FOUNDATION OF THE MECHANICS OF ORIENTED POINT

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The mechanics of an oriented point (point with "spin") based on 3D and 4D Frenet equations is considered. In such mechanics there is an opportunity to describe formally any physical trajectory of a particle with own rotation. We use anholonomic rotational coordinates (Euler angles) as elements of internal space of the mechanics which generate a rotational relativity. The groups of transformations of the mechanics of an oriented point form Poincare’s group with semidirect product of translations and rotations, so translational and rotational momentums appear dependent from each other. Connection of the curve torsion with Ricci rotational coefficients is shown and rotational metric is entered. Equivalence between equations of motion 4D oriented point and geodesic equations of absolute parallelism geometry is established. The space of events an arbitrary accelerated 4D frame of reference, which has 10 degrees of freedom, is described by Cartan structure equations of absolute parallelism geometry $A_4(6)$. It represent 10D coordinate space in which 4 translational coordinates $x_0 = ct$, $x_1 = x$, $x_2 = y$, $x_3 = z$ describe motion of the origin $O$ 4D oriented point and 6 angular coordinates $\varphi_1 = \varphi$, $\varphi_2 = \psi$, $\varphi_3 = \theta$, $\varphi_4 = \vartheta_x$, $\varphi_5 = \vartheta_y$, $\varphi_6 = \vartheta_z$ describe change of its orientation. The structural equations of absolute parallelism geometry $A_4(6)$, represent an extended set of Einstein-Yang-Mills equations with the gauge translations group $T_4$, defined on the base $x^i$ and with the gauge rotational group $O(1,3)$, defined in the fibre $e^i_a$. The sources in these equations are defined through the torsion (torsion field) of $A_4(6)$ geometry. The received system of the equations represents generalization vacuum Einstein’s equations on a case when sources have geometrical nature. On the basis of the Vaidya-like solution of the Einstein-Yang-Mills equations correspondence with the Einstein’s equations is established.

Keywords: torsion, spin

I. INTRODUCTION

Einstein proposed the following approach to the construction of the fundamental theory of things in the microworld. On the left-hand side of his famous equations

$$R_{jm} - \frac{1}{2}g_{jm}R = \frac{8\pi G}{c^4} T_{jm} \tag{1}$$

one finds a purely geometrical quantity (the Einstein tensor $G_{jm} = R_{jm} - \frac{1}{2}g_{jm}R$), and on the right-hand side, the energy-momentum tensor $T_{jm}$ for mater, which was, so to speak, introduced "manually." In Einstein’s picture, mater appears thus against the background of a curved space-time as an entity independent of space-time.

Einstein was not satisfied with the phenomenological representation of $T_{jm}$, since: "The right-hand side includes all that cannot be described so far in the unified field theory. Of course, not for a fleeting moment I have had any doubt that such a formulation is just a temporary answer, undertaken to give to general relativity some closed expression. This formulation has been in essence nothing more that the theory of the gravitation field, which has been separated in a somewhat artificial manner from the unified field of a yet unknown nature."

A way to remove arbitrariness in the selection of the energy-momentum tensor was seen by Einstein in the geometrization of the energy-momentum tensor of matter on the right-hand side of Einstein’s equations (1). Einstein believed that the geometrization of the energy-momentum tensor of matter should result in the geometrization of the matter field that make it up. For Einstein the geometrization of matter fields implied the construction of a fundamental theory of phenomena in the microworld that is in conformity with relativity principle. 30 years he tried constructing a "reasonable general relativistic theory," and within its framework a "more advanced quantum theory." In the present work we offer to geometrize the right part of the equations (1), using instead of a material point of general relativistic Einstein’s mechanics more the general object - an oriented material point. We shall understand any accelerated reference frame, formed by unit orthogonal vectors as an oriented material point. In 3D translational coordinates space the oriented point has 6 degrees of freedom, in 4D translational coordinate space - ten. Generalization of the Einstein’s theory offered by us allows analytically to describe Descartes’s approved idea, that any real motion is a rotation.

II. 3D ORIENTED POINT AND GENERALIZATION OF THE EQUATIONS OF NEWTON'S MECHANIC

In 1847 French mathematician Jean F. Frenet in his thesis has written equations, describing motion of oriented point in the 3D space along an arbitrary curve $x = x(s)$, where $s$ – the arc length. Equations are written for
the three orthogonal unit vectors \( \mathbf{t}, \mathbf{n} \) and \( \mathbf{b} \) with orthogonality conditions \( \mathbf{t}^2 = \mathbf{n}^2 = \mathbf{b}^2 = 1, \mathbf{tn} = \mathbf{nb} = \mathbf{bt} = 0 \).

The tangent unit vector \( \mathbf{t} \) is choused as tangent to the curve at point \( M \) (fig.1), pointing the direction of motion. The normal unit vector \( \mathbf{n} \) is the derivative of \( \mathbf{t} \) with respect to the arc length parameter of the curve, divided by its length and the binormal unit vector \( \mathbf{b} \) is defined as the cross product \( \mathbf{b} = \mathbf{t} \times \mathbf{n} \).

For these vectors Frenet’s equations look like\(^3\)

\[
\frac{dx}{ds} = \mathbf{t},
\]

\[
\frac{dt}{ds} = \kappa(s)\mathbf{n},
\]

\[
\frac{dn}{ds} = -\kappa(s)\mathbf{t} + \chi(s)\mathbf{b},
\]

\[
\frac{db}{ds} = -\chi(s)\mathbf{n},
\]

where \( \kappa(s) \) - curvature of the curve and \( \chi(s) \) - torsion of the curve. Frenet was the first who has shown that arbitrary curve in 3D flat space is determined by two scalar parameters - curvature \( \kappa(s) \) and torsion \( \chi(s) \).

For comparison of the equations (6) (7) with the equations of Newton’s mechanics, it is convenient to pass in them to time parameter \( t \)

\[
\frac{d^2x}{dt^2} = \mathbf{at} + \kappa v^2 \mathbf{n},
\]

\[
\frac{d^3x}{dt^3} = (\frac{d}{dt} - \kappa^2v^3)\mathbf{t} + (3\alpha \kappa v + \kappa^2 v^2 \frac{dv}{dt})\mathbf{n} + \kappa \chi v^3 \mathbf{b},
\]

where \( v = ds/dt \) - absolute velocity and \( a = dv/dt \) - tangent acceleration.

Multiplying these equations on mass \( m \), we shall receive the translational equations of motion of an oriented point with the law of transformation infinitesimal vector \( dx_\alpha \)

\[
dx_\alpha = \frac{\partial x_\alpha}{\partial x_\alpha} dx_\alpha, \quad \alpha = 1, 2, 3,
\]

where matrixes \( \frac{\partial x_\alpha}{\partial x_\alpha} \) form the group 3D translations \( \mathbb{t}(3) \). The equations (8) are similar to the equations of Newton mechanics, but have geometrical nature. A choice of curvature \( \kappa \) and parameter \( s \) it is possible to describe formally any physical trajectory of a particle in 3D space, moving under action of force \( \mathbf{F} = m(\mathbf{at} + \kappa v^2 \mathbf{n}) \). The equations (9) have no analogues in the Newton mechanics as contain the third derivative of coordinate on time. The system of the equations (8) (9) describes the motion of the origin of an oriented material point taking into account of its spin and, certainly, generalizes the equations of Newton’s mechanics.

