Abstract

We propose to address in a natural manner, the modular variable concept explicitly in a Schrödinger picture. The idea of Modular Variables was introduced in 1969 by Aharonov, Pendleton and Petersen to explain certain non-local properties of quantum mechanics. Our approach to this subject is based on Schwinger’s finite quantum kinematics and its continuous limit.

An explicit Schrödinger picture for Aharonov’s Modular Variable concept

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1 Introduction

In [1], Aharonov and collaborators introduced the concept of modular variables to explain some peculiar non-local quantum effects as the modular variable exchange between particles and fields in situations like the well-known Aharonov-Bohm (AB) effect where a beam of electrons suffers a phase shift from the magnetic field of a solenoid even without having had any direct contact with the field. This is contrary to the usual view where the AB phenomenon is explained by a local interaction between the particles and field potentials, even if the potentials are somewhat unphysical because they are determined only up to a gauge transformation. More recently, Aharonov and collaborators have argued for an even wider application of modular variables to explain certain non-local quantum effects that arise in what may be considered as paradigmatic phenomena for quantum theory as wave-particle interference phenomena. Their approach is based on a Heisenberg picture, while we shall address explicitly the same problem within a Schrödinger-picture approach with the help of the mathematical structure behind Schwinger’s Quantum Kinematical Phase Space. We review in the next section, Schwinger’s Finite Quantum Kinematics for those who which it is unfamiliar and also to fix our own notation. In section III, we carry out the continuous limit of this finite structure in two distinct ways to make it easy to understand the transition between our finite analogue definition of modular variables and the one introduced by Aharonov. In section IV we discuss the concept of pseudo-degrees of freedom, an idea introduced by one of the authors on how one attributes quantum degrees of freedom to tensor product spaces that is essential to our proposal of a finite setting for modular variables [2]. In section V we finally discuss our conception of a modular variable and we present some examples. We conclude with section VI where we make some additional comments and set the stage for further work.
2 Schwinger’s Quantum Kinematics

Let $W(N)$ be a $N$-dimensional quantum space together with an orthonormal basis $\{|u_j\rangle\}$, ($j = 0, \ldots N-1$), that is

\[
\langle u^j | u_k \rangle = \delta^j_k \quad \text{(orthonormality)}
\]
\[
|u_j\rangle \langle u^j | = \hat{I} \quad \text{(completeness)}
\]

where we will use from now on the sum convention for repeated lower and upper indices. These states are finite position states. We follow Schwinger [3] and define an unitary translation operator $\hat{V}$ by a cyclic permutation over the $\{|u_j\rangle\}$ basis:

\[
\hat{V}|u_j\rangle = |u_{j-1}\rangle
\]

Clearly one has

\[
\hat{V}^N = \hat{I}
\]

so its spectrum is composed by the $N$-th roots of unity:

\[
v_j = e^{\frac{2\pi i j}{N}} \quad \text{with} \quad (j = 0, \ldots N-1)
\]

The eigenstates of $\hat{V}$ also form an orthonormal basis $\{|v_j\rangle\}$, the finite momentum states. We can repeat the above procedure defining an unitary translation operator $\hat{U}$ that acts upon the momentum basis by the following cyclic permutation:

\[
\hat{U}|v_j\rangle = |v_{j+1}\rangle
\]

Again it follows that the spectrum of $\hat{U}$ are the $N$-th roots of unity $v_j$. It is possible to show that (with an appropriate phase choice) the eigenstates of $\hat{U}$ are the original position states $\{|u_j\rangle\}$. Notice that the indices of the complex phase $v_j = e^{\frac{2\pi i j}{N}}$ and its powers $(v_j)^k = (e^{\frac{2\pi i j}{N}})^k = e^{\frac{2\pi i j k}{N}} = v^k_j$ have a double function both as matrix indices and as integer powers of the $v_j$ phase. Actually, because of the built in $MOD N$ structure of the phases, the indices may be thought as running over the finite ring $Z_N$ of integers $MOD N$ [4]. In fact, this matrix is nothing else but the matrix elements of the Finite Fourier Transform in the position (or momentum) basis. In fact, the overlap between these set of states is given by

\[
\langle u^k | v_j \rangle = \frac{v^k_j}{\sqrt{N}} = \frac{1}{\sqrt{N}} e^{\frac{2\pi i k j}{N}}
\]

The above relation shows that the finite position and momentum bases form a mutually unbiased basis (MUB), a concept that has become important in modern quantum information theory [5]. Also it is not difficult to show that

\[
\hat{V}^j \hat{U}^k = v^{jk} \hat{U}^k \hat{V}^j
\]

The above equation is a kind of finite exponentiated analogue of the usual canonical commutation relations between position and momentum observables known as the Heisenberg-Weyl relation. As we shall see in the next section, essentially the same equation holds for the continuum.
3 The heuristic continuum limit

We shall implement the “continuum limit” in two different manners. One symmetric and the other non-symmetric between the position and momentum states.

