Generalized $n$-locality inequalities in star-network configuration and their optimal quantum violations

Sneha Munshi, Rahul Kumar, and A. K. Pan*
National Institute of Technology Patna, Ashok Rajpath, Patna, Bihar 800005, India

Network Bell experiments reveal a form of nonlocality conceptually different from standard Bell nonlocality. Standard multiparty Bell experiments involve a single source shared by a set of observers. In contrast, network Bell experiments feature multiple independent sources, and each of them may distribute physical systems to a set of observers who perform randomly chosen measurements. The $n$-locality scenario in star-network configuration involves $n$ number of edge observers (Alices), a central observer (Bob), and $n$ number of independent sources having no prior correlation. Each Alice shares an independent state with the central observer Bob. Usually, in network Bell experiments, one considers that each party measures only two observables. In this work, we propose a non-trivial generalization of $n$-locality scenario in star-network configuration, where each Alice performs some integer $m$ number of binary-outcome measurements, and the central party Bob performs $2^{m-1}$ binary-outcome measurements. We derive a family of generalized $n$-locality inequalities for any arbitrary $m$. Using blue elegant sum-of-squares approach, we derive the optimal quantum violation of the aforementioned inequalities can be attained when each and every Alice measures $m$ number of mutually anticommuting observables. For $m = 2$ and 3, one obtains the optimal quantum value bluefor qubit system local to each Alice, and it is sufficient to consider the sharing of a two-qubit entangled state between each Alice and Bob. We further demonstrate that the optimal quantum violation of $n$-locality inequality for any arbitrary $m$ can be obtained when every Alice shares $[m/2]$ copies of two-qubit maximally entangled state with the central party Bob. We also argue that for $m > 3$, a single copy of a two-qubit entangled state may not be enough to exhibit the violation of $n$-locality inequality but multiple copies of it can activate the quantum violation. We discuss the implications of our study and raise some open questions.

I. INTRODUCTION

Bell’s theorem [1] lies at the heart of quantum foundations. This no-go proof asserts that any ontological model satisfying the locality condition cannot account for all quantum statistics. Apart from its immense impact in quantum foundations research, this theorem paved the path for the development of cutting-edge quantum technologies (see review [2]).

In a conventional Bell experiment, a source distributes a physical system to a set of observers who randomly perform measurements on the respective sub-systems in their possession. The simplest bipartite Bell scenario consists of two distant parties, Alice and Bob, who apply respective measurements $x$ and $y$, producing outcomes $a$ and $b$. In classical physics, the system is represented by a classical random variable $\lambda$, which predetermines the local outcomes (reality) independent of the other party’s settings and outcomes (locality). The joint probability distribution of the outcomes $a$ and $b$ can then be written as

$$P(a, b|x, y) = \int \rho(\lambda) P(a|x, \lambda) P(b|y, \lambda) d\lambda \quad (1)$$

where $\rho(\lambda)$ is the probability distribution of $\lambda$. In quantum theory, source can produce an entangled quantum state. In such a case if Alice and Bob perform the measurements of locally incompatible observables, not every joint probability in quantum theory admits the factorized form as in Eq. (1).

This feature is referred to as quantum nonlocality and is commonly witnessed via the quantum violation of a suitable Bell’s inequality [2, 3].

The conventional multipartite nonlocality is a straightforward generalization of bipartite Bell nonlocality. The study of multipartite nonlocality has been a vibrant research area for last two decades [2, 4, 5]. In standard multipartite Bell experiments, three or more distant observers share an entangled state distributed by a single source. A novel approach was recently proposed [6, 7] through the network Bell experiments which demonstrate a form of multipartite nonlocality that conceptually goes beyond the conventional multipartite Bell nonlocality. In contrast to the standard Bell scenario, the network Bell experiments feature many node observers who hold physical systems originating from different sources. Notably, the sources are assumed to be independent of each other, and therefore a priori share no correlations among them. There may be different topological structures of the network, and quantum correlation across the network would manifest in various possible ways.

The simplest network scenario is the bilocality scenario [6, 7] involving three parties and two independent sources, as depicted in Fig.1. Two edge parties, Alice$_1$ and Alice$_2$, each share an independent physical system with the central party Bob. In quantum theory, each source can produce an entangled pair of particles. If Bob performs suitable joint measurement on the two particles in his possession, entanglement can be developed between the particles of two distant edge parties Alice$_1$ and Alice$_2$. Such a protocol is widely known as entanglement swapping [8]. The correlations between Alice$_1$ and Alice$_2$ can then violate traditional Bell inequalities upon postselecting on the outcome of the joint measurement by the central party Bob. However, there exists nonlinear inequali-
ties that witness a form of quantum nonlocality of the tripartite correlation in a network devoid of the postselection, known as bilocality inequalities \[6, 7\]. Such inequalities are violated in quantum theory, thereby exhibiting quantum non-bilocality. It has also been shown that such inequalities are violated for all pure entangled quantum states \[9\].

The networks beyond the bilocality scenario feature many independent sources, and each of them distributes physical systems to a set of observers who perform randomly chosen measurements. Despite the initial independence of different sources, a suitably chosen set of measurements can give rise to nonlocal quantum correlations across the whole network. In recent times, the nonlocal quantum correlations in networks have been studied for various topological configurations \[12–36\]. In the triangle network, an interesting form of genuine quantum nonlocality is demonstrated \[25\] without any inputs, only by considering the output statistics of fixed measurement settings. The network scenario also allows for nonlocality activation \[36\] and less stringent detection efficiencies\[26\]. It has been shown that an arbitrarily small level of independence is capable of revealing the quantum nonlocality in networks \[30\]. The potential of exploiting quantum networks for device-independent information processing has also been discussed \[21\].

One of the well-studied generalizations of the bilocality scenario is the \(n\)-locality scenario in star-network configuration \[10, 11\], involving \(n\) independent sources. Each source distributes a state to one of the \(n\) edge parties (Alices) and the central party (Bob). Experimental tests of the bilocality inequality \[37–40\] and the \(n\)-locality inequalities in star-network configuration \[41\] have also been reported. The nonlocality in the star-network scenario is commonly studied for two binary-outcome observables (say, \(m = 2\)) per party.

In this work, we provide a non-trivial generalization of the \(n\)-locality scenario in star-network configuration, as depicted in Fig. 2. Instead of two observables per party, we consider that each of the \(n\) numbers of Alices performs an arbitrary \(m\) number of binary-outcome measurements, and Bob performs \(2^{m-1}\) binary-outcome measurements. We then derive a family of generalized \(n\)-locality inequalities for arbitrary \(m\) and demonstrate optimal quantum violations. We further show that for any given \(m \geq 2\), the optimal quantum violation can be attained when every Alice chooses \(m\) number of mutually anticommuting observables.

We note here that, for \(m = 2\), the optimal quantum violation of \(n\)-locality inequality has been achieved when each Alice shares a two-qubit maximally entangled state with the central party \[18\]. We first demonstrate that a two-qubit entangled state can also provide optimal violation for the \(m = 3\) case. Since the number of mutually anticommuting observables for a qubit system is restricted to at most three, one requires a higher dimensional system for \(m > 3\). We show that the optimal quantum violation of the family of generalized \(n\)-locality inequalities is attained if \([\frac{n}{2}]\) copies of two-qubit maximally entangled states are shared between each Alice and Bob. We further argue that for \(m > 3\), a single copy of a two-qubit entangled state may not violate the generalized \(n\)-locality inequality, but multiple copies of it may activate the quantum violation.

