SHELLINGS FROM RELATIVE SHELLINGS, WITH AN APPLICATION TO NP-COMPLETENESS

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Abstract. Shellings of simplicial complexes have long been a useful tool in topological and algebraic combinatorics. Shellings of a complex expose a large amount of information in a helpful way, but are not easy to construct, often requiring deep information about the structure of the complex. It is natural to ask whether shellings may be efficiently found computationally. In a recent paper, Goaoc, Paták, Patáková, Tancer and Wagner gave a negative answer to this question (assuming P \( \neq \) NP), showing that the problem of deciding whether a simplicial complex is shellable is NP-complete.

In this paper, we give simplified constructions of various gadgets used in the NP-completeness proof of these authors. Using these gadgets combined with relative shellability and other ideas, we also exhibit a simpler proof of the NP-completeness of the shellability decision problem. Our method systematically uses relative shellings to build up large shellable complexes with desired properties.

1. Introduction

A *shelling* is a certain way of building up (or equivalently, tearing down) a simplicial complex, facet by facet. A precise definition may be found in Section 2. Shellings have found considerable application by combinatorialists and combinatorial algebraists. In the 1970s, work of Hochster, Reisner, and Stanley showed how to use shellings [18, 27, 30] to prove that certain rings are Cohen-Macaulay; this is still a useful tool [17, 24, 34]. The existence of a shelling makes computing homotopy type easy, and the topology (up to homeomorphism) tractable in many cases [5, 9, 36]. In certain cases, the existence of a shelling is equivalent to the existence of other interesting structure: for example, the order complex of a finite group is shellable if and only if the group in question is solvable [28].

In a pair of papers [11, 12] from the 1970s, Danaraj and Klee consider the decision problem SHELLABILITY, that is, the problem of determining whether a simplicial complex is shellable. They showed that SHELLABILITY is in P when restricted to 2-dimensional pseudomanifolds, and suggested that the general problem might be NP-complete. See also [19]. This problem sat open for 40 years, until Goaoc, Paták, Patáková, Tancer and Wagner, in a significant recent advance [13, 14], verified the problem to be NP-complete:

**Theorem 1.1** (Goaoc, Paták, Patáková, Tancer, and Wagner [14, Theorem 1]). SHELLABILITY is NP-complete, even when restricted to 2-dimensional simplicial complexes.

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The proof is by polynomial reduction from 3SAT. As is typical in such a reduction, the construction in [14] proceeds by building “choice gadgets” (corresponding to variables in a 3SAT instance), “constraint gadgets” (corresponding to clauses), and some other needed gadgets for consistency. As an essential building block, these authors require simplicial complexes that are shellable, but where the shelling order is ‘rigid’ in a certain precise sense.

One such building block is a shellable complex with a single free face \( \tau \); in such a complex, every shelling order must end with the facet containing \( \tau \). The authors of [14] use a construction based on Bing’s house with 2 rooms. This construction is originally due to Malgouyres and Francés in [23]. As observed in [14, Remark 9], there is also a somewhat more concrete construction due to Hachimori [15]. We mention that a construction with similar properties was earlier considered by Simon [29, Appendix F], and that Hachimori’s construction was extended to arbitrary dimensions by Adiprasito, Benedetti and Lutz [2, Theorem 2.3].

Another building block in [14] is a shellable complex with 3 nonadjacent free faces, and satisfying some other conditions needed for gluing it to a larger construction. The authors in [14] roughly sketch a construction based on Bing’s house for the latter building block, but do not provide all details. They instead give a reference to another paper of Tancer [33, Section 4], which provides more details of the construction. The resulting simplicial complex is fairly large and complicated.

The first result on this paper will be to give a simple and explicit construction of shellable complexes with an arbitrary number of nonadjacent free faces.

**Theorem 1.2.** For any positive integer \( n \), there is a simplicial complex \( T^{(n)} \) and subcomplex \( \Upsilon \) such that:

1. \( T^{(n)} \) is a shellable and contractible 2-dimensional complex, having exactly \( n \) free edges and no other free faces. Denote by \( F \) the set of free edges.
2. \( \Upsilon \cup \langle F \rangle \) is a shellable and contractible 1-dimensional complex (that is, a tree), having exactly \( n \) leaves; and so that for each leaf vertex, the unique edge containing it is in \( F \).
3. For any proper subset \( \mathcal{E} \subsetneq F \) of free edges, the relative complex \( (T^{(n)}, \Upsilon \cup \langle \mathcal{E} \rangle) \) is shellable.

An easy homology calculation gives (in the notation of the theorem) that \( (T^{(n)}, \Upsilon \cup \langle F \rangle) \) is not shellable, so the word “proper” in Theorem 1.2 (3) cannot be removed. We remark also that Theorem 1.2 essentially restates in terms of relative shellability (and in a more general context) the conditions required by [14].

Our second result will be an improved construction for and proof of Theorem 1.1. Our construction improves on that of [14] in several ways. We significantly simplify the needed choice gadgets, eliminating a consistency gadget needed in the earlier paper, and markedly reducing the size and complexity. See Remark 4.1. Of course, we also use the improved building blocks of Theorem 1.2.

