Abstract. We construct a new family, indexed by the odd integers $N \geq 1$, of $(2+1)$-dimensional quantum field theories called quantum hyperbolic field theories (QHFT), and we study its main structural properties. The QHFT are defined for (marked) $(2+1)$-bordisms supported by compact oriented 3-manifolds $Y$ with a properly embedded framed tangle $L_F$ and an arbitrary $PSL(2, \mathbb{C})$-character $\rho$ of $Y \setminus L_F$ (covering, for example, the case of hyperbolic cone manifolds). The marking of QHFT bordisms includes a specific set of parameters for the space of pleated hyperbolic structures on punctured surfaces. Each QHFT associates in a constructive way to any triple $(Y, L_F, \rho)$ with marked boundary components a tensor built on the matrix dilogarithms, which is holomorphic in the boundary parameters. We establish surgery formulas for QHFT partitions functions and describe their relations with the quantum hyperbolic invariants of [15] (either defined for unframed links in closed manifolds and characters trivial at the link meridians, or hyperbolic cusped 3-manifolds). For every $PSL(2, \mathbb{C})$-character of a punctured surface, we produce new families of conjugacy classes of “moderately projective” representations of the mapping class groups.
Keywords: hyperbolic geometry, quantum field theory; mapping class groups, quantum invariants, Cheeger-Chern-Simons class, dilogarithms.

1. Introduction

In this paper we construct a new family \{\mathcal{H}_N\}, indexed by the odd integers \(N \geq 1\), of \((2 + 1)\)-dimensional quantum field theories (QFT) that we call Quantum Hyperbolic Field Theories (QHFT). Here, following [2, 30], by QFT we mean a functor from a \((2 + 1)\)-bordism category, possibly non purely topological, to the tensorial category of finite dimensional complex linear spaces.

The QHFT bordism category is based on triples \((Y, L_F, \rho)\), where \(Y\) is a compact oriented 3-manifold, possibly with non-empty boundary \(\partial Y\), \(L_F\) is a properly embedded framed tangle (ie. a framed 1-dimensional non oriented submanifold) in \(Y\), and \(\rho\) is a flat sl(2, \mathbb{C})-connection on \(Y \setminus L_F\) up to gauge equivalence (ie. a \(PSL(2, \mathbb{C})\)-character of \(Y \setminus L_F\)), with arbitrary holonomy at the meridians of the tangle components. We require furthermore that \(L_F\) is non-empty when \(N > 1\), and that it intersects each of the boundary component, if any.

We will also consider a variant, denoted QHFT\(^0\), such that the tangles \(L\) are unframed, while the characters \(\rho\) are defined on the whole of \(Y\), that is the meridian holonomies are trivial. Finally we consider a “fusion” of QHFT and QHFT\(^0\) (still denoted QHFT) that incorporates both, by considering tangles having a framed part \(L_F\) as well as an unframed one \(L^0\) (see Section 5.3).

The objects of the bordism category are suitably marked surfaces. Every such a QHFT surface is a diffeomorphism \(f : (S, \mathcal{T}, p(\beta)) \rightarrow \Sigma\), where \(\mathcal{T}\) is a so called “efficient triangulation” of a fixed base oriented surface \(S\) with genus \(g\) and \(r\) marked framed points \(p_i\), represented by points \(p(\beta)\) in specific parameter spaces for \(\text{Hom}(\pi, PSL(2, \mathbb{C}))\), built on \(\mathcal{T}\) and particularly suited to the QHFT. In fact we construct several such parameter spaces with small “residual gauge groups” acting on them, and we point out the relations to each other. One of them, the so called \((-)\)-exponential \(L\)-parameter space, is defined in terms of cross-ratios and incorporates the Bonahon-Thurston shearbend coordinates for pleated hyperbolic surfaces with punctures.

Every QHFT bordism has marked boundary and is considered as a “transition” from its input QHFT surfaces towards the output ones. We understand that the characters \(\rho\) and \(\beta\) are compatible. Every QHFT functor associates to such a transition a tensor called the amplitude, defined up to a sign and multiplication by \(N\)th roots of unity.

When \(Y = W\) is closed (that is \(\partial Y = \emptyset\)), the amplitudes \(\mathcal{H}_N(W, L_F, \rho)\) are numerical invariants called partition functions. The QHFT\(^0\) partition functions \(H_N(W, L, \rho)\) coincide with the “quantum hyperbolic invariants” constructed in [4, 5], while the QHFT ones yield new wide families of numerical invariants, covering interesting geometric situations, such as compact hyperbolic cone manifolds. We will analyze the relations between \(\mathcal{H}_N\) and \(H_N\) partition functions. In [5] we defined also quantum hyperbolic invariants \(H_N(M)\) for non-compact complete hyperbolic 3-manifolds \(M\) of finite volume, ie. for cusped manifolds. Although these invariants are not immediately QHFT partition functions, we will show how they can be obtained in terms of these last. For that we establish a “surgery formula” for quantum hyperbolic invariants of cusped manifolds and QHFT partition functions that generalizes the one for Cheeger-Chern-Simons classes, and makes a crucial use of some of W. Neumann’s arguments in [25], sections 11 and 14.

By restricting QHFT to the trivial bordisms (the cylinders) we get a new family of conjugacy classes of “moderately projective” representations of the mapping class groups of punctured surfaces, that is, defined up to a sign and multiplication by \(N\)th roots of unity.

We stress that we need that any bordism includes a non-empty link, intersecting each boundary component (so that the QHFT surfaces have punctures), in order to build a consistent functor when \(N > 1\) (see [5], Lemma 6.4). Even when the holonomy is trivial around the punctures, we cannot forget them, in particular for what concerns the mapping class groups.

We show that QHFT are in fact restrictions to a geometric bordism category of “universal functors” called Quantum Hyperbolic Geometry (QHG). QHG includes the definition of a specific category of triangulated 3-dimensional pseudomanifolds equipped with additional structures, and modeled on the
functional properties of the \textit{matrix dilogarithms} studied in [5]. The QHG functors associate determined tensors to every such a decorated triangulation, obtained by tracing the matrix dilogarithms supported by each tetrahedron. The key point is that such tensors are invariant up to QHG triangulated pseudomanifold isomorphism. The main step in order to construct specializations with a strong geometric content, such as QHFT, consists in converting each QHFT (marked) bordism to a QHG triangulated pseudomanifold, unique up to QHG triangulated pseudomanifold isomorphism.

Hence we view this paper as a kind of achievement of the fundation of the theory initiated in [4, 5].

A main interest in the QHFT comes from the fact that they relate classical 3-dimensional hyperbolic geometry to the world of quantum field theories, two main themes of low-dimensional topology that remained essentially disjoint since their spectacular developments in the early eighties. In particular, the celebrated Kashaev’s Volume Conjecture for hyperbolic knots in $S^3$ [21] appears as a special instance of the challenging general problem of understanding the relations between the asymptotic behaviour of QHFT partition functions and fundamental invariants coming from differential geometry, like the Cheeger-Chern-Simons class (see [4], section 5, [5], section 7, and section 6.1 of the present paper). We plan to face the asymptotics of QHFT partition functions in future works.

In [3] the spectrum of the mapping class group representations of Section 5.4 is studied by using geometric quantization of the Bonahon-Thurston complex intersection 2-form.

A natural problem left unsettled is to determine the relations between the QHFT and Turaev’s Homotopic QFT [31]. This and formulas describing the behaviour of the QHFT amplitudes under framing changes will be treated in a sequel to this paper, as they rely mainly on $R$-matrix computations.

We refer to [7] for a discussion about QHG in the framework of gravity in dimension 3.

Remark 1.1. The results of this paper can be repeated almost verbatim to define Cheeger-Chern-Simons invariants for QHFT bordisms, by replacing the matrix dilogarithms for $N > 1$ with Neumann’s extended Rogers dilogarithm, corresponding to $N = 1$ (see Remark 5.5). This is described with all details in [5] for the case of cusped manifolds and triples $(W, L, \rho)$, and hence in the following we concentrate on the quantum theory $N > 1$, which is technically harder.

Here is the content of the paper.

The universal QHG functors are defined in section 2, where we recall also from [4, 5] and [24, 25] (section 2.4 in particular) the notions and results we need.

The QHFT bordism category is described in section 4, while its objects, the QHFT surfaces, are developed starting with section 3.

The QHFT functors are defined in section 5. This includes the construction of the distinguished QHG triangulated pseudomanifolds associated to any triple $(Y, L_F, \rho)$ with marked boundary components, and of the trace tensors computed on them. The conjugacy classes of moderately projective representations of the mapping class groups are treated in section 5.4.

The partition functions $H_N(W, L_F, \rho)$ are considered in section 6. We show in section 6.1 that when $\rho$ is defined on the whole of $W$, $H_N(W, L_F, \rho)$ coincides with $H_N(W, L \cup L', \rho)$, where $L'$ is a parallel copy of the unframed link $L$ given by the framing $F$. In section 6.2 we prove the surgery formula for quantum hyperbolic invariants of cusped manifolds and QHFT partition functions. In fact the QHG pseudomanifolds used to compute the trace tensors carry certain cohomological weights (see Section 2.4) and the partition function values actually depend also on them. These weights play indeed a subtle role in the surgery formulas. This eventually leads to realize $H_N(M)$ as the limit of $H_N(M_n, L_n, \rho_n)$, where $M_n$ is a sequence of closed hyperbolic manifolds converging geometrically to $M$, $L_n$ is the link of geodesic cores of the hyperbolic Dehn fillings of $M$ that produce $M_n$, and $\rho_n$ is the hyperbolic holonomy of $M_n$. In section 6.3 we discuss alternative computations of the QHFT partition functions for manifolds that fiber over $S^1$. For fibred cusped manifolds, this allows in particular to identify each $H_N(M)$ with a special instance of QHFT partition function. We conclude with an example in section 6.4.
Convention. Unless otherwise stated all manifolds are oriented, and the boundary is oriented via the convention: last is the ingoing normal. Often we denote “$\equiv_N$” the equality of tensors up to sign and multiplication by $N$th roots of unity.

2. Universal QHG

2.1. Building blocks. The building blocks of QHG are flat/charged $I$-tetrahedra $(\Delta, b, w, f, c)$, and matrix dilogarithms $R_N(\Delta(b, w, f, c)) \in \text{Aut}(\mathbb{C}^N \otimes \mathbb{C}^N)$ defined for every odd positive integer $N$ [5].

Flat/charged- $I$-tetrahedra. Consider the half-space model of the oriented hyperbolic space $\mathbb{H}^3$, with the group of direct isometries identified with $PSL(2,\mathbb{C})$ by the conformal action on $\partial \mathbb{H}^3 = \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ by Möbius transformations. An $I$-tetrahedron is an oriented ideal tetrahedron $\Delta$ in $\mathbb{H}^3$ with distinct ordered vertices $v_0, v_1, v_2$ and $v_3$ on $\partial \mathbb{H}^3$.

In fact we consider $\Delta$ as an abstract oriented simplex equipped with an additional decoration. The ordering of the vertices is encoded by a branching $b$, that is, edge orientations obtained via the rule: each edge points towards the biggest end-point. Each 2-face has an induced branching, and a $b$-orientation, which is just compatible with that of two edges on the boundary. We order the 2-faces $\delta_0, \ldots, \delta_3$ by the opposite vertices, and the edges $e_0, e_1, e_2$ of $\delta_3$ by stipulating that for $j = 0, 1, v_j$ is the first end-point of $e_j$. For exactly two 2-faces the $b$-orientation and the boundary orientation are the same. The $b$-orientation of $\Delta$ coincides with the given one if the $b$-orientation of $\delta_3$ looks anti-clockwise from $v_3$. We give $\Delta$ and each 2-face $\delta$ a $b$-sign $* b$ and $\sigma(\delta)$ respectively, which is 1 if the two orientations agree, and $-1$ otherwise.

The hyperbolic structure is encoded by the cross-ratio moduli that label the edges of $\Delta$. Recall that opposite edges share the same cross-ratio moduli. We set $w = (w_0, w_1, w_2)$ with $w_j = w(e_j) \in \mathbb{C} \setminus \{0, 1\}$. Hence $w_{j+1} = 1/(1 - w_j)$ (indices mod($\mathbb{Z}/3\mathbb{Z}$), and

$$w_0 = (v_2 - v_1)(v_3 - v_0)/(v_2 - v_0)(v_3 - v_1).$$

We say that the $I$-tetrahedron $(\Delta, b, w)$ is non-degenerate if it is of non zero volume, that is if the imaginary part of each $w_i$ is not zero; then they have the same sign $w_i = \pm 1$.

It is very convenient to encode $(\Delta, b, w)$ in dual terms. In Figure 1 we show the 1-skeleton of the dual cell decomposition of $\text{Int}(\Delta)$ ($x$ and the indices $i, j, k$ and $l$ are considered below). It is understood that (dual) edges without arrows are incoming at the crossing. Note that an oriented edge is outgoing exactly when the $b$-sign of the dual 2-face is 1.

Figure 1. $I$-tetrahedra and dual encoding.

A flat/charged $I$-tetrahedron is an $I$-tetrahedron equipped with a flattening $f$ and a charge $c$, two notions first introduced in [24] and [25]. Flattenings and charges are $\mathbb{Z}$-valued functions defined on the edges of $\Delta$ that take the same value on opposite edges and satisfy the following properties, respectively (where $\log$ has the imaginary part in $]-\pi, \pi]$):

F. Flattening condition: $l_0 + l_1 + l_2 = 0$, where

$$l_j = l_j(b, w, f) = \log(w_j) + \sqrt{-1}\pi f_j,$$

C. Charge condition: $c_0 + c_1 + c_2 = 1$. 

$$j = 1$$
We call $l_j$ a (classical) log-branch, and for every odd $N > 1$ we define the (level $N$) quantum log-branch as

\begin{equation}
 l_{j,N} = \log(w_j) + \sqrt{-1}\pi(N + 1)(f_j - \ast b\epsilon_j).
\end{equation}

The bijective map

\[
(l_0, l_1, l_2) \mapsto \left( w_0; \frac{l_0 - \log(w_0)}{\sqrt{-1}\pi}, \frac{l_1 - \log(w_1)}{\sqrt{-1}\pi} \right)
\]

yields an identification of the set of log-branches on $(\Delta, b)$ with the Riemann surface $\hat{\mathbb{C}}$ of the maps $w_0 \mapsto (\log(w_0) + \varepsilon \pi \sqrt{-1}, \log((1 - w_0)^{-1}) + \varepsilon' \pi \sqrt{-1})$, with $\varepsilon, \varepsilon' \in \{0, 1\}$. Similarly, the set of triples $(w_0', w_1', w_2')$ with $w_j' = \exp(l_{j,N}/N)$ gets identified with the quotient covering by $NZ \times NZ$. These two spaces are the domains of definition for the matrix dilogarithms to be described next, when $N = 1$ and $N > 1$ respectively.

**Matrix dilogarithms.** Denote by $\log$ the standard branch of the logarithm, which has the imaginary part in $]-\pi, \pi]$. Recall that the space of triples of log-branches on a branched oriented tetrahedron is identified with the Riemann surface $\hat{\mathbb{C}}$.

For $N = 1$, we forget the integral charge $c$, so that $R_1$ is defined on flattened $I$-tetrahedra:

\begin{equation}
 R_1(\Delta, b, w, f) = \exp \left( \frac{\ast b}{\pi \sqrt{-1}} R(w_0; f_0, f_1) \right) = \exp \left( \frac{\ast b}{\pi \sqrt{-1}} \left( -\pi \frac{1}{1} - 2 \int_0^{w_0} \frac{l_0(t)}{1 - t} dt \right) \right)
\end{equation}

where $l_0(t) = \log(t) + \sqrt{-1}\pi f_0$ and $l_1(t) = \log((1 - t)^{-1}) + \sqrt{-1}\pi f_1$. The map $R : \hat{\mathbb{C}} \to \mathbb{C}/\pi^2\mathbb{Z}$ is holomorphic, and takes values in $\mathbb{C}/2\pi^2\mathbb{Z}$ on the component with even valued flattenings (25, Proposition 2.5).

For $N = 2m + 1 > 1$ and every complex number $x$ set $x^{1/N} = \exp(\log(x)/N)$ ($0^{1/N} = 0$ by convention). Put

\begin{equation}
 g(x) := \prod_{j=1}^{N-1} (1 - x\zeta^j)^{1/N}.
\end{equation}

The function $g$ is defined over $\mathbb{C}$, and analytic over the complement of the rays from $x = \zeta^k$ to infinity, $k = 1, \ldots, N - 1$. Set $h(x) := g(x)/g(1)$ (we have $|g(1)| = N^{1/2}$). For any $u', v' \in \mathbb{C}$ satisfying $(u')^N + (v')^N = 1$ and any $n \in \mathbb{N}$, let

\[
 \omega(u', v'|n) = \prod_{j=1}^{n} \frac{v'}{1 - u'\zeta^j}
\]

with $\omega(u', v'|0) = 1$ by convention. The functions $\omega$ are periodic in their integer argument, with period $N$. Write $[x] = N^{-1}(1 - x^N)/(1 - x)$. Given a flat/charged $I$-tetrahedron $(\Delta, b, w, f, c)$, set

\begin{equation}
 w_j' = \exp(l_{j,N}/N)
\end{equation}

with $l_{j,N}$ as in (2). Put the standard tensor product basis on $\mathbb{C}^N \otimes \mathbb{C}^N$. The matrix dilogarithm of level $N > 1$ is the tensor valued function of flat/charged $I$ tetrahedra defined by

\begin{equation}
 R_N(\Delta, b, w, f, c) = \left( (w_0')^{-c_1} (w_1')^{c_2} \right) \frac{N-1}{\mathcal{L}_N} (\mathcal{L}_N)^{\ast b}(w_0', (w_1')^{-1}) \in \text{Aut}(\mathbb{C}^N \otimes \mathbb{C}^N)
\end{equation}

where (recall that $N = 2m + 1$)

\[
 \mathcal{L}_N(w', v')_{[j]}^{[k]} \frac{h(w')}{h(u')} = h(u') \zeta^{k+j+(m+1)k^2} \omega(u', v'|i - k) \delta(i + j - l)
\]

\[
 (\mathcal{L}_N(w', v')^{-1})_{[j]}^{[k]} \frac{h(w')}{h(u')} = h(u') \zeta^{-i-(m+1)i^2} \frac{h(w')}{h(u')} \omega(u'/\zeta, v'|k - i) \delta(k + l - j)
\]

with $\delta$ the Kronecker symbol with period $N$, that is $\delta(n) = 1$ if $n \equiv 0 \mod(N)$, and $\delta(n) = 0$ otherwise. Note that we use the branching in order to associate an index among $i, j, k$ and $l$ to each 2-face of $\Delta$. The rule is shown in Figure 1.
Up to multiplication by $N$th roots of unity, the map $\mathcal{R}_N$ is holomorphic on $\hat{C}$, because $w'_j = \exp((1/N)(l_j - \ast_N c_j)) \exp((f_j - \ast_N c_j)\pi i)$ and the parity of $f_j$ is unaltered when we move $l_j$ continuously. The ambiguity of $\mathcal{R}_N$ is a consequence of the jumps of $g$ along the cuts from the $\zeta^k$ to infinity.

**On matrix dilogarithms via geometric quantization.** As explained in Remark 1.1 in this paper we concentrate on the quantum matrix dilogarithms (ie. with $N > 1$). These were derived in [5] from the Kashaev’s 6j-symbols for the cyclic representation theory of a Borel quantum subalgebra $B_{\zeta}$ of $U_{\zeta}(sl(2, \mathbb{C}))$, where $\zeta = \exp(2\pi i/N)$. Let us outline here very briefly an alternative construction based on geometric quantization of $\mathcal{R}_1$, thus clarifying their geometric origin (for details see [3]).

Consider an abstract oriented quadrilateral $Q$, triangulated by two triangles. Order the triangles of $Q$, and associate to each a copy $X_i$, $i = 1, 2$, of $X = \{(u,v,w) \in (\mathbb{C}^*)^3 | u w = -1\}$, where $u$, $v$ and $w$ correspond to the corners, ordered cyclically by using the orientation. Let us regard $\hat{C}$ as the Riemann surface of log, ie.

$$\hat{C} = \{(z; p) \in \mathbb{C}^* \times \mathbb{Z}^2 | ((z + i 0; p) \sim (z - i 0; p + 2), \forall z \in (\infty; 0))\}.$$ 

As in [1], we set $l(z; p) = \log(z) + \sqrt{-1} \pi p$. Let

$$\hat{X} = \{(u; p),(v; q), (w; r) \in \hat{X} | l(u; p) + l(v; q) + l(w; r) = 0\}$$

a subspace of the universal covering $\hat{X}$ of $X$, and put

$$\hat{C}(z; p,q) = \{(l(z; p), l(1-z^{-1}; q)) \in \hat{X} \times \hat{X} \}.$$ 

Denote $b_Q = d \log u_1 \wedge d \log v_1 + d \log u_2 \wedge d \log v_2$ the canonical complex symplectic form on $X_1 \times X_2$, and

$$\theta_Q = (1/2) ((l(u_1; p_1)d \log v_1 - l(v_1; q_1)d \log u_1) + (l(u_2; p_2)d \log v_2 - l(v_2; q_2)d \log u_2))$$

the symplectic potential for the lift of $b_Q$ to $\hat{X}_1 \times \hat{X}_2$. Consider the four punctured sphere $S_Q^2$ obtained by gluing along the boundary in the natural way $Q$ and the quadrilateral $Q'$ obtained from it by exchanging the diagonal. The $S_4$-action on the vertices of $S_Q^2$ reorders the copies $X_i$ ($i = 1, \ldots, 4$) attached to the triangles, and induces the usual action of $PSL(2, \mathbb{Z})$ on each $\hat{X}_i \cong \mathbb{C}^2$, with standard basis $l(u_i; p_i)$ and $l(v_i; q_i)$, by symplectomorphisms.

