Research on the Dissipation Field of a Nonlinear Continuum Based on the Operational Irreversible Thermodynamics

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Abstract. The dissipation field of a nonlinear continuum is studied by a new operational theory of irreversible thermodynamics, the evolution equations of the dissipation field can be obtained under the condition of knowing the phenomenological nonlinear constitutive relation of the continuum. In this theory, the dissipation equation of the basic state, which is corresponding to the minimum dissipation principle, is utilized to solve the distribution of dissipation field regarding the equilibrium problem of quasi-statics, and the dissipation equations of higher order are directly related to the dynamics of dissipation fields. The paper also presents a method showing the dissipation force operator by using Helmholtz’s free energy function. This paper will also show important phenomena of the localized deformation, such as the plastic instability of necking and shear band, which are predicted in the form of an analytical formula.

1. Introduction
The dissipation field of a nonlinear continuum, such as plastic field or damage field, is a key problem of modern material mechanics. Although plastic mechanics or damage mechanics can give a good description of nonlinear mechanical behaviors of materials, and an adequate prediction of plastic field or damage field with uniform properties in the light of the inner variable theory of irreversible thermodynamics, the distribution properties and evolitional manner of non-uniform dissipation field still remains a difficult subject. The main stream method in this field today is the nonlinear continuum theory of defects, which is based on the differential geometry, and descried by the gauge theory[1,2]. However, the equation form obtained by the gauge theory is more complicated, and includes a lot of meaningless coefficients because the theory combines the density of the dissipation field with the curvature or torsion of non-Leman space, which make it practically useless. On the other hand, Hill[3] and Peirce[4] established the critical condition of plastic instability by means of the bifurcation theory of uniform elasto-plastics, but it can not point out the location of concentrated deformation, and even give the prediction that there were no concentrated deformations because of using an ideal uniform model[5].

In this paper, the fundamental equations of dissipation field of a nonlinear continuums are obtained by means of the operational theory of irreversible thermodynamics, which is based on the concepts of eigen theory of solid mechanics under standard spaces[6-8]. The equations satisfy the restriction of second law of irreversible thermodynamics and arranges in state of quantized energy. Thus, the
distribution law of non-uniform dissipation field can be obtained by solving the fundamental equations, from which we can determine the location and condition of concentrated deformation. The theory presented here provides a new method to study the dissipation field and instability of a nonlinear continuum.

2. The operational theory of dissipation field

Suppose that there exists a dissipation field in an isotropic nonlinear continuum, and it can be denoted by a tensor of inner variables. Projecting it to the standard space[6-8], we get a set of scalar inner variables. So, according to the second law of irreversible thermodynamics, the dissipation power of the system can be written as follows.

\[
\Phi = \sum_i \dot{Q}_i^* \dot{q}_i^* \geq 0
\]  

(1)

here, \(Q_i^*\) \((i = 1, 2, \cdots, 6)\) are the generalized inner friction force, which are corresponding to the inner variable increments. For convenience sake, rewriting Eq.(1) in the form of the generalized force \(X_i = Q_i^*\) and the generalized stream \(J_i = \dot{q}_i^*\). We have

\[
\Phi = \sum_i X_i J_i \geq 0
\]  

(2)

According to Onsager’s principle, the generalized stream \(J_i\) is a linear function of the generalized force \(X_i\), that is

\[
J_i = \sum_j L_{ij} X_j \quad j = 1, 2, \cdots, 6
\]  

(3)

in which there exist \(L_{ij} = L_{ji}\). Because the modal mechanical quantities are independent of each other in the stage of small deformation, the modal dissipation quantities are also independent of each other as well[8].

Due to the universality of the operational principle[9], Eq.(2) also can be written in the operational form as follows

\[
\hat{\Phi} = \sum_i \hat{X}_i \hat{J}_i \geq 0
\]  

(4)

here, \(\hat{X}_i\) is just \(i\)th operator of dissipation force, and also differential operators with respect to coordinate.

According to the minimal dissipation principle, the dissipation field of quasi-static system will exist in the form of minimal dissipation. So we have

\[
\hat{X}_i J_i = 0 \quad i = 1, 2, \cdots, 6
\]  

(5)

They are the partial differential equations of order two, from which we can obtain the field distribution solution of dissipation increments.

