The chopthin algorithm for resampling

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Abstract

Resampling is a standard step in particle filters and more generally sequential Monte Carlo methods. We present an algorithm, called chopthin, for resampling weighted particles. In contrast to standard resampling methods the algorithm does not produce a set of equally weighted particles; instead it merely enforces an upper bound on the ratio between the weights. A simulation study shows that the chopthin algorithm consistently outperforms standard resampling methods. The algorithms chops up particles with large weight and thins out particles with low weight, hence its name. It implicitly guarantees a lower bound on the effective sample size. The algorithm can be implemented very efficiently, making it practically useful. We show that the expected computational effort is linear in the number of particles. Implementations for C++, R (on CRAN), Python and for Matlab are available.

Key words: chopthin; effective sample size; importance sampling; particle filter; resampling;

1 Introduction

Particle filters and more generally sequential Monte Carlo methods have gained importance and widespread use (Doucet et al., 2001). One of their key steps is resampling, which is intended to prevent weight degeneracy. Broadly speaking, resampling starts with a set of particles $X_1, \ldots, X_n$ with associated weights $w_1, \ldots, w_n$ and produces a new set of particles (a subset of the original set with potentially duplicates) with less uneven weights (often equal weights).

A commonly used resampling algorithm is multinomial sampling, which selects a new set of particles by sampling $n$ times with replacement from $X_1, \ldots, X_n$ with probabilities proportional to $w_1, \ldots, w_n$. Other resampling schemes have been proposed, for example systematic resampling (Carpenter et al., 1999), stratified resampling (Kitagawa, 1996), residual resampling (Liu and Chen, 1998) and branching resampling (Bain and Crisan, 2009, p. 278). All of these algorithms return a set of particles with equal weights.

The general consensus seems to be that, whilst it is possible to outperform multinomial resampling, the more advanced methods such as residual, stratified and systematic resampling are comparable in terms of their performance in particle filters (Douc and Cappé, 2005; Hol et al., 2006).

In this article we show that it is possible to improve the performance of the resampling step significantly. We do this by presenting a new resampling method that consistently outperforms the aforementioned methods.

The new algorithm, called chopthin, ensures that the weights are not too uneven by enforcing an upper bound, $\eta$, on the ratio between the resulting weight. Choppin can outperform other methods because it does not return particles with equal weights.

The chopthin algorithm enforces the upper bound, $\eta$, on the ratio between the weights, as follows: Particles with large weights, above a threshold $a$, are potentially “chopped”, i.e. replicated with the original weight spread equally among the replicates. This does not introduce any randomness to the particle approximation. Particles with small weights, below the threshold $a$, are “thinned” by randomly deciding whether they should be deleted or kept, adjusting the
weights by the selection probability to ensure unbiasedness. A similar approach to the thinning part of chopthin is used in Fearnhead and Clifford (2003) where the optimality of such a resampling method is shown in a certain sense.

Particle filters often only perform the resampling step if a criterion of the unevenness of the weights, such as the effective sample size (ESS), drops below a fixed threshold. This avoids resampling if the weights are relatively even and thus reduces the noise being introduced through the resampling. This results in measures of the evenness of the particles such as the ESS to fluctuate over time.

In contrast to this, chopthin can be executed at every step of a particle filter. This is because chopthin evens out the weights less than existing schemes. It will not alter the weights much (or at all) if they are already relatively even. Using it at every step will lead to less fluctuation in the unevenness of the weights over time. Figure 3 (later in the paper) illustrates this in an example by looking at the ESS over time.

Chopthin can be implemented efficiently. Indeed, we present one version of chopthin, which can be implemented in expected constant linear effort in the number of particles.

We begin by presenting the generic chopthin algorithm in Section 2. In Section 3 we present a version of the algorithm that has expected linear effort and show in a simulation that its effort is comparable to other standard resampling methods. A simulation study is conducted in Section 4 that compares the chopthin algorithm to other resampling schemes within a particle filter. The results show that our new algorithm consistently outperforms the other resampling methods. In Section 5 we prove that the algorithm implicitly controls the ESS.

