Quantum Stabilizer Codes and Classical Linear Codes

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Abstract

We show that within any quantum stabilizer code there lurks a classical binary linear code with similar error-correcting capabilities, thereby demonstrating new connections between quantum codes and classical codes. Using this result—which applies to degenerate as well as nondegenerate codes—previously established necessary conditions for classical linear codes can be easily translated into necessary conditions for quantum stabilizer codes. Examples of specific consequences are: for a quantum channel subject to a δ-fraction of errors, the best asymptotic capacity attainable by any stabilizer code cannot exceed $H\left(\frac{1}{2} + \sqrt{2\delta(1-2\delta)}\right)$; and, for the depolarizing channel with fidelity parameter δ, the best asymptotic capacity attainable by any stabilizer code cannot exceed $1 - H(\delta)$.

89.80.+h, 03.65.Bz

Typeset using REVTEX

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The theory of error-correcting codes for classical information has been extensively studied for almost fifty years. A fundamental question in coding theory concerns what capacity of information can be successfully transmitted through a noisy channel. Call a code that maps $k$-bit inputs into $n$-bit codewords an $(n, k)$ code, and define its capacity to be $k/n$. For any specific $n$-bit channel, let its capacity be the maximum capacity of all $(n, k)$ codes that successfully transmit information through it. This capacity is often taken as an asymptotic limit as $n$ tends towards infinity.

We consider two basic kinds of $n$-bit noisy channels. The first is one that flips any subset of up to $t$ bits of each codeword that passes through it. In this model, transmission through the channel is considered successful if the $k$ bits of data can always be perfectly recovered. Call a code that achieves this a $t$-error-correcting code. Asymptotically, it is natural to take $t$ as some fixed fraction $\delta$ of $n$, written as $\delta n$ and understood to mean $\lfloor \delta n \rfloor$. A second model of a noisy channel is one where each bit of each codeword that passes through it is flipped independently with probability $\delta$. In this model, commonly referred to as the binary symmetric channel, there is no absolute bound on the number of possible errors that occur. Therefore, a probabilistic definition of successful transmission is required. Call a code for which the probability of successful recovery for any $k$ bits of data is at least $1 - \varepsilon$ an $(\varepsilon, \delta)$-error-correcting code.

The subject of error-correcting codes for quantum information is much younger, developing within the past couple of years, though it has received considerable attention during this time [1–14]. Much of the above terminology extends naturally to quantum information by considering qubits instead of bits. Call a quantum code that maps $k$-qubit data to $n$-qubit codewords an $(n, k)$ code. We need to specify the behavior of noisy quantum channels. A natural quantum analogue of the first model is to allow any $t$ qubits of each codeword that passes through it to be altered. We can take this to mean: apply an arbitrary unitary transformation to all the qubits selected for alteration. An apparently stronger definition
allows the unitary transformation to also involve another set of qubits, representing an “external environment”, thereby simulating the effect of “decoherence”. An apparently weaker definition limits the unitary operations to being among: \(\sigma_x, \sigma_y, \sigma_z\), the standard Pauli spin matrices, and \(I\), the unit matrix. It turns out that, by reasoning similar to that in [6,10,11], these three definitions of “alter” can be shown to be equivalent, in the sense that a code that is \(t\)-error-correcting with respect to the apparently weaker one will automatically be \(t\)-error-correcting with respect to the apparently stronger one. A quantum analog of the binary symmetric channel is the *depolarizing channel*, where each bit of each codeword that passes through it is independently subjected to: \(I\) with probability \(1 - \delta\), and \(\sigma_x, \sigma_y, \sigma_z\) each with probability \(\delta/3\). Call a code that achieves a fidelity of at least \(1 - \varepsilon\) on such a channel a \((\varepsilon, \delta)\)-error-correcting code.

It has previously been shown [2,3,14] how to take some special classical linear codes (binary and over \(GF(4)\)) with certain properties and transform them into quantum codes with error-correcting capabilities. In the present paper, we show how to transform quantum codes with certain properties into classical codes with error-correcting capabilities. We do not propose this as a means for constructing new classical codes; rather, this is a means for translating existing proofs of the nonexistence of certain classical codes into new proofs of the nonexistence of certain quantum codes. Our specific results, which apply to the class of stabilizer quantum codes (defined in the next section), are:

**Theorem 1:** If there exists a \(t\)-error-correcting \((n, k)\) quantum stabilizer code then there exists a \(t\)-error-correcting \((n - 1, k)\) classical binary linear code.

