PASCAL TRIANGLE AND RESTRICTED WORDS

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Abstract. We continue to investigate combinatorial properties of functions $f_m$ and $c_m$ considered in our previous papers. They depend on an initial arithmetic function $f_0$. In this paper, the values of $f_0$ are the binomial coefficients.

We first consider the case when the values of $f_0$ are the binomial coefficients from a row of the Pascal triangle. The values of $f_0$ consider next are the binomial coefficients from a diagonal of the Pascal triangle. In two final cases, the values of $f_0$ are the central binomial coefficients and its adjacent neighbors. In each case, we derive an explicit formula for $c_1(n, k)$ and give its interpretation in terms of restricted words. In the first two cases, we also consider the functions $f_m$ and $c_m$, for ($m > 0$).

1. Introduction

In this paper, we continue to investigate the problem of the enumeration of restricted words. Previously, the functions $f_m(n)$ and $c_m(n, k)$ were defined as follows. For an initial arithmetic function $f_0$, $f_m(m > 0)$ is the $m$th invert transform of $f_0$. The function $c_m(n, k)$ was defined in the following way:

$$(1) \quad c_m(n, k) = \sum_{i_1 + i_2 + \ldots + i_k = n} f_{m-1}(i_1) \cdot f_{m-1}(i_2) \cdot \ldots \cdot f_{m-1}(i_k),$$

where the sum is over positive $i_1, i_2, \ldots, i_k$. Its connections with the problem of the enumeration of restricted words were consider. A number of results of this kind is obtained in Janjić [4, 5, 6]. Some results have also been considered by other authors, for instance, in [1, 2, 3, 7].

In this paper, we derive results for four types of initial functions, which have different kind of the binomial coefficients as values. In the first case, the values of $f_0$ are the binomial coefficients from a row of the Pascal triangle. In the second case, the values are from a diagonal of the Pascal triangle. In the remaining cases, the values of $f_0$ are the central binomial coefficients and its adjacent neighbors.

In the first two cases, we describe the restricted words counted by $f_m$ and $c_m$, and derive explicit formulas for these functions. In the last two cases, we consider the function $c_1$ only.

2. Rows of the Pascal Triangle

For a positive integer $a$, we want to enumerate words over the finite alphabet \{0, 1, \ldots, a - 1, \ldots\}, in which letters in words over \{0, 1, \ldots, a - 1\} are arranged into ascending order. We let $P_1$ denote this property.
We define \( f_0(n) = \left( \frac{a}{n-1} \right) \), \((n = 1, 2, \ldots)\).

It is clear that \( f_0(n) \) is the number of words of length \( n - 1 \) over the alphabet \( \{0, 1, \ldots, a-1\} \) satisfying \( P_1 \). Since \( f_0(1) = 1 \), using Janjić [6, Proposition 12], we obtain the following combinatorial meaning of \( c_m(n, k) \).

**Corollary 1.** The number of words of length \( n - 1 \) over the alphabet \( \{0, 1, \ldots, a-1, \ldots, a+m-1\} \) having \( k-1 \) letters equal to \( a+m-1 \) and satisfying \( P_1 \) is \( c_m(n, k) \).

According to [6, Corollary 2], we get

**Corollary 2.** The number of words of length \( n - 1 \) over the alphabet \( \{0, 1, \ldots, a-1, \ldots, a+m-1\} \) satisfying \( P_1 \) is \( f_m(n) \).

We next derive an explicit formula for \( c_1(n, k) \).

**Proposition 1.** We have

\[
c_1(n, k) = \binom{n-1}{k-1}.
\]

**Proof.** We use the induction on \( k \). From Janjić [5, Equation (3)], we have \( c_1(n, 1) = f_0(n) \), which means that the statement holds for \( k = 1 \). Suppose that the statement is true for \( k-1, \) \((k > 1)\). Using the recurrence [6, Equation (3)], and the induction hypothesis, we obtain

\[
c_1(n, k) = \sum_{i=1}^{n-k+1} \binom{a}{i-1} \binom{ak-a}{n-i-k+1},
\]

and the statement follows from the Vandermonde convolution.

