Tensor Product States in the Synthetic Space of Quadratic Bosonic Systems: Emergent Interplay Between Diabolic and Exceptional Degeneracy

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(Dated: June 30, 2022)

Tensor product states (TPS), which enable decomposing the states of large-dimensional Hilbert space in terms of lower-dimensional elementary tensors, are fundamental tools to capture the quantum nature of condensed matter phenomena. Here, we show how TPS can naturally emerge in the synthetic space of operator moments of bosonic systems described by quadratic (non-)Hermitian Hamiltonians. Beyond their theoretical interest, allowing to simulate the many-body physics of nontrivial large-dimensional Hamiltonians, we show how such construction explicitly reveals the interplay between the diabolic (DPs) and exceptional points (EPs) in the quantum regime. That is, the emergent diabolic degeneracy due to the noncommutative nature of the bosonic creation and annihilation operators profoundly impacts the structure of the evolution matrices governing higher-order field moments. This results in the appearance of highly degenerate DPs and EPs, whose interplay enables the existence of hybrid DPs-EPs, which cannot be captured by a “classical” (i.e., commutative) treatment of the bosonic fields. These findings can also be utilized for constructing effective Hamiltonians with similar spectral characteristics, possessing hybrid DPs-EPs. Our results can be exploited in various quantum protocols based on EPs, and can pave a new direction of research in this field.

I. INTRODUCTION

Hermiticity is not a necessary requirement for Hamiltonians to exhibit real spectra, as shown by Bender and Boettcher in 1998 [1]. Their introduction of non-Hermitian quantum mechanics has triggered very active theoretical and experimental research in various areas of classical and quantum physics to study open systems with loss and gain, as described by non-Hermitian Hamiltonians (NHHs) with real eigenvalues if they satisfy parity-time ($PT$) symmetry [2–5]. Although NHHs were applied already in the early 1990s, as a key concept in, e.g., the Monte Carlo wave-function (or quantum trajectory) formalisms [6–8], Ref. [1] has revealed the importance of the $PT$-symmetry to maintain a real spectrum of NHHs.

By contrast to EPs, diabolical points (DPs) also correspond to the eigenvalue degeneracies of NHHs that, however, are not associated with coalescent eigenmodes and, thus, can occur for Hermitian systems as well. Both the EPs and DPs possess remarkable features. EPs have been predicted and observed in different platforms including: classical quantum electrodynamics (QED) [9–12], circuit QED [13–17], electronics [18], plasmonics [19, 20], photonic lattices [21], hybrid metamaterials [22], acoustics [23–26], cavity optomechanics [24, 27–29], optoelectronics [30], spintronics [31–33], cavity magnonics [34–38], atom optics with Bose-Einstein condensates [39], trapped ions [40], and in the interaction of THz light with collective molecular vibrations [41]. EPs can induce counterintuitive effects such as: loss-induced optical transparency [42], loss-induced suppression and revival of lasing [43], anomalous absorption [44, 45], unidirectional invisibility [46], topological chirality [13, 28, 30, 47], topological insulator lasers [48], or state switching [47, 49, 50]. On the other hand, the reported demonstrations of a Berry phase acquired by encircling a DP [51–53], can lead to applications in topological photonics [54], quantum metrology, and geometric quantum computation in the spirit of Refs. [55–57]. Moreover, DPs and EPs are useful in testing and classifying phases and phase transitions [58–60]. A deeper understanding of the emergence of EPs and DPs, and their physical consequences, are required to, e.g., find and classify new types of phase transitions, to achieve a better control of state switching and Berry’s phase around them, to explore new effects, and, in a longer term, to develop new practical devices for quantum technologies. Thus, we believe that exploring the interplay between EPs and DPs and predicting

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the emergence of a hybrid doubly degenerated EP-DP, which are the main results of this article, can pave a new direction of research in this field.

In this article, we reveal the coexistence and merging of DPs and EPs in non-Hermitian quadratic bosonic systems. Such a nontrivial effect emerges naturally in the synthetic space spanned by high-order field moments (FMs). Moreover, we show that this synthetic space is constructed by tensor product states (TPS), which exhibit highly-degenerate DPs. This TPS structure explicitly takes into account the non-commutative nature of the bosonic operators, and demonstrate the quantum nature of the phenomenon. TPS are an indispensable theoretical and numerical tool in condensed-matter physics, as they allow studying complex many-body quantum systems by effectively renormalizing otherwise intractable extremely large Hilbert spaces [61, 62]. In particular, we exploit such emerging TPS in the synthetic space of Bose-Einstein condensates, and to superconducting circuits, mentioned above. That is, the synthetic space of quantum parametric subharmonic generation processes.

Our predictions can be tested in a plethora of configurations, ranging from optomechanical resonators, to Bose-Einstein condensates, and to superconducting circuits, mentioned above. That is, the synthetic space of low-dimensional quadratic systems can be mapped to effective Hamiltonians describing lattice systems of higher dimensions [63, 64]. Our work, thus, paves the road to an experimental observation of quantum effects in EPs, and can prompt further research to explore novel effects arising from the competition between DPs and EPs.

