Generalized local Morrey spaces and fractional integral operators with rough kernel

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Abstract

Let $M_{\Omega,\alpha}$ and $I_{\Omega,\alpha}$ be the fractional maximal and integral operators with rough kernels, where $0 < \alpha < n$. In this paper, we shall study the continuity properties of $M_{\Omega,\alpha}$ and $I_{\Omega,\alpha}$ on the generalized local Morrey spaces $LM_{p,\phi}^{\{x_0\}}$. The boundedness of their commutators with local Campanato functions is also obtained.

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1 Introduction

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x, r)$ denote the open ball centered at $x$ of radius $r$ and $|B(x, r)|$ is the Lebesgue measure of the ball $B(x, r)$. Let $\Omega \in L^s(S^{n-1})$ be homogeneous of degree zero on $\mathbb{R}^n$, where $S^{n-1}$ denotes the unit sphere of $\mathbb{R}^n$ ($n \geq 2$) equipped with the normalized Lebesgue measure $d\sigma$ and $s > 1$. For any $0 < \alpha < n$, then the fractional integral operator with rough kernel $I_{\Omega,\alpha}$ is defined by

$$I_{\Omega,\alpha}f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x - y)}{|x - y|^{n-\alpha}} f(y) dy$$

and a related fractional maximal operator with rough kernel $M_{\Omega,\alpha}$ is defined by

$$M_{\Omega,\alpha}f(x) = \sup_{t > 0} |B(x, t)|^{-1 + \frac{\alpha}{n}} \int_{B(x, t)} |\Omega(x - y)| |f(y)| dy.$$
If $\alpha = 0$, then $M_{\Omega} \equiv M_{\Omega,0}$ is the Hardy-Littlewood maximal operator with rough kernel. It is obvious that when $\Omega \equiv 1$, $I_{\Omega,\alpha}$ is the Riesz potential $I_{\alpha}$ and $M_{\Omega,\alpha}$ is the maximal operator $M_{\alpha}$.

**Theorem A** Suppose that $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, be a homogeneous of degree zero. Let $0 < \alpha < n$, $1 \leq p < \frac{n}{\alpha}$, and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. If $s' < p$ or $q < s$, then the operators $M_{\Omega,\alpha}$ and $I_{\Omega,\alpha}$ are bounded bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$.

Let $b$ be a locally integrable function on $\mathbb{R}^n$, then for $0 < \alpha < n$, we shall define the commutators generated by fractional maximal and integral operators with rough kernels and $b$ as follows.

$$M_{\Omega,b,\alpha}(f)(x) = \sup_{t > 0} |B(x,t)|^{-\frac{\alpha}{n}} \int_{B(x,t)} |b(x) - b(y)||f(y)||\Omega(x-y)|dy,$$

$$[b, I_{\Omega,\alpha}]f(x) = b(x)I_{\Omega,\alpha}f_1(x) - I_{\Omega,\alpha}(bf)(x)$$

$$= \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}}[b(x) - b(y)]f(y)dy.$$

**Theorem B** Suppose that $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, be a homogeneous of degree zero. Let $0 < \alpha < n$, $1 \leq p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $b \in BMO(\mathbb{R}^n)$. If $s' < p$ or $q < s$, then the operators $M_{\Omega,b,\alpha}$ and $[b, I_{\Omega,\alpha}]$ are bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$.

The classical Morrey spaces $M_{p,\lambda}$ were first introduced by Morrey in [35] to study the local behavior of solutions to second order elliptic partial differential equations. For the boundedness of the Hardy-Littlewood maximal operator, the fractional integral operator and the Calderón-Zygmund singular integral operator on these spaces, we refer the readers to [1] [11] [39]. For the properties and applications of classical Morrey spaces, see [12] [13] [22] [23] and references therein.

In the paper, we prove the boundedness of the operators $I_{\Omega,\alpha}$ from one generalized local Morrey space $LM_{p,\varphi_1}^{\{x_0\}}$ to $LM_{q,\varphi_2}^{\{x_0\}}$, $1 < p < q < \infty$, $1/p - 1/q = \alpha/n$, and from the space $LM_{p,\varphi_1}^{\{x_0\}}$ to the weak space $WLM_{q,\varphi_2}^{\{x_0\}}$, $1 < q < \infty$, $1 - 1/q = \alpha/n$. In the case $b \in CBMO_{p_2}$, we find the sufficient conditions on the pair $(\varphi_1, \varphi_2)$ which ensures the boundedness of the commutator operators $[b, I_{\Omega,\alpha}]$ from $LM_{p_1,\varphi_1}^{\{x_0\}}$ to $LM_{q,\varphi_2}^{\{x_0\}}$, $1 < p < \infty$, $1/p = 1/p_1 + 1/p_2$, $1/q = 1/p - \alpha/n$, $1/q_1 = 1/p_1 - \alpha/n$.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant $C$ independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that $A$ and $B$ are equivalent.
2 Generalized local Morrey spaces

We find it convenient to define the generalized Morrey spaces in the form as follows.

**Definition 2.1.** Let \( \varphi(x, r) \) be a positive measurable function on \( \mathbb{R}^n \times (0, \infty) \) and \( 1 \leq p < \infty \). We denote by \( M_{p, \varphi} \equiv M_{p, \varphi}(\mathbb{R}^n) \) the generalized Morrey space, the space of all functions \( f \in L^p_{\text{loc}}(\mathbb{R}^n) \) with finite quasinorm
\[
\|f\|_{M_{p, \varphi}} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x, r)^{-1} |B(x, r)|^{-\frac{1}{p}} \|f\|_{L^p(B(x, r))}.
\]
Also by \( W M_{p, \varphi} \equiv W M_{p, \varphi}(\mathbb{R}^n) \) we denote the weak generalized Morrey space of all functions \( f \in W L^p_{\text{loc}}(\mathbb{R}^n) \) for which
\[
\|f\|_{W M_{p, \varphi}} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x, r)^{-1} |B(x, r)|^{-\frac{1}{p}} \|f\|_{W L^p(B(x, r))} < \infty.
\]
According to this definition, we recover the Morrey space \( M_{p, \lambda} \) and weak Morrey space \( W M_{p, \lambda} \) under the choice \( \varphi(x, r) = r^{\lambda - n/p} \):
\[
M_{p, \lambda} = M_{p, \varphi}\big|_{\varphi(x, r) = r^{\lambda - n/p}}, \quad W M_{p, \lambda} = W M_{p, \varphi}\big|_{\varphi(x, r) = r^{\lambda - n/p}}.
\]

**Definition 2.2.** Let \( \varphi(x, r) \) be a positive measurable function on \( \mathbb{R}^n \times (0, \infty) \) and \( 1 \leq p < \infty \). We denote by \( LM_{p, \varphi} \equiv LM_{p, \varphi}(\mathbb{R}^n) \) the generalized local Morrey space, the space of all functions \( f \in L^p_{\text{loc}}(\mathbb{R}^n) \) with finite quasinorm
\[
\|f\|_{LM_{p, \varphi}} = \sup_{r > 0} \varphi(0, r)^{-1} |B(0, r)|^{-\frac{1}{p}} \|f\|_{L^p(B(0, r))}.
\]
Also by \( W LM_{p, \varphi} \equiv W LM_{p, \varphi}(\mathbb{R}^n) \) we denote the weak generalized Morrey space of all functions \( f \in W L^p_{\text{loc}}(\mathbb{R}^n) \) for which
\[
\|f\|_{W LM_{p, \varphi}} = \sup_{r > 0} \varphi(0, r)^{-1} |B(0, r)|^{-\frac{1}{p}} \|f\|_{W L^p(B(0, r))} < \infty.
\]

