Almost canonical ideals and GAS numerical semigroups

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ABSTRACT
We propose the notion of GAS numerical semigroup which generalizes both almost symmetric and 2-AGL numerical semigroups. Moreover, we introduce the concept of almost canonical ideal which generalizes the notion of canonical ideal in the same way almost symmetric numerical semigroups generalize symmetric ones. We prove that a numerical semigroup with maximal ideal \( M \) and multiplicity \( e \) is GAS if and only if \( M - e \) is an almost canonical ideal of \( M - M \). This generalizes a result of Barucci about almost symmetric semigroups and a theorem of Chau, Goto, Kumashiro, and Matsuoka about 2-AGL semigroups. We also study the transfer of the GAS property from a numerical semigroup to its gluing, numerical duplication and dilatation.

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0. Introduction

The notion of Gorenstein ring turned out to have great importance in commutative algebra, algebraic geometry and other mathematics areas and in the last decades many researchers have developed generalizations of this concept obtaining rings with similar properties in certain respects. With this aim, in 1997 Barucci and Fröberg [3] introduced the notion of almost Gorenstein ring, inspired by numerical semigroup theory. We recall that a numerical semigroup \( S \) is simply an additive submonoid of the set of the natural numbers \( \mathbb{N} \) with finite complement in \( \mathbb{N} \). The simplest way to relate it to ring theory is by associating the so-called numerical semigroup ring \( k[[S]] = k[[t^a | s \in S]] \) with \( S \), where \( k \) is a field and \( t \) is an indeterminate. Actually it is possible to associate a numerical semigroup \( \nu(R) \) with every one-dimensional analytically irreducible ring \( R \). In this case a celebrated result of Kunz [17] ensures that \( R \) is Gorenstein if and only if \( \nu(R) \) is a symmetric semigroup. In [3] Barucci and Fröberg first defined the notion of almost symmetric numerical semigroup in order to generalize symmetric semigroups and, then, they introduced almost Gorenstein rings in the case of analytically unramified rings; it turns out that \( k[[S]] \) is almost Gorenstein if and only if \( S \) is almost symmetric.

More recently, the notion of almost Gorenstein ring has been generalized in the case of one-dimensional local rings [13] and in higher dimension [14]. After that, other classes of rings have been introduced in order to find rings that are close to be almost Gorenstein. Among them, in [5] it is proposed the notion of 2-AGL ring, which in the numerical semigroup ring case leads to a new class of numerical semigroups that are close to be almost symmetric in some respects.
In this paper we introduce the class of Generalized Almost Symmetric numerical semigroups, briefly GAS numerical semigroups, which includes symmetric, almost symmetric and 2-AGL numerical semigroups. Moreover, if $S$ has maximal embedding dimension and it is GAS, then it is either almost symmetric or 2-AGL. Our original motivation to introduce this class is a result on 2-AGL numerical semigroups that partially generalizes a property of almost symmetric semigroups. More precisely, let $S$ be a numerical semigroup with multiplicity $e$ and let $M$ be its maximal ideal. In [3, Corollary 8] it is proved that $M - M$ is symmetric if and only if $S$ is almost symmetric with maximal embedding dimension. If we do not assume that $S$ has maximal embedding dimension, it holds that $S$ is almost symmetric if and only if $M - e$ is a canonical ideal of $M - M$ (indeed $S$ has maximal embedding dimension exactly when $M - e = M - M$, see [1, Theorem 5.2]). In [5, Corollary 5.4] it is shown that $S$ is 2-AGL if and only if $M - M$ is almost symmetric and not symmetric, provided that $S$ has maximal embedding dimension.

Hence, it is natural to investigate what happens to $M - M$, for a 2-AGL semigroup, if we do not make any assumptions on its embedding dimension. It turns out that $M - e$ is an ideal of $M - M$ that satisfies some equivalent conditions, which are the analogue for ideals to the defining conditions of almost symmetric semigroup (cf. Definition 2.1 and Proposition 2.4); for this reason we called the ideals in this class almost canonical ideals. However, the converse is not true: there exist numerical semigroups $S$ such that $M - e$ is an almost canonical ideal of $M - M$, but that are not 2-AGL. This fact lead us to look for those numerical semigroup satisfying this property, and we found that these semigroups naturally generalize 2-AGL semigroups (cf. Proposition 3.1 and Definition 3.2); moreover, as we said above this class coincides with the union of 2-AGL and almost symmetric semigroups, if we assume maximal embedding dimension; hence we called them Generalized Almost Symmetric (briefly GAS). It turns out that GAS semigroups are interesting under many aspects; for example, if $S$ is GAS, it is possible to control both the semigroup generated by its canonical ideal (which plays a fundamental role in [5]; cf. Theorem 3.7) and its pseudo-Frobenius numbers (cf. Proposition 3.8).

The definitions of GAS numerical semigroup and almost canonical ideal can be trivially extended to numerical semigroup rings and their monomial ideals, but it is more difficult to extend them to a general one-dimensional Cohen-Macaulay local ring $(R, \mathfrak{m})$. This is done in the forthcoming paper [7], where these definitions are generalized by means of Proposition 2.4(4) and Proposition 3.5, even if the generalizations are not straightforward. For instance, the facts that $\mathfrak{m} : \mathfrak{m}$ is not necessarily local and that its residue fields are not necessarily isomorphic to $R/\mathfrak{m}$ are issues that need to be carefully addressed. On the other hand, the numerical semigroup case is a good starting point and it is interesting by itself, allowing to deepen the theory with more concrete results and to explicitly construct examples and counterexamples.

We now explain the structure of the paper. After recalling the basic definitions and notations, in Section 2 we introduce the concept of almost canonical ideal. We show under which respect it is a generalization of the notion of canonical ideal and we notice that, similarly to the canonical case, a numerical semigroup $S$ is almost symmetric if and only if it is an almost canonical ideal of itself. Moreover, we prove several equivalent conditions for a semigroup ideal to be almost canonical (cf. Proposition 2.4) and we show how to find all the almost canonical ideals of a numerical semigroup and to count them (Corollary 2.6).

In Section 3 we develop the theory of GAS semigroups proving many equivalent conditions (see Proposition 3.5), exploring their properties (cf. Theorem 3.7 and Proposition 3.8) and relating them with other classes of numerical semigroups that have been recently introduced to generalize almost symmetric semigroups. The main result is Theorem 3.13, where it is proved that $S$ is GAS if and only if $M - e$ is an almost canonical ideal of $M - M$.

Finally, in Section 4 we study the transfer of the GAS property from $S$ to some numerical semigroup constructions: gluing in Theorem 4.1, numerical duplication in Theorem 4.7 and dilatation in Proposition 4.9.
Several computations are performed by using the GAP system [9] and, in particular, the NumericalSgps package [8].

1. Notation and basic definitions

A numerical semigroup $S$ is a submonoid of the natural numbers $\mathbb{N}$ such that $|\mathbb{N} \setminus S| < \infty$. Therefore, there exists the maximum of $\mathbb{N} \setminus S$ which is said to be the Frobenius number of $S$ and it is denoted by $F(S)$. Given $s_1, \ldots, s_n \in \mathbb{N}$ we set $(s_1, \ldots, s_n) = \{\lambda_1s_1 + \ldots + \lambda_ns_n | \lambda_1, \ldots, \lambda_n \in \mathbb{N}\}$ which is a numerical semigroup if and only if $\gcd(s_1, \ldots, s_n) = 1$. We say that $s_1, \ldots, s_n$ are minimal generators of $(s_1, \ldots, s_n)$ if it is not possible to delete one of them obtaining the same semigroup.

It is well known that a numerical semigroup have a unique system of minimal generators, which is finite, and its cardinality is called embedding dimension of $S$. The minimum non-zero element of $S$ is said to be the multiplicity of $S$ and we denote it by $e$. It is always greater than or equal to the embedding dimension of $S$ and we say that $S$ has maximal embedding dimension if they are equal. Unless otherwise specified, we assume that $S \neq \mathbb{N}$.

A set $I \subseteq \mathbb{Z}$ is said to be a relative ideal of $S$ if $I + S \subseteq I$ and there exists $z \in S$ such that $z + I \subseteq S$. If it is possible to choose $z = 0$, i.e. $I \subseteq S$, we simply say that $I$ is an ideal of $S$. Two very important relative ideals are $M(S) = S \setminus \{0\}$, which is an ideal and it is called the maximal ideal of $S$, and $K(S) = \{x \in \mathbb{N} | F(S) - x \not\in S\}$. We refer to the latter as the standard canonical ideal of $S$ and we say that a relative ideal $I$ of $S$ is canonical if $I = x + K(S)$ for some $x \in \mathbb{Z}$. If the semigroup is clear from the context, we write $M$ and $K$ in place of $M(S)$ and $K(S)$. Given two relative ideals $I$ and $J$ of $S$, we set $I - J = \{x \in \mathbb{Z} | x + J \subseteq I\}$ which is a relative ideal of $S$. For every relative ideal $I$ it holds that $K - (K - I) = I$, in particular $K - (K - S) = S$. Moreover, an element $x$ is in $I$ if and only if $F(S) - x \not\in K - I$, see [16, Hilfssatz 5]. As a consequence we get that the cardinalities of $I$ and $K - I$ are equal. Also, if $I \subseteq J$ are two relative ideals, then $|J \setminus I| = |(K - I) \setminus (K - J)|$. We now collect some important definitions that we are going to generalize in the next section.

**Definition 1.1.** Let $S$ be a numerical semigroup.

1. The pseudo-Frobenius numbers of $S$ are the elements of the set $PF(S) = (S - M) \setminus S$.
2. The type of $S$ is $t(S) = \vert PF(S) \vert$.
3. $S$ is symmetric if and only if $S = K$.
4. $S$ is almost symmetric if and only if $S - M = K \cup \{F(S)\}$.

