Isoperimetric estimates for solutions to the p-Laplacian with variable Robin boundary conditions

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Abstract
In this paper we study the $p$-Poisson equation with Robin boundary conditions, where the Robin parameter is a function. By means of some weighted isoperimetric inequalities, we provide various sharp bounds for the solutions to the problems under consideration. We also derive a Faber-Krahn-type inequality.

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1 Introduction
Symmetrization methods have been revealed to be a very effective and flexible tool in the study of partial differential equations (in the sequel just PDEs). The related bibliography is very wide and one of the cornerstone of this theory is the Talenti's Theorem (see [Tal76]), which makes use of the Schwarz symmetrization. The technique he introduced has been refined through the years and it has been adapted to treat a large class of linear and nonlinear elliptic and parabolic PDEs (see, e.g., the survey papers [Tal16], [Tro00] and the monographs [Kaw85], [Kes06] and [Bae19]). However, in most of the subsequent works, the authors, in order to follow Talenti’s original idea, consider problems whose solutions have level sets that do not intersect the boundary of the domain where the problem is defined.

Recently it has been discovered a method to adapt symmetrization techniques to a class of problems that does not exhibit this feature. More precisely, in [ANT], the authors were able to obtain a Talenti-type comparison
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result for the Laplacian with Robin boundary conditions. For further developments in this direction see, e.g., [AGM22; CGNT], where the \( p \)-Laplace and Hermite operator are studied, respectively, while in [ACNT21] and [ACNT] the authors investigate linear problems where the Robin parameter is allowed to be a function.

Here we generalize the above results by considering the following problem

\[
\begin{aligned}
- \text{div} \left( |\nabla u|^{p-2} \nabla u \right) &= f(x)|x|^\ell \quad \text{in } \Omega \\
|\nabla u|^{p-2} \frac{\partial u}{\partial \nu} + \beta(x)|u|^{p-2}u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where, here and throughout, \( n \geq 2 \), \( \nu \) denotes the outer unit normal to \( \partial \Omega \) and \( \Omega \) is a bounded and Lipschitz domain of \( \mathbb{R}^n \) such that \( 0 \notin \partial \Omega \).

Furthermore we will assume that

\[
\begin{aligned}
p &\geq n, \quad (H_1) \\
- n < \ell < 0, \quad (H_2) \\
m &= \inf_{\partial \Omega} \beta(x) > 0 \quad \text{and} \quad M = \sup_{\partial \Omega} \beta(x) < +\infty, \quad (H_3)
\end{aligned}
\]

and, finally,

\[
0 \leq f(x) \in L^{p'}(\Omega, |x|^\ell \, dx), \quad (H_4)
\]

where \( p' = \frac{p}{p-1} \).

Hypotheses \((H_1)\) and \((H_2)\) ensure the validity of certain isoperimetric inequalities, where two different weights (which are suitable powers of the distance from the origin) appear in the perimeter and in the area element, respectively (see, e.g., [How15; CH15; DHHT12; BBMP99; Csa15; ABCMP17] and the references therein). Indeed we need to use, in place of the more common Schwarz symmetrization, some weighted rearrangement, based on these “double density” isoperimetric inequalities.

A weak solution to problem \((1.1)\) is a function \( u \in W^{1,p}(\Omega) \) such that

\[
\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx + \int_{\partial \Omega} \beta(x)|u|^{p-2} u \varphi d\mathcal{H}^{n-1}(x) = \int_\Omega f(x)|x|^\ell \varphi \, dx \quad (1.2)
\]

for any \( \varphi \in W^{1,p}(\Omega) \).

Now we need to construct the so-called symmetrized problem. That is a radial problem, of the same type of \((1.1)\), whose solution will estimate the one to problem \((1.1)\).
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To this aim we introduce the set $\Omega^\sharp$ which is the ball centered at the origin (in the sequel just centered ball) of radius $r^\sharp$, with $r^\sharp$ such that

$$|\Omega^\sharp|_\ell = \int_{\Omega^\sharp} |x|^{\ell} \, dx = \int_{\Omega} |x|^{\ell} \, dx = |\Omega|_\ell,$$

and the function $f^\sharp(x)$ defined by the following relation

$$|\{ x \in \Omega : |f(x)| > t \}|_\ell = |\{ x \in \Omega : f^\sharp(x) > t \}|_\ell \text{ for a.e. } t \geq 0.$$

Our symmetrized problem is the following

$$\begin{cases}
-\text{div}\left(|\nabla v|^{p-2} \nabla v\right) = f^\sharp(x)|x|^{\ell} & \text{in } \Omega^\sharp \\
|\nabla v|^{p-2} \frac{\partial v}{\partial \nu} + \tilde{\beta}(r^\sharp)^{p^*} |v|^{p-2}v = 0 & \text{on } \partial \Omega^\sharp
\end{cases} \tag{1.3}$$

where

$$\tilde{\beta} = \inf_{\partial \Omega} \beta(x)|x|^{-\frac{\ell}{p^*}} > 0, \tag{1.4}$$

since $0 \notin \partial \Omega$ and $H_3$ holds. From the (1.4), it immediately follows that

$$\beta(x) \geq \tilde{\beta}|x|^{\frac{\ell}{p^*}}. \tag{1.5}$$

Our main results are contained in the following theorems.

\textbf{Theorem 1.1.} Assume that assumptions $(H_1)$-$(H_4)$ are in force and let $u$ and $v$ be the solutions to problems (1.1) and (1.3), respectively. Then we have

$$\|u\|_{L^1(\Omega, |x|^\ell \, dx)} = \int_{\Omega} |u(x)||x|^{\ell} \, dx \leq \int_{\Omega^\sharp} |v(x)||x|^{\ell} \, dx = \|v\|_{L^1(\Omega^\sharp, |x|^\ell \, dx)} \tag{1.6}$$

and

$$\|u\|_{L^p(\Omega, |x|^\ell \, dx)}^p = \int_{\Omega} |u(x)|^p |x|^{\ell} \, dx \leq \int_{\Omega^\sharp} |v(x)|^p |x|^{\ell} \, dx = \|v\|_{L^p(\Omega^\sharp, |x|^\ell \, dx)}^p. \tag{1.7}$$

If the source term $f(x)$ is constant we get a stronger result, namely a pointwise comparison between $u^\sharp$ and $v$.

\textbf{Theorem 1.2.} Assume that hypotheses $(H_1)$-$(H_4)$ are in force and suppose that $f(x) \equiv 1$ in $\Omega$. If either $p = n = 2$ or if $p > 2$, $n \geq 2$ and

$$\ell \leq -n + \frac{p-n}{p-2} \tag{1.8}$$

then

$$u^\sharp(x) \leq v(x) \quad x \in \Omega^\sharp, \tag{1.9}$$

where $u$ and $v$ are the solutions to (1.1) and (1.3) respectively.
Remark 1.1. It is not straightforward to compare our estimates with the ones, see [ANT; AGM22], obtained by means of the classical Schwarz symmetrization (which corresponds to choosing $\ell = 0$). Nevertheless, in the forthcoming paper [ACGM], the authors will address the asymptotic as $p \to +\infty$ of some inequalities proved in the present note. From such an analysis, it will be clear that, at least for $p$ large enough and for a special class of domains, the weighted ($\ell < 0$) and the unweighted ($\ell = 0$) estimates are comparable and the first ones are sharper.

