Fundamental limitations on the device-independent quantum conference key agreement

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We provide several general upper bounds on the rate of a key secure against a quantum adversary in the device-independent conference key agreement (DI-CKA) scenario. They include bounds by reduced entanglement measures and those based on multipartite secrecy monotones such as a multipartite squashed entanglement-based measure, which we refer to as reduced c-squashed entanglement. We compare the latter bound with the known lower bound for the protocol of conference key distillation based on the parity Clauser-Horne-Shimony-Holt game. We also show that the gap between the DI-CKA rate and the device-dependent rate is inherited from the bipartite gap between device-independent and device-dependent key rates, giving examples that exhibit the strict gap.

I. INTRODUCTION

Building a quantum secure internet is one of the most important challenges in the field of quantum technologies [1, 2]. It would ensure worldwide information-theoretically secure communication. The idea of quantum repeaters [3–5] gives hope that this dream will come true. However, the level of quantum security proposed originally in a seminal article by Bennett and Brassard [6] seems to be insufficient due to the fact that on the way between an honest manufacturer and an honest user, an active hacker can change with the inner workings of a quantum device, making it totally insecure [7]. Indeed, the hardware Trojan-horse attacks on random number generators are known [8], and the active hacking on quantum devices became a standard testing approach since the seminal attack by Makarov [9]. The idea of device-independent (DI) security overcomes this obstacle [7, 10] (see also [11] and references therein). Although difficult to be done in practice, it has been demonstrated quite recently in several recent experiments [12–14].

In parallel, the study of the limitations of this approach in terms of upper bounds on the distillable key has been put forward [15–18]. However, these approaches focus on point-to-point quantum device-independent secure communication. In this paper we introduce the upper bounds on the performance of the device-independent conference key agreement (DI-CKA) [19, 20]. The task of the conference agreement is to distribute to $N > 2$ honest parties the same secure key for one-time-pad encryption. A protocol achieving this task in a device-independent manner has been shown in Ref. [20]. We set an upper bound on the performance of such protocols in a network setting.

We focus on physical behaviors with $N$ users (for arbitrary $N > 2$), where each user is both the sender and receiver of the behavior treated as a black box. This situation is a special case of a network describable with a multiplex quantum channel where inputs and outputs are classical with quantum phenomena going inside the physical behavior [21]. All $N$ trusted parties have the role of both the sender to and receiver from the channel and their goal is to obtain a secret key in a device-independent way against a quantum adversary. Aiming at upper bounds on the device-independent key, we narrow the consideration to the independent and identically distributed case. In this scenario, the honest parties share $n$ identical devices. All the $N$ parties set (classical) inputs $x = (x_1, \ldots, x_N)$ to each of the $n$ shared devices $P(a|x)$ and receive (classical) outputs $a = (a_1, \ldots, a_N)$ from each of them. We restrict our consideration to quantum devices. Such devices are realized by certain measurements $\mathcal{M} \equiv \otimes_{i=1}^{N} M_{A_i}^{a_i}$ on quantum states $\rho_{A_1,\ldots,A_N} \equiv \rho_{N(A)}$. We define these devices $(\rho_{N(A)}, \mathcal{M}) = \Tr [\rho_{N(A)}(\otimes_{i=1}^{N} M_{A_i}^{a_i})]$.

In this work we provide upper bounds on the device-independent conference key distillation rates for arbitrary multipartite states. As the first main result, we introduce a multipartite generalization of the $cc$-squashed entanglement provided in Ref. [22] and developed in Ref. [18]. With a little abuse of notation with respect to that used
in Refs. [16, 18], for the sake of the reader, we will omit the fact that the measure is multipartite as well as reduce the abbreviation cc in its name and here call it just reduced c-squashed entanglement, denoting it by $E_{sq,dev}^{c}$. We show that $E_{sq,dev}^{c}$ upper bounds the device-independent key rate in the independent and identically distributed setting, achieved by protocols which use a single input to generate the key $\hat{K}$, denoted by $K_{DI,dev}^{iid,\hat{x}}$. The subscript dev in the notation refers to the fact that the adversary has to mimic the statistics of the honestly implemented device. We then generalize this to the case when only some parameters of the device have to be reproduced by the attack and refer to quantities with the subscript par. Typical parameters are the level of violation of a Bell inequality and the quantum bit error rate. In the above finding, we use the notion of multipartite squashed entanglement given in Ref. [23]. Therein, the abbreviation c stands for classical as the systems of the honest parties are classical due to the measurement accordingly to the definition. The bound reads

$$K_{DI,dev}^{iid,\hat{x}}(ρ_{N(A)};\mathcal{M}) \leq E_{sq,dev}^{c}(ρ_{N(A)};\mathcal{M})$$

$$= \inf_{(σ_{N(A)},\mathcal{N})\equiv(ρ_{N(A)},\mathcal{M})} \frac{1}{N-1} I(A_1:...:A_N \downarrow E)_{N(\hat{x})\otimes σ_{N(A)}}.$$ (1)

In the above $\downarrow$ denotes the action of any channel transforming $E$ to some system $E'$ and $I(A_1 : : : A_N|E)σ_{N(A)} = \sum_{i=1}^{N} S(A_i|E)σ_{N(A)} - S(A_1,\ldots,A_N|E)σ_{N(A)}$, with $S(X|Y)σ_{X,Y}$ the conditional von Neumann entropy of the state $σ_{X,Y}$. The quantity $I(A_1 : : : A_N|E')$ (after the action of the channel on $E$) is evaluated on the classical-quantum state emerging from the measurement $N_\hat{x}$ corresponding to input $\hat{x}$ of the device $(σ_{N(A)},\mathcal{N})$, on systems $A_1,\ldots,A_N$. Let us note here that, due to the findings of Ref. [18], for $N = 2$ and the case when the system $E'$, as the output of a channel, is classical, the above bound is equal to the intrinsic information given in Ref. [17] for the case of a single measurement generating the key.

All quantum states considered in this paper are $N$-partite unless it is stated otherwise. Therefore, for the sake of the conciseness of the notation we omit the subscript $N(A)$ in some places.

Our technique is based on the approach of Ref. [24], where the upper bounds on a key distillable against a quantum adversary via local operations and public communication (LOPC) were studied. To achieve this, we generalize the upper bound via (quantum) intrinsic information to the case in which the adversary’s system can be of infinite dimension. This technical contribution was necessary, as in the case of a device-independent attack, the dimension of the attacking state can be infinite. Indeed, while measurements $x$ produce from the attacking state $σ$ finite-dimensional results $a$ yielding a quantum behavior $P(a|x) = Tr[σM_a\hat{x}]$, the system of the adversary which may hold purification of the state $σ$, can still be of infinite dimension.¹

We then compare the obtained upper bounds with the lower bound on the DI-CKA provided in Ref. [20] (see Fig. 2). We obtain the plot by considering simplification of the reduced c-squashed entanglement. Namely, the extension to the adversarial system of the state attacking the honest parties device is classical, i.e., diagonal in the computational basis. For that reason, the bound which we use is in fact a secrecy monotone, called (multipartite) intrinsic information [25].

As the second main result, we show how to construct multipartite states with a strict gap between the rate

¹ In that we have filled in the gap in the proofs of Corollaries 3 and 4 of Ref. [18], where implicit assumption of the adversary holding finite-dimensional state has been made.
of the quantum device-independent conference key rates $K_{DI}$ and quantum device-dependent conference key rates $K_{DD}$. As a proxy, we use a bipartite state that satisfies $K_{DD} > K^*$, where $K^*$ is the reduced distillable key introduced in Ref. [16]. The reduced distillable key of $\rho_{N(A)}$ is the maximum value of the choice of measurements $M$ of the distillable key of the adversarial state $\sigma_{N(A)}$ minimized over the choices of the state $\sigma_{N(A)}$ and measurements $\mathcal{N}$ so that the device $(\sigma_{N(A)}, \mathcal{N}) \equiv \{ \text{Tr} [\sigma_{N(A)}(S_i^N)] \}_{a|x}$ is equal to the honestly implemented device $(\rho, M)$. The mentioned gap between $K_{DI}$ and $K_{DD}$ means that also in a multipartite case for some states $\rho_{A_1,...,A_N}$ (and any $N > 2$), there is neither a Bell-like inequality that can be used for testing nor a distillation protocol based on LOPC that can achieve $K_{DI}(\rho_{A_1,...,A_N}) = K_{DD}(\rho_{A_1,...,A_N})$. See Fig. 1.

Finally, we discuss the issue of genuine nonlocality [11] and genuine entanglement in the context of the DI-CKA [26]. As the third main result, we provide a non-trivial bound on the device-independent key achievable in a parallel measurement scenario, when all the parties set all values of the inputs $x_i$ in parallel. Furthermore, generalizing reduced bipartite entanglement measures in Ref. [18] to multipartite entanglement measures, we show that the reduced regularized relative entropy of genuine entanglement [21] upper bounds the DI-CKA rate of multipartite quantum states. We further focus on the performance of protocols using a single input for key generation, as using such protocols is standard practice (see, e.g., [27]).

The remainder of this paper is organized as follows. Section II is devoted to basic facts and provides bounds on the DI conference key via entanglement measures. In Section III, we develop an upper bound on the DI-CKA via reduced $c$-squared entanglement. In Section IV, we provide particular examples for the performance of upper bounds considered in the paper. In Section V, we provide examples of multipartite states which exhibit a fundamental gap between the device-dependent and -independent secure key rates. Section VI discusses the connection between genuine nonlocality, entanglement, and DI-CKA. We conclude in Section VII with a summary and some directions for future study.

**Note 1.** Theorem 7 of Ref. [28] states that the two multipartite entanglement measures, multipartite squared entanglement $E_{sq}$ (Definition 4 in [29]) and dual $E_{sq}$ (Definition 2 in [29]), of a multipartite state are the same. As a consequence, the reduced $c$-squared entanglement $E_{sq}^c(\rho, M)$ (Definition 5 in [29]) and the dual $c$-squared entanglement $E_{sq}^c(\rho, M)$ (Definition 8 in [29]) of a device $(\rho, M)$ are also the same. For this reason, Section IV in the published version of the manuscript [Phys. Rev. A 105(2), 022604 (2022)] [29] can be skipped in reading.

