The longstanding question of how stochastic behaviour arises from deterministic Hamiltonian dynamics is of great importance, and any truly holistic theory must be capable of describing this transition. Using the Koopman von-Neumann formulation of classical mechanics, we present here the **Classical Influence Functional** (CIF) – a method to bridge the gap between deterministic and stochastic dynamics, and unify the formalisms describing these disparate behaviours. Using this new technique, we demonstrate how irreversible behaviour arises generically from the reduced microscopic dynamics of a system-environment amalgam. The classical influence functional is then used to rigorously derive a generalised Langevin equation from a microscopic Hamiltonian. In this method stochastic terms are not identified heuristically, but instead arise from an exact mapping only available in the path-integral formalism of classical mechanics. As a consequence of the CIF, we are also able to show that the proper classical limit of stochastic quantum dynamics corresponds to this Langevin equation, providing a further unifying link to quantum theories. These derivations highlight the utility of CIFs, and its potential as a tool in both fundamental and applied research.

**I. INTRODUCTION**

The predictive power of physics rests on the presumption of universal laws. These include global spatial and temporal symmetries which demand momentum and energy conservation [1], while time reversal symmetry arises as a consequence of Hamiltonian dynamics [2]. Problematically however, we do not see the conservation implied by fundamental symmetries in mundane experience. Energy leaks, structure deteriorates, and lifetimes (both correlative and biological) are finite. This is an altogether antique notion - “all human things are subject to decay/And when fate summons, monarchs must obey” [3] - but its essential truth is not time dependent. In order to model systems displaying the characteristics of dissipation and fluctuation familiar to us in everyday life, one must use a statistical description.

Stochastic descriptions of physics were originally motivated by a desire to prove the existence of atoms. Einstein’s description of Brownian motion was framed as an experimentally observable consequence of an atomistic picture [4]. This result inspired a proliferation of stochastic methods in physics, with a variety of formalisms used to describe them [5 6]. In particular, stochastic thermodynamics [7] predicts thermodynamic behaviour at both macro and microscopic scales [8] using microscopic stochastic models. This approach has been enormously successful, generalising the laws of thermodynamics [9 10] and providing rich links with information theory [11].

One feature of stochastic theories is their ability to capture the aforementioned phenomena of dissipation and fluctuation, which renders them intrinsically irreversible. This approach stands in marked contrast to the presumption of global spatial and temporal symmetries in the microscopic description of physical systems [1]. The apparent contradiction between statistical and microscopic mechanics is made explicit by the Loschmidt paradox [12], raising the question of how irreversible behaviour may arise from reversible dynamics. This is a problem of fundamental importance, and its ultimate resolution requires a rigorous mapping from a microscopic Hamiltonian to effective irreversible dynamics.

Here, we provide an exact derivation of just such a mapping. To do so, we develop a **Classical Influence Functional** (CIF) technique that may be applied to Hilbert space representations of classical mechanics. As the name suggests, the CIF is the classical analogue to a powerful path-integral technique known as the Feynman-Vernon influence functional [13]. This formalism allows one to characterise the effect of an environmental coupling to an open system without reference to the environment. It is a powerful and flexible formalism that can be used to attack the problem of open quantum systems, yielding a number of both exact [14 15] and approximate [16 21] results. Influence functionals have been deployed in the study of both real and imaginary time path integrals. In real time, influence functionals have been used to rigorously derive quantum Langevin equations [22-27], stochastic Schrödinger [28 29] and Liouville von-Neumann [30] equations, as well as quasiadiabatic path integrals [31]. Additionally, the models derived via influence functionals have also been used successfully in both imaginary and real time numerical simulations of dissipative systems [32 39].

Given the tremendous utility of the influence functional in a quantum mechanical context, the development of a classical analogue is of great importance, both as a method to explore fundamental physics and as a tool for practical calculation. We now outline the structure of the paper, which mirrors the schematic derivation of stochastic dynamics shown in Fig. [1]. We preface this with a brief outline of the “standard” derivation of stochas-
II. THE LANGEVIN EQUATION

While there are many equivalent descriptions for stochastic processes, such as the Fokker-Planck equation \[40\] or the stochastic path integral \[41\], the archetypical example is the Langevin equation \[42\], which can be mapped to other methods as shown in Fig. 2. Naturally, the question arises as to how such an equation of motion may be derived in a manner consistent with Hamiltonian dynamics. The solution lies in considering an extended system described by the Caldeira-Leggett (CL) Hamiltonian \[22\] \[13\]:

\[
H = H_0 + \frac{1}{2} \sum_n \left( m_n \dot{x}_n^2 + m_n \omega_n^2 x_n^2 \right) - q \sum_n c_n x_n + \frac{q^2}{2} \sum_n \frac{c_n^2}{m_n \omega_n^2}.
\]

(1)

This model couples an arbitrary open system with Hamiltonian $H_0$ (described by the coordinate $q$) to an environment of independent harmonic oscillators (masses $m_n$, frequencies $\omega_n$, and displacement coordinates $x_n$), with each oscillator being coupled to the open system with a strength $c_n$. The final term is a counter-term included to enforce translational invariance on the system and eliminate quasi-static effects \[44\].

Using this model, one solve the equation of motion for each oscillator, and formally express the equation of motion for the $Q$ subsystem in terms of the initial conditions of the environment oscillators \[45\]:

\[
m\ddot{q}(t) = -V'(q(t)) - \int_0^t dt' K(t-t')\dot{q}(t') + F(t),
\]

\[
K(t) = \sum_n \frac{c_n^2}{\omega_n^2} \cos(\omega_n t),
\]

\[
F(t) = \sum_n \left[ \frac{m_n c_n}{\omega_n} \dot{q}(0) \sin(\omega_n t) + c_n \left( x_n(0) - q(0) \frac{c_n}{\omega_n^2} \right) \cos(\omega_n t) \right].
\]

(3)

Contact may be made with the Langevin equation by assuming the environment initial conditions are unknown, but that as $F(t)$ contains a large number of independent terms it will be statistically simple via the central limit theorem \[45,47\]. This term is therefore replaced by a noise term $\eta(t)$ to give

\[
m\ddot{q}(t) = -V'(q(t)) - \int_0^t dt' K(t-t')\dot{q}(t') + \eta(t).
\]

(4)

While this approach motivates the form of Eq. (1), it begins from a point particle description, where stochastic terms arise from an ad hoc assertion that the environment initial conditions and statistically distributed, and the statistical character of the noise must be inserted by hand. Figure 2 sketches this broken link between microscopic models and the various formalisms describing dissipative behaviour.

The Langevin equation can be more rigorously derived by using a statistical model from the outset, describing the evolution of probability distributions rather than point particles. This in turn necessitates the use of a Hilbert space formalism for classical dynamics, known as Koopman von-Neumann dynamics.
The sole distinction between KvN and quantum mechanics is therefore the square of its wavefunction $\rho = |\psi|^2$. The Born rule. The probability distribution for a state is therefore the square of its wavefunction $\rho = |\psi|^2$. The sole distinction between KvN and quantum mechanics is

in the choice of commutation relation [59]. In the quantum case $[\hat{x}, \hat{p}] = i\hbar$. When this commutation relation is applied to the Ehrenfest theorems, it uniquely specifies the dynamics of a quantum mechanical wavefunction, yielding the familiar Schrödinger equation:

$$i\hbar \dot{\psi}_{qm} = H \psi_{qm}. \quad (5)$$

In KvN mechanics $[\hat{x}, \hat{p}] = 0$. As a result, the $\hat{x}$ and $\hat{p}$ operators have a common set of eigenstates. These form an orthonormal eigenbasis, furnished with the usual relationships:

$$\hat{x} |x, p\rangle = x |x, p\rangle, \quad \langle x, p | x', p'\rangle = \delta (x - x') \delta (p - p'),$$

$$\hat{p} |x, p\rangle = p |x, p\rangle, \quad \int dx dp \ |x, p\rangle \langle x, p| = 1. \quad (6)$$

