FINITE TIME BLOWUP FOR THE FOURTH-ORDER NLS

YONGGEUN CHO, TOHRU OZAWA, AND CHENGBO WANG

Abstract. We consider the fourth-order Schrödinger equation with focusing inhomogeneous nonlinearity (−|x|−2|u|^3 u) in high space dimensions. Extending Glassey’s virial argument, we show the finite time blowup of solutions when the energy is negative.

1. Introduction

In this paper we consider the fourth-order Schrödinger equations:

\[ i\partial_t u = (-\Delta + \mu \Delta^2)u + F(x, u)u, \]

which was introduced by Karpman, Karpman and Shagalov (see [15, 16] and references therein) to take account of small fourth-order dispersion in the propagation of intense laser beams in a Kerr medium. The function \( F(x, u) \) can be understood as a nonlinear potential affected by electron density [4]. Recently, Carles and Moulay also considered the high-order Schrödinger type equations to describe particles whose velocity is below \( c/\sqrt{2} \) and interpreted it as an interpolation equation between Schrödinger and relativistic equations. They considered a Hartree or Hartree-Fock type \( F \). For details see [6, 7].

In cases involving pure power type (\( F = |u|^{p-1} \)) or inhomogeneous power type (\( F = a(x)|u|^{p-1} \)), there have been various results on the local and global well-posedness, stability and blowup phenomena (see [1, 3, 9, 12, 17, 19, 20, 21, 22, 23, 24]). Contrary to these results, blowup theories of fourth-order nonlinear Schrödinger equations (NLS) have not been elucidated, except for the exact blowup solutions, numerical results [2, 3, 12], or dispersive blowup [5] of fourth-order NLS. This is because it is hard to demonstrate a virial argument for the blowup of fourth-order equations. In this paper we show that a virial argument works and thus the blowup can occur.

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To implement a virial argument we introduce the following Cauchy problem of the focusing fourth-order NLS:

\[
\begin{aligned}
&i\partial_t u = (-\Delta + \mu \Delta^2) u - |x|^{-2}|u|^4 u, \quad \text{in} \quad \mathbb{R}^{1+n}, \\
&u(x,0) = \varphi(x), \quad \text{in} \quad \mathbb{R}^n.
\end{aligned}
\]

Here \( \mu > 0 \) and \( n \geq 3 \). By \( (\mathcal{S}_0) \) we denote the equation \( (\mathcal{S}) \) without \( -\Delta u \) term temporarily. Then \( (\mathcal{S}_0) \) is of \( L^2 \)-scaling invariance (so-called mass-critical).

That is, if \( u \) is a solution of \( (\mathcal{S}_0) \), then for any \( \lambda > 0 \) the scaled function \( u_\lambda \), given by

\[
u_\lambda(t, x) = \lambda^\frac{n}{2} u(\lambda^4 t, \lambda x),
\]

is also a solution to \( (\mathcal{S}_0) \). We note that all results in this paper hold true for \( (\mathcal{S}_0) \).

In Sections 2 and 5, we will show the local well-posedness of \( (\mathcal{S}) \) in 

\[
C([0, T_\ast); H^2) \cap L^2_{loc}([0, T_\ast); H^3) \cap L^2([0, T_\ast); H^n)
\]

by using regularizing argument and Strichartz estimate. Here \( T_\ast \) is the maximal existence time. The solution \( u \) satisfies the mass and energy conservation laws:

\[
m(u) = \|u(t)\|^2_{L^2} = m(\varphi),
\]

\[
E(u) = K(u) + V(u) = E(\varphi),
\]

where

\[
K(u) = \frac{1}{2} \int (|\nabla u|^2 + \mu |\Delta u|^2) \, dx, \quad V(u) = -\frac{n}{2n+4} \int |x|^{-2}|u|^{2+n} \, dx.
\]

Glassey [14] introduced first a virial argument to show the finite time blow-up of solutions to NLS with pure power type nonlinearity. In [10, 13] the authors extended the finite time blowup to the fractional or high-order NLS with Hartree type nonlinearity. In this paper we establish a virial argument and the finite time blowup theory of the fourth-order NLS with inhomogeneous nonlinearity, as in \( (\mathcal{S}) \).

Let \( A \) be the dilation operator \( x \cdot \nabla \), \( M \) be the virial operator \( (1 - 2\mu \Delta)^{-\frac{1}{2}} x \) and \( M(u) = \langle Mu; Mu \rangle \) (see notations below). Then we have the following.

**Theorem 1.1.** Suppose that \( \varphi \in H^2 \), \( x_\varphi \in L^2 \) and \( E(\varphi) < 0 \). Then \( T_\ast < \infty \) if \( n \geq 7 \).

For the proof we present a virial inequality which guarantees the finite time blowup. Indeed, if \( \varphi \) is sufficiently smooth and has fast decay, then \( M \) satisfies the following inequality at least local in time:

\[
M(u(t)) \leq 16t^2 E(\varphi) + 4t \left( \text{Im} \langle \varphi, A\varphi \rangle + Cm(\varphi)^{1+\frac{2}{n}} \right) + M(\varphi).
\]

As we will demonstrate in Section 4, this implies that if \( E(\varphi) < 0 \), then \( T_\ast < \infty \). In order to show (1.2) directly, one may require some regularity of solutions (e.g. \( L^3([0, T]; H^3) \)). See Lemma 3.1 below). However due to the singularity and low power of nonlinearity it is hard to expect such regularity to
the solutions\(^1\). Instead, we implement approximation with regularized solutions to be used for the proof of existence of weak solutions. Since it is important to maintain \(H^2\)-regularity when approximating, the regularized solutions \(u_\varepsilon\) will be shown to be bounded in \(H^2\) uniformly in \(\varepsilon\) on a compact time interval, that is one of the main ingredients. Our method can also be applied to NLS with generalized dispersion \((\mu - \Delta)^{\alpha}(\alpha > 2, \mu \geq 0)\). As for pure power type problem our argument can be applied only up to the approximation of dilation operator (actually an approximation scheme may not be necessary for the pure power type). For the access to pure type one may implement a localization of dilation operator to get a localized virial inequality as in [18]. This issue will be treated somewhere.

The focusing nonlinearity serves as an attracting potential. If the energy is negative (i.e., the magnitude of the potential energy \(V(u)\) is larger than that of kinetic part \(K(u)\)), then self-attracting power overwhelms the dynamics and so it results in a collapse in a finite time.

The term \(m(\varphi)^{1 + \mu}\) in (1.2) appears due to the estimate of \(\text{Im} \langle u, M' \cdot MVu \rangle\), where \(M' = x(1 - 2\mu \Delta)^{-\frac{\mu}{2}}\) and \(V = |x|^{-2} |u|^\mu\). In [10], when \(V = |x|^{-\gamma} |u|^2\), the estimate was controlled by the Stein-Weiss inequality and weighted convolution inequality for which the radial assumption is crucial. In the present case we get the finite time blowup by using only the Stein-Weiss inequality without radial assumption, which comes from the locality of nonlinearity. However, if we assume the radial symmetry, we can implement more subtle estimate to get the following.

**Theorem 1.2.** Suppose that \(\varphi\) is radial and satisfies the same condition as in Theorem 1.1. Let \(u\) be the radial \(H^2\)-solution of (\(\mathcal{G}\)). Then \(T_\ast < \infty\) if \(n \geq 5\).

Theorem 1.1 and Theorem 1.2 can be readily extended to more general power type nonlinearity, such that \(F(x, u) = -\rho(|x|) |x|^{-2} |u|^\mu\) with non-increasing \(\rho\) or Hartree type \(F(x, u) = -\rho(|x|) (|x|^{-2} * |u|^2 + \frac{4}{\mu})\), which are essentially mass critical nonlinearities. We will not pursue these topics in this paper.

The rest of paper is organized as follows: In Section 2 we show the existence of weak solutions and local well-posedness. In Section 3 a virial inequality (1.2) will be considered for regularized solutions. To this end, we derive the estimates of the first moment and dilation operator. The last section is devoted to proving the main theorem.

**Notations**

We will use the notations:

- \(|\nabla| = \sqrt{-\Delta}\), \(\mathcal{D} = \sqrt{1 - \Delta}\), \(\hat{H}_s^r = |\nabla|^{-s} L^r\), \(\hat{H}^s = \hat{H}_2^s\), \(H^s = \mathcal{D}^{-s} L^r\), \(H^s = H_2^s\), \(L^r = L^r_{\infty}(\mathbb{R}^n)\) for some \(s \in \mathbb{R}\) and \(1 \leq r \leq \infty\).

- \((1 - \mu \Delta)^s = \mathcal{F}^{-1}((1 + \mu |\xi|^2)^s \mathcal{F})\) for \(s \in \mathbb{R}\). \(\mathcal{F} f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) \, dx\).

\(^1\)Using Strichartz estimates (5.1) with \(q_1 = q_2 = 2\), one can get the regularity of \(L^2([0, T]; H^3_{\text{rad}})\). See Section 5.3.
We denote the inner product 
\[ \langle f; g \rangle := \sum_{j=1}^{m} \langle f_j, g_j \rangle \] 
and \[ \langle f_j, g_j \rangle = \int f_j(x)g_j(x) \, dx \]
for some \( C^m \)-valued functions \( f = (f_1, \ldots, f_m) \) and \( g = (g_1, \ldots, g_m) \).

- Let \( H^2 = \{ u \in L^2 : \|u\|_{H^2}^2 := \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \mu \|\Delta u\|_{L^2}^2 < \infty \} \). Then \( \|u\|_{H^2} \sim \|u\|_{H^2} \).
- As usual different positive constants are denoted by the same letter \( C \).
- \( Q = O_X(\varepsilon) \) means that \( |Q| \leq CX \) and \( Q \to 0 \) as \( \varepsilon \to 0 \) and \( Q = O_X(\varepsilon) \) that \( |Q| \leq C\varepsilon X \).
- \( \partial^i = \prod_{k=1}^{n} \frac{\partial}{\partial x_k}^{i_k} \) and \( |i| = \sum_{k=1}^{n} j_k \) for multi-index \( i = (j_1, \ldots, j_n) \).
- \([A, B] \) denotes the commutator \( AB - BA \) for any operators \( A \) and \( B \) defined on suitable Banach spaces.

2. Local well-posedness

In this section we show the existence and uniqueness of weak solutions to \((\mathcal{G})\).
Due to the singularity of nonlinear term it is not easy to show the contraction on a Strichartz space. Instead, we use regularizing method to find a weak solution and Strichartz estimate to show the uniqueness. In order to show the existence of weak solutions in \( H^2 \) we follow the same line of arguments as in \[8\] and give only an outline of proof.

We first need the Lipschitz continuity of nonlinearity \( g(u) = V(u)u \) with \( V(u) = |u|^{-2}u \) on a neighborhood of Lebesgue spaces, which guarantees the existence of weak solutions in \( H^2 \). We append its proof in the last section.

