A CHARACTERIZATION OF THE FINITE MULTIPLICITY
OF A CR MAPPING

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1. INTRODUCTION

In this paper, we give a characterization of the finite multiplicity of a CR mapping between real analytic hypersurfaces. The finite multiplicity of a CR mapping was defined algebraically by Baouendi and Rothschild in [BR1] (see the definition below). We will prove that under certain conditions on hypersurfaces the finite multiplicity of a CR mapping is equivalent to that the preimage of the map is finite. More precisely,

Theorem 1. Suppose that $M_1, M_2$ are real analytic hypersurfaces of essential finite type in $\mathbb{C}^n$ and further $M_2$ contains no complex variety of positive dimension. Then a smooth CR mapping $f: M_1 \to M_2$ is of finite multiplicity at $z_0 \in M_1$ if and only if $f^{-1}(f(z_0))$ is finite.

The proof of Theorem 1 relies on the real analyticity result of [BR1] and the following Theorem 2 that we shall prove. In [BR1], Baouendi and Rothschild proved that a smooth CR mapping of finite multiplicity from a real analytic hypersurface of essential finite type to another real analytic hypersurface is real analytic. This result with the proof of Theorem 1 implies the following.

Corollary 1. A smooth CR mapping of finite multiplicity between real analytic hypersurfaces of essential finite type is the restriction of a locally proper holomorphic mapping in $\mathbb{C}^n$.

Theorem 2. Suppose that $f: M_1 \to M_2$ is a smooth CR mapping between real analytic hypersurfaces in $\mathbb{C}^n$. Suppose further that $M_1$ is essentially finite and $M_2$ contains no complex variety of positive dimension. If $f^{-1}(f(z_0)) \setminus \{z_0\}$ is discrete for a point $z_0 \in M_1$, then $f$ extends holomorphically to a neighborhood of $z_0$ in $\mathbb{C}^n$.

A simple example shows that the condition that $M_2$ contains no complex variety of positive dimension is necessary in Theorem 1 and 2.

Corollary 2. Suppose that $f: M_1 \to M_2$ is a smooth CR mapping between real analytic hypersurfaces of finite type of D’Angelo in $\mathbb{C}^n$. If $f^{-1}(f(z_0)) \setminus \{z_0\}$ is
discrete for a point $z_0 \in M_1$, then $f$ extends holomorphically to a neighborhood of $z_0$ in $\mathbb{C}^n$.

A well-known problem in the study of real analyticity of CR mappings is whether every smooth CR mapping between real analytic hypersurfaces of finite type of D'Angelo in $\mathbb{C}^n$ is real analytic.

**Corollary 3.** Let $f: M_1 \to M_2$ is a smooth CR mapping between real analytic hypersurfaces of finite type of D'Angelo in $\mathbb{C}^n$. If $f$ is real analytic on $M_1 \setminus \{p\}$, then $f$ is also real analytic at $p$.

This can be reviewed as a "Removable Singularity Theorem" for the real analyticity of CR mappings. As another corollary of the proof of Theorem 2, one has the following.

**Corollary 4.** A finite to one smooth CR mapping from a real analytic hypersurface of essential finite type to another real analytic hypersurface is real analytic.

Here a map $f: M_1 \to M_2$ is said to be finite to one if $f^{-1}(q)$ is finite for any $q \in M_2$. The proofs of these results depend on the work of Baouendi-Rothschild [BR1] and Diederich-Fornaess [DF] on real analyticity, the Hopf Lemma of [BR3] and the work of Tumanov [T] on holomorphic extension of CR functions. However, we will directly prove the holomorphic extension of CR mappings whenever their work does not apply. For earlier results, see [L], [Pi], [BJT], [DW], [B], [BB] and [BBR]. Theorem 1 will be proved in Section 2 and Theorem 2 along with its corollaries in Section 3. The work of this paper is in part inspired by a paper of Pincuk [Pi2].

