From limits of quantum nonlinear operations to multicopy entanglement witnesses and state spectrum estimation

Pawel Horodecki
Faculty of Applied Physics and Mathematics
Technical University of Gdansk, 80–952 Gdansk, Poland

The limits of nonlinear in quantum mechanics are studied. The impossibility of physical implementation of the transformation $\rho^\otimes n \rightarrow \rho^n$ in quantum mechanics is proved. For sake of further analysis the simplest notion of structural completely positive approximation (SCPA) and structural physical approximations (SPA) of unphysical map are introduced. Both always exist for linear hermitian maps and can be optimised under natural assumptions. However it is shown that some intuitively natural SPA of the nonlinear operation $\rho^\otimes 2 \rightarrow \rho^2$ that was already proven to be unphysical is impossible. It is conjectured that there exist no SPA of the operation $\rho^\otimes n \rightarrow \rho^n$ at all. It is pointed out that, on the other hand, it is physically possible to measure the trace of the second power of the state $\text{Tr}(\rho^2)$ if only two copies of the system are available. This gives the interpretation of one of Tsallis entropy as mean value of some “multicopy” observable. The (partial) generalisation of this idea shows that each of higher order Tsallis entropies can be measured with help of only two multicopy observables. Following this observations the notion of multicopy entanglement witnesses is defined and first example is provided. Finally, with help of multicopy observables simple method of spectrum state estimation is pointed out and discussed.

PACS numbers: 03.65 Bz, 03.67.-a

I. INTRODUCTION

The limits of nonlinear operations within quantum mechanics is an interesting question. It has been shown that, for example the operation

$$\rho \otimes \rho \rightarrow \begin{bmatrix} \rho_{11}^2 & \rho_{12}^2 \\ \rho_{21}^2 & \rho_{22}^2 \end{bmatrix},$$

(1)

can be performed with the finite probability by means of quite simple network with two copies of $\rho$ as an input. On the other hand the limits for other nonlinear operations has been shown resulting in “no-disentanglement” rule in quantum mechanics [2], [3]. In this work we want to show both further limits and advantages of nonlinear transformations in context of quantum entanglement theory.

It can be easily seen that if the state in (1) is diagonal then we get the square of it. However to get this we have to know the eigenvectors of the state. It is interesting that if square power were possible for unknown state we would be able to distill entanglement from large classes of the states with little previous information about them [4]. To some extent it would be similar to the situation in the compression protocol of Ref. [5] where no measurement of the source state is needed if one of its parameters (entropy) is known. We shall show that it is impossible to produce any power of the state if we do not know its eigenvectors. Namely it is impossible to perform the operation providing “n-th” power of the unknown state from $n$-copies of it.

However, one can weaken requirements: sometimes it is impossible to perform some operation but is is possible to perform it approximately. So one can try to perform such approximation. The well known example are cloning operation [6], [7] and transposition (or universal NOT gate) [8] and “two-qubit fidelity” map [9]. Recently more careful study of approximations of one qubit maps has been carried out [10, 11].

It is natural to expect that physical approximations of unphysical maps could help in solving physical problems in general. To study this we use the notion of structural completely positive approximation (SCPA) and structural physical approximation (SPA) of the unphysical operation. Those are very restrictive approximations - the key feature of them is that they always have the direction of generalised Bloch vector of the output state the same as the output state of the original unphysical map. Only the length of the vector is rescaled by some factor.

We point out that SCPA and SPA always exist for linear unphysical hermitian maps. We also prove that there is natural optimisation giving the best SPA. However, we give the proof that the most natural trace preserving SPA of unphysical map $\rho^\otimes 2 \rightarrow \rho^2$ is impossible. We rise the question of whether SPA of such nonlinear maps in general.

However, as we shall see, the trace of the $n$-th power of the state i. e. the value $\text{Tr}(\rho^n)$ can be simply measured if $n$ copies of the system are available. We show how to do it in practice by means of generalised “swap” operator. We
show several interesting applications of that fact. Namely one can apply what we propose to call **multicopy observables** of the system: mean value of such observables is measurable if joint measurement on several copies of the system in the same state is achievable. We point out that, following the latter, that the Tsallis entropy $S_q$ can be treated as two-copy observable. Further all Tsallis entropies $S_q$ of natural index $q = n > 2$ can be measured with help of only two multicopy observable. Then applying the separability conditions in terms of entropic inequalities (initiated in $[14]$, developed in $[16,19,21]$ and completed in an elegant way in $[22]$) we show that some Tsallis entropic separability conditions (equivalent to quantum Renyi ones) can be checked directly (or almost directly) with help of multicopy observables. This is remarkable as only finite (not more than the dimension of single system Hilbert space) number of copies is needed. Finally we point out how to estimate the spectrum of unknown state using the idea of multicopy observables. This observation is in a sense complementary to the optimal procedure of Ref. $[23]$. The advantage is that collective measurements on only finite (not more than the dimension of single system Hilbert space) number of copies is needed.

The paper is organised as follows: In section II we prove the following “no go” result: if the state $\varrho$ is unknown than the operation $\varrho \otimes n \rightarrow \varrho^n$ is impossible. In sect. III we provide general idea of structural completely positive approximation (SCPA) and its slight modification - structural physical approximation (SPA). In sect. IV we show that those approximations always exists for any hermitian map. We also show that the most natural SPA is optimal.

In section V we apply the concept of SPA to nonlinear quantum maps showing that the intuitively most natural tracepreserving SPA of unphysical operation $\varrho \otimes 2 \rightarrow \varrho^2$ does not exist. The general “no go” conjecture is also formulated.