Our proof of Theorem 1.1 is also somewhat differently structured from that of [14]. The authors of this paper give all proofs in terms of collapsibility, rather than shellability (using
a result of Hachimori [16, Theorem 8]). We phrase our proof in the language of shellability and relative shellability, which we believe some readers may prefer.

The main innovation that we introduce is the systematic use of relative shellings to build up large shellable complexes with desired properties. We use this approach in both the proof of Theorem 1.2 as well as in our new proof of Theorem 1.1. We believe that our techniques may find application to other problems.

We mention that Danaraj and Klee asked a more specific question than that answered by [14]: is SHELLABILITY NP-complete when restricted to \(d\)-pseudomanifolds (for some \(d > 2\))? We don’t know the answer to this, but believe that relative shellability ideas similar to those we use here may be helpful in addressing the problem. A similar question that might be interesting to consider: does SHELLABILITY remain NP-complete when restricted to complexes with an embedding in \(\mathbb{R}^3\) \(\mathbb{R}^4\)? Other spaces?

Although SHELLABILITY of a 2-dimensional complex is NP-complete, the problem restricted to a 2-dimensional ball or sphere is trivial, and in this situation a shelling can be computed in linear time [11]. Is SHELLABILITY for a 3-dimensional ball or sphere in P? We remark that one quite general construction of nonshellable 3-balls uses nontrivial knots as an essential ingredient (see e.g. [4, Section XIV.6]). As the KNOTTEDNESS problem has been shown to be in \(\text{coNP}\) [20, 21], it seems plausible that the restriction of SHELLABILITY to 3-balls is also in \(\text{coNP}\).

The paper is organized as follows. In Section 2 we give general background on shellability and relative shellability. We believe that some of the lemmas on relative shellability in this section may be of broader use. In Section 3 we construct the complexes as in Theorem 1.2. In Section 4 we use these complexes and other ideas to give our new proof of Theorem 1.1.

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2. Background

As usual, a simplicial complex \(\Delta\) is a family of sets (called faces) that is closed under inclusion. The simplicial complex generated by a family of sets \(\mathcal{E}\) consists of all subsets of sets in \(\mathcal{E}\), and is denoted \(\langle\mathcal{E}\rangle\). The \(f\)-vector of a simplicial complex \(\Delta\) is \((f_0, f_1, \ldots, f_c)\) where \(f_i\) is the number of faces of \(\Delta\) with \(i\) vertices, and the \(h\)-vector of \(\Delta\) is determined by \(\sum h_i x^{c-i} = \sum f_i (x-1)^{c-i}\). We denote as \(f(\Delta)\) and \(h(\Delta)\), respectively. Note that some authors index the \(f\)-vector by dimension, rather than cardinality.

A free face in \(\Delta\) is a face \(\tau \neq \emptyset\) which is properly contained in exactly one facet (maximal face). It is easy to show that if \(\tau\) is a free face, then \(\Delta\) deformation retracts to the complex \(\Delta \setminus \tau\) obtained by removing from \(\Delta\) all faces containing \(\tau\).

A simplicial complex \(\Delta\) is shellable if there is an ordering \(\sigma_1, \sigma_2, \ldots, \sigma_m\) (a shelling) of the facets such that each \(\sigma_j\) with \(j > 1\) intersects with the complex \(\Delta_{j-1}\) generated by
\sigma_1, \ldots, \sigma_{j-1} \text{ in a pure } (\dim \sigma_j - 1 \text{ complex. A useful equivalent condition is as follows:}
\begin{equation}
\forall j \forall i < j \exists k < j \text{ such that } \sigma_i \cap \sigma_j \subseteq \sigma_k \cap \sigma_j = \sigma_j \setminus \{x\}, \text{ some } x \in \sigma_j.
\end{equation}

Each facet \sigma_j in a shelling contains a minimal “new” face, given by \{x \in \sigma_j : \sigma_j \setminus \{x\} \subseteq \sigma_k, \text{ some } k < j\}. If the minimal new face is \sigma_j itself, then we say \sigma_j is a homology facet, otherwise, \sigma_j contains a face that is free in some earlier \Delta_k.

A relative simplicial complex is a pair \((\Delta, \Gamma)\) of simplicial complexes, where \Gamma is a subcomplex of \Delta. The faces of \((\Delta, \Gamma)\) are the faces of \Delta that are not faces of \Gamma. It may be helpful to recall from algebraic topology that there is a homology theory for relative complexes, and that \(\tilde{H}_i(\Delta, \Gamma)\) is isomorphic to the homology \(\tilde{H}_i(\Delta/\Gamma)\) of the quotient of \Delta by \Gamma.

A relative simplicial complex \(\Psi = (\Delta, \Gamma)\) is (relatively) shellable if there is an ordering \sigma_1, \sigma_2, \ldots, \sigma_m (a shelling) of the facets of \Psi such that each \sigma_j contains a unique minimal “new” face [31, Chapter III.7]. That is, the set of subsets of each \sigma_j that are not in a preceding \sigma_i or in \Gamma has a unique minimal element under inclusion. Previous work on relative shellability includes [1, 3, 10, 32, 37].

