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TAUTNESS AND APPLICATIONS
OF THE ALEXANDER-SPANIER COHOMOLOGY OF K-TYPES

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Abstract

The aim of the present work is centered around the tautness property for the two K-types of Alexander-Spanier cohomology given by the authors. A version of the continuity property is proved, and some applications are given.

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0 Introduction

It is well-known that in the Alexander-Spanier cohomology theory [17], [18] or in the isomorphic theory of Čech [9], if the coefficient group $G$ is topological then either the theory does not take into account the topology on $G$ [9], [18], or considers only the case when $G$ is compact to obtain a compact cohomology [5], [8]. Continuous cohomology naturally arises when the coefficient group of a cohomology theory is topological [6], [7], [11]. The partially continuous Alexander-Spanier cohomology theory [14] can be considered as a variant of the continuous cohomology of a space with two topologies in the sense of Bott-Haefliger [15]; also it is isomorphic to the continuous cohomology of a simplicial space defined by Brown-Szczarba [6].

The idea of $K$-groups [1], [2] where $K$ is a locally-finite simplicial complex, is used to introduce the $K$-types of Alexander-Spanier cohomology with coefficients in a pair $(G, G')$ of topological abelian groups [3], [4]; namely, $K$-Alexander-Spanier and partially continuous $K$-Alexander-Spanier cohomologies $H^*_K$, $\hat{H}^*_K$. It is proved that these $K$-types satisfied the seven Eilenberg-Steenrod axioms [9]; the excision axiom for the second $K$-type is verified for compact Hausdorff spaces when $(G, G')$ are absolutely retract. Therefore the uniqueness theorem of the cohomology theory on the category of compact polyhedral pairs [9], asserts that our Alexander-Spanier $K$-types over a pair of absolute retract coefficient abelian groups are naturally isomorphic.

The present work is centered around the tautness property for the Alexander-Spanier $K$-types cohomology. Roughly speaking, we prove that the $K$-Alexander-Spanier cohomology of a closed subset in a paracompact space is isomorphic to the direct limit of the $K$-Alexander-Spanier cohomology of its neighborhoods; and that the partially continuous $K$-Alexander-Spanier cohomology of a neighborhood retract closed subspace of a Hausdorff space is isomorphic to the direct limit of the partially continuous $K$-Alexander-Spanier cohomology of its neighborhoods. Also a version of the continuity property is proved. Moreover, we study some application of the $K$-type cohomologies.

1 Alexander-Spanier Cohomology of $K$ Types

Here we mention the notations which will be used throughout the present work [3], [4].

For an object $(X, A)$ of the category $Q$ of the pairs of topological spaces and their continuous maps, denote by $\Omega(X, A)(\hat{\Omega}(X, A))$ the set of the pairs $\bar{\alpha} = (\alpha, \alpha')$, where $\alpha$ is an open covering of $X$ and $\alpha'$ is a subcollection of $\alpha$ covering $A[\alpha' = \alpha \cap A]$; it is directed with respect to the refinement relation $\bar{\alpha} < \bar{\beta}$, i.e. $\alpha < \beta$ and $\alpha' < \beta'$ [9]. Denote by $C^{q(\tau)}(\hat{X})$ the group of the functions $\varphi^\tau : \hat{X}^{q(\tau)+1} \to G$, where $\tau$ is a simplex in $K$, $q(\tau) = q + \dim \tau$, $q \geq 0$, and $\hat{X}$ denotes either a space $X$ or a $\alpha \in \Omega(X)$. Let $C^{q(\tau)}(\hat{X})$ be the subgroup of the direct product $\prod_{\tau \in K} C^{q(\tau)}(\hat{X})$ consisting of such $\varphi = \{\varphi^\tau\}$ for which the condition $(k)$ is satisfied, which states that there is a cofinite subset $\hat{\tau}(\varphi)$ of $K$, i.e. $K - \hat{\tau}(\varphi)$ is finite, such that $(\varphi^\tau)^{-1}(G') = \hat{X}^{q(\tau)+1}$,
∀τ ∈ τ(φ). The coboundary δq : Cq(X) → Cq+1(X) is

\[(δ^qφ)^\tau = \sum_{i=1}^{q(\tau)+1} (-1)^i \varphi^\tau p_i^{(q(\tau)+1)} + (-1)^q(\tau)+1 \sum_{\sigma \in st(\tau)} |\sigma : \tau| \varphi^\sigma,\]

where st(τ) = {σ ∈ K : τ is (dimσ − 1)-face of σ}, p_i^{(τ)} : X^{\tau+1} → X^\tau is the projection defined by: if \(i_\tau\) is the \(τ\)-tuple consisting of \(t = (x_0, \ldots, x_\tau) \in X^{\tau+1}\) with \(x_i\) omitted, then \(p_i^{(τ)}(t) = i_\tau\), \(0 ≤ i ≤ \tau\). The cohomology groups of the cochain complex \(C^*(X) = \{C^q(X), \delta^q\}\) is, in general, uninteresting, as shown in the following theorem [3].

