Vacuum polarization around stars: nonlocal approximation

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We compute the vacuum polarization associated with quantum massless fields around stars with spherical symmetry. The nonlocal contribution to the vacuum polarization is dominant in the weak field limit, and induces quantum corrections to the exterior metric that depend on the inner structure of the star. It also violates the null energy conditions. We argue that similar results also hold in the low energy limit of quantum gravity. Previous calculations of the vacuum polarization in spherically symmetric spacetimes, based on local approximations, are not adequate for newtonian stars.

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I. INTRODUCTION

In quantum field theory in curved spaces, the mean value of the energy momentum tensor \( \langle T_{\mu\nu}(x) \rangle \) plays a main role. When evaluated around black holes or collapsing stars, it contains the information about Hawking radiation. As a source, since this would involve the knowledge of the vacuum polarization for at least a family of metrics. For these reasons, it is of interest to develop approximation and numerical schemes. If the quantum field is very massive, the DeWitt-Schwinger approximation is adequate to describe the vacuum polarization \( \mathbf{1} \). However, if the mass is small or vanishes, the situation is more complex. There are many papers dealing with this problem, starting from those concerned with conformal fields in Schwarzschild spacetime \( \mathbf{2} \mathbf{3} \mathbf{4} \mathbf{5} \mathbf{6} \). Analytic approximations have been proposed for Schwarzschild, Reissner Nordstrom, and general static spherically symmetric spacetimes, for both conformal and non conformal fields \( \mathbf{6} \). The approach \( \mathbf{6} \) contains as particular cases the previous analytic approximations \( \mathbf{4} \mathbf{4} \mathbf{2} \) to \( \langle T_{\mu\nu}(x) \rangle \) for black holes. It has been shown \( \mathbf{6} \) that the analytic approximation reproduces the exact result for Schwarzschild spacetime with high accuracy in the case of massless scalar fields (however this is not the case for spinor fields, see \( \mathbf{8} \)).

The different approximations to \( \langle T_{\mu\nu}(x) \rangle \) depend of course on the quantum state of the field. The states usually considered are the Hartle-Hawking, Unruh and Boulware vacua \( \mathbf{8} \). The Hartle-Hawking state describes a black hole in equilibrium with its thermal radiation. The Unruh state is the one that best mimics the gravitational collapse of a star. The Boulware state, being ill defined on the horizon, is unphysical for black holes. However, it is widely believed to describe the quantum vacuum around a non-collapsed object like a star or a planet. For these reasons, analytic approximations in the Boulware state have been used to compute vacuum polarization around stars \( \mathbf{3} \) and to check the validity of the energy conditions \( \mathbf{10} \).

In the approximations mentioned above, the energy momentum tensor is determined by local functions of the metric and its derivatives. However, as we already pointed out, on general grounds one expects \( \langle T_{\mu\nu}(x) \rangle \) to be a nonlocal function. In particular, in the weak field approximation, a covariant perturbative approximation to \( \langle T_{\mu\nu}(x) \rangle \) is explicitly nonlocal \( \mathbf{11} \mathbf{12} \). In this paper we will compute \( \langle T_{\mu\nu}(x) \rangle \) around static spherical stars using this nonlocal approximation. We will consider massless fields with arbitrary coupling to the curvature in the Boulware quantum state. We will compare the result with the analytic approximations, and show that the nonlocal part dominates. Moreover, we will show that from the nonlocal result one can derive the well known \( 1/r^3 \) - quantum correction to the Newtonian potential \( \mathbf{13} \mathbf{14} \). Conversely, we will see that the nonlocal approximation can be easily understood from this quantum correction and the superposition principle, valid in the weak field approximation.

The paper is organized as follows. In the next section, as a warm-up, we discuss the local and nonlocal approximations to \( \langle \phi^2 \rangle \). We expand the nonlocal approximation in a multipolar expansion, which shows explicitly its dependence with the internal structure of the star. We also show that there are surface divergences on the boundary of the star unless the Ricci tensor is sufficiently smooth. In Section III we compute \( \langle T_{\mu\nu}(x) \rangle \) in the weak field approximation for...
an arbitrary, static, spherically symmetric star. We point out again the dependence on the inner structure of the star and the existence of surface divergences. Section IV contains some applications: first we show that \( T_{\mu\nu}(x) \) violates both the null energy and the average null energy conditions. Then we compute quantum corrections to the exterior metric, and find that they depend on the internal structure of the star. Section V contains a short discussion and conclusions.