As a consequence of [6] Equation (9), Proposition 7, we obtain the following explicit formulas for \( c_m(n, k) \) and \( f_m \).

**Corollary 3.** We have

\[
c_m(n, k) = \sum_{i=k}^{n} (m-1)^{i-k} \binom{i-1}{k-1} \binom{ia}{n-i},
\]

\[
f_m(n) = \sum_{i=1}^{n} m^{i-1} \binom{ia}{n-i}.
\]

**Corollary 4.** In the case \( a = 2 \), the sequence \( f_1(1), f_1(2), \ldots \) is seqnumA002478, which is the bisection of the Narayana’s cows sequence (seqnumA000930).

3. **Diagonals of Pascal triangle**

The following problem which we investigate is: For a positive integer \( a \), find the numbers of words over the alphabet \( \{0, 1, \ldots, a-1, \ldots\} \) such that subwords from \( \{0, 1, \ldots, a-1\} \) have no rises. We let \( P_2 \) denote this property. We show that for this problem, the values of the initial function are figurate numbers, that is, the numbers on a diagonal of the Pascal triangle. We let \( d(n-1, a) \) denote the number words length \( n - 1 \) satisfying \( P_2 \).
Proposition 2. The following equation holds:

\[ d(n - 1, a) = \binom{n + a - 2}{a - 1}. \]

Proof. We first have \( d(0, a) = 1 \), since the empty word has no a rise. Assume that \( n > 1 \). The following recurrence holds:

\[ d(n, a + 1) = d(n, a) + d(n - 1, a + 1), \quad (n > 0). \]

Namely, each word of length \( n - 1 \) over \( \{0, 1, \ldots, a - 1, a\} \) that has no rises may begin with a letter from \( \{0, 1, \ldots, a - 1\} \). The letter \( a \) does not appear in a word, which yields that there are \( d(n - 1, a) \) such words. There remains the words of length \( n - 1 \) beginning with \( a \). Obviously, there are \( d(n - 2, a + 1) \) such words. It follows that

\[
\begin{align*}
    d(n - 1, a + 1) - d(n - 2, a + 1) &= d(n - 1, a), \\
    d(n - 2, a + 1) - d(n - 3, a + 1) &= d(n - 2, a), \\
    & \quad \vdots \\
    d(1, a + 1) - d(0, a + 1) &= d(1, a).
\end{align*}
\]

Adding expressions on the left-hand sides and the right-hand sides, we obtain the following recurrence:

\[ d(n - 1, a + 1) = \sum_{i=1}^{n} d(i - 1, a). \]

To prove (2), we use induction on \( a \). If \( a = 1 \), then \( d(n - 1, 1) = 1 \), since the alphabet consists of the empty word. Assume that the statement holds for \( a \geq 1 \). Then (3) takes the form

\[ d(n - 1, a + 1) = \sum_{i=1}^{n} \binom{i + a - 2}{a - 1}, \]

and the statement holds according to the horizontal recurrence for the binomial coefficients.

Therefore, in the case \( f_0(n) = \binom{n+a-2}{a-1}, (n = 1, 2, \ldots) \), since \( f_0(1) = 1 \), using Janjić [6, Proposition 12], we obtain

Corollary 5. The number \( c_m(n, k), (m \geq 1) \) is the number of words of length \( n - 1 \) over the alphabet \( \{0, 1, \ldots, a - 1, \ldots, a + m - 1\} \) having \( k - 1 \) letters equal to \( a + m - 1 \) and satisfying \( P_2 \).

According to [6] Corollary 2, we get

Corollary 6. The number \( f_m(n) \) equals the number of words of length \( n - 1 \) over the alphabet \( \{0, 1, \ldots, a - 1, \ldots, a + m - 1\} \) satisfying \( P_2 \).

To derive an explicit formula for \( c_1(n, k) \), we need the following lemma.

