Spectral properties of Schrödinger-type operators and large-time behavior of the solutions to the corresponding wave equation

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Abstract. Let \( L \) be a linear, closed, densely defined in a Hilbert space operator, not necessarily selfadjoint.

Consider the corresponding wave equations

\[
\begin{align*}
(1) \quad \ddot{w} + Lw &= 0, \quad w(0) = 0, \quad \dot{w}(0) = f, \quad \dot{w} = \frac{dw}{dt} \quad f \in H. \\
(2) \quad \ddot{u} + Lu &= fe^{-ikt}, \quad u(0) = 0, \quad \dot{u}(0) = 0,
\end{align*}
\]

where \( k > 0 \) is a constant. Necessary and sufficient conditions are given for the operator \( L \) not to have eigenvalues in the half-plane \( \text{Re} z < 0 \) and not to have a positive eigenvalue at a given point \( k^2 d > 0 \). These conditions are given in terms of the large-time behavior of the solutions to problem (1) for generic \( f \).

Sufficient conditions are given for the validity of a version of the limiting amplitude principle for the operator \( L \).

A relation between the limiting amplitude principle and the limiting absorption principle is established.

Keywords and phrases: elliptic operators, wave equation, limiting amplitude principle, limiting absorption principle

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1. Introduction

Let \( L \) be a linear, densely defined, closed operator in a Hilbert space \( H \). Our results and techniques are valid in a Banach space also, but we wish to think about \( L \) as of a Schrödinger-type operator in a Hilbert space and, at times, think that \( L \) is selfadjoint. For a Schrödinger operator \( L = -\nabla^2 + q(x) \) the resolvent \( (L - k^2)^{-1} \), \( \text{Im} k > 0 \), is an integral operator with a kernel \( G(x, y, k) \), its resolvent kernel. If \( q \) is a real-valued function, sufficiently rapidly decaying then \( L \) is selfadjoint, \( G(x, y, k) \) is analytic with respect to \( k \) in the half-plane \( \text{Im} k > 0 \), except, possibly, for a finitely many simple poles \( ik_j, k_j > 0 \), the

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semiaxis $k \geq 0$ is filled with the points of absolutely continuous spectrum of $L$, and there exists a limit

$$\lim_{\epsilon \to 0} G(x, y, k + i\epsilon) = G(x, y, k)$$

for all $k > 0$.

Sufficient conditions for $k^2 = 0$ not to be an eigenvalue of $L$ are found in papers [5], [6]. Spectral analysis of the Schrödinger operators is presented in many books (see, for example, [2] and [11]). In papers [3], [4], such an analysis was given in a class of domains with infinite boundaries apparently for the first time, see also [8]. In [7] an eigenfunctions expansion theorem was proved for non-selfadjoint Schrödinger operators with exponentially decaying complex-valued potential $q$. The operator $L$ in this paper is not necessarily assumed to be selfadjoint.

In [1] the validity of the limiting amplitude principle for some class of selfadjoint operators $L$ has been established.

This principle says that, as $t \to \infty$, the solution to problem

$$\ddot{u} + Lu = fe^{-ikt}, \quad u(0) = 0, \quad \dot{u}(0) = 0, \quad \dot{u} = \frac{du}{dt},$$

(1.1)

has the following asymptotics

$$u = e^{-ikt}v + o(1), \quad t \to \infty,$$

(1.2)

where $k$ is a real number and $v \in H$ solves the equation

$$Lv - k^2v = f.$$  

(1.3)

The $v$ is called the limiting amplitude. It turns out that a more natural definition of the limiting amplitude is:

$$v = \lim_{t \to \infty} \frac{1}{t} \int_0^t u(s)e^{iks}ds,$$

(1.4)

if this limit exists and solves equation (1.3).

Why is this definition more natural than (1.2)? There are good reasons for this. One of the reasons is: if (1.2) and (1.3) hold, then the limit (1.4) exists and solves equation (1.3). The other reason is: the limit (1.4) may exist and solve equation (1.3) although the limit (1.2) does not exist.

**Example.** If $u = e^{ikt}v + e^{ik_1t}v_1$, then the limit (1.2) does not exist, while the limit (1.4) does exist and is equal to $v$.

To describe our assumptions and results, some preparation is needed.

