Aspects of Symmetry for Sparse Reflexive Generalized Inverses

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Fundamental in matrix algebra and its applications, a generalized inverse of a real matrix $A$ is a matrix $H$ that satisfies the Moore-Penrose (M-P) property $AHA = A$. If $H$ also satisfies the additional useful M-P property, $HAH = H$, it is called a reflexive generalized inverse. We consider aspects of symmetry related to the calculation of a sparse reflexive generalized inverse of $A$. As is common, and following Lee and Fampa (2018) for calculating sparse generalized inverses, we use the vector 1-norm as a proxy for sparsity.

When $A$ is symmetric, we may naturally desire a symmetric $H$; while generally such a restriction on $H$ may not lead to a 1-norm minimizing reflexive generalized inverse. Letting the rank of $A$ be $r$, and seeking a 1-norm minimizing symmetric reflexive generalized inverse $H$, we give (i) a closed form when $r = 1$, (ii) a closed form when $r = 2$ and $A$ is non-negative, and (iii) an approximation algorithm for general $r$.

Other aspects of symmetry that we consider relate to the other two M-P properties: $H$ is ah-symmetric if $AH$ is symmetric, and ha-symmetric if $HA$ is symmetric. Here we do not assume that $A$ is symmetric, and we do not impose symmetry on $H$. Seeking a 1-norm minimizing ah-symmetric (or ha-symmetric) reflexive generalized inverse $H$, we give (i) a closed form when $r = 1$, (ii) a closed form when $r = 2$ and $A$ satisfies a technical condition, and (iii) an approximation algorithm for general $r$.

Key words: generalized inverse; Moore-Penrose pseudoinverse; reflexive generalized inverse; sparse optimization; linear programming; approximation algorithm; local search

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1. Introduction. Generalized inverses are essential tools in matrix algebra and its applications. In particular, the Moore-Penrose (M-P) pseudoinverse can be used to calculate the least-squares solution of an over-determined system of linear equations and the solution with minimum 2-norm of an under-determined system of linear equations. Considering our motivating use case of a very large matrix and multiple right-hand sides, we can see the value of having at hand a sparse generalized inverse. So we apply techniques of sparse optimization, aiming at balancing the tradeoff between properties of the M-P pseudoinverse and alternative sparser generalized inverses. Recently, [4, 2, 3] used sparse-optimization techniques to develop tractable left and right sparse pseudoinverses. Particulary relevant to what we present here, [7] (also see [8]) derived and analyzed other tractable sparse generalized inverses based on relaxing some of the “M-P properties”. [6] investigated one such kind of sparse generalized inverse, with particular interest in rank-deficient matrices; these reduce to the sparse right (resp., left) sparse pseudoinverses in [4, 2, 3], when the matrix has full row (resp., column) rank.

In what follows, for succinctness, we use vector-norm notation on matrices: we write $\|H\|_1$ to mean $\|\text{vec}(H)\|_1$, and $\|H\|_{\text{max}}$ to mean $\|\text{vec}(H)\|_{\text{max}}$ (in both cases, these are not the usual induced/operator matrix norms). We use $I$ for an identity matrix and $J$ for an all-ones matrix. Matrix dot product is indicated by $\langle X, Y \rangle = \text{trace}(X^TY) := \sum_{ij} x_{ij}y_{ij}$. We use $A[S,T]$ to represent the submatrix of $A$ with row indices $S$ and column indices $T$; additionally, if $A$ is symmetric and $S = T$, we use $A[S]$ to represent the principal submatrix of $A$ with row/column indices $S$. We also
use $A[S,:]$ (resp., $A[:,T]$) to represent the submatrix of $A$ formed by the rows $S$ (resp., columns $T$).

When a real matrix $A \in \mathbb{R}^{m \times n}$ is not square or is square but not invertible, we consider “pseudoinverses” of $A$ (see [13]). The most well-known pseudoinverse is the $M$-$P$ pseudoinverse, independently discovered by A. Bjerhammar, E.H. Moore and R. Penrose (see [1, 5, 12]). If $A = UV^T$ is the real singular-value decomposition of $A$ (see [10], for example), where $U \in \mathbb{R}^{m \times m}$, $V \in \mathbb{R}^{n \times n}$ are orthogonal matrices and $\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_p) \in \mathbb{R}^{m \times n}$ ($p = \min(m, n)$) with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0$, then the $M$-$P$ pseudoinverse of $A$ can be defined as $A^+ = V \Sigma^+ U^T$, where $\Sigma^+ := \text{diag}(\sigma_1^+, \sigma_2^+, \ldots, \sigma_p^+)$ is a generalized inverse (i.e., a solution of $AXB = C$ for all matrices $X$ and $B$ subject to various subsets of $\{P_2, P_3, P_4\}$) having the minimum number of nonzeros, subject to various subsets of $\left\{P_2, P_3, P_4\right\}$. Following [6], we define different tractable sparse “generalized inverses”, based on the following fundamental characterization of the $M$-$P$ pseudoinverse.

**Theorem 1 (see [12]).** For $A \in \mathbb{R}^{m \times n}$, the $M$-$P$ pseudoinverse $A^+$ is the unique $H \in \mathbb{R}^{n \times m}$ satisfying:

\begin{align*}
AHA &= A \quad \text{(P1)} \\
HHA &= H \quad \text{(P2)} \\
(AH)^\top &= AH \quad \text{(P3)} \\
(HA)^\top &= HA \quad \text{(P4)}
\end{align*}

Following [14], a *generalized inverse* is any $H$ satisfying P1. Because we are interested in sparse $H$, P1 is particularly important to enforce, because the completely sparse zero-matrix always satisfies P2+P3+P4. A generalized inverse is reflexive if it satisfies P2 (again, see [14]). Theorem 3.14 in [14] tells us two very useful facts: (1) if $H$ is a generalized inverse of $A$, then $\text{rank}(H) \geq \text{rank}(A)$, and (2) a generalized inverse $H$ of $A$ is reflexive if and only if $\text{rank}(H) = \text{rank}(A)$. A low-rank $H$ can be viewed as being more interpretable (say in the context of the least-squares problem), so we are naturally prefer reflexive generalized inverses which have the least rank possible.

As a convenient mnemonic, if $H$ satisfies P3, we say that $H$ is *ah-symmetric (with respect to $A$)*, and if $H$ satisfies P4, we say that $H$ is *ha-symmetric (with respect to $A$)*. Note that not all of the M-P properties are required for a generalized inverse to exactly solve key problems. For example, if $H$ is an ah-symmetric generalized inverse, then $\hat{x} := Hb$ solves $\min \{\|Ax - b\|_2 : x \in \mathbb{R}^n\}$; if $H$ is a ha-symmetric generalized inverse, then $\hat{x} := Hb$ solves $\min \{\|x\|_2 : Ax = b, x \in \mathbb{R}^n\}$ (see [7]).

It is hard to find a generalized inverse (i.e., a solution of P1) having the minimum number of nonzeros, subject to various subsets of $\{P_2, P_3, P_4\}$ (but not all of them). We let $\|H\|_0$ (resp., $\|x\|_0$) be the number of nonzeros in the matrix $H$ (resp., vector $x$). [3] established that $\min\{\|H\|_0 : \text{P1}\}$ is NP-hard as follows: for full row-rank $A \in \mathbb{R}^{m \times n}$ ($m < n$),

$$
\min\{\|H\|_0 : AHA = A\} = \min\{\|H\|_0 : AH = I\},
$$

and computing an element from this set can be expressed column-wise, as a collection of sparse optimization problems $\min\{\|x\|_0 : Ax = e_i\}$. These latter sparse optimization problems are known to be NP-hard (see [11]) for a general right-hand side $b \neq 0$. But with $A$ having full row rank, we can reduce any general right-hand side $b \neq 0$, to a problem with $b = e_i$. Using the same idea, we can show the following hardness result.

**Proposition 1.** The following problems are also NP-hard.

$\min\{\|H\|_0 : \text{P1} + \text{P2}\}$ \hspace{1cm} (SGI12)

$\min\{\|H\|_0 : \text{P1} + \text{P3}\}$ \hspace{1cm} (SGI13)

$\min\{\|H\|_0 : \text{P1} + \text{P2} + \text{P3}\}$ \hspace{1cm} (SGI123)

$\min\{\|H\|_0 : \text{P1} + \text{P4}\}$ \hspace{1cm} (SGI14)

$\min\{\|H\|_0 : \text{P1} + \text{P2} + \text{P4}\}$ \hspace{1cm} (SGI124)
Proof. For full row-rank $A \in \mathbb{R}^{m \times n}$ ($m < n$), we have $AHA = A \Leftrightarrow AH = I$, and thus $HAH = H$ and $(AH)^\top = AH$ are also satisfied. Therefore, $(\text{SGI12})$, $(\text{SGI13})$, $(\text{SGI123})$ are all equivalent to $\min\{\|H\|_0 : AH = I\}$, which is $\text{NP}$-hard. Similarly, with full column-rank $A$, we have that $(\text{SGI14})$, $(\text{SGI124})$ are $\text{NP}$-hard.

Because of this hardness, we take the standard approach of minimizing $\|H\|_1$ to induce sparsity, subject to $P_1$ and various subsets of $\{P_2, P_3, P_4\}$ (but not all of them). Considering tractability of the optimization that we will have to carry out, we see that $P_1$, $P_3$ and $P_4$ are linear constraints, which are easy to handle, while $P_2$ is a non-convex quadratic, hence rather nasty. Therefore, we are particularly interested in situations where an $H$ that minimizes the 1-norm, subject to $P_1$ and one or none of $P_3$ and $P_4$, also satisfies $P_2$ for free.

[6] gave some answers when neither $P_3$ nor $P_4$ is enforced. [6] gave a “block construction” of a reflexive generalized inverse $H$ of rank-$r$ $A$ that is “somewhat-sparse”, having at most $r^2$ nonzeros. [6] also demonstrates that there exists an easy-to-find block construction of a 1-norm minimizing reflexive generalized inverse, for rank-1 matrices and rank-2 non-negative matrices. Finally, for general rank-$r$ matrices, [6] gave an efficient local-search based approximation algorithm, which efficiently finds a reflexive generalized inverse following the block construction, that has its 1-norm within a factor of (almost) $r^2$ of the minimum 1-norm of any generalized inverse.

In what follows, we are interested in two directions. One direction aims at finding a 1-norm minimizing symmetric reflexive generalized inverse $H$ of a symmetric matrix $A$. Because the inverse of a nonsingular symmetric matrix is also symmetric, it is natural to ask for a symmetric reflexive generalized inverse. [14, Section 3.3] demonstrates that if $A$ is symmetric, then it is not necessarily the case that a reflexive generalized inverse is symmetric; but there always does exist a symmetric reflexive generalized inverse. Proposition 2 below establishes that for a symmetric matrix $A$, finding a symmetric generalized inverse with minimum number of nonzeros is $\text{NP}$-hard. So we aim at construction of a symmetric reflexive generalized inverse with minimum (or approximately minimum) 1-norm.

**Proposition 2.** For symmetric matrix $A$, the following problem is $\text{NP}$-hard.

$$\min\{\|H\|_0 : P_1, \ H^\top = H\} \quad (\text{symSGI})$$

Proof. We reduce $\min\{\|H\|_0 : P_1\}$ to an instance of $(\text{symSGI})$ as follows. Let

$$\tilde{A} := \begin{bmatrix} 0 & A \\ A^\top & 0 \end{bmatrix} \quad \text{and} \quad H := \begin{bmatrix} X & Z^\top \\ Z & Y \end{bmatrix}; \quad \text{then} \quad \tilde{A}H\tilde{A} = \begin{bmatrix} AYA^\top & AZA \\ A^\top Z^\top A^\top & A^\top XA \end{bmatrix}.$$  

Thus $\tilde{A}$ is symmetric, and $(\text{symSGI})$ for $\tilde{A}$ is equivalent to

$$\min\{\|X\|_0 + \|Y\|_0 + 2\|Z\|_0 : A^\top XA = 0, AYA^\top = 0, AZA = A, X^\top = X, Y^\top = Y\}.$$  

Clearly, $(X = 0, \ Y = 0, \ Z)$ is optimal to $(\text{symSGI})$ for $\tilde{A}$ if and only if $Z$ is optimal to $\min\{\|H\|_0 : AH = A\}$; thus $(\text{symSGI})$ is $\text{NP}$-hard. Unfortunately, we do not know the complexity of $\min\{\|H\|_0 : P_1 + P_2, \ H^\top = H\}$.

The second direction aims at finding 1-norm minimizing ah-symmetric (or ha-symmetric) reflexive generalized inverses. Note that if $A$ is symmetric, and we require that $H$ is a symmetric ah-symmetric (or ha-symmetric) reflexive generalized inverse, then $H$ is already the M-P pseudoinverse (see [14]). Therefore, there is no interest in enforcing symmetry on $H$ in this context.

In §2, we consider the situation where $A$ is symmetric, and we show that the same block construction method from [6], but over only the principal submatrices, gives us a symmetric reflexive generalized inverse with minimum 1-norm for rank-1 matrices and rank-2 non-negative matrices. Interestingly, our proof for rank-2 non-negative matrices is quite different than the proof of the
corresponding result in [6]. We also give a local-search based (almost) \( r^2 \)-approximation algorithm for finding a 1-norm minimizing symmetric reflexive generalized inverse. Along the way, we repair a proof of a key result from [6], concerning the correctness of the approximation algorithm. In §3, we consider minimizing the 1-norm of \( H \) over ah-symmetric reflexive generalized inverses, for the case of rank-1 \( A \) and for the case of rank-2 \( A \) under a technical condition. Also we provide a local-search based (almost) \( r \)-approximation algorithm for general rank \( r \). In §4, we observe a connection between ah-symmetric (reflective) generalized inverses and ha-symmetric (reflective) generalized inverses, which extends all the results in §3 to the ha-symmetric case; that is, minimizing the 1-norm of \( H \) over ha-symmetric reflexive generalized inverses. In §5, we make some brief concluding remarks.