### 2. Proof of Theorem 1

To prove Theorem 1, we first recall some basic definitions. Let $M$ be a real analytic hypersurface in $\mathbb{C}^n$ containing the origin and defined locally by $\rho(z, \overline{z}) = 0$, $\nabla \rho \neq 0$, $z \in \mathbb{C}^n$, where $\rho$ is a real valued analytic function, $\rho(0) = 0$. As introduced in [BJT], $M$ is said to be essentially finite at 0 if for any sufficiently small $z \in \mathbb{C}^n \setminus \{0\}$ there exists an arbitrarily small $\zeta \in \mathbb{C}^n$ satisfying $\rho(z, \zeta) \neq 0$, $\rho(0, \zeta) = 0$. We point out that if $M$ does not contain any complex variety of positive dimension through 0, then $M$ is essentially finite at 0. Consequently, a real analytic hypersurface of finite type of D'Angelo is essentially finite. The finite multiplicity of a CR mapping is introduced by Baouendi and Rothschild in [BR1] as follows. If $f: M_1 \to M_2$ is a smooth CR mapping between two smooth real analytic hypersurfaces in $\mathbb{C}^n$, there exist $n$ CR functions $f_1, \ldots, f_n$ defined on $M_1$ such that $f = (f_1, \ldots, f_n)$. On the other hand if $j$ is a smooth CR function defined on $M_1$ near 0, there exists a formal holomorphic power series $J(Z) = \sum a_\alpha Z^\alpha$ in $n$ indeterminates, such that $U \in u \to Z(u) \in \mathbb{C}^n$ ($U$ an open neighborhood of 0 in $\mathbb{R}^{2n-1}$, $Z(0) = 0$) is a parametrization of $M_1$, then the Taylor series of $j(Z(u))$ at 0 is given by $J(Z(u))$. We can choose holomorphic coordinates $Z$ such that $\rho(Z, 0) = \alpha(Z)Z_n$, $\alpha(0) \neq 0$. With $Z = (z', z_n)$ and $z' = (z_1, \ldots, z_{n-1})$, the mapping $f$ is said to be of finite multiplicity at 0 if

$$
\dim_{\mathbb{C}}O[[z']]/(F(z', 0)) < \infty
$$

where $(F(z', 0))$ is the ideal generated by $F_1(z', 0), \ldots, F_n(z', 0)$, the power series associated to the CR functions $f_1, \ldots, f_n$ and $O[[z']]$ the ring of formal power series in $n-1$ indeterminates and the dimension is taken in the sense of vector spaces.
After a holomorphic change of coordinates near 0, we may assume that $M_1$ is given by an equation

$$\Re z_n = \psi(z', \overline{z}', \Re z_n), \quad \psi(0) = d\psi(0) = 0$$

with $(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C}$. We assume that $M_2$ is another real analytic hypersurface defined by

$$\Re z_n = \phi(z', \overline{z}', \Re z_n), \quad \phi(0) = d\phi(0) = 0$$

with $(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C}$. Let $f = (f', f_n)$ be a CR map from $M_1$ to $M_2$ with $f(0) = 0$. We say that $f_n$ is the normal component of $f$ and $z_n$ in the normal direction at 0.

**Proof of Theorem 1.** We actually prove that if $f: M_1 \to M_2$ is a smooth CR mapping of finite multiplicity between real analytic hypersurfaces of essential finite type, then $f^{-1}(0)$ is finite. By Theorem 1 of [BR1], $f$ is real analytic at 0. Let $F = (F_1(z), ..., F_n(z))$ be the holomorphic extension of $f$ to $\mathbb{C}^n$ near 0. If $f^{-1}(0)$ is not finite, then $S = F^{-1}(0)$ must be a complex variety of positive dimension. By Theorem 4 of [BR2], we have

$$\partial F_n \partial z_n(0) \neq 0. \tag{2}$$

We claim that $S$ lies in $M_1$. Indeed, by (2),

$$\Re F_n(z) - \phi(F'(z), \overline{F'}(z), \Re F_n(z)) = 0$$

defines a real analytic hypersurface in $\mathbb{C}^n$ which clearly coincides with $M_1$ near the origin where $F'(z) = (F_1, ..., F_{n-1})$. This proves the claim.

