Kinetic damping in the spectra of the spherical impedance probe

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Abstract

The impedance probe is a measurement device to measure plasma parameters, such as electron density. It consists of one electrode connected to a network analyzer via a coaxial cable and is immersed into a plasma. A bias potential superposed with an alternating potential is applied to the electrode and the response of the plasma is measured. Its dynamical interaction with the plasma in an electrostatic, kinetic description can be modeled in an abstract notation based on functional analytic methods. These methods provide the opportunity to derive a general solution, which is defined by the matrix elements of the resolvent of an appropriate dynamical operator. Based on the general solution, a residual damping for vanishing pressure can be predicted and can only be explained by kinetic effects. In this paper, an explicit response function of the spherical impedance probe is derived. Therefore, the resolvent is determined by its algebraic representation based on an expansion in orthogonal basis functions. This allows one to compute an approximated response function and its corresponding spectra. These spectra show additional damping due to kinetic effects and are in good agreement with former kinetically determined spectra.

Keywords: active plasma resonance spectroscopy, impedance probe, kinetic theory, functional analysis, multipole resonance probe, kinetic damping

1. Introduction

Initial investigations of the so called resonance probe (RP) go back to the year 1960 and were conducted by Takayama et al [1]. They applied a negative bias potential superposed with a small alternating potential at a planar disk electrode immersed into a plasma, and then measured the current. Sweeping the frequency of the alternating potential, they observed a resonance phenomenon close to the electron plasma frequency $\omega_p$. Further theoretical and experimental works showed that the resonance frequency $\omega_r$ has to be below the electron plasma frequency [2, 3].

In subsequent years, the RP was intensively investigated both experimentally and theoretically. Many researchers have attempted this task, especially the RP with a spherical electrode [4–15]. The theoretical works of the cited papers share a common feature in that the underlying models are based on an electrostatic approximation, but the plasma description is of a different complexity. They range from cold and warm fluid models to kinetic descriptions.

An extensive kinetic analysis was conducted by Buckley [8]. He derived and solved an integral equation of the alternating electric field to determine the impedance and the admittance as response functions of the excited plasma. The calculated and measured spectra of the admittance were in good agreement with measurements for certain moderate collision frequencies $\nu_0$ [10]. Besides the collisional damping, the spectra also showed collisionless damping due to kinetic effects, which was found to be dominant for $\nu_0 \lesssim 0.1 \omega_p$ [11]. However, the physical mechanism of the kinetic damping was not explained.

Several years later, Morin and Balmain compared the kinetic admittances of Buckley with admittances determined by a fluid model [16]. The resonance frequencies were in good agreement for both: a continuous equilibrium density profile and a single step density profile. A meaningful difference was observed in the half width of the resonance peaks due to kinetic damping, but they also did not explain its physical mechanism.
Within the last decade, the spherical RP gained new interest and is now referred to as a spherical impedance probe (sIP). Two slightly different designs are proposed, analyzed, and characterized [17, 18]. The sIP is also discussed in the context of industry compatible plasma diagnostics [19], but is still not a standard tool in industry. One possible reason for that might be the kinetic influence on the spectra, which is not yet fully understood.

It is also of interest to study the generic features of such probes independently of any particular realization. Using methods of functional analysis, a general investigation of such probes in electrostatic approximation is given in [20]. Based on the cold plasma model, the main result was that, for any possible probe design, the spectral response function could be expressed as a matrix element of the resolvent of the dynamical operator.

A fully kinetic generalization of the study of [20], i.e. an abstract kinetic model of electrostatic resonance valid for all pressures, is presented in [21]. It turned out that the main result could be directly transferred. In particular, it still holds that, for any possible probe design, the spectral response of the probe–plasma system can be expressed as a matrix element of the resolvent of the dynamical operator. Furthermore, it was shown that the corresponding resonances exhibit a residual damping in the limit of vanishing pressure which cannot be explained by Ohmic dissipation but only by kinetic effects. An analysis of the abstract model allows one to interpret this kinetic damping as loss of kinetic free energy \( \delta \). The free energy is produced by the probe, transported through the plasma to a large distance, where the probe is unable to detect it. This loss of free energy is recorded in the spectrum of the probe as damping.

Functional analytic methods are very useful to study generic features, but they can also be applied to determine explicit spectra for a specific probe design. In spherical geometry, it is possible to solve the analytic solution of the fluid dynamical response, which was applied to the idealized sIP and the multipole resonance probe (MRP), respectively [22]. In the kinetic description, an analytic solution is impossible, but an efficient algorithm can be derived to determine an adequate approximation. Such an algorithm is applied to the parallel electrode probe (PEP) [23], which is not used for real measurements. It is meant as a toy model because it provides the simplest available geometry. The calculated spectra of the PEP show kinetic damping as predicted by the general analysis.