Before presenting our main results, we note that it is useful to consider relaxing P2, arriving at \( \min \{\|H\|_1 : P1\} = \min \{\|H\|_1 : AHA = A\} \), which we re-cast as a linear-optimization problem (P) and its dual (D):

\[
\begin{align*}
& \text{minimize} \quad \langle J, H^+ \rangle + \langle J, H^- \rangle \\
& \text{subject to} \quad A(H^+ - H^-)A = A, \\
& \quad H^+, H^- \succeq 0. \tag{P}\\
& \text{maximize} \quad \langle A, W \rangle \\
& \text{subject to} \quad -J \preceq A^\top W A \preceq J. \tag{D}
\end{align*}
\]

More compactly, we can recast (D) as: \( \max \{\langle A, W \rangle : \|A^\top W A \|_{\max} \leq 1\} \). In what follows, our approach is always to construct a feasible solution to (P) such that \( H := H^+ - H^- \) satisfies P2, and measure the quality of the solution to (P) against a feasible solution that we construct for (D).

2. Symmetric results. We note that considerable effort has been made for tuning hardware to efficiently handle “matrix-vector multiply”: the multiplication of a vector by a sparse symmetric matrix (for example, see [9] and the references therein). Considering that virtually any use of a generalized inverse \( H \) would involve matrix-vector multiply, it can be very useful to prepare a sparse symmetric generalized inverse \( H \) from a symmetric \( A \).

In this section, we assume that \( A \in \mathbb{R}^{n \times n} \) is symmetric, and we seek to obtain an optimal solution to \( \min \{\|H\|_1 : P1 + P2, \quad H^\top = H\} \). Using [6], we could first seek a 1-norm minimizing reflexive generalized inverse \( H \) of \( A \) that is not necessarily symmetric. If \( H \) is not symmetric, then the natural symmetrization \( (H + H^\top)/2 \) is a symmetric generalized inverse with minimum 1-norm, because doing this symmetrization cannot increase the convex function \( \|\cdot\|_1 \). However, symmetrization is very likely to increase the rank and thus violate P2. Therefore, we will seek to develop a new recipe for constructing a symmetric reflexive generalized inverse. Our symmetric block construction in the following theorem is the same block construction as from [6], but only over the principal submatrices of \( A \).

**Theorem 2 (Follows from [6]).** For a symmetric matrix \( A \in \mathbb{R}^{n \times n} \), let \( r := \text{rank}(A) \). Let \( \tilde{A} := A[S] \) be any \( r \times r \) nonsingular principal submatrix of \( A \). Let \( H \in \mathbb{R}^{n \times n} \) be equal to zero, except its submatrix with row/column indices \( S \) is equal to \( \tilde{A}^{-1} \). Then \( H \) is a symmetric reflexive generalized inverse of \( A \).

**2.1. Rank 1.** Next, we demonstrate that when \( \text{rank}(A) = 1 \), construction of a 1-norm minimizing symmetric reflexive generalized inverse can be based on the symmetric block construction over the diagonal elements of \( A \).

**Theorem 3.** Let \( A \) be an arbitrary rank-1 symmetric matrix, which is, without loss of generality, of the form \( A := uu^\top \), where \( 0 \neq u \in \mathbb{R}^n \). If \( i^* := \arg \max \{\|u_i\| : \|(a_{ii})\|\} \), then \( H := \frac{1}{\|u_i\|} e_i e_i^\top \), where \( e_i \in \mathbb{R}^n \) is a standard unit vector, is a symmetric reflexive generalized inverse of \( A \) with minimum 1-norm.
Proof. We consider (P) and (D). A feasible solution for (P) is \( H^+ = \frac{1}{u^*_i} e_i e_i^\top \), \( H^- = 0 \). A feasible solution for (D) is \( W = \frac{1}{u^*_i} e_i e_i^\top \), because \( \| A^\top W A^\top \|_{\max} = \frac{1}{u^*_i} \| A \|_{\max} = 1 \). And the objective value of the dual solution is \( \langle A, W \rangle = u^2_i \cdot \frac{1}{u^*_i} = 1/u^2_i \), which is the objective value of the primal solution. Therefore, by the weak-duality theorem of linear optimization, we have that \( H := H^+ - H^- \) is a generalized inverse of \( A \) with minimum 1-norm. Now we only need to prove the claim. By our construction, \( H \) is symmetric and reflexive. Therefore, \( H \) is a symmetric reflexive generalized inverse with minimum 1-norm. 

Another way to view the rank-1 case is by using the Kronecker product to transform the constraint \( AHA = A \) into \( [A^\top \otimes A] \text{vec}(H) = \text{vec}(A) \). Note that \( \text{vec}(A) = \text{vec}(uu^\top) = u \otimes u \), and \( A^\top \otimes A = uu^\top \otimes uu^\top = [u \otimes u][u^\top \otimes u^\top] \). So the constraint becomes

\[
[u \otimes u] ([u^\top \otimes u^\top] \text{vec}(H)) = u \otimes u \iff [u \otimes u]^\top \text{vec}(H) = 1.
\]

Thus the 1-norm minimization may be re-cast as \( \min \{ \| \text{vec}(H) \|_1 : [u \otimes u]^\top \text{vec}(H) = 1 \} \), or \( \min \{ \| H \|_1 : u^\top Hu = 1 \} \), or \( \min \{ \| H \|_1 : \langle uu^\top, H \rangle = 1 \} \). By using the inequality \( x^\top y \leq \| x \|_\infty \| y \|_1 \), or \( \langle X, Y \rangle \leq \| X \|_{\max} \| Y \|_1 \), we have \( \| H \|_1 \geq 1/\| uu^\top \|_{\max} \), and the equality holds when \( H = [u \otimes u][u^\top \otimes u^\top] \).

### 2.2. Rank 2

Generally, when \( \text{rank}(A) = 2 \), we cannot construct a 1-norm minimizing symmetric reflexive generalized inverse based on the symmetric block construction. For example, with

\[
A := \begin{bmatrix}
5 & 4 & 2 \\
4 & 5 & -2 \\
2 & -2 & 8
\end{bmatrix},
\]

we have a symmetric reflexive generalized inverse

\[
H := \frac{1}{81} A \quad \text{(because } A^2 = 9A \text{)}
\]

with \( \| H \|_1 = \frac{34}{81} \). While the three symmetric reflexive generalized inverses based on the symmetric block construction have 1-norm equal to \( \frac{17}{36}, \frac{17}{36}, 2 \), all greater than \( \frac{34}{81} \).

Next, we demonstrate that under the natural but restrictive condition that \( A \) is non-negative, when \( \text{rank}(A) = 2 \), construction of a 1-norm minimizing symmetric reflexive generalized inverse can be based on the symmetric block construction over the 2 \( \times \) 2 principal submatrix of \( A \).

**Theorem 4.** Let \( A \) be an arbitrary rank-2 non-negative symmetric matrix. For any \( i_1, i_2 \in \{1, \ldots, n\} \), with \( i_1 < i_2 \), let \( \tilde{A} := A[\{i_1, i_2\}] \). If \( i_1, i_2 \) are chosen to minimize the 1-norm of \( \tilde{A}^{-1} \) among all nonsingular 2 \( \times \) 2 principal submatrices, then the \( n \times n \) matrix \( H \) constructed by Theorem 2 over \( \tilde{A} \), is a symmetric reflexive generalized inverse of \( A \) with minimum 1-norm.

**Proof.** If \( i_1, i_2 \) are chosen to minimize the 1-norm of \( \tilde{A}^{-1} \) among all 2 \( \times \) 2 principal submatrices, we first claim that \( \tilde{A} \) also minimizes \( \| \tilde{A}^{-1} \|_1 \) among all nonsingular 2 \( \times \) 2 submatrices (not necessarily principal); then by [6, Theorem 7], we have that \( H \) is a reflexive generalized inverse of \( A \) with minimum 1-norm. By our construction, \( H \) is symmetric, therefore, \( H \) is a symmetric reflexive generalized inverse with minimum 1-norm. Now we only need to prove the claim.

Because \( A \) is a symmetric rank-2 matrix, we can assume that

\[
A = UU^\top = \begin{bmatrix} u_1^\top \\
u_2^\top \\
\vdots \\
u_n^\top \end{bmatrix} \begin{bmatrix} u_1 & u_2 & \ldots & u_n \end{bmatrix},
\]
where \( u_i = (r_i \cos \theta_i, r_i \sin \theta_i)^\top \) and \( a_{ij} = u_i^\top u_j = r_i r_j \cos (\theta_i - \theta_j) \geq 0 \), which implies that \( |\theta_i - \theta_j| \leq \frac{\pi}{2} \).

Without loss of generality, we can assume that \( \theta_1 \leq \theta_2 \leq \cdots \leq \theta_n \leq \theta_1 + \frac{\pi}{2} \).

Next, we calculate \( \left\| A^{-1} \right\|_1 \) for principal submatrix \( A[i, j] \) (denoted as \( A[ij] \) for simplicity) and general submatrix \( A[i, j, \{k, l\}] \) (denoted as \( A[ij, kl] \) for simplicity). For principal submatrix \( A[ij] \), we have

\[
\left\| (A[ij])^{-1} \right\|_1 = \frac{a_{ij} + a_{ji} + a_{ik} + a_{jk}}{|a_{ij} a_{jj} - a_{ij} a_{ji}|} = \frac{r_i^2 + 2r_i r_j \cos (\theta_i - \theta_j) + r_j^2}{r_i^2 r_j^2 \sin^2 (\theta_i - \theta_j)}.
\]

For general submatrix \( A[ij, kl] \), we have

\[
\left\| (A[ij, kl])^{-1} \right\|_1 = \frac{a_{ik} + a_{il} + a_{jk} + a_{jl}}{|a_{ik} a_{jl} - a_{il} a_{jk}|} = \frac{r_i r_j r_k r_l \cos (\theta_i - \theta_k) \cos (\theta_j - \theta_l) - \cos (\theta_i - \theta_j) \cos (\theta_k - \theta_l)}{a_{ik} + a_{il} + a_{jk} + a_{jl}} \\
= \frac{r_i r_j r_k r_l \left( \frac{1}{2} (\cos (\theta_i - \theta_k + \theta_j - \theta_l) + \cos (\theta_i - \theta_k - \theta_j + \theta_l)) - \frac{1}{2} \cos (\theta_i - \theta_l + \theta_j - \theta_k) + \cos (\theta_i - \theta_l - \theta_j + \theta_k) \right)}{r_i r_j r_k r_l \left( \frac{1}{2} (\cos (\theta_i - \theta_k + \theta_j - \theta_l) + \cos (\theta_i - \theta_k - \theta_j + \theta_l)) - \frac{1}{2} \cos (\theta_i - \theta_l + \theta_j - \theta_k) + \cos (\theta_i - \theta_l - \theta_j + \theta_k) \right)} \\
= \frac{r_i r_j r_k r_l \sin (\theta_i - \theta_j) \sin (\theta_k - \theta_l)}{r_i r_j r_k r_l \sin (\theta_i - \theta_j) \sin (\theta_k - \theta_l)}.
\]

We consider the following cases.

