INFORMATION SET DECODING OF LEE-METRIC CODES OVER
FINITE RINGS

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Abstract. Information set decoding (ISD) algorithms are the best known procedures to solve the decoding problem for general linear codes. These algorithms are hence used for codes without a visible structure, or for which efficient decoders exploiting the code structure are not known. Classically, ISD algorithms have been studied for codes in the Hamming metric. In this paper we switch from the Hamming metric to the Lee metric, and study ISD algorithms and their complexity for codes measured with the Lee metric over finite rings.

1. Introduction

The task of decoding a given code, also known as syndrome decoding problem (SDP), is a fundamental issue in coding theory. In formula, given a parity-check matrix \( H \), a syndrome \( s \) and an integer \( w \), solving the syndrome decoding problem (SDP) consists in finding a vector \( e \), of weight not larger than \( w \), such that \( He^\dagger = s \), where \( ^\dagger \) denotes vector transposition. Note that such a formulation does not depend on the metric with which the code is embedded. A well-studied case is that of the Hamming metric, for which the SDP has been proven to be NP-hard \([1, 2]\); recently, the same hardness result has been extended to the rank metric case \([3]\).

The best known solvers for the Hamming SDP are known as information set decoding (ISD) algorithms, originally proposed by Prange in 1962 \([4]\). Prange’s idea consists in iteratively testing randomly chosen information sets, until a set which does not overlap with the support of the unknown vector is found; the expected number of required iterations is given by the reciprocal of the probability that a randomly chosen set is indeed valid. Prange’s information set decoding (ISD) has been improved through several subsequent works \([5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]\). One of these variants, due to Stern \([9]\), is widely used in the literature and will be considered in the following, together with the original ISD by Prange. All these approaches increase the cost of one iteration but, on average, require a smaller number of iterations. Note that, to solve the rank metric SDP, the best known algorithms \([21, 22]\) share the same principle, since they are based on an iterative procedure where the number of iterations depends on the probability of making an initial correct guess.

The study of ISD algorithms finds several applications in coding theory. For example, the ISD principle is at the basis of ordered statistics decoders that are used for soft-decision decoding of linear block codes \([23, 24]\). Another important field of application of ISD algorithms is that of code-based cryptography, since the security of many code-based public-key cryptosystems relies on the hardness of solving the decoding problem for a general linear block code \([25, 26]\). Moreover, code-based
signature and identification schemes exploit similar principles, since the adversary’s ability of forging signatures and proving knowledge depends on the difficulty of finding low-weight vectors associated to given syndromes [9, 27, 28, 29]. Code-based cryptosystems are nowadays characterized by a renewed interest, because of their intrinsic resistance against quantum attacks. In fact, there is no known way to exploit quantum algorithms to efficiently solve the SDP: quantum versions of ISD algorithms are still characterized by a complexity that grows exponentially in the weight of the unknown vector [15]. This well-assessed security makes code-based cryptosystems among the most promising solutions for the post-quantum world [30].

All the above examples, however, rely on codes in the Hamming metric. The rationale of this work is to introduce techniques to solve the SDP for general codes in the Lee metric. To the best of our knowledge, this has not been done yet, except for some preliminary work in [31]. In particular, our study can be useful to address potential applicability of the Lee metric to design new code-based cryptosystems. For such a purpose, starting from [31], where Stern’s ISD algorithm is converted to the ring \( \mathbb{Z}_4 \), we extend this work and propose algorithms inspired by Prange’s and Stern’s ISD to solve the Lee metric variant of SDP for any integer residue ring whose size is a prime power. A detailed complexity analysis of these algorithms is provided.

The most relevant conclusion of our analysis is that, under certain assumptions, the complexity of Prange’s ISD algorithm in the Lee metric is lower bounded by that of Prange’s ISD algorithm in the Hamming metric, reduced by a relatively small polynomial factor.

The paper is organized as follows. In Section 2 we introduce the notation used throughout the paper and give some preliminary notions on the Lee metric. In Section 3 we formulate some general properties of the Lee metric. In Section 4 we extend ISD algorithms to \( \mathbb{Z}_{p^m} \), considering the Lee metric and carry out a complexity analysis of these algorithms. In Section 5 we provide numerical results and in Section 6 we draw some concluding remarks.

