SHARP RESULTS FOR THE WEYL PRODUCT ON MODULATION SPACES

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Abstract. We give sufficient and necessary conditions on the Lebesgue exponents for the Weyl product to be bounded on modulation spaces. The sufficient conditions are obtained as the restriction to $N = 2$ of a result valid for the $N$-fold Weyl product. As a byproduct, we obtain sharp conditions for the twisted convolution to be bounded on Wiener amalgam spaces.

0. Introduction

In the paper we prove necessary and sufficient conditions for the Weyl product to be continuous on modulation spaces, and for the twisted convolution to be continuous on Wiener amalgam spaces. We relax the sufficient conditions in [26] and we prove that the obtained conditions are also necessary.

The Weyl calculus is a part of the theory of pseudo-differential operators. For an appropriate distribution $a$ (the symbol) defined on the phase space $T^*\mathbb{R}^d \simeq \mathbb{R}^{2d}$, the Weyl operator $\text{Op}^w(a)$ is a linear map between spaces of functions or distributions on $\mathbb{R}^d$. (See Section 1 for definitions.) Weyl operators appear in various fields. In mathematical analysis they are used to represent linear operators, in particular linear partial differential operators, acting between appropriate function and distribution spaces. Weyl operators also appear in quantum mechanics where a real-valued observable $a$ in classical mechanics corresponds to the self-adjoint Weyl operator $\text{Op}^w(a)$ in quantum mechanics. For this reason $\text{Op}^w(a)$ is often called the Weyl quantization of $a$. In time-frequency analysis pseudo-differential operators are used as models of non-stationary filters.

In the Weyl calculus operator composition corresponds on the symbol level to the Weyl product, or the twisted product, denoted by $\#$. This means that the Weyl product $a_1 \# a_2$ of appropriate functions or distributions $a_1$ and $a_2$ satisfies

$$\text{Op}^w(a_1 \# a_2) = \text{Op}^w(a_1) \circ \text{Op}^w(a_2).$$

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A basic problem is to find conditions that are necessary or sufficient for the bilinear map
\[(a_1, a_2) \mapsto a_1 \# a_2\] (0.1)
to be well-defined and continuous. Here we investigate these questions when the factors belong to modulation spaces, a family of Banach spaces of distributions which appear in time-frequency analysis, harmonic analysis and Gabor analysis.

The modulation spaces were introduced by Feichtinger [6], and their theory was further developed and generalized by Feichtinger and Gröchenig [8, 10, 15] into the theory of coorbit spaces.

The modulation space \(M_{p,q}^{\omega}(\mathbb{R}^d)\), where \(p, q \in [1, \infty]\) and \(\omega\) is a weight on \(\mathbb{R}^d \times \mathbb{R}^d \simeq \mathbb{R}^{2d}\), consists of all tempered distributions, or ultra-distributions, on \(\mathbb{R}^d\), whose short-time Fourier transforms have finite \(L_{p,q}^{\omega}(\mathbb{R}^{2d})\) norm. Thus the Lebesgue exponents \(p\) and \(q\), and above all the weight \(\omega\), give a scale of function spaces \(M_{p,q}^{\omega}\) with respect to phase space concentration. The definition of modulation spaces resembles that of Besov spaces, and narrow embeddings between modulation and Besov spaces have been found (cf. [14, 25, 31, 44, 46, 50, 51]).

Depending on the assumptions on the weights, the modulation spaces are subspaces of the tempered distributions or ultra-distributions (cf. [2, 33, 34, 48, 49]).

Since the early 1990s modulation spaces have been used in the theory of pseudo-differential operators (cf. [38]). Sjöstrand [35] introduced the modulation space \(M_{\infty,1}^{\omega}(\mathbb{R}^{2d})\), which contains non-smooth functions, as a symbol class. He proved that \(M_{\infty,1}^{\omega}\) corresponds to an algebra of \(L^2\)-bounded operators.

Gröchenig and Heil [16, 19] proved that each operator with symbol in \(M_{\infty,1}^{\omega}\) is continuous on all modulation spaces \(M_{p,q}^{\omega}\), \(p, q \in [1, \infty]\). This extends Sjöstrand’s \(L^2\)-continuity result since \(M_{2,2}^{\omega} = L^2\). Some generalizations to operators with symbols in unweighted modulation spaces were obtained in [20, 44], and [45, 47, 49] contain extensions to weighted modulation spaces.

Concerning the algebraic properties of the Weyl calculus (cf. [11, 13, 27]) with respect to modulation spaces, Sjöstrand’s results [35, 36] were refined in [41], and new results were found by Labate [29], Gröchenig and Rzeszotnik [22], and by Holst and two of the authors [26].

Our main result in this paper is a multi-linear version of a generalization of [26, Theorem 0.3′] which concerns sufficient conditions for continuity of the Weyl product on modulation spaces. We also prove that the sufficient conditions are necessary in the bilinear case, for a certain family of weight functions.

The Weyl product (0.1) is continuous \(\mathcal{S}_s(\mathbb{R}^{2d}) \times \mathcal{S}_s(\mathbb{R}^{2d}) \mapsto \mathcal{S}_s(\mathbb{R}^{2d})\), where \(\mathcal{S}_s(\mathbb{R}^{2d})\) denotes the Gelfand–Shilov space of order \(s\), for every \(s \geq 0\). In order to explain our extension of this result to modulation
spaces, we introduce the Hölder–Young exponent function
\[
R_2(p) = \sum_{j=0}^{2} \frac{1}{p_j} - 1, \quad p = (p_0, p_1, p_2) \in [1, \infty]^3 \tag{0.2}
\]
and consider weights \(\omega_j, j = 0, 1, 2\), in \(\mathcal{P}_E(\mathbb{R}^{4d})\), the set of moderate weights on \(\mathbb{R}^{4d}\). We suppose that
\[
C \leq \omega_0(X_2 + X_0, X_2 - X_0) \prod_{j=1}^{2} \omega_j(X_j + X_{j-1}, X_j - X_{j-1}),
\]
\(X_0, X_1, X_2 \in \mathbb{R}^{2d}\), \(\tag{0.3}\)
holds for some \(C > 0\).

With these terms our result in the bilinear case on sufficient conditions for continuity of the Weyl product reads as follows. Here \(M^{p,q}_{\omega}(\mathbb{R}^{2d})\), as opposed to \(M^{p,q}_{\omega}(\mathbb{R}^{2d})\), is the modulation space defined with the symplectic Fourier transform instead of the usual Fourier transform.

**Theorem 0.1.** Let \(p_j, q_j \in [1, \infty], j = 0, 1, 2\), and suppose
\[
\max (R_2(q_j'), 0) \leq \min_{j=0,1,2} \left(\frac{1}{p_j}, \frac{1}{q_j'}, R_2(p)\right). \tag{0.4}
\]

Let \(\omega_j \in \mathcal{P}_E(\mathbb{R}^{4d}), j = 0, 1, 2\), and suppose \((0.3)\) holds. Then the map \((0.1)\) from \(S_{1/2}(\mathbb{R}^{2d}) \times S_{1/2}(\mathbb{R}^{2d})\) to \(S_{1/2}(\mathbb{R}^{2d})\) extends uniquely to a continuous map from \(M^{p_1,q_1}_{\omega_1}(\mathbb{R}^{2d}) \times M^{p_2,q_2}_{\omega_2}(\mathbb{R}^{2d})\) to \(M^{p_0,q_0}_{(1/\omega_0)}(\mathbb{R}^{2d})\).

This result is the restriction to \(N = 2\) of a multi-linear result treating the Weyl product of \(N\) factors \(a_1 \# \ldots \# a_N\) proved in Section 2 (see Theorem 0.1)'. Theorem 0.1 extends all results in the literature, familiar to us, on the Weyl product acting on modulation spaces, in particular [26, Theorem 0.3'] and its slight extension [48, Theorem 6.4]. In Section 4 we present a table which explains the difference between [26, Theorem 0.3'] and Theorem 0.1 in the important cases when the Lebesgue exponents \(p_j, q_j\) belong to \(\{1, 2, \infty\}\).

In Section 2 we also present a parallel result to Theorem 0.1' on sufficient conditions for continuity of the Weyl product on modulation spaces. It gives continuity in certain cases not covered by Theorem 0.1 with \(N > 2\), e.g. when several of the Weyl operators are Hilbert–Schmidt operators (cf. Theorem 2.9). Section 2 ends with a continuity result for the twisted convolution on Wiener amalgam spaces (cf. Theorem 2.12).

In Section 3 we prove that Theorem 0.1 is sharp with respect to the conditions on the Lebesgue exponents \(p_j\) and \(q_j\), for triplets \((\omega_0, \omega_1, \omega_2)\) of polynomially moderate weights that are interrelated in a certain way (see (3.1)) which implies that (0.3) is automatically satisfied. The sharpness means that (0.4) must hold when the map (0.1) from \(S_{1/2} \times \)}
$S_{1/2}$ to $S_{1/2}$ is extendable to a continuous map from $\mathcal{M}_{(\omega_1)}^{p_1,q_1} \times \mathcal{M}_{(\omega_2)}^{p_2,q_2}$ to $\mathcal{M}_{(1/\omega_0)}^{p_0,q_0}$ (cf. Theorem 3.1).

1. Preliminaries

In this section we introduce notation and discuss the background on Gelfand–Shilov spaces, pseudo-differential operators, the Weyl product, twisted convolution and modulation spaces. Most proofs can be found in the literature and are therefore omitted.

Let $0 < h, s \in \mathbb{R}$ be fixed. The space $S_{s,h}(\mathbb{R}^d)$ consists of all $f \in C^\infty(\mathbb{R}^d)$ such that

$$\|f\|_{S_{s,h}} \equiv \sup_{\||\alpha|+|\beta|s\|} \frac{|x^\beta \partial^\alpha f(x)|}{h^{|\alpha|+|\beta|s}}$$

is finite, with supremum taken over all $\alpha, \beta \in \mathbb{N}^d$ and $x \in \mathbb{R}^d$.

The space $S_{s,h} \subseteq \mathcal{S}$ ($\mathcal{S}$ denotes the Schwartz space) is a Banach space which increases with $h$ and $s$. Inclusions between topological spaces are understood to be continuous. If $s > 1/2$, or $s = 1/2$ and $h$ is sufficiently large, then $S_{s,h}$ contains all finite linear combinations of Hermite functions. Since the space of such linear combinations is dense in $\mathcal{S}$, it follows that the topological dual $(S_{s,h})'(\mathbb{R}^d)$ of $S_{s,h}(\mathbb{R}^d)$ is a Banach space which contains $\mathcal{S}'(\mathbb{R}^d)$.

The Gelfand–Shilov spaces $S_s(\mathbb{R}^d)$ and $\Sigma_s(\mathbb{R}^d)$ (cf. [12]) are the inductive and projective limits, respectively, of $S_{s,h}(\mathbb{R}^d)$, with respect to the parameter $h$. Thus

$$S_s(\mathbb{R}^d) = \bigcup_{h>0} S_{s,h}(\mathbb{R}^d) \quad \text{and} \quad \Sigma_s(\mathbb{R}^d) = \bigcap_{h>0} S_{s,h}(\mathbb{R}^d), \quad (1.1)$$

where $S_s(\mathbb{R}^d)$ is equipped with the the strongest topology such that the inclusion map from $S_{s,h}(\mathbb{R}^d)$ into $S_s(\mathbb{R}^d)$ is continuous, for every choice of $h > 0$. The space $\Sigma_s(\mathbb{R}^d)$ is a Fréchet space with seminorms $\|\cdot\|_{S_{s,h}}$, $h > 0$. We have $\Sigma_s(\mathbb{R}^d) \neq \{0\}$ if and only if $s > 1/2$, and $S_s(\mathbb{R}^d) \neq \{0\}$ if and only if $s \geq 1/2$. From now on we assume that $s > 1/2$ when we consider $\Sigma_s(\mathbb{R}^d)$, and $s \geq 1/2$ when we consider $S_s(\mathbb{R}^d)$.

The Gelfand–Shilov distribution spaces $S'_s(\mathbb{R}^d)$ and $\Sigma'_s(\mathbb{R}^d)$ are the projective and inductive limit respectively of $S_s(\mathbb{R}^d)$. This means that

$$S'_s(\mathbb{R}^d) = \bigcap_{h>0} S'_{s,h}(\mathbb{R}^d) \quad \text{and} \quad \Sigma'_s(\mathbb{R}^d) = \bigcup_{h>0} S'_{s,h}(\mathbb{R}^d). \quad (1.1)'$$

In [12][23][32] it is proved that $S'_s(\mathbb{R}^d)$ is the topological dual of $S_s(\mathbb{R}^d)$, and $\Sigma'_s(\mathbb{R}^d)$ is the topological dual of $\Sigma_s(\mathbb{R}^d)$.

For each $\varepsilon > 0$ and $s > 1/2$ we have

$$S_{1/2}(\mathbb{R}^d) \subseteq \Sigma_{s}(\mathbb{R}^d) \subseteq S_{s}(\mathbb{R}^d) \subseteq \Sigma_{s+\varepsilon}(\mathbb{R}^d)$$

and

$$S'_{s+\varepsilon}(\mathbb{R}^d) \subseteq S'_s(\mathbb{R}^d) \subseteq \Sigma'_s(\mathbb{R}^d) \subseteq S'_{1/2}(\mathbb{R}^d). \quad (1.2)$$
The Gelfand–Shilov spaces are invariant under several basic operations, e.g., translations, dilations, tensor products and (partial) Fourier transformations.

We normalize the Fourier transform of \( f \in L^1(\mathbb{R}^d) \) as

\[
(\mathcal{F}f)(\xi) = \hat{f}(\xi) \equiv (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x)e^{-i(x,\xi)} \, dx,
\]

where \( \langle \cdot , \cdot \rangle \) denotes the scalar product on \( \mathbb{R}^d \). The map \( \mathcal{F} \) extends uniquely to homeomorphisms on \( \mathcal{S}'(\mathbb{R}^d) \), \( \mathcal{S}_s'(\mathbb{R}^d) \) and \( \Sigma_s(\mathbb{R}^d) \), and restricts to homeomorphisms on \( \mathcal{S}(\mathbb{R}^d) \), \( \mathcal{S}_s(\mathbb{R}^d) \) and \( \Sigma_s(\mathbb{R}^d) \), and to a unitary operator on \( L^2(\mathbb{R}^d) \).

Next we recall some basic facts from pseudo-differential calculus (cf. [27]). Let \( s \geq 1/2 \), \( a \in \mathcal{S}_s(\mathbb{R}^d) \), and \( t \in \mathbb{R} \) be fixed. The pseudo-differential operator \( \text{Op}_t(a) \) defined by

\[
\text{Op}_t(a)f(x) = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} a((1-t)x+ty,\xi)f(y)e^{i(x-y,\xi)} \, dyd\xi \quad (1.3)
\]

is a linear and continuous operator on \( \mathcal{S}_s(\mathbb{R}^d) \). For \( a \in \mathcal{S}'(\mathbb{R}^d) \) the pseudo-differential operator \( \text{Op}_t(a) \) is defined as the continuous operator from \( \mathcal{S}(\mathbb{R}^d) \) to \( \mathcal{S}_s'(\mathbb{R}^d) \) with distribution kernel given by

\[
K_{a,t}(x,y) = (2\pi)^{-d/2}(\mathcal{F}_2^{-1}a)((1-t)x+ty,x-y). \quad (1.4)
\]

Here \( \mathcal{F}_2F \) is the partial Fourier transform of \( F(x,y) \in \mathcal{S}'(\mathbb{R}^{2d}) \) with respect to the variable \( y \in \mathbb{R}^d \). This definition generalizes (1.3) and is well defined, since the mappings

\[
\mathcal{F}_2 \quad \text{ and } \quad F(x,y) \mapsto F((1-t)x+ty,y-x) \quad (1.5)
\]

are homeomorphisms on \( \mathcal{S}'(\mathbb{R}^{2d}) \). The map \( a \mapsto K_{a,t} \) is hence a homeomorphism on \( \mathcal{S}'(\mathbb{R}^{2d}) \).

