Convergence of GANs Training: A Game and Stochastic Control Methodology

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Abstract
Training of generative adversarial networks (GANs) is known for its difficulty to converge. This paper first confirms analytically one of the culprits behind this convergence issue: the lack of convexity in GANs objective functions, hence the well-posedness problem of GANs models. Then, it proposes a stochastic control approach for hyper-parameters tuning in GANs training. In particular, it presents an optimal solution for adaptive learning rate which depends on the convexity of the objective function, and builds a precise relation between improper choices of learning rate and explosion in GANs training. Finally, empirical studies demonstrate that training algorithms incorporating this selection methodology outperform standard ones.

1. Introduction
Generative adversarial networks (GANs), debuted in (Goodfellow et al., 2014), have enjoyed a great empirical success. The key idea behind GANs is to interpret the process of generative modeling as a competing game between two neural networks: a generator network $G$ and a discriminator network $D$. The generator network $G$ attempts to fool the discriminator network $D$ by converting random noise into sample data, while the discriminator network $D$ tries to identify whether the input sample is fake or true.

This game idea behind GANs provides a versatile class of generative models with a wide range of applications from image generation and natural language processing (Denton et al., 2015; Radford et al., 2015; Yeh et al., 2016; Ledig et al., 2017; Zhu et al., 2016; Reed et al., 2016; Vondrick et al., 2016; Luc et al., 2016; Kulharia et al., 2017), to economics and finance (Wiese et al., 2019; 2020; Zhang et al.; Chen et al., 2019).

Despite the empirical success of GANs, there are well recognized issues in GANs training, such as the vanishing generator gradients when the discriminator approximation gets better (Arjovsky & Bottou, 2017), and the mode collapse that happens when the generator learns to produce only a specific range of samples (Salimans et al., 2016). All these issues point out the same problem: the difficulty for stochastic gradient algorithm (SGD) and its variants to guarantee an accurate training of GANs parameters (Barnett, 2018).

In response, there have been growing research interests in the theoretical understanding of GANs training. (Berdard et al., 2020) proposes a visualization method for the GANs training process through the gradient vector field of loss functions. (Mescheder et al., 2018) demonstrates that regularization improves the convergence performance of GANs. (Conforti et al., 2020) and (Domingo-Enrich et al., 2020) analyse general minimax games including GANs, and connect the mixed Nash equilibrium of the game with the invariant measure of Langevin dynamics. In the same spirit, (Hsieh et al., 2019) proposes a sampling algorithm that converges towards mixed Nash equilibria. Then, (Kamalaruban et al., 2020) shows how this algorithm escapes saddle points. Recently, (Cao & Guo, 2020) establishes the continuous-time approximation for the discrete-time GANs training by coupled stochastic differential equations (SDE), enabling the convergence analysis of GANs parameters towards invariant measures.

Our work. This paper revisits vanilla GANs from the original work of (Goodfellow et al., 2014). In this context, we conduct a convexity and a saddle-point analysis to identify one of culprits behind the convergence issue of GANs: the lack of convexity in GANs objective function, hence the well-posedness issue in GANs minimax games.

It then follows a stochastic control methodology to tune some crucial hyper-parameters in GANs training, namely the learning rate, the time scales, and the batch size. In particular, it presents a simple expression for the optimal adaptive learning rate which depends on the convexity of the objective function, builds a precise relation between improper choices of learning rate and explosion in GANs training, and highlights the connection with the standard Newton algorithm. Finally, numerical experiments show that training algorithms incorporating this selection methodology outperform standard ones.
2. GANs: Well-posedness and Convexity

GANs as generative models. GANs fall into the category of generative models. The procedure of generative modeling is to approximate an unknown true distribution \( P_X \) of a random variable \( X \) from a sample space \( \mathcal{X} \) by constructing a class of suitable parametrized probability distributions \( P_\theta \). That is, given a latent space \( \mathcal{Z} \), define a latent variable \( Z \in \mathcal{Z} \) with a fixed probability distribution and a family of functions \( G_\theta : \mathcal{Z} \to \mathcal{X} \) parametrized by \( \theta \). Then, \( P_\theta \) can be seen as the probability distribution of \( G_\theta(Z) \).

To approximate \( P_X \), GANs use two competing neural networks: a generator network for the function \( G_\theta \), and a discriminator network \( D_w \) parametrized by \( w \). The discriminator \( D_w \) assigns a score between 0 and 1 to each sample. A higher score indicates that the sample is more likely to be from the true distribution. Moreover, GANs are trained by optimizing \( G_\theta \) and \( D_w \) iteratively until \( D_w \) can no longer distinguish between true samples and generated samples (i.e., with a score close to 0.5).

