Soliton Cellular Automata Associated With Crystal Bases

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Abstract

We introduce a class of cellular automata associated with crystals of irreducible finite dimensional representations of quantum affine algebras \(U_q(\hat{g}_n)\). They have solitons labeled by crystals of the smaller algebra \(U_q(\hat{g}_{n-1})\). We prove stable propagation of one soliton for \(\hat{g}_n = A^{(2)}_{2n-1}, A^{(2)}_{2n}, B^{(1)}_n, C^{(1)}_n, D^{(1)}_n\) and \(D^{(2)}_{n+1}\). For \(\hat{g}_n = C^{(1)}_n\), we also prove that the scattering matrices of two solitons coincide with the combinatorial \(R\) matrices of \(U_q(C^{(1)}_{n-1})\)-crystals.

1 Introduction

Cellular automata are the dynamical systems in which the dependent variables assigned to a space lattice take discrete values and evolve under a certain rule. They exhibit rich behavior, which have been widely investigated in physics, chemistry, biology and computer sciences [M]. When the space lattice is one dimensional, there are several examples known as the soliton cellular automata [FPS, PF, PAS, PST, TS, T]. They possess analogous features to the solitons in integrable non-linear partial differential equations. For example, some patterns propagate with fixed velocity and they undergo collisions retaining their identity and only changing their phases.

There is a notable progress recently in understanding the integrable structure in the soliton cellular automata. In the papers [TTMS, MSTTT, TNS] it was shown that a class of soliton cellular automata can be derived from the known soliton equations such as Lotka-Volterra and Toda equations through a limiting procedure called ultra-discretization. The method enables one to construct the explicit solutions and the conserved quantities of the former from that of the latter. A key in the ultra-discretization is the identities: \((a, b \in \mathbb{R})\)

\[
\lim_{\epsilon \to +0} \epsilon \log(e^{a \epsilon} + e^{b \epsilon}) = \max(a, b),
\]

\[
\lim_{\epsilon \to +0} \epsilon \log(e^{a \epsilon} \times e^{b \epsilon}) = a + b.
\]

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In a sense they change $+$ into $\max$ and $\times$ into $+$. This is a transformation of the continuous operations into piecewise linear ones preserving the distributive law:

$$(A + B) \times C = (A \times C) + (B \times C) \rightarrow \max(a, b) + c = \max(a + c, b + c).$$

The non-uniqueness of the distributive structure is noted by Schützenberger in combinatorics, where the procedure corresponding to the inverse of the ultradiscretization is called ‘tropical variable change’ [Ki].

There is yet further intriguing aspect in the soliton cellular automata (called ‘box and ball systems’) in [T, TNS]. There the scattering of two solitons is described by the rule which turns out to be identical with the $U'_q(A^{(1)}_n)$ combinatorial $R$ matrix [NY] from the crystal base theory. The latter has an origin in the quantum affine algebras at $q = 0$, where the representation theory is piecewise linear in a certain sense.

Motivated by these observations we formulate in this paper and [HHIKT] a class of cellular automata directly in terms of crystals and link the subject to the 1+1 dimensional quantum integrable systems. The theory of crystals is invented by Kashiwara [Kas] as a representation theory of the quantized Kac-Moody algebras at $q = 0$. It is a powerful tool that reduces many essential problems into combinatorial questions on the associated crystals. Irreducible decomposition of tensor products and the Robinson-Schensted-Knuth correspondence are typical such problems [DJM, KN, N]. By connecting the classical and affine crystals, it also explains [KMN1, KMN2] the appearance of the affine Lie algebra characters [DJKM] in Baxter’s corner transfer matrix method in solvable lattice models [B].

Here we shall introduce a cellular automaton associated with crystals of irreducible finite dimensional representations of quantum affine algebras $U'_q(\mathfrak{g}_n)$. The basic idea is to regard the time evolution in the automaton as the action of a row-to-row transfer matrix of integrable $U'_q(\mathfrak{g}_n)$ vertex models at $q = 0$. The essential point is to consider the tensor product of crystals not around the ‘anti-ferromagnetic vacuum’ as in [KMN1, KMN2], but rather in the vicinity of the ‘ferromagnetic vacuum’.

Let $B$ be a classical crystal of irreducible finite dimensional representations of the quantum affine algebra $U'_q(\mathfrak{g}_n)$. It is a finite set having a weight decomposition and equipped with the maps $\hat{e}_i, \hat{f}_i : B \rightarrow B \cup \{0\}$ and $\varepsilon_i, \varphi_i : B \rightarrow \mathbb{Z}_{\geq 0}$ ($i \in \{0, 1, \ldots, n\}$) satisfying certain axioms. (cf. Definition 2.1 in [KKM].) For two crystals $B$ and $B'$ the tensor products $B' \otimes B$ and $B \otimes B'$ are again crystals which are canonically isomorphic. The isomorphism $B' \otimes B \cong B \otimes B'$ is called the combinatorial $R$ matrix [4]. Suppose that there are special elements denoted

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1 More precisely, the isomorphism combined with the data on the energy function is called the combinatorial $R$ matrix [KMN].
by 1 ∈ B and u_2 ∈ B' with the properties

(I) \( u_2 \otimes 1 \simeq 1 \otimes u_2 \),

(II) For any \( u \in B' \) there exists \( k \in \mathbb{Z}_{\geq 0} \) such that

\[ u \otimes 1 \otimes \cdots \otimes 1 \simeq b_1 \otimes \cdots \otimes b_k \otimes u_2, \quad (b_i \in B). \]

We take dynamical variables of our automaton from the crystal B and regard their array \( b_{t-1}, b_t, b_{t+1}, \ldots \) at time \( t \) as an element of the tensor product of crystals

\[ \cdots \otimes b_{t-1} \otimes b_t \otimes b_{t+1} \otimes \cdots \in \cdots \otimes B \otimes B \otimes \cdots, \]

where we assume the boundary condition \( b_j = 1 \in B \) for \( |j| \gg 1 \). See Section 2.2 for a precise treatment. The 'ferromagnetic' state \( \forall b_t^i = 1 \) is understood as the vacuum of the automaton. The infinite tensor product with such a boundary condition does not admit a crystal structure. Nevertheless one can make sense of the construction below thanks to the properties (I) and (II). The time evolution is induced by sending \( u_2 \) from left to right via the repeated application of the combinatorial \( R \) matrix as

\[ \begin{align*}
B' \otimes &(\cdots \otimes B \otimes B \otimes B \otimes \cdots) \simeq (\cdots \otimes B \otimes B \otimes B \otimes \cdots) \otimes B' \\
&u_2 \otimes (\cdots \otimes b_{t-1}^i \otimes b_t^j \otimes b_{t+1}^k \otimes \cdots) \simeq (\cdots \otimes b_{t-1}^{i+1} \otimes b_t^{j+1} \otimes b_{t+1}^{k+1} \otimes \cdots) \otimes u_2,
\end{align*} \]

which is well-defined as long as the above properties and the boundary conditions are fulfilled. In the language of the quantum inverse scattering method [STF, KS], this is the action of the \( q = 0 \) row-to-row transfer matrix whose auxiliary and quantum spaces are labeled by \( B' \) and \( \cdots \otimes B \otimes B \otimes \cdots \), respectively. Note that the transfer matrix has effectively reduced to the \((u_2, u_2)\)-component of the monodromy matrix since its action is considered under the ferromagnetic boundary condition. The fundamental case \( \hat{g}_n = A^{(1)}_n \) will be studied in a more general setting in [HHIKT]. In this paper we concentrate on the other non-exceptional series

\[ \hat{g}_n = A^{(2)}_{2n-1}, A^{(2)}_{2n}, B^{(1)}_n, C^{(1)}_n, D^{(1)}_n, D^{(2)}_{n+1}, \]

with the following choice of crystals:

\[ B = B_1 \ni 1, \quad B' = B_2 \ni u_2. \]

Here \( B_1 \) is the crystal associated with the vector representation of the classical subalgebra of \( \hat{g}_n \) except for \( A^{(2)}_{2n} \) and \( D^{(2)}_{n+1} \). Their cardinalities are \( \sharp B = 2n, 2n+1, 2n+1, 2n, 2n+1, 2n+2 \), respectively. The element \( 1 \in B_1 \) is the highest weight one \( \mathbb{1} \). To explain \( B_2 \) and \( u_2 \), recall the coherent family \( \{ B_l \mid l \in \mathbb{Z}_{\geq 1} \} \) of

\[ \text{For } (\hat{g}_n, B_1) \text{ treated in this paper, a parallel construction seems possible also with the choice } 1 = \text{lowest weight element}. \]
the perfect crystals obtained in KKM. It contains the $B_1$ as its first member. The $B_l$ with higher $l$ corresponds to an $l$-fold symmetric fusion of $B_1$. Then $B_l$ in question is an infinite set corresponding to a certain $l \to \infty$ limit of $B_l$ and $u_2$ is its highest weight element. We shall call the resulting dynamical system $U'_q(\hat{\mathfrak{g}}_n)$ automaton. They are essentially solvable trigonometric vertex models at $q = 0$ in the vicinity of the ferromagnetic vacuum. A peculiarity here is the extreme anisotropy with respect to the relevant fusion degrees; $B_l$ is the simplest one, while $B' = B_2$ corresponds to an infinite fusion.

Once the automata are constructed the first question will be if they are solitonic. We prove a theorem that

- the $U'_q(\hat{\mathfrak{g}}_n)$ automaton has the patterns labeled by the crystals $\{B_l\}$ of the algebra $U'_q(\hat{\mathfrak{g}}_{n-1})$ that propagate stably with velocity $l$.

Computer experiments indicate that they indeed behave like solitons. For instance, the initially separated patterns labeled by the $U'_q(\hat{\mathfrak{g}}_{n-1})$-crystal elements $b \in B_l$ and $c \in B_k$ ($l > k$) undergo a scattering into two patterns labeled again by some $c' \in B_k$ and $b' \in B_l$. Let $S : B_l \otimes B_k \to B_k \otimes B_l$ be the two-body scattering matrix of such collisions, namely, $S(b \otimes c) = c' \otimes b'$. Let $R : B_l \otimes B_k \to B_k \otimes B_l$ denote the combinatorial $R$ matrix of $U'_q(\hat{\mathfrak{g}}_{n-1})$. Then we prove

- $S = R$

for $\hat{\mathfrak{g}}_n = C_n^{(1)}$ and conjecture it for all the other $\hat{\mathfrak{g}}_n$. Similarly the scattering of multi-solitons labeled by $B_{l_1}, \ldots, B_{l_N}$ ($l_1 > \cdots > l_N$) is given by the isomorphism $B_{l_1} \otimes \cdots \otimes B_{l_N} \simeq B_{l_N} \otimes \cdots \otimes B_{l_1}$ experimentally. Thus the solitonic nature is guaranteed by the Yang-Baxter equation obeyed by $S = R$. A precise formulation of these claims is done through an injection

$$u_l : U'_q(\hat{\mathfrak{g}}_{n-1}) \text{-crystal } B_l \to (U'_q(\hat{\mathfrak{g}}_n) \text{-crystal } B_1)^{\otimes l},$$

which will be described in Section 3.