For any \( K \in \mathcal{S}_s'(\mathbb{R}^{d_1+d_2}) \), let \( T_K \) be the linear and continuous mapping from \( \mathcal{S}_s(\mathbb{R}^{d_1}) \) to \( \mathcal{S}_s'(\mathbb{R}^{d_2}) \) defined by

\[
(T_Kf,g)_L^2(\mathbb{R}^{d_2}) = (K,g \otimes \overline{f})_{L^2(\mathbb{R}^{d_1+d_2})}, \quad f \in \mathcal{S}_s(\mathbb{R}^{d_1}), \quad g \in \mathcal{S}_s(\mathbb{R}^{d_2}). \quad (1.6)
\]

It is a well known consequence of the Schwartz kernel theorem that if \( t \in \mathbb{R} \), then \( K \mapsto T_K \) and \( a \mapsto \text{Op}_t(a) \) are bijective mappings from \( \mathcal{S}_s(\mathbb{R}^{2d}) \) to the space of linear and continuous mappings from \( \mathcal{S}(\mathbb{R}^d) \) to \( \mathcal{S}'(\mathbb{R}^d) \) (cf. e.g. [27]).

Likewise the maps \( K \mapsto T_K \) and \( a \mapsto \text{Op}_t(a) \) are uniquely extendable to bijective mappings from \( \mathcal{S}_s(\mathbb{R}^{2d}) \) to the set of linear and continuous mappings from \( \mathcal{S}_s(\mathbb{R}^d) \) to \( \mathcal{S}_s'(\mathbb{R}^d) \). In fact, the asserted bijectivity for the map \( K \mapsto T_K \) follows from the kernel theorem [30, Theorem 2.3] (cf. [12, vol. IV]). This kernel theorem corresponds to the Schwartz kernel theorem in the usual distribution theory. The other assertion follows from the fact that \( a \mapsto K_{a,t} \) is a homeomorphism on \( \mathcal{S}_s'(\mathbb{R}^{2d}) \).
In particular, for each \( a_1 \in \mathcal{S}'(\mathbb{R}^{2d}) \) and \( t_1, t_2 \in \mathbb{R} \), there is a unique \( a_2 \in \mathcal{S}'(\mathbb{R}^{2d}) \) such that \( \text{Op}_{t_1}(a_1) = \text{Op}_{t_2}(a_2) \). The relation between \( a_1 \) and \( a_2 \) is given by

\[
\text{Op}_{t_1}(a_1) = \text{Op}_{t_2}(a_2) \iff a_2(x, \xi) = e^{i(t_2-t_1)(D_x, D_\xi)}a_1(x, \xi).
\]

(Cf. \[27\].) Note that the right-hand side makes sense, since it means \( \widehat{a_2}(x, \xi) = e^{i(t_2-t_1)(x, \xi)}\widehat{a_1}(x, \xi) \), and since the map \( a(x, \xi) \mapsto e^{it(x, \xi)}a(x, \xi) \) is continuous on \( \mathcal{S}'(\mathbb{R}^{2d}) \).

Next we discuss the Weyl product, twisted convolution and related operations (see \[11, 27\]). Let \( s \geq 1/2 \) and let \( a, b \in \mathcal{S}_s(\mathbb{R}^{2d}) \). The Weyl product \( a \# b \) between \( a \) and \( b \) is the function or distribution which satisfies \( \text{Op}^w(a \# b) = \text{Op}^w(a) \circ \text{Op}^w(b) \), provided the right-hand side makes sense as a continuous operator from \( \mathcal{S}_s(\mathbb{R}^{d}) \) to \( \mathcal{S}_s(\mathbb{R}^{d}) \).

The Wigner distribution is defined by

\[
W_{f,g}(x, \xi) = \mathcal{F}(f(x + \cdot/2)g(x - \cdot/2))(\xi), \quad f, g \in \mathcal{S}_{1/2}'(\mathbb{R}^d),
\]

and takes the form

\[
W_{f,g}(x, \xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x + y/2)\overline{g(x - y/2)}e^{-i(y, \xi)} dy,
\]

when \( f, g \in \mathcal{S}_{1/2}(\mathbb{R}^d) \). The Wigner distribution appears in the Weyl calculus in the formula

\[
(\text{Op}^w_a(f), g)_{L^2(\mathbb{R}^d)} = (2\pi)^{-d/2}(a, W_{g,f})_{L^2(\mathbb{R}^{2d})},
\]

\[
a \in \mathcal{S}_{1/2}'(\mathbb{R}^{2d}), \quad f, g \in \mathcal{S}_{1/2}(\mathbb{R}^d).
\]

The Weyl product can be expressed in terms of the symplectic Fourier transform and the twisted convolution. The *symplectic Fourier transform* of \( a \in \mathcal{S}_s(\mathbb{R}^{2d}) \), where \( s \geq 1/2 \), is defined by

\[
(\mathcal{F}_\sigma a)(X) = \pi^{-d} \int_{\mathbb{R}^{2d}} a(Y)e^{2i\sigma(X, Y)} dY,
\]

where \( \sigma \) is the symplectic form

\[
\sigma(X, Y) = \langle y, \xi \rangle - \langle x, \eta \rangle, \quad X = (x, \xi) \in \mathbb{R}^{2d}, \quad Y = (y, \eta) \in \mathbb{R}^{2d}.
\]

We note that \( \mathcal{F}_\sigma = T \circ (\mathcal{F}^{-1} \otimes \mathcal{F}) \), when \( (Ta)(x, \xi) = 2^d a(2\xi, 2x) \). The symplectic Fourier transform \( \mathcal{F}_\sigma \) is continuous on \( \mathcal{S}_s(\mathbb{R}^{2d}) \) and extends uniquely to a homeomorphism on \( \mathcal{S}'(\mathbb{R}^{2d}) \), and to a unitary map on \( L^2(\mathbb{R}^{2d}) \), since similar facts hold for \( \mathcal{F} \). Furthermore \( \mathcal{F}_\sigma^2 \) is the identity operator.

Let \( s \geq 1/2 \) and \( a, b \in \mathcal{S}_s(\mathbb{R}^{2d}) \). The *twisted convolution* of \( a \) and \( b \) is defined by

\[
(a \ast_\sigma b)(X) = (2/\pi)^{d/2} \int_{\mathbb{R}^{2d}} a(X - Y)b(Y)e^{2i\sigma(X, Y)} dY.
\]

The definition of \( \ast_\sigma \) extends in different ways. For example it extends to a continuous multiplication on \( L^p(\mathbb{R}^{2d}) \) when \( p \in [1, 2] \), and to a
continuous map from $\mathcal{S}'_s(\mathbb{R}^{2d}) \times \mathcal{S}_s(\mathbb{R}^{2d})$ to $\mathcal{S}'_s(\mathbb{R}^{2d})$. If $a, b \in \mathcal{S}'_s(\mathbb{R}^{2d})$, then $a \ast b$ makes sense if and only if $a \ast \hat{b}$ makes sense, and

$$a \ast b = (2\pi)^{-d/2}a \ast_\sigma (\mathcal{F}_\sigma b). \quad (1.9)$$

For the twisted convolution we have

$$\mathcal{F}_\sigma(a \ast_\sigma b) = (\mathcal{F}_\sigma a) \ast_\sigma b = \hat{a} \ast_\sigma (\mathcal{F}_\sigma b), \quad (1.10)$$

where $\hat{a}(X) = a(-X)$ (cf. [42]). A combination of (1.9) and (1.10) gives

$$\mathcal{F}_\sigma(a \# b) = (2\pi)^{-d/2}(\mathcal{F}_\sigma a) \ast_\sigma (\mathcal{F}_\sigma b). \quad (1.11)$$

If $\tilde{a}(X) = \overline{a(-X)}$ then

$$(a_1 \ast_\sigma a_2, b) = (a_1, b \ast_\sigma \tilde{a}_2) = (a_2, \tilde{a}_1 \ast_\sigma b), \quad (a_1 \ast_\sigma a_2) \ast_\sigma b = a_1 \ast_\sigma (a_2 \ast_\sigma b),$$

for appropriate $a_1, a_2, b$, and furthermore (cf. [26])

$$(a_1 \# a_2, b) = (a_2, \overline{\sigma \# b}) = (a_1, b \# \overline{\sigma}). \quad (1.12)$$

Next we turn to the basic properties of modulation spaces, and start by recalling the conditions for the involved weight functions. Let $0 < \omega, v \in L^{\infty}_{\text{loc}}(\mathbb{R}^d)$. Then $\omega$ is called moderate or $v$-moderate if

$$\omega(x + y) \lesssim \omega(x)v(y), \quad x, y \in \mathbb{R}^d. \quad (1.13)$$

Here the notation $f(x) \lesssim g(x)$ means that there exists $C > 0$ such that $f(x) \leq Cg(x)$ for all arguments $x$ in the domain of $f$ and $g$. If $f \lesssim g$ and $g \lesssim f$ we write $f \asymp g$. The function $v$ is called submultiplicative if it is even and (1.13) holds when $\omega = v$. We note that if (1.13) holds then

$$v^{-1} \lesssim \omega \lesssim v.$$

For such $\omega$ it follows that (1.13) is true when

$$v(x) = C e^{c|x|},$$

for some positive constants $c$ and $C$. In particular, if $\omega$ is moderate on $\mathbb{R}^d$, then

$$e^{-c|x|} \lesssim \omega(x) \lesssim e^{c|x|},$$

for some $c > 0$.

The set of all moderate functions on $\mathbb{R}^d$ is denoted by $\mathcal{P}_v(\mathbb{R}^d)$. If $v$ in (1.13) can be chosen as $v(x) = \langle x \rangle^s = (1 + |x|^2)^{s/2}$ for some $s \geq 0$, then $\omega$ is said to be of polynomial type or polynomially moderate. We let $\mathcal{P}(\mathbb{R}^d)$ be the set of all polynomially moderate functions on $\mathbb{R}^d$.

Let $\phi \in \mathcal{S}_s(\mathbb{R}^d) \setminus \emptyset$ be fixed. The short-time Fourier transform (STFT) $V_\phi f$ of $f \in \mathcal{S}'_s(\mathbb{R}^{2d})$ with respect to the window function $\phi$ is the Gelfand–Shilov distribution on $\mathbb{R}^{2d}$ defined by

$$V_\phi f(x, \xi) \equiv \mathcal{F}(f \phi(\cdot - x))(\xi).$$
For $a \in S_{1/2}(\mathbb{R}^{2d})$ and $\Phi \in S_{1/2}(\mathbb{R}^{2d}) \setminus \{0\}$ the symplectic short-time Fourier transform $\mathcal{V}_\Phi a$ of $a$ with respect to $\Phi$ is defined similarly as

$$\mathcal{V}_\Phi a(X, Y) = \mathcal{F}_\sigma\left(a \overline{\Phi(\cdot - X)}\right)(Y), \quad X, Y \in \mathbb{R}^{2d}.$$ 

We have

$$\mathcal{V}_\Phi a(X, Y) = 2^d \mathcal{V}_\Phi a(x, \xi, -2\eta, 2y), \quad X = (x, \xi) \in \mathbb{R}^{2d}, \quad Y = (y, \eta) \in \mathbb{R}^{2d}, \quad (1.14)$$

which shows the close connection between $\mathcal{V}_\Phi a$ and $\mathcal{V}_\Phi a$. The Wigner distribution $W_{f,\phi}$ and $V_{\phi}f$ are also closely related.

If $f, \phi \in \mathcal{S}_s(\mathbb{R}^d)$ and $a, \Phi \in \mathcal{S}_s(\mathbb{R}^{2d})$ then

$$V_{\phi}f(x, \xi) = (2\pi)^{-d/2} \int f(y) \overline{\phi(y - x)} e^{-i(y, \xi)} dy$$

and

$$\mathcal{V}_\Phi a(X, Y) = \pi^{-d} \int a(Z) \overline{\Phi(Z - X)} e^{2i\sigma(Y, Z)} dZ.$$ 

Let $\omega \in \mathcal{P}_E(\mathbb{R}^{2d})$, $p, q \in [1, \infty]$ and $\phi \in S_{1/2}(\mathbb{R}^d) \setminus \{0\}$ be fixed. The modulation space $M_{\omega}^{p,q}(\mathbb{R}^d)$ consists of all $f \in S_{1/2}(\mathbb{R}^d)$ such that

$$\|f\|_{M_{\omega}^{p,q}} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_{\phi}f(x, \xi)\omega(x, \xi)|^p d\xi \right)^{q/p} dx \right)^{1/q} \quad (1.15)$$

is finite, and the Wiener amalgam space $W_{\omega}^{p,q}(\mathbb{R}^d)$ consists of all $f \in S_{1/2}(\mathbb{R}^d)$ such that

$$\|f\|_{W_{\omega}^{p,q}} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_{\phi}f(x, \xi)\omega(x, \xi)|^q d\xi \right)^{p/q} dx \right)^{1/p} \quad (1.16)$$

is finite (with obvious modifications in (1.15) and (1.16) when $p = \infty$ or $q = \infty$).

**Remark 1.1.** The literature contains slightly different conventions concerning modulation and Wiener amalgam spaces. Sometimes our definition of a Wiener amalgam space is considered as a particular case of a general class of modulation spaces (cf. [5-7]). Our definition is adapted to give the relation (1.19) that suits our purpose to transfer continuity for the Weyl product on modulation spaces to continuity for twisted convolution on Wiener amalgam spaces.

On the even-dimensional phase space $\mathbb{R}^{2d}$ we may define modulation spaces based on the symplectic STFT. Thus if $\omega \in \mathcal{P}_E(\mathbb{R}^{2d})$, $p, q \in [1, \infty]$ and $\Phi \in S_{1/2}(\mathbb{R}^{2d}) \setminus \{0\}$ are fixed, then the symplectic modulation spaces $M_{\omega}^{p,q}(\mathbb{R}^{2d})$ and Wiener amalgam spaces $W_{\omega}^{p,q}(\mathbb{R}^{2d})$ are obtained by replacing the STFT $a \mapsto V_{\phi}a$ by the corresponding symplectic version $a \mapsto \mathcal{V}_\Phi a$ in (1.15) and (1.16). (Sometimes the word...
Proposition 1.2. Let \( M \subseteq \mathbb{R}^{2d} \). Recall that \( p, q \) are symplectic before modulation space is omitted for brevity.) By (1.14) we have

\[
M^{p,q}(\mathbb{R}^{2d}) = M^{p,q}_{(\omega)}(\mathbb{R}^{2d}), \quad \omega(x, \xi, y, \eta) = \omega_0(x, \xi, -2\eta, 2y).
\]

It follows that all properties which are valid for \( M^{p,q}_{(\omega)} \) carry over to \( M^{p,q} \).

From

\[
M \ni \omega \quad \Rightarrow \quad \hat{f}(\xi, -x) = e^{i(x,\xi)} V_\phi f(x, \xi)
\]

it follows that

\[
f \in W^{p,q}_{(\omega)}(\mathbb{R}^d) \quad \iff \quad \hat{f} \in M^{p,q}_{(\omega)}(\mathbb{R}^d), \quad \omega_0(\xi, -x) = \omega(x, \xi).
\]

In the symplectic situation these formulas read

\[
V_{\mathcal{F}_a\phi}(\mathcal{F}_a)(X, Y) = e^{2i \sigma(Y, X)} V_\phi a(Y, X)
\]

and

\[
\mathcal{F}_a M^{p,q}_{(\omega)}(\mathbb{R}^{2d}) = W^{p,q}_{(\omega)}(\mathbb{R}^{2d}), \quad \omega_0(X, Y) = \omega(Y, X).
\]

For brevity we denote \( M^{p}_{(\omega)} = M^{p}_{(\omega)} = W^{p}_{(\omega)} \), and when \( \omega \equiv 1 \) we write \( M^{p} = M^{p}_{(1)} \) and \( W^{p} = W^{p}_{(1)} \). We also let \( M^{p,q}_{(\omega)}(\mathbb{R}^{2d}) \) be the completion of \( S_\omega(\mathbb{R}^{2d}) \) with respect to the norm \( \| \cdot \|_{M^{p,q}_{(\omega)}} \).

