The Graph of Equivalence Classes of Zero Divisors

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Received 13 December 2013; Accepted 5 February 2014; Published 8 May 2014

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We introduce a graph $G_G(L)$ of equivalence classes of zero divisors of a meet semilattice $L$ with 0. The set of vertices of $G_G(L)$ are the equivalence classes of nonzero zero divisors of $L$ and two vertices $[x]$ and $[y]$ are adjacent if and only if $[x] \cap [y] = \{0\}$. It is proved that $G_G(L)$ is connected and either it contains a cycle of length 3 or $G_G(L) \cong K_2$. It is known that two Boolean lattices $L_1$ and $L_2$ have isomorphic zero divisor graphs if and only if $L_1 \cong L_2$. This result is extended to the class of SSC meet semilattices. Finally, we show that Beck’s Conjecture is true for $G_G(L)$.

1. Introduction

The idea of a zero divisor graph was introduced by Beck in [1] to investigate the interplay between ring theoretic properties and graph theoretic properties. The concept of zero divisor graph is also well studied in ordered structures; see Alizadeh et al. [2, 3], Halaˇs and Jukl [4], Halaˇs and L ¨anger [5], Joshi [6], Joshi et al. [7–11], and Lu and Wu [12].

The graph of equivalence classes of zero divisors of commutative rings is well studied in ordered structures; see Alizadeh et al. [2, 3], Halaˇs and Jukl [4], Halaˇs and L ¨anger [5], Joshi [6], Joshi et al. [7–11], and Lu and Wu [12].

Further, we consider the equivalence relation $\sim$ on a meet semilattice $L$ with 0. The main aspect of the graph $G_G(L)$ of equivalence classes of zero divisors of $L$ is the connection to the associated primes of $L$. In general, every vertex of $G_G(L)$ either corresponds to an associated prime or is adjacent to an associated prime of $L$. Further, we consider the equivalence relation $\sim$ on a meet semilattice $L$ with 0, $x \sim y$ if and only if $\text{ann}(x) = \text{ann}(y)$ for $x, y \in L$ and construct the meet semilattice $L' = \{[x] | x \in Z_{\{0\}}(L)\}$. It is proved that $L'$ is an SSC meet semilattice with the property that $G_G(L) \cong G_G(L')$, the zero divisor graph of $L'$. Hence it is clear that the study of $G_G(L)$ is nothing, but the study of zero divisor graph of SSC meet semilattices. This observation helps us to prove that for bounded SSC meet semilattices $L_1$ and $L_2$, $G_{\{0\}}(L_1) \cong G_{\{0\}}(L_2)$ if and only if $L_1 \cong L_2$. This extends the result of LaGrange [16] for Boolean algebra. It is proved that for an SSC meet semilattice $L$, $G_{\{0\}}(L)$ is connected and either it contains a cycle of length 3 or $G_{\{0\}}(L) \cong K_2$. In the class of a finite semicomplemented meet semilattice $L$, $G_G(L)$ is isomorphic to the zero divisor graph $G_{\{0\}}(L)$ if and only if $L$ is SSC.

2. Properties of the Graph $G_G(L)$

We begin with the necessary definitions and terminology.

A nonempty subset $I$ of a meet semilattice $L$ is said to be semi-ideal, if $y \leq x \in I$ implies that $y \in I$. A semi-ideal $I$ of a meet semi-lattice $L$ is said to be an ideal, if $a, b \in I$ implies that $a \vee b \in I$. Dually we have the concept of a filter. A proper semi-ideal (ideal) is said to be prime, if $a \wedge b \in I$ implies either $a \in I$ or $b \in I$.

A prime semi-ideal (ideal) $P$ of a meet semilattice $L$ is said to be minimal prime semi-ideal (ideal), if there is no prime ideal $Q$ such that $Q \subsetneq P$.

The concept of SSC lattices is introduced by Janowitz [17]. We write this definition for a meet semilattice with 0.

A meet semilattice $L$ with 0 is said to be semicomplemented if, for $a \neq 1$, there exists $b \neq 0$ such that $a \land b = 0$ and $L$ is said to be section semicomplemented (in brief SSC) if for $a, b \in L$ with $a < b$, there exists nonzero $c$ such that $0 < c \leq b$ and $a \land c = 0$. It is clear that every SSC lattice is a semicomplemented lattice, but not conversely. More details about SSC poset can be found in Thakare et al. [18, 19], Joshi [20].

Let $L$ be a meet semilattice with 0. The set of all zero divisors of $L$ is denoted by $Z_{\{0\}}(L) = \{x \in L | x \land y = 0$
for some \( y \in L \setminus \{0\} \) and the set of all nonzero zero divisors is denoted by \( Z_{\{0\}}(L)^* \). Clearly, \( Z_{\{0\}}(L)^* \cup \{0\} = Z_{\{0\}}(L) \).

Let \( G \) be a graph, and let \( x, y \) be distinct vertices in \( G \). We denote by \( d(x, y) \) the length of a shortest path from \( x \) to \( y \) if it exists and put \( d(x, y) = \infty \) if no such path exists. The \textit{diameter} of \( G \), denoted by \( \text{diam}(G) \), is zero if \( G \) is the graph on one vertex and is \( \text{diam}(G) = \sup\{d(x, y) \mid x \text{ and } y \text{ are distinct vertices of } G\} \) otherwise. A \textit{cycle} in a graph \( G \) is a path that begins and ends at the same vertex. The \textit{girth} of \( G \), written \( \text{girth}(G) \), is the length of the shortest cycle in \( G \) (and \( \text{girth}(G) = \infty \) if \( G \) has no cycles).

