Funnel Control for the Fokker-Planck Equation Corresponding to the Ornstein-Uhlenbeck Process

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Abstract. In this paper the feasibility of funnel control techniques for the Fokker-Planck equation corresponding to the Ornstein-Uhlenbeck process on an unbounded spatial domain is explored. First, using weighted Lebesgue and Sobolev spaces, an auxiliary operator is defined via a suitable coercive sesquilinear form. This operator is then transformed to the desired Fokker-Planck operator. We show that any weak solution of the controlled Fokker-Planck equation (which is a probability density) has a variance that exponentially converges to a constant. After a simple feedforward control approach is discussed, we show feasibility of funnel control in the presence of disturbances by exploiting an energy estimate. We emphasize that the closed-loop system is a nonlinear and time-varying PDE. The results are illustrated by some simulations.

Key words. Adaptive control, Fokker-Planck equation, Ornstein-Uhlenbeck process, funnel control, bilinear control systems, robust control

AMS subject classifications. 35K55, 93C40

1. Introduction. In this work we study output tracking control for the Fokker-Planck equation that corresponds to the Ornstein-Uhlenbeck process. The latter is a continuous-time stochastic process which was originally used to describe the motion of a massive Brownian particle under the influence of friction [39]. Although its investigation was mainly driven by physics and mathematics, several other important applications emerged, such as in neurobiology [35] and in finance [36]. The Ornstein-Uhlenbeck process is often considered in the context of optimal control, see e.g. [2, 3, 18, 19]. The Fokker-Planck equation is a parabolic partial differential equation (PDE) which describes the evolution of the probability density function of the solution of a stochastic differential equation, see e.g. [30]. It will be the main tool to treat the output tracking control problem.

In this context, control means that we assume that the drift term of the stochastic differential equation can be manipulated by an external signal, which is called the control input. The resulting Fokker-Planck equation can be viewed as an abstract bilinear control system in terms of the state and the input, cf. [14, 22]; see also the monograph [29] for several topics on bilinear control systems. The mean value (or expected value) of the Ornstein-Uhlenbeck process is chosen as the output and measurements of it are assumed to be available. For a given reference signal, we then seek to achieve that the difference between the mean value and the reference stays within a prescribed error margin for all times, thus allowing to control the mean value of the process as desired. At the same time, we do not require knowledge of the system parameters or the initial probability density. Furthermore, controlling the mean value of the process is indeed sufficient to influence the entire probability density function. Since only the drift term in the Fokker-Planck equation is influenced by the control input, the variance of the process is independent of it. We will show that it is the sum of a constant and an exponentially decaying term. Indeed, simulations show that the shape of the probability density does not change after some initial time, and is essentially only shifted according to the movement of the mean value.

The control law to achieve this is based on the funnel control methodology developed in [24]. The funnel controller is an output-error feedback of high-gain type. Its advantages are that it is model-free (i.e., it requires no knowledge of the system parameters or the initial
value), it is robust and of striking simplicity – for the Fokker-Planck equation we will show that robustness can be guaranteed w.r.t. additive disturbances “with zero mass”. The funnel controller has been successfully applied e.g. in temperature control of chemical reactor models [26], control of industrial servo-systems [20] and underactuated multibody systems [8], voltage and current control of electrical circuits [13], DC-link power flow control [37] and adaptive cruise control [11, 12].

Funnel control for infinite-dimensional systems is a hard task in general. A simple class of systems with relative degree one and infinite-dimensional internal dynamics has been considered in the seminal work [24]. Linear infinite-dimensional systems for which an integer-valued relative degree exists have been considered in [25]. In fact, it has been observed in the recent work [10] that the existence of an integer-valued relative degree is essential to apply known funnel control results as formulated e.g. in [7]. It is then shown that the set of natural numbers is denoted by \( \mathbb{N} \) and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \).

For a Banach space \( X \), \( X' \) stands for its dual. For a measurable set \( \Omega \subseteq \mathbb{R} \), a measurable function \( w : \Omega \to [0, \infty) \) and \( p \in [1, \infty] \), \( L^p(\Omega; w) \) denotes the \( w \)-weighted Lebesgue space of \((\text{equivalence classes of)})\ measurable and \( p \)-integrable functions \( f : \Omega \to \mathbb{R} \) with norm

\[
||f||_{L^p(\Omega; w)} = \left( \int_{\Omega} w(x) |f(x)|^p dx \right)^{1/p}, \quad f \in L^p(\Omega; w),
\]

if \( p < \infty \) and \( ||f||_{L^\infty(\Omega; w)} = \text{ess sup}_{x \in \Omega} w(x)|f(x)| \) if \( p = \infty \). Additionally, for \( k \in \mathbb{N}_0 \), \( W^{k,p}(\Omega; w) \) denotes the \( w \)-weighted Sobolev space of \((\text{equivalence classes of)})\ \( k \)-times weakly differentiable functions \( f : \Omega \to \mathbb{R} \) with \( f, f', \ldots, f^{(k)} \in L^p(\Omega; w) \). If \( w \equiv 1 \), then we write \( L^p(\Omega; 1) = L^p(\Omega) \), \( W^{k,p}(\Omega; 1) = W^{k,p}(\Omega) \) and, when \( \Omega \) is clear from the context, \( \|\cdot\|_{L^p(\Omega; 1)} = \|\cdot\|_{L^p(\Omega)} \).

For an interval \( J \subseteq \mathbb{R} \), a Banach space \( X \) and \( p \in [1, \infty] \), we denote by \( L^p(J; X) \) the vector space of equivalence classes of strongly measurable functions \( f : J \to X \) such that \( ||f(\cdot)||_X \in L^p(J) \); the distinction between \( L^p(J; X) \) and \( L^p(\Omega; w) \) should be clear from the context. Note that if \( J = (a, b) \) for \( a, b \in \mathbb{R} \), the spaces \( L^p((a, b); X) \), \( L^p([a, b]; X) \), \( L^p((a, b]; X) \) and \( L^p([a, b]; X) \) coincide, since the points at the boundary have measure zero. We will simply write \( L^p(a, b; X) \), also for the case \( a = -\infty \) or \( b = \infty \). We refer to [1] for further details on Sobolev and Lebesgue spaces.

By \( C(J; X) \) we denote the space of continuous functions \( f : J \to X \). For \( p \in [1, \infty] \), \( W^{1,p}(J; X) \) stands for the Sobolev space of \( X \)-valued equivalence classes of weakly differentiable and \( p \)-integrable functions \( f : J \to X \) with \( p \)-integrable weak derivative, i.e., \( f, f' \in L^p(J; X) \). Thereby, integration (and thus weak differentiation) has to be understood in the Bochner sense, see [16, Sec. 5.9.2]. The spaces \( L^p_{\text{loc}}(J; X) \) and \( W^{1,p}_{\text{loc}}(J; X) \) consist of all \( f \) whose restriction to any compact interval \( K \subseteq J \) are in \( L^p(K; X) \) or \( W^{1,p}(K; X) \), respectively.

For an interval \( J \subseteq \mathbb{R} \) we denote by \( C^\infty(J) \) the set of all infinitely times continuously differentiable functions \( f : J \to \mathbb{R} \), and by \( C^\infty_c(J) \) the set of all functions \( f \in C^\infty(J) \) with compact support on \( J \).
1.2. The Fokker-Planck equation for a controlled stochastic process. We consider a controlled stochastic process described by the Itô stochastic differential equation (cf. [30, Sec. 11.])

\[
\frac{dX_t}{dt} = b(t, X_t, u(t))dt + \sigma(t, X_t, u(t))dW_t, \quad X(t) = X_0,
\]

where \(X_t : \Omega \to \mathbb{R}^n, t \geq 0\), are random vectors and \(\Omega\) is the sample space of a probability space \((\Omega, \mathcal{F}, P)\). \((W_t)_{t \geq 0}\) denotes a \(d\)-dimensional Wiener process with zero mean value and unit variance, \(b : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n\) is a drift function and \(\sigma : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n \times d}\) is a covariance matrix. The function \(u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m\) is the control input.

Using the framework presented in [2] we can formulate the control problem for the probability density function of the stochastic process \((X_t)_{t \geq 0}\) as a partial differential equation, the Fokker-Planck equation. This approach is feasible under appropriate assumptions on the functions \(b\) and \(\sigma\) as shown in [31, 32]. Define

\[
C : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n \times n}, \quad (t, x, u) \mapsto \frac{1}{2}\sigma(t, x, u)\sigma(t, x, u)^	op,
\]

then the probability density function \(p : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}\) associated with the process \((X_t)_{t \geq 0}\) evolves according to the Fokker-Planck equation

\[
\begin{aligned}
\frac{\partial p}{\partial t}(t, x) &= - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left( b_i(t, x, u(t))p(t, x) \right) \\
&\quad + \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} \left( C_{ij}(t, x, u(t))p(t, x) \right), \quad \text{in } (0, \infty) \times \mathbb{R}^n, \\
p(0, x) &= p_0(x), \quad \text{in } \mathbb{R}^n,
\end{aligned}
\]

and additionally, since \(p\) is a probability density, we require

\[
p(t, x) \geq 0, \quad \text{in } [0, \infty) \times \mathbb{R}^n,
\]

\[
\int_{\mathbb{R}^n} p(t, x)dx = 1, \quad \text{in } [0, \infty).
\]

The second condition in (1.3) is the conservation of probability, while the first requires any probability to be non-negative. Some conditions for the existence of nonnegative solutions of the Fokker-Planck equation are given in [3, 14, 19] for instance.

1.3. The Ornstein-Uhlenbeck process. As a specific stochastic process, in this work we consider the Ornstein-Uhlenbeck process and we assume that it can be controlled via the drift term only. Then it is modelled by an equation of the form (1.1) with \(n = d = m = 1\) and

\[
b(t, x, u) = u - \gamma x, \quad \sigma(t, x, u) = \sigma > 0, \quad \gamma > 0.
\]

This form is often encountered in the literature, see e.g. [2, 3, 18] and the references therein. However, let us stress that the equation is restricted to a bounded spatial domain in many works such as [2, 3], and Dirichlet boundary conditions are used; this is not the natural framework, cf. also Section 2. Let \(c := \frac{1}{2}\sigma^2\), then the associated Fokker-Planck equation (1.2) is given in the form

\[
\begin{aligned}
\frac{\partial p}{\partial t}(t, x) &= c \frac{\partial^2 p}{\partial x^2}(t, x) + \gamma \frac{\partial}{\partial x} \left( xp(t, x) \right) - u(t) \frac{\partial p}{\partial x}(t, x), \quad \text{in } (0, \infty) \times \mathbb{R}, \\
p(0, x) &= p_0(x), \quad \text{in } \mathbb{R}.
\end{aligned}
\]

For later use we define the function

\[
\phi : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \frac{\gamma x^2}{2c}.
\]
Since it is unrealistic to assume that we can measure \( p(t, x) \) for all \( t \geq 0 \) and all \( x \in \mathbb{R} \), we associate an output function \( y : \mathbb{R}_{\geq 0} \to \mathbb{R} \) with (1.4). The output should be chosen in such a way that, by manipulating it via the control input, it is possible to influence the collective behavior of the process. As mentioned in [2], the mean value \( E[X_t] \) “is omnipresent in almost all stochastic optimal control problems considered in the scientific literature”. Therefore, it is a reasonable choice for the output, i.e.,

\[
y(t) = E[X_t] = \int_{-\infty}^{\infty} x p(t, x) dx.
\]

We assume that the measurement of the output \( y(t) \) is available to the controller at each time \( t \geq 0 \). In practice, the corresponding integral cannot be calculated exactly, thus the mean value will typically be approximated by data-driven methods such as Monte Carlo integration.

Note that controlling the Fokker-Planck equation via the drift term with mean value as output is indeed sufficient to influence the shape of the solution density, since the variance of the process is independent of the control input. In fact, we will show in Proposition 3.3 that the variance of the solution is of the form

\[
\int_{-\infty}^{\infty} (x - y(t))^2 p(t, x) dx = \frac{c}{\gamma}(1 + Ke^{-2\gamma t}), \quad t \geq 0,
\]

for some \( K \in \mathbb{R} \), provided (1.3) holds.

**1.4. Control objective.** The objective is to design a robust output error feedback \( u(t) = F(t, e(t)) \), where \( e(t) = y(t) - y_{ref}(t) \) for some reference trajectory \( y_{ref} \in W^{1, \infty}(\mathbb{R}_{\geq 0}) \), such that in the closed-loop system the tracking error \( e(t) \) evolves within a prescribed performance funnel

\[
\mathcal{F}_\varphi := \{ (t, e) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \mid \varphi(t)|e| < 1 \},
\]

which is determined by a function \( \varphi \) belonging to

\[
\Phi := \left\{ \varphi \in W^{1, \infty}(\mathbb{R}_{\geq 0}) \mid \varphi(s) > 0 \text{ for all } s > 0 \text{ and } \lim \inf_{s \to \infty} \varphi(s) > 0 \right\}.
\]

The robustness requirement on the control essentially means that it is feasible under bounded additive disturbances “with zero mass”, which influence the Fokker-Planck equation. This is made precise in Section 6.

The boundary of the funnel \( \mathcal{F}_\varphi \) is given by the reciprocal of \( \varphi \), see Fig. 1.1. We explicitly allow for \( \varphi(0) = 0 \), meaning that no restriction on the initial value is imposed since \( \varphi(0)|e(0)| < 1 \); the funnel boundary \( 1/\varphi \) has a pole at \( t = 0 \) in this case.

An important property of the class \( \Phi \) is that the boundary of each performance funnel \( \mathcal{F}_\varphi \) with \( \varphi \in \Phi \) is bounded away from zero, i.e., because of boundedness of \( \varphi \) there exists \( \lambda > 0 \) such that \( 1/\varphi(t) \geq \lambda \) for all \( t > 0 \).