II. THEORY

We set the stage by first studying the eigendecomposition of the evolution matrices governing the dynamics of higher-order operator moments of any quadratic Markovian system. We will demonstrate that the eigenspace of such evolution matrices is comprised by the TPS of the first-order moment eigenstates. We start from the consideration of a general nonlinear $N$-mode-coupled quadratic bosonic system, which is not necessarily Hermitian. The bosonic quadratic system can be described by a Nambu operator vector $\hat{\Psi} = \left[\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_N, \hat{a}_1^+, \hat{a}_2^+, \ldots, \hat{a}_N^+\right]^T$, where $\hat{a}_k$ ($\hat{a}_k^+$) is the annihilation (creation) operator of a mode $k$, obeying $[\hat{a}_k, \hat{a}_l^+] = \delta_{kl}$ and $[\hat{a}_k, \hat{a}_l] = 0$. The quadratic (non-) Hermitian Hamiltonian, which determines the dynamics of the $N$-mode system takes a general form $\hat{H} = \sum_{m} H_m \hat{\Psi}_m^\dagger \hat{\Psi}_m$, with $\hat{\Psi}_j$ being the $j$th element of the Nambu vector $\hat{\Psi}$. From the Heisenberg equations of motions, one can easily write down the equations for the dynamics of first-order FMs $\langle \hat{\Psi}\rangle$ as $\frac{d}{dt} \langle \hat{\Psi}\rangle = M_1 \langle \hat{\Psi}\rangle$, where $M_1$ is the corresponding evolution matrix for the first-order FMs.

A remarkable feature of quadratic systems is that by knowing the form of the evolution matrix $M_1$ for the first-order FMs, one can immediately obtain the analytical form of an evolution matrix ruling the dynamics of any higher-order FMs [63]. This can be done by exploiting properties of matrices formed by Kronecker sums. Namely, any $m$th order FM vectors, constructed from the moments of the tensor products of the Nambu vector $\hat{\Psi}$, i.e., $\langle \bigotimes_1^m \hat{\Psi}\rangle$, is ruled by the evolution matrix $M_m$, which is obtained from $M_1$ by iteratively applying a Kronecker sum

$$\frac{d}{dt} \langle \bigotimes_1^m \hat{\Psi}\rangle = M_m \langle \bigotimes_1^m \hat{\Psi}\rangle, \quad M_m = \bigoplus M_1,$$

where symbols $\bigotimes$ and $\bigoplus$ stand for the Kronecker tensor product and sum, respectively. The Kronecker sum of two matrices $A$ and $B$ is defined as $A \bigoplus B = A \bigotimes I_B + I_A \bigotimes B$, where $I_A, B$ is the identity matrix (for more details see Ref. [65] and Appendix A). We neglect the effects of quantum fluctuations on the FMs dynamics, which give rise to an inhomogeneous part in the equations of motion, as we solely focus on the spectral properties of evolution matrices. This assumption, in general, is valid when damping is negligibly small compared to a system Hamiltonian. And it is always the case for linear systems when dealing with non-Hermitian FMs [63, 64]. Here, we mainly focus on the spectral properties of evolution matrices. Naturally assuming that the FMs spectrum is not affected by quantum noise. The effects of quantum noise on the field moments dynamics, for certain quadratic systems, can be consulted in Ref. [66].

III. TENSOR PRODUCT STATES AND THEIR DEGENERACY IN THE HIGHER-ORDER OPERATOR MOMENTS EIGENSPACE

Equation (1) implies that one can determine the complete eigendecomposition of the matrix $M_m$, governing the $m$th-order field moments, from the eigenvalues and eigenvectors of the matrix $M_1$. This is indeed a very nice
property, given that the size of the evolution matrix $M_m$ scales up as $(2N)^m \times (2N)^m$, which thus renders the numerical eigendecomposition of the evolution matrices of higher-order field moments a formidable, if not unrealistic, task.

The right eigenvectors of the matrix $M_m$ are found via all $(2N)^m$ combinations of tensor products of the eigenvectors $\psi_j^{(1)}$, $j = 1, \ldots, 2N$ of the matrix $M_1$ (see Fig. 1). We are primarily interested in the right eigenvectors of evolution matrices, since we are dealing with c-number eigenvectors, not q-vectors. As such, we do not invoke the left vectors in our study (for more detail concerning left-right eigenvectors see Ref. [67] and Appendix F). The $m$th-order moments eigenspace is, thus, spanned by $(2N)^m$ eigenvectors of the form

$$\psi_{i_1, i_2, \ldots, i_m}^{(m)} = \psi_1^{(1)} \otimes \psi_2^{(1)} \otimes \cdots \otimes \psi_m^{(1)},$$

where each index $i_k = 1, \ldots, 2N$, for each $k = 1, \ldots, m$. The eigenvector $\psi_{i_1, i_2, \ldots, i_m}^{(m)}$ corresponds to the eigenvalue

$$\lambda_{i_1, i_2, \ldots, i_m}^{(m)} = \sum_k \lambda_k^{(1)},$$

where $\lambda_k^{(1)}$ are the eigenvalues of the matrix $M_1$. Intriguingly, but this unfolding TPS structure in the FMs eigenspace can be used for solving the eigendecomposition problem for other tridiagonal matrices, so-called Sylvester-shaped matrices [68] (see also Appendix B).

