Projective toric polynomial generators in the unitary cobordism ring

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Abstract. According to Milnor and Novikov’s classical result, the unitary cobordism ring is isomorphic to a graded polynomial ring with countably many generators: \( \Omega^U_* \simeq \mathbb{Z}[a_1, a_2, \ldots], \deg(a_i) = 2i \). In this paper we solve the well-known problem of constructing geometric representatives for the \( a_i \) among smooth projective toric varieties, \( a_n = [X^n] \), \( \dim \mathbb{C} X^n = n \). Our proof uses a family of equivariant modifications (birational isomorphisms) \( B_k(X) \rightarrow X \) of an arbitrary complex manifold \( X \) of complex dimension \( n \) \((n \geq 2, k = 0, \ldots, n - 2)\). The key fact is that the change of the Milnor number under these modifications depends only on the dimension \( n \) and the number \( k \) and does not depend on the manifold \( X \) itself.

Bibliography: 22 titles.

Keywords: unitary cobordism, toric varieties, blow-ups, convex polytopes.

§ 1. Introduction

Unitary cobordism theory is an extraordinary cohomology theory, which plays an important role in algebraic topology and its applications (see [5]). The construction of distinguished representatives in every class \( a \in \Omega^U_* \) is an important problem in the theory of unitary cobordisms. In the late 1950s Hirzebruch raised a question in [10] which can be reformulated as follows in terms of Chern numbers: given a tuple of integers, when is it equal to the tuple of Chern numbers of an irreducible complex algebraic variety? Already in real dimension 4 this problem is connected with deep questions in algebraic geometry and is still open in general.

A challenging problem closely related to Hirzebruch’s original question is to describe particular manifolds representing the multiplicative generators of the ring \( \Omega^U_* \). There are classical theorems (see, for example, Stong’s monograph [18]) as well as very recent results (see [22]) concerning this problem. First, recall that over \( \mathbb{Q} \) the unitary cobordism ring is a polynomial ring generated by the classes of projective spaces \( \{\mathbb{C}P^k\}_{k=1}^\infty \) (see [18], Ch. VII):

\[
\Omega^U_* \otimes \mathbb{Q} \simeq \mathbb{Q}[[\mathbb{C}P^1], [\mathbb{C}P^2], \ldots].
\]

G. D. Solomadin’s research was supported by a grant from the Russian Science Foundation (project no. 14-11-00414) in the Steklov Mathematical Institute of the Russian Academy of Sciences. Sections 1, 2.2, 3, 4.1, 5.2 and 6 are the work of Yu. M. Ustinovskiy. The other sections are due to G. D. Solomadin.

AMS 2010 Mathematics Subject Classification. Primary 14M25; Secondary 55N22, 57R77, 52B20.

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So, over the rational numbers the ring $\Omega_*^U \otimes \mathbb{Q}$ has an explicit and simple set of generators. However, the search for multiplicative generators for $\Omega_*^U$ itself turns out to be difficult due to the divisibility relations among the characteristic numbers of stably complex manifolds. As a result, many approaches to this problem lead to subtle questions in number theory [22], [15]. In accordance with a general result proved independently by Milnor and Novikov, finding a sequence of stably complex manifolds to generate the cobordism ring $\Omega_*^U$ is equivalent to constructing manifolds whose distinguished characteristic number $s_n$ takes certain special values.

**Theorem 1** (Milnor and Novikov; see [17] and [18]). The cobordism class of a stably complex manifold $X^{2n}$, $\dim \mathbb{R} X^{2n} = 2n$, can be taken to be a multiplicative generator of degree $2n$ if and only if

$$s_n(X^{2n}) = \begin{cases} \pm p & \text{if } n = p^k - 1 \text{ for some prime number } p, \\ \pm 1 & \text{otherwise.} \end{cases}$$

Here $s_n(X^{2n})$ is the Milnor number of $X^{2n}$ (also sometimes referred to as the top characteristic number):

$$s_n(X^{2n}) := \langle [X^{2n}], t_1^n + \cdots + t_n^n \rangle,$$

where $t_1, \ldots, t_n$ are the Chern roots of the stably complex tangent bundle $TX^{2n}$.

An explicit set of generators of the ring $\Omega_*^U$ was first given by Milnor in the 1960s. He proved that suitable linear combinations of cobordism classes of bidegree $(1, 1)$ hypersurfaces $H_{i,j} \subset \mathbb{CP}^i \times \mathbb{CP}^j$ and projective spaces $\mathbb{CP}^n$ generate the unitary cobordism ring. As a result of Milnor’s construction we can produce (generally speaking, not connected) algebraic generators of the ring $\Omega_*^U$; see, for example, [16] and [18]. In the category of stably complex manifolds we can define an operation of connected sum. This allows us to construct connected stably complex generators of $\Omega_*^U$. The corresponding stably complex manifolds are not necessarily algebraic. In view of these observations, it is natural to ask: is it possible to choose connected algebraic varieties as polynomial generators of the ring $\Omega_*^U$? A positive answer to this question was given by Johnston [14] in 2004. He produced connected algebraic generators of $\Omega_*^U$ via sequences of blow-ups of projective varieties along complete intersections in the corresponding exceptional divisors.

Another problem related to the construction of ‘nice’ generators of the ring $\Omega_*^U$ is the search for polynomial generators with a large symmetry group. Toric varieties constitute a rich source of such manifolds. A normal complex algebraic variety is called toric if it admits an effective action of an algebraic torus $(\mathbb{C}^*)^{\dim X}$ with an open dense orbit. In what follows we shall study smooth projective toric varieties. Buchstaber and Ray [3] and Buchstaber, Panov and Ray [4] used a wider class of stably complex manifolds to construct a representative in any class of the ring $\Omega_*^U$. More precisely, they considered quasitoric manifolds, that is, toric manifolds in the sense of Davis and Januszkiewicz [8]. These are connected stably complex manifolds $M^{2n}$ with an effective action of a half-dimensional torus $(S^1)^n$. The orbit space of such an action is naturally identified with a simple $n$-dimensional polytope. The main result in [4] is the following.
**Theorem 2** (Buchstaber, Panov and Ray; see [4], Theorem 5.9). For dimensions of 3 or more, each unitary cobordism class can be represented by a connected quasitoric manifold $X^{2n}$ endowed with an $(S^1)^n$-invariant stably complex structure. The stably complex structure on $X^{2n}$ is given by the omniorientation on the simple polytope corresponding to $X^{2n}$.

The key tool in proving the above theorem is the equivariant box sum operation.

Various problems concerned with constructing representatives for classes of unitary cobordisms and finding generators of $\Omega_*^U$ in the class of smooth projective toric varieties lead to more subtle questions in convex geometry and number theory. Wilfong [21] fully explored toric representability for elements of toric varieties lead to more subtle questions in convex geometry and number theory.

**Theorem 3** (Wilfong [21], Theorem 5.1). Let $[M] \in \Omega_*^U$.

1) If $c_1c_2[M] \neq 24$ or $c_3[M] \notin \{4, 6, 8, \ldots \}$, then $[M]$ cannot be represented by a smooth projective toric variety.

2) Suppose $c_1c_2[M] = 24$ and $c_3[M] = 4$. Then $[M]$ is represented by a smooth projective toric variety if and only if $c_2^3[M] = 64$.

3) Suppose $c_1c_2[M] = 24$ and $c_3[M] = 6$. Then $[M]$ is represented by a smooth projective toric variety if and only if the equality $c_2^3[M] = 2a^2 + 54$ holds for some integer $a \in \mathbb{Z}$.

4) Let $c_1c_2[M] = 24$ and $c_3[M] \in \{8, 10, 12, \ldots \}$. Then $[M]$ can be represented by a smooth projective toric variety.

Wilfong [22] made significant advances in the question of constructing smooth toric generators of the ring $\Omega_*^U$. Throughout what follows, $n$ denotes the complex dimension of any complex manifold under consideration (unless otherwise stated). The main result in [22] is given as follows.

**Theorem 4** (Wilfong [22], Theorem 1.2). Let $n$ be odd or have the form $n = p^m - 1$ for some prime $p$ and integer $m \in \mathbb{N}$. Then the class of some projective toric variety can be taken as the multiplicative generator of degree $2n$ of the unitary cobordism ring.

The toric varieties in the above theorem are blow-ups of the generalized Bott towers of height 1 and 2 at fixed points and along invariant rational curves.

Thus, to answer the question of the existence of smooth projective toric multiplicative generators of the ring $\Omega_*^U$ it remains to find such varieties $X^n$ satisfying the assumptions of Theorem 1 in all even dimensions $n$ such that $n + 1$ is not a power of a prime (the least such $n$ is 14). This paper aims to construct such varieties in the remaining dimensions. Our approach to this problem combines the ideas of [14] and [22]. Similarly to Johnston and Wilfong, we study blow-ups of complex algebraic manifolds. However, unlike [22] we define a whole family of birational modifications $B_k(X) \to X$ over any complex manifold $X$. Equivariant versions
of these morphisms are well-defined in the category of toric varieties (see §3.3). The modification $B_k$ is the sequential blow-up

$$B_k(X) \to \text{Bl}_x X \to X,$$

where $\text{Bl}_x X \to X$ is the blow-up at a point $x \in X$ and $B_k(X) \to \text{Bl}_x X$ is the blow-up along a submanifold $\mathbb{C}P^k \subset E_x$ of the exceptional divisor $E_x \subset \text{Bl}_x X$. There is an ambiguity in the choice of the point $x \in X$ and the subvariety $\mathbb{C}P^k \subset E_x$ of the exceptional divisor $E_x \subset \text{Bl}_x X$.

There is an ambiguity in the choice of the point $x \in X$ and the subvariety $\mathbb{C}P^k \subset E_x$ to blow up. It turns out that the difference $[B_k(X)] - [X]$ in $\Omega_*^U$ depends only on the numbers $n$ and $k$; in particular, it does not depend on $X$. Hence the numbers $s_{k,n} = s_n(B_k(X)) - s_n(X)$ are well defined. They measure the change of the Milnor number of an $n$-dimensional manifold $X$ under the operations $B_k$.