(I) \( \min_{(s, t) \subseteq \{i, j, k, l\}} \| (A[st])^{-1} \|_1 = \| (A[ij])^{-1} \|_1, \theta_i \leq \theta_j, \theta_k \leq \theta_l \). By comparing \( \| (A[ij])^{-1} \|_1 \) with \( \| (A[ik])^{-1} \|_1 \), we have

\[
r_i r_j \sin (|\theta_i - \theta_k| - |\theta_j - \theta_l|) \leq r_i r_j \sin |\theta_i - \theta_j|.
\]

Similarly, we have

\[
r_i r_j \sin (|\theta_i - \theta_k| - |\theta_j - \theta_l|) \leq r_i r_j \sin |\theta_i - \theta_j|.
\]

(ii) \( \theta_i \geq \theta_k \) and \( \theta_k \leq \theta_j \), i.e., \( \theta_i, \theta_j \) intersects with \( \theta_k, \theta_l \). Then we can simplify \( (I'_{ik}) \) and \( (I'_{il}) \) to get

\[
r_i r_j \sin (\theta_j - \theta_k) - r_i \sin (\theta_k - \theta_l) \leq r_i r_j \sin (\theta_j - \theta_i)
\]

Using these two inequalities to lower bound \( \| (A[ij, kl])^{-1} \|_1 \), we have

\[
r_i r_j \sin (\theta_j - \theta_k) \leq r_i r_j \sin (\theta_j - \theta_i) \\
+ r_k r_l (r_i \cos (\theta_i - \theta_l - \theta_k) + r_j \cos (\theta_j - \theta_l - \theta_k)) \\
\geq r_i r_j r_k r_l (r_i \cos (\theta_i - \theta_l - \theta_k) + r_j \cos (\theta_j - \theta_l - \theta_k)) (r_i \sin (\theta_i - \theta_l) - r_j \sin (\theta_j - \theta_l)) \\
= r_k r_l (r_i^2 r_j^2 \sin (\theta_i - \theta_l - \theta_k) + r_i^2 \sin (\theta_i - \theta_k)) \\
+ r_j r_i (r_j^2 r_i^2 \sin (\theta_j - \theta_l - \theta_k) + r_j^2 \sin (\theta_j - \theta_k)) \\
= r_k r_l (r_i^2 r_j^2 + r_i^2 \sin (\theta_i - \theta_k) + r_j^2 \sin (\theta_j - \theta_k) + 2r_i r_j \cos (\theta_j - \theta_i)) \sin (\theta_j - \theta_k),
\]

which implies that \( \| (A[ij, kl])^{-1} \|_1 \geq \| (A[ij])^{-1} \|_1 \).
\[(ii) \; \theta_k \leq \theta_i < \theta_j \leq \theta_j. \] Then we can simplify \((I'_{jk})\) and \((I'_{kl})\) to get

\[
\begin{align*}
& r_k(r_j \sin(\theta_j - \theta_k) - r_i \sin(\theta_k - \theta_i)) \leq r_i r_j \sin(\theta_j - \theta_i), \\
& r_i(r_j \sin(\theta_j - \theta_i) - r_i \sin(\theta_i - \theta_i)) \leq r_i r_j \sin(\theta_j - \theta_i).
\end{align*}
\]

Using these two inequalities to lower bound \(\| (A[ij,kl])^{-1} \|_1 \), we have

\[
\begin{align*}
& r_i r_j \sin(\theta_j - \theta_i) X \\
& = r_i r_j \sin(\theta_j - \theta_i) r_k(r_i \cos(\theta_i - \theta_k) + r_j \cos(\theta_j - \theta_k)) \\
& \quad + r_i r_j \sin(\theta_j - \theta_i) r_i(r_i \cos(\theta_i - \theta_i) + r_j \cos(\theta_j - \theta_i)) \\
& \geq r_k r_i(r_i \cos(\theta_i - \theta_k) + r_j \cos(\theta_j - \theta_k)) (r_j \sin(\theta_j - \theta_i) - r_i \sin(\theta_i - \theta_i)) \\
& \quad + r_k r_i r_i (r_i \cos(\theta_i - \theta_i) + r_j \cos(\theta_j - \theta_i)) (r_j \sin(\theta_j - \theta_k) - r_i \sin(\theta_i - \theta_i)) \\
& = r_k r_i [r_i^2 \sin(2\theta_i - \theta_i - \theta_k) + r_j^2 \sin(2\theta_j - \theta_i - \theta_k) + 2r_i r_j \sin(\theta_i + \theta_j - \theta_i - \theta_k)] \\
& \geq r_k r_i [r_i^2 \sin(\theta_i - \theta_k) + r_j^2 \sin(\theta_j - \theta_k) \\
& \quad + r_i r_j \sin((\theta_i - \theta_k) - (\theta_i - \theta_i)) - \sin((\theta_i - \theta_i) - (\theta_i - \theta_k))))] \\
& = r_k r_i \sin(\theta_i - \theta_k) [r_i^2 + r_j^2 + 2r_i r_j \cos(\theta_i - \theta_i)].
\end{align*}
\]

The last inequality follows from that \(\sin(\alpha + \beta) \geq \sin(\alpha - \beta)\) when \(\alpha, \beta \in [0, \frac{\pi}{2}]\).

\[(iii) \; \theta_i \leq \theta_j < \theta_k \leq \theta_i. \] Similar to case (I)(ii), using \((I''_{ik})\) and \((I''_{jl})\) instead.

\[(II) \; \min_{(x,y) \in (i,j,k,l)} \| (A[s,t])^{-1} \|_1 = \| (A[i,k])^{-1} \|_1, \; \theta_i \leq \theta_j, \; \theta_k \leq \theta_i. \] Similarly, we have

\[
\begin{align*}
& r_j(r_i \sin(\theta_i - \theta_i) - r_k \sin((\theta_i - \theta_k) - (\theta_i - \theta_i))) \leq r_i r_k \sin(\theta_i - \theta_k) \\
& r_j(r_i \sin(\theta_i - \theta_i) - r_k \sin((\theta_i - \theta_k) - (\theta_i - \theta_i))) \leq r_i r_k \sin(\theta_i - \theta_k) \\
& r_i(r_k \sin(\theta_i - \theta_i) - r_i \sin((\theta_i - \theta_k) - (\theta_i - \theta_i))) \leq r_k r_i \sin(\theta_i - \theta_k) \\
& r_i(r_k \sin(\theta_i - \theta_i) - r_i \sin((\theta_i - \theta_k) - (\theta_i - \theta_i))) \leq r_k r_i \sin(\theta_i - \theta_k).
\end{align*}
\]

\[(i) \; \theta_k \leq \theta_i (\leq \theta_j). \] Then we can simplify \((I''_{kj})\) and \((I''_{kl})\) to get

\[
\begin{align*}
& r_j(r_k \sin(\theta_j - \theta_k) - r_i \sin(\theta_i - \theta_j)) \leq r_i r_k \sin(\theta_i - \theta_k) \\
& r_i(r_k \sin(\theta_i - \theta_k) - r_i \sin(\theta_i - \theta_i)) \leq r_k r_i \sin(\theta_i - \theta_k).
\end{align*}
\]

Using these two inequalities to lower bound \(\| (A[ij,kl])^{-1} \|_1 \), we have

\[
\begin{align*}
& r_i r_j r_k \sin^2(\theta_i - \theta_k) X \\
& = r_i r_j r_k \sin^2(\theta_i - \theta_k) r_i r_j \cos(\theta_i - \theta_k) \\
& \quad + r_i r_j r_k \sin^2(\theta_i - \theta_k) r_i r_j \cos(\theta_i - \theta_i) + r_i r_j r_k \cos(\theta_i - \theta_k) \\
& \geq r_i r_j r_k \sin(\theta_i - \theta_k) - r_i \sin(\theta_i - \theta_i)) r_i (r_i \sin(\theta_i - \theta_k) - r_i \sin(\theta_i - \theta_i)) r_i \sin(\theta_i - \theta_k) \\
& \quad + r_i r_j r_k \sin(\theta_i - \theta_i) r_j r_i \sin(\theta_i - \theta_k) - r_i \sin(\theta_i - \theta_k) r_i \sin(\theta_i - \theta_i) r_j \cos(\theta_j - \theta_i) \\
& \quad + r_i r_j r_k \sin(\theta_i - \theta_i) r_k (r_k \sin(\theta_i - \theta_k) - r_i \sin(\theta_i - \theta_i)) r_k r_j \cos(\theta_j - \theta_k) \\
& \quad + r_i r_j r_k \sin(\theta_i - \theta_i) r_j r_k \sin(\theta_i - \theta_k) + r_k r_j \sin(\theta_i - \theta_k) \\
& \quad + r_i r_j r_k \sin(\theta_i - \theta_i) r_k (r_k \sin(\theta_i - \theta_k) - r_i \sin(\theta_i - \theta_k)) r_k r_j \cos(\theta_j - \theta_k) \\
& \quad + r_i r_j r_k \sin(\theta_i - \theta_i) r_j r_k \sin(\theta_i - \theta_k) + r_k r_j \sin(\theta_i - \theta_k) \\
& \quad + r_i r_j r_k \sin(\theta_i - \theta_i) r_k (r_k \sin(\theta_i - \theta_k) - r_i \sin(\theta_i - \theta_k)) r_k r_j \cos(\theta_j - \theta_k) \\
& \quad + r_i r_j r_k \sin(\theta_i - \theta_i) r_j r_k \sin(\theta_i - \theta_k) + r_k r_j \sin(\theta_i - \theta_k) \\
& \quad + r_i r_j r_k \sin(\theta_i - \theta_i) r_k (r_k \sin(\theta_i - \theta_k) - r_i \sin(\theta_i - \theta_k)) r_k r_j \cos(\theta_j - \theta_k) \\
& \quad + r_i r_j r_k \sin(\theta_i - \theta_i) r_j r_k \sin(\theta_i - \theta_k) + r_k r_j \sin(\theta_i - \theta_k) \\
& \quad + r_i r_j r_k \sin(\theta_i - \theta_i) r_k (r_k \sin(\theta_i - \theta_k) - r_i \sin(\theta_i - \theta_k)) r_k r_j \cos(\theta_j - \theta_i) \\
& \quad + r_i r_j r_k \sin(\theta_i - \theta_i) r_j r_k \sin(\theta_i - \theta_k) + r_k r_j \sin(\theta_i - \theta_k) \\
& \quad + r_i r_j r_k \sin(\theta_i - \theta_i) r_k (r_k \sin(\theta_i - \theta_k) - r_i \sin(\theta_i - \theta_k)) r_k r_j \cos(\theta_j - \theta_i) \\
& \quad + r_i r_j r_k \sin(\theta_i - \theta_i) r_j r_k \sin(\theta_i - \theta_k) + r_k r_j \sin(\theta_i - \theta_k) \\
& \quad + r_i r_j r_k \sin(\theta_i - \theta_i) r_k (r_k \sin(\theta_i - \theta_k) - r_i \sin(\theta_i - \theta_k)) r_k r_j \cos(\theta_j - \theta_i) \\
& \quad + r_i r_j r_k \sin(\theta_i - \theta_i) r_j r_k \sin(\theta_i - \theta_k) + r_k r_j \sin(\theta_i - \theta_k) \\
& \quad + r_i r_j r_k \sin(\theta_i - \theta_i) r_k (r_k \sin(\theta_i - \theta_k) - r_i \sin(\theta_i - \theta_k)) r_k r_j \cos(\theta_j - \theta_i).
\end{align*}
\]


\[-r_r r_k \sin(\theta_i - \theta_k)(\sin(\theta_i - \theta_k) \cos(\theta_j - \theta_k) - \sin(\theta_i - \theta_k) \cos(\theta_j - \theta_i))\]

\[= r_r r_k r_i r_1 [r_i^2 \sin(\theta_i - \theta_j) \sin(\theta_i - \theta_k) + r_i^2 \sin(\theta_i - \theta_k) \sin(\theta_i + \theta_j - 2\theta_k) + 2r_r r_k \sin(\theta_j - \theta_i) \cos(\theta_i - \theta_k) + r_r r_k \cos(\theta_j - \theta_i) \sin(\theta_i - \theta_k) \sin(\theta_i - \theta_k)]

\[= r_r r_k r_i r_1 [r_i^2 \sin(\theta_i - \theta_j) \sin(\theta_i - \theta_k) + r_i^2 \sin(\theta_i - \theta_k) \sin(\theta_i + \theta_j - 2\theta_k) + 2r_r r_k \sin(\theta_j - \theta_i) \cos(\theta_i - \theta_k) + r_r r_k \cos(\theta_j - \theta_i) \sin(\theta_i - \theta_k) \sin(\theta_i - \theta_k)]

\geq r_r r_k r_i r_1 \sin(\theta_j - \theta_i) \sin(\theta_i - \theta_k)(r_i^2 + r_k^2 + 2r_r r_k \cos(\theta_i - \theta_k)).\]

The penultimate equality follows from

\[\sin(\theta_i - \theta_i) \cos(\theta_j - \theta_k) - \sin(\theta_i - \theta_k) \cos(\theta_j - \theta_i)\]

\[= \frac{1}{2}(\sin(\theta_i - \theta_j + \theta_k) - \sin(\theta_i - \theta_j - \theta_k))

\[= \frac{1}{2}(\sin(\theta_i - \theta_j + \theta_k) + \sin(\theta_i - \theta_j - \theta_k))

\[= \frac{1}{2}(\sin(\theta_i - \theta_j - \theta_k) - \sin(\theta_i - \theta_j + \theta_k))

\[= \cos(\theta_i - \theta_j) \sin(\theta_k - \theta_l).

(ii) \(\theta_i \leq \theta_k(\leq \theta_j)\). Similar to case (II)(i), using \((I''_i)\) and \((I''_k)\) instead.

(iii) \(\min_{s,t} \|(A[ijt])^{-1}\|_1 = \|(A[ijt])^{-1}\|_1\), \(\theta_i \leq \theta_j, \theta_k \leq \theta_l\). Similarly, we have

\[r_j(r_i \sin(\theta_i - \theta_j) - r_i \sin(\theta_i - \theta_j)) \leq r_i r_j \sin(\theta_i - \theta_l)

\[r_k r_i \sin(\theta_i - \theta_k) - r_i \sin(\theta_i - \theta_k)) \leq r_i r_k \sin(\theta_i - \theta_l).