Admitting that they are soliton cellular automata, the second question is if there exist classical integrable equations governing them, possibly via the ultradiscretization. Here we only confirm this for $A_1^{(2)}$ case by relating the associated automaton to the known $A_1^{(1)}$ example. This observation is due to TS.

The layout of the paper is as follows. In Section 2 we first explain our construction of the $U'_q(\hat{\mathfrak{g}}_n)$ automata concretely along the $\hat{\mathfrak{g}}_n = A_2^{(2)}$ example. It is valid for any $U'_q(\hat{\mathfrak{g}}_n)$ and any finite crystals having the properties (I) and (II). In Section 3, we formulate the theorem and the conjecture for $\hat{\mathfrak{g}}_n$ precisely. We sketch a proof of $S = R$ for $C_n^{(1)}$ case. In principle the idea used in the proof can also be used for the other $\hat{\mathfrak{g}}_n$. We will specify $B_1$ as an infinite set with the actions $\tilde{e}_i, \tilde{f}_i : B_2 \to B_2 \cup \{0\}$ but without the maps $\varepsilon_i, \varphi_i$. In Section

3 For $C_n^{(1)}$, the family in KKM does not contain $B_1$. See HKKOT.
4 They are different from the limits $B_\infty$ and $b_\infty$ in KKM.
5 To take $B' = B_2$ with finite $l$ is an interesting generalization. See HHIKTT for $A_1^{(1)}$ case.
concluding remarks are given. Appendix A is devoted to an explanation of what is meant by ‘$B_2 \otimes B_1 \simeq B_1 \otimes B_2$’, which shows up when the infinite set $B_2$ is substituted into the finite crystal $B'$. This is actually abuse of notation meaning an invertible map $R' : B_2 \times B_1 \to B_1 \times B_2$ between the sets. We state a conjecture on a stability of the combinatorial $R$ matrix $B_l \otimes B_1 \simeq B_1 \otimes B_l$ when $l$ gets large, which ensures the well-definedness of the map $R'$. It assures that we may regard $B_2$ as a finite crystal $B_l$ with a sufficiently large $l$ to define our automata.

Our construction here and in [HIKTT] is a crystal interpretation of the $L$-operator approach [HIK] for a $\hat{g}_n = A^{(1)}_1$ case. The $U'_q(A^{(1)}_n)$ automaton in this sense coincides with the ones in [TS, I, INS]. As in the $C^{(1)}_1$ case in this paper, the properties stated in the above can actually be proved by means of the crystal theory. The detail will appear elsewhere along with the results on a more general choice of the crystals $B$ and $B'$ [HIKTT].

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2 $A^{(2)}_2$ example

Let us explain our automata concretely along the case $\hat{g}_n = A^{(2)}_2$. This simple example is helpful to gain the idea for the general $\hat{g}_n$ case treated in the next section.

As a peculiarity in the rank 1 situation, the $U'_q(A^{(2)}_2)$ automaton turns out to be an ‘even time sector’ of [TS].

2.1 $A^{(2)}_2$ crystals

For $l \in \mathbb{Z}_{\geq 1}$ set

$$B_l = \{(x, y) \mid x, y \in \mathbb{Z}_{\geq 0}, x + y \leq l\}. \quad (2.1)$$

The action of the Kashiwara operators $\tilde{e}_i, \tilde{f}_i (i = 0, 1) : B_l \to B_l \cup \{0\}$ are given by

$$\tilde{e}_0(x, y) = \begin{cases} (x - 1, y) & \text{if } x > y, \\ (x, y + 1) & \text{if } x \leq y, \end{cases} \quad \tilde{e}_1(x, y) = (x + 1, y - 1),$$

$$\tilde{f}_0(x, y) = \begin{cases} (x + 1, y) & \text{if } x \geq y, \\ (x, y - 1) & \text{if } x < y, \end{cases} \quad \tilde{f}_1(x, y) = (x - 1, y + 1).$$

In the above, the right hand sides are to be understood as 0 if they are not in $B_l$. Setting $\varepsilon_i(b) = \max_k \{\tilde{e}_i^k b \neq 0 \mid k \geq 0\}$ and $\varphi_i(b) = \max_k \{\tilde{f}_i^k b \neq 0 \mid k \geq 0\}$,
one has
\[ \varepsilon_0(b) = l - x - y + 2(x - y)_+, \quad \varepsilon_1(b) = y, \]
\[ \varphi_0(b) = l - x - y + 2(y - x)_+, \quad \varphi_1(b) = x, \]
for \( b = (x, y) \in B_l \). Here the symbol \((\cdot)_+\) stands for
\[ (x)_+ = \max(0, x). \]
These results are obtained by extrapolating the \( A_{2n}^{(2)} \) result [KKM] to \( n = 1 \).
For \( l = 1 \) we use a simpler notation as
\[ B = B_1, \quad 1 = (1, 0), \quad 2 = (0, 0), \quad 3 = (0, 1). \] (2.2)
Given two crystals \( B \) and \( B' \), one can form another crystal (tensor product) \( B \otimes B' \) [Kas]. The crystal \( B_l \otimes B_1 \) is connected and so is \( B_1 \otimes B_l \). Calculating the map \( B_l \otimes B_1 \to B_1 \otimes B_l \) commuting with \( \tilde{e}_i \) and \( f_i \), one has

**Proposition 2.1.** The combinatorial \( R \) matrix \( B_l \otimes B \simeq B \otimes B_l \) is given by

\[
(x, y) \otimes 1 \simeq \begin{cases} 
1 \otimes (l, 0) & \text{if } (x, y) = (l, 0), \\
3 \otimes (x + 1, y - 1) & \text{if } x + y = l, \ y \geq 1, \\
2 \otimes (x + 1, y) & \text{if } x + y = l - 1, \\
1 \otimes (x + 1, y + 1) & \text{otherwise},
\end{cases}
\]

\[
(x, y) \otimes 2 \simeq \begin{cases} 
1 \otimes (l - 1, 0) & \text{if } (x, y) = (l, 0), \\
3 \otimes (x, y - 1) & \text{if } x + y = l, \ y \geq 1, \\
2 \otimes (x, y) & \text{otherwise},
\end{cases}
\]

\[
(x, y) \otimes 3 \simeq \begin{cases} 
3 \otimes (0, l) & \text{if } (x, y) = (0, l), \\
2 \otimes (0, l) & \text{if } (x, y) = (0, l - 1), \\
1 \otimes (0, 1) & \text{if } (x, y) = (1, 0), \\
1 \otimes (x - 2, y) & \text{if } 2 \leq x \leq l, \ y = 0, \\
1 \otimes (x, y + 2) & \text{if } x = 0, \ 0 \leq y \leq l - 2, \\
3 \otimes (x - 1, y - 1) & \text{otherwise}.
\end{cases}
\]

In Section 2.2 we shall use formal \( l \to \infty \) limits of \( B_l \) and the combinatorial \( R \) matrix \( B_l \otimes B \simeq B \otimes B_l \). In the present case the prescription is to simply shift the coordinate \((x, y)\) to \((x - l, y)\) and to consider
\[ B_2 = \{(x, y) \mid x \in \mathbb{Z}_{\leq 0}, \ y \in \mathbb{Z}_{\geq 0}, \ x + y \leq 0\}, \]
without specifying a crystal structure. The map \( R' : B_2 \otimes B \simeq B \otimes B_2 \) in the sense of Appendix A is deduced from Proposition 2.1 by concentrating on those
(x, y) in the vicinity of (0, 0). Thus it reads

\[
(x, y) \otimes 1 \simeq \begin{cases} 
1 \otimes (0, 0) & \text{if } (x, y) = (0, 0), \\
3 \otimes (x + 1, y - 1) & \text{if } x + y = 0, y \geq 1, \\
2 \otimes (x + 1, y) & \text{if } x + y = -1, \\
1 \otimes (x + 1, y + 1) & \text{otherwise},
\end{cases}
\quad (2.3)
\]

\[
(x, y) \otimes 2 \simeq \begin{cases} 
1 \otimes (-1, 0) & \text{if } (x, y) = (0, 0), \\
3 \otimes (x, y - 1) & \text{if } x + y = 0, y \geq 1, \\
2 \otimes (x, y) & \text{otherwise},
\end{cases}
\quad (2.4)
\]

\[
(x, y) \otimes 3 \simeq \begin{cases} 
1 \otimes (x - 2, y) & \text{if } y = 0, \\
3 \otimes (x - 1, y - 1) & \text{otherwise}.
\end{cases}
\quad (2.5)
\]

To depict this in a figure we put $b_b' \text{ beside the arrow } (x, y) \rightarrow (x', y') \text{ to signify the relation}

\[
R' : (x, y) \otimes b \xrightarrow{\sim} b' \otimes (x', y').
\]

We call $b$ and $b'$ the upper index and the lower index, respectively. Now (2.3)–(2.5) are summarized in the semi-infinite triangle in Figure 1.

### 2.2 Cellular automaton

By applying $B_b \otimes B \simeq B \otimes B$ successively one has

\[
\overbrace{B_b \otimes B \otimes \cdots \otimes B}^{L} \simeq \overbrace{B \otimes \cdots \otimes B}^{L} \otimes B \otimes B
\]

\[
w \otimes b_i \otimes \cdots \otimes b_j \simeq b'_i \otimes \cdots \otimes b'_j \otimes u'
\quad (2.6)
\]

for any $L \in \mathbb{Z}_{\geq 1}$, where $i = -\lfloor \frac{L}{2} \rfloor$, $j = i + L - 1$. Denote the element $u \in B$ by $u_b$. The map $R' : B_b \otimes B \simeq B \otimes B$ has the properties (I) and (II) in Section 1. Set

\[
\mathcal{P} = \{ b : \mathbb{Z} \to B \mid b_j = 1 \in B \text{ for } |j| \gg 1 \}.
\]

We shall regard $\mathcal{P}$ as a subset of the tensor product which is formally infinite in both directions, i.e.,

\[
\mathcal{P} = \{ \cdots \otimes b_{-1} \otimes b_0 \otimes b_1 \otimes \cdots \in \cdots \otimes B \otimes B \otimes B \otimes \cdots \mid b_j = 1 \text{ for } |j| \gg 1 \}.
\quad (2.7)
\]

In the latter picture one should distinguish the elements even though they are the same under translations. For example,

\[
\cdots 1 \otimes 1 \otimes 1 \otimes 2 \otimes 3 \otimes 1 \otimes 1 \otimes 1 \otimes \cdots,
\cdots 1 \otimes 1 \otimes 1 \otimes 2 \otimes 3 \otimes 1 \otimes 1 \otimes \cdots
\]

\[\text{The symbol } [x] \text{ denotes the largest integer not exceeding } x.\]
Figure 1:
The Semi-infinite triangle representing $R'$. There are 6 different patterns depicted by circles, squares and diagonal squares which are filled or empty.
are distinct elements in $\mathcal{P}$. The set $\mathcal{P}$ is not equipped with a crystal structure. Nevertheless the properties (I) and (II) enable us to define an invertible map $T : \mathcal{P} \to \mathcal{P}$ that formally corresponds to an $L \to \infty$ limit of (2.6). To describe it precisely, note that any element in $\mathcal{P}$ has the form

$$p = \cdots \otimes 1 \otimes b_i \otimes \cdots \otimes b_j \otimes 1 \otimes \cdots \quad (i \leq j),$$

(2.8)

where $b_i, b_{i+1}, \ldots, b_j \in B$. Owing to the properties (I) and (II) there exists $k_0 \in \mathbb{Z}_{\geq 0}$ such that

$$u_i \otimes b_i \otimes \cdots \otimes b_j \otimes 1 \otimes \cdots \otimes T^k \simeq b'_i \otimes \cdots \otimes b'_j \otimes b'_{j+1} \otimes \cdots \otimes b'_{j+k} \otimes u_i$$

for all $k \geq k_0$. Then $T(p) \in \mathcal{P}$ is defined by

$$T(p) = \cdots \otimes 1 \otimes b'_i \otimes \cdots \otimes b'_j \otimes b'_{j+1} \otimes \cdots \otimes b'_{j+k} \otimes 1 \otimes 1 \otimes \cdots,$$

which is $k$-independent as long as $k \geq k_0$. The inverse $T^{-1}$ can be described similarly.

