Novel Geometrical Models of Relativistic Stars
III. The Point Particle Idealization

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We describe a novel class of geometrical models of relativistic stars. Our approach to the static spherically symmetric solutions of Einstein equations is based on a careful physical analysis of radial gauge conditions. It brings us to a two parameter family of relativistic stars without stiff functional dependence between the stellar radius and stellar mass.

As a result, a point particle idealization – a limiting case of bodies with finite dimension, becomes possible in GR, much like in Newtonian gravity. We devote this article to detailed mathematical study of this limit.

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I. INTRODUCTION

This is the third one of the series of articles, in which we describe novel geometrical models of general relativistic stars (GRS). The preliminary knowledge of the first two ones is highly recommended for the right understanding of the present article. There one can find the basic principles, equations, notations and general results for construction of the new models [1], as well as the detailed description of the simplest model of this type – incompressible GRS [2].

In the third article we describe the point particle limit of bodies of finite dimension in GR. Thus we obtain a new ground for the massive point particle solutions to Einstein equations (EE), described earlier in [3].

At present the vast majority of relativists do not accept the consideration of point particles in GR as an incompatible with EE idealization. There are different reasons: some doubts about consistency of the theory of mathematical distributions (like Dirac delta function $\delta(r)$) with the nonlinear character of EE; the understanding of the drastic change of geometry of the Riemannian space-time $\delta$ discussed by Brillouin [4] as early as in 1923. This geometry ultimately forces one to consider the standard Hilbert gauge (HG) form of the solution outside the source:

$$g_{\mu\nu}$$

in vicinity of a point with infinite concentration of energy (this problem was for the first time discussed by Brillouin [4]), etc.

On the other hand, it is obvious that in Nature very distant objects like stars look like ”points” of finite mass and finite luminosity. This fact has a proper mathematical description in Newton theory of gravity in the language of mathematical distributions. A formal mathematical problem is to find the corresponding idealized treatment of such objects in GR, as well, but up to recently no reasonable approach was known.

In articles [3] we showed that a correct mathematical solutions to the EE with $\delta(r)$ term in the rhs do exist. Such solutions describe a two parameter family of analytical space-times $M^{(1,3)}\{g_{\mu\nu}\}$ with specific strong singularity at the place of the massive point source with bare mechanical mass $M_0 > 0$ and Keplerian mass $M < M_0$.

The price, one is forced to pay for such enlargement of standard GR framework, is:

1) To consider the corresponding metric coefficients $g_{\mu\nu}(x)$ as functions of class $C^0$ of the coordinates $x$. Some of them are to have a fixed finite jumps in their first derivatives at the place of the point source, necessary to reproduce by the Einstein tensor $G_\mu^\nu$ the $\delta(r)$ term of the energy-momentum tensor $T_\mu^\nu$ in the rhs of EE.

2) To accept the unusual geometry of space-time around the matter point in GR, which appeared at first in the original Schwarzschild article [5]. It has been discussed by Brillouin [4] as early as in 1923. This geometry is essentially different from the geometry around space-time points with finite energy density in them.

It turns out that in perfect accord with Dirac intuition [6], the presence of massive point matter source in rhs of EE ultimately forces one to consider the standard Hilbert gauge (HG) form of the solution outside the source:

$$g_{\mu\nu} = 1 - \rho_G/\rho, \; g_{\rho\rho} = -1/g_\mu^\mu (\rho)$$

only on the physical interval of the luminosity variable $\rho \in (\rho_0, \infty)$, where

$$\rho_0 = 2M/(1 - \varrho^2) > \rho_G$$

Here $\rho_G = 2M$ is the Schwarzschild radius, $\varrho = M/M_0 \in (0, 1)$ is the ratio of the Keplerian mass $M$ and the bare mechanical mass $M_0$ of the matter point. This ratio describes the gravitational defect of the mass of the point particle, introduced in [3], in scale invariant way [1].

In the standard approach to GRS [7] a consideration of the limit $R_* \to 0$ of the geometrical radius $R_*$, for a fixed stellar mass $m_*$, is impossible. Indeed, in this approach the extra condition $p_C(m_*, R_*) = 0$ on the central value of Hilbert luminosity variable $\rho$ yields a stiff mass-radii
relation \( m_\ast = m_\ast(R_\ast) \) [1]. Because of this relation, it becomes impossible to vary the radius \( R_\ast \), and, at the same time, to prescribe a constant value to the mass \( m_\ast \).

Here we shall show that our generalized treatment of the GRS with Keplerian mass \( m_0 \) and proper mass \( m_\alpha \), described in [1, 2], allows a correct limit \( R_\ast \rightarrow 0 \) of the stelar radius \( R_* \) under one of the following two extra conditions:

i) Fixed Keplerian mass \( m_* \);

ii) Fixed proper mass \( m_\alpha \).