**Definition 2.3.** Let \( \varphi(x, r) \) be a positive measurable function on \( \mathbb{R}^n \times (0, \infty) \) and \( 1 \leq p < \infty \). For any fixed \( x_0 \in \mathbb{R}^n \) we denote by \( LM_{p, \varphi}^{(x_0)} \equiv LM_{p, \varphi}^{(x_0)}(\mathbb{R}^n) \) the generalized local Morrey space, the space of all functions \( f \in L^p_{\text{loc}}(\mathbb{R}^n) \) with finite quasinorm
\[
\|f\|_{LM_{p, \varphi}^{(x_0)}} = \|f(x_0 + \cdot)\|_{LM_{p, \varphi}}.
\]
Also by \( W LM_{p, \varphi}^{(x_0)} \equiv W LM_{p, \varphi}^{(x_0)}(\mathbb{R}^n) \) we denote the weak generalized Morrey space of all functions \( f \in W L^p_{\text{loc}}(\mathbb{R}^n) \) for which
\[
\|f\|_{W LM_{p, \varphi}^{(x_0)}} = \|f(x_0 + \cdot)\|_{W LM_{p, \varphi}} < \infty.
\]
According to this definition, we recover the local Morrey space \( LM_{p,\lambda}^{(x_0)} \) and weak local Morrey space \( WLM_{p,\lambda}^{(x_0)} \) under the choice \( \varphi(x_0, r) = r^{\frac{\lambda-n}{n}} \):

\[
LM_{p,\lambda}^{(x_0)} = LM_{p,\varphi}^{(x_0)} \bigg|_{\varphi(x_0, r) = r^{\frac{\lambda-n}{n}}}, \quad WLM_{p,\lambda}^{(x_0)} = WLM_{p,\varphi}^{(x_0)} \bigg|_{\varphi(x_0, r) = r^{\frac{\lambda-n}{n}}}. 
\]

Wiener [45, 46] looked for a way to describe the behavior of a function at the infinity. The conditions he considered are related to appropriate weighted \( L_q \) spaces. Beurling [4] extended this idea and defined a pair of dual Banach spaces \( B_{\mu} \) between central \( \mu \in \mathbb{R} \) spaces. The conditions he considered are related to appropriate weighted \( \text{mean oscillation estimates for their commutators.} \)

Let \( \tilde{B}_q(\mathbb{R}^n) \) and \( \tilde{A}_q(\mathbb{R}^n) \) be the homogeneous versions of \( B_q(\mathbb{R}^n) \) and \( A_q(\mathbb{R}^n) \) by taking \( k \in \mathbb{Z} \) in (2.1) and (2.2) instead of \( k \geq 0 \) there.

If \( \lambda < 0 \) or \( \lambda > n \), then \( LM_{p,\lambda}^{(x_0)}(\mathbb{R}^n) = \Theta \), where \( \Theta \) is the set of all functions equivalent to 0 on \( \mathbb{R}^n \). Note that \( LM_{p,0}(\mathbb{R}^n) = L_p(\mathbb{R}^n) \) and \( LM_{p,n}(\mathbb{R}^n) = \tilde{B}_p(\mathbb{R}^n) \).

\[
\tilde{B}_{p,\mu} = LM_{p,\varphi} \bigg|_{\varphi(0, r) = r^{\mu n}}, \quad W\tilde{B}_{p,\mu} = WLM_{p,\varphi} \bigg|_{\varphi(0, r) = r^{\mu n}}.
\]

Alvarez, Guzman-Partida and Lakey [3] in order to study the relationship between central \( BMO \) spaces and Morrey spaces, they introduced \( \lambda \)-central bounded mean oscillation spaces and central Morrey spaces \( \tilde{B}_{p,\mu}(\mathbb{R}^n) \equiv LM_{p,n+np\mu}(\mathbb{R}^n) \), \( \mu \in \left[ -\frac{1}{p}, 0 \right] \). If \( \mu < -\frac{1}{p} \) or \( \mu > 0 \), then \( \tilde{B}_{p,\mu}(\mathbb{R}^n) = \Theta \). Note that \( \tilde{B}_{p,-\frac{1}{p}}(\mathbb{R}^n) = L_p(\mathbb{R}^n) \) and \( \tilde{B}_{p,0}(\mathbb{R}^n) = \tilde{B}_p(\mathbb{R}^n) \). Also define the weak central Morrey spaces \( W\tilde{B}_{p,\mu}(\mathbb{R}^n) \equiv WLM_{p,n+np\mu}(\mathbb{R}^n) \).

Inspired by this, we consider the boundedness of fractional integral operator with rough kernel on generalized local Morrey spaces and give the central bounded mean oscillation estimates for their commutators.
Fractional integral operator with rough kernels in the spaces $L^p_{\{x_0\}}$

In this section we are going to use the following statement on the boundedness of the weighted Hardy operator

$$H_w^*g(t) := \int_t^{\infty} g(s)w(s)ds, \ 0 < t < \infty,$$

where $w$ is a fixed function non-negative and measurable on $(0,1)$.

**Theorem 3.1.** Let $v_1$, $v_2$ and $w$ be positive almost everywhere and measurable functions on $(0,1)$. The inequality

$$\text{ess sup}_{t>0} v_2(t)H_w^*g(t) \leq C \text{ess sup}_{t>0} v_1(t)g(t) \quad (3.1)$$

holds for some $C > 0$ for all non-negative and non-decreasing $g$ on $(0,1)$ if and only if

$$B := \text{ess sup}_{t>0} v_2(t) \int_t^{\infty} \frac{w(s)ds}{\text{ess sup}_{s<\tau<\infty} v_1(\tau)} < 1. \quad (3.2)$$

Moreover, if $C^*$ is the minimal value of $C$ in (3.1), then $C^* = B$.

