Generalized McKay quivers of rank three

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Abstract. For each finite subgroup $G$ of $SL_n(\mathbb{C})$, we introduce the generalized Cartan matrix $C_G$ in view of McKay correspondence from the fusion rule of its natural representation. Using group theory, we show that the generalized Cartan matrices have similar favorable properties such as positive semi-definiteness as in the classical case of affine Cartan matrices (the case of $SL_2(\mathbb{C})$). The complete McKay quivers for $SL_3(\mathbb{C})$ are explicitly described and classified based on representation theory.

1. Introduction

McKay correspondence reveals deep relations and connections among subgroups $G$ of $SL_2(\mathbb{C})$, classical Lie groups of simply-laced types, and resolution of singularities of $\mathbb{C}^2/G$ [18, 20]. It has long been expected that such a correspondence can be generalized to the special linear group $SL_n(\mathbb{C})$, and there are already several important works in this direction [15, 11, 10, 4]. However the complete picture is still not clear, and most results in this direction have focused on some special subgroups such as abelian subgroups. In this paper we give an algebraic description of the complete generalized McKay correspondence of rank three.

One formulation is from representation theoretic viewpoint. We take the original elementary approach of McKay [14] to study the fusion rule of subgroups of $SL_n(\mathbb{C})$, and in particular we focus on the case of $n = 3$. We find that to certain degree some of the beautiful phenomena in two dimensional case are still visible in $n$ dimensional case. The diagrams we constructed are in general some digraphs, and the associated undirected graphs, when looking from a distance, are similar to ADE cases.

Let $G$ be a finite nontrivial subgroup of $SL_n(\mathbb{C})$. We denote the embedding of $G$ into $SL_n(\mathbb{C})$ as $\pi$, and let $\rho_0, \cdots, \rho_s$ be the complete list of inequivalent irreducible representations of $G$ over $\mathbb{C}$. We then construct a
\(\pi_G\)-digraph with the irreducible representations of \(G\) as nodes and \(m_{i,j}\) directed edges from \(\rho_i\) to \(\rho_j\), where \(\pi \otimes \rho_i = \bigoplus_j m_{ij} \rho_j\). We make a convention that an undirected edge between \(\rho_i\) and \(\rho_j\) represents the pair of arrows between \(\rho_i\) and \(\rho_j\). This digraph is called the generalized McKay quiver associated with \(\pi\). We prove in general that all the corresponding matrices are positive semi-definite and also a non-trivial eigenvector is the dimension vector, which defines the specialty of affine Cartan matrices in rank two case.

Our method can produce the generalized McKay correspondence in any dimension. In the second part of the paper we determine the complete list of generalized McKay quivers in rank three, including all the exceptional graphs in this case. Of course this list also includes the usual extended affine Dynkin diagrams as special cases, and if one blurs some of the nodes and edges in the quivers they also look much like the rank two cases.

We remark that a different approach to McKay correspondence in rank 3 was recently given in [5] where a partial list of McKay graphs is produced. Our approach is completely different and we have obtained the complete list of McKay quivers in rank three, and we also indicate how to produce similar McKay graphs in higher dimensions.

One profound problem for McKay correspondence in dimension three is to bridge the geometric approach [3, 4] and the algebraic method in this paper. We only discuss the algebraic picture of the so-called McKay correspondence in rank 3 in this paper. Nevertheless, there seems to be some partial progress between algebra representation theory [1] and the generalized McKay quivers, most results so far are about some subcases (such as abelian subgroups of \(SL_3(\mathbb{C})\)), it would be extremely important to understand the intrinsic connection between our digraphs and quiver theory of representation theory.

2. The generalized McKay-Cartan matrices

Let \(G\) be a finite nontrivial subgroup of \(SL_n(\mathbb{C})\), and let \(\gamma_i\) be its irreducible characters afforded by \(\rho_i\) on the space \(V_i\) \((i = 0, \cdots, |G^*| = s)\). The space \(R(G)\) of complex class functions on \(G\) has \(\{\gamma_i\}\) as a canonical basis. For \(f, g \in R(G)\), the inner product is defined by

\[
\langle f, g \rangle = \frac{1}{|G|} \sum_{x \in G} f(x)g(x^{-1}) = \sum_{C \in G_*} \xi_C^{-1} f(C)g(C^{-1}),
\]

where as a convention we denote by \(\xi_C = |Z_G(C)|\), and \(Z_G(C)\) is the centralizer of one of the elements in \(C\).

As usual we denote by \(G_*\) the set of conjugacy classes: \(\{C_0, C_1, \cdots, C_s\}\), where \(C_0 = \{1\}\). The orthogonality relation of irreducible characters now
reads
\[
\sum_C \xi_C^{-1} \gamma_i(C) \gamma_j(C^{-1}) = \delta_{ij}, \\
(2.2)
\]
\[
\sum_j \gamma_j(C') \gamma_j(C^{-1}) = \delta_{CC'} \xi_C.
\]

Let \( \pi \) be the natural representation of \( G \) on \( \mathbb{C}^n \). The tensor product \( \pi \otimes \rho_i \) decomposes itself into irreducible representations
\[
(2.3) \quad \pi \otimes \rho_i \cong \bigoplus_j m_{ij} \rho_j.
\]

Then,
\[
(2.4) \quad (n \text{Id} - \pi) \otimes \rho_i \cong \bigoplus_j b_{ij} \rho_j,
\]
where \( b_{ij} = n \delta_{ij} - m_{ij} \). The matrix \( M = (m_{ij}) \) is called the adjacency matrix, and \( B = (b_{ij}) \) the pre-Cartan matrix for the finite group \( G \) associated to \( \pi \).

**Definition 2.1.** The matrix \( A = B + B^t \) is called the generalized Cartan matrix for the finite group \( G \) associated with \( \pi \).

We remark that \( A \) is twice of the extended Cartan matrix in the case of \( n = 2 \).