Now we let $S'$ be any complex curve in $S$ parametrized by

$$z(\zeta) = (z_1(\zeta), ..., z_n(\zeta))$$

passing through 0. We claim that $z_n(\zeta) \equiv 0$. Indeed, in the chosen coordinates above, by Lemma (3.7) of [BR1], we have

$$F_n(z) = z_n G(z).$$

By (2), we see $G(0) \neq 0$. On $S'$, it follows $F_n(z(\zeta)) = z_n(\zeta) G(z(\zeta)) = 0$, which implies $z_n(\zeta) = 0$. Therefore, $F_1(z', 0), ..., F_n(z', 0)$ have common zeros near 0 and hence the dimension

$$\dim_{\mathbb{C}} \mathcal{O}[[z']]/(F_1(z', 0), ..., F_n(z', 0))$$

is infinite, a contradiction to the finite multiplicity of $f$ at 0.

As proved above, $S$ lies in $M_1$ and hence $f^{-1}(0) = F^{-1}(0)$. This means that $F$ is a locally proper holomorphic mapping, which gives a proof of Corollary 1.

Now we prove that under the conditions in Theorem 1 if $f^{-1}(0)$ is finite then $f$ is of finite multiplicity. Indeed, by Theorem 2, whose proof does not depend on Theorem 1, $f$ is real analytic at 0. As before, let $F$ be the holomorphic extension of $f$. We notice $F_n(z) = 0$ since $M_2$ contains no complex variety of positive dimension.
and by Theorem 4 of [BR2], \( f \) is of finite multiplicity at 0. This could also proved directly. Indeed, by Theorem 4 of [BR2], (2) holds. As above, this implies that \( F^{-1}(0) \) is finite and therefore \( F \) is locally proper which implies the finite multiplicity of \( f \).

We close this section by an example. Let \( M_1 = \{ \Im z_3 = |z_1|^2 + |z_2|^2 \} \) and \( M_2 = \{ \Im z_3 = |z_1|^2 - |z_2|^2 \} \). Consider \( f = (g, g, 0) \), which is holomorphic in \( \Im z_3 > 0 \) and smooth up the boundary. It is easy to see \( f^{-1}(0) = 0 \) but \( f \) is not finite multiplicity. Note that \( M_2 \) contains a complex curve and both \( M_1, M_2 \) are of essential finite type.

2. Proof of Theorem 2

Following Tumanov [T], we say that a real hypersurface \( M_1 \) is minimal at \( z_0 \) if there is no germ of complex holomorphic hypersurface contained in \( M_1 \) and passing through \( z_0 \). By a theorem of Trepreau [Tr], \( f \) extends holomorphically to one side of \( M_1 \). The main result of [BBR] [BR1] [DF] can be stated as

**Theorem.** ([BBR][BR1][DF]) Let \( M_1 \) is a real analytic hypersurface that is essentially finite at \( 0 \in M_1 \). If \( M_2 \) is another real analytic hypersurface and \( f: M_1 \to M_2 \) is a smooth CR mapping with \( f(0) = 0 \) and \( \frac{\partial f_n}{\partial z_n}(0) \neq 0 \), then \( f \) extends holomorphically to a neighborhood of \( 0 \) in \( \mathbb{C}^n \).

The above theorem has many important applications to global proper holomorphic mappings. For example, it was proved in [BR1] [DF] that every proper holomorphic mapping between bounded pseudoconvex domains with real analytic boundaries extends holomorphically across the boundary. In [BR2], Baouendi and Rothschild showed that if the normal component of \( f \) is not flat (i.e., if there exists a number \( k > 0 \) so that \( \frac{\partial^k f_n}{\partial z_n^k}(0) \neq 0 \) in the normal direction at 0 then the condition \( \frac{\partial f_n}{\partial z_n}(0) \neq 0 \) holds automatically. As an application of this result, it was proved in [HP] that the unique continuation property holds for proper holomorphic mappings between bounded domains with real analytic boundaries. This result in turn proves that every proper holomorphic mapping between bounded real analytic domains that is smooth up to the boundary extends holomorphically across the boundary.