The crucial property of the numbers $s_{k,n}$ in the dimensions we are looking at is that $\gcd(s_{0,n}, \ldots, s_{n-2,n}) = 1$. This fact allows us to start from a smooth projective toric variety $X$ with $s_n(X)$ large enough and apply a sequence of modifications $B_{k_i}$ to get a smooth projective toric variety with Milnor number 1.

The main result of this paper can be formulated as follows.

**Theorem 5.** There exist smooth projective toric varieties $\{X^n\}_{n=1}^\infty$ of complex dimension $n$ such that

$$\Omega_*^U = \mathbb{Z}[[X_1], [X_2], \ldots].$$

In the category of toric varieties the modifications $B_k$ have an interesting property. Namely, the toric varieties $B_k(X)$ and $B_{n-k-2}(X)$ have combinatorially equivalent moment polytopes, provided that the blown-up invariant subvarieties $\mathbb{C}P^k \subset E$ and $\mathbb{C}P^{n-k-2} \subset E$ of the exceptional divisor $E \subset \text{Bl}_x X$ are properly chosen (see Claim 2). This property allows us to formulate a theorem which characterises the 2-parameter Todd genus in terms of combinatorial rigidity (see Theorem 9).

§2. The basic concepts

This section contains a brief review of the basic concepts and also some results in unitary cobordism, toric varieties and blow-ups of complex manifolds. For a more comprehensive treatment of these subjects see [18], [6] and [9].

2.1. Unitary cobordism. Let $M_1^n$ and $M_2^n$ be $C^\infty$-smooth real closed compact manifolds of dimension $n$. $M_1^n$ and $M_2^n$ are called (nonoriented) bordant if there exists a smooth real compact manifold $W^{n+1}$ of dimension $n+1$ such that $\partial W^{n+1} = M_1^n \sqcup M_2^n$. Bordism is an equivalence relation. The disjoint union operation endows the set $\Omega_*^O$ of nonoriented bordism classes for $n$-dimensional manifolds with the structure of an (abelian) group. The direct sum $\Omega_*^O = \bigoplus_{i=0}^\infty \Omega_i^O$ of abelian groups is a commutative ring with respect to the Cartesian product. To define the ring of oriented cobordisms $\Omega_*^{SO}$ we have to take oriented $C^\infty$-smooth real closed compact manifolds and choose the orientation on $\partial W^{n+1}$ inducing compatible orientations on the $M_i^n$. For complex manifolds a cobordism relation is defined in a more subtle way.
**Definition 1.** Let $M$ be a $C^\infty$-smooth closed $n$-dimensional real manifold. A *stably complex structure* on $M$ is an isomorphism class of real vector bundles

$$\iota: TM \oplus \mathbb{R}^{2m-n} \simeq \xi,$$

where $m \in \mathbb{Z}$, $2m \geq n$, and $\xi \to M$ is some rank $m$ complex vector bundle over $M$. Any pair $(M, \iota)$ of a manifold $M$ endowed with a stably complex structure (or $M$ itself in this case) is called a *stably complex manifold*.

**Definition 2.** Closed stably complex compact manifolds $(M_1^n, \iota_1)$ and $(M_2^n, \iota_2)$ of real dimension $n$ are called *unitary cobordant* if there exists a compact stably complex manifold $(W^{n+1}, \iota)$ with boundary such that $\partial W^{n+1} = M_1^n \sqcup M_2^n$, $\iota|_{M_1^n} = \iota_1$, and $\iota|_{M_2^n} = -\iota_2$.

Consider the full Chern class of the complex vector bundle $\xi$ over the stably complex manifold $(M^{2k}, \iota)$:

$$c(\xi) = 1 + c_1(\xi) + \cdots + c_k(\xi), \quad c_i(\xi) \in H^{2i}(M; \mathbb{Z}).$$

**Definition 3.** Let $k = i_1 + \cdots + i_t$ be a partition. A *characteristic Chern number* (or simply a Chern number) of the stably complex manifold $M$ is the pairing

$$\langle c_{i_1}(\xi) \cdots c_{i_t}(\xi), [M] \rangle \in \mathbb{Z}$$

of the cohomology class $c_{i_1}(\xi) \cdots c_{i_t}(\xi) \in H^{2k}(M; \mathbb{Z})$ with the fundamental class $[M] \in H_{2k}(M; \mathbb{Z})$.

Formally, we can write the full Chern class of the bundle $\xi$ as:

$$c(\xi) = \prod_{i=1}^k (1 + t_i),$$

where the $t_i$ are Chern roots of the bundle $\xi$. As $c_i(\xi)$ is the elementary symmetric polynomial $\sigma_i(t_1, \ldots, t_k)$, using the symmetric polynomial theorem, for any symmetric polynomial $P(t_1, \ldots, t_k)$ of degree $2k$ (deg $t_i = 2$, $i = 1, \ldots, k$) the pairing $\langle P(t_1, \ldots, t_k), [M] \rangle$ with the fundamental class is well-defined.

### 2.2. Convex polytopes.

**Definition 4.** A *convex polytope* in an affine (real) space $V \simeq \mathbb{R}^n$ is the convex hull of a non-empty finite set of points of $V$. Another equivalent definition is as follows: a convex polytope in $V$ is a bounded nonempty intersection of a finite number of semispaces:

$$P = \{ v \in V \mid \langle v, a_i \rangle + b_i \geq 0, \; i = 1, \ldots, m \}, \quad (2.1)$$

where the $a_i \in V^*$ are linear functions on $V$ and $b_i \in \mathbb{R}$. The *dimension* of a convex polytope $P$ is the dimension of the linear span of its vertices and is denoted by $\dim P$. In what follows we assume tacitly that system $(2.1)$ is reduced, that is, none of the inequalities $\{ \langle v, a_i \rangle + b_i \geq 0 \}$ follows from the others. In this case the intersection of the polytope $P$ with any of the hyperplanes $\{ \langle v, a_i \rangle + b_i = 0 \}$ is called a *facet* of $P$. A nonempty intersection of facets of $P$ is called a *face* of $P$. A convex polytope $P$ is called *simple* if any vertex of $P$ is contained in exactly $\dim P$ facets.
Fix a lattice $N \simeq \mathbb{Z}^n$ in $V$. Assume that the convex polytope $P$ is rational, that is, $a_i \in \mathbb{N}^*$ and $b_i \in \mathbb{Q}$. Let $\nu_1, \ldots, \nu_m$ be primitive normal vectors to all facets of $P$.

**Definition 5.** A simple $n$-dimensional convex polytope $P \subset V$ is called a Delzant polytope if for any vertex $v \in P$ the normal vectors $\nu_1, \ldots, \nu_m$ form a basis of the dual lattice $N^* \subset V^*$ for all the facets of $P$ containing $v$.

### 2.3. Toric varieties.

**Definition 6.** Let $M$, $\dim \mathbb{C} M = n$, be a normal complex algebraic variety containing the Zariski open dense algebraic torus $(\mathbb{C}^*)^n$. $M$ is called a toric variety if the shift action of the torus $(\mathbb{C}^*)^n$ on itself extends to an action of $(\mathbb{C}^*)^n$ on $M$.

It is a well-known fact that there exists a bijection between $n$-dimensional toric varieties and rational fans in the Lie algebra $\mathfrak{t}$ of the compact torus $(S^1)^n \subset (\mathbb{C}^*)^n$. Here the lattice dual to the character lattice is fixed in $\mathfrak{t}$.

Let $P \subset V$ be an $n$-dimensional Delzant polytope with vertices in the integer lattice $N \subset V$. Then we can construct the rational fan $\Sigma_P$ in the dual space $V^*$ (a normal fan of the polytope $P$). The fan $\Sigma_P$ corresponds to some smooth toric variety $X_P$. The polytope $P$ induces an embedding of $X_P$ into some projective space and vice versa (up to translations in $V$). The polytope $P$ is often called a moment polytope of the variety $X_P$ with a fixed projective embedding. For more information about these correspondences see [6], Ch. 5.

Let $X_P$ be a smooth projective toric variety. There is only a finite number of orbits for the action of the torus $(\mathbb{C}^*)^n$ on $X_P$. In particular, the number of invariant divisors $D_i \subset X_P$ is finite. The stabilizer of an invariant divisor $D_i$ is a one-dimensional subgroup $\mathbb{C}^* \subset (\mathbb{C}^*)^n$. The co-character of this subgroup determines the normal vector to the facet $H_i$ of the polytope $P^n$. Any invariant $k$-dimensional subvariety $Z^{n-k} \subset X_P$ is the intersection of $k$ invariant divisors $D_{ij}$, $j = 1, \ldots, k$. The subvariety $Z^{n-k}$ then corresponds to the $(n-k)$-dimensional face $\bigcap_{j=1}^{k} H_{ij} \subset P$, which is the intersection of the facets $H_{ij}$.

### 2.4. Blow-ups of complex manifolds.

Let $D \subset \mathbb{C}^n$ be a unit polydisc of complex dimension $n$ with centre at the origin. We introduce holomorphic and homogeneous coordinates $z_i$ and $L_i$ in $D$ and $\mathbb{C}P^n$, respectively. The space

$$\text{Bl}_0 D = \{(z, L) \in D \times \mathbb{C}P^{n-1} \mid z_i L_j = z_j L_i \text{ for all } i, j\},$$

is called a blow-up of $D$ at the origin.

Clearly,

$$\text{Bl}_0 D = \{(z, L) \in D \times \mathbb{C}P^{n-1} \mid z \in L\},$$

where $L \in \mathbb{C}P^{n-1}$ is regarded as a line in $\mathbb{C}^n$. This implies that the projection $\pi \colon \text{Bl}_0 D \to D$ onto the first factor is an isomorphism away from the origin and $\pi^{-1}(0) \simeq \mathbb{C}P^{n-1}$.

Let $M^n$ be a smooth complex manifold of complex dimension $n$. Fix a point $x \in M$. The space obtained from $M$ by the above construction in some neighbourhood of $x \in M$ is called a blow-up of $M$ at $x$ and is denoted by $\text{Bl}_x M$. 