We note that $M - M = S \cup PF(S)$. Given $0 \leq i \leq e - 1$, let $\omega_i$ be the smallest element of $S$ that is congruent to $i$ modulo $e$. A fundamental tool in numerical semigroup theory is the so-called Apéry set of $S$ that is defined as $Ap(S) = \{\omega_0 = 0, \omega_1, \ldots, \omega_{e-1}\}$. In $Ap(S)$ we define the partial ordering $x \leq y$ if and only if $y = x + s$ for some $s \in S$ and we denote the maximal elements of $Ap(S)$ with respect to $\leq_S$ by $\max_{\leq_S}(Ap(S))$. With this notation $PF(S) = \{\omega - e|\omega \in \max_{\leq_S}(Ap(S))\}$, see [21, Proposition 2.20]. We also recall that $S$ is symmetric if and only if $t(S) = 1$, which is also equivalent to say that $k[[S]]$ has type 1 for every field $k$, i.e. $k[[S]]$ is Gorenstein. Also for almost symmetric semigroups many useful characterizations are known, for instance it is easy to see that our definition is equivalent to $M + K \subseteq M$, but see also [19, Theorem 2.4] for another useful characterization related to the Apéry set of $S$ and its pseudo-Frobenius numbers.

2. Almost canonical ideals of a numerical semigroup

If $I$ is a relative ideal of $S$, the set $\mathbb{Z} \setminus I$ has a maximum and we denote it by $F(I)$. We set $\bar{I} = I + (F(S) - F(I))$, which is the unique relative ideal $J$ isomorphic to $I$ for which $F(S) = F(J)$, and we note that $\bar{I} \subseteq K \subseteq \mathbb{N}$ for every $I$. The following is a generalization of Definition 1.1.
Definition 2.1. Let \( I \) be a relative ideal of a numerical semigroup \( S \).

(1) The pseudo-Frobenius numbers of \( I \) are the elements of the set \( \text{PF}(I) = (I - M) \setminus I \).
(2) The type of \( I \) is \( t(I) = |\text{PF}(I)| \).
(3) \( I \) is canonical if and only if \( \bar{I} = K \).
(4) \( I \) is almost canonical if and only if \( \bar{I} - M = K \cup \{F(S)\} \).

Remark 2.2. 1. \( S \) is an almost canonical ideal of itself if and only if it is an almost symmetric semigroup.
2. \( M \) is an almost canonical ideal of \( S \) if and only if \( S \) is an almost symmetric semigroup.
   Indeed, \( M - M = S - M \), since \( S \neq \mathbb{N} \). Moreover, \( t(M) = t(S) + 1 \).
3. \( M \) is canonical if and only if \( t(M) = 1 \).
4. \( M \) is an almost canonical ideal of itself if and only if it is an almost symmetric semigroup.

Proposition 2.3. Let \( I \) be a relative ideal of \( S \). The following statements hold:

(1) \( \text{PF}(I) = \{i - e | i \in \text{Max}_{\leq_S}(\text{Ap}(I))\} \);
(2) \( I \) is canonical if and only if its type is 1.

Proof. (1) An integer \( i \in I \) is in \( \text{Max}_{\leq_S}(\text{Ap}(I)) \) if and only if \( i - e \not\in I \) and \( s + i \not\in \text{Ap}(I) \), i.e. \( s + i - e \not\in I \), for every \( s \in M \). This is equivalent to say that \( i - e \in (I - M) \setminus I = \text{PF}(I) \).

(2) Since \( F(S) \subseteq \bar{I} - M \), we have \( t(\bar{I}) = t(I) = 1 \) if and only if \( \bar{I} - M = \bar{I} \cup \{F(S)\} \). Therefore, a canonical ideal has type 1 by Remark 2.2.3. Conversely, assume that \( t(\bar{I}) = 1 \) and let \( x \not\in \bar{I} \). Since \( \bar{I} \subseteq K \), we only need to prove that \( x \not\in K \). By (1), there is a unique maximal element in \( \text{Ap}(\bar{I}) \) with respect to \( \leq_S \) and, clearly, it is \( F(S) + e \). Let \( 0 \neq \lambda \in \mathbb{N} \) be such that \( x + \lambda e \in \text{Ap}(\bar{I}) \). Then, there exists \( y \in S \) such that \( x + \lambda e + y = F(S) + e \) and \( x = F(S) - (y + (\lambda - 1)e) \not\in K \), since \( y + (\lambda - 1)e \in S \).

Let \( g(S) = |\mathbb{N} \setminus S| \) denote the genus of \( S \) and let \( g(I) = |\mathbb{N} \setminus \bar{I}| \). We recall that \( 2g(S) \geq F(S) + t(S) \) and the equality holds if and only if \( S \) is almost symmetric, see, e.g., [19, Proposition 2.2 and Proposition-Definition 2.3].

Proposition 2.4. Let \( I \) be a relative ideal of \( S \). Then \( g(I) + g(S) \geq F(S) + t(I) \). Moreover, the following conditions are equivalent:

(1) \( I \) is almost canonical;
(2) \( g(I) + g(S) = F(S) + t(I) \);
(3) \( \bar{I} - M = K - M \);
(4) \( K - (M - M) \subseteq \bar{I} \);
(5) \( \text{If } x \in \text{PF}(I) \setminus \{F(I)\}, \text{ then } F(I) - x \in \text{PF}(S) \).

Proof. Clearly, \( t(I) = t(\bar{I}) \) and \( g(I) - t(\bar{I}) = |\mathbb{N} \setminus \bar{I}| - |(\bar{I} - M) \setminus \bar{I}| = |\mathbb{N} \setminus (\bar{I} - M)| \). Moreover, since \( F(S) + 1 - g(S) \) is the number of the elements of \( S \) smaller than \( F(S) + 1 \), it holds that
F(S) - g(S) = |N \setminus K| - 1 = |N \setminus (K \cup \{F(S)\})|. We have \( \tilde{I} - M \subseteq K \cup \{F(S)\} \) by Remark 2.2.4, then \( g(I) - t(I) \geq F(S) - g(S) \) and the equality holds if and only if \( \tilde{I} - M = K \cup \{F(S)\} \), i.e. \( I \) is almost canonical. Hence, (1) \( \iff \) (2).

(1) \( \iff \) (3) We have already proved that \( K - M = K \cup \{F(S)\} \) in Remark 2.2.3.

(1) \( \Rightarrow \) (4) The thesis is equivalent to \( M - \tilde{I} \subseteq K - \tilde{I} \). Let \( x \in K - \tilde{I} \) and assume by contradiction that there exists \( m \in M \) such that \( x + m \notin M \). Then, \( F(S) - x - m \in K \cup \{F(S)\} = \tilde{I} - M \) and, so, \( F(S) - x \notin \tilde{I} \). Since \( x \in K - \tilde{I} \), this implies \( F(S) \in K \), which is a contradiction.

(4) \( \Rightarrow \) (1) It is enough to prove that \( x \in \tilde{I} - M \). Suppose by contradiction that there exists \( m \in M \) such that \( x + m \notin \tilde{I} \). In particular, \( x + m \notin \tilde{I} \). Since \( F(S) - (x + m) \in M - M \). This implies \( F(S) - x \in M \), which is a contradiction because \( x \in K \).

(1) \( \Rightarrow \) (5) We notice that \(PF(\tilde{I}) = \{x + F(S) - F(I) \mid x \in PF(I)\}\). Let \( x \in PF(I) \setminus \{F(I)\}\) and let \( y = x + F(S) - F(I) \in PF(\tilde{I}) \setminus \{F(S)\}\). We first note that \( F(S) - y \notin S \), otherwise \( F(S) = y + (F(S) - y) \in \tilde{I}\). Assume by contradiction that \( F(S) - y \notin PF(S) \), i.e. there exists \( m \in M \) such that \( F(S) - y + m \notin S \). This implies that \( y - m \in K \subseteq \tilde{I} - M \) by (1) and, thus, \( y = (y - m) + m \in \tilde{I} \) yields a contradiction. Hence, \(F(I) - x = F(S) - y \notin PF(S)\).

(5) \( \Rightarrow \) (4) Assume by contradiction that there exists \( x \in (K - (M - M)) \setminus \tilde{I} \). It easily follows from the definition that there is \( s \in S \) such that \( x + s \in PF(\tilde{I}) \). Then, \( F(S) - x - s \in PF(S) \cup \{0\} \subseteq M - M \) by (5) and \( F(S) - s = x + (F(S) - x - s) \in (K - (M - M)) + (M - M) \subseteq K \) gives a contradiction. \( \Box \)

**Remark 2.5.** 1. In [19, Theorem 2.4] it is proved that a numerical semigroup \( S \) is almost symmetric if and only if \( F(S) - f \in PF(S) \) for every \( f \in PF(S) \setminus \{F(S)\} \). Hence, the last condition of Proposition 2.4 can be considered a generalization of this result.

2. Almost canonical ideals naturally arise characterizing the almost symmetry of the numerical duplication \( S \bowtie bI \) of \( S \) with respect to the ideal \( I \) and \( b \in S \), a construction introduced in [6]. Indeed [6, Theorem 4.3] says that \( S \bowtie bI \) is almost symmetric if and only if \( I \) is almost canonical and \( K - \tilde{I} \) is a numerical semigroup.

3. Let \( T \) be an almost symmetric numerical semigroup with odd Frobenius number (or, equivalently, odd type). Let \( b \) be an odd integer such that \( 2b \in T \) and set \( I = \{x \in \mathbb{Z} \mid 2x + b \in T\} \). Then, [22, Proposition 3.3] says that \( T \) can be realized as a numerical duplication \( T = S \bowtie bI \), where \( S = T/2 = \{y \in \mathbb{Z} \mid 2y \in T\} \), while [22, Theorem 3.7] implies that \( I \) is an almost canonical ideal of \( S \). In general this is not true if the Frobenius number of \( T \) is even.

Since \( F(K - (M - M)) = F(\tilde{I}) = F(K) \) and \( \tilde{I} \subseteq K \) for every relative ideal \( I \), Condition (4) of Proposition 2.4 allows to find all the almost canonical ideals of a numerical semigroup. Clearly it is enough to focus on the relative ideals with Frobenius number \( F(S) \).

**Corollary 2.6.** Let \( S \) be a numerical semigroup with type \( t \). If \( I \) is almost canonical, then \( t(I) \leq t + 1 \). Moreover, for every integer \( i \) such that \( 1 \leq i \leq t + 1 \), there are exactly \( \binom{t}{i-1} \) almost canonical ideals of \( S \) with Frobenius number \( F(i) \) and type \( i \). In particular, there are exactly \( 2^t \) almost canonical ideals of \( S \) with Frobenius number \( F(S) \).

