Controlling the quantum computational speed

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Abstract.

The speed of quantum computation is investigated through the time evolution of the speed of the orthogonality. The external field components for classical treatment beside the detuning and the coupling parameters for quantum treatment play important roles on the computational speed. It has been shown that the number of photons has no significant effect on the speed of computation. However, it is very sensitive to the variation in both detuning and the interaction coupling parameters.

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1. Introduction

Recently, there are a great interest for developing the computer device. This is mainly due to the complications of the problems which we are usually facing and urgently need to find their solutions [1, 2, 3]. To overcome these problems we have to find a computer device with high capacity and speed or we have to develop a new kind of computer has such properties. This in fact means that we need one has a large memory, an adequate processor and a large hard disk. This stimulated and encouraged the researcher to seek for another kind of computer different to that of the classical one, that is the quantum computer. However, in the quantum information and more precisely in the quantum computer there is an important question would be raised, what is the speed of sending information from nod to the other one that to reach the final output. Since the information is coded in a density operator, therefore we ask how fast the density operator will change its orthogonality. In other words, we search for a minimum time needed for a quantum system to pass from one orthogonal state to another [4]. To perform this task we need qubit contains the information to evolve through a unitary operator where the carrier transforms it from one nod to the other.

For entangled qubit pair, one can see the operators cause a decay of entanglement [5, 6, 7]. Further, it has been shown that the classical noise leads to what is called entanglement sudden death [8]. Moreover, the time-dependent interaction of a single qubit with a field can also produce such a phenomena as well as along-lived entanglement [12]. This means that there are different factors would be involved and affect the transmission speed and consequently the information. Recently the efforts in quantum information research are directed towards improving the performance of single qubit interaction. Also, evolution speed (maximum transition rate between orthogonal state) and the time evolution of some models has been discussed in Ref. [9].

The main purpose for the present communication is to consider the interaction between a single qubit and an external field for the classical treatment and the interaction between a single qubit and cavity field for the quantum treatment. This is to shed some light on the general behavior of the interaction process and its relationship with the speed of the computation [4] (maximum number of orthogonal states that the system can pass through per unit time), speed of orthogonality [10] (minimum time for a quantum state $|\psi_i\rangle$ to evolve into orthogonal state $|\psi_f\rangle$ where $\langle \psi_i | \psi_f \rangle = 0$) and speed of evolution [11] (maximum transition rate between orthogonal state).

The paper is organized as follows. In Sec. 2, we consider the classical treatment, where we calculate the general form of the time evaluation of the density operator. The quantum interaction of the qubit will be considered in Sec. 3, where we introduce the unitary operator in an adequate form. Also, we obtain the final state by means of the Bloch vectors. Also we study the effect of the field parameters on the speed of the quantum computation. Finally, our conclusion is given in Sec 4.
2. Classical treatment of Qubit

Let us start out with a short reminder on a general form of the density operator of a qubit with the aid of analogs of Pauli’s spin vector operator $\vec{\sigma}$. This row vector refers to the three dimensional vector $-\vec{e}\sigma$.

$$\vec{\sigma} = \sum_{\alpha=x,y,z} \vec{\sigma}_\alpha e_\alpha = (\sigma_x, \sigma_y, \sigma_z) \begin{pmatrix} \vec{e}_x \\ \vec{e}_y \\ \vec{e}_z \end{pmatrix},$$  \tag{1}$$

where $\vec{e}_{x,y,z}$ are orthonormal vectors of the three coordinate axes to which the components $\sigma_i$ are referred and $\sigma_i$ are Pauli matrices satisfying the commutation relation $[\sigma_i, \sigma_j] = 2i\sigma_k$, where $i, j, k$ form an even permutation of $x, y, z$. In this case, the density operator can be represented in the following form $\rho_a = \frac{1}{2}(1 + \vec{s} \otimes \sigma^\dagger)$.

$$\begin{align*}
\rho_a &= \frac{1}{2}(1 + \vec{s} \otimes \sigma^\dagger) \\
&= \frac{1}{2}(1 + s_x\sigma_x + s_y\sigma_y + s_z\sigma_z) \\
&= \frac{1}{2} \begin{pmatrix} 1 + s_z & s_x + is_y \\ s_x - is_y & 1 - s_z \end{pmatrix}, \tag{2}
\end{align*}$$

where $\vec{s} = \langle \vec{\sigma} \rangle$.