In section IV we investigate further possibility of direct measurement of nonlinear parameters. We introduce the notion of “multicopy observables” and show how Tsallis entropies can be measured with help of such observables. We utilise entropic separability criteria from the literature of the subject we introducing the notion of “multicopy observable” and show how Tsallis entropies can be measured with help of such observables.

Finally in short Section V we point out a simple method of estimation of spectrum of unknown state defined on $C^n$ which requires collective measurement of some number of copies, but only estimation of $2m - 3$ parameters is needed (instead of $m^2 - 1$ due to usual quantum tomography) that are mean values of some multicopy observables.

### II. PROOF OF IMPOSSIBILITY OF OPERATION $\varrho \otimes n \rightarrow \varrho^n$

Consider an arbitrary quantum state defined on $C^d$ space. We shall show that there is no quantum transformation of the kind

$$\Lambda(\varrho \otimes n) \rightarrow \varrho^n$$

which works for unknown quantum state. Let us note first that such operation would be probabilistic i.e. it would give the required output with the probability $p = \text{Tr}(\varrho^n)$ depending on the input state. Suppose that such operation existed. Then because of complete positivity it would be of the form completely positive map $\Lambda(\varrho) = \sum_{i=1}^{d^n} V_i \varrho V_i^\dagger$. So we would rewrite (2) as follows

$$\sum_{i=1}^{N} V_i \varrho \otimes n V_i^\dagger = \varrho^n$$

Taking trace of both sides of the above we get that there would exist the positive state independent operator $A = \sum_{i=1}^{N} V_i^\dagger V_i$, with the property $0 \leq A \leq I$ such that

$$\text{Tr}(A\varrho \otimes m) = \text{Tr}(\varrho^n)$$

for any state $\varrho$. In particular for any pure projector $P_{\varphi} = |\varphi\rangle\langle\varphi|$ corresponding to normalised vector $|\varphi\rangle$ we would have $\text{Tr}(A\varrho \otimes m) = \text{Tr}(P_{\varphi}) = 1$. But, because all eigenvalues of $A$ belong to the interval $[0, 1]$, this means that any vector of the form $|\Psi\rangle = |\varphi\rangle \otimes m$ must be an eigenvector of $A$. However all $|\Psi\rangle$-s of that form span the completely symmetric subspace $H_{\text{SYM}}$ of $(C^n)^{\otimes m}$. Thus the subspace is an eigenspace of $A$ corresponding to eigenvalue 1. Now for any $\varrho$ the support of the operator $\varrho \otimes m$ belongs to the $H_{\text{SYM}}$. So $A\varrho \otimes m = \varrho \otimes m$ which means that LHS of the equation (4) should be always 1 which clearly leads to the contradiction for any $\varrho$ which is not pure.

---

1If $A, B$ are hermitian operators then the notation $A \leq B$ means that for any vector $|\psi\rangle$ one has $\langle\psi|A|\psi\rangle \leq \langle\psi|B|\psi\rangle$.  

2
III. STRUCTURAL PHYSICAL APPROXIMATIONS OF UNPHYSICAL MAPS: THE CONCEPT

A. Definition

Consider now the possibility of obtaining the completely positive approximation \( \Theta \) of the physically impossible operation \( \Theta : B(C^d) \to B(C^{d'}) \). We shall require that the following “error” operator

\[
\Delta(\rho) = \Theta(\rho) - \gamma(\rho)\Theta(\rho)
\]

satisfies the invariance condition

\[
\Delta(\rho) = \delta(\rho)I
\]

for identity operator \( I \) on \( C^{d'} \) and some scaling parameters \( \gamma(\rho) \geq 0, \delta(\rho) \geq 0 \) depending in general on \( \rho \).

More precisely we require what can be represented by the following proposed definition

Definition 1.- The structural completely positive approximation (SCPA) of unphysical map \( \Theta \) is any completely positive operation of the form

\[
\Theta(\rho) = \delta(\rho)I + \gamma(\rho)\Theta
\]

with the functions \( \delta, \gamma \geq 0 \) and \( \gamma \) strictly positive for all \( \rho \) such that \( \Theta(\rho) > 0 \). The structural physical approximation (SPA) of unphysical map \( \Theta \) is such SCPA \( \Theta \) that for any state \( \rho \) TR(\Theta(\rho)) \leq 1 \ i.e. that can be implemented experimentally.

Remark.- The first SCPA (and as we shall see also SPA) was optimal NOT gate from Ref. [9]. The approximated cloning machine ([6–8]) was not because one the output of the machine is entangled while according to the present definition it should be separable.

The essence of SCPA of any \( \Theta \) is that it (i) is completely positive and (ii) keeps the structure of the output of the unphysical operator \( \Theta \). In other words for any argument the direction of generalised Bloch vector of the output matrix is the same as the direction of the output matrix of the original unphysical map. The output is however “shrunk” by factor \( \gamma \) (a kind of “Black Cow” factor, see [9]) and the additional portion (quantified by \( \delta \)) of completely random noise is admixed. The SPA is such SCPA that can be probabilistically implemented in lab (see Appendix). Note that for finitedimensional systems some SPA can be obtained form nonzero SCPA by normalisation

\[
\Theta_{SPA} = t^{-1}\Theta, t = \max_{\rho} TR[\Theta(\rho)].
\]

where strict positivity of \( t \) is given by complete positivity of nonzero SCPA \( \Theta \).

Now one can ask: what is the optimal approximation in sense of the above definitions? Let us recall that the question of optimality was frequently posed in context of approximate cloning machines etc. In this context we can discuss the proposal of the following notion of optimality:

Definition 2.- The best (or the optimal) SPA of \( \Theta \) is such SPA map \( \Theta_{opt} \) that (i) minimizes the ratio \( \delta(\rho)/\gamma(\rho) \) for any \( \rho \) in \( [3] \) (ii) maximizes \( TR[\Theta(\rho)] \leq 1 \) over all SPA-s satisfying property (i).

