Remarks on Hot QCD

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ABSTRACT

After briefly reviewing work I’ve done recently with Krishna Rajagopal on the nature of the equilibrium phase transition in QCD at finite temperature and zero baryon number and on the possibility of forming misaligned vacuum by a quench, I discuss – also briefly – two new items. These are extension of the arguments concerning the order of the transition to to 2+1 dimensions, relating in this case gauge to conformal field theories, that could be explored in numerical experiments; and the energy budget at the chiral transition and its relevance to the nature of the transition and to quenching.
The bulk of the material in the talk as delivered was taken from two papers written with Krishna Rajagopal. Here I will only give a telegraphic summary of those papers \([1, 2]\) in Sections 1 and 2. Sections 3-4 are devoted to the new items mentioned in the Abstract.

1. Equilibrium Phase Transition

Because of asymptotic freedom, one expects that at sufficiently high temperature hadronic matter in thermal equilibrium will form a weakly interacting plasma of quarks and gluons. Such a plasma differs qualitatively from hadronic matter at low temperature in two important respects. First, of course at low temperatures one does not observe quarks and gluons but rather color neutral objects – the property of confinement. Second, a large body of phenomenology connected with soft pion theorems indicates that \(SU(2) \times SU(2)\) chiral symmetry suffers a large spontaneous breakdown, in addition to being intrinsically but relatively weakly violated by the existence of small but \(u\) and \(d\) quark masses. It may also be useful to consider \(SU(3) \times SU(3)\) chiral symmetry, though in that case it is unclear when it is appropriate to consider the intrinsic violation as small. In any case, the spontaneous portion of the breaking clearly is not contained in the standard description of QCD as a weakly interacting quark-gluon plasma at high temperature.

It is natural to ask whether these qualitative changes are marked by actual phase transitions, and if so what is the nature of these transitions. Is there a first order transition with discontinuities in thermodynamic functions and release of latent heat, or a second order transition where the primary thermodynamic functions are continuous but not analytic – or perhaps no true phase transition at all? In this connection let me remind you that the passage of ordinary neutral gas into an ionized plasma at high temperature, which in some respects is analogous to deconfinement, although it connects two very different kinds of matter, is not marked by any singular behavior in the thermodynamic functions.
How does one approach such questions theoretically? A powerful method is to construct order parameters, usually characterizing the state of symmetry of the system, that are zero in one phase but not in the other. Classic examples of order parameters are the magnetization or staggered magnetization which characterize ferromagnetic respectively antiferromagnetic states. These order parameters necessarily vanish in a state with rotational symmetry, such one expects above the Curie or Neel point. If one has an order parameter of this kind one can be sure that there is a singularity in passing between the states, because an analytic function cannot vanish in a finite range without vanishing everywhere. On the other hand one may have phase transitions with no order parameter – the thermodynamic functions on the two sides of the transition temperature may just be different functions. This is usually the case for liquid-gas transitions. It is not unusual to find that for a given substance the passage from liquid to gas is or is not marked by a true phase transition depending on the value of the ambient pressure, which could not happen if there were an order parameter marking the difference.

It would appear that the partition function \(Z = \text{Tr} e^{-H/T}\), being a sum of analytic terms, must be an analytic function of the temperature – and indeed, in a finite volume, it is. However one is interested in the limit of large volumes, where the sum in the trace may contain an infinite number of terms and analyticity may be lost. If the partition function is not analytic, this must be due to some subtlety in taking the infinite volume limit. One expects that such a subtlety must be associated with the development of correlations which extend over an infinite range. If the transition is second order, the correlation length must go continuously to infinity at the transition point. In particular, sufficiently close the transition point the correlation length will greatly exceed any “microphysical” length associated with the problem, e.g. the lattice spacing (or, for QCD, \(\Lambda^{-1}\)). These observations lead to the concepts of scale invariance and universality, which together form the foundation of the modern theory of second order phase transitions.

