Learning Structure of Partial Markov Random Field via Partitioned Ratio

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Abstract

A new concept, partitioned ratio is proposed to find the partial connectivity of the Markov random field. First we argue this partitioned ratio has a profound link with the Markov properties of random variables via its factorization. Specifically, partitioned ratio may be further decomposed into Bridges, a novel subgraph structure, capturing the partial connectivity of the Markov random field, which can be roughly considered as the “link” structure between two partitions. Second, a simple one-shot optimization is illustrated to learn the sparse factorizations of partitioned ratio efficiently, regardless the Gaussianity of the joint distribution or the marginal distributions. Third, we show the sufficient conditions for the proposed algorithm recovering the correct pairwise bridge factorizations.

1 Introduction

An undirected graphical model, or a Markov random field (MRF) [Koller and Friedman (2009)] has wide range of applications in real world, such as natural language processing, computer vision, and computational biology. The structure of MRF, which encodes the interactions among random variables, is one of the key interests of MRF learning tasks. Recently, machine learning
community has seen huge progresses on learning the *sparse* structure of MRF, thanks to the pioneer works on sparsity inducing norms \cite{Tibshirani(1996), Zhao and Yu(2006), Wainwright(2009)}.

According to Hammersley-Clifford Theorem \cite{Hammersley and Clifford(1971)}, the joint distribution of an MRF factorizes over smaller subgraph structures, cliques. By learning a sparse factorization of the joint distribution, we may recover the corresponding structure of an MRF.

A majority of the previous works fall into the category of regularized maximum likelihood approach which maximizes the likelihood function of a probability model under sparsity constraints. Graphical lasso \cite{Friedman et al.(2008), Banerjee et al.(2008)} considers a joint Gaussian density model parameterized by inverse covariance matrix, where zero elements indicate the conditional independence among random variables, while others have developed useful variations of graphical lasso in order to loosen the Gaussianity assumed on data \cite{Liu et al.(2009, 2012), Loh and Wainwright(2012)}.

The latest advance along this line of research has been made by considering a node-wise conditional probability model. Instead of learning all the structures in one shot, they focus on learning the neighborhood structure of one single random variable at a time. Maximizing the conditional likelihood leads to simple logistic regression (in the case of Ising model) \cite{Ravikumar et al.(2010)} or linear regression (in the case of Gaussian model) \cite{Meinshausen and Bühlmann(2006)}.

Unfortunately, maximum (conditional) likelihood method is difficult to be extended to general non-Gaussian graphical models, since computing the normalization/partition function for continuous distributions is in general intractable. Though one may use sampling such as Monte Carlo methods \cite{Robert and Casella(2005)} to approximate the normalization term, there is no universal guideline telling how to choose sampling parameters so that the approximation error is minimized.

Is there a way to obtain the MRF structure without learning a probability model? Intuitively, the probability density/mass function itself completely characterizes a distribution, i.e., knowing density/mass means knowing everything about a distribution. Therefore, a density/mass function can be very hard to learn. On the other hand, the conditional independence is a very fundamental property among random variables. Learning a quantity that is sufficient to tell the structure of MRF should be easier than modelling the entire distribution.

The above intuition leads to a novel concept of the partitioned ratio (PR) \( \frac{P(X)}{P(X_1)P(X_2)} \), which is a ratio between the joint probability \( P(X) \) and the product between its marginals \( P(X_1)P(X_2) \). In this paper, we argue that this quantity factorizes over the graph structure of an MRF. In the same way that the joint distribution can be decomposed into clique potentials, we prove PR also factorizes over subgraph structures called *bridges*, which indicates the *connectivity* between two partitions of random variables \( X_1 \) and \( X_2 \) in an MRF.

Conventionally, PR is a measure of the independence between two random variables. In this paper, we show that the factorization of this quantity indicates the linkage between two partitions of MRF, which is a very natural extension of the regular usage of PR.

Most importantly, we show the sparse factorization of this quantity may be learned via a one shot convex optimization procedure, which can be computed efficiently even for the most gen-
eral, non-Gaussian distributions. The correct recovery of sparse bridge structure is guaranteed under mild assumptions.

We organize the rest of this paper as follows. Section 2 gives the background and motivation of this research, as well as some preliminary concepts. Section 3 states the factorization theorems of PR and a pairwise simplification. We illustrate an algorithm for learning a PR with a sparse factorization in Section 4, followed by Section 5 where we state a support consistency theorem of the proposed algorithm. Some related works are reviewed in Section 6. We further validate our theorems using experiments in Section 7.

2 Background

It has been well known that a joint probability distribution $P$ can be factorized into smaller factors if and only if $P$ satisfies certain Markov properties. Here we restate such a theorem for our convenience.

2.1 Background and Motivation

To avoid confusion, we limit our discussion on strictly positive distributions.

Definition 1. For a joint density function $P(X)$ of random variables $X = \{X_1, X_2, \ldots, X_m\}$, if for all $i$, $P(X_i|X_{\setminus i}) = P(X_i|X_{N(i)})$, where $N(i)$ is the neighbours of node $X_i$ in graph $G$, then $G$ is called a Markov random field.

Definition 2. For a joint distribution $P$ on a set of random variables $X = \{X_{C_1 \cup C_2 \ldots C_s}\}$, where $C_1, C_2, \ldots, C_s$ are indices of nodes corresponding to all the cliques in graph $G$, if the joint density can be factorized as

$$P(X) = \frac{1}{Z} \prod_{i=1}^{s} \phi_{C_i}(X_{C_i}),$$

where $Z$ is the normalization constant and each factor $\phi_{C_i}$ is defined only on the subset of $X$, then $P(X)$ is called a Gibbs distribution.

Theorem 1 (Hammersley and Clifford (1971)). A probability distribution $P$ is a Gibbs distribution over $G$ (Definition 2) if and only if $G$ is an Markov random field (Definition 1).

Theorem 1 is the keystone of many MRF structure learning methods. It states, by learning a sparse factorization of a joint distribution, we are able to recover the structure of the underlying graphical model. However, learning a joint distribution has never been an easy task.

A prominent issue is that the normalization constant is difficult to compute for joint distributions in general. The Gaussian distribution is an exception where the normalization term has a closed form.

Therefore, an intriguing question is, whether there is an alternative to the joint distribution, whose factorization also relates to the structure of MRF? Ideally, such factorization should be
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Figure 1: Comparison between Cliques and Bridges. Bridges are generalizations to inter-partition cliques, and therefore its structure is usually larger than inter-partition cliques.

efficiently estimated from samples and the estimation procedure must provide good statistical guarantees.

The partitioned ratio (PR) \( \frac{P(X)}{P(X_1)P(X_2)} \), is the ratio of between the joint probability distribution \( P(X) \) and the product of its marginals \( P(X_1)P(X_2) \), and is used as a basic measure of independence between two random variables. Surprisingly, we can also show PR has the desired properties to indicate the structure of an MRF, since it is:

- factorized over the structure of an MRF (see Section 3),
- easy to compute for non-Gaussian distributions (see Section 4),
- guaranteed to give consistent estimation even under high-dimensional settings (see Section 5).

Let us begin with some key concepts.

2.2 Definitions

Notations. Sets are denoted using upper-case letters, e.g., \( X, C \). An upper-case letter with an upper-case subscript \( X_A \) means a subset of \( X \) indexed by the elements in \( A \). \( P(X = x) \) is the probability of an event \( X = x \).

First we show how to divide the entire set of random variables into two non-overlapping subsets.

