Geometrical aspects on the dark matter problem

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Abstract In the present paper we apply the Nash’s theory of perturbative geometry to the study of dark matter gravity in a higher-dimensional space-time. It is shown that the dark matter gravitational perturbations at local scale can be explained by the extrinsic curvature of the standard cosmology. In order to test our model, we use a spherically symmetric metric embedded in a five-dimensional bulk. As a result, considering a sample of 10 low surface brightness and 6 high surface brightness galaxies, we find a very good agreement with the observed rotation curves of smooth hybrid alpha-HI measurements.

1 Introduction

Since the first observations of an apparently additional gravitational effect for the anomalous movement of COMA cluster in 1930 by Zwicky \cite{1} and in the 70’s with the rotation curve discrepancy \cite{2}, dark matter has been a long stand problem in cosmology and particle physics. Due to the most physical properties remain unknown, the dark matter problem has motivated several attempts to obtain a sought-after well based explanation. Interesting reviews and a check list for dark matter candidates and some models can be found in \cite{3–14}.

As for its origin, the current thinking is that dark matter may have appeared in the very early universe, at the inflationary period and that its gravitational field induced the perturbations of matter density, eventually producing today’s observed large structures such as galaxies and clusters. Hence, gravitational perturbation theory has been playing an essential role in our understanding of how the universe came to be populated by stars and clusters and other large structures. The presently accepted theory is that dark matter was the cause of that perturbation, but it is not too clear how dark matter appeared in the first place. As it is well known, perturbation mechanisms in relativistic cosmology are plagued by coordinate gauges, inherited from the group of diffeomorphism of general relativity, making it difficult to differentiate between a physical perturbation from those resulting from a mere coordinate transformation. In spite of its successes in cosmology, the traditional gravitational perturbation theory neglects the fact that it also means perturbations (or deformations) of the space-time geometry by a continuous sequence of local infinitesimal increments of the metric.

In this paper we try to avoid considerations on exotic dark matter and concentrate only on the understanding of the dark matter gravitational field and how it perturbs the gravitational field of baryons. We show that at least in local scale dark matter can be neglected in the rotation curves problem, regardless of any attributes it may possess.

A concept of smooth perturbations is introduced based on the generation of geometries by continuous perturbation introduced by John Nash \cite{15} in 1956. Due to the importance of this concept in cosmology, we naturally start at the cosmological scale. Afterwards, we see how such perturbations affect the local structures. It is also shown that Nash’s embedding theorem essentially translates the original braneworld program as a embedded space-time. Despite its very common use in most braneworld models, particular junction conditions are not considered in this paper.

This paper is organized as follows. In the next section, we present the smooth perturbation based on Nash’s theorem \cite{15, 16}. The section 3 is devoted to discussion of the standard Friedman-Lemaître-Robertson-Walker (FLRW) cosmology, regarded as submanifold embedded in a five dimensional deSitter bulk that can be perturbed leading to a modified Friedman’s equation. In section 4 we apply the same
procedure to a four dimensional spherically symmetric geometry embedded in the five dimensional space-time (the bulk) and obtain the rotation curves of a sample that contains 10 low surface brightness (LSB) galaxies and 6 high surface brightness (HSB) galaxies. Final comments and foresee future works are presented in the conclusion section.

2 Smooth perturbations

As it seems evident, the study of dark matter gravity and its implications to the formation of structure in the early universe must naturally start with gravitational perturbation theory. The traditional perturbation mechanisms in relativistic cosmology are plagued by coordinate gauges, which are inherited from the group of diffeomorphism of general relativity. These gauges make it difficult to differentiate between a true physical perturbation from those resulting from a mere coordinate transformation. Fortunately there are some very successful criteria to filter out the latter perturbations in cosmology [17–19] but they still depend on a choice of a model. A lesser known, but far more general approach to gravitational perturbation was developed by J. Nash, showing that any Riemannian geometry can be generated by a continuous sequence of local infinitesimal increments of a given geometry [15, 20].

Since this theorem sits at the very foundation of how geometrical structures are formed and compared, it seems reasonable that it must be somehow related to the formation of structures in cosmology. In principle, these deformations should be associated with shapes or forms of large structures in the early universe. In this case, the uniqueness and structure of those forms have to do with Riemann’s geometry. Rather than being just semantics, this touches a fundamental issue, namely the lack of precision of Riemannian geometry to determine the curvature of a manifold endowed with a metric geometry.

Nash’s theorem solves an old dilemma of Riemannian geometry, namely that the Riemann tensor is not sufficient to make a precise statement about the local shape of a geometrical object or a manifold. The simplest example is given by a 2-dimensional Riemannian manifold, where the Riemann tensor has only one component $R_{1212}$ which coincides with the Gaussian curvature. Thus, a flat Riemannian 2-manifold defined by $R_{1212} = 0$ may be interpreted as a plane, a cylinder or even a helicoid, in the sense of Euclidean geometry, which is the basis of our astronomical observations. Of course, this ambiguity was known by Riemann, who regarded the concept of flatness as defining an equivalent class of manifolds instead of an specific one [21].

However, when we try to apply this concept to relativistic cosmology, specially referring to structure formation, we require a less ambiguous notion of shape of the observed object. The solution to this ambiguity problem was originally proposed in 1873 by Schlaefli [22], conjecturing that if a Riemannian manifold could be embedded in another one, then a decision on its real shape could be made by comparing the Riemann curvatures of the embedded surface with the one of the embedding space. The formal solution of the problem took a long time to appear, only after the derivations of the conditions that guarantee the embedding of any Riemannian geometry into another, the well known Gauss-Codazzi-Ricci equations of geometry. The Gauss-Codazzi-Ricci equations are non-linear and difficult to solve in the general case. Some simplifications were obtained by assuming that the metric is analytic in the sense that it is a convergence of a positive power series [23, 24]. The most general solution appeared only in 1956 with Nash’s theorem.

2.1 The Nash’s embedding theorem

Nash’s theorem innovated the embedding problem by introducing the notion of differentiable, perturbative geometry: using a continuous sequence of small perturbations of a simpler embedded geometry along the extra dimensions, the theorem shows how to construct any other Riemannian manifold1. (a brief, updated description of Nash’s result can be found in [16, 27]). Thus, Nash’s approach to geometry not only solves the ambiguity problem of the Riemannian curvature, but also gives a prescription on how to construct geometrical structures by deforming simpler ones.

Reviewing some basic ideas of Nash’s geometric perturbation theorem, we suppose we have an arbitrarily given Riemannian manifold $V_n$ with metric $\bar{g}_{\mu\nu}$, which is embedded into another Riemannian manifold $V_D$, the bulk space. Of course, $D$ must be sufficiently large to accommodate the embedded $V_n$ and the bulk geometry $\bar{g}_{AB}$ must be given. Assuming these conditions, we generate another geometry by a small perturbation $\bar{g}_{\mu\nu} + \delta\bar{g}_{\mu\nu}$ where\(^{2}\)

$$\delta\bar{g}_{\mu\nu} = -2k_{\mu\nu\alpha}\delta y^\alpha, \quad a = N + 1...D$$

where $\delta y^\alpha$ denotes an infinitesimal parameter and where $k_{\mu\nu\alpha}$ denote the extrinsic curvature components of $\bar{V}_n$ relative to the extra dimension $y^\alpha$. Using this perturbation we obtain new extrinsic curvature $k_{\mu\nu\alpha}$, and by repeating the process we obtain a continuous sequence of perturbations:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta y k_{\mu\nu} + \delta^2 y \bar{g}^{\rho\sigma} k_{\mu\rho} k_{\nu\sigma} \cdots$$

In this way, Nash showed that any Riemannian manifold embedded in the bulk can be generated even if we have a flat space-time.

