Sharp stability of a string with local degenerate Kelvin–Voigt damping

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This paper is on the asymptotic behavior of the elastic string equation with localized degenerate Kelvin–Voigt damping

\[ u_{tt}(x, t) - [u_x(x, t) + b(x)u_{xt}(x, t)]_x = 0, \quad x \in (-1, 1), \quad t > 0, \tag{0.1} \]

where \( b(x) = 0 \) on \( x \in (-1, 0] \), and \( b(x) = x^\alpha > 0 \) on \( x \in (0, 1) \) for \( \alpha \in (0, 1) \). It is known that the optimal decay rate of solution is \( t^{-2} \) in the limit case \( \alpha = 0 \) and exponential for \( \alpha \geq 1 \). When \( \alpha \in (0, 1) \), the damping coefficient \( b(x) \) is continuous, but its derivative has a singularity at the interface \( x = 0 \). In this case, the best known decay rate is \( t^{-\frac{3-\alpha}{2(1-\alpha)}} \), which fails to match the optimal one at \( \alpha = 0 \). In this paper, we obtain a sharper polynomial decay rate \( t^{-\frac{3-\alpha}{2(1-\alpha)}} \). More significantly, it is consistent with the optimal polynomial decay rate at \( \alpha = 0 \) and uniform boundedness of the resolvent operator on the imaginary axis at \( \alpha = 1 \) (consequently, the exponential decay rate at \( \alpha = 1 \) as \( t \to \infty \)). This is a big step toward the goal of obtaining eventually the optimal decay rate.

1 INTRODUCTION

In this paper, we consider elastic string (one-dimensional wave equation) with local viscoelastic damping of Kelvin–Voigt type. The mathematical model is the following partial differential equation:

\[
\begin{align*}
    u_{tt}(x, t) - [u_x(x, t) + b(x)u_{xt}(x, t)]_x &= 0 & \text{in} & \quad (-1, 1) \times \mathbb{R}^+, \\
    u(-1, t) &= u(1, t) = 0 & \text{in} & \quad \mathbb{R}^+, \\
    u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_0'(x) & \text{in} & \quad [-1, 1].
\end{align*}
\tag{1.1}
\]

The damping coefficient function \( b(\cdot) : (-1, 1) \to \mathbb{R}^+ \cup \{0\} \), belongs to \( L^\infty((-1, 1)) \) and satisfies

\[
b(x) = 0 \quad \text{for} \quad x \in (-1, 0) \quad \text{and} \quad b(x) = a(x) = x^\alpha \quad \text{with} \quad \alpha \in [0, 1) \quad \text{for} \quad x \in [0, 1).
\tag{H1}
\]

In the theory of elasticity, Kelvin–Voigt damping is a type of viscoelastic damping, which assumes that the stress is a linear function of strain and strain rate. When it is globally distributed, the solution to the corresponding elastic equation (string, beam, plate) is exponentially stable and analytic [12]. In 1998, Chen et al. [9, 17] discovered that the semigroup associated with the above system (1.1) is not exponentially stable if the damping is locally distributed and \( a(x) \) is
proportional to the characteristic function of any subinterval of the domain. This surprising result revealed that, unlike the viscous damping, the Kelvin–Voigt damping does not follow the well-known geometric optics condition [3]. In 2002, it was proved in Liu and Liu [18] that system (1.1) is exponentially stable if \( b'(\cdot) \in C^{0,1}[-1,1] \). Later, the smoothness condition on \( b(\cdot) \) was weakened to \( b(\cdot) \in C^1[-1,1] \) and the following conditions [33]:

\[
a'(0) = 0, \quad \int_0^x \frac{|a'(s)|^2}{a(s)} \, ds \leq C|a'(x)|, \quad \forall x \in [0,1], \ C > 0. \tag{1.2}
\]

