Ultra stable charging of fastest scrambling quantum batteries

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Collective behaviors strongly influence charging dynamics of quantum batteries (QBs). Here, we study the impact of non-local correlations on the energy stored in a system of \( N \) QBs. A unitary charging protocol based on a Sachdev-Ye-Kitaev (SYK) quench Hamiltonian is thus introduced and analyzed. SYK models describe strongly interacting systems with non-local correlations and fast thermalization properties. Here, we demonstrate that, once charged, the average energy stored in the QB is very stable, realizing an ultraprecise charging protocol. By characterizing fluctuations of the average energy stored, we show that temporal fluctuations are strongly suppressed by the presence of non-local correlations at all time scales. Comparison with other paradigmatic examples of many-body QBs shows that this is linked to the fast rise of collective dynamics of SYK model and its high level of entanglement. We argue that such feature relies on the fastest scrambling property of the SYK Hamiltonian, and on its fast thermalization properties, promoting this as an ideal model for the ultimate temporal stability of a generic quantum battery. Finally, we show that the temporal evolution of the ergotropy, a quantity that characterizes the amount of extractable work from a QB, can be a useful probe to infer the thermalization properties of a many-body quantum system.

I. INTRODUCTION

Recent advances in technological miniaturization and fabrication processes have led to the emergence of a new branch of research, dubbed “quantum thermodynamics” [1–6]. The study of thermodynamic concepts, such as work and heat, at the nanoscale, and the interplay with the laws of quantum mechanics, is crucial both from a fundamental and an applicable point of view. A key goal here is to find new strategies to precisely control, store, and manipulate work and energy [1, 7], with improved performances, eventually thanks to the presence of quantum coherences [8–14].

In this framework, quantum batteries (QBs) have been introduced [15, 16], as small quantum systems able to temporarily store energy, to be used at a later stage. Different figures of merit, such as charging time and associated power, have been analyzed [17–23] and bounds on their performances have been inspected, depending on the precise charging protocol [24–28]. These usually rely on an external charger, interacting with one or more cells of the QB [21, 22, 30], or on unitary (local or global) evolution of the closed system in a non-equilibrium setting, i.e., by exciting degrees of freedom with a quantum quench [16, 18–20, 24, 29].

It has been shown that the presence of correlations and entanglement between quantum cells can have non trivial impact on both charging power and extractable work of QBs [18, 19, 24, 27, 31]. Although many-body effects can enhance charging performances [24, 29], strong and non-local correlations are required to achieve a true quantum advantage for QBs. Recently, the impact of random disorder on charging performances of QBs have been also investigated, showing that QBs exhibit typical behavior in the large \( N \) limit given the spectral properties of the driving systems [32, 33].

Moreover, after an initial growth, the average energy stored in a QB during the charging protocol inevitably undergoes fluctuations, which usually undermine its subsequent utility. It is thus of great importance to find protocols able to stabilize energy storage [25, 34] or, even better, systems which intrinsically suppress these unwanted fluctuations.
In this work, we show that non-local correlations greatly help in improving charging stability of QBs, by suppressing temporal fluctuations associated to the average energy stored in a QB. To elucidate this point, we investigate a paradigmatic example of strongly correlated systems with non-local interactions: we introduce and characterize QBs based on the so-called Sachdev-Ye-Kitaev (SYK) model.