It is of utmost importance to notice that the function \( \varphi \in \Phi \) is a design parameter in the control law (stated in Section 6), thus its choice is completely up to the designer. Typically, the specific application dictates the constraints on the tracking error and thus indicates suitable choices. We stress that the funnel boundary is not necessarily monotonically decreasing, while such a choice may be convenient in most situations. However, widening the funnel over some later time interval might be beneficial, for instance in the presence of strongly varying reference signals or periodic disturbances. A variety of different funnel boundaries are possible, see e.g. [23, Sec. 3.2].
1.5. Organization of the present paper. In Section 2 we introduce the mathematical framework around the Fokker-Planck operator associated to the equation (1.4). We emphasize that we consider an unbounded spatial domain in (1.4), without any boundary conditions. Using weighted Lebesgue and Sobolev spaces, first an auxiliary operator is defined via a suitable coercive sesquilinear form. This operator, along with the spaces, is then transformed to the desired Fokker-Planck operator. Special emphasis is put on the spaces after the transformation, which are again weighted Lebesgue and Sobolev spaces, but the latter is not equipped with the natural norm. We highlight that this seems to be the natural framework for the equation (1.4). The definition of a weak solution is given in Section 3 and it is shown that any solution satisfies (1.3) and that its variance exponentially converges to \( c/\gamma \). A simple feedforward control approach is then discussed in Section 4, which may be favourable when the system parameters are known and no disturbances are present. As a basis for the feasibility proof of the robust funnel controller in Section 6, we consider Galerkin approximations of possible solutions of (1.4) under disturbances in Section 5 and derive an energy estimate. We emphasize that the closed-loop system corresponding to the application of the funnel controller is a nonlinear and time-varying PDE, thus proving existence and uniqueness of solutions is a nontrivial task. We illustrate our results by some simulations in Section 7.

2. The Fokker-Planck operator. In this section we introduce an operator which can be associated with the PDE (1.4) in the uncontrolled case, i.e., \( u = 0 \). To this end, we invoke form methods for which we frequently refer to [4] and [5]. Consider the system (1.4) with \( c > 0, \gamma > 0 \) and \( \phi \) as defined in (1.5). To begin with, let

\[
H := L^2(\mathbb{R}; e^{-\phi}) \quad \text{and} \quad V := W^{1,2}(\mathbb{R}; e^{-\phi})
\]

and use the short-hand notation \( v' = \frac{\partial v}{\partial z} \) for the weak derivative of \( v \in V \). Define the sesquilinear form

\[
a : V \times V \to \mathbb{R}, \quad (v_1, v_2) \mapsto \langle v_1', v_2' \rangle_H,
\]

which we may associate an operator as follows.

**Proposition 2.1.** Consider the form (2.1), then there exists exactly one operator \( A : \mathcal{D}(A) \subset V \to H \) with

\[
\mathcal{D}(A) = \{ v \in V \mid \exists u \in H \forall z \in V : a(v, z) = \langle u, z \rangle_H \}
\]

and

\[
\forall v \in \mathcal{D}(A) \forall z \in V : a(v, z) = \langle Av, z \rangle_H.
\]

Moreover,
(i) A is self-adjoint, positive and has compact resolvent,
(ii) there exists a monotonically increasing sequence \((\lambda_j)_{j \in \mathbb{N}_0} \in (\mathbb{R}_{\geq 0})^{\mathbb{N}_0}\), which satisfies 
\[ \lim_{j \to \infty} \lambda_j = \infty \] and the spectrum of \(A\) reads \(\sigma(A) = \{ \lambda_j \mid j \in \mathbb{N}_0 \}\),
(iii) there exists a complete orthonormal system \((v_j)_{j \in \mathbb{N}_0} \in H\) such that \(v_j \in \mathcal{D}(A)\) and 
\[ A v_j = \lambda_j v_j \] for all \(j \in \mathbb{N}_0\),
(iv) we have 
\[ \forall v \in \mathcal{D}(A) : \quad A v = \sum_{j \in \mathbb{N}_0} \lambda_j \langle v, v_j \rangle_H v_j. \]

Proof. We show that the operator \(A\) exists as stated. By the Cauchy-Schwarz inequality we have 
\[ a(v, u) \leq ||v'||_H ||u'||_H \leq ||v||_V ||u||_V, \quad v, u \in V, \]
and hence the form \(a\) is bounded. Since the injection \(j : V \to H\) is clearly continuous with dense range, it follows from [4, Prop. 5.5] that \(A\) exists and is positive since \(a\) is positive.

We show (i): As above, there exists an operator \(B : \mathcal{D}(B) \subset V \to H\) associated to the sesquilinear form 
\[ b : V \times V \to \mathbb{R}, \quad (v_1, v_2) \mapsto a(v_1, v_2) + \langle v_1, v_2 \rangle_H \]
which satisfies \(\mathcal{D}(B) = \mathcal{D}(A)\) and \(B = A + I\), cf. [4, Rem. 5.6]. The form \(b\) is obviously bounded and symmetric and satisfies \(b(v, v) = ||v||_V\), thus it is coercive. Further observe that by [28, Prop. 6.2] the injection \(j : V \to H\) is additionally compact. Hence it follows from [4, Cor. 6.18] that the operator \(B\) is self-adjoint, positive and has compact resolvent.

As a consequence, \(A = B - I\) is also self-adjoint and has compact resolvent.

Statements (ii)–(iv) then follow from [4, Thm. 6.17] together with [38, Prop. 3.2.9].

In the following we explicitly derive the eigenvalues and eigenfunctions of \(A\). To this end, recall the Hermite polynomials defined by 
\[ H_n(x) = (-1)^n e^{x^2} \left( \frac{d^n}{dx^n} e^{-x^2} \right), \quad x \in \mathbb{R}, \quad n \in \mathbb{N}_0. \]
It is well known that these polynomials have, for all \(x \in \mathbb{R}\) and all \(n, m \in \mathbb{N}_0\), the properties
\begin{enumerate}[(i)]  
  
i. \quad H_{n+1}(x) = 2xH_n(x) - H'_n(x),
  
  
ii. \quad H'_n(x) = 2nH_{n-1}(x), \quad \text{where} \quad H_{-1}(x) := 0,
  
  
iii. \quad \int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) \, dx = \sqrt{\pi} 2^n n! \delta_{n,m}, \quad \text{where} \quad \delta_{n,m} \text{ denotes the Kronecker delta.}
\end{enumerate}
We obtain the following representation.

**Proposition 2.2.** Use the notation from Proposition 2.1. Then we have, for all \(j \in \mathbb{N}_0\) and all \(x \in \mathbb{R}\), that
\[ \lambda_j = 2j \theta^2, \quad v_j(x) = \alpha_j H_j(\theta x), \quad \alpha_j := \sqrt{\frac{\theta}{\sqrt{\pi} 2^j j!}}, \quad \theta := \sqrt{\frac{\gamma}{2e}}. \]

Furthermore,
\begin{enumerate}[(i)]  
  
i. \quad (v_j)_{j \in \mathbb{N}_0} \text{ is an orthogonal system in } V,
  
  
ii. \quad v_j'(x) = \sqrt{\lambda_j} v_{j-1}(x) \text{ for all } x \in \mathbb{R} \text{ and } j \in \mathbb{N}_0, \text{ where } v_{-1}(x) := 0,
  
  
iii. \quad \left( e^{-\phi(x)} v_j(x) \right)' = -\sqrt{\lambda_j} e^{-\phi(x)} v_{j+1}(x) \text{ for all } x \in \mathbb{R} \text{ and } j \in \mathbb{N}_0,
  
  
iv. \quad \lim_{x \to \pm \infty} e^{-\phi(x)} v_j(x)v(x) = 0 \text{ for all } v \in V.
\end{enumerate}

**Proof.** Define \(w_j(x) := \alpha_j H_j(\theta x)\) and \(\kappa_j := 2j \theta^2\) for \(j \in \mathbb{N}_0\) and \(x \in \mathbb{R}\).

**Step 1:** We show that \(\lim_{x \to \pm \infty} e^{-\phi(x)} w_j(x) = 0\) for all \(x \in V\) and all \(j \in \mathbb{N}_0\). Fix \(j \in \mathbb{N}\). Since \(w_j\) is a polynomial we have that \(e^{-\phi/2} w_j \in L^\infty(\mathbb{R})\). Furthermore, \(z \in V\) yields that \(e^{-\phi/2} z \in L^2(\mathbb{R})\) and \(e^{-\phi/2} z' \in L^2(\mathbb{R})\). Hence \(e^{-\phi/2} w_j = \left( e^{-\phi/2} w_j \right) \left( e^{-\phi/2} z \right) \in L^2(\mathbb{R})\).
and \( e^{-\theta}w_jz' = \left( e^{-\phi/2}w_j \right) (e^{-\phi/2}z') \in L^2(\mathbb{R}) \). Moreover, we compute

\[
(2.2) \quad \left( e^{-\phi(x)}w_j(x) \right)' = \alpha_j \theta e^{-\phi(x)} \left( H'_j(\theta x) - 2(\theta x)H_j(\theta x) \right) = -\alpha_j \theta e^{-\phi(x)}H_{j+1}(\theta x) = -\sqrt{k_{j+1}}e^{-\phi(x)}w_{j+1}(x),
\]

where we have used that \( \sqrt{2(j+1)}\alpha_{j+1} = \alpha_j \). Therefore,

\[
(e^{-\phi}w_j)' z = -\sqrt{k_{j+1}}e^{-\phi}w_{j+1}z = -\sqrt{k_{j+1}} \left( e^{-\phi/2}w_{j+1} \right) \left( e^{-\phi/2}z \right) \in L^2(\mathbb{R}),
\]

and hence

\[
(e^{-\phi}w_j)' z = (e^{-\phi}w_j)' z + e^{-\phi}w_jz' \in L^2(\mathbb{R}).
\]

Since \( e^{-\phi}w_jz \in L^2(\mathbb{R}) \) and \( (e^{-\phi}w_jz)' \in L^2(\mathbb{R}) \), it follows from Barbálat’s Lemma (see e.g. [17, Thm. 5]) that \( \lim_{x \to \pm \infty} e^{-\phi(x)}w_j(x)z(x) = 0 \).

Step 2: We show that \( w_j \in D(A) \) and \( Aw_j = \kappa_j w_j \) for all \( j \in N_0 \). First note that \( w_j \in V \) is clear since \( H'_j = 2jH_{j-1} \). By definition of \( A \) the two assertions hold if, and only if, \( a(w_j, z) = \kappa_j \langle w_j, z \rangle_H \) for all \( z \in V \). For \( j = 0 \) this is clear since \( \kappa_0 = 0 \) and \( w_0 \) is constant, thus \( a(w_0, z) = 0 \) for all \( z \in V \). Now, fix \( j \in N \) and \( z \in V \). Then we have, invoking the properties of the Hermite polynomials,

\[
a(w_j, z) = \int_{-\infty}^{\infty} e^{-\phi(x)}w_j(x)z'(x)dx = \sqrt{2j} \theta \int_{-\infty}^{\infty} e^{-\phi(x)}w_{j-1}(x)z'(x)dx
\]

\[
\overset{\text{Step 1}}{=} -\sqrt{k_j} \int_{-\infty}^{\infty} \left( e^{-\phi(x)}w_{j-1}(x) \right)' z(x)dx \overset{(2.2)}{=} \kappa_j \langle w_j, z \rangle_H
\]

as claimed.

Step 3: We show that \( \{w_j\}_{j \in N_0} \) is an orthonormal system in \( H \). Invoking the properties of the Hermite polynomials, this follows immediately from

\[
\langle w_j, w_k \rangle_H = \int_{-\infty}^{\infty} e^{-\phi(x)}w_j(x)w_k(x)dx
\]

\[
= \frac{\alpha_j \alpha_k}{\theta} \int_{-\infty}^{\infty} e^{-y^2}H_n(y)H_m(y)dy
\]

\[
= \frac{\alpha_j \alpha_k}{\theta} \sqrt{2} \sqrt{2} j! \delta_{j,k} = \delta_{j,k}
\]

for \( j, k \in N_0 \).

Step 4: From Steps 1–3 it follows that the sequence \( \{\kappa_j\}_{j \in N_0} \) is a subsequence of \( \{\lambda_k\}_{k \in N_0} \) and \( \{w_j\}_{j \in N_0} \) is a subsequence of \( \{v_k\}_{k \in N_0} \). We show equality. Seeking a contradiction, assume that there exist \( \lambda \in \mathbb{R}_{>0} \) and \( v \in V \setminus \{0\} \) such that \( Av = \lambda v \) and \( \lambda \neq \kappa_j \) for all \( j \in N_0 \). It is well known that the Hermite polynomials constitute a complete orthogonal system in \( L^2(\mathbb{R}; w) \) for \( w(x) = e^{-x^2} \). Therefore, by Step 3, \( \{w_j\}_{j \in N_0} \) is a complete orthonormal system in \( H \). Since \( Av = \lambda v \) we find that \( a(v, z) = \lambda \langle v, z \rangle_H \) for all \( z \in V \) by definition of \( A \). We consider the special choice \( z = w_k \) \( (w_k \in V) \) by Step 2 for some \( k \in N_0 \) in the following and obtain, as in Step 2,

\[
\lambda \langle v, w_k \rangle_H = a(v, w_k) = \kappa_k \langle v, w_k \rangle_H.
\]

Therefore, since \( \lambda \neq \kappa_k \),

\[
\forall k \in N_0: \langle v, w_k \rangle_H = 0,
\]

which by completeness of \( \{w_j\}_{j \in N_0} \) implies \( v = 0 \), a contradiction.

