A CLASSIFICATION RESULT FOR THE QUASI-LINEAR LIOUVILLE EQUATION

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Abstract. Entire solutions of the $n$–Laplace Liouville equation in $\mathbb{R}^n$ with finite mass are completely classified.

1. Introduction

We are concerned with the following Liouville equation

$$\begin{cases}
-\Delta_n U = e^U & \text{in } \mathbb{R}^n \\
\int_{\mathbb{R}^n} e^U < +\infty
\end{cases}$$

(1.1)

involving the $n$–Laplace operator $\Delta_n(\cdot) = \text{div}(|\nabla(\cdot)|^{n-2} \nabla(\cdot))$, $n \geq 2$. Here, a solution $U$ of (1.1) stands for a function $U \in C^{1,\alpha}(\mathbb{R}^n)$ which satisfies

$$\int_{\mathbb{R}^n} |\nabla U|^{n-2} \langle \nabla U, \nabla \Phi \rangle = \int_{\mathbb{R}^n} e^U \Phi \quad \forall \Phi \in H = \{ \Phi \in W^{1,n}_0(\Omega) : \Omega \subset \mathbb{R}^n \text{ bounded} \}.$$ (1.2)

As we will see, the regularity assumption on $U$ is not restrictive since a solution in $W^{1,n}_{\text{loc}}(\mathbb{R}^n)$ is automatically in $C^{1,\alpha}(\mathbb{R}^n)$, for some $\alpha \in (0, 1)$.

Problem (1.1) has the explicit solution

$$U(x) = \log \frac{c_n}{(1 + |x|^{\frac{2}{n-2}})^n} \quad x \in \mathbb{R}^n,$$

where $c_n = n(\frac{2}{n-2})^{n-1}$. Due to scaling and translation invariance, a $(n + 1)$–dimensional family of explicit solutions $U_{\lambda,p}$ to (1.1) is built as

$$U_{\lambda,p}(x) = U(\lambda(x - p)) + n \log \lambda = \log \frac{c_n \lambda^n}{(1 + \lambda \frac{2}{n-2} |x - p|^\frac{2}{n-2})^n}$$

(1.3)

for all $\lambda > 0$ and $p \in \mathbb{R}^n$. Notice that

$$\int_{\mathbb{R}^n} e^{U_{\lambda,p}} = \int_{\mathbb{R}^n} e^U = c_n \omega_n$$

(1.4)

where $\omega_n = |B_1(0)|$. Our aim is the following classification result:

Theorem 1.1. Let $U$ be a solution of (1.1). Then

$$U(x) = \log \frac{c_n \lambda^n}{(1 + \lambda \frac{2}{n-2} |x - p|^\frac{2}{n-2})^n} \quad x \in \mathbb{R}^n$$

(1.5)

for some $\lambda > 0$ and $p \in \mathbb{R}^n$.

In a radial setting Theorem 1.1 has been already proved, among other things, in [19]. For the semilinear case $n = 2$ such a classification result is known since a long ago. The first proof goes back to J. Liouville [28] who found a formula—the so-called Liouville formula— to represent a solution $U$ on a simply-connected domain in terms of a suitable meromorphic function. On the whole $\mathbb{R}^2$ the finite-mass condition $\int_{\mathbb{R}^2} e^U < +\infty$ completely determines such meromorphic function.

A PDE proof has been found several years later by W. Chen and C. Li [2]. The fundamental point is to represent a solution $U$ of (1.1) in an integral form in terms of the fundamental solution and then deduce the precise asymptotic behavior of $U$ at infinity to start the moving plane technique. Such idea has revealed very powerful and has been also applied to higher-order versions of (1.1) involving the operator $(-\Delta)^2$. Overall, the integral equation satisfied by $U$ can be used to derive asymptotic properties of $U$ at infinity or can be directly studied through the method of moving planes/spheres. Since these methods are very well suited for integral equations, a research line has flourished about qualitative properties of integral equations, see [13, 15, 24, 41, 52] and [14, 15, 36].

The quasi-linear case $n > 2$ is more difficult. Very recently, the classification of positive $D^{1,n}(\mathbb{R}^N)$–solutions to $-\Delta_n U = U^{\frac{2n}{n-2}} - 1$, a PDE with critical Sobolev polynomial nonlinearity, has been achieved for $n < N$, see also some previous somehow related results [14, 15, 36]. The strategy is always based on the moving plane method and
the analytical difficulty comes from the lack of comparison/maximum principles on thin strips. Moreover for $n < N$ it is not available any Kelvin type transform, a useful tool to "gain" decay properties on a solution.

When $n = N$ the classical approach [7] breaks down since an integral representation formula for a solution $U$ of (1.4) is not available, due to the quasi-linear nature of $\Delta_n$. It becomes a delicate issue to determine the asymptotic behavior of $U$ at infinity and overall it is not clear how to carry out the method of moving planes/spheres. However, when $n = N$ there are some special features we aim to exploit to devise a new approach which does not make use of moving planes/spheres, providing in two dimensions an alternative proof of the result in [2]. During the completion of this work, we have discovered that such an approach has been already used in [8] for Liouville systems, where the maximum principle can possibly fail. See also [20] for a somewhat related approach to symmetry questions in a ball.

The case $n = N$ is usually referred to as the conformal situation, since $\Delta_n$ is invariant under Kelvin transform: $\hat{U}(x) = U(\frac{x}{|x|^2})$ formally satisfies

$$\Delta_n \hat{U} = \frac{1}{|x|^{2n}}(\Delta_n U)(\frac{x}{|x|^2}),$$

so that

$$\begin{cases} 
-\Delta_n \hat{U} = F(x) := \frac{e^U}{|x|^2} & \text{in } \mathbb{R}^n \setminus \{0\} \\
\int_{\mathbb{R}^n} \frac{\hat{U}}{|x|^{2n}} < +\infty.
\end{cases}$$

Equation has to be interpreted in the weak sense

$$\int_{\mathbb{R}^n} |\nabla U|^2 - 2 \langle \nabla \hat{U}, \nabla \Phi \rangle = \int_{\mathbb{R}^n} \frac{\hat{U}}{|x|^{2n}} \Phi \quad \forall \Phi \in \dot{H} = \{ \Phi : \Phi \in H \}.$$
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory function so that
\begin{equation}
|a(x, p)| \leq c(a(x) + |p|^{p-1}) \quad \forall p \in \mathbb{R}^n, \ a.e. \ x \in \Omega
\end{equation}
\begin{equation}
(a(x, p) - a(x, q), p - q) \geq d|p - q|^n \quad \forall p, q \in \mathbb{R}^n, \ a.e. \ x \in \Omega
\end{equation}
for some $c, d > 0$ and $a \in L^{\frac{n}{n-1}}(\Omega)$. Given $f \in L^1(\Omega)$, let $u \in W^{1,n}(\Omega)$ be a weak solution of
\[ -\text{div} \, a(x, \nabla u) = f \quad \text{in} \ \Omega. \]

Thanks to (2.1) equation (2.3) is interpreted in the following sense:
\[ \int_\Omega (a(x, \nabla u), \nabla \phi) = \int_\Omega f \phi \quad \forall \phi \in W^{1,n}_0(\Omega) \cap L^\infty(\Omega). \]
Since $u \in W^{1,n}(\Omega)$ let us consider the weak solution $h \in W^{1,n}(\Omega)$ of
\[ \begin{cases} 
\text{div} \, a(x, \nabla h) = 0 & \text{in} \ \Omega \\
h = u & \text{on} \ \partial \Omega.
\end{cases} \tag{2.5} \]
Introduce the truncation operator $T_k$, $k > 0$, as
\[ T_k(u) = \begin{cases} 
\frac{u}{k} & \text{if} \ |u| \leq k \\
0 & \text{if} \ |u| > k.
\end{cases} \tag{2.6} \]
According to [1] [2] [3] we have the following estimates.