**Lemma 2.1.** Let \( g_1(u) = \chi_{|u|<1}g(u) \) and \( g_2(u) = \chi_{|u|\geq1}g(u) \). Then there exist exponents \( \rho_i, r_i \) \( (i = 1, 2) \) \((2, 4)^* \) \((4^* = \frac{2n}{n-4} \) if \( n \geq 5 \) and \( \infty \) if \( n = 3, 4) \) such that for any \( u, v \) with \( \|u\|_{H^2} + \|v\|_{H^2} \leq L \)
\[ \|g(u) - g(v)\|_{L^{r_i}} \leq CL^{\frac{\rho_i}{r_i}} \|u - v\|_{L^{r_i}}. \]

Now let \( J_\varepsilon = (1 - \varepsilon\Delta)^{-\frac{n-1}{2}} \) for \( \varepsilon > 0 \) and \( g_\varepsilon(u) = J_\varepsilon (V(J_\varepsilon u)J_\varepsilon u) \). Then consider regularized equation:

\[ \begin{cases} 
\partial_t u^\varepsilon = (-\Delta + \mu\Delta^2)u^\varepsilon - g_\varepsilon(u^\varepsilon), & \text{in } \mathbb{R}^{1+n}, \quad n \geq 3, \\
u^\varepsilon(0) = \varphi^\varepsilon & \text{in } \mathbb{R}^n, 
\end{cases} \]

where \( \varphi^\varepsilon = \eta\varepsilon \ast \varphi \) and \( \eta\varepsilon(x) = \varepsilon^{-n}\eta(x/\varepsilon) \) for a fixed non-negative \( \eta \in C_0^\infty(\mathbb{R}^n) \) with \( \int \eta \, dx = 1 \). Since from Lemma 2.1 we have

\[ \|g(u)\|_{H^{-2}} \leq C\|u\|_{H^2}^{\frac{4}{n+4}} \quad \text{for } n \geq 3, \]

one can readily show that there exists a unique solution \( u^\varepsilon \in C(\mathbb{R}; H^{2n}) \) to (2.1) with \( \partial_t u \in C(\mathbb{R}; H^{2n-4}) \), provided \( \varphi \in H^2 \). Also \( u^\varepsilon \) satisfies the mass and energy conservations:

\[ \begin{align*}
\text{(2.3) } \quad m(u^\varepsilon(t)) &= m(\varphi^\varepsilon) \quad \text{and} \quad E_\varepsilon(u^\varepsilon(t)) := K(\varphi^\varepsilon) + V(J_\varepsilon u^\varepsilon(t)) = E_\varepsilon(\varphi^\varepsilon).
\end{align*} \]
For details see 3.3 of [8]. Suppose that \( \| \varphi \|_{H^2} \leq L \) for some \( L > 0 \) and thus \( \| \varphi^\varepsilon \|_{H^2} \leq L \) for all \( \varepsilon > 0 \). Let

\[
\mathfrak{T}_\varepsilon = \sup\{ \tau > 0 : \| u^\varepsilon(\tau) \|_{H^2} \leq 2L \text{ on } [0, \tau) \}.
\]

Then by (2.2) and (2.1) we deduce that

\[
(4.4) \quad \sup_{\tau > 0} \| \partial_t u^\varepsilon \|_{L^\infty([0, \mathfrak{T}_\varepsilon]; H^{n-2})} \leq C(L)
\]

for some generic constant \( C(L) \) varying from lines to lines but depending only on \( L \). From (4.4) and embedding it follows that

\[
(4.5) \quad \| u^\varepsilon(t) - u^\varepsilon(t') \|_{L^2} \leq \| u^\varepsilon(t) - u^\varepsilon(t') \|_{H^{n-2}} \| u^\varepsilon(t) - u^\varepsilon(t') \|_{H^2}^\frac{1}{2}
\]

\[
\leq C(L)|t - t'|^{\frac{1}{4}}
\]

for all \( t, t' \in [0, \mathfrak{T}_\varepsilon) \) and thus from Lemma 2.1 that

\[
(4.6) \quad \| \partial_t u^\varepsilon(t) - \partial_t u^\varepsilon(t') \|_{L^2} \leq C(L)|t - t'|^{a_i},
\]

\[
|\nabla(J_\varepsilon u^\varepsilon(t)) - \nabla(J_\varepsilon u^\varepsilon(t'))| \leq C(L)(|t - t'|^{b_1} + |t - t'|^{b_2})
\]

for some \( 0 < a_i, b_i < 1 (i = 1, 2) \) and for all \( t \in [0, \mathfrak{T}_\varepsilon) \). By energy conservation (2.3) we have

\[
\| u^\varepsilon \|_{H^2}^2 \leq \| \varphi^\varepsilon \|_{L^2}^2 + 2K(u^\varepsilon) + 2\nabla(J_\varepsilon u^\varepsilon) - 2\nabla(J_\varepsilon u^\varepsilon)
\]

\[
\leq \| \varphi^\varepsilon \|_{L^2}^2 + 2K(\varphi^\varepsilon) + 2\nabla(J_\varepsilon \varphi^\varepsilon) - \nabla(J_\varepsilon u^\varepsilon)
\]

\[
\leq \| \varphi^\varepsilon \|_{L^2}^2 + C(L)(|t|^{b_1} + |t|^{b_2})
\]

for all \( t \in [0, \mathfrak{T}_\varepsilon) \). Let \( T(L) \) be such that \( C(L)(T(L))^{b_1} + T(L)^{b_2} = 2L^2 \). Then \( T(L) \leq \mathfrak{T}_\varepsilon \). This implies that \( \| u^\varepsilon \|_{L^\infty([0, T(L)]; H^2)} \leq 2L \) and also

\[
\| \partial_t u^\varepsilon \|_{L^\infty([0, T(L)]; H^{n-2})} \leq C(L).
\]

By the weak solution \( u \) in \( H^2 \) on time interval \( [0, T(L)] \) we mean that \( u \in L^\infty([0, T(L)]; H^2), \partial_t u \in L^\infty([0, T; H^{-2}]) \) and \( \langle i\partial_t u - (\Delta + \mu \Delta^2)u + g(u), \zeta \rangle = 0 \)

for all \( \zeta \in H^2 \) for a.e. \( t \in [0, T(L)] \). Now it is an easy matter to show the regularized solutions \( u^\varepsilon(t) \) of (2.1) converges weakly up to a subsequence as \( \varepsilon \to 0 \) to a weak solution \( u(t) \) in \( H^2 \), \( u^\varepsilon \to u \) in \( C([0, T(L)]; L^2) \) and thus \( \nabla(x; u) \to \nabla(u(t)) \) for each \( t \in [0, T(L)] \). Since \( \varphi \in H^2, g \in L^\infty([0, T(L)]; H^{-2}) \) and \( u \in L^\infty([0, T(L)]) \cap W^{1, \infty}([0, T(L)]; H^{-2}) \), by Remark 1.6.1(iii) of [8] we deduce that

\[
(4.7) \quad u(t) = U(t)\varphi + i \int_0^t U(t - t')g(u) \, dt'
\]

for all \( t \in [0, T(L)] \),

where \( U(t) = e^{-it(-\Delta + \mu \Delta^2)} \).

Summarizing this we have the following:
Proposition 2.2. For any $L > 0$, there exists $T(L) > 0$ with the following property: For every $\varphi \in H^2$ with $\|\varphi\|_{H^2} \leq L$, there exist a sequence $\{u^\varepsilon\}$ to (2.1) and a weak solution $u$ of (S) on $[0, T(M))$ satisfying (2.7) and
\[
\sup_{\varepsilon > 0} \|u^\varepsilon\|_{L^\infty([0, T(L)); H^2_\mu)} \leq 2L, \quad \sup_{\varepsilon > 0} \|\partial_t u^\varepsilon\|_{L^\infty([0, T(L)); H^1)} \leq C(L),
\]
\[
\|u\|_{L^\infty([0, T(L)); H^2_\mu)} \leq 2L, \quad \|\partial_t u\|_{L^\infty([0, T(L)); H^1)} \leq C(L),
\]
\[
u^\varepsilon(t) \to u(t) \text{ in } H^2, v^\varepsilon(u_{\varepsilon}) \to V(u(t)) \text{ for all } t \in [0, T(L)).
\]
Moreover, $m(u(t)) = m(\varphi)$ and $E(u(t)) \leq E(\varphi)$ for each $t \in [0, T(L))$.

The uniqueness of the weak solutions will be shown in Section 5.2. By assuming the uniqueness we will show the well-posedness. Let $I = [0, T(L))$ and $M = \sup\{\|u(t)\|_{H^2} : t \in I\}$. For a fixed $t' \in I$ let $\varphi = u(t')$ and $v$ be the weak solution to (S) given by Proposition 2.2. Then $v(t - t')$ is defined on $I$ and by the uniqueness $v(t - t') = u(t)$ for all $t \in I$. Hence we deduce from Proposition 2.2 that
\[
\|u(t)\|_{L^2} = \|u(t')\|_{L^2}, \quad E(u(t)) \leq E(u(t')).
\]
By the same way we can show that the reverse energy inequality so that
\[
(2.8) \quad m(u(t)) = m(u(t')), \quad E(u(t)) = E(u(t')) \text{ for all } t \in I.
\]
By (2.5) $u \in C^{0,1/2}(\mathcal{T}, L^2)$ and thus the function $t \mapsto V(u(t))$ is continuous on $\mathcal{T}$ by (2.6). This together with (2.8) implies $\|u(t)\|_{H^2}$ is continuous from $\mathcal{T}$ to $\mathcal{R}$. Therefore from equation (S) $u \in C([0, T(L)]; H^2) \cap C^1([0, T(L)]; H^{-2})$. Let $T_*$ be the maximal existence time of $H^2$-solution. Then from the local theory as above one can easily get that
\[
(2.9) \quad \text{if } T_* < \infty, \text{ then } \lim_{t \to T_*} \|u(t)\|_{H^2} = \infty.
\]
Now we show the continuous dependence on $[0, T]$ for any $T < T_*$. Suppose $\varphi_k \to \varphi$ in $H^2$ and let $L = 2\sup\{\|u(t)\|_{H^2} : t \in [0, T]\}$. Since $\|\varphi_k\|_{H^2} \leq L$ for sufficiently large $k$, $[0, T(L)] \subset [0, T_*)$. Thus $u_k$ is bounded in
\[
L^\infty([0, T(L)]; H^2) \cap W^{1,\infty}([0, T(L)]; H^{-2}).
\]
Since $\|u_k(t)\|_{L^2} = \|\varphi_k\|_{L^2} \to \|\varphi\|_{L^2} = \|u(t)\|_{L^2}$ uniformly on $[0, T(L)]$, we have $u_k(t) \to u(t)$ uniformly in $C([0, T(L)]; L^2)$. From (2.6) with $\varepsilon = 0$ it follows that $V(u_k) \to V(u(t))$ and thus $\|u_k\|_{H^2} \to \|u\|_{H^2}$ uniformly on $[0, T(L))$ by energy conservation. Therefore we deduce that $u_k \to u$ in $C([0, T(L)]; H^2)$. Since $T(L)$ depends only on $L$, we can repeat this argument to cover the interval $[0, T]$ within a finite number of steps.

3. Virial inequality

In this section we show the virial inequality (1.2). We show the inequality by approximate approach with solutions regularized equation (2.1).
3.1. Approximate for the first moment

**Lemma 3.1.** Let \( u^\varepsilon \) be the solution to (2.1) for \( n \geq 3 \) with \( \varphi \in H^2 \) and \( x\varphi \in L^2 \). Then we have \( \|xu^\varepsilon(t)\|_{L^2} < \infty \) for all \( 0 < \varepsilon < 1 \) and \( t \in [0, \infty) \).

**Proof.** Let \( \psi_\lambda(x) = e^{-\lambda|x|^2} \) and \( m_\lambda = \langle x\psi_\lambda u^\varepsilon; x\psi_\lambda u^\varepsilon \rangle \) for \( \lambda > 0 \). Then

\[
\frac{d}{dt} m_\lambda = 2\mathrm{Re}\langle x\psi_\lambda u^\varepsilon; x\psi_\lambda u^\varepsilon \rangle
\]

\[
= 2\mathrm{Re}\langle x\psi_\lambda u^\varepsilon; x\psi_\lambda(-i(-\Delta + \mu\Delta^2)u^\varepsilon + ig_\varepsilon(u^\varepsilon)) \rangle
\]

\[
= 2\mathrm{Im}\langle x\psi_\lambda u^\varepsilon; x\psi_\lambda(-\Delta + \mu\Delta^2)u^\varepsilon \rangle - 2\mathrm{Im}\langle x\psi_\lambda u^\varepsilon; x\psi_\lambda g_\varepsilon(u^\varepsilon) \rangle
\]

\[
= 2\mathrm{Im}\langle x\psi_\lambda u^\varepsilon; [x\psi_\lambda, -\Delta + \mu\Delta^2]u^\varepsilon \rangle - 2\mathrm{Im}\langle x\psi_\lambda u^\varepsilon; x\psi_\lambda[x, J_\varepsilon]V(u_\varepsilon)u_\varepsilon \rangle
\]

\[
- 2\mathrm{Im}\langle x\psi_\lambda u^\varepsilon; x\psi_\lambda J_\varepsilon V(u_\varepsilon)u_\varepsilon \rangle
\]

\[
= 2\mathrm{Im}\langle x\psi_\lambda u^\varepsilon; \nabla(x\psi_\lambda) \cdot \nabla u^\varepsilon \rangle
\]

\[
+ 2\mathrm{Im} \sum_{|i|_1 + |i|_2 = 4 \atop 1 \leq |i|_1 \leq 3} c_{i_1, i_2} \langle x\psi_\lambda u^\varepsilon; \nabla^{i_1}(x\psi_\lambda) \nabla^{i_2} u^\varepsilon \rangle
\]

\[
- 2\mathrm{Im}\langle x\psi_\lambda u^\varepsilon; \psi_\lambda[x, J_\varepsilon]V(u_\varepsilon)u_\varepsilon \rangle - 2\mathrm{Im}\langle x\psi_\lambda u^\varepsilon; \psi_\lambda J_\varepsilon V(u_\varepsilon)u_\varepsilon \rangle
\]

\[
\leq C\|u^\varepsilon\|_{H^2} \sqrt{m_\lambda} - 2\mathrm{Im}\langle x\psi_\lambda u^\varepsilon; \psi_\lambda[x, J_\varepsilon]V(u_\varepsilon)u_\varepsilon \rangle
\]

\[
- 2\mathrm{Im}\langle x\psi_\lambda u^\varepsilon; \psi_\lambda J_\varepsilon V(u_\varepsilon)u_\varepsilon \rangle.
\]

Here \( u_\varepsilon \) denoted \( J_\varepsilon u^\varepsilon \).

