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Types of Linkage of Quadratic Pfister Forms

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Abstract

Given a field $F$ of positive characteristic $p$, $\theta \in H^{n-1}_p(F)$ and $\beta, \gamma \in F^\times$, we prove that if the symbols $\theta \wedge \frac{d\beta}{\beta}$ and $\theta \wedge \frac{d\gamma}{\gamma}$ in $H^p_p(F)$ share the same factors in $H^1_p(F)$ then the symbol $\theta \wedge \frac{d\beta}{\beta} \wedge \frac{d\gamma}{\gamma}$ in $H^{n+1}_p(F)$ is trivial. We conclude that when $p = 2$, every two totally separably $(n - 1)$-linked $n$-fold quadratic Pfister forms are inseparably $(n - 1)$-linked. We also describe how to construct non-isomorphic $n$-fold Pfister forms which are totally separably (or inseparably) $(n - 1)$-linked, i.e. share all common $(n - 1)$-fold quadratic (or bilinear) Pfister factors.

Keywords: Kato-Milne Cohomology, Fields of Positive Characteristic, Quadratic Forms, Pfister Forms, Quaternion Algebras, Linkage

2010 MSC: 11E81 (primary); 11E04, 16K20, 19D45 (secondary)

1. Introduction

Linkage of Pfister forms is a classical topic in quadratic form theory. We say that two $n$-fold Pfister forms over a field $F$ are separably (inseparably, resp.) $m$-linked if there exists an $m$-fold quadratic (bilinear) Pfister form which is a common factor of both forms. When $\text{char}(F) \neq 2$, there is no difference between quadratic and bilinear factors, so the terms coincide, and we simply say $m$-linked.

We say that two quadratic $n$-fold Pfister forms are totally separably (inseparably) $m$-linked if every quadratic (bilinear) $m$-fold Pfister factor of one of them is also a factor of the other. The following facts were proven in [4]:

- Two $n$-fold quadratic Pfister forms can be totally separably 1-linked, inseparably 1-linked, or even both, without being isometric. (The special case of quaternion algebras over fields of characteristic not 2 was covered in [9].)
- Total separable 1-linkage and total inseparable 1-linkage of $n$-fold quadratic Pfister forms are independent properties, i.e. do not imply each other.

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Clearly total separable (or inseparable) \((n-1)\)-linkage implies (nontotal) separable (inseparable, resp.) \((n-1)\)-linkage. It is known that inseparable \((n-1)\)-linkage of quadratic \(n\)-fold Pfister forms implies separable \((n-1)\)-linkage, but the converse is in general not true (see [12], [8], [5], [6], [3] and [1] for references).

**Question 1.1.** Does total separable \((n-1)\)-linkage of \(n\)-fold quadratic Pfister forms imply (nontotal) inseparable \((n-1)\)-linkage?

We answer this question in the positive in Section 3. We conclude it from deeper results on linkage of symbols in the Kato-Milne cohomology groups.

There are several other natural questions that arise in this setting:

**Question 1.2.** Do there exist totally separably (inseparably) \(m\)-linked quadratic \(n\)-fold Pfister forms which are not isometric for a given \(m \in \{1, \ldots, n-1\}\)?

**Question 1.3.** Over fields of characteristic 2, are total separable \(m\)-linkage and total inseparable \(m\)-linkage independent properties?

**Question 1.4.** Given \(1 \leq \ell < m < n-1\), are there totally separably (or inseparably) \(\ell\)-linked \(n\)-fold quadratic Pfister forms which are not totally separably (inseparably, resp.) \(m\)-linked?

We answer Question 1.2 in full generality (in the positive), and Question 1.4 in the case of fields of characteristic not 2 and \(m = n-1\) (see Section 4). Question 1.3 was answered in the negative in [4] for \(m = 1\), but it remains open for arbitrary \(m\).