Suppose \( \pi \) is afforded by the representation \( V \), the dual representation \( \pi^* \) is the \( G \)-module afforded by \( V^* = \text{Hom}_\mathbb{C}(V, \mathbb{C}) \) and the action is given by
\[
(2.5) \quad x.f(v) = f(x^{-1}v), x \in G, v \in V.
\]

Let \( \chi_\pi \) be the character of \( \pi \), it is well-known that
\[
(2.6) \quad \chi_\pi(x) = \chi_{\pi^*}(x^{-1}).
\]

We also recall a well-known fact: if \( U, V, W \) are \( G \)-modules, then
\[
(2.7) \quad \text{Hom}(U^* \otimes V, W) \cong \text{Hom}(V, U \otimes W)
\]
as \( G \)-modules. Hence \( \langle U^* \otimes V, W \rangle = \langle V, U \otimes W \rangle \).

**Lemma 2.2.** \( \pi^* \otimes \rho_i \cong \bigoplus_j m_{ij} \rho_j \).

**Proof:** Suppose \( \pi^* \otimes \rho_i \cong \bigoplus_j m'_{ij} \rho_j \), then
\[
m'_{ij} = \langle \pi^* \otimes \rho_i, \rho_j \rangle = \langle \rho_i, \pi \otimes \rho_j \rangle = m_{ji}.
\]

**Proposition 2.3.** Let \( \rho \) be a representation of \( G \), and suppose \( \rho \otimes \rho_i \cong \bigoplus_j b_{ij} \rho_j \). Let \( p_i = (\gamma_0(C_i), \ldots, \gamma_s(C_i))^t \in \mathbb{C}^s \), then \( p_i \) is an eigenvector of \( B = (b_{ij}) \) with eigenvalue \( \chi_\rho(C_i) \).
Proof: By the orthogonality of $\gamma_i$ one has that

$$b_{ij} = \langle \rho \otimes \rho_i, \rho_j \rangle = \sum_C \xi_C^{-1} \chi_\rho(C) \gamma_i(C) \gamma_j(C^{-1}).$$

We compute that

$$\sum_j b_{ij} \gamma_j(C_k) = \sum_{j,C} \xi_C^{-1} \chi_\rho(C) \gamma_i(C) \sum_j \gamma_j(C_k) \gamma_j(C_k)$$

$$= \sum_C \chi_\rho(C) \gamma_i(C) \delta_{C,C_k}$$

which means that $B p_k = \chi_\rho(C_k) p_k$.

As a special case, the dimension vector $\delta = (\dim \gamma_0, \cdots, \dim \gamma_s)^t$ is an eigenvector with eigenvalue $\dim \rho$.

**Theorem 2.4.** The generalized Cartan matrix $A = B + B^t$ is positive semi-definite, and the dimension vector $\delta$ is an eigenvector with eigenvalue zero.

Proof: Since $A$ is the matrix of $\rho \otimes \gamma_i = \sum a_{ij} \gamma_j$, where $\rho = (nId - \pi) \oplus (nId - \pi^*)$, we claim that $A$ is positive semi-definite if and only if $\langle \rho \otimes \gamma, \gamma \rangle \geq 0$ for any $\gamma \in R(G)$. In fact, let $\gamma = \sum x_i \gamma_i$, $X^T = (x_0, \cdots, x_s)$, then

$$\langle \chi_\rho \otimes \gamma, \gamma \rangle = \sum_{i,j} x_i \langle \chi_\rho \otimes \gamma_i, \gamma_j \rangle x_j = \sum_{i,j} x_i a_{ij} x_j = X^T A X.$$  

On the other hand, let $\chi$ be the any real character of degree $n$, then

$$\langle \chi \otimes \gamma, \gamma \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \gamma(g) \gamma(g^{-1}) \leq n \langle \gamma, \gamma \rangle.$$  

Hence $\langle \chi_\rho \otimes \gamma, \gamma \rangle = 2n \langle \gamma, \gamma \rangle - \langle (\chi_\pi + \chi_{\pi^*}) \otimes \gamma, \gamma \rangle \geq 0$.

### 3. The $SL_3(\mathbb{C})$ McKay digraphs

The classification of finite subgroups of $SL_3(\mathbb{C})$ (up to isomorphism) was given by Blichfeldt [2] in 1917 and completed in [21] (see also [13]). There are 12 types of nontrivial subgroups of $SL_3(\mathbb{C})$, and each type consists of several non-isomorphic subgroups that bear similar properties. We will construct the $\pi_G$-graphs as defined in Section 1 for all types. For each group we first find its irreducible representations and then decompose their tensor products with the natural representation to construct the corresponding *generalized Cartan matrix* and the $\pi_G$-graph. Except for the type of subgroups that are essentially the same as the $SL_2(\mathbb{C})$ case, we show each $\pi_G$-graph together with the corresponding matrix. Some types may have more than one graphs.

The following result will be useful in construction of irreducible representations of semidirect products.
Lemma 3.1. (Wigner-Mackey’s Little Group method) Suppose $G$ is the semi-direct product of a subgroup $T$ by a normal abelian subgroup $H$.

Let $X = \text{Hom}(H, \mathbb{C}^*)$. Then $G$ acts on $X$ by $(s\chi)(h) = \chi(s^{-1}hs)$ for $s \in G, \chi \in X, h \in H$. Let $(\chi_i)_{i \in X/T}$ be a system of representatives for the orbits of $T$. For each $i \in X/T$, let $T_i = \{t \mid t\chi_i = \chi_i\}$ and $G_i = HT_i$ be the corresponding subgroups of $G$. Extend the function $\chi_i$ to $G_i$ by setting $\chi_i(at) = \chi_i(a)$ for $a \in H, t \in T_i$. Let $\rho$ be an irreducible representation of $T_i$ and $\tilde{\rho}$ be the composition of $\rho$ with the canonical projection $G_i \to T_i$. Finally, let $\theta_{i,\rho} = (\chi_i \otimes \tilde{\rho}) \uparrow^{G_i}_{T_i}$. Then

(a) $\theta_{i,\rho}$ is irreducible;
(b) If $\theta_{i,\rho}$ and $\theta_{i',\rho'}$ are isomorphic, then $i = i'$, and $\rho \simeq \rho'$;
(c) Every irreducible representation of $G$ is isomorphic to one of the $\theta_{i,\rho}$.