The proof of the following proposition can be found in [1, 2, 4, 5, 6, 8, 8]. Recall that \( p, p' \in [1, \infty] \) satisfy \( 1/p + 1/p' = 1 \). Since our main results are formulated in terms of symplectic modulation spaces, we state the result for them instead of the modulation spaces \( M^{p,q}_{(\omega)}(\mathbb{R}^{2d}) \).

**Proposition 1.2.** Let \( p, q, p_j, q_j \in [1, \infty] \) for \( j = 1, 2 \), and \( \omega, \omega_1, \omega_2, v \in \mathcal{F}_E(\mathbb{R}^{2d}) \) be such that \( v = \tilde{v}, \omega \) is \( v \)-moderate and \( \omega_2 \subseteq \omega_1 \). Then the following is true:

1. \( a \in M^{p,q}_{(\omega)}(\mathbb{R}^{2d}) \) if and only if (1.15) holds for any \( \phi \in M^{p}_{(\omega)}(\mathbb{R}^{2d}) \setminus 0 \). Moreover, \( M^{p,q}_{(\omega)} \) is a Banach space under the norm in (1.15) and different choices of \( \phi \) give rise to equivalent norms;
2. if \( p_1 \leq p_2 \) and \( q_1 \leq q_2 \) then
   \[
   \Sigma_1(\mathbb{R}^{2d}) \subseteq M^{p_1,q_1}_{(\omega)}(\mathbb{R}^{2d}) \subseteq M^{p_2,q_2}_{(\omega)}(\mathbb{R}^{2d}) \subseteq \Sigma_1(\mathbb{R}^{2d}).
   \]
3. the \( L^2 \) inner product \( \langle \cdot, \cdot \rangle_{L^2} \) on \( S_{1/2} \) extends uniquely to a continuous sesquilinear form \( \langle \cdot, \cdot \rangle \) on \( M^{p,q}_{(\omega)}(\mathbb{R}^{2d}) \times M^{p',q'}_{(1/\omega)}(\mathbb{R}^{2d}) \).
   On the other hand, if \( \| a \| = \sup \| [a, b] \| \), where the supremum is taken over all \( b \in S_{1/2}(\mathbb{R}^{2d}) \) such that \( \| b \|_{M^{p',q'}_{(1/\omega)}} \leq 1 \), then \( \| \cdot \| \) and \( \| \cdot \|_{M^{p,q}_{(\omega)}} \) are equivalent norms;
4. if \( p, q < \infty \), then \( S_{1/2}(\mathbb{R}^{2d}) \) is dense in \( M^{p,q}_{(\omega)}(\mathbb{R}^{2d}) \) and the dual space of \( M^{p,q}_{(\omega)}(\mathbb{R}^{2d}) \) can be identified with \( M^{p',q'}_{(1/\omega)}(\mathbb{R}^{2d}) \), through the form \( \langle \cdot, \cdot \rangle \). Moreover, \( S_{1/2}(\mathbb{R}^{2d}) \) is weakly dense
in $\mathcal{M}^\prime_{\omega}(\mathbb{R}^{2d})$ with respect to the form $(\cdot, \cdot)$ provided $(p, q) \neq (\infty, 1)$ and $(p, q) \neq (1, \infty)$;

(5) if $p, q, r, s, u, v \in [1, \infty]$, $0 \leq \theta \leq 1$,

\[
\frac{1}{p} = \frac{1 - \theta}{r} + \frac{\theta}{u} \quad \text{and} \quad \frac{1}{q} = \frac{1 - \theta}{s} + \frac{\theta}{v},
\]

then complex interpolation gives

\[
(\mathcal{M}^r_{\omega}, \mathcal{M}^u_{\omega})[\theta] = \mathcal{M}^{p,q}_{\omega}.
\]

Similar facts hold if the $\mathcal{M}^{p,q}_{\omega}$ spaces are replaced by the $\mathcal{W}^{p,q}_{\omega}$ spaces.

Remark 1.3. Let $\mathcal{P}$ be the set of all $\omega \in \mathcal{P}_E(\mathbb{R}^{2d})$ such that

\[
\omega(X, Y) = e^{c(|X|^{1/s} + |Y|^{1/s})},
\]

for some $c > 0$. Then

\[
\bigcap_{\omega \in \mathcal{P}} \mathcal{M}^{p,q}_{\omega}(\mathbb{R}^{2d}) = \Sigma_s(\mathbb{R}^{2d}), \quad \bigcup_{\omega \in \mathcal{P}} \mathcal{M}^{p,q}_{1/\omega}(\mathbb{R}^{2d}) = \Sigma'_s(\mathbb{R}^{2d})\]

\[
\bigcup_{\omega \in \mathcal{P}} \mathcal{M}^{p,q}_{\omega}(\mathbb{R}^{2d}) = S_s(\mathbb{R}^{2d}), \quad \bigcap_{\omega \in \mathcal{P}} \mathcal{M}^{p,q}_{1/\omega}(\mathbb{R}^{2d}) = S'_s(\mathbb{R}^{2d}),
\]

and for $\omega \in \mathcal{P}$

\[
\Sigma_s(\mathbb{R}^{2d}) \subseteq \mathcal{M}^{p,q}_{\omega}(\mathbb{R}^{2d}) \subseteq S_s(\mathbb{R}^{2d}) \quad \text{and} \quad S'_s(\mathbb{R}^{2d}) \subseteq \mathcal{M}^{p,q}_{1/\omega}(\mathbb{R}^{2d}) \subseteq \Sigma'_s(\mathbb{R}^{2d}).
\]

(Cf. [3] Prop. 4.5], [24] Prop. 4], [33] Cor. 5.2] and [40] Thm. 4.1]. See also [48] Thm. 3.9 for an extension of these inclusions to broader classes of Gelfand–Shilov and modulation spaces.)

We have the following result for the map $e^{it(D_x, D_y)}$ in (1.7) when the domains are modulation spaces. We refer to [47] Proposition 1.7 for the proof (see also [48] Proposition 6.14]).

Proposition 1.4. Let $\omega_0 \in \mathcal{P}_E(\mathbb{R}^{2d})$, $p, q \in [1, \infty)$, $t, t_1, t_2 \in \mathbb{R}$, and set

\[
\omega_t(x, \xi, \eta, y) = \omega_0(x - ty, \xi - t\eta, \eta, y).
\]

The map $e^{it(D_x, D_y)}$ on $S_{1/2}^\prime(\mathbb{R}^{2d})$ restricts to a homeomorphism from $M^{p,q}_{\omega_0}(\mathbb{R}^{2d})$ to $M^{p,q}_{\omega_t}(\mathbb{R}^{2d})$.

In particular, if $a_1, a_2 \in S_{1/2}^\prime(\mathbb{R}^{2d})$ satisfy (1.7), then $a_1 \in M^{p,q}_{\omega_0}(\mathbb{R}^{2d})$, if and only if $a_2 \in M^{p,q}_{\omega_t}(\mathbb{R}^{2d})$.

(Note that in the equality of (2) in [48] Proposition 6.14], $y$ and $\eta$ should be interchanged in the last two arguments in $\omega_0$.)

By Proposition 1.2 (4) we have norm density of $S_{1/2}$ in $M^{p,q}_{\omega}$ when $p, q < \infty$. We may relax the assumptions on $p$, provided we replace the norm convergence with narrow convergence. This concept, that allows us to approximate elements in $M^{\infty,q}_{\omega}(\mathbb{R}^{2d})$ for $1 \leq q < \infty$, is treated in [35][44][10], and, for the current setup of possibly exponential weights,
in [35]. (Sjöstrand’s original definition in [35] is somewhat different.) Narrow convergence is defined by means of the function

\[ H_{a, \omega, p}(Y) \equiv \| \mathcal{V}_p a(\cdot, Y) \omega(\cdot, Y) \|_{L^p(\mathbb{R}^d)}, \quad Y \in \mathbb{R}^d, \]

for \( a \in S'_{1/2}(\mathbb{R}^d) \), \( \omega \in \mathcal{P}_E(\mathbb{R}^d) \), \( \Phi \in S_{1/2}(\mathbb{R}^d) \setminus \{0\} \) and \( p \in [1, \infty] \).

**Definition 1.5.** Let \( p, q \in [1, \infty] \), and \( a, a_j \in \mathcal{M}_{(\omega)}^{p,q}(\mathbb{R}^d) \), \( j = 1, 2, \ldots \).

Then \( a_j \) is said to converge narrowly to \( a \) with respect to \( p, q \), \( \Phi \in S_{1/2}(\mathbb{R}^d) \setminus \{0\} \) and \( \omega \in \mathcal{P}_E(\mathbb{R}^d) \), if there exist \( g_j, g \in L^q(\mathbb{R}^d) \) such that:

1. \( a_j \to a \) in \( S'_{1/2}(\mathbb{R}^d) \) as \( j \to \infty \);
2. \( H_{a_j, \omega, p} \leq g_j \) and \( g_j \to g \) in \( L^q(\mathbb{R}^d) \) and a.e. as \( j \to \infty \).

**Proposition 1.6.** If \( \omega \in \mathcal{P}_E(\mathbb{R}^d) \) and \( 1 \leq q < \infty \) then the following is true:

1. \( S_{1/2}(\mathbb{R}^d) \) is dense in \( \mathcal{M}_{(\omega)}^{\infty,q}(\mathbb{R}^d) \) with respect to narrow convergence;
2. \( \mathcal{M}_{(\omega)}^{\infty,q}(\mathbb{R}^d) \) is sequentially complete with respect to the topology defined by narrow convergence.

**Proof.** Assertion (1) is a consequence of [35] Definition 2.12 and Theorem 4.19.

To prove (2), let \( \{a_n\}_{n=1}^\infty \subseteq S'_{1/2}(\mathbb{R}^d) \) be a Cauchy sequence with respect to narrow convergence. This means that

\[ (a_n - a_k, \varphi) \to 0, \quad n, k \to \infty, \quad \varphi \in S_{1/2}(\mathbb{R}^d), \tag{1.20} \]

and there exists a sequence \( \{g_n\} \subseteq L^q(\mathbb{R}^d) \) such that \( H_{a_n, \omega, \infty} \leq g_n \), and \( \|g_n - g_k\|_{L^q} \to 0 \) as well as \( g_n - g_k \to 0 \) a.e., as \( n, k \to \infty \). By (1.20) and the completeness of \( S'_{1/2}(\mathbb{R}^d) \) there exists \( a \in S'_{1/2}(\mathbb{R}^d) \) such that \( a_n \to a \) in \( S'_{1/2}(\mathbb{R}^d) \) as \( n \to \infty \), and by the completeness of \( L^q(\mathbb{R}^d) \) there exists \( g \in L^q(\mathbb{R}^d) \) such that \( g_n \to g \) in \( L^q(\mathbb{R}^d) \) and a.e. as \( n \to \infty \). This shows that conditions (1) and (2) of Definition 1.5 are satisfied.

To show \( a_n \to a \) narrowly as \( n \to \infty \) it remains to prove \( a \in \mathcal{M}_{(\omega)}^{\infty,q}(\mathbb{R}^d) \). We have for \( Y \in \mathbb{R}^d \)

\[ H_{a, \omega, \infty}(Y) = \lim_{n \to \infty} \| \mathcal{V}_p a_n(\cdot, Y) \omega(\cdot, Y) \|_{L^\infty} \leq \limsup_{n \to \infty} \| \mathcal{V}_p a_n(\cdot, Y) \omega(\cdot, Y) \|_{L^\infty} \leq \limsup_{n \to \infty} \| \mathcal{V}_p a_n(\cdot, Y) \omega(\cdot, Y) \|_{L^\infty} = H_{a, \omega, \infty}(Y). \]

Since

\[ H_{a, \omega, \infty}(Y) \leq \liminf_{n \to \infty} H_{a_n, \omega, \infty}(Y), \quad Y \in \mathbb{R}^d, \]

\[ H_{a, \omega, \infty}(Y) \leq \liminf_{n \to \infty} H_{a_n, \omega, \infty}(Y), \quad Y \in \mathbb{R}^d, \]
the limit \( \lim_{n \to \infty} H_{a_n, \omega, \infty}(Y) \) exists, so for almost all \( Y \in \mathbb{R}^{2d} \) it follows that
\[
H_{a, \omega, \infty}(Y) = \lim_{n \to \infty} H_{a_n, \omega, \infty}(Y) \leq \limsup_{n \to \infty} g_n(Y) = g(Y).
\]
Since \( g \in L^q(\mathbb{R}^{2d}) \) we conclude that \( H_{a, \omega, \infty} \in L^q(\mathbb{R}^{2d}) \) which means that \( a \in M^{\infty, q}_\omega(\mathbb{R}^{2d}) \).

2. CONTINUITY FOR THE WEYL PRODUCT ON MODULATION SPACES

In this section we deduce results on sufficient conditions for continuity of the Weyl product on modulation spaces, and the twisted convolution on Wiener amalgam spaces. The main results are Theorems 0.1 and 2.9 concerning the Weyl product, and Theorem 2.12 concerning the twisted convolution.

The first main result Theorem 0.1 together with Theorem 2.9 is equivalent to Theorem 2.12. In the bilinear case, Theorem 0.1 is the same as Theorem 0.1 in the introduction, and contains [26, Theorem 0.3] and Theorem 2.9. On the other hand, in the multi-linear case with \( N > 2 \), Theorems 0.1 and 2.9 are distinct results with none of them included in the other.

When proving Theorem 0.1 we first need norm estimates. Then we prove the uniqueness of the extension, where generally norm approximation not suffices, since the test function space may fail to be dense in several of the domain spaces. The situation is saved by a comprehensive argument based on narrow convergence. First we prove the important special cases Propositions 2.2 and 2.5 and then we state and prove Theorem 0.1.

For \( N \geq 2 \) we let \( R_N \) be the Hölder–Young exponent function
\[
R_N(p) = (N-1)^{-1} \left( \sum_{j=0}^{N} \frac{1}{p_j} - 1 \right), \quad (0.2)
\]
and we consider mappings of the form
\[
(a_1, \ldots, a_N) \mapsto a_1 \# \cdots \# a_N. \quad (0.1)
\]

We first show a formula for the STFT of \( a_1 \# \cdots \# a_N \) expressed with
\[
F_j(X, Y) = \mathcal{V}_{\Phi_j} a_j(X + Y, X - Y). \quad (2.1)
\]

**Lemma 2.1.** Let \( \Phi_j \in \mathcal{S}_{1/2}(\mathbb{R}^{2d}), \ j = 1, \ldots, N, \ a_k \in \mathcal{S}'_{1/2}(\mathbb{R}^{2d}) \) for some \( 1 \leq k \leq N \), and \( a_j \in \mathcal{S}_{1/2}(\mathbb{R}^{2d}) \) for \( j \in \{1, \ldots, N\} \setminus k \). Suppose
\[
\Phi_0 = \pi^{(N-1)d} \Phi_1 \# \cdots \# \Phi_N \quad \text{and} \quad a_0 = a_1 \# \cdots \# a_N.
\]
If \( F_j \) are given by (2.1) then

\[
F_0(X_N, X_0) = \int \cdots \int_{\mathbb{R}^{2(N-1)d}} e^{2iQ(X_0, \ldots, X_N)} \prod_{j=1}^{N} F_j(X_j, X_{j-1}) \, dX_1 \cdots dX_{N-1} \tag{2.2}
\]

with

\[
Q(X_0, \ldots, X_N) = \sum_{j=1}^{N-1} \sigma(X_j - X_0, X_{j+1} - X_0).
\]

**Proof.** The result follows in the case \( N = 2 \) by letting \( X = X_2 + X_0 \) and \( Y = X_2 - X_0 \) in [26, Lemma 2.1]. For \( N > 2 \) the result follows from straightforward computations and induction. \( \square \)

Next we use the previous lemma to find sufficient conditions for the extension of (0.1)' to modulation spaces. The integral representation of \( V_{\Phi_0}a_0 \) in the previous lemma leads to the weight condition

\[
1 \lessgtr \omega_0(X_N + X_0, X_N - X_0) \prod_{j=1}^{N} \omega_j(X_j + X_{j-1}, X_j - X_{j-1}),
\]

\( X_0, X_1, \ldots, X_N \in \mathbb{R}^{2d}. \tag{0.3}' \)

The following result is a generalization of [26, Proposition 0.1].