The chromatic number of \( G \) is denoted by \( \chi(G) \). Thus, \( \chi(G) \) is the minimum number of colors which can be assigned to the elements of \( L \) such that adjacent elements receive different colors. If this number is not finite, write \( \chi(G) = \infty \). A subset \( C \) of \( G \) is a clique if any two distinct vertices of \( C \) are adjacent. If \( G \) contains a clique with \( n \) elements and every clique has at most \( n \) elements then the clique number of \( G \) is \( \text{Clique}(G) = n \). If the sizes of the cliques are not bounded, then \( \text{Clique}(G) = \infty \). We always have \( \chi(G) \geq \text{Clique}(G) \).

Joshi [6] introduced the concept of the zero divisor graph of a poset \( P \) having the smallest element \( 0 \) with respect to an ideal \( I \) of \( P \). We mention below this definition when the corresponding poset is a meet semilattice and \( I = \{0\} \). Note that this definition coincides with the definition given by Lu and Wu [12] when \( I = \{0\} \).

\textit{Definition 1 (Joshi [6], Lu and Wu [12])}. Let \( L \) be a semilattice with 0. We associate a simple undirected graph with \( L \), called the \textit{zero divisor graph} of \( L \) with respect to \( \{0\} \), denoted by \( G_{\{0\}}(L) \), with the set of vertices is \( V(G_{\{0\}}(L)) = Z_{\{0\}}(L)^* \times \{x \in L \setminus \{0\} \mid x \wedge y = 0 \text{ for some } y \in L \setminus \{0\}\} \) and two distinct vertices \( x, y \) are adjacent if and only if \( x \wedge y = 0 \).

For any \( x, y \in L \), we say that \( x \rightarrow y \) if and only if \( \text{ann}(x) = \text{ann}(y) \), where \( \text{ann}(a) = \{b \mid a \wedge b = 0\} \). Note that \( \sim \) is an equivalence relation on \( L \). Furthermore, if \( x_1 \sim x_2 \) and \( x_1 \wedge y = 0 \), then \( y \in \text{ann}(x_1) = \text{ann}(x_2) \) and hence \( x_2 \wedge y = 0 \). We define \( [x] \), the equivalence class of \( x \), as follows: \( [x] = \{z \in L \mid \text{ann}(z) = \text{ann}(x)\} \).

\textit{Lemma 2}. Let \( L \) be a meet semilattice with 0 and \( L' = \{[x] \mid x \in L\} \), where \( [x] = \{z \in L \mid \text{ann}(z) = \text{ann}(x)\} \). Then \( L' \) is a meet semilattice under the partial order \( [x] \leq [y] \text{ if and only if } \text{ann}(y) \subseteq \text{ann}(x) \).

\textit{Proof}. Since \( \text{ann}(x), \text{ann}(y) \subseteq \text{ann}(x \wedge y) \), we have \( [x \wedge y] \subseteq [x], [y] \). We claim that \( [x] \wedge [y] = [x \wedge y] \). Let \( [t] \) be another lower bound of \( [x] \) and \( [y] \). Then \( \text{ann}(x), \text{ann}(y) \subseteq \text{ann}(t) \).

Now, we claim that \( \text{ann}(x \wedge y) \subseteq \text{ann}(t) \). Let \( z \in \text{ann}(x \wedge y) \). Then \( z \wedge x \in \text{ann}(y) \subseteq \text{ann}(t) \). This gives \( z \wedge x \wedge t = 0 \); that is, \( z \wedge t = \text{ann}(x) \subseteq \text{ann}(t) \). Hence \( z \wedge t = 0 \); that is, \( z \in \text{ann}(t) \). This proves that \( \text{ann}(x \wedge y) \subseteq \text{ann}(t) \). Thus \( [t] \subseteq [x \wedge y] \) and this yields \( \text{ann}(x \wedge y) = [x \wedge y] \). Clearly, if \( L \) is a finite lattice, then \( L' \) is also a lattice.

\textit{Lemma 3}. \( L' \) is an SSC meet semilattice.

\textit{Proof}. Let \( 0 < [x] \notin [y] \). Then \( \text{ann}(y) \subseteq \text{ann}(x) \). Hence there exists \( z \in \text{ann}(x) \) such that \( z \notin \text{ann}(y) \). Thus \( 0 \neq z \wedge y = t \). But then \( [0] < [t] \leq [y] \) and \([t] \wedge [x] = [0] \). Thus \( L' \) is an SSC meet semilattice.

Now, we introduce the graph \( G_E(L) \) of equivalence classes of zero divisors of a meet semilattice \( L \) with 0, denoted by \( G_E(L) \).

\textit{Definition 4}. Let \( L \) be a meet semilattice with 0. We associate a simple undirected graph with \( L \) whose vertices are the equivalence classes of elements in \( Z_{\{0\}}(L)^* \) and with each pair of distinct classes \( [x] \) and \( [y] \) joined by an edge if and only if \( [x] \wedge [y] = [0] \).

We illustrate the concept of the graph of equivalence classes of zero divisor with an example, see Figure 1.

Consider the following two infinite lattices \( L \) and \( L_1 \).

We can see that the zero divisor graph of lattices \( L, G_{\{0\}}(L) \) is infinite, but the graph of equivalence classes of zero divisors of the lattice \( L \), namely, \( G_E(L) \), is finite. Also for the lattices \( L_1, G_{\{0\}}(L_1) \), and \( G_E(L_1) \) both are infinite, see Figure 2.