The diabolic (i.e., algebraic) degeneracy $D$ of a given eigenvalue $\lambda_{i_1, i_2, \ldots, i_m}^{(m)}$ in Eq. (3) equals $D(\lambda_{i_1, i_2, \ldots, i_m}^{(m)}) = m! / n_{i_1}! n_{i_2}! \cdots n_{i_m}!$, where $n_{i_k}$ denotes the number of times the index $i_k$ appears in both Eqs. (2) and (3). For example, in the case of second-order moments, the moments vector $\langle \Psi \otimes \Psi \rangle$ possesses moments $\langle \hat{a}_j \hat{a}_k \rangle$ and $\langle \hat{a}_i \hat{a}_j \rangle$, as well as $\langle \hat{a} \hat{a}^\dagger \rangle$ and $\langle \hat{a}^\dagger \hat{a} \rangle$, which result in a degeneracy. In total, there are $2N_m = S_m(2N)$ degenerate eigenvalues of the system, where $S_m(2N)$ denotes the rank of the matrix $M_m$ and is calculated via the formula for combination with repetitions, i.e., $S_m(2N) = (2N + m - 1)!/[m!(2N-1)!]$. In the case of commutative fields, i.e., quantum linear systems or any quadratic classical fields, this diabolic degeneracy is necessarily eliminated, and one deals with only rank-sized evolution matrices (see Appendices B and C). For nonlinear quadratic systems, the degeneracy can be kept, since one can treat noncommutative moments as “nonequivalent”, allowing to capture additional information arising in the TPS moments eigenspace. Note that even though the non-commutative field moments can be treated as nonequivalent, in order to detect extra features which occur in the TPS eigenspace, it seems impossible to experimentally access them directly. Nevertheless, one can indirectly detect such nontrivial spectral properties in lattice photonic systems instead [63, 64].

IV. HYBRID DIABOLIC-EXCEPTIONAL POINTS IN THE TPS EIGENSPACE

Clearly, as it stems from Eq. (3), there can be a large diabolic degeneracy in the spectrum. This degeneracy originates from the TPS structure of the moments eigenspace, which captures the noncommutativity of the quantum fields, and can lead to an interplay between DPs and EPs, resulting even in hybrid DPs-EPs. Because of this, the high-order field moments space can possess, in general, two types of EPs. The first type are rank-EPs. These EPs exist within the rank of a given evolution matrix. The second type of EPs constitute hybrid DPs-EPs, and they only exist in the degenerate eigenspace of moments TPS, i.e., they are defined by the coalescence of eigenvectors corresponding to diabolically degenerate eigenvalues. Indeed, if a matrix $M_1$ of a moments vector $\langle \Psi \rangle$ exhibits $k$ EPs each of order $l_i, i = 1, \ldots, k$, then for any $m$th-order field moments space there are $k$ rank-EPs of order $m(l_i - 1) + 1$ each [69]. However, the eigenvalues of $M_1$, which form a $l_i$th-order rank-EP, also produce diabolically degenerate eigenvalues of matrix $M_m$, according to Eq. (3), whose merge forms hybrid DPs-EPs. In general, the order and degeneracy of such hybrid DPs-EPs are model dependent.

V. EXAMPLE: QUANTUM PARAMETRIC SUBHARMONIC GENERATION PROCESSES

To explicitly illustrate the existence of the TPS structure and two types of EPs in the synthetic space of quadratic nonlinear systems, we consider the field-moments dynamics (up to third order) for the parametric subharmonic generation processeses, which serve as a minimal model. Let us first start from a quadratic Hermitian Hamiltonian $\hat{H}$ describing a second subharmonic generation with a classical pump, and working in the reference frame rotating with the pump frequency $\omega_p$: $\hat{H} = \hat{a}^\dagger \hat{a} + ig/2(\hat{a}^2 - \hat{a}^2)$, where $\Delta = \omega_p - \omega$ is the resonance detuning, i.e., the difference between the frequencies of the pump and quantum field $\omega$, the parameter $g$ is assumed to be a real-valued coupling constant, which involves the amplitude of the pump field [70].

The dynamics of the first-order moments of the Nambu operator vector $\hat{\Psi} = [\hat{a},\hat{a}^\dagger]^T$ is $\frac{d}{dt}\langle \hat{\Psi} \rangle = M_1\langle \hat{\Psi} \rangle$, where the evolution matrix $M_1 = \begin{pmatrix} -i\Delta & -i\Delta & g \\ -g & -i\Delta & -i\Delta \end{pmatrix}$ is $PT$-symmetric [3, 71]. The matrix $M_1$ is invariant under the action $\mathcal{PT}M_1(\mathcal{PT})^{-1}$, where the $\mathcal{PT}$ operator is defined as $\mathcal{PT} = \hat{\sigma}_x \mathcal{K}$, where the operator $\mathcal{K}$ accounts for the complex conjugate operation. The eigenvalues of $M_1$ are $\lambda_{1,2} = \pm \frac{\sqrt{g^2 - \Delta^2}}{2}$. The corresponding first-order moment eigenvectors become

$$\psi_{1,2} \equiv \begin{pmatrix} \mp \exp(\mp i\phi) \\ 1 \end{pmatrix}, \quad \phi = \arctan \frac{\Delta}{\sqrt{g^2 - \Delta^2}},$$

(4)
which satisfy biorthogonality (see Appendix F). Both the eigenvalues and eigenvectors coalesce at the second-order EP, defined by the condition: \( g_{\text{EP}} = \Delta \), with the phase \( \phi_{\text{EP}} = \pi(2k + 1)/2, k \in \mathbb{Z} \), in Eq. (4).

The evolution matrix \( M_2 \), governing the vector \( \langle \hat{\Psi} \otimes \hat{\Psi} \rangle = [(\hat{a}^2), (\hat{a}^4), (\hat{a}^4), (\hat{a}^2)]^T \), reads

\[
M_2 = \begin{pmatrix}
-2i\Delta & -g & -g & 0 \\
-g & 0 & 0 & -g \\
-g & 0 & 0 & -g \\
0 & -g & -2i\Delta & 0
\end{pmatrix}.
\tag{5}
\]

Its eigenvalues are \( \lambda_{1k}^{(2)} = 2\sqrt{g^2 - \Delta^2}\sigma_2 \), where \( \sigma_2 \) is the Pauli matrix. i.e., all four eigenvalues are listed in the matrix form, with \( j, k = 1, 2 \).