We start with some notations. For a natural number $n$ we denote the primitive $n$th root of unity by $\xi_n = e^{2\pi i/n}$. Then a generator of the center of $\text{SL}_3(\mathbb{C})$ is given by

\begin{equation}
W = \text{diag}(\xi_3, \xi_3, \xi_3).
\end{equation}

For positive integers $m$ and $n$, the group

\begin{equation}
H_{m,n} = \{\text{diag}(\xi_m^k, \xi_n^l, \xi_m^{-k} \xi_n^{-l}) \mid k, l \in \mathbb{Z}\}
\end{equation}

is an abelian group of order $mn$. Now we study the 12 types one by one.

Type 1 Abelian Group $H_{m,n}$. It is well-known that $H_{m,n} \simeq \mathbb{Z}_m \otimes \mathbb{Z}_n$. Set $g_{k,l} = \text{diag}(\xi_m^k, \xi_n^l, \xi_m^{-k} \xi_n^{-l}) \in H_{m,n}$ and define

\begin{equation}
\rho_{i,j}(g_{k,l}) = \xi_m^{ik} \xi_n^{jl}, \text{ for } i = 1, \ldots, m \text{ and } j = 1, \ldots, n.
\end{equation}

Then $\rho_{i,j}$ realize all $mn$ (1-dimensional) irreducible representations of $H_{m,n}$.

Let $\pi = \rho_{1,0} \oplus \rho_{0,1} \oplus \rho_{-1,-1}$ be the embedding of $H_{m,n}$ into $\text{SL}_3(\mathbb{C})$. It follows that

\begin{equation}
\pi \otimes \rho_{i,j} = \rho_{i+1,j} \oplus \rho_{i,j+1} \oplus \rho_{i-1,j-1}.
\end{equation}

So we get the following $\pi_{H_{m,n}}$-graph.

Here we label each node by the dimension of the corresponding representation. Note that the upper and lower parts of the graph are identified, similarly the left and right parts are identified, thus we can visualize the $\pi_{H_{m,n}}$-graph as a torus.
We remark that in the special case of $m = 1$, the graph is the extended Dynkin diagram of $A_n^{(1)}$ if we erase all self directing loops, i.e., Fig. 1-2 degenerates into a cycle.

For the generalized Cartan matrix $C_{H_m,n} = (3I - M) + (3I - M)^T$, where $M$ is the adjacency matrix of $\pi_{H_m,n}$-graph, we have the following examples.

(3.5) $C_{H_2,2} = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$.

(3.6) $C_{H_3,2} = \begin{pmatrix} 6 & -2 & -1 & -1 & -1 \\ -2 & 6 & -1 & -1 & -1 \\ -1 & -1 & 6 & -2 & -1 \\ -1 & -1 & -2 & 6 & -1 \\ -1 & -1 & -1 & 6 & -2 \\ -1 & -1 & -1 & -1 & 2 \\ 6 & -1 & -1 & -1 & 0 \end{pmatrix}$.

(3.7) $C_{H_3,3} = \begin{pmatrix} 6 & -1 & -1 & -1 & 0 & -1 & 0 & 0 & -1 \\ -1 & 6 & -1 & 0 & -1 & -1 & -1 & 0 & 0 \\ -1 & -1 & 6 & -1 & 0 & -1 & 0 & -1 & -1 \\ -1 & 0 & -1 & 6 & -1 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 6 & -1 & 0 & -1 & -1 \\ -1 & -1 & -1 & -1 & 6 & -1 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 6 & -1 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 & -1 & -1 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 & -1 & -1 & -1 & 6 & -1 \end{pmatrix}$.
we can get its irreducible representations by Lemma 3.1. Firstly, the orbits

\[
(3.8) \quad C_{H_{3,4}} = \begin{pmatrix}
6 & -1 & 0 & -1 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & -1 \\
-1 & 6 & -1 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 \\
0 & -1 & 6 & -1 & 0 & 0 & -1 & 1 & 0 & -1 & 0 & 0 \\
-1 & 0 & -1 & 6 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\
-1 & 0 & 0 & -1 & 6 & -1 & 0 & -1 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 6 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & -1 & -1 & 0 & 0 & -1 & 6 & -1 & 0 & 0 & -1 & 0 \\
0 & 0 & -1 & -1 & 0 & 0 & -1 & 6 & -1 & 0 & 0 & -1 \\
-1 & 0 & 0 & -1 & 0 & 0 & -1 & 6 & -1 & 0 & 0 & -1 \\
-1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & -1 \\
-1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & -1 & 6 
\end{pmatrix}.
\]

(Type 2) Group $G_m^3$. The group is generated by $H_m := H_{m,m}$ and $T$, where

\[
(3.9) \quad T = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0 
\end{pmatrix}.
\]

It is easy to see that $G_m^3 = H_m \rtimes \langle T \rangle$ and $G_m^3 = H_m \uplus TH_m \uplus T^2 H_m$, so we can get its irreducible representations by Lemma 3.1. Firstly, the orbits of $X = Hom(H_m, \mathbb{C}^*)$ under the action of $\langle T \rangle$ are listed as follows:

\[
O(r_{i,j}) = \{ r_{i,j}, r_{-j,i-j}, r_{j-i,-i}(i,j) \neq (0,0), (\frac{m}{3}, \frac{2m}{3}), (\frac{2m}{3}, \frac{m}{3}) \},
\]

\[
(3.10) \quad (*)O(r_{\frac{m}{3}, \frac{2m}{3}}) = \{ r_{\frac{m}{3}, \frac{2m}{3}} \}, \quad O(r_{\frac{2m}{3}, \frac{m}{3}}) = \{ r_{\frac{2m}{3}, \frac{m}{3}} \},
\]

