FINITE GROUPS GENERATED IN LOW REAL CODIMENSION

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Abstract. We study the intersection lattice of the arrangement $A^G$ of subspaces fixed by subgroups of a finite linear group $G$. When $G$ is a reflection group, this arrangement is precisely the hyperplane reflection arrangement of $G$. We generalize the notion of finite reflection groups. We say that a group $G$ is generated (resp. strictly generated) in codimension $k$ if it is generated by its elements that fix point-wise a subspace of codimension at most $k$ (resp. precisely $k$).

If $G$ is generated in codimension two, we show that the intersection lattice of $A^G$ is atomic. We prove that the alternating subgroup $\text{Alt}(W)$ of a reflection group $W$ is strictly generated in codimension two; moreover, the subspace arrangement of $\text{Alt}(W)$ is the truncation at rank two of the reflection arrangement $A^W$.

Further, we compute the intersection lattice of all finite subgroups of $\text{GL}_3(\mathbb{R})$, and moreover, we emphasize the groups that are “minimally generated in real codimension two”, i.e., groups that are strictly generated in codimension two but have no real reflection representations. We also provide several examples of groups generated in higher codimension.

Let $G$ be a finite subgroup of $\text{GL}(V)$, $G \xrightarrow{\rho} \text{GL}(V)$, and consider the quotient map $\pi : V \to V/G =: X$. In this paper, we study the Luna stratification [Lun73] of $V/G$. In this case, the Luna stratification coincides with the isotropy stratification: each stratum $X_H$ of $V/G$ consists of the irreducible component of closed orbits having isotropy group in a specified conjugacy class $H$ of subgroups of $G$.

In other words, if $v$ is a vector of $V$ then

$$\text{Stab}_\rho(v) \overset{\text{def}}{=} \{ g \in G \mid gv = v \}$$

is the isotropy group at $v$. We denote by $V_H^\rho$ the locus of points in $V$ whose isotropy group is precisely $H$

$$V_H^\rho \overset{\text{def}}{=} \{ v \in V \mid H = \text{Stab}_\rho(v) \}$$

and by $V_L^\rho$ the union of $V_L^\rho$ for all $L$ in the conjugacy class $H$, i.e.,

$$V_H^\rho \overset{\text{def}}{=} \bigcup_{L \in H} V_L^\rho.$$ 

If $V_H^\rho \neq \emptyset$, than we say $H$ is a stabilizer subgroup with respect to $\rho$. The quotient map $\pi$ restricted to $V_H^\rho$ is surjective onto $X_H$; further, $\pi \mid_{V_H^\rho} : V_H^\rho \to X_H$ is a principal $(\mathcal{N}_G(H)/H)$-bundle where $\mathcal{N}_G(H)$ is the normalizer of $H$ in $G$, see [VPS89, §6.9] and [Sch80, §1.5].

As $H$ varies among the stabilizer subgroups, the subvarieties $V_H^\rho$ from a stratification. The closure of each stratum $\overline{V_H^\rho}$ is the union of open strata, i.e.,

$$V_H^\rho = \overline{V_H^\rho} = \bigcup_{H' \supset H} V_H^{\rho'}.$$
where the disjoint union is over the stabilizer subgroups $H'$ that contain $H$. It is easy to check that the closed strata are linear subspaces of $V$ and that

$$V^H_\rho = \{ v \in V \mid H \subseteq \text{Stab}_\rho(v) \}.$$  

The main goal of this article is to study the collection $\mathcal{A}_\rho$ of these subspaces.

**Definition.** The arrangement of subspaces of the representation $\rho$ is

$$\mathcal{A}_\rho \equiv \{ V_H^\rho \mid \{ e \} \neq H \text{ is a stabilizer subgroup w.r.t. } \rho \}.$$  

We remark that this arrangement depends by the representation $\rho$ or, equivalently, it is associated to the linear group $G \hookrightarrow \text{GL}(V)$ and not to an abstract group.

The principal stratum $V^e_\rho$ of the Luna stratification is the stratum with trivial isotropy group. It is the complement of the arrangement $\mathcal{A}_\rho$,

$$V^e_\rho = V \setminus \bigcup_{V^H_\rho \in \mathcal{A}_\rho} V^H_\rho.$$  

This is a very well studied object in literature when $\rho$ is a reflection representation [Bes15, BBR02, Bri73, Del72, FT97].

A reflection $r \in \text{GL}(V)$ is a finite order element whose fixed point set $V^{(r)}$ is a hyperplane, i.e., $V^{(r)}$ has codimension one. We say that $W \hookrightarrow \text{GL}(V)$ is a reflection representation if the finite linear subgroup $W$ is generated by its reflections, i.e., $W$ is a reflection subgroup.

The subspace arrangement $\mathcal{A}_W \equiv \mathcal{A}_\rho$ of a reflection representation has been studied extensively (see for instance [DCP11, OT92]); it is a collection of hyperplanes, called the reflection hyperplane arrangement. The maximal subspaces in $\mathcal{A}_W$ are precisely the reflection hyperplanes $V^{(r)}$, for $r$ a reflection in $W$.

In this work, we study a class of finite groups that naturally generalizes the above description. We focus on finite linear groups $G$ which are generated by elements fixing subspaces of codimension one or two.

Before introducing the definition of the groups (strictly) generated in codimension $i$, and before going further with the presentation of our results, let us motivate this effort.

**Intrinsic stratification.** Kuttler and Reichstein [KR08] studied if every automorphism of $V/G$ maps a Luna stratum to another stratum. When this happens, the stratification is said to be intrinsic. In other words, a Luna stratification of $X$ is intrinsic if for every automorphism $f$ of $X$ and every conjugacy class $H$ of stabilizer subgroups in $G$, we have $f(X_H) = X_{H'}$ for some conjugacy class $H'$ of stabilizer subgroups in $G$. Many reflection representations give rise to non-intrinsic stratifications; on the other hand if one removes the reflections, the Luna stratification is intrinsic.

**Theorem.** (cf. [KR08 Theorem 1.1]) Let $G$ be a finite subgroup of $\text{GL}(V)$. Assume that $G$ does not contain reflections. Then, the Luna stratification of $V/G$ is intrinsic.

Kuttler and Reichstein [KR08] also exhibited several positive examples of linear reductive groups with intrinsic Luna stratification. Later, Schwarz [Sch13] proved that only finitely many connected simple groups $G$ have stratifications that fail to be intrinsic. More results about this can be also found in [Kut11].

**Mixed Tate property.** Ekedahl [Eke09] used the arrangement $\mathcal{A}^G$ to compute the class of the classifying stack $BG$ for certain finite groups $G$, for instance the cyclic group $\mathbb{Z}/n\mathbb{Z}$ and the Symmetric group $S_n$. Later, understanding the Luna stratification of $V/G$ was instrumental towards finding a recursive formula [Mar15] for
geometric invariants belonging to the split Grothendieck group of abelian groups $L_0(Ab)$, see [Mar13]. The combinatorics of Ekedahl computations [Eke09] is also the main interest of [DM15].

Totaro has used the combinatorics of $A^G$, to show recursively that certain quotient substacks of $[V/G]$, and the classifying stack $BG$, are mixed Tate, see [Tot16, Section 9].

The open complement $V^o_e$. Bessis [Bes15] has shown that the open complement $V^o_e$ of $A^W$ is $K(\pi,1)$. The principal stratum plays a crucial role even when the representation is not a reflection representation. Indeed, a common theme in invariant theory is that for sufficiently large $s$ the tensor product representation $V^s$ describes the action of $G$ better than the representation $V$ itself. In the most ambiguous way possible, this is due to the fact that $(V^s)_e$ is a nicer principal open stratum than $V_e$.

There are many examples of this principle. For instance the class of the classifying stack $BG$ in the Motivic ring of algebraic variety $\hat{K}_0(Var_k)$ can be approximated as

$$\{BG\} = \lim_{t} \{V^t/G\} L^{-s \dim V} \in \hat{K}_0(Var_k)$$

where $L$ is the Lefschetz class in $\hat{K}_0(Var_k)$, see [Mar13, Proposition 2.5]. Other examples can be found in [Lor06, Pop07, KR08].

We also mention that whenever $V\setminus V^o$ has codimension at least two, the principal stratum $V^o/G$ can be used as a model for the classifying space $BG$ and, therefore, the group cohomology of $G$ and the unramified cohomology of $BG$ can be computed using $V_e/G$, see [Tot16] and [GMS03, Part 1, Appendix C].

Linear Coxeter Groups and Alternating Subgroups. Any finite reflection group can be equipped with a Coxeter system [Cox34]. For a linear finite reflection groups $W$, the kernel of the determinant map $\det : W \to \mathbb{R}$ is called the alternating subgroup $\text{Alt}(W)$ of $W$. The combinatorics of reflection groups, Coxeter groups and their alternating subgroups is well-studied, see [Bou02, BRR08]. In this work we study $\text{Alt}(W)$ from a different point of view, see Proposition 4.3.

**Main Definitions**

As the reader might sense, in this work we generalize reflection groups to groups that fit the framework of the literature above. We assume from now on that $G$ is a finite linear group with representation $G \hookrightarrow GL(V)$ for some finite vector space $V$.

More generally, all groups (including Coxeter groups) are implicitly assumed to be finite linear groups; otherwise we call them abstract groups and we write $|G|$ for the underlying abstract group of $G$.

**Definition.** Let $g$ be a finite order element of $GL(V)$. We say that $g$ is generated in codimension $i$ if $\text{codim} \langle V^e/g \rangle = i$.

Reflections (resp. rotations) in $GL(V)$ are precisely the elements generated in codimension one (resp. two). Observe that if $V$ is a real vector space, the reflections have order two.

We now introduce the main object of this work.

**Definition.** We say that a group $G$ is generated in codimension $k$ if $G$ is generated by elements of codimension at most $k$, i.e.,

$$G = \langle g \mid \text{codim} \langle V^e/g \rangle \leq k \rangle$$

It is clear that groups generated in codimension one are precisely the finite Coxeter groups.
Definition. We say that a group $G$ is strictly generated in codimension $k$ if $G = \langle g \mid \text{codim}(V^g) = k \rangle$.