III. INTERNAL SPACE OF THE ANHOLONOMIC ROTATIONAL COORDINATES

During infinitesimal displacement of point \( M \) along the curve the triad of Frenet’s vectors simultaneously change their orientation in space. For description of the change it is convenient to introduce anholonomic angular coordinates

\[
\varphi = \angle(\mathbf{e}_1, \mathbf{e}_\xi), \quad \psi = \angle(\mathbf{e}_\xi, \mathbf{e}_\nu), \quad \theta = \angle(\mathbf{e}_3, \mathbf{e}_y),
\]

\[0 \leq \varphi \leq 2\pi, \quad 0 \leq \psi \leq 2\pi, \quad 0 \leq \theta \leq \pi,\]

– Euler angles (see (fig.2a)).

Let’s assume, that with a point \( M \) of a curve the triad with motionless unit vectors \( \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \) is connected. Let’s designate components of motionless Frenet triad as

\[
\mathbf{t} = \mathbf{e}_1', \quad \mathbf{n} = \mathbf{e}_2', \quad \mathbf{b} = \mathbf{e}_3'.
\]

At displacement of the origin \( O \) of Frenet triad along a curve from a point \( M \) in a point \( M' \), there is a rotation vectors of Frenet triad (fig. 2 a). Projecting the axes of a mobile triad \( \mathbf{t}, \mathbf{n}, \mathbf{b} \), located in a point \( M' \) on the motionless triad connected with a point \( M \), we find

![FIG. 1. Trajectory of the 3D oriented point](https://example.com/figure1.png)
FIG. 2. a) Changing of the orientation of an oriented point at displacement of point M on dM; b) according to Euler's theorem an infinitesimal rotations around the three axes it is possible to replace by one rotation with the an infinitesimal angle

\[
\begin{align*}
t &= e'_1 = e_1 (\cos \varphi \cos \psi - \sin \varphi \sin \psi \cos \theta) + e_2 (\sin \varphi \cos \psi + \cos \varphi \sin \psi \cos \theta) + e_3 \sin \psi \sin \theta, \\
n &= e'_2 = -e_1 (\cos \varphi \sin \psi + \sin \varphi \cos \psi \cos \theta) - e_2 (\sin \varphi \sin \psi - \cos \varphi \cos \psi \cos \theta) + e_3 \cos \psi \sin \theta, \\
b &= e'_3 = e_1 \sin \varphi \sin \theta - e_2 \cos \varphi \sin \theta + e_3 \cos \theta.
\end{align*}
\]

Expressing the components of tangent vector \(t = e'_1\) = \(dx/ds\) through angular variables, we have

\[
\begin{align*}
\frac{dx}{ds} &= \cos \varphi \cos \psi - \sin \varphi \sin \psi \cos \theta, \quad (10) \\
\frac{dy}{ds} &= \sin \varphi \cos \psi + \cos \varphi \sin \psi \cos \theta, \quad (11) \\
\frac{dz}{ds} &= \sin \psi \sin \theta. \quad (12)
\end{align*}
\]

Differentiating the third components of vectors \(t\) and \(n\) and second component of the vector \(b\), we get "rotational equations of motion" as follows

\[
\begin{align*}
\frac{d\varphi}{ds} &= \chi \sin \psi \sin \theta, \quad (13)
\end{align*}
\]

According to the equations (13)-(15) curvature \(\chi\) and torsion \(\chi\) cause rotation of Frenet triad, therefore more correctly to name their first \(\chi_1 = \kappa\) and second \(\chi_2 = \chi\) torsion of a curve.

The system of the equations (10)-(15) represents system Cauchy for six unknown functions \(x, y, z, \varphi, \psi, \theta\) and supposes one and only one solution in the form of regular functions

\[
\begin{align*}
x &= x(s), \quad y = y(s), \quad z = z(s), \quad \varphi = \varphi(s), \quad \psi = \psi(s), \quad \theta = \theta(s)
\end{align*}
\]

satisfying to the equations (10)-(15) and entry conditions

\[
\begin{align*}
x &= x_0, \quad y = y_0, \quad z = z_0, \quad \varphi = \varphi_0, \quad \psi = \psi_0, \quad \theta = \theta_0
\end{align*}
\]
for \( s = s_0 \). Entry conditions have simple geometrical sense. Initial coordinates \( x = x_0, y = y_0, z = z_0 \) define position of an original point \( M_0 \) a curve, and Euler’s angles \( \varphi = \varphi_0, \psi = \psi_0, \theta = \theta_0 \) - initial orientation of the attached triad. Three Euler’s angles form in each point \( M \) of a curve internal space anholonomic rotational coordinates, which, as it follows from the equations (10)-(15), define the physical dynamics of an oriented material point. Passing to the time parameter \( t \) in the equations (10)-(12), we will get

\[
\begin{align*}
  v_\alpha(\varphi, \psi, \theta) &= \frac{dx}{dt} = v(\cos \varphi \cos \psi - \sin \varphi \sin \psi \cos \theta), \\
  v_\psi(\varphi, \psi, \theta) &= \frac{dy}{dt} = v(\sin \varphi \cos \psi + \varphi \sin \psi \cos \theta), \\
  v_\varphi(\varphi, \psi, \theta) &= \frac{dz}{dt} = v(\sin \psi \sin \theta), 
\end{align*}
\]

where \( v = ds/dt \) – absolute velocity. Dependence of the components of linear velocity from Euler’s angels in these equations allows us to approve, that the system of the equations (10)-(15) gives the analytical description of Descartes idea that any physical motion is rotation. Let’s write down a Frenet triad as

\[
e^A_\alpha, \quad \alpha = 1, 2, 3,
\]

where holonomic index \( \alpha \) accepts values 1, 2, 3, and index \( A \) – local anholonomic index accepts values 1, 2, 3, designating numbers of a triad vectors.

It will be transformed in group of local three-dimensional rotations. If on the holonomic coordinate index \( \alpha \) the triad \( e^A_\alpha \) has the tensor law of transformation in group of translations \( T(3) \)

\[
e^A_\alpha = \frac{\partial x^A_{\alpha'}}{\partial x^{\alpha'}}, \quad \alpha = 1, 2, 3, \quad \frac{\partial x^A_{\alpha'}}{\partial x^{\alpha'}} \in T(3), \tag{19}
\]

that on the anholonomic local index \( A \) triad \( e^A_\alpha \) it will be transformed in group of local three-dimensional rotations \( O(3) \)

\[
e^{A'}_\alpha = \Lambda^{A'}_{\alpha} e^A_\alpha, \quad A = 1, 2, 3, \quad \Lambda^{A'}_{\alpha} \in O(3). \tag{20}
\]

At the description of motion of the Frenet triad the group \( T(3) \) and \( O(3) T \) form Poincare’s group with semidirect product of translations and rotations, as rotation of vectors of a triad causes translation of its origin – \( M \) points of \( M \) and vice versa. This fact substantially distinguishes mechanics of an oriented point from the Newtonian mechanics, as in the new mechanics translational and rotational energies are not additive.

IV. CONNECTION OF \( \chi_1(s) \) AND \( \chi_2(s) \) WITH RICCI ROTATIONAL COEFFICIENTS AND ROTATIONAL METRIC

Statement 1. Torsion \( \chi_1 \) and \( \chi_2 \) are independent components of Ricci rotation coefficients.

Proof. Let’s consider six-dimensional manifold of coordinates \( x_1, x_2, x_3, \varphi_1, \varphi_2, \varphi_3 \). It is convenient to present it as a vector bundle with the base formed by translational coordinates \( x_1, x_2, x_3 \) (let it be Cartesian coordinates) and fibre, specified at each point \( x_\alpha (\alpha = 1, 2, 3) \) by three orthonormalized Frenet’s reference vectors

\[
e_A, \quad A = 1, 2, 3, \tag{21}
\]

where \( A \) means number of the reference vector.

