MIRROR SYMMETRY AND GENERALIZED COMPLEX MANIFOLDS

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Abstract. In this paper we develop a relative version of T-duality in generalized complex geometry which we propose as a manifestation of mirror symmetry. Let $M$ be an $n$–dimensional smooth real manifold, $V$ a rank $n$ real vector bundle on $M$, and $\nabla$ a flat connection on $V$. We define the notion of a $\nabla$–semi-flat generalized complex structure on the total space of $V$. We show that there is an explicit bijective correspondence between $\nabla$–semi-flat generalized complex structures on the total space of $V$ and $\nabla^\vee$–semi-flat generalized complex structures on the total space of $V^\vee$. Similarly we define semi-flat generalized complex structures on real $n$–torus bundles with section over an $n$-dimensional base and establish a bijective correspondence between semi-flat generalized complex structures on pairs of dual torus bundles. Along the way, we give methods of constructing generalized complex structures on the total spaces of vector bundles and torus bundles with sections. We also show that semi-flat generalized complex structures give rise to a pair of transverse Dirac structures on the base manifold. We give interpretations of these results in terms of relationships between the cohomology of torus bundles and their duals. We also study the ways in which our results generalize some well established aspects of mirror symmetry as well as some recent proposals relating generalized complex geometry to string theory.

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Mirror symmetry is often thought of as relating the very different worlds of complex geometry and symplectic geometry. It was recently shown by Hitchin [17] that symplectic and complex structures on a manifold have a simple common generalization called a generalized complex structure. This is a complexified version of Dirac geometry [10] along with an extra non-degeneracy condition. It is expected that Mirror Symmetry should give rise to an involution on sectors of the moduli space of all Generalized Complex Manifolds of a fixed dimension. One of the most concrete descriptions of the mirror correspondence is the Strominger-Yau-Zaslow picture [33] in which Mirror Symmetry is interpreted as a relative T-duality along the fibers of a special Lagrangian torus fibration. This is sometimes referred to as “T-Duality in half the directions”. In our previous work [2], we investigated the linear algebraic aspects of T-duality for generalized complex structures. See also [37] for the analogous story in Dirac geometry. In this paper, we go one step further and construct an explicit mirror involution on certain moduli of generalized complex manifolds. Similarly to the case of Calabi-Yau manifolds the definition of our mirror involution depends on additional data. In our set up, we will consider generalized complex manifolds equipped with a compatible torus fibration. This involution, when applied to such a manifold, gives another with the same special properties, which we propose to identify as its mirror partner. In the special cases of a complex or symplectic structure on a semi-flat Calabi-Yau manifold our construction reproduces the standard T-duality of [31, 25, 30]. In addition we get new examples of mirror symmetric generalized complex manifolds, e.g. the ones coming from B-field transforms of complex or symplectic structures.

If $V$ is a real vector space then [17] a generalized complex structure on $V$ is a complex subspace $E \subseteq (V \oplus V^\vee) \otimes \mathbb{C}$ that satisfies $E \cap \overline{E} = (0)$ and is maximally isotropic with respect to the canonical quadratic form on $(V \oplus V^\vee) \otimes \mathbb{C}$. Let

$$f : V \oplus V^\vee \to W \oplus W^\vee$$

be a linear isomorphism which is compatible with the canonical quadratic forms. Then $f$ induces a bijection between generalized complex structures on $V$ and generalized complex structures on $W$. Transformations of this type can be viewed as linear analogues of the T-duality transformations investigated in the physics literature (see [21, 35] and references therein). Mathematically they were studied in [37] for Dirac structures and in [2] for generalized complex structures. In this paper, the relevant case is where $V = A \oplus B, W = A^\vee \oplus B$, and $f : A \oplus B \oplus A^\vee \oplus B^\vee \to A^\vee \oplus B \oplus A \oplus B^\vee$ is the obvious shuffle map.
A generalized complex structure on a manifold $X$ is a maximally isotropic sub-bundle of $(T_X \oplus T_X') \otimes \mathbb{C}$ that satisfies $E \cap \overline{E} = (0)$ and that $E$ is closed under the Courant Bracket. In this paper, we shall preform a relative version of this T-duality for pairs of manifolds that are fibered over the same base and where the two fibers over each point are “dual” to each other. In other words we will find a way to apply the linear ideas above to the torus fibered approach. On each fiber, this process will agree with the linear map described above.

We relate the integrability of semi-flat (see definition 7.2) generalized almost complex structures on torus and vector bundles to data which lives only on the base manifold. We show that a semi-flat generalized almost complex structure is integrable if and only its mirror structure is integrable.

Using a natural connection on a torus bundle $Z \to M$ with zero section $s$, we will construct semi-flat generalized complex structures $\mathcal{J}$ on $Z$ from generalized almost complex structures $\mathcal{J}$ on the vector bundle $s^*T_{Z/M} \oplus T_M$. The definition of semi-flat includes the condition that

$$\mathcal{J}(s^*T_{Z/M} \oplus s^*T_{Z/M}') = T_M \oplus T_M'.$$

Then we have the following two results:

**Theorem 1.1.** $(8.4)$ A semi-flat generalized almost complex structure $\mathcal{J}$ on a torus bundle $Z \to M$ with zero section $s$ is integrable if and only if

$$[\mathcal{J}(\mathcal{S} \oplus \mathcal{S}'), \mathcal{J}(\mathcal{S} \oplus \mathcal{S}')] = 0$$

where $\mathcal{S}$ is the sheaf of flat sections of $s^*T_{Z/M}$.

**Corollary 1.2.** $(8.5)$ A semi-flat generalized almost complex structure $\mathcal{J}$ on a torus bundle $Z \to M$ is integrable if and only if its mirror structure $\tilde{\mathcal{J}}$ on the dual torus bundle $\tilde{Z} \to M$ is integrable.

These statements set the stage for understanding mirror symmetry and the mirror transform of D-branes in generalized Calabi-Yau geometry. Our results are a direct generalization of the setup employed by Arinkin and Polishchuk [31] in ordinary mirror symmetry. Explicit examples of this fact can be found in section 11.

We relate this transformation of geometric structures to a purely topological map on differential forms which descends to a map from the de Rham cohomology of $Z$ to the de Rham cohomology of $\tilde{Z}$. In particular, the map on differential forms exchanges the pure spinors associated to the generalized complex structure on $Z$ with the ones associated to the mirror generalized complex structure on $\tilde{Z}$. This type of transformation was also discussed in [1].
Throughout the paper we comment on how our results relate to some of the well established results and conjectures of mirror symmetry [31, 33, 25, 30, 24] and also what they say in regards to the new developments in generalized Kähler geometry [15] and the relationships between generalized complex geometry and string theory [21, 35, 15] which have appeared recently. As mentioned in [21] we may interpret these dualities as being a generalization of the duality between the $A-$model and $B-$model in topological string theory. In the generalized Kähler case, they can be interpreted as dualities of supersymmetric nonlinear sigma models [14]. To this end, in section 5 we sketch a relationship between branes in the sense of [15, 21] in a semi-flat generalized complex structure and branes in its mirror structure. For some simple examples of branes, we give the relationship directly. We also show in section 4 that the Buscher rules [6, 7] for the transformation of metric and B-field hold between the mirror pairs of generalized Kähler manifolds that we consider.

It will be very interesting to extend the discussion in section 5 to a full-fledged Fourier-Mukai transform on generalized complex manifolds. Unfortunately, the in-depth study of branes in generalized complex geometry is obstructed by the complexity of the the behavior of sub-manifolds with regards to a generalized complex structure. Several subtle issues of this nature were analyzed in our previous paper [2]. In particular we investigated in detail the theory of sub and quotient generalized complex structures, described a zoo of sub-manifolds of generalized complex manifolds and studied the relations among those. We also gave a classification of linear generalized complex structures and constructed a category of linear generalized complex structures which is well adapted to the question of quantization. In a future work we plan to incorporate the structure of a torus bundle in this analysis and construct a complete Fourier transform for branes.

For the benefit of the reader who may not be familiar with generalized complex geometry, we have included §2 which introduces the linear algebra and some basics on generalized complex manifolds. More details on these basics may be found in [2, 15, 17, 18, 19].

2. Notation, conventions, and basic definitions

Overall, we will retain the notation and conventions from our previous paper [2], and so we only recall the most important facts for this paper as well as some changes. The dual of a vector space $V$ will be denoted as $V^\vee$. We will often use the annihilator of a subspace $W \subseteq V$, which we will denote

$$\text{Ann}(W) = \{ f \in V^\vee \mid f|_W \equiv 0 \} \subseteq V^\vee.$$
We will need the pairing $\langle \bullet, \bullet \rangle$ on $V \oplus V^\vee$, given by (following [18])

$$\langle v + f, w + g \rangle = -\frac{1}{2} \left( f(w) + g(v) \right)$$

for all $v, w \in V, f, g \in V^\vee$.

Given $v \in V$ and $f \in V^\vee$, we will write either $\langle f \mid v \rangle$ or $\langle v \mid f \rangle$ for $f(v)$. This pairing corresponds to the quadratic form $Q(v + f) = -f(v)$.

We will tacitly identify elements $B \in \bigwedge^2 V^\vee$ with linear maps $V \to V^\vee$. When thought of in this way, we have that the map is skew-symmetric: $B = -B^\vee$.

We will often consider linear maps of $V \oplus W \to V' \oplus W'$. Sometimes, these be written as matrices

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix},$$

with the understanding that $T_1 : V \to W', T_2 : W \to V', T_3 : V \to W'$ and $T_4 : W \to W'$ are linear maps. All of these conventions will be extended to vector bundles and their sections in the obvious way.

If $M$ is a manifold, we let $C^\infty_M$ denote the sheaf of real-valued $C^\infty$ functions on $M$. We will use the same notation for a vector bundle and for its sheaf of sections. The tangent and cotangent bundles of $M$ will be denoted by $T_M$ and $T_M^\vee$. For a vector bundle $V$ over a manifold $M$ and a smooth map $f : N \to M$, we denote the pullback bundle by $f^*V$. A section of $f^*V$ which is a pullback of a section $e$ of $V$ will be denoted $f^*(e)$. If $f$ is an isomorphism onto its image or the projection map of a fiber bundle, the sections of this form give the sub-sheaf $f^{-1}V \subseteq f^*V$. We will sometimes replace $\bigwedge^\bullet T_M^\vee$ by $\Omega^\bullet_M$.

Now we will give some basic facts on generalized complex geometry that we will need in the paper. For more information the reader may see [2, 17, 15].

2.1. **Generalized almost complex manifolds.** Let $M$ be a real manifold. A **generalized almost complex structure** on a real vector bundle $V \to M$ has been defined [15, 17, 18] in the following equivalent ways:

- A sub-bundle $E \subseteq V_C \oplus V_C^\vee$ which is maximally isotropic with respect to the standard pairing $\langle \bullet, \bullet \rangle$ and satisfies $E \cap \overline{E} = 0$
- An automorphism $\mathcal{J}$ of $V \oplus V^\vee$ which is orthogonal with respect to $\langle \bullet, \bullet \rangle$ and satisfies $\mathcal{J}^2 = -1$.

**Example 2.1.** Let $V$ be a real vector bundle.
(a) Let $J$ be an almost complex structure on $V$. Then

$$J = \begin{pmatrix} J & 0 \\ 0 & -J^\vee \end{pmatrix}$$

is a generalized almost complex structure on $V$. If $J$ is a generalized complex structure on $V$ that can be written in this form, we say that $J$ is of complex type.

(b) Let $\omega$ be an almost symplectic form on $V$ (i.e., a non-degenerate section $\omega$ of $\bigwedge^2 V^\vee$). Then

$$J = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

is a generalized complex structure on $V$. We say that such a $J$ is of symplectic type.

There is also a way of describing generalized almost complex structures on $V$ in terms of line sub-bundles of $\bigwedge V^\vee \otimes \mathbb{C}$ or spinors. This interpretation is very convenient for some purposes.

**Definition 2.2.** [18, 15] Let $J$ be a generalized almost complex structure on a vector bundle $V$ over $M$. Define the canonical bundle to be the complex line bundle $L \subseteq \bigwedge^\cdot V^\vee \otimes \mathbb{C}$ consist of the sections $\phi$ satisfying $\iota_v \phi + \alpha \wedge \phi$ for all sections $v + \alpha$ of the $+i$ eigenbundle $E$ corresponding to the generalized almost complex structure on $V$. Sections of $L$ will be called representative spinors.

For the case of an almost symplectic manifold with two form $\omega$, this line bundle is generated by $\exp(-i\omega)$. For an almost complex manifold, one gets the usual canonical bundle. Spinor bundles can also be understood intrinsically in terms of the sheaves of modules over appropriate sheaf of Clifford algebras. The sections will satisfy certain restrictions over each fiber. They are known as pure spinors [9, 15, 18]. We have listed some of their features and examined their restriction to submanifolds in [2].

**Definition 2.3.** [18] In the special case that $V = T_M$ has a generalized almost complex structure, we call $M$ a generalized almost complex manifold.

In this case the spinor sections are differential forms. Such a manifold is always even dimensional as a real manifold. This can be shown by constructing two almost complex structures on $M$ out of the generalized almost complex structure [18]. This also follows from the classification of generalized complex structures on a vector space which was done in our previous paper [2]. For the case of manifolds, a local structure theorem for generalized complex manifolds has been proven by Gualtieri [15].
Consider a real vector bundle $V$ and an automorphism $\mathcal{J}$ of $V \oplus V^\vee$, written in matrix form as

$$\mathcal{J} = \begin{pmatrix} \mathcal{J}_1 & \mathcal{J}_2 \\ \mathcal{J}_3 & \mathcal{J}_4 \end{pmatrix}.$$ 

Let us record, for future use, the restrictions on the $\mathcal{J}_i$ coming from the conditions that $\mathcal{J}$ preserves the pairing $\langle \bullet, \bullet \rangle$ and satisfies $\mathcal{J}^2 = -1$. They are:

\begin{align*}
\mathcal{J}_1^2 + \mathcal{J}_2 \mathcal{J}_3 &= -1; \quad (2.1) \\
\mathcal{J}_1 \mathcal{J}_2 + \mathcal{J}_2 \mathcal{J}_4 &= 0; \quad (2.2) \\
\mathcal{J}_3 \mathcal{J}_1 + \mathcal{J}_4 \mathcal{J}_3 &= 0; \quad (2.3) \\
\mathcal{J}_1^2 + \mathcal{J}_3 \mathcal{J}_2 &= -1; \quad (2.4) \\
\mathcal{J}_4 &= -\mathcal{J}_1^\vee; \quad (2.5) \\
\mathcal{J}_2^\vee &= -\mathcal{J}_2; \quad (2.6) \\
\mathcal{J}_3^\vee &= -\mathcal{J}_3. \quad (2.7)
\end{align*}

2.2. $B$- and $\beta$-field transforms. [16, 17, 18, 15] Consider a real vector bundle $V$ and a global section $B$ of $\bigwedge^2 V^\vee$. Consider the transformation of $V \oplus V^\vee$

$$\exp(B) := \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}. $$

It is easy to see that $\exp(B)$ is an orthogonal automorphism of $V \oplus V^\vee$. Thus $\exp(B) \cdot E$ is a generalized almost complex structure on $V$ for any generalized almost complex structure $E \subseteq (V \oplus V^\vee) \otimes \mathbb{C}$ on $V$. We will call $\exp(B) \cdot E$ the $B$-field transform of $E$ defined by $B$. We should note here that these type of transformations are sometimes called gauge-transformations and were introduced with that name into real Dirac geometry [10] in [34]. For an overview of these transformations in the Dirac geometry context, see [3]. Similarly, if $\beta \in \bigwedge^2 V$, then

$$\exp(\beta) := \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$$

then $\exp(\beta) \cdot E$ will be called the $\beta$-field transform of $E$ defined by $\beta$. One can also write these transformations in terms of the orthogonal automorphisms $\mathcal{J}$ of $V \oplus V^\vee$. In this case, the actions of $B$ and $\beta$ are given by $\mathcal{J} \mapsto \exp(B)\mathcal{J}\exp(-B)$ and $\mathcal{J} \mapsto \exp(\beta)\mathcal{J}\exp(-\beta)$, respectively. We can also describe $B$-field transforms in terms of local spinor representatives: if a generalized almost complex structure on a real vector bundle $V$ is defined by a pure spinor
\[ \phi \in \bigwedge^\bullet V_C^\vee, \text{ and } B \in \bigwedge^2 V^\vee \text{ then the } B\text{-field transform of this structure corresponds to the pure spinor } \exp(-B) \wedge \phi \, [17, 18]. \] The \( \beta \)-field transform corresponds to the pure spinor \( \iota \exp(\beta) \phi \, [15, 18] \).

