New maverick coset theories

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Abstract
We present new examples of maverick coset conformal field theories. They are closely related to conformal embeddings and exceptional modular invariants.

Introduction
The coset construction [1,2] provides a powerful tool to construct explicit examples of conformal field theories. It has found numerous applications: in superstring theory coset conformal field theories with extended supersymmetry have been used to construct string compactifications [3,4,5], and recently the use of the coset construction has been proposed to relate different conformal field theory descriptions of the fractional Quantum Hall Effect ( [6,7], and references therein, see also [8]).

The structure of coset theories turns out to be amazingly rich. It is by now well understood that in the correct treatment of coset conformal field theories group theoretical selection rules lead to field identifications, and that field identification fixed points require a special procedure called fixed point resolution [9,10].
The phenomenon of field identification can be traced back to the following fact: group theoretical selection rules for the embedding

\[(g')_{Ik} \hookrightarrow (g_k)\]  

(1)
typically imply that certain branching functions vanish and that non-vanishing branching functions coincide. (Here subscripts denote the level, \(k\) is an integer and \(I\) is the Dynkin index of the embedding.) In particular, there are different realizations of states of conformal weight zero in the coset theory. All these states should be identified to provide a theory with a unique vacuum. It turns out that this can be done using simple current techniques \([9]\).

It came as a surprise that group theoretical selection rules are not the only reason for the existence of additional copies of the vacuum. First examples of this phenomenon were exhibited in \([11]\) and \([12]\). Following these references, we will call maverick coset theory any coset conformal field theory which has a non-vanishing branching function of conformal weight \(\Delta = 0\) that is not a consequence of group theoretical selection rules. An ADE-classification for mavericks was proposed in \([12]\); however, one more maverick coset theory was found in \([10]\). The known mavericks share the following features (see more below): they appear at level \(k = 2\), and for level 1 the embedding is a conformal embedding \([13, 14]\). In all cases the coset theory can be cured by use of exceptional modular invariants.

The purpose of this note is twofold. We first exhibit a different class of maverick coset conformal field theories which have rather different features, but are still closely related to exceptional modular invariants. Secondly, we present a new maverick coset at level \(k = 2\) and make some general remarks on this type of maverick coset theories.

**New maverick coset theories at level \(k = 1\)**

We start with the embedding

\[(A_1)_{10} \hookrightarrow (A_3)_1,\]  

(2)
induced by the four-dimensional representation of \(A_1\). The corresponding coset conformal field theory has the identification current \((J^2/J')\) of order two and central charge \(c = 1/2\). Hence, if there is a conformal field theory corresponding to this coset theory, it can only be the critical Ising model. Applying the standard procedure of field identification \([9, 10]\) to the theory in question one obtains 11 primary fields; the computation \(\Delta\) of the conformal weights \(\Delta\) for the branching functions \(\Phi_{\lambda}^{\lambda'}\) (where \(\lambda\) is an integrable highest weight of \((g)_k = (A_3)_1\) and \(\lambda'\) is an integrable highest weight of the subalgebra \((g')_{Ik} = (A_1)_{10}\) gives the following values for \(\Delta\) mod \(\mathbb{Z}\):

\[
\begin{array}{ll}
\Phi_0^{000} \cong \Phi_{10}^{010} \quad & \Delta = 0 \\
\Phi_2^{000} \cong \Phi_8^{010} \quad & \Delta = 5/6 \\
\Phi_4^{000} \cong \Phi_{10}^{010} \quad & \Delta = 1/2 \\
\Phi_6^{000} \cong \Phi_4^{010} \quad & \Delta = 0 \\
\Phi_8^{000} \cong \Phi_{10}^{010} \quad & \Delta = 1/3 \\
\Phi_{10}^{000} \cong \Phi_0^{010} \quad & \Delta = 1/2 \\
\Phi_1^{100} \cong \Phi_{10}^{001} \quad & \Delta = 5/16 \\
\Phi_3^{100} \cong \Phi_{7}^{001} \quad & \Delta = 1/16 \\
\Phi_5^{100} \cong \Phi_{5}^{001} \quad & \Delta = 31/48 \\
\Phi_7^{100} \cong \Phi_{3}^{001} \quad & \Delta = 1/16 \\
\Phi_9^{100} \cong \Phi_{1}^{001} \quad & \Delta = 5/16
\end{array}
\]  

(3)

\(^1\)The computations in this paper have partly been performed using the computer program KAC written by Bert Schellekens. KAC is available at [http://norma.nikhef.nl/~t58/kac.html](http://norma.nikhef.nl/~t58/kac.html).
The conformal weights expected for the Ising model \((0, 1/2, 1/16)\) appear twice, and moreover conformal weights appear that do not show up in the Kac table for the Ising model. Using fermionization, it is not hard to show that the following character identities hold:

\[
\begin{align*}
\chi_{000}(\tau) &= \hat{\chi}_0(\tau) \left( \chi_0'(\tau) + \chi'_6(\tau) \right) + \hat{\chi}_{1/2}(\tau) \left( \chi_4'(\tau) + \chi'_{10}(\tau) \right) \\
\chi_{010}(\tau) &= \hat{\chi}_0(\tau) \left( \chi_{10}'(\tau) + \chi'_4(\tau) \right) + \hat{\chi}_{1/2}(\tau) \left( \chi_6'(\tau) + \chi'_0(\tau) \right) \\
\chi_{100}(\tau) &= \chi_{001}(\tau) = \hat{\chi}_{1/16}(\tau) \left( \chi'_3(\tau) + \chi'_7(\tau) \right)
\end{align*}
\]