Implementations of chopthin are available: as an R-package (chopthin) on CRAN, a python package (available on the python package index) and as C++ code and a Matlab extension file from the hompage of the first author.

2 The Generic Algorithm

This section introduces a generic version of the chopthin algorithm (Algorithm 1). As input it receives $n$ weighted particles given by a vector of weights $(w_i)_{1:n}$. Further input parameters are $\eta$, the desired upper bound on the ratio between weights, and $N$, the expected number of particles to be returned.

Every particle gets a (potentially) random number of descendants. For a particle with weight $w$, the expected number of offspring will be $h_\eta^a(w)$, where $h_\eta^a : [0, \infty) \to [0, \infty)$ is a given function which may depend on $\eta$ and on a further threshold parameter $a$. To ensure that $N$ particles are returned (in expectation), we need to find $a$ such that

$$\sum_{i=1}^{n} h_\eta^a(w_i) = N. \quad (1)$$

The mechanism that generates the descendants depends on the weight of the particle as well as on the parameter $a$. Particles with weights below $a$ get “thinned”, i.e. either have one descendant (with weight $a$) or zero descendants. Particles with weights above $a$ get “chopped”, which means that they get subdivided into smaller pieces, the total weight remaining unchanged; the number of descendants will either be $\lfloor h_\eta^a(w) \rfloor$ or $\lceil h_\eta^a(w) \rceil$.

We want chopthin to respect the bound $\eta$ on the ratio of the weights and we want it to be unbiased, in the sense that for every particle the expected weight of its descendants is equal to its original weight. To ensure the bound on the ratio of the weights, we require chopthin to only return weights between $a$ and $\eta a$.

For large weights, i.e. $w \geq a$, the chopping ensures that the total weight of the descendants is equal to the weight of the particle and thus unbiasedness is automatically satisfied. The
Algorithm 1: Generic chopthin

Input: particle weights \((w_i)_{1:n}\); maximal weight ratio \(\eta\); target number of particles \(N\); function \(h^\eta_a : [0, \infty) \rightarrow [0, \infty)\)

Output: vector of ancestors \(I\) and vector of new weights \(\tilde{w}\)

Let \(a\) be a solution to \(\sum_{i=1}^n h^\eta_a(w_i) = N\)
Let \(u \sim U(0, 1)\) and set \(I = ()\), \(\tilde{w} = ()\)
for \(i = 1, \ldots, n\) do
  \(u = u + h^\eta_a(w_i)\)
  if \(u \geq 1\) then
    \(k = \lceil u \rceil\)
    append \((i, \ldots, i) \in \mathbb{N}^k\) to \(I\)
    if \(h^\eta_a(w_i) < 1\) then append \((a) \in \mathbb{R}^1\) to \(\tilde{w}\)
    else append \((w_i/k, \ldots, w_i/k) \in \mathbb{R}^k\) to \(\tilde{w}\)
  \(u = u - k\)
return \(I, \tilde{w}\)

| \(u\) | \(h^\eta_a(w_1)\) | \(h^\eta_a(w_2)\) | \(h^\eta_a(w_3)\) | \(h^\eta_a(w_4)\) | \(h^\eta_a(w_5)\) |
|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 |

Figure 1: Illustration of systematic resampling in Algorithm 1.

restriction that the weights are between \(a\) and \(\eta a\) requires
\[
a \leq \frac{w}{\lceil h^\eta_a(w) \rceil} \quad \text{and} \quad \frac{w}{\lceil h^\eta_a(w) \rceil} \leq \eta a \quad \forall w \geq a,
\]
which implies the following condition on \(h^\eta_a\):
\[
\left\lceil \frac{w}{\eta a} \right\rceil \leq h^\eta_a(w) \leq \left\lfloor \frac{w}{a} \right\rfloor \quad \forall w \geq a. \tag{2}
\]

For small weights \(w < a\), descending particles will have weight \(a\), thus ensuring the range condition on the weights. The unbiasedness condition requires \(h^\eta_a(w) = w/a\), uniquely determ
ing \(h^\eta_a\) in this range.