**Theorem 1.1.** If \(dim K = 0\), then \(H^q(C^f(X)) \cong G^*\) (the subgroup of \(G^K = \prod_{τ \in K} G^τ\), \(G^* = G\), consisting of those elements having all but a finite number of their \(τ\)-coordinates in \(G^*\)), and \(H^q(C^f(X)) = 0\), when \(q ≠ 0\).

To pass to more interesting cohomology groups, the topology of the space \(X\) will be used to define that \(φ \in C^q(X)\) is said to be \(K\)-locally zero on \(M\) if there is \(α \in \Omega(X)\) (the set of external covering of \(M\) by open subsets of \(X\)) such that \(φ\) vanishes on \(α \cap M\), i.e. each \(φ^\tau\) vanishes on \((α \cap M)^q(\tau)+1\), where \(α^\tau = \bigcup \{u_\alpha^\tau : u_\alpha \in α\}\). The subgroups of \(C^q(X)\) consisting of those elements which are \(K\)-locally zero on \(X, A\) respectively are denoted by \(C^q_0(X, A)\), \(C^q(X, A)\). The \(K\)-Alexander-Spanier cohomology of \((X, A)\) over \((G, G')\), denoted by \(H^*_{K}(X, A)\), is the cohomology of the quotient cochain complex \(C^{\tau,\alpha,q}(X, A) = C^q(X, A)/C^q_0(X)\).

If \(f : (X, A) → (Y, B)\) is in \(Q\), \(β \in Ω(Y, B)\) and \(α = f^{-1}(β)\), then \(f\) defines a cochain map \(f^\#: C^\#_{K}(Y, B) → C^\#_{K}(X, A)\), where \(τ(f^\#φ) = τ(φ)\) for each \(φ \in C^\#(Y)\). In turn, \(f^\#\) induces the homomorphism \(f^* : H^*_{K}(Y, B) → H^*_{K}(X, A)\).

On the other hand, for \(α \in Ω(X, A)\), denote by \(C^q_α\). The subgroup of \(C^q_α = C^q(α)\) consisting of those \(φ\) which vanishes on \(α' \cap A\). Then we obtain a direct system \(\{C^q_0\}_{Ω(X, A)}\) such that any map \(f ∈ Q\) constitutes a map \(F : \{C^q_0\}_{Ω(Y, B)} → \{C^q_0\}_{Ω(X, A)}\) [9]; its limit is \(F^{∞}\).

**Theorem 1.2.** The \(K\)-Alexander-Spanier cohomology functor \(\{H^*_{K}, f^*\}\) is naturally isomorphic to the functor \(\{lim\{H^q(C^\#_{K})\}_{Ω(X, A)}, F^{∞}\}\) [4].

In the previous part, the topology on \((G, G)\) plays no role; to pass to the second cohomology of \(K\)-type we characterize an element \(φ \in C^q(X)\) to be \(K\)-partially continuous if it is continuous on some \(α \in Ω(X)\), i.e. \(φ^\tau|α^q(\tau)+1\) are continuous functions. Let \(L^q(X)\) be the group of all such elements, and \(M^q_{K}(X) = L^q(X)/C^q_{0}(X)\). The subgroup of \(C^q_{α}\), where \(α \in Ω(X)\), consisting of the \(K\)-continuous elements \(φ\), i.e. \(φ^\tau\) are continuous, is denoted by \(M^q_{α}\). Let \(i : A → X\), define \(M^q_{K}(X, A)\) to be the mapping cone of \(i^\#: M^q_{K}(X) → M^q_{K}(A)\), [13],[18], assuming that \(M^q_{K}(X, A) = M^q_{K}(X) ⊕ M^{q-1}_{K}(A)\), and the coboundary is \(Δ^q(φ, ψ) = (−δ^qφ, i^qφ + δ^{q-1}ψ)\). The cohomology of \(M^q_{K}(X, A)\) is the partially continuous \(K\)-Alexander-Spanier cohomology of \((X, A)\) over the topological pair \((G, G')\) of coefficient groups; it is denoted by \(H^q_{K}(X, A)\).

On the other hand, if \(α ∈ Ω(X, A)\), then \(i\) defines a cochain map \(i^\#: M^q_{α} → M^q_{α}\); its mapping cone is denoted by \(M^q_{α}\).
Theorem 1.3. For a pair \((X, A) \in Q\) with \(A\) is closed, \(M^q_K(X, A)\) is naturally isomorphic to \(\lim_{\to} \{M^q_k\}_{\overline{\Omega}(X, A)}\) [4].

Theorem 1.4. For a discrete space, and \(q \geq 0\), \(\check{H}_K^q(X) \approx \check{H}_K^q(X)\).