We show in Proposition 3.3 that alternating subgroups of Coxeter groups are strictly generated in codimension two. In fact, we verify that in dimensions up to three, any group that strictly generated in codimension two is the alternating subgroup of some Coxeter group; and further, we can recover the Coxeter group from its alternating group, i.e., if $\text{Alt}(W_1) = \text{Alt}(W_2)$ (as linear groups), then $W_1 = W_2$ (see Table 3).

In Proposition 5.3 we list the subgroups of $GL_3(\mathbb{R})$ which are strictly generated in codimension two. We present an infinite class of groups strictly generated in codimension two in $GL_{3n}(\mathbb{R})$ in Proposition 2.8. Finally in Proposition 5.4 we present another class of groups strictly generated in codimension two. We plan to study the complex case in [MS].

Definition. An abstract group $G$ is minimally generated in codimension $k$ if $k$ is the minimal integer for which there exists a faithful representation $\rho : G \to GL(V)$ such that the linear group $\rho(G)$ is strictly generated in codimension $k$.

We study several interesting examples of abstract groups that are generated in codimension two, but not generated in codimension one, i.e., they have no reflection representation. For instance, the cyclic groups $\mathbb{Z}/n\mathbb{Z}$, for $n \geq 3$ are minimally generated in (real) codimension two. In particular, they are not reflection groups.

A representation $\mathbb{Z}/n\mathbb{Z} \subseteq GL_2(\mathbb{R})$ is shown in Corollary 3.2.

MAIN RESULTS.

We work exclusively with real representations and we focus primarily on groups generated in codimension two. We plan to study the complex case in [MS].

The perk of groups strictly generated in codimension two is that they save many of the nice combinatorial features of the reflection groups. For instance, while the intersection lattice $\mathcal{L}(\mathcal{A}^p)$ of the arrangements $\mathcal{A}^p$ is no longer geometric, it is still atomic, see Theorem 4.3. Further if $G$ is strictly generated in codimension two, the maximal subspaces in the arrangement are of codimension two, see Proposition 4.2. This is not the case for groups strictly generated in higher codimensions and we provide examples in Section 4.

In Section 3 we show that the ‘strictly generated in codimension two’ subgroups of $GL_3(\mathbb{R})$ are precisely the cyclic groups $\mu_3(n)$, see Proposition 3.1.

In Sections 5 and 6 we completely classify the arrangements $\mathcal{A}^G$ of finite linear subgroups $G$ in $GL_3(\mathbb{R})$. Figure 4 shows the inclusion order on linear subgroups of $GL_3(\mathbb{R})$ from left to right.

In our first result we list all the groups $G$ in $GL_3(\mathbb{R})$, for which the underlying abstract group $|G|$ is minimally generated in codimension two. In particular, we find that there exist abstract groups minimally generated in codimension two, besides the toy example $\mathbb{Z}/n\mathbb{Z}$.

Theorem A. The subgroups of $GL_3(\mathbb{R})$ which are minimally generated in codimension two are precisely the following: the Cyclic group $\mu_3(n)$ for $n \geq 3$, the Tetrahedral group $T_3$, and the Icosahedral group $Ico_3$.

Next, we study the intersection lattice of the subspace arrangement of a group strictly generated in codimension two in $GL_3(\mathbb{R})$.

Theorem B. Suppose $G$ is a group strictly generated in codimension two in $GL_3(\mathbb{R})$. Then $\mathcal{L}(\mathcal{A}^G)$ is isomorphic, as a poset, to $\mathcal{L}_n$, see Figure 2 for some $n \neq 2$.

More precisely, for any positive integer $n \neq 2$, there exists $G \subseteq GL_3(\mathbb{R})$, strictly generated in codimension two such that $\mathcal{L}(\mathcal{A}^G) \cong \mathcal{L}_n$. 
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The combinatorics of the quotient arrangement $A^G/G$ encodes the combinatorics of the Luna stratification.

**Theorem C.** Suppose $G \subset GL_3(\mathbb{R})$ is strictly generated in codimension two. The intersection lattice of the closures of the Luna strata of $V/G$ is isomorphic to $\mathcal{L}_n$ for $n \in \{2, 3\}$.

![Figure 1](image1)

**Figure 1.** We denote this poset as $\mathcal{L}_n$.

The Luna stratification of $V/G$ and the arrangement $A^G$ are described for all subgroups of $GL_3(\mathbb{R})$ in Section 6.

Finally, as a byproduct we describe the cohomology of the open complement $U_G$ of the arrangement $A^G$ in the vector space $V$, see Theorems 7.9 and 7.16.

**Theorem D.** A group $G \subset GL_3(\mathbb{R})$ is strictly generated in codimension two if and only if the cohomology of $U_G$ is concentrated in degree one and $h^1(U_G) = 2N - 1$, where $N$ is the number of lines in the subspace arrangement $A_G$.

**Questions**

In Proposition 4.3, we prove that the alternating subgroup of a Coxeter group is strictly generated in codimension two. We wonder if any group strictly generated in real codimension two is the alternating group of a Coxeter group.

**Question 1.** Suppose $G$ is strictly generated in codimension two. Does there exist a Coxeter group $W$ such that $G = \text{Alt}(W)$?

If $G = \text{Alt}(W)$, then $A^G$ is the truncation of $A^W$ at codimension 2. With this in mind, we ask a weaker version of Question 1:

**Question 2.** Let $G$ be a group strictly generated in codimension two. Is there a reflection group $W$ such that $A^G \subseteq A^W$?

We have verified that Question 1 hence also Question 2 has an affirmative answer for subgroups of $GL_d(\mathbb{R})$ for $d \leq 3$. 
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Figure 3. The figure shows the inclusion order of linear subgroups of $GL_3(\mathbb{R})$. In Section 6, we discuss the inclusion of the corresponding subspace arrangements.

Table 4. Rank 3 Linear Groups, rearranged in order of $\mathcal{R}_2G$. We specify with $\text{if}$ if the group is strictly generated in a certain codimension.

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1. Preliminaries and generalities

In this section we briefly review some basic terminology and results.

1.1. Posets. The mathematical objects we are going to study are partially ordered sets, called posets. A standard reference for a complete and comprehensive introduction is \cite[Chapter 2]{Sta12}.

A poset \((P, \leq)\) is a set \(P\) together with a partial order relation \(\leq\). In what follows the set \(P\) will always be a finite set. If \(p, q \in P\) and \(p \leq q\), the closed interval is the subset \([p, q] \defeq \{ r \in P \mid p \leq r \leq q \}\). A lattice is a poset where every pair of elements has a unique minimal upper bound as well as a unique maximal lower bound.

A chain in \(P\) is an ordered sequence \(p_1 < p_2 < \ldots < p_k\) of elements in \(P\). We denote by \(\Delta(P)\) the set of all chains of \(P\), ordered by inclusion. If the poset \(P\) has a unique minimal element \(\hat{0}\) and a unique maximal element \(\hat{1}\), we define the reduced order complex of \(P\) as \(\hat{\Delta}(P) \defeq \Delta(P \setminus \{\hat{0}, \hat{1}\})\), which is naturally a simplicial complex.

1.2. Subspace Arrangements. In this work, we study posets arising from linear subspace arrangements. Let \(V\) be a real vector space of dimension \(d\). An arrangement of linear subspaces in \(V\) is a finite collection \(A\) of linear subspaces of \(V\). The poset of intersections associated to \(A\) is the set \(L(A) \defeq \{ \bigcap_{S \subseteq A} S \mid S \subseteq A \}\) ordered by reverse inclusion: for \(x, y \in L(A)\), \(x \leq y\) if \(x \supseteq y\). The poset \(L(A)\) is a lattice and it has a unique minimal element \(\hat{0} = V\), that is the intersection of the empty family, and a unique maximal element \(\hat{1} = \bigcap_{s \in A} s\). (A subspace arrangement with a unique maximal element is called central.) We say that the arrangement \(A\) is essential if \(\dim \hat{1} = 0\).

1.3. The Subspace Arrangement of a Representation. Let \(G \xrightarrow{\rho} GL(V)\) be a finite dimensional real representation of \(G\). The normalizer \(N_G(H)\) of a subgroup \(H \subseteq G\) (resp. the stabilizer of a subset \(S \subseteq V\)) is denoted by

\[
N_G(H) \defeq \{ g \in G \mid gH = Hg \} \\
\text{Stab}_\rho(S) \defeq \{ g \in G \mid gS \subseteq S \}
\]

If \(v\) is a vector in \(V\) and \(S\) the singleton \(\{v\}\), then we omit the set brackets by writing \(\text{Stab}_\rho(\{v\})\) as \(\text{Stab}_\rho(v)\). The latter is the isotropy group in \(v\), that is the subgroup of \(G\) that stabilizes the point \(v\).

Following \cite{Eke09}, for any subgroup \(H\) of \(G\) we define:

\[
V^H_\rho \defeq \{ v \in V \mid H \subseteq \text{Stab}_\rho(v) \} = \{ v \in V \mid Hv = v \} \subseteq V.
\]

In other words, \(V^H_\rho\) is the subspace containing all the points fixed (at least) by all elements of \(H\).

**Definition 1.4.** For \(L\) a subspace of \(V\), we denote by \(\text{Fix}_\rho(L)\) the fixator of \(L\):

\[
\text{Fix}_\rho(L) \defeq \{ g \in G \mid \forall l \in L, gl = l \} \subseteq G.
\]
Lemma 2.2. Let $d$ be a positive definite bilinear form $\omega$ on $V$. In particular, we may assume $G \subset O(V)$. 

Definition 2.1. A subgroup $H \subset G$ is called a stabilizer subgroup with respect to the representation $G \hookrightarrow GL(V)$ if $\text{Fix}_\rho(V^H) = H$. Equivalently, $H$ is the largest subgroup of $G$ fixing $V^H$ point-wise.

If $H$ is a stabilizer subgroup with respect to the representation $\rho$, then $\text{Fix}_\rho(L) \subseteq \text{Stab}_\rho(V^H) = N_G(H)$.

Definition 1.6. The subspace arrangement $A^\rho$ corresponding to the representation $\rho$ is

$$A^\rho \overset{\text{def}}{=} \{V^H_\rho \mid \{e\} \neq H\text{ is a stabilizer subgroup with respect to }\rho\}.$$ 

We also denote set of codimension-$i$ subspaces by

$$A^\rho_i \overset{\text{def}}{=} \{V^H_\rho \in A^\rho \mid \text{codim}V^H_\rho = i\}.$$ 

Moreover we denote by $U_\rho$ the open complement of $A^\rho$ in the vector space $V$.