It is easy to see that \( a(x) = x^\alpha, \ \alpha > 1 \) satisfies condition (1.2), but not when \( \alpha = 1 \). The exponential stability of this case was confirmed later in Liu and Zhang [21]. On the other hand, Liu and Rao in 2005 [20] proved that the semigroup corresponding to system (1.1) is polynomially stable of order at least 2 if \( a(\cdot) \in C(0,1) \) and \( a(x) \geq \tilde{a} > 0 \) on (0,1), which includes the case \( \alpha = 0 \) in Equation (1.1). Actually, the order can be improved to 2 without any modification to the proof, thanks to the necessary and sufficient conditions for polynomial stability by Borichev and Tomilov [6]. More recently, the optimality of this order was confirmed in Liu and Zhang [21]. On the other hand, Liu and Rao in 2005 [20] proved that the semigroup corresponding to system (1.1) is polynomially stable if \( a(\cdot) \in C(0,1) \) and \( a(x) \geq \tilde{a} > 0 \) on (0,1), which includes the case \( \alpha = 0 \) in Equation (1.1). Actually, the order can be improved to 2 without any modification to the proof, thanks to the necessary and sufficient conditions for polynomial stability by Borichev and Tomilov [6]. More recently, the optimality of this order was confirmed in Liu and Zhang [21]. The same optimal polynomial decay rate was obtained in Ghader et al. [10] for the damping Kelvin–Voigt mechanism acting in any subinterval of the one-dimensional domain.

Before stating the main results, we introduce the notions of stability that we encounter in this work.

**Definition 1.1.** Let \( A : D(A) \subset H \to H \) generate a \( C_0 \)-semigroup \( e^{tA} \) on Hilbert space \( H \). The semigroup \( e^{tA} \) is said to be polynomially stable of order \( \gamma > 0 \) if there exists a positive constant \( C \) such that

\[
\|e^{tA}x_0\|_H \leq C t^{-\gamma} \|x_0\|_{D(A)}, \quad \forall t \geq 1, \ x_0 \in D(A),
\]

where \( \|x_0\|_{D(A)} = \|x_0\|_H + \|Ax_0\|_H \).

In this paper, we shall give a sharper decay rate for system (1.1) than that in Han et al. [13]. In fact, we obtain the sharper polynomial decay rate \( t^{-\frac{2-\alpha}{1-\alpha}} \). More significantly, this rate is consistent with the optimal polynomial decay rate at \( \alpha = 0 \) and the exponential decay rate at \( \alpha = 1 \). The qualitative change of the decay rate at \( \alpha = 1 \) is because the norm of the resolvent operator of the infinitesimal generator associated with the system on the imaginary axis becomes bounded at \( \alpha = 1 \). However, whether this rate is optimal is still an open question. We would like to point out here that showing optimal polynomial decay rate often relies on some knowledge about the spectrum of the system, for example in Section 3.2 of Batty et al. [5]. However, the spectral analysis for the asymptotic behavior of the eigenvalues of system (1.1) is a formidable task due to the degeneracy of the highest order term in its eigensystem. In fact, although there have been a few results on the asymptotic behavior of the spectrum for the system with global Kelvin–Voigt damping (see Refs. [11, 31]), the asymptotic expressions of the eigenvalues for the system with locally Kelvin–Voigt damping remains as an difficult open problem as commented in Xu and Mastorakis [31].

This paper is organized as follows. First, we present our main results and some preliminaries in Section 2. Section 3 is devoted to the proofs of the main results.