2.3. Integrability. Let \( M \) be a generalized almost complex manifold. The Courant bracket ([10], p. 645) is defined on sections of \((T_M \oplus T_M^\vee) \otimes \mathbb{C}\) by

\[
[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi + \frac{1}{2} \cdot d(t_Y \xi - t_X \eta).
\]

or equivalently

\[
[X + \xi, Y + \eta] = [X, Y] + t_X d\eta + \frac{1}{2} d t_X \eta - t_Y d\xi - \frac{1}{2} t_Y \xi.
\]

**Definition 2.4** (cf. [15], [17], [18]). Let \( M \) be a real manifold equipped with a generalized almost complex structure defined by \( E \subseteq (T_M \oplus T_M^\vee) \otimes \mathbb{C}\). We say that \( E \) is integrable if the sheaf of sections of \( E \) is closed under the Courant bracket. If that is the case, we also say that \( E \) is a generalized complex structure on \( M \), and that \( M \) is a generalized complex manifold.

**Remark 2.5** (cf. [10],[18]). As we noted in [2], the integrability condition for a generalized almost complex structure \( J \) is equivalent [2] to the vanishing of the Courant-Nijenhuis tensor

\[
N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]
\]

where \( X, Y \) are sections of \( T_M \oplus T_M^\vee \).

Integrability can also be expressed in terms of spinors [18, 15, 39]. If \( L \subseteq \bigwedge^\bullet T_M^\vee \otimes \mathbb{C} \) is the line bundle of spinors associated to a generalized almost complex structure \( J \) on a manifold \( M \) then \( J \) is integrable if and only if all sections \( \phi \) of \( L \) satisfy

\[
d\phi = \mathcal{L}_v \phi + \alpha \wedge \phi
\]

for some section \( v + \alpha \) of \((T_M \oplus T_M^\vee) \otimes \mathbb{C}\).

**Examples 2.6** (cf. [18, 15]). In the case that a generalized almost complex structure comes from an almost complex structure, it will be integrable if and only if the almost complex structure is integrable, giving a complex structure to the manifold. In the case that a generalized almost complex structure comes from a nondegenerate differential 2-form (almost symplectic structure), it will be integrable if and only if the form is closed, i.e. gives a symplectic structure to the manifold.
General $B$–field and $\beta$–field transformations need not preserve integrability. However [17], a closed 2-form $B$ acts on generalized complex structures on $M$ in the same way as described in §2.2. In fact, a $B$-transform by a 2-form on $M$ is an automorphism of the Courant bracket if and only if the 2-form is closed [18]. Note further that a $B$-field transform of a particular generalized complex structure can be integrable even if the 2-form is not closed. In fact, for any specific generalized complex manifold $(M, J)$, one can write down explicitly the conditions that need to be satisfied by a two form, $B$ or a bi-vector field $\beta$ in order for the $B$-field or $\beta$-field transform of $(M, J)$ to be integrable. We will study examples of this phenomenon in Section 11.

2.4. Generalized almost Kähler manifolds. We will need the notion [18, 15] of a generalized almost Kähler structure.

Definition 2.7. [15] A generalized almost Kähler structure on a manifold $M$ is specified by one of the equivalent sets of data.

1) A pair $(J, J')$ of commuting generalized almost complex structures whose product, $G = -JJ'$ is positive definite with respect to the standard quadratic form $\langle \cdot, \cdot \rangle$ on $T_M \oplus T_M^\vee$.

2) A quadruple $(g, b, J_+, J_-)$ consisting of a Riemannian metric $g$, two-form $b$, and two almost complex structures $J_+$ and $J_-$ such that the isomorphisms $\omega_+ = gJ_+: T_M \to T_M^\vee$ and $\omega_- = gJ_- : T_M \to T_M^\vee$ are anti-symmetric and hence correspond to non-degenerate two-forms.

The two sets of data are related explicitly as follows. The $(+1)$ eigenbundle of $G$ is the graph of $g + b : T_M \to T_M^\vee$. Denote this vector bundle by $C_+$, and the $(-1)$ eigenbundle (which is the graph of $b - g$) by $C_-$. Then

$$J_\pm = \pi_{T_M} \circ J \circ (\pi_{T_M}|_{C_\pm})^{-1}.$$ 

Conversely, given $(g, b, J_+, J_-)$, one defines

$$J = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} J_+ + J_- & -(\omega_+^{-1} - \omega_-^{-1}) \\ \omega_+ - \omega_- & -(J_+^\vee + J_-^\vee) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}$$

and

$$J' = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} J_+ - J_- & -(\omega_+^{-1} + \omega_-^{-1}) \\ \omega_+ + \omega_- & -(J_+^\vee - J_-^\vee) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}$$

Using this same notation we have that
$G = \begin{pmatrix} -g^{-1}b & g^{-1} \\ \ g - bg^{-1}b & bg^{-1} \end{pmatrix}.
$(2.8)

**Examples** 2.8. [15] Notice that this definition naturally generalizes the linear algebraic data of an Kähler manifold. We will refer to this as the ordinary Kähler case. There is an important family of examples which include the ordinary Kähler as a special case. They come from transforming both the complex and symplectic structures which occur in the ordinary Kähler case by the $B$-field $B$.

$$J = \begin{pmatrix} J & 0 \\ BJ + J^\lor B & -J^\lor \end{pmatrix}$$
$(2.9)$

and

$$J' = \begin{pmatrix} \omega^{-1}B & -\omega^{-1} \\ \omega + B\omega^{-1}B & -B\omega^{-1} \end{pmatrix}$$
$(2.10)$

where $\omega J = -J^\lor \omega$. The ordinary Kähler case of course comes about from setting $B$ to zero.

3. **T-duality**

Our main goal is to extend the usual T-Duality transformation of geometric structures on families of tori in a way that will allow us to incorporate generalized (almost) complex structures.

3.1. **T-duality in all directions.** In its simplest form, T-Duality exchanges geometric data on a torus $T \cong (S^1)^n$ with geometric data on the dual torus $T^\lor$. For instance if the torus $T$ is a complex manifold, then the dual torus is also naturally a complex manifold. This immediately generalizes to translation invariant (hence integrable) generalized complex structures on $T$.

Indeed, choose a realization of $T$ as a quotient $T = V/\Lambda$ of a real $n$-dimensional vector space $V$ by a sub-lattice $\mathbb{Z}^n \cong \Lambda \subseteq V$. Then specifying a translation invariant generalized complex structure on $T$ is equivalent to specifying a constant generalized complex structure $J \in GL(V \oplus V^\lor)$ on the vector space $V$. Now the dual torus $T^\lor$ has a natural realization as the quotient $T^\lor = V^\lor/\text{Hom}(\Lambda, \mathbb{Z})$. Thus, in order to describe the $T$-dual generalized complex structure on $T^\lor$ it suffices to produce a constant generalized complex structure on $V^\lor$. This can be done in a simple way: Let $\tau : V \oplus V^\lor \to V^\lor \oplus V$ be the transposition of the
two summands. Using the natural identification of $V^{\vee\vee}$ with $V$, we can also view $\tau$ as an isomorphism between $V \oplus V^{\vee}$ and $V^{\vee} \oplus V^{\vee\vee}$. We will continue to denote by $\tau$ the induced isomorphism $V_C \oplus V^{\vee}_C \rightarrow V^{\vee}_C \oplus V^{\vee\vee}_C \cong V^{\vee}_C \oplus V_C$. With this notation one has the following proposition.

**Proposition 3.1.** [2] The isomorphism $\tau$ induces a bijection between generalized complex structures on $V$ and generalized complex structures on $V^{\vee}$. If $E$ corresponds to $J \in \text{Aut}_\mathbb{R}(V \oplus V^{\vee})$, then $\tau(E)$ corresponds to $\tau \circ J \circ \tau^{-1}$.

**Remark 3.2.** Below, we will see that the transformation of the spinor representatives is a Fourier-Mukai type of transformation. The precise form of this transformation is given in equation 6.1. Notice that this proposition applies equally to generalized complex structures on the vector space $V$ and to constant generalized complex structures (which are automatically integrable) on $V$ thought of as a manifold. These in turn give generalized complex structures on tori which are quotients of the vector space.

We also have the following remark from [2]:

**Remark 3.3.** Suppose that $E$ is a generalized complex structure on a real vector space $V$ and $E'$ is the $B$-field transform of $E$ defined by $B \in \bigwedge^2 V^{\vee}$. Then, obviously, $\tau(E')$ is the $\beta$-field transform of $\tau(E)$, defined by the same $B \in \bigwedge^2 V^{\vee}$ (but viewed now as a bi-vector on $V^{\vee}$). Thus, the operation $\tau$ interchanges $B$- and $\beta$-field transforms.

The relationship from this last remark was exploited in [21] to produce an interesting conjectural relationship to non-commutative geometry.

### 3.2. More general T-duality.

It has been known for some time that the previous example of T-duality generalizes immediately to a whole family of T-duality transformations. This can be found for example [22] and the references therein. More recently Tang and Weinstein [37] applied this observation to Dirac structures to investigate the group of Morita equivalences of real non-commutative tori.

By analogy with the Tang-Weinstein construction we note that if $V = \bigoplus_{i=1}^m V_i$ and $W = \bigoplus_{i=1}^m W_i$, where each $W_i$ equals either $V_i$ or $V_i^{\vee}$, then the obvious isomorphism $\tau$ from $V \oplus V^{\vee}$ to $W \oplus W^{\vee}$ intertwines the canonical quadratic forms and hence it similarly gives a bijection between generalized complex structures on $V$ with those on $W$. Notice that these transformations are all real and so there is no problem with the transversality condition. In general, one could also consider as duality transformations, isometries $\tau$, from $V_C \oplus V^{\vee}_C$ to $W_C \oplus W^{\vee}_C$ such that $\tau \circ J \circ \tau^{-1}$ is a generalized complex structure on $W$ for all (or a family of) generalized complex structures $J$ on $V$. A special case of this duality can easily
be seen to be the right starting point in generalizing the symplectic/complex correspondence in [31]. To see this, let $M$ be a real manifold with trivial tangent bundle, $X$ a real torus with its normal group structure and $V$ the tangent space to $X$ at the identity, thought of as a trivial bundle on $M$. Let $\hat{X}$ be the dual torus to $X$. Then $T_{M \times X} \cong \pi^*(T_M \oplus V)$, and $T_{M \times \hat{X}} \cong \hat{\pi}^*(T_M \oplus V^\vee)$, so for any isomorphism $L : T_M \to V$ we have that

$$\pi^* \begin{pmatrix} 0 & L \\ -L^{-1} & 0 \end{pmatrix}$$

is a complex structure on $M \times X$ and

$$\hat{\pi}^* \begin{pmatrix} 0 & L \\ -L^\vee & 0 \end{pmatrix}$$

is a symplectic structure on $M \times \hat{X}$. Before pulling back, these structures, thought of as generalized complex structures as in example 2.1 on $V \oplus T_M$ and $V^\vee \oplus T_M$, are related by the obvious map

$$V \oplus T_M \oplus V^\vee \oplus T_M^\vee \to V^\vee \oplus T_M \oplus V \oplus T_M^\vee.$$ 

4. Mirror partners of generalized almost complex and generalized almost Kähler structures

In this section we consider a manifold $M$ equipped with a real vector bundle $V$ where the rank of $V$ equals the dimension of $M$. For any connection $\nabla$ on $V$ we show how to build generalized almost complex structures on $X = \text{tot}(V)$ in terms of data on the base manifold $M$. We show that there is a bijective correspondence between generalized almost complex structures built in this way on $X$ and generalized almost complex structures of the same type on $\hat{X} = \text{tot}(V^\vee)$ built using $\nabla^\vee$.

Let $X$ be the total space of any vector bundle $V$ over a manifold $M$. Then we have the exact tangent sequence

$$0 \to \pi^*V \xrightarrow{j} T_X \xrightarrow{d\pi} \pi^*T_M \to 0$$

A connection on the bundle $V$ is by definition a map of sheaves

$$V \xrightarrow{\nabla} V \otimes T_M^\vee.$$
satisfying $\nabla (f \sigma) = \sigma \otimes df + f \nabla (\sigma)$ for all local sections $f$ of $C^\infty_M$ and $\sigma$ of $V$. We can use any such connection to give a splitting of the above tangent sequence. Namely, let

$$\pi^* \nabla : \pi^* V \to \pi^* V \otimes T^\vee_X$$

be the pullback of $\nabla$ and let $S$ be the tautological global section of $\pi^* V$ on $X$. Then $D = (\pi^* \nabla)(S)$ provides a map of vector bundles $\pi^* V \leftarrow T_X$. Now it’s easy to see that this map is a splitting of 4.1. Indeed, given a local frame $\{e_i\}$ of $V$ over an open set $U \subseteq M$, define smooth functions $\xi_i$ on $\pi^{-1}(U)$ by $\xi_i(a_j e_j(m)) = a_i$ for each $m$ in $M$. Together with the functions $x_i \circ \pi$, for $\{x_i\}$ coordinates on $U \subseteq M$, these form a coordinate system in $\pi^{-1}(U)$ in which we have $j(e_i) = \partial/\partial \xi_i$. In these coordinates we have that on $\pi^{-1}(U)$,

$$S = \xi_i \pi^{-1} e_i$$

and so if we define $D$ by

$$D = (\pi^* \nabla)(S) = \pi^{-1} e_i \otimes d\xi_i + \xi_i \pi^{-1} e_j \otimes \pi^* A_{ji}. \quad (4.2)$$

then since $\pi^* A_{ji}$ annihilates the image of $j$ we have that

$$D(j(\pi^{-1} e_k)) = (\pi^{-1} e_i)(d\xi_i j(\pi^{-1} e_k)) = \pi^{-1} e_k$$

and so $D \circ j$ is the identity. We will write this splitting on $X$ as

$$0 \to \pi^* V \xrightarrow{j} T_X \xrightarrow{d\pi} \pi^* T_M \to 0,$$

Consider the isomorphism

$$F : T_X \oplus T^\vee_X \to \pi^* V \oplus \pi^* T_M \oplus \pi^* V \vee \oplus \pi^* T^\vee_M, \quad F = \begin{pmatrix} D & 0 \\ d\pi & 0 \\ 0 & j^\vee \\ 0 & \alpha^\vee \end{pmatrix}. \quad (4.3)$$

with inverse

$$F^{-1} : \pi^* V \oplus \pi^* T_M \oplus \pi^* V \vee \oplus \pi^* T^\vee_M \to T_X \oplus T^\vee_X, \quad F^{-1} = \begin{pmatrix} j & \alpha & 0 & 0 \\ 0 & 0 & D^\vee & (d\pi)^\vee \end{pmatrix}. \quad (4.4)$$

These maps intertwine the obvious quadratic forms and therefore if $J$ is a generalized almost complex structure on $V \oplus T_M$ then $J = F^{-1}(\pi^* J)F$ is a generalized almost complex structure on $X$. 


Definition 4.1. If $\nabla$ is any connection on $V$ then we define a $\nabla$-lifted generalized almost complex structure to be a generalized almost complex structure on $X = \text{tot}(V)$ which can be expressed as $\mathcal{J} = F^{-1}(\pi^*J)F$ where $\mathcal{J}$ is a generalized almost complex structure on $X$ and $F$ depends on $\nabla$ as explained above.

Now using the dual connection $\nabla^\vee$, we may split the sequence tangent sequence of $\hat{X}$ as

$$0 \to \hat{\pi}^*V^\vee \xrightarrow{\hat{j}} T_{\hat{X}} \xrightarrow{\hat{\pi}^*T_M} 0,$$

Of course we will also need the maps

$$\hat{F} : T_{\hat{X}} \oplus T_{\hat{X}}^\vee \to \pi^*V^\vee \oplus \hat{\pi}^*T_M \oplus \hat{\pi}^*V \oplus \hat{\pi}^*T_M^\vee,$$

$$\hat{F} = \begin{pmatrix} \hat{D} & 0 \\ d\hat{\pi} & 0 \\ 0 & \hat{j}^\vee \\ 0 & \hat{\alpha}^\vee \end{pmatrix} (4.5)$$

with inverse

$$\hat{F}^{-1} : \pi^*V^\vee \oplus \pi^*T_M \oplus \pi^*V \oplus \hat{\pi}^*T_M^\vee \to T_{\hat{X}} \oplus T_{\hat{X}}^\vee,$$

$$\hat{F}^{-1} = \begin{pmatrix} \hat{j} & \hat{\alpha} & 0 & 0 \\ 0 & \hat{\alpha} & 0 & (d\hat{\pi})^\vee \end{pmatrix} (4.6)$$

Now if we take any

$$\mathcal{J} \in GL(V \oplus T_M \oplus V^\vee \oplus T_M^\vee)$$

we can apply the duality transformation along the fibers to get

$$\hat{\mathcal{J}} \in GL(V^\vee \oplus T_M \oplus V \oplus T_M^\vee).$$

Clearly this transformation intertwines the quadratic forms and so $\hat{\mathcal{J}}$ is a generalized almost complex structure on $V \oplus T_M$ if and only if $\hat{\mathcal{J}}$ is a generalized almost complex structure on $V^\vee \oplus T_M$. Therefore $\mathcal{J} = F^{-1}(\pi^*\mathcal{J})F$ is a generalized almost complex structure on $X$ if and only $\hat{\mathcal{J}} = \hat{F}^{-1}(\hat{\pi}^*\hat{\mathcal{J}})\hat{F}$ is a generalized almost complex structure on $\hat{X}$. At this point we will impose an extra constraint on these structures.