According to Euler’s theorem, an infinitesimal rotations around the three axes of reference vector (21) is equivalent to one rotation with angle \( d\chi \) around a definite axis passing through the origin of the axis \( O \) (see (fig.2b)). It is possible to define the infinitesimal rotation as

\[
d\chi = d\chi e_\alpha, \tag{22}
\]

where vector \( e_\alpha \) is directed along instantaneous rotation axis of reference system. This direction is selected so that, if one looks from the end of the vector \( e_\alpha \) at a fixed point \( O \), then the rotation is made counterclockwise (right-hand reference system).

Let’s note, that the vector \( \chi \) does not exist, as turn on a finite angle is not commutative. Therefore for an infinitesimal rotation we have entered a designation \( d\chi \) instead of \( d\chi \). Unlike a polar vector holonomic translational coordinates \( d\chi \), infinitesimal rotation

\[
d\chi = e_\alpha d\varphi + e_\beta d\theta + e_\gamma d\psi \tag{22}
\]

is an axial vector.

An infinitesimal rotation of Frenet’s reference vectors \( e_\alpha \) upon rotation \( d\chi \) has the form

\[
de_A = [d\chi e_A], \tag{23}
\]

If we divide (23) by \( ds \), then we shall get

\[
\frac{de_A}{ds} = \frac{d\chi}{ds} e_A = [\omega, e_A], \tag{24}
\]

where \( \omega = d\chi/ds \) - three-dimensional angular velocity of Frenet’s triad with respect to the instantaneous axis.

Writing down the orthogonality conditions for Frenet’s reference vectors in the form

\[
a) \quad e_\alpha e_\beta = \delta^A_B = \begin{cases} 1 & A = B, \\ 0 & A \neq B, \end{cases} \tag{25}
\]

\[
b) \quad e_\alpha e_\beta = \delta^A_B = \begin{cases} 1 & \alpha = \beta, \\ 0 & \alpha \neq \beta, \end{cases} \tag{25}
\]

\[
A, B = 1, 2, 3, \quad \alpha, \delta, \beta = 1, 2, 3,
\]

where \( \alpha, \delta, \beta \) – holonomic coordinate indices, and \( A, B \) – anholonomic triad local indices; it is possible to write down relations (23) and (24) as follows
Let’s rewrite equations (30) in local indices. Using the orthogonality conditions (25) and the coefficients. Since then they have been called Ricci rotation coefficients. While deducing (3-5) from (27), we used the following relations

\[ T^\alpha_{\beta\gamma} = e^A_\alpha T^A_{\beta\gamma} e^\beta_B, \]

let’s rewrite equations (30) in local indices

\[ \frac{de^A_\alpha}{ds} = T^\beta_{\alpha\gamma} e^\beta_B. \]  

Multiplying (26) and (27) by \( e^\beta_A \), we get

\[ d\chi_\alpha = T^\beta_{\alpha\gamma} dx^{\gamma}, \]

\[ \frac{de^A_\alpha}{ds} = T^\beta_{\alpha\gamma} \frac{dx^{\gamma}}{ds} e^A_\beta, \]

where we have defined the designation

\[ T^p_{\beta\gamma} = e^A_{\alpha p} T^A_{\beta\gamma} e^\beta_B, \]

\[ \frac{de^A_\alpha}{ds} = T^A_{B\gamma} \frac{dx^{\gamma}}{ds} e^B_\alpha. \]  

The quantities (30) were first introduced by G. Ricci and since then they have been called Ricci rotation coefficients. Using the orthogonality conditions (25) and the rule of transformation to local indices

\[ T^A_{B\gamma} = e^A_{\alpha} T^\alpha_{\beta\gamma} e^{\beta}_B, \]

let’s rewrite equations (30) in local indices

\[ \frac{de^A_\alpha}{ds} = T^A_{B\gamma} \frac{dx^{\gamma}}{ds} e^B_\alpha. \]  

Let’s chose vectors \( e^{(1)} \), \( e^{(2)} \) and \( e^{(3)} \) so, that they coincide with Frenet’s vectors, and thus the vector \( e^{(1)}_\alpha = dx/\tau \). Then the equations (27) become the well-known Frenet’s equations (3-5), in which

\[ \kappa = \chi_1(s) = T_{(1)}(2) \frac{dx^{\gamma}}{ds}, \quad \chi = \chi_2(s) = T_{(2)}(2) \frac{dx^{\gamma}}{ds}. \]

While deducing (3-5) from (27), we used the following relations

\[ \frac{dx^{\gamma}}{ds} = e^{\gamma}_{(1)}(1), \quad e^{\gamma}_{(1)} e^{\gamma}(1) = 1. \]

From the relations (30) it is clear, that in Frenet’s equations curvature and torsion are expressed through components of Ricci rotation coefficients (26), that proves the Statement 1.

The Ricci rotation coefficients are the part of the connection of absolute parallelism geometry\(^3\) and have an anti-symmetry on the two lower indices

\[ T^\alpha_{\beta\gamma} = -\Omega^\alpha_{\beta\gamma}, \]

\[ \Omega^\alpha_{\beta\gamma} = -\frac{1}{2} e^A_\gamma(e^A_{\beta\gamma} - e^A_{\gamma\beta}), \]

which it is possible to call Ricci torsion. Let’s note, ones more, that the curvature \( \kappa \) and torsion \( \chi \) of Frenet’s curve would be more correctly called the first and second torsion, as they are both expressed through components of Ricci torsion (34).

From (33) we can find \( ds = e^{(1)}_\alpha dx^{\alpha} \) and

\[ ds^2 = e^{(1)}_\alpha dx^{\alpha} e^{(1)}_\alpha dx^{\alpha} = dx^{\alpha} dx^{\alpha} = dx^2 + dy^2 + dz^2. \]

This translational metrics is set on group \( T(3) \) of translational coordinates and defines geometry of the 3D euclidean space, in which the curve is embedded. Besides as follows from (22) and (31), on the group of rotational coordinates \( O(3) \) the rotational metrics is define.

\[ d\chi^2 = d\phi^2 + d\psi^2 + d\theta^2 = d\chi^{\alpha} d\chi^\alpha = T^p_{\beta\gamma} T^q_{\alpha\delta} dx^{\gamma} dx^{\delta} = d\tau^2, \]

This metrics addresses in zero if the first and second torsions (32 address to zero.

V. 4D ORIENTED POINT AND ABSOLUTE PARALLELISM GEOMETRY

A 3D oriented material point is mathematical representation of an arbitrary accelerated three-dimensional system of reference. Motion of such system of reference is described by six equations as it has six degrees of freedom. It would be possible to put and solve the problem on what geometry possess the space of events of an arbitrary accelerated 3D systems of reference. However, we consider as more important question — what geometry possess space of events of an arbitrary accelerated 4D systems of reference or, that is the same, what the space of events form the relative coordinates of the 4D oriented material points? It is in advance possible to tell, and it is obvious, that 4D an arbitrary accelerated system of reference has 10 degrees of freedom, therefore, for the description of its motion, it is necessary to use 10 coordinates. Leaning on experience, which we have received at the description of the dynamics of an arbitrary accelerated 3D system, we shall consider 10D coordinate space in which 4 translational coordinates \( x_0 = c t, x_1 = x, x_2 = y, x_3 = z \) describe motion of the origin \( O \) 4D oriented point and 6 angular coordinates \( \phi_1 = \varphi, \phi_2 = \psi, \phi_3 = \theta, \phi_4 = \varphi_5 = \varphi_6 = \partial_\varphi \) describe change of its orientation.