When considering the process of dynamics, the minimal dissipation principle will be replaced by the eigen dissipation principle in calculating the dissipation field of a nonlinear continuum. The definite state of the dissipation field will emerge in the form of quantization, that is

\[
\hat{X}_i J_i = k_i J_i \quad i = 1, 2, \cdots, 6
\]  

(6)
here, $k_i$ is dissipation energy of order $i$, and determined by the boundary condition. The basic state equation of Eq.(6) is just Eq.(5).

3. Determination of dissipation force operator

According to the theory of irreversible thermodynamics with inner variables, Helmholtz’s free energy function of a nonlinear continuum under standard spaces[8] can be written as follows

$$\psi = \psi(\varepsilon^*_i, q^*_i)$$

(7)

here, $\varepsilon^*_i (i = 1, 2, \cdots, 6)$ are a modal strain. Under the condition of small deformation, the Helmholtz’s free energy function can be expanded in the form of quadratic.

$$\psi = \frac{1}{2} A_{ij} \varepsilon^*_i \varepsilon^*_j + B_{ij} \varepsilon^*_i q^*_j$$

(8)

here, $A_{ij}$ are the elastic coefficients and $B_{ij}$ the coupling coefficients between elastic and dissipation fields. Because the modal dissipation fields are independent of each other[8], the $i$ th dissipation force of the system can be obtained by following equation

$$X_i = \frac{\partial \psi}{\partial q^*_i} = B_{ij} \varepsilon^*_j \quad i = 1, 2, \cdots, 6$$

(9)

For most materials, the $i$ th non-linear constitutive equation can be described approximately in the form of function, that is

$$\varepsilon^*_i = f(\sigma^*_i) \quad i = 1, 2, \cdots, 6$$

(10)

Substituting the above equation into Eq.(9), and the mechanical quantities are replaced with the corresponding mechanical operators, therefore, we have

$$\hat{X}_i = F(\Delta^*_i) \quad i = 1, 2, \cdots, 6$$

(11)

here, $\Delta^*_i$ and $\hat{X}_i$ are the $i$ th stress operator[6] and dissipation force operator respectively. Thus, the dissipation force operator of the system can be represented by the stress operator provided the phenomenological relationship of non-linear constitutive equation of materials is known.

4. The boundary condition of dissipation fields

The distribution pattern and quantum property of the dissipation field for quasi-static systems are strongly dependent on the boundary condition. In common case, the dissipation field is often presented by the inner variables, which make the mathematical presentation of the boundary condition a difficult task because of the concealment of the inner variables, especially for a non-uniform stress field. Considering the truth that the concentrated deformation and damage development generally emerge from the interior of materials under condition of uniform stress, we suppose that the boundary condition of the dissipation inner variables can be expressed with the first type of the boundary conditions, that is

$$q^*_i \bigg|_{s} = c_i \quad i = 1, 2, \cdots, 6$$

(12)

here, $c_i$ are constants, and can be assumed as zero in general.
A possible physical explanation for Eq. (12) is that the restraint of boundary (no matter the
displacement restraint or stress one) confine the development of dissipation variables at its boundary,
such as the block of dislocation and the stop of crack extension.

5. Application
Considering the isotropic and incompressible plastic materials with non-linear property of two order
power, its standard space structure is following

\[ W = W_1^{(i)} (\phi) \oplus W_2^{(s)} (\phi, \cdots, \phi) \]  

(13)

here

\[ \phi = \frac{\sqrt{3}}{3} [1,1,1,0,0,0]^T, \quad \phi_i = \frac{\sqrt{2}}{2} [0,1,-1,0,0,0]^T \]  

\[ \phi = \frac{\sqrt{6}}{6} [2,-1,-1,0,0,0]^T, \quad \phi_i = \xi_i, \quad i = 4,5,6 \]  

(14)

in which \( \xi_i \) is the unit vector that its \( i \) th element is 1, and others zero. So, it is seen that there are two
standard spaces for isotropy, one is the volumetric subspace, another the shear one.