This would mean that any other SPA has for any \( \rho \) the ratio greater than the one of the best SPA. The point (ii) would have similar meaning. The idea of the above notion would be that the ratio of the noise to the approximated map \( \Theta \) in \( [3] \) would “prefer” the latter as much as possible. For the best SPA one requires in addition as much probability of implementation as possible (see Appendix).

The problem with the above notion of optimality is that it is difficult to check whether such optimal maps always exist. In particular it is quite probable that the best SPA may not exist in some peculiar cases (for example when \( \Theta, \delta \) or \( \gamma \) are not continuous). However, as we shall see below, for linear hermitian maps they always exist and under natural conditions they can be easily optimised.

IV. PHYSICAL APPROXIMATIONS OF HERMITIAN MAPS

A. Natural construction

By hermitian we shall regard any map that preserves hermicity property. In what follows we shall briefly prove very simple

Proposition 1 - For any hermitian linear map \( \Theta : B(C^d) \to B(C^{d'}) \) there exists SPA \( \Theta \) defined by

\[
\Theta_{SPA} = t^{-1}\Theta, t = \max_{\rho} TR[\Theta(\rho)].
\]
Θ = t^{-1}(aI_{d'} + Θ) \tag{9}

with the parameter $a \geq \lambda d \equiv \max \{0, -X'd\}$ where $X'$ is the minimal eigenvalue of the operator $[I \otimes Θ](P_+)$. Here $P_+ = |Ψ_+⟩⟨Ψ_+|$ corresponds to the “isotropic” maximally entangled $d \otimes d$ state $|Ψ_+⟩ = \frac{1}{d} \sum_{i=1}^{d} |i⟩⟨i|$, $t \equiv \max_{ε} Tr([aI_{d'} + Θ](g))$.

In the above one must remember that $I_{d'}$ stands for identity matrix acting on $C^d$ and we identify with this matrix “maximal noise” map that replaces every operator with $I_{d'}$. Note also that $[I \otimes I_{d'}](P_+) = \frac{1}{\sqrt{d}} I_d \otimes I_{d'}$. Subsequently we shall omit indices at $I_{d'}$, $I_d$.

The complete positive character of approximation [6] follows immediately from the well-known fact that linear hermitian operation $Λ$ is completely positive iff $[I \otimes Λ](P_+)$ has nonnegative spectrum. The approximation is also the most natural, as it does not involve any nonlinear function of $g$ and relies on adding only “backround noise” to the original map.

### B. Question of optimality

Let us call any hermitian map nontrivial if it is not of the form $Λ(g) = c(g)I$ (with $c(g)$ being some real function) i.e. it does not map all states into (possibly rescaled) identity matrix. Let also call SCPA and SPA regular if it has the shrinking function $γ$ continuous. We have the following

**Proposition 2.** From all regular SPA-s of nontrivial linear hermitian map $Θ$ the approximation $[6]$ with minimal $a$ ($a = λ/d$) is the best one.

**Proof.** Let us recall that the spaces $B(C^d)$, $B(C^d')$ are Hilbert spaces (known as Hilbert-Schmidt spaces) with a scalar product $(A|B) = Tr(A^†B)$. Let us take support of original $Θ$. This is that subspace of $B(C^d)$ on which $Θ$ does not vanish. In this subspace we can find the orthonormal basis consisting of $r$ elements $\{X_i\}_{i=1}^{r}$. There exists its complement called kernel of $Θ$ represented by the set of $k$ elements $\{X_{i}\}_{i=r+1}^{k}$ so that the domain $B(C^d)$ is spanned by all $r+k = d^2$ elements. Because linearily independent (even orthonormal) elements of support $X_i$ must be mapped into linearily independent ones we get that the original map is of the form

$$Θ(A) = \sum_{i=1}^{r} Y_i Tr[X_iA] \tag{10}$$

on arbitrary $A$ with linearily independent operators $Y_i \in B(C^{d'})$. Because $Θ$ is nontrivial at least one of $Y_i$, say $Y_{i_0}$, must be linearily independent on operator $I$. An arbitrary operator $X$ in the domain of $Θ$ is of the form $X = αX_{i_0} + βX'$ where $X'$ is linear combination of $X_i$-s orthogonal to $X_{i_0}$. Let us consider the action of $[\overline{Θ}]$ (which is linear) on $X$. Form linearity of $Θ$, $[\overline{Θ}]$ and analysis of the coefficient at $Y_{i_0}$ in the corresponding expansion we get $γ(αX + βX_{i_0}) = γ(X_{i_0})$ for any nonzero $α$.

From regularity of $[\overline{Θ}]$ (which means continuity of $γ$) we get that $γ$ is a constant function equal to $γ(X_{i_0})$. Note that it was not obvious because $γ$ (and also $δ$) in principle could be nonlinear functions of the state. Because of hermicity of $[\overline{Θ}]$ constant $γ$ is represented by real number that we shall denote also by $γ$. This immediately implies linearity of function $δ$. Hence, because of Riesz theorem applied to Hilbert-Schmidt space $δ$ is uniquely determined by some hermitian operator $D$ in the following way: $δ(ε) = Tr(Dg)$. Now the complete positivity of $[\overline{Θ}]$ results in condition $δ(ε) \geq γλd$ where $d$ is a dimension of Hilbert space and $a \geq λ$ is defined as in Proposition 2. Minimisation of the rate $δ/γ$ results here immediately in (i) minimal value of $a = λ$, (ii) constant character of $δ$ (it is simply a number) (iii) the equality