The map $T$ plays the role of the ‘time evolution’ operator. It is a $q = 0$ analogue of the row-to-row transfer matrix of a solvable lattice model in the vicinity of the ferromagnetic vacuum.

Given $p \in \mathcal{P}$ in (2.8), define $u_m \in B_2$ for all $m \in \mathbb{Z}$ by the recursion relation and the boundary condition

$$u_{m-1} \otimes b_m \simeq b'_m \otimes u_m \quad \text{for all } m \in \mathbb{Z},$$

$$u_m = u_2 \quad \text{for } m \leq i - 1.$$ 

Plainly, $u_i \otimes (\cdots \otimes b_{m-1} \otimes b_m \otimes b_{m+1} \otimes \cdots) \simeq \cdots \otimes b'_{m-1} \otimes b'_m \otimes u_m \otimes b_{m+1} \otimes \cdots$.

Due to the properties (I) and (II) the sequence $u_m, u_{m+1}, \ldots$ tends to $u_2 = (0, 0) \in B_2$. In this way any element $p \in \mathcal{P}$ specifies a trajectory $\{u_m\}_{m=-\infty}^{\infty}$ in the semi-infinite triangle (Figure 1) that starts at the origin $(0, 0)$ and returns to it finally. This picture is useful in calculating $T(p)$. Namely, the trajectory is determined by following the arrows with the upper indices $\ldots, b_{m-1}, b_m, b_{m+1}, \ldots$ appearing in $p = \cdots \otimes b_{m-1} \otimes b_m \otimes b_{m+1} \otimes \cdots$. Then $T(p)$ is constructed by tracing their lower indices as $T(p) = \cdots \otimes b'_{m-1} \otimes b'_m \otimes b'_{m+1} \otimes \cdots$.

For $p = \cdots \otimes b_j \otimes b_{j+1} \otimes \cdots \in \mathcal{P}$ and $t \in \mathbb{Z}$ define $b'_t \in B$ by $T'(p) = \cdots \otimes b'_t \otimes b'_{t+1} \otimes \cdots$. Then the time evolution of the cellular automaton is displayed with the arrays

$$\begin{array}{cccccccc}
\cdots & b^0_{-2} & b^0_{-1} & b^0_0 & b^0_1 & b^0_2 & \cdots \\
\cdots & b^1_{-2} & b^1_{-1} & b^1_0 & b^1_1 & b^1_2 & \cdots \\
\cdots & b^2_{-2} & b^2_{-1} & b^2_0 & b^2_1 & b^2_2 & \cdots \\
\end{array}$$

Let us present a few examples.

**Example 2.2.**
Example 2.3.

\[
t=0 : \cdots 11233311111111111111111111111\cdots \\
t=1 : \cdots 11333311111111111111111111111\cdots \\
t=2 : \cdots 11333311111111111111111111111\cdots \\
t=3 : \cdots 11333311111111111111111111111\cdots \\
\]

Example 2.4.

\[
t=0 : \cdots 11333311111111111111111111111\cdots \\
t=1 : \cdots 11333311111111111111111111111\cdots \\
t=2 : \cdots 11333311111111111111111111111\cdots \\
t=3 : \cdots 11333311111111111111111111111\cdots \\
t=4 : \cdots 11333311111111111111111111111\cdots \\
\]

Example 2.5.

\[
t=0 : \cdots 11233311111111111111111111111\cdots \\
t=1 : \cdots 11333311111111111111111111111\cdots \\
t=2 : \cdots 11333311111111111111111111111\cdots \\
t=3 : \cdots 11333311111111111111111111111\cdots \\
t=4 : \cdots 11333311111111111111111111111\cdots \\
t=5 : \cdots 11333311111111111111111111111\cdots \\
t=6 : \cdots 11333311111111111111111111111\cdots \\
t=7 : \cdots 11333311111111111111111111111\cdots \\
\]

Example 2.6.

\[
t=0 : \cdots 11233311111111111111111111111\cdots \\
t=1 : \cdots 11233311111111111111111111111\cdots \\
t=2 : \cdots 11233311111111111111111111111\cdots \\
t=3 : \cdots 11233311111111111111111111111\cdots \\
t=4 : \cdots 11233311111111111111111111111\cdots \\
t=5 : \cdots 11233311111111111111111111111\cdots \\
t=6 : \cdots 11233311111111111111111111111\cdots \\
t=7 : \cdots 11233311111111111111111111111\cdots \\
\]

The last two show the independence of the order of collisions. These examples suggest that the following patterns are stable \((Q \in \mathbb{Z}_{\geq 1}, R = 0, 1)\):

\[
\cdots \otimes 2 \otimes 3 \otimes \cdots 3 \otimes \cdots R = 1,
\]

\[
\cdots \otimes 3 \otimes 3 \otimes \cdots 3 \otimes \cdots R = 0.
\]
The both patterns should not be followed by 3. The former pattern can be preceded by any element in $B$ while the latter should only be preceded by 1. $Q$ is the size of the soliton and $R$ is the number of occurrences of 2 in its front. They move to the right with the velocity $2Q - R$ when separated sufficiently. These features are consistent with $\tilde{g}_n = A_2^{(2)}$ case of Theorem 3.1. See also Section 3.2.

In fact the $U'_q(A_2^{(2)})$ automaton described above can be interpreted [HI] as an ‘even time sector’ of the automaton in [TS]. Replace the array of $\{1, 2, 3\}$ by that of $\{1, 2\}$ with double length via the rule $1 \rightarrow 11, 2 \rightarrow 12$ and $3 \rightarrow 22$. In the resulting array, play the ‘box and ball game’ as in [TS]. Namely, we regard the array as a sequence of cells which contains a ball or not according to the array variable is 2 or 1, respectively. In each time step, we move each ball once to the nearest right empty box starting from the leftmost ball. Then the 2 time steps in the box and ball system yield the 1 time step in our $U'_q(A_2^{(2)})$ automaton.

In terms of crystals, this can be explained as follows. First, the box and ball game in [TS] is known [HHIKTT] to be equivalent to the $U'_q(A_1^{(1)})$ automaton. Let

$$\hat{\mathcal{B}}_k = \{m_1 \ldots m_k | m_i \in \{1, 2\}, m_1 \leq \cdots \leq m_k\}$$

denote the $U'_q(A_1^{(1)})$-crystal corresponding to the $k$-fold symmetric tensor representation. (We have omitted the frame of the usual semistandard tableaux.) Consider the maps $h_\natural$ and $h_1$ defined by

$$h_\natural : \quad \hat{\mathcal{B}}_k \otimes \hat{\mathcal{B}}_k \quad (k \geq 1)$$

$$(x, y) \quad \mapsto \quad 11 \ldots 12 \ldots 2 \otimes 11 \ldots 12 \ldots 2,$$

$$h_1 : \quad B_1 \quad \mapsto \quad \hat{B}_1 \otimes \hat{B}_1$$

$$1 \quad \mapsto \quad 1 \otimes 1$$

$$2 \quad \mapsto \quad 1 \otimes 2$$

$$3 \quad \mapsto \quad 2 \otimes 2.$$ 

Then for $k$ large enough, we have the commutative diagram:

$$\begin{array}{ccc}
B_2 \otimes B_1 & \xrightarrow{h_\natural \otimes h_1} & \hat{B}_k \otimes \hat{B}_k \otimes \hat{B}_1 \otimes \hat{B}_1 \\
\downarrow r' & & \downarrow \\
B_1 \otimes B_2 & \xrightarrow{h_1 \otimes h_\natural} & \hat{B}_1 \otimes \hat{B}_1 \otimes \hat{B}_k \otimes \hat{B}_k,
\end{array}$$

where the down arrow in the right column is the crystal isomorphism. This asserts that the square of the $T$ in the $U'_q(A_1^{(1)})$ automaton coincides with the $T$ of the $U'_q(A_2^{(2)})$ automaton.
3 $U_q'(\hat{g}_n)$ automaton

3.1 Theorem and conjecture

Let us proceed to the $U_q'(\hat{g}_n)$ automata for $\hat{g}_n = A^{(2)}_{2n-1} (n \geq 3), A^{(2)}_{2n} (n \geq 1), B^{(1)}_n (n \geq 3), C^{(1)}_n (n \geq 2), D^{(1)}_n (n \geq 4)$ and $D^{(2)}_{n+1} (n \geq 2)$. Our aim here is to formulate the theorem and the conjecture stated in Section 1 precisely. In principle construction of the automata is the same as the $A^{(2)}_2$ case explained in Section 2.2. The time evolution operator $T$ is constructed from the invertible map $R' : B \otimes B \rightarrow B \otimes B$ in the sense of Appendix A. However its analytic form like (2.3)–(2.5) is yet unknown for general $\hat{g}_n$. Thus we have generated the combinatorial $R$ matrix $R : B_l \otimes B \rightarrow B \otimes B_l$ directly by computer, and investigated the automata associated with $R$ instead of $R'$ for several large $l$. Consistently with Conjecture A.1, their behaviour becomes stable when $l$ gets large, yielding our automata associated with $R'$.

What we present in Section 3.2 is the list of the data $B_l, B_0, u_l$ and $i_l$ for each $\hat{g}_n$. We change the notation slightly from Section 1 and 2, representing elements in $B = B_1$ with symbols inside a box or $\phi$. For example the special (highest weight) element $1 \in B_1$ in the properties (I)–(II) is denoted by $1$.

Consistently with Conjecture A.1, their behaviour becomes stable when $l$ gets large, yielding our automata associated with $R'$.

