Generalized Path Dependent Representations for Gauge Theories

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Abstract

A set of differential operators acting by continuous deformations on path dependent functionals of open and closed curves is introduced. Geometrically, these path operators are interpreted as infinitesimal generators of curves in the base manifold of the gauge theory. They furnish a representation with the action of the group of loops having a fundamental role. We show that the path derivative, which is covariant by construction, satisfies the Ricci and Bianchi identities. Also, we provide a geometrical derivation of covariant Taylor expansions based on particular deformations of open curves. The formalism includes, as special cases, other path dependent operators such as end point derivatives and area derivatives.
1 Introduction

The importance attributed to the notion of path dependence goes back to Dirac’s work on the non-integrability of the phase exhibited by wave functionals in the presence of electromagnetic fields in Quantum Mechanics [1]. Since then, the idea of incorporating path dependence in gauge formulations, including gravity, has been explored and studied in various physical contexts [2, 3, 5, 8, 10]. In the framework of gauge theory it was originally Mandelstam who elaborate a gauge invariant formulation of electrodynamics coupled to a scalar field with a definite open path dependence [3]. Extensions for the non-Abelian case were developed in [4]. The global and geometrical aspects of non abelian gauge theories were highlighted in the integral formulation of Yang [5] and Wu-Yang [6]. Moreover, the loop representation [10, 12] based on closed curves is one essential tool in the loop quantum gravity approach.

However, and in spite of the advantages that may present path dependence in gauge theories we believe there is still a gap to appropriately understand in which manner the existent path dependent operators are related. The tendency until now has been to adapt each definition of path derivative to a specific domain of considerations which are believed to be relevant for the construction. In general all these considerations result to be different [13]. Efforts to study end point derivatives and area derivatives from a rigorous mathematical viewpoint had been carried out in [13], however a different definition for both path derivatives is formulated. Several definitions of path dependent operators can be found in [7, 8, 9, 15, 16]. The definition of path dependent operators had been made essentially depending on:

i) The space where path dependent functionals take values is either the space of open or closed curves and if the space includes base points or not.

ii) The nature of the variation is due to a point or many points, which have been usually called end point derivatives and area derivatives respectively.

iii) The place where the variation is appended, is on the curve or in other place on the manifold.

In this paper, we address the three points above. We introduce a covariant path derivative acting by continuous deformations on a general class of open and closed curves. We show that the definition of path derivative generalizes all types of end points and area derivatives. Geometrically, the path derivative is an infinitesimal generator of curves which under some assumptions enables a representation with the action of the group of loops deeply involved. We show that the path derivative satisfies the Ricci and Bianchi identities. The work is organized as follows. In section 2 we define the path derivative emphasizing the role of the group of loops. Next, in the third section we compare the direct consequences of the action of the path operator with well known equations, this amounts to compute the path derivative of phase factors and scalar fields. Area and end point derivatives are identified and related therein. In section 4 we calculate the finite variation of a functional when its argument is changed by successive infinitesimal deformations. This change may be interpreted through the action of the group of loops on arbitrary paths which we represent by a set of differential operators.
In section 5 we use the path derivative to obtain covariant Taylor series. These series arise when using particular deformations which do not enclose area and collapse to a point in the original curve, so only open curves can be considered.

Finally we discuss some aspects related to the loop derivative defined in [10]. The loop derivative emerges in our approach when we restrict to spatial curves and to deformations with end points fixed.

2 The Path Derivative

The gauge theory is introduced by considering the principal fiber \( P(G, M) \) with a one form connection \( A \) valued in the algebra of the gauge group \( G \). We will denote by \( \Gamma(M) \) the space of all open and closed smooth curves in the base manifold \( M \) equipped with the usual properties of path composition [11]. Let us consider a general class of path-dependent matrix functionals \( \Psi \) taking values in the space \( \Gamma(M) \) and transforming covariantly under the gauge group \( G \).

