Time-Frequency Localization and the Gabor Transform

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Abstract

The time-frequency content of a signal can be measured by the Gabor transform or windowed Fourier transform. This is a function defined on phase space that is computed by taking the Fourier transform of the product of the signal against a translate of a fixed window. The problem of finding signals with Gabor transform that are maximally concentrated within a given region of phase space is discussed. It is well known that such problems give rise to an eigenvalue problem for an associated self-adjoint, positive concentration operator that has its spectrum contained in the unit interval. In this paper, the asymptotic behavior of these eigenvalues as the concentration region gets large is studied. The smoothness of the eigenfunctions is also examined.

1 Introduction

A time-frequency shift of a square integrable function \( f(t) \in L^2(\mathbb{R}) \) is defined by:

\[
\rho(\tau, \sigma)f(t) = \exp(\pi i \tau \sigma) \exp(2\pi i \sigma t)f(t + \tau).
\]

The first factor is not essential and is present in order to simplify many of the subsequent formulae. Ignoring this, the formula clearly has the structure

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of a time shift of \( \tau \) units together with a frequency shift of \( \sigma \) units. The Gabor transform or windowed Fourier transform depends on the choice of a window function \( \phi(t) \in L^2(\mathbb{R}) \) and is defined by

\[
S_{\phi} f(\tau, \sigma) = \langle f, \rho(\tau, \sigma) \phi \rangle = \int f(t)\rho(\tau, \sigma)\phi(t) \, dt
\]

for any \( f \in L^2(\mathbb{R}) \). Here \( \langle \cdot, \cdot \rangle \) denotes the standard inner product on \( L^2(\mathbb{R}) \).

The Fourier transform of \( f(t) \) is defined by

\[
\hat{f}(\sigma) = \mathcal{F} f(\sigma) = \int f(t)e^{-2\pi it\sigma} \, dt.
\]

One easily checks that

\[
\mathcal{F} \rho(\tau, \sigma)f = \rho(-\sigma, \tau) \hat{f}.
\]

The following proposition collects some formulae that will be useful in the sequel.

**Proposition 1** Let \( f, \phi \in L^2(\mathbb{R}) \).

1. \[ \int \int |S_{\phi} f(\tau, \sigma)|^2 \, d\tau d\sigma = \| f \|^2 \| \phi \|^2. \]

2. \( S_{\phi} f(\tau, \sigma) = S_{\hat{\phi}} \hat{f}(-\sigma, \tau) \) where \( \hat{f} \) and \( \hat{\phi} \) denote the Fourier transform.

**PROOF:** The first item is a well known special case of Moyal's identity. See [3]. The second item is proved by using Plancherel’s theorem together with equation 1:

\[
S_{\phi} f(\tau, \sigma) = \langle f, \rho(\tau, \sigma) \phi \rangle = \langle \mathcal{F} f, \mathcal{F} \rho(\tau, \sigma) \phi \rangle = S_{\hat{\phi}} \hat{f}(-\sigma, \tau)
\]


Let \( \Omega \subset \mathbb{R}^2 \) be a region within the time-frequency plane. For the remainder of the exposition, \( \phi \) will be assumed to be normalized to have \( L^2 \) norm equal to one: \( \| \phi \|^2 = 1 \). We will measure the energy of \( f \) contained within \( \Omega \) by

\[
E_{\Omega}(f) = \int \int_{\Omega} |S_{\phi} f(\tau, \sigma)|^2 \, d\tau d\sigma.
\]
Associated to the energy is the Hermitian symmetric form defined by:

\[ \mathcal{E}_\Omega(f, g) = \int \int_{\Omega} \langle f, \rho(\tau, \sigma)\phi \rangle \langle \rho(\tau, \sigma)\phi, g \rangle \, d\tau d\sigma \]

for any \( f, g \in L^2(\mathbb{R}) \). Proposition 1 insures that \( \mathcal{E} \) is bounded in the following sense:

\[ \mathcal{E}_\Omega(f, g) \leq \|f\| \|g\|. \tag{2} \]