Since \( \|e^{t\frac{\varepsilon}{2} J_\varepsilon} \nabla f \|_{L^2} \leq \left\{ \begin{array}{rl}
\|e^{t\frac{\varepsilon}{2} \nabla} \nabla^{-\frac{1}{2}} f \|_{L^2} & \leq C \|\nabla^{-\frac{1}{2}} f \|_{L^2} \\
\|e^{\frac{\varepsilon}{2} \nabla} J_\varepsilon \nabla f \|_{L^2} & \leq C \|f \|_{L^2}
\end{array} \right. \) for all \( 0 < \varepsilon < 1 \) and \( \ell \geq 1 \) (\( C \) does not depend on \( \varepsilon \)), from (3.5) below it follows that for some \( r \) arbitrarily close to but bigger than \( \frac{2n+2}{n-2} \)

\[
\|e^{t\frac{\varepsilon}{2} J_\varepsilon} \nabla \|_{L^2} \leq \left\{ \begin{array}{rl}
\|\nabla f \|_{L^2} & \leq C \|\nabla f \|_{L^2} \\
\|\nabla J_\varepsilon \nabla f \|_{L^2} & \leq C \|f \|_{L^2}
\end{array} \right. \)

Also for such \( r \) we have

\[
\|xV(u_\varepsilon)u_\varepsilon \|_{L^2} \leq \|\chi_{\{|x| \leq 1\}} xV(u_\varepsilon)u_\varepsilon \|_{L^2} + \|u_\varepsilon \|_{\dot{H}^{1+\frac{4}{n-2}}(\mathbb{R}^n)}.
\]
Thus we have
\[ \frac{d}{dt} m_\lambda \leq C(\|u\|_{H^3} + \|u\|_{H^2_{\frac{n}{2}, \frac{n}{2}}}) \sqrt{m_\lambda}. \]

Integrating this over \([0, t]\) and letting \(\lambda \to 0\), by Fatou’s lemma we get\(^2\)
\[ \|xu^\varepsilon(t)\|_{L^2} \leq \|xu^\varepsilon\|_{L^2} + C \int_0^t (\|u^\varepsilon\|_{H^3} + \|u^\varepsilon\|_{H^2_{\frac{n}{2}, \frac{n}{2}}}) \, dt' \]
for all \(t \in [0, \infty)\).

### 3.2. Approximate for dilatation operator

For fixed \(t > 0\) let us set
\[ L_t := \sup_{0 \leq s \leq t} (1 + \|u^\varepsilon(s)\|_{H^2})^{2 + \frac{n}{4}}, \]
\[ K_t := \sup_{0 \leq s \leq t} (1 + \|u^\varepsilon(s)\|_{H^4} + \|xu^\varepsilon(s)\|_{L^2})^{2 + \frac{n}{4}} \]
for the solution \(u^\varepsilon\) of (2.1). Then we have the following.

**Proposition 3.2.** Let \(\varphi \in H^2\) and \(n \geq 3\). Then the solution \(u^\varepsilon\) of (2.1) satisfies that for all \(t \in [0, \infty)\)
\[ \Im \langle u^\varepsilon(t), Au^\varepsilon(t) \rangle = \Im \langle \varphi^\varepsilon, A\varphi^\varepsilon \rangle + 8tE_\varepsilon(\varphi^\varepsilon) + L_t \int_0^t \text{o}_{\|u^\varepsilon(s)\|_{H^2}}(\varepsilon) \, ds. \]

**Proof.** Let \(\tilde{A} = \nabla \cdot x, A' = \frac{1}{2}(A + \tilde{A})\) and \(A_\lambda = \langle \psi_\lambda u^\varepsilon, A' \psi_\lambda u^\varepsilon \rangle\). Then the adjoint \((A')^*\) is also \(A'\) and \(A_\lambda = \Im \langle \psi_\lambda u^\varepsilon, A' \psi_\lambda u^\varepsilon \rangle\). From the regularity of the solution \(u^\varepsilon\) it follows that
\[ (3.2) \]
\[ A'_\lambda(t) = 2\Re \langle \psi_\lambda u^\varepsilon, A' \psi_\lambda u^\varepsilon \rangle = 2\Re \langle \psi_\lambda(-i(-\Delta + \mu \Delta^2)u^\varepsilon + ig_\varepsilon(u^\varepsilon)), A' \psi_\lambda u^\varepsilon \rangle \]
\[ = -2\Im \langle \psi_\lambda(-\Delta + \mu \Delta^2)u^\varepsilon, A' \psi_\lambda u^\varepsilon \rangle + 2\Im \langle \psi_\lambda g_\varepsilon(u^\varepsilon), A' \psi_\lambda u^\varepsilon \rangle \]
\[ =: 2\Im \langle I_\lambda \rangle. \]

For \(I_\lambda\) we have
\[ I_\lambda = \langle [-\Delta + \mu \Delta^2, \psi_\lambda] u^\varepsilon, A' \psi_\lambda u^\varepsilon \rangle - \langle \psi_\lambda u^\varepsilon, [-\Delta + \mu \Delta^2, A'] \psi_\lambda u^\varepsilon \rangle \]
\[ - \langle \psi_\lambda u^\varepsilon, A' [-\Delta + \mu \Delta^2, \psi_\lambda] u^\varepsilon \rangle + \text{Im} \lambda. \]

\(^2\)In Section 5 it will be shown that \(\int_0^t \|u^\varepsilon\|_{H^3_{\frac{n}{2}, \frac{n}{2}}}^2 \, dt' \leq C(t, L)\), which implies
\[ \int_0^t \|u^\varepsilon\|_{H^2_{\frac{n}{2}, \frac{n}{2}}}^{1 + \frac{n}{4}} \, dt' \leq C(t, L) \] by Hölder’s inequality or interpolation. However we do not know how to control \(\int_0^t \|u^\varepsilon\|_{H^3} \, dt'\) in terms of \(\|u^\varepsilon\|_{H^2}\).
Thus
\[ 2\text{Im} I_\lambda = -i(\left( -\Delta + \mu \Delta^2, \psi_\lambda \right) u^\varepsilon, A' \psi_\lambda u^\varepsilon) - \left( \psi_\lambda u^\varepsilon, \left( -\Delta + \mu \Delta^2, A' \right) \psi_\lambda u^\varepsilon) \]
\[ - \left( \psi_\lambda u^\varepsilon, A' \left( -\Delta + \mu \Delta^2, \psi_\lambda \right) u^\varepsilon) \right) \]
\[ = 2\text{Im} \left( \left( -\Delta + \mu \Delta^2, \psi_\lambda \right) u^\varepsilon, A' \psi_\lambda u^\varepsilon \right) + i \left( \psi_\lambda u^\varepsilon, \left( -\Delta + \mu \Delta^2, A' \right) \psi_\lambda u^\varepsilon) \right). \]
Using the identity
\[ \left( -\Delta, A' \right) = 2i\Delta \text{ and } \left( \Delta^2, A' \right) = -4i\Delta^2, \]
we have
\[ 2\text{Im} I_\lambda \]
\[ = 2\text{Im} \left( \left( -\Delta + \mu \Delta^2, \psi_\lambda \right) u^\varepsilon, A' \psi_\lambda u^\varepsilon \right) + 4 \left( \psi_\lambda u^\varepsilon, \left( -\Delta + \mu \Delta^2, \psi_\lambda \right) u^\varepsilon \right) \]
\[ + 4 \left( \psi_\lambda u^\varepsilon, \psi_\lambda (-\Delta + \mu \Delta^2) u^\varepsilon \right) + 2 \left( \psi_\lambda u^\varepsilon, \left[ \Delta, \psi_\lambda \right] u^\varepsilon \right) + 2 \left( \psi_\lambda u^\varepsilon, \psi_\lambda \Delta u^\varepsilon \right) \]
\[ = 2\text{Im} \left( \left( -\Delta + \mu \Delta^2, \psi_\lambda \right) u^\varepsilon, \psi_\lambda u^\varepsilon \right) + 2 \left( \psi_\lambda u^\varepsilon, \left[ \Delta, \psi_\lambda \right] u^\varepsilon \right) + 2 \left( \psi_\lambda u^\varepsilon, \psi_\lambda \Delta u^\varepsilon \right) \]
\[ = \Re \left( \left( (n + 2x \cdot \nabla) \psi_\lambda + 2\nabla \psi_\lambda \cdot \nabla \right) u^\varepsilon, \psi_\lambda u^\varepsilon \right) \]
\[ + \mu \sum_{[l_1] + [l_2] = 1, 2} c_{l_1, l_2} \left( \Re \left( (2x \cdot \nabla + n) \partial^{l_1} \psi_\lambda \partial^{2l_2} u^\varepsilon, \psi_\lambda u^\varepsilon \right) + 4 \left( \psi_\lambda u^\varepsilon, \partial^{l_1} \psi_\lambda \partial^{2l_2} u^\varepsilon \right) \right) \]
\[ + 2 \left( \psi_\lambda u^\varepsilon, \left( \Delta \psi_\lambda + 2\nabla \psi_\lambda \cdot \nabla \right) u^\varepsilon \right) + 4 \left( \psi_\lambda u^\varepsilon, \psi_\lambda (-\Delta + \mu \Delta^2) u^\varepsilon \right) \]
\[ + 2 \left( \psi_\lambda u^\varepsilon, \psi_\lambda \Delta u^\varepsilon \right). \]
Since \[ |\partial^{l_1} \psi_\lambda(x)| \leq \lambda^{|l_1|/2} (1 + |x|^2)^{|l_1|/4} \psi_\lambda(x) \] by integration by parts we obtain
\[ 2\text{Im} I_\lambda \leq C\lambda^\frac{7}{2} \| u^\varepsilon \|_H^2 + \mu \sum_{[l_1] = 1, [l_2] = 3} c_{l_1, l_2} \left| \left( x \partial^{l_1} \psi_\lambda \partial^{2l_2} u^\varepsilon, \psi_\lambda \nabla u^\varepsilon \right) \right| \]
\[ + 4 \left| \left( \psi_\lambda u^\varepsilon, \psi_\lambda (-\Delta + \mu \Delta^2) u^\varepsilon \right) + 2 \left( \psi_\lambda u^\varepsilon, \psi_\lambda \Delta u^\varepsilon \right) \right|. \]
For the first inner product we have by integration by parts and Lemma 3.1 that
\[ \left| \left( x \partial^{l_1} \psi_\lambda \partial^{2l_2} u^\varepsilon, \psi_\lambda \nabla u^\varepsilon \right) \right|_{[l_1] = 1, [l_2] = 3} \leq C\lambda^\frac{7}{2} \| u^\varepsilon \|_H^2 + \left| \left( x \partial^{l_1} \psi_\lambda u^\varepsilon; \psi_\lambda \partial^{2l_2} \nabla u^\varepsilon \right) \right|_{[l_1] = 1, [l_2] = 3} \]
\[ = C\lambda^\frac{7}{2} \| u^\varepsilon \|_H^2 + C\lambda^\frac{7}{2} \| x u^\varepsilon \|_L^2 \| u^\varepsilon \|_H^4 \]
\[ = O_{K_1}(\lambda^\frac{7}{2}). \]
Thus we get
\[ 2\text{Im} I_\lambda = O_{K_1}(\lambda^\frac{7}{2}) + 4 \left( \psi_\lambda u^\varepsilon, \psi_\lambda (-\Delta + \mu \Delta^2) u^\varepsilon \right) + 2 \left( \psi_\lambda u^\varepsilon, \psi_\lambda \Delta u^\varepsilon \right). \]
Now for \[ I_\Delta \] we have
\[ I_\Delta = \left( A' \psi_\lambda g_\varepsilon(u^\varepsilon), \psi_\lambda u^\varepsilon \right) \]
\[ = \left( \left[ A', \psi_\lambda \right] g_\varepsilon(u^\varepsilon), \psi_\lambda u^\varepsilon \right) + \left( \psi_\lambda \left[ A', J_\varepsilon \right] (V(u_c)u_c), \psi_\lambda u^\varepsilon \right) \]
By direct calculation we have for the last inner product the following to be

\[ \langle \psi_\lambda, J_\varepsilon \rangle = -i(x \cdot \nabla f), \]