A complete description of such arrangements can be found in [DM15].

It is a simple observation (see [DM15, Section 1.2]) that the lattice of intersection $L(A^\rho)$ of $A^\rho$ is precisely $A^\rho$ together with $V = V^e$; thus

$$(1.7)\quad L(A^\rho) = \{V^H_\rho \mid H\text{ is a stabilizer subgroup of }G\text{ w.r.t. }\rho\}.$$ 

The atoms of such lattice are precisely the maximal subspace by inclusion.

The reduced order complex of $L(A^\rho)$ has important geometric and group theoretic properties, see [DM15, Eke09, Section 3]. We are going to show in Section 7.6 that $\Delta L(A^\rho)$ has a crucial role in the computation of the cohomology of $U_\rho$.

2. Finite groups generated in high codimension

In this section we describe the class of finite groups that we study in this work. These groups are a natural generalization of finite reflection groups.

All along this section, $G$ is a finite linear group $G \hookrightarrow GL(V)$. We denote by $|G|$ the underlying abstract group of $G$.

Let us set some useful notations. The rank $rk(G)$ of $G$ is defined to be the dimension of $V$. Given two linear groups $G_1 \overset{\rho_1}{\hookrightarrow} GL(V_1)$ and $G_2 \overset{\rho_2}{\hookrightarrow} GL(V_2)$ we define the direct product $G_1 \times G_2$ as a linear group in $GL(V_1 \times V_2)$ with representation $\rho_1 \times \rho_2$. We denote by $0_n$ the trivial group $\{e\}$, viewed as a linear group in $GL_n(\mathbb{R})$, i.e.,

$$0_n := \left\{ \begin{bmatrix} 1 & & \\ \vdots & \ddots & \\ & & 1 \end{bmatrix} \right\} \subset GL_n(\mathbb{R}).$$

Observe that $rk(0_n) = n$, and $rk(H \times G) = rk(H) + rk(G)$.

Definition 2.1. We say that $G$ is essential if the associated subspace arrangement $A^\rho$ is essential.

It is clear that $0_n$ is non-essential. Further, if a non-trivial group $G$ is non-essential, there exists an essential subgroup $H \subset G$ such that $G = H \times 0_d$ for some $d$, and $|G| \cong |H|$.

Lemma 2.2. Let $G$ be a finite linear group. Then there exists a $|G|$-equivariant positive definite bilinear form $\omega$ on $V$. In particular, we may assume $G \subset O(V)$. 

Proof. The group $G$ acts on the set of bilinear forms on $V$ via:

$$(g \cdot \theta)(v, w) = \theta(g^{-1}v, g^{-1}w)$$

Fix some positive definite bilinear form $\theta$ on $V$. Then

$$\omega = \sum_{g \in G} g \cdot \theta$$

is a $G$-equivariant positive definite bilinear form on $V$. \qed

**Definition 2.3.** An element $r \in GL(V)$ is called a reflection if it has finite order and it fixes a subspace of codimension one, called the reflection hyperplane. Following the notations given in Section 1, the reflection hyperplane is denoted $V(r)$.

**Definition 2.4.** An element $g \in GL(V)$ is called a rotation if it has finite order, and further $\text{codim } V(g) = 2$.

**Definition 2.5.** A group $G \subseteq GL(V)$ is a reflection group if it is generated by reflections, i.e., $G = \langle g \mid \text{codim } V(g) = 1 \rangle$.

**Definition 2.6.** We say that a finite linear group $G$ in $GL(V)$ is generated in codimension $k$ if $G$ is generated by elements of codimension at most $k$, i.e., $G = \langle g \mid \text{codim } V(g) \leq k \rangle$.

We want to emphasize that being a reflection is a property of the representation $\rho$, i.e., of the linear group $G$ and not of the abstract group $|G|$ itself. For instance, the group $\mathbb{Z}/2\mathbb{Z}$, viewed as the subgroup of $GL_1(\mathbb{R})$ generated by $-1$, is a reflection group; on the contrary, $\mathbb{Z}/2\mathbb{Z}$ is not a reflection group if we embed it in $GL_2(\mathbb{R})$ with generator $-\text{Id}$, the linear antipodal map. In the latter case $\mathbb{Z}/2\mathbb{Z}$ is generated in codimension two. Other interesting examples of this phenomena are discussed later; nevertheless we suggest the reader have a look at Sections 6.6 and 6.8.

For our goal, we want to distinguish between abstract groups, for example $\mathbb{Z}/2\mathbb{Z}$, which admit a representation as a linear group generated in low codimension, and abstract groups, like $\mathbb{Z}/n\mathbb{Z}$ that only admit representations in higher codimensions.

**Definition 2.7.** We say that $G$ is strictly generated in codimension $k$ if $G = \langle g \mid \text{codim } V(g) = k \rangle$.

**Proposition 2.8.** Let $G_1, \ldots, G_r$ be finite linear groups (strictly) generated in codimension $k$. Then the product group $G_1 \times \cdots \times G_r$ is (strictly) generated in codimension $k$.

**Proof.** Let $G_1, \ldots, G_r$ be finite linear groups of rank $j_1, \ldots, j_r$ respectively (strictly) generated in codimension $k$, and let $V_i$ denote the vector space underlying the linear group $G_i$. We identify the product group $G = G_1 \times \cdots \times G_r$ as a subgroup of $GL(V)$, where $V = V_1 \times \cdots \times V_r$. Observe that

$$\text{codim}_{V_i} V_i^{(g_i)} = \text{codim}_{V^{([1, \ldots, g_i, \ldots, 1])}}$$

It follows that the subgroup $H_i = 0_{j_1} \times \cdots \times G_i \times 0_{j_r}$ is (strictly) generated in codimension $i$. Since $G$ is generated by its subgroups $H_i$, we deduce that $G$ is also (strictly) generated in codimension $i$. \qed

**Definition 2.9.** An abstract group $|G|$ is minimally generated in codimension $k$ if $k$ is the minimal integer for which there exists a faithful representation $\rho : |G| \to GL(V)$ such that $\rho(|G|)$ is strictly generated in codimension $k$. 
Observe that the essential groups of rank 2. The non-essential ones are  

The linear group $GL_2$ is the unique non-trivial reflection group in rank 1.

### 2.10. Strictly Generated Subgroups

Following Definition 1.6, we set $A_i^G \overset{\text{def}}{=} \{ W \in A^G \mid \text{codim } W = i \}$. We define the subgroups $R_iG$,

$$R_iG \overset{\text{def}}{=} \left\{ g \in G \mid \dim V^{(g)} = i \right\}.$$  

In particular, $R_1G$ (resp. $R_2G$) is the subgroup of $G$ generated by the set of reflections (resp. rotations). Observe that for $g, h \in G$, we have $V^{(gh^{-1})} = hV^{(g)}$, hence $\dim V^{(g)} = \dim V^{(gh^{-1})}$. It follows that the $R_iG$ are normal subgroups of $G$.

### 2.11. Reflection Groups of Small Rank

Linear reflection groups were classified up to conjugacy by Coxeter [Cox34]. We list here all the linear reflection groups of $GL_d(\mathbb{R})$ for $d = 1, 2, 3$, following the notation of [Bou02].

The linear group $A_1 = \{ \pm 1 \} \subset GL(\mathbb{R})$, generated by reflection across the origin $O$ is the unique non-trivial reflection group in rank 1. 

Consider the set $P_n$ containing $n$ equi-distant points on the circle of radius one. The linear group $I_2(n)$, for $n \geq 1$, is defined to be the stabilizer subgroup of $P_n$ in $O_2(\mathbb{R})$, i.e.,

$$I_2(n) = \{ g \in O_2(\mathbb{R}) \mid gP_n = P_n \}.$$  

Observe that $I_2(1) = A_1 \times O_1$ and $I_2(2) = A_1 \times A_1$. Following convention, we set $A_2 = I_2(3)$ and $BC_2 = I_2(4)$. The groups $I_2(n)$, for $n \geq 2$, are precisely all the essential groups of rank 2. The non-essential ones are $I_2(1)$ and $O_2$.

### 3. Groups of Rank Two

Consider a non-trivial group $G$ of rank 1. Any element $g \in G$ either fixes the whole ambient space $\mathbb{R}$, or is a reflection. We deduce that $G = A_1$. Therefore, we start by studying groups of rank two. We use the classification of linear Coxeter groups by Coxeter diagrams [Bou02]; we have set our notations in Section 2.11.

**Proposition 3.1.** Let $G \subset GL_2(\mathbb{R})$ be a finite linear group which is not generated in codimension 1. Then $R_1G = 0_2$, and further

$$G = \mu_2(n) \overset{\text{def}}{=} R_2I_2(n),$$

for some $n \geq 2$. The arrangement $A^G$ contains only the origin and the full space $V = \mathbb{R}^2$, i.e., $A^G = \{ O < V \}$.

**Proof.** Observe that $|\mu_2(n)| = 2^n$. If $R_1G = 0_2$, then $G$ is the cyclic group generated by a single rotation $\theta$. Consequently, we have $G = R_2I_2(n)$, where $n$ is the order of $\theta$.

It remains to show that $R_1G = 0_2$. Suppose this is false. By Section 2.11 we have $G = I_2(n)$ for some $n \geq 1$. Then $A_i^n = A_1^{R_1G}$ contains $n$ lines, with consecutive lines being separated by angle $\pi/n$. Since $G$ preserves both the circle of radius one (since $G \subset O_2(\mathbb{R})$), and the sub-arrangement $A_i^G$, it also preserves their intersection $P_{2n}$. Now, the group of symmetries of $P_{2n}$ being $I_2(2n)$, we have $I_2(n) \subseteq G \subset I_2(2n)$. It follows that $G = I_2(2n)$, since $I_2(n)$ has index 2 in $I_2(2n)$, hence is a maximal proper subgroup. This contradicts the assumption that $G = I_2(n)$. $\square$
For \( n > 2 \), \(|\mu_2(n)| \cong \mathbb{Z}/n\mathbb{Z}\) is not a Coxeter group; hence, \(\mathbb{Z}/n\mathbb{Z}\) has no faithful real reflection representation. On the other hand, the previous proposition shows that there exists a faithful representation for which \(\mathbb{Z}/n\mathbb{Z}\) is strictly generated in codimension two. Therefore this is the first example of a finite group that is minimally generated in real codimension two.