In order to prove Theorem 2, we need the following lemmas. First we recall the definition of a correspondence. Let \( \Omega \) be a domain in \( \mathbb{C}^n \) and \( f: \Omega \to \mathbb{C}^n \) be a holomorphic mapping. Denote by \( \Gamma_f \) as the graph of \( f \)

\[
\Gamma_f = \{(z, w): w = f(z), z \in \Omega \}.
\]

Let

\[
B((z_0, w_0), \epsilon) = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^n: |z - z_0| < \epsilon, |w - w_0| < \epsilon \}.
\]

We say that \( f \) extends as a correspondence to a neighborhood of \( (z_0, w_0) \) if there exist \( \epsilon > 0 \) and a pure n-dimensional subvariety

\[
V \subset B((z_0, w_0), \epsilon)
\]

such that

\[
\Gamma_f \cap B((z_0, w_0), \epsilon) \subset V \cap B((z_0, w_0), \epsilon).
\]

Now we state a lemma due to Bedford and Bell [BB].
Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ with smooth boundary near $z_0 \in \partial \Omega$, and let $f: \Omega \to \mathbb{C}^n$ be a holomorphic mapping that is $C^\infty$ smooth up to the boundary of $\Omega$ near $z_0$. Then $f$ extends holomorphically to a neighborhood of $z_0$ in $\mathbb{C}^n$.

Let $M_1$ and $M_2$ be smooth real hypersurfaces in $\mathbb{C}^n$ and let $\Omega_1, \Omega_2$ be two domains in $\mathbb{C}^n$ with defining functions $r_i$ for $i = 1, 2$ such that $\nabla r_i \neq 0$ on $\Omega_i$ for $i = 1, 2$. Set $\Omega_i^+ = \{z \in \Omega_i : r_i(z) > 0\}$ and $\Omega_i^- = \{z \in \Omega_i : r_i(z) < 0\}$ for $i = 1, 2$.

If $F: \Omega_1^- \to \mathbb{C}^n$ is a holomorphic mapping, we denote by $\text{Jac} F$ the determinant of the Jacobian matrix of $F$.

As will become clear, in order to prove Theorem 2, one has to only consider the case when $w_0$ is a minimal but not minimally convex point in the sense of [BR3]. For this matter, we prove the following result.

**Lemma 2.** Let $f: M_1 \to M_2$ be a smooth CR mapping between smooth real hypersurfaces $M_1, M_2$ in $\mathbb{C}^n$. Suppose that $f$ extends holomorphically to an one-sided neighborhood of $M_1$, say $\Omega_1^-$. Given a point $z_0 \in M_1$, if $M_2$ contains no non-trivial complex variety through $f(z_0)$ and if $f(z_0)$ is not minimally convex and $f^{-1}(f(z_0)) \setminus \{z_0\}$ is discrete, then $f$ extends holomorphically to a neighborhood of $z_0$ in $\mathbb{C}^n$.

We remark that no real analyticity on hypersurfaces is assumed above.

**Proof of Lemma 2.** Let $F(z): \Omega_1^- \to \mathbb{C}^n$ be the extension of $f$. First we prove two facts to be used later.