Consider the problem

$$\ddot{w} + Lw = 0, \quad w(0) = 0, \quad \dot{w}(0) = f.$$  

(1.5)

Assuming that $\|u(t)\| \leq ce^{at}$, where $c > 0$ stands throughout the paper for various generic constants, and $a \geq 0$ is a constant, one can define the Laplace transform of $u(t)$,

$$U := U(p) := \int_0^\infty e^{-pt}u(t)dt, \quad \sigma > a,$$

where $p = \sigma + i\tau$, $\text{Re}p = \sigma$.

Let us take the Laplace transform of (1.1) and of (1.5) to get

$$LLU + p^2U = \frac{f}{p + ik},$$

(1.6)

and

$$LW + p^2W = f,$$

(1.7)
where
\[ W = W(p) = \int_0^\infty w(t)e^{-pt}dt. \]

We also denote \( W(p) := \hat{w}(t). \)

The complex plane \( p \) is related to the complex plane \( k \) by the formula
\[ p = -ik, \quad k = k_1 + ik_2, \quad k_2 \geq 0, \quad \sigma = k_2 \geq 0. \]  

(1.8)

We assume throughout that \( f \) is generic in the following sense:

If \( I \) is the identity operator and a point \( p \) is a pole of the kernel of the operator \((L + p^2I)^{-1}f = W\).

If \( k^2 \) is an eigenvalue of \( L \) and \( \Re k^2 < 0 \), then \( \Im k > 0 \), where \( k = |k|e^{i\arg k^2}, p = -ik, \) so \( \sigma = \Re p > 0. \)

Let \( k > 0 \) and assume that \(-k^2 < 0 \) is an eigenvalue of \( L \). Then \( ik \) is a pole of the resolvent kernel \( G(x, y, k) \), and \( p = -i(ik) = k \) is a pole of the kernel of the operator \((L + p^2I)^{-1}. \) If \( k^2 > 0 \) is an eigenvalue of \( L \), then \( p = -ik \) is a pole of the operator \((L + p^2I)^{-1}. \)

The following known facts from the theory of Laplace transform will be used.

**Proposition 1.1.** An analytic in the half-plane \( \sigma > \sigma_0 \geq 0 \) function \( F(p) \) is the Laplace transform of a function \( f(t) \), such that \( f(t) = 0 \) for \( t < 0 \) and
\[ \int_0^\infty |f(t)|^2e^{-2\sigma t}dt < \infty \]

if and only if
\[ \sup_{\sigma > \sigma_0} \int_{-\infty}^\infty |F(\sigma + i\tau)|^2d\tau < \infty. \]

(1.10)

**Proposition 1.2.** If \( F(p) = \overline{f(t)} \), then
\[ \frac{F(p)}{p} = \int_0^t f(s)ds. \]

(1.11)

Let us now formulate the main Assumptions A and B standing throughout this paper.

**Assumption A.** For a generic \( f \) the \( W(p) = (L + p^2)^{-1}f \) is analytic in the half-plane \( \sigma > 0 \), except, possibly, at a finitely many simple poles at the points \(-ik_j, \ 1 \leq j \leq J, \ k_j \) are real numbers, and at the points \( \kappa_m, \Re \kappa_m > 0, \)

\[ W(p) = \sum_{j=1}^J \frac{v_j}{p + ik_j} + W_1(p) + \sum_{m=1}^M \frac{b_m}{p - \kappa_m}, \]

where \( v_j \) and \( b_m \) are some elements of \( H, \ W_1(p) \) is an analytic function in the half-plane \( \Re p = \sigma > 0, \)
continuous up to the imaginary axis \( \sigma = 0, \) and satisfying the following estimate
\[ ||W_1(p)|| \leq \frac{c}{1 + |p|^{\gamma}}, \quad \gamma > \frac{1}{2}. \]

(1.13)

**Assumption B.** There exists the limit
\[ \lim_{\sigma \to 0} ||W_1(\sigma - ik) - W_1(-ik)|| = 0 \]

(1.14)

for all real numbers \( k. \)
Theorem 1.3. Let the Assumption A hold. Then a necessary and sufficient condition for the operator $L$ to have no eigenvalues in the half-plane $\Re k^2 < 0$ is the validity of the estimate

$$\left\| \int_0^t w(s) ds \right\| = O(e^{\epsilon t}), \quad t \to \infty,$$  \hspace{1cm} (1.15)

for an arbitrary small $\epsilon > 0$.