The idea of the blow-up of a manifold at a point can be generalized to blowing up submanifolds of positive dimension. To do this we first define the blow-up of the subset
\[ V = \{(z_1, \ldots, z_n) \in D \mid z_{k+1} = \cdots = z_n = 0\} \]
of the unit polydisc \( D \subset \mathbb{C}^n \). Let \([L_{k+1}: \ldots: L_n]\) be the homogeneous coordinates of \( \mathbb{C}P^{n-k-1} \). The space
\[ \text{Bl}_V D = \{(z, L) \in D \times \mathbb{C}P^{n-k-1} \mid z_iL_j = z_jL_i \text{ for any } i, j = k+1, \ldots, n\}, \]
is called a blow-up of \( D \) along the submanifold \( V \). This operation can be applied to any complex manifold \( M \) and a submanifold of it, \( V \subset M \). The resulting space is called the blow-up of \( M \) along \( V \) and is denoted by \( \text{Bl}_V M \).

Given any manifolds \( V \subset M \), \( \dim \mathbb{C} M = n \), \( \dim \mathbb{C} V = k \), there is a projection map \( \pi: \text{Bl}_V M \rightarrow M \). Away from \( V \) the map \( \pi \) is an isomorphism. For any point \( v \in V \) the corresponding fibre is \( \pi^{-1}(v) \simeq \mathbb{C}P^{n-k-1} \).

§ 3. Modifications of complex manifolds and relations in the unitary cobordism ring

3.1. Projectivizations of complex vector bundles. We start this section with a brief review of important facts about the projectivizations of vector bundles and their stably complex structures. Throughout this section we tacitly assume that \( B \) is a smooth complex compact manifold.

**Theorem 6** (Leray-Hirsch; see [1], §15). Suppose that \( \xi \rightarrow B \) is a complex \((n-k+1)\)-dimensional vector bundle over \( B \), \( \dim \mathbb{C} B = k \). Consider the fibrewise projectivization \( p: \mathbb{P}(\xi) \rightarrow B \) of the bundle \( \xi \). Let \( v = c_1(\gamma) \in H^2(\mathbb{P}(\xi), \mathbb{Z}) \) denote the first Chern class of the fibrewise line bundle \( \gamma = \mathcal{O}(1) \) over \( \mathbb{P}(\xi) \). Then there is an isomorphism of graded rings:
\[ H^*(\mathbb{P}(\xi)) \simeq H^*(B)[v]/(v^{n-k+1} + v^{n-k}c_1(\xi) + \cdots + c_{n-k+1}(\xi)). \tag{3.1} \]

The projection map \( p \) induces the cohomology homomorphism \( p^*: H^*(B; \mathbb{Z}) \rightarrow H^*(\mathbb{P}(\xi); \mathbb{Z}) \), which endows the ring \( H^*(\mathbb{P}(\xi); \mathbb{Z}) \) with the natural structure of a module over the ring \( H^*(B; \mathbb{Z}) \). It follows from (3.1) (see [20], §2.2) that
\[ \langle \omega \cdot v', [\mathbb{P}(\xi)] \rangle = \langle \omega \cdot c^{-1}(\xi), [B] \rangle, \tag{3.2} \]
where \( \omega \in H^{2n-2l}(B; \mathbb{Z}) \) is a cohomology class and \( c^{-1}(\xi) \in H^*(B; \mathbb{Z}) \) is the full Segre class, that is, the multiplicative inverse of the full Chern class \( c(\xi) = 1 + c_1(\xi) + \cdots + c_{n-k+1}(\xi) \).

There is a canonical stably complex structure on the projective bundle \( \mathbb{P}(\xi) \) given by the isomorphism
\[ T\mathbb{P}(\xi) \oplus \mathbb{C} \simeq (p^*\xi \otimes \gamma) \oplus p^*TB. \tag{3.3} \]
Suppose that \( \xi \) is a holomorphic vector bundle. Then the stably complex structure (3.3) is (stably) equivalent to the complex structure on the complex manifold \( \mathbb{P}(\xi) \). However, in what follows we will use non-standard stably complex structures on the manifolds \( \mathbb{P}(\xi) \).
Definition 7. Consider a splitting vector bundle $\xi = \zeta \oplus \mathbb{C}$ over the base $B$. Define a nonstandard stably complex structure given by the isomorphism of real vector bundles:

$$TP(\zeta \oplus \mathbb{C}) \oplus \mathbb{C} \simeq (p^* \zeta \otimes \gamma) \oplus \gamma^* \oplus p^* TB.$$  \hfill (3.4)

We will denote the manifold $\mathbb{P}(\zeta \oplus \mathbb{C})$ endowed with the stably complex structure (3.4) by $\mathbb{P}(\zeta \oplus \overline{\mathbb{C}})$.

The stably complex structure provided above differs from the standard one because the bundle $\gamma$ is replaced by $\gamma^*$. The nonstandard stably complex structure on $\mathbb{P}(\xi)$ induces the stably complex structure on the fibre $\mathbb{C}P^{n-k}$ of the projective bundle $\mathbb{P}(\xi) \to B$ given by the isomorphism of real vector bundles:

$$T\mathbb{C}P^{n-k} \oplus \mathbb{C} \simeq \gamma \oplus (n-k) \oplus \gamma^*.$$  

3.2. Blow-ups of complex submanifolds (continued). Consider a smooth compact complex manifold $X$ and a complex submanifold $Z \subset X$ of it. Recall that $\text{Bl}_Z X$ denotes the blow-up of the manifold $X$ along the submanifold $Z$. Now, blow-up is a local operation, that is, it depends only on the tubular neighbourhood of $Z \subset X$. Hence it is reasonable to expect that the difference $[\text{Bl}_Z X] - [X]$ is determined by the normal vector bundle $\nu(Z \subset X)$.

Proposition 1 (Hitchin [13], §4.5). Consider the blow-up $\pi: \text{Bl}_Z X \to X$ along $Z$. Then the difference of the classes of manifolds $\text{Bl}_Z X$ and $X$ in the unitary cobordism ring is given by

$$[\text{Bl}_Z X] - [X] = -[\mathbb{P}(\nu(Z \subset X) \oplus \overline{\mathbb{C}})],$$  \hfill (3.5)

where $\nu(Z \subset X)$ is a normal bundle to $Z$, and the projectivization $\mathbb{P}(\nu(Z \subset X) \oplus \overline{\mathbb{C}})$ is equipped with the nonstandard stably complex structure (3.4).

Example 1 (blow-up at a point $\text{Bl}_x X \to X$). Apply Proposition 1 to the blow-up $\text{Bl}_x X$ of a manifold $X$ at a point $x \in X$. In this case the normal bundle is trivial $\nu = \mathbb{C}^{\dimc} X$. So (3.5) reduces to

$$[\text{Bl}_x X] - [X] = -[\mathbb{P}(\mathbb{C}^{\dimc} X \oplus \overline{\mathbb{C}})].$$  \hfill (3.6)

It is well known that the manifold $\text{Bl}_Z X$ is obtained from $X$ under the blow-up map $\pi: \text{Bl}_Z X \to X$ by adding the exceptional divisor $E = \pi^{-1}(Z) \simeq \mathbb{P}(\nu(Z \subset X))$ with the normal vector bundle $\gamma^* = \mathcal{O}(-1)$ (for example, see [9], §6). In particular, for the blow-up $\pi: \text{Bl}_x X \to X$ of an $n$-dimensional complex manifold at a point $x \in X$ the corresponding exceptional divisor $E = \pi^{-1}(x) \simeq \mathbb{C}P^{n-1}$ has the normal vector bundle $\nu(E \subset \text{Bl}_x X) \simeq \mathcal{O}(-1)$.

Now we define our key tool, a family of birational modifications $B_k(X) \to X$.

Definition 8 (modifications $B_k(X)$). Let $X$ be a smooth complex manifold of complex dimension $n$. Consider the blow-up $\pi: \text{Bl}_Z X \to X$ at a point $x \in X$. Fix a number $0 \leq k \leq n-2$ and choose a submanifold $Z^k \simeq \mathbb{C}P^k$ of the exceptional divisor $E = \pi^{-1}(x) \simeq \mathbb{C}P^{n-1}$ with the normal vector bundle $\nu(Z \subset E) \simeq \mathcal{O}(1)^{(n-k-1)}$. In other words, the submanifold $Z \subset E$ is a $k$-dimensional projective space in $E$. We call the blow-up $\text{Bl}_Z(\text{Bl}_x X)$ of the manifold $\text{Bl}_x X$ along $Z$ a $k$-modification of the manifold $X$ and denote it by $B_k(X)$. 

Remark 1. The notation $B_k(X)$ is ambiguous since it does not specify the blown-up point $x \in X$ or the submanifold $Z \subset E$, but different choices of $x \in X$ and $Z \subset E$ result in different complex manifolds $B_k(X)$. However, in this paper we are interested mainly in the cobordism class of $[B_k(X)]$. By Proposition 1 the latter does not depend on the choice of $x$ and $Z$. Thus we shall not label the blown-up point or submanifold in using the $k$-modification, unless otherwise specified.

The normal bundle $\nu(Z \subset \text{Bl}_x X)$ is isomorphic to $\mathcal{O}(-1) \oplus (n-k-1)$, where $\dim_{\mathbb{C}} \mathbb{C} = n$, $\dim_{\mathbb{C}} Z = k$. Proposition 1 gives the following formula for the difference of the cobordism classes $[B_k(X)]$ and $[\text{Bl}_x X]$:

$$[B_k(X)] - [\text{Bl}_x X] = -[\mathbb{P}(\mathcal{O}(-1) \oplus (n-k-1) \oplus \mathbb{C})].$$

(3.7)

The stably complex manifold $D_{k,n} := \mathbb{P}(\mathcal{O}(-1) \oplus (n-k-1) \oplus \mathbb{C})$ is the projectivization of the $(n-k+1)$-dimensional vector bundle $\mathcal{O}(-1) \oplus (n-k-1) \oplus \mathbb{C}$ over $Z \simeq \mathbb{C}P^k$. It follows from the definition of $D_{k,n}$ that

$$s_n(B_k(X)) - s_n(\text{Bl}_x X) = -s_n(D_{k,n}).$$

(3.8)

In §4.2 we will compute the Milnor numbers $s_n(D_{k,n})$.

