Anisotropic hydrodynamics for conformal Gubser flow

By: Mohammad Nopoush

Collaborators: R. Ryblewski and M. Strickland

Primary reference: arXiv:1410.6790
Dissipative hydrodynamics is commonly used to describe ultra-relativistic heavy-ion collisions

- Ideal hydrodynamics
- 2\textsuperscript{nd}-order hydrodynamics (multiple approaches)
- Anisotropic hydrodynamics

We derive the equations of (1+1)d aHydro for a system subject to Gubser flow by taking moments of the Boltzmann equation in relaxation-time approximation.

The obtained solution has both longitudinal and transversal expansion.

To accomplish this, we use a clever method introduced by Gubser that uses symmetries to construct a static flow in Weyl-rescaled de Sitter space.

Once the solution is determined in de Sitter space, we then map it back to Minkowski space which gives the full spatiotemporal evolution.

We then compare the predictions of aHydro against a recent obtained exact solution to the Boltzmann equation in relaxation-time approximation.
Weyl rescaling & Gubser flow

- Conformality \(\leftrightarrow\) Weyl rescaling invariance\(^{[1]}\)

- Gubser flow\(^{[2]}\)

\[
\begin{align*}
\tilde{u}^\tau &= \cosh \left[ \tanh^{-1} \left( \frac{2q^2\tau r}{1 + q^2\tau^2 + q^2r^2} \right) \right] \\
\tilde{u}^r &= \sinh \left[ \tanh^{-1} \left( \frac{2q^2\tau r}{1 + q^2\tau^2 + q^2r^2} \right) \right] \\
\tilde{u}^\phi &= \tilde{u}^\zeta = 0
\end{align*}
\]

\[
\begin{align*}
\tau &= \sqrt{x^2 + y^2} \\
\varphi &= \tan^{-1} \left( \frac{y}{x} \right) \\
\zeta &= \tanh^{-1} \left( \frac{z}{t} \right)
\end{align*}
\]

---

\(^{[1]}\) S. S. Gubser, A. Yarom, arXiv: 1012.1314

\(^{[2]}\) S. S. Gubser, arXiv: 1006.0006
We defined the most general basis (LRF) and its parametrization for boost-invariant and cylindrically symmetric flow \( x^\mu = (t, x, y, z) \)

\[
\begin{align*}
    u^\mu_{LRF} &\equiv (1, 0, 0, 0) \\
    \mathcal{X}^\mu_{LRF} &\equiv (0, 1, 0, 0) \\
    \mathcal{Y}^\mu_{LRF} &\equiv (0, 0, 1, 0) \\
    \mathcal{Z}^\mu_{LRF} &\equiv (0, 0, 0, 1)
\end{align*}
\]

\[
\begin{align*}
    u^\mu &\equiv (\cosh \theta_\perp \cosh \varsigma, \sinh \theta_\perp \cos \phi, \sinh \theta_\perp \sin \phi, \cosh \theta_\perp \sinh \varsigma) \\
    \mathcal{X}^\mu &\equiv (\sinh \theta_\perp \cosh \varsigma, \cosh \theta_\perp \cos \phi, \cosh \theta_\perp \sin \phi, \sinh \theta_\perp \sinh \varsigma) \\
    \mathcal{Y}^\mu &\equiv (0, -\sin \phi, \cos \phi, 0) \\
    \mathcal{Z}^\mu &\equiv (\sinh \varsigma, 0, 0, \cosh \varsigma)
\end{align*}
\]

**Polar Milne space** \( \tilde{x}^\mu = (\tau, r, \phi, \varsigma) \)

**De Sitter space** \( \tilde{x}^\mu = (\rho, \theta, \phi, \varsigma) \)
De Sitter-space basis vectors

- **De Sitter** $\hat{x}^\mu = (\rho, \theta, \phi, \varsigma)$ vs. **Milne** $\tilde{x}^\mu = (\tau, r, \phi, \varsigma)$ coordinates\(^1\) (Figure from Ref. \([2]\))

\[
\sinh \rho = -\frac{1 - q^2 \tau^2 + q^2 r^2}{2q\tau} \\
\tan \theta = \frac{2q r}{1 + q^2 \tau^2 - q^2 r^2}
\]

