Faithful Simulation of Distributed Quantum Measurements with Applications in Distributed Rate-Distortion Theory

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Abstract

We investigate faithful simulation of distributed quantum measurements as an extension of Winter’s measurement compression theorem. We characterize a set of communication and common randomness rates needed to provide faithful simulation of distributed measurements. To achieve this, we introduce binning and mutual packing lemma for distributed quantum measurements. These techniques can be viewed as the quantum counterpart of their classical analogues. Finally, using these results, we develop a distributed quantum-to-classical rate distortion theory and characterize a rate region analogous to Berger-Tung’s in terms of single-letter quantum mutual information quantities.

I. INTRODUCTION

Measurements are the interface between the intricate quantum world and the perceivable macroscopic classical world. A measurement associates to a quantum state a classical attribute. However, quantum phenomena, such as superposition, entanglement and non-commutativity contribute to uncertainty in the measurement outcomes. A key concern, from an information-theoretic standpoint, is to quantify the amount of “relevant information” conveyed by a measurement about a quantum state.

Winter’s measurement compression theorem (as elaborated in [1]) quantifies the “relevant information” as the amount of resources needed to simulate the output of a quantum measurement applied to a given state. Imagine that an agent (Alice) performs a measurement $M$ on a quantum state $\rho$ and sends a set of classical bits to a receiver (Bob). Bob intends to faithfully recover the outcomes of Alice’s measurements without having access to $\rho$. The measurement compression theorem states that at least quantum mutual
information \((I(X; R))\) amount of classical information and conditional entropy \((S(X|R))\) amount of common shared randomness are needed to obtain a *faithful simulation*. 

The measurement compression theorem finds its applications in several paradigms including local purity distillation [1] and private classical communication over quantum channels [2]. This theorem was later used by Datta, et al. [3] to develop a quantum-to-classical rate-distortion theory. The problem involved lossy compression of a quantum information source into classical bits, with the task of compression performed by applying a measurement on the source. In essence, the objective of the problem was to minimize the storage of the classical outputs resulting from the measurement while ensuring sufficient reliability so as to be able to recover the quantum state (from classical bits) within a fixed level of distortion from the original quantum source. To achieve this, the authors in [4] advocated the use of measurement compression protocol and subsequently characterized the so called rate-distortion function in terms of single-letter quantum mutual information quantities. The authors further established that by employing a naive approach of measuring individual output of the quantum source, and then applying Shannon’s rate-distortion theory to compress the classical data obtained is insufficient to achieve optimal rates.

![Diagram](image)

**Fig. 1.** The diagram of a distributed quantum measurement applied to a bipartite quantum system \(AB\). A tensor product measurement \(M_A \otimes M_B\) is performed on many copies of the observed quantum state. The outcomes of the measurements are given by two classical bits. The receiver functions as a classical-to-quantum channel \(\beta\) mapping the classical data to a quantum state.

In this work, we seek to quantify “relevant information” for quantum measurements performed in a distributed fashion. In this setting, as shown in Fig. [1] a composite bipartite quantum system \(AB\) is made available at two separate agents, named Alice and Bob. Alice and Bob have access only to sub-system \(A\) and \(B\), respectively. Two separate measurements, one for each sub-system, are performed in a distributed fashion with no communication taking place between Alice and Bob. Imagine that there is a third party (named Eve) who tries to simulate the action of the measurements without any access to the quantum systems. To achieve this objective, Alice and Bob send classical bits to Eve at rate \(r_1\) and \(r_2\), respectively.
Eve on receiving these pairs of classical bits from Alice and Bob wishes to reconstruct the joint quantum state $\rho_{AB}$ using a classical-to-quantum channel. The reconstruction has to satisfy a fidelity constraint characterized using a distortion observable or a norm-distance.

A naive strategy is to apply Winter’s measurement theorem [5] to compress each individual measurement $M_A$ and $M_B$ separately into $\tilde{M}_A$ and $\tilde{M}_B$. As a result, faithful simulation of $M_A$ by $\tilde{M}_A$ is possible when at least $nI(X;R)$ classical bits of communication and $nH(X|R)$ bits of common randomness are available between Alice and Eve. Similarly, a faithful simulation of $M_B$ by $\tilde{M}_B$ is possible with $nI(X;R)$ classical bits of communication and $nH(Y|R)$ bits of common randomness between Eve and Bob. However, we can further reduce the amount of classical communication by exploiting the statistical correlations between Alice’s and Bob’s measurement outcomes.

As one of the major contributions of this work, we develop a method to simulate Alice’s and Bob’s measurements with lower number of communication bits. In addition, we characterize a set of sufficient communication and common randomness rates in terms of single-letter quantum information quantities (Theorem 2). To prove this theorem, we develop two techniques: 1) binning of quantum measurements, and 2) mutual packing lemma for distributed quantum measurements. These techniques can be viewed as the quantum counterpart of their classical analogues [6]. The idea of binning in quantum setting has been used in [7] and [8] for quantum data compression involving side information (similar to Slepian-Wolf problem). However, in this paper we introduce a novel binning technique for measurements which is different from these works. The binning in this work is used to construct measurements for Alice and Bob with fewer outcomes compared to the above individual measurements, i.e., $\tilde{M}_A$ and $\tilde{M}_B$. The mutual packing lemma is used to ensure that the binned measurements performs as good as these individual measurements.

Secondly, we use our results on the simulation of distributed measurements to develop a distributed quantum-to-classical rate distortion theory (Theorem 3). For the achievability part, we characterize a rate region analogous to Berger-Tung’s [9] in terms of single-letter quantum mutual information quantities. Further, we can show that, as in the classical setting, the n-letter regularization of this rate region is optimal.

The paper is organized as follows: In Section II basic definitions and formulations are provided. Section III contains our results on distributed simulation of quantum measurements. In Section IV we derive the quantum counterpart of Berger-Tung rate region for quantum-classical distributed source coding. Finally, Section V concludes the paper.
II. Preliminaries

We here establish all our notations, briefly state few necessary definitions, and also provide Winter’s theorem on measurement compression. Let $B(\mathcal{H})$ denote the algebra of all linear operators acting on a finite dimensional Hilbert space $\mathcal{H}$. Further, let $D(\mathcal{H})$ denote the set of positive operators of unit trace acting on $\mathcal{H}$. By $I$ denote the identity operator. The trace distance between two operators $A$ and $B$ is defined as $\|A - B\|_1 \equiv \text{tr}|A - B|$, where for any operator $\Lambda$ we define $|\Lambda| \equiv \sqrt{\Lambda^\dagger \Lambda}$. The Von Neumann entropy of a density operator $\rho$ is denoted by $S(\rho)$. The quantum mutual information and conditional entropy for a bipartite density operator $\rho_{AB}$ are defined, respectively, as

$$I(\rho_{AB}) \triangleq S(\rho_A) + S(\rho_B) - S(\rho_{AB}),$$

$$S(A|B)_{\rho} \triangleq S(\rho_{AB}) - S(\rho_B).$$

A positive-operator valued measure (POVM) is a collection of $\Lambda_x$ of positive operators that form a resolution of the identity:

$$\Lambda_x \geq 0, \forall x, \quad \sum_x \Lambda_x = I.$$

Let $\Psi^R_{\rho}$ denote a purification of a density operator $\rho_A \in D(\mathcal{H}_A)$. Given a POVM $M \triangleq \{\Lambda^A_x\}$ acting on $\mathcal{H}_A$ define

$$(\text{id} \otimes M)(\rho) \equiv \sum_x |x\rangle\langle x| \otimes \text{tr}_A\{(I^R \otimes \Lambda^A_x)\Psi^R_{\rho}\}$$

(1)

Consider two POVMs $M_A = \{\Lambda^A_x\}$ and $M_B = \{\Lambda^B_y\}$ acting on $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Define $M_A \otimes M_B$ as the collection of all observables of the form $\Lambda^A_x \otimes \Lambda^B_y$ for all $x, y$. With this definition, $M_A \otimes M_B$ is a POVM acting on $\mathcal{H}_A \otimes \mathcal{H}_B$. By $M^{\otimes n}$ denote the $n$-fold tensor product of the POVM $M$ with itself.