The effective Hamiltonian for one qubit can be defined as $H = \alpha_x\sigma_x + \alpha_y\sigma_y + \alpha_z\sigma_z$, \tag{3}$ where $\alpha_i$ are the external field components. The unitary evolution operator can be obtained from the Hamiltonian, thus,

$$U = \sum_i \exp\{-i\alpha_i t\sigma_i\}, \tag{4}$$

Using the density operator (2), Alice qubit can be transformed as

$$\rho_a \rightarrow \tilde{\rho}_a = \frac{1}{2}(1 + \vec{\tilde{s}} \otimes \sigma^\dagger), \tag{5}$$

where the component of $\vec{\tilde{s}}$ are given by

$$\begin{align*}
\tilde{s}_x &= \frac{1}{3}[s_x(1 + \cos(2t\alpha_2) + \cos(2t\alpha_3)) - s_y \sin(2t\alpha_3) + s_z \sin(2t\alpha_2)] \\
\tilde{s}_y &= \frac{1}{3}[s_y(1 + \cos(2t\alpha_1) + \cos(2t\alpha_3)) + s_x \sin(2t\alpha_3) - s_z \sin(2t\alpha_1)], \\
\tilde{s}_z &= \frac{1}{3}[s_z(1 + \cos(2t\alpha_1) + \cos(2t\alpha_2)) - s_x \sin(2t\alpha_2) + s_y \sin(2t\alpha_1)].
\end{align*} \tag{6}$$

Having obtained the above analytical expressions for $\tilde{s}_x, \tilde{s}_y$ and $\tilde{s}_z$, we are therefore in position to investigate the speed of the orthogonality and hence the speed of computation. To clarify our idea let us assume that the user Alice has encoded some information in her qubit which is defined by

$$\rho_a = \frac{1}{2}(1 + s_x\sigma_x). \tag{7}$$
Figure 1. The speed of orthogonality of qubit as a function of the scaled time, where, the component \( \langle u_i | v_1 \rangle \) is represented by the solid curve, while \( \langle u_i | v_2 \rangle \) is represented by the dotted-curve. The other parameters are \( s_x = 1 \) and \( s_y = s_z = 0 \), (a) \( \alpha_1 = \alpha_2 = \alpha_3 = \frac{\pi}{2} \), (b) \( \alpha_1 = \alpha_2 = \alpha_3 = \frac{\pi}{3} \), (c) \( \alpha_1 = \alpha_2 = \alpha_3 = \frac{\pi}{4} \) and (d) \( \alpha_1 = \frac{\pi}{2}, \alpha_2 = \frac{\pi}{5} \) and \( \alpha_3 = \frac{\pi}{4} \).

Using the time evolution of the unitary operator, (4) one can transform \( \rho_a \) into \( \tilde{\rho}_a \) from which the new Bloch vectors take the form,

\[
\tilde{s}_x = \frac{s_x}{3} (1 + \cos 2t\alpha_2 + \cos 2t\alpha_3), \quad \tilde{s}_y = \frac{s_x}{3} \sin 2t\alpha_3, \quad \tilde{s}_z = -\frac{s_x}{3} \sin 2t\alpha_2, \tag{8}
\]

Let us assume that Alice has prepared her qubit such as \( s_x = 1 \) and \( s_y = s_z = 0 \). Then the eigenvectors of the (7) state can be written as

\[
v_1 = [-1, 1], \quad \text{and} \quad v_2 = [1, -1]. \tag{9}
\]

Thus, it will be easy to get the eigenvectors for the final state \( \tilde{\rho}_a \), which is described by Bloch vectors (8). After some algebraic calculations, we can explicitly write \( u_i \) as

\[
u_1 = u_2 = \Gamma \left\{ \frac{\sin^2 2t\alpha_2 - (3 + \cos 2t\alpha_2 + \cos 2t\alpha_3 + \cos 2t\alpha_2 \cos 2t\alpha_3)}{i \sin 2t\alpha_3} \right\}, \tag{10}
\]

where \( \Gamma = (1 + \cos 2t\alpha_2 + \cos 2t\alpha_3 - i \sin 2t\alpha_3)^{-1} \). In order to facilitate our discussion let us define the scalar product of the vectors \( u_i \) and \( v_i \) such as

\[
Sp_{ij} = \langle v_i | u_j \rangle. \tag{11}
\]

It should be noted that in our calculations we have taken into account all the possible products of \( u_i \) and \( v_i \). In figure (1) we have plotted the amplitude values of \( Sp_{ij} \) against the scaled time to display its behavior for different values of the control parameter \( \alpha_i \). In figure (1a) we have considered the case in which \( \alpha_i = \pi/2 \) where one can see both \( \langle v_1 | u_j \rangle \) and \( \langle v_2 | u_j \rangle \) are coincides on the horizontal axis at different period of time.
However, when we change the value of the parameters \( \alpha_i \) such as \( \alpha_i = \pi/3 \) it is noted that there is decreasing in the number of coincidences points which refer to reduction in the computation speed, see figure (1b). This phenomenon gets more pronounced for the case in which \( \alpha_i = \pi/4 \), see figure (1c). Thus we may conclude that as the value of the control parameters \( \alpha_i \) increases as the speed of the computation increases and vice versa.