Using these two inequalities to lower bound \(\|(A[ijkl])^{-1}\|_1\), we have

\[r_i^2 r_j^2 \sin^2(\theta_i - \theta_j) X

\[= r_i^2 r_j^2 \sin^2(\theta_i - \theta_j) r_i r_j \cos(\theta_i - \theta_l)

\[+ r_i r_j r_k \sin(\theta_i - \theta_k) r_i r_k \sin(\theta_i - \theta_k) (r_i r_k \cos(\theta_i - \theta_k) + r_i r_k \cos(\theta_i - \theta_k))

\[+ r_i^2 r_j^2 \sin^2(\theta_i - \theta_k) r_i r_k \cos(\theta_i - \theta_k)

\geq r_i^2 r_j^2 \sin(\theta_i - \theta_j) r_i r_k \sin(\theta_i - \theta_k) - r_i \sin(\theta_i - \theta_k)) r_i r_j \cos(\theta_i - \theta_l)

\[+ r_i^2 r_k \sin(\theta_i - \theta_k) r_i r_k \sin(\theta_i - \theta_k) - r_i \sin(\theta_i - \theta_k)) r_i r_k \cos(\theta_i - \theta_k)

\[+ r_i^2 r_k \sin(\theta_i - \theta_k) r_i r_k \sin(\theta_i - \theta_k) - r_i \sin(\theta_i - \theta_k)) r_i r_k \cos(\theta_i - \theta_l)

\[+ r_i^2 r_j^2 \sin^2(\theta_i - \theta_k) r_i r_j \cos(\theta_i - \theta_k)

\[= r_i^2 r_j^2 r_k r_1 [r_i^2 \sin(\theta_i - \theta_j) \sin(\theta_i - \theta_k) \cos(\theta_i - \theta_l) - \sin(\theta_i - \theta_l) \sin(\theta_i - \theta_j) \cos(\theta_i - \theta_k)]

\[+ r_i^2 [\sin(\theta_i - \theta_j) \sin(\theta_i - \theta_k) \cos(\theta_i - \theta_l) - \sin(\theta_i - \theta_l) \sin(\theta_i - \theta_j) \cos(\theta_i - \theta_k)]

\[+ r_i^2 r_l \sin(\theta_i - \theta_k) \sin(\theta_i - \theta_k) \cos(\theta_i - \theta_l) + \sin(\theta_i - \theta_l) \sin(\theta_i - \theta_k) \cos(\theta_i - \theta_l)

\[+ \sin(\theta_i - \theta_k) \sin(\theta_i - \theta_k) \cos(\theta_i - \theta_l) + \sin(\theta_i - \theta_k) \sin(\theta_i - \theta_k) \cos(\theta_i - \theta_l)

\[+ \sin^2(\theta_i - \theta_k) \cos(\theta_i - \theta_l))

\geq r_i^2 r_j^2 r_k r_1 \sin(\theta_i - \theta_j) \sin(\theta_i - \theta_k) (2r_i r_k \sin(\theta_i - \theta_k) + r_i r_k \sin(\theta_i - \theta_k)) r_i r_k \cos(\theta_i - \theta_k)

\[+ r_i^2 r_l \sin(\theta_i - \theta_k) \sin(\theta_i - \theta_k) \cos(\theta_i - \theta_l) + \sin(\theta_i - \theta_l) \sin(\theta_i - \theta_k) \cos(\theta_i - \theta_l)

\[+ r_i^2 r_k r_1 \sin(\theta_i - \theta_j) \sin(\theta_i - \theta_k) (r_i^2 + r_k^2 + 2r_i r_k \cos(\theta_i - \theta_k)).

The last inequality follows from \(\sin(2\theta_i - \theta_k - \theta_l) \geq \sin(\theta_i - \theta_k), \sin(\theta_i + \theta_j - 2\theta_k) \geq \sin(\theta_j - \theta_l), \sin(\theta_j - \theta_l) \geq \sin(\theta_j - \theta_i) \geq 0 \) and \(\sin(\theta_i - \theta_k) \geq \sin(\theta_i - \theta_k) \geq 0\).
(ii) \( \theta_i < \theta_l \). Then we can simplify (I') and (I'') to get

\[
\begin{align*}
& r_j(\sin(\theta_j - \theta_i) - r_i \sin(\theta_l - \theta_i)) \leq r_j r_i \sin(\theta_l - \theta_i) \\
& r_k(\sin(\theta_i - \theta_k) - r_i \sin(\theta_l - \theta_i)) \leq r_i r_j \sin(\theta_l - \theta_i).
\end{align*}
\]

Using these two inequalities to lower bound \( \| (A[ij, kl])^{-1} \|_1 \), we have

\[
\begin{align*}
& r_j^2 r_i^2 \sin^2(\theta_i - \theta_l) X \\
& = r_j^2 r_i^2 \sin^2(\theta_i - \theta_l) r_j r_i \cos(\theta_i - \theta_l) \\
& + r_j r_i \sin(\theta_i - \theta_l) r_j r_i \sin(\theta_l - \theta_i) (r_i r_j \cos(\theta_i - \theta_k) + r_i r_j \cos(\theta_l - \theta_i)) \\
& + r_j^2 r_i^2 \sin^2(\theta_i - \theta_l) r_j r_i \cos(\theta_i - \theta_k) \\
& \geq r_j^2 r_i^2 \sin(\theta_i - \theta_l) r_i \sin(\theta_i - \theta_l) (r_i r_j \cos(\theta_i - \theta_k) + r_i r_j \cos(\theta_l - \theta_i)) \\
& + r_j r_i \sin(\theta_i - \theta_l) r_j r_i \sin(\theta_l - \theta_i) (r_i r_j \cos(\theta_i - \theta_k) + r_i r_j \cos(\theta_l - \theta_i)) \\
& + r_j^2 r_i^2 \sin^2(\theta_i - \theta_l) r_j r_i \cos(\theta_l - \theta_i) \\
& = r_j^2 r_i^2 \sin(\theta_i - \theta_l) \sin(\theta_i - \theta_k) \cos(\theta_l - \theta_i) \cos(\theta_i - \theta_l) \sin(\theta_i - \theta_l) \cos(\theta_l - \theta_i) \\
& + r_j^2 r_i^2 \sin(\theta_i - \theta_l) \sin(\theta_l - \theta_i) \cos(\theta_l - \theta_i) \cos(\theta_i - \theta_l) \sin(\theta_i - \theta_l) \cos(\theta_l - \theta_i) \\
& + r_j^2 r_i^2 \sin(\theta_i - \theta_l) \sin(\theta_l - \theta_i) \cos(\theta_l - \theta_i) \cos(\theta_i - \theta_l) \sin(\theta_i - \theta_l) \cos(\theta_l - \theta_i) \\
& + r_j^2 r_i^2 \sin(\theta_i - \theta_l) \sin(\theta_l - \theta_i) \cos(\theta_l - \theta_i) \cos(\theta_i - \theta_l) \sin(\theta_i - \theta_l) \cos(\theta_l - \theta_i) \\
& = r_j^2 r_i^2 \sin(\theta_i - \theta_l) \sin(\theta_i - \theta_k) (r_i r_j \cos(\theta_i - \theta_k) + r_i r_j \cos(\theta_l - \theta_i)) + r_i r_j \sin(\theta_i - \theta_l) \sin(\theta_i - \theta_k) \cos(\theta_l - \theta_i) \\
& + 2r_i r_j \sin(\theta_i - \theta_l) \sin(\theta_i - \theta_k) \cos(\theta_l - \theta_i) \\
& = r_j^2 r_i^2 \sin(\theta_i - \theta_l) \sin(\theta_i - \theta_k) (r_i r_j \cos(\theta_i - \theta_k) + r_i r_j \cos(\theta_l - \theta_i)) + 2r_i r_j \sin(\theta_i - \theta_l) \sin(\theta_i - \theta_k) \cos(\theta_l - \theta_i).
\end{align*}
\]

Because of the symmetry, the cases \( \min_{i \leq k, l} \| (A[ij, kl])^{-1} \|_1 = \| (A[kl])^{-1} \|_1, \| (A[jl])^{-1} \|_1 \) follow from (I), (II), (III), respectively. Therefore, for any submatrix \( A[ij, kl] \), we have

\[
\| (A[ij, kl])^{-1} \|_1 \geq \min_{i \leq k, l} \| (A[st])^{-1} \|_1,
\]

i.e., \( \tilde{A} \) also minimizes \( \| \tilde{A}^{-1} \|_1 \) over all nonsingular 2 \times 2 submatrices. \( \square \)

**Remark 1.** The rank-2 result also applies to any symmetric matrix \( A \) that is equivalent to a nonnegative matrix under symmetric signing of rows and columns, i.e., if there exists a diagonal matrix \( D = \text{diag}(d) \) with \( d_i \in \{ \pm 1 \} \) such that \( \tilde{A} = DAD \geq 0 \). This is because \( \tilde{H} \) is a symmetric reflexive generalized inverse of \( \tilde{A} \) if and only if \( H = D\tilde{H}D \) is a symmetric reflexive generalized inverse of \( A \).

### 2.3. Approximation

For general \( r := \text{rank}(A) \), we will efficiently find a symmetric reflexive generalized inverse following our symmetric block construction that is within approximately a factor of \( r^2(1 + \epsilon) \) of the 1-norm of the symmetric reflexive generalized inverse having minimum 1-norm. Before presenting the approximation result, we first establish a useful lemma.

**Lemma 1.** For a symmetric matrix \( A \in \mathbb{R}^{n \times n} \), let \( r := \text{rank}(A) \). Let \( A[S] \) be a \( r \times r \) nonsingular principal submatrix of \( A \) with indices \( S \), and let \( A[T] \) be a principal submatrix obtained by swapping an element of \( S \) with one from its complement. If \( |\det(A[T])| \leq (1 + \epsilon) |\det(A[S])| \), then we have \( |\det(A[S,T])| \leq \sqrt{(1 + \epsilon)} |\det(A[S])| \).
Proof. Without loss of generality, assume that $S = \{1, \ldots, r\}$ and $T = \{1, \ldots, r - 1, r + 1\}$. Suppose that matrix $A$ is of the form
\[
\begin{bmatrix}
\hat{A} & a_S & a_T & * \\
{a_S}^\top & b & c & * \\
{a_T}^\top & c & d & * \\
* & * & * & *
\end{bmatrix},
\]
where $\hat{A} \in \mathbb{R}^{(r-1) \times (r-1)}$ is nonsingular and symmetric, $a_S, a_T \in \mathbb{R}^{r-1}$, and $b, c, d \in \mathbb{R}$. By the condition $|\det(A[S])| \leq (1 + \epsilon) |\det(A[S])|$, we have
\[
|\det \begin{bmatrix} \hat{A} & a_T \\ a_T^\top & d \end{bmatrix}| \leq (1 + \epsilon) |\det \begin{bmatrix} \hat{A} & a_S \\ a_S^\top & b \end{bmatrix}|.
\]
Because \(\text{rank} \begin{bmatrix} \hat{A} & a_S \\ a_S^\top & b \end{bmatrix} = \text{rank} \begin{bmatrix} \hat{A} & a_T \\ a_T^\top & c \end{bmatrix} = r\), using the Schur complement, we have
\[
d = [a_T^\top c] \left[ \begin{bmatrix} \hat{A} & a_S \\ a_S^\top & b \end{bmatrix} \right]^{-1} \begin{bmatrix} a_T \\ c \end{bmatrix}.
\]
Also by Schur complementation, we have
\[
\det \begin{bmatrix} \hat{A} & a_T \\ a_T^\top & d \end{bmatrix} = (d - a_T^\top \hat{A}^{-1} a_T) \det(\hat{A})
\]
\[
\det \begin{bmatrix} \hat{A} & a_S \\ a_S^\top & b \end{bmatrix} = (b - a_S^\top \hat{A}^{-1} a_S) \det(\hat{A})
\]
\[
\det(A[S,T]) = \det \begin{bmatrix} \hat{A} & a_T \\ a_T^\top & c \end{bmatrix} = (c - a_S^\top \hat{A}^{-1} a_T) \det(\hat{A}).
\]
Because $\hat{A}$ is nonsingular and $b - a_S^\top \hat{A}^{-1} a_S \neq 0$, using the block-matrix inverse, we get
\[
\begin{bmatrix} \hat{A} & a_S \\ a_S^\top & b \end{bmatrix}^{-1} = \begin{bmatrix} \hat{A}^{-1} & + \hat{A}^{-1} a_S (b - a_S^\top \hat{A}^{-1} a_S)^{-1} a_S^\top \hat{A}^{-1} \\
-a_S^\top \hat{A}^{-1} a_S^{-1} & (b - a_S^\top \hat{A}^{-1} a_S)^{-1} \end{bmatrix}.
\]
Therefore, plugging into (1), we have
\[
d - a_T^\top \hat{A}^{-1} a_T = (b - a_S^\top \hat{A}^{-1} a_S)^{-1} [a_T^\top \hat{A}^{-1} a_S a_S^\top \hat{A}^{-1} a_T - 2 c a_T^\top \hat{A}^{-1} a_S + c^2]
\]
\[
= (b - a_S^\top \hat{A}^{-1} a_S)^{-1} (c - a_T^\top \hat{A}^{-1} a_S)^2,
\]
which leads to
\[
|\det(A[S,T])|^2 = |\det(A[S])| |\det(A[T])| \leq (1 + \epsilon) |\det(A[S])|^2.
\]

Definition 1. Let $A$ be an arbitrary $n \times n$, rank-$r$ matrix. For $S$ an ordered subset of $r$ elements from $\{1, \ldots, n\}$ and fixed $\epsilon \geq 0$, if $|\det(A[S])| > 0$ cannot be increased by a factor of more than $1 + \epsilon$ by swapping an element of $S$ with one from its complement, then we say that $A[S]$ is a $(1 + \epsilon)$-local maximizer for the absolute determinant on the set of $r \times r$ nonsingular principal submatrices of $A$. 
\[
\square
\]
Theorem 5. For a symmetric matrix $A \in \mathbb{R}^{n \times n}$, let $r := \text{rank}(A)$. Choose $\epsilon \geq 0$, and let $\tilde{A} := A[S]$ be a $(1 + \epsilon)$-local maximizer for the absolute determinant on the set of $r \times r$ nonsingular principal submatrices of $A$. The $n \times n$ matrix $H$ constructed by Theorem 2 over $\tilde{A}$, is a symmetric reflexive generalized inverse (having at most $r^2$ nonzeros), satisfying $\|H\|_1 \leq r^2(1 + \epsilon)\|H_{opt}\|_1$, where $H_{opt}$ is a $1$-norm minimizing symmetric reflexive generalized inverse of $A$.