\textit{Corollary 5}. Let \( L \) be a meet semilattice with 0. Then there exists an SSC meet semilattice \( L' \) (as constructed in Lemma 2) such that \( G_E(L) = G_{\{0\}}(L') \).
Proof. From the construction of $L'$, it is clear that $G_E(L) = G_{\{0\}}(L')$. \hfill $\square$

Hence from Corollary 5, it is clear that the study of the graph of equivalence classes of zero divisors of $L$ is nothing, but the study of zero divisor graphs of SSC meet semilattices.

Remark 6. Recall that a prime semi-ideal $P$ of a meet semilattice $L$ is an associated prime if it is the annihilator of some elements of $L$; that is, $P = \text{ann}(x)$ for some nonzero $x \in L$. The set of associated primes is denoted by $\text{Ass}(L)$.

Note that every zero divisor is contained in an annihilator ideal and maximal annihilators are associated primes; we have $Z_{\{0\}}(L) \subseteq \bigcup_{P \in \text{Ass}(L)} P$. Clearly, the reverse inclusion is trivial. Hence $Z_{\{0\}}(L) = \bigcup_{P \in \text{Ass}(L)} P$; that is, the set of zero-divisors of $L$ equals the union of all associated primes of $L$.

It is obvious to prove that a map $\phi : \text{Ass}(L) \rightarrow V(G_E(L))$ defined by $\phi(P) = \{x\}$, where $P = \text{ann}(x)$, is an injective map. We adopt the conventions of Spiroff and Wickham [15], and by a slight abuse of terminology we will refer to the vertex $\{y\}$ as an associated prime if $\text{ann}(y) \in \text{Ass}(L)$. It will be clear from context whether $\{y\}$ refers to an equivalence class, a vertex, or a specific annihilator.

Theorem 7. Let $L$ be a meet semilattice with $0$, and let $\text{Clique}(G_{\{0\}}(L)) < \infty$. If $\text{ann}(x)$ and $\text{ann}(y)$ are distinct prime semi-ideals of $L$, then $\{x\}$ and $\{y\}$ are adjacent in $G_E(L)$. Furthermore, every vertex of $G_E(L)$ is either an associated prime semi-ideal or adjacent to a maximal semi-ideal in $\mathcal{L} = \{\text{ann}(x) \mid (0 \neq x \in L)\}$.

Proof. Since $\text{ann}(x)$ and $\text{ann}(y)$ are distinct, we have $x \wedge y = 0$. This further gives that $\{x\} \wedge \{y\} = \{x \wedge y\} = \{0\}$. Thus $\{x\}$ and $\{y\}$ are adjacent. Furthermore, suppose that $\{x\} \in V(G_E(L))$ is not an associated prime semi-ideal. Then there exists $\{z\} \in V(G_E(L))$ such that $\{x\} \wedge \{z\} = \{0\}$. Hence $x \in \text{ann}(z)$. We claim that $\text{ann}(z)$ is maximal. Otherwise, since $\text{Clique}(G_{\{0\}}(L)) < \infty$, $\mathcal{L}$ has a maximal element, say $\text{ann}(z') (z' \neq 0)$ containing $\text{ann}(z)$, then $x \in \text{ann}(z) \subseteq \text{ann}(z')$. It is clear that every maximal element of $\mathcal{L}$ is prime. Therefore $x \wedge z' = 0$. Thus there is an edge between $\{x\}$ and $\{z'\}$. This completes the proof. \hfill $\square$

Remark 8. For a Boolean lattice, $\{a\} = \{a\}$ for every $a \in Z_{\{0\}}(L)$, and hence in this case, $G_E(L)$ and $G_{\{0\}}(L)$ are isomorphic. This fact is illustrated in Figure 3.

In view of Remark 8, we raise the following problem.

**Question 1.** Find a class of meet semilattices $\mathcal{L}$ such that $G_{\{0\}}(L) \cong G_E(L)$ for $L \in \mathcal{L}$.

Before we answer this question, we need the following result which follows from the definition of SSC meet lattice.

**Lemma 9.** Let $L$ be a meet semilattice with $0$. Then $L$ is SSC if and only if for $a, b \in L$, $\text{ann}(a) = \text{ann}(b)$ implies $a = b$.

Proof. Suppose that $L$ is SSC. Further, assume that there exist $a, b \in L$ such that $\text{ann}(a) = \text{ann}(b)$ and $a \neq b$. Since $L$ is SSC, there exist $c \in L$ such that $0 < c \leq a$ and $c \wedge b = 0$. This gives $c \in \text{ann}(b) = \text{ann}(a)$, a contradiction to the fact that $c \neq 0$. Hence $a = b$. \hfill $\square$

**Theorem 10.** Let $L$ be a finite semicomplemented meet semilattice. Then $\phi : G_{\{0\}}(L) \rightarrow G_E(L)$ is a graph isomorphism if and only if $L$ is SSC.

Proof. Let $L$ be a SSC meet semilattice. Then $G_E(L) \cong G_{\{0\}}(L)$ follows from Lemma 9.

Conversely, assume that $G_E(L) \cong G_{\{0\}}(L)$. First, we prove that $[a] = \{a\}, \forall a \in Z_{\{0\}}(L)^*$. Let $b \in [a]$ for $a \neq b \in Z_{\{0\}}(L)^*$. Then we have $[a] = \{b\}$ for $a \neq b$, a contradiction to $|V(G_E(L))| = |V(G_{\{0\}}(L))|$. Now, let $a < b (a \neq 0)$. Since $L$ is semicomplemented, $a, b \in Z_{\{0\}}(L)^*$. This gives $\text{ann}(b) \not\subseteq \text{ann}(a)$. Otherwise, if $\text{ann}(a) = \text{ann}(b)$, then $b \in [a] = \{a\}$, a contradiction. From the proof of Lemma 9, it is clear that $L$ is SSC. \hfill $\square$

If we consider that the $\phi : G_{\{0\}}(L) \rightarrow G_E(L)$ is a graph isomorphism defined by $\phi(a) = [a]$, then we do not need even the finiteness of $L$.

