Abstract. We show that any (1, 2)-rational function with a unique fixed point is topologically conjugate to a (2, 2)-rational function or to the function $f(x) = \frac{ax}{x^2 + a}$. The case (2, 2) was studied in our previous paper, here we study the dynamical systems generated by the function $f$ on the set of complex $p$-adic field $\mathbb{C}_p$. We show that the unique fixed point is indifferent and therefore the convergence of the trajectories is not the typical case for the dynamical systems. We construct the corresponding Siegel disk of these dynamical systems. We determine a sufficiently small set containing the set of limit points. It is given all possible invariant spheres. We show that the $p$-adic dynamical system reduced on each invariant sphere is not ergodic with respect to Haar measure on the set of $p$-adic numbers $\mathbb{Q}_p$. Moreover some periodic orbits of the system are investigated.

1. Introduction

We study $p$-adic dynamical systems generated by a rational function. For motivation of such investigations see [1]-[5] and references therein. The paper is organized as follows: First we give some necessary definitions and facts. Then in Section 2 show that any (1, 2)-rational function with a unique fixed point is topologically conjugate to a (2, 2)-rational function or to the function $f(x) = \frac{ax}{x^2 + a}$. In [4] the case of (2, 2)-rational function with a unique fixed point is studied. In this paper for $f$ we show that the unique fixed point is indifferent. We give a Siegel disk of the dynamical system. We give a localization of the set of limit points. Section 3 contains a description of all invariant spheres with respect to $f$. We study ergodicity properties of the dynamical system reduced on each invariant sphere with respect to Haar measure and show that the $p$-adic dynamical system reduced on each invariant sphere is not ergodic. In Section 4 we find 2-periodic orbit $\{t_1, t_2\}$ and show that it can only be either an attracting or an indifferent. We shall prove that if the cycle is attracting then it attracts each trajectory which starts from an element of a open ball of radius $h = |t_1 - t_2|_p$ centered at $t_1$ or at $t_2$. If the 2-periodic cycle is an indifferent one then every iteration maps either of the two aforementioned balls to another one. All other spheres of radius $> h$ and center $t_1$ and $t_2$ are invariant independently of the attractiveness of the cycle.

1.1. $p$-adic numbers. Let $\mathbb{Q}$ be the field of rational numbers. The greatest common divisor of the positive integers $n$ and $m$ is denoted by $(n, m)$. Every rational number $x \neq 0$ can
be represented in the form \( x = p^n \frac{a}{m} \), where \( r, n \in \mathbb{Z} \), \( m \) is a positive integer, \( (p, n) = 1 \), \( (p, m) = 1 \) and \( p \) is a fixed prime number.

The \( p \)-adic norm of \( x \) is given by

\[
|x|_p = \begin{cases} 
p^{-r}, & \text{for } x \neq 0, \\
0, & \text{for } x = 0.
\end{cases}
\]

It has the following properties:

1) \( |x|_p \geq 0 \) and \( |x|_p = 0 \) if and only if \( x = 0 \),
2) \( |xy|_p = |x|_p|y|_p \),
3) the strong triangle inequality

\[
|x + y|_p \leq \max\{|x|_p, |y|_p\},
\]
3.1) if \( |x|_p \neq |y|_p \) then \( |x + y|_p = \max\{|x|_p, |y|_p\} \),
3.2) if \( |x|_p = |y|_p \) then \( |x + y|_p \leq |x|_p \),

this is a non-Archimedean one.

The completion of \( \mathbb{Q} \) with respect to \( p \)-adic norm defines the \( p \)-adic field which is denoted by \( \mathbb{Q}_p \) (see [2]).

The algebraic completion of \( \mathbb{Q}_p \) is denoted by \( \mathbb{C}_p \) and it is called complex \( p \)-adic numbers. For any \( a \in \mathbb{C}_p \) and \( r > 0 \) denote

\[
U_r(a) = \{ x \in \mathbb{C}_p : |x - a|_p < r \}, \quad V_r(a) = \{ x \in \mathbb{C}_p : |x - a|_p \leq r \},
\]

\[
S_r(a) = \{ x \in \mathbb{C}_p : |x - a|_p = r \}.
\]

A function \( f : U_r(a) \to \mathbb{C}_p \) is said to be analytic if it can be represented by

\[
f(x) = \sum_{n=0}^{\infty} f_n(x-a)^n, \quad f_n \in \mathbb{C}_p,
\]
which converges uniformly on the ball \( U_r(a) \).

