Self-Intersection Times for Random Walk, and Random Walk in Random Scenery in dimensions $d \geq 5$.

Amine Asselah & Fabienne Castell
C.M.I., Université de Provence,
39 Rue Joliot-Curie,
F-13453 Marseille cedex 13, France
asselah@cmi.univ-mrs.fr & castell@cmi.univ-mrs.fr

Abstract

Let $\{S_k, k \geq 0\}$ be a symmetric random walk on $\mathbb{Z}^d$, and $\{\eta(x), x \in \mathbb{Z}^d\}$ an independent random field of centered i.i.d. with tail decay $P(\eta(x) > t) \approx \exp(-t^\alpha)$. We consider a Random Walk in Random Scenery, that is $X_n = \eta(S_0) + \cdots + \eta(S_n)$. We present asymptotics for the probability, over both randomness, that $\{X_n > n^\beta\}$ for $\beta > 1/2$ and $\alpha > 1$. To obtain such asymptotics, we establish large deviations estimates for the self-intersection local times process $\sum l_n^2(x)$, where $l_n(x)$ is the number of visits of site $x$ up to time $n$.

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Running: Random Walk in Random Scenery.

1 Introduction.

We study transport in divergence free random velocity fields. For simplicity, we discretize both space and time and consider the simplest model of shear flow velocity fields:

$$\forall x, y \in \mathbb{Z} \times \mathbb{Z}^d, \quad V(x, y) = \eta(y) e_x,$$

where $e_x$ is a unit vector in the first coordinate of $\mathbb{Z}^{d+1}$, and $\{\eta(y), y \in \mathbb{Z}^d\}$ are i.i.d. real random variables. Thus, space consists of the sites of the cubic lattice $\mathbb{Z}^{d+1}$ and the direction of the shear flow is $e_x$. We wish to model a pollutant evolving by two mechanisms: a passive transport by the velocity field, and collisions with the other fluid particles modeled by random centered and independent increments $\{(\alpha_n, \beta_n) \in \mathbb{Z} \times \mathbb{Z}^d, n \in \mathbb{N}\}$, independent of the velocity field. Thus, if $R_n \in \mathbb{Z} \times \mathbb{Z}^d$ is the pollutant’s position at time $n$, then

$$R_{n+1} - R_n = V(R_n) + (\alpha_{n+1}, \beta_{n+1}), \quad \text{and} \quad R_0 = (0,0).$$

(1)
When solving by induction for \( R_n \), (1) yields

\[
R_n = \left( \sum_{k=1}^{n} \alpha_k + \eta(0) + \sum_{k=1}^{n} \eta\left( \sum_{i=1}^{k} \beta_i \right) \right).
\]

The sum \( \beta_1 + \cdots + \beta_n \) is denoted by \( S_n \), and called the Random Walk (RW). The displacement along \( e_x \) consists of two independent parts: a sum of i.i.d. random variables \( \alpha_1 + \cdots + \alpha_n \), and a sum of dependent random variables \( \eta(S_0) + \cdots + \eta(S_n) \), which we denote by \( X_n \) and call the Random Walk in Random Scenery (RWRS). Writing it in terms of local times of the RW, say \( \{l_n(x), n \in \mathbb{N}, x \in \mathbb{Z}^d \} \), we get

\[
X_n = \sum_{k=0}^{n} \eta(S_k) = \sum_{x \in \mathbb{Z}^d} l_n(x) \eta(x), \quad \text{where} \quad l_n(x) = \sum_{k=0}^{n} \mathbb{I}\{S_k = x\}.
\]

The process \( \{X_n, n \in \mathbb{N}\} \) was studied at about the same time by Kesten & Spitzer [13], Borodin [4, 6], and Matheron & de Marsily [17]. The fact that in dimension 1, \( E[X_n^2] \sim n^{3/2} \) made the model popular and led the way to examples of superdiffusive behaviour. However, the typical behaviour of \( X_n \) resembles that of a sum of \( n \) independent variables all the more when dimension is large.

Our goal is to estimate the probability that \( X_n \) be large. By probability, we consider averages with respect to the two randomness, and \( P = \mathbb{P}_0 \otimes P_\eta \), where \( \mathbb{P}_0 \) is the law of the nearest neighbors symmetric random walk \( \{S_k, k \in \mathbb{N}\} \) on \( \mathbb{Z}^d \) with \( S_0 = 0 \), and \( P_\eta \) is the law of the velocity field.

Now, when \( d \geq 3 \), Kesten & Spitzer established in [13] that \( X_n/\sqrt{n} \) converges in law to a Gaussian variable. Thus, by large, we mean \( \{X_n > n^\beta\} \) with \( \beta > 1/2 \). We expect \( P(X_n > n^\beta) \approx \exp(-In^\xi) \) with constant rate \( I > 0 \), and we characterize in this work the exponent \( \xi \). For this purpose, the only important feature of the \( \eta \)-variables is the \( \alpha \)-exponent in the tail decay:

\[
\lim_{t \to \infty} \frac{\log P_\eta(\eta(0) > t)}{t^\alpha} = -c, \quad \text{for a positive constant} \quad c.
\]

Let us now recall the classical estimates for \( P(Y_1 + \cdots + Y_n > n^\beta) \), where \( \beta > 1/2 \) and the \( \{Y_n, n \in \mathbb{N}\} \) are centered i.i.d. with tail decay \( P(Y_n > t) \approx \exp(-t^a) \), with \( a > 0 \). There is a dichotomy between a “collective” and an “extreme” behaviour. In the former case, each variable contributes about the same, whereas in latter case, only one term exceeds the level \( n^\beta \), when the others remain small. Thus, it is well known that \( P(Y_1 + \cdots + Y_n > n^\beta) \sim \exp(-In^\xi) \) with three regimes for the exponent \( \xi \).

- When \( \beta \geq 1 \) and \( a > 1 \), a large collective contribution yields \( \xi = (\beta - 1)a + 1 \).
- When \( \beta < 1 \) and \( \beta(2-a) < 1 \), a small collective contribution yields \( \xi = 2\beta - 1 \).
- When \( \beta > 1/(2-a) \) and \( a < 1 \), an extreme contribution yields \( \xi = \beta a \).

For the RWRS, one expects a rich interplay between the scenery and the random walk. To get some intuition about the expression of \( \xi \) in terms of \( \alpha \) and \( \beta \), we propose simple scenarii leading to Figure [1] Here also, we focus on the exponent, and constants are omitted.
\[ \begin{align*}
\text{Region I. } & \text{No constraint is put on the walk. When } d \geq 3, \text{ the range of the walk is of order } n \text{ and visited sites are typically visited once. Thus, } \{X_n > n^\beta\} \sim \{\eta_1 + \cdots + \eta_n > n^\beta\}. \text{ When } \beta < 1, \text{ the latter sum performs a moderate deviations of order } n^\beta. \text{ Since the } \eta\text{-variables satisfy Cramer’s condition, we obtain } P(X_n > n^\beta) \geq \exp(-n^{2\beta-1}). \text{ Thus, the } \zeta\text{-exponent in Region I is } \zeta_I = 2\beta - 1. \\
\text{Region II, V. } & \text{A few sites are visited often, so that } X_n \sim \eta(0)l_n(0). \text{ Now, using the tail behaviour of } \eta(x), \text{ and the fact that in } d \geq 3, l_n(x) \text{ is bounded by an exponential variable, we obtain}\n
P\left(X_n \geq n^\beta\right) & \geq P\left(l_n(0)\eta(0) \geq n^\beta\right) \sim \sup_{k \leq n} \left(\mathbb{P}_0(l_n(0) = k) P_\eta \left(\eta(0) \geq \frac{n^\beta}{k}\right)\right) \\
& \sim \exp\left(-\inf_{k \leq n} \left(k + \frac{n^\beta}{k}\right)^{\alpha}\right).
\end{align*} \]

Now, the minimum of } k \mapsto k + n^{\beta\alpha}/k^\alpha \text{ is reached for } k^* = n^{\beta\alpha}/(\alpha + 1). \text{ Since, we impose also that } k \leq n, \text{ two different exponents prevail according to the value of } \beta:

(\text{II}) \ \beta < (\alpha + 1)/\alpha, \text{ and } \zeta_\| := \beta\alpha/(\alpha + 1) < 1. \text{ The RW spends a time of order } n^{\zeta_\|} \text{ on favorite sites.}

(\text{V}) \ \beta \geq (\alpha + 1)/\alpha, \text{ and } \zeta_V := \alpha(\beta - 1). \text{ The RW spends a time of the order of } n \text{ on favorite sites.}

• Region III, IV. The random walk is localized a time } T \text{ in a ball } B_r \text{ of radius } r, \text{ with } r^2 \ll T: \text{ this costs of the order of } \exp(-T/r^2). \text{ Then, during this period, each site of}
$B_r$ is visited about $T/r^d$, and we further assume that $r^d \ll T$. Thus

$$P(X_n \geq n^\beta) \geq \exp(-T/r^2)P_\eta \left( \frac{1}{\sqrt{r^d}} \sum_{B_r} \eta_j \geq \frac{n^\beta r^{d/2}}{T} \right). \quad (5)$$

Two different exponents prevail according to $\beta$:

(III) $\beta \leq 1$. The condition $1 \ll n^\beta r^{d/2}/T \leq r^{d/2}$ means that the sum of $\eta$-s performs a moderate (up to large) deviations and this costs of the order of $\exp(-n^{2\beta} r^d/T^2)$. When the two costs are equalized and the parameter $r$ and $T$ optimized, we obtain that the walk is localized a time $T = n^\beta$ on a ball of volume $r^d = n^{\zeta_\|}$, with $\zeta_\| := d\beta/(d+2)$.

(IV) $\beta > 1$. Here $T = n$ and we deal with a very large deviations for a sum of i.i.d. This has a cost of order $\exp(-n^{\alpha(\beta-1)} r^d)$. Choosing $r$ so that $n/r^2 = n^{\alpha(\beta-1)} r^d$, we obtain $\zeta_{IV} := (d+2\alpha(\beta-1))/(d+2)$. The condition $r \gg 1$ is equivalent to $\beta < 1 + 1/\alpha$. The walk is localized all the time on a ball of radius $r$ satisfying $r^{d+2} = n^{1-\alpha(\beta-1)}$.

The following regions have already been studied.

- $\alpha = +\infty$ (bounded scenery) and $\beta = 1$ in [1] (actually Brownian motion is considered there instead of RW).
- $\alpha = 2$ (Gaussian scenery) and $\beta \in [1, 1 + 1/\alpha]$ in [7, 8].
- Region IV ($\alpha > d/2, 1 \leq \beta < 1 + 1/\alpha$) in [12].
- $0 < \alpha < 1$ and $\beta > \frac{1+\alpha}{2}$, in $d \geq 3$, in [11]. This region is outside Figure 1.
- $\beta = 1$ and $\alpha < d/2$ in [2]. Contrary to the previous cases, distinct lower and upper bounds with the same exponent are obtained in [2].

This paper is devoted to regions I, II and III. Henceforth, we consider $d \geq 5$, unless explicitly mentioned.

**Proposition 1.1** Upper Bounds for the RWRS.

1. **Region I.** We assume $\beta < \min(\frac{\alpha+1}{\alpha+2} \frac{d/2+1}{d/2+2})$. There exists an explicit $y_0$, such that for $y > y_0$, there exists a constant $\bar{c}_1 > 0$ such that

$$P(X_n \geq n^\beta y) \leq \exp(-\bar{c}_1 n^{2\beta-1}). \quad (6)$$

2. **Region II.** Let $\alpha < d/2$, and $\beta \geq \frac{\alpha+1}{\alpha+2}$. For $y > 0$, there exists a constant $\bar{c}_2 > 0$, such that

$$P(X_n \geq n^\beta y) \leq \exp(-\bar{c}_2 n^{\beta\alpha/(\alpha+1)}). \quad (7)$$

For the case $\beta = \frac{\alpha+1}{\alpha+2}$, we further assume that $y > y_0$. 