3.1 The symmetric approach

Let \( \dim W^{(N)} = N \) be an odd integer (with no loss of generality) so that the index variation is symmetric in relation to “zero”: \( j = -\frac{N-1}{2}, ... + \frac{N-1}{2} \). We introduce the “scaled” variables

\[
x_j = \left( \frac{2\pi}{N} \right)^{1/2} j \quad \text{and} \quad y_k = \left( \frac{2\pi}{N} \right)^{1/2} k
\]

so that their “variation” is given by

\[
\Delta x_j = \Delta y_k = \left( \frac{2\pi}{N} \right)^{1/2}
\]

with \( \Delta j = \Delta k = 1 \) as \( \Delta x_j \to 0 \) and \( \Delta y_k \to 0 \) for \( N \to \infty \). We also “scale” the position and momentum eigenvets as:

\[
|q(x_j)\rangle = \left( \frac{N}{2\pi} \right)^{1/4} |u_j\rangle \quad \text{and} \quad |p(y_k)\rangle = \left( \frac{N}{2\pi} \right)^{1/4} |v_k\rangle
\]

so that we can write the completeness relation as

\[
\hat{I} = |u_j\rangle\langle u^j| = |v_k\rangle\langle v^k| = \sum_{j=-\frac{N-1}{2}}^{\frac{N-1}{2}} |q(x_j)\rangle\langle q(x_j)| \Delta x_j = \sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} |p(y_k)\rangle\langle p(y_k)| \Delta y_k
\]

One can give a natural heuristic interpretation of the \( N \to \infty \) limit for the above equation as

\[
\hat{I} = \int_{-\infty}^{+\infty} |q(x)\rangle\langle q(x)| \, dx = \int_{-\infty}^{+\infty} |p(y)\rangle\langle p(y)| \, dy
\]

and the inner product between these continuous eigenvets may be written as

\[
\langle q(x_j)|p(y_k)\rangle = \left( \frac{N}{2\pi} \right)^{1/2} \langle u^j|v_k\rangle = \frac{1}{\sqrt{2\pi}} u^j_k = \frac{1}{\sqrt{2\pi}} e^{ix_j y_k}
\]

so that for \( N \to \infty \) comes:

\[
\langle q(x)|p(y)\rangle = \frac{1}{\sqrt{2\pi}} e^{ix y}
\]

Note that we use a slightly different notation than usual in the sense that we distinguish between the “type” of the eigenvector \((q \text{ or } p)\) from the actual \( x \) eigenvalue [6]. The norm of the \(|q(x)\rangle\) and \(|p(y)\rangle\)
states are clearly “infinite” so the usual orthonormalization must be treated with care in a non-usual manner:

\[
\langle q(x_j) | q(x_k) \rangle = \langle p(x_j) | p(x_k) \rangle = \begin{cases} 0 & \text{for } j \neq k \\ \left(\frac{N}{2\pi}\right)^{1/2} & \text{for } j = k = (N \to \infty) \\ \infty & \text{for } x_j = x_k \end{cases}
\]

which is usually written in a more simplified form as

\[
\langle q(x) | q(x') \rangle = \langle p(x) | p(x') \rangle = \delta(x - x')
\]

(14)

One may even consider this to be an heuristic definition for the Dirac Delta “function”. If we are to insert the Planck constant \( \hbar \) explicitly in equation (13), one obtains the well-known plane wave equation for the inner product between position and momentum eigenkets:

\[
\langle q(x) | p(y) \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ixy/\hbar}.
\]

(15)

The wave function in the position and momentum basis for an arbitrary state \( |\psi\rangle \in W^{(\infty)} \) is given respectively by the amplitudes

\[
\psi_q(x) = \langle q(x) | \psi \rangle \\
\psi_p(y) = \langle p(y) | \psi \rangle
\]

(16)

The action of an arbitrary operator \( \hat{O} \) upon arbitrary wave functions are defined as

\[
\hat{O}\psi_q(x) = \langle q(x) | \hat{O} | \psi \rangle \\
\hat{O}\psi_p(y) = \langle p(y) | \hat{O} | \psi \rangle
\]

(17)

The acting of the translation operators over the position and momentum basis are given by

\[
\hat{V}_\xi |p(y)\rangle = e^{i\xi y} |p(y)\rangle \\
\hat{U}_\eta |q(x)\rangle = e^{i\eta x} |q(x)\rangle.
\]

(18)

and

\[
\hat{V}_\xi |q(x)\rangle = |q(x - \xi)\rangle \\
\hat{U}_\eta |p(y)\rangle = |p(y + \eta)\rangle
\]

(19)

where

\[
\hat{V}_\xi = e^{i\xi \hat{P}} \\
\hat{U}_\eta = e^{i\eta \hat{Q}}
\]