Equilibrium of GANs as minimax games. Under a fixed network architecture, the parametrized version of GANs training can be viewed as the following minimax game:

\[
\min_{\theta \in \mathbb{R}^N} \max_{w \in \mathbb{R}^M} \ g(w, \theta),
\]

with \( g : \mathbb{R}^M \times \mathbb{R}^N \to \mathbb{R} \) the cost function. In vanilla GANs,

\[
g(w, \theta) = \mathbb{E}_X \left[ \log(D_w(X)) \right] + \mathbb{E}_Z \left[ \log \left( 1 - D_w(G_\theta(Z)) \right) \right],
\]

where \( D_w : \mathbb{R}^M \to \mathbb{R} \) is the discriminator network, \( G_\theta : \mathbb{R}^N \to \mathbb{R} \) is the generator network, \( X \) represents the unknown data distribution, and \( Z \) is the latent variable.

From a game theory viewpoint, the objective in (1), when attained, is in fact the upper value of the two-player zero-sum game of GANs. If there exists a locally optimal pair of parameters \((\theta^*, w^*)\) for (1), then \((\theta^*, w^*)\) is a Nash equilibrium, i.e., no player can do better by unilaterally deviating from her strategy.

To guarantee the convergence towards a Nash equilibrium, strict convexity or concavity conditions are required at least locally. For example, the function \( g(x, y) = xy \) clearly admits the point \((0, 0)\) as a unique Nash equilibrium. However, as illustrated in Figure 1, SGD fails to converge since \( g \) is neither strictly concave nor strictly convex in \( x \) or \( y \).

From an optimization perspective, these convexity/concavity conditions also ensure the absence of a duality gap, see minimax theorems in (Von Neumann, 1959; Sion et al., 1958). Moreover, in terms of GANs training, if \( g \) is convex (resp. concave) in \( \theta \) (resp. \( w \)), then it is possible to decrease (resp. increase) the objective function \( g \) by moving in the opposite (resp. same) direction of the gradient. Finally, it is worth noting that such conditions are not usually satisfied in many common GANs models, as pointed out in (Zhu et al., 2020).

Figure 1. Plot of the parameters values when using SGD to solve the minimax problem \( \min_{w \in \mathbb{R}^M} \max_{x \in \mathbb{R}} xy \). The label “Start” (resp. “End”) indicates the initial (resp. final) value of the parameters.

Convexity analysis. To further understand this convexity issue, we revisit vanilla GANs with \( g \) in (2). The analysis confirms analytically one of the culprits behind the convergence issue behind GANs training: the lack of convexity in the objective function, hence the well-posedness problem of GANs models.

To start, take two normal distributions for \( X \) and \( Z \) such...
that 
\[ \mathbf{X} \sim N(m, \sigma^2), \quad \mathbf{Z} \sim N(0, 1), \]
with \((m, \sigma) \in \mathbb{R} \times \mathbb{R}^+\). Now suppose the following parametrization of the discriminator and the generator networks:
\[
\begin{align*}
D_w(x) &= D_{(w_1,w_2,w_3)}(x) = \frac{1}{1 + e^{-(w_3/2x^2+w_2x+w_1)}}, \\
G_\theta(z) &= G_{(\theta_1, \theta_2)}(z) = \theta_2 z + \theta_1,
\end{align*}
\]
where \(w = (w_1, w_2, w_3) \in \mathbb{R}^3\), and \(\theta = (\theta_1, \theta_2) \in \mathbb{R} \times \mathbb{R}^+\). To find the optimal choice for the parameters \(w\) and \(\theta\), we denote by \(f_X\) and \(f_G\) respectively the density functions of \(X\) and \(G_\theta(Z)\). Then, the density \(f_{G^*}\) of the optimal generator is given by \(f_{G^*} = f_X\), meaning \(\theta^* = (m, \sigma)\). Moreover, the optimal value for the discriminator is
\[
D_{w^*}(x) = \frac{f_X(x)}{f_X(x) + f_{G^*}(x)} = \frac{1}{1 + (f_{G^*}/f_X)(x)} = \frac{1}{2},
\]
for any \(x \in \mathbb{R}\) which gives \(w^* = (0, 0, 0)\). Note that the parametrization of the discriminator and the generator networks are standard since the generator is a simple linear layer and the discriminator can be seen as the composition of the sigmoid activation function with a linear layer in the variable \((x, x^2)\).

We now demonstrate that the function \(g\) may not satisfy the convexity/concavity requirement.

- To see this, let us first study the concavity of the function \(g_{\theta^0} = g(\cdot, \theta^0) : w \to g(w; \theta^0)\) with \(\theta^0\) fixed. We aim at showing that \(g_{\theta^0}\) is actually concave with respect to \(w\).

For this, write \(D_w\) as the composition of two functions \(D_1\) and \(L\) where \(D_1\) and \(L\) are given by
\[
D_1(x) = 1/(1+e^{-x}), \quad L(w; x) = w_3/2 \cdot x^2 + w_2x + w_1,
\]
for any \(x \in \mathbb{R}\) and \(w = (w_1, w_2, w_3) \in \mathbb{R}^3\). Note that a straightforward computation of the second derivatives shows that the functions
\[
g^1 : x \to \log(D_1(x)), \quad g^2 : x \to \log(1 - D_1(x)),
\]
with \(x \in \mathbb{R}\) are both concave. Thus, by composition and linearity, the function \(g_{\theta^0} = \mathbb{E}_X[g^1(L(w; X))] + \mathbb{E}_Z[g^2(L(w; G_{\theta^0}(Z)))]\) is concave with respect to \(w\).