**Proposition 2.2.** Let \( p_j, q_j \in [1, \infty], j = 0, 1, \ldots, N, \) and suppose \( R_N(q') \leq p \leq R_N(p) \).

Let \( \omega_j, j = 0, 1, \ldots, N, \) and suppose (0.3)' holds. Then the map (0.1)' from \( \mathcal{S}_{1/2}(\mathbb{R}^{2d}) \times \cdots \times \mathcal{S}_{1/2}(\mathbb{R}^{2d}) \) to \( \mathcal{S}_{1/2}(\mathbb{R}^{2d}) \) extends uniquely to a continuous and associative map from \( \mathcal{M}_{(\omega_1)}^{p_1, q_1}(\mathbb{R}^{2d}) \times \cdots \times \mathcal{M}_{(\omega_N)}^{p_N, q_N}(\mathbb{R}^{2d}) \) to \( \mathcal{M}_{(1/\omega_0)}^{p_0, q_0}(\mathbb{R}^{2d}) \).

The associativity means that for any product (0.1)', where the factors \( a_j \) satisfy the hypotheses, the subproduct

\[
a_{k_1} \# a_{k_1+1} \# \cdots \# a_{k_2}
\]

is well defined as a distribution for any \( 1 \leq k_1 \leq k_2 \leq N \), and

\[
a_1 \# \cdots \# a_N = (a_1 \# \cdots \# a_k) \# (a_{k+1} \# \cdots \# a_N),
\]

for any \( 1 \leq k \leq N - 1 \).

To prove the uniqueness claim we need the following two lemmas, the first of which is a generalization of Lebesgue’s dominated convergence theorem.
Lemma 2.3. Let $0 < q < \infty$, let \( \{f_n\}_{n \geq 0} \) and \( \{g_n\}_{n \geq 0} \) be sequences in \( L^q(\mathbb{R}^d) \) such that
\[
\lim_{n \to \infty} \|g_n - g\|_{L^q(\mathbb{R}^d)} = 0, \quad \lim_{n \to \infty} g_n = g \; \text{a. e.,}
\]
\[
|f_n| \leq g_n, \quad \text{and} \quad \lim_{n \to \infty} f_n = f \; \text{a. e.,}
\]
for some measurable functions \( f \) and \( g \). Then \( f \in L^q(\mathbb{R}^d) \) and
\[
\lim_{n \to \infty} \|f_n - f\|_{L^q(\mathbb{R}^d)} = 0.
\]

Proof. The result follows from an argument based on Fatou’s lemma applied on
\[
\int 2^q g(x)^q \, dx = \int \liminf_{n \to \infty} ((g_n(x) + g(x))^q - |f_n(x) - f(x)|^q) \, dx.
\]
\( \square \)

Lemma 2.4. Let $1 < q \leq \infty$, let \( f \in L^q(\mathbb{R}^d) \), and let \( \{g_n\}_{n \geq 0} \) be a sequence in \( L^q(\mathbb{R}^d) \) such that
\[
\sup_n \|g_n\|_{L^q(\mathbb{R}^d)} < \infty \quad \text{and} \quad \lim_{n \to \infty} g_n = g \; \text{a. e.,}
\]
for some measurable function \( g \). Then \( g \in L^q(\mathbb{R}^d) \) and
\[
\lim_{n \to \infty} \int_{\mathbb{R}^d} (g_n(x) - g(x)) f(x) \, dx = 0.
\]

Proof. The result follows from a combination of Egorov’s theorem and the facts that for any \( \varepsilon > 0 \) there is a ball \( B \subseteq \mathbb{R}^d \) such that
\[
\|f\|_{L^{q'}(\mathbb{R}^d \setminus B)} < \varepsilon,
\]
and
\[
\lim_{|E| \to 0} \|f\|_{L^{q'}(E)} = 0,
\]
where \( |E| \) denotes the volume of the measurable set \( E \subseteq \mathbb{R}^d \). The details are left for the reader. \( \square \)

Proof of Proposition 2.2. By Proposition 1.2 (2) we may assume that \( R_N(p) = R_N(q') = 0 \), which will allow us to use Hölder’s and Young’s inequalities.

Let \( a_1, \ldots, a_N \in S_{1/2}(\mathbb{R}^{2d}) \). By replacing \( X_j \) with \( X_j + X_0 \) in (0.3)', \( j = 1, \ldots, N \), and then replacing \( 2X_0 \) with \( X_0 \), we get
\[
1 \lesssim \omega_0(X_N + X_0, X_N) \omega_1(X_1 + X_0, X_1) \prod_{j=2}^N \omega_j(X_j + X_{j-1} + X_0, X_j - X_{j-1}),
\]
\[
X_0, \ldots, X_N \in \mathbb{R}^{2d}. \tag{2.3}
\]

Let \( \Phi_j, j = 0, \ldots, N \), be as in Lemma 2.1. Set
\[
G_j(X, Y) \equiv |V_{\Phi_j} a_j(X, Y)| \omega_j(X, Y), \quad g_j(Y) \equiv \|G_j(\cdot, Y)\|_{L^p},
\]
\[
14
\]
for $j = 1, \ldots, N$, and

$$K(X_0, \ldots, X_N) = G_1(X_1 + X_0, X_1) \prod_{j=2}^N G_j(X_j + X_{j-1} + X_0, X_j - X_{j-1}).$$

Then Lemma 2.1 gives

$$|V_{a_0}(X_N + X_0, X_N)|/\omega_0(X_N + X_0, X_N)$$

$$\lesssim \int \cdots \int_{\mathbb{R}^{2(N-1)d}} K(X_0, \ldots, X_N) \, dX_1 \cdots dX_{N-1}.$$  

Taking the $L^{p_0}$ norm in the first variable gives, using Minkowski’s and Hölder’s inequalities,

$$\|V_{a_0}(\cdot, X_N)|/\omega_0(\cdot, X_N)\|_{L^{p_0}}$$

$$\lesssim \int \cdots \int_{\mathbb{R}^{2(N-1)d}} \|K(\cdot, X_1, \ldots, X_N)\|_{L^{p_0}} \, dX_1 \cdots dX_{N-1}$$

$$\leq \int \cdots \int_{\mathbb{R}^{2(N-1)d}} g_1(X_1) \prod_{j=2}^N g_j(X_j - X_{j-1}) \, dX_1 \cdots dX_{N-1}$$

$$= (g_1 \ast \cdots \ast g_N)(X_N).$$

Applying the $L^{q_0}$ norm and using Young’s inequality we get

$$\|a_0\|_{\mathcal{M}_{(1/\omega_0)}^{p_0, q_0}} \lesssim \|g_1\|_{L^{p_0}} \cdots \|g_N\|_{L^{p_0}} = \|a_1\|_{\mathcal{M}_{(1/\omega_1)}^{p_1, q_1}} \cdots \|a_N\|_{\mathcal{M}_{(1/\omega_N)}^{p_N, q_N}}. \quad (2.4)$$

The result now follows in the case when $p_j, q_j < \infty$ for $j = 1, \ldots, N$, from the estimate (2.4) and the fact that $S_{1/2}$ is dense in $\mathcal{M}_{(\omega_j)}^{p_j, q_j}$. In the case when at least one $p_j$ or $q_j$ attain $\infty$ for some $j = 1, \ldots, N$, (2.4) still holds when $a_j \in S_{1/2}$, $j = 1, \ldots, N$, and the Hahn–Banach theorem and duality guarantee the existence of a continuous extension.

We must prove its uniqueness and associativity. First we observe that the assumption $R_N(\mathbf{q}) = 0$ is equivalent to $\sum_{j=0}^N 1/q_j = N$, so $q_k = \infty$ may hold for at most one $k$, and in that case $q_j = 1$ must hold for $j \in \{0, \ldots, N\} \setminus k$. If $q_0 > 1$ then $q_j < \infty$ for $1 \leq j \leq N$. So either the uniqueness concerns the inclusion

$$\mathcal{M}_{(\omega_1)}^{p_1, q_1} \# \cdots \# \mathcal{M}_{(\omega_N)}^{p_N, q_N} \subseteq \mathcal{M}_{(1/\omega_0)}^{p_0, q_0}, \quad q_j < \infty, \quad 1 \leq j \leq N, \quad q_0 > 1, \quad (2.5)$$

or

$$\mathcal{M}_{(\omega_1)}^{p_1, q_1} \# \cdots \# \mathcal{M}_{(\omega_N)}^{p_N, q_N} \subseteq \mathcal{M}_{(1/\omega_0)}^{p_0, \infty}, \quad q_j < \infty, \quad 1 \leq j \leq N, \quad (2.6)$$

or, for a unique $k$ such that $1 \leq k \leq N$,

$$\mathcal{M}_{(\omega_1)}^{p_1, 1} \# \cdots \# \mathcal{M}_{(\omega_k)}^{p_k, 1} \# \cdots \# \mathcal{M}_{(\omega_N)}^{p_N, 1} \subseteq \mathcal{M}_{(1/\omega_0)}^{p_0, \infty}. \quad (2.7)$$
First we consider \((\ref{2.5})\). For all \(j\) such that \(p_j < \infty\) we may extend the Weyl product uniquely from \(a_j \in S_{1/2}\) to \(a_j \in M_{\omega_j}^{p_j, q_j}\) as in the first part of the proof, and for the remaining \(j\) we extend the Weyl product from \(a_j \in S_{1/2}\) to \(a_j \in M_{\omega_j}^{\infty, q_j}\) using narrow convergence, as follows. By induction it suffices to perform the extension for some \(j \in \{1, \ldots, N\}\) from \(a_j \in S_{1/2}\) to \(a_j \in M_{\omega_j}^{\infty, q_j}\).

Assume for simplicity that \(j = 1\). We may assume \(q_1 > 1\). In fact, our aim is to prove uniqueness only, so if \(q_1 = 1\) we may by Proposition 1.2 (2) consider the first factor \(a_1\) as an element in \(M_{\omega_1}^{\infty, q_1}\) with the exponent \(q_1\) modified as \(1/q_1 = 1/q_1 - \varepsilon < 1\), where \(\varepsilon > 0\) is so small that we still have \(1/q_0 = 1/q_0 + \varepsilon < 1\) for the modification \(q_0\) of the exponent \(q_0\).

Take a sequence \(\{a_{1,n}\}_{n=1}^{\infty}\) that converges narrowly to \(a_1 \in M_{\omega_1}^{\infty, q_1}\). By Definition 1.5 this means that \(a_{1,n} \rightarrow a_1\) in \(S_{1/2}'(\mathbb{R}^{2d})\) as \(n \rightarrow \infty\), and the existence of \(g_{1,n}, g_1 \in L^q(\mathbb{R}^{2d})\) such that

\[
\|V_{\Phi_j} a_{1,n}(\cdot, Y)\omega_j(\cdot, Y)\|_{L^\infty} \leq g_{1,n}(Y)
\]

and \(g_{1,n} \rightarrow g_1\) in \(L^q(\mathbb{R}^{2d})\) as well as a.e. as \(n \rightarrow \infty\). Set

\[
g_j(Y) = \|V_{\Phi_j} a_j(\cdot, X)\omega_j(\cdot, Y)\|_{L^\infty}, \quad j = 2, \ldots, N,
\]

and define \(a_{0,n} = a_{1,n} # a_2 # \cdots # a_N\). Lemma 2.2 and the definitions above yield

\[
\|V_{\Phi_0} a_{0,n}(\cdot, X) / \omega_0(\cdot, X)\|_{L^\infty} \leq g_{1,n} * g_2 * \cdots * g_N(X_N).
\]

From \(g_{1,n} \rightarrow g_1\) in \(L^q\) as \(n \rightarrow \infty\) and Young’s inequality, we may conclude that \(g_{1,n} * g_2 * \cdots * g_N \rightarrow g_1 * g_2 * \cdots * g_N\) in \(L^{q_0}(\mathbb{R}^{2d})\).

The assumption \(\sum_{j=0}^{N} 1/q_j = N\) implies \(1/q_j \geq 1/q_0'\) for any \(1 \leq j \leq N\). Due to Proposition 1.2 (2) we may therefore assume that \(g_2 \in L^q\) with \(1/q = 1/q_2 - 1/q_0'\). Then Young’s inequality guarantees that \(g_2 * \cdots * g_N \in L^{q_0}(\mathbb{R}^{2d})\). It now follows from Lemma 2.4 that \(g_{1,n} * g_2 * \cdots * g_N \rightarrow g_1 * g_2 * \cdots * g_N\) a.e. So we have shown that the sequence \(\{a_{0,n}\}\) satisfies condition (2) in Definition 1.5 for the modulation space \(M_{\omega_1}^{\infty, q_0}(\mathbb{R}^{2d})\).

Let \(\varphi \in S_{1/2}(\mathbb{R}^{2d})\). Our plan is to show that \((a_{0,n} - a_{0,k}, \varphi) \rightarrow 0\) as \(n, k \rightarrow \infty\). Together with the conclusions above this will imply that \(\{a_{0,n}\}\) is a Cauchy sequence with respect to narrow convergence. Proposition 1.6 (2) then guarantees that it has a narrow limit \(a_0 \in M_{\omega_1}^{\infty, q_0}(\mathbb{R}^{2d})\), which we use as the definition of \(a_1 # \cdots # a_N\). It follows that the Weyl product extends uniquely from \(a_1 \in S_{1/2}\) to \(a_1 \in M_{\omega_1}^{\infty, q_1}\).
By Lemma 2.1 we have

\[ (a_{0,n} - a_{0,k}, \varphi) = C_{\Phi_0} (\mathcal{V}_{\Phi_0}(a_{0,n} - a_{0,k}), \mathcal{V}_{\Phi_0}\varphi) \]

\[ = C_{\Phi_0} \int \cdots \int_{\mathbb{R}^{2(N+1)d}} e^{2iQ(X_0,\ldots,X_N)} H_{n,k}(X_0,\ldots,X_N) \, dX_0 \cdots dX_N, \]

(2.8)

where

\[ H_{n,k}(X_0,\ldots,X_N) = 4^d \mathcal{V}_{\Phi_1}(a_{1,n} - a_{1,k})(X_1 + X_0, X_1 - X_0) \times \left( \prod_{j=2}^{N} F_j(X_j, X_{j-1}) \right) \frac{\mathcal{V}_{\Phi_0}\varphi(X_N + X_0, X_N - X_0)}{\mathcal{V}_{\Phi_0}\varphi(X_N + X_0, X_N - X_0)}. \]

By the narrow convergence we have \( \mathcal{V}_{\Phi_1} a_{1,n} \to \mathcal{V}_{\Phi_1} a_1 \) pointwise as \( n \to \infty \), which implies that

\[ \lim_{n,k \to \infty} H_{n,k}(X_0,\ldots,X_N) = 0, \quad (X_0,\ldots,X_N) \in \mathbb{R}^{2(N+1)d}. \]

(2.9)

If we define

\[ G(X, Y) = |\mathcal{V}_{\Phi_0}\varphi(X + Y, X - Y)| \omega_0(X + Y, X - Y), \]

then \( |H_{n,k}| \lesssim K_{n,k} \), where

\[ K_{n,k}(X_0,\ldots,X_N) \equiv (g_{1,n}(X_1 - X_0) + g_{1,k}(X_1 - X_0)) \left( \prod_{j=2}^{N} g_j(X_j - X_{j-1}) \right) |G(X_N, X_0)|. \]

By Young’s inequality and the assumption \( g_{1,n} \to g_1 \) in \( L^{q_1} \), \( K_{n,k} \) has a limit in \( L^1(\mathbb{R}^{2(N+1)d}) \), denoted \( K \). By the assumption \( g_{1,n} \to g_1 \) a. e., \( K_{n,k} \to K \) a. e. as \( n, k \to \infty \). Hence (2.8), (2.9) and Lemma 2.3 imply that \( (a_{0,n} - a_{0,k}, \varphi) \to 0 \) as \( n, k \to \infty \).

By the same arguments it follows that the integral formula (2.2) holds for the extension for almost all \( (X_N, X_0) \in \mathbb{R}^{4d} \). This finishes the proof of the uniqueness of the extended Weyl product inclusion (2.5).