**Theorem 2:** If there exists a \((\varepsilon, \delta)\)-error-correcting \((n, k)\) quantum stabilizer code then there exists a \((\varepsilon, \delta)\)-error-correcting \((n - 1, k)\) classical binary linear code.

By these results, we can immediately assert that, when we restrict our attention to stabilizer codes, the classical upper bounds in [7,8] apply. In particular, when a quantum channel is subject to \(\delta n\) errors, the asymptotic capacity is bounded above
by \( H \left( \frac{1}{2} + \sqrt{2\delta(1-2\delta)} \right) \), where \( H \) is the binary entropy function defined as \( H(p) = -p \log_2 p - (1-p) \log_2 (1-p) \). In fact, a slightly stronger but more complicated upper bound is proven in [17]. This stronger bound is plotted in FIG. 1.

**FIG. 1 here**

This upper bound is stronger than the previously established \( 1 - 4\delta \) bound in [11], though the latter bound has the advantage that it applies to nonstabilizer codes as well. It is noteworthy that all of the quantum codes proposed to date for the channels described above are stabilizer codes. Other upper bounds exist for nondegenerate quantum codes (see [10] for a definition of nondegenerate). One is \( 1 - H(\delta) - p \log_2 3 \), and is based on an analogue of the classical “sphere packing bound” [10], and another asserts that the asymptotic capacity is zero if \( \delta > 1/6 \) [20]. It remains an open question whether the \( 1 - H(\delta) - \delta \log_2 3 \) bound also applies to degenerate codes. Very recently, it has been announced that the \( \delta > 1/6 \) threshold bound does extend to nondegenerate codes, and this will appear in a forthcoming paper [22]. It is interesting to note that there exist some degenerate stabilizer codes that outperform all known nondegenerate codes on the depolarizing channel, for some values of \( \delta \) [5]. The best lower bound for this channel that we are aware of is \( 1 - H(2\delta) - 2\delta \log_2 3 \) [8].

For the depolarizing channel with error probability \( \delta \), our results imply that, for stabilizer codes, the capacity is upper bounded by \( 1 - H(\delta) \), the bound for the classical binary symmetric channel [18].

**FIG. 2 here**

For some values of \( \delta \), this is stronger than the previously established upper bound of \( 1 - 4\delta \) [8], though the latter bound applies to nonstabilizer codes as well. The best lower bound that we are aware of for this channel is \( 1 - H(\delta) - \delta \log_2 3 \) [8], and a slightly larger value for some values of \( \delta \) [5].

Should any improvements to the upper bounds in [17] for classical coding occur, they will automatically apply to quantum stabilizer codes. Our results demonstrate interesting
connections between quantum stabilizer codes and classical linear codes, and, for some instances of channels, yield stronger upper bounds than those that have appeared to date.

In Sections II and III, we provide a brief overview of quantum stabilizer codes and classical linear codes. In Section IV, we describe how to construct a binary linear code from a quantum stabilizer code, and, in Section V, we show that this construction yields the error-correcting properties required for Theorems 1 and 2.

II. STABILIZER QUANTUM CODES

In [7,8] it has been shown that many quantum codes can be described in terms of stabilizers. Define a stabilizer as a set of \( n \)-qubit unitary operators such that: each operator is a tensor product of \( n \) matrices of the form \( \sigma_x, \sigma_y, \sigma_z, \) and \( I \), with a global phase factor of \( \pm 1 \); and, the set of operators is an abelian group. The code that is defined by a stabilizer is the set of all \( n \)-qubit quantum states that are fixed points of each element of the stabilizer. A stabilizer can be most easily described by a set of operators that generate it. If one negates the phase factor of some of the generators, the resulting code will change, but will have identical characteristics to the original code. Thus, one can always take the phase of each generator to be +1 without any loss of generality.