We define the path derivative of the functional \( \Psi(\alpha) \) for a given path \( \alpha \in \Gamma(M) \) by

\[
\mathcal{D}\Psi(\alpha) = \Delta \Psi(\alpha) - \Psi(\alpha),
\]

such that the action \( \Delta : \Psi(\alpha) \rightarrow \Psi'(\alpha') \), is to displace infinitesimally and continuously the initial curve \( \alpha \) to a deformed curve \( \alpha' \) with some transforming action on \( \Psi \) specified below. To begin the construction we consider the same parametrization for the curves \( \alpha(\sigma) \) and \( \alpha'(\sigma) \) with \( \sigma \in [0,1] \), and corresponding end points \( (x,y) \) and \( (x',y') \) as indicated in Fig 1. Also, let us represent the trajectories followed by the points of the initial curve \( \alpha(\sigma) \) along the deformation as the family of diffeomorphism defined by \( x^\mu(\sigma, t) = x^\mu(\sigma) + \delta x^\mu(\sigma, t) \) and parameterized with \( t \in [0,1] \). The initial and final curves being respectively, \( x^\mu(\sigma,0) = x^\mu(\sigma) \) and \( x^\mu(\sigma,1) = x'^\mu(\sigma) \).

We adopt the view that the operator \( \mathcal{D} \) generates a vector field \( \vec{N}(\sigma,t) \) with

\[
N^\mu(\sigma,t) = \frac{\partial x^\mu(\sigma,t)}{\partial t},
\]

and where \( x^\mu(\sigma,t) \) are the integral curves associated to the deformation of \( \alpha(\sigma) \). With this in mind \( \mathcal{D} \) will be denoted alternatively by \( \mathcal{D}(N) \). We assume a transformation of a matrix functional \( \Psi_{AB}(\alpha) \) under the action of the deformation by

\[
\Delta_t \Psi_{AB}(\alpha) = U_A^A(\delta y^{-1}) \Psi_{A'B'}(\alpha') U_B^{B'}(\delta x),
\]

where the elements \( U_B^{B'}(\delta x) \) and \( U_A^A(\delta y^{-1}) \) transform internal indices \( \{A,B,...\} \) to one system to another and are functions of the paths which connect the end points of both curves \( \alpha \) and \( \alpha' \), see Fig 1. The notation introduced is \( \delta x(0,t) = \delta x \) and \( \delta y^{-1} \) for \( \delta x(1,t) = \delta y \) traversed in the opposite direction. To ensure covariance in the definition (2.1) we define the \( U \) objects to transform covariantly and for the property
Figure 1: Deformation of the curve $\alpha \rightarrow \alpha'$. 

$\Delta t \Delta t' = \Delta_{t \rightarrow t'}$ to hold, their composition rules

$$
U_A^{\alpha'}(\delta x') U_A^\alpha(\delta x) = U_A^{\alpha''}(\delta x \circ \delta x'),
$$

$$
U_B^{\beta'}(\delta y') U_B^{\beta}(\delta y) = U_B^{\beta''}(\delta y \circ \delta y'),
$$

which is the reflection in functional space of the geometric property of path composition. This suggest to identify $U$ with ordered phase factors of the local connection $A$.

Phase factors are defined for a given path $\gamma$,

$$
U(\gamma) = \mathcal{P}_s \left( \exp \int -A_\mu(s) \frac{d\gamma^\mu(s)}{ds} ds \right),
$$

such that $\mathcal{P}_s$ means path ordered. The usual composition properties are,

$$
U(\gamma') U(\gamma'') = U(\gamma' \circ \gamma''),
$$

$$
U(\gamma) U(\gamma^{-1}) = 1,
$$

where the path composition $\gamma = \gamma' \circ \gamma''$ is to be read with the convention that takes $\gamma'$ followed by $\gamma''$. Now, the phase factors in (2.3) approximated to the first order in $N(\sigma, t)$ are

$$
U(\delta x) = 1 - \int_0^1 dt \, N^\mu(0, t) A_\mu(x(t)),
$$

$$
U(\delta y^{-1}) = 1 + \int_0^1 dt \, N^\mu(1, t) A_\mu(y(t)).
$$

From (2.8), (2.3) and (2.1) we write for any matrix path dependent quantity $\Psi_{y,x}$ connecting the points $x$ and $y$, not necessarily distinct,

$$
\delta \Psi_{y,x} = \Psi_{y,x} \left[ \int_0^1 N^\mu(0, t) A_\mu(x(t)) \right] - \int_0^1 N^\mu(1, t) A_\mu(y(t)) \Psi_{y,x}
$$

$$
+ \int_0^1 \int_0^1 d\sigma dt \, N^\mu(\sigma, t) \mathcal{D}_\mu(\sigma, t) \Psi_{y,x},
$$