**Note 2.** The current version differs from [Phys. Rev. A 105(2), 022604 (2022)] [29] by removing Section IV, references to it, along with including updated Fig. 2 and fixing typographic mistakes in notation and explanation of Eq. (58). There is no change with respect to the results and proofs of the [Phys. Rev. A 105(2), 022604 (2022)] [29].

II. BOUNDS ON DEVICE-INDEPENDENT KEY DISTILLATION RATE OF STATES

In this section, we introduce the scenario of device-independent conference key distillation from $n$ identical devices, each shared by $N$ honest users. We then introduce definitions and facts used in subsequent sections.

Consider a setup wherein $N$ multiple trusted spatially separated users (allies) have to extract a secret key, i.e., conference key, against the quantum adversary. Since we aim at upper bounds on the device-independent conference key, we assume that the parties share $n$ identical devices. The device has its honest implementation, which is reflected by the state and measurement, denoted by $(\rho, M)$, that were intended to be delivered by a provider. The adversary may replace this honest implementation with a different device $(\sigma, \mathcal{N})$, however, such that it yields the same input-output statistics as the honest one. Typically, the statistics tested by the allies are the level of violation of some Bell inequality, and the quantum bit error rate, i.e., the probability that the outputs of the honest parties are not equal to each other given the raw key has been generated. In some cases, we will also consider the full statistics reflected by the pair $(\rho, M)$. We note here that the state $\sigma$ can be finite or infinite-dimensional, as we do not restrict the strategies of the adversary in that respect.

The honest device is given by $M \equiv \{ M_{a_1}^x \otimes M_{a_2}^x \otimes \ldots \otimes M_{a_N}^x \}_{a|x}$, where $x := (x_1, x_2, \ldots, x_N)$ and $a := (a_1, a_2, \ldots, a_N)$, for some $N \in \mathbb{N}$. For each $i \in \{1, 2, \ldots, N\}$, the set $\{a_i\}$ denotes the finite set of measurement outcomes for measurement choices $x_i$. The measurement outcomes, i.e., outputs of the device, are secure from the adversary and assumed to be in the possession of the receivers (allies). The joint probability distribution is given as

$$p(a|x) = \text{Tr}[M_{a_1}^x \otimes M_{a_2}^x \otimes \ldots \otimes M_{a_N}^x \rho_{A_1A_2\ldots A_N}]$$

for measurement $M$ on $N$-partite state $\rho_{N(A)}$ defined on the separable Hilbert space $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \ldots \mathcal{H}_{A_N}$; in what follows we will use $N(A) \equiv A_1 \ldots A_N$ for the ease of notation. The tuple $(\rho, M)$, where $M := \{ \{ M_{a_1}^x \}_{x_1}, \{ M_{a_2}^x \}_{x_2}, \ldots, \{ M_{a_N}^x \}_{x_N} \}$, is called the quantum strategy of the distribution. The number of inputs $\{x_i\}$ and corresponding possible outputs $\{a_i\}$ of the local measurement at $A_i$ are arbitrarily finite in general. We denote the identity superoperator by $1$ and the identity operator by $I$.

Let $\omega(\rho, M)$ denote the violation of the given multipartite Bell-type inequality $\mathcal{B}$ by state $\rho$ when the measurement settings are given by $M$. We note that by
multpartite Bell-type inequality we mean any inequality derived using locally realistic hidden variable (LRHV) theories (see, e.g., [30–38]) such that any violation of a given inequality by a density operator implies the nonexistence of an LRV model for the device represented by this state and some measurements. There are families of Bell-type inequalities directly based on the joint probability distribution of local measurements that get violated by all pure multipartite (genuinely) entangled states [36, 37]. On the other hand, there are Bell-type inequalities based on correlation functions of local measurements for which some families of pure multipartite (genuinely) entangled states satisfy the inequalities [39]. Let \( P_{\text{err}}(\rho, M) \) denote the expected quantum bit error rate (QBER). Both the Bell violation and the QBER are functions of the probability distribution of the behavior. In addition, \( \Phi_{N}^{GHZ} := \left| \Phi_{N}^{GHZ} \right| \) denotes the \( N \)-partite Greenberger–Horne–Zeilinger (GHZ) state.

\[
\left| \Phi_{N}^{GHZ} \right| = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle_{A_1} \otimes |i\rangle_{A_2} \otimes \cdots \otimes |i\rangle_{A_N} \tag{3}
\]

for \( d = \min \dim(\mathcal{H}_{A_i}) \). For \( N = 2 \), \( \Phi_{2}^{GHZ} \) is a maximally entangled (Bell) state \( \Phi_{2} \).

If \( \{p(a|x)\}_{a,x} \) obtained from \( (\rho, M) \) and another pair of states and measurements \( (\sigma, \mathcal{N}) \) are the same, we write \( (\sigma, \mathcal{N}) = (\rho, M) \). In most DI-CKA protocols, instead of using the statistics of the full correlation, we use the Bell violation and the QBER to test the level of security of the observed statistics. In this way, for practical reasons, the protocols coarse grain the statistics, and we only use partial information of the full statistics to extract the device-independent key. In this context, the notation \( (\sigma, \mathcal{N}) = (\rho, M) \) also implies that \( \omega(\sigma, \mathcal{N}) = \omega(\rho, M) \) and \( P_{\text{err}}(\sigma, \mathcal{N}) = P_{\text{err}}(\rho, M) \). When conditional probabilities associated with \( (\rho, M) \) and \( (\sigma, \mathcal{N}) \) are \( \varepsilon \)-close to each other, then we write \( (\rho, M) \approx_{\varepsilon} (\sigma, \mathcal{N}) \). For our purpose, it suffices to consider the distance

\[
d(p, p') = \sup_{x} \| p(\cdot|x) - p'(\cdot|x) \|_1 \leq \varepsilon. \tag{4}
\]

The device-independent distillable key rate of a device is informally defined as the supremum of the finite key rates \( \kappa \) achievable by the best protocol on any device compatible with \( (\rho, M) \), within an appropriate asymptotic blocklength limit and security parameter. Another approach taken is to minimize the key rate of the statistics compatible with the Bell parameter and a QBER (see, e.g., [22]). For our purpose, we constrain ourselves to the situation when the compatible devices are supposedly independent and identically distributed. This constraint is because, as noted in Ref. [16], the upper bound on the key in the independent and identically distributed scenario is automatically the upper bound on the device-independent conference key in the general scenario since the independent and identically distributed attack is just one of the possible attacks in the general device-independent scenario.

An ideal conference key state \( \tau^{(K)} \), with \( \log_2 K \) secret key bits for \( N \) allies, is

\[
\tau^{(K)}_{N(A)E} := \frac{1}{K} \sum_{k=0}^{K-1} |k\rangle_{A_1} \otimes |k\rangle_{A_2} \otimes \cdots \otimes |k\rangle_{A_N} \otimes \tau_{\mathcal{E}}, \tag{5}
\]

where \( \tau_{\mathcal{E}} \) is a state of the only system \( \mathcal{E} \) accessible to an adversary, i.e., the adversary is uncorrelated with trusted users and gets no information about their secret bits. Consider the relations

\[
(\rho, \mathcal{M}) \approx_{\varepsilon} (\sigma, \mathcal{N}), \tag{6}
\]

\[
\omega(\rho, \mathcal{M}) \approx_{\varepsilon} \omega(\sigma, \mathcal{N}), \tag{7}
\]

\[
P_{\text{err}}(\rho, \mathcal{M}) \approx_{\varepsilon} P_{\text{err}}(\sigma, \mathcal{M}), \tag{8}
\]

where Eq. (6) implies Eqs. (7) and (8).

Formally, the definition of device-independent quantum key distillation rate in the independent and identically distributed scenario is given as follows.

**Definition 1** (cf. [16]). The maximum (multipartite) device-independent quantum key distillation rate of a device \( (\rho, M) \) with independent and identically distributed behavior is defined as

\[
K_{\text{DI,dev}}^{\text{iid}}(\rho, M) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \inf_{\rho} \kappa^{(n)}_{n}(\hat{\mathcal{P}} ((\sigma, \mathcal{N})^{\otimes n})), \tag{9}
\]

where \( \kappa^{(n)}_{n} \) is the rate of a key distillation protocol \( \hat{\mathcal{P}} \) producing \( \varepsilon \)-secure output, acting on \( n \) copies of the state \( \sigma \), measured with \( \mathcal{N} \). Here \( \hat{\mathcal{P}} \) is a protocol composed of classical local operations and public (classical) communication (CLOPC) acting on \( n \) identical copies of \( (\sigma, \mathcal{N}) \) which, composed with the measurement, results in a quantum local operations and public (classical) communication (QLOPC) protocol.

The following lemma follows from the definition of \( K_{\text{DI,dev}}^{\text{iid}} \) (generalizing statements from bipartite DI quantum key distillation in Refs. [16, 18] to the DI-CKA).

**Lemma 1.** The maximum (multipartite) device-independent quantum key distillation rate \( K_{\text{DI,dev}}^{\text{iid}} \) of a device \( (\rho, M) \) is equal to the maximum (multipartite) device-independent quantum key distillation rate of a device \( (\sigma, \mathcal{N}) \) when \( (\rho, M) = (\sigma, \mathcal{N}) \):

\[
(\rho, M) = (\sigma, \mathcal{N}) \implies K_{\text{DI,dev}}^{\text{iid}}(\rho, M) = K_{\text{DI,dev}}^{\text{iid}}(\sigma, \mathcal{N}). \tag{10}
\]

The maximal DI-CKA rate \( K_{\text{DI,dev}}(\rho, M) \) for the device \( (\rho, M) \) is upper bounded by the maximal device-dependent conference key agreement (DD-CKA) rate \( K_{\text{DD}}(\sigma) \) for all \( (\sigma, \mathcal{N}) \) such that \( (\sigma, \mathcal{N}) = (\rho, M) \) (cf. [16]), i.e.,

\[
K_{\text{DI,dev}}(\rho, M) \leq \inf_{(\sigma, \mathcal{N}) = (\rho, M)} K_{\text{DD}}(\sigma). \tag{11}
\]
The device-dependent quantum key distillation rate $K_{DD} (\rho)$ (cf. [24]) is the maximum secret key (against quantum eavesdropper) that can be distilled between allies using local operations and classical communication (LOCC) (see, e.g., [21]),

$$K_{DD} (\rho) := \inf_{\varepsilon > 0} \lim_{n \to \infty} \sup_{N \in \text{QLOPC}} \left\{ \frac{\log_2 d_n}{n} \left| \Lambda((\psi^\rho)^\otimes n) \approx \frac{1}{\varepsilon} \rho, (\log d_n) \right. \right\},$$

where $\psi^\rho$ is a purification of $\rho$ and $\rho \approx \frac{1}{\varepsilon} \sigma \iff \frac{1}{2} \|\rho - \sigma\|_1 \leq \varepsilon$.