Step 5: Assertion (ii) is an immediate consequence of the properties of the Hermite polynomials, (iii) is shown in (2.2) and (iv) in Step 1. It remains to show (i), i.e., that \( \{v_j\}_{j \in N_0} \) is an orthogonal system in \( V \). First observe that for \( j, k \in N \) with \( j \neq k \) we have

\[
\langle v_j, v_k \rangle_V = \langle v_j, v_k \rangle_H + \langle v'_j, v'_k \rangle_H \overset{(ii)}{=} \sqrt{\lambda_j \lambda_k} \langle v_{j-1}, v_{k-1} \rangle_H = 0.
\]

If \( j = 0 \) or \( k = 0 \), then \( v_j \) or \( v_k \) is constant and hence \( \langle v'_j, v'_k \rangle_H = 0 \). This finishes the proof. \( \square \)
Now we turn to transform the operator $A$ so that it becomes a suitable Fokker-Planck operator. To this end, define the spaces
\[
\mathcal{H} := \{ e^{-\phi} f \mid f \in H \} = L^2(\mathbb{R}; e^\phi),
\]
and the bijection
\[
T : H \rightarrow \mathcal{H}, \quad f \mapsto e^{-\phi} f,
\]
together with the inner products
\[
\langle z_1, z_2 \rangle_{\mathcal{H}} := \langle T^{-1}(z_1), T^{-1}(z_2) \rangle_H = \langle e^\phi z_1, e^\phi z_2 \rangle_H, \quad z_1, z_2 \in \mathcal{H},
\]
\[
\langle z_1, z_2 \rangle_{\mathfrak{V}} := \langle T^{-1}(z_1), T^{-1}(z_2) \rangle_V = \langle e^\phi z_1, e^\phi z_2 \rangle_H + \langle (e^\phi z_1)', (e^\phi z_2)' \rangle_H, \quad z_1, z_2 \in \mathfrak{V}.
\]
We will show that $\mathfrak{V}$ is equal to $W^{1,2}(\mathbb{R}; e^\phi)$, although not equipped with the natural Sobolev norm. We postpone the proof of this equality and first transform the operator $A$ and collect some results on it. To this end, define the sesquilinear form
\[
\tag{2.3}
\mathfrak{A} : \mathfrak{V} \times \mathfrak{V} \rightarrow \mathbb{R}, \quad (z_1, z_2) \mapsto a(T^{-1}(z_1), T^{-1}(z_2)) = \langle (e^\phi z_1)', (e^\phi z_2)' \rangle_H,
\]
as well as $D(\mathfrak{A}) := T(D(A))$ and the operator
\[
\mathfrak{A} := T \circ A \circ T^{-1} : D(\mathfrak{A}) \subset \mathfrak{V} \rightarrow \mathcal{H}.
\]
Then we have that, for $v \in D(\mathfrak{A})$ and $y \in \mathcal{H},$
\[
y = \mathfrak{A} v \iff T^{-1}(y) = AT^{-1}(v) \iff \forall z \in V : a(T^{-1}(v), z) = \langle T^{-1}(y), z \rangle_H
\]
\[
\iff w = T(z) \quad \forall w \in \mathfrak{V} : \mathfrak{A}(v, w) = \langle y, w \rangle_{\mathcal{H}}.
\]
From Proposition 2.2 we immediately obtain the following result on the eigenvalues and eigenfunctions of $\mathfrak{A}.$

**Proposition 2.3.** The operator $\mathfrak{A}$ satisfies
(i) $\sigma(\mathfrak{A}) = \sigma(A),$
(ii) $z$ is an eigenfunction of $\mathfrak{A}$ if, and only if, $e^\phi z$ is an eigenfunction of $A,$
(iii) for $z_j := e^{-\phi} v_j,$ where $v_j$ is as in Proposition 2.2, $(z_j)_{j \in \mathbb{N}_0}$ constitutes a complete orthonormal system of eigenfunctions in $\mathcal{H},$
(iv) $a(z_j, z) = \lambda_j \langle z_j, z \rangle_{\mathfrak{V}}$ for all $z \in \mathfrak{V},$
(v) $(z_j)_{j \in \mathbb{N}_0}$ is an orthogonal system in $\mathfrak{V},$
(vi) $z_j'(x) = -\sqrt{\lambda_{j+1}} z_{j+1}(x)$ for all $x \in \mathbb{R}$ and $j \in \mathbb{N}_0,$
(vii) $\lim_{x \to \pm \infty} e^{\phi(x)} z_j(x) = 0$ for all $z \in \mathfrak{V}.$

We now turn to the proof of $\mathfrak{V} = W^{1,2}(\mathbb{R}; e^\phi),$ highlighting that $\|v\|_{\mathfrak{A}} \neq \|v\|_{W^{1,2}(\mathbb{R}; e^\phi)}$ in general. Nevertheless, we will additionally show that the norm $\|v\|_{W^{1,2}(\mathbb{R}; e^\phi)}$ can be estimated by the norm $\|v\|_{\mathfrak{A}}.$

**Proposition 2.4.** The following statements are true:
(i) For any $v \in V$ we have that $\phi' v \in H.$
(ii) For any $z \in W^{1,2}(\mathbb{R}; e^\phi)$ we have that $\phi' z \in \mathcal{H}.$
(iii) $\mathfrak{V} = W^{1,2}(\mathbb{R}; e^\phi).$
(iv) $\|v\|_{\mathfrak{V}} \leq (1 + 2\sqrt{1 + \theta^2}) \|v\|_{\mathfrak{A}}$ for all $v \in \mathfrak{V},$ where $\theta$ is as in Proposition 2.2.
(v) There exists $C > 0$ such that $\|v\|_{W^{1,2}(\mathbb{R}; e^\phi)} \leq C \|v\|_{\mathfrak{A}}$ for all $v \in \mathfrak{V}.$

**Proof.** We show (i), so let $v \in V.$ Use the notation from Propositions 2.1 and 2.2. Since $(v_j)_{j \in \mathbb{N}_0}$ is a complete orthonormal system in $H$ we have
\[
v = \sum_{j \in \mathbb{N}_0} \beta_j v_j, \quad \beta_j := \langle v, v_j \rangle_H, \quad j \in \mathbb{N}_0,
\]
and, by Parseval’s identity, \( \|v\|^2_H = \sum_{j \in \mathbb{N}_0} \beta_j^2 \). Furthermore, we have
\[
\langle v', v_j \rangle_H = \int_{-\infty}^{+\infty} e^{-\frac{\phi(x)}{2}} v'(x) v_j(x) \, dx \text{ Prop. 2.2 (iv)} - \int_{-\infty}^{+\infty} \left( e^{-\frac{\phi(x)}{2}} v_j(x) \right)' v(x) \, dx
\]
and this implies, again invoking Parseval’s identity,
\[
\|v'\|^2_H = \sum_{j \in \mathbb{N}_0} |\langle v', v_j \rangle_H|^2 = \sum_{j \in \mathbb{N}_0} \lambda_j |\langle v, v_{j+1} \rangle_H|^2 = \sum_{j=1}^{\infty} \lambda_j \beta_j^2.
\]

Now consider the sequence \( w_n := \phi' \sum_{j=0}^{n} \beta_j v_j, n \in \mathbb{N} \). We show that \( (w_n)_{n \in \mathbb{N}} \) has a weakly convergent subsequence in \( H \). To this end, first observe that for all \( x \in \mathbb{R} \) we have
\[
\phi'(x)v_j(x) = 2\alpha_j \theta x H_j(x) = \alpha_j \theta \left( H_{j+1}(\theta x) + H_{j-1}(\theta x) \right)
= \alpha_j \theta \left( H_{j+1}(\theta x) + 2j H_{j-1}(\theta x) \right) = \sqrt{\lambda_{j+1}} v_{j+1}(x) + \sqrt{\lambda_j} v_{j-1}(x),
\]
where we have used the properties of the Hermite polynomials, \( \sqrt{2(j+1)} \alpha_{j+1} = \alpha_j \) and the convention \( v_{-1}(x) := 0 \). Therefore,
\[
w_n = \sum_{j=0}^{n} \beta_j (\sqrt{\lambda_{j+1}} v_{j+1} + \sqrt{\lambda_j} v_{j-1})
= \beta_1 \sqrt{\lambda_1} v_0 + \beta_n \sqrt{\lambda_{n+1}} v_{n+1} + \sum_{j=1}^{n} (\beta_{j-1} \sqrt{\lambda_j} + \beta_{j+1} \sqrt{\lambda_{j+1}}) v_j,
\]
so we may compute
\[
\|w_n\|^2_H = \beta_1^2 \lambda_1 + \beta_n^2 \lambda_{n+1} + \sum_{j=1}^{n} (\beta_{j-1} \sqrt{\lambda_j} + \beta_{j+1} \sqrt{\lambda_{j+1}})^2
\leq \beta_2^2 \lambda_1 + \beta_n^2 \lambda_{n+1} + 2 \sum_{j=1}^{n} (\beta_{j-1} \lambda_j + \beta_{j+1} \lambda_{j+1}) \leq 2 \sum_{j=1}^{n+1} (\beta_{j-1}^2 \lambda_j + \beta_{j+1}^2 \lambda_{j+1})
= 2 \sum_{j=1}^{n+1} \beta_{j-1}^2 (\lambda_{j-1} + 2\theta^2) + 2 \sum_{j=1}^{n+1} \beta_{j+1}^2 \lambda_j \leq 4 \|v'\|^2_H + 4\theta^2 \|v\|^2_H.
\]
As a consequence, \( (w_n)_{n \in \mathbb{N}} \) is a bounded sequence in \( H \) and hence has a weakly convergent subsequence, which we again denote by \( (w_n)_{n \in \mathbb{N}} \). This means that there exists \( w \in H \) such that
\[
\forall z \in H : \lim_{n \to \infty} \langle w_n, z \rangle_H = \langle w, z \rangle_H.
\]
We show that \( w = \phi' v \), which proves the claim \( \phi' v \in H \). To this end, let \( j \in \mathbb{N}_0 \) and observe that
\[
\langle \phi' v, v_j \rangle_H = \langle v, \phi v_j \rangle_H = \sqrt{\lambda_{j+1}} \langle v, v_{j+1} \rangle_H + \sqrt{\lambda_j} \langle v, v_{j-1} \rangle_H = \sqrt{\lambda_{j+1}} \beta_{j+1} + \sqrt{\lambda_j} \beta_{j-1},
\]
where \( \beta_{-1} := 0 \), and, for any \( n \geq j + 1 \),
\[
\langle w_n, v_j \rangle_H = \sum_{k=0}^{n} \beta_k \langle v_k, \phi v_j \rangle_H = \sum_{k=0}^{n} \beta_k \langle v_k, \sqrt{\lambda_{j+1}} v_{j+1} + \sqrt{\lambda_j} v_{j-1} \rangle_H
= \sqrt{\lambda_{j+1}} \beta_{j+1} + \sqrt{\lambda_j} \beta_{j-1} = \langle \phi' v, v_j \rangle_H.
\]
Therefore,
\[ \forall j \in \mathbb{N}_0 : \langle w, v_j \rangle_H = \lim_{n \to \infty} \langle w_n, v_j \rangle_H = \langle \phi'v, v_j \rangle_H, \]
by which \( w = \phi'v \) since \( (v_j)_{j \in \mathbb{N}_0} \) is a complete orthonormal system in \( H \).

We show (ii), so let \( z \in W^{1,2}(\mathbb{R}; e^\phi) \). The proof of this statement is analogous to the proof of (i), utilizing the complete orthonormal system \((z_j)_{j \in \mathbb{N}_0}\) in \( H \) from Proposition 2.3. Since \( z, z' \in L^2(\mathbb{R}; e^\phi) = H \) we may compute, for \( \beta_j := \langle z, z_j \rangle_H \), that \( \|z'\|^2_H = \sum_{j=1}^\infty \lambda_j \beta_j^2 - 1 \).

Then, as above, \( w_n := \phi' \sum_{j=1}^n \beta_j z_j \) is bounded by \( 4\|z'\|^2_B + 4\theta^2 \|z\|^2_B \) and hence has a weakly convergent subsequence, the limit of which is \( \phi'z \), which is hence an element of \( H \).

We show (iii). First observe that
\[ z \in \mathfrak{U} \iff e^\phi z \in V \iff e^\phi z, (e^\phi z)' \in H \text{ and } z, z' \in L^2(\mathbb{R}; e^\phi) = \mathfrak{H}, \]
then the claim follows from
\[
\begin{align*}
e^\phi z, (e^\phi z)' &\in H \quad \overset{(i)}{\Rightarrow} \quad e^\phi z, (e^\phi z)', \phi' e^\phi z \in H \\
&\Rightarrow e^\phi z, e^\phi z' = (e^\phi z)' - \phi' e^\phi z \in H \quad \Rightarrow \quad z, z' \in \mathfrak{H} \quad \text{and} \\
z, z' \in \mathfrak{H} &\quad \overset{(ii)}{\Rightarrow} \quad z, z', \phi' z \in \mathfrak{H} \quad \Rightarrow \quad e^\phi z, e^\phi z', \phi' e^\phi z \in H \\
&\Rightarrow e^\phi z, (e^\phi z)' = e^\phi z' + \phi' e^\phi z \in H.
\end{align*}
\]

We show (iv). As shown in (ii), for any \( v \in \mathfrak{U} \) we have that \( v' \in \mathfrak{H} \). Furthermore, we may apply the findings of (i) to \( e^\phi v \in H \), so that with \( w = \phi' e^\phi v \in H \) and the corresponding sequence \((w_n)_{n \in \mathbb{N}} \) we have, since \( \lim_{n \to \infty} \langle w_n, z \rangle_H = \langle w, z \rangle_H \) for all \( z \in H \),
\[
\|w\|^2_H = \lim_{n \to \infty} \langle w_n, w \rangle_H = \lim_{n \to \infty} \langle w_n, w_k \rangle_H = \lim_{n \to \infty} \|w_n\|^2_H \\
\leq 4\|e^\phi v\|^2_H + 4\theta^2\|e^\phi v\|^2_H \leq 4(1 + \theta^2)\|e^\phi v\|^2_V = 4(1 + \theta^2)\|v\|^2_{\mathfrak{U}}.
\]

Therefore,
\[
\|v'\|_{\mathfrak{H}} \leq \|v' + \phi' v\|_{\mathfrak{H}} + \|\phi' v\|_{\mathfrak{H}} \leq \|v\|_{\mathfrak{U}} + 2\sqrt{1 + \theta^2}\|v\|_{\mathfrak{U}}.
\]

Finally, (v) is a consequence of (iv) upon observing that \( \|v\|^2_{W^{1,2}(\mathbb{R}; e^\phi)} = \|v\|^2_\mathfrak{H} + \|v'\|^2_\mathfrak{H} \leq \|v\|^2_\mathfrak{U} + \|v'\|^2_\mathfrak{H} \).