The uniqueness in the cases (2.6) and (2.7) follow from the uniqueness in the case (2.5) and duality.

It remains to prove the asserted associativity, and first we need to prove that any subproduct of \( a_1 \# \cdots \# a_N \) is well defined. We observe that (10.3)’ can be written as

\[ 1 \lesssim \omega_0(X_N + X_0, X_N - X_0) \vartheta_1(X_0,\ldots,X_k) \vartheta_2(X_k,\ldots,X_N) \]

for

\[ \vartheta_1(X_0,\ldots,X_k) = \prod_{j=1}^{k} \omega_j(X_j + X_{j-1}, X_j - X_{j-1}) \]

and

\[ \vartheta_2(X_k,\ldots,X_N) = \prod_{j=k+1}^{N} \omega_j(X_j + X_{j-1}, X_j - X_{j-1}) \]

for some \( \omega_j \geq 0 \) in \( \mathbb{R}^{4d} \).
and
\[ \vartheta_2(X_k, \ldots, X_N) = \prod_{j=k+1}^{N} \omega_j(X_j + X_{j-1}, X_j - X_{j-1}), \]
and any \(1 \leq k \leq N - 1\). If \(\vartheta\) is defined by
\[ \vartheta(X_k + X_0, X_k - X_0) \equiv \inf \vartheta_1(X_0, \ldots, X_k) \]
where the infimum is taken over all \(X_1, \ldots, X_{k-1} \in \mathbb{R}^{2d}\), it follows from (0.3) that
\[ 1 \lesssim \vartheta(X_k + X_0, X_k - X_0)^{-1} \prod_{j=1}^{k} \omega_j(X_j + X_{j-1}, X_j - X_{j-1}), \]
\[ X_0, X_1, \ldots, X_k \in \mathbb{R}^{2d}, \]
and
\[ 1 \lesssim \omega_0(X_N + X_0, X_N - X_0) \vartheta(X_k + X_0, X_k - X_0) \prod_{j=k+1}^{N} \omega_j(X_j + X_{j-1}, X_j - X_{j-1}), \]
\[ X_0, X_k, X_{k+1}, \ldots, X_N \in \mathbb{R}^{2d}. \]

Note that \(\vartheta \in \mathcal{P}_E(\mathbb{R}^{4d})\) by the assumptions.

It now follows from the first part of the proof that
\[ a_1 \# \cdots \# a_k \in \mathcal{M}_{(\vartheta)}^{r,s} \]
and
\[ b \# a_{k+1} \# \cdots \# a_N \in \mathcal{M}_{(1/\omega)}^{p_0,q_0}, \]
when \(a_j \in \mathcal{M}_{(\omega_j)}^{p_j,q_j}, b \in \mathcal{M}_{(\vartheta)}^{r,s}, \) and \(r, s \in [1, \infty]\) are defined by
\[ \frac{1}{r} = \sum_{j=1}^{k} \frac{1}{p_j} \quad \text{and} \quad \frac{1}{s'} = \sum_{j=1}^{k} \frac{1}{q_j}. \]

This shows that \(a_1 \# \cdots \# a_k\) and \(a_{k+1} \# \cdots \# a_N\) are well-defined as elements in appropriate modulation spaces.

The asserted associativity now follows from the density arguments in the proof of the uniqueness, and the fact that the Weyl product is associative on \(\mathcal{S}_{1/2}\). \(\square\)

For appropriate weights \(\omega\) the space \(\mathcal{M}^2_{(\omega)}(\mathbb{R}^{2d})\) consists of symbols of Hilbert–Schmidt operators acting between certain modulation spaces (cf. [47, 49]). The following proposition, with \(p_j = q_j = 2\) for \(j = 0, \ldots, N\), is a manifestation of the fact that Hilbert–Schmidt operators are closed under composition. The result in that special case is a consequence of [26, Proposition 0.2], which concerns \(N = 2\), with \(p_j = q_j = 2, j = 0, 1, 2, \) and induction. The general result relaxes
the assumption on the exponents, and is an essential step towards the improvement Theorem 1.1 below.

**Proposition 2.5.** Let \( p_j, q_j \in [1, \infty], \ j = 0, 1, \ldots, N, \) and suppose

\[
\max \left( R_N(q'), 0 \right) \leq \min_{j=0,1,\ldots,N} \left( \frac{1}{p_j}, \frac{1}{q_j}, R_N(p) \right). \tag{2.10}
\]

Let \( \omega_j \in \mathcal{P}(\mathbf{R}^d), \ j = 0, 1, \ldots, N, \) and suppose (1.3)' holds. Then the map (0.1) from \( \mathcal{S}_{1/2}(\mathbf{R}^d) \times \cdots \times \mathcal{S}_{1/2}(\mathbf{R}^d) \) to \( \mathcal{S}_{1/2}(\mathbf{R}^d) \) extends uniquely to a continuous and associative map from \( M^{p_{1,q_1}}(\mathbf{R}^d) \times \cdots \times M^{p_{N,q_N}}(\mathbf{R}^d) \) to \( M^{p_{0,q_0}}(1/\omega_0)(\mathbf{R}^d) \).

**Proof.** First we prove the result for \( p_j = q_j = 2 \) for all \( 0 \leq j \leq N. \) Let \( a_j \in \mathcal{S}_{1/2}, \ j = 1, \ldots, N, \) and let

\[
G_j(X, Y) = |F_j(X, Y)\omega_j(X + Y, X - Y)|, \ j = 1, \ldots, N,
\]

where \( F_j \) are given by (2.1). Lemma 2.1 and repeated application of Hölder’s inequality give

\[
|F_0(X_N, X_0)|/\omega_0(X_N + X_0, X_N - X_0)
\]

\[
\lesssim \int \cdots \int_{\mathbf{R}^{2(N-1)d}} \left( \prod_{j=1}^{N} G_j(X_j, X_{j-1}) \right) dX_1 \cdots dX_{N-1}
\]

\[
\lesssim \|G_1(\cdot, X_0)\|_{L^2(\mathbf{R}^{2d})} \|G_N(X_N, \cdot)\|_{L^2(\mathbf{R}^{2d})} \prod_{j=2}^{N-1} \|G_j\|_{L^2(\mathbf{R}^{2d})}. \]

Taking the \( L^2(\mathbf{R}^{2d}) \) norm gives

\[
\|a_0\|_{M_2^{2,q_0}(1/\omega_0)} \lesssim \prod_{j=1}^{N} \|G_j\|_{L^2} \asymp \prod_{j=1}^{N} \|a_j\|_{M_2^{2,q_j}(\omega_j)}. \]

The claim follows from this estimate and the fact that \( \mathcal{S}_{1/2} \) is dense in \( M_2^{2}(\omega_j) \).

The proof of the general case is based on multi-linear interpolation between the case \( p_j = q_j = 2 \) for \( 0 \leq j \leq N \) and Proposition 2.2.

More precisely, by Proposition 2.2 and the first part of this proof we have

\[
M_{r_{1,q_1}}(\omega_1) \# \cdots \# M_{s_N,q_N}(\omega_N) \subseteq M_{r,q_0}(1/\omega_0)
\]

and

\[
M_{r_{1,q_1}}^{2,2}(\omega_1) \# \cdots \# M_{s_N,q_N}^{2,2}(\omega_N) \subseteq M_{r,q_0}^{2,2}(1/\omega_0),
\]

when \( r_j, s_j \in [1, \infty], \ j = 0, 1, \ldots, N, \) and

\[
R_N(s') \leq 0 \leq R_N(r). \tag{2.11}
\]

By multi-linear interpolation, using [1, Theorem 4.4.1] and Proposition 1.2 (5), we get

\[
M_{r_{1,q_1}}^{p_{1,q_1}} \cdots M_{s_N,q_N}^{p_{N,q_N}}(\omega_N) \subseteq M_{r,q_0}^{p,q_0}(1/\omega_0). \tag{2.12}
\]
when
\[ \frac{1}{p_j} = \frac{1 - \theta}{r_j} + \frac{\theta}{2} \quad \text{and} \quad \frac{1}{q_j} = \frac{1 - \theta}{s_j} + \frac{\theta}{2}, \quad 0 \leq \theta \leq 1, \quad \text{(2.13)} \]

\( j = 0, 1, \ldots, N. \)

Suppose \( p_j, q_j \in [1, \infty], \ j = 0, 1, \ldots, N, \) satisfy \( (2.10) \). We have to show that there exist \( 0 \leq \theta \leq 1, r_j \in [1, \infty] \) and \( s_j \in [1, \infty], \ j = 0, 1, \ldots, N, \) such that \( (2.11) \) and \( (2.13) \) are satisfied, after which \( (2.12) \) follows by multi-linear interpolation. Our plan is to first find an appropriate \( \theta \), and then find \( r \) and \( s \) with the right properties.

We have \( r_j \in [1, \infty] \) if and only if
\[ 0 \leq \frac{1 - \theta}{r_j} = \frac{1}{p_j} - \frac{\theta}{2} \leq 1 - \theta, \]
i.e. \( \theta/2 \leq \min(1/p_j, 1/p'_j) \), and likewise \( s_j \in [1, \infty] \) if and only if \( \theta/2 \leq \min(1/q_j, 1/q'_j) \). Since \( R_N(q') \leq 1/2 \) as a consequence of \( (2.10) \), there exists \( \theta \in [0, 1] \) such that
\[ R_N(q') \leq \frac{\theta}{2} \leq \min_{j=0,1,\ldots,N} \left( \frac{1}{q_j}, \frac{1}{p_j}, \frac{1}{p'_j}, R_N(p) \right) \]
again by the assumption \( (2.10) \). With such a choice of \( \theta \) we have \( r_j, s_j \in [1, \infty] \) for \( j = 0, 1, \ldots, N, \) and
\[ R_N(q') \leq \theta/2 \leq R_N(p). \]
This gives
\[ R_N(r) = \frac{1}{1 - \theta} \left( R_N(p) - \frac{\theta}{2} \right) \geq 0 \]
and
\[ R_N(s') = \frac{1}{1 - \theta} \left( R_N(q') - \frac{\theta}{2} \right) \leq \frac{1}{1 - \theta} \left( \frac{\theta}{2} - \frac{\theta}{2} \right) = 0. \]
Hence \( (2.11) \) and \( (2.13) \) are satisfied, and \( (2.12) \) follows. Thus \( (2.10) \) implies \( (2.12) \).

It remains to prove the associativity. If \( R_N(q') \leq 0 \leq R_N(p) \) the associativity follows from Proposition \( (2.11) \), and if \( p_j = q_j = 2, \ j = 0, \ldots, N, \) the associativity follows from the associativity of the Weyl product on \( S_{1/2} \), and the fact that \( S_{1/2} \) is dense in \( M_{(\omega_j)}^{2,2} \) for every \( j \).

The associativity now follows in general from the fact that the general case is an interpolation between the latter two cases. \( \square \)

**Remark 2.6.** A crucial step in the proof is the fact that \( (2.10) \) implies that \( \theta \) and \( r, s \in [1, \infty]^{N+1} \) can be chosen such that \( (2.11) \) and \( (2.13) \) holds. On the other hand, by straight-forward computations it follows that if \( (2.11) \) and \( (2.13) \) are fulfilled, then \( (2.10) \) holds.
Remark 2.7. We note that Proposition 2.5 extends [26, Theorem 0.3′]. The latter result asserts that if $N = 2$,
\[
R_2(p) = R_2(q'), \quad q_1, q_2 \leq q_0',
\]
and \(q_j, j = 0, 1, 2\), are weights that satisfy (0.3), then the map (0.1) extends to a continuous map from \(M_{(\omega_1)}(p) \times M_{(\omega_1)}(q)\) to \(M_{(1/\omega_0)}(p)\). We claim that (2.14) implies (2.10) when $N = 2$, which means that Proposition 2.5 extends [26, Theorem 0.3′].

In fact, by the last inequality in (2.14) we get
\[
R_2(q') \leq \frac{1}{p_j}, \frac{1}{q_j}, \frac{1}{p_0'}, \frac{1}{q_0'}, \quad j = 1, 2.
\]
A combination of these inequalities gives
\[
R_2(q') = R_2(p) \leq \min_{j=0,1,2} \left( \frac{1}{p_j}, \frac{1}{q_j}, \frac{1}{p_0'}, \frac{1}{q_0'} \right),
\]
and it follows that the hypothesis (2.10) in Proposition 2.5 is fulfilled for $N = 2$.

Next we prove that the conclusion of Proposition 2.5 holds under assumptions that are weaker than (2.10). The following lemma shows that we may omit the condition $R_N(q') \leq \min_{0 \leq j \leq N}(1/q_j)$ in (2.10).

Lemma 2.8. Let $N \geq 2$, $x_j \in [0,1]$, $j = 0, \ldots, N$ and consider the inequalities:
\[
\begin{align*}
(1) \quad & (N - 1)^{-1} \left( \sum_{k=0}^{N} x_k - 1 \right) \leq \min_{0 \leq j \leq N} x_j; \\
(2) \quad & x_j + x_k \leq 1, \text{ for all } k \neq j; \\
(3) \quad & (N - 1)^{-1} \left( \sum_{k=0}^{N} x_k - 1 \right) \leq \min_{0 \leq j \leq N} (1 - x_j).
\end{align*}
\]
Then
\[
(1) \implies (2) \implies (3).
\]
If $N = 2$ then (1) and (2) are equivalent.

Proof. Assume that (1) holds but (2) fails. Then $x_j + x_k > 1$ for some $j \neq k$. By renumbering we may assume that $x_2 \leq x_j$ for every $j$, and that $x_0 + x_1 > 1$. Then (1) gives
\[
(N - 1)x_2 \leq \sum_{k=2}^{N} x_k < \sum_{k=0}^{N} x_k - 1 \leq (N - 1)x_2
\]
which is a contradiction. Hence the assumption $x_0 + x_1 > 1$ must be wrong and it follows that (1) implies (2).
Now assume that (2) holds. Then

\[ x_j \leq 1 - x_k, \quad k \neq j. \]

This gives

\[ \sum_{j \neq k} x_j \leq N(1 - x_k) \iff \sum_{j=0}^N x_j - 1 \leq (N - 1)(1 - x_k) \]

for any \( k \), so (3) holds.

Finally, if \( j \neq k, N = 2 \) and (2) holds, then \( x_j + x_k \leq 1 \), which gives \( x_j + x_k + x_l - 1 \leq x_l, \ l = 0, 1, 2 \). In particular,

\[ \sum_{k=0}^2 x_k - 1 \leq x_l, \quad l = 0, 1, 2, \]

and (1) follows. \( \square \)

The next result is one of two principal theorems on sufficient conditions for continuity. It shows that one can eliminate some conditions on the Lebesgue exponents in Proposition 2.5. In particular the result extends [26, Theorem 0.3'] in view of Remark 2.7.

**Theorem 0.1.** Let \( p_j, q_j \in [1, \infty] \), \( j = 0, 1, \ldots, N \), and suppose

\[ \max (R_N(q'), 0) \leq \min_{j=0,1,\ldots,N} \left( \frac{1}{p_j}, \frac{1}{q_j}, R_N(p) \right). \] \( \text{(0.4')} \)

Let \( \omega_j \in \mathscr{P}_E(R^d) \), \( j = 0, 1, \ldots, N \), and suppose (0.3') holds. Then the map (0.1)' from \( S_{1/2}(R^d) \times \cdots \times S_{1/2}(R^d) \) to \( S_{1/2}(R^d) \) extends uniquely to a continuous and associative map from \( M^{p_j,q_j}_\omega(R^d) \times \cdots \times M^{p_N,q_N}_\omega(R^d) \) to \( M^{p_0,q_0}_{1/\omega_0}(R^d) \).

