The asymptotic distribution of the isotonic regression estimator over a countable pre-ordered set

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Abstract

We study the isotonic regression estimator over a general countable pre-ordered set. We obtain the limiting distribution of the estimator and study its properties: It is proved that, under some general assumptions, the limiting distribution of the isotonised estimator is given by the concatenation of the separate isotonic regressions of the restrictions of an underlying estimator’s asymptotic distribution to the comparable level sets of the underlying estimator’s probability limit. Also, we show that the isotonisation preserves the rate of convergence of the underlying estimator. We apply these results to the problems of estimation of a bimonotone regression function and estimation of a bimonotone probability mass function.

1 Introduction

Let \( \mathcal{X} \) be a countable set \( \{x_1, x_2, \ldots\} \) with \( |\mathcal{X}| \leq \infty \), with a pre-order \( \preceq \) defined on it. We begin with the definitions of the order relations on an arbitrary set \( \mathcal{X} \) and of an isotonic regression over it, c.f.

\[ \text{Definition 1} \] A binary relation \( \preceq \) on \( \mathcal{X} \) is a simple order if

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(i) it is reflexive, i.e. \( x \preceq x \) for \( x \in \mathcal{X} \);

(ii) it is transitive, i.e. \( x_1, x_2, x_3 \in \mathcal{X}, x_1 \preceq x_2 \) and \( x_2 \preceq x_3 \) imply \( x_1 \preceq x_3 \);

(iii) it is antisymmetric, i.e. \( x_1, x_2 \in \mathcal{X}, x_1 \preceq x_2 \) and \( x_2 \preceq x_1 \) imply \( x_1 = x_2 \);

(iv) every two elements of \( \mathcal{X} \) are comparable, i.e. \( x_1, x_2 \in \mathcal{X} \) implies that either \( x_1 \preceq x_2 \) or \( x_2 \preceq x_1 \).

A binary relation \( \preceq \) on \( \mathcal{X} \) is a partial order if it is reflexive, transitive and antisymmetric, but there may be noncomparable elements. A pre-order is reflexive and transitive but not necessary antisymmetric and the set \( \mathcal{X} \) can have noncomparable elements. Note, that in some literature the pre-order is called as a quasi-order.

Let us introduce the notation \( x_1 \sim x_2 \), if \( x_1 \) and \( x_2 \) are comparable, i.e. if \( x_1 \preceq x_2 \) or \( x_2 \preceq x_1 \).

**Definition 2** A function \( f : \mathcal{X} \to \mathbb{R} \) is isotonic if \( x_i, x_j \in \mathcal{X} \) and \( x_i \preceq x_j \) imply \( f(x_i) \leq f(x_j) \).

Let \( \mathcal{F}^{\text{is}} = \mathcal{F}^{\text{is}}(\mathcal{X}) \) denote the family of real valued bounded functions \( f \) on a set \( \mathcal{X} \), which are isotonic with respect to the pre-order \( \preceq \) on \( \mathcal{X} \). In the case when \( |\mathcal{X}| = \infty \) we consider the functions from the space \( \ell^w_2 \), the Hilbert space of real-valued functions on \( \mathcal{X} \), which are square summable with some given non-negative weights \( w = \{w_1, w_2, \ldots \} \), i.e. any \( g \in \ell^w_2 \) satisfies \( \sum_{i=1}^{\infty} g(x_i)^2 w_i < \infty \). We use the same notation \( \mathcal{F}^{\text{is}} \) to denote the functions from \( \ell^w_2 \) which are isotonic with respect to the pre-order \( \preceq \).

**Definition 3** A function \( g^* : \mathcal{X} \to \mathbb{R} \) is the isotonic regression of a function \( g : \mathcal{X} \to \mathbb{R} \) over the pre-ordered set \( \mathcal{X} \) with weights \( w \in \mathbb{R}^s_+ \), with \( s \leq \infty \), if

\[
g^* = \arg\min_{f \in \mathcal{F}^{\text{is}}} \sum_{x \in \mathcal{X}} (f(x) - g(x))^2 w_x,
\]

where \( w_x = w_i \), for \( i = 1, \ldots, s \).

Conditions for existence and uniqueness of \( g^* \) will be stated below.

Similarly one can define an isotonic vector in \( \mathbb{R}^s \), with \( s \leq \infty \), and the isotonic regression of an arbitrary vector in \( \mathbb{R}^s \). Let us consider a set of indices \( \mathcal{I} = \{1, \ldots, s\} \), with \( s \leq \infty \), with some pre-order \( \preceq \) defined on it.
Definition 4 A vector \( \theta \in \mathbb{R}^s \), with \( s \leq \infty \), is isotonic if \( i_1, i_2 \in I \) and \( i_1 \ll i_2 \) imply \( \theta_{i_1} \leq \theta_{i_2} \).

We denote the set of isotonic vectors in \( \mathbb{R}^s \), with \( s \leq \infty \), by \( \mathcal{F}^{is} = \mathcal{F}^{is}(I) \).

In the case of an infinite index set we consider the square summable vectors (with weights \( w \)) from \( l_2^w \), the Hilbert space of all square summable vectors with weights \( w \).

Definition 5 A vector \( \theta^* \in \mathbb{R}^s \), with \( s \leq \infty \), is the isotonic regression of an arbitrary vector \( \theta \in \mathbb{R}^s \) (or \( \theta \in l_2^w \), if \( s = \infty \)) over the pre-ordered index set \( I \) with weights \( w \in \mathbb{R}_+^s \) if

\[
\theta^* = \arg \min_{\xi \in \mathcal{F}^{is}} \sum_{i \in I} (\xi_i - \theta_i)^2 w_i.
\]

Given a set \( X \) with a pre-order \( \ll \) on it one can generate a pre-order on the set \( I = \{1, 2, \ldots \} \) of indices of the domain in \( X \) as follows. For \( i_1, i_2 \in I \), \( i_1 \ll i_2 \) if and only if \( x_{i_1} \ll x_{i_2} \). This pre-order on the index set \( I \) will be called the pre-order induced by the set \( X \) and will be denoted by the same symbol \( \ll \). Conversely, if one starts with the set \( I \) consisting of the indices of the elements in \( X \), and \( \ll \) is a pre-order on \( I \), the above correspondence defines a pre-order on \( X \). Therefore, in the sequel of the paper a bold symbol, e.g. \( g \), will denote a vector in \( \mathbb{R}^s \), with \( s \leq \infty \), whose \( i \)-th component is given by \( g_i = g(x_i) \), for \( i = 1, \ldots, s \), where \( g(x) \) is a bounded real valued function on \( X \). In this case we will say that the vector \( g \) corresponds to the function \( g(x) \) on \( X \) and vice versa.

Corollary 1 A real valued function \( f(x) \) on the countable set \( X \) with the pre-order \( \ll \), defined on it, is isotonic if and only if its corresponding vector \( f \in \mathbb{R}^s \), with \( s \leq \infty \), is an isotonic vector with respect to the corresponding pre-order \( \ll \) on its index set \( I = \{1, 2, \ldots \} \), induced by the pre-order on \( X \). A real valued function \( g^*(x) \) on the set \( X \) is the isotonic regression of a function \( g(x) \) with weights \( w \) if and only if its corresponding vector \( g^* \in \mathbb{R}^s \) is the isotonic regression of the vector \( g \in \mathbb{R}^s \) with respect to the corresponding pre-order \( \ll \) on its index set \( I = \{1, 2, \ldots \} \) with weights \( w \).

To state the inference problem treated in this paper, suppose that \( X \) is a finite or an infinite countable pre-ordered set and \( \hat{g} \in \mathcal{F}^{is} \) is a fixed unknown function. Suppose we are given observations \( z_i, i = 1, \ldots, n \), independent or not, that depend on the (parameter) \( \hat{g} \) in some way. In the sequel we will treat in detail two important cases:

The data \( z_1, \ldots, z_n \) are observations of either of
(i) \( Z_i, \ i = 1, \ldots, n \) independent identically distributed random variables taking values in \( \mathcal{X} \), with probability mass function \( \hat{g} \).

(ii) \( Z_i = (x_i, Y_i), \ i = 1, \ldots, n, \) with \( x_i \) deterministic (design) points in \( \mathcal{X} \) and \( Y_i \) real valued random variables defined in the regression model

\[
Y_i = \hat{g}(x_i) + \varepsilon_i, \quad i = 1, \ldots, n,
\]

where \( \varepsilon_i \) is a sequence of identically distributed random variables with \( \mathbb{E}[\varepsilon_i] = 0, \ \text{Var}[\varepsilon_i] = \sigma^2 < \infty \).