Throughout this paper we use units in which \( \hbar = c = 1 \), while retaining \( G = \ell_p^2 \neq 1 \). Our sign conventions are \(- - - -\) in the nomenclature of [15].

II. NONLOCAL APPROXIMATION FOR \( \langle \phi^2 \rangle \)

In this section we compute the quantity \( \langle \phi^2 \rangle \), for a massless scalar field with arbitrary coupling \( (\xi) \) to the curvature, in a weak background gravitational field. We assume the classical source that generates the field to be static and non-relativistic, i.e., its stress energy tensor takes the form \( T_{\mu\nu}(x) = \rho(x)\delta^\mu_\mu \delta^\nu_\nu \).

An expression for \( \langle \phi^2 \rangle \) can be obtained by taking the coincidence limit of the Feynman propagator,

\[
\langle \phi^2(x) \rangle = -\text{Im} \left( \lim_{x' \to x} G_F(x, x') \right),
\]

and then renormalizing it. With this purpose we first find the Feynman Green function by solving the equation

\[
[\Box_x + \xi R(x)] G_F(x, x') = -\frac{1}{\sqrt{g(x)}} \delta(x - x').
\]

Since we are working in a weak gravitational field, we assume that both the metric and the Green function differ only slightly from their Minkowski space counterparts: \( g_{\mu\nu} = \eta_{\mu\nu} + \tilde{h}_{\mu\nu} \) and \( G_F(x, x') = G_F^{(0)}(x, x') + G_F^{(1)}(x, x') \). Expanding Eq. (2) in these small quantities and transforming Fourier in \( x \), it is straightforward to solve it to first order as

\[
G_F(x, x') = G_F^{(0)}(x, x') + \frac{1}{(2\pi)^4} \int \text{d}^4k \int \text{d}^4k' \int \text{d}^4\tilde{x} \frac{e^{ik(x-\tilde{x})}}{\xi R(\tilde{x}) - \tilde{h}^{00} \partial_0^2} \frac{e^{ik'(x-x')}}{k'^2}. \tag{3}
\]

where \( \tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \text{Tr}(h) \), and now \( R(x) \) is the Ricci scalar calculated to first order in \( h_{\mu\nu} \). After changing variables to \( p = k - k' \) and \( q = k + k' \), the \( q \)-integrals in Eq. (3) can be performed by standard dimensional regularization techniques. Integrating in \( d \)-dimensional \( q \)-space and taking the coincidence limit leads to

\[
G_F^{(1)}(x, x) = \left( \xi - \frac{1}{6} \right) \left[ \frac{i}{8\pi^2} \frac{R(x)}{4 - d} - \frac{i}{256\pi^6} \int \text{d}^4p \int \text{d}^4\tilde{x} \frac{R(\tilde{x})}{\Delta} \right] \ln \frac{\mu^2}{\mu_0^2},
\]

where \( \mu \) is an introduced arbitrary mass scale. The first term in the square bracket in Eq. (4) is the divergent part of the Hadamard propagator which must be removed from the result to obtain a finite—and renormalized—value for \( G_F(x, x) \). The second term, on the other hand, contains two pieces: one local and arbitrary proportional to \( R(x) \ln \mu^2 \) and with no physical significance, and other non-local and with physical meaning to \( \langle \phi^2 \rangle \).

Once dropped the \( \xi R(x) \) divergent part in Eq. (4), the integration in \( \ell \) and \( p^0 \) for a static situation yields

\[
\langle \phi^2(x) \rangle = \frac{1}{128\pi^5} \left( \xi - \frac{1}{6} \right) \int \text{d}^4p \frac{\tilde{R}(p)}{\Delta p^2} \ln \frac{\mu^2}{\mu_0^2} = \frac{1}{128\pi^5} \left( \xi - \frac{1}{6} \right) \ln \frac{-\Delta^2}{\mu^2} R,
\]

where \( \tilde{R}(p) \) is the Fourier transform of \( R(x) \). This result is finite everywhere for any smooth and asymptotically flat \( R(x) \). Notice, however, that for the external region of the source, i.e. \( R(x) = 0 \), Eq. (5) can be taken to a more friendly form if the \( \text{d}^4p \) integral is performed first in Eq. (5), namely

\[
\langle \phi^2(x) \rangle = -\frac{1}{32\pi^3} \left( \xi - \frac{1}{6} \right) \int \text{d}^3\tilde{x} \frac{R(\tilde{x})}{|\tilde{x}|^3} \quad \text{(external region)}.
\]