- **Metric (curved space)**

\[
\hat{g}_{\mu\nu} = \text{diag}(-1, \cosh^2 \rho, \cosh^2 \rho \sin^2 \theta, 1)
\]

- **Basis**

\[
\begin{align*}
u^\mu & \rightarrow \hat{\nu}^\mu & \hat{\nu}^\mu &= (1, 0, 0, 0) \\
\chi^\mu & \rightarrow \hat{\Theta}^\mu & \hat{\Theta}^\mu &= (0, (\cosh \rho)^{-1}, 0, 0) \\
\gamma^\mu & \rightarrow \hat{\Phi}^\mu & \hat{\Phi}^\mu &= (0, 0, (\cosh \rho \sin \theta)^{-1}, 0) \\
\zeta^\mu & \rightarrow \hat{\varsigma}^\mu & \hat{\varsigma}^\mu &= (0, 0, 0, 1)
\end{align*}
\]

\(^1\) S. S. Gubser, arXiv: 1006.0006

\(^2\) G. Denicol, U. Heinz, M. Martinez, J. Noronha, M. Strickland, arXiv: 1408.7048

\[
\lim_{\rho \to -\infty} \frac{q r}{\tau} \to 0^+ 
\]
Boltzmann Eq. (BE) and distribution function

- The de Sitter-space Boltzmann equation in relaxation-time approximation is:

\[
\hat{p} \cdot Df = \frac{\hat{p} \cdot \hat{u}}{\hat{\tau}_{eq}} (f - f_{iso}) \quad \hat{\tau}_{eq} = \frac{5 \eta}{\hat{T} s}
\]

- Leading-order aHydro → Ellipsoidal distribution function

\[
f(\hat{x}, \hat{p}) = f_{iso} \left( \frac{1}{\hat{\lambda}} \sqrt{\hat{p}_\mu \hat{\Xi}^{\mu\nu} \hat{p}_\nu} \right) \quad \hat{\Xi}^{\mu\nu} = \hat{u}^\mu \hat{u}^\nu + \hat{\xi}^{\mu\nu}
\]

\[
\hat{\xi}^{\mu\nu} = \hat{\xi}_\theta \hat{\Theta}^\mu \hat{\Theta}^\nu + \hat{\xi}_\phi \hat{\Phi}^\mu \hat{\Phi}^\nu + \hat{\xi}_\varsigma \hat{\varsigma}^\mu \hat{\varsigma}^\nu
\]

- Notation \( \hat{\alpha}_i \equiv (1 + \hat{\xi}_i)^{-1/2} \)

- SO(3) symmetry (rotational symmetry in de Sitter space) requires \( \hat{\alpha}_\theta = \hat{\alpha}_\phi \)

- \( n^{th} \) moment of the distribution function \((\hat{g} \equiv \text{det} \hat{g}_{\mu\nu})\)

\[
\hat{\mathcal{I}}^{\mu_1 \ldots \mu_n} \equiv \frac{1}{(2\pi)^3} \int \frac{d^3\hat{p}}{\sqrt{-\hat{g} \hat{p}^0}} \hat{p}^{\mu_1} \ldots \hat{p}^{\mu_n} f(\hat{x}, \hat{p})
\]
Energy-momentum tensor

- 1st-moment of BE needs:
  \[
  \hat{T}^{\mu\nu} = \frac{1}{(2\pi)^3} \int \frac{d^3 \hat{p}}{\sqrt{-\hat{g}}} \hat{p}^\mu \hat{p}^\nu f(\hat{x}, \hat{p})
  \]

- Energy-Momentum Tensor
  \[
  \hat{T}^{\mu\nu} = \hat{\varepsilon} \hat{u}^\mu \hat{u}^\nu + \hat{P}_\theta \hat{\Theta}^\mu \hat{\Theta}^\nu + \hat{P}_\phi \hat{\Phi}^\mu \hat{\Phi}^\nu + \hat{P}_\zeta \hat{\zeta}^\mu \hat{\zeta}^\nu
  \]