Scale invariance follows from the idea that non-trivial correlations on length scales much larger than the microscopic one can only depend on the ratio of the
length in question to one overall diverging correlation length. (Generically there should be just one fundamental diverging length, since there is one parameter – the temperature – which must be tuned to reach the critical point. Another way of saying this is that all divergent quantities should be given as functions of the reduced temperature $t = |\frac{T - T_c}{T_c}|$.) Universality follows from the idea that only modes with long-wavelength fluctuations in equilibrium and their low-dimension interactions among themselves are important for the singularities, and that the only long-wavelength modes that exist are likely to be those required by the symmetries of the order parameter. Together these principles tell us that in constructing models of the singularities of thermodynamic functions near second order phase transitions, it is necessary and sufficient to look for the simplest scale-invariant models in the appropriate dimension with the appropriate symmetry.

How does all this apply to QCD? What are the available order parameters?

For confinement, the only known order parameter is the Polyakov loop [3]

$$P = \langle \exp \int iA_\tau d\tau \rangle$$

where one is taking a thermal average, $A_\tau$ is the imaginary-time component of the gauge field written as a matrix in the fundamental representation, and the trace is over one period $\beta = T^{-1}$ of the imaginary time. $\ln P$ is basically the free energy associated with adding a static color source with the quantum numbers of a quark. It vanishes when quarks are confined. However it has no reason to vanish when quarks are not confined, and indeed it does not vanish in perturbation theory.

The Polyakov loop is not expected to vanish when the color flux associated with the color source can be screened. Indeed the reason why the energy cost of inserting the color source in the confined phase is infinite is because there is a non-trivial flux extending from the source that cannot terminate, and this flux disrupts the vacuum wherever it threads, costing a finite energy per unit length. On the other hand if we are considering a gauge theory with dynamical quarks, then the quarks that are inevitably present at non-zero temperature need only re-arrange
themselves a bit to soak up the flux. Strictly speaking one then cannot distinguish confinement from screening. There is no strict order parameter for confinement in the presence of dynamical quarks at finite temperature (similar to the case of ionization of an ordinary gas).

For an $SU(N)$ gauge group, the Polyakov loop parameter transforms non-trivially under the center of the gauge group, that is $Z_N$. When it vanishes the global $Z_N$ symmetry under gauge transformations which approach these central elements at infinity is unbroken, while if it does not vanish this symmetry is broken. (Note that local gauge invariance is never broken.)

The universality class of the deconfinement transition for pure glue $SU(N)$ is therefore described by a simple model with a discrete global symmetry group. For $N = 2$ it is the Ising model and for $N = 3$ it is the three-state Potts model. These are well studied models. The first of them has a second-order transition in 3 dimensions, while the second, for quite fundamental reasons, does not. This led Svetitzky and Yaffe [4] to predict that pure glue $SU(2)$ would probably have a second-order, but pure glue $SU(3)$ must certainly have a first-order, transition. After some struggles, their prediction was verified by direct numerical simulation [5].

For the chiral transition with $N_f$ massless flavors, the simplest order parameter one can imagine is given by an $N_f \times N_f$ matrix $M$ of scalar fields, which parametrizes the condensate $\langle \bar{q}_L q_R \rangle$. This matrix is complex. For $N_f = 2$ one can require that $M$ is of the form $\sigma + i \vec{\pi} \cdot \vec{\tau}$, with four independent real components. It turns out that $N_f = 2$ is a very special case [6]. It is in the universality class of a four-component magnet, a model which has been extensively studied in the condensed matter literature [7]. This model is known to have a second-order transition. Thus there is a scale invariant candidate theory to describe the singular behavior near a second-order chiral symmetry restoration transition in 2 massless flavor QCD. For more than two flavors the situation is very different. The more complicated models appropriate to this case have not been studied so extensively.
However all the evidence, both analytical [8] and numerical [9], is that they do not support a second-order transition. Thus there is no candidate theory to describe the singular behavior that would occur near a second-order chiral symmetry restoration in QCD with 3 or more massless flavors. A probable interpretation [6] is that in these cases near the transition fluctuations grow so large that they induce a first-order transition.

These considerations suggest the possibility of a second-order transition uniquely in the case $N_f = 2$. Remarkably, this pattern is consistent with existing numerical evidence [10].