\[
O(r_{0,0}) = \{ r_{0,0} \},
\]

where $i, j \in \{1, 2, \ldots, m\}$. Note: the rows labeled with $(*)$ are excluded when $m$ is not divisible by 3, and the same for (3.13), (3.15), (3.17) and (3.21) in the following. Hence there are $\frac{m^2-1}{3} + 1$ orbits when $3 \nmid m$ and $\frac{m^2-1}{3} + 3$ orbits when $3|\mid m$. Let

\[
T_1 = \left\{ s \in \langle T \rangle \mid s \cdot r_{\frac{m}{3}, \frac{2m}{3}} = r_{\frac{m}{3}, \frac{2m}{3}} \right\} = \langle T \rangle,
\]

\[
T_2 = \left\{ s \in \langle T \rangle \mid s \cdot r_{\frac{2m}{3}, \frac{m}{3}} = r_{\frac{2m}{3}, \frac{m}{3}} \right\} = \langle T \rangle,
\]

\[
T_3 = \left\{ s \in \langle T \rangle \mid s \cdot r_{0,0} = r_{0,0} \right\} = \langle T \rangle,
\]

and let $\rho^j : T \rightarrow \mathbb{C}^*$ ($j = 1, 2, 3$) be the 1-dimensional irreducible representations of subgroup $\langle T \rangle$. Therefore the Little Group method gives that

\[
(3.12) \quad \theta_{0,0,j} = r_{0,0} \otimes \rho^j,
\]

\[
(3.13) \quad (*)\theta_{\frac{m}{3}, \frac{2m}{3}, j} = r_{\frac{m}{3}, \frac{2m}{3}} \otimes \rho^j, \quad \theta_{\frac{2m}{3}, \frac{m}{3}, j} = r_{\frac{2m}{3}, \frac{m}{3}} \otimes \rho^j
\]

are irreducible representations of $G_m^3$ of degree 1. We can check that
(3.14) \[ \{ s \in < T > \mid s \cdot \rho_{i,j} = \rho_{i,j}, (i,j) \neq (0,0), (\frac{m}{3}, \frac{2m}{3}), (\frac{2m}{3}, \frac{m}{3}) \} = Id, \]

then \( \theta_{i,j} = (\rho_{i,j} \otimes id) \uparrow^{G_m^3}_{\text{id}_m} \) are 3-dimensional irreducible representations of \( G_m^3 \), where \( id \) is the trivial representation. Hence there are nine 1-dimensional and \( \frac{m^2-3}{3} \) 3-dimensional irreducible representations in the case of \( 3 \mid m \), and three 1-dimensional and \( \frac{m^2-1}{3} \) 3-dimensional irreducible representations if \( 3 \nmid m \). For \( n = 1, 2, 3 \) and the indexes \( (i, j) \in \{1, 2, \cdots, m\} \times \{1, 2, \cdots, m\} \setminus \{(0,0), (\frac{m}{3}, \frac{2m}{3}), (\frac{2m}{3}, \frac{m}{3})\} \), let

\[
(3.15) \begin{align*}
\theta_{0,0,n}(T) &= \xi_n^3, \quad \theta_{0,0,n}(g_{k,l}) = 1, \\
* \theta_{\frac{m}{3}, \frac{2m}{3}, n}(T) &= \xi_n^3, \quad \theta_{\frac{m}{3}, \frac{2m}{3}, n}(g_{k,l}) = \xi_{k+2l}^3, \\
* \theta_{\frac{2m}{3}, \frac{m}{3}, n}(T) &= \xi_n^3, \quad \theta_{\frac{2m}{3}, \frac{m}{3}, n}(g_{k,l}) = \xi_{2k+l}^3, \\
\theta_{i,j}(T) &= T, \quad \theta_{i,j}(g_{k,l}) = \text{diag}(\xi_{m}^{ik+jl}, \xi_{m}^{j(k-i)l}, \xi_{m}^{-j(k-i)l}).
\end{align*}
\]

This gives all the irreducible representations of \( G_m^3 \). Recall that \( \pi \), the natural representation from \( G_m^3 \) to \( SL_3(\mathbb{C}) \), can be given in this type by

\[
(3.16) \quad \pi(g_{k,l}) = \text{diag}(\xi_{m}^{k}, \xi_{m}^{l}, \xi_{m}^{-k-l}), \quad \pi(T) = T.
\]

By computing characters, we find the following decompositions of the tensor product of the natural representation and the irreducible representations of \( G_m^3 \):

\[
(3.17) \quad \pi \otimes \theta_{0,0,n} = \theta_{0,1,n}, \quad \pi \otimes \theta_{i,j} = \theta_{i-1,j-1} \oplus \theta_{i,j+1} \oplus \theta_{i+1,j}, \\
* \pi \otimes \theta_{\frac{m}{3}, \frac{2m}{3}, n} = \theta_{\frac{m}{3}+1, \frac{2m}{3}, n}, \quad \pi \otimes \theta_{\frac{2m}{3}, \frac{m}{3}, n} = \theta_{\frac{2m}{3}+1, \frac{m}{3}, n}.
\]

We remark that the decomposition can also be confirmed by computing the inner product \( \langle \pi \otimes \rho_{i}, \rho_{j} \rangle \). In the following this can always be used in getting the fusion product.