(20)

The Hermitian infinitesimal generators of translations are then identified with the usual position and momentum observables \( \hat{Q} \) and \( \hat{P} \) such that

\[
\hat{Q}|q(x)\rangle = x |q(x)\rangle \\
\hat{P}|p(y)\rangle = y |p(y)\rangle
\]

(21)
obeying the usual Heisenberg canonical commutation relation

\[ [\hat{Q}, \hat{P}] = i\hat{I} \quad \text{with} \quad \hbar = 1 \quad (22) \]

together with its “exponentiated” form:

\[ \hat{V}_\xi \hat{U}_\eta = e^{i\xi \eta} \hat{U}_\eta \hat{V} \quad (23) \]

which is the continuous analogue of the finite Heisenberg-Weyl relation [7].

3.2 The non-symmetric continuum limit

We shall now set a different scale for position and momentum as

\[ x_j = \frac{\xi}{N} j \]
\[ y_k = \frac{2\pi}{\xi} k \quad (24) \]

so that their “variation” is given respectively as

\[ \Delta x_j = \frac{\xi}{N} \]
\[ \Delta y_k = \frac{2\pi}{\xi} \quad (25) \]

Since \( \Delta j = \Delta k = 1 \), then \( \Delta x_j \to 0 \) for \( N \to \infty \) (but the same not occurring for \( \Delta y_k \)). We also introduce the “scaled” states

\[ |q(x_j)\rangle = \left(\frac{N}{\xi}\right)^{1/2} |u_j\rangle \]
\[ |p(y_k)\rangle = \left(\frac{\xi}{2\pi}\right)^{1/2} |v_k\rangle \quad (26) \]

so that

\[ \hat{I} = |u_j\rangle \langle u^j| = \sum_{j=-\frac{N}{2}}^{\frac{N}{2}-1} |q(x_j)\rangle \langle q(x_j)| \Delta x_j = |v_k\rangle \langle v^k| = \sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} |p(y_k)\rangle \langle p(y_k)| \Delta y_k \]

such that we may write for the “continuum limit”

\[ \hat{I}_\xi = \int_{-\xi/2}^{+\xi/2} |q(x)\rangle \langle q(x)| dx = \frac{2\pi}{\xi} \sum_{k=-\infty}^{+\infty} |p(y_k)\rangle \langle p(y_k)| \]

with the position and momentum basis orthonormalization given by

\[ \langle q(x)|q(x')\rangle = \delta(x - x') \]
\[ \langle p(y_j)|p(y_k)\rangle = \frac{\xi}{2\pi} \delta^j_k \quad (27) \]

where \( \hat{I}_\xi \) is the identity operator over the space of states that represent periodic functions with period \( \xi \) and the inner product between position and momentum eigenstates are essentially given by the same expression as the one obtained from the symmetric continuum limit equation [13].
\[ (q(x)|p(y_k)) = \frac{1}{\sqrt{2\pi}}e^{ixy_k} \] (28)

In this case, the position basis is a set with the power of the continuum and the states have infinite norm (though with the eigenvalue spectrum bounded between $-\xi/2$ and $+\xi/2$) while the momentum states form an infinite but countable set with finite norm for each ket. These two different limit procedures carry essentially the structure of the usual Fourier analysis theory. The first (symmetric case) embodies the Fourier transform theory and the second is equivalent to the Fourier Series expansion for periodic functions. Note also that we could have reversed the process in the non-symmetric limit making the moment states to be continuous infinite normed eigenkets while the position states would be finite normed eigenvectors with eigenvalues taking discrete values in an infinite countable set.

4 Pseudo degrees of freedom

If one wants to deal with a system with two degrees of freedom, for the continuum limit, it suffices to construct the tensor product \( W = W_1 \otimes W_2 \) with the position and momentum states given respectively by \( |q(\vec{x})\rangle = |q(x^1)\rangle \otimes |q(x^2)\rangle \) and \( |p(\vec{y})\rangle = |p(y^1)\rangle \otimes |p(y^2)\rangle \) together with unitary translation operators \( \hat{V}_\xi = \hat{V}_{\xi_1} \otimes \hat{V}_{\xi_2} \equiv e^{i\xi \cdot \vec{P}} \) and \( \hat{U}_\eta = \hat{U}_{\eta_1} \otimes \hat{U}_{\eta_2} \equiv e^{i\eta \cdot \vec{Q}} \) that act upon the eigenbasis as

\[
\begin{align*}
\hat{V}_\xi |q(\vec{x})\rangle &= |q(\vec{x} - \vec{\xi})\rangle \\
\hat{U}_\eta |p(\vec{y})\rangle &= |p(\vec{y} + \vec{\eta})\rangle
\end{align*}
\] (29)