2. Notation and preliminaries

In the rest of the paper we set \( q = p^m \), where \( p \) is a prime number and \( m \) a positive integer. We also denote with \( \mathbb{Z}_q \) the ring of integers modulo \( q \), and with \( \mathbb{F}_q \) the finite field with \( q \) elements. The cardinality of a set is denoted as \( |V| \). We use bold lower case (respectively upper case) letters to denote vectors (respectively matrices). The identity matrix with size \( k \) is denoted as \( I_k \). Given a length-\( n \) vector \( \mathbf{x} \) and a set \( S \subset \{1, \ldots, n\} \), we denote with \( \mathbf{x}_S \) the vector formed by the entries of \( \mathbf{x} \) indexed by \( S \); similarly, given a matrix \( \mathbf{M} \) with \( n \) columns, \( \mathbf{M}_S \) denotes the matrix formed by the columns of \( \mathbf{M} \) that are indexed by the elements in \( S \). The support of a vector \( \mathbf{a} \) is defined as \( \mathcal{S}(\mathbf{a}) = \{j \text{ s.t. } a_j \neq 0\} \). For \( S \subset \{1, \ldots, n\} \), we denote by \( \mathbb{Z}_q^n(S) \) the vectors having support in \( S \).

Classically, an \([n,k]\) linear code \( \mathcal{C} \) is a linear subspace of \( \mathbb{F}_q^n \) of dimension \( k \), endowed with the Hamming metric. The size of the code, denoted as \( |\mathcal{C}| \), is the number of its codewords, i.e. \( |\mathcal{C}| = q^k \). For an \([n,k]\) linear code \( \mathcal{C} \) over \( \mathbb{F}_q \) and \( I \subset \{1, \ldots, n\} \) we denote by \( \mathcal{C}_I = \{\mathbf{c}_I : \mathbf{c} \in \mathcal{C}\} \) and we say that \( I \) of size \( k \) is an information set if \( |\mathcal{C}_I| = |\mathcal{C}| \). In other words, \( \forall \mathbf{c} \in \mathcal{C} \), we take the vectors formed by the entries of each \( \mathbf{c} \) indexed by \( I \) and put them in a set \( \mathcal{C}_I \). We call \( I \) an information
set if $|\mathcal{I}| = |\mathcal{C}|$. Finally, we say that two codes are permutation equivalent iff their codewords coincide, except for a permutation of their symbols.

The definitions above can be extended on a finite ring $R$. For such a purpose, let $h$ and $n$ be positive integers and let $R$ be a finite ring. $C$ is called an $R$-linear code of length $n$ and type $h$, if $C$ is a submodule of $R^n$, with $|C| = h$. Throughout this paper we restrict to the case $R = \mathbb{Z}_q$ and call $C$ a ring linear code of length $n$ iff $C$ is an additive subgroup of $\mathbb{Z}_q^n$.

For a ring linear code $C$ over $\mathbb{Z}_q$ of length $n$ and type $(p^m)^k_1 (p^{m-1})^k_2 \cdots (p)^k_m$, where $k_1, \ldots, k_m$ is a sequence of $m$ integers such that $\sum_{i=1}^{m} k_i = K$, we call a set $I \subseteq \{1, \ldots, n\}$ of size $K$ a (ring linear) information set if $|\mathcal{I}| = |\mathcal{C}|$.

**Proposition 1.** Let $C$ be a linear code over $\mathbb{Z}_{p^m}$ of length $n$ and type

$$|C| = (p^m)^{k_1} (p^{m-1})^{k_2} \cdots (p)^{k_m}.$$ 

Then $C$ is permutation equivalent to a code having the following systematic parity-check matrix $H$ of size $(n - k_1) \times n$

$$H = \begin{pmatrix}
B_{1,1} & B_{1,2} & \cdots & B_{1,m-1} & B_{1,m} & I_{n-K} \\
pB_{2,1} & pB_{2,2} & \cdots & pB_{2,m-1} & pI_{k_m} & 0 \\
p^2B_{3,1} & p^2B_{3,2} & \cdots & p^2B_{3,m-1} & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
p^{m-1}B_{m,1} & p^{m-1}B_{m,2} & \cdots & 0 & 0 & 0
\end{pmatrix},$$

where $B_{1,j} \in \mathbb{Z}_{p^{m+1-k_j}}^{(n-K) \times k_j}$ and $B_{i,j} \in \mathbb{Z}_{p^{m+1-k_j}}^{k_{m+1-i} \times k_j}$ for $i > 1$.