We will sometimes refer to shellability of a simplicial complex as absolute shellability, in order to contrast with shellability of a relative complex. We see that absolute shellability is the special case of relative shellability where \Gamma = \emptyset.

We find it more convenient to work with a formulation of relative shellability that is closer to the standard definition for the absolute case. The proof is essentially the same as in the absolute case [71 Section 2].

**Lemma 2.1.** Let \(\Gamma \neq \emptyset\). An ordering \(\sigma_1, \ldots, \sigma_m\) of the facets of a relative simplicial complex \(\Psi = (\Delta, \Gamma)\) is a shelling if and only if each \sigma_j intersects in a pure \((\dim \sigma_j - 1\) complex with the complex \(\Delta_{j-1}\) generated by \(\sigma_1, \ldots, \sigma_{j-1}\) together with \(\Gamma\).

**Proof.** If \(\Psi\) satisfies the condition, then any face of \sigma_j either contains \(\gamma = \{x \in \sigma_j : x \in \Delta_{j-1}\}\), or else is a face of \(\Delta_{j-1}\). Thus, \(\gamma\) is the required minimal new face. Conversely, if \(\Psi\) is shellable with minimal new face \(\gamma\), then the intersection with \(\Delta_{j-1}\) is generated by the faces \(\sigma_j \setminus v\) over \(v \in \gamma\). It follows that the intersection is pure of codimension one, as required.

Now the immediate analogue of (2.1) is as follows. Let \(\Delta_{j-1}\) be as in the statement of Lemma 2.1. Then an ordering \(\sigma_1, \ldots, \sigma_m\) is a relative shelling if and only if the following holds:
\begin{equation}
\forall j \forall \tau \in \Delta_{j-1} \exists \tau_* \in \Delta_{j-1} \text{ such that } \tau \cap \sigma_j \subseteq \tau_* \cap \sigma_j = \sigma_j \setminus \{x\}, \text{ some } x \in \sigma_j.
\end{equation}

We notice that we may restrict \(\tau_*\) to be a facet of \(\Delta_{j-1}\) in (2.2) without loss of generality, indeed, \(\tau_*\) may be selected to be either one of \(\sigma_1, \ldots, \sigma_{j-1}\) or a facet of \(\Gamma\). Since facets of \(\Gamma\) are not necessarily facets of \(\Delta\), however, we cannot necessarily select \(\tau_*\) to be a facet of \(\Delta\).

**Remark 2.2.** The condition of (2.2) and Lemma 2.1 are exactly the same as those for extending a partial shelling to a full shelling of a simplicial complex. Thus, we may think of relative shellability as giving a setting where we may pretend we have already shelled \(\Gamma\).
relative simplicial complexes, where Lemma 2.3 following lemma. (absolutely) shellable simplicial complexes. The key observation for this endeavor is the facets (subject to the usual facet-attachment conditions). Whether that is possible or not, and must continue the shelling order for the remaining ∆ supercomplex. If ∆ contains at least 1 edge of ∆, we see that the two-triangle complex ∆ is shellable relative to any tree having at least 1 edge. For without loss of generality, we may assume that ∆ contains at least 1 edge of Γ, and that ∆ has at most 1 edge of Γ not in ∆. Now Γ = (∆ ∩ ∆) ′ ∪ (Γ ∩ ∆) is a tree in ∆. Taking Γ = Δ ∩ Γ, we see that the conditions of the lemma are met.

It is worthwhile to observe explicitly that, since absolute shellings form a special case of relative shellings, Lemma 2.3 may be used to combine an absolute shelling and a relative shelling into an absolute shelling.

In order to apply Lemma 2.3, we will need various relative shellings. It will often be more convenient for us to find absolute shellings, and apply the following lemma.

Lemma 2.5. Let ∆ be a pure d-dimensional simplicial complex with shelling order σ₁, . . . , σₘ. Furthermore, let Γ be a pure (d − 1)-dimensional subcomplex of ∆ such that for any facet τ of Γ and any σₖ in the shelling order, one of the following holds:

1. τ ∩ σₖ ⊆ τ′ ∩ σₖ for some facet τ′ of Γ, or
2. τ ∩ σₖ ⊆ σᵢ ∩ σₖ for some i < j, or
3. τ ⊆ σₖ.

Then σ₁, . . . , σₘ is a shelling of the relative complex (∆, Γ).

Proof. Let τ be a facet of the complex ∆ generated by σ₁, . . . , σₖ together with Γ. It suffices to show that there is another facet τ′ of ∆ such that τ′ ∩ σₖ is a (d − 1)-face. If τ ∩ σₖ ⊆ σᵢ for i < j, then this follows from the definition of shelling. Otherwise, τ is a facet of Γ for which [2] does not hold, and the desired is immediate by (1) or (3).