(2.9)
where \( \delta \Psi = \Psi(\alpha') - \Psi(\alpha) \) and the definition of the functional derivative \( D_\mu(\sigma, t) \) is through the expression

\[
D(N) = \int_0^1 \int_0^1 d\sigma dt N^\mu(\sigma, t) D_\mu(\sigma, t).
\] (2.10)

Note that for the particular choice \( x^\mu(\sigma, t) = x^\mu(\sigma) + t \delta x^\mu(\sigma) \) which considers straight paths connecting both curves, the equation (2.9) reduces to one obtained in reference [16]. It is the generality of equation (2.1) together with the geometrical and global character of (2.3) that allow for the generalization introduced here and the use of the group of loops in the underlying structure of equations exhibited further.

Now, for the curve deformation that just moves one point along a straight line, which we call a point deformation, we define, for example for the end point \( y \),

\[
D_\delta y \Psi_{y,x} = \delta \Psi_{y,x} + \delta y^\mu A_\mu(y) \Psi_{y,x},
\] (2.11)

analogously for the initial point \( x \)

\[
D_\delta x \Psi_{y,x} = \delta \Psi_{y,x} - \Psi_{y,x} \delta x^\mu A_\mu(x).
\] (2.12)

And for the curve deformation with \( x \) and \( y \) fixed but that encloses some area we define the loop deformation by

\[
D_L \Psi_{y,x} = \delta \Psi_{y,x}.
\] (2.13)

As mentioned before the construction can be understood in terms of the action of the group of loops \( L \) on arbitrary paths \( \gamma \) belonging to \( \Gamma(M) \). The action defined by usual path composition. Let us consider the same path \( \alpha(\sigma) \) as before and focus on the loop \( l = \delta x o \alpha' o \delta y^{-1} o \alpha^{-1} \) with composition \( l o \alpha = \delta x o \alpha' o \delta y^{-1} \). Also, the variation of a functional \( \Delta \Psi(\alpha) \) will be represented by an operator \( U(l) \) with \( l \in L \) as,

\[
\Psi(l o \alpha) = U(l) \Psi(\alpha),
\] (2.14)

and therefore given (2.3) we have

\[
\Psi(\alpha') = U(\delta y) \left[ U(l) \Psi(\alpha) \right] U(\delta x^{-1}).
\] (2.15)

In the next sections we justify eq (2.14) and we give a precise meaning to the operator \( U(l) \) in terms of the operator \( D \).

Moreover, we see that the expression

\[
\Psi(l o \alpha) = U(\delta y^{-1}) \Psi(\alpha') U(\delta x),
\] (2.16)

behaves as expected with respect to the composition of loops. To see this, consider the action of the two loops \( l_1 = \delta x_1 o \alpha' o \delta y_1^{-1} o \alpha^{-1} \) and \( l_2 = \delta x_2 o \delta x_1 o \alpha'' o \delta y_2^{-1} o \delta y_1^{-1} o \alpha'^{-1} \) on \( \alpha \) which gives \( l_2 o l_1 o \alpha = \delta x_1 o \delta x_2 o \alpha'' o \delta y_1^{-1} o \delta y_2^{-1} \). Thus eq (2.16) is with respect to the composition of loops

\[
\Psi(l_2 o l_1 o \alpha) = U(\delta y_1^{-1} o \delta y_2^{-1}) \Psi(\alpha'') U(\delta x_1 o \delta x_2),
\] (2.17)

where (2.4) and (2.7) have been used.
3 Covariant Differentiation of Gauge Objects

Here we compute the action of the path derivative on variables arising in gauge theories such as phase factors, path dependent matter fields and local gauge fields. At the end of the section we provide a relation between the path derivative introduced here and the so called end point and area derivatives.