Attention now turns to the operator \( -c\mathcal{A} \), which will serve as the Fokker-Planck operator. In view of the right-hand side in (1.4), this is justified by the following property.

**Lemma 2.5.** Let \( z \in \mathfrak{U} \) be such that \( (e^\phi z)' \in V \). Then we have that
\[
\mathcal{A}z = -\left( e^{-\phi} (e^\phi z)' \right)' = -z'' - (\phi' z)'.
\]

**Proof.** We calculate that for any \( j \in \mathbb{N}_0 \)
\[
\langle \mathcal{A}z, z_j \rangle_{\mathfrak{H}} = \mathfrak{a}(z, z_j) = \int_{-\infty}^{\infty} e^{-\phi(x)} \left( e^{\phi(x)} z(x) \right)' \left( e^{\phi(x)} z_j(x) \right)' dx \\
= \left[ e^{\phi(x)} z(x) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{\phi(x)} \left( e^{-\phi(x)} \left( e^{\phi(x)} z(x) \right)' \right)' z_j(x) dx \\
= \left[ -e^{-\phi} (e^\phi z)' \right]' - (e^\phi z)' \big|_{-\infty}^{\infty}.
\]

(2.4)
where the last equality follows from
\[(e^\phi z)^j z_j = e^{-\phi} (e^\phi z)^j v_j,\]
the assumption \((e^\phi z)^j \in V\) and Proposition 2.2 (iv). Since (2.4) is true for all \(j \in \mathbb{N}_0\), we have proved the first equality in the statement. The second is a straightforward calculation. \(\square\)

Recall that \(\phi' = 2\theta^2 x\), and hence \(c\phi'(x) = \gamma x\). Therefore, with the operator
\[(2.5)\]
\[\mathcal{B} : \mathcal{W} \times \mathbb{R} \to \mathcal{H}, \quad (v, u) \mapsto u \cdot v',\]
the Fokker-Planck equation (1.4) can be rewritten as
\[(2.6)\]
\[
\begin{align*}
\dot{p}(t, x) &= -ca \mathcal{B} (p(t, x), u(t))(x), \quad \text{in } (0, \infty) \times \mathbb{R}, \\
p(0, x) &= p_0(x), \quad \text{in } \mathbb{R},
\end{align*}
\]
with state space \(\mathcal{H}\), where \(\dot{p} = \frac{\partial p}{\partial t}\). System (2.6) fits into the framework of bilinear control systems as considered for the Fokker-Planck equation e.g. in [14, 22]. However, we stress that the assumptions required in the aforementioned works are not satisfied for (2.6).

We also note that (2.6) admits solutions with potentially less regularity requirements than equation (1.4), because of Lemma 2.5. The specific definition of a weak solution that we use in the present paper is given in the following section.

3. Solution properties. In this section we introduce solutions of the Fokker-Planck equation (2.6) using the weak formulation, where we closely follow [16, Sec. 7.1]. Thereafter, we derive a set of properties that each solution exhibits, including a variance independent of the control input and properties (1.3).

**Definition 3.1.** Consider the system (1.4) with \(c > 0\), \(\gamma > 0\) and \(\phi\) as defined in (1.5). Recall the spaces \(\mathcal{H}\) and \(\mathcal{W}\) from Section 2 and let \(p_0 \in \mathcal{H}\), \(T > 0\) and \(u \in C([0, T]; \mathbb{R})\). A function \(p\) is called a solution of (2.6) on \([0, T]\), if
(i) \(p \in L^2(0, T; \mathcal{W}) \cap C([0, T]; \mathcal{H})\) with \(p(0) = p_0\),
(ii) \(p\) is weakly differentiable in \(L^1(0, T; \mathcal{W}')\) and satisfies \(\dot{p} \in L^2(0, T; \mathcal{W}')\),
(iii) for all \(v \in \mathcal{W}\) and almost all \(t \in [0, T]\) we have
\[
\langle \dot{p}(t), v \rangle_{\mathcal{H}} = -ca(\mathcal{B}(p(t), u(t)), v)_{\mathcal{H}},
\]
where \(a : \mathcal{W} \times \mathcal{W} \to \mathbb{R}\) is the sesquilinear form defined in (2.3) and \(\mathcal{B} : \mathcal{W} \times \mathbb{R} \to \mathcal{H}\) is the operator defined in (2.5).

A function \(p\) is called a solution of (2.6) on \(\mathbb{R}_{\geq 0}\), if \(p|_{[0, T]}\) is a solution of (2.6) on \([0, T]\) for all \(T > 0\), in particular \(p \in L^2_{\text{loc}}(0, \infty; \mathcal{W})\) and \(\dot{p} \in L^2_{\text{loc}}(0, \infty; \mathcal{W}')\).

**Remark 3.2.** We like to note that since \(\dot{p} \in L^2(0, T; \mathcal{W}')\), where \(\mathcal{W}'\) is the dual of \(\mathcal{W}\) with respect to the pivot space \(\mathcal{H}\), i.e., \(\mathcal{W} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{W}'\) is a Gelfand triple, the duality pairing between \(\mathcal{W}\) and \(\mathcal{W}'\) is compatible with the inner product in \(\mathcal{H}\), which means
\[
\langle \dot{p}(t), v \rangle_{\mathcal{W} \times \mathcal{W}} = \langle \dot{p}(t), v \rangle_{\mathcal{H}},
\]
using an appropriate identification via the Riesz representation theorem. For brevity, we will always use the latter expression.

We may infer the following properties of a solution of (2.6).

**Proposition 3.3.** Use the assumptions from Definition 3.1 and let \(p\) be a solution of (2.6) on \([0, T]\). Then the following statements are true:
(i) \(\int_{-\infty}^{\infty} p(t, x) \, dx = \int_{-\infty}^{\infty} p_0(x) \, dx\) for all \(t \in [0, T]\).
(ii) If \(p_0(x) \geq 0\) for almost all \(x \in \mathbb{R}\), then \(p(t, x) \geq 0\) for all \(t \in [0, T]\) and almost all \(x \in \mathbb{R}\).
(iii) If \( \int_{-\infty}^{\infty} p_0(x)dx = 1 \), then for \( y : [0, T] \to \mathbb{R} \) as in (1.6) there exists \( K \in \mathbb{R} \), which is independent of \( T \), such that

\[
\forall t \in [0, T] : \int_{-\infty}^{\infty} (x - y(t))^2 p(t, x)dx = \frac{c}{\gamma} (1 + Ke^{-2\gamma t}).
\]

**Proof.** We show (i). First observe that \( z_0 = e^{-\theta} v_0 = e^{-\theta} \alpha_0 \) for \( \alpha_0 > 0 \) as in Proposition 2.2. Then

\[
\int_{-\infty}^{\infty} p(t, x)dx = \langle p(t), e^{-\theta} \rangle _{\mathfrak{B}} = \frac{1}{\alpha_0} \langle p(t), z_0 \rangle _{\mathfrak{B}}
\]

for all \( t \in [0, T] \). Moreover, since \( p \) is a solution of (2.6),

\[
\langle \dot{p}(t), z_0 \rangle _{\mathfrak{B}} = -\alpha a(p(t), z_0) - \langle \mathfrak{B} (p(t), u(t)), z_0 \rangle _{\mathfrak{B}}\text{ Prop. 2.3 (iv)} = -u(t) \langle p'(t), z_0 \rangle _{\mathfrak{B}} = -\alpha a_0 u(t) \int_{-\infty}^{\infty} p'(t, x)dx = - u(t) e^{\theta(x)} z_0(x)p(t, x)|_{-\infty}^{\infty} = 0
\]

since \( p(t) \in \mathfrak{B} \) and, by Proposition 2.3 (vii), \( \lim_{x \to \pm \infty} e^{\theta(x)} z_0(x) p(t, x) = 0 \). Hence, it follows from [21, Thm. 1.32] that

\[
\forall t \in [0, T] : \langle p(t), z_0 \rangle _{\mathfrak{B}} - \langle p(0), z_0 \rangle _{\mathfrak{B}} = \int_{0}^{t} \langle \dot{p}(s), z_0 \rangle _{\mathfrak{B}}ds = 0,
\]

by which, together with \( p(0) = p_0 \), the assertion is shown.

We show (ii). First we define the positive and negative part of \( p \) in the usual way by

\[
p^+(t, x) := \max\{p(t, x), 0\}, \quad p^-(t, x) := \max\{-p(t, x), 0\}
\]

for \((t, x) \in [0, T] \times \mathbb{R}\). It is then clear that \( \|p^\pm\|_{L^2([0, T]; \mathfrak{B})} \leq \|p\|_{L^2([0, T]; \mathfrak{B})} \), thus \( p^+, p^- \in L^2(0, T; \mathfrak{B}) \). Furthermore, for all \( t \in [0, T] \) we have that \( p^-(t) \) is weakly differentiable, cf. e.g. [15, Thm. 2.8], and \( \frac{\partial}{\partial x} p^-(t) = 1_{\{p < 0\}} p'(t) \), by which \( \|\frac{\partial}{\partial x} p^-(t)\|_{\mathfrak{B}} \leq \|p'(t)\|_{\mathfrak{B}} \) and hence \( p^-(t) \in \mathfrak{B} \). Then, similar as in [15, Lem. 11.2], we may show that

\[
\forall t \in [0, T] : \frac{1}{2} (\|p^-(t)\|_{\mathfrak{B}}^2 - \|p^-(0)\|_{\mathfrak{B}}^2) = \int_{0}^{t} \langle \dot{p}(s), p^-(s) \rangle _{\mathfrak{B}}ds.
\]

Furthermore, we can estimate

\[
\langle \dot{p}(s), p^-(s) \rangle _{\mathfrak{B}} = - c\alpha (p(s), p^-(s)) - u(s)(p'(s), p^-(s)) _{\mathfrak{B}}
\]

\[
= - c\alpha (p^+(s), p^-(s)) - u(s)(\frac{\partial}{\partial x} p^-(s), p^-(s)) _{\mathfrak{B}}\text{ Prop. 2.4}
\]

\[
\leq - c \|p^-(s)\|_{\mathfrak{B}}^2 + c \|p^-(s)\|_{\mathfrak{B}}^2 + \|u\|_{\infty} \frac{\partial}{\partial x} p^-(s) \|p^-(s)\|_{\mathfrak{B}}^2
\]

\[
\leq - c \|p^-(s)\|_{\mathfrak{B}}^2 + c \|p^-(s)\|_{\mathfrak{B}}^2 + \frac{1}{c} \|p^-(s)\|_{\mathfrak{B}}^2 + \frac{1}{2c} (1 + 2\sqrt{1 + \theta^2})^2 \|u\|_{\infty} \|p^-(s)\|_{\mathfrak{B}} \|p^-(s)\|_{\mathfrak{B}}
\]

\[
\leq \frac{1}{2c} (1 + 2\sqrt{1 + \theta^2})^2 \|u\|_{\infty} \|p^-(s)\|_{\mathfrak{B}}^2 \leq D \|p^-(s)\|_{\mathfrak{B}}^2
\]

for all \( s \geq 0 \), where

\[
D := c + \frac{1}{2c} (1 + 2\sqrt{1 + \theta^2})^2 \|u\|_{\infty}^2.
\]

Hence \( \|p^-(t)\|_{\mathfrak{B}}^2 \leq e^{2Dt} \|p^-(0)\|_{\mathfrak{B}}^2 \) for all \( t \geq 0 \) by Grönwall’s lemma. Since

\[
p^-(0, x) = \max\{-p(0, x), 0\} = \max\{-p_0(x), 0\} = 0
\]
for almost all \( x \in \mathbb{R} \), it follows that \( p^-(t) = 0 \in \mathcal{F} \) for all \( t \geq 0 \), thus the claim is shown.