**Corollary 3.2.** For \( n > 2 \), the abstract group \(\mathbb{Z}/n\mathbb{Z}\) is minimally generated in codimension two.

### 4. Groups of Higher Rank

In this section we study the subspace arrangement of a group of higher rank. We start with a simple observation.

**Lemma 4.1.** Let \( g \) be a finite order element of \( GL_n(\mathbb{R}) \). Then \( g \) is a product of reflections and rotations.

**Proof.** Since \( \langle g \rangle \) is finite, we may assume \( g \in O_n(\mathbb{R}) \). It follows that \( g \) is conjugate to some block matrix:

\[
g = \begin{pmatrix}
R_1 & & \\
& \ddots & \\
& & R_d \\
& & -I_k \\
& & & I_l
\end{pmatrix}
\]

where \( I_k, I_l \) are the identity matrix of size \( k, l \) resp., and

\[
R_i = \begin{pmatrix}
\cos \theta_i & \sin \theta_i \\
-\sin \theta_i & \cos \theta_i
\end{pmatrix}
\]

We can now write \( g \) as product of matrices of the form

\[
\begin{pmatrix}
I_a & \\
& -1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
I_{a'} & R_i \\
& I_{b'}
\end{pmatrix}
\]

which correspond to reflections and rotations respectively. (Observe that \( a + b = n - 1 \) and \( a' + b' = n - 2 \).) \( \square \)

We note that the arrangement of a group generated in codimension \( i > 2 \) might contain subspaces in lower codimension. Take for instance the group \( G \) generated by the ten diagonal matrices having in the diagonal only two 1’s and three \(-1\)’s. This group is generated in codimension three, but

\[
\begin{pmatrix}
-1 & & \\
& -1 & \\
& & 1
\end{pmatrix}
\quad \begin{pmatrix}
1 & & \\
& -1 & \\
& & -1
\end{pmatrix}
= \begin{pmatrix}
-1 & & \\
& 1 & \\
& & -1
\end{pmatrix}
\]

is a rotation and therefore the subspace arrangement contains a codimension two subspace.

One of the perks of groups strictly generated in codimension two is that there are no reflection hyperplanes in the corresponding subspace arrangement.

**Proposition 4.2.** If \( G \subseteq GL_n(\mathbb{R}) \) and \( G \) is strictly generated in codimension two, then \( \mathcal{A}_i^G = \emptyset \).
Proof. By Lemma 2.2 we may assume $G \subseteq O_n(\mathbb{R})$. Now, since $G$ is strictly generated in codimension two, it follows from Lemma 5.1 that $G \subseteq SO_n(\mathbb{R})$. Suppose that there exists a hyperplane $V^H$ in $A^G$, where $H$ is a stabilizer subgroup of $G$. Any non-trivial element $r \in H$ is a reflection, hence satisfies $\det(r) = -1$. This contradicts the assumption that $H \subset SO_3(\mathbb{R})$. \hfill $\Box$

Alternating subgroups of Coxeter groups are strictly generated in codimension two.

**Proposition 4.3.** For $W$ a Coxeter group, we have $R_2W = \operatorname{Alt}(W)$.

Proof. Recall that the alternating subgroup $\operatorname{Alt}(W)$ is the kernel of the determinant map $\det : W \to \mathbb{R}$. In particular, $R_2W \subseteq SO_n(\mathbb{R})$ implies that $R_2W \subseteq \operatorname{Alt}(W)$. Conversely, any element of $\operatorname{Alt}(W)$ is the product of an even number of reflections. Since the product of two reflections is a rotation, it follows that $\operatorname{Alt}(W)$ is generated by rotations, hence $\operatorname{Alt}(W) \subset R_2W$. \hfill $\Box$

As we highlight in the introduction, we wonder in Question 1 if the class of alternating subgroups of Coxeter groups coincides with the class of groups generated in codimension two. Right after Proposition 5.8 we have verified that this is the case in $GL_3(\mathbb{R})$.

Recall that the lattice of intersection $L(A^G)$ of $A^G$ is the poset $A^G$ together with a minimal element $\emptyset$ corresponding to $V^G$, see eq. (1.7). A lattice $L$ is atomic if every $x$ in $L$ is the join of some atoms. It is a classical result that the lattice of intersections of a reflection arrangement is geometric, hence also atomic.

**Theorem 4.4.** Let $G$ be an essential group generated in codimension two. Then the lattice of intersection $L(A^G)$ is atomic.

Proof. Let $x$ be in $L(A^G)$ and consider the closed interval $I = [\emptyset, x]$. If the atoms of $I$ are only hyperplanes, $h_1, \ldots, h_k$, than there is nothing to prove, because as we said above the lattice of intersection of a reflection hyperplanes is a geometric lattice and $x$ is the joins of $h_1, \ldots, h_k$.

Assume now that some of the atoms of $I$ are subspaces $S_1, \ldots, S_h$ of codimension two. Consider the sub-lattice $B$ of $I$ defined by all joins of reflection hyperplanes in $I$.

Since $x \notin B$, we call $y$ the maximal element in $B$ and $y$ properly contains $x$. In other words, $y$ corresponds geometrically to the intersection of all hyperplanes in $I$ and, since, $y$ is not $B$ then $y$ is properly contained in $x$. Then, $x$ is the join of such $y$ with the remaining atoms $S_1, \ldots, S_h$ in $I$. \hfill $\Box$

There is no hope that these arrangements are geometric lattices; a quick glance at Section 6 reveals several pathological examples, see for example Table S

5. Groups of Rank Three

In this section and in Section 6 we study the finite subgroups of $GL_3(\mathbb{R})$. In particular, we identify the subgroups $G \subset GL_3(\mathbb{R})$ for which the abstract group $|G|$ is minimally generated in codimension two.

We remark that by Lemma 2.2 we may assume $G \subseteq O_3(\mathbb{R})$; moreover, if $G$ is strictly generated in codimension two, then $G \subseteq SO_3(\mathbb{R})$.

**Lemma 5.1.** A subgroup $G \subseteq O_3(\mathbb{R})$ is strictly generated in codimension two if and only if $G \subseteq SO_3(\mathbb{R})$.

Proof. Any rotation $g \in O_3(\mathbb{R})$ satisfies $\det(g) = 1$, i.e., $g \in SO_3(\mathbb{R})$. It follows that $G \subseteq SO_3(\mathbb{R})$. Conversely, since every non-identity element of $SO_3(\mathbb{R})$ is a reflection, any subgroup $G \subseteq SO_3(\mathbb{R})$ is strictly generated in codimension two. \hfill $\Box$
In this section, we prove some theoretical results. In Section 6 we compute the subspace arrangements for all $G \subset O_3(\mathbb{R})$. We occasionally refer to some examples presented in the final section.

**Lemma 5.2.** Following [Bou02], the following are the essential rank three linear reflection groups:

1. The product group $A_1 \times I_2(n)$.
2. The group of symmetries of a regular tetrahedron, denoted $A_3$.
3. The group of symmetries of a cube, denoted $BC$.
4. The group of symmetries of a regular icosahedron, denoted $H_3$.

**Proposition 5.3** (cf. [Kle56]). A (finite) subgroup $G \subset O_3(\mathbb{R})$, strictly generated in codimension 2 is, up to conjugation, one of the following:

1. The Cyclic group $\mu_2(n) \times 0_1$.
2. The Dihedral group $D_3(n)$; it is the group of automorphisms in $SO_3(\mathbb{R})$ of a regular $n$-gon.
3. The Tetrahedral group $T_3$; it is the group of automorphisms in $SO_3(\mathbb{R})$ of the regular tetrahedron.
4. The Octahedral group $Oct_3$; it is the group of automorphisms in $SO_3(\mathbb{R})$ of the cube.
5. The Icosahedral group $Ico_3$; it is the group of automorphisms in $SO_3(\mathbb{R})$ of the regular icosahedron.

For convenience, we write $\mu_3(n) \overset{\text{def}}{=} \mu_2(n) \times 0_1$.

Observe that for each group $G$ in the previous proposition, there exists a reflection group $W$ such that $G = \mathbb{R}_2W$. Indeed, we can verify

$$
\mu_3(n) = \mathbb{R}_2(I_2(n) \times 0_1), \quad D_3(n) = \mathbb{R}_2(I_2(n) \times A_1), \\
T_3 = \mathbb{R}_2A_3, \quad Oct_3 = \mathbb{R}_2BC_3, \quad Ico_3 = \mathbb{R}_2H_3.
$$

This settles Question 1 for rank three groups.

We can use the previous proposition along with Proposition 2.8 to construct linear groups of higher rank which strictly generated in codimension two. Indeed, if $G_1, \ldots, G_n$ are any groups in Proposition 5.3, the product group $G_1 \times \cdots \times G_n$ is a linear group of rank $3n$, strictly generated in codimension two.

Moreover, by tensoring the $G$-representation $\rho$, we obtain higher rank linear groups strictly generated in higher codimension.

**Proposition 5.4.** Let $G = ([G], \rho)$ be a rank three linear group strictly generated in codimension two, i.e., one of the linear groups in Proposition 5.3. The $n^{th}$-tensor $\text{diag}_n G$, given by the faithful representation

$$[G] \xrightarrow{\rho \times \cdots \times \rho} \text{GL}(V^n)$$

is a rank $3n$ linear group strictly generated in codimension $2n$.

**Proof.** It is sufficient to observe that $\text{codim}(V^n)^{(\rho \times \cdots \times \rho)} = n \cdot \text{codim} V^{(\rho)}$. \qed

Note that the abstract group underlying $\text{diag}_n G$ is $[G]$, the same as the abstract group underlying $G$; but $\text{diag}_n G$ is, of course, different from $G$ as a linear group.

We can now identify the abstract groups underlying the linear groups in Proposition 5.3 which are minimally generated in codimension two.

**Theorem A.** Let $G$ be a rank three linear group. Then $[G]$ is minimally generated in codimension two precisely when $G$ is one of the following: the Cyclic group $\mu_3(n)$ for $n \geq 3$, the Tetrahedral group $T_3$, or the Icosahedral group $Ico_3$. 

The proof of Theorem A is by inspection of all groups strictly generated in codimension two in Proposition 5.3 compared with the list of all reflection groups, classified in [Bou02].

The rest of this section is devoted to classifying the finite linear groups of rank three.