### 2. Preliminaries

#### 2.1. Quadratic Forms

For general reference on symmetric bilinear forms and quadratic forms see [7]. Throughout, let \(F\) be a field and \(V\) an \(F\)-vector space. A quadratic form over \(F\) is a map \(\varphi: V \to F\) such that \(\varphi(av) = a^2 \varphi(v)\) for all \(a \in F\) and \(v \in V\) and the map defined by \(B_\varphi(v, w) = \varphi(v+w) - \varphi(v) - \varphi(w)\) for all \(v, w \in V\) is a bilinear form on \(V\). The bilinear form \(B_\varphi\) is called the polar form of \(\varphi\) and is clearly symmetric. Two quadratic forms \(\varphi: V \to F\) and \(\psi: W \to F\) are isometric if there exists an isomorphism \(M: V \to W\) such that \(\varphi(v) = \psi(Mv)\) for all \(v \in V\). We are interested in the isometry classes of quadratic forms, so when we write \(\varphi = \psi\) we actually mean that they are isometric.

We say that \(\varphi\) is singular if \(B_\varphi\) is degenerate, and that \(\varphi\) is nonsingular if \(B_\varphi\) is nondegenerate. If \(F\) is of characteristic 2, every nonsingular form \(\varphi\) is even dimensional and can be written as

\[\varphi = [\alpha_1, \beta_1] \perp \cdots \perp [\alpha_n, \beta_n].\]
for some $\alpha_1,\ldots,\beta_n \in F$, where $[\alpha,\beta]$ denotes the two-dimensional quadratic form $\psi(x,y) = \alpha x^2 + xy + \beta y^2$ and $\perp$ denotes the orthogonal sum of quadratic forms. If the characteristic of $F$ is different from 2, symmetric bilinear forms and quadratic forms are equivalent objects, and we do not distinguish between them in this case. The unique nonsingular two-dimensional isotropic quadratic form is $H = [0,0]$, called the hyperbolic plane. A hyperbolic form is an orthogonal sum of hyperbolic planes. Any quadratic form $\varphi$ over $F$ decomposes into an orthogonal sum of a uniquely determined anisotropic quadratic form and a number of hyperbolic planes. The number of hyperbolic planes appearing in this decomposition is called the Witt index and denoted $i_W(\varphi)$.

We denote by $\langle \alpha_1,\ldots,\alpha_n \rangle$ the diagonal bilinear form given by $(x,y) \mapsto \sum_{i=1}^n \alpha_i x_i y_i$. A bilinear $n$-fold Pfister form over $F$ is a symmetric bilinear form isometric to a bilinear form

$$\langle 1,\alpha_1 \rangle \otimes \langle 1,\alpha_2 \rangle \otimes \cdots \otimes \langle 1,\alpha_n \rangle$$

for some $\alpha_1,\alpha_2,\ldots,\alpha_n \in F^\times$. We denote such a form by $\langle \alpha_1,\alpha_2,\ldots,\alpha_n \rangle$. By convention, the bilinear 0-fold Pfister form is $1$. The 2-fold Pfister forms generate the fundamental ideal $IF$ in the Witt ring of nondegenerate symmetric bilinear forms $WF$. Powers of $IF$ are denoted by $I^r F$, and are generated by $n$-fold Pfister forms respectively.

Let $B : V \times V \to F$ be a symmetric bilinear form over $F$ and $\varphi : W \to F$ be a quadratic form over $F$. We may define a quadratic form $B \otimes \varphi : V \otimes_F W \to F$ determined by the rule that $(B \otimes \varphi)(v \otimes w) = B(v,v) \cdot \varphi(w)$ for all $w \in W, v \in V$. We call this quadratic form the tensor product of $B$ and $\varphi$. A quadratic $n$-fold Pfister form over $F$ is a tensor product of a bilinear $(n-1)$-fold Pfister form $\langle \alpha_1,\alpha_2,\ldots,\alpha_{n-1} \rangle$ and a two-dimensional quadratic form $[1,\beta]$ for some $\beta \in F$. We denote such a form by $\langle \alpha_1,\ldots,\alpha_{n-1},\beta \rangle$. Quadratic $n$-fold Pfister forms are isotropic if and only if they are hyperbolic (see [7, (9.10)]). The 2-fold quadratic Pfister forms generate the fundamental ideal, denoted $I_q F$ or $I^1_q F$, of the Witt group of nonsingular quadratic forms. Let $I_q^n F$ denote the subgroup generated by scalar multiples of quadratic $n$-fold Pfister forms.

Given a symmetric bilinear form $B$, we denote by $Q(B)$ the quadratic form given by the map $v \mapsto B(v,v)$.