**Proof. Sufficiency.** Assume that (3.2) holds. Whenever $F$, $G$ are non-negative functions on $(0,1)$ and $F$ is non-decreasing, then

$$\text{ess sup}_{t>0} F(t)G(t) = \text{ess sup}_{t>0} F(t) \text{ess sup}_{s>t} G(s), \ t > 0. \quad (3.3)$$

By (3.3) we have

$$\text{ess sup}_{t>0} v_2(t)H_w^*g(t) = \text{ess sup}_{t>0} v_2(t) \int_t^{\infty} g(s)w(s) \frac{\text{ess sup}_{s<\tau<\infty} v_1(\tau)}{\text{ess sup}_{s<\tau<\infty} v_1(\tau)} ds$$

$$\leq \text{ess sup}_{t>0} v_2(t) \int_t^{\infty} \frac{w(s)ds}{\text{ess sup}_{s<\tau<\infty} v_1(\tau)} \text{ess sup}_{t<\tau<1} g(t) \text{ess sup}_{t<\tau<1} v_1(\tau)$$

$$= \text{ess sup}_{t>0} v_2(t) \int_t^{\infty} \frac{w(s)ds}{\text{ess sup}_{s<\tau<\infty} v_1(\tau)} \text{ess sup}_{t<\tau<1} g(t)v_1(t)$$

$$\leq B \text{ess sup}_{t>0} g(t)v_1(t).$$

**Necessity.** Assume that the inequality (3.1) holds. The function

$$g(t) = \frac{1}{\text{ess sup}_{t<\tau<1} v_1(\tau)}, \ t > 0$$
is nonnegative and non-decreasing on (0, 1). Thus
\[ B = \text{ess sup}_{t>0} v_2(t) \int_{t}^{\infty} \frac{w(s) ds}{\text{ess sup}_{s<\tau<1} v_1(\tau)} \leq C \text{ess sup}_{t>0} \frac{v_1(t)}{\text{ess sup}_{t<\tau<1} v_1(\tau)} \leq C, \]

hence \( C^n = B \).

In [17] the following statement was proved by fractional integral operator with rough kernels \( I_{\Omega, \alpha} \), containing the result in [34, 36].

**Theorem 3.2.** Suppose that \( \Omega \in L_s(S^{n-1}), 1 < s \leq \infty \), be a homogeneous of degree zero. Let \( 0 < \alpha < n, 1 \leq s' < p < \frac{n}{\alpha}, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \) and \( \varphi(x, r) \) satisfy conditions
\[ c^{-1}\varphi(x, r) \leq \varphi(x, t) \leq c \varphi(x, r) \tag{3.4} \]
whenever \( r \leq t \leq 2r \), where \( c (\geq 1) \) does not depend on \( t, r, x \in \mathbb{R}^n \) and
\[ \int_{r}^{\infty} t^{\alpha p} \varphi(x, t)^p \frac{dt}{t} \leq C r^{\alpha p} \varphi(x, r)^p, \tag{3.5} \]
where \( C \) does not depend on \( x \) and \( r \). Then the operators \( M_{\Omega, \alpha} \) and \( I_{\Omega, \alpha} \) are bounded from \( M_p, \varphi \) to \( M_q, \varphi \).

The following statements, containing results obtained in [34], [36] was proved in [26, 28] (see also [5]-[8], [27, 29]).

**Theorem 3.3.** Let \( 0 < \alpha < n, 1 \leq p < \frac{n}{\alpha}, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \) and \( (\varphi_1, \varphi_2) \) satisfy the condition
\[ \int_{r}^{\infty} t^{\alpha - 1} \varphi_1(0, t) dt \leq C \varphi_2(0, r), \tag{3.6} \]
where \( C \) does not depend on \( r \). Then the operators \( M_{\alpha} \) and \( I_{\alpha} \) are bounded from \( LM_{p, \varphi_1} \) to \( LM_{q, \varphi_2} \) for \( p > 1 \) and from \( LM_{1, \varphi_1} \) to \( WLM_{q, \varphi_2} \) for \( p = 1 \).

**Lemma 3.4.** Suppose that \( x_0 \in \mathbb{R}^n, \Omega \in L_s(S^{n-1}), 1 < s \leq \infty \), be a homogeneous of degree zero. Let \( 0 < \alpha < n, 1 \leq s' < p < \frac{n}{\alpha}, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \). Then, for \( p > 1 \) and \( s' \leq p \) or \( q < s \) the inequality
\[ \| I_{\Omega, \alpha} f \|_{L_q(B(x_0, r))} \lesssim r^{\frac{n}{q}} \int_{2r}^{r} t^{-\frac{n}{q} - 1} \| f \|_{L_p(B(x_0, t))} dt \]
holds for any ball \( B(x_0, r) \) and for all \( f \in L^\text{loc}_p(\mathbb{R}^n) \).

Moreover, for \( p = 1 < q < s \) the inequality
\[ \| I_{\Omega, \alpha} f \|_{W L_q(B(x_0, r))} \lesssim r^{\frac{n}{q}} \int_{2r}^{r} t^{-\frac{n}{q} - 1} \| f \|_{L_1(B(x_0, t))} dt, \tag{3.7} \]
holds for any ball \( B(x_0, r) \) and for all \( f \in L^\text{loc}_1 \).
Proof. Let $0 < \alpha < n$, $1 \leq s' \leq p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Set $B = B(x_0, r)$ for the ball centered at $x_0$ and of radius $r$. We represent $f$ as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2B}(y), \quad f_2(y) = f(y)\chi_{(2B)}(y), \quad r > 0, \quad (3.8)$$

and have

$$\|I_{\Omega,\alpha}f\|_{L_q(B)} \leq \|I_{\Omega,\alpha}f_1\|_{L_q(B)} + \|I_{\Omega,\alpha}f_2\|_{L_q(B)}.$$ 

Since $f_1 \in L_p(\mathbb{R}^n)$, $I_{\Omega,\alpha}f_1 \in L_q(\mathbb{R}^n)$ and from the boundedness of $I_{\Omega,\alpha}$ from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ it follows that:

$$\|I_{\Omega,\alpha}f_1\|_{L_q(B)} \leq \|I_{\Omega,\alpha}f_1\|_{L_q(\mathbb{R}^n)} \leq C\|f_1\|_{L_p(\mathbb{R}^n)} = C\|f\|_{L_p(2B)},$$

where constant $C > 0$ is independent of $f$.

It’s clear that $x \in B$, $y \in B(2B)$ implies $\frac{1}{2}|x_0 - y| \leq |x - y| \leq \frac{3}{2}|x_0 - y|$. We get

$$|I_{\Omega,\alpha}f_2(x)| \leq 2^{n-\alpha}c_1 \int_{B(2B)} |f(y)||\Omega(x-y)| \frac{dy}{|x_0 - y|^{n-\alpha}}.$$

By Fubini’s theorem we have

$$\int_{B(2B)} |f(y)||\Omega(x-y)| \frac{dy}{|x_0 - y|^{n-\alpha}} \approx \int_{B(2B)} |f(y)||\Omega(x-y)| \int_{|x_0-y|}^{t} \frac{dt}{t^{n+1-\alpha}} dy$$

$$\approx \int_{2r}^{t} \int_{|x_0-y|}^{t} |f(y)||\Omega(x-y)| dy \frac{dt}{t^{n+1-\alpha}}$$

$$\lesssim \int_{2r}^{t} \int_{B(x_0,t)} |f(y)||\Omega(x-y)| dy \frac{dt}{t^{n+1-\alpha}}.$$

Applying Hölder’s inequality, we get

$$\int_{B(2B)} |f(y)||\Omega(x-y)| \frac{dy}{|x_0 - y|^{n-\alpha}}$$

$$\lesssim \int_{2r}^{t} \|f\|_{L_p(B(x_0,t))} \|\Omega(\cdot - y)\|_{L_q(B(x_0,r))} |B(x_0,t)|^{1-\frac{1}{p} - \frac{1}{q}} \frac{dt}{t^{n+1-\alpha}} \quad (3.9)$$