II. \(n\)-LOCALITY SCENARIO IN STAR-NETWORK CONFIGURATION FOR \(m = 2\)

Let us first encapsulate the essence of the simplest network scenario, i.e., the bilocality scenario \[7\] for \(m = 2\). As depicted in Fig. 1, this network involves three parties and are two independent sources \(S_1\) and \(S_2\). While the source \(S_1\) sends particles to Alice\(_1\) and Bob, the source \(S_2\) sends particles to Alice\(_2\) and Bob. Alice\(_1\) and Alice\(_2\) perform measurements on their respective sub-systems according to the inputs \(x_1, x_2 \in \{1, 2\}\) producing the outputs \(a_1, a_2 \in \{0, 1\}\) respectively. Bob performs the measurements on the joint sub-systems according to the inputs \(i \in \{1, 2\}\) producing outputs \(b \in \{0, 1\}\). Let us now assume that two different hid-

![FIG. 1: Standard bilocality Scenario (See text)](image-url)
only if the following nonlinear bilocality inequality
\[ (a^2)_{bt} = \sqrt{\sum_{i=1}^{4} |J_{i,2}|^4} \leq 2 \] (3)
is satisfied. Here \( J_{i,2} \) and \( J_{2,2} \) are the linear combinations of suitably chosen tripartite correlations, defined as
\[ J_{i,2} = \langle A_i^1 + A_i^2 \rangle B_1 (A_i^1 + A_i^2 + A_i^3) \] (4)
\[ J_{2,2} = \langle A_i^1 - A_i^2 \rangle B_2 (A_i^1 - A_i^2 - A_i^3) \] (5)
where \( A_i^k \) denotes observable corresponding to the input \( x_k \) of the \( k \)th Alice \( (k = 1, 2) \) and \( \langle A_i^1 B_i A_i^2 \rangle = \sum_{a_i, b_i} (-1)^{a_i+b_i} p(a_i, b_i, x_i) \).

The assumption of independent sources is crucial to derive the bilocality inequality in Eq. (3). Analogously, in quantum theory, two sources produce entangled states. It has been shown [7] that the optimal quantum value \( (a_{bt})_{opt}^2 = 2 \sqrt{2} \) can be obtained when Alice1’s as well as Alice2’s observables are anticommuting.

The 3-locality scenario in star-network configuration for arbitrary \( n \) is one of the straightforward generalizations of bilocality scenario \((n = 2)\). It involves \( n \) independent sources and \( n + 1 \) parties. Each of the \( n \) number of edge observers (Alices) shares a physical state with the central observer (Bob), originating from independent sources. Let \( A_i^1 \) and \( A_i^2 \) be the observables of \( k \)th Alice with \( k = 1, 2 \ldots n \). The 3-locality inequality for \( m = 2 \) is given by [11]
\[ (a_{bt})_{m} = |J_{2,1}|^{1/n} + |J_{2,2}|^{1/n} \leq 2 \] (6)
where
\[ J_{2,1} = \langle A_1^1 + A_2^1 \rangle (A_1^2 + A_2^2) \cdots (A_n^1 + A_n^2) B_1 \] (7)
\[ J_{2,2} = \langle A_1^1 - A_2^1 \rangle (A_1^2 - A_2^2) \cdots (A_n^1 - A_n^2) B_2 \] (8)
The optimal quantum value \( (a_{bt})_{opt}^2 = 2 \sqrt{2} \), i.e., same as bilocality case [11, 18] which is obtained when every Alice chooses anticommuting observables and shares a maximally two-qubit entangled state with Bob. In this work, we go beyond the \( m = 2 \) case and derive a family of generalized 3-locality inequalities for arbitrary \( m \) and demonstrate their optimal quantum violation.

III. \( n \)-LOCALITY SCENARIO IN STAR-NETWORK FOR \( m = 3 \)

For the sake of better understanding, we first demonstrate the bilocality scenario by considering \( m = 3 \). Alice1 and Alice2 now perform three dichotomic measurements, and Bob performs four dichotomic measurements. In this context, we propose a nonlinear bilocality inequality of the form
\[ (a_{bt})_{3} = \sum_{i=1}^{4} |J_{i,3,2}|^{1/2} \leq 6 \] (9)
where \( J_{i,3,2} \) (with \( i = 1, 2, 3, 4 \)) are suitably defined linear combinations of correlations, can explicitly be written as
\[ J_{1,3,2} = \langle (A_1^1 + A_2^1 + A_3^1) B_1 (A_1^1 + A_2^1 + A_3^1) \rangle \]
\[ J_{2,3,2} = \langle (A_1^1 + A_2^1 - A_3^1) B_2 (A_1^1 + A_2^1 - A_3^1) \rangle \]
\[ J_{3,3,2} = \langle (A_1^1 + A_2^1 + A_3^1) B_3 (A_1^1 + A_2^1 + A_3^1) \rangle \]
\[ J_{4,3,2} = \langle (A_1^1 + A_2^1 + A_3^1) B_4 (A_1^1 + A_2^1 + A_3^1) \rangle \] (10)
Let us first prove the inequality in Eq. (9) here. Using this bilocality assumption and defining \( A_i^1 | x_i \rangle = \sum_{x_i} (-1)^{x_i} p(a_i, b_i, x_i, t_i, x_i) \), and similarly \( B_i | x_i \rangle \), we can write
\[ J_{1,1} = \int \int dA_i dA_2 \rho_1(A_1) \rho_2(A_2) \times \langle (A_1^1 + A_2^1 a) + (A_1^2 a) + (A_1^3 a) \rangle \]
\[ \langle B_1 | x_i \rangle \langle B_2 | x_i \rangle (A_1^1 + A_2^1 + A_2^2) \]
Since, \( |\langle B_1 | x_i \rangle \langle B_2 | x_i \rangle| \leq 1 \), we can write
\[ J_{1,1} \leq \int \int dA_i dA_2 \rho_1(A_1) \rho_2(A_2) \times \langle (A_1^1 + A_2^1 a) + (A_1^2 a) + (A_1^3 a) \rangle \]
\[ \langle B_1 | x_i \rangle \langle B_2 | x_i \rangle (A_1^1 + A_2^1 + A_2^2) \]
The terms \( J_{2,1} \), \( J_{2,3} \), and \( J_{4,2} \) given in Eq. (10) can also be written in similar manner. Using the inequality, \( \sum_{i=1}^{4} r_i \sqrt{s_i} \leq \sqrt{\sum_{i=1}^{4} r_i^2 s_i} \) (for \( r_i, s_i \geq 0, i \in [4] \)), we find from Eq. (9) that
\[ (a_{bt})_{3} \leq \sqrt{\int dA_i \rho_1(A_i) \delta_i \times \int dA_i \rho_2(A_i) \delta_i} \] (11)
where \( \delta_1 = \left| \langle A_1^1 \rangle A_1 + \langle A_1^2 \rangle A_2 + \langle A_1^3 \rangle A_3 + |A_1^4\rangle A_4 - \langle A_1^1 \rangle A_1 + \langle A_1^2 \rangle A_2 + \langle A_1^3 \rangle A_3 + |A_1^4\rangle A_4 \right| \]
and \( \delta_2 = \left| \langle A_2^1 \rangle A_1 + \langle A_2^2 \rangle A_2 + \langle A_2^3 \rangle A_3 + |A_2^4\rangle A_4 - (A_2^1 A_1 + A_2^2 A_2 + A_2^3 A_3 + A_2^4 A_4) \right| \]
Since all the observables are dichotomic having values \( \pm 1 \), it is simple to check that \( \delta_1 = \delta_2 \leq 6 \). Integrating over \( A_1 \) and \( A_2 \), we obtain \( (a_{bt})_{3} \leq 6 \), as claimed in Eq. (9).

