Neural Observer With Lyapunov Stability Guarantee for Uncertain Nonlinear Systems

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Abstract—In this article, we propose a novel nonlinear observer based on neural networks (NNs), called neural observers, for observation tasks of linear time-invariant (LTI) systems and uncertain nonlinear systems. In particular, the neural observer designed for uncertain systems is inspired by the active disturbance rejection control, which can measure the uncertainty in real time. The stability analysis (e.g., exponential convergence rate) of LTI and uncertain nonlinear systems (involving neural observers) are presented and guaranteed, where it is shown that the observation problems can be solved only using the linear matrix inequalities (LMIs). Also, it is revealed that the observability and controllability of the system matrices are required to demonstrate the existence of solutions for LMIs. Finally, the effectiveness of neural observers is verified in three simulation cases, including the X-29A aircraft model, the nonlinear pendulum, and the four-wheel steering vehicle.

Index Terms—Active disturbance rejection control (ADRC), linear matrix inequalities (LMIs), neural network (NN), nonlinear observer, observability and controllability, uncertain systems.

I. INTRODUCTION

With the success of machine learning (ML) algorithms in various complex tasks, such as computer vision and natural language processing, connecting ML with control theory has become a hot topic in recent years and is attracting more and more researchers [1], [2], [3]. On the one hand, the data-driven ML methods have been widely used to deal with nonlinear control problems, which can be traced back to early years when the NN (NN) theory was proposed [4], [5], [6], [7]. However, it is not easy to utilize the model information to construct the control input when the model itself contains uncertainties (e.g., unmodeled dynamics). To tackle this challenge, more effective modeling methods based on deep NNs are gradually coming into our vision recently, such as physical-informed-ML [8], stable deep dynamics learning [9], and neural operator learning [10]. There are also many works proposed to address high-dimensional control problems (e.g., solving the Hamilton–Jacobi–Bellman equation) and state observation problems via learning methods [11], [12], [13], [14], [15], [16], [17], indicating that the ML methods can be successfully employed in control and identification problems. On the other hand, the classical control theories are conversely applied to explain why and how the ML algorithms work [18], [19], [20], [21]. For example, the convergence performance of optimization algorithms can be analyzed via linear matrix inequalities (LMIs) [19].

Despite the aforementioned advances, there are still some intractable challenges in learning-based control via deep NNs. For example, how to directly analyze the control performance (e.g., stability, optimality, and so on) of a system equipped with NN mappings remains a problem [2], [3]. It is not straightforward to apply the nonlinear control theory [22], [23], as there are various types of nonlinear activation functions and numerous parameters in an NN mapping. Additionally, such systems are generally vulnerable to various malicious perturbations [24] due to the black-box nature of deep NNs. Furthermore, the training process is highly dependent on the data, and thus, one needs to appropriately select the sampling method for the system state and consider the training data distribution, reducing the impact on the closed-loop system [25], [26]. Finally, how to interpret that the trained NNs are applicable is also an open question in the ML community.

In this article, we mainly focus on the state observation tasks, where the observers are designed based on NNs. Following [14], [15], and [16], the following dynamical model and the corresponding observer are considered

**Dynamical model:**

\[
\begin{align*}
\dot{x} &= Ax + g(x, u) \\
y &= Cx
\end{align*}
\]

**NN-based Observer:**

\[
\begin{align*}
\dot{\hat{x}} &= A\hat{x} + \pi_0(\hat{x}, u) + G(y - C\hat{x}) \\
\hat{y} &= C\hat{x}
\end{align*}
\]
where $(C, A)$ is observable and the NN $\pi_\theta(x, u)$ in the observer is trained to approximate the uncertainty $g(x, u)$. Then, one could provide an NN-based observer to achieve $\hat{x} \to x$. In this context, we are motivated to ask a question: how can we find a concise condition to verify the availability of an NN for the system, with which the performance is not limited by the sampling method and can be directly analyzed? If the condition exists, most of the aforementioned challenges can be addressed.

Recently, [27] and [28] proposed an efficient method, based on quadratic constraints (QCs) and LMI, to analyze the robust stability in equilibrium points of systems controlled by one state-feedback NN mapping controller $u(k) = \pi_\theta(x(k))$. However, the analysis of robust stability in [27, Th. 2] highly depends on the assumption that the perturbation $\Delta$ is bounded and depends on the skilled construction of filter $\Psi_\Delta$, which is applied to capture the correlation between the input and output signals of $\Delta$ against time. As for the LMIs conditions that guarantee the stability in [27] and [28], they do not explicitly indicate whether the solutions of LMIs exist or not. Moreover, the filter $\Psi_\Delta$ may also complicate solving the LMI condition [27, Th. 2]. In our work, instead of constructing a filter $\Psi_\Delta$ for the uncertain systems, we design neural observers inspired by the essential philosophy of active disturbance rejection control (ADRC) proposed in [29], where the basic idea is to regard the “total uncertainty” as an extended state of the system. By applying ADRC, one can estimate the uncertainty and compensate for it in the control input in real-time. The theorems about ADRC can be found in [30] and [31].

A. Contributions

The contributions of this article can be summarized as follows.

1) This work belongs to the category of using MT in control problems. We introduce a specially structured NN mapping $\pi_\theta(\cdot)$ to design nonlinear neural observers for the observation task of dynamical systems. We first propose two relative definitions: neural observable and neural exponentially observable. Ideologically, for a controllable and observable linear time-invariant (LTI) system, we construct a Luenberger-form neural observer derived from the feedback of errors and employ the estimated state $\hat{x}$ to design the feedback NN control law $u = \pi_\theta(\hat{x})$. For two classes of nonlinear systems (i.e., integrator chain nonlinear systems and MIMO nonlinear systems consisting of a linear dynamic part and the uncertainty), we, respectively, design the corresponding neural observers to measure the state and the “total uncertainty” by inheriting the idea of ADRC [29]. More details are given in Sections II and III.

2) We develop the NN isolation method and QCs (see Lemma 2) for an NN mapping vector $\pi_\theta(\cdot)$, which is composed of $K$ NN mappings, i.e., $\pi_\theta(\cdot) = [\pi_{\theta_1}(\cdot), \ldots, \pi_{\theta_K}(\cdot)]^T$ with parameters $\theta = (\theta_1, \ldots, \theta_K)$. In addition, we point out that Lemma 2 can be used to deduce the LMI of closed-loop dynamics under a feedback interconnection. More details can be found in Section IV.

3) We provide a verification framework for the availability of NNs in NN-based systems via LMIs. The first and the second results (see Theorems 1 and 2) provide LMI conditions to guarantee the neural exponential observability and globally exponential stability for LTI systems, respectively. Furthermore, in these cases, we reveal the relationship between the existence of solutions for LMIs and the observability and controllability of LTI systems (see Propositions 2 and 3). Under this fundamental framework, we provide the third and the fourth results (see Theorems 3 and 4), which achieve the neural observability for integrator chain nonlinear systems and a class of MIMO nonlinear systems, respectively. Different from Theorem 1 and 2, Theorem 3 and 4 can not only guarantee the observability but also measure the uncertainty in real-time.

This article is organized as follows. In Section II, we present the key ideas of the neural observer. In Section III, we propose two definitions of observability as well as the formulation of neural observers for different kinds of systems. Moreover, relevant observation problems are also defined. Section IV discusses the NN isolation and QCs method for the NN mapping and the NN mapping vector. Then, the convergence analysis associated with the observation problems is provided in Section V. Finally, in Section VI, we provide the simulation results to verify the efficiency of our framework.

B. Mathematical Notation

1) $R^n$ denotes $n$-dimensional real linear space. In this paragraph, only the real linear- and finite-dimensional spaces are considered. Each space, $\mathcal{M}$, holds an inner product $\langle w, v \rangle = w^T v$ and a norm $\|w\|_2 = (w, w)$. We use pointwise orders $\geq, >$ for any vectors $w, v \in \mathcal{M}$, i.e., $w \geq (>) v \iff w_i \geq (>) v_i, i \in \{1, \ldots, \dim(M)\}$. The set of real numbers in the interval $[a, b] \subset R$ is denoted by $T_{[a, b]}$, and the set of real numbers in the interval $[a, \infty) \subset R$ is $T_{a, \infty}. \mathcal{L}(\mathcal{M}, \mathcal{N})$ accounts for the space of all linear mappings (matrices) from “$\mathcal{M}$” to “$\mathcal{N}$.” For any mappings $T \in \mathcal{L}(\mathcal{M}, \mathcal{N})$, the induced two-norm is defined by $\|T\|_2 = \sup_{x \in \mathcal{M}, \|x\|_2=1} \|Tx\|_2$. Specialy, when the linear spaces $\mathcal{M}$ and $\mathcal{N}$ are identical, $\mathcal{L}(\mathcal{M}, \mathcal{N})$ can be abbreviated to $\mathcal{L}(\mathcal{M})$. Given a mapping $T \in \mathcal{L}(\mathcal{M}), T > 0$ represents $T = T^T$ and $(Tw, w) \geq 0$ for all $w \in \mathcal{M}$, where “$\geq$” is true if and only if $w = 0$; and $\lambda_{\max}(T)$ and $\lambda_{\min}(T)$ denote the maximum and minimum eigenvalue, respectively. In addition, diag$(A_1, \ldots, A_n)$ represents a diagonal block matrix, where the $i$th diagonal block is $A_i$. $0_n$ and $I_m$ are $n$-dimensional zero vector and $m$-dimensional vector whose entries are all ones, respectively.