### 3. Properties of the Lee metric

For $x \in \mathbb{Z}_q$ we define the Lee value to be

$$|x|_L = \min\{x, q - x\}.$$

Then, for $x \in \mathbb{Z}_q^n$, we define the Lee weight as

$$\text{wt}_L \{x\} = \sum_{i=1}^{n} |x_i|_L.$$

For $x, y \in \mathbb{Z}_q^n$, the Lee distance is defined as

$$d_L(x, y) = \text{wt}_L \{x - y\}.$$

A code embedded with the Lee distance is called a Lee code. Because of the linearity, the minimum distance $d_L$ of a Lee code is defined as the minimum Lee weight of a non-zero codeword, that is

$$d_L = \min\{\text{wt}_L \{x\} \text{ s.t. } 0_n \neq x \in \mathcal{C}\}.$$

**Lemma 2.** [32, Problem 10.15] Let $x$ be the random variable with uniform distribution over $\mathbb{Z}_q$; its average Lee weight is

$$\mu_q = \begin{cases} 
\frac{q}{2} & \text{if } q \text{ is even}, \\
\frac{q^2 - 1}{4q} & \text{if } q \text{ is odd}.
\end{cases}$$
Lemma 3. Let $n \in \mathbb{N}$, $q$ be a prime power, $w \in \mathbb{N}$ such that $w \in \{1, \ldots, n\lfloor \frac{q}{2}\rfloor\}$, $s \in \mathbb{N}$ such that $s \leq \min\{n, w\}$. Then, $f(n, w, q, s)$ is equal to

\[
\begin{cases}
0 & \text{if } w > s\lfloor \frac{q}{2}\rfloor, \\
\binom{n}{s} & \text{if } w = s\lfloor \frac{q}{2}\rfloor, \\
\binom{n}{s}2^s\binom{w-1}{s-1} & \text{if } w < s + \lfloor \frac{q}{2}\rfloor - 1,
\end{cases}
\]

if $q$ is even, and to

\[
\begin{cases}
0 & \text{if } w > s\lfloor \frac{q}{2}\rfloor, \\
\binom{n}{s}2^s\binom{w-1}{s-1} & \text{if } w \leq s + \lfloor \frac{q}{2}\rfloor - 1, \\
\binom{n}{s}2^s\binom{w-1}{s-1} - \sigma_1 & \text{if } w > s + \lfloor \frac{q}{2}\rfloor - 1,
\end{cases}
\]

if $q$ is odd, where \(\sigma_1 = \sum_{i=\lfloor \frac{w}{2}\rfloor + 1}^{w-s+1} 2nf(n-1, w-i, q, s-1)\) and \(\sigma_2 = nf(n-1, w-\lfloor \frac{q}{2}\rfloor, q, s-1)\).

Proof. If $s > w$, a vector having a support of size $s$ has at least Lee weight $s$ and can have at most Lee weight $s\lfloor \frac{q}{2}\rfloor$, which implies that there are no vectors such that $w > s\lfloor \frac{q}{2}\rfloor$. In the case where $q$ is even, there exists only one element in $\mathbb{Z}_q$ having Lee value $\lfloor \frac{q}{2}\rfloor$, thus if $w = s\lfloor \frac{q}{2}\rfloor$, we can only choose this element in the non-zero positions, which can be done in $\binom{n}{s}$ different ways. Now we check whether $s-1 > w - \lfloor \frac{q}{2}\rfloor$ or $s-1 \leq w - \lfloor \frac{q}{2}\rfloor$. In the first case the vector cannot have an entry of Lee value $\lfloor \frac{q}{2}\rfloor$, thus we can choose $s$ non-zero positions, compose the wanted Lee weight $w$ into $s$ parts and for each choice of a part $x$, there exists also the choice $q-x$, hence $2^s$ many. In the other case, firstly, an entry of the vector could have Lee value $\lfloor \frac{q}{2}\rfloor$, so we cannot simply multiply by $2^s$ anymore and, secondly, the compositions of $w$ into $s$ parts also consists of parts being greater than $\lfloor \frac{q}{2}\rfloor$ which, however, is the largest possible Lee value. For this reason, we have to define $f(q, n, s, w)$ recursively. We start with all possible orderings of the desired Lee weight $w$ into $s$ parts and then take away the orderings that we cannot have, which are starting from a part being $i = \lfloor \frac{w}{2}\rfloor + 1$ and proceed until the largest part being $i = s - w + 1$. Thus, we have to take away $f(q, n-1, s-1, w-i)$, repeating this $2n$ times: factor 2 is justified by the fact that we have assumed there are always two choices for an element having Lee value $i$, and $n$ times for the position of the entry having Lee value $i$. The case $i = \lfloor \frac{w}{2}\rfloor$ has to be taken away only once, since, in the case where $q$ is even, we only have one element having Lee value $\lfloor \frac{q}{2}\rfloor$. The case in which $q$ is odd is simpler, since an element having Lee value $\lfloor \frac{q}{2}\rfloor$ does not need to be treated as a special case. \(\square\)
Corollary 4. Let \( n \in \mathbb{N} \), and \( 1 \leq w \leq n \left\lfloor \frac{q}{2} \right\rfloor \). Then
\[
F(n, w, q) = \min\{n, w\} \sum_{s=1}^{\min\{n, w\}} f(n, w, q, s).
\]