As an application of the results in Eqs. (5-6) we now compute the vacuum polarization for a spherical star of radius \( R_0 \) and constant density \( \rho_0 \) (we assume \( \rho_0 GR_0^2 \ll 1 \) to endorse the weak field approximation). In this case the scalar
The curvature takes the form \( R(x) = 8\pi G \Theta(R_0 - r) \) and Eq. (5) yields
\[
\langle \phi^2(r) \rangle = -\frac{G\rho_0}{2\pi} \left( \xi - \frac{1}{6} \right) \left\{ \ln\frac{C(R_0^2 - r^2)}{r^2} + \frac{2R_0}{r} + \ln\left( \frac{r + R_0}{r - R_0} \right) \right\} \quad r < R_0,
\]
\[
\langle \phi^2(r) \rangle = -\frac{G\rho_0}{4\pi^2} \left( \xi - \frac{1}{6} \right) \left( 1 + \frac{3R_0^2}{r^2} + \cdots \right) \quad (r > R_0),
\]
where \( C \) is an arbitrary constant proportional to \( \mu^2 \). Notice that this expression has a logarithmic divergence at the surface of the star due to the discontinuity of the density; this is because the Fourier transform of the Heaviside function falls only as \( p^{-1} \) when \( p \to \infty \). In effect, any star model with a non-continuous \( \rho \) at the surface will have a logarithmic divergence in \( \langle \phi^2 \rangle \) at the point of discontinuity. On the other hand, for a continuous density the Fourier transforms falls faster to zero and the divergence disappears. The surface divergence is similar to the divergences that appear when computing the Casimir effect for perfect conductors of arbitrary shape. See for instance [11].

It should be noted that the expression for \( \langle \phi^2 \rangle \) outside the star in Eq. (4), depends not only on the mass of the star but also on its radius; in fact, expanding this expression for \( r > R_0 \) yields
\[
\langle \phi^2(r) \rangle = -\frac{G\rho_0}{4\pi^2} \left( \xi - \frac{1}{6} \right) \sum_{n=1}^{\infty} \frac{M_n}{r^{2n+1}},
\]
which is exact in the external region for any spherically symmetric distribution of mass. The multipolar coefficients in Eq. (9) are defined as
\[
M_n = 4\pi \int_0^{R_0} d\tilde{r} \rho(\tilde{r}) \tilde{r}^{2n}, \quad (r > R_0),
\]
so that \( M_1 = M \) and in general \( M_n \sim MR_0^{2(n-1)} \). Therefore, the vacuum polarization in the external region depends at the sub-dominant order on the internal structure of the star.

In contrast to this internal structure dependence, we point out that the local approximation for \( \langle \phi^2 \rangle \) in static spherically symmetrical spaces previously developed [17], for a massless field in the Schwarzschild metric leads to
\[
\langle \phi^2(r) \rangle_{\text{loc}} = \frac{M^2G^2}{48\pi^2r^3(2MG - r)}, \quad (r > R_0),
\]
which, in the weak field approximation, goes as \( M^2G^2/r^4 \) since we are assuming \( r \gg 2MG \) in the external region. A comparison between Eq. (11) and Eq. (9) for large \( r \) gives
\[
\frac{\langle \phi^2(r \to \infty) \rangle_{\text{nonloc}}}{\langle \phi^2(r) \rangle_{\text{loc}}} \sim \left( \xi - \frac{1}{6} \right) \frac{r}{MG},
\]
and hence, except in the conformal case \( \xi = 1/6 \), the nonlocal approximation is dominant. Moreover, since the nonlocal expression in Eq. (9) increases when \( r \to R_0 \) outside the star, the nonlocal expression is dominant everywhere in the external region.

We have therefore shown through an easy object, as \( \langle \phi^2 \rangle \), how the quantum effects on a weak gravitational field can induce dominant nonlocal behaviors, providing a conceptual difference with previous works in the subject. Our next step is to use the light thrown by this calculation to obtain and study analogous results for a more complicated and important object, the stress-energy tensor.