When one has a second-order transition, it is possible to make some precise predictions for the nature of the singularities near the transition, using the powerful concepts of universality and scale invariance as mentioned above. Let me emphasize again that these are precise predictions. The most characteristic results are predictions of the critical exponents, which specify the power dependence of various quantities such as the magnetization (which in QCD language translates to the magnitude of the chiral condensate) the specific heat, or many others, on $t = |T - T_c|$. These predictions are similar in spirit to the famous predictions for the behavior of moments of QCD structure functions as calculable powers of logarithms of $Q^2$, but the calculations involved are much more arduous. The point is that whereas the ultraviolet fixed point of QCD occurs at zero coupling, the infrared fixed point governing the chiral phase transition occurs at a very large value of the coupling, and extraordinary methods have to be used to extract quantitative results. Fortunately the requisite work, involving calculating hundreds of graphs in perturbation theory up to six loops, using analytic methods to estimate the asymptotic behavior of high orders of perturbation theory, and joining the two by sophisticated resummation methods, was performed in a tour de force by Baker, Meiron, and Nickel.

If we accept that for two massless quarks the transition is second order while for three massless quarks it is first order, there is an interesting question how one
behavior goes over into the other as the mass of the third quark varies. A very pretty possibility is that as the third (strange) quark mass is raised continuously from zero the discontinuities associated with the first-order transition get smaller and smaller, eventually vanishing at the so-called tricritical point, after which the transition becomes second-order. The behavior in the the immediate neighborhood of the tricritical point is governed by a four-component asymptotically free massless \((\phi^2)^3\) theory, which features logarithmic corrections to mean field theory. The tricritical point is characterized by a particular value \(m_{\text{crit}}\) of the strange quark mass. Existing numerical evidence while very crude suggests that the physical strange quark mass is larger than \(m_{\text{crit}}\). However it is not impossible that the strange quark mass is quite close to the critical value, and of course in numerical experiments one can in principle vary the masses to probe the suspected tricritical region. It is noteworthy that the specific heat, which generically has a cusp at the second-order transition, develops an actual discontinuity at the tricritical point.

Another direction in which the analysis may be extended is to the consideration of time-dependent behavior near the critical point. One can analyze how the transport equations change near the critical transition. The main qualitative phenomenon is critical slowing: there is a diverging correlation time as well as a diverging correlation length, as the system finds it difficult to relax the very long-wavelength fluctuations that occur. Thus transport coefficients generally acquire singularities at the transition. Dynamical critical behavior probes more physics than the static behavior: models in the same static universality class can have different dynamical critical exponents; for example, ferromagnets and antiferromagnets are in the same static universality class but different dynamic universality classes, basically because the ferromagnetic order parameter, the magnetization, becomes a conserved quantity in the long-wavelength limit – unlike the staggered magnetization – which makes long-wavelength fluctuations relax more slowly. QCD with two massless quarks appears to be in the dynamic universality class of the four-component Heisenberg antiferromagnet. A major quantitative consequence of the analysis is that the correlation time is predicted to scale as the \(3/2\) power of
the correlation length. Note that if we translate this behavior into a prediction for the form of real-time Green functions we get funny cuts at zero frequency and wave vector, which is very different from a naive particle exchange model and of course from any naive extrapolation of the behavior of weakly interacting plasma.

If we take the indications for a second-order phase transition with two massless quarks and the effective “massiveness” of the physical strange quark at face value, what are the consequences? For numerical experiments, the we find ourselves in a happy situation. There is a wealth of detailed, quantitative predictions waiting to be tested. For cosmology the situation is either dull or reassuring, depending on your point of view. The expansion of the universe is very slow indeed on the relevant strong-interaction timescales, and equilibrium should be closely maintained. The non-zero masses of the up and down quarks mean that there is really no phase transition at all. Interesting relics that might have occurred for a strongly first-order transition, including gravity waves, inhomogeneous nucleosynthesis, and possible production of exotic matter, do not occur. For heavy ion collisions, if thermal equilibrium is produced and maintained, the conclusion is similar. Even the interesting long-range correlations that might have been expected to arise near a second order transitions, and to give rise to interesting phenomenological signatures as I shall discuss in a moment, are not quantitatively significant due to the finite pion masses (that is, inverse correlation lengths) that are not much smaller, and maybe not at all smaller, than the temperature at the transition. However that’s a very big if as you know, and it is interesting to consider another quite different idealization of the physics.
2. Quenching and Misaligned Patches

If instead of cooling a bar of iron slowly through its Curie point one suddenly plunges it into ice water, one is said to have quenched the magnet. It may be that it is not inappropriate to model what occurs following a heavy ion collision, as the hot plasma comes into contact with cold empty space outside, as a quench. Insofar as it is plausible to map the important degrees of freedom for QCD into a magnet model, i.e. if the pions and the chiral condensate are mainly what we must keep track of, the analogy is quite close.