SYK models [36–41] are currently receiving a lot of attention from different communities. They describe strongly correlated quantum systems of (Majorana or Dirac) fermions with random all-to-all interactions. It has been shown [36, 39, 40] that SYKs represent unique examples of exactly solvable strongly interacting models in the limit of large number of fermions. Subsequently, the non-Fermi liquid behavior of SYK models has been studied [42–45] and, in a completely different context, intriguing and promising connections with black-hole physics and quantum gravity via holography have been explored [46–49]. Moreover, and in parallel, it has been shown that non-local correlations and random disorder result in highly chaotic dynamics, making these models extremely popular in the quantum chaos community too. The chaotic properties of the SYK models have been extensively investigated both from a random matrix theory point of view, starting from Ref. [50, 51], and by studying the so-called out-of-time-order correlators [36, 39]. The latter result in the saturation of the Maldacena-Shenker-Stanford bound on the Lyapunov exponent [52], thus promoting the SYK models as concrete candidates of this quantity it is possible to infer thermalization time scale of a quantum system. In Sec. V we analyze energy fluctuations of different kinds, i.e. disorder, quantum, and temporal fluctuations, showing comparison between SYK and spin-chain (in the Anderson or MBL phase) based QBs. We demonstrate that SYK-based QBs result in exponentially suppressed temporal fluctuations at all times, a peculiar feature that can be linked to the collective and non-local nature of the system and to its fastest scrambling, and fast thermalizing, property. Sec. VI contains a summary of our main findings.

II. UNITARY CHARGING PROTOCOL

We study charging of QBs with a unitary protocol based on a double-sudden quench [16, 29]. The system is initially assumed to be in the ground state of a given time-independent Hamiltonian, $\hat{H}_0$ (empty battery). We focus on the evolution of the ground state, $|0\rangle$, under the perturbed Hamiltonian

$$\hat{H} = \hat{H}_0 + \kappa \lambda(t) \hat{H}_1,$$

where $\hat{H}_1$ is a time independent driving Hamiltonian and the dimensionless parameter $\kappa$ controls the relative strength between $\hat{H}_0$ and $\hat{H}_1$. The function $\lambda(t)$ describes the charging time interval, and is defined by

$$\lambda(t) = 0, \quad t < 0 \text{ and } t > \tau,$$

$$\lambda(t) = 1, \quad 0 < t < \tau,$$

with $\tau$ being the charging time. Denoting with $|\psi(t)\rangle \equiv \exp(-i\hat{H}t)|0\rangle$ the evolved state under the total hamiltonian (in this work we set $\hbar = 1$), the average energy stored in the QB at the end of the charging time is

$$E(\tau) \equiv \langle \psi(\tau)|\hat{H}_0|\psi(\tau)\rangle - \langle 0|\hat{H}_0|0\rangle.$$

This quantity shows some universal behaviors as a function of $\tau$: it displays an initial growth for $\tau < \bar{\tau}$, with $\bar{\tau}$ being a model-dependent time scale, after that it fluctuates erratically in time around an average value $\bar{E}$ [16, 29], whose value depends on the specific model of QB considered. Charging precision of energy stored in a QB and its subsequent utility are influenced by different (and independent) factors that cause temporal (erratic), disorder and quantum fluctuations. The first kind of fluctuations can be quantified by computing

$$(\sigma_N^{(t)})^2(T_2, T_1) \equiv \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} d\tau \left[ \langle \psi(\tau)|\hat{H}_0|\psi(\tau)\rangle^2 - \bar{E}^2 \right],$$

(4)
where we denoted with $N$ the number of cells (e.g. the number of qubits) of the battery and the time integration is taken in the window defined by $T_2 > T_1 > \tau$. Disorder fluctuations can be thought as an indetermination in $E(\tau)$ due to some randomness or imperfections present in the QB, usually schematized by some random parameters entering the full hamiltonian $\hat{H}$. These fluctuations can be evaluated by

$$
(\sigma_N^{(d)})^2(\tau) \equiv \left\langle \left[ \left\langle \psi(\tau)|\hat{H}_0|\psi(\tau) \right\rangle \right]^2 \right\rangle - \left\langle \left\langle \left\langle \psi(\tau)|\hat{H}_0|\psi(\tau) \right\rangle \right\rangle^2 ,
$$

where the notation $\langle \ldots \rangle$ stands for the average over many disorder realizations. Finally, quantum fluctuations can be thought as caused by quantum indetermination, intrinsically present in the charging process since $|\psi(\tau)\rangle$ is not an eigenstate of the constant hamiltonian, $\hat{H}_0$. These can be quantified by evaluating

$$
(\sigma_N^{(q)})^2(\tau) \equiv \left\langle \left[ \left\langle \psi(\tau)|\hat{H}_0^2|\psi(\tau) \rangle \right\rangle - \left( \left\langle \left\langle \psi(\tau)|\hat{H}_0|\psi(\tau) \right\rangle \right\rangle \right)^2 \right\rangle .
$$