Thus, much like as in Newtonian theory of gravity, where these two masses are identical, in GR one is able to obtain a natural massive-point-particle idealization of objects of finite dimension.

II. POINT PARTICLE LIMIT IN NEWTON THEORY OF STARS

Let us first consider the theory of stars in Newton gravity. To be specific, we will examine here the most well known polytropic stars with EOS:

\[
p = K \varepsilon^\gamma, \quad \text{or } w := p/\varepsilon = K \varepsilon^{\gamma - 1}, \quad \gamma \geq 6/5. \tag{II.1}
\]

See, for example, the book by S. Weinberg in [7].

In this case the non-relativistic energy density in the star is described by the function

\[
\varepsilon^{NR}(r) = \begin{cases} \frac{m_0}{4\pi r^2} \frac{\xi}{[\theta(\xi, r)]} \left( \frac{\varepsilon}{\vartheta(\xi, r)} \right)^{1/(\gamma - 1)}, & \text{if } r \in [0, r_*]; \\ 0, & \text{if } r \in ]0, r_*], \end{cases} \tag{II.2}
\]

where \( r \equiv \rho \) is the standard radial variable in the Euclidean 3D space, \( \theta(\xi) = \text{const} \times p/\varepsilon = \text{const} \times w \) is the Lane-Emden function, i.e. the solution of the Cauchy problem

\[
\xi^2 \frac{d}{d\xi} \left( \xi^2 \frac{d\theta}{d\xi} \right) + \theta^{1/(\gamma - 1)} = 0,
\]

\[
\theta(0) = 1, \quad d\theta/d\xi(0) = 0; \tag{II.3}
\]

\[
\xi := (\varepsilon^{NR}(0))^{2-\gamma} \sqrt{4\pi(\gamma - 1)/K} r, \quad \xi_* = \xi_*(\gamma) \text{ is the minimal positive root of the equation } \theta(\xi) = 0 \text{ and } r/r_* = \xi/\xi_*. \tag{II.4}
\]

Then the mass of the star is

\[
m_* = \int_0^{r_*} dr 4\pi r^2 \varepsilon^{NR}(r). \tag{II.4}
\]

Let us consider the function

\[
F(\xi; \xi_*, 0) = \begin{cases} \frac{\xi^2}{[\theta(\xi)]^{1/(\gamma - 1)}}, & \text{if } \xi \in [0, \xi_*]; \\ 0, & \text{if } \xi \notin [0, \xi_*]. \end{cases} \tag{II.5}
\]

One easily obtains from Eq. (II.3):

\[
\int_{-\infty}^{\xi_*} F(\xi; \xi_*, 0) d\xi = \int_0^{\xi_*} F(\xi; \xi_*, 0) d\xi = 1. \tag{II.6}
\]

Now we apply the Basic Lemma, proved in the Appendix, using \( \epsilon = r_\ast/\xi_* \), and values \( x_0 = x_1 = 0, x_2 = \xi_* \), \( (x - x_0)/\epsilon = \frac{\xi}{\xi_*} r_\ast \). Then in the limit \( r_\ast \rightarrow 0 \), \( m_* = \text{const} \), \( \gamma = \text{const} \) we have:

\[
4\pi r^2 \varepsilon^{NR}(r) = m_* \frac{\xi_*}{r_*} F \left( \frac{\xi_*}{r_*}; \xi_*, 0 \right) \rightarrow m_\ast \delta(r). \tag{II.7}
\]

This way we obtain the massive point particle idealization of bodies of finite dimension in Newtonian gravity [9].

As expected, during the limiting process \( r_\ast \rightarrow 0 \), \( m_* = \text{const} \), \( \gamma = \text{const} \) the central energy density

\[
\varepsilon^{NR}_C = \varepsilon^{NR}(0) = \frac{m_*}{4\pi r_*^2} \frac{\xi_*}{[\theta(\xi_*)]^2} \rightarrow \infty \tag{II.8}
\]

diverges as \( 1/r_\ast^2 \). The limiting value of the central pressure diverges stronger than \( \varepsilon^{NR}_C \):

\[
p^{NR}_C = p^{NR}(0) = \frac{m_*^2}{4\pi r_*^4} \frac{1 - 1/\gamma}{[\theta(\xi_*)]^2} \rightarrow \infty. \tag{II.9}
\]

Hence,

\[
w^{NR}_C = w^{NR}(0) = \frac{1 - 1/\gamma}{\xi_* [\theta(\xi_*)]} \frac{m_*}{r_*} \rightarrow \infty \tag{II.10}
\]

diverges only as \( 1/r_\ast \).