A standard duality argument produces a bounded, positive, self-adjoint operator \( C_\Omega : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) such that

\[ \mathcal{E}_\Omega(f, g) = \langle C_\Omega f, g \rangle \]

for all \( f, g \in L^2(\mathbb{R}) \). Let \( \psi_1, \psi_2, \cdots \) be a complete orthonormal basis of \( L^2(\mathbb{R}) \) and \( \lambda_1, \lambda_2, \cdots \) be a decreasing sequence of nonnegative real numbers for which

\[ C_\Omega \psi_k = \lambda_k \psi_k. \]

The bound in equation 2 forces \( 1 \geq \lambda_1 \geq \lambda_2 \geq \cdots \).

It is clear that the unit norm signal with the maximum energy within \( \Omega \) is \( \psi_1 \). Moreover, this maximum energy is given by \( \lambda_1 \). More generally, the quantity

\[ n(\lambda) = \text{card}\{k : \lambda_k > \lambda\} \]

determines the maximum dimension of a subspace \( V \in L^2(\mathbb{R}) \) of signals for which

\[ \mathcal{E}_\Omega(f) \geq \lambda \|f\|^2 \]

for all \( f \in V \). Thus the distribution of these eigenvalues as well as the smoothness of the corresponding eigenvalues are of interest within the context of signal processing.

2 A Reproducing Kernel Hilbert Space

It will be convenient to analyse the distribution of the eigenvalues using the reproducing kernel Hilbert space setting described in this section.

Proposition 1 implies that \( S_\phi \) is an isometry of \( L^2(\mathbb{R}) \) onto a closed subspace \( \mathcal{V}_\phi \subset L^2(\mathbb{R}^2) \). Let \( P_\phi \) be the orthogonal projection from \( L^2(\mathbb{R}^2) \) to \( \mathcal{V}_\phi \). Let \( F \in \mathcal{V}_\phi \) and \( f = S_\phi^{-1}F \). Then, applying the Cauchy-Schwarz inequality, one has

\[ |F(\tau_0, \sigma_0)| = \langle f, \rho(\tau_0, \sigma_0)\phi \rangle \leq \|f\| \|\rho(\tau_0, \sigma_0)\phi\| = \|F\|. \]
Therefore point evaluation at a given \((\tau_0, \sigma_0)\) of functions in \(V_\phi\) is a bounded linear functional. Let \(K_{(\tau_0, \sigma_0)}(\tau, \sigma) = K(\tau_0, \sigma_0; \tau, \sigma)\) denote that element of \(V_\phi\) for which
\[
F(\tau_0, \sigma_0) = \int \int F(\tau, \sigma)\overline{K(\tau_0, \sigma_0)(\tau, \sigma)} \, d\tau d\sigma.
\]
In order to identify the kernel explicitly one only has to notice that
\[
F(\tau_0, \sigma_0) = \langle S_{\phi}^{-1}F, \rho(\tau_0, \sigma_0)\phi \rangle.
\]
Since \(S_{\phi}\) is an isometry,
\[
F(\tau_0, \sigma_0) = \langle F, S_{\phi} \rho(\tau_0, \sigma_0)\phi \rangle = \int \int F(\tau, \sigma)\overline{\rho(\tau_0, \sigma_0)\phi, \rho(\tau, \sigma)\phi} \, d\tau d\sigma
\]
Consequently,
\[
K(\tau_0, \sigma_0; \tau, \sigma) = \langle \rho(\tau_0, \sigma_0)\phi, \rho(\tau, \sigma)\phi \rangle.
\]
Finally, if \(F \in V_\phi^\perp\) we have
\[
\int \int F(\tau, \sigma)\overline{K(\tau_0, \sigma_0)(\tau, \sigma)} \, d\tau d\sigma = 0
\]
since \(K_{(\tau_0, \sigma_0)}(\tau, \sigma) \in V_\phi\). These remarks are summarized in the following proposition.