We finally note that there is a vast literature on Talenti’s type estimates obtained via weighted rearrangement. Typically, the weight fulfills an isoperimetric inequality and it appears in the ellipticity of the differential operator that one wants to study. In our case, see also [ACNT], the differential operator is the classical $p$-Laplacian and the weight is somehow “hidden” in the boundary condition. This feature could represent one of the novelties of our approach.

Remark 1.2. A straightforward computation shows that the function $v$, appearing in (1.9), has the following explicit expression

$$v(x) = \frac{p - 1}{(\ell + p)(n + \ell)} \left[ (R^\ell)^{\frac{\ell + p}{p - 1}} - \left| x \right|^{\frac{\ell + p}{p - 1}} \right] + \left( \frac{(R^\ell)^{\frac{\ell + 1}{p - 1}}}{\beta (n + \ell)} \right)^{\frac{1}{p - 1}}.$$

The first eigenvalue of the $p$-Laplace operator with the boundary conditions as in (1.1) is the minimum of the following Rayleigh quotient

$$\lambda_{1,\beta(x)}(\Omega) = \min_{\substack{\psi \in W^{1,p}(\Omega) \\ \psi \neq 0}} \frac{\int_{\Omega} |\nabla \psi|^p \, dx + \int_{\partial \Omega} \beta(x)|\psi|^p \, dH^{n-1}(x)}{\int_{\Omega} |\psi|^p |x|^{\ell} \, dx}.$$

Finally, Theorem 1.1 allows us to prove the following Faber-Krahn inequality (see also [ANT; Bos88; BD10; BG10; BG15; BGT19; Csa15; FK15; FNT13] and the references therein).

Theorem 1.3. Assume hypotheses $(H_1)$–$(H_5)$ and let $\bar{\beta}$ be the constant defined in (1.4), then we have

$$\lambda_{1,\beta(x)}(\Omega) \geq \lambda_{1,\bar{\beta}}(\Omega^\sharp).$$

The paper is organized as follows. In the next section, we recall some basic notion from the theory of rearrangements and the weighted isoperimetric
inequalities needed in the sequel. Moreover, we list some properties of the solutions to problems (1.1) and (1.3). Section 3 is devoted to the proofs of Theorem 1.1 and Theorem 1.2. Finally, in Section 4, we prove the Faber-Krahn inequality.

2 Preliminary result

Definition 2.1. Let \( \Omega \) be a Lebesgue measurable subset of \( \mathbb{R}^n \) and let \( \ell \in (-n, 0) \). We define the \( \ell \)-weighted measure of \( \Omega \) as

\[
|\Omega|_\ell := \int_\Omega |x|^{\ell} \, dx,
\]

and the \( k \)-weighted perimeter of \( \Omega \) as

\[
P_k(\Omega) = \begin{cases} 
\int_{\partial\Omega} |x|^k \, d\mathcal{H}^{n-1} & \text{if } \Omega \text{ is } (n-1)\text{-rectifiable} \\
+\infty & \text{otherwise}.
\end{cases}
\]

For this family of measures, an isoperimetric inequality holds true if \( k, \ell \) and \( n \) are suitably related. Here we need the following result (see either [CH15] Theorem 1.3 or [ABCMP17] Theorem 1.1 case ii).

Theorem 2.1. If assumptions \((H_1)\) and \((H_2)\) hold true, then

\[
P_{\frac{\ell}{p}}(\Omega) \geq P_{\frac{\ell}{p}}(\Omega^\circ),
\]

where \( \Omega^\circ \) is the centered ball with same \( \ell \)-measure as \( \Omega \).

Remark 2.1. Since \( |\Omega^\circ|_\ell \) and \( P_k(\Omega^\circ) \) can be explicitly computed, is it possible to rewrite (2.3) in the following equivalent way

\[
P_{\frac{\ell}{p}}(\Omega) \geq \gamma_{n,\ell,p} |\Omega|_\ell^{\frac{(\ell+\rho)(n-1)p}{p(n+\ell)}},
\]

where

\[
\gamma_{n,\ell,p} = (n\omega_0)^\frac{\ell+\rho}{p(n+\ell)}(\ell+n)^{\frac{(\ell+\rho)(n-1)p}{p(n+\ell)}}
\]

and \( \omega_0 \) is the Lebesgue measure of the unit ball of \( \mathbb{R}^n \).

Definition 2.2. Let \( u : \Omega \to \mathbb{R} \) be a measurable function, the weighted distribution function of \( u \) is the function \( \mu_\ell : [0, +\infty] \to [0, +\infty] \) defined by

\[
\mu_\ell(t) = |\{ x \in \Omega : |u(x)| > t \}|_\ell.
\]
In the following, we will omit the $\ell$ and just write $\mu$.

**Definition 2.3.** Let $u : \Omega \to \mathbb{R}$ be a measurable function, the *weighted decreasing rearrangement* of $u$, denoted by $u^*$, is the distribution function of $\mu$.

The *weighted rearrangement* of $u$ is the function $u^\sharp$ whose level sets are centered balls with the same $\ell$-measure as the level sets of $|u|$. More precisely,

$$u^\sharp(x) = u^*(|B_{|x|}| \ell) = u^*(\frac{n\omega_n}{\ell + n} |x|^{\ell + n}).$$

with $B_{|x|}$ the centered ball of radius $|x|$.

**Definition 2.4.** If $p \in [1, +\infty)$ we will denote by $L^p(\Omega, |x|^{\ell} dx)$ the space of all measurable functions such that

$$\|u\|_{L^p(\Omega, |x|^{\ell} dx)} := \left(\int_{\Omega} |u|^p |x|^{\ell} dx\right)^{\frac{1}{p}} < +\infty.$$

It is easily checked that $u, u^* \text{ e } u^\sharp$ are equi-distributed, hence

$$\|u\|_{L^p(\Omega, |x|^{\ell} dx)} = \|u^*\|_{L^p(0, |\Omega|_\ell)} = \|u^\sharp\|_{L^p(\Omega^\sharp, |x|^{\ell} dx)} \quad \forall 1 \leq p < +\infty. \quad (2.5)$$

We also recall the definition of weighted Lorentz spaces.

**Definition 2.5.** Let $0 < p < +\infty$ and $0 < q \leq +\infty$. The weighted Lorentz space $L^{p,q}(\Omega, |x|^{\ell} dx)$ is the space of those functions $f(x)$ such that the quantity

$$\|g\|_{L^{p,q}(\Omega, |x|^{\ell} dx)} := \begin{cases} p^{\frac{1}{q}} \left(\int_{\Omega} t^{\frac{p}{q}} \mu(t) dt\right)^{\frac{1}{q}} & 0 < q < \infty \cr \sup_{t > 0} (t^p \mu(t)) & q = \infty \end{cases} \quad (2.6)$$

is finite.

Let us observe that for $p = q$ the Lorentz space coincides with the $L^p(\Omega, |x|^{\ell} dx)$ space, as a consequence of the well-known *Cavalieri’s Principle*

$$\int_{\Omega} |g|^p |x|^{\ell} dx = p \int_{0}^{+\infty} t^{p-1} \mu(t) dt.$$  

By the definition of decreasing rearrangement, the following result holds true (see for instance [CR71], [Kaw85] and [Kes06]).
Proposition 2.2. Let \( u \in L^1(\Omega, |x|^\ell \, dx) \) be a non negative function and let \( E \subset \Omega \) be a measurable set, then it holds
\[
\int_E u(x)|x|^\ell \, dx \leq \int_0^{\|E\|} u^*(s) \, ds. \tag{2.7}
\]

Now we turn our attention on problems \((1.1)\) and \((1.3)\).