The optimal quantum violation of the inequality in Eq. (9) is derived as \( (a_{bt})_{opt}^2 = 4 \sqrt{3} > (a_{bt})_{3} \) which requires all three observables of Alice1 (and Alice2) to be mutually anticommuting. The detailed derivation of optimal quantum value using an improved version of the sum-of-square (SOS) approach is placed in Appendix A.

The bilocality scenario can be straightforwardly extended to \( n \)-local scenario. Let \( A_i^1, A_i^2, A_i^3 \) be the observables of \( k \)th Alice where \( k = 1, 2 \ldots n \). We can then define
\[ J_{1,3,2} = \langle (A_1^1 + A_2^1 + A_3^1) (A_1^2 + A_2^2 + A_3^2) \cdots (A_1^n + A_2^n + A_3^n) B_1 \rangle \]
\[ J_{2,3,2} = \langle (A_1^1 + A_2^1 - A_3^1) (A_1^2 + A_2^2 - A_3^2) \cdots (A_1^n + A_2^n - A_3^n) B_2 \rangle \]
\[ J_{3,3,2} = \langle (A_1^1 + A_2^1 + A_3^1) (A_1^2 + A_2^2 + A_3^2) \cdots (A_1^n + A_2^n + A_3^n) B_3 \rangle \]
\[ J_{4,3,2} = \langle (A_1^1 + A_2^1 + A_3^1) (A_1^2 + A_2^2 + A_3^2) \cdots (A_1^n + A_2^n + A_3^n) B_4 \rangle \] (12)
Using the similar procedure adopted for deriving Eq. (9), we find the $n$-locality inequality for $m = 3$ is given by

$$\Delta_n^3 = \sum_{i=1}^{2^{n-1}} |\rho_{1}^{m,1} - \rho_{2}^{m,2}|^{1/n} \leq 6 \quad (13)$$

The optimal quantum violation remain same as $(\Delta_{Q}^n)^{opt}$. We show in the Appendix A that the optimal quantum violation of $n$-locality inequality is attained when every Alice chooses mutually anticommuting observables and shares a two-qubit maximally entangled state with Bob.

IV. GENERALIZED Bilocality and $n$-locality Scenario in Star-Network for Arbitrary $m$

We first generalize the bilocality scenario for any arbitrary $m$ and derive the bilocality inequality. The two edge parties Alice$_1$ and Alice$_2$ receive respective inputs $x_1, x_2 \in \{1, 2, 3, \cdots, m\}$ producing outputs $a_1, a_2 \in \{0, 1\}$. Bob receives input $i \in \{1, 2, \cdots, 2^{m-1}\}$ and produces output $b \in \{0, 1\}$. Let us propose a generalized bilocality expression for arbitrary $m$ as

$$\Delta_m^2 = \sum_{i=1}^{2^{n-1}} \sqrt{\rho_{1}^{m,1} - \rho_{2}^{m,2}} \quad (14)$$

where $J_{m,i}^2$ is the linear combinations of suitable correlations are defined as

$$J_{m,i}^2 = \left( \sum_{x_1=1}^{m} (-1)^{y_{A}} A_{x_1} \right) B_i \left( \sum_{x_2=1}^{m} (-1)^{y_{A}} A_{x_2} \right) \quad (15)$$

Here $y_{A,i}$ takes value either 0 or 1 and same for $y_{B,i}$. For our purpose, we fix the values of $y_{A,i}$ and $y_{B,i}$ by invoking the encoding scheme used in Random Access Codes (RACs) [43–46] as a tool. This will fix 1 or $-1$ values of $(-1)^{y_{A,i}}$ and $(-1)^{y_{B,i}}$ in Eq. (15) for a given $i$. Let us consider a random variable $y^{\delta} \in \{0, 1\}^m$ with $\delta \in \{1, 2, \cdots m\}$. Each element of the bit string can be written as $y^{\delta} = y^{\delta}_1 y^{\delta}_2 y^{\delta}_3 \cdots y^{\delta}_m$ with $y^{\delta}_i \in \{0, 1\}$. For example, if $y^{\delta} = 011\cdots00$ then $y^{\delta}_1 = 0$, $y^{\delta}_2 = 1$, $y^{\delta}_3 = 1$, and so on. Now, we denote the length $m$ binary strings as $y^{\delta}$ those have 0 as the first digit in $y^{\delta}$. Clearly, we have $i \in \{1, 2, \cdots, 2^{m-1}\}$ constituting the inputs for Bob. If $i = 1$, we get all zero bit in the string $y^{\delta}$ leading us $(-1)^{y_{A,i}} = 1$ for every $x_1 \in \{1, 2, \cdots, m\}$.

An example for $m = 2$ could be useful here. In this case, we have $y^{\delta} \in \{00, 01, 10, 11\}$ with $\delta = 1, 2, 3, 4$. We then denote $y^{\delta} = y^{\delta}_1 y^{\delta}_2 \in \{00, 01\}$ with $y^{\delta}_1 = 0$ and $y^{\delta}_2 = 1$. This also means $y^{\delta}_1 y^{\delta}_3 = 0$, $y^{\delta}_2 y^{\delta}_2 = 0$, $y^{\delta}_1 y^{\delta}_3 = 0$, and $y^{\delta}_2 y^{\delta}_2 = 1$. Putting those in Eq. (15), we can recover the expressions $J_{2,1}^2$ and $J_{2,2}^2$ in Eqs. (4) and (5) respectively.

Following the similar procedure used earlier, by considering the bilocality assumption $\rho(A_1, A_2) = \rho_1(A_1) \rho_2(A_2)$, we can write $J_{m,i}^2$ in a hidden variable model as

$$|J_{m,i}^2| = \int \int d\lambda_1 d\lambda_2 \rho_1(A_1) \rho_2(A_2) \times \left| \sum_{x_1=1}^{m} (-1)^{y_{A}} A_{x_1} \right| \left( \sum_{x_2=1}^{m} (-1)^{y_{A}} A_{x_2} \right) \quad (16)$$

where to avoid notational clumsiness, we denote

$$J_{m,i}^{2k} = \sum_{x_2=1}^{m} (-1)^{y_{A}} A_{x_2} \quad (17)$$

with $k = 1, 2$. Using $|\langle B_{i} \rangle_{A_{m}A_{k}}| \leq 1$ for dichotomic observable and further arranging, we have

$$\sqrt{|J_{m,i}^2|} \leq \sqrt{\int d\lambda_1 \rho_1(A_1) |J_{m,i}^{2k}| \times \int d\lambda_2 \rho_2(A_2) |J_{m,i}^{2k}|} \quad (17)$$

