Axial ring-down modes in general relativity and in its pseudo-complex extension

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Abstract

We calculate the axial ring-down frequencies of the merger of two black holes, using a modified version of the pseudo-complex General Relativity (pc-GR) and comparing it with the standard General Relativity (GR). The path, on how to determine the axial modes, serve as a starting point for more general extensions of GR, involving additions of $r$-dependent functions to $g_{00}$.

KEYWORDS
general relativity, gravitational waves, modified theories

1 | INTRODUCTION

General Relativity (GR) is well tested in solar system experiments (Will 2006) and also the predicted gravitational waves which have been observed (Abbott et al. 2016a, 2016b), stemming from the merger of two black holes. Many more observed gravitational wave events since 2016 can be found on the LIGO website.

In this contribution, we calculate the axial ring-down modes of a black hole after merger, within GR and pc-GR, initially proposed in Hess & Greiner (2009) and we are interested in the differences between GR and the pseudo-complex General Relativity (pc-GR) (Hess et al. 2015; Hess & Greiner 2009). In Hess (2016), a simplified model was used to estimate the frequencies in the initial phase of the merger, with the result that the mass deduced is larger than in GR and, as a consequence, the source is shifted to larger distances. In Hess & López-Moreno (2019a), a first attempt of a description of gravitational waves is published. In Hess & López-Moreno (2019a), still problems arise in the convergence of the solutions using a particular numerical method used, probably due to a deficient ansatz for the asymptotic limit of the wave function. We hope that the method, presented here, will improve these results, though we will concentrate on the axial modes only.

The pc-GR depends on a parameter $n$ related to the fall-off of a dark energy as a function of the radial distance

$$\phi_{\Lambda} \sim \frac{B_n}{r^{n+2}},$$

where $B_n$ describes the coupling between the mass and the vacuum fluctuations. This also leads to the conjecture that mass not only curves space but also generates changes in the vacuum structure, which are of quantum origin.

In Hess & López-Moreno (2019a), the parameter $n = 3$ was used, which had to be modified later on to $n = 4$ (Hess 2019, 2020; Hess & López-Moreno 2019b), due to the...
observation made in Nielsen & Birnholz (2018, 2019)). In this contribution, a different, improved asymptotic limit for the gravitational wave solution is introduced.

The path of solving the modified gravitational theory, using pc-GR, can be immediately applied to other extensions which add to the $g_{00}$ component an arbitrary function in $r$. The form of the function restricted by observations in the solar system (Will 2006), requires a leading dependence of $\sim 1/r^4$ ($n > 2$) and by analyzing the first gravitational wave event Nielsen & Birnholz (2018, 2019)) requires a leading contribution of at least $1/r^4$. The form of the function restricted to observations within the Solar System (Will 2006), requires a leading dependence of $1/r^6$ ($n > 2$), and by analyzing the first gravitational wave event Nielsen & Birnholz (2018, 2019)) requires a leading contribution of at least $1/r^4$. Therefore, the most general correction is proportional to $f(r)/r^4$, where an expansion of $f(r)$ starts with a constant value. Here, we will start with a correction in $1/r^4$ only and, thus, it can be regarded as the first correction one can think of.

The origin of the leading correction will not be discussed here, but we refer to the literature (Hess 2020; Hess et al. 2015).

Only axial modes are considered, because the equation for polar modes involves a more complicated potential, decreasing the convergence of the particular iteration method applied. In GR, polar and axial modes provide the identical spectrum, which for a generalized metric is not necessarily satisfied. The study of the possible equivalence of both types of modes will be addressed in future, and represents an interesting question.

This contribution is organized as follows: In Section 2, the main ingredients of the Schwarzschild solution in pc-GR is resumed, emphasizing the differences between GR and pc-GR. The Regge-Wheeler equation (Regge & Wheeler 1957) is constructed using the path exposed in (Chandrasekhar 1983). A very brief description is given of the Asymptotic Iteration Method (AIM) (Ciftci et al. 2003, 2005), with its improved version (Cho et al. 2012), used to solve numerically the differential equation. In Section 3, an application is presented and a comparison to earlier results. Finally, in Section 4 conclusions are drawn.