One consequence of allowing the phase space operators to commute is that it is impossible to construct an operator that satisfies the Ehrenfest theorems purely out of $\hat{x}$ and $\hat{p}$. It is therefore necessary to introduce two new operators, $\lambda$ and $\theta$ with the commutation relations:

$$[\hat{x}, \lambda] = [\hat{p}, \theta] = i, \quad [\lambda, \theta] = [\lambda, \hat{p}] = [\theta, \hat{x}] = 0. \quad (7)$$

The new operators are Bopp operators [62, 63], and may be physically interpreted as the operational equivalent of Lagrange multipliers. Specifically, each operator acts as the Lagrange multiplier enforcing one of Hamilton’s equations, an interpretation which follows from Eq. (A24). With these new operators, one is able to derive the propagator for classical states

$$\hat{U}_{cl} (t) = e^{-it K}, \quad (8)$$

where $K$ is the Koopman operator

$$\hat{K} = \hat{p} \lambda/m - \hat{V}' (\hat{x}) \hat{\theta}. \quad (9)$$

B. Liouville’s Theorem for KvN Classical Mechanics

We now show that the Koopman operator is consistent with more standard formulations of classical dynamics. Taking the evolution equation

$$i \frac{d}{dt} |\psi\rangle = -\hat{K} |\psi\rangle \quad (10)$$

we pick a specific representation:

$$\psi (x, p) = \langle x, p | \psi\rangle, \quad \hat{x} \rightarrow x, \quad \hat{p} \rightarrow p,$$

$$\lambda \rightarrow -i \frac{\partial}{\partial x}, \quad \theta \rightarrow -i \frac{\partial}{\partial p}, \quad (11)$$

which leads to

$$i \dot{\psi} = \frac{p}{m} \frac{\partial \psi}{\partial x} - V' (x) \frac{\partial \psi}{\partial p}. \quad (12)$$

III. KOOPMAN VON-NEUMANN DYNAMICS

We now introduce the KvN formalism for classical mechanics. This is in a sense the adjoint to formulations of quantum mechanics in phase space [48-51]. While the latter theories are “classicalised” descriptions of quantum phenomena, KvN mechanics casts classical systems in a quantum language. KvN mechanics is the reformulation of classical mechanics in a Hilbert space formalism [52].

A. The Koopman Operator

As a Hilbert space theory, KvN is similar to quantum mechanics, where elements of the Hilbert space $\psi$ are interpreted as probability density amplitudes which obey the Born rule. The probability distribution for a state
This evolution equation may be expressed via the Poisson bracket
\[
\dot{\psi}(x, p) = i\mathcal{K}\psi(x, p) = \{H, \psi(x, p)\},
\]
\[
\{H, \odot\} = \frac{\partial H}{\partial x} \odot \frac{\partial}{\partial p} - \frac{\partial H}{\partial p} \odot \frac{\partial}{\partial x}.
\]
The phase space representation of the Koopman operator is the Poisson bracket. The evolution equation for the classical wavefunction is therefore identical to that for the associated probability density
\[
\dot{\rho}(q, p) = \{H, \rho(q, p)\}.
\]
This fact is particularly helpful, as it means that a classical wavefunction and its equivalent probability density are evolved by the same propagator
\[
U_{cl}(x_f, p_f; t_f; x_i, p_i, 0) = \langle x_f, p_f | e^{-it\mathcal{K}} | x_i, p_i \rangle,
\]
leading to the evolution equations:
\[
\psi(x, p_f) = \int dx_i dp_i U_{cl}(x, p_f; t_f; x_i, p_i, 0) \psi(x_i, p_i),
\]
\[
\rho(x_f, p_f) = |\psi(x_f, p_f)|^2 = \int dx_i dp_i U_{cl}(x, p_f; t_f; x_i, p_i, 0) \rho(x_i, p_i).
\]
\[\text{(17)}\]
\[\text{(18)}\]

C. KvN Path Integral

We close this section with a discussion of path integral formulations of KvN. It is possible in this formalism to construct both deterministic and stochastic classical path integrals [58, 68, 69], including generalisations with geometric forms [70]. These path integral formulations may be usefully applied with classical many-body diagrammatic methods [71], but in our case, they shall be used to derive the influence functional.

A full derivation of the KvN path integral is available in Appendix A and we quote the result here. The classical propagator (dropping its arguments for brevity) may be expressed as
\[
U_{cl} = \int Dx(t) D\theta(t) e^{i \int_0^t dt \theta(t) [m\dot{x}(t) + V'(x(t))]},
\]
with a functional measure given by
\[
Dx(t) D\theta(t) = \lim_{N \to \infty} \left(\frac{m}{2 \pi \Delta}\right)^N \prod_n dx_n d\theta_n.
\]
\[\text{(19)}\]
\[\text{(20)}\]

It is easy to see that the exponent in the classical path integral is a delta functional which enforces precisely the classical equations of motion, where the kernel of the exponent is the KvN equivalent to the action in the quantum path integral. If we consider the limit of localised probability distributions \(\rho_0(x_i, p_i) = \delta(x_i - x_0) \delta(p_i - p_0)\), the distribution at later times is described by
\[
\rho(x_f, p_f, t_f) = \int dx_i dp_i U_{cl} \rho_0(x_i, p_i)
\]
\[\text{(21)}\]

Hence, the particle remains localised with its trajectory \(x_{cl}(t)\) described by the classical equation of motion. The KvN propagator in this special case is simply a formally excessive representation of single-particle Newtonian mechanics. Clearly, applying this formalism to single-particle classical mechanics recovers well known results, but by expressing the composite of an open system and its environment in this form, we are able to construct an influence functional to integrate out the environment explicitly.

IV. INFLUENCE FUNCTIONALS

In this section we detail the construction of influence functionals, which allow one to re-express a many particle problem in terms of a modified one-body equation. For both the quantum and classical cases, the aim is to produce an effective propagator \(U_Q\) that describes the evolution of a reduced system. The role of this propagator is shown in Fig. 3 describing the evolution of an open system without reference to the environment. We begin with the quantum case, the Feynman-Vernon influence functional [13].

A. Quantum Mechanical

Consider an open system \(Q\) and an environment \(X\) respectively characterised by collective coordinates \(q\) and
x, with an interaction $H_I = H_I(q, x)$. The total Hamiltonian is described by

$$H_{\text{tot}} = H_Q + H_X + H_I.$$  \hfill (22)

Let us say we are only interested in the dynamics of the open system $Q$. The expectation of an operator $A$ acting only on the $Q$ subsystem is

$$\langle \hat{A} \rangle = \int dq dq' dx dx' \ \rho(q, x; q', x'; t) A(q, q') \delta(x-x').$$  \hfill (23)

This expression can be simplified by defining a reduced density matrix $\rho_Q(t)$ which describes subsystem $Q$

$$\rho_Q(q'; t) = \int \rho(q, x; q', x'; t) \delta(x-x') \ dx dx'.$$  \hfill (24)

Additionally, when we incorporate time evolution, the density matrix at time $t_f$ is

$$\rho(q, x; q', x'; t_f) = \int dq dq' dx dx' U(q, x; q_0, x_0; t_f) \times \rho_0(q_0, q'_0; x_0, x'_0) U^\dagger(q', x'; q'_0, x'_0; t_f).$$  \hfill (25)

Notice that for density matrices there are two propagators acting on the unprimed and primed coordinates at either side of the density matrix, which can be interpreted as forward and reversed time trajectories respectively\footnote{72}.

If we insert the quantum path integral representation for the propagators we obtain:

$$\rho(q, x; q', x'; t_f) = \int DQ DX \ e^{\frac{i}{\hbar} S_{\text{tot}}(q_0, x_0; q'_0, x'_0; t_f)},$$  \hfill (26)

where in the interests of concision we have made the abbreviations:

$$DQ = dq dq_0 \ Dq(t) \ Dq'(t)$$  \hfill (27)

$$DX = dx dx_0 \ Dx(t) \ Dx'(t)$$  \hfill (28)

$$S_{\text{tot}} = S_Q[q(t)] - S_Q[q'(t)] + S_X[x(t)] - S_X[x'(t)]$$

$$\quad + S_I[q(t), x(t)] - S_I[q'(t), x'(t)].$$  \hfill (29)

$S_{Q,X}$ are the actions derived from the isolated $Q$ and $X$ subsystem Hamiltonians, while $S_I$ is the component due to the coupling $H_I$. This last equality is somewhat misleading, given the action is a functional of both the coordinates and their time derivatives. The functional arguments should be therefore be thought of purely as labels denoting whether a particular component of the action is due to the forward or backward propagator trajectories.