- Next we investigate the convexity of the function \(g_{w^0} = g(w^0, \cdot) : \theta \to g(w^0, \theta)\) with \(w^0\) fixed. We will show that \(g_{w^0}\) is not necessarily convex with respect to \(\theta\).

To see this, first note that the mapping \(\theta \to \mathbb{E}_X[\log(D_w(X))]\) does not depend on the parameter \(\theta\). Therefore, one can simply focus on the function \(g^3 : \theta \to \mathbb{E}_Z[\log(1 - D_{w^0}(G_{\theta}(Z)))].\) This function is not necessarily convex with respect to \(\theta\). To show it we take \(\theta_2 = 0\) for simplicity\(^1\) and study the function \(g^3|_{\theta_2=0} : \theta_1 \to \log(1 - D_{w^0}(G_{(\theta_1, 0)}(z))) = \log (1 - 1/(1 + e^{-(w_3/2\theta_1^2+w_2\theta_1+w_1)}))\), with \(\theta_1 \in \mathbb{R}\). On one hand, the convexity of \(g^3|_{\theta_2=0}\) depends on the choice of the parameter \(w_3^0\), as demonstrated in Figure 2. On the other hand, a direct computation of the second derivative gives
\[
\begin{align*}
g_{3}^{2}\mid_{\theta_{2}=0}^{(2)}(\theta_{1}) & = -e^{h(\theta_{1})+w_{1}}(w_{3}(e^{h(\theta_{1})+w_{1}}+2h(\theta_{1})+1)+w_{2}^{3})
\end{align*}
\]
with \(\theta_1 \in \mathbb{R}\), and \(h(\theta_1) = \theta_1(w_3^0/2 \times \theta_1 + w_2^0)\). The sign of this function depends on the choice of the parameter \(w^0\) which shows that \(g_{w^0}\) is not necessarily convex with respect to \(\theta\).

\[\text{Figure 2. Plot of }g^3|_{\theta_2=0} \text{ with } (w_1, w_2, w_3) = (1, 1, 2) \text{ in (a) and } (w_1, w_2, w_3) = (1, 1, -2) \text{ in (b).}\]

**“Saddle” point analysis.** Given the non-convexity of \(g\), we next demonstrate the existence of a point \((\tilde{w}, \tilde{\theta}) \neq (w^*, \theta^*)\) for which \(\nabla g(\tilde{w}, \tilde{\theta}) = 0\). The existence of such points may be the predication for solving the minmax games of GANs using SGD and its variants, as the algorithm may be trapped in such non-optimal points.

The analysis consists of two steps. In the first step, we compute the gradient of the vanilla cost function \(g\). In the second one, we show that \(\nabla g(\tilde{w}, \tilde{\theta}) = 0\) for some \(\tilde{w} = (\tilde{w}_1, \tilde{w}_2, \tilde{w}_3) \in \mathbb{R}^2 \times \mathbb{R}^*\) and \(\theta = (\tilde{\theta}, 0)\) with \(\tilde{\theta}_1 = -(\tilde{w}_2/\tilde{w}_3)\).

Let us compute \(\nabla w g(w, \theta)^2\) for any \((w, \theta) \in \mathbb{R}^3 \times (\mathbb{R}^+)^2\).

\(^1\)This enables us to get rid of the expectation with respect to \(Z\).

\(^2\)The expression of \(\nabla_w g\) is not needed for the second part of the proof. However, it is interesting in itself because it provides the expression of the gradient.
Optimal GANs training

Since $\nabla_w \log(D_w(X))$ and $\nabla_w \log(1 - D_w(G_\theta(Z)))$ are both uniformly bounded by 1, we can interchange the expectation and the gradient operator to get

$$\nabla_w g(w, \theta) = \mathbb{E}_X [\nabla_w \log(D_w(X))] + \mathbb{E}_Z [\nabla_w \log(1 - D_w(G_\theta(Z)))]$$

$$= \mathbb{E}_X [\nabla D_w(X) - \mathbb{E}_Y [\nabla D_w(Y)]] \tag{6}$$

where $Y = G_\theta(Z)$. Using (3), we obtain

$$\nabla D_w(x) = D_w(x) \times (1 - D_w(x)) \tilde{x}, \quad \forall x \in \mathbb{R}, \tag{7}$$

with $\tilde{x} = (1, x, x^2/2)$. By combining (6) and (7), we deduce that

$$\nabla_w g(w, \theta) = \mathbb{E}_X [(1 - D_w(X))\tilde{x}] - \mathbb{E}_Y [D_w(Y)\tilde{Y}].$$

We move now to the computation of $\nabla_\theta g(w, \theta)$ for any $(w, \theta) \in \mathbb{R}^3 \times (\mathbb{R} \times \mathbb{R}_+).$ By following the same methodology of (6) and (7), we also get

$$\nabla_\theta g(w, \theta) = -\mathbb{E}_Z [D_w(G_\theta(Z))(w_3 G_\theta(Z) + w_2) Z^*], \tag{8}$$

with $z^* = (1, z)$ for any $z \in \mathbb{R}$.