We show (iii). To this end, consider the system of eigenfunctions \( (z_j)_{j \in \mathbb{N}_0} \) of \( \mathcal{A} \) from Proposition 2.3 and define \( \mu_i(t) := \langle p(t), z_i \rangle_{\mathcal{F}} \) for \( t \in [0, T] \) and \( i \in \{0, 1, 2\} \). Then it follows from [21, Thm. 1.32] that

\[
\dot{\mu}_i(t) = -c\alpha(p(t), z_i) - \langle \mathcal{B}(p(t), u(t)), z_i \rangle_{\mathcal{F}}
\]

where the last equality follows from the assumption. Invoking that the second Hermite polynomial is given by \( H_2(x) = 4x^2 - 2 \) for all \( x \in \mathbb{R} \), it follows that

\[
x^2 = \frac{1}{4\theta^2} \left( \frac{\nu_2(x)}{\alpha_2} + \frac{2\nu_0(x)}{\alpha_0} \right), \quad x \in \mathbb{R}.
\]

Inserting this gives

\[
\int_{-\infty}^{\infty} x^2 p(t, x) dx = \frac{\mu_2(t)}{4\theta^2\alpha_2} + \frac{\mu_0}{2\theta^2\alpha_0}, \quad t \in [0, T].
\]

Furthermore, invoking \( H_1(x) = 2x \) for all \( x \in \mathbb{R} \), we find that

\[
y(t) = \int_{-\infty}^{\infty} xp(t, x) dx = \frac{1}{2\theta\alpha_1} \langle p(t), z_1 \rangle_{\mathcal{F}} = \frac{\mu_1(t)}{2\theta\alpha_1}, \quad t \in [0, T].
\]

From the assumption it also follows that

\[
1 = \int_{-\infty}^{\infty} p_0(x) dx = \langle p_0, e^{-\phi} \rangle_{\mathcal{F}} = \frac{1}{\alpha_0} \langle p_0, z_0 \rangle_{\mathcal{F}} = \frac{\mu_0}{\alpha_0},
\]

and, by definition of \( \theta \), we have \( 2\theta^2 = \gamma/c \). Therefore, we obtain

\[
\sigma(t)^2 = \frac{\gamma}{c} \left( 1 + \frac{\mu_2(t)}{2\alpha_2} - \frac{\mu_1(t)^2}{2\alpha_1^2} \right), \quad t \in [0, T].
\]

Now, define \( g(t) := \frac{\nu_2(t)}{2\alpha_2} - \frac{\nu_1(t)^2}{2\alpha_1^2} \) for \( t \in [0, T] \) and calculate

\[
\dot{g}(t) = \frac{\dot{\mu}_2(t)}{2\alpha_2} - \frac{\mu_1(t)\mu_1(t)}{\alpha_1^2} = -c\lambda_2 \frac{\mu_2(t)}{2\alpha_2} + \sqrt{\lambda_2} \frac{\mu_1(t)}{2\alpha_2} u(t) \mu_1(t) + \frac{c\lambda_1}{\alpha_1} \mu_1(t)^2 - \sqrt{\lambda_1} \frac{\mu_0}{\alpha_1^2} u(t) \mu_1(t)
\]

\[
= -c\lambda_2 g(t) + \left( \frac{\sqrt{\lambda_2}}{2\alpha_2} - \sqrt{\lambda_1} \frac{\mu_0}{\alpha_1^2} \right) u(t) \mu_1(t),
\]
where we have used that $\lambda_1 = \frac{1}{2} \lambda_2$. Furthermore, invoking $\alpha_1 = 2\alpha_2$ and $\alpha_0 = \sqrt{2} \alpha_1$, we find that

$$\frac{\sqrt{2} \lambda_1}{2 \alpha_2} = \frac{\sqrt{2} \lambda_1}{\alpha_1} = \frac{\sqrt{2} \lambda_1 \mu_0}{\alpha_1^2} = \frac{\lambda_1 \mu_0}{\alpha_1^2},$$

thus $\dot{y}(t) = -c \lambda_2 g(t)$ for almost all $t \in [0, T]$. Finally, with $c \lambda_2 = 4c \theta^2 = 2 \gamma$ and $K := g(0)$ (which is independent of $T$) we obtain that

$$\sigma(t)^2 = \frac{c}{\gamma} \left(1 + Ke^{-2\gamma t}\right), \quad t \in [0, T].$$

4. A simple feedforward controller. In this section we present a very simple, yet effective feedforward control strategy. We stress that the presented control law does not achieve the control objective – it is not robust and does not guarantee error evolution within the prescribed performance funnel. Nevertheless, we will show that it guarantees fast (exponential) convergence of the tracking error to zero, provided the system parameters are known, no disturbances are present and the derivative of the reference signal is available to the controller. For $\gamma > 0$ as in (1.4) and reference signal $y_{\text{ref}} \in W^{1,\infty}(\mathbb{R}_0)$ the controller is given by

$$u(t) = y_{\text{ref}}(t) + \gamma y_{\text{ref}}(t). \quad (4.1)$$

Note that (4.1) is not a feedback controller, it is completely determined by $y_{\text{ref}}$. We show that (2.6) with (4.1) admits a solution.

**PROPOSITION 4.1.** Use the assumptions from Definition 3.1 and let $y_{\text{ref}} \in W^{1,\infty}(\mathbb{R}_0)$. Then there exists a unique solution $p$ of (2.6) with (4.1) on $\mathbb{R}_0$ such that

(i) $p \in L^{\infty}(0, \infty; \mathcal{H})$ and

(ii) for the output $y$ defined in (1.6) and $P_0 := \int_{-\infty}^{\infty} p_0(x)dx$ we have that

$$\forall t \geq 0 : \ y(t) = P_0 y_{\text{ref}}(t) + \left(y(0) - P_0 y_{\text{ref}}(0)\right)e^{-\gamma t}.$$

The proof can be found in Appendix A.

We like to emphasize that the feasibility result of Proposition 4.1 is independent of the initial value $p_0 \in \mathcal{H}$. Moreover, if $p_0$ satisfies $\int_{-\infty}^{\infty} p_0(x)dx = 1$, then the control (4.1) achieves exponential convergence of the tracking error $e(t) = y(t) - y_{\text{ref}}(t)$ to zero for all initial probability densities. Furthermore, the solution $p$ exhibits the properties derived in Proposition 3.3; thus its mean value and variance exponentially converge to $y_{\text{ref}}$ and $\frac{c}{\gamma}$, resp.

Although the controller (4.1) requires knowledge of $\gamma$ and $y_{\text{ref}}$ and the absence of disturbances, its simplicity may justify its application in real-world examples. On the other hand, in the presence of uncertainties and disturbances, a feedback control strategy is more suitable, for which we refer to Section 6.

5. Energy estimate. In this section we consider the Galerkin approximation of a possible solution of (2.6) under the influence of a disturbance (the definition of a solution in this case will be given Section 6) and show that these approximations satisfy a certain energy estimate. These arguments follow the spirit of [16, Sec. 7.1]. Let $n \in \mathbb{N}$, $p_0 \in \mathcal{H}$ and

$$d \in L^{\infty}(0, \infty; \mathcal{H}), \quad u \in C(\mathbb{R}_0; \mathbb{R}) \cap L^{\infty}(0, \infty), \quad \mu_i \in C(\mathbb{R}_0; \mathbb{R}), \quad i = 0, \ldots, n,$$

such that $\mu_i$ is absolutely continuous on every interval $[0, T]$, $T > 0$, the disturbance has “zero mass” in the sense

$$\int_{-\infty}^{\infty} d(t, x)dx = 0 \quad \text{for almost all } t \geq 0 \quad (5.1)$$
and with
\[ p_n(t) := \sum_{i=0}^{n} \mu_i(t)z_i \in \mathfrak{B}, \quad t \geq 0, \]  
we have, for all \( i = 0, \ldots, n, \)
(A) \( \mu_i(0) = \langle p_0, z_i \rangle_{\mathfrak{B}}, \)
(B) \( \dot{\mu}_i(t) = -ca\langle p_n(t), z_i \rangle - \langle \mathfrak{B}(p_n(t), u(t)), z_i \rangle_{\mathfrak{B}} \) for almost all \( t \geq 0, \)
where \( \dot{p}_n(t) = \sum_{i=0}^{n} \mu_i(t)z_i \) and \( (z_j)_{j \in \mathbb{N}_0} \) are the eigenfunctions of \( \mathfrak{B} \) from Proposition 2.3.

Observe that, compared to Section 3, here \( d \) acts as an additive disturbance on the Fokker-Planck equation (2.6). Although, admittedly, the set of disturbances \( d \in L^{\infty}(0, \infty; \mathfrak{H}) \) satisfying (5.1) is chosen such that “things work out”, it is nevertheless very large: If \( d \) is chosen such that
\[ d(t) \in \left( \text{span} \{ z_0 \} \right)^\perp \subset \mathfrak{H} \quad \text{for almost all} \; t \geq 0, \]
then \( \int_{-\infty}^{\infty} d(t, x)dx = \frac{1}{c_0} \langle d(t), z_0 \rangle_{\mathfrak{H}} = 0, \) i.e., (5.1) is satisfied. Thus, \( d(t) \) is only restricted to the orthogonal complement of a one-dimensional subspace of \( \mathfrak{H}. \)

**Proposition 5.1.** Under the assumptions stated above, for all \( T > 0 \) there exists a constant \( C > 0 \) which only depends on \( T, p_0, d, u \) and \( c, \gamma, c \) such that
\[ \forall n \in \mathbb{N} : \sup_{t \geq 0} \left| p_n(t) \right|_{\mathfrak{H}} + \left| p_n \right|_{L^2(0, T; \mathfrak{H})} + \left| \dot{p}_n \right|_{L^2(0, T; \mathfrak{H})} \leq C. \]
Furthermore, \( \mu_0 \) is constant and \( \mu_i \in L^{\infty}(0, \infty) \) for all \( i \in \mathbb{N}. \)

**Proof.** Let \( T > 0 \) and \( n \in \mathbb{N}. \) Define \( d_i(t) := \langle d(t), z_i \rangle_{\mathfrak{H}} \) for all \( t \geq 0 \) and \( i = 0, \ldots, n \) and observe that
\[ d_0(t) = \langle d(t), z_0 \rangle_{\mathfrak{H}} = c_0 \int_{-\infty}^{\infty} d(t, x)dx = 0 \quad \text{for almost all} \; t \geq 0. \]

We proceed in several steps.

**Step 1:** We show that \( \mu_0 \) is constant and \( \mu_i \in L^{\infty}(0, \infty) \) for all \( i \in \mathbb{N}. \) Observe that by (B) and Proposition 2.3 we have for all \( i = 1, \ldots, n \) that
\[ \dot{\mu}_i(t) = \langle \dot{p}_n(t), z_i \rangle_{\mathfrak{H}} = -ca\langle p_n(t), z_i \rangle - \langle \mathfrak{B}(p_n(t), u(t)), z_i \rangle_{\mathfrak{H}} + \langle d(t), z_i \rangle_{\mathfrak{H}} \]
\[ = -c\lambda_i \langle p_n(t), z_i \rangle - u(t) \sum_{j=0}^{n} \mu_j(t)\langle z_j, z_i \rangle_{\mathfrak{H}} + d_i(t) \]
\[ = -c\lambda_i \sum_{j=0}^{n} \mu_j(t)\langle z_j, z_i \rangle_{\mathfrak{H}} + u(t) \sum_{j=0}^{n} \sqrt{\lambda_{j+1}} \mu_j(t)\langle z_j, z_i \rangle_{\mathfrak{H}} + d_i(t) \]
\[ = -c\lambda_i \mu_i(t) + u(t) \sqrt{\lambda_{i-1}} \mu_{i-1}(t) + d_i(t) \]
for almost all \( t \geq 0. \) Furthermore, for \( i = 0 \) we have that
\[ \dot{\mu}_0(t) = -c\lambda_0 \sum_{j=0}^{n} \mu_j(t)\langle z_j, z_0 \rangle_{\mathfrak{H}} + u(t) \sum_{j=0}^{n} \sqrt{\lambda_{j+1}} \mu_j(t)\langle z_j, z_0 \rangle_{\mathfrak{H}} + d_0(t) \]
\[ = 0 \quad \text{for all} \; t \geq 0. \]

As a consequence \( \mu_0(t) = \langle p_0, z_0 \rangle_{\mathfrak{H}} \) for all \( t \geq 0 \) and is hence bounded. A simple induction, invoking \( \lambda_i > 0 \) for \( i \in \mathbb{N} \), \( u \in L^{\infty}(0, \infty) \) and \( |d_i(t)| \leq \|d(t)\|_{\mathfrak{H}} \leq \|d\|_{L^{\infty}(0, \infty; \mathfrak{H})}, t \geq 0, \) then shows that \( \mu_i \in L^{\infty}(0, \infty) \) for all \( i \in \mathbb{N}_0. \)

**Step 2:** We show that \( \sup_{t \geq 0} \|p_n(t)\|_{\mathfrak{H}} \leq C_1 \) for \( C_1 > 0 \) independent of \( n \) and \( T. \) First observe that
\[ \forall t \geq 0 : \|p_n(t)\|_{\mathfrak{H}}^2 = \sum_{i=0}^{n} \mu_i(t)^2. \]
We calculate that, writing $\|d\|_\infty := \|d\|_{L^\infty(0,\infty;\mathcal{B})}$ for brevity,

$$\frac{1}{2} \frac{d}{dt} \|p_n(t)\|_\mathcal{B}^2 = \sum_{i=0}^{n} \mu_i(t) \dot{\mu}_i(t) = -c \sum_{i=1}^{n} \frac{1}{2} \|u\|_\infty \sqrt{\lambda_i} \mu_i(t) + \sum_{i=1}^{n} \mu_i(t) d_i(t) \leq \sum_{i=1}^{n} \left( -c \lambda_i \mu_i(t)^2 + \frac{1}{2} \|u\|_\infty \sqrt{\lambda_i} \left( \mu_i(t)^2 + \mu_{i-1}(t)^2 \right) \right) + \langle d(t), p_n(t) \rangle_{\mathcal{B}}$$

$$\leq \sum_{i=1}^{n} \left( -c \lambda_i + \frac{1}{2} \|u\|_\infty \left( \sqrt{\lambda_i} + \sqrt{\lambda_{i+1}} \right) \right) \mu_i(t)^2 + \frac{1}{2} \|u\|_\infty \sqrt{\lambda_i} \mu_i^0 + \|d\|_\infty \|p_n(t)\|_{\mathcal{B}}$$

$$\leq \sum_{i=1}^{n} \left( -c \lambda_i + \|u\|_\infty \left( \sqrt{\lambda_i} + \frac{\theta}{\sqrt{2}} \right) \right) \mu_i(t)^2 + \frac{1}{2} \|u\|_\infty \sqrt{\lambda_i} \mu_i^0 + \|d\|_\infty \|p_n(t)\|_{\mathcal{B}}.$$

Set $\hat{c}_1 := \frac{1}{2} \|u\|_\infty \sqrt{\lambda_i} \mu_i^0$ and $\beta := -c \lambda_i + \|u\|_\infty \left( \sqrt{\lambda_i} + \frac{\theta}{\sqrt{2}} \right)$ for $i \in \mathbb{N}$ as well as $\beta_0 := 0$. Let $i_0 \in \mathbb{N}$ be the smallest index such that $\beta_{i_0} < 0$. Set

$$\hat{c}_2 := \sum_{i=0}^{i_0-1} (\beta_i - \beta_{i_0}) \|\mu_i\|_\infty^2 + \hat{c}_1,$$

then

$$\frac{1}{2} \frac{d}{dt} \|p_n(t)\|_{\mathcal{B}}^2 \leq \sum_{i=1}^{n} \beta_i \mu_i(t)^2 + \hat{c}_1 + \|d\|_\infty \|p_n(t)\|_{\mathcal{B}}$$

$$\leq \sum_{i=1}^{i_0-1} \beta_i \mu_i(t)^2 + \beta_{i_0} \sum_{i=i_0}^{n} \mu_i(t)^2 + \hat{c}_1 + \|d\|_\infty \|p_n(t)\|_{\mathcal{B}}$$

$$= \sum_{i=0}^{i_0-1} (\beta_i - \beta_{i_0}) \mu_i(t)^2 + \beta_{i_0} \sum_{i=0}^{n} \mu_i(t)^2 + \hat{c}_1 + \|d\|_\infty \|p_n(t)\|_{\mathcal{B}}$$

$$\leq \beta_{i_0} \sum_{i=0}^{n} \mu_i(t)^2 + \hat{c}_2 - \frac{\beta_{i_0}}{2} \|p_n(t)\|_{\mathcal{B}}^2 - \frac{1}{2} \frac{d}{dt} \|p_n(t)\|_{\mathcal{B}}^2 = \frac{\beta_{i_0}}{2} \|p_n(t)\|_{\mathcal{B}}^2 + \hat{c}_3$$

for almost all $t \geq 0$, where $\hat{c}_3 := \hat{c}_2 - \frac{1}{2} \beta_{i_0} \|d\|_\infty^2$. Then Grönwall’s lemma implies that

$$\forall t \geq 0 : \|p_n(t)\|_{\mathcal{B}}^2 \leq \|p_n(0)\|_{\mathcal{B}}^2 e^{\beta_{i_0} t} + \int_0^t 2 \hat{c}_3 e^{\beta_{i_0} (t-s)} ds$$

$$\leq \sum_{i=0}^{n} \langle p_0, z_i \rangle_{\mathcal{B}}^2 - \frac{2 \hat{c}_3}{\beta_{i_0}} \leq \|p_0\|_{\mathcal{B}}^2 - \frac{2 \hat{c}_3}{\beta_{i_0}} =: C_1^2,$$

where the last inequality follows from Parseval’s identity.