5.5. **Mixed Groups.** Let \( J : V \to V \) denote the antipodal map \( v \mapsto -v \). For \( G \) a linear group, we denote by \( JG \) the linear group generated by \( G \) and \( J \). Let \( H, K \) be subgroups of \( \text{SO}_3(\mathbb{R}) \) such that the index of \( H \) in \( K \) is 2. We define the mixed group

\[
(K, H) \overset{\text{def}}{=} H \sqcup \{ Jg \mid g \in K \setminus H \}.
\]

Observe that there exists an isomorphism \( (K, H) \cong |K| \), given by

\[
g \mapsto \begin{cases} 
g & \text{if } g \in H, \\
Jg & \text{otherwise.}
\end{cases}
\]

**Lemma 5.6.** Let \( G \) denote the mixed group \((K, H)\). Then \( \mathcal{R}_2G = H \).

**Proof.** It is clear that \( H \subset \mathcal{R}_2G \). Further, since \( G \not\subset \text{SO}_3(\mathbb{R}) \), we have \( \mathcal{R}_2G \neq G \), i.e., \( \mathcal{R}_2G \) is a proper subgroup of \( G \). Since \( H \) is a maximal proper subgroup of \( G \), it follows that \( \mathcal{R}_2G = H \). \( \square \)

**Proposition 5.7.** Let \( G \) be a finite subgroup of \( \text{O}_3(\mathbb{R}) \). Exactly one of the following holds:

1. \( G \subset \text{SO}_3(\mathbb{R}) \).
2. \( G = JH \) for some \( H \subset \text{SO}_3(\mathbb{R}) \).
3. \( G \) is a mixed group.

**Proof.** Let \( G \) be a finite subgroup of \( \text{O}_3(\mathbb{R}) \) not contained in \( \text{SO}_3(\mathbb{R}) \). Consider the map \( \phi : \text{G} \to \text{SO}_3(\mathbb{R}) \) given by

\[
\phi(A) = \begin{cases} A & \text{if } \det A = 1, \\
JA & \text{otherwise.}
\end{cases}
\]

We set \( H = \ker \phi \). If \( J \in \text{G} \), then \( G = JH \). Suppose now that \( J \not\in \text{G} \). Then for any \( g \in G \setminus H \), we have \( Jg \not\in H \). Consider the subgroup

\[
K = H \sqcup \{ Jg \mid g \in G \setminus H \}.
\]

It is straightforward to verify that \( G = (K, H) \). \( \square \)

**Proposition 5.8.** The finite subgroups of \( \text{O}_3(\mathbb{R}) \) are, up to conjugation, precisely the following:

1. The finite subgroups of \( \text{SO}_3(\mathbb{R}) \).
2. \( JH \) where \( H \) is some linear group generated in codimension 2.
3. The mixed group \((\text{Oct}_3, T_3) = A_3\).
4. The mixed group \((\mu_3(2n), \mu_3(n))\).
5. The mixed group \((\text{D}_3(n), \mu_3(n)) = I_2(n) \times 0_1\).
6. The mixed group \((\text{D}_3(2n), \text{D}_3(n))\).

We show in Section 6 that the reflection groups mentioned in Lemma 5.2 occur in the above list as follows:

\[
A_3 = (\text{Oct}_3, T_3), \quad BC_3 = J\text{Oct}_3, \quad H_3 = J\text{Ico}_3, \quad I_2(n) \times 0_1 = (\text{D}_3(n), \mu_3(n)), \quad \text{and}
\]

\[
I_2(n) \times A_1 = \begin{cases} J\text{D}_3(n) & \text{if } n \text{ is even,} \\
(\text{D}_3(2n), \text{D}_3(n)) & \text{if } n \text{ is odd.}
\end{cases}
\]
| $G$    | $A^G$ | $A^G/G$ |
|--------|-------|---------|
| $\mu_3(n)$ | $\mathbb{R}^3$ | $\mathbb{R}^3/G$ |
| $J\mu_3(n)$ for even $n$ | $\mathbb{R}^3$ | $\mathbb{R}^2/G$ |
| $O$ for odd $n$ | $\mathbb{R}^3$ | $\mathbb{R}^2/G$ |
| $(\mu_2(2n), \mu_2(n))$ for even $n$ | $\mathbb{R}^3$ | $\mathbb{R}^2/G$ |
| $(\mu_2(2n), \mu_2(n))$ for odd $n$ | $\mathbb{R}^3$ | $\mathbb{R}^2/G$ |

Table 5. The subspace arrangements of $\mu_3(n)$, $J\mu_3(n)$, and $(\mu_2(2n), \mu_2(n))$. 
| $G$ | $A^G$ | $A^G/G$ |
|-----|-------|--------|
| $D_3(n)$, $\mu_3(n)$ | $\mathbb{R}^3$ | $\mathbb{R}^3/G$ |
| for odd $n$ | $l_0$ | $l_0/G$ |
| $\pi_1$ | $\pi_n$ |
| $\pi_0$ | $\pi_n/G$ |

| $D_3(n)$, $\mu_3(n)$ | $\mathbb{R}^3$ | $\mathbb{R}^3/G$ |
| for even $n$ | $l_0$ | $l_0/G$ |
| $\pi_1$ | $\pi_n$ |
| $\pi_{even}/G$ | $\pi_{odd}/G$ |

| $O$ | $O$ |
| $D_3(n)$ | $\mathbb{R}^3$ | $\mathbb{R}^3/G$ |
| for even $n$ | $l_0$ | $l_0/G$ |
| $l_1$ | $l_n$ |
| $l_{even}/G$ | $l_{odd}/G$ |

| $O$ | $O$ |
| $D_3(n)$ | $\mathbb{R}^3$ | $\mathbb{R}^3/G$ |
| for odd $n$ | $l_0$ | $l_0/G$ |
| $l_1$ | $l_n$ |
| $l_{>0}/G$ |

Table 6. The subspace arrangements of $(D_3(n), \mu_3(n))$ and $D_3(n)$. 
| $G$ | $\mathcal{A}^G$ | $\mathcal{A}^{G/G}$ |
|-----|----------------|-----------------|
| $T_3$ | ![Diagram for $T_3$](image1) | ![Diagram for $\mathcal{A}^{G/G}$](image2) |
| $JT_3$ | ![Diagram for $JT_3$](image3) | ![Diagram for $\mathcal{A}^{G/G}$](image4) |
| $G = (\text{Oct}_3, T_3) = A_3$ | ![Diagram for $G = (\text{Oct}_3, T_3) = A_3$](image5) | ![Diagram for $\mathcal{A}^{G/G}$](image6) |
| $G = \text{Oct}_3$ | ![Diagram for $G = \text{Oct}_3$](image7) | ![Diagram for $\mathcal{A}^{G/G}$](image8) |

Table 7. The subspace arrangements of $T_3$, $JT_3$, $A_3$, and $\text{Oct}_3$. 
\[ G = JD_3(n) \]
for even \( n \)

\[ G = JD_3(n) \]
for odd \( n \)

\[ (D_3(2n), D_3(n)) \]
for odd \( n \)

| Table 8. The subspace arrangements of \( JD_3(n) \), and \( (D_3(2n), D_3(n)) \) for \( n \) odd. |
Table 9. The subspace arrangements of $\textit{JOct}_3$, $\textit{Ico}_3$, $H_3$, and $(D_3(2n), D_3(n))$ for $n$ even.
 completes the classification in rank three

Theorem B. Suppose \( G \) is strictly generated in codimension two. Then \( \mathcal{L}(\mathcal{A}^G) \) is isomorphic, as a poset, to \( \mathcal{L}_n \) for \( n \neq 2 \). More precisely, for any positive integer \( n \neq 2 \), there exists a finite linear group \( G \) in \( \text{GL}_3(\mathbb{R}) \) strictly generated in codimension two such that \( \mathcal{L}(\mathcal{A}^G) \cong \mathcal{L}_n \). (\( \mathcal{L}_n \) is shown in Figure 2.)

Proof. By Proposition 4.2, if \( G \) is strictly generated in codimension two, then there are no planes in the arrangement \( \mathcal{A}^G \). Moreover, there is at least a line in \( \mathcal{A}^G \) otherwise \( G \) is not generated in codimension two. Hence, \( \mathcal{A}^G \) is poset-wise isomorphic to \( \mathcal{L}_n \).

If \( n = 1 \), then set \( G = J\mu_2(2k+1) \), see Section 6.4. If \( n > 2 \), we set \( G = D_3(n-1) \), see Section 6.6.

Finally, we observe that \( \mathcal{A}^G \) cannot have only two lines. Indeed, if there are two rotations \( r_1, r_2 \in G \) the product \( r_1r_2 \) is a rotation distinct from \( r_1 \) and \( r_2 \).

Moreover, we also classify the intersection lattice of the quotient arrangement.

Theorem C. Let \( G \) be a rank three linear group strictly generated in codimension two. The intersection lattice of the closures of the Luna strata of \( V/G \) is isomorphic to \( \mathcal{L}_n \) for \( n \in \{2, 3\} \).

Proof. If \( G \) is a rank three linear group strictly generated in codimension two, then it is one of the group in the list of Proposition 5.3. Hence, \( G \) is a subgroup of the group of symmetries of a regular polygon or a regular polyhedron. Thus we have two facts:

- The group \( G \) acts transitively on the vertices, edges, and in the case of the polyhedron, facets;
- Any axis of a rotation in \( \mathcal{A}^G \) is an axis of symmetry, therefore associate to the regular polygon (or the regular polyhedron).

Thus, we can only have three possible orbits in \( \mathcal{A}^G/G \) and the latter is isomorphic to \( \mathcal{L}_n \) for \( n \leq 3 \). It is a simple check of Section 6 that \( \mathcal{L}_2 \) and \( \mathcal{L}_3 \) occur: see Table 7.

The rest of the section is devoted to the study of the arrangement of subspaces of a finite linear groups in \( \text{GL}_3(\mathbb{R}) \). Moreover, we describe every inclusion in Figure 3 case by case, compute the normal subgroups \( \mathcal{R}_1G, \mathcal{R}_2G \), and fill Table 4. As a byproduct, we prove that \( A_3 = (\text{Oct}_3, I_3) \), \( BC_3 = J\text{Oct}_3 \), \( H_3 = J\text{ico}_3 \), \( I_2(n) \times 0_1 = (D_3(n), \mu_3(n)) \), and

\[
I_2(n) \times A_1 = \begin{cases} 
JD_3(n) & \text{if } n \text{ is even,} \\
(D_3(2n), D_3(n)) & \text{if } n \text{ is odd.}
\end{cases}
\]

We start with two useful lemmas.