Definition 4.2. A $\nabla$-lifted generalized almost complex structure $\mathcal{J} = F^{-1}(\pi^*\mathcal{J})F$ will be called adapted if

$$\mathcal{J}(V \oplus V^\vee) = T_M \oplus T_M^\vee.$$
We will assume that $J$ is an adapted, $\nabla$-lifted generalized almost complex structure from now on.

Remark 4.3. It is clear from the construction above that $J$ is adapted if and only if $\hat{J}$ is.

Finally, let us record the explicit formulas for the operators $J, \hat{J}, \hat{J}$ and $\hat{J}$. The adapted condition together with the fact that $J^2 = -1$ and that $J$ preserves the quadratic form ensure that it is of the form

$$J = \begin{pmatrix} 0 & \mathcal{I}_{12} & 0 & \mathcal{I}_{22} \\ \mathcal{I}_{13} & 0 & -\mathcal{I}^\vee_{22} & 0 \\ 0 & \mathcal{I}_{31} & 0 & -\mathcal{I}^\vee_{13} \\ -\mathcal{I}^\vee_{31} & 0 & -\mathcal{I}^\vee_{12} & 0 \end{pmatrix}, \quad \mathcal{J} \in GL(V \oplus T_M \oplus V^\vee \oplus T_M^\vee) \quad (4.7)$$

subject to

$$\begin{align*}
\mathcal{I}_{12}\mathcal{I}_{13} - \mathcal{I}_{22}\mathcal{I}_{31}^\vee &= -1; \\
\mathcal{I}_{12}\mathcal{I}_{22}^\vee + \mathcal{I}_{22}\mathcal{I}_{12}^\vee &= 0; \\
\mathcal{I}_{13}\mathcal{I}_{12} - \mathcal{I}_{22}\mathcal{I}_{31} &= -1; \\
\mathcal{I}_{13}\mathcal{I}_{22} + \mathcal{I}_{22}\mathcal{I}_{13}^\vee &= 0; \\
\mathcal{I}_{31}\mathcal{I}_{13} + \mathcal{I}_{13}\mathcal{I}_{31}^\vee &= 0; \\
\mathcal{I}_{31}\mathcal{I}_{12} + \mathcal{I}_{12}\mathcal{I}_{31} &= 0. 
\end{align*} \quad (4.8)$$

With this notation we have

$$\hat{J} = \begin{pmatrix} 0 & \mathcal{I}_{31} & 0 & -\mathcal{I}^\vee_{13} \\ -\mathcal{I}^\vee_{22} & 0 & \mathcal{I}_{13} & 0 \\ 0 & \mathcal{I}_{12} & 0 & \mathcal{I}_{22} \\ -\mathcal{I}_{12}^\vee & 0 & -\mathcal{I}_{31}^\vee & 0 \end{pmatrix}, \quad \hat{J} \in GL(V^\vee \oplus T_M \oplus V \oplus T_M^\vee) \quad (4.14)$$

and so

$$J = \begin{pmatrix} j(\pi^*\mathcal{I}_{12})(d\pi) + \alpha(\pi^*\mathcal{I}_{13})D & j(\pi^*\mathcal{I}_{22})\alpha^\vee - \alpha(\pi^*\mathcal{I}_{22}^\vee)j^\vee \\ D^\vee(\pi^*\mathcal{I}_{13})(d\pi) - (d\pi)^\vee(\pi^*\mathcal{I}_{31}^\vee)D & -D^\vee(\pi^*\mathcal{I}_{13}^\vee)\alpha^\vee - (d\pi)^\vee(\pi^*\mathcal{I}_{12}^\vee)j^\vee \end{pmatrix}. \quad (4.15)$$
\[
\mathcal{J} = \begin{pmatrix}
\hat{j}(\hat{\pi}^*J_{31})(d\hat{\pi}) - \hat{\alpha}(\hat{\pi}^*J_{22})\hat{D} - \hat{j}(\hat{\pi}^*J_{13})\hat{\alpha} + \hat{\alpha}(\hat{\pi}^*J_{13})\hat{j} \\
\hat{D}^\vee(\hat{\pi}^*J_{12})(d\hat{\pi}) - (d\hat{\pi})^\vee(\pi^*J_{12}^\vee)\hat{D} - \hat{D}^\vee(\hat{\pi}^*J_{22})\hat{\alpha}^\vee - (d\hat{\pi})^\vee(\hat{\pi}^*J_{31}^\vee)\hat{j}^\vee
\end{pmatrix}.
\] (4.16)

Remark 4.4. Notice that the mirror symmetry transformation “exchanges” \(J_{12}\) with \(J_{31}\) and \(J_{22}\) with \(-J_{13}^\vee\).

We have written down the bijective correspondence between \(\nabla\)-lifted, adapted, generalized almost complex structures on \(X\) and \(\nabla^\vee\)-lifted, adapted, generalized almost complex structures on \(\hat{X}\). We will show below, in the case that \(\nabla\) is flat, that \(J\) is integrable if and only if \(\hat{J}\) is.

4.1. Associated almost Dirac structures. For each of the generalized complex structures on \(X\) that we consider, there is a natural almost Dirac structure that appears on the base manifold \(M\). It does not depend on the connection used to split the tangent sequence of \(X \to M\). An almost Dirac structure on \(M\) is just [10] a maximally isotropic sub-bundle of \(T_M \oplus T_M^\vee\). Now the isomorphism \(\mathcal{J}\), given in equation (4.7), preserves the quadratic form and when restricted to \(V \oplus V^\vee\), gives an isomorphism \(V \oplus V^\vee \to T_M \oplus T_M^\vee\) which preserves the obvious quadratic forms. Hence the image of \(V\) is a maximally isotropic subspace of \(T_M \oplus T_M^\vee\). In other words it is an almost Dirac structure on \(M\) which we will call \(\Delta\), where

\[
\Delta = J(V) = J_{13}(V) - J_{31}^\vee(V) = \hat{J}(V).
\]

Examples 4.5.

1. Suppose we use our method to construct an almost complex structure on \(X = \text{tot}(V)\) out of some arbitrary connection on \(V\). Then we necessarily have that \(J_{13}\) is an isomorphism and \(J_{31} = 0\). Hence \(\Delta = T_M\).

2. If instead we put an almost symplectic structure on \(X = \text{tot}(V)\) then \(\Delta = T_M^\vee\).

Notice that the almost Dirac structure

\[
\hat{\Delta} = \hat{J}(V^\vee) = -J_{22}(V^\vee) - J_{12}^\vee(V^\vee) = J(V^\vee)
\]

arising from the mirror generalized almost complex structure is always transverse to \(\Delta\). Hence we always get a pair

\[
\Delta \oplus \hat{\Delta} = T_M \oplus T_M^\vee
\]
of complementary almost Dirac structures. Later we will return to these structures and study their integrability and the existence of flat connections on them.

### 4.2. Mirror symmetry for generalized almost Kähler manifolds.

In this section, we study the case of a pair of $\nabla$-lifted, adapted generalized almost complex structures on the total space of a vector bundle which form a generalized almost Kähler structure as described in 2.4. Under these conditions, we write down the mirror transformation rule that allows us to relate the generalized almost Kähler metric $G$ on $X$ and the mirror generalized almost Kähler metric $\hat{G}$ on $\hat{X}$. We observe that in general, the local transformation rules for the pair $(g, b)$ that exist in the physics literature, continue to hold in this setting, even though here neither $J$ nor $J'$ needs to be a B-field transform of a generalized complex structure of complex type.

First of all notice that the mirror transform of a generalized almost Kähler pair $(J, J')$ is also generalized almost Kähler. Indeed, if we let $J = F^{-1}(\pi^*J)F$ and $J' = F^{-1}(\pi^*J')F$, then $J$ and $J'$ commute if and only if $\hat{J}$ and $\hat{J}'$ commute. This, in turn, is equivalent to $\hat{J}$ and $\hat{J}'$ commuting which happens if and only if $\hat{J} = \hat{F}^{-1}(\hat{\pi}^*\hat{J})\hat{F}$ and $\hat{J}' = \hat{F}^{-1}(\hat{\pi}^*\hat{J}')\hat{F}$ commute. Similarly, $G = -F^{-1}\pi^*(JJ')F$ is positive definite if and only if $-JJ'$ is. This is equivalent to $-\hat{J}\hat{J}'$ being positive definite, which happens if and only if $\hat{G} = -\hat{F}^{-1}\hat{\pi}^*(\hat{J}\hat{J}')\hat{F}$ is positive definite. By our assumptions on $J$ and $J'$ we may write $G = -J\bar{J}'$ as

$$G : V \oplus T_M \oplus V^\vee \oplus T_M^\vee \to V \oplus T_M \oplus V^\vee \oplus T_M^\vee,$$

where, $G_{21} = G_{21}^\vee, G_{24} = G_{24}^\vee, G_{31} = G_{31}^\vee, G_{34} = G_{34}^\vee$. Finally, using the fact that this matrix squares to the identity, we get:

$$G_{11} = -J_{12}J_{21}^\vee + J_{22}J_{31}^\vee$$

$$G_{21} = J_{12}J_{21}^\vee + J_{22}J_{11}^\vee$$

$$G_{14} = -J_{13}J_{11}^\vee + J_{22}J_{31}^\vee$$

$$G_{24} = -J_{13}J_{21}^\vee - J_{22}J_{12}^\vee$$

$$G_{31} = -J_{31}J_{12}^\vee - J_{13}J_{31}^\vee$$

$$G_{34} = J_{31}J_{21}^\vee + J_{13}J_{31}^\vee.$$
Therefore $G' = -\tilde{\mathcal{J}}\tilde{\mathcal{J}}'$ comes out to be

\[
G' = -\tilde{\mathcal{J}}\tilde{\mathcal{J}}' = \begin{pmatrix}
G_{11}^\vee & 0 & G_{31} & 0 \\
0 & G_{14} & 0 & G_{24} \\
G_{21} & 0 & G_{11} & 0 \\
0 & G_{34} & 0 & G_{14}^\vee
\end{pmatrix}.
\]

(4.18)

Remark 4.6. Notice that the mirror symmetry transformation “exchanges” $G_{11}$ with $G_{11}^\vee$, $G_{21}$ with $G_{31}$, and “preserves” $G_{14}$ and $G_{34}$.

Now writing $G$ in terms of $g$ and $b$, ([15])

\[
G = \begin{pmatrix}
-g^{-1}b & g^{-1} \\
g^{-1}b & bg^{-1}
\end{pmatrix}
\]

(4.19)

and similarly writing $\tilde{G}$ in terms of $\hat{g}$ and $\hat{b}$ we can easily manipulate the resulting equations to yield the following formulas for the metrics and B-fields in terms of the vector bundle maps $G_{ij}$ on the base manifold.

\[
g = D^\vee\pi^*G_{21}^{-1}D + (d\pi)^\vee\pi^*G_{24}^{-1}d\pi
\]

\[
b = D^\vee\pi^*(G_{14}G_{21}^{-1})D + (d\pi)^\vee\pi^*(G_{14}G_{21}^{-1})d\pi
\]

\[
\hat{g} = \tilde{D}^\vee\hat{\pi}^*G_{31}^{-1}\tilde{D} + (d\hat{\pi})^\vee\hat{\pi}^*G_{24}^{-1}d\hat{\pi}
\]

\[
\hat{b} = \tilde{D}^\vee\hat{\pi}^*(G_{14}G_{31}^{-1})\tilde{D} + (d\hat{\pi})^\vee\hat{\pi}^*(G_{14}G_{31}^{-1})d\hat{\pi}
\]

Notice that our assumptions on the compatibility of the generalized complex structures, and the foliation and transverse vector bundle, imply that the metric $g$ and B-field $b$ do not mix the horizontal and vertical directions.

Now if we chose local vertical coordinates adapted to the flat connection, $y^\alpha$ on $X$ and $\hat{y}^\alpha$ on $\tilde{X}$ and $x^i$ on the base then the above just means that locally we have

\[
g = g_{ij}(x)dx^idx^j + h_{\alpha\beta}(x)dy^\alpha dy^\beta
\]

\[
b = b_{ij}(x)dx^idx^j + B_{\alpha\beta}(x)dy^\alpha dy^\beta
\]

\[
\hat{g} = g_{ij}(x)dx^idx^j + \hat{h}_{\alpha\beta}(x)d\hat{y}^\alpha d\hat{y}^\beta
\]

\[
\hat{b} = b_{ij}(x)dx^idx^j + \hat{B}_{\alpha\beta}(x)d\hat{y}^\alpha d\hat{y}^\beta
\]

where of course, $x^i$ means $x^i \circ \pi$ on $X$ and $x^i \circ \hat{\pi}$ on $\tilde{X}$.

Then the Buscher transformation rules [6, 7] (we used [1] as a reference)

\[
(h + B)\hat{h}(h - B) = h \quad \text{and} \quad (h + B)\hat{B}(h - B) = -B
\]
are verified from the easily checked identities
\[(G_{21}^{-1} + G_{11}^{\vee}G_{21}^{-1})G_{31}^{-1}(G_{21}^{-1} - G_{11}^{\vee}G_{21}^{-1}) = G_{21}^{-1}\]
and
\[(G_{21}^{-1} + G_{11}^{\vee}G_{21}^{-1})G_{11}G_{31}^{-1}(G_{21}^{-1} - G_{11}^{\vee}G_{21}^{-1}) = -G_{11}^{\vee}G_{21}^{-1}\]
respectively.

We now work out the transformation rules relating the two almost complex structures, \(J_+, J_-\), and their mirror partners \(\hat{J}_+, \hat{J}_-\). We have
\[J_+ = J_1 + J_2(g + b) \quad \text{and} \quad J_- = J_1 + J_2(b - g).\]
By combining the results above we can easily compute that
\[J_+ = j(\pi^*(J_{12} + J_{22}(G_{14}^{\vee} + 1)G_{24}^{-1}))d\pi + \alpha(\pi^*(J_{13} - J_{22}(G_{11}^{\vee} + 1)G_{21}^{-1}))D\]
and
\[J_- = j(\pi^*(J_{12} + J_{22}(G_{14}^{\vee} - 1)G_{24}^{-1}))d\pi + \alpha(\pi^*(J_{13} - J_{22}(G_{11}^{\vee} - 1)G_{21}^{-1}))D.\]
Hence
\[\hat{J}_+ = \hat{j}(\hat{\pi}^*(J_{31} - J_{13}(G_{14}^{\vee} + 1)G_{31}^{-1}))d\hat{\pi} + \hat{\alpha}(\hat{\pi}^*(-J_{22} + J_{13}(G_{11}^{\vee} + 1)G_{31}^{-1}))\hat{D}\]
and
\[\hat{J}_- = \hat{j}(\hat{\pi}^*(J_{31} - J_{13}(G_{14}^{\vee} - 1)G_{31}^{-1}))d\hat{\pi} + \hat{\alpha}(\hat{\pi}^*(-J_{22} + J_{13}(G_{11}^{\vee} - 1)G_{31}^{-1}))\hat{D}.\]

5. Branes

In this section, we give some ideas of how one can transfer branes [15, 21] from a generalized almost complex manifold to its mirror partner. We will present in detail only a very restricted case. This construction closely parallels that in [30, 25]. Consider the following definition from [21] which also appears in a more general form in [15].

**Definition 5.1.** [21] Let \((X, J)\) be a generalized (almost) complex manifold. Consider triples \((Y, L, \nabla_L)\)

where \(f : Y \hookrightarrow X\) is a sub-manifold of \(X\), \(L\) is a Hermitian line bundle on \(Y\), and \(\nabla_L\) is a connection on \(L\). Such a triple is said to be a *generalized complex brane* if the bundle

\[\{(v, \alpha) \in T_Y \oplus (T_X^\vee|_Y) \mid f^*(df)^\vee\alpha = \ell_v \mathcal{F}\}\]

is preserved by the restriction of \(J\) to \(Y\), where \(\mathcal{F}\) is the curvature two-form of \(\nabla_L\).
We studied some special cases [15] of these branes in [2] under the name of generalized Lagrangian sub-manifolds and found some interesting relationships to sub-manifolds of $X$ which inherit generalized complex structures (which we call generalized complex sub-manifolds).

