COMPUTATION OF NIELSEN NUMBERS FOR CERTAIN MAPS OF HYPERBOLIC SURFACES

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Abstract. Let $X$ be a closed surface for which the Euler characteristic $\chi(X)$ is negative, and let $f : X \to X$ be a self-map that is not surjective. In this short paper, we prove that we can compute the Nielsen number of $f$, $N(f)$, under some algebraic conditions.

1. Introduction

Let $X$ be a hyperbolic surface, that is, a compact connected surface for which the Euler characteristic $\chi(X)$ is negative and let $f : X \to X$ be a self-map. The Nielsen number of $f$, $N(f)$, is a homotopy invariant and provides a lower bound for the minimum number of fixed points over all maps homotopic to $f$. See [1, 5, 8] for the background.

Unfortunately, computing the Nielsen number is difficult and it is particularly difficult on hyperbolic surfaces. See [2, 7, 12] for the details. But recently, for $X$ a hyperbolic surface with boundary, many methods are developed in the papers [3, 4, 9, 10, 13, 14] to compute the Nielsen number $N(f)$.

In this paper, we will first introduce briefly these methods. Then for $X$ a closed hyperbolic surface, we will show that we can apply those methods to compute the Nielsen number $N(f)$ on $X$. The result in this paper is a partial answer to one of open problems in [7]. The question is that is there an algorithm for the calculation of the Nielsen number for a self-map of a surface?

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2. Nielsen number on hyperbolic surfaces with boundary

Let $X$ be a hyperbolic surface with boundary. Then $X$ is homotopy equivalent to a wedge of a finite number of circles and has fundamental group $\pi_1(X)$ that is a finitely generated free group. Let $\{a_1, \cdots, a_n\}$ be a set of generators of $\pi_1(X)$. Let $f : X \to X$ be a self-map and let $f_# : \pi_1(X) \to \pi_1(X)$ be the induced endomorphism of $f$. Since the Nielsen number is a homotopy type invariant [5, p. 21], we may assume that $X$ is a wedge of a finite number of circles if necessary.

Let $G$ be a group and let $\varphi : G \to G$ be an endomorphism. Two elements $u, v \in G$ are Reidemeister equivalent, also called twisted conjugate, if there exists $z \in G$ such that

\[(2.1)\quad u = \varphi(z)vz^{-1}.\]

The challenge in computing $N(f)$ on $X$ is determining whether two elements in $\pi_1(X)$ are Reidemeister equivalent with $\varphi = f_#$. See [2] for the background.

In 1999, Wagner introduced an algorithm, which is now called Wagner’s algorithm, for computing $N(f)$ on $X$. Wagner’s algorithm applies to maps with remnant. For each $i$, the word $f_#(a_i)$ has remnant if there is a nontrivial subword of $f_#(a_i)$ which does not cancel in any product of the form

\[f_#(a_i)^{\pm 1}f_#(a_i)f_#(a_k)^{\pm 1}\]

except if $j$ or $k$ equals $i$ and the exponent is $-1$. The map $f$ has remnant if every word $f_#(a_i)$ has remnant.

**Theorem 2.1** ([13]). *If $f : X \to X$ has remnant, we can compute the Nielsen number $N(f)$ by Wagner’s algorithm.*

Wagner’s remnant was extended to $k$-remnant ($k \in \mathbb{N}$) in [4]. The remnant condition requires that there is limited cancellation in each product $f_#(u)f_#(v)$ when $u$ and $v$ have length 1. Roughly, a map has $k$-remnant if there is limited cancellation in each product $f_#(u)f_#(v)$ when $u$ and $v$ have length $k$ in $\pi_1(X)$.

**Theorem 2.2** ([4]). *If $f : X \to X$ has $k$-remnant, there is an algorithm for computing the Nielsen number $N(f)$.*

Hart in [2, 3] developed also two other algebraic methods, MRN maps (when $\pi_1(X)$ is free on two generators) and 2C3 maps, for determining the Nielsen equivalence classes. Roughly, these maps have partial remnant and have some restrictions on the cancellation in the word product among images of generators $a_i$ under $f_#$. 
Theorem 2.3 ([3]). For MRN maps and 2C3 maps, we can determine the Nielsen equivalence classes, so we can compute the Nielsen numbers.

In the paper [10], Wagner’s idea was extended in a different way. The possible lengths of solutions were considered to Equation (2.1). Let $\varphi : F \to F$ be an endomorphism. A pair $(u, v)$ of two elements of $F$ has bounded solution length (or BSL) if there exists an integer $n$ such that there is no solution $z \in F$ with $|z| > n$ to the equation (2.1)

$$u = \varphi(z)vz^{-1}.$$  

The smallest such $n$ is called the solution bound (or SB) for $(u, v)$.

Given any pair $(u, v)$ of elements of $F$, if $(u, v)$ has BSL, we can algorithmically determine whether or not $u$ and $v$ are Reidemeister equivalent by checking for equality of $u = \varphi(z)vz^{-1}$ where $z$ ranges over all elements of $F$ with $|z| \leq SB$. For a map $f : X \to X$, if any pair of two elements in $\pi_1(X)$, each of which represents a fixed point class of $f$, has BSL for the endomorphism $\varphi = f_\#$ then we say that $f$ has bounded solution length (BSL). The maximum of all SB for such pairs is called the solution bound (SB) for $f$.

Theorem 2.4 ([10]). If $f : X \to X$ has BSL (and we know the SB for $f$), then we can algorithmically determine the Nielsen equivalence classes, so we can compute $N(f)$.