An upper bound, also observed in [32], and a lower bound on (3.3) can easily be derived as reported next.

Corollary 5. Let \( n \in \mathbb{N} \), and \( 1 \leq w \leq n \left\lfloor \frac{q}{2} \right\rfloor \). Then, \( F(n, w, q) \) is not larger than
\[
u(n, w) = \min\{n, w\} \sum_{s=1}^{\min\{n, w\}} \binom{n}{s} 2^s \binom{w-1}{s-1}
\]
and not smaller than
\[
l(n, w) = \begin{cases} \binom{n}{w} 2^w & \text{if } w < n, \\ 2^n & \text{if } w \geq n. \end{cases}
\]
The proof of the upper bound is given in [32]. Observe that the upper bound is exact for \( w \leq \left\lfloor \frac{q}{2} \right\rfloor \). For the lower bound, we only consider the vectors in \( \mathbb{Z}_q^n \) having Lee weight \( w \), with entries in \( \{1, q-1\} \), i.e., those with maximum support size.

Simple computations show that the addends of the sum in (3.4) are monotonically increasing iff, for \( w > 2 \),
\[
n \geq \frac{w^2 + w - 2}{2}.
\]
Under these assumptions, the following relation holds
\[
u(n, w) \leq w \binom{n}{w} 2^w.
\]
The above properties are exploited in the following section, in order to compute the complexity of ISD algorithms.

4. Information set decoding over \( \mathbb{Z}_q \)

All ISD algorithms are characterized by the same approach of first randomly choosing a set of positions in the code and then applying some operations that, if the chosen set has a relatively small intersection with the error vector, allow to retrieve the error vector itself. For each ISD variant, the average computational cost is estimated by multiplying the complexity of each iteration by the expected number of performed iterations; the latter quantity corresponds to the reciprocal of the probability that a random choice of the set leads to a successful iteration. Then, for all ISD algorithms, we have a computational cost that is estimated as \( O(C_{\text{iter}} P_{\text{guess}}^{-1}) \), where \( C_{\text{iter}} \) is the expected number of (binary) operations that are performed in each iteration and \( P_{\text{guess}} \) is the probability that the choice of the set of positions is indeed successful. We now derive some formulas for the complexity of Prange’s and Stern’s ISD algorithms, when adapted to the Lee metric.
4.1. Adaptation of Prange’s ISD to the Lee metric. The idea of Prange’s algorithm is to first find an information set that does not overlap with the support of the searched error vector \( e \); when such a set is found, permuting \( H \) and computing its row echelon form is enough to reveal the error vector. Our proposed adaptation of Prange’s ISD is reported in Algorithm 1. We first find an information set \( I \), and then bring the matrix \( H \) into a systematic form, by multiplying it by an invertible matrix \( U \). For the sake of clarity, we assume that the information set is \( I = \{1, \ldots, K\} \), such that

\[
UH = \begin{pmatrix} A & I_{n-K} \\ pB & 0 \end{pmatrix},
\]