A necessary and sufficient condition for the operator $L$ not to have any positive eigenvalues $k^2 > 0$ is the validity of the estimate

$$\left| \frac{1}{t} \int_0^t e^{iks} w(s) ds \right| = o(1), \quad t \to \infty, \quad \forall k \in \mathbb{R}. \hspace{1cm} (1.16)$$

A point $ik_0 > 0, k_0 > 0$, is not a pole of the resolvent kernel of the operator $(L - k^2 - i0)^{-1}$ if and only if estimate (1.16) holds with $k = k_0 > 0$.

Remark. If condition (1.16) holds for $k = 0$, then \(\left| \int_0^t w(s) ds \right| = o(t)\), so condition (1.15) holds, and the operator $L$ has no eigenvalues in the half-plane $\Re k^2 < 0$.

Theorem 1.4. Let the Assumptions A and B hold. Suppose that estimates (1.14) and (1.15) hold. Then the limiting amplitude principle (1.4) holds for every $k \in \mathbb{R}, k \neq k_j, 1 \leq j \leq J$.

In section 2, proofs are given.

2. Proofs

2.1. Proof of Theorem 1.3

From the Assumption A and Proposition 1.1, it follows that $W(p)$ is a Laplace transform of a function $w(t)$ such that

$$w(t) = \sum_{j=1}^J v_j e^{-ik_j t} + \sum_{m=1}^M b_m e^{\kappa_m t} + w_1(t), \hspace{1cm} (2.1)$$

where

$$w_1(t) = \frac{1}{2\pi i} \int_{c_0 - i\infty}^{c_0 + i\infty} e^{pt} W_1(p) dp, \hspace{1cm} (2.2)$$

and the integral in (2.2) converges in $L^2$-sense due to the assumption (1.13). It is clear from formula (2.1) that all $b_m = 0$ if and only if estimate (1.15) holds with $0 < \epsilon < \min_{1 \leq m \leq M} \Re \kappa_m$. This proves the first conclusion of Theorem 1.3.

Let us calculate the expression on the left side of formula (1.16) and show that this expression is $o(1)$ unless $k = k_j$ for some $1 \leq j \leq J$. In this calculation it is assumed that $L$ does not have any eigenvalues in the half-plane $\Re k^2 < 0$, in other words, that all $b_m = 0$. Otherwise the expression on the left of formula (1.16) tends to infinity as $t \to \infty$ at an exponential rate.

If all $b_m = 0$ in (2.1), then

$$\sum_{j=1}^J \frac{1}{t} \int_0^t e^{i(k-k_j)t} dt + \frac{1}{t} \int_0^t w_1(t)e^{ikt} dt := I_1 + I_2. \hspace{1cm} (2.3)$$

If $k$ and $k_j$ are real numbers, then

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t e^{i(k-k_j)t} dt = \begin{cases} 1, & k = k_j, \\ 0, & k \neq k_j. \end{cases} \hspace{1cm} (2.4)$$
Thus, \( I_1 = 0 \) if and only if \( k \) does not coincide with any of \( k_j, 1 \leq j \leq J \).

Let us prove that
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t w_1(t) e^{ikt} dt = 0.
\]
(2.5)

By proposition (1.2) and the Mellin inversion formula, one has
\[
I := \frac{1}{t} \int_0^t w_1(t) e^{ikt} dt = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} W_1(p-ik) \frac{e^{pt}}{pt} dp,
\]
(2.6)

where \( \text{Re} \sigma > 0 \) can be chosen arbitrarily small.

Let \( pt = q \), take \( \sigma = \frac{1}{t} \), write \( q = 1 + is \), and \( \text{write the integral on the right side of (2.6)} \) as:
\[
I = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} W_1(q) e^{q} \frac{e^{qy}}{qy} dq.
\]
(2.7)

If one uses estimate (1.13) and formula \( |q| = (1 + s^2)^{1/2} \), then one obtains the following inequality
\[
||I|| \leq \frac{1}{2\pi t} \int_{-\infty}^{\infty} 1 \frac{ed\sigma}{(1 + s^2)^{1/2} (1 + [1/s^2] - ik)^\gamma} = c \frac{1}{2\pi t^{1-\gamma}} \int_{-\infty}^{\infty} 1 \frac{ds}{(1 + s^2)^{1/2} (t^\gamma + [1 + (s-kt)^2]^{\gamma/2})}.
\]
(2.8)

