Domain wall thickness and deformations of the field model

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Abstract. We consider the change in the asymptotic behavior of solutions of the type of flat domain walls (i.e. kink solutions) in field-theoretic models with a real scalar field. We show that when the model is deformed by a bounded deforming function, the exponential asymptotics of the corresponding kink solutions remain exponential, while the power-law ones remain power-law. However, the parameters of these asymptotics, which are related to the wall thickness, can change.

1. Introduction and motivation

Domain wall in three-dimensional physical space is a boundary between domains (areas of space) with different properties \cite{1}. In particular, in a field-theoretic model with a real scalar field with potential having at least two minima (vacua of the model), domain wall separates regions with different vacuum values of the scalar field. Thus, domain wall is a transition region in which the field continuously changes from one vacuum value to another. In the direction perpendicular to the wall, the field dependence on the spatial coordinate can be described by a kink solution of the corresponding differential equation \cite{1, 2, 3}. Domain walls arise in a variety of physical models. In particular, the appearance of domain walls and their subsequent collapse in the Early Universe could lead to the formation of primordial black holes \cite{4}. Emphasize that the study of many properties of domain walls is reduced to the study of kink solutions of the corresponding field-theoretic models. In this context, the wall thickness is related to the rate of asymptotic approach of the field to the vacuum value when moving away from the wall deep into the domain.

This our study is at the intersection of two areas of research related to kinks. On the one hand, we study asymptotic properties of kink solutions. On the other hand, we apply the so-called deformation procedure \cite{5, 6, 7} and investigate how this affects on the kink asymptotics. In short, in this paper, we present some preliminary results for transformational properties of the kink asymptotics with respect to deformations of the field-theoretic model.

Note that this short letter is based on the material presented by Tatiana Gani at the conference ICPPA-2020.
2. Deformation procedure and two types of deforming functions
Consider a real scalar field $\varphi(x, t)$ in (1 + 1)-dimensional space-time. Assume that dynamics of the field is described by the Lagrangian density
\[
\mathcal{L} = \frac{1}{2} \left( \frac{\partial \varphi}{\partial t} \right)^2 - \frac{1}{2} \left( \frac{\partial \varphi}{\partial x} \right)^2 - V(\varphi)
\]
with the potential $V(\varphi)$ having two or more denenerate minima. Without loss of generality, we assume that the potential is a non-negative function that vanishes at its minima. Such model can have kink solutions, see, e.g., [2, Part I], [3, Chap. 5].

The essence of the deformation procedure is briefly as follows. Assume that we have a field-theoretic model with the potential $V^{(0)}(\varphi)$ and its known kink solution $\varphi^{(0)}_K(x)$. Also assume that there is a strictly monotonically increasing function $f(\varphi)$ called deforming function. Then we can introduce a new model with the potential $V^{(1)}(\varphi)$ and kink solution $\varphi^{(1)}_K(x)$:
\[
V^{(1)}(\varphi) = \frac{V^{(0)}(f(\varphi))}{[f'(\varphi)]^2} \quad \text{and} \quad \varphi^{(1)}_K(x) = f^{-1}[\varphi^{(0)}_K(x)].
\]

In this paper, we briefly discuss how the asymptotic behavior of kink changes under deformations by two different types of deforming functions:
- type 1 — a strictly monotonic with a finite derivative;
- type 2 — a function which has infinite derivative at the vacuum of the deformed model.

We study transformational properties of two different types of the kink’s asymptotics: exponential and power-law. On the one hand, there is a vast variety of field-theoretic models having kinks with exponential asymptotic behavior, for example, $\varphi^4$ [8, 9, 10, 11, 12, 13], $\varphi^6$ [14, 15, 16, 17, 18], $\varphi^8$ [19, 20], double sine-Gordon [21, 22], etc. [6, 23, 24]. On the other hand, many models admit kink solutions with power-law asymptotics, see, e.g., [25, 26, 27, 28, 29, 30, 31].