The \( \pi_{G_m^3} \)-graphs are shown in Fig.2-1 for \( m = 6 \) and Fig.2-2 for \( m = 7 \), one can draw the \( \pi_{G_m^3} \)-graphs for any \( m \) according to our recipe. Here we label each node corresponding to the representation \( \theta_{i,j} \) (resp. \( \theta_{i,j,k} \)) by its subscript \((ij)\) (resp. \((ijk))\). (For example: \( \theta_{0,0,1} \) is labeled by \((001))\.)
(Type 3) Group $G^6_m$. The group $G^6_m$ is generated by $H_m$, $T$ and $R$, where $T$ was defined in (3.9) and

\[
(3.18) \quad R = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & c & 0 \end{pmatrix}, \quad abc = -1.
\]

For simplicity we only consider the case of $a = b = c = -1$, which is enough to show the main feature of this type. It is not difficult to see that $G^6_m = H_m \rtimes S_3$, so its irreducible representations are obtained again by using Lemma 3.1. The orbits of $X = \text{Hom}(H_m, \mathbb{C}^*)$ under the action of $S_3$
are listed as follows:

\[
\mathcal{O}(\rho_{i,j}) = \{\rho_{i,j}, \rho_{j,i}, \rho_{-j,i}-j, \rho_{i,-j}, \rho_{j-i, i}, \rho_{i,-j-i}|i,j = 0, \ldots, m-1 \}
\]

and \((i, j) \notin \{(k, 0), (0, l), (h, h), (\frac{m}{3}, \frac{2m}{3}), (\frac{2m}{3}, \frac{m}{3})\}\),

\[
\mathcal{O}(\rho_{i,0}) = \{\rho_{i,0}, \rho_{0,i}, \rho_{i,-i}|i \in \{1, 2, \ldots, m-1\}\},
\]

\(\ast\) \[ \mathcal{O}(\rho_{\frac{m}{3}, \frac{2m}{3}}) = \{\rho_{\frac{m}{3}, \frac{2m}{3}}, \rho_{\frac{2m}{3}, \frac{m}{3}}\}, \]

\(\mathcal{O}(\rho_{0,0}) = \{\rho_{0,0}\}, \]

where \(k, l, h = 1, \ldots, m-1\). The irreducible representations of \(G^6_m\) can be similarly constructed as in the case of \(G^3_m\), and we obtain that (a) If \(3|m\), there are two 1-dimensional, four 2-dimensional, 2\((m-1)\) 3-dimensional and \(\frac{m^2-3m+2}{6}\) 6-dimensional irreducible representations. (b) If \(3 \nmid m\), there are two 1-dimensional, one 2-dimensional, 2\((m-1)\) 3-dimensional and \(\frac{m^2-3m+2}{6}\) 6-dimensional irreducible representations.

The irreducible representations corresponding to the orbit \(\mathcal{O}(\rho_{0,0}):\)

\[
\begin{align*}
\theta_{0,0,1}(g_{k,l}) &= 1, \quad \theta_{0,0,1}((12)) = \theta_{0,0,1}((23)) = -1. \\
\theta_{0,0,2}(g_{k,l}) &= 1, \quad \theta_{0,0,2}((12)) = \theta_{0,0,2}((23)) = 1. \\
\theta_{0,0,3}(g_{k,l}) &= E_{2 \times 2}, \theta_{0,0,3}(12) = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \theta_{0,0,3}(23) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. 
\end{align*}
\]

The irreducible representations corresponding to the orbit \(\mathcal{O}(\rho_{\frac{m}{3}, \frac{2m}{3}}):\)

\[
\begin{align*}
\ast \theta_{\frac{m}{3}, \frac{2m}{3}, n_1}(g_{k,l}) &= \text{diag}(\xi_3^{(k+2)}, \xi_3^{(2k+1)}) \quad (n_1 = 1, 2, 3), \\
\ast \theta_{\frac{m}{3}, \frac{2m}{3}, n_1}(12) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \theta_{\frac{m}{3}, \frac{2m}{3}, n_1}(23) = \begin{pmatrix} 0 & \xi_3^{n_1} \\ \xi_3^{n_1} & 0 \end{pmatrix}, 
\end{align*}
\]

The irreducible representations corresponding to the orbit \(\mathcal{O}(\rho_{i,0}):\)

\[
\begin{align*}
\theta_{i,0,n_2}(g_{k,l}) &= \text{diag}(\xi_m^{ik}, \xi_m^{il}, \xi_m^{i-k-l}) \quad (n_2 = 1, 2), \\
\theta_{i,0,n_2}((12)) &= (-1)^{n_2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \theta_{i,0,n_2}((23)) = (-1)^{n_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. 
\end{align*}
\]
The irreducible representations corresponding to the orbit $O(\rho_{i,j})$:

\[
\theta_{i,j}(g_{k,l}) = \text{diag}(\epsilon^{ik+jl}, \epsilon^{jk+il}, \epsilon^{-(ik+jl)})\text{,} \]

\[
\theta_{i,j}((12)) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\end{pmatrix}, \quad \theta_{i,j}((23)) = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

Computing the characters of the tensor product of each irreducible representation by the natural representation $\pi = \theta_{1,0,2}$, we find that

\[
\pi \otimes \theta_{0,0,1} = \theta_{1,0,1}, \quad \pi \otimes \theta_{0,0,2} = \theta_{1,0,2}, \quad \pi \otimes \theta_{0,0,3} = \theta_{1,0,1} + \theta_{1,0,2}, \\
\pi \otimes \theta_{1,0,2n} = \theta_{1+1,0,n2} \oplus \theta_{i,1-i} \text{ for } i \neq 0, \quad n2 = 1, 2,
\]

\[
\pi \otimes \theta_{m,2m,1} = \theta_{m+1,2m,1}, \quad \pi \otimes \theta_{m,2m,2} = \theta_{m,2m+1}, \\
\pi \otimes \theta_{m,2m,3} = \theta_{m-1,2m-1},
\]

\[
\pi \otimes \theta_{i,j} = \theta_{i-1,j-1} \oplus \theta_{i+1,j} \oplus \theta_{i,j+1}, \text{ for } i, j \notin \{0, \frac{2m}{3}, \frac{2m}{3}\}.
\]

These fusion rules completely determine the $\pi_{G_m^6}$-graph. We draw the $\pi_{G_m^6}$-graph (Fig.3.1) and $\pi_{G_6^7}$-graph (Fig.3.2) as follows.

![Fig.3-1](image1)

![Fig.3-2](image2)

From the graph we get the adjacency matrix $M$ associated to $\pi_{G_m^6}$-graph. Then $C(G_m^6) = (3I-M) + (3I-M)^T$ is the corresponding generalized Cartan matrix.