**Proposition 2.1.** There is a canonical family $\{\phi_Q^n\}_{n \in \mathbb{Z}} : (\hat{X}_1 \times \hat{X}_2, b_Q) \to (\hat{X}_3 \times \hat{X}_4, b_Q')$ of analytic symplectomorphisms equivariant for the $S_4$-action on $S_Q^2$, given by (see Figure 3)

$$\phi_Q^0(l(u_1; p_1), l(v_1; q_1), l(u_2; p_2), l(v_2; q_2)) = ((l(u'_1; p'_1), l(v'_1; q'_1)), (l(u'_2; p'_2), l(v'_2; q'_2))),$$

where we identify $\hat{X}$ with $\mathbb{C}^2$ (first two coordinates) and

$$\left\{ \begin{array}{ll} l(u'_1; p'_1) = l(u_1; p_1) - l(u_2; p_2) & \quad l(u'_2; p'_2) = l(u_2; p_2) - l(1-u_1v_2^{-1}; n) \\
1(v'_1; q'_1) = l(v_1; q_1) + l(u_2; p_2) & \quad l(v'_2; q'_2) = l(v_2; q_2) + l(u'_1; p'_1) + l(v'_1; q'_1). \end{array} \right.$$

The maps $\tilde{\phi}_Q^0$ satisfy the pentagon relation, that is $\phi_Q^n \circ \phi_Q^{n_2} \circ \phi_Q^{n_3} \circ (\phi_Q^{n_4})^{-1} \circ (\phi_Q^{n_5})^{-1} = \text{Id}_{\hat{X}_3}$, where $Q_i$ has diagonal the $i$th edge exchanged in Figure 4 for the positive cyclic ordering starting from the top left pentagon. Moreover $\theta_Q - \tilde{\phi}_Q^n \theta_{Q'} = -dR$, where $R : \hat{C}(u_1v_2; m; n) \to \hat{C}/2\pi \mathbb{Z}$ is the extended dilogarithm of $Q$, and $\hat{C}(u_1v_2; m; n) = \{(l(u_1; p_1) + l(v_2; q_2), l((1-u_1v_2^{-1}; n)) \}$ is attached to the diagonal of $Q$.

By “canonical” we mean that $\{\phi_Q^n\}$ is the unique satisfying some natural properties related to configuration spaces of points in $\mathbb{C}P^1$. In fact, $\hat{C}(z; p,q)$ is isomorphic to the moduli space of similarity classes of triangles in the complex plane endowed with lifts to $\overline{R}$ of the angles, or, equivalently, to the moduli space of isometry classes of hyperbolic ideal tetrahedra with even valued flattenings. The complex symplectic form $d \log u \wedge d \log v$ restricts to the real symplectic form $w = -d \log(z) \wedge d \log(1-z)$ on $\hat{C}(z; p,q)$. It is the differential version of the extended complex Dela invariant $\hat{\delta} : \hat{C}(z; p,q) \to \mathbb{C} \wedge \mathbb{C}$, $(z; p,q) \to l(z; p) \wedge_\mathbb{C} l(1-z; q)$ of [25].
When \( \text{Im}(z) \neq 0 \) the form \( w \) is Kähler for the usual complex structure on \( \mathbb{R}^2 \). Replace \( \hat{C} \) with the ramified covering \( \hat{C}_N \) of \( C^* \) obtained by taking the quotient with \( NZ \times NZ \). Put the forms \( Nw \) and \( N\theta \) on \( \hat{C}_N \). Standard half-form quantization of \( \hat{C}_N \) produces a \( N \)-dimensional vector space \( \Gamma^N \) of sections of a line bundle over \( \hat{C}_N \) (the coherent states), and the maps \( \hat{\Phi}^n_Q : \Gamma^N_1 \otimes \Gamma^N_2 \to \Gamma^N_3 \otimes \Gamma^N_4 \) induced by pull back via \( \{ \hat{\phi}^n_Q \} \), that is such that \( \hat{\Phi}^n_Q(s) = s \circ \hat{\phi}^n_Q \) for any \( s \in \Gamma^N_1 \otimes \Gamma^N_2 \), coincide with the matrix dilogarithms \( \mathcal{R}_N \) for some suitable basis. In particular, the pentagon relation for \( \hat{\phi}^n_Q \) lifts to the five term identities mentioned after Proposition 2.5 below.

\[
\begin{array}{c}
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\text{Figure 2. A diagonal exchange.}
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\text{Figure 3. The pentagon relation.}
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\]

2.2. QHG triangulated pseudomanifolds. We restrict the discussion to pseudomanifolds for simplicity, but all what follows makes sense for arbitrary singular 3-cycles whose non manifold locus is of codimension \( \geq 2 \). By a pseudomanifold \( Z \), possibly with non-empty boundary \( \partial Z \), we mean a compact oriented polyhedron with at most a finite set of non manifold points. The boundary is a pseudo-surface.

A QHG triangulated pseudomanifold \((Z, T)\) is a pseudomanifold \( Z \) obtained as the quotient of a finite family \( Z = \{ (\Delta^i, b^i, w^i, f^i, c^i) \} \) of flat/charged \( \mathcal{I} \)-tetrahedra, via a system of orientation reversing simplicial identifications of pairs of 2-faces such that the branchings match. The resulting triangulation \( T \) of \( Z \) is endowed with a global branching \( b \), and is possibly singular (multiply adjacent as well as self adjacent tetrahedra are allowed). The set of non manifold points of \( Z \) is contained in the set of vertices of \( T \). We do not impose for the moment any global constraint on the moduli, flattening and charges. Hence \( Z \) is equipped with a rough flat/charged \( \mathcal{I} \)-triangulation \( T = (T, b, w, f, c) \), where \( w = \{ w^i \} \) and so on.

Next we define the QHG triangulated pseudomanifold isomorphisms. Fix a QHG triangulated pseudomanifold \((Z, T)\), and let \( \epsilon_T : E(Z) \to E(T) \) be the identification map of edges. We define the total
modulus, log-branch and charge of an edge \( e \), respectively, by:

\[
\begin{align*}
W_T(e) &= \prod_{h \in \tau^{-1}(e)} w(h)^{\ast_b} \\
L_T(e) &= \sum_{h \in \tau^{-1}(e)} \ast_b l(h) \\
C_T(e) &= \sum_{h \in \tau^{-1}(e)} c(h)
\end{align*}
\]  

(7)

where \( \ast_b = \pm 1 \) according to the \( b \)-orientation of the tetrahedron in \( Z \) that contains \( h \), \( w(h) \) is the cross-ratio modulus at \( h \), \( l(h) \) the log-branch at \( h \), and \( c(h) \) the charge at \( h \).

**Remark 2.2.** It is easily seen that \( W_T(e) \) is a cross ratio for the four ‘extremal’ points on \( \partial \mathbb{H}^3 \) determined by gluing oriented hyperbolic ideal tetrahedra with moduli \( w(h)^{\ast_b} \) along a common edge (by continuity \( W_T(e) = 1 \) for a degenerate quadrilateral with three distinct vertices).

It is well-known that any two arbitrary naked triangulations \( T, T' \) of \( Z \) with the same boundary triangulation can be connected (keeping the boundary triangulation fixed) by a finite sequence of the local moves shown in Figure 4, the \( 2 \leftrightarrow 3 \) move (top) and the bubble move (bottom).

![Figure 4](image)

**Figure 4.** The moves on naked singular triangulations.

For any such a local move \( T \leftrightarrow T' \) we have two triangulations of a same portion of a polyhedron \( Q \). Assume that both \( T \) and \( T' \) extend to portions \( (Q, T) \) and \( (Q, T') \) of QHG triangulated pseudomanifolds. We have to specify the admissible QHG transits \( T \leftrightarrow T' \). In any case we require that they are local (that is, the portions complements remain unchanged), and that the branchings coincide at every common edge of \( T \) and \( T' \).

For the \( 2 \leftrightarrow 3 \) move we also require that at every common edge \( e \) as above the total modulus, log-branch and charge coincide

\[
W_T(e) = W_{T'}(e), \quad L_T(e) = L_{T'}(e), \quad C_T(e) = C_{T'}(e).
\]

(8)

The same rule restricted to the total modulus and total log-branch holds also for the bubble move; however, the total charge behaves in a different way: any bubble transit \( T \leftrightarrow T' \) includes a marked edge \( e \) common to \( T \) and \( T' \). Referring to the bottom of Figure 4 we require that \( C_T(e) = C_{T'}(e) - 2 \), while for the other two common edges the total charges are unchanged.

**Remark 2.3.** For every QHG transit supported by a \( 2 \leftrightarrow 3 \) move (top of Figure 4), if \( E_0 \) denotes the new edge in \( T' \) then

\[
W_{T'}(E_0) = 1, \quad L_{T'}(E_0) = 0, \quad C_{T'}(E_0) = 2.
\]

(9)
For every QHG bubble transit, let $f'$ be the unique new 2-simplex of $T'$ that contains the marked edge $e$. Denote by $E_1$ and $E_2$ the other edges of $f'$, and by $E_3$ the further new edge of $T'$. Then we have

$$W_{T'}(E_j) = 1, \quad L_{T'}(E_j) = 0, \quad j = 1, 2, 3$$

$$C_{T'}(E_1) = C_{T'}(E_2) = 0, \quad C_{T'}(E_3) = 2.$$ 

**Definition 2.4.** A QHG isomorphism between QHG triangulated pseudomanifolds is any finite composition of QHG transit configurations and oriented simplicial homeomorphisms that preserve the whole decoration.

### 2.3. QHG universal functor.

For every odd $N \geq 1$, we associate to every QHG triangulated pseudomanifold $(Z, T)$ a trace tensor $H_N(T)$, as follows. Define an $N$-state of $T$ as a function that gives every 2-simplex an index, with values in $\{0, \ldots, N - 1\}$. Every $N$-state determines an entry for each matrix dilogarithm $R_N(\Delta, b, w, f, c)$. As two tetrahedra induce opposite orientations on a common 2-face, an index is down for the $R_N$ of one tetrahedron while it is up for the other (see Figure 1). By summing over repeated indices we get the total contraction of the tensors $\{R_N(\Delta, b, w, f, c)\}$, that we denote $\prod_{\Delta \subset T} R_N(\Delta, b, w, f, c)$. Let $v_I$ and $v_\delta$ be the number of vertices of $T \setminus \partial T$ and $\partial T$, respectively, that correspond to manifold points. We set

$$H_N(T) = N^{-(v_\delta/2 + v_I)} \prod_{\Delta \subset T} R_N(\Delta, b, w, f, c)$$

The type of a trace tensor $H_N(T)$ depends on the $b$-signs of the boundary triangles of $(T, b)$. The matrix dilogarithms themselves are special instances of trace tensors. We have

**Proposition 2.5.** For every odd $N \geq 1$, up to sign and multiplication by $N$th roots of unity the trace tensor $H_N(T)$ is invariant up to QHG isomorphism.

This result is a restatement of Theorem 2.1 (2) and Lemma 6.7 of [5], and summarizes the fundamental functional relations satisfied by the matrix dilogarithms. In particular those corresponding to $2 \leftrightarrow 3$ QHG transits are usually called five terms identities. In Figure 5 we show one instance in dual terms.

The normalization factor $N^{-v_I}$ in (11) is due to the bubble move, that changes by 1 the number of internal vertices.

![Figure 5. A 2 ↔ 3 QHG move (\(x_1 = y/x, x_2 = y(1-x)/x(1-y)\), and \(x_3 = (1-x)/(1-y)\)).](image)

The notions of “boundary” (also allowing only portions of the standard boundary - see eg. [30]), bordism, and bordism gluing are well defined for the category of QHG triangulated pseudomanifolds considered up to QHG isomorphisms. Hence, for every odd $N \geq 1$, the association of the trace tensor to each bordism defines a functor. Note that the identities between trace tensors hold up to the phase ambiguity of Proposition 2.5. Also, the normalization factor $N^{-v_\delta/2}$ in (11) compensates the change in the number of internal vertices when gluing along complete connected components of the (usual) boundary (this is easily adapted to more general gluings).
Any bordism is the gluing of flat/charged I-tetrahedra, considered as elementary bordisms between the couples of 2-faces having b-sign equal to −1 or +1. The matrix dilogarithms can be interpreted as the amplitudes of a diagonal exchange that relates two quadrilateral triangulations (recall the above discussion about the geometric quantization derivation of matrix dilogarithms.)

2.4. Towards geometric specializations. The category of QHG triangulated pseudomanifolds is built ad hoc on the functional properties of the matrix dilogarithms. In order to eventually get specializations with a geometric content, such as QHFT, it is necessary to refine more and more our rough flat/charged I-triangulations. In fact we have to impose some global constraints that would be preserved by (possibly refined) QHG isomorphisms. It is useful to recall at once some of these refinements, and related matter. We adopt the notations introduced above.

Definition 2.6. A triple \( T = (T, b, w) \) is an I-triangulation if at each edge \( e \) not contained in \( \partial T \) we have the edge compatibility relation \( W_T(e) = 1 \). We say that \( T = (T, b, w, f) \) is a flattened I-triangulation (and \( f \) a global flattening) if moreover \( L_T(e) = 0 \).

It is easily seen that the collection of cross-ratio moduli of an I-triangulation \( T \) of a pseudomanifold \( Z \) defines a PL pseudo developing map \( d : \tilde{Z} \to \mathbb{P}^1 \), unique up to post composition with the action of \( PSL(2, \mathbb{C}) \), and a holonomy representation \( h : \pi_1(Z) \to PSL(2, \mathbb{C}) \) such that \( d(\gamma(y)) = h(\gamma)(d(y)) \) for every \( \gamma \in \pi_1(Z), y \in \tilde{Z} \).

Our main tool for producing I-triangulations is the following idealization procedure:

Definition 2.7. Let \( T = (T, b, z) \) be a branched triangulation of a pseudomanifold \( Z \) equipped with a \( PSL(2, \mathbb{C}) \)-valued 1-cocycle \( z \), where the cocycle relation is \( z(e_0)z(e_1)z(e_2)^{-1} = 1 \) on a branched triangle with edges \( e_0, e_1 \) and \( e_2 \). A 3-simplex \( \Delta \) of \( (T, b, z) \) with vertices \( x_0, x_1, x_2, x_3 \) is idealizable if

\[
\begin{align*}
\Delta & = (\Delta, b, z) \\
\Delta & \text{ is a branched tetrahedron} \\
\Delta & \text{ with \( \Delta \) as vertices} \\
\Delta & \text{ equipped with \( \Delta \)-branchings with the same} \\
\Delta & \text{ b-orientation} \\
\Delta & \text{ that is, a conformal transformation of \( \mathbb{C}P^1 \). We deduce that triples (}i_l, j_l, k_l\text{) are invariant under a change of} \\
\Delta & \text{ conformal transformation of \( \mathbb{C}P^1 \). We deduce that triples (}i_l, j_l, k_l\text{) are invariant under a change of}
\end{align*}
\]

\[\text{Moreover, replacing at each edge of } T \text{ the standard log with any other log determination still makes it flattened.}
\]

Proof. These expressions are just (corrected) signed sums of the standard logs of the edge vectors in the cross-ratio moduli \( w_0 = (u_2 - u_1)u_3/ u_2(u_3 - u_1) \), \( w_1 = u_2(u_3 - u_1)/u_0(u_3 - u_2) \) and \( w_2 = -(u_3 - u_2)u_0/(u_2 - u_1)u_3 \). They clearly define triples of log-branches. The idealizations for two distinct branchings with the same b-orientation are related by an element in \( PSL(2, \mathbb{C}) \), that is, a conformal transformation of \( \mathbb{C}P^1 \). We deduce that triples \( (l_0, l_1, l_2) \) are invariant under a change of
branching, because the angle formed by any pair of vectors is preserved (for instance \( u_2 - u_1 \) and \( u_3 \) or \( u_3 - u_1 \)). Then, the edge compatibility relations \( L_T(e) = 0 \) follow easily by developing the tetrahedra around \( e \) with branchings such that \( e = e_0 \), similarly as for Lemma 2.8. For the last claim we note that any log correction appears for each 3-simplex in two distinct \( I_1 \), with opposite signs. □

**Distinguished flat/charged \( T \)-triangulations.** Given an arbitrary non empty and properly embedded tangle \( L \) in \( Z \), we say that \((T,H)\) and \( T = (T,H,b,w,f,c) \) are **distinguished** if \( H \) is a subcomplex of the 1-skeleton of \( T \) isotopic to \( L \) and passing through all the vertices that are manifold points, and containing no singular vertices (we say that \( H \) is Hamiltonian).

**Definition 2.10.** A distinguished \( T \)-triangulation \( T = (T,H,b,w,f,c) \) is **flat/charged** if \( f \) is a global flattening, and

\[
C_T(e) = \begin{cases} 
0 & \text{if } e \subset H \\
2 & \text{if } e \subset T \setminus (H \cup \partial T)
\end{cases}
\]

In such a case \( c \) is said a **global charge**.

The existence of global flattenings of an arbitrary \( T \)-triangulation with empty boundary, and of global charges (such that \( C_T(e) = 2 \) at all edges \( e \)) on any topological ideal triangulation of an oriented 3-manifold whose boundary consists of tori, was proved by Neumann in [25], section 9, and [24], section 6, respectively. This last result is easily adapted to the existence of global charges on distinguished triangulations \((T,H)\) (see [4], Theorem 4.7).

We have to refine the QHG isomorphisms in order to deal with distinguished flat/charged \( T \)-triangulations. First we have to incorporate the Hamiltonian tangles into the bare moves. Any positive \( 2 \to 3 \) move \( T \to T' \) naturally specializes to a move \((T,H) \to (T',H')\); in fact \( H' = H \) is still Hamiltonian. For positive bubble moves, we assume that an edge \( e \) of \( H \) lies in the boundary of the involved 2-simplex \( f \); then \( e \) lies in the boundary of a unique 2-simplex \( f' \) of \( T' \) containing the new vertex of \( T' \). We define the Hamiltonian subcomplex \( H' \) of \( T' \) just by replacing \( e \) with the other two edges of \( f' \). The inverse moves are defined in the same way; in particular, for negative \( 3 \to 2 \) moves we require that the edge disappearing in \( T \) belongs to \( T' \setminus H' \). The \( 2 \leftrightarrow 3 \) QHG transit specializes verbatim. For the bubble transit we just impose that the marked edge \( e \) (see section 2.2) coincides with the above edge of \( H \). Thanks to (9) and (11) we see that distinguished flat/charged \( T \)-triangulations are closed under such refined QHG transits.

**Remark 2.11.** The residue mod\( (2\sqrt{-1}\pi N) \) of the classical log-branches of a flattened \( T \)-triangulation are equivalently given by \( N \)th roots \( w'(h) \) of the cross-ratio moduli \( w(h) \) such that \( w'(e_1)w'(e_2)w'(e_3) = 1 \) at each 3-simplex and \( \prod_{h \in c_2^{-1}(e)} w'(h)^{\gamma}(e) = 1 \) at each edge, with the notations of (7). For level \( N \) quantum log-branches of distinguished flat/charged \( T \)-triangulations this is replaced with

\[
w'(e_1)w'(e_2)w'(e_3) = \exp\left(-\gamma_b \sqrt{-1}\pi/N\right)\]

and

\[
\prod_{h \in c_2^{-1}(e)} w'(h)^{\gamma}(e) \begin{cases} 1 & \text{if } e \subset H \\
\exp\left(-2\sqrt{-1}\pi/N\right) & \text{otherwise.}
\end{cases}
\]

**Cohomological weights and structural facts.** Given an \( T \)-triangulation \( T = (T,b,w) \) without boundary, let \( T_0 \) be the complement of an open cone neighborhood of each 0-simplex. This is a disjoint union of triangulated closed oriented surfaces, the links of the vertices in \( T \). To each flattening \( f \) of \( T \) a class \( \gamma(f) \in H^1(\partial T_0; \mathbb{C}) \) is associated as follows. Represent any non zero integral 1-homology class \( a \) of \( \partial T_0 \) by “normal paths”, that is, a disjoint union of oriented essential simple closed curves transverse to the triangulation and such that no component enters and exits from the same face of a 2-simplex. Such a curve selects a vertex for each 2-simplex. The value \( \gamma(f)(a) \) is defined as the signed sum of the log-branches of the edges ending at the vertices selected by \( a \). For each vertex \( v \) the sign is \( \gamma_b \) if the path goes in the direction given by the orientation of \( \partial T_0 \) as viewed from \( v \), and \( -\gamma_b \) otherwise. Using the edge compatibility relations of Definition 2.6 it is easily checked that \( \gamma(f) \) does not depend on the choice of normal path representative. If the holonomy of \( T \) is trivial or parabolic about each (non manifold) point this class takes values in \( 2\mathbb{Z} \) [25, Proposition 5.2]. Similarly we can
define \( \gamma_2(f) \in H^1(T_0; \mathbb{Z}/2\mathbb{Z}) \) by using normal paths in \( T_0 \) and taking modulo 2 sum of the flattenings we meet along the paths. We call \( (\gamma(f), \gamma_2(f)) \) the (cohomological) weight of \( f \). These definitions extend immediately to global charge, replacing log-branches with charges; then branchings are not needed to compute the signed sums (ie. put \( \kappa_0 = 1 \)).

In [24], section 4 (see also [23], section 9), Neumann defines an integral chain complex \( J \) such that global charges are defined from cycles at level 3. The maps \( \gamma \) and \( \gamma_2 \) above are well-defined on the third homology group \( H_3(J) \) and satisfy:

**Theorem 2.12.** ([24], Theorem 5.1) The sequence

\[
0 \rightarrow H_3(J) \xrightarrow{\gamma_2} H^1(T_0; \mathbb{Z}/2\mathbb{Z}) \oplus H^1(\partial T_0; \mathbb{Z}) \xrightarrow{i^*} H^1(\partial T_0; \mathbb{Z}/2\mathbb{Z}) \rightarrow 0
\]

is exact, where \( r : H^1(\partial T_0; \mathbb{Z}) \rightarrow H^1(\partial T_0; \mathbb{Z}/2\mathbb{Z}) \) is the coefficient map and \( i^* : H^1(T_0; \mathbb{Z}/2\mathbb{Z}) \rightarrow H^1(\partial T_0; \mathbb{Z}/2\mathbb{Z}) \) is induced by the inclusion \( \partial T_0 \rightarrow T_0 \).

In particular, any pair \((h, k) \in H^1(T_0; \mathbb{Z}/2\mathbb{Z}) \times H^1(\partial T_0; \mathbb{Z})\) with \( r(k) = i^*(h) \) is a weight for some global charges. This fact extends to log-branches as follows. Assume that the holonomy of \( T \) restricted to each component of \( \partial T_0 \) takes values in a Borel subgroup of \( PSL(2, \mathbb{C}) \), so that it fixes some point in \( \partial T \). Let \( \gamma' \) be defined as above, except that for each 3-simplex log-branches are taken with flattenings \( f_0 = f_1 = 0 \). It can be checked that \( \gamma'(a) \) is the logarithm of the derivative of the holonomy of \( a \) (a similarity), up to multiples of \( 2\sqrt{-1}\pi \). Then, any pair \((h, k) \in H^1(T_0; \mathbb{Z}/2\mathbb{Z}) \times H^1(\partial T_0; \mathbb{C})\) such that

\[
\begin{cases}
(k - \gamma')/\sqrt{-1}\pi \in H^1(\partial T_0; \mathbb{Z}) \\
r((k - \gamma')/\sqrt{-1}\pi) = i^*(h)
\end{cases}
\]

is a weight for some flattening. For instance, when \( \gamma'(a) \in 2\sqrt{-1}\pi \mathbb{Z} \) for all \( a \) the first condition means that \( k \) is integral.