**Theorem 11.** Let $L$ be a semicomplemented meet semilattice. Then $\phi : G_{\{0\}}(L) \rightarrow G_E(L)$ is a graph isomorphism defined by $\phi(a) = [a]$ if and only if $L$ is SSC.

Proof. Let $L$ be a SSC meet semilattice. Then $G_E(L) \cong G_{\{0\}}(L)$ follows from Lemma 9.

Conversely, assume that $\phi : G_{\{0\}}(L) \rightarrow G_E(L)$ is a graph isomorphism defined by $\phi(a) = [a]$. This gives $[a] = \{a\}$, for all $a \in Z_{\{0\}}(L)^*$. Let $a < b (a \neq 0)$. Since $L$ is semicomplemented, $a, b \in Z_{\{0\}}(L)^*$. This gives $\text{ann}(b) \not\subseteq \text{ann}(a)$. From the proof of Lemma 9, it is clear that $L$ is SSC. \hfill $\square$

From Figure 4, it is clear that for the posets $P_1, P_2$, $G_{\{0\}}(P_1) \cong G_{\{0\}}(P_2) \Rightarrow P_1 \cong P_2$. Hence, it is worth to study the following problem.

**Problem 12.** Find a class of posets $\mathcal{P}$ such that $G_{\{0\}}(P_1) \cong G_{\{0\}}(P_2)$ if and only if $P_1 \cong P_2$ for the posets $P_1, P_2 \in \mathcal{P}$.
In [8], Joshi and Khiste answered this problem in the case of Boolean posets. More details about Boolean posets can be found in Waphare and Joshi [21].

**Theorem 13** (Joshi and Khiste [8, Theorem 2.11]). Let $P_1$ and $P_2$ be Boolean posets. Then $G_0(P_1) \cong G_0(P_2)$ if and only if $P_1 \equiv P_2$.

It is well known that there is a $1 \rightarrow 1$ correspondence between Boolean algebras and Boolean rings.

A consequence of the above theorem is the following result.

**Corollary 14** (LaGrange [16, Theorem 4.1], Mohammadian [22, Theorem 7]). Let $P_1$ and $P_2$ be Boolean algebras. Then $G_0(P_1) \cong G_0(P_2)$ if and only if $P_1 \equiv P_2$.

Now, we are ready to extend this result to the class of SSC meet semilattices. Note that every Boolean lattice is SSC but not conversely. Consider the lattice $L_1$ depicted in Figure 2 is SSC but not a Boolean lattice.

**Theorem 15.** Let $L_1$ and $L_2$ be bounded SSC meet semilattices. Then the following statements are equivalent:

1. $G_0(L_1) \cong G_0(L_2)$;
2. $G_0(L_1) \equiv G_0(L_2)$;
3. $L_1 \equiv L_2$.

**Proof.** The equivalence of statements (1) and (2) follows from Lemma 9.

(2) $\Rightarrow$ (3) : Since $L_1$ and $L_2$ are semicomplemented, $V(G_0(L_i)) = L_i \setminus \{0_L, 1_L\}$ for $i = 1, 2$. Let $\phi : V(G_0(L_1)) \rightarrow V(G_0(L_2))$ be a graph isomorphism. Define a map $f : L_1 \rightarrow L_2$ such that $f(a) = \phi(a)$, for all $a \in V(G_0(L_1))$ along with $f(0_{L_1}) = 0_{L_2}$, and $f(1_{L_1}) = 1_{L_2}$, where $0_{L_1}, 1_{L_1}$ is the smallest (largest) element of $L_1$. Since $\phi$ is bijective, $f$ is bijective. We claim that $f$ is a biderer preserving map.

Let $a \leq b$. First, we show that $f(a) \leq f(b)$. If any of $a, b$ is either $0_{L_1}$ or $1_{L_1}$, then we are through. Hence assume that $a, b \notin \{0_{L_1}, 1_{L_1}\}$. Suppose that $f(a) \leq f(b)$; that is, $\phi(a) \leq \phi(b)$. Since $L_1$ is SSC, there exists $\phi(c) \in L_2$ such that $0_{L_2} < \phi(c) \leq \phi(a)$ and $\phi(a) \leq \phi(b) = 0_{L_2}$. Since $b \neq 0_{L_2}$, we have $\phi(b) \neq 0_{L_2}$. This gives that $\phi(c)$ and $\phi(b)$ are adjacent in $G_0(L_2)$. Since $\phi$ is a graph isomorphism, we have that $c$ and $b$ are adjacent which further imply that $c \wedge b = 0_{L_1}$. Hence, $c \wedge a = 0_{L_1}$ in $L_1$. Thus $a$ and $c$ are adjacent in $G_0(L_1)$. Hence, we have $\phi(c) \land \phi(a) = 0_{L_2}$. But $\phi(c) \leq \phi(a)$ gives $\phi(c) = 0_{L_2}$, a contradiction. Hence $f(a) \leq f(b)$ in $L_2$.