We notice that $F(\Omega_1^-) \not\subset M_2$ since $M_2$ contains no complex variety of positive dimension. Now we claim that $\text{Jac} F(z) \not\equiv 0$. Indeed, if $\text{Jac} F(z) \equiv 0$ in $\Omega_1^-$, we let $\mu$ be the maximal rank of the Jacobian matrix of $F$ in $\Omega_1^-$. We have $0 < \mu < n$ and the set

$$\{z \in \Omega_1^- : \text{Rank} F = \mu\}$$

is an open dense subset of $\Omega_1^-$. By the rank theorem and the fact $F(\Omega_1^-) \not\subset M_2$, we may find a sequence of points $z_k \in \Omega_1^-$ converging to $z_0$ such that $F(z_k) \not\in M_2$ and for each $k$ the analytic set

$$\{z \in \Omega_1^- : F(z) = F(z_k)\}$$

has an irreducible component $V_k \subset \Omega_1^-$ of dimension $n - \mu > 0$ passing through $z_k$. Since $F(z_k) \not\in M_2$, it follows that for each $k$, $V_k$ does not have limit points on $M_1$. Therefore $\overline{V_k}$ is a closed analytic variety in $\Omega_1$. Now let $z' \in \overline{V_k} \setminus \overline{V_k}$, and we see that

$$f(z') = F(z') = \lim F(z_k) = F(z_0) = w_0$$

This implies that $z' \in f^{-1}(w_0)$. But $f^{-1}(w_0) \setminus \{z_0\}$ is discrete, we see that the sequence of the sets $\overline{V_k}$ clusters on $M_1$ only at discrete points near $z_0$. Thus by the generalized continuity principle we conclude that $F(z)$ extends holomorphically to a neighborhood of $z_0$ in $\mathbb{C}^n$. As before, it implies that $\text{Jac} F(z) \not\equiv 0$ since $F$ is locally proper.

Using these facts we will prove that $F$ extends holomorphically to a neighborhood of $z_0$ in $\mathbb{C}^n$.

When $w_0 \in M_2$ is not a minimally convex point, an important fact is that (see Theorem 7 of [BR4], Theorem 1 of [BR2], [T]) every holomorphic function.
defined on one side of $M_2$, which admits a distribution limit up to $M_2$, extends holomorphically to a small open neighborhood of $w_0$. This fact has been used in [HP] [P].

To be able to prove the holomorphic extension of $F$ when $w_0$ is not minimally convex, we will construct pieces of proper holomorphic mappings near $z_0$.

Since $f^{-1}(w_0) \setminus \{z_0\}$ is discrete and $f^{-1}(w_0)$ is closed, we may choose an open neighborhood $\Omega_1$ of $z_0$ such that

$$\partial \Omega_1 \cap \{f^{-1}(w_0)\} = \emptyset.$$ 

So we have that $\text{dist}(\partial \Omega_1, \{f^{-1}(w_0)\}) = \delta > 0$.

Now consider

$$V = \{z \in \Omega_1^-, F(z) = w_0\}.$$ 

Then $V$ is an analytic variety in $\Omega_1^-$. If $\dim V \geq 1$, let $V'$ be an irreducible component of $V$. Since $V$ only has limit points $f^{-1}(w_0)$ on $M_1$, by Shiffman’s theorem, $V'$ is an analytic variety in $\Omega_1$. The continuity principle implies that $F$ extends holomorphically to a neighborhood of $z_0$.

Now we may assume that $\dim V = 0$. This means $V$ is a discrete set in $\Omega_1^-$. We may shrink $\Omega_1$ slightly so that $\partial \Omega_1 \cap V = \emptyset$. Therefore, we have

$$\text{dist}(w_0, F(\partial \Omega_1^- \setminus M_1)) > 0.$$ 

Then we can choose a very small open neighborhood $\Omega_2$ of $w_0$ such that

$$(\#) \quad \text{dist}(\partial \Omega_2, F(\partial \Omega_1^- \setminus M_1)) > 0.$$ 

Since $F(\Omega_1^-) \not\subset M_2$, $F(\Omega_1^-)$ intersects at least one side of $M_2$. Therefore there are two possibilities as follows. (I) For any small neighborhood $\Omega_2$ of $w_0$ we have

$$F(\Omega_1^-) \cap \Omega_2^- \neq \emptyset \quad \text{and} \quad F(\Omega_1^-) \cap \Omega_2^+ \neq \emptyset.$$ 

(II) There is an arbitrarily small neighborhood $\Omega_2$ of $w_0$ such that

$$F(\Omega_1^-) \subset \overline{\Omega_2^-} \quad \text{or} \quad F(\Omega_1^-) \subset \overline{\Omega_2^+}.$$ 

We consider the case (I) first, the case (II) can be dealt with similarly.