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1It is worth stressing that this geometric perturbation method was apparently introduced by J. Campbell in a posthumous edition of a textbook on differential geometry [25] but, unfortunately, with by the use of analytic conditions[26].

2Greek indices $\mu, \nu...$ refer to the $n$-dimensional embedded geometries; Small case Latin indices $a, b...$ refer to $N$ extra dimensions; Capital Latin indices $A, B...$ refer to the bulk. Hereon, we adopt natural units where the speed of light $c$ is set as $c = 1$. 
Nash’s original theorem used a flat D-dimensional Euclidean space but this was soon generalized to any Riemannian Manifold, including those with positive and negative signatures [20]. Although the theorem could be generalized to include perturbations on arbitrary directions in the bulk, it would make interpretations more difficult so that we retain Nash’s choice of independent orthogonal perturbations. The embedding apparently introduces fixed background geometry as opposed to a completely intrinsic and self-contained geometry in general relativity. This can be solved by defining the geometry of the embedding space by the Einstein-Hilbert variational principle, which has the meaning that the embedding space has the smoothest possible curvature. This is compatible with Nash’s theorem which requires a differentiable embedding structure [27]. Another aspect of Nash’s theorem is that the extrinsic curvature are the generator of the perturbations of the gravitational field along the extra dimensions, as indicate the eqs.(1) and (2). The symmetric rank-2 tensor structure of the extrinsic curvature lends the physical interpretation of an independent spin-2 field on the embedded space-time.

2.2 Brane-world as an embedded space-time

If one tries to adapt the mathematical embedding structure to a physical model, the original brane-world idea seems to comply with this intent. Brane-world models have its root in a bi-dimensional model [28] proposed an explanation of why the compact space in Kaluza-Klein theory was not observed. In other words, a confined 4-d observer does not verify a fifth dimension because it depends on the energy of observation. In that model, the physical space should be confined to a potential well embedded in a larger space (the bulk). In 1998, Arkani-Hamed, G. Dvali and S. Dimopoulos (ADD for short) proposed a solution of the hierarchy problem of the fundamental interactions[29] giving the basis of the brane-world theory. It contains essentially three fundamental postulates: (a) the space-time or brane-world is an embedded differentiable sub manifold of another space (the bulk) whose geometry is defined by the Einstein-Hilbert action (therefore this should not be confused with the “brane-worlds” of string/M-theory); (b) all gauge interactions are confined to the four-dimensional brane-world (this is a consequence of the Poincaré symmetry of the electromagnetic field and in general of the dualities of Yang-Mills fields, which are consistent in four-dimensional space-time only); and (c) gravitation is defined by Einstein’s equations for the bulk, propagating along the extra dimensions at tev energy. It follows from (b) that ordinary matter must also be confined because it interacts with the confined gauge fields.

The original ADD paper refers to graviton probes to the extra dimensions, but classically it means that the bulk is locally foliated by a family brane-world sub manifolds, whose metric depend on the extra-dimensional coordinates in the bulk.

A different approach to compactification of extra-dimensions was proposed in the Randall-Sundrum brane-world model [30, 31] where the 4-d space-time is embedded in the five dimensional anti deSitter space AdS5. In this model the extrinsic geometry of the universe is passive and reduced to a boundary condition depending only on the confined sources. Thus, the extrinsic curvature becomes an algebraic function of the energy-momentum tensor of matter confined to the four-dimensional embedded space-time well known as Israel-Darmois-Lanczos condition [32]. Despite its success this condition is not unique [33] and cannot be apply to more than a noncompact dimension which is not the brane-world in general [34, 35].

2.2.1 The Confinement condition

The four-dimensionality of the space-time manifold is an experimentally established fact, associated with the Poincaré invariance of Maxwell’s equations and their dualities, later extended to all gauge fields. Consequently, all matter which interacts with these gauge fields for consistency must also be defined in the four-dimensional space-times. On the other hand, in spite of all efforts made so far, gravitation has failed to fit into a similar gauge interaction scheme, so that the gravitational field does not necessarily have the same four-dimensional limitations.

The confinement of ordinary matter and gauge fields implies that the tangent components of the general source term $\alpha T_{\mu \nu}^A$ in the bulk space must coincide with the usual 4-d dimensional term $8\pi GT_{\mu \nu}$ where $T_{\mu \nu}$ is the energy-momentum tensor of the confined sources. Admitting only the gravitational interaction of dark matter, the above discussion implies that dark matter cannot be confined for the same reasons for the confinement of baryons. In fact, we do not know any other property of dark matter and its propagation in the bulk, except its gravitational field. It is tempting to suppose that dark matter also propagates in the bulk, so that the transverse and/or normal components of $T_{\mu \nu}^A$ could contribute to an observable effect in four dimensions. However, this would necessarily lead us to an exercise on dark matter modeling. For example, the WIMP model would give the normal components $T_{ab} = p_{DM}U_a U_b$. This has the important consequence that the diffeomorphism symmetry is also confined to the brane-world. As a result, the confinement can be generally set as a condition on the embedding map such that

$$8\pi GT_{\mu \nu} = G_{\star}Z_{\mu}^A Z_{\nu\nu} T_{AB}^\star,$$

$$Z_{\mu}^A \eta^{B\nu} T_{AB}^\star = 0,$$

$$\eta^A \eta^B T_{AB}^\star = 0.$$

According to Nash’s theorem the gravitational field, regardless of where its source is located, it is geometrically
perturbable along the extra dimensions. All that we use is Nash’s theorem together with the four-dimensionality of gauge fields and the Einstein-Hilbert principle. Some similar approaches [35, 36] have been developed with no need of particular junction conditions and/or with different junction conditions which led to a plethora of brane-world models widely studied in literature [37–42].

2.2.2 Brane-world equations in 5-D

The natural choice for the bulk space is Einstein-Hilbert principle, if for no other reason, but because that principle represents a statement on the smoothness of the embedding space. Admitting that the perturbations are smooth (differentiable) then the embedded geometry will be also differentiable. This smoothness of the embedded geometry was a primary concern of the theorem. The Einstein-Hilbert principle leads to the D-dimensional Einstein’s equations for the bulk metric $g_{AB}$ in arbitrary coordinates

$$\mathcal{R}_{AB} - \frac{1}{2} \mathcal{R} g_{AB} = \alpha_s T_{AB}$$

where $T_{AB}$ denotes the energy-momentum tensor of the known matter and gauge fields. The constant $\alpha_s$ determines the D-dimensional energy scale.