Finally, the structure of the spaces of flattenings and integral charges is given by Theorem 2.4 in [24]. For each fixed weight \((h, k)\) they form an affine space over an integral lattice. Generators have the following combinatorial realization: for each 3-simplex in the star of an edge \( e \), add +1 to the flat/charges of one of the two other pairs of opposite edges, and −1 for the other pair, so that the total log-branches or charges stay equal everywhere. In particular, for flattenings of an idealization \( T_k \) any generator is obtained by adding +1 to the log determination at some edge. Hence any flattening of \( T_k \) inducing the weight of the canonical flattening of Lemma 2.9 differs from it as described in the statement.

We note that the above refined QHG isomorphisms preserve the weights (see eg. Lemma 4.12 in [4]).

The difference in considering global flattenings or charges with different mod(2) weights in \( H^1(T_0; \mathbb{Z}/2\mathbb{Z}) \) seems to carry not so essential information (see Theorem 5.4 and Remark 5.5). This contrasts with boundary weights in \( H^1(\partial T_0; \mathbb{Z}) \), which play a key role in surgery formulae (see [24], theorems 14.5 and 14.7, and section 6.2 below). We note that a process involving 2-handle surgery allows to define explicit isomorphisms between lattices of flat/charges with different boundary weights (see [24], section 11, p. 457).

3. Parameters for \( PSL(2, \mathbb{C}) \)-characters of surfaces

Fix a compact closed oriented surface \( S \) of genus \( g \) with a non empty set \( V = \{v_1, \ldots, v_r\} \) of marked points, and negative Euler characteristic \( \chi(S \setminus V) < 0 \). Denote by \( \pi \) the fundamental group of \( S \setminus V \), and by

\[
\mathcal{R}(g, r) = \text{Hom}(\pi, \text{PSL}(2, \mathbb{C}))/\text{PSL}(2, \mathbb{C})
\]

the set of all conjugacy classes of \( \text{PSL}(2, \mathbb{C}) \)-valued representations of \( \pi \). The group \( \pi \) is free of rank \( \kappa = 2g+r−1 \). Any choice of free generators of \( \pi \) identifies the set \( \text{Hom}(\pi, \text{PSL}(2, \mathbb{C})) \) with \( \text{PSL}(2, \mathbb{C})^\kappa \).

Different such identifications are related by algebraic automorphisms of \( \text{PSL}(2, \mathbb{C})^\kappa \). Moreover, the isomorphism \( \text{PSL}(2, \mathbb{C}) \cong \text{SO}(3, \mathbb{C}) \) induced by the adjoint action \( \text{Ad} : \text{PSL}(2, \mathbb{C}) \rightarrow \text{Aut}(\text{sl}(2, \mathbb{C})) \) implies that \( \text{Hom}(\pi, \text{PSL}(2, \mathbb{C})) \) is an affine complex algebraic set, with the complex algebraic action of \( \text{PSL}(2, \mathbb{C}) \) by conjugation.
3.1. Efficient triangulations. Fix a surface $F$ with $r$ boundary components, obtained by removing from $S$ the interior of small 2-disks $D_i$ such that $v_i \in \partial D_i$. A triangulation $T'$ of $S$ with the set of vertices equal to $V$ is called a topological ideal triangulation of $S \setminus V$. Given such a $T'$, we need a marking of corners of the 2-simplices. The best suited to 3-dimensional extension are induced by global branchings $b'$ of $T'$, and it is known that pairs $(T', b')$ always exist. Then, as in section 2, we have a sign function $\sigma = \sigma(T', b')$. A corner map $v \mapsto c_v$ associates to each vertex $v$ of $T'$ the corner at $v$ of a triangle, say $t_v$, in its star. We say that $v \mapsto c_v$ is $t$-injective if $v \mapsto t_v$ is injective.

**Lemma 3.1.** For every $(g, r) \neq (0, 3)$, every triangulation $T'$ of $S$ with $r$ vertices admits $t$-injective corner maps.

**Proof.** For $(g, r) = (0, 4)$ or $g > 0$ and $r = 1$, it is immediate to construct such a triangulation. Subdividing a triangle by taking the cone from an interior point preserves the existence of $t$-injective corner maps. By induction on $r$ we deduce that for every $(g, r) \neq (0, 3)$ there exist triangulations of $S$ as in the statement.

In Figure 6 the corner selection is specified by a $\ast$, and the rows show all the possible flip moves on triangulations of $S$, up to obvious symmetries, that preserve the injectivity of corner maps. Consider triangulations $T', T''$ of $S$ with $r$ vertices, such that $T''$ supports a $t$-injective corner map. It is well known that $T''$ is connected to $T'$ via a finite sequence of flips. The $t$-injective corner map for $T''$ yields a $t$-injective corner map for $T'$ by decorating these flips as in Figure 6. \hfill \Box
There are no obstructions to use arbitrary corners maps in what follows. Specializing to injective ones just simplifies the exposition. For \((g, r) = (0, 3)\), triangulations have two 2-simplices, one with two selected corners. From now on, we assume that \((g, r) \neq (0, 3)\), the extension to the \((0, 3)\) case being straightforward.

Given a pair \((g, r) \neq (0, 3)\) and \((T', b')\) as in Lemma 3.1 fix a \(t\)-injective map \(v \mapsto c_v\). In the interior of each triangle \(t_v\) consider a bigon \(D_v\) with one vertex at \(v\), and call \(v'\) the other vertex. Remove from \(t_v\) the interior of \(D_v\), and triangulate the resulting cell \(s_v = t_v \setminus \text{Int}(D_v)\) by taking the cone with base \(v'\). Repeating this procedure for each \(t_v\), we get a triangulation \(T\) of \(F\) with \(2r\) vertices and \(p + 2r\) triangles, where \(p\) denotes the number of triangles of \(T'\). The set of edges of \(T\), \(E(T)\), contains \(E(T')\) in a natural way, and \(|E(T)| = |E(T')| + 4r\). We extend \(b'\) to a branching \(b\) on \(T\) as in Figure 7 and the sign function \(\sigma_{(T', b')}\) to the triangles of \((T, b)\) in the natural way. Note that the figure shows only one of the possible branching configurations. In general we extend \(b'\) to \(b\) so that we can recover \((T', b')\) from \((T, b)\) by “zipping” and “collapsing”, as suggested at the bottom of Figure 7.

In what follows, for simplicity we will refer to this configuration, as the treatment of the others is similar.

**Figure 7.** The branched triangulated cell \(s_v\).

**Definition 3.2.** We call the pair \((T, b)\) an efficient triangulation (for short: \(e\)-triangulation) of \(F\). For each vertex \(v\), the preferred inner triangle at \(v\) is the triangle \(\tau_v\) in \(s_v\) with an edge on \(\partial F\) whose \(b\)-orientation coincides with the boundary orientation of \(F\).

**3.2. Cocycle parameters.** The inclusions of \(\text{Int}(F)\) into \(F\) and \(S \setminus V\) induce identifications of the respective fundamental groups.

Fix an \(e\)-triangulation \((T, b)\) of \(F\). Denote by \(Z(T, b)\) the space of \(\text{PSL}(2, \mathbb{C})\)-valued 1-cocycles on \((T, b)\); we stipulate that on a triangle with ordered \(b\)-oriented edges \(e_0, e_1, e_2\) the cocycle relation is \(z(e_0)z(e_1)z(e_2)^{-1} = 1\). In this section we construct a parametrization of \(\mathcal{R}(g, r)\) based on cocycle coefficients by specifying subsets of \(Z(T, b)\) with small ‘residual gauge groups’, that make principal algebraic bundles over the “strata” of a suitable partition of \(\mathcal{R}(g, r)\). These strata are determined by the holonomies around the boundary components of \(F\).

Write \(C(T, b)\) for the space of \(\text{PSL}(2, \mathbb{C})\)-valued 0-cochains on \((T, b)\), that is the \(\text{PSL}(2, \mathbb{C})\)-valued functions defined on the set of vertices of \(T\). Two 1-cocycles \(z\) and \(z'\) are said equivalent up to gauge transformation if there is a 0-cochain \(\lambda\) such that, for every oriented edge \(e = [x_0, x_1]\), we have

\[
    z'(e) = \lambda(x_0)^{-1}z(e)\lambda(x_1).
\]

It is well known that the quotient set \(H(T, b) = Z(T, b)/C(T, b)\) is in one-one correspondence with \(\mathcal{R}(g, r)\). Indeed, for any fixed \(x_0 \in T\) we have a natural surjective map

\[
f_{x_0} : Z(T, b) \to \text{Hom}(\pi, \text{PSL}(2, \mathbb{C})),
\]

where \(\pi = \pi_1(F, x_0)\), and two representations \(f_{x_0}(z)\) and \(f_{x_0}(z')\) define the same point in \(\mathcal{R}(g, r)\) if and only if \(z\) and \(z'\) are equivalent up to gauge transformation. Note that the complex dimension of \(H(T, b)\) is \(3(|E(T)| - (p + 2r) - 2r) = -3\chi(F)\), which is the dimension of \(\mathcal{R}(g, r)\). This is because the complex dimension of \(C(T, b)\) is equal to \(6r\), the set \(Z(T, b)\) is defined by \(3(p + 2r)\) polynomial relations.
on $3|E(T)|$ variables, and $\text{PSL}(2, \mathbb{C})$ has trivial centre. (Recall that $p$ is the number of triangles of the initial triangulation $T'$ of $S$ with $r$ vertices.)

Denote by $B^+(2, \mathbb{C})$ (respectively $B^-(2, \mathbb{C})$) the Borel subgroup of $\text{SL}(2, \mathbb{C})$ of upper (respectively lower) triangular matrices. Let $PB^\pm(2, \mathbb{C}) = B^\pm(2, \mathbb{C})/ \pm I$ and put

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$ 

Define a map

$$(15) \quad \Psi : \text{SL}(2, \mathbb{C}) \to \text{SL}(2, \mathbb{C}), \quad \Psi(A) = PAP$$

and denote by $\Psi$ the induced automorphism of $\text{PSL}(2, \mathbb{C})$. We have $\Psi(B^\pm(2, \mathbb{C})) = B^\mp(2, \mathbb{C})$. For any $g \in \text{PSL}(2, \mathbb{C})$ we distinguish the type of $g$ as trivial, parabolic or generic (the latter for elliptic or loxodromic) with the obvious meaning. We denote

$$C^+(g) \in PB^+(2, \mathbb{C})$$

the canonical upper triangular matrix (up to sign) representative of the conjugacy class of $g$ (for generic $g$ we normalize $C^+(g)$ by stipulating that the top diagonal entry has absolute value $> 1$), and we set

$$C^-(g) = \Psi(C^+(g)).$$

We define the type of $\rho \in \mathcal{R}(g, r)$ as the $n$-uple of types of the $\rho$-holonomies of the oriented boundary components $\gamma_1, \ldots, \gamma_r$ of $F$. Put:

$$\mathcal{R}(g, r, t) = \{\rho \in \mathcal{R}(g, r) \mid \rho \text{ has type } t\}$$

$$\mathcal{R}(g, r, C^\pm) = \{\rho \in \mathcal{R}(g, r, t) \mid C^\pm(\rho(\gamma_i)) = C^\pm_i\}$$

where $C^\pm = (C^+_1, \ldots, C^+_r), C^\mp = (C^-_1, \ldots, C^-_r)$ is an arbitrary diagonal or unipotent element in $PB^\pm(2, \mathbb{C})$, and $C^\pm$ has type $t$. For any $g \in PB^\pm(2, \mathbb{C})$ let us write $g = [a, b]^\pm$, where $a$ is the top diagonal entry of $g$ and $b$ is the non diagonal one (we do the abuse of confusing $g$ with its projective class). For every vertex $v_i$ of $(T, b)$, let $e_i^+$ be the boundary edge of the preferred inner triangle $\tau_{v_i}$ (see Definition 3.2), and $\gamma_i$ the oriented boundary loop of $F$ based at $v_i$. Recall the projections $f_{v_i} : Z(T, b) \to \text{Hom}(\pi, \text{PSL}(2, \mathbb{C}))$.

**Definition 3.3.** The set of $(\pm)$-cocycle parameters for $\mathcal{R}(g, r, C^\pm)$ is

$$\mathcal{Z}(T, b, C^\pm) = \{z \in Z(T, b) \mid \forall i = 1, \ldots, r, \quad f_{v_i}(z(\gamma_i)) = C^\pm_i, \quad z(e_i^+) = [1, 1/2]^\pm\}.$$ 

Clearly $\mathcal{Z}(T, b, C^\pm)$ is non empty. In fact, any $z \in Z(T, b)$ with $\text{conj} \circ f(z) \in \mathcal{R}(g, r, C^\pm)$ is equivalent to one in $\mathcal{Z}(T, b, C^\pm)$ via some gauge transformation. Moreover, given a vertex $v_i$ and a 0-cochain $s$ with support at $v_i$ and $v'_i$, $s$ maps $\mathcal{Z}(T, b, C^\pm)$ onto itself if and only if

$$s(v_i) \in \text{Stab}(C^\pm)$$

and

$$s(v'_i)^{-1}[1, 1/2]^\pm s(v_i) = [1, 1/2]^\pm.$$ 

As the triangles $\tau_{v_i}$ are in one-one correspondence with the $v_i$, we deduce that there is a projection

$$(16) \quad p^\pm_C : \mathcal{Z}(T, b, C^\pm) \to \mathcal{R}(g, r, C^\pm)$$

with fiber isomorphic to the group

$$\mathcal{G}(T, b, C^\pm) := \text{Stab}(C^\pm) \times \cdots \times \text{Stab}(C^\mp)$$

that we call the *residual gauge transformations*. Denote by $\mathcal{Z}(T, b, t)^\pm$ (respectively $p^\pm C$) the union of the $\mathcal{Z}(T, b, C^\pm)$ (respectively $p^\pm_C$) over all $C^\pm$ with type $t$, and put

$$\mathcal{Z}(T, b)^\pm = \bigsqcup \mathcal{Z}(T, b, t)^\pm, \quad p^\pm = \bigsqcup p^\pm C.$$ 

For future reference, we summarize the above constructions in the following proposition.
Proposition 3.4. For every $C^\pm$, the projection $p^\pm : Z(T,b)^\pm \to R(g,r)$ restricts to a complex affine algebraic principal $G(T,b,C^\pm)$-bundle

$$Z(T,b,C^\pm) \to R(g,r).$$

The map $\Psi$ in (15) yields an isomorphism $Z(T,b,C^\pm) \cong Z(T,b,C^\mp)$. Moreover we have:

$$\dim(Z(T,b,C^\pm)) = -3\chi(F),$$

$$\dim(R(g,r,C)) = -3\chi(F) - \dim(G(T,b,t_C)),$$

$$\dim(R(g,r,t_C)) = \dim(R(g,r,C)) + \alpha(t_C)$$

where $t_C$ is the type of $C$ and $\alpha(t_C)$ is the number of generic entries of $t_C$.

Observe that: if $t_{\text{gen}}$ is the purely generic type, then $\dim(R(g,r,t_{\text{gen}})) = -3\chi(F)$; if $t_{\text{par}}$ is purely parabolic, then $\dim(R(g,r,t_{\text{par}})) = -3\chi(F) - r$; if $t_I$ is purely trivial, then $\dim(R(g,r,t_I)) = -3\chi(F) - 3r = -3\chi(S) = 6g - 6$. Hence $R(g,r,t_{\text{gen}})$ is a dense open subset of $R(g,r)$. Moreover, via the inclusion of closures, we have a filtration of $R(g,r)$ for which $R(g,r,t_I)$ is the ‘deepest’ part. It would be interesting to study the singularities of the closure of each $R(g,r,t)$ in $R(g,r)$, in order to check if this filtration induces a “stratification” of $R(g,r)$.

Remark 3.5. For parabolic elements $g \in PSL(2,\mathbb{R})$ we have two conjugacy classes, that can be distinguished by a sign. So, replacing $PSL(2,\mathbb{C})$ with $PSL(2,\mathbb{R})$, the constructions of this section still work by associating a sign to each parabolic end of the surface $S \setminus V$. Note that for $PSL(2,\mathbb{R})$-valued cocycles the idealization procedure described in Section 3.3 below gives only degenerate triangles and tetrahedra (with real shapes or cross-ratio moduli).

Example: the Fricke space. Suppose that $S$ has a unique marked point $v$. Choose a standard curve system $S = \{a_i, b_i\}_{i=1}^g$ based at $v$, so that

$$\pi_1(S,v) = \langle a_1, b_1, \ldots, a_g, b_g ; [a_1, b_1] \ldots [a_g, b_g] = 1 \rangle.$$

Cutting open $S$ along $S$ we get a 4g-gon $P$ with oriented boundary edges. Taking the cone to a vertex it is easy to construct a branched triangulation of $P$, which induces one, say $(T',b')$, for $(S,v)$. Denote by $(T,b)$ any $e$-triangulation obtained from $(T',b')$.

Recall that the Teichmüller space $T(S)$ can be identified with the set of conjugacy classes of $PSL(2,\mathbb{R})$-valued discrete faithful representations of $\pi_1(S)$, and that we have the well-known (real-analytic) Fricke parametrization $T(S) \cong \mathbb{R}^{6g-6}$ (see e.g. [1]). For each $z \in T(S)$ the Fricke coordinates of $z$ are matrix entries of the $\tilde{z}(\gamma)$ for all $\gamma \in S$, where $\tilde{z}$ is a representative of $z$ specified by fixing once and for all three of the fixed points of $z(a_j)$ and $z(b_j)$. So $T(S)$ embeds in the space of real cocycle parameters for $F$ with trivial type $t_I$. This embedding is generalized easily to the Teichmüller space of arbitrary bordered Riemann surfaces, by considering the spaces $Z(T,b,t)$ for all types $t$ and $e$-triangulations of $(S,V)$ with arbitrary $V$ (see Remark 3.3).

3.3. Cross-ratio parameters. In this section we derive from the cocycle parameters $Z(T,b)^\pm$ other parameters for $R(g,r)$ which are related to the shear-bend coordinates for pleated hyperbolic surfaces. These parameters are obtained via an idealization procedure that includes the choice of a base point on the Riemann sphere. We fix this base point as 0. Note that the $B^+(2,\mathbb{C})$-orbit of 0 is the whole of $\mathbb{C}$, while $B^-(2,\mathbb{C})$ fixes 0. Hence the symmetry between $Z(T,b)^+$ and $Z(T,b)^-$ given by the map $\Psi$ in [15] shall be broken.

Definition 3.6. (Compare with Definition 2.4.) Let $(K,b)$ be any oriented surface branched triangulation. Let $z$ be any $PSL(2,\mathbb{C})$-valued cocycle on $(K,b)$. We say that $z$ is idealizable if for any triangle $t$ of $(T,b)$ with $b$-ordered edges $e_0$, $e_1$ and $e_2$, the points $u_0 = 0$, $u_1 = z(e_0)(0)$ and $u_2 = z(e_2)(0)$ are distinct in $\mathbb{C}$. We say that the complex triangle with vertices $u_0$, $u_1$ and $u_2$ is the idealization of $t$.

Definition 3.7. Let $(T,b)$ be an $e$-triangulation of $F$ obtained from a branched ideal triangulation $(T',b')$ of $S \setminus V$. A cocycle $z \in Z(T,b)^+$ is (strongly) idealizable if:

(a) $z$ is idealizable;

(b) $\Psi(z) \in Z(T,b)^-$, which cannot be idealizable at the triangles having an edge on $\partial F$, is nevertheless idealizable at every other triangle.
We denote by $Z_I(T, b)^+$ the set of (strongly) idealizable cocycles, and we put $Z_I(T, b)^- = \Psi(Z_I(T, b)^+)$ and $R_I(T, b) = p^+(Z_I(T, b)^+) = p^-(Z_I(T, b)^-)$.

Clearly, $Z_I(T, b)^+$ is a non-empty dense open subset of $Z(T, b)^+$. If every edge of $T$ has distinct endpoints (in case we say that $T$ quasi-regular), then $R_I(T, b) = R(g, r)$. In general, characters of representations with a free action on a non-empty domain of $\mathbb{C}P^1$ (such as quasi-Fuchsian representations) always belong to $R_I(T, b)$, for any $(T, b)$. By using the arguments of [22], Theorem 1, it can be shown that for any character $\rho \in R(g, r)$ of irreducible representations, there exists an $e$-triangulation $(T, b)$ with idealizable cocycles representing $\rho$. In fact, the union of a finite number of spaces $Z_I(T, b)^+$ cover the whole of $R(g, r)^{irr}$.

**Exponential $I$-parameters.** For any cocycle $z \in Z_I(T, b)^+$, we associate a non zero complex weight $W^+(z)(e)$ to each edge $e$ of $T$ that is not contained in $\partial F$, as follows. Let $p_e$ be the initial endpoint of $e$, and $t_l$ and $t_r$ the left and right adjacent triangles (as viewed from $e$). Locally modify the branching on $t_l \cup t_r$ by cyclically reordering the vertices on each triangle, so that $p_e$ is eventually the source of the new branching on both $t_l$ and $t_r$. The $(+)$-exponential $I$-parameter $W^+(z)(e)$ is the cross-ratio modulus at $e$ of the (possibly degenerate) branched oriented hyperbolic ideal tetrahedron spanned by the idealization of $t_r \cup t_l$, where the branching completes the one of $t_l \cup t_r$ so that $*_{b} = 1$.

Let us assume now that $\Psi(z) \in Z_I(T, b)^-$. For each $v \in V$, denote by $e_v^l$ and $e_v^r$ the edges of the triangle $t_v$ of $T'$ having $v$ as corner. At an edge $e$ of $T'$ distinct from any of the $e_v^l$, we define the $(−)$-exponential $I$-parameter $W^-(\Psi(z))(e)$ in the same way as $W^+(z)(e)$, but taking the idealization of $\Psi(z)$ instead of $z$. If $e$ is one of the $e_v^l$, the formula works as well, except that each left/right triangle with an edge on $\partial F$ is replaced with the innermost triangle in the corresponding triangle $t_v$ of $T'$. Note that $t_r \cup t_l$ is again a quadrilateral, because $0$ is a fixed point of the cocycle values at the boundary edges of $F$.