**Proof.** Since \(X^{q+1}\) admits a discrete topology, it follows that each \(\tau\)-coordinate \(\varphi^\tau\) of \(\varphi \in C_K^q(X)\) is continuous [16]. Then \(\varphi\) is \(K\)-partially continuous with respect to any \(\alpha \in \Omega(X)\). Therefore \(L^q(X) = C_K^q(X)\) and \(M^q_K(X) = C_K^q(X)\).

2 Tautness and Continuity Properties

This article is devoted to study the tautness property for both Alexander-Spanier cohomology of \(K\)-types. One of its applications is the continuity property.

The star of a subset \(A\) in a space \(X\) with respect to \(\alpha \in \Omega(X)\) is

\[
\text{st}(A, \alpha) = \bigcup \{U_\alpha \in \alpha : U_\alpha \cap A \neq \emptyset\}
\]

The star of \(\alpha\) is

\[
\alpha^* = \{\text{st}(U_\alpha, \alpha) : u_\alpha \in \alpha\}
\]

**Definition 2.1** Let \(\alpha, \beta \in \Omega(X)\), then \(\beta\) is a star-refinement of \(\alpha\), written \(\alpha <^* \beta\), if \(\alpha < \beta^*\).

Denote by \(\mathcal{N}(A)\) the collections of neighborhoods \(\{N\}\) of \(A\) in \(X\); it is directed downward by inclusion. If \(N_1 < N_2\), then the inclusion \(\pi_{N_1N_2} : N_2 \hookrightarrow N_1\) induces the homomorphisms \(\pi^*_{N_1N_2} : H^q_K(N_1) \to H^q_K(N_2)\). Also \(i_N : A \hookrightarrow N\) induces \(i_N^* : H^q_K(N) \to H^q_K(A)\), and they define a homomorphism

\[
I^\infty : \lim_{\to} \{H^q_K(N), \pi^*_{N_1N_2}\}_{\mathcal{N}(A)} \to H^q_K(A).
\]

**Theorem 2.1** (Tautness). A closed subspace of a paracompact space is a taut subspace relative to the \(K\)-Alexander-Spanier cohomology, i.e. \(I^\infty\) is an isomorphism for each \(q\) and any pair \((G, G')\) of coefficient groups.

**Proof.** (1) \(I^\infty\) is an epimorphism. Actually let \(h \in \check{H}_K^q(A)\) with representative \(\varphi \in C_K^q(A)\), written \(h = [\varphi]\). Let \(\varphi \in C^q(A)\) such that \(\varphi \in \tilde{\varphi}\). Then there is \(\alpha = \{u_\alpha = \nu_\alpha \cap A : \nu_\alpha \subseteq X\text{ is open}\} \in \Omega(A)\) such that

\[
(\delta^q \varphi)^\tau \alpha^{q+2} = 0 \tag{2.1}
\]

Since \(A\) is closed, it follows that \(\beta = \{\nu_\alpha\} \cup \{X - A\} \in \Omega(X)\). The paracompactness of \(X\) is equivalent to the existence of such \(\gamma \in \Omega(X)\) that \(\beta <^* \gamma\) [21], and a neighborhood \(N\) of \(A\) and an extension \(f : N \to A\) (not necessarily continuous) of the identity map \(\text{id}_A\) of \(A\), i.e. \(f|_N = \text{id}_A\), such that \(f(u_\gamma \cap N) \subseteq \text{st}(u_\gamma, \gamma)\) for each \(u_\gamma \in \gamma\) [18]. One can show that \(f\) defines a cochain map \(f^\#: C^\#(A) \to C^\#(N)\) by \((f^\# \varphi)^\tau = \varphi^\tau f^{q(q^\tau + 1)}\) with \(\tau(f^q \varphi) = \tau(\varphi)\), where
Let $f^r : N^r \to A^r$ given by $f(x_0, \ldots, x_{r-1}) = (f(x_0), \ldots, f(x_{r-1}))$. The relation $\beta < \gamma^r$ yields that for each $u_\gamma \in \gamma$ there is $u_\beta \in \beta$ such that $f(u_\gamma \cap N) \subseteq \text{st}(u_\gamma, \gamma) \subseteq u_\beta$. Because $f(N) = A$, then $f(u_\gamma \cap N) \subseteq u_\beta \cap A \subseteq u_\alpha$ for some $u_\alpha \in \alpha$. By using (2.1), we get $(\delta^q f^q \varphi)^r |(\gamma \cap N)^q r+1 = 0$, i.e. $\delta^q (f^q \varphi) \in C^q_{r+1}(N)$. Then $f^q \varphi$ represents a cocycle $\bar{f}^q \varphi \in C^q_{K}(N)$ which, in turn, defines $h_N \in H^q_K(N)$, i.e. $h_N = [\bar{f}^q \varphi]$. Let $t \in A^q r+1$, then

$$(i^q_N (f^q \varphi))^r (t) = \varphi^r f^q (r+1) i^q_N (f^q (r+1)) (t) = \varphi^r (t),$$

and therefore $i^N_N h_N = [(f|N)^q \varphi] = [\varphi] = h$.