**Proposition 2.1.** Let $f \in L^1(\Omega)$ and assume (2.1)–(2.2). Let $u$ be a weak solution of (2.3) in the sense (2.4), and set
\[ \Lambda_q = \left( \frac{S_q^\frac{p}{1-p}}{\left\| f \right\|_1} \right)^\frac{n}{n-q} \]
where $S_q$ is the Sobolev constant for the embedding $D^{1,1}(\mathbb{R}^n) \hookrightarrow L^{\frac{n}{n-q}}(\mathbb{R}^n)$, $1 \leq q < n$. Then, for every $0 < \lambda < \Lambda_1$ there hold
\[ \int_\Omega e^{\lambda |u-h|} \leq \frac{|\Omega|}{1-\lambda \Lambda_1^{-1}}, \quad \int_\Omega |\nabla (u-h)|^q \leq 2S_q \Lambda_q \left( 1 + \frac{2^{\frac{q}{q-n}}}{(n-1)^{\frac{n-q}{q}}} \right)^\frac{q}{n} |\Omega|^{\frac{n-q}{n}}. \tag{2.7} \]

**Proof.** Fix $k \geq 0$, $a > 0$. Since $T_{k+a}(u-h) - T_k(u-h) \in W^{1,n}_0(\Omega) \cap L^\infty(\Omega)$, by (2.3)–(2.4) we get that
\[ \begin{align*}
\int_\Omega (a(x, \nabla u) - a(x, \nabla h), \nabla [T_{k+a}(u-h) - T_k(u-h)]) &= \int_\Omega f[T_{k+a}(u-h) - T_k(u-h)],
\end{align*} \tag{2.8} \]
yielding to
\[ \frac{1}{a} \int_{\{ k < |u-h| \leq k+a \}} |\nabla (u-h)|^q \leq \frac{\| f \|_1}{d} \tag{2.9} \]
in view of (2.2). By (2.9) and the following Lemma we deduce the validity of (2.7) and the proof of Proposition 2.1 is complete. \hfill \Box

**Lemma 2.2.** Let $w$ be a measurable function with $T_k(w) \in W^{1,n}_0(\Omega)$ so that for all $k \geq 0$, $a > 0$
\[ \frac{1}{a} \int_{\{ k < |w| \leq k+a \}} |\nabla w|^q \leq C_0 \tag{2.10} \]
for some $C_0 > 0$. Then there hold
\[ \int_\Omega e^{\lambda |w|} \leq \frac{|\Omega|}{1-\lambda \Lambda_1^{-1}}, \quad \int_\Omega |\nabla w|^q \leq 2C_0 \Lambda_q \left( 1 + \frac{2^{\frac{q}{q-n}}C_0}{(n-1)^{\frac{n-q}{q}}} \right)^\frac{q}{n} |\Omega|^{\frac{n-q}{n}} \tag{2.11} \]
for every $0 < \lambda < \Lambda = \left( \frac{S_q^\frac{p}{1-p}}{\left\| f \right\|_1} \right)^\frac{n}{n-q}$ and $1 \leq q < n$, where $k_0$ is given in (2.15).

**Proof.** Let $\Phi(k) = \{|x \in \Omega : |w(x)| > k\}$ be the distribution function of $|w|$. We have that
\[ \Phi(k+a)^{\frac{n}{n-q}} \leq \frac{1}{a} \left( \int_{\{ k+a \leq |w| \leq k+a \}} |\nabla w|^{n-q} \right)^{\frac{1}{n-q}} \leq \frac{1}{aS_1} \int_{\{ k+a \leq |w| \leq k+a \}} |\nabla w| \]
where $S_1$ is the Sobolev constant of the embedding $D^{1,1}(\mathbb{R}^n) \hookrightarrow L^{\frac{n}{n-q}}(\mathbb{R}^n)$. By the Hölder’s inequality and (2.10) we deduce that
\[ \Phi(k+a) \leq \frac{\Phi(k) - \Phi(k+a)}{a\Lambda} \]
and, as \( a \to 0^+ \),

\[
\Phi(k) \leq -\frac{1}{\Lambda} \Phi'(k)
\]

(2.12)

for a.e. \( k > 0 \). Since \( \Phi \) is a monotone decreasing function, an integration of (2.12)

\[
\ln \frac{\Phi(k)}{\Phi(0)} \leq \int_0^k \frac{\Phi'(s)}{\Phi(s)} \, ds \leq -\Lambda k
\]

provides that

\[
\Phi(k) \leq |\Omega|e^{-\Lambda k}
\]

for all \( k > 0 \), and then

\[
\int_{\Omega} e^{\lambda|u|} = |\Omega| + \lambda \int_{\Omega} \int_0^{u(x)} e^{\lambda k} \, dk = |\Omega| + \lambda \int_0^\infty e^{\lambda k} \Phi(k) \, dk
\]

\[
\leq |\Omega| + \lambda |\Omega| \int_0^\infty e^{-(\Lambda - \lambda)k} \, dk = \frac{|\Omega|}{1 - \lambda \Lambda^{-1}}
\]

for all \( 0 < \lambda < \Lambda \). Given \( k_0 \in \mathbb{N} \) introduce the sets

\[
\Omega_{k_0} = \{ x \in \Omega : |u(x)| \leq k_0 \}, \quad \Omega_k = \{ x \in \Omega : k - 1 < |u(x)| \leq k \} \quad (k > k_0),
\]

and by the Hölder’s inequality write for \( 1 \leq q < n \)

\[
\int_{\Omega_{k_0}} |\nabla w|^q \leq (C_0k_0)^\frac{q}{n} |\Omega|^\frac{n-q}{n}, \quad \int_{\Omega_k} |\nabla w|^q \leq C_0^\frac{q}{n} |\Omega_k|^\frac{n-q}{n} \leq \left( \int_{\Omega_k} |w|^\frac{nq}{n-q} \right)^\frac{n-q}{n}
\]

thanks to (2.10). For \( N \in \mathbb{N} \) let us sum up to get by the Hölder’s inequality

\[
\int_{\Omega} |\nabla T_{k_0+N}(w)|^q = \sum_{k=k_0}^{k_0+N} \int_{\Omega_k} |\nabla w|^q \leq (C_0k_0)^\frac{q}{n} |\Omega|^\frac{n-q}{n} + C_0^\frac{q}{n} \left( \sum_{k=k_0+1}^{k_0+N} \frac{1}{(k-1)^n} \nu \left( \int_{\Omega_k} |w|^\frac{nq}{n-q} \right) \right)^\frac{n-q}{n}
\]

(2.13)

Letting

\[
k_0 = 1 + \left( 2^\frac{q}{n} C_0 \right)^{\frac{n-q}{n}},
\]

(2.14)

we have that

\[
\sum_{k=k_0}^{\infty} \frac{1}{k^q} \leq \int_{k_0}^\infty \frac{dt}{t^q} = \frac{(k_0 - 1)^{-(q-1)}}{q - 1} = \frac{1}{C_0 \frac{q}{n}} \left( \frac{S_q}{2} \right)^\frac{n-q}{n}
\]

(2.15)

By using the Sobolev embedding \( \mathcal{D}^{1,q}(\mathbb{R}^n) \hookrightarrow L^{\frac{nq}{n-q}}(\mathbb{R}^n) \) on the L.H.S. of (2.13) and by (2.14) we deduce that

\[
S_q \left( \int_{\Omega} |\nabla w|^\frac{nq}{n-q} \right)^\frac{n-q}{n} \leq 2(C_0k_0)^\frac{q}{n} |\Omega|^\frac{n-q}{n}
\]

which inserted into (2.13) gives in turn

\[
\int_{\Omega} |\nabla T_{k_0+N}(w)|^q \leq 2(C_0k_0)^\frac{q}{n} |\Omega|^\frac{n-q}{n}
\]

(2.16)

Letting \( N \to +\infty \) we finally deduce that

\[
\int_{\Omega} |\nabla w|^q \leq 2(C_0k_0)^\frac{q}{n} |\Omega|^\frac{n-q}{n} = 2C_0^\frac{q}{n} \left( 1 + \frac{2^\frac{q}{n} C_0}{(n-1)S_q^{\frac{n-q}{n}}} \right)^\frac{1}{n-q} |\Omega|^\frac{n-q}{n}
\]

in view of (2.14) and the proof is complete.

As a first by-product of Proposition 2.1 we have that

**Theorem 2.3.** Let \( U \in W^{1,\alpha}_{loc}(\mathbb{R}^n) \) be a weak solution of (1). Then \( \sup_{\mathbb{R}^n} U < +\infty \) and \( U \in C^{1,\alpha}(\mathbb{R}^n) \), \( \alpha \in (0,1) \).

**Proof.** Assume that for \( 0 < \epsilon \leq 1 \)

\[
\int_{B_{\epsilon}(x)} e^{U} \leq \frac{S_n \epsilon}{3^{n-1} - \epsilon}.
\]

(2.16)

Thanks to Proposition 2.1 by (2.16) we deduce that

\[
\int_{B_{\epsilon}(x)} e^{2(U-H)} \leq 3\omega_n
\]

(2.17)
where $H$ is a $n$–harmonic function in $B_r(x)$ with $H = U$ on $\partial B_r(x)$. Since $H \leq U$ on $B_r(x)$ by the comparison principle, we have that
\[
\int_{B_r(x)} H_n^+ \leq \int_{B_r(x)} U_n^+ \leq n! \int_{R^n} e^U
\] (2.18)
where $u_+$ denotes the positive part of $u$. Since Theorem 2 in [34] is easily seen to be valid for $H^+$ too (simply by replacing $|H|$ with $H^+$ in the proof), by (2.18) we have that
\[
\sup_{B_{2r}(x)} H_n^+ \leq C_0(e)
\] (2.19)
for some $C_0(e) > 0$ independent on $x$. By (2.19) and (2.20) we deduce that
\[
\int_{B_{2r}(x)} e^{2U} = \int_{B_{2r}(x)} e^{2|U-H|} \leq 3e^{2C_0(e)\omega_n}.
\] (2.20)
Still thanks to the elliptic estimates in [34] on $U^+$, by (2.18) and (2.20) we have that
\[
\sup_{B_{2r}(x)} U_n^+ \leq C_1(e)
\] (2.21)
for some $C_1(e) > 0$ independent on $x$. To complete the proof, we argue as follows. Since $\int_{R^n} e^U < +\infty$ we can find $R > 0$ so that
\[
\int_{R^n \setminus B_R(0)} e^U \leq S_1^d/3^{n-1}.
\] (2.22)
Given $|x| > R + 1$, by (2.10) we have the validity of (2.10) with $\epsilon = 1$. For all $|x| \leq R + 1$ we can find $\epsilon_x > 0$ small so that (2.10) holds. By the compactness of the set $\{ |x| \leq R + 1 \}$ we can find points $x_1, \ldots, x_L$ so that
\[
\{ |x| \leq R + 1 \} \subset \bigcup_{i=1}^L B_{\epsilon_x}(x_i).
\] (2.23)
Therefore, by (2.21) we deduce that
\[
\sup_{R^n} U \leq \max\{C_1(1), C_1(\epsilon_{x_1}), \ldots, C_1(\epsilon_{x_L})\} < +\infty
\]
in view of (2.20). Since $e^U \in L^\infty(R^n)$ and $U \in L^1_{\text{loc}}(R^n)$, we can use the elliptic estimates in [16] [24] [37] to show that $U \in C^{1,\alpha}(R^n)$, for some $\alpha \in (0, 1)$. \hfill \Box