Consider a four-dimensional differentiable manifold of 4D oriented points with translational coordinates \( x_i \) \( (i = 0, 1, 2, 3 \quad a = 0, 1, 2, 3) \). What each point of the manifold we connect four vectors \( e^a_i \) \( (i = 0, 1, 2, 3) \) and four covectors \( e^b_j \) with the orthogonality conditions

\[ e^a_i e^a_j = \delta^a_i, \quad e^a_i e^b_j = \delta^b_j. \]

Anholonomic tetrad \( e^a_i \) defines the metric tensor of the space

\[ g_{ik} = \eta_{ab} e^a_i e^b_k, \quad \eta_{ab} = \eta^{ab} = \text{diag}(1 - 1 - 1 - 1). \]
and the translational Riemannian metric

\[ ds^2 = \eta_{ab} e^a_i e^b_j dx^i dx^j. \]  

(38)

Using the tensor (40), we can construct the Christoffel symbols

\[ \Gamma^j_{ik} = \frac{1}{2} g^{jm} (g_{jm,k} + g_{km,j} - g_{jk,m}). \]  

(39)

that transform following a nontensor law of transformation

\[ \Gamma'^j_{i'k'} = \frac{\partial^2 x^k}{\partial x^{i'} \partial x^{j'}} + \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} \Gamma^k_{ij}. \]  

(40)

with respect to the coordinate transformations

\[ dx^i' = \frac{\partial x^i}{\partial x^{i'}} dx^i, \quad \|\partial x^i/\partial x^{i'}\| \in T(4). \]

were \( T(4) \) – group of 4D translations. Now the Ricci rotation coefficients (30) can be represented in the form

\[ T^i_{jk} = e^i_a \nabla_k e^a_j, \quad T^i_{jk} = -e^i_j \nabla_k e^a_j, \quad T_{ijk} = -T_{jik}, \]  

(41)

where \( \nabla_k \) is a covariant derivative with respect to the Christoffel \( \Gamma^i_{jk} \) symbols. The rotational metric in the new space can be written as

\[ d\tau^2 = d\chi^a dx^b = T^a_{\ i} T^b_{\ am} dx^i dx^m, \]

\[ i, j, k... = 0, 1, 2, 3, \quad a, b, c... = 0, 1, 2, 3. \]

Let us have an arbitrary curve in four-dimensional Riemannian space with translational coordinates \( x^i, (i = 0, 1, 2, 3) \). Then the curve is defined by three scalar invariants \( \chi_1, \chi_2 \chi_3 \), and in our case the four-dimensional Frenet’s equations have

\[ \frac{De^0_k}{ds} = \chi_1 e^1_k, \]

\[ \frac{De^1_k}{ds} = \chi_1 e^0_k + \chi_2 e^2_k, \]

\[ \frac{De^2_k}{ds} = -\chi_2 e^1_k + \chi_3 e^3_k, \]

\[ \frac{De^3_k}{ds} = -\chi_3 e^2_k. \]

Here vectors \( e^0_k, e^1_k, e^2_k \) and \( e^3 \) form a tetrad, and \( D \) is the absolute differential with respect to the four-dimensional Christoffel symbols (40).

**Statement 2.** Any curve of Riemannian space can be considered as the geodesics of space of absolute parallelism, with equations of the form

\[ \frac{d^2 x^i}{ds^2} = -\Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} - T^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds}. \]  

(47)

**Proof.** Connection of absolute parallelism is defined as

\[ \Delta^i_{jk} = \Gamma^i_{jk} + T^j_{ik} = e^i_a e^a_j = -e^a_i e^a_k. \]  

(48)

These relations can be rewritten as follows

\[ T^i_{jk} = e^i_a \nabla_k e^a_j = -e^a_j \nabla_k e^a_j, \]

(49)

where \( \nabla_k \) - covariant derivative with respect to Christoffel symbols. Multiplying equality (49) on \( e^a_i (e^a_j) \) and using the orthogonality conditions (36) let’s present (49) as follows

a) \( \nabla_k e^a_j = 0 \) or b) \( \nabla_k e^a = -T^i_{jk} e^a_i. \)

(50)

Multiplying (41a) and (41b) on \( dx^k/ds \), we shall obtain

\[ \frac{De^a_k}{ds} = T^a_{bk} e^b_j \frac{dx^k}{ds}. \]

(51)

Uncovering in equations (52) the absolute differential and supposing in them \( e^i(0) = dx^i/ds \), we shall obtain geodesics equations (47).

Changing in equations (51) indices on which there is a contraction, we find

\[ \frac{De^a_k}{ds} = T^a_{bj} e^b_j \frac{dx^i}{ds}. \]

Choosing these equations the Frenet’s tetrad and writing down them component by component, we have

\[ \frac{De^0_k}{ds} = T^0_{(1)j} e^1_k \frac{dx^j}{ds}, \]

\[ \frac{De^1_k}{ds} = T^1_{(0)j} e^0_k \frac{dx^j}{ds} + T^1_{(2)j} e^2_k \frac{dx^j}{ds}, \]

\[ \frac{De^2_k}{ds} = T^2_{(1)j} e^1_k \frac{dx^j}{ds} + T^2_{(3)j} e^3_k \frac{dx^j}{ds}, \]

\[ \frac{De^3_k}{ds} = T^3_{(2)j} e^2_k \frac{dx^j}{ds}. \]

Comparing equations (62)-(65) with equations (74)-(77), we shall obtain

\[ \chi_1 = T^0_{(1)j} \frac{dx^j}{ds}, \quad \chi_2 = T^1_{(2)j} \frac{dx^j}{ds}, \quad \chi_3 = T^2_{(3)j} \frac{dx^j}{ds}. \]

Since the quantities \( T^i_{jk} \) are defined through Ricci torsion (see (49)), then, as it follows from relations obtained
above, is possible to geometrize any curves of Riemannian space, using Ricci torsion.

The common symmetries os space of events of 4D oriented point are determined as:

a) by transformation of the four holonomic translation coordinates $x_i$, describing the motion of the origin of an arbitrary accelerated 4D frame

$$
e^a_i = \frac{\partial x'_i}{\partial x^a} e^a_i, \quad i = 0, 1, 2, 3, \quad \| \frac{\partial x'_i}{\partial x^a} \| \in T(4), \quad (57)$$

where $T(4)$ is a local group of 4D translations;

b) by transformation of the six anholonomic rotational coordinates $\chi_{ab} = -\chi_{ba}$, describing rotation of 4D oriented point (or an arbitrary accelerated 4D frame)

$$e^a_i = \Lambda^a_a e^a_i, \quad a = 0, 1, 2, 3, \quad \Lambda^a_a \in O(1,3), \quad (58)$$

where $O(1,3)$ is a local Lorenz group of 4D rotations. Term "local group" means, that the parameters of the group depends on the point of the curve.