where \( A \in \mathbb{Z}_{p^m}^{(n-K)\times K} \) and \( B \in \mathbb{Z}_{p^m-1}^{(K-k_1)\times K} \). Since we assume that no errors occur in the information set, we have that \( e = (0, e_1) \), with wt\(_L\) \( \{e_1\} = t \). Thus, if we also partition the new syndrome \( Us \) into parts of the same sizes as the (row-)parts of \( UH \), and we multiply \( UH \) by the unknown \( e^\top \), we get the following situation

\[
UHe^\top = \begin{pmatrix} A & I_{n-K} \\ pB & 0 \end{pmatrix} \begin{pmatrix} 0 \\ e_1 \end{pmatrix} = \begin{pmatrix} s_1 \\ 0 \end{pmatrix} = Us.
\]

It follows that \( e_1 = s_1 \), hence we are only left to check the weight of \( s_1 \).

**Algorithm 1** Prange’s Algorithm over \( \mathbb{Z}_{p^m} \) in the Lee metric

Input: \( H \in \mathbb{Z}_{p^m}^{(n-k_1)\times n} \), \( s \in \mathbb{Z}_{p^m}^{n-k_1} \), \( t \in \mathbb{N} \).

Output: \( e \in \mathbb{Z}_{p^m}^{n} \) with \( He^\top = s \) and \( wt_L(e) = t \).

1. Choose an information set \( I \subset \{1, \ldots, n\} \) of size \( K \) and define \( J = \{1, \ldots, n\} \setminus I \).
2. Compute \( U \in \mathbb{Z}_{p^m}^{(n-k_1)\times n-k_1} \) such that

\[
(UH)_I = \begin{pmatrix} A \\ pB \end{pmatrix} \quad \text{and} \quad (UH)_J = \begin{pmatrix} I_{n-K} \\ 0 \end{pmatrix}
\]

where \( A \in \mathbb{Z}_{p^m}^{(n-K)\times K} \) and \( B \in \mathbb{Z}_{p^m-1}^{(K-k_1)\times K} \).
3. Compute \( Us = \begin{pmatrix} s_1 \\ 0 \end{pmatrix} \) with \( s_1 \in \mathbb{Z}_{p^m}^{n-K} \).
4. if \( wt_L \{s_1\} = t \) then

5. Return \( e \) such that \( e_I = 0 \) and \( e_J = s_1 \).
6. Start over with Step 1 and a new selection of \( I \).

4.2. Complexity analysis: Prange’s ISD in the Lee metric. In this section we provide a complexity estimate of our adaptation of Prange’s ISD to the Lee metric. First of all, we assume that adding two elements in \( \mathbb{Z}_q \) costs \( \lambda_{\text{sum}} = \log_2 q \) binary operations and multiplying two elements costs \( \lambda_{\text{mul}} = (\log_2 q)^2 \) binary operations [33, 34]. An iteration of Prange’s ISD only consists in bringing \( H \) into systematic form and to apply the same row operations on the syndrome; thus, the cost can be assumed equal to that of computing \( U(H | s) \), from which we obtain a broad estimate as

\[
C_{\text{iter}} = O((n - k_1)^2(n + 1)\lambda_{\text{mul}}).
\]
The success probability is given by having chosen the correct weight distribution of $e$; in this case, we require that $S(e)$ does not overlap with the chosen information set, hence

$$ P_{\text{guess}} = \frac{F(n-K,t,q)}{F(n,t,q)}. $$

The estimated computational cost of Prange’s ISD in the Lee metric is

$$ OC_{\text{Prange}} = C_{\text{iter}}P_{\text{guess}}^{-1}. $$

We now analytically compare the complexity of Prange’s ISD in the Lee and Hamming metric, exploiting the properties derived in Section 3. Under the assumption that $n - K \geq \frac{t^2 + t - 2}{2}$, with $2 < t < n - K$, from Corollary 5 we derive the following chain of inequalities

$$ P_{\text{guess}}^{-1} = \frac{F(n,t,q)}{F(n-K,t,q)} \geq \frac{t(n,t)}{u(n-K,t)} $$

$$ \geq \frac{\binom{n}{t}2^t}{t(n-K)2^t} = \frac{1}{t} \binom{n}{t} = \frac{(P^{H}_{\text{guess}})^{-1}}{t}, $$

where $P^{H}_{\text{guess}}$ corresponds to the success probability of an iteration of Prange’s ISD over the Hamming metric, seeking for an error vector of Hamming weight $t$, in a code with length $n$ and dimension $K$. A crude approximation, which however is particularly tight when $t \ll n - K$, shows that $P^{H}_{\text{guess}} \approx (1 - \frac{K}{n})^t$ [35]. Then, we have

$$ P_{\text{guess}}^{-1} \geq \frac{(P^{H}_{\text{guess}})^{-1}}{t} \approx 2^{-t\log_2(1-\frac{K}{n})-\log_2 t}. $$