Now assume that \( \hat{g}_n = \hat{g}_n(z_1, \ldots, z_n) \) is a \( \mathbb{R}^s \)-valued statistic. We will call the sequence \( \{\hat{g}_n\}_{n \geq 1} \) the basic estimator of \( \hat{g} \). In order to discuss consistency and asymptotic distribution result we introduce the following basic topologies: When \( s < \infty \), we study the Hilbert space with the inner product \( <g_1, g_2> = \sum_{i=1}^{s} g_{1,i} g_{2,i} w_i \), for \( g_1, g_2 \in \mathbb{R}^s \), endowed with its Borel \( \sigma \)-algebra \( \mathcal{B} = \mathcal{B}(\mathbb{R}^s) \) and when \( s = \infty \) we study the space \( l_w^2 \) with the inner product \( <g_1, g_2> = \sum_{i=1}^{\infty} g_{1,i} g_{2,i} w_i \), for a fixed weight vector \( w \) satisfying

\[
\begin{align*}
\inf_i \{w_i\} &> 0 \\
\sup_i \{w_i\} &< \infty,
\end{align*}
\]

and we equip \( l_w^2 \) with its Borel \( \sigma \)-algebra \( \mathcal{B} = \mathcal{B}(l_w^2) \).

Now define the isotonized estimator \( \hat{g}_n^* \) by

\[
\hat{g}_n^* = \arg\min_{\xi \in \mathcal{F}^{is}} \sum_{i \in \mathcal{I}} (\xi_i - \hat{g}_{n,i})^2 w_i.
\]

The main goal of this paper is to study the asymptotic behaviour of \( \hat{g}_n^* \), as \( n \to \infty \).

We make the following assumptions on the basic estimator \( \hat{g}_n \), for the finite, \( s < \infty \), and the infinite, \( s = \infty \), support case, respectively.

**Assumption 1** Suppose that \( s < \infty \). Assume that \( \hat{g}_n \overset{P}{\to} \hat{g} \) for some \( \hat{g} \in \mathcal{F}^{is} \) and \( B_n(\hat{g}_n - \hat{g}) \overset{d}{\to} \lambda \), where \( \lambda \) is a random vector in \( (\mathbb{R}^s, \mathcal{B}) \) and \( B_n \) is a diagonal \( s \times s \) matrix with elements \( [B_n]_{ii} = n q_i \) with \( q_i \) being real positive numbers.
Assumption 2 Suppose that $s = \infty$. Let $\hat{g}_n$, for $n = 1, 2, 3, \ldots$, be a tight sequence of random vectors taking values in the Hilbert space $l^w_2$. Assume that $\hat{g}_n \xrightarrow{p} \hat{g}$ for some $\hat{g} \in \mathcal{F}^{is}$, and $B_n(\hat{g}_n - \hat{g}) \xrightarrow{d} \lambda$, where $\lambda$ is a random vector in $(l^w_2, \mathcal{B})$ and $B_n$ is a linear operator $l^w_2 \to l^w_2$, such that for any $g \in l^w_2$ it holds that $(B_n g)_i = n^{q_i} g_i$, with $q_i$ being the real positive numbers. Suppose also that any finite $s$-dimensional cylinder set in $l^w_2$ is a continuity set for the law of $\lambda$.

Note that the matrix $B_n$ in Assumption 1 and the operator $B_n$ in Assumption 2 allow for different rates of convergence for different components of $\hat{g}_n$ and the values of $q_i$ will be specified later.

For a general introduction to the subject of constrained inference we refer to the monographs: Barlow R. E. et al. [4], Robertson T. et al. [16], Silvapulle M. J. [17], and Groeneboom P et al. [12]. In these monographs the problem of an isotonic regression has been considered in different settings, and in particular basic questions such as existence and uniqueness of the estimators have been addressed. In Lemmas 1 and 7 below we list those properties which will be used in the proofs.

The asymptotic behaviour of the regression estimates over a continuous setup under monotonic restriction was first studied in [9, 19] where it was shown that the difference of the regression function and its estimate multiplied by $n^{1/3}$, at a point with a positive slope, has a nondegenerate limiting distribution. In [11] the authors studied a general asymptotic scheme for an order constrained inference. The problem of a probability density estimation was studied, for example, in [10, 11, 15]. In the discrete case some recent results are [5, 6, 13].

This work is mainly motivated by the results obtained in [6, 13]. In [13] the problem of estimation of a discrete monotone distribution was studied in detail. It was shown that the limiting distribution of the constrained maximum likelihood estimator (mle) of a probability mass function (pmf) is a concatenation of the isotonic regressions of Gaussian vectors over a periods of constancy of a true pmf $p$, c.f. Theorem 3.8 in [13]. In the derivation of the limiting distribution in [13] the authors used the strong consistency of the empirical estimator of $p$ as well as the fact that the constrained MLE in the case of decreasing constraints is given by the the least concave majorant (lcm) of the empirical cumulative distribution function (ecdf).

The problem of mle of a unimodal pmf was studied in [6]. That problem is different from the one being considered here, since [6] treats only pmf on $Z$, whereas we are able to treat multivariate problems with our approach. Also the article [6] contains substantial references to the applications where one deals with discrete or discretized data.
Recall, that in our work we do not require strong consistency of a basic estimator $\hat{g}_n$ and we consider a general pre-order constraints where the expression for an isotonic regression is more complicated than lcm of the ecdf, c.f Assumptions 1 and 2. Also it turns out that the limiting distribution of the isotonised estimator $\hat{g}_n^*$ can be split deeper than to, the analogue of the periods of constancy of $\hat{g}$ in the univariate case, the level sets of $\hat{g}$.

The remainder of this paper is organised as follows. In Section 2 we consider the finite dimensional case, i.e. $s < \infty$. Theorem 1 gives the asymptotic distribution of the isotonised estimator. Next, in Section 3 we consider the infinite case, which is quite different from the finite one. Theorem 3 describes the asymptotic behaviour of the isotonised estimator for the infinite dimensional case. In Section 4 we discuss the application of the obtained results to the problems of estimation of a bimonotone regression function and of a bimonotone probability mass function, respectively.

2 Case of finitely supported functions

Let us assume that $s < \infty$, i.e. that the basic estimator $\{\hat{g}_n\}_{n \geq 1}$ is a sequence of finite-dimensional vectors. The next lemma states some well-known general properties of the isotonic regression of a finitely supported function.

**Lemma 1** Suppose Assumption 1 holds. Let $\hat{g}_n^* \in \mathbb{R}^s$ be the isotonic regression of the vector $\hat{g}_n$, defined in (2), for $n = 1, 2, 3, \ldots$. Assume also that $a \leq \hat{g}_{n,i} \leq b$ holds for some constants $-\infty < a < b < \infty$, for all $n = 1, 2, \ldots$ and $i = 1, \ldots, s$. Then the following hold:

(i) $\hat{g}_n^*$ exists and it is unique.

(ii) $\sum_{i=1}^s \hat{g}_{n,i} w_i = \sum_{i=1}^s \hat{g}_{n,i}^* w_i$, for all $n = 1, 2, \ldots$.

(iii) $\hat{g}_n^*$, viewed as a mapping from $\mathbb{R}^s$ into $\mathbb{R}^s$, is continuous. Moreover, it is also continuous if it is viewed as a function on the $2s$-tuples of real numbers $(w_1, w_2, \ldots, w_s, g_1, g_2, \ldots, g_s)$, with $w_i > 0$.

(iv) $\hat{g}_n^*$ satisfies the same bounds as the basic estimator, i.e. $a \leq \hat{g}_{n,i}^* \leq b$, for all $n = 1, 2, \ldots$ and $i = 1, \ldots, s$.

(v) $\hat{g}_n^*$ is a consistent estimator of $\hat{g}$, i.e. $\hat{g}_n^* \overset{p}{\rightarrow} \hat{g}$.

(vi) $(\hat{g}_n + c)^* = \hat{g}_n^* + c$ for all constant vectors $c \in \mathbb{R}^s$; 
$c \hat{g}_n^* = c^* \hat{g}_n^*$ for all $c \in \mathbb{R}_+$.  

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Proof. The statements (i), (ii), (iii) and (iv) are from [16] (Theorems 1.3.1, 1.3.3, 1.4.4 and 1.3.4). The statements (v) and (vi) are proved in [4] (Theorems 2.2 and 1.8).

Note that statement (ii) means that if the basic estimator \( \hat{g}_n \) satisfies a linear restriction, e.g. \( \sum_{i=1}^s w_i \hat{g}_{n,i} = c \), with some positive reals \( w_i \), then the same holds for its isotonic regression with the weights \( w \), i.e. for \( \hat{g}_n^* \) one has \( \sum_{i=1}^s w_i \hat{g}_{n,i}^* = c \). □

We make a partition of the original set into comparable sets \( X^{(1)}, \ldots, X^{(k)} \)

\[
X = \bigcup_{v=1}^k X^{(v)},
\]

where each partition set \( X^{(v)} \) contains elements such that if \( x \in X^{(v)} \), then \( x \) is comparable with at least one different element in \( X^{(v)} \) (if there are any), but not with any other element in \( X^{(v)} \) for any \( \mu \neq v \). In fact, the partition can be constructed even for an infinite set \( X = \{x_1, x_2, \ldots\} \) in the following way. We first construct the set \( X^{(1)} \) containing \( x_1 \), iteratively, as follows: Define \( X^{(1)}_0 = \{x_1\} \) and append to \( X^{(1)}_0 \) all points \( x_i \in X \setminus X^{(1)}_0 \) that are comparable to \( x_1 \), i.e.

\[
X^{(1)}_1 = X^{(1)}_0 \cup \left( \bigcup_{x_i \in X \setminus X^{(1)}_0, x_i \sim x_j} \{x_i\} \right).
\]

Note that \( x_i \sim x_j \) for some \( x_j \in X^{(1)}_0 \) in the above statement, means exactly, \( x_j \sim x_1 \). Next, in the second iteration, we append to \( X^{(1)}_1 \) all points \( x_i \in X \setminus X^{(1)}_0 \) that are comparable to some \( x_j \in X^{(1)}_1 \), i.e.

\[
X^{(1)}_2 = X^{(1)}_1 \cup \left( \bigcup_{x_i \in X \setminus X^{(1)}_1, x_i \sim x_j} \{x_i\} \right).
\]

We iterate the construction to obtain \( X^{(1)}_3, X^{(1)}_4, \ldots \), in the obvious way. This construction is countable, and in the final step gives us a (finite or infinite) set which we call \( X^{(1)} \).

