OPTIMAL PRODUCTION SCHEDULE IN A SINGLE-SUPPLIER MULTI-MANUFACTURER SUPPLY CHAIN INVOLVING TIME DELAYS IN BOTH LEVELS

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ABSTRACT. This paper considers an optimal production scheduling problem in a single-supplier-multi-manufacturer supply chain involving production and delivery time-delays, where the time-delays for the supplier and the manufacturers can have different values. The objective of both levels is to find an optimal production schedule so that their production rates and their inventory levels are close to the ideal values as much as possible in the whole planning horizon. Each manufacturer’s problem, which involves one time-delayed argument, can be solved analytically by using the necessary condition of optimality. To tackle the supplier’s problem involving $n+1$ different time-delayed arguments (where $n$ is the number of manufacturers) by the above approach, we need to introduce a model transformation technique which converts the original system of combined algebraic/differential equations with $n+1$ time-delayed arguments into a sum of $n$ sub-systems, each of which consists of only two time-delayed arguments. Thus, the supplier’s problem can also be solved analytically. Numerical examples consisting of a single supplier and four manufacturers are solved to provide insight of the optimal strategies of both levels.

1. Introduction. Many researchers have devoted effort in modeling the supply chain. One of the earliest researchers, Simon, applied servomechanism theory to study the production control in [17]. The development of Industrial Dynamics was influenced to a great extent by the research work pursued in [7]. By modeling the dynamical systems by sets of differential equations, he set up a highly complicated nonlinear model for simulation purpose. Since then, many researchers explored other mathematical modeling techniques to formulate various kinds of supply chain problems. Beginning in the 1970s, some researchers applied optimal control theory to formulate supply chain problems. For example, Bradshaw and Daintith in [2], Porter and Bradshaw in [12] and Porter and Taylor in [13] studied modal control.
theory involving piecewise constant control policies. The modal control theory concepts in [2], [12] and [13] were further investigated by Axskater in [1]. More recently, Kogan, Leu, and Perkins in [9] used the maximum principle to model continuous production scheduling problem. They derived algorithms for multiple parallel machines with variable planning horizon. Disney and Towill in [6] and Dejonckheere et al in [5] used control theory to analyze the bullwhip effect. Riddalls and Bennett in [14] studied the batch size effect on demand amplification and Riddalls and Bennett in [15] studied the effects of trust in a multi-level supply chain system. These two papers examined the conditions of stability in supply chain models with time-delayed differential equations. Wong et. al. in [20] and Chan et. in [3] further extended the model formulations of [14] and [15] to obtain the vendor’s optimal production policy for time-delayed problems. Miranbeigi et. al. in [10] and [11] applied optimal controls to maximize customer satisfaction and minimize supply chain operating cost in a constrained inventory-level supply chain. Zaher and Zaki in [25] described how to control the inventory production system with Weibull distributed deterioration items.

In this paper, we consider the supply chain system in a single-supplier-multiple-manufacturer with time-delayed arguments in the production processes. These time-delayed arguments correspond to the production and delivery delays once the demand order is initiated. The objective of both the supplier and the manufacturers is to find an optimal production schedule so that their production rates and their inventory levels are close to the ideal values as much as possible in the whole planning horizon. We assume that all the time-delayed arguments corresponding to the production processes of the manufacturers and the single supplier can have different values. These time-delayed arguments greatly complicated the structure of the problem. Thus, the problem cannot be solved by standard optimal control software, such as MISER in [23].

Each manufacturer’s problem, which involves only one time-delayed argument, can be solved analytically by using the necessary condition of optimality for time-delayed control problems. However, the supplier’s problem cannot be solved easily by using the necessary condition of optimality because the method involves solving a system of combined algebraic/differential equations (involving state and co-state variables) with \( n + 1 \) different time-delay arguments, where \( n \) is the number of manufacturers. To overcome the above difficulty, we first employ a model transformation technique which involves the partitions of both the optimal production rate and the optimal inventory level of the supplier into \( n \) portions, so that each portion can be regarded either as the artificial production rate or the artificial inventory level reserved for a particular manufacturer. In this way, the original system of combined algebraic/differential equations with \( n + 1 \) time-delayed arguments can be converted into a sum of \( n \) sub-systems, each of which consists of only two time-delayed arguments and hence can also be solved analytically. Thus, both the optimal inventory level and the optimal production level for the two echelons can be analytically determined. Numerical examples consisting of a single supplier and four manufacturers are solved to provide insight of the optimal strategies of the supplier and the manufacturers.

To solve the \( i \)th sub-system (\( i = 1, \ldots, n \)) described in the previous paragraph analytically, we need to partition the common planning horizon of the supplier and the \( i \)th manufacturer, \([0,1]\), into three sub-intervals, namely, \([0, h_{sup}]\), \([h_{sup}, 1 - h_i]\) and \([1 - h_i, 1]\), where \( h_{sup} \) and \( h_i \) are the time-delayed arguments of the supplier.
and the \( i^{th} \) manufacturer, respectively. In this way, the co-state equations (i.e., the differential equations governed by the co-state variable \( \lambda^*_i(t) \)) can be reduced into a form not involving any time-delayed argument in each sub-interval. Then, the general solution of the co-state variable in each sub-interval involving two arbitrary constants can be obtained first. These arbitrary constants can be easily found by using the facts that both the co-state variable \( \lambda^*_i(t) \) and its derivative \( \dot{\lambda}^*_i(t) \) need to be continuous at the two connecting points \( h_{sup} \) and \( 1 - h_i \), and also \( \lambda^*_i(1) = 0 \) (because there is no final state cost in the objective function of our problem) and \( \dot{\lambda}^*_i(0) \) equal to a given quantity. The continuities of \( \lambda^*_i(t) \) and \( \dot{\lambda}^*_i(t) \) for all \( t \in [0, 1] \) arise from the fact that the state variable (i.e. the inventory level of the supplier) needs to be continuous for all \( t \in [0, 1] \). After we have solved the co-state equations, we can easily solve the state equations. The above approach of solving time-delayed system analytically has seldom been discussed in the optimal control literature.

The method of using the necessary condition of optimality for solving optimal control problem with one single time-delayed argument has been discussed by Kaji and Wong in [8], Teo et.al. in [18] and [19], Wong et.al. in [21] and [22]. More recently, based on the idea of Miser [23], Yu et. al. in [24] developed a hybrid time-scaling transformation method which can transform a constrained optimal control problem with multiple time-delayed arguments into a canonical form solvable by standard gradient-based optimization software, such as NLPPQ in [4] and [16]. On the other hand, we employ a hybrid technique in this paper to develop a method which can analytically solve unconstrained optimal control problems with multiple time-delayed arguments.