We aim now to establish some bounds on $U$ at infinity. Let us recall that the Kelvin transform $\hat{U}(x) = U(\frac{x-x_0}{|x-x_0|^n})$ of $U$ satisfies
\[
\begin{cases}
-\Delta \hat{U} = \frac{e^0}{|x|^{n-2}} & \text{in } R^n \setminus \{0\} \\
\int_{R^n} \frac{e^0}{|x|^{n+2}} < +\infty,
\end{cases}
\] (2.24)
where the equation is meant in the weak sense
\[
\int_{R^n} |\nabla \hat{U}|^{n-2} (\nabla \hat{U}, \nabla \Phi) = \int_{R^n} \frac{\hat{U}}{|x|^{n+2}} \Phi \quad \forall \Phi \in H = \{ \Phi : \hat{\Phi} \in \tilde{H} \}
\]
with $H$ given in (2.12). By Theorem 2.3 we know that $\hat{U} \in C^{1,\alpha}(R^n \setminus \{0\})$. Here and in the sequel, $\alpha \in (0, 1)$ will denote an Hölder exponent which can varies from line to line.

In order to understand the behavior of $\hat{U}$ at 0, we fix $r > R$ small and, for all $0 < \epsilon < r$, let $H_\epsilon \in W^{1,n}(A_\epsilon)$ satisfy
\[
\begin{cases}
\Delta_n H_\epsilon = 0 & \text{in } A_\epsilon := B_r(0) \setminus B_{\epsilon}(0) \\
H_\epsilon = \hat{U} & \text{on } \partial A_\epsilon.
\end{cases}
\] (2.26)
Regularity issues for quasi-linear PDEs involving $\Delta_n$ are well established since the works of DiBenedetto, Evans, Lewis, Serrin, Tolksdorff, Uhlenbeck, Uraltseva. For example, local Hölder estimates on $H_\epsilon$ can be found in [33] and then it follows by [16] [37] that $H_\epsilon \in C^{1,\alpha}(A_\epsilon)$. Thanks to [26] such regularity can be pushed up to the boundary to deduce that $H_\epsilon \in C^{1,\alpha}(\overline{A_\epsilon})$. By (2.20) the function $U_\epsilon = \hat{U} - H_\epsilon \in C^{1,\alpha}(\overline{A_\epsilon})$ satisfies
\[
\begin{cases}
\Delta_n (\hat{U} - U_\epsilon) = 0 & \text{in } A_\epsilon \\
U_\epsilon = 0 & \text{on } \partial A_\epsilon.
\end{cases}
\] (2.27)
We aim to derive estimates on $H_\epsilon$ and $U_\epsilon$ on the whole $A_\epsilon$ by using Proposition 2.1 with
\[
a(x,p) = |\nabla \hat{U}(x)|^{n-2} \nabla \hat{U}(x) - |\nabla \hat{U}(x) - p|^{n-2} (\nabla \hat{U}(x) - p).
\] (2.28)
Remark 2.4. Let us notice that $a(x, p)$ in (2.24) satisfies (2.4) with $a = |\nabla U|^{-1}$. Since $\hat{U}$ is expected to be singular at 0, it is likely true that $\|a\|_{L^\infty(\mathbb{T}^n)} \to +\infty$ as $\epsilon \to 0$. In order to get uniform estimates in $\epsilon$, it is crucial that the estimates in Propositions 2.1 do not depend on $\|a\|_{L^\infty(\mathbb{T}^n)}$. Assumption (2.1) is just necessary to make meaningful the notion of $W^{1,n}$-weak solution for (2.24). The same remark is in order for Proposition 4.1, when we will use it in Section 4 to show the logarithmic behavior of $\hat{U}$ at 0.

As a second by-product of Proposition 2.1, we have that

**Theorem 2.5.** There holds

$$\hat{U} \in W^{1,q}_{\text{loc}}(\mathbb{R}^n)$$

for all $1 \leq q < n$.

**Proof.** Since (2.24) does hold in $A_\epsilon$, (2.27) can be re-written as

$$\begin{align*}
\Delta_n(\hat{U} - U_\epsilon) - \Delta_n U_\epsilon &= 0 \\
U_\epsilon &= 0
\end{align*}$$

(2.30)

Since

$$d = \inf_{v \neq w} \frac{|v|^{n-2}v - |w|^{n-2}w}{|v - w|^n} > 0,$$

we can apply Proposition 2.1 to $a(x, p)$ in (2.28). Since $|A_\epsilon| \leq \omega_n r^2$ and $a(x, 0) = 0$, we deduce that

$$\int_{A_\epsilon} |\nabla U_\epsilon|^p + \int_{A_\epsilon} e^{U_\epsilon} \leq C$$

(2.32)

for all $1 \leq q < n$ and all $p \geq 1$ if $r$ is sufficiently small, where $C$ is uniform in $\epsilon$. Notice that

$$\int_{B_r(0)} \frac{e^U}{|x|^{2n}} = \int_{\mathbb{R}^n \setminus B_r(0)} e^U \to 0$$

as $r \to 0$. By the Sobolev embedding $D^{1,2}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ estimate (2.32) yields that

$$\int_{A_\epsilon} |U_\epsilon|^p \leq C$$

(2.33)

for some $C$ uniform in $\epsilon$. Since $H_\epsilon = \hat{U} - U_\epsilon$ with $\hat{U} \in C^{1,\alpha}(\mathbb{R}^n \setminus \{0\})$, by (2.33) we deduce that

$$\|H_\epsilon\|_{L^\infty(A)} \leq C(A) \quad \forall \ A \subset \subset B_r(0) \setminus \{0\}$$

(2.34)

for $\epsilon$ sufficiently small. Arguing as before, by [16, 22, 34, 37] it follows that

$$\|H_\epsilon\|_{C^{1,\alpha}(\mathbb{R}^n \setminus \{0\})} \leq C(A) \quad \forall \ A \subset \subset B_r(0) \setminus \{0\}$$

for $\epsilon$ small. By the Ascoli-Arzela’s Theorem and a diagonal process, we can find a sequence $\epsilon \to 0$ so that $H_\epsilon \to H_0$ in $C^{1,\alpha}_{\text{loc}}(B_r(0) \setminus \{0\})$, where $H_0$ satisfies

$$\begin{align*}
\Delta_n H_0 &= 0 & \text{in } B_r(0) \setminus \{0\} \\
H_0 &= \hat{U} & \text{on } \partial B_r(0).
\end{align*}$$

Since $H_\epsilon \leq \hat{U}$ in $A_\epsilon$ by the comparison principle, we have that $U_\epsilon \to U_0 := \hat{U} - H_0$ in $C^{1,\alpha}_{\text{loc}}(B_r(0) \setminus \{0\})$, where $U_0$ satisfies

$$U_0 \geq 0 \text{ in } B_r(0) \setminus \{0\}, \quad \partial_r U_0 \leq 0 \text{ on } \partial B_r(0).$$

Moreover, by (2.32) we get that

$$U_0 \in W^{1,q}_{\text{loc}}(B_r(0)), \quad e^{U_0} \in L^p(B_r(0))$$

(2.35)

for all $1 \leq q < n$ and all $p \geq 1$ if $r$ is sufficiently small.

Since $H_0$ is a continuous $n$-harmonic function in $B_r(0)$, we have

$$H_0 \leq \sup_{\mathbb{R}^n \setminus \{0\}} \hat{U} = \sup_{\mathbb{R}^n} U < \infty$$

in view of Theorem 2.2, we can apply the result in [41] about isolated singularities: either $H_0$ has a removable singularity at 0 or

$$\frac{1}{C} \leq \frac{H_0(x)}{\ln |x|} \leq C$$

near 0 for some $C > 1$. According to [35], in both situations we have that

$$H_0 \in W^{1,q}(B_r(0))$$

(2.36)

for all $1 \leq q < n$. The combination of (2.34) and (2.36) establishes the validity of (2.29) for $\hat{U} = U_0 + H_0$. □

In terms of $U$, Theorem 2.5 simply gives that
Corollary 2.6. There holds
\[ \int_{\mathbb{R}^n \setminus B_1(0)} \frac{|\nabla U|^q}{|x|^{2(n-q)}} < +\infty \] (2.36)
for all \(1 \leq q < n\).

