The Librating Companions in HD 37124, HD 12661, HD 82943, 47 Uma and GJ 876: Alignment or Antialignment?

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ABSTRACT

We investigated the apsidal motion for the multi-planet systems. In the simulations, we found that the two planets of HD 37124, HD 12661, 47 Uma and HD 82943 separately undergo apsidal alignment or antialignment. But the companions of GJ 876 and \upsilon And are only in apsidal lock about 0\degree. Moreover, we obtained the criteria with Laplace-Lagrange secular theory to discern whether a pair of planets for a certain system are in libration or circulation.

Subject headings: methods:N-body simulations — celestial mechanics — planetary systems — stars:individual(HD 37124, HD 12661, HD 82943)

1. Introduction

The number of known extrasolar planetary systems are quickly growing (Butler et al. 2003, Paper I) in recent years. The analysis and orbital fitting of the curves of radial velocity can reveal the fact that the main-sequence stars may host one or more planets. Recently, Fischer et al. (2003) (Paper II) reported the properties of ten multiple-planet systems (see Table 8 in Paper II) and pointed out that most of the systems with the ratio of orbital periods less than 5:1 are characterized by mean motion resonance (MMR), such as GJ 876(2:1) and HD 82943(2:1). The other important feature for these systems is the apsidal lock of the orbiting companions, which indicates that the relative apsidal longitudes of two orbits librate about a constant, such that two planets have common time-averaged rate of apsidal precession. A number of researchers studied this dynamical mechanism in the planetary

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systems. The work of Ji et al. (2003a) confirmed that the inner two companions of 55 Cnc (Marcy et al. 2002) are in a 3:1 resonance and still experience the apsidal phase-locking. In fact, one may be familiar to the presence of the apsidal lock in our solar system, such as the Galilean satellites of Io-Europa where the difference of their apsidal longitudes $\Delta \varpi$ are anti-aligned (Lee & Peale 2002). In the case of the pulsar PSR 1257+12, Wolszczan & Frail (1992) pointed out that there exists almost a $\Delta \varpi$-libration about $180^\circ$ for the two planets in a 3:2 MMR. Then a question arises: are all discovered multiple-planet systems in apsidal lock? If so, how frequently does this mechanism take place for these systems?

According to Laplace-Lagrange (L-L) linear secular theory (Murray & Dermott 1999, MD99), if $\Delta \varpi$ is a libration point, then $360^\circ - \Delta \varpi$ could also be the other equilibrium due to the symmetry of planetary configurations (also see Beauge et al. 2003). So the second question is that if two planets undergo alignment or antialignment, could the mirror geometry occur? In this Letter, our goal is to examine the apsidal motion for HD 37124, HD 12661, HD 82943, 47 Uma, GJ 876.

2. Results

In the present work, we performed a preliminary study of various systems by using N-body codes (Ji, Li & Liu 2002). We take the masses of the parent star and companions (with $\sin i = 1$) from Table 1 for each system. The time step is adopted to be 1% to 2.5% of the orbital period of the innermost planet. Here we simply account for coplanar configurations for these systems. We remain the values of the semi-major axes unchanged for all orbits. The eccentricities and arguments of periapse are generated in the orbital parameters web based on the best-fit orbital solutions given the nominal observation errors (see Table 1). And the remaining arguments of nodal longitudes and mean anomalies are randomly made between $0^\circ$ and $360^\circ$. Finally, five two-planet systems were individually produced 100 pairs of coplanar orbits for integration from 1 Myr to 10 Myr.

Throughout the paper, $\lambda$ and $\varpi$ denote the mean and periastron longitudes of a planet respectively. The subscripts $b$ and $c$ separately represent the inner and outer companions. Consequently, the relative apsidal longitude is $\Delta \varpi = \varpi_b - \varpi_c$.

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1Here we study the possible motion near the best-fit orbital solutions. This may reveal some important dynamical features for a certain investigated system.