Suppose that $M$ is an $n$-manifold, $V$ is a rank $n$ vector bundle on $M$, $\nabla$ is a connection on $V$, $X$ is the total space of $V$, $\hat{X}$ is the total space of $V^\vee$ and $J$ is an adapted, $\nabla$-lifted (see section 4) generalized almost complex structure on $V$. Let $S$ be a sub-manifold of $M$, $W \subseteq V|_S$ a sub-bundle, $Y$ the total space of $W$, and $\hat{Y}$ the total space of the sub-bundle $Ann(W) \subseteq V^\vee|_S$. Then we propose that the relationship between $Y$ and $\hat{Y}$ is a special case of a potential generalization of the relationship between $A$--cycles and $B$--cycles in mirror symmetry (see e.g. [25] and references therein). We justify this with the following lemma.

**Lemma 5.2.** Under the conditions of the preceding paragraph, the triple $(Y, \mathbb{C} \otimes C^\infty_Y, d)$ is a generalized complex brane of $(X, J)$ if and only if the triple $(\hat{Y}, \mathbb{C} \otimes C^\infty_{\hat{Y}}, d)$ is a generalized complex brane of $(\hat{X}, \hat{J})$.

**Proof.**

In this proof, we will be using the notation of section 4. We need to show that

\[ J(T_Y \oplus \text{Ann}(T_Y)) = T_Y \oplus \text{Ann}(T_Y) \]  

if and only if

\[ \hat{J}(T_{\hat{Y}} \oplus \text{Ann}(T_{\hat{Y}})) = T_{\hat{Y}} \oplus \text{Ann}(T_{\hat{Y}}), \]  

where it is to be understood that we are restricting $J$ to $Y$ and $\hat{J}$ to $\hat{Y}$. Observe that when understood as bundles on $Y$, we have

\[ T_Y = j(\pi^*W) \oplus \alpha(\pi^*T_S), \quad \text{Ann}(T_Y) = D^\vee(\pi^*(\text{Ann}(W))) \oplus (d\pi)^\vee(\pi^*(\text{Ann}(T_S))) \]

and when understood as bundles on $\hat{Y}$, we have

\[ T_{\hat{Y}} = \hat{j}(\hat{\pi}^*(\text{Ann}(W))) \oplus \hat{\alpha}(\hat{\pi}^*T_S), \quad \text{Ann}(T_{\hat{Y}}) = \hat{D}^\vee(\hat{\pi}^*(W)) \oplus (d\hat{\pi})^\vee(\hat{\pi}^*(\text{Ann}(T_S))). \]

From this perspective it is clear that both 5.1 and 5.2 are both equivalent simply to the conditions (understanding that $\hat{J}$ is restricted to $S$)

\[ J_{13}(W) \subseteq T_S \]  

\[ J_{12}(T_S) \subseteq W \]  

\[ J_{22}(\text{Ann}(T_S)) \subseteq W \]  

\[ J_{31}(T_S) \subseteq \text{Ann}(W). \]
and therefore we are done. A more general treatment will involve replacing $W$ by an affine sub-bundle which will result in a non-trivial line bundle on the mirror side. A more general story will be the result of upcoming work. The extension of the results in this section to the case of torus bundles will be left to the reader, and should be clear upon reading section 8. We hope that in a suitable extended version of the homological mirror symmetry conjecture [24], generalized complex manifolds would be assigned categories in a natural way, and branes would be related to objects in these categories.

Remark 5.3. On a torus bundle $Z \to M$, where $Z$ is an orientable compact manifold, it is plausible that the correspondence which we are describing here, when thought of as a correspondence between homology classes on $Z$ to homology classes on the dual torus bundle $\hat{Z} \to M$ agrees, upon using Poincaré Duality, with the correspondence in cohomology given in section 8.

6. The mirror transformation on spinors and the Fourier transform

In this section we study a map from certain complex valued differential forms on the total space of a vector bundle to complex valued differential forms on the total space of the dual vector bundle. We show that the line sub-bundle of the bundle of differential forms associated to an adapted, $\nabla$-lifted generalized almost complex structure has a sub-sheaf which goes under this correspondence to the sub-sheaf associated to the mirror generalized almost complex structure. The idea of using a Fourier transform in the context of T-duality for generalized complex structures has appeared in a slightly different context in both [15] (based on ideas appearing in [27]) and also [1], [35] and the references therin.

Consider a vector bundle $V$ of rank $n$ on a manifold $M$. There is an isomorphism

$$\bigwedge V^\vee \otimes \mathbb{C} \longrightarrow \left( \bigwedge V \otimes \bigwedge^n V^\vee \right) \otimes \mathbb{C}$$

given by

$$\phi \mapsto \int (\phi \wedge \exp(\kappa))$$

(6.1)

where $\kappa$ is the canonical global section of $(V \otimes V^\vee) \otimes \mathbb{C} \subseteq \bigwedge^2 (V \oplus V^\vee) \otimes \mathbb{C}$ and

$$\int : \bigwedge (V \oplus V^\vee) \otimes \mathbb{C} \to \bigwedge V \otimes \bigwedge^n V^\vee \otimes \mathbb{C}$$

is the projection map. Furthermore, this map decomposes into isomorphisms

$$\bigwedge^p V^\vee \otimes \mathbb{C} \to \bigwedge^{n-p} V \otimes \bigwedge^n V^\vee \otimes \mathbb{C}.$$
and also induces a Fourier transform isomorphism, which we will call $F.T.$

\[
\bigwedge V^\vee \otimes \bigwedge T_M^\vee \otimes \mathbb{C} \xrightarrow{F.T.} \bigwedge V \otimes \bigwedge T_M^\vee \otimes \bigwedge^n V^\vee \otimes \mathbb{C}
\]

which in turn decompose into isomorphisms

\[
(\bigwedge^q T_M^\vee \otimes \bigwedge^p V^\vee \otimes \mathbb{C}) \rightarrow (\bigwedge^q T_M^\vee \otimes \bigwedge V \otimes \bigwedge^n V^\vee \otimes \mathbb{C}).
\]

**Lemma 6.1.** Let $V$ be a rank $n$ orientable vector bundle on an $n$–manifold, $\mathcal{J}$ a generalized almost complex structure on the vector bundle $V \oplus T_M$ satisfying $\mathcal{J}(V \oplus V^\vee) = T_M \oplus T_M^\vee$, and $\mathcal{J}$ the mirror structure. Then the composition of the map $F.T.$ with any trivialization of $\bigwedge^n V^\vee$ takes the line bundle $L \subseteq \bigwedge(V \oplus T_M)^\vee \otimes \mathbb{C}$ which represents $\mathcal{J}$ to the line bundle $\hat{L} \subseteq \bigwedge(V \oplus T_M)^\vee \otimes \mathbb{C}$ which represents $\mathcal{J}$.

**Proof.**

First of all notice that changing the trivialization of $\bigwedge^n V$ multiplies the image of $F.T.$ by a non-zero function on the base manifold. This is an automorphism of image of the composed map along with its inclusion into $\bigwedge(V \oplus T_M)^\vee \otimes \mathbb{C}$. The $(+i)$ eigenbundle of $\mathcal{J}$ is the graph of the map

\[
-i\mathcal{J}|_{(V \oplus V^\vee) \otimes \mathbb{C}} : (V \oplus V^\vee) \otimes \mathbb{C} \rightarrow (T_M \oplus T_M^\vee) \otimes \mathbb{C}.
\]

Similarly, the $(+i)$ eigenbundle of $\mathcal{J}$ is the graph of

\[
-i\mathcal{J}|_{(V^\vee \oplus V) \otimes \mathbb{C}} : (V \oplus V^\vee) \otimes \mathbb{C} \rightarrow (T_M \oplus T_M^\vee) \otimes \mathbb{C}.
\]

The sections $\phi$ of $L$ therefore satisfy

\[
L_{-i\mathcal{J}_1^\vee} \alpha + i(\mathcal{J}_1^\vee \alpha) \wedge \phi = 0 \tag{6.2}
\]

\[
L_{i\mathcal{J}_2^\vee} \alpha + (\alpha + i\mathcal{J}_2^\vee \alpha) \wedge \phi = 0 \tag{6.3}
\]

for all sections $\alpha$ of $V^\vee$.

We need to show that the section $F.T.(\phi)$ of $\bigwedge(V^\vee \oplus T_M)^\vee \otimes \bigwedge^n V^\vee \otimes \mathbb{C}$ satisfies

\[
L_{-i\mathcal{J}_1^\vee} F.T.(\phi) + (\phi + i\mathcal{J}_1^\vee \phi) \wedge F.T.(\phi) = 0 \tag{6.4}
\]

\[
L_{\alpha + i\mathcal{J}_2^\vee} F.T.(\phi) + (i\mathcal{J}_2^\vee \alpha) \wedge F.T.(\phi) = 0 \tag{6.5}
\]

for all sections $\alpha$ of $V^\vee$. These equations hold for the map $F.T.$ if and only if they hold for the composition of $F.T.$ with any trivialization. This can be seen by writing $F.T.$
as the composed map followed by the action of “wedging” with a global section of $\bigwedge^n V^\vee$. The equations 6.4 and 6.5 will follow immediately from taking the Fourier transform of both sides of 6.2 and 6.3 and using the following lemma.

\begin{lemma}
For any sections $\zeta$ of $\bigwedge (V \oplus T_M)^\vee \otimes \mathbb{C}$, $v$ of $V \otimes \mathbb{C}$, $w$ of $T_M \otimes \mathbb{C}$, $\alpha$ of $V^\vee \otimes \mathbb{C}$ and $\beta$ of $T_M^\vee \otimes \mathbb{C}$ we have

(i) $\text{F.T.}(l_v \zeta) = v \wedge \text{F.T.}(\zeta)$

(ii) $\text{F.T.}(l_w \zeta) = l_w \text{F.T.}(\zeta)$

(iii) $\text{F.T.}(\alpha \wedge \zeta) = l_{\alpha} \text{F.T.}(\zeta)$

(iv) $\text{F.T.}(\beta \wedge \zeta) = \beta \wedge \text{F.T.}(\zeta)$

\end{lemma}

\begin{proof}
It clearly suffices to prove this in the case that $\zeta$ is a section of $\bigwedge^p (V \oplus T_M)^\vee$. Notice also that $\int l_v = 0$ and $l_v \kappa = -v$ for any section $v$ of $V$ and $l_{\alpha} \kappa = \alpha$ for any section $\alpha$ of $V^\vee$. Then we have

\begin{align*}
\text{F.T.}(l_v \zeta) &= \int (l_v \zeta) \wedge \exp(\kappa) = \int l_v (\zeta \wedge \exp(\kappa)) = (-1)^p \int \zeta \wedge l_v \exp(\kappa) \\
&= (-1)^p \int \zeta \wedge l_v \exp(\kappa) = (-1)^p \int \zeta \wedge (l_v(\zeta)) \wedge \exp(\kappa) \\
&= (-1)^p \int \zeta \wedge v \wedge \exp(\kappa) = \int v \wedge \zeta \wedge \exp(\kappa) = v \wedge \int \zeta \wedge \exp(\kappa) \\
&= v \wedge \text{F.T.}(\zeta)
\end{align*}

\begin{align*}
\text{F.T.}(l_w \zeta) &= \int (l_w \zeta) \wedge \exp(\kappa) = \int l_w (\zeta \wedge \exp(\kappa)) = l_w \int (\zeta \wedge \exp(\kappa)) \\
&= l_w \text{F.T.}(\zeta)
\end{align*}

\begin{align*}
\text{F.T.}(\alpha \wedge \zeta) &= \int (\alpha \wedge \zeta \wedge \exp(\kappa)) = (-1)^p \int (\zeta \wedge \alpha \wedge \exp(\kappa)) \\
&= (-1)^p \int (\zeta \wedge \alpha \exp(\kappa)) = l_{\alpha} \int (\zeta \wedge \exp(\kappa)) \\
&= l_{\alpha} \text{F.T.}(\zeta)
\end{align*}

\begin{align*}
\text{F.T.}(\beta \wedge \zeta) &= \int \beta \wedge \zeta \wedge \exp(\kappa) = \beta \wedge \int \zeta \wedge \exp(\kappa) \\
&= \beta \wedge \text{F.T.}(\zeta)
\end{align*}

\end{proof}
Let $M$ an $n$-manifold and $X \overset{\pi}{\to} M$ be the total space of an orientable vector bundle $V$ on $M$, with connection $\nabla$ and $\mathcal{J}$ a $\nabla$-lifted, adapted generalized almost complex structure on $X$. Let $\hat{X} \overset{\hat{\pi}}{\to} M$ be the total space of $V^\vee$. Using $\nabla$ we may realize $\pi^*(\bigwedge V^\vee \otimes T_M^\vee \otimes \mathbb{C})$ as a sub-bundle of $\bigwedge T^\vee \otimes \mathbb{C}$. Now $\mathcal{J}$ determines a spinorial line bundle $L \subseteq \bigwedge T^\vee \otimes \mathbb{C}$ which is simply the image under this isomorphism of the pullback $\pi^*L$. Similarly, $\hat{\mathcal{J}}$ determines a spinorial line bundle $\hat{L} \subseteq \bigwedge T^\vee \hat{X} \otimes \mathbb{C}$ isomorphic to $\hat{\pi}^*\hat{L}$. Therefore, interpreting the Fourier transform maps as isomorphisms $\pi^* L \to \hat{\pi}^* \hat{L}$ we can map certain sections of $L$ over open sets of the form $\pi^{-1}(U)$ to sections of $\hat{L}$ over open sets of the form $\hat{\pi}^{-1}(U)$. We have shown the following lemma.

**Lemma 6.3.** Let $V$ is an orientable rank $n$ vector bundle on an $n$-manifold $M$ and $\mathcal{J}$ an adapted, $\nabla$-lifted generalized almost complex structure on $X = \text{tot}(V)$ with associated line bundle $L$. Let the mirror generalized almost complex structure have a ssociated line bundle $\hat{L}$. Then their are sub-sheaves, $\pi^{-1}L \subseteq L$ and $\hat{\pi}^{-1}\hat{L} \subseteq \hat{L}$ such that if we compose the isomorphism $\pi^* \pi^{-1}(\bigwedge T_M^\vee \otimes \bigwedge V^\vee \otimes \mathbb{C}) \overset{\text{FT}}{\to} \hat{\pi}^* \hat{\pi}^{-1}(\bigwedge T^\vee_M \otimes \bigwedge V \otimes \bigwedge^n V^\vee \otimes \mathbb{C})$ with any trivialization of $\bigwedge^n V^\vee$, the resulting isomorphism $\pi^* (\bigwedge T_X^\vee \otimes \mathbb{C}) \supseteq \pi^{-1}L \to \hat{\pi}^* \hat{\pi}^{-1}\hat{L} \subseteq \hat{\pi}^* \hat{L}$ restricts to an isomorphism $\pi^* L \supseteq \pi^{-1}L \to \hat{\pi}^* \hat{\pi}^{-1}\hat{L} \subseteq \hat{\pi}^* \hat{L}$

This is useful because, from the $\nabla$-lifted property, its easy to see that $L = \pi^{-1}L \otimes C_X^\infty$ and $\hat{L} = \hat{\pi}^{-1}\hat{L} \otimes C_X^\infty$. Therefore for $U$ small enough, representative spinors for $\mathcal{J}$ over $\pi^{-1}(U)$ and $\hat{\mathcal{J}}$ over $\hat{\pi}^{-1}(U)$ exist and can be chosen as pullbacks of sections of $L$ and $\hat{L}$ over $U$. They are exchanged under the Fourier transform even though we have not written down a map between the pushforwards of $L$ and $\hat{L}$. The situation will be much more simple in the case of torus bundles.

**Remark 6.4.** It is important to remember that the geometry of $\mathcal{J}$ is not just captured by the abstract line bundle $L$ up to isomorphism, but rather, by $L$ together with its embedding into the differential forms.

Understanding mirror symmetry in terms of a relationship between pure spinors was approached with similar techniques in [1].
7. The question of integrability

The purpose of this section is to express the integrability of $\nabla$-lifted, adapted generalized almost complex structures $\mathcal{J}$ on the total space of vector bundles in terms of data on the base manifold $M$. We do this only in the case where $\nabla$ is flat (in which case we call the structures $\mathcal{J}$ semi-flat). Once we do this it will be clear that $\mathcal{J}$ on $X = \text{tot}(V)$ is integrable if and only if the mirror structure $\widehat{\mathcal{J}}$ on $\widehat{X} = \text{tot}(V^\vee)$ is integrable.

Recall that the choice of a connection $\nabla$ gives rise to a splitting $(D, \alpha)$:

$$0 \xrightarrow{} \pi^*V \xrightarrow{j} TX \xrightarrow{\partial/\partial y_i} \pi^*T_M \xrightarrow{\alpha} 0,$$

of the tangent sequence of $X \to M$.