Since $C_{\text{iter}}$ does not depend on the considered metric, this simple analysis shows that the complexity of Prange’s algorithm over the Lee metric and over the Hamming metric differ at most by a polynomial factor. For all known ISD variants, the complexity grows asymptotically as $2^{ct(1+o(1))}$, where $c$ is a constant that depends on the code rate [36]; different ISD variants essentially differ only in the value of $c$. Our analysis shows that, for the Lee metric, Prange’s algorithm leads to an analogous expression. Thus, our results indicate that some SDP instances in the Lee metric are as hard as their corresponding Hamming counterparts, except for a relatively small polynomial factor; we leave further studies (such as, for instance, NP-hardness results) for future works.

4.3. **Stern’s ISD adaptation to the Lee metric.** As a further contribution of this paper, we improve upon the basic algorithm by Prange by adapting the idea of Stern’s ISD to the Lee metric. In this algorithm, we relax the requirements on the weight distribution, by allowing an information set with small Lee weight and the existence of a (small) set of size $\ell$, called zero-window, within the redundant set, where no errors occur. Our proposed adaptation of Stern’s algorithm to the Lee metric is reported in Algorithm 2.

For the sake of readability, in the following explanation we consider an information set $I = \{1, \ldots, K\}$ and a zero-window given by $\{K + 1, \ldots, K + \ell\}$, such that
metric; to this end, we make the following considerations.

We derive the computational cost of our adapted Stern’s ISD algorithm in the Lee metric. Complexity analysis: Stern’s ISD in the Lee metric.

4.4. We want to choose \( \mathbf{e}_1 \) such that it has support in the information set \( I \) and Lee weight \( 2v \), whereas \( \mathbf{e}_2 \) should have a support disjoint from that of \( \mathbf{e}_1 \), and the remaining Lee weight \( t - 2v \). More precisely, we test \( \mathbf{e}_1 = \mathbf{e}_X + \mathbf{e}_Y \), where \( \mathbf{e}_X \) and \( \mathbf{e}_Y \) have disjoint supports of respective maximal sizes \( m_1 \) and \( m_2 \) and equal weight \( v \). In order for (4.5) and (4.7) to be satisfied we construct two sets \( S \) and \( T \), where \( S \) contains the equations regarding \( \mathbf{e}_X \) and \( T \) contains the equations regarding \( \mathbf{e}_Y \). For all choices of \( \mathbf{e}_X \) and \( \mathbf{e}_Y \), we check whether the entries of \( S \) and \( T \) coincide, if they do we call this a collision. For each collision, we construct from (4.6) \( \mathbf{e}_2 = \mathbf{e}_2 - \mathbf{Be}_1 = \mathbf{s}_2 - \mathbf{Be}_X - \mathbf{Be}_Y \) and check if \( \mathbf{e}_2 \) has the missing Lee weight \( t - 2v \): if this occurs, we have found the error vector \( \mathbf{e} = (\mathbf{e}_X + \mathbf{e}_Y, 0, \mathbf{s}_2 - \mathbf{Be}_X - \mathbf{Be}_Y) \).

All these considerations are incorporated in Algorithm 2, where we allow any choice of \( I \) and \( Z \).