3.3. Equivariant modifications $B_k$ of toric varieties. In §3.2 we defined the family of modifications $B_k(X) \rightarrow X$, $k = 0, \ldots, n-2$, for any complex manifold $X$. Now we describe the equivariant analogues of the operations $B_k(X)$ in the category of smooth projective toric varieties.

Let $X = X_P$ be a smooth projective toric variety corresponding to a Delzant polytope $P$. It is a well-known fact (see [6]) that the blow-up along an invariant submanifold $Z \subset X$ gives the equivariant modification $\text{Bl}_Z X \rightarrow X$. Recall that there exists a one-to-one correspondence between invariant $k$-dimensional subvarieties of $X$ and $k$-dimensional faces of the polytope $P$ (see §2.3). Consider an invariant $k$-dimensional submanifold $Z \subset X$ and the corresponding $k$-dimensional face $G$ of the polytope $P$. The face $G$ is the intersection of $n-k$ facets $H_i \subset P$. Let $\nu_i \in \mathbb{Z}^n$ be the primitive normal vector to the facet $H_i$. We choose $\nu_i$ to point into $P$. Let $H \in \mathbb{R}^n$ be an affine hyperplane with normal vector $\nu_1 + \cdots + \nu_{n-k}$ such that it does not contain any vertex of $P$ and separates all the vertices of the face $G$ from the other vertices of $P$. The polytope cut$_G P := P \cap H_{\nu_1}$ which is the intersection of $P$ with the positive semispace $H_{\nu_1}$, is called a truncation of the polytope $P$ along the face $G$.

Proposition 2 (see [6], §7.4). The polytope cut$_G P$ is Delzant. It corresponds to the toric manifold $\text{Bl}_Z X$.

Recall that the modification $B_k(X)$ is defined by the choice of a point $x \in X$ and a $k$-dimensional submanifold $Z = \mathbb{C}P^k$ of the exceptional divisor $E \subset \text{Bl}_x X$. The variety $\text{Bl}_x X$ corresponds to the polytope cut$_p P$. The latter is obtained from $P$
by a truncation at the vertex \( p \) corresponding to the fixed point \( x \in X \). The exceptional divisor \( E \subset \text{Bl}_x X \) is a \((\mathbb{C}^*)^n\)-invariant subvariety of \( \text{Bl}_x X \). Similarly, the blow-up \( B_k(X) = \text{Bl}_Z(\text{Bl}_x X) \to X \) is an equivariant modification, provided that the fixed submanifold \( Z \simeq \mathbb{C}P^k \), \( Z \subset E \subset \text{Bl}_x(X) \) is \((\mathbb{C}^*)^n\)-invariant. The blow-up \( B_k(X) = \text{Bl}_Z(\text{Bl}_x X) \to X \) is an equivariant modification and \( B_k(X) \) is a toric variety corresponding to the polytope \( P_k = \text{cut}_{S_k}(\text{cut}_p P) \). To put it another way, the polytope \( P_k \) is obtained from \( P \) by two sequential truncations: at the vertex \( p \) and then at one of the new \( k \)-dimensional simplicial faces \( S_k \simeq \Delta^k \).

So we have the following.

Claim 1. In the category of toric varieties the manifold \( B_k(X) = \text{Bl}_Z(\text{Bl}_x X) \) is toric and the modification \( B_k(X) \to X \) is equivariant, provided that \( x \) is a fixed point and \( Z \) is an invariant submanifold.

Below we shall only consider equivariant blow-ups and modifications of toric varieties.

§4. The construction of toric polynomial generators of the ring \( \Omega^U_* \)

4.1. Proof of the main theorem. In this section we construct a toric variety of complex dimension \( n \) with Milnor number \( s_n(X) = 1 \) for each even dimension \( n \) such that \( n+1 \) is not a power of a prime. By Theorem 1 this will imply the existence of toric polynomial generators of the ring \( \Omega^U_* \). Recall that toric varieties in all other dimensions with the necessary Milnor numbers were constructed by Wilfong (see Theorem 4).

Claim 1 implies that the equivariant modifications \( B_k(X) \to X, k = 0, \ldots, n-2 \), are defined in the category of toric varieties. Set

\[
s_{k,n} := s_n(B_k(X)) - s_n(X).
\]

It follows from equations (3.6) and (3.7) that the numbers \( s_{k,n} \) do not depend on the choice of the manifold \( X \).

Definition 9. Integers \( t_1, \ldots, t_l \in \mathbb{Z} \) are called coprime if the ideal they generate in the ring \( \mathbb{Z} \) coincides with \( \mathbb{Z} \).

We need the following key lemma to prove the existence of toric polynomial generators of the unitary cobordism ring.

Lemma 1. Let \( n \) be an even integer such that \( n+1 \) is not an integer power of a prime. Then the integers \( s_{0,n}, s_{1,n}, \ldots, s_{n-2,n} \) are coprime.

We will prove this lemma in §4.3, but first we use it to derive our main theorem.

Theorem 7. There exists a sequence of smooth projective toric varieties \( \{X^n\}_{n=1}^\infty \), representing polynomial generators of \( \Omega^U_* \).

The proof of this theorem is based on several lemmas.

Lemma 2. Let \( t_0, \ldots, t_l \) be coprime integers such that \( t_0 > 0 \). Then there exists a natural number \( N = N(t_0, \ldots, t_l) \) such that for any integer \( x > N \) there exists a representation \( x = \sum_{i=0}^l a_it_i \), where the \( a_i \geq 0, i = 0, \ldots, l \), are non-negative integers.
Proof. When all the $t_i$ are nonnegative this is an elementary fact from number theory, related to the classical Frobenius problem, see [2], for example.

The general case (that is, when some $t_i$ are negative) will be reduced to this case. Namely, there exists a number $N = N(|t_0|, |t_1|, \ldots, |t_l|)$ satisfying the assumptions of the lemma for the absolute values $\{|t_i|\}_{i=0}^l$. We claim that any integer $x > N$ can be represented as a linear combination of the numbers $\{t_i\}_{i=0}^l$ with non-negative integer coefficients. Let $x > N$. From the definition of $N$ there exists a linear combination

$$x = \sum_{i=0}^l a_i \cdot |t_i| = \sum_{i=0}^l (a_i \cdot \text{sgn}(t_i)) \cdot t_i$$

with non-negative integer coefficients $a_i$, where $\text{sgn}(t)$ denotes the sign of the number $t$. Recall that $t_0 > 0$ by hypothesis. Hence $\text{sgn}(t_0) = 1$. For any index $j$ such that $t_j$ is negative we replace $a_0$ by $a'_0 = a_0 - kt_j$ and $a_j \cdot \text{sgn}(t_j)$ by $a'_j = a_j \cdot \text{sgn}(t_j) + kt_0$, where $k \in \mathbb{N}$. Obviously, this does not change the linear combination. For $k$ large enough ($k > a_j/t_0$) the coefficients of both $t_0$ and $t_j$ become positive. After performing this procedure for all the negative $t_j$ we obtain the desired representation.

Lemma 3. Let $n > 2$. Then for any natural number $N \in \mathbb{N}$ there exists an $n$-dimensional smooth projective toric variety $X$, $\dim_{\mathbb{C}} X = n$, with Milnor number $s_n(X) > N$.

Proof. Consider a vector bundle $\xi = \pi_1^* \mathcal{O}(-1) \oplus \pi_2^* \mathcal{O}(a) \oplus \mathbb{C}^{n-3}$ over $\mathbb{C}P^1 \times \mathbb{C}P^1$, where $\pi_1, \pi_2 : \mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}P^1$ are projections onto the first and second factors, respectively. The elements $u_1 = \pi_1^* c_1(\mathcal{O}(1))$ and $u_2 = \pi_2^* c_1(\mathcal{O}(1))$ are the generators of $H^2(\mathbb{C}P^1 \times \mathbb{C}P^1; \mathbb{Z})$. Clearly, $u_1^2 = u_2^2 = 0$, $\langle u_1 u_2, [\mathbb{C}P^1 \times \mathbb{C}P^1] \rangle = 1$. The full Chern and Segre classes of the bundle $\xi$ are

$$c(\xi) = 1 - u_1 + au_2 - au_1 u_2 \quad \text{and} \quad c^{-1}(\xi) = 1 + u_1 - au_2 - au_1 u_2.$$ 

respectively.

Consider the projectivization $\mathbb{P}(\xi)$. Similarly to Theorem 6 denote the first Chern class of the fibrewise vector bundle $\gamma = \mathcal{O}(1)$ over $\mathbb{P}(\xi)$ by $v = c_1(\gamma)$. The ring $H^*(\mathbb{P}(\xi))$ is an $H^*(\mathbb{C}P^1 \times \mathbb{C}P^1)$-module. The isomorphism (3.3) implies that

$$c(T\mathbb{P}(\xi)) = (1 - u_1 + v)(1 + au_2 + v)(1 + v)^{n-3} c(T(\mathbb{C}P^1 \times \mathbb{C}P^1))$$

$$= (1 - u_1 + v)(1 + au_2 + v)(1 + v)^{n-3}(1 + 2u_1)(1 + 2u_2).$$

Consequently, the Milnor number of $\mathbb{P}(\xi)$ is given by

$$s_n(\mathbb{P}(\xi)) = \langle (v - u_1)^n + (au_2 + v)^n + (n - 3)v^n + (2u_1)^n + (2u_2)^n, [\mathbb{P}(\xi)] \rangle.$$

We rewrite the expression for $s_n$ above by expanding the brackets and using the relation $u_1^2 = u_2^2 = 0$ in the following way:

$$s_n(\mathbb{P}(\xi)) = \langle (n - 1)v^n - nu_1 v^{n-1} + nau_2 v^{n-1}, [\mathbb{P}(\xi)] \rangle.$$
Finally, from (3.2) we obtain

\[ s_n(\mathbb{P}(\xi)) = \langle (n - 1)v^n - nu_1v^{n-1} + nau_2v^{n-1}, [\mathbb{P}(\xi)] \rangle \]

\[ = \langle (n - 1 + n(au_2 - u_1)) \cdot c^{-1}(\xi), [\mathbb{C}P^1 \times \mathbb{C}P^1] \rangle \]

\[ = \langle -(n - 1)au_1u_2 - n(au_2 - u_1)^2, [\mathbb{C}P^1 \times \mathbb{C}P^1] \rangle \]

\[ = 2na - (n - 1)a = (n + 1)a. \]

Thus, \( s_n(\mathbb{P}(\xi)) = (n + 1)a \). Hence, if \( a > N/(n + 1) \), then \( s_n(\mathbb{P}(\xi)) > N \). The total space of the fibre bundle \( \xi \to (\mathbb{C}P^1 \times \mathbb{C}P^1) \) admits a natural action of the torus \((\mathbb{C}^\ast)^{n+1}\). Consequently, the variety \( \mathbb{P}(\xi) \) is toric.