### 2 MAIN RESULTS AND PRELIMINARIES

In this section, we shall recall some results concerning well-posedness and the weighted Hardy inequality, as well as present the main results.
Let $H^1_0(-1,1)$ be the Sobolev space $\{u \in H^1(-1,1) \mid u(-1) = u(1) = 0\}$ with norm $\| u \|_{H^1_0(-1,1)} = \| u_x \|_{L^2(-1,1)}$ for any $u \in H^1_0(-1,1)$. We introduce a Hilbert space

$$H = H^1_0(-1,1) \times L^2(-1,1),$$

(2.1)

whose inner product induced norm is given by

$$\| U \|_H = \sqrt{\| u \|_{H^1_0(-1,1)}^2 + \| v \|_{L^2(-1,1)}^2}, \quad \forall U = (u, v) \in H.$$ 

(2.2)

Define an unbounded operator $A : D(A) \subset H \to H$ by

$$AU = \begin{pmatrix} u \\ v \\ (u' + bv')' \end{pmatrix} \quad \forall U = \begin{pmatrix} u \\ v \end{pmatrix} \in D(A),$$

(2.3)

and

$$D(A) = \left\{ (u, v) \in H \mid v \in H^1_0(-1,1), \ (u' + bv')' \in L^2(-1,1) \right\}.$$ 

(2.4)

Then system (1.1) can be written as

$$\frac{dU}{dt} = AU, \quad \forall \ t > 0, \quad U(0) = (u_0, v_0)^T.$$ 

(2.5)

It is easy to check that $A$ is dissipative. Indeed, a direct calculation yields

$$Re\langle AU, U \rangle_H = -\int_0^1 x^\alpha |v'|^2 \, dx \leq 0.$$ 

(2.6)

If the coefficient function $b(\cdot) \geq 0$ satisfies assumption (H1), it is known in Chen et al. [9] that the following result on well-posedness of the system (2.5) holds by employing semigroup theories (see Pazy [25] or Lemma 2.1 in Chen et al. [9]). In fact, by applying the abstract semigroup formulation in Chen et al. [9], Section 2) to the system (1.1), the assumptions (H1)–(H3) in Chen et al. [9] are verified, and so the Lemma 2.1 in Chen et al. [9], based on the well-known Lumer–Phillips theorem, yields the well-posedness of system (2.1).

**Lemma 2.1.** $A$ generates a contractive $C_0$-semigroup on $H$ and

$$i\mathbb{R} \subset \rho(A).$$

(2.7)

We also have the following lemma, which is from Lemmas 2.2 to 2.3 in Liu and Zhang [21] or Han et al. [14] deduced by the weighted Hardy’s inequality (see Stepanov [29]).

**Lemma 2.2.** Assume that $\beta > -1$ and $\alpha < 1$ are two constants. Then, there exists a $C = C(\alpha, \beta) > 0$ so that

$$\int_0^1 x^\beta |\xi(x)|^2 \, dx \leq C \int_0^1 x^\alpha |\xi(x)|^2 \, dx,$$

(2.8)

for any $\xi(x) \in W^{1,1}(0,1)$ satisfying $x^\frac{\alpha}{2} \xi(x) \in L^2(0,1)$ and $\xi(1) = 0$.

Our subsequent findings on polynomial stability will rely on the following result from Theorem 2.4 in Borichev and Tomilov [6] or Refs. [4, 20], which gives necessary and sufficient conditions for a semigroup to be polynomially stable.
Lemma 2.3. Let $A : D(A) \subset H \rightarrow H$ generate a bounded $C_0$-semigroup $e^{tA}$ on Hilbert space $H$. Assume that

\[ i \mathbb{R} \subset \rho(A). \quad (2.9) \]

Then, the semigroup $e^{tA}$ is polynomially stable of order $\frac{1}{\theta}$ if and only if

\[ \lim_{\omega \in \mathbb{R}, |\omega| \to \infty} |\omega|^{-\theta} \| (i\omega I - A)^{-1} \|_{L(H)} < \infty. \quad (2.10) \]

Our main result in this paper is the following.

Theorem 2.1. Assume that the damping coefficient $b(x)$ in Equation (1.1) satisfies the condition $(H1)$. Then system (1.1) is polynomially stable of order $\frac{1 - \alpha}{2 - \alpha}$, that is

\[ \| e^{At}U_0 \|_H \leq C t^{-\frac{2 - \alpha}{1 - \alpha}} \| U_0 \|_{I(A)}, \quad \forall U_0 \in D(A), \quad t \geq 1, \quad (2.11) \]

where the constant $C > 0$ is independent of $U_0$.