Proposition 2.3. If the assumptions \((H_2)\) and \((H_3)\) are fulfilled and if \( \Omega \) is a bounded and Lipschitz domain of \( \mathbb{R}^n \) then \( W^{1,p}(\Omega) \) is compactly embedded in \( L^p(\Omega, |x|^\ell \, dx) \).

Proposition 2.4. Problems \((1.1)\) and \((1.3)\) admit a unique solution.

The proofs of the above propositions will be postponed in the Appendix, since they require rather standard arguments.

Let us observe that \( u \geq 0 \) in \( \Omega \). Indeed, choosing \( \psi = u^- = \max\{ -u, 0 \} \) as test function in \((1.2)\), we have
\[
0 \geq \int_\Omega |
abla u^-|^p \, dx - \int_{\partial \Omega} \beta(x)(u^-)^p \, d\mathcal{H}^{n-1}(x) = \int_\Omega (u^-) f |x|^\ell \, dx,
\]
thus \( u^- = 0 \) a.e. in \( \Omega \).

Furthermore we have
\[
u_m = \inf_{\Omega} u \leq \min_{\Omega^\sharp} v = v_m. \tag{2.8}
\]
Indeed, using the fact that \( v(x) = v_m \ \forall \ x \in \partial \Omega^\sharp \), taking into account of \((2.5)\) and choosing \( \psi \equiv 1 \) in the weak formulation of \((1.1)\) and \((1.3)\), respectively, we get
\[
w_m = \frac{1}{P_{\frac{\ell}{p}}(\Omega^\sharp)} = \frac{1}{\beta} \int_{\partial \Omega^\sharp} |x|^\ell v(x)^{p-1} \, d\mathcal{H}^{n-1}(x) = \frac{1}{\beta} \int_{\Omega^\sharp} f|x|^\ell \, dx = \frac{1}{\beta} \int_{\partial \Omega^\sharp} \beta(x) u(x)^{p-1} \, d\mathcal{H}^{n-1}(x).
\]
In turn, by \((1.5)\) and using the isoperimetric inequality \((2.3)\), we get
\[
\frac{1}{\beta} \int_{\partial \Omega} \beta(x) u(x)^{p-1} \, d\mathcal{H}^{n-1}(x) \geq w_m^{p-1} \int_{\partial \Omega} |x|^\ell \, d\mathcal{H}^{n-1}(x)
\]
\[
= w_m^{p-1} P_{\frac{\ell}{p}}(\Omega) \geq w_m^{p-1} P_{\frac{\ell}{p}}(\Omega^\sharp).
\]

Gathering the inequalities above, we conclude that
\[
v_m^{p-1} P_{\frac{\ell}{p}}(\Omega^\sharp) \geq w_m^{p-1} P_{\frac{\ell}{p}}(\Omega^\sharp)
\]
and the claim follows.

Furthermore inequality \((2.8)\) implies that
\[
\mu(t) \leq \phi(t) = |\Omega|_\ell \ \forall t \leq v_m. \tag{2.9}
\]
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2.1 Some useful Lemmata

Let $u$ be the solution to (1.1). For $t \geq 0$, we define

$$U_t = \{ x \in \Omega : u(x) > t \} \quad \partial U_t^{int} = \partial U_t \cap \Omega, \quad \partial U_t^{ext} = \partial U_t \cap \partial \Omega,$$

and

$$\mu(t) = |U_t|_{\ell} \quad P_u(t) = P_{\frac{\mu(t)}{p}}(U_t).$$

Analogously, if $v$ is the solution to (1.3), we set

$$V_t = \{ x \in \Omega^f : v(x) > t \}, \quad \phi(t) = |V_t|_{\ell}, \quad P_v(t) = P_{\frac{\phi(t)}{p}}(V_t).$$

As it is easy to check $v(x) \equiv v^f(x)$, therefore for $0 \leq t \leq v_m$, $V_t = \Omega^f$, while, for $t > v_m$ $V_t$ is a centered ball contained in $\Omega^f$.

**Lemma 2.5 (Gronwall).** Let $\xi(t) : [\tau_0, +\infty[ \to \mathbb{R}$ be a continuous and differentiable function satisfying, for some non negative constant $C$, the following differential inequality

$$\tau \xi'(\tau) \leq (p-1)\xi(\tau) + C \quad \forall \tau \geq \tau_0 > 0. \quad (2.10)$$

Then we have

(i) $\xi(\tau) \leq \left( \xi(\tau_0) + \frac{C}{p-1} \right) \left( \frac{\tau}{\tau_0} \right)^{p-1} - \frac{C}{p-1} \quad \forall \tau \geq \tau_0$;

(ii) $\xi'(\tau) \leq \frac{(p-1)\xi(\tau_0) + C}{\tau_0} \left( \frac{\tau}{\tau_0} \right)^{p-2} \quad \forall \tau \geq \tau_0$.

**Proof.** Dividing both sides of the differential inequality (2.10) by $\tau^p$, we get

$$\frac{\xi'(\tau)}{\tau^{p-1}} - (p-1)\frac{\xi(\tau)}{\tau^p} = \left( \frac{\xi(\tau)}{\tau^{p-1}} \right)' \leq C \frac{\tau-1}{\tau^p}.$$

Integrating the last inequality on $(\tau_0, \tau)$ we obtain

$$\int_{\tau_0}^{\tau} \left( \frac{\xi(\tau)}{\tau^{p-1}} \right)' \, d\tau \leq \int_{\tau_0}^{\tau} \frac{C}{\tau^p} \, d\tau,$$

and therefore

$$\xi(\tau) \leq \left( \xi(\tau_0) + \frac{C}{p-1} \right) \left( \frac{\tau}{\tau_0} \right)^{p-1} - \frac{C}{p-1},$$

which gives (i).

The second statement of the Lemma, claim (ii), is a direct consequence of the first one. \qed
Lemma 2.6. Let $u$ and $v$ be the solutions to $(1.1)$ and $(1.3)$ respectively. Then, for almost every $t > 0$, it holds

$$
\gamma_{n,\ell,p} \mu(t) \left( \frac{(p-1)+p(n-1)}{p(n+\ell)} \right)^{\frac{p}{p-1}} \leq \left( \int_0^{\mu(t)} f^*(s) \, ds \right)^{\frac{1}{p-1}} \left( -\mu'(t) + \frac{1}{\beta^{p-1}} \int_{\partial U_t^{\text{ext}}} |x|^{\frac{p}{p-1}} \, dH^{n-1}(x) \right),
$$

(2.11)

where

$$
\gamma_{n,\ell,p} = \left( n^\omega \right)^{\frac{\ell}{p(n+\ell)}}. \frac{(p-1)+(n-1)}{p(n+\ell)}
$$

and

$$
\gamma_{n,\ell,p} \phi(t) \left( \frac{(p-1)+p(n-1)}{p(n+\ell)} \right)^{\frac{p}{p-1}} = \left( \int_0^{\phi(t)} f^*(s) \, ds \right)^{\frac{1}{p-1}} \left( -\phi'(t) + \frac{1}{\beta^{p-1}} \int_{\partial U_t^{\text{ext}}} |x|^{\frac{p}{p-1}} \, dH^{n-1}(x) \right).
$$

(2.12)

Proof. Let $t > 0$ and $h > 0$, we use the following test function in $(1.2)$

$$
\varphi(x) = \begin{cases} 
0 & \text{if } u < t \\
\frac{u-t}{h} & \text{if } t < u < t + h \\
h & \text{if } u > t + h.
\end{cases}
$$