Definition 3 (Partitions on Random Variables). For a set of random variables, \( X = \{X_1, \ldots, X_m\} \), \( M \) is the index set, and \( \{T_1, T_2\} \) is a non-trivial partition on \( M = \{1, \ldots, m\} \), i.e., \( T_1 \cup T_2 = M, T_1 \cap T_2 = \emptyset \) and \( T_1, T_2 \neq \emptyset \).
Thus, by marginalizing the joint distribution $P(X)$ over either $X_{T_1}$ or $X_{T_2}$, we will obtain two marginal distributions $P(X_1)$ or $P(X_2)$

Then we introduce a graph, whose nodes correspond to the random variables in $P$.

**Definition 4 (Partitions on Cliques).** For a graph $G = (V, E)$, whose node set $V$ shares the same index set $M$ as $X$, we define $C$ as the set of clique indices in $G$, i.e.

$$C = \{C|C \subseteq M, V_C \text{ are nodes in a clique of } G\}.$$ 

Moreover, $C = \{C_\lor, C_\land\}$, where $C_\lor$ is defined as the inter-partition clique set,

$$C_\lor = \{C|C \in C, C \cap T_1 \neq \emptyset, C \cap T_2 \neq \emptyset\},$$

and $C_\land = \setminus C_\lor$. Similarly, we can split the edge set into disjoint set $E = \{E_\lor, E_\land\}$, where $E_\lor$ have nodes across two partitions, while $E_\land$ have both nodes within the same partition.

The cliques in such a graph are also separated into two groups: The group of cliques that is “across” two partitions, and the group is not. These inter-partition cliques are the building blocks of more sophisticated subgraphs: Bridges.

**Definition 5 (Bridge).** For a graph $G = (V, E)$, we define a bridge $B$ as a subgraph $B = \bigcup C$ for some $C \in C_\lor$, and this subgraph become a clique if and only if a subset $E \subseteq E_\land$ is added to this subgraph.

In plain words, the Bridge structure is the subgraph that is full connected excepts edges across the nodes on the same side may be missing.

**Proposition 1.** A inter-partition clique $C \in C_\lor$ is a bridge.

**Proof.** By definition, any $C \in C_\lor$ become a clique if and only if an empty edge set $E = \emptyset, \emptyset \subset E_\land$ added to this subgraph. 

Here we illustrate some bridge structures of three simple graphs in Figure 1.

Now, by using the novel graph structure, a bridge, we are able to define a new quantity: Gibbs partitioned ratio. This is an analogy to a Gibbs distribution used in Hammersley-Clifford Theorem.

**Definition 6 (Gibbs Partitioned Ratio).** For a joint distribution $P$ on a set of random variables $X = \{X_1, X_2\}$, if the partitioned ratio has the form

$$\frac{P(X_1, X_2)}{P(X_1)P(X_2)} = \frac{1}{Z_B} \prod_{B \in B} \phi_B(X_B),$$

where $B$ is the set of all bridges in $G$, then $\frac{P(X_1, X_2)}{P(X_1)P(X_2)}$ is called Gibbs partitioned ratio (GPR) over $G$.

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1From now on, we use $X_s$ as short for $X_{T_s}$ whenever there is no ambiguity.

2We do not distinguish the indices of subgraph from subgraph itself.
3 Factorization over Bridges

In this section, we will investigate a curious question: can we have a similar factorization property like Theorem 1 for the partitioned ratio? Not surprisingly, the answer is yes.

3.1 Fundamental Properties

There are two steps in introducing our factorization theorems. The first step is to establish Markov property of random variables using the factorization of GPR.

**Theorem 2.** Given a graph $G$, the fact that $GPR \frac{P(X_1, X_2)}{P(X_1)P(X_2)}$ factorizes over bridge set $B$ in $G$ implies the following partition Markov properties:

$$P(X_i | X_{T_1} \setminus i, X_{B_{i,2}}) = P(X_i | X_{T_1} \setminus i), \forall i \in T_1,$$

(1)

$$P(X_i | X_{T_2} \setminus i, X_{B_{i,1}}) = P(X_i | X_{T_2} \setminus i), \forall i \in T_2,$$

(2)

where $B_{i,2} = \cup(B \cap T_2), \forall B \in B, i \in B$ and $B_{i,1}$ is defined similarly.

This theorem can be simply understood as, whenever the pair $\langle X_u, X_v \rangle, u \in T_1, v \in T_2$ does not co-occur in any bridge factor, then $X_u, X_v$ are conditionally independent given the rest of the random variables.

**Proof.** Here we only shows the proof for Eq. (1). Define a new subset of $B$: $B_i$ where each member bridge contains $i$ as a member, i.e., $B_i = \{B | B \in B, i \in B\}$. Then

$$P(X_i | X_{T_1} \setminus i, X_{B_{i,2}}) = \frac{1}{Z} \int_{X_{B_{i,2}}} P(X_1) P(X_2) \prod_{B \in B} \phi_B$$

$$\frac{1}{Z} \int_{X_{T_1} \cap B_{i,2}} P(X_1) P(X_2) \prod_{B \in B} \phi_B$$

$$= \frac{1}{Z} \int_{X_i} P(X_1) \prod_{B \in B_i} \phi_B \cdot \left( \int_{X_{B_{i,2}}} P(X_2) \prod_{B \in B_i} \phi_B \right)$$

$$= \frac{1}{Z} \int_{X_i} P(X_1) \prod_{B \in B_i} \phi_B \cdot \left( \int_{X_{B_{i,2}}} P(X_2) \prod_{B \in B_i} \phi_B \right)$$

$$= P(X_i | X_{\setminus i})$$

From which, we obtain the desired equality. \qed

Next, let’s prove the other direction: From the Markov property to the factorization.
Theorem 3. Given a graph $G$, if a joint distribution has a partition Markov property
\[ P(X_i | X \setminus i) = P(X_i | X_{T_1 \setminus i}, X_{B_{i,2}}), \forall i \in T_1, \]
and
\[ P(X_i | X \setminus i) = P(X_i | X_{T_2 \setminus i}, X_{B_{i,1}}), \forall i \in T_2, \]
then its partitioned ratio is a GPR that factorizes over bridge set $B$ in $G$:
\[ \frac{P(X_1, X_2)}{P(X_1)P(X_2)} = \frac{1}{Z} \prod_{B \in B} \phi_B(X_B). \] (3)

Proof. This proof is constructive. We use the following potential function:
\[ \psi_S(X_S = x_S) = \prod_{Z \subseteq S} \Delta_Z(X_Z = x_Z)^{-1^{\|Z\| - |Z|}}, \]
where $X_S$ is a subset of $X$. Let’s denote $Z \cap T_1 = Z_1$ and $Z \cap T_2 = Z_2$. Then
\[ \Delta_Z(x_Z) = \begin{cases} \frac{P(x_Z, 0)}{P(x_{Z_1}, 0_{Z_1})P(x_{Z_2}, 0_{Z_2})}, & \exists C \subseteq C, C \subseteq Z, \\ 1, & \text{otherwise}, \end{cases} \] (4)
where $P(x_A, 0_B)$ is short for $P(X_A = x_A, X_B = 0)$. Due to the inclusion-exclusion principle (see, e.g. Koller and Friedman 2009, 4.4.2.1), it can be shown that
\[ \prod_{S \subseteq G} \psi_S(X_S = x_S) = \Delta_G(x). \]

If the graph $G$ includes one or more inter-partition cliques, then $\Delta_G(x) = \frac{P(x)}{P(x_{T_1})P(x_{T_2})}$, which is exactly the GPR. However, if $G$ does not include one or more inter-partition cliques, meaning $X_1$ are completely independent of $X_2$, then $\Delta_G(x) = 1$, which is the exact value that a GPR would take in such case.