Let us consider the ordered phase factor of the same path as before \( \alpha(\sigma) \),

\[
U_{y,x}(\alpha) = \mathcal{P}_{\sigma} \left( \exp \int_{0}^{1} -A_\mu(\sigma) \frac{d\alpha^\mu(\sigma)}{d\sigma} \, d\sigma \right), \tag{3.18}
\]

from definition (2.1) and (2.3),

\[
\mathcal{D}U(\alpha) = U(\delta y^{-1}) U(\alpha') U(\delta x) - U(\alpha). \tag{3.19}
\]

We partition the paths \( \alpha \) and \( \alpha' \) in \( N \) segments. Each segment \( \alpha_{i+1,i} \) defines a phase factor joining the points \( x_i = x(\sigma_i) \) and \( x_{i+1} = x(\sigma_{i+1}) \), similarly for \( \alpha' \). By the composition property of phase factors we have

\[
U(\alpha') = \prod_{i=0}^{N} U(\alpha'_{i+1,i}) = U'_{N+1,N} \cdots U'_{1,0}, \tag{3.20}
\]

and considering each segment \( \alpha'_{i+1,i} = \delta x_i^{-1} o l_i o \alpha_i o \delta x_{i+1} \) in terms of the loop \( l_i = \delta x_i o \alpha' o \delta x_{i+1}^{-1} o \alpha^{-1} \) we may write

\[
U(\alpha'_{i+1,i}) = U(\delta x_{i+1}) U(\alpha_{i+1,i}) U(l_i) U(\delta x_i^{-1}). \tag{3.21}
\]

Therefore from eqs (3.21) and (3.20), \( \mathcal{D}U(\alpha) \) is written as a product of paths given by

\[
\mathcal{D}U(\alpha) = \prod_{i=0}^{N} U(\alpha_{i+1,i}) H(x_i) - U(\alpha), \tag{3.22}
\]

where we have replaced \( U(l_i) \) by the holonomy \( H(x_i) \) evaluated along the line \( x_i = x(\sigma_i,t) \). Now, \( H(x_i) \) can be written using the non abelian Stokes theorem to lowest order as

\[
H(x_i) = 1 - \int_{0}^{1} \mathcal{F}_{\mu\nu}(x_i) N^\mu(\sigma_i,t) \frac{\partial x_i^\nu}{\partial \sigma_i} d\sigma_i dt, \tag{3.23}
\]

where \( \mathcal{F}_{\mu\nu}(x_i) = U(\delta x_i) F_{\mu\nu}(x_i) U(\delta x_i^{-1}) \) is the parallel transported curvature, see [17, 20]. Replacing, we have

\[
\mathcal{D}(N) U(\alpha) = - \int_{0}^{1} \sum_{i=0}^{N} U_{i+1,i} \mathcal{F}_{\mu\nu}(x_i) N^\mu(\sigma_i,t) \frac{\partial x_i^\nu}{\partial \sigma_i} d\sigma_i dt. \tag{3.24}
\]
The continuum limit of the above equation gives
\[ D(N) U(\alpha) = - \int_0^1 dt \int_0^1 d\sigma U_{y,x(\sigma,t)} F_{\mu\nu}(x(\sigma,t)) U_{x(\sigma,t),x} N^\mu(\sigma,t) \frac{\partial x^\nu(\sigma,t)}{\partial \sigma}. \tag{3.25} \]

It is easy to show that the result of applying the path derivative on phase factors gives the usual covariant derivative on the path introduced in \cite{16} if both curves are connected by straight line segments. From the definition (2.10) we read
\[ D_\mu(\sigma,t) U(\alpha) = - U_{y,x(\sigma,t)} F_{\mu\nu}(x(\sigma,t)) U_{x(\sigma,t),x} \frac{\partial x^\nu(\sigma,t)}{\partial \sigma}. \tag{3.26} \]

It is straightforward to compute the point deformation of the phase factor \( U(\gamma) \) that changes \( z \) to \( z + \delta z \). This is
\[ D_{\delta z} U_{AB}(\gamma) = 0, \tag{3.27} \]
which is consequence of using the retracing property \cite{7}. Analogously, the point deformation on \( x \) of the path dependent scalar field defined by \( \Phi_{AB}(\gamma,x) = U_{AC}(\gamma) \phi_{CB}(x) \) is
\[ D_{\delta x} \Phi_{AB}(\gamma,x) = U_{AC}(\gamma) \left( U_{CD}(\delta x^{-1}) \phi_{DF}(x') U_{E'B}(\delta x) - \phi_{CB}(x) \right) \]
\[ = U_{AC}(\gamma) \delta x^\mu \nabla_\mu \phi_{CB}(x), \tag{3.28} \]
where \( \nabla_\mu = \partial_\mu + [A_\mu, \ ] \) is the usual covariant derivative defined on point functions. We emphasize that although the action of \( D_{\delta x} \) is the same of the end point derivative on path dependent scalar fields, their definitions are formulated rather different.