Consider a POVM $M = \{\Lambda_x\}_{x \in \mathcal{X}}$ with classical outputs $\mathcal{X}$. Given a mapping $\beta : \mathcal{X} \rightarrow \mathcal{Y}$, define $\beta(M)$ as a new POVM $M_Y$ with observables $\Lambda_x$ but with classical outputs $\beta(x)$ where $x \in \mathcal{X}$. For this POVM equation (1) can be written as

$$(\text{id} \otimes \beta(M))(\rho) = \sum_x |\beta(x)\rangle\langle \beta(x)| \otimes \text{tr}_A\{(I^R \otimes \Lambda^A_x)\Psi^R_{\rho}\}.$$

A. Quantum Information Source

Consider a family of quantum states $\rho_i, i \in [1, m]$ acting on a Hilbert space $\mathcal{H}$. For each state assign a priori probability $p_i$. We denote such a setup by the ensemble $\{p_i, \rho_i, i \in [1 : m]\}$. For such an ensemble, a quantum source is a sequence of states each equal to $\rho_i$ with probability $p_i$, $i \in [1, m]$. Each realization of the source, after $n$ generations of states, is represented by $\rho_x^n \triangleq \bigotimes_{j=1}^n \rho_{x^j}$, where $x^n$ is a vector with
elements in $[1, m]$. Let $\rho \triangleq \sum_{i} p_{i} \rho_{i}$, then the average density operator of the source after $n$ generations is $\rho^{\otimes n}$.

B. Measurement Compression Theorem

Here, we provide a brief overview of the measurement compression theorem [5].

**Definition 1** (Faithful simulation). Given a POVM $M \triangleq \{\Lambda_{x}\}_{x \in X}$ and a density operator $\rho$ acting on a Hilbert space $\mathcal{H}$, a POVM $\tilde{M}$ acting on $\mathcal{H}^{\otimes n}$ is $\epsilon$-faithful if for $\epsilon > 0$ the following holds:

$$\sum_{x} \frac{1}{2} \left\| \sqrt{\rho^{\otimes n}} \left( \Lambda_{x^{n}} - \tilde{\Lambda}_{x^{n}} \right) \sqrt{\rho^{\otimes n}} \right\|_{1} \leq \epsilon, \quad (2)$$

where $\tilde{\Lambda}_{x^{n}}$ are the observables of $\tilde{M}$.

**Theorem 1.** [5] For $\epsilon > 0$, a density operator $\rho$ and a POVM $M$, there exist a collection of POVMs $\tilde{M}^{(\mu)}$ for $\mu \in [1, N]$, each acting on $\rho^{\otimes n}$, and having at most $2^{nR}$ outcomes where

$$R \geq I(X; R)_{\rho} + \epsilon, \quad \text{and} \quad \frac{1}{n} \log_{2} N \geq S(X|R)_{\rho} + \epsilon \quad (3)$$

such that $\tilde{M} \triangleq \frac{1}{N} \sum_{\mu} \tilde{M}^{(\mu)}$ is $\epsilon$-faithful.

III. APPROXIMATION OF DISTRIBUTED POVMs

We provide our extension to the Winter’s measurement compression protocol for a distributed setting. Consider a bipartite composite quantum system $(A, B)$ represented by Hilbert Space $\mathcal{H}_{A} \otimes \mathcal{H}_{B}$. Let $\rho_{AB}$ be a quantum information source on $\mathcal{H}_{A} \otimes \mathcal{H}_{B}$. Imagine that three parties, traditionally named Alice, Bob and Eve, are trying to collectively implement two measurements, one applied to each sub-system. Eve has no access to the quantum system; while Alice and Bob have access to sub-system $A$ and $B$, respectively. Alice and Bob perform a measurement $M_{A}$ and $M_{B}$ on sub-systems $A$ and $B$, respectively. The measurements are performed in a distributed fashion with no communication taking place between Alice and Bob. In this context, the overall measurement is characterized by the tensor product measurement $M_{A} \otimes M_{B}$. The objective of Eve is to reconstruct an asymptotically faithful simulation of $M_{A} \otimes M_{B}$ when acted on $\rho_{AB}$. For that, Alice and Bob send a number of classical bits to Eve. Then, Eve applies a decoding map to the received bits and reconstructs the original measurement outcomes. The design objective is to minimize the amount of the classical bits that Eve needs to simulate the measurements.

We consider a special class of quantum information sources. In particular, the focus of this section is on non-entangled quantum information sources defined as in the following.
Definition 2. A non-entangled quantum state $\rho_{AB}$ is characterized by an ensemble $\{p_{a,b}, \rho_a \otimes \rho_b\}_{a \in A, b \in B}$ where $\rho_{AB} = \sum_{a,b} p_{a,b} \rho_a \otimes \rho_b$.

In what follows, we formulate the problem.

Definition 3. For a given Hilbert space $\mathcal{H}_{AB}$, a distributed protocol with classical communication rates $(R_1, R_2)$ and common randomness rate $C$ is characterized by a collection of $2^nC$ POVM-pairs $(M_A^{(n,\mu)}, M_B^{(n,\mu)}), \mu \in [1, 2^nC]$, acting on $\mathcal{H}_{AB}^{\otimes n}$ and with at most $2^nR_1$ and $2^nR_2$ outcomes.

In the above definition, $(R_1, R_2)$ determines the amount of classical bits communicated from Alice and Bob to Eve. The amount of common randomness is $C$ bits and $\mu$ can be viewed as the common randomness bits distributed among the parties. In the following, we define a measure for faithful simulation.

Definition 4. A distributed protocol with POVM-pairs $(M_A^{(n,\mu)}, M_B^{(n,\mu)}), \mu \in [1, N]$ is $\epsilon$-faithful for simulation of the POVM $M_A \otimes M_B$ on the source $\rho_{AB}$, if there exist a collection of mappings $\beta^{(\mu)}$ such that the average POVM $\tilde{M}_{AB}^{(n)} = \frac{1}{N} \sum_{\mu=1}^{N} \beta^{(\mu)}(M_A^{(n,\mu)} \otimes M_B^{(n,\mu)})$ is $\epsilon$-faithful according to Definition 1.

In the above definition, the mappings $\beta^{(\mu)}$ represent the action of Eve on the received classical bits.

Theorem 2. Let $\rho_{AB}$ denote a non-entangled quantum information source measured by a POVM of the form $M_A \otimes M_B$. For any $\epsilon > 0$ and sufficiently large $n$, there exists an $\epsilon$-faithful distributed protocol with classical communication rates $(R_1, R_2)$ and common randomness rate $C$ provided that

$$R_1 \geq I(U; R)_{\sigma} - I(U; V)_{\sigma},$$
$$R_2 \geq I(V; R)_{\sigma} - I(U; V)_{\sigma},$$
$$R_1 + R_2 \geq I(U; R)_{\sigma} + I(V; R)_{\sigma} - I(U; V)_{\sigma},$$
$$C \geq S(V|R)_{\sigma} + S(U|R)_{\sigma},$$

where the mutual information and conditional entropy terms are calculated for the state

$$\sigma_{UV} \triangleq \sum_{u,v} |u,v\rangle\langle u,v| \otimes \text{tr}_{AB}\{(I^R \otimes \Lambda_u^A \otimes \Lambda_v^B)\Psi^{RAB}_{\rho}\},$$

where $\Psi^{RAB}_{\rho}$ is a purification of $\rho_{AB}$ and $\{\Lambda_u^A \otimes \Lambda_v^B\}$ denote the observables of $M_A \otimes M_B$.

Proof. The proof is given in Appendix.
Fig. 2. Figure shows the achievable rate region with two different schemes. The Naive compression scheme is where each quantum source is independently compressed, while the other scheme, in order to exploit the correlation among the measurement outcomes, bins the POVMs before applying the measurements. As a result, the rate achieved by the latter is lower than the naive compression which translates into a larger rate region.

Fig. 2 demonstrates the region in Theorem 2 in terms of the quantum information quantities. It also shows the gains achieved by employing such an approach as opposed to independently compressing the two sources $\rho_A$ and $\rho_B$.

As for the converse, one can extend the single letter characterization to an n-letter regularized formulation and provide an optimal rate region with a converse to its achievability. Define $\mathcal{R}$ as the set of all rates $(R_1, R_2, C)$ for which there exist $n$ such that

$$R_1 \geq \frac{1}{n}(U^n; R^n)_\sigma - I(U^n; V^n)_\sigma,$$

$$R_2 \geq \frac{1}{n}I(V^n; R^n)_\sigma - \frac{1}{n}I(U^n; V^n)_\sigma,$$

$$R_1 + R_2 \geq \frac{1}{n}I(U^n; R^n)_\sigma + \frac{1}{n}I(V^n; R^n)_\sigma - \frac{1}{n}I(U^n; V^n)_\sigma,$$

$$C \geq \frac{1}{n}S(V^n|R^n)_\sigma + \frac{1}{n}S(U^n|R^n)_\sigma.$$

We can show that $\mathcal{R}$ is the optimal rate-region for this problem. Note that characterizing the optimal rate region in terms of single letter information quantities is still an open problem even in classical setting.