Usually when calculating dynamical properties of the reduced system, it is assumed that the density matrix is initially in a product state, that is:

$$\rho_0 = \rho_Q(q_0; q'_0) \rho_X(x_0; x'_0).$$  \hfill (30)

For an initial product state, the reduced density matrix is

$$\rho_Q(q; q'; t) = \int DQ \ F[q(t), q'(t)] \rho_Q(q_0; q'_0) \times e^{\frac{i}{\hbar} (S_Q[q(t)] - S_Q[q'(t)])}.$$  \hfill (31)

Here $F[q(t), q'(t)]$ is the influence functional,

$$F[q(t), q'(t)] = \int DX \ \rho_X(x_0; x'_0) e^{\frac{i}{\hbar} S_X},$$  \hfill (32)

which explicitly integrates out the $X$ system, leaving it a pure function of the $Q$ system coordinates. If the influence functional is expressed as a complex phase

$$F[q(t), q'(t)] = e^{\frac{i}{\hbar} \Phi[q(t), q'(t)]},$$  \hfill (34)

then it is possible to describe the evolution of the $Q$ system

$$\rho_Q(q; q'; t) = \int DQ \ U_Q \rho_Q(q_0; q'_0),$$  \hfill (35)

defining an effective propagator $U_Q$:

$$U_Q = e^{\frac{i}{\hbar} (S_Q[q(t)] - S_Q[q'(t)] + \Phi[q(t), q'(t)])}.$$  \hfill (36)

If one is able to disentangle $U_Q$ into a product of the form

$$U_Q[q(t), q'(t')] = \tilde{U}_Q[q(t)] \tilde{U}_Q^\dagger[q'(t')],$$  \hfill (37)

then effective Hamiltonians for forward and backward evolutions can also be defined, capturing exactly the dynamics of the $Q$ system, but without any reference to the $X$ system it is interacting with. This is the power of the influence functional, as it allows for the mapping of an interacting subsystem to an isolated system with a modified Hamiltonian. In the context of open systems, the dimensionality of the environment is incomparably large as compared to the system of interest. Being able to use the influence functional to characterise the environmental effect on the open system is highly desirable (even if only for numerical efficiency), however explicit expressions for even the simplest environmental models require extensive derivations.

\subsection{Classical Influence Functionals}

Using the path integral KvN formulation, it is possible to directly import many of the results derived for the quantum path integral. Principal among these is the ability to describe the reduced dynamics of an open system + environment amalgam with an equivalent influence functional formalism. For a global system described
with canonical coordinates \( q, p \) and \( x, k \), the total Hamiltonian may characterised as in Eq. \([22]\) using
\[
H_Q = \frac{p^2}{2m} + V_Q, \quad H_X = \frac{k^2}{2m_k} + V_X, \quad H_I = V_{QX}.
\]
(38)

This system is initially described by the probability density
\[
\rho_{0}^{\text{tot}} = \rho_Q(q_0, p_0)\rho_X(x_0, k_0, q_0, p_0).
\]
(39)

where to retain full generality, the initial environment state may also depend on the open system coordinates.

Using Eq. \([19]\), the classical reduced probability density may be expressed in a similar manner to Eq. \([31]\):
\[
\rho_Q(q_f, p_f) = \int \mathcal{D}q(t)\mathcal{D}\theta_Q(t) dq_0 dp_0 \mathcal{F}_{cl}(q_0, p_0) \times e^{i\int_{t_0}^{t_f} dt \theta_Q(t)(m\dddot{q}(t) + aV_Q)}(40)
\]

where \( \mathcal{F}_{cl} \equiv \mathcal{F}_{cl}[q(t), p(t), \theta_Q(t)] \) is the classical influence functional given by
\[
\mathcal{F}_{cl} = \int \mathcal{D}F \rho_X(x_0, k_0, q_0, p_0) e^{i\int_{t_0}^{t_f} dt \Gamma}, \quad \Gamma = \theta_X(t) \left( m\ddot{q}(t) + \frac{\partial V_{QX}}{\partial q} \right) + \theta_Q(t) \frac{\partial V_{QX}}{\partial q}, \quad \mathcal{D}F = dx_0 dk_0 dq_1 dp_1 \mathcal{D}x(t) \mathcal{D}\theta_X(t).
\]
(41)

It is worth taking a moment to consider the physical implications of the CIF. First, the tracing out of the environment corresponds physically to ignorance of the environment after the initial time \( t_0 \). We denote the effective propagator evolving the reduced system from \( t_0 \) to \( t_1 \) as \( \hat{U}_Q[t_0, t_1, \rho_X(t_0)] \), where the final argument indicates the dependence on the environment state at time \( t_0 \), as is clear from Eq. \([41]\). Given that in general \( \rho_X(t_0) \neq \rho_X(t_1) \), the time reversal symmetry between forwards and backwards propagations from time \( t_0 \) is broken, as
\[
\hat{U}_Q^{-1}[t_0, t_1, \rho_X(t_0)] \neq \hat{U}_Q^\dagger[t_0, t_1, \rho_X(t_0)].
\]
(44)

and the inverse of the effective propagator instead depends on the environment state at the later time
\[
\hat{U}_Q^{-1}[t_0, t_1, \rho_X(t_0)] = \hat{U}_Q^\dagger[t_0, t_1, \rho_X(t_1)].
\]
(45)

Thus, the evolution of the \( Q \) system is itself now dependent on initial conditions. The reverse evolution for the effective propagator from \( t_1 \) to \( t_0 \) now depends on the (unknown) environment state at that time. This phenomenon is illustrated in Fig. 4 where the reversing the reduced system dynamics require the state of the environment at a later time. This marginalisation of the environment at time \( t_0 \) represents the moment in which information about the environment is lost. Clearly, if knowledge of an unobserved part of the system (the environment) is lost at an earlier time, it is impossible to reconstruct the correct reversed effective propagator at a later time a priori, and the observed dynamics will appear to break time reversiblity. This is in effect an arrow of time, forced upon the observed system as a consequence of ignorance of the later environment state. Reversibility is only restored under the the two trivial cases that the system and environment are uncoupled, or the total system begins in thermal equilibrium and the total Hamiltonian is time independent, such that \( \rho_X(t_0) = \rho_X(t_1) \).

Finally, under certain circumstances an effective equation of motion may be defined from the influence functional. This is the case when it is possible to express \( \mathcal{F}_{cl} \) as
\[
\mathcal{F}_{cl} = e^{i\int_{t_0}^{t_f} dt \theta_Q(t)\chi[q(t), p(t)]}, \quad \chi[q(t), p(t)] \text{ is an arbitrary functional of the phase space coordinates only. In the case, the effective equation of motion reads}
\]
\[
m\ddot{q} = -\frac{\partial V_Q}{\partial q} - \chi[q(t), p(t)].
\]
(47)

\[ V. \quad \text{GENERALISED LANGEVIN EQUATION} \]

Using the CIF, it is possible to derive a generalise Langevin equation without resorting to the insertion of stochastic terms by hand. We shall take a simplified version of Eq. \([1]\) as the model Hamiltonian:
\[
H_{\text{tot}} = H_Q(q) + \frac{1}{2} \sum_n \left( k_n^2 + \omega_n^2 x_n^2 \right) - q \sum_n c_n x_n.
\]
(48)

We also implement the initial condition
\[
\rho_0^{\text{tot}} = \rho_Q(q_0, p_0)\rho_X(x_0, k_0), \quad \rho_X(x_0, k_0) = \prod_n \frac{\beta \omega_n}{2\pi} \exp\left( -\frac{\beta}{2} \left( k_n^2 + \omega_n^2 x_n^2 \right) \right).
\]
(49)