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Moreover, when $\beta > \frac{\alpha + 1}{\alpha + 2}$, the main contribution to $\{X_n \geq n^\beta y\}$ comes from the level sets

$$D_\beta := \{x : n^{b-\delta} < l_n(x) < n^{b+\delta}\} \quad \text{with} \quad b = \frac{\beta}{\alpha + 1}, \quad \text{and any } \delta > 0.$$

In other words, for any $y > 0$

$$\lim_{n \to \infty} \frac{1}{n^{\zeta_\beta}} \log P \left( \sum_{x \not\in D_\beta} \eta(x)l_n(x) > yn^\beta \right) = -\infty. \quad (8)$$

3. Region III. Let $\alpha \geq d/2$ and $\beta \geq \frac{d/2+1}{d/2+2}$. For $y > 0$ and $\epsilon > 0$ small, there exists a constant $\bar{c}_3 > 0$, such that

$$P(X_n \geq n^\beta y) \leq \exp(-\bar{c}_3 n^{\beta d/2+\epsilon}). \quad (9)$$

For the case $\beta = \frac{d/2+1}{d/2+2}$, we further assume that $y > y_0$.

Moreover, if we define $D_\beta := \{x : 0 < l_n(x) < n^{b+\delta}\}$ with $b := \beta/(d/2+1)$, then we have, for any $\delta > 0$ small enough

$$\lim_{n \to \infty} \frac{1}{n^{\zeta_\beta}} \log P \left( \sum_{x \not\in D_\beta} \eta(x)l_n(x) > yn^\beta \right) = -\infty. \quad (10)$$

**Remark 1.2** Note that the control in Region III is less satisfactory than in Regions I and II. An inspection of the proof makes it clear that our techniques actually yield a logarithmic artifact $P(X_n \geq n^\beta y) \leq \exp(-n^{\zeta_\beta}/\log(n))$.

We indicate below lower bounds for $P(X_n \geq n^\beta y)$, which prove that we obtain the correct rates of the logarithmic decay of $P(X_n \geq n^\beta y)$. These lower bounds are given under an additional symmetry assumption on the scenery, which is not crucial, but simplifies the proofs. Hence, we say that a real random variable is bell-shaped, if its law has a density with respect to Lebesgue which is even, and decreasing on $\mathbb{R}^+$.  

**Proposition 1.3** Lower Bounds for the RWRS. Assume $d \geq 3$, and that the random variables $\{\eta(x), x \in \mathbb{Z}^d\}$ are bell-shaped.

1. Region I. Let $1 \geq \beta > 1/2$. For any $y > 0$, there exists a constant $c_1 > 0$, such that

$$P(X_n \geq n^\beta y) \geq \exp(-c_1 n^{2\beta-1}). \quad (11)$$

2. Region II. Let $\beta \leq 1 + 1/\alpha$. For any $y > 0$, there exists a constant $c_2 > 0$, such that

$$P(X_n \geq n^\beta y) \geq \exp(-c_2 n^{\beta\alpha/(\alpha+1)}). \quad (12)$$

3. Region III. Let $\beta \leq 1$. For any $y > 0$, there exists a constant $c_3 > 0$, such that

$$P(X_n \geq n^\beta y) \geq \exp(-c_3 n^{\beta d/2+1}). \quad (13)$$
In the process of establishing Proposition 1.1, one faces the problem of evaluating the chances the random walk visits often the same sites. More precisely, a crucial quantity is the self-intersection local time process (SILT):

\[
\Sigma^2_n = \sum_{x \in \mathbb{Z}^d} l^2_n(x) = n + 1 + 2 \sum_{0 \leq k < k' \leq n} \mathbb{I}\{S_k = S_{k'}\}.
\] (14)

It is expected that \(\Sigma^2_n\) would show up in the study of RWRS. Indeed, \(\Sigma^2_n\) is the variance of \(X_n\) when averaged over \(P_\eta\). If we assume for a moment that the \(\eta\)-variables are standard Gaussian, then conditionally on the random walk, \(X_n\) is a Gaussian variable with variance \(\Sigma^2_n\), so that

\[
P_\eta\left(\sum_{x \in \mathbb{Z}^d} \eta(x) l_n(x) > n^\beta\right) \leq \exp\left(-\frac{n^{2\beta}}{2 \sum_{x \in \mathbb{Z}^d} l^2_n(x)}\right)
\] (15)

It is well known that an inequality similar to (15) holds for any tail behaviour with \(\alpha \geq 2\).

Now, if we average with respect to the random walk law, then for any \(\gamma > 0\)

\[
P(\sum_{x \in \mathbb{Z}^d} l_n(x) > n^\gamma) \leq \mathbb{E}_0 \left[\exp\left(-\frac{n^{2\beta}}{2 \sum_{x \in \mathbb{Z}^d} l^2_n(x)}\right)\right]
\]

\[
\leq \exp(-n^{2\beta-\gamma}) + \mathbb{P}_0\left(\sum_{x \in \mathbb{Z}^d} l^2_n(x) > n^\gamma\right).
\] (16)

Hence, at least for large \(\alpha\), we have to evaluate the logarithmic decay of quantities such as \(\mathbb{P}_0\left(\sum_{x \in \mathbb{Z}^d} l^2_n(x) > n^\gamma\right)\). Note first that for \(d \geq 3\), and \(n \to \infty\),

\[
\mathbb{E}_d \left[\sum_{x \in \mathbb{Z}^d} l^2_n(x)\right] \approx n(2G_d(0) - 1),
\] (17)

where \(G_d\) is the Green kernel

\[
G_d(x) \triangleq \mathbb{E}_0[l_\infty(x)].
\]

Therefore, we have to take \(\gamma \geq 1\) to be in a large deviations scaling. For large deviations of SILT in \(d = 1\), we refer the reader to Mansmann [16], and Chen & Li [9], while in \(d = 2\), this problem is treated in Bass & Chen [4], and in Bass, Chen & Rosen [3].

We first present large deviations estimates for the SILT.

**Proposition 1.4** Assume \(d \geq 5\). For \(y > 1 + 2 \sum_{x \in \mathbb{Z}^d} G_d(x)^2\), there are positive constants \(\tilde{c}, \bar{c}\) such that

\[
\exp(-\bar{c}\sqrt{n}) \geq \mathbb{P}_0\left(\sum_{x \in \mathbb{Z}^d} l^2_n(x) \geq ny\right) \geq \exp(-\tilde{c}\sqrt{n}).
\] (18)

**Proposition 1.4** is a corollary of the next result where we prove that the main contribution in the estimates comes from the region where the local time is of order \(\sqrt{n}\).

**Proposition 1.5**

1. For \(\epsilon > 0\), and \(y > 1 + 2 \sum_{x \in \mathbb{Z}^d} G_d(x)^2\),

\[
\limsup_{n \to \infty} \frac{1}{\sqrt{n}} \log \mathbb{P}_0\left(\sum_{x: l_n(x) \leq n^{3/2-\epsilon}} l^2_n(x) \geq ny\right) = -\infty.
\] (19)
2. For \( y > 0 \) and \( \epsilon > 0 \), there exists a constant \( \tilde{c} > 0 \), such that

\[
\limsup_{n \to \infty} \frac{1}{\sqrt{n}} \log \mathbb{P}_0 \left( \sum_{x : l_n(x) > n^{1/2-\epsilon}} l_n^2(x) \geq ny \right) \leq -\tilde{c}. \tag{20}
\]

We present now estimates for \( \mathbb{P}_0(\sum l_n^p(x) > n^\gamma) \).

**Proposition 1.6**  
(i) Assume \( p = \frac{d}{d-2}, \) and \( p > \gamma > 1 + \frac{(p-2)^+}{(d-2)p+4} \). There are \( c_1, c_2 > 0 \) such that

\[
e^{-c_1n^{\gamma/p}} \leq \mathbb{P}_0 \left( \sum_{x \in \mathbb{Z}^d} l_n^p(x) > n^\gamma \right) \leq e^{-c_2n^{\gamma/p}}. \tag{21}\]

(ii) Assume \( 1 < p \leq \frac{d}{d-2} \), and \( p > \gamma > 1 \). For any \( \epsilon > 0 \), there are \( d_1, d_2 > 0 \) such that

\[
e^{-d_1 n^{\epsilon}} \leq \mathbb{P}_0 \left( \sum_{x \in \mathbb{Z}^d} l_n^p(x) > n^\gamma \right) \leq e^{-d_2 n^{\epsilon - \epsilon}}, \quad \text{with} \quad \zeta = 1 - \frac{2(p - \gamma)}{d(p - 1)}. \tag{22}\]

Let us give some heuristics on the proof of Proposition 1.5. First of all, we decompose \( \Sigma^2_n \) using the level sets of the local time. Note that it is not useful to consider \( \{ x : l_n(x) \gg \sqrt{n} \} \), since \( l_n(x) \) is bounded by an exponential variable. Now, for a subdivision \( \{ b_i \}_{i \in \mathbb{N}} \) of \( [0, 1/2] \), let \( D_n = \{ x \in \mathbb{Z}^d : \ n^b_i < l_n(x) < n^{b_{i+1}} \} \). Denoting by \( |\Lambda| \) the number of sites in \( \Lambda \subset \mathbb{Z}^d \), we then have

\[
\Sigma^2_n = \sum_i \sum_{x \in D_n} l_n^2(x) \leq \sum_i n^{2b_{i+1}} |D_n|. \]

Hence, choosing \( (y_{b_i})_{i \in \mathbb{N}} \) such that \( \sum_i y_{b_i} \leq y \),

\[
\mathbb{P}_0 (\Sigma^2_n \geq ny) \leq \sum_i \mathbb{P}_0 \left( \sum_{x \in D_n} l_n^2(x) \geq ny_{b_i} \right) \leq \sum_i \mathbb{P}_0 (|D_n| \geq n^{1-2b_{i+1}}y_{b_i}).
\]

A first estimate of the right hand term is given by Lemma 1.2 of [2], that we now recall.  
**Lemma 1.2 of [2].** Assume \( d \geq 3 \). There is a constant \( \kappa_d > 0 \) such that for any \( \Lambda \subset \mathbb{Z}^d \), and any \( t > 0 \)

\[
P (l_\infty(\Lambda) > t) \leq \exp \left( -\kappa_d \frac{t}{|\Lambda|^{2/d}} \right), \quad \text{where} \quad l_\infty(\Lambda) = \sum_{x \in \Lambda} l_\infty(x).
\]

Hence, if we drop the index \( i \), and set \( b = b_{i+1} \approx b_i \), for \( L = n^{1-2b}y_b \), we have

\[
\mathbb{P}_0 (|D_b| \geq L) \leq \sum_{\Lambda \subset [-n^{1/2}, n^{1/2}]^d, |\Lambda| = L} \mathbb{P}_0 (D_b = \Lambda, l_n(\Lambda) \geq n^b L) \leq (2n)^{dL} \exp (-\kappa_d b y_b^{1-2/d}n^\zeta) \quad \text{with} \quad \zeta = 1 - b - \frac{2}{d}(1 - 2b). \tag{23}
\]