The second step is to show that $\nabla \tilde{g}(\tilde{w}, \tilde{\theta}) = 0$ for some $\tilde{w} = (\tilde{w}_1, \tilde{w}_2, \tilde{w}_3) \in \mathbb{R}^2 \times \mathbb{R}^2$ and $\tilde{\theta} = (\tilde{\theta}_1, 0)$ with $\tilde{\theta}_1 = -(\tilde{w}_2/\tilde{w}_3)$. Since $\tilde{\theta} = (\tilde{\theta}_1, 0)$, we have $G_\tilde{\theta}(Z) = \tilde{\theta}_1$ almost surely and therefore Equation (8) reads

$$\nabla_\theta \tilde{g}(\tilde{w}, \tilde{\theta}) = \begin{pmatrix} D_w(\tilde{\theta}_1)(\tilde{w}_3 \tilde{\theta}_1 + \tilde{w}_2) \\ D_w(\tilde{\theta}_1)(\tilde{w}_3 \tilde{\theta}_1 + \tilde{w}_2) \tilde{\theta}_1 \end{pmatrix} \tag{9}$$

Given that $\tilde{\theta}_1 = -(\tilde{w}_2/\tilde{w}_3)$, Equation (9) guarantees that $\nabla g(\tilde{w}, \tilde{\theta}) = 0$. Now, we fix $\tilde{\theta} = (\tilde{\theta}_1, 0)$ and maximise $g_\tilde{\theta} : w \rightarrow g(w, \tilde{\theta})$. We have

$$g_\tilde{\theta}(w) = \mathbb{E}_X [\log(D_w(X))] + \log(1 - D_w(\tilde{\theta}_1)).$$

Let $D^1$ and $L$ be the functions introduced in (4). According to Equation (5), the functions

$$g^1 : x \rightarrow \log(D^1(x)) \text{ and } g^2 : x \rightarrow \log(1 - D^1(x)),$$

with $x \in \mathbb{R}$ are both concave. Moreover, the function

$$\bar{L} : (w_1, w_2, w_3) \rightarrow w_2 \tilde{\theta}_1^2 + w_3 \tilde{\theta}_1 + w_1 = -\frac{w_2^2}{w_3} + w_1,$$

is concave as well. Thus, by linearity and composition, the function $g_\bar{L}(w) = \mathbb{E}_X [g^1(L(w; X))] + g^2(L(w))$ is also concave which ensures the existence of a maximum value $\tilde{w}$ satisfying $\nabla_w g(\tilde{w}, \tilde{\theta}) = 0$.

3. Optimization of GANs Training

Having discussed the convexity and well-posedness issue of GANs as minimax games, we now turn to GANs training and parameters tuning.

Hyper-parameters in GANs training. There are three critical hyper-parameters in stochastic gradient algorithms for GANs training: the learning rate, the batch size, and the time scale choice.

- The learning rate determines how far to move along the gradient direction. Higher learning rate accelerates the convergence, yet with a higher chance of explosion; while lower learning rate yields slower convergence.

- The time scale parameters monitor the number of updates of the variables $w$ and $\theta$. In the context of GANs training, there are generally two different time scales: a finer time scale for the discriminator and a coarser one for the generator. Note that more updates means faster convergence, however, also computationally more costly.

In fact, we will establish the equivalence between the time scale and the learning rate optimization problems in a stochastic control framework for GANs training (see Section A of the supplementary manuscript).

- The batch size refers to the number of training samples used in the estimate of the gradient. The more training samples used in the estimate, the more accurate this estimate, yet with a higher computational cost. Smaller batch size, while providing cruder estimate, offers a regularizing effect and lowers the chance of overfitting.

Clearly each hyper-parameter tuning is not independent, and GANs training often involves choices between adjusting learning rates or changing sample sizes. Nevertheless, the basic example below demonstrates how poor choices of parameters tuning, in this case the learning rate, lead to the failure of convergence.