To determine the number of descendants, chopthin uses systematic resampling on \(h^\eta_a(w_1), \ldots, h^\eta_a(w_n)\).
This first ensures that the number of descendants for every particle will either be \(\lceil h^\eta_a(w) \rceil\) or \(\lfloor h^\eta_a(w) \rfloor\) and second ensures that the total number of particles returned is exactly \(N\). Instead of systematic resampling, chopthin could also use stratified resampling.

Figure 1 is an illustration of the systematic resampling in Algorithm 1 in the for-loop, where \(h^\eta_a(w)\) denotes the expected number of offspring for a particle with current weight \(w\). This depends on the threshold \(a\). Particles 1, 3, and 5 have \(h^\eta_a(w_i) > 1\) and thus are subject to chopping. Particle 1 is not divided at all, particle 3 gets chopped into 2 equally weighted particles, particle 5 into 3 equally weighted particles. Particles 2 and 4 have \(h^\eta_a(w_i) < 1\) and thus are subject to thinning. Particle 2 receives one descendant with weight \(a\) and particle 4 receives no descendant.

We have considerable freedom in choosing \(h^\eta_a\) for \(w \geq a\). One natural choice would be
\[
h^\eta_a(w) = \begin{cases} w/a & \text{if } w < a \\ \left\lfloor \frac{w}{\eta a} \right\rfloor & \text{if } w \geq a \end{cases} \tag{3}
\]
see Figure 2 for an illustration. We will call the resulting algorithm \textit{step-chopthin}. The upper inequality of (2) for values slightly less than \(2a\) implies \(\eta \geq 2\) for this algorithm.
Algorithm 2: Fast determination of $a$

**Input:** particle weights $w_i$; maximal weight ratio $\eta$; target number of particles $N$

**Output:** $a > 0$ such that $\sum_{i=1}^n h^n_\eta(w_i) = N$

- $w^u = w^l = w$, $s^l = 0$, $c^m = 0$, $s^u = 0$, $c^u = 0$

while $w^u \neq \emptyset$ or $w^l \neq \emptyset$ do

- if $|w^l| \geq |w^u|$ then sample $a$ uniformly from $w^l$ and let $b = \eta a/2$
  else sample $b$ uniformly from $w^u$ and let $a = 2b/\eta$

- $h = s^l/a + \sum_{v \in w^l} \min(v/a,1) + c^m + \sum_{v \in w^u} \max(v/b - 1,0) + s^u/b - c^u$

if $h = N$ then return $a$

if $h > N$ then

- $s^l = s^l + \sum_{v \in w^l} I(v \leq a)$
  $w^l = \{v \in w^l; v > a\}$, $w^u = \{v \in w^u; v > b\}$

else

- $c^m = c^m + \sum_{v \in w^l} I(v \geq a)$
  $s^u = s^u + \sum_{v \in w^u} vI(v \geq b)$, $c^u = c^u + \sum_{v \in w^u} I(v \geq b)$

- $w^l = \{v \in w^l; v < a\}$, $w^u = \{v \in w^u; v < b\}$

return $a = \frac{s^l + 2s^u}{N - c^m + c^u}$

There are two problems with this choice — one is that the function is discontinuous (in $a$) and as such it will not guarantee the existence of a solution $a$ of (1). Instead of having an exact solution, one could use an approximate solution, using a numerical root finding algorithm. However, it is not obvious what stopping criterion to use.

3 Implementation in expected linear time

In this section we present our main version of the algorithm, which we simply call chopthin. For this we choose $h^n_\eta$ such that it is continuous (in $a$) and such that (1) can be solved for $a$ in expected linear effort. Consider the function

$$h^n_\eta(w) = \begin{cases} w/a & \text{if } w < \eta \frac{a}{2} \\ 1 & \text{if } a \leq w < \eta \frac{a}{2} \\ 2w/\eta a & \text{if } w \geq \eta \frac{a}{2} \end{cases}$$  \hspace{1cm} (4)$$

which is depicted in Figure 2. The upper inequality of (2) for $w$ slightly less than $2a$ implies that chopthin is only valid for $\eta \geq 4$. 