1.2. Dynamical systems in \( \mathbb{C}_p \). Recall some known facts concerning dynamical systems \( (f, U) \) in \( \mathbb{C}_p \), where \( f : x \in U \to f(x) \in U \) is an analytic function and \( U = U_r(a) \) or \( \mathbb{C}_p \) (see for example [2]).

Now let \( f : U \to U \) be an analytic function. Denote \( f^n(x) = f \circ \cdots \circ f(x) \).

If \( f(x_0) = x_0 \) then \( x_0 \) is called a fixed point. The set of all fixed points of \( f \) is denoted by \( \text{Fix}(f) \). A fixed point \( x_0 \) is called an attractor if there exists a neighborhood \( U(x_0) \) of \( x_0 \) such that for all points \( x \in U(x_0) \) it holds \( \lim_{n \to \infty} f^n(x) = x_0 \). If \( x_0 \) is an attractor then its basin of attraction is

\[
A(x_0) = \{ x \in \mathbb{C}_p : f^n(x) \to x_0, \ n \to \infty \}.
\]

A fixed point \( x_0 \) is called repeller if there exists a neighborhood \( U(x_0) \) of \( x_0 \) such that \( |f(x) - x_0|_p > |x - x_0|_p \) for \( x \in U(x_0), x \neq x_0 \).

Let \( x_0 \) be a fixed point of a function \( f(x) \). Put \( \lambda = f'(x_0) \). The point \( x_0 \) is attractive if \( 0 < |\lambda|_p < 1 \), indifferent if \( |\lambda|_p = 1 \), and repelling if \( |\lambda|_p > 1 \).
The ball $U_r(x_0)$ (contained in $V$) is said to be a Siegel disk if each sphere $S_\rho(x_0)$, $\rho < r$ is an invariant sphere of $f(x)$, i.e. if $x \in S_\rho(x_0)$ then all iterated points $f^n(x) \in S_\rho(x_0)$ for all $n = 1, 2, \ldots$. The union of all Siegel disks with the center at $x_0$ is said to a maximum Siegel disk and is denoted by $SI(x_0)$.

Let $f : U \to U$ and $g : V \to V$ be two maps. $f$ and $g$ are said to be topologically conjugate if there exists a homeomorphism $h : U \to V$ such that, $h \circ f = g \circ h$. The homeomorphism $h$ is called a topological conjugacy. Mappings that are topologically conjugate are completely equivalent in terms of their dynamics. For example, if $f$ is topologically conjugate to $g$ via $h$, and $x_0$ is a fixed point for $f$, then $h(x_0)$ is fixed for $g$. Indeed, $h(x_0) = hf(x_0) = gh(x_0)$.

2. $(1, 2)$-Rational $p$-adic dynamical systems

In this paper we consider the dynamical system associated with the $(1, 2)$-rational function $f : \mathbb{C}_p \to \mathbb{C}_p$ defined by

$$f(x) = \frac{ax + b}{x^2 + cx + d}, \quad a \neq 0, \quad a, b, c, d \in \mathbb{C}_p. \quad (2.1)$$

where $x \neq x_{1,2} = \frac{-c \pm \sqrt{c^2 - 4d}}{2}$.

We can see that for $(1, 2)$-rational function (2.1) the equation $f(x) = x$ for fixed points is equivalent to the equation

$$x^3 + cx^2 + (d - a)x - b = 0. \quad (2.2)$$

Since $\mathbb{C}_p$ is algebraic closed the equation (2.2) may have three solutions with one of the following:

(i). One solution having multiplicity three;
(ii). Two solutions, one of which has multiplicity two;
(iii). Three distinct solutions.

In this paper we investigate the behavior of trajectories of an arbitrary $(1, 2)$-rational dynamical system in complex $p$-adic field $\mathbb{C}_p$ when there is unique fixed point for $f$, i.e., we consider the case (i).

The following lemma gives a criterion on parameters of the function (2.1) guaranteeing the uniqueness of its fixed point.