(3.5)

\[ [x, J_\varepsilon] = -2(n+1)\varepsilon J_\varepsilon^{\frac{n-2}{n+2}} \nabla, \]

\[ [A', J_\varepsilon] = -\frac{i}{2} \left( [x, J_\varepsilon] \cdot \nabla + \nabla \cdot [x, J_\varepsilon] \right). \]

Then it follows that

(3.6)

\[ II_\lambda = i \langle (x \cdot \nabla \psi_\lambda) g_\varepsilon (u^\varepsilon), \psi_\lambda u^\varepsilon \rangle - 2i(n+1) Re \langle \psi_\lambda (\varepsilon \Delta) J_\varepsilon^{\frac{n+2}{n+4}} (V(u_\varepsilon)u_\varepsilon), \psi_\lambda u^\varepsilon \rangle \]

\[ + \langle V(u_\varepsilon)u_\varepsilon, A'[J_\varepsilon, \psi_\lambda] \psi_\lambda u^\varepsilon \rangle - i\langle x \cdot \nabla \psi_\lambda (V(u_\varepsilon)u_\varepsilon), [J_\varepsilon, \psi_\lambda] u^\varepsilon \rangle \]

\[ + \langle \psi_\lambda V(u_\varepsilon)u_\varepsilon, A'[J_\varepsilon, \psi_\lambda] u^\varepsilon \rangle - \text{Im} \langle \psi_\lambda x \cdot \nabla u_\varepsilon, \psi_\lambda V(u_\varepsilon)u_\varepsilon \rangle \]

\[ - i\langle V(u_\varepsilon)u_\varepsilon, \psi_\lambda (x \cdot \nabla \psi_\lambda) u_\varepsilon \rangle + \frac{i}{2} \langle \psi_\lambda (x \cdot \nabla V(u_\varepsilon))u_\varepsilon, \psi_\lambda u_\varepsilon \rangle. \]

By direct calculation we have for the last inner product the following to be shown later:

(3.7)

\[ \langle \psi_\lambda (x \cdot \nabla V(u_\varepsilon))u_\varepsilon, \psi_\lambda u_\varepsilon \rangle = -\frac{4n}{n+2} \int \psi_\lambda^2 |x|^{-2}|u_\varepsilon|^{2+\frac{4}{n}} dx - \frac{4}{n+2} \int |x|^{-2}|u_\varepsilon|^{2+\frac{4}{n}} x \cdot \nabla \psi_\lambda \psi_\lambda dx. \]

Substituting this into (3.6) and taking imaginary part, we get

(3.8)

\[ 2 \text{Im} II_\lambda = O_K(\lambda) - 2(n+1) \text{Re} \langle \psi_\lambda (\varepsilon \Delta) J_\varepsilon^{\frac{n+2}{n+4}} (V(u_\varepsilon)u_\varepsilon), \psi_\lambda u^\varepsilon \rangle \]

\[ + 2 \text{Im} \langle V(u_\varepsilon)u_\varepsilon, A'[J_\varepsilon, \psi_\lambda] \psi_\lambda u^\varepsilon \rangle - 2 \text{Re} \langle x \cdot \nabla \psi_\lambda (V(u_\varepsilon)u_\varepsilon), [J_\varepsilon, \psi_\lambda] u^\varepsilon \rangle \]

\[ + 2 \text{Im} \langle \psi_\lambda V(u_\varepsilon)u_\varepsilon, A'[J_\varepsilon, \psi_\lambda] u^\varepsilon \rangle - \frac{2n+8}{n+2} \langle V(u_\varepsilon)u_\varepsilon, \psi_\lambda (x \cdot \nabla \psi_\lambda) u_\varepsilon \rangle \]

\[ - \frac{4n}{n+2} \int \psi_\lambda^2 |x|^{-2}|u_\varepsilon|^{2+\frac{4}{n}} dx. \]

Since \(|x \cdot \nabla \psi_\lambda g_\varepsilon (u^\varepsilon)| = 2\lambda|u_\varepsilon|^{1+\frac{4}{n}}\), we have

(3.8)

\[ 2 \text{Im} II_\lambda = O_K(\lambda) - 2(n+1) \text{Re} \langle \psi_\lambda (\varepsilon \Delta) J_\varepsilon^{\frac{n+2}{n+4}} (V(u_\varepsilon)u_\varepsilon), \psi_\lambda u^\varepsilon \rangle \]

\[ + 2 \text{Im} \langle V(u_\varepsilon)u_\varepsilon, A'[J_\varepsilon, \psi_\lambda] \psi_\lambda u^\varepsilon \rangle + \langle \psi_\lambda V(u_\varepsilon)u_\varepsilon, A'[J_\varepsilon, \psi_\lambda] u^\varepsilon \rangle \]

\[ - \frac{4n}{n+2} \int \psi_\lambda^2 |x|^{-2}|u_\varepsilon|^{2+\frac{4}{n}} dx. \]
For the 3rd and 4th terms we need the followings to be shown later:

\begin{align}
(3.9) & \langle V(u_\varepsilon)u_\varepsilon, A'[J_\varepsilon, \psi_\lambda] \psi_\lambda u^\varepsilon \rangle \\
& = O_{K_1}(\lambda^z) + i2(n + 1)\langle (1 - \Delta)^{-1}(V(u_\varepsilon)u_\varepsilon), \psi_\lambda J_\varepsilon^{\frac{n+2}{2}}(\varepsilon \Delta)\psi_\lambda (1 - \Delta)u^\varepsilon \rangle \\
& \quad - i\langle (1 - \Delta)^{-1}(V(u_\varepsilon)u_\varepsilon), [J_\varepsilon, x\psi_\lambda] \psi_\lambda \cdot \nabla(1 - \Delta)u^\varepsilon \rangle,
\end{align}

\begin{align}
(3.10) & \langle \psi_\lambda V(u_\varepsilon)u_\varepsilon, A'[J_\varepsilon, \psi_\lambda] u^\varepsilon \rangle \\
& = O_{K_1}(\lambda^z) + i2(n + 1)\langle (1 - \Delta)^{-1}(\psi_\lambda V(u_\varepsilon)u_\varepsilon), \psi_\lambda J_\varepsilon^{\frac{n+2}{2}}(\varepsilon \Delta)(1 - \Delta)u^\varepsilon \rangle \\
& \quad - i\langle (1 - \Delta)^{-1}(\psi_\lambda V(u_\varepsilon)u_\varepsilon), [J_\varepsilon, x\psi_\lambda] \cdot \nabla(1 - \Delta)u^\varepsilon \rangle.
\end{align}

Substituting (3.4) and (3.8) together with (3.9) and (3.10) into (3.2), and integrating it over \([0, t]\), we then get

\begin{align}
A_\lambda(t) = & \langle \psi_\lambda \varphi, A'[\psi_\lambda \varphi] \rangle + \int_0^t O_{K_1}(\lambda^z)\, ds + 2\langle \psi_\lambda u^\varepsilon, \psi_\lambda \Delta u^\varepsilon \rangle \\
& + 8\int_0^t \left( \frac{1}{2} \langle \psi_\lambda u^\varepsilon, \psi_\lambda (\mu \Delta^2 - \Delta)u^\varepsilon(s) \rangle - \frac{n}{2(n+4)} \int \psi_\lambda^2 |x|^{-2} |u_\varepsilon|^2 + \frac{4}{n+4} \, dx \right) \, ds \\
& - 2(n + 1)\text{Re} \int_0^t \langle \psi_\lambda(s^\varepsilon), J_\varepsilon^{\frac{n+2}{2}}(V(\varepsilon u_\varepsilon), \psi_\lambda u^\varepsilon) \rangle \, ds \\
+ 2(n + 1)\text{Re} \int_0^t \langle (1 - \Delta)^{-1}(\psi_\lambda V(\varepsilon u_\varepsilon), [J_\varepsilon, x\psi_\lambda] \psi_\lambda \cdot \nabla(1 - \Delta)u^\varepsilon) \rangle \, ds \\
- \text{Re} \int_0^t \langle (1 - \Delta)^{-1}(\psi_\lambda V(\varepsilon u_\varepsilon), [J_\varepsilon, x\psi_\lambda] \cdot \nabla(1 - \Delta)u^\varepsilon) \rangle \, ds.
\end{align}

Since \(\psi_\lambda(1 - \Delta)^{-1}(V(\varepsilon u_\varepsilon)) \to (1 - \Delta)^{-1}(V(\varepsilon u_\varepsilon))\) in \(L^2\) and \(J_\varepsilon^{\frac{n+2}{2}}(\varepsilon \Delta) : L^2 \to L^2\) uniformly on \(\varepsilon\), one can easily show that the inner product of (3.11) converges to \(\langle (1 - \Delta)^{-1}(V(\varepsilon u_\varepsilon), J_\varepsilon^{\frac{n+2}{2}}(\varepsilon \Delta)(1 - \Delta)u^\varepsilon) \rangle\) for each \(t\) as \(\lambda \to 0\). For (3.12) we use

\begin{align}
\langle (1 - \Delta)^{-1}(V(\varepsilon u_\varepsilon), [J_\varepsilon, x\psi_\lambda] \psi_\lambda \cdot \nabla(1 - \Delta)u^\varepsilon) \rangle \\
& = \langle (1 - \Delta)^{-1}(V(\varepsilon u_\varepsilon), ([J_\varepsilon, x] \psi_\lambda^2 + x [J_\varepsilon, \psi_\lambda] \psi_\lambda \cdot \nabla(1 - \Delta)u^\varepsilon) \\
& = \langle [x, J_\varepsilon](1 - \Delta)^{-1}(V(\varepsilon u_\varepsilon)u_\varepsilon); \psi_\lambda \psi_\lambda \cdot \nabla(1 - \Delta)u^\varepsilon \rangle \\
& \quad + \langle x(1 - \Delta)^{-1}(V(\varepsilon u_\varepsilon)u_\varepsilon); [J_\varepsilon, \psi_\lambda] \psi_\lambda \cdot \nabla(1 - \Delta)u^\varepsilon \rangle \\
& = \langle [x, J_\varepsilon](1 - \Delta)^{-1}(V(\varepsilon u_\varepsilon)u_\varepsilon); \psi_\lambda \psi_\lambda \cdot \nabla(1 - \Delta)u^\varepsilon \rangle.
\end{align}
Then we can handle it similarly to (3.12) and (3.13). We have

$$\| (1 - \Delta)^{-1}(x V(u_\varepsilon)u_\varepsilon) \|_{L^2} \leq C \| \nabla \psi \|_{L^2} \| u^\varepsilon \|_{H^1} = O_K(\lambda^{\frac{\nu}{2}}).$$
Since for some $\frac{2n+2}{n+2} < \gamma < \frac{2n}{n-2}$ we have

$$\| (1 - \Delta)^{-1}(x V(u_\varepsilon)u_\varepsilon) \|_{L^2} \leq C \| |x| \leq 1 |x|^{-1} |u_\varepsilon| \|_{L^\infty} + \| |u_\varepsilon| \|_{L^{n+2}}$$