The paper is organized as follows. In Section 2, we formulate the manufacturer’s problem. In Section 3, we derive a formula for the manufacturer’s production rate. In Section 4, we formulate the supplier’s problem. In Section 5, we derive a formula for the supplier’s optimal production rate. In Section 6, we derive formulae for the optimal inventory levels of the manufacturer and the supplier. In Section 7, three numerical examples consisting of a single supplier and four manufacturers are solved to provide insight about the optimal strategies of the supplier and the manufacturers. In Section 8, we provide some concluding remarks and suggestions for further studies.

2. Formulation of the manufacturer’s problem. Suppose that there are two levels. The higher level is the supplier and the lower level consists of \( n \) manufacturers. Let \( P_i(t) \), \( I_i(t) \) and \( D_i(t) \), \( i = 1, \cdots, n \), be, respectively, the rate of production, the inventory level and the demand rate of manufacturer \( i \) at any time \( t \). Let \( \hat{P}_i \), \( \hat{I}_i \) and \( \hat{D}_i \) be, respectively, the ideal production rate, the ideal inventory level and the saturated demand rate of manufacturer \( i \). Let \( I_{i,0} \) and \( D_{i,0} \) be, respectively, the initial inventory level and the initial demand rate of manufacturer \( i \). Let \( c_{P,i} \) (respectively \( c_{I,i} \)) be the unit penalty cost associated with the deviation of the production rate (respectively, the inventory level) of manufacturer \( i \) away from its ideal value. Let \( h_i \) be the time-delayed argument for the production and delivery processes of manufacturer \( i \). Suppose the planning horizon of each of the manufacturers is \([0, 1]\). Then the problem facing the manufacturer \( i \) can be stated as follows:

\[
\text{Min } J_i = \int_0^1 \left[ c_{P,i}(P_i(t) - \hat{P}_i)^2 + c_{I,i}(I_i(t) - \hat{I}_i)^2 \right] dt
\]

subject to
where \( r_{D,i} \) is a given constant.

3. **The optimal production schedule of the manufacturer.** In this section, we shall find an explicit formula for the optimal production rate of manufacturer \( i \). For the sake of simplicity, we shall first delete the subscript \( i \) in our description.

Let the Hamiltonian \( H : R^{5} \to R \) be defined by

\[
H(I(t), \lambda(t), \lambda(t + h), P(t), P(t - h)) = \lambda(t)[P(t - h) - D(t)] + \lambda(t + h)P(t) + c_{p}(P(t) - \hat{P})^{2} + c_{I}(I(t) - \hat{I})^{2}. \tag{7}
\]

Let \( P^{*}(t) \) and \( I^{*}(t) \) be, respectively, the optimal production rate and the optimal inventory level of the manufacturer. Let \( \lambda^{*}(t) \) be the co-state variable corresponding to the above optimal solutions. To get the optimal production rate, we set

\[
\frac{\partial H}{\partial P(t)} \bigg|_{P^{*}(t)} = 0, \quad t \geq 0.
\]

\[
\Rightarrow 2c_{p}(P^{*}(t) - \hat{P}) + \lambda^{*}(t + h) = 0, \quad t \geq 0. \tag{8}
\]

\[
\Rightarrow P^{*}(t) = \hat{P} - \frac{\lambda^{*}(t + h)}{2c_{p}}, \quad t \geq 0,
\]

where

\[
\lambda^{*}(t) = -\frac{\partial H}{\partial I^{*}(t)} = -2c_{I}(I^{*}(t) - \hat{I}), \quad t \in [0, 1], \tag{9}
\]

\[
\lambda^{*}(1) = 0, \tag{10}
\]

\[
\lambda^{*}(t) = 0, \quad t > 1. \tag{11}
\]

To obtain an explicit formula for \( P^{*}(t), t \in [0, 1] \), we need to solve for \( \lambda^{*}(t), t \in [h, 1 + h] \) using the following procedure:

**Step 1.** Find \( \lim_{t \to h^{-}} \lambda^{*}(t) \). (The purpose of finding \( \lim_{t \to h^{-}} \lambda^{*}(t) \) is to obtain an initial condition for solving \( \lambda^{*}(t) \) in the time interval \([h, 1] \).

(i) Solve for \( I^{*}(t), t \in [0, h) \) by using (2), (6), (3), (5) and (4). Hence, we obtain \( \lim_{t \to h^{-}} I^{*}(t) \).

(ii) Obtain \( \lim_{t \to h^{-}} \lambda^{*}(t) \) from (9) and the value of \( \lim_{t \to h^{-}} I^{*}(t) \) just computed.

**Step 2.** Solve for \( \lambda^{*}(t), t \in [h, 1] \).

(i) Obtain a formula for \( \lambda^{*}(t), t \in [h, 1] \) (as a function of \( \lambda^{*}(t) \) and \( t \) with constant coefficients) by differentiating (9) and using (2), (8), (3) and (5).

(ii) Obtain the boundary condition \( \lambda^{*}(1) \) from (10) and the boundary condition \( \lambda^{*}(h) \) from the fact that \( \lambda^{*}(h) = \lim_{t \to h^{-}} \lambda^{*}(t) \), due to the continuity of \( \lambda^{*}(t) \) for all \( t \in [0, 1] \).
(iii) Hence solve for \( \lambda^*(t), t \in [h,1] \).

**Step 3.** Obtain the formula for \( \lambda^*(t), t \in [h,1+h] \)

Extend the formula of \( \lambda^*(t), t \in [h,1] \) to \( \lambda^*(t), t \in [h,1+h] \) by using (11).

A more detailed explanation of the method of obtaining \( \lambda^*(t), t \in [h,1+h] \) is as follows:

**Step 1.** From (2), (6), (3), (5) and (4), we have

\[
\dot{I}^*(t) = -D(t), \quad t \in [0,h).
\]

\[
\Rightarrow \dot{I}^*(t) = (D_0 - \dot{D})e^{-\rho dt} - \dot{D}, \quad t \in [0,h).
\]

\[
\Rightarrow I^*(t) = \frac{(\dot{D} - D_0)(1 - e^{-\rho dt})}{r_D} - \dot{D}t + I_0, \quad t \in [0,h).
\]

Thus, from (9) and (12), we get

\[
\lim_{t \to h^-} \dot{\lambda}^*(t) = -2c_I \left( \lim_{t \to h^-} I^*(t) - \dot{I} \right).
\]

\[
\Rightarrow \lim_{t \to h^-} \dot{\lambda}^*(t) = k_1,
\]

where

\[
k_1 = \frac{2c_I(D_0 - \dot{D})(1 - e^{-\rho Dh})}{r_D} + 2c_I(\dot{D}h + \dot{I} - I_0).
\]

