -D_B \left(\frac{\partial C}{\partial y}\right) \) at \( y = 0 \) is the surface mass flux.

\[ C_f R e_s^{1/2} = (1 + \beta) \left( f''(0) + \frac{R}{2} f''(0) \right), \quad -\left(1 + \frac{4}{3} R \right) \theta'(0) = N u_s R e_s^{1/2}, \quad -\phi(0) = S h_x R e_s^{1/2}, \]  
(21)

where \( R e_s^{1/2} = \frac{x q_w(x)}{\nu} \) is local Reynolds number.
3 | SOLUTION METHODOLOGY

3.1 | Keller box method

The transmuted ordinary differential Equations (15) to (17) subject to boundary conditions (18) and (19) are solved numerically using an implicit finite difference method (Keller box) in combination with the Newton linearization techniques. The key features of this method are:

1. Only to some extent more arithmetic to solve than the Crank-Nicolson method.
2. Second-order correctness with arbitrary (nonuniform) $x$ and $y$ spacing’s.
3. Tolerates very speedy $x$ variations.
4. Tolerates easy programming of the solution of hefty numbers of coupled equation.

Four steps are considered to solve an equation by this method.

1. Reduce the equation or equations to a first-order system.
2. Using central differences write difference equations.
3. Linearize the resulting algebraic equations (if they are nonlinear), and write them in matrix-vector form.
4. Solve the linear system by the block-tridiagonal-elimination method.

Accuracy and strength of this method has been confirmed by different investigators. We have compared our results with the investigators Krishnamurthy et al.\(^{17}\) for a further check on the exactness of our numerical computations and we have found an admirable agreement.

3.2 | The finite difference scheme

We write the governing third-order momentum Equation (15) and second-order energy and concentration Equations (16) and (17) in terms of a first-order equations. For this purpose we introduce new dependent variable $u, v, t, \theta = s(x, \eta), \phi(\eta) = g(x, \eta)$ such that $f' = u, u' = v, s' = t$, and $g' = z$.

Thus, Equations (15) to (17) can be written as

$$v' + f u - u^2 + \gamma v v' - (M + d)u + \beta(2f uv - f^2 v') = 0,$$

$$\left(1 + \frac{4}{3} R\right) t' + Pr f t + Pr Nb z t + Pr N t^2 + Pr E c v^2 = 0,$$

$$z t + L e f z + \frac{N t}{N b} t' - K Leg = 0.$$

The boundary conditions are

$$u(\eta) = 1, Qt(0) + Pr f (0) = 0, s(0) = 0, g(\eta) = 0, \text{ at } \eta = 0,$$

$$u(\eta) \to 0, s(\eta) \to 1, g(\eta) \to 1, \text{ as } \eta \to \infty.$$

where prime denotes the differentiation with respect to $\eta$.

We know consider the net rectangle in the $x - \eta$ plane shown in Figure 2 and the net points defined as below.

$$x^0 = 0, \quad x^i = x^{i-1} + k_i, i = 1, 2, 3, \ldots, I,$$

$$\eta_0 = 0, \eta_i = \eta_{i-1} + h_j, j = 1, 2, 3, \ldots, J, \eta_j = \eta_{\infty},$$

where $k_i$ is the $\Delta x$-spacing and $h_j$ is the $\Delta \eta$-spacing. Here $i$ and $j$ are the sequence of numbers that indicate the coordinate location, not tensor indices, or exponents.
Since only first derivatives appear in the governing equations, centered differences, and two-point averages can be constructed involving only the four corner nodal values of the “box.” For example, if \( p \) represents any of the dependent variables \( u, v, s, \) and \( t \) then

\[
[p]_{j-\frac{1}{2}} = 0.5 \left(p_{j-1} + p\right),
\]

\[
[p]_{j+\frac{1}{2}} = 0.5 \left( [p]_{j-\frac{1}{2}} + [p]_{j+\frac{1}{2}} \right),
\]

\[
\frac{\partial p}{\partial \eta}_{j-\frac{1}{2}} = 0.5 \left( \frac{\partial p}{\partial \eta}_{j-\frac{1}{2}} + \frac{\partial p}{\partial \eta}_{j+\frac{1}{2}} \right),
\]

\[
\frac{\partial p}{\partial \eta}_{j+\frac{1}{2}} = 0.5 \frac{\left( p_{j} - p_{j-1} \right)}{(\eta_j - \eta_{j-1})},
\]

\[
\frac{\partial p}{\partial x}_{j+\frac{1}{2}} = 0.5 \frac{\left( p_{j} - p_{j-1} \right)}{(x_j - x_{j-1})}.
\]

Now write the finite difference approximations for first-order ordinary differential equation for the mid-point \((x^i, \eta_{j-\frac{1}{2}})\) of the segment \(P_1P_2\) using centered difference derivatives. This process is called centering about \((x^i, \eta_{j-\frac{1}{2}})\). We get

\[
f_j - f_{j-1} - \frac{h_j}{2} (u_j + u_{j-1}) = 0,
\]

\[
u_j - u_{j-1} - \frac{h_j}{2} (v_j + v_{j-1}) = 0,
\]

\[
s_j - s_{j-1} - \frac{h_j}{2} (t_j + t_{j-1}) = 0,
\]

\[
g_j - g_{j-1} - \frac{h_j}{2} (z_j + z_{j-1}) = 0,
\]