**Proposition 2** For any \(F \in L^2(\mathbb{R}^2)\) the projection \(\mathcal{P}_\phi F\) is computed by
\[
\mathcal{P}_\phi F(\tau, \sigma) = \int \int F(\tau', \sigma')\overline{K(\tau, \sigma)(\tau', \sigma')} \, d\tau' d\sigma'.
\]

Our next task is to use the isometry \(S_{\phi}\) to identify the concentration operator \(C_\Omega\) explicitly. In particular, the next proposition provides a formula for \(S_{\phi}C_\Omega S_{\phi}^{-1}\) on \(V_\phi\).

**Proposition 3** For any \(F \in V_\phi\),
\[
S_{\phi}C_\Omega S_{\phi}^{-1} F(\tau, \sigma) = \int_{\Omega} F(\tau', \sigma')\overline{K(\tau, \sigma)(\tau', \sigma')} \, d\tau' d\sigma'.
\]
Proof: Let $S_\phi f = F$ and $S_\phi g = G$. Then
\[
\mathcal{E}_\Omega(f, g) = \int \int_\Omega \langle f, \rho(\tau, \sigma) \phi \rangle \langle \rho(\sigma, \tau) \phi, g \rangle d\tau d\sigma \\
= \int \int_\Omega S_\phi f \overline{S_\phi g} d\tau d\sigma \\
= \int \int_\Omega FG d\tau d\sigma \\
= \int \int F|_\Omega \overline{G} d\tau d\sigma \\
= \int \int \mathcal{P}_\phi (F|_\Omega) \overline{G} d\tau d\sigma.
\]

On the other hand, $\mathcal{E}_\Omega(f, g) = \langle \mathcal{C}_\Omega f, g \rangle$. Putting this together easily gives the formula in the proposition. □

In view of proposition 3, the spectral properties of the concentration operator $\mathcal{C}_\Omega$ are identical to those of the operator from $\mathcal{V}_\phi$ to itself defined by

$$F \mapsto \mathcal{P}_\phi \chi_\Omega F$$

where $\chi_\Omega$ is the operator that restricts functions to the set $\Omega$.

Since the function $\phi$ is fixed in the ensuing discussion, its use as a subscript will often be suppressed in the sequel.

3 Distribution of the Eigenvalues

In this section we examine the behavior of the eigenvalues as the concentration region $\Omega$ gets large. In particular, we show that the number of eigenvalues that cluster around one grows like the area of the concentration region. Both the philosophy and proofs in this section bear a great resemblance to the work of H.J. Landau in [7, 8].

Theorem 1 Let $\Omega$ be a bounded measurable set and

$$n(\lambda, \Omega_r) = \text{card}\{ k : \lambda_k(\Omega_r) \geq \lambda \}$$

where $\Omega_r = \{ (r\tau, r\sigma) : (\tau, \sigma) \in \Omega \}$. Then

$$\liminf_{r \to \infty} \frac{n(\lambda, \Omega_r)}{\text{area}(\Omega_r)} = 1$$

as $r \to \infty$. 

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The essential idea behind this theorem, as in the bandlimited case treated by Landau, is that \( \sum \lambda_k \) and \( \sum |\lambda_k|^2 \) grow at the same asymptotic rate as \( r \to \infty \). The following lemma enables us to work with the operator \( \chi_\Omega P \) on \( L^2(\Omega) \) as opposed to the operator \( P\chi_\Omega \) on \( V \).

**Lemma 1** The operator \( \chi_\Omega P \) on \( L^2(\Omega) \) and the operator \( P\chi_\Omega \) have the same nonzero eigenvalues with multiplicity.