**III. NONLOCAL APPROXIMATION FOR \( \langle T_{\mu\nu} \rangle \)**

A nonlocal formal expression for \( \langle T_{\mu\nu} \rangle \) in the weak field limit, similar in character to our expression for \( \langle \phi^2 \rangle \) in Eqs. (4)-(5), has been found in [11][12]. In the massless case it reads
\[
\langle T_{\mu\nu}(x) \rangle = -\frac{1}{32\pi^2} \left[ \frac{1}{2} \left( \xi - \frac{1}{6} \right)^2 - \frac{1}{90} \right] \ln \frac{\Box}{\mu^2} H^{(1)}_{\mu\nu}(x) + \frac{1}{60} \ln \frac{\Box}{\mu^2} H^{(2)}_{\mu\nu}(x),
\]
where \( C \) is an arbitrary constant proportional to \( \mu^2 \). Notice that this expression has a logarithmic divergence at the surface of the star due to the discontinuity of the density; this is because the Fourier transform of the Heaviside function falls only as \( p^{-1} \) when \( p \to \infty \). In effect, any star model with a non-continuous \( \rho \) at the surface will have a logarithmic divergence in \( \langle \phi^2 \rangle \) at the point of discontinuity. On the other hand, for a continuous density the Fourier transforms falls faster to zero and the divergence disappears. The surface divergence is similar to the divergences that appear when computing the Casimir effect for perfect conductors of arbitrary shape. See for instance [16].

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\langle T_{\mu\nu}(x) \rangle = -\frac{1}{32\pi^2} \left[ \frac{1}{2} \left( \xi - \frac{1}{6} \right)^2 - \frac{1}{90} \right] \ln \frac{\Box}{\mu^2} H^{(1)}_{\mu\nu}(x) + \frac{1}{60} \ln \frac{\Box}{\mu^2} H^{(2)}_{\mu\nu}(x),
\]
where the tensors \( H^{(i)}_{\mu \nu} \) are the two independent higher-order tensors that appear in the Einstein equations when derived from an action including \( R^2 \) and \( R_{\mu \nu} R^{\mu \nu} \). In the weak field limit they reduce to

\[
H^{(1)}_{\mu \nu} = 4 \nabla_\mu \nabla_\nu R - 4g_{\mu \nu} \Box R + O(R^2)
\]

\[
H^{(2)}_{\mu \nu} = 2 \nabla_\mu \nabla_\nu R - g_{\mu \nu} \Box R - 2\Box R_{\mu \nu} + O(R^2).
\]

(14)

The action of the nonlocal operator \( F(\Box) = \ln \frac{\Box}{\pi^2} \) in Eq. (13) has been described in detail in several papers (see for example [11]). For time dependent situations it involves an integral in the past null cone of \( x \), and therefore \( \langle T_{\mu \nu} \rangle \) is nonlocal and causal. On the other hand, for time independent situations, it can be shown that \( F(\Box) = F(-\nabla^2) \) [18].

From Eq. (13), assuming a static situation and performing the relevant integrations, it is possible to derive a nonlocal expression analogous to (6), valid in the external region where the classical source vanishes. It is given by

\[
\langle T_{\mu \nu}(x) \rangle = \frac{1}{128\pi^3} \left\{ \left( \xi - \frac{1}{6} \right)^2 - \frac{1}{90} \right\} \int d^3\tilde{x} \frac{H^{(1)}_{\mu \nu}(\tilde{x})}{|x - \tilde{x}|^3} + \frac{1}{30} \int d^3\tilde{x} \frac{H^{(2)}_{\mu \nu}(\tilde{x})}{|x - \tilde{x}|^3}.
\]

(15)

Inserting the definitions Eq. (14) into this equation, after integration by parts we obtain

\[
\langle T_{00} \rangle = \frac{1}{32\pi^3} \left[ \left( \xi - \frac{1}{6} \right)^2 - \frac{1}{90} \right] \nabla^2 \int d^3\tilde{x} \frac{R(\tilde{x})}{|x - \tilde{x}|^3},
\]

\[
\langle T_{0i} \rangle = 0,
\]

\[
\langle T_{ij} \rangle = \frac{1}{32\pi^3} \left[ \left( \xi - \frac{1}{6} \right)^2 + \frac{1}{180} \right] (\partial_i \partial_j - \delta_{ij} \nabla^2) \int d^3\tilde{x} \frac{R(\tilde{x})}{|x - \tilde{x}|^3},
\]

