\textbf{Δ−Convergence Theorems for Multivalued Non-expansive Mappings in Hyperbolic Spaces}

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\textbf{Abstract}

The purpose of this paper is that iteration scheme of multivalued non-expansive mappings in Banach spaces is extended to hyperbolic spaces and to prove some \( \Delta \)-convergence theorems of the mixed type iteration process to approximating a common fixed point for two multivalued non-expansive mappings and two non-expansive mappings in hyperbolic spaces. The results presented in the paper extend and improve some recent results announced in the current literature.

\textbf{Keywords:} \( \Delta \)-convergence theorems; Multivalued non-expansive mapping; Common fixed point; Hyperbolic space

\textbf{Introduction}

The study for fixed point problem involve that multivalued contractions and non-expansive mappings used the Hausdorff metric was initiated by Markin \cite{1,2} later, different iterative processes have used to approximate the fixed points of multivalued non-expansive mappings in Banach space, many scholars have made extensive research in \cite{1-17}. An interesting and rich fixed point theory for such mappings was developed which has applications in control theory, convex optimization, differential inclusion, and economics \cite{3}. But the hyperbolic space has no set up the theory of multivalued non-expansive mappings fixed point. In order to define the concept of multivalued non-expansive mapping in the general setup of Banach spaces, we first collect some basic concepts.

Let \( E \) be a real Banach space. A subset \( K \) is called proximinal if for each \( x \in E \), there exists an element \( k \in K \) such that
\[
\|x - k\| = \inf \{\|x - y\| : y \in K\} = d(x, K)
\]
It is known that weakly compact convex subsets of a Banach space and closed convex subset of a uniformly convex Banach space are proximinal sub- set of \( K \) by \( P(K) \). Consistent with \cite{1}, let \( CB(K) \) be the class of all nonempty bounded and closed subset of \( K \). Let \( H \) be a Hausdorff metric induced by the metric \( d \) of \( E \), that is
\[
H(A, B) = \max \{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}
\]
for every \( A, B \in CB(E) \). A multivalued mapping \( T: K \to P(K) \) is said to be a contraction if there exists a constant \( k \in [0, 1) \) such that for any \( x, y \in K \),
\[
H(Tx, Ty) \leq k \|x - y\|
\]
Definition 1.1 \cite{15} A multivalued mapping \( T: K \to P(K) \) is said to be non-expansive, if
\[
H(Tx, Ty) \leq \|x - y\|, \quad \forall n \geq 1, x, y \in K
\]
(1.1)
Lemma 1.2 \cite{12} Let \( T: K \to P(K) \) be a multivalued mapping and \( P_n(x) = \{y \in T \| x - y\| = d(x, Tx)\} \). Then the following are equivalent.

\( \text{(1)} \) \( x \in F(T) \);

\( \text{(2)} \) \( P_n(x) = \{x\} \);

\( \text{(3)} \) \( x \in F(P_n) \).

Moreover, \( F(T) = F(P) \).

Throughout this paper, we work in the setting of hyperbolic spaces introduced by Kohlenbach \cite{18}, defined below, which is restrictive than the hyperbolic type introduced in \cite{19} and more general than the concept of hyperbolic space in \cite{20}.

A hyperbolic space is a metric space \((X, d)\) together with a mapping \( W: X^2 \times [0, 1] \to X \) satisfying
\[
(1) \quad d(u, W(x, y, \alpha)) \leq \alpha d(u, x) + (1 - \alpha)d(u, y);
\]
\[
(2) \quad d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta|d(x, y);
\]
\[
(3) \quad W(x, y, \alpha) = W(y, x, (1 - \alpha));
\]
\[
(4) \quad d(W(x, z, \alpha), W(y, w, \beta)) \leq (1 - \alpha)d(x, y) + \alpha d(z, w).
\]
for all \( x, y, z, w \in X \) and \( \alpha, \beta \in [0, 1] \). A nonempty subset \( K \) of a hyperbolic space \( X \) is convex if \( W(x, y, \alpha) \in K \) for all \( x, y \in K \) and \( \alpha \in [0, 1] \). The class of hyperbolic spaces contains normed spaces and convex subsets thereof, the Hilbert ball equipped with the hyperbolic metric \cite{21}, Hadamard manifolds as well as CAT(0) spaces in the sense of Gromov \cite{22}.

A hyperbolic space is uniformly convex \cite{23} if for any \( r > 0 \) and \( \varepsilon \in (0, 2] \) there exists a \( \delta = \delta(r, \varepsilon) \) such that for all \( u, v \in X \), we have
\[
d(W(x, y, 1/2)) \leq (1 - \delta)r
\]
provided \( d(x, u) \leq r, d(y, u) \leq r \) and \( d(x, y) \geq \varepsilon r \).