The matrix $\Lambda^a_a$ can be represented as

$$\Lambda^a_a = R^a_b L^b_a,$$

where

$$R^a_b = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \varphi_{xx} & \cos \varphi_{xy} & \cos \varphi_{xz} \\ 0 & \cos \varphi_{yx} & \cos \varphi_{yy} & \cos \varphi_{yz} \\ 0 & \cos \varphi_{zx} & \cos \varphi_{zy} & \cos \varphi_{zz} \end{pmatrix}, \quad (59)$$

is the matrix of the spatial rotations and

$$L^b_a = \begin{pmatrix} \gamma & -\beta_x \gamma & -\beta_y \gamma & -\beta_z \gamma \\ -\beta_x \gamma & 1 + \frac{(\gamma - 1) \beta_x^2}{\beta^2} & \frac{(\gamma - 1) \beta_x \beta_y}{\beta^2} & \frac{(\gamma - 1) \beta_x \beta_z}{\beta^2} \\ -\beta_y \gamma & \frac{(\gamma - 1) \beta_y \beta_x}{\beta^2} & 1 + \frac{(\gamma - 1) \beta_y^2}{\beta^2} & \frac{(\gamma - 1) \beta_y \beta_z}{\beta^2} \\ -\beta_z \gamma & \frac{(\gamma - 1) \beta_z \beta_x}{\beta^2} & \frac{(\gamma - 1) \beta_z \beta_y}{\beta^2} & 1 + \frac{(\gamma - 1) \beta_z^2}{\beta^2} \end{pmatrix}, \quad (60)$$

– is the matrix, which describes rotation in space-time planes. Here

$$\gamma = \sqrt{1 - \beta^2}, \quad \beta^2 = \beta_x^2 + \beta_y^2 + \beta_z^2$$

– is relativistic factor, in which 3D velocity $v_\alpha = dx_\alpha / dt$ of 4D frame connects whit $\beta_\alpha$ and space-time angle $\vartheta_\alpha$ as

$$\frac{v_\alpha}{c} = \beta_\alpha = \frac{\partial \vartheta_\alpha}{\partial x_\alpha}, \quad (61)$$

were $c$ - velocity of light.

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### VI. GENERALIZATION OF THE EINSTEIN’S MECHANICS

Einstein’s General Relativity assumes the description of laws of physics in an arbitrary accelerated 4D frames.

As we have shown, an arbitrary accelerated 4D frame has 10 digress of freedom and describes by 10 equations of motions: four equations of motions of the origin of 4D frame (47) and six rotational equations of motion (52).

Einstein used only four equations

$$\frac{d^2 x^i}{ds^2} = -\Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds}.$$ \quad (62)

---

### A. Generalization of the equations of motion of accelerated 4D frame

The equations of motion of the origin of 4D oriented point (or an arbitrary accelerated 4D frame) coincide with the equations of the geodesics of the space of absolute parallelism

$$\frac{d^2 x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} + T^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0, \quad (63)$$

which differ from the equations of motion in Einstein’s theory of gravitation (62) by the additional term

$$T^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds}.$$  

The name of the quantities

$$T^i_{jk} = \epsilon_{ak} \nabla \epsilon_{ij}$$

– the Ricci rotation coefficients suggests that they describe rotation. It follows from our analysis, that the quantities $T^i_{jk}$ describe the change in the orientation of the tetrad vectors $e^a_i$ when the origin of tetrad shifts by an infinitesimal distance $dx^i$.

Einstein interpreted symbols $\Gamma^i_{jk}$ in his equations (62) as intensity of a gravitational field. The object $\Gamma^i_{jk}$ get transformed relative to the transformations in $T(4)$ group as nontensor, whist respect formula (40). So, using normal coordinates, we can make $\Gamma^i_{jk}$ equal to zero. The Ricci rotational coefficients under transformation of translation coordinates in $T(4)$ transform as tensor

$$T^k_{ji} = \frac{\partial x^i}{\partial x^j} \frac{\partial x^j}{\partial x^k} T^k_{ji} \quad (64)$$

Writing down the equations (63) in normal coordinates, we have

$$\frac{d^2 x^i}{ds^2} + T^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0, \quad (65)$$

Using the Ricci rotation coefficients we can form the 4D angular velocity of the tetrad vector

$$\Omega^i_{jk} = T^i_{jk} \frac{dx^k}{ds} \quad (66)$$
with the symmetry properties
\[
\Omega_{ij} = -\Omega_{ji}. \quad (67)
\]

Suppose now that the tetrad vectors coincide with the vectors of a 4D arbitrarily accelerated reference frame, then, by (66), the rotation of the reference frame is fully determined by the torsion field \( T^i_{jk} \). Since the field \( T^i_{jk} \) transforms following a tensor law relative to the coordinates transformations \( x_i \), the rotation of reference frames relative to the coordinate transformations is absolute. The nontensor transformation law of \( T^i_{jk} \) is valid for transformations in the angular coordinates \( \varphi_1, \varphi_2, \varphi_3, \vartheta_1, \vartheta_2, \vartheta_3 \), therefore rotation is only relative for the group of rotations \( O(1.3)^3 \).

Let us now write the nonrelativistic equations of motion of a mass \( m \) under inertia forces alone, assuming that at a given moment of time it passes through the origin of an accelerated system

\[
\frac{d}{dt}(mv) = m(-W + 2[v\omega]), \quad (68)
\]

where \(-mW\) – force of inertia, arising at forward acceleration and \(2m[v\omega]\) – Coriolis force of inertia.

These equations can be written in the form

\[
\frac{d}{dt}(mv_\alpha) = m(-W_\alpha + 2\omega_\alpha^\beta \frac{dx^\beta}{dt}), \quad \alpha, \beta = 1, 2, 3, \quad (69)
\]

where \( W = (W_1, W_2, W_3) = (W_{10}, W_{20}, W_{30}), \omega = (\omega_1, \omega_2, \omega_3) \),

\[
\begin{align*}
\Omega_{ij} & = \frac{1}{c^2} \begin{pmatrix}
O & -W_1 & -W_2 \\
W_1 & 0 & -\omega_3 \\
W_2 & \omega_3 & 0 \\
W_3 & -\omega_2 & \omega_1
\end{pmatrix}, \\
\omega_{\alpha\beta} & = -\omega_{\beta\alpha} = -\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\
\omega_3 & 0 & -\omega_1 \\
-\omega_2 & \omega_1 & 0
\end{pmatrix}, \quad (70)
\end{align*}
\]

On the other hand, equations (65), if we take into account (66), can be represented as

\[
\frac{d^2x^i}{ds^2} + \Omega^i_j \frac{dx^j}{ds} = 0. \quad (71)
\]

Multiplying these equations by mass \( m \), we will write the nonrelativistic three-dimensional part of these equations in the form

\[
m \frac{dv_\alpha}{ds_0} = -m\Omega_{\alpha0} \frac{dx^0}{ds_0} - 2m\omega_\alpha^\beta \frac{dx^\beta}{ds_0}. \quad (72)
\]

Since in a nonrelativistic approximation

\[
ds_0 = cdt, u_\alpha = \frac{v_\alpha}{c}
\]

and \( dx_0 = cdt \), then the equations (72) become

\[
m \frac{dv_\alpha}{dt} = -m\omega_\alpha^\beta \frac{dx^\beta}{dt}. \quad (73)
\]

Comparing (73) with (69) gives

\[
\begin{align*}
\Omega_{10} & = \frac{W_1}{c^2}, \quad \Omega_{20} = \frac{W_2}{c^2}, \\
\Omega_{30} & = \frac{W_3}{c^2}, \quad \Omega_{12} = -\frac{\omega_3}{c}, \\
\Omega_{13} & = \frac{\omega_2}{c}, \quad \Omega_{23} = -\frac{\omega_1}{c}.
\end{align*}
\]