(16)

therefore \( \langle T_{\mu \nu} \rangle \) can be written in terms of derivatives of \( \langle \phi^2 \rangle \). For a spherically symmetric situation, using the series expansion Eq. (9) we find

\[
\langle T^t_t \rangle = \frac{G}{2\pi^2} \left[ \left( \xi - \frac{1}{6} \right)^2 - \frac{1}{90} \right] \sum_{n=1}^{\infty} (2n + 1) \frac{M_n}{r^{2n+3}},
\]

\[
\langle T^r_r \rangle = -\frac{G}{2\pi^2} \left[ \left( \xi - \frac{1}{6} \right)^2 + \frac{1}{180} \right] \sum_{n=1}^{\infty} (2n + 1) \frac{M_n}{r^{2n+3}},
\]

\[
\langle T^\theta_\theta \rangle = \frac{G}{4\pi^2} \left[ \left( \xi - \frac{1}{6} \right)^2 + \frac{1}{180} \right] \sum_{n=1}^{\infty} (2n + 1)^2 \frac{M_n}{r^{2n+3}}.
\]

(17)

Note that the nonlocal contribution to \( \langle T^\nu_\nu \rangle \) does not vanish for any value of \( \xi \).

The components of \( \langle T^\nu_\nu \rangle \) fall off as \( MG/r^5 \) for \( r \gg R_0 \). On the other hand, in the local approximation for the Boulware vacuum (see for instance [8, 11]), they fall off as \( M^2G^2/r^6 \). So the nonlocal approximation dominates for \( \langle T_{\mu \nu} \rangle \) as well as for \( \langle \phi^2 \rangle \), by a factor of order \( r/MG \).

Inside the star, \( \langle T_{\mu \nu} \rangle \) can be found by taking derivatives of the finite expression for \( \langle \phi^2 \rangle \) (Eq. 15) in the same fashion. As an example, the complete \( \langle T^\nu_\nu \rangle \) for a star of constant density is, for \( r < R_0 \),

\[
\langle T^t_t \rangle = -\frac{G\rho_0}{\pi} \left[ \left( \xi - \frac{1}{6} \right)^2 - \frac{1}{90} \right] \frac{3R_0^2 - r^2}{(R_0^2 - r^2)^2},
\]

\[
\langle T^r_r \rangle = -\frac{2G\rho_0}{\pi} \left[ \left( \xi - \frac{1}{6} \right)^2 + \frac{1}{180} \right] \frac{1}{R_0^2 - r^2},
\]

\[
\langle T^\theta_\theta \rangle = -\frac{2G\rho_0}{\pi} \left[ \left( \xi - \frac{1}{6} \right)^2 + \frac{1}{180} \right] \frac{R_0^2}{(R_0^2 - r^2)^2},
\]

(18)
while for $r > R_0$

$$\langle T^t_t \rangle = \frac{G\rho_0}{\pi} \left[ \left( \xi - \frac{1}{6} \right)^2 - \frac{1}{90} \right] \frac{2R_0^3}{r(r^2 - R_0^2)^2}$$

$$\langle T^r_r \rangle = -\frac{2G\rho_0}{\pi} \left[ \left( \xi - \frac{1}{6} \right)^2 + \frac{1}{180} \right] \frac{R_0^2}{r^3(r^2 - R_0^2)}$$

$$\langle T^\theta_\theta \rangle = \frac{2G\rho_0}{\pi} \left[ \left( \xi - \frac{1}{6} \right)^2 + \frac{1}{180} \right] \frac{R_0^3(3r^2 - R_0^2)}{2r^3(r^2 - R_0^2)^2}.$$  (19)

This result is quadratically divergent on the surface of the star. To remove the logarithmic divergence in $\langle \phi^2 \rangle$ it was enough to require $R(x)$ to be a continuous function; as $\langle T_{\mu\nu} \rangle$ is constructed with second derivatives of $\langle \phi^2 \rangle$, to ensure that no divergence arises on the surface it is necessary to require the continuity of the second derivatives of $R(x)$.