By varying the edge in $T$ or $T'$ for every $C^\pm$, we have two $(±)$-exponential $I$ parameter maps (recall that the number of edges of $T'$ is $−3\chi(F)$, and $p$ denotes the number of number of triangles of $T'$):

$$W^+ : Z_I(T, b, C^+) \to (\mathbb{C} \setminus \{0\})^{-3\chi(F)+2p}$$
$$W^- : Z_I(T, b, C^-) \to (\mathbb{C} \setminus \{0\})^{-3\chi(F)}.$$

**Definition 3.8.** We call $W^\pm(T, b, C) = W^\pm(Z_I(T, b, C^\pm))$ the $(±)$-exponential $I$-parameter space of $R_I(T, b, C)$.

We stress again that the result of the idealization strongly depends on the choice of the base point, here $0$. In particular, the exponential $I$-parameters are not invariant under arbitrary gauge transformations of cocycles. A remarkable exception is for gauge transformations associated to $0$-cochains $\lambda$ with values in $PB^−(2, \mathbb{C})$. Indeed, these act on the idealization of quadrilaterals as conformal transformations of the four vertices (this is because $0$ is fixed by every $\lambda(v), v \in V$), and cross-ratios are conformal invariants. This makes a big difference between $W^+$ and $W^−$. In fact the whole of the residual gauge group $G(T, b, C^−)$ acts on $W^+(T, b, C)$ via the map $W^+$. On the other hand, consider the subgroup of $G(T, b, C^−)$ defined as

$$BG(T, b, C^-) := \prod_{i=1, \ldots, r} \text{Stab}(C_i^-) \cap PB^-(2, \mathbb{C}).$$

Any fiber $(W^-)^{-1}(\Psi(z))$ is given by the $BG(T, b, C^-)$-orbit of $z$ in $Z_I(T, b, C^-)$. Hence the actual residual gauge transformations of $W^-, W^+$ are in one-one correspondence with the quotient set $G(T, b, C^-)/BG(T, b, C^-)$, where the equivalence class of $\lambda$ is $BG(T, b, C^-)\lambda$. For the matter of notational convenience, formally put $BG(T, b, C^+) := \prod_{i=1, \ldots, r} \text{Id}$. The map

$$\Theta^\pm : W^\pm(T, b, C) \to R(g, r, C^\pm)$$

given by $\Theta^\pm(W^\pm(z)) = p^\pm_z(z)$, where $p^\pm_z$ defined in [10], is a principal $G(T, b, C^\pm)/BG(T, b, C^\pm)$-bundle.

The situation is particularly clean when $C$ has no trivial entries:

**Proposition 3.9.** If $C$ has no trivial entries the $(−)$-exponential $I$-parameter map $W^- : Z_I(T, b, C^-) \to W^-(T, b, C)$ is invariant under gauge transformations.
Geometry and computation of $\Theta^\pm$. The maps $\Theta^\pm$ can be defined directly in terms of $I$-parameters. A way to see this is to consider $I$ triangulations of cylinders $C = F \times [0,1]$, as discussed in Section 6.3 and the corresponding pseudo-developing maps. We give here another description. Let us just consider $\Theta^-$. Recall that $PSL(2, \mathbb{C})$ is isomorphic to $\text{Isom}^+(\mathbb{H}^3)$, with the natural conformal action on $\mathbb{CP}^1 = \partial \mathbb{H}^3$ via linear fractional transformations. For any $\rho \in R(g,r,C^-)$, take a representative $\tilde{\rho} : \pi_1(F,q) \to PSL(2, \mathbb{C})$ in the conjugacy class. Consider the associated flat principal $PSL(2, \mathbb{C})$-bundle $F_\rho$. A trivializing atlas defines a cocycle $z \in Z(T,b,C^-)$. If $\rho \in R_I(T;b,C^-)$, we can take $z \in Z_I(T,b,C^-)$, so that the trivializing atlas of $F_\rho$ associated to the cellulation $T'^* \text{ dual to } T'$, with edges oriented by using the orientation of $F$ and the branching orientation of the edges of $T'$, has non trivial transition functions. These can be viewed as transition functions for the fiber bundle associated to $F_\rho$ and with fibre $\mathbb{H}^3$. For each 2-simplex $t$ of $T'$, there is a unique $g_t \in \text{Isom}(\mathbb{H}^3)$ (possibly reversing the orientation) mapping the vertices $u_0$, $u_1$ and $u_2$ of the idealization of $t$ to $0$, $\infty$ and $-1$ respectively. Then, the transition function along the edge of $T'^* \text{ positively transverse to } a$ given edge $e$ of $T'$ is of the form $(g_t)^{-1} \circ \varphi(z)(e) \circ g_t \in \text{Isom}^+(\mathbb{H}^3)$, where $t_i$ is the triangle on the left of $e$, and $\varphi(z)(e)$ is the isometry of $\mathbb{H}^3$ of hyperbolic type fixing $0$ and $\infty$ and mapping $1$ to $W^-(z)(e)$. Analytic continuation defines the parallel transport of $F_\rho$ along paths transverse to $T'$, whence a representation into $PSL(2, \mathbb{C})$ of the groupoid of such paths, well-defined up to homotopy $\text{rel}(\partial)$). In particular, it gives a practical recipe to compute $\tilde{\rho}$, that we describe now.

Let the base point $q$ be not in the 1-skeleton of $T'$. Given an element of $\pi_1(F,q)$, represent it by a closed curve $\gamma$ in $F$ transverse to $T'$, and which do not departs from an edge it just entered. Assume that $\gamma$ intersects an edge $e$ of $T'$ positively with respect to the orientation of $F$. Figure 8 shows three possible branching configurations for the two triangles glued along $e$. Fix arbitrarily a square root $W^-(z)(e)^{1/2}$ of $W^-(z)(e)$. Consider the elements of $PSL(2, \mathbb{C})$ given by

$$\gamma(e) = \begin{pmatrix} W^-(z)(e)^{1/2} & 0 \\ 0 & W^-(z)(e)^{-1/2} \end{pmatrix}, \quad p = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad l = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

and $r = l^{-1}$. The matrix $\gamma(e)$ represents the isometry with fixed points $0, \infty \in \mathbb{CP}^1$ and mapping $1$ to $W^-(z)(e)$. The elliptic elements $p$ and $l$ send $(0,1,\infty)$ to $(\infty,1,0)$ and $(\infty,0,1)$ respectively. For the portion of $\gamma$ on the left of Figure 8 if $\gamma$ turns to the left after crossing $e$ the parallel transport operator is $\gamma(e) \cdot p \cdot l$, while it is $\gamma(e) \cdot p \cdot r$ if $\gamma$ turns to the right. (The matrix multiplication is on the right, as is the action of $PSL(2, \mathbb{C})$ on the total space of $F_\rho$). In the middle and right pictures the parallel transport operators along the portion of $\gamma$ are given by $\gamma(e) \cdot l$ or $\gamma(e) \cdot p \cdot l$, and $\gamma(e) \cdot p \cdot r$ or $\gamma(e) \cdot r$ respectively. We see that the action of $p$, $l$ and $r$ depends on the reordering of the vertices after the mapping $\gamma(e)$. If $\gamma$ intersects $e$ negatively, we replace $\gamma(e)$ with $\gamma(e)^{-1}$ in the above expressions. A similar recipe applies for any other branching of the two triangles glued along $e$.

Continuing this way each time $\gamma$ crosses an edge of $T'$ until it comes back to $q$, we get an element $[W(z)]_q(\gamma) \in PSL(2, \mathbb{C})$ that depends only on the homotopy class of $\gamma$ based at $q$ and coincides with $\tilde{\rho}(\gamma)$, because of the identity $D_z(\gamma \cdot x) = [W(z)]_q(\gamma) \cdot D_z(x)$ for any $x \in \tilde{F}$, where $D_z : \tilde{F} \to \mathbb{CP}^1$ is a $\tilde{\rho}$-equivariant pseudo-developing map from the universal cover $\tilde{F}$ of $F$. Varying $\gamma$, we eventually get the representation $[W(z)]_q = \tilde{\rho} : \pi_1(F,q) \to PSL(2, \mathbb{C})$.
(-)-exponential $\mathcal{I}$-parameters and pleated surfaces. When $\mathcal{C}^-$ has no trivial entries, there is a nice interpretation of the parameter space $W^-(T, b, \mathcal{C})$ in terms of pleated hyperbolic surface structures on $S \setminus V$. (see eg. [10]; compare also with [9], Section 8.)

As before, let $T'$ be a triangulation of $S$ with vertices $V$, viewed as an ideal triangulation of $S^o = S \setminus V$. A pleated surface (with pleating locus $T'$) is a pair $(\bar{f}, r)$, where $r: \pi_1(S^o) \to \text{PSL}(2, \mathbb{C})$ is a group homomorphism (not up to conjugacy), and $\bar{f}: S^o \to \mathbb{H}^3$ is a map from the universal cover $\hat{S}^o$ of $S^o$ such that:

- $\bar{f}$ sends homeomorphically each component of the preimage $\bar{T}'$ of $T'$ in $\hat{S}^o$ to a complete geodesic in $\mathbb{H}^3$;
- $\bar{f}$ sends homeomorphically the closure of each component of $\hat{S}^o \setminus \bar{T}'$ to an ideal triangle in $\mathbb{H}^3$;
- $\bar{f}$ is $r$-equivariant, that is, for all $x \in S^o$, $\gamma \in \pi_1(S^o)$ we have $\bar{f}(\gamma x) = r(\gamma)\bar{f}(x)$.

Two pleated surfaces $(\bar{f}, r)$ and $(\bar{f}', r')$ are said isometric if there exists an isometry $A \in \text{PSL}(2, \mathbb{C})$ and a lift $\tilde{A}: \mathbb{H}^3 \to S^o$ of an isometry of $S^o$ such that $\tilde{A} = A \circ \bar{f}$ and $r'(\gamma) = Ar(\gamma)A^{-1}$ for all $\gamma \in \pi_1(S^o)$.

From [8] it is known that isometry classes of pleated surfaces $(\bar{f}, r)$ are in one-one correspondence with arrays $\{x_e\}_e$ of non zero complex numbers $x_e$ associated to the edges $e$ of $T'$, the exponential shear-bend parameters.

To a pleated surface $(\bar{f}, r)$ we can associate the type $\mathcal{C}^- = (C_1, \ldots, C_r)$ of the conjugacy class of $r$, defined as in Section 3.2. For instance, it is shown in [29] that the set of isometry classes of pleated surfaces given by real positive shear-bend parameters is real-analytic diffeomorphic to the Teichmüller space of hyperbolic metrics on $S^o$, with totally geodesic boundary completion or non compact finite area completion at $v_i$, according to the type of $C_i$, loxodromic or parabolic.

In general, we can also associate a sign to each loxodromic puncture. Namely, if $A$ is a small annulus neighborhood of $v_i$, each connected component $\tilde{A}$ of the preimage of $A$ in $\hat{S}^o$ is the fixed point set of a subgroup $\pi$ of $\pi_1(S^o)$. This subgroup is the image of the fundamental group of $A$ for some choice of base points and paths between these base points. All the edges of $\bar{T}'$ that meet $A$ are sent by $\bar{f}$ to geodesics lines that meet at one of the two fixed points of $r(\pi)$. We specify this fixed point by a sign, as it determines an orientation (whence a generator) for the axis of the group $r(\pi)$. Since any two subgroups $\pi$ as above are conjugated, for each puncture the fixed point assignment is $r$-equivariant, so that the sign is canonically associated to the puncture.

Recall that in Definition 3.3 the types $C_i$ where assigned to the components of $\partial F$ endowed with the boundary orientation. Let us remove this constraint, and associate to each boundary component with loxodromic type an orientation that we specify by a sign, positive for the boundary orientation, and negative otherwise. Then, for each type $C^-$ with $l$ loxodromic entries and each $l$-uple of signs $s$, we get a space $\mathcal{Z}_l(T, b, C^-, s)$. We define $W^-(T, b, C, s) = W(\mathcal{Z}_l(T, b, C^-, s))$.

**Proposition 3.10.** For each type $C$ with non trivial entries $C_i$, $l$ being loxodromic, and for each $l$-uple of signs $s$, the space $W^-(T, b, C, s)$ coincides with the exponential shear-bend parameter space $\mathcal{PS}_C^s$ of isometry classes of pleated surfaces of type $C$ and signs $s$:

$$\mathcal{PS}_C^s = \{ \{x_e\}_e \in (\mathbb{C} \setminus \{0\})^{-3x(s^o)} \mid \forall i = 1, \ldots, r, \prod x_e = \mu_i \}. \tag{17}$$

Here the product is over all edges $e$ with $v_i \in \partial e$ (counted with multiplicities), $\mu_i$ is 1 if $C_i$ is parabolic, and, if $C_i$ is loxodromic, $\mu_i$ is the dilation factor of the generator of $< C_i >$ specified by the sign of $v_i$. (Hence, in either case this is an eigenvalue of $C_i$.)

**Proof.** By the results of [8] recalled above, each point of $W^-(T, b, C, s)$ is identified with the family of exponential shear-bend parameters of an isometry class of pleated hyperbolic surface on $S^o$ with pleated locus $T'$. The definition of the map $\Theta: W^-(T, b, C, s) \to \mathcal{R}_l(T, b, C)$ shows that this isometry class has type $C$. Furthermore, by using the recipe given above for computing $\Theta$, we check that the holonomy of any positively oriented (with respect to the boundary orientation) small loop about the puncture $v_i$ is exactly $C_i$. The upper left diagonal entry is just the product of a square root of the exponential $\mathcal{I}$-parameters at the edges with endpoint $v_i$. From (17), which is an easy consequence of results in [8] (see sections 12.2-12.3 in that paper), we deduce that $W^-(T, b, C, s) \subset \mathcal{PS}_C^s$. 
Conversely, for any $(\tilde{f}, r) \in \mathcal{PS}_c^2$ we have $\text{conj}(r) \in \mathcal{R}(T, b, C)$, because $r$ is injective. Also, by [9], Proposition 33, it is known that the isometry class of $(\tilde{f}, r)$ is determined by $r$ and the signs $s$. As the map $\Theta$ is onto, let us take $z \in \mathbb{Z}_{\mathcal{F}}(T, b, C, -s)$ with $\text{conj} \circ f(z) = \text{conj}(r)$ and consider the isometry class of pleated hyperbolic surfaces associated to $W^-(z)$. There is a representative $(\tilde{f}', r)$ with the same holonomy $r$. As the signs of $(\tilde{f}', r)$ are $s$, the same as for $(\tilde{f}, r)$, the two pleated surfaces coincide. So $\mathcal{PS}_c^2 \subset W^-(T, b, C, s)$.

**Remark 3.11.** Exponential shear-bend parameters do not depend on branchings but the orientation of $F$ (taking the opposite branching edge orientation simultaneously exchanges the left and right triangles). However, branchings govern all choices in QHFT tensors. Also, they allow us to interpret exponential shear-bend parameters as exponential $I$-parameters, thus coming from 1-cocycles representing arbitrary $PSL(2, \mathbb{C})$-characters on the triangulated boundary of arbitrary compact orientable 3-manifolds. When $C$ has no trivial entries, the maps $W^\pm$ define decorated shear-bend parameter spaces similar to those occurring in [22, 26].

**Remark 3.12.** For types $C$ with no trivial entries, we can use simpler e-triangulations of $F$ (see Figure 9): for each of the triangles $t_v$ of the base surface $S$ we remove the interior of a monogon inside $v$, and triangulate the resulting quadrilateral by adding an edge $e_v$ with endpoints $v$ and the $b$-output vertex of the opposite edge. We extend $b$ by orienting $e_v$ from that vertex to $v$. Proposition 3.3 applies to the cocycle parameters based on such e-triangulations, for points of $\mathcal{R}(g, r)$, and also the treatment of exponential $I$-parameters works as well.

![Figure 9. More economic e-triangulation.](image)

**4. The QHFT bordism category**

We define first a topological $(2 + 1)$-bordism category. Then we will give it more structure, including the parameter spaces of the previous sections.

**4.1. Marked topological bordisms.** Like in Section 3 for every $(g, r) \in \mathbb{N} \times \mathbb{N}$ such that $g \geq 0$, $r > 0$, and $r > 2$ if $g = 0$, fix a compact closed oriented base surface $S$ of genus $g$ with a set $V = \{v_1, \ldots, v_r\}$ of $r$ marked points. Denote by $-S$ the same surface with the opposite orientation, and write $*S$ for $S = +S$ or $-S$. Moreover, fix a set of disjoint embedded closed segments $a_{v_i}$ in $S$ such that $a_{v_i}$ has one end point at $v_i$. We say that $v_i$ is framed by $a_{v_i}$.

We say that two orientation preserving diffeomorphisms $\phi_1 : *S \to \Sigma_1$ and $\phi_2 : *S \to \Sigma_2$ are equivalent if there is an orientation-preserving diffeomorphism $h : \Sigma_1 \to \Sigma_2$ such that $(\phi_2)^{-1} \circ h \circ \phi_1$ pointwise fixes the segments $a_{v_i}$ and is isotopic to the identity automorphism of $S$ relatively to $\{a_{v_i}\}$. We write $[*S, \phi]$ for such an equivalence class.

Let $Y$ be an oriented compact 3-manifold with (possibly empty) boundary $\partial Y$, with an input/output bipartition $\partial Y = \partial_- Y \cup \partial_+ Y$ of the boundary components (we say that $\partial_- Y$ is “at the bottom” of $Y$, while $\partial_+ Y$ is “on the top”). Each boundary component inherits the boundary orientation, via the usual convention last is the ingoing normal. Let $L_{\mathcal{F}}$ be a properly embedded non-empty framed tangle in $Y$. This means that $L_{\mathcal{F}}$ is a disjoint union of properly embedded orientable ribbons. We split $L_{\mathcal{F}} = (L_{\mathcal{F}})_i \cup (L_{\mathcal{F}})_h$, where $(L_{\mathcal{F}})_i$ is the internal part of $L_{\mathcal{F}}$ made by its closed connected components, homeomorphic to the annulus $S^1 \times [0, 1]$, while $(L_{\mathcal{F}})_h$ is the union of the components homeomorphic to the quadrilateral $I \times [0, 1]$. We consider $L_{\mathcal{F}}$ up to proper ambient isotopy. For every boundary
component $\Sigma$ of $Y$, we assume that $(L_F)_b \cap \Sigma \neq \emptyset$, and that $(L_F)_b \cap \Sigma$ consists of at least two segments if $g(\Sigma) = 0$ (we do not require that every component of $(L_F)_b$ goes from $\partial_-$ towards $\partial_+$). Hence, on each boundary component $\Sigma$ of $Y$ we have a set of marked points framed by $L_F \cap \Sigma$. Also, associated to $\partial_{\pm}$ we have a finite disjoint union $\alpha_{\pm} = \bigsqcup_{\phi \in \partial_{\pm}(S,\phi)} [S,\phi]$ of equivalence classes of diffeomorphisms as above.

Consider the topological $(2+1)$-bordism category with objects the empty set and any finite union of the $\{S,\phi\}$, and morphisms the triples $(Y,L_F,\alpha_{\pm})$ as above. We say that $(Y,L_F,\alpha_{\pm})$ is a bordism from $\alpha_-$ to $\alpha_+$ with support $(Y,L_F)$. We allow the case when $Y$ is a closed manifold, so that $\partial Y = \emptyset$ and $(Y,L_F)$ is a morphism from the empty set to itself. We stress that $L_F$ is non-empty in any case.

We can reformulate this category in a setup closer to that of the phase space parameters of Section 8, as follows. If we cut open each $a_{\nu}$ in $S$ we get an oriented surface $F$ with $r$ boundary bigon components. This is the domain of an elementary object $\{S,\phi\}$, where the target surfaces $\Sigma$ have now $r$ boundary components, and the diffeomorphisms $\phi$ are considered up to isotopy rel($\partial$).

Consider a bordism $(Y,L_F,\alpha_{\pm})$. On the boundary of each ribbon component of $L_F$ we keep track of a tangle line $X \times \{0\}$ ($X = S^1$ or $I$) for the corresponding component of the unframed tangle $L$, and there is a longitudinal line $X \times \{1\}$ that specifies the framing of the normal bundle of the parallel tangle line. These make a pair $\lambda = (\lambda,\lambda')$ of parallel unframed tangles in $Y$. Cutting open each ribbon we get a 3-manifold with corners $\tilde{Y}$. The boundary $\partial Y$ has two “horizontal” parts $\partial_{\pm} \tilde{Y}$ contained in $\partial_{\pm} Y$, and a “vertical” tunnel part $\tilde{L}_F$. The horizontal parts intersect the tunnel part at the corner locus; this is a union of bigons contained in $\partial Y$. Each boundary component $\Sigma$ of $Y$ corresponds to a horizontal boundary component of $\tilde{Y}$, still denoted by $\Sigma$. Each tunnel boundary component is made by the union of two copies of $X \times (0,1)$, glued each to the other at $\lambda \cup \lambda'$. The horizontal boundary components are the targets of elementary objects $\{S,\phi\}$, and each triple $(\tilde{Y},\tilde{L}_F)$ supports a morphism between such objects. Clearly, we can recover $(Y,L_F)$ from $(\tilde{Y},\tilde{L}_F)$, so that we have two equivalent settings to describe the same topological bordism category.

### 4.2. Boundary structures.

We equip a marked topological bordism with additional boundary structures by using the notions of section 9. Fix an $e$-triangulation $(T,b)$ of $F$. Following Remark 9.2 and the definition of log-branches, we put:

**Definition 4.1.** Let $z \in Z_1(T,b)^{\pm}$, and $e$ be a non boundary edge of $T$. Denote $l(e)$ the canonical log-branch of $W^{\pm}(z)(e)$, computed from the idealization of $\text{Star}(e)$ as in Lemma 2.9 and before Definition 3.8. For any collection $m = \{m_{\lambda}\}_{\lambda}$ of integers, one for each edge $\lambda$ in $T$, the classical Log-$L$-parameter of $(z,m)$ at $e$ is

$$l_{(z,m)e}(e) = l(e) + \sqrt{-1}\pi (m_a + m_c - m_b - m_d),$$

where $a,\ldots,d$ make $\text{Link}(e)$ with $a$ opposite and $e$, $a$ have coherent branching orientations (see Figure 10). Similarly, for every $N > 1$ and any other collection $n = \{n_{\lambda}\}_{\lambda}$ of integers, the quantum Log-$L$-parameter at $e$ is

$$l_{(z,m)e}(e) + \sqrt{-1}\pi N (m_a + m_c - m_b - m_d) - s_b\sqrt{-1}\pi (N + 1)(n_a + n_c - n_b - n_d).$$

We call the collections

$$f = \{f_e\}_e = \{(l_{(z,m)e}(e) - \log(W^{\pm}(z)(e)))/\pi \sqrt{-1}\}_e$$

and

$$c = \{c_e\}_e = \{n_a + n_c - n_b - n_d\}_e$$

the flattenings of $(z,m)$ and the charge of $n$ respectively, and we denote generically by $L$ any such a system of classical or quantum $(\pm)$-Log-$L$-parameters.