On the other hand when we consider different values for the control parameters \( \alpha_i \) such that \( \alpha_1 = \pi/2, \alpha_2 = \pi/3, \alpha_3 = \pi/4 \), then more decreasing can be seen in the computation speed. In the meantime we can observe irregular fluctuations in both functions \( S_{P_{1j}} \) and \( S_{P_{2j}} \) in addition to the intersection at different points, see figure (1d). This is contrary to the previous cases where regular oscillations can be realized in each case.

3. Quantum treatment of Qubit

Now let us turn our attention to consider the quantum treatment of the computation speed, taking into account the quantized field interacting with a single qubit. In this case the Hamiltonian can be written as,

\[
\hat{H}_{\text{int}} = \lambda (\hat{a}^\dagger \sigma_- + \hat{a} \sigma_+) + \frac{\Delta}{2} \sigma_z, \tag{12}
\]

where \( \hat{a}^\dagger \) and \( \hat{a} \) are the creation and annihilation operators satisfy the commutation relation \([\hat{a}, \hat{a}^\dagger] = 1\). We denote by \( \lambda \) the coupling constant and \( \Delta \) the detuning parameter while \( \sigma_+ (\sigma_-) \), \( \sigma_z \) are usual raising (lowering) and inversion operators for the two-level atomic system, satisfying \([\sigma_z, \sigma_\pm] = \pm 2\sigma_\pm\) and \([\sigma_+, \sigma_-] = 2\sigma_z\). The time-dependent density operator \( \rho(t) \) is given by

\[
\rho(t) = U(t) \rho(0) U(t)^\dagger, \tag{13}
\]

where \( \rho(0) = \rho_a(0) \otimes \rho_f(0) \) is the initial state of the system. We assume that the initial state of the qubit \( \rho_a(0) \) is given by equation (7) while the field starts from a Fock state. Under the above assumptions the unitary evolution operator \( U(t) \) can be defined as

\[
U(t) = u_{11}|0\rangle \langle 0| + u_{12}|0\rangle \langle 1| + u_{21}|1\rangle \langle 0| + u_{22}|1\rangle \langle 1|, \tag{14}
\]

where

\[
u_{11} = C_{n+1} - i\frac{\Delta}{2} S_{n+1}, \quad u_{12} = -i\eta S_n a, \quad u_{21} = u_{12}^\dagger, \quad u_{22} = u_{11}^\dagger,
\]

and

\[
S_n = \frac{\sin(\mu_n \gamma t)}{\mu_n}, \quad C_n = \cos(\mu_n \gamma t), \quad \mu_n = \sqrt{\frac{\Delta^2}{4\gamma^2} + \eta^2 n}.
\]

It should be noted that in the above equations, we have introduced the parameters \( \eta \) and \( \gamma \) to connect up the coupling parameter \( \lambda \) such that \( \lambda = \eta \gamma \). This in fact would enable us to discuss the effect of the coupling parameter using \( \eta \) instead of \( \lambda \) regarding \( \gamma \) as a dimensionless parameter. Using Eqs. (13) and (14) one can obtain the explicitly
time-dependent density operator $\rho(t)$ in Bloch vectors representation, thus

$$\tilde{S}_x = -i\eta S_{n+1} \frac{1 + s_z}{2} [\sqrt{n + 1}(C_{n+1} + \frac{\Delta}{2} S_{n+1}) + (C_{n+1} - \frac{\Delta}{2} S_{n+1})]$$

$$+ \eta^2 \sqrt{n + 1} \sqrt{n + 2} S_n S_{n+1} s_x + i\eta \sqrt{n + 1} \frac{1 - s_z}{2} S_n C_{n+2}$$

$$+ (C_{n+1} - \frac{\Delta^2}{2} S_{n+1}^2) s_x - \Delta S_{n+1} C_{n+1} s_y,$$

$$\tilde{S}_y = \eta S_{n+1} \frac{1 + s_z}{2} [\sqrt{n + 1}(C_{n+1} + \frac{\Delta}{2} S_{n+1}) + (C_{n+1} - \frac{\Delta}{2} S_{n+1})]$$