**Proof:** Let \( F \in V \) be a nontrivial eigenfunction with a nonzero eigenvalue of the operator \( P\chi_\Omega \): \( P\chi_\Omega F = \lambda F \). Since both \( F \) and \( \lambda \) are nonzero, \( \chi_\Omega F \not\equiv 0 \).

On the other hand, applying the restriction operator to both sides we have \( \chi_\Omega P\chi_\Omega F = \lambda \chi_\Omega F \). A similar argument shows that every nonzero eigenvalue of \( \chi_\Omega P \) is also an eigenvalue of \( P\chi_\Omega \). \( \square \)

**Lemma 2** \( \sum \frac{|\lambda_k(\Omega_r)|^2}{\text{area}(\Omega_r)} \to 1 \) as \( r \to \infty \).

**Proof:** By lemma 1, it suffices to consider the trace operator \( (\chi_\Omega P)^2 \). The kernel of this operator on \( L^2(\Omega_r) \) is determined by the formula

\[
(\chi_\Omega \mathcal{P})^2 F(\tau, \sigma) = \int \int \int_{\Omega_r \times \Omega_r} F(\tau'', \sigma'') K_{(\tau'', \sigma'')}(\tau', \sigma') d\tau'' d\sigma'' d\tau' d\sigma'.
\]

The kernel is continuous on the bounded set \( \Omega_r \times \Omega_r \) so by Mercer’s theorem the trace of \( (\chi_\Omega \mathcal{P})^2 \) is

\[
\int \int_{\Omega_r} \int \int_{\Omega_r} |K(\tau, \sigma; \tau', \sigma')|^2 d\tau d\sigma d\tau' d\sigma'.
\]

First note that

\[
K(\tau, \sigma; \tau', \sigma') = (\rho(\tau, \sigma) \phi, \rho(\tau', \sigma') \phi) = (\rho(-\tau', -\sigma') \rho(\tau, \sigma) \phi, \phi) = \exp(\pi i (\tau' \sigma' - \sigma' \tau')) \rho(\tau - \tau', \sigma - \sigma') \phi, \phi) = \exp(\pi i (\tau' \sigma' - \sigma' \tau')) H(\tau - \tau', \sigma - \sigma').
\]

Therefore, it is enough to estimate

\[
\int \int_{\Omega_r} \int \int_{\Omega_r} |H(\tau - \tau', \sigma - \sigma')|^2 d\tau d\sigma d\tau' d\sigma'.
\]

By proposition the \( L^2 \) norm of \( H \) is one. The theorem then follows from the following lemma. \( \square \)
Lemma 3  Let $f \in L^1(\mathbb{R}^n)$ with $f(x) \geq 0$ and $\int f(x) dx = 1$. Let $Q \subset \mathbb{R}^n$ is a bounded set of positive measure. Then
\[ r^{-n} \int_{rQ} \int_{rQ} f(x-y) dx \, dy \to |Q| \]
as $r \to \infty$.

Proof: Apply the following change of variables:
\[ x = \xi + r\eta \quad y = r\eta \]
to get
\[ r^{-n} \int_{rQ} \int_{rQ} f(x-y) dx \, dy = \int_Q \int_{r(Q-\eta)} f(\xi) d\xi \, d\eta. \quad (3) \]
Recall that almost every point of a set of positive measure is a point of density:
\[ \lim_{\epsilon \to 0} \frac{|B_\epsilon(\eta) \cap Q|}{\pi \epsilon^2} = 1 \quad \text{for almost all } \eta \in Q. \]
Hence for almost all $\eta \in Q$ and arbitrary $R > 0$,
\[ \frac{|r(Q-\eta) \cap B_R(0)|}{\pi R^2} \to 1 \]
as $r \to \infty$. It is then straight-forward to argue that
\[ \int_{r(Q-\eta) \cap B_R(0)} f(\xi) d\xi \to \int_{B_R(0)} f(\xi) d\xi \]
as $r \to \infty$ for almost all $\eta \in Q$. Since
\[ \int_{r(Q-\eta) \cap B_R(0)} f(\xi) d\xi \leq \int_{r(Q-\eta)} f(\xi) d\xi \leq 1 \]
it follows that, for almost all $\eta \in Q$:
\[ \int_{B_R(0)} f(\xi) d\xi \leq \liminf_{r \to \infty} \int_{r(Q-\eta)} f(\xi) d\xi \leq \limsup_{r \to \infty} \int_{r(Q-\eta)} f(\xi) d\xi \leq 1. \]
Since $R > 0$ was chosen arbitrarily, it follows that
\[ \lim_{r \to \infty} \int_{r(Q-\eta)} f(\xi) d\xi = 1 \]
for almost all $\eta \in Q$. The result then follows from Lebesgue’s dominated convergence theorem. □