Let $\pi$ be an $n$-fold quadratic Pfister form over $F$. For $m \in \{1,\ldots,n\}$, we say an $m$-fold quadratic (resp. bilinear) Pfister form $\psi$ (resp. $B$) is a factor of $\pi$ if there exists an $(n-m)$-fold bilinear (resp. quadratic) Pfister form $B'$ (resp. $\psi'$) such that $\pi = B' \otimes \psi$ (resp. $\pi = B \otimes \psi'$).

Let $\omega$ be an $n$-fold quadratic Pfister form over $F$. We say $\pi$ and $\omega$ are separably (resp. inseparably) $m$-linked if there exists an $m$-fold quadratic (resp. bilinear) Pfister form $\psi$ such that $\psi$ is a factor of both $\pi$ and $\omega$. We say $\pi$ and $\omega$ are totally separably (resp. inseparably) $m$-linked if every quadratic (resp. bilinear) $m$-fold Pfister form is a factor of $\pi$ if and only if it is a factor of $\omega$. This terminology comes from the fact that in characteristic 2, the function fields of quadratic (resp. bilinear) Pfister forms are separable (resp. inseparable) extensions of the ground field.

2.2. Kato-Milne Cohomology

In this section, assume $F$ is a field of characteristic $p > 0$. For $n > 0$, the Kato-Milne Cohomology group $H^{n+1}_p(F)$ is defined to be the cokernel of the Artin-Schreier
map
\[ \phi : \Omega^n_F \to \Omega^n_F / d\Omega^{n-1}_F \]
\[ \frac{d\beta_1}{\beta_1} \wedge \cdots \wedge \frac{d\beta_n}{\beta_n} \mapsto (\alpha^p - \alpha) \frac{d\beta_1}{\beta_1} \wedge \cdots \wedge \frac{d\beta_n}{\beta_n}. \]
We also fix \( H^1_p(F) \) to be \( F/\mathcal{F}(F) \). The group \( \nu_F(n) \) is defined to be the kernel of this map. By [2], \( \nu_F(n) \cong \mathcal{K}_n F/p^\mathcal{K}_n F \), with the isomorphism given by
\[ \frac{d\beta_1}{\beta_1} \wedge \cdots \wedge \frac{d\beta_n}{\beta_n} \mapsto \{\beta_1, \ldots, \beta_n\}. \]
It is known that \( H^2_p(F) \cong p \text{Br}(F) \) and for \( p = 2 \), \( H^2_p(F) \cong F_p/F_{p+1} \). The first isomorphism is given by the map
\[ \alpha \frac{d\beta}{\beta} \mapsto [\alpha, \beta]_p, \]
where \([\alpha, \beta]_p,F \) stands for the symbol \( p \)-algebra \( F\langle x, y : x^p - x = \alpha, y^p = \beta, yxy^{-1} = x + 1 \rangle \).
The second isomorphism is given by the map
\[ \alpha \frac{d\beta}{\beta} \wedge \cdots \wedge \frac{d\beta_n}{\beta_n} \mapsto \langle [\beta_1, \ldots, \beta_n, \alpha] \rangle. \]
We call the logarithmic differentials \( \alpha \frac{d\beta_1}{\beta_1} \wedge \cdots \wedge \frac{d\beta_n}{\beta_n} \) in \( H^n_F(F) \) and \( \frac{d\gamma_1}{\gamma_1} \wedge \cdots \wedge \frac{d\gamma_m}{\gamma_m} \) in \( \nu_F(m) \) “symbols”. There is a natural map
\[ H^n_F(F) \times \nu_F(m) \to H^{n+m}_p(F) \]
defined by the wedge product
\[ \left( \frac{d\beta_1}{\beta_1} \wedge \cdots \wedge \frac{d\beta_{n-1}}{\beta_{n-1}}, \frac{d\gamma_1}{\gamma_1} \wedge \cdots \wedge \frac{d\gamma_m}{\gamma_m} \right) \mapsto \alpha \frac{d\beta_1}{\beta_1} \wedge \cdots \wedge \frac{d\beta_{n-1}}{\beta_{n-1}} \wedge \frac{d\gamma_1}{\gamma_1} \wedge \cdots \wedge \frac{d\gamma_m}{\gamma_m}. \]
We define the linkage of symbols in an analogous manner to the linkage of Pfister forms. If a symbol \( \omega \) in \( H^{m+n}_p(F) \) is a wedge product \( \theta \wedge \psi \) where \( \theta \) is a symbol in \( H^n_p(F) \) and \( \psi \) is a symbol in \( \nu_F(m) \), then \( \theta \) and \( \psi \) are called factors of \( \omega \). We say that two symbols \( \pi \) and \( \omega \) are separably \( k \)-linked if they have a common factor in \( H^k_p(F) \), and inseparably \( k \)-linked if they have a common factor in \( \nu_F(k) \). We say that two symbols \( \pi \) and \( \omega \) are totally separably \( k \)-linked if they share all factors in \( H^k_p(F) \), and inseparably \( k \)-linked if they share all factor in \( \nu_F(k) \).