First we claim stable propagation of the 1-soliton as

**Theorem 3.1.** For any $l \in \mathbb{Z}_{>1}$ and $b \in U_q'(\hat{g}_{n-1})$-crystal $B_l$, the time evolution map $T$ acts on the state

$$\cdots \otimes 1 \otimes 1 \otimes i_l(b) \otimes 1 \otimes 1 \otimes \cdots$$

as the overall translation to the right by $l$ slots.

Thus $l$ is the velocity of the soliton. Even when $n$ is the minimal possible value for $\hat{g}_n$, Theorem 3.1 makes sense if $i_l$ and $B_l$ for $U_q'(\hat{g}_{n-1})$ are interpreted

\[\text{For $C^{(1)}_n$ we have a concrete description of the combinatorial $R$ matrix $B_l \otimes B_0$ in terms of an insertion algorithm. See [HKOT].}\]
appropriately by an extrapolation of the data given in Section 3.2. For example, the solitons for $\hat{g}_n = A^{(1)}_n$ case argued in Section 2.2 agree with those coming from $U'_q(A^{(2)}_1)$-crystal $B_l$ in such a sense. The proof is given in Section 3.3.

For scattering of two solitons we have

**Theorem 3.2.** Let $\hat{g}_n = C^{(1)}_n (n \geq 3)$. For fixed positive integers $l > k$, let $B_l \otimes B_k \supseteq b_1 \otimes b_2 \simeq c_2 \otimes c_1 \in B_k \otimes B_l$ be an isomorphism under the combinatorial $R$ matrix of the $U'_q(\hat{g}_{n-1})$-crystals. Then for $m \gg l$, there exists $t_0 > 0$ such that for any $t \geq t_0$ and some $m'$

$$T^t : \cdots \otimes 1 \otimes 1 \otimes u_l(b_1) \otimes 1 \otimes \cdots \otimes u_k(b_2) \otimes 1 \otimes 1 \otimes 1 \cdots \cdots \quad \mapsto \cdots \otimes 1 \otimes 1 \otimes 1 \otimes u_k(c_2) \otimes 1 \otimes \cdots \otimes u_l(c_1) \otimes 1 \otimes 1 \otimes 1 \cdots \ .$$

In other words, we have the combinatorial $R$ matrix as the scattering matrix of the ultra-discrete solitons. Compared with $u_l(b_1), u_k(c_2)$ is shifted to the right, but we do not concern the precise distance. A sketch of a proof of Theorem 3.2 will be given in Section 3.4.

In fact we have a conjecture on $N$-soliton case for general $\hat{g}_n$.

**Conjecture 3.3.** Let $N \in \mathbb{Z}_{\geq 2}$. Fix positive integers $k_1, \ldots, k_N$ ($k_1 > \cdots > k_N$) and the elements $b_i \in B_{k_i}$ ($i = 1, \ldots, N$) of the $U'_q(\hat{g}_{n-1})$-crystals. Define $c_i \in B_{k_i}$ by $b_1 \otimes \cdots \otimes b_N \simeq c_N \otimes \cdots \otimes c_1$ under the isomorphism $B_{k_1} \otimes B_{k_2} \otimes \cdots \otimes B_{k_N} \simeq B_{k_N} \otimes \cdots \otimes B_{k_2} \otimes B_{k_1}$. For $m_1, \ldots, m_{N-1} \gg k_1$, there exists $t_0 > 0$ such that for any $t \geq t_0$,

$$T^t : \cdots \otimes 1 \otimes u_{k_1}(b_1) \otimes 1 \otimes \cdots \otimes 1 \otimes u_{k_N}(b_N) \otimes 1 \otimes \cdots \quad \mapsto \cdots \otimes 1 \otimes 1 \otimes 1 \otimes u_{k_N}(c_N) \otimes 1 \otimes \cdots \otimes 1 \otimes 1 \otimes 1 \otimes \cdots \ .$$

holds for some $m'_1, \ldots, m'_{N-1}$.

In particular, $c_1, \ldots, c_N$ do not depend on $m_i$’s (i.e., the order of collisions) as long as $m_i \gg k_i$. Again we do not concern the precise distance between $u_{k_1}(b_1)$ and $u_{k_N}(c_N)$ in the above.

The rank $n$ in Conjecture 3.3 should be taken greater than the minimal possible values, i.e., $A^{(2)}_{2n-1}$ ($n \geq 4$), $A^{(2)}_{2n}$ ($n \geq 2$), $B^{(1)}_{n}$ ($n \geq 4$), $C^{(1)}_n$ ($n \geq 3$), $D^{(1)}_n$ ($n \geq 5$) and $D^{(2)}_{n+1}$ ($n \geq 3$). We have checked the $N = 2, 3$ cases by computer with several examples for $\hat{g}_n = A^{(1)}_7, A^{(2)}_4, A^{(2)}_6, B^{(1)}_4, D^{(1)}_5, D^{(2)}_4$ and the $N = 3$ case for $C^{(1)}_4$.

Let us present a few examples from the $U'_q(C^{(1)}_3)$ automaton. We use the notation that will be introduced in Section 3.3. The dynamical variables are taken from the crystal $B_1 = \{ 1 \ 2 \ 3 \ 3 \ 2 \ 1 \}$. In the following examples we drop the boxes for simplicity.

**Example 3.4.**
Example 3.6.

Under the isomorphism $B_4 \otimes B_2 \simeq B_2 \otimes B_4$ of $U_q'(C_2^{(1)})$-crystals.

Example 3.5.

Compare this with

$$(1, 1, 0, 2) \otimes (0, 1, 1, 0) \simeq (0, 2, 0, 0) \otimes (0, 1, 2, 1)$$

under the isomorphism $B_4 \otimes B_2 \simeq B_2 \otimes B_4$ of $U_q'(C_2^{(1)})$-crystals.

Example 3.6.

Compare this with

$$(0, 1, 0, 2) \otimes (1, 0, 1, 1) \simeq (0, 1, 0, 0) \otimes (1, 0, 1, 3)$$

under the isomorphism $B_5 \otimes B_3 \simeq B_3 \otimes B_5$ of $U_q'(C_2^{(1)})$-crystals.
3.2 Data on crystals

Let us present the data $B_l, B_2, u_2$ and $u_l$ for each $\mathfrak{g}_n$. $B_l$ and $B_2$ will be specified only as sets. The crystal structure of $B_l$ is available in [KKM]. (See [HKKOT] for $C_n^{(1)}$ case.) For all the $\mathfrak{g}_n$, $u_2 \in B_1$ is given by

$$u_2 = (\forall x_i, \bar{x}_i = 0)$$

in the notation employed below. As a set $B_l$ can be embedded into $B_2$ by

$$g_l : B_l \rightarrow B_2$$

$$(x_i, \bar{x}_i) \rightarrow (x_i - l\delta_{1i}, \bar{x}_i).$$

Obviously any $u \in B_2$ has the inverse image in each $B_l$ if $l$ is large enough. It is easy to see that the composition $B_2 \xrightarrow{g_l^{-1}} B_l \xrightarrow{\tilde{e}_i, \tilde{f}_i} B_l \cup \{0\} \xrightarrow{g_l} B_2 \cup \{0\}$ is independent of $l$ when $l$ is large enough. Here we understand that $g_l(0) = 0$. Thus one can endow $B_2$ with the actions of $\tilde{e}_i, \tilde{f}_i$ defined by this with $l$ sufficiently large. (However it will not be used in this paper.)

$\mathfrak{g}_n = A_{2n-1}^{(2)}$:

$$B_l = \{(x_1, \ldots, x_n, \bar{x}_n, \ldots, \bar{x}_1) \in \mathbb{Z}^{2n} | x_i, \bar{x}_i \geq 0, \sum_{i=1}^n (x_i + \bar{x}_i) = l\},$$

$$B_2 = \{(x_1, \ldots, x_n, \bar{x}_n, \ldots, \bar{x}_1) \in \mathbb{Z}^{2n} | x_1 \leq 0, x_i \geq 0 (i \neq 1), \bar{x}_i \geq 0, \sum_{i=1}^n (x_i + \bar{x}_i) = 0\}.$$

For $B_1$ we use a simpler notation

$$(x_1, \ldots, x_n, \bar{x}_n, \ldots, \bar{x}_1) = \begin{cases} \mathbb{I} & \text{if } x_i = 1, \text{ others } = 0, \\ \mathbb{I} & \text{if } \bar{x}_i = 1, \text{ others } = 0. \end{cases}$$

For $b = (x_1, \ldots, x_{n-1}, \bar{x}_{n-1}, \ldots, \bar{x}_1)$ in $U'_q(A_2^{(2)}_{2n-3})$-crystal $B_l$,

$$u_l(b) = 2 \otimes \bar{x}_1 \otimes \ldots \otimes 2 \otimes \bar{x}_{n-1} \otimes n \otimes x_{n-1} \otimes \ldots \otimes 2 \otimes x_1.$$

$\mathfrak{g}_n = A_{2n}^{(2)}$:

$$B_l = \{(x_1, \ldots, x_n, \bar{x}_n, \ldots, \bar{x}_1) \in \mathbb{Z}^{2n} | x_i, \bar{x}_i \geq 0, \sum_{i=1}^n (x_i + \bar{x}_i) \leq l\},$$

$$B_2 = \{(x_1, \ldots, x_n, \bar{x}_n, \ldots, \bar{x}_1) \in \mathbb{Z}^{2n} | x_1 \leq 0, x_i \geq 0 (i \neq 1), \bar{x}_i \geq 0, \sum_{i=1}^n (x_i + \bar{x}_i) \leq 0\}.$$

For $B_1$ we use a simpler notation

$$(x_1, \ldots, x_n, \bar{x}_n, \ldots, \bar{x}_1) = \begin{cases} \mathbb{I} & \text{if } x_i = 1, \text{ others } = 0, \\ \mathbb{I} & \text{if } \bar{x}_i = 1, \text{ others } = 0, \\ \varnothing & \text{if } x_i = 0, \bar{x}_i = 0 \text{ for all } i. \end{cases}$$
For $b = (x_1, \ldots, x_{n-1}, x_{n-1}, \ldots, x_1)$ in $U_q'(A_{2n-2}^{(2)})$-crystal $B_l$, we define $s(b) = \sum_{i=1}^{n-1} (x_i + \bar{x}_i)$, $s'(b) = [(l - s(b))/2]$. 

If $l - s(b)$ is odd,

$$u_l(b) = \phi \otimes 1 \otimes s'(b) \otimes 2 \otimes x_1 \otimes \cdots \otimes n \otimes \bar{x}_{n-1} \otimes \cdots \otimes 2 \otimes x_1 \otimes 1 \otimes s'(b),$$

otherwise

$$u_l(b) = 1 \otimes s'(b) \otimes 2 \otimes x_1 \otimes \cdots \otimes n \otimes \bar{x}_{n-1} \otimes \cdots \otimes 2 \otimes x_1 \otimes 1 \otimes s'(b).$$

When $n = 1$, the above notation for $B_1$ and $B_2$ are related by $1 = [1], 2 = \phi$ and $3 = [1]$. The solitons and their velocity mentioned in the beginning of Section 2.2 agree with the $n = 1$ case here. One interprets $s(b) = 0$ and $s'(b) = [l/2]$ for $b$ from $U_q'(A_{2n}^{(2)})$-crystal $B_l$. Then, under the identification $R = (1 - (-1)^l)/2$ and $Q = R + s'(b)$, the velocity is indeed $l = 2Q - R$. 