3 ESTIMATE FOR THE RESOLVENT OPERATOR (PROOF OF THEOREM 2.1)

In this section, we shall prove Theorem 2.1. Due to Lemma 2.3, along with Lemma 2.1, this is equivalent to showing that there exists constant $r > 0$ such that

\[ \inf \{ |\omega|^\theta \| i\omega U - AU \|_H \mid U \in D(A), \| U \|_H = 1, \omega \in \mathbb{R}, |\omega| > 1 \} \geq r \quad (3.1) \]

for the parameter

\[ \theta = \frac{1 - \alpha}{2 - \alpha}. \quad (3.2) \]

Suppose that Equation (3.1) fails. Then there exist a sequence of real numbers $\omega_n$ and a sequence of functions $\{U_n\}_{n=1}^{\infty} = \{(u_n, v_n)\}_{n=1}^{\infty} \subset D(A)$ with $\|U_n\|_H = 1$ such that

\[ |\omega_n|^\theta \| i\omega_n U_n - AU_n \|_H = o(1). \quad (3.3) \]

In what follows we assume $\omega_n > 1$ since the proof is similar for negative $\omega_n$. One notes that $\omega_n \to \infty$ due to Equation (2.7) (otherwise, if $\omega_n \to \omega_0$, one can deduce that $\omega_0 \in \sigma(A)$, which contradicts Equation (2.7)). It follows from Equation (3.3) that

\[ \omega_n^\theta (i\omega_n u_n - v_n) = o(1) \quad \text{in} \quad L^2(-1, 1), \quad (3.4) \]

\[ \omega_n^\theta (i\omega_n v_n - (u_n + b(x)v_n')) = o(1) \quad \text{in} \quad L^2(-1, 1). \quad (3.5) \]

Let

\[ u_{1,n} \equiv u_n \chi_{[-1,0]}, \quad v_{1,n} \equiv v_n \chi_{[-1,0]}, \quad (3.6) \]

and

\[ u_{2,n} \equiv u_n \chi_{[0,1]}, \quad v_{2,n} \equiv v_n \chi_{[0,1]}, \quad T_n = u_{2,n}' \chi_{[0,1]} + x^\alpha v_{2,n}' \chi_{[0,1]}, \quad (3.7) \]
Then, by Equation (3.3), one has
\[ g_{1,n} = \omega_n^\delta (i\omega_n u_{1,n} - v_{1,n}) = o(1) \quad \text{in} \quad H^1(-1,0), \tag{3.8} \]
\[ g_{2,n} = \omega_n^\delta (i\omega_n v_{1,n} - u_{1,n}') = o(1) \quad \text{in} \quad L^2(-1,0), \tag{3.9} \]
\[ f_{1,n} = \omega_n^\delta (i\omega_n u_{2,n} - v_{2,n}) = o(1) \quad \text{in} \quad H^1(0,1), \tag{3.10} \]
\[ f_{2,n} = \omega_n^\delta (i\omega_n v_{2,n} - T_n') = o(1) \quad \text{in} \quad L^2(0,1), \tag{3.11} \]
and the following connecting boundary conditions:
\[ T_n(0) = u_{1,n}'(0), \quad u_{1,n}(0) = u_{2,n}(0), \quad v_{1,n}(0) = v_{2,n}(0). \tag{3.12} \]

By Equations (2.6) and (3.3), we obtain that
\[ \omega_n^\delta \left( i\omega_n U_n - A U_n \right)_H = \omega_n^\delta x^2 u_{2,n}' \parallel_{L^2(0,1)}^2 = o(1), \tag{3.13} \]
which along with Equation (3.10) yields
\[ \parallel x^2 u_{2,n}' \parallel_{L^2(0,1)} = o(1). \tag{3.14} \]