A map \( \eta: (0, \infty) \times (0, 2] \to (0, 1] \) which provides such a \( \delta = \delta(r, \varepsilon) \) for For given \( r > 0 \) and \( \varepsilon \in (0, 2] \), is known as a modulus of uniform convexity of \( X \). We call \( \eta \) monotone if \( \eta \) decreases with \( r \) (for a fixed \( \varepsilon \)), i.e., \( \forall \varepsilon > 0, \forall r_2 \geq r_1 > 0, (\eta(r_2, \varepsilon) \leq \eta(r_1, \varepsilon)) \).

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In the sequel, let \((X, d)\) be a metric space and let \(K\) be a nonempty subset of \(X\). We shall denote the fixed point set of a mapping \(T\) by \(F(T) = \{x \in K : Tx = x\}\).

A mapping \(T : K \to K\) is said to be non-expansive, if \(d(Tx, Ty) \leq d(x, y), \forall x, y \in K\).

A mapping \(T : K \to K\) is said to be uniformly \(L\)-Lipschitzian, if there exists a constant \(L > 0\) such that \(d(T^n x, T^n y) \leq Ld(x, y), \forall x, y \in K, n \geq 0\).

The purpose of this paper is to study the iteration scheme of multivalued non-expansive mappings in Banach spaces and to prove some \(\Delta\)-convergence theorems of the mixed type iteration process for approximating a common fixed point of two non-expansive mappings in hyperbolic spaces. The results presented in the paper extend and improve some recent results given in [6-9, 13-15, 16, 19, 21-24]. The main results of the paper are presented below.

**Main Results**

**Theorem 2.1:** Let \(K\) be a nonempty closed convex subset of a complete uniformly convex hyperbolic space \(X\) with monotone modulus of uniform convexity \(\eta\). Let \(T_i : K \to P(K), i=1, 2\) be non-expansive mappings and 
\[
S_i, i=1,2 \text{ is a multivalued nonexpansive mapping, Assume that } \mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset, \text{ and for arbitrarily chosen } x_1 \in K, \{x_n\} \text{ is defined as follows}
\[
\begin{align*}
x_{n+1} &= W(S_i x_n, T_i u_n, \alpha_n) \\
y_n &= W(S_i x_n, T_i v_n, \beta_n)
\end{align*}
\]
where \(v_n \in T_i x_n, u_n \in S_i y_n, d(v_n, u_n) \leq H(T_i x_n, S_i y_n) + r_i\) and \(i = 1, 2, \{\alpha_n\} \text{ and } \{\beta_n\} \text{ satisfy the following conditions:}
\begin{enumerate}
  \item \(\lim_{n \to \infty} r_n = 0, \sum_{n=1}^{\infty} r_n < \infty\);
  \item There exist constants \(a, b \in (0, 1)\) with \(0 < b(1-\alpha) \leq \frac{1}{2}\) such that \(\{\alpha_n\} \subset [a, b] \text{ and } \{\beta_n\} \subset [a, b] \).
  \item \(\|x_n - p\| = d(x_n, p) = d(y_n, p)\); \(d(x, T_i y) \leq d(S_i x, T_i y)\) for all \(x, y \in K\) and \(i = 1, 2\).
\end{enumerate}

Then the sequence \(\{x_n\}\) defined by (2.1) \(\Delta\)-converges to a common fixed point of \(\mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i) \cap F(S_i)\).