**Proof.** We may assume that \( R_N(q') > 0 \) since otherwise the result follows from Proposition 2.2. By Lemma 2.8 the conditions (0.4') imply

\[ R_N(q') \leq \min_{j=0,1,\ldots,N} \left( \frac{1}{q_j}, \frac{1}{q_j} \right). \] \( \text{(2.15)} \)

Hence, if \( r \) is defined by

\[ \frac{1}{r} \equiv R_N(q'), \]

then \( r \geq 2. \)

By Proposition 2.5 and (2.15) we have

\[ M^{u_1,q_1}_\omega \# \cdots \# M^{u_N,q_N}_\omega \subseteq M^{u_0,q_0}_{1/\omega_0}, \]

when \( u_j \in [1, \infty] \) for \( 0 \leq j \leq N \) and

\[ \frac{1}{r} \leq \min_{j=0,1,\ldots,N} \left( \frac{1}{u_j}, \frac{1}{u_j}, \frac{1}{q_j}, R_N(u) \right). \] \( \text{(2.16)} \)
Due to Proposition 1.2 (2) the result follows if we can prove that $p_j \leq u_j$ for some $u_j \in [1, \infty]$, $j = 0, \ldots, N$, that satisfy (2.10). We claim that $u_j = \max(p_j, r^i)$, $j = 0, \ldots, N$, satisfy (2.16).

To wit, for such a choice we have

$$\frac{1}{u_j} = \max \left( \frac{1}{p_j}, \frac{1}{r^i} \right) \geq \frac{1}{r^i}, \quad j = 0, \ldots, N,$$

and

$$\frac{1}{u_j} = \min \left( \frac{1}{p_j}, \frac{1}{r^i} \right) \geq \frac{1}{r^i}, \quad j = 0, \ldots, N,$$

where (2.17) follows from $r \geq 2$ and the assumption $p_j \leq r$.

Let $I$ be the set of all $j$ such that $r^i \leq p_j$. If $I = \{0, 1, \ldots, N\}$ the result follows from Proposition 2.5. Therefore we may assume that there exists $k \in \{0, 1, \ldots, N\}$ such that $k \notin I$. Then $u_k = r^i$, and (2.17) gives

$$(N - 1)R_N(u) = \frac{1}{u_k} + \sum_{j \neq k} \frac{1}{u_j} - 1 = \frac{1}{r^i} + \sum_{j \neq k} \frac{1}{u_j} - 1$$

$$= -\frac{1}{r} + \sum_{j \neq k} \frac{1}{u_j} \geq -\frac{1}{r} + \frac{N}{r} = \frac{N - 1}{r}.$$

Hence

$$R_N(u) \geq \frac{1}{r},$$

and the continuity assertion follows.

The uniqueness and associativity follows from Proposition 2.5 and the inclusions above.

In the next section we prove that Theorem 0.1’ is sharp for $N = 2$ with respect to the conditions on the Lebesgue exponents. On the other hand, for $N \geq 3$, the result cannot be sharp. In fact, Theorem 0.1’ with $N = 2$ gives in particular that every unweighted modulation space $\mathcal{M}^{p,q}$ is an $\mathcal{M}^{\infty,1}$-module. This property combined with the fact that $\mathcal{M}^{2,2}$ is an algebra under the Weyl product give the inclusion

$$\mathcal{M}^{\infty,1} \# \mathcal{M}^{2,2} \# \mathcal{M}^{2,2} \subseteq \mathcal{M}^{2,2}. \quad (2.18)$$

Theorem 0.1 does however not contain this inclusion.

The next result gives another sufficient condition for the map (0.1)’ to be continuous that contains the inclusion (2.18). In the bilinear case $N = 2$ the result follows from Theorem 0.1, because of the sharpness of the latter result in that case.

**Theorem 2.9.** Let $p_j, q_j \in [1, \infty]$, $j = 0, 1, \ldots, N$, and suppose

$$R_N(p) \geq 0 \quad \text{and} \quad \frac{1}{q_j} \leq \frac{1}{p_j} \leq \frac{1}{2}. \quad (2.19)$$
Let \( \omega_j \in \mathcal{P}_E(\mathbb{R}^{4d}) \), \( j = 0, 1, \ldots, N \), and suppose \([0,1]'\) holds. Then the map \( (0.1)' \) from \( \mathcal{S}_{1/2}(\mathbb{R}^{2d}) \times \cdots \times \mathcal{S}_{1/2}(\mathbb{R}^{2d}) \) to \( \mathcal{S}_{1/2}(\mathbb{R}^{2d}) \) extends uniquely to a continuous and associative map from \( \mathcal{M}^{p,q}_{(\omega_1)}(\mathbb{R}^{2d}) \times \cdots \times \mathcal{M}^{p,q}_{(\omega_N)}(\mathbb{R}^{2d}) \) to \( \mathcal{M}^{p,q}_{(1/\omega_0)}(\mathbb{R}^{2d}) \).

The proof is by induction over \( N \), and we need the existence of certain intermediate weights. The following lemma guarantees the existence of such weights.

**Lemma 2.10.** Let \( \omega_0, \ldots, \omega_N \in \mathcal{P}_E(\mathbb{R}^{4d}) \) satisfy \([0,3]'\). Then there exists a weight \( \vartheta \in \mathcal{P}_E(\mathbb{R}^{4d}) \) such that

\[
1 \lesssim \frac{\omega_0(X_2 + X_0, X_2 - X_0) \omega_N(X_2 + X_1, X_2 - X_1)}{\vartheta(X_1 + X_0, X_1 - X_0)}, \quad X_0, X_1, X_2 \in \mathbb{R}^{2d},
\]

\[
1 \lesssim \vartheta(X_{N-1} + X_0, X_{N-1} - X_0) \prod_{j=1}^{N-1} \omega_j(X_j + X_{j-1}, X_j - X_{j-1}),
\]

\( X_0, \ldots, X_{N-1} \in \mathbb{R}^{2d} \).

**(2.20)**

**Proof.** Let \( X = X_{N-1} + X_0, Y = X_{N-1} - X_0 \), define the linear mappings from \( \mathbb{R}^{4d} \) to \( \mathbb{R}^{4d} \) given by

\[
T_{j,k}(X, Y, Z) = \left( \frac{X + (-1)^j Y}{2} + Z, (-1)^k \left( \frac{X + (-1)^j Y}{2} - Z \right) \right),
\]

for \( j, k = 1, 2 \), and set

\[
H_1(X_1, \ldots, X_{N-2}) \equiv \prod_{j=2}^{N-2} \omega_j(X_j + X_{j-1}, X_j - X_{j-1}),
\]

\[
H_2(X_1, X_{N-2}, X, Y) \equiv \omega_1(T_{1,1}(X, Y, X_1)) \omega_{N-1}(T_{2,2}(X, Y, X_{N-2}))
\]

and

\[
H_3(X_N, X, Y) \equiv \omega_0(T_{1,1}(X, Y, X_N)) \omega_N(T_{2,1}(X, Y, X_N)).
\]

Then \([0,3]'\) is equivalent to

\[
(H_2(X_1, X_{N-2}, X, Y) H_1(X_1, \ldots, X_{N-2}))^{-1} \lesssim H_3(X_N, X, Y).
\]

The left hand side is independent of \( X_N \) and the right hand side is independent of \( X_1, \ldots, X_{N-2} \).

If we define

\[
\vartheta(X, Y) \equiv \sup_{X_1, \ldots, X_{N-2} \in \mathbb{R}^{2d}} (H_2(X_1, X_{N-2}, X, Y) H_1(X_1, \ldots, X_{N-2}))^{-1}
\]

then (2.20) holds.
then
\[ \vartheta(X_{N-1} + X_0, X_{N-1} - X_0) \lesssim H_3(X_N, X_{N-1} + X_0, X_{N-1} - X_0), \]
\[ X_0, X_{N-1}, X_N \in \mathbb{R}^{2d}, \]
and (2.20) holds.

It remains to show \( \vartheta \in \mathcal{P}_E(\mathbb{R}^{4d}) \). For \( \varepsilon > 0 \) arbitrary we have
\[ \vartheta(Z_1 + Z_2, Y_1 + Y_2) \leq (H_2(X_1, X_{N-2}, Z_1 + Z_2, Y_1 + Y_2) H_1(X_1, \ldots, X_{N-2}))^{-1} + \varepsilon \]
for some choice of \( X_1, \ldots, X_{N-2} \). Since each \( \omega_j \) is moderate we have
\[ (H_2(X_1, X_{N-2}, Z_1 + Z_2, Y_1 + Y_2))^{-1} \leq C (H_2(X_1, X_{N-2}, Z_1, Y_1))^{-1} v(Z_2, Y_2), \]
for \( C > 0 \) and some submultiplicative function \( v \in L_{\text{loc}}^\infty(\mathbb{R}^{4d}) \), which depends on \( \omega_1 \) and \( \omega_{N-1} \).

This estimate yields
\[ \vartheta(Z_1 + Z_2, Y_1 + Y_2) \leq C (H_2(X_1, X_{N-2}, Z_1, Y_1) H_1(X_1, \ldots, X_{N-2}))^{-1} v(Z_2, Y_2) + \varepsilon \]
\[ \leq C \vartheta(Z_1, Y_1)v(Z_2, Y_2) + \varepsilon. \]

Since \( \varepsilon > 0 \) is arbitrary and \( C \) does not depend on \( \varepsilon \), it follows that \( \vartheta \) is \( v \)-moderate, and we may conclude that \( \vartheta \in \mathcal{P}_E(\mathbb{R}^{4d}) \). \( \square \)

**Proof of Theorem 2.9.** By Proposition 1.2 (2) we may assume \( q'_j = p_j \), \( j = 0, \ldots, N \).

We start by proving the result for \( N = 2 \). Assume (2.19) for \( N = 2 \).

Then for every fixed \( j \in \{0, 1, 2\} \) we get
\[ R_2(p) = R_2(q') = \sum_{k=0}^{2} \frac{1}{p_k} - 1 \leq \frac{1}{p_j}. \]

The continuity statement now follows from Theorem 0.1.

Next we perform the induction step. We assume that \( N \geq 3 \) and the result holds for lower values of \( N \). In particular we assume the inclusion
\[ \mathcal{M}^{r_1, r'_1}_{(\omega_1)} \# \cdots \# \mathcal{M}^{r_{N-1}, r'_{N-1}}_{(\omega_{N-1})} \subseteq \mathcal{M}^{r_0, r_0}_{(1/\vartheta)} \]
whenever \( r_j \geq 2 \),
\[ \sum_{j=0}^{N-1} \frac{1}{r_j} \geq 1, \]
and \( (\vartheta, \omega_1, \ldots, \omega_{N-1}) \in \mathcal{P}_E(\mathbb{R}^{4d})^N \) satisfy (0.3)'.

We now distinguish two cases.
In the first case we suppose that
\[ \frac{1}{p_0} + \frac{1}{p_N} \leq \frac{1}{2} \quad \text{or} \quad \frac{1}{p_j} + \frac{1}{p_{j+1}} \leq \frac{1}{2} \] for some \( j \in \{0, \ldots, N - 1\} \).

By (1.12) and duality it suffices to consider the case when the first inequality in (2.22) holds. We define \( r_0 \) as
\[ \frac{1}{r_0} = \frac{1}{p_0} + \frac{1}{p_N} \leq \frac{1}{2}, \]
and the result follows if we prove the inclusions
\[ M_{(\omega_1)}^{p_1, p'_1} \# \cdots \# M_{(\omega_{N-1})}^{p_{N-1}, p'_{N-1}} \subseteq M_{(1/\vartheta)}^{r_0, r_0} \] (2.23)
and
\[ M_{(1/\vartheta)}^{r_0, r_0} \# M_{(\omega_N)}^{p_{N-1}, p'_{N-1}} \subseteq M_{(1/\omega_0)}^{p_0, p_0}, \] (2.24)
where \( \vartheta \) is chosen according to Lemma 2.10.

Since
\[ \frac{1}{r_0} + \sum_{k=1}^{N-1} \frac{1}{p_k} = \sum_{k=0}^{N} \frac{1}{p_k} \geq 1 \quad \text{and} \quad \frac{1}{r_0}, \frac{1}{p_j} \leq \frac{1}{2}, \quad j = 1, \ldots, N - 1, \]
the inclusion (2.23) follows from the induction assumption (2.21).

By letting \( s_0 = p_0, s_1 = r_0', s_2 = p_N \), it follows from the choice of \( r_0 \) that \( R_2(s) = 0 \), and the inclusion (2.24) follows from Proposition 2.2 since \((\omega_0, 1/\vartheta, \omega_N)\) satisfy (0.3)' for \( N = 2 \) by Lemma 2.10. The induction hypothesis (2.21) thus gives
\[ M_{(1/\vartheta)}^{p_1, p'_1} \# \cdots \# M_{(\omega_{N-1})}^{p_{N-1}, p'_{N-1}} \subseteq M_{(1/\omega_0)}^{2, 2}. \] (2.27)
Combining (2.26) and (2.27) gives the induction step in the second case. The induction step is thus complete so the continuity statement holds for any integer \( N \geq 2 \).

Finally, the uniqueness and associativity of the extension follows as in the proof of Proposition 2.2. In fact, if \( p_j = \infty \) then by the assumptions \( q_j = 1 \), and a factor \( a_j \in \mathcal{M}_{\infty,1}^{\omega_j} \) can be approximated narrowly by elements in \( \mathcal{S}_{1/2} \). If \( p_j < \infty \) then the assumption \( 2 \leq q_j \) implies that a factor \( a_j \in \mathcal{M}_{\omega_j}^{p_j, q_j} \) can be approximated in norm by elements in \( \mathcal{S}_{1/2} \). □

We may use (1.7) and Proposition 1.4 to extend Theorems 0.1 and 2.9 to concern not only the Weyl product but general products arising in the pseudo-differential calculi (1.3) indexed by \( t \in \mathbb{R} \). More precisely, for every \( t \in \mathbb{R} \), the \( \#_t \) product with \( N \) factors

\[
(a_1, \ldots, a_N) \mapsto a_1 \#_t \cdots \#_t a_N \quad \tag{2.28}
\]

from \( \mathcal{S}_{1/2}(\mathbb{R}^{2d}) \times \cdots \times \mathcal{S}_{1/2}(\mathbb{R}^{2d}) \) to \( \mathcal{S}_{1/2}(\mathbb{R}^{2d}) \) is defined by the formula

\[
\text{Op}_t(a_1 \#_t \cdots \#_t a_N) = \text{Op}_t(a_1) \circ \cdots \circ \text{Op}_t(a_N).
\]

By (1.7) we have

\[
a_1 \#_t \cdots \#_t a_N = e^{i \omega_0(D_x, D_t)}((e^{-i \omega_0(D_x, D_t)} a_1) \#_t \cdots \#_t (e^{-i \omega_0(D_x, D_t)} a_N)),
\]

\[
t_0 = t - \frac{1}{2}.
\]

If we combine this relation with Proposition 1.4, Theorems 0.1 and 2.9 we get the following result. The condition on the weight functions is

\[
1 \lesssim \omega_0(T_t(X_N, X_0)) \prod_{j=1}^N \omega_j(T_t(X_j, X_{j-1})), \quad X_0, \ldots, X_N \in \mathbb{R}^{2d}, \quad \tag{2.29}
\]

where

\[
T_t(X, Y) = ((1 - t)x + ty, t\xi + (1 - t)\eta, \eta - \xi, y - x),
\]

\[
X = (x, \xi) \in \mathbb{R}^{2d}, \quad Y = (y, \eta) \in \mathbb{R}^{2d}, \quad \tag{2.30}
\]

**Theorem 2.11.** Let \( p_j, q_j \in [1, \infty], j = 0, 1, \ldots, N, \) be as in Theorems 0.1 or 2.9. Let \( t \in \mathbb{R} \), \( \omega_j \in \mathcal{P}_E(\mathbb{R}^{2d}), j = 0, 1, \ldots, N, \) and suppose (2.29) and (2.30) hold. Then the map (2.28) from \( \mathcal{S}_{1/2}(\mathbb{R}^{2d}) \times \cdots \times \mathcal{S}_{1/2}(\mathbb{R}^{2d}) \) to \( \mathcal{S}_{1/2}(\mathbb{R}^{2d}) \) extends uniquely to a continuous and associative map from \( \mathcal{M}_{\omega_1}^{p_1, q_1}(\mathbb{R}^{2d}) \times \cdots \times \mathcal{M}_{\omega_N}^{p_N, q_N}(\mathbb{R}^{2d}) \) to \( \mathcal{M}_{\omega_1}^{p_1, q_1}(\mathbb{R}^{2d}) \).