2For the sake of convenience, we exchange the order of the subscripts for two planets of HD 82943 such that the inner planet can be always described by the symbol $b$. 
2.1. HD 37124

Paper I revealed that two companions orbit HD 37124, but indicated that with only 30 observations the eccentricity of the outer planet was poorly determined, with any eccentricity between 0.3 and 0.8 fitting the observations within measurement errors (Butler 2003, private communication). In our simulations, we let $e_c$ initially be in the interval $[0.20, 0.60]$ in an attempt to find likely dynamical constraints for $e_c$. However, all orbits can survive for $10^6$ yr, indicating the system is extremely stable for any initial $e_c$ no greater than 0.60. Moreover, we noticed that $e_c$ ranges from 0 to 0.62 in secular evolution for all cases, coinciding with the outcomes by Paper I and Kiseleva-Eggleton et al.(2002). They pointed out that the system self-destructed for the critical initial value of $e_c$ above 0.65. In addition, there are two typical secular variations of both eccentricities for stable orbits: (a) Case of large oscillations: $e_b$ is up to 0.85 and $e_c$ is less than 0.62. (b) Case of small modulations, both $e_b$ and $e_c$ are less than 0.40 (see Figure 1). The dynamical origin of high eccentricity for Case (a) may result from Kozai mechanism (Kozai 1962; Wu & Murray 2003) that implies the coupled relationship of the maximum eccentricity and minimum inclination.

Furthermore, it is noteworthy that 43% and 1% of the stable orbits are separately in the periapse alignment and antialignment. To the best of our knowledge, such results are a new finding for this system. In Figure 1, the upper and low panels separately exhibit the variations of eccentricities and relative apsidal longitudes. The results of L-L theory (dash lines) confirm the numerical results (thick lines) in the amplitudes and periods of libration for both eccentricities and $\Delta \varpi$.

2.2. HD 12661

The orbital fits revealed that the HD 12661 system is close to 11:2 MMR (Paper II; Gozdiewski 2003a) or 6:1 MMR (Gozdziewski & Maciejewski 2003b). With the best-fit data from paper II, our simulations exhibit all the orbits are stable for 1 Myr. We observe that 46% and 49% of the orbits are in $\Delta \varpi$-libration about $0^\circ$ and $180^\circ$ respectively. Moreover, Gozdiewski (2003a) indicated that the possibility of $\Delta \varpi$-circulation is much less likely than $\Delta \varpi$-libration about $0^\circ$ or $180^\circ$ in the numerical investigations. Here the two types of the apsidal librations almost share an equal opportunity to be acted as one of the dynamical mechanisms to remain the stability of HD 12661, because the two planets with moderate

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3The percentage of the aligned and antialigned cases may be affected by the uncertainty of $e_c$, but imply that two planets of this system likely tend to be aligned.
eccentricities are always far from each other with isolated orbits and protected from close encounters. Still, the fact of $\Delta\varpi$-libration indicates that one planet travels through the periapse or apoapse when the other moves to the opposite side at the same time.

The critical arguments for 11:2 MMR $\phi_k$ can be written:

$$\phi_k = 2\lambda_b - 11\lambda_c + k\varpi_b + (9 - k)\varpi_c, k = 0...9.$$  

Here we define $\theta_1 = \phi_3 = 2\lambda_b - 11\lambda_c + 3\varpi_b + 6\varpi_c$, $\theta_2 = \phi_2 = 2\lambda_b - 11\lambda_c + 2\varpi_b + 7\varpi_c$ and $\theta_3 = \theta_1 - \theta_2 = \Delta\varpi$. Most systems in a 11:2 MMR are only in behavior of $\theta_1$-libration. A particular case is exhibited in Figure 2, which not only ten resonant arguments all librate but $\theta_3$ librates about 180° for 1 Myr. Using octupole-level secular theory, Lee & Peale (2003) found that if the ratio of the maximum orbital angular momenta is approximately equal to a critical value, then the $\Delta\varpi$-libration about 0° or 180° could happen with large amplitude variations of the eccentricities. Additionally, we further found that the status of $\Delta\varpi$-libration are relevant to the ratio of the initial eccentricities and the initial relative apsidal longitudes of two planets (see §3).