The rest of the proof tries to show that if $S$ is not a bridge, then $\psi_S(X_S = x_S) = 1$. Obviously, for $S \cap T_1 = \emptyset$ or $S \cap T_2 = \emptyset$, by definition, $\psi_S = 1$.

Otherwise, if $S$ is not a bridge, we should be able to find two nodes, indexed by $a \in T_1 \cap S$ and $b \in T_2 \cap S$, that are not connected by an edge. Therefore we may write the potential function for a subgraph $S$ as
\[ \psi_S(X_S = x_S) = \prod_{W \subseteq S \setminus \{a,b\}} \left( \frac{\Delta_W(x_W)\Delta_{W \cup \{a,b\}}(x_W)}{\Delta_{W \cup \{a\}}(x_{W \cup \{a\}})\Delta_{W \cup \{b\}}(x_{W \cup \{b\}})} \right)^*, \]
where $*$ means we do not care the exact power which can be either -1 or 1, and
\[ \frac{\Delta_W(x_W)\Delta_{W \cup \{a,b\}}(x_W)}{\Delta_{W \cup \{a\}}(x_{W \cup \{a\}})\Delta_{W \cup \{b\}}(x_{W \cup \{b\}})} = \frac{P_W P_{W \cup \{a\}}}{P_{W \cup \{a\}} P_W(b)} \cdot \frac{P_{W \cup \{a\}} P_{W \cup \{b\}}}{P_W(b) P_{W \cup \{b\}}} = \frac{P_W P_{W \cup \{a\}} P_{W \cup \{b\}}}{P_{W \cup \{a\}} P_W(b) P_{W \cup \{b\}}}, \] (5)
where we have simplified the notation $P(x_A, 0_{A})$ as $P_A$, $P(x_A, 0_{T_1 \setminus A})$ as $P'_A$, and $W \cap T_1$ as $W_I$.\footnote{Note that in this context that the superscript does not mean the power.} The second term in the RHS is apparently 1. For the first term in the RHS, we may divide both the numerator and denominator by $P_W \cdot P_W$. Then it yields

$$\frac{P(x_a, x_b | x_W, 0)}{P(x_a | x_W, 0) P(x_b | x_W, 0)} = 1.$$ 

The fact that $a$ and $b$ are not connected in $G$ is implied by the Markov property, and the equality therefore holds under conditional independence between $X_a$ and $X_b$. \hfill $\square$

Simply, the factorization of a GPR is only related to the “links” (or rigorously, bridges) between two partitions. One interesting consequence is that if we have two partitions of an MRF which are linked via a few “bottleneck” links, then the factorization is simply over those sparse bridges, no matter how densely the other parts of the graph are connected.

Theorem 2 points out a promising direction for structural learning in a partitioned MRF. Once the sparse factorizations of a GPR are learned, we are able to recover the sparse connections between two partitions of an MRF. However, Theorem 3 implies a challenge for us to model such a GPR in practice: If a bridge involves many nodes (see Figure 1(b)), its parameterization will take up an enormous amount of memory resources.

The above issue motivates us to explore some simplifications of general GPRs.

### 3.2 Simplification of General GPR

A popular subclass of MRF that has been extensively studied in recent years is called pairwise MRF, where the MRF decomposes into the smallest cliques, the edge set $E$ of $G$ by assuming the higher-order interactions between random variables, such as triples or quadruples do not exist. It is natural to ask, under the same assumption, whether our GPR model also factorizes into smaller components, such as edges.

Unfortunately, the answer is not entirely positive, as we may see directly from the definition of a potential function $\psi$ on a bridge $B$. Recall

$$\psi_B(X_B = x_B) = \prod_{Z \subseteq B} \Delta_Z(X_Z = x_Z)^*,$$

and by definition $4$, $Z$ must be a bridge. Otherwise, if a subset of $B$ is only “one-sided”, $\Delta_Z(x_Z)$ is nullified. The factor $\Delta_Z$ is defined as a ratio between a partially specified joint distribution and a partially specified marginal distribution. As the joint distribution factorizes only over the pairwise factors, we may obtain the following decomposition of $\Delta_Z(X_Z)$:

$$\Delta_Z(X_Z) = \frac{\prod_{i,j \in Z} P(x_i, x_j, 0)}{P(x_{Z_1}, 0_{T_1 \setminus Z_1}) P(x_{Z_2}, 0_{T_2 \setminus Z_2})}.$$
However, a pairwise joint density function does not guarantee a pairwise marginal distribution in general. Therefore, it is difficult to further factorize marginal distributions $P(x_{Z_1}, o_{T_1 \setminus Z_1}), P(x_{Z_2}, o_{T_2 \setminus Z_2})$. Finally, the bridge potential factorizes as

$$
\psi_B(x_B) = \psi_{B_1}(x_{B_1})\psi_{B_2}(x_{B_2}) \prod_{i \in B_1, j \in B_2} \psi_{i,j}(x_{i,j}),
$$

where we denote subgraph indices as $B_1 = B \cap T_1, B_2 = B \cap T_2$.

Although in general we cannot get the pairwise factorization of a bridge, we list two useful assumptions on the topology of the graph and the model representation under which the bridges do factorize pairwisely:

- (Topology) If $|B_1| \leq 2$ and $|B_2| \leq 2$;
- (Model) The marginal distributions $P(X_1)$ and $P(X_2)$ over the bridge nodes are also factorized in pairwise fashion, which can be formalized in the following definition:

**Definition 7.** Two partitions of a random variable $X_1$ and $X_2$ in an MRF are called **pairwisely connected** iff. the following conditions are met: for any $C \in C_V$, there exist pairwise potentials in joint distributions such that

$$
P(x_C, o_C) \propto \prod_{i,j \in C} \phi_{i,j}(X_{i,j}),
$$

and for any $B \in B$, there exist pairwise potentials in marginal distributions such that

$$
P(x_{B_1}, o_{T_1 \setminus B_1}) \propto \prod_{i,j \in B_1} \phi^1_{i,j}(X_{i,j})$$
$$
P(x_{B_2}, o_{T_2 \setminus B_2}) \propto \prod_{i,j \in B_2} \phi^2_{i,j}(X_{i,j})
$$

If either of the above two criteria holds for partitions of random variables in an MRF, these partitions are said to be **simply connected**.

The question is, however, how strong these two assumptions are? Though we focus on theoretical analysis in this paper, we list two possible scenarios where graphical models do satisfy the above assumptions:

- Conditional random fields used in natural language processing: $X_1$ are words while $X_2$ are labels (e.g., grammatic tags). Each word may link to only one of the labels. Though we may use $n$-gram models for modeling long-range interactions among words, for each bridge $B$, $|B_1| = 1$ and $|B_2| = 1$. Therefore, the assumption on topology of the graph is satisfied.

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4Gaussian distribution is an exception.
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Figure 2: Two illustrative examples of MRFs where two partitions are “simply connected”. Left: a conditional random field for NLP tagging. Right: an image and its sub-images.

- Image and sub-images: the pairwise MRF assumptions are widely used in image processing. For an image that is partitioned into two sub-images, it is natural to consider each of these images as a pairwise MRF, i.e., pixels only affect each other in pairs.

Illustrative examples of these “simply connected” MRFs are given in Figure 2.2.

As we have already mentioned at the end of the last section, factorization of GPRs is simple as long as the links between two partitions are sparse. The above analysis tells that we may further break down bridges into pairwise pieces under certain assumptions. Such simplification helps us build computationally efficient algorithms.