Remark that if $z \in Z_1(T,b,\mathcal{C}^-)$ and $\mathcal{C}$ has no trivial entries, then the $(\pm)$-Log-$L$-parameters depend on $W^-(z)$, not on $z$, because of Proposition 3.9.

**Definition 4.2.** The QHFT category is the $(2+1)$-bordism category with objects the empty set and any finite union of the $\{S,\phi\}$, where $\mathcal{L}$ is a system of $(\pm)$-Log-$L$-parameters, and morphisms the 4-uples $(\tilde{Y},\tilde{L}_F,\rho,\alpha_{\pm})$, where: $\rho$ is a conjugacy class of $\text{PSL}(2,\mathbb{C})$-valued representations of $\pi_1(\tilde{Y}\backslash \tilde{L}_F)$; $\alpha_{\pm}$ are QHFT objects with targets $\partial_{\pm} \tilde{Y}$, such that for every $\{S,\phi\}$, the character $\phi^*(\rho)$
coincides with $\Theta(W^+(z))$. We say that $(\tilde{Y}, \tilde{L}_Y, \rho)$ is the support of a QHFT bordism from $\alpha_-$ to $\alpha_+$, and that $\alpha_{\pm}$ is a QHFT surface.

Given bordisms $B$ and $B'$ from $\alpha_-$ to $\alpha_+$ and $\alpha'_-$ to $\alpha'_+$, respectively, assume that $\beta_+$ and $\beta'_-$ are subobjects of $\alpha_+$ and $\alpha'_-$ that coincide up to the change of orientation.

**Definition 4.3.** The bordism $B''$ from $\alpha''_+ = \alpha_- \cup (\alpha'_- \setminus \beta'_-) \cup (\alpha_+ \setminus \beta_+)$ to $\alpha''_- = \alpha'_- \cup \alpha_+ \setminus \beta_+$ obtained by gluing $B$ and $B'$ along $\beta_+$ is called the composition of $B$ followed by $B'$. We write $B'' = B' \ast B$.

**Examples from hyperbolic geometry.** Any topologically tame hyperbolic 3-manifold $Y$ with hyperbolic holonomy $\rho$ and a tangle $L_\rho$ of singularities makes a QHFT bordism. More specifically, any geometrically finite non compact complete hyperbolic 3-manifold $Y$ defines a triple $(Y', L_\rho, \rho)$ with a non empty link $L_\rho$, as follows. The manifold $Y$ has a natural compactification $\tilde{Y}$, with $\partial \tilde{Y} \cong \text{Int}(\hat{Y})$, which is a “pared” manifold $(\tilde{Y}, \hat{P})$. Here $\hat{P}$ is a union of disjoint tori or annuli embedded in the boundary of $\tilde{Y}$. The tori correspond to the cusps of $Y$. Each annulus $A$ of $\hat{P}$ comes from a couple of cusps on some boundary component of $\text{(a small neighborhood of)}$ the convex core of $Y$: $\hat{A}$ is fibered by geodesic arcs. If $A$ is separating the cusps belong to different components. Define $Y''$ as the result of attaching a 2-handle to $\tilde{Y}$ at each annulus $A$, so that $\partial Y''$ is contained in the interior of $\partial Y''$ and is transverse to $\partial Y''$. Equivalently, $Y''$ contains a properly embedded framed 1-tangle $L'_\rho$ made by the cocores of the 2-handles, the framing being determined by the fibration by intervals of the annuli of $\hat{P}$. Let us choose a framing at each torus of $\hat{P}$. By Dehn filling we get a manifold $Y'$, and $Y''$ is the exterior in $Y'$ of the union $L''_\rho$ of the framed cores of the filling solid tori. Hence, if every boundary component of the convex core of $Y$ contains at least one cusp, associated to $Y$ and the cusp framings we have $(Y', L_Y, \rho)$, where $L_Y = L'_\rho \cup L''_\rho$ intersects all the boundary components of $Y'$, and $\rho$ is a $PSL(2, \mathbb{C})$-character of $Y' \setminus L_Y \cong Y$. If furthermore $\rho$ is the holonomy of a complete hyperbolic metric on $Y$ and $Y$ has infinite volume ends, then $\partial Y''$ is non empty. We can give the triple $(Y', L_Y, \rho)$ a natural boundary structure $\alpha_- \cup \alpha_+$, induced by exponential $I$-parameters of the pleated surfaces in the boundary of the convex core (see eg. [14]).

5. **The QHFT functor**

Consider a QHFT bordism $B = (\tilde{Y}, \tilde{L}_Y, \rho, \alpha_{\pm})$. For every odd integer $N \geq 1$, we associate a finite dimensional complex linear space $V(\alpha_{\pm})$ and a linear map $H_N(B) : V(\alpha_-) \to V(\alpha_+)$ to $B$, well-defined up to sign and multiplication by $N$th roots of unity. This defines a (moderately projective) functor $H_N : \text{QHB} \to \text{Vect}$, where Vect is the tensor category of complex linear spaces. The construction immediately implies that $H_N$ is a modular functor, in the sense of [30]. III.1.2.

5.1. **From QHFT bordisms to QHG-triangulated pseudomanifolds.** First we associate to $B$ a pseudomanifold $Z(B)$. Fill each tunnel boundary component of $\hat{Y}$ with a solid tube, thus recovering a copy of the manifold $Y$. The cores of the solid tubes make a parallel unframed copy $\lambda''$ of $L$. We define $Z(B)$ as the result of collapsing to one point each component of $\lambda''$. In other words, we glue to each tunnel component of $\hat{Y}$ the oriented topological cusp $C = B \times [0, +\infty]/(B \times \{\infty\})$ with base equal to either $B = S^1 \times [-1, 1]$ or $B = S^1 \times S^1$.

Next we describe a procedure to convert $Z(B)$ to a distinguished QHG-triangulated pseudomanifold. We refer to the notions introduced in Subsection 2.3.

We say that a branched triangulation $(T, b)$ of $B$ as above is *admissible* if $B \cap \partial \pm \hat{Y}$ and the tangles $\lambda$, $\lambda'$ are covered by the 1-skeleton. We denote $(\bar{T}, \bar{b})$ the branched triangulation of $C$, where $\bar{T}$ is...
the cone over \( T \) from the non manifold point, say \( \infty \), and \( \hat{b} \) extends \( b \) so that \( \infty \) is a pit for every branched tetrahedron of \( \hat{T} \). Assume we are given an idealizable \( PB^+(2, \mathbb{C}) \)-valued cocycle \( z \) on an admissible triangulation of \( B \). The idealization of \( z \) determines for each 2-simplex of \( B \) a face of an ideal hyperbolic tetrahedron with further vertex at \( \infty \) (see Figure 12) where opposite vertical triangles are identified). Since the fundamental group of \( B \) is Abelian, the resulting family of oriented ideal hyperbolic tetrahedra actually makes an \( \mathcal{L} \)-triangulation \((\hat{T}, \hat{b}, w)\), which we call an \( \mathcal{L} \)-cusp. By conjugating if necessary, we see that \( \mathcal{L} \)-cups make sense also when \( z \) takes values more generally in \( \text{PSL}(2, \mathbb{C}) \). We get flattenings similarly as in Lemma 2.9 at a corner of a 2-simplex formed by edges \( e_l \) and \( e_r \), we put the difference of the logarithms of the vectors in \( \mathbb{C} \) associated by the idealization to \( e_l \) and \( e_r \), respectively.

\[\text{Figure 11. An } \mathcal{L} \text{-cusp.}\]

**Definition 5.1.** A \( \mathcal{D} \)-triangulation of \( B = (\hat{Y}, \hat{L}_\mathcal{F}, \rho, \alpha_{\pm}) \) consists of a 4-uple \( K = (K, \hat{H}, b, z) \) where:

(a) \((K, b)\) is a branched triangulation of \( \hat{Y} \) extending that on \( \partial_\pm \hat{Y} \), and inducing an admissible cusp base triangulation at each tunnel component of \( \hat{L}_\mathcal{F} \).

(b) The 1-dimensional subcomplex \( \hat{H} = H \cup H' \) of \( K \) is ambiently isotopic to the tangle \( \hat{\lambda} = \lambda \cup \lambda' \), and \( \hat{H} \) contains all the vertices of \( K \).

(c) \( z \) is an idealizable \( \text{PSL}(2, \mathbb{C}) \)-valued 1-cocycle on \((K, b)\) such that:

(i) the conjugacy class of \( \text{PSL}(2, \mathbb{C}) \)-representations of \( \pi_1(\hat{Y}) \) associated to \( z \) coincides with \( \rho \);

(ii) the \((+)\)-exponential \( \mathcal{L} \)-parameters given by the restriction of \( z \) to \( \partial_\pm \hat{Y} \) coincide with that of the objects \( \alpha_- \cup \alpha_+ \) (see Definition 4.1).

(d) the restriction of \( z \) to each vertical tunnel component of \( \hat{L}_\mathcal{F} \) takes values in the Borel subgroup \( PB^+(2, \mathbb{C}) \) of \( \text{PSL}(2, \mathbb{C}) \).

For any \( \mathcal{D} \)-triangulation \( \mathcal{K} = (K, \hat{H}, b, z) \) of \( B \) we get a distinguished \( \mathcal{L} \)-triangulation \( \mathcal{K}_\mathcal{L} = (K, \hat{H}, b, w) \) of \((Z(B), \hat{\lambda})\) by gluing the idealization of \( \mathcal{K} \) with the \( \mathcal{L} \)-cusp given by the cocycle at each tunnel component. Note that \( \hat{H} \) contains all the vertices of \( K \) that are manifold points.

**Definition 5.2.** We say that \( \mathcal{T}(B) = (\mathcal{K}_\mathcal{L}, f, c) = (K, \hat{H}, b, w, f, c) \) is a distinguished flat/charged \( \mathcal{L} \)-triangulation of \((Z(B), \hat{\lambda})\) if it satisfies Definition 2.10 and at every boundary edge of \( Z(B) \) the total (classical or quantum) log-branch of \((\mathcal{K}_\mathcal{L}, f, c)\) of (7) coincides with the (classical or quantum) Log-\( \mathcal{L} \)-parameter of the boundary object \( \alpha_- \cup \alpha_+ \).

Recall the cohomological weights from section 2.4. These notions still make sense for distinguished flat/charged \( \mathcal{L} \)-triangulations, where the homology of \( \partial \mathcal{T}_0 \) is replaced with that of the tunnel components \( \hat{L}_\mathcal{F} \). Since we have weights \((h_f, k_f)\) for log-branches and \((h_c, k_c)\) for charges at the same time, we will denote them \((h, k) = ((h_f, h_c), (k_f, k_c))\). We can also define, in the very same way, boundary weights \( k_f \in H^1(\partial \hat{Y}; \mathbb{C}) \), but these are completely encoded by \( \alpha_{\pm} \). We have:

**Theorem 5.3.** For every bordism \( B = (\hat{Y}, \hat{L}_\mathcal{F}, \rho, \alpha_{\pm}) \) and every \((h, k) \in H^1(\hat{Y}; \mathbb{Z}/2\mathbb{Z}) \times H^1(\hat{L}_\mathcal{F}; \mathbb{C})\) satisfying the properties (14), there are distinguished flat/charged \( \mathcal{L} \)-triangulations \( \mathcal{T}(B) \) of \( Z(B) \) with weight \((h, k)\), and any two are QHG-isomorphic.
Proof. The existence of distinguished $\mathcal{I}$-triangulations $K_{\mathcal{I}}$ of $Z(\mathcal{B})$ follows from a tedious but straightforward generalization of Theorem 4.13 in [11]. Global flattenings and integral charges with arbitrary weight exist on the double $DK_{\mathcal{I}}$ of $K_{\mathcal{I}}$ by the results recalled with Theorem 2.12. Consider the $(+)$-Log-$\mathcal{I}$-parameters $\{W^{\pm}(z)\}_{t}$ at $\alpha_\pm$. They are in one-one correspondence with the interior edges of the corresponding $e$-triangulations, which is less than the cardinality of the families $m$ and $n$ used to defined flattenings and charges in Definition 5.4. Hence any family of determinations of the logarithms of the $W^{\pm}(z)\{e\}$ is a system of $(+)$-Log-$\mathcal{I}$-parameters. Also, Lemma 2.9 implies that any system of $(+)$-Log-$\mathcal{I}$-parameters at $\alpha_\pm$ extends to a distinguished flat/charge $\mathcal{I}$-triangulation of the pseudo manifold obtained from the trivial cylinders over $\partial (\partial_\pm \hat{Y})$ by collapsing to a point each annulus of $\partial_\pm \hat{Y} \times [-1, 1]$.

This means that any QHFT surface bounds a QHFT bordism, and that for the bordism $\mathcal{B}$ there are flat/charges on $DK_{\mathcal{I}}$ whose restriction to $K_{\mathcal{I}}$ induce the Log-$\mathcal{I}$-parameters of $\alpha_\pm$. Hence we get global flat/charges as in Definition 5.2. In fact the affine spaces of flat/charges on $DK_{\mathcal{I}}$ project onto that on $K_{\mathcal{I}}$ compatible with $\alpha_\pm$ (see the end of section 2). Then, the Mayer-Vietoris exact sequence in cohomology for the triad $(DK_{\mathcal{I}}, K_{\mathcal{I}}, -K_{\mathcal{I}})$ shows that $(h, k)$ is induced by some weight on $DK_{\mathcal{I}}$. As we can choose the latter arbitrarily, this concludes the proof of the first claim.

The second is harder, but follows strictly from the arguments in the proof of Theorem 6.8 (2) in [5]. The only new ingredient is the presence of $\mathcal{I}$-cusps, which mimic the ends of cusped manifolds treated in that paper.

An alternative characterization of classical/quantum log-branched flat/charged $\mathcal{I}$-triangulations $T(\mathcal{B})$ follows from Remark 2.11.

5.2. Amplitudes. Fix an odd positive integer $N$. Write $V = \mathbb{C}^N$, with the canonical basis $\{e_i\}$, and $V^{-1}$ for the dual space. Both are endowed with the hermitian inner product with orthonormal basis the vectors $e_0$ and $(e_i + e_{N-i})/\sqrt{2}$, $i = 1, \ldots, N - 1$.

Recall the notations of section 3. For each base surface $F$ fix an $e$-triangulation $(T, b)$, an ordering of the set $T^{(2)}$ of $2$-simplices, and let $V(T, b) = \otimes_{t \in T^{(2)}} V^{\sigma_t(t)}$. Given a QHFT bordism $\mathcal{B} = (\hat{Y}, \hat{L}_F, \rho, \alpha_\pm)$ with a distinguished flat/charged $\mathcal{I}$-triangulation $T(\mathcal{B})$, the trace tensor in (11) is a morphism $\mathcal{H}_N(T(\mathcal{B})) \in \text{Hom}(V(\alpha_\pm), V(\alpha_\pm))$, where $V(\alpha_\pm)$ is the tensor product of isomorphic copies of the spaces $V(T, b)$ over the (ordered) components of $\alpha_\pm$.

Theorem 5.4. The morphism $\mathcal{H}_N(\mathcal{B}, h, k) = \mathcal{H}_N(T(\mathcal{B}))$ does not depend on the choice of $T(\mathcal{B})$ up to sign and multiplication by $N$th roots of unity, and $\mathcal{H}_N(\mathcal{B}, h, k) = \mathcal{H}_N(\mathcal{B}, h', k')$ if the mod($N\mathbb{Z}$) reductions of $h$ and $h'$ (resp. $k$ and $k'$) are the same, that is, if we have $k - k' \in H^1(\hat{L}_F; \mathbb{Z})$ and $k - k' = 0 \in H^1(\hat{L}_F; \mathbb{Z}/N\mathbb{Z})$, and similarly for $h$ and $h'$. Moreover, there is no sign ambiguity if we restrict to even valued flattenings as in Lemma 2.9 (hence with $h = 0$). We call $\mathcal{H}_N(\mathcal{B}, h, k)$ the amplitude of $(\mathcal{B}, h, k)$.

Proof. The result up to sign is an immediate consequence of Proposition 2.5 and the last claim in Theorem 5.2. For the dependance with respect to the mod($N\mathbb{Z}$) reductions of weights, we note that the associated systems of $N$th roots of moduli (see Remark 2.11) are connected by QHG isomorphisms. Indeed, the difference $k - k' \in H^1(\hat{L}_F; N\mathbb{Z})$ coincides with $\gamma(f) - \gamma(f')/\sqrt{-1}\pi$ and $\gamma(c) - \gamma(c')/\sqrt{-1}\pi$ for some $f$, $f'$ and $c$, $c'$, and similarly for $h - h'$ and the $\gamma_2$ maps. By first considering $(k - k')/N$ we can eventually take the collections of values of $f - f'$ and $c - c'$ in $N\mathbb{Z}$, and equal. Hence the conclusion follows from [5] and [9]. For even valued flattenings, the claim follows from the fact that Proposition 2.5 has no sign ambiguity. For clarity let us state the result with some details.

Consider all the possible branching configurations of $2 \leftrightarrow 3$ QHG transits, up to obvious symmetries. They are obtained from a single one by composing with the transpositions $(01)$, $(12)$, $(23)$ and $(34)$ of the vertices (ordered in accordance with the branching). Any such a transposition changes the matrix dilogarithm of each tetrahedron by matrix conjugation and multiplication by a determined scalar factor (see Corollary 5.6 of [5]). Now, there is a preferred “basic” $2 \leftrightarrow 3$ QHG transit, for which even flattenings give no sign ambiguity in Proposition 2.5 (see the proof of Theorem 5.7 in [5]). The corresponding branching configuration is defined by ordering the vertices as $1$ and $3$ on the bottom and top of the central edge, and $0$, $2$, $4$ counterclockwise as viewed from the 3rd vertex. The following
table describes for each tetrahedron $\Delta^i$, opposite to the $i$-th vertex, the scalar factors induced by the above transpositions (we put $v = \exp(\sqrt{-1} \pi (1 - N)/2N)$, which is $-\zeta^{-(m+1)(1-N)/2}$ in the notations of [5], and the $c_k$ are charge values):

|   | $\Delta^1$ | $\Delta^3$ | $\Delta^0$ | $\Delta^2$ | $\Delta^4$ |
|---|------------|------------|------------|------------|------------|
| 01 | 0          | $v^{c_0}$  | 0          | $v^{c_2}$  | $v^{c_4}$  |
| 12 | 0          | $v^{c_3}$  | $v^{c_1}$  | 0          | $v^{c_4}$  |
| 23 | $v^{c_4}$  | 0          | $v^{c_1}$  | 0          | $v^{c_2}$  |
| 34 | $v^{c_3}$  | 0          | $v^{c_0}$  | 0          | 0          |

Because of [8], we see that the scalars at both sides are equal. Hence, for any $2 \leftrightarrow 3$ log-branch transit with even flattenings, the only ambiguity in Proposition [25] is by multiplication by $N$th roots of unity, which is due to the definition of the function $g$ in [41]. The conclusion follows as before by using QHG-isomorphisms preserving the parity of flattenings (whence based on even multiples of the generators of flat/charged lattices). □

**Remark 5.5.** (Cheeger-Chern-Simons invariants and $H_1$.) By the results of [25], there is an injective homomorphism from $H_3(PSL(2, \mathbb{C}); \mathbb{Z})$ (discrete homology) to a scissors congruence group $\hat{P}(\mathbb{C})$, such that the dilogarithm $\hat{\rho}$, defined on $\hat{P}(\mathbb{C})$, restricts to the universal Cheeger-Chern-Simons class $\hat{C}^2 : H_3(PSL(2, \mathbb{C}); \mathbb{Z}) \to \mathbb{C}/\pi i \mathbb{Z}$. Hence $H_1$ is a natural extension of $\exp(\hat{C}^2/2\pi i)$, the exponential of a constant times $\text{Vol} + iCS$, to classes representing QHFT bordisms. Recently J. Dupont and C. Zickert produced dilogarithmic formulas for the lift $\hat{C}^2$ of $C^2$ to $H_3(SL(2, \mathbb{C}); \mathbb{Z})$, such that $\exp(\hat{C}^2/2\pi i)$ coincides with the lift of $H_1$ determined by even flattenings in Theorem [57, 15]. We are indebted to their work for pointing out the existence of such flattenings.

Recall Definition [4, 3]. Assume that $B'' = B' \ast B$ exists, and let $T(B)$ and $T(B')$ be given weights $(h, k)$ and $(h', k')$, respectively. Then $T(B') \ast T(B)$ is a distinguished flat/charged $I$-triangulation $T(B'')$ with some weight $(h'', k'')$. It follows from the Mayer-Vietoris exact sequence for $(B'', B', B)$ that even if $h = h' = 0$, it can happen that $h'' \neq 0$. However, if the glued part of the boundary is connected, or is a boundary in $B \ast B'$, then $h = h' = 0$ implies $h'' = 0$.)

**Proposition 5.6.** (Functoriality) For any composition $B'' = B' \ast B$ of bordisms, $\mathcal{H}_N(B'', h'', k'')$ coincides with $\mathcal{H}_N(B', h', k') \circ \mathcal{H}_N(B, h, k)^*$, the adjoint for the hermitian structure of $\mathcal{V}(\alpha_{\pm})$, coincides up to sign and multiplication by $N$th roots of unity.

This is a direct consequence of Theorem [5, 1]. Also, we prove as in Proposition 4.29 of [5]:

**Proposition 5.7.** (Polarity) Write $\bar{B}$ for the QHFT bordism with opposite orientation and complex conjugate holonomy $\bar{p}$. Then $\mathcal{H}_N(\bar{B}, -h, -k)$ and $\mathcal{H}_N(B, h, k)^*$, the adjoint for the hermitian structure of $\mathcal{V}(\alpha_{\pm})$, coincide up to sign and multiplication by $N$th roots of unity.

