Single Index Regression Models with Randomly Left-truncated Data

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Abstract

In this paper, based on the kernel estimator proposed by Ould-Saïd and Lemdani (Ann. Instit. Statist. Math. 2006), we develop some new generalized M-estimator procedures for single index regression models with left-truncated responses. The consistency and asymptotic normality of our estimators are also established. Some simulation studies are given to investigate the finite sample performance of the proposed estimators.

Keywords: Semiparametric regression, single index model, left-truncated data, the product-limit estimator

1. Introduction

In order to avoid the so-called ”curse of dimensionality” in the high dimensional data analysis, many powerful semiparametric models have been developed to reduce the complexity of high dimensional data. One of the popular semiparametric models is the single index model, which takes the form

\[ Y = g(\theta_0^T X; \theta_0) + \epsilon, \]  

where \( Y \) is the response variable, \( X \in \mathbb{R}^d (d \geq 2) \) is a covariate vector, \( g(\cdot) \) is an unknown univariable measurable link function, \( \epsilon \) is the random error with \( \mathbb{E}(\epsilon|X) = 0, \theta_0 \in \mathbb{R}^d \) is the unknown index parameter with \( \|\theta_0\| = 1 \) (where \( \|\cdot\| \) denotes the Euclidean metric) and the first nonzero component of \( \theta_0 \) is positive for model identification. In recent years the single index model has been considered by many authors. Different methods have been carried out to estimate the index parameter, such as the average derivative approach (Stoker [25], Härdle and Tsybakov [8]), semiparametric least squares estimation (Härdle et al. [9], Ichimura [13]), semiparametric maximum likelihood estimation (Delecroix et al. [4]), the sliced inverse regression method (Duan and Li [5], Yin and Cook [38]), spline estimation (Wang and Yang [34]), and so on. Moreover, for the model (1.1), Xia et al. [36] considered the goodness-of-fit test. Kong and Xia [14] and Wang [32] studied the variable selection. Xue and Zhu [37] established the empirical likelihood confidence regions for

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In practice, the response variable in the model (1.1) may be left-truncated, that is, the variable \((X, Y)\) is interfered by another independent variable \(T\) (the truncation variable) in such a way that we may observe \((X, Y)\) and \(T\) only when \(Y \geq T\), and nothing is observed if \(Y < T\). Truncated data may be encountered in many fields, such as astronomy, economics, biostatistics and other fields. Truncated data issues have been investigated extensively (e.g. Lynden-Bell [19], Woodroofe [35], Stute [26], He and Yang [10, 11, 12], Stute and Wang [29], Moreira et al. [20], and among others). Compared with random censored data (or random missing data), random truncation seems to be more difficult, since the censored data (or random missing data) at least can provide the information on the censored lifetime, while, in the truncated case, we observe nothing given \(Y < T\). In this paper, we study the single index model with left-truncated response. By extending a kernel estimator for the nonparametric regression with left-truncated response in Ould-Saïd and Lemdani [21], we establish the generalized semiparametric least squares estimators for the model (1.1) in the truncation framework. The consistency and the asymptotic normality for our estimators are also provided.

The rest of this paper is organized as follows. In Section 2, we first recall the truncation framework and then construct the estimators for \(\theta_0\) and the link function \(g(\cdot)\) of the model (1.1) when the response is left-truncated. In Section 3, we present the consistency and the asymptotic normality of the estimators. Section 4 is devoted to present some simulation studies to test the quality of the estimators with finite samples. The proofs of our results are collected in Appendix.

2. Preliminary

2.1. Background for left-truncated data

\((X_j, Y_j, T_j), 1 \leq j \leq N,\) is a sequence of i.i.d. random vectors from \((X, Y, T),\) where \(T\) is the truncation variable. Throughout this paper, we assume that \(T\) is independent of \((X, Y).\) Due to the truncation, we are unable to observe the complete data. Let

\[(X_{ki}, Y_{ki}, T_{ki}) =: (U_i, V_i, W_i), 1 \leq i \leq n,\]

denote the observed sample. It is obvious that the potential sample size \(N\) is unknown and the observed sample size \(n\) is a random variable satisfying \(n \leq N.\) We use \(\alpha\) to denote the probability we may observe \(Y,\) that is, \(\alpha = \mathbb{P}(Y \geq T).\) Without loss of generality, we assume \(\alpha > 0,\) since \(\alpha = 0\) means that no data can be observed. For any distribution function \(L,\) we use \(a_L\) and \(b_L\) to stand for the left and right support endpoints of \(L,\) respectively. Define

\[F(y) = \mathbb{P}(Y \leq y), \quad G(t) = \mathbb{P}(T \leq t),\]
and

\[ H(x, y) = P(X \leq x, Y \leq y). \]

Let \( F^*, G^* \) and \( H^* \) be the corresponding conditional distributions of \( Y, T \) and \( (X, Y) \) given \( Y \geq T \), respectively, that is,

\[
F^*(v) = \alpha^{-1} \int_{-\infty}^{v} G(y) F(dy), \\
G^*(v) = \alpha^{-1} \int_{-\infty}^{v} G(y \land v) F(dy), \\
H^*(u, v) = \alpha^{-1} \int_{-\infty}^{u} \int_{-\infty}^{v} G(y) H(dx, dy).
\]

It follows from Stute [26] and He and Yang [12] that we can estimate \( F^*, G^* \) and \( H^* \) by \( F^*_n(v), G^*_n(v) \) and \( H^*_n(u, v) \), respectively, where

\[
F^*_n(v) = n^{-1} \sum_{i=1}^{n} I(V_i \leq v), \\
G^*_n(v) = n^{-1} \sum_{i=1}^{n} I(T_i \leq v), \\
H^*_n(u, v) = n^{-1} \sum_{i=1}^{n} I(U_i \leq u, V_i \leq v).
\]

