STRONG COINCIDENCE AND OVERLAP COINCIDENCE

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Abstract. We show that strong coincidences of a certain many choices of control points are equivalent to overlap coincidence for the suspension tiling of Pisot substitution. The result is valid for dimension ≥ 2 as well, under certain topological conditions. This result gives a converse of the paper [2] and elucidates the tight relationship between two coincidences.

1. Introduction. Self-affine tiling dynamical system in $\mathbb{R}^d$ is a generalization of substitution dynamical system on letters, which gives a nice model of self-inducing structures appear in dynamical systems, number theory and the mathematics of aperiodic order. Pure discreteness of self-affine tiling dynamics is long studied from many points of views. The idea of coincidence$^1$ appeared firstly in Kamae [8], and then in a comprehensive form in Dekking [6] for constant length substitution (see also [16]). Generalizing a pioneer work of Rauzy [17], Arnoux-Ito [3] gave a geometric realization of irreducible Pisot unit substitution of degree $d$. They defined strong coincidence, which ensures that their geometric substitution gives rise to a domain exchange of $\mathbb{R}^{d-1}$, which is also semi-conjugate to the toral rotation of $T^{d-1}$. It is remarkable that in many cases, it is even conjugate to the toral rotation, which immediately implies that the system is pure discrete (see [4, 7, 5, 18] for further developments). On the other hand, overlap coincidence introduced by Solomyak [19] is an equivalent condition for pure discreteness of a given self-affine tiling dynamical system. This is also described as a geometric/combinatorial condition which guarantees that the tiling and its translation by return vectors become exponentially close if we iteratively enlarge return vectors by substitution. Lee [11] showed deeper characterizations that overlap coincidence is equivalent to algebraic coincidence, and the fact that the corresponding point set is an inter-model set.

Until now, the relation between strong coincidence and overlap coincidence is not fully understood. Motivated by the claim of Nakaishi [15], Akiyama-Lee [2] generalized the notion of strong coincidence to $\mathbb{R}^d$ and showed that overlap coincidence implies strong coincidence, and moreover simultaneous coincidence, provided that

$^1$In their notation, the column number is one.
the associated point set is admissible and its height group is trivial. In this paper we shall give a converse statement for the suspension tiling of Pisot substitution at the expense of assuming many strong coincidences at a time, that is, strong coincidences on a certain many choices of control points imply overlap coincidence and vice versa. If every tile is connected and the tiling is not a collection of unbounded connected identical colored patches, then the same result holds for \( d \geq 2 \) (Theorem 3.1). This result elucidates the tight relationship between two coincidences.

2. Terminologies.

2.1. Tiles and tilings. We shall briefly recall basic definitions used in this paper. A tile in \( \mathbb{R}^d \) is defined as a pair \( T = (A, i) \) where \( A \) is a compact set in \( \mathbb{R}^d \) which is the closure of its interior, and \( i = \ell(T) \in \{1, \ldots, m\} \) is the color of \( T \). We call \( A \) the support of \( T \) and denote \( \text{supp}(T) = A \). The translate of \( T \) is defined by \( g + T = (g + A, i) \) for \( g \in \mathbb{R}^d \). Let \( \mathcal{A} = \{T_1, \ldots, T_m\} \) be a finite set of tiles in \( \mathbb{R}^d \) such that \( T_i = (A_i, i) \); we will call them prototiles. A tiling \( \mathcal{T} \) is a collection of translates of prototiles which covers \( \mathbb{R}^d \) without interior overlaps. A finite collection of tiles which appear in \( \mathcal{T} \) is called a patch. A generalized patch is a collection of tiles in \( \mathcal{T} \) whose cardinality is not necessarily finite. Its support is defined to be the union of the supports of tiles. The diameter of a generalized patch is the supremum of Euclidean distance of two points lie within the support of the patch. A map \( \Omega \) from \( \mathcal{A} \) to the set of patches is called a substitution with a \( d \times d \) expansive matrix \( Q \) if there exist finite sets \( \mathcal{D}_{ij} \subset \mathbb{R}^d \) for \( i, j \leq m \) such that

