Kinetic modeling of 3D equilibria in a tokamak

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Abstract. External resonant magnetic perturbations (RMPs) can modify the magnetic topology in a tokamak. In this case the magnetic field cannot generally be described by ideal MHD equilibrium equations in the vicinity of resonant magnetic surfaces where parallel and perpendicular relaxation timescales are comparable. Usually, resistive MHD models are used to describe these regions. In the present work, a kinetic model is used for this purpose. Within this model, plasma response, current and charge density are computed with help of a Monte Carlo method, where guiding center orbit equations are solved using a semianalytical geometrical integrator. Besides its higher efficiency in comparison to usual integrators this method is not sensitive to noise in field quantities. The computed charges and currents are used to calculate the electromagnetic field with help of a finite element solver. A preconditioned iterative scheme is applied to search for a self-consistent solution. The discussed method is aimed at the nonlinear kinetic description of RMPs in experiments on Edge Localized Mode (ELM) mitigation by external perturbation coil systems without simplification of the device geometry.

1. Introduction
Resonant magnetic perturbations (RMPs) is presently one of the prospective tools for the control of edge localized mode (ELM) in a future tokamak-reactor [1]. Together with other external magnetic perturbations which violate the toroidal symmetry of a tokamak magnetic field such as toroidal field ripples and error fields, these perturbations may have a significant effect on plasma parameters, in particular, on plasma toroidal rotation velocity. It should be noted that, in most of the plasma volume away from resonant flux surfaces, RMPs can be well described by ideal MHD theory. In particular, if RMPs are strongly shielded by plasma response currents, ideal MHD computations using 3D equilibrium codes [2, 3] agree well with models that take non-ideal plasma response into account (e.g., resistive MHD models) in most of the plasma except for narrow resonant layers around resonant flux surfaces. This permits, in particular, to use externally perturbed ideal 3D MHD equilibria for computations of the non-resonant toroidal torque (neoclassical toroidal viscosity, NTV) resulting from the violation of the toroidal magnetic field symmetry [4]. At the same time, the resonant torque, which is comparable to NTV but, in contrast to NTV, is localized within resonant layers, is much more sensitive to the details of the electromagnetic field distribution in these layers and cannot be treated within ideal MHD. The same is true also for modification in the resonant layers of temperature and density...
profiles by RMPs that, together with the change in toroidal rotation (or, equivalently, of the 
equilibrium radial electric field), affects the RMP shielding by the plasma. The general trend of 
this modification is to bring plasma parameter profiles to the penetration point where the main 
shielding by electron component disappears (in the two fluid MHD theory this point corresponds 
to the zero of unperturbed perpendicular electron fluid velocity at the resonant flux surface [5, 6]), 
and, if RMP amplitude is above a certain threshold value, a bifurcation occurs [7] which results 
in the equilibrium state with nearly unshielded perturbation field and significant modification 
of the magnetic field topology (creation of islands). Clearly, the threshold RMP amplitude 
value predicted by the theory is sensitive to the accuracy of the non-ideal effects modeling, and 
often the very question whether RMPs are shielded or not in a particular experiment cannot be 
definitely answered taken the uncertainty of existing numerical models and the fact that typical 
RMP amplitudes are of the same magnitude order as the threshold amplitude values (see, e.g., 
Ref. [8]).

Presently the most advanced numerical models in terms of treatment of realistic 3D geometries 
and nonlinear effects are based on the MHD theory [9, 10, 11]. This theory, however, is not 
ever universally applicable to modern and, especially, future reactor scale devices where plasma 
collisionality may be very low. In this case MHD models using the realistic values of anomalous 
diffusion and shear viscosity often predict the linear width of resonant layers much below the 
ion Larmor radius in contrast to the kinetic model where these scales are comparable [12]. 
Besides that, collisionless effects in wave-particle interaction start to be important in resonant 
layers. A particular consequence of that is the shift of the field penetration point away 
from the zero of perpendicular electron fluid velocity, as predicted by the kinetic theory in 
presence of electron temperature gradient [8]. The magnitude of this shift depends on plasma 
collisionality. Finally, the diffusive ansatz in the description of anomalous transport employed 
in the MHD approach fails if the resonant layer width becomes comparable or smaller than 
the perpendicular correlation length of the turbulence (and that is not an exceptional case). 
Therefore, development of the 3D kinetic model of RMP interaction with the plasma which can 
address the above issues remains to be of interest.