Now, by using Eq. (17) and the inequality $\sum_{i=1}^{2^{n-1}} r_i s_i \leq \sqrt{\sum_{i=1}^{2^{n-1}} r_i \sum_{j=1}^{2^{n-1}} s_j}$ for $r_i, s_i \geq 0$, the expression for $\Delta_m^2$ in Eq. (14) becomes,
The term \( J_m \) scheme for Alice \((\alpha\) we take in Eq.(20), we derive demonstrating the quantum violation of bilocality inequality considering that each of the he receives from \( n \) and Alice \( n \), then without loss of generality we take \( \alpha_m = \alpha^*_m \equiv \alpha_m \). Plugging \( \alpha_m \) in Eq. (18) and integrating over \( \lambda_1 \) and \( \lambda_2 \), we obtain the bilocality inequality for arbitrary \( m \) is

\[
(\Delta^2_m)_{bl} \leq \alpha_m.
\]  

Hence, the upper bound \( \alpha_m \) is the key quantity whose maximum value has to be determined for arbitrary \( m \). We derive

\[
(\Delta^2_m)_{bl} \leq \alpha_m = \sum_{j=0}^{|\frac{m}{2}|} \binom{m}{j} (m - 2j)
\]  

The detailed derivation of \( \alpha_m \) is placed in Appendix B. Note that, for \( m = 2 \) and \( m = 3 \) the respective bilocality inequalities (\( \Delta^2_m)_{bl} \leq 2 \) and (\( \Delta^2_m)_{bl} \leq 6 \) can be recovered. Before demonstrating the quantum violation of bilocality inequality in Eq.(20), we derive \( n \)-locality inequality for arbitrary \( m \).

In our \( n \)-locality scenario in star-network configuration we consider that each of the \( n \) number of Alcles shares a state with Bob, generated by the independent sources \( S_k \) with \( k \in [n] \), as depicted in Fig.2. Alice \( k \) performs \( m \) number of binary-outcome measurements \( A^k_{x_k} \) \((x_k \in [m])\), for any \( k \). Bob receives fixed number of inputs \( i \in \{1, 2, \ldots, 2^{m-1}\} \) and performs binary-outcome measurements on \( n \) number of systems he receives from \( n \) independent sources. We define the following expression,

\[
(\Delta^2_m)_{nl} = \sum_{i=1}^{2^{m-1}} |J^n_{m,i}|^{\frac{1}{2}}
\]  

where \( J^n_{m,i} \) for given \( i, n, \) and \( m \) is given by

\[
J^n_{m,i} = \prod_{k=1}^{n} \left[ \sum_{x_k=1}^{m} (-1)^{x_k} A^k_{x_k} B_i \right] = \prod_{k=1}^{n} J^n_{m,i} B_i
\]  

The term \( J^n_{m,i} \) is same as given in (16). Using \( n \)-locality assumption \( \rho(\lambda_1, \lambda_2, \ldots, \lambda_n) = \prod_{k=1}^{n} \rho_k(\lambda_k) \) and by defining \( \langle A^k_{x_k}\rangle_{\lambda_k} = \sum_{x_k=1}^{m} (-1)^{x_k} P(\lambda_k|x_k, \lambda_k) \), we can write

\[
|J^n_{m,i}|^{\frac{1}{2}} \leq \left[ \prod_{k=1}^{n} \int d\lambda_k \rho_k(\lambda_k)|J^n_{m,i}| \right]^{\frac{1}{2}}
\]  

where we consider \( |\langle B_i\rangle_{\lambda_1, \lambda_2, \ldots, \lambda_i}| \leq 1 \). From Eq.(22), we get

\[
(\Delta^2_m)_{nl} \leq \sum_{i=1}^{2^{m-1}} \left[ \prod_{k=1}^{n} \int d\lambda_k \rho_k(\lambda_k)|J^n_{m,i}| \right]^{\frac{1}{2}} (\Delta^2_m)_{nl} \leq \sum_{i=1}^{2^{m-1}} \left[ \prod_{k=1}^{n} \int d\lambda_k \rho_k(\lambda_k)|J^n_{m,i}| \right]^{\frac{1}{2}}
\]  

Assuming \( \int d\lambda_k \rho_k(\lambda_k)|J^n_{m,i}| = \zeta^i_k \) and by using the inequality [11]

\[
\forall \ z^i_k \geq 0: \sum_{i=1}^{2^{m-1}} \left( \prod_{k=1}^{n} \zeta^i_k \right)^{\frac{1}{2}} \leq \prod_{k=1}^{n} \sum_{i=1}^{2^{m-1}} \zeta^i_k^k \frac{1}{2}
\]  

evidently the Eq. (25) can be written as

\[
(\Delta^2_m)_{nl} \leq \sum_{i=1}^{n} \left[ \prod_{k=1}^{n} \int d\lambda_k \rho_k(\lambda_k)|J^n_{m,i}| \right]^{\frac{1}{2}}
\]  

As in Eq. (19), by putting \( \sum_{i=1}^{2^{m-1}} |J^n_{m,i}| = \alpha_m \) for \( k \in [n] \) in Eq. (25), and integrating over \( \lambda_k \), we finally get

\[
(\Delta^2_m)_{nl} \leq \prod_{k=1}^{n} (\alpha_m)^{n} = \alpha_m
\]  

Since we have used the same encoding scheme for each Alice, thus we have \( \forall k, \alpha_m = \alpha_m \). Eq. (28) provides the family of generalized \( n \)-locality inequalities for any arbitrary \( n \geq 2 \) and \( m \geq 2 \). The value of \( \alpha_m \) is given in Eq. (21) and explicitly derived in Appendix B.

To find the optimal quantum value of the expression (\( \Delta^2_m \)) in Eq.(22), we use an improved version of sum-of-squares (SOS) approach, so that, (\( \Delta^2_m \)) \( \leq \beta^m \) for all possible quantum states \( \rho_k(\lambda_k) \) and measurement operators \( A^k_{x_k} \) and \( B_i \). This is equivalent to showing that there is a positive semi-definite operator \( \gamma^m_{\lambda_k} \geq 0 \), which can be expressed as \( \gamma^m_{\lambda_k} = \langle A^k_{x_k}\rangle_{\lambda_k} + \beta^m \). This can be proven by considering a set of suitable positive operators \( M^n_{m,i} \), which are polynomial functions of \( A^k_{x_k} \) and \( B_i \), such that

\[
\langle \gamma^m_{\lambda_k} \rangle = \sum_{i=1}^{2^{m-1}} \left( \omega^m_{n,i} \right)^{1/2} \langle \rho(M^n_{m,i}) \rangle \rho^*_{m,i} \psi \rangle
given \text{where} \omega^m_{n,i} \text{ is a positive number with} \omega^m_{n,i} = \prod_{k=1}^{n} \omega_{m,i}^n. \text{The optimal quantum value of} \( \langle \gamma^m_{\lambda_k} \rangle \text{is obtained if} \( \langle \gamma^m_{\lambda_k} \rangle = 0 \text{i.e.,} \forall i, \ M^n_{m,i} \langle \rho(A_{\lambda_k} A_{\lambda_k} B_i) = 0 \text{ (30)} \text{where} \langle \rho(A_{\lambda_k} A_{\lambda_k} B_i) = \langle \rho(A_{\lambda_k} B_i) \otimes \langle \rho(A_{\lambda_k} B_i) \otimes \ldots \otimes \langle \rho(A_{\lambda_k} B_i) \rangle_{A_{\lambda_k} B_i} \text{are originating from independent sources} S_k. \text{By keeping in mind the expression given by Eq. (23), the operators} M^n_{m,i} \text{are suitably chosen as}