Remark 2.6. In simple language, the condition of Lemma 2.5 requires that every maximal intersection of σₖ with Γ is either a facet of Γ (so (d − 1)-dimensional), or else contained in some earlier facet in the shelling.
**Example 2.7.** A shedding vertex \(v\) of \(\Delta\) has the property that if \(v\) is in a face \(\sigma\), then there is some other vertex \(w\) so that \((\sigma \setminus v) \cup w\) is a face. It is well-known that if \(v\) is a shedding vertex such that \(\Delta \setminus v\) and \(\text{link}_\Delta v\) are both shellable, then also \(\Delta\) is shellable \([8, 35]\). This fact for pure complexes follows also from Lemma 2.3 where we take \(\Delta_a = \Delta \setminus v\), \(\Delta_b = v \ast \text{link}_\Delta v\), and \(\Gamma_a = \Gamma_b = \emptyset\). Now \((v \ast \text{link}_\Delta v, \text{link}_\Delta v)\) is shellable by Lemma 2.5 and the shelling order on \(v \ast \text{link}_\Delta v\), where we take \(\tau' = \sigma_j \setminus v\) whenever \(\Box\) fails. We recover that \(\Delta\) is shellable relative to \(\Gamma_a \cup \Gamma_b = \emptyset\), i.e., \(\Delta\) is (absolutely) shellable.

A similar argument applies to a shedding face \(\gamma\), setting \(\Delta_a = \Delta \setminus \gamma\) and \(\Delta_b = \gamma \ast \text{link}_\Delta \gamma\), so that \(\Delta_a \cap \Delta_b = (\gamma \ast \text{link}_\Delta \gamma) \setminus \gamma\). As we will not use this result, we omit the details.

We use the following observation on gluing simplicial complexes freely and without explicit reference, but state it here for clarity. The proof is immediate from definitions.

**Lemma 2.8.** Let \(\Delta_a\) and \(\Delta_b\) be simplicial complexes on disjoint vertex sets \(V(\Delta_a)\) and \(V(\Delta_b)\). Let \(\{v_1, \ldots, v_k\} \subseteq V(\Delta_a)\) and \(\{w_1, \ldots, w_k\} \subseteq V(\Delta_b)\), and let \(\Gamma_a\) and \(\Gamma_b\) respectively be the subcomplexes induced by these vertex subsets. By identifying each \(v_i\) with \(w_i\), we form a simplicial complex \(\Sigma\), where topologically \(\Sigma\) is formed by gluing \(\Delta_a\) and \(\Delta_b\) along \(\Gamma_a \cap \Gamma_b\).

We refer to \([3]\) for additional background and definitions on simplicial combinatorics.

### 3. Turbines and Blades

In this section, we construct the complexes \(T^{(n)}\) of Theorem 1.2. We call the complex \(T^{(n)}\) the \(n\)-turbine, for reasons that will be apparent from Figure 3.3.

**3.1. The 1-turbine.** First, Hachimori’s example (pictured in Figure 3.1) will be \(T^{(1)}\). Hachimori verified his example to be shellable in his thesis \([15]\), and it is clear from inspection that it has a single free face. It is immediate from Lemma 2.5 that if \(\Upsilon\) is generated by any edge of the initial facet in a shelling, then \((T^{(1)}, \Upsilon)\) is relatively shellable. It is straightforward to find a shelling that begins with a facet intersecting with the free edge at a vertex; we have shown one such in Figure 3.1 relative to the subcomplex \(\Upsilon = \langle sw \rangle\). This proves Theorem 1.2 for the case \(n = 1\). As previously mentioned, this was already substantively observed in \([14]\).

**Remark 3.1.** The underlying topological space of Hachimori’s example is a main ingredient of our constructions. Since Hachimori based his construction on the dunce cap space, we propose the tricorne cap as a name for this space.

**3.2. Construction of \(T^{(n)}\).** We will construct \(T^{(n)}\) for higher \(n\) by gluing together several copies of the tricorne cap space. We will need a triangulation of \(B\) having 3 adjacent free edges, shown in Figure 3.2. As our \(n\)-turbines will comprise \(n\) copies of \(B\) glued around a central “shaft”, we call \(B\) the blade complex.

Having constructed the blade, we now construct the \(n\)-turbine \(T^{(n)}\) for \(n \geq 3\). We begin with an \(n\)-cycle, having vertices \(y_1, y_2, \ldots, y_n\). Subdivide each edge of the cycle by adding a vertex \(a_i\) between \(y_i\) and \(y_{i+1}\) (index considered mod \(n\)). Cone over the subdivided \(n\)-gon to get a 2-dimensional disc. Finally, for every subdivided edge \(y_i, a_i, y_{i+1}\), glue a copy \(B_i\) of \(B\)
Figure 3.1. The tricorne cap space, with a slight variation of the triangulation of Hachimori. The arrows indicate a shelling order that begins with the facet labeled $\mathbf{1}$ and ends with that labeled $\bullet$. The dashed edge is (a possible choice for) $\Upsilon$.

Figure 3.2. The blade complex $B$, pictured with two shelling orders. Each shelling order begins with the facet labeled $\mathbf{1}$ and ends with that labeled $\bullet$ in the respective diagram. See also Corollary 3.4.

We also need to construct the tree subcomplex $\Upsilon$. Denote by $w$ the apex vertex of the cone over the subdivided $n$-gon in $T^{(n)}$. For each blade $B_i$ let $x_i$ be the vertex of a free edge that is not glued to the central disc. The complex $\Upsilon$ will be generated by all edges of the form $a_iw$ together with those of the form $a_ix_i$; it is pictured with bold dark edges in Figure 3.3.