The corresponding limit of the coefficient \( K \) depends on the polytropic index \( \gamma \):

\[
K \sim r_\ast^{3\gamma - 4} \rightarrow \begin{cases} \infty, & \text{if } \gamma \in [6/5, 4/3); \\ \text{const} \neq 0, \infty, & \text{if } \gamma = 4/3; \\ 0, & \text{if } \gamma > 4/3. \end{cases} \tag{II.11}
\]

Hence, during the above limiting process this parameter in EOS (II.1) is changing if \( \gamma \neq 4/3 \). It is well known that the values \( \gamma < 4/3 \) lead to unstable solutions, the value \( \gamma = 4/3 \) corresponds to stars, build of ultra-relativistic matter, and only the values \( \gamma > 4/3 \) describe stable Newtonian stars. In the limit \( \gamma \rightarrow \infty \) one obtains the following results for incompressible non-relativistic stars:

\[
\theta = 1 - \xi^2/\xi^2_* \nmid \xi_* = \sqrt{6}, \tag{II.12a}
\]

\[
w^{NR}_C = w^{NR}_C \left( 1 - \xi^2/\xi^2_* \right), \tag{II.12b}
\]

\[
\varepsilon^{NR}(r) = \frac{3m_*}{4\pi r_*^4} = \text{const}, \tag{II.12c}
\]

\[
p^{NR}(r) = p^{NR}_C \left( 1 - r^2/r_*^2 \right), \quad p^{NR}_C = \frac{3m_*^2}{8\pi r_*^4}. \tag{II.12d}
\]

\[
w^{NR}_C = \frac{1}{2} \frac{m_*}{r_*}, \tag{II.12e}
\]

Hence, as one is expecting, the point particle idealization can be reached considering incompressible Newtonian stars, too.

Thus we see in full details how one can reach a point particle idealization in Newton gravity. It is obvious that the resulting point particle limit does not depend on the specific form of the energy density distribution \( \varepsilon(r) \), determined by some fixed EOS.
III. POINT PARTICLE LIMIT IN GR THEORY OF STARS

A. Point Particle Limit in Hilbert Gauge

According to article [1], the solution of GRS problem in HG can be formulated as a boundary problem for extended Tolman-Openheimer-Volkov (ETOV) system of ordinary differential equations. The very boundary is unknown. It can be fixed by the unknown values of the luminosity variable $\rho$ at the stellar center $C$: $\rho_C \geq 0$ and at the stellar edge: $\rho_* > \rho_C$. For given EOS $\varepsilon = \varepsilon(\rho)$ the result can be represented in the form of the relations:

$$
\varepsilon(\rho; \rho_*, \rho_C) = \begin{cases} 
\varepsilon(\rho(\rho; \rho_*, \rho_C)), & \text{if } \rho \in [\rho_C, \rho_*]; \\
0, & \text{if } \rho \notin [\rho_C, \rho_*],
\end{cases}
$$

$$
m_* := m(\rho_*, \rho_C) = \int_{-\infty}^{\infty} 4\pi \varepsilon(\rho; \rho_*, \rho_C) \rho^2 d\rho, \quad (\text{III.2})
$$

$$
m_0* := m_0(\rho_*, \rho_C) = \int_{-\infty}^{\infty} \frac{4\pi \varepsilon(\rho; \rho_*, \rho_C) \rho^2 d\rho}{\sqrt{1 - 2m(\rho; \rho_*, \rho_C)/\rho}}, \quad (\text{III.3})
$$

$$
R_* := R(\rho_*, \rho_C) = \int_{\rho_C}^{\rho_*} \frac{\rho d\rho}{\sqrt{\rho - 2m(\rho; \rho_*, \rho_C)}}. \quad (\text{III.4})
$$

The last integration is over the finite interval $[\rho_C, \rho_*]$ and the integrand is a continuous function of the variable $\rho$. Then the function $R(\rho_*, \rho_C)$ is a continuous one with respect to both variables $\rho_C$ and $\rho_*$. As a result the condition $R_* \rightarrow 0$ entails simultaneously two limits:

$$
\rho_C \rightarrow \rho_0 \quad \text{and} \quad \rho_* \rightarrow \rho_0, \quad (\text{III.5})
$$

where $\rho_0 \geq 0$ is some unknown limiting value, which must be the same both for $\rho_C$ and $\rho_*$

Let us consider this limit under the two possible extra conditions, which are physically different:

i) Under the extra condition $m_* = m(\rho_*, \rho_C) = \text{const}$:

To use the standard procedure, which gives the point-particle limit, in this case we construct the function

$$
F(\rho; \rho_*, \rho_C) := \frac{\rho}{m_*} 4\pi \rho^2 \varepsilon(\rho; \rho_*, \rho_C).
$$

According to our Basic Lemma (see the Appendix), we obtain from Eq. (III.2) $\frac{\rho}{m_*} \int F(\rho(\rho_*, \rho_C)) \rightarrow \delta(\rho - \rho_0)$.