Figure 1: 2 degrees of freedom

In the \( x^1 - x^2 \) plane, one can easily visualize the translations of the ket \( |q(\vec{x})\rangle \) acted repeatedly upon with \( \hat{V}_\xi \) as in figure 1 where the resulting position kets can be represented on a straight line in the plane that contains point \( \vec{x} \) but with slope given by the \( \vec{\xi} \) direction. Of course, to reach an arbitrary point in the plane, one needs at least two linear independent directions. This is precisely what one means when it is said that the plane is two-dimensional. But things for finite quantum spaces are not quite
so simple. Let us consider first a 4-dimensional system given by the product of two 2-dimensional spaces (two qubits) $W^{(4)} = W_{1}^{(2)} \otimes W_{2}^{(2)}$. (It is important to notice here that one must not confuse the dimension of space, the so called degree of freedom with the dimensionality of the quantum vector spaces). We shall discard in the following discussion the upper indices that indicate dimensionality to eliminate excessive notation. So let $\{|u_0\rangle, |u_1\rangle\}$ be the position basis for each individual qubit space so that computational (unentangled) basis of the tensor product spaces is $\{|u_0\rangle \otimes |u_0\rangle, |u_0\rangle \otimes |u_1\rangle, |u_1\rangle \otimes |u_0\rangle, |u_1\rangle \otimes |u_1\rangle\}$. One may represent such finite 2-space as the discrete set formed by the four points depicted in figure 2.

![Figure 2: Finite 2-space for 2 qubits](image2)

One may even construct distinct "straight lines" in this discrete two-dimensional space acting upon the computational basis $|u_j\rangle \otimes |u_\sigma\rangle$ with the $\hat{V} \otimes \hat{V}$ operator as shown below:

![Figure 3: Discrete parallel lines (0,0); (1,1) and (0,1); (1,0)](image3)

Each of the two parallel “straight lines” above are geometric invariants of the discrete 2-plane under the action of $\hat{V} \otimes \hat{V}$. Consider now, a six-dimensional quantum space $W^{(6)} = W_{1}^{(2)} \otimes W_{2}^{(3)}$ given by
the product of a qubit and a qutrit space with finite position basis respectively given by \{\ket{u_0}, \ket{u_1}\} and \{\ket{u_0}, \ket{u_1}, \ket{u_2}\}. In this case, the fact the dimensions of the individual are coprime means that the action of the \( \hat{V} \otimes \hat{V} \) operator on the product basis \{\ket{u_j} \otimes \ket{u_\sigma}\} \((j = 0, 1, 2)\) can be identified with the action of \( \hat{V}^{(6)} = \hat{V} \otimes \hat{V} \) on the same basis relabeled as \{\ket{u_0}, \ket{u_1}, \ket{u_2}, \ket{u_3}, \ket{u_4}, \ket{u_5}\}. One can start with the \( \ket{u_0} \otimes \ket{u_0} \) state and cover the whole space with one single line as shown in figure 4.

![Figure 4: \( W^{(6)} = W^{(2)} \otimes W^{(3)} \)](image)

This reduction of two degrees of freedom to only one single effective degree of freedom is a general fact for all product spaces when the dimensions of the factor spaces are coprime. In fact we may state

**Theorem 1**

Let \( W^{(N)} = W^{(N_1)} \otimes W^{(N_2)} \) be the tensor product of two spaces with \( \text{MDC}(N_1, N_2) = 1 \) and let each factor space have its own pair of translation operators \((\hat{U}^{(N_\alpha)}, \hat{V}^{(N_\alpha)})\) together with a pair of position and momentum basis \( \{\ket{u_{j_\alpha}}^{(N_\alpha)}\}, \{\ket{v_{k_\alpha}}^{(N_\alpha)}\}, (j_\alpha, k_\alpha = 0, \ldots, N_\alpha - 1) \) with \( (\alpha = 1, 2) \) obeying the relations given by (6) and (7) then if one defines finite position states \( \ket{u_j} \in W \) by

\[
\ket{u_j} = (\hat{V}^T)^j \ket{u_0}
\]

with

\[
\hat{V} = \hat{V}^{(N_1)} \otimes \hat{V}^{(N_2)}
\]

and

\[
\ket{u_0} = \ket{u_0^{(N_1)}} \otimes \ket{u_0^{(N_2)}}
\]

then there exists a single pair \( r_1 \in Z_{N_1} \) and \( r_2 \in Z_{N_2} \) such that the operator

\[
\hat{U} = \hat{U}^{r_1(N_1)} \otimes \hat{U}^{r_2(N_2)}
\]

defines finite momentum states by

\[
\ket{v_k} = \hat{U}^k \ket{v_0}
\]

with

\[
\ket{v_0} = \ket{v_0^{(N_1)}} \otimes \ket{v_0^{(N_2)}}
\]

such that \( \hat{V} \) and \( \hat{U} \) obey (7) and the finite position and momentum basis \( \{\ket{u_k}\}\{\ket{v_k}\} \) obey relation (6).
As a second example, let us consider again the six-dimensional case:

Example 2 Let us consider again the six-dimensional case: $N = 6$ with $N_1 = 2$ and $N_2 = 3$. We may write

$$|u_0^{(6)}⟩ = |u_0⟩ ⊗ |u_0⟩; \quad |u_1^{(6)}⟩ = |u_1⟩ ⊗ |u_2⟩; \quad |u_2^{(6)}⟩ = |u_0⟩ ⊗ |u_1⟩;$$

$$|u_3^{(6)}⟩ = |u_1⟩ ⊗ |u_0⟩; \quad |u_4^{(6)}⟩ = |u_0⟩ ⊗ |u_2⟩; \quad |u_5^{(6)}⟩ = |u_1⟩ ⊗ |u_1⟩$$

for the finite position states and compute $r_1 = 3^{-1}(mod 2) = 3^{φ(2)−1}(mod 2) = 1(mod 2)$ and $r_2 = 2^{-1}(mod 3) = 2^{φ(3)−1}(mod 3) = 2(mod 3)$, so that $\hat{U} = \hat{U}^2 ⊗ \hat{U}^2$ and the momentum states given by:

$$|v_0^{(6)}⟩ = |v_0⟩ ⊗ |v_0⟩; \quad |v_1^{(6)}⟩ = |v_1⟩ ⊗ |v_2⟩; \quad |v_2^{(6)}⟩ = |v_0⟩ ⊗ |v_1⟩;$$

$$|v_3^{(6)}⟩ = |v_1⟩ ⊗ |v_0⟩; \quad |v_4^{(6)}⟩ = |v_0⟩ ⊗ |v_2⟩; \quad |v_5^{(6)}⟩ = |v_1⟩ ⊗ |v_1⟩.$$

Example 3 As a second example, let us consider $N = 15$ with $N_1 = 3$ and $N_2 = 5$. We can write

$$|u_0^{(15)}⟩ = |u_0⟩ ⊗ |u_0⟩; \quad |u_1^{(15)}⟩ = |u_2⟩ ⊗ |u_4⟩; \quad |u_2^{(15)}⟩ = |u_1⟩ ⊗ |u_3⟩;$$

$$|u_3^{(15)}⟩ = |u_0⟩ ⊗ |u_2⟩; \quad |u_4^{(15)}⟩ = |u_2⟩ ⊗ |u_1⟩; \quad |u_5^{(15)}⟩ = |u_1⟩ ⊗ |u_0⟩;$$

$$|u_6^{(15)}⟩ = |u_0⟩ ⊗ |u_4⟩; \quad |u_7^{(15)}⟩ = |u_2⟩ ⊗ |u_3⟩; \quad |u_8^{(15)}⟩ = |u_1⟩ ⊗ |u_2⟩;$$

$$|u_9^{(15)}⟩ = |u_0⟩ ⊗ |u_1⟩; \quad |u_{10}^{(15)}⟩ = |u_2⟩ ⊗ |u_0⟩; \quad |u_{11}^{(15)}⟩ = |u_1⟩ ⊗ |u_4⟩;$$

$$|u_{12}^{(15)}⟩ = |u_0⟩ ⊗ |u_3⟩; \quad |u_{13}^{(15)}⟩ = |u_2⟩ ⊗ |u_2⟩; \quad |u_{14}^{(15)}⟩ = |u_1⟩ ⊗ |u_1⟩.$$

for the finite position states and compute $r_1 = 5^{-1}(mod 3) = 5^{φ(3)−1}(mod 3) = 2(mod 3)$ and $r_2 = 3^{-1}(mod 5) = 3^{φ(5)−1}(mod 5) = 2(mod 5)$, so that $\hat{U} = \hat{U}^2 ⊗ \hat{U}^2$ and the finite momentum states are

$$|v_0^{(15)}⟩ = |v_0⟩ ⊗ |v_0⟩; \quad |v_1^{(15)}⟩ = |v_2⟩ ⊗ |v_2⟩; \quad |v_2^{(15)}⟩ = |v_1⟩ ⊗ |v_1⟩;$$

$$|v_3^{(15)}⟩ = |v_0⟩ ⊗ |v_1⟩; \quad |v_4^{(15)}⟩ = |v_2⟩ ⊗ |v_3⟩; \quad |v_5^{(15)}⟩ = |v_1⟩ ⊗ |v_0⟩;$$

$$|v_6^{(15)}⟩ = |v_0⟩ ⊗ |v_2⟩; \quad |v_7^{(15)}⟩ = |v_2⟩ ⊗ |v_4⟩; \quad |v_8^{(15)}⟩ = |v_1⟩ ⊗ |v_1⟩;$$

$$|v_9^{(15)}⟩ = |v_0⟩ ⊗ |v_3⟩; \quad |v_{10}^{(15)}⟩ = |v_2⟩ ⊗ |v_0⟩; \quad |v_{11}^{(15)}⟩ = |v_1⟩ ⊗ |v_2⟩;$$

$$|v_{12}^{(15)}⟩ = |v_0⟩ ⊗ |v_4⟩; \quad |v_{13}^{(15)}⟩ = |v_2⟩ ⊗ |v_1⟩; \quad |v_{14}^{(15)}⟩ = |v_1⟩ ⊗ |v_3⟩.$$