The eigenvectors of \( M_2 \) are then found via all \( 2^2 \) combinations of tensor products of the eigenvectors \( \psi_{1,2} \in \mathbb{C} \) in Eq. (4), according to Eq. (2):

\[
\psi_{1,2}^{(2)} = \psi_{1,2}^{(2)} \otimes \psi_{1,2}^{(2)},
\]

where each index \( i_k = 1, 2 \), for each \( k = 1, 2 \). Namely, the right eigenvectors \( \psi_{1,1}^{(2)} \), \( \psi_{1,2}^{(2)} \), \( \psi_{2,1}^{(2)} \), \( \psi_{2,2}^{(2)} \), and \( \psi_0^{(2)} \) are presented as columns of the following matrix, respectively:

\[
\psi^{(2)} = \begin{pmatrix}
e^{-2i\phi} & -1 & -1 & e^{2i\phi} \\
e^{-i\phi} & -e^{-i\phi} & e^{i\phi} & e^{i\phi} \\
e^{-i\phi} & e^{i\phi} & -e^{-i\phi} & e^{i\phi} \\
1 & 1 & 1 & 1
\end{pmatrix}.
\tag{6}
\]

The zero-energy eigenvectors \( \psi_{1,2}^{(2)} \) and \( \psi_{21}^{(2)} \) form a subspace of the moments eigenspace due to the degeneracy of \( \lambda = 0 \). Thus, any of their linear combinations is also an eigenstate of \( M_2 \), i.e., \( \psi_0^{(2)} = a\psi_{1,2}^{(2)} + b\psi_{21}^{(2)} \), \( a, b \in \mathbb{C} \).

The rank of the matrix \( M_2 \) is four, there might exist rank-1 eigenvectors, which are then found via all \( 3 \) eigenvectors of third-order field moments. The corresponding eigenvalues \( \pm \sqrt{g^2 - \Delta^2} \) and \( \lambda_{122} \) are two hybrid DPs-EPs of second order (presented by the merge of two eigenvalues – two dashed and two dotted curves, respectively, which are characterized by the square root dependence on perturbation, \( \sqrt{\epsilon} \)). Apart from it, there are also two hybrid DPs-EPs of second order (presented by the merge of two eigenvalues – two dashed and two dotted curves, respectively, which are characterized by the square root dependence on perturbation, \( \sqrt{\epsilon} \)). These hybrid DPs-EPs only exist in the degenerate moments TPS eigenspace.

![FIG. 2. Real (a) and imaginary (b) parts of eigenvalues of the matrix \( M_3 \), governing third-order FMs, at the EP \( g = \Delta \), under the perturbation \( \epsilon \), which serves for the resolving of emerging hybrid DPs-EPs (see the main text). The matrix \( M_3 \) has a single rank-EP of fourth order (presented by the coalescence of four solid curves, characterized by the quartic root dependence on perturbation, \( \sqrt{\epsilon} \)). Apart from it, there are also two hybrid DPs-EPs of second order (presented by the merge of two eigenvalues – two dashed and two dotted curves, respectively, which are characterized by the square root dependence on perturbation, \( \sqrt{\epsilon} \)). These hybrid DPs-EPs only exist in the degenerate moments TPS eigenspace.](image-url)
VI. CONCLUSIONS

We have demonstrated that a highly-degenerate synthetic TPS space emerges in quadratic bosonic systems, stemming from the non-commutative nature of the bosonic operators. The emergent TPS structure of the FMs eigenspace leads to a high-order diabolic degeneracy in the quantum regime. The coexistence of DPs and EPs in the high-order FMs space can induce hybrid DPs-EPs. Our results, thus, reveal the nontrivial quantum nature of the spectral degeneracy in quadratic bosonic systems, which can be exploited in various quantum protocols based on EPs. Namely, the observed interplay between DPs and EPs is not limited to the synthetic space of the FMs of low-dimensional systems, but can be simulated in photonic lattice systems [63, 64]. The prediction of the emergence of such hybrid degenerated EPs-DPs can, thus, ignite further research in this field.

ACKNOWLEDGMENTS

I.A. acknowledges funding by the Ministry of Education, Youth and Sports of the Czech Republic Project no. CZ.02.1.010.00.016_019000754. A.M. is supported by the Polish National Science Centre (NCN) under the Maestro Grant No. DEC-2019/34/A/ST2/00081. F.N. is supported in part by: Nippon Telegraph and Telephone Corporation (NTT) Research, the Japan Science and Technology Agency (JST) [via the Quantum Leap Flagship Program (Q-LEAP) program, and the Moonshot R&D Grant Number JPMJMS2061], the Japan Society for the Promotion of Science (JSPS) [via the Grants-in-Aid for Scientific Research (KAKENHI) Grant No. JP20H00134], the Army Research Office (ARO) (Grant No. W911NF-18-1-0358), the Asian Office of Aerospace Research and Development (AOARD) (via Grant No. FA2386-20-1-4069), and the Foundational Questions Institute Fund (FQXi) via Grant No. FQXi-IAF19-06. S.K.O. acknowledges support from Air Force Office of Scientific Research (AFOSR) Multidisciplinary University Research Initiative (MURI) Award on Programmable systems with non-Hermitian quantum dynamics (Award No. FA9550-21-1-0202).