**Step 2.** Differentiating (9) and using (2), (8), (3) and (5), we have

\[
\dot{\lambda}^*(t) = -2c_I \dot{I}^*(t), \quad t \in [h,1].
\]

\[
\Rightarrow \dot{\lambda}^*(t) = -2c_I \left[ (\dot{P} - \frac{\lambda^*(t)}{2c_P}) - D(t) \right], \quad t \in [h,1].
\]

\[
\Rightarrow \dot{\lambda}^*(t) = \frac{c_I}{c_P} \lambda^*(t) + 2c_I(D_0 - \dot{D})e^{-\rho dt} - 2c_I(\dot{P} - \dot{D}), \quad t \in [h,1].
\]

From (10), (13) and the continuity of \( \dot{\lambda}^*(t), t \in [0,1], \) we have

\[
\lambda^*(1) = 0
\]

and

\[
\dot{\lambda}^*(h) = k_1.
\]

Equations (15)-(17) constitute a linear second order boundary value problem with constant coefficients which can be easily solved to obtain \( \lambda^*(t), t \in [h,1] \), as follows:

\[
\lambda^*(t) = A e^{\sqrt{\gamma} t} + B e^{-\sqrt{\gamma} t} + \bar{W}e^{-\rho t} + Y, \quad t \in [h,1],
\]

where

\[
\bar{W} = \frac{2c_I c_P (D_0 - \dot{D})}{c_P r_D^2 - c_I},
\]

\[
\bar{Y} = 2c_P (\dot{P} - \dot{D}),
\]

\( \bar{A} \) and \( \bar{B} \) are solutions of the following simultaneous equations:

\[
\bar{A} e^{\sqrt{\gamma} h} - \bar{B} e^{-\sqrt{\gamma} h} = \sqrt{\frac{c_P}{c_I}} (r_D \bar{W} e^{-\rho Dh} + k_1),
\]

\[
\bar{A} e^{\sqrt{\gamma} h} - \bar{B} e^{-\sqrt{\gamma} h} = -(\bar{W} e^{-\rho Dh} + \bar{Y}).
\]
Step 3. From (18) and (11), we have
\[
\lambda^*(t) = \begin{cases} 
\bar{A}e^{\sqrt{c_I c_P} t} + \bar{B}e^{-\sqrt{c_I c_P} t} + \bar{W}e^{-r_D t} + \bar{Y}, & t \in [h, 1], \\
0, & t \in [1, 1 + h].
\end{cases}
\]
(23)
(Note that \(\lambda^*(t)\) is continuous at \(t = 1\).)

From (8) and (23), we have
\[
P^*(t) = \begin{cases} 
A^* e^{\sqrt{c_{I,i} c_{P,i}} t} + B^* e^{-\sqrt{c_{I,i} c_{P,i}} t} + W^* e^{-r_{D,i} t} + Y^*, & t \in [0, 1 - h_i], \\
\hat{P}, & t \in [1 - h_i, 1],
\end{cases}
\]
where
\[
A^* = -\frac{\bar{A} e^{\sqrt{c_{I} c_{P}}} h}{2c_P},
\]
(25)
\[
B^* = -\frac{\bar{B} e^{\sqrt{c_{I} c_{P}}} h}{2c_P},
\]
(26)
\[
W^* = -\frac{\bar{B} e^{\sqrt{c_{I} c_{P}}} h}{2c_P}
\]
(27)
and
\[
Y^* = \hat{P} - \frac{\bar{Y}}{2c_P}.
\]
(28)
(Note that \(P^*(t)\) is continuous at \(t = 1 - h_i\).)

Recovering the optimal production rate for manufacturer \(i\), we have
\[
P^*_i(t) = \begin{cases} 
A^*_i e^{\sqrt{c_{I,i} c_{P,i}} t} + B^*_i e^{-\sqrt{c_{I,i} c_{P,i}} t} + W^*_i e^{-r_{D,i} t} + Y^*_i, & t \in [0, 1 - h_i], \\
\hat{P}_i, & t \in [1 - h_i, 1],
\end{cases}
\]
where \(A^*_i, B^*_i, W^*_i, Y^*_i\) are obtained from \(A^*, B^*, W^*, Y^*\) by replacing \(\hat{P}, \hat{I}, \hat{D}, D_0, c_P, c_I, r_D\) and \(h\) by \(\hat{P}_i, \hat{I}_i, \hat{D}_i, D_{i,0}, c_{P,i}, c_{I,i}, r_{D,i}\) and \(h_i\) respectively in the above discussion.

4. Formulation of the supplier’s problem. Let \(P_{\text{sup}}(t)\), \(I_{\text{sup}}(t)\) and \(D_{\text{sup}}(t)\) be the production rate, the inventory level and the demand rate at any time \(t\) of the supplier. Then, we have
\[
D_{\text{sup}}(t) = \sum_{i=1}^{n} \alpha P^*_i(t),
\]
(30)
where
\[
\alpha = \frac{\text{amount of raw material supplied to the manufacturer}}{\text{amount of new product manufactured from the raw material}}
\]
and \(P^*_i(t)\) is the optimal production rate of manufacturer \(i\) as defined in (29).

Let \(\hat{P}_{\text{sup}}\) and \(\hat{I}_{\text{sup}}\) be, respectively, the ideal production rate and the ideal inventory level of the upper level. Let \(I_{\text{sup},0}\) be the initial inventory level of the supplier. Let \(c_{P,\text{sup}}\) (respectively, \(c_{I,\text{sup}}\)) be the penalty cost associated with the deviation of the production rate (respectively, the inventory level) of the supplier. Let \(h_{\text{sup}}\) be the time-delayed argument for the production and delivery processes of the supplier. We assume that
\[
h_{\text{sup}} + h_i < 1, \quad i = 1, \ldots, n,
\]
(31)
where $h_i$ is the time-delayed argument of manufacturer $i$ as defined in Section 2. Suppose the planning horizon of the supplier is also $[0,1]$. Then the problem of the supplier can be stated as follows:

$$\min J = \int_0^1 \left[ c_{P,\text{sup}}(P_{\text{sup}}(t) - \dot{P}_{\text{sup}})^2 + c_{I,\text{sup}}(I_{\text{sup}}(t) - \dot{I}_{\text{sup}})^2 \right] dt$$  \hspace{1cm} (32)

subject to

$$\dot{I}_{\text{sup}}(t) = P_{\text{sup}}(t - h_{\text{sup}}) - \alpha \sum_{i=1}^{n} P_i^*(t), \quad t \geq 0,$$  \hspace{1cm} (33)

$$I_{\text{sup}}(0) = I_{\text{sup},0},$$  \hspace{1cm} (34)

$$P_{\text{sup}}(t) = 0, \quad t < 0,$$  \hspace{1cm} (35)

and $P_i^*(t)$ is as defined in (29).