It is convenient to denote the generators of a stabilizer in the language of binary vector spaces, as in [8]. Denote the generator

\[
G = U_1 \otimes U_2 \otimes \cdots \otimes U_n
\]

as the \( 2n \) bit vector \((a|b)\), where, for \( i \in \{1, \ldots, n\} \),

\[
a_i = \begin{cases} 
1 & \text{if } U_i = \sigma_x \text{ or } \sigma_y \\
0 & \text{if } U_i = I \text{ or } \sigma_z
\end{cases}
\]

and

\[
b_i = \begin{cases} 
1 & \text{if } U_i = \sigma_z \text{ or } \sigma_y \\
0 & \text{if } U_i = I \text{ or } \sigma_x
\end{cases}
\]

For example, \( \sigma_x \otimes I \otimes \sigma_x \otimes I \otimes \sigma_z \otimes \sigma_y \otimes \sigma_z \otimes \sigma_y \) is denoted as \((1 | 0 1 0 0 1 0 1 | 0 0 0 0 1 1 1 1)\). In this notation, the product of any two generators \((a|b)\) and \((a'|b')\) is equivalent (modulo...
a phase factor of $\pm 1$ to $(a \oplus a' | b \oplus b')$, where $\oplus$ denotes the bit-wise sum in modulo two arithmetic. Also, $(a|b)$ and $(a'|b')$ commute if and only if

$$(a \cdot b') \oplus (a' \cdot b) = 0,$$  \hspace{1cm} (1)$$

where $\cdot$ denotes the inner product in modulo two arithmetic.

A stabilizer can then be written as an $m \times 2n$ matrix whose rows represent the generators. For example,

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1
\end{pmatrix}$$  \hspace{1cm} (2)$$

represents the eight generators of the stabilizer of a specific $(8,3)$ code that is 1-error-correcting (see [7] for a detailed analysis of this code).

In general, if there are $n$ qubits and $m$ generators, we can encode $k = n - m$ data qubits (in the above example, $3 = 8 - 5$). The error correcting capabilities of the code are related to commutativity relationships between the error operators and the generators [7,8].

### III. CLASSICAL BINARY LINEAR CODES

An $(n,k)$ binary linear code is a $k$ dimensional subspace of $\{0,1\}^n$ over modulo two arithmetic. It is sufficient to specify a basis $M_1, \ldots, M_k$ for such a code. Then the codeword for the $k$-bit string $x_1 \ldots x_k$ can be taken as the linear combination $x_1M_1 \oplus \cdots \oplus x_kM_k$ (this mapping is a bijection between $\{0,1\}^k$ and the code). A natural way of specifying such a code is by an $k \times n$ generator matrix, whose rows are $M_1, \ldots, M_k$. An example of such a code is

$$\begin{pmatrix}
1 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 1
\end{pmatrix}$$  \hspace{1cm} (3)$$

which is a $(5,2)$ code that is 1-error-correcting.
IV. CONSTRUCTION OF LINEAR CODES FROM STABILIZER CODES

Consider a quantum stabilizer code specified by a $2n \times m$ matrix $(X|Z)$. We shall show how to construct the generator matrix of a classical binary linear code with similar error-correcting capabilities.

Our construction involves a transformation of the generator matrix into a useful standard form along the lines of that in [13]. This conversion is accomplished by applying a series of basic transformations of the following two types, each of which leaves the error-correcting characteristics of the code unchanged. The first is a row addition, where the $j$th row is added to the $i$th row, where $i \neq j$. This corresponds to replacing the $i$th generator with the product of the $i$th and $j$th generator, and setting its phase to +1. The second is a column transposition, where $i$th column is transposed with the $j$th column in the submatrices $X$ and $Z$ simultaneously. This corresponds to transposing the $i$th qubit position with the $j$th qubit position.

We begin by applying transformations of the above types to the matrix $(X|Z)$ in order to obtain

$$s \begin{pmatrix} I & A \\ 0 & 0 \end{pmatrix}$$

$$r \begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix}$$

where $s$ is the rank of the submatrix $X$, and $r = n - k - s$. This is like performing Gaussian elimination on the $X$ submatrix. Next, by performing row additions among the last $r$ generators and column transpositions among the last $n - s$ qubit positions, the matrix can be further converted to the form

$$s \begin{pmatrix} I & A_1 & A_2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$r_1 \begin{pmatrix} B_1 & B_2 & B_3 \\ C_1 & C_2 & I \end{pmatrix}$$

$$r_2 \begin{pmatrix} D & 0 & 0 \end{pmatrix}$$
where \( r_1 \) is the rank of \( E_4 \), \( r_2 = r - r_1 \), and \( t = n - s - r_1 \). This is like performing Gaussian elimination on the submatrix \( E_4 \) of (4). Note that if \( r_2 > 0 \) and \( D \neq 0 \) and then one of the last \( r_2 \) generators would not commute with one of the first \( s \) generators. Therefore, we can set \( r_2 = 0 \), \( r_1 = r \), and \( t = k \). Thus, the form (3) becomes

\[
\begin{bmatrix}
I & A_1 & A_2 \\
0 & 0 & 0 \\
C_1 & C_2 & I
\end{bmatrix}
\]