2 | THEORY

The length element of the Schwarzschild solution depends on the metric component $g_{00} = e^{2\nu}$ and using the notation of Chandrasekhar (1983):

$$ e^{2\nu} = g_{00} = \left(1 - \frac{2m(r)}{r}\right), $$

$$ m(r) = m_0 \left(1 - \frac{27}{32} \left( \frac{m_0}{r^5} \right) \right). $$

The $m_0$ is the mass measured at infinite distance and $\chi$ is a parameter which for $\chi = 0$ corresponds to the limit of GR and for $\chi = 1$ the pc-GR limit. The expression of $m(r)$ guaranties that for $a > 0$ ($a$ is the Kerr rotational parameter with $0 \leq a < 1$) there is no event-horizon and only at $a = 0$ there is still one at $r = \frac{2}{3}m_0$. This is the latest form as used in pc-GR (Hess 2019, 2020; Hess & Lópe Moreno 2019b).

Arbitrary values between 0 and 1 were also considered in Hess & Lópe Moreno (2019b)), were the transition from GR and pc-GR was studied and related to a phase transition. Other dependencies on $r$ can also be used as can be found in other contributions in this volume.

For the study of stability of a Schwarzschild black hole in GR and pc-GR, we follow the path proposed in (Chandrasekhar 1983). The metric is defined as

$$ ds^2 = e^{2\nu} dt^2 - e^{2\nu} (d\phi - o dt - q_r dr - q_{\theta} d\theta)^2 - e^{2\mu} dr^2 - e^{2\nu} d\theta^2. $$

The Schwarzschild metric is obtained, setting $\omega$, $q_r$, $q_{\theta}$ to zero and the exponential factors to the known expressions. The metric Equation (3) allows small deviations, where changing only $\omega$, $q_r$, $q_{\theta}$ leads to the axial (negative parity) solutions, also known as the Regge-Wheeler solutions (Regge & Wheeler 1957). When $\omega$, $q_r$, $q_{\theta}$ are set to zero but the $\nu$, $\psi$, $\mu_r$, and $\mu_{\theta}$ are varied, one obtains the polar (positive parity) solutions, also known as the Zerilli solutions (Zerilli 1970).

Following Chandrasekhar (1983) we are led for the Regge-Wheeler modes (Regge & Wheeler 1957)) to the equation

$$ \left(\frac{d^2}{dr_*^2} + \omega^2 \right) Z^{(-)} = V^{(-)} Z^{(-)}, $$

with $\omega$ being the complex frequency $\omega_R + i\omega_I$, we are interested in. The real part provides the frequency of the gravitational wave while the imaginary component describes its damping. The $r_*$ is related to $r$ via the relation$^1$

$$ \frac{d}{dr_*} = \left(1 - \frac{2m(r)}{r}\right) \frac{d}{dr}. $$

The potential $V^{(-)}$, for $\chi = 1$, is given by

$$ V^{(-)} = \frac{\Delta}{r^5} \left( \mu_r^2 r + 2r + 2mr - 6m + 2m' r \right), $$

which agrees in form with GR, save the last term which depends on the derivative of $m(r)$ with respect to $r$. 

$^1$r. is also called the Tortoise coordinate.
The corresponding potential for the polar modes is more involved and the solution of the corresponding wave equation is difficult to obtain, due to a poorer convergence of the iteration method applied.

2.1 The asymptotic limit

The solutions have to satisfy the asymptotic conditions of an outgoing wave to infinity and an ingoing wave toward the event horizon:

\[
Z^\pm (r) = \begin{cases} \displaystyle e^{-i\omega r}, r_s \to -\infty \\ \displaystyle e^{i\omega r}, r_s \to +\infty \end{cases} .
\] (7)

The Tortoise coordinate has an analytic solution, as exposed next, and thus also the asymptotic limit in Equation (7) has an analytic expression. Defining the dimensionless coordinate \( y = \frac{r}{m_0} \) this solution is given by

\[
y_s = y + 2 \ln \left( \frac{r}{2} - 1 \right) \quad (8)
\]

For pc-GR, surprisingly also an analytic solution exists, namely

\[
y_s = y + 2 \ln \left( \frac{y}{2} - 1 \right) + 2 \ln (3) - \frac{9}{2 (12 - 8y)} - \frac{\arctan \left( \frac{1 + 2y}{\sqrt{2}} \right)}{4\sqrt{2}} .
\] (9)

Note, that when \( m(r) \) in Equation (2) is slightly changed, no analytic solution could be found. This is a surprising fact which we still would like to understand.