Since \( \zeta > 1/2 \) when \( b < 1/2 \) and \( d > 4 \), this estimate would suffice if the combinatorial factor \( (2n)^{dL} \) were negligible. This case corresponds to “large” \( b \). For “small” \( b \), we need to
get rid of the combinatorial term. Inspired by Le Gall’s work [15], we propose a reduction to intersection local times of two independent random walks. Assume indeed for a moment that we can compare
\[ \sum_{x \in D} b^2 l_n(x) \] with
\[ \sum_{x \in \tilde{D}} l_n(x) \tilde{l}_n(x) \text{,} \]
where \(( \tilde{l}_n(x) )_{x \in \mathbb{Z}^d} \) is an independent copy of \(( l_n(x) )_{x \in \mathbb{Z}^d} \). Then, using Lemma 1.2 of [2], we obtain
\[ P_0 \left( \sum_{x \in D} b^2 l_n(x) \geq n y_b \right) \leq P_0 \left( \sum_{x \in \tilde{D}} l_n(x) \tilde{l}_n(x) \geq n y_b \right) \approx P_0 \otimes \tilde{P}_0 \left( \sum_{x \in \tilde{D}} \tilde{l}_n(x) \tilde{b} \tilde{n}(x) \geq n y_b \right) \leq \exp \left( -\kappa_d \frac{n^{1-b} y_b}{|D|^2/d} \right) + P_0 \left( b \tilde{n}(\tilde{D}) \geq n^{1-\epsilon} \right) \leq \exp \left( -\kappa_d n^{\zeta-2\epsilon/d} y_b \right) + P_0 \left( \sum_{x \in \tilde{D}} b^2 \tilde{l}_n(x) \geq n^{1+\epsilon} \right) \quad (24) \]

Now the last term in (24) should be negligible compared to the left-hand term of (24), so that we obtain
\[ P_0 \left( \sum_{x \in D} b^2 l_n(x) \geq n y_b \right) \leq \exp \left( -\kappa_d n^{\zeta-2\epsilon/d} y_b \right) \cdot \exp \left( -\kappa_d n^{\zeta-2\epsilon/d} y_b \right) + P_0 \left( \sum_{x \in \tilde{D}} b^2 \tilde{l}_n(x) \geq n^{1+\epsilon} \right) \leq P_0 \left( \sum_{x \in \tilde{D}} b^2 \tilde{l}_n(x) \geq n^{1+\epsilon} \right) \leq \exp \left( -c n^{\zeta-2\epsilon/d} \right) \quad (25) \]

This in turns, motivates the next result, interesting on its own. Define, for \( 0 < b < a \),
\[ D_{b,a} := \{ x : n^b \leq l_n(x) < n^a \} \text{.} \]

**Proposition 1.7** Fix positive numbers \( b, \gamma, \zeta \) and \( \delta \) small. The following inequality holds for large \( n \)
\[ P_0( |D_{b,b+\delta}| > n^\gamma y) \leq \exp(-n^{\zeta}), \quad \text{with} \quad \zeta < (1 - \frac{2}{d}) \gamma + b - \delta, \quad (26) \]
provided we assume either
1. \( (i) \) \( \gamma + 2b > 1 \), and \( y > 0 \), or \( (ii) \) \( \gamma + 2b = 1 \), and \( y > 1 + \sum_{x \in \mathbb{Z}^d} G_d^2(x) \).
2. \( b > \frac{2}{d} \gamma \), in which case, we can take \( \zeta = (1 - \frac{2}{d}) \gamma + b \).

The paper is organized as follows. We gather the technical Lemmas, and the proof of Proposition 1.7 in Section 2. The results of Section 2 are applied to the problem of large deviations for SILT in Section 3. We give also in Section 3 the proof of Proposition 1.6 as well as large deviations estimates for \( \sum_{x \in \mathbb{D}} b^2 l_n(x) \) for \( p \neq 2 \), where \( \mathcal{D} \) are subsets of the range of the random walk. In Section 4 we treat the problem of large and moderate deviations upper bounds for the RWRS, and prove Proposition 1.1. Finally, the corresponding lower bounds (Proposition 1.3) are shown in Section 5.

## 2 Technical Lemmas.

### 2.1 Estimates for low level sets

**Lemma 2.1** Assume \( d \geq 5 \), and fix positive real numbers \( b, \gamma, z \). Let
\[ D_b = \{ x \in \mathbb{Z}^d : l_n(x) \leq n^b \} \text{.} \]
Assume that either (i) $\gamma = 1$ and $z > 1 + 2 \sum_{x \in \mathbb{Z}^d} G_d(x)^2$ or (ii) $\gamma > 1$ and $z > 0$. Then, for $\zeta < \gamma - b - \frac{2}{d}(\gamma - 2b)$ there is a constant $c$ (depending also on $b$, $\gamma$, $z$) such that for $n$ large enough,

$$\mathbb{P}_0 \left( \sum_{x \in \mathbb{D}_n} l_n^2(x) \geq n^\gamma z \right) \leq \exp(-cn^\zeta). \quad (27)$$

**Proof:** We first prove the case $\gamma = 1$. The case $\gamma > 1$ is less delicate and will follow the same pattern. We then indicate the necessary changes for the case $\gamma > 1$.

**Case $\gamma = 1$.** The strategy is to rewrite the restricted sum of the self-intersection times in terms of intersections of independent random walks. Also, we assume for simplicity that $n$ is a power of 2, $n = 2^N$; the easy generalisation is left to the reader. First, note that,

$$\sum_{x \in \mathbb{D}_n} l_n^2(x) = \sum_{x \in \mathbb{D}_n} \sum_{1 \leq k, k' \leq n} \mathbb{I}\{S_k = S_{k'} = x\} \leq n + 2Z^{(0)}, \quad (28)$$

with $Z^{(0)} = \sum_{x \in \mathbb{D}_n} \sum_{0 \leq k < k' \leq 2^N} \mathbb{I}\{S_k = S_{k'} = x\}$. \quad (29)

Now, the estimate (27) is equivalent to showing that $\mathbb{P}_0(Z^{(0)} \geq y2^N) \leq \exp(-2N^{(0)})$, with $y > \sum_x G_d(x)^2$. We now bound $Z^{(0)} \leq Z_1^{(1)} + Z_2^{(1)} + J_1^{(1)}$, with

$$Z_1^{(1)} = \sum_x \mathbb{I}\{l_{2N-1}(x) \leq 2^N\} \sum_{0 \leq k < k' \leq 2^{N-1}} \mathbb{I}\{S_k = S_{k'} = x\},$$

$$Z_2^{(1)} = \sum_x \mathbb{I}\{l_{2N}(x) - l_{2N-1}(x) \leq 2^N\} \sum_{2^{N-1} \leq k < k' \leq 2^N} \mathbb{I}\{S_k = S_{k'} = x\},$$

$$J_1^{(1)} = \sum_x \mathbb{I}\{l_{2N-1}(x) \leq 2^N\} \sum_{0 \leq k \leq 2^{N-1} \leq k' \leq 2^N} \mathbb{I}\{S_k = S_{k'} = x\}.$$

We can express $Z_1^{(1)}$, $Z_2^{(1)}$ and $J_1^{(1)}$ in terms of the two independent random walks

$$\forall k \in \{0, \ldots, 2^{N-1}\}, \quad S_{k,1} = S_{2N-1} - S_{2N-1-k}, \quad \text{and} \quad S_{k,2} = S_{2N-1} - S_{2N-1+k}.$$

Indeed, denoting by $\{l_{k,i}(x), k \in \mathbb{N}, x \in \mathbb{Z}^d\}$ the local times of the random walk $(S_{k,i})_{k \in \mathbb{N}}$, we have on the event $\{S_{2N-1} = y\}, l_{2N-1}(x) = l_{2N-1,1}(y-x)$, and $l_{2N}(x) - l_{2N-1}(x) = l_{2N-1,2}(y-x)$. We obtain therefore $Z^{(0)} \leq Z_1^{(1)} + Z_2^{(1)} + I_1^{(1)}$ with for $i = 1, 2$

$$Z_i^{(1)} = \sum_{y \in \mathbb{Z}^d} \mathbb{I}\{S_{2N-1} = y\} \sum_x \mathbb{I}\{l_{2N-1,i}(y-x) \leq 2^N\} \sum_{0 \leq k < k' \leq 2^{N-1}} \mathbb{I}\{S_{k,i} = S_{k',i} = y - x\}.$$

Changing $x$ in $y - x$ in the second summation, we obtain for $i = 1, 2$

$$Z_i^{(1)} = \sum_x \mathbb{I}\{l_{2N-1,i}(x) \leq 2^N\} \sum_{0 \leq k < k' \leq 2^{N-1}} \mathbb{I}\{S_{k,i} = S_{k',i} = x\}.$$

Finally,

$$J_1^{(1)} = \sum_x \mathbb{I}\{l_{2N-1,1}(x) \leq 2^N\} l_{2N-1,1}(x) l_{2N-1,2}(x).$$
We now denote \( \{y_1, \ldots, y_N\} \) positive reals summing up to \( \bar{y} < y \), and \( \{b_0, \ldots, b_M\} \) a regular subdivision of \([0, b]\) of mesh \( \delta > 0 \), such that \( b_0 = 0, b_M = M\delta = b \). The precise form of \( \{y_i, b_i\} \) is given later. From \( Z^{(0)} \leq Z^{(1)}_1 + Z^{(1)}_2 + J^{(1)}_1 \), we deduce
\[
\mathbb{P}_0(Z^{(0)} > y2^N) \leq \mathbb{P}_0(Z^{(1)}_1 + Z^{(1)}_2 > \bar{y}_2 2^N) + \mathbb{P}_0(J^{(1)}_1 > y_1 2^N), \quad \text{with} \quad \bar{y}_2 = y_2 + \cdots + y_N. \quad (30)
\]
If we define for \( i = 0, \ldots, M - 1 \)
\[
\mathcal{D}^{(1)}_{i,1} = \left\{ x : 2^{Nb_i} < l^{2N-1,1}(x) \leq 2^{N_b+1} \right\},
\]
then, the idea is to replace \( \{|\mathcal{D}^{(1)}_{i,1}| \text{ large } \} \) by a condition on \( Z^{(1)}_1 \). Thus, we introduce
\[
\mathcal{G}^{(1)} = \{ \forall i = 0, \ldots, M - 1; |\mathcal{D}^{(1)}_{i,1}| \leq 4\bar{y}_2 2^{N(1-2b_i)} \}. \quad (31)
\]
The symbol \( \mathcal{G}^{(1)} \) stands for \textit{good set} at the first generation. Now, note that only sites visited more than once appear in \( \{\mathcal{D}^{(1)}_{i,1}, i = 0, \ldots, M - 1\} \) and contribute to \( Z^{(1)}_1 \). Thus, using \( k(k - 1) \geq k^2/2 \) for an integer \( k \geq 2 \), we have
\[
Z^{(1)}_1 = \sum_{x \in \mathcal{D}^{(1)}_{i,1}} \sum_{1 \leq k < k' \leq 2N-1} \mathbb{I} \{ S_{k,1} = S_{k',1} = x \} = \frac{1}{2} \sum_{x \in \mathcal{D}^{(1)}_{i,1}} l^{2N-1,1}(x)(l^{2N-1,1}(x) - 1) \\
\geq \frac{1}{4} \sum_{x \in \mathcal{D}^{(1)}_{i,1}} l^{2N-1,1}(x) \geq \frac{|\mathcal{D}^{(1)}_{i,1}|}{4} 2^{2b_N}. \quad (32)
\]
Thus, from \( (32) \)
\[
(\mathcal{G}^{(1)})^c \subset \{ Z^{(1)}_1 + Z^{(2)}_2 > \bar{y}_2 2^N \}. \quad (33)
\]
Thus, \( (30) \) becomes
\[
\mathbb{P}_0(Z^{(0)} > y2^N) \leq 2\mathbb{P}_0(Z^{(1)}_1 + Z^{(1)}_2 > \bar{y}_2 2^N) + \mathbb{P}_0(\mathcal{G}^{(1)}, J^{(1)}_1 > y_1 2^N). \quad (34)
\]
We proceed now by induction, and at generation \( l \), we have \( 2^l \) independent strands whose local times we denote by \( l^{2N-1, k} \). We introduce for \( k = 1, \ldots, 2^{l-1} \)
\[
\mathcal{D}^{(l)}_{i,k} = \{ x : 2^{Nb_i} \leq l^{2N-1,2k-1}(x) < 2^{N_b+1} \}, \quad (35)
\]
and for \( i = 0, \ldots, M - 1 \)
\[
J^{(l)}_{l,k,i} = \sum_{x} \mathbb{I} \{ x \in \mathcal{D}^{(l)}_{i,k} \} l^{2N-1,2k-1}(x)l^{2N-1,2k}(x), \quad \text{and} \quad J^{(l)}_{k} = \sum_{i=0}^{M-1} J^{(l)}_{l,k,i}. \quad (36)
\]
The \textit{good} sets at generation \( l \) are as follow. We first set \( \bar{y}_{l+1} = y_{l+1} + \cdots + y_N \), and for \( k = 1, \ldots, 2^{l-1} \),
\[
\mathcal{G}^{(l)}_k = \{ \forall i = 0, \ldots, M - 1; |\mathcal{D}^{(l)}_{i,k}| < 4\bar{y}_{l+1} 2^{N(1-2b_i)} \}, \quad \mathcal{G}^{(l)} = \bigcap_k \mathcal{G}^{(l)}_k. \quad (38)
\]
As in \( (38) \), we obtain
\[
(\mathcal{G}^{(l)})^c \subset \{ Z^{(l)}_1 + \cdots + Z^{(l)}_2 > \bar{y}_{l+1} 2^N \}. \quad (37)
\]
It is easy, after \( N \) inductive steps, to obtain