Let

\[ C(y) = G^*(y) - F^*(y) = \alpha^{-1} G(y)[1 - F(y)]. \]

Then it can be consistently estimated by the following empirical estimator

\[ C_n(y) = n^{-1} \sum_{i=1}^{n} I(W_i \leq y \leq V_i). \]

Next, we introduce the estimators for \( F(y) \), \( G(t) \) and \( H(x, y) \), respectively. From Lynden-Bell [19], \( F \) and \( G \) can be estimated by the so-called Lynden-Bell product-limit estimators \( F_n(y) \) and \( G_n(y) \), respectively, where

\[
F_n(y) = 1 - \prod_{V_i \leq y} \left( 1 - \frac{1}{nC_n(V_i)} \right)
\]

and

\[
G_n(t) = \prod_{W_i > t} \left( 1 - \frac{1}{nC_n(W_i)} \right).
\]

On the other hand, He and Yang [11] established the following strong consistent estimator \( \alpha_n \) for \( \alpha \),

\[ \alpha_n = \frac{G_n(y)[1 - F_n(y - )]}{C_n(y)}, \]

where \( F_n(y - ) \) denotes the left-continuous version of \( F_n(y) \). Based on the above estimators, He and Yang [12] got the following nonparametric estimator for \( H(x, y) \),

\[
H_n(x, y) = \alpha_n \int_{-\infty}^{x} \int_{-\infty}^{y} \frac{1}{G_n(v)} H^*_n(du, dv). \tag{2.1}
\]
2.2. Estimators

We now come back to our main problem: Estimate the index parameter $\theta_0$ and the link function $g(\cdot)$ in the model (1.1) when the response variable is left-truncated. When data are fully observed, $\theta_0$ and $g(\cdot)$ may be estimated in the following two stages: (i) Estimate the coefficient vector $\hat{\theta}_0$; (ii) Establish the estimator of the link function $g(\cdot)$ with the estimator of $\theta_0$ in Step (i). When data are left-truncated, we can still follow the same steps as in the full-data case. Note that the link function $g$ is unknown. Due to the left-truncation, we can estimate the link function $g(\theta^T u; \theta)$ by the following extended Nadraya-Watson estimator:

$$
\hat{g}_n(\theta^T u; \theta) = \frac{\sum_{i=1}^n V_i G_n^{-1}(V_i) K_h(\theta^T u - \theta^T U_i)}{\sum_{i=1}^n G_n^{-1}(V_i) K_h(\theta^T u - \theta^T U_i)},
$$

(2.2)

where $K_h(\cdot) = K(\cdot/h)$ with $h$ being a bandwidth, and $K(\cdot)$ being a symmetric kernel function with support on $(-1, 1)$. Ould-Saïd and Lemdani [21] and Moreira et al. [20] constructed two similar nonparametric estimators for the regression function with left-truncated and doubly-truncated responses, respectively.

Similar to the full-data case, we first estimate $\hat{\theta}_0$ in the model (1.1). For any measurable function $\varphi(u, v)$, under Condition (C1) (see Section 3 below), He and Yang [12] Theorem 3.2 showed that

$$
\int \varphi(u, v) H_n(du, dv) \to \int \varphi(u, v) H(du, dv) \text{ a.s.}
$$

Hence, we define the estimator $\hat{\theta}_n$ of $\theta_0$ by minimizing $M_n(\theta, \hat{g}_n)$ with

$$
M_n(\theta, \hat{g}_n) = \frac{\alpha_n}{n} \sum_{i=1}^n G_n^{-1}(V_i) \left[ V_i - \hat{g}_n(\theta^T U_i; \theta) \right]^2 J(U_i),
$$

(2.3)

where $J(u) = I(u \in \mathcal{A})$, $\mathcal{A} \subset \mathbb{R}^d$, is the trimming function used to guarantee that the denominator of $\hat{g}_n(\theta^T U_i; \theta)$ is not close to zero. In the second stage, with $\hat{\theta}_n$, the estimator of the link function $g$ is given by

$$
\hat{g}_n^*(s; \hat{\theta}_n) = \frac{\sum_{i=1}^n V_i G_n^{-1}(V_i) K_h(s - \hat{\theta}_n^T U_i)}{\sum_{i=1}^n G_n^{-1}(V_i) K_h(s - \hat{\theta}_n^T U_i)}.
$$

Remark 2.1 If there is no truncation, that is, $\alpha = 1$, $n = N$ and $(U_i, V_i) = (X_i, Y_i), 1 \leq i \leq N$, then our estimators reduce to the ordinary semiparametric least squares estimators.

3. Main Results

In this section, we state the consistency and asymptotic properties of $\hat{\theta}_n$. We first introduce some notations. Let $\Theta$ be the set of all unit $d$-vectors with first nonzero component positive. For any function $f$, let $\nabla_x f$ (resp. $\nabla^2_{x,x} f$) denote the vector (resp. matrix) of partial derivatives with respect to $x$.

In order to establish our results, we need the following regularity conditions:

(C1) $F$ and $G$ are continuous with $a_G < a_F$.

(C2) $E[Y^2] < \infty$. 

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(C3) The set $X = \text{Supp}(X)$ is a compact subset of $\mathbb{R}^d$.

(C4) The link function $g(\theta^T u)$ is continuous with respect to $\theta$ and $u$. Furthermore, $g(\theta^T u)$ is twice continuously differentiable with respect to $\theta$, and $\nabla_\theta g$, $\nabla^2_{\theta, \theta} g$ are bounded as functions of $\theta$ and $u$.

(C5) The kernel function $K$ is a symmetric, positive and twice continuously differentiable function. Furthermore, $K^n$ is a Lipschitz continuous function.

(C6) For all $\theta \in \Theta$, the joint density function $f_{\theta^T X, Y}$ of $(\theta^T X, Y)$ is twice continuously differentiable with respect to the first variable.