\[
\Omega(T_j) = \{T_i + u : u \in \mathcal{D}_{ij}, \ i = 1, \ldots, m\} \tag{1}
\]

with

\[
QA_j = \bigcup_{i=1}^m (A_i + \mathcal{D}_{ij}) = \bigcup_{i=1}^m \bigcup_{u \in \mathcal{D}_{ij}} (A_i + u) \quad \text{for} \ j \leq m, \tag{2}
\]

and the last union has mutually disjoint interiors. The substitution (1) extends to all translates of prototiles and patches in a natural way. A substitution tiling of \( \Omega \) is a tiling \( \mathcal{T} \) that all the patches of \( \mathcal{T} \) is a sub-patch of \( \Omega^n(T) \) for some \( n \in \mathbb{N} \) and \( T \in \mathcal{T} \). A substitution tiling \( \mathcal{T} \) is a fixed point of \( \Omega \) if \( \Omega(T) = T \) holds. We say that a substitution tiling is primitive if the corresponding substitution matrix \( M = (\mathcal{D}_{ij}) \) is primitive, and irreducible if the characteristic polynomial of \( M \) is irreducible. We say that \( \mathcal{T} \) has finite local complexity (FLC) if for any \( R \) there are only finitely many patches of diameter less than \( R \) up to translation. A tiling \( \mathcal{T} \) is repetitive if every patch is relatively dense in \( \mathcal{T} \). A FLC substitution tiling of a primitive substitution is called a self-affine tiling. Every self-affine tiling is repetitive, which follows from the primitivity of substitution. Let \( \lambda > 1 \) be the Perron-Frobenius eigenvalue of the substitution matrix \( M \) and \( D \) be the set of eigenvalues of \( Q \). By the tiling criterion of Lagarias-Wang [9], \( \lambda \) is the element of \( D \) of maximum modulus. We say that \( Q \) fulfills Pisot family condition if every algebraic conjugate \( \mu \) of an element of \( D \) with \( |\mu| \geq 1 \) is contained in \( D \).

The set of all substitution tilings of \( \Omega \) forms a tiling space. By using a fixed point \( \mathcal{T} \) of \( \Omega \), we can describe this space as the orbit closure of \( \mathcal{T} \) under the translation action: \( X_\mathcal{T} = \{(\mathcal{T} - g : g \in \mathbb{R}^d)\} \), the closure is taken by ‘local topology’. The FLC assumption implies \( X_\mathcal{T} \) is compact and we get a topological dynamical system \( (X_\mathcal{T}, \mathbb{T}^d) \) where \( \mathbb{T}^d \) acts by translations. This system is minimal and uniquely ergodic ([19, 12]), and we are interested in the spectra of self-affine tiling dynamical
systems. Tiling dynamical system $X_T$ has pure discrete spectrum if the eigenfunctions for the $\mathbb{R}^d$-action forms a complete orthonormal basis of $L^2(X_T, \mu)$ [19].

2.2. Control points. A Delone set is a relatively dense and uniformly discrete subset of $\mathbb{R}^d$. We say that $\Lambda = (\Lambda_i)_{i \leq m}$ is a Delone multi-color set in $\mathbb{R}^d$ if each $\Lambda_i$ is Delone and $\bigcup_{i=1}^m \Lambda_i \subset \mathbb{R}^d$ is Delone. We say that $\Lambda \subset \mathbb{R}^d$ is a Meyer set if it is a Delone set and $\Lambda - \Lambda$ is uniformly discrete in $\mathbb{R}^d$ [10]. $\Lambda = (\Lambda_i)_{i \leq m}$ is called a substitution Delone multi-color set if $\Lambda$ is a Delone multi-color set and there exist an expansive matrix $Q$ and finite sets $D_{ij}$ for $i, j \leq m$ such that

$$\Lambda_i = \bigcup_{j=1}^m (Q \Lambda_j + D_{ij}), \quad i \leq m,$$

where the union on the right side is disjoint.