This gives

$$
\frac{4\pi}{\epsilon} \left( \frac{\rho - \rho_0}{\epsilon} \right)^2 \left( \frac{\rho - \rho_0}{\epsilon} \right) \rightarrow \frac{4\pi}{\epsilon} \left( \frac{\rho - \rho_0}{\epsilon} \right) \rightarrow m_* \delta(\rho - \rho_0).
$$

Then, to calculate in this limit the integral (III.3), we consider the function $\varphi(\rho) = 1/\sqrt{1 - 2m(\rho; \rho_*, \rho_C)/\rho}$ as a test function in the meaning of integral (A.1). This way we obtain from integral (III.3) the relation

$$
m_{0*} = \frac{m_*}{\sqrt{1 - 2m_*/\rho_0}}.
$$

ii). Under the extra condition $m_{0*} = m_0(\rho_*, \rho_C) = \text{const}$:

Now the point particle limit requires consideration of the function $F(\rho; \rho_*, \rho_C) := \frac{1}{m_{0*}} \frac{4\pi \varepsilon(\rho; \rho_*, \rho_C)}{\sqrt{1 - 2m(\rho; \rho_*, \rho_C)/\rho}}$. Then the application of the Basic Lemma and Eq. (III.3) gives

$$
\frac{4\pi}{\epsilon} \left( \frac{\rho - \rho_0}{\epsilon} \right)^2 \left( \frac{\rho - \rho_0}{\epsilon} \right) \rightarrow m_0 \delta(\rho - \rho_0). \quad (\text{III.7})
$$

To calculate the integral (III.2) in the present limit, we have to introduce in it the function $\varphi(\rho) = 1/\sqrt{1 - 2m(\rho; \rho_*, \rho_C)/\rho}$ as a test function in the meaning if integral (A.1). After simultaneous multiplication and division of the integrand by the last function $\varphi(\rho)$, and using Eq. (III.7), we obtain from integral (III.2) the relation

$$
m_* = m_{0*} \sqrt{1 - 2m_*/\rho_0}.
$$

Hence, in both cases i) and ii) we arrive at the same relation, which defines the value of $\rho_0$ via the masses $m_*$ and $m_{0*}$:

$$
\rho_0 = 2m_*/(1 - g_2) > \rho_{G*}. \quad (\text{III.8})
$$

Here $g_* := m_*/m_{0*} \in (0, 1)$ is the stellar mass defect ratio. This way we prove that the limit value $\rho_0$ is greater then the Schwarzschild radius of the star $\rho_{G*}$.

The presence of $\rho_0$ in delta functions (III.6) and (III.7) is a consequence of the fact that the mass ratio $g_*$ is in the open interval $(0, 1)$. As a result, in point particle limit we obtain strictly positive value of the luminosity variable $\rho_0 > \rho_{G*} > 0$.

The comparison with the relation (I.2) and the more complete description of gravitational field of massive point particle in GR, recently found in [3], shows that in both cases we have obtained point particle limit of the GRS solutions with arbitrary EOS. In addition, we see that $M = m_*$, and $M_0 = m_{0*}$, as one has to expect.

The physical difference between the two limiting procedures i) and ii) is clear:

The mass $m_{0*}$ is defined by the initial amount of matter, the star is build of. During the limiting procedure ii) this amount of matter is preserved, but the mass $m_*$ changes, depending on EOS.

The mass $m_*$ depends on the concentrations of the matter. Therefore it will be different for different concentrations of the same amount of matter $m_{0*}$.

Vice versa, during the limiting procedure i) we are preserving the same value of Keplerian mass $m_*$, changing the volume of the contracting star. This is possible only via corresponding change of the very amount of stellar matter, depending on EOS.

Clearly, such physically different limiting procedures are possible only in the theory of GR stars. In Newtonian gravity the masses $m_{0*}$ and $m_*$ are identical and we have a unique limiting transition to the massive point particle idealization.
The point particle limit of body of finite dimension may look a little bit tricky in HG, because the luminosity variable $\rho$ is not a true radial variable in the problem at hand. Much more nature is the consideration of point particle idealization in basic regular gauge (BRG), or in physical regular gauge (PRG) [1, 3]. For simplicity we shall consider this limiting procedure in BRG. In PRG we do not meet some new features of the problem. The transition from BRG to PRG is a simple fractional-linear transformation of the radial variable $r$, preserving the place and the properties of the center of the coordinate system $C$ at which $r = 0$ in both gauges [1, 3].