Finally we prove a continuity result for the twisted convolution. The map (0.1)' is then replaced by

\[
(a_1, a_2, \ldots, a_N) \mapsto a_1 \ast_\sigma a_2 \ast_\sigma \cdots \ast_\sigma a_N. \quad \tag{2.31}
\]
The following result follows immediately from Theorem 0.1 and Theorem 2.9. Here the condition (0.3)' is replaced by

\[ 1 \lesssim \omega_0(X_N - X_0, X_N + X_0) \prod_{j=1}^{N} \omega_j(X_j - X_{j-1}, X_j + X_{j-1}), \]

\[ X_0, X_1, \ldots, X_N \in \mathbb{R}^{2d}. \quad (2.32) \]

**Theorem 2.12.** Let \( p_j, q_j \in [1, \infty], \ j = 0, 1, \ldots, N, \) and suppose that

\[ \max (R_N(p'), 0) \leq \min_{j=0,1,\ldots,N} \left( \frac{1}{p_j'}, \frac{1}{q_j} \right) \]

or

\[ R_N(q') \geq 0 \quad \text{and} \quad \frac{1}{p_j'} \leq \frac{1}{q_j} \leq \frac{1}{2}. \]

Suppose \( \omega_j \in \mathcal{P}(\mathbb{R}^{2d}), \ j = 0, 1, \ldots, N, \) satisfy (2.32). Then the map (2.31) from \( S_{1/2}(\mathbb{R}^{2d}) \times \cdots \times S_{1/2}(\mathbb{R}^{2d}) \) to \( S_{1/2}(\mathbb{R}^{2d}) \) extends uniquely to a continuous and associative map from \( \mathcal{W}^{p_1,q_1}_{(\omega_1)}(\mathbb{R}^{2d}) \times \cdots \times \mathcal{W}^{p_N,q_N}_{(\omega_N)}(\mathbb{R}^{2d}) \) to \( \mathcal{W}^{p_0,q_0}_{(1/\omega_0)}(\mathbb{R}^{2d}). \)

### 3. Necessary boundedness conditions

In this section we prove necessary conditions for continuity of the Weyl product when \( N = 2 \) and certain polynomially moderate weight triplets that satisfy (0.3).

More precisely, for weights of the form

\[ \omega_0(X,Y) = \vartheta_0(X + Y), \quad \omega_1(X,Y) = \vartheta_2(X - Y), \]

\[ \omega_2(X,Y) = \vartheta_1(X - Y), \]

where \( \vartheta_j \in \mathcal{P}(\mathbb{R}^{2d}), \ j = 0, 1, 2, \) we have the following result. Note that the necessary condition (3.2) equals the sufficient condition (0.3) of Theorem 0.1.

**Theorem 3.1.** Let \( p_j, q_j \in [1, \infty], \vartheta_j \in \mathcal{P}(\mathbb{R}^{2d}), \) and define \( \omega_j \in \mathcal{P}(\mathbb{R}^{2d}) \) by (3.1) for \( j = 0, 1, 2. \) If the map (1.1) from \( S_{1/2}(\mathbb{R}^{2d}) \times S_{1/2}(\mathbb{R}^{2d}) \) to \( S_{1/2}(\mathbb{R}^{2d}) \) is extendable to a continuous map from \( \mathcal{M}^{p_1,q_1}_{(\omega_1)}(\mathbb{R}^{2d}) \times \mathcal{M}^{p_2,q_2}_{(\omega_2)}(\mathbb{R}^{2d}) \) to \( \mathcal{M}^{p_0,q_0}_{(1/\omega_0)}(\mathbb{R}^{2d}), \) then

\[ \max(R_2(q'), 0) \leq \min_{j=0,1,2} \left( \frac{1}{p_j'}, \frac{1}{q_j'} \right) R_2(p'). \quad (3.2) \]

**Remark 3.2.** The conditions (3.1) on the weights appear naturally when Weyl operators with symbols in modulation spaces act on modulation spaces. For example, if (3.1) is fulfilled, \( p, q \in [1, \infty] \) and \( a_1 \in \)
$\mathcal{M}_{_{(\omega)}}^1(\mathbb{R}^{2d})$, then $\text{Op}^w(a_1)$ is continuous from $M^{p,q}_{\mathcal{S}_{(\vartheta_2)}}(\mathbb{R}^d)$ to $M^{p,q}_{\mathcal{S}_{(\vartheta_1)}}(\mathbb{R}^d)$ (cf. [15, Theorem 6.2]).

We need some preparations for the proof. The first result is a reduction to trivial weights. For $\omega \in \mathcal{P}(\mathbb{R}^{2d})$ we use the notation $S_{(\omega)}(\mathbb{R}^{2d})$ for the symbol space of all $a \in C^\infty(\mathbb{R}^{2d})$ such that $(\partial^a a)/\omega \in L^\infty$ for all $a \in \mathbb{N}^{2d}$.

**Lemma 3.3.** Let $\vartheta, \vartheta_1, \vartheta_2 \in \mathcal{P}(\mathbb{R}^{2d})$, $\omega_1, \omega_2 \in \mathcal{P}(\mathbb{R}^d)$ and let $p, q \in [1, \infty]$. There exist $b_j \in S_{(\vartheta_j)}(\mathbb{R}^{2d})$ and $c_j \in S_{(1/\vartheta_j)}(\mathbb{R}^{2d})$ such that

$$\text{Op}^w(b_j) \circ \text{Op}^w(c_j) = \text{Op}^w(c_j) \circ \text{Op}^w(b_j)$$

is the identity map on $\mathcal{S}'(\mathbb{R}^d)$, for $j = 1, 2$, and the following holds:

1. $\text{Op}^w(b_j)$ is continuous and bijective from $M^{p,q}_{\mathcal{S}_{(1)}}(\mathbb{R}^d)$ to $M^{p,q}_{\mathcal{S}_{(\vartheta_j)}}(\mathbb{R}^d)$ with inverse $\text{Op}^w(c_j)$, $j = 1, 2$;
2. If $\omega_2(X, Y) \lesssim \omega_1(X, Y)/\vartheta(X + Y)$, then the map (\ref{eq:lem3.3.1}) on $\mathcal{S}'(\mathbb{R}^{2d})$ extends uniquely to a continuous map from $M^{p,q}_{\mathcal{S}_{(\omega_1)}}(\mathbb{R}^{2d}) \times S_{(\vartheta)}(\mathbb{R}^{2d})$ to $M^{p,q}_{\mathcal{S}_{(\omega_2)}}(\mathbb{R}^{2d})$;
3. If $\omega_2(X, Y) \lesssim \omega_1(X, Y)/\vartheta(X - Y)$, then the map (\ref{eq:lem3.3.1}) on $\mathcal{S}'(\mathbb{R}^{2d})$ extends uniquely to a continuous map from $S_{(\vartheta)}(\mathbb{R}^{2d}) \times M^{p,q}_{\mathcal{S}_{(\omega_1)}}(\mathbb{R}^{2d})$ to $M^{p,q}_{\mathcal{S}_{(\omega_2)}}(\mathbb{R}^{2d})$;
4. If $\omega(X, Y) = \vartheta_2(X - Y)/\vartheta_1(X + Y)$, then the map $a \mapsto b_2 a - a c_1$ is continuous on $\mathcal{S}'(\mathbb{R}^{2d})$, and extends uniquely to a continuous and bijective map from $M^{p,q}_{\mathcal{S}_{(\omega_1)}}(\mathbb{R}^{2d})$ to $M^{p,q}_{\mathcal{S}_{(\omega_2)}}(\mathbb{R}^{2d})$.

**Proof.** The assertion (1) follows immediately from [23, Theorem 3.1]. In order to prove (2) and (3), we first use the assumption that $\omega_1$ and $\vartheta$ are moderate, which gives

$$\omega_1(X, Y) \lesssim \omega_1(X - Y + Z, Z)\langle Y - Z \rangle^N,$$

$$\omega_1(X, Y) \lesssim \omega_1(X + Z, Y - Z)\langle Z \rangle^N,$$

$$\frac{1}{\vartheta(X + Y)} \lesssim \frac{\langle Y - Z \rangle^N}{\vartheta(X + Z)}, \quad \frac{1}{\vartheta(X - Y)} \lesssim \frac{\langle Z \rangle^N}{\vartheta(X - Y + Z)},$$

for some $N \geq 0$. The assumption in (2) leads to

$$\omega_2(X, Y) \lesssim \omega_1(X - Y + Z, Z)\frac{\langle Y - Z \rangle^{2N}}{\vartheta(X + Z)},$$

and the assumption in (3) gives

$$\omega_2(X, Y) \lesssim \frac{\langle Z \rangle^{2N}}{\vartheta(X - Y + Z)} \omega_1(X + Z, Y - Z).$$

If we set $\omega(X, Y) = \langle Y \rangle^{2N}/\vartheta(X)$ then Theorem 0.1 with \ref{eq:lem3.3.1} and \ref{eq:lem3.3.3}, respectively, shows that the map (\ref{eq:lem3.3.1}) from $\mathcal{S} \times \mathcal{S}$ to $\mathcal{S}$ extends
uniquely to a continuous map from $M_{p,q}^{\infty,1} \times M_{\omega}^{\infty,1} \times M_{\omega}^{p,q}$, respectively, to $M_{\omega}^{p,q}$. The assertions (2) and (3) now follow from $S(\emptyset) \subseteq M_{\omega}^{\infty,1}$, which is a consequence of [23, Proposition 2.7 (3)].

Finally (4) follows by a straight-forward combination of (1)–(3). □

Lemma 3.3 shows that for weights $\omega_j$, $j = 0, 1, 2$, satisfying (3.1) we have

$$\|a \# b\|_{M_{p_0,q_0}^{p_0',q_0'}} \lesssim \|a\|_{M_{\omega_1}^{p_1,q_1}} \|b\|_{M_{\omega_2}^{p_2,q_2}}, \quad a, b \in S_{1/2}(\mathbb{R}^{2d}),$$

if and only if

$$\|a \# b\|_{M_{p_0,q_0}^{p_0',q_0'}} \lesssim \|a\|_{M_{p_1,q_1}} \|b\|_{M_{p_2,q_2}}, \quad a, b \in S_{1/2}(\mathbb{R}^{2d}),$$

thus reducing the problem to the case of trivial weights.

It remains to show Theorem 3.1 with trivial weights. By the last part of Lemma 2.8, the Condition (3.2) is equivalent to the inequalities

$$1 \leq \frac{1}{p_0} + \frac{1}{p_1} + \frac{1}{p_2},$$

$$1 \leq \frac{1}{q_j}, \quad 0 \leq j \neq k \leq 2,$$

$$2 - \frac{1}{q_0} - \frac{1}{q_1} - \frac{1}{q_2} \leq \frac{1}{p_j}, \quad j = 0, 1, 2,$$

$$3 \leq \frac{1}{q_0} + \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{p_0} + \frac{1}{p_1} + \frac{1}{p_2},$$

which we prove in the following sequence of lemmas.

The first lemma shows that (3.5) and (3.6) are necessary for the requested continuity.

**Lemma 3.4.** Let $1 \leq p_j, q_j \leq \infty$ for $j = 0, 1, 2$, and suppose

$$\|a \# b\|_{M_{p_0,q_0}^{p_0',q_0'}} \lesssim \|a\|_{M_{p_1,q_1}} \|b\|_{M_{p_2,q_2}} \quad \text{for every} \quad a, b \in S_{1/2}(\mathbb{R}^{2d}).$$

Then (3.5) and (3.6) hold.

**Proof.** First we observe that the assumption (3.9) combined with (1.12), duality and $a \# b = b \# a$ (cf. [20]) imply that

$$\|a \# b\|_{M_{r_0,s_0}^{r_0',s_0'}} \lesssim \|a\|_{M_{r_1,s_1}} \|b\|_{M_{r_2,s_2}} \quad \text{for every} \quad a, b \in S_{1/2}(\mathbb{R}^{2d}),$$

when $r_j = p_{\sigma(j)}$ and $s_j = q_{\sigma(j)}$, where $\sigma$ is any permutation $\{0, 1, 2\}$. By [20, Corollary 3.4] we therefore have

$$1 \leq \frac{1}{q_j} + \frac{1}{q_k}, \quad 0 \leq j \neq k \leq 2,$$

i.e. (3.6).
In order to show (3.5), we consider the family of functions \( a_\lambda (X) = e^{-\lambda |X|^2}, X \in \mathbb{R}^d, \lambda > 0 \). Straight-forward computations show that (cf. [26, Proposition 3.1])
\[
\|a_\lambda\|_{M^{r,s}}^{1/d} = \pi^{\frac{1}{2}+\frac{1}{r}-\frac{1}{s}}(1 + \lambda)^{1/r+1/s-1}
\]
and
\[
a_\lambda \# a_\lambda(X) = (1 + \lambda^2)^{-d} \exp \left(-\frac{2\lambda^2}{1 + \lambda^2} |X|^2 \right).
\]
This gives
\[
\|a_\lambda \# a_\lambda\|_{M^{r,s}}^{1/d} = \pi^{1/r+1/s-1} \|\pi\|_{1/p, 1/q} = 1 + \lambda^2 \|\pi\|_{1/p, 1/q}^{1/r+1/s-1}. (3.12)
\]
From the assumption (3.9) we obtain
\[
\pi^{1/p_0+1/q_0-1} \|\pi\|_{1/p_0, 1/q_0} \leq C \pi^{1/p_1+1/q_1-1} \|\pi\|_{1/p_1, 1/q_1}
\]
\[
\leq C \pi^{1/p_2+1/q_2-1} \|\pi\|_{1/p_2, 1/q_2}.
\]
Letting \( \lambda \to 0 \) gives the inequality (3.5). \( \square \)

Next we introduce more general Gaussians of the form
\[
a_{\lambda, \mu}(x, \xi) = e^{-\lambda |x|^2-\mu |\xi|^2}, \ x, \xi \in \mathbb{R}^d, \ \lambda, \mu > 0.
\]
We consider \( a_{\lambda_1, \mu_1} \# a_{\lambda_2, \mu_2} \), where \( \lambda_j, \mu_j > 0 \) satisfy
\[
\frac{\lambda_1}{\mu_1} = \frac{\lambda_2}{\mu_2}.
\]
so that
\[
\nu \sim \nu_0 \quad \text{when} \quad \nu = 1 + \lambda_1 \mu_2 = 1 + \lambda_2 \mu_1, \ \nu_0 = 1 + \lambda_1 \mu_2 + \lambda_2 \mu_1. (3.15)
\]
The first part of the following result follows by straight-forward computations and (1.9). The other statements follow from the first part and [44, Lemma 1.8].

**Lemma 3.5.** Let \( r, s \in [1, \infty] \), let \( \lambda, \mu, \lambda_j, \mu_j > 0, j = 1, 2 \), satisfy (3.14), and define \( \nu \) and \( \nu_0 \) by (3.15). Then
\[
a_{\lambda_1, \mu_1} \# a_{\lambda_2, \mu_2}(x, \xi) = (2\pi)^{-d/2} \nu^{-d/2} e^{-(\lambda_1 + \lambda_2)|x|^2/\nu} e^{-(\mu_1 + \mu_2)|\xi|^2/\nu},
\]
\[
\|a_{\lambda, \mu}\|_{M^{r,s}}^{1/d} = c_{r,s}(\lambda \mu)^{-1/2r} ((1 + \lambda)(1 + \mu))^{(1/r+1/s-1)/2},
\]
and
\[
\|a_{\lambda_1, \mu_1} \# a_{\lambda_2, \mu_2}\|_{M^{r,s}}^{1/d} = c_{r,s} \nu^{-1/s} ((\lambda_1 + \lambda_2)(\mu_1 + \mu_2))^{-1/(2r)} ((\nu + \lambda_1 + \lambda_2)(\nu + \mu_1 + \mu_2))^{(1/r+1/s-1)/2},
\]
for some constants \( c_{r,s}, C_{r,s} > 0 \) which only depend on \( r, s \).
Lemma 3.5 is used in the proof of the following result, which shows that (3.8) is a consequence of the requested continuity.