(27)

\[
(v_j - v_{j-1}) + \frac{h_j}{4} (f_j + f_{j-1})(v_j + v_{j-1}) - \frac{h_j}{4} (u_j + u_{j-1})^2 + \frac{h_j \beta}{4} (f_j + f_{j-1})(u_j + u_{j-1})(v_j + v_{j-1})
\]

\[
- \frac{\beta}{4} (f_j + f_{j-1})(f_j + f_{j-1})(v_j - v_{j-1}) + \frac{\gamma}{2} ((v_j + v_{j-1})(v_j - v_{j-1})) - \frac{h_j}{2} (M + d)(u_j + u_{j-1}) = Q_{j-\frac{1}{2}},
\]
\[
\begin{align*}
(1 + \frac{4R}{3}) (t_j - t_{j-1}) &+ \frac{h_j Pr}{4} (f'_j + f_{j-1})(t_j + t_{j-1}) + \frac{h_j PrNb}{4} (z_j + z_{j-1})(t_j + t_{j-1}) \\
+ \frac{h_j PrNt}{4} (t_j + t_{j-1})(t_j + t_{j-1}) &+ \frac{h_j PrEc}{4} (v_j + v_{j-1})^2 = T_{j-\frac{1}{2}},
\end{align*}
\]

\[
(z_j - z) + \frac{h_j Le}{4} (f'_j + f_{j-1})(z_j + z_{j-1}) + \frac{Nt}{Nb} (t_j - t_{j-1}) - \frac{Le h_j}{2} (g_j + g_{j-1}) = S_{j-\frac{1}{2}},
\]

where

\[
Q_{1-\frac{1}{2}} = -(v_j - v_{j-1}) - h_j (f v)_{j-\frac{1}{2}} + h_j (u^2)_{j-\frac{1}{2}} - 2 \beta h_j (f u v)_{j-\frac{1}{2}} + \frac{\delta}{4} (f^2)_{j-\frac{1}{2}} (v_j - v_{j-1}) - \gamma v_j (v_j - v_{j-1})
\]

\[+(M + d) h_j (u)_{j-\frac{1}{2}},\]

\[
T_{j-\frac{1}{2}} = -\left(1 + \frac{4R}{3}\right) (t_j - t_{j-1}) - h_j \text{Pr} (f t)_{j-\frac{1}{2}} - h_j \text{Pr} \text{Nb} (z t)_{j-\frac{1}{2}} - \beta h_j (t^2)_{j-\frac{1}{2}} - h_j \text{Pr} \text{Ec} (v^2)_{j-\frac{1}{2}},
\]

\[
S_{j-\frac{1}{2}} = -(z_j - z_{j-1}) - h_j (f z)_{j-\frac{1}{2}} - \frac{Nt}{Nb} (t_j - t_{j-1}) + Le h_j K g_{j-\frac{1}{2}}.
\]

We note \(Q_{1-\frac{1}{2}}, T_{j-\frac{1}{2}},\) and \(S_{j-\frac{1}{2}}\) involve only known quantities if we assume that the solution is known on \(x = x^{i-1}\). In terms of the new dependent variables, the boundary conditions become

\[
u(x, 0) = 1, Q(x, 0) + Pr f(x, 0) = 0, s(x, 0) = 0, g(x, 0) = 0. \tag{28}\]

\[
u(x, \infty) \rightarrow 0, s(x, \infty) \rightarrow 1, g(x, \infty) \rightarrow 1. \tag{29}\]

Equations (27) are imposed for \(j = 1, 2, 3, \ldots, J\) and the transformed boundary layer thickness \(\eta_j\) is sufficiently large so that it is beyond the edge of the boundary layer. The boundary conditions yields at \(x = x^j\) are

\[
u_0' = 1, 0 = Q_0^j, s_0' = 0, g_0' = 0, u_j = 0, s_j = 1, g_j = 1. \tag{30}\]

### 3.3 Newton’s method

Equation (21) are nonlinear algebraic equations and therefore have to be linearized before the factorization scheme can be used. Let us write the Newton iterates as follows: For \((k + 1)\)th iterates, we write

\[
\begin{align*}
\delta f_j^{(k+1)} &= \delta f_j^{(k)}, \\
\delta u_j^{(k+1)} &= \delta u_j^{(k)}, \\
\delta v_j^{(k+1)} &= \delta v_j^{(k)}, \\
\delta t_j^{(k+1)} &= \delta t_j^{(k)}, \\
\delta s_j^{(k+1)} &= \delta s_j^{(k)}, \\
\delta g_j^{(k+1)} &= \delta g_j^{(k)}, \\
\delta z_j^{(k+1)} &= \delta z_j^{(k)}.
\end{align*}
\]
Equation (27) can be written as

\begin{align*}
  f_j + \delta f_j - f_{j-1} - \delta f_{j-1} &= \frac{h_j}{2}(u_j + \delta u_j + u_{j-1} + \delta u_{j-1}), \\
  u_j + \delta u_j - u_{j-1} - \delta u_{j-1} &= \frac{h_j}{2}(v_j + \delta v_j + v_{j-1} + \delta v_{j-1}), \\
  g_i + \delta g_i - g_{i-1} - \delta g_{i-1} &= \frac{h_j}{2}(z_i + \delta z_i + z_{i-1} + \delta z_{i-1}), \\
  s_j + \delta s_j - s_{j-1} - \delta s_{j-1} &= \frac{h_j}{2}(t_j + \delta t_j + t_{j-1} + \delta t_{j-1}).
\end{align*}

