DISTRIBUTIVE AND LOWER-MODULAR ELEMENTS OF THE LATTICE OF MONOID VARIETIES

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Abstract. The sets of all neutral, distributive and lower-modular elements of the lattice of semigroup varieties are finite, countably infinite and uncountably infinite, respectively. In 2018, we established that there are precisely three neutral elements of the lattice of monoid varieties. In the present work, it is shown that the neutrality, distributivity and lower-modularity coincide in the lattice of monoid varieties. Thus, there are precisely three distributive and lower-modular elements of this lattice.

1. Introduction and summary

An element $x$ of a lattice $L$ is

neutral if $\forall y, z \in L: (x \lor y) \land (y \lor z) \land (z \lor x) = (x \land y) \lor (y \land z) \lor (z \land x)$;

standard if $\forall y, z \in L: (x \lor y) \land z = (x \land z) \lor (y \land z)$;

distributive if $\forall y, z \in L: x \lor (y \land z) = (x \lor y) \land (x \lor z)$;

modular if $\forall y, z \in L: y \leq z \rightarrow (x \lor y) \land z = (x \land z) \lor y$;

cancellable if $\forall y, z \in L: x \lor y = x \lor z \land x \land y = x \land z \rightarrow y = z$;

lower-modular if $\forall y, z \in L: x \leq y \rightarrow x \lor (y \land z) = y \land (x \lor z)$.

Costandard, codistributive and upper-modular elements are defined dually to standard, distributive and lower-modular ones respectively. Neutral, cancellable and modular elements are self-dual. It is evident that a neutral element is both standard and costandard; a standard or costandard element is cancellable; a cancellable element is modular; a [co]distributive element is lower-modular [upper-modular]. It is well known also that a [co]standard element is [co]distributive (see [2, Theorem 253], for instance). These types of special elements play an important role in general lattice theory; significant information about special elements in a lattice can be found in [2, Section III.2].

Many articles were devoted to special elements of different types in the lattice $SEM$ of all semigroup varieties; an overview of results published before 2015 can be found in the survey [11]. In 2018, the study of special elements in the lattice $MON$ of all monoid varieties was started (referring to monoid varieties, we consider monoids as algebras of type $(2,0)$). Now there are three articles on this topic.
In [3], neutral and costandard elements of the lattice $\text{MON}$ are described. In [4], it is shown that an element of $\text{MON}$ is standard if and only if it is neutral. Finally, cancellable elements of the lattice $\text{MON}$ are completely determine in [5]. In the present work, we continue these investigations. We describe all distributive and lower-modular elements in the lattice $\text{MON}$.

Let $\text{T}$, $\text{SL}$, and $\text{MON}$ denote the variety of trivial monoids, the variety of semilattice monoids, and the variety of all monoids, respectively. Our main result is the following theorem.

**Theorem.** For a monoid variety $V$, the following are equivalent:

(i) $V$ is a lower-modular element of the lattice $\text{MON}$;
(ii) $V$ is a distributive element of the lattice $\text{MON}$;
(iii) $V$ is a standard element of the lattice $\text{MON}$;
(iv) $V$ is a neutral element of the lattice $\text{MON}$;
(v) $V$ is one of the varieties $\text{T}$, $\text{SL}$ or $\text{MON}$.

Note that the equivalence of items (iv) and (v) is proved in [3, Theorem 1.1], while the equivalence of items (iii) and (v) is established in [4, Theorem 1].

The theorem differs sharply from earlier results on special elements of the lattice $\text{SEM}$. The set of all lower-modular elements of $\text{SEM}$ is uncountably infinite (this fact easily follows from Theorem 3.2 in [10]); the set of all standard elements of $\text{SEM}$ is countably infinite (see [10, Theorem 3.3]); the set of all neutral elements of $\text{SEM}$ is finite (see [10, Theorem 3.4]). Moreover, an element of $\text{SEM}$ is standard if and only if it is distributive (see [10, Theorem 3.3]). The theorem implies that these four types of special elements coincide in the lattice $\text{MON}$ and the number of such elements is finite.