5 Modular Variables and pseudo-degrees of freedom

Consider the single degree of freedom associated with the one-dimensional motion of a particle. Note that because of equation (23), the $V_L$ and $U_{2π/L}$ operators commute for all values of $L$:

$$[V_L, U_{2π/L}] = 0 \quad (36)$$

Since these operators are unitary, their eigenvalues are necessarily complex phases. Aharonov and collaborators named the phases of the simultaneous eigenvalues of these pairs of operators as modular variables. They have shown convincingly the importance of this object for Quantum Mechanics and, in particular,
have argued in favor of the necessity of the modular variable concept to describe correctly the quantum particle interference phenomena, a problem that may be considered as paradigmatic for the general world-vision introduced by Quantum Physics [7], [8]. Consider the n-slit diffraction experiment where a beam of particles (electrons for instance) goes through an n-slit lattice and strikes a screen behind it.

![n-slit interference experiment](image)

Figure 5: n-slit interference experiment

The initial state of the incident particle is $|p(\vec{y})\rangle = |p_\xi(0)\rangle \otimes |p_\eta(\xi)\rangle$. Immediately after the interaction with the two-slit apparatus, the particle will be in a state $|\psi\rangle \otimes |p_\eta(\xi)\rangle$ where $|\psi\rangle$ is a linear combination of different moment eigenstates in the $x$ direction. This happens because the particle exchanges modular momentum with the $n$-slit screen in the $x$ direction, while leaving unperturbed the particle’s $y$ degree of freedom. So, from now on, we will concentrate only on the $x$ degree of freedom. Aharonov et al have shown that this state must be an eigenstate of both the commuting unitary translations $\hat{V}_L = e^{iL\hat{P}}$ and $\hat{U}_{2\pi/L} = e^{i\pi\hat{Q}/L}$ (equation 36). This means that the state $|\psi\rangle$ is simultaneously an eigenstate of modular momentum and modular position. They presented an example of a phase space description of such a state as the one given by figure 6 below. This state has definite values of modular position $q_{\text{mod}} = 2L/3$ and modular momentum $p_{\text{mod}} = \pi/L$. This means that the state is represented in each cell by an exact point with sharp values of $q_{\text{mod}}$ and $p_{\text{mod}}$, but there is a complete uncertainty about which cell it belongs to. This is a basic feature of the modular variable description. We propose a mathematical description of the finite analogue of this phenomenon in terms of the pseudo-degrees of freedom described in the previous section. In fact, let $W^{(N)} = W^{(N_1)} \otimes W^{(N_2)}$ be a state space for a quantum mechanical system with $\dim W^{(N)} = N = N_1 \cdot N_2$ and $MDC(N_1, N_2) = 1$. Also each individual factor space carries their finite position and momentum base states $\{|v_{j_\alpha}^{(N_\alpha)}\rangle\}$, $\{|v_{k_\alpha}^{(N_\alpha)}\rangle\}$, $(j_\alpha, k_\alpha = 0, \ldots, N_\alpha - 1)$ with $\alpha = 1, 2$ and so we may define finite base states for $W^{(N)}$ as $|u_j^{(N)}\rangle = (\hat{V}^t)^j|u_0^{(N)}\rangle$ with $|u_0^{(N)}\rangle = |u_0\rangle \otimes |u_0\rangle$, $\hat{V}^{(N)} = \hat{V} \otimes \hat{V}$ and $|v_k^{(N)}\rangle = \hat{U}^k|v_0^{(N)}\rangle$, $|v_0^{(N)}\rangle = |v_0\rangle \otimes |v_0\rangle$, $\hat{U} = \hat{U}_{r_1} \otimes \hat{U}_{r_2}$ where $r_1$ and $r_2$ are given by theorem 7. We can then offer an interpretation for this single degree of freedom of $W^{(N)}$ as a degree composed of “$N_2$ periods of size $N_1$” (or vice-versa). In fact, we may define the following state of $W^{(N)}$:

$$|j_1, k_2^{(N)}\rangle = |v_{j_1}\rangle \otimes |u_{k_2}\rangle$$ (37)
This state is simultaneously an eigenstate of finite momentum of \( W^{(N_1)} \) and finite position of \( W^{(N_2)} \) and clearly represents a finite analogue of the state represented in figure 4. We may also define the following operators