**Corollary 1.** For entanglement measures $\text{Ent}$ which upper bound the maximum device-dependent key distillation rate, i.e., $K_{DD} (\rho) \leq \text{Ent} (\rho)$ for a density operator $\rho$, we have

$$K_{DI, dev} (\rho, \mathcal{M}) \leq \inf_{(\sigma, \Lambda) = (\rho, \mathcal{M})} K_{DD} (\sigma) \leq \inf_{(\sigma, \Lambda) = (\rho, \mathcal{M})} \text{Ent} (\sigma).$$

**Remark 1.** We do not make any assumption about the dimension of the Hilbert space on which the state $\sigma$ (Definitions 1–3) is defined as the systems can be finite-dimensional or infinite dimensional. We only assume that the systems $A_i$ accessible by the allies for key distillation upon measurement are finite dimensional in an honest setting. The systems $E$ accessible to an adversary can be finite dimensional or infinite dimensional, depending on the cheating strategy (see Lemma 7, Appendix B).

As discussed above, a large class of device-independent quantum key distillation protocols relies on the Bell violation and the QBER of the device $p(a | x)$. For such protocols, we can define the device-independent key distillation protocol as follows.

**Definition 2** (cf. [22]). The maximal (multipartite) device-independent quantum key distillation rate of a device $(\rho, \mathcal{M})$ with independent and identically distributed behavior, Bell violation $\omega (\rho, \mathcal{M})$, and QBER $P_{\text{err}} (\rho, \mathcal{M})$ is defined as

$$K_{\text{DD}, \text{par}}^{\text{id}} (\rho, \mathcal{M}) := \inf_{\varepsilon > 0} \lim_{n \to \infty} \sup_{\mathcal{P}} \left\{ K_{\text{DD}}^n (x, \hat{N}) \right\},$$

where $\mathcal{P}$ reads [41]

$$I (A_1 : \ldots : A_N | E \rho) = \sum_{i=1}^N S (A_i | E \rho) - S (A_1, \ldots, A_N | E \rho).$$

**Remark 2.** As Eq. (6) implies Eqs. (7) and (8), it follows from the definitions of $K_{\text{DD}, \text{par}}^{\text{id}} (\rho, \mathcal{M})$ and $K_{\text{DD}, \text{par}}^{\text{id}} (\rho, \mathcal{M})$ that $K_{\text{DD}, \text{par}}^{\text{id}} (\rho, \mathcal{M}) \leq K_{\text{DD}, \text{par}}^{\text{id}} (\rho, \mathcal{M})$.

**Remark 3** (cf. [11]). We note that there may exist states $\rho$ for which $K_{\text{DD}, \text{par}}^{\text{id}} (\rho) = 0$ but $K_{\text{DD}, \text{par}}^{\text{id}} (\rho^{\otimes k}) > 0$ for some $k \in \mathbb{N}$.

**III. REDUCED C-SQUASHED ENTANGLEMENT BOUND**

In this section we generalize the notion of the cc-squashed entanglement [18, 22] to the multipartite form. Next we prove that the properly scaled reduced c-squashed entanglement serves as an upper bound on the device-dependent conference key of the classical-quantum state. Furthermore, via Lemma 2 and Proposition 1 we prove that the reduced c-squashed entanglement is convex. This result may be further applied to generate numeric upper bounds with the convexification technique [18, 40]. Then we prove the main result of this section. Namely, we prove that the independent and identically distributed quantum device-independent conference key is upper bounded by the reduced c-squashed entanglement. Finally, we show that similar results hold when the honest parties broadcast the inputs to their devices so that the adversary can learn them.

In what follows, we first prove that the “measured” version of the multipartite squashed entanglement $E_{\text{sq}}^{\text{M}}$ defined in Ref. [23], if properly scaled, upper bounds the conference key secure against the quantum adversary. Let us first recall facts and definitions. The multipartite conditional mutual information of a state $\rho_{A_1 \ldots A_N E}$ reads

$$I (A_1 : \ldots : A_N | E \rho) = \sum_{i=1}^N S (A_i | E \rho) - S (A_1, \ldots, A_N | E \rho).$$

Here, the conditional entropy $S(A_i | E \rho) = S(A_i E \rho) - S(E \rho)_\rho$, with $S(AB) := - \text{Tr} [\rho_{AB} \log_2 \rho_{AB}]$ and $S(A) := - \text{Tr} [\rho_A \log_2 \rho_A]$ being von Neumann entropies. The von Neumann entropy reduces to the Shannon entropy $H(X)$ for classical register $X$, $H(X) := - \sum_x p(x) \log_2 p(x)$, where $\{p(x)\}_x$ is the probability distribution associated with the random variable $X$. It will be crucial to note that the following identity holds [23]

$$I (A_1 : \ldots : A_N | E \rho) = I (A_1 : A_2 | E \rho) + I (A_3 : A_1 A_2 | E \rho) + \cdots + I (A_N : A_1 \ldots A_{N-1} | E \rho).$$

Here $I (A : B | C \rho) = S(AC \rho) + S(BC \rho) - S(C \rho) - S(ABC \rho)$ is the conditional mutual information.

**Remark 4.** Let $\Sigma$ be any permutation of indices $1, \ldots, N$. Then

$$I (A_1 : \ldots : A_N | E \rho) = I (A_{\Sigma (1)} : \ldots : A_{\Sigma (N)} | E \rho)$$

$$= I (A_{\Sigma (1)} : A_{\Sigma (2)} | E \rho) + I (A_{\Sigma (3)} : A_{\Sigma (1)} A_{\Sigma (2)} | E \rho) + I (A_{\Sigma (4)} : A_{\Sigma (3)} A_{\Sigma (2)} A_{\Sigma (3)} | E \rho) + \cdots + I (A_{\Sigma (N)} : A_{\Sigma (1)} \ldots A_{\Sigma (N-1)} | E \rho).$$

Further, the multipartite squashed entanglement of a quantum state $\rho_{A_1 \ldots A_N}$ is defined as follows (Definition 3 of Ref. [23]).
Definition 3 ([23]). For an N-partite state \( \rho_{A_1, \ldots, A_N} \),

\[
E^\sigma_{sq}(\rho_{A_1, \ldots, A_N}) := \inf_{\sigma} I(A_1 : A_2 : \ldots : A_N | E)_\sigma,
\]

where the infimum is taken over states \( \sigma_{A_1, \ldots, A_N E} \) that are extensions of \( \rho_{A_1, \ldots, A_N} \), i.e., \( \text{Tr}_E[\sigma_{A_1, \ldots, A_N E}] = \rho_{A_1, \ldots, A_N} \).

We will need to generalize the notion of the cc-squashed entanglement [18, 22] to the multipartite form.

Definition 4. A reduced c-squashed entanglement of a state \( \rho_{A_1, \ldots, A_N} \) is defined as

\[
E^\sigma_{sq}(\rho_{A_1, \ldots, A_N}, M) := \inf_{A_i : E \rightarrow E_i} I(A_1 : \ldots : A_N | E^i)_{M_{N(A)} \otimes \Lambda_{\psi^\rho_N(E_i)}},
\]

where \( M_{N(A)} \) is an N-tuple of positive-operator-valued measures (POVMs) \( M_{A_1}, \ldots, M_{A_N} \) and state \( \psi^\rho_N(E) \) is a purification of \( \rho_{N(A)} \).

The first theorem comes with the following fact, which is a multipartite generalization of Theorem 5 from Ref. [18], where \( N = 2 \). Namely, the device-dependent conference key of the classical-quantum state is upper bounded by the properly scaled reduced c-squashed entanglement. We are ready to state a theorem that shows that the c-squashed entanglement, when properly scaled, upper bounds the device-dependent key.

Theorem 1. For an N-partite state \( \rho_{N(A)} \), its purification \( \psi^\rho_N(E) \), and an N-tuple of POVMs \( M_{N(A)} \), there is

\[
K_{DD}(M_{N(A)} \otimes \text{id}_E, \psi^\rho_N(E)) \leq \frac{1}{N-1} E^\sigma_{sq}(\rho_{N(A)}, M_{N(A)}).
\]

Proof. We closely follow the proof of Theorem 3.5 of Ref. [24], however, based not on Theorem 3.1 of Ref. [24], but its generalization to the case where system \( E \) need not be finite (see Lemma 7, Appendix B). Namely, any function which satisfies (i) monotonicity under LOPC, (ii) asymptotic continuity, (iii) normalization, and (iv) subadditivity is, after regularization, an upper bound on the distillable key secure against the quantum adversary \( \langle K_{DD} \rangle \).

We first show the monotonicity. The LOPC consist of local operation and public communication. A local operation consists of adding a local ancilla, performing a unitary transformation, and a partial trace. It is easy to see that adding a local ancilla at one system does not alter this quantity. The same holds for the unitary transformation. The partial trace does not increase it as it can be rewritten in terms of the conditional mutual information terms as in Eq. (17). Then the same argument as in the proof of Theorem 3.5 of Ref. [24] [see Eq. (57) therein] applies.

Finally, for classical communication, we use the form given in Eq. (17) to verify the inequality stated below for the case when \( A_i \) produces locally the variable \( C_i \) and then broadcasts it to all the parties in the form of \( C_i \) for \( j \neq i \) and to the adversary in the form of \( C_{N+1} \) (note that broadcasting followed by a partial trace, if needed, can simulate any classical communication among \( N \) parties):

\[
I(A_1 : \ldots : A_i C_i : \ldots : A_N | E)_{\rho} \leq \frac{1}{N-1} E^\sigma_{sq}(\rho_{N(A)}, M_{N(A)}).
\]

In Eq. (22) [step (I)], we transposed the labels \( i \) and \( 1 \) (see Remark 4), and step (II) [inequality (24)] follows (termwise) from the monotonicity of the (tripartite) mutual information function proved in Ref. [24] (see the proof of Theorem 3.5 therein).