In the same way the action of the derivative on the local gauge field \( A_\mu(x) \) is
\[ (D_{\delta x} A_\mu)_{AB} = (\delta x^\alpha \partial_\alpha A_\mu + \delta x^\alpha [A_\mu, A_\alpha] )_{AB}, \tag{3.29} \]
where we have considered an arbitrary path contracted to the point \( x \). From here we have
\[ \delta x^\mu D_{\delta x} A_\mu(x) = \delta x^\mu \delta x^\alpha \partial_\alpha A_\mu(x), \tag{3.30} \]
which is just the directional derivative of the gauge field.

4 The Path Derivative as the Generator of Curves

Eq (2.14) defines a representation based on differential operators associated to a family of deformed curves. Let us proceed to explicitly calculate these operators and relate them to the action of the group of loops.

Consider the functional \( \Psi(\alpha(\sigma)) \) and a finite variation of the path \( \alpha(\sigma) \to \alpha'(\sigma) \). Using the same notation as before we introduce a family of deformed curves \( \alpha_t(\sigma) \)
parametrized with \( t \in [0, 1] \). For an \( N \) partition of the \( t \)-interval \([0, 1]\) the infinitesimal deformations are

\[
\Psi(\alpha_{n+1}) = \Psi(\alpha_n) + \Psi(\alpha_n)A_{x_n} - A_{y_n}\Psi(\alpha_n) + D(N_n)\Psi(\alpha_n),
\]

(4.31)

with the definitions,

\[
A_{x_n} = \int_{t_n}^{t_{n+1}} dt N^\mu(0, t) A_\mu(x(t)),
\]

(4.32)

\[
A_{y_n} = \int_{t_n}^{t_{n+1}} dt N^\mu(1, t) A_\mu(y(t)),
\]

(4.33)

\[
D(N_n) = \int_{t_n}^{t_{n+1}} dt \int_0^1 d\sigma N^\mu(\sigma, t) D_\mu(\sigma, t),
\]

(4.34)

and \( \alpha_n = \alpha_{t_n}(\sigma) \) together with \( x = \alpha_1(0) \), \( x' = \alpha_N(0) \), and \( y = \alpha_1(1) \), \( y' = \alpha_N(0) \).

Iterating \( n \) times the equation (4.31) and using the identities

\[
U_{x'x} = \lim_{N \to \infty} (1 - A_{x_N})(1 - A_{x_{N-1}}) \ldots (1 - A_{x_1}),
\]

(4.35)

\[
U^{-1}_{x'x} = \lim_{N \to \infty} (1 + A_{x_1}) \ldots (1 + A_{x_{N-1}})(1 + A_{x_N}),
\]

(4.36)

and

\[
\int_0^1 dt \int_t^t' f(t, t') + \int_0^1 dt' \int_t^{t'} dt f(t, t') = \int_0^1 dt \int_0^1 dt' f(t, t'),
\]

(4.37)

it can be shown by taking the limit \( n \to \infty \) that,

\[
\Psi(\alpha') = U(\alpha(1)) \left[ P_t \left( \exp \int_0^1 dt D_t \right) \Psi(\alpha) \right] U(\alpha^{-1}(0)),
\]

(4.38)

where,

\[
P_t \left( \exp \int_0^1 dt D_t \right) \Psi(\alpha) = \Psi(\alpha) + \int_0^1 dt D_t \Psi(\alpha) + \int_0^1 dt_1 \int_0^{t_1} dt_2 D_{t_1}D_{t_2} \Psi(\alpha) + \ldots,
\]

(4.39)

and where the notation is

\[
U(\alpha(\sigma)) = P_t \left( \exp \int_0^1 dt D_t \right) N^\mu(\sigma, t) dt,
\]

(4.40)

\[
D_{t_n} = \int_0^1 d\sigma N^\mu(\sigma, t_n) D_\mu(\sigma, t_n),
\]

(4.41)
Thus comparing (4.38) with (2.15) we identify

$$U(l) = \mathcal{P}_t \left( \exp \int_0^1 dt \mathcal{D}_t \right),$$

(4.42)

which tell us that each deformation may be associated to the loop described in the previous section. Furthermore, it can be shown that when one considers only spatial curves and their loop deformations the operator $D_L$ is the loop derivative $\Delta$ introduced in [10, 11], since both are curves generators.