In the proof of Theorem [5, 3] we have seen that the space $\hat{W}^+(T, b)$ of $(+)$-Log-$I$-parameters over $W^+(T, b)$ (the disjoint union of spaces $W^+(T, b, \mathbb{C})$) is isomorphic to $\hat{C}^{-3\chi(F) + 2p}$, where $\hat{C}$ is the universal cover of $\mathbb{C} \setminus \{0\}$. Similarly, for any admissible triangulation $\tau$ of a topological cusp with $n$ 2-simplices, we have the analytic subspace $\text{Def}(\tau)$ of $\hat{C}^n$ made of the $n$-uples of log-branches for the tetrahedra of $I$-cusps with base triangulation $\tau$, where $\hat{C}$ is defined in Section [2, 1]. Such log-branches satisfy the compatibility relations $L_\tau(e) = 0$ at interior edges.

**Definition 5.8.** Let $X = (Y, L_\tau, \alpha_{\pm})$ be a marked topological bordism with $e$-triangulated or admissibly triangulated boundary components. The **phase space** of $X$ is the (analytic) subset $\text{Def}(X)$ in the product of the spaces $\hat{W}^+(T, b)$ and $\text{Def}(\tau)$ over the components of $\partial Y$ and $L$, determined by the family of distinguished flat/charged $I$-triangulations of QHFT bordisms supported by $(Y, L_\tau, \alpha_{\pm})$. 


When \( Y \) has empty boundary, \( \text{Def}(X) \) is a generalization of the well known deformation space of hyperbolic structures supported by ideal triangulations of \( Y \mid L \), introduced in \([29]\), and recently studied in \([11]\) and \([12]\). When \( \tilde{Y} \) is the mapping cylinder of a diffeomorphism \( \phi \) of \( F \), the amplitudes of QHFT bordisms supported by \( \tilde{Y} \) define a morphism of the trivial vector bundle

\[
E(F)^+ = E(T, b)^+ : \tilde{W}(T, b) \times V(T, b) \to \tilde{W}(T, b).
\]

(Note that any two choices of \( e \)-triangulations \((T_1, b_1)\) and \((T_2, b_2)\) of \( F \) yield isomorphic bundles \( E(T_1, b_1) \) and \( E(T_2, b_2) \), with birationally equivalent bases.) The resulting mapping of sections of \( E(F) \), the states of \( F \), are studied in \([3]\).

**Proposition 5.9.** (Analyticity) For every \( N \geq 1 \), the amplitudes of QHFT bordisms supported by \( X = (Y, L_F, \alpha, \pm) \) vary analytically with the boundary structure in \( \text{Def}(X) \), up to sign and multiplication by \( N \)th roots of unity.

This follows immediately from the fact that the matrix dilogarithms are analytic, together with the fact that any path in \( \text{Def}(X) \) lifts to a path of log-branches via the relations induced by Definition 5.2 (4).

5.3. QHFT variants. By varying the bordism category we can vary the corresponding QFT.

**QHFT\(^0\):** Consider the bordism category supported by triples \((Y, L, \rho)\), where \( L \) is an non-empty unframed tangle in \( Y \) and \( \rho \) is a \( PSL(2, \mathbb{C})\)-character on the whole of \( Y \) (ie. \( \rho \) is trivial at the meridians of \( L \)). In fact, we restrict to holonomies \( \rho \) such that \((Y, L, \rho)\) admits \( \mathcal{D}\)-triangulations that extend a topological branched ideal triangulation \((T', b')\) of each boundary component, say \((S, V)\), and for which the link \( L \) is realized as a Hamiltonian subcomplex (hence with no \( \mathcal{L}\)-cusp). In particular the objects of this bordism category incorporate the idealization of (necessarily idealizable) cocycles on \((T', b')\), that represent the restriction of \( \rho \) to \( S \). The arguments of Theorem 5.4 can be easily adapted to produce tensors \( \mathcal{H}_N(B, h, k) \) associated to such a bordism \( B \), and eventually the so called QHFT\(^0\) variant of quantum hyperbolic field theory.

**Fusion of QHFT and QHFT\(^0\):** We can consider triples \((Y, L_F, L^0, \rho)\), where \( L = L_F \cup L^0 \) is a tangle with a framed part \( L_F \) and an unframed one \( L^0 \). We also stipulate that \( \rho \) is trivial at each meridian of \( L^0 \). For every object support \((S, V)\), we have a partition \( V = V_F \cup V_0 \) and we use "mixed" triangulations that looks like an efficient one at \( p \in V_F \) and like an ideal one at \( p \in V_0 \). A similar mixed behaviour holds for the adapted \( \mathcal{D}\)-triangulations of such bordisms. We eventually get tensors still denoted \( \mathcal{H}_N(B, h, k) \) giving variants, still denoted QHFT, that extend both the previous one \((L^0 = \emptyset)\), and QHFT\(^0\) \((L_F = \emptyset)\).

**QHFT\(^\pm\):** Let \((Y, L_F, L^0, \rho)\) be as above, and let us specialize to \( \rho \) that, as usual, are trivial at the meridians of \( L^0 \), but are not trivial at the meridians of \( L_F \). Now we use mixed triangulations of each object support \((S, V_F \cup V_0)\) that look like an economic triangulation (see Remark 3.12) at each \( p \in V_F \). Concerning the adapted \( \mathcal{D}\)-triangulations, each component of \( L_F \) contributes the hamiltonian subcomplex with just a copy of the parallel curve specifying the framing (recall that by using ordinary efficient triangulations, it contributed with two parallel curves). We get tensors now denoted \( \mathcal{H}_N'(B, h, k) \), and a variant denoted QHFT\(^\pm\).

Of course, there are no deep structural differences between these variants; nevertheless each one has its own interest (see also Section 6).

5.4. Mapping class group representations. Fix \( F \). Set \( \hat{Y}_F = F \times [-1, 1], \hat{L} = \partial F \times [-1, 1] \) with trivial vertical framing, and let \( \text{Mod}(g, r) \) be the mapping class group of \( F \), that is, the group of homotopy classes of orientation-preserving diffeomorphisms of \( F \) fixing pointwise each boundary component. Given \( \psi : \pm(F, (T, b, L)) \to F \times \{ \pm1 \} \), put \( \psi = \psi_+^{-1} \psi_- \) and \( [\psi] \) for the corresponding element in \( \text{Mod}(g, r) \). Denote by \( \hat{W}_[\psi] \) the mapping torus \((F \times [-1, 1])/\{x,-1\} \sim (\psi(x), 1) \) of \( \psi \), with tunnel boundary \( \hat{L}_[\psi] \). Let \( \rho \) be the conjugacy class of \( PSL(2, \mathbb{C})\)-valued representations of \( \pi_1(F \times [-1, 1]) \) (identified with \( \pi_1(F) \)) associated to \( L \). To simplify notations, in all statements of this section we do not mention the weights (we understand they are fixed).
Lemma 5.10. Up to sign and multiplication by Nth roots of unity (denoted "\(=_{N}\)") we have:

1. For any fixed \((T, b, \mathcal{L})\) the amplitudes \(\mathcal{H}_N(\tilde{Y}, \tilde{L}_F, \rho, (\pm F, (T, b, \mathcal{L})), [\psi_{\pm}])\) depend only on \([\psi]\). We denote them \(\mathcal{H}_N([\psi])\).

2. \(\mathcal{H}_N([id])\) is the identity map from \(E_N(\alpha_-)\) to \(E_N(\alpha_+)\). and \(\mathcal{H}_N([h_2]) \circ \mathcal{H}_N([h_1]) = \mathcal{H}_N([h_2 h_1])\). In particular, for any \([\psi] \in \text{Mod}(g, r)\) the QHFT tensor \(\mathcal{H}_N([\psi])\) is invertible, with inverse \(\mathcal{H}_N([\psi^{-1}])\), and for a homotopically d-periodic \([\psi]\) the QHFT tensor \(\mathcal{H}_N([\psi])\) is of finite order less than or equal to \(d\).

3. If \(\psi(\rho)\) coincides with \(\rho\), then \(\text{Trace}(\mathcal{H}_N([\psi])) = \mathcal{H}_N(\tilde{W}([\psi], \tilde{L}([\psi], \rho))\).

Proof. Point (1) follows from Theorem 5.2 because \(\mathcal{H}_N(\tilde{Y}, \tilde{L}_F, \rho, (\pm F, (T, b, \mathcal{L})), [id], [\psi])\) (the homeomorphism \(\psi^{-1} \times \text{id}\) sends the first mapping cylinder to the second). By Proposition 5.6 we have \(\mathcal{H}_N^2([id]) = \mathcal{H}_N([id])\), so \(\mathcal{H}_N([id])\) is an idempotent. It is invertible because the matrix dilogarithms are. Both facts imply the first claim in (2). The rest is a direct consequence of Proposition 5.6 and formula (11).

The arguments in the proof of Lemma 5.10 imply also that, letting \([\psi] = [id]\) and \(\rho\) fixed, the amplitudes of any marking variation \((T, b, \mathcal{L}) \to (T_1, b_1, \mathcal{L}_1)\) are invertible. Hence \(\mathcal{H}_N([\psi])\) is conjugated to the tensor \(\tilde{\mathcal{H}}_N(\tilde{Y}, \tilde{L}_F, \rho, (\pm F, (T_1, b_1, \mathcal{L}_1)), [\psi_{\pm}])\). Moreover \(\mathcal{H}_N(-\tilde{Y}, \tilde{L}_F, \rho, (\mp F, (T, b, \mathcal{L})), [\psi_{\pm}])\), the amplitude with reversed orientation, clearly coincides with \(\mathcal{H}_N([\psi^{-1}])\). Using Proposition 5.7 we deduce:

Corollary 5.11. For any fixed \(\rho \in \mathcal{R}(g, r)\), the homomorphisms \(\psi \mapsto \mathcal{H}_N([\psi])\) induce a conjugacy class of linear representations of \(\text{Mod}(g, r)\), well-defined up to sign and multiplication by \(N\)th roots of unity. For \(SL(2, \mathbb{R})\)-valued characters \(\rho\) these representations are unitary.

5.5. Tunneling the \((+)/(-)\) states. We use \((+)-\text{Log-}\mathcal{L}\)-parameters to define the QHFT because of the existence of strongly idealizable cocycles on QHFT bordism triangulations, which makes functoriality easy to check. Here we exhibit a family of tensors correlating the \((\pm)-\text{Log-}\mathcal{L}\)-parameters, thus recovering, in particular, the direct and nice interpretation of boundary structures having non trivial holonomy at the punctures in terms of pleated hyperbolic surfaces (see Section 3.3). These tensors are also used in Section 6.3.

For any base surface \(F\) with an \(e\)-triangulation \((T, b)\), let \(Z(F)\) be the pseudo-manifold obtained by collapsing to a point each boundary annulus of the cylinder \(C(F) = F \times [-1, 1]\). Recall the bundle \(E(F)^+\) in [18], and consider similarly \(E(F)^-\). We have:

Proposition 5.12. There exists a canonical family \(\mathcal{F}\) of flat/charged \(\mathcal{L}\)-triangulations covering a portion of \(Z(F)\), with invertible trace tensor \(\mathcal{H}_N(\mathcal{F}) : E(F)^+ \to E(F)^-\).

Proof. Orient \(C(F)\) so that \(\pm F\) is identified with \(F \times \{\pm 1\}\). Let \((P(T, b))\) be the cell decomposition of \(C(F)\) made by the prisms with base the 2-simplices of \(T\). Orient all the “vertical” (ie. parallel to \([-1, 1]\)) edges of \(P(T, b)\) towards \(+F\). For every abstract prism \(P\), every vertical boundary quadrilateral \(R\) has both the two horizontal and the two vertical edges endowed with parallel orientations. So exactly one vertex of \(R\) is a source (that belongs to \(-F\), and exactly one is a pit (that belongs to \(+F\)). Triangulate each \(R\) by the oriented diagonal going from the source to the pit. Finally extend the resulting triangulation of \(\partial P\) to a triangulation of \(P\) made of 3 tetrahedra, by taking the cone from the \(b\)-first vertex of the bottom base triangle of \(P\) (note that no further vertices nor further edges have been introduced). Repeating this for every prism, we get a branched distinguished triangulation \((C(T, b), H)\) of \(C(F)\), where the vertical edges make the Hamiltonian tangle \(H\). As in the proof of Theorem 5.3 there exists integral charges on \((C(T, b), \mathcal{H})\).

Let \(F \times \{1, 2\}\) be triangulated by two adjacent copies of \((C(T, b), \mathcal{H})\), glued each to the other at \(F \times \{1\}\). For any \(z \in \tilde{Z}_I(T, b)^+\), consider the unique cocycle \(C_0(z)\) on the composition \(C(T, b) * C(T, b)\) that:

1. Extends \(z \cup \Psi(z)\), given on \((F \times \{1\}) \cup (F \times \{3\})\); takes the value \(P\) of [13] on each vertical edge contained in \(F \times \{1, 2\}\); takes the value 1 on each vertical edge contained in \(F \times \{1\}\). Perturb \(C_0(z)\) with a 0-cochain \(s\) that: takes the value 1 on \((F \times \{1\}) \cup (F \times \{3\})\); takes values in \(PB^+ (2, \mathbb{C})\) at
each vertical boundary annulus; restricts to an idealizable cocycle on $F \times [−1, 1]$ and to a maximally idealizable cocycle on $F \times [1, 3]$ (see Definition 3.7). Finally, glue $I$-cusps to the idealization. Note that the only non idealizable tetrahedra are those in the star of a boundary edge of the triangulation of $F \times \{3\}$. Lemma 2.39 gives flattenings for the idealizable tetrahedra.

Look at the ideal triangulation $(T', b')$ of $S \setminus V$ corresponding to the copy of the triangulation $(T, b)$ for the boundary component $F \times \{3\}$. For every cocycle $s$ as above and every edge $e$ of $T'$ we have two complex numbers: the ($−$)-exponential $I$-parameter $W^−(Ψ(z))(e)$, and, as in (7), the total product $W(e)$ of the cross-ratio moduli at the edges of $T$ that enter the definition of $W^−(Ψ(z))(e)$. Recall that there are two distinct such edges only when $e$ contributes to make a marked corner. It is possible to normalize $s$ so that for every edge $e$ of $T'$ we have $W^−(Ψ(z))(e) = W(e)^{-1}$.

Varying the cocycle $z \in \mathcal{Z}_I(T, b)^+$, this choice determines the family $F$ in the statement. By perturbing the initial cocycle $z$ with 0-cochains $t$ with values in $PSL(2, C) \setminus PB^+(2, C)$, the same construction leads to families $F_t$ of flat/charged $I$-triangulations covering the whole of $F \times [−1, 3]$.

Note that for suitable flat/charges ($f_1 = 0$ in (3) and $c_1 = 0$ in (6)) the matrix dilogarithms have well-defined finite limits when the cross-ratio modulus $w_0 \to 0$. From the symmetry relations of the matrix dilogarithms (see [5], Corollary 5.6), this is true more in general for any degenerating sequence of $I$-tetrahedra, that is when $w_0$ goes to 0, 1 or $∞$. Now, we can choose in a continuous way the flat/charges of $F_t$ so that they satisfy the above constraints on the tetrahedra of $F_t$ that become non idealizable in $F$, when $t \to id$. Then $\mathcal{H}_N(F) := \lim_{t \to id} \mathcal{H}_N(F_t)$ exists. As in Lemma 3.41 (2) we see that $\mathcal{H}_N(F_t)$ is invertible, with inverse $\mathcal{H}_N(−F_t)$. Since $\mathcal{H}_N(F) \circ \mathcal{H}_N(−F_t) = \mathcal{H}_N(F_t \circ \mathcal{H}_N(−F_t)) = N \circ id$ (Proposition 5.9), we deduce that $\mathcal{H}_N(F)$ is invertible.

Figure 12. Pasting opposite vertical sides yields an instance of $C(T, b)$ for the once-punctured torus $S$, based on an economic triangulation of $S$ as in Remark 3.12.

6. Partition functions

Assume that $W$ is a closed oriented 3-manifold, and that $L$ is a link in $W$ with a framed part $L_F$ and an unframed one $L^0$. Each variant of quantum hyperbolic field theory (see Section 5.3) leads to the respective partition functions.

If $\rho$ is trivial at each meridian of $L^0$, we have the QHFT partition functions

$$\mathcal{H}_N(W, L_F, L^0, \rho, h, k)$$

that specialize to the QHFT$^0$ ones when $L = L^0$:

$$H_N(W, L, \rho, h, k) = \mathcal{H}_N(W, \emptyset, L^0, h, k) .$$

If $\rho$ is also assumed to be non trivial at each meridian of $L_F$, we have also

$$\mathcal{H}_N^\omega(W, L_F, L^0, \rho, h, k) .$$

These partition functions are scalars, well-defined up to sign and multiplication by $N$th-roots of unity. Typical examples of triples $(W, L_F, \rho)$ are given by hyperbolic cone manifolds $W$ with framed cone locus $L_F$ and hyperbolic holonomy $\rho$ on $W \setminus L_F$. The partition functions can be expressed in terms of manifolds $Y$ with toric boundary and containing an unframed link $L^0$ in the interior. By
fixing an ordered basis \((m_i, l_i)\) for the integral homology of each boundary torus, let \(W\) be obtained from \(Y\) by Dehn filling along the \(m_i\) and \(L_F\) be the disjoint union of the cores of the filling solid tori, framed by the \(l_i\). Then the partition functions of \((W, L_F, L^0, \rho, h, k)\) are in fact invariants of \((Y, \{m_i, l_i\}_i), L^0, \rho, h, k)\).

6.1. **QHFT vs QHFT**\(^0\) **partition functions.** For \(\mathcal{B} = (W, L, \rho), L = L^0\), with weights \(h = k = 0, H_N(W, L, \rho, 0, 0)\) coincide with the invariants \(H_N(W, L, \rho)\) constructed in [4, 5]. Let us consider more generally \((W, L_F, L^0, \rho, 0, 0)\). Fix also a framing \(\mathcal{F}_0\) for \(L^0\). Then we can consider the partition function \(\mathcal{H}_N(W, L_F \cup L^0_{\mathcal{F}_0}, \emptyset, \rho, 0, 0)\). Let us denote by \(\tilde{\lambda}\) the unframed link obtained by splitting each component of \(L^0\) in the two corresponding parallel boundary components of the ribbon link \(L^0_{\mathcal{F}_0}\). We have:

**Proposition 6.1.**

\[
\mathcal{H}_N(W, L_F \cup L^0_{\mathcal{F}_0}, \emptyset, \rho, 0, 0) = \mathcal{H}_N(W, L_F, \tilde{\lambda}, \rho, 0, 0) .
\]

**Proof.** For simplicity, assume that \(L = L^0\). Fix a \(D\)-triangulation of \((\tilde{W}, \tilde{L}_F, \rho)\) where each tunnel component \(B\) has a symmetric admissible triangulation as in Figure 13 (opposite sides of the quadrilateral are identified). The tangle \(\tilde{\lambda}\) cuts open \(B\) into symmetric annuli, left and right to the central vertical line in Figure 13.

**Figure 13.** A special admissible triangulation of \(B\).

Because \(\rho\) has trivial holonomy at the meridians of \(L_F\), we can assume that the cocycle takes the same values on symmetric edges. Identifying the annuli we thus get a \(D\)-triangulation for the QHFT\(^0\) triple \((W, \tilde{\lambda}, \rho)\). Since \(H_N(W, \tilde{\lambda}, \rho)\) is computed from the idealization and symmetric tetrahedra in the cusps have opposite branching orientation, the result will follow if we show the existence of symmetric flat/charges. Then each component will be the identity map.

The existence of flattenings with this property can be shown using Remark 2.9 but for charges we need to take another route. Recall from the end of section 2 that flat/charges form affine spaces over an integral lattice generated by vectors attached to the edges. For an edge of \(B\), such vectors can be represented as adding +1 at one of the adjacent corners and −1 at the other, and the inverse for the right adjacent corners. Using these rules and 29, it is straightforward (though tedious) to check that any given flat/charge can be turned into one with equal quantum log-branches on symmetric tetrahedra.

6.2. **Invariants of cusped manifolds and surgery formulas.** Let us recall the QHG pseudo-manifold triangulations \(T\) used in 5 (see Definition 6.2 and Definition 6.3 in that paper) to define the quantum hyperbolic invariant trace tensors \(\mathcal{H}_N(T)\) for oriented cusped hyperbolic manifolds.

Let \(M\) be a cusped manifold. Denote \(Z\) the pseudo-manifold obtained by taking the one point compactification of each cusp of \(M\). \(M\) admits a triangulation by positively embedded hyperbolic ideal tetrahedra, possibly including some degenerate ones of null volume (ie. having real cross-ratios). Such a triangulation can be obtained by subdividing the canonical Epstein-Penner cell decomposition of \(M\). This gives rise to triangulation \((T_0, z_0)\) of \(Z\), where \(z_0\) is the cross-ratio function of the abstract edges of \(T\), the imaginary part of every cross-ratio being \(\geq 0\). We call it a quasi-geometric ideal triangulation of \(Z\). If some quasi-geometric triangulation admits a global branching, we say that \(M\) is gentle. More generally, \(M\) is said weakly-gentle if there is an \(I\)-triangulation \((T, b, w)\) of \(Z\) such that \((T, z), z = w^\infty\), is obtained via a (possibly empty) finite sequence \((T_0, z_0) \to \ldots \to (T_i, z_i) \to \ldots \to (T, z)\) of
positive $2 \to 3$ transits, where $(T_0, z_0)$ is a quasi-geometric triangulation of $Z$ as above. Each transit $(T_i, z_i) \to (T_{i+1}, z_{i+1})$ is defined by $W_{T_i}(e) = W_{T_{i+1}}(e)$, with all exponents $w_b = 1$ (see (7)). Every such a $(T, b, w)$ can be enhanced to flat/charged $I$-triangulations $T = (T, b, w, f, c)$.