A. Proof Techniques

**Binning for POVMs:** We introduce a quantum-counterpart of the classical binning technique used to prove Theorem 2. Here, we describe this technique.
Consider a POVM $M$ with observables $\{\Lambda_{\alpha_1}, \Lambda_{\alpha_2}, \ldots, \Lambda_{\alpha_N}\}$. Given $K$ for which $N$ is divisible, partition $[1, N]$ into $K$ equal bins and for each $i \in [1, K]$ let $B(i)$ denote the $i^{th}$ bin. The binned POVM $\tilde{M}$ is given by the collection of operators $\{\tilde{\Lambda}_{\beta_1}, \tilde{\Lambda}_{\beta_2}, \ldots, \tilde{\Lambda}_{\beta_K}\}$ where $\tilde{\Lambda}_{\beta_i}$ is defined as

$$
\tilde{\Lambda}_{\beta_i} = \sum_{j \in B(i)} \Lambda_{\alpha_j}.
$$

Using the fact that $\Lambda_{\alpha_i}$ are self-adjoint and positive $\forall i \in [1, N]$ and $\sum_{i=1}^{N} \Lambda_{\alpha_i} = I$, (which is because $M$ is a POVM); it follows that $\tilde{M}$ is a valid POVM.

**Mutual Packing Lemma for POVMs:** Another technique used to prove Theorem 2 is a quantum version of mutual packing lemma. In what follows, we describe the mutual packing lemma for quantum measurements. For a Hilbert Space $\mathcal{H}_{AB}$ consider a POVM of the form $M_A \otimes M_B$, where $(M_A, M_B)$ are two POVMs each acting on one sub-system. The observables for $M_A$ and $M_B$ are denoted, respectively, by $\Lambda_u^A \in \mathcal{B}(\mathcal{H}_A), u \in \mathcal{U}$ and $\Lambda_v^B \in \mathcal{B}(\mathcal{H}_B), v \in \mathcal{V}$, where $\mathcal{U}$ and $\mathcal{V}$ are finite sets. Fix a joint-distribution $P_{UV}$ on the set of all outcomes $\mathcal{U} \times \mathcal{V}$. For each $l \in [1, 2^{nr_1}]$, let $U^n(l)$ be a random sequence generated according to $\prod_{i=1}^{n} P_{U}$. Similarly, let $V^n(k)$ be a random sequence distributed according to $\prod_{i=1}^{n} P_{V}$, where $k \in [1, 2^{nr_2}]$. Suppose $U^n(l)$’s and $V^n(k)$’s are independent. Define the following random observables:

$$
A_u^n \triangleq |l : U^n(l) = u^n|\Lambda_u^A, \quad B_v^n \triangleq |k : V^n(k) = v^n|\Lambda_v^B
$$

where $\Lambda_u^A = \otimes_i \Lambda_u^{A_i}$ and $\Lambda_v^B = \otimes_i \Lambda_v^{B_i}$.

**Lemma 1.** For any $\epsilon > 0$ and sufficiently large $n$, with high probability

$$
\left\| \sum_{(u^n,v^n) \in T_{\epsilon}^{(n)}(U,V)} A_u^n \otimes B_v^n \right\|_\infty \leq \epsilon
$$

provided that $r_1 + r_2 < I(U;V) - \delta(\epsilon)$.

**Proof.** From the triangle-inequality and the definition of $A_u^n$ and $B_v^n$, the norm in the lemma does not exceed the following

$$
\sum_{l,k} \sum_{(u^n,v^n) \in T_{\epsilon}^{(n)}(U,V)} 1\{U^n(l) = u^n, V^n(k) = v^n\} \left\| \Lambda_u^A \otimes \Lambda_v^B \right\|_\infty 
\leq \sum_{l,k} \mathbb{P}\{(U^n(l), V^n(k)) \in T_{\delta}^{(n)}(U,V)\}
$$

where the last inequality holds since $\Lambda_u^A \otimes \Lambda_v^B \leq I$. The proof completes from the classical mutual packing lemma. \hfill \Box
IV. Q-C DISTRIBUTED RATE DISTORTION THEORY

As an application to the above theorem on faithful simulation of distributed measurements (Theorem 2), we investigate the distributed extension of quantum-to-classical rate distortion coding [3]. This problem is a quantum counterpart of the classical distributed source coding. In this setting, many copies of a bipartite quantum information source $\rho_{AB}$ are generated. Alice and Bob have access to the partial trace of the copies related to $\rho_A$ and $\rho_B$. They perform measurements on their marginalized source. The goal is to generate a classical description of the source so that a third party, namely Eve, would be able to reconstruct the source within a distortion threshold. First, we formulate the problem in the following.

**Definition 5.** A q-c source coding setup is characterized by an input Hilbert space $\mathcal{H}$, an information source $\rho$ acting on $\mathcal{H}$, a reconstruction Hilbert space $\hat{\mathcal{H}}$, and a distortion observable $\Delta$ acting on $\mathcal{H} \otimes \hat{\mathcal{H}}$ such that it is non-negative and bounded, i.e., $0 \leq \Delta \leq kI_d$ for some $k \in \mathbb{R}$.

The distortion defined for the action of a POVM $M$ on a state $\rho$ is measured by

$$\bar{d}(\rho, M) = \text{tr}\{\Delta((\text{id}_R \otimes M)(\Psi_{RA}))\}.$$  

For n-letter distortion, we use average distortion observable defined as

$$\Delta^{(n)} = \frac{1}{n} \sum_{i=1}^{n} \Delta_{R,A} \otimes \hat{I}_{RA}^{[n]}.$$  

This gives the n-letter average distortion as

$$\bar{d}(\rho^{\otimes n}, M^{(n)}) = \text{tr}\{\Delta^{(n)}(\text{id}_{R^n} \otimes M^{(n)})(\Psi_{R^nA^n})\}$$

where $M^{(n)}_A$ is the measurement map acting on the source state $\rho^{\otimes n}$ and $\Psi_{R^nA^n}$ is the purification of the state $\rho^{\otimes n}$.

The authors in [3] studied the point-to-point version of the above formulation. They considered a special distortion observable of the form

$$\Delta = \sum_{x \in \mathcal{X}} \Delta_x \otimes |x\rangle\langle x|$$  

where $\Delta_x \geq 0$ acts on the reference Hilbert space and $\mathcal{X}$ is the reconstruction alphabet. In this paper, we allow $\Delta$ to be any non-negative and bounded operator acting on the appropriate Hilbert spaces. Moreover, we allow for the use of any c-q mapping as a decoder.

In the distributed setting, we define the distortion observable as any operator $0 \leq \Delta \leq kI_d$ acting on the Hilbert space $\mathcal{H}_R \otimes \hat{\mathcal{H}}_{AB}$. In what follows we provide a definition for a distributed quantum-classical compression code.
**Definition 6.** An \((n, \theta_1, \theta_2)\) Quantum-to-Classical (q-c) code is defined by POVMs \(M_A^{(n)}\) and \(M_B^{(n)}\) with \(\theta_1\) and \(\theta_2\) outcomes, respectively, and a mapping
\[
\beta : [1 : \theta_1] \times [1 : \theta_2] \rightarrow \mathcal{D}(\mathcal{H}_{XY})
\]

**Definition 7.** Given a source \(\rho_{XY}\), a pair of rates \((R_1, R_2)\) and a distortion \(D\) are said to be achievable if for all \(\epsilon > 0\) and sufficiently large \(n\), there exists an \((n, \theta_1, \theta_2)\) q-c code such that
\[
\frac{1}{n} \log_2 \theta_1 \leq R_1 + \epsilon, \quad \frac{1}{n} \log_2 \theta_2 \leq R_2 + \epsilon,
\]
\[
d(\rho^{\otimes n}, M_A^{(n)} \otimes M_B^{(n)}) \leq D + \epsilon.
\]

**Definition 8.** For a given distortion threshold \(D\), the set of all rate pairs \((R_1, R_2)\) such that \((R_1, R_2, D)\) is achievable is called as the rate distortion region.

**Theorem 3.** For a bipartite quantum source \((\rho_{AB})\), a distortion observable \(\Delta_{RAB}\) and any given distortion \(D > 0\), the q-c rate distortion region is the union of all rate-pairs \((R_1, R_2)\) that satisfy
\[
R_1 \geq I(U; R)_\sigma - I(U; V)_\sigma,
\]
\[
R_2 \geq I(V; R)_\sigma - I(U; V)_\sigma,
\]
\[
R_1 + R_2 \geq I(U; R)_\sigma + I(V; R)_\sigma - I(U; V)_\sigma
\]
where the union is taken over all POVMs \((M_A, M_B)\) with \(d(\rho_{AB}, M_A \otimes M_B) \leq D\).