(2) $I^\infty$ is a monomorphism. Actually, let $h_1 \in H^q_K(N_1), \varphi_1 \in C^q_{K}(N_1)$ and $\varphi_1 \in C^q(N_1)$ such that $\varphi_1 \in \varphi_1, \varphi_1 \in h_1$, and $[h_1] \in \text{Ker} I^\infty$.

First, one can consider that the neighborhood $N_1$ of $A$ is a paracompact subset of $X$. For, if $N_1$ is not so, then there is a paracompact subset $M_1$ of $X$ such that $M_1 < N_1$ (e.g., take $M_1 = X$) [10]. The inclusion $\pi_{M_1 N_1}$ induces an epimorphism $\pi_{M_1 N_1}^q [3]$, let $\pi_{M_1 N_1}^q \tilde{\psi}_1 = \varphi_1$. Thus the cohomology class of $H^q_K(M_1)$ represented by $\tilde{\psi}_1$ is $[h_1]$, which shows that $N_1$ can be taken paracompact.

Now, $\varphi_1 \in \text{Ker} \delta^q$, or equivalently, there is $\alpha = \{u_\alpha = \nu_\alpha \cap N_1 : \nu_\alpha \subseteq X \text{ is open} \} \in \Omega(N_1)$ such that

$$(\delta^q \varphi_1)^r |\alpha^{q r+2} = 0 . \tag{2.2}$$

On the other hand, the assumption $i^q_{N_1} h_1 = 0$ asserts that there exists $\varphi \in C^q_{K-1}(A)$ such that $i^q_{N_1} \varphi_1 - \delta^q - 1 \varphi \in C^q_0(A)$, where $\varphi \in \varphi$. This means that there is such $\beta = \{u_\beta = \omega_\beta \cap A : \omega_\beta \subseteq X \text{ is open} \} \in \Omega(A)$ that

$$(i^q_{N_1} \varphi_1)^r = (\delta^q - 1 \varphi)^r \text{ on } \beta^{q r+1} \tag{2.3}$$

Assume that $\beta_1 = \{u_\beta = \omega_\beta \cap N_1 \} \cup \{N_1 \cap A \}$. The paracompactness of $N_1$ asserts the existence of $\gamma_1, \gamma_2 \in \Omega(N_1)$ for which $\alpha <^r \gamma_1$ and $\beta_1 <^r \gamma_2$. The directedness of $\Omega(N_1)$ implies that there $\gamma \in \Omega(N_1)$ for which $\gamma_1, \gamma_2 < \gamma$; and so for each $u_\gamma \in \gamma$ there are $u_\gamma \in \gamma_i, i = 1, 2$ and $u_\alpha \in \alpha$, $u_\beta_1 \in \beta_1$ such that

$$u_\gamma \subseteq u_\gamma \subseteq \text{st}(u_\gamma, \gamma_i) \subseteq u_\alpha \cap u_\beta_1 , \tag{2.4}$$

Then

$$\text{st}(u_\gamma, \gamma) \subseteq u_\alpha \cap u_\beta_1$$

i.e. $\alpha, \beta_1 <^r \gamma$. According to Lemma 6.6.1 in [18], there is a neighborhood $N_2$ of $N_1$ and $f : N_2 \to A$ (not necessarily continuous) such that $fi_{N_2} = \text{id}_A$, and $u_\beta_1 \in \beta_1$ such that

$$f(u_\gamma \cap N_2) \subseteq \text{st}(u_\gamma, \gamma) \subseteq u_\beta_1 \subseteq u_\beta_1 \cap A = u_\beta$$

Then, by (2.3), we get

$$(\delta^q - 1 f^q - 1 \varphi)^r = (f^q i^q_{N_1} \varphi_1) \text{ on } (\gamma \cap N_2)^{q r+1} \tag{2.5}$$
Define \( D^q : C^{q+1}(N_1) \to C^q(N_2) \) by:

\[
\text{if } t = (x_0, \ldots, x_q(\tau)) \in N_2^{q(\tau)+1} \text{ and } \psi_1 \in C^{q+1}(N_1) \text{ then }
\]

\[
(D^q\psi_1)^T(t) = \sum_{r=0}^{q(\tau)} (-1)^r \psi_1^T(y_0, \ldots, y_r, z_r, \ldots, z_q(\tau)),
\]

where

\[
y_j = \pi_{N_1N_2}(x_j), \quad z_j = (i_{N_1}f)(x_j) = f(x_j),
\]

and \( \tau(D^q\psi_1) = \tau(\psi_1) \). A similar calculation as given in [4], we get

\[
(\delta^q D^q - \varphi_1)^T = (f^q i_{N_1} \varphi_1)^T - (\pi_{N_1N_2}^q \varphi_1)^T - (D^q \delta^q \varphi_1)^T
\]