2) The space of $k$th continuously differentiable functions from $\mathcal{M}$ to $\mathcal{N}$ is denoted by $C^k(\mathcal{M}, \mathcal{N})$. Similarly, when the spaces $\mathcal{M}$ and $\mathcal{N}$ are identical, the notation $C^k(\mathcal{M})$ is used for short. For any differentiable vector function $F(x)$, $F_{x_i}$ represents the partial derivative of $F$ with respect to $x_i$. $O(\alpha^2)$ is said to be the infinitesimal of $k$-order of $\alpha$ if $\lim_{\alpha \to 0}(O(\alpha^k)/\alpha^k) = c, c \neq 0$.Authorized licensed use limited to the terms of the applicable license agreement with IEEE. Restrictions apply.
Fig. 1. Residual NN: a feed-forward NN with L hidden layers and a shortcut connection.

II. NEURAL OBSERVER

In this article, we focus on an underlying, but not necessarily single-input single-output (SISO) or multiple-input multiple-output (MIMO), continuous-time nonlinear and uncertain system formulated by an ordinary differential equation (ODE)

$$\dot{x}(t) = f(x(t), u(t), w(t), t)$$
$$y(t) = Cx(t)$$

where \(x(t) \in \mathbb{R}^n\), \(u(t) \in \mathbb{R}^n\), and \(y(t) \in \mathbb{R}^m\) denote the state, the control input, and the control output, respectively. Generally, the external disturbances \(w \in \mathbb{R}^n\) satisfying \(\sup_{t \in T_w} \|w(t)\| < \infty\) are considered in dynamical systems. \(f \in C^1(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)\) is a nonlinear function called the total uncertainty, might be partially unknown or totally unknown. \(C \in \mathbb{L}(\mathbb{R}^n, \mathbb{R}^m)\) denotes the observation matrix of the system.

Due to uncertainty and disturbance, the direct measurement of the state would be costly and less credible. Nevertheless, the output measurement is convenient to obtain. Hence, our observation objective is to design an NN-based output-feedback observer (neural observer), such that the state of system (1) is globally observable for any initial state \(x(0)\) and total uncertainty.

To present the structure of the neural observer, we first introduce an output-feedback NN (NN) mapping \(\pi_\theta(\cdot)\) with a parameter \(\theta\). We consider \(\pi_\theta(\cdot)\) as a feed-forward NN with \(L\) hidden layers and activation functions \(\sigma(\cdot) \in C(\mathbb{R}|\sigma(0) = 0)\) that are identical in all layers. It should be noted that the input is the 0th-layer and the output is given by the \((L + 1)\)th-layer, i.e., \(\pi_\theta^{[0]}(x) = x\) and \(\pi_\theta^{[L+1]}(x) = \pi_\theta(x)\). Let \(n_l\) be the number of neurons in \(l\)th-layer. By given weights matrix \(W^l \in \mathbb{L}(\mathbb{R}^{n_{l-1}}, \mathbb{R}^{n_l})\), the \((L + 4)\)-tuple parameter \(\theta\) and the NN mapping \(\pi_\theta(\cdot)\) are defined as follows:

\[
\theta = (L, n_0, W^1, \ldots, W^{L+2}), \quad n_0 \triangleq \sum_{i=1}^{L} n_i
\]

\[
\pi_\theta^{[l]}(x) = \sigma^{[l]}\left(W^l \pi_\theta^{[l-1]}(x)\right), \quad l = 1, \ldots, L
\]

\[
\pi_\theta(x) = W^{L+1} \pi_\theta^{[L]}(x) + W^{L+2} \pi_\theta^{[0]}(x)
\]

where \(\sigma^{[l]}(x) = [\sigma(x_1), \ldots, \sigma(x_{n_l})]^T\). We note that when \(W^{L+2} \neq O\), the NN mapping is called the residual NN proposed by He et al. [32] and is shown in Fig. 1.

Our work focuses on such an NN mapping, which facilitates revealing the existence of the parameter \(\theta\) in observer \(\pi_\theta(\cdot)\) that would be demonstrated in Remark 5. Based on the defined NN, we construct the neural observer as follows:

\[
\hat{x}(t) = g(\hat{x}(t), u(t), \pi_\theta(y - \hat{y}))
\]

\[
\hat{y} = C\hat{x}
\]

where \(\hat{x} \in \mathbb{R}^n\) and \(\hat{y} \in \mathbb{R}^m\) are estimated state and estimated output, respectively. \(\pi_\theta(y - \hat{y}) = [\pi_\theta^{[T]}(y - \hat{y}), \ldots, \pi_\theta^{[T]}(y - \hat{y})]^T\) represents the NN mapping vector, where \(\theta_j = (L_i, n_i, W_i^1, \ldots, W_{i+2}^{L+2})\). Moreover, \(g\) is a known function. The block diagram of the neural observation framework is shown in Fig. 2.

Remark 1 (Keys to Neural Observers):

1) It is worth noting that the dimension of \(\hat{x}\) is designed to be no lower than the state \(x\), i.e., \(n_\hat{x} \geq n_x\), since the relatively higher dimensional information may improve the estimation performance, which is similar to the well-known kernel trick in ML algorithms (such as Gaussian process regression [33]).

2) Furthermore, the construction of the continuous function \(g\) follows the “white box modeling” information, including but not limited to the known system matrices \(A, B, C\) in the LTI system (see details in Section III-A), the integrator chain structure and the order of models \(n\) in the integrator chain system (see Section III-B), the known system matrices \(A, B, C, B_w\) in the nonlinear system (see Section III-C), and so on.

3) Moreover, in order to reduce the computational complexity, the parameters of different NNs in an NN mapping vector \(\pi_\theta(\cdot) = [\pi_\theta^{[1]}(\cdot), \ldots, \pi_\theta^{[K]}(\cdot)]^T\) could be identical. For example, \(\theta_{n_i} = \theta_{n_j}\) for \(i, j \in I\), where \(I\) is an index subset of \(\{1, \ldots, K\}\).

III. PROBLEM FORMULATION

Before presenting the main results, we first clarify the main problems that we will consider in this article. We aim to find a family of architectures, including \(\pi_\theta(\cdot)\) and \(g(\hat{x}, u, \pi_\theta(\cdot))\), for output-feedback observation tasks. Moreover, the architectures are expected to ensure the existence of NN parameter \(\theta\) and enable the implementation of neural observers. To analyze the neural observers theoretically, we propose the following definitions.

Definition 1 (Neural Observable): We suppose that there exists the NN mapping vector \(\pi_\theta(\cdot)\) such that the closed-loop system composed of (1) and (3) satisfies for all \(x(0)\), \(\hat{x}(0)\)

\[\|x(t) - T\hat{x}(t)\|_2 \to 0\]

in some sense, for example, \(t \to +\infty\) where \(T \in \mathbb{L}(\mathbb{R}^n, \mathbb{R}^m)\) with \(T_{ij} = 1, i = j; T_{ij} = 0\). Then, we say that the system (1) is neural observable.

We note that \(T = I\) if \(n_\hat{x}\) is equal to \(n_x\). And \(T\) is also applied in the following definition.

Definition 2 (Neural Exponentially Observable): We consider the aforementioned closed-loop system. If there exists

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two constants $M > 0$, $\kappa > 0$, and the NN mapping vector $\pi_0$, such that for any initial state $x(0)$ and $\hat{x}(0)$, the closed-loop system satisfies that for all $t > 0$
\[
\|x(t) - T\hat{x}(t)\|_2 \leq M \exp^{-\kappa t} [\|x(0)\|_2 + \|\hat{x}(0)\|_2]
\]
then the system (1) is called neural exponentially observable.

In this article, the canonical observation problems for three specific dynamical models of system (1) are taken into account, including the linear systems without uncertainty, the integrator chain systems and the MIMO nonlinear systems (consisting of a linear dynamic part and the general uncertainty). Based on the above-mentioned definitions, we would like to post the question: under what conditions are these systems neural observable?

### A. Neural Observers for Linear Systems

We first consider the following continuous-time LTI system without uncertainty, which is a typical case in (1):
\[
\begin{align*}
\dot{x} &= Ax + Bu, \quad x(0) = x_0 \\
y &= Cx
\end{align*}
\] (4)
where $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^{n_x}, \mathbb{R}^n)$ are known system matrices. For the neural observable problem, we employ the standard assumption.

*Assumption 1*: $(A, B)$ is controllable, and $(C, A)$ is observable.

Due to the availability of system matrices $(A, B, C)$, we can construct a neural observer corresponding to (4), which is consistent with Remark 1, as follows:
\[
\begin{align*}
\dot{\hat{x}} &= A\hat{x} + Bu + \pi_0(\hat{y} - \hat{y}) , \quad \hat{x}(0) = \hat{x}_0 \\
\dot{\hat{y}} &= C\hat{x}
\end{align*}
\] (5)
where $\pi_0(\cdot)$ is an NN mapping. The concrete neural observation diagram for the system (4) is shown in Fig. 3. Accordingly, we propose the following intuitive questions.

*Problem 1.1*: Under Assumption 1, what is the necessary condition for the system (4) to be neural exponentially observable?

We note that in analogy with classical control problems, if the system (4) is neural exponentially observable, the estimated state $\hat{x}(t)$ can be utilized to design a feedback controller for system (4). Inspired by NN controllers for discrete-time LTI systems in [27] and [28], we construct the following observer-based NN controller:
\[
u(t) = \pi_0(\hat{x}).
\] (6)

Consequently, a second question for system (4) is raised.

*Problem 1.2*: By applying the control law $u(t) = \pi_0(\hat{x})$, the problem is to find a suitable NN mapping $\pi_0(\cdot)$ such that $\lim_{t \to \infty} \|x(t)\|_2 = 0$, and in the meanwhile, the system (4) is neural exponentially observable under the given Assumption 1.