Given $(T_0, z_0)$, it is certainly possible to get an $I$-triangulation $(T, b, w)$ by performing also some bubble moves (hence introducing new interior vertices). The authors do not know any example of non weakly-gentle cusped manifold, that is, such that we are forced to do it. Anyway, dealing with bubble moves is a technical difficulty which will appear also in the proof of Theorem 6.3 (1) below. We overcome it as follows. We fix an edge $a$ of the canonical Epstein-Penner cell decomposition of $M$, and take $A$ made by two copies of $a$ that intersect at non manifold points of $Z$; the second copy runs parallel to $a$ within an open cell of the decomposition. Hence, $A$ is a circle covered by two arcs. We need to enlarge the notions introduced in Definition 2.10. We say that $(T, H)$ is a distinguished triangulation of $(Z, a)$ if $H$ is a subcomplex of the 1-skeleton of $T$ isotopic to $A$, that contains all the regular vertices of $T$, and such that one arc of $A$ is covered by an edge $l$ of $H$. We say that $c$ is a global charge on $(T, H)$ if

$$C_T(e) = \begin{cases} 4 & \text{if } e = l \\ 0 & \text{if } e \in H \setminus l \\ 2 & \text{if } e \in T \setminus H \end{cases}$$

If $H = \emptyset$ this reduces to the usual notion of global charge on a closed triangulated pseudo manifold whose non manifold points have toric links. By using bubble moves and the existence of such usual global charges, it is easily seen that $(T, H)$ supports global charges as in (19) (see the proof of Theorem 6.8 in [5] for the details).

We say that $T = (T, H, b, w, f, c)$ is a flat/charged $I$-triangulation of $(Z, a)$ if $(T, H, b, c)$ is a branched, charged and distinguished triangulation of $(Z, a)$, and $(T, z)$, $z = w^n$, is obtained from a quasi geometric $(T_0, z_0)$ via a finite sequence $(T_0, z_0) \to \ldots \to (T, z)$ of transits supported by positive $2 \to 3$ moves and bubble moves. By setting $a = \emptyset$ and $H = \emptyset$, this definition incorporates that for the weakly-gentle case.

In [3] it is shown that flat/charged $I$-triangulations $T$ of $(Z, a)$ (with arbitrary weights) do exist and that, for every odd $N \geq 1$,

$$H_N(M, a) = \mathcal{H}_N(T)$$

is a well defined invariant of $(M, a)$, providing the weights of flat/charges to be 0. To simplify the exposition, below we continue with this normalization. When $M$ is weakly-gentle we get invariants $H_N(M)$. In fact, as a by product of the following discussion, we will realize that $H_N(M, a)$ does not depend on the choice of $a$, so that $H_N(M)$ is always well defined (see Corollary 6.4).

Let us recall now a few facts related to hyperbolic Dehn filling (see [29], [6], [27]). A quasi geometric triangulation $(T_0, z_0)$ as above corresponds to the complete structure of $M$. It can be deformed in a complex variety of dimension equal to the number of cusps. If $z'$ is close enough to $z_0$, $(T_0, z')$ is a triangulation by (possibly negative - see [27]) embedded hyperbolic ideal tetrahedra in a non-complete hyperbolic structure, say $M'$, close to $M$. In some case the completion of $M'$ gives rise to a compact closed hyperbolic manifold $W$, topologically obtained by Dehn filling of the (truncated) cusps of $M$. The core of each attached solid torus is a "short" simple closed geodesic $L_j$ of $W$, so that we have the (geodesic) link $L = \bigsqcup L_j$. Moreover, there are sequences $(W^n, L^n)$ obtained in this way such that the length of $L^n$ goes to 0 when $n \to +\infty$. Hence $(W^n, L^n)$ converges to the cusped manifold $M$ (in a neat geometric sense). From now on we will consider small deformations $z'$ leading to such closed completions.

As well as $(T_0, z_0)$ gives rise to a triangulation $T = (T, H, b, w, f, c)$ of $(Z, a)$, $z'$ close to $z_0$ gives rise to another flat/charged $I$-triangulation $T' = (T, H, b, w', f', c)$, where $w'$ is close to $w$ and the log-branch associated to the global flattening $f'$ corresponds to a continuous deformation of the one for $f$.

**Lemma 6.2.** (See [25], p. 469) Let $z'$ be a small deformation of $z_0$ producing $(W, L)$, and $m_j$ be a meridian of each link component $L_j$. Then there exist flattening $f''$ for the deformed triangulation $(T, b, w')$ such that the weight $\gamma(f'')$ associated to the collection of log-branches of $T'' = (T, H, b, w', f'', c)$ satisfies $\gamma(f'')(m_j) = 0$ for all $j$. 

Quantum Hyperbolic Geometry

**Theorem 6.3.** [Cusped manifold surgery formula] Let \((W, L)\) be obtained by completion of a small deformation \(z'\) of \(z_0\), and \(T', T''\) be associated triangulations. Denote by \(\rho\) the hyperbolic holonomy of \(W\). Then we have \(H_N(W, L, \rho) = N H_N(T'')\). Moreover, associated to each cusp \(C_j\) of \(M\) there is an explicitly known map \(\Lambda_N^j(T'') : \{N \text{- states of } T\} \to \mathbb{C}\) such that the following surgery formula holds:

\[
H_N(W, L, \rho) = N \sum_s \prod_{\Delta \subset T} R_N(\Delta, b, w', f', c)_s \prod_j \Lambda_N^j(T'')(s)
\]

where \(s\) runs over the \(N\)-states of \(T\) and \(R_N(\Delta, b, w', f', c)_s\) is the matrix dilogarithm entry determined by \(s\), for the tetrahedron with the continuously deformed structure.

**Corollary 6.4.** If \(\{(W_n, L_n, \rho_0)\}\) is a sequence of closed hyperbolic Dehn fillings converging to the cusp manifold \(M\), then for every arc \(a\) we have \(\lim_n H_N(W_n, L_n, \rho_0) = N H_N(M, a)\). Hence \(H_N(M) = H_N(M, a)\) is always a well defined invariant of \(M\) (beyond the weakly-gentle case).

**Remarks 6.5.**
1. **(1)** Theorem 6.3 is the analog for \(N > 1\) of Theorems 14.7 and Theorem 14.5 in [25], which describe surgery formulas for the volume, \(\text{Vol}(W)\), and Chern-simons invariant, \(\text{CS}(W)\), of \(W\):

\[
\sqrt{-1} (\text{Vol}(W) + \sqrt{-1} \text{CS}(W)) = \sum_{\Delta \subset T} R(\Delta, b, w', f') - \frac{\pi \sqrt{-1}}{2} \sum_j \lambda(L_j)
\]

where \(R\) is given by [3] and \(\lambda(L_j)\) is the complex length of \(L_j\), that is, the logarithm of the dilation factor of its holonomy, which is a loxodromic transformation of \(\mathbb{H}^3\). The technical complications due to the bubble moves disappear for \(N = 1\).

2. **(2)** If \(M\) is gentle and has a geometric branched ideal triangulation \((T, b, w)\) without degenerate tetrahedra, then for each 3-simplex the flattennings of \(T'\) for a sufficiently small deformation are just \(-s_t\) times integral charges. It follows from the proof of Theorem 6.3 that the scissors congruence class \(c_{\mathbb{H}^3}(W, L, \rho)\) of [3], section 7, coincides with Neumann’s deformed scissors congruence class \(\beta(M')\) in [25], Theorem 14.7 (see also Remark 6.12 and Conjecture 7.9 in [3], where the undeformed \(\beta(M)\) is denoted \(c_{\mathbb{H}^3}(M)\)).

3. **(3)** In general there are small deformations \(z'\) of \(z_0\) leading to complete manifolds that are still cusped, that is only some cusps of \(M\) have supported a hyperbolic Dehn filling. There are also sequences of such cusped manifolds \(M^n\), with (short) geodesic links \(L^n\), converging to \(M\). Similarly to the fusion of QHFT with QHFT⁰ (see Section 7.3) we can define quantum hyperbolic invariants \(H_N(M^n, L^n)\) for which the natural extensions of Theorem 6.3 and Corollary 6.4 hold.

Let us consider now \((W, L_F, L^0, \rho)\). Let \(L_j\) be a component of \(L_F\), \(\lambda_j = L_j \cup L'_j\), \(L'_j\) being the longitude of \(L_j\) specifying the framing. Let \(U = U(L_j)\) be a tubular neighbourhood of \(L_j\) in \(W\), and \(l \subset \partial U\) be a non separating simple closed curve. Let \(W(l)\) be obtained from \(W\) by the Dehn filling of \(W \setminus \text{Int}(U)\) along \(l\). Denote by \(l^*\) the core of the attached solid torus.

**Theorem 6.6.** [Closed manifold surgery formula] Assume that \(\rho(l) = \text{id} \in \text{PSL}(2, \mathbb{C})\) and the weight \(k\) satisfies \(k(\|l\|) = 0\). Denote: \(\rho'\) the natural extension of \(\rho_{W \setminus U}\) to \(W(l)\); \(L_F \subset L_j\); \(k'\) the restriction of \(k\) to \(W(l)\). We have

\[
H_N(W, L_F, L^0, \rho, 0, k) = N H_N(W(l), L_F, L^0 \cup l^* \cup \lambda_j, \rho', 0, k')
\]

If moreover \(\rho\) is not trivial at the meridians of \(L_F\) and \(l\) is a longitude of \(\partial U\), then
Let us assume now that $L^0$ is made by $r$ parallel copies of $L_j$ along the ribbon $L_j$ that encodes the framing. So denote $L^0$ by $\lambda_j$; with this notation, $\hat{\lambda} = \lambda_2$. Assume furthermore that $l = m$ is a meridian of $L_j$, so that $l^* = L_j$. By applying inductively both \ref{assertion} and Proposition \ref{assertion} to this situation we get

**Corollary 6.7.** For every $r \geq 2$ we have

$$H_N(W, \lambda, \rho) = H_N(W, 2\lambda, \rho).$$

**Remarks 6.8.** (1) Though disjoint and complementary by hypothesis, formula (21) is formally the same as that of Proposition \ref{assertion}, when replacing $l$ by $m$.

(2) Assume (for simplicity) that $L = L_\infty$. When $l$ is a longitude of $L_j$, $l^*$ inherits a natural framing in $W(l)$. Hence we get a triple $(W(l), L_j, \rho)$. It follows from the very definition of the QHFT tensors that $H_N(W(l), L_j, \rho, 0, 0) = H_N(W(l), L_j, \rho, 0, 0)$ and the same with $H_N$ (when defined) replacing $H_N$.

(3) We have seen that both $H_N$ and $H_N'$ partition functions display interesting features of QHG. A main advantage of the $H_N$ ones is the possibility to set in a same “holomorphic family” the QHG tensors associated to characters that are both trivial and non trivial at link meridians. Consider for example a hyperbolic knot $L$ in $S^3$, endowed with the canonical framing $F$. Kashaev’s volume conjecture concerns the asymptotic behaviour of $H_N(S^3, L, \rho_{triv})$ when $N \to +\infty$. A reasonable variant of it is in terms of the partition functions $H_N(S^3, L_\infty, \rho_{triv}, 0, 0) = H_N(S^3, \hat{\lambda}, \rho_{triv})$. A family as above could be useful in order to establish connections with the $H_N$ partition functions of $(S^3, L_\infty, \rho_{hyp})$, where $\rho_{hyp}$ is the hyperbolic holonomy of the cusped manifold $M = S^3 \setminus L$.

The rest of the section is devoted to the proof of these results. This goes in several steps.

A main tool is the *simplicial blowing up/down* procedure considered by Neumann in \cite{Neumann}, section 11. We use it just to get a simplicial version of (topological) Dehn filling. Let $Z$ be a pseudo manifold without boundary such that every non-manifold point has toric link. Let $v$ be a non-manifold point. Consider a closed cone neighborhood $N(v)$ of $v$, and a non separating simple closed curve $C$ on the torus $\partial N(v)$. The topological Dehn filling of $Z$ at $v$ along $C$ is the pseudo manifold $Z'$ obtained by gluing a 2-handle to $Z \setminus \text{Int}(N(v))$ along $C$, and then collapsing to one point the resulting boundary component.

Now, let $T$ be a pseudo manifold triangulation of $Z$. Consider the *abstract star* $\text{Star}^0(v)$ of $v$ in $T$. The boundary of $\text{Star}^0(v)$ is the abstract link $\text{Link}(v)$ which is homeomorphic to $\partial N(v)$. Assume that the curve $C$ is realized as a simplicial curve on $\text{Link}(v)$. Then the cone from $v$ over $C$ in $\text{Star}^0(v)$ is a triangulated disk $D^0$. The interior of $\text{Star}^0(v)$ embeds onto the interior of the actual star of $v$ in $T$, $\text{Star}(v)$, which is made of the union of the 3-simplices having $v$ as a vertex. In this way $D^0$ maps onto a triangulated singular disk $D$ in $Z$, that has embedded interior and singular boundary immersed in the boundary of $\text{Star}(v)$. Cut open $T$ along $\text{Int}(D)$ and glue the double cone $CD$ of $D$ (this is a triangulated singular 3-ball, see Figure 14) so that the top and the bottom get identified with the two copies of $D$ resulting from slicing. This gives a triangulation $T'$ of the pseudo-manifold $Z'$ obtained by Dehn filling along $C$. It has the property that every (abstract) tetrahedron of $T$ persists in $T'$. Referring to the topological description, the interior of the co-core of the 2-handle attached to $Z \setminus \text{Int}(N(v))$ is isotopic to the interior of the union $H'$ of two edges, each joining $v$ to the new vertex $v'$ at the “center” of $CD$. In fact $H'$ is the core of the solid torus added by the Dehn filling.

**Remark 6.9.** Note that in general there are very few simplicial curves on a given $\text{Link}(v)$. Hence to get such a simplicial description of an arbitrary Dehn filling, we will usually have to modify a given triangulation. For the peculiar QHG pseudo-manifold triangulations considered in this section, retriangulating will be possible by using QHG isomorphisms, hence without altering the trace tensors. In fact, any two triangulations of $\partial N(v)$ are connected by a finite sequence of 2-dimensional $1 \leftrightarrow 1$ “flip” moves (see Figure 6), and $1 \leftrightarrow 3$ moves obtained by replacing a 2-simplex with the cone of its boundary to a point. Since $N(v)$ is homeomorphic to $\text{Link}(v)$, any such a sequence is the boundary trace of a sequence of $2 \leftrightarrow 3$ moves and bubble moves in $\text{Star}(v)$. (Note, in particular, that $H$ passes
through the new interior vertices). By using the arguments of Theorem 6.8 in [5], we will always be able to choose that sequence so that it lifts to a sequence of QHG transits.

Let us consider now a distinguished triangulation ($T$, $H$) by letting the knot $H$ be the result of a simplicial Dehn filling of $Z$ along a curve $C$. Denote $H''$ the graph union of the knot $H'$ (the core of the solid torus) and the image of $H$ in $T'$. We define the notion of global charge on ($T', H''$) by formally replacing $H$ by $H''$ in (12).

Lemma 6.10. Let $c$ be a global charge on ($T, H$) such that the charge weight of the curve $C$ is 0. Then $c$ extends to a global charge $c'$ on ($T', H'$).

Proof. The complex $CD$ is made of pairs of adjacent 3-simplices, respectively above and below the disk $D$. For a 3-simplex of the top layer with charges $c_0$ and $c_1$ at the edges in $D$ (ordered by using an orientation of $D$, say), we will put the charges $-c_0$ and $2 - c_1$ at these edges for the symmetric 3-simplex in the bottom layer. Then the other charges are $c_2 = 1 - c_0 - c_1$ and $-c_2$, respectively. We have the charge sum $C_{T'}(e) = 2$ at each interior edge of $D$, and $C_{T'}(e) = C_T(e)$ at the edges $e$ of $\partial D$. For the top edges $e'$ of $CD$ we can also choose the charges so that $C_{T'}(e')$ equals $C_T(e)$, where $e$ is the copy of $e'$ in $D \subset M$. Indeed, there are $n$ degrees of freedom in doing this, where $n$ is the number of 1-simplices in the curve $C$ used for blowing down. Then we check that $C_{T'}(e') = 2$ at the bottom edges. In particular, the subcomplex $H$ survives in $T'$.

Note that $C_{T''}(e_0) = -C_{T'}(e_1)$ at the edges $e_0$ and $e_1$ of $H'$. We have to check that $C_{T''}(e_0) = 0$, so that (12) is satisfied on ($T', H'' \setminus l$). In fact, $C_{T''}(e_0)$ is $n$ minus the sum of the $2n$ charges at the bottom edges of $CD$, which is also the sum of charges in $T' \setminus CD$ at these edges, minus $n$. We can form $n$ pairs of such charges corresponding to the 3-simplices of the ideal triangulation $T$ of $M$ having a 2-simplex in $D$. Replacing for each of them the pair with 1 minus the last charge, we get that $C_{T''}(e_0)$ is equal to $\gamma(a)$, with $\gamma$ defined in section 2 and $a$ is a normal path in $\text{Link}(v)$ that runs parallel to $C$ on one side (see Figure 15). Because the weight of $C$ is zero, we deduce $C_{T''}(e_0) = 0$.

If $H \neq \emptyset$, we have to show that it can be deleted from $H''$. As the two components $l$ and $H \setminus l$ are isotopic and satisfy $C_T(l) = 4$ and $C_T(e) = 0$ for each edge $e \in H \setminus l$, we can retriangulate the surgered pseudo-manifold $Z'$ so as to delete them, by using a sequence of charge transits starting from ($T', H''$) and terminating with a negative bubble move (see [4], Proposition 4.27 and [5], proof of Theorem 6.8). Retriangulating $Z'$ backward, we eventually find a sequence of charge transits terminating at ($T', H', c'$). The result follows.

Proof of Theorem 6.3. To simplify assume that $M$ has only one cusp. Take $T'' = (T, H, b, w', f'', c)$. For the first claim we assume that the meridian $m$ of $L$ is a simplicial path in $\text{Link}(v)$, where the vertex $v$ of $T$ corresponds to the filled cusp. This is possible due to Remark 6.9.

Lemma 6.14 implies that $c'$ extends to ($T', H'$) after the Dehn filling along $m$. Extend the branching $b$ by letting the new vertex $v'$ be a pit of the double cone $CD$ we splice in $T$. By using Lemma 6.2 arguments similar to that of Lemma 6.10 show that we can give the same log-branches on the 3-simplices of $CD$, in a pair above and below the disk $D$ (see [24], p. 454). Hence we get a distinguished
flat/charged $\mathcal{I}$-triangulation for $(W, L, \rho)$. The weight $h \in H^1(W; \mathbb{Z}/2\mathbb{Z})$ is clearly 0 because of the epimorphism $H_1(M; \mathbb{Z}/2\mathbb{Z}) \to H_1(W; \mathbb{Z}/2\mathbb{Z})$ induced by inclusion. As in the first claim of Lemma 5.10 (2) we see that the (unnormalized) trace tensor for $CD$ is $N$ times the identity map from the linear space attached to the top copy of $D$ to that for the bottom one. Combining this with the normalization of trace tensors in (11) gives

$T$ linear space attached to the top copy of $D$.

By Lemma 6.2 we know that $\gamma$ is an epimorphism (deduced from Corollary 5.10 (2)) we see that the (unnormalized) trace tensor for $CD$ is $N$ times the identity map from the linear space attached to the top copy of $D$ to that for the bottom one. Combining this with the normalization of trace tensors in (11) gives $H_N(W, L, \rho) = H_N(T'')$.

By Lemma 6.2 we know that $\gamma(f'') - \gamma(f') \in H^1(\partial M; \mathbb{Z}/2\mathbb{Z})$ is non zero only at the class of $m$, where it is $-2\sqrt{-1}\pi$. Hence the collection of values of $f'' - f'$ determines a path $l$ normal to the cusp triangulation induced by $T$, that intersects $m$ once and whose homology class is Poincaré dual to $(\gamma(f'') - \gamma(f'))/2\sqrt{-1}\pi$. Denote

$\Delta(l_j) = (\Delta^1, \ldots, \Delta^{\lvert \Delta(l_j) \rvert})$

the sequence of flat/charged $\mathcal{I}$-tetrahedra (possibly with repetitions) determined by the 2-simplices met by $l$. Each time $l$ goes through a 2-simplex it selects one of its vertices, whence a cross-ratio modulus, say $z_i$, of the tetrahedron $\Delta^i$ corresponding to the 2-simplex. The values of $f''$ on $\Delta^i$ are obtained from those of $f'$ by adding or subtracting 1 at the edges corresponding to the other two vertices, as indicated in Fig. 16. For any fixed tetrahedron $\Delta$ of $T$ all three flattenings may be eventually altered, and/or differ from those of $f'$ by adding or subtracting $n_i \in \mathbb{Z}$ with $n_i \neq -1$ or 1, exactly when $\Delta = \Delta^i = \Delta^j$ for some $i \neq j$. Now, recall from (11) that

$H_N(M, a) = H_N(T) = \sum_s \prod_{\Delta \subset T} \mathcal{R}_N(\Delta, b, w, f, c)_s$.

Put $\zeta = \exp(2\pi\sqrt{-1}/N)$. For any $x \in \mathbb{C}\setminus\{\zeta^j, j = 1, \ldots, N-1\}$ the function $g$ defined in the Appendix satisfies [5], Lemma 8.2:

$g(x\zeta^k) = g(x) \prod_{j=1}^k \frac{(1 - x\zeta^j)^1/N}{1 - x\zeta^j}$.

Using this formula, it is easily checked that given a flat/charged $\mathcal{I}$-tetrahedron $(\Delta, b, w', f', c)$ with $f' = (f'_0, f'_1, f'_2)$ and positive branching orientation, if $f'' = (f'_0 + n, f'_1, f'_2 - n)$ we have

\[ \mathcal{R}_N(\Delta, b, w', f'', c)_s = \mathcal{R}_N(\Delta, b, w', f', c)_s \prod_{j=1}^n \frac{1}{1 - w'_j \zeta^{i-k+j}} \]

up to multiplication by $N$th roots of unity, where $i$ and $k$ are as in (6). For each 2-simplex met by $l$ we can apply (22) to the corresponding tetrahedron, or the similar formula (deduced from Corollary 5.6 of [5]) for any other branching. This defines the function $\Lambda_N(T'')$, so that we get

$H_N(T'') = \sum_s \prod_{\Delta \subset T} \mathcal{R}_N(\Delta, b, w', f'', c)_s$

$= \sum_s \prod_{\Delta \subset T} \mathcal{R}_N(\Delta, b, w', f', c)_s \Lambda_N(T'')$.