We get

$$
\int_{U_t \setminus U_{t+h}} |\nabla u|^p \, dx + h \int_{\partial U_t^{\text{ext}} \setminus \partial U_{t+h}^{\text{ext}}} \beta(x) u^{p-1} \, dH^{n-1}(x) \\
+ \int_{\partial U_t^{\text{int}} \setminus \partial U_{t+h}^{\text{int}}} \beta(x) u^{p-1}(u-t) \, dH^{n-1}(x) \\
= \int_{U_t \setminus U_{t+h}} f|x|^{\ell} (u-t) \, dx + h \int_{U_{t+h}} f|x|^{\ell} \, dx.
$$

Dividing by $h$, using coarea formula and letting $h$ go to $0$, we have for a.e. $t > 0$

$$
\int_{\partial U_t} g(x) \, dH^{n-1}(x) = \int_{U_t} f|x|^{\ell} \, dx,
$$

with

$$
\begin{align*}
g(x) = \begin{cases} 
|\nabla u|^{p-1} & \text{if } x \in \partial U_t^{\text{int}}, \\
\beta(x) u^{p-1} & \text{if } x \in \partial U_t^{\text{ext}}.
\end{cases}
\end{align*}
$$

(2.13)
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Using isoperimetric inequality (2.4) and Hölder inequality, for a.e. \( t > 0 \), we obtain

\[
\gamma_{n,\ell,p} \mu(t) \left( \frac{p^*(p-1)+(p-1)}{p(n+1)} \right) 
\leq P_u(t) = \int_{\partial U_t} |x|^{\ell} \, d\mathcal{H}^{n-1}(x)
\leq \left( \int_{\partial U_t} g \, d\mathcal{H}^{n-1}(x) \right)^{\frac{1}{p}} \left( \int_{\partial U_t} \frac{|x|^\ell}{u} \, d\mathcal{H}^{n-1}(x) \right)^{1-\frac{1}{p}}.
\]

(2.14)

Since, for a.e. \( t > 0 \) it holds that

\[-\mu'(t) = \int_{\partial U_t^v} \frac{|x|^\ell}{|\nabla u|} \, d\mathcal{H}^{n-1},\]

by (1.5), (2.7), (2.13) and (2.14), we get

\[
\gamma_{n,\ell,p} \mu(t) \left( \frac{p^*(p-1)+(p-1)}{p(n+1)} \right) 
\leq \left( \int_{\partial U_t} g \, d\mathcal{H}^{n-1}(x) \right)^{\frac{1}{p}} \left( \int_{\partial U_t^v} \frac{|x|^\ell}{\nabla u} \, d\mathcal{H}^{n-1}(x) + \frac{1}{\beta^{p-1}} \int_{\partial U_t^v} \frac{|x|^\ell}{u} \, d\mathcal{H}^{n-1}(x) \right)^{\frac{1}{p}}
\leq \left( \int_0^{\mu(t)} f^*(s) \, ds \right)^{\frac{1}{p}} \left( -\mu'(t) + \frac{1}{\beta^{p-1}} \int_{\partial U_t^v} \frac{|x|^\ell}{u} \, d\mathcal{H}^{n-1}(x) \right)^{\frac{1}{p}}
\]

for a.e. \( t \in [0, \text{sup}_\Omega u) \).

Then (2.11) follows. Replacing \( u \) with \( v \), the solution to (1.3), all the inequalities are verified as equalities, so we obtain (2.12).

**Lemma 2.7.** Let \( u \) and \( v \) be the solutions to (1.1) and (1.3) respectively. Then for all \( \tau \geq v_m \) we have

\[
\int_0^\tau \ell^{p-1} \left( \int_{\partial U_t^v} \frac{|x|^\ell}{u(x)} \, d\mathcal{H}^{n-1}(x) \right) \, dt \leq \frac{1}{p\beta} \int_0^{\mu(t)} f^*(s) \, ds
\]

(2.15)

and

\[
\int_0^\tau \ell^{p-1} \left( \int_{\partial U_t \cap \partial \Omega^v} \frac{|x|^\ell}{v(x)} \, d\mathcal{H}^{n-1}(x) \right) \, dt = \frac{1}{p\beta} \int_0^{\mu(t)} f^*(s) \, ds.
\]

(2.16)

**Proof.** Firstly we show (2.15). Fubini’s Theorem ensures that
Choosing $\varphi \equiv 1$ as test function in (1.2) and using (1.5), we obtain

$$\int_{\partial\Omega} \hat{\beta} |x|^p \mathcal{H}^{n-1} \leq \int_{\partial\Omega} \beta(x) u^{p-1} d\mathcal{H}^{n-1} = \int_{\Omega} f |x|^t \, dx. \quad (2.18)$$

From (2.17) and (2.18) we immediately get

$$\int_0^\infty \tau^{p-1} \left( \int_{\partial U^{ext}_\tau} |x|^{p-1} u(x) d\mathcal{H}^{n-1}(x) \right) d\tau \leq \frac{1}{p\hat{\beta}} \int_0^{|\Omega|_t} f^*(s) \, ds,$$

and, arguing in the same way, we deduce

$$\int_0^\infty \tau^{p-1} \left( \int_{\partial V^{ext}_\tau \cap \partial \Omega^\sharp} |x|^{p-1} v(x) d\mathcal{H}^{n-1}(x) \right) d\tau = \frac{1}{p\hat{\beta}} \int_0^{|\Omega|_t} f^*(s) \, ds.$$

Hence for any $t \geq 0$, we have

$$\int_0^t \tau^{p-1} \left( \int_{\partial U^{ext}_\tau \cap \partial \Omega^\sharp} |x|^{p-1} u(x) d\mathcal{H}^{n-1}(x) \right) d\tau \leq \frac{1}{p\hat{\beta}} \int_0^{|\Omega|_t} f^*(s) \, ds.$$

Since $\partial V_t \cap \partial \Omega^\sharp$ is empty for any $t \geq \tau_m$, we get

$$\int_0^t \tau^{p-1} \left( \int_{\partial U^{ext}_\tau \cap \partial \Omega^\sharp} |x|^{p-1} u(x) d\mathcal{H}^{n-1}(x) \right) d\tau = \frac{1}{p\hat{\beta}} \int_0^{|\Omega|_t} f^*(s) \, ds.$$

Lemma 2.7 is hence proved. \hfill \Box

## 3 Main results

In this Section we prove Theorem 1.1 and Theorem 1.2.
3 MAIN RESULTS

Proof of Theorem 1.1. Let

\[ \delta_1 = 1 - \left( \frac{\ell(p - 1) + p(n - 1)}{(p - 1)(\ell + n)} \right). \]

Note that, since \( p \geq n \), \( \delta_1 \) is a positive constant.