Proof. Since
\[ |\det D_x U(x)| = \frac{1}{|x|^n}, \]
and
\[ |\nabla U(x)| = \frac{1}{|x|^2} |\nabla U|(\frac{x}{|x|^2}), \]
we have that
\[ \int_{B_r(0)} |\nabla U|^q = \int_{\mathbb{R}^n \setminus B_1(0)} \frac{|\nabla U|^q}{|x|^{2(n-q)}}. \]
By Theorems 2.3 and 2.5 we then deduce that
\[ \int_{\mathbb{R}^n \setminus B_1(0)} \frac{|\nabla U|^q}{|x|^{2(n-q)}} < +\infty \]
for all \(1 \leq q < n\), as desired. \(\square\)

3. An isoperimetric argument

The aim is to classify all the solutions \(U\) of (1.1) with small “mass”. The following isoperimetric approach leads to:

Theorem 3.1. Let \(U\) be a solution of (1.1) with \(\int_{\mathbb{R}^n} e^U \leq c_n \omega_n\). Then \(U\) is given by (1.3).

Proof. Since \(U \in C^{1,\alpha}(\mathbb{R}^n)\), we can use Theorem 3.1 in [12] to get that \(Z_k = \{x \in B_1(0) : \nabla U(x) = 0\}\) is a null set for all \(k \in \mathbb{N}\). By the Lipschitz continuity of \(U\) on \(B_1(0)\), we deduce that
\[ \{t \in \mathbb{R} : \exists x \in \mathbb{R}^n \text{ s.t. } U(x) = t, \ \nabla U(x) = 0\} = \bigcup_{k \in \mathbb{N}} U(Z_k) \]
is a null set in \(\mathbb{R}\). Therefore \(\Omega_t = \{U > t\}\) is a smooth set for a.e. \(t \leq t_0, t_0 = \sup U\), and has bounded Lebesgue measure in view of \(\int_{\mathbb{R}^n} e^U < +\infty\).

Let \(t \leq t_0\) and \(r > 0\). Given \(\delta, \eta > 0\), let us define the following functions:
\[ \chi_\delta(s) = \begin{cases} 0 & \text{if } s \leq t \\ \frac{s-t}{\delta} & \text{if } t \leq s \leq t+\delta \\ 1 & \text{if } s \geq t+\delta \end{cases} \]
and
\[ \chi_\eta(x) = \begin{cases} 1 & \text{if } x \in B_t(0) \\ \frac{1 + \eta}{\eta} & \text{if } x \in B_{t+\eta}(0) \setminus B_t(0) \\ 0 & \text{if } x \notin B_{t+\eta}. \end{cases} \]

We can use \(\chi_\delta(U)\chi_\eta(x)\) as a test function in (3.1) to get
\[ \int_{\mathbb{R}^n} e^U \chi_\delta(U)\chi_\eta(x) = \frac{1}{\delta} \int_{\Omega_t \setminus \Omega_{t+\delta}} \chi_\eta |\nabla U|^n - \frac{1}{\eta} \int_{B_{t+\eta}(0) \setminus B_t(0)} \chi_\delta(U) |\nabla U|^{n-2} \langle U, \frac{x}{|x|}\rangle. \] (3.1)

By the Lebesgue’s monotone convergence theorem for the first term in the R.H.S. of (3.1) we have that
\[ \frac{1}{\delta} \int_{\Omega_t \setminus \Omega_{t+\delta}} \chi_\eta |\nabla U|^n \rightarrow \frac{1}{\eta} \int_{\Omega_t \setminus \Omega_{t+\delta}} |\nabla U|^n \]
as \(\eta \rightarrow 0\). Since by the co-area formula we can write
\[ \int_{\Omega_t \setminus \Omega_{t+\delta}} |\nabla U|^n = \int_t^{t+\delta} ds \int_{\partial\Omega_s \cap \partial B_r(0)} |\nabla U|^{n-1} d\sigma, \]
it results that the function \(t \rightarrow \int_{\partial\Omega_s \cap \partial B_r(0)} |\nabla U|^{n-1} d\sigma\) is in \(L^1_{\text{loc}}(\mathbb{R})\), and as \(\delta \rightarrow 0\) by the Lebesgue’s differentiation Theorem we conclude that for a.e. \(t \leq t_0\)
\[ \frac{1}{\delta} \int_{\Omega_t \setminus \Omega_{t+\delta}} \chi_\eta |\nabla U|^n \rightarrow \int_{\partial\Omega_s \cap \partial B_r(0)} |\nabla U|^{n-1} d\sigma \] (3.2)
as \(\eta \rightarrow 0\) and \(\delta \rightarrow 0\). The second term in the R.H.S. of (3.1) writes in radial coordinates as
\[ \frac{1}{\eta} \int_{B_{t+\eta}(0) \setminus B_t(0)} \chi_\delta(U) |\nabla U|^{n-2} \langle U, \frac{x}{|x|}\rangle = \frac{1}{\eta} \int_r^{r+\eta} ds \int_{\partial B_s(0)} \chi_\delta(U) |\nabla U|^{n-2} \langle U, \frac{x}{|x|}\rangle d\sigma, \]
and by the fundamental Theorem of calculus we get that for all \( r > 0 \)
\[
\frac{1}{\eta} \int_{B_{r+\eta}(0) \setminus B_r(0)} \chi_s(U) |\nabla U|^{n-2} \langle \nabla U, \frac{x}{|x|} \rangle \to \int_{\partial B_r(0)} \chi_s(U) |\nabla U|^{n-2} \langle \nabla U, \frac{x}{|x|} \rangle d\sigma
\]
as \( \eta \to 0 \). By the Lebesgue’s monotone convergence theorem we deduce that for all \( r > 0 \)
\[
\frac{1}{\eta} \int_{B_{r+\eta}(0) \setminus B_r(0)} \chi_s(U) |\nabla U|^{n-2} \langle \nabla U, \frac{x}{|x|} \rangle \to \int_{\partial B_r(0)} |\nabla U|^{n-2} \langle \nabla U, \frac{x}{|x|} \rangle d\sigma
\]as \( \eta \to 0 \) and \( \delta \to 0 \). Letting \( \eta \to 0 \) and \( \delta \to 0 \) in (3.1) and letting (3.10) and the Hölder’s inequality we deduce the crucial estimate
\[
\int_{\Omega_t \cap \partial B_r(0)} e^U = \int_{\partial \Omega_t \cap \partial B_r(0)} |\nabla U|^{n-1} d\sigma - \int_{\Omega_t \cap \partial B_r(0)} |\nabla U|^{n-2} \langle \nabla U, \frac{x}{|x|} \rangle d\sigma.
\]for all \( r > 0 \) and a.e. \( t \leq t_0 \) (possibly depending on \( r \)) in view of the Lebesgue’s monotone convergence theorem.

**Remark 3.2.** We aim to let \( r \to +\infty \) in (3.1). In [2] no special care is required since for \( n = 2 \) \( U \) has a logarithmic behavior at infinity and then \( \Omega_t \) is a bounded set. When \( n > 2 \) we still don’t know that \( U \) behaves logarithmically at infinity and the validity of Theorem [3,1] is crucial in the next Section to establish such a property. Our argument relies instead on (2.36) and on the finite measure property of \( \Omega_t \), compare with [22].

In radial coordinates we can write
\[
|\Omega_t| = \int_0^r \frac{dr}{r} \int_{\Omega_t \cap \partial B_r} d\sigma < +\infty, \quad \int_{\mathbb{R}^n \setminus B_1} \frac{|\nabla U|^q}{|x|^{2(n-q)}} = \int_1^\infty \frac{dr}{r^{2(n-q)}} \int_{\partial B_r(0)} |\nabla U|^q d\sigma < +\infty
\]in view of (3.5). We claim that for all \( M \geq 1 \) there exists \( r \geq M \) so that
\[
\int_{\Omega_t \cap \partial B_r(0)} d\sigma \leq \frac{1}{r} \quad \text{and} \quad \frac{1}{r^{2(n-q)}} \int_{\partial B_r(0)} |\nabla U|^q d\sigma \leq \frac{1}{1}
\]Indeed, if the claim were not true, we would find \( M \geq 1 \) so that for all \( r \geq M \) there holds either
\[
\int_{\Omega_t \cap \partial B_r(0)} d\sigma > \frac{1}{r}
\]or
\[
\frac{1}{r^{2(n-q)}} \int_{\partial B_r(0)} |\nabla U|^q d\sigma > \frac{1}{r^q}.
\]
Setting \( I = \{ r \geq M : \text{3.5 holds} \} \) and \( II = [M, \infty) \setminus I \), we have that
\[
\int_I \frac{dr}{r} < \int_M^\infty \frac{dr}{r^{2(n-q)}} \int_{\partial B_r(0)} |\nabla U|^q d\sigma \leq \int_{\mathbb{R}^n \setminus B_1} \frac{|\nabla U|^q}{|x|^{2(n-q)}}
\]since (3.7) does hold for all \( r \in II \). Summing up (3.8)-(3.9) we get that
\[
\int_{M^\infty} \frac{dr}{r} \leq |\Omega_t| + \int_{\mathbb{R}^n \setminus B_1} \frac{|\nabla U|^q}{|x|^{2(n-q)}}
\]in contradiction with (3.5), and the claim is established.

