Research Article

Dissipativity Analysis of Linear State/Input Delay Systems

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This paper discusses dissipativity problem for system of linear state/input delay equations. Motivated by dissipativity theory of control systems, we choose a new quadratic supply rate. Using the concept of dissipativity, necessary and sufficient conditions for the linear state/input delay systems to be dissipative and exponentially dissipative are derived. The connection of dissipativity with stability is also considered. Finally, passivity and finite gain are explored, correspondingly. The positive-real and bounded-real lemmas are derived.

1. Introduction

Dissipativity theory is one of major issues in control system theory. It was developed by Willems [1] and further extended by Hill and Moylan [2–4]. And it has become one of the major approaches to study complex systems [1, 5, 6]. In some particular systems with certain physical meaning, for example, electrical network systems, physical system, and so forth, dissipativity, just as the name implies, shows that the energy stored inside the system is no more than energy supplied from outside the system. In the area of dynamical system theory, storage function and supply rate are introduced to express the generalized abstract energy, correspondingly.

For dissipative systems, the storage functions usually provide natural candidates for Lyapunov functions [6]. Therefore, in many cases, stability and stabilization problems can be solved once the dissipativity property is assured [7–14]. Dissipativity theory is applied to impulsive hybrid dynamical systems [8], discontinuous dynamical systems [9], and switched systems [10]. Also, dissipativity theory is an efficient tool for the analysis and
design of composite systems. Especially, it is used to deal with some problems of stability and stabilization for interconnected systems [5, 6, 15]. Passivity and finite gain are two special cases of dissipativity which are important in the stability analysis of dynamical systems. The fact that system is passive is equivalent to that system satisfies positive-real or Kalman-Yakubovich-Popov (KYP) condition [12]. Many versions of positive-real lemmas are given for nonlinear [16, 17] and linear systems [18, 19]. The results about $L_2$-gain are given in recent references [15, 20, 21]. In papers [22, 23], bounded real lemmas are summarized and are applied to $H_{\infty}$ control.

On the other hand, time delay is often encountered in various practical systems [24]. The existence of delays is often one main reason of instability and poor performance of systems. A great number of results on the stability and stabilization problems of control systems with time delays have been reported in the literature (see [25, 26], and the references therein). Lyapunov-Krasovskii functional approach is usually used to cope with various problems [27]. Dissipative theory about time-delay systems is discussed in references [28–31]. Sufficient conditions that guarantee robustly stable and strictly passive for uncertain time-delay systems are derived [28]. Dissipativity of linear and nonlinear time-delay systems is, respectively, discussed in [29, 30]. Dissipative problems of singular time-delay systems are considered in [31]. Unlike the systems discussed in the literatures, we consider a special class of linear state/input delay systems. This paper considers the dissipative problem of linear state/input delay systems with a new supply rate and with a storage function substituted by Lyapunov-Krasovskii functional.

There is no systematic dissipativity theory about time-delay systems because of the existence of delays. We consider linear state/input delay systems in this paper. The cause is not only that it has a simply form but it contains state delay and input delay. Motivated by dissipative theory, the variants of supply rate contain not input and output but delay input.

A new quadratic supply rate developed in our paper is a function of input, delay input, dissipative theory, the variants of supply rate contain not input and output but delay input. Motivated by Lyapunov-Krasovskii functional.

state/input delay systems with a new supply rate and with a storage function substituted by Lyapunov-Krasovskii functional. This problem is, respectively, discussed in

$$L_\infty$$

problems.

Finally, we discuss passivity and $L_2$-gain and derive position-real lemma and bounded-real lemma for the system.

**Notation.**

$\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space, and $\| \cdot \|$ stands for the Euclidean norm of any vector. $\mathbb{R}^{n \times m}$ is the set of $n \times m$ real matrices, and $\lambda(), \lambda_{\max}(), \lambda_{\min}()$ denote, respectively, the eigenvalues, and the maximum eigenvalues, the minimum eigenvalues of any square matrix. We use $W > 0$ to denote a symmetric positive-definite matrix. In a symmetric block matrix, the expression $\ast$ will be used to denote the submatrices that lie under the diagonal. Given a positive number $r > 0$, $C_r = C([0, r], \mathbb{R}^n)$ denotes the Banach space of continuous functions that maps the interval $[0, r]$ into $\mathbb{R}^n$ with the topology of uniform convergence and $\| \cdot \|_C$ denotes the norm of continuous function, that is, for $\phi \in C_r$, $\| \phi \|_C = \sup_{0 \leq r \leq r} \| \phi(r) \|$. The space of square integrable functions is denoted by $L_2$. And, for any $g(t) \in L_2[0, \infty)$, its norm is defined as

$$\| g(t) \| = \left( \int_{t_0}^{\infty} \| g(t) \|^2 dt \right)^{1/2}.$$  
(1.1)
The rest of the paper is organized as follows. Problem formulation is presented in Section 2. We give some results about dissipativity analysis of linear state/input delay systems in Section 3. In Section 4, positive-real lemma and bounded-real lemma are derived. Finally, a brief conclusion is provided to summarize the paper.