The results for the above-mentioned questions are considered the most fundamental ones, which could be served as the baselines in Sections III-B and III-C.

### B. Neural Observers for Integrator Chain Nonlinear Systems

We consider a class of SISO uncertain systems described by the following differential equation with an order of $n$:
\[
\begin{align*}
\dot{x}^{(n)}(t) &= \mathcal{F}(t, x, \ldots, x^{(n-1)}(t), w(t)) + bu(t) \\
y(t) &= x(t)
\end{align*}
\] (7)
where $\mathcal{F}(\cdot)$ is an unknown and continuously differentiable function, and $b$ is a known constant. The above-mentioned system (7) is an integral-chain system that can be rewritten as a controller canonical form
\[
\begin{align*}
\dot{x} &= Ax + B(\mathcal{F}(t, x, w) + bu), \quad x(0) = x_0 \\
y &= cx
\end{align*}
\] (8)
where $x = [x_1, \ldots, x_n]^\top$, $A = (a_{ij})_{n \times n}$ is defined by
\[
a_{ij} = \begin{cases} 
1, & i + 1 = j \\
0, & \text{else} 
\end{cases}, \quad c = [1, 0, \ldots, 0]
\]
\[
B = [0, \ldots, 0, 1]^\top.
\]
Note that $(A, B, c)$ is a canonical form representation of a chain of $n$ integrators. We note that if $\mathcal{F}(\cdot)$ in the integral-chain system is known, one can prove that (8) is a flat and controllable system [34]. When the order $n = 2$, the model (7) can describe most of the common physical systems via Newton’s second law, including the inverted pendulum model shown in Section VI. And due to the differentiability of $\mathcal{F}(\cdot)$, the system (8) is a case of (1). Then, we make some basic assumptions for nonlinear systems (7).

From the Remark 1, we can regard the matrix $A$ and $c$ as the knowledge that is used to describe the corresponding neural observer for systems (7)
\[
\begin{align*}
\dot{\hat{x}}_i &= \mathcal{F}(\hat{x}_{i+1}) + \epsilon^{n-i} \pi_0(\epsilon^{-n} (y - \hat{y})), \quad \hat{x}_0 = \hat{x}_{i,0} \\
\dot{\hat{y}} &= c\hat{x}, \quad i = 1, \ldots, n - 1
\end{align*}
\]
\[
\begin{align*}
\dot{\hat{x}}_n &= \mathcal{F}(\hat{x}_{n+1}) + \epsilon^{n} \pi_0(\epsilon^{-n} (y - \hat{y})), \quad \hat{x}_n = \hat{x}_{n,0} \\
\dot{\hat{x}}_{n+1} &= \epsilon^{-1} \pi_0(\epsilon^{-n} (y - \hat{y})), \quad \hat{x}_{n+1,0} = \hat{x}_{n+1,0} \\
\dot{\hat{y}} &= c\hat{x}, \quad n \leq T_{\epsilon,0}
\end{align*}
\] (9)
where $\epsilon$ is a positive constant, $\hat{x} = [\hat{x}_1, \ldots, \hat{x}_n]^\top$.

*Assumption 2*: [30], [35]: First, there exists a continuous function $\chi(x, w)$ such that $\sup_{t \in T_{\epsilon,0}} \|\mathcal{F}(\cdot, \nabla \mathcal{F})\|_2 \leq \chi(x, w)$. 

2) Second, there exists a bounded control \( u(t) \) such that 
\[ \sup_{t \in \mathbb{T}_{x_0}} (|\mathbf{x}(t)|_2 + |u(t)|) < \infty. \]

We note that the first assumption imposed the differentiability of noise \( w \) and uncertainty \( F \) with respect to time. Based on the neural observer, we note that the simplest way to satisfy Assumption 2 is to design the bounded control in the linear form 
\[ u(t) = (\rho \sum_{\sigma=1}^{n} k_{\sigma} \text{sat}_{M_{\sigma}}(\rho_{\sigma}^\gamma_{\sigma} \hat{x}_{\sigma}(t)) - \text{sat}_{M_{\sigma}}(K_{\sigma} + \hat{w}_{\sigma}(t))) / (b) \]
with parameters \( \rho, k_{\sigma}, \) and \( M_{\sigma}, \) where the control gain \( K = [k_1, \ldots, k_n] \) is designed by the Hurwitz matrix \( A + c_0 K \) with \( c_0 = [0, \ldots, 1]^T, \) the details of which can be found in [36].

Therefore, we informally introduce the observation problem for systems (7).

\textbf{Problem 2.1:} Consider the neural observer-based closed system (7) and (9), with the Assumption 2, we need to investigate the necessary conditions for the underlying result.

1) the system (7) is neural observable.

\textbf{Remark 2:} We need to point out that the formulation of the neural observer (9) resembles the extended state observer (ESO) in ADRC [35], [37]. But in fact, the construction of nonlinear functions in ESO is complicated in industrial processes. Hence, due to the approximating capability of NNs, we can take advantage of this property to relieve these pressures.

\subsection*{C. Neural Observers for MIMO Systems}

As another case of system (1), the following MIMO nonlinear system (composed of a linear dynamic and general uncertainty) is taken into account

\[ \begin{align*}
\dot{x} &= Ax + Bu + B_wK(x, w, t), \quad x(0) = x_0 \\
y &= Cx 
\end{align*} \tag{10} \]

where \( A \) and \( B \) are defined in the same way as those in (4), \( K \in C^1(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}; \mathbb{R}^n) \) represents the uncertainty with respect to \( x(t), w(t), \) and \( t. \) \( B_w \in L(\mathbb{R}^n, \mathbb{R}^n) \) is a known matrix.

\textbf{Remark 3:} Since system (8) is an affine control system, system (8) is one case of (10). In other words, (10) is a more general formulation compared with (7). We also note that (10) is not restricted to the integral chain form and may be subject to the mismatched uncertainty and disturbances [38].

\textbf{Assumption 3:} 1) \( A, C, \) and \( B_w \) satisfy an \textit{extending observable condition}, that is,

\( (C, A) \triangleq \left[ (C, O), \begin{bmatrix} A & Bw \\ O & O \end{bmatrix} \right] \) is observable.

2) \( \sup_{t \in \mathbb{T}_{x_0}} (|K|_2, |\nabla K|_2) \leq \sigma(x, w), \) where \( \sigma(x, w) \in C(\mathbb{R}^{n+1}, \mathbb{R}). \)

3) There exists a bounded control law \( u = u(t) \) such that the state \( x(t) \) is bounded.

Now, we give an example to illustrate the \textit{extending observable condition} and a necessary condition of extending observability.

\textbf{Example 1:} We consider that 
\[ A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad B_w = [1, 0]^T \]
and \( C = [1, 0]. \) It is easy to check that 
\[ \begin{bmatrix} C^\top, (CA)^\top, (CA^2)^\top \end{bmatrix} = 3 \iff (C, A) \text{ is observable}. \]

\textbf{Proposition 1:} \( (C, A) \) is observable if \( A, C, \) and \( B_w \) satisfy the \textit{extending observable condition} defined in Assumption 3.

\textbf{Proof:} The proof is given in Appendix I.

\textbf{According to Remark 1, we hold} \( (A, B, C) \) and \( B_w \) as the knowledge to design the neural observer, as shown in the following:

\[ \begin{align*}
\dot{x}_1 &= B_w \hat{x}_2 + A\hat{x}_1 + Bu + \pi_0(x)(\epsilon^{-1}(y - \hat{y})), \quad \hat{x}_1(0) = \hat{x}_{1,0} \\
\dot{x}_2 &= \epsilon^{-1}\pi_0(x)(\epsilon^{-1}(y - \hat{y})), \quad \hat{x}_2(0) = \hat{x}_{2,0} \\
\hat{y} &= C\hat{x} 
\end{align*} \tag{11} \]

where \( \epsilon \) is positive, and \( \hat{\mathbf{x}} = [\hat{x}_1^T, \hat{x}_2^T]^T. \) In the end, a problem is raised accordingly.

\textbf{Problem 3.1:} We need to seek out the necessary condition for system (10) to be neural observable under Assumption 3.

\section*{IV. NN Representation and NN Mapping Vector}

We will present the main theorems of this work in a later section: Theorems 1–4, which directly solve each of the problems mentioned earlier. As a necessary prelude, however, we introduce the following two definitions. The first definition is about the isolation of nonlinear activation function from the linear operation of NNs defined in (2) and QCs for activation functions, similarly done in [22], [27], and [39], respectively.

In the second, we define the concept of NN mapping vector and QCs for NN mapping vector.