5. The optimal production schedule of the supplier. In this section, we shall find an explicit formula for the optimal production rate of the supplier. Let the Hamiltonian $H_{\text{sup}} : R^5 \rightarrow R$ be defined by

$$H_{\text{sup}}(I(t), \lambda_{\text{sup}}(t), \lambda_{\text{sup}}(t + h), P_{\text{sup}}(t), P_{\text{sup}}(t - h)) = \lambda_{\text{sup}}(t) \left[ P_{\text{sup}}(t - h_{\text{sup}}) - \alpha \sum_{i=1}^{n} P_i^*(t) \right] + \lambda_{\text{sup}}(t + h)P_{\text{sup}}(t) + c_{P,\text{sup}}(P_{\text{sup}}(t) - \dot{P}_{\text{sup}})^2 + c_{I,\text{sup}}(I_{\text{sup}}(t) - \dot{I}_{\text{sup}})^2.$$  \hspace{1cm} (36)

Let $P_{\text{sup}}^*$ and $P_{\text{sup}}^*(t)$ be the optimal production rate and the optimal inventory level of the supplier. Let $\lambda_{\text{sup}}^*(t)$ be the co-state variable corresponding to the above optimal solutions. To find the optimal production rate, we set

$$\left. \frac{\partial H_{\text{sup}}}{\partial P_{\text{sup}}(t)} \right|_{P_{\text{sup}}^*(t)} = 0, \quad t \geq 0.$$  \hspace{1cm} (37)

$$\Rightarrow P_{\text{sup}}^*(t) = \dot{P}_{\text{sup}} - \frac{\lambda_{\text{sup}}^*(t + h_{\text{sup}})}{2c_{P,\text{sup}}}, \quad t \geq 0,$$  \hspace{1cm} (38)

where

$$\dot{\lambda}_{\text{sup}}^*(t) = -\frac{\partial H}{\partial \lambda_{\text{sup}}^*(t)} = -2c_{I,\text{sup}}(I_{\text{sup}}^*(t) - \dot{I}_{\text{sup}}), \quad t \in [0,1],$$  \hspace{1cm} (39)

$$\lambda_{\text{sup}}^*(1) = 0,$$  \hspace{1cm} (40)

$$\lambda_{\text{sup}}^*(t) = 0, \quad t > 1.$$  \hspace{1cm} (41)

To find $P_{\text{sup}}^*(t)$ from (37)-(40) and (33)-(35) is a very difficult task, because the above system consists of both algebraic and differential equations with $(n+1)$ time-delayed arguments, namely, $h_{\text{sup}}$ and $h_1, \ldots, h_n$, where $h_1, \ldots, h_n$ come from the terms $P_i^*(t), \ldots, P_n^*(t)$ in (29), respectively.

Thus, in order to find $P_{\text{sup}}^*(t)$ more easily, we need to perform a simple model transformation on the above system. This model transformation technique involves the partitions of both the supplier’s optimal production rate $P_{\text{sup}}^*(t)$ and the supplier’s optimal inventory level $I_{\text{sup}}^*(t)$ into $n$ portions, so that each portion can be regarded as either the artificial production rate or the artificial inventory level reserved for a particular manufacturer. Let these two partitions be $\{P_{\text{sup}}^*1(t), \ldots, P_{\text{sup}}^*n(t)\}$ and $\{I_{\text{sup}}^1(t), \ldots, I_{\text{sup}}^n(t)\}$, respectively. (Note that the artificial production rate $P_{\text{sup}}^0(t)$
(respectively, the artificial inventory level \(I_{\text{sup}}^*(t)\)) reserved for manufacturer \(i\) is not the same as the true optimal production rate \(P^*_i(t)\) (respectively, the true optimal inventory level \(I^*_i(t)\)) of manufacturer \(i\). Therefore, we use superscript \(i\) to denote artificial quantities reserved for manufacturer \(i\), but use subscript \(i\) to denote true optimal quantities of manufacturer \(i\). In this way, we can easily distinguish between artificial quantities and true optimal quantities.)

As a consequence, we also need to divide the co-state variable \(\lambda_{\text{sup}}^*(t)\) into \(n\) components, which are denoted by \(\lambda_{\text{sup}}^{*,1}(t), \ldots, \lambda_{\text{sup}}^{*,n}(t)\), respectively. In this way, we can obtain \(n\) sub-problems, each of which involves solving differential equations with two time-delayed arguments only and hence can be solved analytically more easily. The detail is as follows:

From (33)-(34), (38)-(40), (37) and (35), it is clear that \(P_{\text{sup}}^{*,i}(t), I_{\text{sup}}^{*,i}(t)\) and \(\lambda_{\text{sup}}^{*,i}(t)\) \((i = 1, \ldots, n)\) are solutions of the following algebraic/differential equations:

\[
\dot{I}_{\text{sup}}^{*,i}(t) = \frac{P_{\text{sup}}^{*,i}(t - h_{\text{sup}}) - \alpha P^*_i}{n}, \quad t \in [0, 1],
\]

\[
I_{\text{sup}}^{*,i}(0) = I_{\text{sup},0}, \quad (42)
\]

\[
\dot{\lambda}_{\text{sup}}^{*,i}(t) = -2c_{I,\text{sup}} \left( I_{\text{sup}}^{*,i}(t) - \frac{\dot{I}_{\text{sup}}^{*,i}}{n} \right), \quad t \in [0, 1],
\]

\[
\lambda_{\text{sup}}^{*,i}(1) = 0, \quad (44)
\]

\[
\lambda_{\text{sup}}^{*,i}(t) = 0, \quad t > 1, \quad (45)
\]

\[
P_{\text{sup}}^{*,i}(t) = \frac{\dot{P}_{\text{sup}}}{n} - \frac{\lambda_{\text{sup}}^{*,i}(t + h_{\text{sup}})}{2c_{P,\text{sup}}}, \quad t \in [0, 1],
\]

\[
P_{\text{sup}}^{*,i}(t) = 0, \quad t < 0. \quad (47)
\]

For each \(i\), the system of algebraic/differential equations (41)-(47), which consists of only two time-delayed arguments, can be solved much easily than the system (37)-(40) and (33)-(35), which consists of \((n + 1)\) time-delayed arguments. After having solved the system (41)-(47) for \(i = 1, \ldots, n\), we can easily obtain the optimal production rate \(P_{\text{sup}}^*(t)\) and the optimal inventory level \(I_{\text{sup}}^*(t)\) of the supplier as follows:

\[
P_{\text{sup}}^*(t) = \sum_{i=1}^{n} P_{\text{sup}}^{*,i}(t) \quad (48)
\]

and

\[
I_{\text{sup}}^*(t) = \sum_{i=1}^{n} I_{\text{sup}}^{*,i}(t). \quad (49)
\]

Similar to the discussion for the manufacturer’s problem, in order to obtain an explicit formula for \(P_{\text{sup}}^{*,i}(t)\), \(t \in [0, 1]\) from (41)-(47), we first need to solve for \(\lambda_{\text{sup}}^{*,i}(t), t \in [h_{\text{sup}}, 1 + h_{\text{sup}}]\) using the following procedure.