Then either \( X^{(1)} = X \), in which case we call the set \( X \) non-decomposable, and we are done with the partition construction, with \( k = 1 \). Or else, there is a point \( x_l \in X \setminus X^{(1)}_0 \), and we can construct the next set \( X^{(2)} \) in the partition, iteratively, as follows: Define \( X^{(2)}_0 = \{x_l\} \), and append to this all points outside of \( X^{(1)}_0 \) that are comparable to \( x_l \), i.e.

\[
X^{(2)}_1 = X^{(2)}_0 \cup \left( \bigcup_{x_i \in X \setminus X^{(1)}_0, x_i \sim x_j} \{x_i\} \right).
\]

The next iteration in the construction of \( X^{(2)} \) is

\[
X^{(2)}_2 = X^{(2)}_1 \cup \left( \bigcup_{x_i \in X \setminus X^{(1)}_1, x_i \sim x_j} \{x_i\} \right).
\]
This procedure can be iterated, producing sets $X_2^{(2)}, X_3^{(2)}, \ldots$, and is a countable iterative construction which will terminate with a final set, which we call $X^{(2)}$.

Then either $X = X^{(1)} \cup X^{(2)}$, and we are done with the partition construction, with $k = 2$. Or else there is a point $x_u \in X \setminus (X^{(1)} \cup X^{(2)})$, and we can continue the construction of the set $X^{(3)}$ containing $x_u$.

The above procedure can be iterated in an obvious way, will exhaust the countable set $X$, and therefore is a countable construction of the desired partition (3). Note, that it is possible for a partitioning set $X^{(v)}$ to have only one element. Also, the partition of $X$ in (3) is unique and $X^{(v)} \cap X^{(v')} = \emptyset$, if $v \neq v'$.

Now assume that $X$ is finite, that we are given the partition (3) and let $g^{(v)}(x)$, for $v = 1, \ldots, k$, be real valued functions defined on the sets $X^{(v)}$ as $g^{(v)}(x) = g(x)$, whenever $x \in X^{(v)}$, i.e. $g^{(v)}(x)$ is the restriction of the function $g(x)$ to the set $X^{(v)}$. The family of functions $g^{(v)}(x)$ defined on the set $X^{(v)}$, which are isotonic with respect to the pre-order, will be denoted by $F^{is}_v$, for $v = 1, \ldots, k$.

The next lemma states a natural result of an isotonic regression on $X$, that it can be obtained as a concatenation of the individual isotonic regressions the comparable sets of the restrictions to the comparable sets.

**Lemma 2** Let $g(x)$ be an arbitrary real valued function on the finite set $X$ with a pre-order $\preceq$ defined on it, and assume that the partition (3) is given. Then the isotonic regression of $g(x)$ with any positive weights $w$ with respect to the pre-order $\preceq$ is equal to

$$g^*(x) = g^{*(v)}(x), \text{ whenever } x \in X^{(v)}, \quad (4)$$

where $g^{*(v)}(x)$ is the isotonic regression of the function $g^{(v)}(x)$ over the set $X^{(v)}$ with respect to the pre-order $\preceq$.

**Proof.** Let $g(x)$ be an arbitrary real-valued function defined on $X$. From the definition of the isotonic regression

$$g^* = \arg\min_{f \in F^{is}} \sum_{x \in X} (f(x) - g(x))^2 w_x$$

$$= \arg\min_{f \in F^{is}} \sum_{v=1}^k \sum_{x \in X^{(v)}} (f(x) - g(x))^2 w_x$$

$$= \sum_{v=1}^k \arg\min_{f^{(v)} \in F^{is}_v} \sum_{x \in X^{(v)}} (f^{(v)}(x) - g^{(v)}(x))^2 w_x,$$
where \( f^{(v)} \) is the restriction of the function \( f : \mathcal{X} \to \mathbb{R} \) to the set \( \mathcal{X}^{(v)} \). The second equality follows from (4) and the last equality follows from the fact that since the elements from the different partition sets \( \mathcal{X}^{(v)} \) are non-comparable, then any function \( f \in \mathcal{F}^{(v)} \) can be written as a concatenation of \( f^{(v)} \in \mathcal{F}^{(v)}_{(v)} \), with no restrictions imposed on the values of \( f^{(v_1)} \) and \( f^{(v_2)} \) for \( v_1 \neq v_2 \).

\[ \square \]

Now let \( \hat{g}(x) \) be the fixed function defined in Assumption 1; assume that we are given the partition (3) of \( \mathcal{X} \) and for an arbitrary but fixed \( v \in \{1, \ldots, k\} \) let \( \hat{g}_v(x) \) be the restriction of \( \hat{g}(x) \) to \( \mathcal{X}^{(v)} \). Then if \( N_v = |\mathcal{X}^{(v)}| \) we can introduce the vector \( \hat{g}_v = (\hat{g}_{v,1}, \ldots, \hat{g}_{v,N_v}) = (\hat{g}_v(x_{i_1}), \ldots, \hat{g}_v(x_{i_{N_v}})) \), where \( x_{i_1}, \ldots, x_{i_{N_v}} \) are the unique points in \( \mathcal{X}^{(v)} \). Given \( \hat{g}_v(x) \) we can partition the set \( \mathcal{X}^{(v)} \) into \( m_v \) sets

\[ \mathcal{X}^{(v)} = \bigcup_{l=1}^{m_v} \mathcal{X}^{(v,l)}. \tag{5} \]

The partition is constructed in the following way: We note first that the \( N_v \) values in the vector \( \hat{g}_v \) are not necessarily all unique, so there are \( \hat{m}_v \leq N_v \) unique values in \( \hat{g}_v \). Then in a first step we construct \( \hat{m}_v \) level sets

\[ \hat{\mathcal{X}}^{(v,l)} = \{ x \in \mathcal{X}^{(v)} : \hat{g}_v(x) = \hat{g}_{v,l} \} \]

with \( l = 1, \ldots, \hat{m}_v \).

Next we note that for any non-singleton level set \( \hat{\mathcal{X}}^{(v,l)} \) there might be non-comparable points, i.e. \( x_i, x_j \in \hat{\mathcal{X}}^{(v,l)} \) can be such that neither \( x_i \ll x_j \) nor \( x_j \ll x_i \) hold. Therefore, in the second step for each fixed \( l \) we can partition (if necessary) the level set \( \hat{\mathcal{X}}^{(v,l)} \) into sets with comparable elements, analogously to the construction of (5). We can do this for every \( v \) and end up in a partition (5) with \( \hat{m}_v \leq m_v \leq N_v \).

In the partition (5) each set \( \mathcal{X}^{(v,l)} \) is characterised by

(i) for every \( x \in \mathcal{X}^{(v,l)} \) we have \( \hat{g}_v(x) = \hat{g}_{v,l} \).
(ii) if \( |\mathcal{X}^{(v,l)}| \geq 2 \) then for every \( x \in \mathcal{X}^{(v,l)} \) there is at least one \( x' \in \mathcal{X}^{(v,l)} \) such that \( x \sim x' \).

We have therefore proved the following lemma.

\textbf{Lemma 3} For any countable set \( \mathcal{X} \) with the pre-order \( \ll \) and any isotonic function \( \hat{g}(x) \), defined on it, there exists a unique partition \( \mathcal{X} = \bigcup_{v=1}^{k} \bigcup_{l=1}^{m_v} \mathcal{X}^{(v,l)} \), satisfying the statements (i) and (ii) above. For the index set \( \mathcal{I} \) with the pre-order \( \ll \) generated by the set \( \mathcal{X} \) and any isotonic function \( \hat{g}(x) \), defined on \( \mathcal{X} \), there exists a unique partition \( \mathcal{I} = \bigcup_{v=1}^{k} \bigcup_{l=1}^{m_v} \mathcal{I}^{(v,l)} \), satisfying conditions analogous to (i) and (ii) stated above.
Definition 6 The set $X$ will be called decomposable if in the partition, defined in (3), $k > 1$. In the partition (3) the sets $X^{(v,l)}$ will be called the comparable level sets of $\hat{g}(x)$. In the corresponding partition of the index set $I$ the sets $I^{(v,l)}$ will be called the comparable level index sets of $\hat{g}(x)$.

