On the imbalance lattice of path-length sequences of binary trees

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Abstract

The existence of greatest lower bounds in the imbalance order of path-length sequences of binary trees is seen to be a consequence of a joint monotonicity property of the greater and lower expansion operations. Path length sequences that are join-irreducible in the imbalance lattice are characterized.

1 Introduction, terminology, notation

Generally the framework and terminology of Stott Parker and Prasad Ram [SPPR] is followed in what follows.

For any sequence $x = (x_1, ..., x_n)$ of real numbers, $n \geq 1$, we write $\exp x$ for the sequence $(2^{-x_1}, ..., 2^{-x_n})$, we write $\Sigma x$ for the sum $x_1 + ... + x_n$, and last $x$ (resp first $x$) for $x_n$ (resp. $x_1$). We also write $Sx$ for the sequence of partial sums $(x_1, x_1 + x_2, ..., x_1 + ... + x_n)$. The suffix length $\text{suf } x$ of $x$ is the largest integer $k \leq n$ such that the last $k$ components of $x$ are equal. A sequence $x$ of non-negative integers is a path-length sequence if $x_1 \leq ... \leq x_n$ and $\Sigma \exp = 1$. By Kraft’s Theorem, this means that there is binary tree whose root-to-leaf paths are in order of increasing lengths from left to right, the lengths being $x_1, ..., x_n$ in that order [K]. Path-length sequences have even suffix length. Between path-length sequences $l$ and $h$ with the same number of components, $l$ is said to be more balanced than $h$, in symbols $l \leq h$, if $S \exp l \leq S \exp$ in the componentwise order of vectors. This defines a partial order relation on the set of path-length sequences with a given number $n$ of components.

For a path-length sequence $l = (l_1, ..., l_n)$, the expansion in position $i$ is defined as the path-length sequence $(l_1, ..., l_{i-1}, l_i + 1, l_i + 1, l_{i+1}, ..., l_n)$, for any $1 \leq i \leq n$, the upper expansion $l^+$ is the expansion in position $n$, while the
lower expansion $l_+$ is the expansion in position $\max(1, n - \text{suf } l)$. Thus lower expansion is defined even for constant sequences, and a sequence is constant if and only if its lower and upper expansion coincide. Also $l \leq h$ implies $l_+ \leq h_+$ in all cases, just as it implies $l^+ \leq h^+$.

For path-length sequences with $n \geq 2$ components and suffix length $k$, the contraction $\hat{l} = (l_1, \ldots, l_{n-k}, l_{n-k+1} - 1, l_n, \ldots, l_n)$ is defined, it has $n - 1$ components, and it is also a path-length sequence, satisfying $\hat{l}_+ \leq l \leq \hat{l}^+$.  

2 Uniqueness of the lattice meet

Stott Parker and Prasad Ram [SPPR] state the fact that the imbalance order on the set of path-length sequences with a given number of components is a lattice. We show that this is can be verified as a consequence of the Lemma below. It is of course enough to prove that the imbalance order on path-length sequences $(l_1, \ldots, l_n)$ with $n$ components is a meet semilattice.

Observe first that for any path-length sequences if $l \leq h$ then last $l \leq$ last $h$, and if last $l = $ last $h$ then $\text{suf } l \leq \text{suf } h$.

Lemma. For path-length sequences $l \leq h$, if last $l <$ last $h$ then $l^+ \leq h_+$.  

Proof. Let $l = (l_1, \ldots, l_n)$ and $h = (h_1, \ldots, h_n)$. If the suffix length of $h$ is $2k$, then for $i = n - 2k$ we have $h_1 \leq \ldots \leq h_i < h_{i+1} = \ldots = h_n > l_n \geq \ldots \geq l_{i+1}$.

Recall that $l^+ = (l_1, \ldots, l_{n-k}, l_{n-k+1}, l_{n-1+1})$ and $h_+ = (h_1, \ldots, h_{i-1}, h_i + 1, h_i + 1, h_{i+1}, \ldots, h_n)$.

We have

$$2^{-h_i} \leq \Sigma \exp(h_{i+1}, \ldots, h_n) \leq \frac{1}{2} \Sigma \exp(l_{i+1}, \ldots, l_n) \leq \Sigma \exp(l_{i+1}, \ldots, l_{n-k})$$

It can be deduced that $\Sigma \exp(h_1, \ldots, h_{i-1}, h_i + 1) > \Sigma \exp(h_1, \ldots, h_{i-1}) \geq \Sigma \exp(l_1, \ldots, l_{i-1}, l_i)$.

Also, for every $2 \leq j \leq 2k - 1$, we have $\Sigma \exp(h_{n-j+1}, \ldots, h_n) \leq \Sigma \exp(l_{n-j+1}, \ldots, l_n) = \Sigma \exp(l_{n-j+2}, \ldots, l_n + 1, l_{n} + 1)$.  

It follows that $l^+ \leq h_+$.  