In BRG the coordinate radius of the star is $r_*$ and in stelar interior we have [1]:

$$\rho(r) = \rho\left(\frac{r}{r_*}; \rho_*, \rho_C\right), \quad \text{for } r \in [0, r_*], \quad \text{(III.9)}$$

with following basic properties:

i) $\rho(\eta; \rho_*, \rho_C) \in [\rho_*, \rho_*]$ for $\eta = \frac{r}{r_*} \in [0, 1]$;

ii) $\rho(0; \rho_*, \rho_C) = \rho_C, \rho(1; \rho_*, \rho_C) = \rho_*$;

iii) $\rho(1; \rho_*, \rho_C) = \rho_*$;

iv) $\rho(\eta; \rho_0, \rho_0) = \rho_0$ for all $\rho_0 \geq 0$ and $\eta \in [0, 1]$;

v) $d\rho/ d\eta \geq 0$.

In addition:

$$\varepsilon(r) = \begin{cases} \varepsilon(\frac{r}{r_*}; \rho_*, \rho_C), & \text{if } r \in [0, r_*], \\ 0, & \text{if } r \in [0, r_*], \end{cases} \quad \text{(III.10a)}$$

$$m(r) = \begin{cases} m(\frac{r}{r_*}; \rho_*, \rho_C), & \text{if } r \in [0, r_*], \\ m_*, & \text{if } r \geq r_*, \end{cases} \quad \text{(III.10b)}$$

$$m_0(r) = \begin{cases} m_0(\frac{r}{r_*}; \rho_*, \rho_C), & \text{if } r \in [0, r_*], \\ m_*, & \text{if } r \geq r_*, \end{cases} \quad \text{(III.10c)}$$

and

$$m_* = \int_{-\infty}^{r_*} \frac{dr}{r_*} 4\pi\varepsilon \left(\frac{r}{r_*}; \rho_*, \rho_C\right) \rho\left(\frac{r}{r_*}; \rho_*, \rho_C\right) \frac{d\rho}{d\eta}\left(\frac{r}{r_*}; \rho_*, \rho_C\right), \quad \text{(III.11a)}$$

$$m_* = \int_{-\infty}^{r_*} \frac{dr}{r_*} 4\pi\varepsilon \left(\frac{r}{r_*}; \rho_*, \rho_C\right) \rho\left(\frac{r}{r_*}; \rho_*, \rho_C\right) \frac{d\rho}{d\eta}\left(\frac{r}{r_*}; \rho_*, \rho_C\right) \sqrt{1 - 2m\left(\frac{r}{r_*}; \rho_*, \rho_C\right)/\rho\left(\frac{r}{r_*}; \rho_*, \rho_C\right)}, \quad \text{(III.11b)}$$

As we see, in BRG the expressions for the corresponding quantities, although more complicated, automatically appear in the form, suitable for study of the point particle limit $r_* \to 0$, much like in Newtonian theory of stars, see Section II.

Besides, we have the following two condition

$$m_*^2 \int_{\rho_C}^{\rho_*} \frac{dp}{p^2} \sqrt{\frac{-9\rho - r_* c^2}{gu}} - r_* c^2 = 0, \quad \text{(III.12a)}$$

$$\left(m_* \frac{m_0}{m_*}\right)^2 \exp\left(\frac{2r_*}{m_*}\right) + \frac{2m_*}{\rho_*} - 1 = 0, \quad \text{(III.12b)}$$

for matching the interior solution of EE with the exterior one in BRG [1]. In Eq. (III.12a) $F \in (-\infty, \infty)$ is an arbitrary fixed parameter, which describes the stellar surface tension. This equation shows that in the limit $r_* \to 0$ we have $\rho_* \to \rho_0, \rho_C \to \rho_0$. The boundary value $\rho_0$ of the luminosity variable is determined by the second equation (III.12b), which obviously coincides with Eq. (III.8) in the limit under consideration.

Now we are ready to consider the two physically different limits $r_* \to 0$, which lead to point particle idealization in BRG.

i) For fixed $m_*$:

We will obviously satisfy the requirements, described in the Note to the Basic Lemma (see the Appendix), if we consider the function

$$F(\eta; \rho_*, \rho_C; 1, 0) := \frac{4\pi}{m_*^2} \varepsilon(\eta; \rho_*, \rho_C) \rho(\eta; \rho_*, \rho_C)^2 \frac{d\rho}{d\eta}(\eta; \rho_*, \rho_C) \quad \text{for } \eta \in [0, 1],$$

which is identically zero for $\eta \in [0, 1]$ and corresponds to the following choice of the parameters: $x = \eta, x_1 = 0, x_2 = 1, x_0 = r_0, y_1 = r_*, y_2 = \rho_0$. As a result of the last definition $1 F(\eta; \rho_*, \rho_C; 1, 0) d\eta = 1$ and

$$\int_{r_*}^{\infty} \frac{dr}{r_*} 4\pi\varepsilon \left(\frac{r}{r_*}; \rho_*, \rho_C\right) \rho\left(\frac{r}{r_*}; \rho_*, \rho_C\right) \frac{d\rho}{d\eta}\left(\frac{r}{r_*}; \rho_*, \rho_C\right) \rightarrow m_* \delta(r). \quad \text{(III.14)}$$