Recall that $g^{(v,l)}(x)$ is the restriction of the function $g(x)$ to the comparable level set $X^{(v,l)}$, for $l = 1, \ldots, m_v$ and $v = 1, \ldots, k$.

In the case of a non-decomposable set, the full partition will be written as $X = \bigcup_{l=1}^{m} X^{(l)} = \bigcup_{l=1}^{m} X^{(v,l)}$, so we may then drop the index $v = 1$. Similarly, in this case $g^{(1,l)}(x) = g^{(l)}(x)$ denotes the restriction of a function $g(x)$ to the comparable level set $X^{(l)} = X^{(v,1)}$.

Next, suppose that $X$ is a non-decomposable set, and let us consider an arbitrary function $\hat{g}(x) \in F^*$. Assume that for $\hat{g}(x)$ there has been made a partition $X = \bigcup_{l=1}^{m} X^{(l)}$ in (3), satisfying (i) and (ii). Define the smallest comparable level distance of $\hat{g}$ as
\begin{equation}
\tilde{\varepsilon} = \inf \{|\hat{g}_l - \hat{g}_{l'}|\},
\end{equation}
for $l, l' = 1, \ldots, m, l \neq l'$, provided that there exist at least one $x_1 \in X^{(l)}$ and at least one $x_2 \in X^{(l')}$, such that $x_1$ and $x_2$ are comparable. Note, that $\tilde{\varepsilon}$ is always finite and for the finite support case, $s < \infty$, also $\tilde{\varepsilon} > 0$.

Lemma 4 Consider an arbitrary real valued function $g(x)$ on a non-decomposable finite set $X$ with the pre-order $\ll$ and let $\tilde{\varepsilon}$ be defined in (6). If
\begin{equation}
\sup_{x \in X} \{|g(x) - \hat{g}(x)|\} < \tilde{\varepsilon}/2,
\end{equation}
then the isotonic regression of $g(x)$ is given by
\begin{equation}
g^*(x) = g^{*\, (l)}(x), \text{ whenever } x \in X^{(l)},
\end{equation}
where $g^{*\, (l)}(x)$ is the isotonic regression of the function $g^{(l)}(x)$ over the set $X^{(l)}$ with respect to the pre-order $\ll$. Therefore, the function $g^*(x)$ is a concatenation of the isotonic regressions of the restrictions of $g(x)$ to the comparable level sets of $\hat{g}(x)$.

Proof. First, note that if the condition of the lemma is satisfied, then the function $g^*(x)$ defined in (6) on the set $X$ is isotonic. This follows from Lemma 1 statement (iv). Second, assume that the function $g^*(x)$ defined in (6) is not an isotonic regression of $g(x)$. This means that there exists another function $\hat{g}(x)$, such that
\begin{equation}
\sum_{x \in X} (\hat{g}(x) - g(x))^2 w_x < \sum_{x \in X} (g^*(x) - g(x))^2 w_x,
\end{equation}
for
Using the partition of $X$, (9) can be rewritten as

$$
\sum_{l=1}^{m} \sum_{x \in X(l)} (\hat{g}(x) - g(x))^2 w_x < \sum_{l=1}^{m} \sum_{x \in X(l)} (\hat{g}^*(x) - g(x))^2 w_x.
$$

Therefore, for some $l'$ we must have

$$
\sum_{x \in X(l')} (\hat{g}(x) - g(x))^2 w_x < \sum_{x \in X(l')} (\hat{g}^*(x) - g(x))^2 w_x
$$

or, equivalently,

$$
\sum_{x \in X(l')} (\hat{g}^{(l')}(x) - g^{(l')}(x))^2 w_x < \sum_{x \in X(l')} (\hat{g}^{*(l')}(x) - g^{(l')}(x))^2 w_x,
$$

with $g^{(l')}(x)$, $\hat{g}^{(l')}(x)$ and $g^{*(l')}(x)$ the restrictions to the comparable level set $X^{(l')}_{(v,l)}$ of $g(x)$, $\hat{g}(x)$ and $g^*(x)$, respectively. Since the function $g^{*(l')}(x)$ is the isotonic regression of the function $g^{(l')}(x)$ on the set $X^{(l')}$, the last inequality contradicts the property of the uniqueness and existence of the isotonic regression $g^{*(l')}(x)$ (statement (i) of Lemma 1). \(\square\)

The next lemma is an auxiliary result which will be used later in the proof of the asymptotic distribution of $\hat{g}_n$.

**Lemma 5** Assume $X_n$ and $Y_n$ are sequences of random vectors, taking values in the space $R^s$, for $s \leq \infty$, with some metric on it, endowed with its Borel $\sigma$-algebra. If $X_n \overset{d}{\to} X$ and $\lim_{n \to \infty} P(X_n = Y_n) = 1$, then $Y_n \overset{d}{\to} X$.

**Proof.** This result was proved in [2]. \(\square\)

Let us consider the sequence $B_n(\hat{g}_n^* - g)$, where $\hat{g}_n^*$ is the isotonic regression of $\hat{g}_n$, which was defined in Assumption 11 and with a specified matrix $B_n$. As mentioned in Assumption 11 we allow different rates of convergence $n^q_i$ for different components of $\hat{g}_n$. We however require $q_i$, for $i = 1, \ldots, s$, to be equal on the comparable level index sets $I^{(v,l)}$ of $\hat{g}$, i.e. $q_{i_1}$, for $i_1 = 1, \ldots, s$, are real positive numbers such that $q_{i_1} = q_{i_2}$, whenever $i_1, i_2 \in I^{(v,l)}$.

We introduce an operator $\varphi : \mathbb{R}^s \to \mathbb{R}^s$ defined in the following way. First, for any vector $\theta \in \mathbb{R}^s$ we define the coordinate evaluation map $\theta(x) : X \to \mathbb{R}$, corresponding to the vector $\theta$, by $\theta(x_i) = \theta_i$, for $i = 1, \ldots, s$.\(11\)
Then, let $\theta^{(v',l')}(x)$ be the isotonic regression of the restriction of $\theta(x)$ to the comparable level set $X^{(v',l')}$ of $\hat{g}(x)$, and define

$$\varphi(\theta)_i = \theta^{(v',l')}(x_i),$$

for $i = 1, \ldots, s$, with $(v', l')$ the (unique) indices such that $x_i \in X^{(v',l')}$. The asymptotic distribution of $B_n(\hat{g}_n^* - \hat{g})$ is given in the following theorem.

**Theorem 1** Suppose that Assumption 1 holds. Then

$$B_n(\hat{g}_n^* - \hat{g}) \xrightarrow{d} \varphi(\lambda),$$

where $\varphi$ is the operator, defined in (10).

**Proof.**

First, from Lemma 3 we have that any pre-ordered set $X$ can be uniquely partitioned as

$$X = \bigcup_{v=1}^k X^{(v)},$$

$$X^{(v)} = \bigcup_{l=1}^m X^{(v,l)},$$

and with the partition (12) of $X^{(v)}$ determined by the isotonic vector $\hat{g}$.

Second, as shown in Lemma 2, the isotonic regression of $g(x)$ on the original set $X$ can be obtained as a concatenation of the separate isotonic regressions of the restrictions of $g(x)$ to the non-decomposable sets in the partition (3). Therefore, without loss of generality, we can assume that the original set $X$ is non-decomposable. Thus, any $x \in X$ is comparable with at least one different element of $X$, $k = 1$, and

$$X = \bigcup_{l=1}^m X^{(1,l)}$$

and $\hat{g}_{1,l} \equiv \hat{g}_l$. Note, that we have dropped the index $v$.

Third, since $\hat{g}_n$ is consistent, by Assumption 11 for any $\varepsilon > 0$,

$$\mathbb{P}[\sup_{x \in X} \{ |\hat{g}_n(x) - \hat{g}(x)| \} < \varepsilon] \to 1,$$

as $n \to \infty$. Note that the comparable level distance $\hat{\varepsilon}$ of $\hat{g}$, defined in (6), satisfies $\hat{\varepsilon} > 0$, and take $\varepsilon = \hat{\varepsilon}/2$. Then from Lemma 4 we obtain

$$\{ \sup_{x \in X} |\hat{g}_n(x) - \hat{g}(x)| < \hat{\varepsilon}/2 \} \subseteq \{ \hat{g}_n^* = \varphi(\hat{g}_n) \}.$$
Therefore, (13) and (14) imply
\[ P[\hat{g}_n^* = \varphi(\hat{g}_n)] \to 1, \] as \( n \to \infty \).

Next, since the isotonic regression is a continuous map (statement (iii) of Lemma 1), the operator \( \varphi \) is a continuous map from \( \mathbb{R}^s \) to \( \mathbb{R}^s \). Therefore, using the continuous mapping theorem [18], we get
\[ \varphi(B_n(\hat{g}_n - \hat{\hat{g}})) \overset{d}{\to} \varphi(\lambda). \] (16)

Furthermore, using statement (vi) of Lemma 1 and taking into account the definition of the matrix \( B_n \), we get
\[ \varphi(B_n(\hat{g}_n - \hat{\hat{g}})) = B_n(\varphi(\hat{g}_n) - \hat{\hat{g}}). \] (17)

Then (15), (16) and (17) imply that
\[ P[B_n(\hat{g}_n^* - \hat{\hat{g}}) = B_n(\varphi(\hat{g}_n) - \hat{\hat{g}})] \to 1, \] as \( n \to \infty \). Finally, using Lemma 5 from (16) and (18) we prove that
\[ B_n(\hat{g}_n^* - \hat{\hat{g}}) \overset{d}{\to} \varphi(\lambda), \]
as \( n \to \infty \).