However, the predicted kinetic damping from the general model in the functional analytic description is not verified for an existing probe design and is not yet quantitatively determined. In this paper, the author focuses on the idealized sIP and determines its approximated response function by means of functional analytic methods. Therefore, he follows the approximation algorithm presented in [23] and compares the spectra with the spectra of Buckley [11]. It will be shown that both are in good agreement: the spectra of the admittance and impedance.

![Figure 1. Illustration of the idealized spherical impedance probe with the powered electrode \( \mathcal{E} \), the dielectric \( D \), and the perturbed plasma \( \mathcal{P} \). The radius of the electrode is \( R - d \) and the thickness of the dielectric is \( d \).](image-url)

2. Model of the idealized sIP

As depicted in figure 1, the idealized sIP consists of one spherical electrode \( \mathcal{E} \) of radius \( R - d \). In a general case, it can be surrounded by a dielectric \( D \) of thickness \( d \). Applying an RF voltage \( V \) at the electrode, the plasma will be dynamically disturbed in the surrounding of the probe. Then \( \mathcal{P} \) is the disturbed plasma and \( \mathcal{V} = \mathcal{P} \cup \mathcal{D} \) the influence domain of the probe. The former interface \( \mathcal{F} \) between the perturbed and unperturbed plasma, which was presented in the general model of active plasma resonance spectroscopy (APRS) [21], is treated as a grounded spherical surface at a large distance \( R_\infty \)—theoretically in an infinite distance.

In general, the IP belongs to the class of APRS probes which excite surface wave modes of long electrostatic surface waves along a homogeneous plasma boundary sheath thickness \( \delta \). These modes occur below \( \omega_p \), which represents the cut-off frequency of the probes’ electromagnetic radiation. Thus, the interaction between the probe and the plasma is dominated by the corresponding electric field, and the dynamical behavior of the probe–plasma system in \( \mathcal{P} \) can be described by the linearized and normalized Boltzmann equation in electrostatic approximation. In the geometry of the idealized sIP, the six-dimensional distribution function reduces to three dimensions due to symmetry. It depends on the radial distance \( r \in [R, R_\infty] \), the absolute value of the velocity \( \nu \in [0, \infty) \), and the projection angle \( \chi \in [0, \pi] \) of \( \nu \) to the \( r \) direction and is given by

\[
\frac{\partial g}{\partial t} + \nu \cos(\chi) \frac{\partial g}{\partial r} - \frac{\partial \Phi}{\partial r} + \frac{\partial \Phi}{\partial r} \cos(\chi) \frac{\partial g}{\partial \nu} - \frac{\sin(\chi)}{\nu} \frac{\partial g}{\partial \chi} = \frac{\nu}{r} \int_0^\pi \cos(\chi) d\chi - \nu_0 g.
\]

The perturbed distribution function \( g \) of the electrons is described in \( \mathcal{P} \) with homogeneous boundary conditions at the surface of the probe \( g(R, \nu, \chi, t) = 0 \) and the outer grounded surface \( g(R_\infty, \nu, \chi, t) = 0 \). Pure elastic collisions with a
constant collision frequency \( \nu_0 \) are taken into account between electrons and the neutral background.

\( \Phi \) is the inner potential. It is a linear functional of \( g \) and obeys Poisson’s equation

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \varepsilon(r) \frac{\partial \Phi}{\partial r} \right) = \frac{0}{2\pi} \int_0^{2\pi} \int_0^\infty w g \sin(\chi) v^2 \, dv \, d\chi \quad r \in \mathcal{D}
\]

with homogeneous boundary conditions \( \Phi(R - d) = \Phi(R_{\infty}) = 0 \). \( \varepsilon(r) \) is the dielectric constant and is defined as \( \varepsilon(r) = \varepsilon_D \) in \( \mathcal{D} \) and \( \varepsilon(r) = 1 \) in \( \mathcal{P} \). In addition, \( w \) is a positive weighting function. It is defined as the negative derivative of the equilibrium distribution \( F(\epsilon) \) with respect to the total energy \( \epsilon = \frac{1}{2} v^2 - \Phi \) in equilibrium. Assuming a Maxwellian distribution, \( w \) is equal to

\[
w(r, v) = \frac{1}{(2\pi)^3} e^{-\frac{v^2}{2} + \Phi(r)},
\]

where \( \Phi \) is the equilibrium potential.