- Taking different projections

\[
\hat{\varepsilon} = \frac{6 \hat{\alpha}_\theta \hat{\alpha}_\phi \hat{\lambda}^4}{(2\pi)^3} \int_0^{2\pi} d\phi \hat{\alpha}_\perp^2 H_2(y)
\]

\[
\hat{P}_\theta = \frac{6 \hat{\alpha}_\theta^3 \hat{\alpha}_\phi \hat{\lambda}^4}{(2\pi)^3} \int_0^{2\pi} d\phi \cos^2 \phi H_{2T}(y)
\]

\[
\hat{P}_\phi = \frac{6 \hat{\alpha}_\theta \hat{\alpha}_\phi^3 \hat{\lambda}^4}{(2\pi)^3} \int_0^{2\pi} d\phi \sin^2 \phi H_{2T}(y)
\]

\[
\hat{P}_\zeta = \frac{6 \hat{\alpha}_\theta \hat{\alpha}_\phi \hat{\lambda}^4}{(2\pi)^3} \int_0^{2\pi} d\phi \hat{\alpha}_\perp^2 H_{2L}(y)
\]

\[
H_{2L}(y) = \frac{y^2}{(y^2 - 1)^{3/2}} \left( y \sqrt{y^2 - 1} - \tanh^{-1} \sqrt{y^2 - 1} \right)
\]

\[
H_{2T}(y) = \frac{y}{(y^2 - 1)^{3/2}} \left( 2y^2 - 1 \tanh^{-1} \sqrt{y^2 - 1} - y \sqrt{y^2 - 1} \right)
\]

\[
H_2(y) = H_{2T}(y) + H_{2L}(y)
\]

\[
\hat{\alpha}_\perp \equiv \sqrt{\hat{\alpha}_\theta^2 \cos^2 \phi + \hat{\alpha}_\phi^2 \sin^2 \phi}
\]

\[
y \equiv \hat{\alpha}_\zeta / \hat{\alpha}_\perp
\]

\[
\bar{y} \equiv \hat{\alpha}_\zeta / \hat{\alpha}_\theta
\]
1st-moment of BE

\[ D_\mu \hat{T}^{\mu\nu} = 0 \]

\[ \hat{T}^{\mu\nu} = \hat{\varepsilon} \hat{u}^{\mu} \hat{u}^{\nu} + \hat{P}_\Theta \hat{\Theta}^{\mu} \hat{\Theta}^{\nu} + \hat{P}_\Phi \hat{\Phi}^{\mu} \hat{\Phi}^{\nu} + \hat{P}_\xi \hat{\xi}^{\mu} \hat{\xi}^{\nu} \]

- Dynamical Landau-matching condition can be used to fix the “effective temperature”

\[ \hat{T} = \frac{\hat{\alpha}_\xi}{\bar{y}} \left( \frac{H_2(\bar{y})}{2} \right)^{1/4} \hat{\lambda} \]

\[ \partial_\rho \hat{\varepsilon} + \tanh \rho (2\hat{\varepsilon} + \hat{P}_\Theta + \hat{P}_\Phi) = 0 \]

\[ \partial_\theta \hat{P}_\Theta + (\hat{P}_\Theta - \hat{P}_\Phi) \cot \theta = 0 \]

\[ \partial_\phi \hat{P}_\Phi = 0 \]

\[ \partial_\xi \hat{P}_\xi = 0 \]

SO(3)-symmetry:

\[ \hat{\alpha}_\theta = \hat{\alpha}_\phi \]

\[ \partial_\rho \hat{\varepsilon} + 2 \tanh \rho (\hat{\varepsilon} + \hat{P}_\Theta) = 0 \]

\[ \partial_\theta \hat{P}_\Theta = \partial_\phi \hat{P}_\Phi = \partial_\xi \hat{P}_\xi = 0 \]

\[ 4 \frac{d \log \hat{\lambda}}{d \rho} + \frac{3\hat{\alpha}_\xi^2 \left( \frac{H_2L(\bar{y})}{H_2(\bar{y})} + 1 \right) - 4}{3\hat{\alpha}_\xi^2 - 1} \frac{d \log \hat{\alpha}_\xi}{d \rho} + \tanh \rho \left( \frac{H_2T(\bar{y})}{H_2(\bar{y})} + 2 \right) = 0 \]
2\textsuperscript{nd}-moment of the BE