\[
\begin{align*}
\hat{U}^{N_1} &= \hat{I} \otimes \hat{U}^{r_2 N_1} \\
\hat{V}^{N_2} &= \hat{V}^{N_2} \otimes \hat{I}
\end{align*}
\] (38)

which clearly commute:

\[
[\hat{U}^{N_1}, \hat{V}^{N_2}] = 0
\] (39)

which is the finite analogue of equation (36). In fact, it is easy to compute the eigenvalues:

\[
\begin{align*}
\hat{V}^{N_2}|j_1, k_2^{(N)}\rangle &= v_{j_1, k_2}^{(N)}|j_1, k_2^{(N)}\rangle \\
\hat{U}^{N_1}|j_1, k_2^{(N)}\rangle &= v_{k_2 N_1}^{(N)}|j_1, k_2^{(N)}\rangle
\end{align*}
\] (40)

Let us illustrate this with some examples, starting again with the six-dimensional case:

**Example 4** \( N = 6, \ N_1 = 2 \) and \( N_2 = 3 \). we can represent the state \( |1, 2^{(6)}\rangle = |v_1\rangle \otimes |u_2\rangle \) in the finite phase space given by figure 7 below.

**Example 5** Let us consider the \( N = 15 \) case with \( N_1 = 3, \ N_2 = 5 \) and the ket \( |1, 2^{(15)}\rangle = |v_1\rangle \otimes |u_2\rangle \) represented in the finite phase space given by figure 8 below.

In this case the finite momentum degree of freedom is composed of 5 periods of size 3 and the finite position degree of freedom is composed of 3 periods of size 5.
States with these peculiar mathematical structure have been described independently by Zak to study systems with periodic symmetry in quantum mechanics \[9, 10\]. We may call them Aharonov-Zak states (AZ).

This AZ state can be thought as obtained by an ideal projective measurement of modular variables. In fact, the AZ state \(|\psi\rangle\) can be obtained starting from \(|p_x(0)\rangle\) (we only consider the \(x\) degree of freedom) and the hamiltonian

\[
\hat{H}(t) = \frac{\hat{P}_x^2}{2m} + V(\hat{Q})\delta(t) \quad \text{with} \quad V(\hat{Q} + L) = V(\hat{Q}) \quad (41)
\]

where the particle "hits the screen" at \(t = 0\) so that the time evolution is given by \(e^{-iV(\hat{Q})|p_x(0)\rangle}\).
Expanding $e^{-i\hat{V}(\hat{Q})}$ in a Fourier series gives us

$$e^{-i\hat{V}(\hat{Q})} = \sum_n c_n e^{2\pi in\hat{Q}} = \sum_n c_n \hat{U} \frac{2\pi n}{L}$$

(42)

so that $e^{-i\hat{V}(\hat{Q})}|p_\tau(0)\rangle$ is clearly an eigenstate both of $\hat{U} \frac{2\pi n}{L}$ and $\hat{V}_L$. This mathematical structure is behind the non-locality involved in the n-slit interferometric experiment as thoroughly discussed in [8].

The Continuum limit of the AZ state can be constructed in the following manner: the position and momentum basis of the subsystem $W^{(N_a)}$ are scaled in the non-symmetric way

$$|q(x)\rangle = \left(\frac{1}{L}\right)^{1/2} |u^{(N_a)}_j\rangle \quad \text{and} \quad |p(y)\rangle = \left(\frac{N_a L}{2\pi}\right)^{1/2} |v^{(N_a)}_k\rangle$$

(43)

with

$$x_j = L j \quad \text{so that} \quad \Delta x_j = L$$

$$y_k = \frac{2\pi}{L N_a} k \quad \text{so that} \quad \Delta y_k = \frac{2\pi}{L N_a}$$

and in a similar way (but with the opposite construction) we have for $W^{(N_b)}$

$$|q(x)\rangle = \left(\frac{N_b}{L}\right)^{1/2} |u^{(N_b)}_\sigma\rangle \quad \text{and} \quad |p(y)\rangle = \left(\frac{L}{2\pi}\right)^{1/2} |v^{(N_b)}_\lambda\rangle$$

(45)

with

$$x_\sigma = \frac{L}{N_b} \sigma \quad \text{so that} \quad \Delta x_\sigma = \frac{L}{N_b}$$

$$y_\lambda = \frac{2\pi}{L} \lambda \quad \text{so that} \quad \Delta y_\lambda = \frac{2\pi}{L}$$

And the unitary translation operators:

$$\hat{V}_{x_j}^{(N_a)} |q(x)\rangle = |q(x_j - x_j)\rangle \quad \hat{V}_{x_j}^{(N_b)} |p(y)\rangle = e^{ix_j y_k} |p(y)\rangle$$