Moreover, for all $p \in [1,1)$ the inequality

$$\|I_{\Omega,\alpha}f_2\|_{L_q(B)} \lesssim r^\frac{n}{q} \int_{2r}^{t} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}}. \quad (3.10)$$

is valid. Thus

$$\|I_{\Omega,\alpha}f\|_{L_q(B)} \lesssim \|f\|_{L_p(2B)} + r^\frac{n}{q} \int_{2r}^{t} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}}.$$
On the other hand,
\[ \| f \|_{L_p(2B)} \approx r^{\frac{n}{q}} \| f \|_{L_p(2B)} \int_{2r}^{r} \frac{dt}{t^{\frac{n}{q}+1}} \]
\[ \leq r^{\frac{n}{q}} \int_{2r}^{r} \| f \|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}} . \] (3.11)

Thus
\[ \| I_{\Omega,\alpha} f \|_{L_q(B)} \lesssim r^{\frac{n}{q}} \int_{2r}^{r} \| f \|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}} . \]

When \( 1 < q < s \), by Fubini's theorem and the Minkowski inequality, we get
\[ \| I_{\Omega,\alpha} f_2 \|_{L_q(B)} \leq \left( \int_B \left( \int_{2r}^{r} \int_{B(x_0,t)} |f(y)| \| \Omega(x-y) \|_{L_q(B)} dy \frac{dt}{t^{n+1-\alpha}} \right)^q \right)^{\frac{1}{q}} \]
\[ \leq \int_{2r}^{r} \int_{B(x_0,t)} |f(y)| \| \Omega(x-y) \|_{L_q(B)} dy \frac{dt}{t^{n+1-\alpha}} \]
\[ \leq r^{\frac{n}{q}} \int_{2r}^{r} \| f \|_{L_1(B(x_0,t))} \frac{dt}{t^{n+1-\alpha}} \]
\[ \lesssim r^{\frac{n}{q}} \int_{2r}^{r} \| f \|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}} . \] (3.12)

Let \( p = 1 < q < s \leq 1 \). From the weak \((1,q)\) boundedness of \( I_{\Omega,\alpha} \) and (3.11) it follows that:
\[ \| I_{\Omega,\alpha} f \|_{W L_q(B)} \leq \| I_{\Omega,\alpha} f \|_{W L_q(\mathbb{R}^n)} \lesssim \| f \|_{L_1(\mathbb{R}^n)} \]
\[ = \| f \|_{L_1(2B)} \lesssim r^{\frac{n}{q}} \int_{2r}^{r} \| f \|_{L_1(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}} . \] (3.13)

Then from (3.10) and (3.13) we get the inequality (3.7). \( \square \)

**Theorem 3.5.** Suppose that \( x_0 \in \mathbb{R}^n \), \( \Omega \in L_s(S^{n-1}) \), \( 1 < s \leq \infty \), be a homogeneous of degree zero. Let \( 0 < \alpha < n \), \( 1 \leq p < \frac{n}{\alpha} \), \( \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \), and \( s' \leq p \) or \( q < s \). Let also, the pair \((\varphi_1, \varphi_2)\) satisfy the condition
\[ \int_r^{\infty} \inf_{t < r < \infty} \varphi_1(x_0, t) \left( \frac{t}{r} \right)^{\frac{\alpha}{q}} dt \leq C \varphi_2(x_0, r) . \] (3.14)

where \( C \) does not depend on \( r \). Then the operators \( M_{\Omega,\alpha} \) and \( I_{\Omega,\alpha} \) are bounded from \( LM_{p,\varphi_1}^{(x_0)} \) to \( LM_{q,\varphi_2}^{(x_0)} \) for \( p > 1 \) and from \( LM_{1,\varphi_1}^{(x_0)} \) to \( WLM_{q,\varphi_2}^{(x_0)} \) for \( p = 1 \). Moreover, for \( p > 1 \)
\[ \| M_{\Omega,\alpha} f \|_{LM_{q,\varphi_2}^{(x_0)}} \lesssim \| I_{\Omega,\alpha} f \|_{LM_{q,\varphi_2}^{(x_0)}} \lesssim \| f \|_{LM_{p,\varphi_1}^{(x_0)}} . \]
and for $p = 1$

$$\|M_{\Omega, \alpha}f\|_{WLM_{q, \varphi_2}^{(x_0)}} \lesssim \|I_{\Omega, \alpha}f\|_{WLM_{q, \varphi_2}^{(x_0)}} \lesssim \|f\|_{LM_{1, \varphi_1}}.$$ 

Proof. By Lemma 3.4 and Theorem 3.1 with $v_2(r) = \varphi_2(x_0, r)^{-1}$, $v_1(r) = \varphi_1(x_0, r)^{-1}r^{-\frac{n}{p}}$ and $w(r) = r^{-\frac{n}{p}}$ we have for $p > 1$

$$\|I_{\Omega, \alpha}f\|_{LM_{q, \varphi_2}^{(x_0)}} \lesssim \sup_{r > 0} \varphi_2(x_0, r)^{-1} \int_r^1 \|f\|_{L_p(B(x_0, t))} \frac{dt}{t^{\frac{n}{q} + 1}}$$

$$\lesssim \sup_{r > 0} \varphi_1(x_0, r)^{-1} r^{-\frac{n}{p}} \|f\|_{L_p(B(x_0, r))} = \|f\|_{LM_{1, \varphi_1}}$$

and for $p = 1$

$$\|I_{\Omega, \alpha}f\|_{WLM_{q, \varphi_2}^{(x_0)}} \lesssim \sup_{r > 0} \varphi_2(x_0, r)^{-1} \int_r^1 \|f\|_{L_1(B(x_0, t))} \frac{dt}{t^{\frac{n}{q} + 1}}$$

$$\lesssim \sup_{r > 0} \varphi_1(x_0, r)^{-1} \|f\|_{L_1(B(x_0, r))} = \|f\|_{LM_{1, \varphi_1}}.$$ 

\[\square\]

Corollary 3.6. Suppose that $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, be a homogeneous of degree zero. Let $0 < \alpha < n$, $1 \leq p < \frac{n}{\alpha}$, \(\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}\), and $s' \leq p$ or $q < s$. Let also, the pair $(\varphi_1, \varphi_2)$ satisfy the condition

$$\int_r^\infty \text{ess inf}_{t < \tau < \infty} \frac{\varphi_1(x, \tau) \tau^n}{t^{\frac{n}{q} + 1}} dt \leq C \varphi_2(x, r),$$

where $C$ does not depend on $x$ and $r$. Then the operators $M_{\Omega, \alpha}$ and $I_{\Omega, \alpha}$ are bounded from $M_{p, \varphi_2}$ to $M_{q, \varphi_2}$ for $p > 1$ and from $M_{1, \varphi_1}$ to $WLM_{q, \varphi_2}$ for $p = 1$. Moreover, for $p > 1$

$$\|M_{\Omega, \alpha}f\|_{M_{q, \varphi_2}} \lesssim \|I_{\Omega, \alpha}f\|_{M_{q, \varphi_2}} \lesssim \|f\|_{M_{p, \varphi_1}},$$

and for $p = 1$

$$\|M_{\Omega, \alpha}f\|_{WLM_{q, \varphi_2}^{(x_0)}} \lesssim \|I_{\Omega, \alpha}f\|_{WLM_{q, \varphi_2}^{(x_0)}} \lesssim \|f\|_{M_{1, \varphi_1}}.$$ 

Corollary 3.7. Let $1 \leq p < \infty$, $0 < \alpha < \frac{n}{p}$, \(\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}\) and $(\varphi_1, \varphi_2)$ satisfy condition (3.14). Then the operators $M_\alpha$ and $I_\alpha$ are bounded from $LM_{p, \varphi_2}^{(x_0)}$ to $LM_{q, \varphi_2}^{(x_0)}$ for $p > 1$ and from $M_{1, \varphi_1}$ to $WLM_{q, \varphi_2}^{(x_0)}$ for $p = 1$.