(32)

\begin{align*}
  v_j + \delta v_j - v_{j-1} - \delta v_{j-1} &= \frac{h_j}{4}(f_j + \delta f_j + f_{j-1} + \delta f_{j-1})(v_j + \delta v_j + v_{j-1} + \delta v_{j-1}) - \frac{h_j}{4}(u_j + \delta u_j + u_{j-1} + \delta u_{j-1})^3 \\
  + \frac{h_j \beta}{4}(f_j + \delta f_j + f_{j-1} + \delta f_{j-1})(u_j + \delta u_j + u_{j-1} + \delta u_{j-1})(v_j + \delta v_j + v_{j-1} + \delta v_{j-1}) \\
  - \frac{\beta}{4}(f_j + \delta f_j + f_{j-1} + \delta f_{j-1})^2(v_j + \delta v_j - v_{j-1} - \delta v_{j-1}) - \frac{h_j}{4}(M + d)(u_j + \delta u_j + u_{j-1} + \delta u_{j-1}) \\
  + \frac{1}{2}(v_j + \delta v_j + v_{j-1} + \delta v_{j-1})(v_j + \delta v_j - v_{j-1} - \delta v_{j-1}) = Q_{j-\frac{1}{2}}, \\

  \left(1 + \frac{4}{3}R\right)(t_j + \delta t_j - t_{j-1} - \delta t_{j-1}) + \frac{h_j \Pr}{4}(f_j + \delta f_j + f_{j-1} + \delta f_{j-1})(t_j + \delta t_j + t_{j-1} + \delta t_{j-1}) \\
  + \frac{h_j \Pr N}{4}(z_j + \delta z_j + z_{j-1} + \delta z_{j-1})(t_j + \delta t_j + t_{j-1} + \delta t_{j-1}) + \frac{h_j \Pr N t}{4}(t_j + \delta t_j + t_{j-1} + \delta t_{j-1})^2 \\
  + \frac{h_j \Pr Ec}{4}(v_j + \delta v_j + v_{j-1} + \delta v_{j-1})^2 = T_{j-\frac{1}{2}}, \\

  (z_j + \delta z_j - z_{j-1} - \delta z_{j-1}) + \frac{h_j Le}{4}(f_j + \delta f_j + f_{j-1} + \delta f_{j-1})(z_j + \delta z_j + z_{j-1} + \delta z_{j-1}) \\
  + \frac{N t}{N b}(t_j + \delta t_j - t_{j-1} - \delta t_{j-1}) - \frac{L e h_j K}{2}(g_j + \delta g_j + g_{j-1} + \delta g_{j-1}) = S_{j-\frac{1}{2}}.
\end{align*}

By dropping the quadratic and higher order terms in \(\delta f_j^{(i)}, \delta u_j^{(i)}, \delta v_j^{(i)}, \delta s_j^{(i)}, \delta g_j^{(i)}, \delta t_j^{(i)},\) and \(\delta z_j^{(i)}\) a linear tridiagonal system of equations will be obtained, as follows:

\begin{align*}
  \delta f_j - \delta f_{j-1} &= \frac{h_j}{2}(\delta u_j + \delta u_{j-1}) = (r_1)_{j-\frac{1}{2}}, \\
  \delta u_j - \delta u_{j-1} &= \frac{h_j}{2}(\delta v_j + \delta v_{j-1}) = (r_2)_{j-\frac{1}{2}}, \\
  \delta s_j - \delta s_{j-1} &= \frac{h_j}{2}(\delta t_j + \delta t_{j-1}) = (r_3)_{j-\frac{1}{2}}, \\
  \delta g_j - \delta g_{j-1} &= \frac{h_j}{2}(\delta z_j + \delta z_{j-1}) = (r_4)_{j-\frac{1}{2}}, \\
  a_1 \delta v_j + a_2 \delta v_{j-1} + a_3 \delta f_j + a_4 \delta f_{j-1} + a_5 \delta u_j + a_6 \delta u_{j-1} &= (r_5)_{j-\frac{1}{2}}, \\
  b_1 \delta t_j + b_2 \delta t_{j-1} + b_3 \delta f_j + b_4 \delta f_{j-1} + b_5 \delta z_j + b_6 \delta z_{j-1} + b_7 \delta v_j + b_8 \delta v_{j-1} &= (r_6)_{j-\frac{1}{2}}, \\
  c_1 \delta z_j + c_2 \delta z_{j-1} + c_3 \delta f_j + c_4 \delta f_{j-1} + c_5 \delta t_j + c_6 \delta t_{j-1} + c_7 \delta g_j + c_8 \delta g_{j-1} &= (r_7)_{j-\frac{1}{2}},
\end{align*}