(6)

where \( s + k + r = n \). Call any set of generators in this form (6) in standard form.

For a generator matrix in standard form, consider the classical binary linear code generated by the \( k \times (n - r) \) matrix

\[
\begin{bmatrix}
A^T_1 \\
I
\end{bmatrix}
\]

(7)

We claim that this classical code has similar characteristics to the original quantum code. Before stating this precisely, consider as an example the aforementioned \((8,3)\) code, that corrects one error, whose stabilizer was given by (2). Converting it to standard form yields

\[
\begin{bmatrix}
1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1
\end{bmatrix}
\]

(8)

which is an equivalent \((8,3)\) code. The resulting binary linear code is

\[
\begin{bmatrix}
1 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}
\]

(9)

which is a well-known \((7,3)\) code that corrects one error. Thus, in this example, we obtain a classical code with slightly better capacity. The reader may recall that previous constructions of \((7,1)\) quantum codes have been based on the same classical code (2) (23). It should be noted that the present connection is quite different, as it involves an \((8,3)\) quantum code.
In the next section we shall show that, for any $t$-error-correcting ($n, k$) quantum stabilizer code that is in standard form (6), the matrix (7) generates a $t$-error-correcting ($n - r, k$) binary linear code. Also, for the case of the depolarizing channel, we shall show that, for any $(\varepsilon, \delta)$-error-correcting ($n, k$) quantum stabilizer code in standard form (6), the matrix (4) generates an $(\varepsilon, \delta)$-error-correcting ($n - r, k$) binary linear code.

Furthermore, by applying operations of the types below, which also do not affect the error-correcting capabilities of the code [12], we can guarantee the additional property that $r \geq 1$, which slightly sharpens the result. The first operation is the column switch, in which the $i$th column of submatrix $X$ is transposed with the $i$th column of submatrix $Z$. This corresponds to: in the $i$th qubit position, changing each instance of a $\sigma_x$ in each generator to a $\sigma_z$, and each instance of a $\sigma_z$ to a $\sigma_x$ (while leaving each $I$ and $\sigma_y$ intact). The second operation is the column addition, in which the $i$th column of submatrix $Z$ is added to the $i$th column of submatrix $X$. This corresponds to: in the $i$th qubit position, changing each instance of a $\sigma_y$ in each generator to a $\sigma_z$, and each instance of a $\sigma_z$ to a $\sigma_y$ (while leaving each $I$ and $\sigma_x$ intact). See [12] for an explanation of why these two operations do not affect the characteristics of the code.

V. PROOF OF ERROR-CORRECTING PROPERTIES OF CONSTRUCTION

In this section, we show that the constructions of the previous section satisfy the claimed error-correcting properties.

The classical code whose generator matrix is given by (7) consists of $2^k$ codewords in the space $\{0, 1\}^{n-r}$. We shall construct an isomorphism between this code and a restricted version of the quantum code specified by (9). The restricted version of the quantum code consists of $2^k$ codewords that are contained in a special set $S$ of $2^{n-r}$ distinct $n$-qubit states. This set $S$ has the property that it is closed with respect to $\sigma_z$ errors among the first $n - r$ qubit positions. Intuitively, the effect of bit errors on classical codewords within the space $\{0, 1\}^{n-r}$ is equivalent to the effect of $\sigma_z$ errors in the first $n - r$ qubit positions on quantum
codewords within the space $S$. Formally, the isomorphism that we shall construct is a mapping $\phi : \{0,1\}^{n-r} \rightarrow S$, such that:

1. $\phi$ is bijective.

2. For each $y_1 \ldots y_{n-r} \in \{0,1\}^{n-r}$ that is a codeword of the classical code, $\phi(y_1 \ldots y_{n-r})$ is a codeword of the quantum code.