Conversely, assume that $f(a) \leq f(b)$ in $L_2$. We claim that $a \leq b$. If any of $f(a)$ or $f(b)$ is $0_{L_2}$ or $1_{L_2}$, then we are through. Let us assume that $f(a), f(b) \notin \{0_{L_2}, 1_{L_2}\}$. This yields $a, b \notin \{0_{L_1}, 1_{L_1}\}$. Now, suppose on the contrary that $a \neq b$ for $a, b \notin \{0_{L_1}, 1_{L_1}\}$. Since $L_1$ is SSC, there exists $c \in L_1$ such that $0_{L_1} < c \leq a$ and $c \wedge b = 0_{L_1}$. Thus $c$ and $b$ are adjacent in $G_0(L_1)$. Then $\phi(c)$ and $\phi(b)$ are adjacent in $G_0(L_2)$. Then $\phi(c) \land \phi(b) = 0_{L_2}$ in $L_2$. Replacing the role of $a, b$ in the above proof by $c, a$, respectively, we have $\phi(c) \leq \phi(a)$. This implies that $\phi(c) \leq \phi(a) \leq \phi(b)$. But this gives $\phi(c) = \phi(c) \land \phi(b) = 0_{L_2}$, a contradiction. Hence $a \leq b$. Thus $L_1 \equiv L_2$.

(3) $\Rightarrow$ (2) : Obvious.

**Lemma 16.** Let $L_1$ be a bounded SSC meet semilattice, and $L_2$ be a bounded semicomplemented meet semilattice such that $G_0(L_1) \cong G_0(L_2)$. Then $L_2$ is also SCC.

**Proof.** Since $L_1$ and $L_2$ are semicomplemented, $V(G_0(L_i)) = L_i \setminus \{0_{L_i}, 1_{L_i}\}$ for $i = 1, 2$. Let $\phi : V(G_0(L_1)) \rightarrow V(G_0(L_2))$ be a graph isomorphism. Define a map $f : L_1 \rightarrow L_2$ such that $f(a) = \phi(a)$, for all $a \in V(G_0(L_1))$ along with $f(0_{L_1}) = 0_{L_2}$, and $f(1_{L_1}) = 1_{L_2}$, where $0_{L_1}, 1_{L_1}$ is the smallest (largest) element of $L_1$. Since $\phi$ is a bijective map, $f$ is bijective. Let $f(a) < f(b)$ in $L_2$. Since $L_2$ is semicomplemented, and if $f(a) = 0_{L_2}$ or $f(b) = 1_{L_2}$, then we are through. Hence assume that $f(a) \neq 0_{L_2}$ and $f(b) \neq 1_{L_2}$. Then clearly, $a \neq b$ and $b \neq 1_{L_1}$. Since $L_1$ is SSC and $a \wedge b \lt b$, there exists $0_{L_1} \lt c \lt b$ and $c \wedge a = 0_{L_1}$. Since $c$ and $b$ are adjacent in $G_0(L_1)$, we have $\phi(c)$ and $\phi(b)$ are adjacent in $G_0(L_2)$; that is, $\phi(c) \land \phi(a) = 0_{L_2}$. Further $b$ and $c$ are nonadjacent in $G_0(L_1)$. Then $\phi(b)$ and $\phi(c)$ are nonadjacent in $G_0(L_2)$. Consider $0_{L_2} \neq \phi(b) \land \phi(c) = f(b) \land f(c) = f(d)$ say. Clearly, $0_{L_2} < f(d) \leq f(b)$ and $f(d) \land f(a) = 0_{L_2}$ (as $\phi(c) \land \phi(a) = 0_{L_2}$ implies $f(c) \land f(a) = 0_{L_2}$). Thus $L_2$ is SCC.

**Theorem 17.** Let $L_1$ and $L_2$ be bounded semicomplemented meet semilattices. If $G_E(L_1) \equiv G_E(L_2)$, then $L_2$ is SSC.

**Proof.** For the given lattice $L_1$, we can construct an SSC meet semilattice $L_1'$ (as given in Lemma 2), such that $G_E(L_1) \equiv G_0(L_1')$. This gives $G_0(L_1') \equiv G_0(L_2)$. By Lemma 16, $L_2$ is SSC.

The above result is an analogue to the following result of Anderson and LaGrange [14].

**Theorem 18** (Anderson and LaGrange [14, Theorem 2.6]). Let $R$ and $S$ be commutative reduced rings with $1 \neq 0$ and $Z(S) \neq \{0\}$. If $\Gamma(R) \equiv \Gamma(S)$, then $S$ is a Boolean ring.

We quote the result of Joshi [6, Theorem 2.4] when the corresponding poset is a meet semilattice.

**Theorem 19** (Joshi [6, Theorem 2.4]). Let $L$ be a meet semilattice with 0. Then the graph $G_0(L)$ is connected and $\text{diam}(G_E(L)) \leq \text{diam}(G_0(L)) \leq 3$.

**Theorem 20.** Let $L$ be a meet semilattice with 0. Then the graph $G_E(L)$ is connected and $\text{diam}(G_E(L)) \leq \text{diam}(G_0(L)) \leq 3$. 

![Figure 4: Nonisomorphic lattices with isomorphic zero divisor graphs.](image-url)
Proof. Follows from Corollary 5, Theorem 19, and the fact that adjacency in $G_E(L)$ implies adjacency in $G(G_0(L))$. Hence $\text{diam}(G_E(L)) \leq \text{diam}(G(G_0(L))) \leq 3$. \hfill $\blacksquare$

**Theorem 21.** Let $L$ be an SSC lattice. Then $G(G_0(L))$ is either $K_2$ or contains a cycle of length 3. Hence $\text{girth}(G(G_0(L))) = 3$.