(47)

$$\hat{U}_{y_j}^{(N_a)} |q(x)\rangle = e^{ix_j y_j} |q(x)\rangle \quad \hat{U}_{y_j}^{(N_b)} |p(y)\rangle = |p(y + y_j)\rangle$$

(48)

$$\hat{V}_{x_\lambda}^{(N_a)} |q(x)\rangle = |q(x_\lambda - x_\lambda)\rangle \quad \hat{V}_{x_\lambda}^{(N_b)} |p(y)\rangle = e^{ix_\lambda y_r} |p(y)\rangle$$

(49)

$$\hat{U}_{y_r}^{(N_a)} |q(x)\rangle = e^{ix_\lambda y_r} |q(x)\rangle \quad \hat{U}_{y_r}^{(N_b)} |p(y)\rangle = |p(y_r + y_\lambda)\rangle$$

(50)

with identity operators

$$\hat{I}^{(N_a)} = \int_0^L dy |p(y)\rangle \langle p(y)| = L \sum_{j=0}^\infty |q(x_j)\rangle \langle q(x_j)|$$

(51)

$$\hat{I}^{(N_b)} = \int_0^L dx |q(x)\rangle \langle q(x)| = 2\pi \sum_{\tau=0}^\infty |p(y_\tau)\rangle \langle p(y_\tau)|$$

(52)
The continuum limit is then obtained asymptotically with both $N_a, N_b \rightarrow \infty$.

The finite analogue of the ideal state represented in fig.5 (the Dirac comb state) can be understood as obtained through an ideal projective measurement carried on by the n-slit apparatus on the incident particle (in the $x$ degree of freedom):

$$|v_0\rangle \otimes |v_0\rangle \xrightarrow{\text{measurement}} |v_0\rangle \otimes |u_0\rangle$$

(53)

Note that the first subspace is left untouched while the second subspace is projected to a position eigenstate. This state can be expanded in the following two ways:

$$|v_0\rangle \otimes |u_0\rangle = \frac{1}{\sqrt{N}} \sum_j |v_0\rangle \otimes |v_j\rangle = \frac{1}{\sqrt{N}} \sum_j |u_j\rangle \otimes |u_0\rangle$$

(54)

The first expansion allows us to read the state as a sum over many momentum states and the second expansion is precisely the finite analogue of the Dirac comb state as shown in the figure below:

![Figure 9: Dirac Comb](image)

6 Conclusion

With the advent of Quantum Information Theory, the foundations of Quantum Mechanics have come back to the main stage of Physics after decades where this kind of discussion was almost abandoned to what most physicists saw as a more philosophical kind of concern. Our feeling is that many concepts - like modular variables, the two-state formalism and weak values developed by Aharonov and many important collaborators are of major importance to help clarify the understanding of Quantum Physics and Quantum Information and in particular to the comprehension of the still very elusive phenomena of quantum non-locality [11], [12], [13]. In [8], the authors have addressed the foundational problem of interference of a particle with a two-slit or multi-slit apparatus. They offer an explanation of the inherent non-locality in this experiment in terms of modular variables. How can one conceive that the electron passing through one slit “knows” that another slit is open or has been closed by a capricious
experimentalist? Their answer is analogous to the one given in [1] for the AB effect. Instead of thinking of this experiment as a non-local Schrödinger (physically fictitious) wave function interacting locally with the slits they prescribe an ontology where localized particles interact non-locally (by exchanging modular momentum with the slit screen) in a Heisenberg representation. And though this non-local interaction cannot violate causality, with the concept of weak measurements and weak values, it should be possible to observe in a certain sense, the modular variable exchange. They also affirm that this non-locality is dynamical and should not be confused with the more kinematic kind of non-locality that happens with EPR like experiments where the non-locality arises because of the entanglement of the common state of two distant particles. In fact, they discuss this issue entirely in a Heisenberg-picture framework. We believe that our analysis implies that, in a certain sense, both kinds of non-locality arise from the same kind of tensor product space, that can be carried-out explicitly in a Schrödinger-picture, so that these apparently different kind of non-local quantum phenomena may not be so unsimilar after all. Recently it has come to our knowledge that the relation between Schwinger’s Finite QM and modular variables has been noticed before [14].

We intend to conduct further investigations on this issue and also on the possibility to extend the above analysis to the modular energy concept [7]. We expect difficulties with this last task because of the lack of symmetry between time and energy in the usual formulation of Quantum Mechanics if compared with the perfectly symmetric roles played by position and momentum. The power and flexibility of finite quantum mechanics may be of great help here. The origin of these ideas may be mostly credited to Schwinger, but further investigations of the mathematical structure of finite phase spaces have been conducted by many authors since then. See [15] and [16] for some early results on this subject and also [17] for some more recent developments.

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