Thanks to the claim we can construct a sequence \( r_k \to +\infty \) so that
\[
\int_{\Omega_t \cap \partial B_{r_k}} d\sigma \leq \frac{1}{r_k} \frac{1}{r_k^{2(n-q)}} \int_{\partial B_{r_k}} |\nabla U|^q d\sigma \leq \frac{1}{r_k}
\]By (3.10) and the Hölder’s inequality we deduce the crucial estimate
\[
\int_{\Omega_t \cap \partial B_{r_k}} |\nabla U|^{n-1} d\sigma \leq \left( \int_{\Omega_t \cap \partial B_{r_k}} |\nabla U|^q d\sigma \right)^{\frac{1}{q}} \left( \int_{\Omega_t \cap \partial B_{r_k}} d\sigma \right)^{\frac{n-1}{n}} \leq \frac{1}{r_k^{1-2 \frac{n-q}{n-1}}} \to 0
\]by choosing \( q \in (n-1, n) \) sufficiently close to \( n \).

Choosing \( r = r_k \) in (3.4) and letting \( k \to +\infty \) we get that
\[
\int_{\Omega_t} e^U = \int_{\partial \Omega_t} |\nabla U|^{n-1} d\sigma
\]for a.e. \( t \leq t_0 \) in view of (3.11). Arguing as previously, by the co-area formula and the Lebesgue’s differentiation theorem we have that
\[
|\Omega_t| = \lim_{r \to +\infty} |\Omega_t \cap B_r(0)| = \lim_{r \to +\infty} \int_r^\infty ds \int_{\partial \Omega_t \cap \partial B_r(0)} \frac{d\sigma}{|\nabla U|} = \int_t^\infty ds \int_{\partial \Omega_t} \frac{d\sigma}{|\nabla U|}.
\]
and then
\[ -\frac{d}{dt} |\Omega| = \int_{\partial \Omega_t} \frac{d\sigma}{|\nabla U|} \] (3.13)
for a.e. \( t \leq t_0 \). Thanks to (3.11), (3.12) by the Hölder’s and the isoperimetric inequalities we can now compute
\[ -\frac{d}{dt} \left( \int_{\Omega_t} e^U \, dx \right)^{\frac{n-1}{n}} = -\frac{n}{n-1} \left( \int_{\Omega_t} e^U \, dx \right)^{\frac{n-1}{n}} e^t \frac{d}{dt} |\Omega_t| \]
\[ = \frac{n}{n-1} \left( \int_{\Omega_t} |\nabla U|^{n-1} \, d\sigma \right)^{\frac{n-1}{n}} \frac{d}{dt} \int_{\partial \Omega_t} |\nabla U|^{n-1} \, d\sigma \]
\[ \geq \frac{n}{n-1} \left( \frac{d}{dt} |\partial \Omega_t| \right)^{\frac{n-1}{n}} \geq (c_n \omega_n)^{\frac{n-1}{n}} e^t |\Omega_t| \] (3.14)
for a.e. \( t \leq t_0 \). Since \( t \to \int_{\Omega_t} e^U \, dx \) is a monotone decreasing function, we get that
\[ \left( \int_{\mathbb{R}^n} e^U \, dx \right)^{\frac{n-1}{n}} \geq \frac{1}{n-1} \left( \int_{\mathbb{R}^n} e^U \, dx \right)^{\frac{n-1}{n}} \geq (c_n \omega_n)^{\frac{n-1}{n}} \int_{\mathbb{R}^n} e^U \, dx. \] (3.15)
Since by assumption \( \int_{\mathbb{R}^n} e^U \, dx \leq c_n \omega_n \), we get that
\[ \int_{\mathbb{R}^n} e^U \, dx = c_n \omega_n \]
and the inequalities in (3.14)-(3.15) are actually equalities. We have that for a.e. \( t \leq t_0 \)
- \( \Omega_t = B_R(t)(x(t)) \) for some \( R(t) > 0 \) and \( x(t) \in \mathbb{R}^n \), since \( \Omega_t \) in an extremal of the isoperimetric inequality
- \( |\nabla U|^{n-1} \) is a multiple of \( \frac{1}{|\partial \Omega_t|} \) on \( \partial \Omega_t \),
- the function \( M(t) = \int_{\Omega_t} e^U \, dx \) is absolutely continuous in \((-\infty, t_0)\) with
\[ \frac{1}{n-1} M^{\frac{n-1}{n}}(t) = \frac{1}{n} \frac{d}{dt} M(\Omega) = -(c_n \omega_n)^{\frac{n-1}{n}} e^t R^n(t). \] (3.16)
The aim now is to derive an equation for \( M(t) \) by means of some Pohozaev identity. Let us emphasize that \( U \in C^{1,\alpha}(\mathbb{R}^n) \) and the classical Pohozaev identities usually require more regularity. In \( \text{124} \) a self-contained proof is provided in the quasilinear case, which reads in our case as

**Lemma 3.3.** Let \( \Omega \subset \mathbb{R}^n, n \geq 2 \), be a smooth bounded domain and \( f \) be a locally Lipschitz continuous function. Then, there holds
\[ n \int_\Omega F(U) = \int_\Omega \left[ F(U)(x-y, \nu) + |\nabla U|^{n-2}(x-y, \nabla U) \partial_x U - \frac{|\nabla U|^n}{n}(x-y, \nu) \right] \]
for all \( y \in \mathbb{R}^n \) and all weak solution \( U \in C^{1,\alpha}(\Omega) \) of \(-\Delta U = f(U) \) in \( \Omega \), where \( F(t) = \int_0^t f(s) \, ds \) and \( \nu \) is the unit outward normal vector at \( \partial \Omega \).

Let us re-write (3.12) as
\[ M(t) = n \omega_n |\nabla U|^{n-1} R^{n-1}(t) \] (3.17)
and use Lemma 3.3 on \( \Omega_t = B_R(t)(x(t)) \) with \( y = x(t) \) to deduce
\[ M(t) = \omega_n e^t R^n(t) + \frac{n-1}{n} \omega_n |\nabla U|^n R^n(t) \] (3.18)
in view of \( |\nabla U| = -\partial_x U \) constant on \( \partial \Omega_t \). By (3.17)-(3.18) we have that
\[ \omega_n e^t R^n(t) = M(t) - (c_n \omega_n)^{\frac{n-1}{n}} e^t M^{\frac{n-1}{n}}(t), \] (3.19)
which, inserted into (3.18), gives rise to
\[ \frac{1}{n} \int_{\mathbb{R}^n} \frac{dM}{M - (c_n \omega_n)^{\frac{n-1}{n}} M^{\frac{n-1}{n}}(t)} = \ln |M^{\frac{n-1}{n}} - (c_n \omega_n)^{\frac{n-1}{n}}|, \]
we can integrate (3.20) to get
\[ M(t) = c_n \omega_n \left[ 1 - e^{\frac{t-t_0}{n}} \right]^{\frac{n-1}{n}} \] (3.21)
for all \( t \leq t_0 \), in view of \( M(t_0) = 0 \). Inserting (3.21) into (3.19) we deduce that
\[ R^n(t) = c_n \left[ 1 - e^{\frac{t-t_0}{n}} \right]^{\frac{n-1}{n}} e^{-\frac{(t-t_0)\omega_n}{n}}. \] (3.22)
for a.e. \( t \leq t_0 \). Since \( R(t) \) is monotone, notice that (3.22) is valid for all \( t \leq t_0 \) and can be re-written as

\[
e^t = \frac{c_0 \lambda^n}{(1 + \lambda \overline{\omega}^{\pi + 1}(R \overline{\omega})^{\pi + 1})^n}
\]

where \( \lambda = \left( \frac{u}{\overline{\omega}} \right)^{\frac{1}{n}} \). To conclude, we just need to show that \( x(t) = x_0 \). First notice that a.e. \( t_1, t_2 \leq t_0 \) either \( x(t_1) = x(t_2) \) or, assuming for example \( t_2 < t_1 \), \( B_{R(t_2)}(x(t_1)) \subset B_{R(t_2)}(x(t_2)) \) and \( x(t_2) - R(t_2) \frac{x(t_2) - x(t_1)}{\|x(t_2) - x(t_1)\|} \in \partial B_{R(t_2)}(x(t_2)) \) implies

\[
R(t_2) - \|x(t_1) - x(t_2)\| = \|x(t_2) - x(t_1)\| - R(t_2) = \|x(t_2) - R(t_2) \frac{x(t_2) - x(t_1)}{\|x(t_2) - x(t_1)\|} - x(t_1)\| > R(t_1).
\]