The radio frequency excitation of the probe is represented by the electrode function \( \Psi(r) \). The following Laplace’s equation

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Psi^{(D)}(r)}{\partial r} \right) = 0
\]

and fulfills the boundary conditions \( \Psi^{(D)}(R - d) = 1 \) and \( \Psi^{(D)}(R_{\infty}) = 0 \) and the transition conditions \( \Psi^{(D)}(R) = \Psi^{(D)}(R_{\infty}) \) and \( \Psi^{(D)}(R) = \Psi^{(D)}(R_{\infty}) \). The solutions are easily determined

\[
\Psi^{(D)}(r) = \frac{(d - R)(R_{\infty}(\varepsilon_D - 1) - R \varepsilon_D) + R \varepsilon_D}{\varepsilon_D (d - R)(R_{\infty} - R) - drR_{\infty}},
\]

\[
\Psi^{(P)}(r) = \frac{R \varepsilon_D (R - d)(R_{\infty} - R) - drR_{\infty}}{R \varepsilon_D (d - R)(R_{\infty} - R) - drR_{\infty}}.
\]

### 3. Inner admittance in functional analytic description

Since the model of the idealized sIP is defined, the results of the general analysis presented in [21] can be applied. Thus, the vacuum current \( i_{\text{vac}} \), which is present even without plasma, and the inner current \( i_{\text{in}} \):

\[
I = i_{\text{vac}} + i_{\text{in}} = (Y_{\text{vac}} + Y) U = Y_{\text{IP}} U.
\]

The vacuum admittance \( Y_{\text{vac}} \) is determined by the characteristic function \( \Psi(r) \) and is defined as

\[
Y_{\text{vac}} = -4\pi \varepsilon_D (R - d)^2 i \omega \frac{\partial}{\partial r} \Psi^{(D)} \bigg|_{r=R-d}.
\]

\( i_{\text{in}} \) can be written in a Hilbert space notation as

\[
i_{\text{in}} = \langle e \vert (i\omega - T_V - T_0) \vert e \rangle U \equiv Y U.
\]

The inner admittance \( Y \) is determined by the scalar product of the excitation and observation vector \( e = \nu \cos(\chi) \frac{\partial}{\partial r} \psi \) and the resolvent of the dynamical operator \( T_V + T_0 \)

\[
Y = \langle e \vert (i\omega - T_V - T_0)^{-1} e \rangle.
\]

\( T_V \) and \( T_0 \) are the Vlasov and the collision operator, respectively. They, applied to a dynamical state vector \( g \), are defined as follows:

\[
T_V g = \nu \cos(\chi)(\frac{\partial g}{\partial r} - \frac{\partial \Phi}{\partial r}) + \frac{\partial \Phi}{\partial r} \frac{\sin(\chi)}{\nu} \frac{\partial g}{\partial \chi},
\]

\[
T_0 g = \nu_0 \int_0^\pi \int_0^\infty g \sin(\chi) \, dv \, d\chi,
\]

Of particular importance in the functional analytic description is the scalar product, which is used in (10). To allow for physical interpretations, it has to be connected to the system dynamics and is motivated by the kinetic free energy \( \mathcal{F} \).

Generally, the scalar product of the two dynamical state vectors \( g' \) and \( g \) is given by

\[
\langle g' \vert g \rangle = \langle g' \vert g \rangle v + \langle g' \vert g \rangle v
\]

\[
= \sqrt{2} \int_R^{R_D} \int_0^\pi \int_0^\infty g' g (v^2 \sin(\chi) v^2 - 2) dv \, d\chi \, dr + 4\pi \int_{R-D}^{R-D} \frac{\partial g}{\partial r} \frac{\partial g^*}{\partial r} v^2 \, dr.
\]

Based on this scalar product, \( Y \) can be expanded by means of a complete orthonormal basis \( \{ a \} \) of the Hilbert space. Introducing the corresponding completeness relation twice into equation (10) yields

\[
Y = \sum_{a'} \langle e \vert a' \rangle \sum_a \langle a' \vert (i\omega - T_V - T_0)^{-1} a \rangle \langle a \vert e \rangle.
\]

This equates to a vector–matrix–vector multiplication which is determined by the algebraic representation of the resolvent. The algebraic representation of the resolvent can be calculated by the inverse of the algebraic representation of \( i\omega - T_V - T_0 \) [22].

### 4. Explicit expansion of the inner admittance

In the previous section, the general expansion of \( Y \) is shown. Now, one can follow the algorithm presented in [23] to determine an explicit expansion for the sIP with finite dimension

\[
Y = e^T \cdot (i\omega \lambda - \mathbf{T}_V - \mathbf{T}_0)^{-1} \cdot e.
\]

The three matrices in (15) are: the identity matrix \( I \), the collision matrix \( \mathbf{T}_c \), and the Vlasov matrix \( \mathbf{T}_V \). \( e \) is the explicitly expanded excitation vector.