**Proof.** The proof follows from Theorem 2. Fix POVMs \((M_A, M_B)\) as in the statement of the theorem. Hence,
\[
d(\rho_{AB}, M_A \otimes M_B) \leq D.
\]
According to Theorem 2, there exists a set of POVMs \((M_A^{(n,\mu)}, M_B^{(n,\mu)}), \mu \in [1, N]\) each having at most \((2^{nR_1}, 2^{nR_2})\) outcomes, where
\[
R_1 \geq I(U; R)_\sigma - I(U; V)_\sigma,
\]
\[
R_2 \geq I(V; R)_\sigma - I(U; V)_\sigma,
\]
\[
R_1 + R_2 \geq I(U; R)_\sigma + I(V; R)_\sigma - I(U; V)_\sigma
\]
and a mapping \(\beta\) such that \(\beta(M_A^{(n)} \otimes M_B^{(n)}\) is a faithful simulation of \((M_A \otimes M_B)\), where
\[
M_A^{(n)} = \frac{1}{N} \sum_\mu M_A^{(n,\mu)}, M_B^{(n)} = \frac{1}{N} \sum_\mu M_B^{(n,\mu)},
\]
Therefore, from Definition 1, \( \forall \epsilon > 0 \) and large enough \( n \), the following condition is satisfied:
\[
\| (I \otimes M_A^{\otimes n} \otimes M_B^{\otimes n}) \Psi_{\rho}^{R^n_{A^nB^n}} - (I \otimes \beta(M_A^{(n)} \otimes M_B^{(n)})) \Psi_{\rho}^{R^n_{A^nB^n}} \|_1 \leq \epsilon
\]

Suppose \((i, j)\) represents the outcomes of the POVMs \((M_A^{(n,\mu)} \otimes M_B^{(n,\mu)})\). Then the mapping \(\beta\) can viewed as the classical to quantum decoder for such measurements. In particular the decoder is \(|\beta(i, j)\rangle\langle\beta(i, j)|\).

With this notation, it suffices to show the following bounds
\[
\bar{d}(\rho_A^{\otimes n}, \beta(M_A^{(n)} \otimes M_B^{(n)}))
\]
\[
= \frac{1}{N} \sum_{\mu=1}^{N} \sum_{i=1}^{2^{nR_1}} \sum_{j=1}^{2^{nR_2}} \text{tr} \left\{ \Delta^n \text{tr}_{AB} \left\{ I \otimes (\gamma_i^A,\mu) \otimes \Gamma_j^B,\mu \right\} \Psi_{R^n_{A^nB^n}} \right\} \otimes |\beta(i, j)\rangle\langle\beta(i, j)|
\]
\[
= \frac{1}{N} \sum_{\mu=1}^{N} \sum_{i=1}^{2^{nR_1}} \sum_{j=1}^{2^{nR_2}} d_{\max} \text{tr} \left\{ \frac{\Delta^n}{d_{\max}} \text{tr}_{AB} \left\{ I \otimes (\gamma_i^A,\mu) \otimes \Gamma_j^B,\mu \right\} \Psi_{R^n_{A^nB^n}} \right\} \otimes |\beta(i, j)\rangle\langle\beta(i, j)|
\]
\[
\leq \text{tr} \left\{ \Delta^n \left\{ I \otimes (M_A^{\otimes n} \otimes M_B^{\otimes n}) \Psi_{R^n_{A^nB^n}}^p \right\} + \epsilon d_{\max} \right\}
\]
\[
\leq D + \epsilon d_{\max}
\]

One can observe that the rate-region in Theorem 3 matches exactly with the classical Berger-Tung region when \(\rho_{AB}\) is a mixed state of a collection of orthogonal pure states. Note that the rate-region is an inner bound for the set of all achievable rates. The single-letter characterization of the set of achievable rates is still an open problem even in the classical setting.

V. Conclusions

We established a distributed extension of Winter’s measurement compression theory. A set of communication rate-pairs and common randomness rate is characterized for faithful simulation of distributed measurements. We further investigated distributed quantum-to-classical rate-distortion theory and derived a quantum counterpart of Berger-Tung rate-region.
A. Proof of the Theorem

Suppose the observables of $M_A$ and $M_B$ are denoted by $\{W_u^A: u \in \mathcal{U}\}$ and $\{W_v^B: v \in \mathcal{V}\}$, respectively, where $\mathcal{U}$, $\mathcal{V}$ are two finite subsets. The canonical ensembles corresponding to $M_A$ and $M_B$ are defined as

$$\{\lambda_u^A, \hat{\rho}_u^A\}_{u \in \mathcal{U}}, \quad \{\lambda_v^B, \hat{\rho}_v^B\}_{v \in \mathcal{V}}, \quad \text{and} \quad \{\lambda_{uv}^{AB}, \hat{\rho}_{uv}^{AB}\}_{(u,v) \in \mathcal{U} \times \mathcal{V}}$$

where

$$\lambda_u^A \triangleq \text{tr}(W_u^A \rho_A), \quad \lambda_v^B \triangleq \text{tr}(W_v^B \rho_B), \quad \lambda_{uv}^{AB} \triangleq \text{tr}(W_u^A \otimes W_v^B \rho_{AB}),$$

$$\hat{\rho}_u^A \triangleq \frac{1}{\lambda_u^A} \sqrt{\rho_A W_u^A \sqrt{\rho_A}}, \quad \hat{\rho}_v^B \triangleq \frac{1}{\lambda_v^B} \sqrt{\rho_B W_v^B \sqrt{\rho_B}}, \quad \hat{\rho}_{uv}^{AB} \triangleq \frac{1}{\lambda_{uv}^{AB}} \sqrt{\rho_{AB} W_u^A \otimes W_v^B \sqrt{\rho_{AB}}}$$

where $\rho_A = \text{tr}_B(\rho_{AB})$ and $\rho_B = \text{tr}_A(\rho_{AB})$. Note that $\{\lambda_{uv}^{AB}\}$ is a joint probability distribution on $\mathcal{U} \times \mathcal{V}$ with $\{\lambda_u^A\}$ and $\{\lambda_v^B\}$ as the marginals. With this notation, corresponding to each of the probability distributions, we can associate a $\delta$-typical set. Let us denote $\mathcal{T}_\delta^{(n)}(A)$, $\mathcal{T}_\delta^{(n)}(B)$ and $\mathcal{T}_\delta^{(n)}(AB)$ as the $\delta$-typical sets defined for $\{\lambda_u^A\}$, $\{\lambda_v^B\}$ and $\{\lambda_{uv}^{AB}\}$, respectively.

Let $\Pi_{\rho_A}$ and $\Pi_{\rho_B}$ denote the $\delta$-typical projectors (as in [10]) for marginal density operators $\rho_A$ and $\rho_B$, respectively. Also, for any $u^n \in \mathcal{U}^n$ and $v^n \in \mathcal{V}^n$, let $\Pi_{\rho_u}^n$ and $\Pi_{\rho_v}^n$ denote the conditional typical projectors (as in [5]) for the canonical ensembles $\{\lambda_u^A, \hat{\rho}_u^A\}$ and $\{\lambda_v^B, \hat{\rho}_v^B\}$, respectively. For each $u^n \in \mathcal{U}^n$ and $v^n \in \mathcal{V}^n$ define

$$\Lambda_{u^n}^A = \Pi_{\rho_u} \Pi_{\rho_u}^A \hat{\rho}_{u^n}^A \Pi_{\rho_u} \Pi_{\rho_u}^A, \quad \Lambda_{v^n}^B = \Pi_{\rho_v} \Pi_{\rho_v}^B \hat{\rho}_{v^n}^B \Pi_{\rho_v} \Pi_{\rho_v}^B$$

(6)

where $\hat{\rho}_{u^n}^A \triangleq \otimes_i \hat{\rho}_{u_i}^A$ and $\hat{\rho}_{v^n}^B \triangleq \otimes_i \hat{\rho}_{v_i}^B$.