2. Problem Formulation

Consider the continuous-time systems with a state/input delay

\[ \dot{x}(t) = A_1 x(t) + A_2 x(t-d) + B_1 u(t) + B_2(t-d), \]
\[ y(t) = C_1 x(t) + C_2(t-d), \]
\[ x(\theta) = \varphi(\theta), \quad \theta \in [t_0 - d, t_0], \]  \hspace{1cm} (2.1)

where \( x(t) \in \mathcal{X} \subset \mathbb{R}^n \) is the state vector, \( x(t-d) = x_t \in \mathcal{C}_d \) is the delay state vector, \( u(t) \in \mathcal{U} \subset \mathbb{R}^m \) is exogenous input, where \( \mathcal{U} \) is the set of admissible input, \( y(t) \in \mathcal{Y} \subset \mathbb{R}^p \) is the measured output vector. \( A_1, A_2, B_1, B_2, C_1, \) and \( C_2 \) are constant system matrices with appropriate dimensions. \( d > 0 \) is an unknown constant delay. \( \varphi(t) \in \mathcal{C}_d \) which is a real-valued continuous function on \([t_0 - d, t_0] \) with the topology of uniform convergence is the initial function. For any continuous initial function \( \varphi(t) \) and any admissible input \( u(t) \), there exists a unique solution \( x(t, t_0, \varphi, u) \) to system (2.1) on \([t_0, \infty) \) which continuously depends on the initial data. If \( t \geq t_0 \), we denote the segment of trajectory by \( x_t(\varphi, u) \)

\[ x_t(\varphi, u) : \theta \rightarrow x(t - t_0 + \theta, \varphi, u), \quad \theta \in [t_0 - d, t_0]. \]  \hspace{1cm} (2.2)

Assume that a real-valued function \( w \) about input and output variants is the supply rate of system (2.1). We assume that for all admissible input and for any \( t_0, t \in \mathbb{R}^+ \), the output \( y(t) = y(t, t_0, \varphi, u) \) of (2.1) is such that \( w(t) \) satisfies the following:

\[ \int_{t_0}^{t} w(\tau)d\tau < \infty, \quad \forall t > t_0. \]  \hspace{1cm} (2.3)

**Definition 2.1.** The linear time-delay system (2.1) with supply rate \( w \) is said to be dissipative if there exists a nonnegative functional \( S : \mathcal{X} \times \mathcal{C}_d \rightarrow \mathbb{R}^+ \), called the storage functional, such that for all \( t, t_0 \in \mathbb{R}^+ \) \( (t \geq t_0) \), \( (x, x_t) \in \mathcal{X} \times \mathcal{C}_d \), and \( u \in \mathcal{U} \),

\[ s(x, x_t) - s(x_0, x_{t_0}) \leq \int_{t_0}^{t} w(\tau)d\tau, \]  \hspace{1cm} (2.4)

where \( x = x(t, t_0, \varphi, u) \), and \( w(t) \) is a function about input and output, with \( y = y(t_0, \varphi, u) \).

**Remark 2.2.** The inequality (2.4) is called the dissipation inequality. The dissipation inequality implies that the increase in generalized system energy over a given time interval cannot exceed the generalized energy supply delivered to the system (2.1) during this time interval. Thus, some of the supplied generalized energy to the system is stored, and some is dissipative. The dissipated energy is nonnegative and equals the difference of the right-hand of dissipation inequality and the left-hand.
Remark 2.3. If using equality sign as a substitution for inequality sign in the dissipation inequality, then Definition 2.1 of dissipativity becomes the definition of losslessness. We call system (2.1) lossless with respect to supply rate $w$.

The next definition shows the notion of available storage, which is the maximum amount of storage energy which may be extracted from a dynamical system.

Definition 2.4. The available storage, $s_a$, of system (2.1) with supply rate $w$ is the functional $s_a: \mathcal{X} \times C_d \to \mathbb{R}$ defined by

$$s_a(x, x_t) = -\inf_{w \in \mathcal{H},t \geq t_0} \left\{ \int_{t_0}^{t} w(\tau) d\tau \right\}. \quad (2.5)$$

Note that the available storage, whenever defined, is nonnegative, that is, for all $(x, x_t) \in \mathcal{X} \times C_d$, $s_a(x, x_t) \geq 0$. The available storage plays an important role in determining whether a given system is dissipative or not. This is shown in the following proposition (see [1]).