(C7) There exist two Donsker classes $H_1$ and $H_2$ such that

$$u \rightarrow g(\theta_0^T u) \in H_1 \text{ and } u \rightarrow \nabla_\theta g(\theta_0^T u) \in H_2.$$ 

(C8) As $n \to \infty$, $nh^{-5}(\log n)^{-1} \to \infty$ and $nh^7 \to 0$.

The continuity of $F$ and $G$ in Condition (C1) is commonly used in truncated models, see for example [10, 11, 12, 21, 39]. In fact, the continuity guarantees that there is no ties in the observed data. $a_G < a_F$ in Condition (C1) is needed for deriving the representation of $\int \varphi(u, v)H_n(du, dv)$ for any measurable function $\varphi$. See Proposition A.1 in Appendix. Conditions (C2)-(C6) have been widely used by many authors in the single index model, for example, [2, 4, 16, 17]. Conditions (C6)-(C8) are used to prove the consistency and the asymptotic normality of our estimator. The Donsker class in Condition (C7) is also used by [2, 16, 17]. For some typical examples of Donsker class, see van der Vaart and Wellner [31, Section 2.10] and (A.4), (A.5) in [17].

Theorem 3.1 Under Conditions (C1)-(C6) and (C8), we have

$$\hat{\theta}_n \rightarrow \theta_0 \text{ in probability.}$$

To state the asymptotic normality of $\hat{\theta}_n$, we first introduce some notations. For any measurable function $\varphi(u, v)$, set

$$\Gamma(u, v, \varphi) = \int_{\{v < y\}} [\varphi(u, v) - \varphi(u, y)]F(dy).$$  \hspace{1cm} (3.1)

Moreover, define

$$\Lambda = \mathbb{E} \left[ \nabla_\theta g(\theta_0^T U) \nabla_\theta g(\theta_0^T U)^T J(U) \right]$$ \hspace{1cm} (3.2)

and

$$\psi(u, v) = [v - g(\theta_0^T u)]\nabla_\theta g(\theta_0^T u) J(u).$$  \hspace{1cm} (3.3)
Theorem 3.2 Under Conditions (C1) to (C8), we have
\[ \hat{\theta}_n - \theta_0 = n^{-1/2} \Lambda^{-1} W_n + o_p(n^{-1/2}), \]
where \( W_n = n^{-1/2} \sum_{i=1}^{n} \zeta_i(\psi) \) is a random vector and for \( 1 \leq i \leq n, \)
\[ \zeta_i(\psi) = \frac{\Gamma(U_i, V_i, \psi)}{C(V_i)} - \int_{T_i}^{V_i} \frac{\Gamma(U, v, \psi)}{C^2(v)} F^*(dv). \] (3.4)

Hence, as a consequence, we have
\[ \hat{\theta}_n - \theta_0 \overset{d}{\to} \mathcal{N}(0, \Lambda^{-1} \Omega \Lambda^{-1}) \]
with
\[ \Omega = \text{Var} \left\{ \frac{\Gamma(U, V, \psi)}{C(V)} - \int_{T}^{V} \frac{\Gamma(U, v, \psi)}{C^2(v)} F^*(dv) \right\}. \]

The detailed proofs of Theorems 3.1 and 3.2 are given in Appendix.

4. Simulation Study

In this section, we conduct a simulation study to check the finite sample performance of our estimators. We conducted the simulation study with the following three different models:

Model 1: \( Y = -((\theta^T X - 1/\sqrt{2})^2 + 1) + \epsilon, \) where \( X \sim U[-2, 2] \otimes U[-2, 2], \epsilon \sim \mathcal{N}(0, 0.2^2), \) the truncated variable \( T_1 \sim \mathcal{N}(\lambda, 1) \) and the true value of the parameter is
\[ \theta_0 = (b_1, b_2)^T = \frac{1}{\sqrt{2}}(1, 1)^T. \]

This model comes from Härdle et al. \[9\] and Lu and Burke \[18\].

Model 2: \( Y = \sin(\theta^T X) + \epsilon, \) where \( X \sim \mathcal{N}(0, 1) \otimes \mathcal{N}(0, 1), \epsilon \sim \mathcal{N}(0, 0.5^2), \) the truncated variable \( T_2 \sim U(-1.5, \lambda) \) and the true value of the parameter is
\[ \theta_0 = (b_1, b_2)^T = \frac{1}{\sqrt{5}}(1, 2)^T. \]

The second model can be found in Wang et al. \[33\].

Model 3: \( Y = \exp\{2\theta^T X\} + \epsilon, \) where \( X \sim \mathcal{N}(0, 1) \otimes \mathcal{N}(0, 1), \epsilon \sim \mathcal{N}(0, 1), \) the truncated variable \( T_2 \sim \mathcal{N}(\lambda, 1) \) and the true value of the parameter is
\[ \theta_0 = (b_1, b_2)^T = (0.6, 0.8)^T. \]

Moreover, in the above three models, the variables \( X, \epsilon \) and \( T \) are mutually independent.

Here, we should point out that, from Section 2.1, the estimators \( F_n \) and \( G_n \) depend highly on the behavior of the estimator \( C_n \), while \( C_n \) may be zero with the truncated data. To overcome this problem, similar to Woodroofe \[35\] and Zhou \[39\], in the simulation study, we replaced \( C_n(y) \) by
\[ \tilde{C}_n(y) = \max \left\{ C_n(y), \frac{1}{n} + \frac{1}{n^2} \right\}, \] for any \( y \in (V_1(V)), \)
where \( V_{(1)}, V_{(n)} \) are the ordered statistics. Moreover, Stute and Wang [29] proved that the corresponding estimators based on \( C_n(y) \) and \( \tilde{C}_n(y) \) are asymptotically equivalent at the \( \sqrt{n} \)-rate.