4.4. Complexity analysis: Stern’s ISD in the Lee metric. In this section we derive the computational cost of our adapted Stern’s ISD algorithm in the Lee metric; to this end, we make the following considerations.

i) The cost of bringing \( \mathbf{H} \) in systematic form is as in Section 4.2 and it requires

\[
\chi_U = (n - k_1)^2(n + 1)\lambda_{\text{mul}}
\]

binary operations.

ii) To build the set \( S \), we need to compute \( \mathbf{Ae}_X \) and \( p\mathbf{Ce}_X \) for all \( \mathbf{e}_X \in \mathbb{Z}_{pm}^K(X) \) with Lee weight \( v \); since \( X \) is fixed, such vectors have a cardinality \( F(m_1, v, q) \). The cost of building \( S \) is given by

\[
\chi_S = F(m_1, v, q) [(K - k_1 + \ell)m_1\lambda_{\text{mul}} + (K - k_1 + \ell)(m_1 - 1)\lambda_{\text{sum}}]
\]

binary operations.

iii) The set \( T \) is constructed similarly, but in the first two entries we need to subtract the vector \( \mathbf{s}_1 \) (resp. \( p\mathbf{s}_3 \)) from each resulting vector. Thus, constructing the set \( T \) costs

\[
\chi_T = F(m_2, v, q)(K - k_1 + \ell)m_2 (\lambda_{\text{mul}} + \lambda_{\text{sum}})
\]

binary operations.
Algorithm 2 Stern’s Algorithm over $\mathbb{Z}_p^{m}$ in the Lee metric

Input: $H \in \mathbb{Z}_q^{(n-k) \times n}$, $s \in \mathbb{Z}_q^n$, $v, m_1, m_2, \ell \in \mathbb{Z}$, such that $K = m_1 + m_2$, $v \leq \min\{m_1 \frac{q}{2}, m_2 \frac{q}{2}\}$, $\ell \leq n - K$ and $t - 2v \leq (n - K - \ell)\frac{q}{2}$.

Output: $e \in \mathbb{Z}_q^n$ with $He^T = s$ and $wt_L(e) = t$.

1. Choose an information set $I \subset \{1, \ldots, n\}$ of size $K$.
2. Choose a set $Z \in \{1, \ldots, n\} \setminus I$ of size $\ell$ and define $J = \{1, \ldots, n\} \setminus (I \cup Z)$.
3. Choose a uniform random partition of $I$ into disjoint sets $X$ and $Y$ of size $m_1$ and $m_2 = K - m_1$, respectively.
4. Find an invertible matrix $U \in \mathbb{Z}_p^{(n-k_1) \times (n-k_1)}$ such that

\[
(UH)_I = \begin{pmatrix} A \\ B \\ pC \end{pmatrix}, \quad (UH)_Z = \begin{pmatrix} I_a \\ 0 \\ 0 \end{pmatrix}, \quad (UH)_J = \begin{pmatrix} 0 \\ I_{n-K-\ell} \end{pmatrix}
\]

where $A \in \mathbb{Z}_p^{\ell \times K}$, $B \in \mathbb{Z}_p^{(n-K-\ell) \times K}$ and $C \in \mathbb{Z}_p^{K \times K}$.

5. Compute $Us = \begin{pmatrix} s_1 \\ s_2 \\ ps_3 \end{pmatrix}$ with $s_1 \in \mathbb{Z}_p^{\ell}$, $s_2 \in \mathbb{Z}_p^{n-K-\ell}$ and $s_3 \in \mathbb{Z}_p^{K-k_1}$.

6. Compute the set $S$ consisting of all triples $(Ae_X, pCe_X, e_X)$, where $e_X \in \mathbb{Z}_p^K(X)$, $wt_L(e_X) = v$.

7. Compute the set $T$ consisting of all triples $(s_1 - Ae_Y, ps_3 - pCe_Y, e_Y)$, where $e_Y \in \mathbb{Z}_p^K(Y)$, $wt_L(e_Y) = v$.

8. For each $(a, b, e_X) \in S$ do
9. \hspace{1em} For each $(a, b, e_Y) \in T$ do
10. \hspace{2em} If $wt_L(s_2 - B(e_X + e_Y)) = t - 2v$: then
11. \hspace{3em} Return $e_I = e_X + e_Y, e_Z = 0, e_J = s_2 - B(e_X + e_Y)$.
12. \hspace{1em} Start over with Step 1 and a new selection of $I$.

iv) The average amount of collisions in the two entries of the set $S$ and $T$ is given by

\[
\frac{|S| \cdot |T|}{(p^n)^{t+K-k_1}} = \frac{F(m_1, v, q)F(m_2, v, q)}{(p^n)^{t+K-k_1}}.
\]