(33)
Where

\[(r_1)_j = f_j - f_j + \frac{h_j}{2}(u_j + u_{j-1}),\]

\[(r_2)_j = u_j - u_j + \frac{h_j}{2}(v_j + v_{j-1}),\]

\[(r_3)_j = s_j - s_j + \frac{h_j}{2}(t_j + t_{j-1}),\]

\[(r_4)_j = g_j - g_j + \frac{h_j}{2}(z_j + z_{j-1}),\]

\[(r_5)_j = -(v_j - v_{j-1}) - h_j(fv)_{j-\frac{1}{2}} + h_ju^2_{j-\frac{1}{2}} - 2h_j(\beta)(uv)_{j-\frac{1}{2}} + \beta f^2_{j-\frac{1}{2}}(v_j - v_{j-1}) - \gamma v_{j-1}(v_j - v_{j-1}) + h_j(M + d)u_{j-\frac{1}{2}},\]

\[(r_6)_j = -\left(1 + \frac{4}{3}R\right)(t_j - t_{j-1}) - Pr h_j(fT)_{j-\frac{1}{2}} - Pr h_jNb(zt)_{j-\frac{1}{2}} - h_jPrNt^2_{j-\frac{1}{2}} - Pr h_jEc v^2_{j-\frac{1}{2}},\]

\[(r_7)_j = -(z_j - z_{j-1}) - Le h_j(fz)_{j-\frac{1}{2}} - \frac{N_t}{Nb}(t_j - t_{j-1}) - Le h_jKg_{j-\frac{1}{2}},\]

\[(a_1)_j = 1 + \frac{h_j}{2}f_{j-\frac{1}{2}} + h_j(\beta)(u)_{j-\frac{1}{2}} - \beta f^2_{j-\frac{1}{2}} + \frac{\gamma}{2}(v_j + v_{j-1}) + \frac{\gamma}{2}(v_j - v_{j-1}),\]

\[(a_2)_j = -1 + \frac{h_j}{2}f_{j-\frac{1}{2}} + h_j(\beta)(u)_{j-\frac{1}{2}} + \beta f^2_{j-\frac{1}{2}} - \frac{\gamma}{2}(v_j + v_{j-1}) + \frac{\gamma}{2}(v_j - v_{j-1}),\]

\[(a_3)_j = (a_4)_j = \frac{h_j}{2}v_{j-\frac{1}{2}} + h_j(\beta)(uv)_{j-\frac{1}{2}} - \beta f_{j-\frac{1}{2}}(v_j - v_{j-1}),\]

\[(a_5)_j = (a_6)_j = -\frac{h_j}{2}u_{j-\frac{1}{2}} + h_j(\beta)(fv)_{j-\frac{1}{2}} + \frac{h_j}{2}(M + d),\]

\[(b_1)_j = \left(1 + \frac{4}{3}R\right) + \frac{h_j Pr}{2}f_{j-\frac{1}{2}} + \frac{h_j Pr Nb}{2}z_{j-\frac{1}{2}} + h_j Pr Nt^2_{j-\frac{1}{2}},\]

\[(b_2)_j = \left(1 + \frac{4}{3}R\right) + \frac{h_j Pr}{2}f_{j-\frac{1}{2}} + \frac{h_j Pr Nb}{2}z^2_{j-\frac{1}{2}} + h_j Pr Nt t_{j-\frac{1}{2}},\]

\[(b_3)_j = (b_4)_j = \frac{h_j Pr}{2}t_{j-\frac{1}{2}},\]

\[(b_5)_j = \frac{h_j Pr Nb}{2}t_{j-\frac{1}{2}}, (b_6)_j = (b_8)_j = \frac{h_j Pr Ec}{2} v_{j-\frac{1}{2}},\]

\[(c_1)_j = 1 + \frac{h_j Le}{2}f_{j-\frac{1}{2}},\]

\[(c_2)_j = -1 + \frac{h_j Le}{2}f_{j-\frac{1}{2}}, (c_3)_j = (c_4)_j = \frac{h_j Le}{2} f_{j-\frac{1}{2}},\]

\[(c_5)_j = \frac{N_t}{Nb}(c_6)_j = -(c_5)_j, (c_7)_j = (c_8)_j = \frac{-h_j Le K}{2},\]

To complete the system (33) we recall the boundary conditions (30) which can be satisfied exactly with no iteration. Therefore, in order to maintain these correct values in all the iterates, we take

\[\delta f_0 = 0, \delta u_0 = 0, \delta s_0 = 0, \delta g_0 = 0, \delta u_j = 0, \delta s_j = 0, and \delta g_j = 0.\]
In general case Equation (33) in vector matrix form:

\[
[A][\delta] = [r]
\]  

(34)

where

\[
A = \begin{bmatrix}
[A_1] & [C_1] \\
[B_2] & [A_2] & [C_2] \\
[B_3] & [A_3] & [C_3] \\
& \cdots & \cdots \\
[B_{J-1}] & [A_{J-1}] & [C_{J-1}] \\
& [B_J] & [A_J]
\end{bmatrix}, \quad
r = \begin{bmatrix}
[\delta_1] \\
[\delta_2] \\
\vdots \\
[\delta_{J-1}] \\
[\delta_J]
\end{bmatrix}, \quad
r = \begin{bmatrix}
[r_1] \\
[r_2] \\
\vdots \\
[r_{J-1}] \\
[r_J]
\end{bmatrix},
\]