Figure 4. A forward propagation \( U_f = U_Q[t_1, t_0, \rho_X(t_0)] \) is no longer a unitary transformation. The proper reversal of this propagation is \( U_b = U_I^{-1} = U_Q[t_0, t_1, \rho_X(t_1)] \), while the naive assumption that \( \rho_X(t_0) = \rho_X(t_1) \) corresponds to \( U_f^\dagger = U_Q[t_0, t_1, \rho_X(t_0)] \).
This initial environment state is the Gibbs distribution for the bath of harmonic oscillators. It is actually possible to take the initial condition \( \rho_0^{\text{tot}} = e^{-\beta H_{\text{tot}}} \) and include the interaction \(-q \sum_n c_n x_n \) in \( \rho_0 \). In this case we would complete the square in the \( \rho_0 \) exponent, redefining \( x_{0n} \rightarrow x_{0n} - \frac{2\omega_n c_n}{\omega_n} \). This would result in an extra constant term \( \beta \sum_n \frac{c_n^2}{\omega_n^2} \) which could itself be cancelled by the inclusion of the counter-term often found in the CL model. We have neglected this term, and any other term solely dependent on \( q \), as the only effect due to these are modifications of the \( Q \) system potential and distribution, which are arbitrary to begin with. Critically, including this interaction, even when it is arbitrarily strong, does not introduce extra noise sources to the final result, or affect the correlations of these noises. With this setup, we are able to insert the CL terms into Eq. (41). Suppressing the functional arguments of the influence functional, we have

\[
\mathcal{F}_{\text{cl}} = \prod_n \int dx_{0n} dk_{0n} dx_{n} \, d\theta_{n} \, d\phi_{n} \, e^{i\int_0^t dt \, \theta_{n}(t)c_{n}x_{n}(t)}
\]

where we have replaced the integrations over \( \theta_{X} \) with their equivalent delta functionals. This delta functional will force the trajectory to obey \( x_n(t) = x_{n0}^2(t) \), which solves the equation of motion \( x_{n}^2(t) = -\omega_n^2 x_n^2(t) + c_n q(t) \). Appendix [3] details one method of obtaining the solution below

\[
x_n^2(t) = \frac{k_{0n}}{\omega_n} \sin(\omega_n t) + x_0n \cos(\omega_n t)
\]

Inserting this into the influence functional

\[
\mathcal{F}_{\text{cl}} = \prod_n \int dx_{0n} dk_{0n} \rho_{\beta}(x_{0n}, k_{0n})
\]

\[
\times \exp \left[ i \int_0^t dt \, \theta_{n} (t) \left( c_n \frac{k_{0n}}{\omega_n} \sin(\omega_n t) + x_0n \cos(\omega_n t) \right) \right]
\]

\[
\times \exp \left[ -i \int_0^t dt \, c_n \frac{k_{0n}}{\omega_n} \sin(\omega_n t) + x_0n \cos(\omega_n t) \right]
\]

and using Eq. (50) to substitute for \( \rho_\beta \), we find that the integrals over initial positions and momenta are of a Gaussian form. Integrations over the initial phase space coordinates yields

\[
\int dx_{0n} \, e^{-\frac{1}{2} \omega_n^2 (x_{0n} + 2Ax_{0n})} = \sqrt{\frac{2\pi}{\beta}} \omega_n e^{-\frac{1}{2} \omega_n^2 A^2},
\]

\[
\int dk_{0n} \, e^{-\frac{1}{2} \left( \frac{k_{0n}}{\omega_n} + 2Bk_{0n} \right)^2} = \sqrt{\frac{2\pi}{\beta}} e^{-\frac{1}{2} B^2},
\]

using

\[
A = \frac{ic_n}{\beta \omega_n^2} \int_0^t dt \, \theta_{Q} (t) \cos(\omega_n t),
\]

\[
B = \frac{ic_n}{\beta \omega_n} \int_0^t dt \, \theta_{Q} (t) \sin(\omega_n t).
\]

Combining these we obtain

\[
\exp \left( -\frac{\beta}{2} (\omega_n^2 A^2 + B^2) \right) = \exp \left( -\frac{c_n^2 k_B T}{2\omega_n} \int_0^t dt \, \theta_{Q} (t) \gamma_n (t-t') \theta_{Q} (t') \right)
\]

using \( \gamma_n (t-t') = \frac{1}{\omega_n} \cos(t-t') \). Collecting these results, we are able to express the influence functional

\[
\Phi_{\text{cl}} = -2i \int_0^t dt \, \theta_{Q} (t) \int_0^t dt' q(t') \sin(\omega_n (t-t'))
\]

\[
+ k_B T \int_0^t dt \, \theta_{Q} (t) \gamma_n (t-t') \theta_{Q} (t').
\]

At this point we take the continuum limit for the oscillators,

\[
\sum_n \frac{c_n^2}{2\omega_n} \rightarrow \int_0^\infty d\omega \, \frac{1}{2\pi} I(\omega)
\]

such that our final influence functional is given by

\[
\mathcal{F}_{\text{cl}} = \exp \left[ 2i \int_0^t dt \, \theta_{Q} (t) \int_0^t dt' q(t') \frac{d\gamma (t-t')}{dt'} \right]
\]

\[
- \int_0^t dt \, \theta_{Q} (t) \frac{d\gamma (t-t')}{dt'} \theta_{Q} (t') \]

\[
\gamma (t-t') = \int_0^\infty \frac{d\omega}{\omega\pi} I(\omega) \cos(t-t').
\]

Having evaluated the influence functional, we would like to work backwards from the path integral to obtain an effective equation of motion. Unfortunately the integral over the forward and backwards \( \theta_{Q} \) trajectories prevents the functional being brought to the form required by Eq. (40).

To remedy this, we employ the Hubbard-Stratonovich (HS) transformation. This equates a deterministic non-local integral exponent to one involving local stochastic terms that must be averaged over a distribution \( W \). In a more physical sense, we can consider the HS transformation as converting a system of two body potentials into
interaction in our original thermal density, we would have

Putting all of this together, we find that the KvN propagator and performing the path integral over the trajectories these propagators generate, one is able to obtain the true reduced probability distribution evolved by the effective propagator.

had an extra term cancelling \( q(0) \gamma(t) \) here, while including the counterterm in the open system Hamiltonian would cancel \( q(t) \gamma(0) \). Substituting this back into the propagator and performing the path integral over \( \theta(t) \) we obtain:

\[
\tilde{U}_{cl} = \int_{q_0}^{q_f} Dq(t) \delta \left[ m\dot{q}(t) + V'(q,t) \right. \\
+ 2\int_0^t dt' \dot{q}(t')\gamma(t-t') - \eta_{cl}(t) \right] 
\]  

This brings us to the ultimate result of this section, namely that the equation of motion for a single trajectory is a generalised Langevin equation:

\[
m\ddot{q}(t) = -V'(q,t) - 2\int_0^t dt' \dot{q}(t')\gamma(t-t') + \eta_{cl}(t) .
\]  

In the particular case where \( I(\omega) = D\omega \), we recover a Markovian Langevin equation, with \( \langle \eta_{cl}(t) \eta_{cl}(t') \rangle_r = 2k_BT\delta(t-t') \), \( \gamma(t) = D\delta(t) \).

In order to obtain the reduced probability density, we must average over the stochastic propagations of the initial density:

\[
\rho_Q(q_f,p_f) = \left\langle \tilde{U}_{cl}\rho_Q(q_0,p_0) \right\rangle_r .
\]  

This effectively corresponds to constructing the distribution from the average number of trajectories that end at each point in the phase space. Figure 6 illustrates some sample trajectories, together with the probability distribution their average describes.