In our simulations, we assumed that the complete data size \( N \) is fixed and the observed data size \( n \) is random for convenience (you may also set \( n \) be fixed and \( N \) be random). For each model, we performed 500 repetitions for each setting \((N, \alpha)\), where the sample size \( N \in \{50, 100, 200\} \) and the proportions of truncated data \( 1 - \alpha = \mathbb{P}(Y < T) \in \{10\%, 20\%, 40\%\} \). We chose the Epanechnikov kernel function \( K(u) = \frac{3}{4}(1-u^2)I\{u \leq 1\} \) and the bandwidth sequence \( h = n^{-1/5}(\log n)^{1/5} \) to compute \( \hat{\theta}_n \). The bias and the mean squared error (MSE) for \( \hat{\theta}_n \) were computed. The corresponding results are presented in Tables 1-3.

Table 1: Simulation results for Model 1.

| \( \lambda \) | \( 1 - \alpha \) | \( N \) | \( b_1 \) | \( b_2 \) | \( b_1 \) | \( b_2 \) |
|---|---|---|---|---|---|---|
| -0.72 | 0.4 | 50 | \( 7.38 \times 10^{-3} \) | \(-4.01 \times 10^{-3} \) | \( 3.20 \times 10^{-3} \) | \( 1.57 \times 10^{-3} \) |
| | | 100 | \( 1.02 \times 10^{-3} \) | \( 3.58 \times 10^{-5} \) | \( 6.92 \times 10^{-4} \) | \( 7.10 \times 10^{-4} \) |
| | | 200 | \(-3.95 \times 10^{-4} \) | \( 7.83 \times 10^{-4} \) | \( 2.75 \times 10^{-4} \) | \( 2.73 \times 10^{-4} \) |
| -2 | 0.2 | 50 | \( 1.63 \times 10^{-3} \) | \(-8.39 \times 10^{-4} \) | \( 5.82 \times 10^{-4} \) | \( 5.48 \times 10^{-4} \) |
| | | 100 | \( 3.42 \times 10^{-4} \) | \(-2.63 \times 10^{-5} \) | \( 2.27 \times 10^{-4} \) | \( 2.19 \times 10^{-4} \) |
| | | 200 | \(-7.93 \times 10^{-4} \) | \( 9.21 \times 10^{-4} \) | \( 9.01 \times 10^{-5} \) | \( 9.20 \times 10^{-5} \) |
| -3.5 | 0.1 | 50 | \(-2.20 \times 10^{-4} \) | \( 8.69 \times 10^{-4} \) | \( 4.53 \times 10^{-4} \) | \( 4.65 \times 10^{-4} \) |
| | | 100 | \( 1.42 \times 10^{-3} \) | \(-1.26 \times 10^{-3} \) | \( 1.60 \times 10^{-4} \) | \( 1.57 \times 10^{-4} \) |
| | | 200 | \( 3.10 \times 10^{-4} \) | \( 4.06 \times 10^{-4} \) | \( 6.74 \times 10^{-5} \) | \( 6.84 \times 10^{-5} \) |

Table 2: Simulation results for Model 2.

| \( \lambda \) | \( 1 - \alpha \) | \( N \) | \( b_1 \) | \( b_2 \) | \( b_1 \) | \( b_2 \) |
|---|---|---|---|---|---|---|
| 0.92 | 0.4 | 50 | \(-2.29 \times 10^{-2} \) | \(-8.76 \times 10^{-3} \) | \( 2.87 \times 10^{-2} \) | \( 7.41 \times 10^{-3} \) |
| | | 100 | \(-1.50 \times 10^{-2} \) | \(-3.30 \times 10^{-3} \) | \( 1.52 \times 10^{-2} \) | \( 4.11 \times 10^{-3} \) |
| | | 200 | \(-4.35 \times 10^{-3} \) | \(-2.25 \times 10^{-3} \) | \( 6.36 \times 10^{-3} \) | \( 1.57 \times 10^{-3} \) |
| -0.13 | 0.2 | 50 | \(-1.47 \times 10^{-2} \) | \(-6.33 \times 10^{-2} \) | \( 1.93 \times 10^{-2} \) | \( 5.16 \times 10^{-3} \) |
| | | 100 | \(-3.29 \times 10^{-3} \) | \(-5.70 \times 10^{-3} \) | \( 1.03 \times 10^{-2} \) | \( 2.85 \times 10^{-3} \) |
| | | 200 | \(-5.98 \times 10^{-3} \) | \( 7.52 \times 10^{-5} \) | \( 4.20 \times 10^{-3} \) | \( 1.01 \times 10^{-3} \) |
| -0.75 | 0.1 | 50 | \(-1.19 \times 10^{-2} \) | \(-5.10 \times 10^{-3} \) | \( 1.60 \times 10^{-2} \) | \( 3.82 \times 10^{-3} \) |
| | | 100 | \(-3.97 \times 10^{-3} \) | \(-3.41 \times 10^{-3} \) | \( 7.60 \times 10^{-3} \) | \( 2.07 \times 10^{-3} \) |
| | | 200 | \(-5.60 \times 10^{-3} \) | \( 2.05 \times 10^{-4} \) | \( 3.74 \times 10^{-3} \) | \( 9.06 \times 10^{-4} \) |

From Tables 1-3 we can see that our estimator \( \hat{\theta}_n \) performs well. Moreover, the performance of \( \hat{\theta}_n \) become better and better as the sample size \( N \) increases. We also observe that the quality of our estimator in each model is slightly affected by the proportion of the truncated data, \( 1 - \alpha \), and shrinks as the proportion becomes larger.