\[
\mathbb{P}_0(Z^{(0)} > y2^N) \leq 2^N \sum_{l=1}^{N} \mathbb{P} \left( G^{(l)} \sum_{k=1}^{2^{l-1}} J_k^{(l)} > yl2^N \right),
\]

where for each \( l \), the random variables \( (J_k^{(l)}, k = 1, \ldots, 2^{l-1}) \) are i.i.d and are distributed as a variable, say \( J^{(l)} \), with

\[
J^{(l)} = \sum_{x} 1 \{ \tilde{l}_{2N-l}(x) \leq 2^{Nl} \} \tilde{l}_{2N-l}(x),
\]

where \( \{\tilde{l}_n(x), x \in \mathbb{Z}^d\} \) is an independent copy of \( \{l_n(x), x \in \mathbb{Z}^d\} \). The strategy is now the following:

- When \( l \) is large, we use the trivial bound \( J_k^{(l)} \leq 2^{2(N-l)} \), and the classical Cramer’s estimates for sums of i.i.d. In that case, we need to center the \( J_k^{(l)} \)’s, i.e. to have \( yl2^N > 2^{l-1} E[J^{(l)}] \).
- When \( l \) is small, the trivial bound \( J_k^{(l)} \leq 2^{2(N-l)} \) is to crude. To use Cramer’s estimates, we need the existence of some exponential moments \( J^{(l)} \).

First, we specify the \( \{y_l\} \). They have to be chosen in order to center the variables \( J^{(l)} \). Set \( I_\infty = \sum_x \tilde{l}_\infty(x)\tilde{l}_\infty(x) \), where \( l \) and \( \tilde{l} \) are the local times of two independent walks. Note that for \( d \geq 5 \), \( m_1 = \mathbb{E}_0[I_\infty] = \sum_{x \in \mathbb{Z}^d} G_d(x)^2 < \infty \). A convenient choice is the following. Set \( l^* = N(1 - \delta_0) \) (\( \delta_0 \) small enough), and choose

- \( y_l = y/(2N) \) for \( l < l^* \).
- \( y_l = y/2^{N-l+2} \), for \( l \geq l^* \).

It is easy to check that \( \sum_{l=1}^{N} y_l \leq (1 - \delta_0)y/2 + y/2 \leq y \). We obtain for large enough \( N \), a decomposition of \( \mathbb{P}_0(Z^{(0)} > y2^N) \leq 2^N (R_1 + R_2) \)

\[
R_1 = \sum_{l<l^*} \mathbb{P}_0 \left( G^{(l)} \sum_{k=1}^{2^{l-1}} J_k^{(l)} \geq \frac{2N}{4N}y \right), \quad \text{ and } \quad R_2 = \sum_{l \geq l^*} \mathbb{P}_0 \left( \sum_{k=1}^{2^{l-1}} \tilde{J}^{(l)}_k \geq 2^{l-1}(y - m_1) \right),
\]

with centered variables \( \tilde{J}_k^{(l)} = J_k^{(l)} - \mathbb{E}_0[J_k^{(l)}] \).

About \( R_2 \). Note that for all \( l \), \( J^{(l)} \leq 2^{2(N-l)} \). Using Markov inequality, for any \( \lambda > 0 \),

\[
\mathbb{P}_0 \left( \sum_{k=1}^{2^{l-1}} \tilde{J}^{(l)}_k \geq 2^{l-1}(y - m_1) \right) \leq \exp \left( -\lambda 2^{l-1}(y - m_1) \right) \mathbb{E}_0 \left[ \exp \left( \lambda \tilde{J}^{(l)} \right) \right]^{2^{l-1}}.
\]

We choose \( \lambda = 1/2^{2(N-l)} \) and use the fact that \( \exp(x) \leq 1 + x + x^2 \) for \( |x| \leq 1 \), to obtain

\[
\mathbb{E}_0 \left[ \exp \left( \lambda \tilde{J}^{(l)} \right) \right] \leq 1 + 2\lambda^2 E[\tilde{J}^{(l)}]^2 \leq 1 + 2m_2\lambda^2,
\]

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where $m_2 = \mathbb{E}_0 [T_\infty^2] < \infty$ by [14]. Thus,
\[
R_2 \leq \sum_{l \geq l^*} e^{-2l^{-1} \lambda (y-m_1) - 2m_2 \lambda} \leq \sum_{l \geq l^*} e^{-2M^{-2N-1}(y-m_1-2m_2 \lambda)} \leq N e^{-c2^N(1-3\delta_0)} .
\] (40)

Hence, for $\delta_0 < 1/(2d)$, $R_2$ is much smaller than $\exp(-2^N \zeta)$ with $\zeta = 1 - 2/d - b(1 - 4/d) < 1 - 2/d < 1 - 3\delta_0$.

About $R_1$. We first obtain the existence of some exponential moments for $J^{(l)}$. For each $l \leq l^*$, $k = 1, \ldots, 2^{l-1}$ and any $u > 0$, we use Lemma 1.2 of [21], and independence between $l_{2N-1,2k-1}$ and $l_{2N-1,2k}$,
\[
\mathbb{P}_0(J_k^{(l)} > u, G_k^{(l)}) \leq \sum_{i=0}^{M-1} \mathbb{P}_0 \left( \sum_{i_{2N-l}} \{D_{i,k}^{(l)} \leq 4g 2^{N(1-2b)} \} \right) \leq \sum_{i=0}^{M-1} \mathbb{P}_0 \left[ \exp \left( \frac{-c_k u}{2^{N+1} M} \right) \right] \leq \exp \left( \frac{-C}{2^{N+1} M} u \right),
\] (41)

with $\zeta_i = b_{i+1} + \frac{2}{d}(1 - 2b_i) = \delta(1 - 4/d) + 2/d + \delta$. Thus,
\[
\max_i \zeta_i = M \delta(1 - 4/d) + 2/d + \delta \leq b(1 - 4/d) + 2/d + \delta,
\]

and for any $\epsilon > 0$, we can choose $\delta$ such that $\max_i \zeta_i \leq \epsilon + \frac{2}{d}(1 - 2b) + \frac{\epsilon}{2}$. Thus, we have a constant $C$ such that
\[
\mathbb{P}_0(J_k^{(l)} > u, G_k^{(l)}) \leq \exp(-\xi N u), \quad \text{with} \quad \xi_N = \frac{C}{M} 2^{-N(b+(1-2b)/2d+\epsilon/2)}.
\] (42)

Note that, when $u > \kappa / \xi_N^2$, this estimate is better than an estimate obtained from [14],
\[
\mathbb{P}_0(J_k^{(l)} > u, G_k^{(l)}) \leq \mathbb{P}_0(I_\infty > u) \leq \exp(-\kappa \sqrt{u}).
\] (43)

However, it permits us to consider exponential moment $E[\exp(\lambda J_k)]$ for $\lambda < \xi_N$. We now go back to the standard Cramer's method. For simplicity of notations, we drop the indices $l$ and $k$ when unambiguous. Returning now to evaluating $R_1$, for any $0 \leq \lambda < \xi_N$,
\[
\mathbb{P}_0 \left( \sum_{k=1}^{2^{l-1}} J_k^{(l)} \geq \frac{2N y}{4N} \cap G_k^{(l)} \right) \leq \exp \left( -\lambda \frac{2N y}{4N} \right) \mathbb{E}_0 \left[ \exp(\lambda J^{(l)}); G_k^{(l)} \right] \leq \exp \left( -\lambda \frac{2N y}{4N} \right) \mathbb{E}_0 \left[ \exp(\lambda J^{(l)}); G_k^{(l)} \right] 2^{l-1}.
\] (44)