**Proof.** If $V(G(G_0(L))) = 2$, then $G(G_0(L)) \cong K_2$, by Theorem 19. Now assume that $|V(G(G_0(L)))| \geq 3$. Choose $a, b, c \in V(G(G_0(L)))$ such that $a - b - c$ is a path with $a \neq c$ and $a \wedge c \neq 0$. Since $L$ is SSC and without loss of generality assume that $a \wedge c < a$, there exists $x$ such that $0 < x \leq a$ and $c \wedge x = 0$. Clearly, $x \in V(G(G_0(L)))$. Since $a \wedge b = 0$, we have $x \wedge b = 0$. Thus we have a cycle $b - x - c - b$ of length 3. \hfill $\blacksquare$

An immediate consequence of the above theorem is the following corollary.

**Corollary 22.** Let $L$ be a meet semilattice with 0; then $G_E(L)$ is either $K_2$ or contains a cycle of length 3. Hence, if $G(E(L)) \neq K_2$, then $\text{girth}(G_E(L)) = 3 = \text{girth}(G(G_0(L)))$.

**Proof.** It follows from Corollary 5 and Theorem 21. \hfill $\blacksquare$

**Remark 23.** In the above result, the condition of existence of a cycle in $G_E(L)$ is necessary. In Figure 1, we can observe that $G(G_0(L))$ has a cycle but $G_E(L)$ has no cycle.

### 3. Correlation between Diameter of $G(G_0(L))$ and $G_E(L)$

The following result gives the correlation between diameter of $G(G_0(L))$ and $G_E(L)$.

**Theorem 24.** The following statements are true for a meet semilattice $L$ with 0.

(a) If $\text{diam}(G(G_0(L))) = 1$, then $\text{diam}(G_E(L)) = 1$.

(b) If $\text{diam}(G(G_0(L))) = 2$, then $\text{diam}(G_E(L)) = 1$ or 2.

(c) If $\text{diam}(G(G_0(L))) = 3$, if and only if $\text{diam}(G_E(L)) = 3$.

(d) If $\text{diam}(G_E(L)) = 1$, then $\text{diam}(G(G_0(L))) = 1$ or 2.

(e) If $\text{diam}(G_E(L)) = 2$, then $\text{diam}(G(G_0(L))) = 2$.

**Proof.** (a) It is clear that $\text{diam}(G_E(L)) \neq 0$. Hence the result follows from Theorem 20.

(b) Suppose that $\text{diam}(G(G_0(L))) = 2$. So we have a path $a - b - c$ in $G(G_0(L))$. We have two possibilities.

**Case 1.** If $[a]$, $[b]$, and $[c]$ are distinct vertices of $G_G(L)$, we get the path $[a] - [b] - [c]$. Hence by Theorem 20, $\text{diam}(G_E(L)) = 2$.

**Case 2.** The only possibility in this case is $[a] = [c]$, and hence, $\text{diam}(G_E(L)) = 1$.

(c) Let $\text{diam}(G(G_0(L))) = 3$. Then there exists a path $a - c - d - b$. Since $c \in \text{ann}(a)$ gives $[a] \neq [b]$, otherwise $d(a, b) = 2$, a contradiction. Now assume that $d([a], [b]) = 2$ in $G_E(L)$. Then there exists $[x] \in V(G_E(L))$ such that $[a] - [x] - [b]$. This yields that $d(a, b) = 2$, again a contradiction. Thus by

**Theorem 20**, $\text{diam}(G_E(L)) = 3$. The converse follows from Theorem 20.

(d) Suppose that $\text{diam}(G_E(L)) = 1$. Then $\text{diam}(G(G_0(L))) \neq 3$, by (c). Hence the result.

(e) Suppose that $\text{diam}(G_E(L)) = 2$; then there exists a path $[a] - [c] - [b]$ in $G_G(L)$. This yields a path $a - c - b$ in $G_G(L)$. Thus $\text{diam}(G_G(L)) \neq 1$. The result follows from (c) and $\text{diam}(G(G_0(L))) \neq 3$. \hfill $\blacksquare$

**Remark 25.** The assertions (a) and (d) of Theorem 24 are justified in Figures 1 and 5, whereas the assertions (b) and (e) are justified in Figures 5 and 6. Figure 3 verifies the assertion (c).

**Corollary 26.** Let $L$ be a meet semilattice with 0. Then $\text{diam}(G_E(L)) \leq 2$ if and only if $\text{diam}(G(G_0(L))) \leq 2$.

### 4. Cut Vertices in $G_E(L)$

In this section, we examine the properties of cut vertices of $G_E(L)$.

**Definition 27.** A lattice $L$ with 0 is said to be $0$-distributive, if $a \wedge b = a \wedge c = 0$ implies that $a \wedge (b \vee c) = 0$; Varlet [23]. More details about 0-distributive posets can be found in Joshi and Waphare [24]; see also Joshi and Mundlik [25].

**Theorem 28.** Let $L$ be a 0-distributive lattice. If $[a]$ is a cut vertex of $G_E(L)$, then $[a] \cup \{0\}$ forms an ideal of $L$.

**Proof.** Let $[a]$ be a cut vertex of $G_E(L)$. Let $a_1$ and $a_2$ be nonzero elements of $[a]$. Since $a_1, a_2 \in [a]$, $\text{ann}(a_1 \wedge a_2) = \text{ann}(a_1) = \text{ann}(a_2)$. Now, it is enough to show that $\text{ann}(a_1 \vee a_2) = \text{ann}(a_1 \wedge a_2)$. Clearly, $\text{ann}(a_1 \vee a_2) \subseteq \text{ann}(a_1 \wedge a_2)$.