For a given pre-order \( \ll \) on \( X \) there exists a matrix \( A \) such that \( Ag \geq 0 \) is equivalent to \( g \) is isotonic with respect to \( \ll \), c.f. Proposition 2.3.1 in [17]. Therefore, if there are no linear constraints imposed on the basic estimator \( \hat{g}_n \), Theorem 1 can also be established by using the results on estimation when a parameter is on a boundary, in Section 6 in [3].

Assume that each vector \( \hat{g}_n \) has the following linear constraint \( \sum_{i=1}^s \hat{g}_{n,i}w_i = c \) (for example, in the case of estimation of a probability mass function it would be \( \sum_{i=1}^s \hat{g}_{n,i} = 1 \)). Then, the expression for a limiting distribution in Theorem 1 does not follow directly from the results in [3] in the case when \( \hat{g}_n \) is linearly constrained. However, the result of Theorem 1 holds, because, as it established in statement (ii) of Lemma 1 isotonic regression with weights \( w \) preserves the corresponding linear constraint.

Next we consider the case when the vector of weights \( w \) is not a constant, i.e. we assume that some non-random sequence \( \{w_n\}_{n \geq 1} \), where each vector \( w_n \) satisfies the condition (1), converges to some non-random vector \( w \), which
also satisfies (1). We denote by \( \hat{\theta^\ast w}(x) \) the isotonic regression of \( \theta(x) \) with weights \( w \) and analogously to (10) we introduce the notation \( \varphi^w(\theta) \)

\[
\varphi^w(\theta)_i = \theta^{*w(v',l')} (x_i),
\]

where \( \theta^{*w(v',l')}(x) \) is the isotonic regression, with weights \( w \), of the restriction of \( \theta(x) \) to the comparable level set \( X(v',l') \) of \( \hat{g}(x) \), where the indices \( v' \) and \( l' \) are such that \( x_i \in X(v',l') \). Define the isotonic regression \( \hat{g}^{*w_n} \) of the basic estimator \( \hat{g}_n \). The next theorem gives the limiting distribution of \( \hat{g}^{*w_n} \).

**Theorem 2** Suppose that Assumption 1 holds. Then the asymptotic distribution of the isotonic regression \( \hat{g}^{*w_n} \) of the basic estimator \( \hat{g}_n \) is given by

\[
B_n(\hat{g}^{*w_n} - \hat{g}) \overset{d}{\to} \varphi^w(\lambda),
\]

where \( \varphi^w \) is the operator, defined in (19).

**Proof.** Without loss of generality, we can assume that the original set \( \mathcal{X} \) is non-decomposable.

First, since the sequence \( \hat{g}_n \) is consistent, then for any

\[
P\left[ \sup_{x \in \mathcal{X}} |\hat{g}_n(x) - \hat{g}(x)| < \tilde{\varepsilon}/2 \right] \to 1,
\]

as \( n \to \infty \), with \( \tilde{\varepsilon} \) taken from Lemma 4. Using the statement of Lemma 4, we obtain

\[
\{ \sup_{x \in \mathcal{X}} |\hat{g}_n(x) - \hat{g}(x)| < \tilde{\varepsilon}/2 \} \subseteq \{ \hat{g}^{*w_n} = \varphi^{w_n}(\hat{g}_n) \}. \tag{22}
\]

Note that the result of Lemma 4 holds for any weights \( w_n \).

Therefore, from (21) and (22) we have

\[
P[\hat{g}^{*w_n} = \varphi(\hat{g}_n)] \to 1, \tag{23}
\]

as \( n \to \infty \).

Second, from statement (iii) of Lemma 4 the operators \( \varphi^{w_n} \), \( \varphi^w \) are continuous maps from \( \mathbb{R}^{2s} \) to \( \mathbb{R}^s \), for all weights \( w_n \), \( w \) satisfying (1). Using the (extended) continuous mapping theorem [18], we get

\[
\varphi^{w_n}(B_n(\hat{g}_n - \hat{g})) \overset{d}{\to} \varphi^w(\lambda), \tag{24}
\]

where \( w \) is the limit of the sequence \( \{ w_n \}_{n \geq 1} \).
Third, using statement (vi) of Lemma 1 and the definition of the matrix $B_n$ we obtain

$$
\varphi^{wn}(B_n(\hat{g}_n - \hat{g})) = B_n(\varphi^{wn}(\hat{g}_n) - \hat{g}).
$$

(25)

Therefore, (25) gives us

$$
P[B_n(\hat{g}_n^{*wn} - \hat{g}) = B_n(\varphi^{wn}(\hat{g}_n) - \hat{g})] \to 1,
$$

(26)

as $n \to \infty$. Finally, using Lemma 5 from (24) and (26) we prove that

$$
B_n(\hat{g}_n^{*wn} - \hat{g}) \overset{d}{\to} \varphi^w(\lambda),
$$

as $n \to \infty$. □

3 Case of infinitely supported functions

In this section we assume that the original set $\mathcal{X} = \{x_1, x_2, \ldots\}$ is an infinite countable enumerated set with a pre-order $\ll$ defined on it.

In the case of the infinitely supported functions the isotonic regression’s properties are similar to the ones in the finite case, but the proofs are usually different. For completeness we state these properties in the following lemma.

Lemma 6 Suppose Assumption 1 holds. Let $\hat{g}_n^* \in L^w_2$ be the isotonic regression of the vector $\hat{g}_n \in L^w_2$, for $n = 1, 2, 3, \ldots$. Assume also that $a \leq \hat{g}_{n,i} \leq b$ holds for some constants $-\infty < a < b < \infty$, for all $n = 1, 2, \ldots$ and $i = 1, \ldots, \infty$. Then statements (i) - (vi) of Lemma 1 hold, with (iii) suitably changed to the mapping from $L^w_2$ to $L^w_2$.

Proof. Statements (i), (ii) and (iii) follow from Theorem 8.2.1, Corollary B of Theorem 8.2.7 and Theorem 8.2.5, respectively, in [16], statements (iv), (v) and (vi) follow from Corollary B of Theorem 7.9, Theorems 2.2 and Theorems 7.5 and 7.8, respectively, in [3]. □

We partition the original set $\mathcal{X}$ in the same way as it was done in the finite case, i.e., first, let

$$
\mathcal{X} = \bigcup_{v=1}^k \mathcal{X}^{(v)}.
$$

(27)
where $k \leq \infty$ is the number of sets and each set $\mathcal{X}^{(v)}$ is such that if $x \in \mathcal{X}^{(v)}$, then $x$ is comparable with at least one different element in $\mathcal{X}^{(v)}$ (if there are any), but not with any other elements which belong to other terms in the partition. Note, that $\mathcal{X}^{(v)}$ can have only one element. The partition of $\mathcal{X}$ is unique and $\mathcal{X}^{(v)} \cap \mathcal{X}^{(v')} = \emptyset$, for $v \neq v'$.

Furthermore, for the fixed function $\hat{g} \in \mathcal{F}^{is}$, defined in Assumption 2 (or, equivalently, its corresponding isotonic vector $\hat{g} \in \mathcal{F}$), we partition each set $\mathcal{X}^{(v)}$ in (27) into the comparable level sets of $\hat{g}$, i.e.

$$\mathcal{X}^{(v)} = \bigcup_{l=1}^{m_v} \mathcal{X}^{(v,l)}, \quad (28)$$

in the same way as it was done in the finite case in (4).

Note, that since $\hat{g} \in \mathcal{L}^{w_2}$ and condition (1) is satisfied, the cardinality of any set $\mathcal{X}^{(v,l)}$ is less than infinity whenever $\hat{g}_{v,l} \neq 0$, otherwise we would have $\sum_{i=1}^{\infty} (\hat{g}_i)^2 w_i = \infty$, which would mean that $\hat{g} \notin \mathcal{L}^{w_2}$. The set $\mathcal{X}^{(v,l)}$ can have infinitely many elements only if $\hat{g}_{v,l} = 0$.

For the partition in (27) we obtain a result similar to the one obtained in Lemma 2 for the finite case.

**Lemma 7** Let $g(x)$ be an arbitrary real valued function in $l^{w}_2$ on the set $\mathcal{X}$ with a pre-order $\ll$ defined on it. Then the isotonic regression of $g(x)$ with any positive weights $w$ is equal to

$$g^*(x) = g^{*(v)}(x), \quad \text{whenever } x \in \mathcal{X}^{(v)}, \quad (29)$$

where $g^{*(v)}(x)$ is the isotonic regression of the restriction of the function $g(x)$ to the set $\mathcal{X}^{(v)}$ over this set with respect to the pre-order $\ll$.