In both cases, we have that \( |x(t_2) - x(t_1)| \leq |R(t_2) - R(t_1)| \) for a.e. \( t_1, t_2 \leq t_0 \). Since \( R \in C(-\infty, t_0) \cap C^1(-\infty, t_0) \), \( x(t) \) can be uniquely extended as a map \( \tilde{x}(t) \) which is continuous in \((-\infty, t_0)\) and locally Lipschitz in \((-\infty, t_0)\). Given \( t < t_0 \) we can always find \( t_n \downarrow t \) so that \( \Omega_{t_n} = B_{R(t_n)}(x(t_n)) \), \( x(t_n) = \tilde{x}(t_n) \), and then there holds

\[
\Omega_t = \bigcup_{n \in \mathbb{N}} \Omega_{t_n} = \bigcup_{n \in \mathbb{N}} B_{R(t_n)}(x(t_n)) = B_{R(t)}(\tilde{x}(t))
\]

by the continuity of \( R(t) \) and \( \tilde{x}(t) \). Identifying and \( \tilde{x} \), we can assume that \( x \in C(-\infty, t_0) \cap Lip_{loc}(-\infty, t_0) \) and \( \Omega_t = B_{R(t)}(x(t)) \) for all \( t \leq t_0 \). Use now the property \( t = U(x(t) + R(t)\omega) \), \( \omega \in S^n \), to deduce

\[
h = U(x(t + h) + R(t + h)\omega) - U(x(t) + R(t)\omega) = (\nabla U)(x(t) + R(t)\omega, x(t) - x(t)) + [R(t + h) - R(t)](\nabla U)(x(t) + R(t)\omega, \omega) + o(|x(t + h) - x(t)| + |R(t + h) - R(t)|)
\]

as \( h \to 0 \), uniformly in \( \omega \in S^n \). Since \( \nabla U \) is a non-zero constant on \( \partial \Omega_t \) for a.e. \( t \leq t_0 \) and \( \Omega_t = B_{R(t)}(x(t)) \), we have that

\[
\nabla U(x(t) + R(t)\omega) = -\nabla U|\omega|
\]

and then, applied to \(-\omega\) and \( \omega \), it yields that

\[
h = -|\nabla U|(x(t + h) - x(t), \omega) - |R(t + h) - R(t)||\nabla U| + o(|x(t + h) - x(t)| + |R(t + h) - R(t)|)
\]

as \( h \to 0 \), uniformly in \( \omega \in S^n \). If \( x(t + h) \neq x(t) \), the choice \( \omega = \frac{x(t + h) - x(t)}{|x(t + h) - x(t)|} \) leads to

\[
\frac{|x(t + h) - x(t)|}{h} \leq o\left(\frac{|R(t + h) - R(t)|}{h}\right) \to 0
\]

as \( h \to 0 \). So we have shown that \( x'(t) = \lim_{h \to 0} \frac{x(t + h) - x(t)}{h} = 0 \) for a.e. \( t \leq t_0 \). Since \( x \in Lip_{loc}(-\infty, t_0) \), by integration we deduce that \( x(t) \) is constant for all \( t \leq t_0 \), say \( x(t) = x_0 \).

Given \( x \in \mathbb{R}^n \setminus \{x_0\} \), by (3.22) we can find a unique \( t < t_0 \) so that \( R(t) = |x - x_0| \) and then

\[
e^{-|x|} = \frac{c_0 \lambda^n}{(1 + \lambda \overline{\omega}^{\pi + 1}(|x - x_0|^{\pi + 1}))^n}
\]

in view of (3.24) and \( U = t \) on \( \partial B_{R(t)}(x_0) \). The proof is complete since we have shown that \( U = U_{\lambda, x_0} \) for some \( \lambda > 0 \) and \( x_0 \in \mathbb{R}^n \).

\[ \square \]

4. Behavior of \( U \) at infinity

The estimates in Proposition 4.1 are not sufficient to establish the logarithmic behavior of \( U \) at infinity but are essentially optimal in the limiting case \( f \in L^1(\Omega) \). According to (2.31) [8], a bit more regularity on \( f \) gives \( L^\infty \)-bounds as stated in

Proposition 4.1. Let \( f \in L^p(\Omega), p > 1 \), and assume (2.31) - (2.24). Let \( u \in W_0^{1,n}(\Omega) \) be a weak solution of \(-div a(x, \nabla u) = f \). Then

\[
\|u\|_\infty \leq C \|f\|_p^{\alpha_0} + 1)^{\alpha_0}(\Omega + 1)^{\beta_0} \|u\|_{L^p(\Omega)}^{\beta_0}
\]

for some constants \( C, \alpha_0, \beta_0, \bar{q} > 0 \) just depending on \( n, p \) and \( \Omega \).

Proof. Given \( q \geq 1 \) and \( k > 0 \) set

\[
F(s) = \begin{cases} s^q & \text{if } 0 \leq s \leq k \\ qk^{q-1} - (q - 1)k^q & \text{if } s > k \end{cases}
\]

and \( G(s) = F(s)[F'(s)]^{-1} \). Notice that \( G \) is a piecewise \( C^1 \)–function with a corner just at \( s = k \) so that

\[
[F'(s)]^{-1} \leq G'(s), \quad G(s) \leq s^{-1}F^{-\frac{q(k-1)}{q-1}}(s).
\]

(4.1)
Since $G(|u|) \in W^{1,n}_0(\Omega)$ for $G$ is linear at infinity, use $\text{sign}(u)G(|u|)$ as a test function in the equation of $u$ to get
\[ \int_{\Omega} |\nabla F(|u|)|^n \leq \frac{1}{d} \int \Omega G'(|u|)(a(x,\nabla u), \nabla u) = \frac{1}{a} \int \Omega f \text{sign}(u)G(|u|) \quad (4.2) \]

in view of (2.2) and (4.1). Setting $m = \frac{1}{n-1}$ in view of $p > 1$, by (4.1) and the Hölder's inequality we deduce that
\[ |\int \Omega f \text{sign}(u)G(|u|)| \leq q^{n-1} \int \Omega |f|^\frac{n(q-1)+1}{mnq} \leq q^{n-1} |\Omega|^{\frac{1}{n-1}} \cdot |\int \Omega F^{mn}(|u|)|^{\frac{n(q-1)+1}{mnq}}. \quad (4.3) \]

The Sobolev embedding Theorem applied on $F(|u|) \in W^{1,n}_0(\Omega)$ now implies that
\[ \int_{\Omega} F^{2mn}(|u|) \leq C \int_{\Omega} |\nabla F(|u|)|^n \leq \frac{C}{d} \int_{\Omega} |\nabla u|^n \leq |\Omega|^{\frac{1}{n-1}} \cdot |\int \Omega F^{mn}(|u|)|^{\frac{n(q-1)+1}{mnq}} \]

for some $C \geq 1$ in view of (3.1), (4.3). Since $F(s) \to s^q$ in a monotone way as $k \to +\infty$, we have that
\[ \int_{\Omega} |u|^{2mnq} \leq \left[ \frac{1}{d} \ln \frac{C\|\|\|}{d} \right] \left( \frac{(n-1)\ln |\Omega|}{mnq} + (n-1)\ln q \right) \left( \int_{\Omega} |u|^{mnq} \right)^{\frac{1}{1-\frac{1}{n-1}}} \quad (4.4) \]

Assume now that $u \in L^{mnq}(|\Omega|)$ for some $q_i \geq 1$. Setting $q_i = 2^{-1}q_i$, $j \in \mathbb{N}$, by iterating (4.4) we deduce that
\[ \int_{\Omega} |u|^{mnq_{j+1}} \leq \exp \left[ \frac{1}{q_j} \ln \frac{C\|\|\|}{d} \right] \left( \frac{(n-1)\ln |\Omega|}{mnq_{j+1}} + (n-1)\ln q_j \right) \left( \int_{\Omega} |u|^{mnq_j} \right)^{\frac{1}{1-\frac{1}{n-1}}} \]

\[ \cdot \leq \exp \left[ \frac{1}{q_j} \ln \frac{C\|\|\|}{d} \right] \left[ \frac{(n-1)\ln |\Omega|}{mn} \sum_{k=1}^j \frac{1}{q_k} + (n-1)\ln q_j \right] \left( \int_{\Omega} |u|^{mnq_{j+1}} \right)^{\frac{1}{1-\frac{1}{n-1}}} \]

where
\[ a_i^j = \begin{cases} \frac{1}{j+1} & \text{if } 0 \leq k < j \\ \frac{1}{j} & \text{if } k = j. \end{cases} \]

Since $a_i^j \leq 1$ for all $k = 0, \ldots, j$, we have that
\begin{align*}
\alpha_0 &= \frac{1}{n} \sup_{j \in \mathbb{N}} \sum_{k=1}^j a_i^j \leq \frac{1}{n} \sup_{j \in \mathbb{N}} \sum_{k=1}^j \frac{1}{q_k} = \frac{2}{n} \sum_{k=1}^\infty \frac{1}{q_k 2^k} < \infty \\
\beta_0 &= \frac{n-1}{mn^2} \sup_{j \in \mathbb{N}} \sum_{k=1}^j a_i^j \leq \frac{4(n-1)}{mn^2} \sum_{k=1}^\infty \frac{1}{q_k 4^k} < +\infty \\
\gamma_0 &= \frac{n-1}{n} \sup_{j \in \mathbb{N}} \sum_{k=1}^j a_i^j \leq \frac{2}{n} \frac{n-1}{\sum_{k=1}^\infty (k-1) \ln 2 + \ln q_k} \frac{1}{q_k 2^k} < +\infty,
\end{align*}

and then it follows that
\[ \left( \int_{\Omega} |u|^{mnq_{j+1}} \right)^{\frac{1}{mnq_{j+1}}} \leq \exp \left[ \alpha_0 \ln C(\frac{\|\|\|}{d} + 1) + \beta_0 \ln (|\Omega| + 1) + \gamma_0 \right] \left( \int_{\Omega} |u|^{mnq_j} \right)^{\frac{1}{mnq_j}}. \quad (4.5) \]