Appendix A: Tensor product states in the higher-order field moments dynamics of quadratic systems: Kronecker products and sums

In this section we describe in detail the construction of the evolution matrices for higher-order field moments based on the Kronecker sum operations. For completeness, we repeat some of the lines from in the main text of the Article.

The bosonic quadratic system can be described by a Nambu operator vector \( \hat{\Psi} = \left[ \hat{a}_1, \hat{a}_2, \ldots, \hat{a}_N, \hat{a}_1^\dagger, \hat{a}_2^\dagger, \ldots, \hat{a}_N^\dagger \right]^T \), where \( \hat{a}_k \) (\( \hat{a}_k^\dagger \)) is the annihilation (creation) operator of a mode \( k \), obeying: \( [\hat{a}_k, \hat{a}_l^\dagger] = \delta_{kl} \) and \( [\hat{a}_k, \hat{a}_l] = 0 \). The quadratic (non-) Hermitian Hamiltonian, which determines the dynamics of an \( N \)-mode system takes the general form

\[
\hat{H} = \sum_{mn} H_{mn} \hat{\Psi}_m^\dagger \hat{\Psi}_n, \tag{A1}
\]

with \( \hat{\Psi}_j \) being the \( j \)th element of the Nambu vector \( \hat{\Psi} \).

From the Heisenberg equations of motions, one can easily write down the equations for the dynamics of first-order field moments \( \langle \hat{\Psi} \rangle \) as

\[
\frac{d}{dt} \langle \hat{\Psi} \rangle = M_1 \langle \hat{\Psi} \rangle, \tag{A2}
\]

where \( M_1 \) is the corresponding evolution matrix for the first-order field moments.

For open systems, the benefit of the moments space (compared to the operators space) is that it allows to discard quantum noise for odd-order field moments, which is presented via Langevin forces in the Heisenberg equations of motion [70, 72, 73]. This is always true for linear systems, when dealing with moments comprised by only annihilation or creation operators. However, for general Markovian systems, there exists a noise vector, stemming from quantum fluctuations [66], thus, rendering the equation of motion for higher-order field moments inhomogeneous. Nevertheless, it is natural to assume that quantum noise has no effect of the very spectrum of the FMs dynamics.

One of the remarkable features of quadratic systems is that one can obtain the analytical form of an evolution matrix ruling the dynamics of any higher-order field moments comprised by only annihilation or creation operators. However, for general Markovian systems, there exists a noise vector, stemming from quantum fluctuations [66], thus, rendering the equation of motion for higher-order field moments inhomogeneous. Nevertheless, it is natural to assume that quantum noise has no effect of the very spectrum of the FMs dynamics.

\textit{Theorem.} A square complex matrix \( C \in \mathbb{C}^{(m+n)\times(m+n)} \), which is obtained as a Kronecker sum of two square complex matrices \( A \in \mathbb{C}^{m\times m} \) and \( B \in \mathbb{C}^{n\times n} \), i.e., \( C = A \oplus B = A \otimes I_n + I_m \otimes B \), has the eigenvalues, which are sums of the eigenvalues of \( A \) and \( B \), and the corresponding right eigenvectors are the tensor products of the right eigenvectors of \( A \) and \( B \), i.e.,

\[
\lambda(C) = \lambda(A) + \lambda(B), \quad \psi_C = \psi_A \otimes \psi_B. \tag{A3}
\]

\textit{Proof.} The proof is straightforward. We start from the eigenvalue-eigenvector equation for the matrix \( C \), by feeding into the equation the right eigenvector, which is a tensor product of two eigenvectors \( \psi_A \) and \( \psi_B \), with

\[
\frac{d}{dt} \langle \hat{\Psi} \rangle = M_1 \langle \hat{\Psi} \rangle,
\]

where \( M_1 \) is the corresponding evolution matrix for the first-order field moments.
eigenvalues $\lambda(A)$ and $\lambda(B)$, respectively. Namely,
\[
C(\psi_A \otimes \psi_B) = (A \otimes B)(\psi_A \otimes \psi_B) = (A \otimes I_B)(\psi_A \otimes \psi_B) + (I_A \otimes B)(\psi_A \otimes \psi_B)
\]
\[
= (A\psi_A \otimes I_B\psi_B) + (I_A\psi_A \otimes B\psi_B)
\]
\[
= (\lambda_A\psi_A \otimes \psi_B) + (\psi_A \otimes \lambda_B\psi_B)
\]
\[
= (\lambda_A + \lambda_B)\psi_A \otimes \psi_B, \quad (A4)
\]
where we used the tensor and dot product properties of matrices and vectors. In other words, the eigenvector of the matrix $C = A \otimes B$, corresponding to the eigenvalue $\lambda(C)$, is indeed just the tensor product of the two eigenvectors of the matrices $A$ and $B$ with eigenvalues $\lambda(A)$ and $\lambda(B)$.

According to Eq. (A4), the matrix, whose eigenvectors are formed by the tensor products of eigenvectors of two matrices can be utilized in the construction of the evolution matrices of any higher order. To reveal how this works in practice, let us consider the second-order field moments. The various combinations of second-order field moments are obtained from the tensor product of the Nambu vector on itself, i.e., from the $4N^2$ dimensional vector $\langle \hat{\Psi} \otimes \Psi \rangle$. According to Eq. (A4), the evolution matrix $M_2$, governing the vector of the second-order field moments
\[
\frac{d}{dt} \langle \hat{\Psi} \otimes \Psi \rangle = M_2 \langle \hat{\Psi} \otimes \Psi \rangle, \quad (A5)
\]
attains the form
\[
M_2 = M_{1} \oplus M_{1} = M_{1} \otimes I_{2N} + I_{2N} \otimes M_{1}. \quad (A6)
\]

The form of the evolution matrix $M_2$ coincides, as it should, with that derived from the Lyapunov equation for the covariance matrix, when the latter is presented as a column vector [65]. This procedure can, thus, be iteratively continued to any $n$th order field moment vectors $\langle \otimes \Psi \rangle$, thus obtaining the Eq. (1) in the main text.