Remark 3.8. Note that, in the case $s = 1$ Corollary 3.6 was proved in [29]. The condition (3.14) in Theorem 3.5 is weaker than condition (3.6) in Theorem 3.3 (see [29]).
4 Commutators of fractional integral operator with rough kernels in the spaces $LM_{p,\varphi}^{x_0}$

Let $T$ be a linear operator, for a function $b$, we define the commutator $[b, T]$ by

$$[b, T]f(x) = b(x) Tf(x) - T(bf)(x)$$

for any suitable function $f$. If $\tilde{T}$ be a Calderón-Zygmund singular integral operator, a well known result of Coifman, Rochberg and Weiss [14] states that the commutator $[b, \tilde{T}]f = b\tilde{T}f - \tilde{T}(bf)$ is bounded on $L_p(\mathbb{R}^n)$, $1 < p < \infty$, if and only if $b \in BMO(\mathbb{R}^n)$. The commutator of Calderón-Zygmund operators plays an important role in studying the regularity of solutions of elliptic partial differential equations of second order (see, for example, [12, 13, 22]). In [9], Chanillo proved that the commutator $[b, I_\alpha]\hat{f} = b I_\alpha f - I_\alpha (bf)$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$, $(1 < p < q < \infty, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n})$ if and only if $b \in BMO(\mathbb{R}^n)$.

The definition of local Campanato space as follows.

**Definition 4.1.** Let $1 \leq q < 1$ and $0 \leq \lambda < \frac{1}{n}$. A function $f \in L^0_{loc}(\mathbb{R}^n)$ is said to belong to the $CBMO^{(x_0),\lambda}_{q}\mathbb{R}^n$ (central Campanato space), if

$$\|f\|_{CBMO^{(x_0),\lambda}_{q}\mathbb{R}^n} = \sup_{r>0} \left( \frac{1}{|B(x_0, r)|^{1+\lambda}} \int_{B(x_0, r)} |f(y) - f_{B(x_0, r)}|^{q} dy \right)^{1/q} < \infty,$$

where

$$f_{B(x_0, r)} = \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} f(y) dy.$$

Define

$$CBMO^{(x_0)}_{q,\lambda}(\mathbb{R}^n) = \{ f \in L^0_{loc}(\mathbb{R}^n) : \|f\|_{CBMO^{(x_0),\lambda}_{q}\mathbb{R}^n} < 1 \}.$$

In [30], Lu and Yang introduced the central BMO space $CBMO_q(\mathbb{R}^n) = CBMO^0_{q,0}(\mathbb{R}^n)$. Note that, $BMO(\mathbb{R}^n) \subset CBMO^0_{q}(\mathbb{R}^n)$, $1 \leq q < 1$. The space $CBMO^0_{q} (\mathbb{R}^n)$ can be regarded as a local version of $BMO(\mathbb{R}^n)$, the space of bounded mean oscillation, at the origin. But, they have quite different properties. The classical John-Nirenberg inequality shows that functions in $BMO(\mathbb{R}^n)$ are locally exponentially integrable. This implies that, for any $1 \leq q < 1$, the functions in $BMO(\mathbb{R}^n)$ can be described by means of the condition:

$$\sup_{r>0} \left( \frac{1}{|B|} \int_{B} |f(y) - f_{B}|^{q} dy \right)^{1/q} < \infty,$$

where $B$ denotes an arbitrary ball in $\mathbb{R}^n$. However, the space $CBMO^0_{q}(\mathbb{R}^n)$ depends on $q$. If $q_1 < q_2$, then $CBMO^0_{q_1}(\mathbb{R}^n) \not\subset CBMO^0_{q_2}(\mathbb{R}^n)$. Therefore, there is no analogy of the famous John-Nirenberg inequality of $BMO(\mathbb{R}^n)$ for the space $CBMO^0_{q}(\mathbb{R}^n)$. One can imagine that the behavior of $CBMO^0_{q}(\mathbb{R}^n)$ may be quite different from that of $BMO(\mathbb{R}^n)$. 

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Lemma 4.2. Let $b$ be a function in $\text{CBMO}_q^{(\alpha)}(\mathbb{R}^n)$, $1 \leq q < \infty$, $0 \leq \lambda < \frac{1}{n}$ and $r_1, r_2 > 0$. Then
\[
\left( \frac{1}{|B(x_0, r_1)|^{1+\lambda q}} \int_{B(x_0, r_1)} |b(y) - b_{B(x_0, r_2)}|^q dy \right)^{\frac{1}{q}} \leq C \left( 1 + \left| \ln \frac{r_1}{r_2} \right| \right) \|b\|_{\text{CBMO}^{(\alpha)}_q},
\]
where $C > 0$ is independent of $b$, $r_1$ and $r_2$.

In [17] the following statement was proved for the commutators of fractional integral operators with rough kernels, containing the result in [34, 36].

Theorem 4.3. Suppose that $x_0 \in \mathbb{R}^n$, $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, be a homogeneous of degree zero and $b \in \text{BMO}(\mathbb{R}^n)$. Let $0 < \alpha < n$, $1 \leq s' < p < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $\varphi(x, r)$ which satisfies the conditions (3.4) and (3.5). Then the operator $[b, I_{\Omega, \alpha}]$ is bounded from $M_{p, \varphi}$ to $M_{q, \varphi}$.

Lemma 4.4. Suppose that $x_0 \in \mathbb{R}^n$, $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, be a homogeneous of degree zero. Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $b \in \text{CBMO}^{(\alpha)}_q(\mathbb{R}^n)$, $0 \leq \lambda < \frac{1}{n}$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $q_1 = \frac{1}{p_1} - \frac{\alpha}{n}$.

Then, for $s' \leq p$ or $q_1 < s$ the inequality
\[
\| [b, I_{\Omega, \alpha}] f \|_{L_q(B(x_0, r))} \lesssim \| b \|_{\text{CBMO}^{(\alpha)}_q} r^\frac{n}{q} \int_{2r}^r \left( 1 + \ln \frac{t}{r} \right)^{n\lambda - \frac{n}{q_1} - 1} \| f \|_{L_{p_1}(B(x_0, t))} dt
\]
holds for any ball $B(x_0, r)$ and for all $f \in L_{p_1}^{\text{loc}}(\mathbb{R}^n)$.