$\hat{g}_n = B_n^{(1)}$:

$$B_1 = \left\{ (x_1, \ldots, x_n, x_0, \bar{x}_n, \ldots, \bar{x}_1) \in \mathbb{Z}^{2n} \times \{0, 1\} \mid \begin{array}{l} x_0 = 0 \text{ or } 1, x_i, \bar{x}_i \geq 0, \\
x_0 + \sum_{i=1}^{n} (x_i + \bar{x}_i) = l \end{array} \right\},$$

$$B_2 = \left\{ (x_1, \ldots, x_n, x_0, \bar{x}_n, \ldots, \bar{x}_1) \in \mathbb{Z}^{2n} \times \{0, 1\} \mid \begin{array}{l} x_0 = 0 \text{ or } 1, x_i \leq 0, \\
x_i \geq 0 \ (i \neq 1), \bar{x}_i \geq 0, \\
x_0 + \sum_{i=1}^{n} (x_i + \bar{x}_i) = 0 \end{array} \right\}.$$

For $B_1$ we use a simpler notation

$$(x_1, \ldots, x_n, x_0, \bar{x}_n, \ldots, \bar{x}_1) = \begin{cases} 1 & \text{if } x_i = 1, \text{ others } = 0, \\
1 & \text{if } \bar{x}_i = 1, \text{ others } = 0. \end{cases}$$

For $b = (x_1, \ldots, x_{n-1}, x_0, \bar{x}_{n-1}, \ldots, \bar{x}_1)$ in $U_q'(B_{n-1}^{(1)})$-crystal $B_l$,

$$u_l(b) = 2 \otimes x_1 \otimes \cdots \otimes n \otimes \bar{x}_{n-1} \otimes \cdots \otimes 2 \otimes x_1 = \hat{g}_n = C_n^{(1)}$$. 

$$B_1 = \left\{ (x_1, \ldots, x_n, x_0, \bar{x}_n, \ldots, \bar{x}_1) \in \mathbb{Z}^{2n} \mid \begin{array}{l} x_i, \bar{x}_i \geq 0, \\
l \geq \sum_{i=1}^{n} (x_i + \bar{x}_i) \in l - 2\mathbb{Z} \end{array} \right\},$$

$$B_2 = \left\{ (x_1, \ldots, x_n, x_0, \bar{x}_n, \ldots, \bar{x}_1) \in \mathbb{Z}^{2n} \mid \begin{array}{l} x_1 \leq 0, x_i \geq 0 \ (i \neq 1), \bar{x}_i \geq 0, \\
0 \geq \sum_{i=1}^{n} (x_i + \bar{x}_i) \in 2\mathbb{Z} \end{array} \right\}.$$

For $B_1$ we use a simpler notation

$$(x_1, \ldots, x_n, x_0, \bar{x}_n, \ldots, \bar{x}_1) = \begin{cases} 1 & \text{if } x_i = 1, \text{ others } = 0, \\
1 & \text{if } \bar{x}_i = 1, \text{ others } = 0. \end{cases}$$
For \( b = (x_1, \ldots, x_{n-1}, \bar{x}_{n-1}, \ldots, \bar{x}_1) \) in \( U_q'(C_{n-1}^{(1)}) \)-crystal \( B_l \), we define \( s(b) = \sum_{i=1}^{n-1} (x_i + \bar{x}_i) \), \( s'(b) = (l - s(b))/2 \).

\[
\mathfrak{g}_n = D_n^{(1)}:
\]

\[
B_l = \left\{ (x_1, \ldots, x_n, \bar{x}_n, \ldots, \bar{x}_1) \in \mathbb{Z}^{2n} \mid \begin{array}{l}
x_n = 0 \text{ or } \bar{x}_n = 0, \quad x_i, \bar{x}_i \geq 0, \\
\sum_{i=1}^{n}(x_i + \bar{x}_i) = l
\end{array} \right\},
\]

\[
B_2 = \left\{ (x_1, \ldots, x_n, \bar{x}_n, \ldots, \bar{x}_1) \in \mathbb{Z}^{2n} \mid \begin{array}{l}
x_n = 0 \text{ or } \bar{x}_n = 0, \quad x_i \geq 0 (i \neq 1), \quad \bar{x}_i \geq 0, \\
\sum_{i=1}^{n}(x_i + \bar{x}_i) = 0
\end{array} \right\}.
\]

For \( B_1 \) we use a simpler notation

\[
(x_1, \ldots, x_n, \bar{x}_n, \ldots, \bar{x}_1) = \begin{cases} 
1 & \text{if } x_i = 1, \text{ others } = 0, \\
1 & \text{if } \bar{x}_i = 1, \text{ others } = 0.
\end{cases}
\]

For \( b = (x_1, \ldots, x_{n-1}, \bar{x}_{n-1}, \ldots, \bar{x}_1) \) in \( U_q'(D_{n-1}^{(1)}) \)-crystal \( B_l \),

\[
\mathfrak{g}_n = D_{n+1}^{(2)}:
\]

\[
B_l = \left\{ (x_1, \ldots, x_n, x_0, \bar{x}_n, \ldots, \bar{x}_1) \in \mathbb{Z}^{2n+1} \times \{0, 1\} \mid \begin{array}{l}
x_0 = 0 \text{ or } 1, \quad x_i, \bar{x}_i \geq 0, \\
x_0 + \sum_{i=1}^{n}(x_i + \bar{x}_i) \leq l
\end{array} \right\},
\]

\[
B_2 = \left\{ (x_1, \ldots, x_n, x_0, \bar{x}_n, \ldots, \bar{x}_1) \in \mathbb{Z}^{2n+1} \times \{0, 1\} \mid \begin{array}{l}
x_0 = 0 \text{ or } 1, \quad x_i \leq 0, \\
x_i \geq 0 (i \neq 1), \quad \bar{x}_i \geq 0, \\
x_0 + \sum_{i=1}^{n}(x_i + \bar{x}_i) \leq 0
\end{array} \right\}.
\]

For \( B_1 \) we use a simpler notation

\[
B_1 \ni (x_1, \ldots, x_n, x_0, \bar{x}_n, \ldots, \bar{x}_1) = \begin{cases} 
1 & \text{if } x_i = 1, \text{ others } = 0, \\
1 & \text{if } \bar{x}_i = 1, \text{ others } = 0, \\
0 & \text{if } x_i = 0, \quad \bar{x}_i = 0 \text{ for all } i.
\end{cases}
\]

For \( b = (x_1, \ldots, x_{n-1}, x_0, \bar{x}_{n-1}, \ldots, \bar{x}_1) \) in \( U_q'(D_n^{(2)}) \)-crystal \( B_l \), we define \( s(b) = \sum_{i=1}^{n-1} (x_i + \bar{x}_i) \), \( s'(b) = [(l - s(b))/2] \).

If \( l - s(b) \) is odd,

\[
\nu_l(b) = \phi \otimes 1 \otimes s'(b) \otimes 2^x \otimes \ldots \otimes 2^n \otimes \bar{x}_{n-1} \otimes 0^x \otimes 2^{x_0} \otimes 2^x_{x_{n-1}} \otimes \ldots \otimes 2^{x_1} \otimes 1 \otimes s'(b),
\]

otherwise

\[
\nu_l(b) = \begin{cases} 
1 \otimes s'(b) \otimes 2^x \otimes \ldots \otimes 2^n \otimes \bar{x}_{n-1} \otimes 0^x \otimes 2^{x_0} \otimes 2^x_{x_{n-1}} \otimes \ldots \otimes 2^{x_1} \otimes 1 \otimes s'(b) & \text{for } l - s(b) \text{ is even},
\end{cases}
\]

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3.3 Proof of Theorem 3.1

Our proof below uses the crystal structure of $B_l$ given in [HKKOT] for $\hat{g}_n = C_n^{(1)}$ and in [KKM] for the other types.

**Lemma 3.7.** For $i \in \{1, 2, \ldots, n - 1\}$, the following diagram is commutative:

\[
\begin{array}{ccc}
U'_q(\hat{g}_{n-1})\text{-crystal } B_l & \xrightarrow{\tilde{e}_i} & U'_q(\hat{g}_n)\text{-crystal } B_l^\otimes_l \\
\downarrow \quad \tilde{e}_i & & \downarrow \quad \tilde{e}_{i+1} \\
U'_q(\hat{g}_{n-1})\text{-crystal } B_l \sqcup \{0\} & \xrightarrow{\tilde{f}_i} & U'_q(\hat{g}_n)\text{-crystal } B_l^\otimes_l \sqcup \{0\}.
\end{array}
\]

The same relation holds between $\tilde{f}_i$ and $\tilde{f}_{i+1}$.

The Kashiwara operators of $U'_q(\hat{g}_{n-1})$ and $U'_q(\hat{g}_n)$-crystals should not be confused although we use the same notation. The proof of the lemma is due to the explicit rules for $\tilde{e}_i$ [KKM, HKKOT] and the embedding of $U'_q(\hat{g}_{n-1})$-crystal $B_l$ into $U'_q(\hat{g}_{n-1})$-crystal $B_l^\otimes_l$ as $U_q(\hat{g}_{n-1})$-crystals [KN]. Here $\hat{g}_{n-1}$ is the classical subalgebra of $\hat{g}_n$. According to $\hat{g}_n = A_{2n-1}^{(2)}, A_{2n}^{(2)}, B_{n}^{(1)}, C_{n}^{(1)}, D_{n}^{(1)}$ and $D_{n+1}^{(2)}$, it is given by $\hat{g}_n = C_n, B_n, C_n, D_n$ and $B_n$ respectively.

**Proof of Theorem 3.1.**

$\hat{g}_n = A_{2n-1}^{(2)}$:

Since $B_l$ of $U'_q(A_{2n-3}^{(2)})$-crystal is isomorphic to $B(l\Lambda_1)$ as $U_q(C_{n-1})$-crystals, for any $b \in B_l$ there exists a sequence $i_1, \ldots, i_p \in \{1, \ldots, n - 1\}$ such that $b_0 = \tilde{e}_{i_p} \cdots \tilde{e}_{i_1} b$ with $b_0 = (l, 0, \ldots, 0) \in B_l$. From Lemma 3.7, $\tilde{u}(b_0) = \tilde{e}_{i_{p+1}} \cdots \tilde{e}_{i_{i+1}} \tilde{u}(b)$ is valid.