- 2\textsuperscript{nd} moment of BE needs:

\[ \hat{I}^{\lambda \mu \nu} = \int \frac{d^3 \hat{p}}{\sqrt{-\hat{g} \hat{p}^0}} \hat{p}^\lambda \hat{p}^\mu \hat{p}^\nu f(\hat{x}, \hat{p}) \]

- Expanding over non-zero components gives:

\[ \hat{I} = \hat{I}_\rho \left[ \hat{u} \otimes \hat{u} \otimes \hat{u} \right] \]
\[ + \hat{I}_\theta \left[ \hat{u} \otimes \hat{\theta} \otimes \hat{\theta} + \hat{\theta} \otimes \hat{u} \otimes \hat{\theta} + \hat{\theta} \otimes \hat{\theta} \otimes \hat{u} \right] \]
\[ + \hat{I}_\phi \left[ \hat{u} \otimes \hat{\phi} \otimes \hat{\phi} + \hat{\phi} \otimes \hat{u} \otimes \hat{\phi} + \hat{\phi} \otimes \hat{\phi} \otimes \hat{u} \right] \]
\[ + \hat{I}_\varsigma \left[ \hat{u} \otimes \hat{\varsigma} \otimes \hat{\varsigma} + \hat{\varsigma} \otimes \hat{u} \otimes \hat{\varsigma} + \hat{\varsigma} \otimes \hat{\varsigma} \otimes \hat{u} \right] \]

- Taking projections gives:

\[ \hat{I}_i = \hat{\alpha} \hat{\alpha}_i^2 \hat{I}_{\text{iso}} \quad (i \in \theta, \phi, \varsigma) \]
\[ \hat{I}_\rho = \hat{\alpha} \left[ \sum_{i=\theta, \phi, \varsigma} \hat{\alpha}_i^2 \right] \hat{I}_{\text{iso}} \]

\[ \hat{\alpha} \equiv \hat{\alpha}_\theta \hat{\alpha}_\phi \hat{\alpha}_\varsigma \]
\[ \hat{I}_{\text{iso}} \equiv 4 \hat{\lambda}^5 / \pi^2 \]
\[ \hat{I}_\rho = \sum_{i=\theta, \phi, \varsigma} \hat{I}_i \]
Taking the 2\textsuperscript{nd}-moment of BE

\[ D_{\lambda} \hat{I}^{\lambda\mu\nu} = -\frac{1}{\hat{\tau}_{eq}} \left( \hat{u}_\lambda \hat{I}^{\lambda\mu\nu}_{\text{iso}} - \hat{u}_\lambda \hat{I}^{\lambda\mu\nu} \right) \]

Using the expanded form of \( \hat{I}^{\lambda\mu\nu} \)

\[
\begin{align*}
\partial_\rho \hat{I}_\theta + 4 \tanh \rho \hat{I}_\theta &= \frac{1}{\hat{\tau}_{eq}} \left[ \hat{I}_{\theta,\text{iso}} - \hat{I}_\theta \right] \\
\partial_\rho \hat{I}_\phi + 4 \tanh \rho \hat{I}_\phi &= \frac{1}{\hat{\tau}_{eq}} \left[ \hat{I}_{\phi,\text{iso}} - \hat{I}_\phi \right] \\
\partial_\rho \hat{I}_\zeta + 2 \tanh \rho \hat{I}_\zeta &= \frac{1}{\hat{\tau}_{eq}} \left[ \hat{I}_{\zeta,\text{iso}} - \hat{I}_\zeta \right] \\
\partial_\theta \hat{I}_\theta + \cot \theta (\hat{I}_\theta - \hat{I}_\phi) &= 0 \\
\partial_\phi \hat{I}_\phi &= 0 \\
\partial_\zeta \hat{I}_\zeta &= 0
\end{align*}
\]