To determine the identity matrix, a complete orthonormal basis is required. An appropriate basis function in velocity space is given by

\[
g_{\nu}^{\lambda} \nu(v, \chi) = \pi^4 \Lambda_{\nu}^\lambda(v) \mathcal{P}_\nu(\cos(\chi)).
\]

\( \lambda \in \mathbb{N}_0 \) is the expansion index for the projection angle and \( \nu \in \mathbb{N}_0 \) for the absolute value of the velocity. \( \mathcal{P}_\nu(\cos(\chi)) \) are the normalized Legendre polynomials, which are orthonormal
and complete on the interval $\chi \in [0, \pi]$
\[
P_\chi(\chi) = \sqrt{\frac{2\lambda + 1}{2}} P_\chi(\cos(\chi)).
\] (17)

$\Lambda_0^\nu(v)$ are based on the generalized Laguerre polynomials $L_{\kappa + \frac{1}{2}}(\frac{1}{2}v^2)$. Due to the exponential part in the weighting function $w$, they are an adequate choice on the interval $v \in [0, \infty)$. They become the orthonormal functions $\Lambda_0^\nu(v)$ with an additional factor
\[
\Lambda_0^\nu(v) = \sqrt{\frac{\kappa!}{\Gamma(\kappa + \frac{1}{2})}} \left(\frac{v^2}{2}\right)^{\kappa/2} L_{\kappa+\frac{1}{2}}(\frac{v^2}{2}).
\] (18)

In physical space, it is difficult to determine an orthogonal function due to the two different parts of the scalar product. Therefore, the focus is placed on the first part $(g'_k | g)_T$ of (13). An appropriate basis function can be defined as
\[
g_k^k(r) = \frac{\sin(k \pi - R/R_\infty - R)}{\sqrt{2\pi} e^{k\varphi(r)}(R_\infty - R)} r
\] (19)
with $k \in \mathbb{N}$. It is orthonormal and complete on the interval $[R, R_\infty]$ and fulfills the boundary conditions $g_k^k(R) = g_k^k(R_\infty) = 0$. In summary,
\[
g_k^{\nu, \lambda}(r, v, \chi) = g_k^k(r) \Lambda_k^\nu(v, \chi)
\] (20)
is an orthonormal and complete basis function based on the scalar product $(g'_k | g)_T$.

Indeed, this basis function is neither orthonormal nor complete based on the complete scalar product (13). However, it can be used to determine a non-diagonal basis matrix $\mathbb{B}$, which can be diagonalized afterwards. To do so, the complete scalar product of two basis functions has to be computed, and thus the derivative of the inner potential is required. It is given by
\[
\frac{\partial \Phi_k'}{\partial \nu(v)} = \frac{\delta_{\nu,0} \delta_{\nu,0}}{r^2} \left\{ A_k(D) \right\}^r + \int_R^r d'r' \frac{\partial \Phi_k'}{\partial r'}
\] (21)
($\delta_{\nu,0}$ and $\delta_{\nu,0}$ are Kronecker deltas. The derivation is presented in appendix A). Equation (21) shows that the inner potential $\Phi_k'$ of a basis function is zero for all $\lambda = 0 \neq \nu$. The same holds for the inner potential $\Phi_k'$ with $\lambda' = 0 \neq \nu'$. Thus, the second part of the scalar product simplifies to
\[
\langle g_k^{\nu, \lambda'} | g_k^{\nu'} \rangle = 4\pi \int_{R - d}^{R \infty} e^{k\varphi(r')} g_k^{\nu'} \frac{\partial \Phi_k'}{\partial r} \frac{d'r'}{d'r} \delta_{\nu,0} \delta_{\nu,0} \delta_{\nu,0} \delta_{\nu,0} 
\] (22)
and is not zero only if $\lambda = \lambda' = \kappa' = 0$. This leads to two different results for the complete scalar product of the basis functions
\[
\langle g_k^{0, 0} | g_k^{0, 0} \rangle = B_k^{00} + 6
\] (23)
\[
\langle g_k^{0, \lambda'} | g_k^{0, \lambda} \rangle = 0
\] (24)
These expressions are the elements of the basis matrix $\mathbb{B}$, which is a block diagonal matrix. It is almost diagonal. The only non-diagonal block is the first block $B_k^{00}$ with the elements in equation (23). This block can be diagonalized with a rotation matrix $\mathbb{C}$ to find the diagonal matrix $\mathbb{B}_k^{00} = \mathbb{C} B_k^{00} \mathbb{C}^T$. Multiplying this diagonal matrix with its inverse leads to the identity matrix $I_k^{00} = \mathbb{D}_k^{00} \mathbb{D}_k^{00}$, then, $\mathbb{B}_k^{00}$ turns into a pure identity matrix.