The local approximation for $\langle T_{\mu\nu} \rangle$ inside a star of constant density, presented in [2], gives an expression with an additional factor of $MG/R_0$. Therefore the nonlocal approximation is, for nonrelativistic stars, dominant over the local one both in the internal as well as the external region.

IV. APPLICATIONS

A. Energy conditions

One important question to ask is whether $\langle T_{\mu\nu} \rangle$ calculated within semiclassical gravity satisfies the energy conditions or not. It is well known that quantum fields often do not satisfy the energy conditions, at least not in the same way as classical fields do. This fact has raised considerable discussion regarding its implications for singularity theorems, black hole dynamics, existence of macroscopic traversable wormholes, creation of closed time-like curves, etc. In other words, violation of the energy conditions imply that the semiclassical Einstein equations, with $\langle T_{\mu\nu} \rangle$ as a source, could in principle admit solutions qualitatively different from classical solutions.

In this section we will discuss the validity of two energy conditions, the Null Energy Condition (NEC) and the Averaged Null Energy Condition (ANEC) for the massless scalar field $\langle T_{\mu\nu} \rangle$ found in last section. The discussion will be restricted to the exterior region, since inside the star all energy conditions are satisfied by the classical source. These conditions have been studied in Ref. [10] using the local approximation for the Boulware vacuum in the conformal case. The conclusion was that neither of them holds in the exterior of a non-collapsed star. However, since the nonlocal contribution to $\langle T_{\mu\nu} \rangle$ dominates in the weak field limit, it is necessary to revise the calculation.

The NEC states that $T_{\mu\nu}K^\mu K^\nu \geq 0$ for every null vector $K^\mu$. For a static spherically symmetric situation this condition is equivalent to the following inequalities

$$\langle T^t_t \rangle \geq \langle T^r_r \rangle \quad \langle T^\theta_\theta \rangle \geq \langle T^\phi_\phi \rangle,$$  (20)

to which we will refer as radial and tangential conditions respectively. Using the series expansions, we find that

$$\langle T^t_t \rangle - \langle T^r_r \rangle = \frac{G}{2\pi^2} \sum_{n=1}^\infty \frac{M_n}{r^{2n+3}} (2n + 1) \times \left[ -\frac{1}{90} \left( n - \frac{1}{2} \right) + \left( \xi - \frac{1}{6} \right)^2 (n + 1) \right]$$

$$\langle T^t_t \rangle - \langle T^\theta_\theta \rangle = -\frac{G}{2\pi^2} \sum_{n=1}^\infty \frac{M_n}{r^{2n+3}} (2n + 1) \times \left[ \frac{1}{360} (6n + 1) + \frac{1}{360} \left( \xi - \frac{1}{6} \right)^2 \right].$$  (21)

It is easily seen that the tangential condition is violated for any $\xi$, which is enough to ensure NEC violation. The radial condition is also violated in the conformal case, but it holds in the minimal coupling case. Therefore the possibility remains that in this case the average of the NEC condition over a null geodesic may be positive. We will show now that this is not the case: the ANEC is also violated for arbitrary $\xi$.

The ANEC states that $\int_{-\infty}^{\infty} dp \rho K^\mu K^\nu \geq 0$ for any null geodesic of tangent vector $K^\mu$ and affine parameter $p$. To prove that the ANEC is also violated, we consider a null geodesic external to the star with impact parameter $b > R_0$, defined by the following functions $x^\mu(p)$

$$t(p) = p \quad r(p) = \sqrt{p^2 + b^2}$$

$$\theta(p) = \pi/2 \quad \phi(p) = \arctan(p/b).$$  (23)
The geodesic is taken to be a straight line, to keep the analysis consistently at first order. The integration of \( \langle T_{\mu\nu} \rangle K^\mu K^\nu \) over the whole trajectory (with \( K^\mu = \frac{dx^\mu}{dp} \)) can be performed using the series expansion for \( \langle T_{\mu\nu} \rangle \) in Eq.(17). The result is

\[
\int_{-\infty}^{\infty} dp \, \langle T_{\mu\nu} \rangle K^\mu K^\nu = -\frac{G}{2880 \beta^2} \sum_{n=1}^{\infty} \frac{M_n}{(2b)^{2n}} \times \frac{2(4n(n+1)-3) \Gamma(2n+2) + \Gamma(2n+4)}{\Gamma(n+3/2) \Gamma(n+5/2)} . \tag{24}
\]

This quantity is clearly negative and independent of \( \xi \). We conclude that the ANEC is violated for a massless scalar field in the exterior of a nonrelativistic star independently both of the coupling of the field and of the internal structure of the star.