**Proof of Theorem 1:** By Mercer’s theorem, the trace of the operator $\mathcal{P}_\chi$ is

$$\sum \lambda_k = \int \int_{\Omega_r} K(\tau, \sigma, \tau, \sigma) \, d\tau d\sigma = \int \int_{\Omega_r} \langle \rho(\tau, \sigma) \phi, \rho(\tau, \sigma) \phi \rangle \, d\tau d\sigma = \text{area}(\Omega_r).$$

Choose $r$ so large that $\sum |\lambda_k(\Omega_r)|^2 \geq (1 - \epsilon)\text{area}(\Omega_r)$. Then

$$(1 - \epsilon)\text{area}(\Omega_r) \leq \sum |\lambda_k(\Omega_r)|^2 \leq \sum_{k=1}^n \lambda_k + \lambda \sum_{k>n} \lambda_k \leq (1 - \lambda) \sum_{k=1}^n \lambda_k + \lambda \text{area}(\Omega_r) \leq (1 - \lambda)n + \lambda \text{area}(\Omega_r)$$

As a consequence,

$$n \geq \frac{1 - \lambda - \epsilon}{1 - \lambda} \text{area}(\Omega_r).$$

Since $\epsilon > 0$ was chosen arbitrarily, we have that

$$\liminf \frac{n(\lambda, \Omega_r)}{\text{area}(\Omega_r)} \geq 1.$$  

□

The following refinement of lemma 3, in the case when $Q$ is a bounded domain with $C^1$ boundary and $f$ has sufficiently strong decay, is useful in order to estimate the size of the plunge region.

**Lemma 4** Let $Q$ be a domain with $C^1$ boundary and $f(x)$ as in lemma 3. Moreover, assume that

$$|1 - \int_{B_r(0)} f(x) \, dx| \leq \frac{C}{r^p}$$

for some constants $p, C > 0$. Then the error term

$$\frac{1}{r^n} \int_{rQ} \int_{rQ} f(x - y) \, dx \, dy - |Q|$$

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is $O(\frac{1}{r^2})$ as $r \to \infty$.

**Proof:** Given a $\delta > 0$, let $Q_\delta = \{x \in Q : \text{dist}(x, \partial Q) > \delta\}$. The second iterated integral in equation 3 can then be expressed as follows:

$$\int_0^{\delta_0} \int_{\partial Q_\rho} I_r(\eta) \, d\mathcal{H}(\eta) \, d\rho$$

where

$$I_r(\eta) = \int_{r(Q-\eta)} f(\xi) \, d\xi,$$

d$\mathcal{H}$ is Hausdorff $n-1$ dimensional measure, and $\delta_0$ is any positive number larger than the diameter of $Q$. Using equation 4 to estimate $I_r(\eta)$ yields

$$\left| \int_0^{\delta_0} \int_{\partial Q_\rho} I_r(\eta) \, d\mathcal{H}(\eta) \, d\rho - |Q_{1/r}| \right| \leq \int_0^{\delta_0} \int_{\partial Q_\rho} |I_r(\eta) - 1| \, d\mathcal{H}(\eta) \, d\rho \leq C \int_0^{\delta_0} \int_{\partial Q_\rho} \frac{1}{(r\rho)^p} \, d\mathcal{H}(\eta) \, d\rho. \quad (6)$$