Lemma 6.1. Consider a non-trivial finite order element \( g \in \text{SO}_3(\mathbb{R}) \). Then \( g \) is a rotation by some angle \( \theta \) around the axis \( V^{(\theta)} \). Then

\[
V^{(Jg)} = \begin{cases} 
(V^{(g)})^\perp & \text{if } \theta = \pi, \\
0 & \text{otherwise.}
\end{cases}
\]

In particular, \( Jg \) is a reflection if and only if \( \theta = \pi \).
Proof. Let \( p \) be a non-zero vector in \( V^g \). Let \( \{e,f\} \) be an orthonormal basis of \( p^\perp \). Then we have
\[
g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}
\]
\[
\implies Jg = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \cos(\theta + \pi) & -\sin(\theta + \pi) \\ 0 & \sin(\theta + \pi) & \cos(\theta + \pi) \end{bmatrix}
\]
in the basis \( \{p,e,f\} \). It follows that \( Jg \) fixes a nontrivial subspace if and only if \( \theta = \pi \). Further, if \( \theta = \pi \), then \( V^{(Jg)} \) is the plane spanned by \( \{e,f\} \), i.e., the plane \( (V^g)^\perp \).

\[\square\]

**Proposition 6.2.** Let \( H \) be a linear subgroup of \( SO_3(\mathbb{R}) \). The linear group \( JH \) is generated in codimension two if and only if \( H \) contains a rotation of angle \( \pi \).

*Proof.* We know from Lemma 5.1 that \( H \) is strictly generated in codimension two, hence \( H \subset R_2 JH \). Further, since \( H \) has index two in \( JH \), and \( J \not\in R_2 JH \), we deduce
\[
H \subset R_2 JH \subsetneq JH \implies R_2 JH = H.
\]
Let \( K \) be the subgroup of \( JH \) generated by the rotations and reflections in \( JH \). Then \( K = JH \) if and only if \( J \in K \).

Suppose \( H \) contains a rotation \( r \) of angle \( \pi \). Then \( Jr \) is a reflection, hence \( Jr \in K \). Further, since \( r \in H \subset K \), we have \( J \in K \).

Conversely, suppose \( J \in K \). Then \( J = sr \) where \( r \) is a rotation, and \( s \) is a product of reflections in \( H \). Suppose \( s = s_1 \ldots s_k \), and set \( r_1 = Js_1 \). Then \( \det r_1 = \det(Js_1) = 1 \); hence \( r_1 \) is a rotation in \( H \), and \( Jr_1 = J^2 s_1 = s_1 \) is a reflection. It follows from Lemma 6.1 that \( r_1 \) is a rotation of angle \( \pi \). \[\square\]

6.3. The group \( \mu_3(n) \). The group \( G = \mu_3(n) \) is not essential. The subspace arrangement of \( G \), in Table 5 is deduced from the arrangement of the essential rank 2 group \( \mu_2(n) \). We note that \( R_1 G \) is trivial, \( R_2 G = G \), and also \( \mu_2(n) = R_2 I_2(n) \), and \( \mu_3(n) = R_2 (I_2(n) \times 0_1) \).

6.4. The group \( J\mu_3(n) \). The generator \( J \) makes the representation essential and fixes the origin. Let \( l_0 \) be the one-dimensional subspace on which the group acts trivially.

Suppose first that \( n \) is even, so that the group \( G = J\mu_3(n) \) contains a rotation of angle \( \pi \); by Lemma 6.1 there is a reflection in \( J\mu_3(n) \) across the plane \( l_0^\perp \). We deduce from Lemma 6.1 that \( J\mu_3(n) = \mu_2(n) \times A_1 \), \( R_1(\mu_2(n) \times A_1) = 0_2 \times A_1 \), and \( R_2(\mu_2(n) \times A_1) = \mu_3(n) \).

We now consider the case where \( n \) is odd, so that there are no rotations with angle \( \pi \) in \( \mu_2(n) \). It follows from Proposition 6.2 that \( J\mu_3(n) \) is not generated in codimension 2, and that the subspace arrangement of \( J\mu_3(n) \) is made by \( O < l_0 < \mathbb{R}^3 \), see Table 5. Further, \( R_1 G = 0_3 \), and \( R_2 G = \mu_3(n) \).

6.5. The mixed group \((\mu_3(2n), \mu_3(n))\). Suppose first that \( n \) is even. The group \( G = (\mu_3(2n), \mu_3(n)) \) has the same arrangements as \( J\mu_3(n) \), see Table 5. By Section 6.5 we have
\[
(\mu_3(2n), \mu_3(n)) = \mu_3(n) \sqcup \{Jg \mid g \in \mu_3(2n)\} \mu_3(n) \}.
\]
The rotation by the angle \( \pi \) is not among the elements \( g \) that we use for \( Jg \). It is clear from the arrangement that \( G \) is not generated in codimension two. Indeed \( R_1 G = 0_4 \), and \( R_2 G = \mu_3(n) \).
Suppose now that \( n \) is odd, so that \( (\mu_3(2n), \mu_3(n)) \) has a reflection across the plane \( l_0^+ \). This group is generated in codimension two, with \( \mathcal{R}_1 G = 0_2 \times A_1 \), and \( \mathcal{R}_2 G = \mu_3(n) \). The arrangement \( \mathcal{A}^G \) is given in Table 6. It is similar to the arrangement of \( J\mu_3(k) \) for even \( k \).

6.6. The group \( D_3(n) \). The linear group \( G = D_3(n) \) is strictly generated in codimension 2, so that \( \mathcal{R}_1 G = 0_3 \), and \( \mathcal{R}_2 G = G \).

Let \( l_1, l_2, \ldots, l_n \) be the lines passing through a pair of opposite vertices of a regular \( 2n \)-gon \( P \); and \( l_0 \) the line perpendicular to the plane containing \( P \). There are \( n \) rotations \( r_1, \ldots, r_n \) of angle \( \pi \) around the line \( l_1, \ldots, l_n \). Further there is a rotation \( r_0 \) of angle \( \pi/n \) around the polar line \( l_0 \).

- If \( n \) is even, the arrangement splits into 3 orbits: a single orbit for the polar line \( l_0 \), the orbit \( l_{\text{even}} = \{ l_i | i > 0, i \text{ even} \} \), and the orbit \( l_{\text{odd}} = \{ l_i | i \text{ odd} \} \).
- If \( n \) is odd, the arrangement splits into 2 orbits: a single orbit for the polar line \( l_0 \), and the orbit \( l = \{ l_i | i \neq 0 \} \), and the orbit \( l_{\text{odd}} = \{ l_i | i \text{ odd} \} \).

The arrangements are shown in Table 6.

6.7. The group \( G = JD_3(n) \). When we add \( J \) to the Dihedral group, we need to distinguish again two cases.

Suppose first that \( n = 2k \) for some \( k \). We want to show that \( JD_3(2k) \) is a reflection group, and precisely this is \( I_2(2k) \times A_1 \).

Let \( l_1, l_2, \ldots, l_{2k} \) be the lines passing through a pair of opposite vertices of the \( 4k \)-gon. There are \( 2k \) rotations \( r_1, \ldots, r_{2k} \) by \( \pi \) around the line \( l_1, \ldots, l_{2k} \). Further, there is a rotation \( r_0 \) by \( \pi \) around the polar line \( l_0 \), because \( n \) is even. It follows that \( J r_i \), with \( 0 \leq i \leq 2k \), is a reflection with respect the plane \( l_i^+ \), which contains exactly two lines from the arrangement: \( l_0 \) and \( l_{i \pm k} \). Thus, the group \( I_2(2k) \times A_1 \) is a subgroup of \( JD_3(2k) \), with the same order.

The arrangement \( \mathcal{A}^G \) contains the plane \( l_0^+ \) and the lines \( l_1, \ldots, l_{2k} \), see Table 8. The rotation around the line \( l_0 \) acts transitively on the reflection planes \( l_i^+ \) with \( 2k \neq 0 \) and similarly on \( l_{\text{odd}}^+ \). The group \( G \) is generated in codimension 1. Further, we have

\[
\mathcal{R}_1 G = I_4(2k) \times A_1, \\
\mathcal{R}_2 (G) = D_3(2k).
\]

Hence, we can read the one dimensional orbits directly from the arrangements of \( D_3(2k) \). The orbit arrangements of \( D_3(2k) \) for \( k \) odd is also shown in Table 6. The quotient arrangement depends on the parity of \( k \) as it is clear from Table 6, see Table 10.

| Table 10. The arrangement quotients of \( G = JD_{2k} \) for \( k \) odd and \( k \) even respectively. |
Let now consider the linear group \( G = JD_{3}(n) \), where \( n = 2k + 1 \) for some \( k \).

Let \( l_1, \ldots, l_n \) be the lines determined by the vertices of a regular \( n \)-gon and the origin; let \( l_0 \) denote the polar line. For \( 1 \leq i \leq n \), we denote by \( r_i \) the rotation of angle \( \pi \) around the axis \( l_i \). Then \( J r_i \) is a reflection across the plane \( l_i^\perp \). The polar line \( l_0 \) is contained in \( l_i^\perp \). We note that \( l_i^\perp \) does not contain \( l_j \) for any \( j \), because \( n \) is odd. The subspace arrangement is shown in Table 5. This group is generated in codimension 2, with

\[
\mathcal{R}_1 G = I_2(n) \times 0_1.
\]

\[
\mathcal{R}_2 G = D_3(n).
\]

6.8. The mixed group \((D_3(n), \mu_3(n))\). By Section 6.5

\[
(D_3(n), \mu_3(n)) = \mu_3(n) \cup \{ J g | g \in D_3(n) \setminus \mu_3(n) \}
\]

Observe that this group equals \( I_2(n) \times 0_1 \). It follows that \( \mathcal{R}_1 G = G \) and \( \mathcal{R}_2 G = \mu_3(n) \).