As in Kaluza-Klein and in the brane-world theories, the embedding space $V_n$ has a metric geometry defined by the higher-dimensional Einstein’s equations

$$^5 \mathcal{R}_{AB} - \frac{1}{2} ^5 \mathcal{R} g_{AB} = G_{AB} T_{AB}$$

where $G_{AB}$ is the new gravitational constant and where $T_{AB}$ denotes the components of the energy-momentum tensor of the known gauge fields and material sources. From these dynamical equations we may derive the gravitational field in the embedded space-times. Taking the tangent, vector and scalar components of eq.(5) and using the previous confinement conditions eq.(3) one can obtain (see [16] for the derivation of these equations in the high-dimensional case)

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - Q_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$k_{\mu\nu}^B - h_{\mu\nu} = 0 ,$$

where the term $Q_{\mu\nu}$ in the first equation results from the expression of $R_{AB}$ in eq.(5), involving the orthogonal and mixed components of the Christoffel symbols for the metric $g_{AB}$. Explicitly this new term is

$$Q_{\mu\nu} = g^{\rho\sigma} k_{\mu\rho} k_{\nu\sigma} - k_{\mu\nu} h - \frac{1}{2} \left( K^2 - h^2 \right) g_{\mu\nu} .$$

where $h^2 = g^{\mu\nu} k_{\mu\nu}$ is the squared mean curvature and $K^2 = k^{\mu\nu} k_{\mu\nu}$ is the squared Gauss curvature. This quantity is therefore entirely geometrical and it is conserved in the sense of $\mathcal{Q}^{\mu\nu} \cdot \nu = 0 . \tag{9}$

It is important to stress that the Israel-Darmois-Lanczos condition does not follow from Einstein’s equations by themselves. To see how it works, consider a Riemannian manifold $V_n$ with metric $g_{\mu\nu}$, and its local isometric embedding in a D-dimensional Riemannian manifold $V_D$, $D = n + 1$, given by a differentiable and regular map $X : V_n \rightarrow V_D$ satisfying the embedding equations

$$\mathcal{D}_A^X \mathcal{D}_B^X g_{AB} = g_{\mu\nu} , \quad \mathcal{D}_A^X \eta^B g_{AB} = 0 , \quad \eta^A \eta^B g_{AB} = 1 . \tag{10}$$

where $A,B = 1...D$ and we have denoted by $g_{AB}$ the metric components of $V_D$ in arbitrary coordinates, and where $\eta$ denotes the unit vector field orthogonal to $V_n$. The extrinsic curvature of $V_n$ is by definition the projection of the variation of $\eta$ on the tangent plane [43]:

$$\tilde{k}_{\mu\nu} = - \mathcal{D}_A^X \eta^B g_{AB} = \mathcal{D}_A^X \eta^A g_{AB} \tag{11}$$

The integration of the system of equations eq.(10) gives the required embedding map $X$.

Construct the one-parameter group of diffeomorphism defined by the map $h_t (p) : V_D \rightarrow V_D$, describing a continuous curve $\alpha(y) = h_t (p)$, passing through the point $p \in V_n$, with unit normal vector $\alpha'(y) = \eta(p)$ [44]. The group is characterized by the composition $h_t \circ h_{t' \eta} \equiv h_{t + t'}(p)$, $h_0 \equiv I$. Applying this diffeomorphism to all points of a small neighborhood of $p$, we obtain a congruence of curves (or orbits) orthogonal to $V_n$. It does not matter if the parameter $y$ is time-like or not, nor if it is positive or negative.

Given a geometric object $\varnothing$ in $V_n$, its Lie transport along the flow for a small distance $\delta y$ is given by $\Omega = \Omega + \delta y \varnothing \Omega$, where $\varnothing$ denotes the Lie derivative with respect to $\eta$ [44]. In particular, the Lie transport of the Gaussian frame $\{ X^A, \eta^A \}$, defined on $V_n$ gives

$$Z^A \cdot = X^A \cdot + \delta y \left[ \eta^A X^A \cdot + \delta y \eta^A \right]$$

$$\eta^A = \eta^A + \delta y \left[ \eta \eta^A \right] = \eta^A \tag{12}$$

However, from eq.(11) we note that in general $\eta^A \neq \eta^A$.

The set of coordinates $Z^A$ obtained by integrating these equations does not necessarily describe another manifold. In order to be so, they need to satisfy embedding equations similar to eq.(10):

$$\mathcal{D}_A^X \mathcal{D}_B^X g_{AB} = g_{\mu\nu} , \quad \mathcal{D}_A^X \eta^B g_{AB} = 0 , \quad \eta^A \eta^B g_{AB} = 1 . \tag{14}$$

Replacing eq.(12) and eq.(13) in eq.(14) and using the definition eq.(11) we obtain the metric and extrinsic curvature of the new manifold

$$g_{\mu\nu} = \delta_{\mu\nu} - 2 k_{\mu\nu} + \eta^{\rho\sigma} k_{\mu\rho} k_{\nu\sigma}$$

$$k_{\mu\nu} = \tilde{k}_{\mu\nu} - 2 \eta^{\rho\sigma} k_{\mu\rho} k_{\nu\sigma}$$

$$\eta^A = \eta^A + \delta y \left[ \eta \eta^A \right] = \eta^A \tag{16}$$

The third gravitational equation was omitted here due to the fact that it vanishes in 5-D, but when the higher dimensional space-time is considered, one can obtain the equation $R - \left( K^2 - h^2 \right) \mathcal{D}(D - 5) = 0$, sometimes called gravitational scalar equation.
Taking the derivative of eq. (15) with respect to y we obtain Nash’s deformation condition eq. (1).

Now we can write Einstein’s equations as

\[ 5\mathcal{R}_{AB} = G_s \left( T_{AB} - \frac{1}{3T} T^{\gamma \delta} \delta_{AB} \right). \] (17)

The Ricci tensor in five-dimension \( 5\mathcal{R}_{AB} \) may be evaluated in the embedded space-time by contracting it with the Gaussian frame \( Z^A_{\mu}, Z^B_{\mu}, Z^\alpha_{\mu} \eta^\beta \) and \( \eta^\alpha \eta^\beta \). Using eq. (1), eq. (14) and the confinement conditions eq. (3), Einstein’s equations in eq. (17) become

\[ 5\mathcal{R}_{\mu \nu} = R_{\mu \nu} + \frac{\partial k_{\mu \nu}}{\partial y} - 2k_{\mu \rho} k_{\nu}^\rho + h h_{\mu \nu} \] (18)

\[ 5\mathcal{R}_{\mu 5} = k_{\mu \rho} + \frac{\partial \Gamma_{\mu 5}^\rho}{\partial y} \] (19)

In this sense, it becomes necessary that the embedded geometry satisfies particular conditions such that Ricci curvature of the embedding space coincides with the extrinsic curvature of the embedded space-time, that is \( 5\mathcal{R}_{\mu \nu} = k_{\mu \nu} \), which is not generally true. One of these conditions is that the embedded space-time acts as a mirror boundary between two regions of the embedding space (see e.g. [32]). In this case we may evaluate the difference of \( 5\mathcal{R}_{\mu \nu} \) from both sides of the space-times and the above mentioned boundary condition holds. However, in doing so the deformation given by eq. (1) ceases to be. Therefore, to find the deformations caused by the extrinsic curvature such special conditions are not applied and they are not needed. A more detailed discussion can be found in [16, 45].