Let $U^n$ and $V^n$ be random sequences generated independently and according to

$$\mathbb{P}(U^n = u^n) = \frac{\lambda_u^{u^n}}{1 - \varepsilon}, \quad \forall u^n \in \mathcal{T}_\delta^{(n)}(A),$$

(7)

$$\mathbb{P}(V^n = v^n) = \frac{\lambda_v^{v^n}}{1 - \varepsilon'}, \quad \forall v^n \in \mathcal{T}_\delta^{(n)}(B),$$

(8)

and $\mathbb{P}(U^n = u^n) = \mathbb{P}(V^n = v^n) = 0$ for any $u^n \notin \mathcal{T}_\delta^{(n)}(A)$ and $v^n \notin \mathcal{T}_\delta^{(n)}(B)$. Here $\varepsilon > 0$ and $\varepsilon' > 0$ are chosen such that $\mathbb{P}(\mathcal{T}_\delta^{(n)}(A)) = \mathbb{P}(\mathcal{T}_\delta^{(n)}(B)) = 1$. With the notation above, define $\sigma^{A'} \triangleq \mathbb{E}[\Lambda_{u^n}^A]$ and $\sigma^{B'} \triangleq \mathbb{E}[\Lambda_{v^n}^B]$, where the expectation is taken with respect to $U^n$ and $V^n$, respectively. Let $\hat{\Pi}_A$ and $\hat{\Pi}_B$ be the projector onto the subspace spanned by the eigen-states of $\sigma^{A'}$ and $\sigma^{B'}$ corresponding to eigenvalues that are larger than $\varepsilon 2^{-nS(\rho_A)}$ and $\varepsilon' 2^{-nS(\rho_B)}$, respectively. Lastly, define

$$\Lambda_{u^n}^A \triangleq \hat{\Pi}_A \Lambda_{u^n}^A \hat{\Pi}_A \quad \text{and} \quad \sigma^A \triangleq \mathbb{E}[\Lambda_{u^n}^A],$$

(9)

$$\Lambda_{v^n}^B \triangleq \hat{\Pi}_B \Lambda_{v^n}^B \hat{\Pi}_B \quad \text{and} \quad \sigma^B \triangleq \mathbb{E}[\Lambda_{v^n}^B].$$

(10)
B. Construction of Random POVMs

In what follows, we construct two random POVMs one for each encoder. Fix a positive integer $N$. For each $\mu \in [1, N]$, randomly and independently select $2^{nR_1}$ and $2^{nR_2}$ sequences according to $P_{U^n}$ and $P_{V^n}$ (as in (7) and (8)), respectively. Let $(U^{n,\mu}(l), V^{n,\mu}(k))$ represent the randomly selected sequences for each $\mu$, where $l \in [1, 2^{nR_1}]$ and $k \in [1, 2^{nR_2}]$. Construct observables

$$A_{\mu}^{(n)} \triangleq \gamma_{\mu}^{(n)} \left( \sqrt{\rho_A^{-1}} A_{\mu}^{A} \sqrt{\rho_A^{-1}} \right) \text{ and } B_{\mu}^{(n)} \triangleq \zeta_{\mu}^{(n)} \left( \sqrt{\rho_B^{-1}} A_{\mu}^{B} \sqrt{\rho_B^{-1}} \right)$$

where

$$\gamma_{\mu}^{(n)} \triangleq \frac{1 - \varepsilon}{1 + \eta} 2^{-\tilde{R}_1} |\{ l : U^{n,\mu}(l) = u^n \}| \quad \text{and} \quad \zeta_{\mu}^{(n)} \triangleq \frac{1 - \varepsilon'}{1 + \eta} 2^{-\tilde{R}_2} |\{ k : V^{n,\mu}(k) = v^n \}|,$$

where $\eta \in (0, 1)$ is a parameter to be determined. Then, for each $\mu \in [1, N]$ construct $M_B^{(n,\mu)}$ and $M_A^{(n,\mu)}$ as in the following

$$M_1^{(n,\mu)} \triangleq \{ A_{\mu}^{(n)} : u^n \in \mathcal{T}_1^{(n)}(A) \}, \quad M_2^{(n,\mu)} \triangleq \{ B_{\mu}^{(n)} : v^n \in \mathcal{T}_1^{(n)}(B) \}. \quad (13)$$

As a first step, one can show that with probability sufficiently close to one, $M_B^{(n,\mu)}$ and $M_A^{(n,\mu)}$ form sub-POVMs for all $\mu \in [1, N]$. More precisely the following Lemma holds.

**Lemma 2.** For any $\varepsilon, \varepsilon', \eta \in (0, 1)$, as in (12), and any $\zeta \in (0, 1)$, there is $n(\varepsilon, \varepsilon', \eta, \zeta)$ such that for all $n \geq n(\varepsilon, \varepsilon', \eta, \zeta)$ the collection of observables $M_A^{(n,\mu)}$ and $M_B^{(n,\mu)}$ form sub-POVMs for all $\mu \in [1, N]$ with probability at least $1 - \zeta$, provided that

$$\tilde{R}_1 > I(X; R)_\rho, \quad \text{and} \quad \tilde{R}_2 > I(Y; R)_\rho.$$

**Proof.** The proof uses a similar argument as in the proof of Theorem 2 in [5]. Hence it is omitted. \(\square\)

C. Binning of POVMs

We introduce the quantum counterpart of the so-called binning technique which has been widely used in the context of classical distributed source coding. Fix binning rates $(R_1, R_2)$. Partition the sets $[1, 2^{nR_1}]$ and $[1, 2^{nR_2}]$ into $2^{nR_1}$ and $2^{nR_2}$ equal size bins, respectively. For each $i \in [1, 2^{nR_1}]$ and $j \in [1, 2^{nR_2}]$, let $B_1(i)$ and $B_2(j)$ denote the $i$th and the $j$th bins for $[1, 2^{nR_1}]$ and $[1, 2^{nR_2}]$, respectively. Define the following observables:

$$\Gamma_i^{A,(\mu)} \triangleq \sum_{l \in B_1(i)} A_{U^{n,\mu}(l)}^{(n)} \quad \text{and} \quad \Gamma_j^{B,(\mu)} \triangleq \sum_{k \in B_2(j)} B_{V^{n,\mu}(k)}^{(n)}$$

for all $i \in [1, 2^{nR_1}]$ and $j \in [1, 2^{nR_2}]$. The above observables generate the following POVMs

$$M_A^{(n,\mu)} \triangleq \{ \Gamma_i^{A,(\mu)} : i \in [1, 2^{nR_1}] \}, \quad M_B^{(n,\mu)} \triangleq \{ \Gamma_j^{B,(\mu)} : j \in [1, 2^{nR_2}] \}. \quad (14)$$
Note that if $M_1^{(n,\mu)}$ and $M_2^{(n,\mu)}$ are sub-POVMs, then so are $M_A^{(n,\mu)}$ and $M_B^{(n,\mu)}$. This is due to the relations
\[
\sum_i \Gamma_{i}^{A,(\mu)} = \sum_{u^n} A_{u^n}^{(\mu)}, \quad \text{and} \quad \sum_j \Gamma_{j}^{B,(\mu)} = \sum_{v^n} B_{v^n}^{(\mu)}.
\]
Note that the effect of the binning is in reducing the communication rates from $(\tilde{R}_1, \tilde{R}_2)$ to $(R_1, R_2)$.

D. Decoder mapping

Note that the observables $A_{u^n}^{(\mu)} \otimes B_{v^n}^{(\mu)}$ are used to simulate the original measurement. The effect of the binning is to reduce the communication rates. The binning can be viewed as partitioning the set of classical outcomes into a number of bins. Suppose an outcome $(U^n, V^n)$ occurred after the measurement. Then, if the bins are small enough one might be able to recover the outcomes by knowing the bin numbers. For that we will create a decoder that takes as an input the bin numbers and produces a pair of sequences $(u^n, v^n)$. More precisely, we define a mapping $\beta(\mu)$ acting on the outputs of $M_A^{(n,\mu)} \otimes M_B^{(n,\mu)}$. For each $\mu$ and bin numbers $(i, j)$, let us define
\[
D_{i,j}^{(\mu)} \triangleq \{(l, k) : (U^n,(\mu) (l), V^n,(\mu) (k)) \in T_\delta^{(n)}(AB) \text{ and } (l, k) \in B_1(i) \times B_2(j)\}.
\]
For $i \in [1 : 2^{nR_1}]$ and $j \in [1, 2^{nR_2}]$ define the function $F(\mu)(i, j) = (U^n,(\mu) (l), V^n,(\mu) (k))$ if $(k, l)$ is the only element of $D_{i,j}^{(\mu)}$; otherwise $F(\mu)(i, j) = (u^n_0, v^n_0)$, where $(u^n_0, v^n_0)$ are arbitrary sequences. For classical outcomes $(u, v)$, denote the original measurement reconstruction state as $\Phi_{u,v}$. For more convenience, we can assume that $\Phi_{u,v} = |u, v\rangle\langle u, v|$. By $\Phi_{u^n,v^n}$ denote $\otimes_i \Phi_{u_i,v_i}$. For each measurement outcome $(i, j)$, define the reconstruction state as $S_{i,j}^{(\mu)} \triangleq \Phi_{F(\mu)(i,j)}$. With these notations we define the decoding mapping is $\beta(\mu)$ as a c-q channel given by $\beta(\mu)(i, j) \triangleq S_{i,j}^{(\mu)}$, where $i \in [0, 2^{nR_1}], j \in [0, 2^{nR_2}]$.