Lemma 1. Let \( u \geq v \geq w \geq 1 \) be integers. Then

\[ \binom{u}{v} = \sum_{i=w}^{u-v+w} \binom{i-1}{w-1} \binom{u-i}{v-w}. \]

We know that \( \binom{u}{v} \) is the number of binary words of length \( u \) with \( v \) zeros. We let \( i \) denote the position of the \( u \)th zero in a word. It is clear that \( w \leq i \leq u-v+w \). For a fixed \( i \), the number of words is \( \binom{i-w}{u-w} \). Summing over all \( i \), we obtain (4).

**Note 1.** The identity (4) generalizes the horizontal recurrence for the binomial coefficients, which we obtain for either \( w = 1 \) or \( w = v \).

Next, we derive a formula for \( c_1(n, k) \).

**Proposition 3.** We have

\[ c_1(n, k) = \binom{n + ak - k - 1}{ak - 1}. \]

**Proof.** We use induction on \( k \). From (9, Equation (3)), we have

\[ c_1(n, 1) = f_0(n) = \binom{n + a - 2}{a - 1}, \]

which means that the statement is true for \( k = 1 \). Using induction, we conclude that the statement is equivalent to the following binomial identity:

\[ \binom{n + ak - k - 1}{ak - 1} = \sum_{i=1}^{n-k+1} \binom{i + a - 2}{a - 1} \binom{n + ak - k - i}{ak - a - 1}. \]

We prove that this identity follows from Identity (4). Namely, taking \( w+1 \) instead of \( w \) in Identity (4) and then replacing \( i-1 \) by \( j \) yields

\[ \binom{u}{v} = \sum_{j=w}^{u-v+w} \binom{j}{w} \binom{u-j-1}{v-w-1}. \]

Taking, in particular, \( w = a - 1 \) and replacing \( j \) by \( i + a - 2 \) implies

\[ \binom{u}{v} = \sum_{i=1}^{u-v+1} \binom{i + a - 2}{a - 1} \binom{u-1-i}{v-a}. \]

Finally, taking \( u = n + ak - k - 1, v = ak - 1 \), we obtain the desired result. □

**Corollary 7.** The following formulas holds:

1. \( c_m(n, k) = \sum_{i=k}^{n} (m - 1)^{i-k} \binom{i-1}{k-1} \binom{n + ai - i - 1}{ai - 1}. \)

2. \( f_m(n) = \sum_{k=1}^{n} m^{i-1} \binom{n + ai - i - 1}{ai - 1}. \)

**Proof.** The proof follows from (9, Equation (9), Proposition 7). □

We state some particular cases. Note that the case \( a = 2 \) was considered in [6, Example 31].

**Example 1.** In the case \( a = 3 \), we have \( f_0(n) = \binom{n+1}{2}, (n \geq 1) \). Hence, \( f_0(n) \) is the \( n \)th triangular number. We thus obtain

1. The number \( \binom{n+2k-1}{3k-1} \) is the number of ternary words of length \( n-1 \) having \( k-1 \) letters equal to 2, and avoiding 01, 02, 12.
(2) The number
\[
\sum_{i=k}^{n} (m-1)^{i-k} \binom{i-1}{k-1} \binom{n+2i-1}{3i-1}
\]
is the number of words of length \(n - 1\) over the alphabet \(\{0, 1, \ldots, m + 2\}\) having \(k - 1\) letters equal to \(m + 2\) and avoiding 01, 02, 12.

(3) The number
\[
\sum_{i=1}^{n} m^{i-1} \binom{n+2i-1}{3i-1}
\]
is the number of words of length \(n - 1\) over the alphabet \(\{0, 1, \ldots, m + 2\}\) avoiding 01, 02, 12.