3. Separably \((n - 1)\)-linked Symbols in \( H^n_p(F) \)

In this section, assume \( F \) is a field of characteristic \( p > 0 \). One of the main goals is to show that total separable \((n - 1)\)-linkage implies inseparable \((n - 1)\)-linkage for quadratic \( n \)-fold Pfister forms when \( p = 2 \).
Lemma 3.1. For $\alpha \in F$ and $\beta \in F^*$, let 
\[ t = \alpha + \frac{(\alpha - \beta)}{\gamma}. \]
The symbol $p$-algebra $[\alpha, \gamma]_{p,F}$ contains the étale extension $F[x : x^p - x = t^p\gamma + \beta]$ of $F$.

Proof. Let $i$ and $j$ be a pair of generators of $[\alpha, \gamma]_{p,F}$ with $i^p - i = \alpha$, $j^p = \gamma$ and $ji^{-1} = i + 1$. Take $x = t + tj + ij$ in $[\alpha, \gamma]_{p,F}$. Then $x^p - x$ is equal to 
\[ \gamma\alpha^p + \gamma 
^{-1} p \alpha \gamma - \gamma 
^{-1} p \beta^p + \beta = t^p\gamma + \beta \]
by [3, Lemma 3.1]. Hence the subalgebra $F[x]$ of $[\alpha, \gamma]_{p,F}$ is as required. \hfill \Box

Proposition 3.2. Consider two separably $(n - 1)$-linked symbols 
\[ \pi = \alpha \frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\beta}{\beta} \wedge dy \quad \text{and} \quad \omega = \alpha \frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{dy}{\gamma} \]
in $H^1_p(F)$ and let $t = \alpha + \frac{\alpha \beta}{\gamma}$. If $t^p\gamma + \beta$ is a factor in $H^1_p(F)$ of $\pi$, then the class of $\alpha \frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{dy}{\gamma}$ in $H^{n+1}_p(F)$ is trivial.