Remark 3.2. We found the triangulation $B$ of the tricorne cap space by first subdividing the free edge in $T^{(1)}$, and then applying cross-flips and bistellar reductions in the sense of Pachner [25]. See also the systematic application of bistellar reductions developed by Lutz in his thesis [22], and applied by Björner and Lutz in [6]. The authors find it interesting that
Figure 3.3. The $n$-turbine space $T^{(n)}$ and its component blades $B_i$. For clarity, the blades are shown schematically (with the details of the triangulation of the blades omitted).

there is a triangulation of the tricorne cap with 3 free edges and only 6 vertices, while they have been unable to find a triangulation with single free edge and fewer than 7 vertices.

The case $n = 2$ will require a slight variation of our main construction. The complex obtained by gluing two copies of $B$ to the cone over a 4-gon in the above manner is not simplicial, since we do not wish to identify the edges $y_1y_2$ that are in each copy of $B$. To fix this, we subdivide the $y_1y_2$ edges in both copies. All other details for $n = 2$ proceed in exactly the same way as for higher $n$. We picture $T^{(2)}$ in Figure 3.4 where we take $\Upsilon$ to be the path formed by the bolded dark edges.

3.3. **Proof of Theorem 1.2 for $n = 2$.** Although it is not difficult to verify that the order in Figure 3.4 that begins with facet 1 is a shelling, we prefer to break the complex down using relative shellability. Our strategy is to break $T^{(2)}$ into several simpler subcomplexes, relatively shell each of them, and use Lemma 2.3 to glue the relative shellings together.

First, the two facets beginning with 1 clearly form a shelling, and the order is also a shelling relative to $\Upsilon$. Second, the (modified) blade $B_1$ is shellable with the indicated order (beginning with 2, indicated with the solid arrow). It follows from Lemma 2.5 that $B_1$ is shellable relative to the following subcomplexes: $\langle a_1y_1 \rangle$, $\langle a_1y_1 \rangle \cup \Upsilon$, and $\langle a_1y_1, x_1 y_2 \rangle \cup \Upsilon$. By Lemma 2.3, the union of the initial two facets and $B_1$ are shellable, and also shellable relative to $\Upsilon$ and $\langle a_1y_1, x_1 y_2 \rangle \cup \Upsilon$. In a similar way, the two facets beginning with 3 are shellable relative to $\langle a_1y_2 \rangle$ or $\langle a_1y_2 \rangle \cup \Upsilon$. Thus, we can use Lemma 2.3 to verify that the concatenation of the first two facets, the shelling of $B_1$, and that of 3 and its successor is a shelling. Finally, the pictured order of $B_2$ beginning with 4 is a shelling (indicated with the hollow arrow). By Lemma 2.5, this is also a shelling relative to $\langle a_2y_2 \rangle$ or $\langle a_2y_2 \rangle \cup \Upsilon$. Another application of Lemma 2.3 gives that the pictured order on $T^{(2)}$ is a shelling, and
also a shelling relative to \( \Upsilon \) and \( \Upsilon \cup \langle x_1 y_2 \rangle \). Since the shelling of \( T(2) \) has no homology facet, the complex is contractible. The \( n = 2 \) case of Theorem 1.2 follows from these (relative) shellings and symmetry of the complex.

### 3.4. Proof of Theorem 1.2 for \( n \geq 3 \)

Our proof of Theorem 1.2 for \( n \geq 3 \) will be entirely similar to the \( n = 2 \) case. We first need two shellings of \( B \), one for the final copy of \( B \) that is built up in the shelling order, and one for all other copies of \( B \). By inspection, we observe:

**Proposition 3.3.** The facet orders pictured in Figure 3.2 are shellings of \( B \).

**Lemma 2.5** lets us easily move to desired relative shellings:

**Corollary 3.4.** In the following list of facets orders and relative simplicial complexes, each order is a shelling of the associated relative complexes.

1. The left-pictured facet order of Figure 3.2 for the relative complexes \((B, \langle a_n y_n, a_n y_1 \rangle)\) and \((B, \langle a_n y_n, a_n y_1, a_n x_n \rangle)\).

2. The right-pictured facet order of Figure 3.2 for the relative complexes \((B, \langle a_i y_i \rangle)\), \((B, \langle a_i y_i, a_i x_i \rangle)\), and \((B, \langle a_i y_i, a_i x_i, x_i y_{i+1} \rangle)\).

The subcomplexes of Corollary 3.4 are pictured in bold and/or dashed edges in Figure 3.2.

For each \( i \), let \( \alpha_i \) and \( \beta_i \) be respectively the facets \( a_{i-1} y_i w \) and \( a_i y_i w \) of \( T(\alpha) \), and let \( \Sigma_i \) be the complex spanned by \( \alpha_i \) and \( \beta_i \). We begin the shelling with \( \alpha_1, \beta_1 \). By Lemma 2.3, we can continue with \( B_1 \) in the ordering given by the first relative shellings of Corollary 3.4 (2). Now
by applying the simple argument from Example 2.4, we can follow that with the shelling \( \alpha_2, \beta_2 \) of \( \Sigma_2 \) (relative to \( \langle a_1y_2 \rangle \)).