In what follows, we shall reach a contradiction by showing \( \parallel U_n \parallel_H = o(1). \)

**Lemma 3.1.** For \( n \to \infty \), one has the following estimates
\[ |v_{2,n}(0)| = \omega_n^{-\frac{\delta}{2}} o(1), \quad \parallel v_{2,n} \parallel_{L^2(0,1)} = \omega_n^{-\frac{\delta}{2}} o(1). \tag{3.15} \]

**Proof.** A direct computation gives that when \( 0 \leq \alpha < 1 \),
\[ |v_{2,n}(y)| = \left( \int_y^1 u_{2,n}' dx \right) \leq \left( \int_y^1 x^\alpha |u_{2,n}'|^2 dx \right)^{\frac{1}{2}} \left( \int_y^1 x^{-\alpha} dx \right)^{\frac{1}{2}}. \tag{3.16} \]

Combining this with Equation (3.13) and letting \( y = 0 \), we obtain the first estimate in Equation (3.15).

Moreover, by Equation (3.16), along with Equation (3.13), we have
\[ \int_0^1 |v_{2,n}|^2 dx \leq \int_0^1 \left( \int_y^1 x^\alpha |u_{2,n}'|^2 dx \right) \left( \int_y^1 x^{-\alpha} dx \right)^{\frac{1}{2}} dx = \omega_n^{-\frac{\delta}{2}} o(1). \tag{3.17} \]

Thus, the second estimate in Equation (3.15) follows. \( \square \)

**Lemma 3.2.** Set \( \xi_n = \omega_n^{-\delta} \) and \( Q_n = [\xi_n/2, \xi_n] \), where \( \delta = \frac{1}{2-\alpha} \). Then there exists a sequence \( \eta_n \in Q_n \) such that
\[ |T_n(\eta_n)| = o(1). \tag{3.18} \]

**Proof.** By Equations (3.13) and (3.14), a direct computation gives
\[ \min_{x \in Q_n} |T_n(x)| \leq \left[ 4\omega_n^\delta \int_{Q_n} (|u_{2,n}'|^2 + |\alpha v_{2,n}'|^2)dx \right]^{\frac{1}{2}} \leq 2\omega_n^\delta \left( \max_{x \in Q_n} |x^{-\alpha}| \int_{Q_n} x^\alpha |u_{2,n}'|^2 dx + \max_{x \in Q_n} |x^\alpha| \int_{Q_n} x^{\alpha} |v_{2,n}'|^2 dx \right)^{\frac{1}{2}} \leq \omega_n^{\frac{1}{2}(\delta + 5\alpha - 2 - \delta)} \left( \frac{1}{2}(\delta - 5\alpha - \delta) o(1) + \omega_n^{\frac{1}{2}(\delta - 5\alpha - \delta)} o(1) = o(1). \tag{3.19} \right] \]
Here we have used $\delta = \frac{1}{2-\alpha}$ and $\theta = \frac{1-\alpha}{2-\alpha}$ as given in Equation (3.2). Consequently, we are able to select $\eta_n \in Q_n$ such that Equation (3.18) holds.

Integrating Equation (3.11) on $(0, \eta_n)$ yields

$$i\omega_n \int_0^{\eta_n} v_{2,n} dx + T_n(0) - T_n(\eta_n) = \omega_n^{-\theta} o(1).$$

(3.20)

We now claim that

$$\omega_n \int_0^{\xi_n} |v_{2,n}| dx = o(1),$$

(3.21)

and prove it later.