**Proof.** Let \( C = \{s \in S \mid s > F(S)\} = K - \mathbb{N} \) be the conductor of \( S \) and let \( n(S) = |\{s \in S \mid s < F(S)\}| \). It is straightforward to see that \( g(S) + n(S) = F(S) + 1 \). If \( I \) is almost canonical, Proposition 2.4 implies that

\[
\begin{align*}
t(I) &= g(I) + g(S) - F(S) \leq |N \setminus (K - (M - M))| - n(S) + 1 = \\
&= |(M - M) \setminus (K - N)| - n(S) + 1 = |(M - M) \setminus C| - n(S) + 1 = \\
&= |(M - M) \setminus S| + |S \setminus C| - n(S) + 1 = t + n(S) - n(S) + 1 = t + 1.
\end{align*}
\]

By Proposition 2.4 an ideal \( I \) with \( F(I) = F(S) \) is almost canonical if and only if \( K - (M - M) \subseteq I \subseteq K \) and we notice that \(|K \setminus (K - (M - M))| = |(M - M) \setminus S| = t \). Let \( A \subseteq (K \setminus (K - (M - M))) \) and
consider \( I = (K - (M - M)) \cup A \). We claim that \( I \) is an ideal of \( S \). Indeed, let \( x \in A, m \in M \) and \( y \in (M - M) \). It follows that \( m + y \in M \) and, then, \( x + m + y \in K \), since \( K \) is an ideal. Therefore, \( x + m \in K - (M - M) \) and \( I \) is an ideal of \( S \). Moreover, by [6, Lemma 4.7], \( t(I) = |(K - I) \setminus S| + 1 = |K \setminus I| + 1 = t + 1 - |A| \) and the thesis follows, because there are \( \binom{t}{i-1} \) subsets of \( K \setminus (K - (M - M)) \) with cardinality \( t + 1 - i \). 

If \( S \) is a symmetric semigroup, the only almost canonical ideals with Frobenius number \( F(S) \) are \( M \) and \( S \). In this case \( t(M) = t(S) + 1 = 2 \). If \( S \) is pseudo-symmetric, the four almost canonical ideals with Frobenius number \( F(S) \) are \( M, S, M \cup \{F(S)/2\} \) and \( K \). In this case \( t(M) = 3, t(S) = t(M \cup \{F(S)/2\}) = 2 \) and \( t(K) = 1 \).

### 3. GAS numerical semigroups

In [5] (see also [11]) it is introduced the class of 2-almost Gorenstein local rings, briefly 2-AGL rings, as a class of rings that are close to be almost Gorenstein. Using an equivalent condition proved in [5, Theorem 1.4], given a canonical module \( \omega \) of a Cohen-Macaulay one-dimensional local ring \( R \) such that \( R \subseteq \omega \subseteq R \), we say that \( R \) is 2-AGL if \( \ell_R(R[\omega]/R) = 2 \), where \( \ell_R(\cdot) \) denotes the length as \( R \)-module. Note that \( R \) is Gorenstein if and only if \( \ell_R(R[\omega]/R) = 0 \) and it is almost Gorenstein exactly when \( \ell_R(R[\omega]/R) \leq 1 \).

Given a numerical semigroup \( S \) with standard canonical ideal \( K \) we denote by \( \langle K \rangle \) the numerical semigroup generated by \( K \). Generalizing the definition in the case of numerical semigroups, it is natural to say that \( S \) is \( n \)-AGL if \( |\langle K \rangle \setminus K| = n \). It follows that \( S \) is symmetric if and only if it is 0-AGL, whereas it is almost symmetric and not symmetric if and only if it is 1-AGL.

It is easy to see that a numerical semigroup is 2-AGL if and only if \( 2K = 3K \) and \( 2K \setminus K = 2 \), see [5, Theorem 1.4] for a proof in the general case. We now give another easy characterization which will lead us to generalize this class.

**Proposition 3.1.** A numerical semigroup \( S \) is 2-AGL if and only if \( 2K = 3K \) and \( 2K \setminus K = \{F(S) - x, F(S)\} \) for a minimal generator \( x \) of \( S \).

**Proof.** One implication is trivial, so assume that \( S \) is 2-AGL. Since \( S \) is not symmetric, there exists \( k \in \mathbb{N} \) such that \( k \) and \( F(S) - k \) are in \( K \) and so \( F(S) \in 2K \setminus K \). Let now \( a \in (2K \setminus K) \setminus \{F(S)\} \). Since \( a \notin K \), we have \( F(S) - a \in S \). Assume that \( F(S) - a = s_1 + s_2 \) with \( s_1, s_2 \in S \setminus \{0\} \). It follows that \( F(S) - s_1 = a + s_2 \in 2K \), since \( 2K \) is a relative ideal, and by definition \( F(S) - s_1 \notin K \).

Therefore, \( \{a, F(S) - s_1, F(S)\} \subseteq 2K \setminus K \) and this is a contradiction, since \( S \) is 2-AGL. Hence, \( a = F(S) - x \), where \( x \) is a minimal generator of \( S \). \( \square \)

In light of the previous proposition we propose the following definition.

**Definition 3.2.** We say that \( S \) is a \emph{generalized almost symmetric} numerical semigroup, briefly GAS numerical semigroup, if either \( 2K = K \) or \( 2K \setminus K = \{F(S) - x_1, ..., F(S) - x_r, F(S)\} \) for some \( r \geq 0 \) and some minimal generators \( x_1, ..., x_r \) of \( S \) such that \( x_i - x_j \notin PF(S) \) for every \( i, j \).

The last condition could seem less natural, but these semigroups have a better behavior. For instance, in Theorem 3.7 we will see that this condition ensures that every element in \( \langle K \rangle \setminus K \) can be written as \( F(S) - x \) for a minimal generator \( x \) of \( S \).

We recall that \( S \) is symmetric if and only if \( 2K = K \) and it is almost symmetric exactly when \( 2K \setminus K \subseteq \{F(S)\} \).

**Examples 3.3.** 1. Let \( S = \langle 9, 24, 39, 43, 77 \rangle \). Then, \( PF(S) = \{58, 73, 92, 107\} \) and \( 2K \setminus K = \{107 - 77, 107 - 43, 107 - 39, 107 - 24, 107 - 9, 107\} \). Hence, \( S \) is a GAS semigroup.
2. If \( S = \langle 7, 9, 15 \rangle \), we have \( 2K = 3K \) and \( 2K \setminus K = \{ 26 - 14, 26 - 7, 26 \} \). Hence, \( S \) is 3-AGL but it is not GAS because 14 is not a minimal generator of \( S \).

3. Consider the semigroup \( S = \langle 8, 11, 14, 15, 17, 18, 20, 21 \rangle \). We have \( 2K \setminus K = \{ 13 - 11, 13 - 8, 13 \} \), but \( S \) is not GAS because \( 11 - 8 \in \text{PF}(S) \). In this case \( 2K = 3K \) and thus \( S \) is 3-AGL.

The last example shows that in a numerical semigroup \( S \) with maximal embedding dimension there could be many minimal generators \( x \) such that \( F(S) - x \in 2K \setminus K \). This is not the case if we assume that \( S \) is GAS.

**Proposition 3.4.** If \( S \) has maximal embedding dimension \( e \) and it is GAS, then it is either almost symmetric or 2-AGL with \( 2K \setminus K = \{ F(S) - e, F(S) \} \).

**Proof.** Assume that \( S \) is not almost symmetric and let \( F(S) - x = k_1 + k_2 \in 2K \setminus K \) with \( x \neq 0 \) and \( k_1, k_2 \in K \). Let \( x \neq e \) and consider \( F(S) - e = k_1 + k_2 + x - e \). Since \( x - e \leq F(S) - e < F(S) \) and \( S \) has maximal embedding dimension, \( x - e \in \text{PF}(S) \setminus \{ F(S) \} \subseteq K \) and, therefore, \( F(S) - e \in 3K \setminus K \). Moreover, \( F(S) - e \) cannot be in \( 2K \), because \( S \) is GAS and \( x - e \in \text{PF}(S) \), then \( k_1 + x - e \in 2K \setminus K \). Hence, we have \( F(S) - (F(S) - k_1 - x + e) \in 2K \setminus K \) and, thus, \( F(S) - k_1 - x + e \) is a minimal generator of \( S \). Since \( S \) has maximal embedding dimension, this implies that \( F(S) - k_1 - x \in \text{PF}(S) \) and, then, \( F(S) - k_1 \in S \) yields a contradiction, since \( k_1 \in K \). This means that \( x = e \) and \( 2K \setminus K = \{ F(S) - e, F(S) \} \).

Suppose by contradiction that \( 2K \neq 3K \) and let \( F(S) - y \in 3K \setminus 2K \). In particular, \( F(S) - y \not\in K \) and, therefore, \( y \in S \). If \( F(S) - y = k_1 + k_2 + k_3 \) with \( k_i \in K \) for every \( i \), then \( k_1 + k_2 \in 2K \setminus K \) and, thus, \( k_1 + k_2 = F(S) - e \). This implies that \( F(S) - e < F(S) - y \), i.e. \( y < e \), which is a contradiction.

In particular, we note that in a 2-AGL semigroup with maximal embedding dimension it always holds that \( 2K \setminus K = \{ F(S) - e, F(S) \} \).

**Proposition 3.5.** Given a numerical semigroup \( S \), the following conditions are equivalent:

1. \( S \) is GAS;
2. \( x - y \not\in (M - M) \) for every different \( x, y \in M \setminus (S - K) \);
3. either \( S \) is symmetric or \( 2M \subseteq S - K \subseteq M \) and \( M - M = ((S - K) - M) \cup \{ 0 \} \).

**Proof.** If \( S \) is symmetric, then \( M \subseteq S - K \) and both (1) and (2) are true, so we assume \( S \neq K \).

(1) \( \Rightarrow \) (2) Note that \( K - S = K \) and \( K - (S - K) = K - ((K - K) - K) = K - (K - 2K) = 2K \). Thus, \( x \in S \setminus (S - K) \) if and only if \( F(S) - x \in (K - (S - K)) \setminus (K - S) = 2K \setminus K \). Hence, if \( S \) is GAS, then \( x - y \not\in S \cup \text{PF}(S) = M - M \) for every \( x, y \in M \setminus (S - K) \).