Regarding asymptotic continuity, we consider two states \( \rho_{N(A)} \) and \( \sigma_{N(A)} \) such that \( \| \rho_{N(A)} - \sigma_{N(A)} \|_1 \leq \epsilon \). Then, as in the proof of Theorem 3.5 of Ref. [24], for any map \( \Lambda : E \rightarrow E' \) there is \( \| \rho'_{N(A)} - \sigma'_{N(A)} \|_1 \leq \epsilon \), where \( \rho'_{N(A)} := \text{id}_{N(A)} \otimes \Lambda \rho_{N(A)} \) and \( \sigma'_{N(A)} := \text{id}_{N(A)} \otimes \Lambda \sigma_{N(A)} \). Then, by the expansion Eq. (17) we obtain

\[
|I(A_1 : A_1 \ldots A_i-1 | E')_{\rho} - I(A_1 : A_1 \ldots A_i-1 | E')_{\sigma}| \leq 2 \log_2 d_{A_i} + 2g(\epsilon),
\]

with

\[
d_{A_i} := \text{dim}(\mathcal{H}_{A_i}) \quad \text{and} \quad g(\epsilon) := (1 + \epsilon) \log_2 (1 + \epsilon) - \epsilon \log_2 \epsilon,
\]

where we use Lemma 5 from Appendix A, provided in Ref. [42]. Hence, in total we get

\[
|I(A_1 : \ldots : A_N | E')_{\rho} - I(A_1 : \ldots : A_N | E')_{\sigma}| \leq 2 \epsilon \sum_{i=1}^{N-1} \log d_{A_i} + (N-1)2g(\epsilon) \leq (N-1)[2\epsilon \max_{i \in \{1, \ldots, N\}} \log d_{A_i} + 2g(\epsilon)].
\]

For a finite natural \( N \), the right-hand side of the above approaches 0, with \( \epsilon \to 0 \).

The subadditivity follows again from the fact that we can split the term \( I(A_1 : \ldots : A_N | E) \) into \( N-1 \) terms of the form \( I(A_1 : A_1 \ldots A_i | E) \). Further treating \( A_1 \ldots A_i \) together as \( B_i \) (equivalent of \( B \) in the proof of Theorem 3.5 in Ref. [24]), we can prove the additivity of the form

\[
I(A_1 A_1' : B_i B_i'| EE') = I(A_1 : B_i | E) + I(A_1' : B_i'| E')
\]
for each term and notice the subadditivity from the fact that in the infimum in the definition of $E_{sq}^c(p,M)$ there are product channels; hence in general the formula can be lower than the above.

Finally, we consider normalization. It is straightforward to see that, assuming $d_A = d_A$ for each $i \in \{1, \ldots, N\}$, on the state representing the ideal key $\tau_{N(A)} = \frac{1}{d_A} \sum_{i=0}^{d_A-1} |ii_i⟩⟨ii_i| ⊗ E$ [Eq. (5) there is $E_{sq}^c(\tau,M) = (N-1) \log_2 d_A$, by noticing that on the product state $I(A_1 : A_1 \ldots A_{i-1}) = I(A_1 : A_1 \ldots A_{i-1}) = \log_2 d_A$, and there are $(N-1)$ of such terms in the definition of $E_{sq}^c$. We assume here also that the measurement $M$ is generating the key in the computational basis. $\square$

We have further an analog of Observation 4 of Ref. [18]. Its proof goes along similar lines. Indeed, it does not depend on either the type of objective function that is minimized or the number of parties; hence we omit it here.

**Observation 1.** For an $N$-partite state $\rho_{N(A)}$ and a POVM $M_{N(A)} = M_A, \ldots, M_{AN}$ there is

$$E_{sq}^c(\rho,M) = \inf_{\rho_{N(A)} = \text{Ext} (\rho_{N(A)})} I(A_1 : \ldots : A_N | E)_{M_{N(A)} \otimes \text{id}_E \rho_{N(A)}},$$

where $\text{Ext}(\rho_{N(A)})$ stands for the state extension of $\rho_{N(A)}$, i.e., $\rho_{N(A)} | E$ is a density operator such that $\text{Tr}[\rho_{N(A)} | E] = \rho_{N(A)}$.

Owing to Observation 1, we can obtain the analog of Lemma 6 of Ref. [18], which states that $E_{sq}^c$ is convex.

**Lemma 2.** For a tuple of POVMs $M_{N(A)}$, two states $\rho_{N(A)}^{(1)}$ and $\rho_{N(A)}^{(2)}$, and $0 < p < 1$, there is

$$E_{sq}^c(\rho_{N(A)}) \leq p E_{sq}^c(\rho_{N(A)}^{(1)}) + (1-p) E_{sq}^c(\rho_{N(A)}^{(2)}),$$

where $\rho_{N(A)} = p \rho_{N(A)}^{(1)} + (1-p) \rho_{N(A)}^{(2)}$.

**Proof.** The proof is due to the fact that the function $E_{sq}^c(\rho_{N(A)})$ is upper bounded by $I(A_1 : \ldots : A_N | E)$ evaluated on a state $\rho_{N(A)} = M_{N(A)} \otimes \text{id}_E (p \rho_{N(A)}^{(1)} \otimes |0⟩⟨0|_F + (1-p) \rho_{N(A)}^{(2)} \otimes |1⟩⟨1|_F$. Further, by Eq. (17) there is

$$I(A_1 : \ldots : A_N | E)_{\rho} = p I(A_1 : \ldots : A_N | E)_{M_{N(A)} \otimes \text{id}_E \rho_{N(A)}^{(1)}},$$

$$+ (1-p) I(A_1 : \ldots : A_N | E)_{M_{N(A)} \otimes \text{id}_E \rho_{N(A)}^{(2)}}.$$  

(32)

Since the states $\rho^{(1)}$ and $\rho^{(2)}$ were arbitrary, we get the thesis. $\square$

We further note that switching from a bipartite key distillation task to the conference key distillation does not alter the formulation or the proof of Lemma 7 of Ref. [18]. We state it below for the sake of the completeness of the further proofs.

**Lemma 3.** The independent and identically distributed quantum device-independent key achieved by protocols using (for generating the key) a single tuple of measurements $(\hat{x}_1, \ldots, \hat{x}_N) \equiv \hat{x}$ applied to $M$ of a device $(\rho_{N(A)}, M)$ is upper bounded as

$$K_{D,\hat{x}}^{\text{id}}(\rho_{N(A)}, M) := \inf_{\epsilon > 0} \limsup_{n \to \infty} \sup_{\mathcal{P} \in \text{LOPC}} \inf_{\sigma_{N(A),\mathcal{L}}} \kappa_n^{\epsilon,\hat{x}}(\mathcal{P}(L(\sigma_{N(A)}))^\otimes n),$$

where $L = \mathcal{L}(\hat{x})$ is a single pair of measurements induced by inputs $\hat{x}$ and $\sigma_{N(A),\mathcal{L}} = \rho_{N(A),\mathcal{L}}^{(1)}$ is the rate of the $\epsilon$-perfect conference key achieved and classified labels from local classical operations in $\mathcal{P} \in \text{LOPC}$ are possessed by the allies holding systems $A_i$ for $i \in \{1, \ldots, N\}$.

Combining Theorem 1 with Lemma 3, we obtain the main result of this section. This is a bound by the reduced reduced c-squashed entanglement.

**Theorem 2.** The independent and identically distributed quantum device-independent conference key achieved by protocols using a single tuple of measurements $(\hat{x}_1, \ldots, \hat{x}_N) \equiv \hat{x}$ applied to $M$ of a device $(\rho_{N(A)}, M)$ is upper bounded as

$$K_{D,\hat{x}}^{\text{id}}(\rho_{N(A)}, M) \leq \frac{1}{N-1} \inf_{\mathcal{L}} \inf_{\rho_{N(A),\mathcal{L}}} E_{sq}^c(\sigma_{N(A),\mathcal{L}}(\hat{x}))$$

$$=: E_{sq,\text{dev}}^c(\rho_{N(A)}, M(\hat{x})).$$

(35)

We have an analogous result for a key which is a function only of the tested parameters, that of Bell inequality violation and the quantum bit error rate.

**Theorem 3.** The independent and identically distributed quantum device-independent key achieved by protocols using (for generating the key) a single tuple of measurements $(\hat{x}_1, \ldots, \hat{x}_N) \equiv \hat{x}$ applied to $M$ of a device $(\rho_{N(A)}, M)$ is upper bounded as

$$K_{D,\hat{x}}^{\text{id}}(\rho_{N(A)}, M)$$

$$:= \inf_{\epsilon > 0} \limsup_{n \to \infty} \sup_{\mathcal{P} \in \text{LOPC}} \inf_{\omega(\sigma_{N(A),\mathcal{L}})} \kappa_n^{\epsilon,\hat{x}}(\mathcal{P}(L(\sigma_{N(A)}))^\otimes n),$$

$$\leq \frac{1}{N-1} \inf_{\mathcal{L}} \inf_{\rho_{N(A),\mathcal{L}}} E_{sq}^c(\sigma_{N(A),\mathcal{L}}(\hat{x}))$$

$$=: E_{sq,\text{par}}^c(\rho_{N(A)}, M(\hat{x})).$$

(37)
For $N = 2$, the above bound recovers the result of Ref. [18].

In the definition of $E^c_{sq,par}(\bar{\mathcal{M}})$, one can take the infimum only over the classical extensions to Eve [23]. In that case, for a single input $\bar{x}$ this bound reads

$$\frac{1}{N-1} I(N(A) \downarrow E)$$

as given in Ref. [23] (see Ref. [17, 24, 43] for the bipartite case). We have the following immediate corollary.

**Corollary 2.** The independent and identically distributed quantum device-independent key achieved by protocols using a tuple of measurements $\bar{x}$ applied to a device $(\rho_{N(A)}, \mathcal{M})$ is upper bounded as

$$K_{DI,dev}^{id,\bar{x}}(\rho_{N(A)}, \mathcal{M}) \leq \inf_{(\sigma_{N(A)}, \mathcal{L})=(\rho_{N(A)},\mathcal{M})} \frac{1}{N-1} I(N(A) \downarrow E)P(A_1, \ldots, A_N | E).$$

(38)

where $P(A_1, \ldots, A_N | E)$ is a distribution coming from measurement $\mathcal{L}(\bar{x})$ on purification of $\rho_{N(A)}$ to system $E$, and the infimum is taken over classical channels transforming a random variable $E$ to a random variable $F$.