(2.7)

By (2.4), (2.5) for each \( u_\gamma \in \gamma \), there is \( u_\alpha \in \alpha \) such that

\[
(w_\gamma \cap N_2) \cup f(u_\gamma \cap N_2) \subseteq u_\alpha
\]

Then, by (2.7), (2.2), (2.6) consequently, we have

\[
(\delta^q - D^q - \varphi_1)^T = (f^q i_{N_1} \varphi_1)^T - (\pi_{N_1N_2}^q \varphi_1)^T \text{ on } (\gamma \cap N_2)^{q(\tau)+1},
\]

and so

\[
(\pi_{N_1N_2}^q \varphi_1)^T = (\delta^q (f^q - D^q - \varphi_1))^T \text{ on } (\gamma \cap N_2)^{q(\tau)+1}.
\]

Therefore

\[
\psi_2 = f^q - D^q - \varphi_1 \in C^{q-1}(N_2) \text{ such that }
\]

\[
(\pi_{N_1N_2}^q \varphi_1)^T = (\delta^q \psi_2)^T \text{ on } (\gamma \cap N_2)^{q(\tau)+1},
\]

i.e. \( \pi_{N_1N_2}h_1 = 0 \) which completes the proof.

**Corollary 2.2.** Any one-point subset of a paracompact is a taut subspace relative to \( \bar{H}^*_{K_0} \).

The next part is devoted to study the tautness property for \( \bar{H}^*_{K_0} \), which is also valid for \( \bar{H}^*_{K} \).

The idea and results of \( \alpha - \beta \)-contiguous maps, introduced in [4] plays an essential role in this study.

The inclusions \( \pi_{N_1N_2} : N_2 \hookrightarrow N_1 \) corresponding to the relations \( N_1 < N_2 \) in \( \mathcal{N}(A) \), define the direct system \( \{ \bar{H}^q_{K_0}(N), \pi_{N_1N_2}^* \} \). Also the inclusion \( i_N : A \hookrightarrow N \), where \( N \in \mathcal{N}(A) \), define a map of direct systems [9]:

\[
I_N : \{ \bar{H}^q(M^\alpha_{\bar{\alpha}}); \pi^*_{\alpha,\bar{\alpha}} \}_{\Omega(N)} \longrightarrow \{ \bar{H}^q(M^\alpha_{\bar{\alpha}}); \pi^*_{\alpha,\bar{\alpha}} \}_{\Omega(A)}
\]

where \( \alpha \in \Omega(N), \bar{\alpha} = i_N^{-1}(\alpha) = \alpha \cap A \). On the other hand, \( \{ i_N^* \} \) define a homomorphism

\[
\bar{I}^\infty : \varprojlim \{ \bar{H}^q_{K_0}(N), \pi_{N_1N_2}^* \}_{\Omega(A)} \to \bar{H}^q_{K_0}(A)
\]
Theorem 2.3 (Tautness). If \( A \) is a closed subset in a Hausdorff space \( X \) such that \( A \) is a neighborhood retract, then \( A \) is a taut subspace relative to the cohomology \( H^*_K \).

Proof. 1) \( \tilde{I}^\infty \) is an epimorphism. Actually, let \( h \in \tilde{H}^*_K(A) \). Without loss of generality, the neighborhood retractness of \( A \) in \( X \) yields that \( A \) has an open neighborhood \( U \) (in \( X \)) such that \( U \subseteq N \) and a retraction \( \tau_1 : U \to A \) (If \( U_1 \) is an open neighborhood of \( A \) of which \( A \) is retract but \( U_1 \nsubseteq N \), take \( U = U_1 \cap \text{Int}(N) \). Let \( \iota_U : A \hookrightarrow U \) then, \( \tilde{I}^\infty(\tilde{\tau}_1^*(h)) = \tilde{I}^\infty_\alpha(\tilde{\tau}_1^*(h)) = \tilde{id}_A^*(h) = h \).

2) \( \tilde{I}^\infty \) is a monomorphism. Let \( [h] \in \ker \tilde{I}^\infty \), it is sufficient to construct \( V \in \mathcal{N}(A) \) satisfying \( N < V \) and \( \tilde{\pi}^*_N h = 0 \).