\[
M^n_{m,i} \langle \rho(A_{\lambda_k} B_i) = \prod_{k=1}^{n} \left( \frac{1}{\omega^m_{n,i}^{1/2}} \left[ \sum_{x_k=1}^{m} (-1)^{x_k} A^k_{x_k} \right] \right)^{1/2} \langle \rho(B_i) \rangle^{1/2}
\]  

and

\[
|J^n_{m,i}|^{1/2} = \left[ \prod_{k=1}^{n} \int d\lambda_k \rho_k(\lambda_k)|J^n_{m,i}| \right]^{1/2}
\]  

where \( J^n_{m,i} \).
Here, for notational convenience we write \( |\psi\rangle_{A_1 A_2 \cdots A_m B} = |\psi\rangle \). Note that \( \omega_{m,i}^{n,k} = \|J_{m,i}^{n,k}\|_2 = \| \sum_{x_k=1}^n (-1)^{y_k} A_x^{k} |\psi\rangle \|_2 \) and
\[
(\omega_{m,i}^{n,k})^2 = \langle \psi | (J_{m,i}^{n,k})^\dagger (J_{m,i}^{n,k}) |\psi\rangle = \langle \psi | (J_{m,i}^{n,k})^2 |\psi\rangle. \quad (32)
\]
We can then write,
\[
\langle \psi | (M_{m,i}^n)^\dagger M_{m,i}^n |\psi\rangle = \prod_{k=1}^n \frac{1}{(\omega_{m,i}^{n,k})} \left( \frac{1}{(\omega_{m,i}^{n,k})} \right)^{\frac{1}{2}}
- 2 \prod_{k=1}^n \left( \omega_{m,i}^{n,k} \right)^{\frac{1}{2}} \left( \langle \psi | (J_{m,i}^{n,k}) B_i |\psi\rangle \right) \frac{1}{2} \left( \langle \psi | (J_{m,i}^{n,k})^2 |\psi\rangle \right)^{\frac{1}{2}}
= 2 - 2 \prod_{k=1}^n \left( \omega_{m,i}^{n,k} \right)^{\frac{1}{2}} \left( \langle \psi | (J_{m,i}^{n,k}) B_i |\psi\rangle \right) \frac{1}{2} \left( \langle \psi | (J_{m,i}^{n,k})^2 |\psi\rangle \right)^{\frac{1}{2}} \quad (33)
\]
where we have used Eq. (32) and \( (A^k_x)^i A^k_x = B^i_x B_i = 1 \). Putting Eq. (33) in Eq. (29) we get
\[
\langle \psi | (M_{m,i}^n)^\dagger M_{m,i}^n |\psi\rangle = \sum_{i=1}^{2^n-1} \left( \omega_{m,i}^{n,k} \right)^{\frac{1}{2}} - \sum_{i=1}^{2^n-1} \left( \omega_{m,i}^{n,k} \right)^{\frac{1}{2}} \prod_{k=1}^n \left( \langle \psi | (J_{m,i}^{n,k}) B_i |\psi\rangle \right) \frac{1}{2} \left( \langle \psi | (J_{m,i}^{n,k})^2 |\psi\rangle \right)^{\frac{1}{2}} \quad (34)
\]
Since \( \omega_{m,i}^{n,k} = \prod_{k=1}^n \omega_{m,i}^{n,k} \) we have
\[
\langle \psi | (M_{m,i}^n)^\dagger M_{m,i}^n |\psi\rangle = \sum_{i=1}^{2^n-1} \left( \omega_{m,i}^{n,k} \right)^{\frac{1}{2}} - \sum_{i=1}^{2^n-1} \left( \omega_{m,i}^{n,k} \right)^{\frac{1}{2}} \prod_{k=1}^n \left( \langle \psi | (J_{m,i}^{n,k}) B_i |\psi\rangle \right) \frac{1}{2} \left( \langle \psi | (J_{m,i}^{n,k})^2 |\psi\rangle \right)^{\frac{1}{2}} \quad (35)
\]
\[
= \sum_{i=1}^{2^n-1} \left( \omega_{m,i}^{n,k} \right)^{\frac{1}{2}} - (\Delta_{m,i}^{n})_Q \quad (36)
\]
Since \( \gamma_{m,i}^n \) is positive semi-definite, the maximum value of \( (\Delta_{m,i}^{n})_Q \) is obtained when \( \gamma_{m,i}^n = 0 \), i.e.,
\[
(\Delta_{m,i}^{n})_Q^{\text{opt}} = \max \left( \sum_{i=1}^{2^n-1} \left( \omega_{m,i}^{n,k} \right)^{\frac{1}{2}} \right) \quad (37)
\]
Using again the inequality (26), we can write \( \sum_{i=1}^{2^n-1} \left( \omega_{m,i}^{n,k} \right)^{\frac{1}{2}} \leq \prod_{k=1}^n \left( \sum_{i=1}^{2^n-1} \omega_{m,i}^{n,k} \right)^{\frac{1}{2}} \) and since \( A_{x,i} \)'s are dichotomic, the quantity \( \omega_{m,i}^{n,k} \) can explicitly be written as
\[
\omega_{m,i}^{n,k} = \left[ m + \sum_{x_i=1}^m (-1)^{y_i} A_x \right] + \left[ (-1)^{y_i} A_x \right] \quad (38)
\]
\[
\sum_{x_i=1}^m (-1)^{y_i} A_x \right] \quad \cdots \quad \left[ (-1)^{y_i} A_x \right] \quad (39)
\]
Following the procedure adopted for \( m = 3 \) in Appendix A, we find \( \omega_{m,i}^{n,k} \leq \sqrt{m} \) for every \( i \) and \( k \), and the equality holds only when every anticommutator is zero. We then have
\[
\sum_{i=1}^{2^n-1} \left( \omega_{m,i}^{n,k} \right)^{\frac{1}{2}} \leq \prod_{k=1}^n \left( \sum_{i=1}^{2^n-1} \omega_{m,i}^{n,k} \right)^{\frac{1}{2}} = \prod_{k=1}^n \left( 2^{m-1} \sqrt{m} \right)^{\frac{1}{2}} . \quad (30)
\]
\[
(\Delta_{m,i}^{n})_Q^{\text{opt}} = 2^{m-1} \sqrt{m} \quad (38)
\]
which is larger than \( \alpha_m \) in Eq. (21) for any arbitrary \( m \), thereby implying the quantum violation of the family of \( n \)-locality inequalities given by Eq. (28). The ratio \( R_m = (\Delta_{m,i}^{n})_Q^{\text{opt}} / (\Delta_{m,i}^{n})_Q^{\text{opt}} \) between quantum and classical upper bounds for a given \( m \) is plotted in Fig. 3. We find that the ratio saturates to 1.25 for sufficiently large values of \( m \).