Multiplying both sides of (2.11) by \( t^{p-1} \mu(t)^{\delta_1} \) and integrating over \((0, \tau)\) with \( \tau \geq v_m \), by Lemma 2.7, we have

\[
\gamma_{n,\ell,p} \int_0^\tau t^{p-1} \mu(t) \, dt \leq \int_0^\tau \left( -\mu'(t) \right) t^{p-1} \mu(t)^{\delta_1} \left( \int_0^t f^*(s) \, ds \right)^{\frac{1}{p-1}} \, dt \\
+ \frac{|\Omega|^{\delta_1}}{p^{\frac{1}{p-1}} \beta} \left( \int_0^\tau f^*(s) \, ds \right)^{\frac{1}{p-1}}. \tag{3.1}
\]

Setting

\[ F(\ell) = \int_0^\ell \omega^{\delta_1} \left( \int_0^\omega f^*(s) \, ds \right)^{\frac{1}{p-1}} \, d\omega, \]

and, integrating by parts both sides of the last inequality, we obtain

\[
\tau^{p-1} \left( \gamma_{n,\ell,p} \int_0^\tau \mu(t) \, dt + F(\mu(\tau)) \right) \\
\leq (p - 1) \int_0^\tau t^{p-2} \left( \gamma_{n,\ell,p} \int_0^t \mu(s) \, ds + F(\mu(t)) \right) \, dt \\
+ \frac{|\Omega|^{\delta_1}}{p^{\frac{1}{p-1}} \beta} \left( \int_0^\tau f^*(s) \, ds \right)^{\frac{1}{p-1}}. \]

We observe that all the hypotheses of Gronwall’s Lemma 2.5 are satisfied with

\[ \xi(\tau) = \xi_1(\tau) = \int_0^\tau t^{p-2} \left( \gamma_{n,\ell,p} \int_0^t \mu(s) \, ds + F(\mu(t)) \right) \, dt \]

and

\[ C = C_1 = \frac{|\Omega|^{\delta_1}}{p^{\frac{1}{p-1}} \beta} \left( \int_0^\tau f^*(s) \, ds \right)^{\frac{1}{p-1}}. \]

Applying such a Lemma with \( \tau_0 = v_m \), we infer that for any \( \tau \geq v_m \) it holds

\[
\tau^{p-2} \left( \gamma_{n,\ell,p} \int_0^\tau \mu(s) \, ds + F(\mu(\tau)) \right) \leq \left( \frac{(p - 1)\xi_1(v_m) + C_1}{v_m} \right) \left( \frac{\tau}{v_m} \right)^{p-2}. \]
Replacing $\mu(t)$ with $\phi(t)$ the previous inequality holds as an equality. Therefore for any $\tau \geq v_m$ it holds that

$$
\gamma_{n,\ell,p} \int_0^\tau \mu(s) \, ds + F(\mu(\tau)) \leq \gamma_{n,\ell,p} \int_0^\tau \phi(s) \, ds + F(\phi(\tau)). \tag{3.2}
$$

On the other hand, since $\mu(t) \leq \phi(t) = |\Omega|_\ell, \forall t \leq v_m$, and $F(\ell)$ is an increasing function, we have that for any $\tau \leq v_m$ it holds that

$$
\gamma_{n,\ell,p} \int_0^\tau \mu(s) \, ds + F(\mu(\tau)) \leq \gamma_{n,\ell,p} \int_0^\tau \phi(s) \, ds + F(\phi(\tau)) \tag{3.3}
$$

Combining inequalities (3.2) and (3.3) we conclude that for any $\tau \geq 0$ it holds

$$
\gamma_{n,\ell,p} \int_0^\tau \mu(s) \, ds + F(\mu(\tau)) \leq \gamma_{n,\ell,p} \int_0^\tau \phi(s) \, ds + F(\phi(\tau)).
$$

Since

$$
\lim_{\tau \to +\infty} F(\mu(\tau)) = \lim_{\tau \to +\infty} F(\phi(\tau)) = 0,
$$

letting $\tau$ go to $+\infty$, we conclude that

$$
\int_0^\infty \mu(t) \, dt \leq \int_0^\infty \phi(t) \, dt,
$$

and hence

$$
\|u\|_{L^1(\Omega,|x|^\ell \, dx)} \leq \|v\|_{\tilde{L}^1(\Omega,|x|^\ell \, dx)}.
$$

Now let us show (1.7). Firstly let us observe that it is enough to verify that

$$
\int_0^\infty t^{p-1} \mu(t) \, dt \leq (p-1) \int_0^\infty t^{p-2} F(\mu(t)) \, dt + \frac{|\Omega|_\ell}{p \beta^{p-1}} \left( \int_0^{\delta_1} f^\ast(s) \, ds \right)^\frac{p}{p-1}, \tag{3.4}
$$

Integrating by parts the first term on the right-hand side in (3.1), and then letting $\tau$ go to $+\infty$, we obtain

$$
\gamma_{n,\ell,p} \int_0^\infty t^{p-1} \mu(t) \, dt \leq (p-1) \int_0^\infty t^{p-2} F(\mu(t)) \, dt + \frac{|\Omega|_\ell}{p \beta^{p-1}} \left( \int_0^{\delta_1} f^\ast(s) \, ds \right)^\frac{p}{p-1},
$$

and similarly we have

$$
\gamma_{n,\ell,p} \int_0^\infty t^{p-1} \phi(t) \, dt = (p-1) \int_0^\infty t^{p-2} F(\phi(t)) \, dt + \frac{|\Omega|_\ell}{p \beta^{p-1}} \left( \int_0^{\delta_1} f^\ast(s) \, ds \right)^\frac{p}{p-1}.
$$
Thus if we prove
\[ \int_0^\infty t^{p-2} F(\mu(t)) \, dt \leq \int_0^\infty t^{p-2} F(\phi(t)) \, dt \]
we get the claim (3.4).

Let
\[ \theta_1 = -\frac{\ell(p - 1) + p(n - 1)}{(p - 1)(\ell + n)}. \]

Multiplying both sides of (2.11) by \( t^{p-1} F(\mu(t)) \mu(t)^{\theta_1} \) and then integrating over \((0, \tau)\), taking into account that the function \( h(\ell) = F(\ell)\ell^{\theta_1} \) is non decreasing, we get
\begin{align*}
\gamma_{n,\ell,p} & \int_0^\tau t^{p-1} F(\mu(t)) \, dt \\
& \leq \int_0^\tau (-\mu'(t)) t^{p-1} \mu(t)^{\theta_1} F(\mu(t)) \left( \int_0^{\mu(t)} f^*(s) \, ds \right)^{\frac{1}{p-1}} \, dt \\
& + F(|\Omega|_{\ell}) \frac{|\Omega|_{\ell}^{\theta_1}}{p\beta^{\frac{p-1}{p}}} \left( \int_0^{\beta|\Omega|_{\ell}} f^*(s) \, ds \right)^{\frac{p}{p-1}}.
\end{align*}

Let
\[ C_2 = F(|\Omega|_{\ell}) \frac{|\Omega|_{\ell}^{\theta_1}}{p\beta^{\frac{p-1}{p}}} \left( \int_0^{\beta|\Omega|_{\ell}} f^*(s) \, ds \right)^{\frac{p}{p-1}}. \]

Integrating by parts both sides of (3.6) we derive
\begin{align*}
\gamma_{n,\ell,p} \tau & \int_0^\tau t^{p-2} F(\mu(t)) \, dt + \tau H_\mu(\tau) \leq \gamma_{n,\ell,p} \int_0^\tau \int_0^\tau r^{p-2} F(\mu(r)) \, dr \, dt \\
& + \int_0^\tau H_\mu(t) \, dt + C_2,
\end{align*}
where
\[ H_\mu(\tau) = -\int_{\tau}^{+\infty} t^{p-2} \mu(t)^{\theta_1} F(\mu(t)) \left( \int_0^{\mu(t)} f^*(s) \, ds \right)^{\frac{1}{p-1}} \, d\mu(t). \]