Here \(\hat{\chi}_\Delta\) denote characters of the Ising model, \(\chi_\lambda\) characters of the ambient algebra \(g = A_3\) and \(\chi'_\lambda\) of the subalgebra \(g' = A_1\). We observe that those branching functions that have conformal weights that do not appear in the Kac table are indeed vanishing, although they are perfectly allowed by the group theoretical selection rules. Moreover, each Ising primary is realized by different branching functions that are not related by the action of simple currents. In particular, there are more representatives of the vacuum, \(\Phi_{000}^0 \cong \Phi_{410}^0\), so that the coset is clearly maverick.

However, there is indeed an exceptional modular invariant that is exactly suited to make sense of the coset: at level \(k = 10\) the chiral algebra of \(A_1\) allows for an exceptional extension which leads to an exceptional modular invariant of extension type, the so-called \(E_6\) exceptional modular invariant in the ADE-classification of modular invariants \([13]\).

Another coset which exhibits similar features is provided by the embedding

\[
(A_1)_{28} \hookrightarrow (B_3)_1,
\]

which has an identification current \((1/\mathcal{J}')\) of order two. Since it has central charge \(c = 7/10\), it should be the tricritical Ising model. It is straightforward to check that in this case the extension leading to the \(E_8\)-modular invariant of \(A_1\) at level \(k = 28\) has to be used.

What is the general pattern behind these mavericks? Suppose that the embedding \((g')_{1k} \hookrightarrow (g)_k\) allows an intermediate algebra \(h\),

\[
(g')_{1k} \hookrightarrow (h)_k \hookrightarrow (g)_k,
\]

such that the embedding \((g')_{1k} \hookrightarrow (h)_k\) is a conformal embedding at level \(k = 1\). We will show that in this case the coset theory \((g'/g')_k\) is maverick at level \(k = 1\) if and only if the conformal embedding does not correspond to an extension by integer spin simple currents.

In the two examples, the relevant conformal embeddings are \((A_1)_{10} \hookrightarrow (B_2)_1 = (C_2)_1\) and \((A_1)_{28} \hookrightarrow (G_2)_1\), respectively. Other examples, e.g. for \(g' = A_2\), are easily obtained from the conformal embeddings

\[
\begin{align*}
(A_2)_5 & \hookrightarrow (A_5)_1 \\
(A_2)_9 & \hookrightarrow (E_6)_1 \\
(A_2)_{21} & \hookrightarrow (E_7)_1
\end{align*}
\]

which correspond to the exceptional extensions of the chiral algebra of \(A_2\) that give rise to modular invariants of \(A_2\) of extension type.

We now present an explicit argument which shows that all cosets of the form \([8]\) contain non-vanishing branching functions of conformal weight zero. To this end we consider the decomposition of the adjoint representation of the horizontal subalgebra \(\mathfrak{h}\) of \(h\) into \(g'\) irreducible representations. This decomposition certainly contains the adjoint representation of \(g'\), and at least one more irreducible representation. Let \(\lambda'\) be the highest weight of one such irreducible
representation. We claim that in the coset $g/g'$ the branching function for $\Phi^{\Omega}_{\lambda}$, where $\Omega$ is the vacuum of $g$, is non-vanishing and has conformal weight zero.

The conformal weight $\hat{\Delta}_{\lambda'/\lambda}$ of a coset primary field $\Phi^\lambda_{\lambda'}$ is computed according to the formula

$$\hat{\Delta}_{\lambda'/\lambda} = \Delta_{\lambda} - \Delta_{\lambda'} + n,$$

where $n$ is the lowest degree in the affine module $H_{\lambda}$ on which the $g'$-module $H_{\lambda'}$ appears. In the present situation, we compute $n$ by decomposing the vacuum $H_{\Omega}$ of the $g$-theory in terms of $g'$ modules. The lowest degree contains a single state and therefore only contributes to the vacuum $H_{\Omega'}$ of the $g'$-theory. On degree one, the adjoint of $\bar{g}$ appears. In its decomposition in $\bar{h}$-modules the adjoint of $\bar{h}$ appears. Decomposing the adjoint representation of $\bar{h}$ further into $\bar{g}'$ modules, we find $\lambda'$ among the $\bar{g}'$ modules. This module therefore appears at degree one, $n = 1$, and moreover, we see that the branching function for $\Phi^{\Omega}_{\lambda}$ is non-vanishing. Since $\lambda = \Omega$ is the vacuum, we obviously have $\Delta_{\lambda} = 0$.