3 The cosmological model

We naturally start at the cosmological scale. In coordinates \( (x^1, x^2, x^3, x^4) \) the FLRW cosmological model can be expressed as \(4\):

\[ ds^2 = -dt^2 + a^2 [\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2)] \] (20)

In coordinates \( (r, \theta, \phi, t) \) the Friedman-Robertson-Walker (FLRW) model can be expressed as [46]:

\[ ds^2 = -dt^2 + a^2(t) [dr^2 + f_k^2(r)(d\theta^2 + \sin^2 \theta d\phi^2)] \]

where \( f_k(r) = r, \sin r, \sinh r \) corresponding to \( k = 0, +1, -1 \) (spatially flat, closed, open respectively).

One can start finding the solution of Codazzi’s equations for the FLRW metric in the deSitter bulk

\[ k_{\mu \nu} = 0 \]

This is the same as in [16], but written in a better known coordinate system, where \( dr^2 \rightarrow dr^2/(1 - kr^2) \) and \( f(r)^2 \rightarrow r^2 \).

of which (7) is just its trace. The general solution of this equation is

\[ k_{ij} = \frac{b}{a^3} g_{ij}, \quad k_{44} = -\frac{1}{a} \frac{d}{dt} \left( \frac{b}{a^3} \right), \quad i, j = 1 \ldots 3 \] (21)

Defining \( B = \frac{b}{a} \), we may express the components of \( Q_{\mu \nu} \) as

\[ Q_{ij} = \frac{B^2}{a^3} \left( 2B - 1 \right) g_{ij}, \quad Q_{44} = -\frac{3B^2}{a^3}, \] (22)

\[ Q = -\frac{6B^2}{a^3} H, \quad i, j = 1, 3 \] (23)

where \( H = a/\dot{a} \) is the usual Hubble parameter. After replacing in (6) we obtain Friedmann’s equation modified by the extrinsic curvature [16]:

\[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{\kappa}{a^2} = \frac{8}{3} \pi G \rho + \frac{1}{3} + \frac{b^2}{\epsilon a^4}. \] (24)

where \( a(t) \) is the usual expansion parameter of the Universe. The same results were independently obtained by the authors in [35] also without assumption of any junction condition. It is important to note that the perturbed term \( \frac{b^2}{\epsilon a^4} \) in eq. (24) is differently from that one provided by the Randall-Sundrum model with the quadratic density term that leads to a serious cosmological constraints [47]. In ref. [16]) it was shown that once the Israel-Darmois-Lanczos condition is imposed on eq. (24), the quadratic density term is obtained.

It was shown in [48] that the contribution of the extrinsic curvature, \( b(t) \), can be given by

\[ b(t) = a_0 (a)^{\beta_0} e^{\frac{1}{3} b(t)} \] (25)

where all integration constants were combined in \( a_0 \) and \( \beta_0 \) and the term \( \gamma(t) \) denoted by

\[ \gamma(t) = \sqrt{4\eta_0^2 a^4 - 3 - \sqrt{3} \arctan \left( \frac{\sqrt{3}}{3} \sqrt{4\eta_0^2 a^4 - 3} \right)} \] (26)

Alternatively, the modified Friedmann equation can be written as

\[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{\kappa}{a^2} = \frac{4}{3} \pi G \rho + \frac{1}{3} + \kappa_0 a \beta_0 e^{-\epsilon \gamma(t)} \] (27)

where \( \eta_0 \) and \( \kappa_0 = \frac{\beta_0}{\epsilon a_0} \) are integration constants. In addition, this result in first approximation can be compared with the phenomenological x-fluid model (XCDM), with state equation \( p_x = \omega_x \rho_x \), which corresponds to the geometric equation on \( b(t) \) which in the particular case when \( \omega_x = \omega_0 \)-constant, one obtain a simple solution

\[ b(t) = b_0 \left( \frac{a}{a_0} \right)^{\frac{1}{3} \left( 1 - 3\omega_0 \right)} \] (28)

where \( a_0 \) and \( b_0 \neq 0 \) are integration constants. This solution is consistent with the most recent observations within the range \(-1 \leq \omega_0 \leq -1/3 \) [16, 48], for \( \omega = -1 \) results in ΛCDM cosmological model. Furthermore, the theoretical power spectrum obtained from the extrinsic curvature perturbation of the FLWR model, is not very different from the observed power spectrum in the Planck experiment [49].
4 Local Dark Matter Gravity and rotation curves

It is possible that the gravitational field of young galaxies which are still in the process of formation [50]; in galaxies with active galaxy nuclei [51]; or even in cluster collisions, Nash’s perturbations similar to the cosmological case could be applied, where the metric symmetry is taken to be local, instead of the homogeneous and isotropic conditions which lead to the $G_2$ metric symmetry of the standard cosmological model.

These equations are be understood in the context of the embedded space-times and with the confinement conditions for ordinary matter and gauge fields. They do not represent the whole of general relativity because the principle of general covariance does not necessarily apply to the bulk geometry. This follows from the fact that Nash’s perturbations are restricted to be along the orthogonal directions only.

In order to see how realistic such description is, we apply it and compare to the observational data. Since in eq.(6) and eq.(7), the metric and extrinsic curvature are independent variables, thus non-trivial cosmological perturbations are obtained only from a non-vanishing extrinsic curvature.

4.1 The model

Consider a four dimensional spherically symmetric metric in a form

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 .$$

We can straightforwardly obtain the components of Ricci tensor as

$$R_{tr} = \frac{B'}{2B} - \frac{1}{4} \left( \frac{A'}{A} + \frac{B'}{B} \right) - \frac{1}{r} A' ,$$

$$R_{\theta\theta} = -1 + \frac{r}{2A} \left( \frac{A'}{A} + \frac{B'}{B} \right) + \frac{1}{A} ,$$

$$R_{\phi\phi} = \sin^2 \theta R_{\theta\theta} ,$$

and also

$$R_{tt} = \frac{B'}{2A} - \frac{1}{4} \left( \frac{A'}{A} + \frac{B'}{B} \right) - \frac{1}{r} B' ,$$

where we denote $\frac{dA}{dr} = A'$ and $\frac{dB}{dr} = B'$.

The gravi-tensor equation in five dimensions can be written as

$$R_{\mu\nu} - \frac{1}{2} Q g_{\mu\nu} = Q_{\mu\nu} ,$$

(29)

where $Q = g^{\mu\nu} Q_{\mu\nu}$.