For each collision we need to compute $s_2 - B(e_X + e_Y)$ and check that its Lee weight is not larger than $t - 2v$. We exploit the concept of early abort [15], i.e., stop the computation as soon as the maximum Lee weight is reached. Since a random element over $\mathbb{Z}_p^n$ has average Lee weight $\mu_{p^n}$, on average we need to compute $\mu_{p^n}^{-1}(t - 2v + 1)$ entries of the vector, each one costing $K(\lambda_{\text{sum}} + \lambda_{\text{mul}})$ binary operations. This implies a further cost term

\[
\chi_{ST} = \frac{F(m_1, v, q)F(m_2, v, q)}{(p^n)^{t+K-k_1}} \cdot \mu_{p^n}^{-1}(t - 2v + 1)K(\lambda_{\text{sum}} + \lambda_{\text{mul}}).
\]

So, the number of binary operations that, on average, are performed by an iteration of Algorithm 2 is estimated as

\[
C_{\text{iter}} = \chi U + \chi S + \chi T + \chi_{ST}.
\]
The success probability of one iteration corresponds to the probability of correctly guessing the weight distribution in the unknown $e$, which in this case is given by

$$P_{\text{guess}} = \frac{F(m_1, v, q)F(m_2, v, q)F(n - K - \ell, t - 2v, q)}{F(n, t, q)}.$$ 

The estimate of the overall complexity is given by

$$(4.8) \quad OC_{\text{Stern}} = C_{\text{iter}} P_{\text{guess}}^{-1}. $$

5. Numerical Results

In this section we assess the complexity of ISD algorithms over a finite ring $\mathbb{Z}_q$ endowed with the Lee metric, by using (4.3) and (4.8), and we compare it with that of ISD algorithms over a finite field $\mathbb{F}_q$ endowed with the Hamming metric. Notice that we need to define $0 \leq k_1 < K$: the cost of the ISD algorithms decreases with increasing $k_1$, thus the lowest cost in the Lee metric is given for $k_1 = K - 1$. Some numerical examples are reported in Table 1, where many different values of the code block length and dimension are considered and the cost is expressed in bits, i.e., as the exponent of 2 which provides the work factor of the attack. Notice that, for space reasons, Hamming, Lee, Prange and Stern were denoted as H., L., P. and S., respectively.

**Table 1.** Cost of Stern’s and Prange’s ISD algorithms in the Hamming and Lee metric, for different parameter sets.

| $q$ | $n$ | $K$ | $t$ | H.-P. | L.-P. | H.-S. | L.-S. |
|-----|-----|-----|-----|-------|-------|-------|-------|
| 256 | 1000| 500 | 40 | 75.08 | 73.88 | 64.13 | 59.83 |
| 256 | 1000| 600 | 40 | 87.91 | 86.10 | 75.76 | 70.68 |
| 1024| 1000| 600 | 40 | 88.55 | 86.74 | 78.78 | 70.80 |
| 243 | 200 | 100 | 50 | 88.9  | 75.94 | 78.30 | 60.01 |
| 256 | 200 | 100 | 50 | 88.93 | 75.97 | 78.45 | 59.93 |
| 256 | 1000| 700 | 40 | 104.70| 101.84| 91.30 | 85.14 |
| 343 | 300 | 150 | 75 | 121.95| 102.33| 109.34| 82.86 |
| 256 | 1000| 500 | 100| 141.85| 133.86| 125.69| 113.53|
| 2401| 2000| 1600| 60 | 179.99| 174.41| 166.23| 151.80|
| 512 | 500 | 250 | 125| 186.60| 153.67| 171.44| 128.44|
| 2401| 2000| 1600| 100| 283.35| 266.74| 267.69| 233.79|

We observe from Table 1 that, when (3.6) is satisfied (entry marked with *), according with the prediction in Section 4.2, the complexity of Prange’s ISD in the Lee metric is smaller than that of Prange’s ISD in the Hamming metric by a factor not larger than $t$, for the considered parameters, which span finite fields and finite rings with size $q = p^n$, where $p \in \{2, 3, 7\}$. The difference is more significant when error vectors with relatively large weights are considered. Notice that many of the parameters have been chosen in such a way to reach, or even exceed, the security levels recommended in [37].
6. Conclusion

We generalized Prange’s and Stern’s ISD algorithms considering codes over finite rings endowed with the Lee metric. We analyzed the complexity of these ISD algorithms. The assessed values of the complexity make ring linear codes endowed with the Lee metric a promising choice in cryptographic settings.

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