### 2.3. 47 Uma

The two companions of 47 Uma is a close analog to the duo of Jupiter and Saturn in our solar system in the ratios of the planet masses and orbital periods. Laughlin et al.(2002) found the system is in $\Delta\varpi$-libration about 0° with low eccentricities of the companions. Further, Gozdziewski (2002) used MEGNO technique to investigate in wide ranges of the orbital parameters for the system. Using the updated parameters by Paper II, we still found this system can experience not only a $\Delta\varpi$-libration about 0°, but about 180°. We note that all the experiments can last for 10 Myr. The fractions in alignment, antialignment and circulation are respectively 20%, 10% and 70%. As well known, Jupiter and Saturn are near 5:2 MMR, but the $\Delta\varpi$ circulates at present-day. As for 47 Uma, we did find one of the critical arguments $3\lambda_b - 8\lambda_c + 2\varpi_b + 3\varpi_c$ temporarily librates for hundreds of years but $\Delta\varpi$ simultaneously circulates in the secular evolution. However, we argue that 47 Uma may be captured into 8:3 MMR in the past during the process of migration due to the planet-disk interaction.

### 2.4. HD 82943 and GJ 876

The celebrated nature for HD 82943 and GJ 876 is that the ratio of the orbital periods of two companions is respectively close to 2:1. The critical arguments for 2:1 resonance are
\[ \theta_1 = \lambda_b - 2\lambda_c + \varpi_b, \quad \theta_2 = \lambda_b - 2\lambda_c + \varpi_c, \quad \text{and the relative apsidal longitude is} \quad \theta_3 = \theta_1 - \theta_2. \]

In the simulations, we found three types of stable orbits for HD 82943: (I) Only \( \theta_1 \) librates about \( 0^\circ \) (II) Case of alignment, \( \theta_1 \approx \theta_2 \approx \theta_3 \approx 0^\circ \) (III) Case of antialignment, \( \theta_1 \approx 180^\circ, \theta_2 \approx 0^\circ, \theta_3 \approx 180^\circ \). In addition, we noticed that 2\%, 3\% and 2\% of the cases belong to Classes I, II and III respectively. The geometry of Class I was also discovered by Gozdiewski & Maciejewski (2001). But Classes II and III are quite a new discovery that exhibits the HD 82943 system can undertake apsidal lock (see Figure 3). Recently, Lee M. H. (2003, private communication) pointed out that the best-fit solution\(^4\) makes the orbits closer to be aligned than anti-aligned in the study of the 2:1 resonance. We used a semi-analytical (Ji et al. 2003b) model to investigate the apsidal motion for this system and found that although the two planets have high eccentricities up to 0.54 and 0.41, the system remain stable for 10 Myr because of both dynamical mechanisms of the 2:1 MMR and apsidal lock.

Several best-fit orbital solutions for GJ 876 appeared in literature (Marcy et al. 2001; Laughlin & Chambers 2001; Rivera & Lissauer 2001). Here we presented numerical results based on Laughlin-Chambers fit. Two families of orbits are found to be stable: Classes I (14\%) and II (7\%). In summary, for above two systems, we stress that the 2:1 MMR always takes place for stable cases and the apsidal alignment (Laughlin & Chambers 2001; Lee & Peale 2002) seems to have more likely chances.

On the other hand, as the two orbits of GJ 876 (or HD 82943) are not well separated, repeated interplanetary close approaches may lead to disruption of the systems: 79\% for GJ 876 rapidly become unstable, and 93\% for HD 82943.

### 2.5. Other systems

Butler et al. (1999) reported the first triple-planet system of \( \upsilon \) And, consisting of a 51 Peg-like planet with a 4.6-day orbit and two librating outer planets (Rivera & Lissauer 2000; Chiang, Tabachnik & Tremaine 2001; Chiang & Murray 2002). With the updated data (Paper II), we again examined several tests for integration of \( 10^6 \) yr and discovered that two outer components can still remain in the apsidal lock about \( 0^\circ \). As for 55 Cnc, we revealed that the inner two companions perform the asymmetric \( \Delta \varpi \)-libration about 250\° or 110\° in the secular evolution (Ji et al. 2003a).