Proof. We prove a stronger result $\|H\|_1 \leq r^2(1 + \epsilon)\|H_{opt}\|_1$, where $H_{opt}$ is an optimal solution to (P), which implies $\|H\|_1 \leq r^2(1 + \epsilon)\|H_{opt}\|_1 \leq r^2(1 + \epsilon)\|H_{opt}'\|_1$. Without loss of generality, we assume that $\tilde{A}$ is in the north-west corner of $A$. So we take $A$ to have the form \[
\begin{bmatrix}
\tilde{A} & B \\
B^T & D
\end{bmatrix}.
\] Let $M := \text{sign}(\tilde{A}^{-1})$, where $\text{sign}(x) = \begin{cases} 1, & x > 0, \\
0, & x = 0, \\
-1, & x < 0. \end{cases}$

Now we let \[
W := \begin{bmatrix}
\tilde{W} & 0 \\
0 & 0
\end{bmatrix} := \begin{bmatrix}
\tilde{A}^{-T} & M \tilde{A}^{-T} \\
0 & 0
\end{bmatrix}.
\]

The dual objective value is \[
\langle A, W \rangle = \text{trace}(A^TW) = \text{trace}(\tilde{A}^T\tilde{W}) = \text{trace}(M\tilde{A}^{-T}) = \langle M, \tilde{A}^{-1} \rangle = \|\tilde{A}^{-1}\|_1,
\]
i.e., $\langle A, W \rangle = \|H\|_1$.

Also, \[
A^TW\tilde{A} = \begin{bmatrix}
M \\
B^T \tilde{A}^{-T}M
\end{bmatrix} \begin{bmatrix}
M \tilde{A}^{-T}B \\
B^T \tilde{A}^{-T}M \tilde{A}^{-T}B
\end{bmatrix}.
\]

Clearly $\|M\|_{\text{max}} \leq 1$. Next, we consider $\tilde{\gamma} := M\tilde{A}^{-T}\gamma = M\tilde{A}^{-1}\gamma$ ($\tilde{A}$ is symmetric), where $\gamma$ is an arbitrary column of $B$. By Cramer’s rule, where $\tilde{A}_i(\gamma)$ is $\tilde{A}$ with column $i$ replaced by $\gamma$, we have \[
\tilde{\gamma} = M \frac{1}{\text{det}(\tilde{A})} \begin{bmatrix}
\text{det}(\tilde{A}_1(\gamma)) \\
\vdots \\
\text{det}(\tilde{A}_r(\gamma))
\end{bmatrix}.
\]

And for $j = 1, \ldots, r$, using Lemma 1, we have \[
|\tilde{\gamma}_j| = \sum_{i=1}^r \text{sign}(\tilde{A}^{-1}_{ji}) \frac{\text{det}(\tilde{A}_i(\gamma))}{\text{det}(\tilde{A})} \leq \sum_{i=1}^r \left|\frac{\text{det}(\tilde{A}_i(\gamma))}{\text{det}(\tilde{A})}\right| \leq r\sqrt{1+\epsilon},
\]
i.e., $\|M\tilde{A}^{-T}B\|_{\text{max}} \leq r\sqrt{1+\epsilon}$. Finally, we have \[
\|B^T \tilde{A}^{-T}M \tilde{A}^{-T}B\|_{\text{max}} \leq r^2 \|B^T \tilde{A}^{-1}\|_{\text{max}} \|\tilde{A}^{-1}B\|_{\text{max}} \leq r^2(1 + \epsilon).
\]

Therefore, $\|A^TW\tilde{A}\|_{\text{max}} \leq r^2(1 + \epsilon)$; so then $\frac{1}{r^2(1+\epsilon)}W$ is dual feasible. By the weak duality for linear optimization, we have $\langle A, \frac{1}{r^2(1+\epsilon)}W \rangle = \frac{1}{r^2(1+\epsilon)}\|H\|_1 \leq \|H_{opt}\|_1$. \hfill $\Box$

Remark 2. In Theorem 5, we could have required the stronger condition that $\tilde{A}$ is a global maximizer for the absolute determinant on the set of $r \times r$ nonsingular principal submatrices of $A$. But we prefer our hypothesis, both because it is weaker and because we can find an $\tilde{A}$ satisfying our hypothesis by a simple finitely-terminating local search. Moreover, if $A$ is rational, and we choose $\epsilon$ positive and fixed, then our local search is efficient:
Theorem 6. Let $A$ be rational. We have an FPTAS (fully polynomial-time approximation scheme; see [15]) for calculating a symmetric reflexive generalized inverse $H$ of $A$ that has $\|H\|_1$ within a factor of $r^2$ of $\|Hopt\|_1$, where $Hopt$ is a 1-norm minimizing symmetric reflexive generalized inverse of $A$.

Proof. Following the proof in [6, Theorem 10], we know that the local search reaches a $(1+\epsilon)$-local maximizer for the absolute determinant on the set of $A$ in at most $O(poly(size(A))(1+\frac{1}{r}))$ iterations, where $size(A)$ is the number of bits in a binary encoding of $A$. Along with Theorem 5, we conclude that the local search is an FPTAS. □

Remark 3. The general idea of our proof follows the scheme of [6, Theorem 9] (the nonsymmetric situation). However, there is a mistake in the proof of [6, Theorem 9]. For this, we give a new column block construction for producing an ah-symmetric reflexive generalized inverse.

3. ah-symmetric results. In this section, let $A$ be an arbitrary $m \times n$ real matrix. We seek to obtain a solution to $\min\{\|H\|_1 : P1 + P2 + P3\}$ (that is, a 1-norm minimizing ah-symmetric reflexive generalized inverse). For this, we give a new column block construction for producing an ah-symmetric reflexive generalized inverse.

Theorem 7. For $A \in \mathbb{R}^{m \times n}$, let $r := \text{rank}(A)$. For any $T$, an ordered subset of $r$ elements from $\{1, \ldots, n\}$, let $\hat{A} := A[:,T]$ be the submatrix of $A$ formed by columns $T$. If $\text{rank}(\hat{A}) = r$, let

$$H := \hat{A}^+ = (\hat{A}^+)^{-1}\hat{A}^T.$$  

The $n \times m$ matrix $H$ with all rows equal to zero, except rows $T$, which are given by $\hat{H}$, is an ah-symmetric reflexive generalized inverse of $A$.

Proof. Without loss of generality, let us assume that $T = (1,2,\ldots,r)$, so we may write

$$A = \begin{bmatrix} \hat{A} & \hat{B} \end{bmatrix}, \quad H = \begin{bmatrix} \hat{H} & 0 \end{bmatrix}.$$  

We have that $H$ satisfies:

- $P1$, as

$$AH \hat{A} = [\hat{A} \hat{H} \hat{A} \hat{H} \hat{B}] = [\hat{A} \hat{B}] = A,$$

where $\hat{A} \hat{H} \hat{A} = \hat{A}$ because $\hat{H} \hat{A}$ is the $r \times r$ identity matrix, and $\hat{A} \hat{H} \hat{B} = \hat{B}$ because, as $A$ (and $\hat{A}$) has rank $r$, the columns of $\hat{B}$ are in the range of $\hat{A}$ and $\hat{A} \hat{H}$ is the projection matrix on the range of $\hat{A}$.

- $P2$, as

$$HAH = \begin{bmatrix} \hat{H} \hat{A} \hat{H} \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{H} \\ 0 \end{bmatrix} = H,$$

where we again use the fact that $\hat{H} \hat{A}$ is the $r \times r$ identity matrix.

- $P3$, as

$$AH = \hat{A} \hat{H} = \hat{A} (\hat{A}^+ \hat{A})^{-1} \hat{A}^T$$

is symmetric. □
Similarly as before, we note that it is useful to consider relaxing $P_2$, arriving at \( \min\{\|H\|_1 : P_1 + P_3\} = \min\{\|H\|_1 : AHA = A, (AH)^\top = AH\} \), which we re-cast as a linear-optimization problem \((P_{ah})\) and its dual \((D_{ah})\):

\[
\begin{align*}
\text{minimize} & \quad \langle J, H^+ \rangle + \langle J, H^- \rangle \\
\text{subject to} & \quad A(H^+ - H^-)A = A, \\
& \quad (H^+ - H^-)^\top A^\top = A(H^+ - H^-), \\
& \quad H^+, H^- \geq 0. \\
\end{align*}
\]

\((P_{ah})\)

\[
\begin{align*}
\text{maximize} & \quad \langle A, W \rangle \\
\text{subject to} & \quad -J \leq A^\top W A^\top + A^\top (V^\top - V) \leq J \\
\end{align*}
\]

\((D_{ah})\)

More compactly, we can see \((D_{ah})\) also as: \(\max\{\langle A, W \rangle : \|A^\top W A^\top + A^\top U\|_{\max} \leq 1, U^\top = -U\}\).

### 3.1. Rank 1

Next, we demonstrate that when \(\text{rank}(A) = 1\), construction of a 1-norm minimizing ah-symmetric reflexive generalized inverse can be based on the column block construction.

**Theorem 8.** Let \(A\) be an arbitrary \(m \times n\), rank-1 matrix. For any \(j \in \{1, \ldots, n\}\), let \(\hat{a}\) be column \(j\) of \(A\). If \(j\) is chosen to minimize the 1-norm of \(\hat{a}^+\) among all columns except the zero columns, then the \(n \times m\) matrix \(H\) constructed by Theorem 7 over \(\hat{a}\), is an ah-symmetric reflexive generalized inverse of \(A\) with minimum 1-norm.

**Proof.** We prove a stronger result — that our constructed \(H\) is a 1-norm minimizing ah-symmetric reflexive generalized inverse. By our construction, \(H\) is reflexive, thus \(H\) is an ah-symmetric reflexive generalized inverse with minimum 1-norm. To establish the minimum 1-norm of \(H\), we consider the linear-optimization problems \((P_{ah})\) and \((D_{ah})\). As verified in Theorem 7, \(H\) is a feasible solution for \((P_{ah})\), and its objective value is

\[\|H\|_1 = \|\hat{a}^+\|_1\]

(it also satisfies the nonlinear equations \((P_2)\)).

The objective function of \((D_{ah})\) only depends on the variable \(W\). Feasibility of a \(W\) is equivalent to the existence of a skew-symmetric matrix \(U\) so that

\[\|A^\top W A^\top + A^\top U\|_{\max} \leq 1.\]  \hspace{1cm} (2)

Next, we are going to construct a dual feasible solution \(W\) with objective value \(\langle A, W \rangle = \|H\|_1\); then by the weak duality for linear optimization, we establish that \(H\) is optimal to \((P_{ah})\).

Let \(\hat{e} = \text{sign}(\hat{a}^+)\). Suppose that \(\hat{a}_i\) is a non-zero element in \(\hat{a}\) with index \(i\). Let \(W\) be a \(m \times n\) matrix with all elements equal to zero, except the one in row \(i\) and column \(j\), which is given by \(\hat{w}\). Let \(U\) be a \(m \times m\) skew-symmetric matrix, with only row \(i\) and column \(i\) different from zero.

If \(\hat{w}\) and \(U\) are chosen to be

\[
\begin{align*}
\hat{w} := \frac{1}{\hat{a}_i}\hat{e}(\hat{a}^+)^\top, \\
u_{ki} := -u_{ik} := \frac{1}{\hat{a}_i}(\hat{a}_k\hat{e}(\hat{a}^+)^\top - \hat{e}_k), \quad \forall \, k \neq i,
\end{align*}
\]

then they satisfy

\[
\hat{a}_i\hat{w}\hat{a}_i^\top + \hat{a}_i^\top U = \hat{e}.
\]

\hspace{1cm} (3)

This is because for \(k \neq i\), \(\hat{a}_i\hat{w}\hat{a}_k + \hat{a}_i u_{ik} = \hat{e}(\hat{a}^+)^\top \hat{a}_k + \hat{e}_k - \hat{a}_k\hat{e}(\hat{a}^+)^\top = \hat{e}_k\), and

\[
\hat{a}_i\hat{w}\hat{a}_k + \sum_{k \neq i} \hat{a}_k u_{ki} = \hat{e}(\hat{a}^+)^\top (\hat{a}_i + \sum_{k \neq i} \hat{a}_k^2) - \sum_{k \neq i} \hat{a}_k \hat{e}_k = \frac{1}{\hat{a}_i}(\hat{e}(\hat{a}^+)^\top (\hat{a}_i^\top \hat{a}_i) - \hat{e}_i) + \hat{e}_i = \hat{e}_i,
\]

\[
\hat{a}_i\hat{w}\hat{a}_k + \sum_{k \neq i} \hat{a}_k u_{ki} = \hat{e}(\hat{a}^+)^\top (\hat{a}_i + \sum_{k \neq i} \hat{a}_k^2) - \sum_{k \neq i} \hat{a}_k \hat{e}_k = \frac{1}{\hat{a}_i}(\hat{e}(\hat{a}^+)^\top (\hat{a}_i^\top \hat{a}_i) - \hat{e}_i) + \hat{e}_i = \hat{e}_i,
\]

\[
\hat{a}_i\hat{w}\hat{a}_k + \sum_{k \neq i} \hat{a}_k u_{ki} = \hat{e}(\hat{a}^+)^\top (\hat{a}_i + \sum_{k \neq i} \hat{a}_k^2) - \sum_{k \neq i} \hat{a}_k \hat{e}_k = \frac{1}{\hat{a}_i}(\hat{e}(\hat{a}^+)^\top (\hat{a}_i^\top \hat{a}_i) - \hat{e}_i) + \hat{e}_i = \hat{e}_i.
\]
and
\[
\text{trace}(A^T W) = \hat{a}_i \hat{w} = \hat{e}(\hat{a}^+)^T = \|\hat{a}^+\|_1 = \|H\|_1.
\]