**VI. THE CIF AS A CLASSICAL LIMIT**

As a final application of the CIF, we demonstrate its utility in establishing the classical limit of a dynamical equation derived using the quantum influence functional for the CL model. Specifically, we shall consider...
the Stochastic Liouville von-Neumann Equation (SLE) \cite{30, 39, 73}:

\[ i\hbar \frac{d\hat{\rho}(t)}{dt} = \left[ \hat{H}_Q(t), \hat{\rho}(t) \right] - \eta(t) \left[ \hat{q}(t), \hat{\rho}(t) \right] + \frac{1}{2} \nu(t) \left\{ \hat{q}(t), \hat{\rho}(t) \right\}. \] (72)

The SLE evolves a single-trajectory density matrix \( \hat{\rho}(t) \) which upon stochastic averaging gives the physical reduced density matrix. The two noises obey the following statistical relationships

\[ \langle \eta(t) \eta(t') \rangle_r = \hbar \int_0^\infty \frac{d\omega}{\pi} I(\omega) \coth \left( \frac{1}{2} \omega \hbar \beta \right) \cos(\omega (t - t')), \] (73)

\[ \langle \eta(t) \nu(t') \rangle_r = -2i\hbar \Theta(t - t') \frac{d\gamma(t - t')}{dt}. \] (74)

The SLE corresponds to the propagation of the density matrix \( \hat{\rho}(t) \) in the following manner

\[ \hat{\rho}(t) = \hat{U}^+(t) \hat{\rho}(0) \hat{U}^-(t), \] (75)

using the propagator

\[ \hat{U}^\pm(t_f) = \hat{T}^\pm \exp \left( \pm i \frac{\hbar}{\hbar} \int_0^{t_f} \hat{H}_Q(t) - \left[ \eta(t) \pm \frac{1}{2} \nu(t) \right] \hat{q}(t) \right) \] (76)

where \( \hat{T}^- \) is the (anti) time ordering operator.

Taking the classical limit of this (and indeed any) system is by no means straightforward (see, e.g., \cite{74}), particularly given the stochastic term correlations also involve the factor of \( \hbar \) which formally must be limited to zero. As we shall see in the next section, difficulties arise from this which require the CIF to resolve.

### A. Heuristic Limit

To understand some of the difficulties associated with taking the classical limit of the SLE, we first make a heuristic calculation, noting that in the classical limit, the only path from propagator that contributes is that with the action \( \hat{S}^\pm_\text{cl} \)

\[ \hat{S}^\pm_\text{cl} = \int_0^{t_f} dt \left[ \int_0^{t_f} dt \left[ L_Q(q(t)) + \left[ \eta_\text{cl}(t) \pm \frac{1}{2} \nu_\text{cl}(t) \right] q(t) \right) \right], \] (77)

where \( L_Q(q(t)) \) indicates the system Lagrangian. The correlation functions for the noises will also be affected in the classical limit, hence we define new noises that obey these limiting correlation functions:

\[ \lim_{\hbar \to 0} \langle \eta(t) \eta(t') \rangle_r = \langle \eta_\text{cl}(t) \eta_\text{cl}(t') \rangle_r = 2k_B T \gamma(t - t'), \] (78)

\[ \lim_{\hbar \to 0} \langle \eta(t) \nu(t') \rangle_r = 0. \] (79)

The statistics of the \( \eta \) noise becomes identical to that derived previously, and since the \( \nu \) noise is now entirely uncorrelated, it will have no effect on the average dynamics and can be dropped from the action. This restores symmetry to the forwards and backwards propagations

\[ \hat{S}_\text{cl} = \int_0^{t_f} dt \left[ L_Q(q(t)) + \eta_\text{cl}(t) q(t) \right]. \] (80)

The classical equation of motion we obtain for a single trajectory is therefore a type of Langevin equation:

\[ \ddot{q} = -\frac{\partial V(q)}{\partial q} - \eta_\text{cl}(t). \] (81)

It is not a surprise that the classical limit of the SLE corresponds to a Langevin equation, but Eq. \cite{81} appears to lack some essential features. Most concerning, the \( \nu \) noise appears to have no effect on the dynamics. This is a consequence of incorporating the dynamic response of the bath into the HS transformation, and this information appears to be lost in the classical limit.

In order to understand what has happened, we return to Eq. \cite{62}, applying the HS transform to both the \( \theta_Q \) and \( q \) variables (see Eq. \cite{C13} for detail). This modified influence functional is then:

\[ \mathcal{F}_\text{cl} = \left\langle \exp \left( i \int_0^{t_f} dt \left[ -\theta_Q(t) \eta_\text{cl}(q(t) + q(t) \nu_\text{cl}(t)) \right) \right) \right\rangle, \] (82)

with the \( \nu_\text{cl} \) noise defined by its correlations

\[ \langle \eta_\text{cl}(t) \nu_\text{cl}(t') \rangle_r = -2i\hbar \Theta(t - t') \frac{d\gamma(t - t')}{dt}, \] (83)

\[ \langle \nu_\text{cl}(t) \nu_\text{cl}(t') \rangle_r = 0. \] (84)

The classical propagator for a single realisation is now expressible as:

\[ \hat{U}_\text{cl} = \int_{\theta_0, q_0}^{\theta_{t_f}, q_{t_f}} D\theta(t) D\hat{q}(t) \delta[m\hat{q}(t) + V'(q,t) - \eta_\text{cl}(t)], \] (85)

\[ D\hat{q}(t) = D(q(t)) e^{i\int_0^{t_f} dt q(t)\nu_\text{cl}(t)}. \] (86)

Just like in the heuristic classical limit of the SLE, the equation of motion for an individual trajectory is a frictionless Langevin equation. The friction component has not vanished, but its influence on the expectations is to introduce a stochastic weighting on each trajectory. Clearly, the equations of motion for individual trajectories are affected by the presence or absence of a friction kernel, but the expectations of the two systems must be identical, provided the appropriate stochastic weighting is used in the averaging of the frictionless propagator. The heuristic classical limit therefore reproduces the dynamics of a frictionless Langevin system, but obscures the resultant non-trivial weighting on trajectories for expectations required to obtain the correct averaging.

This interpretation is not entirely satisfying, as it implies a critical loss of information when taking the classical limit of the SLE that must be restored with a post hoc
prescription for the weighting of trajectories. Clearly, it would be more desirable to formulate the SLE in such a way that its classical equation of motion corresponds to Eq. (70) rather than Eq. (81). We now detail precisely how to achieve this reformulation.

B. Alternative SLE classical limit

In order to derive a classical limit consistent with a frictional Langevin equation, we must alter the form of the influence phase [see Eq. (34)] used to derive the SLE before employing the HS transform. For the CL model, the influence phase reads [73,74]:

\[ \Phi[q, q'] = \frac{1}{2} \int_0^{t_f} dt \int_0^{t_f} dt' K^R(t - t') \epsilon(t) \epsilon(t') + 2i \int_0^{t_f} dt \int_0^{t_f} dt' K^I(t - t') \epsilon(t) y(t') \]  

(87)

using the kernels

\[ K^R(t - t') = \hbar \int_0^\infty \frac{d\omega}{\pi} I(\omega) \coth \left( \frac{1}{2} \omega \hbar \beta \right) \cos(\omega(t - t')) , \]

(88)

\[ K^I(t) = \frac{d\gamma(t - t')}{dt} , \]

(89)

and introducing sum-difference coordinates:

\[ \epsilon(t) = q(t) - q'(t) , \quad y(t) = \frac{1}{2} (q(t) + q'(t)) . \]

(90)

Rather than utilising the HS transformation for both \( \epsilon \) and \( y \), we perform it only over \( \epsilon \):

\[ \Phi[q, q'] = \frac{1}{2} \int_0^{t_f} dt \int_0^{t_f} dt' \eta(t) \epsilon(t) + 2i \int_0^{t_f} dt \int_0^{t_f} dt' K^I(t - t') \epsilon(t) y(t') \]

(91)

where \( \eta \) has the autocorrelation \( \langle \eta(t) \eta(t') \rangle_r = K^R(t - t') \). For the \( K^I \) term, we integrate by parts with respect to \( t' \) to obtain

\[ \int_0^{t_f} dt' K^I(t - t') y(t') = [\gamma(t - t') y(t')]_0 - \int_0^t dt' \gamma(t - t') \dot{y}(t') . \]

(92)

The term \( 2i \int_0^{t_f} dt \epsilon(t) [\gamma(t - t') y(t')]_0 \) when expressed in the original \( q, q' \) coordinates is decoupled between the \( q \) and \( q' \) coordinates, and just as in the classical case may be absorbed into the open system potentials for the forward and backward propagators separately. As a result, the reduced density matrix for the system is evolved in the following manner:

\[ \hat{\rho}_t(q; q') = \int dq'' dq'' \hat{U}_{\text{eff}}(q, q', t; \bar{q}, \bar{q}, 0) \hat{\rho}_0(\bar{q}; \bar{q}') , \]