Now, using $e^x \leq 1 + x + 2x^2$ for $x \leq 1$, and $\mathbb{E}_0 [\hat{J}] = 0$, we have
\[
\mathbb{E}_0 [e^{\lambda \hat{J}}; G_k^{(l)}] = \mathbb{E}_0 [e^{\lambda \hat{J}}; \{J < \lambda \} \cap G_k^{(l)}] + \mathbb{E}_0 [e^{\lambda \hat{J}}; \{J \geq \lambda \} \cap G_k^{(l)}]
\leq \mathbb{E}_0 [e^{\lambda \hat{J}}; \{J < \lambda \}] + \mathbb{E}_0 [e^{\lambda \hat{J}}; \{J \geq \lambda \} \cap G_k^{(l)}]
\leq \mathbb{E}_0 [1 + \lambda \hat{J} + 2\lambda^2 (\hat{J})^2; \{J < \lambda \}] + \mathbb{E}_0 [e^{\lambda \hat{J}}; \{J \geq \lambda \} \cap G_k^{(l)}]
\leq 1 + \lambda \mathbb{E}_0 [\hat{J}; \{J \geq \lambda \}] + 2\lambda^2 \mathbb{E}_0 [\hat{J}^2] + \mathbb{E}_0 [e^{\lambda \hat{J}}; \{J \geq \lambda \} \cap G_k^{(l)}].
\]
Now,\[
    \mathbb{E}_0 [J; \{J \geq 1/\lambda\}] \leq \mathbb{E}_0 [(J)^2]^{1/2} \mathbb{P}_0 (J \geq 1/\lambda)^{1/2} \leq \lambda \mathbb{E}_0 [J^2] \leq \lambda \mathbb{E}_0 (I^2_\infty).
\]
Note that by the results of [14], $$\mathbb{E}_0 (I^2_\infty) < \infty.$$ Hence, for some constant \(c\),\[
    \mathbb{E}_0 [e^\lambda J; G^{(l)}_k] \leq 1 + c \lambda^2 + \mathbb{E}_0 [e^\lambda J; \{J \geq 1/\lambda\} \cap G^{(l)}_k].
\]
We now show that for some constant \(C\), \(E [e^\lambda J; \{J \geq 1/\lambda\} \cap G^{(l)}_k] \leq C \lambda^2\). We decompose this last expectation into\[
    \mathbb{E}_0 [e^\lambda J; \{J \geq 1/\lambda\} \cap G^{(l)}_k] = e^1 \mathbb{P}_0 (\lambda J \geq 1) + I \leq e^1 \mathbb{E}_0 [I^2_\infty] \lambda^2 + I,
\]
with\[
    I = \int_{1/\lambda}^{\infty} \lambda e^{\lambda u} \mathbb{P}_0 (J^{(l)}_k \geq u; G^{(l)}_k) du, \quad \text{and we choose } \lambda = \frac{\lambda_i}{2 \log(1/\xi^3_N)}.
\]
To bound \(I\), we use estimate (12), \(\lambda < \xi_N/2\) and \(N\) large enough\[
    I \leq \int_{1/\lambda}^{\infty} \lambda e^{\lambda u - \xi_N u} du \leq \frac{2\lambda}{\xi_N} \int_{1/\lambda}^{\infty} (\xi_N/2)e^{-(\xi_N/2)u} du \leq \frac{2\lambda}{\xi_N} \exp\left(\frac{\xi_N}{2\lambda}\right) \leq \frac{\xi_N^3}{\log(1/\xi_N^3)} \leq 4\xi_N \log(1/\xi_N^3) \lambda^2 \leq \lambda^2. \tag{45}
\]
Thus, there is a constant \(C\) such that\[
    \mathbb{E}_0 [\exp(\lambda J^{(l)}_k; G^{(l)}_k)] \leq 1 + C \lambda^2 \leq \exp(C \lambda^2),
\]
which together with (14), yield\[
    2^N R_1 \leq N 2^N \exp\left(-\frac{2N y \xi_N}{4N \log(1/\xi_N^3)} + \frac{C \xi_N^2 2^l}{4 \log^2(1/\xi_N^3)}\right) \leq N 2^N \exp\left(-\frac{2N y \xi_N}{16 N \log(1/\xi_N^3)}\right), \tag{46}
\]
where we used that \(2^N y > 4CN \xi_N 2^l/\log(1/\xi_N^3)\) for any \(l \leq l^*\) and \(N\) large enough, as soon as \(\epsilon\) is chosen so that \(1 - b - \frac{\epsilon}{d}(1 - 2b) = \epsilon/2 > 0\). Now, we can use an extra \(\epsilon/2\) to swallow the denominator \(N \log(1/\xi_N^3)\) in the exponential, as well as the \(N 2^N\) factor in front of the exponential in (46). We obtain then for large enough \(N\),\[
    \mathbb{P}_0 (Z^{(l)} > y 2^N) \leq \exp\left(-C 2^N \zeta\right), \quad \text{with } \zeta = 1 - b - \frac{\epsilon}{d}(1 - 2b) - \epsilon. \tag{47}
\]
\[\text{Case } \gamma > 1. \] The sequence \(\{y_l\}\) and the good sets \(\{G^{(l)}, l = 1, \ldots, N\}\) are different here. Since recentring the \(J^{(l)}_k\) poses no problem, we can choose \(y_l = y/N\) for \(l = 1, \ldots, N\). Next, we set for \(l = 1, \ldots, N\)\[
    G^{(l)} = \{\forall i = 0, \ldots, M - 1; \forall k = 1, \ldots, 2^{l-1}; |D^{(l)}_{i,k}| \leq 4 \bar{y}_{l+1} 2^{N(\gamma - 2b)}\}. \tag{48}
\]
As in (37), we obtain
\[(G^{(l)})^c \subset \{ Z_1^{(l)} + \cdots + Z_2^{(l)} > \bar{y}t + 2^{N\gamma} \}. \quad (49)\]

By induction, we obtain an inequality similar to (38) with \( y_l2^N \) replacing \( y_l2^N \). The proof follows exactly the same pattern yielding the desired result.

**Proof of Proposition 1.7(1):** Note that
\[\{|D_{b,b+\delta}| > n^\gamma y\} \subset \left\{ \sum_{D_{b,\delta}} l_n(x)^2 > n^\gamma + 2b y \right\}.\]

We invoke Lemma 2.1 to conclude the proof.

**2.2 Estimates for high level sets**

We only need an improvement of Lemma 1.2 of [2].

**Lemma 2.2** Assume \( d \geq 3 \). There exists a constant \( \kappa_d > 0 \) such that for any \( t > 0, L \geq 1 \),
\[\mathbb{P}_0 \left( \{|x : l_n(x) \geq t\} \geq L \right) \leq (2n)^dL \exp(-\kappa_d t L^{1-2/d}). \quad (50)\]

**Proof:** The proof is a simple application of Lemma 1.2 of [2].
\[P(\{|x : l_n(x) \geq t\} \geq L) \leq \sum_{\Lambda \subset \mathbb{Z}^d, |\Lambda| = L} P(\forall x \in \Lambda, l_n(x) \geq t) \leq \sum_{\Lambda \subset \mathbb{Z}^d, |\Lambda| = L} P(l_n(\Lambda) \geq Lt) \leq n^{dL} \exp(-\kappa_d t L^{1-2/d}). \quad (51)\]

**Proof of Proposition 1.7(2):** Note that
\[\{|D_{b,b+\delta}| > n^\gamma y\} \subset \{|\{x : l_n(x) \geq n^b\} \geq y n^{\gamma}\}.\]

We use Lemma 2.2 with \( L = yn^\gamma \) and \( t = n^b \). The combinatorial term is negligible when \( b > (2/d)\gamma \), and this gives the correct exponent \( b + (1-2/d)\gamma \).

**3 Estimates for SILT.**

We first prove Proposition 1.5, then the lower bound of Proposition 1.4, and finally estimates on \( \mathbb{P}_0(\sum l_n(x) > n^\gamma) \).
3.1 Proof of Proposition 1.5

Note that 1. of Proposition 1.5 is a direct corollary of Lemma 2.1. Thus, we focus on point 2. of Proposition 1.5. Note first that

\[
\mathbb{P}_0 \left( \sum_{x: l_n(x) \geq \sqrt{n}} l_n^2(x) \geq ny \right) \leq \mathbb{P}_0 \left( \exists x; l_n(x) \geq \sqrt{n} \right) \leq \sum_{x \in [-n,n]^d} \mathbb{P}_0 \left( l_n(x) \geq \sqrt{n} \right)
\]

\[
\leq \sum_{x \in [-n,n]^d} \mathbb{P}_0(H_x < \infty) \mathbb{P}_x(l_n(x) \geq \sqrt{n})
\]

\[
\leq cn^d \mathbb{P}_0(l_n(0) \geq \sqrt{n}) \leq cn^d \exp(-\epsilon \sqrt{n}).
\]

Thus, it is enough to prove that for any \( y > 0 \) and any \( \epsilon \in [0,1/2-1/d], \exists \bar{c} > 0 \) such that

\[
\limsup_{n \to \infty} \frac{1}{\sqrt{n}} \log \mathbb{P}_0 \left( \sum_{\mathcal{D}_{1/2-\epsilon,1/2}} l_n^2(x) \geq ny \right) \leq -\bar{c}, \tag{52}
\]

where for any \( a, b \), with \( 0 < b < a \), we have defined

\[
\mathcal{D}_{b,a} = \left\{ x : n^b \leq l_n(x) \leq n^a \right\}. \tag{53}
\]

We write \( \mathcal{D}_{1/2-\epsilon,1/2} \subset \bigcup_{i=0}^{M-1} \mathcal{D}_i \), with \( b_0 \leq 1/2 - \epsilon, b_M = 1/2 \). However, this time, \( M \) will depend on \( n \) (actually \( M \approx \log(\log(n)) \)). Let \( (y_i, i = 0 \cdots M - 1) \) be positive numbers such that \( \sum_i y_i \leq 1 \). Then, using Lemma 2.2

\[
\mathbb{P}_0 \left( \sum_{\mathcal{D}_{1/2-\epsilon,1/2}} l_n^2(x) \geq ny \right) \leq \sum_{i=0}^{M-1} \mathbb{P}_0 \left( \sum_{x \in \mathcal{D}_i} l_n^2(x) \geq ny_i y \right) \leq \sum_{i=0}^{M-1} \mathbb{P}_0 \left( |\mathcal{D}_i| \geq n^{1-2b_i+1} y_i y \right)
\]

\[
\leq \sum_{i=0}^{M-1} n_i^{d_n^{1-2b_i+1} y_i y} \exp \left( -\kappa d n_i^{1+(1-2/d)(1-2b_i+1)} (y_i y)^{1-2/d} \right).
\]

Therefore, we need to choose \( (y_i, b_i, 0 \leq i \leq M - 1) \) such that for some \( \beta > 0 \),

\[
\left\{ \begin{array}{l}
n_i^{1-2b_i+1} y_i \log(n) \ll n_i^{b_i+(1-2/d)(1-2b_i+1)} y_i^{1-2/d} \\
n_i^{b_i+(1-2/d)(1-2b_i+1)} y_i^{1-2/d} \geq \beta \sqrt{n}
\end{array} \right\} \iff \left\{ \begin{array}{l}(n_i^{1-2b_i+1} y_i)^{2/d} \log(n) \ll n_i^{b_i} \\
\beta n_i^{1-2b_i} \leq n_i^{2(1-2/d)(1-2b_i+1)} y_i^{1-2/d}
\end{array} \right\}
\]

For \( i = M - 1 \), the second condition in (54) is \( \beta n_i^{1-2b_{M-1}} \leq y_{M-1}^{1-2/d} \), so that we have to take

\[
1/2 - b_{M-1} = 1/\log(n), \quad \text{and} \quad y_{M-1} = (\beta \epsilon)^{d/2d}
\]

For this choice of \( b_{M-1}, y_{M-1} \), the first condition in (54) is satisfied.

For the others \( b_i \ (i \leq M-2) \), we take \( b_{i+1} - 1/2 = a(b_i - 1/2) \), with \( d_{2(d-2)} < a < 1 \). Hence for \( i \leq M-1, \frac{1}{2} - b_i = a_{1/2 - \epsilon}^{M-1} \frac{1}{\log(n)} \). To have \( b_0 \leq 1/2 - \epsilon < b_1 \), we take \( M - 1 = \left\lceil \frac{\log(\epsilon \log(n))}{\log(1/a)} \right\rceil \).

With these choices, the second condition in (54) becomes

\[
\forall i \leq M - 2, \quad y_i \geq \beta \frac{d}{2(d-2)} \exp \left( -2(1/a)^{M-i-1}(a - \frac{d}{2(d-2)}) \right).
\]
and we take $y_i$ to satisfy the equality. Now, the first condition in (54) is for $i \leq M - 2$,
\[ \beta^{\frac{d}{d-2}} \exp \left( \frac{d}{d-2} \left( \frac{1}{a} \right)^{M-i-1} \right) \ll \frac{\sqrt{n}}{\log(n)} \iff \beta^{\frac{d}{d-2}} \exp \left( \frac{d}{d-2} \left( \frac{1}{a} \right)^{M-1} \right) \ll \frac{\sqrt{n}}{\log(n)}. \] (55)

Recalling the value of $M$, this is satisfied as soon as
\[ \frac{\epsilon}{a} \left( \frac{d}{d-2} \right) < \frac{1}{2}. \] (56)

But for $\epsilon < 1/2 - 1/d$, one can find $a \in \left[ \frac{d}{2(d-2)}, 1 \right]$ such that (56) holds.