Now if that $\nabla$ is flat it is known [23] that we may find in a neighborhood of any point of $M$ a frame, $\{e_i\}$ such that $\nabla e_i = 0$. We will call $\{e_i\}$ a flat frame. Given a flat frame, along with the corresponding vertical coordinates $\{\xi_i\}$ we have that for any choice of coordinates $x_i$ on the base, the functions $\xi_i$ together with $y_i = x_i \circ \pi$ form a coordinate system on $X$ and $\alpha(\pi^*\partial/\partial x_i) = (1 - j \circ D)\partial/\partial y_i = \partial/\partial y_i$ follows from the expression in this frame (see 4.2) for $D$. We define a frames $\{f_i\}$ for $\pi^*V$ and $\{f^i\}$ for $\pi^*V^\vee$ by using the pullbacks $f_i = \pi^*e_i$ and $f^i = \pi^*e^i$, where $\{e^i\}$ is a dual frame to $\{e_i\}$.

Remark 7.1. Notice that $\nabla$ is flat if and only if the image of $\alpha$ is involute. Hence in this case we have a horizontal foliation instead of just a horizontal distribution. We will consider the geometry of a pair of transverse foliations and its interaction with a generalized complex structure in section 10.

Definition 7.2. If $\nabla$ is a flat connection on a rank $n$ vector bundle $V$ over a real $n$-manifold then a $\nabla$-semi-flat generalized almost complex structure on $X = \text{tot}(V)$ is an adapted, $\nabla$-lifted (see 4.2) generalized almost complex structure.

Let $S$ be the sub-sheaf of flat sections of $V \oplus V^\vee$. Consider the isomorphism of vector bundles

$$\mathcal{M} : V \oplus V^\vee \to T_M \oplus T_M^\vee, \quad \mathcal{M} = \begin{pmatrix} \mathcal{J}_{13} & \mathcal{J}_{22} \\ -\mathcal{J}_{31} & \mathcal{J}_{12} \end{pmatrix}, \quad \mathcal{M}^{-1} = \begin{pmatrix} -\mathcal{J}_{12} & -\mathcal{J}_{22} \\ -\mathcal{J}_{31} & \mathcal{J}_{13} \end{pmatrix}. \quad (7.1)$$

With this notation we have
Theorem 7.3. If $V$ is a vector bundle on a manifold $M$, then a semi-flat generalized almost complex structure $\mathcal{J} = F^{-1}(\pi^*\mathcal{J})F$ on $X = \text{tot}(V)$ is integrable if and only if all pairwise Courant brackets of sections of the sheaf $\mathcal{M}(S)$ vanish.

Notice that this condition is expressed entirely in terms of data on the base manifold $M$. Furthermore, we will see that this theorem implies the following corollary.

Corollary 7.4. The generalized almost complex structure $\mathcal{J} = F^{-1}(\pi^*\mathcal{J})F$ on $X = \text{tot}(V)$ is integrable if and only if the generalized almost complex structure $\mathcal{J} = \tilde{F}^{-1}(\tilde{\pi}^*\tilde{\mathcal{J}})\tilde{F}$ on $\tilde{X} = \text{tot}(V^\vee)$ is integrable.

Remark 7.5. In other words the mirror symmetry transformation is a bijective correspondence between $\nabla$-semi-flat generalized complex structures on $X$ and $\nabla^\vee$-semi-flat generalized complex structures on $\tilde{X}$.

Example 7.6. If $\nabla$ is any flat, torsion-free connection on $T_M$, we can put a canonical complex structure on $\text{tot}(T_M)$. See section 11 for more details. This construction was first done in [11]. It is easy to see that the mirror structure is always the canonical symplectic structure on $\text{tot}(T_M^\vee)$.

Proof of Theorem 7.3.

Let us analyze the condition that the $(+i)$ eigenbundle $E$ be involute. The bundle $E$ is the graph of the isomorphism

$$-i\mathcal{J}|_{\text{image}(j \oplus D^\vee) \otimes \mathbb{C}} : \text{image}(j \oplus D) \otimes \mathbb{C} \rightarrow \text{image}(\alpha \oplus d\pi^\vee) \otimes \mathbb{C}.$$  

It suffices to analyze involutivity it locally on the base manifold. Note that in the local frame and coordinates which we have chosen, we have the following formulas.

$$j(f_i) = \partial/\partial \xi_i, \quad D(\partial/\partial \xi_i) = f_i, \quad D(\partial/\partial y_i) = 0$$

$$\alpha(\partial/\partial x_i) = \partial/\partial y_i, \quad d\pi(\partial/\partial y_i) = \pi^*\partial/\partial x_i, \quad d\pi(\partial/\partial \xi_i) = 0$$

$$D^\vee(f^i) = d\xi_i, \quad \alpha^\vee(dy_i) = \pi^*dx_i, \quad \alpha^\vee(d\xi_i) = 0$$

$$(d\pi)^\vee(\pi^*dx_i) = dy_i, \quad j^\vee(d\xi_i) = f^i, \quad j^\vee(dy_i) = 0$$

Furthermore, an isotropic sub-bundle of $(T_X \oplus T_X^\vee) \otimes \mathbb{C}$ is involute if and only if it has a basis of sections whose pairwise Courant brackets are themselves sections of the original
bundle. This follows immediately from the Leibniz property of the Courant Bracket see e.g. [39, 10]. This property says that

\[ [v_1 + \alpha_1, f(v_2 + \alpha_2)] = f[v_1 + \alpha_1, v_2 + \alpha_2] + v_1(f)(v_2 + \alpha_2) + (v_1 + \alpha_1, v_2 + \alpha_2)df \]  

(7.2)

for all vector fields \( v_1 \) and \( v_2 \), one-forms \( \alpha_1 \) and \( \alpha_2 \) and functions \( f \). Let \( U \) is the coordinate neighborhood of the base. We will analyze involutivity in \( \pi^{-1}(U) \), using the coordinate system and frame described above. Involutivity of \( E \) is equivalent to the condition that \([a_i, a_j]\), \([a_i, b_j]\), and \([b_i, b_j]\) are all sections of \( E \) where

\[ a_i = j(f_i) - i\mathcal{J}(j(f_i)) \]

and

\[ b_i = D^\nu(f^i) - i\mathcal{J}(D^\nu(f^i)). \]

Using the special form of \( \mathcal{J} \) we have

\[ a_i = \partial/\partial \xi_i - i\alpha\pi^*(\mathcal{J}_{13}e_i) + i(d\pi)^\nu\pi^*(\mathcal{J}_{31}^\nu e_i) \]

and

\[ b_i = d\xi_i + i\alpha\pi^*(\mathcal{J}_{22}^\nu e^i) + id\pi^\nu\pi^*(\mathcal{J}_{12}^\nu e^i) \]

Hence we have that

\[ [a_i, a_j] = [\partial/\partial \xi_i - i\alpha\pi^*(\mathcal{J}_{13}e_i) + i(d\pi)^\nu\pi^*(\mathcal{J}_{31}^\nu e_i), \partial/\partial \xi_j - i\alpha\pi^*(\mathcal{J}_{13}e_j) + i(d\pi)^\nu\pi^*(\mathcal{J}_{31}^\nu e_j)] \]

\[ = [\partial/\partial \xi_i - i\alpha\pi^*(\mathcal{J}_{13}e_i), \partial/\partial \xi_j - i\alpha\pi^*(\mathcal{J}_{13}e_j)] \]

\[ + t_{\partial/\partial \xi_i - i\alpha\pi^*(\mathcal{J}_{13}e_i)}d(d\pi)^\nu\pi^*(\mathcal{J}_{31}^\nu e_j) - t_{\partial/\partial \xi_j - i\alpha\pi^*(\mathcal{J}_{13}e_j)}d(d\pi)^\nu\pi^*(\mathcal{J}_{31}^\nu e_i) \]

\[ + (1/2)dt_{\partial/\partial \xi_i - i\alpha\pi^*(\mathcal{J}_{13}e_i)}i(d\pi)^\nu\pi^*(\mathcal{J}_{31}^\nu e_j) - (1/2)dt_{\partial/\partial \xi_j - i\alpha\pi^*(\mathcal{J}_{13}e_j)}i(d\pi)^\nu\pi^*(\mathcal{J}_{31}^\nu e_i) \]

\[ = -[\alpha\pi^*(\mathcal{J}_{13}e_i), \alpha\pi^*(\mathcal{J}_{13}e_j)] \]

\[ + t_{\alpha\pi^*(\mathcal{J}_{13}e_i)}d(d\pi)^\nu\pi^*(\mathcal{J}_{31}^\nu e_j) - t_{\alpha\pi^*(\mathcal{J}_{13}e_j)}d(d\pi)^\nu\pi^*(\mathcal{J}_{31}^\nu e_i) \]

\[ + (1/2)dt_{\alpha\pi^*(\mathcal{J}_{13}e_i)}(d\pi)^\nu\pi^*(\mathcal{J}_{31}^\nu e_j) - (1/2)dt_{\alpha\pi^*(\mathcal{J}_{13}e_j)}(d\pi)^\nu\pi^*(\mathcal{J}_{31}^\nu e_i) \]

\[ = -\alpha[\pi^*(\mathcal{J}_{13}e_i), \pi^*(\mathcal{J}_{13}e_j)] \]

\[ + t_{\alpha\pi^*(\mathcal{J}_{13}e_i)}(d\pi)^\nu\pi^*d(\mathcal{J}_{31}^\nu e_j) - t_{\alpha\pi^*(\mathcal{J}_{13}e_j)}(d\pi)^\nu\pi^*d(\mathcal{J}_{31}^\nu e_i) \]

\[ + (1/2)dt_{\alpha\pi^*(\mathcal{J}_{13}e_i)}(d\pi)^\nu\pi^*d(\mathcal{J}_{31}^\nu e_j) - (1/2)dt_{\alpha\pi^*(\mathcal{J}_{13}e_j)}(d\pi)^\nu\pi^*d(\mathcal{J}_{31}^\nu e_i) \]
\[= -\alpha \pi^*[J_{13}e_i, J_{13}e_j] \]
\[+ (d\pi)^\vee \pi^*l_{J_{13}e_i}d(J_{31}^\vee e_j) - (d\pi)^\vee \pi^*l_{J_{13}e_j}d(J_{31}^\vee e_i) \]
\[+ (1/2)(d\pi)^\vee \pi^*dl_{J_{13}e_i}(J_{31}^\vee e_j) - (1/2)(d\pi)^\vee \pi^*dl_{J_{13}e_j}(J_{31}^\vee e_i) \]
\[= -(\alpha + (d\pi)^\vee)[(\pi^*J_{13})e_i - (\pi^*J_{31})e_i, \pi^*(J_{13}e_j) - \pi^*(J_{31}^\vee e_j)] \]
\[= -(\alpha + (d\pi)^\vee)\pi^*[J_{13}e_i - J_{31}^\vee e_i, J_{13}e_j - J_{31}^\vee e_j] \]
or
\[[a_i, a_j] = -(\alpha + (d\pi)^\vee)\pi^*[J_{13}e_i - J_{31}^\vee e_i, J_{13}e_j - J_{31}^\vee e_j] \quad (7.3) \]

Similarly we have
\[[a_i, b_j] = (\alpha + (d\pi)^\vee)\pi^*[J_{13}e_i - J_{31}^\vee e_i, J_{22}^\vee e_j + J_{12}^\vee e_j] \quad (7.4) \]
and
\[[b_i, b_j] = -(\alpha + (d\pi)^\vee)\pi^*[J_{22}^\vee e_i + J_{12}^\vee e_i, J_{22}^\vee e_j + J_{12}^\vee e_j]. \quad (7.5) \]

The right hand sides of all three of these expressions are sections of the vector bundle \(image(\alpha + (d\pi)^\vee) \otimes \mathbb{C}\). Therefore, the right hand sides are sections of \(E\) and in particular be sections of the graph of a map of vector bundles from \(image(j + D^\vee) \otimes \mathbb{C}\) to \(image(\alpha + (d\pi)^\vee) \otimes \mathbb{C}\) if and only if \([a_i, a_j], [a_i, a_j], \text{ and } [b_i, b_j]\) all vanish. This is precisely the statement of 7.3: that all pairwise Courant brackets between sections of \(M(S)\) vanish.

\[\square\]

Notice now that if we replace the vector bundle \(V\) by \(V^\vee\) and \(J\) by \(\hat{J}\) (see 4.14, 4.4) then \(M\) gets replaced by
\[\hat{M} = \begin{pmatrix} -J_{22}^\vee & -J_{13}^\vee \\ -J_{12}^\vee & J_{31}^\vee \end{pmatrix}. \quad (7.6) \]
but \(M(S) = \hat{M}(S^\vee)\). Therefore we have also proven 7.4.

\[\square\]

It is also clear from this proof and using equation (7.2), that if \(J\) is integrable, then the two almost Dirac structures \(\Delta = \mathcal{J}(V) = \hat{\mathcal{J}}(V)\) and \(\hat{\Delta} = \hat{\mathcal{J}}(V^\vee) = \mathcal{J}(V^\vee)\) are as well. The
vector bundle $\Delta$ inherits the same flat connection from $V$ via $J$ or $\hat{J}$. Similarly, $\hat{\Delta}$ inherits the same flat connection from $J$ or $\hat{J}$.

**Corollary 7.7.** For a flat connection $\nabla$ on a vector bundle $V$ over $M$, a $\nabla$-semi-flat generalized complex structure on the total space of $V$ induces a pair of transverse Dirac structures on $M$. These Dirac structures inherit flat connections.

\[
\begin{proof}
\end{proof}
\]

**Remark 7.8.** The geometry of a pair of transversal Dirac subbundles was recently studied by A. Wade and found to be equivalent to a generalized paracomplex structure as defined in [36]. Furthermore using the analysis of the integrability condition in terms of local systems above, the two Dirac structures that we have identified above form a pair of Dirac structures (see e.g. [12]) in the sense of Gelfand and Dorfman and therefore leads to method of constucting integrable hierarchies with respect to the two Poisson structures conming from the two Dirac structures. This remark also applies to the torus bundle case below. Another overlap with the mathematics of integrable systems is also noted in section 10 and these overlaps will be the subject of future work.

Note that a generalized Kähler structure is defined [15] to be a generalized almost Kähler structures where both of the two generalized almost complex structures are integrable. Therefore we have also proven (using the results of subsection 4.2) the following.

**Corollary 7.9.** The correspondence in Corollary 7.4, gives a bijective correspondence between $\nabla$-semi-flat generalized Kähler structures on $X$ and $\nabla^\vee$-semi-flat generalized Kähler structures on $\hat{X}$.

\[
\begin{proof}
\end{proof}
\]

8. From vector bundles to torus bundles

In this section we describe generalized complex structures on (real) torus bundles with sections and their mirrors. The base of our torus bundles will turn out to support a pair of complimentary Dirac structures.

8.1. The geometry of torus bundles and the dual of a torus bundle. Let $Z \xrightarrow{p} M$ be a fiber bundle over a real manifold $M$ for which the fibers have the diffeomorphism type of a real torus of dimension $n$. We call this a torus bundle. So we have for $U$ small in the base, local isomorphisms of fiber bundles $p^{-1}(U) \cong U \times T$. Assume that this fiber bundle possesses
a global (smooth) section $s$. This is equivalent to assuming that the structure group of the bundle is $Diff(T,0)$ as opposed to $Diff(T)$. However, since the connected component of $Diff(T,0)$ is contractible, the structure group of the bundle may be reduced to those diffeomorphisms which respect the group structure: $Aut(T) \cong GL(n,\mathbb{Z})$. Recently this issue was discussed in [20]. We consider this to have been done and regard $s$ as the zero section.

We therefore consider $X$ as a (Lie) group bundle or bundle of (Lie) groups. Recall that for a bundle of Lie groups modeled on a Lie group $G$ (we sometimes call this simply a $G$-bundle) we have local maps $\rho^{-1}(U) \cong U \times G$ and the transition maps $U \cap V \times G \cong U \cup V \times G$ restrict to a Lie group isomorphism of $G$ on each fiber.

Consider the tangent sequence

$$0 \longrightarrow T_{Z/M} \longrightarrow T_Z \xrightarrow{dp} p^*T_M \longrightarrow 0$$

As in case of vector bundles, $T_{Z/M}$ is a pullback. Indeed, let $V = s^*T_{Z/M}$, then we have that $T_{Z/M} \cong p^*V$. This follows from the following simple observation.

**Lemma 8.1.** Let $G$ be a Lie group and $\mathcal{Y} \xrightarrow{\rho} N$ a $G$-bundle. Call the zero-section $s$. Then we have $\rho^*s^*T_{Y/N} \cong T_{Y/N}$.