(93)

with an effective propagator, \( \hat{U}_{\text{eff}} \):

\[ \hat{U}_{\text{eff}}(q, q', t; \bar{q}, \bar{q}, 0) = \int Dq(t) Dq'(t) \exp \left[ \frac{i}{\hbar} S_{\text{eff}} \right] , \]

(94)

defined by the effective action

\[ S_{\text{eff}} = \int_0^{t_f} dt \left( L_Q(q(t)) - L_Q(q'(t)) + \eta(t) \epsilon(t) - 2\epsilon(t) \int_0^t dt' \gamma(t - t') \dot{y}(t') \right) . \]

(95)

In this formulation, the propagator is no longer decoupled between the forward and backward trajectories, preventing the straightforward identification of a classical limit as in Eq. (76). To address this, we express \( L_Q(q(t)) - L_Q(q'(t)) \) in the sum-difference coordinates

\[ S_{\text{eff}} = \int_0^{t_f} dt \left[ m \ddot{y}(t) + \eta(t) \epsilon(t) - V(y(t) + \frac{\epsilon(t)}{2}) + V(y(t) - \frac{\epsilon(t)}{2}) - 2\epsilon(t) \int_0^t dt' \gamma(t - t') \dot{y}(t') \right] . \]

(96)

To obtain the classical result, we note that the average size of the fluctuating coordinate \( \epsilon(t) \) will be proportional to \( \hbar \) [15]. The crucial step in obtaining the classical limit is approximating \( \hbar \) as small before taking the limit:

\[ V(y(t) + \frac{\epsilon(t)}{2}) - V(y(t) - \frac{\epsilon(t)}{2}) \approx V'(y(t)) \epsilon(t) , \]

(97)

and \( \eta \approx \eta_{cl} \). This becomes exact in the \( \hbar \rightarrow 0 \) limit. Note that this approach implicitly adopts the definition of the classical limit as that in which observable operators commute [74]. Integration of the first term in the effective action by parts yields:

\[ S_{\text{eff}} = \int_0^{t_f} dt \epsilon(t) \left[ - m \ddot{y}(t) - V'(y(t)) \right] + \eta_{cl}(t) - 2 \int_0^t dt' \gamma(t - t') \dot{y}(t') . \]

(98)

To perform the \( \hbar \rightarrow 0 \) limit, we must examine the path integral measure [70] in its discrete form:

\[ \mathcal{D}y(t) \mathcal{D}\epsilon(t) = \lim_{N \rightarrow \infty} \left( \frac{m}{2\pi \hbar \Delta} \right)^N \prod_n^N dy_n d\epsilon_n . \]

(99)

Making the substitution \( \theta(t) = \epsilon(t)/\hbar \), the measure now reads

\[ \mathcal{D}y(t) \mathcal{D}\theta(t) = \lim_{N \rightarrow \infty} \left( \frac{m}{2\pi \Delta} \right)^N \prod_n^N dy_n d\theta_n . \]

(100)
Comparison to Eq. \([20]\) reveals this is the KvN measure. Furthermore, the effective propagator is now

\[
\tilde{U}_{\text{eff}} = \int_{y_0, \dot{y}_0}^{y_f, \dot{y}_f} \mathcal{D}y(t) \mathcal{D}\theta(t) \ e^{i \int_0^t dt \ [\theta(t)R(t) - \dot{y}(t)]},
\]

\[ \gtrsim \int_{y_0, \dot{y}_0}^{y_f, \dot{y}_f} \mathcal{D}y(t) \ h[R(t)], \quad (101) \]

\[ R(t) = m \ddot{y}(t) + V'(y, t) - \eta \dot{y}(t) + 2 \int_0^t dt' \dot{y}(t') \gamma(t - t'). \quad (102) \]

There is now no \(\hbar\) dependence in this path integral\([77]\), and we have recovered the KvN propagator found in Eq. \([69]\). This demonstrates that when the friction kernel is explicitly included in the quantum mechanical path integral, the classical limit corresponds exactly to the KvN path integral, providing a valuable consistency check for both of these results. Furthermore, this result emphasises that in order to take a consistent classical of results derived with the quantum influence functional, the theory of the CIF is required to make sense of the classical path integral this limit produces.

**VII. CONCLUSIONS**

We have used the Koopman von-Neumann representation of classical dynamics to derive an analogue to the Feynman-Vernon influence functional. This has the great advantage of allowing one to make direct contact between a microscopic model, and the equivalent stochastic description. We have demonstrated that the irreversible behaviour of an open system is a natural consequence of its description using the CIF. Using this technique, we have also rederived the generalised Langevin equation in a formally exact manner.

This derivation may potentially be generalised in a number of ways. For example, a recent development is the incorporation of a driven environment within the CL model \([78]\). Specifically, it is possible to take a Rubin model (consisting of two chains of oscillators coupled to a central system) \([79]\) with a universal driving term and map this to the CL model. Using CIFs, novel stochastic representations of such a system could be derived.

The CIF also allows one to make contact with the classical limit of quantum dynamics derived via the quantum influence functional. In the case of the SLE, the correct classical limit is found to be of a form only obtainable using the CIF. The CIF has a more easily evaluable form than its quantum equivalent. For this reason it may be possible to find analytic expressions for a larger class of environment models than in the quantum case. Recent progress in evaluating path integrals of singular potentials \([14]\) may enable the assessment of CIFs for environments with \(r^{-1}\) or \(r^{-2}\) potentials. Another potential avenue of extension is in the study of quantum-classical hybrids \([54, 56]\), where a quantum system interacting with a classical environment could be modeled with the use of the CIFs.

In the quantum case, imaginary time influence functionals have been used to describe a reduced system equilibrium state, even when the environment coupling is arbitrarily strong \([80]\). This is important, as the stationary distribution of dissipative systems with finite couplings has been shown to deviate from that expected under partitioned conditions \([81]\), with the Gibbs distribution now being described by a “Hamiltonian of mean force” \(\hat{H}_{MF} \quad [82]\). A similar result could be achieved in the classical case by using the CIF to derive an effective Hamiltonian for the reduced system. This effective Hamiltonian would then be identical to the Hamiltonian of mean force required to describe the thermal state of the reduced system.

Finally, while the CIF is a specific tool, the underlying formalism it is built on a statistical interpretation of all physics. Randomness is not an \(ad \ hoc\) model addition, but an essential, irreducible component in our description of reality. Its existence always reflects imperfect information, whether that is due to unobserved interactions with other systems, or a fundamentally non-commutative algebraic structure. This idea was famously articulated in Max Born’s Nobel speech: “Ordinary mechanics must also be statistically formulated: the determinism of classical physics turns out to be an illusion, it is an idol, not an ideal in scientific research” \([83]\).

This notion is why standard derivations of the Langevin equations are methodologically flawed. By starting from the equations of motion for a localised distribution (a point particle), the logical hierarchy of theories is reversed, as one attempts to construct a statistical theory from the deterministic limit of classical dynamics. This transition from the specific to the general is what necessitates the \(ad \ hoc\) identification of stochastic terms. As we have demonstrated, the proper starting point is the explicitly statistical, Hilbert space formulation of classical physics, from which the Langevin equation may be unambiguously derived.

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**Appendix A: Path integrals in KvN**

Deriving the KvN path integral follows the same procedure as its quantum equivalent. Before embarking on this, it is worth considering how a change of basis is achieved in KvN.
1. Basis overlaps

In quantum mechanics, the position and momentum bases form a complementary pair, and it is often useful to transform between them. To do so, one must derive the overlap between them. This is particularly helpful when specifying a representation of an operator in its conjugate basis. In the classical case there are four “canonical” sets of simultaneous eigenbases, these are

\[ |x, p\rangle, \quad |x, \theta\rangle, \quad |\lambda, p\rangle, \quad |\lambda, \theta\rangle. \]

Here we outline the procedure for deriving the overlap between two bases of non-commuting operators. Take two Hermitian operators \( \hat{x} \) and \( \hat{y} \) with the commutation relation:

\[ [\hat{x}, \hat{y}] = 1, \quad (A1) \]

it follows that

\[ [\hat{x}, e^{a\hat{y}}] = ae^{a\hat{y}}. \quad (A2) \]

Applying this commutator to an eigenstate of \( \hat{x} \) we obtain

\[ \hat{x}e^{a\hat{y}} |x\rangle = (x + a) e^{a\hat{y}} |x\rangle, \quad (A3) \]

indicating \( e^{a\hat{y}} |x\rangle \) is an eigenstate of the \( \hat{x} \) operator with eigenvalue \( x + a \). From this we can conclude that \( e^{a\hat{y}} \) is a translation

\[ e^{a\hat{y}} |x\rangle = |x + a\rangle. \quad (A4) \]

Furthermore, it is possible give an explicit form for \( \hat{y} \) in this basis:

\[ \langle x | \hat{y} | \psi \rangle = \lim_{a \to 0} \frac{1}{a} \langle x | e^{a\hat{y}} - 1 | \psi \rangle = \lim_{a \to 0} \frac{1}{a} (\psi(x + a) - \psi(x)) = \frac{\partial}{\partial x} \psi(x), \quad (A5) \]

i.e., \( \hat{y} \) is given by \( \frac{\partial}{\partial x} \) in the \( x \) representation.