Of course, It would be highly desirable to find models of QBs able to reach high values of $\bar{E}$ and, at the same time, very low values of the various fluctuations.

Let us focus for the moment on temporal fluctuations. While it is easy to see that both $\bar{E}$ and its relative temporal fluctuations are not affected by an overall rescaling of the whole hamiltonian ($\hat{H} \to \alpha \hat{H}$), less trivial is the role played by the relative strength of the quench hamiltonian. This is governed by the dimensionless parameter $\kappa$. A small value of $\kappa$ makes the quench hamiltonian to be a small perturbation, thus resulting in a low value of $\bar{E}$. On the other hand, by increasing $\kappa$ the quench hamiltonian becomes a strong perturbation and induces transitions from $|0\rangle$ to the highly excited states of $\hat{H}_0$.

To grasp the qualitative behavior of the charging protocol by varying the parameter $\kappa$, we have studied numerically a simple model of QB under the unitary protocol, i.e. a one-dimensional spin chain. Its Hamiltonian reads

$$
\hat{H} = \hat{H}_0 + \kappa \lambda(\tau) \hat{H}_1 =
$$

$$
h \sum_{j=1}^{N} \sigma_j^x + \kappa \lambda(\tau) \left( \sum_{j=1}^{N} -J_j \sigma_j^+ \sigma_{j+1}^x + J_2 \sigma_j^y \sigma_{j+2}^x \right),
$$

where $\sigma_j^x, \sigma_j^y$ are the usual spin 1/2 operators located at the site $j$ of the spin chain. In the above equation, $h$ corresponds to a transverse magnetic field, while the coupling constants $J_j$ are composed by a constant piece plus a random fluctuation term, $J_j = J + \delta J_j$. Finally, the last term describes next-to-nearest-neighbors interactions with coupling constant $J_2$. Hereafter, all quantities are measured in units of $h$, and the random variables $\delta J_j$ are sampled over a uniform distribution with support $[-\delta J, \delta J]$. For sake of definiteness, we take $\delta J = 8.33 h$ and $J = 1.67 h$. In Fig. 2 we report representative results for the charging dynamics of a QB in the so-called Anderson localized phase (setting $J_2 = 0$). Here, we plot the ratio between the energy stored in the battery and the bandwidth of $\hat{H}_0$, denoted with $\Delta_{\hat{H}_0}$,

$$
R(\tau) \equiv \frac{\bar{E}(\tau)}{\Delta_{\hat{H}_0}},
$$

for different values of the dimensionless parameter $\kappa$. All curves grow as a function of time $\tau$ until saturating to $\bar{E}$, whose precise value depends on $\kappa$. We clearly see that by increasing $\kappa$, $\bar{E}$ grows until saturating to the value $R(\tau) \sim 1/2$, while any larger value of $\kappa$ does not further increase $\bar{E}$ and simply reduces $\bar{E}$. This means that, when $\kappa$ is strong enough, the quench hamiltonian induces a transition from $|0\rangle$ to a superposition involving several eigenstates of $\hat{H}_0$, symmetrically distributed around the center of the bandwidth of $\hat{H}_0$, thus ensuring $\bar{E} \sim \frac{1}{2} \Delta_{\hat{H}_0}$.