Nagasawa et al. (2003) showed that the two planets of HD 168443 and HD 74156 are

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\(^4\) The velocities for HD 82943 have not yet been published, and the best-fit orbital parameters taken from [http://obswww.unige.ch](http://obswww.unige.ch).
both in circulation. Moreover, we performed additional experiments for HD 38529, but simply found the planets circulate. The systems of HD 168443, HD 74156 and HD 38529 are similar (Table 2), because each of them has a more massive outer planet and a close inner companion. The reason for circulation will be explained in §3. Nevertheless, if one of three systems can host an extra middle planet with new observations, there may exist librating planets similar to those of 55 Cnc and υ And.

3. Discussion on apsidal motion of $\Delta \varpi$

However, one should make clear the conditions for libration or circulation of $\Delta \varpi$ for two planets. Several theoretical works (Malhotra 2002; Lee & Peale 2003; Nagasawa et al. 2003; Zhou & Sun 2003) were concentrated on the problem. Beauge et al. (2003) presented analytical and numerical outcomes on the existence and location of stable equilibrium solutions about the systems in mean motion resonance. Here we present our results based on L-L theory. Following MD99, we have:

$$\frac{d\varpi_b}{dt} = -\frac{e_c}{e_b} A_{12} \cos(\varpi_b - \varpi_c) - A_{11}$$  \hspace{1cm}\hspace{1cm} (2)$$

$$\frac{d\varpi_c}{dt} = -\frac{e_b}{e_c} A_{21} \cos(\varpi_b - \varpi_c) - A_{22}$$  \hspace{1cm}\hspace{1cm} (3)$$

$$\frac{d\Delta \varpi}{dt} = A_{22} - A_{11} + \left(\frac{e_b}{e_c} A_{21} - \frac{e_c}{e_b} A_{12}\right) \cos \Delta \varpi$$  \hspace{1cm}\hspace{1cm} (4)$$

where $A_{11}, A_{12}, A_{21}$ and $A_{22}$ are initially determined constants (see MD99) and $e_b, e_c$ are the eccentricities of each planet. Here we rewrite the coefficients in equation (4), such that $B_1 = A_{22} - A_{11}$ and $B_2 = (e_b/e_c) A_{21} - (e_c/e_b) A_{12}$. For hierarchical planetary systems (HD 168443, HD 74156 and HD 38529, the ratio of the semi-major axes $\alpha = a_b/a_c \ll 1$), we have $|B_1| \gg |B_2|$ (see Table 2), thus the part of $|B_1|$ is dominant and $\Delta \varpi$ circulates (see also Nagasawa et al. 2003). According to (4), the critical values of $e_b/e_c$ for $\Delta \varpi$-libration:

$$\left(\frac{e_b}{e_c}\right)_{\pm} = \pm(\gamma - 1) - \sqrt{(\gamma - 1)^2 + 4\gamma\beta^2}$$  \hspace{1cm}\hspace{1cm} (5)$$

where the plus sign in $\pm$ corresponds to the case for $\Delta \varpi$-libration about $0^\circ$, and the minus sign to that of $\Delta \varpi$-libration about $180^\circ$, with $\beta = -b_{3/2}^{(2)}(\alpha)/b_{3/2}^{(1)}(\alpha)$ ($b_{3/2}^{(j)}(\alpha)$ are Laplace coefficients, $j = 1, 2$) and $\gamma = (1/\sqrt{\alpha})(\mu_c/\mu_b)\sqrt{(1 + \mu_c)/(1 + \mu_b)}$ ($\mu_{b,c}$ are respectively the ratio of the masses of Companion b (c) and host star). Hence, there are two requirements for apsidal libration of a pair of planets in the absence of mean motion commensurabilities. Firstly, the ratio of initial eccentricities of two planets should not be far from the critical
values; secondly, the starting relative apsidal longitude should be satisfied to approach 0° or 180°. As a paradigm of HD 37124 (see Figure 1), the initial ratio of the eccentricities $(e_b/e_c)_0 \simeq 0.30$ is close to $(e_b/e_c)_+ = 0.34$, and $\Delta \varpi_0 \simeq 7^\circ \simeq 0^\circ$, thus the two planets are aligned. We can apply the criteria to other systems.