The lemma is proved.

**Proof of Theorem 7.** As we mentioned at the beginning of this section, it is enough to construct toric manifolds with Milnor number 1 in all even dimensions \( n \) such that \( n + 1 \) is not a power of a prime. Fix such an \( n \).

The numbers \( s_{0,n}, \ldots, s_{n-2,n} \) are coprime due to Lemma 1. Next,

\[ s_{0,n} = s_n(B_0(X)) - s_n(X) = s_n(\text{Bl}_{x'} \text{Bl}_x X) - s_n(X) \]

\[ = s_n(\text{Bl}_{x'} \text{Bl}_x X) - s_n(\text{Bl}_x X) + s_n(\text{Bl}_x X) - s_n(X) = -2(n + 1) < 0 \]

where the last equality follows from (4.3) below. Now, we apply Lemma 2 to \(-s_{0,n}, \ldots, -s_{n-2,n}\) to find a natural number \( N = N(-s_{0,n}, \ldots, -s_{n-2,n}) \) such that any integer \( m > N \) can be represented as a linear combination of \(-s_{0,n}, \ldots, -s_{n-2,n}\) with some nonnegative integer coefficients \( a_0, \ldots, a_{n-2} \).

Now we take a toric manifold \( X, \dim_{\mathbb{C}} X = n \), with Milnor number \( s_n(X) > N + 1 \) and express \( s_n(X) - 1 \) as an integer linear combination

\[ s_n(X) - 1 = - \sum_{i=0}^{n-2} a_i \cdot s_{i,n}, \]

where all the \( a_i \) are nonnegative.

Then we apply \( a_i \) modifications \( B_i \) for each \( i = 0, \ldots, n - 2 \), starting from the variety \( X \). As a result, we obtain an \( n \)-dimensional toric variety \( Y \) with Milnor number given by

\[ s_n(Y) = s_n(X) + \sum_{i=0}^{n-2} a_i s_{i,n} = 1, \]

as required. The theorem is proved.

**Remark 2.** If we had started from a projective toric variety \( X \), then the construction in the above proof leads to a projective variety \( Y \) as well. Let \( X \) be the projectivization of the bundle \( \xi = \pi^* \mathcal{O}(1) \oplus \pi^* \mathcal{O}(a) \oplus \mathbb{C}^{n-3} \) over \( \mathbb{C}P^1 \times \mathbb{C}P^1 \) as in Lemma 3. Then the corresponding moment polytope of \( X \) is combinatorially equivalent to the product of simplices \( P = \Delta^1 \times \Delta^1 \times \Delta^{n-2} \). In this case, according to the description of equivariant modifications in §3.3, the polytope corresponding to the variety \( Y \) is obtained from \( P \) by successive vertex and simplicial face truncations.
Example 2. Now we will give a construction of a smooth projective toric variety of (complex) dimension 14 by following the method above closely. Recall that 14 is the lowest dimension that does not satisfy Theorem 4, that is, the least even number \( n \) such that \( n + 1 \) is not a power of a prime. The numbers \( s_{k,14} \) in increasing order for \( k = 0, \ldots, 12 \) are

\[
-30, -15, -435, 2010, -10100, 31779, -79593, 140520, -190768, 174195, -120879, 36858, -16398.
\]

The semigroup \( S \) generated by the moduli \( |s_{k,14}| \) for all possible \( k \) has the minimal generators \( |s_{1,14}|, |s_{4,14}|, |s_{5,14}| \) and \( |s_{12,14}| \). The conductor of this semigroup, that is, the smallest natural number \( N \) such that all bigger numbers belong to \( S \), is 68363. Now consider the toric variety \( X = \mathbb{P}(\pi_1^*\Theta(1) \oplus \pi_2^*\Theta(a) \oplus \mathbb{C}^{n-3}) \) from Remark 2 with the parameter \( a = 4558 \). According to the same remark, \( s_{14}(X) = 68370 \). Consider the decomposition of the number \( s_{14}(X)-1 = 68369 \) with respect to the semigroup \( S \): 68369 = 1766 \cdot 15 + 10100 + 31779. In order to obtain a similar decomposition of \( s_{14}(X)-1 \) with respect to the elements of the semigroup generated by the integers \( -s_{k,14} \) for all possible \( k \), we apply the procedure from Lemma 2 to the previous decomposition:

\[
68369 = (1766 + 31779d) \cdot 15 + 10100 + (-1 + 15d) \cdot (-31779),
\]

where \( d \) is an integer. To have nonnegative coefficients in the equality obtained we can take \( d = 1 \). The corresponding decomposition is

\[
68369 = 33545 \cdot 15 + 10100 + 14 \cdot (-31779).
\]

According to Theorem 7, we must apply the operation \( B_1 \) 33545 times, the operation \( B_4 \) once and \( B_5 \) 14 times to the variety \( X \) to obtain a toric variety with the Milnor number 1.

4.2. Computation of Milnor numbers \( s_n(D_{k,n}) \). This and the following subsections are devoted to the proof of Lemma 1, which is used in the proof of our main theorem.

First, we deduce an explicit formula for the Milnor numbers \( s_n(D_{k,n}) \) of the stably complex manifolds \( D_{k,n} = \mathbb{P}(\Theta(-1) \oplus \Theta(1) \oplus \mathbb{C}^{n-k-1} \oplus \mathbb{C}) \). Consider the natural projection \( p: D_{k,n} \to \mathbb{C}P^k \). Let \( u \) denote the positive generator of \( H^2(\mathbb{C}P^k, \mathbb{Z}) \), that is, \( u = c_1(\Theta(1)) \). Let \( v = c_1(\gamma) \in H^2(D_{k,n}, \mathbb{Z}) \), where \( \gamma = \Theta(1) \) is the line bundle along the fibres of the projection \( p \). According to the isomorphism (3.1), the cohomology ring \( H^*(D_{k,n}; \mathbb{Z}) \) is generated multiplicatively by \( u \) and \( v \):

\[
H^*(D_{k,n}; \mathbb{Z}) \cong \mathbb{Z}[u,v]/(u^{k+1}, v(v + u)^{n-k-1}(v - u)).
\]

From the definition of the nonstandard stably complex structure (3.4) it follows that on the manifold \( D_{k,n} := \mathbb{P}(\Theta(-1) \oplus \Theta(1) \oplus \mathbb{C}^{n-k-1} \oplus \mathbb{C}) \)

\[
c(TD_{k,n}) = (1 + u + v)^{n-k-1}(1 - u + v)(1 - v)(1 + u)^{k+1},
\]

where the last factor corresponds to \( c(T\mathbb{C}P^k) \). Since \( u^n = 0 \), the above formula implies that

\[
s_n(D_{k,n}) = \langle (n - k - 1)(u + v)^n + (-u + v)^n + (-v)^n, [D_{k,n}] \rangle. \tag{4.1}
\]

We need some auxiliary lemmas to complete the computation of \( s_n(D_{k,n}) \).
Lemma 4. For any $n, 0 \leq k \leq n - 2$, 
\[
\langle v^n, [D_{k,n}] \rangle = \sum_{i=0}^{k} (-1)^i 2^{k-i} \binom{n-1}{i}.
\]

Proof. According to (3.2),
\[
\langle v^n, [D_{k,n}] \rangle = \langle c^{-1}(\xi), [\mathbb{C}P^k] \rangle.
\]
The total Segre class of the bundle $\xi = \mathcal{O}(-1) \oplus \mathcal{O}(n-k-1) \oplus \mathbb{C}$ over $\mathbb{C}P^k$ is equal to $c^{-1}(\xi) = (1+u)^{-(n-k-1)}(1-u)^{-1}$. Hence
\[
\langle c^{-1}(\xi), [\mathbb{C}P^k] \rangle = \left\langle (1+u)^{-(n-k-1)}(1-u)^{-1}, [\mathbb{C}P^k] \right\rangle
\]
\[
= \left\langle (1+u)^{-(n-k)} \left(1 - \frac{2u}{1+u}\right)^{-1}, [\mathbb{C}P^k] \right\rangle
\]
\[
= \left\langle \sum_{i=0}^{\infty} 2^i u^i (1+u)^{-(n-k+i)}, [\mathbb{C}P^k] \right\rangle = \sum_{i=0}^{k} 2^i (-1)^{k-i} \binom{n-1}{k-i},
\]
where in the last equality we have use the fact that the coefficient of $u^{k-i}$ in the series expansion of $(1+u)^{-(n-k+i)}$ is equal to $(-1)^{k-i} \binom{n-1}{k-i}$.