This case is also related with the enumeration of some Dyck paths. Namely, the sequence \(f_0(1), f_0(2), \ldots\) makes the second column of the Narayana triangle. It follows that \(f_0(n)\) is the number of Dyck paths of semilength \(n + 1\) having exactly two peaks. In the formula
\[
c_1(n, k) = \sum_{i_1+i_2+\cdots+i_k=n} f_0(i_1) \cdot f_0(i_2) \cdot \cdots \cdot f_0(i_k),
\]
the product \(f_0(i_1) \cdot f_0(i_2) \cdot \cdots \cdot f_0(i_k)\) equals the number of Dyck paths of semilength \(n + k\) obtained by the concatenation of \(k\) Dyck paths with exactly two peaks. We thus obtain the following Euler-type identity.

**Identity 1.** The following sets have the same number of elements

1. The set of quaternary words of length \(n - 1\) in which \(k - 1\) letters equal 3 and which avoid 01, 02, 12.
2. The set Dyck paths of semilength \(n + k\) obtained by the concatenation of \(k\) Dyck paths with exactly two peaks.

4. Central binomial coefficients ant its adjacent neighbors

In two concluding examples, we give combinatorial properties for \(c_1(n, k)\) only.
We start with a slight generalization of [6, Proposition 12]. Let \(W\) be a set of words with a property \(\mathcal{P}\), over a finite alphabet \(\alpha\). Assume that the empty word has the property \(\mathcal{P}\). For a positive integer \(i\), we denote by \(W_{i-1}\) the set of words of length \(l(i-1)\). In particular, \(W_0 = \emptyset\) yields \(l(0) = 0\). We let \(f_0(i)\) denote the number of words from \(W_{i-1}\). In particular, we have \(f_0(1) = 1\). Consider the equation
\[
i_1+i_2+\cdots+i_k=n, (i_t>0, t=1,2,\ldots,k).
\]
For \(x \not\in \alpha\), we want to count words from the alphabet \(\alpha \cup \{x\}\) of the form:
\[
w_{i_1-1}, x, w_{i_2-1}, x, \ldots, w_{i_k-1}-1, x, w_{i_k-1},
\]
where \(w_{i_t-1} \in W_{i_t-1}\). We let \(N\) denote the number of such words. For fixed \(i_1, i_2, \ldots, i_k\), the word has length
\[
l(i_1-1) + \cdots + l(i_k-1) + k - 1,
\]
and its \(k - 1\) letters equal to \(x\). Choosing suitable \(i_1, i_2, \ldots, i_k\), each of \(k - 1\) letters \(x\) may be put at any prescribed place in a word. Summing over all \(i_1, i_2, \ldots, i_k\), implies
Proposition 4. We have
\[ N = \sum_{i_1+i_2+\cdots+i_k=n} f_0(i_1) \cdots f_0(i_k). \]

Remark 1. Taking, in particular, \( l(i-1) = i-1 \) for all \( i \), we obtain [6, Proposition 12].

In the following two cases, we restricted our investigation to find explicit formulas for \( c_1(n, k) \) and its combinatorial interpretations. We first assume that the values of \( f_0 \) are the central binomial coefficients, that is
\[ f_0(n) = \binom{2n-2}{n-1}. \]

It is clear that \( f_0(n) \) is the number of binary words of length \( 2n-2 \), in which the number of zeros equals the number of ones. We let \( P_3 \) denote this condition. In this case, we take \( l(n-1) = 2n-2 \). Since the empty word satisfies \( P_3 \), we have \( l(0) = 0 \). Hence, Proposition 4 may be applied to obtain Corollary 8.

The number \( c_1(n, k) \) is the number of ternary words of length \( 2n-k-1 \) having \( k-1 \) letters equal 2, and all binary subwords satisfy \( P_3 \).

We derive an explicit formula for \( c_1(n, k) \).