![Figure 2: Expected number $h^n_\eta$ of offspring as a function of the weight. Solid: chopthin (4); dashed: step-chopthin (3).](image-url)
expected effort of Algorithm 2 is linear in \( n \)

\[
\sum_{i=1}^{n} (|w_i^l| + |w_i^u|)
\]

Consider iteration \( i \). The following statements are conditional on the sets \( w_{i-1}^l, w_{i-1}^u \). Suppose that \( |w_{i-1}^l| \geq |w_{i-1}^u| \). We show that \( \mathbb{E}(|w_i|) \leq (3/4)|w_{i-1}^l| \). Let \( a \) be the randomly selected element from \( w_{i-1}^l \). Let \( a^* \) be such that \( \sum_{i=1}^{n} h_i^0(w_i) = N \). Let \( M^l = |\{v \in w_{i-1}^l : v < a^*\}| \), \( M^u = |\{v \in w_{i-1}^l : v > a^*\}| \). We then have

\[
\mathbb{E}(|w_i|) = \mathbb{E}\left(|w_i| : a < a^*\right) + \mathbb{E}\left(|w_i| : a > a^*\right) = \left(\frac{M^l}{2} + M^u\right) \frac{M^l}{|w_{i-1}^l|} + \left(\frac{M^u}{2} + M^l\right) \frac{M^u}{|w_{i-1}^l|} = \left(\frac{M^l + M^u}{2}\right) \frac{M^l}{|w_{i-1}^l|} + \left(\frac{M^l + M^u}{2}\right) \frac{M^u}{|w_{i-1}^l|} \leq \frac{1}{2} |w_{i-1}^l| + \frac{1}{4} |w_{i-1}^l| = \frac{3}{4} |w_{i-1}^l|
\]
Table 2: Effort of resampling \(N\) particles divided by the effort to generate \(N\) exponentially distributed random variables in R

| Method      | 1000  | 10000 | 1e+05 | 1e+06 |
|-------------|-------|-------|-------|-------|
| chophthin   | 1.62  | 1.39  | 1.40  | 1.51  |
| systematic  | 0.45  | 0.43  | 0.39  | 0.39  |
| multinomial | 0.92  | 0.89  | 1.03  | 1.45  |

Hence, \(\mathbb{E}(|w^t_i| + |w^u_{i}|) \leq 7/8 |w_{i-1}^t| + |w_{i-1}^u| \leq 7/8 (|w_{i-1}^t| + |w_{i-1}^u|)\) as \(|w_{i-1}^t| \geq |w_{i-1}^u|\). Similarly, it can be seen that the above also holds if \(|w_{i-1}^t| < |w_{i-1}^u|\).

Thus,

\[
\mathbb{E}(|w_i^t| + |w_i^u|) = \mathbb{E} \left( \mathbb{E} \left[ |w_i^t| + |w_i^u| \mid w_{i-1}^t, w_{i-1}^u \right] \right) \leq \frac{7}{8} \mathbb{E}(|w_{i-1}^t| + |w_{i-1}^u|)
\]

\[
\leq \ldots \leq \left( \frac{7}{8} \right)^{i-1} \mathbb{E}(|w_1^t| + |w_1^u|) = \left( \frac{7}{8} \right)^{i-1} 2n
\]

and therefore

\[
\mathbb{E} \left( \sum_{i=1}^{\infty} (|w_i^t| + |w_i^u|) \right) \leq \sum_{i=1}^{\infty} \left( \frac{7}{8} \right)^{i-1} 2n = 2n \frac{1}{1 - 7/8} = 16n.
\]

This shows that the expected effort of Algorithm 2 is \(O(n)\). The remainder of Algorithm 1 entails generating the output of length \(N\) and it runs through all \(n\) particles with an overall effort of \(O(\max(n, N))\). Thus the expected effort of the combined Algorithms 1, 2 is \(O(\max(n, N))\).

We now compare the effort of chophthin to the effort of sampling with replacement (multinomial sampling), via the in-built function sample.int in R, and a (fast) C++-based implementation of systematic resampling.