In general, a distributive element in a lattice need not be costandard. The theorem together with Theorem 1.2 in [3] implies the following interesting fact.

**Corollary 1.** Each lower-modular and so distributive element is costandard and so cancellable, codistributive, modular and upper-modular one in the lattice $\text{MON}$. □

An element of $\text{SEM}$ is modular whenever it is lower-modular (see [10, Corollary 3.9]); is cancellable whenever it is distributive (compare Theorem 1.1 in [9] and Theorem 3.3 in [10]). However, the costandardity does not follow from the distributivity (compare Theorems 3.3 and 3.4 in [10]) and the lower-modularity does not imply the cancellability in $\text{SEM}$ (compare Theorem 1.1 in [9] and Theorem 3.2 in [10]).

It is well known that the set of all neutral [standard] elements of a lattice forms a sublattice (see [2, Theorem 259]). In general, the sets of distributive or lower-modular elements in a lattice need not form a sublattice. Nevertheless, the following fact easily follows from the theorem.

**Corollary 2.** The set of all distributive [lower-modular] elements of the lattice $\text{MON}$ forms a sublattice. □

Note that the set of all distributive elements of $\text{SEM}$ forms a sublattice (this easily follows from Theorem 3.3 in [10]) but the set of all lower-modular elements of $\text{SEM}$ does not form a sublattice (this can be easily deduced from Theorem 3.2 in [10]).

The article consists of three sections. Section 2 contains definitions, notation and auxiliary results. Section 3 is devoted to the proof of the theorem.
2. Preliminaries

Let $X^*$ denote the free monoid over a countably infinite alphabet $X$. Elements of $X$ are called variables and elements of $X^*$ are called words. Words unlike variables are written in bold. An identity is written as $u \approx v$, where $u, v \in X^*$. Let $\lambda$ denote the empty word. An identity is written as $u \approx v$, where $u, v \in X^*$. Let $\lambda$ denote the empty word. We denote by $\text{End}(X^*)$ the endomorphism monoid of the monoid $X^*$. An identity is written as $u \approx v$, where $u, v \in X^*$. For any identity system $\Sigma$, let $\text{var} \Sigma$ denote the variety of monoids defined by $\Sigma$.

The following assertion is a specialization for monoids of a well-known universal-algebraic fact (see [1, Theorem II.14.19]).

Lemma 1. An identity $u \approx v$ holds in $\text{var} \Sigma$ if and only if there exists some finite sequence $u = w_0, w_1, \ldots, w_n = v$ of distinct words such that for any $i \in \{0, 1, \ldots, n-1\}$ there exist the words $a_i, b_i \in X^*$, the endomorphism $\xi_i \in \text{End}(X^*)$ and the identity $s_i \approx t_i \in \Sigma$ for which $\{w_i, w_{i+1}\} = \{a_i \xi_i(s_i) b_i, a_i \xi_i(t_i) b_i\}$. □

A variety of monoids is called completely regular if it consists of completely regular monoids, that is, unions of groups. Let $C = \text{var}\{x^2 \approx x^3, xy \approx yx\}$. The equivalence of items a) and b) of the following lemma is well known and can be easily verified, while the equivalence of items a) and c) is established in [6, Corollary 2.6].

Lemma 2. For a monoid variety $V$, the following are equivalent:

a) $V$ is completely regular;

b) $V$ satisfies the identity $x \approx x^{1+n}$ for some $n \geq 1$;

c) $C \not\subseteq V$. □

A monoid variety is combinatorial if all its groups are trivial. The following statement is well known and can be easily verified.

Lemma 3. A monoid variety $V$ is combinatorial if and only if it satisfies the identity $x^n \approx x^{n+1}$ for some $n \geq 1$. □

Let $\text{LRB} = \text{var}\{xy \approx xyx\}$ and $\text{RRB} = \text{var}\{xy \approx yxy\}$. The following statement can be easily deduced from Proposition 4.7 in [11].