In Equation (34) the elements are defined by

\[
[A_1] = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-\frac{h_1}{2} & 0 & 0 & 0 & -\frac{h_1}{2} & 0 & 0 \\
0 & -\frac{h_2}{2} & 0 & 0 & 0 & -\frac{h_2}{2} & 0 \\
0 & 0 & -\frac{h_3}{2} & 0 & 0 & 0 & -\frac{h_3}{2} \\
a_2 & 0 & 0 & a_3 & a_1 & 0 & 0 \\
b_8 & b_2 & b_6 & b_3 & b_7 & b_1 & b_5 \\
0 & c_6 & c_2 & c_3 & 0 & c_5 & c_1
\end{bmatrix},
\]

\[
[A_j] = \begin{bmatrix}
-\frac{h_j}{2} & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & -\frac{h_j}{2} & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & -\frac{h_j}{2} & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & -\frac{h_j}{2} \\
a_6 & 0 & 0 & a_3 & a_1 & 0 & 0 \\
b_8 & b_2 & b_6 & b_3 & b_7 & b_1 & b_5 \\
0 & c_8 & c_3 & 0 & c_5 & c_1
\end{bmatrix}, \quad 2 \leq j \leq J,
\]

\[
[B_j] = \begin{bmatrix}
0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{h_j}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{h_j}{2} & 0 \\
0 & 0 & 0 & 0 & -\frac{h_j}{2} & 0 \\
a_4 & 0 & 0 & a_2 & 0 & 0 \\
b_4 & b_8 & b_2 & b_6 \\
0 & c_4 & 0 & c_6 & c_2
\end{bmatrix}, \quad 2 \leq j \leq J,
\]

\[
[C_j] = \begin{bmatrix}
-\frac{h_j}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
a_5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c_7 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad 1 \leq j \leq J.
\]
\[
\begin{bmatrix}
\delta v_0 \\
\delta t_0 \\
\delta z_0 \\
\delta v_1 \\
\delta t_1 \\
\delta z_1
\end{bmatrix}
\quad \begin{bmatrix}
\delta u_{j-1} \\
\delta s_{j-1} \\
\delta g_{j-1} \\
\delta v_j \\
\delta t_j \\
\delta z_j
\end{bmatrix}
\quad \begin{bmatrix}
(r_1)_{j-\frac{1}{2}} \\
(r_2)_{j-\frac{1}{2}} \\
(r_3)_{j-\frac{1}{2}} \\
(r_4)_{j-\frac{1}{2}} \\
(r_5)_{j-\frac{1}{2}} \\
(r_6)_{j-\frac{1}{2}} \\
(r_7)_{j-\frac{1}{2}}
\end{bmatrix}
\]

\[
[\delta] = [\delta f_1], \quad [\delta_j] = [\delta f_j], \quad 2 \leq j \leq J, \quad [r_j] = [r_{j-\frac{1}{2}}], \quad 1 \leq j \leq J.
\]  

(35)

To solve Equation (34), we assume that \( A \) is nonsingular and can be factored into

\[
[A] = [L][U]
\]  

(36)

where

\[
A =
\begin{bmatrix}
[a_1] \\
[B_2] & [a_1] \\
& \ddots & \ddots \\
& & \ddots & [a_{j-1}] \\
& & & [B_{j-1}] & [a_j]
\end{bmatrix}
\quad \text{and}

U =
\begin{bmatrix}
[I] & [\Gamma_1] \\
[I] & [\Gamma_2] \\
& \ddots & \ddots \\
& & \ddots & [I] & [\Gamma_{j-1}] \\
& & & [I]
\end{bmatrix}
\]

Equation (36) can now be substituted into Equation (34), and we get

\[
[L][U][\delta] = [r].
\]  

(41)

If we define

\[
[U][\delta] = [W].
\]  

(42)

Equation (41) becomes

\[
[L][W] = [r].
\]  

(43)
where

\[
W = \begin{bmatrix}
W_1 \\
W_2 \\
\vdots \\
W_{J-1} \\
W_J
\end{bmatrix},
\]

and the \([ W_j ]\) are \(7 \times 1\) column matrices. The elements \(W\) can be solved from Equation (43):

\[
[a_1][W_1] = [r_1],
\]

\[
[a_j][W_j] = [r_j] - [B_j][W_{j-1}], \quad 2 \leq j \leq J.
\]

The step in which \(\Gamma_j, a_j,\) and \(W_j\) are calculated is usually referred to as the forward sweep. Once the element of \(W\) are found, Equation (38) then gives the solution in the so-called backward sweep, in which the elements are obtained by the following relations:

\[
[\delta_j] = [W_j],
\]

\[
[\delta_j] = [W_j] - [\Gamma_j][\delta_{j+1}], \quad 1 \leq j \leq J - 1.
\]

These calculations are repeated until some convergence criterion is satisfied and calculations are stopped when

\[
|\delta^{(i)}_0| \leq \varepsilon_1,
\]

where \(\varepsilon_1\) is small arranged value.