(2) \( \Rightarrow \) (1) If \( x, y \in M \setminus (S - K) \), then \( F(S) - x, F(S) - y \in 2K \setminus K \) and \( x - y \not\in \text{PF}(S) \), since it is not in \( M - M \). We only need to show that \( x \) is a minimal generator of \( S \). If by contradiction \( x = s_1 + s_2 \), with \( s_1, s_2 \in M \), it follows that also \( s_1 \) is in \( M \setminus (S - K) \). Therefore, \( s_2 = x - s_1 \in M \) yields a contradiction since \( x - s_1 \not\in M - M \) by hypothesis.

(2) \( \Rightarrow \) (3) Since \( S \) is not symmetric, \( S - K \) is contained in \( M \). Moreover, if \( 2M \) is not in \( S - K \), then there exist \( m_1, m_2 \in M \) such that \( m_1 + m_2 \in 2M \setminus (S - K) \). Clearly also \( m_1 \) is not in \( S - K \) and \( (m_1 + m_2) - m_1 = m_2 \in M \subseteq M - M \) yields a contradiction.

It always holds that \( ((S - K) - M) \cup \{ 0 \} \subseteq M - M \), then given \( x \in (M - M) \setminus \{ 0 \} \) and \( m \in M \), we only need to prove that \( x + m \in S - K \). If \( m \in M \setminus (S - K) \) and \( x + m \not\in S - K \), then \( (x + m) - m = x \in M - M \) gives a contradiction. If \( m \in (S - K) \setminus 2M \) and \( k \in K \), then \( 0 \neq m + k \in S \) and, so, \( x + m + k \in M \), which implies \( x + m \in S - K \). Finally, if \( m \in 2M \), then \( x + m \in 2M \subseteq S - K \).
(3) ⇒ (2) Let \( x, y \in M \setminus (S - K) \) with \( x \neq y \) and assume by contradiction that \( x - y \in (M - M) = ((S - K) - M) \cup \{0\} \). By hypothesis \( y \in M \), then \( x = (x - y) + y \in S - K \) yields a contradiction. □

In the definition of GAS semigroup we required that in \( 2K \setminus K \) there are only elements of the type \( F(S) - x \) with \( x \) minimal generator of \( S \). In general, this does not imply that the elements in \( 3K \setminus 2K \) are of the same type. For instance, consider \( S = \{8, 12, 17, 21, 26, 27, 30, 31\} \), where \( 2K \setminus K = \{23 - 21, 23 - 17, 23 - 12, 23 - 8, 23\} \) and \( 3K \setminus 2K = \{23 - 20, 23 - 16\} \). However, by Proposition 3.4, this semigroup is not GAS. In fact, this never happens in a GAS semigroup as we are going to show in Theorem 3.7. First we need a lemma.

**Lemma 3.6.** Assume that \( 2K \setminus K = \{F(S) - x_1, ..., F(S) - x_r, F(S)\} \) with \( x_1, ..., x_r \) minimal generators of \( S \). If \( F(S) - x \in nK \setminus (n - 1)K \) for some \( n > 2 \) and \( x = s_1 + s_2 \) with \( s_1, s_2 \in M \), then \( F(S) - s_1 \in (n - 1)K \).

**Proof.** Let \( F(S) - (s_1 + s_2) = k_1 + ... + k_n \in nK \setminus (n - 1)K \) with \( k_i \in K \) for \( 1 \leq i \leq n \). Since \( F(S) - (s_1 + s_2) \notin (n - 1)K \), we have \( F(S) \neq k_1 + k_2 \in 2K \setminus K \) and, then, \( F(S) - (k_1 + k_2) \) is a minimal generator of \( S \). Since \( F(S) - (k_1 + k_2) = s_1 + s_2 + k_3 + ... + k_n \), this implies that \( s_1 + k_3 + ... + k_n \notin S \). That is \( k_1 + k_2 + s_2 = F(S) - (s_1 + k_3 + ... + k_n) \in K \). Therefore, \( F(S) - s_1 = (k_1 + k_2 + s_2) + k_3 + ... + k_n \in (n - 1)K \) and the thesis follows. □

**Theorem 3.7.** Let \( S \) be a GAS numerical semigroup that is not symmetric. Then, \( \langle K \rangle \setminus K = \{F(S) - x_1, ..., F(S) - x_r, F(S)\} \) for some minimal generators \( x_1, ..., x_r \) with \( r \geq 0 \) and \( x_i - x_j \notin PF(S) \) for every \( i \) and \( j \).

**Proof.** We first prove that \( x_i - x_j \notin PF(S) \) for every \( i \) and \( j \) without assuming that \( x_i \) and \( x_j \) are minimal generators. We can suppose that \( x_i = x_1 \) and \( x_j = x_2 \).

Let \( F(S) - x_1 = k_1 + ... + k_n \in nK \setminus (n - 1)K \) with \( k_i \in K \) for every \( i \) and assume by contradiction that \( x_1 - x_2 \in PF(S) \). We note that \( F(S) - x_2 = k_1 + ... + k_n + (x_1 - x_2) \) and \( k_1 + (x_1 - x_2) \in K \). Indeed, if \( F(S) - k_1 - (x_1 - x_2) = s \in S \), then \( s \neq 0 \) and \( F(S) - k_1 = (x_1 - x_2) + s \in S \) yields a contradiction. If \( k_1 + k_2 + (x_1 - x_2) \notin K \), then it is in \( 2K \setminus K \) and, since also \( k_1 + k_2 \in 2K \setminus K \), we get a contradiction because their difference is a pseudo-Frobenius number. Hence, \( k_1 + k_2 + (x_1 - x_2) \in K \).

We proceed by induction on \( n \). If \( n = 2 \), it follows that \( F(S) - x_2 = k_1 + k_2 + (x_1 - x_2) \in K \), which is a contradiction. So, let \( n \geq 3 \) and let \( i \) be the minimum index for which \( k_1 + ... + k_i + (x_1 - x_2) \notin K \). It follows that \( k_1 + ... + k_i + (x_1 - x_2) \in 2K \setminus K \) and, since also \( k_1 + k_2 \in 2K \setminus K \), this implies that \( k_3 + ... + k_i + (x_1 - x_2) \notin PF(S) \). Moreover, it cannot be in \( S \), because it is the difference of two minimal generators, since \( S \) is GAS. Therefore, there exists \( m \in M \) such that \( k_3 + ... + k_i + (x_1 - x_2) + m \notin S \), which means \( F(S) - (k_3 + ... + k_i + (x_1 - x_2) + m) = k' \in K \). Thus, \( F(S) - ((x_1 - x_2) + m) = k' + k_3 + ... + k_i \in jK \setminus K \) for some \( 1 < j < n \). Moreover, \( F(S) - m = k' + k_3 + ... + k_i + (x_1 - x_2) \notin K \) and by induction \( (x_1 - x_2) + m \notin PF(S) \), which is a contradiction. Hence, \( x_1 - x_2 \notin PF(S) \).

Let now \( h \geq 3 \). To prove the theorem it is enough to show that, if \( F(S) - x \in hK \setminus (h - 1)K \), then \( x \) is a minimal generators of \( S \). We proceed by induction on \( h \). Using the GAS hypothesis, the case \( h = 3 \) is very similar to the general case, so we omit it (the difference is that also \( F(S) \in 2K \setminus K \)). Suppose by contradiction that \( x = s_1 + s_2 \) and \( F(S) - (s_1 + s_2) = k_1 + ... + k_n \in hK \setminus (h - 1)K \) with \( k_1, ..., k_n \in K \) and \( s_1, s_2 \in M \). Clearly, \( F(S) - s_1 \notin K \) and by Lemma 3.6 we have \( F(S) - s_1 \in (h - 1)K \); in particular, \( s_1 \) is a minimal generator of \( S \) by induction. Let \( 1 < i < h \) be such that \( F(S) - s_1 \in iK \setminus (i - 1)K \). Since \( F(S) - (s_1 + s_2) \notin (h - 1)K \), we have \( k_1 + ... + k_i \in iK \setminus (i - 1)K \) and, by induction, \( F(S) - (k_1 + ... + k_i) \) is a minimal generator of \( S \) and \( F(S) - (k_1 + ... + k_i - s_1) \notin PF(S) \) by the first part of the proof. This means that there exists \( s \in M \) such
For every $f (2)$, for every $i$, there exist $f_j (1)$ such that $F = (k_1 + \ldots + k_i) - s_i + s \not\in S$, i.e. $k_1 + \ldots + k_i + s_1 - s \in K$. This implies that $F(S) - (s_2 + s) = (k_1 + \ldots + k_i + s_1 - s) + k_{i+1} + \ldots + k_h$ $(h - i + 1)K$ and, since $h - i + 1 < h$, the induction hypothesis yields a contradiction because $s_2 + s$ is not a minimal generator of $S$. □

We recall that in an almost symmetric numerical semigroup $F(S) - f \in PF(S)$ for every $f \in PF(S) \setminus \{F(S)\}$, see [19, Theorem 2.4]. The following proposition generalizes this fact.

**Proposition 3.8.** Let $S$ be a numerical semigroup with $2K \setminus K = \{F(S) - x_1, \ldots, F(S) - x_r, F(S)\}$, where $x_i$ is a minimal generator of $S$ for every $i$.

1. For every $i$, there exist $f_j, f_k \in PF(S)$ such that $f_j + f_k = F(S) + x_i$.
2. For every $f \in PF(S) \setminus \{F(S)\}$, it holds either $F(S) - f \in PF(S)$ or $F(S) - f + x_i \in PF(S)$ for some $i$.

**Proof.** Let $F(S) - x_i = k_1 + k_2 \in 2K \setminus K$ for some $k_1, k_2 \in K$ and let $s \in M$. Since $x_i + s \in S$, we have $F(S) - x_i - s \not\in K$ and then $F(S) - x_i - s = k_1 + k_2 - s \not\in 2K$ because $x_i + s$ is not a generator of $S$. In particular, $k_1 - s$ and $k_2 - s$ are not in $K$. This means that $F(S) - k_1 + s$ and $F(S) - k_2 + s$ are in $S$ and, thus, $F(S) - k_1, F(S) - k_2 \in PF(S)$. Moreover, $F(S) - k_1 + F(S) - k_2 = 2F(S) - (F(S) - x_i) = F(S) + x_i$ and (1) holds.