Lemma 5. For any $n, 0 \leq k \leq n - 2$, 
\[
\langle (u+v)^n, [D_{k,n}] \rangle = 2^{k+1} - 1.
\]

Proof. We use (3.2) again and substitute the expression for the Segre class $c^{-1}(\xi) = (1+u)^{-(n-k-1)}(1-u)^{-1}$:
\[
\langle (u+v)^n, [D_{k,n}] \rangle = \left\langle \sum_{i=0}^{n} \binom{n}{i} u^i v^{n-i}, [D_{k,n}] \right\rangle = \left\langle \sum_{i=0}^{n} \binom{n}{i} u^i c^{-1}(\xi), [\mathbb{C}P^k] \right\rangle
\]
\[
= \left\langle (1+u)^n c^{-1}(\xi), [\mathbb{C}P^k] \right\rangle = \left\langle (1+u)^{k+1} (1+u+u^2+\cdots), [\mathbb{C}P^k] \right\rangle
\]
\[
= \left\langle \sum_{i=0}^{k} \binom{k+1}{i} u^k, [\mathbb{C}P^k] \right\rangle = 2^{k+1} - 1.
\]

Lemma 6. For any $n, 0 \leq k \leq n - 2$, 
\[
\langle (-u+v)^n, [D_{k,n}] \rangle = \sum_{i=0}^{k} (-2)^i \binom{n-1}{i}.
\]
Proof. As in the previous lemmas:

\[ \langle (-u + v)^n, [D_{k,n}] \rangle = \left\langle \sum_{i=0}^{n} (-1)^i \binom{n}{i} u^i v^{n-i}, [D_{k,n}] \right\rangle \]

\[ = \left\langle \sum_{i=0}^{n} (-1)^i \binom{n}{i} u^i c^{-1}(\xi), [\mathbb{C}P^k] \right\rangle = \langle (1 - u)^{n-1}(1 + u)^{-(n-k-1)}, [\mathbb{C}P^k] \rangle \]

\[ = \langle ((1 + u) - 2u)^{n-1}(1 + u)^{-(n-k-1)}, [\mathbb{C}P^k] \rangle \]

\[ = \left\langle \sum_{i=0}^{n-1} (-2)^i \binom{n-1}{i} u^i(1 + u)^{n-1-i}(1 + u)^{-(n-k-1)}, [\mathbb{C}P^k] \right\rangle \]

\[ = \sum_{i=0}^{k} (-2)^i \binom{n-1}{i}. \]

Collecting the expressions given by Lemmas 4–6 and using (4.1) we obtain the following.

**Proposition 3.** The Milnor number of the manifold \( D_{k,n} \) satisfies

\[ s_n(D_{k,n}) = (n - k - 1)(2^{k+1} - 1) + \sum_{i=0}^{k} (-1)^i(2^i + (-1)^n 2^{k-i}) \binom{n-1}{i}. \]  (4.2)

**Example 3.** In the particular case \( k = 0 \) we obtain a classical formula for the change in the Milnor number under blow-up at a point, see (3.6):

\[ s_n(\text{Bl}_x X) - s_n(X) = -s_n(D_{0,n}) = -(n + (-1)^n). \]  (4.3)

For \( k = 1 \) and \( k = n - 2 \) we have:

\[ s_n(D_{1,n}) = \begin{cases} 0 & \text{for even } n, \\ 2(n-3) & \text{for odd } n; \end{cases} \]

\[ s_n(D_{n-2,n}) = \begin{cases} 2^n - 1 & \text{for even } n, \\ 0 & \text{for odd } n. \end{cases} \]  (4.4)

**Corollary 1.** For any compact complex manifold \( X^n \), the change in the Milnor number under the sequential blow-up \( B_k(X) = \text{Bl}_{Z^k}(\text{Bl}_x X) \) satisfies

\[ s_{k,n} = s_n(B_k(X)) - s_n(X) \]

\[ = (s_n(B_k(X)) - s_n(\text{Bl}_x X)) + (s_n(\text{Bl}_x X) - s_n(X)) \]

\[ = -s_n(D_{k,n}) - (n + (-1)^n) \]

\[ = \left( (n - k - 1)(2^{k+1} - 1) \right. \]

\[ + \sum_{i=0}^{k} (-1)^i(2^i + (-1)^n 2^{k-i}) \binom{n-1}{i} + n + (-1)^n \right). \]  (4.5)

where \( Z^k \simeq \mathbb{C}P^k \subset E \) is a projective subspace in the exceptional divisor of the blow-up \( \text{Bl}_x X \rightarrow X \).
4.3. The changes in the Milnor number $s_{k,n}$ are coprime. In § 4.2 we deduced formula (4.2) for the Milnor numbers $s_n(D_{k,n})$. Using this formula we computed the change in the Milnor numbers under the modifications $B_k(X) \to X$, see (4.5).

In this subsection we prove Lemma 1, that is, we show that $s_{0,n}, s_{1,n}, \ldots, s_{n-2,n}$ are coprime, provided that $n$ is even and $n+1$ is not a power of a prime. First note that $s_{1,n} = -(n+1)$ for any even $n$ (see Example 3 and formula (4.5)). To complete the proof we will show that for any prime divisor $p$ of $n+1$ a certain integer linear combination of $s_{k,n}$ is not divisible by $p$.

We introduce a family of linear combinations of the numbers $s_{k,n}$ which are given by a more compact formula than (4.5). To do this we make use of an auxiliary computation.

**Proposition 4.** For any $n \geq 2$ and $k = 2, \ldots, n-2$ let $L_{k,n} = s_{k,n} - 3s_{k-1,n} + 2s_{k-2,n}$. Then

$$L_{k,n} = 2^k + 1 - (-2)^k \binom{n}{k} - (-1)^{n+k} \binom{n}{k}.$$  

In particular, for even $n$

$$L_{k,n} = (2^k + 1) \left( 1 + (-1)^{k+1} \binom{n}{k} \right).$$

**Proof.** A direct computation of $-s_{k,n} + 2s_{k-1,n}$ ($k = 1, \ldots, n-2$) involving (4.5) yields

$$-s_{k,n} + 2s_{k-1,n}
= \left( (n-k+1)(2^{k+1} - 1) + \sum_{i=0}^{k} (-1)^i (2^i + (-1)^n 2^{k-i}) \binom{n-1}{i} \right) + n + (-1)^n
- \left( (n-k)(2^{k+1} - 2) + \sum_{i=0}^{k-1} (-1)^i (2^{i+1} + (-1)^n 2^{k-i}) \binom{n-1}{i} \right) + 2n + 2(-1)^n
= -k - 2^{k+1} + 1 - (-1)^n + \sum_{i=0}^{k-1} (-1)^i (2^i - 2^{i+1}) \binom{n-1}{i}
+ (-1)^k (2^k + (-1)^n) \binom{n-1}{k}
= -k - 2^{k+1} + 1 - (-1)^n + \sum_{i=0}^{k} (-2)^i \binom{n-1}{i}
+ \sum_{i=1}^{k} (-2)^i \binom{n-1}{i-1} + (-1)^{n+k} \binom{n-1}{k}
= -k - 2^{k+1} + 1 - (-1)^n + \sum_{i=0}^{k} (-2)^i \binom{n}{i} + (-1)^{n+k} \binom{n-1}{k}, \quad (4.6)$$
where the last equality follows from the binomial identity \( \binom{n}{i} = \binom{n-1}{i} + \binom{n-1}{i-1} \).

Applying (4.6) twice we obtain
\[
s_{k,n} - 3s_{k-1,n} + 2s_{k-2,n} = (-s_{k-1,n} + 2s_{k-2,n}) - (-s_{k,n} + 2s_{k-1,n})
\]
\[
= \left(-k - 2^k + 2 - (-1)^n + \sum_{i=0}^{k-1}(-2)^i\binom{n}{i} + (-1)^{n+k-1}\binom{n-1}{k-1}\right)
\]
\[
- \left(-k - 2^{k+1} + 1 - (-1)^n + \sum_{i=0}^{k}(-2)^i\binom{n}{i} + (-1)^{n+k}\binom{n-1}{k}\right)
\]
\[
= 2^k + 1 - (-2)^k\binom{n}{k} - (-1)^{n+k}\binom{n}{k},
\]
where the last equality again follows from the binomial identity.

We need a theorem from number theory.

**Theorem 8** (Lucas’s Theorem [19], Lemma 2.6). Let \( p \) be prime, and let
\[
n = n_0 + n_1p + \cdots + n_{r-1}p^{r-1} + n_rp^r,
\]
\[
m = m_0 + m_1p + \cdots + m_{r-1}p^{r-1} + m_rp^r
\]
be the base \( p \) expansions of the positive integers \( n \) and \( m \). Then
\[
\binom{n}{m} \equiv \binom{n_0}{m_0}\binom{n_1}{m_1}\cdots\binom{n_r}{m_r} \pmod{p}.
\]

Now we can carry out the proof of Lemma 1.

**Proof of Lemma 1.** We recall that \( s_{1,n} = -(n+1) \). Therefore, it suffices to show that for any prime divisor \( p \) of \( n+1 \) some linear combination of \( s_{0,n}, \ldots, s_{n-2,n} \) is not divisible by \( p \). Using Proposition 4 we will look for such a combination among the numbers
\[
L_{k,n} = (1 + 2^k)\left(1 + (-1)^{k+1}\binom{n}{k}\right), \quad k = 2, \ldots, n-2.
\]
Now we will find an integer \( k \) such that neither of the congruences
\[
2^k \equiv -1 \pmod{p},
\]
\[
\binom{n}{k} \equiv (-1)^k \pmod{p}
\]
holds. Consider the base \( p \) expansion of \( n \), \( n = n_0 + n_1p + \cdots + n_rp^r \), where \( 0 \leq n_i \leq (p-1), n_r \neq 0 \). Notice that \( n_0 = p-1 \) (because \( n+1 \) is divisible by \( p \)). Moreover, there exists an index \( j \leq r \) such that \( n_j < p-1 \) because \( n+1 \) is not a prime power. Let \( j \) be the smallest index with this property. Consider \( k = p^j \). According to Lucas’ Theorem,
\[
\binom{n}{k} \equiv \binom{n_j}{1} = n_j \pmod{p}.
\]
The congruence (4.8) does not hold for this $k$, since $k$ is odd and $0 \leq n_j < p - 1$. There are two cases.