Setting
\[ \xi_2(\tau) = \gamma_{n,\ell,p} \int_0^\tau \int_0^\tau r^{p-2} F(\mu(r)) \, dr + \int_0^\tau H_\mu(t) \, dt, \]
then (3.7) reads as
\[ \tau \xi_2'(\tau) \leq \xi_2(\tau) + C_2. \]
Lemma 2.5, with \( \tau_0 = v_m \), ensures that the following inequality holds true for any \( \tau \geq v_m \)

\[
\gamma_{n,\ell,p} \int_0^\tau t^{p-2} F(\mu(t)) \, dt + H_\mu(\tau) \leq \frac{(p-1) \int_0^{v_m} t^{p-2} F(\mu(t)) \, dt + H_\mu(v_m) + C_2}{v_m} \left( \frac{\tau}{v_m} \right)^{p-2}.
\]

The previous inequality becomes an equality if we replace \( \mu(t) \) with \( \phi(t) \), so, recalling that \( \mu(t) \leq \phi(t) = |\Omega|_\ell \) for \( t \leq v_m \), we obtain

\[
\gamma_{n,\ell,p} \int_0^\tau t^{p-2} F(\mu(t)) \, dt + H_\mu(\tau) \leq \gamma_{n,\ell,p} \int_0^\tau t^{p-2} F(\phi(t)) \, dt + H_\phi(\tau).
\]

Letting \( \tau \to +\infty \), we have

\[
\int_0^\infty t^{p-2} F(\mu(t)) \, dt \leq \int_0^\infty t^{p-2} F(\phi(t)) \, dt,
\]

since, as we will show,

\[
\lim_{\tau \to 0} H_\mu(\tau) = \lim_{\tau \to 0} H_\phi(\tau) = 0. \tag{3.8}
\]

This proves (3.5), and hence (1.7).

To prove (3.8), recalling that \( p \geq n \), we observe that

\[
t^{p-2} \mu(t) = \int_{\{u > t\}} t^{p-2} |x|^\ell \, dx \leq \int_{\{u > t\}} u^{p-2} |x|^\ell \, dx \leq \|u\|^{p-2}_{L^p(\Omega, |x|^\ell \, dx)} \mu(t)^{\frac{2}{p}},
\]

therefore

\[
|H_\mu(\tau)| = \int_\tau^{+\infty} t^{p-2} F(\mu(t)) \mu(t)^{\theta_1} \left( \int_0^{\mu(t)} f^*(s) \, ds \right) (-\mu'(t)) \, dt \\
\leq \left( \int_{|\Omega|_\ell} f^*(s) \, ds \right) \|u\|^{p-2}_{L^p(\Omega, |x|^\ell \, dx)} \int_\tau^{+\infty} F(\mu(t)) \mu(t)^{\frac{2}{p} + \theta_1 - 1} (-\mu'(t)) \, dt.
\]

The claim follows by observing that the right-hand side of the above inequality goes to 0 as \( \tau \to +\infty \).

**Remark 3.1.** We emphasize that, with the same argument in [ANT; AGM22], it is possible to obtain a stronger comparison in Theorem 1.1. It can be shown

\[
\|u\|_{L^k,1(\Omega, |x|^\ell \, dx)} \leq \|v\|_{L^k,1(\Omega, |x|^\ell \, dx)} \quad \forall 0 < k \leq \frac{(\ell + n)(p - 1)}{\ell(p - 1) + p(n - 1)}
\]
and

$$\|u\|_{L^p(\Omega,|x|^\ell dx)} \leq \|v\|_{L^p(\Omega,|x|^\ell dx)} \quad \forall 0 < k \leq \frac{(\ell + n)(p - 1)}{\ell(p - 1) + p(n - 2) + n},$$

where $\|\cdot\|_{L^p(\Omega,|x|^\ell dx)}$ is defined in (2.6).

**Proof of Theorem 1.2.** Let

$$\theta_2 = \frac{\ell(p - 1) + p(n - 1)}{p(\ell + n)}$$

and

$$\delta_2 = -\left(\theta_2 - \frac{1}{p}\right) \frac{p}{p - 1}.$$

We point out that $\delta_2 \geq 0$ by assumption (1.8).

Obviously in this case we have

$$\int_0^{\mu(t)} f^*(s) ds = \mu(t).$$

Hence (2.11) can be written as

$$\gamma_{n,\ell,p} \mu(t)^{(\theta_2 - \frac{1}{p})\frac{p}{p - 1}} \leq -\mu'(t) + \frac{1}{\beta^{n-1}} \int_{\partial U_{1,n}} \frac{|x|^\ell}{u} dH^{n-1}(x). \quad (3.9)$$

Multiplying both sides of (3.9) by $t^{p-1}\mu(t)^{\delta_2}$ and, then, integrating from 0 to $\tau \geq v_m$, we obtain

$$\gamma_{n,\ell,p} \int_0^\tau t^{p-1} dt \leq \int_0^\tau t^{p-1}\mu(t)^{\delta_2}(-\mu'(t)) dt$$

$$+ \frac{1}{\beta^{n-1}} \int_0^\tau t^{p-1}\mu(t)^{\delta_2} \int_{\partial U_{1,n}} \frac{1}{u} dH^{n-1}(x) \quad (3.10)$$

$$\leq \int_0^\tau t^{p-1}\mu(t)^{\delta_2}(-\mu'(t)) dt + \frac{|\Omega|^{1+\delta_2}}{p\beta^{n-1}}.$$

Taking into account of Lemma 2.7, if we replace $\mu(t)$ with $\phi(t)$, the previous inequality holds as equality. Hence, we get

$$\int_0^\tau t^{p-1}\mu(t)^{\delta_2}(-\mu'(t)) dt \geq \int_0^\tau t^{p-1}\phi(t)^{\delta_2}(-\phi'(t)) dt.$$
In turn an integration by parts yields

\[- \tau^{p-1} \frac{\mu(\tau)^{1+\delta_2}}{1 + \delta_2} + (p - 1) \int_0^\tau t^{p-2} \frac{\mu(t)^{1+\delta_2}}{1 + \delta_2} dt \geq - \tau^{p-1} \frac{\phi(\tau)^{1+\delta_2}}{1 + \delta_2} + (p - 1) \int_0^\tau t^{p-2} \frac{\phi(t)^{1+\delta_2}}{1 + \delta_2} dt.\]

The above inequality allows us to use claim (ii) of the Gronwall’s Lemma, this time to the function

\[\xi_3(\tau) = \int_0^\tau s^{p-2} \left( \frac{\mu(s)^{1+\delta_2} - \phi(s)^{1+\delta_2}}{1 + \delta_2} \right) ds.\]

Hence for any \(\tau \geq v_m\) it holds

\[\tau^{p-2} \left( \frac{\mu^{1+\delta_2}(\tau) - \phi^{1+\delta_2}(\tau)}{1 + \delta_2} \right) \leq (p - 1) \frac{\tau^{p-2}}{v_m^{p-2}} \int_0^{v_m} s^{p-2} \left( \frac{\mu^{1+\delta_2}(s) - \phi^{1+\delta_2}(s)}{1 + \delta_2} \right) ds.\]

Inequality (2.8) ensures us that the right-hand side of the inequality above is non-positive, therefore

\[\mu(\tau) \leq \phi(\tau) \quad \forall \tau \geq v_m.\]

Finally, since by (2.8) we have

\[\mu(\tau) \leq \phi(\tau) = |\Omega|_\ell \quad \forall \tau \leq v_m,\]

and, therefore, the inequality above holds true in \([0, +\infty)\). Claim (1.9) is hence proven.

\[\square\]

4 Some applications

In this section we give some applications of our previous results. First of all, as already mentioned in the introduction, it is possible to derive a Faber-Krahn inequality for the Robin \(p\)-Laplacian operator with non-constant boundary parameter.