We will exemplify Corollary 2 for $N = 3$ parties and the scenario considered in Ref. [20]. For the results, see Fig. 2. Let us also note that when one restricts the infimum in Eq. (39), the channel $\Lambda : E \rightarrow F$ has only a classical output and the above bound is a multipartite generalization of the intrinsic information bound given in Ref. [17]. An analogous corollary holds for the case of $K_{ID,dev}^{id,\bar{x}}$.

We finally note, that $E^c_{sq,par}$ is convex also in the multipartite case. This may prove important when one finds upper bounds, as any convexification of two plots obtained from optimization of $E^c_{sq,par}$ is then an upper bound on $K_{DI,par}$, as it was used in Ref. [18]. We state it below following Lemma 8 of Ref. [18].

**Proposition 1.** The $E^c_{sq,par}$ is convex, i.e., for every device $(\bar{\rho}, \mathcal{M})$ and an input tuple $\bar{x}$ there is

$$E^c_{sq,par}((\bar{\rho}, \mathcal{M}(\bar{x})) \leq p_{1}E^c_{sq,par}(\bar{\rho}, \mathcal{M}(\bar{x})) + p_{2}E^c_{sq,par}(\bar{\rho}, \mathcal{M}(\bar{x})),$$

(40)

where $\bar{\rho} = p_{1}\rho_{1} + p_{2}\rho_{2}$ and $p_{1} + p_{2} = 1$ with $0 \leq p_{1} \leq 1$.

**Proof.** The proof goes the same way as that for the bipartite case of Lemma 8 in Ref. [18], with the only change that we base it on the convexity of its multipartite version $E^c_{sq}$, i.e., Lemma 2 here, and the fact that

$$I(A_1, A'_1; \ldots; A_N, A'_N | E | \rho_{A_1, \ldots, A_N} \otimes | i, \ldots, i \rangle \langle i, \ldots, i | A'_1, \ldots, A'_N) = I(A_1; \ldots; A_N | E | \rho_{A_1, \ldots, A_N})$$

(41)

where $i \in \{0, 1\}$, $\rho_{A_1, \ldots, A_N}$ is arbitrary state of systems $A_1, \ldots, A_N$, and we define $I(A_1, A'_1; \ldots; A_N | E | \rho_{A_1, \ldots, A_N}) = I(A_1; \ldots; A_N | E | \rho_{A_1, \ldots, A_N})$. This is because a pure product state alters neither the entropy of marginals nor the global entropy of the state.

We note that the multipartite function $E^c_{sq}$ can be defined for multiple measurements as in Ref. [18] and the analogous results (e.g., Corollary 6 of Ref. [18]) to the bipartite case would hold for the multipartite case.

**Definition 5.** The reduced $c$-squashed entanglement of the collection of measurements $\mathcal{M}$ with probability distribution $p(x)$ of the input reads

$$E^c_{sq}(\rho_{N(A)}, \mathcal{M}(x)) := \sum_{x} p(x) E^c_{sq}(\rho_{N(A)}, M_{x}).$$

(42)

Usually, the parties broadcast their inputs used to generate the key during the protocol. One can therefore consider a version of the distillable device-independent key achieved by such protocols which do this broadcasting. We then consider the quantum device-independent key rate

$$K_{DI,dev}^{id,broad}(\rho_{N(A)}, \mathcal{M}(x)) := \inf_{\epsilon > 0} \lim_{n \to \infty} \sup_{\rho \in \text{LOPC}(\sigma_{N(A)}, \sigma_{N}(\rho_{N(A)}), \mathcal{M})} \kappa_{\epsilon}^n(\hat{P}(\sum_{x} p(x) N_{x} \otimes \text{id}_{E}(|\psi_{\sigma} \rangle \langle \psi_{\sigma} | \otimes | x \rangle \langle x | E_{x}) \otimes n)),$$

(43)

where by broad we mean that $x := (x_1, \ldots, x_N)$ are broadcasted and we make it explicit by adding classical registers $E_{x} := E_{x_1}, \ldots, E_{x_N}$ held by Eve. We have then a generalization of Theorem 2 to the case of more measurements that are revealed during the protocol of key distillation.

**Proposition 2.** The independent and identically distributed quantum device-independent key achieved by protocols using measurements of a device $(\rho_{N(A)}, M)$ with probability $p(x)$ is upper bounded as

$$K_{DI,dev}^{id,broad}(\rho_{N(A)}, \mathcal{M}(x)) := \inf_{\epsilon > 0} \lim_{n \to \infty} \sup_{\rho \in \text{LOPC}(\sigma_{N(A)}, \sigma_{N}(\rho_{N(A)}), \mathcal{M})} \kappa_{\epsilon}^n(\hat{P}(\sum_{x} p(x) N_{x} \otimes \text{id}_{E}(|\psi_{\sigma} \rangle \langle \psi_{\sigma} | \otimes | x \rangle \langle x | E_{x}) \otimes n))$$

(44)

$$\leq \frac{1}{N-1} \inf_{(\sigma_{N}), (\rho, \mathcal{M})} E^c_{sq}(\sigma_{N(A)}, N, p(x))$$

(45)

$$=: E^c_{sq,dev}(\rho_{N(A)}, \mathcal{M}(x)),$$

(46)

where $N_{x}$ are measurements induced by $x$ on $N$.  

**Proof.** The proof follows straightforwardly from generalization of Lemma 10 and Theorem 10 from Ref. [18] for the case of $E^c_{sq}$, taking as the argument the measurements as in Eq. (42), composed with a broadcast map which for the choice of inputs $x$ creates systems $E_{x}$ in state $| x \rangle \langle x |$.
IV. Bound on the Rate of a Parity CHSH Based Protocol by the Reduced C-Squashed Entanglement

In this section we consider the scenario of $N = 3$ parties and compare the known lower bound on the conference key rate [20] with the upper bounds introduced in previous sections.

Below we exemplify the use of the bound by the reduced c-squared entanglement $E_{\text{nc}}$ in the case with classical Eve, that is, when the infimum in its definition runs over the extensions of the form $\sum_i p_i |i\rangle \otimes |i\rangle_E$ (or equivalently the channels acting on system $E$ have only classical outputs). We restrict ourselves to the standard protocols with a single pair of inputs generating the key [17]. We exemplify the bound given in Corollary 2 by means of $I(N(A) \downarrow E)$. It then is in essence a matter of checking the value of the multipartite intrinsic information measure of a distribution which is the output of a key-generating measurement on the attacking state (as it is done in the bipartite case in Ref. [17]).

To compare the introduced upper bounds with the known lower bound, for the honest implementation, we focus on the GHZ state, on which depolarizing noise acts locally on three qubits [20]. Having this state, and playing a tripartite game on it [11], called the parity Clauser-Horne-Shimony-Holt (CHSH) game, one can obtain (in the low-noise regime) a secure conference key. More precisely, we have the following.

Definition 6 (parity CHSH game [20]). The parity CHSH inequality extends the CHSH inequality to $N$ parties as follows. Let Alice and Bob$_1$, ..., Bob$_{N-1}$ be the $N$ players of the following game (the parity CHSH game). Alice and Bob$_1$ are asked uniformly random binary questions $x \in \{0,1\}$ and $y \in \{0,1\}$, respectively. The other Bobs are each asked a fixed question, e.g., always equal to 1. Alice will answer bit $a$, and for all $i \in \{1, \ldots, N-1\}$, Bob$_i$ answers bit $b_i$. We denote by $\hat{b} = \bigotimes_{2 \leq i \leq N-1} b_i$ the parity of all the answers of Bob$_2$, ..., Bob$_{N-1}$. The players win if and only if

$$a + b_1 = x(y + \hat{b}) \mod 2. \quad (47)$$

As for the CHSH inequality, the winning probability $p_{\text{win}}^{\text{Parity-CHSH}}$ for the classical strategies of the parity CHSH game must satisfy

$$p_{\text{win}}^{\text{Parity-CHSH}} \leq \frac{3}{4}. \quad (48)$$

The above inequality can be violated with the $\Phi^\text{GHZ}_3$ state, with the maximal (quantum) value of $\frac{1}{2} + \frac{1}{\sqrt{2}}$.

We adopt the same model of noise as in Ref. [20], which is represented by qubit depolarizing channels acting the same way on each qubit of the GHZ state:

$$D_\nu(r) = (1 - \nu)r + \nu \frac{1}{2}. \quad (49)$$

Below we explain the result of applying this global channel to the GHZ state $|\Phi^\text{GHZ}_N\rangle$ in the case of $N = 3$.

Observation 2. The GHZ state after the action of depolarizing noise on each qubit reads

$$D_\nu \otimes \mathbb{1}_{B_1B_2} (|\Phi^\text{GHZ}_3\rangle |\Phi^\text{GHZ}_3\rangle_{AB_1B_2}) = (1 - \nu)|\Phi^\text{GHZ}_3\rangle |\Phi^\text{GHZ}_3\rangle_{AB_1B_2} + \nu \frac{1}{2} \otimes \kappa_{B_1B_2}, \quad (50)$$

where the $\kappa_{B_1B_2} = \frac{1}{2} \otimes \kappa_{B_1B_2}$ is separable.

Remark 5. The fully separable state originating from a depolarizing channel (single party), i.e., $\frac{1}{2} \otimes \kappa_{B_1B_2}$, cannot violate the parity CHSH inequality.

After applications of the depolarizing channel to each of three qubits we obtain the following.