B. Quantum corrections to the metric

In this section we shall solve the linearized semiclassical Einstein equations, which take into account the backreaction of the quantum field over the spacetime metric. The results will be found to agree with those obtained by [13] but disagree with those found relying on the local approximation for \( \langle T_{\mu\nu} \rangle \) [9].

Disregarding both cosmological constant and higher-order terms, the semiclassical Einstein equations read

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -8\pi G \left( T_{\mu\nu}^{(cl)} + \langle T_{\mu\nu} \rangle \right). \tag{25}
\]

We take the source term to be separated in a classical part, the ordinary density of the star, and a quantum part, which is the \( \langle T_{\mu\nu} \rangle \) we have calculated in the previous section. The gravitational field is similarly separated in a classical term (the background spacetime) and a correction that is produced by the quantum

\[
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}^{(cl)} + h_{\mu\nu}^{(q)} . \tag{26}
\]

We express the correction to the gravitational field in isotropic form. In Cartesian coordinates

\[
h_{00}^{(q)} = f(r) , \quad h_{0i}^{(q)} = 0 , \quad h_{ij}^{(q)} = g(r) \delta_{ij} . \tag{27}
\]

It is a simple matter to solve the equations [23], linearized in \( h_{\mu\nu} \), using the \( \langle T_{\mu\nu} \rangle \) given by Eq. [14] as a source. The result is

\[
f(r) = -\frac{G}{4\pi^2} \left[ (\xi - \frac{1}{6})^2 + \frac{1}{45} \right] \int d^3\tilde{x} \frac{\tilde{R}(\tilde{x})}{|\tilde{x} - \tilde{x}|^3} \tag{28}
\]

\[
g(r) = -\left[ (\xi - \frac{1}{6})^2 + \frac{1}{45} \right] f(r) . \tag{29}
\]

Note that the correction \( h_{00}^{(q)} \) is proportional to \( \langle \phi^2 \rangle \) (except in the conformal case \( \xi = 1/6 \), in which \( \langle \phi^2 \rangle \) vanishes but \( h_{00}^{(q)} \) does not). Eq. [28] is correct only outside the star, where \( \langle \phi^2 \rangle \) is given by Eq. [10]. Inside the star, \( h_{00}^{(q)} \) has the same proportionality with the finite \( \langle \phi^2 \rangle \) given in Eq. [4].

In the exterior region we can replace Eq. [28] by its multipolar expansion

\[
f(r) = -\frac{2G^2}{\pi} \left[ (\xi - \frac{1}{6})^2 + \frac{1}{45} \right] \sum_{n=1}^{\infty} \frac{M_n}{r^{2n+1}} . \tag{30}
\]

Here we see an explicit dependence of the external metric with the internal structure of the star, which can be thought as a "quantum violation" of Birkhoff’s theorem. This is possible because the space surrounding the star, although empty at the classical level, contains a quantum vacuum energy given by \( \langle T_{\mu\nu} \rangle \).
As an example, the quantum correction to the Schwarzschild metric external to a star of constant density is predicted by our nonlocal approximation to be

\[ f(r) = -\frac{2}{\pi} \left( \left( \frac{\xi - 1}{6} \right)^2 + \frac{1}{45} \right) \frac{MG^2}{r^3} \left( 1 + \frac{3}{5} \frac{R_0^2}{r^2} + \cdots \right), \]

with \( g(r) \) given by Eq. (24).

The leading term in Eqs. (31) and (32) for \( r \gg R_0 \) is the quantum correction to the Newtonian potential of a point mass, since it contains no information about the internal structure. Therefore the complete Newtonian potential of a point mass when the effect of a quantum massless scalar field is taken into account reads

\[ \Phi(r) = -\frac{MG}{r} - \frac{1}{\pi} \left( \left( \frac{\xi - 1}{6} \right)^2 + \frac{1}{45} \right) \frac{MG^2}{r^3}. \]  

This result agrees with previous ones [12]. It also has the same form that the long-distance quantum corrections to \( \Phi \) due to gravitons, calculated in the low energy limit of quantum gravity [14]. By contrast, the correction to the Schwarzschild metric found in [3] using the local approximation for \( T_{\mu\nu} \) is

\[ g_{00}(r \to \infty) = 1 - \frac{2MG}{r} - \frac{M^2G^2}{60\pi r^4} + \cdots, \]

which does not agree with previous results.