□
Then let us prove by induction on $n$ the following:

**Proposition 1** Path-length sequences with the same number of components that is at most $n$ always have a greatest lower bound (meet) in the imbalance order, for which last $(s \land t) = \min(\text{last } s, \text{last } t)$. For any such sequences $s$ and $t$ with at least two components we have $s \land t = (\widehat{s} \land \widehat{t})^+$ if $(\widehat{s} \land \widehat{t})^+ \leq s, t$, otherwise $s \land t = (\widehat{s} \land \widehat{t})_+$.

**Proof.** The statement is obvious for $n = 1$. The inductive step from $n - 1$ to $n$ is as follows.

**Case 1:** $(\widehat{s} \land \widehat{t})^+ \leq s, t$.

To show that $(\widehat{s} \land \widehat{t})^+$ is the greatest lower bound of $s, t$, let $l \leq s, t$. Then $l \leq s, t$ and thus $\widehat{l} \leq \widehat{s} \land \widehat{t}$. From this $l \leq (\widehat{l})^+ \leq (\widehat{s} \land \widehat{t})^+$.

From $(\widehat{s} \land \widehat{t})^+ \leq s, t$ we get

$$\min(\text{last } \widehat{s}, \text{last } \widehat{t}) + 1 \leq \min(\text{last } s, \text{last } t)$$

If last $s = \text{last } t$ then suf $s$ or suf $t$ is 2, because if both were larger, then last $\widehat{s} = \text{last } s$ and last $\widehat{t} = \text{last } t$, contradicting the above inequality.

But then last $\widehat{s} + 1 = \text{last } s$ or last $\widehat{t} + 1 = \text{last } t$, and by the inductive hypothesis last $(\widehat{s} \land \widehat{t}) = (\text{last } s) - 1 = (\text{last } t) - 1$ and last $(\widehat{s} \land \widehat{t})^+ = \text{last } s = \text{last } t = \min(\text{last } s, \text{last } t)$.

If last $s < \text{last } t$ then suf $s$ is 2 because otherwise last $\widehat{s} = \text{last } s \leq \text{last } \widehat{t}$ and (last $s) + 1 = (\text{last } \widehat{s}) + 1 = \text{last } (\widehat{s} \land \widehat{t})^+ \leq \text{last } s$, which is impossible. Also last $\widehat{s} < \text{last } \widehat{t}$.

And also (last $\widehat{s}) + 1 = \text{last } s$ implying last $(\widehat{s} \land \widehat{t})^+ = \text{last } (\widehat{s} \land \widehat{t}) + 1 = \min(\text{last } \widehat{s}, \text{last } \widehat{t}) + 1 = (\text{last } \widehat{s}) + 1 = \text{last } s = \min(\text{last } s, \text{last } t)$.

**Case 2:** $(\widehat{s} \land \widehat{t})^+ \not\leq s$ or $(\widehat{s} \land \widehat{t})^+ \not\leq t$.

Certainly still $(\widehat{s} \land \widehat{t})_+ \leq s, t$. Note that in this case $(\widehat{s} \land \widehat{t})$ cannot be constant.

By the induction hypothesis last $(\widehat{s} \land \widehat{t})_+ = \text{last } (\widehat{s} \land \widehat{t}) = \min(\text{last } \widehat{s}, \text{last } \widehat{t})$. 

3
Subcase 2.1: $last s \neq last t$
Without loss of generality, we may suppose that $last s < last t$.

We claim that $suf s > 2$. For if $suf s = 2$ then $s = (\hat{s})^+$ and $(\hat{s} \wedge \hat{t})^+ \not\leq s$ but $(\hat{s} \wedge \hat{t})^+ \not\geq t$. Also $last \hat{s} < last \hat{t}$, and thus $last (\hat{s} \wedge \hat{t}) = last \hat{s}$ is less then $last \hat{t}$,

implying $(\hat{s} \wedge \hat{t})^+ \not\leq (\hat{t})_+$ by the Lemma, a contradiction proving that $suf s > 2$.

Clearly then $last (\hat{s} \wedge \hat{t}) = \min(last \hat{s}, last \hat{t}) = last \hat{s} = last s$.

Subcase 2.2: $last s = last t$

Now the suffixes of both $s$ and $t$ cannot be 2, because in that case $(\hat{s} \wedge \hat{t})^+ \not\leq s, t$.

If one of the suffix lengths, say $suf s$ were 2, then $last (\hat{s} \wedge \hat{t}) = \min(last \hat{s}, last \hat{t}) = last \hat{s} < last \hat{t}$. Then $(\hat{s} \wedge \hat{t})^+ \not\leq (\hat{s})^+ = s$ and by the Lemma

$(\hat{s} \wedge \hat{t})^+ \not\leq (\hat{t})_+ = t$,
a contradiction.

Thus both suffix lengths are greater than 2, we have $(\hat{s})_+ = s$ and $(\hat{t})_+ = t$ and $last (\hat{s} \wedge \hat{t}) = \min(last \hat{s}, last \hat{t}) = last \hat{s} = last t = last \hat{t} = last t$.

In conclusion, in both subcases, if $last s \leq last t$ then $last (\hat{s} \wedge \hat{t}) = \min(last \hat{s}, last \hat{t}) = last \hat{s} = last s$.