With Equation (3.21) at hand, let us substitute Equations (3.18) and (3.21) into Equation (3.20), and note that $\eta_n \leq \xi_n$ to get

$$|T_n(0)| = o(1).$$

(3.22)

Then, multiplying Equation (3.11) with $\omega_n^{-\theta} u_{2,n}$ and integrating it on $(0,1)$, one gets

$$i\omega_n \int_0^1 v_{2,n} u_{2,n} dx + \int_0^1 (u'_{2,n} + x^\alpha v'_{2,n}) u'_{2,n} dx + T_n(0) u_{2,n}(0) = o(1).$$

(3.23)

Similarly, it follows from Equation (3.10) that

$$i\omega_n \int_0^1 v_{2,n} u_{2,n} dx + \int_0^1 |v_{2,n}|^2 dx = o(1).$$

(3.24)

Combining the above two equations with Equations (3.13) and (3.22), we obtain

$$\|u'_{2,n}\|_{L^2(0,1)}^2 - \|v_{2,n}\|_{L^2(0,1)}^2 = o(1),$$

(3.25)

which together with the second estimate in Equation (3.15) implies that

$$\|u'_{2,n}\|_{L^2(0,1)} = o(1).$$

(3.26)

Taking $L^2(-1,0)$—inner product of Equation (3.9) with $\omega_n^{\theta} (x+1) u'_{1,n}$, and using Equation (3.8), we have

$$\int_{-1}^0 \left( i\omega_n v_{1,n} - u''_{1,n} \right)(x+1) u'_{1,n} dx = - \int_{-1}^0 \left( (x+1)v_{1,n} u_{1,n} + (x+1)u'_{1,n} u_{1,n} \right) dx + o(1) = o(1).$$

(3.27)

On the one hand, by integration by parts, we have

$$\int_{-1}^0 (x+1)v_{1,n} u'_{1,n} dx = |v_{1,n}(0)|^2 - \int_{-1}^0 ((x+1)v'_{1,n} + v_{1,n}) u_{1,n} dx,$$

(3.28)

and hence

$$2 \text{Re} \int_{-1}^0 (x+1)v_{1,n} u'_{1,n} dx = |v_{1,n}(0)|^2 - \int_{-1}^0 |v_{1,n}|^2 dx.$$

(3.29)

On the other hand, similarly, using integration by parts, we get

$$2 \text{Re} \int_{-1}^0 (x+1)u''_{1,n} u'_{1,n} dx = |u'_{1,n}(0)|^2 - \int_{-1}^0 |u'_{1,n}|^2 dx.$$

(3.30)
Hence
\[
2\text{Re}(-\int_{-1}^{0} \left( (x+1)v'_{1,n}u_{1,n} + (x+1)u''_{1,n}u_{1,n} \right) dx) \quad (3.31)
\]
\[
= -|v_{1,n}(0)|^2 + \|v_{1,n}\|_{L^2(-1,0)}^2 - |u'_{1,n}(0)|^2 + \|u'_{1,n}\|_{L^2(-1,0)}^2 = o(1). \quad (3.31)
\]

Thus, by Equation (3.22), the first estimate in Equation (3.15) and the transmission conditions (3.12) at the interface, we get
\[
\|v_{1,n}\|_{L^2(-1,0)}^2 + \|u'_{1,n}\|_{L^2(-1,0)}^2 = o(1). \quad (3.32)
\]
Finally, by the second estimate in Equations (3.15), (3.26), and (3.32), we have achieved the contradiction
\[
\|U_n\|_H = o(1). \quad (3.33)
\]

Now, in order to complete the proof, it is sufficient to show Equation (3.21) holds.
Indeed, by the Hölder inequality, we have
\[
\omega_n \int_{0}^{1} |v_{2,n}| dx \leq C \omega_n^{1-\frac{\delta}{2}} \left( \int_{0}^{1} |v_{2,n}|^2 dx \right)^{\frac{1}{2}} \\
\leq C \omega_n^{1-\frac{\delta}{2}} \left| \text{Re} \int_{0}^{1} x v_{2,n} \overline{v_{2,n}} dx \right|^{\frac{1}{2}} \\
\leq C \omega_n^{1-\frac{\delta}{2}} \left( \int_{0}^{1} x^\alpha |v'_{2,n}|^2 dx \right)^{\frac{1}{4}} \left( \int_{0}^{1} x^{2-\alpha} |v_{2,n}|^2 dx \right)^{\frac{1}{4}}. \quad (3.34)
\]
If we can show the estimate (3.37) in the following Lemma 3.3 holds, then above inequality leads to
\[
\omega_n \int_{0}^{1} |v_{2,n}| dx \leq \omega_n^{1-\frac{\delta}{2}} \omega_n^{-\frac{2+\theta}{4}} \omega_n^{-2-\theta} o(1) = o(1) \quad (\text{using (3.13) and (3.37))}
\]