We simulated \(N\) weights from an Exponential distribution, i.e. \(w_i \sim \text{Exp}(1), i = 1, \ldots, N\) independently. We then applied the resampling procedures to the simulated weights.

Table 2 reports the mean effort of the resampling procedures over 10000 repetitions. The reported effort is relative to the effort to generate the weights, which is only a single use of the in-built R function rexp. Constant values indicate that the effort is linear in \(N\), as the effort of generating the random variables is linear in \(N\).

Systematic resampling and chophthin are both approximately linear in \(N\). The implementation of resampling with replacement in R is not linear. As expected, chophthin is more computationally demanding than systematic resampling as part of the chophthin algorithm is a systematic resampling step. Nevertheless, the computational effort of chophthin is very moderate, only slightly more expensive than generating exponentially distributed random variables.

4 Simulations

In this section, we compare the chophthin algorithm with other resampling methods. We conduct a simulation using various resampling methods in a particle filter and compare the accuracy of the results. Further, we discuss how the results of chophthin are affect by varying the bound of the ratio on the weights, \(\eta\), and illustrate that that chophthin results in a less variable ESS.
Algorithm 3: Particle filter

**Input:** target number of particles $N$; ESS threshold $\beta$; resampling scheme $r$; observations $y_1, \ldots, y_T$.

**Output:** weighted particles $(w_i, X_t^{(i)})_{1:n_t}$ for $t = 1, \ldots, T$

Sample $X_0^{(i)} \sim N(0,1)$, $i = 1, \ldots, N$

$w_i = 1$, $i = 1, \ldots, N$

Let $n_0 = N$

for $t = 1, \ldots, T$ do

Sample $X_t^{(i)} \sim N(X_{t-1}^{(i)}, \sigma^2_X)$, $i = 1, \ldots, n_{t-1}$

$w_i = w_i \phi(y_t; X_t^{(i)}, \sigma^2_Y)$, $i = 1, \ldots, n_{t-1}$, where $\phi(\cdot; \mu, \sigma^2)$ is the pdf of a $N(\mu, \sigma^2)$ distr.

if $\text{ESS}(w) \leq \beta$ then

Run $r$ with a target of $N$ particles to get a set of particles $(w_i, X_t^{(i)})_{1:n_t}$

Normalise weights such that $\sum_{i=1}^{n_t} w_i = N$

end if

end for

4.1 Model

Consider a model with hidden Markov process $X_t \in \mathbb{R}$ and observed process $Y_t \in \mathbb{R}$ for $t \in \mathbb{N}$.

In this section, we are interested in the model

$$
\begin{align*}
X_t &= X_{t-1} + \epsilon_t, \quad \epsilon_t \overset{iid}{\sim} N(0,1) \\
Y_t &= X_t + \xi_t, \quad \xi_t \overset{iid}{\sim} N(0, \sigma^2_Y)
\end{align*}
$$

with $X_0 \sim N(0,1)$. For this model, we can use the Kalman filter [Kalman 1960] to obtain the exact conditional distribution — this will be our benchmark when comparing the resampling methods.

4.2 Simulation

We shall use the particle filter presented in Algorithm 3 to give estimates of the hidden states $X_1, \ldots, X_T$ based upon some observations $y_1, \ldots, y_T$. More precisely, we are interested in the posterior $f(x_t|y_1, \ldots, y_t)$ for $t = 1, \ldots, T$. These observations shall be generated from the model introduced in Section 4.1. A parameter of the particle filter is $\beta \in [0, N]$ which determines when resampling occurs based on the ESS. The ESS is a measure of the unevenness of the weights and it is typically used in particle filters to trigger the resampling step. We define the ESS for a weight vector, $w = (w_1, \ldots, w_n)$, as

$$
\text{ESS}(w) = \frac{\left(\sum_{i=1}^{n} w_i\right)^2}{\sum_{i=1}^{n} w_i^2}.
$$

In Algorithm 3, resampling is performed if the ESS drops below $\beta$. If $\beta = N$ then resampling is performed at every step as $\text{ESS}(w) \leq N$. Lastly, potentially any resampling scheme $r$ can used in Algorithm 3.