$$\leq C \left( \| |u_\varepsilon| \|_{L^\infty}^{\frac{n+2}{n+2}} + \| |u_\varepsilon| \|_{L^{n+2}}^{\frac{n+2}{n+2}} \right)$$

$$\leq CL,$$

from Lemma 2.1 it follows that

$$x(1 - \Delta)^{-1}(V(u_\varepsilon)u_\varepsilon) = -2\nabla (1 - \Delta)^{-2}(V(u_\varepsilon)u_\varepsilon) + (1 - \Delta)^{-1}(x V(u_\varepsilon)u_\varepsilon) \in L^2$$
and $[x, J_\varepsilon](1 - \Delta)^{-1}(V(u_\varepsilon)u_\varepsilon) \in L^2$. These imply that as $\lambda \to 0$

$$\langle (1 - \Delta)^{-1}(x V(u_\varepsilon)u_\varepsilon), [J_\varepsilon, x \psi] \nabla (1 - \Delta)u^\varepsilon \rangle$$

$$\to \langle (1 - \Delta)^{-1}(V(u_\varepsilon)u_\varepsilon), [J_\varepsilon, x] \cdot \nabla (1 - \Delta)u^\varepsilon \rangle.$$
\[ - \text{Re} \int_0^t \langle (1 - \Delta)^{-1} (V(u_\varepsilon), [J_\varepsilon, x] \cdot \nabla (1 - \Delta) u_\varepsilon) \rangle \, ds \]
\[ + 2(n + 1) \text{Re} \int_0^t \langle (1 - \Delta)^{-1} (V(u_\varepsilon), J_\varepsilon^{n+2}(\varepsilon \Delta)(1 - \Delta) u_\varepsilon) \rangle \, ds \]
\[ - \text{Re} \int_0^t \langle (1 - \Delta)^{-1} (V(u_\varepsilon), [J_\varepsilon, x] \cdot \nabla (1 - \Delta) u_\varepsilon) \rangle \, ds. \]

Since
\[ |\langle g_\varepsilon(u_\varepsilon), (\varepsilon \Delta)(1 - \varepsilon \Delta)^{-1} u_\varepsilon \rangle| \leq \|g_\varepsilon(u_\varepsilon)\|_{H^{-1}} \| (\varepsilon \Delta)(1 - \varepsilon \Delta)^{-1} u_\varepsilon \|_{H^2}, \]
\[ \|g_\varepsilon(u_\varepsilon)\|_{H^{-2}} \leq C \|u_\varepsilon\|_{H^2}^{1 + \frac{\varepsilon}{2}} \]
uniformly on \( \varepsilon \), and \( \| (\varepsilon \Delta)(1 - \varepsilon \Delta)^{-1} u_\varepsilon \|_{H^2} \sim \| \varepsilon \| \xi^2 (1 + \| \xi \|^2) (1 + \| \xi \|^2) \| u_\varepsilon \|_{L^2} \)
\[ \to 0 \text{ as } \varepsilon \to 0 \text{ by LDCT, we get} \]
\[ \int_0^t \langle g_\varepsilon(u_\varepsilon), (\varepsilon \Delta)(1 - \varepsilon \Delta)^{-1} u_\varepsilon \rangle \, ds = L_t \int_0^t o_{\|w_0(x)\|_{H^2}}(\varepsilon) \, ds. \]

By utilizing (3.5) and the previous argument, we get the desired result. It remains to show (3.7), (3.9) and (3.10).

By the differentiation of \( x \cdot \nabla V(u_\varepsilon)u_\varepsilon = r \partial_r (r^{-2}|u_\varepsilon|^2 u_\varepsilon) = -2V(u_\varepsilon)u_\varepsilon + r^{-1}\partial_r (|u_\varepsilon|^2 u_\varepsilon) \) we have
\[ \langle \psi_\lambda(x \cdot \nabla V(u_\varepsilon))u_\varepsilon, \psi_\lambda u_\varepsilon \rangle \]
\[ = -2 \langle \psi_\lambda u_\varepsilon, \psi_\lambda V(u_\varepsilon)u_\varepsilon \rangle + \frac{4}{n} \text{Re} \langle \psi_\lambda V(u_\varepsilon)x \cdot \nabla u_\varepsilon, \psi_\lambda u_\varepsilon \rangle. \]

By the way we have
\[ \text{Re} \langle \psi_\lambda V(u_\varepsilon)x \cdot \nabla u_\varepsilon, \psi_\lambda u_\varepsilon \rangle \]
\[ = -\frac{n}{2} \langle \psi_\lambda V(u_\varepsilon)u_\varepsilon, \psi_\lambda u_\varepsilon \rangle - \frac{1}{2} \langle (x \cdot \nabla \psi_\lambda^2) V(u_\varepsilon)u_\varepsilon, u_\varepsilon \rangle \]
\[ - \frac{1}{2} \langle \psi_\lambda (x \cdot \nabla V(u_\varepsilon))u_\varepsilon, \psi_\lambda u_\varepsilon \rangle \]
and thus
\[ \langle \psi_\lambda (x \cdot \nabla V(u_\varepsilon))u_\varepsilon, \psi_\lambda u_\varepsilon \rangle \]
\[ = -2 \langle \psi_\lambda u_\varepsilon, \psi_\lambda V(u_\varepsilon)u_\varepsilon \rangle - 2 \langle \psi_\lambda V(u_\varepsilon)u_\varepsilon, \psi_\lambda u_\varepsilon \rangle \]
\[ - \frac{2}{n} \langle (x \cdot \nabla \psi_\lambda^2) V(u_\varepsilon)u_\varepsilon, u_\varepsilon \rangle - \frac{2}{n} \langle \psi_\lambda (x \cdot \nabla V(u_\varepsilon))u_\varepsilon, \psi_\lambda u_\varepsilon \rangle. \]

This implies (3.7).

To treat (3.9) and (3.10) we need to estimate \( (1 - \Delta) A' [J_\varepsilon, \phi] f \) for any \( f \in H^4 \) and Schwartz function \( \phi \). Actually, we see that
\[ A' = \frac{1}{2i} (x \cdot \nabla + \nabla \cdot x) = -ix \cdot \nabla - \frac{n}{2} i, \]
\[ [(1 - \Delta), A'] = [-\Delta, -ix \cdot \nabla] = i[\Delta, x \cdot \nabla] = i[\partial_j \partial_j, x^k \partial_k] = 2i \Delta. \]
Thus we have
\[
(1 - \Delta)A'[J_{\varepsilon}, \phi]f = [(1 - \Delta), A'] [J_{\varepsilon}, \phi] f + A'(1 - \Delta) [J_{\varepsilon}, \phi] f \\
= 2i\Delta [J_{\varepsilon}, \phi] f + A'[(1 - \Delta), [J_{\varepsilon}, \phi]] f + A'[J_{\varepsilon}, \phi](1 - \Delta)f \\
= 2i\Delta [J_{\varepsilon}, \phi] f + 2i [J_{\varepsilon}, \phi] \Delta f \\
- A'[\Delta, [J_{\varepsilon}, \phi]] f + A'[J_{\varepsilon}, \phi](1 - \Delta)f \\
= (2i - A')\Delta [J_{\varepsilon}, \phi] f + 2i [J_{\varepsilon}, \phi] \Delta f \\
+ A'[J_{\varepsilon}, \phi](1 - \Delta)f.
\]

For \([\Delta, [J_{\varepsilon}, \phi]]\), we know that
\[
[\Delta, [J_{\varepsilon}, \phi]] = -[\phi, [\Delta, J_{\varepsilon}]] - [J_{\varepsilon}, [\phi, \Delta]] \\
= [J_{\varepsilon}, [\Delta, \phi]] \\
= [J_{\varepsilon}, \Delta\phi] + 2[J_{\varepsilon}, \nabla\phi] \cdot \nabla.
\]

So we have
\[
(1 - \Delta)A'[J_{\varepsilon}, \phi] f \\
= (2i - A')\Delta [J_{\varepsilon}, \phi] f + 2i [J_{\varepsilon}, \phi] \Delta f + A'[J_{\varepsilon}, \phi](1 - \Delta)f \\
= (2i - A')\Delta [J_{\varepsilon}, \phi] f + 2i [J_{\varepsilon}, \phi] f - (2i - A') [J_{\varepsilon}, \phi](1 - \Delta)f \\
= 2i\left(\frac{n+4}{2} + x \cdot \nabla\right)([J_{\varepsilon}, \Delta\phi] f + 2[J_{\varepsilon}, \nabla\phi] \cdot \nabla f - [J_{\varepsilon}, \phi](1 - \Delta)f) \\
+ 2i [J_{\varepsilon}, \phi] f.
\]

Let \(\tilde{\phi}\) be one of \(\Delta\phi, \nabla\phi, \phi\). Then
\[
x \cdot \nabla [J_{\varepsilon}, \tilde{\phi}] = [x, J_{\varepsilon}] \cdot (\nabla \tilde{\phi}) + [J_{\varepsilon}, (x \cdot \nabla \tilde{\phi})] + [x, J_{\varepsilon}] \tilde{\phi} \cdot \nabla + [J_{\varepsilon}, x \tilde{\phi}] \cdot \nabla.
\]

From (3.5) it follows that
\[
x \cdot \nabla [J_{\varepsilon}, \tilde{\phi}] \\
= -2(n + 1)\varepsilon \xi J_{\varepsilon}^{\frac{n+4}{2}} (\Delta \tilde{\phi} + 2\nabla \tilde{\phi} \cdot \nabla + \tilde{\phi} \Delta) + [J_{\varepsilon}, (x \cdot \nabla \tilde{\phi})] + [J_{\varepsilon}, x \tilde{\phi}] \cdot \nabla \\
= -2(n + 1)\varepsilon ([J_{\varepsilon}^{\frac{n+4}{2}}, \Delta \tilde{\phi}] + 2[J_{\varepsilon}^{\frac{n+4}{2}}, \nabla \tilde{\phi}] \cdot \nabla + [J_{\varepsilon}^{\frac{n+4}{2}}, \tilde{\phi} \Delta]) \\
- 2(n + 1)\varepsilon (\Delta \tilde{\phi} J_{\varepsilon}^{\frac{n+4}{2}} + 2\nabla \tilde{\phi} \cdot \nabla J_{\varepsilon}^{\frac{n+4}{2}} + \tilde{\phi} J_{\varepsilon}^{\frac{n+4}{2}} \Delta) \\
+ [J_{\varepsilon}, (x \cdot \nabla \tilde{\phi})] + [J_{\varepsilon}, x \tilde{\phi}] \cdot \nabla.
\]

If \(\tilde{\phi} = \partial^j \psi_\lambda\) with \(|j| \geq 0\) or \(x \cdot \nabla \psi_\lambda\), then
\[
\|\nabla \partial^j \psi_\lambda\|_{L^1} \leq C\lambda^{\frac{|j|}{2}}, \|\nabla x \nabla \partial^j \psi_\lambda\|_{L^1} \\
\leq C\lambda^{\frac{|j|}{2}}.
\]