**Proof.**

Write any section $\sigma$ of $(s \circ \rho)^{-1}T_{Y/N}$ over $U \subseteq \mathcal{Y}$ as $\sigma = \sigma_0 \circ s \circ \rho$ where $\sigma_0$ is a section of $T_{Y/N}$ over $s(\rho(U))$. Now using the local group structure we may push $\sigma_0$ forward along the fibers. The transition maps respect the group structure of $G$ and therefore these vector fields patch to a section of $T_{Y/N}$ over $\rho^{-1}(\rho(U))$ and then we can restrict this section to $U$. This gives a morphism of vector bundles

$$(s \circ \rho)^*T_{Y/N} \xrightarrow{\psi} T_{Y/N}.$$  

Over a point $y \in \mathcal{Y}$, when we look in one of the trivial neighborhoods, $\rho^{-1}(U) \cong U \times G$ where $y = (u,g)$, the map becomes just the obvious map $\text{Lie}(G) \rightarrow T_yG$ which is clearly an isomorphism. Hence we can conclude that the map $\psi$ gives an isomorphism $\rho^*s^*T_{Y/N} \cong T_{Y/N}$. \hfill $\square$

Furthermore, if we use the fact that the torus compact and connected, the sheaf of sections, $V$ of $V = s^*T_{Z/M}$ is isomorphic to $(R^1p_*\mathbb{R})^\vee \otimes C^\infty_{\mathcal{M}}$.

**Lemma 8.2.** If $Z \xrightarrow{p} M$ is a $T = \text{Lie}(T)/\Gamma$ bundle with structure group $Aut(\Gamma)$ then $V \cong (R^1p_*\mathbb{R})^\vee \otimes C^\infty_{\mathcal{M}}$. 

Notice that this is just a relative version of the natural isomorphism \( \text{Lie}(\mathbf{T})^\vee \cong H^1(\mathbf{T}, \mathbb{R}) \) which is described for example in [4].

**Proof.**

Notice that \( V = s^*T_{Z/M} \) is a \( \text{Lie}(\mathbf{T}) \)-bundle on \( M \) with structure group \( \text{Aut}(\Gamma) \). Let \( \Lambda \subseteq \text{tot}(V) \) be the lattice induced by \( \Gamma \) and \( \mathcal{S}_\Lambda \) be its sheaf of sections. There is a morphism of presheaves of abelian groups

\[
\mathcal{S}_\Lambda \to [U \mapsto H_1(p^{-1}(U), \mathbb{Z})].
\]

It is given (for \( U \) connected) by sending \( \lambda \in \mathcal{S}_\Lambda(U) \) to the homology class of the image in \( Z \) of the line in \( \text{tot}(V) \) connecting the point 0 over \( m \) to the point \( \lambda(m) \) for some \( m \in U \). For \( U \) small enough this is an isomorphism using the Künneth theorem

\[
\mathcal{S}_\Lambda(U) \cong H_1(p^{-1}(U), \mathbb{Z}) = \text{Hom}_{\mathbb{Z}}(H^1(p^{-1}(U), \mathbb{Z}), \mathbb{Z}).
\]

Hence we have isomorphisms of sheaves

\[
\mathcal{S}_\Lambda \cong \text{Hom}_{\mathbb{Z}}(R^1p_*\mathbb{Z}, \mathbb{Z}), \quad \mathcal{S}_\Lambda \otimes \mathbb{R} \cong (R^1p_*\mathbb{R})^\vee, \quad \text{and} \quad V \cong (R^1p_*\mathbb{R})^\vee \otimes C_\infty^M.
\]

Therefore we have a flat Gauss-Manin connection \( \nabla \) on \( V \) given as the image of \( 1 \otimes d \) under this isomorphism. Recall that in the case of vector bundles we had for each flat connection on \( V \) a (potentially) inequivalent mirror symmetry transformation. By contrast, in the case of torus bundles, the topology of a torus bundle has given us a natural flat connection on \( V \) and so we need not make any additional choices.

The multisection of \( V \) given by \( \Lambda = \text{tot}(\mathcal{S}_\Lambda) \) acts on \( X = \text{tot}(V) \) and the orbits are the fibers of the natural map from \( X \) to \( Z \). Hence we have a diffeomorphism \( X/\Lambda \cong Z \). Under this quotient, the multisection goes to \( s(M) \). The sheaf of sections of \( Z \) becomes the sheaf of groups \( V/\mathcal{S}_\Lambda \cong ((R^1p_*\mathbb{R})^\vee \otimes C_\infty^M)/(R^1p_*\mathbb{Z})^\vee \) where the zero section has image \( s(M) \). The isomorphism \( X/\Lambda \cong Z \) is an isomorphism of \( GL(n, \mathbb{Z}) \) fiber bundles over \( M \).

Now the **dual torus bundle** is defined to be \( \tilde{Z} = \tilde{X}/\tilde{\Lambda} \xrightarrow{\tilde{p}} M \) where \( \tilde{\Lambda} = \text{tot}(\mathcal{S}_\Lambda^\vee) \) and \( \mathcal{S}_\Lambda^\vee = \text{Hom}_{\mathbb{Z}}(\mathcal{S}_\Lambda, \mathbb{Z}) \subseteq V^\vee \). Furthermore, \( \mathcal{S}_\Lambda^\vee \cong R^1\tilde{p}_*\mathbb{Z} \) and \( V^\vee \cong R^1\tilde{p}_*\mathbb{R} \otimes C_\infty^M \). This gives a flat connection on \( V^\vee \) which is of course just the dual connection \( \nabla^\vee \). Also \( \mathcal{S}_\Lambda^\vee \otimes \mathbb{R} \) is the sheaf of flat sections of \( V^\vee \) with respect to \( \nabla^\vee \). The sheaf of sections of \( \tilde{Z} \) over \( M \) is a sheaf of groups given by \( V^\vee/\mathcal{S}_\Lambda^\vee \cong ((R^1\tilde{p}_*\mathbb{R})) \otimes C_\infty^M)/(R^1\tilde{p}_*\mathbb{Z}) \). We then have a global section \( \tilde{s} \) of \( \tilde{Z} \) over \( M \) which is the zero section and satisfies that \( \tilde{s}(M) \) is the image of the multisection \( \tilde{\Lambda} \) under the quotient map.
We saw in section 4 that if we let $X = \text{tot}(V)$, then $\nabla$ gives us a splitting $D$ of the tangent sequence of the map $X \xrightarrow{\pi} M$. We can use this to split the tangent sequence of the map $Z \xrightarrow{p} M$. Consider the following diagram where we have decomposed $\pi$ as $p \circ q$.

\[
\begin{array}{c}
X \xrightarrow{q} Z \xrightarrow{p} M,
\end{array}
\]

We may push forward the exact sequence

\[
0 \longrightarrow \pi^*V \xrightarrow{\ } T_X \xrightarrow{d\pi} \pi^*TM \longrightarrow 0
\]

which is split by $\pi^*(V) \xleftarrow{D} T_X$ to the exact sequence

\[
0 \longrightarrow q_\ast\pi^*V \xrightarrow{\ } q_\ast T_X \xrightarrow{q_\ast d\pi} q_\ast\pi^*TM \longrightarrow 0
\]

which is split by $q_\ast\pi^*(V) \xleftarrow{D} q_\ast T_X$. Furthermore, $\Lambda$ naturally acts on all three of these sheaves and if we take the $\Lambda$ invariants of each term of this sequence we recover precisely the exact sequence that we want to split, namely:

\[
0 \longrightarrow p^*V \xrightarrow{\ } T_Z \xrightarrow{dp} p^*TM \longrightarrow 0
\]

Therefore, the only thing to check in order to split this sequence is that the map $D$ satisfies $(dt_\lambda)(Dw) = D((dt_\lambda)(w))$ where for some small $U \subseteq M$ and small $U' \subseteq p^{-1}(U)$ $w$ is a section of $T_X$ over $q^{-1}(U')$, $\lambda$ is a component of $\Lambda \cap \pi^{-1}(U)$ and $t_\lambda : X \rightarrow X$ is the action of addition of $\lambda$. However,

\[
(dt_\lambda)(Dw) = (dt_\lambda)(((\pi^*\nabla)S)w) = ((\pi^*\nabla)(S + \lambda))(dt_\lambda)w = ((\pi^*\nabla)S)((dt_\lambda)w) = D((dt_\lambda)w)
\]

due to the fact that that the sections of the lattice are flat.

8.2. **Generalized complex structures on torus bundles and the mirror transformation.** We will now use the same names as in the vector bundles case for the splittings of the tangent sequences of $Z$ and $\hat{Z}$. That is:

\[
0 \longrightarrow \hat{p}^*V \xrightarrow{\ } \hat{T}_Z \xrightarrow{d\hat{p}} \hat{p}^*TM \longrightarrow 0,
\]

and

\[
0 \longrightarrow \hat{p}^*V \xrightarrow{\ } \hat{T}_Z \xrightarrow{d\hat{p}} \hat{p}^*TM \longrightarrow 0,
\]
Since we will be using only one connection in the case of torus bundles, we will drop $\nabla$ from the notation.

**Definition 8.3.** If $M$ is an $n$-dimensional real manifold and $Z \to M$ is a real torus bundle with fiber dimension $n$ and zero section $s$ then we call a generalized almost complex structure $\mathcal{J}$ on $Z$ which comes from (see section 4) an adapted generalized almost complex structure $\mathcal{J}$ on $s^*T_{Z/M} \oplus T_M = V \oplus T_M$ a semi-flat generalized almost complex structure.

Recall that “adapted” just means that

$$\mathcal{J}(s^*T_{Z/M} \oplus s^*T_{Z/M}^\vee) = T_M \oplus T_M^\vee$$

As in the vector bundle case, there is a bijective correspondence between semi-flat generalized almost complex structures on $Z$ and $\hat{Z}$. The proof is precisely the same, except the choice of local coordinates is local along the base and the fiber, instead of just along the base.

**Theorem 8.4.** A semi-flat generalized almost complex structure $\mathcal{J}$ on a torus bundle $Z \to M$ with zero section $s$ is integrable if and only if

$$[\mathcal{J}(\mathcal{S} \oplus \mathcal{S}^\vee), \mathcal{J}(\mathcal{S} \oplus \mathcal{S}^\vee)] = 0$$

where $\mathcal{S}$ is the sheaf of flat sections of $s^*T_{Z/M}$.

**Corollary 8.5.** A semi-flat generalized almost complex structure $\mathcal{J}$ on a torus bundle $Z \to M$ is integrable if and only if its mirror structure $\hat{\mathcal{J}}$ on the dual torus bundle $\hat{Z} \to M$ is integrable.

**Remark 8.6.** This means that we have given a bijective correspondence between semi-flat generalized complex structures on $Z$ and semi-flat generalized complex structures on $\hat{Z}$. The same holds true for generalized Kähler structures in which both of the generalized complex structures are semi-flat.

**Corollary 8.7.** A semi-flat generalized almost complex structure $\mathcal{J}$ on a torus bundle $Z \to M$ induces a pair of almost Dirac structures

$$\Delta = \mathcal{J}(s^*T_{Z/M}), \quad \hat{\Delta} = \mathcal{J}(s^*T_{Z/M}^\vee) \subseteq T_M \oplus T_M^\vee$$

Each carries its own flat connection and these Dirac structures are exchanged under mirror symmetry. If $\mathcal{J}$ is integrable then $\Delta$ and $\hat{\Delta}$ are integrable.
Now in the case when the generalized complex structure on the torus bundle is of symplectic type and the torus fibers are Lagrangian this result reproduces the starting point of the work [31] where the torus bundle is written as $\text{tot}(T_M^\vee)/\Lambda$ and $\Delta$ is the Dirac structure $T_M^\vee$, which inherits a flat connection $\nabla$. The mirror manifold $\text{tot}(T_M)/\Lambda^\vee$ inherits a complex structure as explained in [31] constructed using the dual connection $\nabla^\vee$ which is both flat and torsion-free. This corresponds to the canonical almost complex structure on $\text{tot}(T_M)$ associated to a connection on $T_M$ which is known [11] to be integrable if and only if the connection is flat and torsion-free.

9. THE COHOMOLOGY OF TORUS BUNDLES

Consider the diagram

Now the space $Z \times_M \hat{Z}$ is endowed with a global closed two form given as $\Xi = \frac{1}{2\pi i} F$ where $F$ is the curvature of a connection on the relative Poincaré (line) bundle. See [30] for an explanation of the relative Poincaré bundle in this context. Now we would like to introduce a relative version of a map given [29] in the context of mirror symmetry of abelian varieties as introduced by Mukai [28]. This idea has appeared in various ways in [1, 35, 15, 27] and the references therin.

Lemma 9.1. If the bundle $\hat{q}^* T_{Z/M}$ is orientable then we have a morphism (independent of the choice of orientation) of sheaves of $C^\infty_M$ modules $p_* \Omega_Z^\bullet \to \hat{p}_* \Omega_{\hat{Z}}^\bullet$ is a morphism of the de Rham complexes. Therefore this morphism gives a map of presheaves

$$[U \mapsto H^\bullet(p^{-1}(U), \mathbb{R})] \to [U \mapsto H^\bullet(\hat{p}^{-1}(U), \mathbb{R})].$$

This map of presheaves induces an isomorphism of the sheafifications $R^j p_* \mathbb{R} \to R^j \hat{p}_* \mathbb{R}$ which decomposes into isomorphisms $R^j p_* \mathbb{R} \to R^{n-j} \hat{p}_* \mathbb{R}$ for $j = 1, \ldots, n$.

Proof.
We have a map \( \tilde{q}^\flat = (dq)^\vee \circ \tilde{q}^* \) from \( p_\ast \Omega^j_Z \) to \( p_\ast \pi_\ast \Omega^j_{Z \times_M \tilde{Z}} \) given by pulling back differential forms. Clearly, \( \tilde{q}^\flat \) commutes with the de Rham differentials. Observe that the map \( q \) makes \( Z \times_M \tilde{Z} \) into a torus bundle over \( \tilde{Z} \). The relative tangent bundle of the tangent sequence of the map \( q \) is isomorphic to \( \tilde{q}^* T_{Z/M} \). Therefore we also have a map \( q_\ast \) which integrates along the fibers and maps \( q_\ast \Omega^k_{Z \times_M \tilde{Z}} \) to \( \Omega^{k-n}_{\tilde{Z}} \). Explicitly, if we take our global section \( s \) over \( Z \times_M \tilde{Z} \) of \( \wedge^n \tilde{q}^* T_{Z/M} \) and the corresponding global section \( t \) of \( \wedge^n q_\ast T_{Z/M} \) then \( q_\ast (Y) = \int (Z \times_M \tilde{Z}) ((t_\ast Y) \wedge t) \). This does not depend on the choice of \( s \) but we do need the fibers of \( q \) to be orientable manifolds to integrate over them. Since the torus fibers of \( \tilde{q} \) are manifolds without boundary we have that \( q_\ast \) and also its pushforward, \( \tilde{p}_\ast [q_\ast] \) commutes with the de Rham differentials. Now we can define the map \( F \cdot T : p_\ast \Omega^j_Z \to \tilde{p}_\ast \Omega^j_{\tilde{Z}} \) by

\[
F \cdot T(\mu) = \tilde{p}_\ast [q_\ast] (\tilde{q}^\flat (\mu) \wedge \exp(\Xi))
\]

Now since \( d\Xi = 0 \) we have \( F \cdot T(d\mu) = dF \cdot T(\mu) \) and hence we get a map of presheaves \( [U \mapsto H^\ast (p^{-1}(U), \mathbb{R})] \to [U \mapsto H^\ast (\tilde{p}^{-1}(U), \mathbb{R})] \). In particular we have a natural \( \mathbb{R} \)-linear map \( H^\ast (Z, \mathbb{R}) \to H^\ast (\tilde{Z}, \mathbb{R}) \). In order to write the map on differential forms locally on the base, chose a trivializing open neighborhood \( U \subseteq M \) and \( \mu \in \Omega^\ast (p^{-1}(U)) \). Then let \( \xi_i \) be the flat vertical coordinates on \( p^{-1}(U) \) and \( \eta_k \) be the dual flat vertical coordinates on \( \tilde{p}^{-1}(U) \). In these coordinates we may assume without loss of generality that

\[
s = \partial / \partial \xi_n \wedge \cdots \wedge \partial / \partial \xi_1
\]
on \( p^{-1}(U) \). (Every section may be extended to a global section.) Let us now express \( \mu \) in local coordinates.

\[
\mu = \sum_{|J|=1, \ldots, c} f_J \Theta_J \wedge d\xi_{j_1} \wedge \cdots \wedge d\xi_{j_b}
\]