For a given $\sigma^2_Y$, resampling scheme $r$, target number of particles $N$ and resampling trigger $\beta$, a single iteration of the simulation is conducted as follows: simulate from the model $T = 1000$ observations: $y_1, \ldots, y_T$. Using this realisation of observations, run the particle filter to give estimates of the hidden states $X_1, \ldots, X_T$. Lastly, the Kalman filter is run to obtain the exact conditional distribution. This is repeated $M = 1000$ iterations. The simulation is conducted using combinations of the parameters: $\sigma_Y$, $N$, $\beta$, $\eta$ (for chopthin) and various resampling schemes. See later in Table 3.
Figure 3: ESS before and after resampling for selected resampling schemes (one realisation).

4.3 Illustration of One Run

Figure 3 considers the effect of different resampling schemes on the ESS during the first 50 steps of one realisation of the particle filter (Algorithm 3) with $N = 10000$ target particles. It plots the ESS before and after resampling. As resampling for the multinomial and branching algorithm is only occurs if the ESS has dropped below $0.5N$, the ESS is far more variable than in the chopthin algorithm. For both $\eta = 3 + \sqrt{8}$ and $\eta = 10$, it seems that the chopthin algorithm after resampling stays significantly above its theoretical lower bound (given in Section 5), which is $0.5N$ and $0.33N$, respectively. Also the different choice in $\eta$ within the chopthin algorithm does not lead to vastly different behaviour.

4.4 Results

The results of the full simulation are presented in Table 3. For each iteration, we obtain the estimated posterior mean of $X_t$ for $t = 1, \ldots, T$. For a given $\sigma_Y$, $N$ and $\beta$, denote the estimated posterior mean from iteration $i$, at time $t$, for resampling scheme $r$ as $\tilde{\mu}_{i,t,r}$. Further, denote the true posterior mean at time $t$ given by the Kalman filter as $\mu_{i,t}$. We report the approximate mean squared error (MSE) for resampling scheme $r$ as

$$
\frac{1}{M} \sum_{i=1}^{M} \left\{ \frac{1}{T} \sum_{t=1}^{T} (\tilde{\mu}_{i,t,r} - \mu_{i,t})^2 \right\}.
$$

The MSE values, presented in Table 3, are divided by the MSE given by the systematic resampling. The results show that using the chopthin algorithm at every step ($\beta = N$) and using the trigger ($\beta = 0.5N$) with various values for the ratio bound $\eta$ consistently achieves a lower MSE than the other resampling methods. The simulations using $\sigma_Y = 1/3$ is based on a setting where there is a small amount of noise between the state and observation. In this case, the particle filter will be resampling at nearly every step for all methods.
The conditional distribution

\[ p(y_{1}, \ldots, y_{T}) \]

\[ = \prod_{t=1}^{T} p(y_{t} | y_{1:t-1}). \]

The conditional distribution \( p(y_{t} | y_{1:t-1}) \) can be approximated from these simulations the average of the weights; that is

\[ \hat{p}(y_{t} | y_{1:t-1}) = \frac{1}{N} \sum_{k=1}^{N} w_{k}, \]

where the \( w_{k} \) are the weights after the conditioning on the observation \( y_{t} \).

For the model, presented in Section 4.1, the exact marginal likelihood can be computed using the Kalman filter, providing a comparison with the estimates given by the particle filter. For the same run of the simulation conducted in Section 4.4, we estimate the conditional likelihood as follows. Let \( y_{i,t} \) denote the observations simulated in iteration \( i \) for \( t = 1, \ldots, T \). Then denote the estimate of \( p(y_{t}^{i} | y_{1:t-1}^{i}) \) as \( \hat{p}(y_{t}^{i} | y_{1:t-1}^{i}) \). In Table 4, we report the following MSE

\[ \frac{1}{M} \left\{ \sum_{i=1}^{M} \frac{1}{T} \sum_{t=1}^{T} \left( \log \hat{p}(y_{t}^{i} | y_{1:t-1}^{i}) - \log \hat{p}(y_{t} | y_{1:t-1}) \right)^{2} \right\}. \]

Based on the MSE results, the chopthin method approximates the log likelihood better than systematic and consistently for other resampling methods. Accurate estimation of the conditional \( p(y_{t} | y_{1:t-1}) \) is particularly important in particle MCMC, see for example Andrieu et al. (2010).