Consequently, the matrix of the 4D angular velocity of rotation of an arbitrarily accelerated reference frame (matrix of the 4D "classical spin") has the form

\[
\begin{pmatrix}
O & -\Theta_1 & -\Theta_2 \\
\Theta_1 & 0 & -\omega_3 \\
\Theta_2 & \omega_3 & 0 \\
\Theta_3 & -\omega_2 & \omega_1
\end{pmatrix} = \frac{1}{c} \begin{pmatrix}
O & -\Theta_1 & -\Theta_2 \\
\Theta_1 & 0 & -\omega_3 \\
\Theta_2 & \omega_3 & 0 \\
\Theta_3 & -\omega_2 & \omega_1
\end{pmatrix}, \quad (74)
\]

by the inertial fields \( T^i_{jk} \) is associated with the torsion

\[
\Delta^{i}[jk] = T^i_{j[k]} = -\Omega_{jk}^i = -\epsilon_{\alpha}^{\epsilon_{[i \alpha}, c_{\epsilon_{k,j], \alpha}}} = \frac{-1}{2} \epsilon_{\alpha}^{\epsilon_{[i \alpha}, c_{\epsilon_{k,j], \alpha}}} \quad (75)
\]

of a space of absolute parallelism, since

\[
T^i_{jk} = -\Omega_{jk}^i + g^{im}(g_{jm}^{\cdot k} \Omega_{mk}^i + g_{km}^{\cdot j} \Omega_{mj}^i). \quad (76)
\]

Fields determined by the rotation of space came to be known as torsion fields. Accordingly, the torsion field \( T^i_{jk} \) represents the inertial field engendered by the torsion of a space of absolute parallelism\(^6\). A. Einstein used
the torsion of absolute parallelism determining torsion as
\[ \Lambda^i_{jk} = \frac{1}{2} (\Delta^i_{jk} - \Delta^i_{kj}), \]
where
\[ \Delta^i_{jk} = e^i_{a} e^a_{j,k}, \quad k = \frac{\partial}{\partial x_k}, \quad i,j,k... = 0,1,2,3, \]
-connection of absolute parallelism,
\[ \Lambda^i_{jk} = -\Omega^i_{jk} \]
anholonomity object in J. Schouten definition. In the same article A. Einstein has specified, that when torsion \( \Lambda^i_{jk} \) (anholonomity object) is equal to zero the space becomes Minkovski space.

**B. Generalization of the Einstein’ vacuum equations**

An empty, but curved space in Einstein’s theory obeys the equations
\[ R_{ik} = 0, \quad (77) \]
whose Schwarzschild’s solution is supported by experiment (the shift of Mercury’s perihelion, the deviation of a light ray in the Solar gravitational field, the delay of radio signals in a gravitational field, etc.).

Note that Einstein’s vacuum equations do not contain any physical constants. They are purely field nonlinear equations, and Einstein held that a correct generalization of these equations would lead us to equations of the unified field theory. He wrote: “I believe, further, that the equations of gravitation for empty space are the only rationally justified case of field theory that can claim to be rigorous (considering nonlinear terms as well). This all leads to an attempt to generalize the gravitation theory for empty space.”

Einstein believed that one of the main problems in unified field theory is the one of the geometrization of the energy-momentum tensor of matter on the right-hand side of his equations (1). This problem can be solved using the concept of 4D oriented point and the space of events with the geometry of absolute parallelism and Cartan’s structural equations in this geometry.

Displacement of the origin and changing of the orientation of 4D oriented point can be presented by differentials
\[ dx^i = e^a_i dx^a, \quad (78) \]
\[ de^i_a = \Delta^a_{b} e^i_a, \quad (79) \]
where
\[ e^a_a = e^a_i dx^i, \quad (80) \]
\[ \Delta^a_{b} = e^a_i de^i_{a} = \Delta^a_{bk} dx^k \quad (81) \]
are differential 1-forms of tetrad \( e^a_i \) and connection of absolute parallelism \( \Delta^a_{bk} \). Differentiating the relationships (78), (79) externally, we have, respectively,
\[ d(dx^i) = (d e^a_i - e^c_i \Delta^a_{c}) e^i_a = -S^a e^i_a, \quad (82) \]
\[ d(de^i_a) = (d \Delta^b_{a} - \Delta^c_{a} \Delta^b_{c}) e^i_b = -S^b_a e^i_b. \quad (83) \]

Here \( S^a \) denotes the 2-form of Cartanian torsion, and \( S^b_a \) – the 2-form of the curvature tensor. The sign \( \wedge \) signifies external product, e.g.,
\[ e^a \wedge e^b = e^a e^b - e^b e^a. \quad (84) \]
By definition, a space has a geometry of absolute parallelism, if the 2-form of Cartanian torsion \( S^a \) and the 2-form of the Riemann-Christoffel curvature \( S^b_a \) of this space vanish
\[ S^a = 0, \quad (85) \]
\[ S^b_a = 0. \quad (86) \]

At the same time, these equalities are the integration conditions for the differentials (85) and (86).

Equations
\[ de^a - e^c \wedge \Delta^a_{c} = -S^a, \quad (87) \]
\[ d\Delta^b_{a} - \Delta^c_{a} \wedge \Delta^b_{c} = -S^b_a, \quad (88) \]
which follow from (82) and (83), are Cartan’s structural equations for an appropriate geometry. For the geometry of absolute parallelism hold the conditions (85) and (86), therefore Cartan’s structural equations for \( A_4 \) geometry have the form
\[ de^a - e^c \wedge \Delta^a_{c} = 0, \quad (89) \]
\[ d\Delta^b_{a} - \Delta^c_{a} \wedge \Delta^b_{c} = 0. \quad (90) \]

Considering (48), we will represent 1-form \( \Delta^a_{b} \) as the sum
\[ \Delta^a_{b} = \Gamma^a_{b} + T^a_{b}. \quad (91) \]
Substituting this relationship into (89) and noting that
\[ e^c \wedge \Delta^a_{c} = e^c \wedge T^a_{c}, \]
we get the first of Cartan’s structural equations for space of events of the 4D oriented points
\[ de^a - e^c \wedge T^a_{c} = 0. \]
In matrix form these equations will look like
\[ \nabla_{[k} e^a_{m]} - e^b_{[k} T^a_{b|m]} = 0. \quad (A) \]
Substituting (91) into (90) gives the second of Cartan’s equations for the space.

\[ R^a_b + dT^a_b - T^a_b \wedge T^c_c = 0, \]

or, in matrix form

\[ R^a_{bkm} + 2\nabla [kT^a_{b[m]} + 2T^a_{c[k}T^c_{b]}m] = 0. \quad (B) \]

In the coordinate indexes the equations (B), written as

\[ R^{i}_{jkm} + 2\nabla [kT^i_{j[m]} + 2T^i_{s[k}T^s_{j]}m] = 0. \quad (92) \]

Forming, using (92), the Einstein tensor

\[ G_{jm} = R_{jm} - \frac{1}{2}g_{jm}R, \]

we obtain the 10 equations

\[ R_{jm} - \frac{1}{2}g_{jm}R = \nu T_{jm}, \quad (93) \]

which are similar to Einstein’s equations, but with the geometrized right-hand side defined as

\[ T_{jm} = -\frac{2}{\nu} \left( \nabla^i [i T^i_{jm} + T^i_s[i T^s_{jm}]] - \frac{1}{2} g_{jm} g^{pn} \left( \nabla^i [i T^i_{jm} + T^i_s[i T^s_{jm}]] \right) \right) \quad (94) \]

Let us now decompose the Riemann tensor \( R_{ijkm} \) into irreducible parts

\[ R_{ijkm} = C_{ijkm} + g_{i[k} R_{jm]} + g_{j[k} R_{m]i} + \frac{1}{3} R g_{i[m} g_{k]j}, \quad (95) \]

where \( C_{ijkm} \) is the Weyl tensor; the second and third terms are the traceless part of the Ricci tensor \( R_{jm} \) and \( R \) is its trace.