Proposition 2.5. The available storage, $s_a$, is finite for all $(x, x_t) \in \mathcal{X} \times C_d$ if and only if system (2.1) with respect to $w$ is dissipative. Moreover, any possible storage function $s$ satisfies $0 \leq s_a \leq s$ and if $s_a$ is continuous, then $s_a$ itself is a possible storage function.

Since supply rate is a function of input and output and system (2.1) contains delay input, the delay input should be considered as an argument of supply rate. The supply rate should be functional about input, delay input, and output, that is, $w(t) = w(u(t), u(t - d), y(t))$. Motivated by Reference [1, 2], we are interested in studying dissipativity of system (2.1) with the following new quadratic supply rate:

$$w(u(t), u(t - d), y(t)) = y^T(t) Ty(t) + 2y^T(t) S_1 u(t) + 2y^T(t) S_2 u(t - d)$$

$$+ u^T(t) R_1 u(t) + u^T(t - d) R_2 u(t - d) + 2u^T(t) R_3 u(t - d), \quad (2.6)$$

where $T \in \mathbb{R}^{p \times p}$, $S_i \in \mathbb{R}^{p \times p}$ ($i = 1, 2$), and $R_j \in \mathbb{R}^{m \times m}$ ($j = 1, 2, 3$) are known constant matrices with $T$ and $R_i$ being symmetric. Specially, two important special cases of dissipative system (2.1) are the following.

(i) When $w(u(t), u(t - d), y(t)) = y^T(t) u(t) + y^T(t) u(t - d)$, $m = p$, dissipative system (2.1) is also called passive system.

(ii) When $w(u(t), u(t - d), y(t)) = y^2(u^T(t) u(t) + u^T(t - d) u(t - d) - y(t)y^T(t))$, $\gamma > 0$, dissipative system (2.1) is also called finite gain system.

Remark 2.6. Generally speaking, the storage functional $s$ is nonnegative and needs not to satisfy differentiability. Lyapunov-Krasovskii functional $V(x, x_t)$ can be chosen instead of $s$ because it is an energy functional representing the abstract storage energy of system. If choose $s = V(x, x_t)$, then the dissipation inequality becomes

$$V(x, x_t) - V(x_0, x_{t_0}) \leq \int_{t_0}^{t} w(\tau) d\tau. \quad (2.7)$$
If functional $V$ is continuous differentiable, then the integral version of dissipativity inequality (2.7) can be transformed into a differential version. Differentiating both sides of (2.7), we can get the following:

$$\frac{d}{dt} V(x, x_t) \leq w(u(t), u(t-d), y(t)). \quad (2.8)$$

In this paper, we assume that $s$ has the following form of Lyapunov-Krasovskii functional:

$$V(x_t) = x^T(t)Px(t) + \int_{t-d}^{t} x^T(\tau)Qx(\tau) d\tau,$$  

(2.9)

where $P, Q$ are positive-definite matrices.

Before concluding this section, we introduce two definitions which will be used in the development of our results.

**Definition 2.7.** The linear time-delay system (2.1) is reachable from the origin, if for any given $x_1$ and $t_1$, there exist $t_2 \leq t_1$ and an admissible control $u(t) \in \mathcal{U}$ such that the state can be driven from $x(t_1) = x_1$ to $x(t_2) = 0$.

**Remark 2.8.** We suppose that the system (2.1) is reachable from the origin.

**Definition 2.9.** The linear time-delay system (2.1) with $u(t) \equiv 0$ and $u(t-d) \equiv 0$ is zero-state detectable, if for any trajectory $x(t, t_0, \varphi, 0)$, there holds

$$y \equiv 0 \implies \lim_{t \to \infty} x(t, t_0, \varphi, 0) = 0. \quad (2.10)$$

### 3. Dissipativity Analysis

Notice that dissipativity defined according to the Definition 2.1 represents an input-output property of the system. The following theorem, which is the central result of our paper, shows that dissipativity can also be characterized in terms of the coefficient matrices of system (2.1).

