Separable and entangled states
of composite quantum systems; Rigorous description.

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Abstract

We present a general description of separable states in Quantum Mechanics. In particular, our result gives an easy proof that inseparability (or entanglement) is a pure quantum (noncommutative) notion. This implies that distinction between separability and inseparability has sense only for composite systems consisting of pure quantum subsystems. Moreover, we provide the unified characterization of pure-state entanglement and mixed-state entanglement.

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Among the most emblematic concepts in Quantum Mechanics there is the idea of entanglement. Let us remind that this concept enters in description of quantum correlations between subsystems, so in particular plays a crucial role in quantum information theory and quantum computation. This explains why the interest in better understanding of this concept is so important.

From a formal point of view, a state of a composite quantum system is called *inseparable* (or *entangled*) if it cannot be represented as tensor products of states of its subsystems. On the contrary, a density matrix describes a *separable* state if it can be expressed as a convex combination of tensor products of its subsystem states. These definitions stem from the mathematical fact that, in general, the convex hull of product states is not a dense subset in the state space of the tensor product of two von Neumann algebras (cf. [9], [12]).

The purpose of this paper is to discuss, within the framework of quantum mechanics, a general, rigorous description of separable states. To this end we will need some preliminaries. Let us consider two physical systems $\Sigma_1$ and $\Sigma_2$ respectively. As we wish to study quantum mechanical problems, the systems $\Sigma_i$, $i = 1, 2$ will be described by $(\mathcal{H}_i, \mathcal{B}(\mathcal{H}_i), \varrho_i)$, $i = 1, 2$, where $\mathcal{H}_i$ denotes the set of all pure states of $\Sigma_i$, $\mathcal{B}(\mathcal{H}_i)$ the set of all linear bounded operators on $\mathcal{H}_i$ (thus describing the set of all bounded observables of $\Sigma_i$), and finally $\varrho_i$ is a density matrix describing a state of $\Sigma_i$. Here and subsequently we shall identify a density matrix with the notion of state. In the sequel, we assume that $\varrho_i$ is an invertible operator (for example, an equilibrium state of the system $\Sigma_i$).

Let us emphasize that the above assumptions mean that we restrict ourselves to Dirac’s approach to Quantum Mechanics. A more complete theory may be obtained by an application of a general local quantum physics approach (cf. [6]). The advantage of using such a general approach lies in the fact that only within this scheme one can discuss properly a relation between inseparability and “non-locality”. However, this topic demands more advanced mathematics and therefore exceeds the scope of this paper. We wish only to mention that the presented description of separable states can be generalized, without any problem, to this general framework. Furthermore, we shall treat only two-particle entanglement (cf.
although a generalization to multipartile entanglement is straightforward too.

Let us consider the following family of states of the complex system \( \Sigma = \Sigma_1 + \Sigma_2 \)

\[
\sigma = \sum_{l}^{N<\infty} a_l \sigma_l^{(1)} \otimes \sigma_l^{(2)} \quad (1)
\]

where \( a_l \geq 0, \sum_l a_l = 1, \) and \( \sigma_l^{(1)} (\sigma_l^{(2)}) \) is an arbitrary state of \( \Sigma_1 (\Sigma_2 \text{ respectively}) \).

This family (or more precisely, its closure) constitutes the set of separable states and it will be denoted by \( S_{\text{sep}} \). In many recent papers (e.g \[13\], \[7\], \[4\], \[11\]) the question of a characterization of separable states was posed and treated. From the existing rigorous results we would like also to mention the work (\[1\]) in which the fine structure of the states of a composite system was studied in the Hilbert space language. Here, we want to give a simple description of \( S_{\text{sep}} \) (which follows from Tomita-Takesaki theory) and to discuss some physical properties of non-separable states. For a comprehensive account of Tomita-Takesaki theory addressed to physicists we refer Haag’s book \[3\] while the mathematical description can be found in \[3\], and \[14\].