Since
\[ \tilde{q} = \lim_{j \to +\infty} a_i^j = \prod_{k=1}^\infty \left( 1 - \frac{n-1}{nq_k} \right) < \infty, \]

letting $j \to +\infty$ in (4.5) we finally deduce that
\[ \|u\|_\infty \leq e^{\alpha_0 \ln C + \gamma_0 \ln (\frac{\|\|\|}{d} + 1) + \beta_0 (|\Omega| + 1) \ln n \|u\|_{mnq_1}^2} \]

and the proof is complete. \qed

Thanks to Theorem 3.1 we are just concerned with the range
\[ \int_{|\Omega|} e^U \geq c_n \omega_n. \quad (4.6) \]

By Proposition 4.1 we can improve the estimates in Section 2 to get
Theorem 4.2. Let $U$ be a solution of \((\text{4.11})\) which satisfies \((\text{4.10})\). Then $\hat{U}(x) = U(\frac{r}{|x|^n})$ satisfies

$$\hat{U}(x) - \left(\frac{\gamma_0}{n\omega_n}\right)^{\frac{1}{n-1}} \ln |x| \in L^\infty_{\text{loc}}(\mathbb{R}^n)$$

(4.7)

and

$$\sup_{|x|=r} |\nabla \left(\hat{U}(x) - \left(\frac{\gamma_0}{n\omega_n}\right)^{\frac{1}{n-1}} \ln |x|\right)| \to 0$$

(4.8)

for a sequence $r \to 0$, where $\gamma_0 = \int_{\mathbb{R}^n} e^U$.

Proof. We adopt the same notations as in Theorem 4.1 and we try to push more the analysis thanks to \((\text{4.10})\). Given $r > 0$, recall that $\hat{U}$ has been decomposed in $B_r(0)$ as $\hat{U} = U_0 + H_0$, $U_0, H_0 \in C^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$, where $H_0$ is a $n-$harmonic function in $B_r(0) \setminus \{0\}$ with $\sup_{B_r(0) \setminus \{0\}} H_0 < +\infty$ and $U_0 \geq 0$ satisfies \((\text{4.9})\) with $U_0 = 0$, $\partial_r U_0 \leq 0$ on $\partial B_r(0)$.

The description of the behavior of $H_0$ at $0$, as established in \((\text{4.9})\), has been later improved in \((\text{21})\) to show that there exists $\gamma \geq 0$ with

$$H_0(x) - \left(\frac{\gamma_0}{n\omega_n}\right)^{\frac{1}{n-1}} \ln |x| \in L^\infty(B_r(0)),$$

$$\Delta H_0 = \gamma \delta_0 in B_r(0))$$

(4.9)

Since $\hat{U} \in W^{1,n-1}(B_r(0))$ according to Theorem \((\text{4.2})\) we can extend \((\text{2.23})\) at $0$ as

$$- \Delta_\nu \hat{U} = \frac{e^U}{|x|^n} - \gamma \delta_0$$

(4.10)

in the sense

$$\int_{\mathbb{R}^n} |\nabla \hat{U}|^{n-2} \langle \nabla \hat{U}, \nabla \Phi \rangle = \int_{\mathbb{R}^n} \frac{e^U}{|x|^n} \Phi - \gamma \phi(0)$$

(4.11)

for all $\Phi \in C^1(\mathbb{R}^n)$ so that $\Phi \in W^{1,n}_{\text{loc}}(\mathbb{R}^n)$. Indeed, let us consider a smooth function $\eta$ so that $\eta = 0$ for $|x| \leq \delta$, $\eta = 1$ for $|x| \geq 2\delta$ and $|\nabla \eta| \leq \frac{2}{\delta}$. Use $\eta \Phi - \Phi(0) \in \hat{H}$ as a test function in \((\text{2.23})\) to provide

$$\int_{\mathbb{R}^n} \eta |\nabla \hat{U}|^{n-2} \langle \nabla \hat{U}, \nabla \Phi \rangle + O(\int_{\mathbb{R}^n} |\nabla \hat{U}|^{n-1} |\nabla \eta||\Phi - \Phi(0)|) = \int_{\mathbb{R}^n} \eta \frac{e^U}{|x|^n} (\Phi - \Phi(0)).$$

Since

$$\int_{\mathbb{R}^n} |\nabla \hat{U}|^{n-1} |\nabla \eta||\Phi - \Phi(0)| \leq C \int_{B_{2\delta}(0)} |\nabla \hat{U}|^{n-1} \to 0$$

as $\delta \to 0$, we can let $\delta \to 0$ in \((\text{4.11})\) and get the validity of \((\text{4.11})\) in view of $\gamma_0 = \int_{\mathbb{R}^n} \frac{e^U}{|x|^n} = \int_{\mathbb{R}^n} e^U$.

Since $U_0 \geq 0$, the singularity of $\hat{U} = U_0 + H_0$ at $0$ should be weaker than that of $H_0$. Via an approximation procedure, it is easily seen that equations \((\text{4.10})\), \((\text{4.19})\) can be re-written as

$$\gamma \Phi(0) = \int_{\partial B_r(0)} |\nabla H_0|^{n-2} \partial_r H_0 \Phi - \int_{B_r(0)} |\nabla H_0|^{n-2} \langle \nabla H_0, \nabla \Phi \rangle$$

(4.13)

$$\gamma_0 \phi(0) = \int_{B_r(0)} \frac{e^U}{|x|^n} \Phi + \int_{\partial B_r(0)} |\nabla \hat{U}|^{n-2} \partial_r \hat{U} \Phi - \int_{B_r(0)} |\nabla \hat{U}|^{n-2} \langle \nabla \hat{U}, \nabla \Phi \rangle$$

(4.14)

for all $\Phi \in C^1(B_r(0))$. We claim that

$$|\nabla H_0|^{n-2} \partial_r H_0 \geq |\nabla \hat{U}|^{n-2} \partial_r \hat{U} \quad \text{on} \ \partial B_r(0)$$

(4.15)

and then, by taking $\Phi = 1$ in \((\text{4.13})\), \((\text{4.14})\), we deduce that

$$\gamma = \int_{\partial B_r(0)} |\nabla H_0|^{n-2} \partial_r H_0 \geq \int_{\partial B_r(0)} |\nabla \hat{U}|^{n-2} \partial_r \hat{U} = \gamma_0 - \int_{B_r(0)} \frac{e^U}{|x|^n}.$$
we can use $r$ along the sequence $r$ in view of (4.6), by (4.9) and (4.16) we have that
\[ e^{\frac{H_0}{|x|^{2n}}} \in L^q(B_r(0)) \] (4.17)
for all $1 \leq q < \frac{n^2}{n-2}$ if $r$ is sufficiently small. By (2.34) and (4.17) it follows that
\[ e^{\frac{\hat{U}}{|x|^{2n}}} = e^{\frac{H_0}{|x|^{2n}}} \in L^q(B_r(0)) \] (4.18)
for all $1 \leq q < \frac{n^2}{n-2}$ if $r > 0$ is sufficiently small. Thanks to (4.3) we can apply Proposition 4.1 to $U_r$ on $A_r$ (see (2.20)-(2.27)) with $a(x,p)$ given by (2.28) to get
\[ \|U_r\|_{\infty,A_r} \leq C \]
for some uniform $C > 0$. We have used that
\[ \sup_{t} \|U_t\|_{p,A_t} < +\infty \]
for all $p \geq 1$ in view of (2.32) and the Sobolev embedding Theorem. Letting $\epsilon \to 0$ we get that $\|U_0\|_{\infty,B_r(0)} < +\infty$ and then
\[ \hat{U} = U_0 + H_0 = (\frac{\gamma}{n\omega_n})^{\frac{1}{n-1}} \ln |x| + H(x), \quad H \in L^\infty_{\text{loc}}(\mathbb{R}^n) \] (4.19)
in view of (4.14). Notice that now $\gamma$ does not depend on $r$ and then satisfies
\[ \gamma \geq C_n \omega_n \]
in view of (4.9) and (4.10). Given $r > 0$ small, let us define the function \[ V_r(y) = \hat{U}(ry) - (\frac{\gamma}{n\omega_n})^{\frac{1}{n-1}} \ln r = (\frac{\gamma}{n\omega_n})^{\frac{1}{n-1}} \ln |y| + H(ry). \]
Since \[ \Delta_n V_r = -e^{\frac{\hat{U}(ry)}{r^n|y|^{2n}}} = -\frac{r^{n+\alpha} e^{H(ry)}}{|y|^{n+\alpha - n}} \]
in view of (4.19) with $\alpha = (\frac{\gamma}{n\omega_n})^{\frac{1}{n-1}} - \frac{n^2}{n-1} \geq 0$, we have that $V_r$ and $\Delta_n V_r$ are bounded in $L^\infty_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$, uniformly in $r$. By (10) (34) (37) we deduce that $V_r$ is bounded in $C^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$, uniformly in $r$. By the Ascoli-Arzelà’s Theorem and a diagonal process we can find a sequence $r \to 0$ so that $V_r \to V_0$ in $C^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$, where $V_0$ is a $n$-harmonic function in $\mathbb{R}^n \setminus \{0\}$. Setting $H_r(y) = H(ry)$, we deduce that $H_r \to H_0$ in $C^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$, where $H_0 \in L^\infty(\mathbb{R}^n)$ in view of (4.19). Since $V_0 = (\frac{\gamma}{n\omega_n})^{\frac{1}{n-1}} \ln |y| + H_0$ with $H_0 \in L^\infty(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{0\})$, we can apply Lemma 4.3 below to show that $H_0$ is a constant function. In particular we get that
\[ \sup_{|x|=r} |\nabla (\hat{U}(x) - (\frac{\gamma}{n\omega_n})^{\frac{1}{n-1}} \ln |x|)| \quad \sup_{|y|=1} |\nabla H_r(y)| \to \sup_{|y|=1} |\nabla H_0(y)| = 0 \] (4.20)
along the sequence $r \to 0$. The proof of (4.17)-(4.8) now follows by (4.19)-(4.20) once we show that $\gamma = \gamma_0$. Indeed, by (4.10) we have that
\[ \gamma_0 = \int_{B_r(0)} \frac{e^{\hat{U}}}{|x|^{2n}} + \int_{\partial B_r(0)} |\nabla \hat{U}|^{n-2} \partial_r \hat{U} = o(1) + \frac{\gamma}{n\omega_n} \int_{\partial B_r(0)} \frac{1}{|x|^{n-1}} (1 + o(1)) \to \gamma \]
where $r \to 0$ is any sequence with property (4.20). The proof is complete. \(\square\)