**Proof:** The proof of Theorem 2.1 is divided into three steps:

**Step:** First we prove that \(\lim_{n \to \infty} d(x_n, p)\) exists for each \(p \in F\).

For any given \(p \in F\), since \(T_i, i = 1, 2\), is a multivalued nonexpansive mapping, \(S, i=1,2\), is a non-expansive mapping, by condition (2) and (2.1), we have
\[
d(x_{n+1}, p) = d(W(S_i x_n, T_i u_n, \alpha_n), p)
\leq (1 - \alpha_n) d(S_i x_n, p) + \alpha_n d(T_i u_n, p)
\leq (1 - \alpha_n) d(S_i x_n, p) + \alpha_n d(T_i u_n, T_i p)
\leq (1 - \alpha_n) d(x_n, p) + \alpha_n d(p, T_i p)
\leq (1 - \alpha_n) d(x_n, p) + \alpha_n H(S_i y_n, p) + \alpha_n r_n
\leq (1 - \alpha_n) d(x_n, p) + \alpha_n \|y_n - p\| + \alpha_n r_n
= (1 - \alpha_n) d(x_n, p) + \alpha_n d(y_n, p) + \alpha_n r_n
\]
where
\[
d(y_n, p) = d(W(S_i x_n, T_i v_n, \beta_n), p)
\leq (1 - \beta_n) d(S_i x_n, p) + \beta_n d(T_i v_n, p)
\leq (1 - \beta_n) d(S_i x_n, p) + \beta_n d(T_i u_n, T_i p)
\leq (1 - \beta_n) d(x_n, p) + \beta_n d(p, v_n)
\leq (1 - \beta_n) d(x_n, p) + \beta_n d(x_n, p) + \beta_n r_n
\leq (1 - \beta_n) d(x_n, p) + \beta_n \|x_n - p\| + \beta_n r_n
= (1 - \beta_n) d(x_n, p) + \beta_n d(x_n, p) + \beta_n r_n
= d(x_n, p) + \beta_n r_n
\]
Substituting (2.3) into (2.2) and simplifying it, we have
\[
d(x_{n+1}, p) \leq d(x_n, p) + (1 - \alpha_n) \alpha_n r_n
\]
where \(\delta_n = 0, b_n = (1 - \beta_n) \alpha_n r_n\). Since \(\sum_{n=1}^{\infty} r_n < \infty\) and condition it follows from Lemma 1.4 that \(\lim_{n \to \infty} d(x_n, p)\) exist for \(p \in F\).

**Step 2:** We show that...
\[
\lim_{n \to \infty} d(x_n, T_{x_n}) = 0, \quad \lim_{n \to \infty} d(y_n, S_{x_n}) = 0, \quad i = 1, 2
\]  
(2.5)

For each \( p \in F \), from the proof of Step 1, we know that \( \lim_{n \to \infty} d(x_n, p) \) exists.

We may assume that \( \lim_{n \to \infty} d(x_n, p) = c \geq 0 \). If \( c = 0 \), then the conclusion is trivial. Next, we deal with the case \( c > 0 \). From (2.3), we have
\[
d(y_n, p) \leq d(x_n, p) + \beta_n \tau_n
\]  
(2.6)

Taking \( \limsup \) on both sides in (2.6), we have
\[
\limsup_{n \to \infty} d(y_n, p) \leq c
\]  
(2.7)

In addition, since \( d(T_{x_n}, p) = d(T_{x_n}, T_{x_n}) \leq d(y_n, p) \) and
\[
d(S_{x_n}, p) \leq d(S_{x_n}, x_n) + d(x_n, p) \leq d(x_n, p) + \beta_n \tau_n
\]  
(2.8)

and
\[
\limsup_{n \to \infty} d(S_{x_n}, p) \leq c
\]  
(2.9)

Since \( \limsup_{n \to \infty} d(x_n, p) = c \), it is easy prove that
\[
\lim_{n \to \infty} d(W(S_{x_n}, T_{x_n}, \alpha_n), p) = c
\]  
(2.10)

It follows from (2.8)-(2.10) and Lemma 1.5 that
\[
\lim_{n \to \infty} d(S_{x_n}, T_{x_n}) = 0
\]  
(2.11)

By the same method, we can also prove that
\[
\lim_{n \to \infty} d(y_n, T_{x_n}) = 0
\]  
(2.12)

By virtue of the condition (4), it follows from (2.11) and (2.12) that
\[
\lim_{n \to \infty} d(x_n, T_{x_n}) = 0
\]  
(2.13)

and
\[
\lim_{n \to \infty} d(y_n, S_{x_n}) = 0
\]  
(2.14)

From (2.1) and (2.12) we have
\[
d(y_n, S_{x_n}) = d(W(S_{x_n}, T_{x_n}), S_{x_n}) \\
\leq \beta_n d(T_{x_n} S_{x_n}, x_n) \to 0 \quad (a \to \infty)
\]  
(2.15)

and
\[
d(y_n, S_{x_n}) = d(W(S_{x_n}, T_{x_n}, \beta_n), S_{x_n}) \\
\leq \beta_n d(T_{x_n} S_{x_n}, x_n) \to 0 \quad (a \to \infty)
\]  
(2.16)

Observe that
\[
d(x_n, y_n) = d(x_n, T_{x_n}) + d(T_{x_n} x_n, S_{x_n}) + d(S_{x_n}, y_n)
\]

It follows from (2.14) and (2.15) that
\[
\lim_{n \to \infty} d(x_n, y_n) = 0
\]  
(2.17)

This together with (2.13) implies that
\[
d(x_n, T_{x_n}) \leq d(x_n, T_{x_n}) + d(T_{x_n}, y_n) \\
\leq d(x_n, T_{x_n}) + d(y_n, x_n) \to 0 \quad (n \to \infty)
\]  
(2.18)