On the other hand, the temporal fluctuations are mostly unaffected by $\kappa$ and one has to find smarter ways to reduce them. This set of fluctuations will be the main focus of the analysis, and we will show how they can be efficiently suppressed. As we will see, the internal structure of $\hat{H}_1$ will play a crucial role for this task.

### III. From Local to Non-Local Interactions: How to Reduce Temporal Fluctuations

In Ref. [29] it was argued that interactions between quantum cells in the quench hamiltonian $\hat{H}_1$ can help in reducing the erratic temporal fluctuations during the charging of QBs. Here, we show that, while local interactions have only limited effects on the fluctuations, non-local correlations display much more prominent results, and they allow to build models of QBs with high temporal stability of $\bar{E}$.

As a first hint in this direction, in Ref. [29] it was studied a model of QB based on the Hamiltonian of Eq. (7),
with next-to-nearest-neighbors coupling constant set to $J_2 = 0.5 \, \hbar$, i.e. moving from the Anderson localized to a MBL phase (hereafter denoted by $\hat{H}_{1\text{MBL}}$). Here, we inspect highly non-local Hamiltonians and their impact in suppressing the temporal fluctuations. To this end, we will consider a model of quantum battery inspired by the so-called “Kourkoulou-Maldacena” SYK model, [61], which, for reasons which will become clear in a moment, we will call “local” SYK model (l-SYK). This model describes a strong interacting system of 2N Majorana fermions, $\tilde{\gamma}_i$, interacting via a fully non-local, all-to-all, interaction characterized by coupling constants randomly distributed with Gaussian profile. The Hamiltonian is

$$\hat{H} = \hat{H}_0 + \kappa \lambda(t) \hat{H}_4 = \hbar \left( \hat{H}_{\text{loc}} + \kappa \lambda(t) \hat{H}_4 \right),$$

where $\hbar$ is the same energy constant defined in eq. (7), $\hat{H}_{\text{loc}}$ reads

$$\hat{H}_{\text{loc}} = -2i \sum_{i \text{ odd}} \tilde{\gamma}_i \tilde{\gamma}_{i+1},$$

and $\hat{H}_4$ is the quartic Hamiltonian

$$\hat{H}_4 = \sum_{i<j<k<l} J_{ijkl} \tilde{\gamma}_i \tilde{\gamma}_j \tilde{\gamma}_k \tilde{\gamma}_l.$$

The random couplings $J_{ijkl}$ have null mean values and variances

$$\langle \langle J_{ijkl} J_{ijkl} \rangle \rangle = \frac{3}{4 N^3}.$$

Interestingly, the local term $\hat{H}_{\text{loc}}$ can be rewritten, via a Jordan-Wigner transformation, as

$$\hat{H}_{\text{loc}} = \sum_{j=1}^N \hat{\sigma}_j^z,$$

therefore $h \hat{H}_{\text{loc}}$ can be mapped to the same constant Hamiltonian, $\hat{H}_0$, for the spin chain, of length $N$, of Eq. (7).

As a first step, we have verified that the l-SYK model is indeed able to charge the battery by studying – for a given value of $\kappa$ and as a function of the length $N$ of the associated spin chain – the optimal charging time, $\langle \langle \bar{\tau} \rangle \rangle$, defined as the time at which the energy stored in the battery reaches a value equals to 97% [62] of the maximal energy, as well as the energy stored in the battery at the optimal time, $\langle \langle \bar{E} \rangle \rangle$, for the l-SYK battery. These quantities are reported in Fig. 3. From the plot, we see that the optimal charging time is a decreasing function while increasing the number of lattice sites. It is worth to underline that this result is in qualitative agreement with the analogous result obtained in Ref. [29] for the MBL battery. Moreover, the averaged energy stored in the battery at the optimal time $\langle \langle \bar{E} \rangle \rangle$ scales linearly with the number of sites. This property is easy to understand since, as already pointed out in Sec. II, when $\kappa$ is large enough $\langle \langle \bar{E} \rangle \rangle$ is determined by the bandwidth of $\hat{H}_0$, which scales linearly with $N$.