$$δ = γλd \tag{11}$$

This results in $γ$-parameter family SPA maps of the form $[\overline{Θ}] ≡ [\overline{Θ}]_γ = γ(λdI + Θ)$. On the other hand to get “the best” SPA we have to maximise the value $Tr([\overline{Θ}](g))$ under the condition $Tr([\overline{Θ}](g)) \leq 1$. If for original $Θ$ one defines $α_{Θ}$ as the value of $Tr[Θ(g)]$ maximised over all states $g$ ($Θ_{opt} = \max_{g} Tr[Θ(g)]$) then we get immediately (i) optimal $γ = \frac{1}{α_{Θ}}$, (ii) optimal $δ = \frac{λd}{α_{Θ} + α_{Θ}d}$, and the best SPA:

$$Θ_{opt} = \frac{λd}{α_{Θ} + α_{Θ}d}I + \frac{1}{α_{Θ} + α_{Θ}d}Θ \tag{12}$$

One concludes the proof by checking that the above is identical with [6] after putting $a = λd$. 

4
C. Interpretation and example

There is a simple interpretation of the optimal formula (12) if \( \alpha_\Theta \neq 0 \). To get the best SPA \( \overline{\Theta} \) one need the following two operations

(i) rescale given linear hermitian \( \Theta \) by taking \( \Theta_1 = \alpha_\Theta^{-1} \Theta \), \( \alpha_\Theta = \max_\varphi \text{Tr}[\Theta(\varphi)] \). (Then \( \Theta_1 \) already satisfies \( \text{Tr}[\Theta_1(\varphi)] \leq 1 \).

(ii) take the following convex combination

\[
\overline{\Theta}_{\text{opt}} = p_\ast \frac{I}{d^2} + (1 - p_\ast) \Theta_1
\]

with probability \( p_\ast = \frac{\lambda dd' \alpha_\Theta^{-1}}{\lambda dd' \alpha_\Theta^{-1} + 1} \) and \( \lambda = \max[0, -\lambda'] \) where \( \lambda' \) is the minimal eigenvalue of the operator \([I \otimes \Theta](P_+)\).

In the case of tracepreserving maps (or in general all maps that satisfy \( \alpha_\Theta = 1 \)) the above protocol gives \( \Theta_1 = \Theta \) so the only step (ii) taking convex combination is important. This means that in those cases to get optimal SPA one have to perform probabilistic mixture of \( \Theta \) with depolarising channel \( \Lambda_{\text{dep}} : \mathcal{B}(\mathcal{C}^d) \to \mathcal{B}(\mathcal{C}^d) \) defined by \( \Lambda_{\text{dep}}(\varphi) = \frac{I}{d} \) for any \( \varphi \). In particular we have the

Example .- Consider the best SPA of the transposition map \( T : \mathcal{B}(\mathcal{C}^d) \to \mathcal{B}(\mathcal{C}^d) \) which transposes the matrix \([T(A)]_{mn} = A_{nm}\). Applying the prescription above for \( \Theta = T \) we get \([I \otimes T](P_+) = \frac{1}{d} V \) where \( V \) is a “flip” or “swap” operator \([13]\). As \( V \) has spectrum \( \pm 1 \) this gives \( \lambda = -\frac{1}{d}, \alpha = 1 \) which results in the optimal parameter \( p_\ast = \frac{1}{d} \). In this way we get tracepreserving SPA which is the following quantum channel

\[
\overline{T} : \varphi \to \frac{d}{d + 1} I + \frac{1}{d} \varphi^T
\]

or \( \overline{T} = \frac{d}{d + 1} I + \frac{1}{d} T \). It has already been realised as a byproduct of optimal quantum cloning machines \([3, 8]\) and is closely related to universal quantum NOT gate \([1]\).

V. SOME LIMIT OF APPROXIMATIONS OF NONLINEAR MAPS

What about our map making the power of the state? Can we approximate it in a way described above? We conjecture that it is impossible in general. We shall consider the case which seems to be natural at least. Namely consider the hypothetical map \( \Lambda_\gamma : \mathcal{B}(\mathcal{C}^d \otimes \mathcal{C}^d) \to \mathcal{B}(\mathcal{C}^d) \) defined in the form

\[
\Lambda_\gamma(\varphi \otimes \varphi) = \frac{I - \text{Tr}(\varphi^2)}{d} I/d + \varphi^2
\]

Here we have the tracepreserving property manifestly taken into account and the error parameter at the noise depends on the purity of the state being equal \( 1 - \text{Tr}(\varphi^2) \). Is it possible to build such map? The answer is negative. Namely we have

Proposition 3 .- For qubit case the map \((14)\) is not completely positive.

To prove the above statement let us first observe that putting two copies of pure qubit states into \((13)\) we get

\[
\Lambda_\gamma(|\varphi\rangle\langle \varphi| \otimes |\varphi\rangle\langle \varphi|) = |\varphi\rangle\langle \varphi|.
\]

Taking this into account and putting two copies of mixed states we get

\[
\Lambda_\gamma(|\varphi\rangle\langle \varphi| \otimes |\varphi^\perp\rangle\langle \varphi^\perp| + |\varphi^\perp\rangle\langle \varphi^\perp| \otimes |\varphi\rangle\langle \varphi|) = \frac{I}{2}
\]

for all pairs of orthogonal vectors \(|\varphi\rangle, |\varphi^\perp\rangle\). But the required action of the map will not change if we compose it with the symmetrisation as a first step. After such modification we have

\[
\Lambda_\gamma(|\varphi\rangle\langle \varphi| \otimes |\varphi^\perp\rangle\langle \varphi^\perp|) = \frac{I}{2}
\]