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Taking sufficiently large $L$ and $f = \tilde{f}_2 \cdots \tilde{f}_{n-1} \tilde{f}_n$, one has

$$u_L \otimes \varepsilon_l(b) \otimes 1 \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \cdots \otimes 1 \otimes u_L$$

Thus the isomorphism $B_L \otimes B_1 \otimes 2 \otimes 1 \otimes B_L$ sends $u_L \otimes \varepsilon_l(b) \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \otimes u_L$ to $1 \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes (L, 0, \ldots, 0)$.

Thus the isomorphism $B_L \otimes B_1 \otimes 2 \otimes 1 \otimes B_L$ sends $u_L \otimes \varepsilon_l(b) \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \otimes u_L$ to $1 \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes (L, 0, \ldots, 0)$. This proves Theorem 3.1.

$\phi_n = A_{2n}^{(2)}$:

Let $b_0^{(m)} = (m, 0, \ldots, 0) \in B_{l}$ of $U_{q}^{(2)}(A_{2n-2})$, where $0 \leq m \leq l$. First we consider $b = b_0^{(l)}$ case. Taking sufficiently large $L$ and $f = \tilde{f}_2 \cdots \tilde{f}_{n-1} \tilde{f}_n \tilde{f}_{n-1} \cdots \tilde{f}_2$, one
has

\[ u_L \otimes \iota_L(b_{0}^{(l)}) \otimes 1^{\otimes L} \]

\[ \overset{f}{\rightarrow} \]

\[ (L, 0, \ldots, 0) \otimes 2^{\otimes L} \otimes 1^{\otimes L} \]

\[ \overset{f}{\rightarrow} \]

\[ (L, 0, \ldots, 0) \otimes 2^{\otimes L} \otimes 1^{\otimes L} \]

Thus the isomorphism \( B_L \otimes B_1^{\otimes 2L} \rightarrow B_1^{\otimes 2L} \otimes B_L \) sends \( u_L \otimes \iota_L(b_{0}^{(l)}) \otimes 1^{\otimes L} \) to \( 1^{\otimes L} \otimes \iota_L(b_{0}^{(l)}) \otimes u_L \), verifying Theorem 3.1 for \( b = b_0^{(l)} \). Explicitly it reads

\[ (L, 0, \ldots, 0) \otimes 2^{\otimes L} \otimes 1^{\otimes L} \sim 1^{\otimes L} \otimes 2^{\otimes L} \otimes (L, 0, \ldots, 0) \]  \hspace{1cm} (3.1)

Let us proceed to arbitrary \( b \in B_l \) case. We will attribute the proof to the \( b = b_0^{(m)} \) case for some \( 0 \leq m \leq l \) and eventually to the \( b_0^{(l)} \) case shown above. Set \( k = \lceil \frac{l-2m}{2} \rceil \), \( k' = \lfloor \frac{l-m+1}{2} \rfloor \) so that \( \iota_l(b_{0}^{(m)}) = \phi^{\otimes k'-k} \otimes 1^{\otimes k} \otimes 2^{\otimes m} \otimes 1 \) according to Section 3.2. Since \( B_l \) of \( U_q(A_{\frac{n-2}{2}}) \)-crystal is isomorphic to \( B(l-1) \oplus B_l \oplus \cdots \oplus B(0) \) as \( U_q(C_{n-1}) \)-crystals, for any \( b \in B_l \) there exists a sequence \( i_1, \ldots, i_p \in \{1, \ldots, n-1\} \) such that \( b_{0}^{(m)} = \epsilon_{i_p} \cdots \epsilon_{i_1} b \) for some \( 0 \leq m \leq l \). From Lemma 2.7 one has

\[ u_L \otimes \iota_L(b) \otimes 1^{\otimes L+k'} \]

\[ \overset{\epsilon_{i_p+1} \cdots \epsilon_{i_1+1}}{\rightarrow} \]

\[ u_L \otimes \iota_L(b_{0}^{(m)}) \otimes 1^{\otimes L+k'} \]

where in the bottom we have used (3.1) and \( u_L \otimes 1 \sim 1 \otimes u_L \). Thus the isomorphism \( B_L \otimes B_1^{\otimes 2L} \rightarrow B_1^{\otimes 2L} \otimes B_L \) sends \( u_L \otimes \iota_L(b) \otimes 1^{\otimes L} \) to \( 1^{\otimes L} \otimes \iota_L(b) \otimes u_L \), proving Theorem 3.1.
\[ \hat{g}_n = B_n^{(1)}: \]
The proof is similar to the case \( \hat{g}_n = A_{2n-1}^{(2)} \) with \( \mathfrak{f} = \hat{f}_{t} \cdots \hat{f}_{n-1} \hat{f}_{n} \hat{f}_{t} \cdots \hat{f}_{t} \).
\[ \hat{g}_n = C_n^{(1)}: \]
The proof is similar to the case \( \hat{g}_n = A_{2n}^{(2)} \).
\[ \hat{g}_n = D_n^{(1)}: \]
The proof is similar to the case \( \hat{g}_n = A_{2n-1}^{(2)} \) with \( \mathfrak{f} = \hat{f}_{t} \cdots \hat{f}_{n-1} \hat{f}_{n} \hat{f}_{t} \cdots \hat{f}_{t} \).
\[ \hat{g}_n = D_n^{(2)}: \]
The proof is similar to the case \( \hat{g}_n = A_{2n}^{(2)} \) with \( \mathfrak{f} = \hat{f}_{t} \cdots \hat{f}_{n-1} \hat{f}_{n} \hat{f}_{t} \cdots \hat{f}_{t} \).

\[ \square \]

### 3.4 Proof of Theorem 3.2

We divide the proof into Part I and Part II. The statements in Part I are valid not only for \( C_n^{(1)} \) but for any \( \hat{g}_n \). Apart from the separation into two solitons in the final state, this already proves Theorem 3.2 for those \( \hat{g}_n \) in which the decomposition of \( B_l \otimes B_k \) is multiplicity-free. Among the list of \( \hat{g}_n \) in question, such cases are \( A_{2n-1}^{(2)}, B_n^{(1)} \) and \( D_n^{(1)} \). Since the \( C_n^{(1)} \) case has the multiplicity, we need Part II, which relies on the explicit result on the combinatorial \( R \) matrices for \( U'_q(C_n^{(1)}) \) in [HKT]. In the rest of Section 3 we shall write \( U'_q(\hat{g}_n) \)-crystals as \( B_l \) and \( U'_q(\hat{g}_{n-1}) \)-crystals as \( \hat{B}_l \) for distinction.

**Part I.** For sufficiently large \( L \), let \( u_L = (L, 0, \ldots, 0) \in B_L \) be the highest weight element. Let \( \mathcal{R} : \hat{B}_l \otimes \hat{B}_k \simeq B_k \otimes \hat{B}_l \) be the combinatorial \( R \) matrix of \( U'_q(\hat{g}_{n-1}) \)-crystals. The main claim in Part I is

**Proposition 3.8.** For \( b_1 \in \hat{B}_l, b_2 \in \hat{B}_k (l > k) \) and \( L_1, L_2 \in \mathbb{Z}_{\geq 1} \), suppose there exist \( t_0 \in \mathbb{Z}_{\geq 1}, c_1 \in \hat{B}_l \) and \( c_2 \in \hat{B}_k \) such that

\[
\begin{align*}
    u_{L_1}^{\otimes t} &\otimes \hat{1}^{\otimes L_2} \otimes u(b_1) \otimes \hat{1}^{\otimes L_2} \otimes u(b_2) \otimes \hat{1}^{\otimes L_1} \\
    &\simeq \hat{1}^{\otimes L_1} \otimes u(c_2) \otimes \hat{1}^{\otimes L_2} \otimes u(c_1) \otimes \hat{1}^{\otimes L_1} \otimes u_{L_2}^{\otimes t} \quad (3.2)
\end{align*}
\]

holds for any \( t \geq t_0 \) for some \( L_1', L_2', L_3 \) and \( L_4 \) under the isomorphism \( B_{L_1'}^{\otimes t} \otimes B_1 \otimes \cdots \otimes B_3 \simeq B_1 \otimes \cdots \otimes B_1 \otimes B_{L_4}^{\otimes t} \). Define \( \mathcal{S} : \hat{B}_l \otimes \hat{B}_k \rightarrow \hat{B}_k \otimes \hat{B}_l \) through this relation by \( \mathcal{S}(b_1 \otimes b_2) = c_2 \otimes c_1 \). Then \( \mathcal{R}(b_1 \otimes b_2) = \mathcal{S}(b_1 \otimes b_2) \) implies \( \mathcal{R}(\hat{f}_i(b_1 \otimes b_2)) = \mathcal{S}(\hat{f}_i(b_1 \otimes b_2)) \) for each \( i = 1, \ldots, n-1 \). Similarly, \( \mathcal{S}(b_1 \otimes b_2) = 0 \) implies \( \hat{e}_i(b_1 \otimes b_2) = 0 \) for each \( i = 1, \ldots, n-1 \).

**Lemma 3.9.** For each \( i = 1, \ldots, n-1 \), we have a commutative diagram:

\[
\begin{array}{ccc}
\hat{B}_l \otimes \hat{B}_k & \xrightarrow{\mathfrak{f}_i \otimes \mathfrak{f}_k} & (B_1)^{\otimes \ell + k} \\
\mathcal{S} & \downarrow & \hat{e}_{i+1} \\
(\hat{B}_l \otimes \hat{B}_k) \sqcup \{0\} & \xrightarrow{\mathfrak{f}_i \otimes \mathfrak{f}_k} & (B_1)^{\otimes \ell + k} \sqcup \{0\}.
\end{array}
\]

The same relation holds also between \( \hat{f}_i \) and \( \hat{f}_{i+1} \).
This is a simple corollary of Lemma 3.7. Similarly we have

**Lemma 3.10.** Let \( a_1, \ldots, a_L \in B_1 \), \( i_1, \ldots, i_m \) be the subsequence of \( 1, \ldots, L \) satisfying \( a_j = \mathbb{1} \in B_1 \) if \( j \notin \{i_1, \ldots, i_m\} \) \((m \leq L)\), and \( p_k = a_{i_k} \) \((k = 1, \ldots, m)\). Then for any \( t, t' \in \mathbb{Z}_{\geq 0} \) and \( L \in \mathbb{Z}_{\geq 1} \),

1. \( \tilde{f}_{i + 1}(p_1 \otimes \cdots \otimes p_m) = p'_{i + 1} \otimes \cdots \otimes p'_{m} \Rightarrow \tilde{f}_{i + 1}(u_L^{\otimes t} \otimes a_1 \otimes \cdots \otimes a_L \otimes u_L^{\otimes t'} = u_L^{\otimes t} \otimes a'_1 \otimes \cdots \otimes a'_L \otimes u_L^{\otimes t'}, \)
   where \( a'_j = \begin{cases} p'_k & \text{if } j = i_k, \\ 1 & \text{otherwise,} \end{cases} \)

2. \( \tilde{f}_{i + 1}(u_L^{\otimes t} \otimes a_1 \otimes \cdots \otimes a_L \otimes u_L^{\otimes t'}) = u_L^{\otimes t} \otimes a'_1 \otimes \cdots \otimes a'_L \otimes u_L^{\otimes t'}, \)
   \( \Rightarrow \tilde{f}_{i + 1}(p_1 \otimes \cdots \otimes p_m) = p'_1 \otimes \cdots \otimes p'_m, \) where \( p'_k = a'_{i_k} \).