**Proof.** The proof is exactly the same as in the finite case (Lemma 2). $\square$

As a consequence of Lemma 7 without loss of generality in the sequel of the paper we can assume that the original set $\mathcal{X}$ is non-decomposable and use the same notation as in the finite case, i.e. $\mathcal{X} = \bigcup_{l=1}^{m} \mathcal{X}^{(l)} \equiv \bigcup_{l=1}^{m_1} \mathcal{X}^{(1,l)}$ and, respectively, $g^{(l)}(x) = g^{(1,l)}(x)$ for the restriction of the function $g(x)$ to the set $\mathcal{X}^{(l)}$.

In the case of an infinite support the result of Lemma 4 is generally not applicable, because the value of $\hat{\varepsilon}$ can be zero. We therefore make the following slight modification of Lemma 4. Thus, assume that for a function $\hat{g}(x) \in \mathcal{F}^{is}$ we have made a partition $\mathcal{X} = \bigcup_{l=1}^{m} \mathcal{X}^{(l)}$ with $m \leq \infty$. Furthermore, for any finite positive integer number $m' < m \leq \infty$ we choose $m'$
comparable level sets $\mathcal{X}^{(l_i)}$, such that the values of the function $\hat{g}(x)$ on them satisfy $|g_1| \geq |g_2| \geq \cdots \geq |g_n|$. Next, we rewrite the partition as

$$\mathcal{X} = \mathcal{X}^{(l_1)} \cup \mathcal{X}^{(l_2)} \cup \cdots \cup \mathcal{X}^{(l_{m'})} \cup \mathcal{X}^{(l_{m'+1})},$$

where $\mathcal{X}^{(l_{m'+1})} = \mathcal{X} \setminus \mathcal{X}^{(l_1)} \cup \mathcal{X}^{(l_2)} \cup \cdots \cup \mathcal{X}^{(l_{m'})}$. Define

$$\varepsilon' = \inf\{|\hat{g}_p - \hat{g}_l| : l' \in \{l_1, \ldots, l_{m'}\}, \, l \in \{1, \ldots, m\}, \exists x_1 \in \mathcal{X}^{(l)}, \, \exists x_2 \in \mathcal{X}^{(l')}, \text{ such that } x_1 \sim x_2\}.$$  

**Lemma 8** Consider an arbitrary real valued function $g(x) \in L^w_2$ on a non-decomposable infinite countable set $\mathcal{X}$ with the pre-order $\ll$ defined on it. If for some $\hat{g}(x) \in \mathcal{F}^w$ we have

$$\sup_{x \in \mathcal{X}}\{|g(x) - \hat{g}(x)|\} < \varepsilon'/2,$$

then the isotonic regression of $g(x)$ is given by

$$g^*(x) = g^{*l''}(x), \text{ whenever } x \in \mathcal{X}^{(l'')}, \text{ for } l'' \in \{l_1, \ldots, l_{m'}, l_{m'+1}\},$$

where $g^{(l'')}(x)$ is the isotonic regression of the function $g^{(l')}(x)$ over the set $\mathcal{X}^{(l')}$ with respect to the pre-order $\ll$. Therefore, the function $g^*(x)$ is a concatenation of the isotonic regressions of the restrictions of $g(x)$ to the sets $\mathcal{X}^{(l_1)}, \mathcal{X}^{(l_2)}, \ldots, \mathcal{X}^{(l_{m'})}$ and $\mathcal{X}^{(l_{m'+1})}$.

**Proof.** The proof is exactly the same as in the case of a finite support (Lemma 4). \qed

Next we state and prove an auxiliary lemma which will be used in the final theorem.

**Lemma 9** Let $Z_n, \text{ for } n = 1, \ldots, \infty$, be a tight sequence of random vectors in $L^w_2$, endowed with its Borel $\sigma$-algebra $\mathcal{B}$. Consider the set of indices $I = \{1, 2, \ldots, \infty\}$ of the component of the vectors $Z_n$. Assume that for some random vector $Z$ in $(L^w_2, \mathcal{B})$ and some rearrangement $\tilde{I}$ of the original index set $I$ the following holds: For any positive finite integer $s$ we have $\hat{Z}_n^{(1,s)} \Rightarrow \hat{Z}^{(1,s)}$, where $\hat{Z}_n^{(1,s)}$ and $\hat{Z}^{(1,s)}$ are vectors in $\mathbb{R}^s$ constructed from the elements of the vectors $Z_n$ and $Z$ in the way that $j$-th elements of $\hat{Z}_n^{(1,s)}$ and $\hat{Z}^{(1,s)}$ are equal to $\hat{i}_j$-th elements of the vectors $\hat{Z}_n$ and $Z$, respectively, with $\hat{i}_j$ being the $j$-th index from the rearranged index set $\tilde{I}$. In addition, assume that any cylinder set in $L^w_2$ is a continuity set for the law of $\hat{Z}^{(1,s)}$. Then $Z_n \Rightarrow Z$. \qed
Proof. The space $l^w_2$ is separable and complete. Then, from Prokhorov’s theorem [21], it follows that the sequence $Z_n$ is relatively compact, which means that every sequence from $Z_n$ contains a subsequence, which converges weakly to some vector $Z$. If the limits of the convergent subsequences are the same, then the result of the lemma holds.

Since the space $l^w_2$ is separable, the Borel $\sigma$-algebra equals the $\sigma$-algebra generated by open balls in $l^w_2$ [8]. Therefore, it is enough to show that the limit laws agree on the open ball, since the finite intersections of the finite balls in $l^w_2$ constitute a $\pi$-system. To show this, we note that the open ball in $l^w_2$ can be written as

$$B(z, \varepsilon) = \cap_{M \geq 1} B_M,$$

where

$$B_M = \cup_{n \geq 1} A^M_n,$$
$$A^M_n = \{ y \in l^w_2 : \sum_{j \in i_1, \ldots, i_M} |z_j - y_j|^2 w_j < \varepsilon^2 - \frac{1}{n}\},$$

where indices $\tilde{i}_1, \ldots, \tilde{i}_M$ are the $M$ first indices from $\tilde{I}$.

The sequence of vectors $\tilde{Z}_n^{(1,M)}$ converges weakly to $\tilde{Z}^{(1,M)}$ for all finite $M$, therefore any subsequence of $\tilde{Z}_n^{(1,M)}$ converges weakly to $\tilde{Z}^{(1,M)}$. That means that, with $\mathbb{P}^M_n$ the laws of an arbitrary but fixed subsequence of $\tilde{Z}_n^{(1,M)}$, and $\mathbb{P}^M$ the law of $\tilde{Z}^{(1,M)}$, $\mathbb{P}^M_n(A) \to \mathbb{P}^M(A)$ for any $\mathbb{P}^M$-continuity set $A$. Therefore, since $A^M_n$ is a continuity set for the limit law $P(M)$, and by the continuity properties of a probability measure, we obtain

$$\mathbb{P}(B(z, \varepsilon)) = \lim_{M \to \infty} \mathbb{P}(B_M) = \lim_{M \to \infty} \lim_{n \to \infty} \mathbb{P}(A^M_n) = \lim_{M \to \infty} \lim_{n \to \infty} \mathbb{P}^M(A^M_n),$$

where $\mathbb{P}$ is the law of $Z$.

Thus, we have shown that the limit laws, $\mathbb{P}$, of the convergent subsequences of $\{Z_n\}$ agree on the open balls $B(z, \varepsilon)$, and, therefore, also on the finite intersections of these open balls. Since the laws agree on the $\pi$-system (they are all equal to $\mathbb{P}$), they agree on the Borel $\sigma$-algebra. □

Finally, the next theorem gives the limiting distribution of $\hat{g}^*_n$. Similarly to the finite case we introduce the operator $\varphi : l^w_2 \to l^w_2$, defined in the
following way. For any vector $\theta \in B^w$ we consider the coordinate evaluation map $\theta(x) : X \to \mathbb{R}$ defined as $\theta(x_i) = \theta_i$, for $i = 1, \ldots, \infty$. Then, let

$$\varphi(\theta)_i = \theta^*(v',l')(x_i),$$

(32)

where $\theta^*(v',l')(x)$ is the isotonic regression of the restriction of $\theta(x)$ to the set $X(v',l')$ in the partition of $X$. The indices $v'$ and $l'$ are such that $x_i \in X(v',l')$. The restriction of $\varphi(\theta)$ to the comparable index level set $\mathcal{I}^{(v,l)}$ will be denoted by $[\varphi(\theta)]^{(v,l)}$

**Theorem 3** Suppose the Assumption $\mathcal{A}$ holds. Then the asymptotic distribution of the isotonised estimator $\hat{\mathcal{g}}^*_n$ is given by

$$B_n(\hat{\mathcal{g}}^*_n - \mathcal{g}) \overset{D}{\to} \varphi(\lambda),$$

(33)

where $\varphi$ is the operator defined in (32).