Applying the collision operator to the basis function and computing the scalar product leads to the matrix elements of the collision matrix $\mathbb{T}_S$
\[
\langle g_k^{0, \nu} | \mathbb{T}_S g_k^{\nu} \rangle = \langle g_k^{0, \nu} | \mathbb{T}_S g_k^{\nu} \rangle = i\nu \delta_{\nu,0} \delta_{\nu,0} \delta_{\nu,0} \delta_{\nu,0}
\] (25)
$\mathbb{T}_S$ is a diagonal matrix with zero elements on the main diagonal if $\lambda = \lambda' = 0$. As a result, no diagonalization is required.

The computation of the Vlasov matrix $\mathbb{V}_S$ is more complicated. Its elements are determined by the scalar product between the basis functions and the Vlasov operator $\langle g_k^{0, \nu} | \mathbb{V}_S g_k^{\nu} \rangle$. As shown in [23], $\mathbb{V}_S$ is an anti-symmetric block matrix. As a result, only two inner block matrices with the indices $\kappa = \kappa' = \lambda = \lambda' = 1$ and $\kappa = \kappa' = \lambda = \lambda' = 0$ have to be multiplied with the rotation matrices $\mathbb{C}$ or $\mathbb{C}^T$ to obtain the correct expanded Vlasov matrix. Detailed calculations for the Vlasov matrix can be found in appendix B.

Finally, the excitation vector $e$ has to be determined. Its elements are defined by
\[
\langle g_k^{0, \nu} | e \rangle = 4\pi \int_{R}^{R_{\infty}} r^2 e^{k\varphi(r')} g_k^{0, \nu} \frac{\partial \Phi_k}{\partial r} dr' \delta_{\nu,0} \delta_{\nu,0}
\] (26)
Obviously, $e$ has non-vanishing elements only for $\nu' = 0$ and $\lambda' = 1$ and is given by
\[
e = (0, \mathbb{D}_k^{00}, \mathbb{C} e, 0, ...)^T.
\] (27)
Since all matrices and the excitation vector in equation (15) are defined, the explicit expansion of the admittance can be calculated to compute different spectra of the sIP.

5. Spectra of the sIP

In the last section, an explicit expansion of the inner admittance of the idealized sIP is derived and can be used to compute approximated spectra. To compare the first calculated kinetic spectra based on functional analytic methods for a real probe design, all parameters are taken from Buckley [11]. His spectra are calculated for an sIP without a dielectric $d = 0, \varepsilon_p = 1$ and a probe radius of $R = 5.15 \lambda_p$. He used the equilibrium potential $\Phi(r)$ of a spherical electrode in a plasma presented by Bernstein and Rabinowitz [24]. Buckley varied the collision frequency $\nu_0 \in \{0.05, 0.15, 0.25, 0.4\} \omega_p$. 

and normalized the admittance to $4\pi \varepsilon_0 \omega_p R$. The distance to the outer grounded surface is chosen to be $R_\infty = 150 \lambda_p$, where the plasma frequency is also normalized to $\omega_p$.

In figure 2(a), the real part of the admittance $Y_{IP}$ is depicted for the maximum expansion indices in velocity space $\kappa_{max} = \lambda_{max} = 25$ and a maximum expansion index $k_{max} = 500$ in physical space. All spectra are almost converged and are in good agreement with the spectra of Buckley, but the resonance frequencies and the half widths are not identical. Buckley’s resonance frequencies increase from $\omega_{prB} = 0.59 \omega_p$ to $\omega_{prB} = 0.62 \omega_p$ by increasing the collision frequency. The resonance frequencies calculated in this paper also increase $\omega_{vr} \in \{0.557, 0.572, 0.584, 0.598\} \omega_p$, but are smaller than Buckley’s. The half width $\Delta \omega_{ip}$ from the functional analytic calculations is broader and increases less than Buckley’s, as shown in figure 2(b).

Another resonance behavior can be observed in the spectra of the impedance $Z_{IP} = Y_{IP}^{-1}$, which are shown in figure 3(a). A clear resonance close to—but smaller than—the plasma frequency appears. Its frequency decreases with an increase in the collision frequency ($\omega_{pr} \in \{0.989, 0.961, 0.934, 0.884\} \omega_p$). These resonance frequencies are larger than Buckley’s, which decrease from $\omega_{prB} = 0.9 \omega_p$ to $\omega_{prB} = 0.8 \omega_p$. The half widths in the impedance spectra increase by increasing the collision frequency (see figure 3(b)). Thus, the behavior is identical to Buckley’s results, but the half widths are smaller. The half width for the largest collision frequency could not be determined due to strong damping.