Let now $f \in PF(S) \setminus \{F(S)\}$ and assume that $F(S) - f \not\in PF(S)$. Then, there exists $s \in M$ such that $F(S) - f + s \in PF(S)$. In particular, $f - s \in K$ and $F(S) - s = (F(S) - f) + (f - s) \in 2K \setminus K$; thus, $s$ has to be equal to $x_i$ for some $i$ and $F(S) - f + x_i \in PF(S)$. □

**Examples 3.9.** 1. Let $S = \langle 28, 40, 63, 79, 88 \rangle$. We have $2K \setminus K = \{281 - 28, 281\}$ and $S$ is $2$-$AGL$. In this case $PF(S) = \{100, 132, 177, 209, 281\}$ and $100 + 209 = 132 + 177 = 281 + 28$.
2. Consider $S = \langle 67, 69, 76, 78, 86 \rangle$. Here $2K \setminus K = \{485 - 86, 485\}$ and the semigroup is $2$-$AGL$. Moreover, $PF(S) = \{218, 226, 249, 259, 267, 322, 485\}$, $218 + 267 = 226 + 259 = 485$ and $249 + 322 = 485 + 86$.
3. If $S = \langle 9, 10, 12, 13 \rangle$, then $2K \setminus K = \{17 - 13, 17 - 12, 17 - 10, 17 - 9, 17\}$ and $PF(S) = \{11, 14, 15, 16, 17\}$. Hence, $S$ is GAS and, according to the previous proposition, we have

\[
F(S) + 9 = 11 + 15 F(S) + 12 = 14 + 15
\]

\[
F(S) + 10 = 11 + 16 F(S) + 13 = 14 + 16.
\]

4. Conditions (1) and (2) in Proposition 3.8 do not imply that every $x_i$ is a minimal generator. For instance, if we consider the numerical semigroup $S = \{15, 16, 19, 20, 24\}$, we have $2K \setminus K = \{42 - 40, 42 - 36, 42 - 32, 42 - 24, 42 - 20, 42 - 19, 42 - 16, 42 - 15, 42\}$ and $PF(S) = \{28, 29, 33, 37, 41, 42\}$. Moreover,

\[
F(S) + 40 = 41 + 41 F(S) + 20 = 29 + 33
\]

\[
F(S) + 36 = 37 + 41 F(S) + 19 = 28 + 33
\]

\[
F(S) + 32 = 37 + 37 F(S) + 16 = 29 + 29
\]

\[
F(S) + 24 = 33 + 33 F(S) + 15 = 28 + 29
\]

and, so, it is straightforward to see that the conditions in Proposition 3.8 hold, but 32, 36 and 40 are not minimal generators.

We recall that $L(S)$ denotes the set of the gaps of the second type of $S$, i.e. the integers $x$ such that $x \not\in S$ and $F(S) - x \not\in S$, i.e. $x \in K \setminus S$, and that $S$ is almost symmetric if and only if $L(S) \subseteq PF(S)$, see [3].
Lemma 3.10. Let $S$ be a numerical semigroup with $2K \setminus K = \{F(S) - x_1, ..., F(S) - x_r, F(S)\}$, where $x_i$ is a minimal generator of $S$ for every $i$. If $x \in L(S)$ and $F(S) - x \not\in PF(S)$, then both $x$ and $F(S) - x + x_i$ are pseudo-Frobenius numbers of $S$ for some $i$.

Proof. Assume by contradiction that $x \not\in PF(S)$. Therefore, there exists $s \in M$ such that $x + s \not\in S$ and, then, $F(S) - x - s \in K$. Moreover, since $F(S) - x \not\in PF(S)$, there exists $t \in M$ such that $F(S) - x + t \not\in S$ and then $x - t \in K$. Consequently, $F(S) - s - t = (F(S) - x - s) + (x - t) \in 2K$ and $F(S) - s - t \not\in K$, since $s + t \in S$. This is a contradiction, because $s + t$ is not a minimal generator of $S$. Hence, $x \in PF(S)$ and, since $F(S) - x \not\in PF(S)$, Proposition 3.8 implies that $F(S) - x + x_i \in PF(S)$ for some $i$.

Lemma 3.11. As ideal of $M - M$, it holds $\overline{M - e} = M - e$ and

$$K(M - M) \setminus (M - e) = \{x - e \mid x \in L(S) \text{ and } F(S) - x \not\in PF(S)\}.$$ 

Proof. We notice that $F(S) - e \not\in (M - M)$ and, if $y > F(S) - e$ and $m \in M$, we have $y + m > F(S) - e + m \geq F(S)$. Therefore, $F(M - M) = F(S) - e = F(M - e)$ and, then, $M - e = M - e$.

We have $x - e \in K(M - M) \setminus (M - e)$ if and only if $x \not\in M$ and $(F(S) - e) - (x - e) \not\in (M - M)$ which is in turn equivalent to $x \not\in M$ and $F(S) - x \not\in S \cup PF(S)$. Since $x \not= 0$, this means that $x \in L(S)$ and $F(S) - x \not\in PF(S)$.

The following corollary was proved in [1, Theorem 5.2] in a different way.

Corollary 3.12. $S$ is almost symmetric if and only if $M - e$ is a canonical ideal of $M - M$.

Proof. By definition $M - e$ is a canonical ideal of $M - M$ if and only if $K(M - M) = (M - e)$. In light of the previous lemma, this means that there are no $x \in L(S)$ such that $F(S) - x \not\in PF(S)$, which is equivalent to say that $L(S) \subseteq PF(S)$, i.e. $S$ is almost symmetric.

In [3, Corollary 8] it was first proved that $S$ is almost symmetric with maximal embedding dimension if and only if $M - M$ is a symmetric semigroup. In general it holds $M - M \subseteq M - e \subseteq K(M - M)$ and the first inclusion is an equality if and only if $S$ has maximal embedding dimension, whereas the previous corollary says that the second one is an equality if and only if $S$ is almost symmetric. Moreover, if $S$ has maximal embedding dimension, in [5, Corollary 5.4] it is proved that $S$ is 2-AGL if and only if $M - M$ is an almost symmetric semigroup which is not symmetric. If we want to generalize this result in the same spirit of Corollary 3.12, it is not enough to consider the 2-AGL semigroups, but we need that $S$ is GAS. More precisely, we have the following result.

Theorem 3.13. The semigroup $S$ is GAS if and only if $M - e$ is an almost canonical ideal of the semigroup $M - M$.

Proof. In the light of Remark 2.2.4 and Lemma 3.11, $M - e$ is an almost canonical ideal of $M - M$ if and only if

$$K(M - M) \setminus (M - e) \subseteq ((M - e) - ((M - M) \setminus \{0\})).$$

(1)

Assume that $S$ is GAS with $2K \setminus K = \{F(S) - x_1, ..., F(S) - x_r, F(S)\}$. By Lemma 3.11 the elements of $K(M - M) \setminus (M - e)$ can be written as $x - e$ with $x \in L(S)$ and $F(S) - x \not\in PF(S)$. In addition, Lemma 3.10 implies that both $x$ and $F(S) - x + x_i$ are pseudo-Frobenius numbers of $S$ for some $i$. Let $0 \not= z \in (M - M)$. We need to show that $x - e + z \in M - e$, i.e. $x + z \in M$. Assume by contradiction $x + z \not\in M$, which implies $F(S) - x - z \not\in K$. Since $x + z \not\in M$ and $x \in PF(S)$, it follows that $z \not\in M$ and, then, $z \in PF(S)$; hence, $x + x_i \in M$ and $F(S) - z - x_i \not\in K$. We also have that $x - x_i \in K$, since $F(S) - x + x_i \in PF(S)$. Therefore,
and this yields a contradiction because \((z + x_i) - x_i \in \text{PF}(S)\) and \(S\) is a GAS semigroup.

Conversely, assume that the inclusion (1) holds. An element in \(2K \setminus K\) can be written as \(F(S) - s\) for some \(s \in S\), since it is not in \(K\). Assume by contradiction that \(s \neq 0\) is not a minimal generator of \(S\), i.e. \(F(S) - s_1 - s_2 = k_1 + k_2 \in 2K \setminus K\) for some \(s_1, s_2 \in M\) and \(k_1, k_2 \in K\). It follows that \(F(S) - k_1 - s_1 = k_2 + s_2 \notin S\), otherwise \(F(S) - s_1 \in K\). Moreover, \(k_1 + s_1 \notin \text{PF}(S) \cup S\), since \(k_1 + s_1 + s_2 = F(S) - k_2 \notin S\). Hence, Lemma 3.11 and our hypothesis imply that

\[
k_2 + s_2 - e = F(S) - k_1 - s_1 - e \in ((M - e) - ((M - M) \setminus \{0\})).
\]

Therefore, \(F(S) - k_1 - e = (k_2 + s_2 - e) + s_1 \in M - e\) and, thus, \(k_1 \notin K\) yields a contradiction. This means that \(2K \setminus K = \{F(S) - x_i, \ldots, F(S) - x_r, F(S)\}\) with \(x_i\) minimal generator of \(S\) for every \(i\). Now, assume by contradiction that \(z = x_i - x_j \in \text{PF}(S)\) for some \(i, j\) and let \(F(S) - x_i = F(S) - x_j - z = k_1 + k_2\) for some \(k_1, k_2 \in K\). Since \(k_2 + z + x_j = F(S) - k_1 \notin S\), it follows that \(k_2 + z \notin S \cup \text{PF}(S)\). Moreover, \(F(S) - k_2 - z \notin S\), otherwise \(F(S) - k_2 \in S\). Therefore, Lemma 3.11 and inclusion (1) imply that \(F(S) - k_2 - z - e \in ((M - e) - ((M - M) \setminus \{0\}))\) and, since \(z \in M - M\), it follows that \(F(S) - k_2 \in M\) which is a contradiction because \(k_2 \in K\).