As already mentioned, the optimal quantum value is obtained when for every \( k \), Alice chooses \( m \) number of anticommutating observables \( (A^k_x) \). Bob’s observables can be fixed from Eq. (30), so that, \( B_i |\psi\rangle = \prod_{k=1}^n \frac{1}{\sqrt{m}} \omega_{m,i}^{n,k} |\psi\rangle \). This in turn implies that the required state \( |\psi\rangle \) has to satisfy \( B_i \otimes B_j |\psi\rangle = |\psi\rangle \) for every \( i \).

Note here that for \( m = 2 \) and \( m = 3 \) cases, optimal quantum value can be achieved when every Alice shares a single two-qubit maximally entangled state with Bob. However, at most three mutually anticommutating observables are available for qubit system. Hence, for \( m \geq 4 \), every Alice requires higher dimensional system to achieve the optimal value. Note that, there exists \( 2^m + 1 \) mutually anticommutating observables for \( l \) \( l \in \mathbb{N} \) qubit system [52]. We find that to obtain the optimal value for arbitrary \( m \), the local dimension of every Alice to be \( d = 2^{m/2} \). In other words, every Alice shares at least \( [m/2] \) copies of two-qubit maximally entangled state with Bob. The total state for \( n + 1 \) parties can be written as
\[
|\psi_{A_1 A_2 \cdots A_m B}^{\text{opt}}\rangle = |\phi_{A_B}^{\text{opt}}\rangle \otimes |\phi_{A_B}^{\text{opt}}\rangle \otimes \cdots \otimes |\phi_{A_B}^{\text{opt}}\rangle \otimes |\phi_{A_B}^{\text{opt}}\rangle \otimes |\phi_{A_B}^{\text{opt}}\rangle \quad (39)
\]
where \( |\phi_{A_B}^{\text{opt}}\rangle \otimes |\phi_{A_B}^{\text{opt}}\rangle \) is the state originated from each independent source \( S_k \). The above discussion clearly indicates the possibility of witnessing the dimension of Hilbert space in network scenario and activation of non-\( n \)-locality by using multiple copies of entangled states.

**V. SUMMARY AND DISCUSSION**

In summary, we have provided a non-trivial generalization of the \( n \)-locality scenario in the star-network configuration.
Such a network involves $n$ number of edge observers (Alices), a central observer (Bob), and $n$ number of independent sources. Instead of two dichotomic measurements per party, we considered that every Alice performs an arbitrary $m$ number of dichotomic measurements, and the central observer Bob performs $2^{m-1}$ number of dichotomic measurements. We derived a family of generalized $n$-locality inequalities.

As a case study, we first considered the bilocality scenario for $m = 3$ case and derived the bilocality inequality, when each of the two Alices performs three binary-outcome measurements, and Bob performs four binary-outcome measurements. We demonstrated the optimal quantum violation using an improved version of the usual sum-of-squares (SOS) approach. The optimal quantum value is attained when observables of each Alice are mutually anticommuting. Bob’s observables and the required entangled states are then fixed by the optimization condition. We showed that each Alice has to share a two-qubit maximally entangled state with Bob to obtain the optimal value. We further extended the argument to $n$-locality scenario in star-network configuration.

We then extended our study for arbitrary $m$ and derived a family of generalized $n$-locality inequalities in star-network configuration. We demonstrated that the optimal quantum violation is attained when all $m$ observables of every Alice has to be mutually anticommuting. Since there are at most three mutually anticommuting observables for a qubit system, for $m > 3$, the local dimension of Hilbert space of each Alice needs has to be more than two. We found that the local dimension of every Alice needs to be $d = 2^{[m/2]}$. We argued that every Alice and Bob share $[m/2]$ copies of two-qubit maximally entangled state to achieve the optimal quantum violation of the family of generalized $n$-locality inequalities.

Hence, a single copy of a two-qubit entangled state may not reveal the non-$n$-locality for $m > 3$ cases, but multiple copies of it may activate non-$n$-locality in the network. Such a feature indicates witnessing the dimension of Hilbert space in network scenario. Suppose for $m = 4$, we find $(\Delta^3_n)_Q > (\Delta^4_n)_nl$ for a single two-qubit maximally entangled state, thereby providing an upper bound for qubit system. To obtain the optimal value $(\Delta^4_n)_{opt}$, a pair of two-qubit maximally entangled state is required to be shared by $k^{th}$ Alice and Bob, where $k \in [n]$. Instead of a two-qubit maximally entangled state, one may use a noisy two-qubit entangled state, and Bob performs four binary-outcome measurements, and the central observer Bob performs $2^{m-1}$ number of dichotomic measurements. We demonstrated the optimal quantum violation using an improved version of the usual sum-of-squares (SOS) approach. The optimal quantum value is attained when observables of each Alice are mutually anticommuting. Since there are at most three mutually anticommuting observables for a qubit system, for $m > 3$, the local dimension of Hilbert space of each Alice needs has to be more than two. We found that the local dimension of every Alice needs to be $d = 2^{[m/2]}$. We argued that every Alice and Bob share $[m/2]$ copies of two-qubit maximally entangled state to achieve the optimal quantum violation of the family of generalized $n$-locality inequalities.

### Appendix A: Optimal value of bilocality inequality ($n = 2$) for $m = 3$

To derive the optimal quantum value of $(\Delta^3_2)_Q$, we use a variant of the known sum-of-squares (SOS) approach. We show that there is a positive semidefinite operator $\gamma^3_2$, that can be expressed as $\langle \gamma^3_2 \rangle_Q = -(\Delta^3_2)_Q + \beta^2_3$. Here $\beta^2_3$ is the optimal value, can be obtained when $\langle \gamma^3_2 \rangle_Q$ is equal to zero. This can be proven by considering a set of suitable positive operators $M^2_{ij}$, which is polynomial functions of $A_{11}$, $B_i$ and $A_{12}$, so that

$$\langle \gamma^3_2 \rangle = \sum_{i=1}^{4} \frac{\sqrt{\omega^3_{ij}}}{2} \langle \psi | (M^2_{ij})^\dagger (M^2_{ij}) | \psi \rangle$$  \hspace{1cm} (A1)
where $\omega_{3,j}^2$ is suitable positive numbers and $\omega_{3,j}^2 = \omega_{3,j}^{2,1} \cdot \omega_{3,j}^{2,2}$. We choose a suitable set of positive operators $M_{3,j}$ as

$$M_{3,1}^{2,1} |\psi\rangle_{A_1} = \sqrt{\frac{A_1^3 + A_1^2 A_2^1 + A_2^2}{(2)_{3,j}^2}} \otimes \frac{A_1^3 + A_1^2 A_2^1 + A_2^2}{(2)_{3,j}^2} |\psi\rangle_{A_1} - \sqrt{B_1^2} |\psi\rangle_{A_1}$$  \hspace{1cm} (A2)

$$M_{3,2}^{2,2} |\psi\rangle_{A_1} = \sqrt{\frac{A_1^3 + A_1^2 A_2^1 + A_2^2}{(2)_{3,j}^2}} \otimes \frac{A_1^3 + A_1^2 A_2^1 + A_2^2}{(2)_{3,j}^2} |\psi\rangle_{A_1} - \sqrt{B_2^2} |\psi\rangle_{A_1}$$  \hspace{1cm} (A3)