Since the cohomology functor commutes with the direct limit \([18]\), Theorem 1.3 asserts that one may assume that \( h \) belongs to \( \lim \{H^q(M^\alpha_\lambda), \tilde{\pi}^*_\alpha \beta \} \Omega(N) \), with representative \( h_\alpha \in H^q(M^\alpha_\lambda) \), where

\[
\alpha = \{ u_\alpha = \omega_\alpha \cap N : \omega_\alpha \subseteq X \text{ is open} \} \in \Omega(N)
\]

Let \( \alpha_1 = \{ \omega_\alpha \} \cup \{ X - A \} \), \( \tilde{\alpha} = \alpha_1 \cap A \),

\[
\beta = \{ u_\beta = \tau_1^{-1}(u_\tilde{\alpha}) \cap (u_\alpha \cap U) : \phi \neq u_\tilde{\alpha} \in \tilde{\alpha} \},
\]

\( V = \cup u_\beta \), \( \tau = \tau_1 |V \) : \( V \hookrightarrow A \), and \( \alpha' = \alpha_1 \cap V \). Then \( \tilde{\alpha} \in \Omega(A), \alpha' = \alpha \cap V \in \Omega(V), u_\tilde{\alpha} \subseteq u_\beta \) for each \( u_\tilde{\alpha} \neq \phi, \beta \) is a family of open subsets in \( U \) and so open in \( V \), \( V \) is an open neighborhood of \( A \) such that \( V \subseteq U \), and \( \beta \in \Omega(V) \). Since \( u_\beta = u_\tilde{\alpha} \cap u_\alpha \subseteq V \cap u_\alpha = u_{\tilde{\alpha}'}, \) it follows that \( \alpha' < \beta \). Also \( \alpha' \cap A = \alpha \cap A = \tilde{\alpha} \) and \( j^{-1} \beta = \tilde{\alpha} \), where \( j : A \hookrightarrow V \). If \( \ell : V \hookrightarrow N, \) and \( [\varphi] \in H^q(M^\alpha_\lambda) \), then

\[
\tilde{\gamma}^{\alpha}_\beta \tilde{\pi}^*_\alpha \beta \tilde{\epsilon}^\alpha_\beta [\varphi] = \tilde{\gamma}^{\alpha}_\beta (\{(\varphi^\alpha' \gamma(q + 1) + \beta \gamma(q + 1))\})
\]

\[
= \{ (\varphi^\alpha' | \tilde{\alpha} \gamma(q + 1)) \},
\]

i.e.

\[
\tilde{\gamma}^{\alpha}_\beta \tilde{\pi}^*_\alpha \beta \tilde{\epsilon}^\alpha_\beta = \tilde{\gamma}^{\alpha}_N \alpha
\]

where \( \tilde{\gamma}^{\alpha}_N : M^\alpha_\lambda \to M^\alpha_\beta \) is induced by \( \iota_N : A \hookrightarrow N \).

On the other hand, \( (j \tau) u_\beta \subseteq u_\beta \) and so \( j \tau, \iota \delta \ell : V \to V \) are \( \beta - \beta \) contiguous \([4]\).

It follows that \( (\tilde{id}_V)^\alpha_\beta, (\tilde{j} \tau)^\alpha_\beta, \tilde{M}^\alpha_\beta \to \tilde{M}^\alpha_\beta \) are cochain homotopic \([4]\). Then \( (\tilde{id}_V)^\alpha_\beta = (\tilde{j} \tau)^\alpha_\beta \), \( \tilde{\epsilon}^\alpha_\beta = \tilde{\gamma}^{\alpha}_\beta \), which yields that \( \tilde{j} \beta \) is a monomorphism. Because \( \tilde{\gamma}^{\alpha}_N \alpha h_\alpha = 0 \), \( (2.8) \) yields that \( \tilde{\pi}^*_\alpha \beta \tilde{\epsilon}^\alpha_\beta h_\alpha = 0 \). Since \( \tilde{\epsilon}^\alpha_\beta h_\alpha, \tilde{\pi}^*_\alpha \beta (\tilde{\epsilon}^\alpha_\beta h_\alpha) \) represent the zero element of \( \lim \{H^q(M^\alpha_\lambda), \tilde{\pi}^*_\alpha \beta \} \Omega(N) \), it follows that \( \tilde{\pi}^*_N h = [\tilde{\epsilon}^\alpha_\beta h_\alpha] = 0 \).

The rest of this article is centered around a special case of the continuity property for \( \tilde{H}^*_K \). As an application of the continuity property the cohomology groups satisfy a much stronger form of the excision axiom.

The following results can be deduced from those given in \([9]\).
Lemma 2.4. Let $X$ be the intersection of a nested system \(\{X_{\alpha}, \pi_{\beta}\}_{\Lambda}\), then (i) $X$ and \(\lim\{X_{\alpha}, \pi_{\beta}\}_{\Lambda}\) are homeomorphic

(ii) If the nested system consists of compact Hausdorff spaces then $X$ is a closed subset of each $X_{\alpha}$.

(iii) If $N$ is an open neighborhood of $X$ in $X_{\alpha}$ (for some $\alpha \in \Lambda$), then there is $\beta > \alpha$ in $\Lambda$ such that $X_{\beta} \subseteq N$.

The inclusions $i_{\alpha}: X \hookrightarrow X_{\alpha}$ define a map

\[ I: \{H^{q}_{K}(X_{\alpha}), \pi^{\ast}_{\alpha\beta}\}_{\Lambda} \to H^{q}_{K}(X) , \]

its direct limit is denoted by $I^\infty$.