The first line in Equation (9) is the analog to the solution in GR, where \( \frac{r}{2} \) in the logarithm is changed to \( \frac{r}{3} \), now the \( \frac{3}{2} \) referring to the position of the event horizon. In the second line, only the term \( \frac{9}{12 - 8y} \) is of importance, because its contribution at \( y = \frac{3}{2} \) dominates. The other terms are either constant or approach a constant value near the horizon or at infinity.

In the next section, the numerical method, the so-called Asymptotic Iteration Method (AIM) (Cho et al. 2012) is briefly resumed. For that we need to change the variable \( y \) to another one \( \xi \) with a compact support. We define

\[
\xi = 1 - \frac{2}{y}, \text{ for } \chi = 0,
\]

\[
\xi = 1 - \frac{3}{2y}, \text{ for } \chi = 1.
\] (10)

The variable \( \xi \) is constrained to the region \([0, 1]\).

Extracting the asymptotic limit for axial modes, defining \( \tilde{\omega} = m_0 \omega \), leads to an ansatz for the asymptotic behavior of the gravitational wave, namely

\[
Z^\pm (\xi) = (1 - \xi)^{-2m_0} e^{2m_0 (\frac{1}{m_0} - 1) \tilde{\omega} \xi} P^\pm (\xi), \text{ for } \chi = 0
\]

\[
Z^\pm (\xi) = (1 - \xi)^{-2m_0} e^{-2m_0 (\frac{1}{m_0} - 1) \tilde{\omega} \xi} e^{\frac{m_0 - 1}{2(1 - m_0)} \tilde{\omega} \xi} P^\pm (\xi), \text{ for } \chi = 1.
\] (11)

The exponents in Equation (11) are obtained, substituting in the plane wave Equation (7) for \( r \), the dominant contribution in \( y_s = \frac{r}{m_0} \), of Equations (8), for GR, and Equation (9), for pc-GR, at the limits \( y_s \to \infty \) and \( y_s \to 0 \) toward the event horizon.

2.2 The asymptotic iteration method (AIM)

H. Ciftci, R. L. Hall, and N. Saad introduced in Ciftci et al. (2003, 2005)) the AIM. This method is quite effective in resolving second-order differential equations of the form (Cho et al. 2012)

\[
f''(x) = \lambda_0 (x) f'(x) + s_0 (x) f(x) .
\] (12)

Deriving both sides several times leads to an equivalent differential equation

\[
f^{(p+1)}(x) = \lambda_{p-1} (x) f'(x) + s_{p-1} (x) f(x) ,
\] (13)

with \( p \) as the iteration number, and

\[
\lambda_p (x) = \lambda_{p-1} (x) + s_{p-1} (x) + \lambda_0 (x) \lambda_{p-1} (x)
\]

\[
s_p (x) = s'_{p-1} (x) + s_0 (x) \lambda_{p-1} (x) .
\] (14)

Convergence is achieved, when the ratio of \( s_p(x) \) and \( \lambda_p(x) \) does not change with \( p \). This condition is fulfilled, when

\[
s_p (x) \lambda_{p-1} (x) - s_{p-1} (x) \lambda_p (x) = 0.
\] (15)

A practical obstacle is the dependence on the variable \( x \), which makes it difficult to get a rapid convergence. In Cho et al. (2012)), this problem was partially resolved by expanding the \( \lambda_p \) and \( s_p \) in a Taylor series around a point \( x \), leading to new equations which only depend on the frequency \( \omega \). Defining

\[
\lambda_p (\rho) = \sum_{i=0}^{\infty} \lambda^i_p (x - \rho) i
\]

\[
s_p (\rho) = \sum_{i=0}^{\infty} s^i_p (x - \rho) i.
\] (16)
and substituting this into Equation (14) leads to a new recursion relation for the coefficients $c_p^0$ and $d_p^0$, namely

$$d_p^0 c_{p-1}^0 - d_{p-1}^0 c_p^0 = 0. \quad (17)$$

One problem remains, i.e., the result still depends on the point $x$ in the Taylor series Equation (16).

In order to obtain convergence, the following rules should be observed, which are the result of experience:

- A compact support for the range of the coordinate has to be used, i.e. the coordinate $\xi$, which is zero at the event horizon and approaches 1 for $r \to \infty$. This leads to a better approximation near the asymptotic limits.