**Step 3:** We show that $\|p_n\|_{L^2(0,T;\mathcal{B})} \leq C_2$ for $C_2 > 0$ independent of $n$. This is a direct consequence of Step 2 as

$$\|p_n\|_{L^2(0,T;\mathcal{B})} = \left( \int_0^T \|p_n(t)\|_{\mathcal{B}}^2 dt \right)^{1/2} \leq C_1 \sqrt{T} =: C_2.$$

**Step 4:** We show that $\|p_n\|_{L^2(0,T;\mathcal{B})} \leq C_3$ for $C_3 > 0$ independent of $n$. Multiplying
with $\mu_i(t)$ in condition (B) and summing for $i = 0, \ldots, n$ gives
\[
\frac{1}{2} \frac{d}{dt} \|p_n(t)\|_\beta^2 = \langle p_n(t), p_n(t) \rangle_\beta
\]
\[
= -ca(p_n(t), p_n(t)) - (\mathfrak{B}(p_n(t), u(t)), p_n(t))_\beta + (d(t), p_n(t))_\beta
\]
\[
= -c\|p_n(t)\|_\beta^2 + c\|p_n(t)\|_\beta^2 - u(t)p_n'(t), p_n(t))_\beta + (d(t), p_n(t))_\beta
\]
\[
\leq -c\|p_n(t)\|_\beta^2 + c\|p_n(t)\|_\beta^2 + \|u\|_\infty \|p_n'(t)\|_\beta \|p_n(t)\|_\beta + \|d\|_\infty \|p_n(t)\|_\beta
\]
\[
\leq -c\|p_n(t)\|_\beta^2 + c\|p_n(t)\|_\beta^2 + c\|p_n(t)\|_\beta^2 + \frac{1}{2} \|p_n(t)\|_\beta^2 + \frac{1}{2} \|p_n(t)\|_\beta^2
\]
\[
+ \frac{1}{2} \|p_n(t)\|_\beta^2 + \frac{1}{2} \|d\|_\infty^2.
\]
for almost all $t \geq 0$ and hence
\[
\frac{d}{dt} \|p_n(t)\|_\beta^2 + c\|p_n(t)\|_\beta^2 \leq \left(1 + 2c + \frac{1}{c}(1 + 2\sqrt{1 + \theta^2})^2 \|u\|_\infty^2\right) \|p_n(t)\|_\beta^2 + \|d\|_\infty^2.
\]
Integration gives
\[
\|p_n\|_{L^2(0,T;\mathfrak{B})}^2 = \int_0^T \|p_n(t)\|_\beta^2 dt
\]
\[
\leq \frac{1}{c} \|p_n(0)\|_\beta^2 + \left(1 + 2 + \frac{\|u\|_\infty^2}{c^2}(1 + 2\sqrt{1 + \theta^2})^2\right) \int_0^T \|p_n(t)\|_\beta^2 dt + \frac{\|d\|_\infty^2}{c} T
\]
\[
\leq \frac{1}{c} \|p_0\|_\beta^2 + \left(1 + 2 + \frac{\|u\|_\infty^2}{c^2}(1 + 2\sqrt{1 + \theta^2})^2\right) C_n^2 + \frac{\|d\|_\infty^2}{c} T =: C_n^2.
\]

Step 5: We show that $\|\hat{p}_n\|_{L^2(0,T;\mathfrak{B})} \leq C_4$ for $C_4 > 0$ independent of $n$. To this end, let $v \in \mathfrak{B}$ with $\|v\|_\mathfrak{B} \leq 1$ and write $v = v^1 + v^2$ with
\[
v^1 \in \mathfrak{B}_n := \text{span}\{z_0, \ldots, z_n\} \subset \mathfrak{B} \quad \text{and} \quad v^2 \in \mathfrak{B}_n^\perp \subset \mathfrak{B}
\]
by the orthogonal decomposition $\mathfrak{B} = \mathfrak{B}_n \oplus \mathfrak{B}_n^\perp$. Therefore, we have
\[
\forall i = 0, \ldots, n : \langle v^2, z_i \rangle_\mathfrak{B} = 0.
\]
Moreover, for all $i \in \{0, \ldots, n\}$ we compute
\[
0 = \langle v^2, z_i \rangle_\mathfrak{B} = \langle v^2, z_i \rangle_\beta + a(v^2, z_i)
\]
and since $a(v^2, z_i) = \lambda_i \langle v^2, z_i \rangle_\beta$ by Proposition 2.3 we obtain
\[
0 = (1 + \lambda_i)\langle v^2, z_i \rangle_\beta,
\]
whence $\langle v^2, z_i \rangle_\beta = 0$ for all $i = 0, \ldots, n$. Now, $\|v^1\|_\mathfrak{B} \leq \|v\|_\mathfrak{B} \leq 1$ and since $\hat{p}_n(t) = \sum_{i=0}^n \hat{\mu}_i(t) z_i$ we obtain
\[
\langle \hat{p}_n(t), v \rangle_\beta = \langle \hat{p}_n(t), v^1 \rangle_\beta = \langle p_n(t), v^1 \rangle_\beta
\]
\[
\leq -ca(p_n(t), v^1) - u(t)\langle p_n'(t), v^1 \rangle_\beta + (d(t), v^1)\beta
\]
\[
= -c(p_n(t), v^1) + c(p_n(t), v^1)\beta + u(t)\langle p_n'(t), v^1 \rangle_\beta + (d(t), v^1)\beta.
\]
Therefore,
\[
\|\langle \hat{p}_n(t), v \rangle_\beta\| \leq c\|p_n(t)\|_\mathfrak{B}\|v^1\|_\mathfrak{B} + c\|p_n(t)\|_\mathfrak{B} - u(t)p_n'(t) + d(t)\|\|v^1\|_\beta
\]
\[
\leq c\|p_n(t)\|_\mathfrak{B} + c\|p_n(t)\|_\beta + c\|u\|_\infty \|p_n'(t)\|_\beta + \|d\|_\infty
\]
\[
\leq C_4.
\]
for almost all \( t \geq 0 \) and hence
\[
\|\hat{p}_n(t)\|_{\mathcal{V}'} = \sup_{v \in \mathcal{V}, \|v\|_{\mathcal{V}} \leq 1} |\langle \hat{p}_n(t), v \rangle_{\mathcal{B}}| \leq \hat{C}_4\|p_n(t)\|_{\mathcal{V}} + c\|d\|\infty.
\]
Finally, this implies
\[
\|\hat{p}_n\|_{L^2(0,T;\mathcal{V}')}^2 = \int_0^T \|\hat{p}_n(t)\|_{\mathcal{V}'}^2 dt \leq 2\hat{C}_4^2 \int_0^T \|p_n(t)\|_{\mathcal{V}}^2 dt + 2c^2\|d\|\infty^2 T
\]
\[
\leq 2\hat{C}_4^2 C_3^2 + 2c^2\|d\|\infty^2 T =: C_4^2,
\]
which finishes the proof.

6. Funnel control. The controller that we propose in order to achieve the control objective formulated in Subsection 1.4 is the funnel controller. It has the advantage that it is model-free, i.e., we may state the control law without any further information about the equation (1.4). Therefore, it is inherently robust and hence able to handle both uncertainties in the system parameters as well as disturbances in the PDE itself. In particular, we do not need any knowledge of the parameters \( c > 0 \) and \( \gamma > 0 \), or of the initial probability density \( p_0(\cdot) \). Utilizing the version from [7], we only require the relative degree in order to state the appropriate control law. For finite dimensional systems we refer to [27] for a definition of the relative degree; this notion can be extended to systems with infinite-dimensional internal dynamics, see e.g. [10]. However, for general infinite-dimensional systems a concept of relative degree is not available. Nevertheless, for the system (2.6), (1.6) we may calculate – formally at this point –
\[
\dot{y}(t) = \int_{-\infty}^\infty x \hat{p}(t, x) dx \cdot \frac{e^{-\mu y(t)}}{2\mu} \mu(-c\mu p(t) - u(t)p'(t), z_1)_{\mathcal{B}}
\]
\[
= -c\mu(p(t), z_1) - \mu u(t) \langle p'(t), z_1 \rangle_{\mathcal{B}}
\]
\[
= -c\lambda_1(p(t), z_1) + \mu \sqrt{\lambda_1} u(t) \langle p(t), z_0 \rangle_{\mathcal{B}} = -\gamma y(t) + \eta u(t),
\]
where \( \mu = \frac{1}{2\mu}, \eta = \mu \sqrt{\lambda_1}(p_0, z_0)_{\mathcal{B}} \) and we have used that \( \langle p(t), z_0 \rangle_{\mathcal{B}} = \alpha_0 \int_{-\infty}^\infty p(t, x) dx = \alpha_0 \int_{-\infty}^\infty p_0(x) dx = \langle p_0, z_0 \rangle_{\mathcal{B}} \) by Proposition 3.3. The input appears explicitly in the above equation for \( \dot{y} \), which suggests that (2.6), (1.6) at least exhibits an input-output behavior similar to that of a relative degree one system. This justifies to investigate the application of the funnel controller
\[(6.1) \quad u(t) = -k(t)e(t), \quad e(t) = y(t) - y_{ref}(t), \quad k(t) = \frac{1}{1 - \varphi(t)^2} = \frac{1}{1 - \varphi(t)^2} e(t)^2
\]
to (2.6), (1.6), where \( \varphi \in \Phi \). We stress that the control gain \( k \) in (6.1) is not dynamically generated and it is not monotone. It is only large when the error \( e(t) \) is close to the funnel boundary \( 1/\varphi(t) \) at some time \( t > 0 \).

For feasibility we seek to show that for any \( y_{ref} \in W^{1,\infty}(\mathbb{R}_{\geq 0}) \), \( \varphi \in \Phi \) and any initial probability density \( p_0 \) such that \( \varphi(0)|e(0)| < 1 \) we have that the closed-loop system consisting of (2.6), (1.6) and (6.1) has a global and bounded solution \( p \) which satisfies the conditions (1.3) and the tracking error \( e \) evolves uniformly within the funnel boundaries, i.e.,
\[
\exists \varepsilon > 0 \forall t > 0 : |e(t)| \leq \varphi(t)^{-1} - \varepsilon.
\]
Hence, even if a solution exists on a finite time interval \( [0, T] \), it is not clear that it can be extended to a global solution. Moreover, the closed-loop system (2.6), (1.6) and (6.1) is a time-varying and nonlinear PDE. This renders the solution of the above problem a challenging task.
Furthermore, we study the robustness of the controller (6.1) w.r.t. disturbances $d \in L^\infty(0,\infty;\mathcal{F})$ that satisfy the zero-mass condition (5.1) and influence the Fokker-Planck equation (2.6) in an additive way, cf. also Section 5. That is, we consider the system

\[
\begin{align*}
\dot{p}(t, x) &= -cA p(t, x) - \mathcal{B} \left( p(t, \cdot), u(t) \right)(x) + d(t, x), \quad \text{in } (0, \infty) \times \mathbb{R}, \\
p(0, x) &= p_0(x), \quad \text{in } \mathbb{R},
\end{align*}
\]

where $a : \mathcal{W} \times \mathcal{W} \to \mathbb{R}$ is the sesquilinear form defined in (2.3) and $\mathcal{B} : \mathcal{W} \times \mathbb{R} \to \mathcal{H}$ is the operator defined in (2.5). Note that, in the presence of disturbances, it cannot be expected that the solution $p(t)$ is a probability density function for any $t \geq 0$ in general, i.e., conditions (1.3) will typically not hold. We introduce solutions of the closed-loop system resulting from the application of the funnel controller (6.1) to system (6.2) with output (1.6) as follows.