Corollary 3. We have

$$D_\nu^{\otimes 3} (|\Phi^\text{GHZ}_3\rangle |\Phi^\text{GHZ}_3\rangle_{AB_1B_2}) = (1 - \nu)^3|\Phi^\text{GHZ}_3\rangle |\Phi^\text{GHZ}_3\rangle_{AB_1B_2} + [1 - (1 - \nu)^3] \chi_\nu, \quad (51)$$

where $\chi_\nu$ is a fully separable state which reads

$$\chi_\nu := \frac{1}{1 - (1 - \nu)^3} \left( (1 - \nu)^2 \nu_{AB_1B_2} \otimes \frac{1}{2} + (1 - \nu)^2 \nu_{AB_2B_3} \otimes \frac{1}{2} + (1 - \nu)^2 \nu_{AB_3B_1} \otimes \frac{1}{2} \right). \quad (52)$$

In Ref. [20], the expected winning probability for the parity CHSH game (with respect to the depolarizing noise parameter) is calculated:

$$p_{\text{exp}} := \frac{1}{2} + \frac{(1 - \nu)^N}{2\sqrt{2}} + \frac{(1 - \nu)^2(1 - (1 - \nu)^{N-2})}{8\sqrt{2}}. \quad (53)$$

From the above equality for $N = 3$, the state in Eq. (51) violates the classical bound of $\frac{1}{2}$ for $0 \leq \nu < \nu_{\text{crit}}$, where $\nu_{\text{crit}} \approx 0.1189$.

In this place, we start the construction of the eavesdropper strategy. According to the DI-CKA protocol in Ref. [20], the ranges of inputs and outputs are $x \in \{0,1\}$, $y \in \{0,1,2\}$, $y_2 \in \{0,1\}$, and $a, b_1, b_2 \in \{0,1\}$. The setting $(x, y, y_2) = (0, 2, 0)$ associated with measurements of $\sigma_z$ observable is the key-generating round:

$$P_\nu(a, b_1, b_2|x, y, y_2) = \text{Tr} \left[ M_{a|x} \otimes M_{b_1|y_1} \otimes M_{b_2|y_2} D_\nu^{\otimes 3} (|\Phi^\text{GHZ}_3\rangle |\Phi^\text{GHZ}_3\rangle_{AB_1B_2}) \right] \quad (54)$$

$$= (1 - \nu)^3 \text{Tr} \left[ M_{a|x} \otimes M_{b_1|y_1} \otimes M_{b_2|y_2} |\Phi^\text{GHZ}_3\rangle \langle \Phi^\text{GHZ}_3|_{AB_1B_2} \right] + (1 - (1 - \nu)^3) \text{Tr} \left[ M_{a|x} \otimes M_{b_1|y_1} \otimes M_{b_2|y_2} \chi_\nu \right] \quad (55)$$

$$= (1 - \nu)^3 P_{\text{GHZ}}(a, b_1, b_2|x, y_1, y_2) + (1 - (1 - \nu)^3) P_{\text{L}}(a, b_1, b_2|x, y_1, y_2). \quad (56)$$
Eve prepares a convex combination attack \[ P^\text{CC}_{\nu}(a, b_1, b_2, e|x, y_1, y_2) \]
\[ = (1 - \nu)^3 P_{\text{GHZ}}(a, b_1, b_2|x, y_1, y_2) \delta_{e, ?} \]
\[ + [1 - (1 - \nu)^3]P^\text{L}_{\nu}(a, b_1, b_2|x, y_1, y_2) \delta_{e,(a,b_1,b_2)}. \] (57)

This attack might not be optimal as it uses a particular decomposition of \( P_{\nu} \). In order to optimize the attack, Eve should find a decomposition with a maximal weight of local behavior \([1 - (1 - \nu)^3]\) here.

We now consider a particular strategy of post-processing the data which is in Eve’s possession, represented by a channel \( E \rightarrow F \) in Corollary 2. Following Ref. [17], we consider only the distribution coming from a key-generating measurement, which according to the protocol of Ref. [20] is \( X = 0 \) for Alice and \( B_1 = 2 \) and \( B_2 = 0 \) for the Bobs in the case of \( N = 3 \),

\[ P_{\text{ATTACK}}(a, b_1, b_2, f|020) = \Lambda_{E \rightarrow F} P^\text{CC}_{\nu}(a, b_1, b_2, e|020) \]
\[ = (1 - \nu)^3 P_{\text{GHZ}}(a, b_1, b_2|020) \delta_{f, ?} \]
\[ + [1 - (1 - \nu)^3]P^\text{L}_{\nu}(a, b_1, b_2|020) \]
\[ \times [\delta_{a, b_1, b_2} \delta_{f, a} + (1 - \delta_{a, b_1, b_2}) \delta_{f, ?}], \] (58)

where \( \delta_{a, b_1, b_2} \) is 1 if all three indices have the same value and 0 otherwise. The above attack strategy is therefore a direct three-partite generalization of strategy proposed in Ref. [17]. The eavesdropper aims to be correlated only with the events \((a, b_1, b_2) = (0, 0, 0)\) or \((a, b_1, b_2) = (1, 1, 1)\), whenever they originate from the local behavior \( P^\text{L}_{\nu} \), and maps all other events to \( f = ? \). By applying the above attack strategy, we are ready to plot an upper bound on the reduced c-squeezed entanglement shown in Corollary 2. The latter bound is a multipartite version of the intrinsic information [46, 47], used first for the bipartite case in [45] against non-signaling adversary (see in this context [15, 40, 48]). Here the strategy of Eve to process her classical variable \( E \) to \( F \) is based on [17] as shown above.

V. GAP BETWEEN DI-CKA AND DD-CKA

In this section, we provide a bound on the conference key agreement of \( N \) parties in terms of the bounds for groupings of these parties into groups of fewer than \( N \) users. We further show that there is a gap between the device-independent and device-dependent conference key agreement rates. This gap implies that there are states for which there are no measurements used for testing and no CLOPC protocol that can achieve the same number of keys as in the device-dependent case. The gap is inherited from the analogous gap shown for the bipartite case [16].

In what follows, by a (nontrivial) partition \( P \) of the set of systems \( \{A_1, \ldots, A_N\} \), we mean any grouping of the systems into at least two but no more than \( N - 1 \) subsets such that each \( A_i \) belongs to exactly one subset and each of them belongs to some subset. Let us now generalize the definition of the reduced device-dependent key to the case of the conference key agreement. We will further also show the fact that the latter quantity bounds the device-independent conference key (i.e., Theorem 6 of Ref. [16]).

**Definition 7.** The reduced device-dependent conference key rate of an \( N \)-partite state \( \rho_{\mathcal{N}(A)} \) reads

\[ K^\text{D}(\rho_{\mathcal{N}(A)}) := \sup_{\mathcal{M}} \inf_{(\mathcal{N}(A), \mathcal{L}) = (\rho_{\mathcal{N}(A)}, \mathcal{M})} K_{\text{DD}}(\sigma_{\mathcal{N}(A)}). \] (59)

A direct analog of Theorem 6 of Ref. [16] (with an analogous proof which we omit here) states that the reduced device-dependent key upper bounds the device independent key.

**Theorem 4.** For any \( N \)-partite state \( \rho_{\mathcal{N}(A)} \) and any \( \mathcal{M} \), there is

\[ K_{\text{DI}}(\rho_{\mathcal{N}(A)}, \mathcal{M}) \leq \inf_{(\mathcal{N}(A), \mathcal{L}) = (\rho_{\mathcal{N}(A)}, \mathcal{M})} K_{\text{DD}}(\sigma_{\mathcal{N}(A)}), \] (60)

and in particular,

\[ K_{\text{DI}}(\rho_{\mathcal{N}(A)}) \equiv \sup_{\mathcal{M}} K_{\text{DI}}(\rho_{\mathcal{N}(A)}, \mathcal{M}) \leq K^\text{D}(\rho_{\mathcal{N}(A)}). \] (61)

We first observe the following bound.

**Proposition 3.** For any \( N \)-partite quantum behavior \((\rho_{\mathcal{N}(A)}, \mathcal{M})\) there is

\[ K_{\text{DI, dev}}^{\text{id}}(\rho_{\mathcal{N}(A)}, \mathcal{M}) \leq \min_{\mathcal{P}} \left\{ K_{\text{DI, dev}}^{\text{id}}(\rho_{\mathcal{P}(\mathcal{N}(A))}) \right\}, \] (62)

where \( \mathcal{P} \) is any non-trivial partition of the set of systems \( A_1, \ldots, A_N \).

**Proof.** The proof of the bound by \( K_{\text{DI, dev}}^{\text{id}}(\rho_{\mathcal{P}(\mathcal{N}(A))}) \) follows from the fact that any protocol of distillation of the DI conference key from the \( N \)-partite state is a special case of a protocol that distills the DI conference key from a non-trivial partition \( \mathcal{P} \). This is because the class of LOPC protocols in these two scenarios is in relation to \( \text{LOPC}(A_1, \ldots, A_N) \subseteq \text{LOPC}(\mathcal{P}(A_1, \ldots, A_N)) \). The other bound follows from the fact that for any grouping \( \mathcal{P} \), by Theorem 4 above,

\[ K_{\text{DI, dev}}^{\text{id}}(\rho_{\mathcal{P}(\mathcal{N}(A))}, \mathcal{M}) \leq \inf_{(\mathcal{P}(\mathcal{N}(A)), \mathcal{L}) = (\rho_{\mathcal{P}(\mathcal{M}(\mathcal{N}(A)))}, \mathcal{M})} K_{\text{DD}}(\sigma_{\mathcal{P}(\mathcal{N}(A))}). \] (63)
An analogous fact to the above holds for $K^{id}_{DD,par}$ as well.

Following Ref. [16], we show now that there is a gap between the numbers of conference keys and device-independent conference keys. We will use the fact that there it has been proven that there are states with $K^i(\rho_{AB}) < K_{DD}(\rho_{AB})$. From such state $\rho_{AB}$ we construct a multipartite state with the property that $K_{DI}(\rho_{N(A)}) < K_{DD}(\rho_{N(A)})$, as it is described in the proof of the following theorem.

**Theorem 5.** Let $\rho_{AB} \in B(\mathcal{H}_A \otimes \mathcal{B})$, where $\dim(\mathcal{H}_A) = d_A$ and $\dim(\mathcal{H}_B) = d_B$, be a bipartite state which admits a gap $K_{DD}(\rho_{AB}) - K^i(\rho_{AB}) \geq c > 0$ for some constant $c$. Then for any $N$ there is a multipartite state $\rho_{N(A)}$ with local dimensions at most $d_A \times d_B$ with $K_{DD}(\rho_{N(A)}) - K_{DI}(\rho_{N(A)}) \geq c$.