3. \( \tilde{f}_{i + 1}(u_L^{\otimes t} \otimes a_1 \otimes \cdots \otimes a_L \otimes u_L^{\otimes t'}) = 0 \iff \tilde{f}_{i + 1}(p_1 \otimes \cdots \otimes p_m) = 0, \)
   for each \( i = 1, \ldots, n - 1 \). The same is true also for \( \tilde{c}_{i + 1} \).

**Proof of Proposition 3.8.** Suppose \( R(b_1 \otimes b_2) = S(b_1 \otimes b_2) = c_2 \otimes c_1 \). Thus \((c_2, c_1) \in B_k \times B_k^1\) is related to \((b_1, b_2)\) via (3.2). Set \( b'_i \otimes b''_i = f_i(b_1 \otimes b_2) \) for some \( i = 1, \ldots, n - 1 \) and put \( c'_2 \otimes c'_1 = S(b'_1 \otimes b''_1) \). We are to show \( R(b'_1 \otimes b''_1) = c'_2 \otimes c'_1 \).

From Lemma 3.9 one has

\[
\tilde{f}_{i + 1}(u(b_1) \otimes t_k(b_2)) = u(b'_1) \otimes t_k(b'_2). \tag{3.3}
\]

Setting \( t' = 0, a_1 \otimes \cdots \otimes a_L = \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes t_k(b_2) \otimes \mathbb{1} \otimes \mathbb{1} \) and \( p_1 \otimes \cdots \otimes p_m = u(b_1) \otimes t_k(b_2) \) in Lemma 3.10, one has

\[
\tilde{f}_{i + 1}
= u_L^{\otimes t} \otimes \mathbb{1} \otimes t_k(b_1) \otimes \mathbb{1} \otimes t_k(b_2) \otimes \mathbb{1} \otimes \mathbb{1}
\]

where (3.3) is used. By sending \( u_L \) to the right as in (3.4), this is equivalent to

\[
\tilde{f}_{i + 1}
= \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}
\]

in terms of the \((c'_2, c'_1)\) specified above. With the help of Lemma 3.10 and 3.9 we can now go backwards to see that this implies \( \tilde{f}_{i + 1}(t_k(c_2) \otimes t_k(c_1)) = t_k(c'_2) \otimes t_k(c'_1) \) hence \( \tilde{f}_i(c_2 \otimes c_1) = c'_2 \otimes c'_1 \). Thus we have

\[
R(b'_1 \otimes b''_1) = R(f_i(b_1 \otimes b_2)) = f_i R(b_1 \otimes b_2) = f_i S(b_1 \otimes b_2) = c'_2 \otimes c'_1.
\]

It is very similar to verify that \( \tilde{c}_i(b_1 \otimes b_2) = 0 \) implies \( \tilde{c}_i S(b_1 \otimes b_2) = 0 \) for any \( i = 1, \ldots, n - 1 \).}

The above proof is also valid in \( A_n^{(1)} \) case [Hik1]. Viewed as \( U_q(\mathfrak{g}_{n-1}) \)-crystals with Kashiwara operators \( \tilde{f}_i, \tilde{c}_i \) \((1 \leq i \leq n - 1)\), both crystals \( B_i \otimes B_k \).
and $\tilde{B}_k \otimes \tilde{B}_l$ decompose into connected components. Note that (3.2) obviously tells that $U_q(\mathfrak{g}_{n-1})$-weights of $b_1 \otimes b_2$ and $S(b_1 \otimes b_2) = c_2 \otimes c_1$ are equal. Therefore apart from the separation into two solitons in the final state, Proposition 3.8 reduces the proof of Theorem 3.2 to showing $R(b_1 \otimes b_2) = S(b_1 \otimes b_2)$ only for the $U_q(\mathfrak{g}_{n-1})$-highest weight elements $b_1 \otimes b_2$. In particular, if the tensor product decomposition of $\tilde{B}_l \otimes \tilde{B}_k$ is multiplicity-free, it only remains to check the separation.

Part II. Here we concentrate on the $U_q'(C_{n}^{(1)})$ automaton and prove the separation and $R(b_1 \otimes b_2) = S(b_1 \otimes b_2)$ directly for $U_q(\mathfrak{g}_{n-1})$-highest weight elements $b_1 \otimes b_2$ ($n \geq 3$). In the notation $(x_1, \ldots, x_{n-1}, \bar{x}_{n-1}, \ldots, \bar{x}_1) \in \tilde{B}_k$ or $\tilde{B}_l$, they have the form

$$(f, 0, \ldots, 0) \otimes (d, c, 0, \ldots, 0, b) \in \tilde{B}_l \otimes \tilde{B}_k, \quad f \geq b + c. \quad (3.4)$$

In what follows we always assume $l > k$ and use the non-negative integers $e$ and $a$ defined by $l = f + 2e$ and $k = 2a + b + c + d$.

**Proposition 3.11** ([HKOT]). Under the isomorphism $R : \tilde{B}_l \otimes \tilde{B}_k \simeq \tilde{B}_k \otimes \tilde{B}_l$ of $U_q'(C_{n-1}^{(1)})$-crystals, the image of (3.4) is given by

$$(k - 2e, 0, \ldots, 0) \otimes (d + l - k - y, c, 0, \ldots, 0, b - y), \quad y = \min(l - k, (b - d)_+), \quad (3.5)$$

if $a \geq e$. In case $a < e$, it is given by

$$(k - 2a, 0, \ldots, 0) \otimes (d + l - k - e + a - z, c, 0, \ldots, 0, b + e - a - z), \quad z = \min(b - d + e - a, l - k - e + a),$$

if $l - k > e - a \geq d - b$, and

$$(k - 2e + 2w, 0, \ldots, 0) \otimes (d + l - k - w, c, 0, \ldots, 0, b + w), \quad w = \min(l - k, (2e - 2a - d + b)_+),$$

otherwise.

See Section 2.1 for the definition of the symbol $(x)_+$. Below we employ the notation

$$[s, t, u] = (x_1 = L - s - t - 2u, x_2 = s, \bar{x}_1 = t, \text{ other } x_i, \bar{x}_i = 0) \in B_L,$$

and always assume $L \gg s, t, u$. To save the space, the tensor product of $U_q'(C_{n}^{(1)})$-crystals such as

$$[s, t, u] \otimes 1 \otimes \left(2^{\otimes q} \otimes [s'', t'', u''] \otimes 3^{\otimes r} \otimes [s', t', u']^{\otimes j'} \right)$$

will simply be denoted by

$$[s, t, u]^{j} 1^{p} 2^{q} [s'', t'', u''] 3^{r} [s', t', u']^{j'}. \quad \text{From the results in } [\text{HKOT}], \text{ we further derive two lemmas given below.}$$

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Lemma 3.12. If $L \gg p, q, s, t, u$, we have

$$[0, 0, 0] \tilde{I}^p 2^q \simeq 1^{p+q} [g, 0, p],$$

$$[s, t, u] \simeq \begin{cases} 
1 [s, t + 1, u - 1] & u > 0, \\
\bar{1} [s, t - 1, 0] & u = 0, t > 0, \\
2 \{s - 1, 0, 0\} & u = t = 0, s > 0, \\
1 [0, 0, 0] & u = t = s = 0.
\end{cases}$$

Lemma 3.13. If $L \gg s, t, u, a, b, c, d$, we have

$$[s, t, u] \tilde{I}^a \tilde{I}^b 3^c 2^d 1^d \cdot$$

$$\simeq \tilde{I}^a 2^{s+t-a} 1^{u-t-s+2a+b+c+d} \tilde{I}^a + u 2^{t-a+b} 3^c 2^d \cdot \{0, 0, 0\}$$

if (I),

$$\simeq \tilde{I}^a 2^{t-a+2b+c} 1^{u-t+2a-b+d} \tilde{I}^a + u 2^{t-a+b} 3^c 2^{s-2b-c+d} \cdot \{0, 0, 0\}$$

if (II),

$$\simeq \tilde{I}^a 2^{b+c+d} 1^{u+t+2a+b+d} \tilde{I}^a + u 2^{t-a+b} 3^c 2^{d} \cdot \{0, 0, 0\}$$

if (III),

$$\simeq \tilde{I}^a 2^{b+c+d} 1^{a+u} \tilde{I}^a + u 2^{t-a+b} 3^c 2^{s-2b-c+d} \cdot \{0, 0, 0\}$$

if (IV),

$$\simeq \tilde{I}^a 2^{s-t+2a+b+c+d} 1^{a+u} 2^{b} 3^c 2^{d} \cdot \{0, 0, 0\}$$

if (V),

$$\simeq \tilde{I}^a 2^{u+2b+c-t} 1^{a+u-b+d} \tilde{I}^a + u 2^{b} 3^c 2^{s+t-a-2b-c+d} \cdot \{0, 0, 0\}$$

if (VI),

$$\simeq \tilde{I}^a 2^{a+b+c+d-t} 1^{a+u} \tilde{I}^a + u 2^{t-b-c-d} 2^a + 2b + c + d - s - t 3^c 2^{d} \cdot \{0, 0, 0\}$$

if (VII),

$$\simeq \tilde{I}^a 2^{a+b+c+d-t} 1^{a+u} \tilde{I}^a + u + a+b-d 2^{d} 3^c 2^{s+t-a-2b-c+d} \cdot \{0, 0, 0\}$$

if (VIII),

where the conditions (I)-(VIII) are given by

(I) : $t \geq a$, $s \leq 2b + c$, $s + t \leq a + b + c + d$,

(II) : $t \geq a$, $s > 2b + c$, $t \leq a - b + d$,

(III) : $t \geq a$, $s \leq 2b + c$, $s + t > a + b + c + d$,

(IV) : $t \geq a$, $s > 2b + c$, $t > a - b + d$,

(V) : $t < a$, $s + t \leq \min(a + 2b + c, a + b + c + d)$,

(VI) : $t < a$, $b \leq d$, $s + t > a + 2b + c$,

(VII) : $t < a$, $b > d$, $a + b + c + d < s + t \leq a + 2b + c$,

(VIII) : $t < a$, $b > d$, $s + t > a + 2b + c$.

On the boundaries, the formulae in the adjacent domains are equal. The domain (II) is absent if $b > d$. $\delta = a + b + c + d + t + 2a$ for (I, III), $a - b + d + s + t + 2a$ for (II, IV, VI, VIII), $2a + b + c + d + 2a$ for (V, VII).