Corresponding to \( N = 200 \) and \( N = 500 \), the curves of \( \hat{g}_n \) for three models with
Table 3: Simulation results for Model 3.

| $\lambda$ | $1 - \alpha$ | N  | Bias $b_1$ | Bias $b_2$ | MSE $b_1$ | MSE $b_2$ |
|-----------|--------------|----|------------|------------|-----------|-----------|
| 0.97      | 0.4          | 50 | $-7.38 \times 10^{-3}$ | $7.81 \times 10^{-5}$ | $5.52 \times 10^{-3}$ | $3.20 \times 10^{-3}$ |
|           |              | 100| $-7.08 \times 10^{-3}$ | $2.47 \times 10^{-3}$ | $3.11 \times 10^{-3}$ | $1.42 \times 10^{-3}$ |
| -0.20     | 0.2          | 50 | $-6.83 \times 10^{-3}$ | $6.10 \times 10^{-5}$ | $5.11 \times 10^{-3}$ | $2.99 \times 10^{-3}$ |
|           |              | 100| $-2.39 \times 10^{-3}$ | $-3.18 \times 10^{-3}$ | $3.08 \times 10^{-3}$ | $2.30 \times 10^{-3}$ |
| -4.3      | 0.1          | 50 | $3.79 \times 10^{-5}$  | $5.33 \times 10^{-4}$ | $4.01 \times 10^{-4}$ | $4.05 \times 10^{-4}$ |
|           |              | 100| $5.38 \times 10^{-4}$  | $-3.61 \times 10^{-4}$ | $1.25 \times 10^{-4}$ | $1.24 \times 10^{-4}$ |
|           |              | 200| $-1.87 \times 10^{-4}$ | $2.77 \times 10^{-4}$ | $6.27 \times 10^{-5}$ | $6.41 \times 10^{-5}$ |

$1 - \alpha = 20\%$ are graphed in Figures 1 and 2, respectively. The appearance of the estimated curves is very similar to that of the true curves. Figures 1 and 2 suggest that our estimators work well too.

Figure 1: Curve estimations for Models 1~3 with $N = 200$.

5. Concluding Remarks

In this paper, we have considered the single index models under randomly truncated framework. The estimators of the index parameter and the link function are established based on the kernel estimator proposed by [21]. Our estimators possess the consistency and the asymptotic normality. Simulations indicate that the proposed method performs well.

Most of statistical methods dealing with the truncated data, including the Lynden-Bell estimator [19], rely heavily on the quasi-independence (independence) between the truncated random $T$ and the interest variable $Y$. See, for example, [10, 11, 12, 21, 29, 39]. However, the quasi-independence (independence) may fail in many situations. For example, Chaieb et al. [3] introduced a copular dependency between $T$ and $Y$, and established some modified estimators for the distribution functions. Thus, it will be interesting to extend some similar ideas to our setting. We will investigate this in the future.
A. Appendix: Technical Proofs

A.1. Representation of \( \int \varphi(u, v)H_n(du, dv) \)

In this subsection, we study the representation of \( \int \varphi(u, v)H_n(du, dv) \) for any measurable function \( \varphi(u, v) \), which is essential for the proof of Theorem 3.2.

**Proposition A.1** Let \( \varphi(u, v) \) be any measurable function satisfying

\[
\int \frac{\varphi^2(u, v)}{G(v)}H(du, dv) < \infty. \tag{A.1}
\]

Then, under Condition (C1), we have

\[
\int \varphi(u, v)[H_n(du, dv) - H(du, dv)]
= \int \frac{\Gamma(u, v, \varphi)}{C(v)}[H_n^*(du, dv) - H^*(du, dv)]
- \int \frac{C_n(v) - C(v)}{C^2(v)}\Gamma(u, v, \varphi)H_n^*(du, dv) + o_P(n^{-\frac{1}{2}}),
\]

where \( \Gamma(u, v, \varphi) \) is defined by \( \text{(3.1)} \).

**Remark A.1** Proposition A.1 extends Theorem 1.1 in Stute and Wang [29] which dealt with the representation when the covariables are absent. It is obvious that the inclusion of covariables will enlarge the class of possible applications.

**Proof:** The proof of Proposition A.1 is similar to that of Stute and Wang [29, Theorem 1.1] which studies the representation without covariables. The rest of the proof is devoted to some modifications.

We first introduce an asymptotically equivalent estimator \( \hat{H}_n \) of \( H_n \), which is defined by

\[
\int \varphi(u, v)\hat{H}_n(du, dv) := \int \frac{\varphi(u, v)\lambda_n(v)}{C_n(v)}H_n^*(du, dv),
\]

where

\[
\lambda_n(v) = \exp \left\{ n \int_{-\infty}^{v} \log \left[ 1 - \frac{1}{1 + nC_n(y)} \right] F_n(dy) \right\}.
\]
with \( \int_{-\infty}^{v} \) denoting the integral on the interval \((-\infty, v)\). Set

\[ \lambda(v) = 1 - F(v) = \exp \left\{ - \int_{-\infty}^{v} \frac{F^*(dy)}{C(y)} \right\}. \]

Similar to the proofs of Lemma 3.2 and Corollaries 3.1-3.3 in Stute and Wang [29], we obtain that

\[
\int \varphi(u, v) \hat{H}_n(du, dv) = L_{n1} + L_{n2} + L_{n3} + o_P(n^{-1/2}) \tag{A.2}
\]

with

\[
L_{n1} = \int \varphi(u, v) \frac{\lambda(v)}{C(v)} H_n^+(du, dv) + \int \varphi(u, v) \frac{\lambda(v)(C(v) - C_n(v))}{C^2(v)} H_n^+(du, dv),
\]

\[
L_{n2} = -\int \varphi(u, v) \frac{\lambda(v)}{C(v)} \int_{-\infty}^{v} \frac{F_n^*(dy) - F^*(dy)}{C(y)} H_n^+(du, dv),
\]

\[
L_{n3} = \int \varphi(u, v) \frac{\lambda(v)}{C(v)} \int_{-\infty}^{v} \frac{C_n(y) - C(y)}{C^2(y)} F_n^*(dy) H_n^+(du, dv).
\]

It follows from (A.2) and Theorems 5.3.2 and 5.3.3 in Serfling [23] that Proposition A.1 holds for \( \hat{H}_n \) instead of \( H_n \). By applying the SLLN for U-statistics, we obtain that Proposition A.1 also holds for \( H_n \).