Example. Consider an $\mathbb{R}$-valued function

$$f(x) = (a/2) x^2 + bx, \quad \forall x \in \mathbb{R},$$

where $(a, b) \in \mathbb{R} \times \mathbb{R}$. Finding the minimum $x^* = -(b/a)$ of $f$ via the gradient algorithm goes as follows:

$$x_{n+1} = x_n - \eta(ax_n + b), \quad \forall n \geq 0, \tag{10}$$

with $x_0 \in \mathbb{R}$ given and $\eta$ the learning rate. Let us study the behaviour of the error $e_n = |x_n - x^*|^2$. Using (10) and
\[ ax^* + b = 0, \] we have
\[
e_{n+1} = |x_{n+1} - x^*|^2 = |x_n - x^*|^2 + 2(x_{n+1} - x_n)e_n
+ |x_{n+1} - x_n|^2
= (1 - \eta a(2 - \eta a))|x_n - x^*|^2.
\] (11)

Thus, when \( \eta > 2/a \), the factor \( \eta a(2 - \eta a) < 0 \) which means \( r = (1 - \eta a(2 - \eta a)) > 1 \). In such case, Equation (11) becomes
\[
e_{n+1} = r e_n,
\]
ensuring \( e_n \to +\infty \), and leading to the failure of the convergence for the gradient algorithm.

This issue of divergence for GANs training with improper learning rate choices can be shown in a more general setting via the following stochastic control approach.

### 3.1. Optimal Learning Rate

The stochastic control formulation is built on a continuous-time approximation framework of the discrete-time GANs training.

**SDE approximation of GANs training.** Recall that in GANs training, some gradient algorithms for the optimal parameters \( \theta^* \) and \( w^* \) start with an initial guess \((w_0, \theta_0)\) and apply the following (simultaneous) update rule:
\[
\begin{align*}
    w_{t+1} &= w_t + \eta^w_t g_w(w_t, \theta_t), \\
    \theta_{t+1} &= \theta_t - \eta^\theta_t g_\theta(w_t, \theta_t),
\end{align*}
\] (12)

with \( g_w = \nabla w g \), \( g_\theta = \nabla \theta g \), and \((\eta^w_t, \eta^\theta_t) \in \mathbb{R}^2_+ \) the learning rate.

The continuous-time approximation of GANs training via functional central limit theorem (Cao & Guo, 2020) is to replace the update rule in (12) with coupled stochastic differential equations (SDEs)
\[
\begin{align*}
    dw(t) &= g_w(q(t))dt + \sqrt{\eta} \sigma_w(q(t))dW^1(t), \\
    d\theta(t) &= -g_\theta(q(t))dt + \sqrt{\eta} \sigma_\theta(q(t))dW^2(t),
\end{align*}
\]

where \( q(t) = (w(t), \theta(t)) \), the functions \( \sigma_w : \mathbb{R}^M \times \mathbb{R}^N \to \mathcal{M}_R(M) \) and \( \sigma_\theta : \mathbb{R}^M \times \mathbb{R}^N \to \mathcal{M}_R(N) \) are approximated by the covariance of \( g_w \) and \( g_\theta \), and the Brownian motions \( W^1 \) and \( W^2 \) are independent. Note that in this SDE approximation, the learning rates for the generator and the discriminator \((\eta, \eta)\) are fixed constants.

**Adaptive learning rate.** The idea of choosing an optimal learning rate goes as follows: a learning rate at time \( t \) consists of two components, the first predefined component being a base learning rate \( \eta_t^* = (\eta^w_t(t), \eta^\theta_t(t)) \) fixed by the controller using its favorite learning rate selection method, and the second component being an adapted learning rate \( u_t = (u^w(t), u^\theta(t)) \) that is adjusted around \( \eta_t^* \). The adjustment is adaptive to the training process. The adapted learning rates \( u^w(t) \) and \( u^\theta(t) \) are further assumed to be bounded by a fixed constant \( u^{\text{max}} \geq 0 \). This “clipping” parameter \( u^{\text{max}} \geq 0 \) is introduced to handle the convexity issue discussed earlier for GANs training and to avoid explosion, as will be clear from the discussion later.

More specifically, let \( \sigma_w : \mathbb{R}^M \times \mathbb{R}^N \to \mathcal{M}_R(M) \) and \( \sigma_\theta : \mathbb{R}^M \times \mathbb{R}^N \to \mathcal{M}_R(N) \) be two continuous functions, and consider the following form for the adaptive learning rate process \( \eta(t) \):
\[
\eta(t) = (\eta^w(t), \eta^\theta(t)) = (u^w(t) \times \eta^w(t), u^\theta(t) \times \eta^\theta(t))
= u(t) \cdot \eta(t), \quad \forall t \geq 0,
\]

where \( \eta_t^* = (\eta^w(t), \eta^\theta(t)) \in [0, 1]^2 \) and \( u_t = (u^w(t), u^\theta(t)) \) is a \([0, u^{\text{max}}]^2\)-valued process and the symbol \( \bullet \) refers to the component-wise product between vectors. With the incorporation of the adaptive learning rate (13), the corresponding SDE for GANs training becomes
\[
\begin{align*}
    dw(t) &= u^w(t)g_w(q(t))dt + (u^w \sqrt{\eta^w})T(t)\sigma_w(q(t))dW^1(t), \\
    d\theta(t) &= -u^\theta(t)g_\theta(q(t))dt + (u^\theta \sqrt{\eta^\theta})T(t)\sigma_\theta(q(t))dW^2(t),
\end{align*}
\]

with \( q(t) = (w(t), \theta(t)) \) for any \( t \geq 0 \).