We now turn to the main focus of the paper: the temporal fluctuations of the average energy stored in the charging protocol. We discuss the behavior of both the MBL spin chain and the l-SYK Hamiltonian. Notice that the two models differ for the driving terms only and that, once rewritten after the Jordan-Wigner transformation, the driving term for the l-SYK model looks highly non-local and highly interacting, contrary to the quench term of the MBL spin chain which instead couples next-to-nearest neighbors, at most.

To characterize the charging performance, we have performed extensive numerical calculations, computing the ratio $R(\tau)$ in Eq. (8). In order to make fair comparisons between these models, we have set the constant $\kappa$ in the MBL battery equal to $1$, while for the l-SYK Hamiltonian we have fixed it such that the two quench Hamiltonians have exactly the same bandwidth, $\Delta_{\hat{H}_{1\text{MBL}}} = \Delta_{\hbar \hat{H}_4}$. In the upper panels of Fig. 4(a) and Fig. 4(b) we plot the ratios $R(\tau)$, defined in (8), for a single ensemble realization for the MBL spin chain and for the l-SYK battery, respectively.

We immediately see that, compared to the analogous plot for the Anderson spin chain, presented in Fig. 2, the MBL battery (see blue curve) is indeed able to partially reduce the temporal fluctuations at late times (denoted with a dark grey background in the figure), i.e. for times much larger than the optimal charging time $\tau \gg \langle \langle \bar{\tau} \rangle \rangle$. However, at early times (light grey background in the figure), i.e. for times roughly included in $\langle \langle \bar{\tau} \rangle \rangle < \tau < 100 \langle \langle \bar{\tau} \rangle \rangle$, the fluctuations are still very large. On the contrary, the plot clearly shows that the l-SYK battery (see red curve) is extremely precise and stable at any time scale: all the temporal fluctuations,
after reaching $\langle \langle \tau \rangle \rangle$, are completely removed. It should be stressed that the early time window is very relevant, since one would like to have a great control of the charging precision immediately after reaching the saturation of the energy stored. This plot clearly shows the qualitative advantage of the l-SYK model and confirms the intuition that non-local correlations play a crucial role in the charging dynamics. Thus, a strongly interacting, non-local, quench (like the l-SYK model) represents a perfect candidate to build models of very stable QBs with high charging precision. Moreover, we show in App. A that non-locality alone, in $\hat{H}_1$, is not enough and that the highly chaotic dynamics of the l-SYK system is necessary to efficiently suppress the temporal fluctuations. Finally, in App. B we show that the constant term of the Hamiltonian, $\hat{H}_0$, does not play a major role in the charging dynamics, by making a comparison with another kind of SYK-like QB with a non-local $\hat{H}_0$.

We now discuss the microscopic origin of this better efficiency and charging stability. In general, temporal energy fluctuations are caused by transitions of the probability amplitudes $c_{k, i} \equiv \langle k, i | \psi(t) \rangle$, with $|k, i\rangle$ being the eigenstates of $\hat{H}_0$ (we denote with $k$ the energy levels and the index $i$ accounts for the different degenerate eigenstates), between eigenstates with different energies. Such transitions cause a large fluctuation of the energy stored if both the following conditions are met: (I) the two eigenstates have very different energies and (II) the probability amplitudes of being in the eigenstates involved in the transition are large. The first condition is immediate to understand: if the two eigenstates have similar energies, the energy stored in the battery will not vary a lot after the transition, with the extreme case of a transition between degenerate eigenstates.