E. Trace Distance

We will show that $\tilde{M}_A^{(n)} \triangleq \frac{1}{N} \sum_{\mu=1}^{N} \beta(\mu)(M_A^{(n,\mu)} \otimes M_B^{(n,\mu)})$ is a faithful simulation (according to Definition 1) for $(M_A^{(n)}, M_B^{(n)})$. Let $C(\mu)$ denote the set of all the pairs of the codewords $(U^n,(\mu) (l), V^n,(\mu) (k))$. For $i \in [1 : 2^{nR_1}]$ and $j \in [1, 2^{nR_2}]$ and $(l, k) \in B_1(i) \times B_2(j)$ define $e^{(\mu)}(U^n,(\mu) (l), V^n,(\mu) (k)) \triangleq F^{(\mu)}(i, j)$. For any $(u^n, v^n) \notin C(\mu)$ define $e^{(\mu)}(u^n, v^n) = (u_0, v_0)$. Let
\[
A_{u^n} \otimes B_{v^n} \otimes S_{u^n,v^n}^{\mu} \triangleq \frac{1}{N} \sum_{\mu} A_{u^n}^{(\mu)} \otimes B_{v^n}^{(\mu)} \otimes \Phi_{e^{(\mu)}(u^n,v^n)}.
\]
Similarly, define
\[
A_{u^n} \otimes B_{v^n} = \frac{1}{N} \sum_{\mu} A_{u^n}^{(\mu)} \otimes B_{v^n}^{(\mu)}.
\]
We will show that given any \( \epsilon > 0 \) the following inequality holds with probability sufficiently close to 1.

\[
\sum_{u^n \in U^n \atop v^n \in V^n} \| \lambda^{AB}_{u^n,v^n} \tilde{\rho}^{(n)}_{u^n,v^n} \otimes \Phi_{u^n,v^n} - (\sqrt{\rho}^{(n)}(A_{u^n} \otimes B_{v^n}) \sqrt{\rho}^{(n)}) \otimes S_{u^n,v^n} \|_1 \leq \epsilon \tag{16}
\]

For each \( (u^n, v^n) \) we add and subtract \((\sqrt{\rho}^{(n)}(A_{u^n} \otimes B_{v^n}) \sqrt{\rho}^{(n)}) \otimes \Phi_{u^n,v^n}\) to the terms inside the summation above. Therefore, from the triangle inequality, each term of the summation in \((16)\) does not exceed

\[
\| (\lambda^{AB}_{u^n,v^n} \tilde{\rho}^{(n)}_{u^n,v^n} - \sqrt{\rho}^{(n)}(A_{u^n} \otimes B_{v^n}) \sqrt{\rho}^{(n)}) \otimes \Phi_{u^n,v^n} \|_1 + \| (\sqrt{\rho}^{(n)}(A_{u^n} \otimes B_{v^n}) \sqrt{\rho}^{(n)}) \otimes (\Phi_{u^n,v^n} - S_{u^n,v^n}) \|_1.
\]

Using the triangle inequality, the second term above is further upper-bounded by

\[
\frac{1}{N} \sum_{\mu} \left( \| (\sqrt{\rho}^{(n)}(A_{u^n}^{(\mu)} \otimes B_{v^n}^{(\mu)}) \sqrt{\rho}^{(n)}) \otimes (\Phi_{u^n,v^n} - S_{u^n,v^n}) \|_1 \times \| \Phi_{u^n,v^n} - S_{u^n,v^n} \|_1 \right)
\]

The last inequality follows, since \( A_{u^n}^{(\mu)} \leq \frac{\epsilon}{\lambda^{u^n}_A} M_{u^n}^A \) and \( B_{v^n}^{(\mu)} \leq \frac{\epsilon}{\lambda^{v^n}_B} M_{v^n}^B \). This is because, from \((9)\) \( \Lambda_{u^n}^{A} \leq \tilde{\rho}_{u^n}^{A} \) and from \((10)\) \( \Lambda_{v^n}^{B} \leq \tilde{\rho}_{v^n}^{B} \). As a result of the above argument, \((16)\) is bounded by

\[
\sum_{u^n,v^n} \| \lambda^{AB}_{u^n,v^n} \tilde{\rho}^{(n)}_{u^n,v^n} - (\sqrt{\rho}^{(n)}(A_{u^n} \otimes B_{v^n}) \sqrt{\rho}^{(n)}) \|_1 + \frac{1}{N} \sum_{u^n,v^n} \sum_{\mu} \gamma_{u^n}^{(\mu)} \zeta_{v^n}^{(\mu)} \frac{\lambda^{AB}_{u^n,v^n}}{\lambda^{A}_{u^n} \lambda^{B}_{v^n}} \| \Phi_{u^n,v^n} - S_{u^n,v^n} \|_1. \tag{17}
\]

Next, we derive an upper bound on the first summation in \((17)\). Note that by definition

\[
A_{u^n} \otimes B_{v^n} = \left( \frac{1}{N} \sum_{\mu=1}^{N} \gamma_{u^n}^{(\mu)} \zeta_{v^n}^{(\mu)} \right) \left( \sqrt{\rho} - 1 A_{u^n} \sqrt{\rho} \right) \otimes \left( \sqrt{\rho} - 1 B_{v^n} \sqrt{\rho} \right).
\]

We show that \( \left( \frac{1}{N} \sum_{\mu=1}^{N} \gamma_{u^n}^{(\mu)} \right) \approx \lambda_{u^n}^{A} \lambda_{u^n}^{B} \) with probability sufficiently close to one. Note that from \((12)\) we have

\[
\gamma_{u^n}^{(\mu)} = \frac{1 - \epsilon}{1 + \eta} \sum_{l \in [1,2^n,\bar{h}_1]} 1\{U^{(\mu)}(l) = u^n\}, \quad \zeta_{v^n}^{(\mu)} = \frac{1 - \epsilon'}{1 + \eta'} \sum_{k \in [1,2^n,\bar{h}_2]} 1\{V^{(\mu)}(k) = v^n\}.
\]

Taking the expectation gives \( \mathbb{E}[\gamma_{u^n}^{(\mu)} \zeta_{v^n}^{(\mu)}] = \lambda_{u^n}^{A} \lambda_{v^n}^{B} (1 + \eta) \) for all \( u^n \in U^n, v^n \in V^n \) and \( \mu \in [1, N] \). Therefore, using Chernoff bound

\[
\mathbb{P} \left\{ \bigcup_{u^n,v^n} \left\{ \frac{1}{N} \sum_{\mu=1}^{N} \gamma_{u^n}^{(\mu)} \zeta_{v^n}^{(\mu)} \neq \frac{\lambda_{u^n}^{A} \lambda_{v^n}^{B}}{(1 + \eta)^2} (1 \pm \epsilon) \right\} \right\} \leq |T_\delta^{(n)}(A)||T_\delta^{(n)}(B)| \exp \left\{ -\frac{2^{n(R_1 + R_2)} N \epsilon^2 \alpha}{2 \ln 2} \right\}, \tag{18}
\]

where \( \alpha \triangleq \min \{ \frac{\lambda_{u^n}^{A} \lambda_{v^n}^{B}}{(1 + \eta)^2} : u^n \in T_\delta^{(n)}(A), v^n \in T_\delta^{(n)}(B) \} \). Hence, \( \alpha \geq 2^{-n(H(\lambda^A) + H(\lambda^B) + 2\delta)} \). As a result, the probability in \((18)\) can be made arbitrary small, if

\[
\frac{1}{n} \log_2 N \geq H(\lambda^A) + H(\lambda^B) + 2\delta - R_1 - R_2 \geq S(X|R) + S(Y|R) + 2\delta,
\]

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where the last inequality follows by Lemma 2. As a result, for the first term in (17) we have

\[ \sum_{u^n,v^n} \| \lambda^{AB}_{u^n,v^n} \rho^{\otimes n}_{u^n,v^n} - (\sqrt{\rho^{\otimes n}} (A_{u^n} \otimes B_{v^n}) \sqrt{\rho^{\otimes n}}) \|_1 \]

\[ \leq \sum_{u^n,v^n} \| \lambda^{AB}_{u^n,v^n} \rho^{\otimes n}_{u^n,v^n} - \frac{\lambda^{AB}_{u^n,v^n} (1 + \eta)^2 \sqrt{\rho^{\otimes n}}}{} \left( \sqrt{\rho A \otimes \rho B}^{-1} (A_{u^n} \otimes B_{v^n}) \sqrt{\rho A \otimes \rho B}^{-1} \right) \sqrt{\rho^{\otimes n}} \|_1 \]

\[ + \sum_{u^n,v^n} \varepsilon \frac{\lambda^{AB}_{u^n,v^n}}{(1 + \eta)^2} \| \sqrt{\rho^{\otimes n}} (A_{u^n} \otimes B_{v^n}) \sqrt{\rho A \otimes \rho B}^{-1} \sqrt{\rho^{\otimes n}} \|_1 \]  

The summation in (21) is bounded by

\[ \sum_{u^n,v^n} \varepsilon \frac{\lambda^{XY}_{u^n,v^n}}{(1 + \eta)^2} \leq \varepsilon. \]