Theorem 2.5 (Weak continuity). If $X$ is the intersection of a nested system \(\{X_{\alpha}, \pi_{\beta}\}_{\Lambda}\) of compact Hausdorff spaces, then $I^\infty$ is an isomorphism.

Proof. Since each $X_{\alpha}$ is a paracompact Hausdorff space \cite{10} and $X_{\alpha}$ is closed in $X$ (Lemma 2.4), it follows, by Theorem 2.1, that $X$ is a taut subspace in $X_{\alpha}$ relative to $H^{q}_{K}$.

(1) $I^\infty$ is an epimorphism. Let $h \in H^{q}_{K}(X)$, then, according to Theorem 2.1, there exists an open neighborhood $N$ of $X$ in $X_{\alpha}$ and $h_{N} \in H^{q}_{K}(N)$, such that $i^{\ast}_{N}(h_{N}) = h$. By Lemma 2.4, there is $\beta > \alpha$ in $\Lambda$ such that $X_{\beta} \subseteq N$. Let $i_{\beta}: X \hookrightarrow X_{\beta}$, $j_{\beta}: X_{\beta} \hookrightarrow N$. Because $i^{\ast}_{\beta}(j^{\ast}_{\beta}h_{N}) = (j^{\ast}_{\beta}i^{\ast}_{\beta})^{\ast}h_{N} = \tilde{i}^{\ast}_{N}h_{N} = h$, then $I^\infty[j^{\ast}_{\beta}h_{N}] = h$.

(2) $I^\infty$ is a monomorphism. Let $[h_{\alpha}] \in \operatorname{Ker}I^\infty$, i.e. $i^{\ast}_{\alpha}h_{\alpha} = 0$. The tautness of $X$ in $X_{\alpha}$ yields, by Theorem 2.1, an open neighborhood $N$ of $X$ in $X_{\alpha}$ such that $h_{N}$ is the unique element for which $i^{\ast}_{N}h_{N} = 0$, where $i^{\ast}_{N}: X \hookrightarrow N$. Because $i^{\ast}_{N}(i^{\ast}_{N}h_{\alpha}) = i^{\ast}_{\alpha}h_{\alpha} = 0$, then $\tilde{i}^{\ast}_{N}h_{\alpha} = 0$. Let $\beta > \alpha$ in $\Lambda$ such that $X_{\beta} \subseteq N$, then $\pi^{\ast}_{\alpha\beta}h_{\alpha} = (\tilde{i}^{\ast}_{N}\tilde{i}^{\ast}_{\beta})^{\ast}h_{\alpha} = \tilde{j}^{\ast}_{\beta}(\tilde{i}^{\ast}_{N}h_{\alpha}) = 0$, i.e. $[h_{\alpha}] = 0$.

3 Applications

One of the good applications of the Alexander-Spanier cohomology of $K$-types is the study of the 0-dimensional cohomology groups and their relation with the connectedness of the space \cite{4}. In this article two applications are given. In a next work we hope to give more applications. The first application is concentrated to define the partially continuous $K$-Alexander-Spanier cohomology of an excision map and calculate its value for some dimensions.

Let $\tilde{f}^{\#}: M^{q}_{K}(Y, B) \to M^{q}_{K}(X, A)$ be the cochain map induced by the map $f$ in $Q$. Define $M^{q}_{K}(f)$ to be the mapping cone of $\tilde{f}^{\#}$ by:

\[ M^{q}_{K}(f) = M^{q}_{K}(Y, B) \oplus M^{q-1}_{K}(X, A) , \]

\[ = M^{q}_{K}(Y) \oplus M^{q-1}_{K}(B) \oplus M^{q-1}_{K}(X) \oplus M^{q-2}_{K}(A) , \]

and the coboundary is

\[ \Delta^{q}(\varphi_{2}, \psi_{2}, \varphi_{1}, \psi_{1}) = \]
Then there is a short exact sequence

\[ 0 \to M^p(X, A) \xrightarrow{\Delta} M^p(Y, B) \to \Omega \]

where \( \Delta \) is injection, projection respectively; \( M^p(X, A) \) is the complex with the dimensions all raised by one, and \( M^p(Y, B) \) is the complex with the sign of the coboundary changed [12]. Note that \( H^q(M^p(Y, B)) = \tilde{H}^q(Y, B) \). Let \( V \) be an open subset of \( X \) such that \( V \subseteq \text{Int}A, B = X - V, \) and \( C = A - V \). Put the excision map \( e : (B, C) \hookrightarrow (X, A) \) in (3.1) instead of \( f \), and then apply the cohomology functor, we get the long exact sequence:

\[ \cdots \to \tilde{H}^q(e) \xrightarrow{\Delta} \tilde{H}^q(X, A) \xrightarrow{e} \tilde{H}^q(B, C) \]

(3.2)

Thus the groups \( \tilde{H}^q(e), \tilde{H}^{q+1}(e) \) measure how much the cohomological groups deviate from the excision axiom.