**Lemma 1.** The function (2.1) has unique fixed point if and only if

$$-\frac{c}{3} = -\sqrt[3]{\frac{d - a}{3}} = \sqrt[3]{b} \quad \text{or} \quad -\frac{c}{3} = \sqrt[3]{\frac{d - a}{3}} = \sqrt[3]{b}. \quad (2.3)$$

**Proof.** Necessariness. Assume (2.1) has a unique fixed point, say $x_0$. Then the LHS of equation (2.2) (which is equivalent to $f(x) = x$) can be written as

$$x^3 + cx^2 + (d - a)x - b = (x - x_0)^3.$$

Consequently,

$$\begin{cases} 3x_0 = -c \\ 3x_0^2 = d - a \\ x_0^3 = b \end{cases}$$
which gives
\[ x_0 = \frac{-c}{3} = \pm \sqrt{\frac{d-a}{3}} = \sqrt{b}. \]

**Sufficiency.** Assume the coefficients of (2.1) satisfy (2.3). Then it can be written as
\[
f(x) = \frac{ax - \frac{c^3}{27}}{x^2 + cx + \frac{c^2}{3} + a}, \quad a \neq 0, \quad a, c \in C_p.
\]

(2.4)

In this case the equation \( f(x) = x \) can be written as
\[
(x + \frac{c}{3})^3 = 0.
\]

Thus \( f(x) \) has unique fixed point \( x_0 = -\frac{c}{3}. \) □

It follows from this lemma that if the function (2.1) has unique fixed point then it has the form (2.4). Thus we study the dynamical system \( (f, C_p) \) with \( f \) given by (2.4).

Let homeomorphism \( h : C_p \to C_p \) is defined by \( x = h(t) = t + x_0. \) So \( h^{-1}(x) = x - x_0. \) Note that, the function \( f \) is topologically conjugate to function \( h^{-1} \circ f \circ h. \) We have
\[
(h^{-1} \circ f \circ h)(t) = \frac{c^2 t^2 + (a + \frac{c^2}{3})t}{t^2 + \frac{c^2}{3}t + a + \frac{c^2}{3}}.
\]

(2.5)

In (2.5), the case \( c \neq 0 \) is studied in [4].

Thus in this paper we consider the case \( c = 0 \) in (2.5). Therefore, in this paper we study dynamical systems of the following function
\[
f(x) = \frac{ax}{x^2 + a}, \quad a \neq 0, \quad a \in C_p.
\]

(2.6)

where \( x \neq \hat{x}_1, \hat{x}_2 = \pm \sqrt{-a}. \)

It is easy to see that function (2.6) has unique fixed point \( x_0 = 0. \) For (2.6) we have
\[
f'(x_0) = f'(0) = 1,
\]
i.e., the point \( x_0 \) is an indifferent point for (2.6).

It follows from (2.1) that
\[
|f(x) - x_0|_p = |f(x)|_p = |x|_p \cdot \frac{|a|_p}{|a^2 + a|_p}.
\]

(2.7)

Letting \( A = |a|_p, \) we have
\[
|f(x)|_p = |x|_p \cdot \frac{A}{|x^2 + a|_p}.
\]

Denote:
\[
P = \{ x \in C_p : \exists n \in \mathbb{N} \cup \{0\}, f^n(x) \in \{ \hat{x}_1, \hat{x}_2 \} \}.
\]

(2.8)
Let the function $\varphi_A : [0, +\infty) \to [0, +\infty)$ be defined by:

$$\varphi_A(r) = \begin{cases} r, & \text{if } r < \sqrt{A} \\ A^*, & \text{if } r = \sqrt{A} \\ \frac{A}{r}, & \text{if } r > \sqrt{A} \end{cases}$$

where $A^*$ is a positive number such that $A^* \geq \sqrt{A}$.

**Lemma 2.** If $x \in S_r(x_0)$, then the following holds for the function (2.6):

$$|f^n(x)|_p = \varphi_A^n(r).$$

We now see that the real dynamical system compiled from $\varphi^n_A$ is directly related to the $p$-adic dynamical system compiled by the function (2.6).

**Lemma 3.** The function $\varphi_A$ has the following properties

1. $\text{Fix}(\varphi_A) = \{ r : 0 \leq r < \sqrt{A} \} \cup \{ \sqrt{A} : \text{if } A^* = \sqrt{A} \}$
2. If $r > \sqrt{A}$ then $\varphi^n_A(r) = \frac{A}{r}$ for all $n \geq 1$.
3. If $r = \sqrt{A}$ and $A^* > \sqrt{A}$, then $\varphi_A^n(\sqrt{A}) = \frac{A}{r}$ for all $n \geq 2$.

**Proof.**

1) This is a simple observation of the equation $\varphi_A(r) = r$.