$$M_{3,3}^{2,3} |\psi\rangle_{A_1} = \sqrt{\frac{A_1^3 + A_1^2 A_2^1 + A_2^2}{(2)_{3,j}^2}} \otimes \frac{A_1^3 + A_1^2 A_2^1 + A_2^2}{(2)_{3,j}^2} |\psi\rangle_{A_1} - \sqrt{B_3^2} |\psi\rangle_{A_1}$$  \hspace{1cm} (A4)

$$M_{3,4}^{2,4} |\psi\rangle_{A_1} = \sqrt{\frac{A_1^3 + A_1^2 A_2^1 + A_2^2}{(2)_{3,j}^2}} \otimes \frac{A_1^3 + A_1^2 A_2^1 + A_2^2}{(2)_{3,j}^2} |\psi\rangle_{A_1} - \sqrt{B_4^2} |\psi\rangle_{A_1}$$  \hspace{1cm} (A5)

where $\omega_{2,k}^2$ is defined as $\omega_{2,k}^2 = ||(A_k^1 + A_k^2) |\psi\rangle_{A_1}||_2$ and similarly for other $\omega_{2,k}^2$, $s, k = 1, 2$. Here $||\cdot||_2$ denotes the norm of the vector. Plugging Eqs. (A2-A5) in Eq. (A1), we get $(\gamma_{3,j}^2)_Q = - (\Delta_{3,j}^2)_Q + \frac{4}{\sum_{i=1}^{4} \sqrt{\omega_{3,j}^2}}$. The optimal quantum value of $(\Delta_{3,j}^2)_Q$ is obtained if $(\gamma_{3,j}^2)_Q = 0$, implying that $\forall i, M_{3,j}^{2,i} |\psi\rangle_{A_1} = 0$. Hence,

$$(\Delta_{3,j}^2)_{Q}^{opt} = \max \left( \left( \frac{4}{\sum_{i=1}^{4} \sqrt{\omega_{3,j}^2}} \right) \right)$$  \hspace{1cm} (A6)

As defined, $\omega_{3,1}^{2,1} = ||(A_1^1 + A_1^2 + A_1^3) |\psi\rangle_{A_1}||_2 = \sqrt{3 + \langle (|A_1^1, A_1^2| + |A_1^2, A_1^3| + |A_1^3, A_1^1|) \rangle}$. Similarly, we can write

$$\omega_{3,1}^{2,2} = \sqrt{3 + \langle (|A_1^1, A_1^2| + |A_1^2, A_1^3| + |A_1^3, A_1^1|) \rangle}, \omega_{3,2}^{2,1} = \sqrt{3 + \langle (|A_1^1, A_1^2| - |A_1^2, A_1^3| - |A_1^3, A_1^1|) \rangle}, \omega_{3,3}^{2,2} = \sqrt{3 + \langle (|A_1^1, A_1^2| - |A_1^2, A_1^3| + |A_1^3, A_1^1|) \rangle}, \omega_{3,4}^{2,1} = \sqrt{3 + \langle (|A_1^1, A_1^2| + |A_1^2, A_1^3| - |A_1^3, A_1^1|) \rangle}.$$  \hspace{1cm} (A7)

Since $\omega_{3,j}^{2,1} \cdot \omega_{3,j}^{2,2}$, by using the inequality $\sum_{i=1}^{4} \sqrt{r_i s_i} \leq \sum_{i=1}^{4} r_i \sum_{i=1}^{4} s_i$ for $r_i, s_i \geq 0$, $i = 1, 2, 3, 4$, we get $\frac{4}{\sum_{i=1}^{4} \sqrt{\omega_{3,j}^2}} \leq \sqrt{\frac{1}{\sum_{i=1}^{4} \omega_{3,j}^{2,1} \sum_{i=1}^{4} \omega_{3,j}^{2,2}}}$, by using the identity $\sqrt{b} + a + \sqrt{b} - a = \sqrt{2b + 2 \sqrt{b^2 - a^2}}$, we obtain

$$\sum_{i=1}^{4} \sqrt{\omega_{3,j}^2} \leq \left( \frac{2 + \sqrt{3 + \langle (|A_1^1, A_1^2| + |A_1^2, A_1^3| + |A_1^3, A_1^1|) \rangle}}{2 + \sqrt{3 + \langle (|A_1^1, A_1^2| + |A_1^2, A_1^3| + |A_1^3, A_1^1|) \rangle}} \right)^{\frac{1}{2}}$$

$$+ \left( \frac{2 + \sqrt{3 - \langle (|A_1^1, A_1^2| + |A_1^2, A_1^3| + |A_1^3, A_1^1|) \rangle}}{2 + \sqrt{3 - \langle (|A_1^1, A_1^2| + |A_1^2, A_1^3| + |A_1^3, A_1^1|) \rangle}} \right)^{\frac{1}{2}}$$

$$+ \left( \frac{2 + \sqrt{3 + \langle (|A_1^1, A_1^2| + |A_1^2, A_1^3| + |A_1^3, A_1^1|) \rangle}}{2 + \sqrt{3 + \langle (|A_1^1, A_1^2| + |A_1^2, A_1^3| + |A_1^3, A_1^1|) \rangle}} \right)^{\frac{1}{2}}$$

$$+ \left( \frac{2 + \sqrt{3 - \langle (|A_1^1, A_1^2| + |A_1^2, A_1^3| + |A_1^3, A_1^1|) \rangle}}{2 + \sqrt{3 - \langle (|A_1^1, A_1^2| + |A_1^2, A_1^3| + |A_1^3, A_1^1|) \rangle}} \right)^{\frac{1}{2}}$$

Clearly, $(\Delta_{3,j}^2)_{Q}^{opt} = max \left( \left( \frac{4}{\sum_{i=1}^{4} \sqrt{\omega_{3,j}^2}} \right) \right) = 4 \sqrt{3}$. This is obtained when $A_1^1, A_1^2$ and $A_1^3$ are mutually anticommuting and same for $A_2^1, A_2^2$ and $A_2^3$. It is straightforward to find Bob’s observables from $M_{3,j}^{2,1} |\psi\rangle_{A_1} = 0$, $\forall i$ which, in turn, fixes the state $|\psi\rangle_{A_1} = 0$, to be a two-qubit maximally entangled state. The same approach is used in the main text to derive the optimal quantum violations of the family of $n$-local inequalities for arbitrary $m$. 