**Optimal control of adaptive learning rate.** Let \( T < \infty \) be a finite time horizon. Define the reward function \( J \) as
\[
J(T, t, q; u) = \mathbb{E}[g(q(T))|q(t) = q, u],
\]
where \( q = (w, \theta) \in \mathbb{R}^M \times \mathbb{R}^N \) is the value of the process \( q(t) \) at \( t \in [0, T] \). Then the control problem for the adaptive learning rate is formulated as follows:
\[
v(t, q) = \min_{u_t \in U^w} \max_{u_\theta \in U^\theta} J(T, t, q; u),
\] (14)

for any \((t, q) \in [0, T] \times \mathbb{R}^M \times \mathbb{R}^N\), with \( U^w \) the set of all appropriate admissible controls for \( u^w \), and \( U^\theta \) the set of all appropriate admissible controls for \( u^\theta \).

The value function \( v \), see (14), of this control problem can be analysed based on classical control techniques (for details, see Section B of the supplementary manuscript). In particular, under proper regularity assumptions, the value function \( v \) is a solution (classical or viscosity) to the following Hamilton-Jacobi-Bellman equation:
\[
\begin{align*}
    v_t + \max\{u^w_{t, u^w}v_{u^w} - u^\theta_{t, u^\theta}v_{u^\theta}\} &+ \frac{1}{2} \{[u^w]_T^2(\Sigma^w : v_{u^w}) + [u^\theta]_T^2(\Sigma^\theta : v_{u^\theta})\} = 0,
\end{align*}
\]
Moreover, we have
Proposition 1. Under simple regularity conditions, the optimal adaptive learning rate \( \tilde{u}^w \) and \( \tilde{u}^\theta \) are given by

\[
\tilde{u}^w = \begin{cases} 
0 \vee \left( -\frac{g_w^T v_w}{\Sigma^w : v_{ww}} \wedge u_{\text{max}}, \right), & \text{if } \Sigma^w : v_{ww} < 0, \\
\text{otherwise}, & \text{if } \Sigma^w : v_{ww} = 0,
\end{cases}
\]

and

\[
\tilde{u}^\theta = \begin{cases} 
0 \vee \left( \frac{g_\theta^T v_\theta}{\Sigma^\theta : v_{\theta\theta}} \wedge u_{\text{max}}, \right), & \text{if } \Sigma^\theta : v_{\theta\theta} > 0, \\
\text{otherwise}, & \text{if } \Sigma^\theta : v_{\theta\theta} = 0.
\end{cases}
\]

Here the matrices \( \Sigma^w \) and \( \Sigma^\theta \) satisfy

\[
\Sigma^w(t, q) = \{\sigma^w(t), q\}, \quad \Sigma^\theta(t, q) = \{\sigma^\theta(t), q\},
\]

\[\tilde{\sigma}^w(q) = \sqrt{\tilde{g}^w(t)} \sigma^w(q), \quad \tilde{\sigma}^\theta(q) = \sqrt{\tilde{g}^\theta(t)} \sigma^\theta(q),\]

for any \( t \in \mathbb{R}_+ \), and \( q = (u, \theta) \in \mathbb{R}^M \times \mathbb{R}^N \). Recall that \( A : B = \text{Tr}[A^T B] \) for any real matrices \( A \) and \( B \).

Remarks. Through this control framework, we can see that the optimal learning rate depends on the convexity of the objective function in the minimax game of GANs, and is closely related to Newton’s algorithm. Specifically,

- Proposition 1 provides a two steps scheme for the selection of the optimal learning rate

  Step 1. Use the HJB equation to get \( \tilde{u} = (\tilde{u}^w, \tilde{u}^\theta) \)

  Step 2. Given \( \tilde{u} \), apply the gradient algorithm with the optimal learning rate \( \tilde{u} \cdot \tilde{\eta} \).

- To get the expression of the optimal adaptive learning rate in Proposition 1, we need some regularity conditions on the value function \( v \), and it suffices that \( v \) belongs to \( C^{1,2}([0, T], \mathbb{R}^M \times \mathbb{R}^N) \). Moreover, it is possible to show that \( v \in C^{1,2}([0, T], \mathbb{R}^M \times \mathbb{R}^N) \) when \( g \) and the volatility \( \sigma \) satisfy simple Lipschitz continuity conditions, see (Evans, 1983) for more details.

When the value function \( v \) does not satisfy the regularity requirement \( v \in C^{1,2}([0, T], \mathbb{R}^M \times \mathbb{R}^N) \), it is standard to use discrete time approximations, see for example (Barles & Souganidis, 1991; Kushner et al., 2001; Wang & Forsyth, 2008). Note that these approximations produce almost optimal controls since they are shown to converge towards the value function.