Continue inductively in this manner, alternately adding \( B_i \), followed by \( \Sigma_i+1 \), using Lemma 2.3 to glue the shellings at each step. After \( n-1 \) such steps, we have all of \( T(n) \) except for \( B_n \).

Now as \( B_n \) intersects the central cone in the two edges \( a_ny_n \) and \( a_ny_1 \), we use the relative shelling of Corollary 3.4 (1) with Lemma 2.3 to complete the absolute shelling of \( T(n) \). Since this shelling has no homology facets, the complex is contractible.

The relative statement follows with exactly the same proof, but using the 2nd or 3rd relative complex in each application of Corollary 3.4 on \( B_i \), and relative to \( a_iw \) in each \( \Sigma_i \). Part (3) follows by this relative shelling, together with rotational symmetry of \( T(n) \).

3.5. Additional remarks on turbines. It is straightforward to count the faces of \( B \). The \( f \)-vector and \( h \)-vector are

\[
\begin{align*}
    f(B) &= (1, 6, 14, 9), & h(B) &= (1, 3, 5, 0).
\end{align*}
\]

Using this calculation, together with direct counting for \( T^{(1)} \) and \( T^{(2)} \), we see that

\[
\begin{align*}
    f(T^{(1)}) &= (1, 7, 19, 13), & h(T^{(1)}) &= (1, 4, 8, 0),
    f(T^{(2)}) &= (1, 13, 38, 26), & h(T^{(2)}) &= (1, 10, 15, 0),
    f(T^{(n)}) &= (1, 5n + 1, 16n, 11n), & h(T^{(n)}) &= (1, 5n - 2, 6n + 1, 0), \text{ for } n \geq 3.
\end{align*}
\]

We have several remarks about variations of our construction. First, if we were only interested in a shelling of \( T^{(n)} \), and not also in the more delicate relative shellability property of Theorem 1.2 (3), then we could somewhat simplify the construction. Indeed, we could simplify the blade construction to have only 2 free edges, and glue a single free edge of each blade to the each facet of a cone over an \( n \)-gon. A similar argument to that in Section 3.4 gives shellability. Relative shellability, on the other hand, seems to require each blade to have two facets adjacent to the central \( 2n \)-gon: an “in” face and an “out” face.

We also remark that higher-dimensional analogues of our construction are possible, at least in some special cases. Adiprasito, Benedetti and Lutz [2, Section 2] have generalized Hachimori’s construction to arbitrary dimension \( d \geq 2 \). The \( d \)-dimensional blade analogue will be formed from their example, by subdividing the free face into 3 free faces. There are some technical difficulties in forming the central “shaft” portion of a \( d \)-dimensional analogue of a turbine, as the cyclic symmetry of the \( n \)-gon does not cleanly generalize to higher dimensions. Special cases are easy to construct, such as an analogue of \( T^{(2)} \).

The restriction of Theorem 1.1 to any dimension higher than 2 follows immediately by observing that coning preserves shellability, so we do not need complexes with such properties for NP-completeness. As we at present have no other application for such complexes, we do not here pursue further higher-dimensional analogues of Theorem 1.2.
4. NP-completeness of SHELLABILITY

In this section, we assume general familiarity with the theory of NP-completeness and polynomial reductions, on the level of [26].

The decision problem SHELLABILITY asks, given a list of facets of an abstract simplicial complex $\Delta$, whether there is a shelling of $\Delta$. Given an ordering of the facets, checking the condition (2.1) can certainly be done in polynomial time, so SHELLABILITY is in NP. It is well-known that the restriction of the 3SAT problem where every literal occurs at most twice is NP-complete [26, Proposition 9.3], and we will give a polynomial reduction from this restricted 3SAT to SHELLABILITY. As our reduction will involve only 2-dimensional facets, this will prove Theorem 1.1. Our reduction will be simpler in several aspects than that of [14], and our proofs will be phrased directly in terms of shellings (rather than in terms of collapsings).

We begin with an overview of our reduction. It is usual to divide NP-hardness proofs into building blocks called gadgets. We will have a choice gadget, corresponding to a variable, which will consist of a triangulated sphere with a $T^{(1)}$ glued along its free face to a portion of the equator. The choice gadgets are glued together by identifying an edge in the $T^{(1)}$'s, so that the triangulated spheres intersect at a single vertex. We will also have a constraint gadget, corresponding to a clause, which will be a $T^{(2)}$ or $T^{(3)}$ (according to the number of literals in the clause). These constraint gadgets are glued to the choice gadgets of the corresponding variables by gluing the central vertex of the tree $\Upsilon$ to the common vertex of the triangulated spheres in the choice gadgets, by wrapping each branch of $\Upsilon$ around a choice gadget’s equator, and by gluing the free face to the upper or lower hemisphere (depending on whether the literal in question is negated). Precise details are in Section 4.2.

An assignment of variables will correspond with a selection of upper/lower hemispheres, one from each choice gadget. We will show that such an assignment is satisfying if and only there is a shelling that has homology facets exactly in the selected hemispheres.