Let \( \pi_0 \) be the plane on which \( I_2(n) \) acts, and let \( P \) be the regular \( 2n \)-polygon in \( \pi_0 \). For \( 1 \leq i \leq n \), let \( \pi_i \) be the plane containing \( l_0 = \pi_0^\perp \), and the \( i \)th vertex of \( P \). The subspace arrangement \( \mathcal{A}^G \) contains the line \( l_0 \) and the \( n \) planes \( \pi_1, \ldots, \pi_n \), see Table 6.

6.9. The mixed group \((D_3(2n), D_3(n))\). Suppose first that \( n \) is odd. We show that in this case \((D_3(2n), D_3(n)) = I_2(n) \times A_1 \). Since \( n \) is odd, the rotation \( r_0 \) with angle \( \pi \) and axis \( l_0 \) belongs to \( D_3(2n) \setminus D_3(n) \). Hence the reflection \( J r_0 \) belongs to \( G \), i.e., \( 0_2 \times A_1 \subset (D_3(2n), D_3(n)) \). Let \( l_1, \ldots, l_{2n} \) be the lines in the arrangement of \( J D_3(2n) \), and \( r_i \) the rotation of angle \( \pi \) with axis \( l_i \). For \( i \) even, \( r_i \) is a reflection across \( l_i^\perp = \langle l_0, l_{1+n} \rangle \). Hence the subgroup generated by the \( l_i^\perp \) for \( i \) even is precisely \( I_2(n) \times 0_1 \). It follows by comparing order that \((D_3(2n), D_3(n)) = I_2(n) \times A_1 \). The lattice of intersection of the related subspace arrangement is computed in Table 7. We leave to the reader to check that the quotient arrangement is the one described in Table 8; we just remark that \( l_i^\perp = \langle l_0, l_{1+n} \rangle \).

When \( n \) is even, \( r_0 \) does not belong in \( D_3(2n) \). Moreover, the reflecting hyperplane of any reflection in \( \{ J g | g \in D_3(2n) \setminus D_3(n) \} \) contains only the line \( l_0 \) in the arrangement of \( D_3(2n) \setminus D_3(n) \).

Hence, the group \((D_3(2n), D_3(n))\) is generated in codimension 2, and we have \( \mathcal{R}_1 G = I_2(n) \times 0_1 \), and \( \mathcal{R}_2 G = D_3(n) \).

This argument, along with the observation that the lines \( l_2j \) in the arrangement of \( D_3(n) \) are missing, explain the poset of the quotient arrangement.

6.10. The group \( T_3 \). The group \( T_3 \) is the group generated by the rotational symmetries of the regular tetrahedron \( P \) in \( \mathbb{R}^3 \). Let \( v_1, v_2, v_3 \) be the vertices of \( P \); and let \( e_1, e_2, e_3 \) be the planes passing through the midpoints of two opposite edges of \( P \). The arrangements of the linear group \( T_3 \) are shown in Table 7. Since \( G \) is generated by rotations, we have \( \mathcal{R}_1 G = 0_3 \), and \( \mathcal{R}_2 G = G \).

6.11. The group \( JT_3 \). The rotations \( r_i \) in \( T_3 \) with axes \( e_i \) has angle \( \pi \), hence \( J r_i \) is a reflection. The arrangements of \( JT_3 \) is shown in Table 8, and it contains the arrangement of \( T_3 \). The quotient arrangement is obtained again by the one of \( T_3 \) and by the observation that all planes \( e_i^\perp \) are in the same orbit, because of rotation around a line containing a vertex of the tetrahedron sends a reflection plane to another one. We also note that \( \mathcal{R}_1 JT_3 = A_1 \times A_1 \times A_1 \), and \( \mathcal{R}_2 G = T_3 \).
6.12. The group $Oct_3$. The linear group $Oct_3$ is generated strictly in codimension 2. In particular, $R_1G = 0_3$, and $R_2G = G$. It is the group of rigid symmetries of the octahedron, i.e., dual polytopes of the cube. Its subspace arrangement and the respective quotient arrangement are in Table 7.

6.13. The group $JOct_3$. The group $Oct_3$ has six order-two rotations with axes $f_1, \ldots, f_6$, and three order two rotations with axes $e_1, e_2, e_3$ respectively. It follows that $JOct_3$ contains nine reflections, across the planes $e_i^\perp$ and $f_i^\perp$. The subspace arrangement of $JOct_3$ is in Table 9. By definition, see Lemma 5.2, $BC_3$ is the group of symmetry of a cube, so we observe that $JOct_3 = BC_3$, $R_1G = G$, and $R_2G = Oct_3$.

6.14. The Mixed Group $(Oct_3, T_3)$. We are going to show that the linear group $(Oct_3, T_3)$ is precisely $A_3$, the symmetry group of the regular tetrahedron $P$. Indeed, $T_3 \subset (Oct_3, T_3)$, and further $(Oct_3, T_3)$ preserves $P$; by counting elements, we deduce that $(Oct_3, T_3)$ is the symmetry group of $P$. In particular, we have $R_1G = G$ and $R_2G = T_3$.

Now, by Section 5.5, the group $(Oct_3, T_3)$ contains the elements $Jg$ where $g \in Oct_3 \setminus T_3$; by Lemma 6.1, this $Jg$ is a reflection if and only if $g$ is a rotation by an angle $\pi$; these rotations are precisely the six rotations along the lines passing through middle points of edges. Hence, the arrangement of $(Oct_3, T_3)$, see Table 7, is the subset of $A^{JOct_3}$ spanned by $e_i, v_i$ and $f_i^\perp$.

The quotient arrangement of $(Oct_3, T_3)$ is a sub-arrangement of the quotient arrangement of $JOct_3$.

6.15. The group $Ico_3$. The construction of the arrangements for $Ico_3$ is very similar to the one of $Oct_3$ and $T_3$. These arrangements are shown in Table 9. Here we shortly explain how to get these arrangements, because it will be useful for the next group, $JIco_3$.

There are three types of rotations in the symmetries of the icosahedron. The first one are around the lines through opposite vertices $v_1, \ldots, v_6$; then, we consider the rotations around the axes through two opposite edges $e_1, \ldots, e_{15}$; and finally the rotations around lines through central points of opposite facets $f_1, \ldots, f_{10}$. Hence all those lines are in the arrangements $A^{Ico_3}$.

To construct the quotient arrangement, we note that there are only three orbits of lines, the $f$-orbits, the $e$-orbits and the $v$-orbits. Indeed, there are three $Ico_3$-orbits in the icosahedron corresponding to vertices, edges, and facets. Finally this is a group strictly generated in codimension two: $R_1G = 0_3$, and $R_2G = G$.

6.16. The group $H_3 = JIco_3$. We are going to show that $JIco_3$ is the full icosahedral group; in our notation this is identified with the Coxeter group $H_3$, hence strictly generated in codimension 1.

As we said above, there are three types of rotations in the icosahedral group $Ico_3$: $v_i, e_i$ and $f_i$. (In the previous subsection they are described in details.) Only the rotations $e_1, \ldots, e_{15}$ have order 2. So, $Je_1, \ldots, Je_{15}$ are reflection with respect the planes $\pi_1 = e_i^{\perp}, \ldots, \pi_{15}^{\perp}$. This description, and an order check, already shows that $JIco_3$ is the Coxeter group $H_3$ with $R_1H_3 = H_3$, and $R_2H_3 = Ico_3$.

Nevertheless, we want to convince the reader that the two subspace (actually hyperplane) arrangements coincide. The presentation of the hyperplane of $H_3$ is, for instance, in Exercise 26 of Section 7.3 of [GM12].

Let us pick a rotation $e_i$: this means picking four vertices among the 12 of the icosahedron. (Those vertices identify two opposing edges.) Consider the induced simplicial complex on the remaining eight vertices: This is made of four antipodal 2-simplexes and two antipodal edges $a$ and $b$. We are going to show that each $e_i^{\perp}$
contains six lines. Indeed, the plane $\pi_i = e^+_i$ contains these edges, $a$ and $b$; so it contains the two lines through pairs of antipodal vertices, and lines through the middle points of $a$ and $b$). The plane $\pi_i$ also contains lines through the middle points of the common edges of the two antipodal 2-simplexes; in addition, the lines through the pairs of opposite facets of the latter 2-simplexes also belongs to $\pi_i$. These arrangements are shown in Table 9.

7. Cohomological computations

In this section, we compute the singular reduced cohomology of $U_{AG}$, the open complement of the arrangement $\mathcal{A}^G$ in $\mathbb{R}^n$. Firstly, we recall the Goresky–MacPherson formula and we set some useful notations; later in Section 7.6 we show our results.

7.1. Some Notations and Results from the Literature. In 1988, Goresky and MacPherson [GM88] gave a formula to express the group cohomology of the open complement of a real subspace arrangement.

Given a real central subspace arrangement $\mathcal{A}$, we denote by $\mathcal{L}(\mathcal{A})$ its lattice of intersection. We set $U_\mathcal{A} \overset{\text{def}}{=} \mathbb{R}^n \setminus \cup_{x \in \mathcal{L}(\mathcal{A})} x$, that is the open complement of the arrangement in $\mathbb{R}^n$. For every, $x \in \mathcal{L}(\mathcal{A})$, the interval $[0, x]$ is the subposet of $\mathcal{L}(\mathcal{A})$ made by the elements $y \in \mathcal{L}(\mathcal{A})$ such that $0 \leq y \leq x$. We are going to denote $\Delta_x \overset{\text{def}}{=} \tilde{\Delta}(0, x)$, the (reduced) order complex, see Section 11. We mainly care about the lattice of intersection associated to the arrangement $\mathcal{A}^G$ and we are going to show that the simplicial complex

$$\Delta_G \overset{\text{def}}{=} \tilde{\Delta}(0, \hat{1}) = \Delta\mathcal{L}(\mathcal{A})$$

plays a crucial role. For simplicity, we write $U_G$ instead of $U_{AG}$.

With an abuse of notation, we denote the rational singular reduced cohomology and the rational simplicial reduced homology by $H^\ast(U_\mathcal{A})$ and $H_\ast(\Delta(0, x))$. We write $h^\ast(U_\mathcal{A})$ and $h_\ast(\Delta(0, x))$ for their dimensions as $\mathbb{Q}$-vector space.

We are going to compute the singular reduced cohomology of $U_\mathcal{A}$ via the simplicial reduced homologies of the order complex of the interval $[0, x]$ in the lattice $\mathcal{L}(\mathcal{A})$.