- The introduction of the clipping parameter \( u_{\text{max}} \) is closely related to the convexity issue discussed for GANs minimax games. When the convexity condition \( \Sigma^w : v_{ww} < 0 \) is violated, explosion in GANs training can be prevented by fixing an upper bound \( u_{\text{max}} \) for the learning rate.

- The control \((\tilde{u}^w, \tilde{u}^\theta)\) in Proposition 1 is closely related to the standard Newton algorithm. To see this, let us take \( \eta = (1, 1) \), \( M = N = 1 \), \( \sigma^w = g_w, \sigma^\theta = g_\theta \), \( R = 0 \), and replace the value function \( v \) by the sub-optimal choice \( \tilde{g} \). For such a choice of the parameters and when the convexity conditions

\[
\Sigma^w : v_{ww} = |g_w|^2 g_w < 0, \quad (\Sigma^\theta : v_{\theta\theta}) = |g_\theta|^2 g_\theta > 0,
\]

are satisfied, the controls \( \tilde{u}^w \) and \( \tilde{u}^\theta \) become

\[
\tilde{u}^w(t) = -\frac{1}{\tilde{g}^w(t) g_w(t)} \wedge u_{\text{max}}, \quad \tilde{u}^\theta(t) = -\frac{1}{\tilde{g}^\theta(t) g_\theta(t)} \wedge u_{\text{max}}.
\]

In absence of the clipping parameter \( u_{\text{max}} \), these parameters \( \tilde{u}^w \) and \( \tilde{u}^\theta \) are exactly the ones used for Newton’s algorithm.

3.2. Learning Rate and Divergence

Let us further analyse the impact of the learning rate on the convergence of GANs algorithms, generalizing the example of Section 3.1 in which poor choices of the learning rate destroy the convergence.

Let \( \epsilon > 0 \), and \( \tilde{u} = (\tilde{u}^w, \tilde{u}^\theta) \) be the process

\[
\tilde{u}(t) = \left( \frac{-2|g_w|^2}{\Sigma^w : g_w}, \frac{2|g_\theta|^2}{\Sigma^\theta : g_\theta} \right)(t, q(t)), \quad \forall t \geq 0.
\]

We assume that there exists \( \gamma > 0 \) such that

\[
\gamma \leq -(\Sigma^w : g_w)(t, q), \quad \gamma \leq (\Sigma^\theta : g_\theta)(t, q),
\]

for every \( t \geq 0 \), and \( q = (w, \theta) \in \mathbb{R}^M \times \mathbb{R}^N \). Then, for any control process \( u = (u^w, u^\theta) \in U^w \times U^\theta \) such that

- \( u^w \geq (\tilde{u}^w \vee 1) + \epsilon, \ a.s., \) there exists \( \tilde{\epsilon} > 0 \) satisfying

\[
J(T, 0, q_0; u) \leq -\tilde{\epsilon} \times T + \mathbb{E} \left[ \int_0^T \left\{ -u^\theta|g_\theta|^2 \right. \\
+ \frac{1}{2} (u^\theta)^2 (\Sigma^\theta : g_\theta) \right\} (s, q(s)) ds | q_0 \right]
\]

\[
= -\tilde{\epsilon} \times T + L^1(T, q_0; u),
\]

for any \( (T, q_0) \in \mathbb{R}_+ \times \mathbb{R}^M \times \mathbb{R}^N \).

- \( u^\theta \geq (\tilde{u}^\theta \vee 1) + \epsilon, \ a.s., \) there exists \( \tilde{\epsilon} > 0 \) such that

\[
J(T, 0, q_0; u) \geq \tilde{\epsilon} \times T + \mathbb{E} \left[ \int_0^T u^w |g_w|^2 \\
+ \frac{1}{2} (u^w)^2 (\Sigma^w : g_w) \right\} (s, q(s)) ds | q_0 \right]
\]

\[
= \tilde{\epsilon} \times T + L^1(T, q_0; u),
\]

for any \( (T, q_0) \in \mathbb{R}_+ \times \mathbb{R}^M + \mathbb{N} \).
At the end of the training, the discriminator accuracy is with numerical experiments. For example, Figure 5 shows that the smallest batch size guarantees the fastest convergence.

3.3. Optimal Batch Size

The problem of optimal batch size can be similarly formulated and analysed. Our analysis shows that the optimal batch size is a simple bang-bang type. That is, it is optimal to choose either the smallest batch size, modeled by the unit value 1, or the biggest batch size, corresponding to a maximum bound $m^{\text{max}}$ (see the supplementary manuscript, Section C). Moreover, using the analogy with Newton’s algorithm, we remark that it is always optimal to choose the smallest (resp. biggest) batch size when solving a minimisation (resp. maximisation) problem. This result is consistent with numerical experiments. For example, Figure 5 shows that the smallest batch size guarantees the fastest convergence.

4. Numerical Experiment

In this section, we use the vanilla GANs problem reviewed in Section 2 to show the relevance of our adaptive learning rate, and the influence of the batch size on GANs training.