The second condition is more subtle: let us consider the extreme case in which the evolved ket, $|\psi(t)\rangle$, can be written as a superposition of all the eigenstates of $\hat{H}_0$ with approximately the same probability amplitudes

$$c_{k, i} \sim \frac{1}{\sqrt{\mathcal{D}}} \sim O(2^{-N/2}) \ ,$$

where $\mathcal{D}$ is the dimension of the Hilbert space, which for a QB is exponentially large in the number of cells, $N$. Given that $\mathcal{D}$ is very large, all the coefficients $c_{k, i}$ result to be very small. In this case, a transition between eigenstates, even with very different energies, will not be reflected in large fluctuations. Indeed, since the bandwidth of $\hat{H}_0$ scales linearly in $N$, the fluctuation will be, at most, around

$$\Delta E(t) \sim c_{k, i}^2 N \sim O(2^{-N}) \ ,$$

a quantity exponentially small in $N$.

On the other hand, let us assume that just few (of order $N$) eigenstates of $\hat{H}_0$ are involved in the expansion of the evolved state. In this case, some of the coefficients $c_{k, i}$ will be relatively large

$$c_{k, i} \sim \frac{1}{\sqrt{\mathcal{N}}} \ , \text{ for some } k, i \ ,$$

and a transition including one of these states will cause a large fluctuation in the energy stored.

Given these considerations, we expect that in the MBL case the evolved state at early times should have non vanishing overlap with just few eigenstates of $\hat{H}_0$ (for each energy level), while involving more and more states at late times, thus reducing the associated fluctuations. On the contrary, for the l-SYK model the evolved state should involve a large portion of the Hilbert space of $\hat{H}_0$ from the very early times.

To corroborate this hypothesis, we first notice that the energy spectrum of $\hat{H}_0$ is formed by several lines,
well separated from each other. Each line in the spectrum is degenerate with the degeneracy degree counted by the number of configurations with the right number of aligned spins.

Hence, to estimate if, for a given energy level, the evolved state has non vanishing overlap with just few or many eigenstates of $\mathcal{H}_0$ we have computed, as a function of time, the quantities

$$a_k^i(\tau) \equiv |\langle k, i | \psi(\tau) \rangle|^2,$$ (17)

which express the probability of measuring the evolved state in the eigenstate $|k, i\rangle$. We have then considered all the eigenstates $|k, i\rangle$ with a given energy, i.e. with fixed $k$, and we have computed the standard deviation associated to the $a_k^i$s for that eigenstates divided by its average

$$\sigma_k \equiv \frac{\text{STD}(a_k^i)}{\overline{a}_k^i},$$ (18)

where the mean values and the standard deviations are taken among all the $a_k^i$ with the same index $k$ and different indices $i$. More explicitly, we have

$$\overline{a}_k^i \equiv \frac{1}{\text{deg} E_k} \sum_{i=1}^{\text{deg} E_k} a_k^i,$$

$$\text{STD}(a_k^i) \equiv \left( \frac{1}{\text{deg} E_k - 1} \sum_{i=1}^{\text{deg} E_k} (a_k^i - \overline{a}_k^i)^2 \right)^{1/2},$$ (19)

with $\text{deg} E_k$ being the degeneracy degree of the energy level $E_k$.

We can thus determine if the expansion of $|\psi(\tau)\rangle$ in the degenerate eigenstates for a given energy level is involving many or few of the eigenstates, with the former case represented by small values $\sigma_k$ and the latter represented by large values. The results are reported in the two lower panels of Fig. 4.

The figure confirms our hypothesis: the MBL system shows at early times huge values of $\sigma_k$ for each energy sector, and in correspondence with these peaks we can clearly trace a huge temporal fluctuation of the average energy stored in the battery (see upper panels of the figure). This behavior gets reduced by increasing the time and, after bouncing for a while, the system reaches low values for all the $\sigma_k$s. On the other hand, the l-SYK model shows from the very beginning low values for the $\sigma_k$s (around an order of magnitude smaller), clearly showing that in this model many more eigenstates of $\mathcal{H}_0$, for each energy level, are rapidly involved in the expansion of the evolved state. Hence, the charging protocol turns out to be very stable, and this is reflected in the small erratic temporal fluctuations. It should be emphasized that the low values of all the $\sigma_k$s, for the l-SYK model, are reached at a time scale which is even shorter than the optimal charging time, $\langle \tau \rangle$, thus ensuring the total absence of temporal fluctuations.