**Theorem 3.1.** System (2.1) is said to be dissipative with respect to supply rate (2.6) if and only if there exist positive definite matrices $P, Q$ and constant matrices $L_i \in \mathbb{R}^{m \times n}, W_i \in \mathbb{R}^{n \times m} (i = 1, 2)$ such that

\[
\begin{align*}
L_1^T L_1 &= -PA_1 - A_1^T P - Q + C_1^T T C_1, \quad L_2^T L_2 = Q + C_2^T T C_2, \\
L_1^T L_2 &= -PA_2 + C_1^T T C_2, \quad L_1^T W_i = -PB_i + C_i^T S_i, \\
L_2^T W_i &= C_2^T S_i, \quad W_1^T W_2 = R_3, \quad W_1^T W_i = R_i.
\end{align*}
\]

**Proof**

**Sufficiency.** Suppose that $V(x_t)$ of form (2.9), $L_i$ and $W_i$ are given such that (3.1) is satisfied. Then, for any admissible control $u$ and any $t_0$ and $\varphi$, we consider the following performance index:

$$J = \frac{d}{dt} V(x_t) - w(t). \quad (3.2)$$
Using (3.1) and (2.9), we can get the following:

\[ \mathcal{J} = \frac{d}{dt} V(x_t) - \left[ y^T(t) y(t) + 2 y^T(t) S_1 u(t) + 2 y^T(t) S_2 u(t - d) 
\right. \\
\left. + u^T(t) R_1 u(t) + u^T(t - d) R_2 u(t - d) \right] \\
= x^T(t) \left( P A_1 + A_1^T P \right) x(t) + 2 x^T(t) P A_2 x(t - d) + 2 x^T(t) P B_1 u(t) \\
+ 2 x^T(t) P B_2 u(t - d) + x^T(t) Q x(t) - x^T(t - d) Q x(t - d) \\
- x^T(t) C_1^T T C_1 x(t) - 2 x^T(t) C_1^T T C_2 x(t - d) - x^T(t - d) C_2^T T C_2 x(t - d) \\
- 2 x^T(t) C_1^T S_1 u(t) - 2 x^T(t - d) C_2^T S_2 u(t - d) - 2 x^T(t) C_1^T S_2 u(t - d) \\
- 2 x^T(t - d) C_2^T S_2 u(t - d) - u^T(t) R_1 u(t) - u^T(t - d) R_2 u(t - d) \\
= x^T(t) \left( P A_1 + A_1^T P + Q - C_1^T T C_1 \right) x(t) + 2 x^T(t) \left( P A_2 - C_1^T T C_2 \right) x(t - d) \\
+ 2 x^T(t) \left( P B_1 - C_1^T S_1 \right) u(t) + 2 x^T(t) \left( P B_2 - C_2^T S_1 \right) u(t - d) \\
+ x^T(t - d) \left( -Q - C_2^T T C_2 \right) x(t - d) + 2 x^T(t - d) \left( -C_1^T S_1 \right) u(t) \\
+ 2 x^T(t - d) \left( -C_2^T S_2 \right) u(t - d) + u^T(t) \left( -R_1 \right) u(t) \\
+ 2 u^T(t - d) \left( -R_2 \right) u(t - d) \\
\leq -(L_1 x(t) + L_2 x(t - d) + W_1 u(t) + W_2 u(t - d))^T \left( L_1 x(t) + L_2 x(t - d) \\
+ W_1 u(t) + W_2 u(t - d) \right) \\
\leq 0. \]

Obviously,

\[ \frac{d}{dt} V(x_t) \leq w(t). \]  

(3.4)

Integrating both sides of (3.4) from \( t_0 \) to \( t \), it yields

\[ V(x_t) - V(x_{t_0}) \leq \int_{t_0}^{t} w(\tau) d\tau. \]  

(3.5)

**Necessity.** Suppose that system (2.1) is dissipative with respect to the supply rate (2.6). We will show that the available storage, \( V_a \), is a solution of (3.1) for some appropriate matrices
$L_i$ and $W_i$. Due to (2.6) and Definition 2.4, it follows from Proposition 2.5 that the available storage $V_a$ of system (2.1) is finite for all $x_t \in C_d$, $V_a(0) = 0$ and satisfies that

$$V_a(x_t) - V_a(x_{t_0}) \leq \int_{t_0}^{t} w(\tau) d\tau, \quad (6.9)$$

for all $t \geq t_0$ and $u \in U$.

Differentiating both sides of the above inequality,

$$\frac{d}{dt} V_a(x_t) \leq w(u(t), u(t - d), y(t)). \quad (3.7)$$

Defining a functional $d : \mathcal{X} \times C_d \times U \to \mathbb{R}$ by

$$d(x(t), x(t - d), u(t), u(t - d)) = -\frac{d}{dt} V_a(x_t) + w(u(t), u(t - d), y(t)). \quad (3.8)$$

From (3.7), $d(x(t), x(t - d), u(t), u(t - d)) \geq 0$. In addition, it is known from (3.8) that $d(x(t), x(t - d), u(t), u(t - d))$ is quadratic in $x(t), x(t - d), u(t)$ and $u(t - d)$.