Let \( \mathcal{K}_i \) be the set of all Hilbert-Schmidt operators on the Hilbert space \( \mathcal{H}_i \) associated with the system \( \Sigma_i \). It is a Hilbert space with the scalar product \((\cdot, \cdot)\) of the form \( (\sigma, \mu) = Tr\sigma^*\mu \).

Clearly, \( \varrho_i^{\frac{1}{2}} \in \mathcal{K}_i \). Now let us form

\[
\mathcal{P}_i = \text{closure}\left\{ \varrho_i^{\frac{1}{2}} A \varrho_i^{\frac{1}{2}} \right\} \subseteq \mathcal{K}_i \quad (2)
\]

where \( A \in \mathcal{B}(\mathcal{H}_i) \quad A \geq 0 \). We observe that \( \mathcal{P}_i \) is the natural cone from Tomita-Takesaki theory. We note that any operator \( A \in \mathcal{B}(\mathcal{H}_i) \) can be considered as a (left)multiplication of \( A \) with a Hilbert-Schmidt operator \( \mu \in \mathcal{K}_i \). In such a case the set \( \{ A \in \mathcal{B}(\mathcal{H}_i) \} \) will be denoted by \( \mathcal{M}_i \).

So we arrived to

\[
(\mathcal{K}_i, \mathcal{M}_i, \mathcal{P}_i, \varrho_i^{\frac{1}{2}}) \quad (3)
\]

i.e. we got the standard form of \( \mathcal{B}(\mathcal{H}_i) \) in the framework of Tomita-Takesaki theory. The reader non acquainted with this theory can consider \( (\mathcal{K}_i, \mathcal{M}_i, \mathcal{P}_i, \varrho_i^{\frac{1}{2}}) \) as the following
quadruple: (all Hilbert-Schmidt operators on \( \mathcal{H}_i \), \( \mathcal{B}(\mathcal{H}_i) \) acting on the set of Hilbert-Schmidt operators as left multiplications, all positive Hilbert-Schmidt operators, square-root of the distinguished density matrix) where \( \mathcal{H}_i \) is the Hilbert space of all pure states of the system \( \Sigma_i \). We wish to recall one important result of the just mentioned theory. Namely, each state \( \sigma_i \) of \( \Sigma_i \) can be represented (in the unique way) by a vector \( \sigma_i^\frac{1}{2} \in \mathcal{P}_i \) such that

\[
<A_i>_{\sigma_i} \equiv Tr A_i \sigma_i = (\sigma_i^\frac{1}{2}, A_i \sigma_i^\frac{1}{2})
\]  

(4)

where \( \sigma_i \) is a state of the system \( \Sigma_i \), and with the small abuse of notation we used the same letter \( A_i \) for the operator in \( \mathcal{B}(\mathcal{H}_i) \) (left hand side of (4)) representing an arbitrary but fixed observable and for the operator in \( \mathcal{M}_i \) (right hand side of (4)) which represents the same observable.

Now let us turn to the promised characterization of separable states of the composite system \( \Sigma \). Using the prescription that the Hilbert space \( \mathcal{H} \) of all pure states of a composite system \( \Sigma \) is described by the tensor product of Hilbert spaces \( \mathcal{H}_i \) of pure states of its components, \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \), we infer that the standard form associated with \( \Sigma \) is

\[
(\mathcal{K}_1 \otimes \mathcal{K}_2, \mathcal{M}_1 \otimes \mathcal{M}_2, \mathcal{P}, \sigma_1^\frac{1}{2} \otimes \sigma_2^\frac{1}{2})
\]  