To derive the Bianchi identity let us perform six consecutive loop deformations on an arbitrary curve $\gamma$. We consider the loop deformations along the edges $\delta x_1, \delta x_2, \delta x_3$ of a parallelepiped. The functional $\Psi$ of the curve $\gamma$ is affected by

$$\Psi(\gamma) = \Psi \left( \prod_{n=1}^6 L_n \circ \gamma \right) = \prod_{n=1}^6 U(L_n) \Psi(\gamma).$$

(4.43)

From (2.13) we may write

$$\Psi(\gamma) = \prod_{n=1}^6 U(D_{L_n}) \Psi(\gamma),$$

(4.44)

with

$$L_1 = \delta x_1 \delta x_2 \delta x_1^{-1} \delta x_2^{-1},$$
$$L_2 = \delta x_3 \delta x_1 \delta x_3^{-1} \delta x_1^{-1},$$
$$L_3 = \delta x_2 \delta x_3 \delta x_2^{-1} \delta x_3^{-1},$$
$$L_4 = \delta x_1 o (\delta x_3 \delta x_2 \delta x_3^{-1} \delta x_2^{-1}) o \delta x_1^{-1},$$
$$L_5 = \delta x_2 o (\delta x_1 \delta x_3 \delta x_1^{-1} \delta x_3^{-1}) o \delta x_2^{-1},$$
$$L_6 = \delta x_3 o (\delta x_2 \delta x_1 \delta x_2^{-1} \delta x_1^{-1}) o \delta x_3^{-1}.$$  

(4.45)

But first, an intermediate step is required. We need the calculation of the loop deformation $D_L$ along the edges $\delta x_1, \delta x_2$ of a rectangle with one vertex in contact with the curve $\gamma$. Thus, expanding the four point deformations that produces the single loop deformation, to second order in the segments we have

$$U(D_L) \Psi(\gamma) = (1 + [D_{\delta x_1}, D_{\delta x_2}]) \Psi(\gamma).$$

(4.46)

And by expanding the left side of eq (4.46) we obtain $D_L = [D_{\delta x_1}, D_{\delta x_2}]$ which is the operator version of the Ricci identity. We also need the same loop deformation when one vertex of the rectangle is connected to the curve by a retraced segment $\delta x$. It is easy to show that

$$U(D_{\delta x^{-1}} L o \delta x) \Psi(\gamma) = (1 + U_{\delta x}^{-1} [D_{\delta x_1}, D_{\delta x_2}] U_{\delta x}) \Psi(\gamma).$$

(4.47)
Using these expressions, the right side of eq (4.44) turns to be
\[ \prod_{n=1}^{6} U(D_{\varepsilon_n}) \Psi(\alpha) = \left( 1 - [D_{\delta x_1}, D_{\delta x_2}] \right) \times \left( 1 - [D_{\delta x_1}, D_{\delta x_3}] \right) \times \left( 1 - [D_{\delta x_2}, D_{\delta x_3}] \right) \times \left( 1 + U_{\delta x_1}^{-1} [D_{\delta x_1}, D_{\delta x_2}] U_{\delta x_3} \right) \times \left( 1 + U_{\delta x_2}^{-1} [D_{\delta x_1}, D_{\delta x_3}] U_{\delta x_2} \right) \times \left( 1 + U_{\delta x_3}^{-1} [D_{\delta x_2}, D_{\delta x_3}] U_{\delta x_1} \right) \Psi(\alpha), \]
and therefore we obtain
\[ \Psi(\alpha) = \left( 1 + D_{\delta x_1} [D_{\delta x_2}, D_{\delta x_3}] + D_{\delta x_3} [D_{\delta x_1}, D_{\delta x_2}] + D_{\delta x_2} [D_{\delta x_3}, D_{\delta x_1}] \right) \Psi(\alpha), \]
which gives the Bianchi identity for the point deformation \( D_\mu(\sigma) \)
\[ D_\alpha [D_\mu, D_\nu] + D_\mu [D_\nu, D_\alpha] + D_\nu [D_\alpha, D_\mu] = 0, \]
recall the expression (2.10).