The general solution of Codazzi equation can take the form

$$k_{\mu\nu} = f_\mu g_{\mu\nu} \quad (no \ sum \ on \ \mu) ,$$

(30)

where $f_\mu$ represents a set of scalar functions. Thus, considering the definition of $Q_{\mu\nu}$, we have:

$$Q_{\mu\nu} = f_\mu^2 g_{\mu\nu} - \left( \sum_\alpha f_\alpha \right) f_\mu g_{\mu\nu} - \frac{1}{2} \left( \sum_\alpha f_\alpha^2 - \left( \sum_\alpha f_\alpha \right)^2 \right) g_{\mu\nu} ,$$

where we identify

$$U_\mu = f_\mu^2 - \left( \sum_\alpha f_\alpha \right) f_\mu - \frac{1}{2} \left( \sum_\alpha f_\alpha^2 - \left( \sum_\alpha f_\alpha \right)^2 \right) g^{\mu\nu} .$$

Thus, one can rewrite $Q_{\mu\nu}$ in terms of $f_\mu$ as

$$Q_{\mu\nu} = U_\mu g_{\mu\nu} \quad (no \ sum \ on \ \mu) .$$

(31)

Since $Q_{\mu\nu}$ is a conserved quantity, once can find 4 equations from $\sum_\nu g^{\mu\nu} U_{\mu\nu} = 0$ which can be reduced to two equations:

$$\begin{cases}
(f_1 + f_2) (f_3 - f_4) = 0 ; \\
(f_3 + f_1) (f_1 - f_2) = 0 .
\end{cases}$$

and results in

$$f_1 f_3 = f_2 f_4 .$$

We have two options: $f_1 = f_2 e f_3 = f_4 f_1 = f_3 e f_2 = f_4$. Taking the first option, $f_1 = f_2 = \alpha(r)$ that leads us to $f_3 = f_4 = \beta(r)$, where $\alpha(r)$ and $\beta(r)$ are arbitrary functions. In addition, we can find the $Q_{\mu\nu}$ components:

$$\begin{cases}
K^2 = 2 (\alpha^2 (r) + \beta^2 (r)) , \\
h = 2 (\alpha(r) + \beta(r)) \rightarrow H^2 = 4 (\alpha(r) + \beta(r))^2 , \\
K^2 - H^2 = -2 (\alpha^2 (r) + \beta^2 (r)) - 8 \alpha(r) \beta(r) .
\end{cases}$$

And we obtain $Q_{\mu\nu} = Q_{ii} + Q_{jj}$:

$$\begin{cases}
Q_{ii} = (\alpha^2 (r) + 2 \alpha(r) \beta(r)) g_{ii} \quad (for \ (11) \ and \ (22) \ components) , \\
Q_{jj} = (\beta^2 (r) + 2 \alpha(r) \beta(r)) g_{jj} \quad (for \ (33) \ and \ (44) \ components) .
\end{cases}$$

(32)

and the trace

$$Q = 2 \alpha^2 (r) + 2 \beta^2 (r) + 8 \alpha(r) \beta(r) .$$

(33)

From eq.(29), we find the Ricci tensor components:

$$\begin{cases}
R_{ii} = (\alpha^2 (r) + 2 \beta^2 (r) + 6 \alpha(r) \beta(r)) g_{ii} \quad (for \ (11) \ and \ (22) \ components) , \\
R_{jj} = (\beta^2 (r) + 2 \alpha^2 (r) + 6 \alpha(r) \beta(r)) g_{jj} \quad (for \ (11) \ and \ (22) \ components) .
\end{cases}$$

(34)

Using the $R_{11}$ and $R_{44}$ components, we find:

$$\frac{A'}{A} + \frac{B'}{B} = (\beta^2 - \alpha^2) ,$$

(35)

which can be integrated as

$$AB = \exp \left( - \left[ (\beta^2 - \alpha^2) rdr - C \right] \right) ,$$

(36)

where $C$ is a integration constant.
Using the contour
\[ \lim_{r \to 0} A(r) = \lim_{r \to \infty} B(r) = 1, \]
and without loss of generality, we can set \( C = 0 \) and find
\[ A(r) = \frac{\sigma(r)}{B(r)} \tag{37} \]
where we denote \( \sigma(r) = \int (\beta^2 - \alpha^2) r \, dr \).

If we set the other possible option, i.e., \( f_1 = f_2 \) and \( f_2 = f_3 \), we will end up in the same situation. Thus, setting \( \alpha(r) = \beta(r) \), it allows us to do
\[ B'' - \frac{1}{2B} \left( \frac{A'}{A} + \frac{B'}{B} \right) - \frac{1}{r} \frac{A'}{A} = 9\alpha^2(r)A, \tag{38} \]
and also
\[ - \frac{B''}{2A} + \frac{1}{2A} \left( \frac{A'}{A} + \frac{B'}{B} \right) - \frac{1}{r} \frac{B'}{A} = -9\alpha^2(r)B, \tag{39} \]
and one can find
\[ \frac{A'}{A} = -\frac{B'}{B} \]
thus, \( AB = \text{constant} \). Taking the Minkowskian contour
\[ \lim_{r \to 0} A(r) = \lim_{r \to \infty} B(r) = 1, \]
we have
\[ A(r) = \frac{1}{B(r)}. \]

On the other hand, using the former condition, we can write the component \( \theta \theta \) of the Ricci tensor as
\[ R_{\theta \theta} = -1 + B' r + B \]
and using (39), we find
\[ B(r) = 1 + \frac{K}{r} + \frac{9}{r} \int \alpha^2(r) r^2 \, dr, \tag{40} \]
where \( K \) is an integration constant, and also
\[ A(r) = \left[ B(r) \right]^{-1} = \left[ 1 + \frac{K}{r} + \frac{9}{r} \int \alpha^2(r) r^2 \, dr \right]^{-1}. \tag{41} \]

We point out that the \( B(r) \) function remains undetermined due to the fact that the \( \alpha(r) \) and \( \beta(r) \) functions are arbitrary which do not allow the integration of \( B(r) \). It is a consequence of the homogeneity of Codazzi equation (7) which is not possible to be solved in 5-dimensions without an additional equation or a particular condition. Moreover, it was shown that the embedding of Schwarzschild spacetime can only be possible in at least six dimensions [52–56].

As a result, one gets a very constrained embedded geometry which can lead a serious physical consequences depending on how the extrinsic curvature is taken into account.

In order to attenuate the embedding constraint, we can try to obtain a solution with a minimum assumption with the use of the asymptotically conformal flat condition
\[ \lim_{r \to 0} g_{\mu \nu} = \lim_{r \to \infty} \alpha(r) \lim_{r \to \infty} g_{\mu \nu}. \tag{42} \]
In first approximation, this condition is simply derived from the analysis on the behavior of the extrinsic curvature at infinity.

As \( \lim_{r \to \infty} g_{\mu \nu} \to \eta_{\mu \nu} \), where \( \eta_{\mu \nu} \) is Minkowski metric, the extrinsic curvature vanishes as it tends to infinity, so the function \( \alpha(r) \) must comply with this condition. Thus, we can infer conveniently that the function \( \alpha(r) \) must be analytical at infinity such as
\[ \alpha(r) = \frac{\sqrt{|\alpha_0|}}{r^n}, \tag{43} \]
where \( \alpha_0 \) is a constant (that represents the influence of extrinsic curvature) and the index \( n \) represents all the set of scalar fields that fall off with \( r \)-coordinate following the inverse \( n \)th power law and \( n \geq 0 \) in order to comply with eq.(42).

Using the equations (40) and (43), one can obtain
\[ B(r) = 1 + \frac{K(9\alpha_0 + 1)}{r} - \frac{9\alpha_0}{3 - 2n} r^{2(1-n)} \tag{44} \]
In addition, from the eq.(44) one can calculate the gravitational potential and find
\[ \Phi(r) = -1 - \frac{K(9\alpha_0 + 1)}{2r} + (9/2)\alpha_0 r^{(2-2n)}/(3 - 2n). \tag{45} \]
It is worth stressing to the reader on how the obtained gravitational potential on eq.(45), which without the second term it resembles the Schwarzschild solution, is affected by extrinsic curvature resulting in a modification so far of the very symmetric aspect of the original spherical symmetry. This characteristic will be decisive to deal with the rotation curve problem.

