Tree topology analysis of the arterial system model

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Abstract. The optimal network topology and spatial arrangement of the vessels providing a large surface area are necessary for the efficient functioning of the vasculature. In the present study we have developed the algorithm for construction of the arterial system model with physiologically significant geometric properties. The analysis of the effect of the bifurcation exponent on the topology of the arterial network was made using the proposed approach.

1. Introduction
The transport of oxygen and the essential metabolites throughout the organs and the supply of tissues is the main function of the circulatory system. Convection in blood flow and diffusive exchange between the blood and surrounding tissues provide effective organ perfusion. For correct functioning of the system rigid constraints should be imposed on the structure of the vasculature. In particular, the delivery of oxygen and other metabolites by diffusion to brain cells depends significantly on the spatial arrangement of the vessels, while the network topology and the diameters and lengths of the vessels strongly influence the blood flow distribution.

2. Methods
In the present study we have proposed the algorithm for construction of the arterial system of the rat brain, which describes both the main arteries and smaller vessels. In this approach the vasculature is divided into the two parts with different modeling methods. The first one includes the main arteries forming the circle of Willis, which were realized as an ordered set of cylinders. The radii and the coordinates of the centers were set, and the radius of the cylinder was corresponded to the vessel’s radius.

The second part is a stochastic structure that describes smaller vessels branching from the main arteries, which were modeled as a binary tree according to the physiological laws (Figure 1) [1]. The relation of the decreasing vessel radii from proximal to distal branches is governed by a bifurcation law [2]. It relates the radius of the proximal parent branch to the radius of the left and right daughter branches:

$$R_0^\gamma = R_1^\gamma + R_2^\gamma$$

where $R_0$, $R_1$, $R_2$ – the radii of parent and offspring vessels, $\gamma$ – bifurcation exponent, which value can vary from two to three with mean values given by 2.5, 2.7 and 2.9 [3–5].
Moreover, the branching geometry is determined by the bifurcation angles ($\theta_1$ and $\theta_2$), which interrelations are based on the hydrodynamics laws and the conditions for the work and energy dissipation minimization proposed by Murray and Rosen [6, 7]:

$$\theta_{1,2} = \arccos \left( \frac{R_0^4 + R_{1,2}^4 - R_{2,1}^4}{2 \cdot R_0^2 \cdot R_{1,2}^2} \right)$$  \hspace{1cm} (2)

Construction of the vascular system occurs in the space defined by geometric dimensions of the brain phantom, which was approximated by a system of ellipsoids with given parameters:

$$\Omega (\vec{x}) = \bigcup_i \left( \sum_j \left( \frac{x_j - x_{i,j}^{\text{ellipse}}}{R_{i,j}^{\text{ellipse}}} \right)^2 \leq 1 \right)$$ \hspace{1cm} (3)

Note: $x_{i,j}^{\text{ellipse}}$ and $R_{i,j}^{\text{ellipse}}$ - the coordinates of the center and the values of the semi-axes of the $i^{th}$ ellipsoid respectively. The values of these geometric parameters were obtained by digitizing images of the rat brain. The generation of the arterial tree occurs up to the minimum specified radius of the branch, which corresponds to 8 $\mu$m in this model.

3. Results and discussion

The rat brain arterial tree models represented in Figure 2 are generated using the proposed approach with the two boundary values of the bifurcation exponent ($\gamma = 2.3$ and $\gamma = 3.1$). Constructed cerebral vasculature represents the anatomically correct spatial arrangement of the main arteries. It may affect the ability to maintain cerebral blood perfusion, because the circle of Willis can potentially compensate for acute changes in blood flow in the case of occlusion or stenosis of one or more large feeding arteries [8]. It is easy to see that the arterial network with $\gamma = 3.1$ has essentially tight density of small vessels which seems to be more native and physiological. At the level of an isolated branch for the low value of bifurcation exponent, one can also notice that the degree of the volume coverage is insufficient due to the pronounced tree growth direction. Moreover, not only the vessels distribution, but the topology of the whole system determines its functioning efficiency. Various criteria based on the superposition of the structural parameters (radius and length of the vessel, bifurcation angles, etc.) can be used to estimate the topology of the network. However, the experimental evaluation of these values is possible for the large vessels only. The main characteristic of tree branching is the distribution of bifurcation levels, which affects the total volume of the system and the tissue supply of metabolites. The density distributions of bifurcation level for different values of the bifurcation exponent have been shown in Figure 3. These distributions follow the gamma type of
distribution with the small deviations in the region corresponding to the large vessels. Effective tissue perfusion appears under the optimal modal values of the branch order which are achieved when the bifurcation exponent is higher than 2.9. The increase of the branching exponent (γ) yields the maximum value of bifurcation level to be risen from 32 up to 44. It considerably exceeds the limit of the experimental estimation for such a system with advanced imaging technique. The limiting order of the bifurcation level is only 12 in that case [9].

To estimate the symmetry of the branching the radius ratio at bifurcation point was calculated (Figure 4).

\[ S = \frac{\min(R_1, R_2)}{\max(R_1, R_2)} \]  

(4)

Based on the analysis we can conclude that a lower bound of symmetry exists and it is strongly dependent on the mother radius. Moreover, the symmetry significantly increases for the vessels with the small radii. Despite the fact that the lower limit of symmetry is not changed with an increase of the bifurcation exponent, the analysis of the vasculature has shown that the density of vessels with a symmetry value above 0.5 increases significantly. Non-optimal branching geometry has been reported to be associated with ischemic heart disease [10], increased coronary artery calcification [11] and peripheral vascular disease [12] due to deviation from optimal bifurcation symmetry. Thus, the symmetry and the distribution of the bifurcation levels significantly affect the topology of the vasculature as a whole, but the structural properties of single vessels at the different branching levels determine the perfusion of local regions.
Figure 3. Probability density distribution of bifurcation level for the different values of bifurcation exponent $\gamma$. The red lines show the corresponding best-fit gamma distribution.

Figure 4. The symmetry of vascular nodes as a function of the radius of the mother segment for the different values of bifurcation exponent $\gamma$. The scatter plot represents the symmetry distribution. The red lines indicate the minimal symmetry value.

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