Using the fact that the Hausdorff $n-1$ measure of the sets $\partial Q_\rho$ are uniformly bounded, one easily has an estimate of the required type for the first expression in inequality 6. Moreover, since $0 \leq I_r(\eta) \leq 1$,

$$\left| \int_0^{1/r} \int_{\partial Q_\rho} I_r(\eta) \, d\mathcal{H}(\eta) \, d\rho - \text{meas}(Q \setminus Q_{1/r}) \right| = \int_0^{1/r} \int_{\partial Q_\rho} (1 - I_r(\eta)) \, d\mathcal{H}(\eta) \, d\rho \leq \int_0^{1/r} \int_{\partial Q_\rho} d\mathcal{H}(\eta) \, d\rho \leq \text{meas}(Q \setminus Q_{1/r}) \leq C/r. \quad (7)$$

Putting equations 6 and 7 together via the triangle inequality yields the sought after result. \[ \square \]

**Theorem 2** Let $\Omega$ be a bounded domain with $C^1$ boundary and

$$n(\lambda, \mu, \Omega_r) = \text{card}\{k : \lambda_k(\Omega_r) \in [\lambda, \mu]\}$$

where $0 < \lambda < \mu < 1$. Moreover, assume that the function $|S_\sigma(\tau, \sigma)|^2$ satisfies condition 3. Then, there is a constant $C > 0$ for which

$$n(\lambda, \mu, \Omega_r) \leq Cr.$$
Proof: Mercer’s theorem applied to the positive operator $\chi_{\Omega}, P - (\chi_{\Omega}, P)^2$ yields that

$$\sum \lambda_k(\Omega_r) - \lambda_k(\Omega_r)^2 = |\Omega_r| - \int \int_{\Omega_r} \int \int_{\Omega_r} |H(\tau - \tau', \sigma - \sigma')|^2 d\tau d\tau' d\sigma d\sigma'.$$

Applying lemma 4 yields

$$0 < \sum \lambda_k(\Omega_r) - \lambda_k(\Omega_r)^2 \leq C.$$  

Therefore, if $\epsilon \in (0, 1/2)$ then

$$n(\epsilon, 1 - \epsilon, \Omega_r)(\epsilon - \epsilon^2) \leq \sum \lambda_k(\Omega_r) - \lambda_k(\Omega_r)^2 \leq C.$$  

The result follows.  

4 Regularity of the Eigenfunctions

In this section we will be concerned with the regularity of the eigenfunctions of the concentration operator $C_{\Omega}$. In order to do this it will be necessary to work with the integral kernel of this operator.

**Proposition 4** The kernel $k_{\Omega}(x, y)$ of the operator $C_{\Omega}$ is

$$k_{\Omega}(x, y) = \int \int_{\Omega} \rho(\tau, \sigma)\phi(x)\rho(\tau, \sigma)\phi(y) \, d\tau d\sigma.$$  

Proof: Using the definition of the concentration operator, one has

$$\langle C_{\Omega}f, g \rangle = \mathcal{E}_{\Omega}(f, g)$$

$$= \int \int_{\Omega} \langle f, \rho(\tau, \sigma)\phi(x) \rangle \langle \rho(\tau, \sigma)\phi(x), g \rangle \, d\tau d\sigma$$

$$= \int \int f(y) \left( \int \int_{\Omega} \rho(\tau, \sigma)\phi(x)\rho(\tau, \sigma)\phi(y) \, d\tau d\sigma \right) g(x) \, dx dy.$$  

The result follows.  