- The asymptotic behavior plays an important role and has to be extracted as optimal as possible. In this respect, the improved dependence of the Tortoise coordinate $y$ on $y$, as obtained in Equation (9), is of great help, which also explains the deficiency of a former calculation.

- The expansion should be done around the maximum (minimum) of the potential $V^{(\ell)}$. Often, the convergence is increased by choosing a point slightly larger than the position of the maximum/minimum.

- The iteration method was programmed with Wolfram (2020). It is recommended to use only rational numbers and to approximate irrational values by rational ones, otherwise MATHEMATICA develops numerical instabilities and the method does not converge at all.

2.3 Relation of frequency to the object’s mass

This part is important for the extraction of the deduced mass of the black hole. We assume the sample frequency of the order of $v = 250$ Hz, i.e. $(2\pi)v \approx 1,571$, which is in the range of sensitivity of LIGO.

We define

$$\tilde{\omega} = m_0 \omega = \tilde{\omega}_R + i \tilde{\omega}_I, \quad (18)$$

where $m_0$ is the central mass in units of [km] and $\omega$ is in units of [km$^{-1}$]. The gravitational constant is set to $G = 1$ and $c = 1$. With these numbers, the conversion of Hz to km$^{-1}$ is $1$ Hz $= \frac{1}{5} \times 10^{-5}$ km$^{-1}$, thus $\omega = 5.24 \times 10^{-3}$ km$^{-1}$.

In the first observed gravitational wave event reported in Abbott et al. (2016a)) the mass of the final black hole is of 60 solar masses. Taking into account that the Schwarzschild radius of the sun is approximately 1.5 km, a 60 solar mass black hole has a radius of $m = 90$ km. Using Equation (18), the $\tilde{\omega}_R$ value is 0.47 for 250 Hz. This value will be compared to the frequency obtained in pc-GR and in an inverse calculation the new mass can be deduced.

### TABLE 1

| Axial frequency $\tilde{\omega}$ | pc-GR | (Cho et al. 2012) |
|----------------------------------|-------|------------------|
| 0.3737-i 0.0890                 | 0.3887-i 0.0782 | 0.3737-i 0.0896 |
| 0.3467-i 0.2740                 | 0.3613-i 0.2391 | 0.3467-i 0.2739 |
| 0.3011-i 0.4783                 | 0.3092-i 0.4147 | 0.3012-i 0.4785 |
| 0.2515-i 0.7051                 | 0.2430-i 0.6156 | 0.2523-i 0.7023 |
| 0.2075-i 0.9468                 | 0.1809-i 0.8413 | —                |
| 0.1692-i 1.1957                 | 0.1351-i 1.0773 | —                |
| 0.1339-i 1.4483                 | 0.0086-1.1622   | —                |
| 0.0958-i 1.7031                 | 0.1069-i 1.3271 | —                |
| 0.0699-i 2.2970                 | 0.0611-i 1.5612 | —                |
| 0.840-i 2.5639                  | 0.0741-i 1.8065 | —                |

Note: The index $R$ refers to the real part and $I$ to the imaginary one of the complex frequency.

For example, the first entry for GR in Table 1 is 0.37, which would correspond, assuming 250 Hz, to $m_0 = \frac{\tilde{\omega}}{\omega} \approx 71$ solar masses. Caution has to be applied, because in observation one does not observe a single frequency but a frequency distribution. In order to obtain this frequency distribution in theory, one has to perform a numerical approach, as for example done in Rezzolla & Zanotti (2013) based on relativistic hydrodynamics. We are, for the moment, not able to do this type of involved calculations.

3 | APPLICATIONS

In this section, we present the frequencies of the ring down modes, as obtained within GR and pc-GR and comparing both. In Table 1, the first 10 frequencies within GR (first column) and pc-GR (second column) are listed, for 400 iterations. The values of the real and imaginary parts are all of the same range, in some cases the real part of the frequency is larger, while in others it is inverted. The imaginary part, which controls the decay, is of the same order. For comparison, in the third column the frequencies from Cho et al. (2012) are listed, obtained after 15 iterations.