4 Algorithm

Theorems 2 and 3 show that the conditional independence among random variables leads to direct parameterizations of a GPR and direct GPR parameterization also indicates the conditional independence. Therefore, we do not need to learn a model for density, but just a straightforward model of the density ratio. Here we adopt a log-linear density ratio model. For a sample of $X, x \in \mathbb{R}^m$, and we define a pairwise GPR model $g_{12}(x; \theta)$:

$$g_{12}(x; \theta) := \frac{1}{N(\theta)} \exp \left( \sum_{u<v} \theta_{u,v}^T f(x_{u,v}) \right),$$

where $\theta_{u,v} \in \mathbb{R}^b$,

$$\theta = (\theta_{1,2}, \ldots, \theta_{1,m}, \theta_{2,3}, \ldots, \theta_{2,m}, \ldots, \theta_{m-1,m})^T,$$

5We focus on continuous distributions from now on.
6Notationally, $g_{12}(x: \theta)$ is equivalent to $g(x_1, x_2; \theta)$, similarly to $\tilde{g}_{12}(x: \theta)$.
7For simplicity, we limit our discussion on continuous distributions. However, the theorems stated in this paper do not have such restriction.
and $f$ is a vector valued feature function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$. Notice that we still model all pairs of dimensions in $x$, but a significant amount of pairs is going to be nullified as indicated by Theorem 3. Further, we denote the unnormalized density ratio model as

$$\bar{g}_{12}(x; \theta) := \exp \left( \sum_{u<v} \theta_{u,v}^\top f(x_{u,v}) \right) .$$

$N(\theta)$ is defined as a normalization function:

$$N(\theta) := \int p(x_1)p(x_2)\bar{g}_{12}(x; \theta) \, dx,$$

where $p(x_1)$ and $p(x_2)$ are the marginal distributions. Then the following equality holds:

$$\int p(x_1)p(x_2)g_{12}(x; \theta) \, dx = 1.$$

The above normalization term can be approximated via two-sample U-statistics Hoeffding (1963),

$$N(\theta) \approx N^n(\theta) := \frac{1}{\binom{n}{2}} \sum_{i \neq j} \bar{g}(x_1^{(i)}, x_2^{(j)}; \theta),$$

where we denote superscript $^n$ as the indication of the sample approximated integral. Notice that this term can always be normalized even if the ratio model is non-Gaussian.

This model can be learned via the algorithm of maximum likelihood mutual information (MLMI) Suzuki et al. (2008), by simply minimizing the Kullback-leibler divergence between $p(x)$ and $p_\theta(x) = p(x_1)p(x_2)g_{12}(x; \theta)$:

$$\min_\theta \text{KL}[p||p_\theta].$$

Then the estimated parameter $\hat{\theta}$ is obtained as

$$\hat{\theta} = \arg\max_\theta \sum_{i=1}^n \sum_{u<v} \theta_{u,v}^\top f(x_{u,v}^{(i)}) - \log N^n(\theta).$$

As an important consequence of Theorem 3, if the links between two partitions are sparse, the density ratio model is in fact very sparse. Therefore, we may use sparsity inducing norms to encourage the group sparsity within the parameter vector $\theta$. Now we have reached our final optimization problem, which is convex, unconstrained, and can be easily solved by the subgradient methods:

$$\hat{\theta} = \arg\min_\theta \sum_{i=1}^n \sum_{u<v} \theta_{u,v}^\top f(x_{u,v}^{(i)}) - \log N^n(\theta) + \lambda \sum_{u<v} ||\theta_{u,v}||_{\mathcal{L}_{\text{MLMI}}(\theta)}$$

where $\mathcal{L}_{\text{MLMI}}(\theta)$ is a sparsity inducing norm.
where we denote $\ell_{\text{MLMI}}(\theta)$ as the negative likelihood function. Again, as Theorem 2 shows, the sparsity pattern of $\theta$ tells the partial structure of a graphical model: if $\theta_{u,v} = 0$, $u \in T_1$, $v \in T_2$, signaling that $u$, $v$ are not in any bridges, then $X_u$ and $X_v$ are conditionally independent in the MRF. However, for $u, v \in T_1$ or $u, v \in T_2$, the corresponding sparsity does not directly stand for connectivity in the joint MRF. Nonetheless, we still impose the sparsity regularizer on all pairwise nodes for both notational and theoretical simplicity.

5 Support Consistency

In the support consistency, we analyze the sparsistency of the MLMI algorithm with a pairwise model. More specifically, we give sufficient conditions for recovering the correct edge set $E_\vee$ in terms of the sample size $n$, data dimensions $m$, and $d$, the number of pairwise potentials in the factorization.

5.1 Notations

Before introducing our consistency results, we define a few notations. In the previous section, a sub-vector of $\theta$ indexed by $(u, v)$ corresponded to a specific pairwise factor of an GPR. From now on, we use new indices with respect to the “oracle” sparsity pattern of the true parameter $\theta^*$ for notational simplicity. By defining two sets of sub-vector indices $S := \{t' \mid \|\theta^*_t\| \neq 0\}$ and its complement $S^c := \{t'' \mid \|\theta^*_t\| = 0\}$, we rewrite the objective (6) as

$$\hat{\theta} = \arg\min_{\theta} \ell_{\text{MLMI}}(\theta) + \lambda_n \sum_{t' \in S} \|\theta_{t'}\| + \lambda_n \sum_{t'' \in S^c} \|\theta_{t''}\|. \quad (7)$$

The support of estimated parameter $\hat{\theta}$ and its complement are denoted as $\hat{S}$ and $\hat{S}^c$. Sample Fisher information matrix $I \in \mathbb{R}^{b(\frac{m}{2}) \times b(\frac{m}{2})}$ is the Hessian of the log-likelihood: $I = \nabla^2 \ell_{\text{MLMI}}(\theta^*)$. $I_{AB}$ is a sub-matrix of $I$ indexed by two sets of indices $A$ and $B$ on rows and columns.

5.2 Assumptions

**Assumption 1** (Dependency Assumption). The sample Fisher information matrix $I_{SS}$ has bounded eigenvalues:

$$\Lambda_{\min}(I_{SS}) \geq \Lambda_{\min}.$$

**Assumption 2** (Incoherence Assumption). The bridge factors cannot be overly affected by other non-bridge factors:

$$\max_{t'' \in S^c} \|I_{t''S}I_{SS}^{-1}\|_1 \leq 1 - \alpha,$$

where $\|Y\|_1 = \sum_{i,j} \|Y_{i,j}\|_1$, and $\|\cdot\|_1$ is the 1-norm.
Assumption 3 (Smoothness Assumption on $\ell_{\text{MLMI}}(\theta)$). We assume that the main objective $\ell_{\text{MLMI}}(\theta)$ is smooth around optimal value and has a bounded third order derivative:

$$\max_{t \in S \cup S^c} \max_{\delta, ||\delta|| < ||\theta||} \left| \left| \nabla_\theta \nabla_\theta^2 \ell_{\text{MLMI}}(\theta^* + \delta) \right| \right| \leq \Lambda_{\max},$$  \hspace{1cm} (8)

where $|||\cdot|||$ is the spectral norm of a tensor.

Assumption 4 (The Correct Model Assumption). The density ratio model is correct, i.e., there exists $\theta^*$ such that

$$p(x) = r(x; \theta^*)p(x_1)p(x_2).$$

Assumption 5 (Bounded Density Ratio). $\max_t \| f_t(x) \|_\infty \leq f_{\max}$. As a consequence, if the true parameter has a bounded norm, i.e., $\| \theta^* \|_\infty \leq \infty$, then it is clear that $\bar{g}_{\min} \leq \bar{g}(x; \theta^*) \leq \bar{g}_{\max}$, so $g^n(x; \theta^*) \leq g_{\max}$ and $g(x; \theta^*) \leq g_{\max}$.