Thus, using the newtonian expression for a orbiting particle \( v(r) = \sqrt{\frac{GM}{2r}} \), the velocity rotation modified by extrinsic curvature is given by
\[ v(r) = \sqrt{r(1/2) \frac{K(9\alpha_0 + 1)}{r^2} + (9/2)\alpha_0 (2 - 2n)r^{2(2-2n)}/(3 - 2n)r}. \tag{46} \]

In order to make a suitable notation to eq.(46) we set new parameters \( \beta_0 \) and \( \gamma_0 \) with a following correspondence to parameter \( n \) and \( \alpha \) as
\[ \beta_0 = 2 - 2n; \quad \gamma_0 = \frac{9}{2} \frac{\alpha_0 \beta_0}{(1 + \beta_0)}. \tag{47} \]
The appropriate relation to the standard gravity is set when \( K(\beta_0 + 1) = GM \). Thus, the correct units for the rotation velocity in eq.(46) are guaranteed. In addition, the bulge is modeled using Blumenthal’s mass model defined by
\[ M(r) = M_\ast \left[ 1 - (1 + \frac{r}{r_1}) \exp \left( \frac{r}{r_1} \right) \right]. \tag{48} \]
where \( r_1 \) is the scale length parameter and \( M_\ast = M_{gas} + M_{disk} \).
We consider here only the mass of gas and the mass of the galactic disk. The mass scale for the primordial He is given by $M_{\text{gas}} = 1.4M_{\text{HI}}$, where $M_{\text{HI}}$ is the mass of the hydrogenous 21cm lines that go through the galactic disk to the outermost regions. As the rotation curve goes up to the bound of a circular disk one can obtain

$$v(r) = \sqrt{\frac{GM(r)}{r} + \frac{GM(r)}{r_c} \gamma_0 \left( \frac{r}{r_c} \right)^{\beta_0}}, \quad (49)$$

where $r_c$ is the disk scale length.

In terms of the parameters $\beta_0$ and $\gamma_0$, it is important to consider that when the perturbation induced by the extrinsic curvature $k_{\mu\nu}$ vanishes, it implies that $\gamma_0 = 0$ and also $\beta_0 = 0$, as indicates eq.(47). In this case, we recover the Schwarzschild solution. On the other hand, when the parameter $\beta_0$ vanishes the extrinsic curvature decays as $\sim 1/r$ which is not compatible with the observed rotation curves. Moreover, to comply with eq.(47), the parameter $\beta_0$ is constrained by $\beta_0 \neq 1$.

The case when $\beta_0 = 2$ is also neglected even if it induces umbilicus points as expected for a spherical geometry, but the presence of such umbilicus is inconsistent with a rotation curve solution in asymptotic regions where the dark matter dilemma arises. It resembles a Schwarzschild-deSitter solution modified by the extrinsic term with a gravitational potential in a form

$$\Phi = -\frac{M}{r} + C^* r^2,$$

where $C^*$ is an integration constant. Consequently, any point in the braneworld with the condition $Q = -(K^2 - h^2) = 0$ or $Q = \text{constant}$ must be umbilicus, which means that it has a non-trivial solution when $K = \pm h$ or $y = \mp 1$. This is not the case of a braneworld in general, and it may occur in very special situations where all directions $dx^i$ are principal directions.

With these caveats in mind, we are able to deal with the rotation curves problem per se.

### 4.2 The samples

The main criteria used to choose an appropriated sample was to study galaxies with at least 11kpc and also being asymptotic enough to check our model in the region where the discrepancy is most critical. To this end, we chose a sample of 10-LSB galaxies varying from 11.6kpc up to 61.9kpc. As well known, the LSB galaxies are in principle dark dominated and they provide a good test for a gravitational theory [59, 60]. The present sample was extract from [59]. The authors in [59] presented a larger sample of 30-galaxies based on the analysis of high-resolution of smooth hybrid alpha-HI rotation curves. The measurements were obtained with long-slit major axis spectra taken with the 4 m telescope at Kitt Peak in and the 2.5 m telescope at Las Campanas Observatory. The adopted distances were computed assuming Hubble constant $H_0 = 75 \text{ km}^{-1} \text{ Mpc}^{-1}$.

The table (1) shows the main values considered to fit rotation curves. The data of disk scale length $R_*$, HI gas mass $M_{\text{HI}}$ that extends beyond optical disk and the disk mass $M_{\text{disk}}$ were obtained from [60]. In addition, the adopted distances $D$, the maximum radius $R_{\text{max}}$ in kpc for adopted distance and the main velocities $V_{\text{gas}}$ (rotation caused by the observed gas), $V_{\text{disk}}$ (rotation caused by the observed disk stars) and $V$ (the observed smoothed velocity) were obtained from [59].

The rotation curves of 10-LSB galaxies are presented in fig.(1) and (2). It is worth noting that in the fig.(1) we present the theoretical and the observed rotation curves with solid lines and error bars respectively. In addition, the observed quantities of the velocity of the gas (dashed lines) extended far beyond the visible disk and the velocity caused by disk stars (starred lines) are also shown. In fig.(2) we show both theoretical (solid lines) and observed (error bars) rotation

| Galaxies | D (Mpc) | $R_*$ (kpc) | $R_{\text{max}}$ (kpc) | $M_{\text{HI}}(10^{10}M_\odot)$ | $M_{\text{disk}}(10^{10}M_\odot)$ | $V_{\text{gas}}$(km/s) | $V_{\text{disk}}$(km/s) | $V$(km/s) | rotation curve |
|----------|---------|-------------|------------------------|-----------------------------|-----------------------------|------------------|------------------|-------------|----------------|
| F563-1   | 45      | 2.9         | 17.5                   | 0.29                        | 1.35                        | 32.8             | 20.8             | 110.9       | fig.01 (a)     |
| F571-8   | 48      | 5.4         | 14.0                   | 0.16                        | 4.48                        | 33.3             | 34.6             | 143.9       | fig.01 (b)     |
| F579-V1  | 85      | 5.2         | 14.4                   | 0.21                        | 3.33                        | 33.0             | 34.7             | 114.2       | fig.01 (c)     |
| F583-1   | 32      | 1.6         | 14.0                   | 0.18                        | 0.15                        | 38.2             | 9.82             | 86.9        | fig.01 (d)     |
| UGC11454 | 91      | 3.4         | 11.9                   | -                           | 3.15                        | -                | -                | 152.2       | fig.02 (a)     |
| UGC11748 | 73      | 5.1         | 21.0                   | -                           | 6.97                        | -                | -                | 246.5       | fig.02 (b)     |
| UGC11819 | 60      | 4.7         | 11.7                   | -                           | 4.83                        | -                | -                | 262.8       | fig.02 (c)     |
| ESO01400040 | 212  | 10.1        | 29.9                   | -                           | 20.70                       | -                | -                | 262.8       | fig.02 (d)     |
| ESO02060140 | 60   | 5.1         | 11.65                  | -                           | 3.51                        | -                | -                | 118.0       | fig.02 (e)     |
| ESO4250180 | 86   | 7.3         | 14.4                   | -                           | 4.79                        | -                | -                | 144.5       | fig.02 (f)     |

Table 1 Relevant observed properties of 10-LSB galaxies. A larger sample of the data can be obtained in [59, 60] and also with error data in Meguah’s data base (http://astroweb.case.edu/ssm/data/RCsmooth.0701.dat.)
Dof individual adopted distance for evaluating the rotation curves of HSB galaxies. The data its radii varying from 18 kpc up to 49.6 kpc in the Ursa Major galaxies, we study a minor sample of 6-HSB galaxies with attracted from [61]. Following the same criteria as to the LSB from one galaxy to another.