4.1. Gadgets. Our choice gadget will consist of three parts. The first part will be a $T^{(1)}$. The second part will be a 2-dimensional disc $D$ having a boundary vertex $x$ that is incident to at least 2 interior vertices $y$ and $y'$. Such a $D$ may be obtained by subdividing a triangle with vertices $x, w, a$ to get a new interior vertex $y$, then subdividing the edge $ay$ to get a new interior vertex $y'$. See Figure 4.1.

The third part will be an isomorphic copy of $D$, which we label $\neg D$, and in which we label the interior vertices $\neg y$ and $\neg y'$.

We glue the discs $D$ and $\neg D$ along their boundaries, and glue the free edge of the $T^{(1)}$ to the edge $wx$. Here $x$ is as above, and $w$ is a vertex that is adjacent to $x$ in the common boundary of $D$ and $\neg D$. The gluing edge of the choice gadget will be the edge $sw$ from Figure 3.1 in its $T^{(1)}$ complex. The discs $D$ and $\neg D$ we call the (positive and negative) literal hemispheres.

In the variation of 3SAT where each literal occurs at most twice, we must consider clauses with either two or three literals. For clauses with two literals, our constraint gadget will
Figure 4.1. The literal hemisphere (left), and the choice gadget (shown at right in 3D, with the $T^{(1)}$ complex represented schematically). The choice gadgets are glued together along the bold edge $w$s. The remaining bold edges are used to attach free edges from constraint gadgets.

be a $T^{(2)}$; for those with three literals, it will be a $T^{(3)}$. We will need the tree subcomplex $\Upsilon \cup \langle F \rangle$ from Theorem 1.2. We call this subcomplex the *gluing tree* of the constraint gadget. As in Section 3.2, we may take the gluing tree to consist of two or three branches $w, a_i, x_i, y_i$, where each $x_iy_i$ is a free edge in the turbine. (Here $\tilde{y}_i$ corresponds to $y_{i+1} \mod n$ in Figures 3.3 and 3.4; $a_i$ and $x_i$ are as in the figures.)

Remark 4.1. Our choice gadgets consist of three parts, fitting together in a simple way. For comparison, the choice gadgets of [15] have six parts, and there are are some subtleties in how these parts are attached. We also completely avoid the use of their consistency (‘conjunction’) gadget.

4.2. Reduction. The details of the reduction are now straightforward.

We are given a list of clauses, each with 2 or 3 variables. Without loss of generality, each variable appears at most once in each clause.

For each variable appearing in the list, we take a choice gadget. We glue all of these gadgets together by identifying their gluing edges to a single edge $sw$. Thus, the edge $sw$ is shared by all of the choice gadgets, and the vertex $w$ is in every literal hemisphere.

Now for each clause, we attach a $T^{(2)}$ or $T^{(3)}$ constraint gadget, according to whether the clause has 2 or 3 literals. Thus, each literal $\ell$ of the clause corresponds to a branch of the gluing tree. The literal also corresponds to a literal hemisphere $D$ or $\neg D$ in a choice gadget. Let $y^*$ be the vertex $y, y', \neg y, or \neg y'$ of this literal hemisphere, where we decorate $y$ with $\neg$ if $\ell$ is negated and with a prime if this is the second clause containing $\ell$. (Recall that each literal occurs in at most two clauses.) Now we glue the corresponding branch $w, a_i, x_i, \tilde{y}_i$ of the gluing tree to the vertices and edges $w, a, x, y^*$ of the literal hemisphere. That is, we glue
$w$, $a_i$, $x_i$, $\bar{y}_i$ to the choice gadget along a path that begins by wrapping around a portion of the equator (common to both literal hemispheres), and whose final vertex is in the interior of a literal hemisphere.

We remark that, since a given variable appears in a given clause at most once, the vertices of the gluing tree attach to distinct vertices in the union of choice gadgets.

Although we have described this reduction in terms of gluing, the process admits a clear translation into facets, sets, and abstract simplicial complexes. It is straightforwardly implemented in polynomial time.

We denote by $\Delta$ the complex obtained by the polynomial reduction. It is easy to calculate the homotopy type of $\Delta$:

**Lemma 4.2.** The complex $\Delta$ obtained by the polynomial reduction is homotopy equivalent to a bouquet of 2-spheres, where the 2-spheres are in bijective correspondence with the variables in the 3SAT instance.

*Proof.* The union $\Delta_{\text{lit}}$ of the literal hemispheres is exactly a bouquet of simplicial spheres. The complex $\Delta$ can be obtained from $\Delta_{\text{lit}}$ by repeatedly attaching copies of the contractible complexes $T^{(1)}$, $T^{(2)}$, and $T^{(3)}$ along a contractible (tree) subcomplex. The lemma now follows by well-known results on gluing and homotopy type [5, Lemma 10.3].

**4.3. Satisfiability implies shellability.** Suppose that our 3SAT instance has a satisfying assignment, which sets some literals to true and their negations to false. We will show the complex $\Delta$ constructed in Section 4.2 is shellable, using repeated applications of Lemma 2.3.

We will need the following (relative) shellings of the literal hemispheres:

**Lemma 4.3.** The literal hemisphere disc $D$ is shellable relative to the subcomplexes $(wx)$, $\partial D$, and $\partial D \cup (E)$, where $E$ is any subset of the edges $\{xy, xy'\}$.