(5)

where \( \mathcal{P} = \text{closure}\{\sigma_1^\frac{1}{2} \otimes \sigma_2^\frac{1}{2}, A \in \mathcal{M}_1 \otimes \mathcal{M}_2, A \geq 0\} \subset \mathcal{K}_1 \otimes \mathcal{K}_2 \). It is important to note that \( \mathcal{P}_1 \otimes \mathcal{P}_2 \equiv \text{closure}\{\sum_k a_k x_k^{(1)} \otimes x_k^{(2)}, a_k \geq 0, \sum_k a_k = 1, x_k^{(i)} \in \mathcal{P}_i\} \) is, in general, a proper subset of \( \mathcal{P} \). This is a result of the fact that the set of all positive operators in \( \mathcal{M}_1 \otimes \mathcal{M}_2 \) contains, in general as a proper subset, the closure of \( \sum A_k^{(1)} \otimes A_k^{(2)} \) where \( A_k^{(1)} \) is a positive operator in \( \mathcal{M}_1 \) while \( A_k^{(2)} \) is a positive operator in \( \mathcal{M}_2 \). (For a discussion of this fact in physical terms see \[13\] while for mathematical argument see \[3\]).

Consider the tensor product \( \mathcal{M}_1 \otimes \mathcal{M}_2 \). Our characterization of separable states starts with the observation that each convex combination \( \omega_0 = \sum_i \lambda_i \omega_i^{(1)} \otimes \omega_i^{(2)} \) of product states of \( \mathcal{M}_1 \otimes \mathcal{M}_2 \) can be expressed in the form (cf. (4))

\[
\omega_0 = \sum_i \lambda_i \omega_{x_i} \otimes \omega_{y_i}
\]  

(6)
where \( \omega_{x_i}(\cdot) = (x_i, x_i), \) \( x_i \in \mathcal{P}_1, \) \( \omega_{y_i}(\cdot) = (y_i, y_i), \) \( y_i \in \mathcal{P}_2. \) (We recall that \( (\cdot, \cdot) \) is the scalar product in \( \mathcal{K} \).) Thus for \( A \in \mathcal{M}_1, \) \( B \in \mathcal{M}_2 \)

\[
\omega_0 (A \otimes B) = \sum_i \lambda_i (x_i, Ax_i)(y_i, By_i) = \sum_i \lambda_i (x_i \otimes y_i, (A \otimes B) \cdot x_i \otimes y_i)
\]

(7)

Hence

\[
\omega_0 (A \otimes B) = Tr \varrho_0 A \otimes B
\]

(8)

where

\[
\varrho_0 = \sum_i \lambda_i |x_i \otimes y_i > < x_i \otimes y_i |
\]

(9)

In (9) we have written projectors in the Dirac’s notation, i.e. \( |x_i \otimes y_i > < x_i \otimes y_i |\) \( f \otimes g \) \( \equiv \) \( x_i \otimes y_i |f \otimes g > x_i \otimes y_i > \) for any \( f \in \mathcal{K}_1 \) and \( g \in \mathcal{K}_2. \) Consequently, we have the following characterization of separable states:

*The set of separable states is the norm closure of the set*

\[
\{ \varrho_0 = \sum_i \lambda_i |x_i \otimes y_i > < x_i \otimes y_i |\}
\]

(10)

where \( x_i \in \mathcal{P}_1, \) \( y_i \in \mathcal{P}_2, \) \( i = 1, 2, \ldots \)

To avoid the future confusion we emphasize that the set of vectors \( \{ |x_i \otimes y_i > \} \) does not form, in general, an orthogonal system in \( \mathcal{K}_1 \otimes \mathcal{K}_2. \) Consequently, (10) is not a spectral resolution of the corresponding density matrix. Now let us rewrite (7) in the following way

\[
(\varrho_0^{\frac{1}{2}}, (A \otimes B) \varrho_0^{\frac{1}{2}}) \equiv \omega_0 (A \otimes B) = \sum_i \lambda_i (x_i \otimes y_i, (A \otimes B)x_i \otimes y_i)
\]