In Figure 1, the frequency distribution for the axial modes in a limited range of $\tilde{\omega}_I$ is shown. On the left hand side, the result in GR is depicted and on the right hand side the one of pc-GR. Note that the structure is very similar. The same is depicted in Figure 2 with a larger range of $\tilde{\omega}_I$. This shows that the two results are in structure similar, showing a fish-like form, with the tail to the left and the head to the right. This head is not physical because it moves with increasing iteration steadily to the right and no convergence is obtained.
The comparison of axial modes using 20 (red dots) and 40 (blue dots) iterations in the AIM. The left panel shows the result for GR while the right panel is for pc-GR. In the range shown, the frequencies are nearly the same.

The absolute values of \( \tilde{\omega} \) are for values above 15 much larger by various factors. If the center of the frequency distribution is in this range, it would imply that the mass of the source object is also larger and also the luminosity distance.

Which of the cases discussed corresponds to the observed data depends very much on measuring both parts of the wave signal: Its real and imaginary part. Measuring the ratio allows possibly to differentiate between GR and pc-GR.

3.1 | Comparison to a former calculation of axial modes

In Hess & López-Moreno (2019a)) the axial modes in pc-GR were calculated, guessing a particular asymptotic limit, leaned from the GR case. The result is depicted in the Figures 3 and 4.

Due to the new improved asymptotic limit obtained for \( n = 4 \) we also improved the one for \( n = 3 \) and studied the effect of changing the asymptotic limit on the final result for the frequencies.

The asymptotic limits used are

- case a) : \( Z_a^{(-)} = (1 - \xi) \frac{4m}{\pi} \xi^{-\frac{4m}{\pi}} e^{\frac{4m}{\pi} \xi} P(\xi), \) (19)
- case b) : \( Z_b^{(-)} = (1 - \xi)^{-2\bar{\omega}_0} \xi^{-2\bar{\omega}_0} e^{\frac{4m}{\pi} \xi} P(\xi), \) (20)
- case c) : \( Z_c^{(-)} = (1 - \xi)^{-2\bar{\omega}_0} \xi^{-2\bar{\omega}_0} e^{\frac{4m}{\pi} \xi} P(\xi). \) (21)

In Hess & López-Moreno (2019a)) the case a) limit in Equation (19) was used. The asymptotic limit was guessed from the one of GR, changing \( 2m \) by \( \frac{4}{3} m \), the position of the event horizon for \( n = 3 \). The case b) (Equation (20)) corresponds to guess the asymptotic limit for \( n = 3 \) using the one for \( n = 4 \), without the second exponential factor \( e^{-\frac{2\bar{\omega}_0}{\xi}} \) and in the case c) (Equation (21) this exponential factor is included. Note, that going from case a) to case b) the structure is the same but already with changes in the absolute values of \( \omega_{GR} \). Noticeable, including the second exponential factor has an important effect, cutting the low-frequency branch. Now, the figure resembles more the one for \( n = 4 \).

In conclusion, the results in Hess & López-Moreno (2019a)) suffer from taking into account a deficient asymptotic limit, which is now improved.
4 | CONCLUSIONS

In this contribution, results have been presented, on the ring-down frequencies of a Schwarzschild black hole: Within the standard theory of relativity and using a specific example of a more general form of the metric, having added a $1/r^4$ correction, which also serves as a parting point for more general forms. The extended theory used was the pseudo-complex General Relativity. The metric of this extended theory depends on a parameter which describes the coupling between a central mass and the vacuum fluctuations nearby. This parameter of the specific example was chosen such that no event horizon appears, but is not limited to this assumption.

We showed that the axial modes are stable solutions of the extended theory. The values for the frequencies in GR, as obtained in other studies, were reproduced. In general, the structure of the axial modes were presented up to large damping modes. At low $-\tilde{\omega}_I$ both GR and pc-GR show a very similar structure. For large $-\tilde{\omega}_I$ the deviations are strong, though the overall structure in both resembles that of a “fish”, with the tail at low $-\tilde{\omega}_I$. However, the “fish-head” is not a stable structure and moves steadily to larger imaginary frequency values, thus it is not of physical origin.

The results were compared to an earlier calculation, showing that using the correct asymptotic limit is essential in order to obtain better results with an acceptable convergence.

More studies are still to be done, as to apply the method to polar modes, which defy to give a good convergence. Also, if the polar modes show the same frequency spectrum in pc-GR as it does in GR, is a question we would like to answer.

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