Proof. Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, and $q_1 = \frac{1}{p_1} - \frac{\alpha}{n}$. As in the proof of Lemma 3.4, we represent function $f$ in form (3.8) and have
\[
[b, I_{\Omega, \alpha}] f (x) = (b(x) - b_B) I_{\Omega, \alpha} f_1 (x) - I_{\Omega, \alpha} \left( (b(\cdot) - b_B) f_1 \right) (x)
+ (b(x) - b_B) I_{\Omega, \alpha} f_2 (x) - I_{\Omega, \alpha} \left( (b(\cdot) - b_B) f_2 \right) (x)
\equiv J_1 + J_2 + J_3 + J_4.
\]

Hence we get
\[
\| [b, I_{\Omega, \alpha}] f \|_{L_q(B)} \leq \| J_1 \|_{L_q(B)} + \| J_2 \|_{L_q(B)} + \| J_3 \|_{L_q(B)} + \| J_4 \|_{L_q(B)}.
\]

From the boundedness of $[b, I_{\Omega, \alpha}]$ from $L_{p_1}(\mathbb{R}^n)$ to $L_{q_1}(\mathbb{R}^n)$ it follows that:
\[
\| J_1 \|_{L_q(B)} \leq \| (b(\cdot) - b_B) [b, I_{\Omega, \alpha}] f_1 (\cdot) \|_{L_q(\mathbb{R}^n)}
\leq \| (b(\cdot) - b_B) \|_{L_{p_2}(\mathbb{R}^n)} [b, I_{\Omega, \alpha}] f_1 (\cdot) \|_{L_{q_1}(\mathbb{R}^n)}
\leq C \| b \|_{\text{CBMO}^{(\alpha)}_q} r^{\frac{n}{p_2} + \frac{n\lambda}{q_1}} \| f_1 \|_{L_{p_1}(\mathbb{R}^n)}
= C \| b \|_{\text{CBMO}^{(\alpha)}_q} r^{\frac{n}{q_1} + \frac{n\lambda}{q_1}} \| f \|_{L_{p_1}(B)} \int_{2r}^\infty t^{-1 - \frac{n}{q_1}} dt
\lesssim \| b \|_{\text{CBMO}^{(\alpha)}_q} \int_{2r}^\infty \left( 1 + \ln \frac{t}{r} \right) \| f \|_{L_{p_1}(B(x_0, t))} t^{-1 - \frac{n}{q_1}} dt.
\]
For $J_2$ we have

$$
\|J_2\|_{L_q(B)} \leq \| [b, I_{\Omega, \alpha}] (b(\cdot) - b_B) f_1 \|_{L_q(\mathbb{R}^n)}
$$

$$
\lesssim \| (b(\cdot) - b_B) f_1 \|_{L_p(\mathbb{R}^n)}
$$

$$
\lesssim \| b(\cdot) - b_B \|_{L_{p_2}(\mathbb{R}^n)} \| f_1 \|_{L_{p_1}(\mathbb{R}^n)}
$$

$$
\lesssim \| b \|_{CBMO_{p_2, \lambda}} r^{\frac{n}{p_2} + \frac{n}{p_1} + n \lambda} \| f \|_{L_{p_1}(2B)} \int_{2r}^{\infty} t^{-1 - \frac{n}{q_1}} dt
$$

$$
\lesssim \| b \|_{CBMO_{p_2, \lambda}} r^{\frac{n}{p_2} + n \lambda} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \| f \|_{L_{p_1}(B(x_0, t))} t^{-1 - \frac{n}{q_1}} dt.
$$

For $J_3$, it is known that $x \in B$, $y \in c(2B)$, which implies $\frac{1}{2} |x_0 - y| \leq |x - y| \leq \frac{3}{2} |x_0 - y|.$

When $s' \leq p$, by Fubini's theorem and applying Hölder inequality we have

$$
|I_{\Omega, \alpha} f_2(x)| \leq c_0 \int_{c(2B)} \Omega(x - y) \frac{|f(y)|}{|x_0 - y|^{n - \alpha}} dy
$$

$$
\lesssim \int_{2r}^{\infty} \int_{2r < |x_0 - y| < t} \Omega(x - y) |f(y)| dy t^{-1 - n - \alpha} dt
$$

$$
\lesssim \int_{2r}^{\infty} \int_{B(x_0, t)} \Omega(x - y) |f(y)| dy t^{-1 - n - \alpha} dt
$$

$$
\lesssim \int_{2r}^{\infty} \| f \|_{L_{p_1}(B(x_0, t))} \| \Omega(x - \cdot) \|_{L_s(B(x_0, t))} |B(B(x_0, t))|^{1 - \frac{1}{p_1} - \frac{1}{s}} t^{-1 - \frac{n}{q_1} - \alpha} dt
$$

$$
\lesssim \int_{2r}^{\infty} \| f \|_{L_{p_1}(B(x_0, t))} t^{-1 - \frac{n}{q_1}} dt.
$$

Hence, we get

$$
\| J_3 \|_{L_q(B)} = \| (b(\cdot) - b_B) I_{\Omega, \alpha} f_2(\cdot) \|_{L_q(\mathbb{R}^n)}
$$

$$
\leq \| (b(\cdot) - b_B) \|_{L_q(\mathbb{R}^n)} \int_{2r}^{\infty} \| f \|_{L_{p_1}(B(x_0, t))} t^{-1 - \frac{n}{q_1}} dt
$$

$$
\leq \| (b(\cdot) - b_B) \|_{L_{p_2}(\mathbb{R}^n)} r^{\frac{n}{q_1}} \int_{2r}^{\infty} \| f \|_{L_{p_1}(B(x_0, t))} t^{-1 - \frac{n}{q_1}} dt
$$

$$
\lesssim \| b \|_{CBMO_{p_2, \lambda}} r^{\frac{n}{q_1} + n \lambda} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \| f \|_{L_{p_1}(B(x_0, t))} t^{-1 - \frac{n}{q_1}} dt.
$$
When \( q_1 < s \), by Fubini's theorem and the Minkowski inequality, we get

\[
\|J_3\|_{L_q(B)} \leq \left( \int_B \left| \int_{2r}^{r} \int_B \Omega(x - y) |f(y)| dy \right| \right)^{\frac{1}{q}} \cdot \|b(x) - b_B\|_{L_q(B)} \leq \left( \int_B \left( \int_{2r}^{r} \int_B \Omega(x - y) |f(y)| dy \right) \right)^{\frac{1}{q}} \cdot \|b(x) - b_B\|_{L_q(B)}
\]