We are now ready to state the main theorem.

5.3 Sufficient Conditions for Successful Bridge Recovery

The following theorem establishes sufficient conditions for pairwise bridge factor recovery in terms of parameter sparsity. Its proof is provided in Appendix 9.1.

Theorem 4. Suppose that Assumptions 1, 2, 3, 4, and 5 as well as $\min_t \| \theta^*_t \| \geq \frac{10}{\Lambda_{\min}} \sqrt{d} \lambda_n$ are satisfied, where $d$ is the number of pairwise bridge factors. Suppose also that the regularization parameter is chosen so that

$$\frac{12(2 - \alpha)}{\alpha} \sqrt{\frac{M_1 \log \left( \frac{m}{2} \right)}{n}} \leq \lambda_n,$$

where $M_1$ is some positive constant. Then there exist some constants $L$, $K_1$, and $K_2$ such that if $n \geq Ld^2 \log \left( \frac{m}{2} \right)$, with the probability at least $1 - K_2 \exp \left( -K_1 \lambda^2 n \right)$, the following properties hold:

- **Unique Solution**: The solution of Eq. (7) is unique

- **Successful Bridge Recovery**: $\hat{S} = S^*$ and $\hat{S}^c = S^c$.

Remarks This theorem states that under mild assumptions, we are guaranteed to recover the sparse pairwise bridge factorizations with high probability. As a consequence of this theorem, $||\hat{\theta} - \theta^*||$ converges to 0 at the order of $O_p\left( \frac{1}{\sqrt{n}} \right)$ (see Lemma 9.4).
6 Related Works

Recently, a novel approach of learning changes between two MRFs has been reported in Liu et al. (2014, 2015). This work, using the same concept of density ratio estimation, can be regarded as a closely related work. In Liu et al. (2014), the author considered the problem where two sets of data are given, and the task is to learn changes between two MRFs. This work, though only one set of data is used as input, can be naturally thought as a variation of the previous problem.

In this paper, the product between the marginal distributions, $P(X_1)P(X_2)$, can be regarded as another distribution where the links between $X_1$ and $X_2$ are completely removed. Intuitively, the ratio between two probability distributions will “highlight” the “links” between two partitions. In this paper, we give rigorous proofs that the “links” are in fact bridges, and such connectivities are revealed via learning the sparse factorization of a GPR.

As an algorithm that learns the structure of graphical models, it naturally has connections with many recent works on neighborhood selection in Ravikumar et al. (2010); Meinshausen and Bühlmann (2006). As reported in previous literature, the neighborhood of a random variable $X_i$ is selected via learning a model of $P(X_i|X_{\backslash i})$. In fact, if we consider a closely related quantity, $\frac{P(X_i|X_{\backslash i})}{P(X_i)}$, we can see it is exactly a PR, where random variables are partitioned into $\{X_i\}, \{X_{\backslash i}\})$. The key difference between two works are in the algorithms and models. By modeling the “ratio” directly, we may consider a wider range of distributions that has been so far limited to (quasi-)Gaussian or discrete MRFs. Moreover, the partition on the random variables is no longer limited to “one-versus-the-others”.

Next, we verify the correctness of our algorithm and theorems by experiments.

7 Experiments

As we have stated in Section 4, the MLMI algorithm may be used for learning sparse (pairwise) bridge fractionation. We first generate a 50-dimensional Gaussian MRF with random graphical structure, and then we learn a PR $\frac{P(X)}{P(X_1)P(X_2)}$ from 300 samples, where $X_1$ and $X_2$ are two...
partitions with equal number of random variables, as shown in Figure 4(a). As we increase the regularization parameter $\lambda$, the learned factorization over pairwise factors is getting sparser and sparser. We use the plot called the regularization path to capture the trend of the sparsification of a GPR model. We highlight the pairwise factors $\langle u, v \rangle$ that is included in at least one of the bridges in black, while mark other pairwise factors in red. The results show that indeed, the bridge factors last until the end, while the other non-bridge factors are reduced to zeros in earlier stages. Thus it verifies Theorem 3.

Since Theorem 3 should hold for general distributions, and we naturally hope to see if the GPR of non-Gaussian distributions will factorize over pairwise bridge factors. We construct a partitioned Square Wave MRF (Figure 4(b)) manually in order to guarantee it is simply connected. The underlying distribution is a non-Gaussian “diamond distribution” [Liu et al. (2014)]. Thanks to the MLMI algorithm, even for the diamond distribution which does not have a closed-form normalization term, we are still able to estimate its bridge factorization from 400 samples, and similar sparsity patterns can be observed from Figure 3(b).

Finally, we verify the support consistency theorem (Theorem 4) in Figure 3(c). As the theorem states, if all assumptions are met, we may pick a $\lambda_n$ that scales with $\sqrt{\frac{\log \left( \frac{m^2}{n} \right)}{n}}$. Then the effective sample size is $\beta = \frac{K_n}{d^2 \log \left( \frac{m^2}{n} \right)}$ where $n$ is chosen from 100 to 1500. In this experiment, we plot the probability of success of the MLMI algorithm, with fixed bridge structure but different dimensions $m$ on truncated-domain Gaussian MRF. The result shows, all curves align well with respect to $\beta$.

8 Conclusion

We present a novel approach to learn the partial structure of an MRF using the partitioned ratio, which factorizes over subgraph structures of an MRF. We also give a simple learning algorithm which can be efficiently computed even on non-Gaussian MRFs. We prove this algorithm is support consistent, and experiments have validated our conclusions.
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Proof of Theorem, Corollaries and Lemmas

For notational simplicity, we redefine the feature function \( f : \mathbb{R}^m \mapsto \mathbb{R}^{b(m)} \) as
\[
\mathbf{f}(\mathbf{x}) = (\mathbf{f}^T(x_1, x_2), \ldots, \mathbf{f}^T(x_1, x_m), \mathbf{f}^T(x_2, x_3), \ldots, \mathbf{f}^T(x_2, x_m), \ldots, \mathbf{f}^T(x_{m-1}, x_m))^T.
\]

We also review and define a few notations before proceed:

| Notation | Explanation |
|----------|-------------|
| \( \bar{g}(\mathbf{x}; \theta) := \exp(\langle \theta, \mathbf{f}(\mathbf{x}) \rangle) \) | Unnormalized density ratio model |
| \( N(\theta) := \mathbb{E}_{\mathbf{X}_1, \mathbf{X}_2} [\bar{g}_{12}(\mathbf{x}, \theta)] \) | Normalization function of density ratio |
| \( N_n(\theta) := \frac{1}{n} \sum_{i \neq j} \bar{g}(\mathbf{x}^{(i)}, \mathbf{x}^{(j)}; \theta) \) | Empirical normalization function of density ratio |
| \( g(\mathbf{x}; \theta) := \frac{\bar{g}(\mathbf{x}; \theta)}{N(\theta)} \) | Density Ratio Model |
| \( g_n(\mathbf{x}; \theta) := \frac{\bar{g}(\mathbf{x}; \theta)}{N_n(\theta)} \) | Empirical density ratio model |
| \( A(\theta) := \log N(\theta) \) | Log-normalization function of density ratio |
| \( A_n(\theta) := \log N_n(\theta) \) | Empirical log-normalization function of density ratio |

For simplicity, we consider a rearranged true parameter \( \theta^* = [\theta^*_S, 0] \).