Suppose that $\gamma(s)$ is a positive decreasing function defined on $[0, \infty)$ satisfying the following conditions:

- $\gamma(0) = 0$ and $\lim_{s \to \infty} \gamma(s) = 0$, 

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• there is an $\epsilon_0 > 0$ such that $\int |\gamma(s)|^p ds < \infty$ for all $p \in (2 - \epsilon_0, 2)$, and
• for any $s_0 > 0$, $\gamma(s - s_0) = O(\gamma(s)^{1-\epsilon})$ for all sufficiently small $\epsilon > 0$.

Examples of functions $\gamma(s)$ that satisfy such conditions include
• $\gamma(s) = (1 + s^2)^{-q/2}$ where $q \in (1, \infty)$, and
• $\gamma(s) = \exp(-\kappa |s|^q)$ where $\kappa, q > 0$.

**Proposition 5** If $|\phi(t)| \leq \gamma(|x|)$ for all $t$ then any eigenfunction with nonzero eigenvalue $\psi$ of the concentration operator $C_\Omega$ satisfies an estimate of the form
$$|\psi(t)| \leq C\gamma(|t|)^{1-\epsilon}$$
for all $t$.

**Proof:** Suppose that $C_\Omega \psi = \lambda \psi$ with $\lambda > 0$. The first step is to estimate the kernel, $k_\Omega(x, y)$:
$$|k_\Omega(x, y)| = \left| \int_\Omega e^{2\pi i \sigma(x-y)} \phi(x + \tau) \overline{\phi(y + \tau)} d\tau d\sigma \right| \leq C\gamma(||x| - R|)\gamma(||y| - R|) \leq C\gamma(||x||^{1-\epsilon})\gamma(||y||^{1-\epsilon})$$
where $R = \sup\{|(\tau, \sigma)| : (\tau, \sigma) \in \Omega\}$. Consequently, using the Cauchy-Schwarz inequality, we have the estimate:
$$\lambda |\psi(x)| \leq C\gamma(||x||^{1-\epsilon}) \int \gamma(||y||^{1-\epsilon}) \psi(y) dy \leq C\gamma(||x||^{1-\epsilon}).$$

Let $\tilde{\Omega} = \{(\tau, \sigma) : (-\sigma, \tau) \in \Omega\}$. The second item in proposition implies that:

$$E_\Omega(f, g) = \int_\Omega S_\phi f \overline{S_\phi g} = \int_{\tilde{\Omega}} S_\phi \hat{f} \overline{S_\phi \hat{g}} = E_{\tilde{\Omega}}(\hat{f}, \hat{g}).$$
In particular,

\[ E_{\Omega}(\hat{\psi}_k, \hat{\psi}_l) = \delta_{kl} \]

and \( \hat{\psi}_1, \ldots \) are the eigenfunction of the operator \( C_{\Omega} \). Putting this together with the preceding proposition yields the following theorem.

**Theorem 3** If

\[
\begin{align*}
|\phi(t)| &\leq \gamma_1(|t|) \text{ and } \\
|\hat{\phi}(\sigma)| &\leq \gamma_2(|\sigma|)
\end{align*}
\]

where both \( \gamma_1 \) and \( \gamma_2 \) satisfy the conditions described above, then

\[
\begin{align*}
|\psi(t)| &\leq C\gamma_1(|t|)^{1-\epsilon} \text{ and } \\
|\hat{\psi}(\sigma)| &\leq C\gamma_2(|\sigma|)^{1-\epsilon}.
\end{align*}
\]

Various special cases merit mention.