Let \( E = \sqrt{\rho^{\otimes n}} \sqrt{\rho A \otimes \rho B}^{-1} \), and \( \Pi_{u,v,\rho} = (\tilde{\Pi}^A \Pi_{\rho,\lambda} \Pi^A_{u^n}) \otimes (\tilde{\Pi}^B \Pi_{\rho,\lambda} \Pi^B_{v^n}) \). Then, with this notation each term in the summation in (20) can be written as

\[ \| \lambda^{AB}_{u^n,v^n} \rho^{\otimes n}_{u^n,v^n} - \frac{\lambda^{AB}_{u^n,v^n}}{(1 + \eta)^2} E \Pi_{u,v,\rho} (\sqrt{\rho^Y_{u^n,v^n}} \Pi^\dagger_{u,v,\rho}) E^\dagger \|_1 \]

\[ = \| \lambda^{AB}_{u^n,v^n} \rho^{\otimes n}_{u^n,v^n} - \frac{\lambda^{AB}_{u^n,v^n}}{(1 + \eta)^2} E \Pi_{u,v,\rho} E^{-1} \rho^{\otimes n}_{u^n,v^n} E^{-1} \Pi^\dagger_{u,v,\rho} E^\dagger \|_1 \]

\[ \leq (1 - \frac{1}{(1 + \eta)^2}) \lambda^{AB}_{u^n,v^n} + \frac{\lambda^{AB}_{u^n,v^n}}{(1 + \eta)^2} \| \rho^{\otimes n}_{u^n,v^n} - E \Pi_{u,v,\rho} E^{-1} \rho^{\otimes n}_{u^n,v^n} E^{-1} \Pi^\dagger_{u,v,\rho} E^\dagger \|_1, \]

where the first equality follows by as \( \rho^X_{u^n} \otimes \rho^Y_{v^n} = E^{-1} \rho^{\otimes n}_{u^n,v^n} E^{-1} \). The last inequality follows by the triangle inequality. For the second term in (25), by multiple uses of the triangle inequality we have:

\[ \| \rho^{\otimes n}_{u^n,v^n} - E \Pi_{u,v,\rho} E^{-1} (I \pm \Pi_{u,v,\rho}) \|_1 \]

\[ = \| \rho^{\otimes n}_{u^n,v^n} - E \Pi_{u,v,\rho} E^{-1} \Pi_{u,v,\rho} \|_1 \]

\[ + \| E \Pi_{u,v,\rho} E^{-1} (I \pm \Pi_{u,v,\rho}) \|_1 \]

\[ \leq \| \rho^{\otimes n}_{u^n,v^n} - E \Pi_{u,v,\rho} E^{-1} \Pi_{u,v,\rho} \|_1 \]

\[ + \| E \Pi_{u,v,\rho} E^{-1} (I \pm \Pi_{u,v,\rho}) \|_1 \]

\[ \leq \| \rho^{\otimes n}_{u^n,v^n} - E \Pi_{u,v,\rho} E^{-1} \Pi_{u,v,\rho} \|_1 \]

Note that \( E \Pi_{u,v,\rho} E^{-1} \Pi_{u,v,\rho} = \tilde{\Pi}_{u,v,\rho} \), where \( \tilde{\Pi}_{u,v,\rho} \) is the projection onto the support of \( E \Pi_{u,v,\rho} \).

Similarly, \( E(I - \Pi_{u,v,\rho}) E^{-1} (I - \Pi_{u,v,\rho}) = \tilde{\Pi}_{u,v,\rho} \), where \( \tilde{\Pi}_{u,v,\rho} \) is the projection onto the support of \( E(I - \Pi_{u,v,\rho}) \). We need the following lemma to proceed.

**Lemma 3.** For any operator \( A \) and \( B \) acting on a Hilbert space \( \mathcal{H} \) the following inequalities hold.

\[ \| BA \|_1 \leq \| B \|_\infty \| A \|_1, \quad \text{and} \quad \| AB \|_1 \leq \| B \|_\infty \| A \|_1. \]
Proof. According to Theorem 1.3 in [10], A has a polar decomposition of the form \( A = U|A| \), where \( U \) is a unitary operator and \( |A| = \sqrt{A^*A} \). As \( |A| \) is a positive operator, it has an eigenvalue decomposition of the form \( |A| = \sum_{i=1}^{d} \lambda_i |\phi_i\rangle \langle \phi_i| \), where \( \lambda_i \geq 0 \). From triangle-inequality we have

\[
\|BA\|_1 = \|BU|A|\|_1 \leq \sum_i \lambda_i \|BU|\phi_i\rangle \|_1 = \sum_i \lambda_i \tr \langle \phi_i|U^\dagger BU|\phi_i\rangle \leq \sum_i \lambda_i \|BU|\phi_i\rangle \| \leq \sum_i \|B\|_\infty \lambda_i = \|B\|_\infty \|A\|_1,
\]

where the last inequality holds by the definition of \( \| \cdot \|_\infty \) and the fact that \( U \) is unitary. For the second statement of the lemma we have

\[
\|AB\| \leq \sum_i \lambda_i \|U|\phi_i\rangle \langle \phi_i|B\| \leq \sum_i \lambda_i \tr \sqrt{\langle \phi_i|U^\dagger BU|\phi_i\rangle} \leq \|B\|_\infty \sum_i \lambda_i \leq \|B\|_\infty \|A\|_1.
\]

Let \( |\psi_i\rangle \triangleq \frac{B|\phi_i\rangle}{\|B|\phi_i\rangle\|} \). Then

\[
\tr \sqrt{B|\phi_i\rangle \langle \phi_i|B} = \|B|\phi_i\rangle\| \tr \{ \sqrt{|\psi_i\rangle \langle \psi_i|} \} = \|B|\phi_i\rangle\| \leq \|B\|_\infty
\]

Therefore, we obtain \( \|AB\|_1 \leq \sum_i \|B\|_\infty \lambda_i = \|B\|_\infty \|A\|_1 \) \( \square \)

As a result, the term in (27) equals to

\[
\|\hat{\rho}_{u,n}^{\otimes n} - \hat{\Pi}_{u,v,\rho} \hat{\rho}_{u,v,\rho}^{\otimes n} \hat{\Pi}_{u,v,\rho}^\dagger \|_1.
\]

The term in (28) equals

\[
\|\hat{\Pi}_{u,v,\rho} \hat{\rho}_{u,v,\rho}^{\otimes n} (I - \Pi_{u,v,\rho}^\dagger) \Pi_{u,v,\rho}^\dagger \|_1 = \|\hat{\Pi}_{u,v,\rho} \hat{\rho}_{u,v,\rho}^{\otimes n} (I - \Pi_{u,v,\rho}^\dagger) \Pi_{u,v,\rho}^\dagger (\Pi_{u,v,\rho}^\dagger \Pi_{u,v,\rho} + I) \Pi_{u,v,\rho}^\dagger \|_1 \\
\leq \|\hat{\Pi}_{u,v,\rho} \hat{\rho}_{u,v,\rho}^{\otimes n} (I - \Pi_{u,v,\rho}^\dagger) \Pi_{u,v,\rho}^\dagger \|_1 + \|\hat{\Pi}_{u,v,\rho} \hat{\rho}_{u,v,\rho}^{\otimes n} (I - \Pi_{u,v,\rho}^\dagger) \Pi_{u,v,\rho}^\dagger (I - \Pi_{u,v,\rho}^\dagger) \Pi_{u,v,\rho}^\dagger \|_1 \\
= \|\hat{\Pi}_{u,v,\rho} \hat{\rho}_{u,v,\rho}^{\otimes n} (I - \Pi_{u,v,\rho}^\dagger) \Pi_{u,v,\rho}^\dagger \|_1 + \|\hat{\Pi}_{u,v,\rho} \hat{\rho}_{u,v,\rho}^{\otimes n} (I - \Pi_{u,v,\rho}^\dagger) \Pi_{u,v,\rho}^\dagger \|_1 \\
\leq 2 \|\hat{\rho}_{u,v,\rho}^{\otimes n} (I - \Pi_{u,v,\rho}^\dagger) \|_1.
\]
where the last inequality follows by multiple uses of Lemma 3. Similarly, the term in (29) does not exceed the following:

$$2\| (I - \Pi_{u,\nu,\rho}) \hat{\rho}_{u^n,v^n} \|_1.$$ 

Lastly, the term in (30) satisfies

$$ \| E\Pi_{u,\nu,\rho} E^{-1} (I - \Pi_{u,\nu,\rho}) \hat{\rho}_{u^n,v^n} \|_1 = \| E\Pi_{u,\nu,\rho} E^{-1} (I - \Pi_{u,\nu,\rho}) \hat{\rho}_{u^n,v^n} \|_1 \leq \| E\Pi_{u,\nu,\rho} E^{-1} (I - \Pi_{u,\nu,\rho}) \hat{\rho}_{u^n,v^n} \|_1,$$