Given a fixed point $T$ of $\Omega$, we can associate a substitution Delone multi-color set $\Lambda_T = (\Lambda_i)_{i \leq m}$ of $T$ by taking representative points of tiles in the relatively same positions for the same color tiles in the tiling. There is a canonical way to choose representative points, called control points. A tile map $\gamma = \gamma_\Omega$ is a map from $T$ to itself which sends a tile $T$ to the one in $\Omega(T)$ such that $\gamma(T_1)$ and $\gamma(T_2)$ are located in the same relative position in $\Omega(T_1)$ and $\Omega(T_2)$ whenever $\ell(T_1) = \ell(T_2)$. A control point $c(T)$ of $T \in T$ is defined by

$$c(T) = \bigcap_{n=1}^{\infty} Q^{-n}(\text{supp}(\gamma^n(T))).$$

Control points are representative points, i.e., $U - c(U) = V - c(V)$ holds if $\ell(U) = \ell(V)$ with $U, V \in T$. Let $\Lambda_i$ be the set of control points of color $i$. Clearly, $j = \ell(\gamma(T_i))$ implies $Q \Lambda_i \subset \Lambda_j$ and the set of control points $C = \bigcup_{i=1}^m \Lambda_i$ is invariant under the expansion by $Q$, that is, $QC \subset C$. We obtain an associated substitution Delone multi-color set $\Lambda = \Lambda_T = (\Lambda_i)_{i \leq m}$.

In section 3, we have to assume a lot of strong coincidences by changing control points for a given tiling $T$. If we change control points of tiles of $T$ by $\Lambda'_i = \Lambda_i - g_i$, then the set equation will be shifted like

$$\Lambda'_i = \bigcup_{j=1}^m Q \Lambda'_j + D'_{ij}$$

with $D'_{ij} = \{d_{ij} + Q g_j - g_i : 1 \leq i, j \leq m\}$. The corresponding tile equation becomes

$$QA'_j = \bigcup A'_i + D'_{ij}$$

which is satisfied by $A'_j = A_j + g_j$. So we set $\text{supp}(T'_j) = A'_j$ and $\ell(T'_j) = \ell(T_j)$. To avoid heavy notation, we do not distinguish such changes of control points and use the same symbols $\Lambda_i$ and $T_i$.

2.3. Coincidences. The set of return vectors is defined by $\Xi(T) = \{y \in \mathbb{R}^d : U = V - y, \text{where } U, V \in T\}$. A triple $(U, y, V)$, with $U, V \in T$ and $y \in \Xi(T)$, is called an overlap if

$$(\text{supp}(U))^\circ \cap (\text{supp}(V) - y)^\circ \neq \emptyset.$$

An overlap $(U, y, V)$ is a coincidence if $U = V - y$. Let $\mathcal{O} = (U, y, V)$ be an overlap in $T$, we define $\ell$-th inflated overlap

$$\Omega^\ell \mathcal{O} = \{(U', Q^\ell y, V') : U' \in \Omega^\ell (U), V' \in \Omega^\ell (V), \text{and } (U', Q^\ell y, V') \text{ is an overlap}\}.$$
We say that a self-affine tiling $\mathcal{T}$ admits overlap coincidence if there exists $\ell \in \mathbb{Z}_+$ such that for each overlap $O$ in $\mathcal{T}$, $O^\ell O$ contains a coincidence. Two overlaps $(U, y, V)$ and $(U_1, y_1, V_1)$ are equivalent, if there is $x \in \mathbb{R}^d$ that both $U_1 = U - x$ and $V_1 - y_1 = V - y - x$ hold. The equivalence class is denoted by $(U, y, V)$. Hereafter we assume an important condition that $\Xi(\mathcal{T})$ forms a Meyer set. This condition is equivalent to the Pisot family condition for $Q$, if $Q$ is diagonalizable and all its eigenvalues are algebraic conjugate with the same multiplicity [13]. The number of equivalence classes of overlaps is finite, by the Meyer property of $\Xi(\mathcal{T})$. The action of $\Omega$ is well-defined on equivalence classes of overlaps. An overlap graph with multiplicity is a finite directed graph whose vertices are the equivalence classes of overlaps. Multiplicities of the edge from $(U, y, V)$ to $(A, z, B)$ is given by the number of overlaps in $\Xi((U, y, V))$ equivalent to $(A, z, B)$ (c.f. [1]). Overlap coincidence is confirmed by checking whether from each vertex of this graph there is a path leading to a coincidence. Overlap coincidence is equivalent to pure discreteness of self-affine tiling dynamical system $X_{\mathcal{T}}$ [19].