4. Summary

In this work, we investigated the apsidal motion for the multi-planet systems: we find that the two planets of HD 37124, HD 12661, 47 Uma and HD 82943 separately undergo the apsidal alignment or antialignment. But the companions of GJ 876 and υ And are only in apsidal lock about 0°, which means that no mirror configurations occur for them. Moreover, we again used the L-L theory to discern whether two planets of a system librate or not. The status of $\Delta \varpi$-libration depends on the ratio of their initial eccentricities, semi-major axes and planetary masses and original relative apsidal longitudes. And it seems to be a selection effect for the companions in the multi-planet systems that are involved in the apsidal libration. However, this should be further testified by additional observations of the planetary systems.

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Table 1. The astrocentric orbital parameters of seven multi-planet systems\(^1\)

| Planet     | \(M_{\text{star}}\) \((M_{\odot})\) | \(M \sin i\) \((M_{\text{Jup}})\) | \(P\) \((\text{days})\) | \(a\) \((\text{AU})\) | \(e\) | \(\omega\) \((\text{deg})\) | \(\text{MMR}\) | Comments | Align | Anti |
|------------|-------------------------------------|---------------------------------|-----------------|-----------------|-----|-----------------|--------|---------|------|-----|
| GJ 876 b\(^a\) | 0.32 | 1.06 | 29.995 | 0.1294 | 0.314(0.02) | 51.8(10) | 2:1 | Yes | No |
| GJ 876 c\(^a\) | 0.32 | 3.39 | 62.092 | 0.2108 | 0.051(0.02) | 40.0(10) | 2:1 | Yes | No |
| HD 82943 b\(^b\) | 1.05 | 0.88 | 221.6 | 0.73 | 0.54(0.05) | 138(13) | 2:1 | Yes | Yes |
| HD 82943 c\(^b\) | 1.05 | 1.63 | 444.6 | 1.16 | 0.41(0.08) | 96(7) | 2:1 | Yes | Yes |
| HD 37124 b\(^c\) | 0.91 | 0.86 | 153.0 | 0.54 | 0.10(0.06) | 97(40) | Yes | Yes |
| HD 37124 c\(^c\) | 0.91 | 1.01 | 1942.0 | 2.95 | 0.40(0.20) | 265(120) | Yes | Yes |
| 47 UMa b\(^d\) | 1.03 | 2.86 | 1079.2 | 2.077 | 0.05(0.04) | 124.3(12.7) | 8:3? | Yes | Yes |
| 47 UMa c\(^d\) | 1.03 | 1.09 | 2845.0 | 3.968 | 0.00(0.03) | 170.8(15.3) | 8:3? | Yes | Yes |
| HD 12661 b\(^d\) | 1.07 | 2.30 | 263.6 | 0.823 | 0.35(0.03) | 293.1(5.0) | 11:2/6:1 | Yes | Yes |
| HD 12661 c\(^d\) | 1.07 | 1.57 | 1444.5 | 2.557 | 0.20(0.04) | 162.4(18.5) | 11:2/6:1 | Yes | Yes |
| Ups And b\(^d\) | 1.30 | 0.64 | 4.617 | 0.058 | 0.019(0.20) | 200.0(20.0) | Yes | Yes |
| Ups And c\(^d\) | 1.30 | 1.79 | 241.16 | 0.805 | 0.26(0.03) | 249.1(9.3) | 16:3? | Yes | No |
| Ups And d\(^d\) | 1.30 | 3.53 | 1276.15 | 2.543 | 0.25(0.03) | 264.2(17.2) | 16:3? | Yes | No |
| 55 Cnc b\(^e\) | 1.03 | 0.83 | 14.65 | 0.115 | 0.03 | 90.25 | 3:1 | No | No |
| 55 Cnc c\(^e\) | 1.03 | 0.20 | 44.27 | 0.241 | 0.41 | 46.39 | 3:1 | No | No |
| 55 Cnc d\(^e\) | 1.03 | 3.69 | 4780.0 | 5.461 | 0.28 | 218.91 | |