It remains now to check that we can take $\beta$ in order to get \( \sum_{i=0}^{M-1} y_i \leq 1 \). But,
\[
\sum_{i=0}^{M-1} y_i = \beta^d \left[ e^{\frac{d}{d-2}} + \sum_{i=1}^{M-1} \exp \left( -2 \left( a - \frac{d}{2(d-2)} \right) \left( \frac{1}{a} \right)^i \right) \right] \\
\leq \beta^d \left[ e^{\frac{d}{d-2}} + \sum_{i=1}^{\infty} \exp \left( -2 \left( a - \frac{d}{2(d-2)} \right) \left( \frac{1}{a} \right)^i \right) \right].
\]

Since the last series is convergent, one can obviously find $\beta$ such that $\sum_{i=0}^{M-1} y_i \leq 1$.

3.2 Proof of the lower bound of Proposition 1.4

For $k \in \mathbb{N}$, let $T_0^{(k)}$ be the $k$-th return time at 0:
\[ T_0^{(k)} \triangleq T_0^{(0)} \land T_0^{(k)} \triangleq \inf \left\{ n > T_0^{(k-1)}, S_n = 0 \right\}. \]

For $y > 0$,
\[
P \left( \sum_{x \in \mathbb{Z}^d} l_n^2(x) \geq ny \right) \geq P \left( l_n(0) \geq \left\lfloor \sqrt{ny} \right\rfloor + 1 \right) = P \left( T_0^{(\lfloor \sqrt{ny} \rfloor)} \leq n \right) \\
\geq P \left( \forall k \in \{ 1, \ldots, \lfloor \sqrt{ny} \rfloor \}, T_0^{(k)} - T_0^{(k-1)} \leq \frac{n}{\lfloor \sqrt{ny} \rfloor} \right) \\
\geq P \left( T_0 \leq \frac{n}{\sqrt{ny}} \right) \sqrt{ny} \\
= \left( P \left( T_0 < \infty \right) - P \left( \frac{\sqrt{n}}{\sqrt{y}} < T_0 < \infty \right) \right) \sqrt{ny}
\]

This proves the lower bound since $\lim_{n \to \infty} P \left( \frac{\sqrt{n}}{\sqrt{y}} < T_0 < \infty \right) = 0$, and $P(T_0 < \infty) < 1$ for $d \geq 3$. 

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3.3 About $\{\sum_{x} l_n^p(x) > n^\gamma\}$.

We present two results about upper bounds for $\mathbb{P}_0(\sum_{x} l_n^p(x) > n^\gamma)$, where $\mathcal{D}$ is a subset of $\{x : l_n(x) \geq 1\}$. The first estimate concerns sites visited not too often, and is a corollary of Lemma 2.1. The notation $\mathcal{D}_{b,a}$ is defined in (53).

**Proposition 3.1** Fix positive numbers $a, b, \gamma, \zeta, p, y$, with $a > b$. For any $y > 0$, the following inequality holds for some constant $c > 0$

$$\mathbb{P}_0 \left( \sum_{x \in \mathcal{D}_{b,a}} l_n^p(x) \geq n^\gamma y \right) \leq \exp(-cn^\zeta),$$

provided the following conditions are satisfied.

(0) When $p > 2$, $\gamma > 1 + a(p - 2)$. When $p \leq 2$, $\gamma > 1 - b(2 - p)$.

(i) When $p \geq \frac{2d}{d-2}$, $\zeta < \gamma - a(p-1) - \frac{\gamma}{d}(\gamma - ap)$.

(ii) When $1 < p \leq \frac{2d}{d-2}$, $\zeta < \gamma - b(p-1) - \frac{\gamma}{d}(\gamma - bp)$.

The constant $c$ depends on $a, b, p, \gamma, \zeta, y$, and we take $n$ large enough.

**Proof:** The strategy is to partition $\mathcal{D}_{b,a}$ into a finite number $M$ of regions:

$$\mathcal{D}_i = \{x : n^{b_i} \leq l_n(x) < n^{b_{i+1}}\}, \quad \text{where} \quad b_i = b + \frac{i}{M}(a - b), \quad \text{for} \quad i = 0, \ldots, M, \quad (58)$$

where $M$ will be chosen large enough later. Then, using Proposition 1.7 for an arbitrarily small $\epsilon$, one can choose $M$ large enough such that

$$\mathbb{P}_0 \left( \sum_{x \in \mathcal{D}_{b,a}} l_n^p(x) \geq n^\gamma y \right) \leq \sum_{i=0}^{M-1} \mathbb{P}_0 \left( \sum_{x \in \mathcal{D}_i} l_n^p(x) \geq \frac{n^\gamma y}{M} \right) \leq \sum_{i=0}^{M-1} \mathbb{P}_0 \left( |\mathcal{D}_i| \geq \frac{n^\gamma - pb_{i+1}y}{M} \right) \leq \sum_{i=0}^{M-1} e^{-Cn^\zeta_i}, \quad \text{where} \quad \zeta_i = (\gamma - pb_{i+1})(1 - \frac{2}{d}) + b_i - \epsilon$$

$$\leq Me^{-Cn^\zeta}, \quad \text{where} \quad \zeta = \min_i \zeta_i. \quad (59)$$

However, Proposition 1.7 requires that $2b_i + \gamma - pb_{i+1} > 1$ for each $i = 0, \ldots, M - 1$. This is satisfied when $\gamma$ satisfies the condition (0), and when $(a - b)/M$ is small enough. If we set $\delta = (a - b)/M$, then we rewrite $\zeta_i$ as

$$\zeta_i = \gamma(1 - \frac{2}{d}) + b_{i+1}(1 - p + \frac{2}{d}p) - \delta - \epsilon. \quad (60)$$

Thus, in case (i), we have $1 - p + \frac{2}{d}p \leq 0$ and the minimum over the $\zeta_i$ is reached for the largest $b_{i+1}$. In case (ii), the minimum over the $\zeta_i$ is reached for the smallest $b_{i+1}$. It then remains to choose $\delta$ small enough to obtain the desired result. \[ \blacksquare \]
As a corollary of Lemma 2.2, we obtain the following estimates for the regions where the local times are large. We recall that for \( p > 1 \), we denote by \( p^* := p/(1 - p) \) the conjugate exponent of \( p \).

**Proposition 3.2** Assume \( d \geq 3 \), and fix positive numbers \( a, b, \gamma, \zeta, p, y \), with \( a > b \). For any \( y > 0 \), the following inequality holds for some constant \( c > 0 \)

\[
P_0 \left( \sum_{x \in \mathcal{D}_{b,a}} l_n^p(x) \geq n^\gamma y \right) \leq \exp(-cn^{\zeta}) ,
\]

provided \((0)\) \( \zeta \leq \frac{d}{2}b \), and either of the following two conditions.

- \( p \geq d/(d - 2) \); (i) \( \zeta < \frac{\gamma}{p(2/d)+1} \); (ii) \( \zeta < \gamma - a(p - 1) - \frac{2}{d}(\gamma - ap) \).
- \( 1 < p \leq d/(d - 2) \); (iii) \( \zeta < \gamma - b(p - 1) - \frac{2}{d}(\gamma - bp) \).

The constant \( c \) depends on \( a, b, p, \gamma, \zeta, y \), and \( n \) is taken large enough.

**Proof:** As in the proof of Proposition 3.1 we decompose \( \mathcal{D}_{b,a} \) into a finite number \( M \) of regions, as in (58), where \( M \) will be chosen later. Then, as in (59),

\[
P_0 \left( \sum_{x \in \mathcal{D}_{b,a}} l_n^p(x) \geq n^\gamma y \right) \leq \sum_{i=0}^{M-1} P_0 \left( |\mathcal{D}_i| \geq \frac{n^{\gamma-pb_{i+1}+1}y}{M} \right).
\]

We now use Lemma 2.2 with \( t = n^{b_i} \) and \( L = n^{\gamma-pb_{i+1}}y/M \) to get

\[
P_0 \left( \sum_{x \in \mathcal{D}_{b,a}} l_n^p(x) \geq n^\gamma y \right) \leq \sum_{i=0}^{M-1} n^{dn^{\gamma-pb_{i+1}+1}y/M} \exp \left(-\kappa_d n^{b_i+(1-2/d)(\gamma-pb_{i+1})}y/M \right)^{1-2/d} .
\]

To conclude, it is now enough to check that we can find a finite sequence \( (b_i, 0 \leq i \leq M) \), such that \( b_0 = b, b_M > a \) and satisfying the constraints

\[
\begin{aligned}
\gamma - pb_{i+1} &< b_i + (1 - 2/d)(\gamma - pb_{i+1}) \\
\zeta &\leq b_i + (1 - 2/d)(\gamma - pb_{i+1}) \\
b_i &< b_{i+1}
\end{aligned}
\]

\[
\begin{aligned}
\gamma - pb_{i+1} &> b_i + (1 - 2/d)(\gamma - pb_{i+1}) \\
\zeta &\leq b_i + (1 - 2/d)(\gamma - pb_{i+1}) \\
b_i &> b_{i+1}
\end{aligned}
\]

\[
\begin{aligned}
\left\{ \begin{array}{l}
\gamma - pb_{i+1} < b_i + (1 - 2/d)(\gamma - pb_{i+1}) \\
\zeta \leq b_i + (1 - 2/d)(\gamma - pb_{i+1}) \\
b_i < b_{i+1}
\end{array} \right. \Rightarrow \left\{ \begin{array}{l}
b_{i+1} > \frac{\gamma}{p} - \frac{d}{2p}b_i \\
b_{i+1} \leq \frac{\zeta}{p} + \frac{d}{p(d-2)}(b_i - \zeta) \\
b_{i+1} > b_i
\end{array} \right. .
\]

Let \( D_0 \) be the line \( y = x \), \( D_1 \) be the line \( y = \frac{\gamma}{p} + \frac{d}{p(d-2)}(x - \zeta) \), and \( D_2 \) the line \( y = \frac{\zeta}{d/2} \). Assume first that \( p \neq d/(d - 2) \). Let \( a_0 \) (resp. \( a_2 \)) be the abscissa of the intersection of \( D_1 \) with \( D_0 \) (resp. \( D_2 \))

\[
a_0 = \frac{\gamma - \zeta d/(d - 2)}{p - d/(d - 2)} , \quad a_2 = \frac{\zeta}{d/2} .
\]

Case \( p > d/(d - 2) \): In that case, the slope of \( D_1 \) is less than 1. Then, the region of constraints is non empty (see Figure 2) if and only if

\[
a_2 < a_0 \Leftrightarrow \zeta < \frac{\gamma}{1 + 2p/d} \text{ (i.e. condition (i))}.
\]
In that case, it is always possible to construct a finite sequence \((b_i)_{0 \leq i \leq M}\) satisfying the constraints \((C_0),(C_1),(C_2)\) and \(b_0 = b, b_M \geq a\), as soon as \(b \geq a_2\) i.e. (0) and \(a < a_0\), i.e. (ii). A possible choice is to take \(b_{i+1} = \frac{2}{p} + \frac{d}{p(d-2)}(b_i - \zeta)\), \(M\) being defined by \(b_{M-1} < a \leq b_M.\)

**Case \(p < d/(d-2)\):** In that case, the slope of \(D_1\) is greater than 1, and the region of constraints is never empty. It is always possible to construct a finite sequence \((b_i)_{0 \leq i \leq M}\) satisfying the constraints \((C_0),(C_1),(C_2)\) as soon as \(b > a_0\), and \(b \geq a_2\). A possible choice is to take \(b_{i+1} = \frac{2}{p} + \frac{d}{p(d-2)}(b_i - \zeta)\), \(M\) being defined by \(b_{M-1} < a \leq b_M.\)

When \(p = d/(d-2)\), we choose \(b_{i+1} - b_i = \frac{\gamma}{p} - \zeta > 0\) with the appropriate boundaries. Note that conditions (i), (ii) and (iii) are identical and reads \(\zeta < \frac{\gamma}{p} = \frac{\gamma(1-2/d)}{2}.\)

### 3.4 Proof of Proposition 1.6

#### 3.4.1 Point (i) of Proposition 1.6

Note first that we only need to deal with sites in \(D_{0,\gamma/p}\). Fix an arbitrarily small \(\epsilon > 0\). We first focus on \(D_{0,\gamma/p-\epsilon}\). We consider three cases.