(11)

where \( \varrho_0^{\frac{1}{2}} \in \mathcal{P} \) while \( A (B) \) is an arbitrary operator in \( \mathcal{M}_1 (\mathcal{M}_2, \) respectively). We remind the reader that (9) is not a spectral decomposition of \( \varrho_0. \) Hence, \( \varrho_0^{\frac{1}{2}}, \) being the representative vector of \( \omega_0 \) in \( \mathcal{P} \), is not the square root of \( \varrho_0 \) (given by (9)) in the sense that it is not equal to \( \sum_i \lambda_i^{\frac{1}{2}} |x_i \otimes y_i > < x_i \otimes y_i |. \) Equality (11) implies

\[
Tr P_{\varrho_0^{\frac{1}{2}}} A \otimes B = \sum_i \lambda_i Tr P_{x_i \otimes y_i} A \otimes B
\]

(12)
Here, \( P_{\frac{1}{2}^0} \) and \( P_{x_i \otimes y_i} \) are projectors in \( \mathcal{B}(\mathcal{K}) \). Clearly, \( Tr \) is taken with respect to a basis in \( \mathcal{K} \). So

\[
P_{\frac{1}{2}^0} = \sum_i \lambda_i P_{x_i \otimes y_i}.
\] (13)

Hence

\[
\frac{1}{2}^0 = \sum_i \lambda_i P_{x_i \otimes y_i} \frac{1}{2}^0 = \sum_i \lambda_i (x_i \otimes y_i, \frac{1}{2}^0) x_i \otimes y_i
\] (14)

We note that

\[
(x_i \otimes y_i, \frac{1}{2}^0) \geq 0
\]

as \( \mathcal{P}_1 \otimes \mathcal{P}_2 \subset \mathcal{P} \) and \( \mathcal{P} \) is a selfdual cone. Consequently

\[
\frac{1}{2}^0 \in \mathcal{P}_1 \otimes \mathcal{P}_2.
\] (15)

Conversely, let \( 0 \neq \frac{1}{2}^0 \in \mathcal{P}_1 \otimes \mathcal{P}_2 \), i.e.

\[
\frac{1}{2}^0 = \sum_i \lambda_i^0 x_i \otimes y_i
\] (16)

with \( 0 \neq x_i \in \mathcal{P}_1 \), \( 0 \neq y_i \in \mathcal{P}_2 \) and \( \lambda_i^0 > 0 \). This is an easy direction, however for the reader convenience we present all necessary details. We observe

\[
(\frac{1}{2}^0, x_i \otimes y_i) > 0
\] (17)

for all \( i \). To prove this inequality let us assume

\[
(\frac{1}{2}^0, x_j \otimes y_j) = 0
\] (18)

for some fixed \( j \). Thus

\[
0 = \sum_k \lambda_k^0 (x_k \otimes y_k, x_j \otimes y_j)
\] (19)

where \( (x_k \otimes y_k, x_j \otimes y_j) \geq 0 \) by selfduality of the natural cone. So each term of the sum has to be equal to 0. In particular

\[
(x_j \otimes y_j, x_j \otimes y_j) = \|x_j \otimes y_j\|^2 = 0
\] (20)
So $x_j \otimes y_j = 0$ and this is a contradiction. Therefore

$$(\varrho^{\frac{1}{2}}, x_i \otimes y_i) > 0 \quad (21)$$

for each $i$. Consequently, we can rewrite (16) in the following way

$$\varrho^{\frac{1}{2}} = \sum_i \lambda_i x_i \otimes y_i = \sum_i \frac{\lambda_i}{(x_i \otimes y_i, \varrho^{\frac{1}{2}})} (x_i \otimes y_i, \varrho^{\frac{1}{2}}) x_i \otimes y_i \equiv \sum_i \lambda_i (x_i \otimes y_i, \varrho^{\frac{1}{2}}) x_i \otimes y_i \quad (22)$$

where $\lambda_i = \frac{\lambda_i}{(x_i \otimes y_i, \varrho^{\frac{1}{2}})} > 0$. In other words, we arrived to the same form for $\varrho^{\frac{1}{2}}$ as in (14). This justifies the following equality