Proof. Note first that if $t^p\gamma = 0$ then $dy \wedge d\beta = 0$ and the result holds. Assume otherwise. The class of $t^p\gamma + \beta$ in $H^1_p(F)$ is a factor of $\omega$ by Lemma 3.1, so it is a common factor of $\pi$ and $\omega$. We have 
\[ \frac{\alpha d\beta}{\beta} \wedge \frac{dy}{\gamma} = \frac{\alpha \beta \gamma^{-1}}{\beta \gamma^{-1}} \wedge \frac{d(t^p\gamma + \beta)}{t^p\gamma + \beta} \]
(see [6, Lemma 5.1, (e)]). Now, 
\[ \alpha \frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{dy}{\gamma} = \]
\[ \frac{\alpha d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\beta}{\beta} \wedge \frac{dy}{\gamma} = \]
\[ \frac{\alpha d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\beta}{\beta} \wedge \frac{d(t^p\gamma + \beta)}{t^p\gamma + \beta} \]
\[ = \frac{\alpha d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d(t^p\gamma + \beta)}{t^p\gamma + \beta} \]
Since $t^p\gamma + \beta$ is a factor in $H^1_p(F)$ of $\alpha \frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\beta}{\beta}$, we have 
\[ \alpha \frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\beta}{\beta} \wedge \frac{d(t^p\gamma + \beta)}{t^p\gamma + \beta} = (t^p\gamma + \beta) \tau \wedge \frac{d(t^p\gamma + \beta)}{t^p\gamma + \beta} \]
for some $\tau \in \mathfrak{m}_F(n - 1)$. As $(t^p\gamma + \beta) \frac{d(t^p\gamma + \beta)}{t^p\gamma + \beta} = d(t^p\gamma + \beta)$, it is trivial in $H^2_p(F)$. Hence $\alpha \frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{dy}{\gamma} = 0$. Similarly, since $t^p\gamma + \beta$ is a factor in $H^1_p(F)$ of $\alpha \frac{d\gamma}{\gamma}$, we have $\alpha \frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\gamma}{\gamma} = 0$. Therefore $\alpha \frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{dy}{\gamma} = 0$ in $H^{n+1}_p(F)$ as required. \hfill \Box
Corollary 3.3. Let
\[ \pi = \alpha \frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\beta}{\beta} \quad \text{and} \quad \omega = \alpha \frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{dy}{y} \]
be two separably \((n-1)\)-linked symbols in \(H^n_p(F)\). If \(\pi\) and \(\omega\) are totally separably 1-linked then the class of
\[ \alpha \frac{d\delta_1}{\delta_1} \wedge \cdots \wedge \frac{d\delta_{n-2}}{\delta_{n-2}} \wedge \frac{d\beta}{\beta} \wedge \frac{dy}{y} \]
in \(H^{n+1}_p(F)\) is trivial.

When \(p = 2\), by the identification of the symbols with quadratic \(n\)-fold Pfister forms, we obtain the following results:

**Proposition 3.4.** Assume \(p = 2\). Let
\[ \pi = \langle \beta, \delta_{n-2}, \ldots, \delta_1, \alpha \rangle \quad \text{and} \quad \omega = \langle \gamma, \delta_{n-2}, \ldots, \delta_1, \alpha \rangle \]
be two separably \((n-1)\)-linked \(n\)-fold quadratic Pfister forms over \(F\) and let \(t = \alpha + \frac{(\alpha-\beta)}{\gamma}\). If the 1-fold Pfister form \(\langle t^p \gamma + \beta \rangle\) is a factor of \(\pi\), then \(\langle \beta, \gamma, \delta_{n-2}, \ldots, \delta_1, \alpha \rangle\) is trivial. In particular, \(\pi\) and \(\omega\) are inseparably \((n-1)\)-linked.

**Proof.** By [5], the \((n+1)\)-fold Pfister form \(\langle \beta, \gamma, \delta_{n-1}, \ldots, \delta_2, \alpha \rangle\) is trivial if and only if \(\pi\) and \(\omega\) are inseparably \((n-1)\)-linked. \(\Box\)

**Corollary 3.5.** Assume \(p = 2\). If a pair of separably \((n-1)\)-linked \(n\)-fold quadratic Pfister forms over \(F\) are totally separably 1-linked then they are also inseparably \((n-1)\)-linked. In particular, if a pair of \(n\)-fold quadratic Pfister forms over \(F\) are totally separably \((n-1)\)-linked then they are inseparably \((n-1)\)-linked.

**Remark 3.6.** A similar result to Corollary 3.3 holds more straight-forwardly for Milnor \(K\)-groups. Let \(p\) be a prime integer, \(n\) a positive integer and \(F\) an arbitrary field. If \(p = 2\) then further assume that \(\sqrt{-1} \in F\). Then the following is trivial:

- If \(\{\alpha\} \cup \theta \in K_n F/pK_n F\) has \(\{\beta\}\) as a factor in \(K_1 F/pK_1 F\) then \(\{\alpha, \beta\} \cup \theta = 0\) in \(K_{n+1} F/pK_{n+1} F\).
- Therefore, if \(\{\alpha\} \cup \theta\) and \(\{\beta\} \cup \theta\) in \(K_n F/pK_n F\) are totally 1-linked then \(\{\alpha, \beta\} \cup \theta = 0\) in \(K_{n+1} F/pK_{n+1} F\).