Thus there exist constant matrices $L_i \in \mathbb{R}^{n \times n}$, and $W_i \in \mathbb{R}^{n \times m}$ ($i = 1, 2$) such that

$$d(x(t), x(t - d), u(t), u(t - d)) = (L_1 x(t) + L_2 x(t - d) + W_1 u(t) + W_2 u(t - d))^T$$

$$\cdot (L_1 x(t) + L_2 x(t - d) + W_1 u(t) + W_2 u(t - d)). \quad (3.9)$$

Substituting (3.9) into (3.8), we have

$$-\frac{d}{dt} V_a(x_t) + y^T(t) Ty(t) + 2y^T(t) S_1 u(t) + 2y^T(t) S_2 u(t - d)$$

$$+ u^T(t) R_1 u(t) + u^T(t - d) R_2 u(t - d) + 2u^T(t) R_3 u(t - d)$$

$$= (L_1 x(t) + L_2 x(t - d) + W_1 u(t) + W_2 u(t - d))^T$$

$$\cdot (L_1 x(t) + L_2 x(t - d) + W_1 u(t) + W_2 u(t - d)) \quad (3.10)$$

for all $(x, x_t) \in \mathcal{X} \times C_d$ and $u(t) \in U$.

Now, let $V(x_t) = V_a(x_t)$. Equating coefficients of equal powers, (3.1) is obtained.

Theorem 3.1 provides a dissipative criterion for system (2.1). Theorem 3.1 can be represented by linear matrix inequality (LMI). The Corollary 3.2 shows a LMI’s version of dissipative criterion.
Corollary 3.2. System (2.1) is said to be dissipative with respect to supply rate (2.6) if and only if there exist positive definite matrices $P$ and $Q$ such that

\[
\Sigma = \begin{pmatrix}
    PA_2 + C^T TC_2 & PB_1 - C^T S_1 & PB_2 - C^T S_2 \\
    -Q - C^T TC_2 & -C^T S_1 & -C^T S_2 \\
    -R_1 & -R_3 & -R_2
\end{pmatrix} \leq 0, \tag{3.11}
\]

where $\Sigma = PA_1 + A_1^T P + Q - C^T TC_1$.

The theory of dissipative systems is used to investigate stability of system via Lyapunov methods. It is shown in the following theorem.

Theorem 3.3. Suppose that system (2.1) is dissipative with smooth storage functional (2.9) with respect to the supply rate (2.6) and zero-state detectable, then the unforced system $\dot{x}(t) = A_1 x(t) + A_2 x(t - d)$ is Lyapunov stable if $T \leq 0$ and asymptotically stable if $T < 0$.

Proof. Suppose that system (2.1) is dissipative, according to the proof part of Theorem 3.1, and (3.10) is satisfied. Let $u(t) = u(t - d) = 0$, we have

\[
\frac{d}{dt} V(x_i) = -(L_1 x(t) + L_2 x(t - d))^T (L_1 x(t) + L_2 x(t - d)) + y^T(t) Ty(t). \tag{3.12}
\]

If $T \leq 0$, then $(d/dt)V(x_i) \leq 0$. The unforced system is Lyapunov stable (see [22, 23]).

If $T < 0$, then $(d/dt)V(x_i) < 0$. By zero-state detectable and LaSalle’s invariance principle (see [22, 23]), the unforced system is asymptotically stable. \(\square\)

Next, we give some results of exponential dissipativity. First, the definition of exponential dissipativity is represented.

Definition 3.4. The linear time-delay system (2.1) with supply rate $w$ is said to be exponentially dissipative if there exists a nonnegative storage functional $S : \mathcal{X} \times \mathcal{C}_d \to \mathbb{R}^+$ and a constant $\varepsilon > 0$, such that for all $t, t_0 \in \mathbb{R}^+$ ($t \geq t_0$), $(x, x_i) \in \mathcal{X} \times \mathcal{C}_d$, and $u \in \mathcal{U}$,

\[
e^{\varepsilon t} s(x, x_i) - e^{\varepsilon t_0} s(x_0, x_{i_0}) \leq \int_{t_0}^t e^{\varepsilon \tau} w(\tau) d\tau, \tag{3.13}
\]

where $x = x(t, t_0, \varphi, u)$ and $w(t) = w(u(t), u(t - d), y(t))$, with $y = y(t_0, \varphi, u)$.