Using the corollary of the Basic Lemma, we obtain from Eq. (III.11b) and Eq. (A.5):

$$m_0 = m_* \int_{0}^{1} F(\eta; \rho_0, \rho_0; 1, 0) d\eta = \frac{m_*}{\sqrt{1 - 2m(\eta; \rho_0, \rho_0)/\rho(\eta; \rho_0, \rho_0)}},$$

Here we have used the property iv) of the function $\rho(\eta; \rho_*, \rho_C)$, which entails $\rho(\eta; \rho_*, \rho_C) \to \rho(\eta; \rho_0, \rho_0) = \rho_0$ and, in combination with Eq. (III.8), gives in the present limit $m(\eta; \rho_*, \rho_C) \to m(\eta; \rho_0, \rho_0) = m_*$. ii) For fixed $m_0$:

Now we obtain

$$\int_{r_*}^{\infty} \frac{dr}{r_*} 4\pi\varepsilon \left(\frac{r}{r_*}; \rho_*, \rho_C\right) \rho\left(\frac{r}{r_*}; \rho_*, \rho_C\right) \frac{d\rho}{d\eta}\left(\frac{r}{r_*}; \rho_*, \rho_C\right) \sqrt{1 - 2m(\eta; \rho_*, \rho_C)/\rho(\eta; \rho_*, \rho_C)} \rightarrow m_0 \delta(r). \quad \text{(III.15)}$$
using in the Basic Lema the following proper function

\[ F(\eta; \rho_*, \rho_C; 1, 0) := \text{ (III.16)} \]

\[
\frac{4\pi \varepsilon (\eta; \rho_*, \rho_C) \rho (\eta; \rho_*, \rho_C)^2 \frac{d\rho}{d\eta} (\eta; \rho_*, \rho_C)}{m_* \sqrt{1 - 2m(\eta; \rho_*, \rho_C) / \rho (\eta; \rho_*, \rho_C)}},
\]

for \( \eta \in [0, 1] \),

which is identically zero for \( \eta \in [0, 1] \).

In this case the Eq. (III.11a), in combination with relation (III.8), reproduces the limit \( m(\eta; \rho_*, \rho_C) \rightarrow m(\eta; 0, 0) = m_* \). Having in mind that the functions (III.10) are obtained from their HG counterparts (III.1)-(III.4) by replacing the variable \( \eta \) with \( \rho \) with the function (III.9) [1] (for example, \( m(\eta; \rho_*, \rho_C) := m^{HG}(\rho; \rho_*, \rho_C; \rho_*, \rho_C) \)) we see that the last result gives the limit \( m^{HG}(\rho; \rho_*, \rho_C) \rightarrow m^{HG}(0; \rho_0, \rho_0) = m_* \) in both limiting procedures at hand. This entails the limit \( \sqrt{-g_{pp}} \rightarrow 1/\sqrt{1 - 2m_*/\rho_0} \).

The detailed behavior of the physical parameters of the GRS under the above limiting procedures depends on EOS. For example, for incompressible GRS, in framework of general novel models, developed in [2], we obtain the following behavior of the central energy density:

\[ \varepsilon_C = \frac{3m_*}{4\pi(\rho_0^3 - \rho_C^3)} \propto \frac{m_* \rho_0}{4\pi \rho_0^3 \Delta \rho} \rightarrow \infty, \text{ (III.17)} \]

when \( \Delta \rho \rightarrow 0 \). It diverges as \( 1/\Delta \rho \), if \( \rho_C > 0 \). This is much more weak divergency than in Newtonian theory of stars, see Eq. (II.12c). Only under the widely accepted extra condition \( \rho_C = 0 \) one obtains \( \varepsilon_C \propto 1/\Delta \rho^3 \), i.e., a divergency of the same strength as in Newtonian theory of stars.

To get the corresponding limit of the central pressure \( \rho_C \) in GRS of general type, we consider the quantity \( w_C \) [2]:

\[ w_C = \frac{1}{\sqrt{-g_{pp}} - \chi(\rho_*, \rho_C)} - 1. \text{ (III.18)} \]

Using the previous results, we obtain the limit

\[ \chi(\rho_*, \rho_C) = 4\pi \varepsilon \int_{\rho_C}^{\rho_*} \rho \left(\sqrt{-g_{pp}}\right)^3 \rightarrow \frac{m_*}{\rho_0} \left(\sqrt{1 - 2m_*/\rho_0}\right)^{-3}. \text{ (III.19)} \]

Together with Eq. (III.18) and Eq. (III.8) this gives:

\[ w_C \rightarrow \left(\frac{1 - 2m_*/\rho_0}{1 - 3m_*/\rho_0}\right)^3 - 1 = (1 - \rho^2)^2 \frac{2\rho - 1}{3\rho^2 - 1}. \text{ (III.20)} \]

The last result entails the following consequences:

1) In the limits at hand the central pressure of incompressible GRS of general type \( \rho_C \rightarrow \infty \) with the same strength as \( \varepsilon_C \propto 1/\Delta \rho \), if \( m_/\rho_0 < 1/3 \), because of the finite constant limit (III.20) of \( w_C \).