**Remark 3.7.** The result analogous to Corollary 3.3 for inseparable linkage is also straight-forward. For fields \(F\) of characteristic \(p > 0\) and positive integer \(n\),

- If \(\omega \wedge \frac{d\gamma}{\beta} \in H^n_p F\) has \(\frac{dx}{y}\) in \(\nu_F(1)\) as a factor then \(\omega \wedge \frac{d\gamma}{\beta} \wedge \frac{dx}{y} = 0\) in \(H^{n+1}_p(F)\).
- Therefore, if \(\omega \wedge \frac{d\gamma}{\beta} \wedge \frac{dy}{y}\) in \(H^n_p F\) are totally inseparably 1-linked then \(\omega \wedge \frac{d\gamma}{\beta} \wedge \frac{dy}{y} = 0\) in \(H^{n+1}_p(F)\).
4. Totally Linked Quadratic Pfister Forms

In [4] we considered whether total 1-linkage of Pfister forms implied isometry. In general it does not. In this section, we consider whether total m-linkage implies isometry.

**Lemma 4.1.** Let \( n \) be an integer \( \geq 2 \) and \( m \in \{1, \ldots, n-1\} \). Let \( \varphi \) be an \( n \)-fold quadratic Pfister form, \( \pi \) an \( m \)-fold quadratic Pfister form and \( \theta \) an \((m-1)\)-fold quadratic Pfister form. Assume \( \theta \) is a common factor of \( \varphi \) and \( \pi \). Then \( i_W(\varphi \perp -\pi) = 2^m \) if and only if \( \pi \) is a factor of \( \varphi \). Otherwise, \( i_W(\varphi \perp -\pi) \neq 2^m \).

**Proof.** By [7, Corollary 24.3], \( i_W(\varphi \perp -\pi) \) must be a power of 2. Since \( \theta \) is a factor of \( \varphi \), \( i_W(\varphi \perp -\pi) \geq 2^m \). Therefore, the only other possible value is \( 2^m \), in which case \( \pi \) is a subform of \( \varphi \) and therefore a factor of \( \varphi \). \( \square \)

**Lemma 4.2.** Let \( n \) be an integer \( \geq 2 \) and \( m \in \{1, \ldots, n-1\} \). Let \( \varphi \) and \( \psi \) be two non-hyperbolic, separably \((n-1)\)-linked and totally separably \(m\)-linked \(n\)-fold quadratic Pfister forms over \( F \). Let \( \pi \) be an \((m+1)\)-fold quadratic Pfister form such that \( \pi \) is a factor of \( \varphi \) but not \( \psi \). Then there exists a field extension \( L \) such that \( \pi_L \) is a factor of \( \psi_L \) and \( \varphi_L \) is not isometric nor hyperbolic.

**Proof.** Since \( \varphi \) and \( \psi \) are separably \((n-1)\)-linked, the form \( \varphi \perp \psi \) is congruent mod \( F_{q^{n+1}} \) to some anisotropic \( n \)-fold Pfister form \( \phi \). Let \( \pi_0 \) be an \( m \)-fold quadratic Pfister factor of \( \pi \). Since \( \varphi \) and \( \psi \) are totally separably \( m \)-linked, \( \pi_0 \) is a common factor of both forms.

By Lemma 4.1, \( \varphi \perp -\pi \) is Witt equivalent to some anisotropic \( 2^n \)-dimensional form \( \theta \). Write \( L = F(\theta) \) for the function field of \( \theta \) over \( F \). If one of the forms \( \varphi_L, \psi_L \) and \( \phi_L \) were hyperbolic, then \( \theta \) would be similar to a subform of the form by [7, Corollary 22.5]. However, since the forms are of the same dimension, this would imply that \( \theta \) is similar to an \( n \)-fold Pfister form. This is impossible because the \( n \)th cohomological invariant of \( \theta \) is nontrivial. It follows that \( \psi_L \) and \( \varphi_L \) are not isometric as \( \phi_L \) is not hyperbolic. \( \square \)

**Theorem 4.3.** Let \( n \) be an integer \( \geq 2 \) and \( m \in \{1, \ldots, n-1\} \). Let \( \varphi \) and \( \psi \) be two non-hyperbolic, separably \((n-1)\)-linked and totally separably \(m\)-linked \(n\)-fold quadratic Pfister forms over \( F \). Then there exists a field extension \( K \) of \( F \) such that \( \varphi_K \) and \( \psi_K \) are totally separably \((m+1)\)-linked but not isometric nor hyperbolic.