5 Implied control of the Effective Sample Size

The following lemma shows that imposing a bound on the ratio between the weights implicitly results in a lower bound on the ESS. It implies that chopthin has a lower bound on the ESS.
Table 4: Simulation Results: MSE values of log likelihood for various simulation parameters and different resampling methods. Presented MSE values are divide by the MSE from the simulations using systematic resampling.

| $\beta$ | $\eta$ | $N$ | $10^3$ | $10^3$ | $10^3$ | $10^3$ | $10^4$ | $10^4$ | $10^4$ |
|----------|--------|-----|--------|--------|--------|--------|--------|--------|--------|
| chopthin | $N$    | $3 + \sqrt{8}$ | 0.98 | 0.95 | 0.87 | 0.95 | 0.91 | 0.91 | 0.88 | 0.96 |
| multinomial | $0.5N$ | - | 1.03 | 1.07 | 1.37 | 1.22 | 0.94 | 1.07 | 1.17 | 1.34 |
| branching | $0.5N$ | - | 1.02 | 1.00 | 0.99 | 0.99 | 0.98 | 0.99 | 1.00 | 1.01 |
| residual | $0.5N$ | - | 1.00 | 1.00 | 1.00 | 1.00 | 1.01 | 1.00 | 1.00 | 1.02 |
| stratified | $0.5N$ | - | 1.04 | 0.99 | 1.02 | 1.01 | 0.97 | 1.00 | 1.02 | 1.02 |
| systematic | $N$ | - | 0.99 | 0.95 | 1.06 | 1.37 | 1.05 | 0.93 | 1.10 | 1.45 |
| systematic | $0.5N$ | - | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |

after resampling.

**Lemma 2.** Suppose $w_1, \ldots, w_n > 0$. Then

$$ESS(w) = \frac{(\sum_{i=1}^{n} w_i)^2}{\sum_{i=1}^{n} w_i^2} \geq 4 \frac{n+1 - \eta^2}{(\eta+1)^2}$$

where $\eta = \max \frac{w_i}{\min \, w_i}$.

**Proof.** In the case where all weights are equal, i.e. $\eta = 1$, then $ESS(w) = n$, thus inequality holds. From now on consider the case $\eta > 1$.

Let $W_i = \frac{W_i}{\sum_{j} W_j}$ be the normalized weights corresponding to $W$. Then $ESS(w) = ESS(W)$.

The set of possible normalized weights is compact and $ESS$ is a continuous function, thus there exists a $W^\ast$ that minimises $ESS$. Without loss of generality, assume $W_1^\ast \leq \ldots \leq W_n^\ast$.

The normalised weight $W_i^\ast$ has to be of the form $W_i^\ast = \alpha \xi$, $W_i^\ast = \eta a$ for $i < k$, where $k \in \{1, \ldots, n-1\}$, $\alpha > 0$ and $1 \leq \xi < \eta$. To see this let $W$ be a normalised weight vector for which there exist mutually distinct indices $i, j, k, l$ such that $W_i < W_j \leq W_k < W_l$. Define a new weight vector $V$ identical to $W$ except for $V_j = W_j - \Delta$, $V_k = W_k + \Delta$ with $\Delta = \min((W_i - W_k)/2, (W_j - W_l)/2)$. Then

$$\frac{1}{ESS(V)} = \sum_{\nu} V^2_{\nu} = 2\Delta^2 + 2\Delta(W_k - W_j) + \sum_{\nu} W^2_{\nu} > \frac{1}{ESS(W)},$$

which shows that $W$ does not minimise $ESS$. Hence, $W^\ast$ can take at most 3 values, the middle one, if present, appearing exactly once. The two extreme values have to have a ratio of $\eta$, otherwise one could move them further apart and create a weight vector with smaller $ESS$.