The half widths in the spectra represent the damping of the probe–plasma system. In both cases—admittance and impedance—they are larger than those determined by fluid models, which is caused by kinetic effects. Assuming $\Delta \omega = \nu_p + \nu_{kin}$, the kinetic damping can be determined. In the spectra of the admittance, it is about $\nu_{kin} \in \{0.283, 0.336, 0.353, 0.343\} \omega_p$ and in the spectra of the impedance about $\nu_{kin} \in \{0.074, 0.132, 0.206\} \omega_p$.

6. Conclusion

In this paper, a kinetic model of the sIP is presented. Its dynamical interaction with the plasma is given by the inner admittance of the probe–plasma system, which is determined by the resolvent of the dynamical operator $T_f + T_s$. The expanded inner admittance of the sIP is derived by means of a complete basis in its particular geometry. This leads to the matrix representation of the dynamical operator. The truncation of the expansion allows one to approximate the inner admittance and thus to analyze the kinetic damping within its spectra.
To compare the approximated spectra of the functional analytic approach, the parameters in the calculations are taken from Buckley [11]. The spectra for the four different collision frequencies \( v_0 \in \{0.05, 0.15, 0.25, 0.4\} \omega_p \) are in good agreement in the position and in the height of the peaks. The resonance frequencies and the half widths are not identical, which is probably due to the different collision terms. In this paper, the collision term for pure elastic collisions between electrons and neutral atoms is used, which is based on a more physically reasonable derivation than the simple relaxation term used by Buckley.

Besides the admittance spectra, impedance spectra are also shown, where a clear resonance smaller than the plasma frequency can be observed. Compared to Buckley, the resonance frequencies differ slightly, but the half widths and thus the heights of the peaks differ more, which can also be explained by the different collision terms.

All spectra show larger half widths, as in spectra calculated by a fluid model. This additional damping is caused by kinetic effects. Due to the scalar product (13), which is motivated by the kinetic free energy \( \mathcal{F} \), the kinetic loss mechanism can be interpreted: in a periodic state, the probe constantly emits plasma waves which propagate to a large distance. The corresponding distribution \( g \) can, in principle, be calculated but is not square integrable and thus not an element of the Hilbert space. However, one may assume that the projection \( \langle e | g \rangle = i \text{in} \) on the observation vector exists: the free energy simply leaves the ‘observation range’ of the probe [21]. This loss of kinetic free energy is recorded in the spectrum of the probe as damping.

In other words, electrons on trajectories passing the influence region of the probe with a thickness of \( \Delta R \) gain energy due to the probe’s electric field. If the electrons leave this region without any collisions, they keep the gained energy and follow their trajectories to large distances. The gained energy of the electrons equals the lost field energy of the probe and is recorded as damping. Based on this interpretation, the kinetic damping in the pure collisionless case can be assumed to be \( \nu_{\text{kin}} = \frac{\hbar \nu}{\Delta R} \), where \( \nu_{\text{kin}} \) is the thermal velocity of the electrons, as proposed in [25] and similarly by Popov and Godyak for stochastic heating [26]. The influence range should be larger than the sheath around the probe tip and can be assumed to be equal to the probe radius itself \( \Delta R = R = 5.15 \lambda_p \). This yields \( \nu_{\text{kin}} \approx 0.194 \omega_p \) and fits as the limit of vanishing pressure to the kinetic damping of the admittance \( \nu_{\text{g,kin}} \), but has to be proven in further work.

However, pure collisionless damping is not the only part of the complete kinetic damping, or at least it is not constant, which can be seen by the increase of \( \nu_{\text{kin,kin}} \) dependent on the increase of collisions. An additional loss of kinetic free energy is present in the collision process itself. The collision term in the linearized Boltzmann equation (1) drives the perturbed distribution function to an isotropic one \( F(e) \). This means that information about the electron’s velocity directions will be lost. The more the distribution function reaches \( F(e) \), the more information is lost and the kinetic entropy \( \mathcal{S} \) reaches its maximum. Thus, the kinetic free energy \( \mathcal{F} = \mathcal{U} - \mathcal{S} \) decreases to its minimum, which also represents kinetic damping, where \( \mathcal{U} \) is the total energy.

The resonance of the impedance is called parallel resonance and is identical to the lowest Tonks–Dattner resonance, where the applied frequency equals the local plasma frequency \( \omega_p(\rho_p) \) at the resonance location \( \rho_p \). Thus, it represents a local resonance phenomenon, which can excite longitudinal plasma waves. The corresponding collisionless damping mechanism might be explained by Landau damping [27] of these waves, but is also known as the Herlofson paradox [28]. However, the damping mechanism is not yet fully understood, and also a simplified formula for the damping, especially in spherical coordinates, is as yet unknown. A detailed analysis based on the present model will allow both.