- Let

\[
\phi(t) = \begin{cases} 
1 - |t| & \text{if } |t| \leq 1 \\
0 & \text{otherwise.}
\end{cases}
\]

Then \( k_\Omega(x, y) = 0 \) whenever \( |x - y| \geq 2 \). Hence, any eigenfunction, with nonzero eigenvalue, must have support contained in the interval \([-3, 3]\). Moreover, the Fourier transform of \( \phi \) is the Fejer kernel which is bounded by \( \gamma(s) = (1 + s^2)^{-1} \). As a consequence, if \( \psi \) is any eigenfunction with nonzero eigenvalue,

\[
\text{support}(\psi) \subset [-3, 3] \\
\hat{\psi}(\sigma) = \mathcal{O}\left(\frac{1}{1+|\sigma|^{2-\epsilon}}\right).
\]

In particular, \( \psi' \) is square integrable.

- If \( \phi \) is Schwartz class, one can verify directly using proposition 4 that the kernel \( k_\Omega \) is also Schwartz class. A straightforward interchange of the differentiation and integration symbol then shows that \( \psi \) is also Schwartz class:

\[
|t^\alpha D^\beta_t \psi(t)| = |\int t^\alpha D^\beta_t k_\Omega(t, s)\psi(s)ds| \\
\leq \|t^\alpha D^\beta_t k_\Omega(t, s)\| \|\psi\|.
\]

Since the kernel is Schwartz class, the last expression is bounded independently of \( t \). Thus, if \( \phi \) is Schwartz class, so are any eigenfunctions associated to nonzero eigenvalues.

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If $\phi(t) = \exp(-Ct^2)$, theorem 3 implies that the eigenfunctions of the concentration operator with nonzero eigenvalue are $O(\exp(-C(1-\epsilon)t^2))$. Moreover, since $\hat{\phi}(\sigma) = \sqrt{\pi/C} \exp(-\pi/C\sigma^2)$, the Fourier transforms of these eigenfunctions are $O(\exp(-\pi C(1-\epsilon)\sigma^2))$. In particular, these functions are entire by the Paley-Wiener theorem [6]. Furthermore, by differentiating under the integral sign (as above, and in the formula for the kernel in proposition 4), one can show that all derivatives of these eigenfunctions satisfy the same decay estimates.

Theorem 3 is useful in that it provides quantitative bounds on the decay (and eventually regularity) of the eigenfunctions for a wide class of $\phi$. Among the works in the literature that bear on the problems discussed here, we remark that Pietsch [9] provides estimates on the decay of the eigenvalues for given decay and regularity of the integral kernel. In particular, his results imply that when $\phi$ is in Schwartz space, the sequence of eigenvalues decays faster than the inverse of any polynomial in the index $n$. Janssen [4] obtains the same estimate in the Schwartz class case.

For $\phi(t) = \exp(-Ct^2)$, the work of Daubechies [1] is relevant. She solves the eigenvalue problem explicitly for circular (or elliptical) domains, and obtains the Hermite functions as eigenfunctions. She also obtains a formula for the eigenvalues, which decay exponentially in the index $n$. Our approach shows that for any bounded measurable domain, the eigenfunctions are analytic and have quadratic exponential decay on the real line. Since the kernel is Schwartz class, [3] implies that the eigenvalues decay faster than the inverse of any polynomial in the index $n$. In fact, a stronger statement can be obtained using a result of Janssen [5]. Note that the integral kernel is

$$k_{\Omega}(x,y) = \int_{\Omega} \exp(2\pi i\sigma(x-y)) \exp(-C((x+\tau)^2 + (y+\tau)^2)) d\tau d\sigma.$$ 

By expanding the exponentials in power series and integrating term by term, it is easy to show that this kernel is analytic in each variable, and satisfies the bound $O(\exp(-ARe[x]^2 + BIm[x]^2 - CRe[y]^2 + DIm[y]^2))$, for some $A, B, C, D > 0$. Since the kernel is positive definite as well, it follows from theorem A.1 in [5] that the eigenfunctions belonging to nonzero eigenvalue are analytic and satisfy $O(\exp(-ARe[x]^2 + BIm[x]^2))$, some $A, B > 0$. At the same time, the eigenvalues are in $O(\exp(-n\alpha))$, some $\alpha > 0$.

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