**Lemma 3.6.** If (3.9) holds then (3.8) is true.

**Proof.** Let $\lambda_1 = \lambda_2 = 1/\mu_1 = 1/\mu_2 = \lambda > 1$. Then $\nu = 2$, and Lemma 3.5 gives

$$\|a_{\lambda_1, \mu_1} \# a_{\lambda_2, \mu_2}\|_{M^{p_0, q_0}} \asymp \lambda^{d(1/p_0 + 1/q_0 - 1)/2},$$

and

$$\|a_{\lambda_j, \mu_j}\|_{M^{p_j, q_j}} \asymp \lambda^{d(1/p_j + 1/q_j - 1)/2}, \quad j = 1, 2.$$

The assumption (3.9) together with $\lambda \to +\infty$ give

$$\frac{1}{p_0} + \frac{1}{q_0} - 1 \leq \frac{1}{p_1} + \frac{1}{q_1} - 1 + \frac{1}{p_2} + \frac{1}{q_2} - 1,$$

which is the same as (3.8). \qed

Finally we show that (3.9) implies (3.7).

**Lemma 3.7.** If (3.9) holds then (3.7) is true.

**Proof.** By duality it suffices to show that (3.9) implies that

$$\frac{1}{p_0} + \frac{1}{q_0} \leq \frac{1}{p_1} + \frac{1}{q_2}, \quad \text{or} \quad \frac{1}{p_0} \leq \frac{1}{p_1} + \frac{1}{q_2}.$$

The proof is a contradictory argument. We assume that (3.9) holds,

$$\frac{1}{p_0} + \frac{1}{q_0} > \frac{1}{p_0} + \frac{1}{q_2} \quad (3.16)$$

and

$$\frac{1}{p_0} + \frac{1}{q_0} > \frac{1}{p_1} + \frac{1}{q_2} \quad (3.17)$$

This will lead to a contradiction which shows that (3.9) implies

$$\frac{1}{p_0} + \frac{1}{q_0} \leq \frac{1}{p_1} + \frac{1}{q_2},$$

i.e. (3.7) with $j = 0$. The cases $j = 1, 2$ follows by duality.

Thus we assume (3.16), (3.17) and

$$M^{p_1, q_1}(\mathbb{R}^{2d}) \# M^{p_2, q_2}(\mathbb{R}^{2d}) \subseteq M^{p_0, q_0}(\mathbb{R}^{2d}). \quad (3.18)$$

From (3.17) we may conclude that there exists $\varepsilon > 0$ such that

$$\frac{1}{p_0} + \frac{1}{q_0} + \frac{1}{q_2} - \frac{1}{p_2} < 1 - \frac{4\varepsilon}{d}. \quad (3.19)$$

The rest of the proof is an adaptation of the proof of [26, Theorem 3.6] (see also the proof of [21, Theorem 5]).

Let $0 \leq g \in C_0^{\infty}(\mathbb{R}^d) \setminus 0$ be supported in a ball with center in the origin and radius $1/4$. For $n \in \mathbb{Z}^d$ we set

$$d_n = d_{n, \varepsilon} = \begin{cases} 1 & n = 0 \\ |n|^{-(d+\varepsilon)} & n \neq 0 \end{cases} \quad (3.20)$$
so that \( \{ d_n \} \in l^1 \). We also set

\[
\alpha_n = d_n^{1/p_2}, \quad \beta_n = d_n^{1/q_1}, \quad \gamma_n = d_n^{1/q_2}, \quad \eta_n = d_n^{1/q_3},
\]

and we let \( \tau_n \) be the operator \( \tau_n f = f(h - n) \). Our plan is to use the family of functions on \( \mathbb{R}^d \)

\[
f_1 = \sum_n \alpha_n \tau_n g, \quad f_2 = f_{2,N} = \sum_{|n| \leq N} \beta_n \tau_n g, \quad f_3 = \sum_n \gamma_n \tau_n g, \quad f_4 = \sum_n \eta_n \tau_n g,
\]

(3.21)

to construct an element \( b \in M^{p_2,q_2}(\mathbb{R}^d) \) and a sequence \( \{ a_N \} \) in \( \mathcal{S}(\mathbb{R}^d) \) such that \( \{ a_N \} \) is uniformly bounded in \( M^{p_1,q_1}(\mathbb{R}^d) \) but \( \{ a_N \# b \} \) is not a bounded sequence in \( M^{q_0,q_0}(\mathbb{R}^d) \). This is the desired contradiction to \( 3.18 \).

By [26, Remark 1.3] we know that the sequence \( \{ f_{2,N} \} \subseteq \mathcal{S}(\mathbb{R}^d) \) is uniformly bounded in \( M^{q_1,1}(\mathbb{R}^d) \), and that

\[
f_1 \in M^{p_2,1}(\mathbb{R}^d), \quad \hat{f}_3 \in \mathcal{F} M^{q_2,1}(\mathbb{R}^d) \subseteq M^{1,q_2}(\mathbb{R}^d), \]

\[
f_4 \in M^{q_0,1}(\mathbb{R}^d) \subseteq M^{q_0}(\mathbb{R}^d).
\]

In a moment we will prove that if we choose \( \varphi \in C_0^\infty(\mathbb{R}^d) \setminus 0 \) and define \( a_N \) and \( b \) as

\[
a_N = W_{\varphi,f_{2,N}} \quad \text{and} \quad \text{Op}^w(b) h = f_1 \cdot (f_3 * h), \quad h \in C_0^\infty(\mathbb{R}^d),
\]

(3.22) then

\[
\|a_N\|_{M^{p_1,1}} \leq C, \quad b \in M^{p_2,q_2}(\mathbb{R}^d) \quad \text{and}
\]

\[
\text{Op}^w(b) f_4 = \sum_n \lambda_n \tau_n g_0, \quad \text{where} \quad g_0 = g \cdot (g \ast g),
\]

(3.23)

and \( \lambda_n \geq C|n|^{-d(1/q_2+1/p_2-1/q_0')-\varepsilon(1/q_2+1/p_2+1/q_0)} \),

for some constant \( C > 0 \) which is independent of \( N \). We note that \( g \ast g \) is supported in a ball with center at the origin and radius \( 1/2 \).

Assuming this for a while we may proceed as follows. From (3.18) and (3.23) we get that \( \{ a_N \# b \} \) is a bounded sequence in \( M^{p_2,q_2}(\mathbb{R}^d) \), which implies that \( \text{Op}^w(a_N \# b) \) is a uniformly bounded sequence of continuous operators from \( M^{q_0}(\mathbb{R}^d) \) to \( M^{q_0}(\mathbb{R}^d) \). In fact, (3.18) gives \( 2/q_0' \leq 1/q_1 + 1/q_2 \) which combined with (3.16) yield \( 1/q_0' < 1/p_0' \). Now [20, Theorem 7.1] or [44, Theorem 4.3] together with Proposition 1.2 give the assertion. (See also [4, Theorem 1.1].)

On the other hand, since \( f_4 \in M^{q_0}(\mathbb{R}^d) \) and \( \text{Op}^w(f_4) = f_1 \cdot (f_3 \ast f_4) \),

we get

\[
\text{Op}^w(a_N \# b) f_4 = \left( \text{Op}^w(W_{\varphi,f_3}) \right) (f_1 \cdot (f_3 \ast f_4))
\]

33.
which by [26, Lemma 1.6] gives
\[ \text{Op}_w(a_N \# b)f_4 = (2\pi)^{-d/2}(f_1 \cdot (f_3 \ast f_4), f_2)\varphi. \] (3.24)

Now (3.22), (3.23) and the fact that \((g_0, g) > 0\) show that
\[ (f_1 \cdot (f_3 \ast f_4), f_2) \geq C \sum_{|n| \leq N} \lambda_n \beta_n \]
\[ \geq C' \sum_{|n| \leq N} |n|^{-d(1/q_1 + 1/q_2 + 1/p_2 - 1/q_0) - \varepsilon(1/q_1 + 1/q_2 + 1/p_2 + 1/q_0)}, \]
which gives, using (3.19),
\[ (f_1 \cdot (f_3 \ast f_4), f_2) \geq C' \sum_{|n| \leq N} |n|^{-d}. \] (3.25)

Consequently, since the right-hand side can be made arbitrarily large by increasing \(N\), we have obtained a contradiction to the uniformly boundedness of \(\text{Op}_w(a_N \# b)\) as a sequence of operators from \(M^{q_0}(\mathbb{R}^d)\) to \(M^{q_0}(\mathbb{R}^d)\). Hence our assumption, contrary to the statement, is wrong, and the result follows.

It remains to prove (3.23). From the assumptions we have that \(\varphi \in C_0^\infty(\mathbb{R}^d) \subseteq M^1(\mathbb{R}^d)\) and \(f_2 \in M^{n.1}(\mathbb{R}^d)\). From [13, Theorem 4.1] it follows that \(a_N = W_{\varphi, f_2}\) is uniformly bounded in \(M^{1,q_1}(\mathbb{R}^{2d})\), and likewise in \(M^{p_1,q_1}(\mathbb{R}^{2d})\). We have \(\text{Op}_w(b) = \text{Op}_0(c)\) with \(c = f_1 \otimes \hat{f}_3\) and \(\text{Op}_0(c)\) is a pseudo-differential operator of Kohn–Nirenberg type, i.e. \(t = 0\) in (1.3). Since \(f_1, \hat{f}_3 \in M^{p_2,q_2}(\mathbb{R}^d)\), it follows that \(c \in M^{p_2,q_2}(\mathbb{R}^{2d})\). By [26, Remark 1.5] we then obtain \(b \in M^{p_2,q_2}(\mathbb{R}^{2d}) = M^{p_2,q_2}(\mathbb{R}^{2d})\).

In order to prove the last part of (3.23) we note that
\[ f_3 \ast f_4 = \sum_n \mu_n \tau_n (g \ast g), \]
where \(\{\mu_n\}\) is the discrete convolution between \(\{\gamma_n\}\) and \(\{\eta_n\}\), i.e.
\[ \mu_n = \sum_k \gamma_{n-k} \eta_k. \]

By Young’s inequality it follows that \((\mu_n) \in \ell^r\), where \(1/q_2 + 1/q_0 = 1 + 1/r\). Here (3.6) guarantees that \(r \in [1, \infty]\).

From the support properties of \(g\) and \(g \ast g\), it follows that
\[ f_1 \cdot (f_3 \ast f_4) = \sum_n \lambda_n \tau_n g_0 \]
where \(\lambda_n = \alpha_n \mu_n\). We have to estimate \(\lambda_n\). For any \(n \in \mathbb{Z}^d\), let
\[ \Omega_n = \{ k \in \mathbb{Z}^d : |k| \leq |n|, k \neq 0, k \neq n \}. \]
We have
\[ \mu_n = \sum_k \gamma_{n-k} \eta_k \geq \sum_{k \in \Omega_n} |n - k|^{-(d+\epsilon)/q_2} |k|^{-(d+\epsilon)/q_0} \]
\[ \geq C |n|^{d(1-1/q_2-1/q_0) - \epsilon(1/q_2+1/q_0)}, \]
for some \( C > 0 \). Hence
\[ \lambda_n = \alpha_n \mu_n \geq C |n|^{-(d+\epsilon)/p_2} |n|^{d(1-1/q_2-1/q_0) - \epsilon(1/q_2+1/q_0)} \]
\[ = C |n|^{-d(1/q_2+1/p_2-1/q_0') - \epsilon(1/q_2+1/q_0')}. \]
This proves (3.23) and the result follows. \( \square \)

Proof of Theorem 3.1. By Lemma 3.3 we may assume trivial weights. By Lemmas 3.4, 3.6 and 3.7, the inequalities (3.5)–(3.8) hold. Thus Lemma 2.8 shows that (3.2) holds true. \( \square \)

4. Some particular cases

We list in a table some special cases of the inclusion results for the Weyl product on unweighted modulation spaces. More precisely, we compare the inclusion results in Theorem 0.1 and [26, Theorem 0.3′] when the exponents belong to \( p_j, q_j \in \{1, 2, \infty\} \). The table illustrates the generality of Theorem 0.1 as compared to [26, Theorem 0.3′]. In fact the latter result gives no inclusions in modulation spaces for several of the studied cases.

On the other hand [26, Theorem 0.3′] combined with Proposition 1.2 (2) gives the inclusions in Theorem 0.1 for these particular cases.

A corresponding table can be made for the twisted convolution acting on Wiener amalgam spaces \( W_{(\omega)}^{p,q} \), provided the involved Lebesgue exponents \( p_j, q_j \) are interchanged (cf. Theorem 2.12). In the literature it is common that the norm in the Wiener amalgam space \( W_{(\omega)}^{p,q} \) is defined with reversed order of the Lebesgue exponents, i.e. the norm is defined by
\[ \|f\|_{W_{(\omega)}^{p,q}} \equiv \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_{\omega} f(x, \xi) \omega(x, \xi)|^p d\xi \right)^{q/p} dx \right)^{1/q}, \]
instead of (1.16) (cf. [5, 7]).

With this definition of the Wiener amalgam spaces, the table for the twisted convolution is the same as the table below, if the Weyl product \( \# \) is replaced by the twisted convolution \( *_{\sigma} \), and the modulation spaces \( M_{(\omega)}^{p,q} \) are replaced by \( W_{(\omega)}^{p,q} \).

Finally we remark that the relations in the table are valid for weighted spaces provided the involved weights satisfy (0.3).
| No. | $M^{p_1,q_1} \# M^{p_2,q_2}$ | Theorem 0.1 | Theorem 0.3$'$ |
|-----|-----------------------------|-------------|--------------|
| 1   | $M^{1,1} \# M^{1,1}$       | $M^{1,1}$   | —            |
| 2   | $M^{1,1} \# M^{1,2}$       | $M^{1,2}$   | —            |
| 3   | $M^{1,1} \# M^{1,\infty}$  | $M^{1,\infty}$ | —        |
| 4   | $M^{1,1} \# M^{2,1}$       | $M^{1,1}$   | —            |
| 5   | $M^{1,1} \# M^{2,2}$       | $M^{1,2}$   | —            |
| 6   | $M^{1,1} \# M^{2,\infty}$  | $M^{1,\infty}$ | —        |
| 7   | $M^{p,q} \# M^{\infty,1}$  | $M^{p,q}$   | $M^{p,q}$    |
| 8   | $M^{1,1} \# M^{\infty,2}$  | $M^{1,2}$   | $M^{1,2}$    |
| 9   | $M^{1,1} \# M^{\infty,\infty}$ | $M^{1,\infty}$ | $M^{1,\infty}$ |
| 10  | $M^{2,2} \# M^{2,2}$       | $M^{2,2}, M^{1,\infty}$ | $M^{2,2}, M^{1,\infty}$ |
| 11  | $M^{1,2} \# M^{1,2}$       | $M^{2,2}$   | —            |
| 12  | $M^{1,2} \# M^{2,1}$       | $M^{1,2}$   | —            |
| 13  | $M^{1,2} \# M^{2,2}$       | $M^{2,2}, M^{1,\infty}$ | —        |
| 14  | $M^{1,2} \# M^{\infty,2}$  | $M^{1,\infty}$ | $M^{1,\infty}$ |
| 15  | $M^{2,1} \# M^{1,\infty}$  | $M^{1,\infty}$ | —            |
| 16  | $M^{2,1} \# M^{2,1}$       | $M^{1,1}$   | $M^{1,1}$    |
| 17  | $M^{2,1} \# M^{2,2}$       | $M^{1,2}$   | $M^{1,2}$    |
| 18  | $M^{2,1} \# M^{2,\infty}$  | $M^{1,\infty}$ | $M^{1,\infty}$ |
| 19  | $M^{2,1} \# M^{\infty,2}$  | $M^{2,2}$   | $M^{2,2}$    |
| 20  | $M^{2,1} \# M^{\infty,\infty}$ | $M^{2,\infty}$ | $M^{2,\infty}$ |
| 21  | $M^{2,2} \# M^{\infty,2}$  | $M^{2,\infty}$ | $M^{2,\infty}$ |
| 22  | $M^{\infty,2} \# M^{\infty,2}$ | $M^{\infty,\infty}$ | $M^{\infty,\infty}$ |
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