Consider two nonempty open sets in $\Omega_1^-$:

$$U^+ = F^{-1}(\Omega_2^+) \quad \text{and} \quad U^- = F^{-1}(\Omega_2^-).$$

We claim that the restriction of $F$ to $U^+$ (resp. $U^-$) is a proper map from $U^+$ to $\Omega_2^+$ (resp. $\Omega_2^-$). Indeed, let $F^+ = F|_{U^+}$ and let $K \subset \subset \Omega_2^+$ be a compact subset, we want to prove that $(F^+)^{-1}(K)$ is a compact subset in $U^+$. If $(F^+)^{-1}(K)$ is not compact in $U^+$, there exists a point $p \in \partial U^+$ such that $F^+(p) \in K$. Since $K \cap M_2 = \emptyset$, we have $p \not\in M_1$, and by (\#) $p \in \Omega_1^-$. Therefore, there exists a neighborhood $O$ of $p$, such that $F(O) \subset \Omega_2^+$. Hence $p$ cannot be a boundary point of $U^+$, a contradiction.

Now we observe that the open set $U^+ \cup U^-$ is, in general, not connected. We make some simple observations that are crucial to what follows in the proof of Lemma 3.
Claim 1: The set $U^+ \cup U^-$ is an open dense set near $z_0$ in $\Omega_1^-$ along $M_1$. Indeed, if it is not the case, then, for any small neighborhood of $z_0$, there exists a point $p \in \Omega_1^-$ in that neighborhood, and there exists a small neighborhood $O$ of $p$ contained in $\Omega_1^-$ so that $f(O) \subset \partial \Omega_2 \cup M_2$ (since by continuity $F(O) \subset \Omega_2$). This is impossible since $\text{Jac} F(z) \neq 0$ in $\Omega_1^-$.

Claim 2: The open set $U^+ \cup U^-$ has finitely many connected components.

Indeed, if it is not the case, we let $U_j$ be connected components of $U^+ \cup U^-$ for $j = 1, 2, \ldots$. Let $E_j = \partial U_j$ be the boundary of $U_j$. Since $\Omega_1^-$ is bounded, either $\{E_j\}$ accumulates at a neighborhood of an interior point of $\Omega_1^-$ where they are disjoint each other, or at a boundary point or both. We prove that neither is possible. Indeed, if $E_j$ accumulates at $p \in \Omega_1^-$ we can assume that $\text{Jac} F(p) \neq 0$ since $\text{Jac} F \neq 0$ and $\{z \in \Omega_1^-, \text{Jac} F(z) = 0\}$ is an analytic variety of complex dimension of $n - 1$. Therefore $F$ is a local biholomorphism in a neighborhood $O$ of $p$, therefore we may assume $E_j \subset O$ locally near $p$ for all $j$. On the other hand, we have $F(E_j) \subset \partial \Omega_2 \cup M_2$ for all $j$, from which we arrive at a contradiction.

If $E_j$ accumulates at $p \in M_1$ we can assume that $\text{Jac} F(p) \neq 0$ since $\{z \in M_1: \text{Jac} F \neq 0\}$ is a dense open subset of $M_1$. Then the above argument applies since $F$ is a diffeomorphism near $p$ after we extend $F$ smoothly to a neighborhood of $p$ in $\mathbb{C}^n$.

Now let $\{U_j^+\}_{j=1}^k$ be connected components of $U^+$, similarly $\{U_j^-\}_{j=1}^l$ for $U^-$. Let $g_j$ be the restriction of $F$ on $U_j^+$, and $h_j$ on $U_j^-$. Therefore $g_j: U_j^+ \rightarrow \Omega_2^+$ and $h_j: U_j^- \rightarrow \Omega_2^-$ are proper holomorphic mappings.