**Definition 6.1.** Consider the system (6.2) and let $p_0 \in \mathcal{H}$, $y_{\text{ref}} \in W^{1,\infty}(\mathbb{R}_{\geq 0})$, $\varphi \in \Phi$ and $d \in L^\infty(0, \infty; \mathcal{F})$ that satisfies (5.1). For $T > 0$, a triple of functions $(p, u, y)$ is called solution of the closed-loop system (6.2), (1.6), (6.1) on $[0, T]$, if

(i) $p \in L^2(0, T; \mathcal{W}) \cap C([0, T]; \mathcal{F})$ with $p(0) = p_0$,
(ii) $p$ is weakly differentiable in $L^1(0, T; \mathcal{W}')$ and satisfies $\dot{p} \in L^2(0, T; \mathcal{W}')$,
(iii) for all $v \in \mathcal{W}$ and almost all $t \in [0, T]$ we have

\[
\langle \dot{p}(t), v \rangle_{\mathcal{H}} = -cA \langle p(t), v \rangle - \langle \mathcal{B} \left( p(t), u(t) \right), v \rangle_{\mathcal{H}} + \langle d(t), v \rangle_{\mathcal{H}},
\]

(iv) $u, y \in C([0, T]; \mathbb{R})$ and (1.6), (6.1) hold for all $t \in [0, T]$.

The triple $(p, u, y)$ is called solution of (6.2), (1.6), (6.1) on $\mathbb{R}_{\geq 0}$, if $(p, u, y)|_{[0, T]}$ is a solution of (6.2), (1.6), (6.1) on $[0, T]$ for all $T > 0$, in particular $p \in L^2_{\text{loc}}(0, \infty; \mathcal{W})$ and $\dot{p} \in L^2_{\text{loc}}(0, \infty; \mathcal{W}')$.

In the following main result of the present paper we prove feasibility of funnel control for the Fokker-Planck equation corresponding to the Ornstein-Uhlenbeck process.

**Theorem 6.2.** Use the assumptions from Definition 6.1, let $E_0 := \int_{-\infty}^{\infty} x p_0(x)dx$ and assume that

\[
\int_{-\infty}^{\infty} p_0(x)dx > 0 \quad \text{and} \quad \varphi(0)|E_0 - y_{\text{ref}}(0)| < 1.
\]

Then there exists a unique solution $(p, u, y)$ of (6.2), (1.6), (6.1) on $\mathbb{R}_{\geq 0}$ which satisfies

(i) $p \in L^\infty(0, \infty; \mathcal{F})$, $u, y \in W^{1,\infty}(\mathbb{R}_{\geq 0})$ and
(ii) $\exists \varepsilon > 0 \forall t > 0 : |e(t)| \leq \varphi(t)^{-1} - \varepsilon$.

If $d = 0$, then $p$ has the additional properties derived in Proposition 3.3.

**Proof.** We divide the proof into several steps.

**Step 1:** We construct a Galerkin approximation of a solution candidate, which has the properties as in Section 5. To this end, fix $n \in \mathbb{N}$, consider the eigenvalues $(\lambda_j)_{j \in \mathbb{N}_0}$ and the system of eigenfunctions $(z_j)_{j \in \mathbb{N}_0}$ of $\mathcal{A}$ from Propositions 2.2 and 2.3, and define $\mu_0(t) := \langle p_0, z_0 \rangle_{\mathcal{F}}$ as well as $d_1(t) := \langle d(t), z_1 \rangle_{\mathcal{F}}$ for $t \geq 0$ and $i \in \mathbb{N}_0$. Furthermore, consider the one-dimensional initial value problem

\[
\begin{align*}
\dot{\mu}_1(t) &= -c\lambda_1 \mu_1(t) + \sqrt{\lambda_1} \mu_0 u(t) + d_1(t), \quad \mu_1(0) = \langle p_0, z_1 \rangle_{\mathcal{F}}, \\
u(t) &= -\frac{\delta \mu_1(t) - y_{\text{ref}}(t)}{1 - \varphi(t)^2 (\delta \mu_1(t) - y_{\text{ref}}(t))^2}, \quad \delta := \sqrt{\frac{\pi}{2\varphi^2}}.
\end{align*}
\]

Observe that $\mu_0 = \alpha_0 \int_{-\infty}^{\infty} p_0(x)dx > 0$ by assumption, $|d_1(t)| \leq \|d(t)\|_{\mathcal{F}} \leq \|d\|_{L^\infty(0, \infty; \mathcal{F})}$ for almost all $t \geq 0$ and

\[
\delta \mu_1(0) = \langle p_0, \delta z_1 \rangle_{\mathcal{F}} = \int_{-\infty}^{\infty} x p_0(x)dx = E_0.
\]
hence \(\varphi(0)|\delta \mu_1(0) - y_{\text{ref}}(0)| < 1\). Therefore, existence of a solution to (6.3) follows from [7, Thm. 3.1], that is there exists a function \(\mu_1 \in C([0,T]; \mathbb{R})\) which is absolutely continuous on \([0,T]\) for all \(T > 0\) and satisfies the initial condition and differential equation in (6.3) for almost all \(t \geq 0\). Moreover, we have that \(u \in C([0,T]; \mathbb{R})\) and \(\mu_1, u \in L^\infty([0,T])\) as well as the estimate

\[\exists \varepsilon > 0 \forall t > 0: |\delta \mu_1(t) - y_{\text{ref}}(t)| \leq \varphi(t)^{-1} - \varepsilon.\]

Now consider the system of \(n-1\) ordinary differential equations given by

\[\dot{\mu}_i(t) = -c_i \lambda_i(t) + \sqrt{\lambda_i} \mu_{i-1}(t) u(t) + d_i(t), \quad \mu_i(0) = (p_0,z_i)_{\Sigma}, \quad i = 2, \ldots, n.\]

It follows from the theory of ordinary differential equations, see e.g. [40, §10, Thm. XX], that there exists a solution \((\mu_2, \ldots, \mu_n) \in C([0,\omega); \mathbb{R}^{n-1})\) with maximal \(\omega \in (0,\infty)\), which is absolutely continuous on \([0,T]\) for all \(0 < T < \omega\) and satisfies the initial condition and differential equation in (6.4) for almost all \(t \in [0,\omega]\). Furthermore, the closure of the graph of \((\mu_2, \ldots, \mu_n)\) is not a compact subset of \(\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}\).

Since \(\mu_1, u \in L^\infty([0,T])\) and \(d_i \in L^\infty([0,T])\) by \(\text{ess sup}_{t \geq 0} |d_i(t)| \leq \|d\|_{L^\infty([0,\omega];\Sigma)}\) for \(i = 1, \ldots, n\), it follows from a simple induction that \(\mu_i \in L^\infty([0,\omega]\) for \(i = 2, \ldots, n\). Now assume that \(\omega < \infty\). Then, since \((\mu_2, \ldots, \mu_n)\) is bounded, the closure of the graph of \((\mu_2, \ldots, \mu_n)\) is a compact subset of \(\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}\), a contradiction. Therefore, \(\omega = \infty\). Moreover, by boundedness of \(u, \mu_1, \ldots, \mu_n\) and \(d_1, \ldots, d_n\) it follows from (6.3), (6.4) that \(\mu_1 \in W^{1,\infty}(\mathbb{R}_{\geq 0})\) for \(i = 1, \ldots, n\). Finally, since \(y_{\text{ref}}, \varphi \in W^{1,\infty}(\mathbb{R}_{\geq 0})\) and \(\varphi(0)^2 (\delta \mu_1(0) - y_{\text{ref}}(0)) \leq 1 - \varepsilon_0\) for some \(\varepsilon_0 > 0\), it follows that \(u \in W^{1,\infty}(\mathbb{R}_{\geq 0})\). In particular, the Galerkin approximation \(p_n(t)\) as in (5.2) satisfies the properties (A) and (B) in Section 5, cf. also Step 1 of the proof of Proposition 5.1.

**Step 2:** We show that there exists a solution \((\rho, u, y)\) of (6.2), (1.6), (6.1) on \(\mathbb{R}_{\geq 0}\), where \(u\) is the function defined in Step 1. Fix \(T > 0\) and observe that by Step 1 and Proposition 5.1 the sequence \((p_n)_{n \in \mathbb{N}}\) satisfies the energy estimate (5.3). Therefore, the sequence \((p_n)_{n \in \mathbb{N}}\) is both bounded in \(L^2(0,T;\Sigma)\) and \(L^2(0,T;\mathfrak{V})\), and the sequence \((\dot{p}_n)_{n \in \mathbb{N}}\) is bounded in \(L^2(0,T;\mathfrak{V}')\). Hence there exists \(p \in L^2(0,T;\Sigma)\) with \(\dot{p} \in L^2(0,T;\mathfrak{V})\) and subsequences of \((p_n)_{n \in \mathbb{N}}\) and \((\dot{p}_n)_{n \in \mathbb{N}}\), resp., again denoted in the same way, such that

\[p_n \to p \quad \text{weakly in } L^2(0,T;\Sigma),\]

\[p_n \to p \quad \text{weakly in } L^2(0,T;\mathfrak{V}),\]

\[\dot{p}_n \to \dot{p} \quad \text{weakly in } L^2(0,T;\mathfrak{V}').\]

We recall that these properties mean that

\[\forall \chi \in L^2(0,T;\Sigma): \lim_{n \to \infty} \int_0^T \langle p_n(t), \chi(t) \rangle_{\Sigma} dt = \int_0^T \langle p(t), \chi(t) \rangle_{\Sigma} dt,\]

\[\forall \chi \in L^2(0,T;\mathfrak{V}): \lim_{n \to \infty} \int_0^T \langle p_n(t), \chi(t) \rangle_{\mathfrak{V}} dt = \int_0^T \langle p(t), \chi(t) \rangle_{\mathfrak{V}} dt,\]

\[\forall \chi \in L^2(0,T;\mathfrak{V}'): \lim_{n \to \infty} \int_0^T \langle \dot{p}_n(t), \chi(t) \rangle_{\mathfrak{V}'} dt = \int_0^T \langle \dot{p}(t), \chi(t) \rangle_{\mathfrak{V}'} dt.\]

Invoking the Riesz isomorphism \(\mathcal{R}: \mathfrak{V}' \to \mathfrak{V}\) we find that for any \(\chi \in L^2(0,T;\mathfrak{V}'),\) we have that \(\rho(\cdot) := \mathcal{R}(\chi(\cdot))\) satisfies \(\rho \in L^2(0,T;\mathfrak{V})\) and

\[\langle \dot{p}(t), \chi(t) \rangle_{\mathfrak{V}} = \langle \mathcal{R}(\dot{p}(t)), \rho(t) \rangle_{\mathfrak{V}} = \langle \dot{p}(t), \rho(t) \rangle_{\mathfrak{V} \times \Sigma} = \langle \dot{p}(t), \rho(t) \rangle_{\Sigma}.\]

This gives that

\[\forall \rho \in L^2(0,T;\mathfrak{V}): \lim_{n \to \infty} \int_0^T \langle \dot{p}_n(t), \rho(t) \rangle_{\Sigma} dt = \int_0^T \langle \dot{p}(t), \rho(t) \rangle_{\Sigma} dt.\]
Step 2a: We show that \( p \) satisfies Definition 6.1 (iii). Let \( m \in \mathbb{N} \) be arbitrary and set \( \mathfrak{U}_m := \text{span}\{z_0, \ldots, z_m\} \). Further let \( \psi \in C_c^\infty((0, T)) \) and \( v \in \mathfrak{U}_m \). Then \( \psi(\cdot)v \in L^2(0, T; \mathfrak{U}) \) and we have, invoking property (B) from Section 5, that

\[
\int_0^T \langle \dot{p}(t), \psi(t)v \rangle_{\mathcal{S}} \, dt = \lim_{n \to \infty} \int_0^T \langle \bar{p}_n(t), \psi(t)v \rangle_{\mathcal{S}} \, dt
\]

\[
= \lim_{n \to \infty} \int_0^T -ca(p_n(t), \psi(t)v) - \langle \mathfrak{B}(p_n(t), u(t)), \psi(t)v \rangle_{\mathcal{S}} + \langle d(t), \psi(t)v \rangle_{\mathcal{S}} \, dt
\]

\[
= \lim_{n \to \infty} \int_0^T -c(p_n(t), \psi(t)v) + c(p_n(t), \psi(t)v) - \langle p_n'(t), \psi(t)u(t)v \rangle_{\mathcal{S}} + \langle d(t), \psi(t)v \rangle_{\mathcal{S}} \, dt
\]

\[
= \int_0^T -c(p(t), \psi(t)v) + c(p(t), \psi(t)v) + \langle d(t), \psi(t)v \rangle_{\mathcal{S}} \, dt - \lim_{n \to \infty} \int_0^T \langle p_n'(t), \psi(t)u(t)v \rangle_{\mathcal{S}} \, dt.
\]

Observe that \( \psi(\cdot)u(\cdot)v \in L^2(0, T; \mathfrak{U}) \) since \( u \in L^\infty(\mathbb{R}_{\geq 0}) \) and we may compute, invoking Proposition 2.3,

\[
\langle p_n'(t), v \rangle_{\mathcal{S}} = \int_{-\infty}^\infty e^\phi(x)p_n'(t, x)v(x) \, dx = -\int_{-\infty}^\infty p_n(t, x)\left( e^\phi(x)v(x) \right)' \, dx
\]

\[
= -\langle p_n(t), e^{-\phi}(e^\phi v)' \rangle_{\mathcal{S}}
\]

for \( n \geq m \) and all \( t \in [0, T] \). Since \( v \in \mathfrak{U} \) it follows that \( e^{-\phi}(e^\phi v)' \in \mathcal{S} \) and hence \( \psi(\cdot)u(\cdot)e^{-\phi}(e^\phi v)' \in L^2(0, T; \mathcal{S}) \), by which

\[
\lim_{n \to \infty} \int_0^T \langle p_n'(t), \psi(t)u(t)v \rangle_{\mathcal{S}} \, dt = \int_0^T \langle p'(t), \psi(t)u(t)v \rangle_{\mathcal{S}} \, dt.
\]