**Proof.** Let \( S \) be the set of \((m+1)\)-fold quadratic Pfister forms \( \pi \) over \( F \) such that \( \pi \) is a factor of \( \varphi \) but not of \( \psi \). Then as in Lemma 4.2, for each \( \pi \in S \), there exists a \( 2^n \)-dimensional quadratic form \( \theta \) such that \( \theta \) is Witt equivalent to \( \pi \perp \psi \). Let \( F_0 \) be the compositum of the function fields of all such \( \theta \).

Similarly, let \( T \) be the set of \((m+1)\)-fold quadratic Pfister forms \( \pi' \) over \( F \) such that \( \pi \) is a factor of \( \varphi \) but not of \( \psi \). Again, as in Lemma 4.2, for each \( \pi' \in T \), there exists a \( 2^n \)-dimensional quadratic form \( \theta' \) such that \( \theta' \) is Witt equivalent to \( \pi' \perp \psi \). Let \( F_0' \) be the field compositum of the function fields of all such \( \theta' \).

Let \( K_0 \) be the compositum of \( F_0 \) and \( F_0' \). Then by Lemma 4.2, \( \varphi_{K_0} \) and \( \psi_{K_0} \) are not isometric nor hyperbolic, and every \((m+1)\)-fold Pfister form over \( K_0 \) defined over \( F \) is
a factor of $\varphi_{K_n}$ if and only if it is a factor of $\psi_{K_n}$. Using this construction inductively, we obtain the required field extension $K/F$.

**Lemma 4.4.** Assume $\text{char}(F) = 2$. Let $\theta$ be an anisotropic $n$-fold bilinear Pfister form over $F$ and let $\theta'$ be an $(n - 1)$-fold bilinear Pfister form factor of $\theta$. Let $\beta$ be an element represented by $\theta$ but not by $\theta'$. Then $Q(\theta) = Q((\langle \beta \rangle \otimes \theta')$.

**Proof.** This follows easily from [7, Proposition 10.4].

**Lemma 4.5.** Assume $\text{char}(F) = 2$. Let $n$ be an integer $\geq 2$ and $m \in \{1, \ldots, n - 1\}$. Let $\varphi$ be an anisotropic $n$-fold quadratic Pfister form, $\pi$ an anisotropic $m$-fold bilinear Pfister form and $\theta$ an $(m - 1)$-fold bilinear Pfister form. Assume $\theta$ is a common factor of $\varphi$ and $\pi$. Then $iw(\varphi \perp Q(\pi)) = 2^m$ if and only if $\pi$ is a factor of $\varphi$. Otherwise, $iw(\varphi \perp Q(\pi)) = 2^{m-1}$.

**Proof.** Since $\theta$ is a factor of $\varphi$, there exists a $2^{n-m} - 1$ dimensional bilinear form $b$ and a quadratic 1-fold Pfister form $\rho$ such that

$$\varphi = \theta \otimes (\langle 1 \rangle \perp b) \otimes \rho.$$ 

Set $\tau = \theta \otimes b \otimes \rho$ (note that $b \otimes \rho$ is the so-called pure part of the quadratic Pfister form $(\langle 1 \rangle \perp b) \otimes \rho$, see [7, p.66]). Then we have that

$$\varphi \perp Q(\theta) = 2^{m-1} \otimes \mathbb{H} \perp \tau \perp Q(\theta).$$

Note that $\tau \perp Q(\theta)$ is anisotropic as $\varphi$ is anisotropic.

Suppose $\tau \perp Q(\pi)$ is anisotropic, there exists an element $\beta$ represented by $\tau$ and $Q(\pi)$ but not by $Q(\theta)$. As $\beta$ is represented by $Q(\pi)$ but not by $Q(\theta)$, it follows from Lemma 4.4 that $Q(\langle \beta \rangle \otimes \theta) = Q(\pi)$. As $\beta$ is represented by $Q(\tau)$ but not by $Q(\theta)$, it follows from [7, Proposition 15.7] that $\langle \beta \rangle \otimes \theta$ is a factor of $\varphi$. In particular, $\varphi$ becomes isotropic over the function field of $\pi$. Hence $\pi$ is a factor of $\varphi$ by [11, (1.4)] and repeated use of [7, (15.6)].