(\textbf{Type 4}) \textbf{Group} $\overline{G_\alpha}$. For any fixed non-zero $\alpha$ and a finite subgroup $G$ of $SL_2(\mathbb{C})$ let $\overline{G_\alpha} = \{\text{diag}(\alpha^{-2}, \alpha A) | A \in G\} \leq SL_3(\mathbb{C})$. It is obvious that $\overline{G_\alpha}$ and $G$ are isomorphic. So we just need to study the finite subgroups of $SL_2(\mathbb{C})$. It is well known that in the case of $SL_2(\mathbb{C})$ there are five types of finite subgroups and the $\pi_G$-graphs are exactly the Coxeter graphs of affine (simply laced) types [14]. Correspondingly, we have five subtypes here and the five $\overline{G_\alpha}$-graphs are also the Coxeter diagram of affine types but with an extra circle for every node.
Type 5) Group $G_5$. The group $G_5$ is of order 108 and generated by $T, S, V$, where $T$ is defined in type 2 (cf. (3.9)),

$$S = \text{diag}(1, \xi_3, \xi_3^2),$$

and

$$V = \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \xi_3 & \xi_3^2 \\ 1 & \xi_3^2 & \xi_3 \end{pmatrix}.$$ 

The group $G_5$ has 14 conjugate classes. A set of conjugacy representatives is given by $\{I, W, W^2, S, ST\} \cup \{V^iW^k \mid 1 \leq i, j \leq 3, 0 \leq k \leq 2\}$, where $W$ is the generator of $Z(SL_3(\mathbb{C}))$ (see (3.1)). The 14 inequivalent irreducible representations can be described as follows. Set $H_3 = \langle T, S, W \rangle \triangleleft G_5$, then

$$G_5 = H_3 \sqcup VH_3 \sqcup V^2H_3 \sqcup V^3H_3.$$ 

Let $\rho_j (j = 1, 2, 3, 4)$ be the 1-dimensional irreducible representations of $G_5$:

$$\rho_j : S \rightarrow 1, \quad T \rightarrow 1, \quad V \rightarrow \xi_j^4,$$

We also need to use the canonical involution $\sigma$ of $SL_3(\mathbb{C})$ defined by:

$$\sigma : A \rightarrow (A^{-1})^T.$$

In terms of the natural representation $\pi$, we can construct the irreducible representations of $G_5$ as follows. For $j = 1, 2, 3, 4$ we define

$$\rho_{4+j} = \pi \otimes \rho_j, \quad \rho_{8+j} = \sigma \circ \rho_{4+j}, \quad \rho_{12+j} = \psi_j \chi_{H_3}^{G_5}.$$

Again by computing their characters we obtain the following fusion rules, and the $\pi_{G_5}$-graph is drawn in Fig. 5.

$$\pi \otimes \rho_1 = \rho_5, \quad \pi \otimes \rho_2 = \rho_7, \quad \pi \otimes \rho_3 = \rho_9, \quad \pi \otimes \rho_4 = \rho_{11},$$

$$\pi \otimes \rho_5 = \rho_6 \oplus \rho_8 \oplus \rho_{10}, \quad \pi \otimes \rho_6 = \rho_1 \oplus \rho_{13} \oplus \rho_{14},$$

$$\pi \otimes \rho_7 = \rho_6 \oplus \rho_8 \oplus \rho_{12}, \quad \pi \otimes \rho_8 = \rho_2 \oplus \rho_{13} \oplus \rho_{14},$$

$$\pi \otimes \rho_9 = \rho_6 \oplus \rho_8 \oplus \rho_{10}, \quad \pi \otimes \rho_{10} = \rho_3 \oplus \rho_{13} \oplus \rho_{14},$$

$$\pi \otimes \rho_{11} = \rho_8 \oplus \rho_{10} \oplus \rho_{12}, \quad \pi \otimes \rho_{12} = \rho_4 \oplus \rho_{13} \oplus \rho_{14},$$

$$\pi \otimes \rho_{13} = \rho_5 \oplus \rho_7 \oplus \rho_9 \oplus \rho_{11},$$

$$\pi \otimes \rho_{14} = \rho_5 \oplus \rho_7 \oplus \rho_9 \oplus \rho_{11}.$$ 

Similarly, we let $M$ be the adjacency matrix associated with $\pi_{G_5}$-graph, and the generalized Cartan matrix $C(G) = (3I - M) + (3I - M)^T$. 

Fig. 5
(Type 6) Group $G_6$. The group of order 216 is generated by $G_5$ and $K$, where

\[ K = \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 1 & \xi_3^2 \\ 1 & \xi_3 & \xi_3 \\ \xi_3 & 1 & \xi_3 \end{pmatrix}. \]

It is known that there are 16 conjugacy classes in $G_6$, hence $G_6$ has 16 irreducible representations. Since $G_5$ is a normal subgroup of $G_6$, we can find some irreducible representations of $G_6$ by induction. In fact we find two 6-dimensional irreducible representations denoted as $\rho_{14}$ and $\rho_{15}$, one 2-dimensional representation denoted as $\rho_{13}$ and one 8-dimensional irreducible representations denoted as $\rho_{16}$. The other four 1-dimensional and eight 3-dimensional irreducible representations are given as follows: for $i, j = 0, 1$,

\[ \rho^{(i,j)} : T \rightarrow 1, \quad S \rightarrow 1, \quad V \rightarrow \xi_2^i, \quad K \rightarrow \xi_2^j, \]

\[ \rho^{(2+i,2+j)} = \pi \otimes \rho^{(i,j)}, \quad \rho^{(4+i,4+j)} = \sigma \circ \rho^{(2+i,2+j)}. \]

The natural representation is seen to be $\pi = \rho^{(2,2)}$, and we have the following decompositions by comparing characters. The $\pi_{G_6}$-graph is drawn in Fig.6., and the corresponding generalized Cartan matrix is then determined accordingly.

\[ \pi \otimes \rho^{(i,j)} = \rho^{(2+i,2+j)}, \]

\[ \pi \otimes \rho^{(2+i,2+j)} = \rho^{(5-i,5-j)} \oplus \rho^{14}, \]

\[ \pi \otimes \rho^{(4+i,4+j)} = \rho^{(1-i,1-j)} \oplus \rho^{16}, \]

\[ \pi \otimes \rho^{13} = \rho^{15}, \quad \pi \otimes \rho^{14} = 2\rho^{16} \oplus \rho^{13}, \]

\[ \pi \otimes \rho^{15} = \rho^9 \oplus \rho^{10} \oplus \rho^{11} \oplus \rho^{12} \oplus \rho^{14}, \]

\[ \pi \otimes \rho^{16} = \rho^5 \oplus \rho^6 \oplus \rho^7 \oplus \rho^8 \oplus 2\rho^{15}. \]

Fig.6.