**Proof of Theorem 1.3.** Let \(u\) be a positive minimizer of (1.10), then

\[
\begin{cases}
-\Delta_p u = \lambda_{1,\beta}(\Omega) u^{p-1} |x|^{\ell-2} & \text{in } \Omega \\
|\nabla u|^{p-2} \frac{\partial u}{\partial \nu} + \beta(x) u^{p-1} = 0 & \text{on } \partial \Omega.
\end{cases}
\]
Now, let \( z \) be a solution to the following problem

\[
\begin{cases}
-\Delta_p z = \lambda_1(\Omega) (u^t)^{p-1} |x|^{\ell} & \text{in } \Omega^2 \\
|\nabla z|^{p-2} \frac{\partial z}{\partial \nu} + \tilde{\beta}(r^z) \frac{r}{p} z^{p-1} = 0 & \text{on } \partial \Omega^2.
\end{cases}
\] (4.1)

Therefore, Theorem 1.1 ensures that

\[
\int_{\Omega} u^p |x|^{\ell} \, dx = \int_{\Omega^2} (u^t)^p |x|^{\ell} \, dx \leq \int_{\Omega^2} z^p |x|^{\ell} \, dx,
\]

while Hölder inequality gives

\[
\int_{\Omega} (u^t)^{p-1} z |x|^{\ell} \, dx \leq \left( \int_{\Omega^2} (u^t)^p |x|^{\ell} \, dx \right)^{\frac{p-1}{p}} \left( \int_{\Omega^2} z^p |x|^{\ell} \, dx \right)^{\frac{1}{p}} \leq \int_{\Omega^2} z^p |x|^{\ell} \, dx.
\]

Hence, writing \( \lambda_{1,\beta}(\Omega) \) according to (4.1), we get

\[
\lambda_{1,\beta}(\Omega) = \frac{\int_{\Omega^2} |\nabla z|^p \, dx + \int_{\partial \Omega^2} \tilde{\beta}(r^z) \frac{r}{p} z^p \, dH^{n-1}(x)}{\int_{\Omega^2} (u^t)^{p-1} z |x|^{\ell} \, dx} \geq \frac{\int_{\Omega^2} |\nabla z|^p \, dx + \int_{\partial \Omega^2} \tilde{\beta}(r^z) \frac{r}{p} z^p \, dH^{n-1}(x)}{\int_{\Omega^2} z^p |x|^{\ell} \, dx} \geq \lambda_{1,\beta}(\Omega^2). \]

\[
5 \text{ Appendix}
\]

Here we provide the proofs of Propositions 2.3 and 2.4 just for \( p = n \), since the remaining cases can be treated analogously.

**Proof of Proposition 2.3.** We recall that, as well-known, \( W^{1,n}(\Omega) \) is compactly embedded in \( L^q(\Omega) \) for every \( q \) finite. Then Hölder inequality gives

\[
\left( \int_{\Omega} |\psi|^n |x|^{\ell} \, dx \right)^{\frac{1}{n}} \leq \left( \int_{\Omega} |\psi|^q \right)^{\frac{1}{q}} \left( \int_{\Omega} |x|^{n\ell} \right)^{\frac{1}{s}} \quad \text{where} \quad \frac{1}{n} = \frac{1}{q} + \frac{1}{s},
\]
If \( q \geq 0 \), then \( s \geq n \) and so
\[
q > \frac{n^2}{n - |\ell|} \implies \frac{|\ell|}{n}s < n.
\]

Hence
\[
\|\psi\|_{L^n(\Omega, |x|^{\ell} \, dx)} \leq D_1 \|\psi\|_{L^q(\Omega)} \leq C_1 \|\psi\|_{W^{1,n}(\Omega)}^n,
\]
and, therefore, the embedding is at least continuous.

Now, if we take a sequence \( \{\psi_k\}_k \) bounded in \( W^{1,n}(\Omega) \), by the compact embedding of \( W^{1,n}(\Omega) \) in \( L^q(\Omega) \), up to a subsequence, \( \{\psi_k\}_k \) strongly converges in \( L^q(\Omega) \). The continuous embedding of \( L^q(\Omega) \) in \( L^n(\Omega, |x|^{\ell} \, dx) \) guarantees us the strong convergence in the second space as well. \( \square \)

**Proof of Proposition 2.4.** The solution to (1.1) is the unique minimum of the functional

\[
G : \psi \in W^{1,n}(\Omega) \rightarrow \frac{1}{n} \int_{\Omega} |\nabla \psi|^n \, dx + \frac{1}{n} \int_{\partial \Omega} \beta(x)|\psi|^n \, d\mathcal{H}^{n-1}(x) - \int_{\Omega} f\psi|x|^{\ell} \, dx.
\]

(5.1)

Existence can be achieved by means of direct methods of calculus of variation, see for instance [Dac08; Giu94; Lin06; AGM22], while one can prove uniqueness by using the same arguments contained in [BK02; DS87].

Here we just recall the proof of the existence.

First of all we notice that functional \( G \) is bounded from below. Indeed by (1.5) and since \( 0 \notin \partial \Omega \), we have
\[
G(\psi) = \frac{1}{n} \int_{\Omega} |\nabla \psi|^n \, dx + \frac{1}{n} \int_{\partial \Omega} \beta(x)|\psi|^n \, d\mathcal{H}^{n-1}(x) - \int_{\Omega} f\psi|x|^{\ell} \, dx
\]
\[
\geq \frac{1}{n} \int_{\Omega} |\nabla \psi|^n \, dx + C \int_{\partial \Omega} |\psi|^n - \int_{\Omega} f\psi|x|^{\ell} \, dx
\]
\[
\geq C\|\psi\|_{W^{1,n}(\Omega)}^n - \int_{\Omega} f\psi|x|^{\ell} \, dx,
\]

where in the last inequality we have used the well-known trace embedding Theorem. Using Hölder inequality and Proposition 2.3, we have
\[
G(\psi) \geq C\|\psi\|_{W^{1,n}(\Omega)}^n - \|\psi\|_{W^{1,n}(\Omega)}\|f\|_{L^{n'}(\Omega, |x|^{\ell} \, dx)}
\]

By Young parametric inequality we finally get
\[
G(\psi) \geq \left( C - \frac{\varepsilon n}{n} \right) \|\psi\|_{W^{1,n}(\Omega)}^n - \frac{C}{\varepsilon n' n'}\|f\|_{L^{n'}(\Omega, |x|^{\ell} \, dx)}
\]
(5.2)
For $\varepsilon$ small enough we have

$$G(\psi) \geq -\frac{1}{\varepsilon n'} \| f \|_{L^{n'}(\Omega, |x|^{\ell} dx)} > -\infty.$$ 

Now let $\{ u_k \}_k$ a minimizing sequence for $G$, we can assume, without loss of generality, that $G(u_k) \leq m_G + 1$ where

$$m_G = \inf_{u \in W^{1,n}(\Omega)} G(u).$$

So, by (5.2) for a fixed $\varepsilon$ small, we have

$$m_G + 1 + \frac{C}{\varepsilon n'} \| f \|_{L^{n'}(\Omega, |x|^{\ell} dx)} \geq \left( C - \frac{\varepsilon n}{n} \right) \| u_k \|_{W^{1,n}(\Omega)}^{n}.$$ 

So the sequence $\{ u_k \}_k$ is bounded in $W^{1,n}(\Omega)$ hence, up to a subsequence, it converges weakly in $W^{1,n}(\Omega)$ and strongly in $L^n(\Omega, |x|^{\ell} dx)$ and in $L^n(\partial \Omega)$ to a function $u$.