The dual constraint (2) can be written as
\[
\|\hat{a}^+ W A^T + \hat{a}^T U\|_{\max} \leq 1,
\]
and
\[
\|\hat{B}^T W A^T + \hat{B}^T U\|_{\max} \leq 1.
\]
From (3), we have that (4) is satisfied. To verify (5), let \(\hat{b} \in \mathbb{R}^m\) be an arbitrary column of \(\hat{B}\). As \(A\) has rank 1,
\[
\hat{b} = \alpha \hat{a},
\]
and
\[
\hat{b}^T W A^T + \hat{b}^T U = \alpha \hat{a}^T (W A^T + U).
\]
Considering (3), we have that \(\|\hat{a}^T(W A^T + U)\|_{\max} = 1\), and therefore,
\[
\|\hat{b}^T W A^T + \hat{b}^T U\|_{\max} = |\alpha|.
\]
We have
\[
\hat{a}^+ = (\hat{a}^T \hat{a})^{-1} \hat{a}^T = \frac{1}{\hat{a}^T \hat{a}} \hat{a}^T.
\]
We also have
\[
\hat{b}^+ = (\hat{b}^T \hat{b})^{-1} \hat{b}^T = \frac{1}{\hat{b}^T \hat{b}} \hat{b}^T = \frac{1}{\alpha (\hat{a}^T \hat{a})} \hat{a}^T.
\]
From optimality of \(H\), we have
\[
\|\hat{a}^+\|_1 \leq \|\hat{b}^+\|_1.
\]
Therefore
\[
\frac{1}{\|\hat{a}\|_1} \|\hat{a}^+\|_1 \leq \frac{1}{|\alpha| \|\hat{a}\|_1} \|\hat{a}^+\|_1.
\]
So, \(|\alpha| \leq 1\), which implies that (5) is satisfied. \(\square\)

Before moving on to the rank-2 and rank-\(r\) cases, we generalize the choice of \(\hat{w}, U\) satisfying (3) to the general rank-\(r\) case.

**Theorem 9.** Let \(T\) be an ordered subset of \(r\) elements from \{1, \ldots, n\} and \(\hat{A} := A[; T]\) be the \(m \times r\) submatrix of a \(m \times n\) matrix \(A\) formed by columns \(T\), and \(\text{rank}(\hat{A}) = r\). There exists a \(m \times n\) matrix \(W\) and a skew-symmetric \(m \times m\) matrix \(U\) such that
\[
\hat{A}^T W A^T + \hat{A}^T U = E,
\]
where \(E := \text{sign}(\hat{A}^+)\). Furthermore, \((A, W) = \|\hat{A}^+\|_1\).

**Proof.** Suppose that \(\tilde{A} := A[S, T]\) is the nonsingular \(r \times r\) submatrix of \(\hat{A}\) formed by rows \(S := \{i_1, i_2, \ldots, i_r\}\). Let \(W\) be a \(r \times r\) matrix and \(W\) be a \(m \times n\) matrix with all elements equal to zero, except the ones in rows \(S\) and columns \(T\), which are given by the respective elements in \(W\). If we choose \(W\) and \(U\) to be
\[
\tilde{W} := \hat{A}^{-T} E A (\hat{A}^T \hat{A})^{-1} = \hat{A}^{-T} E (\hat{A}^+)^+.
\]
and
\[
U := \hat{A} \tilde{W}^T D - D^T \hat{W} A^T + D^T \hat{A}^{-T} E - E^T \hat{A}^{-1} D,
\]
where \(D\) is a \(r \times m\) matrix with all elements equal to zero, except \(D_{11} = D_{22} = \cdots = D_{rr} = 1\).
Because $D \hat{A} = \tilde{A}$, we have
\[
\hat{A}^\top U = \hat{A}^\top \hat{W}^\top D - \hat{A}^\top \hat{W} \hat{A}^\top + E - \hat{A}^\top E^\top \hat{A}^{-1} D \\
= E - \hat{A}^\top \hat{W} \hat{A}^\top + (\hat{W}(\hat{A} \hat{A}^\top) - \hat{A}^{-\top} E \hat{A})^\top D \\
= E - \hat{A}^\top \hat{W} \hat{A}^\top.
\]
Hence, $\hat{A}^\top W A^\top + \hat{A}^\top U = \hat{A}^\top \hat{W} A^\top + \hat{A}^\top U = E$. Furthermore,
\[
\langle A, W \rangle = \text{trace}(\hat{A}^\top \hat{W}) = \text{trace}(E(\hat{A}^\top)^+) = \langle \hat{A}^+, E \rangle = \|\hat{A}^+\|_1.
\]

3.2. Rank 2. Generally, when rank($A$) = 2, we cannot construct a 1-norm minimizing ah-symmetric reflexive generalized inverse based on the column block construction. Even under the condition that $A$ is non-negative, we have the following example:
\[
A = \begin{bmatrix} 1 & 3 & 8 \\ 2 & 2 & 8 \\ 3 & 1 & 8 \end{bmatrix}.
\]
Note that rank($A$) = 2 because $a_3 = 2a_1 + 2a_2$. We have an ah-symmetric reflexive generalize inverse with 1-norm $\frac{9}{8}$,
\[
H := \begin{bmatrix} -\frac{1}{4} & 0 & -\frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{24} & \frac{1}{24} & \frac{1}{24} \end{bmatrix}.
\]
However, the three ah-symmetric reflexive generalized inverses based on our column block construction have 1-norm $\frac{9}{2}, \frac{31}{24}, \frac{7}{6}$, respectively.

Next, we demonstrate that under an efficiently-checkable technical condition, when rank($A$) = 2, construction of a 1-norm minimizing ah-symmetric reflexive generalized inverse can be based on the column block construction.

Theorem 10. Let $A$ be an arbitrary $m \times n$, rank-2 matrix. For any $j_1, j_2 \in \{1, \ldots, n\}$, with $j_1 < j_2$, let $\hat{A} := [\hat{a}_{j_1}, \hat{a}_{j_2}]$ be the $m \times 2$ submatrix of $A$ formed by columns $j_1$ and $j_2$. Suppose that $j_1, j_2$ are chosen to minimize the 1-norm of $\hat{H} := \hat{A}^\top$ among all $m \times 2$ rank-2 submatrices of $A$. Every column $\hat{b}$ of $\hat{A}$ can be uniquely written in the basis $\hat{a}_{j_1}, \hat{a}_{j_2}$, say $\hat{b} = \alpha \hat{a}_{j_1} + \beta \hat{a}_{j_2}$. Suppose that for each such column $\hat{b}$ of $\hat{A}$, one of the following conditions holds on the associated $\alpha, \beta$:

(i) $|\alpha| + |\beta| \leq 1$;
(ii) $\hat{H}_{1j} \hat{H}_{2j} \leq 0$ for $j = 1, \ldots, m$, and $\alpha \beta \geq 0$;
(iii) $\hat{H}_{1j} \hat{H}_{2j} \geq 0$ for $j = 1, \ldots, m$, and $\alpha \beta \leq 0$.

Then the $n \times m$ matrix $H$ constructed by Theorem 7 based on $\hat{A}$, is an ah-symmetric reflexive generalized inverse of $A$ with minimum 1-norm.

Proof. We prove a stronger result that our constructed $H$ is a 1-norm minimizing ah-symmetric generalized inverse. By our construction, $H$ is reflexive, thus $H$ is an ah-symmetric reflexive generalized inverse with minimum 1-norm. To establish the minimum 1-norm of our constructed $H$, we consider the dual pair of linear-optimization problems ($P_{ah}$) and ($D_{ah}$). As verified in Theorem 7, $H$ is a feasible solution for ($P_{ah}$), and its objective value is
\[
\|H\|_1 = \|\hat{A}^+\|_1
\]
(it also satisfies the nonlinear equations (P2)).
and then

\[ \langle A, W \rangle = \| \hat{A}^+ \|_1 = \| H \|_1. \]

The dual constraint (6) can be written as

\[ \| \hat{A}^+ W A^T + \hat{A}^+ U \|_{\max} \leq 1, \quad (7) \]

and

\[ \| \hat{B}^+ W A^T + \hat{B}^+ U \|_{\max} \leq 1, \quad (9) \]

Next, we are going to construct a dual feasible solution \( W \) with objective value \( \langle A, W \rangle = \| H \|_1 \), then by the weak duality for linear optimization, we prove that \( H \) is optimal to \( (P_{ah}). \)

By Theorem 9, we can choose \( W \) and a skew-symmetric matrix \( U \) such that

\[ \hat{A}^+ W A^T + \hat{A}^+ U = E, \quad (7) \]

\[ \langle A, W \rangle = \| \hat{A}^+ \|_1 = \| H \|_1. \]

\[ \| \hat{A}^+ W A^T + \hat{A}^+ U \|_{\max} \leq 1, \quad (8) \]

\[ \| \hat{B}^+ W A^T + \hat{B}^+ U \|_{\max} \leq 1. \quad (9) \]

From (7), we have that (8) is satisfied. To verify (9), without loss of generality, let \((j_1, j_2) = (1, 2)\), and let \( b \in \mathbb{R}^m \) be an arbitrary column of \( B \) with \( \alpha \) and \( \beta \) such that \( \hat{b} = \alpha \hat{a}_1 + \beta \hat{a}_2 \); thus

\[ \hat{b}^+ W A^T + \hat{b}^+ U = ([\alpha \ \beta])(\hat{A}^+ W A^T + \hat{A}^+ U) = ([\alpha \ \beta]E = [\alpha \ \beta]|\text{sign}(\hat{H}). \]

• For case (i), we have \( \| \hat{b}^+ W A^T + \hat{b}^+ U \|_{\max} \leq (|\alpha| + |\beta|)\| E \|_{\max} \leq 1. \)

• For case (ii), because \( \hat{H}_{1j} \hat{H}_{2j} \leq 0 \) for \( j = 1, \ldots, m \), we have \( \hat{b}^+ W A^T + \hat{b}^+ U = (\alpha - \beta)\text{sign}(\hat{H}_{1j}), \) and thus

\[ \| \hat{b}^+ W A^T + \hat{b}^+ U \|_{\max} = |\alpha - \beta|. \]

Also we have \( \alpha \beta \geq 0 \), so

\[ \| \hat{b}^+ W A^T + \hat{b}^+ U \|_{\max} = |\alpha| - |\beta|. \]

• For case (iii), because \( \hat{H}_{1j} \hat{H}_{2j} \geq 0 \) for \( j = 1, \ldots, m \), we have \( \hat{b}^+ W A^T + \hat{b}^+ U = (\alpha + \beta)\text{sign}(\hat{H}_{1j}), \) and thus

\[ \| \hat{b}^+ W A^T + \hat{b}^+ U \|_{\max} = |\alpha + \beta|. \]

Also we have \( \alpha \beta \leq 0 \); so

\[ \| \hat{b}^+ W A^T + \hat{b}^+ U \|_{\max} = |\alpha| - |\beta|. \]

So to prove the dual feasibility, we only need to show that \( |\alpha| - |\beta| \leq 1. \)

Let \( \hat{A}_{b/1} := [\hat{b} \ \hat{a}_2] \) and \( \delta_{ij} := \hat{a}_i^T \hat{a}_j, \) for \( i, j = 1, 2 \). We have

\[ \hat{A}^+ = (\hat{A}^+ \hat{A})^{-1} \hat{A}^+ = \begin{bmatrix} \hat{a}_1^T \\ \hat{a}_2^T \end{bmatrix} \begin{bmatrix} \hat{a}_1 \ \\ \hat{a}_2 \end{bmatrix}^{-1} \begin{bmatrix} \hat{a}_1^T \\ \hat{a}_2^T \end{bmatrix} \]

\[ = \begin{bmatrix} \delta_{22} - \delta_{12} & \delta_{22} \hat{a}_1 - \delta_{12} \hat{a}_2 \\ -\delta_{12} - \delta_{11} & -\delta_{12} \hat{a}_1 + \delta_{11} \hat{a}_2 \end{bmatrix} \]

\[ = \begin{bmatrix} \delta_{22} \hat{a}_1 - \delta_{12} \hat{a}_2 \\ -\delta_{12} \hat{a}_1 + \delta_{11} \hat{a}_2 \end{bmatrix}, \]
where \( \theta = \delta_{11} \delta_{22} - \delta_{12}^2 \). We also have

\[
\hat{A}^{+}_{b/1} = \left( \hat{A}^{T}_{b/1} \hat{A}_{b/1} \right)^{-1} \hat{A}^{T}_{b/1}
\]

\[
= \frac{1}{\alpha \theta} \left( \alpha \hat{a}_{1}^T + \beta \hat{a}_{2}^T \right) \left( \alpha \hat{a}_{1} + \beta \hat{a}_{2} \right)^{-1} \left( \alpha \hat{a}_{1}^T + \beta \hat{a}_{2}^T \right)
\]

\[
= \frac{1}{\theta} \left[ \alpha \hat{a}_{1}^T + \beta \hat{a}_{2}^T \right] \left[ \alpha \hat{a}_{1} + \beta \hat{a}_{2} \right]^{-1} \left[ \alpha \hat{a}_{1}^T + \beta \hat{a}_{2}^T \right]
\]

\[
\delta = \delta_{11} \delta_{22} + \alpha \beta \delta_{12} + \beta^2 \delta_{22} - (\alpha \delta_{12} + \beta \delta_{22})^2
\]

\[
\frac{1}{\alpha \theta} \left( \alpha \hat{a}_{1}^T - \delta_{12} \hat{a}_{2}^T \right)_{1} + \| \alpha \hat{a}_{1} + \delta_{12} \hat{a}_{2} \|_{1}
\]

\[
\leq \frac{1}{\theta} \left( \alpha \hat{a}_{1}^T - \delta_{12} \hat{a}_{2}^T \right)_{1} + \| \alpha \left( \delta_{12} \hat{a}_{1} - \delta_{11} \hat{a}_{2} \right) - \beta \left( \delta_{22} \hat{a}_{1}^T - \delta_{12} \hat{a}_{2}^T \right) \|_{1}
\]

\[
= \frac{1}{\alpha \theta} \left( \| \alpha \left( \delta_{12} \hat{a}_{1} - \delta_{11} \hat{a}_{2} \right) + \beta \left( \delta_{22} \hat{a}_{1}^T - \delta_{12} \hat{a}_{2}^T \right) \|_{1}
\]

\[
= \frac{1}{\alpha \theta} \left( \| \delta_{22} \hat{a}_{1}^T - \delta_{12} \hat{a}_{2}^T \|_{1} + \| \alpha \| - \delta_{12} \hat{a}_{1}^T - \delta_{11} \hat{a}_{2}^T \|_{1} + \| \beta \| - \delta_{22} \hat{a}_{1}^T + \delta_{12} \hat{a}_{2}^T \|_{1} \right)
\]