**Step 1.** Find \(\lim \dot{\lambda}_{\text{sup}}^{*,i}(t)\). (The purpose of finding \(\lim \dot{\lambda}_{\text{sup}}^{*,i}(t)\) is to obtain an initial condition for solving \(\lambda_{\text{sup}}^{*,i}(t)\) in the time interval \([h_{\text{sup}}, 1 + h_{i}]\). Note that \(h_{\text{sup}} < 1 - h_i\) due to (31).)

1. Solve for \(I_{\text{sup}}^{*,i}(t), t \in [0, h_{\text{sup}}]\) by using (41), (47), the first part of (29) and (42).
Hence we obtain \( \lim_{t \to h_{\text{sup}}} I_{\text{sup}}^{*,i}(t) \).

(ii) Obtain \( \lim_{t \to h_{\text{sup}}} \dot{\lambda}_{\text{sup}}^{*,i}(t) \) from (43) and the value of \( \lim_{t \to h_{\text{sup}}} I_{\text{sup}}^{*,i}(t) \) just computed.

**Step 2.** Obtain a general solution for \( \lambda_{\text{sup}}^{*,i}(t), \ t \in [h_{\text{sup}}, 1 - h_i] \).

(i) Obtain a formula for \( \dot{\lambda}_{\text{sup}}^{*,i}(t), \ t \in [h_{\text{sup}}, 1 - h_i] \) (as a function of \( \lambda_{\text{sup}}^{*,i}(t) \) and \( t \) with constant coefficients) by differentiating (43) and using (41), (46) and the first part of (29).

(ii) Hence obtain a general solution for \( \lambda_{\text{sup}}^{*,i}(t), \ t \in [h_{\text{sup}}, 1 - h_i] \), involving two arbitrary constants, denoted by \( \bar{k}_1^i \) and \( \bar{k}_2^i \), respectively.

**Step 3.** Obtain a general solution for \( \lambda_{\text{sup}}^{*,i}(t), \ t \in [1 - h_i, 1] \).

(i) Obtain a formula for \( \dot{\lambda}_{\text{sup}}^{*,i}(t), \ t \in [1 - h_i, 1] \) (as a function of \( \lambda_{\text{sup}}^{*,i}(t) \) and \( t \) with constant coefficients) by differentiating (43) and using (41), (46) and the second part of (29).

(ii) Hence obtain a general solution for \( \lambda_{\text{sup}}^{*,i}(t), \ t \in [1 - h_i, 1] \) involving another two arbitrary constants, denoted by \( \bar{k}_3^i \) and \( \bar{k}_4^i \), respectively.

**Step 4.** Obtain a formula for \( \lambda_{\text{sup}}^{*,i}(t), \ t \in [h_{\text{sup}}, 1] \).

(i) Find the arbitrary constants \( \bar{k}_1^i, \bar{k}_2^i, \bar{k}_3^i, \bar{k}_4^i \) in step 2 and step 3 by solving

\[
\lim_{t \to h_{\text{sup}}} \dot{\lambda}_{\text{sup}}^{*,i}(t) = \lim_{t \to h_{\text{sup}}} \dot{\lambda}_{\text{sup}}^{*,i}(t),
\]

(50)

\[
\lim_{t \to (1-h_i)^-} \lambda_{\text{sup}}^{*,i}(t) = \lim_{t \to (1-h_i)^+} \lambda_{\text{sup}}^{*,i}(t),
\]

(51)

\[
\lim_{t \to (1-h_i)^-} \dot{\lambda}_{\text{sup}}^{*,i}(t) = \lim_{t \to (1-h_i)^+} \lambda_{\text{sup}}^{*,i}(t),
\]

(52)

\[
\lambda_{\text{sup}}^{*,i}(1) = 0,
\]

(53)

which provide 4 simultaneous equations in 4 unknowns.

(ii) Obtain the entire formula of \( \lambda_{\text{sup}}^{*,i}(t), \ t \in [h_{\text{sup}}, 1] \) from the general solutions of \( \lambda_{\text{sup}}^{*,i}(t) \) in Steps 2 and 3, together with the values of \( \bar{k}_1^i, \bar{k}_2^i, \bar{k}_3^i, \bar{k}_4^i \) just computed.

**Step 5.** Obtain the formula for \( \lambda_{\text{sup}}^{*,i}(t), \ t \in [h_{\text{sup}}, 1 + h_{\text{sup}}] \).

(i) Extend the formula for \( \lambda_{\text{sup}}^{*,i}(t), \ t \in [h_{\text{sup}}, 1] \) to \( \lambda_{\text{sup}}^{*,i}(t), \ t \in [h_{\text{sup}}, 1 + h_{\text{sup}}] \) by using (45).

A more detailed method of obtaining \( \lambda_{\text{sup}}^{*,i}(t), \ t \in [h_{\text{sup}}, 1 + h_{\text{sup}}] \) is as follows:

We first simplify our notations by letting \( E_i \) and \( E_{\text{sup}} \) to be the economic factor of manufacturer \( i \) and the supplier, respectively, where

\[
E_i = \sqrt{\frac{c_{P,i}}{c_{I,i}}},
\]

(54)

and

\[
E_{\text{sup}} = \sqrt{\frac{c_{P,\text{sup}}}{c_{I,\text{sup}}}}.
\]

(55)

