On the Topology of Initial Data Sets with Higher Genus Ends

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Abstract: In this note we study the topology of 3-dimensional initial data sets with horizons of a sort associated with asymptotically locally anti-de Sitter spacetimes. We show that, within this class, those initial data sets that contain no (immersed) marginally outer trapped surfaces in their interior must have simple topology: they are a product of a surface and an interval, or a mild variation thereof, depending on the connectedness of the horizon and on its genus relative to that of the end. The results obtained here extend results in Eichmair et al. (J Differ Geom 95:389–405, 2013) to the case of higher genus ends.

1. Introduction

One of the interesting features of general relativity is that it does not a priori impose any restrictions on the topology of space. In fact, as was shown in [17], given an asymptotically flat initial data set of arbitrary topology, there always exists a solution to the vacuum Einstein constraint equations. However, according to the principle of topological censorship, the topology of the domain of outer communications (DOC), i.e., the region outside of all black holes (and white holes), should, in a certain sense, be simple. Roughly speaking, results on topological censorship [6,12–14] show that, under suitable energy and causality conditions, the topology of the DOC (at the fundamental group level) cannot be more complicated than the topology at infinity. In particular, in the asymptotically Minkowskian case, the DOC must be simply connected. However, the results alluded to here are spacetime results, i.e., they involve conditions that are essentially global in time. In [11] a result on topological censorship was obtained at the pure initial data level for asymptotically flat initial data sets, thereby circumventing difficult questions of global evolution; cf. [11, Theorem 5.1].

The aim of this note is to extend this result to 3-dimensional initial data sets that arise in asymptotically locally anti-de Sitter (ALADS) spacetimes. Examples of such are the generalized Kottler spacetimes [5,7,19], which are solutions to the vacuum Einstein


equations with negative cosmological constant. These solutions have spacetime ends and event horizons with arbitrary (but equal) genus. Cauchy surfaces for the DOC have product topology. Here we will establish conditions on ALADS initial data sets, similar to those in [11], which imply product topology. The proofs rely on existence results for marginally outer trapped surfaces, as well as our current understanding of 3-manifolds. Results in [8] will play a key role in establishing product topology.

In Sect. 2 we present some preliminary material. Our main results are presented in Sect. 3.

2. Preliminaries

We begin with some basic definitions and important facts about 3-manifolds. Let $V$ be a compact 3-manifold without spherical boundary components. $V$ is said to be \textit{irreducible} provided every embedded 2-sphere in $V$ bounds a ball in $V$.

A surface $\Sigma$ embedded in $V$ is said to be \textit{compressible} if there exists an embedded disk $D \subset V$ such that $D \cap \Sigma = \partial D$ and $\partial D$ does not bound a disk in $\Sigma$; such a disk is a \textit{compressing disk} for $\Sigma$. A surface that is not compressible is \textit{incompressible}. It is a fundamental consequence of the Loop Theorem [20] that $\Sigma$ is incompressible if and only if the map on fundamental groups $i_* : \pi_1(\Sigma) \rightarrow \pi_1(V)$ induced by inclusion $i : \Sigma \hookrightarrow V$ is injective.

If $\Sigma$ is a compressible surface in $V$ and $D$ is a compressing disk for $\Sigma$, then $D$ has a collar neighborhood $N(D) \cong \mathbb{D}^2 \times (0, 1)$ such that $\Sigma \cap N(D) \cong \partial D \times (0, 1)$, and the \textit{compression} of $\Sigma$ along $D$ is the surface $\Sigma' = (\Sigma \cup \partial N(D)) - (\Sigma \cap N(D))$. If moreover $\Sigma$ is a component of $\partial V$, then this compression produces the submanifold $V' = V - N(D)$. Observe that $V$ may be recovered from $V'$ by attaching a 3-dimensional 1-handle along $\Sigma'$; the compressing disk is its \textit{co-core}.

From Theorem 10.5 in [15], together with the positive resolution of the Poincaré Conjecture (which ensures that there are no fake 3-cells), we have the following algebraic criterion for $V$ to be an $I$-bundle (where $I$ is the interval $[0, 1]$).

\textbf{Theorem 2.1.} Let $V$ be a compact, connected, orientable 3-manifold without spherical boundary components, and $\Sigma$ an incompressible boundary component. If the index, $[\pi_1(V) : i_\#\pi_1(\Sigma)]$, of $i_\#\pi_1(\Sigma)$ in $\pi_1(V)$ is finite, then either

- $[\pi_1(V) : i_\#\pi_1(\Sigma)] = 1$ and $V$ is diffeomorphic to $\Sigma \times [0, 1]$ with $\Sigma = \Sigma \times \{0\}$, or
- $[\pi_1(V) : i_\#\pi_1(\Sigma)] = 2$ and $V$ is a twisted $I$-bundle over a compact non-orientable surface $\hat{\Sigma}$ with $\Sigma$ the associated 0-sphere bundle.

A group $G$ is said to be \textit{residually finite} if for each non-identity element $g \in G$, there is a normal subgroup $N$ of finite index such that $g \notin N$. It follows from work of Hempel [16], together with the positive resolution of the geometrization conjecture, that the fundamental group of every compact 3-manifold $V$ is residually finite; see e.g. [3]. Hence, by basic covering space theory, if $\pi_1(V) \neq 0$ then $V$ admits a finite cover $\tilde{V}$. [If $\pi_1(V)$ were finite, one could simply take the universal cover. Residual finiteness becomes important when $\pi_1(V)$ is infinite.]

The following lemma about submanifolds representing codimension one homology classes is well established for closed orientable manifolds; see [21]. When the ambient manifold has boundary the result appears well-known to the experts, though we were

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1 Here and elsewhere we assume that boundary components (which physically correspond to horizons or ends) are of genus $\geq 1$. The case of spherical ends is considered in [11].