**Proof.** Let us consider the partition of the original set $X = \cup_{i=1}^{m} X(i)$ made for the function $\hat{\mathcal{g}}(x)$. As it was shown above, the cardinality $|X(i)|$ of each comparable level set in the partition must be less than infinity, unless $\hat{\mathcal{g}}_i = 0$, in which case it can have infinite cardinality. Since that if the number of terms in the partition is less than infinity, i.e. $m < \infty$, then some terms (or just one) in the partition are such that the function $\hat{\mathcal{g}}(x)$ is equal to zero on them, i.e. $\hat{\mathcal{g}}_i = 0$. Therefore, in this case we can use the same approach as in the case of the finite set $X$ (Lemma $\mathcal{B}$), because in this case he smallest comparable level distance $\mathcal{e}$, defined in (34), is greater that zero.

Therefore, further in the proof we assume that $m = \infty$ and write the partition as $X = \cup_{i=1}^{\infty} X(i)$. First, for any positive integer $m' < \infty$ let us take $m'$ terms from the partition of $X$ which satisfy $|\hat{\mathcal{g}}_{i_1}| \geq |\hat{\mathcal{g}}_{i_2}| \geq \cdots \geq |\hat{\mathcal{g}}_{i_{m'}}|$.

Second, since the sequence $\hat{\mathcal{g}}_n$ is consistent, then for any $\mathcal{e} > 0$

$$\lim_{n \to \infty} \mathbb{P}[\sup_{i \in I} \{ |\hat{\mathcal{g}}_{n,i} - \hat{\mathcal{g}}_i| \leq \mathcal{e} \}] = 1.$$  

Therefore, letting $\mathcal{e} = \mathcal{e}'/2$ from Lemma $\mathcal{A}$ and using its result, for the isotonic regression $\hat{\mathcal{g}}^*_n$ of $\hat{\mathcal{g}}_n$ we obtain

$$\lim_{n \to \infty} \mathbb{P}[\hat{\mathcal{g}}^*_i = \hat{\mathcal{g}}^*_i(l''')] = 1,$$

(34)

whenever $i \in I(l''')$, for $l''' \in \{l_1, \ldots, l_{m'+1}\}$, where $I(l')$, for $l' \in \{l_1, \ldots, l_{m'}\}$, are the comparable level sets and $I(m'+1)$ is the index set of $X(m'+1) = X \setminus X(l_1) \cup X(l_2) \cup \cdots \cup X(l_{m'})$.

Third, let us introduce a linear operator $A(m') : B^w \to \mathbb{R}^s$, with $s = \sum_{l \in \{l_1, \ldots, l_{m'}\}} |X(l)|$, such that for any $\mathcal{g} \in B^w$ first $|X(l)|$ elements of the vector
\( A^{(m')} g \) are equal to ones taken from \( g \) whose indices are in \( I(l_1) \), the second \( |X^{(l_2)}| \) elements are the ones from \( g \) whose indices are from \( I(l_2) \) and so on. Therefore, using the result in (34), the definition of \( B_n \) and statement (vi) of Lemma 1, the following holds

\[
\lim_{n \to \infty} P[A^{(m')} \phi(B_n(\hat{g}^*_n - \hat{g}))] = 1.
\] (35)

Next, since \( \phi \) is a continuous map, which follows from statement (iii) of Lemma 7, and \( A^{(m')} \) is a linear operator, then from the continuous mapping theorem and delta method [18] it follows that

\[
A^{(m')} \phi(B_n(\hat{g}^*_n - \hat{g})) \xrightarrow{d} A^{(m')} \phi(\lambda).
\] (36)

and, using Lemma 5 and result in (35), we prove

\[
A^{(m')} B_n(\hat{g}^*_n - \hat{g}) \xrightarrow{d} A^{(m')} \phi(\lambda).
\]

Note, that the number \( m' \) is an arbitrary finite integer. Also, since \( \phi \) is a continuous map, then the law of \( \phi(\lambda) \) has the same continuity sets as \( \lambda \).

Using Lemma 9 we finish the proof of the theorem.

Recall that the cardinality of any comparable level set \( X^{(v,l)} \) is less than infinity whenever \( \hat{g}_{v,l} \neq 0 \). Then, as in the finite case, we note that the order constraints on \( X^{(v,l)} \) can be expressed in the form \( Ag \geq 0 \), for some matrix \( A \). Therefore, one can use the results in [3] to describe the behaviour of \( [\phi(g)]^{(v,l)} \) when \( |X^{(v,l)}| < \infty \). It follows from Theorem 5 in [3] that the distribution of \( [\phi(g)]^{(v,l)} \) is a mixture of \( 2^{|X^{(v,l)}|} \) distributions of the projections of \( g \) onto the cone \( A_t g \geq 0 \), where the matrixes \( A_t \), for \( t = 1, \ldots, 2^{|X^{(v,l)}|} \), are comprised of the rows of the matrix \( A \).

Next, let us consider the case of non-constant weights \( w \). In this section till now we assumed that the vector of weights satisfies the condition in [1], it is fixed, \( w_n = w \), i.e. it does not depend on \( n \), and the random elements \( \hat{g}_n \) in Assumption 2 all take their values in \( (l^w_2, B) \), for some fixed \( w \), with \( B \) the Borel \( \sigma \)-algebra generated by the topology which is generated by the natural norm in \( l^w_2 \).

Now we consider some non-random sequence \( \{w_n\}_{n \geq 1} \), taking values in the space \( \mathbb{R}^\infty \), where each \( w_n \) satisfies the condition in [1], converges in some norm \( \| \cdot \|_R \) on \( \mathbb{R}^\infty \) to some non-random vector \( w \), which also satisfies the condition in [1]. Next, let \( B_n \) denotes the Borel \( \sigma \)-algebra generated by the topology which is generated by the natural norm in \( l^w_2 \). Next Lemma shows that the normed spaces \( l^w_2 \) are all equivalent.
Lemma 10  Let two vectors $w_1$ and $w_2$ satisfy the condition in (1). Then the normed spaces $l^{w_1}_2$ and $l^{w_2}_2$ are equivalent.

Proof. First, we prove that if $w$ satisfies the condition in (1), then $x \in l_w^2$ if and only if $x \in l_2^w$ ($l_2$ is the space of all square summable sequences, i.e. $w = \{1, 1, \ldots\}$). Let $x \in l_w^2$, then $\sum_{i=1}^{\infty} x_i^2 w_i < \infty$ and we have

$$(\inf \{w_i\}) \sum_{i=1}^{\infty} x_i^2 \leq \sum_{i=1}^{\infty} x_i^2 w_i < \infty.$$  

Therefore, since $\inf \{w_i\} > 0$, we have that $\sum_{i=1}^{\infty} x_i^2 < \infty$, which means that $x \in l_2$.

Next, let $x \in l_2$, then $\sum_{i=1}^{\infty} x_i^2 < \infty$ and we have

$$\sum_{i=1}^{\infty} x_i^2 w_i \leq (\sup_i \{w_i\}) \sum_{i=1}^{\infty} x_i^2 < \infty,$$

since $\sup_i \{w_i\} < \infty$. Therefore, $x \in l_w^2$.

Second, let $\| \cdot \|_w$ and $\| \cdot \|$ denote the natural norms in $l_w^2$ and $l_2$. We can prove that if $w$ satisfies the condition in (1), then $l_w^2$ and $l_2$ are equivalent, i.e. there exist two positive constants $c_1$ and $c_2$ such that

$$c_1 \|x\| \leq \|x\|_w \leq c_2 \|x\|.$$  

(37)

If we take, for example, $c_1 = \inf_i \{w_i\}$ and $c_2 = \sup_i \{w_i\}$, we prove (37).

Therefore, since the equivalence of norms is transitive, then $l^{w_1}_2$ and $l^{w_2}_2$ are equivalent, provided $w_1$ and $w_2$ satisfy the condition in (1).

Therefore, since the normed spaces $l^{w_n}_2$ are all equivalent, then the topologies generated by these norms are the same. Then, the Borel $\sigma$-algebras $B_n$ generated by these topology are also the same. Therefore, the measurable spaces $(l^{w_n}_2, B_n)$ are all the same and we will suppress the index $n$.

Next, analogously to the finite case, let us introduce the notation $\varphi^w(\theta)$

$$\varphi^w(\theta)_i = \theta^w(x^{(v',l')})(x_i),$$  

(38)

where $\theta^w(x^{(v',l')})(x)$ is the isotonic regression with weights $w$ of the restriction of $\theta(x)$ to the comparable level set $\mathcal{X}^{(v',l')}$ of $\hat{g}(x)$, where the indices $v'$ and $l'$ are such that $x_i \in \mathcal{X}^{(v',l')}$. Next theorem gives the limiting distribution of $\hat{g}^n_{w_2}$. 

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Theorem 4 Suppose the Assumption 2 holds. Then the asymptotic distribution of the isotonic regression \( \hat{g}_{n}^{w} \) of the basic estimator \( \hat{g}_{n} \) is given by

\[
B_{n}(\hat{g}_{n}^{w} - \hat{g}) \xrightarrow{d} \varphi^{w}(\lambda),
\]

where \( \varphi \) is the operator, defined in (38).

Proof. First, we note that the result of Lemma 9 holds, if we assume that the random vectors \( Z_{n}, \) for \( n = 1, \ldots, \infty \) take their values in \( l_{w}^{n} \), if all elements of \( w_{n} \) and its limit \( w \) satisfy the condition in (1). It follows from the fact that the measurable spaces \( (l_{w}^{w}, B_{n}) \) are equivalent, which was proved in Lemma 10.