Therefore, we have shown that

\[
\int_0^T \langle \dot{p}(t), v \rangle_{\mathcal{S}} \psi(t) \, dt = \int_0^T \left( -ca(p(t), v) - \langle \mathfrak{B}(p(t), u(t)), v \rangle_{\mathcal{S}} + \langle d(t), v \rangle_{\mathcal{S}} \right) \psi(t) \, dt,
\]

and since \( \psi \in C_c^\infty((0, T)) \) was arbitrary and \( \bigcup_{m \in \mathbb{N}} \mathfrak{U}_m \) is dense in \( \mathfrak{U} \), it follows that (invoking [21, Lem. 1.5]) for all \( v \in \mathfrak{U} \) and almost all \( t \in [0, T] \) we have

\[
\langle \dot{p}(t), v \rangle_{\mathcal{S}} = -ca(p(t), v) - \langle \mathfrak{B}(p(t), u(t)), v \rangle_{\mathcal{S}} + \langle d(t), v \rangle_{\mathcal{S}}.
\]

Step 2b: We show that \( p(0) = p_0 \). To this end, let \( m \in \mathbb{N} \) and \( \psi \in C_c^\infty([0, T]) \) be such that \( \psi(0) = 1 \) and \( \psi(T) = 0 \), but otherwise arbitrary. Then \( \psi(\cdot)v \in L^2(0, T; \mathfrak{U}) \) and \( \dot{\psi}(\cdot)v \in L^2(0, T; \mathfrak{U}') \) for all \( v \in \mathfrak{U}_m \). Similar to Step 2a and again invoking [21, Thm. 1.32] we find that

\[
-\langle p(0), v \rangle_{\mathcal{S}} = \int_0^T \langle \dot{p}(t), v \rangle_{\mathcal{S}} \, dt + \langle \dot{\psi}(t)v, p(t) \rangle_{\mathcal{S}} \, dt
\]

\[
= \lim_{n \to \infty} \int_0^T \langle \bar{p}_n(t), v \rangle_{\mathcal{S}} + \langle \psi(t)v, p_n(t) \rangle_{\mathcal{S}} \, dt = -\lim_{n \to \infty} \langle p_n(0), v \rangle_{\mathcal{S}},
\]

thus

\[
\forall v \in \mathfrak{U}_m : \langle p(0), v \rangle_{\mathcal{S}} = \lim_{n \to \infty} \langle p_n(0), v \rangle_{\mathcal{S}}.
\]

For \( v = \sum_{i=0}^m \alpha_i z_i \) and \( n \geq m \) we may finally compute that

\[
\langle p_n(0), v \rangle_{\mathcal{S}} = \sum_{i=0}^m \mu_i(0)\alpha_i = \sum_{i=0}^m \langle p_0, z_i \rangle_{\mathcal{S}} \alpha_i = \langle p_0, v \rangle_{\mathcal{S}},
\]
hence obtaining $\langle p(0), v \rangle_\delta = \langle p_0, v \rangle_\delta$ for all $v \in \bigcup_{m \in \mathbb{N}} \mathfrak{W}_m$. Since the latter set is dense in $\mathfrak{W}$ we may infer that $p(0) = p_0$.

\textbf{Step 2d:} It remains to show that (1.6), (6.1) are satisfied for all $t \in [0, T]$. To this end, it suffices to show that $y$ as in (1.6) satisfies $y(t) = \delta \mu_1(t)$ for $\delta$ as defined in (6.3). Let $t \in [0, T]$ and observe that we have $p_n \to p$ weakly in $L^2(0, t; \delta)$. Therefore,

$$\int_0^t y(s) ds = \int_0^t \int_{-\infty}^\infty xp(s, x) dx ds = \int_0^t \langle p(s), \delta z_1 \rangle_\delta ds$$

and the fundamental theorem of calculus gives that $y(t) = \delta \mu_1(t)$.

\textbf{Step 3:} We show uniqueness of the solution $(p, u, y)$ on $\mathbb{R}_{\geq 0}$. Assume that $(p^1, u^1, y^1)$ and $(p^2, u^2, y^2)$ are two solutions of (6.2), (1.6), (6.1) on $\mathbb{R}_{\geq 0}$ with the same initial values $p^1(0) = p_0$ and $p^2(0) = p_0$. Then, as in Step 2d, for $i = 1, 2$ we may show that $y^i(t) = \delta \mu^i_1(t)$, $t \geq 0$, where $\mu^i_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is the solution of the initial value problem

$$\mu^i_1(t) = -c \lambda_1 \mu^i_1(t) - \frac{\sqrt{\lambda_1} \mu_0 (\delta \mu^i_1(t) - y_{\text{ref}}(t))}{1 - \varphi(t)^2 (\delta \mu^i_1(t) - y_{\text{ref}}(t))^2} + d_1(t),$$

$$\mu^i_1(0) = \langle p_0, z_1 \rangle_\delta.$$

Since the right hand side of the ordinary differential equation above is measurable in $t$ and locally Lipschitz continuous in $\mu^i_1$, its solution is unique, see e.g. [40, § 10, Thm. XX]. Since $\mu^i_1(0) = \mu^j_1(0)$ this implies that $\mu^i_1(t) = \mu^j_1(t)$ for all $t \geq 0$. Therefore, we have that $y^1(t) = y^2(t)$ and $u^1(t) = u^2(t) =: u(t)$ for all $t \geq 0$. Then $P := p^1 - p^2$ is a solution of (2.6) on $\mathbb{R}_{\geq 0}$ with initial value $P(0) = 0 \in \mathfrak{W}$ and input function $u := u^1$. In particular, $P(t) \in \mathfrak{W}$ for all $t \geq 0$ and hence we find that

$$\langle \dot{P}(t), P(t) \rangle_\delta = -ca(P(t), P(t)) - \langle \mathfrak{W}(P(t), u(t)), P(t) \rangle_\delta$$

for almost all $t \geq 0$. Since $\frac{1}{2} \frac{d}{dt} \|P(t)\|_\delta^2 = \langle P(t), P(t) \rangle_\delta$ by [21, Thm. 1.32] we obtain

$$\frac{1}{2} \frac{d}{dt} \|P(t)\|_\delta^2 = -ca(P(t), P(t)) - u(t)\langle P'(t), P(t) \rangle_\delta$$

$$\leq -c\|P(t)\|_\delta^2 + c\|P(t)\|_\delta^2 + \|u(t)\| \|P'(t)\|_\delta \|P(t)\|_\delta$$

$$\leq -c\|P(t)\|_\delta^2 + c\|P(t)\|_\delta^2 + \frac{c}{2} \|P(t)\|_\delta^2 + \frac{1}{2c}(1 + 2\sqrt{1 + \theta^2})^2 |u(t)|^2 \|P(t)\|_\delta^2$$

$$\leq D(t) \|P(t)\|_\delta^2,$$

where

$$D(t) := c + \frac{1}{2c} (1 + 2\sqrt{1 + \theta^2})^2 |u(t)|^2.$$

Then Grönwall’s lemma, continuity of $u$ on $[0, t]$ and $P(0) = 0$ imply that $P(t) = 0$ for all $t \geq 0$, which proves $p^1 = p^2$.

\textbf{Step 4:} We show that $p \in L^\infty(0, \infty; \delta)$. By Proposition 5.1 and the energy estimate (5.3) the sequence $(p_n)_{n \in \mathbb{N}}$ is bounded in $L^\infty(0, \infty; \delta)$. Hence, there exists $\tilde{p} \in L^\infty(0, \infty; \delta)$ and a subsequence of $(p_n)_{n \in \mathbb{N}}$, again denoted in the same way, such that

$$p_n \to \tilde{p} \quad \text{weak}^* \quad \text{in} \quad L^\infty(0, \infty; \delta).$$
Now let \( \psi \in C^\infty(\mathbb{R}_{\geq 0}) \) and \( v \in \mathcal{Y} \) be arbitrary and choose \( T > 0 \) such that \( \text{supp} \psi \subseteq [0, T] \). Then \( \psi(\cdot)v \in L^1(0, T; \mathcal{Y}) \cap L^2(0, T; \mathcal{Y}) \) and hence

\[
\int_0^\infty \langle \dot{p}(t), (t)v \rangle_H dt = \int_0^T \langle \dot{p}(t), (t)v \rangle_H dt = \lim_{n \to \infty} \int_0^T \langle p_n(t), (t)v \rangle_H dt
\]

Thus \( \langle \dot{p}(t), v \rangle_H = \langle p(t), v \rangle_H \) for almost all \( t \geq 0 \) by [21, Lem. 1.5]. Since \( v \in \mathcal{Y} \) was arbitrary it follows that \( p(t) = \dot{p}(t) \) for almost all \( t \geq 0 \), thus \( p \in L^\infty(0, \infty; \mathcal{Y}) \).

7. A numerical example. In this section, we illustrate the applicability of the funnel controller by means of a numerical example. We simulate the evolution of a given initial probability density \( p_0 \) under the Fokker-Planck equation (6.2) with the mean value as output (1.6) and under the influence of the controller (6.1). To show the universality of Theorem 6.2 we consider an initial density that is in \( \mathcal{H} \), but not in \( \mathcal{Y} \), namely a uniform distribution on \([-1, -\frac{1}{2}] \cup [\frac{1}{2}, \frac{3}{4}] \) given by

\[
p_0 : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \begin{cases} 1, & -1 \leq x \leq -\frac{1}{2} \lor \frac{1}{2} \leq x \leq \frac{3}{4}, \\ 0, & \text{otherwise} \end{cases}
\]

The parameters \( c \) and \( \gamma \) in (1.4) are chosen as \( c = 0.1 \) and \( \gamma = 1 \), the reference signal is \( y_{\text{ref}}(t) = \sin t \) and the funnel function \( \varphi \in \Phi \) is \( \varphi(t) = (2e^{-2t} + 0.1)^{-1} \), \( t \geq 0 \). As disturbance we consider

\[
d : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}, \quad (t, x) \mapsto 3\cos(4t) x e^{-3x^2},
\]

which clearly satisfies \( d \in L^\infty(0, \infty; \mathcal{H}) \) and condition (5.1). Since \( E_0 = \int_{-\infty}^\infty xp_0(x)dx = -\frac{1}{8} \) and \( y_{\text{ref}}(0) = 0 \), it follows that \( \varphi(0)|E_0 - y_{\text{ref}}(0)| = \frac{5}{41} < 1 \). Therefore, feasibility of funnel control, i.e., the application of (6.1) to (6.2), (1.6), is guaranteed by Theorem 6.2.

For the simulation the PDE is solved using a finite difference method with a uniform time grid (in \( t \)) with 10,000 points for the interval \([0, 10]\) and a uniform spatial grid (in \( x \)) with 2,000 points for the interval \([-5, 5]\). The simulation has been performed in MATLAB, where in each time step an ODE is solved by using the command \texttt{pdepe} with (artificial) Dirichlet boundary conditions. Relative and absolute tolerance are set to the default values \( 10^{-3} \) and \( 10^{-6} \), resp. Fig. 7.1 (a) shows the error \( e(t) = y(t) - y_{\text{ref}}(t) \) between mean value and reference signal and the input values \( u(t) \) generated by the controller are depicted in Fig. 7.1 (b). Several snapshots of the solution \( p \), are shown in Fig. 7.1 (c) and (d). It can be seen that, in the presence of disturbances, \( p(t) \) is not a probability density function for \( t > 0 \) in general, since it takes negative values. Nevertheless, the controller guarantees that the error stays within the prescribed funnel boundaries, while the control input shows an acceptable performance.

A simulation of the same configuration, but without disturbance can be seen in Fig. 7.2. Here, the simulations of the undisturbed equation show that \( p(t) \) is always a probability density and its variance exponentially converges to \( \frac{2}{\gamma} = 0.2 \), as stated in Proposition 4.1. Video clips of the simulations showing the evolution of \( p(t) \) for \( t \in [0, 10] \) can be found in the supplementary material.

Appendix A. Proof of Proposition 4.1. 

Proof. The existence and uniqueness of a solution \( p \) of (2.6) with (4.1) on \( \mathbb{R}_{\geq 0} \) that satisfies \( p \in L^\infty(0, \infty; \mathcal{Y}) \) follows along the lines of the proof of Theorem 6.2 by observing that the linear ordinary differential equation

\[
\mu_1(t) = -c_1 \lambda_1 \mu_1(t) + \sqrt{\lambda_1} \mu_0 \left( y_{\text{ref}}(t) + \gamma y_{\text{ref}}(t) \right), \quad \mu_1(0) = \langle p_0, z_1 \rangle_H,
\]
has a unique and bounded solution on $\mathbb{R}_{\geq 0}$. Therefore, it remains to show (ii). To this end, define $g(t) := \mu_1(t) - \sqrt{\lambda_1} \mu_0 y_{ref}(t)$ for $t \geq 0$ and observe that

$$\dot{g}(t) = -c\lambda_1 \left( \mu_1(t) - \frac{\gamma \sqrt{\lambda_1}}{c \lambda_1} \mu_0 y_{ref}(t) \right) = -c\lambda_1 g(t)$$

for almost all $t \geq 0$, since $\lambda_1 = \frac{\gamma}{c}$. Thus we find that

$$\mu_1(t) = \sqrt{\lambda_1} \mu_0 y_{ref}(t) + e^{-c\lambda_1 t} g(0), \quad t \geq 0,$$

and with $\delta$ from (6.3) we obtain

$$y(t) = \delta \mu_1(t) = \frac{\sqrt{\lambda_1} \mu_0}{\lambda_1 \alpha_0} y_{ref}(t) + e^{-\gamma t} \delta g(0), \quad t \geq 0,$$

where we have used $c\lambda_1 = \gamma$. Observing $\delta g(0) = y(0) - \frac{\mu_0}{\alpha_0} y_{ref}(0)$ and $\alpha_0 P_0 = \langle p_0, z_0 \rangle_0 = \mu_0$, this finishes the proof.

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(a) Tracking error and funnel boundary
(b) Input function
(c) Snapshots of the solution $p(t_i)$ for $t_i = 0.025 \cdot i$, $i = 0, \ldots, 60$, from red to black.
(d) Snapshots of the solution $p(t_i)$ for $t_i = 1.5 + 0.025 \cdot i$, $i = 0, \ldots, 60$, from black to turquoise.

Fig. 7.2: Simulation of the controller (6.1) applied to (6.2) with (1.6), but without disturbance, i.e., $d = 0$.

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