Case 1. Congruence (4.7) is not satisfied either. Then $k = p^j$ is the desired number.

Case 2. Congruence (4.7) is satisfied: $2^k \equiv -1 \pmod{p}$. Then we replace $k$ with $k' = k + 1$. By Lucas’ Theorem the binomial coefficient $\binom{n}{k}$ is congruent to

$$\binom{n_j}{1}\binom{p-1}{1} = -n_j \pmod{p}$$

and is not equal to $(-1)^{k'} = 1$, because $n_j < p - 1$. Also we have $2^{k'} \equiv -2 \not\equiv -1 \pmod{p}$.

In both cases for at least one of the numbers $k \in \{p^j, p^j + 1\}$ the combination $L_{k,n} = s_{k,n} - 3s_{k-1,n} + 2s_{k-2,n}$ is not divisible by $p$. We have $j \geq 1$, hence $k \geq p > 2$. It remains to show that $k \leq n - 2$. In the base $p$ expansion of $n$ we have $n_0 = p - 1$ and $n_r \geq 1$. Therefore, $n \geq p^r + (p-1) \geq p^j + (p-1) \geq k + (p-2)$. If $p > 3$ or $r > j$, then $k \leq n - 2$, as required. Otherwise, if $j = r$ and $p = 3$, then $n = 3^r + \sum_{i=0}^{r-1} 2 \cdot 3^i$ since $j$ is the smallest number such that $n_j \neq p - 1 = 2$. But then $n$ is odd, while in Lemma 1 we assume that $n$ is even.

The lemma is proved.

Remark 3. The assumptions on $n$ in Lemma 1 are important. If $n = p^m - 1$ for some prime $p$, then all the $s_{k,n}$ are divisible by $p$ and clearly not coprime. This follows from the properties of Milnor numbers: additivity and vanishing on decomposables in $\Omega_*^U \ (\text{see } [18])$, and Theorem 1. Therefore, the $L_{k,n}$ are also divisible by $p$. To give some examples we list the values of $L_{k,n}$ for certain $n$, in the order of increasing $k = 2, \ldots, n - 2$:

- $n = 4 = 5 - 1: -25$;
- $n = 6 = 7 - 1: -70, 189, -238$;
- $n = 8 = 3^2 - 1: -135, 513, -1173, 1881, -1755$.

§ 5. Properties of the modifications $B_k$ and their applications

5.1. The modifications $B_k$ and polytope operations. In this subsection we study a connection between equivariant modifications $B_k$ of toric manifolds and the corresponding operations on Delzant polytopes.

Recall that for any smooth projective toric variety $X$ with underlying $n$-dimensional polytope $P$, the toric variety $B_k(X)$ corresponds to the polytope cut$_{S^k}(\text{cut}_p P)$ obtained from $P$ by successive vertex $p$ and $k$-dimensional simplicial face $S^k$ truncations. The choice of the vertex and the face corresponds to the choice of the fixed point $x \in X$ and the submanifold $Z \subset E \subset \text{Bl}_x(X)$ to blow up. It turns out that for a specific choice of submanifolds $Z_1 = \mathbb{CP}^k$ and $Z_2 = \mathbb{CP}^{n-k-2}$ the modifications $B_k(X^n)$ and $B_{n-k-2}(X^n)$ of the toric variety $X^n$ lead to a pair of toric varieties with combinatorially equivalent moment polytopes.

Claim 2. Consider a truncation of a simple polytope $P$ at a vertex $p$. Denote the new simplicial facet of the truncated polytope $\text{cut}_p P$ by $G \subset \text{cut}_p P$. For any two complementary (that is, disjoint) faces $S_1 = \Delta^k$ and $S_2 = \Delta^{n-k-2}$ of the
simplicial facet $G$ the polytopes $P_1 = \text{cut}_{S_1}(\text{cut}_p P)$ and $P_2 = \text{cut}_{S_2}(\text{cut}_p P)$ are combinatorially equivalent (see Figure 1).

**Proof.** Consider the sets of facets of the polytopes $P_1$ and $P_2$

$$\mathcal{F}_1 = \{ F \mid F \in P_1 \} \quad \text{and} \quad \mathcal{F}_2 = \{ F \mid F \in P_2 \}. $$

Consider the set $\mathcal{F} = \{ F \mid F \in P \}$ of facets of the polytope $P$. Denote the new facet of the polytope $P_i$ obtained by truncating the face $S_i$ by $G_i$ ($i = 1, 2$). Then $\mathcal{F}_1 = \mathcal{F} \cup \{ G_1, G \}$ and $\mathcal{F}_2 = \mathcal{F} \cup \{ G_2, G \}$.

We define the map $f : \mathcal{F}_1 \to \mathcal{F}_2$ as follows:

$$f(G) = G_2,$$

$$f(G_1) = G,$$

$$f(F) = F \quad \text{if} \quad F \neq G, G_1.$$

We will show that the bijection $f$ induces a combinatorial isomorphism of the polytopes $P_1$ and $P_2$, that is, the facets $F_1, \ldots, F_k$ have nonempty intersection in $P_1$ if and only if the facets $f(F_1), \ldots, f(F_k)$ have nonempty intersection in $P_2$. As the facets $P_1$ and $P_2$ are simple, it is enough to consider only maximal (that is, $n$-fold) intersections $F_{i_1} \cap \cdots \cap F_{i_n}$.

Denote the facets containing the vertex $p \in P$ by $H_1, \ldots, H_n$. The simplices $S_1$ and $S_2$ are complementary by hypothesis. Hence, after a proper relabelling of the facets $H_i$ in the polytope $\text{cut}_p P$,

$$S_1 = G \cap H_1 \cap \cdots \cap H_{n-k-1} \quad \text{and} \quad S_2 = G \cap H_{n-k} \cap \cdots \cap H_n.$$ 

The polytopes $P_1$ and $P_2$ are the same outside neighbourhoods of the facets $G, G_1$ and $G, G_2$, respectively. Consequently, $f$ induces a combinatorial isomorphism
\( \mathcal{F}_1 \setminus \{G, G_1\} \rightarrow \mathcal{F}_2 \setminus \{G, G_2\} \). It remains to consider \( n \)-fold intersections \( F_i \cap \cdots \cap F_{i_n} \) including one of the facets \( G, G_1 \) in \( P_1 \) (\( G, G_2 \) in \( P_2 \), respectively).

We can write down all the nonempty intersections \( F_i \cap \cdots \cap F_{i_n} \), which include at least one of the facets \( G \) and \( G_1 \) of \( P_1 \):

\[
G \cap H_1 \cap \cdots \cap \widehat{H}_j \cap \cdots \cap H_n, \quad \text{where } 1 \leq j < n - k,
\]

\[
G_1 \cap H_1 \cap \cdots \cap \widehat{H}_j \cap \cdots \cap H_n, \quad \text{where } n - k \leq j \leq n,
\]

\[
G \cap G_1 \cap H_1 \cap \cdots \cap \widehat{H}_{j_1} \cap \cdots \cap \widehat{H}_{j_2} \cap \cdots \cap H_n, \quad \text{where } 1 \leq j_1 < n - k \leq j_2 \leq n.
\]

Similarly, all the nonempty intersections \( F_i \cap \cdots \cap F_{i_n} \) involving at least one of the facets \( G \) and \( G_2 \) of \( P_2 \) are:

\[
G_2 \cap H_1 \cap \cdots \cap \widehat{H}_j \cap \cdots \cap H_n, \quad \text{where } 1 \leq j < n - k,
\]

\[
G \cap H_1 \cap \cdots \cap \widehat{H}_j \cap \cdots \cap H_n, \quad \text{where } n - k \leq j \leq n,
\]

\[
G \cap G_2 \cap H_1 \cap \cdots \cap \widehat{H}_{j_1} \cap \cdots \cap \widehat{H}_{j_2} \cap \cdots \cap H_n, \quad \text{where } 1 \leq j_1 < n - k \leq j_2 \leq n.
\]

These descriptions and the fact that the bijection \( f \) acts identically on all of the facets, apart from \( G \) and \( G_1 \), show that \( f \) induces a combinatorial isomorphism between \( P_1 \) and \( P_2 \). The claim is proved.

**Remark 4.** It is important to assume that the simplices \( S_1 \) and \( S_2 \) are complementary in \( G \). Otherwise Claim 2 does not hold in general.

**Corollary 2.** Consider an \( n \)-dimensional smooth projective toric variety \( X \). Take a fixed point \( x \in X \) and consider the blow-up \( \pi : \text{Bl}_x X \rightarrow X \). Pick two disjoint \((\mathbb{C}^*)^n\)-invariant subspaces

\[
\mathbb{C}P^k \simeq Z_1 \subset E \subset \text{Bl}_x X \quad \text{and} \quad \mathbb{C}P^{n-k-2} \simeq Z_2 \subset E \subset \text{Bl}_x X
\]

in the exceptional divisor of \( \pi \). Then the blow-ups \( B_k(X) = \text{Bl}_{Z_1}(\text{Bl}_x X) \) and \( B_{n-k-2}(X) = \text{Bl}_{Z_2}(\text{Bl}_x X) \) are toric varieties and their moment polytopes are combinatorially equivalent.

**Example 4.** Let \( P = \Delta \subset \mathbb{R}^3 \) be a Delzant 3-dimensional simplex. Fix a vertex \( p \) of \( P \). The new face of the truncation polytope \( \text{cut}_P P \) is a two-dimensional simplex. The modifications \( B_0 \) and \( B_1 \) of the corresponding toric variety \( \mathbb{C}P^3 \) are the vertex truncation and truncation of an edge in the new face of the polytope \( \text{cut}_P P \), respectively. Choose a vertex \( S_1 \) and an edge \( S_2 \) in the truncation face of the polytope \( \text{cut}_P P \). If \( S_1 \) and \( S_2 \) have empty intersection, then the polytopes \( \text{cut}_{S_0}(\text{cut}_P P) \) and \( \text{cut}_{S_1}(\text{cut}_P P) \) are combinatorially equivalent; see Figure 1.