### 4 RESULTS AND DISCUSSION

Figures 3-5 portray the temperature, velocity, and concentration profiles for different values of dimensionless melting parameter. It is noticed that when dimensionless melting parameter heightens the flow and velocity boundary layer rises, whereas, the thermal and concentration profiles decreases. This is due to the fact that an increase in \(Q\) will upsurge the intensity of melting, which acts as blowing boundary condition at the stretching surface and hence tends to thicken the boundary layer. Melting parameter affects adversely the velocity boundary layer thickness of Williamson nanofluid when compared with upper-convected Maxwell nanofluid. The influence of thermal radiation parameter on temperature and concentration profiles is recounted through Figures 6 and 7. From the figures it is noticed that an increment in the thermal radiation parameter causes the reduction in temperature profiles, but opposite effect on concentration profile. This is because an increase in the radiation parameter \(R\) leads to a decrease in the boundary layer thickness and enhances the heat transfer rate on melting surface in the presence of chemical effect. The behavior of the concentration profiles in contradiction of chemical reaction parameter is offered in Figure 8. From the graph it can be seen that if the chemical reaction parameter increased the concentration boundary layer thickness and concentration profiles reduced. This is due to the fact that chemical reaction in this system results in ingesting of the chemical and hence results in reduction of concentration profile. The most important outcome is that the first-order chemical reaction has a tendency to reduce the overshoot in the profiles of the solute concentration in the solutal boundary layer.

Prandtl number is a ratio of momentum diffusivity to thermal diffusivity. When Prandtl number is high fluid has low thermal conductivity, which decreases the thermal boundary layer thickness and conduction. Accordingly, form Figure 9 it is seen that with increasing Prandtl number the thermal boundary layer and temperature profiles upsurges. The impact of magnetic field parameter on velocity, temperature, and concentration profiles is plotted in Figures 10-12. From the figures it is noticed that as the values of magnetic parameter goes up, the velocity of the fluid, the temperature, and
FIGURE 3  Velocity graph for different values of Q

FIGURE 4  Temperature graph for different values of Q

FIGURE 5  Concentration graph for different values of Q
FIGURE 6  Temperature graph for different values of $R$

FIGURE 7  Concentration graph for different values of $R$

FIGURE 8  Concentration graph for different values of $K$
**Figure 9** Temperature graph for different values of $Pr$

**Figure 10** Velocity graph for different values of $M$

**Figure 11** Temperature graph for different values of $M$
concentration declines. The physical significance of this behavior is an increase in the values of magnetic field parameter develops the force opposite to the flow, which is known as Lorentz force. This body force abridged the boundary layer flow and congeals the velocity boundary layers.

The impact of thermophoresis and Brownian motion parameter on concentration and temperature profiles are portrayed in Figures 13-16. A phenomenon in which small particles are pulled away from the hot surface to cold one is called thermophoresis. So, when the surface is heated large number of nanoparticles is moved away which raises the temperature of fluid. Therefore, the temperature of fluid increases, whereas, the concentration of the fluid declines. Because of more heat is produced an increase in Nb enhances the random motion of fluid particles. This causes an increase of fluid temperature and concentration. Figures 17-19 are conspired to demonstrate the influence of permeability parameter on the flow of the fluid, temperature, and concentration profiles. It is obvious that the presence of porous medium causes higher restriction to the fluid flow, which in turn slows its motion. Therefore, an increment of permeability parameter causes the reduction in boundary layer thickness and the profiles of velocity, temperature, and concentration. In particular, the boundary layer of Williamson nanofluid more affected by permeability parameter when compared with the boundary layer of upper-convected Maxwell nanofluid. From Figure 20 it is noticed that as the value of Lewis number escalated the concentration boundary layer and concentration profiles rises. This is probably due to the fact that higher values of Lewis number create the smaller Brownian diffusion coefficient.
**FIGURE 14**  Concentration graph for different values of $Nt$

**FIGURE 15**  Temperature graph for different values of $Nb$

**FIGURE 16**  Concentration graph for different values of $Nb$
FIGURE 17  Velocity graph for different values of $d$

FIGURE 18  Temperature graph for different values of $d$

FIGURE 19  Concentration graph for different values of $d$
Figure 21 reveals that when the values of Eckert number upsurges the thermal boundary layer and the temperature profiles are enhanced. This is because of the point that heat energy is stowed in the liquid because of the frictional heating.

To check the validity of the numerical solution obtained a comparison of Nusselt and Sherwood number with Krishnamurthy et al.\(^{17}\) for different values of \(\text{d}\) and \(\gamma\) is exhibited in Table 1 and reasonable agreement with them has been found. The changes in physical quantities at various relevant parameters are displayed in Tables 2 and 3. From the tables it is noticed that with an increment in Eckert number \(Ec\) and thermophoresis parameter \(Nt\) a decrement in friction factor and mass transfer rate attained, whereas, the heat transfer rate is amplified. Moreover, when magnetic field parameter \(M\) upsurges skin friction coefficient upsurges but Nusselt number \(-Nu_x(0)\) and Sherwood number \(-Sh_x(0)\) are decreased. Furthermore, from the tables it can be seen that with an increase in Brownian motion parameter \(Nb\) there is a decrement in a friction factor, in contrast, there is an increment in both Sherwood and Nusselt number.