So,

\[
|\alpha| \left( \| \delta_{22} \hat{a}_{1}^T - \delta_{12} \hat{a}_{2}^T \|_{1} + \| - \delta_{12} \hat{a}_{1}^T + \delta_{11} \hat{a}_{2}^T \|_{1} \right)
\]

\[
\leq \| \delta_{22} \hat{a}_{1}^T - \delta_{12} \hat{a}_{2}^T \|_{1} + |\alpha| \| - \delta_{12} \hat{a}_{1}^T + \delta_{11} \hat{a}_{2}^T \|_{1} + |\beta| \| - \delta_{22} \hat{a}_{1}^T + \delta_{12} \hat{a}_{2}^T \|_{1} \right)
\]

and

\[
|\alpha| - |\beta| \leq 1.
\]

Now, considering that

\[
\| \hat{A}^{+} \|_{1} \leq \| \hat{A}^{+}_{b/2} \|_{1} ,
\]
where \( \hat{A}_{b/2} := [\hat{a}_1 \ b] \), we analogously obtain
\[
\frac{1}{|\theta|} \left( \| \delta_2 \hat{a}_1^\top - \delta_2 \hat{a}_2^\top \|_1 + \| - \delta_1 \hat{a}_1^\top + \delta_1 \hat{a}_2^\top \|_1 \right) \\
\leq \frac{|\beta|}{|\theta|} \left( \| - \delta_1 \hat{a}_1^\top + \delta_1 \hat{a}_2^\top \|_1 + \| \alpha(\delta_1 \hat{a}_1^\top - \delta_1 \hat{a}_2^\top) + \beta(\delta_2 \hat{a}_1^\top - \delta_2 \hat{a}_2^\top) \|_1 \right) \\
\leq \frac{1}{|\theta|} \left( \| - \delta_1 \hat{a}_1^\top + \delta_1 \hat{a}_2^\top \|_1 + |\alpha| \| \delta_1 \hat{a}_1^\top - \delta_1 \hat{a}_2^\top \|_1 + |\beta| \| \delta_2 \hat{a}_1^\top - \delta_2 \hat{a}_2^\top \|_1 \right).
\]

So,
\[
|\beta| \left( \| \delta_2 \hat{a}_1^\top - \delta_2 \hat{a}_2^\top \|_1 + \| - \delta_1 \hat{a}_1^\top + \delta_1 \hat{a}_2^\top \|_1 \right) \\
\leq \| - \delta_1 \hat{a}_1^\top + \delta_1 \hat{a}_2^\top \|_1 + |\alpha| \| \delta_1 \hat{a}_1^\top - \delta_1 \hat{a}_2^\top \|_1 + |\beta| \| \delta_2 \hat{a}_1^\top - \delta_2 \hat{a}_2^\top \|_1 ,
\]

and
\[
|\beta| - |\alpha| \leq 1.
\]

From (10) and (11), we have
\[
|\alpha| - |\beta| \leq 1.
\]

**Remark 4.** The following example shows that we can allow different cases for each column \( b \).

Let
\[
A := \begin{bmatrix} 2 & 3 & 1 & 5 \\ 2 & 3 & 1 & 5 \\ 2 & 5 & 2 & 7 \end{bmatrix}.
\]

Then \( \{i_1, i_2\} = \{1, 2\} \) minimizes the 1-norm of \( \hat{H} \) with \( \|\hat{H}\| = 3 \). We have
\[
\hat{H} = \begin{bmatrix} \frac{2}{5} & \frac{2}{5} & -\frac{2}{5} \\ -\frac{1}{5} & -\frac{1}{5} & \frac{1}{5} \end{bmatrix}
\]
satisfying \( \hat{H}_{i_1 j_2} \leq 0 \) for \( j = 1, 2, 3 \), and \( \hat{b}_1 = [1, 1, 2]^\top = -\frac{1}{2} \hat{a}_1 + \frac{1}{2} \hat{a}_2 \) satisfies only case (i), and \( \hat{b}_2 = [5, 5, 7]^\top = \hat{a}_1 + \hat{a}_2 \) satisfies only case (ii).

The technical sufficient condition in Theorem 10, while efficiently checkable, may seem rather complicated. But perhaps surprisingly, if \( H \) with minimum 1-norm of \( \hat{H} \) is an optimal solution to \((\mathcal{P}_{ab})\) (i.e., a 1-norm minimizing ah-symmetric generalized inverse of \( A \), following our column block construction), then the condition is also necessary. So, for rank-2, there is no possibility of further generalizing the condition, in the context of proving the optimality of our chosen column block construction.

**Theorem 11.** Let \( A \) be an arbitrary \( m \times n \), rank-2 matrix. For any \( j_1, j_2 \in \{1, \ldots, n\} \), with \( j_1 < j_2 \), let \( A := [\hat{a}_{j_1}, \hat{a}_{j_2}] \) be the \( m \times 2 \) submatrix of \( A \) formed by columns \( j_1 \) and \( j_2 \). Suppose that \( j_1, j_2 \) are chosen to minimize the 1-norm of \( \hat{H} := \hat{A}^\top \) among all \( m \times 2 \) rank-2 submatrices of \( A \). Suppose that the \( n \times m \) matrix \( H \) constructed by Theorem 7 based on \( A \), is an ah-symmetric generalized inverse of \( A \) with minimum 1-norm. Every column \( \hat{b} \) of \( A \), can be uniquely written in the basis \( \hat{a}_{j_1}, \hat{a}_{j_2} \), say \( \hat{b} = \alpha \hat{a}_{j_1} + \beta \hat{a}_{j_2} \). Then for each such column \( \hat{b} \) of \( A \), one of the following conditions holds on the associated \( \alpha, \beta \):

- (i) \( |\alpha| + |\beta| \leq 1 \);
- (ii) \( \hat{H}_{1 j} \hat{H}_{2 j} \leq 0 \) for \( j = 1, \ldots, m \), and \( \alpha \beta \geq 0 \);
- (iii) \( \hat{H}_{1 j} \hat{H}_{2 j} \geq 0 \) for \( j = 1, \ldots, m \), and \( \alpha \beta \leq 0 \).
Proof. We consider the dual pair of linear-optimization problems \((P_{ah})\) and \((D_{ah})\). Because \(H\) is an optimal solution to \((P_{ah})\), by the complementary slackness, we have

\[
\langle J - A^T W A^T - A^T U, H^+ \rangle = 0 \\
\langle J + A^T W A^T + A^T U, H^- \rangle = 0
\]

where \(H^+ = \max\{H, 0\}\), \(H^- = -\min\{H, 0\}\), \(W, U\) is an optimal solution to \((D_{ah})\) with \(U^+ = -U\). Along with \(H^+, H^- \geq 0\) and dual feasibility, we have

\[
(J - A^T W A^T - A^T U)_{ij} H_{ij}^+ = 0 \\
(J + A^T W A^T + A^T U)_{ij} H_{ij}^- = 0
\]

Thus,

\[
(A^T W A^T + A^T U)_{ij} = \begin{cases} 
1, & H_{ij} > 0, \\
-1, & H_{ij} < 0, \\
[-1, 1], & H_{ij} = 0.
\end{cases}
\]

Without loss of generality, assume that \((j_1, j_2) = (1, 2)\), and \(H = [\hat{H}; 0]\) with \(\hat{H} \in \mathbb{R}^{2 \times m}\). Let

\[
(A^T W A^T + A^T U)[\{1, 2\}; :] = \hat{A}^T W A^T + \hat{A}^T U := E.
\]

For every column \(\hat{b}\) of \(A\), \(\hat{b} = \alpha \hat{a}_1 + \beta \hat{a}_2\) because \(\text{rank}(A) = 2\). Hence

\[
\hat{b}^T W A^T + \hat{b}^T U = [\alpha \beta](\hat{A}^T W A^T + \hat{A}^T U) = [\alpha \beta] E,
\]

and we have \(\|\hat{b}^T W A^T + \hat{b}^T U\|_{\text{max}} \leq 1\).

- If for any \(j \in \{1, \ldots, m\}\), one of \(\hat{H}_{1j}, \hat{H}_{2j}\) is zero, then \(\hat{H}_{1j} \hat{H}_{2j} = 0\) for \(j = 1, \ldots, m\), thus for any \(\alpha, \beta\), either \(\alpha \beta \geq 0\) or \(\alpha \beta \leq 0\) holds.
- If \(\hat{H}_{1j} \hat{H}_{2j} \leq 0\) for \(j = 1, \ldots, m\), and \(\hat{H}_{1k} \hat{H}_{2k} < 0\) for some \(k\), then \(E_{1k} = -E_{2k} \neq 0\), and \(|\alpha - \beta| = |\alpha E_{1k} + \beta E_{2k}| \leq 1\), thus \(|\alpha| + |\beta| \leq 1\) or \(\alpha \beta \geq 0\).
- If \(\hat{H}_{1j} \hat{H}_{2j} \geq 0\) for \(j = 1, \ldots, m\), and \(\hat{H}_{1k} \hat{H}_{2k} > 0\) for some \(k\), then \(E_{1k} = E_{2k} \neq 0\), and \(|\alpha + \beta| = |\alpha E_{1k} + \beta E_{2k}| \leq 1\), thus \(|\alpha| + |\beta| \leq 1\) or \(\alpha \beta \leq 0\).
- Otherwise, we have both \(|\alpha - \beta| \leq 1\) and \(|\alpha + \beta| \leq 1\), which implies \(|\alpha| + |\beta| \leq 1\).

Hence \(\alpha, \beta\) must satisfy one of (i), (ii) and (iii). \(\square\)

3.3. Approximation. For general \(r := \text{rank}(A)\), we will efficiently find an \(ah\)-symmetric reflexive generalized inverse following our column block construction that is within approximately a constant factor \(r(1 + \epsilon)\) of the 1-norm of the \(ah\)-symmetric reflexive generalized inverse having minimum 1-norm.

**Definition 2.** Let \(A\) be an arbitrary \(m \times n\), rank-\(r\) matrix, and let \(S\) be an ordered subset of \(r\) elements from \(\{1, \ldots, m\}\) such that these \(r\) rows of \(A\) are linearly independent. For \(T\) an ordered subset of \(r\) elements from \(\{1, \ldots, n\}\), and fixed \(\epsilon \geq 0\), if \(|\det(A[S, T])|\) cannot be increased by a factor of more than \(1 + \epsilon\) by swapping an element of \(T\) with one from its complement, then we say that \(A[S, T]\) is a \((1 + \epsilon)\)-local maximizer for the absolute determinant on the set of \(r \times r\) nonsingular submatrices of \(A[S, :]\).

**Theorem 12.** Let \(A\) be an arbitrary \(m \times n\), rank-\(r\) matrix, and let \(S\) be an ordered subset of \(r\) elements from \(\{1, \ldots, m\}\) such that these \(r\) rows of \(A\) are linearly independent. Choose \(\epsilon \geq 0\), and let \(\hat{A} := A[S, T]\) be a \((1 + \epsilon)\)-local maximizer for the absolute determinant on the set of \(r \times r\) nonsingular submatrices of \(A[S, :]\). Then the \(n \times m\) matrix \(H\) constructed by Theorem 7 over \(\hat{A} := A[S, T]\), is an \(ah\)-symmetric reflexive generalized inverse of \(A\) satisfying \(\|H\|_1 \leq r(1 + \epsilon)\|H_{opt}\|_1\), where \(H_{opt}\) is a 1-norm minimizing \(ah\)-symmetric reflexive generalized inverse of \(A\).
Proof. We prove a stronger result \(|H_1| \leq r(1 + \epsilon)|H_{\text{opt}}^{ah}|_1\), where \(H_{\text{opt}}^{ah}\) is an optimal solution of \((P_{ah})\), which implies \(|H_1| \leq r(1 + \epsilon)|H_{\text{opt}}^{ah}|_1 \leq r(1 + \epsilon)|H_{\text{opt}}'|_1\). We will construct a dual feasible solution with objective value \(\frac{1}{r(1 + \epsilon)}|H_1|\). By weak duality for linear optimization, we will then have \(\frac{1}{r(1 + \epsilon)}|H_1| \leq |H_{\text{opt}}^{ah}|_1\).