In KvN mechanics, the commutator between operators is always \( i \). Making the assignment \( i\hat{y} = \hat{\lambda} \), we can calculate the overlap between \( \hat{x} \) and \( \hat{\lambda} \):

\[ \lambda \langle \lambda | x \rangle = \langle \lambda | i\hat{y} | x \rangle = i \frac{\partial}{\partial x} \langle \lambda | x \rangle \quad (A6) \]

\[ \implies \langle \lambda | x \rangle = N(\lambda) e^{-i\lambda x}. \quad (A7) \]

The normalisation of the overlap is easily checked using:

\[ \delta(x - x') = \int d\lambda \langle x' | \lambda \rangle \langle \lambda | x \rangle = 2\pi |N(\lambda)|^2 \delta(x - x') \quad (A8) \]

\[ \implies N(\lambda) = \frac{1}{\sqrt{2\pi}} \quad (A9) \]

This generically specifies the form of the overlap between eigenstates. Any eigenbasis of an operator is also an eigenbasis of operators it commutes with. Equipped with this, one may straightforwardly generate the following overlaps for the simultaneous eigenstates

\[ \langle x, \theta | x', p \rangle = \frac{1}{\sqrt{2\pi}} \delta(x - x') e^{-i\theta p}, \quad (A10) \]

\[ \langle \lambda, p | x, p' \rangle = \frac{1}{\sqrt{2\pi}} \delta(p - p') e^{-i\lambda x}, \quad (A11) \]

\[ \langle \lambda, p | x, \theta \rangle = \frac{1}{2\pi} e^{-i\lambda x} e^{i\theta p}. \quad (A12) \]

The mathematics of specifying overlaps is generic between quantum and KvN mechanics, with the only generalisation arising from KvN’s simultaneous eigenbases allowing a greater degree of freedom in representation. Equipped with this information, it is possible to represent the KvN propagator as a path integral.

2. The Propagator As A Path Integral

Take the KvN propagator,

\[ U_{cl} = e^{-it\hat{K}}, \quad (A13) \]

where \( \hat{K} \) is given by Eq. \([3] \). In the phase space representation this propagator is

\[ U_{cl}(x_f, p_f, t_f; x_i, p_i) = \left\langle x_f, p_f \right| e^{-it\hat{K}} \left| x_i, p_i \right\rangle. \quad (A14) \]

Performing a Trotter splitting, this propagator may be decomposed into a product of infinitesimal propagations

\[ U_{cl}(x_f, p_f, t_f; x_i, p_i, 0) = \lim_{N \to \infty} \int dx_1 dp_1 \cdots dx_{N-1} dp_{N-1} \prod_{j=0}^{N-1} (x_{j+1} + p_{j+1}) (x_j, p_j). \quad (A15) \]
Combining these together with Eq. (A12) leads to the following expression for a single infinitesimal propagation:

\[ \langle x_{j+1}, p_{j+1} | e^{-i \Delta K} | x_j, p_j \rangle = \langle x_{j+1}, p_{j+1} | \exp \left(-i \frac{\Delta}{m} \hat{p} \right) \exp \left(-i \Delta \hat{V}'(\hat{x}) \right) | x_j, p_j \rangle, \]  

(A16)

which can be evaluated by inserting resolutions of unity

\[ \exp \left(-i \Delta \hat{V}'(\hat{x}) \right) | x_j, p_j \rangle = \int dx \ exp (x, \theta) \langle x, \theta | x_j, p_j \rangle \exp \left(-i \Delta \theta V' (x) \right), \]  

(A17)

\[ \langle x_{j+1}, p_{j+1} | \exp \left(-i \frac{\Delta}{m} \hat{p} \right) \) \]  

(A18)

\[ \langle x_{j+1}, p_{j+1} | \exp \left(-i \Delta \hat{V}'(\hat{x}) \right) | x_j, p_j \rangle = \frac{1}{\sqrt{2\pi}} \int d\theta \ \langle x_j, \theta | x_{j+1}, p_{j+1} \rangle \exp \left(-i \Delta \theta V' (x_j) - i\theta \right), \]  

(A19)

\[ \langle x_{j+1}, p_{j+1} | \exp \left(-i \frac{\Delta}{m} \hat{p} \right) \) \]  

(A20)

Using the overlaps specified by Eqs. (A10) and (A11), we obtain for Eqs. (A17) and (A18):

\[ \langle x_{j+1}, p_{j+1} | e^{-i \Delta K} | x_j, p_j \rangle = \frac{1}{(2\pi)^2} \int d\lambda d\theta \left[ \exp \left(i \lambda \left( \frac{\Delta}{m} \left( m \frac{x_{j+1} - x_j}{\Delta} - p_j \right) \right) \right) \exp \left(i \Delta \theta \left( \frac{p_{j+1} - p_j}{\Delta} + V'(x_j) \right) \right) \right]. \]  

(A21)

Note we have added a \( j \) subscript to the \( \theta \) and \( \lambda \) variables in anticipation of inserting the appropriate resolutions of the identity. The overall propagator is therefore described by

\[ U_{cl}(x_f, p_f, t_f; x_i, p_i, 0) = \lim_{N \to \infty} \prod_{j=1}^{N-1} \left( \frac{dx_j}{\sqrt{2\pi}} \frac{dp_j}{\sqrt{2\pi}} \frac{d\lambda_j}{\sqrt{2\pi}} \frac{d\theta_j}{\sqrt{2\pi}} \right) \exp \left[i \Delta \sum_{j=0}^{N-1} \left[ \lambda_j \left( x_{j+1} - x_j \right) - p_j \right] + \theta_j \left( \frac{p_{j+1} - p_j}{\Delta} + V'(x_j) \right) \right]. \]  

(A22)

In the limit we can once again describe this with a functional notation (although we have cheated and moved directly to describing a time-dependent potential, which can be justified in the same way as in the quantum case)

\[ U_{cl}(x_f, p_f, t_f; x_i, p_i, 0) = \int_{x_i, p_i}^{x_f, p_f} DxDpD\lambda D\theta \ e^{i R}, \]  

(A23)

\[ R = \int_0^{t_f} dt \ \left[ \lambda (t) \left( \dot{x} (t) - \frac{p (t)}{m} \right) + \theta (t) \left( \dot{p} (t) + V'(x(t), t) \right) \right]. \]  

(A24)

The functional measure for each path variable is

\[ Df = \lim_{N \to \infty} \prod_{n=1}^{N} \frac{df_n}{\sqrt{2\pi}} \]  

(A25)

and compared to the quantum path integral, there is no factor of \( i \) causing the measure to fluctuate. For this reason, the KvN path integral is well behaved in the continuous limit.