*Proof.* All are immediate from Lemma 2.5 using any shelling order beginning with the facets $wxy, xyy'$. 

Given a satisfying assignment, we now build up the same complex $\Delta$ as in Section 4.2 but by gluing together subcomplexes in a specific order that is compatible with Lemma 2.3.

There are three main steps:

**Step 1:** ‘augmented’ false literal hemispheres. We begin with the portion of the choice gadget consisting of $T^{(1)} \cup D^*$, where $D^*$ is either the positive or negative literal hemisphere. By Lemmas 2.3 and 4.3 and the discussion in Section 3.1, these augmented literal hemispheres are both shellable and also shellable relative to $sw$. We take an augmented literal hemisphere for each false literal in our assignment, and glue all copies together along the gluing edge $sw$. By additional applications of Lemma 2.3, this complex is shellable.

**Step 2:** constraint gadgets. Next, for each clause in the 3SAT instance, we attach a constraint gadget along the portion of its gluing tree that is already present in the complex built so far. This portion consists of $\Upsilon$, which attaches to the equators of false literal hemispheres; together with the subset of the free edges $F$ corresponding to the false literals
in the clause, which attach to the interiors of the corresponding false literal hemispheres. (See the description in Section 4.2.) As every clause contains at least one true literal, at least one edge of \( F \) is not glued. Thus, by Lemma 2.3 and Theorem 1.2 shellability is preserved under attaching each constraint gadget.

**Step 3: true literal hemispheres.** Finally, we attach the true literal hemispheres, yielding the full complex \( \Delta \). By Lemma 4.3 and more applications of Lemma 2.3 this complex is shellable, as desired.

We notice that the edge of \( F \) in each constraint gadget that does not attach to an edge in the interior of a literal hemisphere in Step 2 is crucial for this construction. Indeed, it is straightforward to show that a turbine is not shellable relative to \( \Upsilon \cup \langle F \rangle \).

**Remark 4.4.** We observe that the complex built in this manner remains contractible through Step 2, but that attaching each true literal hemisphere in Step 3 creates a homology facet.

### 4.4. Shellability implies satisfiability.

Given a shelling of \( \Delta \), and a subcomplex \( \Gamma \) generated by facets, we say that \( \Gamma \) finishes shelling at \( \sigma \) if \( \sigma \) is the last facet of the shelling that is contained in \( \Gamma \). A subcomplex may finish shelling either at a homology facet of the shelling, or in a facet containing a free face in the partial shelling. In the latter case, the free face \( \tau \) obviously must be free in \( \Gamma \), and \( \Gamma \) must finish shelling before any other facet containing \( \tau \) occurs in the shelling.

Now consider the complex \( \Delta \) constructed by the polynomial reduction. The facets of \( \Delta \) consist of the disjoint union of the facets of choice gadgets and of constraint gadgets. Moreover, the facets of each choice gadget are the disjoint union of those of the \( T^{(1)} \), those of the positive literal hemisphere, and those of the negative literal hemisphere.

By Lemma 4.2, we have a homology facet for each variable. As removing the homology facets leaves a contractible complex, there must be exactly one homology facet in each choice gadget, contained in either the positive or negative literal hemisphere. For each variable, we set the hemisphere containing the homology facet to be the \textit{true} hemisphere, and the other one to be the \textit{false} hemisphere. (Thus, a selection of homology facets corresponds, up to equivalence, to a truth assignment of the variables.)

We now sort the subcomplexes formed by the \( T^{(1)} \)'s, the literal hemispheres, and the constraint gadgets according to when they finish shelling. By results of Björner and Wachs [7, Lemma 2.7], the homology facets of the complex may be taken without loss of generality to come last in any shelling, thus the true literal hemispheres may be assumed to finish shelling after all other considered subcomplexes. Since each \( T^{(1)} \) has a unique free face, it must finish shelling before its associated false literal hemisphere. Each false literal hemisphere has only three free faces, with one attached to a \( T^{(1)} \), and the other two attached to all of the constraint gadgets for clauses containing the literal. Thus, a false literal hemisphere must finish shelling before any constraint gadget for any clause containing the literal. Since each constraint gadget has only two or three free faces, each glued to an edge in a literal hemisphere, it must finish shelling before at least one of the literals in the corresponding clause.
Since each false literal finishes shelling before all the clauses containing it, and each clause finishes shelling before at least one literal contained in it, we must have that each clause contains a true literal.

Remark 4.5. The authors of [14] obtained as a consequence of their main result that checking vertex-decomposability is \( \mathsf{NP} \)-complete, and that SHELLABILITY is \( \mathsf{NP} \)-complete even when restricted to order complexes of posets. We sketch how to recover this result using our techniques, and our simpler construction. Since the barycentric subdivision of a shellable simplicial complex is a vertex-decomposable order complex [8, Section 11], it suffices to show that if any subdivision of the complex \( \Delta \) constructed by the polynomial reduction is shellable, then the underlying formula is satisfiable. But as a free face in a subdivision of a simplicial complex arises only by subdividing a free face in the original complex, an argument entirely similar to the above shows that shellability of a subdivision implies satisfiability.

Data availability. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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