2) If $r > \sqrt{A}$, then by definition of function $\varphi_A$, we have

$$\varphi_A(r) = \frac{A}{r}. \quad (2.9)$$

Moreover, $r > \sqrt{A}$, $\frac{A}{r} < \sqrt{A}$. By part 1 of this lemma, $\frac{A}{r}$ is to be considered a fixed point for $\varphi_A(r)$. Furthermore, $\varphi^n_A = \frac{A}{r}$ for all $n \geq 1$.

3) The proof of part 3 follows part 2 of this lemma.

From this lemma it follows that

$$\lim_{n \to \infty} \varphi^n(r) = \begin{cases} r, & 0 \leq r < \sqrt{A} \\ \sqrt{A}, & r = \sqrt{A}, \ A^* = \sqrt{A} \\ \frac{A}{r}, & r = \sqrt{A}, \ A^* > \sqrt{A} \\ \frac{A}{r}, & r > \sqrt{A} \end{cases},$$

for any $r \geq 0$.

Denote:

$$A^*(x) = |f(x)|_p, \text{ if } x \in S_{\sqrt{A}}(0).$$

By the applying Lemma 2 and 3, and formula (2.10) we get the following properties of the $p$-adic dynamical system compiled by the function (2.6).
Theorem 1. The $p$-adic dynamical system is generated by the function (2.6) has the following properties:

1. 1.1) $SI(x_0) = U_{\sqrt{A}}(0)$.
1.2) $\mathcal{P} \subset S_{\sqrt{A}}(0)$.
2. If $r > \sqrt{A}$ and $x \in S_r(0)$, then
   $$f^n(x) \in S_{\frac{A}{A^{\ast}(x)}}(0) \text{ for all } n \geq 1.$$
3. Let $x \in S_{\sqrt{A}}(0) \setminus \mathcal{P}$.
   3.1) If $A^{\ast}(x) = \sqrt{A}$, then
       $$f(x) \in S_{\sqrt{A}}(0).$$
   3.2) If $A^{\ast}(x) > \sqrt{A}$, then
       $$f^n(x) \in S_{\frac{A}{A^{\ast}(x)}}(0), \ \forall \ n \geq 2.$$

Proof. 1.1 By lemma [2] and part 1 of Lemma [3] sphere $S_r(0)$ is invariant for $f$ if and only if $r < \sqrt{A}$. Consequently, $SI(x_0) = U_{\sqrt{A}}(0)$. 
1.2 Note that $|\hat{x}_1| = |\hat{x}_2| = \sqrt{A}$, i.e., $\{\hat{x}_1, \hat{x}_2\} \subset S_{\sqrt{A}}(0)$. By part 1.1, and 2 of this theorem if $x \in S_r(0), r \neq \sqrt{A}$, then $f(x) \notin S_{\sqrt{A}}(0)$. By definition of set $\mathcal{P}$, we can conclude that $\mathcal{P} \subset S_{\sqrt{A}}(0)$.
2. The proof of part 2 easily follows of Lemma [2] and part 2 of Lemma [3].
3.1 If $x \in S_{\sqrt{A}}(0) \setminus \mathcal{P}$ and $A^{\ast} = \sqrt{A}$, we have $|f(x)|_p = \sqrt{A}$, i.e., $f(x) \in S_{\sqrt{A}}(0)$.
3.2 If $A^{\ast} > \sqrt{A}$, then by part 2 of this theorem we have $f^n(x) \in S_{\frac{A}{A^{\ast}(x)}}(0)$, for all $n \geq 2$. \hfill \square

3. Ergodicity properties of the dynamical system $f(x) = \frac{ax}{x^2 + a}$ in $\mathbb{Q}_p$.

In this section we assume that $\sqrt{-a}$ exists in $\mathbb{Q}_p$. Consider the dynamical system (2.6) in $\mathbb{Q}_p$.

Define the following set:
$$\Delta = \{ r : 0 < r < \sqrt{A} \}.$$ 

From previous section we have

Corollary 1. The sphere $S_r(0)$ is invariant for $f$ if and only if $r \in \Delta$.

In this section we study ergodicity of dynamical system on each invariant sphere.

Lemma 4. For every closed ball $V_{\rho}(c) \subset S_r(0), r \in \Delta$, the following is sufficient to say

$$f((V_{\rho}(c)) = V_{\rho}(f(c)). \quad (3.1)$$

Proof. By inclusion of $V_{\rho}(c) \subset S_r(0)$, we have $|c|_p = r$. 