In summary, it is shown that the approximated spectra based on the functional analytic approach are in good agreement with the former determined spectra of Buckley. They show kinetic damping which can be explained by loss of kinetic free energy. This loss of kinetic free energy is recorded by the probe as damping. On one hand, it is caused by the increase of the kinetic entropy due to elastic collisions, and on the other hand by the escaped free energy to an unobservable distance from the probe. The latter is connected to collisionless damping. Furthermore, the presented algorithm can also be applied without major difficulties to more complicated probe geometries like the MRP.

Of course, electron depletion caused by a probe, which is immersed into a plasma, has to be taken into account in the investigations of APRS probes, as stated in a recent published paper [29]. Initial work in that direction has already been done: Bernstein and Rabinowitz [24] derived the potential of a spherical electrode surrounded by plasma, which was used by Buckley [8] as equilibrium potential. Morin and Balmain [16] compared Buckley’s results with a warm fluid model, where the resonance frequencies were in good agreement. Within their calculations, they used a continuous electron density profile in equilibrium, including electron depletion from Allen et al [30]. Furthermore, they compared their results with a simplified model, including a single step electron density profile, and observed just a small difference in the resonance frequency. Thus, a few works regarding the influence of the equilibrium density profile in the surrounding of a spherical probe were presented and the resonance frequencies are comparable.

However, the author will focus on detailed investigations of the influence of pressure and temperature-dependent equilibrium density profiles on the spectra of APRS probes. This is necessary on one hand to demonstrate the correctness of APRS measurements, and on the other hand to understand, in detail, all parts of kinetic damping and the Herlofson paradox. For practical applications of the probe, especially for monitoring or control purposes, not only qualitative but also quantitative results of the resonance parameters are required. In more detail, the author will derive a relation between the half width of resonance peaks in admittance spectra and the electron temperature to allow for simultaneous measurement of electron density and temperature.
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Appendix A. Potential of a basis function

Entering the basis function (20) into Poisson’s equation (2), the integrals over the velocity space can be solved which yields

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi_k^{\text{le}}}{\partial r} \right) = \delta_{k,0} \delta_{\theta,0} \delta_{\phi,0} \begin{cases} 0 & r \in \mathcal{D} \\ e^{\Phi(r)} g_k^k(r) & r \in \mathcal{P}. \end{cases} \quad (A1)$$

The potentials for the different regions (plasma \(\mathcal{P}\) and dielectric \(\mathcal{D}\)) can be solved by integration and using the boundary conditions \(\Phi_k^{\text{le}}(R_\infty) = 0\) and \(\Phi_k^{\text{le}}(R - d) = 0\):

\[
\Phi_k^{\text{le}}(r) = \begin{cases} A_k^{(P)} \left( \frac{1}{R_\infty} - \frac{1}{r} \right) \delta_{\theta,0} \delta_{\phi,0} \\
+ \int_{R}^{\infty} \left( \frac{1}{r^2} \int_{R}^{r} r'^2 e^{\Phi(r')} g_k^k(r') \, dr' \right) \delta_{\theta,0} \delta_{\phi,0} \\
- \int_{R}^{\infty} \left( \frac{1}{r^2} \int_{R}^{r} r'^2 e^{\Phi(r')} g_k^k(r') \, dr' \right) \delta_{\theta,0} \delta_{\phi,0} & r \in \mathcal{D} \\
A_k^{(P)} \left( \frac{1}{R - d} - \frac{1}{r} \right) \delta_{\theta,0} \delta_{\phi,0} & r \in \mathcal{P}. \end{cases} \quad (A2)
\]

The constants \(A_k^{(P)}\) and \(A_k^{(D)}\) are determined by the transition conditions

\[
\Phi_k^{(D)}(R) = \Phi_k^{(P)}(R), \quad (A4)
\]

\[
\frac{\partial \Phi_k^{(D)}}{\partial r} \bigg|_r = \frac{\partial \Phi_k^{(P)}}{\partial r} \bigg|_r. \quad (A5)
\]

In the scalar product, the derivative of the potential is required and can be written as

\[
\frac{\partial \Phi_k^{\text{le}}}{\partial r} = \begin{cases} \frac{A_k^{(D)}}{r^2} & r \in \mathcal{D} \\
A_k^{(P)} + \int_{R}^{\infty} \left( \int_{R}^{r} r'^2 e^{\Phi(r')} g_k^k(r') \, dr' \right) \delta_{\theta,0} \delta_{\phi,0} & r \in \mathcal{P}. \end{cases} \quad (A6)
\]