\subsection*{A. NN Isolation and QCs for Single NN Mapping}

For a specific NN mapping \( \pi_0 \) and the input \( x \in \mathbb{R}^{\text{input}}, \) we define \( w^0 = x \) and \( \xi^i = W^i w^{n-1}, w^0 = \sigma^{n_1}(\xi^i), \ i = 1, \ldots, L. \) By collecting the input and output of all activation functions, we denote two \( n_n \)-dimensional vectors \( \xi_\sigma \) and \( w_\sigma \) as follows:

\[ \xi_\sigma \triangleq \begin{bmatrix} \xi^1 \\ \vdots \\ \xi^L \end{bmatrix}, \quad w_\sigma \triangleq \begin{bmatrix} w^1 \\ \vdots \\ w^L \end{bmatrix}. \]

Then, by recalling that \( \pi_\sigma(x) = W^{L+1}w^L + W^{L+2}\pi_0^{[L]}(x) \) and \( \pi_0^{[L]}(x) = x, \) the NN mapping \( \pi_0 \) can be rewritten into

\[ \begin{bmatrix} \pi_\sigma \\ \xi^t \end{bmatrix} = \begin{bmatrix} W^{L+1}w^L & O & \cdots & O \\ W^{L+2} & O & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & \cdots & O & W^L \end{bmatrix} \begin{bmatrix} x \\ w^1 \\ \vdots \\ w^L \end{bmatrix} \tag{12} \]

which can be abbreviated in the following formulation:

\[ \begin{bmatrix} \pi_\sigma(x) \\ \xi_\sigma \end{bmatrix} = \begin{bmatrix} N_{\pi_x} & N_{\pi_w} \\ N_{\xi_x} & N_{\xi_w} \end{bmatrix} \begin{bmatrix} x \\ w_\sigma \end{bmatrix}. \tag{13} \]

Now, we define the two following linear mappings:

\[ R_\pi \triangleq \begin{bmatrix} I & 0 \\ N_{\pi_x} & N_{\pi_w} \end{bmatrix}, \quad R_\xi \triangleq \begin{bmatrix} N_{\xi_x} & N_{\xi_w} \\ O & I \end{bmatrix} \tag{14} \]
then derive the corresponding linear transformations for 
\[ [x^T, w_\sigma^T]^T \]
\[
\begin{bmatrix}
x \\
\pi_\sigma(x)
\end{bmatrix} = R_\pi \begin{bmatrix}
x \\
w_\sigma
\end{bmatrix}, \quad \begin{bmatrix}
\xi_\sigma \\
w_\sigma
\end{bmatrix} = R_\xi \begin{bmatrix}
x \\
w_\sigma
\end{bmatrix}.
\]

To avoid confusion, we must emphasize that two identical matrices \( I \) in \( R_\pi \) and \( R_\xi \) belong to different spaces of linear mappings, \( \mathcal{L}(\mathbb{R}^{\text{input}}) \) and \( \mathcal{L}(\mathbb{R}^{\text{input}}) \), respectively.

Remark 4: Due to the existence of a shortcut connection shown in Fig. 1, the matrix \( N_{\pi x} \) in (14) is a nonzero matrix that makes a difference with [27]. In addition, the nonzero matrix \( N_{\pi x} \) plays a crucial role in neural observability, which would be certified in Proposition 2.

Next, we deal with another thorny difficulty in analyzing NNs, which is the composition of nonlinear activation functions. The key is to remove the nonlinearity of activation functions but preserve some geometrical properties.

Consider the activation function \( \sigma(\cdot) \in \mathcal{C}(\mathbb{R})(\sigma(0) = 0) \), then, the function is said to be sector bounded in sector \([\alpha, \beta] \) with \( \alpha \leq \beta < \infty \) if the following inequality holds for all \( s \in \mathbb{R} \):
\[
(\sigma(s) - \alpha s)(\beta s - \sigma(s)) \geq 0.
\]

Intuitively, the above-mentioned inequality implies that the function \( y = \sigma(s) \) lies in the open region of \( y = \alpha s, y = \beta s \), and the origin. For the sector bounded, as mentioned earlier, the nonlinear functions commonly used in practice [37] are of the following form:
\[
\text{fal}(s, \gamma, \delta) = \begin{cases} |s|^\gamma \text{sgn}(s), & |s| > \delta \\ \delta^{1-\gamma}, & |s| \leq \delta \end{cases}
\]
which also satisfies the sector boundedness illustrated in Fig. 4.

In other words, when we take the NN in the neural observer as a single layer, with no shortcut connection \( (W_{L+2} = O) \), and use the \( \text{fal}(s, \gamma, \delta) \) function as the activation function, in this sense, then the extended state observers with the \( \text{fal}(s, \gamma, \delta) \) function can be included in neural observers.

Next, the activation functions \( \sigma(\cdot) \) at each hidden layer are sector bounded in sector \([\alpha_i, \beta_i] \), \( i = 1, \ldots, n_\sigma \), respectively. By denoting sector vectors \( \alpha_\sigma = [\alpha_1, \ldots, \alpha_{n_\sigma}] \) and \( \beta_\sigma = [\beta_1, \ldots, \beta_{n_\sigma}] \), the QCs for one NN mapping are provided as follows.

**Lemma 1 ([27]):** Let \( \alpha_\sigma, \beta_\sigma \in \mathbb{R}^{n_\sigma} \) be defined above with \( \alpha_{n_\sigma} \leq \beta_{n_\sigma} \). If \( \lambda_\sigma \in \mathbb{R}^{n_\sigma} \) and \( \lambda_\sigma \geq 0 \), then
\[
\Psi_\sigma \triangleq \begin{bmatrix}
\text{diag}(\beta_\sigma) - I \\
-\text{diag}(\alpha_\sigma) I
\end{bmatrix}, \quad M_\sigma(\lambda_\sigma) \triangleq \begin{bmatrix}
O & \text{diag}(\lambda_\sigma)
\end{bmatrix}.
\]

**B. NN Isolation and QCs for NN Mapping Vector**

Whereafter, we try to isolate the nonlinearity of NN mapping vector \( \pi_\theta(x) = [\pi_{\theta_1}(x), \ldots, \pi_{\theta_K}(x)]^T \) shown in Fig. 5, which consists of \( K \) NN mappings with different parameters \( \theta_k \).

We denote \( w_0^0 = x \) and \( \xi_k^i = W_k^i w_k^{i-1}, w_k^i = \sigma^{|i|}(\xi_k^i), i = 1, \ldots, L_k, k = 1, \ldots, K \), and two \( n_\sigma \triangleq \sum_{i=1}^{L_k} n_i \) dimensional vectors \( \xi_{\pi_k} \) and \( w_{\pi_k} \) as follows:
\[
\xi_{\pi_k} \triangleq \begin{bmatrix}
\xi_{\pi_k}^1 \\
\vdots \\
\xi_{\pi_k}^{L_k}
\end{bmatrix}, \quad w_{\pi_k} \triangleq \begin{bmatrix}
w_{\pi_k}^1 \\
\vdots \\
w_{\pi_k}^{L_k}
\end{bmatrix}.
\]

With the help of (13), we derive the following transformation:
\[
\begin{bmatrix}
\pi_{\theta_k}(x) \\
\xi_{\pi_k}
\end{bmatrix} = \begin{bmatrix}
N_{\pi_k x} & N_{\pi_k w_k} \\
N_{\pi_k x} & N_{\pi_k w_k}
\end{bmatrix} \begin{bmatrix}
x \\
w_{\pi_k}
\end{bmatrix}.
\]

We suppose \( \xi_{\pi_k}^K = [\xi_{\pi_k}^1, \ldots, \xi_{\pi_k}^K]^T \) and \( w_{\pi_k}^K = [w_{\pi_k}^1, \ldots, w_{\pi_k}^K]^T \), and have
\[
\begin{bmatrix}
\pi_{\theta_k}(x) \\
\xi_{\pi_k}^K
\end{bmatrix} = \begin{bmatrix}
\tilde{N}_{\pi_k x} & \tilde{N}_{\pi_k w_k} \\
\tilde{N}_{\pi_k x} & \tilde{N}_{\pi_k w_k}
\end{bmatrix} \begin{bmatrix}
x \\
w_{\pi_k}^K
\end{bmatrix}.
\]

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where the block matrices equal to
\[
\begin{bmatrix}
N_{\pi w} \\
N_{\pi x}
\end{bmatrix}, \quad \tilde{N}_{\pi w} = \text{diag}(N_{\pi w1}, \ldots, N_{\pi w_k})
\]
\[
\begin{bmatrix}
N_{\xi w} \\
N_{\xi x}
\end{bmatrix}, \quad \tilde{N}_{\xi w} = \text{diag}(N_{\xi w1}, \ldots, N_{\xi w_k})
\]
Then, it is easy to verify that the following transformations derived from (16) are held on:
\[
\begin{bmatrix}
x \\
\pi_\theta(x)
\end{bmatrix} = \tilde{R}_\pi \begin{bmatrix}
x \\
K \sigma w_\sigma
\end{bmatrix}
\]
\[
\begin{bmatrix}
\xi^K_w \\
\xi^K_x
\end{bmatrix} = \tilde{R}_\xi \begin{bmatrix}
\xi^K_w \\
\xi^K_x
\end{bmatrix}
\]
where
\[
\tilde{R}_\pi \triangleq \begin{bmatrix}
I & \frac{O}{N_{\pi w}} \\
N_{\pi x} & \frac{O}{N_{\pi w}}
\end{bmatrix}, \quad \tilde{R}_\xi \triangleq \begin{bmatrix}
\frac{N_{\xi x}}{O} & \frac{N_{\xi w}}{1}
\end{bmatrix}
\]

A. Neural Observers for Systems Without Uncertainty

For the intuitiveness and simplicity of the arguments, we start analyzing the neural observability of linear systems. First, we formally state our main result for Problem 1.1 in the following theorem.

**Theorem 1:** We consider an NN mapping \( \pi_\theta \) with \( \theta \) and a vector \( \lambda_\sigma \) that satisfies the QC in Lemma 1. We update \( W_{1}^{L+2} \) and \( W_{1} \) in (14) to \( W_{1}^{L+2} C \) and \( W_{1} C \), respectively. If there exists a matrix \( P \in \mathbb{L}(\mathbb{R}^n) \) and \( P \succ 0 \) such that
\[
\begin{bmatrix}
A^T P + P A^T & \frac{P}{O} \\
\frac{P}{O} & \frac{P}{O}
\end{bmatrix} + \tilde{R}_\sigma \Psi_\sigma \tilde{R}_\sigma \succ 0
\]
then the LTI system (4) is neural exponentially observable, equivalently
\[
\|x(t) - \hat{x}(t)\| \leq M \exp^{-\lambda t} \|x(0)\| + \|\hat{x}(0)\|
\]
where \( R_\sigma \) and \( \tilde{R}_\sigma \) and \( \Psi_\sigma \) and \( M_\sigma(\lambda_\sigma) \) are defined in (14), (15), respectively.