2) For massive points, obtained as a limits of incompressible GRS, the compactness \( \varsigma := 2m_*/\rho_0 < 2/3 \). The mass defect ratio is to fulfill the restriction \( \varrho > 1/\sqrt{3} \).

IV. CONCLUDING REMARKS

Thus far we were able to prove mathematically the existence of point particle idealization in GR.

This confirms our previous results about solutions of EE with point particle source in energy-momentum tensor [3]. There a proper GR description of gravitational field of massive point particle was reached for the first time.

Now we see that the gravitational field of massive point particle in GR can be considered in mathematically and physically correct way as a proper limiting case of the field of finite body. Moreover, generalizing the corresponding Newtonian treatment of this problem, we see that in GR we can consider two types of such limits:

i) under fixed Keplerian mass of the finite body; and

ii) under fixed proper mass of this body.

In both cases the limit of the energy density is proportional to a Dirac \( \delta \) function and a mass defect of the resulting point particle is inherit from the finite body.

The resulting massive point has a zero radius and zero volume, but it is surrounded by a finite area. This specific unusual property is possible only in curved space-times and reflects the fact that in GR the energy distribution changes the geometry of the space-time.

In the above limit an infinite energy concentration at the place of the resulting massive point emerge. Therefore one must expect some essential deviations from the standard geometry of the 3D balls of infinitesimal radius in an Euclidean space. In GR the geometry around the massive point can not be the same as the geometry around the empty one, or around a point with finite energy density in it. This happens because in GR the energy distribution changes the very space-time geometry by construction. In the extreme case of infinite energy concentration the changes of space-time geometry must be drastic. More details about this subject the reader can find in [3]. Our present consideration gives a more physical understanding of the results, obtained in these articles.

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especially, of present article.

APPENDIX A: THE BASIC LEMMA FOR SEQUENTIAL APPROACH TO DIRAC \(\delta\)-FUNCTION

The sequential approach to mathematical distributions is well known \([8]\). It is suitable just for consideration of the point particle limit in a uniform and mathematically strict way, both in Newton gravity and in GR, starting from corresponding models of stars of finite dimension.

In this approach the distributions, like 1D Dirac function \(\delta(x-x_0)\), are defined as a weak limits of different special sequences of usual continuous functions \(f_\epsilon(x,x_0)\) which, in addition, depend on some parameter \(\epsilon \to 0\). We denote by the symbol \(\rightarrow\) the limit in a weak sense. This means, that when we are considering an arbitrary test function \(\varphi(r)\) (see \([8]\)) we will have

\[
\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} f_\epsilon(x)\varphi(x) = \varphi(x_0).
\]

Then we write down

\[
f_\epsilon(x,x_0) \rightarrow \delta(x-x_0) \text{ for } \epsilon \to 0.
\]

We were not able to find in the literature most general consideration of the class of functions \(f_\epsilon(x,x_0)\) which can be used to define the 1D Dirac \(\delta\)-function in the above sense, although a lot of specific examples are well known. Therefore we give here some general consideration, which is most suitable for our purposes.

**Basic Lemma:** Let us consider a real numbers \(x_2 > x_1\) and an arbitrary real function \(F(x;x_2,x_1)\) with the following properties:

1. \(F(x;x_2,x_1) \in \mathcal{C}^0_{[x_1,x_2]}\); i.e., \(F(x;x_2,x_1)\) is a continuous function of \(x\) on the compact interval \([x_1,x_2]\);
2. \(F(x;x_2,x_1)\) is identically zero outside this interval:
\[
F(x;x_2,x_1) \equiv 0 \text{ for } x \notin [x_1,x_2];
\]
3. \(\int_{x_1}^{x_2} dx F(x;x_2,x_1) = 1;\)

Then the weak limit of the function

\[
f_\epsilon(x,x_0) := \frac{1}{\epsilon} F\left(\frac{x-x_0}{\epsilon};x_2,x_1\right)
\]

is the Dirac \(\delta\)-function \(\delta(x-x_0)\). Here \(x_0\) is an arbitrary real number.