By Theorem 9, we could choose \(W\) and a skew-symmetric matrix \(U\) such that
\[
\hat{A}^TWA^T + \hat{A}^TU = E.
\]
and
\[
\langle A, W \rangle = \|\hat{A}^+\|_1 = \|H\|_1.
\]

So it is sufficient to demonstrate that \(\|A^TWA^T + A^TU\|_{\max} \leq r(1 + \epsilon), \) then \(\frac{1}{r(1 + \epsilon)}W, \frac{1}{r(1 + \epsilon)}U\) is dual feasible and \(\langle A, W \rangle = \frac{1}{r(1 + \epsilon)}\|H\|_1\).

First it is clear that
\[
\|\hat{A}^TWA^T + \hat{A}^TU\|_{\max} = \|E\|_{\max} = 1 \leq r(1 + \epsilon).
\]

Next, we consider any column \(\hat{b}\) of \(\hat{B}\), because \(\text{rank}(\hat{A}) = r = \text{rank}(A)\), we know that \(\hat{b} = \hat{A}\beta, \beta \in \mathbb{R}^r\), which implies \(\hat{b} = \hat{A}\beta\). By Cramer’s rule, where \(\hat{A}(\hat{b})\) is \(\hat{A}\) with column \(i\) replaced by \(\hat{b}\), we have
\[
|\beta_i| = \left|\frac{\det(\hat{A}(\hat{b}))}{\det(\hat{A})}\right| \leq 1 + \epsilon,
\]
because \(\hat{A}\) is a \((1 + \epsilon)\)-local maximizer for the absolute determinant of \(A[S,:]\). Therefore
\[
\|\hat{b}^TWA^T + \hat{b}^TU\|_{\max} = \|\beta^T (\hat{A}^TWA^T + \hat{A}^TU)\|_{\max}
\]
\[
= \|\beta^TE\|_{\max} \leq \sum_{i=1}^n |\beta_i| \leq r(1 + \epsilon).
\]

Remark 5. In Theorem 12, we could have required the stronger condition that \(\hat{A}\) is a global maximizer for the absolute determinant on the set of \(r \times r\) nonsingular submatrices of \(A_\sigma\). But we prefer our hypothesis — the reasons are the same as in Remark 2. And the local search is efficient:

Theorem 13. Let \(A\) be rational. We have an FPTAS for calculating an ah-symmetric reflexive generalized inverse \(H\) of \(A\) that has \(|H|_1\) within a factor of \(|H_{\text{opt}}^{r}|_1\), where \(H_{\text{opt}}^{r}\) is a 1-norm minimizing ah-symmetric reflexive generalized inverse of \(A\).

4. ha-symmetric results. In this section, let \(A\) be an arbitrary \(m \times n\) matrix, we seek to obtain an optimal solution to \(\min\{\|H\|_1 : P1 + P2 + P4\}\) (that is, a 1-norm minimizing ha-symmetric reflexive generalized inverse). First of all, we have the following observation.

Lemma 2. \(H\) is a ha-symmetric (reflexive) generalized inverse of \(A\) if and only if \(H^T\) is an ah-symmetric (reflexive) generalized inverse of \(A^T\).

Proof. Suppose that \(H\) is ha-symmetric with respect to \(A\). Then \(A^TH = (HA)^T = HA = (A^THT)^T\), which is equivalent to \(H^T\) being ha-symmetric with respect to \(A^T\). Also, because \(AHA = A \iff A^TH^TA^T = A^T\), we have that \(H\) satisfies \(P1\) with respect to \(A\) if and only if \(H^T\) satisfies \(P1\) with respect to \(A^T\). Furthermore, because \(HAH = H \iff H^TA^TH^T = H^T\), we have that \(H\) satisfies \(P2\) with respect to \(A\) if and only if \(H^T\) satisfies \(P2\) with respect to \(A^T\). Following this observation, we can extend all the results in section §3 to the ha-symmetric case with short proofs. We have a new recipe for constructing a ha-symmetric reflexive generalized inverse in this section, which we refer to as row block construction.
Theorem 14. For $A \in \mathbb{R}^{m \times n}$, let $r := \text{rank}(A)$. For any $S$, an ordered subset of $r$ elements from $\{1, \ldots, m\}$, let $\hat{A}^\top := A[S,:]$ be the $r \times n$ submatrix of $A$ formed by rows $S$. If $\text{rank}(\hat{A}) = r$, let
\[ \hat{H} := (\hat{A}^\top)^+ = \hat{A}(\hat{A}^\top \hat{A})^{-1}. \]
The $n \times m$ matrix $H$ with all columns equal to zero, except columns $S$, which are given by $\hat{H}$, is a ha-symmetric reflexive generalized inverse of $A$.

Proof. Consider $A^\top$. Then $\hat{A} = A^\top [:, S]$ is formed by columns $S$ of $A^\top$. The $m \times n$ matrix $\hat{H}$ constructed by Theorem 7 over $\hat{A}$, is an ah-symmetric reflexive generalized inverse of $A^\top$. Clearly, $H = \hat{H}^\top$, by Lemma 2, we know that $H$ is a ha-symmetric reflexive generalized inverse of $A$.

Next, we demonstrate that when $\text{rank}(A) = 1$ or when $\text{rank}(A) = 2$ with some technical conditions, construction of a 1-norm minimizing ha-symmetric reflexive generalized inverse can be based on the row block construction.

Theorem 15. Let $A$ be an arbitrary $m \times n$, rank-1 matrix. For any $i \in \{1, \ldots, m\}$, let $a$ be row $i$ of $A$. If $i$ is chosen to minimize the 1-norm of $\hat{a}^\top$ among all rows except the zero rows, then the $n \times m$ matrix $H$ constructed by Theorem 14 over $\hat{a}$, is a ha-symmetric reflexive generalized inverse of $A$ with minimum 1-norm.

Theorem 16. Let $A$ be an arbitrary $m \times n$, rank-2 matrix. For all $i_1, i_2 \in M := \{1, \ldots, m\}$, with $i_1 < i_2$, let $\hat{A}^\top := [\hat{a}_{i_1}^\top; \hat{a}_{i_2}^\top]$ be the $2 \times n$ submatrix of $A$ formed by rows $i_1$ and $i_2$. Suppose that $i_1, i_2$ are chosen to minimize the 1-norm of $\hat{H} := (\hat{A}^\top)^+$ among all $2 \times n$ rank-2 submatrices of $A$. Every row $\hat{b}^\top$ of $A$, can be uniquely written in the basis $\hat{a}_{i_1}^\top, \hat{a}_{i_2}^\top$, say $\hat{b}^\top = \alpha \hat{a}_{i_1}^\top + \beta \hat{a}_{i_2}^\top$. Suppose that for each such row $\hat{b}^\top$ of $A$, one of the following conditions holds on the associated $\alpha, \beta$:

\begin{enumerate}
\item $|\alpha| + |\beta| = 0$;
\item $H_{i_1} \hat{H}_{i_2} \leq 0$ for $i = 1, \ldots, n$, and $\alpha \beta \geq 0$;
\item $H_{i_1} \hat{H}_{i_2} \geq 0$ for $i = 1, \ldots, n$, and $\alpha \beta \leq 0$.
\end{enumerate}

Then the $n \times m$ matrix $H$ constructed by Theorem 14 based on $\hat{A}^\top$, is a ha-symmetric reflexive generalized inverse of $A$ with minimum 1-norm.

Furthermore, for general $r := \text{rank}(A)$, we can also efficiently find a ha-symmetric reflexive generalized inverse following our row block construction that is within approximately a constant factor $r(1 + \epsilon)$ of the 1-norm of the ha-symmetric reflexive generalized inverse having minimum 1-norm.

Definition 3. Let $A$ be an arbitrary $m \times n$, rank-$r$ matrix, and let $T$ be an ordered subset of $r$ elements from $\{1, \ldots, n\}$ such that these $r$ columns of $A$ are linearly independent. For $S$ be an ordered subset of $r$ elements from $\{1, \ldots, m\}$, and fixed $\epsilon \geq 0$, if $|\det(A[S,T])|$ cannot be increased by a factor of more than $1 + \epsilon$ by swapping an element of $S$ with one from its complement, then we say that $A[S,T]$ is a $(1 + \epsilon)$-local maximizer for the absolute determinant on the set of $r \times r$ nonsingular submatrices of $A[;,:T]$.

Theorem 17. Let $A$ be an arbitrary $m \times n$, rank-$r$ matrix, and let $T$ be an ordered subset of $r$ elements from $\{1, \ldots, n\}$ such that these $r$ columns of $A$ are linearly independent. Choose $\epsilon \geq 0$, and let $\hat{A} := A[S,T]$ be a $(1 + \epsilon)$-local maximizer for the absolute determinant on the set of $r \times r$ nonsingular submatrices of $A[;,:T]$. Then the $n \times m$ matrix $H$ constructed by Theorem 14 over $\hat{A}^\top := A[S,:]$ is a ha-symmetric reflexive generalized inverse of $A$ satisfying $\|H\|_1 \leq r(1 + \epsilon)\|H^*\|_1$, where $H^*$ is a 1-norm minimizing ha-symmetric reflexive generalized inverse of $A$.

Proof. Consider $A^\top$. Then $(H^*)^\top$ is a 1-norm minimizing ah-symmetric reflexive generalized inverse of $A^\top$, and $\hat{A}^\top = A^\top[T,S]$ is a $(1 + \epsilon)$-local maximizer of $A^\top[T,:,]$. By Theorem 12, the $m \times n$ matrix $\hat{H}$ constructed by Theorem 7 over $\hat{A}$, is an ah-symmetric reflexive generalized inverse
of $A^\top$ satisfying $\|\tilde{H}\|_1 \leq r(1+\epsilon)\|H_{\text{opt}}^r\|_1$. Clearly, $H = \tilde{H}^\top$, and by Lemma 2, we have that $H$ is a ha-symmetric reflexive generalized inverse of $A$. Also, $\|H\|_1 = \|\tilde{H}\|_1 \leq r(1+\epsilon)\|H_{\text{opt}}^r\|_1 = r(1+\epsilon)\|H_{\text{opt}}^r\|_1$. □

Remark 6. In Theorem 17, we could have required the stronger condition that $\tilde{A}$ is a global maximizer for the absolute determinant on the set of $r \times r$ nonsingular submatrices of $A_\tau$. But we prefer our hypothesis, the reasons are the same as Remark 2. And the local search is efficient:

Theorem 18. Let $A$ be rational. We have an FPTAS for calculating a ha-symmetric reflexive generalized inverse $H$ of $A$ that has $\|H\|_1$ within a factor of $r$ of $\|H_{\text{opt}}^r\|_1$, where $H_{\text{opt}}^r$ is a 1-norm minimizing ha-symmetric reflexive generalized inverse of $A$.

5. Conclusions. Of course giving efficient algorithms to improve any of our approximation factors is a nice challenge. Even for special classes of matrices, this could be interesting. Also, it would even be nice to find families of examples to show that the approximation factors of our algorithms are best possible.

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References
[1] Bjerhammar A (1951) Application of calculus of matrices to method of least squares with special reference to geodetic calculations. Trans. Roy. Inst. Tech. Stockholm (49):86 pp. (2 plates).
[2] Dokmanić I, Gribonval R (2017) Beyond Moore-Penrose Part I: generalized inverses that minimize matrix norms, http://arxiv.org/abs/1706.08349.
[3] Dokmanić I, Gribonval R (2017) Beyond Moore-Penrose Part II: the sparse pseudoinverse, https://hal.inria.fr/hal-01547283/file/pseudo-part2.pdf.
[4] Dokmanić I, Kolundžija M, Vetterli M (2013) Beyond Moore-Penrose: sparse pseudoinverse. ICASSP 2013, pp. 6526–6530.
[5] Dresden A (1920) The fourteenth western meeting of the American Mathematical Society. Bull. Amer. Math. Soc. 26(9):385–396.
[6] Fampa M, Lee J (2018) On sparse reflexive generalized inverses. Operations Research Letters 46(6):605–610.
[7] Fuentes V, Fampa M, Lee J (2016) Sparse pseudoinverses via LP and SDP relaxations of Moore-Penrose. CLAIO 2016, 343–350.
[8] Fuentes V, Fampa M, Lee J (2019) Diving for sparse partially-reflexive generalized inverses. Preprint.
[9] Gkountouvas T, Karakasis V, Kourtis K, Goumas G, Koziris N (2013) Improving the performance of the symmetric sparse matrix-vector multiplication in multicore. 2013 IEEE 27th International Symposium on Parallel and Distributed Processing, 273–283, ISSN 1530-2075.
[10] Golub G, Van Loan C (1996) Matrix Computations (3rd Ed.) (Baltimore, MD, USA: Johns Hopkins University Press).
[11] Natarajan BK (1995) Sparse approximate solutions to linear systems. SIAM journal on computing 24(2):227–234.
[12] Penrose R (1955) A generalized inverse for matrices. Proc. Cambridge Philos. Soc. 51:406–413.
[13] Rao C, Mitra S (1971) Generalized Inverse of Matrices and Its Applications. Probability and Statistics Series (Wiley).
[14] Rohde C (1964) Contributions to the theory, computation and application of generalized inverses. Ph.D. thesis, University of North Carolina, Raleigh, N.C., https://www.stat.ncsu.edu/information/library/mimeo.archive/ISMS_1964_392.pdf.
[15] Williamson D, Shmoys D (2011) The Design of Approximation Algorithms (New York, NY, USA: Cambridge University Press), 1st edition.