We now return to the general conditions of Case 2. Again, without loss of generality, we may suppose that $last s \leq last t$.

Let $l \not\leq s,t$. Obviously $\hat{l} \not\leq \hat{s} \wedge \hat{t}$ and $last \hat{l} \leq last (\hat{s} \wedge \hat{t})$.

If $last \hat{l} < last (\hat{s} \wedge \hat{t})$ then by the Lemma $l \not\leq (\hat{l})_+ \leq (\hat{s} \wedge \hat{t})_+ .$

If $last \hat{l} = last (\hat{s} \wedge \hat{t})$ then $last l \geq last \hat{l} = last (\hat{s} \wedge \hat{t}) = last \hat{s} = last s$ and, since $l \not\leq s$ implies $last l \leq last s$, we have $last l = last s = last \hat{l}$. This means that the suffix length of $l$ is also greater than 2 and $l = (\hat{l})_+ \leq (\hat{s} \wedge \hat{t})_+$.

\[\Box\]

3 Join-irreducible path-length sequences

Stott Parker and Prasad Ram have shown [SPPR] that for path-length sequences the imbalance order relation $\not\leq$ is the transitive-reflexive closure the minimal balancing relation which can be defined as follows. For any path length sequence $l = (l_1, ..., l_n)$ call an index $1 < j < n$ an excess index if
\[ l_{j-1} < l_j = l_{j+1} \] and there is an index \( i \) such that \( l_i \leq l_j - 2 \). For every excess index \( j \) of \( l \) consider the last index \( i \) such that \( l_i \leq l_j - 2 \), and let \( bal[l, j] \) be the path-length sequence obtained from \( l \) by replacing the last occurrence of \( l_i \) with two consecutive entries equal to \( l_i + 1 \) and replacing the first two occurrences of \( l_j \) by a single entry equal to \( l_j - 1 \). The minimal imbalance relation, on the set of path-length sequences \( l \) with \( n \) components, is

\[ \{ ( bal[l, j], l ) : j \text{ is an excess index of } l \} \]

The minimal imbalance relation contains (generally properly) the covering relation of the partial order \( \leq \).

Recall that an element of a finite lattice is \textit{join-irreducible} if it covers a unique element of the lattice (its \textit{unique lower cover}). The following is not difficult to verify:

**Proposition 2** Let \( j \) be the first excess index of a path-length sequence \( l \). Then \( l \) is join-irreducible if and only if for all excess indices \( k \) of \( l \) we have \( bal[l, j] \sqsupseteq bal[l, k] \).

We call a sequence of integers \textit{near-constant} if it contains at most two distinct values, and the difference between these is 1.

**Proposition 3** A path-length sequence \( l = (l_1, ..., l_n) \) is join-irreducible if and only if it is the concatenation of 3 sequences, \( l = uvw \) such that

(i) both \( u \) and \( w \) are near-constant, and \( v \) is strictly increasing,

(ii) \( w \) is not the empty sequence.

(iii) if \( w \) is not constant but its first two components are equal, than the value of these components is at least \((\text{last } uv) + 2 \).

**Proof** The conditions are easily seen to be sufficient for irreducibility.

Suppose on the other hand that \( l \) is irreducible. It is useful to keep in mind that there is a topological tree whose path-length sequence is \( l \). Obviously \( l \) is not near-constant. Let \( u \) be the longest near-constant prefix of \( l \), and \( z \) its complement, \( l = uz \). Let \( w \) be the longest near-constant post-fix of \( z \) and \( v \) its complement, \( z = vw \). Clearly \( w \) cannot be empty and, because \( l = uvw \) is a path-length sequence, \( w \) ends with an even number of equal components, and the index of the first one of these is an excess index \( k \). Also \( uv \) cannot be empty, as required by condition (ii).
If \( v \) had any repeated components, the index \( j \) of the first of the first two repeated components would be the first excess index of \( l \). Then, by Proposition 2, \( l \triangleright \text{bal}[l, j] \triangleright \text{bal}[l, k] \) would have to hold. In the tree corresponding to a given path-length sequence, consider the number of nodes that are on some root-to-leaf path of length at most \( l \). This parameter is monotone, it can never decrease as we go down in the imbalance lattice, but for \( \text{bal}[l, k] \) it is the same as for \( l \) itself. However, for \( \text{bal}[l, j] \) the parameter is actually higher than for \( l \). This shows that \( v \) is strictly increasing, thus condition (i) also holds.

Finally, if condition (iii) failed, the first indices of the two constant runs of components in \( w \), say \( i \) and \( k \) in that order, would both be excess indices, and we would have \( l_i = (\text{last } uv) + 1 \). For the first excess index \( j \) we would have to have \( j \leq i < k \) and thus \( l \triangleright \text{bal}[l, j] \triangleright \text{bal}[l, k] \). Consider now the sum of components of a path-length sequence. This parameter, of integer value, is strictly monotone in the imbalance lattice, it always decreases as we go down [SPPR]. But from \( l \) to \( \text{bal}[l, k] \) it decreases only by 1. This shows that (iii) must also hold.

\[ \square \]

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