**Example 3.14.** Consider \(S = \{9, 13, 14, 15, 19\}\), which is a GAS numerical semigroup with \(2K \setminus K = \{25 - 15, 25 - 13, 25 - 9, 25\}\). Then, \(M - 9\) is an almost canonical ideal of \(M - M\) by the previous theorem. In fact \(M - M = \{0, 9, 13, 14, 15, 17, \rightarrow\}\),

\[
K(M - M) = \{0, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 17 \rightarrow\},
\]

\[
M - 9 = \{0, 4, 5, 6, 9, 10, 13, 14, 15, 17 \rightarrow\},
\]

\[
(M - 9) \setminus ((M - M) \setminus \{0\}) = K(M - M) \cup \{16\} = \{0, 4, 5, 6, 8 \rightarrow\}.
\]

**Remark 3.15.** If \(S\) is GAS, it is possible to compute the type of \(M - e\) seen as an ideal of the semigroup \(M - M\). In fact by Theorem 3.13 and Proposition 2.4 it follows that

\[
t(M - e) = g(M - e) + g(M - M) - F(M - M) = g(M) - e + g(S) - t(S) - F(S) + e = 2g(S) + 1 - t(S) - F(S).
\]

Moreover, we recall that \(2g(S) \geq t(S) + F(S)\) is always true and the equality holds exactly when \(S\) is almost symmetric. Therefore, as \(t(S)\) is a measure of how far \(S\) is from being symmetric, \(t(M - e) = t(M)\) (as ideal of \(M - M\)) can be seen as a measure of how far \(S\) is from being almost symmetric. On the other hand, we note that the type of \(M\) as an ideal of \(S\) is simply \(t(S) + 1\).

If \(S\) has type 2 and \(\text{PF}(S) = \{f, F(S)\}\), in [5, Theorem 6.2] it is proved that \(S\) is 2-AGL if and only if \(3(F(S) - f) \in S\) and \(F(S) = 2f - x\) for some minimal generator \(x\) of \(S\). In the next proposition we generalize this result to the GAS case.

**Proposition 3.16.** Assume that \(S\) is not almost symmetric and that it has type 2, i.e. \(\text{PF}(S) = \{f, F(S)\}\). Then, \(S\) is GAS if and only if \(F(S) = 2f - x\) for some minimal generator \(x\) of \(S\). In this case, if \(n\) is the minimum integer for which \(n(F(S) - f) \in S\), then \(|2K \setminus K| = 2, |3K \setminus 2K| = \ldots = |(n - 1)K \setminus (n - 2)K| = 1\) and \(nK = (n - 1)K\).

**Proof.** Assume first that \(S\) is GAS and let \(F(S) - x, F(S) - y \in 2K \setminus K\). Proposition 3.8 implies that \(F(S) + x = f_1 + f_2\) and \(F(S) + y = f_3 + f_4\) for some \(f_1, f_2, f_3, f_4 \in \text{PF}(S)\). Since \(f_i\) has to be different from \(F(S)\) for all \(i\), it follows that \(F(S) + x = F(S) + y = 2f\) and, then, \(x = y\). In particular, \(F(S) = 2f - x\).
Assume now that $F(S) = 2f - x$ for some minimal generator $x$ of $S$. Clearly, $F(S) - x = 2(F(S) - f) \in 2K \setminus K$. Let $y \neq 0, x$ be such that $F(S) - y \in 2K \setminus K$. Since $2K \setminus K$ is finite, we may assume that $y$ is maximal among such elements with respect to $\leq_S$, that is $F(S) - (y + m) \notin 2K \setminus K$ for every $m \in M$. Let $F(S) - y = k_1 + k_2$ with $k_1, k_2 \in K$. Since $F(S) - y - m = k_1 + k_2 - m \notin 2K \setminus K$, then $k_1 - m$ and $k_2 - m$ are not in $K$, which is equivalent to $F(S) - k_1 + m \in S$ and $F(S) - k_2 + m \in S$ for every $m \in M$. This means that $F(S) - k_1, F(S) - k_2 \in PF(S) \setminus \{F(S)\}$ which implies $F(S) - y = 2(F(S) - f) = F(S) - x$ and, thus, $x = y$. Therefore, $[2K \setminus K] = 2$ and $S$ is GAS.

Moreover, if $S$ is GAS and $F(S) - y = k_1 + ... + k_r \in rK \setminus (r - 1)K$ with $r \geq 2$ and $k_1, ..., k_r \in K$, then $k_1 = ... = k_r = F(S) - f$ because $k_1 + k_j \in 2K \setminus K$ for every $i$ and $j$. Therefore, if $n(F(S) - f) \in S$, then $nK = (n - 1)K$. Assume that $r(F(S) - f) \notin S$. Clearly, it is in $rK$ and we claim that it is not in $K$. In fact, if $r(F(S) - f) \subset K$, it follows that it is in $L(S)$ and, if $F(S) - r(F(S) - f) = f$, then $(r - 1)(F(S) - f) = 0 \in S$ yields a contradiction. Therefore, Lemma 3.10 implies that $F(S) - r(F(S) - f) + x = f$ and, again, $(r - 1)(F(S) - f) = x \in S$ gives a contradiction. This means that $r(F(S) - f) \notin rK \setminus K$. Moreover, if $r(F(S) - f) = k_1 + ... + k_r \in rK \setminus (r' - 1)K$ with $1 < r' < r$ and $k_1, ..., k_r \in K$, we get $k_1 = ... = k_r = F(S) - f$ as above, which is a contradiction. Hence, $|rK \setminus (r - 1)K| = 1$ for every $1 < r < n$. 

Example 3.17. Consider $S = \langle 5, 6, 7 \rangle$. In this case $f = 8$ and $F(S) = 9$. Therefore, the equality $F(S) = 2f - 7$ implies that $S$ is GAS. With the notation of the previous corollary we have $n = 5$ and, in fact, $2K \setminus K = \{2, 9\}, 3K \setminus 2K = \{3\}$ and $4K \setminus 3K = \{4\}$.

In [15] another generalization of almost Gorenstein ring is introduced. More precisely a Cohen-Macaulay local ring admitting a canonical module $\omega$ is said to be nearly Gorenstein if the trace of $\omega$ contains the maximal ideal. In the case of numerical semigroups it follows from [15, Lemma 1.1] that $S$ is nearly Gorenstein if and only if $M \subset K + (S - K)$, see also the arXiv version of [15]. It is easy to see that an almost symmetric semigroup is nearly Gorenstein, but in [5] it is noted that a 2-AGL semigroup is never nearly Gorenstein (see also [4, Remark 3.7] for an easy proof in the numerical semigroup case). This does not happen for GAS semigroups.

Corollary 3.18. Let $S$ be a GAS semigroup, not almost symmetric, with $PF(S) = \{f, F(S)\}$. It is nearly Gorenstein if and only if $3f - 2F(S) \in S$.

Proof. We will use the following characterization proved in [18]: $S$ is nearly Gorenstein if and only if for every minimal generator $y$ of $S$ there exists $g \in PF(S)$ such that $g + y - g' \in S$ for every $g' \in PF(S) \setminus \{g\}$.

By Proposition 3.16 it follows that $F(S) = 2f - x$ with $x$ minimal generator of $S$. Let $y \neq x$ another minimal generator of $S$ and assume by contradiction that $F(S) + y - f \notin S$. Therefore, there exists $s \in S$ such that $F(S) + y - f + s \in PF(S)$. If it is equal to $F(S)$, then $f = y + s \in S$ yields a contradiction. If $F(S) + y - f + s = f$, then $y + s = 2f - F(S) = x$ by Proposition 3.16 and this gives a contradiction, since $x \neq y$ is a minimal generator of $S$. Hence, $F(S) + y - f \in S$ for every minimal generator $y \neq x$. On the other hand, $F(S) + x - f = 2f - x - x - f = f \notin S$ and, therefore, $S$ is nearly Gorenstein if and only if $f + x - F(S) = 3f - 2F(S) \in S$. 

Examples 3.19. 1. In Example 3.17 we have $3f - 2F(S) = 6 \in S$ and, then, the semigroup is both GAS and nearly Gorenstein.

2. Consider $S = \langle 9, 17, 67 \rangle$ that has $PF(S) = \{59, 109\}$. Since $2 \times 59 - 109 = 9$ and $3 \times 59 - 2 \times 109 = -41 \notin S$, the semigroup is GAS but not nearly Gorenstein.

3. If $S = \langle 10, 11, 12, 25 \rangle$, we have $PF(S) = \{38, 39\}$ and $2 \times 38 - 39 = 37$ is not a minimal generator, thus, $S$ is not GAS. On the other hand, it is straightforward to check that this semigroup is nearly Gorenstein.
Remark 3.20. In literature there are other two generalizations of almost Gorenstein ring. One is given by the so-called ring with canonical reduction, introduced in [20], which is a one-dimensional Cohen-Macaulay local ring \((R, \mathfrak{m})\) possessing a canonical ideal \(I\) that is a reduction of \(\mathfrak{m}\). When \(R = k[[S]]\) is a numerical semigroup ring, this definition gives a generalization of almost symmetric semigroup and \(R\) has a canonical reduction if and only if \(e + F(S) - g \in S\) for every \(g \in \mathbb{N} \setminus S\), see [20, Theorem 3.13]. This notion is unrelated with the one of GAS semigroup, in fact it is easy to see that \(S = \langle 4, 7, 9, 10 \rangle\) is GAS and it doesn’t have canonical reductions, while \(S = \langle 8, 9, 10, 22 \rangle\) is not GAS, but has a canonical reduction.

Another generalization of the notion of almost Gorenstein ring is given by the so-called generalized Gorenstein ring, briefly GGL, introduced in [10, 12]. A Cohen-Macaulay local ring \((R, \mathfrak{m})\) with a canonical module \(\omega\) is said to be GGL with respect to \(a\) if either \(R\) is Gorenstein or there exists an exact sequence of \(R\)-modules

\[ 0 \to R^\oplus \omega \to C \to 0 \]

where \(C\) is an Ulrich module of \(R\) with respect to some \(\mathfrak{m}\)-primary ideal \(a\) and \(\varphi \otimes R/a\) is injective. We note that \(R\) is almost Gorenstein and not Gorenstein if and only if it is GGL with respect to \(a\). Let \(S\) be a numerical semigroup and order \(PF(S) = \{f_1, f_2, \ldots, f_i = F(S)\}\) by the usual order in \(\mathbb{N}\). Defining a numerical semigroup GGL if its associated ring is GGL, in [23] it is proved a useful characterization: \(S\) is GGL if either it is symmetric or the following properties hold:

1. there exists \(x \in S\) such that \(f_i + f_{i-1} = F(S) + x\) for every \(i = 1, \ldots, \lceil t/2 \rceil\);
2. \((c - M) \cap S \setminus c = \{x\}\), where \(c = S \setminus \langle K \rangle\).