An effort in this direction has been undertaken in Ref. [13] where parallel plasma current in 
a given perturbed field has been computed using a full-\(f\) guiding center PIC code, this current 
has been used then to compute the updated perturbed field (poloidal magnetic flux) and the 
convergence of this iterative procedure has been enabled by using the damped iteration scheme. 
The approach of the present report is based on the same idea of using the Monte Carlo method 
for evaluation of plasma response current combined with iterative account of plasma response 
field generated by this current. The main difference from Ref. [13] consists in using the \(\delta f\) 
method, which is better suited to weakly perturbed magnetic fields where it strongly reduces the 
requirements to the statistics as compared to the full-\(f\) method, and a preconditioned iteration 
scheme, which enables the convergence in cases where direct iterations are strongly linearly 
divergent, which is actually the case here. For those reasons the approach presented here is 
expected to be more efficient in terms of CPU cost.

The main purpose of the present report is to demonstrate the feasibility of such an approach 
and its relatively low CPU cost, therefore, the physical model has been strongly simplified by 
restricting the consideration only to the linear case (infinitesimal perturbations) and by treating 
the perturbation field in the leading order over the Larmor radius. In addition, modification of 
electrostatic field by the perturbations has been ignored here for simplicity. These and other 
problem simplifications do not principally affect the linear convergence of iterations and the 
CPU cost, which are the main points of interest in this initial study.
2. Theoretical background

Presenting the total tokamak magnetic field as \( \mathbf{B} = \mathbf{B}_0 + \delta \mathbf{B} \) where \( \mathbf{B}_0 \) is the axisymmetric equilibrium magnetic field and \( \delta \mathbf{B} \) is the non-axisymmetric perturbation field fixed by the condition that the toroidal average of its components in cylindrical coordinates, \( x^k = (R, \varphi, Z) \), associated with the main torus axis is zero, the later field is convenient to split into the known vacuum field \( \delta \mathbf{B}_v \) and the unknown plasma response field \( \delta \mathbf{B}_p \). Presenting the non-axisymmetric part of the plasma current as \( \delta j = \hat{P} \delta \mathbf{B} \) where \( \hat{P} \) is the (linear in our example) integral operator corresponding to the solution of the kinetic equation, and presenting the solution to Maxwell equations in a similar form, \( \delta \mathbf{B}_p = \hat{M} \delta \mathbf{j} \), the integral equation for the plasma response field, \( \delta \mathbf{B}_p \), is

\[
\delta \mathbf{B}_p = \hat{K} \delta \mathbf{B}_p + \delta \mathbf{B}_{pv},
\]

where \( \hat{K} = \hat{M} \hat{P} \) and \( \delta \mathbf{B}_{pv} = \hat{K} \delta \mathbf{B}_v \). Eq. (1) formally defines direct iterations, which are divergent if the spectrum \( \hat{K} \) contains eigenvalues of modulus larger than one. As this is the case in the present problem, preconditioned iterations,

\[
\delta \mathbf{B}_p^{(k+1)} = \delta \mathbf{B}_p^{(k)} - \hat{\Pi}^{-1} \left( (\hat{K} - \hat{I}) \delta \mathbf{B}_p^{(k)} + \delta \mathbf{B}_{pv} \right),
\]

are used instead, where \( \hat{I} \) is unity and the preconditioner \( \hat{\Pi} \) is constructed from an approximation of the unstable part of the spectrum of \( \hat{K} \) by a Krylov method (see, e.g., Ref. [14]). This method follows a similar approach as described in Ref. [15]. In addition to stabilization, it can also provide convergence acceleration and is used for this purpose in the NEO-2 code [4].

2.1. Non-axisymmetric plasma response current

In case of small enough perturbations and negligible finite Larmor radius effects (which is well fulfilled for electrons mainly responsible for the RMP shielding) the gyrokinetic equation linearized with respect to the perturbation field is

\[
\frac{\partial \delta f}{\partial t} + V_0^k \frac{\partial \delta f}{\partial z^k} - \hat{L}_C \delta f = -\delta V^k \frac{\partial f_0}{\partial z^k},
\]

where \( \delta f = f - f_0 \) is the perturbation of axisymmetric steady state distribution function \( f_0 \), \( \delta V^k = V^k - V_0^k \) is the non-axisymmetric perturbation of the phase space velocity and \( \hat{L}_C \) is the linearized collision operator (in this case a Fokker-Planck operator described in Ref. [16]). Using for \( f_0 \) a local Maxwellian, \( f_0 = f_M(\psi, W) \) where \( \psi \) is the unperturbed poloidal magnetic flux and \( W \) is the total particle energy (such an \( f_0 \) model approximates the core plasma but is not suited for the scrape-off layer and private flux region where, in turn, plasma response currents are relatively small), and taking \( \delta V^k \) in leading order over the Larmor radius (see, e.g., Ref. [8]), the r.h.s. of (3) is approximated as