5 Applications of the Path Dependent Formalism

In this section, the path dependent formalism is used to deduce the covariant Taylor expansions for non Abelian fields. These expansions were developed as part of a method of calculation to found the effective action in quantum field theories [18]. Generalizations that consider supersymmetric expansions are given in [19]. In the derivation of these series we follow a geometrical approach based on the deformation of open curves instead of using local operators an in the standard method. Thus, in order to compare both methods we review in the first subsection the standard method and in the second the path dependent formalism. This geometrical framework could be a starting point to explore expansions on curved base manifolds and where singularities are present.

5.1 Standard Derivation

We follow the original method [18], which we extend here to accommodate besides gravitational covariant derivatives also gauge covariant ones.

Let us define the covariant derivative along the path \( x(t) \) given by the operator
\[ \frac{D}{dt} = \dot{x}^\mu(t) D_\mu, \]
where \( D_\mu \) is the usual covariant derivative; for a gauge theory we just have to lift the curve \( x(t) \) to the bundle. Let us write the parallel transport equation for the geodesic curve \( x(t) \) that connects the points \( x_1 \) and \( x_2 \),
\[ \frac{D \dot{x}^{\nu}}{dt} = \dot{x}^\mu D_\mu \dot{x}^{\nu}(t) = 0. \]
Using (5.52), for the scalar function \( f(x(t)) \) one has for all \( n \),

\[
\frac{d^n f(x(t))}{dt^n} = [D_{\nu_n} \ldots D_{\nu_1} f(x)]_{x=x(t)} \dot{x}^{\nu_1} \ldots \dot{x}^{\nu_n}. \tag{5.53}
\]

Then, considering the expansion of \( f(x(t)) \)

\[
f(x(t_2)) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \frac{d^n f(x(t))}{dt^n} \right]_{t=t_1} (t_2 - t_1)^n, \tag{5.54}
\]

and defining,

\[
\sigma^\mu(x_1, x_2) = (t_2 - t_1) \left[ \frac{dx^\mu(t)}{dt} \right]_{t=t_1}, \tag{5.55}
\]

we arrive to the expression,

\[
f(x_2) = \sum_{n=0}^{\infty} \frac{1}{n!} \sigma^{\nu_1}(x_1, x_2) \ldots \sigma^{\nu_n}(x_1, x_2) D_{\nu_n} \ldots D_{\nu_1} f(x_1). \tag{5.56}
\]

In order to obtain the covariant Taylor series we consider the field composed with the two parallel propagators as \( U(x', x) \phi(x) U(x, x') \). Since the composition behaves as a scalar with respect to the point \( x \) we can apply the expansion (5.56),

\[
U(x'', x') \phi(x') U(x', x'') = \sum_{n=0}^{\infty} \frac{1}{n!} \sigma^{\nu_1}(x, x') \ldots \sigma^{\nu_n}(x, x') D_{\nu_n} \ldots D_{\nu_1} \times U(x'', x') \phi(x) U(x, x''). \tag{5.57}
\]

Now, we need the identity

\[
\sigma^{\nu_1}(x, x') \ldots \sigma^{\nu_n}(x, x') D_{\nu_n} \ldots D_{\nu_1} U(x', x) = 0, \tag{5.58}
\]

that results by using \( \sigma^\mu D_\mu \sigma^\nu = \sigma^\nu \) and making the operator \( \sigma^\mu D_\mu \) act successively on

\[
\sigma^\mu D_\mu U(x', x) = 0, \tag{5.59}
\]

We finally obtain the covariant Taylor series for the field \( \phi(x) \), taking \( x'' = x' \) and multiplying by \( U^{-1}(x, x') \) and \( U^{-1}(x', x) \),

\[
U(x, x') \phi(x') U(x', x) = \sum_{n=0}^{\infty} \frac{1}{n!} \sigma^{\nu_1}(x, x') \ldots \sigma^{\nu_n}(x, x') D_{\nu_n} \ldots D_{\nu_1} \phi(x). \tag{5.60}
\]
5.2 Geometrical Derivation