We then consider a proper mapping $g$ from $D$ to $G$, where the paring $(D, G)$ is either $(U_j^+, \Omega_2^+)$ or $(U_j^-, \Omega_2^-)$ and $g$ is either $g_j$ or $h_j$. The graph of $g$ is defined to be

$$\Gamma_g = \{(z, w) \in D \times G, w = g(z)\}.$$

By the Proper Mapping Theorem, $g$ is a covering from $D \setminus g^{-1}(g(V_g))$ to $G \setminus g(V_g)$ of multiplicity $m$, where

$$V_g = \{z \in D: \text{Jac} g = 0\}.$$

Let $G_1, G_2, \ldots, G_m$ be the local inverses defined on $G \setminus g(V_g)$. Define over $D \times G \setminus g(V_g)$

$$H_i(z, w) = \Pi_{j=1}^m (z_i - (G_j(w))_i).$$

By the removable singularity result of bounded holomorphic functions, $H_i$ extends to be holomorphic on $D \times G$. Denote

$$A_g = \{(z, w) \in D \times G : H_1 = H_2 = \ldots = H_n = 0\}.$$

It is easy to check that $\Gamma_g = A_g$.

Let $\Gamma_{g_j}, \Gamma_{h_j}$ be the graphs of $g_j, h_j$ respectively, and let $A_{g_j}, A_{h_j}$ be associated with $g_j, h_j$ as defined above. We see that the graph of $F$ over $U^+ \cup U^-$ is given by

$$\bigcup_{j=1}^k \Gamma_{g_j} \cup \bigcup_{j=1}^l \Gamma_{h_j},$$

which is equal to

$$\bigcup_{j=1}^k A_{g_j} \cup \bigcup_{j=1}^l A_{h_j}.$$
As we have observed that the open set $U^+ \cup U^-$ is an open dense set along $M_1$ near $z_0$ (Claim 1). By the continuity, we conclude that the graph of $F$ over a small one-sided neighborhood of $M_1$ near $z_0$ is contained in

$$
\bigcup_{j=1}^k A_{g_j} \cup \bigcup_{j=1}^l A_{h_j}.
$$

Now we want to show that

$$
\bigcup_{j=1}^k A_{g_j} \cup \bigcup_{j=1}^l A_{h_j}
$$

extends to be an analytic variety of pure dimension $n$ in $\mathbb{C}^n \times \mathbb{C}^n$ near $(z_0, w_0)$. Indeed, we notice for each $g$ (either $g_i$ or $h_j$)

$$
H_i(z, w) = z_i^m + S_{m-1}(w) z_i^{m-1} + \ldots + S_0(w),
$$

where $S_j(w)$ is the $j$-th symmetric function of $(G_j(w))i$ for $j = 1, \ldots, m$. Since $S_j(w)$ are bounded, and since $w_0$ is not minimally convex, then $S_j(w)$ extends to be holomorphic in a neighborhood of $w_0$ in $\mathbb{C}^n$ from either side whenever applicable. Therefore $H_i(z, w)$ extends to be holomorphic to a neighborhood of $(z_0, w_0)$ in $\mathbb{C}^n \times \mathbb{C}^n$, this, in turn, implies that

$$
\bigcup_{j=1}^k A_{g_j} \cup \bigcup_{j=1}^l A_{h_j}
$$

is an analytic variety of pure dimension $n$ in a neighborhood of $(z_0, w_0)$, which implies that $F$ extends to be a correspondence to a neighborhood of $z_0$. Lemma 1 then gives the holomorphic extension of $F$ at $z_0$. This completes the proof of Lemma 2 for the case (I). Case (II) can be proved equally.