Lemma 4. If a variety $V$ of idempotent monoids does not contain $\text{LRB}$, then $V$ coincides with one of the varieties $\text{T}$, $\text{SL}$ or $\text{RRB}$. □

For any word $w$, we denote by $\text{ini}(w)$ the word obtained from $w$ by retaining only the first occurrence of each variable. The following statement is well known and can be easily verified.

Lemma 5. An identity $u \approx v$ is satisfied by $\text{LRB}$ if and only if $\text{ini}(u) = \text{ini}(v)$. □

3. Proof of the theorem

To prove the theorem, we need some definitions, notation and one auxiliary lemma. The content of a word $w \in X^*$, that is, the set of all variables occurring in $w$ is denoted by $\text{con}(w)$, while the length of $w$ is denoted by $\ell(w)$. For a word $w \in X^*$ and a variable $x \in X$, let $\text{occ}_x(w)$ denote the number of occurrences
of \( x \) in \( w \). For any variety \( V \) of monoids, let \( \text{FIC}(V) \) denote the fully invariant congruence on \( X^* \) corresponding to \( V \). A word \( w \) is an isoterm for a variety \( V \) if the \( \text{FIC}(V) \)-class of \( w \) is singleton. A monoid variety is proper if it is different from \( \text{MON} \).

**Lemma 6.** Let \( V \) be a proper monoid variety. If \( LRB \subseteq V \), then \( V \) is not a lower-modular element in \( \text{MON} \).

**Proof.** Since \( V \) is a proper monoid variety, \( V \) satisfies some non-trivial identity \( u_1 \approx v_1 \). There are distinct variables \( x \) and \( y \) such that the identity obtained from \( u_1 \approx v_1 \) by retaining only the variables \( x \) and \( y \) is non-trivial. This allows us to assume that the words \( u_1 \) and \( v_1 \) depend on the variables \( x \) and \( y \) only. Clearly, \( V \) satisfies the identities

\[
x^{\text{occ}_x(u_1)+1} \approx x^{\text{occ}_x(v_1)+1} \quad \text{and} \quad y^{\text{occ}_y(u_1)+1} \approx y^{\text{occ}_y(v_1)+1},
\]

and so the identity

\[
u_1 x^{\text{occ}_x(v_1)+1} y^{\text{occ}_y(v_1)+1} \approx v_1 x^{\text{occ}_x(u_1)+1} y^{\text{occ}_y(u_1)+1}.
\]

One can multiply both sides of the last identity on the left by the variables \( x \) and \( y \) and obtain the identity both sides of which contain all these variables exactly \( n \) times for some \( n \geq 2 \). Therefore, \( V \) satisfies a non-trivial identity \( u \approx v \) such that \( \text{con}(u) = \text{con}(v) = \{x, y\} \) and \( \text{occ}_x(u) = \text{occ}_x(v) = \text{occ}_y(u) = \text{occ}_y(v) = n \).

In view of the inclusion \( LRB \subseteq V \) and Lemma 6, we may assume without any loss that \( \text{ini}(u) = \text{ini}(v) = xy \). Let \( u' \) and \( v' \) be words obtained from \( u \) and \( v \), respectively, by performing the substitution \( (x, y) \mapsto (y, x) \). Then \( \text{ini}(u') = \text{ini}(v') = yx \).

Let

\[
X = \text{var}\{ux \approx u'x, vx \approx v'x\} \quad \text{and} \quad Y = \text{var}\{ux \approx vx\}.
\]

Let us show that the set \( \{ux, u'x\} \) forms a \( \text{FIC}(X) \)-class. To do this, it suffices to verify that if \( X \) satisfies \( p \approx q \) and \( p \in \{ux, u'x\} \), then \( q \in \{ux, u'x\} \). In view of Lemma 1 it suffices to consider the case when \( (p, q) = (a\xi(s)b, a\xi(t)b) \), where \( a, b \in X^* \), \( \xi \in \text{End}(X^*) \) and either \( \{s, t\} = \{ux, u'x\} \) or \( \{s, t\} = \{vx, v'x\} \). Evidently, if \( \xi(x) = \lambda \) or \( \xi(y) = \lambda \), then \( \xi(s) = \xi(t) \) and so \( q = p \in \{ux, u'x\} \). Therefore, we may assume that the words \( \xi(x) \) and \( \xi(y) \) are non-empty. Then

\[
\ell(\xi(s)) \geq \ell(s) = 2n + 1 = \ell(p).
\]