For \( x \in B \) by Fubini's theorem and applying Hölder inequality we have

\[
|I_{\Omega,a}(b(x) - b_B) f_2(x)| \lesssim \int_{\ell(2B)} |b(y) - b_B| |\Omega(x - y)| \frac{|f(y)|}{|x - y|^{n-\alpha}} dy
\]

\[
\lesssim \int_{\ell(2B)} |b(y) - b_B| |\Omega(x - y)| \frac{|f(y)|}{|x_0 - y|^{n-\alpha}} dy
\]

\[
\lesssim \int_{2r}^{\infty} \int_{2r < |x_0 - y| < t} |b(y) - b_B| |\Omega(x - y)| |f(y)| dy t^{a-1} dt
\]

\[
\lesssim \int_{2r}^{r} \int_B |b(y) - b_B(x_0,t)| |\Omega(x - y)| |f(y)| dy t^{a-1} dt + \int_{2r}^{r} |b_B(x_0,t) - b_B(x_0)| \int_B |\Omega(x - y)| |f(y)| dy t^{a-1} dt
\]

\[
\lesssim \int_{2r}^{r} \|b(\cdot) - b_B(x_0,t)\|_{L_p(B(x_0,t))} \|\Omega(\cdot - y)\|_{L_s(B(x_0,t))} \|B(x_0,t)\|^{1 - \frac{1}{p} - \frac{1}{q}} dt
\]

\[
+ \int_{2r}^{r} \|b_B(x_0,t)\|_{L_p(B(x_0,t))} \|\Omega(\cdot - y)\|_{L_s(B(x_0,t))} \|B(x_0,t)\|^{1 - \frac{1}{p} - \frac{1}{q}} t^{a-1} dt
\]

\[
\lesssim \int_{2r}^{r} \|b(\cdot) - b_B(x_0,t)\|_{L_p(B(x_0,t))} \|\Omega(\cdot - y)\|_{L_s(B(x_0,t))} \|B(x_0,t)\|^{1 - \frac{1}{p} - \frac{1}{q}} dt
\]

\[
+ \|b\|_{CBMO} \int_{2r}^{r} \left( 1 + \ln \frac{t}{r} \right) \|f\|_{L_p(B(x_0,t))} t^{n\lambda - \frac{n}{q}} dt
\]

\[
\lesssim \|b\|_{CBMO} \int_{2r}^{r} \left( 1 + \ln \frac{t}{r} \right) \|f\|_{L_p(B(x_0,t))} t^{n\lambda - \frac{n}{q}} dt.
\]
Then for $J_4$ we have
\[
\|J_4\|_{L_q(B)} \leq \|I_{\Omega, \alpha} (b(\cdot) - b_B) f_2\|_{L_q(\mathbb{R}^n)}
\]
\[
\lesssim \|b\|_{CBMO_{p_2, \lambda}^{(x_0)}} r^{\frac{n}{q}} \int_0^\infty \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_{p_1}(B(x_0,t))} t^{n\lambda - 1 - \frac{n}{q^{\lambda}}} dt.
\]
When $q_1 < s$, by Fubini’s theorem and the Minkowski inequality, we get
\[
\|I_{\Omega, \alpha} f_2\|_{L_q(B)} \leq \left( \int_B \int_0^1 \int_{B(x_0,t)} |f(y)| \|\Omega(x - y)\| dy \frac{dt}{t^{n-a+1}} \right)^{\frac{1}{q}}
\]
\[
\leq \int_B \int_0^1 \int_{B(x_0,t)} |f(y)| \|\Omega(\cdot - y)\|_{L_q(B)} dy \frac{dt}{t^{n-a+1}}
\]
\[
\leq |B|^{-\frac{1}{q}} \int_0^1 \int_{B(x_0,t)} |f(y)| \|\Omega(\cdot - y)\|_{L_q(B)} dy \frac{dt}{t^{n-a+1}}
\]
\[
\lesssim r^{\frac{n}{q}} \int_0^1 \|f\|_{L_1(B(x_0,t))} \frac{dt}{t^{n-a+1}}
\]
\[
\lesssim r^{\frac{n}{q}} \int_0^1 \|f\|_{L_{p_1}(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}}}.
\] (4.2)

Now combined by all the above estimates, we end the proof of this Lemma 4.3

The following theorem is true.

**Theorem 4.5.** Suppose that $x_0 \in \mathbb{R}^n$, $\Omega \in L_s(S^{n-1})$ with $1 < s \leq \infty$, be a homogeneous of degree zero. Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $b \in CBMO_{p_2, \lambda}^{(x_0)}(\mathbb{R}^n)$, $0 \leq \lambda < \frac{1}{n}$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{p} - \alpha$, $\frac{1}{q_1} = \frac{1}{p_1} - \frac{\alpha}{n}$. Let also, for $s' \leq p$ or $q_1 < s$ the pair $(\varphi_1, \varphi_2)$ satisfy the condition
\[
\int_r^\infty \left(1 + \ln \frac{t}{r}\right) \text{ess inf}_{t < T < \infty} \varphi_1(x_0, T) t^{\frac{n}{q} - n\lambda + 1} dt \leq C \varphi_2(x_0, r),
\] (4.3)
where $C$ does not depend on $r$. Then, the operators $M_{\Omega, b, \alpha}$ and $[b, I_{\Omega, \alpha}]$ are bounded from $LM_{p, \varphi_1}^{(x_0)}$ to $LM_{p, \varphi_2}^{(x_0)}$. Moreover
\[
\|M_{\Omega, b, \alpha} f\|_{LM_{p, \varphi_1}^{(x_0)}} \lesssim \|[b, I_{\Omega, \alpha}] f\|_{LM_{p, \varphi_2}^{(x_0)}} \lesssim \|b\|_{CBMO_{p_2, \lambda}^{(x_0)}} \|f\|_{LM_{p, \varphi_1}^{(x_0)}}.
\]

**Proof.** The statement of Theorem 4.5 follows by Lemma 4.4 and Theorem 3.1 in the same manner as in the proof of Theorem 3.5.

For the sublinear commutator of the fractional maximal operator $M_{b, \alpha}$ and for the linear commutator of the Riesz potential $[b, I_{\alpha}]$ from Theorem 4.3 we get the following new results.
Corollary 4.6. Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $b \in CBMO^{\{x_0\}}_{p_2, \lambda}(\mathbb{R}^n)$, $0 \leq \lambda < \frac{1}{n}$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $\frac{1}{r_1} = \frac{1}{p_1} - \frac{\alpha}{n}$, and $(\varphi_1, \varphi_2)$ satisfies the condition (4.3). Then, the operators $M_{b, \alpha}$ and $[b, I]_{\alpha}$ are bounded from $LM^{\{x_0\}}_{p_1, \varphi_1}$ to $LM^{\{x_0\}}_{q, \varphi_2}$.

5 Some applications

In this section, we shall apply Theorems 3.5 and 4.5 to several particular operators such as the Marcinkiewicz operator and fractional powers of the some analytic semigroups.

5.1 Marcinkiewicz operator

Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere in $\mathbb{R}^n$ equipped with the Lebesgue measure $d\sigma$. Suppose that $x_0 \in \mathbb{R}^n$, $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, be a homogeneous of degree zero and satisfy the cancellation condition.

In 1958, Stein [41] defined the Marcinkiewicz integral of higher dimension $\mu_\Omega$ as

\[
\mu_\Omega(f)(x) = \left( \int_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},
\]

where

\[
F_{\Omega,t}(f)(x) = \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy.
\]

Since Stein’s work in 1958, the continuity of Marcinkiewicz integral has been extensively studied as a research topic and also provides useful tools in harmonic analysis [33, 40, 42, 43].

