One-Time Pad, Arithmetic Coding and Logic Gates: An unifying theme using Dynamical Systems

Nithin Nagaraj and Prabhakar G. Vaidya
School of Natural Sciences and Engineering
National Institute of Advanced Studies
IISc Campus, Bangalore 560012
Email: nithin_nagaraj@yahoo.com

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Abstract

In this letter, we prove that the perfectly secure One-Time Pad (OTP) encryption can be seen as finding the initial condition on the binary map under a random switch based on the perfectly random pad. This turns out to be a special case of Grangetto’s randomized arithmetic coding performed on the Binary Map. Furthermore, we derive the set of possible perfect secrecy systems using such an approach. Since OTP encryption is an XOR operation, we thus have a dynamical systems implementation of the XOR gate. We show similar implementations for other gates such as NOR, NAND, OR, XNOR, AND and NOT. The dynamical systems framework unifies the three areas to which Shannon made foundational contributions: lossless compression (Source Coding), perfect encryption (Cryptography), and design of logic gates (Computation).

1 Shannon’s Legacy: Coding, Cryptography and Computation

Claude Shannon was one of the most important figures in the information revolution of the last century. He made foundational contributions to Coding, Cryptography and Computation. His master’s thesis used Boolean algebra to analyze and synthesize switching and computer circuits [1]. In 1948, he formulated a mathematical theory of communication where among other things, he was the first to define Entropy as the fundamental limit of noiseless lossless source coding [2]. In the following year, he proved the perfect secrecy
of the Vernam cryptographic system, also popularly known as One-Time Pad (OTP) [3]. OTP is the only method to boast of perfect secrecy.

In this paper, we attempt to unify these three themes by a dynamical systems framework. We claim that the three things are deeply related when viewed from a dynamical systems perspective. The letter is organized as follows. In Section 2 we introduce the Binary Map and its skewed cousins, a piece-wise linear dynamical system which is Lebesgue measure (in this case, this is the probability measure) preserving, chaotic and ergodic. Section 3 introduces arithmetic coding as finding the initial condition on the skewed Binary Map. Section 4 introduces Grangetto’s randomized arithmetic coding and establishes its connection with the OTP. Section 5 deals with generalizing OTP to higher alphabets. Section 6 talks about implementation of logic gates using randomized arithmetic coding. We conclude in Section 7.

2 The Binary Map and its Skewed Cousins

The Binary Map (Fig. 1(a)) \( T : [0, 1) \rightarrow [0, 1) \) is defined as:

\[
x \mapsto 2x, \quad 0 \leq x < \frac{1}{2}
\]

\[
x \mapsto 2x - 1, \quad \frac{1}{2} \leq x < 1.
\]

It is well known that the binary map is a non-linear chaotic dynamical system, which preserves the Lebesgue measure [4]. Furthermore, every initial condition in \([0, 1)\) has a unique symbolic sequence and every finite length (\(> 0\)) symbolic sequence corresponds to a subset of \([0, 1)\) of non-zero measure. Since the binary map has the maximum topological entropy for two symbols (\(= \ln(2)\)), all possible arrangements of 0 and 1 can occur in its space of symbolic sequences.

![Figure 1: (a) Binary Map. (b) Skewed Binary Map. Both these are examples of GLS.](image)

The symbol ‘0’ corresponds to the interval \([0, 0.5)\) and the symbol ‘1’ corresponds to the interval \([0.5, 1)\). The binary map belongs to a larger class of dynamical systems known as
Generalized Luröth Series (GLS) \[4\]. GLS is studied for its number theoretical properties. The skewed binary map is shown in Fig. 1(b). Here the symbols ‘0’ and ‘1’ correspond to the intervals \([0, p]\) and \([p, 1)\) respectively \((0 \leq p \leq 1, \ p = 0.5\) corresponds to the binary map).

2.1 Modes of Skewed Binary Map

There are 8 different modes of the skewed binary map as shown in Fig. 2. These are obtained by a combination of swapping the two intervals corresponding to ‘0’ and ‘1’, and changing the sign of the slope of the map in the two intervals. We shall call a map with two alphabets a dual of the another if the two intervals along with their symbols are swapped.