Hence \( \xi(x) \) and \( \xi(y) \) are variables and \( a = b = \lambda \). Taking into account that

\[
\text{occ}_x(p) = \text{occ}_x(s) = n + 1 \quad \text{and} \quad \text{occ}_y(p) = \text{occ}_y(s) = n,
\]

we have that \( \xi(x) = x \) and \( \xi(y) = y \). This is only possible when \( s = p \), whence \( \{s, t\} = \{ux, u'x\} \) and so \( q \in \{ux, u'x\} \).

Thus, the set \( \{ux, u'x\} \) forms a \( \text{FIC}(X) \)-class. By similar arguments we can show that \( \{vx, v'x\} \) is a \( \text{FIC}(X) \)-class and \( u'x \) is an isoterm for \( Y \). These facts, the inclusion \( LRB \subseteq V \) and Lemma 6 imply that the words \( ux, u'x, vx, v'x \) are isoterm for \( V \lor X \). Then \( u'x \) must be an isoterm for \( Y \lor (V \lor X) \). Obviously, the variety \( Y \lor X \) satisfies the identity \( u'x \approx v'x \). Clearly, this identity is also satisfied by the variety \( V \). Therefore, \( V \lor (Y \lor X) \) satisfies \( u'x \approx v'x \). Since \( V \subseteq Y \), this implies that

\[
V \lor (Y \land X) \subseteq Y \land (V \lor X).
\]
Thus, the variety $V$ is not a lower-modular element of the lattice $\mathbb{MON}$, and we are done. □

**Proof of the theorem.** As we have noted in the introduction, implications (iv) $\iff$ (v) are proved in [3, Theorem 1.1], implications (iii) $\iff$ (v) are verified in [3, Theorem 1], implication (iii) $\Rightarrow$ (ii) follows from [2, Theorem 253], while implication (ii) $\Rightarrow$ (i) is obvious. Thus, it remains to verify implication (i) $\Rightarrow$ (v).

Let $V$ be a proper monoid variety, which is a lower-modular element in $\mathbb{MON}$. Then $V$ is periodic, i.e., it consists of periodic monoids by [4, Lemma 1]. It is well known and can be easily verified that any periodic variety satisfies the identity $x^n \approx x^{n+m}$ for some $n, m \geq 1$. Assume that $n$ and $m$ are the least numbers such that $x^n \approx x^{n+m}$ holds in $V$. Two cases are possible.

**Case 1:** $V$ is completely regular. Then $n = 1$ by Lemma [2].

Suppose that $V$ contains a non-trivial group. Then $m > 1$ by Lemma [3]. Put

$$u_1 = x^{4m+1}yx^{m+1}, \quad u_2 = x^{2m+2}yx^{3m+1}, \quad v_1 = x^{3m+1}yx^{2m+1}, \quad v_2 = x^{m+2}yx^{4m+1}.$$ 

Let

$$X = \text{var}\{u_1 \approx u_2, v_1 \approx v_2\} \quad \text{and} \quad Y = \text{var}\{u_2 \approx v_2\}.$$ 