$$P_{\varrho^{\frac{1}{2}}} = \sum_i \lambda_i P_{x_i \otimes y_i} \quad (23)$$

and therefore $P_{\varrho^{\frac{1}{2}}}$ determines the separable state. More precisely, we use the one-to-one correspondence between a product state $\sigma \otimes \mu$ and a simple tensor $x_{\sigma} \otimes y_{\mu} \in \mathcal{P}$. Subsequently, this correspondence is extended by convexity. To summarize, we have obtained the one-to-one correspondence between the set of normalized vectors in $\mathcal{P}_1 \otimes \mathcal{P}_2$ and the set of all separable states $S_{sep}$. It should be noted however that the above correspondence is implemented by the proper choice of projectors (cf. (13)).

**Example:**

To illustrate the presented description let us consider the following example. Let $\eta = \lambda_1 x \otimes y + \lambda_2 y \otimes x$ be a vector in $\mathcal{H} \equiv \mathcal{H}_1 \otimes \mathcal{H}_2$, $x \in \mathcal{H}_1$, $y \in \mathcal{H}_2$, $\lambda_1, \lambda_2 \in \mathbb{C}$. We note that the vector $\eta$ is a prototype of an entangled (pure) state in Dirac’s quantum mechanics. Obviously, this state is uniquely determined by the projector $P_{\lambda_1 x \otimes y + \lambda_2 y \otimes x}$ which belongs to $\mathcal{P}$. However, easy but a little bit tedious calculations (which we left to the reader) show that $P_{\lambda_1 x \otimes y + \lambda_2 y \otimes x} \notin \mathcal{P}_1 \otimes \mathcal{P}_2$. Consequently, we have shown that the standard example of pure entangled state fits well in our characterization.

To see peculiarity of the subset $\mathcal{P}_1 \otimes \mathcal{P}_2$ of $\mathcal{P}$ (which uniquely characterizes the set of all separable states) let us write down explicitly an operator $\sigma$ such that

$$(\sigma, P_x \otimes P_y) \geq 0 \quad (24)$$
for any $x \in \mathcal{H}_1$, $y \in \mathcal{H}_2$ while $\sigma \not\in \mathcal{P}$. This clearly shows that the set $\mathcal{P}_1 \otimes \mathcal{P}_2$ does not have the selfduality property. We remind the reader that this is extremely important property of the natural cone $\mathcal{P}$ (which describes the set of all states). Let us consider (cf. [7])

$$\sigma = (|f><f|) \otimes (|g><g|) + (|g><g|) \otimes (|f><f|)$$

$$-(|f><g|) \otimes (|f><g|) - (|g><f|) \otimes (|g><f|)$$

where $f, g \in \mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_0$ and $(f, g) = 0$. Here, $|g><f| \equiv (g, h)_{\mathcal{H}_0}|f>$. One can show that

$$(\sigma, \xi)_K \geq 0$$

for every $\xi \in \mathcal{P}_1 \otimes \mathcal{P}_2$ while there does exist an element $\theta$ in $\mathcal{P}$

$$\theta = (|f><f|) \otimes (|f><f|) + (|g><g|) \otimes (|g><g|)$$

$$+(|f><g|) \otimes (|f><g|) + (|g><f|) \otimes (|g><f|)$$

such that

$$(\sigma, \theta)_K < 0$$

Thus the set $\mathcal{P}_1 \otimes \mathcal{P}_2$ is a (proper) subset of $\mathcal{P}$ (provided that $\dim \mathcal{H}_1 \geq 2$, $\dim \mathcal{H}_2 \geq 2$) and therefore $\mathcal{P}_1 \otimes \mathcal{P}_2$ is too a small set to describe all states of composite quantum system.