Let us now observe that
\[
\|[J_{\varepsilon}, \psi]h\|_{L^2} \leq C\|\nabla \psi\|_{L^1}\|h\|_{L^2}.
\]
By Plancherel’s theorem it suffices to show that
\[(3.18) \quad \|T_{\varepsilon, \psi}(\hat{h})\|_{L^2} \leq C\|\nabla \psi\|_{L^2}\|\hat{h}\|_{L^2},\]
where
\[T_{\varepsilon, \psi}(\hat{h}) = \int \hat{\psi}(\xi - \eta) \left( (1 + \varepsilon|\xi|^2)^{-\left(\frac{n+1}{2}\right)} - (1 + \varepsilon|\eta|^2)^{-\left(\frac{n+1}{2}\right)} \right) \hat{h}(\eta) \, d\eta.\]
Indeed, by MVT we have
\[|T_{\varepsilon, \psi}(\hat{h})(\xi)| \leq 2(n+1)\int_0^T \int |\xi - \eta||\hat{\psi}(\xi - \eta)|(1 + \varepsilon|\xi|^2)^{\left(\frac{n+2}{2}\right)}|\xi, \hat{h}(\eta)| \, d\eta \, ds \leq C \int |\nabla \hat{\psi}(\xi - \eta)||\hat{h}(\eta)| \, d\eta,
\]
where \(|\xi|_{s} = \min\{|\xi|, (1-s)|\eta|\}\). Young’s convolution inequality yields the estimate (3.18). Therefore we get from (3.15), (3.16) and (3.17) that
\[\langle V(u_{\varepsilon})u_{\varepsilon}, A' [J_{\varepsilon}, \psi_{\lambda}] \lambda_{\lambda} u_{\varepsilon} \rangle = \langle (1 - \Delta)^{-1} (V(u_{\varepsilon})u_{\varepsilon}), (1 - \Delta)A' [J_{\varepsilon}, \psi_{\lambda}] \lambda_{\lambda} u_{\varepsilon} \rangle
\]
\[= O_{K_{\varepsilon}}(\lambda^{\frac{d}{2}}) + i2(n+1)\langle (1 - \Delta)^{-1} (V(u_{\varepsilon})u_{\varepsilon}), \psi_{\lambda} J_{\varepsilon} \frac{\hat{\psi}}{\hat{h}} \Delta((1 - \Delta)\lambda u_{\varepsilon}) \rangle
\]
\[\quad - i\langle (1 - \Delta)^{-1} (V(u_{\varepsilon})u_{\varepsilon}), [J_{\varepsilon}, \lambda_{\lambda} x] \cdot \nabla (1 - \Delta)\lambda u_{\varepsilon} \rangle
\]
\[= O_{K_{\varepsilon}}(\lambda^{\frac{d}{2}}) + i2(n+1)\langle (1 - \Delta)^{-1} (V(u_{\varepsilon})u_{\varepsilon}), \psi_{\lambda} J_{\varepsilon} \frac{\hat{\psi}}{\hat{h}} \Delta(\lambda(1 - \Delta)u_{\varepsilon}) \rangle
\]
\[\quad - i\langle (1 - \Delta)^{-1} (V(u_{\varepsilon})u_{\varepsilon}), [J_{\varepsilon}, \lambda_{\lambda} x] \psi_{\lambda} \cdot \nabla (1 - \Delta)u_{\varepsilon} \rangle.
\]
By the same way one can get (3.10). This completes the proof of Proposition 3.2.

\[\square\]

3.3. Approximate for virial operator

In this section we consider the nonnegative quantity
\[\mathcal{M}(u) = \langle Mu; Mu \rangle.
\]

**Proposition 3.3.** Suppose that \(\varphi \in H^2\) and \(n \geq 7\). Let \(u^\varepsilon\) be the solution of (2.1). Then it satisfies that for \(t \in [0, \infty)\). Then we have for any \(0 < \varepsilon < 1\)
\[\mathcal{M}(u^\varepsilon) \leq 16t^2 E_\varepsilon(\varphi^\varepsilon) + 4t \left( \Im \langle \varphi^\varepsilon, A \varphi^\varepsilon \rangle + Cm(\varphi^\varepsilon) \right) + \mathcal{M}(\varphi^\varepsilon)
\]
\[+ C \varepsilon^{\frac{d}{2}} \int_0^t \int_0^{s'} L_s \, ds \, ds',
\]
for \(t \in [0, \infty)\), where \(C\) is a positive constant depending only on \(n\) but not on \(u^\varepsilon, \varphi\).

If \(u^\varepsilon\) and \(\varphi\) are radial, then the above inequality holds for \(n \geq 5\).
Proof of Proposition 3.3. Let $\mathcal{M}_\lambda := \langle M\psi_\lambda u^\varepsilon; M\psi_\lambda u^\varepsilon \rangle$. Then we have
\[
\frac{d}{dt} \mathcal{M}_\lambda = 2\text{Re}\langle M\psi_\lambda u^\varepsilon; M\psi_\lambda u_t^\varepsilon \rangle \\
= 2\text{Re}\langle M\psi_\lambda u^\varepsilon; M\psi_\lambda \left(-i(-\Delta + \mu \Delta^2)u^\varepsilon + ig_\varepsilon(u^\varepsilon)\right) \rangle \\
= 2\text{Im}\langle M\psi_\lambda u^\varepsilon; M\psi_\lambda (-\Delta + \mu \Delta^2)u^\varepsilon \rangle - 2\text{Im}\langle M\psi_\lambda u^\varepsilon; M\psi_\lambda g_\varepsilon(u^\varepsilon) \rangle \\
=: III_\lambda - IV_\lambda.
\]

We first treat $III_\lambda$ as follows:
\[
(3.19) \quad III_\lambda = 2\text{Im}\langle M\psi_\lambda u^\varepsilon; M\left[\psi_\lambda, -\Delta + \mu \Delta^2\right]u^\varepsilon \rangle \\
+ 2\text{Im}\langle x\psi_\lambda u^\varepsilon; (1 - 2\mu \Delta)^{-1}[x, -\Delta + \mu \Delta^2]\psi_\lambda u^\varepsilon \rangle \\
= 2\text{Im}\langle (1 - 2\mu \Delta)^{-1/2}\psi_\lambda x u^\varepsilon; (1 - 2\mu \Delta)^{-1/2}x[\psi_\lambda, -\Delta + \mu \Delta^2]u^\varepsilon \rangle \\
+ 4\text{Im}\langle \psi_\lambda x u^\varepsilon; (\nabla \psi_\lambda)u^\varepsilon \rangle \\
+ 4\text{Im}\langle \psi_\lambda u^\varepsilon, \psi_\lambda x \cdot \nabla u^\varepsilon \rangle \\
= o_K(\lambda) + 4\text{Im}\langle \psi_\lambda u^\varepsilon, \psi_\lambda x \cdot \nabla u^\varepsilon \rangle \\
= o_K(\lambda) + 4A_\lambda.
\]

Here we used the fact that \( \|\lambda|x|^2\psi_\lambda \partial^1 u^\varepsilon\|_{L^2} \to 0 \) for \( 1 \leq |j| \leq 4 \) as \( \lambda \to 0 \) by LDCT.

From now on we treat $IV_\lambda$. At first we have
\[
IV_\lambda = 2\text{Im}\langle M\psi_\lambda u^\varepsilon; M\psi_\lambda g_\varepsilon(u^\varepsilon) \rangle \\
= 2\text{Im}\langle x\psi_\lambda u^\varepsilon; [J_\varepsilon, x]V(u_\varepsilon)u_\varepsilon \rangle \\
+ 2\text{Im}\langle J_\varepsilon x \cdot \nabla u^\varepsilon; (1 - 2\mu \Delta)^{-1}J_\varepsilon xV(u_\varepsilon)u_\varepsilon \rangle \\
+ 2\text{Im}\langle J_\varepsilon x \cdot \nabla u^\varepsilon; xV(u_\varepsilon)u_\varepsilon \rangle \\
=: V^1_\lambda + V^2_\lambda + V^3_\lambda + V^4_\lambda.
\]

For $V^1_\lambda$ we have from (3.5), (3.1) and similar estimate to (3.7) that
\[
V^1_\lambda \leq C\lambda^{1/2}\|x\|_{L^2} \left( \|\Delta J_\varepsilon^{(\frac{n+2}{2})} \Delta^{-1}V(u_\varepsilon)\|_{L^2} + \|\|x\|^{-1}\|u_\varepsilon\|_{H^{3/2}} \right) \\
\leq C\lambda^{1/2}\|x\|_{L^2} \|u^\varepsilon\|_{1+\frac{1}{3}}^{1/2} \\
\leq C\lambda^{1/2}K_\varepsilon.
\]
For $V^2_\lambda$ we have
\[
V^2_\lambda = 4(n+1)\text{Im}\langle \psi^2_{\lambda} u^\varepsilon, x(1-2\mu\Delta)^{-1}\varepsilon J^\frac{n+2}{2} \cdot \nabla V(u)u \rangle \\
= 4(n+1)\text{Im}\langle \psi^2_{\lambda} u^\varepsilon, \varepsilon J^\frac{n+2}{2} \cdot \nabla (1-2\mu\Delta)^{-1}V(u)u \rangle \\
+ 4(n+1)\text{Im}\langle \psi^2_{\lambda} u^\varepsilon, \varepsilon J^\frac{n+2}{2} (\nabla \cdot x - n)(1-2\mu\Delta)^{-1}V(u)u \rangle \\
= -8(n+1)(n+2)\text{Im}\langle \psi^2_{\lambda} u^\varepsilon, \varepsilon^2 J^\frac{n+2}{2} \Delta (1-2\mu\Delta)^{-1}V(u)u \rangle \\
- 4(n+1)\text{Im}\langle \psi^2_{\lambda} u^\varepsilon, \varepsilon J^\frac{n+2}{2} (1-2\mu\Delta)^{-1}V(u)u \rangle \\
+ 4(n+1)\text{Im}\langle \psi^2_{\lambda} u^\varepsilon, \varepsilon J^\frac{n+2}{2} \nabla \cdot [x, (1-2\mu\Delta)^{-1}]V(u)u \rangle \\
+ 4(n+1)\text{Im}\langle \psi^2_{\lambda} u^\varepsilon, \varepsilon J^\frac{n+2}{2} \nabla \cdot (1-2\mu\Delta)^{-1}xV(u)u \rangle \\
\leq C\varepsilon^2 \|u\|_{L^2} \|V(u)u\|_{H^{-2}} + C\varepsilon^2 \|u\|_{L^2} \|xV(u)u\|_{H^{-2}} \\
\leq C\varepsilon^2 \|u\|_{L^2}^{2+\frac{4}{n}} \\
\leq C\varepsilon^2 L_1.
\]

For $V^3_\lambda$ we have
\[
V^3_\lambda = -2\text{Im}\langle [J_\lambda, x] \psi^2_{\lambda} u^\varepsilon; (1-2\mu\Delta)^{-1}xV(u)u \rangle \\
= -4(n+1)\text{Im}\langle \varepsilon J^\frac{n+2}{2} \nabla \psi^2_{\lambda} u^\varepsilon; (1-2\mu\Delta)^{-1}xV(u)u \rangle \\
\leq C\varepsilon^2 \|u\|_{L^2} \|xV(u)u\|_{H^{-2}} \\
\leq C\varepsilon^2 \|u\|_{L^2}^{2+\frac{4}{n}} \\
\leq C\varepsilon^2 L_1.
\]

Finally we use for $V^4_\lambda$ the Stein-Weiss inequality such that
\[
\left| \int \int \frac{f(x) \overline{g(y)} |x|^{\theta_1} |y|^{\theta_2}}{|x-y|^p |y|^q} \, dx dy \right| \leq C\|f\|_{L^{p_1}} \|g\|_{L^{p_2}},
\]
provided that $1 < p_1, p_2 < \infty$, $\theta_1 + \theta_2 \geq 0^3$, $0 < \theta < n$, $\frac{1}{p_1} + \frac{1}{p_2} + \frac{\theta}{n} = 2$ and $\theta_1 < \frac{\theta}{n}$, $\theta_2 < \frac{\theta}{n}$. Here our case is that $\theta_1 = -1$, $\theta_2 = 1$, $\theta = n - 2$, $p_1 = 2$, $p_2 = \frac{2n}{n+1}$, $f(x) = [J_\lambda \psi^2_{\lambda} u^\varepsilon(x)]$ and $g(y) = |y| \Delta (1-2\mu\Delta)^{-1}(|y|^{-1}|u|)^{1+\frac{4}{n}}$ (since $n \geq 7$, $\theta_2 = 1 < \frac{\theta}{n} = \frac{\theta}{2}$). Therefore
\[
V^4_\lambda \leq C \int f(x) |x| \Delta^{-1} (|y|^{-1}) \, dx \\
\leq C \int \int \frac{f(x) \overline{g(y)} |x|^{-1} |x-y|^{|n-2|} |y|}{|x-y|^p |y|^q} \, dx dy \\
\leq C \|f\|_{L^{p_1}} \|g\|_{L^{p_2}}.
\]

\(^3\theta_1, \theta_2\) need not be non-negative.
For the Hardy-Littlewood-Sobolev inequality we have

\[
(3.21) \quad \langle x, (1 - 2\mu\Delta)^{-1} |x|^2 V(u_c) u_c \rangle 
\]

and

\[
(3.22) \quad C|V_{\lambda}^1|^{2+\frac{\beta}{\beta'}} \leq Cm(\varphi)^{1+\frac{\beta}{\beta'}}.
\]

For the last inequality we used the fact that \(|y|\) is \(A_{2n/n+2}\)-weight.