Let us show that the set $\{u_1, u_2\}$ forms a FIC($X$)-class. To do this, it suffices to verify that if $X$ satisfies $p \approx q$ and $p \in \{u_1, u_2\}$, then $q \in \{u_1, u_2\}$. In view of Lemma [1] it suffices to consider the case when $(p, q) = (a\xi(s)b, a\xi(t)b)$, where $a, b \in X^*, \xi \in \text{End}(X^*)$ and either $(s, t) = \{u_1, u_2\}$ or $(s, t) = \{v_1, v_2\}$. Obviously, if $\xi(x) = \lambda$, then $\xi(s) = \xi(t) = \xi(y)$ and so $q = p \in \{u_1, u_2\}$. If $\xi(y) = \lambda$ and $\xi(x) \neq \lambda$, then $\{\xi(s), \xi(t)\} = \{(\xi(x))^{5m+2}, (\xi(x))^{5m+3}\}$. In either case, $(\xi(x))^{5m+2}$ is a subword of $\xi(s)$. But this is impossible because the words $u_1$ and $u_2$ do not contain any subword that is the $(5m+2)$th power of a non-empty word. Therefore, we may further assume that the words $\xi(x)$ and $\xi(y)$ are non-empty. Then

$$\ell(\xi(s)) \geq \ell(s) \geq 5m + 3.$$ 

Hence $\xi(x)$ is a variable and, moreover, $\xi(x) = x$. If $y \not\in \text{con}(\xi(y))$, then $\text{con}(\xi(y)) = \{x\}$ and so $(\xi(x))^{5m+2}$ is a subword of $\xi(s)$ contradicting the above. Thus, $y \in \text{con}(\xi(y))$. It is easy to see that this is only possible when $\xi(y) = y$ and $p = \xi(s) = s$. Hence $q \in \{u_1, u_2\}$.

Therefore, $\{u_1, u_2\}$ forms a FIC($X$)-class. By similar arguments we can show that $\{v_1, v_2\}$ is a FIC($X$)-class and $u_1$ is an isoterm for $Y$. Note that $V$ violates the identities $u_1 \approx u_2$ and $v_1 \approx v_2$ because any variety satisfying one of these identities must satisfy the identity $x^{5m+3} \approx x^{5m+4}$ and so must be combinatorial by Lemma [3]. In view of the above, this implies that the words $u_1, u_2, v_1, v_2$ are isoterms for $V \lor X$. Then $u_1$ must be an isoterm for $Y \land (V \lor X)$. Obviously, the variety $Y \land X$ satisfies the identity $u_1 \approx v_1$. Clearly, this identity is also satisfied by the variety $V$ because it is a consequence of the identity $x \approx x^{m+1}$. Therefore, $V \lor (Y \land X)$ satisfies $u_1 \approx v_1$. Since $V \subseteq Y$, it follows that

$$V \lor (Y \land X) \subset Y \land (V \lor X).$$

Thus, $V$ is not a lower-modular element of the lattice $\mathbb{MON}$. This contradicts the assumption that $V$ contains a non-trivial group.

So, it remains to consider the case when the variety $V$ is combinatorial. Then $V$ is an idempotent variety because every combinatorial completely regular variety consists of idempotent monoids. It follows from Lemma [3] and the dual to it that
\( LRB, RRB \not\subseteq V \). Then Lemma 3 and the dual to it imply that \( V \) coincides with one of the varieties \( SL \) or \( T \), and we are done.

**Case 2:** \( V \) is not completely regular. Then \( C \subseteq V \) by Lemma 2. Lemma 6 allows us to assume that \( LRB \not\subseteq V \).

According to Lemma 5, \( V \) satisfies some identity \( u \approx v \) with \( \text{ini}(u) \neq \text{ini}(v) \). As in the proof of Lemma 6, one can choose the identity \( u \approx v \) so that \( \text{con}(u) = \{x, y\} \) and \( \text{occ}_x(u) = \text{occ}_x(v) = \text{occ}_y(u) = \text{occ}_y(v) = k \) for some \( k \geq 2 \) and, moreover, \( x^k \) and \( y^k \) are not subwords of \( u \) and \( v \).