SO(3)-symmetry

\[
\begin{align*}
\partial_\rho \hat{I}_\theta + 4 \tanh \rho \hat{I}_\theta &= \frac{1}{\hat{\tau}_{eq}} \left[ \hat{I}_{\theta,\text{iso}} - \hat{I}_\theta \right] \\
\partial_\rho \hat{I}_\phi + 4 \tanh \rho \hat{I}_\phi &= \frac{1}{\hat{\tau}_{eq}} \left[ \hat{I}_{\phi,\text{iso}} - \hat{I}_\phi \right] \\
\partial_\rho \hat{I}_\zeta + 2 \tanh \rho \hat{I}_\zeta &= \frac{1}{\hat{\tau}_{eq}} \left[ \hat{I}_{\zeta,\text{iso}} - \hat{I}_\zeta \right] \\
\partial_\theta \hat{I}_\theta &= \partial_\phi \hat{I}_\phi = \partial_\zeta \hat{I}_\zeta = 0
\end{align*}
\]

\[
\frac{6\hat{\alpha}_\zeta}{1 - 3\hat{\alpha}_\zeta^2} \frac{d\hat{\alpha}_\zeta}{d\rho} - \frac{3}{4\hat{\tau}_{eq}\hat{\alpha}_\zeta^5} \left( \frac{\hat{T}}{\hat{\lambda}} \right)^5 + 2 \tanh \rho = 0
\]
It is possible to solve the aHydro equations analytically in two limiting cases

- **Ideal limit** ($\hat{\alpha}_s \to 1$, $\partial_\rho \hat{\alpha}_s \to 0$, $\hat{\tau}_{eq} \to 0$)\[^1\]

\[
\hat{T}(\rho) = \hat{T}_0 \left( \frac{\cosh \rho_0}{\cosh \rho} \right)^{2/3}
\]

- **Free-streaming limit** ($\hat{\tau}_{eq} \to \infty$)\[^2,3\]

\[
\hat{\varepsilon}_{FS} = \frac{3\hat{\lambda}_0^4 \hat{\alpha}_s^4}{\pi^2} \mathcal{H}_\varepsilon \left( C_{\rho_0,\rho} \right)
\]

\[
(\hat{\pi}_s)_{FS} = \frac{\hat{\lambda}_0^4 \hat{\alpha}_s^4}{\pi^2} \mathcal{H}_\pi \left( C^{-1}_{\rho_0,\rho} \right)
\]

- aHydro gives both the ideal and free streaming limits!

\[^1\] S. S. Gubser and A. Yarom, arXiv:1012.1314.
\[^2\] G. S. Denicol, U. W. Heinz, M. Martinez, J. Noronha, and M. Strickland, arXiv:1408.5646.
\[^3\] G. S. Denicol, U. W. Heinz, M. Martinez, J. Noronha, and M. Strickland, arXiv:1408.7048.
Numerical results (setup)

- Solve two ODEs numerically subject to BCs at $\rho \to -\infty \ (\tau \to 0^+)$

$$
\frac{4}{3} \frac{d \log \lambda}{d \rho} + \frac{3}{2} \frac{d \hat{\alpha}}{d \rho} - \frac{1}{2} \frac{d \hat{\alpha}}{d \rho} + \tanh \rho \left( \frac{H_2(y)}{H_2(y)} + 2 \right) = 0
$$

$$
\left( \frac{T}{\lambda} \right)^5 + 2 \tanh \rho = 0 \quad \hat{T} = \frac{\hat{\alpha} y}{y} \left( \frac{H_2(y)}{2} \right)^{1/4}
$$