**Step 1.** From (41), (47), the first part of (29), (55) and (42), we have

\[
\dot{I}_{\text{sup}}^{*,i}(t) = -\alpha P_i^{*}(t), \quad t \in [0, h_{\text{sup}}),
\]

\[
\Rightarrow \dot{I}_{\text{sup}}^{*,i}(t) = -\alpha (A_i^{*}e^{-E_i t} + B_i^{*}e^{-E_i t} + W_i^{*}e^{-r_{D,i}t} + Y_i^{*}), \quad t \in [0, h_{\text{sup}}),
\]

\[
\Rightarrow I_{\text{sup}}^{*,i}(t) = -\alpha \left( \frac{A_i^{*}e^{-E_i t} - B_i^{*}e^{-E_i t}}{E_i} + \frac{W_i^{*}e^{-r_{D,i}t}}{r_{D,i}} + Y_i^{*} \right) + k_i^1, \quad t \in [0, h_{\text{sup}}),
\]

(56)
where
\[ k_1^i = \frac{I_{\text{sup},0}}{n} + \alpha \left( \frac{A_i^* - B_i^*}{E_i} - \frac{W_i^*}{r_{D,i}} \right). \quad (57) \]

Thus, from (43) and (55), we have
\[
\lim_{t \to h_{\text{sup}}} \hat{\lambda}_{\text{sup}}^{*,i}(t) = -2c_{I,\text{sup}} \left( \lim_{t \to h_{\text{sup}}} \hat{\lambda}_{\text{sup}}^{*,i}(t) - \frac{\hat{I}_{\text{sup}}}{n} \right)
\]
\[
\Rightarrow \quad \lim_{t \to h_{\text{sup}}} \hat{\lambda}_{\text{sup}}^{*,i}(t) = \hat{k}_1^i,
\]
where
\[
\hat{k}_1^i = 2c_{I,\text{sup}} \left[ \alpha \left( \frac{A_i^* e^{E_i h_{\text{sup}}} - B_i^* e^{-E_i h_{\text{sup}}}}{E_i} + \frac{W_i^* e^{-r_{D,i} h_{\text{sup}}}}{r_{D,i}} + Y_i^* h_{\text{sup}} \right) + \frac{\hat{I}_{\text{sup}}}{n} - k_1^i \right].
\]

**Step 2.** Differentiating (43) and using (41), (46), and the first part of (29), we have
\[
\hat{\lambda}_{\text{sup}}^{*,i}(t) = -2c_{I,\text{sup}} \hat{I}_{\text{sup}}^{*,i}(t), \quad t \in [h_{\text{sup}}, 1 - h_i)
\]
\[
\Rightarrow \quad \hat{\lambda}_{\text{sup}}^{*,i}(t) = E_{\text{sup}}^2 \lambda_{\text{sup}}^{*,i}(t) + 2\alpha c_{I,\text{sup}} (A_i^* e^{E_i t} + B_i^* e^{-E_i t} + W_i^* e^{-r_{D,i} t})
\]
\[
+ 2c_{I,\text{sup}} \left( \alpha Y_i^* - \hat{P}_{\text{sup}} \right), \quad t \in [h_{\text{sup}}, 1 - h_i).
\]

From (60) and (55), the general solution of \( \lambda_{\text{sup}}^{*,i}(t), \quad t \in [h_{\text{sup}}, 1 - h_i) \) is
\[
\lambda_{\text{sup}}^{*,i}(t) = \hat{k}_1^i e^{E_{\text{sup}} t} + \hat{k}_2^i e^{-E_{\text{sup}} t} + \hat{k}_3^i e^{E_i t} + \hat{k}_4^i e^{-E_i t}
\]
\[
+ \hat{k}_5^i e^{-r_{D,i} t} + \hat{k}_6^i, \quad t \in [h_{\text{sup}}, 1 - h_i),
\]
where
\[
\hat{k}_2^i = \frac{2\alpha c_{I,\text{sup}} A_i^*}{(E_i^2 - E_{\text{sup}}^2)},
\]
\[
\hat{k}_3^i = \frac{2\alpha c_{I,\text{sup}} B_i^*}{(E_i^2 - E_{\text{sup}}^2)},
\]
\[
\hat{k}_4^i = \frac{2\alpha c_{I,\text{sup}} W_i^*}{(r_{D,i}^2 - E_{\text{sup}}^2)},
\]
\[
\hat{k}_5^i = 2c_{P,\text{sup}} \left( \frac{\hat{P}_{\text{sup}}}{n} - \alpha Y_i \right),
\]
and \( \hat{k}_1^i, \hat{k}_2^i \) are arbitrary constants which will be determined from (50)-(53) later in this discussion.

**Step 3.** Differentiating (43) and using (41), (46) and the second part of (29), we have
\[
\hat{\lambda}_{\text{sup}}^{*,i}(t) = -2c_{I,\text{sup}} \hat{I}_{\text{sup}}^{*,i}(t), \quad t \in [1 - h_i, 1]
\]
\[
\Rightarrow \quad \hat{\lambda}_{\text{sup}}^{*,i}(t) = E_{\text{sup}}^2 \lambda_{\text{sup}}^{*,i}(t) - 2c_{I,\text{sup}} \left( \frac{\hat{P}_{\text{sup}}}{n} - \alpha \hat{P}_i \right), \quad t \in [1 - h_i, 1]
\]

From (66) and (55), the general solution of \( \lambda_{\text{sup}}^{*,i}(t), \quad t \in [1 - h_i, 1] \) is
\[
\lambda_{\text{sup}}^{*,i}(t) = \hat{k}_3^i e^{E_{\text{sup}} t} + \hat{k}_4^i e^{-E_{\text{sup}} t} + \hat{k}_6^i, \quad t \in [1 - h_i, 1],
\]
where

\[
\tilde{k}_6 = 2c_{P, t} \left( \frac{\hat{P}_{\text{sup}}}{n} - \alpha \hat{P}_i \right)
\]  

(68)

and \( \hat{k}_3, \hat{k}_4 \) are arbitrary constants, which will be determined from (50)-(53) later in our discussion.

**Step 4 and Step 5.** From (61), (67), (45), (50)-(53), (13) and (58), the complete solution of \( \lambda^{*,i}_\text{sup}(t), \ t \in [h_{\text{sup}}, 1 + h_{\text{sup}}] \), is as follows:

\[
\lambda^{*,i}(t) = \begin{cases} 
\tilde{k}_1 e^{E_{\text{sup}} t} + \tilde{k}_2 e^{-E_{\text{sup}} t} + \tilde{k}_4 e^{-E_{\text{sup}} t} + \tilde{k}_5 e^{-\tau_D, i t} + \tilde{k}_6, & t \in [h_{\text{sup}}, 1 - h_i], \\
\tilde{k}_3 e^{E_{\text{sup}} t} + \tilde{k}_4 e^{-E_{\text{sup}} t} + \tilde{k}_5, & t \in [1 - h_i, 1], \\
0, & t \in [1, 1 + h_{\text{sup}}], 
\end{cases}
\]  