**Proof.** Consider a state $\rho_{N(A)}$ constructed as a path made of state $\rho_{AB}$ (as, e.g., in a line of a quantum repeater):

$$\tilde{\rho}_{N(A)} := \rho_{A_1^1 A_2^1} \otimes \rho_{A_1^1 A_2^2} \otimes \rho_{A_2^1 A_3^1} \otimes \cdots \otimes \rho_{A_{N-1}^1 A_{N-1}^2}. \tag{64}$$

Here $\rho_{A_i^1 A_i^2} = \rho_{A_{i-1}^1 A_{i-1}^2} = \cdots = \rho_{A_{i-1}^1 A_{i-1}^2} = \rho_{AB}$ and by the way of notation we have $A_1^1 \equiv A_1$ and $A_1^2 A_2^1 \equiv A_2$, $A_2^2 A_3^1 \equiv A_3$, etc. That is, the first party has only system $A_1$ and the last only system $A_{N-1}$, while the $i$th party for $1 < i < N$ has systems $A_{i-1}^1 A_i^2$ at hand.

Since the states $\rho_{A_i^1 A_i^2}$ form a spanning tree of a graph of $N$ systems (in fact a path), we can follow the lower bound given in Section VI A of Ref. [21] and note that

$$K_{DD}(\tilde{\rho}_{N(A)}) \geq \min_i K_{DD}(\rho_{A_i^1 A_i^2}) = K_{DD}(\rho_{AB}). \tag{65}$$

Indeed, the parties can first distill a key at rate $K_{DD}(\rho_{AB})$ along the edges of the path. Denote such distilled keys by $k_{ij}$ between nodes $i$ and $j$. Further, $A_1$ can XOR her key $k_{12}$ with a locally generated private random bit string $r$ of length $K_{DD}(\rho_{AB})$ and send $k_{12} \oplus r$ to $A_2$; further, $A_2$ can obtain $r = k_{12} \oplus (k_{12} \oplus r)$ and send it to the next party by XORing it with the key $k_{23}$. This process repeated $N - 1$ times, leaves all the parties knowing $r$, which remained secret due to one-time pad encryption by the keys $k_{12}, k_{23}, \ldots, k_{N-1,N}$. It then suffices to note that, by Proposition 3,

$$K_{DI}(\tilde{\rho}_{N(A)}, \mathcal{M}) \leq \inf_{(\sigma_{A_1^1 A_2^1 \ldots A_{N-1}^1 A_{N-1}^2} : (\rho_{A_1^1 A_2^1 \ldots A_{N-1}^1 A_{N-1}^2}, \mathcal{M}))} K_{DD}(\sigma_{A_1^1 A_2^1 \ldots A_{N-1}^1 A_{N-1}^2}). \tag{66}$$

This is due to the fact that any distillation protocol between $A_1^1$ and $(A_2^1, \ldots, A_{N-1}^1)$ is a particular protocol distilling key between systems $A_1^2$ and $A_2^2$.

Taking the supremum over $\mathcal{M}$ on both sides of the inequality (67), we obtain

$$K_{DI}(\tilde{\rho}_{N(A)}) = \sup_{\mathcal{M}} K_{DI}(\tilde{\rho}_{N(A)}, \mathcal{M}) \leq K^i(\rho_{A_1^1 A_2^1}) \equiv \inf_{\mathcal{M} (\sigma_{A_1^1 A_2^1} : (\rho_{A_1^1 A_2^1}, \mathcal{M}))} K_{DD}(\sigma_{A_1^1 A_2^1}). \tag{68}$$

Hence we get $K_{DI}(\tilde{\rho}_{N(A)}) \leq K^i(\rho_{AB})$. This fact, by Eq. (65), and the fact that by assumption $K_{DD}(\rho_{AB}) - K^i(\rho_{AB}) \geq c > 0$ imply the following chain of inequalities:

$$K_{DD}(\tilde{\rho}_{N(A)}) \geq K_{DD}(\rho_{AB}) > K^i(\rho_{AB}) \geq K_{DD}(\tilde{\rho}_{N(A)}). \tag{69}$$

The above implies then the desired gap $K_{DD}(\tilde{\rho}_{N(A)}) - K_{DI}(\tilde{\rho}_{N(A)}) > 0$. Moreover, this gap is as large as $c > 0$ due to the assumption that $K_{DD}(\rho_{AB}) - K_{DI}(\rho_{AB}) \geq c > 0$. The claim about dimensions follows from the form of the state given in Eq. (64).

From Ref. [16] we have the immediate corollary that there is a gap between the DI-CKA and DD-CKA.

**Corollary 4.** For any $N$ there is a state $\tilde{\rho}_{N(A)}$ for which there is

$$K^{id}_{DD,par}(\tilde{\rho}_{N(A)}) < K_{DD}(\tilde{\rho}_{N(A)}). \tag{70}$$

**Proof.** Reference [16] shows an example of a bipartite state $\rho_{AB}$ with the gap $K^i(\rho_{AB}) < K_{DD}(\rho_{AB})$. The construction given in Eq. (64) based on this $\rho_{AB}$ proves the thesis via Theorem 5.

We note also that a bound similar to that in the above corollary holds for $K^{id}_{DD,par}$ and $K_{DI}$ itself due to the fact that $K_{DI}^{id} \geq K_{DI}$ by definition [16]. We can modify the proof technique shown above to see the following general remark.

**Remark 6.** In the above construction one need not use only the state $\rho_{A_1^1 A_2^2} \otimes \sigma_{A_2^1 A_3^1}$ having $K^i(\rho_{A_1^1 A_2^2}) < K_{DD}(\rho_{A_1^1 A_2^2})$. In fact the state on systems $A_2^1 A_2^2 \ldots A_{N-1}^1 A_{N-1}^2$ can be an arbitrary state having $K_{DD}(\rho_{A_1^1 A_2^2 \ldots A_{N-1}^1 A_{N-1}^2}) \geq K_{DD}(\rho_{A_1^1 A_2^2})$. It can be even a $\psi_{GHZ}^{N-1}$ state of arbitrary large local dimension. This is with no change in the above proof if only $\rho_{A_1^1 A_2^2}$ is on systems $A_1^2 A_2^1$ with the gap we have mentioned. See Fig. 1 for the tripartite example.

**VI. DI-CKA VERSUS GENUINE NONLOCALITY AND ENTANGLEMENT**

We now discuss the topic of genuine nonlocality and entanglement in the context of the DI-CKA, introducing the notion of quantum locality.
We say that a behavior \( P(a|x) \) is local in a cut \((A_1, ..., A_k) : (A_{k+1}, ..., A_N)\) if it can be written as a product of two behaviors on systems \( A_1, ..., A_k \) and \( A_{k+1}, ..., A_N \), respectively. The behavior \( P(a|x) \) is genuinely non-local if and only if it is not a mixture of behaviors that are a product in at least one cut.

We show that any behavior from which the parties draw the conference key in a single-shot (single run) must exhibit genuine nonlocality. The scenario of a single run was considered in the context of a non-signaling adversary [49, 50]. For that reason, we depart from the traditional definition of DI quantum key distillation rate by considering a single-shot DI quantum key distillation rate obtained by an LOPC post-processing of a distribution obtained from some behavior \( P(a|x) \) when all the parties measure all the inputs \( x \) in parallel at the same time.

For the purpose of Theorem 6 below, by a "local" set we will mean the set of behaviors that are convex mixtures of behaviors that are a product in some cut and both behaviors in the product have quantum realization. We will denote this set by \( LQ \) (locally quantum).

The maximum single-shot device-independent quantum key rate obtained in this setup in full analogy to Definition 1 as follows.

**Definition 8.** The maximum single-shot device-independent quantum key distillation rate of a device \((\rho,M)\) with independent and identically distributed behavior is defined as

\[
K^{single-shot}_{\text{DI,dev}}(\rho,M) := \inf_{\varepsilon > 0} \sup_{\varepsilon > 0} \inf_{\mathcal{P}(\sigma,N)} \kappa^\varepsilon_n \left( \mathcal{P}(\sigma,N) \right),
\]

where \( \kappa^\varepsilon_n \) is the quantum key rate achieved for any security parameter \( \varepsilon \) and measurements \( N \).

Here \( \mathcal{P} \) is a protocol composed of classical local operations and public (classical) communication acting on a single copy of \( (\sigma,N) \) which, composed with the measurement, results in a quantum local operations and public (classical) communication protocol.

We are ready to state the following theorem.

**Theorem 6.** If a behavior \((\rho_N(A),M)\) satisfies \( K^{single-shot}_{\text{DI,dev}}(\rho_N(A),M) > 0 \) then it is not in \( LQ \).

Proof. The proof goes by contradiction. Suppose a behavior \( p(a|x) \equiv (\rho_N(A),M) \) is not genuinely nonlocal.

That is, it can be expressed as a convex combination of behaviors which are a product in at least one cut denoted by \((A_{j_1}, ..., A_{j_k}) : (A_{j_{k+1}}, ..., A_{j_N})\) for the \( i \)-th behavior in the combination. We express this as

\[
\sum_i q_i p_i(a|x)_{(A_{j_1} ... A_{j_k}) : (A_{j_{k+1}} ... A_{j_N})},
\]

where \( p_i(a|x) \) are some quantum behaviors. Consider then a device \( p_i \) as a bipartite one, with parties \((A_{j_1} ... A_{j_k}) \) together forming \( A' \) and \((A_{j_{k+1}} ... A_{j_N}) \) forming \( A'' \). Such a device has zero bipartite DI quantum keys, as it is a product in cut \( A' : A'' \). By virtue of purification, Eve can have access to the mixture (72), knowing which of the mixing terms \( i \) happened. By Proposition 3 we have that from any of such terms, one can not draw a conference key, as Eve has a local hidden variable model for it. Indeed, the right-hand side of (62) is then 0, as Eve can adopt an attack which, e.g., makes zero reduced c-squashed entanglements [18]. We thus obtained the desired contradiction.

\[ \Box \]

Let us now recall the notion of genuine entanglement. We say that a multipartite state \( \rho_{A_1A_2...A_N} \) is separable in a cut \((A_1, ..., A_k) : (A_{k+1}, ..., A_N)\) if it can be written as convex mixtures of product states between systems \( A_1, ..., A_k \) and \( A_{k+1}, ..., A_N \). If a multipartite state \( \rho_{A_1A_2...A_N} \) can be written as a mixture of separable states that are a product in at least one cut then it is called biseparable. We say that \( \rho_{A_1A_2...A_N} \) is genuinely entangled if and only if it is not a mixture of separable states that are a product in at least one cut. It was shown in Ref. [51] that there exist \( N \)-partite states for all \( N > 2 \) where some genuinely entangled states admit a fully LRHV model, i.e., where all parties are separated.