**Theorem 7.2. (Goresky–MacPherson formula)** Let $\mathcal{A}$ be a subspace arrangement in $\mathbb{R}^n$ and $U_\mathcal{A}$ be its open complement in $\mathbb{R}^n$. Then

$$H^\ast(U_\mathcal{A}) \cong \bigoplus_{0 \neq x \in \mathcal{L}(\mathcal{A})} H_{\text{codim}(x) - 2 - 1} \Delta_x.$$ 

We rewrite the previous formula in a more convenient way; the set of intersections in a specific codimension $j$ is $\mathcal{L}_j(\mathcal{A})$:

$$H^\ast(U_\mathcal{A}) \cong \bigoplus_{j \neq 0} \bigoplus_{x \in \mathcal{L}_j(\mathcal{A})} H_{j - 2 - 1} \Delta_x, \quad (7.3)$$

Note that the sum is over $j > 0$ because the is always a unique minimal element $\hat{0}$ corresponding to the intersection $\mathbb{R}^n$. We also remark that $\mathcal{L}_j(\mathcal{A}^G) = \mathcal{A}_j^G$.

Moreover, if the arrangement is central, there is also the unique maximal element $\hat{1}$ corresponding to the origin $O$. In the above direct sum, the contribution for $j = n$ is $H_{n - 2 - 1} \Delta_1$. So we observe that $H^\ast(U_\mathcal{A}) \cong H_{-2} \Delta_1 = 0$.

We say that a subspace arrangement $\mathcal{A}$ is non-trivial if there is at least an intersection different from $\hat{0}$ and $\hat{1}$, i.e., the lattice of intersection $\mathcal{L}(\mathcal{A})$ is different from $\{0 < \hat{1}\}$. In other words, $\tilde{\Delta}(0, \hat{1}) \neq \emptyset$. 


Lemma 7.4. A central arrangement of subspaces $A$ is trivial if and only if $H^{n-1}(U_A) \neq 0$.

Proof. By equation (7.3), we note that $H^{n-1}(U_A) \cong H_{-1}(\hat{\Delta}(0,1))$. This is non-zero if and only if the order complex $\hat{\Delta}(0,1)$ is not empty. □

Lemma 7.5. Let $A$ be a non-trivial central arrangement of subspaces. If $a \in L(A)$, then $H_{-1} \Delta_a$ is non zero if and only if $a$ is an atom of the lattice of intersection $L(A)$.

Proof. We observe that $h_{-1} \Delta_a = 1$ if and only if $\Delta_a$ is empty if and only if $a$ is an atom for $L(A)$. □

7.6. The Cohomology of the Principal Stratum. We assume that our arrangement is non-trivial. In this case the maximal cohomology could be in degree $(n-2)$. Using equation (7.3), one gets:

\begin{equation}
    H^{n-2}(U_A) \cong H_0 \Delta_1 \bigoplus_l H_{-1} \Delta_l
\end{equation}

where the sum is over the lines $l$ in the arrangement.

Proposition 7.8. Let $N$ be the number of atom lines in the arrangements $A$. Then $h^{n-2}(U_A) \geq 2N - 1$. Further, the inequality is strict, i.e., $h^{n-2}(U_A) = 2N - 1$ if and only if $A$ contains precisely $N$ lines.

Proof. We know from equation (7.7) that only $H_0 \Delta_1$ and $H_{-1} \Delta_l$ contribute to the $(n-2)$-cohomology. By Lemma 7.5, each $H_{-1} \Delta_l$ is non-zero (and one dimensional) if and only if $l$ is an atom. On the other hand, $H_0 \Delta_1$ could be made by several disconnected components; among those, there is a singleton vertex $\{l\}$ for each atom line $l$. They contribute by an $(N - 1)$-dimensional vector space in $H_0 \Delta_1$. Thus, $h^{n-2}(U_A) = h_0 \Delta_1 + N \geq 2N - 1$.

Assume now that $A$ is only made by lines, so every line is an atom for the lattice of intersection of $A$. Moreover, $\Delta_1$ has $N$ vertices. So $h^{n-2}(U_A) = 2N - 1$. Vice versa, assume $h^{n-2}(U_A) = 2N - 1$, then $h_0 \Delta_1 = N - 1$; note that $\Delta_1$ contains only $N$ vertices. □

Theorem D. Let $G$ be a finite linear group in $GL_3(\mathbb{R})$. The group $G$ is strictly generated in codimension two if and only if the cohomology of $U_G$ is concentrated in degree 1, and $h^1(U_G) = 2N - 1$, where $N$ is the number of lines in the arrangements.

Proof. Since $G$ is strictly generated in codimension two, then that $A^G$ is non-trivial and $h^2(U_G) = 0$. By Proposition 1.2 we have that $A^G$ is only made by lines. So we apply the Proposition 7.8. □

Finally, we have enough information to fulfill the three dimensional finite linear groups.

Theorem 7.9. Let $G$ be a finite linear subgroup of $GL_3(\mathbb{R})$.

- If $G$ is strictly generated in codimension 3, then $A^G$ is trivial, $U_G = \mathbb{R}^3 \setminus O$, $h^0(U_G) = h^1(U_G) = 0$ and $h^2(U_G) = 1$;
- If $G$ is strictly generated in codimension two, then the arrangement is made by $N$ atom lines, and $h^0(U_G) = h^2(U_G) = 0$ and $h^1(U_G) = 2N - 1$;
- If $G$ is strictly generated in codimension one, then the $G$ is a reflection group, $h^2(U_G) = 0$, $h^1(U_G) = h_0 \Delta_G$, and $h^0(U_G) = C - 1$, where $C$ is the number of chambers of the sub-hyperplane arrangements in $A^G$;
- If $G$ is generated in codimension two or three, then $h^2(U_G) = 0$, $h^1(U_G) = h_0 \Delta_G + N$, and $h^0(U_G) = C - 1$, where $C$ is the number of chambers of the sub-hyperplane arrangements in $A^G$. 


Proof. The proof uses all the facts we have proved along the section. The first item follows easily from the definition of trivial arrangement. The second item follows from Theorem D. The case of reflection groups is widely studied, see [OT92, DCP11]. In this case, let us only mention that there are no atom lines, hence $h^0(U_G) = h_0\Delta_G$. We also remark that $h^0(U_G) + 1$ is the number of chambers $C$ of the hyperplane arrangements.

The last item arises when there are generators in different codimensions. The arrangement is not a reflection arrangement, but it might contain atom lines. Such atom lines change the degree one cohomology, as described by equation (7.7); the degree zero cohomology is unchanged. □

The previous theorem suggests two results in higher dimension.

**Proposition 7.10.** Let $G$ be a finite linear group generated in codimension one. Then, $h^0(U_G) = h^0(U_{R,1}(G))$.

Proof. The proof follows from the same observation given in the proof of the last case of Theorem 7.9. □

**Corollary 7.11.** Let $G$ be a finite linear group generated such that $A_G$ has no hyperplane. Then, $h^0(U_G) = 0$.

In the four dimensional real case, we are particularly interested in the finite groups generated in codimension two. From what we have already proven in this section, $h^i(U_G) = 0$ for $i \neq 1, 2$. Let us have a look at the remaining cohomology group:

\[(7.12) \ H^1(U_A) \cong H_1 \Delta_1 \bigoplus_l H_0 \Delta_l \bigoplus_{\pi} H_{-1} \Delta_\pi \]

Using Lemma 7.5, we know that $\bigoplus_{\pi} H_{-1} \Delta_\pi$ counts the number of planes; all planes are atoms since this the $G$ is strictly generated in codimension two. Set $\sum_{\pi} h_{-1}(\Delta_\pi) = a_G^2 = \#A_G^2$, the number of intersections in codimension two of $A_G^2$. More in general, we set

$$a_i = \#A_i, \text{ and } a_i^G = \#A_i^G.$$

**Proposition 7.13.** If $A$ is a central subspace arrangement made only by lines and planes in $\mathbb{R}^n$, then

$$\sum_l h_0(\Delta_l) = \sum_l M_l - a_{n-1} + N,$$

where $M_l = \{\pi \in A : l \subset \pi\}$ is the number of planes containing $l$, and $N$ is the number of atom lines.

Proof. We note that $a_{n-1} = \#A_{n-1}$ is the number of lines in the arrangements. The order complex $\Delta_i$ is empty if and only if $l$ is an atom lines. In such case $h_0(\Delta_l) = 0$. If $l$ is not an atom, then there exist a plane $\pi$ such that $0 < \pi < l$ and $\Delta_l$ is the union of $M_l$ vertices, precisely, $M_l = \{\pi \in A : l \subset \pi\}$; the number of planes containing $l$; hence $h_0(\Delta_l) = M_l - 1$. We observe that $M_l = 0$, if $l$ is an atom. Hence, $\sum_l h_0(\Delta_l) = \sum_{l'} (M_{l'} - 1)$, where the second sum run over the non atom lines $l'$. Observe that

$$\sum_{l'} (M_{l'} - 1) = \sum_{l'} M_{l'} - \sum_{l'} 1 = \sum_l M_l - \sum_{l'} 1.$$

Observe that $a_{n-1} = \sum_l 1 = \sum_{l'} 1 + N$. □
The finite linear groups generated in codimension two in $GL_3(\mathbb{R})$ and strictly generated in codimension two in $GL_4(\mathbb{R})$ satisfy the hypothesis of the previous proposition.

**Corollary 7.14.** If $G$ is a finite linear group strictly generated in codimension two in $GL_4(\mathbb{R})$, then

$$\sum_l h_0(\Delta_l) = \sum_l M_l - a_3^G + N.$$  

**Proof.** There are no hyperplanes in $\mathcal{A}^G$ because of Proposition 4.2. Observe that lines are in codimension three. □

If $G$ is strictly generated in codimension two in $GL_4(\mathbb{R})$, then

$$h^1(U_G) = h_1(\Delta_G) + \sum_l M_l - a_3^G + N + a_2^G.$$  

where $M_l$, $a_3^G$, $a_2^G$, and $N$ are defined above.

**Theorem 7.16.** If $G$ is a finite linear group strictly generated in codimension two in $GL_4(\mathbb{R})$, then

$$h^1(U_G) = h_1(\Delta_G) + \sum_l M_l - a_3^G + N + a_2^G,$$

$$h^2(U_G) > 2N - 1.$$  

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