Let \( x \in V_p(c) \), i.e. \(|x - c|_p \leq \rho\), then

\[
|f(x) - f(c)|_p = |x - c|_p \cdot \frac{|a|_p \cdot |a - xc|_p}{(x^2 + a)(c^2 + a)} \tag{3.2}
\]

\(|x \cdot c|_p = r^2\), since \( x \in V_p(c) \subset S_r(0) \). Moreover, \(|a|_p = A\). If \( r \in \Delta \) i.e. \( r < \sqrt{A} \), then \(|x^2|_p = |x \cdot c|_p = r^2 < A = |a|_p\). By the equality of (3.2), \(|f(x) - f(c)|_p = |x - c|_p \leq \rho\). \( \square \)

**Lemma 5.** If \( c \in S_r(0), \ r \in \Delta, \ then \ |f(c) - c|_p = \frac{r^3}{A} \).

**Proof.** This follows from the following equality,

\[
|f(c) - c|_p = \left|\frac{-e^3}{c^2 + a}\right|_p = \frac{|c^3|_p}{|c^2 + a|_p} = \frac{r^3}{A}, \text{ because } r^2 < A. \tag{3.3}
\]

By Lemma 5 we have that \(|f(c) - c|_p\) relies on \( r\), but does not rely on \( c \in S_r(0)\), therefore we define \( \rho(r) = |f(c) - c|_p\), if \( c \in S_r(0)\).

The following theorems and respective proofs can be reviewed upon in the references as they are similar to the results of paper [4].

**Theorem 2.** If \( c \in S_r(0), \ r \in \Delta \) then,

1. For any \( n \geq 1 \) the following equality holds \(|f^{n+1}(c) - f^n(c)|_p = \rho(r)\).
2. \( f(V_{\rho(r)}(c)) = V_{\rho(r)}(c)\).
3. If for some \( \theta > 0 \), the ball \( V_\theta(c) \subset S_r(0) \) is an invariant for \( f \), then \( \theta \geq \rho(r)\).

For each \( r \in \Delta \), let us consider a measurable space \((S_r(0), \mathcal{B})\), in this case, \( \mathcal{B} \) is the algebra generated by the closed subsets of \( S_r(0) \). Each element of \( \mathcal{B} \) is a union of some ball \( V_p(c)\).

A measure \( \mu : \mathcal{B} \to \mathbb{R} \) is considered to be Haar Measure if it is defined by \( \mu(V_p(c)) = \rho \). Notice that \( S_r(a) = V_r(a)/V_p(a) \), where \( p \) is a prime.

Consider the normalized (probability) Haar measure:

\[
\mu(V_p(c)) = \frac{\mu(V_p(c))}{\mu(S_r(0))} = \frac{pp}{r(p-1)}.
\]

By Lemma 4 we can conclude that \( f \) preserves the measure \( \mu \), i.e.,

\[
\mu(f(V_p(c))) = \mu(V_p(c)).
\]

The dynamical system, \((X, T, \mu)\), where \( T : X \to X \), is a measure preserving transformation whereas \( \mu \) us a probability measure. The dynamical system is ergodic, if for every invariant set \( V \) we have \( \mu(V) = 0 \) or \( \mu(V) = 1 \).

**Theorem 3.** The \( p \)-adic dynamical system \((S_r(0), f, \mu)\) is not ergodic for all prime \( p \) and all \( r \in \Delta \).
Proof. Consider \((S_r(0), f, \mu)\) a dynamical system where \(\mu\)-normalizer Haar measure. Note that \(V_{\rho(r)}(c) \subset S_r(0)\) is a minimal invariant ball. By Lemma \(\text{Lemma 5}\) we have the following:

\[
\mu(V_{\rho(r)}(c)) = \frac{pp(r)}{r(p-1)} = \frac{p^2}{r(p-1)} = \frac{pr^2}{A(p-1)}.
\]

If \(r \in \Delta\), then \(r < \sqrt{A}\), i.e., \(pr \leq \sqrt{A}\). By this inequality, we have

\[
\mu(V_{\rho(r)}(c)) \leq \frac{1}{p(p-1)} < 1,
\]

for all prime \(p\).

\[\square\]

4. 2-Periodic Points

In this section, we will be interested in finding periodic points of function \((2.6)\).