- Compare to exact solution obtained via iterative method$^{[1,2]}$

- Compare to Israel-Stewart approximation$^{[3]}$

- Compare to Denicol-Niemi-Molnar-Rischke (DNMR) approximation$^{[1]}$

---

1 G. S. Denicol, U. W. Heinz , M. Martinez, M. Strickland, arXiv:1408.7048
2 S. G. Denicol, U. W. Heinz, M. Martinez, J. Noronha, M. Strickland, arXiv:1408.5646
3 H. Marrochio, J. Noronha, G. S. Denicol, M. Luzum, S. Jeon, C. Gale, arXiv:1307.6130
Numerical results (isotropic IC)

\[ \hat{\alpha}_{\xi,0} = 1 \]
\[ \hat{T}_0 = 0.002 \]

\[ 4\pi\eta/s = 1 \]
\[ 4\pi\eta/s = 3 \]
\[ 4\pi\eta/s = 10 \]

\[ \pi_{\xi} = \frac{\pi^s_{\xi}}{\tilde{\varepsilon} + \tilde{P}} \]
Mapping back to Milne coordinates

Taking $4\pi\eta/s = 3$ and $q = (1 \text{fm})^{-1}$

\begin{align*}
\tau &= 1 \text{ fm/c} \\
\tau &= 5 \text{ fm/c} \\
\tau &= 10 \text{ fm/c}
\end{align*}
Slice through the x-y plane;
\[4\pi \eta/s = 3 \quad \text{and} \quad q = (1\, fm)^{-1}\]

Slice through the x-z plane;
\[4\pi \eta/s = 3 \quad \text{and} \quad q = (1\, fm)^{-1}\]
Conclusions and outlook

- Dynamical equations in the aHydro framework were derived
- Weyl rescaling + coordinate transformation makes the velocity profile static and simplifies the problem; 1+1d Minkowski flow $\rightarrow$ 0+1d de Sitter flow
- The exact solution of RTA Boltzmann equation was used to test different frameworks
- aHydro reproduces the exact solution better than both the DNMR and IS approximations
- The aHydro equations analytically reproduce the exact solutions in both the ideal ($\eta/s \rightarrow 0$) and free streaming ($\eta/s \rightarrow \infty$) limits.

- In the future, we plan to derive and test NLO aHydro for Gubser flow.
- In addition, we are working on phenomenological applications of aHydro for general 2+1d and 3+1d systems.
Numerical results (oblate IC)

\[ \hat{\alpha}_{\zeta,0} = 0.6 \]
\[ \hat{T}_0 = 0.002 \]
Numerical results (prolate IC)

\[ \hat{\alpha}_{\xi,0} = 10 \]
\[ \hat{T}_0 = 0.002 \]

(a) 4\(\pi\eta/s = 1\)
(b) 4\(\pi\eta/s = 3\)
(c) 4\(\pi\eta/s = 10\)

(d) \(\tau_{\xi,0}\)
(e) \(\tau_{\xi,0}\)
(f) \(\tau_{\xi,0}\)

Graphs show the comparison between exact solutions and approximations for different values of 4\(\pi\eta/s\). The graphs are labeled with the type of approximation used (Exact, aHydro, DNMR).
Connection to $2^{nd}$-order viscous hydro

$$\partial_\rho \hat{\varepsilon} + \tanh \rho \left( 2\hat{\varepsilon} + \hat{P}_\theta + \hat{P}_\phi \right) = 0$$

$$\hat{\pi}_{\mu \nu} = \hat{\pi}_{\theta} \hat{\Theta}_\mu \hat{\Theta}_\nu + \hat{\pi}_{\phi} \hat{\Phi}_\mu \hat{\Phi}_\nu + \hat{\pi}_\zeta \hat{\zeta}_\mu \hat{\zeta}_\nu$$

$$\hat{P}_i = \hat{P}_{iso} + \hat{\pi}_i$$

$$\hat{T} \hat{s} = 4\hat{\varepsilon}/3$$

$$\hat{P}_{iso} = \hat{\varepsilon}/3$$

- $2^{nd}$ order viscous hydrodynamics$^{[1]}$

$$\frac{\partial_\rho \hat{T}}{\hat{T}} + \frac{2}{3} \tanh \rho = \frac{1}{3} \hat{\pi}_\zeta \tanh \rho$$

$^{[1]}$H. Marrochio, J. Noronha, G. S. Denicol, M. Luzum, S. Jeon, et al., , arXiv:1307.6130.