The conclusion follows from the equality $H_N(W, L, \rho) = H_N(T'')$.  \[ \square \]
Proof of Theorem 6.6. Again for simplicity, assume that $L$ is a knot (one component). We apply the very same arguments as for the first claim of Theorem 6.3. In particular, Lemma 6.10 applies verbatim. Since $\rho(l)$ is trivial, for an arbitrary flattening the weight along $l$ (computed in the flattened $I$-cusps) lies in $\mathbb{Z} \sqrt{-1}$. Hence we can again use Theorem 2.12 to deduce the existence of flattenings with zero weight along $l$. Then we give the same log-branches on the 3-simplices of the singular 3-ball $CD$, in a pair above and below the disk $D$ (see [25], p. 454). Note that if we use $D$ triangulations leading to $\mathcal{H}_N$ partition functions, then both parallel components $L, L'$ that make \( \bar{\lambda} \) survive in the Hamiltonian subcomplex. If we can deal with $\mathcal{H}_N^*$-ones, only the framing longitude $L'$ survives. Hence we eventually get a distinguished flat/charged $I$-triangulation for $(W(l), l^* \cup \bar{\lambda}, \rho)$, or $(W(l), l^* \cup \lambda', \rho)$ respectively. \( \square \)

6.3. Manifolds that fiber over $S^1$ - Examples. Lemma 5.10 (3) gives a practical recipe to compute the QHFT partition functions of mapping tori. A specific class of distinguished flat/charged $I$-triangulations of $(\tilde{W}_{[\psi]}, \tilde{L}_{[\psi]}, \rho)$ is obtained by composing one for the trivial mapping cylinder $F \times [-1,1]$, say $T_{triv}$, with the monodromy action on the $e$-triangulation $(T, b)$ of $F \times \{1\}$, and then gluing the two boundary components. The monodromy action can always be decomposed as a sequence of flip moves: a single flip on the ideal triangulation $T'$ associated to $T$ defines a flip on $(T, b)$ if it is not adjacent to a marked corner, and it lifts to sequences as in Figure 17 otherwise. We view these sequences as the result of gluing tetrahedra. Hence the monodromy action determines a branched triangulated pseudo-manifold $T_s$. This can be completed with global charges, and, as for any $\rho$ we are free to choose the cocycle in $T_{triv}$, we can also complete $T_s$ to a flattened $I$-triangulation. Equivalently we can define a sequence

\begin{equation}
(23) \quad s : (T, b, \mathcal{L}) \to \ldots \to \psi(T, b, \mathcal{L})
\end{equation}

of $e$-triangulations with $(+)$-Log-$I$-parameters compatible with $\rho$. Note that the edges of the associated pattern $T_s$ of flat/charged $I$-tetrahedra are disjoint from the Hamiltonian link $H$. Since $\mathcal{H}_N(T_{triv}) = N$ id, we deduce that $\mathcal{H}_N(\tilde{W}_{[\psi]}, \tilde{L}_{[\psi]}, \rho) = N \mathcal{H}_N(T_s)$.

Using a similar construction we now prove the relationship with the quantum hyperbolic invariants $H_N$ of fibered cusped manifolds [5]. Recall that $W \setminus L$ is homeomorphic to $\text{Int}(\tilde{W}_{[\psi]})$. Denote by $l$ the number of components of $L$. 

---

**Figure 16.** Flat/charge corrections for a Dehn filling.

**Figure 17.** Lifts to $e$-triangulations (economic ones - see Remark 3.12 - at the first row) of flip moves on the corresponding ideal triangulations near marked corners. The tetrahedron associated to the first flip (first and third ones for the second row) degenerates for a sequence $s$ with $(-)$-Log-$I$-parameters.
Proposition 6.11. If $\Int(\tilde{W}_[\psi])$ supports a (necessarily unique) complete hyperbolic structure with holonomy $\rho$, then $\mathcal{H}_N(W_{[\psi]}, \tilde{L}_{[\psi]}, \rho, 0, 0) = N^{2l} \mathcal{H}_N(W \setminus L)$.

Proof. Let $S \setminus V$ be the fiber of $W \setminus L \to S^1$, equipped with an ideal triangulation $T'$. First we show the existence of sequences $s : T' \to \ldots \to \psi(T')$ of flip moves decomposing the monodromy action, such that the associated pseudo-manifolds $T'_s$ are topological ideal triangulations of $W \setminus L$, which moreover have maximal volume.

The first condition follows from the fact that the monodromy is homotopically aperiodic (i.e. pseudo-Anosov), so that $T'_s$ is genuinely three-dimensional. When $S \setminus V$ is a once-punctured torus, the second condition is a consequence of a result of Lackenby [23], showing that the monodromy ideal triangulation of Floyd and Hatcher [17] is isotopic to the canonical Epstein-Penner cellulation. More in general, since no edge of $T'_s$ is homotopically trivial, the results of [15] imply that we can straighten the tetrahedra to oriented geodesic ones, possibly with overlappings, so that the algebraic sum of volumes is $\Vol(W \setminus L)$. This is known to be maximal [16].

As in Section 6.2 we can complete $T'_s$ to a flat/charged $\mathcal{I}$-triangulation $T'$. Hence the invariants $H_N(W \setminus L)$ can be computed as trace tensors $\mathcal{H}_N(T')$. We note that in the case when there are several fibrations of $W \setminus L$, or $T'_s$ is not canonical, the invariance follows from Theorem 6.8 (2) in [5], which shows that any two flat/charged $\mathcal{I}$-triangulations of $W \setminus L$ with maximal volume are QHG-isomorphic.

Let us denote $T'_s$ the result of cutting $T'$ along the fiber. The two boundary copies are marked pleated hyperbolic surfaces $(T', b', \mathcal{L}) \to S' \setminus V$ and $(T', b', \mathcal{L}) \to \psi(S' \setminus V)$, with shear-bend coordinates (i.e. $(-)$-Log-$\mathcal{I}$-parameters) $\mathcal{L}$ that determine completely the log-branches of $T'_s$. Recall from Section 5.5 the families of flat/charged $\mathcal{I}$-triangulations $\mathcal{F}_t$ and $\mathcal{F}$, and let $C \in \mathcal{F}$, $C_t \in \mathcal{F}_t$ have boundary structures $(T, b, \mathcal{L})$, associated to $(T', b', \mathcal{L})$, respectively, at $F \times \{1\}$. We have:

$$
\mathcal{H}_N(T) = \Tr(\mathcal{H}_N(T'_s)) = N^{-2l} \Tr(\mathcal{H}_N(T'_s) \otimes \id^{2l})
$$

$$
= N^{-2l} \Tr(\mathcal{H}_N(C_t) \circ (\mathcal{H}_N(T'_s) \otimes \id^{2l}) \circ \mathcal{H}_N(\psi(C))^{-1})
$$

$$
= N^{-2l} \lim_{t \to \id} \Tr(\mathcal{H}_N(C_t) \circ (\mathcal{H}_N(T'_s) \otimes \id^{2l}) \circ \mathcal{H}_N(\psi(C_t))^{-1})
$$

$$
= N^{-2l} \lim_{t \to \id} \Tr(\mathcal{H}_N(C_t \ast T'_s, t \ast (\psi(C_t))))
$$

Here we use the invertibility of $\mathcal{H}_N(C)$ (Proposition 5.12) and the equality $\mathcal{H}_N(C) = N \mathcal{H}_N(\psi(C))$. We define $T'_{s,t}$ as the continuous deformation of $T'_s$ obtained from the sequence

$$
s_t : (T, b, \mathcal{L}) \to \ldots \to \psi(T, b, \mathcal{L})
$$

similarly as in (23) (see also Figure 17). In the last equality, $C_t \ast T'_s \ast (\psi(C_t))$ is for any fixed $t$ a distinguished flat/charged $\mathcal{I}$-triangulation of the mapping cylinder of $\psi$, Hence $\mathcal{H}_N(C_t \ast T'_s, t \ast (\psi(C_t)))$ does not depend on $t$ up to conjugacy, and we conclude with Lemma 6.10 (3).

By following the above computation backwards, we see more in general that for any $PSL(2, \mathbb{C})$-character $\rho$ that can be realized by $(-)$-Log-$\mathcal{I}$-parameters on some ideal triangulation of the fiber $S' \setminus V$, we have

$$
\mathcal{H}_N(\tilde{W}_{[\psi]}, \tilde{L}_{[\psi]}, \rho, h, k) = N^{2l} \Tr(\mathcal{H}_N(T'_s))
$$

where $T'_s$ is a pattern of flat/charged $\mathcal{I}$-tetrahedra associated to a sequence similar to (23), but with $e$-triangulations equipped with $(-)$-Log-$\mathcal{I}$-parameters compatible with $\rho$.

Remark 6.12. The formula $H_N(W \setminus L) = \Tr(\mathcal{H}_N(T'_s))$ expresses the quantum hyperbolic invariants of fibered cusped manifolds as amplitudes between two markings of the fiber, identified with a pleated hyperbolic surface. For a similar construction based on representations of quantum Teichmuller spaces, see [9].

Example: the figure-eight knot complement Here we compute the QHFT partition functions of $(S^3, K_0)$, where $K_0$ is the 0-framed figure-eight knot in $S^3$. Recall that $S^3 \setminus K$ is fibered over $S^1$, with fiber the once-punctured torus $\Sigma_{1,1}$; the 0-framing of $K$ is induced by the fibration. For
simplicity, below we consider only characters $\rho$ of injective representations $\pi_1(S^3 \setminus K) \to \text{PSL}(2, \mathbb{C})$. The restriction to $\Sigma_{1,1}$ of such representations can be realized by ($-$)-Log-$I$-parameters on any ideal triangulation of $\Sigma_{1,1}$, so that, by \cite{21}, we can determine the corresponding subspace (still denoted $\text{Def}(S^3, K)$) of the phase space of Definition 5.8 by using the monodromy ideal triangulation.

The monodromy $\Phi : \Sigma_{1,1} \to \Sigma_{1,1}$ of $S^3 \setminus K$ is isotopic to the hyperbolic element 

$$
\begin{pmatrix}
2 & 1 \\
1 & 1
\end{pmatrix}
= 
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
1 & 1
\end{pmatrix}
\in \text{SL}(2, \mathbb{Z}).
$$

This description of $[\Phi] \in \text{Mod}(\Sigma_{1,1})$ can be understood in terms of the Diagram of $\text{PSL}(2, \mathbb{Z})$ (see eg. \cite{17}) via the action of $\Phi$ on topological ideal triangulations of $\Sigma_{1,1}$, which can be represented by two flip moves. See Figure 18, where the left picture is a lift to $\mathbb{R}^2 \setminus \mathbb{Z}^2$ of such a triangulation.

![Figure 18](image1)

**Figure 18.** The composition of flip transformations for the monodromy of $S^3 \setminus K$.

The monodromy ideal triangulation $T$ of $S^3 \setminus K$ is obtained by realizing each flip move via the gluing of an ideal tetrahedron, first on a fixed triangulation of $\Sigma_{1,1}$, then on the resulting one. The remaining four free faces are identified under $\Phi$. It is not difficult to see that $T$ is isotopic to the canonical geodesic ideal triangulation of $S^3 \setminus K$ with its complete hyperbolic structure. The gluing pattern of the tetrahedra in $T$ is shown in Figure 19.

![Figure 19](image2)

**Figure 19.** The face and edge identifications for the canonical geodesic ideal triangulation of $S^3 \setminus K$.

It is well-known (see \cite{29}) that the deformation space of smooth (non necessarily complete) hyperbolic structures on $S^3 \setminus K$ is isomorphic to the algebraic set $\text{Def}_{f_{\text{hyp}}}(S^3 \setminus K) \subset \mathbb{H}^2 \times \mathbb{H}^2$ of points $(w_2, z_0)$ such that

$$
\begin{cases}
    w_2 \in \mathbb{H}^2 \setminus \left\{ \frac{1}{2} + \frac{t}{2} i \mid t \geq \sqrt{15} \right\}, \\
z_0 = \frac{1}{2} + \left( \frac{1}{4} + \frac{1}{w_2(w_2 - 1)} \right)^\frac{1}{2}
\end{cases}
$$

where $w_2$ is the cross-ratio modulus of the edge $e_2$ in the tetrahedron $\Delta^+$ with positive branching orientation (back edge in the left tetrahedron of Figure 19), and similarly for $z_0$ in $\Delta^-$. The space $\text{Def}_{f_{\text{hyp}}}(S^3 \setminus K)$ is a subspace of

\begin{equation}
C = \{(w_2, z_0) \in (\mathbb{C} \setminus \{0, 1\})^2 \mid w_1 w_2^2 z_0^{-2} z_1^{-1} = 1\},
\end{equation}

which is isomorphic to the whole set of solutions of the edge compatibility relations for cross-ratio moduli (see Definition 2.6). By an easy computation we find that the edge compatibility relations for log-branches and charges are:

\begin{equation}
\begin{cases}
f_1^- + 2f_0^- + 2f_1^+ + f_2^+ = (\arg(w_1) + 2\arg(w_2) - \arg(z_1) - 2\arg(z_0)) - 2s_+ \\
c_1^- + 2c_0^- + 2c_0^+ + c_1^+ = 0
\end{cases}
\end{equation}
where \( *_+ \) is the sign of the imaginary part of the \( w_i \), and the flattenings \( f_+ \) and \( f_- \) correspond to the cross-ratio moduli \( w_i \) and \( z_i \) of \( \Delta^+ \) and \( \Delta^- \), respectively. Hence, by (27) we see that the QHFT phase space \( \text{Def}(S^3, K) \) is the covering of \( \mathcal{C} \) given by

\[
\text{Def}(S^3, K) = \{(w_2; f_2^+, c_1^+, f_3^+ c_3^-_0), (z_0; f_0^+, c_0^+, f_1^- + c_1^-) \} \in \hat{\mathbb{C}} \times \hat{\mathbb{C}} | (w_2, z_0) \in \mathcal{C}, (S) \text{ is satisfied}\}
\]

Let us point out some geometrically meaningful subspaces (compare with [25], Section 15). The dilation factors of the standard meridian \( \mu(m) = z_2 w_2^{-1} \), \( \mu(l) = z_0^{-2} z_2^{-2} \).

The complete hyperbolic structure of \( S \) is the sign of the imaginary part of the cross-ratio moduli \( w \) and \( \pi \) of \( \Delta^\text{hyp} \). Let us parametrize by \((p, q)\) Dehn filling, we have \( \mu(m) + q \mu(l) = 2 \pi \sqrt{-1} \). Hence the flattenings \( f = f'' \) of Lemma 6.2 are to satisfy

\[
p(f^-_0 - f^+_0) + q(2f^-_0 - 2f^-_2) = -2
\]

Let us fix \( r, s \in \mathbb{Z} \) such that \( ps - qr = 1 \). Solving simultaneously (28) and the last equation gives

\[
f^+_1 = r - 1 - 2f^+_0, \quad \begin{cases} f_0^- = -2s - f_0^+ \\ f^-_1 = -r + 4s + 1 + 2f_0^+ \end{cases}
\]

The parity condition \( f^+_0 + f^+_1 + f^-_0 + f^-_1 \in 2 \mathbb{Z} \) (for the class \( h = 0 \) is automatically satisfied.

Now we can compute the QHFT invariants of \((S^3, K_0)\) as functions on \( \text{Def}(S^3, K) \). Given a \( N \)-state \( s \) of \( T \), put \( \alpha = s(2), \beta = s(\bar{0}), \gamma = s(\bar{3}) \) and \( \delta = s(\bar{1}) \), where \( \bar{1} \) denotes the face of \( \Delta^+ \) opposite to the 7th vertex, by (24) we have:

\[
N^{-2} \mathcal{H}_N(S^3, K_0, \rho) = \sum_{\alpha, \beta, \gamma, \delta = 0}^{N-1} \mathcal{R}_N(\Delta^+, b^+, w, f^+, c^+)_{\alpha, \beta} \mathcal{R}_N(\Delta^-, b^-, z, f^-, c^-)_{\beta, \alpha} \omega(w_0^\prime w_1^\prime | \alpha - \gamma) \delta(\alpha + \beta - \delta)
\]

\[
\times (z_0^\prime c_3^- z_1^\prime)^{N-1} \frac{[z_0^\prime g(1)]}{g(z_0^\prime)} \frac{[z_0^\prime g(1)]}{g(z_0^\prime)} \frac{[z_0^\prime g(1)]}{g(z_0^\prime)} \frac{[z_0^\prime g(1)]}{g(z_0^\prime)}
\]

\[
= (w_0^{\prime c_3^-} w_1^\prime c_3^- z_0^\prime z_1^\prime) \sum_{\alpha, \beta = 0}^{N-1} \mathcal{R}_N(\Delta^+, b^+, w, f^+, c^+)_{\alpha, \beta} \mathcal{R}_N(\Delta^-, b^-, z, f^-, c^-)_{\beta, \alpha} \omega(w_0^\prime w_1^\prime | \alpha - \gamma) \delta(\alpha + \beta - \delta)
\]

\[
\times (z_0^\prime c_3^- z_1^\prime)^{N-1} \frac{[z_0^\prime g(1)]}{g(z_0^\prime)} \frac{[z_0^\prime g(1)]}{g(z_0^\prime)} \frac{[z_0^\prime g(1)]}{g(z_0^\prime)} \frac{[z_0^\prime g(1)]}{g(z_0^\prime)}
\]

\[
= \sum_{\alpha, \beta = 0}^{N-1} \frac{[z_0^\prime g(1)]}{g(z_0^\prime)} \frac{[z_0^\prime g(1)]}{g(z_0^\prime)} \frac{[z_0^\prime g(1)]}{g(z_0^\prime)} \frac{[z_0^\prime g(1)]}{g(z_0^\prime)}
\]
By the proof of Proposition 8.6 in [3] we have
\[
\frac{[z'_0]g(w'_0)}{g(z'_0)} = \frac{g((z'_0)^*)g(w'_0)}{|g(1)|^2}
\]
and
\[
\omega(z'_0/\zeta, z'_{-1}|N - \alpha)^{-1} = \omega((z'_0)^*, (z'_{-1})^*|\alpha)^*
\]
where \(z^*\) is the complex conjugate of \(z\). Thus, setting
\[
S(w'_0, w'_1) = \sum_{\beta=0}^{N-1} \zeta^{\beta^2} \omega(w'_0, w'_{1-1}|\beta) = 1 + \sum_{\beta=1}^{N-1} \zeta^{\beta^2} \prod_{k=1}^\beta \frac{w'_{1-k}^{-1}}{1 - w'_0 \zeta^k}
\]
we get
\[
\mathcal{H}_N(S^3, K_0, \rho) = N^2 \left(\frac{g(c_0^{+}/3N)}{|g(1)|^2}\right)^2 |S(w'_0, w'_1) S((z'_0)^*, (z'_1)^*)|^2.
\]
Using (29) with \(k(m) = k(l) = f_0^+ = 0\) and the global charge with \(c_0^+ = c_0^- = 0\), we see that for the complete hyperbolic structure \(\rho_{\text{comp}}\) on \(S^3 \setminus K\) we have
\[
z'_0 = (w'_0)^* = \exp(i\pi/3N), \quad z'_1 = (w'_1)^* = \exp(-5i\pi/3N).
\]
Hence
\[
\mathcal{H}_N(S^3, K_0, \rho_{\text{comp}}) = N^2 \left|\frac{g(c_0^{+}/3N)}{|g(1)|^2}\right|^2 |S(e^{i\pi/3N}, e^{-5i\pi/3N})|^2.
\]
Let us finally consider hyperbolic \((p, q)\) Dehn filling of \(S^3 \setminus K\). Denote \(S^3(K_{(p,q)})\) the surgered manifold, \(L\) the core of the surgery, and \(\rho_{(p,q)}\) its hyperbolic holonomy. Because of (30) the difference \(f^m - f\) is given on the edges \(e_0, e_1\) and \(e_2\) of \(\Delta^+\) (resp. \(\Delta^-\)) by \(0, r\) and \(-r\) (resp. \(-2s, 4s\) and \(0\)). Put \(N = 2m + 1\). From Theorem 6.3 we deduce
\[
\mathcal{H}_N(S^3(K_{(p,q)}), L, \rho_{(p,q)}) = N^2 \left(\frac{g(c_0^{+}/3N)}{|g(1)|^2}\right)^2 \prod_{\alpha=0}^{N-1} \left(\zeta^{\alpha^2} \omega(w'_0, w'_{1-1}|N - \beta) \omega((z'_0)^*, (z'_{-1})^*|\alpha)^* \zeta^{\alpha(N-\beta)}(m+1) \prod_{j=1}^{N-2s} \frac{(z'_0)^* \zeta^{4s\alpha(m+1)}}{1 - (z'_0)^* \zeta^{4s\alpha}} \right).
\]

**Remark 6.13**. Recall the cycle \(C\) in (25). As already mentioned after Definition 2.6, we have a holonomy map \(\text{hol} : C \to X\) to the character variety \(X = X(\pi_1(S^3 \setminus K))\). (See [28] or [18] for a complete description of the latter). The map \(\text{hol}\) is generically 2 : 1, and is onto the geometric component of \(X\) [12]. We can express the above partition functions in terms of standard generators of \(X\) by the following observation. Considering \(S^3 \setminus K\) as the mapping torus of the monodromy \(\Phi\), the edges \(e_0, e_1\) of \(\Delta^+\) are identified with a longitude \(l\) and meridian \(m\) of the punctured torus \(\Sigma_{1,1}\), and in \(\Delta^-\) we have \((e'_0)\) is opposite to \(e_0\):
\[
e_0 = \Phi(l,m), \quad e_1 = \Phi(l), \quad e'_0 = \Phi(m).
\]
As above, assume that \(\rho\) has non trivial holonomy at \(m, l\) and \(l.m\). Take a flat/charged \(Z\)-triangulation of \((S^3, K_0, \rho)\) as in Section 6.3 with \(PSL(2, \mathbb{C})\)-valued cocycle \(z\). Denote \(z_t\) the value at \(t\), and so on. Note that \(z\Phi(m) = A z_m A^{-1}\), where \(A = z(S^1)\), the cocycle value on the standard meridian of the knot \(K\). Then the cross ratio moduli of \(\Delta^+\) and \(\Delta^-\) are given by
\[
w_0 = [0 : z_0(0) : z_1 z_m(0) : z_0(0)], \quad z_0 = [0 : z_0(0) z_0(0) : z_0(0) z_0(0) : z_m z_1(0)].
\]
(We use the branching to remove the twofold ambiguity of \(\text{hol}\), as it allows to specify an equivariant association of a fixed point for each peripheral subgroup of \(\rho(\pi_1(S^3 \setminus K))\).)

**Remark 6.14**. Formulas for Cheeger-Chern-Simons invariants \(\mathcal{H}_1(S^3, K_0, \rho)\) of arbitrary \(PSL(2, \mathbb{C})\)-characters of \(S^3 \setminus K\) come exactly in the same way (See Remark 1.1 and Remark 5.5). In the peculiar situation of the complete hyperbolic structure and its hyperbolic Dehn fillings, they coincide with those of [25], Section 15.
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