Let \( \text{GE}(N(A)) \), \( \text{BS}(N(A)) \), and \( \text{FS}(N(A)) \) denote the set of all \( N \)-partite states \( \rho_{A_1A_2...A_N} \) that are genuinely entangled, biseparable, and fully separable, respectively (see Ref. [21]). For \( n \) copies of \( N \)-partite state, when we consider partition across designated \( N \) parties, we denote local groupings by \( N(A^{\otimes n}) \).

**Remark 7.** It is necessary to consider a single-shot DI key in Theorem 6 because the set of \( LQ \) behaviors is not closed under tensor product. This is for the same reason that the set of biseparable states is not closed under a tensor product.

The following theorem follows from Corollary 1 as well as Proposition 2 of Ref. [21].

**Theorem 7.** The maximum device-independent conference key agreement rates of a device \((\rho_N(A),M)\) are up-
per bounded by
\[
K_{DI,\text{dev}}(\rho_{N(A)}, \mathcal{M}) \leq \inf_{(\sigma_{N(A)}, \mathcal{L}) = (\rho_{N(A)}, \mathcal{M})} E_{GE}^\infty(\sigma_{N(A)}),
\]
(73)
\[
K_{DI,\text{par}}(\rho_{N(A)}, \mathcal{M}) \leq \inf_{\mathcal{L}} \omega(\sigma_{N(A)}, \omega_{\mathcal{L}} = \omega(\rho_{N(A)}, \mathcal{M})} E_{GE}^\infty(\sigma_{N(A)}),
\]
(74)
where \(E_{GE}^\infty(\cdot)\) is the regularized relative entropy of genuine entanglement \([21]\) of a state \(\varsigma_{A_1, A_2 \ldots A_N}\), with
\[
E_{GE}^\infty(\varsigma) = \inf_{\varphi \in \text{BS}(\mathcal{N}(A \otimes n))} \lim_{n \to \infty} \frac{1}{n} D(\varsigma^n \parallel \varphi)
\]
(75)
where \(D(\rho\parallel\sigma)\) is the relative entropy between two states \(\rho\) and \(\sigma\), with \(D(\rho\parallel\sigma) = \text{Tr}[\rho (\log_2 \rho - \log_2 \sigma)]\) if \(\text{supp} \rho \subseteq \text{supp} \sigma\); otherwise it is \(\infty\) \([32]\).

We note here that there is a trivial bound that can be obtained from Theorem 7 above, which is encapsulated in the following corollary.

**Corollary 5.** For any state \(\rho_{N(A)}\) with \(\min_{i \in \{1 \ldots N\}} d_{A_i} = d\) there is
\[
K_{DI}(\rho_{N(A)}) = \sup_{\mathcal{M}} K_{DI}(\rho_{N(A)}, \mathcal{M}) \leq \min\{p \log_2 d : p \in [0, 1], \rho = p \rho' + (1-p) \rho_{fs}, \rho_{fs} \in \text{FS}(\mathcal{N}(A)), \}
\]
(76)

**Proof.** Given any decomposition of a state \(\rho_{N(A)}\) into \(\rho_{N(A)} = p \rho' + (1-p) \rho_{fs}\), where the state \(\rho_{fs}\) is a fully separable state, we have
\[
K_{DI}(\rho_{N(A)}) \leq \sup_{\mathcal{M}} K_{DI}(\rho_{N(A)}, \mathcal{M}) \leq \sup_{\mathcal{M}} \inf_{(\sigma_{N(A)}, \mathcal{L}) = (\rho_{N(A)}, \mathcal{M})} E_{GE}^\infty(\sigma_{N(A)}) \leq \sup_{\mathcal{M}} E_R(\rho_{N(A)}) \leq p \min \log_2 d_{A_i},
\]
(77)
where we have used Theorem 7 (also see Corollary 6 of \([21]\)) and the fact that \(E_R(\rho) = \inf_{\kappa \in \text{FS}} D(\rho\parallel\kappa)\) is (i) convex, (ii) zero on fully separable states, and (iii) does not exceed the minimum logarithm of dimensions of the input state, which can be proved by noticing that \(E_R(\rho) \leq D(\rho(\parallel\rho_{A_1} \otimes A_{N\setminus i})) = I(A_i : A_{N\setminus i}) \leq \log_2 d\) where \(A_i\) has minimal dimension among systems \(A_1, \ldots, A_N\). \(\Box\)

We presented this bound in Fig. 2 in Section IV and we saw that it is indeed above the upper bounds which we derive in Section III.

**VII. CONCLUSION**

We have demonstrated a number of upper bounds on the quantum secure conference key, generalizing (i) the results of Ref. \([18]\) regarding a relative entropy based bound and (ii) the results of Ref. \([17]\) regarding the reduced c-squashed entanglement.

Interestingly, the approach of Ref. \([17]\) does not result in zero keys in any noise regimes for the parity CHSH game of Ref. \([20]\). It would be important to see if this can be improved by changing Eve’s strategy or the bound needs to be changed.

We have also shown that the fundamental gap between device-independent and device-dependent keys also holds in the multipartite case. We have given an exemplary state which is based directly on the bipartite states given in Ref. \([16]\). It is interesting if such a state exists in lower dimensions or even possibly on \(N\) qubits.

Finally, our results hold for the static case of quantum states. The next step would be to generalize the results of Ref. \([18]\) for the dynamic case of quantum channels to the multipartite scenario.

**Note added.**—The topic of upper bounds on the DICKA is also studied in the parallel work of \([48]\). Comparison between basic approaches (i.e., for the DI quantum key distribution between two honest parties) used in Ref. \([48]\) and in this paper to get upper bounds on DICKA is discussed in Ref. \([18]\).

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Appendix A: Continuity statements

There are the following lemmas.

**Lemma 4** (Alicki-Fannes-Winter continuity bounds [53]). For states $\rho_{AB}$ and $\sigma_{AB}$, if $\frac{1}{2} \| \rho - \sigma \|_1 \leq \varepsilon \leq 1$, then

$$|S(A|B)_{\rho} - S(A|B)_{\sigma}| \leq 2\varepsilon \log_2 d + g(\varepsilon),$$

(A1)

where $d = \text{dim}(\mathcal{H}_A) < \infty$ and $g(\varepsilon) := (1 + \varepsilon) \log_2(1 + \varepsilon) - \varepsilon \log_2 \varepsilon$.

**Lemma 5** (from [42]). If $d = \min \{\text{dim}(\mathcal{H}_A), \text{dim}(\mathcal{H}_B)\} < +\infty$, then

$$|I(A: B|C)_{\rho} - I(A: B|C)_{\sigma}| \leq 2\varepsilon \log_2 d + 2g(\varepsilon)$$

(A2)

for any states $\rho_{ABC}$ and $\sigma_{ABC}$, where $\varepsilon = \frac{1}{2} \| \rho - \sigma \|_1$.

Appendix B: Secrecy monotones

In this Appendix we revisit Theorem 3.1 of Ref. [24] and generalize the result by relaxing the constraints on the Hilbert spaces in the following way. First, we prove an analogy to Lemma A.1 of Ref. [24].

**Lemma 6** (cf. [24]). The maximization in the definition of $K_{DD}$ (12) can be restricted to protocols that use communication at most linear in the number of copies of $\rho_{ABE}$. The eavesdropper system is not necessarily restricted to a finite dimension.

Proof. The proof of Lemma 6 goes along the lines of the proof of Lemma A.1 in Ref. [24]. The change that is necessary to allow the eavesdropper to hold the system of infinite dimension is the use of asymptotic continuity of the conditional mutual information of Ref. [42] (see Lemma 5 herein) instead of the Alicki–Fannes inequality. This results in:

$$I(A: B)_n - I(A: E)_n \geq l_n(1 - 4\varepsilon) - 4g(\varepsilon),$$

(B1)

where $l_n$ is the length of the output of a distillation protocol using $n_0$ copies of the input state. The state $\sigma$ is the output of the latter protocol. The overall key rate of the modified protocol which has linear communication admits then a lower bound

$$\tilde{R} \geq (1 - 4\varepsilon)(R - \varepsilon) - \frac{4g(\varepsilon)}{n_0}.$$  

(B2)

The other parts of the proof are not altered. □

**Lemma 7** (cf. [24]). Let $E(\rho)$ be a function mapping a tripartite quantum state $\rho_{ABE}$ into positive numbers such that the following hold: (a) monotonicity, i.e., $E(\Lambda(\rho)) \leq E(\rho)$ for any LOPC $\Lambda$; (b) asymptotic continuity, i.e., for any states $\rho^n$ and $\sigma^n$ on $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E$, the condition $\| \rho^n - \sigma^n \|_1 \to 0$ implies $\frac{1}{\log_2 n} |E(\rho^n) - E(\sigma^n)| \to 0$ where $r_n = \text{dim}(\mathcal{H}_A^n)$; and (c) normalization, i.e., $E(\rho^n) = 1$.

Then the regularization of the function $E$ given by $E^\infty(\rho) = \limsup_n \frac{E(\rho^n)}{n}$ is an upper bound on the device-dependent key distillation rate $K_{DD}$, i.e., $E^\infty(\rho_{ABE}) \geq K_{DD}(\rho_{ABE})$ for all $\rho_{ABE}$ with $\text{dim}_A < \infty$, if in addition $E$ satisfies (d) subadditivity on tensor products: $E(\rho^{\otimes n}) \leq nE(\rho)$; then $E$ is an upper bound on $K_{DD}$.

Proof. The proof arguments are same as those stated in Ref. [24] with relaxation on the Hilbert space of $E$. We observe that the proof arguments hold even when there is no restriction on the $\text{dim}(\mathcal{H}_E)$, i.e., $E$ can be finite dimensional or infinite dimensional. It suffices to have $\text{dim}(\mathcal{H}_A)$ be finite dimensional. □

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