3. For each codeword $y_1 \ldots y_{n-r} \in \{0,1\}^{n-r}$ of the classical code, and each error vector $e_1 \ldots e_{n-r} \in \{0,1\}^{n-r}$,

$$\phi(y_1 \ldots y_{n-r} \oplus e_1 \ldots e_{n-r}) = \sigma_z^e_1 \otimes \cdots \otimes \sigma_z^e_{n-r} \otimes I \otimes \cdots \otimes I \phi(y_1 \ldots y_{n-r}).$$

The existence of such an isomorphism means that an error in the $i^{th}$ bit of the classical code (for any $i \in \{1, \ldots, n-r\}$) corresponds to a $\sigma_z$ error in the $i^{th}$ qubit of the restricted version of the quantum code. More precisely, if the quantum code can correct any $t$ errors then it can correct any $t \sigma_z$ errors among the first $n-r$ qubit positions, and then the following procedure for correcting any $t$ errors in the classical code exists. Given a codeword $y_1 \ldots y_{n-r}$ subjected to an error vector $e_1 \ldots e_{n-r}$ of weight bounded by $t$, first apply the mapping $\phi$ to it. By the second and third properties of $\phi$, the result is $\phi(y_1 \ldots y_{n-r})$ subjected to at most $t \sigma_z$ errors among the first $n-r$ qubit positions, which can therefore be corrected. By the first property, $\phi^{-1}$ can be applied to this corrected quantum codeword, yielding the correction of the original codeword. Therefore, if we establish that there exists a $\phi$ that satisfies the above three properties then the classical code specified by (7) must correct at least as many errors as the quantum code specified by (6).

For the case of the depolarizing channel with parameter $\delta$, if the quantum code attains fidelity $1 - \varepsilon$ then it attains fidelity $1 - \varepsilon$ for a channel that applies $\sigma_z$ in each of the first $n-r$ qubit positions independently with probability $\delta$ (in fact we may need to slightly modify the code by applying some column switch and column addition operations—defined in Section IV—along the lines of the “twirling” techniques explained in [6]). Therefore,
the corresponding classical code is correcting with probability at least $1 - \varepsilon$ on a binary symmetric channel with parameter $\delta$. Thus, the existence of the above $\phi$ also suffices for this noisy channel model.

In order to construct a bijection $\phi$ with the above properties, we shall construct a useful basis for the quantum code. We begin with the stabilizer specified by the matrix in standard form (I). Call the operators corresponding to the respective rows of this matrix $G_1, \ldots, G_m$. Define the additional operators $L_1, \ldots, L_k$ in terms of the matrix

$$
\begin{pmatrix}
 0 & I & C_2^T & D & 0 & 0 \\
\end{pmatrix},
\tag{10}
$$

where $D = B_2^T + C_2^T B_3^T$, and $N_1, \ldots, N_k$ in terms of the matrix

$$
\begin{pmatrix}
 0 & 0 & 0 & A_1^T & I & 0 \\
\end{pmatrix}.
\tag{11}
$$

By considering (I), (II), (III), and recalling the criterion for commutativity (I), it is straightforward to verify that:

- $G_1, \ldots, G_m, L_1, \ldots, L_k$ is a set of $n$ independent commuting operators.
- $G_1, \ldots, G_m, N_1, \ldots, N_k$ is a set of $n$ independent commuting operators.
- Each $N_i$ and $L_j$ commute if $i \neq j$ and anticommute if $i = j$.

Using these properties, we can construct a basis $\{|C_{x_1 \ldots x_k}\rangle : x_1 \ldots x_k \in \{0,1\}^k\}$ for the code with some useful structural features. First, set $|C_{0 \ldots 0}\rangle$ to be the quantum state stabilized by $G_1, \ldots, G_m, L_1, \ldots, L_k$ (this state is unique up to a global phase factor). In fact,

$$
|C_{0 \ldots 0}\rangle = \frac{1}{\sqrt{2^{r+k}}}(I + G_1) \cdots (I + G_s)(I + L_1) \cdots (I + L_k)|0\ldots 0\rangle
\tag{12}
$$

is a quantum state with this property. Next, for each $x_1 \ldots x_k \in \{0,1\}^k$, set $|C_{x_1 \ldots x_k}\rangle = N_1^{x_1} \cdots N_k^{x_k}|C_{0 \ldots 0}\rangle$. Since $N_i$ commutes with $L_j$ if and only if $i = j$, $|C_{x_1 \ldots x_k}\rangle$ is in the
+1-eigenspace of each \( G_1, \ldots, G_m \) and the \((-1)^{x_1}, \ldots, (-1)^{x_k} \) eigenspaces of \( L_1, \ldots, L_k \), respectively. Therefore, these states are an orthogonal basis for the quantum code.