9.1 The Proof of the Main Theorem

We prove the main theorem with following steps:

- Solve the constrained optimization problem
  \[
  \hat{\theta}_S = \arg\min_{\theta_S} \ell \left( \begin{bmatrix} \theta_S \\ 0 \end{bmatrix} \right) + \lambda_n \sum_{t' \in S_c} \|\theta_{t'}\|;
  \]

- For all \( t' \in S \), set \( \hat{z}_{t'} = \nabla \|\hat{\theta}_{t'}\| \), and let \( \hat{\theta} = [\hat{\theta}_S, 0] \);

- Obtain \( \hat{z}_{t''} \) for all \( t'' \in S_c \) using equality (9) (see below for details);
  \[
  \nabla \ell(\hat{\theta}) + \lambda_n \hat{z} = 0. \tag{9}
  \]

- Show \( \max_{t'' \in S_c} \|z_{t''}\| < 1 \) with high probability under certain conditions. According to Lemma 1, we conclude that for any optimal \( \hat{\theta} \), the correct sparsity pattern is recovered.

Bounding \( \max_{t'' \in S_c} \|z_{t''}\| \) requires obtaining \( z_{t''} \) from (9). More specifically, from (9) we have:

\[
\nabla \ell(\hat{\theta}) + \lambda_n \hat{z} = 0 \\
\Rightarrow \nabla \ell(\hat{\theta}) + \lambda_n \hat{z} - \nabla \ell(\theta^*) = -\nabla \ell(\theta^*).
\]
Applying Mean-value Theorem,

\[
\nabla^2 \ell(\hat{\theta})[\hat{\theta} - \theta^*]^\top + \lambda_n \hat{z} = -\nabla \ell(\theta^*)
\]

\[
\to \frac{\nabla^2 \ell(\theta^*)}{\mathcal{I}}[\hat{\theta} - \theta^*]^\top + \lambda_n \hat{z} = -\nabla (\theta^*) + \frac{\nabla^2 \ell(\theta^*) - \nabla^2 (\hat{\theta})}{\mathcal{I}}[\hat{\theta} - \theta^*]^\top.
\tag{10}
\]

where $\hat{\theta}$ is between $\theta^*$ and $\hat{\theta}$ in a coordinate fashion and $\mathcal{I}$ is the Fisher Information Matrix.

We can rewrite (10) in block-wise fashion:

\[
\mathcal{I}_{SS}[\hat{\theta}_S - \theta^*_S] + \lambda_n \hat{z}_S = w_S + r_S
\]

\[
\mathcal{I}_{t't'}[\hat{\theta}_{t'} - \theta^*_{t'}] + \lambda_n \hat{z}_{t'} = w_{t'} + r_{t'}, \quad t' \in S^c.
\tag{11}
\]

Substitute $\hat{\theta}_S - \theta^*_S = \mathcal{I}^{-1}_{SS}[w_S + r_S - \lambda_n \hat{z}_S]$ into (11), we have

\[
\mathcal{I}_{t't'} \mathcal{I}^{-1}_{SS}[w_S + r_S - \lambda_n \hat{z}_S] + \lambda_n \hat{z}_{t'} = w_{t'} + r_{t'}.
\]

Rearrange terms, we have

\[
\lambda_n \hat{z}_{t'} = w_{t'} + r_{t'} - \mathcal{I}_{t't'} \mathcal{I}^{-1}_{SS}[w_S + r_S - \lambda_n \hat{z}_S].
\]

According to triangle inequality,

\[
\lambda_n \max_{t' \in S^c} \|\hat{z}_{t'}\| \leq \max_{t' \in S^c} \|w_{t'}\| + \max_{t' \in S^c} \|r_{t'}\| + \max_{t' \in S^c} \|\mathcal{I}_{t't'} \mathcal{I}^{-1}_{SS}\|_1 \left(\|w_S\|_\infty + \|r_S\|_\infty + \lambda_n\right).
\]

By assumption, $\max_{t' \in S^c} \|\mathcal{I}_{t't'} \mathcal{I}^{-1}_{SS}\|_1 \leq (1 - \alpha)$, and we obtain

\[
\max_{t' \in S^c} \|\hat{z}_{t'}\| \leq \frac{2 - \alpha}{\lambda_n} \left(\max_{t \in S \cup S^c} \|w_t\| + \max_{t \in S \cup S^c} \|r_t\|\right) + (1 - \alpha).
\]

We will show $\max_{t \in S \cup S^c} \|w_t\|, \max_{t \in S \cup S^c} \|r_t\| \leq \frac{\alpha}{4(2-\alpha)} \lambda_n$ using Lemma 2, Lemma 3 and Lemma 4 below. Therefore

\[
\max_{t' \in S^c} \|\hat{z}_{t'}\| \leq 1 - \frac{\alpha}{2} < 1.
\]

To show the correct non-zero pattern recovery, it suffices to show $\max_{t \in S \cup S^c} \|\hat{\theta}_t - \theta^*_t\| < \frac{1}{2} \min_{t \in S} \|\theta^*_t\|$. Since Lemma 3 shows

\[
\max_{t \in S \cup S^c} \|\hat{\theta}_t - \theta^*_t\| \leq \|\hat{\theta} - \theta^*\| < \frac{10}{\Lambda_{\min}} \sqrt{d} \lambda_n,
\]

we just need $\min_{t \in S} \|\theta^*_t\| > \frac{20}{\Lambda_{\min}} \sqrt{d} \lambda_n$ to ensure such recovery.

First, we show the uniqueness of the solution in Lemma 1.
9.2 Lemma 1

**Lemma 1.** If there exists an optimal \( \hat{\theta} \) of (7) with associated \( \hat{z} \) in (9) such that \( \| \hat{z}_{t'' \in S^c} \| < 1 \). Then any optimal \( \tilde{\theta} \) of (7) should have \( \tilde{\theta}_{t''} = 0 \) for all \( t'' \in S^c \).

**Proof.** Consider two optimals \( \hat{\theta}, \tilde{\theta} \) of (7), \( \hat{\theta} \) has the correct sparsity over zero elements. We have the following equality

\[
\ell(\hat{\theta}) + \lambda_n \sum_{t \in S \cup S^c} \| \hat{\theta}_t \| = \ell(\tilde{\theta}) + \lambda_n \sum_{t \in S \cup S^c} \| \tilde{\theta}_t \|. 
\]

noting \( \hat{z}_t \in \nabla \| \theta_t \| \),

\[
\lambda_n \| \hat{\theta}_t \| \geq \lambda_n \langle \hat{\theta}_t - \tilde{\theta}_t, \hat{z}_t \rangle + \lambda_n \| \tilde{\theta}_t \|,
\]

substituting (13) into (12) we have the following inequality:

\[
\ell(\tilde{\theta}) + \lambda_n \sum_{t \in S \cup S^c} \| \tilde{\theta}_t \| \geq \ell(\hat{\theta}) + \lambda_n \sum_{t \in S \cup S^c} \langle \hat{\theta}_t - \tilde{\theta}_t, \hat{z}_t \rangle + \lambda_n \sum_{t \in S \cup S^c} \| \tilde{\theta}_t \|.
\]

Due to convexity and \( \nabla \ell(\hat{\theta}) = -\lambda_n \hat{z} \) as stated in (9), the leftmost term is further lower bounded as:

\[
\ell(\tilde{\theta}) + \lambda_n \sum_{t \in S \cup S^c} \| \tilde{\theta}_t \| \geq \ell(\hat{\theta}) + \lambda_n \sum_{t \in S \cup S^c} \langle \hat{\theta}_t - \tilde{\theta}_t, -\hat{z}_t \rangle + \lambda_n \sum_{t \in S \cup S^c} \| \tilde{\theta}_t \| \\
\geq \ell(\hat{\theta}) + \lambda_n \sum_{t \in S \cup S^c} \| \tilde{\theta}_t \|
\]

The above suggests all the inequality we have used to lower-bound \( \ell(\hat{\theta}) + \lambda_n \sum_{t \in S \cup S^c} \| \tilde{\theta}_t \| \) should take the exact equality. Therefore (13) should take the exact equality:

\[
\lambda_n \| \hat{\theta}_t \| = \lambda_n \langle \hat{\theta}_t - \tilde{\theta}_t, \hat{z}_t \rangle + \lambda_n \| \tilde{\theta}_t \|
\]

which holds only if \( \tilde{\theta}_t = 0 \), given \( \| \hat{z}_t \| < 1 \).