Using this characterization it is not difficult to see that also this notion is unrelated with the one of GAS semigroup. In fact, the semigroups in Examples 3.9.2 and 3.9.3 are GAS but do not satisfy (1), whereas the semigroup \(S = \langle 5, 9, 12 \rangle\) is not GAS by Proposition 3.16, because \(PF(S) = \{13, 16\}\), but it is easy to see that it is GGL with \(x = 10\).

4. Constructing GAS numerical semigroups

In this section we study the behavior of the GAS property with respect to some constructions. In this way we will be able to construct many numerical semigroups satisfying this property.

4.1. Gluing of numerical semigroups

Let \(S_1 = \langle s_1, \ldots, s_n \rangle\) and \(S_2 = \langle t_1, \ldots, t_m \rangle\) be two numerical semigroups and assume that \(s_1, s_2, \ldots, s_n\) and \(t_1, \ldots, t_m\) are minimal generators of \(S_1\) and \(S_2\) respectively. Let also \(a \in S_2\) and \(b \in S_1\) be not minimal generators of \(S_2\) and \(S_1\) respectively and assume \(\gcd(a, b) = 1\). The numerical semigroup \(\langle aS_1, bS_2 \rangle = \langle as_1, \ldots, as_n, bt_1, \ldots, bt_m \rangle\) is said to be the gluing of \(S_1\) and \(S_2\) with respect to \(a\) and \(b\). It is well known that \(as_1, \ldots, as_n, bt_1, \ldots, bt_m\) are its minimal generators, see [21, Lemma 9.8]. Moreover, the pseudo-Frobenius numbers of \(T = \langle aS_1, bS_2 \rangle\) are

\[ PF(T) = \{af_1 + bf_2 + ab \mid f_1 \in PF(S_1), f_2 \in PF(S_2)\}\]

see [19, Proposition 6.6]. In particular, \(t(T) = t(S_1)t(S_2)\) and \(F(T) = aF(S_1) + bF(S_2) + ab\). Consequently, since \(K(T)\) is generated by the elements \(F(T) - f\) with \(f \in PF(T)\), it is easy to see that \(K(T) = \{ak_1 + bk_2 \mid k_1 \in K(S_1), k_2 \in K(S_2)\}\).

Since \(t(T) = t(S_1)t(S_2)\), it follows that \(T\) is symmetric if and only if both \(S_1\) and \(S_2\) are symmetric, so in the next theorem we exclude this case.
Theorem 4.1. Let $T$ be a gluing of two numerical semigroups and assume that $T$ is not symmetric. The following are equivalent:

1. $T$ is GAS;
2. $T$ is 2-AGL;
3. $T = \langle 2s, b\mathbb{N} \rangle$ with $b \in S$ odd and $S$ is an almost symmetric semigroup, but not symmetric.

Proof. (2) $\Rightarrow$ (1) True by definition.

(1) $\Rightarrow$ (3) Let $T = \langle aS_1, bS_2 \rangle$. Since $T$ is not symmetric, we can assume that $S_1$ is not symmetric and, then, $F(S_1) = k_1 + k_2$ for some $k_1, k_2 \in K(S_1)$. This implies that

$$F(T) - b(F(S_2) + a) = aF(S_1) + bF(S_2) + ab - bF(S_2) - ab = ak_1 + ak_2 \in 2K(T) \setminus K(T)$$

because $F(S_2) + a \in S_2$. Therefore, since $T$ is GAS, $F(S_2) + a$ is a minimal generator of $S_2$. By definition of gluing, $a$ is not a minimal generator of $S_2$, so write $a = s + s'$ with $s, s' \in M(S_2)$. Since $F(S_2) + s + s'$ is a minimal generator of $S_2$, we get $F(S_2) + s = F(S_2) + s' = 0$, i.e. $F(S_2) = -1$ and $a = s + s' = 2$. This proves that $T = \langle 2S_1, b\mathbb{N} \rangle$. Clearly, $b$ is odd by definition of gluing, so we only need to prove that $S_1$ is almost symmetric. Assume by contradiction that it is not almost symmetric and let $s \in M(S_1)$ such that $F(S_1) - s = k_1 + k_2 \in 2K(S_1) \setminus K(S_1)$ with $k_1, k_2 \in K(S_1)$. Then

$$F(T) - (2s + b) = 2F(S_1) - b + 2b - 2s - b = 2k_1 + 2k_2 \in 2K(T) \setminus K(T)$$

and $2s + b$ is not a minimal generator of $T$, contradiction.

(3) $\Rightarrow$ (2) Since $S$ is not symmetric, $\langle K(S) \rangle \setminus K(S) = 2K(S) \setminus K(S) = \{F(S)\}$. Consider an element $z \in \langle K(T) \rangle \setminus K(T)$, that is $z = 2k_1 + \lambda_1 + \ldots + 2k_r + \lambda_r = 2(k_1 + \ldots + k_r) + b(\lambda_1 + \ldots + \lambda_r)$ for some $k_1, \ldots, k_r \in K(S)$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{N}$. Since $z \notin K(T)$, then $k_1 + \ldots + k_r \notin K(S)$ and so $k_1 + \ldots + k_r = F(S)$. Therefore, $z = 2F(S) + b(\lambda_1 + \ldots + \lambda_r) \in 2K(T) \setminus K(T)$ and, since it is not in $K(T)$ and $F(T) = 2F(S) + b$, it follows that either $z = 2F(S)$ or $z = 2F(S) + b$. Hence, $|\langle K(T) \rangle \setminus K(T)| = 2$ and thus $T$ is 2-AGL. 

\[ \square \]

4.2. Numerical duplication

In the previous subsection we have shown that if a non-symmetric GAS semigroup is a gluing, then it can be written as $\langle 2S, b\mathbb{N} \rangle$. This kind of gluing can be seen as a particular case of another construction, the numerical duplication, introduced in [6].

Given a numerical semigroup $S$, a relative ideal $I$ of $S$ and an odd integer $b \in S$, the numerical duplication of $S$ with respect to $I$ and $b$ is defined as $S^{\bowtie}bI = 2 \cdot S \cup \{2 \cdot I + b\}$, where $2 \cdot X = \{2x | x \in X\}$ for every set $X$. This is a numerical semigroup if and only if $I + I + b \subseteq S$. This is always true if $I$ is an ideal of $S$ and, since in the rest of the subsection $I$ will always be ideal, we ignore this condition. In this case, if $S$ and $I$ are minimally generated by $\{s_1, \ldots, s_l\}$ and $\{i_1, \ldots, i_m\}$ respectively, then $S^{\bowtie}bI = \langle 2s_1, \ldots, 2s_l, 2i_1 + b, \ldots, 2i_m + b \rangle$ and these generators are minimal. It follows that $\langle 2S, b\mathbb{N} \rangle = S^{\bowtie}bS$.

Remark 4.2. The Frobenius number of $S^{\bowtie}bI$ is equal to $2F(I) + b$. Moreover, the odd pseudo-Frobenius numbers of $S^{\bowtie}bI$ are $\{2j + b | j \in PF(I)\}$, whereas the even elements in $PF(S^{\bowtie}bI)$ are exactly the doubles of the elements in $((M - M) \cap (I - I)) \setminus S$; see the proof of [6, Proposition 3.5]. In particular, if $2f \in PF(S^{\bowtie}bI)$, then $f \in PF(S)$.

In this subsection we write $K$ in place of $K(S)$. We note that $S - \langle K \rangle \subseteq S$ and $F(S - \langle K \rangle) = F(S)$.

Lemma 4.3. Let $S$ be a numerical semigroup, $b \in S$ be an odd integer, $I$ be an ideal of $S$ with $F(I) = F(S)$ and $T = S^{\bowtie}bI$. The following hold:
(1) If \( k \in K \), then both \( 2k \) and \( 2k + b \) are in \( K(T) \). In particular, if \( F(S) - x \in iK \setminus K \),
then \( F(T) - 2x \in iK(T) \setminus K(T) \).

(2) Let \( k \in K(T) \). If \( k \) is odd, then \( \frac{k-b}{2} \in K \), otherwise \( F(S) - \frac{k}{2} \notin I \).

(3) If \( I = S - \langle K \rangle \) and \( k \in K(T) \) is even, then \( \frac{k}{2} \in jK \) for some \( j \geq 1 \).

(4) Let \( I = S - \langle K \rangle \). If \( F(T) - 2i - b \in \langle K(T) \rangle \setminus K(T) \), then \( F(S) - i \in \langle K \rangle \setminus K \) for every \( i \in I \).
Moreover, \( F(S) - x \notin \langle K \rangle \setminus K \) if and only if \( F(T) - 2x \notin \langle K(T) \rangle \setminus K(T) \).

**Proof.** (1) If \( k \in K \), then \( 2k + b \in K(T) \), since \( F(T) - (2k + b) = 2(F(S) - k) \notin 2 \cdot S \). Moreover, \( F(T) - 2k = 2(F(S) - k) + b \) and \( F(S) - k \notin I \) because it is not in \( S \), so \( 2k \in K(T) \). Therefore, if \( F(S) - x = k_1 + \ldots + k_i \in iK \setminus K \), with \( k_1, \ldots, k_i \in K \), then \( F(T) - 2x = 2k_1 + \ldots + 2k_{i-1} + (2k_i + b) \in iK(T) \) and, clearly, it is not in \( K(T) \), since \( 2x \in T \).

(2) Let \( k \) be odd. Since \( 2(F(S) - \frac{k-b}{2}) = 2(F(S) + b - k = F(T) - k \notin T \), it follows that \( F(S) - \frac{k-b}{2} \notin S \), i.e. \( \frac{k-b}{2} \in K \). If \( k \) is even, then \( 2(F(S) - \frac{k}{2}) + b = F(T) - k \notin T \) and, thus, \( F(S) - \frac{k}{2} \notin I \).

(3) Since \( F(S) - \frac{k}{2} \notin S - \langle K \rangle \) by (2), there exist \( i \geq 1 \) and \( a \in iK \) such that \( F(S) - \frac{k}{2} + a \notin S \), that is \( \frac{k}{2} - a \in K \). Hence, \( \frac{k}{2} = a + \left( \frac{k}{2} - a \right) \in (i+1)K \).