We have used the following simple result:

**Lemma 4.3.** Let $\gamma \ln |x| + H$ be a $n-$harmonic function in $\mathbb{R}^n \setminus \{0\}$ with $H \in C^1(\mathbb{R}^n \setminus \{0\})$. If $H \in L^\infty(\mathbb{R}^n)$, then $H$ is a constant function.

**Proof.** Let $\eta$ be a cut-off function with compact support in $\mathbb{R}^n \setminus \{0\}$. Since
\[ -\Delta_n (\gamma \ln |x| + H) = -\Delta_n (\gamma \ln |x| + H) + \Delta_n (\gamma \ln |x|) = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}, \]
we can use $\eta^n H$ as a test function to get
\[ d \int_{\mathbb{R}^n} \eta^n |\nabla H|^n \quad \leq \quad \int_{\mathbb{R}^n} \eta^n (|\nabla (\gamma \ln |x| + H)|^{n-2} \nabla (\gamma \ln |x| + H) - |\nabla (\gamma \ln |x|)|^{n-2} \nabla (\gamma \ln |x|) \cdot \nabla H)
\]
\[ = -n \int_{\mathbb{R}^n} \eta^{n-1} H (|\nabla (\gamma \ln |x| + H)|^{n-2} \nabla (\gamma \ln |x| + H) - |\nabla (\gamma \ln |x|)|^{n-2} \nabla (\gamma \ln |x|) \cdot \nabla \eta) \]
in view of \(2.31\). Since \(H \in L^\infty(\mathbb{R}^n)\), by the Young’s inequality we get that
\[
d \int_{\mathbb{R}^n} \eta^n |\nabla H|^n \leq Cn ||H||_\infty \int_{\mathbb{R}^n} \eta^{n-1} \left( |\nabla H|^{n-1} + \frac{|\nabla H|}{|x|^{n-2}} \right) |\nabla \eta| \leq \frac{d}{2} \int_{\mathbb{R}^n} \eta^n |\nabla H|^n + C \int_{\mathbb{R}^n} |\nabla \eta|^n + \int_{\mathbb{R}^n} \frac{|\nabla \eta|}{|x|^{\frac{n-2}{n-1}}} \]
\]
in view of \(\eta \leq 1\) and
\[
||v_1 + w||^{n-2}(v_1 + w) - ||w||^{n-2}w \leq C( ||v||^{n-1} + ||v|| ||w||^{n-2}).
\]
Hence, we have found that
\[
\int_{\mathbb{R}^n} \eta^n |\nabla H|^n \leq C \left( \int_{\mathbb{R}^n} |\nabla \eta|^n + \int_{\mathbb{R}^n} \frac{|\nabla \eta|}{|x|^{\frac{n-2}{n-1}}} \right).
\]
Given \(\delta \in (0,1)\), we make the following choice for \(\eta\):
\[
\eta(x) = \begin{cases}
0 & \text{if } |x| \leq \delta^2 \\
\frac{-\ln |x| - 2 \ln \delta}{\ln \delta} & \text{if } \delta^2 \leq |x| \leq \delta \\
\frac{\ln |x| + 2 \ln \delta}{\ln \delta} & \text{if } \delta \leq |x| \leq \frac{1}{\delta^2} \\
1 & \text{if } |x| \geq \frac{1}{\delta^2}.
\end{cases}
\]
Since
\[
\int_{\mathbb{R}^n} |\nabla \eta|^n = \left( \int_{\{x^2 \leq |x| \leq \delta \}^n} \frac{1}{|x|^n} \right) \frac{2 \omega_{n-1}}{\ln \delta} \to 0
\]
and
\[
\int_{\mathbb{R}^n} \frac{|\nabla \eta|}{|x|^{\frac{n-2}{n-1}}} = \frac{2 \omega_{n-1}}{\ln \delta} \int_{\{x^2 \leq |x| \leq \delta \}^n} \frac{1}{|x|^n} \to 0
\]
as \(\delta \to 0\), we deduce that
\[
\int_{\mathbb{R}^n} |\nabla \eta|^n = 0
\]
by letting \(\delta \to 0\) in \(4.21\). Then \(H\) is a constant function.

5. Pohozaev identity

Thanks to Theorem \(4.2\), we aim to apply the Pohozaev identity of Lemma \(3.3\) to show that \(1.6\) automatically implies \(\int_{\mathbb{R}^n} e^U = c_n \omega_n\). Combined with Theorem \(5.1\) it completes the proof of the classification result in Theorem \(1.1\).

To this aim, we show the following:

**Theorem 5.1.** Let \(U\) be a solution of \(1.1\) which satisfies \(1.6\). Then, there holds
\[
\int_{\mathbb{R}^n} e^U = c_n \omega_n.
\]

**Proof.** Since
\[
\partial U(x) = \sum_{k=1}^n \frac{1}{|x|^2} \left( \delta_{ik} - \frac{x_i x_k}{|x|^2} \right) (\partial_k \tilde{U})(x) \frac{|x|}{|x|^2},
\]
we have that
\[
|\nabla U|(x) = \frac{1}{|x|} |\nabla \tilde{U}|(x) \frac{|x|}{|x|^2}, \quad \langle x, \nabla U(x) \rangle = -(\frac{x}{|x|^2}, \nabla \tilde{U}(x) \frac{|x|}{|x|^2}).
\]
We can apply Theorem \(2.2\) and deduce by \(1.8\) that
\[
|\nabla U|(x) = \frac{1}{|x|} \left( \frac{\gamma_0}{n \omega_n} \frac{1}{|x|^2} + o(1) \right), \quad \langle x, \nabla U(x) \rangle = -(\frac{x}{|x|^2}, \nabla \tilde{U}(x) \frac{|x|}{|x|^2}).
\]
uniformly for \(x \in \partial B_R(0)\), for a sequence \(R = \frac{1}{n} \to +\infty\) and \(\gamma_0 = \int_{\mathbb{R}^n} e^U\). By \(5.1\) we have that
\[
\int_{\partial B_R(0)} \left[ |\nabla U|^{n-2}(x, \nabla U) \partial U - \frac{|\nabla U|^n}{n} (x, \partial U) \right] - \omega_{n-1}(1 - \frac{1}{n})(\frac{\gamma_0}{n \omega_n}) \frac{1}{|x|^2} \to 0
\]
as \(R \to +\infty\). Since by \(1.8\)
\[
|x| \left( \frac{\omega_n}{n \omega_n} \right) \frac{1}{|x|^2} e^U \in L^\infty(\mathbb{R}^n \setminus B_1(0))
\]
with \(\left( \frac{\omega_n}{n \omega_n} \right) \frac{1}{|x|^2} \geq \frac{1}{|x|^2} \) in view of \(1.9\), we also get that
\[
\int_{\partial B_R(0)} e^U \langle x, \partial U \rangle \to 0
\]
as \(R \to +\infty\). We apply Lemma \(3.3\) to \(U\) on \(B_R(0)\) with \(y = 0\) and let \(R \to +\infty\) to get
\[
n \gamma_0 = \omega_n \left( \frac{\gamma_0}{n \omega_n} \right) \frac{1}{|x|^2}
\]
in view of \( \text{(52)-(53)} \). It results that

\[
\gamma_n = \int_{\mathbb{R}^n} e^U = c_n \omega_n.
\]

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