Remark 3.5. When we choose $\varepsilon = 0$, the definition of exponential dissipativity becomes the definition of dissipativity.

Assume that $s$ has the following form of Lyapunov-Krasovskii functional:

\[
V(x_i) = x^T(t) P x(t) + \int_{t-d}^t x^T(\tau) e^{\varepsilon(\tau-d)} Q x(\tau) d\tau, \tag{3.14}
\]

where $P, Q$ are positive definite matrices, and $\varepsilon$ is a positive constant.
Theorem 3.6. System (2.1) is said to be exponentially dissipative with respect to supply rate (2.6) if and only if there exist positive definite matrices $P$, $Q$, constant matrices $L_i \in \mathbb{R}^{n \times n}$, $W_i \in \mathbb{R}^{n \times m}$ ($i = 1,2$) and a constant $\epsilon > 0$ such that

$$
L_1^T L_1 = -PA_1 - A_1^T P - \epsilon P - Q + C_1^T T C_1, \\
L_2^T L_2 = \epsilon - \epsilon d_1 Q + C_2^T T C_2,
$$

$$
L_1^T L_2 = -PA_2 + C_1^T T C_2, \\
L_1^T W_i = -PB_i + C_1^T S_i, \\
L_2^T W_1 = C_2^T S_i, \\
W_1^T W_2 = R_3, \\
W_i^T W_i = R_i.
$$

Proof. Using (3.14), there is

$$
\frac{d}{dt} \left[ e^{\epsilon t} V(x_i) - e^{\epsilon t_0} V(x_{i_0}) - \int_{t_0}^{t} e^{\epsilon \tau} \omega(\tau) d\tau \right] = e^{\epsilon t} \frac{d}{dt} V(x_i) + \epsilon e^{\epsilon t} V(x_i) - e^{\epsilon t} \omega(t)
$$

$$
= e^{\epsilon t} \left[ \frac{d}{dt} V(x_i) + \epsilon V(x_i) - \omega(t) \right].
$$

Let performance index be

$$
J_1 = \frac{d}{dt} V(x_i) + \epsilon V(x_i) - \omega(t).
$$

Note that $e^{\epsilon t} \geq 1$. So if $J_1 \leq 0$, we can directly obtain that

$$
e^{\epsilon t} \left[ \frac{d}{dt} V(x_i) + \epsilon V(x_i) - \omega(t) \right] \leq 0.
$$

Substituting $(d/dt)V(x_i)$ and $V(x_i)$ into $J_1$, it yields the following:

$$
J_1 = x(t) \left( PA_1 + A_1^T P \right) x(t) + 2x^T(t)PA_2x(t - d) + 2x^T(t)PB_1u(t)
$$

\begin{align*}
&+ 2x^T(t)PB_2u(t - d) + x^T(t)Qx(t) - x^T(t - d)e^{-\epsilon d}Qx(t - d)
&- \epsilon \int_{t-d}^{t} x^T(\tau)e^{\epsilon(\tau-t)}Qx(\tau) d\tau + \epsilon x^T(t)Px(t)
&+ \epsilon \int_{t-d}^{t} x^T(\tau)e^{\epsilon(\tau-t)}Qx(\tau) d\tau - \omega(t)
&= x^T(t) \left( PA_1 + A_1^T P + Q + \epsilon P \right) x(t)
&+ 2x^T(t)PA_2x(t - d) + 2x^T(t)PB_1u(t)
&+ 2x^T(t)PB_2u(t - d) - x^T(t - d)e^{-\epsilon d}Qx(t - d) - \omega(t).
\end{align*}

The sufficiency and necessity proof is similar to the corresponding part of Theorem 3.1. For the sake of brevity, we omit it.

We show the result of Theorem 3.6 by LMI.
Corollary 3.7. System (2.1) is said to be exponentially dissipative with respect to supply rate (2.6) if and only if there exist positive definite matrices $P$, $Q$ and a constant $\epsilon > 0$ such that

$$\Sigma_2 \begin{pmatrix} PA_2 - C_i^TTC_2 & PB_1 - C_i^TS_1 & PB_2 - C_i^TS_2 \\ -e^{-\epsilon t}Q - C_i^TTC_2 & -C_i^TS_1 & -C_i^TS_2 \\ -R_1 & -R_3 & -R_2 \end{pmatrix} \leq 0,$$  

(3.20)

where $\Sigma_2 = PA_1 + A_1^TP + \epsilon P + Q - C_1^TTC_1$.

Similar to the result about the effect of dissipativity on the stability of unforced system, we will show that the exponentially dissipative system which satisfies that the property of zero-state detectability is exponentially stable.