Let \( s = ty \). Then the integral on the right side of (2.8) can be written as
\[
\frac{e^{ct}}{2\pi t^{1-\gamma}} \int_{-\infty}^{\infty} \frac{dy}{(1 + t^2y^2)^{1/2} (t^\gamma + t^2(y-k)^2)^{\gamma/2}} \leq \frac{c}{2\pi} \int_{-\infty}^{\infty} \frac{dy}{(1 + t^2y^2)^{1/2} (1 + [t^{-2} + (y-k)^2])^{\gamma/2}} \rightarrow 0, \text{ as } t \to \infty,
\]
(2.9)

and the convergence of the last integral to zero is uniform with respect to \( k \in \mathbb{R} \).

Thus
\[
\lim_{t \to \infty} ||I|| = 0.
\]
(2.10)

From (2.3) - (2.5) the last two conclusions of Theorem 1.3 follow. Theorem 1.3 is proved.

\(\square\)

2.2. Proof of Theorem 1.4

Using Proposition 1.2, equation (1.6), and the Mellin formula, one gets
\[
\frac{1}{t} \int_0^t u(t)e^{ikt} dt = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} U(p-ik) \frac{e^{pt}}{p} dp,
\]
(2.11)

where, according to (1.6),
\[
U(p-ik) = \frac{W(p-ik)}{p}.
\]
(2.12)

Let \( \sigma = \frac{1}{t} \) and \( pt = q \). Then
\[
\frac{1}{t} \int_0^t u(t)e^{ikt} dt = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} W \left( \frac{q}{t} - ik \right) \frac{e^{qy}}{qy} dq.
\]
(2.13)

Estimate (1.15) and Theorem 1.3 imply that all \( b_m = 0 \) in formula (2.1). Therefore, using formula (2.1) with \( b_m = 0 \), one gets
\[
W = \sum_{j=1}^{J} \frac{1}{p + ik_j} + \mathcal{W}.
\]

and
\[ \mathcal{W}\left(\frac{q}{l} - ik\right) = \mathcal{W}_1\left(\frac{q}{l} - ik\right) + \sum_{j=1}^{J} u_j \left( \frac{1}{l} - i(k - k_j) \right). \] (2.14)

One has \( \mathcal{W} = \frac{n_l}{p^{\frac{1}{2}}} \). Therefore
\[ \lim_{t \to \infty} \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} e^{\frac{q}{l} \frac{1}{q^2}} dq = 1, \]
and
\[ \lim_{t \to \infty} \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} e^{\frac{q}{l} \frac{1}{q^2}} dq = \begin{cases} \frac{i v_j}{l(k - k_j)}, & k \neq k_j, \\ \frac{1}{\infty}, & k = k_j. \end{cases} \] (2.15)

Furthermore,
\[ \lim_{t \to \infty} \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \mathcal{W}_1\left(\frac{q}{l} - ik\right) e^{\frac{q}{l} \frac{1}{q^2}} dq = \mathcal{W}_1(-ik), \] (2.16)
as follows from assumption (1.14) and the Lebesgue’s dominated convergence theorem if one passes to the limit \( t \to \infty \) under the sign of the integral (2.16). Let us check that this \( v \) solves equation (1.3). This would conclude the proof of Theorem 1.4. We need a lemma.

**Lemma 1.** If \( h \in L_{loc}(0, \infty) \) and the limit \( \lim_{t \to \infty} t^{-1} \int_0^t h(s)ds \) exists, then the limit \( \lim_{p \to 0} p \int_0^\infty e^{-pt} h(t)dt \) exists, and
\[ \lim_{t \to \infty} t^{-1} \int_0^t h(s)ds = \lim_{p \to 0} p \int_0^\infty e^{-pt} h(t)dt. \] (2.17)

**Proof of Lemma 1.** One has
\[ p \int_0^\infty e^{-pt} h(t)dt = pe^{-pt} \int_0^t h(s)ds|_0^\infty + p^2 \int_0^\infty te^{-pt} t^{-1} \int_0^t h(s)ds dt. \]
For any \( p > 0 \) one has
\[ pe^{-pt} \int_0^t h(s)ds|_0^\infty = 0. \]
Let \( q = pt \) and denote \( H(t) := t^{-1} \int_0^t h(s)ds, J := \lim_{t \to \infty} H(t) \). Then
\[ \lim_{p \to 0} p^2 \int_0^\infty te^{-pt} t^{-1} \int_0^t h(s)ds dt = \lim_{p \to 0} \int_0^\infty qe^{-q} H(q(pq)^{-1}) dq. \]
Passing in the last integral to the limit \( p \to 0 \) one obtains (2.17). Lemma 1 is proved.