Proposition 1. If function \((2.6)\) has \(k\)-periodic points \(\{y_0 \rightarrow y_1 \rightarrow ... \rightarrow y_k \rightarrow y_0\}\), then

\[
|y_0|_p = |y_1|_p = ... = |y_k|_p \leq \sqrt{A}.
\]

Proof. Let function \((2.6)\) have \(k\)-periodic points \(\{y_0 \rightarrow y_1 \rightarrow ... \rightarrow y_k \rightarrow y_0\}\).

Assume that \(|y_i|_p > \sqrt{A}\) for some \(i \in \{0, 1, ..., k\}\). Then by part 2 of Theorem \(\text{Theorem 1}\) we have \(y_{i+1} = f(y_i) \in U_{\sqrt{A}}(0)\), i.e., \(|y_{i+1}|_p = r < \sqrt{A}\) and \(S_r(0)\) is invariant sphere of function \((2.6)\). From this

\[
|y_{i+1}|_p = |y_{i+2}|_p = ... = |y_k|_p = |y_0|_p = ... = |y_i|_p < \sqrt{A}.
\]

This contradicts our assumption. Consequently \(|y_i|_p \leq \sqrt{A}\) for any \(i \in \{0, 1, ..., k\}\).

If \(|y_i|_p = r < \sqrt{A}\), then \(|y_0|_p = |y_1|_p = ... = |y_k|_p = r < \sqrt{A}\), because \(S_r(0)\) is invariant for all \(r < \sqrt{A}\).

If \(|y_i|_p = r = \sqrt{A}\), then by the above-mentioned results \(|y_{i+1}|_p \neq \sqrt{A}\) and \(|y_{i+1}|_p \neq \sqrt{A}\). Consequently, \(|y_0|_p = |y_1|_p = ... = |y_k|_p = \sqrt{A}\).

\[\square\]

Let us consider 2-periodic points, i.e. consider the equation

\[
g(x) \equiv f(f(x)) = \frac{ax^3 + a^2x}{x^4 + 3ax^2 + a^2} = x. \tag{4.1}
\]

This equation is equivalent to \(x^2 + 2a = 0\), hence two solutions, \(t_{1,2} = \pm \sqrt{-2a}\). For these points we have

\[
|t_1|_p = |t_2|_p = \begin{cases} 
\sqrt{A}, & \text{if } p \neq 2 \\
\sqrt{\frac{A}{2}}, & \text{if } p = 2.
\end{cases}
\]

It is a coincidence that \(g'(t_1) = g'(t_2) = 9\), i.e. the value does not rely on the parameter \(a\). Therefore we have

\[
|g'(t_1)|_p = |g'(t_2)|_p = \begin{cases} 
1, & \text{if } p \neq 3 \\
\frac{1}{3}, & \text{if } p = 3.
\end{cases}
\]
Theorem 5. Let us define the following:
\[ \mathcal{P}_2 = \{ x \in \mathbb{C}_p : \exists n \in \mathbb{N} \text{ such that } f^n(x) \in \{ \hat{x}_{1,2}, \hat{x}_{1,2,3,4} \} \}. \]

4.1. Case \( p = 2 \). In this case we have \( |t_1|_2 = |t_2|_2 = \sqrt{\frac{A}{2}} < \sqrt{A} \) and \( |t_2 - t_1|_2 = \frac{\sqrt{A}}{2\sqrt{2}} \). By Lemma 5 and part 2 of Theorem 2 we have
\[ \rho \left( \sqrt{\frac{A}{2}} \right) = |f(t_1) - t_1|_2 = \frac{\sqrt{A}}{2\sqrt{2}} \quad \text{and} \quad f(V_{\sqrt{A}/2}(t_1)) = V_{\sqrt{A}/2}(t_1). \]

In this case each fixed point \( t_1, t_2 \) of \( g \) is an indifferent point and is the center of a Siegel disk.

Theorem 4. If \( p = 2 \) then \( f(S_r(t_1) \setminus \mathcal{P}_2) \subseteq S_r(t_2), f(S_r(t_2) \setminus \mathcal{P}_2) \subseteq S_r(t_1) \), for any \( 0 < r \leq \frac{\sqrt{A}}{2\sqrt{2}} \).