The dynamical evolution following a quench is quite different from that for cooling through equilibrium. One expects that broadly speaking whereas near equilibrium the question is the typical size of correlated regions, which individual grow, shrink, come into being and pass away, following a quench the system is racing to the ordered state, the dynamics is unidirectional, and the question is how the domains evolve (grow) in time.

There is a substantial condensed matter literature on problems of this type, including very recent work [11]. Also related problems arise in the cosmological models (“texture models”) where the fluctuations responsible for triggering the formation of structure in the universe are ascribed to the formation, growth, and eventual collapse of domains during a cosmic phase transition [12]. The QCD problem has its own special features, however, and requires a fresh analysis. The main new features that are central to the QCD problem are the fact that the symmetry is intrinsically broken, and that one considers hyperbolic rather than diffusive equations. In many condensed matter problems diffusive equations for the dynamical variables of interest are appropriate, because these variables are in contact with other degrees of freedom (e.g. phonons) whose effect is modelled by appropriate diffusive transport equations. However in the QCD problem, and probably also in some condensed matter problems, one should use the true microscopic equations which of course are hyperbolic.

The fact that the symmetry is intrinsically broken means that every region of
space is heading toward the same ground configuration, and interest focuses on the nature of the approach. Do different regions of space relax independently to the ground configuration (small oscillations around the $\sigma$ direction), or do large aligned regions form and then relax coherently? The latter possibility could have dramatic consequences, because it would mean that large regions of space have a misaligned condensate [13], a classical field configuration that might be expected to emit classical pion radiation – to “lase” pions – as it relaxes.

To model a QCD quench we take a representative configuration of the $\sigma$ and $\pi$ fields from the ensemble weighted by the free energy at the pre-quench temperature $T$, and then evolve it according to the zero temperature equations of motion. We find that for broad ranges of the parameters large coherent structures do form following a quench. We believe that the fundamental mechanism underlying this behavior is the following. The $(mass)^2$ of a Nambu-Goldstone field is zero in the true ground state, due to a cancellation between an intrinsically negative bare $(mass)^2$ and a positive contribution from the interaction with the condensate:

$$m^2 = -\mu^2 + 2\lambda v^2 = 0 \text{ (groundstate)} \quad (2.1)$$

However in a quench the vacuum expectation value $v^2 = \langle \sigma \rangle^2$ starts centered around zero, not its ground state value $\mu^2/2\lambda$. Thus $m^2$ can easily be negative, and according to the dispersion relation

$$\omega^2 = k^2 + \mu^2 \quad (2.2)$$

modes with sufficiently small spatial frequencies will grow – and the longer the wavelength, the more the growth. Though the real situation (that is, our idealized model of a quench) is more complicated in various ways, including the non-linearity of the equations, the phenomenon suggested by this simple picture does occur roughly as expected.

Taking a realistic intrinsic symmetry breaking into account does not destroy the phenomenon, because the intrinsic $(mass)^2$ of the pion is substantially less than
μ². (The ratio of the two is basically the square of the ratio of pion and sigma masses; see section 4 below).

Thus there seems to be a good chance that large regions of misaligned vacuum might form. The radiation from such a region as it relaxes will be quite structured and unusual. The 4-component classical field \((\sigma, \vec{\pi})\) will oscillate in the plane determined by sigma and some definite direction of the vector \(\vec{\pi}\). The radiation is a coherent configuration of pions \textit{in that direction}. These radiated pion clumps should have small relative momentum, and each one will have a fixed ratio of charged to neutral pions. The probability distribution for a given charge ratio is

\[
\text{Prob.}(R) = \frac{1}{2} R^{-\frac{1}{2}} \quad (2.3)
\]

where

\[
R \equiv \frac{\pi^0}{\pi^0 + \pi^+ + \pi^-} \quad (2.4)
\]

This distribution is, of course, highly non-Gaussian. For example the probability of having less than 1% neutral pions is 10% !