Using equation (2.17), one writes \( v = \lim_{p \to 0} pU(p - ik) \), where \( U \) solves equation (1.6). Thus,
\[ LU(p - ik) + (p - ik)^2 U(p - ik) = p^{-1} f. \]
Multiplying both sides of this equation by \( p \) and passing to the limit \( p \to 0 \), one obtains equation (1.8).

In the passage to the limit under the sign of the unbounded operator \( L \) the assumption that \( L \) is closed was used.

Thus, the conclusion of Theorem 1.4 follows. \( \square \)

If the limit (1.14) exists at a point \( p = i\tau \) then one says that the limiting absorption principle holds for the operator \( L \) at the point \( k = -p = i(-ik) = k, k > 0 \).

Thus, Assumption B means that the limiting absorption principle holds for \( L \) at the point \( k > 0 \), that is, \( \lim_{\varepsilon \to 0} (L - k^2 - i\varepsilon)^{-1} f \) exists.
3. Applications

Let $L = -\nabla^2 + q(x)$, where $q(x)$ is a real-valued function, $|q(x)| \leq c(1 + |x|)^{-2-\epsilon}, \epsilon > 0, x \in \mathbb{R}^3$. Then $L$ is selfadjoint on the domain $H^2(\mathbb{R}^3)$. Its resolvent $(L - k^2 - i0)^{-1}$ satisfies Assumptions A and B if one keeps in mind the following.

Let $G(x, y, k)$ be the resolvent kernel of $L$, that is, the kernel of the operator $(L - k^2 - i0)^{-1}$,

$$LG(x, y, k) = -\delta(x - y) \text{ in } \mathbb{R}^3,$$

$G \in L^2(\mathbb{R}^3)$ for $k > 0$. If $f \in L^2(\mathbb{R}^3)$ is compactly supported, then for $k > 0$ the function

$$v(x) := (L - k^2 - i0)^{-1} f = \int_{\mathbb{R}^3} G(x, y, k) f(y) dy$$

does not necessarily belong to $L^2(\mathbb{R}^3)$.

For example, if $q(x) = 0$, then $G(x, y, k) = \frac{e^{ik|x-y|}}{4\pi|x-y|^2}$, and the function

$$v(x, k) = \int_{|y| \leq 1} g(x, y, k) dy = O \left( \frac{1}{|x|} \right) \quad (3.1)$$

does not belong to $L^2(\mathbb{R}^3)$ (except for those $k > 0$ for which $x(k) = 0$ in the region $|y| \geq 1$. These numbers $k > 0$ are the zeros of the Fourier transform of the characteristic function of the ball $|y| \leq 1$, see [1], Chapter 11.

By this reason the abstract results of theorem (1.3) and (1.4) can be used in applications if one defines some subspace of $H$, for example, a subspace of functions with compact support, denote by $P$, a projection operator on this subspace, and replaces $W$ and $W_1$ by $PW$ and $PW_1$ in equations (1.12) and (1.14).

For example, the function (3.1) one replaces by $\eta(x)v(x, k)$, where $\eta(x)$ is a characteristic function of a compact subset of $\mathbb{R}^3$.

The analytic properties of $\eta(x)v(x, k)$ and of $v(x, k)$ as functions of $k$ are the same. A similar suggestion is used in [1].

With the above in mind, one knows (for example, from [2] or [11]) that Assumptions A and B hold for $L = -\nabla^2 + q(x)$.

Consequently, the conclusions of Theorems 1.3 and 1.4 hold.

In addition, the assumptions

$$|q(x)| \leq c(1 + |x|)^{-2-\epsilon}, \epsilon > 0, \text{ Im } q = 0,$$

imply that $L$ does not have positive eigenvalues, so all $v_j = 0$, and zero is not an eigenvalue of $L \geq 0$ if $\epsilon > 0$ (see [3], [4]).

A new method for estimating of large time behavior of solutions to abstract evolution problems is developed in [4], where some applications of this method are given.

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