Let $t \in \text{ann}(a_1 \wedge a_2)$. This gives $t \wedge a_1 = t \wedge a_2 = t \wedge a = 0$. Since $L$ is 0-distributive, we have $t \wedge (a_1 \vee a_2) = 0$; that is, $t \in \text{ann}(a_1 \vee a_2)$. Thus $\text{ann}(a_1 \vee a_2) \subseteq \text{ann}(a_1 \wedge a_2)$. This implies that $a_1 \vee a_2 \in [a]$.

Now suppose that $x \in [a] \cup \{0\}$ and $y \leq x$, to show that $y \in [a] \cup \{0\}$. If $x = 0$ or $y = 0$, then $y \in [a] \cup \{0\}$. Let $y \neq 0$ and $y \leq x$. Then $\text{ann}(x) \subseteq \text{ann}(y)$. Since $[a]$ is a cut vertex of $G_E(L)$,
therefore for any two arbitrary vertices \([b], [c] \in V(G_E(L))\), we have a path \([b] - [a] - [c] \). Therefore \(b \in \text{ann}(a) = \text{ann}(x)\); that is, \(b \land x = 0\), which further yields \(b \land y = 0\). Similarly, \(c \land y = 0\). Therefore, we get a path \([b] - [y] - [c] \). Since \([a] \) is cut vertex, \([a] = [y] \). Thus \([a] \cup \{0\}\) is an ideal.

**Remark 29.** It is clear from the proof of the above theorem that in a general lattice, if \([a] \) is a cut vertex of \(G_E(L)\), then \([a] \cup \{0\}\) is a semi-ideal.

**Lemma 30.** Let \(L\) be a meet semilattice with \(0\). If \([a] \) is a cut vertex of \(G_E(L)\), then \(\text{ann}(a)\) is maximal in \(\mathcal{L} = \{\text{ann}(x) \mid 0 \neq x \in L\}\). Hence \(\text{ann}(a)\) is a prime semi-ideal.

**Proof.** Let \([a] \) be a cut vertex of \(G_E(L)\), and let \(G_1 \) and \(G_2 \) be mutually separated subgraphs of \(G_E(L)\) with \(V(G_1 \cup G_2) = V(G_E(L)) \setminus [a]\). Let \([b] \in G_1\) and \([c] \in G_2\), therefore we have the path \([b] - [a] - [c] \). Suppose that \(\text{ann}(a) \subseteq \text{ann}(x)\) for some \(\text{ann}(x) \in \mathcal{L}\). Since \([b] - [a] - [c] \) is an edge, \(b \in \text{ann}(a) \subseteq \text{ann}(x)\). Then \([b] \land [x] = [0]\). Similarly, \([c] \land [x] = [0]\). Thus we have another path \([b] - [x] - [c]\) passing through \([x]\). Since \([a] \) is a cut vertex of \(G_E(L)\), then \([x] = [a]\). Thus \(\text{ann}(x) = \text{ann}(a)\). It is easy to prove that \(\text{ann}(a)\) is a prime semi-ideal.

**Lemma 31.** If \([a] \) is a cut vertex of \(G_E(L)\), then all other associated primes of \(G_E(L)\) are contained in only one component of \(G_E(L) \setminus [a]\).

**Proof.** Suppose that \(G_1 \) and \(G_2 \) are two mutually separated connected components of \(G_E(L) \setminus [a]\), and each contains an associated prime. It is easy to observe that the associated primes are adjacent, and hence \(G_1 \), \(G_2 \) are connected, a contradiction.

Alizadeh et al. [2] raised the following problem.

**Problem 32.** Characterize those posets \(P\) for which \((\text{diam}(G_{E_0}(P)), \text{girth}(G_{E_0}(P))) = (2,3)\) or \((3,3)\).

In view of Corollary 5 and the following result, it is clear that for an SSC meet semilattice \(L\), \((\text{diam}(G_{E_0}(L)), \text{girth}(G_{E_0}(L))) = (3,3)\).

**Theorem 33.** Let \(L\) be a meet semilattice with \(0\). If \(|V(G_E(L))| \geq 4 \) and \(G_E(L)\) have at least \(2\) cut vertices, then \((\text{diam}(G_E(L)), \text{girth}(G_E(L))) = (3,3)\).

**Proof.** Let \([a] \) and \([b] \) be cut vertices of \(G_E(L)\). Since \([a] \) is a cut vertex of \(G_E(L)\), there is some \([a_1] \) such that any path connecting \([a_1] \) and \([b] \) must include \([a] \). Similarly, since \([b] \) is a cut vertex, there is some \([b_1] \) such that any path connecting \([b_1] \) and \([a] \) must include \([b] \). Clearly, \([a_1] \neq [b_1] \) and any path from \([a_1] \) to \([b_1] \) must include \([a] \) and \([b] \) and so \(\text{diam}([a], [b]) \geq 3\). By Theorem 20, \(\text{diam}(G_E(L)) = 3\). Thus \((\text{diam}(G_E(L)), \text{girth}(G_E(L))) = (3,3)\), by Corollary 22.

It is easy to see that Figure 3 fulfills the conditions of Theorem 33.

5. **Beck’s Conjecture for \(G_E(L)\)**

Joshi [6] (see also Halaš and Jukl [4], Nimbhorkar et al. [26], and Lu and Wu [12]) proved that Beck’s Conjecture is true for the zero divisor graphs of posets. We quote this result when the poset is a meet semilattice and ideal is a zero ideal.