Note that $EE^{-1} = E$, where $E$ is a projection operator. Therefore, the above terms are simplified to the following:

$$\| \hat{\Pi}_{u,\nu,\rho} (I - \Pi_{u,\nu,\rho}) \hat{\rho}_{u^n,v^n} \|_1 + \| \hat{\Pi}_{u,\nu,\rho} (I - \Pi_{u,\nu,\rho}) \hat{\rho}_{u^n,v^n} \|_1 \leq 2\eta + o(\epsilon, \varepsilon),$$

where the last inequality is due to a more general version of the gentle measurement lemma in (11). As a result, from (17), the condition for faithful simulation given in (16) is upper-bounded by

$$2\eta + o(\epsilon, \varepsilon) + \frac{1}{N} \sum_{u^n, v^n} \sum_{\mu} \gamma(u^n, v^n, \mu) \lambda^{AB}_{u^n,v^n} K_{u^n,v^n,\mu} \Phi_{e(\mu)} u^n, v^n + \Phi \|_1.$$
The above summation can be divided into two summations depending whether \((u^n, v^n) \in T^{(n)}(AB)\). Noting that \(\|\Phi_u - \Phi_{e(\mu)}(u^n, v^n)\|_1 \leq 2\), the above term does not exceed the following
\[
\frac{1}{N} \sum_{(u^n, v^n) \in T^n} \sum_{\mu} \gamma_{u^n}^{(\mu)} \zeta_{v^n}^{(\mu)} \frac{\lambda_{u^n}^{AB}}{\lambda_{u^n}^{A}} \frac{\lambda_{v^n}^{AB}}{\lambda_{v^n}^{B}} \Phi_{u^n, v^n} - \Phi_{\{\mu\}(u^n, v^n)} \|_1 + o(\eta, \epsilon, \varepsilon)
\]
\[
\leq \frac{1}{N} \sum_{(u^n, v^n) \in T^n} \sum_{\mu} \gamma_{u^n}^{(\mu)} \zeta_{v^n}^{(\mu)} \frac{\lambda_{u^n}^{AB}}{\lambda_{u^n}^{A}} \frac{\lambda_{v^n}^{AB}}{\lambda_{v^n}^{B}} \|\Phi_{u^n, v^n} - \Phi_{e(\mu)(u^n, v^n)}\|_1 + o(\eta, \epsilon, \varepsilon, \delta)
\]
The last inequality holds because from Chernoff bound \(\sum_{\mu} \gamma_{u^n}^{(\mu)} \zeta_{v^n}^{(\mu)} \leq (1 - \epsilon)\lambda_{u^n}^{A} \lambda_{v^n}^{B}\) with probability sufficiently close to 1. In addition, from the properties of typical sets, the second summation can be upper bounded by
\[
2(1 - \epsilon) \sum_{(u^n, v^n) \in T^n} \lambda_{u^n}^{AB} \lambda_{v^n}^{AB} \leq 2(1 - \epsilon) o(\delta).
\]
From the definition of \(\gamma_{u^n}^{(\mu)}\) and \(\zeta_{v^n}^{(\mu)}\), the terms in (33) can be written as
\[
\frac{(1 - \varepsilon)(1 - \varepsilon')}{(1 + \eta)^2} \frac{2^{-\tilde{R}}}{N} \sum_{(u^n, v^n) \in T^n} \sum_{l, k} 1\{U^{(\mu)}(l) = u^n, V^{(\mu)}(k) = v^n\} \frac{\lambda_{u^n}^{AB}}{\lambda_{u^n}^{A}} \frac{\lambda_{v^n}^{AB}}{\lambda_{v^n}^{B}} \|\Phi_{u^n, v^n} - \Phi_{e(\mu)(u^n, v^n)}\|_1
\]
The norm-1 term above can be replaced by
\[
\|\Phi_{U^{(\mu)}(l), V^{(\mu)}(k)} - \Phi_{e(\mu)(U^{(\mu)}(l), V^{(\mu)}(k))}\|_1
\]
Suppose \((l, k)\) belongs to the bin \(B_1(i) \times B_2(j)\). Then, by the definition, \(e^{(\mu)}(U^{(\mu)}(l), V^{(\mu)}(k)) = F^{(\mu)}(i, j)\). Such a term equals to \(U^{(\mu)}(l), V^{(\mu)}(k)\) if \((l, k)\) is the only element of \(D^{(\mu)}_{i,j}\). Hence, for \((l, k) \in B_1(i) \times B_2(j)\), the above norm-1 quantity can be bounded by
\[
2 \times 1\{3(\tilde{l}, \tilde{k}) : (U^{(\mu)}(\tilde{l}), V^{(\mu)}(\tilde{k})) \in T^{(n)}(UV), (\tilde{l}, \tilde{k}) \in B_1(i) \times B_2(j), (\tilde{l}, \tilde{k}) \neq (l, k)\}
\]
Denote such indicator function by \(1_{i,j}^{(\mu)}(k, l)\). The summation in (34) is bounded by
\[
\frac{(1 - \varepsilon)(1 - \varepsilon')}{(1 + \eta)^2} \frac{2^{-\tilde{R}}}{N} \sum_{i, j} \sum_{l \in B_1(i)} \sum_{k \in B_2(j)} \sum_{(u^n, v^n) \in T^n} \frac{1}{N} \sum_{\mu} \frac{\lambda_{u^n}^{AB}}{\lambda_{u^n}^{A}} \frac{\lambda_{v^n}^{AB}}{\lambda_{v^n}^{B}} 2^1_{i,j}^{(\mu)}(k, l) 1\{U^{(\mu)}(l) = u^n, V^{(\mu)}(k) = v^n\}.
\]
Note that \(1^{(\mu)}_{i,j}(k, l)\) can be viewed as a random variable which is independent of \(1\{U^{(\mu)}(l) = u^n, V^{(\mu)}(k) = v^n\}\). Next, we use the Markov inequality to show that the above quantity is less than \(\epsilon\) with probability sufficiently close to 1. For that, it is sufficient to show that the expectation of the above quantity can be made arbitrary small by taking \(n\) large enough. Taking the expectation gives
\[
\frac{2^{-\tilde{R}}}{(1 + \eta)^2} \sum_{i, j} \sum_{l \in B_1(i)} \sum_{k \in B_2(j)} \sum_{(u^n, v^n) \in T^n} \frac{1}{N} \sum_{\mu} \lambda_{u^n}^{AB} 2E\left[1^{(\mu)}_{i,j}(k, l)\right] \leq 2^{-\tilde{R}} \frac{1}{(1 + \eta)^2} \sum_{i, j} \sum_{l \in B_1(i)} \sum_{k \in B_2(j)} \sum_{\mu} \frac{1}{N} \sum_{(u^n, v^n) \in T^n} \lambda_{u^n}^{AB} 2E\left[1^{(\mu)}_{i,j}(k, l)\right]
\]

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where \((l, k)\) belongs to \(B_1(1) \times B_2(1)\). The first inequality is due to the fact that \(\mathbb{P}(\mathcal{T}_\delta^{(n)}) \leq 1 - o(\delta)\). The last equality holds as \(U^{n,(\mu)}(\tilde{l})\) and \(V^{n,(\mu)}(\tilde{k})\) are IID for different values of \(\tilde{l}, \tilde{k}\) and \(\mu\). Therefore, 
\[
\mathbb{E}[\mathbb{1}^{(\mu)}(k, l)] = \mathbb{E}[\mathbb{1}^{(1)}(l, k)]
\]
which is equal to
\[
\mathbb{P}\left\{ \exists (\tilde{l}, \tilde{k}) : (U^{n,(1)}(\tilde{l}), V^{n,(1)}(\tilde{k})) \in \mathcal{T}_\delta^n(UV), U^{n,(1)}(\tilde{l}) \in B_1(1), U^{n,(1)}(\tilde{k}) \in B_2(1), (\tilde{l}, \tilde{k}) \neq (l, k) \right\}
\]
To bound the above probability we apply the mutual packing lemma (Lemma 12.2 in [6]). Note that the size of the bin \(B_i(1)\) equals to \(2^{nr_i}\), where \(r_i = \tilde{R}_i - R_i, i = 1, 2\). With this notation, the above probability approaches zero as \(n \to \infty\) if \(r_1 + r_2 \leq I(U; V)\).

To sum-up, we showed that the trace distance given in (16) is sufficiently small with probability sufficiently close to 1 if the following bounds hold:
\[
\tilde{R}_1 > I(X; R)_\sigma
\]
\[
\tilde{R}_2 > I(Y; R)_\sigma
\]
\[
\frac{1}{n} \log_2 N > S(X|R) + S(Y|R)
\]
\[
(\tilde{R}_1 - R_1) + (\tilde{R}_2 - R_2) < I(U; V)_\sigma
\]
These bounds can be simplified to the same bound as in the statement of the theorem. Hence, the proof is complete.

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