Proof. We shall use the following equalities:
\[ f(t_1) = t_2, \quad f(t_2) = t_1. \]

Let \( t_1 = \sqrt{-2a} \) and \( x \in S_r(t_1) \setminus \mathcal{P}_2 \subset V_{\sqrt{A}/2}(t_1) \), i.e., \( |x - t_1|_2 = r \leq \frac{\sqrt{A}}{2\sqrt{2}} \). We have
\[ |f(x) - t_2|_2 = |f(x) - f(t_1)|_2 = r \left| \frac{-3a + \sqrt{-2a}(x - t_1)}{|x - t_1 + \sqrt{-a}(\sqrt{2} - 1)||x - t_1 + \sqrt{-a}(\sqrt{2} + 1)|} \right|_2. \] (4.2)

In RHS of equality (4.2) we have \( |-3a|_2 = A, |\sqrt{-2a}(x - t_1)|_2 = r \frac{\sqrt{A}}{2} < A \) and \( |\sqrt{2} - 1|_2 = |\sqrt{2} + 1|_2 = 1 \). So \( |f(x) - t_2|_2 = |f(x) - f(t_1)|_2 = r \), i.e., \( f(x) \in S_r(t_2) \).

If \( x \in S_r(t_2) \setminus \mathcal{P}_2 \subset V_{\sqrt{A}/2}(t_1), \) then we have
\[ |f(x) - t_1|_2 = |f(x) - f(t_2)|_2 = r \left| \frac{-3a - \sqrt{-2a}(x - t_2)}{|x - t_2 + \sqrt{-a}(\sqrt{2} - 1)||x - t_2 + \sqrt{-a}(\sqrt{2} + 1)|} \right|_2. \] (4.3)

Consequently, \( |f(x) - t_1|_2 = |f(x) - f(t_2)|_2 = r \), i.e., \( f(x) \in S_r(t_1) \). \( \square \)

4.2. Case \( p = 3 \). In this case \( |t_1|_3 = |t_2|_3 = |t_2 - t_1|_3 = \sqrt{A} \) and each fixed point \( t_1, t_2 \) of \( g \) is an attractive point of \( g \).

Theorem 5. If \( p = 3 \) and \( r < \sqrt{A} \), then
\( a) \) For any \( x \in S_r(t_1) \setminus \mathcal{P}_2, \lim_{n \to \infty} f^{2n}(x) = t_1 \) and \( \lim_{n \to \infty} f^{2n+1}(x) = t_2 \).
\( b) \) For any \( x \in S_r(t_2) \setminus \mathcal{P}_2, \lim_{n \to \infty} f^{2n}(x) = t_2 \) and \( \lim_{n \to \infty} f^{2n+1}(x) = t_1 \).

Proof. Let \( S_r(t_1) \subset S_{\sqrt{A}/2}(0), r < \sqrt{A} \) and \( x \in S_r(t_1) \setminus \mathcal{P}_2 \), i.e., \( |x - t_1|_3 = r \). We have
\[ |f(x) - t_2|_3 = |f(x) - f(t_1)|_3 = |x - t_1|_3 \cdot \frac{|-3a + \sqrt{-2a}(x - t_1)|_3}{|x - t_1 + \sqrt{-a}(\sqrt{2} - 1)|_3|x - t_1 + \sqrt{-a}(\sqrt{2} + 1)|_3}. \]
By this equality

\[ |f(x) - t_2|_3 = \phi(r) = \begin{cases} 
\frac{r^2}{\sqrt{A}}, & \text{if } \frac{\sqrt{A}}{3} < r < \sqrt{A}, \\
\frac{r^2}{9}, & \text{if } r = \frac{\sqrt{A}}{3}, \\
\frac{r^2}{3}, & \text{if } r < \frac{\sqrt{A}}{3}.
\end{cases} \] (4.4)

For \( f^2(x) \) we have

\[ |f^2(x) - t_1|_3 = |f^2(x) - f^2(t_1)|_3 = |f(x) - t_2|_3 \cdot \frac{|-3a + \sqrt{-2a(f(x) - t_2)|_3}}{|f(x) - t_2 + \sqrt{-a(2 - 1)|_3}} = \frac{(\phi(r))^2}{\frac{\sqrt{A}}{3}}, \]

\[ \text{if } \frac{\sqrt{A}}{3} < \phi(r) < \sqrt{A}, \]

\[ \leq \frac{\sqrt{A}}{9}, \text{ if } \phi(r) = \frac{\sqrt{A}}{3}, \]

\[ \phi(r), \text{ if } \phi(r) < \frac{\sqrt{A}}{3}. \]