Let $X$ be the pants surface, the 2-sphere with three disjoint open disks removed, or more generally, let $X$ be a compact polyhedron that is homotopy equivalent to the figure-eight. Yi and this author in [11, 14] extended Wagner’s work using the concept of the mutant of a map, which had been introduced by Jiang [6], so that an algorithm for computing the Nielsen number on $X$ was completed.

Theorem 2.5 ([11]). Let $X$ be an aspherical figure-eight type finite polyhedron and let $f : X \to X$ be a self-map. There is an algorithm for computing the Nielsen number $N(f)$.

This algorithm is now called the WYK-algorithm.

3. Nielsen number on closed hyperbolic surfaces

Let $X$ be a closed surface of genus $n \geq 2$. Then we have

$$\pi_1(X) = \langle a_1, a_2, \ldots, a_{2n-1}, a_{2n} \mid a_1a_2a_1^{-1}a_2^{-1}\cdots a_{2n-1}a_{2n}a_{2n-1}^{-1}a_{2n}^{-1} \rangle$$
where the relator is the product of $n$ commutators. Let $f : X \to X$ be a self-map and let $f_\# : \pi_1(X) \to \pi_1(X)$ be the induced endomorphism of $f$. Let $F$ be the free group on the generators $\{a_1, a_2, \cdots, a_{2n}\}$. Given a particular representation of $f_\#$, let $f_{\#F} : F \to F$ be the homomorphism for which $f_{\#}(a_i)$ and $f_{\#F}(a_i)$ look identical as strings of letters for each generator $a_i$. The notation $f_{\#F}$ was introduced in [2] and we use the same notation in this section.

**Theorem 3.1.** Let $X$ be a closed hyperbolic surface and let $f : X \to X$ be a self-map that is not surjective. If $f$ satisfies one of the following:

1. $f_{\#F}$ has remnant or $k$-remnant,
2. $f_{\#F}$ is a $2C3$ map,
3. $f_{\#F}$ has $BSL$,

then there is an algorithm for computing the Nielsen number $N(f)$.

**Proof.** Let $X$ be a closed hyperbolic surface of genus $n$ and let $f : X \to X$ be a self-map that is not surjective. Take a point $x$ in $X - f(X)$. Let $A$ be a regular neighborhood of a wedge of $2n$ circles and identify $A$ with a strong deformation retract of $X - \{x\}$. Then for the subspace $A$ of $X$, we may consider that the wedge point is the base point of $X$ and that each circle with fixed orientation represents $a_i$-loop. Since $A$ is a strong deformation retract of $X - \{x\}$, which contains $f(X)$, and the Nielsen number is a homotopy invariant, we may assume that the image of $f$ is into $A$. For instance, we can retract the image of $f$ into $A$ using a strong deformation retraction of $X - \{x\}$ onto $A$.

Let $\bar{f} : X \to A$ be the corestriction of $f$ to $A$. Then

$$f = i \circ \bar{f}$$

where $i : A \to X$ is the inclusion map. Let $f_A : A \to A$ be the map obtained from $\bar{f}$ by commutation, that is

$$f_A = \bar{f} \circ i.$$

Since the Nielsen number has the commutativity property, we have

$$N(f) = N(f_A).$$
The subspace $A$ of $X$ is a hyperbolic surface with boundary and $\pi_1(A) = F$. For each $i$, we have

$$(f_A)_\#(a_i) = \tilde{f}_\# \circ i_\#(a_i) = \tilde{f}_\#(a_i) = f_{\#F}(a_i).$$

Thus we have

$$(f_A)_\# = f_{\#F}.$$ 

Consequently, if $f_{\#F}$ has remnant or $k$-remnant (resp. $f_{\#F}$ is a $2C3$ map, $f_{\#F}$ has $BSL$), then by Theorem 2.1 or Theorem 2.2 (resp. Theorem 2.3, Theorem 2.4), there is an algorithm for computing $N(f_A)$, which equals $N(f)$.

Since $f_{\#F}(F)$ is a subgroup of the free group $F$, the group $f_{\#F}(F)$ is also a free group.

**Theorem 3.2.** Let $X$ be a closed hyperbolic surface and let $f : X \to X$ be a self-map that is not surjective. If the rank of the free group $f_{\#F}(F)$ is 2 then there is an algorithm for computing the Nielsen number $N(f)$.

**Proof.** Using the same arguments in the proof of Theorem 3.1, we have that $N(f) = N(f_A)$ and $(f_A)_\# = f_{\#F}$, where $A$ and $f_A$ are the same as that in the proof of Theorem 3.1. Thus we will show that there is an algorithm for computing $N(f_A)$.

Let $Y$ be the figure-eight and let $F_2$ be the fundamental group of $Y$ that is a free group of rank 2. Since $(f_A)_\#(F) = f_{\#F}(F)$ is also a free group of rank 2, the homomorphism $(f_A)_\#$ factors through $F_2$, that is, there are homomorphisms $\phi : F \to F_2$ and $\psi : F_2 \to F$ such that $(f_A)_\# = \psi \circ \phi$.

Since $A$ and $Y$ are $K(\pi,1)$-spaces, there are maps $g : A \to Y$ and $h : Y \to A$ such that $g_\# = \phi$, $h_\# = \psi$ and $h \circ g$ is homotopic to $f_A$ so that we have

$$N(f_A) = N(h \circ g).$$
Let $f_Y = g \circ h$ be the commutation of $h \circ g$. Then $f_Y$ is a self-map of $Y$ and since the Nielsen number has the commutativity property, we have

$$N(f_A) = N(f_Y).$$

By Theorem 2.5, there is an algorithm for computing $N(f_Y)$. 

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