\[
-\delta V^k \frac{\partial f_0}{\partial z^k} \approx \dot{\psi}_a f_M, \quad \dot{\psi}_a \equiv -v_\parallel \frac{\delta B^\psi}{B_0} \left( A_1 + A_2 \frac{m_\alpha v^2}{2T_\alpha} \right),
\]

where \( v \) and \( v_\parallel \) velocity module and parallel component, respectively, \( \delta B^\psi = \delta \mathbf{B} \cdot \nabla \psi \) is the normal perturbation magnetic field component, and the thermodynamic forces are

\[
A_1 = \frac{\partial \ln n_\alpha}{\partial \psi} + \frac{e_\alpha}{T_\alpha} \frac{\partial \Phi_0}{\partial \psi} - \frac{3}{2} \frac{\partial \ln T_\alpha}{\partial \psi}, \quad A_2 = \frac{\partial \ln T_\alpha}{\partial \psi}.
\]

Here, \( \Phi_0 \) is the unperturbed electrostatic potential, and \( e_\alpha, m_\alpha, n_\alpha \) and \( T_\alpha \) are \( \alpha \) species charge, mass, density and temperature, respectively. Within a \( \delta f \) Monte Carlo algorithm, the
solution to (3), which determines the non-axisymmetric plasma response currents, is given by the expectation value of the test particle distribution function,
\[
\delta f(z) \approx \frac{1}{t_0} \int_0^{t_0} dt \, w \delta (z - Z(t, z_0)) \tag{6}
\]
where \(Z(t, z_0)\) is the stochastic orbit of the unperturbed motion corresponding to the l.h.s. of (3) and originating at the phase space point \(z_0\) being also the random number distributed in accordance with \(f_M\), test particle weight \(w\) is evolving in time in accordance with \(\dot{w} = \dot{w}_s(Z(t, z_0))\), and particle tracing time \(t_0\) is chosen to be much longer than particle correlation time with the perturbation field.

2.2. Non-axisymmetric part of the magnetic field
For the computation of the nonaxisymmetric part of the magnetic field in cylindrical coordinates \(x^k = (R, \varphi, Z)\), it is convenient to use a Fourier expansion of the covariant components of the vector potential in \(\varphi\) with \(\tilde{A}_k = \sum_{n \neq 0} A_{k,n} e^{in\varphi}\). The gauge \(\tilde{A}_\varphi = 0\) is chosen as described e.g. in the appendix of Ref. [12], leading to a particularly simple representation of Fourier amplitudes of the contra-variant components (equal to the physical ones in \(R\) and \(Z\)) of the non-axisymmetric part of the magnetic field perturbation,
\[
\delta B^R_n = \frac{in}{R} A_{Z,n}, \quad \delta B^\varphi_n = (\nabla \times \tilde{A})^\varphi_n = \frac{1}{R} \left( \frac{\partial A_{R,n}}{\partial Z} - \frac{\partial A_{Z,n}}{\partial R} \right), \quad \delta B^Z_n = -\frac{in}{R} A_{R,n}. \tag{7}
\]
From Ampère’s law, a 2-dimensional system of differential equations in \(A_{R,n}, A_{Z,n}\) follows as
\[
-\frac{\partial}{\partial Z} \left( R^2 (\nabla \times \tilde{A})^\varphi_n \right) + \frac{n^2}{R} A_{R,n} = \frac{4\pi}{c} R j^R_n, \quad \frac{\partial}{\partial R} \left( R^2 (\nabla \times \tilde{A})^\varphi_n \right) + \frac{n^2}{R} A_{Z,n} = \frac{4\pi}{c} R j^Z_n, \tag{8}
\]
where \(j^R_n\) and \(j^Z_n\) are Fourier amplitudes of current density components. The equivalent weak form for the homogenous Dirichlet problem (\(\tilde{A}\) orthogonal to the boundary of the computation domain \(\Omega\)) of Eqs. (8) is
\[
\int_\Omega dR dZ R \left( R^2 (\nabla \times \tilde{A})^\varphi_n (\nabla \times W) + \frac{n^2}{R^2} (W R A_{R,n} + W Z A_{Z,n}) \right) \tag{9}
\]
\[
= \frac{4\pi}{c} \int_\Omega dR dZ R \left( W R j^R_n + W Z j^Z_n \right),
\]
where a test function \(W\) with \(W_\varphi = 0\) is chosen from the appropriate function space. If \(\Omega\) is chosen with a large enough current-free region around the actual domain of interest, this description is suited to approximately describe the decay of the magnetic field at infinite distance.