Appendix B: Reduction of the TPS structure and diabolic degeneracy in the field moments space of bosonic quadratic systems

In this section we discuss the elimination of diabolic degeneracy for commutative fields, i.e., classical and linear quantum systems.

The described degeneracy of the eigenspace formed by TPS in the field moments space occurs due to the general non-commutative nature of the quantum fields, which is captured by the Kronecker sum operation, when constructing evolution matrices for any higher-order field moments. Evidently, this degeneracy must be lifted for classical fields and also for quantum fields which do not contain coupled annihilation and creation operators of the same field in the moments dynamics (e.g., linear fields). Hence, the redundant field moment elements should be eliminated. This also means that the corresponding eigenstates are also reduced to some effective eigenstates which are no longer the TPS. In the quantum regime, on the other hand, the degeneracy, and therefore the TPS structure of the eigenstates, can be kept. The $(2N)^m \times (2N)^m$-dimensional eigenspace of the matrix $M_m$ can be reduced to the size $S_m(2N) \times S_m(2N)$ with $S_m(2N) < (2N)^m$, where
\[
S_m(2N) = \frac{(2N + m - 1)!}{m!(2N - 1)!}. \quad (B1)
\]

In particular, for linear dimers, the initial $N^m$-dimension of the eigenvectors of $m$-order field moments can always be reduced to $(m + 1)$, as expected [63] (note that for the linear case, the annihilation operators are decoupled from the creation operators, as such, a general $(2N)^m \times (2N)^m$ dimension of evolution matrices automatically reduces to $N^m \times N^m$). That is, the TPS structure of the eigenstates is always eliminated in that case. Indeed, for the classical case, the fields commute, meaning that the two moments $\langle \hat{a}^\dagger \hat{a}^\dagger \rangle$ and $\langle \hat{a} \hat{a} \rangle$ refer to the same second-order classical field moment $|\alpha|^2$. As a result, these two moments can be merged into one, namely, into their symmetrical sum $\langle (\hat{a}^\dagger \hat{a}^\dagger) + (\hat{a} \hat{a}) \rangle/2 = |\alpha|^2$.

The reduction of the initial degenerate eigenspace of higher-order field moments to the effective one can be performed in the following way. First, the $D(\lambda)$ degenerate eigenvectors, in Eq. (3) of the main text, corresponding to the same eigenvalue $\lambda$, are merged into a symmetric superposition of the degenerate eigenmodes
\[
\psi'^{(m)}(\lambda) = \frac{1}{D(\lambda)} \sum_{i_1, \ldots, i_m}^{D(\lambda)} \psi^{(m)}_{i_1, \ldots, i_m}. \quad (B2)
\]

As such, there remain $S_m(2N)$ non-degenerate eigenvectors of size $(2N)^m$. The subsequent reduction of the $(2N)^m$ elements of those vectors to $S_m(2N)$ elements is performed by merging all equivalent moment elements also into their average. In particular, for linear dimers, in the case of the second-order field moments, after the effective reduction of the four eigenvectors into three, the two elements $\langle \hat{a}_1 \hat{a}_2 \rangle$ and $\langle \hat{a}_2 \hat{a}_1 \rangle$ of the four-dimensional moment of three-left eigenvectors are substituted by the single element $\frac{1}{2} (\langle \hat{a}_1 \hat{a}_2 \rangle + \langle \hat{a}_2 \hat{a}_1 \rangle)$. That is, the moments $\langle \hat{a}_1 \hat{a}_2 \rangle$ and $\langle \hat{a}_2 \hat{a}_1 \rangle$ are removed from the moment vectors, and are substituted by a single moment which is their symmetrical sum. This leads to the reduction of the four-dimensional eigenvectors to three-dimensional eigenvectors. This procedure ensures the reduction of all $(N)^m$ eigenvectors to the $S_m(N)$ effective ones. We elaborate in detail on such a procedure in Section ??.

Interestingly, when dealing with dimer systems, that reduction always leads to the formation of the effective evolution matrices with a Sylvester matrix shape [63, 64, 68]. The Sylvester matrix is a tridiagonal matrix, whose elements obey certain relations (see Ref. [68] for details). Only recently a formula for the eigendecomposition of such matrices
has been proposed [68]. A such, our results offer an alternative solution to the eigendecomposition problem of Sylvester matrices. And, moreover, highlight a possible physical origin of such matrices. That is, they are derived from the moments space of quadratic systems.

**Appendix C: Example of degeneracy reduction in the second-order field moments space generated in parametric subharmonic processes**

In this section, we elaborate on the removal of the diabolic degeneracy for the case of fields generated in the parametric subharmonic processes.