**Comment:**

Note that the function \(f_\epsilon(x,x_0)\) \((A.3)\) is a continuous not-identically-zero one in the interval \([x_0 + \epsilon x_1, x_0 + \epsilon x_2] \rightarrow [x_0,x_0]\). At the same time its magnitude increases to infinity, because of the factor \(1/\epsilon\).

**Proof:**

For our purposes it is enough to consider test functions \(\varphi(x) \in \mathcal{C}^0_{(-\infty,\infty)}\). Then, using the second property of the function \(F(x;x_2,x_1)\) and performing corresponding changes of variables, we obtain:

\[
\int_{-\infty}^{\infty} dx f_\epsilon(x)\varphi(x) = \int_{x_0}^{x_0+\epsilon x_2} dx \frac{1}{\epsilon} F\left(\frac{x-x_0}{\epsilon};x_2,x_1\right) \varphi(x) = \int_{x_1}^{x_2} dx F(x;x_2,x_1) \varphi(x_0 + \epsilon x).
\]

According to first property of the function \(F(x;x_2,x_1)\), when \(\epsilon \to 0\) the functions \(\psi_\epsilon(x_0,x_1) := F(x;x_2,x_1)\varphi(x_0 + \epsilon x) \in \mathcal{C}^0_{[x_1,x_2]}\) have a uniform point limit: \(\psi_\epsilon(x_0,x_1) \rightarrow F(x;x_2,x_1)\varphi(x_0)\) at every point \(x \in [x_1,x_2]\). This allows us to commute the limit \(\epsilon \to 0\) and the integration. Thus, taking into account the third property of the function \(F(x;x_2,x_1)\), we obtain:

\[
\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} dx f_\epsilon(x)\varphi(x) = \int_{x_1}^{x_2} dx \lim_{\epsilon \to 0} \left( F(x;x_2,x_1) \varphi(x_0 + \epsilon x) \right) = \left( \int_{x_1}^{x_2} dx F(x;x_2,x_1) \right) \varphi(0) = \varphi(0).
\]

**Corollary:** For function \(F(x;x_2,x_1)\) with the properties 1-3 in the formulation of the Basic Lemma, and a second function \(G(x) \in \mathcal{C}^0_{[x_1,x_2]}\), with arbitrary properties outside the interval \([x_1,x_2]\), we have

\[
f_\epsilon(x_0)G_\epsilon(x_0) \rightarrow <G>F \delta(x-x_0) \text{ for } \epsilon \to 0,
\]

where \(f_\epsilon(x_0) := \frac{1}{\epsilon} F\left(\frac{x-x_0}{\epsilon};x_2,x_1\right), G_\epsilon(x) := G\left(\frac{x-x_0}{\epsilon}\right)\) and

\[
<G>F = \int_{x_1}^{x_2} dx F(x;x_2,x_1)G(x).
\]

The proof is a simple repetition of the above proof of the Basic Lemma, after inclusion of the function \(G(x)\) as a multiplier of function \(F(x;x_2,x_1)\) in all considerations.

In particular, this corollary permits us to retire the property 3 of the function \(F(x;x_2,x_1)\) and to formulate a more general result for functions without such property.

**Note:**

One more generalization of the Basic Lemma in other direction is needed for our purposes.

We will have the same result and the same proof, if we consider instead of function \(F(x;x_2,x_1) \in \mathcal{C}^0_{[x_1,x_2]}\) a more general one \(F(x;y_1,y_2,...,x_2,x_1) \in \mathcal{C}^0_{[x_1,x_2] \times \mathbb{R}_{y_1} \times \mathbb{R}_{y_2} \times ...}\), i.e., a continuous function of many variables \(x_1,y_1,y_2,...\), and suppose that in it \(y_1 = y_1(\epsilon), y_2 = y_2(\epsilon), \ldots\) are \(\mathcal{C}^0_{\mathbb{R}}\) functions of the variable \(\epsilon\) with definite limits \(y_1(0), y_2(0), \ldots\), when \(\epsilon \to 0\). In addition, it is supposed that the function \(F(x;y_1(\epsilon),y_2(\epsilon),...,x_2,x_1)\) owns the properties 2 and 3 for any small enough \(\epsilon\).
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[9] According to our general scheme, in present series of articles [1, 2], as well as in [3], we are considering the static spherically symmetric solutions of EE as one-dimensional problem. If one wishes to return back to the 3D form of the corresponding results, one has to replace the 1D Dirac \( \delta \)-function \( \delta(r) \) with 3D one: \( \delta^{(3)}(r) \). Formally, the last corresponds to the expression \( \frac{1}{4 \pi r^2} \delta(r) \) in our 1D considerations. As a result, in full 3D notations we obtain
\[
\varepsilon(r) \rightarrow m_\ast \delta^{(3)}(r)
\]
both in Newtonian and in GR theory of gravity.