**Theorem 4.6.** Assume $\text{char}(F) = 2$ and $n$ be an integer $\geq 2$ and $m \in \{1, \ldots, n - 1\}$. Let $\varphi$ and $\psi$ be two non-hyperbolic, separably $(n - 1)$-linked and totally inseparably $m$-linked $n$-fold quadratic Pfister forms over $F$. Then there exists a field extension $K$ of $F$ such that $\varphi_{K}$ and $\psi_{K}$ are totally inseparably $(m + 1)$-linked but neither isometric nor hyperbolic.

**Proof.** The result follows from Lemma 4.5 in a similar way to Theorem 4.3.

Using Theorems 4.3 and 4.6, one can construct examples of non-isometric pairs of $n$-fold quadratic Pfister forms which are totally separably (or inseparably, or both) $m$-linked for any $m \in \{1, \ldots, n - 1\}$.

**Example 4.7.** Start with the field $F = \mathbb{F}_2(\langle x_1 \rangle \ldots \langle x_{n+1} \rangle)$ of iterated Laurent series in $n + 1$ indeterminates over $\mathbb{F}_2$. The forms

$$\varphi = \langle x_1, \ldots, x_{n-1}, x_1 \cdot \ldots \cdot x_n \rangle \quad \text{and} \quad \psi = \langle x_2, \ldots, x_n, x_2 \cdot \ldots \cdot x_{n+1} \rangle$$

over $F$ are $(n - 1)$-linked but not totally 1-linked (see [5, Sections 9&10]). By iterating Theorem 4.3 (or 4.6) $m$ times, we end up with a field $K$ over which $\varphi_{K}$ and $\psi_{K}$ are totally $m$-linked, but neither isometric nor hyperbolic.
Question 4.8. When \( \text{char}(F) = p \), does there exist a similar process that extends two non-equal, nontrivial separably \((n - 1)\) symbols in \( H^n_p(F) \) to two non-equal, nontrivial totally separably (or inseparably) \((n - 1)\)-linked symbols?

Theorem 4.9. Assume \( \text{char}(F) \neq 2 \) and \( n \) be an integer \( \geq 3 \) and \( m \in \{1, \ldots, n - 3\} \). Let \( \varphi \) and \( \psi \) be two non-hyperbolic \((n - 1)\)-linked and totally \( m \)-linked \( n \)-fold quadratic Pfister forms over \( F \). Assume there exists an \((n - 1)\)-fold quadratic Pfister form \( \omega \) such that

(a) \( \omega \) is a factor of \( \varphi \),
(b) \( \omega \) is not a factor of \( \psi \),
(c) there exists an \((n - 2)\)-fold quadratic Pfister form that is a factor of both \( \omega \) and \( \psi \).

Then there exists a field extension \( K \) of \( F \) such that \( \varphi_K \) and \( \psi_K \) are totally \((m + 1)\)-linked but not totally \((n - 1)\)-linked nor hyperbolic.

Proof. This is essentially the same proof as in Theorem 4.3. Here we just need to note the following: The form \( \psi \perp -\omega \) is Witt equivalent to some anisotropic \( 2^n \) dimensional form. The latter remains anisotropic under scalar extension to \( L \) by [10, Theorem 5.4]. Therefore \( \omega_L \) is not a subform of \( \psi_L \), and that completes the proof.

Example 4.10. Start with the field \( F = \mathbb{C}((x_1)) \ldots ((x_{n+1})) \) of iterated Laurent series in \( n + 1 \) indeterminates over \( \mathbb{C} \). The forms \( \varphi = \langle x_1, \ldots, x_n \rangle \) and \( \psi = \langle x_2, \ldots, x_{n+1} \rangle \) over \( F \) are \((n - 1)\)-linked but not totally \( 1 \)-linked. By iterating Theorem 4.3 \( m \) times, we end up with a field \( K \) over which \( \varphi_K \) and \( \psi_K \) are totally \( m \)-linked, but neither hyperbolic nor isometric. If \( m \leq n - 2 \) then by Theorem 4.9, \( \varphi_K \) and \( \psi_K \) are also not totally \((n - 1)\)-linked.

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