Strong coincidence on letter substitution is naturally generalized to self-similar tiling in $\mathbb{R}^d$ in [2]. We adapt this definition to control points. Let $\mathcal{T}$ be a self-affine tiling in $\mathbb{R}^d$ and $\Lambda = \{T_1, \ldots, T_m\}$ be the prototile set of $\mathcal{T}$. We say that the set of the control points is admissible if $\cap_{1 \leq m} (\text{supp}(T_i) - c(T_i))$ has non-empty interior.

Let $\mathcal{T}$ be the fixed point of $\Omega$. Let $c(T_i)$ ($i = 1, \ldots, m$) be the admissible control points and $\Lambda$ be an associated substitution Delone multi-color set for which $\mathcal{T} = \{T_i - c(T_i) + u_i \mid u_i \in \Lambda_i, i \leq m\}$. If for any $1 \leq i, j \leq m$, there is a positive integer $L$ that

$$\Omega^L(T_i - c(T_i)) \cap \Omega^L(T_j - c(T_j)) \neq \emptyset,$$

then we say that $\Lambda$ admits strong coincidence. In other words, strong coincidence means that for every pair of tiles $(U, V) \in \mathcal{T}^2$, $\Omega^L(U - c(U))$ and $\Omega^L(V - c(V))$ share a common tile in the same position for some $L$.

3. Strong coincidence and overlap coincidence. The set of eventually return vectors is defined by

$$\mathcal{G} : = \bigcup_{k=0}^{\infty} Q^{-k}(\Lambda_i - \Lambda_i), \quad \text{for some } i \leq m$$

which is independent of the choice of $i$, by primitivity of $\Omega$. The tiling dynamical system is invariant under replacement of the substitution rule $\Omega$ by $\Omega^n$. We consider control points of $\Omega^n$ as well. Hereafter we put $\Lambda = \bigcup_{i=1}^{m} \Lambda_i$ for $\Lambda = \Lambda_\mathcal{T} = (\Lambda_i)$ to distinguish the multi-color set and its union. Let $\langle \mathcal{G} \rangle$ be the additive subgroup of $\mathbb{R}^d$ generated by $\mathcal{G}$. We say that $\mathcal{T}$ satisfies multiple strong coincidence of level $n$ if all multi-color Delone set $\Lambda$’s generated by admissible control points of $\Omega^n$ with $\Lambda - \Lambda \subset \langle \mathcal{G} \rangle$ admit strong coincidence.

Hereafter when we speak about a topological/metrical property (connected, bounded, diameter) of a generalized patch, it refers to the corresponding property of its support. A rod is an unbounded connected generalized patch of $\mathcal{T}$ whose tiles have an identical color. A rod tiling is a tiling that every tile belongs to a rod. For ease of negation, a non-rod tiling is a tiling which is not a rod tiling. A tiling is called non-periodic if there are no non-trivial period, i.e., $\{p \in \mathbb{R}^d \mid \mathcal{T} + p = \mathcal{T}\} = \{0\}$.

Remark 1. There are many examples of periodic self-affine rod tiling. Consider a tiling of $\mathbb{R}^2$ by squares $[0, 1]^2 + (x, y)$ with $(x, y) \in \mathbb{Z}^2$ and their colors are defined by
y \text{(mod 2)} or \( x + y \text{(mod 2)} \). However we do not know an example of non-periodic self-affine rod tiling.