![Different modes of the skewed binary map. The bottom row are duals of the maps in the first row.](image)

The GLS can be readily extended to larger alphabets.

3 Arithmetic coding seen as a Dynamical System

Recently, we have proposed a method of lossless data compression where the (binary) message \((M)\) is treated as the symbolic sequence on the appropriate GLS \((p\) corresponds to the probability of the alphabet ‘0’ in the message) and the initial condition is determined by iterating backwards. The initial condition serves as the compressed file which can be used to determine the symbolic sequence (message) at the decoder (given \(p\)). We called such a method as GLS-coding. GLS-coding is a generalization of arithmetic coding which achieves the Shannon’s entropy rate for the source. Thus GLS-coding yields optimal noiseless lossless data compression. For full details please refer to \[5\].
4 Randomized Arithmetic Coding and the One Time Pad

Grangetto’s Randomized Arithmetic Coding is one of the earliest attempts to provide both source coding and encryption using Arithmetic Coding [6]. The idea of Randomized Arithmetic Coding is to randomly swap (or not swap) the two intervals corresponding to the symbols (‘0’ and ‘1’) at every iteration based on a random private binary key stream (\(K\)) which is available only to the decoder of the intended party. This randomizes the location of the final interval while retaining compression efficiency. Having already established that Arithmetic Coding is a specific mode of GLS, we can interpret Randomized Arithmetic Coding as a swapping between two modes of the GLS (the two modes are duals of each other) at every iteration based on a private key stream (Fig. 3).

![Diagram showing K=0 and K=1](image)

Figure 3: Grangetto’s Randomized Arithmetic Coding. ‘K’ is the binary key stream. When \(K = 0\), the backward iteration is done on the skewed binary map on the left and for \(K = 1\), the backward iteration is done on its dual on the right.

Two important points that need to be remembered in randomized arithmetic coding are:

1. The key stream \(K\) should be perfectly random to ensure best security.
2. The key stream \(K\) is as long as the uncompressed message \(M\).

4.1 OTP is a special case of randomized arithmetic coding

It turns out that OTP encryption and decryption can be seen as special case of randomized arithmetic coding. Instead of using the skewed binary map, if we used the binary map then we end up with the OTP encryption which Shannon showed in 1949 to be perfectly secure.

**Theorem:** OTP encryption is equivalent to finding the initial condition for the symbolic sequence \(M\) under switching based on key \(K\) on the binary map and its dual.
Figure 4: OTP is a special case of Grangetto’s Randomized Arithmetic Coding performed on the binary map instead of the skewed binary map. Thus OTP can be seen as finding the initial condition under random switching on the binary map and its dual.

Proof: We shall prove that this is equivalent to an XOR operation between the message $M$ and the key $K$. Since XOR operation is equivalent to OTP encryption, we would thus have a proof of the theorem.

Let us consider all possibilities for one bit of the key $K$ and one bit of the message $M$. When $K = 0$ and $M = 0$, the interval is mapped to $[0,0.5)$. The initial condition is going to lie in this interval irrespective of future bits (this is because the map is continuous in each of the intervals). The first bit of the initial condition is going to be 0. When $K = 0$ and $M = 1$, the initial condition will lie in $[0.5,1)$ which would mean that the first bit of the initial condition is 1. When $K = 1$, the outputs are reversed. The output is shown in Table 1. It can be seen that this is equivalent to the XOR operation between $K$ and $M$. Subsequent bits would follow the same logic (one can imagine that the first bit of the initial condition has been flushed as output and the interval has been rescaled to $[0,1)$ to begin encoding the second bit of $M$ with the second bit of $K$ for switching).