**Data.** All numerical experiments samples of $X$ and $Z$ are drawn respectively from the Gaussian distributions $N(m, \sigma^2)$ and $N(0, 1)$ with $(m, \sigma) = (3, 1)$. These samples are then decomposed into two sets: a training set, and a test set.

At the end of the training, the discriminator accuracy is expected to be around 50%, meaning that the generator manages to produce samples that fool the discriminator who is unable to differentiate the original data from the fake ones.

One epoch here refers to the number of gradient updates needed to pass the entire training dataset.

4.1. Learning Rate

**Optimal learning rate algorithm.** For computational efficiency, instead of directly applying the two-steps resolution scheme introduced in Section 3.1 for high dimensional HJB equation, we exploit the connection between the optimal learning rate and Newton’s algorithm by following the methodology illustrated in Equation (15). The main idea is to approximate the function $v$ by $g$, see Equation (14). Furthermore, we divide all the parameters into $M$ sets. For example, for neural networks, one may associate a unique set to each layer of the network. For each set $i$, denote by $x^i_t$ the value at iteration $t \in \mathbb{N}$ of the set’s $i$ parameters and write $g^i_t$ for the gradient of the loss $g$ with respect to the parameters of the set $i$. Then, we replace for each set $i$ the update rule

$$x^i_{t+1} = x^i_t - \eta_t g^i_t(x^i_t),$$

with $\tilde{\eta}_t$ a base reference learning rate that depends on the optimizer, by the following update rule:

$$x^i_{t+1} = x^i_t - u^i_t \eta_t g^i_t(x^i_t),$$

where the adjustment $u^i_t$, derived from Proposition 1, is defined as follows:

$$u^i_t = \frac{\|g^i_t(x^i_t)\|^2}{\text{Tr}(\bar{g}^i_t \bar{g}^i_t x^i_t)} \wedge u^{\text{max}}$$

if $(\text{Tr}(\bar{g}^i_t \bar{g}^i_t x^i_t)) (x^i_t) > 0$; otherwise $u^i_t = u^{\text{max}}$.

Here $\bar{g}^i_t = \sqrt{\text{Tr} g^i_t}$ and $u^{\text{max}}$ is the maximum allowed adjustment.

Note that the same adjustment $u^i_t$ is used for all the parameters of the set $i$. The most expensive operation is then the computation of the term $(\text{Tr}(\bar{g}^i_t \bar{g}^i_t x^i_t)) (x^i_t)$. For an efficient estimation, we first re-write it as:

$$(\text{Tr}(\bar{g}^i_t \bar{g}^i_t x^i_t)) (x^i_t) = [\bar{g}^i_t (x^i_t)] (x^i_t).$$

Thus, instead of approximating the Hessian matrix $g^{x \cdot x}$, one only needs to evaluate the vector $(g^{x^i_t \cdot x^i_t}) (x^i_t)$.

This can be done through a simple adaptation of the standard L-BFGS algorithm, see the supplementary manuscript for a detailed implementation.

**Numerical results.** First, the base learning rate, denoted by “lr” in all relevant figures, takes the values $10^{-2}, 5 \cdot 10^{-3}, 10^{-3},$ and $5 \cdot 10^{-4}$.

We compare the accuracy of the discriminator for two choices of the optimizers, namely the SGD optimizer with fixed learning rates and the SGD optimizer with adaptive learning rates (i.e., LSGD). Figure 3 plots the accuracy of the discriminator when the base learning rate varies for both SGD and LSGD. Evidently, a bigger learning rate yields faster convergence for both SGD and LSGD; LSGD outperforms SGD due to the introduction of the adaptive learning rate, and more importantly LSGD performance is more robust with respect to the initial learning rate since he can adjust using the additional adaptive component.
Next, we analyse the generator loss for both optimizers SGD and LSGD. Figure 4 shows the variations of the generator loss when moving the base learning rate for both SGD and LSGD. First, one can see that the loss decreases faster when using LSGD. Second, Figure 4 confirms the robustness of LSGD performance with respect to the choice of the initial learning rate, again thanks to the additional adaptive learning rate component.

### 4.2. Batch Size

Parallel to the learning rate experiment, we investigate numerically the best choice of the batch size, and apply the SGD algorithm for different values of the batch size.

Figure 5 plots the accuracy of the discriminator when the batch size varies. We can see that the smallest batch size gives the fastest convergence which is consistent with the results of Section 3.3. It is worth noting that the empirical study conducted in (Masters & Luschi, 2018) for neural networks also shows that mini-batch sizes give the best performance, which confirms our findings.

**Figure 3.** Discriminator accuracy for SGD with a base learning rate in (a) and LSGD with an additional adaptive learning rate component in (b).

**Figure 4.** Generator loss for SGD with a base learning rate in (a) and LSGD with an additional adaptive learning rate component in (b).

**Figure 5.** Discriminator accuracy for SGD with different values of the batch size.

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