(69)

where \( \tilde{k}_1, \tilde{k}_2, \tilde{k}_3, \tilde{k}_4, \tilde{k}_5 \) and \( \tilde{k}_6 \) are as defined in (62), (63), (64), (65) and (68), respectively, and \( \tilde{k}_1, \tilde{k}_2, \tilde{k}_3, \tilde{k}_4, \tilde{k}_5 \) are solutions of the matrix equation \( \hat{A} \hat{x} = \hat{B} \), where

\[
\hat{A} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & e^{E_{\text{sup}} (1 - h_i)} & -e^{E_{\text{sup}} (1 - h_i)} & -e^{E_{\text{sup}} (1 - h_i)} \\
e^{E_{\text{sup}} (1 - h_i)} & -e^{E_{\text{sup}} (1 - h_i)} & e^{E_{\text{sup}} (1 - h_i)} & -e^{E_{\text{sup}} (1 - h_i)} & 0 \\
0 & 0 & 0 & e^{E_{\text{sup}}} & e^{E_{\text{sup}}}
\end{pmatrix},
\]  

(71)

\[
\hat{B} = \frac{1}{E_{\text{sup}}} \begin{pmatrix}
\hat{k}_1 + \frac{1}{E_{\text{sup}}} \left[ -E_i \hat{k}_2 e^{E_i (1 - h_i)} + E_i \hat{k}_3 e^{-E_i h_{\text{sup}}} + r_{D,i} \hat{k}_4 e^{-r_{D,i} (1 - h_i)} \right] \\
-\hat{k}_2 e^{E_i (1 - h_i)} - \hat{k}_3 e^{-E_i (1 - h_i)} - \hat{k}_4 e^{-r_{D,i} (1 - h_i)} - \hat{k}_5 + \hat{k}_6
\end{pmatrix}.
\]  

(72)

(Note that \( \lambda^{*,i}_\text{sup}(t) \) is continuous at \( t = 1 - h_i \) and at \( t = 1 \).) Hence, the optimal production schedule of the supplier can be found from (46) and (69) as follows:

\[
P^{*,i}_\text{sup}(t) = \sum_{i=1}^{n} P^{*,i}_\text{sup}(t),
\]  

(73)

where

\[
P^{*,i}_\text{sup}(t) = \begin{cases} 
\frac{\hat{P}_{\text{sup}}}{n} - \frac{1}{2c_{P, t}} \left( \tilde{k}_1 e^{E_{\text{sup}} (t + h_{\text{sup}})} + \tilde{k}_2 e^{-E_{\text{sup}} (t + h_{\text{sup}})} \right) + \tilde{k}_4 e^{-E_{\text{sup}} (t + h_{\text{sup}})} + \tilde{k}_5 e^{-r_{D,i} (t + h_{\text{sup}})} + \tilde{k}_6, & t \in [0, 1 - h_i - h_{\text{sup}}], \\
\frac{\hat{P}_{\text{sup}}}{n} - \frac{1}{2c_{P, t}} \left( \tilde{k}_3 e^{E_{\text{sup}} (t + h_{\text{sup}})} + \tilde{k}_4 e^{-E_{\text{sup}} (t + h_{\text{sup}})} + \tilde{k}_5 \right), & t \in [1 - h_i - h_{\text{sup}}, 1 - h_{\text{sup}}], \\
\frac{\hat{P}_{\text{sup}}}{n}, & t \in [1 - h_{\text{sup}}, 1].
\end{cases}
\]  

(74)

(Note that \( P^{*,i}_\text{sup}(t) \) is continuous at \( t = 1 - h_i - h_{\text{sup}} \) and at \( t = 1 - h_{\text{sup}} \).)
6. Method of finding the optimal inventory levels of the manufacturer and the supplier.

6.1. The optimal inventory level of the manufacturer. After we have obtained the optimal production rate \( P^{*\text{,}i}_\text{sup}(t) \) of manufacturer \( i \), we can find the optimal inventory level of manufacturer \( i \) as follows:

**Case 1.** When \( t \in [0, h_i] \), we obtain from (12) that

\[
I^*_i(t) = \frac{(\hat{D}_i - D_{i,0})(1 - e^{-rD_i t})}{rD_i} - \hat{D}_i t + I_{i,0}.
\]

(75)

**Case 2.** When \( t \in [h_i, 1] \), we get from (9), (18) and (54) that

\[
I^*_i(t) = \hat{I}_i - \frac{\dot{\hat{I}}^*_i(t)}{2c_{I,i}}
\]

\[
\Rightarrow I^*_i(t) = \hat{I}_i - \frac{E_i(\bar{A}_i e^{E_i t} - \bar{B}_i e^{-E_i t}) - r_{D,i} \bar{A}_i e^{-r_{D,i}t}}{2c_{I,i}}.
\]

(76)

6.2. The optimal inventory level of the supplier. In view of (49), in order to find the optimal level \( I^*_{\text{sup}}(t) \) of the supplier, we first need to find \( I^{*\text{,}i}_{\text{sup}}(t) \) for \( i = 1, ..., n \). After we have obtained the optimal production rate \( P^{*\text{,}i}_{\text{sup}}(t) \) (\( i = 1, ..., n \)), we can obtain \( I^*_{\text{sup}}(t) \) as follows:

**Case 1.** When \( t \in [0, h_{\text{sup}}] \), we obtain from (56) that

\[
I^*_{\text{sup}}(t) = -\alpha \left( \frac{A^*_i e^{E_i t} - B^*_i e^{-E_i t}}{E_i} + \frac{W^*_i e^{-r_{D,i} t}}{r_{D,i}} + Y^* t \right) + \lambda_{I,\text{sup}}^i.
\]

(77)

**Case 2.** When \( t \in [h_{\text{sup}}, 1 - h_i] \), we obtain from (43) and the first part of (69) that

\[
I^{*\text{,}i}_{\text{sup}}(t) = \frac{\hat{I}_{\text{sup}}}{n} - \frac{\dot{\lambda}_{I,\text{sup}}^i(t)}{2c_{I,\text{sup}}}
\]

\[
\Rightarrow I^{*\text{,}i}_{\text{sup}}(t) = \frac{\hat{I}_{\text{sup}}}{n} - \frac{E_{\text{sup}}(\hat{k}_1 e^{E_{\text{sup}} t} - \hat{k}_2 e^{-E_{\text{sup}} t}) + E_i(\bar{k}_1 e^{E_i t} - \bar{k}_2 e^{-E_i t}) - r_{D,i} \bar{k}_1 e^{-r_{D,i} t}}{2c_{I,\text{sup}}}.
\]

(78)