Iterating this argument we obtain the following formulas for \( x \in S_r(t_1) \setminus P_2 \):

\[ |f^{2n}(x) - t_1|_3 = \phi^{2n}(r), \quad |f^{2n+1}(x) - t_2|_3 = \phi^{2n+1}(r). \] (4.5)

Thus the dynamics of the radius \( r \) of the spheres is given by the function \( \phi : [0, \sqrt{A}) \rightarrow [0, \sqrt{A}) \), which is defined in formula (4.4). The following properties of \( \phi \) are obvious:

1. The set of fixed points of \( \phi(r) \) is \( \text{Fix}(\phi) = \{0\} \};
2. The fixed point \( r = 0 \) is attractive with basin of attraction \([0, \sqrt{A})\), independently

on the value \( \phi\left(\frac{\sqrt{A}}{3}\right) \leq \frac{\sqrt{A}}{9} \).

Using (4.5) it is easy to see that the assertion (a) and (b) follows from property 2. \( \square \)

4.3. Case \( p \geq 5 \). In this case we have \( |t_1|_p = |t_2|_p = \sqrt{A} \) and each fixed point \( t_1, t_2 \) of \( g \) is an indifferent point and is the center of a Siegel disk.

**Theorem 6.** If \( p \geq 5 \) then \( f(S_r(t_1) \setminus P_2) \subseteq S_r(t_2), f(S_r(t_2) \setminus P_2) \subseteq S_r(t_1), \) for any \( 0 < r < \sqrt{A} \).

**Proof.** Let \( x \in S_r(t_1) \setminus P_2 \subseteq V_{\sqrt{A}}(t_1) \), i.e., \( |x - t_1|_p = r < \sqrt{A} \). In RHS of equality (4.2) we have \( | - 3a|_p = A, |\sqrt{-2a(x - t_1)}|_p = r\sqrt{A} \leq A \) and \( |\sqrt{2} - 1|_p = |\sqrt{2} + 1|_p = 1 \). So \( |f(x) - t_2|_p = |f(x) - f(t_1)|_p = r \), i.e., \( f(x) \in S_r(t_2) \).

If \( x \in S_r(t_2) \setminus P_2 \subseteq V_{\sqrt{A}}(t_1) \), then we have \( |f(x) - t_1|_p = |f(x) - f(t_2)|_p = r \), i.e., \( f(x) \in S_r(t_1) \).

Consequently, \( f(S_r(t_1) \setminus P_2) \subseteq S_r(t_2) \) and \( f(S_r(t_2) \setminus P_2) \subseteq S_r(t_1), \) for any \( 0 < r < \sqrt{A} \). \( \square \)

**Acknowledgements**

The third author was supported by the National Science Foundation, grant number NSF HRD 1302873.
References

[1] S. Albeverio, U.A. Rozikov, I.A. Sattarov. p-adic (2, 1)-rational dynamical systems. *Jour. Math. Anal. Appl.* 398(2) (2013), 553–566.

[2] N. Koblitz, *p-adic numbers, p-adic analysis and zeta-function* Springer, Berlin, 1977.

[3] U.A. Rozikov, I.A. Sattarov. On a non-linear p-adic dynamical system. *p-Adic Numbers, Ultrametric Analysis and Applications*, 6(1) (2014), 53–64.

[4] U.A. Rozikov, I.A. Sattarov. p-adic dynamical systems of (2, 2)-rational functions with unique fixed point. *Chaos, Solitons and Fractals*, 105 (2017), 260–270.

[5] I.A. Sattarov. p-adic (3, 2)-rational dynamical systems. *p-Adic Numbers, Ultrametric Analysis and Applications*, 7(1) (2015), 39–55.

[6] P. Walters, *An introduction to ergodic theory*. Springer, Berlin-Heidelberg-New York, (1982).

U. A. Rozikov, INSTITUTE OF MATHEMATICS, 81, MIRZO ULG’BEK STR., 100125, TASHKENT, UZBEKISTAN.

E-mail address: rozikovu@yandex.ru

I. A. Sattarov, INSTITUTE OF MATHEMATICS, 81, MIRZO ULG’BEK STR., 100125, TASHKENT, UZBEKISTAN.

E-mail address: sattarovi-a@yandex.ru

S. Yam, CALIFORNIA STATE UNIVERSITY, MONTEREY BAY, 100 CAMPUS CENTER, SEASIDE, CALIFORNIA, 93955 USA

E-mail address: syam@csumb.edu