Suppose that the map is completely positive i.e. that it is equal to tracepreserving map of the form \( \Lambda_{CP} \sum_i V_i \sigma_i V_i^\dagger \). Here \( \sigma \) is arbitrary two-qubit state and \( V_i : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \). The map is tracepreserving so it represents a quantum channel. Following the equivalence between quantum states and quantum channels we can assume that the number \( k \) of
the operators $V_i$ is equal to $\dim(C^2 \otimes C^2) \cdot \dim(C^2) = 4 \cdot 2 = 8$. Let us define the following operation $\tilde{U} : C^2 \otimes C^2 \to C^4 \otimes C^2$ defined as

$$\tilde{U} = \sum_{k=1}^{8} |k\rangle V_k$$

(19)

From tracepreserving property of the map $\Lambda_{CP}$ we immediately see that $\tilde{U}^\dagger \tilde{U} = I$ where identity acts on $C^2 \otimes C^2$ so it is isometry and preserves scalar product of its arguments. The isometry has the property that for any $2 \otimes 2$ partial trace of the $4 \otimes 2$ operator $\tilde{U} g \tilde{U}^\dagger$ with respect to the left subsystem reproduces the action of the map $\Lambda_{CP}$. Indeed we have

$$Tr_A(\tilde{U} g \tilde{U}^\dagger) = \Lambda_{CP}(g).$$

(20)

From the conditions [16], [18] we see that $\tilde{U}$ satisfies the two following conditions:

$$\tilde{U}|\phi\rangle|\phi\rangle = |\Phi\rangle|\phi\rangle$$

(21)

$$\tilde{U}|\phi^\perp\rangle|\phi\rangle = |\Psi_{\text{max}}\rangle$$

(22)

where $|\Psi_{\text{max}}\rangle = |\Psi_{\text{max}}(\phi)\rangle$ is some maximally entangled vector in space $C^8 \otimes C^2$. The state $|\Phi\rangle = |\Phi(\phi)\rangle$ and lives in $C^8$. Properties of $\tilde{U}$ implies that it maps the standard basis $|0\rangle|0\rangle, |0\rangle|1\rangle, |1\rangle|0\rangle, |1\rangle|1\rangle$, into the orthonormal basis $|\eta\rangle|0\rangle, |\Psi_{\text{max}}\rangle, |\Psi'_{\text{max}}\rangle, |\eta'\rangle|1\rangle$ where $|\Psi_{\text{max}}\rangle, |\Psi'_{\text{max}}\rangle$ are maximally entangled. Consider now a normalised vector $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$. Then according to the above we have

$$\tilde{U}|\phi\rangle|\phi\rangle = \alpha^2|\eta\rangle|0\rangle + \alpha\beta(|\Psi_{\text{max}}\rangle + |\Psi'_{\text{max}}\rangle) + \beta^2|\eta'\rangle|1\rangle =$$

$$|\Phi\rangle(\alpha|0\rangle + \beta|1\rangle)$$

(23)

We can act on both sides the linear operator

$$A_0 \equiv I \otimes |0\rangle$$

(24)

Note that $A_0 : C^2 \otimes C^2 \to C^2$.

Application of $A_0$ results in equivalence $\alpha^2|\eta\rangle + \alpha\beta(A_0|\Psi_{\text{max}}\rangle + |\Psi'_{\text{max}}\rangle) = \alpha|\Phi\rangle$. This leads to $\alpha|\eta\rangle + \beta|u_0\rangle = |\Phi\rangle$ where $|u_0\rangle \equiv A_0|\Psi_{\text{max}} + \Psi'_{\text{max}}\rangle$ and $\alpha$ is nonzero. The vector $|\Phi\rangle$ is normalised so taking $\alpha$ real, $\beta$ real or purely imaginary we obtain that independently on value of $\alpha$

$$\alpha^2 + (1 - \alpha^2)||u_0||^2 + x\alpha \sqrt{1 - \alpha^2} = 1$$

(25)

where $x$ is equal either real or imaginary part of $\langle \eta|u_0\rangle$. This condition (25) gives immediately

$$||u_0|| = 1$$

$$\langle \eta|u_0\rangle = 0$$

(26)

Consider now the condition (22) again for the vector $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$. Because $\tilde{U}|\phi\rangle|\phi^\perp\rangle$ is supposed to be maximally entangled then the state $A_0\tilde{U}|\phi\rangle|\phi^\perp\rangle$ living on $C^4$ has to have the norm $\frac{1}{2}$, e.

$$||A_0\tilde{U}|\phi\rangle|\phi^\perp\rangle||^2 = \frac{1}{2}$$

(27)

Now we act the operation $\tilde{U}$ on the vector $|\phi\rangle|\phi^\perp\rangle$ with normalised $|\phi\rangle = \alpha|0\rangle + \sqrt{1 - \alpha^2}|1\rangle$. Rewriting the condition (27) with help of conditions (26), and the fact that $|\Psi_{\text{max}}\rangle$ is maximally entangled lead simply to

$$\alpha^2(1 - \alpha^2) + 2\alpha(1 - \alpha^2)^{3/2}\text{Re}(|\Psi_{\text{max}}\rangle|\eta\rangle)$$

$$+ 2\alpha^2(1 - \alpha^2)^{3/2}\text{Re}(|\Psi'_{\text{max}}\rangle|u_0\rangle) - \alpha^4 + 2(1 - \alpha^2)^2 = \frac{1}{2}$$

(28)

This gives the expected contradiction as $\alpha$ can have arbitrary value for the interval $[0, 1]$. This concludes the proof that the map (15) and can not be performed within quantum mechanics.

It seems that the following general conjecture is true.

**Conjecture.** - There is no SPA of the nonlinear map (3).

Note that nonexistence of SPA is equivalent to nonexistence of SCPA as one can be obtained from another by suitable rescaling procedure.