Since least-squares fit applying the Levenberg-Marquardt algorithm.


equation represents the probability of concordance between the model and the data.

For the fits, we used GnuPlot 4.6 to compute non-linear least-squares fit applying the Levenberg-Marquardt algorithm. Since \( r_1, \gamma_0, \beta_0 \) are not universal constants, they must vary from one galaxy to another.

For the HSB galaxies we consider a different sample extracted from [61]. Following the same criteria as to the LSB galaxies, we study a minor sample of 6-HSB galaxies with its radii varying from 18 kpc up to 49.6 kpc in the Ursa Major cluster.

In the table (3) we present the main values considered for evaluating the rotation curves of HSB galaxies. The data of individual adopted distance \( D \), disk scale length \( R_c \), HI gas mass \( M_{HI} \) and disk mass \( M_{disk} \) were obtained from [60].

The specific analysis by the authors in [61] was based on measurements of the HI 21-cm line synthesis image survey using the Westerbork Synthesis Radio Telescope (WSRT). The mean distance adopted by the authors for the cluster was of 18.6 Mpc. As a result, the parameter values \( r_1, \gamma_0, \beta_0 \) are presented in table (4) as well as \( \chi^2_{red} \) for the 6-HSB galaxies.

Concerning values of the obtained \( \chi^2_{red} \), we noted that UGC11454 and NGC4157 are near 1. Thus, these cases would represent only the best-fittings in the sample. Moreover, the critical value for \( \chi^2_{red} \) would represent a failure of the model mainly on the ESO4250180 galaxy. Accordingly, the model could not be used for the rotation curves problems even with the fits of rotation curves presented in figures (1) and (2) are very closed to the observed ones.

On the other hand, we must analyze the obtained values of \( \chi^2_{red} \) with caution. Small values of \( \chi^2_{red} \) does not indicate a poor model, but the uncertainties were conservatively over-estimated. The authors in [59] reported that error bars can easily dominate any model fit and constrain the \( \chi^2_{red} \) values, consequently the goodness-of-fit parameters must assume values smaller than 1 generally. This is due to the difficulty to obtain fair estimates of the errors on many observational quantities. In this sense, \( \chi^2_{red} \) values can only tell us relative merits of different models. As a result, this makes the error distribution not representable as a random error in the usual statistical sense. Thus, different smooth procedures may induce unexpected correlations among data point. For instance, eventual misalignment between hybrid HI and H-alpha measurements could induce a bias in data points leading to a non gaussian distribution and maybe the appearance of eventual outliers.

If we analyze our obtained \( \chi^2_{red} \) we attend to the constraint proposed in [59] that states a probability \( p \) which measures the data and the model compatibilities.

In the next pages, the resulting rotation curves of the sample of 16-galaxies are presented.

| Galaxies   | \( D \) (Mpc) | \( r_1 \) (kpc) | \( \gamma_0 \) | \( \beta_0 \) | \( M_{HI}(10^{10}M_\odot) \) | \( M_{disk}(10^{10}M_\odot) \) | \( v(km/s) \) | rotation curve |
|------------|---------------|----------------|-------------|-------------|-----------------|-----------------|-----------|----------------|
| NGC3726    | 17.5          | 3.2            | 31.47       | 0.60        | 3.82            | 167             | fig.03 (a) |
| NGC3769    | 15.5          | 1.5            | 32.01       | 0.41        | 1.36            | 113             | fig.03 (b) |
| NGC3992    | 25.6          | 5.7            | 49.64       | 1.94        | 13.94           | 237             | fig.03 (c) |
| NGC4100    | 21.4          | 2.9            | 27.07       | 0.44        | 5.74            | 159             | fig.03 (d) |
| NGC4157    | 18.7          | 2.6            | 30.82       | 0.88        | 5.83            | 185             | fig.03 (e) |
| NGC4217    | 19.6          | 3.1            | 18.15       | 0.30        | 5.53            | 178             | fig.03 (f) |

Table 2 Fitting parameters for 10-LSB galaxies. The \( \chi^2_{red} \) and the chance \( p \) are shown for each galaxy.

Table 3 Relevant observed properties of HSB galaxies. A larger sample can be obtained in [61].
Fig. 1 The rotation curves of smoothed hybrid HI and H-alpha measurements. The points with the solid lines represent the model prediction. The velocities of gas extended far the galactic disk are the dashed lines. The starred lines indicate the velocity of stars in the disk. Error bars are uncertainties in velocity measurements and take into account measurement, inclination and asymmetry uncertainties [59].

| Galaxies  | $r_1$ | $y_0$     | $\beta_0$     | $\chi^2_{red}$ | $p$  |
|-----------|-------|-----------|----------------|-----------------|------|
| NGC3726   | 2.768 | 0.363     | 0.335          | 0.634           | 0.7689 |
| NGC3769   | 19.170| 104.681   | -1.553         | 0.185           | 0.9957 |
| NGC3992   | 13.383| 9.887     | -1.120         | 0.390           | 0.8560 |
| NGC4100   | 4.515 | 2.0321    | -0.690         | 1.816           | 0.0141 |
| NGC4157   | 3.193 | 0.695     | -1.60          | 1.121           | 0.3326 |
| NGC4217   | 3.944 | 1.623     | -0.495         | 2.167           | 0.0044 |

Table 4 Fitting parameters for HSB galaxies with their individuals statistical parameters $\chi^2_{red}$ and the chance $p$.

The values of $p$ in tables (2) and (4) were evaluated using the standard $\chi^2$ test. These values indicate that if $p > 0.95$ we have a good data plus model fit and $p < 0.05$ means an inconclusive analysis (due to systematic effect) or a new model must be found. The “average” fit is obtained from a range of $0.05 < p < 0.95$.

As a result, we checked that even if the case of ESO4250 180 with the lowest obtained $\chi^2_{red}$ the model complies with a good values for $p$. The fitting parameters in most cases present a similar pattern as those shown in [59]. However, it is worth pointing out that 1-LSB and 2-HSB galaxies presented a peculiar situation.
Fig. 2 Rotation curves of the second group of 6-LSB galaxies. The solid lines represent the model predictions and the error bars represent the observed values. The dashed lines are the rotation curve without the extrinsic contribution.
Fig. 3 Rotation curves of 6-HSB galaxies. The solid lines represent the model predictions and the error bars represent the observed values. The dashed lines are the rotation curve without the extrinsic contribution.
The three galaxies UGC11748, NGC4100 and NGC4217 have in common a high $\chi^2_{\text{red}}$ and $p < 0.05$. Comparing our results to the data [59], we obtained the same non-representative value for chance $p$. For UGC11748 we obtained a similar chance $p$ compatible with the data. In both cases, $\chi^2_{\text{model}} < \chi^2_{\text{data}}$ indicates that the data were conservatively overestimated.