**Proof of Theorem 2.** Let $z_0 \in M_1$, $w_0 = f(z_0) \in M_2$. Since $M_1$ is minimal at $z_0$, by Treppeau’s Theorem, $f$ extends holomorphically to one side neighborhood of $M_1$, say $\Omega_1^-$, the extension is denoted by $F(z)$. Therefore $F(z): \Omega_1^- \rightarrow \mathbb{C}^n$ is a holomorphic mapping, such that $F = f$ on $M_1$. If $w_0$ is minimally convex, then the complex Hopf Lemma of [BR3] and the theorem of [BR1] and [DF] imply the $f$ extends holomorphically to a neighborhood of $z_0$ since $\text{Jac}F \neq 0$. When $w_0$ is not minimally convex then Lemma 2 applies. This completes the proof.

Corollary 2 is a special case of Theorem 2 since a real analytic hypersurface of finite type of D’Angelo is essentially finite and contains no nontrivial complex varieties.

Now we give a proof of Corollary 3.

**Proof of Corollary 3.** It suffices to prove that $f^{-1}(f(p)) \setminus \{p\}$ is discrete. Let $q \in f^{-1}(f(p)) \setminus \{p\}$ but $q \neq p$. We want to prove that $q$ is an isolated point. Since $f$ is real analytic at $q$ by the assumption, then $f$ extends holomorphically to a neighborhood of $q$, say the extension as $F$. By a result of [BR2], the Hopf Lemma holds at $q$ for the normal component of $F$. Let $\rho$ be a real analytic defining function of $M_2$ near $w_0$. By the Hopf Lemma just mentioned at $q$, it is easy to see, by changes of coordinates at both $q$ and $f(q)$, that $\rho \circ F$ is again a defining function of $M_1$ near $q$. Therefore the equation

$$
\{ z \in \mathbb{C}^n : \rho \circ F(z) = 0 \}
$$

defines a real analytic hypersurface near $q$, which is identical to $M_1$ near $q$. This implies that $F^{-1}(f(q))$ is contained in $M_1$. Since $M_1$ is of finite type of D’Angelo and $F^{-1}(f(q))$ is a complex analytic variety, we conclude that $q$ is a isolated point in $M_1$. Theorem 2 then applies at $p$ since $f^{-1}(f(p)) \setminus \{p\}$ is discrete.

In order to prove Corollary 4, we prove the following first.
Lemma 3. Let \( f : M_1 \to M_2 \) be a finite to one smooth CR mapping between smooth real hypersurfaces that extends holomorphically to \( \Omega_1^- \) as \( F \). Given \( z_0 \in M_1 \) and \( f(z_0) = w_0 \in M_2 \). If \( M_2 \) is minimal but not minimally convex at \( w_0 \), then \( f \) extends holomorphically to a neighborhood of \( z_0 \) in \( \mathbb{C}^n \).

Proof. By the proof of Lemma 2, it suffices to prove the following (I) \( F(\Omega_1^-) \not\subset M_2 \), and (II) \( \text{Jac} f(z) \not\equiv 0 \).

Indeed, if \( F(\Omega_1^-) \subset M_2 \) then \( \text{Jac} F(z) \equiv 0 \) in \( \Omega_1^- \). This implies that the Jacobian matrix of the map \( f : M_1 \to M_2 \) considered as a real map of the real manifolds is of maximal rank \( \mu \) such that \( 0 < \mu < 2n-1 \). Therefore by the rank theorem, there exists a point \( w' \) near \( w_0 \) such that \( f^{-1}(w') \) is a manifold of dimension \( n - \mu \), a contradiction to finite to one. (II) follows too.

Proof of Corollary 4. First we observe that \( F(\Omega_1^-) \not\subset M_2 \) by the proof of Lemma 3. If \( w_0 \) is not minimal, then, by a unique continuation result for holomorphic mappings in (Theorem 2, [P]), \( F \) does not vanish to infinite order at \( z_0 \) in the normal component. Then \( F \) extends holomorphically to a neighborhood of \( z_0 \). The rest of the proof follows as in Theorem 2 by Lemma 3.

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