**Theorem 3.1.** Let \( \mathcal{T} \) be a non-rod self affine tiling by connected tiles such that \( \Xi(\mathcal{T}) \) is a Meyer set. Then there is a constant \( n \) depending only on \( \mathcal{T} \) that \( \mathcal{T} \) satisfies multiple strong coincidence of level \( n \) if and only if \( \mathcal{T} \) satisfies overlap coincidence.

Consider a substitution \( \sigma \) over \( m \) letters \( \{1, 2, \ldots, m\} \) whose substitution matrix is \( M_\sigma = (|\sigma(j)|_i) \), where \( |w|_i \) is the number of letter \( i \) in a word \( w \). We say that \( \sigma \) is a Pisot substitution, if the Perron Frobenius root \( \beta \) of \( M_\sigma \) is a Pisot number. The canonical suspension tiling \( \mathcal{T} \) in \( \mathbb{R} \) of \( \sigma \) with an expansion factor \( \beta \) is defined by associating to the letters the intervals whose lengths are given by a left eigenvector of \( M_\sigma \) corresponding to \( \beta \).

**Corollary 1.** The statement is valid for the suspension tiling of a Pisot substitution.

Indeed, \( 1 \times 1 \) matrix \( Q = (\beta) \) satisfies Pisot family condition, tiles are intervals and the suspension tiling can not be a rod tiling, since it has at least two translationally inequivalent tiles in \( \mathbb{R} \).

**Remark 2.** Multiple strong coincidence of level \( n \) requires many strong coincidences at a time for a fixed tiling \( \mathcal{T} \) even when \( n = 1 \). In dimension one, the claim of Nakaishi [15] reads a single strong coincidence implies overlap coincidence. Theorem 3.1 covers general cases but the requirement is much stronger. It would be interesting is to make smaller the constant \( n \) in Theorem 3.1. For e.g., can we take \( n = 1 \) ?

We prepare a lemma.

**Lemma 3.2.** Let \( G \) be a strongly connected finite directed graph and \( C \) be a set of cycles of \( G \). Then there is a subgraph \( G(C) \) of \( G \) with the following property.

- The set of vertices of \( G(C) \) is equal to that of \( G \).
- Every vertex has exactly one outgoing edge.
- The set of cycles of \( G(C) \) is equal to \( C \).

**Proof.** Put \( H_0 = C \). We inductively construct \( H_i \) for \( i = 0, 1, \ldots \) which satisfies:

- Every vertex has exactly one outgoing edge.
- The set of cycles of \( H_i \) is equal to \( C \).

Assume that the induced graph \( G \setminus H_i \) is non empty and take a vertex \( v \) from \( G \setminus H_i \). Since \( G \) is strongly connected, there is a path from \( v \) leading to \( H_i \). So there is a vertex \( u \in G \setminus H_i \) and an edge from \( u \) to a vertex of \( H_i \). We define \( H_{i+1} \) by adding this \( u \) and the outgoing edge. Then \( H_{i+1} \) clearly satisfies above two conditions. Since \( G \) is finite, we find \( m \) that \( G \setminus H_m \) is empty, i.e., the set of vertices of \( G \) and \( H_m \) are the same. We finish the proof by taking \( G(C) = H_m \).

**Proof of Theorem 3.1.** Theorem 4.3 of [2] shows that overlap coincidence of \( \mathcal{T} \) implies multiple strong coincidence of level \( n \) for any \( n \geq 1 \). We prove that there is a constant \( n \) such that multiple strong coincidence of level \( n \) implies overlap coincidence.