$$- \eta^2 \sqrt{n + 1} \sqrt{n + 2} S_n S_{n+1} s_y + i\eta \Delta \sqrt{n + 1} \frac{1 - s_z}{2} S_n S_{n+2}$$

$$+ \Delta S_{n+1} C_{n+1} s_x + (C_{n+2}^2 - \frac{\Delta^2}{2} S_{n+1}^2) s_y,$$

$$\tilde{S}_z = -i\eta \left[ (1 + \sqrt{n + 1}) C_{n+1} + \frac{\Delta}{2} (1 - \sqrt{n + 1}) S_{n+1} \right] s_x S_{n+1}$$

$$+ \eta \left[ (1 - \sqrt{n + 1}) C_{n+1} + \frac{\Delta}{2} (1 + \sqrt{n + 1}) S_{n+1} \right] s_y S_{n+1}$$

$$- (C_{n+1} + \frac{\Delta^2}{4} S_{n+1}) s_z + \eta^2 S_n^2 \frac{1}{2} - (n + \frac{1}{2}) s_z$$

$$- i\eta \frac{s_z - is_y}{2} \sqrt{n + 1} S_n C_{n+1}.$$ (15)

The parameters $s_x, s_y$ and $s_z$ which appear in the right hand side of equation (15), describe the initial state Bloch vectors.
In fact, these states are widely used in the quantum information tasks. For example, one may consult a recent applications given in Ref. [15]. In the meantime it would be interesting to employ the Hamiltonian given by equation (12) that to discuss the speed of computation. This is extensively used in the field of quantum information to describe the interaction between field and qubit, particularly for investigating the loss of entanglement after propagation in a quantum noisy channel. Furthermore, compare with the classical treatment the interaction Hamiltonian contains three different parameters to control the dynamics of the system, \( \Delta \) the detuning parameter, \( \eta \) the coupling constant, and the mean photon number \( \langle a^\dagger a \rangle \) [16]. These parameters are involved through the Rabi frequency \( \lambda_n \) as well as in the Bloch vectors themselves. This would give us a wide range of variety to discuss the variation in \( S_{pij} \) resultant of change one of these parameters. To do so we have numerically calculated the overlap between the initial and the final states of \( S_{pij} \). For example, to see the effect of the coupling parameter \( \eta \) we have considered the number of photons \( n = 10 \), and the detuning parameter \( \Delta/\gamma = 2 \), while \( \eta = 0.05 \).

In this case and from figure (2a) we can see nearly perfect overlap between both of \( S_{p_1j} \) and \( S_{p_2j} \) as well as coincidences at the horizontal line showing high speed. Increase the value of \( \eta \) such that \( \eta = 0.1 \) leads to slight increase in the speed of computation, beside increases in one of the projectors value, see figure (2b). More increases in the coupling parameter \( \eta = 0.15 \) shows increasing in the speed of computation but with less coincidences between the two projectors, see figure (2c). More increasing in the coupling value \( \eta = 0.2 \) leads to more decreasing in the speed of computation but with regular increasing in both projectors value. This means that there is a certain value (critical value) of the coupling parameter where the speed of computation reaches its maximum and then starts to slow down. To examine the effect of the mean photon number we have considered the case in which \( n = 20 \), keeping the other parameters unchange as in figure (2a). In this case we observe no change in the speed of orthogonality and the behavior in general is the same as before, however, there is increasing in the amplitude for one of the projectors, see figure (3a). However, if we decrease the value of the detuning parameter \( \Delta/\gamma = 1 \) drastic change can be realized. For instance, we can see decreasing

![Figure 3](image.png)

**Figure 3.** The same as figure (2a), but \( n = 20 \), (a) \( \Delta/\gamma = 2 \), (b) \( \Delta/\gamma = 1 \).
in the number of the oscillations period, increasing in one of the projector amplitude, in addition to decrease in the speed of computation, see figure (3b). Thus we come to conclusion if one increases the value of the detuning parameter then the speed of the interaction increases. This result is in agreement with that given by Montangero [17], where they investigated the dynamics of entanglement in quantum computer with imperfections.

4. Conclusion

In the above sections of the present paper we have considered the problem of speed computation in quantum information. The problem has been handled from two different point of view; where we have considered both of classical and quantum treatments. The main concentration was on how to improve and control the computation’s speed in each case separately. For the classical treatment it has been shown that the speed of computation is proportional with the total value of the external field. However, for quantum treatment we have seen that the speed of computation is sensitive to the variation of the coupling parameter and the detuning parameter. In the meantime we found the mean photon number does not play any role with the speed of computation but it is just effect the amplitude of the projectors. This in fact would turn our attention to look for the atom-atom interaction to be discussed in a forthcoming work.

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