Now, define the function \( \phi : \{0, 1\}^{n-r} \rightarrow S \) as

\[
\phi(y_1 \ldots y_{n-r}) = \sigma_z^{y_1} \otimes \cdots \otimes \sigma_z^{y_{n-r}} \otimes I \otimes \cdots \otimes I |C_{0\ldots 0}\rangle,
\]

for each \( y_1 \ldots y_{n-r} \in \{0, 1\}^{n-r} \). We shall show that \( \phi \) satisfies the three required properties. By considering (9) and (10), the operator that applies \( \sigma_z \) to the \( i \)th qubit (and \( I \) to all other qubits) anticommutes with the \( i \)th generator in the sequence \( G_1, \ldots, G_s, L_1, \ldots, L_k \) and commutes with all others. Therefore, \( \phi(y_1 \ldots y_{n-r}) \) is in the \((-1)^{y_1}, \ldots, (-1)^{y_{n-r}} \) eigenspaces of \( G_1, \ldots, G_s, L_1, \ldots, L_k \), respectively (recall that \( n - r = s + k \)). Thus, \( \phi(y_1 \ldots y_{n-r}) \) is orthogonal for each distinct \( y_1 \ldots y_{n-r} \). This proves the first property, that \( \phi \) is a bijection.

Also, due to the close similarity between (7) and (11),

\[
\phi(x_1 M_1 \oplus \cdots \oplus x_k M_k) = N_1^{x_1} \cdots N_1^{x_k} |C_{0\ldots 0}\rangle = |C_{x_1 \ldots x_k}\rangle,
\]

so the second property for \( S \) holds. Finally, the third property for \( S \) holds because, using (13) and (14),

\[
\phi(y_1 \ldots y_{n-r} \oplus e_1 \ldots e_{n-r}) = \sigma_z^{y_1 \oplus e_1} \otimes \cdots \otimes \sigma_z^{y_{n-r} \oplus e_{n-r}} \otimes I \otimes \cdots \otimes I |C_{0\ldots 0}\rangle
\]

\[
= \sigma_z^{e_1} \otimes \cdots \otimes \sigma_z^{e_{n-r}} \otimes I \otimes \cdots \otimes I \phi(y_1 \ldots y_{n-r}).
\]

Thus, \( \phi \) satisfies the three required properties.

ACKNOWLEDGMENTS

I am very grateful to Hans-Benjamin Braun for help in analyzing and plotting the functions in [17], David DiVincenzo for several interesting discussions about quantum coding theory and comments about an earlier draft of this paper, Emanuel Knill for providing references to existing bounds for classical codes, and Juan Paz for interesting discussions about stabilizer representations of codes. I am also grateful for the hospitality of the program on
Quantum Computers and Quantum Coherence at the Institute for Theoretical Physics, University of California at Santa Barbara, where this work was completed. This research was supported in part by NSERC of Canada and the U.S. National Science Foundation under Grant No. PHY94-07194.
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FIG. 1. Asymptotic upper bounds (solid lines), upper bounds for nondegenerate codes (broken lines), and lower bound (dashed line) for the capacity of a quantum channel with $\delta$-bounded fraction of errors. A: Linear upper bound in [11], B: Our new upper bound for stabilizer codes, based on the upper bounds for classical codes in [17] (see also [16]), C: The quantum “sphere-packing” upper bound for the case of nondegenerate codes in [10], D: The upper bound implied in [20] for nondegenerate codes, E: Lower bound in [14].
FIG. 2. Asymptotic upper bounds (solid lines) and lower bound (dashed line) for the capacity of the depolarizing quantum channel with probability parameter $\delta$. A: Linear upper bound in [6], B: Our new upper bound for stabilizer codes, based on the upper bounds for classical codes in [18] (see also [15]), C: Lower bound in [6] that matches the “sphere-packing” bound in [10], very slightly improved by [5].