Here, the \( \Theta_J \) are pullbacks of \((c-b)-\) forms from the base, \( J = (j_1, \ldots, j_b) \) where \( j_1 < \cdots < j_b \) and \( f_J \) are functions on \( p^{-1}(U) \). A simple calculation shows that

\[
\tilde{\mu} = F \cdot T(\mu) = \int_T \mu \wedge \exp(d\xi_i \wedge d\eta_i)
\]
is given by

\[
\tilde{\mu} = \sum_{|J|=1, \ldots, c} (-1)^{k_1 + \cdots + k_n-b} \theta_J \wedge d\eta_{k_1} \wedge \cdots \wedge d\eta_{k_{n-b}} \int_T f_J d\xi_1 \wedge \cdots \wedge d\xi_n
\]
where \( k_1 < \cdots < k_{n-b} \) is the compliment to \( J \). Now suppose that \( \mu \) is closed and that we consider the cohomology class \( [\mu] \in H^j (p^{-1}(U), \mathbb{R}) \). Using the Künneth isomorphism
\[ H^c(p^{-1}(U), \mathbb{R}) \cong \bigoplus_{b=0, \ldots, c} H^{c-b}(U, \mathbb{R}) \otimes H^b(T, \mathbb{R}) \cong H^c(T, \mathbb{R}) \]

we may absorb the \( f_j \) into the \( \Theta_j \) in the above expression and therefore since the cohomology of the tori \( T \) and \( T^\vee \) are generated by the classes \([d\xi_1 \wedge \cdots \wedge d\xi_j]\) and \([d\eta_1 \wedge \cdots \wedge d\eta_k]\) respectively we conclude that the map \( F.T. \) induces isomorphisms \( R^j p_* \mathbb{R} \rightarrow R^{n-j} \hat{p}_* \mathbb{R} \) for \( j = 1, \ldots, n \) as promised.

\[ \blacksquare \]

**Corollary 9.2.** If \( J \) is a semi-flat generalized almost complex structure on a \( n \)-torus bundle with section \( Z \) on an \( n \)-manifold \( M \) with associated spinor line bundle \( L \subseteq \wedge^* T^\vee_Z \otimes \mathbb{C} \), and \( \hat{L} \) is the line bundle associated to the mirror structure \( \hat{J} \) on \( \hat{Z} \), then

\[ F.T.(p_* L) = \hat{p}_* \hat{L} \subseteq \hat{p}_* \wedge T^\vee_{\hat{Z}} \otimes \mathbb{C}. \]

**Proof.**

This follows from tensoring the previous lemma with \( \mathbb{C} \) and using lemma 6.3.

\[ \blacksquare \]

**Remark 9.3.** The Fourier-Mukai transformation for spinors, combined with the formulae given in Lemma 6.2 can easily be used to show again that integrability is preserved by the mirror transformation we have described. As we have already shown this, we do not demonstrate it again with spinors.

**Example 9.4.** Let \( M = S^1 \) or \( \mathbb{R} \), \( Z = V/\Lambda \times M, \hat{Z} = V^\vee/\Lambda^\vee \times M \), where \( V/\Lambda \cong S^1 \). Let \( x, \theta, \hat{\theta} \) be “coordinates” on \( M, V/\Lambda \) and \( V^\vee/\Lambda^\vee \) respectively. Then for \( f \) a complex valued smooth nowhere vanishing function on \( M \).

\[ F.T. (e^{f\theta} dx) = \int_{V/\Lambda} e^{f\theta + dx} \theta \wedge d\theta = \int_{V/\Lambda} d\theta \wedge (f dx + d\theta) = d\theta + f dx \]

When we take \( f = i \), we see the spinor corresponding to a symplectic structure going to one representing a complex structure.

**Remark 9.5.** Let \( Z \) an \( n \)-torus bundle over a compact connected \( n \)-manifold such that \( \hat{q}^* T_{Z/M} \) is orientable. Consider the “Moduli-Space” \( SFGCY(Z) \) of semi-flat generalized Calabi-Yau structures. These are semi-flat generalized complex structures which are generalized Calabi-Yau \([17]\), meaning that the associated spinor line bundles \( L \) have nowhere vanishing, closed, global sections.

Since these sections are known \([9, 15, 17]\) to be either even or odd we may consider a “period map” \([18, 19]\) from this space into \( \mathbb{P}(H^{even}(Z, \mathbb{C})) \coprod \mathbb{P}(H^{odd}(Z, \mathbb{C})) \). Note that we are assuming here that for a fixed structure, different closed, nowhere vanishing global
sections of $L$ define the same cohomology class up to multiplication by constants. Under this assumption, we have shown the existence of a commutative diagram.

$$
\begin{array}{ccc}
\text{SFGCY}(Z) & \longrightarrow & \mathbb{P}(H_{\text{even}}(Z, \mathbb{C})) \coprod \mathbb{P}(H_{\text{odd}}(Z, \mathbb{C})) \\
\downarrow & & \downarrow \\
\text{SFGCY}(\hat{Z}) & \longrightarrow & \mathbb{P}(H_{\text{even}}(\hat{Z}, \mathbb{C})) \coprod \mathbb{P}(H_{\text{odd}}(\hat{Z}, \mathbb{C}))
\end{array}
$$

In the case that the torus bundles are $Z$ and $\hat{Z}$ trivial, the vertical map takes horizontal $i-$th cohomology to itself and vertical $i$-th cohomology to vertical $(n-i)$-th cohomology (both with multiplication by signs).

**Conjecture 9.6.** Let $(Z, \mathcal{J})$ be a compact generalized Calabi-Yau manifold of (real) dimension $2n$. As we have mentioned in the previous remark, it would be desirable to know that there is a unique element in $\mathbb{P}(H^*(Z, \mathbb{C}))$ associated with $Z$. Without this knowledge, the previous diagram would have to be modified by the appropriate restrictions on the left hand side. Therefore we conjecture (without overwhelming evidence) that if $\phi$ is a global, closed, nowhere vanishing differential form, representing $\mathcal{J}$, and $f$ is a nowhere zero smooth complex valued function such that $d(f\phi) = 0$, that $f$ is constant. If we call generalized Calabi-Yau manifolds satisfying this condition **Liouville** then it is easy to see that all symplectic manifolds are Liouville (take $\phi = e^{-i\omega}$), compact Calabi-Yau manifolds are Liouville (take $\phi$ to be a nowhere zero holomorphic $n-$form), and products and $B-$field transformations of Liouville generalized Calabi-Yau manifolds are Liouville.

**10. Transverse foliations and generalized Kähler geometry**

In this section we study in the abstract some of essential geometric details of our construction without reference to the specific context (e.g. the type of bundle).

**Definition 10.1.** Suppose that $X$ is a foliated manifold and let $\mathcal{P} \subseteq T_X$ be the involute sub-bundle tangent to the leaves of the foliation. We say that a generalized complex structure $\mathcal{J}$ and $\mathcal{P}$ are **compatible** if there exists a complementary sub-bundle $\mathcal{Q} \subseteq T_X$ so that

$$
\mathcal{J}\big|_{\mathcal{P} \oplus \text{Ann}(\mathcal{Q})} : \mathcal{P} \oplus \text{Ann}(\mathcal{Q}) \to \mathcal{Q} \oplus \text{Ann}(\mathcal{P})
$$

is an isomorphism of vector bundles. Under this condition, we will call $\mathcal{Q}$ a $\mathcal{J}$-compliment to $\mathcal{P}$. For $\mathcal{Q}$ a $\mathcal{J}$-compliment to $\mathcal{P}$ we will often tacitly identify $\mathcal{P}^\vee$ with $\text{Ann} \mathcal{Q}$, and $\mathcal{Q}^\vee$ with $\text{Ann} \mathcal{P}$.
Notice that the \((+i)\) eigenbundle, \(E\) of \(J\) is in this case necessarily transverse to both \(\mathcal{P}_C \oplus \text{Ann}(\mathcal{Q}_C)\) and \(\mathcal{Q}_C \oplus \text{Ann}(\mathcal{P}_C)\). Hence \(E\) is the graph of a map from \(\mathcal{P}_C \oplus \text{Ann}(\mathcal{Q}_C)\) to \(\mathcal{Q}_C \oplus \text{Ann}(\mathcal{P}_C)\). In fact, it is easy to see that we have \(E = \text{graph}(-iJ)\) where we consider \((-iJ)\) as a map from \(\mathcal{P}_C \oplus \text{Ann}(\mathcal{Q}_C)\) to \(\mathcal{Q}_C \oplus \text{Ann}(\mathcal{P}_C)\).

**Examples 10.2.** (i) Suppose that \(X\) is a manifold equipped with an involute distribution \(\mathcal{P} \subseteq T_X\) of half the dimension of \(X\). Let \(J\) be the generalized almost complex structure on \(X\) corresponding to a non-degenerate real two form \(\omega\). Then \(\mathcal{P}\) and \(J\) are compatible if and only if \(\mathcal{P}\) defines a Lagrangian foliation on \(X\). Indeed, the compatibility shows that \(\omega\) defines an isomorphism from \(\mathcal{P}\) to \(\text{Ann} \mathcal{P}\), which shows that \(\mathcal{P}\) is Lagrangian. Conversely, if \(\mathcal{P}\) is Lagrangian, then by (see [8]) choosing an almost complex structure \(J\) so that the isomorphism \(-\omega J : T_X \rightarrow T_X^\vee\) represents a Riemannian metric on \(X\), it is easy to see that the vector bundle \(J\mathcal{P}\) is a \(J\)-compliment to \(\mathcal{P}\), and so \(J\mathcal{P}\) is also Lagrangian. This example signifies some relationship of the content of this paper with the area of integrable systems.

(ii) On the other hand if \(X\) is a manifold equipped with an involute distribution \(\mathcal{P} \subseteq T_X\) of half the dimension of \(X\) and \(J\) is the generalized almost complex structure on \(X\) corresponding to an almost complex structure \(J\) then \(\mathcal{P}\) and \(J\) are compatible if and only if \(J\mathcal{P} \cap \mathcal{P} = (0)\). In other words the leaves of the foliation are totally real sub-manifolds [5]. In this case the \(J\)-compliment to \(\mathcal{P}\) is fixed uniquely as \(J\mathcal{P}\).

(iii) One of the main classes of examples in this paper is where \(X\) is an \(n\)-torus bundle over an \(n\)-manifold and \(J\) is a semi-flat generalized complex structure, \(\mathcal{P}\) is the vertical foliation tangent to the torus fibers, and \(\mathcal{Q}\) is the horizontal foliation given by the splitting of the tangent sequence given by the connection as in section 8.

**Remark 10.3.** In the above and in much of what follows, the fact that \(\mathcal{P}\) is involute is irrelevant. That is to say, it could just be a sub-bundle of the tangent bundle. However, it will be taken to be involute for the applications that we have in mind, for instance when \(\mathcal{P}\) represents the tangent directions to a torus fibration.

**Definition 10.4.** Suppose that \(J\) and \(J'\) constitute a generalized almost Kähler pair of generalized almost complex structures and \(\mathcal{P} \subseteq T_X\) is a sub-bundle of the tangent bundle of
half the dimension. Then we say that $P$ is compatible with the pair $(\mathcal{J}, \mathcal{J}')$ when

$$J_2(\text{Ann}(P)) \subseteq P \quad (10.1)$$
$$J_3(\mathcal{P}) \subseteq \text{Ann}(P) \quad (10.2)$$
$$J'_2(\text{Ann}(P)) \subseteq P \quad (10.3)$$
$$J'_3(\mathcal{P}) \subseteq \text{Ann}(P) \quad (10.4)$$
$$J_1J'_1(\mathcal{P}) \subseteq P \quad (10.5)$$
$$J'_1J_1(\mathcal{P}) \subseteq P \quad (10.6)$$
$$J_4J'_4(\text{Ann}(P)) \subseteq \text{Ann}(P) \quad (10.7)$$
$$J'_4J_4(\text{Ann}(P)) \subseteq \text{Ann}(P) \quad (10.8)$$

Notice that if there is a sub-bundle $Q$ which is both a $J$-compliment and an $J'$-compliment to $P$ then $P$ is compatible with $(\mathcal{J}, \mathcal{J}')$. The converse will be shown below. In the ordinary Kähler case 2.7 the condition that $P$ is compatible with $(\mathcal{J}, \mathcal{J}')$ simply says that $P$ is Lagrangian with respect to the symplectic structure. In the $B$-transformed almost Kähler case where we have an almost Kähler pair $(J, \omega)$ the conditions are as follows: $P$ must be Lagrangian with respect to the symplectic structure $\omega$ and also $B(\omega^{-1}BP, P) = 0$ and $B(JP, P) = 0$.

**Theorem 10.5.** If $X$ is a $2n$-dimensional real manifold then a rank $n$ bundle $P \subseteq T_X$ is compatible with a generalized almost Kähler pair $(\mathcal{J}, \mathcal{J}')$ if and only if there is a sub-bundle $Q \subseteq T_X$ which is both a $J$-compliment and a $J'$-compliment to $P$. These properties specify $Q$ uniquely.

**Proof.**

If such a $Q$ exists then it is clear that $P$ and $Q$ are both compatible with the pair $(\mathcal{J}, \mathcal{J}')$. Furthermore, the property that $Q$ is a $J$-compliment and a $J'$-compliment to $P$ for a generalized Kähler pair $(\mathcal{J}, \mathcal{J}')$ fixes $Q$ uniquely. Indeed, if we are in this situation and $G = -\mathcal{J}J'$ is the generalized Kähler metric then we have that

$$g - bg^{-1}b = G_3 = -J'_3J_1 - J'_4J_3.$$

Therefore the isomorphism $(g - bg^{-1}b)$ takes $P$ to $\text{Ann}(Q)$ and so $Q$ must be the perpendicular to complement of $P$ with respect to the metric $(g - bg^{-1}b)$. We will now realize $Q$ explicitly as the image of a different automorphism $K_+ \in GL(T_X)$ of the tangent bundle with itself which becomes an almost complex structure in the when $b = 0$. Therefore the proof will be completed via the following lemma.
Lemma 10.6. If $(\mathcal{J}, \mathcal{J}')$ is a generalized almost Kähler pair then the map

$$K_+ = J_+ (1 - g^{-1}b) = J_1 + J'_1$$

is an isomorphism of the tangent bundle with itself. If $P$ is compatible with the pair $(\mathcal{J}, \mathcal{J}')$ then $K_+$ takes $P$ to a sub-bundle $Q$, transversal to $P$, and we have that

$$J\big|_{P \oplus \text{Ann}(Q)} : P \oplus \text{Ann}(Q) \to Q \oplus \text{Ann}(P)$$

and

$$J'\big|_{P \oplus \text{Ann}(Q)} : P \oplus \text{Ann}(Q) \to Q \oplus \text{Ann}(P)$$

are isomorphisms of vector bundles. In other words, $Q$ is both a $\mathcal{J}$-compliment and a $\mathcal{J}'$-compliment to $P$.

Proof.

First of all, notice that $K_+$ is an isomorphism of the vector bundle $T_X$ with itself. Indeed, the vector bundle map $(g - bg^{-1}b)$ from $T_X$ to $T^\vee_X$ corresponds to the metric $g(v, w) + g^{-1}(bv, bw)$ which is positive definite and hence if we consider $f$ to $(g - bg^{-1}b)^{-1}$, we see that $K_+(-(g + b)fJ_+) = 1$.

Now we have

$$G_3J_1 = -(J_3J'_1 + J_4J'_3)J_1 - J_3J'_1J_1 - J_4J'_3J_1$$

$$= -J_3J'_1J_1 - J_4(J_3J'_1 + J_4J'_3 - J'_4J_3)$$

$$= -J_3J'_1J_1 - J_4J_3J'_1 - (J_4)^2 J'_3 + J_4J'_4 J_3$$

$$= -J_3J'_1J_1 - J_4J_3J'_1 - (1 - J_5J_2)J'_3 + J_4J'_4 J_3$$

$$= -J_3J'_1J_1 + J_4J_3J'_1 + J'_3 + J_3 J_2 J'_3 + J_4 J'_4 J_3$$

$$= -J_3J'_1J_1 + J_3 J_1 J'_1 + J'_3 + J_3 J_2 J'_3 + J_4 J'_4 J_3.$$  

By inspection of the definition of compatibility of $P$ with the pair $(\mathcal{J}, \mathcal{J}')$ we have that all of these terms send $P$ into $\text{Ann}(P)$. Thus $G_3J_1(P) \subseteq \text{Ann}(P)$. Since the roles of $\mathcal{J}$ and $\mathcal{J}'$ are interchangeable we have,
\[ G_3J_1' = -J_2J_1'J_1 + J_3J_1'J_3 + J_3J_2J_3 + J_4J_4J_3' \]
as well. Hence \( G_3J_1'(P) \subseteq \text{Ann}(P) \). Therefore the image of \( P \) under the isomorphism \( K_+ = J_1 + J_1' \) is the perpendicular sub-bundle to \( P \) with respect to the metric \( G_3 \).