It remains to compute $\Delta_{\lambda'}$. This is by definition the eigenvalue of the zero mode of the Virasoro algebra obtained by the affine Sugawara construction for $g'$. However, since the embedding $g' \hookrightarrow h$ is conformal, we can equally well use the zero mode of the Virasoro algebra for $h$ to determine the conformal weight. In terms of $h$ modules, however, the relevant state appears at degree 1 of the vacuum module and therefore has conformal weight $\Delta_{\lambda'} = 1$. We conclude that the coset conformal weight is $\hat{\Delta}_{\lambda'/\lambda} = 0 - 1 + 1 = 0$.

So far our argument applies to any conformal embedding. Now we have to consider two different situations: either $\lambda'$ is a simple current of the $g'$-theory or not. The first case happens when the conformal embedding $g'_{I} \hookrightarrow h_{1}$ is described by an integer spin simple current extension of $g'$; the identification current $\Omega/\lambda'$ will then do the appropriate field identification, according to the standard prescription [9, 10].

The second case happens when the embedding does not correspond to a simple current extension. Then clearly the standard prescription does not identify the branching function with vanishing conformal weight with the vacuum of the coset theory. The coset is therefore maverick. However, the coset still yields a consistent conformal field theory, we only have to use the exceptional extension of the chiral algebra of the $g'$-theory that is provided by the conformal embedding.

Coset conformal field theories also have a description as gauged WZW-theories. Field identification and fixed point resolution that are due to group theoretical selection rules can be traced back in this language to the fact that a non-simply connected quotient of the real compact connected Lie group corresponding to the embedded algebra has to be gauged [16]. In this language, the above situation can be described as the rather puzzling observation that at level $k = 1$ the subgroup corresponding to $g'$ cannot be consistently gauged. Rather, one has inevitably to gauge the larger subgroup corresponding to $h$ to obtain a consistent coset theory. A deeper understanding of this fact in a Lagrangian formulation is not known to us; we would like to emphasize that this effect exclusively occurs at level $k = 1$, but not at higher level.

Remarkably enough, the present class of theories shows an intimate relation between maverick coset theories, exceptional extensions of the chiral algebra (and therefore exceptional modular invariants) and conformal embeddings, although here these relations are much less mysterious than in the coset theories of [11, 12].
Some remarks on maverick coset theories at $k = 2$

The mechanism we have presented for maverick coset theories at level $k = 1$ does not extend to the maverick cosets of $[11,12]$ nor does it provide any insight in the classification of mavericks of this type. In fact, a first counterexample to the conjectured ADE-classification $[12]$ of mavericks was provided in $[10]$. One more example is provided by the coset theory

$$(D_6)_2 \oplus (A_1)_2 \hookrightarrow (E_7)_2 \tag{9}$$

which has $c = 4/5$ and turns out to be the tetracritical Ising model. It is therefore in particular a counterexample to the conjecture of $[12]$ that maverick cosets have extended $\mathcal{W}$-symmetry.

The coset theory (9) shares the following four features of the mavericks of $[11,12]$.

1) The ambient algebra $g$ is simply laced and at level $k = 2$; at level $k = 1$ the embedding is a conformal embedding.
2) The pair $(g^c, g'^c)$ is a symmetric Lie algebra of compact type, where $g^c$ and $g'^c$ stand for the compact real forms of $g$ and $g'$, respectively.
3) Identification currents occur that are trivial for the ambient algebra $g$, and the branching functions of vanishing conformal weight $\Delta = 0$ that are not consequences of group theoretical selection rules are fixed points under these identifications.
4) Finally, and most remarkably, the following observation of J. Fuchs $[17]$ is valid also in this maverick: consider the fixed point $\Phi_{\lambda_0}$ that has conformal weight $\Delta = 0$ and hence has to be identified with the vacuum. For the quantum dimensions of the $g$-primary $\lambda_0$ and the $g'$-primary $\lambda'_0$ one has the relation

$$D_{\lambda_0} = \frac{1}{2} D_{\lambda'_0}. \tag{10}$$

We remark that the coset (9) does not share the property of the mavericks of $[11,12]$ that $\lambda_0 = \theta$ is the adjoint representation and that $\lambda'_0$ appears in the decomposition of the adjoint representation of $g$ in $g'$ irreducible representations.

The fourth property of maverick coset theories seems to be a crucial feature of maverick coset theories. For example, consider the coset

$$(A_5)_2 \oplus (A_1)_2 \hookrightarrow (E_6)_2 \tag{11}$$

with central charge $c = 25/28$. It shares the first three properties of mavericks. While it has an identification current of the form $(1/J^0J')$ which does have fixed points, the ratio of the quantum dimensions of the $g$ and $g'$ primaries of the fixed points is in this case not exactly equal to two, but rather equals

$$\frac{\sqrt{2}}{\sqrt{2} - \sqrt{2} \cos(\pi/7)} \approx 2.050858. \tag{12}$$

One can convince oneself that this coset is in fact not a maverick.

The present examples of maverick coset theories show that a general understanding of field identification in coset theories still requires further clarification.

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