(4) If \( F(T) - 2i - b = k_1 + \ldots + k_j + \ldots + k_n \in \langle K(T) \rangle \setminus K(T) \) with \( k_1, \ldots, k_j \in K(T) \) even and \( k_{j+1}, \ldots, k_n \in K(T) \) odd, then \( F(S) - i = \frac{k_1}{2} + \ldots + \frac{k_j}{2} + \frac{k_{j+1} - b}{2} + \ldots + \frac{k_n - b}{2} + \frac{(n-j)b}{2} \in \langle K \rangle \) by (2) and (3). Using (1) the other statement is analogous. \( \square \)

**Example 4.4.** 1. In the previous lemma we cannot remove the hypothesis \( F(I) = F(S) \). For instance, consider \( S = \{3, 10, 11\} \), \( I = \{3, 10\} \) and \( T = S \gg b^2 I \). Then, \( F(I) = 11 \neq 8 = F(S) \) and we have \( F(S) - 6 \notin 2K \setminus K \), but \( F(T) - 12 \notin \langle K(T) \rangle \).

2. In the third statement of the previous lemma, \( j \) may be bigger than 1. For instance, consider \( S = \{6, 28, 47, 97\} \) and \( T = S \gg b^4 T \langle S - \langle K \rangle \rangle = \{12, 56, 71, 94, 115, 153, 159, 194, 197, 241\} \). Then \( 88, 126, 170, 182 \in K(T) \), while \( 44, 63, 91 \in 2K \setminus K \) and \( 85 \in 3K \setminus 2K \).

**Corollary 4.5.** Let \( b \in S \) be odd and let \( I = S - \langle K \rangle \). The following hold:

(1) If \( S \) is not almost symmetric, then \( S \gg b^1 M \) is not GAS;
(2) \( S \) is n-AGL if and only if \( S \gg b^1 I \) is n-AGL.

**Proof.** (1) Let \( T = S \gg b^1 M \) and let \( x \neq 0 \) be such that \( F(S) - x \notin 2K \setminus K \). By Lemma 4.3 (1), \( F(T) - 2x \) and \( F(T) - (2x + b) \) are in \( 2K(T) \setminus K(T) \). Even though \( 2x + b \) and \( 2x \) are minimal generators, their difference \( b \) is a pseudo-Frobenius number of \( T \) by Remark 4.2, because \( 0 \in PF(M) \), hence \( T \) is not GAS.

(2) Let \( T = S \gg b^1 I \). By Lemma 4.3 (4) we have that \( F(S) - x \notin \langle K \rangle \setminus K \) if and only if \( F(T) - 2x \notin \langle K(T) \rangle \setminus K(T) \). Moreover, if \( F(T) - (2i + b) \notin \langle K(T) \rangle \setminus K(T) \), Lemma 4.3 (4) implies that \( F(S) - i \notin \langle K \rangle \) and, since \( i \in (S - \langle K \rangle) \), it follows that \( F(S) \notin S \), which is a contradiction. Hence, \( S \) is n-AGL if and only if \( T \) is n-AGL. \( \square \)

**Remark 4.6.** If \( S \) is almost symmetric with type \( t \), then \( M = K - (M - M) \) and, consequently, \( S \gg b^1 M \) is almost symmetric with type \( 2t + 1 \) by [6, Theorem 4.3 and Proposition 4.8].

If \( R \) is a one-dimensional Cohen-Macaulay local ring with a canonical module \( \omega \) such that \( R \subseteq \omega \subseteq \overline{R} \), in [5, Theorem 4.2] it is proved that the idealization \( R^k \langle R : R(\omega) \rangle \) is 2-AGL if and only if \( R \) is 2-AGL. The numerical duplication may be considered the analogous of the idealization in the numerical semigroup case, since they are both members of a family of rings that share many properties (see [2]); therefore, Corollary 4.5 (2) should not be surprising. In the following proposition we generalize this result for the GAS property.
Theorem 4.7. Let $S$ be a numerical semigroup, let $b \in S$ be an odd integer and let $I = S - \langle K \rangle$. The semigroup $T = S \bowtie b I$ is GAS if and only if $S$ is GAS.

Proof. Assume that $T$ is GAS and let $F(S) - x \in 2K \setminus K$. By Lemma 4.3, $F(T) - 2x \in 2K(T) \setminus K(T)$, so $2x$ is a minimal generator of $T$ and, thus, $x$ is a minimal generator of $S$. Now let $F(S) - x, F(S) - y \in 2K \setminus K$ and assume by contradiction that $x - y \in PF(S)$. In particular, $S$ is not symmetric and, then, $I = M - \langle K \rangle$. Moreover, $F(T) - 2x$ and $F(T) - 2y$ are in $2K(T) \setminus K(T)$. We also notice that $x - y \notin I - I$, indeed, if $i \in I$ and $a \in \langle K \rangle$, it follows that $(x - y) + i + a \in (x - y) + M \subseteq S$. Therefore, Remark 4.2 implies that $2(x - y) \in PF(T)$; contradiction.

Conversely, assume that $S$ is GAS and let $F(T) - z = k_1 + k_2 \in 2K(T) \setminus K(T)$ with $k_1, k_2 \in K(T)$. If $z = 2i + b$ is odd and both $k_1$ and $k_2$ are odd, then $i \in I$ and $F(S) - i = (k_1 - b)/2 + (k_2 - b)/2 + b \in 2K$ by Lemma 4.3.(2); on the other hand, if $k_1$ and $k_2$ are both even, $F(S) - i = k_1/2 + k_2/2 \in \langle K \rangle$ by Lemma 4.3.3. Since $i \in (S - \langle K \rangle)$, in both cases we get $F(S) \in S$, which is a contradiction. Hence, $z = 2x$ is even. If $k_1$ is even and $k_2$ is odd, Lemma 4.3 implies that $F(S) - x = k_1/2 + (k_2 - b)/2 \in (j + 1)K \setminus K$ for some $j \geq 1$ and, therefore, by Theorem 3.7 it follows that $x$ is a minimal generator of $S$, i.e. $z = 2x$ is a minimal generator of $T$. Moreover, let $F(T) - 2x,F(T) - 2y \in 2K(T) \setminus K(T)$ and assume by contradiction that $2x - 2y \in PF(T)$.

Remark 4.2 implies that $x - y \in PF(S) \subseteq K \cup \{ F(S) \}$. Thus, if $F(T) - 2x = k_1 + k_2$ with $k_1, k_2 \in K(T)$ and $k_1$ even, then $F(S) - x = k_1/2 + (k_2 - b)/2 + (x - y) \in \langle K \rangle \setminus K(S)$ by Lemma 4.3 and, so, $F(S) - y = k_1/2 + (k_2 - b)/2 + (x - y) \in \langle K \rangle \setminus K(S)$. Hence, Theorem 3.7 yields a contradiction, because $x - y \notin PF(S)$.

Example 4.8. 1. Consider the semigroup $S$ in Example 4.4.2. It is GAS and, then, the previous theorem implies that also $T = S \bowtie 47(S - \langle K \rangle)$ is GAS. However, we notice that $2K \setminus K = \{ 44, 63, 91 \}, 3K \setminus 2K = \{ 85 \}$ and $4K = 3K$, while $2K(T) \setminus K(T) = \{ 135, 173, 217, 229 \}$ and $2K(T) \setminus K(T) = 3K(T)$.

2. Despite Theorem 4.7, if $S \bowtie b I$ is GAS for an ideal $I$ different form $S - \langle K \rangle$, it is not true that also $S$ is GAS. For instance, the semigroup $S$ in Example 4.4.1 is not GAS, but $S \bowtie 3 I$ is.

4.3. Dilatations of numerical semigroups

We complete this section by studying the transfer of the GAS property in a construction recently introduced in [4]: given $a \in M - 2M$, the numerical semigroup $S + a = \{ 0 \} \cup \{ m + a | m \in M \}$ is called dilatation of $S$ with respect to $a$.

Proposition 4.9. Let $a \in M - 2M$. The semigroup $S + a$ is GAS if and only if $S$ is GAS.

Proof. We denote the semigroup $S + a$ by $T$. Recalling that $F(T) = F(S) + a$, by [4, Lemma 3.1 and Lemma 3.4] follows that $2K(T) = 2K(S)$ and

$2K(S) \setminus K(S) = \{ F(S) - x_1, ..., F(S) - x_r, F(S) \},$

$2K(T) \setminus K(T) = \{ F(T) - (x_1 + a), ..., F(T) - (x_r + a), F(T) \}$

for some $x_1, ..., x_r \in M$.

Assume that $S$ is a GAS semigroup. Then, $x_i$ is a minimal generator of $S$ and it is straightforward to see that $x_i + a$ is a minimal generator of $T$. Moreover, if $(x_i + a) - (x_j + a) \in PF(T)$, then for every $m \in M$ we have $x_i - x_j + m + a \in T$, i.e. $x_i - x_j + m \in M$, that is a contradiction, since $S$ is GAS.

Now assume that $T$ is GAS. Suppose by contradiction that $x_i$ is not a minimal generator of $S$, that is $x_i = s_1 + s_2$ for some $s_1, s_2 \in M$. We have $F(S) - (s_1 + s_2) \in 2K(S) \setminus K(S)$ and so $F(S) - s_1 \in 2K(S) \setminus K(S)$, since $2K(S)$ is a relative ideal. Hence, $s_1 = x_j$ for some $j$ and $(x_j + a) - (x_j + a) = s_2 \in S$. Since $x_j + a$ is a minimal generator, we have that $s_2 \not\in T$. Moreover, for every $m +
Suppose 2\(K(S + a) \setminus K(S + a) = \{ F(S + a) - (x_1 + a), ..., F(S + a) - (x_r + a), F(S + a) \} \) with \(x_1 + a, ..., x_r + a\) minimal generators of \(S + a\), but \(S + a\) is not GAS. Then \(2K(S) \setminus K(S) = \{ F(S) - x_1, ..., F(S) - x_r, F(S) \} \), but it is not necessarily true that \(x_1, ..., x_r\) are minimal generators of \(S\). For instance, consider \(S = \langle 7, 9, 11 \rangle\) and \(S + 7 = \langle 14, 16, 18, 21, 23, 25, 27, 29, 38, 40 \rangle\). In this case \(2K(S + 7) \setminus K(S + 7) = \{ 33 - 29, 33 - 18, 33 \}\) and \(2K(S) \setminus K(S) = \{ 26 - 22, 26 - 11, 26 \}\).

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