### 5.2. Combinatorial rigidity of Hirzebruch genera

In this subsection we give an application of the modifications \( B_k \) to the theory of Hirzebruch genera. More specifically, we give a combinatorial characterization of the two-parameter Todd genus (for more details on Hirzebruch genera see [11] and [12]).

Let \( R \) be a commutative ring with a unit and with no additive torsion. Hirzebruch genus with values in \( R \) is a ring homomorphism \( \varphi : \Omega(U) \rightarrow R \) (the term *multiplicative genus* is also used). Because the ring is torsion-free, any genus \( \varphi \)
is uniquely determined by its extension $\varphi_Q: \Omega_*^U \otimes \mathbb{Q} \to R \otimes \mathbb{Q}$. Each Hirzebruch genus $\varphi: \Omega_*^U \to R$ corresponds to a formal power series $Q(x) = 1 + \sum_{k=1}^{\infty} q_k x^k \in (R \otimes \mathbb{Q})[[x]]$. In this case the value of $\varphi$ on a stably complex manifold $M^{2n}$ is given by $\varphi(M) = (Q(t_1) \cdots Q(t_n), [M])$, where the $t_i$ are Chern roots of $M$.

**Example 5** (two-parameter Todd genus). The two-parameter Todd genus

$$\chi_{a,b}: \Omega_*^U \to \mathbb{Z}[a,b]$$

is given by the $Q$-series $x \cdot \frac{ae^{bx}-be^{ax}}{e^{ax}-e^{bx}}$ (see [6]). Some specializations of the two-parameter Todd genus include the signature ($a = 1$ and $b = -1$), the arithmetic genus ($a = 0$, and $b = -1$) and the Euler characteristic (for $a = -1$ and taking the limit in the $Q$-series as $b \to -1$). With the specialization $a = y$, $b = -1$ the genus $\chi_{a,b}$ turns into the $\chi_{y}$-genus. Similarly to the $\chi_{y}$-genus (see [11]), the value of $\chi_{a,b}$ on any $n$-dimensional complex manifold can be expressed in terms of the dimensions of the cohomology groups of holomorphic differentials:

$$\chi_{a,b}(M) = \sum_{i,j=0}^{n} \dim_{\mathbb{C}} H^i(M; \Omega^j)(-1)^{n-i-j}a^{n-i}b^j. \quad (5.1)$$

Notice that the image of the two-parameter Todd genus is the ring of symmetric polynomials $\mathbb{Z}[\sigma_1, \sigma_2] \subset \mathbb{Z}[a,b]$ in two variables: $\sigma_1 = a + b = -\chi_{a,b}([\mathbb{C}P^1])$ and $\sigma_2 = ab = \chi_{a,b}([\mathbb{C}P^1]^2 - [\mathbb{C}P^2])$. In what follows we consider the genus $\chi_{a,b}$ as a ring homomorphism onto $\mathbb{Z}[\sigma_1, \sigma_2]$.

**Definition 10.** A Hirzebruch genus $\varphi: \Omega_*^U \to R$ is called combinatorially rigid if, for any two smooth projective toric varieties $X_1$ and $X_2$ with combinatorially equivalent moment polytopes $P_1 \simeq P_2$, the corresponding values of $\varphi$ coincide: $\varphi(X_1) = \varphi(X_2)$.

**Example 6.** The Euler characteristic of a toric variety is equal to the number of vertices of the underlying moment polytope. Hence, it is combinatorially rigid.

Using some of Danilov’s results (on the cohomology of toric varieties, see [7], §12) and formula (5.1) we can deduce that the two-parameter Todd genus $\chi_{a,b}$ is given by

$$\chi_{a,b}(XP) = (-1)^n \sum_{i=0}^{n} h_i(P)a^ib^{n-i},$$

where the $h_i(P)$ are components of the $h$-vector of the polytope $P$ (see [6], Ch.1, §3). The numbers $h_i(P)$ are combinatorial invariants of the polytope $P$. Hence, the two-parameter Todd genus is combinatorially rigid.

It turns out that any combinatorially rigid Hirzebruch genus is a specialization of the two-parameter Todd genus.

**Theorem 9.** Let $\varphi: \Omega_*^U \to R$ be a combinatorially rigid Hirzebruch $R$-genus. Then there exists a unique ring homomorphism $f: \mathbb{Z}[\sigma_1, \sigma_2] \to R$ such that $\varphi = f \circ \chi_{a,b}$.
Proof. By assumption the ring $R$ is torsion-free. We construct a homomorphism $f_Q: \mathbb{Q}[\sigma_1, \sigma_2] \to R \otimes \mathbb{Q}$ such that $\varphi \otimes \mathbb{Q} = f_Q \circ (\chi_{a,b} \otimes \mathbb{Q})$.

Consider the ideal $\mathcal{I} \subset \Omega_*^U \otimes \mathbb{Q}$ generated by all the differences $[X_1] - [X_2]$, where $X_1$ and $X_2$ are smooth projective toric varieties with combinatorially equivalent moment polytopes. It follows from the definition of combinatorial rigidity that $\varphi \otimes \mathbb{Q}$ factors through the quotient ring $(\Omega_*^U \otimes \mathbb{Q})/\mathcal{I}$. Recall that the two-parameter Todd genus is combinatorially rigid and maps the ring $\Omega_*^U \otimes \mathbb{Q}$ onto $\mathbb{Q}[\sigma_1, \sigma_2]$. Therefore, there exists an epimorphism $\pi_1: (\Omega_*^U \otimes \mathbb{Q})/\mathcal{I} \to \mathbb{Q}[\sigma_1, \sigma_2]$ such that the composition

$$\Omega_*^U \otimes \mathbb{Q} \to (\Omega_*^U \otimes \mathbb{Q})/\mathcal{I} \xrightarrow{\pi_1} \mathbb{Q}[\sigma_1, \sigma_2]$$

is equal to $\chi_{a,b} \otimes \mathbb{Q}$. Now it remains to check that the projection $\pi_1$ is an isomorphism.

For any $n \geq 3$ consider two equivariant modifications $B_0(\mathbb{C}P^n)$ and $B_{n-2}(\mathbb{C}P^n)$ with combinatorially equivalent moment polytopes (see Corollary 2). It follows from (3.8) and (4.4) that

$$s_n(B_0(\mathbb{C}P^n)) - s_n(B_{n-2}(\mathbb{C}P^n)) = (s_n(\text{Bl}_x \mathbb{C}P^n) - s_n(D_{0,n})) - (s_n(\text{Bl}_x \mathbb{C}P^n) - s_n(D_{n-2,n})) = s_n(D_{n-2,n}) - s_n(D_{0,n}) \neq 0.$$

Hence the elements $[\mathbb{C}P^1]$, $[\mathbb{C}P^2]$ and $\{[B_0(\mathbb{C}P^n)] - [B_{n-2}(\mathbb{C}P^n)]\}_{n \geq 3}$ are polynomial generators of the ring $\Omega_*^U \otimes \mathbb{Q}$. By construction, the elements $\{[B_0(\mathbb{C}P^n)] - [B_{n-2}(\mathbb{C}P^n)]\}_{n \geq 3}$ belong to the ideal $\mathcal{I}$. Hence there exists a natural projection $\pi_2: \mathbb{Q}[\mathbb{C}P^1, \mathbb{C}P^2] \to (\Omega_*^U \otimes \mathbb{Q})/\mathcal{I}$. The composition

$$\mathbb{Q}[\mathbb{C}P^1, \mathbb{C}P^2] \xrightarrow{\pi_2} (\Omega_*^U \otimes \mathbb{Q})/\mathcal{I} \xrightarrow{\pi_1} \mathbb{Q}[\sigma_1, \sigma_2]$$

to $R \subset R \otimes \mathbb{Q}$. Consequently, the image of the specialization $f_Q$ also belongs to $R$.

§ 6. Concluding remarks

In this paper we have proposed a method for constructing polynomial generators of the unitary cobordism ring $\Omega_*^U$ among the projective toric varieties. We note, however, that each variety provided by Theorem 7 is the result of a very large number of modifications $B_{k_i}$. Thus these varieties have quite a complicated topology. So, it would be of interest to look for toric polynomial generators of $\Omega_*^U$ with the minimum number of invariant subvarieties.
Problem 1. Find projective toric polynomial generators of the ring $\Omega^U_\ast$ whose moment polytopes have:

a) the smallest number of vertices;
b) the smallest number of facets;
c) the smallest number of faces of all dimensions.

Analysing the modifications $B_k$ we gave a characterization of the two-parameter Todd genus in terms of combinatorial rigidity (see Theorem 9). The combinatorial rigidity of any genus $\varphi$ requires that the values $\varphi(X_1) = \varphi(X_2)$ coincide for any pair of toric varieties with combinatorially equivalent moment polytopes. It is natural to generalize the notion of combinatorial rigidity in the following way.

Definition 11. Let $\mathcal{P} = \{P_i\}_{i \in I}$ be a family of simple combinatorial polytopes. A Hirzebruch genus $\varphi$ is called combinatorially $\mathcal{P}$-rigid if $\varphi(X_1) = \varphi(X_2)$ for all pairs of toric varieties with moment polytopes combinatorially equivalent to the same polytope in $\mathcal{P}$.

In connection with the definition of combinatorial $\mathcal{P}$-rigidity the following problems are worth investigating.

Problem 2. Given a family of combinatorial polytopes $\mathcal{P}$ (for instance, Stasheff polytopes, permutahedra, nestohedra and so on), describe the combinatorially $\mathcal{P}$-rigid Hirzebruch genera.

Problem 3. Given a Hirzebruch genus $\varphi$, find a maximal family of polytopes $\mathcal{P}$ such that the genus $\varphi$ is combinatorially $\mathcal{P}$-rigid.

We are grateful to V. M. Buchstaber and T. E. Panov for suggesting the problems studied in this paper and for numerous fruitful discussions. We also would like to thank P. Landweber for most helpful remarks and suggestions. Finally, the close attention of the reviewer led to improvements in the paper.

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