**Proof:** The proof is provided in Appendix III.

Furthermore, it is not difficult to imply that the neural exponential observability of system (4) and the existence of \( \theta \) in \( \pi_\theta \) depend heavily on the existence of \( P \), i.e., the solution of LMI (19). Hence, a natural subquestion is: under what conditions does the solution \( P \) in LMI exist? To the best of our knowledge, this question has not been effectively solved in the NN-based closed-loop control (for example, [27, Th. 1] and [28, Th. 1]) at present. Therefore, we present the following proposition to answer this subquestion.

**Proposition 2:** We set \( \alpha_\sigma = 0_\sigma \), \( \tilde{A} = A + N_{\pi x} \) with \( N_{\pi x} = W_{1}^{L+2} C \), \( Q \in \mathbb{L}(\mathbb{R}^n) \) is a diagonal matrix with positive diagonal entries, and \( P = \int_0^\infty e^{A^T t} Q e^A dt \). We suppose that there exists \( \lambda_\sigma > 0_\sigma \) such that the following holds.
1) \( \|M_1\|_\infty \leq \min_i (q_i) \), where \( M_1 = -P N_{\pi w} - N_{\pi x}^T R_1 \) with \( R_1 = \text{diag}(\lambda_\sigma \circ \beta_\sigma)^i \), and \( q_i \) is the ith diagonal entry in \( Q \).
2) \( \|M_2\|_\infty + \|M_2\|_\infty \leq 2 \min_i (\lambda_{o,i}) \), where \( \lambda_{o,i} \) is the ith element of \( \lambda_\sigma \), \( M_2 = R_1 N_{\pi w} + N_{\pi x}^T R_1 \).

Then, LMI (19) has a solution \( P \) if and only if \( (C, A) \) is observable.

**Proof:** The proof is provided in Appendix I.

**Remark 5:** It should be noticed that \( W_{1}^{L+2} \) in the shortcut connection of the NN plays an essential role in the construction of the above-mentioned solution \( P \) by pole assignments. Moreover, from 1) and 2), the solution \( W_{1}^{L+2} \) is Hurwitz. Then, we can take \( \tilde{A} = A \) by setting \( W_{1}^{L+2} = O \) from the NN mapping \( \pi_\theta \), which means that the residual NN defined in (2) will degenerate to a fully connected NN. Therefore, Corollary 1 shows that the LMI (19) solution exists in this case.

**Corollary 1:** Let Assumptions 1 and 2 be still satisfied. We set \( \alpha_\sigma = 0_\sigma \) but \( A = A \), and corresponding matrices \( Q \), \( P \) defined in Proposition 2. Then, \( P \) is a solution for LMI (19) if and only if \( A \) is Hurwitz.

**Proof:** The proof is a direct extension of Proposition 2.
The next theorem gives the necessary conditions to achieve the control target in Problem 1.2 by using the measurement \( \hat{x}(t) \) from (4) and (5), by utilizing the NN controller of the form \( u(t) = \pi_{\theta}(\hat{x}) \), it is not difficult to obtain

\[
\begin{align*}
\dot{\hat{x}} &= Ax + B\pi_{\theta}(\hat{x}) \\
\dot{\hat{x}} &= A\hat{x} + B\pi_{\theta}(\hat{x}) + \pi_{\theta}(C(\hat{x} - x)).
\end{align*}
\]

(20)

By denoting \( x_1 \triangleq x, x_2 \triangleq \hat{x} - x \), and \( x^T = [x_1^T, x_2^T] \), equation (20) turns into

\[
\frac{dx}{dt} = \hat{A}x + v(x)
\]

where \( \hat{A} = \text{diag}(A, A) \) and

\[
v(x) = \begin{bmatrix} B\pi_{\theta}(I, I|x) \\
\pi_{\theta}(O, C|x) \end{bmatrix}.
\]

Correspondingly, as \( K = 2 \), we also treat \( x(t) \) as an input variable of the NN mapping vector \( \pi_{\theta} = [\pi_{\theta}^T, \pi_{\theta}^T]^T \).

**Theorem 2:** We consider two NN mappings \( \pi_{\theta_1} \) and \( \pi_{\theta_2} \) with parameters \( \theta_1 = (L_1, n_m, W^1, \ldots, W^{l_1+2}), i = 1, 2 \). Let parameter \( K \) in Lemma 2 be equal to 2, and \( T_1^1, T_2^1, T_2^2 \) in (17) are equal to \( [I, I], B, [O, C] \), and \( I \), respectively. We suppose that there exists a matrix \( \hat{P} \in \mathcal{L}(\mathbb{R}^{n_0}) \) and \( \hat{P} \succ O \) such that

\[
\begin{bmatrix} \hat{R}_x^\top & \hat{A}^\top \hat{P} + \hat{P} \hat{A}^\top \hat{P} \end{bmatrix} \begin{bmatrix} \hat{R}_x & \Psi(2)^\top \mathbf{M}(2) \Psi(2) \end{bmatrix} \hat{R}_x < 0
\]

(21)

where \( \hat{R}_x \) and \( \hat{R}_x \) and \( \Psi(2) \) and \( \mathbf{M}(2) \) are defined in (17), and (18) as \( K = 2 \), respectively. Then, the LTI system (4) is neural exponentially observable and globally exponentially stable.

**Proof:** The proof is provided in Appendix III.

B. Uncertainty Is Effectively Dealt by Neural Observers

We construct an extended state

\[
x_{n+1}(t) = \mathcal{F}(t, x, u)
\]

for (8) and then redefine system (8) as follows:

\[
\begin{bmatrix} \hat{y} \\ y \end{bmatrix} = \begin{bmatrix} \tilde{A} \hat{x} + \tilde{L}(t, x, u, w) \\ \tilde{c} \end{bmatrix}
\]

(22)

where \( \hat{x} = [x^T, x_{n+1}]^T \), \( \tilde{A} = (a_{ij}(n+1) \times (n+1)) \) is defined by

\[
a_{ij} = \begin{cases} 1, & i + 1 = j \\ 0, & \text{else} \end{cases}
\]

\( \tilde{c} = [1, 0, \ldots, 0] \)

Correspondingly, the output of neural observer (9) is redenfined as \( \tilde{y} = \tilde{c} \hat{x}, \tilde{x}_{n+1} \).

**Theorem 3:** We consider \( n + 1 \) NN mappings \( \pi_{\theta} \) with parameters \( \theta = (L_1, n_m, W^1, \ldots, W^{l_1+2}), i = 1, \ldots, n + 1 \). We assume that the following holds.

1) \( K = n + 1 \) in Lemma 2, and \( T_1^1 = \bar{c}, T_2^1 = \bar{c}, i = 1, \ldots, n + 1 \).

2) There exists a matrix \( P \in \mathcal{L}(\mathbb{R}^{n+1}) \) and \( P \succ O \) such that

\[
\begin{bmatrix} \tilde{R}_x^\top & \tilde{A}^\top P + PA^\top - P \end{bmatrix} \begin{bmatrix} \tilde{R}_x \\ \Psi(n+1)^\top \mathbf{M}(n+1) \Psi(n+1) \end{bmatrix} < 0.
\]

(24)

Then, we have the following results.

1) Neural observability: for all \( x(0) \in \mathbb{R}^n, \|x(t) - \hat{x}(t)\| \rightarrow 0 \) as \( \epsilon \rightarrow 0^+ \) for \( t \in T_{x,T} \) with \( T > 0 \).

2) Total uncertainty \( x_{n+1} = F(t, x, w) \) can be measured by \( \tilde{x}_{n+1} \).

**Proof:** The proof is provided in Appendix III.

Remark 6: For (24), due to the observability of \( \tilde{C}, \tilde{A} \), we can select \( W^{l+2} = [W_1^{l+2}, \ldots, W_{n+1}^{l+2}]^T \) as a pole assignment matrix such that \( \tilde{A} + \tilde{R}_x \tilde{R}_x = \tilde{A} + W^{l+2} \tilde{c} \) is Hurwitz, where \( \tilde{R}_x \) is a block matrix of \( \tilde{R}_x \). Then, by a similar analysis with Propositions 2 and 3, it is not difficult to check that LMI (24) has solutions under some given conditions.

To decrease the computational complexity for LMI (24) and avoid the consequences of sparsity \([40, 41]\), we can take that the NN mappings in (9) are identical, i.e., \( \pi_{\theta}(\cdot) = \pi_{\theta}(\cdot) \).
i = 1, . . . , n + 1. Moreover, the gains of the NN mapping are equal to $e^{n+1-i} b_i$, that is,

\[
\begin{align*}
\hat{x}_i &= \hat{x}_{i+1} + \epsilon^{n-i} \theta_i (\epsilon^{-n}(y - \hat{y})) + \bar{u}_i, i = 1, . . . , n - 1 \\
\hat{x}_n &= \bar{x}_{n+1} + \epsilon^{n-1} \theta_i (\epsilon^{-n}(y - \hat{y})) + b_i, \hat{x}_n(0) = \bar{x}_0, \\
\hat{x}_{n+1} &= \epsilon^{-1} \theta_i (\epsilon^{-n}(y - \hat{y})), \hat{x}_{n+1}(0) = \bar{x}_{n+1}, \\
\hat{y} &= \epsilon \hat{x}.
\end{align*}
\]

Then, we present the following corollary to solve the sparsity of LMI (24).

**Corollary 2:** We consider this NN map $\pi_\theta$ with one parameter $\theta = (L, n_a, W^1, . . . , W^{L+2})$ and reassume the following.