Proof of Theorem 3.4. Let $b_1 \otimes b_2$ be the highest weight element (3.4) and $c_2 \otimes c_1 = R(b_1 \otimes b_2)$ be its image specified in Proposition 3.11. Two solitons labeled by $b_1$ and $b_2$ emerge in our automaton as the patterns $i_k(b_1) = 1^c 2^l 1^a$ and $i_k(b_2) = 1^{a+b} 3^c 2^d 1^1$, respectively. Thus $S(b_1 \otimes b_2)$ can be found by computing the image of

$$[0, 0, 0] \tilde{I}^c 2^l 1^c 2^m \tilde{I}^a 2^b 3^c 2^d 11 \cdots$$

(3.6)
under the isomorphism $B_k^{\otimes j} \otimes (B_1 \otimes B_1 \otimes \cdots) \simeq (B_1 \otimes B_1 \otimes \cdots) \otimes B_k^{\otimes j}$ for an interval $m$. This calculation can be done by using Lemma 3.12 and 3.13 only. It branches into numerous sectors depending on the seven parameters $a, b, c, d, f, e$ and $m$. (We use the dependent variables $l = f + 2e$ and $k = 2a + b + c + d$ simultaneously.) We have checked case by case that the collision always 'completes' within two time steps ($j = 2$) ending with the result $1^{m_1}t_k(c_2)1^{m_2}t_l(c_1)11 \ldots [0, 0, 0]_{j}$ for some $m_1$ and $m_2$. This establishes $R = S$ hence Theorem 3.2. Below we illustrate a typical such calculation in the sector

$$a \geq 2e, \quad b > d, \quad a - e \leq m \leq f + e - a - 2b - c.$$  \hfill (3.7)

In this case $c_2 \otimes c_1 = R(b_1 \otimes b_2)$ is determined by Proposition [3.11] as

$$c_1 = (l - k + b + 2d, c, 0, \ldots, 0, d) \in \tilde{B}_l, \quad c_2 = (k - 2e, 0, \ldots, 0) \in \tilde{B}_k.$$  

Thus we seek the patterns

$$u_k(c_2) = 1^{e} 2^{k-2e} 1^{e}, \quad u_l(c_1) = 1^{a+b-d} 2^{d} 3^{e} 2^{l-k-b+2d} 1^{a+b-d}$$  \hfill (3.8)

to show up after a collision and separate from each other. Starting from (3.6) with $j = 3$, one can rewrite it by means of Lemma 3.12 and (3.7) as

$$[0, 0, 0]_{j} 1^{\delta_1} 1^{e} \cdot 2^{m-e} [s_1, 0, 0] 1^{a} 2^{d} 3^{e} 2^{d} 11 \ldots ,$$

with $s_1 = f + e - m$ and $\delta_1 = f + 2e$. From (3.7) one can apply Lemma 3.13 (VIII) to transform this into

$$[0, 0, 0]_{j} 1^{\delta_1} 1^{e} 2^{s_2} 1^{a} 2^{d} 3^{e} 2^{d} 11 \ldots [0, 0, 0],$$

where $s_2 = m - e + a + b + c + d, a_2 = a + b - d$ and $d_2 = s_1 - a - 2b - c + d$. This completes the calculation of the one time step in the automaton. The second time step can be worked out similarly. Lemma 3.12 with (3.7) transforms the above into

$$[0, 0, 0]_{j} 1^{\delta_2} 1^{e} 2^{s_2} 1^{a} 2^{d} 3^{e} 2^{d} 11 \ldots [0, 0, 0],$$

where $s_2 = s_2' - a + 2e = m + b + c + d + e$ and $\delta_2 = \delta_1 + 2e + s_2'$. This time Lemma 3.13 (VI) is applied, leading to

$$[0, 0, 0]_{j} 1^{\delta_2} 1^{e} 2^{k-2e} 1^{e} 2^{d} 3^{e} 2^{d} 11 \ldots [0, 0, 0]^2,$$  \hfill (3.9)

where $\delta_3 = a_2 - d + d_2 = f + e - m - b - c - d$. This already contains the patterns (3.8). Let us calculate one more time step to confirm that they are separating hereafter. Since $\delta_3 > 2e$, (3.9) is isomorphic to

$$1^{\delta_3} 1^{e} \cdot [k - 2e, 0, 0] 1^{\delta_3 - 2e} 1^{a+b-d} 2^{d} 3^{e} 2^{d} 11 \ldots [0, 0, 0]^2,$$  \hfill (3.10)

where $\delta_4 = \delta_2 + k$. If $k \leq \delta_3$ this becomes

$$1^{\delta_4} 1^{e} 2^{k-2e} 1^{\delta_4 + l-k} 1^{a+b-d} 2^{d} 3^{e} 2^{d} 11 \ldots [0, 0, 0]^3.$$  \hfill (3.11)
Compared with (3.9), the distance of the two patterns here has increased by \( l - k \) in accordance with the velocity of solitons in Theorem 3.1. In the other case \( k > \delta_3 \) one rewrites (3.10) as

\[
1^{\delta_1} e^{2\delta_3 - 2c[k - \delta_3, 0, 0]} 1^{a+b-d\delta_3} e^{-2d-k-b+2d} 11\ldots[0, 0, 0]^2,
\]

for which Lemma 3.13 (V) can be applied. The result coincides with (3.11).

\( \square \)

4 Discussion

In this paper we have proposed a class of cellular automata associated with \( U'_q(\hat{\mathfrak{g}_n}) \). They are essentially solvable vertex models at \( q = 0 \) in the vicinity of the ferromagnetic vacuum. They exhibit soliton behavior stated in Theorem 3.1, 3.2 and Conjecture 3.3.

Behavior of tensor products of crystals in the vicinity of ferromagnetic vacuum has not been investigated in detail so far. To understand it better will be a key to prove Conjecture 3.3 and to clarify the relevance to the subalgebra \( \hat{\mathfrak{g}}_{n-1} \), which has also been observed in the RSOS models in the ferromagnetic-like 'regime II' \([BR, K]\).

The automata considered in this paper are associated with an \( l \to \infty \) limit \( R' : B_2 \otimes B_1 \to B_1 \otimes B_2 \) of the ordinary combinatorial \( R \) matrices \( R : B_1 \otimes B_1 \simeq B_1 \otimes B_1 \). To vary \( B_1 \) from site to site and to replace \( R' \) with the corresponding combinatorial \( R \) matrices is a natural generalization. See \([HHIKTT]\).

We close by raising a few questions. What kind of soliton equations can possibly be related to the \( U'_q(\hat{\mathfrak{g}_n}) \) automata in general? Is it possible to derive them conceptually from the associated vertex models, bypassing the route; vertex models \( q \to 0 \) \( \to \) soliton equations? Is it efficient to seek a piecewise analytic form of the combinatorial \( R \) matrices \( B_1 \otimes B_k \simeq B_k \otimes B_1 \) by ultra-discretizing the relevant solitons solutions? Is it possible to extract informations on the associated energy function from the automata?

A The map \( R' : B_2 \otimes B_1 \to B_1 \otimes B_2 \)

Let \( g_l : B_l \to B_2 \) be the embedding of \( B_l \) as a set as mentioned in Section 3. By the definition, for any \( u \in B_2 \) there exists the unique element \( u_l \in B_l \) such that \( g_l(u_l) = u \) for each \( l \) which is sufficiently large. Consider the composition:

\[
B_2 \times B_1 \xrightarrow{g_l^{-1} \otimes id} B_1 \otimes B_1 \xrightarrow{\bar{e}_l, \bar{f}_l} (B_1 \otimes B_1) \sqcup \{0\} \xrightarrow{g_l \times id} (B_2 \times B_1) \sqcup \{0\}
\]

where of course either \( u'_l = u_l \) or \( b' = b \), and we interpret \( (g_l \times id)(0) = 0 \). For all the \( B_l \) considered in this paper it is easy to see that this is independent of \( l \) if \( l \) is sufficiently large. We let \( B_2 \otimes B_1 \) denote the set \( B_2 \times B_1 \) equipped with the actions \( \bar{e}'_l, \bar{f}'_l \) determined as above with sufficiently large \( l \). We shall simply
write it as $\tilde{e}'_i, \tilde{f}'_i : B_2 \otimes B_1 \to (B_2 \otimes B_1) \sqcup \{0\}$. Similarly we define $B_1 \otimes B_2$ and $\tilde{e}'_i, \tilde{f}'_i : B_1 \otimes B_2 \to (B_1 \otimes B_2) \sqcup \{0\}$.

Given $u \in B_2$ and $b \in B_1$, let $c = c(l, u, b) \in B_1$ and $v = v(l, u, b) \in B_1$ be the elements determined by

$$u_i \otimes b \simeq c \otimes v$$

under the combinatorial $R$ matrix $R : B_l \otimes B_1 \simeq B_1 \otimes B_l$. Then we have

**Conjecture A.1.** For any fixed $u \in B_2$ and $b \in B_1$, the elements $c(l, u, b) \in B_1$ and $g_i(v(l, u, b)) \in B_1$ are independent of $l$ for $l$ sufficiently large.

Assuming the conjecture we define the map $R' : B_2 \otimes B_1 \to B_1 \otimes B_2$ to be the composition:

$$R' : B_2 \otimes B_1 \xrightarrow{g_1^{-1} \otimes \text{id}} B_1 \otimes B_1 \xrightarrow{R} B_1 \otimes B_1 \xrightarrow{\text{id} \otimes g_l} B_1 \otimes B_2$$

with $l$ sufficiently large. Obviously this is invertible. Moreover it commutes with $\tilde{e}'_i$ and $\tilde{f}'_i$, although this fact is not used in the main text. For $\tilde{e}'_i$ this can be seen from the commutative diagram ($\tilde{f}'_i$ case is completely parallel.)

$$\begin{array}{ccc}
\tilde{e}'_i & \to & \tilde{e}'_i \\
\downarrow & & \downarrow \\
B_2 \otimes B_1 & \xrightarrow{g_1^{-1} \otimes \text{id}} & B_1 \otimes B_1 \\
\end{array} \quad \begin{array}{ccc}
\tilde{f}'_i & \to & \tilde{f}'_i \\
\downarrow & & \downarrow \\
B_2 \otimes B_1 & \xrightarrow{g_1^{-1} \otimes \text{id}} & B_1 \otimes B_1 \\
\end{array} \quad \begin{array}{ccc}
\tilde{e}'_i & \to & \tilde{e}'_i \\
\downarrow & & \downarrow \\
B_2 \otimes B_1 & \xrightarrow{g_1^{-1} \otimes \text{id}} & B_1 \otimes B_1 \\
\end{array}$$

where the left and the right squares are the definitions of $\tilde{e}'_i$ (for elements not annihilated by $\tilde{e}'_i$). The map $R'$ indeed has the property (I) in Section 2. The property (II) is also valid for $A_2^{(2)}$ and $C_n^{(1)}$. We conjecture it for all the cases considered in this paper.

**Note:** While writing the paper the authors learned from [FOY] that the energy of combinatorial $R$ matrices for $V_q(A_n^{(1)})$ is encoded in the phase shift of soliton scattering in the automaton. We thank Yasuhiko Yamada for communicating their result.

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