Moreover, if \( I_{SS} \) is invertible, it can be shown that \( \hat{\theta} \) is the unique solution. \( \square \)

9.3 Lemma 2

**Lemma 2.** If \( \lambda_n \geq \frac{12(2-\alpha)}{\alpha} \sqrt{\frac{K \log (\frac{2}{\gamma})}{n}} \), then \( \max_t \| w^n_t \| < \frac{4(2-\alpha)}{\alpha} \lambda_n \) with probability at least \( 1 - (4b+2) \exp (-Bn) \), where \( K, B \) are constants and \( K \) is a constant depends on the boundedness of \( g_{\max} \) and \( \max_t \| f_t \|_\infty \).
Proof. Notice that $w_t^i = -\nabla_{\theta_t} \ell(\theta^*) = \frac{1}{n} \sum_{i=1}^n w_t^{(i)}$ where the empirical sum is over

$$w_t^{(i)} = f(x_t^{(i)}) - \nabla_{\theta_t} A^n(\theta^*)$$

$$= f(x_t^{(i)}) - \frac{1}{n} \sum_{i \neq j} g^n(x_1^{(i)}, x_2^{(i)}; \theta^*) f(x_t^{(i)}).$$

Now, let’s write

$$w_t^n = \frac{1}{n} \sum_{i=1}^n f(x_t^{(i)}) - \nabla_{\theta_t} A(\theta^*) + \nabla_{\theta_t} A(\theta^*) - \nabla_{\theta_t} A^n(\theta^*)$$

and since

$$\|w_t^n\| \leq \left\| \frac{1}{n} \sum_{i=1}^n f(x_t^{(i)}) - \nabla_{\theta_t} A(\theta^*) \right\| + \|\nabla_{\theta_t} A(\theta^*) - \nabla_{\theta_t} A^n(\theta^*)\|,$$  \hspace{1cm} (14)

it is suffice to bound two terms on RHS separately. Now, we show that both terms will converge to zero at the rate $O_p(1/n)$.

Utilizing the fact that

$$\nabla_{\theta_t} A(\theta^*) = E_{X_1, X_2} [g(x_1, x_2; \theta^*) f(x_t)] = E_X [f(x_t)],$$

the first term is the deviation of an empirical average from its population mean, thus we may obtain the following inequality using Hoeffding inequality [Hoeffding (1963)]:

$$P \left[ \left\| \frac{1}{n} \sum_{i=1}^n f(x_t^{(i)}) - \nabla_{\theta_t} A(\theta^*) \right\| \geq \epsilon \right] \leq 2 b \exp \left( -\frac{2 n \epsilon^2}{b f_{\text{max}}^2} \right).$$ \hspace{1cm} (15)

The second term may be written as

$$\nabla_{\theta_t} A(\theta^*) - \nabla_{\theta_t} A^n(\theta^*) =$$

$$\mathbb{E}_{X_1, X_2} \left[ g(x_1, x_2; \theta^*) f(x_{t_1}, x_{t_2}) \right]_{\kappa(\theta^*)} - \frac{N(\theta^*)}{N^n(\theta^*)} \cdot \frac{1}{2} \sum_{i \neq j} g^n(x_1^{(i)}, x_2^{(i)}; \theta^*) f(x_t^{(i)}, x_t^{(j)}).$$ \hspace{1cm} (16)

Since

$$\frac{N(\theta^*)}{N^n(\theta^*)} = 1 + \frac{N(\theta^*) - N^n(\theta^*)}{N^n(\theta^*)},$$

the $\ell_2$ norm of (16) is bounded by the following term

$$\kappa^{(\theta^*)} - \kappa^n(\theta^*) + \|\frac{N^n(\theta^*) - N(\theta^*)}{N(\theta^*)}\| \cdot g_{\text{max}} \cdot f_{\text{max}} \cdot b,$$  \hspace{1cm} (16)

\footnote{Noticing $w_t$ is an empirical sum, we denote it as $w_t^n$ with superscript $n$ in this context.}
where both terms are the deviations of empirical averages from their population means. Thus, we may write the following again using Hoeffding inequalities:

\[
P \left[ \| \kappa(\theta^*) - \kappa^n(\theta) \| \geq \epsilon \right] \leq 2b \exp \left( - \frac{2n\epsilon^2}{b f_{\max}^2 g_{\max}^2} \right), \tag{17}
\]

\[
P \left\{ \left| \frac{1}{n} \sum_{i \neq j} g \left( x_1^{(i)}, x_2^{(j)}; \theta^* \right) - 1 \cdot g_{\max} \cdot f_{\max} \cdot b \geq \epsilon \right| \right\} \leq 2 \exp \left( - \frac{2n\epsilon^2}{b^2 f_{\max}^2 g_{\max}^4} \right), \tag{18}
\]

We may combine inequalities (15), (17) and (18), to get the bound for

\[
P \left[ \left\| \sum_{i=1}^{n} f \left( x_t^{(i)} \right) - \nabla_{\theta_t} A(\theta^*) \right\| + \left\| \nabla_{\theta_t} A(\theta^*) - \nabla_{\theta_t} A^n(\theta^*) \right\| \geq 3\epsilon \right] \leq (4b + 2) \exp \left( - \frac{2n\epsilon^2}{c_3} \right),
\]

where \( c_3(f_{\max}, g_{\max}, b) = \max \left( b^2 f_{\max}^2 g_{\max}^4, b f_{\max}^2 g_{\max}^2, b f_{\max}^2 \right) = b^2 f_{\max}^2 g_{\max}^4 \), since \( b \geq 1 \) and \( g_{\max} \geq 1 \). Finally, due to the inequality Eq. (14), we may write

\[
P \left[ \| w_t^* \| \geq 3\epsilon \right] \leq (4b + 2) \exp \left( - \frac{2n\epsilon^2}{c_3} \right).
\]

Let \( 3\epsilon = \frac{\alpha}{4(2-\alpha)} \lambda_n \),

\[
P \left[ \| w_t^* \| \geq \frac{\alpha}{4(2-\alpha)} \lambda_n \right] \leq (4b + 2) \exp \left( - \frac{2n}{c_3} \cdot \left( \frac{\alpha}{12(2-\alpha)} \lambda_n \right)^2 \right).
\]

Further, using union-bound, the upper-bound for \( \max_t \| w_t^* \| \) is obtained as

\[
P \left[ \max_t \| w_t^* \| \geq \frac{\alpha}{4(2-\alpha)} \lambda_n \right] \leq (4b + 2) \exp \left( - \frac{2n}{c_3} \cdot \left( \frac{\alpha}{12(2-\alpha)} \lambda_n \right)^2 + \log \left( \frac{m}{2} \right) \right).
\]

It can be seen that, as long as \( \lambda_n \geq \frac{12(2-\alpha)}{\alpha} \sqrt{\frac{c_4 m \log m}{n}} \), \( \max_t \| w_t^* \| \) is bounded with probability at least \( 1 - (4b + 2) \exp (-c_4 n) \), where \( c_4 \) is a constant. \( \square \)

### 9.4 Lemma 3

The proof Lemma 3 and Lemma 4 is in a straightforward fashion following Lemma 2.