This microscopic argument confirms that temporal fluctuations get suppressed when an initially localized state (in the eigenbasis of $\mathcal{H}_0$) spreads and covers a large portion of the eigenstates of $\mathcal{H}_0$ and, as such, we think that it could be naturally linked to the physics of scrambling and of thermalization. Indeed, the thermalization properties of the SYK model have been already investigated in Refs. [63, 64], where it has been shown that this model shows thermalization, even without long time averaging, thus suggesting that SYK has to be considered as a mixing more than ergodic model. On the other hand, a MBL system does not thermalize in the thermodynamic limit. Hence, we expect that the huge suppression of the temporal fluctuations at late times, in this case, should be a finite $N$ effect: by increasing the size of the system, the time at which the fluctuations are highly suppressed should tend to infinity.

IV. THE ERGOTROPY AS A MEASURE OF THERMALIZATION

Another important quantity, which characterizes the performance of a QB, is the so-called ergotropy, $\mathcal{E}$ [10, 27]. Let us recall that it quantifies the amount of extractable work from a QB after the charging protocol [10, 16, 27]. If one assumes to have access to just $M < N$ cells of the full QB, part of the energy stored can be locked by internal correlations, thus reducing the efficiency of the QB itself. Given a density matrix $\rho$, representing the evolved state $|\psi(\tau)\rangle$ after tracing out the useless $N - M$ cells, the associated ergotropy is:

$$\mathcal{E}^{(N)}_M \equiv \text{Tr}[\mathcal{H}_0^{(M)} \rho] - \min_{U} \left\{ \text{Tr}[\mathcal{H}_0^{(M)} \hat{U} \rho \hat{U}^\dagger] \right\},$$ (20)

where $\mathcal{H}_0^{(M)}$ denotes the local portion of the Hamiltonian, (1), once restricted to the $M$ cells (we are assuming that $\mathcal{H}_0$ can be written as a sum of local terms, such that it makes sense to define $\mathcal{H}_0^{(M)}$) and the minimization runs over all the possible unitaries, $\hat{U}$, acting on $\rho$.

It is known that the ergotropy is highly affected by the presence of entanglement, [27]: if the evolved state $|\psi(\tau)\rangle$ is highly entangled, the resulting density matrix $\rho$ will be highly mixed, with a very low level of corresponding ergotropy, thus showing that, in this case, the amount of extractable energy from a subset of $M$ cells is low. An interesting quantity to study is the following

$$X_{M,N}(\tau) \equiv \left\langle \mathcal{E}^{(N)}_M(\tau) \frac{E_N(\tau)}{N} \right\rangle^{-1},$$ (21)

which quantifies the fraction of energy, per cell, that can be extracted from a reduced battery of $M$ cells, out of the initial $N$ cells. It is thus interesting to study the behavior of $X_{M,N}$, both as a function of $M$, at fixed $N$, and as a function of $N$, at $M$ fixed.

We have studied, for both the MBL and the l-SYK QBs, $X_{M,14}$ for $N = 14$ and very small values of $M$ and
Moving to the MBL case, the situation is very different: from the upper panel of Fig. 5(b) we see that the amount of extractable energy is by far higher in agreement with the results of Ref. [29], where it was observed that the levels of ergotropy for the MBL system are generally very high, a feature that can be traced back to the low level of entanglement typical of the MBL phase. Much more interesting is the time behavior which can be observed in the lower panel of Fig. 5(b): at early times we clearly see that the amount of extractable energy per cell is, essentially, independent of the value of $N$, a behavior which is in striking contrast with what we observed for the l-SYK battery. However, at later times, the situation changes and the amount of extractable energy per cell becomes $N$-dependent, and in particular it gets reduced by increasing $N$, showing a behavior qualitatively similar to the l-SYK battery.