Let

\[ E = \text{var}\{x^2 \approx x^3, x^2 y \approx xyx, x^2 y^2 \approx y^2 x^2\}. \]

Suppose that \( E \not\subseteq V \). In is verified in [7, Proposition 4.1] that \( E \subseteq C \lor LRB \). This fact and the inclusion \( C \subseteq V \) imply that

\[ (V \lor E) \land (V \lor LRB) = V \lor E. \]

Evidently, \( u \approx v \) holds in \( E \). It follows that \( V \lor E \) satisfies \( u \approx v \). Hence \( LRB \not\subseteq V \lor E \) by Lemma 5 and the fact that \( \text{ini}(u) \neq \text{ini}(v) \). This fact, the evident inclusion \( SL \subseteq V \lor E \) and Lemma 4 imply that

\[ V \lor ((V \lor E) \land LRB) = V \lor SL = V, \]

contradicting the fact that \( V \) is a lower-modular element in \( MON \).

Thus, it remains to consider the case when \( E \subseteq V \). Let

\[ X = \text{var}\{xtu \approx tru, xtv \approx txv\} \quad \text{and} \quad Y = \text{var}\{txu \approx txv\}. \]

Let us show that the set \( \{xtu, tru\} \) forms a \( \text{FIC}(X) \)-class. It suffices to verify that if \( X \) satisfies \( p \approx q \) and \( p \in \{xtu, tru\} \), then \( q \in \{xtu, tru\} \). In view of Lemma 3 it suffices to consider the case when \( (p, q) = (a\xi(s)b, a\xi(t)b) \), where \( a, b \in X^* \), \( \xi \in \text{End}(X^*) \) and either \( \{s, t\} = \{xtu, tru\} \) or \( \{s, t\} = \{xtv, txv\} \). Obviously, if \( \xi \) maps one of the variables \( x \) or \( t \) to the empty word, then \( \xi(s) = \xi(t) \) and so \( q = p \in \{xtu, tru\} \). Therefore, we may assume that the words \( \xi(x) \) and \( \xi(t) \) are non-empty. If \( \xi(y) = \lambda \), then \( \{\xi(s), \xi(t)\} = \{\xi(x)\xi(t)k, \xi(t)\xi(x)k+1\} \).

Since \( \xi(x) \neq \lambda \), \( \text{occ}_x(p) = k \) and \( \text{occ}_t(p) = 1 \), this is only possible when \( \xi(x) = x \). However, the last equality contradicts the assumption that the word \( x^k \) is not a subword of \( u \). Thus, \( \xi(y) \neq \lambda \). Then

\[ \ell(\xi(s)) \geq \ell(s) = 2k + 2 = \ell(p). \]

Hence \( \xi(x), \xi(y) \) and \( \xi(t) \) are variables and \( a = b = \lambda \). Then \( \xi(x) = x, \xi(y) = y \) and \( \xi(t) = t \) because

\[ \text{occ}_x(p) = \text{occ}_x(s) = k + 1, \quad \text{occ}_y(p) = \text{occ}_y(s) = k \quad \text{and} \quad \text{occ}_t(p) = \text{occ}_t(s) = 1. \]

This is only possible when \( s = p \), whence \( \{s, t\} = \{xtu, tru\} \) and so \( q \in \{xtu, tru\} \).

Thus, the set \( \{xtu, tru\} \) forms a \( \text{FIC}(X) \)-class. By similar arguments we can show that \( \{xtv, txv\} \) is a \( \text{FIC}(X) \)-class and \( xtv \) is an isomer for \( Y \). The inclusion \( E \subseteq V \) and Theorem 4.1(i) in [8] imply that \( xtu \) and \( txu \) lie in different \( \text{FIC}(V) \)-classes. In view of the above, this implies that these words are isomers for the variety \( V \lor X \). Then the word \( xtu \) is an isomer for \( Y \lor (V \lor X) \). Obviously, \( Y \lor X \) satisfies the identity \( xtu \approx xtv \). Clearly, this identity is also holds in \( V \). Thus, \( V \lor (Y \lor X) \) satisfies \( xtu \approx xtv \). Since \( V \subseteq Y \), it follows that

\[ V \lor (Y \land X) \subseteq Y \land (V \lor X). \]
This, however, contradicts the fact that $V$ is a lower-modular element of the lattice $\mathbb{M}_{ON}$. Thus, Case 2 is impossible. □

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