On the other hand, from (2.11) and (2.17), we have
\[
d(S_{x_n}, T_{x_n}) \leq d(S_{x_n}, T_{x_n}) + d(T_{x_n}, y_n) \\
\leq d(S_{x_n}, T_{x_n}) + d(y_n, x_n) \to 0 \quad (n \to \infty)
\]  
(2.19)

Hence from (2.18) and (2.19), we have that
\[
d(S_{x_n}, x_n) \leq d(S_{x_n}, T_{x_n}) + d(T_{x_n}, x_n) \to 0 \quad (n \to \infty)
\]  
(2.20)

In addition, since
\[
(\alpha_n) \leq 1 - \alpha_n d(S_{x_n}, x_n) + \alpha_n d(T_{x_n}, x_n)
\]

from (2.13) and (2.20), we get
\[
\lim_{n \to \infty} d(x_{n+i}, x_n) = 0
\]  
(2.21)

Finally, for all \( i = 1, 2 \), we have
\[
d(x_n, T_{x_n}) \leq d(x_n, T_{x_n}) + d(T_{x_n}, x_n) \\
+ d(S_{x_n}, T_{x_n}) + d(T_{x_n}, x_n)
\]

\[
\leq 2d(y_n, x_n) + d(y_n S_{x_n}) + d(S_{x_n}, T_{x_n})
\]

it follows from (2.11) and (2.15) and (2.16) and (2.17) that
\[
\lim_{n \to \infty} d(x_n, T_{x_n}) = 0, \quad i = 1, 2
\]  
(2.22)

Since
\[
d(x_n, S_{x_n}) \leq d(x_n, T_{x_n}) + d(T_{x_n}, S_{x_n})
\]

it follows from (2.12) and (2.19) and (2.22) that
\[
\lim_{n \to \infty} d(x_n, S_{x_n}) = 0, \quad i = 1, 2
\]  
(2.23)

**Step 3:** Now we prove the sequence \( \{x_n\} \) \( \Delta \)-converges to a common fixed point of
\[
F := \bigcap_{\alpha} T_{\alpha}, \quad F(S_T)
\]

In fact, since for each \( p \in F \), \( \lim_{n \to \infty} d(x_n, p) \). \( \exists \). Exist.

This implies that the sequence \( \{d(x_n, p)\} \) is bounded, so is the sequence \( \{x_n\} \). Hence by virtue of Lemma1.3, \( \{x_n\} \) has a unique asymptotic center \( A_{K}(\{x_n\}) \}

Let \( \{u_n\} \) be any subsequence of \( \{x_n\} \) with \( A_{K}(\{u_n\}) \} = \{u \}. \) It follows from (2.5) that
\[
\lim_{n \to \infty} d(u_n, T_{u_n}) = 0
\]

Now, we show that \( u \in F(T_{u_n}) \). For this, we define a sequence \( \{z_n\} \) in \( K \) by \( z_j = T_{u_j} u \). So we calculate
\[
d(z_j, u_j) \leq d(T_{u_j} u, T_{u_j} u) + d(T_{u_j} u, T_{u_j}^{-1} u) + \cdots + d(T_{u_j} u, u_j)
\]

\[
= d(T_{u_j} u, T_{u_j} u) + \sum_{j=1}^{\infty} d(T_{u_j} u, T_{u_j}^{-1} u)
\]

Since \( T_{u_j} \) is a non-expansive mapping, by
\[
d(T_{u_j} u, T_{u_j} u) \leq d(T_{u_j} u, T_{u_j}^{-1} u) \leq \cdots \leq d(u_j, u_j) \leq d(u_j, u_j) \leq d(T_{u_j}^{-1} u, T_{u_j}^{-2} u) \leq \cdots \leq d(T_{u_j} u, u_j)
\]

from (2.25) we have:
\[
d(z_j, u_j) \leq d(u_j, u_j) + jd(T_{u_j} u, u_j)
\]

Taking limsup on the sides of the above estimate and using (2.24), we have
\[
r(z_j, u) = \limsup_{n \to \infty} \sum_{j=1}^{\infty} d(T_{u_j} u, T_{u_j}^{-1} u)
\]

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Proof. Take $\mathcal{S}_i = \{i\}$, $i = 1, 2$ in Theorem 2.1. Since all conditions are satisfied, it follows from Theorem 2.1 that the sequence $\{x_n\}$ converges to a common fixed point of $F := \bigcap_{i=1}^{2} F(T_{\mathcal{S}_i})$. This completes the proof of Theorem 2.2.

Competing Interests

The author declares that he has no competing interests.

Author’s Contributions

Author contributed equally and significantly in this research work.

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