From Proposition A.1 we get an i.i.d. representation of \( \int \varphi(u, v) H_n(du, dv) \) and the asymptotic normality. For similar results of the censored data, refer to Theorem 1.1 in Stute [27].

**Corollary A.1** Under the assumptions of Proposition A.1 we have

\[
\sqrt{n} \int \varphi(u, v)[H_n(du, dv) - H(du, dv)] = n^{\frac{1}{2}} \sum_{i=1}^{n} \zeta_i(\varphi) + o_P(1),
\]

where \( \zeta_i(\cdot) \) is defined by (3.4).

**Corollary A.2** Under the assumptions of Proposition A.1 we have

\[
\sqrt{n} \int \varphi(u, v)[H_n(du, dv) - H(du, dv)] \xrightarrow{d} N(0, \sigma^2),
\]

where

\[
\sigma^2 = \text{Var} \left\{ \frac{\Gamma(U, V, \varphi)}{C(V)} - \int_{T} \frac{\Gamma(U, v, \varphi)}{C^2(v)} F^*(dv) \right\}.
\]

and the function \( \Gamma \) is defined by (3.1).

**Remark A.2** \( a_G < a_F \) in Condition (C1), together with

\[
\int \varphi^2(u, v)H(du, dv) < \infty, \tag{A.3}
\]

means that (A.1) holds. Note that (A.3) is the standard moment condition when the data are complete.
Remark A.3 Sánchez Sellero et al. [22] Theorem 1 introduced an i.i.d. representation for the product-limit integrals under truncation and censoring with covariables. However, Theorem 1 in [22] requires that the following two integrals

\[ \int \Phi(u, v)[1 - F(v)]^{-5}H(du, dv) \quad \text{and} \quad \int \Phi^2(u, v)[1 - F(v)]^{-3}H(du, dv) \quad (A.4) \]

are finite where \( \Phi \) is an envelope for the class \( \{ \varphi(u, v) \} \) (we refer to van der Vaart and Wellner [31, P. 84] for the definition of the envelope function). It is obvious that both the integrals in (A.4) equal infinity when \( \Phi \) is a finite constant. Hence, the representation proposed by [22] can only be applied to the functions converging to zero as \( u, v \to \infty \).

However, Proposition A.1 only requires a finite second moment condition.

A.2. Difference between \( g \) and \( \hat{g}_n \)

To prove the consistency and the asymptotic normality of \( \hat{\theta}_n \), we need to study the difference between \( g \) and \( \hat{g}_n \) defined by [22].

Let

\[ \phi_{\theta^T X}(s) = \int y f_{\theta^T X, Y}(s, y) dy, \]

where \( f_{\theta^T X, Y}(s, y) \) is the joint density function of \( (\theta^T X, Y) \). Hence, we can rewrite the link function \( g(\cdot) \) in (1.1) as

\[ g(s; \theta_0) = \frac{\phi_{\theta^T X}(s)}{f_{\theta^T X}(s)}, \]

where \( f_{\theta^T X}(\cdot) \) is the density function of \( \theta^T X \). Define

\[ \hat{f}_{\theta^T X, n}(s) = \frac{\alpha_n}{nh} \sum_{i=1}^n \frac{1}{G_n(V_i)} K\left( \frac{s - \theta^T U_i}{h} \right), \]

\[ \hat{\phi}_{\theta^T X, n}(s) = \frac{\alpha_n}{nh} \sum_{i=1}^n \frac{V_i}{G_n(V_i)} K\left( \frac{s - \theta^T U_i}{h} \right). \]

Note that

\[ \hat{g}_n(s; \theta) = \frac{\hat{\phi}_{\theta^T X, n}(s)}{\hat{f}_{\theta^T X, n}(s)}. \]

The following lemmas A.1 to A.5 study the distance between \( g \) and \( \hat{g}_n \). Before we state them, we first introduce two equivalent estimators. Set

\[ \tilde{f}_{\theta^T X, n}(s) = \frac{\alpha_n}{nh} \sum_{i=1}^n \frac{1}{G(V_i)} K\left( \frac{s - \theta^T U_i}{h} \right), \]

\[ \tilde{\phi}_{\theta^T X, n}(s) = \frac{\alpha_n}{nh} \sum_{i=1}^n \frac{V_i}{G(V_i)} K\left( \frac{s - \theta^T U_i}{h} \right). \]
Lemma A.1 Under Conditions (C1), (C2), (C5) and (C8), we have
\[
\sup_{u \in \mathcal{X}, \theta \in \Theta} \left| \tilde{f}_{\theta^T X_n}(\theta^T u) - \mathbb{E} \tilde{f}_{\theta^T X_n}(\theta^T u) \right| = O\left( \sqrt{\frac{\log n}{nh}} \right) \text{ a.s.} \quad (A.5)
\]
and
\[
\sup_{u \in \mathcal{X}, \theta \in \Theta} \left| \tilde{\phi}_{\theta^T X_n}(\theta^T u) - \mathbb{E} \tilde{\phi}_{\theta^T X_n}(\theta^T u) \right| = O\left( \sqrt{\frac{\log n}{nh}} \right) \text{ a.s.} \quad (A.6)
\]
Proof: Let \( \mathcal{F}_1 = \{ \frac{y}{G(y)} \} \). From Giné and Guillou [7, Lemma 3(a)], under Condition (C1) the class \( \mathcal{F}_1 \) is a V-C subgraph class (see Giné and Guillou [7, P. 2049]) with the envelope \( \frac{1}{G(a_F)} \). Hence, the assumptions of Theorem 1 in Einmahl and Mason [6] hold under Conditions (C1), C(5) and (C8). Thus, by applying Theorem 1 in [6], we conclude (A.5).