3. Numerical implementation and results
The Monte Carlo procedure for evaluation of the perturbed distribution (6) and corresponding plasma response currents has been realized on the basis of the K2D code [16] which employs a geometrical integrator [17] for advancing the guiding center orbits with a significantly lower CPU cost as compared to usual integration of guiding center equations. Various quantities are discretized in this code on the triangular mesh (left Fig. 1) covering the whole vacuum vessel volume including the divertor region and the scrape-off layer. Eq. (9) is solved by a Galerkin finite element method using 2-dimensional triangular edge elements [18] of lowest order within the package FreeFEM++ [19]. The Dirichlet problem is solved on an extended mesh (right Fig. 1) with a vacuum region around the original domain to emulate the current-free space in the
exterior of the device. While this region is not large enough for a highly accurate representation of vacuum boundary conditions here, it is sufficient for avoiding a strong influence from the domain boundary for the intended proof of concept.

Here, we consider a 3D equilibrium for a case corresponding to a typical medium size tokamak configuration with ELM mitigation coils producing a resonant magnetic perturbation of toroidal mode number $n=2$. Within the linear approximation described by Eq. (3), perturbation fields and corresponding plasma response currents are treated as infinitesimals. Therefore, coupling of toroidal Fourier modes (nonlinear effect) is negligible, and the solution for the currents and fields is fully described by Fourier amplitudes of the toroidal mode $n = 2$ shown in Figs. 2-5. Quasilinear effects of small but finite perturbation field on plasma parameter profiles (see, e.g., Ref. [8]) have been out of interest in the present initial study since they do not affect the linear convergence. Therefore, profiles of temperature, density and electric potential are modeled as linear functions of the poloidal magnetic flux $\psi$ with magnitudes consistent with plasma parameters of present day devices.

The resulting plasma response currents are strongly localized around rational surfaces, which is illustrated in Fig. 2. The currently used mesh is not yet fully capable of resolving all details in the radial direction. However, the results for the magnetic field perturbation (Figs. 3-5) show the typical behavior observed in earlier studies by MHD calculations [20, 21]. While the plasma response leads to strong reduction of the perturbation fields around rational surfaces that are resonant with the perturbation field, amplification is observed outside this region (Figs. 4 and 5). This is in qualitative agreement to the results in Ref. [21].

4. Conclusion and outlook
In this article, the computation of a linear kinetic three-dimensional equilibrium in a tokamak with resonant non-axisymmetrical magnetic field perturbations has been demonstrated. The results were obtained by a kinetic $\delta f$ Monte Carlo code using a geometric orbit integrator in conjunction with a finite element field solver employing two-dimensional edge elements for the computation of non-axisymmetrical toroidal perturbation harmonics. In the iterative scheme resulting from coupling these two models, convergence could be achieved by a preconditioner based on a Krylov subspace method.

The obtained results show typical shielding near resonant rational surfaces and are in qualitative agreement with existing results. Potential features of the presented method that allow to overcome some of the limitations of MHD models, especially in narrow resonant layers
Figure 3. Real part of contra-variant radial component of magnetic field perturbation $\delta B_{m}^{\psi}$ for the vacuum field (left) and for the field including the plasma response (right).

Figure 4. Modules of poloidal harmonics of $\delta B_{m}^{\psi}$ in symmetry flux variables for vacuum field (left) and including plasma response (right) depending on $\rho_{\text{pol}}$ (see Fig. 5). “+” marks resonant surfaces $m + nq = 0$ where shielding occurs.

Figure 5. Module of poloidal modes of the radial contra-variant magnetic field component $\delta B_{m}^{\psi}$ including plasma response (solid) and vacuum field (dashed) over $\rho_{\text{pol}}^{\psi} = (\psi - \psi_{\text{axis}})/(\psi_{\text{sep}} - \psi_{\text{axis}})$. Rational surfaces with $q = 1.5, 2, 2.5, 3, 3.5, 4$ are marked by vertical gray lines. Shielding by plasma currents leads to vanishing perturbation fields at radial positions where $m + nq = 0$.

around rational surfaces, have been pointed out. Since the models do not rely on magnetic coordinates, the approach is well suited for investigations of three-dimensional effects in the region of the divertor and the scrape-off layer.

To obtain quantitative results for comparison to experiments, non-linear modeling will be required. This method will use a non-linear $\delta f$ algorithm combined with a 3D geometric integrator.

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