Proposition 5. We have
\[ c_1(n, n) = 1, \quad c_1(n, k) = \frac{2^{n-k}k(k+2)\cdots[k+2(n-k-1)]}{(n-k)!}, (k < n). \]

Proof. It is well-known that the generating function \( g(x) \), for the sequence
\[ \left\{ \binom{2n-2}{n-1} : n = 1, 2, \ldots \right\}, \]
is \( g(x) = \frac{1}{\sqrt{1-4x}} \). We know that \( c_1(n, k) \) is the coefficient of \( x^n \) in the expansion of \( [xg(x)]^k \) into powers of \( x \). The formula follows by the use of Taylor expansion for the binomial series. \( \square \)

The fact that \( c_1(n, 1) = f_0(n) \) leads to the following identity:

Identity 2. For each \( n \geq 1 \), we have
\[ \prod_{i=1}^{n} (n+i) = 2^n(2n-1)!!. \]

We also consider the particular case \( k = 2 \). Then \( c_1(n, 2) = 4^{n-2}, (n > 1) \). Hence, powers of 4 have the following property.

Corollary 9. For \( n \geq 2 \), the number \( 4^{n-2} \) is the number of ternary words of length \( 2n-3 \) in which one letter is 2 and in each binary subword, the number of ones and zeros are equal.

The number \( c_1(n, k) \) may be interpreted in terms of lattice paths. Namely, It is a well-known fact that \( f_0(n) \) is the number of lattice paths from \((0, 0)\) to \((n-1, n-1)\) using the steps \((0, 1)\) and \((1, 0)\). We may consider the symbol \( x \) in Proposition 4 as the \((1,1)\)-step possible only on the main diagonal. We thus obtain the following Euler-type identity.
Identity 3. The following sets have the same number of elements.

1. The set of ternary words having $k - 1$ letters equal 2, in which each binary subword has the same number of zeros and ones.

2. The set of the lattice paths from $(0, 0)$ to $(n + k - 2, n + k - 2)$ using steps $(0, 1), (1, 0)$ and $k - 1$ steps $(1, 1)$ possible only on the main diagonal.

Our final example is the case $f_0(n) = \binom{2n - 1}{n} (n = 1, 2, \ldots)$. Combinatorially, $f_0(n)$ is the number of binary words of length $2n - 1$ in which the number of ones is by 1 greater than the number of zeros. We let $P_4$ denote this property. Obviously, the empty word does not satisfy this condition, so that, Proposition 4 cannot be applied.

We count the number of words of the form $w_{i_1}, 2, w_{i_2}, 2, \ldots w_{i_k-1}, 2, w_{i_k}$ when $i_1 + i_2 + \cdots + i_k = n$, and $l(w_i) = 2i - 1$, for $i \in \{i_1, i_2, \ldots, i_k\}$. It is easy to see that the following result holds:

Corollary 10. The number $c_1(n, k)$ is number of ternary words of length $n + k - 1$ having the following properties:

1. No word either begins or ends with 2.
2. No two 2 can be adjacent.
3. Each binary subword satisfies $P_4$.

It is known that a generating function for the sequence $f_0(1), f_0(2), \ldots$ is

$$g(x) = \frac{1}{2x\sqrt{1 - 4x}} - \frac{1}{2x}.$$ 

To obtain an explicit formula for $c_1(n, k)$, we have to expand $[xg(x)]^k$ into powers of $x$. Using the binomial formula and the expansion of the binomial series, we obtain

$$[xg(x)]^k = \frac{1}{2^k} \sum_{j=0}^{\infty} \left( \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i(i+2) \cdots (i+2j-2) \right) \frac{2^j}{j!} x^j.$$ 

Since the least power of $x$ on the left-hand side is $k$, we obtain the following:

Identity 4.

$$\sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i(i+2) \cdots (i+2j-2), (j < k).$$

Remark 2. The identity may easily be proved directly.

We thus obtain

Proposition 6. The following formula holds:

$$c_1(n, k) = \frac{2^{n-k}}{n!} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \prod_{t=0}^{n-1} (i + 2t).$$

From equation (6), we obtain

Identity 5. The following formula holds:

$$\prod_{i=1}^{n} (n + i - 1) = 2^{n-1}(2n-1)!!.$$ 

Remark 3. We note that a number of other combinatorial interpretations of our results may be found in Sloane’s [8].
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