The raw form of the KvN propagator is not particularly illuminating, but the integration over the ambiguity variables represents a product of delta functionals enforcing Hamilton’s equations. We can see this most easily by returning to the discrete formulation. Specifically, consider the integration over \( \lambda_j \)

\[ \int d\lambda_j \ \exp \left(i \lambda_j \left( \frac{\Delta}{m} \left( m \frac{x_{j+1} - x_j}{\Delta} - p_j \right) \right) \right) = \frac{m}{2\pi \Delta} \delta \left( m \frac{x_{j+1} - x_j}{\Delta} - p_j \right). \]  

(A26)

If this delta function is now integrated with respect to \( p_j \), the propagator may be expressed with a reduced number of path variables. The functional measure is now

\[ DxD\theta = \lim_{N \to \infty} \left( \frac{m}{2\pi \Delta} \right)^N \prod_{n} dx_n d\theta_n, \]  

(A27)
while the propagator itself is
\[ U_{cl} (x_f, p_f, t_f; x_i, p_i, 0) = \int_{x_i, \dot{x}_i}^{x_f, \dot{x}_f} \mathcal{D}x \mathcal{D}\theta \exp \left[ i \int_0^{t_f} dt \theta (t) (m\ddot{x} (t) + V' (x(t), t)) \right]. \] (A28)

Appendix B: The driven Harmonic oscillator

In order to evaluate the CIF for the CL model, we require the solution to
\[ m\ddot{x}(t) = -m\omega^2 x(t) + f(t). \] (B1)

Solving this equation is not entirely trivial, but can be accomplished in a variety of ways (A Green’s function approach is often used here). In the interest of novelty we shall take a slightly different approach, by re-expressing Eq. (B1) as a first-order matrix equation
\[ m\dot{X}(t) = mA X + F(t), \] (B2)
\[ X = \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} f(t) \end{pmatrix}. \] (B3)

Solving this equation with the integrating factor exp(At) yields
\[ X(t) = e^{-At} X(0) + \frac{1}{m} \int_0^t ds e^{-A(t-s)} f(s). \] (B4)

This solution can be recast by expanding the matrix exponentials. This first requires the evaluation of \( A^n \):
\[ A^n = \begin{cases} \frac{(-1)^{n/2} \omega^n}{(2n)!} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & n \text{ even} \\ \frac{(-1)^{(n-1)/2} \omega^{n-1}}{(-\omega^2)^{n/2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & n \text{ odd} \end{cases} \] (B5)

which can be used to rearrange the exponential expansion into odd and even terms
\[ e^{-At} = \sum_{n=0}^{\infty} \left\{ \frac{(At)^{2n}}{(2n)!} + \frac{(At)^{2n+1}}{(2n+1)!} \right\} \]
\[ = \cos(\omega t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sin(\omega t) \begin{pmatrix} 0 & \omega^{-1} \\ -\omega & 0 \end{pmatrix}. \] (B6)

A nice feature of this method is that once the equation of motion is obtained reading off the top row, there is a free consistency check that its derivative is equal to the bottom row. Using \( A \) and \( B \) for constants and grouping terms produces
\[ x(t) = A \sin(\omega t) + B \sin(\omega (t_f - t)) \]
\[ + \frac{1}{m \omega} \int_0^t ds f(s) \sin(\omega(t-t_s)), \] (B8)
\[ \dot{x}(t) = A \omega \cos(\omega t) - B \omega \cos(\omega (t_f - t)) \]
\[ + \frac{1}{m} \int_0^t ds f(s) \cos(\omega(t-t_s)). \] (B9)

Appendix C: The Hubbard Stratonovich Transformation

Consider a complex Gaussian distribution \( W(z) \):
\[ W[\eta_1, \eta_1^*, \ldots, \eta_N, \eta_N^*] = C \exp \left[ -\frac{1}{2} z^T \Phi z \right]. \] (C1)

Here \( z \) is the vector of all the complex variables and its conjugates, with individual elements labeled as \( z_i^\alpha \)
\[ z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{pmatrix}, \quad z_i = \begin{pmatrix} z_1^\alpha \\ z_2^\alpha \\ \vdots \\ z_N^\alpha \end{pmatrix} = \begin{pmatrix} \eta_i \\ \eta_i^* \end{pmatrix}. \] (C2)

The Fourier transform of this distribution is:
\[ \kappa(k_1, \ldots, k_N) = \int dz \ W(z) e^{z^T k} \]
\[ = \int dz \ \exp \left[ -\frac{1}{2} z^T \Phi z + iz^T k \right]. \] (C3)

Evaluating the Fourier transform is simply a case of completing the square of the exponent and produces
\[ \kappa[k] = e^{-\frac{1}{2} k^T \Phi^{-1} k}. \] (C4)

The exponent may be expanded in terms of the random variable correlations:
\[ k^T \Phi^{-1} k = \sum_{i,j=\alpha,\beta} k_i^\alpha \langle z_i^\alpha z_j^\beta \rangle_r k_j^\beta \] (C5)

We now order each random variable by a parameter \( t_i \), where the value of each parameter is evenly spaced by a gap \( \Delta \). If \( t_N = t_f \), \( t_0 = 0 \), the gap is given by \( \Delta = \frac{t_f}{N} \).

Defining now a single process \( z_i^\alpha = z^\alpha(t_i) \)
\[ k_i^\alpha = \Delta k^\alpha(t_i), \]
we take the continuum limit $N \to \infty$. In this limit, vector and matrix products become integrals

$$\sum_i z_i^n \kappa_i^n = \sum_i \Delta z^n(t_i) k^n(t_i) \to \int_0^{t_f} dt \, z^n(t) k^n(t),$$  \hspace{1cm} (C6)

$$\sum_{ij} k_i^{\alpha} \left( \sum_j \kappa_j^{\beta} \right) = \sum_{i,j} \Delta^2 k^n(t_i) A^{\alpha\beta} (t_i, t_j) k^n(t_j) \to \int_0^{t_f} \int_0^{t_f} dt dt' \, k^n(t) A^{\alpha\beta} (t, t') k^n(t').$$  \hspace{1cm} (C7)

Here the matrix $A^{\alpha\beta}$ is defined in relation to $\Phi^{\alpha\beta}$ as follows:

$$\sum_{\beta} \int_0^{t_f} dt' \, \Phi^{\alpha\beta}(t, t') A^{\beta\gamma}(t', t'') = \delta(t - t'') \delta_{\alpha\gamma},$$  \hspace{1cm} (C8)

$$A^{\alpha\beta}(t, t') = \left\langle z_i^{\alpha}(t) z_j^{\beta}(t') \right\rangle_r.$$  \hspace{1cm} (C9)

Having taken the continuous limit, the measure for the integration is now akin to a path integral, as $\lim_{N \to \infty} \prod_i d^2 z^n = Dz^n(r)$. In the continuous limit, the Fourier transform $\kappa$ becomes

$$\kappa(k(t_f)) = \exp \left[ -\frac{1}{2} \sum_{\alpha\beta} \int_0^{t_f} \int_0^{t_f} dt dt' \, k^n(t) A^{\alpha\beta}(t, t') k^n(t') \right].$$  \hspace{1cm} (C10)

Remembering the original definition of $\kappa$ in Eq. (C3), it is possible to interpret this not just as a Fourier transform but as a functional average

$$\kappa(k(t_f)) = \left\langle \exp \left[ i \sum_{\alpha} \int_0^{t_f} dt \, z^n(t) k^n(t) \right] \right\rangle_r.$$  \hspace{1cm} (C11)

Importantly, the relationship between the $k^n$ is not constrained in the same way as the variables $z^n$ are. This means we are free to choose what, if any, functional dependence there is between $k^1(t)$ and $k^2(t)$.

Putting all of this together gives us the Hubbard-Stratonovich (HS) transformation\cite{44}:

$$\left\langle \exp \left[ i \sum_{\alpha} \int_0^{t_f} dt \, z^n(t) k^n(t) \right] \right\rangle_r = \exp \left[ -\frac{1}{2} \sum_{\alpha\beta} \int_0^{t_f} \int_0^{t_f} dt dt' \, k^n(t) \left\langle z^n(t) z^n(t') \right\rangle k^n(t') \right].$$  \hspace{1cm} (C12)

which is easily generalised to multivariate processes

$$\left\langle \exp \left[ i \sum_{\alpha} \int_0^{t_f} dt \, z^n(t) k^n(t) \right] \right\rangle_r = \exp \left[ -\frac{1}{2} \sum_{i \alpha\beta} \int_0^{t_f} \int_0^{t_f} dt \, k^n_i(t) \left\langle z^n(t) z^n(t') \right\rangle k^n_i(t') \right].$$  \hspace{1cm} (C13)

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