- **Case \(d/(d-2) < p \leq 2\):** Proposition 3.1 with \(b = 0\) and \(a = \gamma/p - \epsilon\) yields
  \[
  \frac{\gamma}{p} < \gamma - a(p-1) - \frac{2}{d} \left(\gamma - pa\right),
  \]  
  (64) since the condition (0) \(\gamma > 1\) holds.

- **Case \(2 < p < \gamma > p/2\):** Note that Proposition 3.1 imposes that \(a < (\gamma - 1)/(p-2)\). Note also that \(\gamma > p/2\) is equivalent to \(\gamma < (\gamma - 1)/(p-2)\). Thus, we can again take \(b = 0\) and \(a = \gamma/p - \epsilon\) in Proposition 3.1 to obtain (64). Condition (0) follows from \(\gamma > p/2\).

- **Case \(2 < p \text{ and } \gamma \leq p/2\):** Proposition 3.1 is used to deal with \(D_{a,\gamma/p-\epsilon}\) with \(a = \gamma/(p-2) - \epsilon\), for \(\epsilon > 0\) arbitrarily small. We use Proposition 3.2 to control the contribution of sites of \(D_{a,\gamma/p-\epsilon}\). Indeed, the three conditions we have to check reads

\[
\begin{align*}
(0) \quad \frac{\gamma}{p} &< \frac{d}{2} \left(\frac{\gamma - 1}{p - 2} - \epsilon\right), \\
(i) \quad \frac{\gamma}{p} &< \frac{\gamma}{2p/d + 1}, \\
(ii) \quad 0 &< \epsilon(p - 1 - \frac{2}{d}p).
\end{align*}
\]  
(65)
Thus, ǫ > 3.4.2 Point (ii) of Proposition 1.6 Condition (i) is equivalent to 
γ > 1 + 2(p − 2)/(4 + p(d − 2)).

The proof that for some constant C > 0
\[ \mathbb{P}_0 \left( \sum_{D \gamma / p - \epsilon} l_n^p(x) > n^\gamma \right) \leq \exp(-C n^{\gamma/p}), \]
is similar to the tedious proof of Proposition 1.5 (2), and is left to the reader.

The lower bound follows trivially from \( \{ l_n(0) > n^{\gamma/p} \} \subset \{ \sum l_n^p(x) > n^\gamma \}. \)

3.4.2 Point (ii) of Proposition 1.6

We assume here that 1 < p ≤ d/(d − 2), and p > γ > 1 and we start with proving the upper bound in (22). Fix an arbitrarily small \( \epsilon > 0 \), set \( q = d/(d - 2) + \epsilon \), and choose \( \alpha \) such that
\[ \frac{1}{p} = \alpha + \frac{1 - \alpha}{q} \iff \alpha = 1 - q^*/p^*. \]

The idea is to interpolate between 1 and q. In other words,
\[ \left( \sum_{x \in \mathbb{Z}^d} l_n^p(x) \right)^{1/p} \leq \left( \sum_{x \in \mathbb{Z}^d} l_n(x) \right)\alpha \left( \sum_{x \in \mathbb{Z}^d} l_n^q(x) \right)^{(1-\alpha)/q} \leq n^\alpha \left( \sum_{x \in \mathbb{Z}^d} l_n^q(x) \right)^{q^*/(qp^*)}. \]
(66)

Thus,
\[ \{ \sum l_n^p(x) > n^\gamma \} \subset \{ \sum l_n^q(x) > n^{\tilde{\gamma}} \}, \quad \text{with} \quad \tilde{\gamma} = \left( \frac{\gamma}{p} - 1 + \frac{q^*}{p^*} \right) \frac{qp^*}{q^*}. \]

A simple computation yields γ > 1 is equivalent to \( \tilde{\gamma} > 1 \). Thus, we can use (i) of Proposition 1.6 to obtain that
\[ \zeta := \frac{\tilde{\gamma}}{q} \left( \frac{\gamma}{p} - 1 + \frac{q^*}{p^*} \right) \frac{p^*}{q^*} = 1 - \frac{p^*(p - \gamma)}{pq^*} = 1 - \frac{(p - \gamma)}{(p - 1)q^*}. \]
(67)

Since this is true for any \( \epsilon > 0 \), we have \( \zeta < 1 - \frac{2(p - \gamma)}{(p - 1)q^*} \).

We prove now the lower bound in (22). We set \( R_n := \{ x : l_n(x) \geq 1 \} \), and use Holder’s inequality
\[ \left( \frac{1}{|R_n|} \sum_{R_n} l_n(x) \right)^p \leq \frac{1}{|R_n|} \sum_{R_n} l_n(x)^p. \]
(68)

Thus, recalling that \( p > \gamma > 1 \), we have from (68)
\[ \mathbb{P}_0 \left( \sum l_n^p(x) > n^\gamma \right) \geq \mathbb{P}_0 \left( |R_n| \leq r^d \right), \quad \text{with} \quad r^d = n^{(p - \gamma)/(p - 1)} \]
\[ \geq \mathbb{P}_0 (\sigma_r \geq n), \quad \text{with} \quad \sigma_r := \inf \{ k \geq 0 : S_k \notin [-r/2; r/2]^d \}. \]
(69)

We use now the classical estimate \( \mathbb{P}_0 (\sigma_r \geq n) \geq \exp(-C n/r^2) \), for some constant \( C \), and if we set \( n/r^2 = n^\xi \), this yields \( \zeta = 1 - \frac{2(p - \gamma)}{d(p - 1)} \).
4 Upper bounds for the deviations of the RWRS

The aim of this Section is to prove Proposition 1.1. Let \( \Lambda \) denote the log-Laplace transform of \( \eta(0) \):
\[
\forall t \in \mathbb{R}, \Lambda(t) = \log E_\eta[\exp(t\eta(0))].
\]

Since \( \eta(0) \) is centered, there exists a constant \( C_0 \) such that for \( |t| \leq 1 \), \( \Lambda(t) \leq C_0 t^2 \). By Tauberian Theorem, for \( \eta(0) \) having the tail behavior \( [1] \), \( \Lambda(t) \) is of order \( t^{\alpha^*} \) for large \( t \), where \( \alpha^* \) is the conjugate exponent of \( \alpha \) \( (\frac{1}{\alpha} + \frac{1}{\alpha^*} = 1) \). Hence, there exists a constant \( C_{\infty} \) such that for \( t \geq 1 \), \( \Lambda(t) \leq C_{\infty} t^{\alpha^*} \).

Our aim is to show that \( P(X_n > y\beta) \leq \exp(-Cn^\zeta) \). In each region, we partition the range into two domains \( \mathcal{D}_b = \{ x \in \mathbb{Z}^d; l_n(x) \geq n^b \} \) and \( \mathcal{D}_b = \{ x \in \mathbb{Z}^d; 0 < l_n(x) \leq n^b \} \), parametrized by a positive \( b \), which will turn out to be \( \beta - \zeta \).

First, for any \( y_1, y_2 > 0 \), such that \( y_1 + y_2 = y \),
\[
P \left( \sum_x \eta(x) l_n(x) \geq n^\beta y \right) \leq P \left( \sum_{x \in \mathcal{D}_b} \eta(x) l_n(x) \geq n^\beta y_1 \right) + P \left( \sum_{x \in \mathcal{D}_b} \eta(x) l_n(x) \geq n^\beta y_2 \right).
\]

Let \( A := \left\{ \sum_{x \in \mathcal{D}_b} l_n^\alpha(x) \geq n^{\beta-b+\alpha^*} \frac{y_1}{2C_{\infty}} \right\} \).
\[
P \left( \sum_{x \in \mathcal{D}_b} \eta(x) l_n(x) \geq n^\beta y_1 \right) \leq P_0(A) + P \left( A^c \sum_{x \in \mathcal{D}_b} \eta(x) l_n(x) \geq n^\beta y_1 \right)
\leq P_0(A) + e^{-n^\beta y_1} E_0 \left[ \mathbb{I}_{A^c} \exp \left( \sum_{x \in \mathcal{D}_b} \Lambda \left( \frac{l_n(x)}{n^b} \right) \right) \right].
\]

Now, on \( \mathcal{D}_b \), \( l_n(x) \geq n^b \), so that using the behaviour of \( \Lambda \) near infinity,
\[
P \left( \sum_{x \in \mathcal{D}_b} \eta(x) l_n(x) \geq n^\beta y_1 \right) \leq P_0(A) + e^{-n^\beta y_1^2} E_0 \left[ \mathbb{I}_{A^c} \exp \left( C_{\infty} \sum_{x \in \mathcal{D}_b} l_n^\alpha(x) \right) \right]
\leq P_0 \left( \sum_{x \in \mathcal{D}_b} l_n^\alpha(x) \geq n^{\beta-b+\alpha^*} \frac{y_1}{2C_{\infty}} \right) + e^{-n^\beta y_1^2}. \tag{71}
\]

Thus, we need to prove in each region that for some constant \( C > 0 \), and the appropriate parameters \( \beta, b, \) and \( \alpha \), we have
\[
P_0 \left( \sum_{x \in \mathcal{D}_b} l_n^\alpha(x) \geq n^{\beta-b+\alpha^*} \frac{y_1}{2C_{\infty}} \right) \leq e^{-Cn^{\beta-b}}. \tag{72}
\]

Similarly, but using this time the behaviour of \( \Lambda \) near 0,
\[
P \left( \sum_{x \in \mathcal{D}_b} \eta(x) l_n(x) \geq n^\beta y_2 \right) \leq P_0 \left( \sum_{x \in \mathcal{D}_b} l_n^2(x) \geq n^{\beta+b} \frac{y_2}{2C_0} \right) + \exp(-n^\beta y_2^2/2). \tag{73}
\]
In this case, we need to prove that for some constant $C > 0$

$$\mathbb{P}_0 \left( \sum_{x \in \mathcal{D}_0} l_n^2(x) > n^{\beta+b} \frac{y_2}{2C_0} \right) \leq \exp(-Cn^{\beta-b})$$  \hfill (74)

**Region I.** We choose $b = 1 - \beta$ in order to have $\zeta_I = \beta - b = 2\beta - 1$. We first prove (74).

Since $\beta + b = 1$, Lemma 2.1 requires that $y_2 > y_0 := 2(1 + 2 \sum_x G_2(x)) C_0$. The condition of Lemma 2.1 on $\zeta_I$ reads

$$\beta - b < \beta + b - \frac{2}{d}(\beta + b - 2b) \implies \beta < b(1 + d/2) \implies \beta < \frac{1 + d/2}{2 + d/2}.$$  \hfill (75)

Secondly, (72) relies on Proposition 3.2 with $p = \alpha^*$, $\gamma = \beta + b(\alpha^* - 1)$ and $\zeta = \beta - b$. Condition (0) is equivalent to $\beta \leq (1 + d/2)b$, already fulfilled in (74).

When $\alpha > d/2$, the condition (iii) of Proposition 3.2 is equivalent to $\beta < (1 + d/2)b$, which we have already taken into account in (75).