Now, in order to show the remaining claims, it suffices to define \( Q = K_+(P) \) and show that \( J_1(Q) \subseteq P \) and \( J_4(\text{Ann}(P)) \subseteq \text{Ann}(Q) \). Indeed suppose that we have shown this. Note that reversing the roles of \( J \) and \( J' \) does not change \( Q \) and so we get that \( J_1'(\text{Ann}(P)) \subseteq \text{Ann}(Q) \), and therefore \( J(Q \oplus \text{Ann}(P)) \subseteq P \oplus \text{Ann}(Q) \) and \( J'(Q \oplus \text{Ann}(P)) \subseteq P \oplus \text{Ann}(Q) \), which is enough since \( J \) and \( J' \) are isomorphisms.

Let \( v \) be an element of a fiber of \( Q \). We may express it as \( (J_1 + J_1')w \) for a unique fiber \( w \) of \( P \) over the same point. Then

\[ J_1v = J_1^2w + J_1J_1'w = -w - J_2J_3w + J_1J_1'w \]

which is an element of the fiber of \( P \) over the same point.

Let \( \mu \) be an element of a fiber of \( \text{Ann}(P) \). Then, if \( v \) is in the fiber of \( Q \) over the same point, we have that \( (J_4\mu)v = -\mu(J_1v) \) which is zero by the previous paragraph. Therefore \( J_4\mu \) is in the fiber of \( \text{Ann}(Q) \) over the same point. \( \square \)

The reader may wonder about the possibility of instead taking

\[ K_- = J_1 - J_1' = J_- (1 + g^{-1}b) \]

**Lemma 10.7.** \( K_- \) is an isomorphism of the tangent bundle with itself. In general it is not equal to \( K_+ \). However, if \( P \) is compatible with the generalized almost Kähler pair \((J, J')\), we have that \( K_+(P) = K_-(P) \). In fact we have that they are both equal to the orthogonal complement of \( P \) with respect to the metric \( G_3 = g - bg^{-1}b \).

**Proof.** To see that \( K_- \) is an isomorphism, simply note that \(-f(g - b)J_-K_- = 1\) where \( f \) is the inverse to \( g - bg^{-1}b \). Define \( Q_+ = K_+(P) \) and \( Q_- = K_-(P) \). By the above arguments it is clear that \( K_- = J_1 - J_1' \) is an isomorphism from \( P \) to the orthogonal complement of \( P \) with respect to the metric \( G_3 = g - bg^{-1}b \). Therefore we have \( K_+(P) = K_-(P) \). \( \square \)

**Remark 10.8.** As an aside, we mention that for any generalized almost complex structure, \( J \) there is another one \( J' \) such that \((J, J')\) are a generalized almost Kähler structure.

Since we will not be using this and since the proof precisely mimics the proof that every almost symplectic manifold has a compatible almost complex structure we do not include the proof here.
The next requirement that one should want to place on \((\mathcal{J}, \mathcal{J}', \mathcal{P})\) is that the distribution \(Q\) be involute. This is the analogue of considering a flat connection in definition 7.2. We plan to return to this analysis in a future paper.

11. Examples

11.1. Mirror images of \(B\)-field and \(\beta\)-field transforms.

Let \(V\) be a vector bundle on \(M\) with connection \(\nabla\), \(X = \text{tot}(V)\) and \(\mathcal{J} = F^{-1}(\pi^*\mathcal{J})F\) a generalized almost complex structure on \(X\), where \(F\) is defined in equation 4.3. We will need to relate \(B\)-field and \(\beta\)-field transforms of generalized complex structures \(\mathcal{J}\) on \(X\) to transformations of their mirror generalized complex structures \(\hat{\mathcal{J}} = \hat{F}^{-1}(\hat{\pi}^*\hat{\mathcal{J}})\hat{F}\) on \(\hat{X}\).

The transformation \(\mathcal{J} \to \exp(\mathcal{B})\mathcal{J}\exp(-\mathcal{B})\), where

\[
\exp(\mathcal{B}) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
B_{31} & B_{32} & 1 & 0 \\
B_{33} & 0 & 0 & 1
\end{pmatrix}, \quad \exp(\mathcal{B}) \in GL(V \oplus T_M \oplus V^\vee \oplus T_M^\vee) \tag{11.1}
\]
corresponds under mirror symmetry to the transformation \(\hat{\mathcal{J}} \to \exp(\hat{\mathcal{B}})\hat{\mathcal{J}}\exp(-\hat{\mathcal{B}})\), where

\[
\exp(\hat{\mathcal{B}}) = \begin{pmatrix}
1 & B_{32} & B_{31} & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & B_{34} & B_{33} & 1
\end{pmatrix}, \quad \exp(\hat{\mathcal{B}}) \in GL(V^\vee \oplus T_M \oplus V \oplus T_M^\vee) \tag{11.2}
\]

Similarly, the transformation \(\mathcal{J} \to \exp(\mathcal{\beta})\mathcal{J}\exp(-\mathcal{\beta})\), where

\[
\exp(\mathcal{\beta}) = \begin{pmatrix}
1 & 0 & \beta_{21} & \beta_{22} \\
0 & 1 & \beta_{23} & \beta_{24} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \exp(\mathcal{\beta}) \in GL(V \oplus T_M \oplus V^\vee \oplus T_M^\vee) \tag{11.3}
\]
corresponds under mirror symmetry to the transformation \(\hat{\mathcal{J}} \to \exp(\hat{\mathcal{\beta}})\hat{\mathcal{J}}\exp(-\hat{\mathcal{\beta}})\), where

\[
\exp(\hat{\mathcal{\beta}}) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
\beta_{23} & 1 & 0 & \beta_{24} \\
\beta_{21} & 0 & 1 & \beta_{22} \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \exp(\hat{\mathcal{\beta}}) \in GL(V^\vee \oplus T_M \oplus V \oplus T_M^\vee) \tag{11.4}
\]
11.2. \textit{B}-complex structures on $X = \text{tot}(TM)$ and their mirror images. Let us examine a very simple “deformation” of the setup from [31]. It should be clear that there are many variants of this that one could easily do instead. For instance one could vary the complex structure constructed on $TM$ from a fixed choice of connection. Let $M$ be any manifold and $\nabla$ a flat and torsion-free connection on $T_M$ (one may drop the torsion free condition, but then the analysis would become more complicated). Let $X = \text{tot}(T_M)$, and $\hat{X} = \text{tot}(T^\vee_M)$. We will investigate $B$-field transforms of the canonical complex structure on $X$, where $B$ is an arbitrary real two-form. We will give the condition for these transforms to be (integrable) $\nabla$-semi-flat (see definition 7.2) generalized complex structures on $X$ and give their integrable mirror structures on $\hat{X}$.

That is to say, consider a generalized almost complex structure on $X$ of the form. The $B$-field transform of the canonical complex structure is $J = F^{-1} (\pi^* J) F$ where

\[ J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -B_{32} - B_{33} & B_{31} - B_{34} & 0 & 1 \\ B_{31} - B_{34} & B_{32} + B_{33} & -1 & 0 \end{pmatrix}. \tag{11.5} \]

and $B_{31}$ and $B_{34}$ represent arbitrary two forms on $M$, $B_{31} = -B_{31}^\vee$ and $B_{34} = -B_{34}^\vee$. For this to be semi-flat, we need it to be adapted (to the splitting, see 4.2) and hence $B_{32} + B_{33} = 0$. Therefore we consider

\[ J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & B_{31} - B_{34} & 0 & 1 \\ B_{31} - B_{34} & 0 & -1 & 0 \end{pmatrix}. \tag{11.6} \]

Now the analysis in section 7 tells us precisely when the generalized almost complex structure $J$ is integrable. Namely, we must have that all Courant Brackets of sections in the image of the subsheaf $S$ of flat sections of $T_M \oplus T^\vee_M$ under

\[ \mathcal{M} = \begin{pmatrix} -1 & 0 \\ B_{31} - B_{34} & 1 \end{pmatrix}. \tag{11.7} \]
must vanish. This, in turn, is equivalent to the following three Courant Brackets vanishing for any choice of flat sections \(X, Y\) of \(T_M\), and flat sections \(\xi, \eta\) of \(T_M^\vee\).

\[
\begin{align*}
[X + (B_{31} - B_{34})X, -Y + (B_{31} - B_{34})Y] & = 0 \\
[-X + (B_{31} - B_{34})X, \eta] & = 0 \\
[\xi, \eta] & = 0
\end{align*}
\]

Notice that when \(B = 0\), we recover no further conditions as expected. The second and third conditions are clearly vacuous. Set \(B' = B_{31} - B_{34}\). The first condition then reads:

\[
\iota - X d(B'Y) - \iota - Y d(B'X) - \frac{1}{2} d(\iota - Y (B'X) - \iota - X (B'Y)) = 0
\]

or

\[
0 = -t_X d(t_Y B') - t_Y d(t_X B') + (t_Y t_X B')
\]

\[
= -t_X (\mathcal{L}_Y - t_Y d)B' + t_Y (\mathcal{L}_X - t_X d)B' + (\mathcal{L}_Y - t_Y d)t_X B'
\]

\[
= -t_{[X,Y]} B' + 2t_X t_Y dB' + \mathcal{L}_Y t_X B' - t_Y \mathcal{L}_X B' - t_Y t_X dB'
\]

\[
= 3t_X t_Y dB'.
\]

Thus \(\mathcal{J}\) is integrable if and only if \(B' = B_{31} - B_{34}\) is closed. The mirror structure on \(\hat{X}\) is given by \(\hat{\mathcal{J}} = \hat{F}^{-1}(\hat{\pi}^* \hat{\mathcal{J}})\hat{F}\) where

\[
\hat{\mathcal{J}} = \begin{pmatrix}
0 & B_{31} - B_{34} & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & B_{31} - B_{34} & 0
\end{pmatrix}
\]

(11.8)

This is the \(\beta\)-field transform of the canonical symplectic structure on \(\hat{X}\), where

\[
\beta = \left( \begin{array}{c}
\hat{j} \\
\hat{\alpha}
\end{array} \right) \begin{pmatrix}
0 & \hat{\pi}^*(B_{31} - B_{34}) \\
0 & 0
\end{pmatrix}
\]

\[
\left( \begin{array}{c}
\hat{j}^\vee \\
\hat{\alpha}^\vee
\end{array} \right) = \hat{j}^\vee \hat{\pi}^*(B_{31} - B_{34}) \hat{\alpha}^\vee
\]

11.3. \textbf{\(B\)-symplectic structures on \(\hat{X} = \text{tot}(T_M^\vee)\) and their mirror transforms.}

Let \(\nabla^\vee\) be the dual of a flat, torsion-free connection \(\nabla\) on \(T_M\). In this section we will compute the conditions for a \(B\)-field transform of the canonical symplectic structure on \(\hat{X}\) to be a \(\nabla^\vee\)-semi-flat generalized complex structure and find the (integrable) mirror image structure on \(X\).

This \(B\)-symplectic generalized almost complex structure \(\hat{\mathcal{J}} = \hat{F}^{-1}(\hat{\pi}^* \hat{\mathcal{J}})\hat{F}\) on \(\hat{X}\) is given by
\[
\hat{\mathcal{J}} = \begin{pmatrix}
-B_{33} & -B_{34} & 0 & 1 \\
B_{31} & -B_{32} & -1 & 0 \\
B_{32}B_{31} + B_{31}B_{33} & 1 + (B_{32})^2 - B_{31}B_{34} & -B_{33}B_{31} & \\
-1 + B_{34}B_{31} - (B_{33})^2 & B_{34}B_{32} - B_{33}B_{34} & -B_{34}B_{31} & B_{33}
\end{pmatrix}.
\] (11.9)

The adapted requirement (see 4.2) forces \(B_{32} = B_{33} = 0\) and so
\[
\hat{\mathcal{J}} = \begin{pmatrix}
0 & -B_{34} & 0 & 1 \\
B_{31} & 0 & -1 & 0 \\
0 & 1 - B_{31}B_{34} & 0 & B_{31} \\
-1 + B_{34}B_{31} & 0 & -B_{34} & 0
\end{pmatrix}.
\] (11.10)

Now using the above analysis on integrability we know that this the generalized almost complex structure on \(\hat{X}\) will be integrable if and only if all Courant Brackets of sections in the image of the subsheaf \(\mathcal{S}\) of flat sections of \(T_M \oplus T_M^\vee\) under
\[
\mathcal{M} = \begin{pmatrix}
-1 & -B_{31} \\
-B_{34} & 1 - B_{34}B_{31}
\end{pmatrix}
\] (11.11)
must vanish. This, in turn, is equivalent to the following three Courant Brackets vanishing for any choice of \(X, Y\) flat sections of \(T_M\), and \(\xi, \eta\) flat sections of \(T_M^\vee\).

\[
\begin{align*}
[X - B_{34}X, -Y + -B_{34}Y] &= 0 \\
[X - B_{34}X, -B_{31}\eta + \eta - B_{34}B_{31}\eta] &= 0 \\
[-B_{31}\xi + \xi - B_{34}B_{31}\xi, -B_{31}\eta + \eta - B_{34}B_{31}\eta] &= 0
\end{align*}
\]

As in the previous subsection, the first equation is equivalent to \(dB_{34} = 0\). The second equation is equivalent to
\[
[X, B_{31}\eta] = 0
\]
\[
t_X dt_{B_{31}\eta}B_{34} - t_{B_{31}\eta}dt_X B_{34} - \frac{1}{2}d(t_{B_{31}\eta}t_X B_{34} - t_X (B_{34}B_{31}\eta)) = 0
\]
The first of these equations simply says that \(B_{31}\) is a flat bivector field. On the other hand, we claim that if \(dB_{34} = 0\) and \(B_{31}\) is a flat bivector field then the second part of the second equation and also the third equation are also satisfied, and hence all the equations are satisfied. Indeed, we have
\[ d\eta = d\xi = d(t_{B_{31} \xi} \eta) = d(t_{B_{31} \xi} \zeta) = 0 \]

Therefore the third equation gives

\[
\begin{align*}
  t_{B_{31} \xi} dt_{B_{31} \eta} B_{34} - t_{B_{31} \eta} dt_{B_{31} \xi} B_{34} &+ \frac{1}{2} (dt_{B_{31} \xi} t_{B_{31} \eta} B_{34} - dt_{B_{31} \eta} t_{B_{31} \xi} B_{34}) \\
  = t_{B_{31} \xi} L_{B_{31} \eta} B_{34} &- t_{B_{31} \eta} L_{B_{31} \xi} B_{34} + dt_{B_{31} \xi} t_{B_{31} \eta} B_{34} \\
  = t_{B_{31} \xi} L_{B_{31} \eta} B_{34} &- t_{B_{31} \eta} L_{B_{31} \xi} B_{34} - t_{B_{31} \xi} dt_{B_{31} \eta} B_{34} + L_{B_{31} \xi} t_{B_{31} \eta} B_{34} \\
  = t_{[B_{31} \xi, B_{31} \eta]} B_{34} &+ t_{B_{31} \xi} L_{B_{31} \eta} B_{34} - t_{B_{31} \eta} dt_{B_{31} \xi} B_{34} \\
  = t_{B_{31} \xi} L_{B_{31} \eta} B_{34} &- t_{B_{31} \xi} dt_{B_{31} \eta} B_{34} = t_{B_{31} \xi} L_{B_{31} \eta} B_{34} - t_{B_{31} \xi} L_{B_{31} \eta} B_{34} \\
  = 0
\end{align*}
\]

Similarly the second part of the second equation gives

\[
\begin{align*}
  t_X dt_{B_{31} \eta} B_{34} - t_{B_{31} \eta} dt_X B_{34} &- \frac{1}{2} d(t_{B_{31} \eta} t_X B_{34} - t_X t_{B_{31} \eta} B_{34}) \\
  = t_X L_{B_{31} \eta} B_{34} &- t_{B_{31} \eta} L_X B_{34} - L_{B_{31} \eta} t_X B_{34} + t_{B_{31} \eta} dt_X B_{34} \\
  = t_{[X, B_{31} \eta]} B_{34} &- t_{B_{31} \eta} t_X dB_{34} \\
  = 0
\end{align*}
\]

The mirror structure \( \mathcal{J} \) to \( \hat{\mathcal{J}} \) is thus given by

\[
\mathcal{J} = \begin{pmatrix}
0 & 1 - B_{31} B_{34} & 0 & B_{31} \\
-1 & 0 & B_{31} & 0 \\
0 & -B_{34} & 0 & 1 \\
-B_{34} & 0 & -1 + B_{34} B_{31} & 0
\end{pmatrix}.
\] (11.12)

Therefore the mirror \( \mathcal{J} \) of \( \hat{\mathcal{J}} \) is the canonical complex structure transformed by the composition of the \( B \)-field

\[
(d\pi)^* (\pi^* B_{34})(d\pi)
\]
and the $\beta$-field

$$j(\pi^*B_{31})j^\vee.$$ 

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