**Lemma 3.** If \( d\lambda_n \leq \frac{\lambda_n^2}{20g_{\max}^3} \) and \( \max_{t \in S^c \cup S} \| w_t \\| \leq \frac{\lambda_n}{4} \), then \( \| \theta^*_S - \hat{\theta}_S \| \leq \frac{10 \lambda_{\min}}{\sqrt{d \lambda_n}} \sqrt{d \lambda_n} \).

**Proof.** Since we are trying to prove that \( \| \hat{\theta}_S - \theta^*_S \| \leq B \), according to Ravikumar et al. (2010); Yang et al. (2012), we may construct the following function:

\[
G(\delta_S) = \ell(\theta^*_S + \delta_S) - \ell(\theta^*_S) + \lambda_n \sum_{t' \in S} (\| \theta^*_{t'} + \delta_{t'} \| - \| \theta^*_{t'} \|), \tag{19}
\]
where $G$ is a convex function, $G(0) = 0$, reaches the minimal at $\delta^*_S = \hat{\theta}_S - \theta^*_S$ and $G(\delta^*_S) \leq 0$. Simple proof in Ravikumar et al. (2010) can show that for a $\delta_S$, $\|\delta_S\| = B$ if $G(\delta_S) > 0$, $\|\theta_S - \theta^*_S\| \leq B$.

We first use Taylor expansion on the first two terms of (19),

$$G(\delta) = \delta^\top \nabla \ell(\theta^*_S) + \frac{1}{2} \delta^\top \nabla^2 \ell(\theta^*_S + \delta_S) \delta_S + \lambda_n \sum_{v' \in S} (\|\theta^*_{v'} + \delta_{v'}\| - \|\theta^*_{v'}\|).$$

(20)

Since for the first and the last term

$$|\langle w_S, \delta_S \rangle| \leq \|w_S\| \|\delta_S\| \leq \sqrt{d} \|\delta_S\| \max_{v' \in S} \|w_{v'}\| \leq \frac{\sqrt{d} \lambda_n}{4} \|\delta_S\|,$$

(21)

and

$$\lambda_n \sum_{v' \in S} (\|\theta^*_{v'} + \delta_{v'}\| - \|\theta^*_{v'}\|) \geq -\lambda_n \sum_{v' \in S} \|\delta_{v'}\| \geq -\sqrt{d} \lambda_n \|\delta_S\|,$$

(22)

we only need to lower-bound the middle term.

Obviously, we need to bound $\Lambda_{\min} \left( \nabla^2 \ell(\theta^*_S + \bar{\delta}_S) \right)$. By applying the Mean-value theorem

$$\nabla^2 \ell(\theta^*_S + \bar{\delta}_S) = \nabla^2 \ell(\theta^*_S) + \nabla^2 \ell(\theta^*_S + \bar{\delta}_S) \bar{\delta}_S,$$

Weyl’s inequality Horn and Johnson (1986) implies:

$$\Lambda_{\min} \left( \nabla^2 \ell(\theta^*_S + \bar{\delta}_S) \right) \geq \Lambda_{\min} \left( \nabla^2 \ell(\theta^*_S) \right) - \left\| \nabla^2 \ell(\theta^*_S + \bar{\delta}_S) \bar{\delta}_S \right\|$$

$$\geq \Lambda_{\min} \left( \nabla^2 \ell(\theta^*_S) \right) - \sqrt{d} \max_{t \in S \cup S^c} \left\| \nabla_{\theta_t} \nabla^2 \ell(\theta^*_S + \bar{\delta}_S) \right\| \|\bar{\delta}_S\|,$$

Given $B \leq \|\theta^*\|$, we can get

$$\Lambda_{\min} \left( \nabla^2 \ell(\theta^*_S + \bar{\delta}_S) \right) \geq \Lambda_{\min} - \sqrt{d} \Lambda_{\max} \|\delta_S\|.$$

(23)

Combining (21), (23) and (22) with (20), we can get

$$G(\delta_S) \geq -\frac{d}{4} (\lambda_n)^2 \|\delta_S\|^2 + \frac{1}{4} \|\delta_S\|^2 \Lambda_{\min} - (\lambda_n)^2 d \|\delta_S\|,$$

(24)

by assuming $\frac{1}{2} \Lambda_{\min} \geq \sqrt{d} \Lambda_{\max} \|\delta_S\|$.

Let $B = \sqrt{d} \lambda_n M$, where $M > 0$, we have

$$G(\delta_S) \geq (\lambda_n)^2 d (- \frac{M}{4} + \Lambda_{\min} \frac{1}{4} M^2 - M),$$

which is strictly positive when $M = \frac{10}{\Lambda_{\min}}$.

In conclusion, given $d\lambda_n \leq \frac{\lambda_n^2}{20 \Lambda_{\max}^2}$ and $\max_{t \in S \cup S^c} \|w_t\| \leq \frac{\lambda_n}{4}$, we have $\|\theta^* - \hat{\theta}\| \leq \frac{10 \lambda_n \sqrt{d}}{\Lambda_{\min}}$.

□
9.5 Lemma 4

**Lemma 4.** If $\lambda_n d \leq \frac{\Lambda_{\min}^2}{100 \Lambda_{\max}} \frac{\alpha}{4(2-\alpha)}$, and $\max_{t \in S \cup S^c} \| w_t \| \leq \frac{\lambda_n}{4}$, then $\max_{t \in S \cup S^c} \| r_t \| \leq \frac{\alpha \lambda_n}{4(2-\alpha)}$.

Since
\[
r_t = [\nabla_{\theta_t} \nabla \ell(\theta^*) - \nabla_{\theta_t} \nabla \ell(\bar{\theta})] \left[ \theta^* - \hat{\theta} \right]
\]
by applying the Mean-value theorem, we get
\[
\| r_t \| \leq \left\| \nabla_{\theta_t} \nabla^2 \ell(\theta^* + \bar{u}) \right\| \| \theta^* - \bar{\theta} \| \| \theta^* - \hat{\theta} \|
\leq \left\| \nabla_{\theta_t} \nabla^2 \ell(\theta^* + \bar{u}) \right\| \| \theta^* - \bar{\theta} \|^2
\leq \frac{100(\lambda_n)^2 d}{\Lambda_{\min}^2} \Lambda_{\max}.
\]

Thus, when
\[
\lambda_n d \leq \frac{\Lambda_{\min}^2}{100 \Lambda_{\max}} \frac{\alpha}{4(2-\alpha)},
\]
we have
\[
\max_{t \in S \cup S^c} \| r_t \| \leq \frac{\alpha \lambda_n}{4(2-\alpha)}.
\]
Note that this condition is stronger than the one in Lemma 2, so the result of Lemma 2 holds automatically.

9.6 Sample Complexity Analysis

In this section, we analyze the sample complexity from above lemmas. First, if we set $\lambda_n$ as
\[
\lambda_n = \frac{16(2-\alpha)}{\alpha} \sqrt{K \log \left( \frac{m}{2} \right) \frac{n}{n}},
\]
the lower bound on $\lambda_n$ is satisfied.

Further, in order to satisfy Lemma 3 and 4, $\lambda_n d$ has to satisfy
\[
\lambda_n d = \frac{16d(2-\alpha)}{\alpha} \sqrt{K \log \left( \frac{m}{2} \right) \frac{n}{n}} \leq \frac{\Lambda_{\min}^2}{100 \Lambda_{\max}} \frac{\alpha}{4(2-\alpha)},
\]
which leads to
\[
n \geq Cd^2 \log \left( \frac{m}{2} \right),
\]
where $C$ is a constant.