**Theorem 3.1.** If \( \dim K = 0 \), \( e : (B, C) \hookrightarrow (X, A) \) is an excision map, where \( A \) is closed and \( (G, G') \) any pair of topological abelian groups, then \( \tilde{H}^q(e) = 0 \) when \( q = 0 \) or \( q = 1 \).

**Proof.** (1) Case \( q = 0 \). We have

\[ M^0_K(e) = M^0_K(X, A) = M^0_K(X) = L^0_K(X) \]

Let \( \varphi \in M^0_K(e) \) such that \( \tilde{\Delta} \varphi = 0 \), then \( i^0 \varphi = 0, \tilde{e} \varphi = 0 \). Then \( \varphi = 0 \) [4], which means that \( \text{Ker} \Delta^0 = 0 \).

(2) Case \( q = 1 \). We have

\[ M^1_K(e) = M^1_K(X) \oplus L^0(A) \oplus L^0(B) \]

It is sufficient to show that \( \text{Ker} \tilde{\Delta}^1 \subseteq \text{Im} \Delta^0 \). Let \( (\varphi_2, \varphi_2, \varphi_1, 0) \in \text{Ker} \tilde{\Delta}^1 \), then

\[ \delta^1 \varphi = 0, \quad \tilde{i} \varphi = -\delta^0 \varphi \]

and

\[ \tilde{e}^1 \varphi = \delta^0 \varphi_1 \]

(3.3)

\[ \tilde{e}^1_1 (-\varphi_2) = \tilde{j} \varphi \]

(3.4)

where \( i : A \hookrightarrow X, j : C \hookrightarrow B \) and \( e_1 = e|C \).

By (3.4), there exists [4], \( \varphi \in M^0_K(X) = L^0(X) \) such that

\[ \tilde{i}^0 \varphi = -\varphi_2, \quad \tilde{e}^0 \varphi = \varphi_1 \]

(3.5)
By (3.3)-(3.5), we get
\[ i^1(\delta^0 \varphi - \varphi_2) = 0, \quad \bar{e}^1(\delta^0 \varphi - \varphi_2) = 0 \] (3.6)

Then \( \delta^0 \varphi = \varphi_2 \) [4], which with (3.6) yield that \( (\varphi, 0, 0, 0) \in M^0_K(c) \) such that \( \bar{\Delta}^0(\varphi, 0, 0, 0) = (\varphi_2, \varphi_2, \varphi_1, 0) \).

Combining the sequence (3.2) and the above theorem, we get the following result.

**Corollary 3.2** Under the assumptions of Theorem (3.1), the map \( \bar{e}^{*0} : \bar{H}^0_K(X, A) \to \bar{H}^0_K(B, C) \) is an isomorphism but \( \bar{e}^{*1} \) is a monomorphism:

The second application is to give attention in our work to use a pair of coefficients groups, an arbitrary locally-finite simplicial complex \( K \), and the condition \( (k) \).

Let \( \eta : (G, G') \to (F, F') \) be a homeomorphism of pairs of (discrete) abelian groups which is an epimorphism, \( (L, L') = \text{Ker} \eta \) and \( \lambda : (L, L') \to (G, G') \). Then for each \( \bar{a} \in \Omega(X, A) \), the maps \( \eta, \lambda \) define, naturally a short exact sequence

\[ 0 \to C^q(\bar{a}, L, L') \to C^q(\bar{a}; G, G') \to C^q(\bar{a}; F, F') \to 0 ; \]

its cohomology is a long exact sequence [12] denoted by \( S_\bar{a} \). One can show that \( \{ S_\bar{a} \}_{\Omega(X, A)} \) is a direct system, its direct limit \([3], [4]\)

\[ \cdots \to \bar{H}^{q-1}_K(X, A; F, F') \to \bar{H}^q_K(X, A', L, L') \to \]

\[ \bar{H}^q_K(X, A; G, G') \to \bar{H}^1_K(X, A, F, F') \to \bar{H}^{q+1}_K(X, A; L, L') \to \cdots \]

Now instead of \( F \) take the factor group \( G/G' \) and so instead of \( F' \) will be the null subgroup of \( G/G' \). Then the above sequence yields the following result.

**Theorem 3.3** Consider that \( (X, A) \) has a trivial \((q - 1)\)-dimensional \( K \)-Alexander-Spanier cohomology group with finite cochains, and a trivial \((q + 1)\)-dimensional \( K \)-Alexander-Spanier cohomology with infinite cochains, taken over the coefficient groups \( G/G' \) and \( G' \) respectively. Then the group \( \bar{H}^q_K(X, A; G, G') \) defined over an arbitrary pair \( (G, G') \) of coefficient groups is the extension of the cohomology group \( \bar{H}^q_K(X, A; G') \) with infinite cochains over \( G' \) by the group \( \bar{H}^q_K(X, A, G/G') \) with finite cochains over \( G/G' \).

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