Theorem 3.8. Suppose that system (2.1) is exponentially dissipative with smooth storage functional (3.14) with respect to the supply rate (2.6) and zero-state detectable, then the unforced system $\dot{x}(t) = A_1x(t) + A_2x(t - d)$ is exponentially stable if $T \leq 0$.

Proof. According to the conditions of Theorem 3.8 and the proof of Theorem 3.6, we can get the following:

$$e^{\epsilon t}\frac{d}{dt}V(x_t) + \epsilon V(x_t) = -(L_1x(t) + L_2x(t - d))^T(L_1x(t) + L_2x(t - d)) + e^{\epsilon t}y^T(t)Ty(t).$$

(3.21)

If $T \leq 0$, then

$$\frac{d}{dt}V(x_t(\varphi, 0)) + \epsilon V(x_t(\varphi, 0)) \leq 0.$$  

(3.22)

Integrating both sides from $t_0$ to $t$, it yields that

$$V(x_t(\varphi, 0)) \leq e^{-\epsilon(t-t_0)}V(\varphi), \quad t \geq t_0.$$  

(3.23)

From (3.14), we get the following inequality:

$$\alpha_1 \|x(t, \varphi, 0)\|^2 \leq V(x_t(\varphi, 0)) \leq \alpha_2 \|x_t(\varphi, 0)\|^2,$$  

(3.24)

where $\alpha_1 = \lambda_{\text{min}}(P)$ and $\alpha_2 = \lambda_{\text{max}}(P) + d\lambda_{\text{max}}(Q)$.

By (3.23) and (3.24), we have

$$\alpha_1 \|x(t, \varphi, 0)\|^2 \leq V(x_t(\varphi, 0)) \leq e^{-\epsilon(t-t_0)}V(\varphi) \leq \alpha_2 e^{-\epsilon(t-t_0)}\|\varphi\|^2_c.$$  

(3.25)
Any solution \( x(t, \varphi, 0) \) of unforced system satisfies that

\[
\|x(t, \varphi, 0)\| \leq \sqrt{\frac{a_2}{a_1}} e^{-(1/2)e(t-t_0)} \|\varphi\|_C, \quad t \geq t_0.
\]  

(3.26)

Thus, the unforced system is exponentially stable.

\[ \square \]

4. Positive-Real and Bounded-Real Conditions

4.1. Positive-Real Lemma

When \( m = p \), we shall focus on studying dissipative systems with supply rate given by the inner product of the following form;

\[
\omega(u(t), u(t-d), y(t)) = y^T(t)u(t) + y^T(t)u(t-d).
\]

(4.1)

For convenience, we characterize this choice by means of a separate definition.

**Definition 4.1.** The linear time-delay system (2.1) with supply rate (4.1) is said to be passive if it is dissipative with supply rate (4.1); that is, there exists a nonnegative storage functional \( S : \mathcal{X} \times C_d \to \mathbb{R}^+ \), such that for all \( t, t_0 \in \mathbb{R}^+ (t \geq t_0), (x, x_t) \in \mathcal{X} \times C_d \), and \( u \in \mathcal{U} \),

\[
s(x, x_t) - s(x_0, x_{t_0}) \leq \int_{t_0}^{t} \left( y^T(\tau)u(\tau) + y^T(\tau)u(\tau-d) \right) d\tau.
\]

(4.2)

The following theorem is positive-real lemma (KYP lemma) for linear time-delay system (2.1).

**Theorem 4.2.** System (2.1) is said to be passive with respect to supply rate (4.1) if and only if there exist positive definite matrices \( P, Q \) and constant matrices \( L_i \in \mathbb{R}^{n \times n} (i = 1, 2) \) such that

\[
L_1^T L_1 = -PA_1 - A_1^T P - Q, \quad L_2^T L_2 = Q,
\]

\[
L_1^T L_2 = -PA_2, \quad -PB_i + \frac{1}{2} C_i^T C_i = 0, \quad C_2 = 0.
\]

(4.3)

**Proof.** Let \( T = R_1 = R_2 = R_3 = 0 \), and \( S_1 = S_2 = (1/2)I_n \) in Theorem 3.1. We can get \( W_1 = W_2 = 0 \). The result can be directly obtained.

\[ \square \]

We rewrite the positive-real condition of (4.3) in terms of LMI.
Corollary 4.3. System (2.1) is said to be passive with respect to supply rate (4.1) if and only if there exists positive definite matrices $P$ and $Q$ such that

\[
\begin{pmatrix}
\Sigma_3 & PA_2 & PB_1 - \frac{1}{2} C_1^T \\
* & -Q & 0 \\
* & * & 0 \\
* & * & 0
\end{pmatrix} \leq 0,
\]

where $\Sigma_3 = PA_1 + A_1^T P + Q$.