To conclude the results the section: according to quantum mechanics not only the map $\rho \otimes \rho \to \rho^2$ but also $\rho \otimes \rho \to (1 - Tr(\rho^2))I/d + \rho$ is physically impossible. However it is very interesting that, as we shall see subsequently, the value of $Tr(\rho^4)$ can be measured in a very simple way.
VI. TSALLIS ENTROPY AS “MULTICOPY” OBSERVABLE AND MULTICOPY ENTANGLEMENT WITNESSES

A. Multicopy observables

We propose to extend the notion of quantum observable \( A \) to \( n \)-copy observable \( A^{(n)} \). Suppose that the system state is defined on Hilbert space \( \mathcal{H} \). Then measurement of \( A \) performed on \( \rho \) leads to the mean value \( \langle A \rangle = \text{Tr}(A\rho) \).

Here we propose the simple definition

**Definition 3** - Let \( A^{(n)} \) be the hermitian operator on \( \mathcal{H}^{\otimes n} \). We interpret it as \( n \)-copy observable with respect to the single system defined on a single Hilbert space \( \mathcal{H} \) by defining “mean value” of \( A^{(n)} \) on \( \rho \) as:

\[
\langle \langle A^{(n)} \rangle \rangle = \langle A^{(n)} \rangle_{\rho^{\otimes n}} = \text{Tr}(A^{(n)} \rho^{\otimes n})
\]

(29)

**Example 1**: “Swap” observable. Consider the “swap” or “flip” operator \( 1 \) on two-system space which has the property \( V|\Phi\rangle \otimes |\Psi\rangle = |\Psi\rangle \otimes |\Phi\rangle \) for any \( \Phi, \Psi \in \mathbb{C}^d \). It has been shown \( 8 \) that:

\[
\text{Tr}(VA \otimes B) = \text{Tr}(AB)
\]

(30)

It can be particularly easy illustrated on product pure state

\[
\text{Tr}(V|\phi\rangle \otimes |\psi\rangle \langle \psi \langle \phi \rangle) = \text{Tr}(|\psi\rangle \langle \psi \langle \phi \rangle) \text{Tr}(|\phi\rangle \langle \phi \rangle) = |\langle \psi \langle \phi \rangle|^2 = \text{Tr}(|\phi\rangle \langle \phi \rangle) (31)
\]

In a similar way if we consider means value of \( V \) in state \( \rho \otimes \rho \) with \( \rho = \sum \lambda_i P_{\rho_i} \). Here \( P_{\rho_i} \) is a projector corresponding to the eigenvector \( |\phi_i\rangle \langle \phi_i| \) then we obtain

\[
\text{Tr}(V\rho \otimes \rho) = \sum_{i,j} \lambda_i \lambda_j \text{Tr}(VP_{\rho_i} \otimes P_{\rho_j}) = \\
\sum_{i,j} \lambda_i \lambda_j |\langle \phi_i \rangle |^2 = \sum_i \lambda_i^2 = \text{Tr} (\rho^2).
\]

(32)

This reproduces the value implied by general formula (30). According to the above “flip” or “swap” \( V \) operator can be viewed as \( 2-copy \) observable: the formula (32) leads to the conclusion that the value \( \text{Tr}(\rho^2) \) is measurable iff two copies of the state are available. From that we get immediately that the Tsallis entropy \( S_q^T (\rho) = \frac{1-\text{Tr}(\rho^q)}{q-1} \) is measurable for \( q = 2 \). Indeed we take the observable \( W = I - V \) and \( S_q^T (\rho) = \langle W \rangle_{\rho^{\otimes n}} = \text{Tr}(W\rho \otimes \rho) \).

**Example 2**: “Shift” operation and related multicopy observables. Consider the well-known natural generalisation of the “swap” \( V \). This is a “shift” operation \( V_n \) which can be interpreted as some cyclic permutation. It is defined as \( V^{(n)} u_1 \otimes u_2 \otimes ... \otimes u_n = u_2 \otimes ... \otimes u_n \otimes u_1 \). It is known that (see [23]) \( \text{Tr}(V^{(n)} A_1 \otimes ... \otimes A_n) = \text{Tr}(A_1...A_n) \). Unfortunately is in general not hermitian (which can be checked directly looking at its decomposition into “swaps” \( V = V^{(2)} \)). But \( \text{Tr}(X\sigma) \) for any operator \( X \) and state \( \sigma \) can be experimentally checked by measuring hermitian (defined as \( X_h \equiv \frac{1}{2}(X + X^\dagger) \)) and antihermitian (defined as \( X_\alpha \equiv \frac{1}{2}(X - X^\dagger) \)) part of \( X \) because

\[
\langle X \rangle = \text{Tr}(X\sigma) = \text{Tr}(X_h\sigma) - i\text{Tr}(X_\alpha \sigma) = \langle X_h \rangle - i\langle X_\alpha \rangle.
\]

(33)

Thus one can determine “mean value” \( \text{Tr}(V^{(n)} \sigma) \) of nonhermitian operator \( V^{(n)} \) by experimental measurement of only two hermitian observables.