Table 1: Switching between the two maps is equivalent to XOR between $K$ and $M$.

| $K$ | $M$ | First bit of initial condition |
|-----|-----|-------------------------------|
| 0   | 0   | 0                             |
| 0   | 1   | 1                             |
| 1   | 0   | 1                             |
| 1   | 1   | 0                             |

We are effectively attempting to compress the message stream $M$ under the switching operation. We are performing GLS-coding (or arithmetic coding) on the binary map and its dual. However, it must be noted that since the two intervals for the binary map and its dual are of equal length, no compression will be achieved by the method. Thus the initial condition when expressed in binary need to have the same length as the message stream $M$ to enable lossless decompression (in this case, decryption). Decryption involves finding the symbolic sequence on the binary map (and its dual under the operation of switching...
Figure 5: Perfect secrecy systems equivalent to OTP. There is a choice of 4 modes for $K = 0$ and $K = 1$ independently. Thus there are 16 possible perfect secrecy systems which are equivalent to OTP.

4.2 Other perfect secrecy systems that are equivalent to OTP

This connection between arithmetic coding, binary map and OTP enables us to find perfect secrecy systems which are all equivalent to OTP. We know that the binary map has 4 possible modes that correspond to choosing either positive or negative slope in the two intervals and 4 other modes that are duals. In order to obtain secrecy systems that are equivalent to OTP, we can choose any of the 4 modes for $K = 0$ and $K = 1$ independently. Thus there are 16 possible secrecy systems which are all perfectly secure. The OTP is in...
5 \textit{n-OTP: generalization to non-binary alphabets}

The dynamical system viewpoint that we have proposed immediately enables us to generalize OTP for non-binary alphabets ($n$-OTP where $n \geq 2$) while retaining perfect secrecy. Suppose we have a message that takes values from the ternary alphabet \{0,1,2\}. We further assume that we have a \textit{perfect} random key stream that also takes values from the ternary alphabet. To perform encryption, we switch between the three GLS maps shown in Fig. 6 depending on the key value.

![Figure 6](image.png)

Figure 6: \textit{n-OTP}: For $n = 3$ and for an input ternary message $M$, the OTP can be implemented as finding the initial condition by switching between the above three dynamical systems based on a random key $K$ drawn from the alphabet \{0,1,2\} with equal probabilities. This is equivalent to addition modulo 3 between $M$ and $K$.

For $n > 2$, there are multiple options for choosing the $n$ dynamical systems to switch. Thus it is possible to generalize OTP encryption to larger alphabets.

6 \textbf{A Dynamical Systems Implementation of XOR and other Logic Gates}

As we noted earlier, it is well known that the OTP encryption is an XOR (exclusive-OR) operation between message stream $M$ and the key stream $K$. This means that we have a dynamical system implementation of the XOR gate. Is it possible to get other logic gates from this framework?

6.1 \textbf{XNOR, OR, AND, NAND, NOR and NOT gates}

We show that it is possible to implement the well known logic gates XNOR, OR, AND, NAND, NOR and NOT gates as finding the initial condition by switching of \textit{appropriate} dynamical systems. Please see Fig. 7 for a description of the implementation of logic gates.
Figure 7: XNOR, OR, NOR, AND and NAND logic gates implemented as finding the initial condition by switching of dynamical systems. NOT gate is just finding the initial condition on the single dynamical system shown.

The NOT gate does not involve a switching operation. The NOT operation can be seen as finding the initial condition on the binary map with the symbols for the two intervals swapped. One can easily extend this method for any logical gate. It is also possible to extend these systems to higher alphabets just like how we did for OTP. The ternary-OTP described in Section 5 would corresponds to addition modulo 3 of $M$ and $K$.

There has already been substantial research on logic gate implementation by a dynamical system approach and the building of a chaotic computer (a computer where the components are non-linear) [7, 8, 9]. An advantage our implementation has is that the input can be a stream of symbols which can be buffered and the logical output which is an initial condition (a real number who’s binary representation is the logical output) can also be stored and the output can be given as a stream instead of performing the operation for every bit. In a sense, the dynamical system can be used both to perform the logical operation and accumulate and store the logical output simultaneously. It remains to be seen
whether the new implementations proposed in this paper can be deployed in the hardware in an efficient manner (fast, precise, compact and low-power implementation).

7 Conclusions

We have established that finding the initial condition under switching of dynamical systems belonging to the class of Generalized Luröth Series acts as an unifying framework for lossless compression, randomized arithmetic coding, perfect secrecy systems and implementation of logic gates.

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