The deduction in the geometrical approach is rather simple. This consists in performing a point deformation on a path dependent field while taking the limit where the curve collapse to a point. To illustrate the idea let us consider them in two separately deformations, both acting on the path-dependent field $\Psi(\gamma)$ defined to be the composition of the path propagator $U(\gamma)$ and the insertion of the field $\phi(x)$ as,

$$\Psi(\gamma) = U(\gamma_2)\phi(x)U(\gamma_1),$$

where $\gamma_1$ and $\gamma_2$ are the curves left in both sides of the insertion, see Fig 2. For the first deformation, drawn in dotted lines, we leave the end points $x_1$ and $x_2$ fixed, so we write

$$\Psi(\gamma') = U(D)\Psi(\gamma).$$

In the next deformation we carry the end points over the paths of $\gamma_1$ and $\gamma_2$ to coincide on the point $x$. This last deformation produces the curve $\gamma''$ made of the two straight line segments connecting $x$ to $x'$ and back. This way the next deformation can be represented by

$$\Psi(\gamma'') = U(\gamma_2^{-1})[U(D')U(\gamma')U(\gamma_1^{-1})],$$

and from (5.62) we have,

$$\Psi(\gamma'') = U(\gamma_2^{-1})[U(D')U(D)\Psi(\gamma)]U(\gamma_1^{-1}).$$

Since the effect of the two deformations can be regarded as a point deformation $D_{\delta x}$, we can write

$$\Psi(\gamma'') = U(\gamma_2^{-1})[U(D_{\delta x})\Psi(\gamma)]U(\gamma_1^{-1}).$$

Therefore using the expression (3.28), we obtain covariant Taylor expansions for a parallel transported field along a straight line segment $\delta x = x' - x$

$$U(x, x')\phi(x')U(x', x) = \sum_{n=0}^{\infty} \frac{1}{n!}\delta x^{\nu_1} \ldots \delta x^{\nu_n} D_{\nu_1} \ldots D_{\nu_n} \phi(x),$$

which coincides with (5.60), identifying $\sigma^\nu(x, x')$ with $\delta x^\nu$. 

Figure 2: Two deformations of the curve $\gamma$. 

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6 Conclusions

We have defined a path dependent operator in gauge theory, which is covariant by construction, and acts by continuous deformations on the space of smooth curves $\Gamma(M)$. The path operator may be seen as generating a vector field associated to the deformation of a given curve. We have adapted the deformation to a one-parameter family of diffeomorphism which allows us to define the path operator manifestly independent of coordinates. Although different in nature, since these diffeomorphisms drag many points at the same time, we found a close analogy with the Lie derivative introduced in general relativity. Therefore, it is in order to check whether the path operator $\mathcal{D}(N)$ belongs to the algebra of diffeomorphism or not. When we restrict to a sector of $\Gamma(M)$ containing only spatial curves, their loop deformations (end points fixed) are simply the loop derivatives defined in [10, 11]. We have also established a clear relation between the path derivative introduced here and the area and end point derivative. This has been done by comparing the action of the path derivative with well-known equations involving phase factors and path dependent scalar fields. We have calculated the finite variation of a functional when its argument is changed by successive infinitesimal deformations. This change has been interpreted through the action of the group of loops on arbitrary paths, which we have represented by the action of the covariant path operators. Geometrically, the path operator is identified with an infinitesimal generator of curves. Ricci and Bianchi identities have been obtained for loop deformations along the edges of a rectangle. We have deduced covariant Taylor expansions for non Abelian fields by considering the deformation of open curves. The reason why closed curves cannot be used in the derivation is because there is no way to deform closed curves to points without enclosing area, which would have produced loop deformations instead of point deformations and local covariant derivatives. Another feature of the derivation is the fact that we have used a deformation that moves all the points of the original curve not considered by other path operators.

It is important to note the close analogies found with the approach [10], not only the identification of the loop derivative as a particular case of deformation, but also the role of the group of loops as the fundamental geometrical structure underlying the gauge theory. An immediate main difference with this approach, besides the space dimension, is the global action of the path derivative. The path derivative action, allows us to deform the entire curve which is not possible using the loop derivative, since its more abstract action consist in attaching infinitesimal loops on curves.

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