B. Multicopy entanglement witnesses

As a separability conditions entropy inequality was initiated in [14] and continued [15–17] as Renyi entropy analysis in context of separability. At present we know that any separable state satisfy the entropy inequality

\[
S_\alpha (\rho_{AB}) - S_\alpha (\rho_X) \geq 0, \quad X = A, B
\]

(34)

they are equivalent to the Tsallis entropy inequalities:

\[
S_q^T (\rho_{AB}) - S_q^T (\rho_X) \geq 0, \quad X = A, B
\]

(35)
The fact that inequalities above are satisfied by separable states was proved in steps (for different parameters $\alpha$) \cite{14,15,16,17,18,19} with proof completing the result for all $\alpha$-s given in Ref. \cite{20}. As such they form necessary conditions for separable states. Now for $q = 2$ we easily show that the condition can be measured directly by the following observable - an example of what we call multicipy entanglement witness. Consider the two copy state $\varrho \otimes \varrho \equiv \varrho_{AA'} \otimes \varrho_{BB'}$ on the systems $AA'BB'$ and the following observables

$$W^A = V_{(AA'),(BB')} - I_{AA'} \otimes V_{BB'} \tag{36}$$

and

$$W^B = V_{(AA'),(BB')} - V_{AA'} \otimes I_{BB'} \tag{37}$$

with $V_{XY}$ is a “swap” or “flip” operator between space $\mathcal{H}_X$ and $\mathcal{H}_Y$ (see \cite{13}. Evidently meanvalue $\langle (W^X) \rangle_{\varrho} = \langle W^X \rangle_{\varrho \otimes \varrho}$ is positive iff the inequalities (35), (34) are satisfied for $X = A, B$ and $q = 2$. Thus, because $\langle (W^X) \rangle_{\varrho}$ is (i) positive for all separable states $\varrho$ (ii) negative for some entangled states $\varrho$ (those that violate (35), (34)) we propose to call them multicipy entanglement witnesses. In general it is likely that multicipy observables like the above entanglement witnesses can be useful not only in quantum information theory but quantum domain in general.

Simple detection of entanglement by multicipy entanglement witnesses.-

The above scheme can be immediately generalised to $n$-copies with one exception - the multicipy generalisations of \cite{30}, \cite{37} are not hermitian, so to estimate their “mean values” we need separate measurements of hermitian and antihermitian parts. The corresponding (nonhermitian) multiparticle operators composed of both parts could be called multicipy entanglement quasi-witnesses as they detect entanglement but they are not hermitian.

VII. REMARKS ON STATE SPECTRUM ESTIMATION

It is remarkable that if we want to determine spectrum of the state $\varrho$ defined on $m$ dimensional Hilbert space than all we need is the $m-1$ values $Tr(\varrho^2), Tr(\varrho^3), ..., Tr(\varrho^m)$. This can be seen by realising that $Tr(\varrho^k) = (p_1)^k + (p_2)^k + ... + (p_m)^k$ where $p_i$ stand for spectrum of $\varrho$. The effectiveness is determined by unique solution because finite discrete random variable is determined by its first $m$ moments.

Thus we have the following

**Conclusion .-** In order to determine the spectrum of (completely unknown) state $\varrho$ it is enough to determine the mean values of $2m - 3$ observables: $V_1^{(2)}, V_1^{(3)}, V_1^{(4)}, ..., V_1^{(m)}$, $V_2^{(k)}, V_2^{(k)}$ where $V_1^{(k)}, V_2^{(k)}$ are hermitian and antihermitian parts of “shift” (or generalised swap) operation $V(k)$.

**Remark .-** This result is complementary to what is known in literature so far we have had either (i) to perform full tomography: estimation of mean values of $m^2$ observables, each via single copy statistics or (ii) asymptotic estimation of Yang frames requiring estimation of sequence of $n$-copy observables with $n$ approaching infinity. The method (ii) was shown to be optimal (see \cite{22}). It involves effectively only $m$ observables (because it requires estimation of probabilities of results of $m$ output observables) but requires possibility of collective measurements on arbitrary number of copies to get arbitrary good accuracy. The former (i) requires only single copy per measurement but $m^2$ observable are needed. In present approach we see that only $2m - 3$ observables we need if collective measurements on finite numbers of $2, 3, ..., m$ copies is possible. In that sense it represents some kind of compromise between to previous two methods (i) and (ii).

Finally note that the Renyi or Tsallis inequalities discussed in previous section involve only spectra of system and subsystem density matrices. Using the above spectrum estimation for system and subsystem state one can check the inequalities just putting the estimated spectra in them.

VIII. CONCLUSIONS

We have considered the possibility of transformation of getting $n$-th power of the state $\varrho$ given $n$ copies of it. We have shown that it is impossible. We analysed possibility of structural physical approximations (SPA) of unphysical maps ones under the condition of complete symmetry of approximation error $\Delta$. We have pointed out that it is possible to approximate any linear hermitian map in such a way. We have optimised SPA of any nontrivial hermitian map under the assumption of continuity of shrinking factor $\gamma$.

On the other hand we have shown that the natural tracepreserving approximation of (unphysical) operation of the power of the state in the above sense is impossible for power 2 in the sense of SPA. It was conjectured that this “no go ” property it is true in general i. e. that no SPA of the transformation $\varrho \otimes \cdots \otimes \varrho \rightarrow \varrho^n$ exists.
It has been pointed out that, however, given two copies of $\rho$ the nonlinear function of the state defined by $Tr(\rho^2)$ can be easily measurable. This leads to the physical “twocopy observable” measuring the Tsallis entropy. This approach allows to consider notion of multcopy observables and, in particular, to leads to the definition of multcopy entanglement witnesses. The examples of the latter has been provid ed, measuring degree of violation of separability conditions can be checked by means of two entropies which can be estimated by means of only $k$ multicopy observables. Thus the corresponding separability conditions can be checked by means of pairs of hermitian observables. Finally the existence of simple method of spectrum estimation for unknown state has been pointed out which requires collective measurements on small number of copies. The number of needed estimated parameters is $2dimH - 3$ which is less than $(dimH)^2 - 1$ required in tomography. One can hope that the multcopy observables idea together with physical interpretation of Tsallis entropy in context of multcopy observables can be useful not only for quantum entanglement theory but also for quantum physics in general.