The two-fold degenerate eigenvalue $\lambda_{12} = \lambda_{21} = 0$, below Eq. (5) in the main text, arises because of the redundant moment elements $\langle \hat{a} \hat{a} \rangle$ and $\langle \hat{a}^\dagger \hat{a} \rangle$ in the moments vector $\langle \hat{\Psi} \rangle$. As such, the $(2^2 = 4)$-dimensional eigenspace of the second-order moments can be decreased to a $(2 + 1) = 3$-dimensional one, according to Eq. (B1). Following the procedure, described at the end of Sec. B, the effective moments space can be obtained by substituting those two redundant elements by a single one, namely, by their symmetrical sum. That is, the second and third rows in the moments vector and evolution matrix are merged into one:

$$\langle \hat{\Psi} \rangle \rightarrow \langle \hat{\Psi} \rangle_{\text{eff}} = \left[ \langle \hat{a} \hat{a} \rangle, \langle \hat{a} \hat{a} \rangle_s, \langle \hat{a} \hat{a} \rangle \right]^T, \quad \text{(C1)}$$

where the symmetrically-ordering moment is defined as

$$\langle \hat{a} \hat{a} \rangle_s = \frac{1}{2} \left( \langle \hat{a} \hat{a} \rangle + \langle \hat{a}^\dagger \hat{a} \rangle \right). \quad \text{(C2)}$$

This reduction directly corresponds to the classical limit of the fields, i.e., with the fields vector $\langle \hat{\Psi} \rangle_{\text{eff}} = [\alpha^2, |\alpha|^2, \alpha^*]^T$. The corresponding effective evolution matrix attains the following Sylvester form:

$$M_2 \rightarrow M_2^{\text{eff}} = \begin{pmatrix} \frac{-2i\Delta}{g} & -2g & 0 \\ -g & 0 & -g \\ 0 & -2g & 2i\Delta \end{pmatrix}. \quad \text{(C3)}$$

The corresponding reduction of the eigenvectors $\psi^{(2)}$ is obtained similarly

$$\psi^{(2)}_{\text{eff}} = \begin{pmatrix} e^{-2i\phi} & -1 & e^{2i\phi} \\ -e^{-i\phi} & i \sin(\phi) & e^{i\phi} \\ 1 & 1 & 1 \end{pmatrix}. \quad \text{(C4)}$$

by combining together the second and third eigenvectors, due to the two-fold degeneracy of the eigenvalue $\lambda = 0$, and accompanied by merging everywhere the second and third elements of the initial eigenvectors in Eq. (C1). All the three eigenvectors in Eq. (C4) coalesce at the EP $g_{\text{EP}} = \Delta$, to the singular vector $\psi^{(2)}_{\text{EP}} = [-1, i, 1]$, implying that the order of the EP is three, and no other eigenstate exists, in drastic contrast with the genuine quantum regime.

**Appendix D: Diabolic degeneracy in the eigenspace of the second-order field moments**

The diabolic degeneracy, emerging in the field moments eigenspace of second-order, enables the existence of eigenvectors at the EP which do not collapse to the singular vector, as in the case of classical fields. Indeed, consider the system at the EP, assuming that the injected field is prepared in the two-photon quantum state $\rho$ (in the Fock representation):

$$\rho = \frac{1}{4} \begin{pmatrix} 3 & 0 & -2\sqrt{2}i \\ 0 & 0 & 0 \\ 2\sqrt{2}i & 0 & 1 \end{pmatrix}. \quad \text{(D1)}$$

It is easy to check that the state $\rho$ generates the vector of the second-order field moments:

$$\psi = \langle \hat{\Psi} \rangle \equiv \begin{pmatrix} i, 3 \quad 1 \quad -i \end{pmatrix}^T, \quad \text{(D2)}$$

which is the zero-energy eigenstate of the evolution matrix $M_2$. This eigenvector can be formed from the superposition of the two degenerate eigenvectors $\psi^{(2)}_{12}$ and $\psi^{(2)}_{21}$, namely

$$\psi = \lim_{\phi \to \pi/2} \psi^{(2)}_0, \quad \text{(D3)}$$

with coefficients $a = \frac{1}{4\cos^2(\phi)} \left(1 - 2ie^{i\phi}\right)$, and $b = -a^*$, in Eq. (7). Even though it may seem that $a$ and $b$ blow up at the EP ($\phi = \pi/2$), the limit gives the finite vector $\psi$. Additionally, the second and third elements of such moments vector obey the commutation relation, i.e., $\langle [\hat{a}, \hat{a}^\dagger] \rangle = 1$.

Remarkably, at the EP, the zero-energy eigenvector $\psi$ is not equal to the singular eigenvector $\psi_{\text{EP}} = [-1, i, 1]^T$. Such an eigenvector is not a generalized eigenvector nor their combination at the EP, since $M_2 \psi = 0$. Recall that the generalized eigenvectors $\xi_k$ of a matrix $M$, characterizing an EP $\lambda_{\text{EP}}$ of order $N$, are found via formula $[M - \lambda_{\text{EP}} I_M] \xi_k = 0$, for $k = 2, \ldots, N$. The existence of such an eigenvector is enabled solely by the presence of the diabolic degeneracy in the field moments eigenspace spanned by TPS.

**Appendix E: Diabolic degeneracy in the eigenspace of the third-order field moments: Hybrid Diabolic-Exceptional Points**

The vector of third-order field moments reads

$$\langle \bigotimes_1^3 \hat{\Psi} \rangle = \left[ \langle \hat{a}^3 \rangle, \langle \hat{a}^2 \hat{a} \rangle, \langle \hat{a}^2 \hat{a} \rangle, \langle \hat{a} \hat{a} \rangle, \langle \hat{a} \hat{a} \rangle, \langle \hat{a} \hat{a} \rangle \right]^T. \quad \text{(E1)}$$
The evolution matrix $M_3$ for the vector $\langle \otimes_1 \hat{\Psi} \rangle$ is obtained as follows

$$M_3 = M_2 \oplus I_2 + I_4 \oplus M_1,$$  \hspace{1cm} (E2)

which results in

$$M_3 = \begin{pmatrix}
-3i\Delta & -g & -g & 0 & -g & 0 & 0 & 0 \\
-g & -i\Delta & 0 & -g & 0 & -g & 0 & 0 \\
-g & 0 & -i\Delta & -g & 0 & 0 & -g & 0 \\
0 & -g & -g & i\Delta & 0 & 0 & 0 & -g \\
-g & 0 & 0 & 0 & -i\Delta & -g & -g & 0 \\
0 & -g & 0 & -g & 0 & i\Delta & -g & 0 \\
0 & 0 & 0 & -g & 0 & -g & -g & 3i\Delta \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.$$  \hspace{1cm} (E3)