4.2. Bounded-Real Lemma

In this subsection, we shall be particularly interested in the quadratic supply rate

\[
w(u(t), u(t - d), y(t)) = \gamma^2 \left( u^T(t) u(t) + u^T(t - d) u(t - d) \right) - y(t) y^T(t),
\]

where positive constant $\gamma > 0$ and $u(t) \in L_2[t_0, \infty)$.

Definition 4.4. The system (2.1) is said to have finite $L_2$-gain from $u(t)$ to $y(t)$ less than or equal to $\gamma > 0$, if it is dissipative with supply rate (4.5): that is, there exists a nonnegative storage functional $S : \mathcal{X} \times \mathcal{C}_d \to \mathbb{R}^+$, such that for all $(t_0, t) \in [0, \infty) (t \geq t_0)$, $(x, x_i) \in \mathcal{X} \times \mathcal{C}_d$, and any $u(t) \in L_2[t_0, t]$

\[
s(x, x_i) - s(x_0, x_i) \leq \int_{t_0}^t \gamma^2 \left( u^T(\tau) u(\tau) + u^T(\tau - d) u(\tau - d) \right) - y(\tau) y^T(\tau) d\tau.
\]

When $u(t) = u(t - d) = 0$, obviously, $s(x, x_i) - s(x_0, x_i) \leq -\int_{t_0}^t y(\tau) y^T(\tau) d\tau \leq 0$. Inequality (4.6) can be rewritten as

\[
\int_{t_0}^t y(\tau) y^T(\tau) d\tau \leq \int_{t_0}^t \gamma^2 \left( u^T(\tau) u(\tau) + u^T(\tau - d) u(\tau - d) \right) d\tau.
\]

The following theorem is bounded-real lemma for linear time-delay system (2.1).

Theorem 4.5. System (2.1) is said to have finite $L_2$-gain from $u(t)$ to $y(t)$ less than or equal to $\gamma > 0$ with respect to supply rate (4.5) if and only if there exist positive definite matrices $P$, $Q$ and constant matrices $L_i \in \mathbb{R}^{n \times n}$, $W_i \in \mathbb{R}^{n \times m}$ ($i = 1, 2$) such that

\[
L_1^T L_1 = -PA_1 - A_1^T P - Q - C_1^T C_1, \quad L_2^T L_2 = Q - C_2^T C_2, \\
L_1^T L_2 = -PA_2 - C_2^T C_2, \quad L_1^T W_i = -PB_i, \\
L_1^T W_i = W_i^T W_i = 0, \quad W_i^T W_i = \gamma^2 I.
\]

Proof. The result can be directly obtained from the consequence of Theorem 3.1 with $T = -I$, $R_1 = R_2 = \gamma^2 I$, and $S_1 = S_2 = R_3 = 0$. \qed
By LMI, Theorem 4.5 can be rewritten as the following corollary.

**Corollary 4.6.** System (2.1) is said to have finite $L_2$-gain from $u(t)$ to $y(t)$ less than or equal to $\gamma > 0$ with respect to supply rate (4.5) if and only if there exist positive definite matrices $P$ and $Q$ such that

$$
\begin{pmatrix}
\Sigma_3 & PA_2 + C_1^T C_2 & PB_1 & PB_2 \\
* & -Q + C_2^T C_2 & 0 & 0 \\
* & * & -\gamma^2 I & 0 \\
* & * & * & -\gamma^2 I \\
\end{pmatrix} \leq 0,
$$

(4.9)

where $\Sigma_4 = PA_1 + A_1^T P + Q - C_1^T C_1$.

**Remark 4.7.** The connections between passive property or finite gain property and unforced systems have not been shown in this section. They can be derived according to the results of Theorem 3.3 and Theorem 3.8.

**5. Conclusion**

This paper considers dissipativity theory for linear state/input delay systems. Motivated by reference [1, 2], we consider a new type quadratic supply rate for linear state/input delay systems. The choice of the new supply rate also depends on the case of system itself. Based on new supply rate, the necessary and sufficient conditions of dissipativity and exponential dissipativity are represented. And, stability results of dissipative linear state/input delay systems are derived. Finally, we focus on passive and finite $L_2$-gain systems and discuss positive-real and bounded-real condition, correspondingly. Positive-real lemma and bounded-real lemma based on new supply rate may help us to analyze the problem of stabilization using output or state feedback for the system with or without external disturbances, that is a future direction of research. Interconnected dissipative delay systems are also considered in future works.

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