Assume that \( \mathcal{T} \) does not admit overlap coincidence. Construct the overlap graph \( G \) of \( \mathcal{T} \) with multiplicity. Since \( \mathcal{T} \) does not admit overlap coincidence, there is a strongly connected component\(^2\) \( S \) of \( G \) such that its spectral radius is equal to

\(^2\)In this assertion, one can take either usual overlaps or potential overlaps as we like.
|

\[ |\det(Q)| \]

and from each overlap of \( S \) there is no path leading to a coincidence in \( G \). Without loss of generality, we may assume that the incidence matrix of \( S \) is primitive\(^3\). Thus we can find a positive integer \( n_0 \) such that for every overlap \( (U, y, V) \), \( \Omega^{n_0}(U, y, V) \) contains an overlap equivalent to \( (U, y, V) \). Since \( \Xi(T) \) is a Meyer set, number of equivalence classes of overlaps is finite and bounded by a constant which depends only on \( T \). Thus there is an upper bound of \( n_0 \) which depends only on \( T \). We further assume multiple strong coincidence of level \( n = n_0 \) on \( T \) and derive a contradiction.

We claim that in the component \( S \) there is an overlap \( (U, y, V) \) with \( \ell(U) \neq \ell(V) \) for any non-rod self-affine tiling by connected tiles. Assume on the contrary that all overlaps in \( S \) are of the form \((A, z, B)\) with \( \ell(A) = \ell(B) \). Since \( S \) does not contain a coincidence, \( z \neq 0 \) for these overlaps. Taking \( k \)-th inflated overlap of \((A, z, B)\), we obtain of patches \( P \) and \( Q \), both contain large balls, say \( B_p(r) \) and \( B_q(r) \), that the tiles of \( P \) close to \( p \) and the tiles of \( Q \) close to \( q \) are in multiple correspondence in the following sense. Putting \( x = Q^k z \), for a tile \( U \in P \) close to \( p \) there are several (at least two) tiles \( V \in Q \) that \( (U, x, V) \) are overlaps in \( S \) and \( \text{supp}(U) \) is contained in the union of \( \text{supp}(V - x) \), and the same statements hold after interchanging the role of \( U \) and \( V \). Take a tile \( U \) with \( p \in \text{supp}(U) \subset B_p(r) \). Then overlaps \((U, x, V)\) with \( V \in Q \) give rise to a patch \( V_1 = \bigcup V \) that \( \text{supp}(U) \subseteq \text{supp}(V_1) - x \). By assumption, \( \ell(V_1) = \ell(U) \) for every \( V_1 \in V_1 \). By using path connectedness of tiles\(^4\), the patch \( V_1 \) is path connected. If \( \text{supp}(V_1) \subset B_q(r) \), then there is a patch \( U_1 = \bigcup U_1 \) where \( U_1 \in P \) are taken from all overlaps of the form \((U_1, x, V_1)\) with some \( V_1 \in V_1 \). This patch is also path connected and satisfies \( \text{supp}(V_1) - x \subseteq \text{supp}(U_1) \) and each tile of \( U_1 \) has the same color as \( U \). In this manner, by taking large \( r \), we obtain a long sequence of path connected patches

\[ \text{supp}(U) \subset \text{supp}(V_1 - x) \subset \text{supp}(U_1) \subset \text{supp}(V_2 - x) \subset \text{supp}(U_2) \subset \ldots . \]

The number of tiles strictly increases and all tiles appear in this sequence has the same color \( \ell(U) \). This shows for any \( M > 0 \), there exists a ball of radius \( R \) that each tile \( U \) in the ball belongs to a connected patch in \( T \) having diameter greater than \( M \), whose tiles have an identical color \( \ell(U) \). Therefore by using FLC, among \( X_T \) we can choose a rod tiling. Being a rod tiling is invariant under translation and closure operation, using minimality of \( X_T \) we see that every tiling in \( X_T \) is a rod tiling. This gives a contradiction, which finishes the proof of the claim.