The constitutive equation under the first subspace is

\[ \varepsilon_1^* = \frac{1}{K} \sigma_1^* \]  

(15)

here, \( K \) is the volumetric modulo of materials. The modal stress and modal strain of order 1 are respectively

\[ \sigma_1^* = \frac{\sqrt{3}}{3} (\sigma_1 + \sigma_2 + \sigma_3) \]  

(16)

\[ \varepsilon_1^* = \frac{\sqrt{3}}{3} (\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \]  

(17)

The constitutive equation under the second subspace is

\[ \varepsilon_2^* = \varepsilon_3 \left( \frac{\sigma_2^*}{\sigma_3} \right)^2 \]  

(18)

here, \( \sigma_3 \) and \( \varepsilon_3 \) are the stress and strain at the yield point of materials. The modal stress and modal
strain of order 2 are respectively

\[ \sigma_2^* = \sqrt{\frac{1}{3} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]} \]  

(19)

\[ \varepsilon_2^* = \sqrt{\frac{1}{3} \left[ (\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2 \right]} \]  

(20)

The stress operators of both the volumetric subspace and the shear one of isotropic materials are
all Laplace’s operator[6].
Two significant problems of the concentrated deformation of a nonlinear continuum are predicted here.

5.1 **The necking phenomenon in the uniaxial tension of bar**

The uniaxial tension of bar can be regarded as a one dimensional problem. By means of Eqns.(15) and (18), the equations of stress field and dissipation field for two subspaces are respectively

\[
\frac{d^2}{dx^2} \sigma_1^* = 0 \quad \frac{d^2}{dx^2} q_1^* = 0
\]

\[
\frac{d^2}{dx^2} \sigma_2^* = 0 \quad \frac{d^4}{dx^4} q_2^* = 0
\]

By the use of the force condition at two ends and first type of boundary conditions of a dissipation field, we get

\[
\sigma_x = \sigma \quad q_1^* = 0
\]

\[
\sigma_x = \sigma \quad q_2^* = -\frac{q_0}{L^2} (x^2 - L^2)
\]

here, \(L\) is the length of bar. It is seen from above solutions that no dissipation field occurs in the volumetric deformation subspace, the dissipation field occurs only in the shear deformation subspace. The distribution pattern of the dissipation field is that the maximal value of dissipation is at the middle of the bar, and then attenuate along the bar axis with a rate of two order power. These results give a theoretical explanation why the necking always occurs at the middle of bar in the uniaxial tension of the bar.

5.2 **The shear band phenomenon in the uniaxial tension of plate**

The uniaxial tension of plate can be regarded as a two dimensional problem. By means of Eqns.(15) and (18), the equations of the stress field and dissipation field for two subspaces are respectively

\[
\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right) \sigma_1^* = 0 \quad \left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right) q_1^* = 0
\]

\[
\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right) \sigma_2^* = 0 \quad \left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right)^2 q_2^* = 0
\]

By the use of the force condition at two ends, the first type of boundary conditions of a dissipation field, and the symmetrical properties of deformation, we get

\[
\sigma_x = \sigma \quad q_1^* = 0
\]

\[
\sigma_x = \sigma \quad q_2^* \bigg|_{x=\pm y} = q_0
\]

It is seen from above solutions that no dissipation field occurs in the volumetric deformation subspace, the dissipation field occurrence is only observed in the shear deformation subspace. The distribution pattern of the dissipation field is that the maximal value of dissipation is at the direction \(x = \pm y\) of the plate. These results also give a theoretical explanation why the shear band response always occurs at the direction \(x = \pm y\) of the plate in the uniaxial tension of a plate.
6. Conclusion

Based on the operational theory of irreversible thermodynamics under standard spaces, the equations of the dissipation field of a nonlinear continuum are obtained when the phenomenological nonlinear constitutive relation of continuum is given, which satisfy both the second law of thermodynamics and quantum property of a field. This new theory gives a good explanation to the localized deformation, such as the plastic instability of necking and shear band response, and it also can be extended to explain the non-uniform dissipation field of anisotropic materials.

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