The principal significance of this characterization is that it offers a clear explanation why the set of separable states of a general complex system is a proper subset of the set of all states. The characterization of the set of separable states $\mathcal{S}_{\text{sep}}$ needs only a proper subset $\mathcal{P}_1 \otimes \mathcal{P}_2$ of the natural cone $\mathcal{P}$ while for the description of all states of $\Sigma$ we have to use the whole set $\mathcal{P}$. In other words, convex combinations of product states do not lead to
the set of all states. To get another important and somewhat surprising consequence of this characterization we recall the following result (see [5]) saying that \( P = P_1 \otimes P_2 \) if and only if \( \Sigma_1 \) or \( \Sigma_2 \) (or both) is a classical subsystem, i.e. the algebra of observables of \( \Sigma_i \) is an abelian algebra. This means that for such a system the set of separable states is equal to the set of all states. In particular, this implies that any approximation of a real quantum atom interacting with the quantum electromagnetic field by a system of quantum atom and classical electromagnetic field kills the distinction between separability and inseparability. For an interpretation of this result, in terms of correlations, see below. As another example in this direction let us mention a system in quantum information theory where one of channels is assumed to be a classical one. Again, for such a model the set of all separable states is equal to the set of all states. Finally we want to note that our characterization partly clarifies the theory of mixed-state entanglement (cf. [4]). Namely, we present the unified characterization of “pure-state entanglement” and mixed-state entanglement”. For other consequences of the presented characterization see [8].

Now let us turn to the following question: What is the basic “physical” difference between the sets of separable and non-separable states? To answer this question let us recall another important property of a natural cone \( P \). Let \( \xi \) be a vector in a fixed natural cone \( P \) such that \( \{ A\xi, A \in \mathcal{M}_1 \otimes \mathcal{M}_2 \} \) is dense in \( \mathcal{K}_1 \otimes \mathcal{K}_2 \). Then a natural cone constructed with such a vector \( \xi \) is equal to the previous one. Let us discuss this property in physical terms. Assume \( \varrho_i \) is an equilibrium state for \( \Sigma_i \), so \( \varrho_i = Z_i^{-1} \exp\{ -\beta H_i \} \) where \( Z_i \) is the partition function, \( H_i \) the hamiltonian of the system \( \Sigma_i \), and \( \beta \) stands for the inverse temperature. Now it is clear that the set \( P_1 \otimes P_2 \) is built on uncorrelated state \( \varrho_1 \otimes \varrho_2 \) in such a way that it does not contain essential quantum correlations (we remind the reader that to form \( P_1 \otimes P_2 \) we need only to consider convex combinations of \( x_i \otimes y_i, x_i \in P_1, y_2 \in P_2 \). To form \( P \) we can use a Gibbs state of the form \( Z^{-1} \exp\{ -\beta (H_1 + H_2 + H_I) \} \) where \( H_I \) describes any interaction between \( \Sigma_1 \) and \( \Sigma_2 \). So \( P \) contains, from the very beginning, also vectors describing all quantum correlations. Clearly, this feature can be also explained by the fact that the set
\{ A \in \mathcal{M}_1 \otimes \mathcal{M}_2, \; A \geq 0 \} \text{ contains all (positive) “correlated” observables. Consequently we arrive to the following conclusion: } \text{the set of nonseparable states contains pure quantum correlations of the complex system } \Sigma, \text{ while separable states do not have this property.}

We wish close this note with a remark that the just given characterization of states can be used in the debate on the Einstein-Podolsky-Rosen paradox and Bell’s inequality (for a deeper discussion on these subjects we refer the reader to \( [2] \)). Namely, considering a composite interacting system \( \Sigma = \Sigma_1 + \Sigma_2 \) in a nonseparable state \( \sigma \) it is difficult to specify the notion of subsystem. The main difficulty in carrying out such a specification is that such a procedure should take into account ”causal ties” which are hidden in \( \sigma \) (cf. \( [3] \)). Of course, this problem does not exist for the pair \( (\Sigma, \rho) \) where \( \rho \in \mathcal{S}_{sep} \).

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