Consider a directed graph \( V \) over \( \{1, \ldots, m\} \) whose edge \( i \to j \) is given if there are \( U, V \in S \) that \( V \in \Omega^m(U) \) with \( i = \ell(U) \) and \( j = \ell(V) \). Clearly \( V \) is strongly connected as well. Pick one overlap \((U, y, V)\) from \( S \) that \( \ell(U) \neq \ell(V) \) and select one of the overlaps equivalent to \((U, y, V)\) in \( \Omega^m(U, y, V) \). We select a tile map \( \gamma = \gamma_{\Omega^m} \) which sends \( \gamma(U) \) to this \( U \) in \((U, y, V)\), and \( \gamma(V) \) to the \( V \) in \((U, y, V)\), which correspond to two cycles \( \ell(U) \to \ell(U) \) and \( \ell(V) \to \ell(V) \) on \( V \). Let \( C \) be the set of these two cycles and take \( V(C) \) by Lemma 3.2. The tile map \( \gamma = \gamma_{\Omega^m} \) is chosen so that \( \ell(U) \to \ell(\gamma(U)) \) for \( U \in \{T_1, \ldots, T_m\} \) forms the set of edges of \( V(C) \). By the choice of the subgraph, every path of length \( m \) on this subgraph must fall into one of the two cycles. Note that by this choice of \( \gamma \), the control points of

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\(^3\)If the incidence matrix of \( S \) is irreducible but not primitive, then take a suitable power of \( \Omega \) by Perron-Frobenius theory.

\(^4\)We say that an overlap belongs to \( S \) if its equivalence class does.

\(^5\)Connectedness and path connectedness are equivalent for self-affine tiles \([14]\).
and $V - y$ are exactly matching, because both of them are equal to a common point $\cap_{k=1}^{\infty} Q^{-nk} (\gamma^k(U) \cap \gamma^k(V - y))$.

We claim that by this $\gamma$, we have $\Lambda - \Lambda \subset \langle \mathcal{G} \rangle$. In fact, since every overlap in the overlap graph is of the form $(A, z, B)$ with $z \in \bigcup_{i=1}^{m} (\Lambda_i - \Lambda)$, and control points of $U$ and $V - y$ are matching on $(U, y, V)$, i.e., $c(U) - c(V) - y$, we have $c(U) - c(V) \in \mathcal{G}$. By construction of $\mathcal{V}$ for any $x, y \in \Lambda$, we have $Q^m x, Q^m y \in \Lambda_{i(U)} \cup \Lambda_{i(V)}$. For instance, if $Q^m x \in \Lambda_{i(U)}$ and $Q^m y \in \Lambda_{i(V)}$, then $Q^m x = c(U) + f, Q^m y = c(V) + g$ hold with $f \in \Lambda_{i(U)} - \Lambda_{i(U)}, g \in \Lambda_{i(V)} - \Lambda_{i(V)}$. Therefore we have $\Lambda - \Lambda \subset \langle \mathcal{G} \rangle$.

We also see that the set of control points $\Lambda = (\Lambda_i)$ associated to $\gamma$ is admissible. In fact, since $(U, y, V)$ is an overlap, $\text{supp}(U) \cap \text{supp}(V - y)$ has an inner point. Since $y = c(V) - c(U)$, we have $(\text{supp}(U - c(U)))^c \cap (\text{supp}(V - c(V)))^c \neq \emptyset$. The admissibility follows from $Q^m x \in \Lambda_{i(U)} \cup \Lambda_{i(V)}$ for any $x \in \Lambda$.

Summing up, from $(U, y, V) \in S$, we have chosen a tile map $\gamma\Omega^c$ which produces a substitution Delone multi-color set of admissible control points with $\Lambda - \Lambda \subset \langle \mathcal{G} \rangle$. By the assumption of multiple strong coincidence of level $n$, we know $\Omega^k(U - c(U)) \cap \Omega^k(V - c(V))$ is non empty for some $k$, which shows that $(U, y, V)$ leads to a coincidence, giving a desired contradiction.

**Remark 3.** We use the assumptions that each tile is connected and $T$ is a non-rod tiling only to show that there is an overlap $(U, y, V) \in S$ that $\ell(U) \neq \ell(V)$, which allows us to define a tile map. It is likely that these assumptions are not necessary, i.e., every non-periodic self-affine tiling that $\Xi(T)$ is a Meyer set, then such overlap must appear in $S$.

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