The proof of (A.6) is similar to that of (A.5), but using the V-C subgraph class \( \mathcal{F}_2 = \{ \frac{y}{G(a_F)} \} \) instead of the class \( \mathcal{F}_1 \). \( \square \)

Lemma A.2 Under Conditions (C1), (C2), (C5), (C6) and (C8), we have
\[
\sup_{u \in \mathcal{X}, \theta \in \Theta} \left| f_{\theta^T X_n}(\theta^T u) - \tilde{f}_{\theta^T X_n}(\theta^T u) \right| = O\left( \sqrt{\frac{\log n}{nh}} + h^2 \right) \text{ a.s.} \quad (A.7)
\]
and
\[
\sup_{u \in \mathcal{X}, \theta \in \Theta} \left| \phi_{\theta^T X_n}(\theta^T u) - \tilde{\phi}_{\theta^T X_n}(\theta^T u) \right| = O\left( \sqrt{\frac{\log n}{nh}} + h^2 \right) \text{ a.s.} \quad (A.8)
\]
Proof: From Lemma A.1 to prove Lemma A.2, we only need to consider the following two bias terms
\[
\sup_{u \in \mathcal{X}, \theta \in \Theta} \left| f_{\theta^T X_n}(\theta^T u) - \mathbb{E} \tilde{f}_{\theta^T X_n}(\theta^T u) \right|
\]
and
\[
\sup_{u \in \mathcal{X}, \theta \in \Theta} \left| \phi_{\theta^T X_n}(\theta^T u) - \mathbb{E} \tilde{\phi}_{\theta^T X_n}(\theta^T u) \right|.
\]
From the classic change of variable, a Taylor expansion and Conditions (C2) and (C6), we get that both the bias terms are of order \( O(h^2) \). Hence, we complete the proof of Lemma A.2. \( \square \)

Lemma A.3 Under Conditions (C1)-(C3), (C5), (C6) and (C8), we have
\[
\sup_{u \in \mathcal{X}, \theta \in \Theta} \left| f_{\theta^T X_n}(\theta^T u) - \tilde{f}_{\theta^T X_n}(\theta^T u) \right| = O\left( \sqrt{\frac{\log n}{nh}} + h^2 \right) \text{ a.s.} \quad (A.7)
\]
and
\[
\sup_{u \in \mathcal{X}, \theta \in \Theta} \left| \phi_{\theta^T X_n}(\theta^T u) - \tilde{\phi}_{\theta^T X_n}(\theta^T u) \right| = O\left( \sqrt{\frac{\log n}{nh}} + h^2 \right) \text{ a.s.} \quad (A.8)
\]
Proof: We first consider (A.7). Similar to the proof of Lemma 2 in Lemdani et al. [15], we get from Theorem 3.2 in He and Yang [11], Theorem 4.1 in He and Yang [10] and the strong law of large numbers that

$$\sup_{u \in \mathcal{X}, \theta \in \Theta} \left| \tilde{f}_{\theta} \mathcal{X}_n \left( \theta^T u \right) - \hat{f}_{\theta} \mathcal{X}_n \left( \theta^T u \right) \right| = O \left( n^{-1/2} \right) \text{ a.s.},$$

which, together with Lemma A.2 and Condition (C8), implies that (A.7) holds.

Following the same lines as the proof of (A.7), we get (A.8) by using Lemma A.2 again. □

Noting that $g(s; \theta) = \phi_{\theta^T \mathcal{X}(s)}$ and $\hat{g}_n(s; \theta) = \hat{\phi}_{\theta^T \mathcal{X}_n(s)}$, we get the following lemmas from Lemma A.3.

**Lemma A.4** Under the assumptions of Lemma A.3, we have

$$\sup_{u \in \mathcal{X}, \theta \in \Theta} \left| g(\theta^T u; \theta) - \hat{g}_n(\theta^T u; \theta) \right| = O \left( \sqrt{\frac{\log n}{nh}} + h^2 \right) \text{ a.s.}$$

Similar to the proof of Lemma A.4, we have

**Lemma A.5** Under the assumptions of Lemma A.3, we have

$$\sup_{u \in \mathcal{X}, \theta \in \Theta} \left| \nabla_{\theta} g(\theta^T u; \theta) - \nabla_{\theta} \hat{g}_n(\theta^T u; \theta) \right| = O \left( \sqrt{\frac{\log n}{nh^3}} + h^2 \right) \text{ a.s.}$$

**A.3. Proofs of Theorems 3.1 and 3.2**

In this subsection, we give the detailed proofs of Theorems 3.1 and 3.2. Define

$$M(\theta, g) = \int \left[ v - g(\theta^T u) \right]^2 J(u) H(du, dv).$$

**Proof of Theorem 3.1:** From Theorem 5.7 in van der Vaart [30], to prove Theorem 3.1 we only need to show that

$$\sup_{\theta \in \Theta} |M_n(\theta, \hat{g}_n) - M(\theta, g)| = o_P(1), \quad (A.9)$$

where $M_n(\theta, g)$ is defined in (2.3). We first consider the difference

$$|M_n(\theta, \hat{g}_n) - M_n(\theta, g)| \
\leq \sup_{u \in \mathcal{X}, \theta \in \Theta} \left| \hat{g}_n(\theta^T u) - g(\theta^T u) \right| \int \left[ \sup_{u \in \mathcal{X}, \theta \in \Theta} \left| g(\theta^T u) + \hat{g}_n(\theta^T u) \right| + 2|v| \right] H_n(du, dv).$$

From Conditions (C2) and (C4), the integral on the righthand side is finite. By Lemma A.4, we deduce that

$$\sup_{\theta \in \Theta} |M_n(\theta, \hat{g}_n) - M_n(\theta, g)| = o_P(1).$$

Moreover, similar to the proof of Theorem 1.1 in Stute [25], we get

$$\sup_{\theta \in \Theta} |M_n(\theta, g) - M(\theta, g)| = o_P(1).$$
Hence (A.9) holds and we end the proof of Theorem 3.1.