The author thanks Vladimir Buzek and Marek Czachor for interesting discussions on nonlinearity in context of quantum mechanics. He is grateful to Artur Ekert for pointing out inconsistency in erlier version of this work and for helpful discussions. Remarks of Joachim Domsta on classical probability theory are also acknowledged. The work is supported by Polish Committee for Scientific Research, contract No. 2 P03B 103 16, and by the European Union, project EQUIP, contract No. IST-1999-11053.

IX. APPENDIX

Here according to well-known quantum mechanical procedure known form Kraus (see [23]) we shall recall how any completely positive map $\Lambda$ satisfying

\[ Tr[\Lambda(\rho)] \leq 1 \]  \hspace{1cm} (38)

can be probabilistically implemented in the laboratory. We know that $\Lambda(\cdot) = \sum_{i=1}^{k} V_i(\cdot) V_i^\dagger$. From (38) remembering that $Tr[\sum_{i=1}^{k} V_i V_i^\dagger] = Tr[\sum_{i=1}^{k} V_i^\dagger V_i]$, we get that the positive operator $A_0 = \sum_{i=1}^{k} V_i^\dagger V_i$ has the form of the interval $[0,1]$. Thus we can define $V_0 = \sqrt{I - A_0}$ an then the extended completely positive map $\Lambda' = \Lambda(\cdot) + V_0(\cdot) V_0^\dagger = \sum_{i=1}^{k} V_i(\cdot) V_i^\dagger$ is tracepreserving because $Tr[\Lambda(\rho)] = Tr[\sum_{i=1}^{k} V_i \rho V_i^\dagger] = Tr[\sum_{i=0}^{k} V_i^\dagger V_i] = Tr[(A_0 + I - A_0)\rho] = Tr(I\rho) = Tr(\rho) = 1$. But any tracepreserving map can be implemented in lab by interaction with some additional quantum system (ancilla) and some von Neumann measurement on that system with outputs $i = 0, 1, ..., k$ (for description see Ref. [24]). In that case the $i$-th “event” corresponds to the single map $V_i \cdot V_i^\dagger$. It can be interpreted as “producing” unnormalised state $V_i \rho V_i^\dagger$. Indeed though its action results in normalised state $\rho_i = V_i \rho V_i^\dagger / p_i$ this occurs only with probability $p_i = Tr(V_i \rho V_i^\dagger)$. In this sense the original map $\Lambda$ can be implemented: we apply some special von Neumann measurement on the ancilla and keep the system if only the i-th event with $i \neq 0$ occurs. If the singed out event corresponding to $i = 0$ occurs we “discard” our system. This gives the new state $\rho' = \Lambda(\rho)/p$ with probability $p = Tr[\Lambda(\rho)]$.

* e-mail: pawel@mif.pg.gda.pl

[1] H. Bachmann-Pasquinucci, B. Huttner, N. Gisin, Phys. Lett. A 242 (1998) 198.
[2] D. R. Terno, Phys. Rev. A 59 (1999) 3320.
[3] T. Mor, D. R. Terno, Phys. Rev. A 60 (1999) 4341.
[4] For mixtures of Bell-diagonal states see T. A. Brunn, C. M. Caves, R. Schack, Phys. Rev. A 63 (2001) 042309 ; this can be generalised to any entangled 2 $\otimes$ 2 state [P. Horodecki, unpublished].
[5] R. Jozsa, M. Horodecki, P. Horodecki and R. Horodecki, Phys. Rev. Lett. 81, 1714 (1998).
[6] V. Buzek and M. Hillery, Phys. Rev. A 54 (1996) 802.
[7] V. Buzek and M. Hillery, Phys. Rev. Lett. 81 (1998) 5003.
[8] R. F. Werner, Phys. Rev. A 58 (1998) 1827.
[9] V. Buzek, M. Hillery and R. F. Werner, Phys. Rev. A, 60 (1999) 2626; V. Buzek, M. Hillery and R. F. Werner, J. Mod. Opt. 47 (2000) 211.
[10] A. Winter, J. Phys. A: Math. Gen. 34 7095 (2001).
[11] M. D. Bowdrey, D. K. L. Oi, A. J. Short, J. A. Jones, “Fidelity of single qubit maps”, quant-ph/0103090v1.
[12] D. K. L. Oi, “A geometry of single qubit maps”, quant-ph/0106035v1.
[13] R. F. Werner, Physical Review A, 40 (1989) 4277.
[14] R. Horodecki, P. Horodecki, Phys. Lett. A, 194 (1994) 146.
[15] R. Horodecki, P. Horodecki and M. Horodecki, Phys. Lett. A 210 (1996) 377.
[16] R. Horodecki and M. Horodecki, Phys. Rev. A 54 (1996) 1838.
[17] N. J. Cerf and C. Adami, Phys. Rev. Lett. 79 (1997) 5194.
[18] N. J. Cerf, C. Adami, R. M. Gingrich, Phys. Rev. A 60 (1999) 893.
[19] P. Horodecki, J. A. Smolin, B. M. Terhal and A. V. Thapliyal, “Rank two bipartite bound entangled states do not exist”,
quant-ph/9910012.
[20] C. Tsallis, S. Lloyd, M. Baranger, Phys. Rev. A, 63 (2001) 041204.
[21] B. M. Terhal, “Detecting quantum entanglement” quant-ph/0101032.
[22] K. G. H. Vollbrecht, M. M. Wolf, “Can spectral and local information decide separability?” quant-ph/0107014.
[23] M. Keyl, R. F. Werner, Phys. Rev. A 64 (2001) 052311.
[24] C. J. Isham, N. Linden, S. Scheckenberg, J. Math. Phys. 35 (1994) 6360.
[25] R. Alicki, K. Lendi, “Quantum Dynamical Semigroups and Applications”, vol. 286 of “Lecture Notes in Physics”, Springer 1987.