Its eigenvalues $\lambda_{ijk}$, according to Eq. (3), can be listed as

$$\lambda^{(3)} = \begin{pmatrix}
\lambda_{111} & \lambda_{112} & \lambda_{121} & \lambda_{122} & \lambda_{211} & \lambda_{212} & \lambda_{221} & \lambda_{222} \\
3s & s & -s & s & -s & -s & -3s & s \\
\end{pmatrix},$$  \hspace{1cm} (E4)

where $s = \sqrt{g^2 - \Delta^2}$. The corresponding eigenvectors are easily found, according to Eq. (2):

$$\psi^{(3)} = \begin{pmatrix}
e^{-3i\phi} & e^{-i\phi} & e^{-i\phi} & -e^{i\phi} & e^{-i\phi} & -e^{i\phi} & e^{3i\phi} \\
e^{-2i\phi} & e^{-2i\phi} & -1 & -1 & -1 & -1 & e^{2i\phi} & e^{2i\phi} \\
e^{-2i\phi} & -1 & e^{-2i\phi} & -1 & -1 & e^{2i\phi} & -1 & e^{2i\phi} \\
e^{-i\phi} & e^{-i\phi} & e^{-i\phi} & e^{i\phi} & e^{i\phi} & e^{i\phi} & e^{i\phi} & e^{i\phi} \\
e^{-i\phi} & -1 & -1 & e^{2i\phi} & e^{-2i\phi} & -1 & -1 & e^{2i\phi} \\
e^{-i\phi} & e^{i\phi} & e^{i\phi} & e^{i\phi} & e^{i\phi} & e^{i\phi} & e^{i\phi} & e^{i\phi} \\
e^{-i\phi} & e^{i\phi} & e^{-i\phi} & e^{i\phi} & e^{i\phi} & e^{-i\phi} & e^{i\phi} & e^{i\phi} \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{pmatrix},$$  \hspace{1cm} (E5)

It is seen, that two eigenvalues $\pm \sqrt{g^2 - \Delta^2}$ are triply degenerate. At the EP $g = \Delta$, the Jordan form of $M_3$ reads

$$M_3^{EP} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},$$  \hspace{1cm} (E6)

which implies the existence of two different types of EPs. One EP, which is of fourth order, corresponds to the rank of the matrix $M_3$. That is, this EP is always present in the field moments dynamics. However, in the eigenspace spanned by the triply degenerate eigenvectors with $\lambda = \pm \sqrt{g^2 - \Delta^2}$, there exists a single doubly degenerate EP of second order, i.e., a hybrid DP-EP.

The eigenvalues $\lambda_{112}$ and $\lambda_{122}$ are triply degenerate. As a result, it is difficult to resolve the appearance of two hybrid DPs-EPs in the third-order moments eigenspace. However, one can induce a fictitious perturbation to the matrix $M_3$ in the form $M_3 + \epsilon P$, where $\epsilon$ denotes the perturbation strength, and the matrix $P$ reads:

$$P = \text{diag}[1, 0, 0, 1, 0, 1, 0, 0].$$  \hspace{1cm} (E7)

The result of such a perturbation on the eigenvalues of $M_3$ is shown in Fig. 2 in the main text. This perturbation is fictitious because it is prohibited in the field moments space. Indeed, by unfolding the evolution matrices $M_3$ from the first-order moments $2 \times 2$ matrix $M_1$, it is impossible to attain such a perturbed matrix $M_3$. Nevertheless, in order to highlight the presence of such hybrid DPs-EPs, one can resort to this allowed mathematical, but physically prohibited, operation. However, the physicality of such a perturbation pertains, when mapping the
evolution matrix $M_3$ to the same photonic-lattice Hamiltonian (e.g., a system of eight coupled cavities [64]).

**Appendix F: Biorthogonality of the eigenvectors**

In order to make the solution of the eigendecomposition of the studied complex matrices complete, we briefly mention the notion of the biorthogonality. Because of the complex nature of the evolution matrices $M_m \in \mathbb{C}$, the right eigenvectors $\psi^{(m)}$ in Eq. (3) are, in general, not orthogonal [67]. However, by introducing the notion of biorthogonality, i.e., when the inner product is defined orthogonally [67].

When determining the decomposition of arbitrary vector by means of both left and right eigenvectors of a complex matrix, one may overcome the difficulty. The left eigenvectors of a complex matrix $M_m$ are defined as

$$M^\dagger_m \xi^{(m)}_j = \lambda_j^{(m)} \xi^{(m)}_j.$$  

And the biorthogonality condition reads

$$\sum_p \xi^{\dagger (m)_j}_p \psi^{(m)}_k (p) = \delta_{jk}.$$  

The biorthogonality condition in Eq. (F2) is necessary when determining the decomposition of arbitrary vector on right eigenvectors. However, in our study, we exclusively focus on the eigenspace spanned by the right eigenvectors of the studied quadratic systems.

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