As $\sum W_i^\ast = \alpha[k - 1 + \xi + \eta(n-k)]$, we have

$$ESS(W^\ast) = \frac{[k - 1 + \xi + \eta(n-k)]^2}{k - 1 + \xi^2 + (n-k)\eta^2} \geq \frac{[k + \eta(n-k)]^2}{k - 1 + (n-k+1)\eta^2} \geq \inf_{x \in [1, n-1]} h(x)$$

where $h(x) = \frac{[x+\eta(n-x)]^2}{x+1+2(n-x+1)\eta^2}$.

It remains to derive the minimum of $h$. Candidates for minimizers of $h$ are $x = \eta \eta/(\eta - 1)$ (which is not in the right range) and $x = (\eta(n+2) + 2)/(\eta + 1)$. Plugging this into $h$ gives

$$h(x) \geq \frac{4n+1 - \eta^2}{(\eta+1)^2}.$$

Large $\eta$ allow for more variability in the weights and thus should lead to lower effective sample sizes. Consistent with this, the lower bound on $ESS$ is decreasing in $\eta$. This can be seen by differentiating it with respect to $\eta$. 


For large $n$, the leading term is $4 \frac{\eta}{(\eta+1)^2}$. Equating this to a desired minimal effective sample size of $\gamma n$ gives

$$
\eta = \frac{2 - \gamma + 2\sqrt{1 - \gamma}}{\gamma}.
$$

For example, for $\gamma = 0.5$, this leads to $\eta = 3 + \sqrt{8}$. Furthermore, for $\eta = 10$, the lower bound on the ESS is $\frac{46}{121} n - \frac{99}{121} \approx 0.33 n$.

6 Discussion

6.1 Why not only impose an upper or a lower threshold on the weights?

The chopthin algorithm imposes a bound on the ratio of the largest and the smallest weight. Alternatively, one could have imposed only a lower or only an upper bound on the normalized weights. The following examples illustrate that there are situations in which these bounds would not lead to resampling despite very uneven weights. However, the chopthin algorithm (with $\eta < n$) would even out the weights in both examples.

**Example 1.** Suppose our weight vector $w$ of length $n$ is produced by one importance sampling step, where the target distribution is a uniform distribution on $[0, 0.5]$ and the importance sampling distribution is a uniform distribution on $[0, 1]$. Then roughly half of the weights will be approximately $2/n$ and half of the weight will be 0. In this situation none of the weights is large, so imposing an upper bound on the weights would not lead to resampling.

**Example 2.** Consider the same setting as in the previous distribution, but now having as target distribution a mixture of two equally probable components, one being a uniform distribution on $[0, 1]$, the other one being a uniform distribution on $[0, 1/n]$. Suppose the first particle is in $[0,1/n]$ and all other samples are greater than $1/n$. Then the weight of the first particle is $(n+1)/2$ and the weight of all other particles is $1/2$. Thus in this case, no weight is small, so imposing a lower bound on the weights would not lead to resampling.

6.2 The Thinning Method

In the chopthin algorithm, presented in Section 3, samples are thinned using systematic resampling. Another method can be used instead of systematic resampling as long as the exact number of samples are resampled. A drawback of systematic resampling is its unknown theoretical behaviour as the number of samples tends to infinity. Therefore, in order to obtain a theoretical result for the chopthin algorithm, systematic resampling may be replaced with another resampler for which the large-sample behaviour is known e.g. stratified resampling.

7 Summary

In this paper we have introduced the chopthin algorithm which bounds the ratio between the weights. We showed, in a simulation, that chopthin consistently outperforms standard resampling schemes used in particle filters. The simulation also demonstrated that chopthin can be used at every iteration in a particle filter with no detrimental effects. The chopthin algorithm can be implemented efficiently and we have proved that its expected effort is linear in the number of samples. Lastly, we have shown that imposing a bound on the ratio between weights implicitly controls the ESS.
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