**Case 3.** When \( t \in [1 - h_i, 1] \), we obtain from (43) and the second part of (69) that

\[
I^{*\text{,}i}_{\text{sup}}(t) = \frac{\hat{I}_{\text{sup}}}{n} - \frac{\dot{\lambda}_{I,\text{sup}}^i(t)}{2c_{I,\text{sup}}}
\]

\[
\Rightarrow I^{*\text{,}i}_{\text{sup}}(t) = \frac{\hat{I}_{\text{sup}}}{n} - \frac{E_{\text{sup}}(\hat{k}_3 e^{E_{\text{sup}} t} - \hat{k}_4 e^{-E_{\text{sup}} t})}{2c_{I,\text{sup}}}.
\]

(79)

Thus, we can obtain the optimal inventory level of the supplier \( I^*_i(t) \) as follows:

\[
I^*_i(t) = \sum_{l=1}^{n} I^{*\text{,}i}_{\text{sup}}(t),
\]

(80)

where \( I^{*\text{,}i}_{\text{sup}}(t) \) is as given in (77)-(79).
7. **Numerical result and discussion.** In this section, three numerical examples are solved to provide insight of the manufacturers and the supplier. Moreover, these examples are also used to test the sensitivity of the changes in the optimal values of the objective functions of the manufacturers and the supplier with respect to the changes in the initial inventory levels.

**Example 7.1.** An optimal production scheduling problem involving four manufacturers and a single supplier has been solved using the following data.

The data for the manufacturers is as follows:

- **Time-Delayed Argument** \( h_i (i = 1, 2, 3, 4) = (0.11, 0.15, 0.07, 0.05) \).
- **Penalty Cost** \( c_{P,i} (i = 1, 2, 3, 4) = (10, 12, 13, 7) \).
- **Penalty Cost** \( c_{I,i} (i = 1, 2, 3, 4) = (13, 14, 16, 9) \).
- **Ideal Production Rate** \( \hat{P}_i (i = 1, 2, 3, 4) = (32, 27, 38, 25) \).
- **Ideal Inventory Level** \( \hat{I}_i (i = 1, 2, 3, 4) = (28, 20, 24, 17) \).
- **Initial Inventory Level** \( I_{i,0} (i = 1, 2, 3, 4) = (15, 17, 19, 10) \).
- **Saturated Demand Rate** \( \hat{D}_i (i = 1, 2, 3, 4) = (35, 35, 30, 22) \).
- **Initial Demand Rate** \( D_{i,0} (i = 1, 2, 3, 4) = (24, 22, 25, 14) \).
- **Constant in the Changes of the Demand Rate** \( r_{D,i} (i = 1, 2, 3, 4) = (5, 5, 7, 3) \).

The data for the supplier is as follows:

- **Time-Delayed Argument** \( h_{sup} = 0.12 \).
- **Penalty Cost** \( c_{P,sup} = 20 \).
- **Penalty Cost** \( c_{I,sup} = 10 \).
- **Ideal Production Rate** \( \hat{P}_{sup} = 75 \).
- **Ideal Inventory Level** \( \hat{I}_{sup} = 64 \).
- **Initial Inventory Level** \( I_{sup,0} = 24 \).
- \( \alpha = 1.7 \).

The optimal production rates of all the manufacturers are as shown in Figure 1. The optimal production rate of the supplier is as shown in Figure 2. The optimal inventory levels of all the manufacturers are as shown in Figure 3. The optimal inventory level of the supplier is as shown in Figure 4.

The optimal values of the objective function of the manufacturers are 187.6, 61.3, 26.7 and 63.7 respectively. The optimal value of the objective function of the supplier is 1846.8.

![Figure 1](image1.png)

**Figure 1.** Optimal Production Rates of the Manufacturers in Example 7.1

**Example 7.2.** Same as Example 7.1, except that for the following changes of data:

The initial inventory levels of the manufacturers are changed to 20, 19, 22, 18 respectively and the initial inventory level of the supplier is changed to 48. (Thus,
the differences between manufacturers’ initial inventory levels and manufacturers’ ideal inventory levels in this example are much smaller than those in Example 7.1; the difference between supplier’s initial inventory level and supplier’s ideal inventory level in this example is also much smaller than that in Example 7.1.)

The optimal production rates of all the manufacturers are as shown in Figure 5. The optimal production rate of the supplier is as shown in Figure 6. The optimal inventory levels of all the manufacturers are as shown in Figure 7. The optimal inventory level of the supplier is as shown in Figure 8.

The optimal values of the objective function of the manufacturers are 89.7, 40.7, 15.0 and 38.9 respectively. The optimal value of the objective function of the supplier is 920.2.
Figure 5. Optimal Production Rates of the Manufacturers in Example 7.2

Figure 6. Optimal Production Rate of the Supplier in Example 7.2

Figure 7. Optimal Inventory Levels of the Manufacturers in Example 7.2

Figure 8. Optimal Inventory Level of the Supplier in Example 7.2
Example 7.3. Same as Example 7.1, except that for the following changes of data:

The initial inventory levels of the manufacturers are changed to 28, 20, 24, 17 respectively and the initial inventory level of the supplier is changed to 64. (Thus, in this example, the initial inventory level of each manufacturer is exactly the same as his ideal inventory level; the initial inventory level of the supplier is also exactly the same as his ideal inventory level.)

The optimal production rates of all the manufacturers are as shown in Figure 9. The optimal production rate of the supplier is as shown in Figure 10. The optimal inventory levels of all the manufacturers are as shown in Figure 11. The optimal inventory level of the supplier is as shown in Figure 12.

The optimal values of the objective function of the manufacturers are 7.8, 32.5, 14.4 and 36.9 respectively. The optimal value of the objective function of the supplier is 824.1.

![Figure 9. Optimal Production Rates of the Manufacturers in Example 7.3](image1)

![Figure 10. Optimal Production Rate of the Supplier in Example 7.3](image2)

Remark 7.1. From the above examples, it is clear that the optimal values of the objective functions of both the manufacturers and the supplier decrease significantly when the differences between the initial inventory level and the ideal inventory level decrease, especially for manufacturer 1.

8. Conclusion. In this paper, an analytic method using a hybrid technique has been devised for finding the optimal production schedule in a single-supplier multi-manufacturer supply chain involving time delays in both levels. Numerical examples consisting of a single supplier and four manufacturers have been solved to provide
insight of the optimal strategies of the supplier and the manufacturers. In the future, we would like to extend this method to finding the optimal production schedule in a multi-supplier multi-manufacturer supply chain involving time delays, or alternatively, to finding the optimal production schedule in a single-supplier multi-manufacturer supply chain involving both time delays and stochastic demands.

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