When $\alpha \leq d/2$, condition (i) of Proposition 3.2 is also equivalent to $\beta < (1 + d/2)b$. Now, note that there is no point in considering sites visited more than $n^{\zeta_I}$. Indeed, for some constant $C > 0$

$$\mathbb{P}_0 \left( \sum_{\mathcal{D}_{\zeta_I}} l_n^{\alpha^*}(x) \geq n^{\beta-b+\alpha^*b}y \right) \leq \mathbb{P}_0 \left( \exists x \in ] - n; n \mid l_n(x) \geq n^{\zeta_I} \right) \leq e^{-Cn^{\zeta_I}}.$$  \hfill (74)

Thus, condition (ii) with $a = \zeta_I$ is equivalent to $\beta < (\alpha + 1)b$, which implies $\beta < \frac{\alpha + 1}{\alpha + 2}$.

**Region II.** We choose $b = \frac{\beta}{\alpha+1}$, to get $\zeta_\beta = \beta - b$. Here $\alpha < d/2$. We start with proving (74). Lemma 2.1 imposes $\beta + b \geq 1$ and $\beta < (1 + d/2)b$. The latter inequality holds true when $\alpha < d/2$, whereas the former requires $\beta \geq \frac{\alpha - 1}{\alpha + 2}$. Note that in case $\beta + b = 1$, we need that $y_2 > y_0$.

In order to prove (72), we use Proposition 3.2 in case $p > (d/2)^*$ and need to check its conditions (0),(i) and (ii). Condition (0) and (i) are equivalent to $\alpha \leq d/2$. Finally, Condition (ii) has to be checked with $\zeta_\beta = \beta - b = \alpha b$ and $\gamma = \beta - b + \alpha^*b = (\alpha + \alpha^*)b = \alpha \alpha^*b$. If we choose $a = \zeta_\beta$, a simple computation yields $\zeta_\beta = \gamma - a(\alpha^* - 1) - 2/d(\gamma - a\alpha^*)$. Thus, Proposition 3.2 allows to conclude that for any $\epsilon > 0$,

$$\mathbb{P}_0 \left( \sum_{\mathcal{D}_{\alpha,ab}} l_n^{\alpha^*}(x) \geq n^{\alpha^*b}y \right) \leq \exp(-Cn^{\zeta_\beta}).$$

Hence, it remains to prove that for $y > 0$, $\epsilon > 0$, and $n$ sufficiently large,

$$\mathbb{P}_0 \left( \sum_{\mathcal{D}_{ab,-\epsilon,ab}} l_n^{\alpha^*}(x) \geq n^{\alpha^*b}y \right) \leq \exp(-Cn^{\zeta_\beta}).$$
We are in the situation of point 2. of Proposition 1.5. The proof is the same, and is left to the reader.

We now prove (8). We need to show that $D_{b-\delta}$ and $\tilde{D}_{b+\delta}$ bring a negligible contribution. If we define for each $\delta > 0$, $B_\delta := \left\{ x \in D_{b-\delta} | l_2^n(x) \geq n^{\beta-b} \frac{y_2}{2C_0} \right\}$, then as in (73), we obtain

$$P\left( \sum_{x \in D_{b-\delta}} \eta(x) l_2^n(x) > y_2 n^\beta \right) \leq P(B_\delta) + e^{-C_n^{\beta-b-\delta}},$$

and we need to show that $P(B_\delta) \leq \exp(-Cn^{\beta-b+\delta'})$ for some $\delta' > 0$. By Lemma 2.1, we need $\delta$ small enough so that $\beta + b - \delta > 1$. We also need to check that

$$\zeta - \delta' < \gamma - \delta - (b - \delta) - \frac{2}{d} (\gamma - \delta - 2(b - \delta)) \implies \delta + \frac{d}{2} \delta' < \frac{d/2 - \alpha}{\alpha + 1} \beta.$$

Now, for the large level sets, let $A_\delta := \left\{ x \in \tilde{D}_{b+\delta} | l_2^\alpha^n(x) \geq n^{\beta-b+\alpha^*b+(\alpha^*-1)\delta} \frac{y_2}{2C_0} \right\}$. As in (74), we obtain

$$P\left( \sum_{x \in \tilde{D}_{b+\delta}} \eta(x) l_2^n(x) > y_1 n^\beta \right) \leq P_0(A_\delta) + e^{-C_n^{\beta-b+\delta}}, \quad (76)$$

and we need to show that $P(A_\delta) \leq \exp(-Cn^{\beta-b+\delta'})$ for some $\delta' > 0$. We invoke again Proposition 3.2 in the case $\alpha < d/2$. Here, condition (0) imposes $b(d/2 - \alpha) \geq \delta' - (d/2)\delta$. Condition (i) imposes

$$b\alpha^*(1 - \frac{\alpha}{d/2}) > \delta'(1 + \frac{\alpha^*}{d/2}) - (\alpha^* - 1)\delta.$$

Finally, condition (ii) yields $(\alpha^* - 1)\delta > \alpha^*\delta'$. Thus, conditions (0),(i) and (ii) are clearly satisfied for $\delta$ and $\delta'$ small enough.

**Region III.** We choose $b = 2\beta/(d+2)$, and obtain $\zeta = \beta - b$. Here, $\alpha \geq d/2$, and we need to prove a result weaker than (74). For any $\epsilon > 0$

$$P_0 \left( \sum_{x \in D_b} l_2^n(x) > n^{\beta+b} \frac{y_2}{2C_0} \right) \leq \exp(-n^{\zeta-g-\epsilon}) \quad (77)$$

This is a direct application of Lemma 2.1 as soon as we check that $\gamma = \beta + b \geq 1$ (and $y_2 > y_0$ in the case of equality). This last condition means that $\beta \geq \frac{d/2+1}{d/2+2}$ which defines precisely Region III.

Now, we prove (72) invoking Proposition 3.2 with $p = \alpha^* < (d/2)^*$. Condition (0) is equivalent to $b(1 + d/2) \geq \beta$, and we have here equality. Condition (iii) holds for any $\zeta = \beta - b - \epsilon$ by a straightforward computation.

We prove now (70). We define $A_\delta$ as in Region II, and (70) follows similarly. We invoke again Proposition 3.2 in the case $\alpha \geq d/2$ with $\gamma = \beta - (\alpha^* - 1)(b + \delta)$, and $\zeta = \beta - b + \delta'$. Condition (0) requires $(d/2)\delta \geq \delta'$, whereas condition (iii) requires $(2/d)\delta > \delta'$. So that a choice $\delta' = \delta/d$ yields (70), for any $\delta > 0$.  

\[\Box\]
5 Lower Bounds for RWRS.

This Section is devoted to the proof of Proposition 1.3. The symmetry assumption simplifies the proof, thanks to the following Lemma.

Lemma 5.1 (Lemma 2.1 of [2]) When \( \{\eta(x), x \in \mathbb{Z}^d\} \) are independent and have bell-shaped densities, then for any \( \Lambda \) finite subset of \( \mathbb{Z}^d \), and any \( y > 0 \)

\[
P\left( \sum_{x \in \Lambda} \alpha_x \eta(x) > y \right) \leq P\left( \sum_{x \in \Lambda} \beta_x \eta(x) > y \right), \quad \text{if } 0 \leq \alpha_x \leq \beta_x \text{ for all } x \in \Lambda. \tag{78}
\]

Region I. Let \( \mathcal{R}_n := \{x : l_n(x) \geq 1\} \). Under the symmetry assumption, \( \forall c > 0 \),

\[
P\left( \sum_{x} \eta(x) l_n(x) \geq n^\beta y \right) \geq P\left( \sum_{x \in \mathcal{R}_n} \eta(x) \geq n^\beta y \right) \geq \mathbb{P}_0(|\mathcal{R}_n| \geq cn) P_\eta \left( \sum_{j=1}^{cn} \eta_j \geq n^\beta y \right).
\]

Now, it is well known, that for \( d \geq 3 \), there is \( c > 0 \) such that \( \lim_{n \to \infty} \mathbb{P}_0(|\mathcal{R}_n| \geq cn) = 1 \). For the other terms, if \( 1/2 < \beta < 1 \), we are in a regime of moderate deviations for a sum of i.i.d., and there is \( C > 0 \) such that

\[
\liminf_{n \to \infty} \frac{1}{n^{2\beta - 1}} \log P_\eta \left( \sum_{j=1}^{cn} \eta_j \geq n^\beta y \right) \geq -C.
\]

Region II. Under the symmetry assumption,

\[
P\left( \sum_{x} \eta(x) l_n(x) \geq n^\beta y \right) \geq P\left( \eta(0) l_n(0) \geq n^\beta y \right) \geq P_\eta \left( \eta(0) \geq n^{\frac{\beta}{\alpha + 1}} y \right) \mathbb{P}_0 \left( l_n(0) \geq n^{\frac{\beta \alpha}{\alpha + 1}} \right).
\]

Now, for \( \frac{\beta \alpha}{\alpha + 1} \leq 1 \), the second probability is of order \( \exp(-Cn^{\frac{\beta \alpha}{\alpha + 1}}) \), which is also the order of the first one. This leads to the lower bound in region II.

Region III. We keep the notations of the heuristic discussion of Region III: \( T = n^\beta \), and \( r^d = T^{d/(d+2)} \). Recall that \( \mathcal{R}_n \) is the range of the walk, and let \( \sigma_r := \inf\{k \geq 0 : S_k \not\in ]-r/2; r/2[^d\} \). Under the symmetry assumption, for any \( \epsilon > 0 \)

\[
P\left( \sum_{x} \eta(x) l_n(x) \geq yn^\beta \right) \geq P\left( \sum_{x} \eta(x) l_T(x) \geq yT \right)
\]

\[
\geq P\left( \{\forall x \in \mathcal{R}_T, \ \eta(x) > y\} \cap \{\epsilon r^d < |\mathcal{R}_T| < r^d\} \right)
\]

\[
\geq P_\eta (\eta(0) > y)^{\epsilon r^d} (\mathbb{P}_0(|\mathcal{R}_T| < r^d) - \mathbb{P}_0(|\mathcal{R}_T| < \epsilon r^d))
\]

\[
\geq P_\eta (\eta(0) > y)^{\epsilon r^d} (\mathbb{P}_0(\sigma_r > T) - \mathbb{P}_0(|\mathcal{R}_T| < \epsilon r^{d/(d+2)})). \tag{79}
\]

It is now well known that for some constant \( C > 0 \), \( \mathbb{P}_0(\sigma_r > T) \geq \exp(-CT/r^2) \). On the other hand, from Donsker and Varadhan [10], there is a constant \( c_{DV} \) such that for \( \lambda > 0 \)

\[
\mathbb{P}_0(|\mathcal{R}_T| < \epsilon r^{d/(d+2)}) \leq \exp(\lambda \epsilon r^{d/(d+2)}) \mathbb{E}_0 \left[ e^{-\lambda |\mathcal{R}_T|} \right]
\]

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\leq \exp\left(-\frac{c_{DV}^2}{2} \frac{\lambda^2}{d+2} \frac{T^d}{(d+2)} \right) \\
\leq \exp\left(-\frac{c_{DV}^2}{2} \frac{2\lambda \epsilon}{2\epsilon} \right), \quad (80)

where we have chosen \( c_{DV}^2 \frac{\lambda^2}{(d+2)} = 2\lambda \epsilon \). Thus, we can choose \( \epsilon \) small enough so that 
\( 2\mathbb{P}_0(|R_T| < r^d) \leq \mathbb{P}_0(|R_T| < r^d) \), and conclude the lower bound.

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