Using the equations (93), written as

\[ R_{jm} = \nu \left( T_{jm} - \frac{1}{2} g_{jm} T \right), \quad (96) \]

we will rewrite the relationship (95) as

\[ R_{ijkm} = C_{ijkm} + 2\nu g_{i[k} T_{jm]} + \frac{1}{3} \nu T g_{i[m} g_{k]j}, \quad (97) \]

where \( T \) is the tensor trace (94).

Now we introduce the tensor current

\[ J_{ijkm} = 2g_{i[k} T_{jm]} - \frac{1}{3} T g_{i[m} g_{k]j} \quad (98) \]

and represent the tensor (97) as the sum

\[ R_{ijkm} = C_{ijkm} + \nu J_{ijkm}. \quad (99) \]

Substituting this relationship into the equations (92), we will arrive at

\[ C_{ijkm} + 2\nabla [kT_{ij|m} + 2T^s_{s[k}T^s_{j]}m] = -\nu J_{ijkm}. \quad (100) \]

Equations (100) are the Yang-Mills equations with a geometrized source, which is defined by the relationship (98). In equations (100) for the Yang-Mills field we have the Weyl tensor \( C_{ijkm} \), and the potentials of the Yang-Mills field are the Ricci rotation coefficients \( T^a_{ij} \).

Summarizing the geometrized Einstein equations (93) and the Yang-Mills equations (100), we can represent the structural Cartan equations (A) and (B) as an extended set of Einstein-Yang-Mills equations

\[ \nabla [k e^a_i] + T^i_s[i e^a_i] = 0, \quad (A) \]

\[ R_{jm} - \frac{1}{2} g_{jm} R = \nu T_{jm}, \quad (B.1) \]

\[ C^i_{jkm} + 2\nabla [kT^i_{j|m} + 2T^i_s[k T^s_{j}]m] = -\nu J^i_{jkm}. \quad (B.2) \]

in which the geometrized sources \( T_{jm} \) and \( J_{ijkm} \) are given by (94) and (98).

For the case of Einstein’s vacuum the equations are much simpler

\[ \nabla [k e^a_i] + T^i_s[i e^a_i] = 0, \quad (i) \]

\[ R_{jm} = 0, \quad (ii) \]

\[ C^i_{jkm} + 2\nabla [kT^i_{j|m} + 2T^i_s[k T^s_{j}]m] = 0. \quad (iii) \]

Thus, the structural equations of absolute parallelism geometry, represent an extended set of Einstein-Yang-Mills equations with the gauge translations group \( T_4 \) defined on the base \( x^i \) with the structural equations (A), and with the gauge rotational group \( O(1,3) \), defined in the fibre \( e^i_a \) with the structural equations in the form of the geometrized Einstein-Yang-Mills equations (B.1) and (B.2).

It is easy to see, that when torsion \( \Omega_{jk}^i \) (and , hence, torsion field \( T^a_{i,j} \)) in the (A) and (B) equations is equal to zero the space of events becomes Minkowski space. The converse proposition, generally, is incorrect. If to put in the equations (A) and (B) Riemannian curvature equal to zero, we shall receive the equations

\[ \nabla [k e^a_i] + T^i_s[i e^a_i] = 0, \quad (101) \]

\[ \nabla [i T^i_{j|m}] + T^i_s[k T^s_{j}]m = 0, \quad (102) \]

which describe so-called primary torsion fields\(^5\).

**VII. CORRESPONDENCE WITH THE EQUATIONS OF EINSTEIN’S THEORY AND DETERMINISTIC QUANTUM MECHANICS**

The equations (63) will be transformed to the equations of motions of Einstein’s theory of gravitation when
the inertia force in (63) becomes zero

$$F^i = mT^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0,$$

(103)

or, using (76) (for $m \neq 0$)

$$-\Omega^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} + g^{im}(g_{gs}\Omega^s_{mk} + g_{ks}\Omega^s_{mj}) \frac{dx^j}{ds} \frac{dx^k}{ds} = 0.$$

(104)

Since $\Omega_{mkj}$ is skew-symmetric in indices $m$ and $k$, then it follows from (104) that in inertial reference frames the torsion $\Omega_{mkj}$ of the space is skew-symmetrical in all the three indices

$$T_{ijk} = -T_{jik} = -T_{ikj},$$

(105)

but not equal to zero and coincides with torsion field $T^j_{jk}$. The energy-momentum tensor (94) in these case is symmetrical in the indices $j$ and $m$ to yield

$$T_{jm} = \frac{1}{\nu}(\Omega^{ij}_{km} \Omega_{ji}^s - \frac{1}{2}g_{jm}\Omega^{ij}_{ks}\Omega_{ij}^s).$$

(106)

In general case torsion $\Omega^i_{jk}$ has 24 independent components and it can be represented as the sum of three irreducible parts as follows

$$\Omega^i_{jk} = \frac{2}{3}\varepsilon^i_{[jk]} + \frac{1}{3}\varepsilon_{jk} \hat{\Omega}^s_i + \hat{\Omega}^i_{jk},$$

(107)

where

$$\Omega^i_{jk} = g^{im}g_{ks}\Omega_{mkj}^s,$$

(108)

and $\Omega$ – the vector, $\hat{\Omega}$ – the pseudovector and $\hat{\Omega}^i_{jk}$ – the traceless part of torsion are given by

$$\Omega^i = \Omega^i_{ji},$$

(109)

$$\hat{\Omega} = \frac{1}{2}\varepsilon_{jns}\Omega^{nsj},$$

(110)

$$\hat{\Omega}^s_{j} = 0, \quad \hat{\Omega}^s_{ij} + \hat{\Omega}^s_{ji} + \hat{\Omega}^s_{ij} = 0,$$

(111)

where $\varepsilon_{ijklm}$ is a fully skew-symmetrical Levi-Civita symbol.

Since in inertial reference frames the torsion $\Omega_{ij}$ is skew-symmetrical in all the three indices, among the irreducible parts of torsion in inertial frames only the pseudovector (110) is nonzero.

We can define the auxiliary pseudovector $h_m$ through the field (110) as follows

$$\Omega^i_{jk} = \varepsilon^{i}_{km}h_m, \quad \hat{\Omega}^{i}_{jk} = \varepsilon_{ijkm}h^m$$

(112)

and write the tensor (106) as

$$T_{jm} = \frac{1}{2\nu}(h_jh_m - \frac{1}{2}g_{jm}h^ih_i).$$

(113)

If the pseudovector $h_m$ is light-like, it can be represented as

$$h_m = \Phi l_m, \quad l^m_m = 0, \quad \Phi = \Phi(x^i).$$

(114)

In this case the matter tensor (113) becomes

$$T_{jm} = \frac{1}{\nu}\Phi^2(x^i)l_jl_m,$$

(115)

and the density of matter is given by

$$\rho = \frac{1}{\nu c^2}\Phi^2(x^i).$$

(116)

If the pseudovector $h_m$ is time-like, it can conveniently be represented as

$$h_m = \varphi(x^i)u_m,$$

(117)

where

$$u_m u^m = 1$$

(118)

and $\varphi(x^i)$ is a scalar quantity.