\(^1\)Recently, Lee & Peale (2003) pointed out the difference between the astrocentric and Jacobi orbits for a hierarchical system. The values are taken from the references:(a) Laughlin et al.(2001) (b) http://obswww.unige.ch (c) Butler et al. (2003) (d) Fischer et al.(2003) (e) Fischer et al.(2003) and Ji et al.(2003)
Table 2. The $\Delta \varpi$-circulation for three hierarchical planetary systems

| System $^a$ | $M_b/M_c$ | $a_b/a_c$ | $e_b/e_c$ | $B_1/B_2$ | $(g_+/g_-)^b$ |
|-------------|-----------|-----------|-----------|-----------|---------------|
| HD 38529    | 0.06      | 0.035     | 0.81      | 18.3      | 87.3          |
| HD 74156    | <0.20     | <0.072    | 1.86      | 23.6      | >19.0         |
| HD 168443   | 0.45      | 0.1027    | 2.65      | 7026.8    | 7.0           |

$^a$The data for the masses, eccentricities and semi-major axes of two planets for three systems are taken from Fischer et al. (2003).

$^b$Here $g_+$, $g_-$ are two eigenfrequencies from L-L secular solutions. For $\alpha = a_b/a_c \ll 1$, we have $g_+/g_- = (1/\sqrt{\alpha})(\mu_c/\mu_b)\sqrt{(1+\mu_c)/(1+\mu_b)} = \gamma$. And $g_+/g_- \simeq (1/\sqrt{\alpha})(\mu_c/\mu_b) \gg 1$, indicating $g_+$ is dominant.
Fig. 1.— The alignment case for HD 37124. The upper two panels show the secular evolution of the eccentricities of Companions b and c. Notice that $e_b$ ranges from 0.08 to 0.14, while $e_c$ varies from 0.28 to 0.30 (Case b). The bottom panel displays the relative apsidal longitude $\Delta \varpi$ librates about 0° with the amplitude of ±15°. The thick lines denote the numerical results (dots for $e_c$) and the dash lines are those from L-L theory. Both of them are in good agreement. The initial values: $a_b = 0.54$ AU, $e_b = 0.089$, $\Omega_b = 199.22^\circ$, $\omega_b = 83.38^\circ$. $a_c = 2.95$ AU, $e_c = 0.299$, $\Omega_c = 344.54^\circ$, $\omega_c = 291.10^\circ$. 
Fig. 2.— The antialignment case for HD 12661. Here \( \theta_1 = 2\lambda_b - 11\lambda_c + 3\varpi_b + 6\varpi_c, \)
\( \theta_2 = 2\lambda_b - 11\lambda_c + 2\varpi_b + 7\varpi_c \) and \( \theta_3 = \Delta\varpi. \) The signs of diamond and plus, respectively, represent \( \theta_1 \) and \( \theta_2. \) And the thick line denotes \( \theta_3, \) which librates about 180° with small amplitudes for 10^5 yr. Notice that \( \theta_1 \) librates about 0°, while \( \theta_2 \) does about 180° for the same timescale. In fact, ten resonant arguments \( \phi_k \) (\( k = 0 \ldots 9 \)) all librate for the same timescale for this case. The scenario can be seen for 1 Myr.
Fig. 3.— The antialignment case for HD 82943 (Class III). Here $\theta_1 = \lambda_b - 2\lambda_c + \varpi_b$, $\theta_2 = \lambda_b - 2\lambda_c + \varpi_c$ and $\theta_3 = \Delta \varpi$. Note that two critical arguments of $\theta_1$ (the sign triangle) and $\theta_2$ (the sign diamond), respectively, librate about $0^\circ$ and $180^\circ$ with slight amplitudes for $10^4$ yr. The thick line represent $\theta_3$, librating about $180^\circ$ in the same timescale. The results can extend to 10 Myr.