3. Equilibrium Phase Transition One Dimension Down

It would be interesting to test the logic of the argument leading to the expectation of a possible second order deconfinement phase transition for color \(SU(2)\) pure glue theory and chiral symmetry restoration phase transition for two flavors of massless quarks, but definitely first order for deconfinement in color \(SU(3)\) pure glue theory or chiral symmetry restoration with three or more flavors, in other examples. A simple possibility lies near at hand: one can study the corresponding questions in one fewer spatial dimension. While the direct relevance of such studies to any achievable laboratory situation is doubtful, they have the advantage of being relatively easy to access by numerical experiments. Also as we shall see some rather striking situations might arise.
Following the same arguments as before, we relate the possible second order transitions of 2+1 dimensional gauge theories to the existence of 2 dimensional scale invariant theories with the same symmetries.

The study of 2 dimensional scale invariant theories has been a large thriving activity in recent years [14]. In two dimensions scale invariance leads to a much larger and more powerful symmetry group of conformal symmetry transformations, and use this symmetry to carry the analysis of scale or conformal invariant theories a long way. In favorable cases critical exponents can be calculated exactly and there are tractable algorithms for calculating just about any correlation function of interest. Thus a question of interest becomes, which conformal field theories correspond to which gauge theories at their critical points.

It seems quite plausible that the $SU(N)$ pure glue theories have deconfinement phase transitions associated with the $Z_N$ central symmetries discussed above. For $N = 2$ one has the symmetry class of the Ising model, for $N = 3$ the three-state Potts model, and so forth. Whereas in 3 dimensions the phase transition was second order for $N = 2$ but necessarily first order for $N = 3$, here they can both be second order. As I said there is a quite a full understanding of the correlation functions of these models, though it may not be entirely trivial to set up the dictionary translating these results into gauge theory language.

The issue of chiral symmetry breaking in 2+1 dimensions is complicated by two factors, which make the situation very different from what one has in 3+1 dimensions, where one has breakdown of a continuous chiral symmetry. First, there is no chirality in 2+1 dimensions; and second, there can be no spontaneous breaking of a continuous symmetry. Let me recall these facts for you.

In 2+1 dimensions we can use (essentially) the ordinary two by two Pauli matrices as gamma matrices, say $(\gamma_0, \gamma_1, \gamma_2) = (\sigma_2, i\sigma_1, i\sigma_3)$. These are chosen in such a way that they are all imaginary, so that the Dirac equation can be solved with real (Majorana) spinors. However we shall focus on fermions with the quantum numbers of quarks, which are necessarily complex anyway. More
important for present purposes is the fact that the product $\gamma_0\gamma_1\gamma_2$ reduces to a pure number. This implies that it is impossible to make a chiral projection. Another fact that will be important to us is that the parity operation of reflection in one axis acts as (say) $\gamma_0\gamma_2$ on the spinors, so that the operator $\bar{\psi}\psi$ is odd under parity (and time reversal).

There is no spontaneous breaking of continuous symmetries in 2 dimensional quantum field theory at zero temperature (Coleman theorem, [15]) – or for 2+1 dimensional theories at non-zero temperature (Mermin-Wagner theorem, [16]). These are very nearly the same theorem, since the zero-frequency components of the putative Nambu-Goldstone fields have the infrared singularities (long wavelength fluctuations) of the 2 dimensional theory. These fluctuations are actually divergent at long distances; their divergence undoes the hypothesized symmetry breaking or long-range order. This shows the internal inconsistency of that hypothesis that the continuous symmetry breaks, leading to the theorems.

Still there is something to investigate. The theory with $f$ flavors of massless quarks has a $U(f)$ flavor symmetry and also a parity symmetry. As we have seen even a common mass terms, or a quark-antiquark condensate without any preferred direction in flavor space, breaks parity. Thus if we assume that a flavor singlet condensate develops at zero temperature, but goes away at high temperature, then there is the possibility of a phase transition with $Z_2$ symmetry, corresponding to parity restoration. If this phase transition is second order, then it could be mapped onto the Ising transition.