1) $K = 1$ in Lemma 2, and $T_1^1 = \tilde{C}, T_1^2 = 1$ in (17).
2) Let $b$ be the vector $[b_1, . . . , b_{n+1}]^\top$. We suppose that there exists a matrix $P \in \mathcal{L}(R^{n+1})$ and $P > O$ such that

\[
\begin{align*}
\hat{R} &= \begin{bmatrix} \hat{A}^\top P + P \hat{A} - P C & -P b \\
-P & O \end{bmatrix} \hat{R} + \hat{R}_1 \Psi(1) M(1) \Psi(1) \hat{R}_2 < 0
\end{align*}
\]

where $\hat{R}_1$, $\hat{R}_2$, $\Psi(1)$, and $M(1)$ are defined in (17) and (18).

Then, we can still obtain three results in Theorem 3, including neural observability and the measurement of the total uncertainty $F(t, x, w)$.

**Proof:** By directly extending the proof of Theorem 3, the proof of Corollary 2 can be obtained trivially.

---

**C. General Uncertainty in Linear Dynamics Can Be Dealt by Neural Observers**

Before showing Theorem 4 for Problem 3.1, we introduce a necessary lemma. Furthermore, finally, we present the last theorem for systems (10).

**Lemma 3:** $(C, A_i)$ is observable if $A$, $C$, and $B_w$ satisfy the extending observable condition defined in Assumption 3, where $C$ is defined in Assumption 3 (1), and $A_i = \begin{bmatrix} \epsilon A & B_w \\ O & 0 \end{bmatrix}$ with $\epsilon > 0$.

**Proof:** The proof is given in Appendix II.

**Theorem 4:** We consider two NN mappings $\pi_{\theta_i}$ and $\pi_{\theta_j}$

\[
\begin{align*}
\theta_i &= (L_i, n_a, W^1, . . . , W^{L+2}), i = 1, 2. \text{ Let parameter } K \text{ in Lemma } 2 \text{ be equal to } 2, \text{ and } T_i^1 \text{ and } T_j^2 \text{ in (17) are equal to } C \text{ and } I, \text{ respectively. We suppose that there exists a positive definite matrix } P \in \mathcal{L}(R^{n+1}) \text{ such that }
\end{align*}
\]

\[
\begin{align*}
D = \begin{bmatrix} \hat{R}_1 \hat{R}_2 \end{bmatrix}^\top \begin{bmatrix} \hat{A}_i \hat{C} & \hat{P} \\ \bar{P} & -O \end{bmatrix} \begin{bmatrix} \hat{R}_1 \hat{R}_2 \end{bmatrix} + \begin{bmatrix} \hat{R}_1 \hat{R}_2 \end{bmatrix} \begin{bmatrix} \Psi(2) \begin{bmatrix} \bar{M} \end{bmatrix} \Psi(2) \end{bmatrix} < 0.
\end{align*}
\]

Under Assumption 3 aforementioned, then for $\epsilon > 0$, the system (10) is neural observable in the following sense.

1) \( \lim_{t \to \infty} \|x_i(t) - \hat{x}_i(t)\| = 0 \) for all $t \in T_{2a}$, $a > 0$.

2) \( \sup_{t \to \infty} \|x_i(t) - \hat{x}_i(t)\| \leq O(\epsilon^{2-i}) \).

---

**Remark 7:** For (26), since $(C, A_i)$ is observable from Lemma 3, we can construct $W^{L+2} = (W^{L+2})^\top, (W^{L+2})^\top$ such that $A_i + \hat{R} \approx A_i + W^{L+2}C$ is Hurwitz. Subsequently, one can verify the existence of LMI (26) solutions $P$ via a similar process with Propositions 2 and 3.

---

**VI. NUMERICAL EXPERIMENTS**

We apply the neural observers for three different dynamical models to demonstrate the effectiveness of our proposed analyses. In these examples, the LMIs (21), (24), and (26) are solved using the LMI Toolbox in MATLAB R2021a.

**A. Linearized Aerodynamic Models of the X-29A Aircraft**

We implement the neural control framework combining the neural observer (5) and the NN controller (6) to the X-29A aircraft, which is formulated in the following state-space form:

\[
\begin{align*}
\dot{x} &= A x + B u + w, \quad x(0) \in \mathbb{R}^4 \\
y &= C x + v
\end{align*}
\]

where the nominal system matrices $A$, $B$, and $C$ satisfying Assumption 1 can be obtained from [42, Table 9]: $w \sim N(0, (1/10)I)$ and $v \sim N(0, (1/10)I)$ are process noises. The NNs $\pi_{\theta_i}, i = 1, 2$ in (5) and (6) are both parameterized by three hidden layers ($n_1 = n_2 = n_3 = 3$) with ReLU/tanh as the activation function for all layers. We further perform a comparison between neural observers with different activation functions ($\hat{x}_{i,R}$ and $\hat{x}_{i,T}$ denote ReLU and tanh activations, respectively) and the Kalman filter ($\hat{x}_{i,K}$), where $i = 1, . . . , 4$. All initial values are set to be $\hat{x}(0) = 0$. The system response and the output of the neural observer are depicted in Fig. 6. It is shown that $x_i(t), i = 1, . . . , 4$ all converge to a tiny neighborhood of 0 and are well estimated by $\hat{x}_{i,R}(t), \hat{x}_{i,T}(t), \hat{x}_{i,K}(t), i = 1, . . . , 4$. In addition, the different choices of activation functions in neural observers only have a slight impact on the observation in this scene. It is
worth mentioning that the state can be also estimated by \( \hat{x}_{i,LMI}(t), i = 1, \ldots, 4 \), which are generated from a neural observer (5) with \( \hat{x}(0) = \Theta_4 \) that dissatisfies the LMI (21), indicating that the LMI criterion (21) for neural observers is overly conservative.

B. Second-Order Nonlinear Model of the Inverted Pendulum

Next, to show the effectiveness of neural observers for integrator chain nonlinear systems, we consider the control of the nonlinear inverted pendulum system formulated by \( \ddot{\theta}(t) = (mgI\sin(\theta(t)) - \zeta_0\dot{\theta}(t) + u(t) + w(t)/mI^2) \), where \( \theta(t) \) is the angular position (rad), and \( w(t) = \sum_{i=1}^n a_i \sin(b_i t + \phi_i) \) is the external disturbance. By denoting \( x_1 = \theta \) and \( x_2 = \dot{\theta} \), we rewrite state-space form for inverted pendulum system

\[
\begin{align*}
\dot{x}_1 &= x_2, \quad x_1(0) = x_{1,0} \\
\dot{x}_2 &= \frac{mgI \sin(x_1) - \zeta_0 x_2 + u(t) + w(t)}{mI^2}, \quad x_2(0) = x_{2,0}
\end{align*}
\]

where \( m, l, \) and \( \zeta_0 \) represent the mass (kg), the length (m), and the friction coefficient (Nms/rad), respectively. However, \( m = m_0 + \delta_m m_0 \) and \( l = l_0 + \delta_l l_0 \) are the uncertain parameters, where \( m_0 \) and \( l_0 \) denote the nominal value and \( \delta_m, \delta_l \) are parameter perturbation coefficients sketching the uncertainty of parameters. Without loss of generality, we consider that \( m_0 = 1, \quad l_0 = 1, \quad [\delta_m, |\delta_l|] \in [0, 0.1], \quad \zeta_0 = 0.5 \), and \( w(t) = 0.1\sin(4\pi t) + 0.2\cos(2\pi t) + 0.2\sin(3\pi t - \pi/7) \). The following neural observer is designed without involving the parameters \( m, l, \) and \( \zeta_0 \):

\[
\begin{align*}
\hat{x}_1 &= \hat{x}_2 + \epsilon^1 \pi_0 \epsilon^{-2}(y - \hat{y}), \quad \hat{x}_1(0) = 0 \\
\hat{x}_2 &= \hat{x}_3 + \pi_0 \epsilon^{-2}(y - \hat{y}) + bu, \quad \hat{x}_2(0) = 0 \\
\hat{x}_3 &= -\epsilon^{-1} \pi_0 \epsilon^{-2}(y - \hat{y}), \quad \hat{x}_3(0) = 0
\end{align*}
\]

and the feedback control law \( u(t) \) (Nm) is designed by \( u(t) = \rho \sum_{i=1}^n k_i \text{sat}_{\rho\pi}(\epsilon^{-1}(y - \hat{y})) - \text{sat}_{\rho\pi}(\epsilon^{-1}(y - \hat{y})) \).

In the corresponding neural observer (9), we design that: 1) the gain \( \epsilon = 0.1 \) and 2) the NN \( \pi_{\theta}, i = 1, \ldots, 3 \) are all parameterized by two hidden layers \( (n_1 = 3 \text{ and } n_2 = 2) \) with tanh as the activation function for all layers. As for the control law, we set \( \rho = 1, \quad k_1 = -25, \quad k_2 = -10, \quad M_1 = M_2 = M_3 = 10 \) (more details about the parameters setting can be seen in [36]). We also compare the above-mentioned neural observer with the gain scheduled Luenberger observers (GSLOs) [43], which is designed by involving \( m_0 \) and \( l_0 \) and selecting \( \delta_m = \delta_l = 0.1 \) and zero initial value.

As illustrated in Figs. 7 and 8, we can conclude that the neural observer is not only more effective than the GSLO in tracking the states \( x_1 \) and \( x_2 \) but also can estimate the extended state \( x_3 \) (total disturbance), which is not possible for GSLO.