Substitution of (117) into the tensor (113) yields the energy-momentum tensor of the form

$$T_{jm} = \frac{1}{\nu}\varphi^2(u_ju_m - \frac{1}{2}g_{jm}) = -\rho c^2(u_ju_m - \frac{1}{2}g_{jm}),$$

(119)

were

$$\rho = -\frac{1}{\nu c^2}\varphi^2(x^i)$$

(120)

density of the matter. Tensor (115) looks like an energy-momentum tensor of isotropic radiation, and the tensor (119) in its structure looks rather like the energy-momentum tensor of an ideal liquid. Thus, in a post-Einstein’s approximation the matter density is defined through squares of torsion fields $\Phi$ and $\varphi$ according to (116) and (120).

VIII. NEW CONCEPT OF MASS DENSITY AND A MASS

The mass of a particle in the equations of Newton mechanics has no structure and does not depend on velocity of its movement. The special theory of a relativity has found out, that the mass depends on the velocity of a particle as

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}},$$

were $m_0$ the rest mass. On the other hand, in the quantum mechanics the mass of a particle depends on the frequency $\omega$ of the wave function $\psi$ as

$$m = E/c^2 = \hbar\omega/c^2.$$
It was shown before that in the general case the purely field energy-momentum tensor of matter \( T_{jm} \) in the equations of (B.1) is defined through the inertial fields \( T_{jk}^i \) by formula (94).

Using (94), we define the matter density \( \rho \) as

\[
\rho = \frac{T}{c^2} = (g^{jm}T_{jm})c^2 = \frac{2g^{jm}}{\nu c^2}(-\nabla_i T_{ij}^l + T_{ik}^l T_{jm}^k),
\]

then the inertial mass \( M \) of a system to be calculated through the integral

\[
M = \int (-g)^{1/2} \rho dV
\]

is defined through the inertial fields \( T_{jk}^i \) created by the torsion of the space. When torsion \( \Omega \) is a measure of the inertial mass in new mechanics, it is equal to zero and the mass (122) disappears.

\[
\frac{\partial}{\partial \tau} - \text{rotation,}
\]

\[
\sigma_{ab} = \nabla (u_a) + \frac{Du(\nu)}{\partial \tau} u_b - \frac{1}{3} \Theta h_{ab},
\]

- shear,

\[
\Theta = \nabla a u^a,
\]

- expansion (or contraction) and

\[
W_a = u_b \nabla b u_a = \frac{D u_a}{\partial \tau}
\]

- the local 4-acceleration vector, are determined through the irreducible parts (109)-(111) of the torsion \( \Omega_{bc} \), because

\[
T_{bc}^a = -\frac{\partial}{\partial \tau} \eta^f (\eta_{bd} \Omega_{fd}^c + \eta_{ed} \Omega_{fc}^d).
\]

Substitution (123) in the equations (B), gives

\[
R_{abc}^d = 2W_a (\omega_{bc} - W_{[bc]}u_a) u^d + 2
\]

\[
-2\nabla \omega_{[bc]} u^d - 2\nabla \sigma_{[bc]} u^d - \frac{2}{3} \Theta \eta_{[bc]} u^d + \frac{2}{3} \Theta [u_a \omega_{bc} -
\]

\[
-\omega_a W_{[bc]} + \omega_{[bc]} u_a + \sigma_{[bc]} u_a + \Theta \eta_{[bc]} u^d,
\]

\[
(B^{1+3})
\]

Using the equations (B.1) and the energy-momentum tensor with the structure \( T_{ab} = \rho c^2 u_a u_b \), we have for the matter energy density

\[
\rho = \frac{1}{c^2} T_{ab} u^a u^b = \frac{1}{\nu c^2} (\nabla_a W^a + 2\omega^2 - 2\sigma^2 - \frac{d \Theta}{d \tau} - \frac{1}{3} \Theta^2)
\]

and for the mass \( M \)

\[
M = \frac{1}{\nu c^2} \int (-g)^{1/2} (\nabla_a W^a + 2\omega^2 - 2\sigma^2 - \frac{d \Theta}{d \tau} - \frac{1}{3} \Theta^2) dV,
\]

\[
(M^{1+3})
\]

\[
\omega^2 = \omega_{ab} \omega^{ab}, \quad \sigma^2 = \sigma_{ab} \sigma^{ab}.
\]

The mass \( (M^{1+3}) \) describes the general case for Alcubierre’s version of the warp bubble. From \( (M^{1+3}) \) follows, that the null energy condition obeys, when \( \nu \geq 3 \sigma^2 \geq 0 \).

\[
\nabla_a W^a + 2\omega^2 - 2\sigma^2 - \frac{d \Theta}{d \tau} - \frac{1}{3} \Theta^2 \geq 0.
\]

The practical interest is represented the constraint equations

\[
\nabla_a \omega^a - W_a \omega^a = 0
\]

connecting acceleration \( W_a \) of the center of mass of system with its internal rotation \( 2\omega_a = \varepsilon_{abc} \omega_{bc} \).
Let \( \sigma = \Theta = 0 \) equal to zero, then the mass is

\[
M(\omega) = \frac{1}{\nu c^2} \int (-g)^{1/2}(\nabla W^a + 2\omega^2)dV.
\]

This equation shows, that we can operate by mass \( M \), i.e. angular velocity \( \omega \) of rotation of masses which mass \( M \) consists of, it will lead us to the creation of an inertial propulsion system of an essentially new type, that will move in space\(^{14}\) like Alcubierre’s bubble\(^{15}\) without action on it of an external forces. The elementary scientific model of such propulsion system - 4D gyroscope was created in Thailand and the patent for the vacuum torsion propulsion system representing a warp drive was granted. Now we shall write down Einstein’s well-known formula \( E = mc^2 \) as

\[
E = m(\omega)c^2,
\]

where the mass of object depends on the internal rotation.

**Conclusion**

The majority fundamental physical theories uses a point manifold and, accordingly, concept of a material point in the basis. Attempts undertaken by theorists to solve such problem as unification the general theory of a relativity with quantum theory of a matter till now have not brought success. The basic result of the given work consists in showing an opportunity for decision of this problem, being based on a manifold of an oriented points. The basis for the new physics has been prepared by outstanding mathematicians – G.Frenet, G. Ricci, L.Euler, E.Cartan and R.Penrose. The basic of the physical ideas have been stated by W.Clifford, E.Mach, in greater degree by A.Einstein, W.Heisenberg and P.Dirac.

More than 300 years we have been applying Newton’s mechanics to explain non-relativistic mechanical experiments. Although Newton’s mechanics has been generalized three times: by the special relativity theory, general relativity theory, and quantum mechanics, there remains a possibility for its further generalization. The fourth generalization of Newtonian mechanics has become possible with regards that new mechanics has been based upon the following: 1) Clifford-Einstein program for geometrization of all physics equations, including classical mechanics, (Unified Field Theory\(^2\)); 2) Cartan’s idea about the connection of the torsion of space with physical rotation\(^{16}\).

Einstein assumed the solution of these problems in the geometrization of the right hand of its equations. Generalizing Einstein’s vacuum equations, we have introduce structural Cartan equations geometry of absolute parallelism as the new vacuum equations. It has allowed us not only to find a general view of geometrized energy-momentum tensor, but also to specify communication torsion of the space of absolute parallelism with a field of inertia and wave function of quantum mechanics according to Einstein ideas. The mass of any object in the generalized theory has cleanly field nature and is defined as a measure of field of inertia. The rest mass of such object can be operated, using rotation of masses of which the object consists. The first experimental acknowledgement of these theoretical conclusions are already received by us at research of the dynamics so called 4D gyroscope\(^{14}\).

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