I think it would be quite interesting to carry out appropriate numerical experiments on lower-dimensional QCD at finite temperature, to check whether second-order transitions with the predicted exponents in fact occur. Hansson and Zahed [17] have also emphasized that the high temperature behavior of the 2+1 dimensional gauge theory appears to be tractable theoretically and to form a good testing ground for analytical methods which can also be applied to the more difficult 3+1 case.
4. Energy Budget for Chirality and Deconfinement

It is an interesting exercise to estimate the energy locked up in the chiral vacuum, and to compare it with the energy of the glue near the phase transition.

To estimate the energy density in the chiral vacuum, begin with the sigma model potential

$$V(\sigma) = -\mu^2\sigma^2 + \lambda\sigma^4.$$  \hspace{1cm} (4.1)

Elementary calculations lead to the vacuum expectation value $\langle \sigma \rangle = \sqrt{\mu^2/2\lambda}$, the sigma mass $m_\sigma = 2\mu$ and the energy density $E = -\mu^4/4\lambda$ at the minimum, where the symmetric vacuum energy density is normalized to zero. Identifying $\langle \sigma \rangle = f_\pi$ we arrive at the estimate

$$E = -\frac{f_\pi^2m_\pi^2}{8}$$  \hspace{1cm} (4.2)

of the condensation energy in terms of observables. Inserting experimental values, one finds

$$|E| \approx 5 \times (100 \text{ Mev})^4.$$  \hspace{1cm} (4.3)

This estimate, based on classical reasoning and on taking the broad observed $\sigma$ resonance as the embodiment of the field in the sigma model, is certainly crude. However it may not be inappropriate to use it to draw one tentative qualitative conclusion, as follows.

The energy density for an ideal gas of $SU(N)$ color gluons at temperature $T$ is

$$E_{\text{glue}} = (N^2 - 1)\frac{\pi^2}{15}T^4 \approx 4.8 \times T^4 \text{ (for } N = 3).$$  \hspace{1cm} (4.4)

There is also a contribution from $N_f$ species of massless quarks, with the prefactor $\frac{7}{4}NN_f$ replacing $N^2 - 1$ in (4.4). For $N = 3$ and two light flavors, this quark contribution is just a bit larger than (4.4). The simple qualitative point I want
to stress is that the sorts of energy densities associated with the chiral condensation are, for $T \sim 100$ Mev and for the stated values of $N$ and $N_f$, roughly comparable to the energy densities of the entire quark-gluon plasma. This comparison is important in thinking about the question whether the singular parts of the thermodynamic functions, which are the universal quantities we can address theoretically, are quantitatively important in the relative to the thermodynamic functions themselves. For example is the predicted cusp in specific heat a major feature or just a tiny dimple on an otherwise smooth curve? This is a non-universal question, whose answer will depend on details of the microscopic theory including the absolute value of the transition temperature (which is very much conditioned on such details). The estimates above suggest that if $T \approx 100$ Mev, as is perhaps suggested by the numerical work, then the energy controlled by the value of the order parameter is not much less than the total energy, so that the singular parts should may not be overly difficult to discern. Existing evidence suggests that $T$ is a little larger than this, but on the other hand probably one should not expect the gluon plasma to acquire its full (non-interacting) energy immediately following the transition. In any case, it is important to emphasize that the foolproof way to look for manifestations of critical singularities is to study the behavior of the order parameters and closely related quantities, which is completely dictated by the universal theory.

The issue of the energy difference is also important in assessing the appropriateness of the quench model as an idealization. The essence of the quench phenomenon is that the modes associated with spontaneous symmetry breaking are slower to relax to equilibrium than the generality of modes. Thus when energy is suddenly drained from the quenched system these modes are trapped in the pre-existing configuration, but at a low temperature. Clearly the most favorable case is when the energy associated with the symmetry-breaking modes is commensurate with the total energy, so that we are not making absurd demands on the cooling process. Expansion of the plasma provides an excellent cooling mechanism, but we should not require miracles of it.
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