To check the validity of the numerical solution obtained a comparison of Nusselt and Sherwood number with Kairi and Murthy et al.\(^{16}\) for different values of \(\text{d}\) and \(\gamma\) is exhibited in Table 1 and reasonable agreement with them has been found. The changes in physical quantities at various relevant parameters are indicated in Tables 2 and 3. From the tables it is noticed that with an increment in Eckert number \(Ec\) and thermophoresis parameter \(Nt\) a decrement in friction factor and mass transfer rate attained, whereas, the heat transfer rate is amplified. Moreover, when
**TABLE 1** Comparison of Nusselt number $-\text{Nu}_x(0)$ and Sherwood number $-\text{Sh}_x(0)$ for different values of permeability parameter $d$ and Williamson parameter $\gamma$ when $Ec = 0$, and $\beta = 0$, $Q = 0.5$, $K = 0.01$

| $d$ | $\gamma$ | Kairi and Murthy $16-\theta(0)$ | Present result | Kairi and Murthy $16-\phi(0)$ | Present result |
|-----|----------|-------------------------------|----------------|-------------------------------|----------------|
| 0.0 | 0.2      | 2.1656                        | 2.16565        | 0.3345                        | 0.33458        |
| 1.0 | 0.2      | 1.9578                        | 1.95781        | 0.2975                        | 0.29757        |
| 2.0 | 0.2      | 1.7914                        | 1.79143        | 0.2698                        | 0.26982        |
| 0.0 | 0.01     | 1.8511                        | 1.85112        | 0.2780                        | 0.27803        |
| 0.0 | 0.1      | 1.8256                        | 1.82567        | 0.2744                        | 0.27448        |
| 0.0 | 0.2      | 1.7914                        | 1.7914         | 0.2698                        | 0.26987        |

**TABLE 2** Calculation of skin friction coefficient $f''(0)$, local Nusselt number $-\text{Nu}_x(0)$ and local Sherwood number $-\text{Sh}_x(0)$ for different values of $\gamma$, $Ec$, $Nb$, $Nt$, and $M$ when $d = 2.0$, $Pr = 3.2$, $Le = 10$, $K = 0.01$, $R = 0.01$, $Q = 0.2$ for Williamson nanofluid

| $\gamma$ | $Ec$ | $Nb$ | $Nt$ | $M$ | $f''(0)$ | $-\text{Nu}_x(0)$ | $-\text{Sh}_x(0)$ |
|----------|------|------|------|-----|----------|-------------------|-------------------|
| 0.1      | 0.1  | 0.25 | 0.25 | 0.3 | -1.872267| 1.969929         | 0.063527         |
| 0.2      |      |      |      |     | -2.047295| 1.946866         | 0.043231         |
| 0.3      |      |      |      |     | -2.404276| 1.916175         | 0.010166         |
| 0.1      | 0.1  |      |      |     | -1.872267| 1.969929         | 0.063527         |
|          | 0.2  |      |      |     | -1.863774| 2.205993         | -0.182168        |
|          | 0.3  |      |      |     | -1.855431| 2.439110         | -0.418728        |
|          | 0.1  | 0.1  |      |     | -1.884558| 1.630553         | -0.934150        |
|          |      | 0.2  |      |     | -1.876417| 1.855036         | -0.082370        |
|          |      | 0.3  |      |     | -1.868094| 2.085765         | 0.148010         |
|          |      | 0.25 | 0.2  |     | -1.875426| 1.882452         | 0.368256         |
|          |      | 0.3  |      |     | -1.868972| 2.061364         | -0.269270        |
|          |      |      | 0.4  |     | -1.861986| 2.255834         | -1.025984        |
|          |      | 0.25 | 0.1  |     | -1.807712| 1.986533         | 0.075362         |
|          |      |      | 0.2  |     | -1.840199| 1.978144         | 0.069415         |
|          |      |      | 0.3  |     | -1.872267| 1.969929         | 0.063527         |

Magnetic field parameter $M$ upsurges skin friction coefficient upsurges but Nusselt number $-\text{Nu}_x(0)$ and Sherwood number $-\text{Sh}_x(0)$ are decreased. Furthermore, from the tables it can be seen that with an increase in Brownian motion parameter $Nb$ there is a decrement in a friction factor, in contrast, there is an increment in both Sherwood and Nusselt number.

### 5 CONCLUSION

This article was aimed at exploring the effect of melting heat transfer, viscous dissipation, and chemical reaction on the flow of upper-convected Maxwell and Williamson nanofluids above a stretching surface. By employing well-known similarity transformations the nonlinear coupled dimensionless equations from the governing equations are achieved. Then, the resulting dimensionless equations are solved computationally by implementing Keller Box method with MATLAB software. The graphs and tables for velocity profiles, skin-friction coefficient, temperature field, local Nusselt number, concentration field, and local Sherwood number for different values of dimensionless melting, magnetic field, thermal radiation, chemical reaction, Brownian motion, thermophoresis, permeability parameter, Prandtl number, Eckert number, and Lewis number are revealed and pondered. The result shows that when the melting parameter enhance the fluid flow and boundary layer thickness increases but opposite trends happens to the temperature and concentration profiles.
Table 3: Calculation of skin friction coefficient \( f''(0) \), local Nusselt number \(-Nux(0)\) and local Sherwood number \(-Shx(0)\) for different values of \( \beta, Ec, Nb, Nt, \) and \( M \) when \( d = 2.0, Pr = 3.2, Le = 10, K = 0.01, R = 0.01, Q = 0.2 \) for Williamson nanofluid.