since
\[
1 - \frac{\delta}{2} - \frac{\theta}{4} - \frac{2 + \theta}{4} = 1 - \frac{1}{2(2-\alpha)} - \frac{1 - \alpha}{2(2-\alpha)} = 0. \quad (3.36)
\]

Hence, Equation (3.21) holds.
The rest is devoted to showing the following Lemma 3.3.

**Lemma 3.3.** Let $0 \leq \alpha < 1$. Then it holds that
\[
\omega_n \int_{0}^{1} x^{2-\alpha} |v_{2,n}|^2 dx = \omega_n^{1-\delta} o(1). \quad (3.37)
\]

**Proof.** Multiplying Equation (3.11) with $\omega_n^{-\delta} x^{2-\alpha} \overline{v_{2,n}}$ and integrating it from 0 to 1, we get
\[
\omega_n \int_{0}^{1} x^{2-\alpha} |v_{2,n}|^2 dx + \int_{0}^{1} T_n \left( (2-\alpha)x^{1-\alpha} v_{2,n} + x^{2-\alpha} v'_{2,n} \right) dx \\
= \omega_n \int_{0}^{1} x^{2-\alpha} |v_{2,n}|^2 dx + \int_{0}^{1} \left[ (2-\alpha)x^{1-\alpha} u'_{2,n} \overline{v_{2,n}} + x^{2-\alpha} u_{2,n} v'_{2,n} \overline{v_{2,n}} \right] dx \\
+ (2-\alpha)x^{\alpha+2-\alpha} |u'_{2,n}|^2 \int_{0}^{1} x^{2-\alpha} f_{2,n} \overline{v_{2,n}} dx. \quad (3.38)
\]
By taking the imaginary parts of the above equality and using the fact that
\[
\omega_n^{-\frac{1}{2}} \left| \int_0^1 x^{2-\alpha} f_{2,n} u_{2,n} d\bar{v}_{2,n} dx \right| \leq \frac{\omega_n}{2} \int_0^1 x^{2-\alpha} |u_{2,n}|^2 dx + \frac{\omega_n^{2\delta-1}}{2} \int_0^1 x^{2-\alpha} |f_{2,n}|^2 dx,
\] (3.39)
we obtain
\[
\omega_n \int_0^1 x^{2-\alpha} |u_{2,n}|^2 dx
\leq 2 \left[ \int_0^1 [(2 - \alpha)x^{1-\alpha} u_{2,n}' \overline{u_{2,n}} + x^{2-\alpha} u_{2,n}' \overline{u_{2,n}} + (2 - \alpha)xv_{2,n}' \overline{u_{2,n}}] dx \right] + \omega_n^{2\delta-1} o(1).
\] (3.40)

In the following, in order to show Equation (3.37), we shall estimate the three terms on the right-hand side of Equation (3.40), respectively.