C. Dynamics of the Four-Wheel Steering Vehicle

Finally, we implement the proposed neural observers (11) to the four-wheel steering vehicle, which is modeled as a linear dynamic with a general uncertainty [44]

\[
\begin{align*}
\dot{x} &= Ax + Bu + B_w K(x, u, t), \quad x(0) = x_0 \\
y &= Cx
\end{align*}
\]

where \( x = [e, \dot{e}, \Delta \psi, \dot{\Delta \psi}] \), with \( (e, \Delta \psi) \) are defined as the perpendicular distance to the lane edge and the angle between the tangent to the straight section of the road; \( C = I \); and \( A, B, B_w, \text{and } K(x, u, t) \) are defined as follows:

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & \frac{C_{af} + C_{ar}}{mU} & -\frac{C_{af} + C_{ar}}{mU} \\
0 & 0 & \frac{aC_{af} - bC_{ar}}{I_U} & -\frac{aC_{af} - bC_{ar}}{I_U} \\
0 & 0 & \frac{aC_{af} + bC_{ar}}{I_U} & \frac{aC_{af} + bC_{ar}}{I_U}
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & \frac{C_{af}}{m} & \frac{C_{ar}}{m} \\
0 & 0 & \frac{bC_{af}}{I_T} & \frac{bC_{ar}}{I_T} \\
0 & 0 & \frac{aC_{af}}{I_T} & \frac{aC_{ar}}{I_T}
\end{bmatrix}, \quad B_w = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
K(\cdot) = \begin{bmatrix}
0.1\sin(4\pi t) + 0.3\cos(2\pi t) + \frac{aC_{af} - bC_{ar} - ml^2}{mp} \\
0.2\cos(5\pi t) + 0.1\cos(6\pi t) + \frac{aC_{af} + bC_{ar}}{I_T}
\end{bmatrix}
\]

For simplicity, we denote that \( x = [x_1, \ldots, x_4]^T \). The parameters \( C_{af}, C_{ar}, m, U, I_T, a, \) and \( b \) represent the front cornering stiffness (N/rad), rear cornering stiffness (N/rad), mass (kg), longitudinal velocity (m/s), the moment of inertia (kgm²), and distances from vehicle center of gravity to the front axle and rear axle, respectively, which are chosen to the nominal values obtained from [44, Appendix A]. The constant road curvature \( \rho_R \) in \( K(\cdot) \) can be chosen to be 400 (meters).

Likewise, the corresponding neural observer (11) is designed by: 1) the NNs \( \pi_{\theta}, i = 1, 2 \) are parameterized by
three hidden layers \((n_1 = n_2 = n_3 = 3)\) with leaky ReLU as the activation function for all layers; 2) the gain \(\epsilon = 0.1\); and 3) \(\hat{x}(0) = \theta_4, \hat{x}_2(0) = \theta_2\). The control input is given by the output-feedback control \(u = Gy\), where \(G\) is designed by the matrix \(A + BG\) and is Hurwitz. Hence, it is easy to check the boundedness of the state \(x\) and the input \(u\). In addition, \((C, A_x)\) is observable, indicating the system complies with the whole Assumption 3. Furthermore, since \(r(CB_w) = r(B_w)\), we can apply the unknown input observer (UIO) for comparison with the neural observer [45], which is described as

\[
\begin{aligned}
\dot{z} &= Nz + Ly + Gu, \quad z(0) = \theta_4 \\
\hat{x}_{\text{UIO}} &= z - Ey
\end{aligned}
\]

where the matrices \(N, L, G,\) and \(E\) are given by [45, (6)–(12)]. Then, the state \(\hat{x}_{\text{UIO}}\) can be the estimate of \(x\). We notice that we can simply set the initial value of \(\hat{x}_{\text{UIO}}(0)\) by adopting \(z(0) = \theta_4\) to reduce the cost of identification of \(x(0)\).

As shown in Fig. 9, the state \(x\) can be well-estimated by \(\hat{x}\) with less response time than \(\hat{x}_{\text{UIO}}\). Moreover, in Fig. 10, the extended states \((x_i, i = 5, 6)\) could be well-estimated by \(\hat{x}_i, i = 5, 6\) very quickly, which cannot be done by the conventional UIO.

VII. CONCLUSION

ML meeting control theory is a hot topic worth investigating. In this article, we creatively introduce the residual NNs into the design of the observer, called neural observer, and provide the necessary proof of the convergence. More specifically, we propose a new framework to design the neural observers for different dynamical systems, including linear systems and two classes of nonlinear systems, with some mild assumptions. The great performance of our proposed observer benefits from the introduction of NNs. Accordingly, we provide specific neural observers for linear systems, integrator chain nonlinear systems, and a class of MIMO nonlinear systems composed of a linear dynamic and general uncertainty. For linear systems, by combining the recent NN controller proposed in [27], we show that the observer could be used in global feedback stabilization. In addition, by using QCs to bound the nonlinear activation functions in NNs, we propose the corresponding LMI conditions for different system settings to achieve neural observability (according to Definition 1). On the other hand, it has also been shown that the observability of system matrices is a necessary condition for the existence of solutions of the aforementioned LMI. To the best of our knowledge, this is the first time that neural observability has been discussed theoretically and connected with the observability of a specific system.

There are some future works that can be done. For instance, we note that the global sector boundedness regarding activation functions introduced in Section IV is relatively “strict” so that some information from activation functions may not be exploited fully. In detail, the left subdiagram in Fig. 11 shows the global sector using the tanh function as an example. Although we can describe the activation function \(y = \tanh(s)\) roughly by using the open region formed by two straight lines \(y = \alpha s\) and \(y = \beta s\) passing through the origin, some geometric information about the activation function, such as \(\lim_{s \to +\infty} \tanh(s) = 1, \lim_{s \to -\infty} \tanh(s) = -1\), \(\lim_{s \to \infty} (d/ds) \tanh(s) = 0\), is not fully extracted in Lemmas 1 and 2. Since the LMIs (19), (21), (24), and (26) are all based on Lemmas 1 and 2, it is obvious that we would ignore some NN architectures that do not satisfy the LMIs but can still be used in the design of neural observers. Intuitively, we could use the piecewise sectors shown in the right subdiagram in Fig. 11 to characterize the nonlinear activation functions in the NN, which may make better use of geometric information to improve the results. The remaining question, therefore, arises whether we can find constraint conditions from the piecewise sectors’ boundedness that can be utilized in neural observers.

APPENDIX I

PROOFS OF PROPOSITIONS

A. Proposition 1

Proof: If \(A, C,\) and \(B_w\) satisfy the extending observable condition, i.e., \((C, A)\) is observable, then for any \(s \in \mathbb{C},\)
we have
\[
\begin{bmatrix}
I - A & -Bw \\
O & I
\end{bmatrix}
\begin{bmatrix}
s & I
\end{bmatrix}
= n_s + n_q.
\]
We suppose that \((C, A)\) is not observable. Hence, there exists
\(s_0 \in \mathbb{C}\), such that \(r[s_0I_n - A - Bw] < n_s\). In the case
of \(s_0 \neq 0\), we have
\[
\begin{bmatrix}
s_0I_n - A & -Bw \\
O & s_0I_n
\end{bmatrix}
= \begin{bmatrix}
O & s_0I_n \\
C & O
\end{bmatrix}
= \begin{bmatrix}
s_0I_n + r[s_0I_n - A] \\
C & O
\end{bmatrix}
< n_s + n_q.
\]
And in the case of \(s_0 = 0\), we have
\[
\begin{bmatrix}
s_0I_n - A & -Bw \\
O & s_0I_n
\end{bmatrix}
= \begin{bmatrix}
O & s_0I_n \\
C & O
\end{bmatrix}
< n_s + n_q.
\]
Therefore, the above-mentioned inequalities lead to a contradiction.

B. Proposition 2

Proof: First, we unfold and directly compute the matrices
on the left-hand side in LMI (19) as follows:
\[
R_\pi^T \begin{bmatrix}
A^TP + PA \\
O
\end{bmatrix}
R_\pi = \begin{bmatrix}
A^TP + PA \\
O
\end{bmatrix}
\]
\[
R_\xi^T \Psi_\sigma^T M_\sigma(\lambda_\sigma) \Psi_\sigma R_\xi = \begin{bmatrix}
O \\
N_{\xi}^T R_1
\end{bmatrix}
+ \begin{bmatrix}
\lambda_\sigma R_1 \\
-2\text{diag}(\lambda_\sigma)
\end{bmatrix}.
\]
To prove the Proposition 2, we need the following steps.
Step 1: A strictly diagonally dominant diagonal matrix
\(T = (t_{ij}) \in \mathcal{L}(\mathbb{R}^n)\) has positive diagonal entries, which means that for all \(i = 1, \ldots, m\), we have \(|t_{ii}| > \sum_{i \neq j}|t_{ij}|\) and \(t_{ii} > 0\). Then, this matrix is positive definite. Specifically, for all \(x \in \mathbb{R}^n\)
\[
x^T T x = \sum_{i=1}^m t_{ii} x_i^2 + \sum_{i < j} t_{ij} x_i x_j
\]
> \sum_{i=1}^m (\sum_{i \neq j} |t_{ij}|) x_i^2 - \sum_{i \neq j} |t_{ij}| |x_i||x_j|
= \sum_{j=1}^m \left( |t_{jj}| x_j^2 + x_j^2 - 2|x_j||x_j| \right) \geq 0.
Step 2: \(\Rightarrow\) For the sufficiency, due to the observability of
\((C, A)\), then \(\tilde{A} = A + N_{\pi x}\) is a Hurwitz matrix by taking the
matrix \(W_{L^2}^T W_{L^2}\) in \(N_{\pi x} = W_{L^2}^T W_{L^2}\) is a pole assignment
matrix for \(\tilde{A}\). Since \(\tilde{A} = A + N_{\pi x}\) is a Hurwitz matrix, we imply
that the Lyapunov equation \((-\tilde{A}^T P + P \tilde{A}) = Q\) has a unique
solution \(P = \int_0^\infty e^{\tilde{A} \tau} Q e^{\tilde{A} \tau} d\tau\) that is finite, i.e., \(\|P\| < \infty\).