| \( \gamma \) | \( Ec \) | \( Nb \) | \( Nt \) | \( M \) | \( f''(0) \) | \(-Nux(0)\) | \(-Shx(0)\) |
|---|---|---|---|---|---|---|---|
| 0.1 | 0.1 | 0.25 | 0.25 | 0.3 | -1.756946 | 1.987314 | 0.077509 |
| 0.2 | --- | --- | --- | --- | -1.758375 | 1.985692 | 0.076464 |
| 0.3 | --- | --- | --- | --- | -1.759786 | 1.984084 | 0.075423 |
| 0.1 | 0.1 | --- | --- | --- | -1.756946 | 1.987314 | 0.077509 |
| 0.2 | --- | --- | --- | --- | -1.749135 | 2.216304 | -0.158271 |
| 0.3 | --- | --- | --- | --- | -1.741450 | 2.442636 | -0.385522 |
| 0.1 | 0.1 | --- | --- | --- | -1.768794 | 1.642032 | -0.910417 |
| 0.2 | --- | --- | --- | --- | -1.760948 | 1.870408 | -0.066449 |
| 0.3 | --- | --- | --- | --- | -1.752921 | 2.105189 | 0.160509 |
| 0.25 | 0.2 | --- | --- | --- | -1.759997 | 1.898168 | 0.383877 |
| 0.3 | --- | --- | --- | --- | -1.753763 | 2.080503 | -0.257463 |
| 0.4 | --- | --- | --- | --- | -1.747012 | 2.255834 | -1.020299 |
| 0.25 | 0.1 | --- | --- | --- | -1.700836 | 2.278734 | 0.088168 |
| 0.2 | --- | --- | --- | --- | -1.729108 | 1.995063 | 0.082807 |
| 0.3 | --- | --- | --- | --- | -1.756946 | 1.987314 | 0.077509 |

In addition upper-convected Maxwell fluids are less affected by melting parameter when compared with Williamson fluids. The interpretations of the conclusion are summarized as follows:

1. Melting and magnetic field parameter affects highly the velocity boundary layer of Williamson nanofluid when compared with upper-convected Maxwell nanofluid.
2. Melting parameter Q have similar effect on both temperature and concentration graphs. That is, when melting parameter increases both temperature and concentration profiles are reduces.
3. Radiation parameter R has opposite effect on temperature and concentration graph. As the values of radiation parameter upsurges the temperature decreases, whereas, concentration rises.
4. An increment of Eckert number causes an increment of temperature profiles.
5. With an increment of Eckert number and thermophoresis parameter a decrement in friction factor and mass transfer rate attained, whereas, heat transfer rate is enhance.
6. An augmentation of permeability parameter causes the reduction of the profiles of velocity, temperature, and concentration.
7. When the Williamson parameter upsurges the velocity, temperature, and concentration declines.
8. When magnetic field parameter M upsurges skin friction coefficient is upsurges but Nusselt number \(-Nux(0)\) and Sherwood number \(-Shx(0)\) are decreased.

Data sharing statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

CONFLICT OF INTEREST

The authors declare that there is no conflict of interest regarding the publication of this article.

NOMENCLATURE

- \( B_0 \): strength of magnetic field
- \( c \): volumetric volume expansion coefficient
- \( c_f \): heat capacity of the fluid
- \( c_s \): heat capacity of the solid
$C_f$ skin friction coefficient

$C_{\infty}$ ambient concentration

$d$ permeability parameter

$D_B$ Brownian diffusion coefficient

$D_T$ thermophoresis diffusion coefficient

$E_c$ Eckert number

$f$ dimensionless velocity stream function

$k$ thermal conductivity

$Le$ Lewis number

$M$ magnetic parameter

$Nb$ Brownian motion parameter

$Nt$ thermophoresis parameter

$Nu_x$ local Nusselt number

$Pr$ Prandtl number

$K$ chemical reaction parameter

$Q$ dimensionless melting parameter

$R$ thermal radiation parameter

$Re_x$ local Reynolds number

$T$ temperature of the fluid

$Sh_x$ local Sherwood number

$Ec$ Eckert number

$T_m$ temperature of the melting surface

$C_w$ uniform concentration over the surface

$T_{\infty}$ ambient temperature

$(u, v)$ velocity component along x- and y direction

$u_e$ free stream velocity

**GREEKS**

$\alpha_m$ thermal diffusivity

$\beta$ Deborah number

$\Gamma$ MAterial constant

$\gamma$ non-Newtonian Williamson parameter

$\varphi$ latent heat of the fluid

$\eta$ dimensionless similarity variable

$\theta$ dimensionless temperature function

$\mu$ dynamic viscosity of the fluid

$\nu$ kinematic viscosity of the fluid

$\rho_f$ density of the fluid

$(\rho c)_f$ heat capacity of the fluid

$(\rho c)_p$ effective heat capacity of a nanoparticle

$\sigma$ electrical conductivity

$\xi$ relaxation time parameter of the fluid

$\phi$ dimensionless concentration function

$\psi$ stream function

$\phi_w$ concentration function at the surface

$\phi_{\infty}$ concentration function at large values of y

$\tau$ parameter defined by $(\rho c)_p/(\rho c)_f$

**SUBSCRIPTS**

$\infty$ condition at the free stream

$m$ condition at the melting surface

**AUTHOR CONTRIBUTIONS**

Wubshet Ibrahim and Mekonnen Negera: Conceptualization; data curation; formal analysis; methodology.
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