If we assume the radial symmetry, then we can estimate \(V_{\lambda}^1\) further.

\[
V_{\lambda}^1 = -2\text{Im}\langle J_{\lambda} \psi_{2\lambda}^2 u^\varepsilon, (1 - 2\mu\Delta)^{-1} |x|^2 V(u_c) u_c \rangle
\]

\[
- 2\text{Im}\langle J_{\lambda} \psi_{2\lambda}^2 u^\varepsilon, [x, (1 - 2\mu\Delta)^{-1}] \cdot x V(u_c) u_c \rangle
\]

\[
= -2\text{Im}\langle J_{\lambda} \psi_{2\lambda}^2 u^\varepsilon, (1 - 2\mu\Delta)^{-1} |x|^2 V(u_c) u_c \rangle
\]

\[
+ 8\mu \text{Im}\langle J_{\lambda} \psi_{2\lambda}^2 u^\varepsilon, (1 - 2\mu\Delta)^{-2} \nabla \cdot x V(u_c) u_c \rangle
\]

\[
=: V_{\lambda}^1 + V_{\lambda}^2.
\]

For \(V_{\lambda}^1\) we have by \((1 - 2\mu\Delta)^{-1} f \leq C \|f\|_{L^\infty}, n > 4\) that

\[
(3.21) \quad V_{\lambda}^1 \leq \|J_{\lambda} \psi_{2\lambda}^2 u^\varepsilon\|_{L^2}(1 - 2\mu\Delta)^{-1} |u_c|^\frac{\beta}{\beta'} u_c \|_{L^2}
\]

\[
\leq C \|u^\varepsilon\|_{L^2} \|u^\varepsilon\|^{1+\frac{\beta}{\beta'}} \|_{L^\infty}
\]

\[
\leq Cm(\varphi)^{1+\frac{\beta}{\beta'}};
\]

where \(C\) does not depend on \(\varepsilon, \lambda\).

Let \(f = (-2\mu\Delta)^2 (1 - 2\mu\Delta)^{-2} J_{\lambda} \psi_{2\lambda}^2 u^\varepsilon\) and \(h = V(u_c) u_c\). Then

\[
V_{\lambda}^2 = \frac{\mu}{2} \text{Im}\langle x \cdot \nabla (-\Delta)^{-2} f, h \rangle.
\]

We use the fact that

\[
x \cdot \nabla (-\Delta)^{-2} f(x) = -c_1 |x|^2 \Delta^{-1} f + c_2 \int |x - y|^{-(n-2)} x \cdot y f(y) \, dy,
\]

where \(c_1, c_2\) are constants depending only on \(n\). Thus

\[
V_{\lambda}^2 = -c_1 \text{Im}\langle \Delta^{-1} f, |x|^2 h \rangle + c_2 \text{Im}\langle \mathcal{B}_{\lambda}, |x|^2 h \rangle,
\]

where \(\mathcal{B}_{\lambda} = |x|^{-2} \int |x - y|^{-(n-2)} x \cdot y f(y) \, dy\). The first term can be treated as (3.21). For fixed \(0 < \delta \ll 1\) we divide \(\mathcal{B}_{\lambda}\) as follows:

\[
\mathcal{B}_{\lambda} = |x|^{-2} \int_{|y| < |x|/\delta} + |x|^{-2} \int_{|y| \geq |x|/\delta} =: \mathcal{B}_{\lambda}^1 + \mathcal{B}_{\lambda}^2.
\]

Then since \(|\mathcal{B}_{\lambda}^2| \leq C \int |x - y|^{-(n-2)} \psi_\lambda(y)|u^\varepsilon(y)| \, dy\), by Hölder’s inequality and Hardy-Littlewood-Sobolev inequality we have

\[
(3.22) \quad \text{Im}\langle \mathcal{B}_{\lambda}^2, |x|^2 V(u_c) u_c \rangle \leq C \|\mathcal{B}_{\lambda}^2\|_{L^{2\mu}} \|u^\varepsilon\|_{L^2}^{1+\frac{\beta}{\beta'}} \leq C \|u^\varepsilon\|^{2+\frac{\beta}{\beta'}} \leq Cm(\varphi)^{1+\frac{\beta}{\beta'}}.
\]
For $\mathbb{S}_1^n$ we use the radial symmetry. More precisely, let $x = rx', y = ry'$ for $x', y' \in S^{n-1}$. Then for each $x \neq 0$
\[
\mathbb{S}_1^n = r^{-1} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} |rx' - ry'|^{-n} |x' \cdot y' d\sigma(y') \psi(x, y) |
\]
Since every function is radial, we may assume that $x' = e_1 = (1, 0, \ldots, 0)$. Thus if $\delta$ is sufficiently small, using spherical coordinates we have
\[
| \mathbb{S}_1^n | \leq C r^{-1} \int_{\mathbb{S}^{n-1}} \left( \int_0^\pi \int_0^{2\pi} (r^2 + \rho^2 - 2r\rho \cos \theta)^{-\frac{n+2}{2}} \cos \theta \sin^{n-2} \theta d\theta \right) \| \psi(x, y) \|_{L^{\frac{n}{2+n}}} d\rho
\]
Summing the estimates (3.19) and (3.20) for nonradial case, and from (3.19), (3.21)-(3.23) for radial case, we have
\[
\frac{d}{dt} \mathcal{M}_\lambda = 4A_\lambda + O(m(\varphi)^{1+\frac{2}{n}}) + o_\lambda(\lambda) + C\varepsilon
\]
Integrating over $[0, t]$ and letting $\lambda \to 0$, since $\mathcal{M}_\lambda = (2\pi)^{-\frac{n}{2}} \| x\psi \|_{L^2}$ and $\psi(x, y) \to x\psi(x, y)$ for a.e. $\psi$, by Fatou’s lemma we get
\[
\mathcal{M}(\varphi(t)) \leq 4 \int_0^t \mathcal{I}(x^\varphi, A x^\varphi) dt' + Ctm(\varphi)^{1+\frac{2}{n}} + C\varepsilon \int_0^t L_a dt'
\]
By Proposition 3.2 we finally get
\[
\mathcal{M}(\varphi(t)) \leq 16\varepsilon E_\varepsilon(\varphi) + 4t \left( \mathcal{I}(x^\varphi, A x^\varphi) + Cm(\varphi)^{1+\frac{2}{n}} + \mathcal{M}(\varphi) \right) + C\varepsilon \int_0^t \int_0^{t'} L_a ds dt'.
\]
This completes the proof of Proposition 3.3.
4. Proof of Theorem 1.1

We show the main theorem. Suppose that \( u \) is the unique solution to (\( \mathcal{S} \)) and \( T_* = \infty \). From the local well-posedness result we deduce that there exist a sequence \( \{t_j\} \), \( j \geq 0 \) with \( t_0 = 0 \) such that \( t_j \to \infty \) as \( j \to \infty \) and for each interval \([t_{j-1}, t_j]\) the regularized solution \( u_j^\varepsilon \) of (2.1) with initial data \( \psi_\varepsilon \ast u(t_{j-1}) \) satisfies the result of Proposition 2.2.

We deduce from Proposition 3.3 that up to subsequence \( u_j^\varepsilon \to u \) in \( \tilde{H}^2 := \{ u \in H^2 : \| u \|_{H^2} := \| u \|_{H^2} + \mathcal{M}(u) < \infty \} \) for each \( t \in [t_{j-1}, t_j], \ j \geq 1 \). Moreover, by uniqueness and energy conservation we can implement the same argument as in the continuous dependence to get \( u_j^\varepsilon \to u \) in \( C([t_{j-1}, t_j]; \tilde{H}^2) \).

Thus
\[
\text{Im} \langle u_j^\varepsilon(t_j), Au_j^\varepsilon(t_j) \rangle = \text{Im} \langle Mu_j^\varepsilon; (1 - 2\mu \Delta)^\frac{1}{2} \nabla u_j^\varepsilon \rangle
\to \text{Im} \langle Mu(t_j); (1 - 2\mu \Delta)^\frac{1}{2} \nabla u(t_j) \rangle =: A_j.
\]

Here \( A_0 \) denotes \( \text{Im} \langle \varphi, A\varphi \rangle \). Since \( \varphi \in H^2, \ x\varphi \in L^2, \ M\varphi \in L^2 \), then one can readily show that
\[
\text{Im} \langle M(\psi_\varepsilon \ast \varphi); (1 - \mu \Delta)^\frac{1}{2} \nabla (\psi_\varepsilon \ast \varphi) \rangle \to \text{Im} \langle M\varphi; (1 - \mu \Delta)^\frac{1}{2} \nabla \varphi \rangle = A_0
\]
as \( \varepsilon \to 0 \). We thus have \( A_j = A_{j-1} + 8(t_j - t_{j-1})E(\varphi), \ j \geq 1 \) from Proposition 3.2 and thus \( A_j = A_0 + 8t_j E(\varphi) \) by induction.

Now since \( \mathcal{M}(\psi_\varepsilon \ast u(t_{j-1})) \to \mathcal{M}(u(t_{j-1})) \), it follows from Proposition 3.3 and lower semi-continuity of \( \mathcal{M} \) that
\[
\mathcal{M}(u(t_j)) \leq 16 (t_j - t_{j-1})^2 E(\varphi) + 4(t_j - t_{j-1})(A_{j-1} + Cm(\varphi)^{1 + \frac{2}{n}})
+ \mathcal{M}(u(t_{j-1})),
\]
or equivalently
\[
\mathcal{M}(u(t_j)) - \mathcal{M}(u(t_{j-1})) \leq 16(t_j - t_{j-1})^2 E(\varphi) + 4(t_j - t_{j-1})(A_0 + 8t_j E(\varphi) + Cm(\varphi)^{1 + \frac{2}{n}})
\]
for some fixed constant \( C \). Since \( E(\varphi) < 0 \) and \( t_j \to \infty \), we can find an integer \( j_0 > 1 \) such that
\[
16t_{j_0}^2 E(\varphi) + 4t_{j_0} \left( |A_0| + Cm(\varphi)^{1 + \frac{2}{n}} \right) + \mathcal{M}(\varphi) < 0.
\]

Summing up both sides of (4.1) up to \( j_0 \), we have
\[
\mathcal{M}(u(t_{j_0})) \leq 16t_{j_0}^2 E(\varphi) + 4t_{j_0} \left( |A_0| + Cm(\varphi)^{1 + \frac{2}{n}} \right) + \mathcal{M}(\varphi) < 0.
\]

This contradicts the non-negativity of \( \mathcal{M} \). Therefore \( T_* < \infty \). This completes the proof of Theorem 1.1.
5. Appendix

5.1. Proof of Lemma 2.1

Let $r_0$ be a real number such that $\max(1, \frac{2^2}{n+4}) < r_0 < n$. Then the straight line of points ($\frac{1}{r}, \frac{1}{r}$) satisfying $1 - \frac{1}{n} = \frac{2^2}{r_0} + \frac{4n}{nr}$ passes through the inside of square with vertices ($\frac{1}{r}, \frac{1}{r}$), ($\frac{1}{r}, \frac{n}{r}$) and ($\frac{1}{r}, \frac{1}{r}$). This implies that by Hölder’s inequality for such $\rho, r$ and Sobolev inequality

$$
\|g_1(u) - g_1(v)\|_{L^{r'}} \leq C \|\chi_{\{|x|<1\}}|x|^{-2}(|u|^\frac{1}{r'} + |v|^\frac{1}{r'})|u - v|\|_{L^{r'}}
$$

$$
\leq C \|\chi_{\{|x|<1\}}|x|^{-2}\|_{L^r} \|u\|_{L^{r'}} + \|v\|_{L^{r'}}\|u - v\|_{L^{r'}}
$$

$$
\leq C(\|\|v\|_{H^2}^\frac{1}{r'} + \|v\|_{H^2}^\frac{1}{r'})\|u - v\|_{L^{r'}} \leq CL^\frac{4}{n}\|u - v\|_{L^{r'}}.
$$