Therefore, we can rewrite the LMI into
\[
R_\pi^T \begin{bmatrix}
A^T P + PA \\
O
\end{bmatrix}
R_\pi + R_\xi^T \Psi_\sigma^T M_\sigma(\lambda_\sigma) \Psi_\sigma R_\xi
= \begin{bmatrix}
Q \\
M_1
\end{bmatrix}
\]
\[
\Delta M_0.
\]
By substituting \(N_{\xi w}\) into \(M_2\) from earlier, we can show that
\(M_2\) is a symmetric matrix with zero diagonal entries. Hence, under Assumptions 1 and 2, the LMI (19) is satisfied due to
\(M_0\) is strictly diagonally dominant.

\(\Leftarrow\) For the sake of necessity, we assume that \((C, A)\) is
unobservable, and there is a matrix \(P\) that makes the LMI (19)
accurate. Since \(\tilde{A}\) is not Hurwitz, we imply that all eigenvalues of
\(-Q = \tilde{A}^T \tilde{P} + \tilde{P} \tilde{A}\) are not negative, which leads to a
contradiction since LMI (19) has no solution. This completes
the proof.

C. Proposition 3

Proof: Sufficiency: Since \((C, A)\) is observable, and
\((A, B)\) is controllable, \(\tilde{A}_1 = A + BW_{L^2}^T\) and \(\tilde{A}_2 = A +
W_{L^2}^T W_{L^2}\) are two Hurwitz matrices by making \(W_{L^1} + W_{L^2}\)
are pole assignment matrices. Subsequently, it is not difficult to verify that \(P_1\) and \(Q_1\) satisfy the Lyapunov equation
\((-\tilde{A}_1^T P + P \tilde{A}_1) = Q_1\), \(i = 1, 2\). The matrices on the left-hand
side of LMI (21) can be expanded to
\[
\begin{bmatrix}
\tilde{A}_1^T \tilde{P} + \tilde{P} \tilde{A}_1 \\
\tilde{P}
\end{bmatrix}
\begin{bmatrix}
R_\pi \\
R_\xi
\end{bmatrix}
= \begin{bmatrix}
\tilde{Q}_1 + M_3 \tilde{P} N_{\xi w}^T \\
O
\end{bmatrix}
\]
\[
\begin{bmatrix}
\tilde{Q}_1 \\
M_2 \Psi(2) \Psi(2) \tilde{R}_\xi
\end{bmatrix}
= \begin{bmatrix}
O \\
N_{\xi w}^T \tilde{R}_1 + R_1 N_{\xi w}^T
\end{bmatrix}
- 2\text{diag}(\lambda_\sigma^2).
\]
Hence, based on Assumptions 3 and 4, the LMI (21) is satisfied due to the property of strict diagonal dominance.

Necessity: The proof is the same as step 2 in
Proposition 2.
APPENDIX III
PROOF OF THEOREM

A. Theorem 1

Proof: First, we suppose that the existence of the matrix $P$ is true. Denote $e(t) = \hat{x}(t) - x(t)$, then from (4) and (5), it is not difficult to obtain

$$\dot{e}(t) = Ae(t) + \pi_\sigma(Ce(t)).$$

We denote $v(t) \triangleq \pi_\sigma(Ce(t))$. Equivalently, the form of $\pi_\sigma$ can be regarded as $e(t)$ by updating $W^{L+2}$ and $W^1$ in (14) to $W^{L+2}C$ and $W^1C$, respectively. Recall that $P > 0$, we define a radially unbounded Lyapunov function $V : R^n \rightarrow R$, $e(t) \mapsto e^T(t)Pe(t)$. Then, the time derivative of $V$ along the trajectories of $e(t)$ is given by

$$\frac{dV}{dt}|_{e(t)} = e^T(t)Pe(t) + e^T(t)\dot{P}e(t) = \left[\begin{array}{c} e^T(t) \end{array}\right] \left[\begin{array}{ccc} A^T + v^T(t) & P & P \\ v(t) & 0 & O \\ P & O & 0 \end{array}\right] \left[\begin{array}{c} e(t) \\ \pi_\sigma(v(t)) \end{array}\right],$$

where $\pi_\sigma$ can be inferred from symmetry. By using the transformation from (14) and the strict LMI (19), we imply that there exists $\varepsilon > 0$ such that the left/right multiplication of the LMI by $\left[\begin{array}{c} e(t) \\ \pi_\sigma(v(t)) \end{array}\right]$ and its transpose yields

$$\left[\begin{array}{cc} A^T + v^T(t) & P \\ v(t) & 0 \end{array}\right] \left[\begin{array}{c} e(t) \\ \pi_\sigma(v(t)) \end{array}\right] \leq -\varepsilon \left[\begin{array}{cc} e(t) \\ \pi_\sigma(v(t)) \end{array}\right].$$

Therefore, by using the Gronwall–Bellman inequality, we deduce that

$$\frac{dV}{dt}|_{e(t)} = \left[\begin{array}{c} e^T(t) \end{array}\right] \left[\begin{array}{ccc} A^T + v^T(t) & P & P \\ v(t) & 0 & O \\ P & O & 0 \end{array}\right] \left[\begin{array}{c} e(t) \\ \pi_\sigma(v(t)) \end{array}\right] \leq -\varepsilon \left[\begin{array}{c} e(t) \\ \pi_\sigma(v(t)) \end{array}\right].$$

As a consequence, we obtain

$$\|x(t) - \hat{x}(t)\|_2 \leq \sqrt{\lambda_{\min}(P)\lambda_{\max}(P)} e^{-\varepsilon\frac{\|x(0)\|_2}{\|\pi_\sigma\|}} \|e(0)\|_2.$$
where $M_1 = \epsilon(2\lambda_2^2/\lambda_3^2)M_0$. To be specific, if $\epsilon \to +0^+$, we obtain
\[
\sqrt{V_0(x(0))}e^{-\frac{\kappa t}{\lambda_1}} \leq \left( \sum_{i=1}^{n} (x_i(0) - \tilde{x}_i(0)) \right)^2 e^{-\frac{\kappa t}{\lambda_1}} \to 0^+.
\]
Therefore, as $\epsilon \to +0^+$, we obtain that for $t \in T > T$
\[
\|x - \tilde{x}\|_2 = \left( \sum_{i=1}^{n} (x_i(t) - \tilde{x}_i(t))^2 \right)^{1/2} \leq \epsilon \|\eta\|_2 \to 0^+.
\]
Moreover, $|x_{n+1} - \tilde{x}_{n+1}| = |\eta_{n+1}| \leq \|\eta\|_2 \to 0^+$. These complete the whole proof.

**D. Theorem 4**

**Proof:** First, we denote $x_1(t) \triangleq x(t)$ and $x_2(t) \triangleq K(t)$, then the system (10) can be rewritten into
\[
\begin{align*}
\dot{x}_1 &= Ax_1 + Bu + Bw, \\
\dot{x}_2 &= \nabla_K, \\
y &= C\begin{bmatrix}x_1 \\ x_2\end{bmatrix}.
\end{align*}
\]
By denoting the errors $e_i(t) = x_i(t) - \tilde{x}_i(t)$ and $\eta_i(t) = (e_i(t)/\epsilon^{t-1})$, we have the following formulation:
\[
\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \epsilon A & Bw \\ O & O \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} + \begin{bmatrix} \pi_0(\theta) \\ \pi_0(\tilde{\theta}) \end{bmatrix} + \frac{0}{\epsilon} \nabla \hat{K}.
\]
In form, $\eta(t)$ can be considered as the input of NN mapping vector $\pi_\theta$. From Assumption 3, it is easy to check that $\|\nabla \hat{K}\|_2 \leq M$ with $M > 0$. We take the Lyapunov function $V(\eta) = \eta^T P \eta$. Then, by applying Lemma 2 and $D_1 > O$, we denote $\lambda_1 = \lambda_{\min}(P)$ and $\lambda_2 = \lambda_{\max}(P)$ and consequently have the following inequality:
\[
\begin{align*}
dV \bigg|_{\eta(t)} &= \begin{bmatrix} A^TP + P \alpha \cdot -P -P \end{bmatrix} \begin{bmatrix} \eta(t) \\ \pi_\theta \end{bmatrix} + \frac{\partial V}{\partial \eta} \epsilon \nabla \hat{K} \\
&\leq -\begin{bmatrix} \Psi(2) \tilde{M}(2) \Psi(2) \end{bmatrix} \begin{bmatrix} \frac{\xi(\lambda_2)}{\lambda_2} \\ \frac{w(\lambda_2)}{\lambda_2} \end{bmatrix} \\
&- \kappa \|\eta(t)\|^2 + 2\epsilon M\lambda_2 \|\eta(t)\|^2 \\
&\leq -\frac{\kappa}{\lambda_2} V(\eta) + \frac{2\epsilon M\lambda_2}{\sqrt{\lambda_1}} \sqrt{V(\eta)}.
\end{align*}
\]
Uniformly, by using the Gronwall–Bellman inequality, we derive $\|e_i(t)\|_2 \to 0^+$ (as $\epsilon \to 0^+$) from the following inequality:
\[
\|e_i(t)\|_2 \leq e^{2\epsilon t} \left( V(\eta(0)) - \frac{\epsilon}{\lambda_1} + \frac{2\epsilon M\lambda_2}{\lambda_1} \right).
\]
This completes the proof.
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