(Type 7) Group $G_7$. The group $G_7$ of order 60 is generated by $T$, $E_2$, and $E_3$, where $T$ was defined as above (3.9) and

\[ E_2 = \text{diag}(1, -1, -1), \quad E_3 = \frac{1}{2} \begin{pmatrix} -1 & \mu_- & \mu_+ \\ \mu_- & \mu_+ & -1 \\ \mu_+ & -1 & \mu_- \end{pmatrix}, \quad \mu_{\pm} = \frac{1}{2}(-1 \pm \sqrt{5}). \]
The group $G_7$ is isomorphic to the simple group $A_5$, whose irreducible characters are known. Table 1 lists its character table, where $\nu_\pm = -\frac{1\pm \sqrt{5}}{2}$. For the natural character $\chi_\pi = \chi_3$ of the imbedding, we have the following decompositions, and the $\pi G_7$-graph is shown in Fig.7.

\[
\begin{align*}
\chi_\pi \otimes \chi_1 &= \chi_3, \\
\chi_\pi \otimes \chi_2 &= \chi_2 \oplus \chi_4 \oplus \chi_5, \\
\chi_\pi \otimes \chi_3 &= \chi_1 \oplus \chi_3 \oplus \chi_4, \\
\chi_\pi \otimes \chi_4 &= \chi_2 \oplus \chi_3 \oplus \chi_4 \oplus \chi_5, \\
\chi_\pi \otimes \chi_5 &= \chi_2 \oplus \chi_4, \\
\end{align*}
\]

Fig.7.

**Table 1. The characters of $G_7$**

| Class | 1   | 2   | 3   | 5   | 5   |
|-------|-----|-----|-----|-----|-----|
| $\chi_1$ | 1   | 1   | 1   | 1   | 1   |
| $\chi_2$ | 3   | -1  | 0   | $\nu_+$ | $\nu_-$ |
| $\chi_3$ | 3   | -1  | 0   | $\nu_-$ | $\nu_+$ |
| $\chi_4$ | 4   | 0   | 1   | -1  | -1  |
| $\chi_5$ | 5   | 1   | -1  | 0   | 0   |

The group $G_7$ is isomorphic to the simple group $A_5$, whose irreducible characters are known. Table 1 lists its character table, where $\nu_\pm = -\frac{1\pm \sqrt{5}}{2}$. For the natural character $\chi_\pi = \chi_3$ of the imbedding, we have the following decompositions, and the $\pi G_7$-graph is shown in Fig.7.

\[
\begin{align*}
\chi_\pi \otimes \chi_1 &= \chi_3, \\
\chi_\pi \otimes \chi_2 &= \chi_2 \oplus \chi_4 \oplus \chi_5, \\
\chi_\pi \otimes \chi_3 &= \chi_1 \oplus \chi_3 \oplus \chi_4, \\
\chi_\pi \otimes \chi_4 &= \chi_2 \oplus \chi_3 \oplus \chi_4 \oplus \chi_5, \\
\chi_\pi \otimes \chi_5 &= \chi_2 \oplus \chi_4, \\
\end{align*}
\]

\[
C_{G_7} = \begin{pmatrix}
3 & 0 & -1 & 0 & 0 \\
0 & 2 & 0 & -1 & -1 \\
-1 & 0 & 2 & -1 & 0 \\
0 & -1 & -1 & 2 & -1 \\
0 & -1 & 0 & -1 & 3
\end{pmatrix}.
\]

**Table 2. The characters of $G_8$**

| Class | 1   | 2   | 3   | 5   | 7   | 72 |
|-------|-----|-----|-----|-----|-----|----|
| $\chi_1$ | 1   | 1   | 1   | 1   | 1   | 1   |
| $\chi_2$ | 6   | 2   | 0   | 0   | -1  | -1  |
| $\chi_3$ | 7   | -1  | -1  | 1   | 0   | 0   |
| $\chi_4$ | 8   | 0   | 0   | -1  | 1   | 1   |
| $\chi_5$ | 3   | -1  | 1   | 0   | $\alpha_+$ | $\alpha_-$ |
| $\chi_6$ | 3   | -1  | 1   | 0   | $\alpha_-$ | $\alpha_+$ |

\[
\begin{align*}
X_7 &= \text{diag}(\beta, \beta^2, \beta^4), \\
U &= \frac{1}{\sqrt{-7}} \begin{pmatrix}
a & b & c \\
b & c & a \\
c & a & b
\end{pmatrix},
\end{align*}
\]

where for $\beta = \xi_7, a = \beta^4 - \beta^3, b = \beta^2 - \beta^5,$ and $c = \beta - \beta^6$. It is known that $G_8$ has 6 irreducible characters, whose character table is given in Table 2, where $\alpha_\pm = \frac{-1\pm \sqrt{7}}{2}$. The natural character $\chi_\pi = \chi_5$, then we have the

\[
\begin{align*}
\chi_\pi \otimes \chi_1 &= \chi_3, \\
\chi_\pi \otimes \chi_2 &= \chi_2 \oplus \chi_4 \oplus \chi_5, \\
\chi_\pi \otimes \chi_3 &= \chi_1 \oplus \chi_3 \oplus \chi_4, \\
\chi_\pi \otimes \chi_4 &= \chi_2 \oplus \chi_3 \oplus \chi_4 \oplus \chi_5, \\
\chi_\pi \otimes \chi_5 &= \chi_2 \oplus \chi_4, \\
\end{align*}
\]
following decompositions of $\chi_{\pi} \otimes \chi_i (i = 1, 2, \ldots, 6)$. The $\pi_{G_8}$-graph is drawn in Fig.8 and the generalized Cartan matrix is $C_{G_8}$.