On the other hand, the straight line $1 - \frac{1}{n} = \frac{n+4}{nr}$ always passes through the square. Thus we have

$$
\|g_2(u) - g_2(v)\|_{L^{r'}} \leq C(\|u\|_{\tilde{H}^s_{r'}} + \|v\|_{\tilde{H}^s_{r'}})\|u - v\|_{L^{r'}} \leq CL^\frac{4}{n}\|u - v\|_{L^{r'}}.
$$

5.2. Proof of uniqueness

We now show the uniqueness of the weak solutions constructed in the previous section. From Proposition 2.2 we know that the weak solution $u$ satisfies Duhamel’s formula: for all $t \in [0, T]$

$$
u(t) = U(t)\varphi + i \int_0^t U(t-t')g(u(t')) dt',
$$

where $U(t) = e^{-it(-\Delta + \alpha \Delta^2)}$. It is well-known that for any Schrödinger admissible pairs $(q_i, r_i)$, $i = 1, 2$ such that $q_i \in [2, \infty]$, $r_i \in [\frac{2}{2q_i}, 1]$ and $\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{n}$ and for any $s \in \mathbb{R}$ the following Strichartz estimate holds

$$
\|D^s U(t)\varphi\|_{L_{q_i}^{r_i} L_{r_i}^{q_i}} \leq C \|D^{s - \frac{3}{n}}\varphi\|_{L^2},
$$

(5.1)

$$
\|D^s \int_0^t U(t-t')f(t') dt'\|_{L_{q_i}^{r_i} L_{r_i}^{q_i}} \leq C \|D^{s - \frac{3}{n}} - \frac{3}{n} f\|_{L_{q_i}^{r_i} L_{r_i}^{q_i}},
$$

where $L_{q_i}^{r_i} L_{r_i}^{q_i} = L_{q_i}^2(0, T; L_{r_i}^{q_i})$ and the constant $C$ does not depend on $T'$. For instance see [11]. Let $u$ and $v$ be weak solutions with the same initial data $\varphi \in H^2$. We first consider the case $n \geq 5$. Let $q_1 = q_2$, $r_1 = r_2$ and $s = \frac{4}{q_i} = n - \frac{2n}{r_1}$. With $\beta \in (0, 1/2)$, take

$$
r_1 = \frac{2n^2}{n^2 - 2n + 4 - \beta} \in \left(2, \frac{2n}{n-2}\right).
$$

For this $r_1$, setting

$$
\frac{1}{r_1} = \frac{1}{r_1} - \frac{s}{n} = \frac{1}{r_1} - \frac{1}{n}(n - \frac{2n}{r_1}) = \frac{3}{r_1} - 1 = \frac{n^2 - 6n + 12 - 3\beta}{2n^2} > 0
$$
\[ r_0 = \frac{n^2}{n + \beta} \in (1, n), \quad p = \frac{n^2}{n - \beta} > n, \]
such that

\[ 1 - \frac{1}{r_1} = \frac{2}{r_0} + \frac{n - 4}{2n} + \frac{1}{r}, \quad 1 - \frac{1}{r_2} = \frac{2}{p} + \frac{n - 4 + 2\beta}{2n} + \frac{1}{r}. \]

Then by Sobolev inequality we have for any \( 0 < T' < T \)

\[ \|u - v\|_{L_{r_1}^{p_1} L_r'} \leq C\|\nabla\|^\beta (u - v)\|_{L_{r_1}^{p_1} L_r'} \]
\[ \leq C\|g_1(u) - g_1(v)\|_{L_{r_1}^{p_1} L_r'} + C\|g_2(u) - g_2(v)\|_{L_{r_1}^{p_1} L_r'} \]
\[ \leq C\|\chi_{\{x|\leq 1\}} |x|^{-1} \|L_{\infty}^p\|((\|u\|^\frac{1}{L_{\infty}^p} + \|v\|^\frac{1}{L_{\infty}^p})\|u - v\|_{L_{r_1}^{p_1} L_r'} \]
\[ + C\|\chi_{\{x|> 1\}} |x|^{-1} \|L_{\infty}^p\|((\|u\|^\frac{1}{L_{\infty}^p} + \|v\|^\frac{1}{L_{\infty}^p})\|u - v\|_{L_{r_1}^{p_1} L_r'} \]
\[ \leq CT^\beta r^\frac{2}{n} \|u - v\|_{L_{r_1}^{p_1} L_r'}. \]

Thus if \( T' \) is sufficiently small, then we get \( u = v \) on \([0, T']\).

Now let \( I = [0, b) \) be the maximal interval of \([0, T]\) with

\[ \|u - v\|_{L_{r_1}^{p_1} L_r'} = 0 \text{ for all } a < b. \]

Suppose that \( b < T \). Then for a small \( \varepsilon > 0 \) we can find \( b < t' := b + \varepsilon < T \) such that

\[ \|u - v\|_{L_{r_1}^{p_1} L_r'} \leq CL^\beta \|u - v\|_{L_{r_1}^{p_1} L_r'}. \]

Since \( \varepsilon \) can be made arbitrarily small, this contradicts the maximality of \( I \).

Thus \( u = v \) on \([0, T']\) for \( n \geq 5 \). Let \( T^* \) be the maximal existence time of weak solution. Then by the same argument as above one gets the uniqueness on \([0, T^*)\).

If \( n = 3, 4 \), then with \( k \geq 4 \) and \( p \in (2k, 2(k + 3)) \), we choose \( r_0 = \frac{p}{p + T}, \quad \tilde{p} = \frac{p}{p^2 + n}, \)

\[ 1 - \frac{1}{r_1} = \frac{n - 1}{2n} + \frac{k}{pn} \quad \frac{1}{r_2} = \frac{n - 1}{2n} - \frac{k + 3}{pn} \in \left( \frac{n - 2}{2n}, \frac{1}{2} \right), \]
\[ \frac{2}{\tilde{p}} + \frac{n}{r_1} = \frac{n}{2}, \quad s = n - \frac{n}{r_1} - \frac{n}{r_2}, \quad \frac{1}{r} = \frac{1}{r_1} - \frac{1}{r_2} = \frac{n + 3}{2n} \quad \text{Then} \]
\[ 1 - \frac{1}{r_2} = \frac{2}{r_0} + \frac{1}{p} + \frac{1}{r} = \frac{2}{p} + \frac{1}{p/2} + \frac{1}{r}. \]

Thus by Strichartz estimate (5.1) and Sobolev inequality we obtain

\[ \|u - v\|_{L_{r_1}^{p_1} L_r'} \leq C\|\nabla\|^\beta (u - v)\|_{L_{r_1}^{p_1} L_r'} \]
\[ \leq C\|g_1(u) - g_1(v)\|_{L_{r_1}^{p_1} L_r'} + C\|g_2(u) - g_2(v)\|_{L_{r_1}^{p_1} L_r'} \]
\[ \leq C\|\chi_{\{x|\leq 1\}} |x|^{-1} \|L_{\infty}^p\|((\|u\|^\frac{1}{L_{\infty}^p} + \|v\|^\frac{1}{L_{\infty}^p})\|u - v\|_{L_{r_1}^{p_1} L_r'} \]
\[ \leq CT^\beta r^\frac{2}{n} \|u - v\|_{L_{r_1}^{p_1} L_r'}. \]
We can treat the remaining thing similarly to the high dimensional cases. This completes the proof of uniqueness.

5.3. $L^2 H^3_{\text{loc}}$--Strichartz estimate

Let $u$ be the solution to $(\mathcal{S})$ or (2.1) in $C([0,T];H^2)$ and let

$$L = \sup_{[0,T]} \|u(t)\|_{H^2}.$$ Notice that for small enough positive $\epsilon > 0$, by using Hardy’s inequality, we have

$$\|g(u)\|_{L^{2n/(n+2)}([0,T];H^{1/2+n}_{\text{loc}})} \lesssim \|\varphi\|_{H^2} + \|g(u)\|_{L^2([0,T];H^{1/2+n}_{\text{loc}})}$$

$$\lesssim \|\varphi\|_{H^2} + \|g(u)\|_{L^2([0,T];L^{2n/(n+2)})} + \|\nabla g(u)\|_{L^2([0,T];L^{2n/(n+2)})}$$

$$\lesssim L + T^{\epsilon/2} L^{4/n+\epsilon} \|u\|^{1-\epsilon}_{L^2([0,T];H^{2n/(n+2)})}$$

$$+ \|\varphi\|_{H^2} + \|u\|^{1-\epsilon}_{L^2([0,T];H^{2n/(n+2)})}.$$ Let us denote the last two terms by $I$ and $II$. Then as for $I$ we have

$$I \lesssim \|\chi(|x|<1)|x|^{-3}|u|^{1+\frac{\epsilon}{n}}_{L^2([0,T];L^{2n/(n+2)})} + \|\chi(|x|\geq1)|x|^{-3}|u|^{1+\frac{\epsilon}{n}}_{L^2([0,T];L^{2n/(n+2)})}$$

$$\lesssim \|u\|^{1+\frac{\epsilon}{n}}_{L^2([0,T];L^{2n/(n+2)})} + \|u\|^{1+\frac{\epsilon}{n}}_{L^2([0,T];L^{2n/(n+2)})}$$

$$\lesssim \|u\|^{1+\frac{\epsilon}{n}}_{L^2([0,T];L^{2n/(n+2)})} + \|u\|^{1+\frac{\epsilon}{n}}_{L^2([0,T];H^{\alpha/2})},$$

where $\frac{1}{\epsilon} = \frac{n+2}{2n} - \frac{n}{4}$ for some real $\beta$ slightly less than $n$ and $\alpha = n(\frac{n-2}{2n} - \frac{n}{(n+4)^2}).$ If $\beta$ is sufficiently close to $n$, then by interpolation we can find max$(0,(4 -$
$n)/n) < \varepsilon < \frac{4}{n}$ such that
\[ \|u\|_{H^{\frac{n+1}{2n}}}^{1+\frac{\varepsilon}{n}} \lesssim \|u\|_{H^2} \|u\|_{H^{\frac{n+4}{2n}}}^{\frac{n-4}{n}+\varepsilon}. \]
Thus we have
\[ I \lesssim T^{\frac{n}{n-4}} L^{\frac{n}{n-4}} \|u\|_{L^2([0,T],H^{\frac{n}{n-4}})} + T^{\frac{n}{n-4}} L^{1+\frac{\varepsilon}{n}}. \]
As for $II$ we have
\[ II \leq \|x\{x<1\}|x|^{-2} |u|^{\frac{\varepsilon}{2}} \|\nabla u\|_{L^2([0,T],L^{\frac{n}{n-4}})} + \|u\|^{\frac{\varepsilon}{2}} \|\nabla u\|_{L^2([0,T],L^{\frac{n}{n-4}})} \]
\[ \lesssim \|u\|^{\frac{\varepsilon}{2}} \|\nabla u\|_{L^2([0,T])} + \|u\|^{\frac{\varepsilon}{2}} \|\nabla u\|_{L^2([0,T])}, \]
where $\frac{1}{r_1} = \frac{n+2}{2n} - \frac{2}{n}$ and $\frac{1}{r_2} = \frac{1}{2} - \frac{n-2}{4n}$ for some $\beta$ slightly less than $n$ and $\frac{1}{r_2} = \frac{1}{2} - \frac{n-2}{4n}$. If $\beta$ is sufficiently close to $n$ then there exists $0 < \varepsilon < \frac{4}{n}+\frac{\varepsilon}{n}$ so that $\alpha = 2 - \frac{12}{n+4} + \varepsilon$. Thus we have
\[ II \lesssim T^{\frac{n}{n-4}} L^{\frac{n}{n-4}} + T^{\frac{n}{n-4}} L^{1+\frac{\varepsilon}{n}}, \]
where $\theta = \frac{n+4}{2n} = 1 - \frac{2}{n} + \frac{(n+4)\varepsilon}{2n}$. Substituting the estimates of $I$ and $II$ into (5.3) and using Young’s inequality, we have
\[ \|u\|_{L^2([0,T],H^{\frac{n}{n-4}})} \leq C(T, L). \]

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