3. Transformational properties of the kink’s asymptotics
First of all, recall that the kink can have exponential or power-law asymptotics, depending on the behavior of the potential $V^{(0)}(\varphi)$ near the vacuum $\varphi = \varphi_0$ (assume that $\lim_{x \to +\infty} \varphi^{(0)}_K(x) = \varphi_0$), for more details see, e.g., [26]. If
\[
V^{(0)}(\varphi) \approx \frac{1}{2} (\varphi_0 - \varphi)^2 v(\varphi_0)
\]
then the asymptotics of $\varphi^{(0)}_K(x)$ is exponential for $k = 1$:
\[
\varphi^{(0)}_K(x) \approx \varphi_0 - \exp \left[ -v(\varphi_0) x \right] \quad \text{at} \quad x \to +\infty;
\]
and power-law for $k > 1$:
\[
\varphi^{(0)}_K(x) \approx \varphi_0 - \frac{A_k^{(0)}}{x^{1/(k-1)}} \quad \text{at} \quad x \to +\infty,
\]
where
\[
A_k^{(0)} = \left[ (k - 1) v(\varphi_0) \right]^{1/(1-k)}.
\]
We have found that after deformation of the model (3) by deforming function $f(\varphi)$ of the first type, the asymptotics of the deformed kink will be the following. The exponential asymptotics (4) changes to

$$\varphi^{(1)}_K(x) \approx \varphi_1 - \exp \left[ -\sqrt{v(\varphi_0)} x \right] \quad \text{at} \quad x \to +\infty,$$

while the power-law asymptotics (5) changes to

$$\varphi^{(1)}_K(x) \approx \varphi_1 - \frac{A^{(1)}_k}{x^{1/(k-1)}} \quad \text{at} \quad x \to +\infty,$$

where

$$A^{(1)}_k = \frac{1}{f'(\varphi_1)} \left[ (k-1) \sqrt{v(\varphi_0)} \right]^{1/(1-k)},$$

so that

$$\frac{A^{(0)}_k}{A^{(1)}_k} = f'(\varphi_1).$$

Here $\varphi_1 = f^{-1}(\varphi_0)$ is the corresponding vacuum of the deformed model.

In the case of deforming function of the second type

$$f(\varphi) \approx f(\varphi_1) - B (\varphi_1 - \varphi)^\beta \quad \text{at} \quad \varphi \to \varphi_1 - 0,$$

where $B > 0$ and $0 < \beta < 1$ are constants, i.e. $f'(\varphi)$ is unbounded at $\varphi \to \varphi_1 - 0$, we obtain the following asymptotics of the deformed kink. The exponential asymptotics (4) changes to

$$\varphi^{(1)}_K(x) \approx \varphi_1 - \exp \left[ -\frac{\sqrt{v(\varphi_0)}}{\beta} x \right] \quad \text{at} \quad x \to +\infty,$$

while the power-law asymptotics (5) changes to

$$\varphi^{(1)}_K(x) \approx \varphi_1 - \frac{B^{(1)}_k}{x^{n(k-1)}} \quad \text{at} \quad x \to +\infty,$$

where

$$B^{(1)}_k = \left[ (k-1) B^{k-1} \sqrt{v(\varphi_0)} \right]^{1/(1-k)}. $$

4. Concluding remarks
To summarize, we have found the following transformational properties of the kink’s asymptotics under the deformation procedure.

A. For strictly monotonic deforming function with finite derivative:

- the exponential asymptotics after deformation remains exponential with the same constant coefficient in front of $x$;
- the power-law asymptotics after deformation remains power-law with the same power; however, depending on $f'(\varphi)$, the numerical coefficient can change, Eq. (10).

B. If the derivative of the deforming function goes to infinity in the vacuum of the deformed model:
• the exponential asymptotics after deformation remains exponential, however, the coefficient in front of \( x \) increases;
• the power-law asymptotics after deformation remains power-law, however, the power of \( x \) in the denominator increases;
• we can say that for both asymptotics the kink solution of the deformed model approaches the vacuum value faster.

A more detailed description of these and some other results can be found in preprint [7] and, we hope, will be published.

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