Appendix B. Matrix elements of the Vlasov operator

The Vlasov operator is defined in (11). Applied to the basis function, \(g_k^{\text{le}}\) yields

\[
T_v g_k^{\text{le}} = v \cos(\chi) \left( \frac{\partial \Phi}{\partial r} - \frac{\partial \Phi^{\text{le}}}{\partial r} \right) + \frac{\partial \Phi}{\partial r} \left( \sin(\chi) \frac{\partial g_k^{\text{le}}}{\partial r} \right)
- \cos(\chi) \frac{\partial g_k^{\text{le}}}{\partial \chi} + \frac{\partial \Phi}{\partial r} \left( \sin(\chi) \frac{\partial g_k^{\text{le}}}{\partial \chi} \right). \quad (B1)
\]

In the scalar product, the derivative of the potential \(\Phi^{\text{le}}\) is required, which is meant as the potential produced by the Vlasov operator applied to the basis function. In [23], it is shown that this derivative is given by the electron particle flux within the plasma \(\mathcal{P}\) and vanishes within the dielectric \(\mathcal{D}\). In the geometry of the sIP, one finds

\[
\frac{\partial \Phi_k^{(D)}}{\partial r} = -\frac{1}{\sqrt{2\pi}} g_k^k e^{\Phi(r)} \int_0^\pi e^{-\pi} \sin^{\chi}(\chi) e^{\Phi(r)} e^{\Phi(r)} \, d\chi. \quad (B2)
\]

As a result, the elements of the Vlasov matrix are given by

\[
\langle \delta_k^{\text{le}} | T_v | \delta_k^{\text{le}} \rangle = \frac{1}{\sqrt{2\pi}} \int_0^\pi e^{-\pi} \sin^{\chi}(\chi) \sin^{\chi}(\chi) \, d\chi V_{kk}^{(1)}
+ \frac{1}{\sqrt{2\pi}} \int_0^\pi e^{-\pi} \sin^{\chi}(\chi) \sin^{\chi}(\chi) \, d\chi V_{kk}^{(2)}
+ \frac{1}{\sqrt{2\pi}} \int_0^\pi e^{-\pi} \sin^{\chi}(\chi) \sin^{\chi}(\chi) \, d\chi V_{kk}^{(3)}
+ \frac{1}{\sqrt{2\pi}} \int_0^\pi e^{-\pi} \sin^{\chi}(\chi) \sin^{\chi}(\chi) \, d\chi V_{kk}^{(4)}
\]

with

\[
V_{kk}^{(1)} = 4\pi \int_{R}^{\infty} r^2 g_k^k \frac{\partial e^{\Phi(r)}}{\partial r} g_k^k \, dr, \quad (B4)
\]

\[
V_{kk}^{(2)} = -4\pi \int_{R}^{\infty} r^2 g_k^k \frac{\partial e^{\Phi(r)}}{\partial r} g_k^k \, dr, \quad (B5)
\]

\[
V_{kk}^{(3)} = 4\pi \int_{R}^{\infty} r^2 g_k^k \frac{\partial e^{\Phi(r)}}{\partial r} g_k^k \, dr, \quad (B6)
\]

\[
V_{kk}^{(4)} = 4\pi \int_{R}^{\infty} r^2 g_k^k \frac{\partial e^{\Phi(r)}}{\partial r} g_k^k \, dr, \quad (B7)
\]

The integrals over the velocity space in (B3) can be solved analytically, but lead to long expressions. The integrals over the physical space in equations (B4)–(B8) usually have to be
solved numerically, depending on the equilibrium potential $\Phi(r)$.

The final Vlasov matrix $\mathbf{V}$ is an anti-symmetric block matrix, where the inner blocks are given by the matrices of the physical space $\mathbf{V}_i^{(0)}$ over the indices $k$ and $k'$. Due to the anti-symmetry, only the block matrices at the positions with the indices $\kappa = \kappa' = \lambda = 0$, $\lambda' = 1$ and $\kappa = \kappa' = \lambda' = 0$, $\lambda = 1$ have to be corrected for the complete orthonormal expansion. The correct block matrices at these positions are $\mathbf{D}_{k0}^{000-1/2} \mathbf{C}_{000}^{000-1/2}$ for $\kappa = \kappa' = \lambda = 0$, $\lambda' = 1$ and $\mathbf{V}_i^{(0)} \mathbf{D}_{k0}^{000-1/2}$ for $\kappa = \kappa' = \lambda' = 0$, $\lambda = 1$. After this correction, the complete Vlasov matrix can be computed as

$$\mathbf{V} = \sum_{i=1}^{5} \mathbf{V}_i.$$  \hspace{1cm} (B9)

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