The rest of the proof is exactly the same as for Theorem 3 with \( \varphi \) and \( \hat{g}_{n} \) suitable changed to \( \varphi^{w} \) and \( \hat{g}_{n}^{w} \). Also, recall that the result of Lemma 8 does not depend on the weights \( w_{n} \).

\( \square \)

4 Examples

In this section we consider the problems of estimation of a bimonotone regression function and of a bimonotone probability mass function.

First, let us introduce a bimonotone order relation \( \ll \) on a set \( X := \{(i, j)^{T} : i = 1, 2, \ldots, r, j = 1, 2, \ldots, s, \} \), with \( r, s \leq \infty \) in the following way. For any \( x_{1} \) and \( x_{2} \) in \( X \) we have \( x_{1} \ll x_{2} \) if and only if \( x_{1,1} \geq x_{2,1} \) and \( x_{1,2} \geq x_{2,2} \). The order relation \( \ll \) is a partial order, because it is reflexive, transitive, antisymmetric, but there are elements in \( X \) which are noncomparable. A real valued function \( g(x) \) is bimonotone if whenever \( x_{1} \ll x_{2} \) one has \( g(x_{1}) \leq g(x_{2}) \), c.f. [7].

Second, note that \( X \) with the order relation \( \ll \) defined above is nondecomposable, because for any \( x_{1} = (x_{1,1}, x_{1,2}) \) and \( x_{2} = (x_{2,1}, x_{2,2}) \) in \( X \) there exist \( x_{3} = (x_{3,1}, x_{3,2}) \) in \( X \) such that \( x_{3,1} \geq x_{1,1}, x_{3,1} \geq x_{2,1} \) and \( x_{3,2} \geq x_{1,2}, x_{3,2} \geq x_{2,2} \), which means that \( x_{1} \sim x_{3} \) and \( x_{2} \sim x_{3} \), or \( x_{4} = (x_{4,1}, x_{4,2}) \) in \( X \) such that \( x_{4,1} \leq x_{1,1}, x_{4,1} \leq x_{2,1} \) and \( x_{4,2} \geq x_{1,2}, x_{4,2} \geq x_{2,2} \), which means that \( x_{1} \sim x_{4} \) and \( x_{2} \sim x_{4} \). Therefore, in a partition \( k \) \( k = 1 \).

Also, following our notations above we denote by \( I \) the set of indices of the domain \( X \) and use the same notation \( \ll \) for the order relation on \( I \) generated by \( X \).
4.1 Estimation of a bimonotone regression function

The problem of estimation of a bimonotone regression function via least squares was studied in detail in [7], where authors described an algorithm for minimization of a smooth function under bimonotone order constraints.

Suppose we have observed 
\[
Z_i = (x_i, Y_i), \quad i = 1, \ldots, n, \quad \text{with } x_i \text{ design points taking values from the set } \mathcal{X} := \{x = (i, j)^T : i = 1, 2, \ldots, r, j = 1, 2, \ldots, s\}, \quad \text{with } r, s < \infty \text{ and } Y_i \text{ real valued random variables defined in the regression model}
\]

\[
Y_i = \hat{g}(x_i) + \varepsilon_i, \quad i = 1, \ldots, n,
\]

where \(\varepsilon_i\) is a sequence of identically distributed random variables with \(\mathbb{E}[\varepsilon_i] = 0, \text{Var}[\varepsilon_i] = \sigma^2 < \infty\).

The least squares estimate of \(\hat{g}(x)\) under bimonotone constraints is given by

\[
g_n^* = \arg\min_{f \in F^{is}} \sum_{x \in \mathcal{X}} (f(x) - \hat{g}_n(x))^2 w^{(n)}_x, \quad (40)
\]

where \(F^{is}\) denotes the set of all bounded bimonotone functions on \(\mathcal{X}\), \(\hat{g}_n(x)\) is the average of \(Y_i\), \(i = 1, \ldots, n\), over the design element \(x\), i.e.

\[
\hat{g}_n(x) = \frac{\sum_{i=1}^n Y_i 1\{x_i = x\}}{\sum_{i=1}^n 1\{x_i = x\}} \quad (41)
\]

and

\[
w^{(n)}_x = \frac{\sum_{i=1}^n 1\{x_i = x\}}{n}. \quad (42)
\]

Note that \(g_n(x)\) in (41) is the unconstrained least squares estimate of \(\hat{g}(x)\). The asymptotic properties of nonlinear least squares estimators were studied in [14, 20]. Assume that the design points \(x_i\), with \(i = 1, \ldots, n\), satisfy the following condition

\[
w^{(n)} \to w, \quad (43)
\]

as \(n \to \infty\), where \(w^{(n)}\) is a sequence of vectors in \(\mathbb{R}^{r \times s}_+\) whose components are from (42), and \(w \in \mathbb{R}^{r \times s}_+\). Given the condition in (43) is satisfied, the basic estimator \(\hat{g}_n(x)\) is consistent and has the following asymptotic distribution

\[
n^{1/2}(\hat{g}_n - g) \xrightarrow{d} Y_{0, \Sigma}, \quad (44)
\]
where $Y_{0,\Sigma}$ is a Gaussian vector with mean zero and diagonal covariance matrix $\Sigma$, whose elements are given by $\Sigma_{ii} = \sigma^2 w_i$, for $i = 1, \ldots, r \times s$, c.f. Theorem 5 in [20].

Next theorem gives the asymptotic distribution of the regression function under bimonotone constraints.

**Theorem 5** Given that the condition (43) on the design points is satisfied, the asymptotic distribution of the regression function $\hat{g}(x)$ under bimonotone constraints is given by

$$n^{1/2}(\hat{g}^*_n - \hat{g}) \xrightarrow{d} \varphi^w(Y_{0,\Sigma}),$$

where $\varphi^w$ is the operator defined in (19) and $Y_{0,\Sigma}$ is a Gaussian vector defined in (44).

**Proof.** The requirements of Assumption 1 are satisfied. Therefore the result follows from Theorem 3. \qed

**Corollary 2** The limiting distribution of $\hat{g}^*_n$ is equal to $Y_{0,\Sigma}$, i.e. it is the same as for the unconstrained least squares estimator $\hat{g}_n$, if and only if the true regression function $\hat{g}(x)$ is strictly bimonotone.

### 4.2 Estimation of a bimonotone probability mass function

Suppose that we have observed $Z_1, Z_2, \ldots, Z_n$ i.i.d. random variables taking values in $\mathcal{X} = \mathbb{Z}^2 = \{(i, j)^T : i = 1, 2, \ldots, \infty, j = 1, 2, \ldots, \infty, \}$ with probability mass function $p$. The empirical estimator of $p$ is then given by

$$\hat{p}_{n,i} = \frac{n_i}{n}, \quad n_i = \sum_{j=1}^n 1\{Z_j = x_i\}, \quad i \in \mathcal{I}$$

and it is also the unrestricted MLE, which generally does not satisfy the bimonotonicity constraints introduced above. However, $\hat{p}_n$ is consistent, i.e. $\hat{p}_n \xrightarrow{p} p$ and asymptotically Gaussian

$$n^{1/2}(\hat{p}_n - p) \xrightarrow{d} Y_{0,C},$$

where $Y_{0,C}$ is a Gaussian process in $l_2$, with mean zero and the covariance operator $C$ such that $<Ce_i, e_{i'}> = \delta_{ii'} p_{ii'}$, with $e_i \in l_2$ the orthonormal basis in $l_2$ such that in a vector $e_i$ all elements are equal to zero but the one with the index $i$ is equal to 1, c.f. [13].
The constrained MLE $\hat{p}_n^*$ of $p$ is then given by the isotonic regression of the empirical estimator $\hat{p}_n$ over the set $\mathcal{X}$ with respect to the pre-order $\preceq$.

\[
\hat{p}_n^* = \arg\min_{\xi \in \mathcal{F}^{bs}} \sum_{x \in \mathcal{X}} (\xi_x - \hat{p}_{n,x})^2,
\]

where $\mathcal{F}^{bs}$ denotes the set of all bimonotone functions on $\mathcal{X}$. This result is the consequence of the problem of maximising the product of several factors, given relations of order and linear side condition, cf. pages 45–46 in [4] and pages 38–39 [16].

Next theorem gives the asymptotic distribution of $\hat{p}_n^*$.

**Theorem 6** The asymptotic distribution of the constrained MLE $\hat{p}_n^*$ of a bimonotone probability mass function $p$ is given by

\[
n^{1/2}(\hat{p}_n^* - p) \xrightarrow{d} \varphi(Y_{0,C}),
\]

where $\varphi$ is the operator defined in (32) and $Y_{0,C}$ is a Gaussian process in $l_2$ defined in (46).

**Proof.** The requirements of Assumption 2 are satisfied. Therefore the result follows from Theorem 3. \hfill \Box

**Corollary 3** The limiting distribution of $\hat{p}_n^*$ is equal to $Y_{0,C}$, i.e. it is the same as for the empirical estimator $\hat{p}_n$, if and only if the true probability mass function is strictly bimonotone.

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