**Theorem 34** (Joshi [6, Theorem 2.9]). Let \(L\) be a meet semilattice with \(0\). If \(\text{Clique}(G_{E_0}(L)) < \infty\), then \(\text{Clique}(G_{E_0}(L)) = \chi(G_{E_0}(L)) = n\), where \(n\) is the number of minimal prime semi-ideals of \(L\).

From Theorem 34 and Corollary 5, it is clear that Beck’s Conjecture is true for the graph \(G_E(L)\). In the sequel, we calculate the chromatic number of \(G_E(L)\). As a preparation, we need the following easy lemma.

**Lemma 35.** Let \(P\) be a minimal prime semi-ideal of a meet semilattice \(L\) with \(0\). Then for any \(x \in P\), there exists \(y \notin P\) such that \(x \land y = 0\).

**Proof.** It follows from the fact that \(L \setminus P\) is a maximal filter of \(L\) and \(x \notin L \setminus P\).

**Lemma 36.** Let \(L\) be a meet semilattice with \(0\), and let \(L' = (\text{Clique}(G_{E_0}(L))\) be a meet semilattice with \(0\) (as constructed in Lemma 2). If \(P\) is a minimal prime semi-ideal of \(L\), then \(P' = \{[x] \mid x \in P\}\) is a minimal prime semi-ideal of \(L'\).

**Proof.** First, we prove that \(P'\) is a semi-ideal. Let \([y] < [x] \in P'\). On the contrary, assume that \([y] \notin P'\). Hence \(\text{ann}(y) \subseteq P\). Further, \([y] < [x] \) gives \(\text{ann}(x) \subseteq \text{ann}(y)\) which yields \(\text{ann}(x) \subseteq P\). Since \(x \in P\), by Lemma 35, there exists \(y \notin P\) such that \(x \land y = 0\); that is, \(y \in \text{ann}(x)\). But then \(y \in \text{ann}(x) \subseteq P\), a contradiction. Thus \(P'\) is a semi-ideal.

Let \([x] \land [y] \in P'\). Then \(x \land y \in P\). By primeness of \(P\), either \(x \in P\) or \(y \in P\). This gives either \([x] \in P'\) or \([y] \in P'\). Thus \(P'\) is prime. Let \(Q'\) be a prime semi-ideal of \(L'\) such that \(Q' \subseteq P'\). Hence there exists \([x] \in P'\) such that \([x] \notin Q'\). Again by Lemma 35, there exists \(y \notin P'\) such that \(x \land y = 0\). This gives \([x] \land [y] = [0] \in Q'\). But then \([y] \in Q' \subseteq P'\), a contradiction. Thus \(P'\) is a minimal prime semi-ideal of \(L'\).

On similar lines, we can prove the following result.

**Lemma 37.** Let \(L\) be a meet semilattice with \(0\), and let \(L' = (\text{Clique}(G_{E_0}(L))\) be a meet semilattice with \(0\) (as constructed in Lemma 2). If \(P'\) is a minimal prime semi-ideal of \(L'\), then \(P = \{[x] \mid [x] \in P'\}\) is a minimal prime semi-ideal of \(L\).

Let us denote the set of all minimal prime semi-ideals of \(L\) by \(\text{Min}(L)\).

**Theorem 38.** Let \(L\) be a meet semilattice with \(0\), and let \(L' = (\text{Clique}(G_{E_0}(L))\) be a meet semilattice with \(0\) (as constructed in Lemma 2). Let \(\phi : \text{Min}(L) \rightarrow \text{Min}(L')\) be a map defined by \(\phi(P) = P' = \{[x] \mid x \in P\}\). Then \(\phi\) is bijective.

**Proof.** Let \(\phi : \text{Min}(L) \rightarrow \text{Min}(L')\) be a map defined by \(\phi(P) = P' = \{[x] \mid x \in P\}\). By Lemma 36, \(P'\) is a minimal prime semi-ideal. First, we show that \(\phi\) is one-to-one. Let
\( \phi(P) = \phi(Q) \). Then for \( x \in P \), we have \( [x] \in P' = \phi(P) = \phi(Q) = Q' \). This gives \( x \in Q \). Hence \( P \subseteq Q \). Similarly, \( Q \subseteq P \). Thus \( \phi \) is one-to-one. Let \( Q' \) be any minimal prime semi-ideal in \( L' \). Then \( Q = \{ x \mid [x] \in Q' \} \) is a minimal prime semi-ideal of \( L \), by Lemma 37. It is clear that \( \phi(Q) = Q' \). Thus \( \phi \) is onto.

With this preparation, we now prove Beck’s Conjecture for \( G_E(L) \).

**Theorem 39.** Let \( L \) be a meet semilattice with 0. If \( \text{Clique}(G_E(L)) < \infty \), then \( \chi(G_E(L)) = \chi(G_{00}(L)) = \text{Clique}(G_{00}(L)) = \text{Clique}(\{0\}|(L)) = n \), where \( n \) is the number of all minimal prime semi-ideals of \( L \).

**Proof.** Let \( \text{Clique}(G_E(L)) < \infty \). Then \( \text{Clique}(G_{00}(L')) < \infty \), by Corollary 5, where \( L' \) is the semilattice constructed as in Lemma 2. By Theorem 34, \( \text{Clique}(G_{00}(L')) = \chi(G_{00}(L')) = n \), where \( n \) is the number of minimal prime semi-ideals of \( L' \). Now, by Theorem 38, the number of minimal prime semi-ideals of \( L \) is also \( n \). Thus we have \( \text{Clique}(G_E(L)) = \chi(G_E(L)) = n \).

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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