The Marcinkiewicz operator is defined by (see [44])

\[
\mu_{\Omega,\alpha}(f)(x) = \left( \int_0^\infty |F_{\Omega,\alpha,t}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},
\]

where

\[
F_{\Omega,\alpha,t}(f)(x) = \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1-\alpha}} f(y) dy.
\]

Note that $\mu_\Omega f = \mu_{\Omega,0} f$.

Let $H$ be the space $H = \{h : \|h\| = (\int_0^\infty |h(t)|^2 dt / t^3)^{1/2} < 1\}$. Then, it is clear that $\mu_{\Omega,\alpha}(f)(x) = \|F_{\Omega,\alpha,t}(x)\|$.

By Minkowski inequality and the conditions on $\Omega$, we get

\[
\mu_{\Omega,\alpha}(f)(x) \leq \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-1-\alpha}} |f(y)| \left( \int_{|x-y|}^\infty \frac{dt}{t^3} \right)^{1/2} dy \leq C I_{\Omega,\alpha}(f)(x).
\]
Corollary 5.1. Suppose that \( x_0 \in \mathbb{R}^n \), \( \Omega \in L_a(S^{n-1}) \), \( 1 < s \leq \infty \), be a homogeneous of degree zero and satisfy the cancellation condition. Let \( 0 < \alpha < n \), \( 1 < p < \frac{n}{\alpha} \), \( \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \) and for \( s' \leq p \) or \( q_1 < s \) the pair \((\varphi_1, \varphi_2)\) satisfy the condition (3.14). Then \( \mu_{\alpha, \alpha} \) is bounded from \( LM_{p, \varphi_1}^{\{x_0\}} \) to \( LM_{q, \varphi_2}^{\{x_0\}} \) for \( p > 1 \) and from \( M_{1, \varphi_1}^{\{x_0\}} \) to \( WLM_{q, \varphi_2}^{\{x_0\}} \) for \( p = 1 \).

Corollary 5.2. Suppose that \( x_0 \in \mathbb{R}^n \), \( \Omega \in L_a(S^{n-1}) \), \( 1 < s \leq \infty \), be a homogeneous of degree zero and satisfy the cancellation condition. Let \( 0 < \alpha < n \), \( 1 < p < \frac{n}{\alpha} \), \( b \in CBMO_{p_2, \lambda}^{\{x_0\}}(\mathbb{R}^n) \), \( 0 \leq \lambda \leq \frac{1}{n} \), \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \), \( \frac{1}{q} = \frac{1}{p_1} - \frac{\alpha}{n} \), \( q_1 = \frac{1}{p_1} - \frac{\alpha}{n} \) and for \( s' \leq p \) or \( q_1 < s \) the pair \((\varphi_1, \varphi_2)\) satisfy the condition (3.14). Then \([a, \mu_{\alpha, \alpha}]\) is bounded from \( LM_{p_1, \varphi_1}^{\{x_0\}} \) to \( LM_{q_1, \varphi_2}^{\{x_0\}} \).

5.2 Fractional powers of the some analytic semigroups

The theorems of the previous sections can be applied to various operators which are estimated from above by Riesz potentials. We give some examples.

Suppose that \( L \) is a linear operator on \( L_2 \) which generates an analytic semigroup \( e^{-tL} \) with the kernel \( p_t(x, y) \) satisfying a Gaussian upper bound, that is,

\[
|p_t(x, y)| \leq \frac{c_1}{t^{n/2}} e^{-c_2 \frac{|x-y|^2}{t}}
\]

for \( x, y \in \mathbb{R}^n \) and all \( t > 0 \), where \( c_1, c_2 > 0 \) are independent of \( x, y \) and \( t \).

For \( 0 < \alpha < n \), the fractional powers \( L^{-\alpha/2} \) of the operator \( L \) are defined by

\[
L^{-\alpha/2} f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-tL} f(x) \frac{dt}{t^{\alpha/2+1}}.
\]

Note that if \( L = -\Delta \) is the Laplacian on \( \mathbb{R}^n \), then \( L^{-\alpha/2} \) is the Riesz potential \( I_\alpha \). See, for example, Chapter 5 in [40].

Theorem 5.3. Let condition (5.1) be satisfied. Moreover, let \( 1 \leq p < \infty \), \( 0 < \alpha < \frac{n}{p} \), \( \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \), \((\varphi_1, \varphi_2)\) satisfy condition (3.14). Then \( L^{-\alpha/2} \) is bounded from \( L_{p_1, \varphi_1}^{\{x_0\}} \) to \( L_{q_1, \varphi_2}^{\{x_0\}} \) for \( p > 1 \) and from \( M_{1, \varphi_1}^{\{x_0\}} \) to \( WLM_{q_1, \varphi_2}^{\{x_0\}} \) for \( p = 1 \).

Proof. Since the semigroup \( e^{-tL} \) has the kernel \( p_t(x, y) \) which satisfies condition (5.1), it follows that

\[
|L^{-\alpha/2} f(x)| \lesssim I_\alpha(|f|)(x)
\]
(see [20]). Hence by the aforementioned theorems we have
\[ \|L^{-\alpha/2}f\|_{M^{(x_0)}_{p_1,\psi_2}} \lesssim \|I_\alpha(|f|)\|_{M^{(x_0)}_{p_1,\psi_2}} \lesssim \|f\|_{M^{(x_0)}_{p_2,\psi_2}}. \]

Let \( b \) be a locally integrable function on \( \mathbb{R}^n \), the commutator of \( b \) and \( L^{-\alpha/2} \) is defined as follows
\[ [b, L^{-\alpha/2}]f(x) = b(x)L^{-\alpha/2}f(x) - L^{-\alpha/2}(bf)(x). \]

In [20] extended the result of [9] from \((-\Delta)\) to the more general operator \( L \) defined above. More precisely, they showed that when \( b \in BMO(\mathbb{R}^n) \), then the commutator operator \([b, L^{-\alpha/2}]\) is bounded from \( L_p(\mathbb{R}^n) \) to \( L_q(\mathbb{R}^n) \) for \( 1 < p < q < \infty \) and \( \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \). Then from Theorem 4.5 we get

**Theorem 5.4.** Let condition (5.1) be satisfied. Moreover, let \( 0 < \alpha < n \), \( 1 < p < \frac{n}{\alpha} \), \( b \in CBMO^{(x_0)}_{p_2,\lambda}(\mathbb{R}^n) \), \( 0 \leq \lambda < \frac{1}{n} \), \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \), \( \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \), and \( \frac{1}{q_1} = \frac{1}{p_1} - \frac{\alpha}{n} \), and \((\psi_1, \psi_2)\) satisfies the condition (4.3). Then \([b, L^{-\alpha/2}]\) is bounded from \( LM^{(x_0)}_{p,\psi_1} \) to \( LM^{(x_0)}_{q,\psi_2} \).

Property (5.1) is satisfied for large classes of differential operators (see, for example [6]). In [6] also other examples of operators which are estimates from above by Riesz potentials are given. In these cases Theorem 3.5 and 4.5 are also applicable for proving boundedness of those operators and commutators from \( LM^{(x_0)}_{p,\psi_1} \) to \( LM^{(x_0)}_{q,\psi_2} \).

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