The NGC4100 galaxy presents strong warps which constrain an accurate measurements. For NGC4217 the presence of a huge bulge, high inclination and HI holes in inner regions [62] that lead to a steeper rotation curve. As a result, in this particular group, the systematic effects seems to be dominant which make the goodness-of-fit test inappropriate to define a clearer analysis.

The overall conclusion is that the shape of the rotation curves present in this sample of 16-galaxies was not heavily affected by the overestimated uncertainties and presents a similar pattern as shown in literature [59–62].

5 Summary and remarks

In the course of our study, we realized that dark matter cannot be explained only from the local observations of galaxies clusters and merging clusters. One possible explanation is that the perturbation is made through the intermediation of the gravitational field of dark matter. If one asks for the origins of dark matter we end up in the early universe, when dark matter was supposed to break the cosmological homogeneity of baryons thus creating the large structures that are observed today, such as galaxies and clusters of galaxies. In this context, the natural starting point to study the dark matter gravitational field is in the epoch of structure formation. The local effects of the dark matter gravitational field are obtained from the local limit of the cosmological gravitational field.

Here we have presented an application of Nash’s theorem on the formation of geometrical structures by metric perturbations as an alternative to the dark matter gravitational perturbation.

In a brief justification, we have discussed that Nash’s theorem does not only eliminate the ambiguity of Riemann’s notion of curvature, but it also provides a mathematically sound way to construct any Riemannian manifold by continuous perturbations along extra dimensions. We have conceded that the existence of these extra dimensions does not represent a breakdown of the very sensible intrinsic view of geometry laid down by Gauss and Riemann. Rather, their existence improve Riemann’s geometry in the sense that it restores the relativity of shapes of manifolds, replacing somewhat the absolute notion of curvature described by the Riemann tensor alone by a relative notion of curvature.

Given the Riemann tensor of the bulk (given by the Einstein-Hilbert principle), then the Riemann tensor of the embedded space-time acquires a reference standard. The four-dimensionality of space-time is regarded here as an experimental fact related to the symmetry properties of Maxwell’s equations or, more generally, of the gauge theories of the standard model. Thus, any matter that interacts with gauge fields, including us as observers, must remain confined to four-dimensions. However, the gravitational interaction does not fit in the same gauge scheme. Therefore, and in accordance with Nash’s theorem, gravity is regarded as different from gauge fields, because it is not confined, but it is perturbable along the extra dimensions.

The metric perturbation and the embedding lead naturally to a brane-world-like higher dimensional structure which is very similar to the brane-world program, except that it is not string inspired as suggested by the adjective “brane”. Furthermore it is completely independent of the many existing brane-world models.

Most of our mathematical tools are based on strong and well known results in differential geometry. In particular, the primary role of the extrinsic curvature in Nash’s perturbations represents an innovation when compared with the traditional perturbative methods in cosmology. In fact, it represents a rank-2 symmetric tensor which is independent of the metric but induces the metric perturbations.

In the cosmological side, we have discussed our results with the present observational data. In doing so we have started with cosmology, applying equations (6) and (7) to the FLRW cosmological model, finding that Friedman’s equation is modified by the presence of the extrinsic curvature. The theoretical power spectrum closely compares with the latest experimental result and our previous results show that it also agrees with the accelerated expansion of the universe. In a previous publication we have found that the same extrinsic curvature perturbs Friedman’s equation in such a way that it is consistent with the observed acceleration of the universe [48]. Since gravitational perturbation means also geometry perturbation, the understanding of structure formation implies in the study on the generation of geometrical forms in geometry. This goes beyond the classical classic perturbation theorems used in general relativity, in the line described by J. Bardeen [17], which admit that the Riemann geometry of space-time is already established together with coordinate gauges resulting from the diffeomorphism invariance of the theory.

All astrophysical observations are made as if we were in the Euclidean 3-space. However, when we finally write a theory to describe those observations we use a different Riemann geometry. The difficulty is that the objects described by the chosen geometry (by Einstein), does not necessarily coincide with the objects described by the observation geometry. In other words, we cannot be sure that our theoretically constructed structures using Riemann geometry correspond to what is actually observed. It appears that the recent
astrophysical observations are telling us about this geometrical difference. We have shown that even we have a flat space or metric, a perturbation can be generated in an embedded space-time. This is essence of Nash’s theorem.

Next we have applied the same equations (6) and (7) to the study of the local dark matter gravitational field of galaxies considering a 4-d spherically symmetric metric as a starting point. It is worth noting that a dark matter component was not necessary to invoke. Even with the problematic group of UGC11748, NGC4100 and NGC4217, where the systematic effect seems to be more strong entangled, the overall result presented in the present sample galaxies were satisfactorily with similar rotation curves present in literature. A possible further step is to extend the sample mainly on the HSB galaxies to large one. Giant and dwarf galaxies should be also verified. Eventually, a different geometrical structure can be used to study possible different effects that such changing can imprint on the results. An oblate spheroid can be used since it naturally carries variations of angles, which can be interesting for dealing with the inclination parameter of galaxies (e.g., NGC4217 presents a high inclination). In order to obtain a more interesting results, we postponed to a later paper a more sophisticated statical analysis as we intend to work with a larger sample.

It is important to point out that the perturbational aspects lie in the rotation velocity obtained which is modified by the extrinsic curvature with the appearance of the term $g_{\mu\nu}(\frac{C_0}{r^2})\delta_0$. It is interesting to note that this correction term in eq.(49) resembles a similar expression if one calculates the rotation velocity for a thin disk using the Weyl coordinates $\phi \mid_{disk} = -\frac{1}{2}(1 + g_{44})_{r=0} = -\frac{1}{2}(1 - e^{2\phi_0})$ where $C_0$ and $\delta_0$ are integration constants. This interpretation seems to be compatible with the fact that the galaxy core is spherically symmetric and the outermost regions are disk-like shape.

There are three related problems which deserve further attention and are presently in progress. The first problem concerns the compatibility with the various ad hoc braneworld models, which use dif ferent bulk geometries and or additional symmetries. The fact that eq.(7) is a homogeneus equation (a consequence of confinement condition), has forced us to appeal to a conformal flat condition (42). Our explanation is that the embedding equations have two independent variables $g_{\mu\nu}$ and $k_{\mu\nu}$ and one missing equation, which is usually supplied as an additional assumption in the mentioned models. Regarding $k_{\mu\nu}$ as a spin-2 field over the braneworld, then it must obey an Einstein-like equation. We already know what is that equation as stated in a previous communication [48] but the source term is still undefined. The second problem concerns inflation. If the extrinsic curvature effect agrees with the power spectrum of the PLANCK collaboration, then we need to explain how it relates to the inflaton. In our view, since our arguments in this paper were all classical, this can only be solved when a proper quantum theory of the brane-world is developed along with Nash’s theory of embedded geometries. The third problem is the explanation of the cosmological constant problem within the context of the brane-world. As the reader may have already guessed, the extrinsic curvature must play a role on that specific problem.

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