To get the asymptotic normality of \( \hat{\theta}_n \), we first consider the case \( g \) is known. From Sherman [24, Theorems 1, 2], to prove our result, we only need to study the representation of \( M_n(\theta, g) \). In fact, we have the following lemma.

**Lemma A.6**

(i) Under Conditions (C1)-(C4), we have, on \( \mathcal{O}(1) \) neighborhoods of \( \theta_0 \),

\[
M_n(\theta, g) = M(\theta, g) + \mathcal{O}_P\left( \frac{\|\theta - \theta_0\|}{\sqrt{n}} \right) + \mathcal{O}_P\left( \|\theta - \theta_0\|^2 \right) + K_n(\theta_0), \tag{A.10}
\]

where

\[
K_n(\theta_0) = \int [v - g(\theta_0^T u)]^2 J(u) \left( H_n(du, dv) - H(du, dv) \right).
\]

(ii) Under Conditions (C1)-(C4), we have, on \( \mathcal{O}(n^{-1/2}) \) neighborhoods of \( \theta_0 \),

\[
M_n(\theta, g) = \frac{1}{2} (\theta - \theta_0)^T \Lambda(\theta - \theta_0) + n^{-1/2} (\theta - \theta_0)^T W_n + \mathcal{O}_P(n^{-1}) + K_n(\theta_0), \tag{A.11}
\]

where \( W_n = n^{-1/2} \sum_{i=1}^{n} \zeta_i(\psi) \) is a random vector, \( \psi(\cdot) \) and \( \zeta_i(\cdot) \) are defined by (3.3) and (3.4), respectively.

**Proof:** We only need to show (A.10). (A.11) can be done in the same way. Note that

\[
M_n(\theta, g) - M(\theta, g) = 2 \int [v - g(\theta_0^T u)] [g(\theta_0^T u) - g(\theta^T u)] J(u) \left( H_n(du, dv) - H(du, dv) \right)
\]

\[
+ \int [g(\theta_0^T u) - g(\theta^T u)]^2 J(u) \left( H_n(du, dv) - H(du, dv) \right)
\]

\[
+ \int [v - g(\theta_0^T u)]^2 J(u) \left( H_n(du, dv) - H(du, dv) \right)
\]

\[
=: A_{1n} + A_{2n} + K_n(\theta_0).
\]

From a Taylor’s expansion, \( A_{1n} \) can be rewritten as

\[
2(\theta_0 - \theta)^T \int \psi(u, v) \left( H_n(du, dv) - H(du, dv) \right)
\]

\[
- (\theta_0 - \theta)^T \int \beta(u, v) \left( H_n(du, dv) - H(du, dv) \right)(\theta_0 - \theta), \tag{A.12}
\]

where \( \psi(u, v) \) is defined by (3.3), and

\[
\beta(u, v) = [v - g(\theta_0^T u)] \nabla_{\theta, \theta} g(\theta_0^T u) J(u)
\]

with \( \theta_1 \) being a vector between \( \theta \) and \( \theta_0 \). It follows from Conditions (C1) to (C4) that (A.1) holds for \( \psi(u, v) \). Hence, by applying Corollary A.1, the first term in (A.12) is

\[
2(\theta_0 - \theta)^T \left[ n^{-1} \sum_{i=1}^{n} \zeta_i(\psi) + \mathcal{O}_P(n^{-1/2}) \right].
\]
It follows from the multivariate central limit theorem that the first term in (A.12) is of order $O_P\left(\frac{\|\theta - \theta_0\|}{\sqrt{n}}\right)$. By the strong consistency of the Lynden-Bell integral (He and Yang [12, Theorem 3.2]) and the boundedness of $\nabla g$ and $\nabla^2 g$ (see Condition (C4)), the second term in (A.12) is $o_P\left(\|\theta - \theta_0\|\right)$. Moreover, by a Taylor’s expansion,

$$A_{2n} = (\theta_0 - \theta)^T \int \left[ \nabla g(\theta^2 u) \nabla g(\theta^2 u)^T \right] J(u) \left( H_n(du, dv) - H(du, dv) \right) (\theta_0 - \theta),$$

where $\theta_2$ is a vector between $\theta$ and $\theta_0$. Using the strong consistency of the Lynden-Bell integral and Condition (C4) again, we obtain that $A_{2n}$ is also of order $o_P\left(\|\theta - \theta_0\|^2\right)$. The proof of (A.10) is completed. \hfill \Box

With Lemmas A.4 to A.6 in hand, we are able to present the proof of Theorem 3.2 in the following.

Proof of Theorem 3.2: The proof of Theorem 3.2 is similar to that of the Main Lemma in Bouaziz and Lopez [2]. From Theorems 1 and 2 of Sherman [24] and Lemma A.6 to prove Theorem 3.2 we only need to show that

$$M_n(\theta, \hat{g}_n) = M_n(\theta, g) + \hat{K}_n(\theta_0) + o_P\left(\frac{\|\theta - \theta_0\|}{\sqrt{n}}\right) + o_P\left(\|\theta - \theta_0\|^2\right), \quad (A.13)$$

where $\hat{K}_n(\theta_0)$ is a term that depends only on $\theta_0$. Following similar ideas as those in the proof of Theorem 3.5 in [17], especially the theory of empirical process, we obtain that (A.13) holds. We end the proof of Theorem 3.2. \hfill \Box

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