**Observation 1.** By Hölder inequality, we have
\[
\left| \int_0^1 x^{1-\alpha} u_{2,n}' \overline{u_{2,n}} dx \right| \leq \left( \int_0^1 x^{2} |u_{2,n}'|^2 dx \right)^{\frac{1}{2}} \left( \int_0^1 x^{2-3\alpha} |u_{2,n}|^2 dx \right)^{\frac{1}{2}}.
\] (3.41)

Note that \(2 - 3\alpha > -1\). Then by Lemma 2.2 and Equation (3.13), one has that
\[
\left( \int_0^1 x^{2-3\alpha} |u_{2,n}|^2 dx \right)^{\frac{1}{2}} \leq C \|x^{\frac{\alpha}{2}} u_{2,n}'\|_{L^2(0,1)} = \omega_n^{\frac{\theta}{2}} o(1).
\] (3.42)

Then, substituting Equations (3.14) and (3.42) into Equation (3.41), we obtain
\[
\left| \int_0^1 x^{1-\alpha} u_{2,n}' \overline{u_{2,n}} dx \right| \leq \omega_n^{1-\delta} o(1).
\] (3.43)

**Observation 2.** By the Hölder inequality, we have
\[
\left| \int_0^1 x^{2-\alpha} u_{2,n}' \overline{u_{2,n}} dx \right| = \left| \int_0^1 x^{1-\frac{2\alpha}{2}} x^{\frac{\alpha}{2}} u_{2,n}' x^{\frac{\alpha}{2}} v_{2,n}' dx \right|
\leq \|x^{\frac{\alpha}{2}} u_{2,n}'\|_{L^\infty(0,1)} \|x^{\frac{\alpha}{2}} v_{2,n}'\|_{L^\infty(0,1)} = \omega_n^{1-\delta} o(1).
\] (3.44)

Here, we have used the fact that \(0 \leq \alpha < 1\).

**Observation 3.** Using Hölder inequality, one has that for a positive constant \(\varepsilon\), there exists \(C_\varepsilon\) such that
\[
\left| \int_0^1 x u_{2,n}' \overline{v_{2,n}} dx \right|
\leq \|x^{\frac{\alpha}{2}} u_{2,n}'\|_{L^\infty(0,1)} \|x^{\frac{2-\alpha}{2}} v_{2,n}\|_{L^\infty(0,1)}
\leq \varepsilon \omega_n \|x^{\frac{2-\alpha}{2}} u_{2,n}\|_{L^\infty(0,1)}^{\frac{2-\alpha}{2}} + C_\varepsilon \omega_n^{1-\delta} \|x^{\frac{\alpha}{2}} v_{2,n}'\|_{L^2(0,1)}^2
= \varepsilon \omega_n \|x^{\frac{2-\alpha}{2}} u_{2,n}\|_{L^\infty(0,1)}^2 + \omega_n^{1-\delta} o(1).
\] (3.45)

By substituting **Observation I, II, III** into the right-hand side of Equation (3.40), we obtained the desired result (3.37). □
CONCLUSION

In this paper, we have obtained a sharper estimate of the decay rate of solution to system (1.1) when the damping coefficient function $a(x)$ is equivalent to $x^\alpha$, $0 \leq \alpha < 1$ near the interface $x = 0$, which is consistent with the existing optimal decay rate when $\alpha = 0$, and with the known exponential decay rate when $\alpha = 1$. We summarize the stability results for Equation (1.1) in the following table.

| $\alpha$ | Damping coefficient $a(x)$ | Decay rate of solution |
|----------|-----------------------------|------------------------|
| 0        | $x^2$                       | Optimal polynomial decay rate $t^{-2}$ |
| (0,1)    | $x^2$                       | Polynomial decay rate $t^{-\frac{\alpha}{1-\alpha}}$ |
| $\geq 1$ | $x^2$                       | Exponential decay rate |

We would like to point out that the above results are proved only for the damped region, which is an interval including one end point of the spatial domain. It remains as an open question for the general case where the damping coefficient function $b(x)$ is supported on a proper interval $(x_1, x_2)$ with $-1 < x_1 < x_2 < 1$, and behaves like $(x - x_1)^\beta$ and $(x - x_2)^\beta$ near the interfaces $x = x_1, x_2$, respectively. The optimality of the polynomial decay rate obtained in this paper is another open question as stated in the Introduction.

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