Appendix B: Derivation of $\alpha_m$ in Eq. (28) of the main text

The $n$-locality upper bound $\alpha_m^k$ for a given $k$ is given by

$$\alpha_m^k = \sum_{i=1}^{2^{n-1}} |J_{m,i}^k| = \sum_{i=1}^{2^{n-1}} \left| \sum_{x_i=1}^{m} (-1)^{y_i} \langle A_{x_i}^k \rangle_{A_i} \right|$$

(B1)

As mentioned in the main text, for a given $i \in \{1, 2, \ldots, 2^{m-1}\}$, the quantity $y_i$ is either 0 or 1, fixed by the encoding scheme of random-access code. Here $y_i$ contains those elements (bit strings) of $x_0 \in \{0, 1\}^m$ having first bit 0. The term $y_i$ then denotes the $y_i^{th}$ bit of $y_i$. By writing the length $m$ bit string $y_i$ as $i^{th}$ column we have the generator matrix of the augmented Hadamard code [47] of order $m \times 2^{m-1}$ as

$$G = \begin{bmatrix}
    y_1^1 & y_1^2 & \cdots & y_1^{2^{m-1}} \\
y_2^1 & y_2^2 & \cdots & y_2^{2^{m-1}} \\
y_3^1 & y_3^2 & \cdots & y_3^{2^{m-1}} \\
\vdots & \vdots & \ddots & \vdots \\
y_{m-1}^1 & y_{m-1}^2 & \cdots & y_{m-1}^{2^{m-1}} \\
y_m^1 & y_m^2 & \cdots & y_m^{2^{m-1}} 
\end{bmatrix} = \begin{bmatrix}
    0 & 0 & \cdots & 0 \\
    0 & 0 & \cdots & 1 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & 1 
\end{bmatrix}_{m \times 2^{m-1}}$$

Since $J_{m,i}^k = \sum_{x_i=1}^{m} (-1)^{y_i} \langle A_{x_i}^k \rangle_{A_i}$, then there is a one-to-one correspondence with the $i^{th}$ column of $G$. We can then write

$$J^k = \begin{bmatrix}
    J_{m,1}^k & J_{m,2}^k & \cdots & J_{m,2^{m-1}}^k 
\end{bmatrix} = \begin{bmatrix}
    \langle A_1^k \rangle_{A_1} & \langle A_1^k \rangle_{A_2} & \cdots & \langle A_1^k \rangle_{A_{2^{m-1}}} \\
    \langle A_2^k \rangle_{A_1} & \langle A_2^k \rangle_{A_2} & \cdots & \langle A_2^k \rangle_{A_{2^{m-1}}} \\
    \vdots & \vdots & \ddots & \vdots \\
    \langle A_{2^{m-1}}^k \rangle_{A_1} & \langle A_{2^{m-1}}^k \rangle_{A_2} & \cdots & \langle A_{2^{m-1}}^k \rangle_{A_{2^{m-1}}} 
\end{bmatrix}$$

For deriving $\alpha_m^k$, we require the modulus of each column of $J^k$ and their sum. We also note that $\langle A_{x_i}^k \rangle_{A_i} = \pm 1$. A bit-flip operation in a row of $G$ corresponds to the change of sign of $\langle A_{x_i}^k \rangle_{A_i}$ in the same row of $J^k$. There are $2^m$ number of permutations of bit-flips are possible for a given $m$. However, the encoding scheme in a random-access-code is so constructed that due to sign change of observables, the modulus value of each column $|J_{m,i}^k|$ may change its place but for every permutation of observable signs just corresponds to the permutation of column. Hence, for every permutation of observable signs, $\sum_{i=1}^{2^{m-1}} |J_{m,i}^k|$ remains invariant.

Hence, without loss of generality, we can derive the upper bound $\alpha_m^k$ by simply taking the outcomes of all observables are +1. Now, note that there are total $m \cdot 2^{m-1}$ number of $\pm 1$ entries in $2^{m-1}$ number of columns in $J^k$. Among them, $\binom{m}{1}$ columns have one $-1$ entry, $\binom{m}{2}$ columns have two $-1$ entries, $\cdots$, $\binom{m}{\left\lfloor \frac{m}{2} \right\rfloor - 1}$ columns have $\left\lfloor \frac{m}{2} \right\rfloor - 1$ number of $-1$ entries. Now, if $m$ is odd, then $\binom{m}{\left\lfloor \frac{m}{2} \right\rfloor}$ columns have $\left\lfloor \frac{m}{2} \right\rfloor$ number of $-1$ entries. But, if $m$ is even, then $\frac{1}{2}\binom{m}{\left\lfloor \frac{m}{2} \right\rfloor}$ columns have $\left\lfloor \frac{m}{2} \right\rfloor$ number of $-1$ entries. Altogether, by taking sum of the modulus of the sum of each column, we have

$$\alpha_m^k = m2^{m-1} - 2 \left( \sum_{j=1}^{\left\lceil \frac{m}{2} \right\rceil} \binom{m}{j} \left( 1 - \frac{m \oplus 1}{2} \right) \right) \left( \frac{m}{\left\lceil \frac{m}{2} \right\rceil} \right)$$

(B2)

Now, in order to match the form in last quantity, we re-write $m2^{m-1}$ in a specific form as

$$m2^{m-1} = \sum_{j=0}^{\left\lceil \frac{m}{2} \right\rceil} \binom{m}{j} \left( 1 - \frac{m \oplus 1}{2} \right) m \left( \frac{m}{\left\lceil \frac{m}{2} \right\rceil} \right)$$

(B3)
Using Eq. (B3) in Eq.(B2), we get the following. If \( m \) is odd, we have
\[
\alpha_m^k = \sum_{j=0}^{\lfloor \frac{j}{2} \rfloor} m(j) + \frac{m}{2} + \lfloor \frac{m}{2} \rfloor = \sum_{j=0}^{\lfloor \frac{j}{2} \rfloor} \left( m(j) − 2 \left( \sum_{j=0}^{\lfloor \frac{j}{2} \rfloor} \left( \frac{m}{2} \right) \right) \right) = \sum_{j=0}^{\lfloor \frac{j}{2} \rfloor} (m − 2j) (B4)
\]
and if \( m \) is even, we have
\[
\alpha_m^k = \sum_{j=0}^{\lfloor \frac{j}{2} \rfloor} m(j) + \frac{m}{2} + \lfloor \frac{m}{2} \rfloor = \sum_{j=0}^{\lfloor \frac{j}{2} \rfloor} \left( m(j) − 2 \left( \sum_{j=0}^{\lfloor \frac{j}{2} \rfloor} \left( \frac{m}{2} \right) \right) \right) = \sum_{j=0}^{\lfloor \frac{j}{2} \rfloor} (m − 2j) (B5)
\]
Thus, for any arbitrary \( m \),
\[
\alpha_m^k = \sum_{j=0}^{\lfloor \frac{j}{2} \rfloor} (m − 2j) (B6)
\]
As we argued in the main text, \( \alpha_m^k \) is same for every \( k \), we then have \( \Delta_m^k \leq \prod_{k=1}^{n} \left( \alpha_m^k \right)^{\frac{1}{2}} = \alpha_m \). This is placed in Eq.(21) in the main text.
[44] S. Ghorai, A. K. Pan, Phys. Rev. A 98, 032110 (2018).
[45] A. K. Pan, and S. S. Mahato, Phys. Rev. A 102, 052221 (2020).
[46] A. Kumari and A. K. Pan, Phys. Rev. A 100, 062130 (2019).
[47] A. Sanjeev, B. Boaz, Computational Complexity: A Modern Approach. Cambridge University Press, ISBN 978-0-521-42426-4.
[48] C. Palazuelos, Phys. Rev. Lett. 109, 190401 (2012).
[49] M. Navascués, T. Vértesi, Phys. Rev. Lett. 106, 060403 (2011).
[50] P. Caban, A. Molenda, K. Trzcińska, Phys. Rev. A 92, 032119 (2015).
[51] P. Contreras-Tejada, C. Palazuelos, and J. I. de Vicente, Phys. Rev. Lett. 126, 040501 (2021).
[52] J. Lawrence, C. Brukner, and A. Zeilinger Phys. Rev. A 65, 032320 (2002).