Thus, we have

$$\liminf_k \int_{\Omega} |\nabla u_k|^n dx \geq \int_{\Omega} |\nabla u|^n dx$$

$$\liminf_k \int_{\Omega} \beta(x) u_k^n dH^{n-1}(x) = \int_{\partial \Omega} \beta(x) u^n dH^{n-1}(x)$$

$$\liminf_k \int_{\Omega} u_k f |x|^{\ell} dx = \int_{\Omega} u f |x|^{\ell} dx$$

and finally

$$m_G = \liminf_k G(u_k) \geq G(u) = m_G. \quad \square$$

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References

[ABCMP17] A. Alvino, F. Brock, F. Chiacchio, A. Mercaldo, and M. R. Posteraro. “Some isoperimetric inequalities on $\mathbb{R}^N$ with respect to weights $|x|^a$”. In: J. Math. Anal. Appl. 451.1 (2017), pp. 280–318.

[ACGM] V. Amato, F. Chiacchio, A. Gentile, and A.L. Masiello. In preparation.
| Reference | Authors | Title and Details |
|-----------|---------|-------------------|
| [ACNT]    | A. Alvino, F. Chiacchio, C. Nitsch, and C. Trombetti. | “Weighted symmetrization results for a problem with variable Robin parameter”. Preprint, (2022), arXiv:2207.14010. |
| [ACNT21]  | A. Alvino, F. Chiacchio, C. Nitsch, and C. Trombetti. | “Sharp estimates for solutions to elliptic problems with mixed boundary conditions”. In: *J. Math. Pures Appl.* 152 (2021), pp. 251–261. |
| [AGM22]   | V. Amato, A. Gentile, and A. L. Masiello. | “Comparison results for solutions to p-Laplace equations with Robin boundary conditions”. In: *Ann. Mat. Pura Appl. (4)* 201.3 (2022), pp. 1189–1212. |
| [ANT]     | A. Alvino, C. Nitsch, and C. Trombetti. | “A Talenti comparison result for solutions to elliptic problems with Robin boundary conditions”. Preprint, (2019), arXiv:2109.10117. To appear on *Comm. Pure Appl. Math.* |
| [Bae19]   | A. Baernstein II. | *Symmetrization in analysis*. Vol. 36. New Mathematical Monographs. With David Drasin and Richard S. Laugesen, With a foreword by Walter Hayman. Cambridge University Press, Cambridge, 2019. |
| [BBMP99]  | M. F Betta, F. Brock, A. Mercaldo, and M. R. Posteraro. | “A weighted isoperimetric inequality and applications to symmetrization”. In: *J. Inequal. Appl.* 4.3 (1999), pp. 215–240. |
| [BD10]    | D. Bucur and D. Daners. | “An alternative approach to the Faber-Krahn inequality for Robin problems”. In: *Calc. Var. Partial Differential Equations* 37.1-2 (2010), pp. 75–86. |
| [BG10]    | D. Bucur and A. Giacomini. | “A variational approach to the isoperimetric inequality for the Robin eigenvalue problem”. In: *Arch. Ration. Mech. Anal.* 198.3 (2010), pp. 927–961. |
| [BG15]    | D. Bucur and A. Giacomini. | “Faber-Krahn inequalities for the Robin-Laplacian: a free discontinuity approach”. In: *Arch. Ration. Mech. Anal.* 218.2 (2015), pp. 757–824. |
| [BGT19]   | D. Bucur, A. Giacomini, and P. Trebeschi. | “Best constant in Poincaré inequalities with traces: a free discontinuity approach”. In: *Ann. Inst. H. Poincaré Anal. Non Linéaire* 36.7 (2019), pp. 1959–1986. |
| [BK02]    | M. Belloni and B. Kawohl. | “A direct uniqueness proof for equations involving the p-Laplace operator”. In: *Manuscripta Mathematica* 109 (2002), pp. 229–231. |
[Bos88] M.H. Bossel. “Membranes élastiquement liées inhomogènes ou sur une surface: une nouvelle extension du théorème isopérimétrique de Rayleigh-Faber-Krahn”. In: Z. Angew. Math. Phys. 39.5 (1988), pp. 733–742.

[CGNT] F. Chiacchio, N. Gavitone, C. Nitsch, and C. Trombetti. Sharp estimates for the Gaussian torsional rigidity with Robin boundary conditions. Preprint, (2021), arXiv:2109.10117. To appear on Potential Analysis.

[CH15] N. Chiba and T. Horiuchi. “On radial symmetry and its breaking in the Caffarelli-Kohn-Nirenberg type inequalities for \( p = 1 \)”. In: Math. J. Ibaraki Univ. 47 (2015), pp. 49–63.

[CR71] K. M. Chong and N. M. Rice. Equimeasurable rearrangements of functions. Queen’s Papers in Pure and Applied Mathematics, No. 28. Queen’s University, Kingston, Ont., 1971.

[Csa15] G. Csató. “An isoperimetric problem with density and the Hardy Sobolev inequality in \( \mathbb{R}^2 \)”. In: Differential Integral Equations 28.9-10 (2015), pp. 971–988.

[Dac08] B. Dacorogna. Direct methods in the calculus of variations. Second. Vol. 78. Applied Mathematical Sciences. Springer, New York, 2008.

[DHHT12] A. Díaz, N. Harman, S. Howe, and D. Thompson. “Isoperimetric problems in sectors with density”. In: Adv. Geom. 12.4 (2012), pp. 589–619.

[DS87] J.I. Díaz and J.E. Saá. “Existence et unicité de solutions positives pour certaines équations elliptiques quasilineaires”. In: C. R. Acad. Sci. Paris Sér. I Math. 305.12 (1987), pp. 521–524.

[FK15] P. Freitas and D. Krejcirik. “The first Robin eigenvalue with negative boundary parameter”. In: Advances in Mathematics 280 (2015), pp. 322–339.

[FNT13] V. Ferone, C. Nitsch, and C. Trombetti. “On a conjectured reverse Faber-Krahn inequality for a Steklov-type Laplacian eigenvalue”. In: Communications on Pure and Applied Analysis 14 (2013).

[Giu94] E. Giusti. Metodi diretti nel calcolo delle variazioni. Unione Matematica Italiana, Bologna, 1994.
REFERENCES

[How15] S. Howe. “The log-convex density conjecture and vertical surface area in warped products”. In: Adv. Geom. 15.4 (2015), pp. 455–468.

[Kaw85] B. Kawohl. Rearrangements and convexity of level sets in PDE. Vol. 1150. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1985.

[Kes06] S. Kesavan. Symmetrization & applications. Vol. 3. Series in Analysis. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006.

[Lin06] P. Lindqvist. Notes on the $p$-Laplace equation. Vol. 102. Report. University of Jyväskylä Department of Mathematics and Statistics. University of Jyväskylä, Jyväskylä, 2006.

[Tal16] G. Talenti. “The art of rearranging”. In: Milan J. Math. 84.1 (2016), pp. 105–157.

[Tal76] G. Talenti. “Elliptic equations and rearrangements”. In: Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 3.4 (1976), pp. 697–718.

[Tro00] G. Trombetti. “Symmetrization methods for partial differential equations”. In: Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) 3.3 (2000), pp. 601–634.

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