It is worth noting that our result satisfies consistently the principle of superposition; if the gravitational field of the star is calculated as a sum of infinitesimal contributions of the form given in Eq. (28), the result takes us back to Eq. (28). In fact, Eq. (28) is correct even if there is no spherical symmetry, as can be seen by noting that the 00 component of the linearized Einstein equations reads (in the Lorentz gauge)

\[ \square h^{(q)}_{00} = -16\pi G \left( T_{00} - \frac{1}{2} \langle T^\lambda_\lambda \rangle \right). \]  

Using the expressions Eq. (10) for \( \langle T_{\mu\nu} \rangle \), it follows immediately that \( h^{(q)}_{00}(x) \) is given by Eq. (28) (if the point \( x \) is located outside the sources, i.e., if \( R(x) = 0 \)).

As an application in a non-spherically symmetric situation, we outline here the calculation of the long-distance quantum correction to the Kerr metric by considering a rotating source. The classical \( T_{\mu\nu} \) has now off-diagonal terms related to the angular velocity of the star, \( T_{0i}(x) = \rho \omega r \cos \theta (-\sin \varphi \hat{x} + \cos \varphi \hat{y})_i \). This produces a nonzero \( H_{0i}^{(2)} \) which in turn causes \( \langle T_{\mu\nu} \rangle \) to have, outside the star, the following components

\[ \langle T_{0i} \rangle = \frac{G\omega}{240\pi^2} \sqrt{2} \int d^3x' \rho(x') r' \cos \theta' (-\sin \varphi' \hat{x} + \cos \varphi' \hat{y})_i \left| \hat{x} - \hat{x}' \right|^3, \]

in addition to those included in Eq. (10), which are not modified by the rotation. Inserting \( \langle T_{0i} \rangle \) in the semiclassical Einstein equations, we find that the long-distance correction to the off-diagonal terms of the Kerr metric are of the form

\[ h_{0i} = \frac{G^2}{16\pi r^2} \langle J \times r \rangle_i + O \left( \frac{G^2J R_0^2}{r^6} \right), \]

where \( J \) is the total angular momentum of the source. Note that the information about internal structure is again contained in the sub-dominant term, which always falls off two powers of \( r \) faster than the dominant one. Once again, similar results hold for the graviton quantum corrections.

V. CONCLUSIONS

In this paper we computed \( \langle \phi^2 \rangle \) and \( \langle T_{\mu\nu}(x) \rangle \) for a massless quantum field with arbitrary coupling to the curvature in the geometry of spherically symmetric Newtonian stars. We obtained expressions for both quantities as multipolar expansions, valid in the weak field limit, that show explicitly the nonlocal dependence of the quantum effects. As a byproduct, we have demonstrated that previous local approximations to the vacuum polarization in spherically
symmetric spacetimes do not apply to Newtonian stars. Indeed, the nonlocal part is the leading contribution that overwhelms the local one both inside and outside the surface of the star. The results are divergent on the surface if the star model is not taken as sufficiently smooth.

We have shown that outside the star \( \langle T_{\mu\nu}(x) \rangle \) violates the NEC and ANEC energy conditions. A local approximation for \( \langle T_{\mu\nu}(x) \rangle \) was used in previous proofs of these results, which stand now on firmer grounds. Our results hold for fields with arbitrary coupling, while previously only conformal fields had been considered. We also computed the quantum corrections to the metric, and found a "quantum violation" to Birkoff’s theorem: the external metric depends not only on the mass but on all the multipolar moments of the distribution within the star. From the same principles it is also easily shown that, due to quantum corrections, the gravitational field inside a spherical shell is slightly different from zero.

We have pointed out that the nonlocal quantum correction to the metric outside the star is a consequence of the \( 1/r^3 \) modification to the Newtonian potential and the superposition principle. This fact can be used to argue that the nonlocal modifications to the metric will be present in any quantum theory of gravity, since gravitons also induce \( 1/r^3 \)-corrections to the Newtonian potential. These corrections are, of course, extremely small.

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