\begin{align}
\chi_{\pi} \otimes \chi_1 &= \chi_5, \\
\chi_{\pi} \otimes \chi_2 &= \chi_3 \oplus \chi_4 \oplus \chi_6, \\
\chi_{\pi} \otimes \chi_3 &= \chi_2 \oplus \chi_3 \oplus \chi_4, \\
\chi_{\pi} \otimes \chi_4 &= \chi_2 \oplus \chi_3 \oplus \chi_4 \oplus \chi_5, \\
\chi_{\pi} \otimes \chi_5 &= \chi_2 \oplus \chi_6, \\
\chi_{\pi} \otimes \chi_6 &= \chi_1 \oplus \chi_4.
\end{align}

\begin{equation}
C_{G_8} = \begin{pmatrix}
6 & 0 & 0 & 0 & -1 & -1 \\
0 & 6 & -2 & -2 & -1 & -1 \\
0 & -2 & 4 & -2 & 0 & 0 \\
0 & -2 & -2 & 4 & -1 & -1 \\
-1 & -1 & 0 & -1 & 6 & -1 \\
-1 & -1 & 0 & -1 & -1 & 6
\end{pmatrix}
\end{equation}

In this case we can obtain the fusion rule easily by observing the character table. For example, the product of the characters $\chi_5 = \chi_{\pi}$ and $\chi_2$ is the sum of the character (row) vectors $\chi_3, \chi_4$ and $\chi_6$ in Table 2.

\section*{(Type 9) Group $G_9$}

The group $G_9$ of order 180 is generated by group $G_7$ and matrix $W$ (cf. 3.1). We have $G_9 \simeq A_5 \times \langle W \rangle$, so it is easy to get all the irreducible representations of $G_9$ from those of $A_5$ and group $\langle W \rangle$. Then the $\pi_{G_9}$-graph is Fig.9. and the generalized Cartan matrix is $C_{G_9}$.

\begin{equation}
C_{G_9} = \begin{pmatrix}
6E & -B & -B \\
-B^t & 6E & -B \\
-B & -B^t & 6E
\end{pmatrix},
B = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 1
\end{pmatrix}
\end{equation}

\section*{(Type 10) Group $G_{10}$}

The group $G_{10}$ of order 504 is generated by $G_8$ and matrix $W$. It is easy to see that $G_{10} \simeq G_8 \times \langle W \rangle$. The $\pi_{G_{10}}$-graph is
drawn in Fig.10 and the generalized Cartan matrix is $C_{G_{10}}$.

\begin{equation}
C_{G_{10}} = \begin{pmatrix}
6E & -B & -B^t \\
-B^t & 6E & -B \\
-B & -B^t & 6E
\end{pmatrix},
B = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0
\end{pmatrix}.
\end{equation}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig10}
\caption{Fig.10.}
\end{figure}

\textbf{(Type 11) Group $G_{11}$}. The $G_{11}$ of order 648 is generated by $G_5$ and the matrix $M$, where

\begin{equation}
M = \text{diag}(\varepsilon, \varepsilon, \varepsilon \xi_3), \quad \varepsilon^3 = \xi_3^2.
\end{equation}

We first obtain three 1-dimensional representations $\rho_i$ by the defining relations of $G_{11}(i = 0, 1, 2)$. We then have three 3-dimensional irreducible representations $\rho_{2+i} = \pi \otimes \rho_i$, where $\pi$ is the natural representation. Furthermore, we get three more non-isomorphic 3-dimensional irreducible representations $\rho_{5+i} = \sigma \cdot \rho_{2+i}$, where $\sigma$ is the action of transpose inverse (3.28). Obviously, $G_6$ is a normal subgroup of $G_{11}$, so we can get eight 9-dimensional representations by inducing the 3-dimensional irreducible representations from $G_6$ to $G_{11}$. By Frobenius reciprocity we observe that among these induced representations, there are just two new irreducible representations which are 9-dimensional, and the rest can be decomposed into direct sums of one 3-dimensional and one new 6-dimensional representations. In short, there are three 1-dimensional, three 2-dimensional, seven 3-dimensional, six 6-dimensional, three 8-dimensional and two 9-dimensional irreducible representations. Finally, we decompose the tensor product of the natural representation $\pi$ with the irreducible representations. The $\pi_{G_{11}}$ - graph is shown in Fig.11.
(Type 12) **Group** $G_{12}$. The group of order 1080 is generated by $G_7$ and matrix $E_4$, where

$$E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -w \\ 0 & -w^2 & 0 \end{pmatrix}.$$  

The center of group $G_{12}$ is $Z(G) = \langle W \rangle$ (cf. 3.1), and $G_{12}/Z(G) \cong A_6$. Thus for an irreducible representation $\rho$ of $A_6$, we can lift it to an irreducible representation of $G_{12}$ through the canonical map. In this way we can find two 5-dimensional, one 9-dimensional, one 10-dimensional and two 8-dimensional irreducible representations of $G_{12}$. Moreover, there are one 1-dimensional and four 3-dimensional irreducible representations. To sum up, $G_{12}$ has one 1-dimensional, four 3-dimensional, two 5-dimensional, two 6-dimensional, two 8-dimensional, three 9-dimensional, one 10-dimensional and two 15-dimensional irreducible representations. Then we decompose the tensor product of the natural representation $\pi$ with the irreducible representations, and the McKay quiver $\pi_{G_{12}}$ is drawn in Fig.12.
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