This confirms the picture we outlined in the previous section: the dynamics of the MBL battery shows a clear change when passing from early times to late times. The behavior at early times is similar to the behavior expected for an integrable system, while at late times it becomes more similar to the behavior of a chaotic system. The crossover between the two behaviors is in correspondence with the thermalization of the system and, once again, we stress that it should tend to infinity in the thermodynamic limit for the MBL system contrary to the SYK, for which the thermalization properties have been studied in the large $N$ limit, see Refs. [63, 64].

V. FLUCTUATIONS OF ENERGY STORED

Let us summarize the results obtained so far. We have argued that, in general, during the charging of a generic QB two different time windows can be uncovered: after reaching the optimal charging time, $\langle \bar{\tau} \rangle$, we have a “early time” window, in which the average energy stored in the battery, $E(\tau)$ shows huge temporal fluctuations, the expansion of the evolved state $|\psi(\tau)\rangle$ on the base of the eigenstates of $\hat{H}_0$ involves just few eigenstates for each energy level and the ratio $X_{M,N}$ is substantially independent of $N$. At later times the dynamics turns to a “late time” window, in which the energy $E(\tau)$ shows suppressed temporal fluctuations, the evolved state has spread to cover a large portion of the eigenstates of $\hat{H}_0$ and $X_{M,N}$ is highly dependent on $N$. We have also argued that the time of crossover, between the early time and the late time behavior, is connected with the thermalization properties of the system under investigation and, as such, it is model-dependent.

By inspecting the lower panels of Fig.5, we can fix the early time window to include all the times from the optimal charging time to the time $\tau \approx 18$, and the late-time window to include all the times from the time $\tau \equiv 120$ to the final time, $\tau \equiv 600$. This is in line with the behavior of $X_{3,N}$, for the MBL system and for the values of $N$ investigated in this work. On the other hand, for

![Image of graphs showing energy stored over time for different battery configurations]
the l-SYK, as already mentioned, the transition between the early and late times dynamics appears at times much earlier than the optimal charging time. Notice that we have extensively checked that the discussed results are not qualitatively affected by possible variations in the choice of these time-interval windows.

We now come back to a precise evaluation of the various sources of energy fluctuations, as written in (4), (5) and (6), in the two time windows just defined. Starting with the temporal fluctuations, we define the following, dimensionless, quantity:

$$\Sigma(t) = \frac{2}{\Delta H_0} \sqrt{\langle (\sigma_N^t)^2(W) \rangle},$$  \hfill (22)

where, with $W$ we mean the time window (early or late) over which we take the temporal integral of (4). In Fig. 6(a) we plot the results for the MBL spin chain. The behavior for different $N$, of (22), at early and late times is qualitatively different: while at late times $\Sigma$ is fastly decreasing with $N$, at early time $\Sigma$ instead shows a much slower decrease. In Fig. 6(b) we plot the results for the l-SYK battery. Here, the situation is different: both the early and late time fluctuations are rapidly suppressed in $N$.

These observations can be made quantitative: in the MBL case, the early time curve is greatly reproduced by the function

$$\Sigma^{(e)} = \frac{a}{\sqrt{N}} + b,$$  \hfill (23)

with $a$ and $b$ being fitting parameters. On the other hand, the late time behavior is well reproduced by

$$\Sigma^{(l)} = a N^2 2^{-N} + b,$$  \hfill (24)

which shows that the late time temporal fluctuations are exponentially suppressed with $N$.

The numerical data for the l-SYK case instead can be reproduced by the function

$$\Sigma(t) = a N^{2.5} 2^{-N} + b.$$  \hfill (25)

In summary, this shows that the temporal fluctuations, in the l-SYK model, both at early and late times, are exponentially suppressed by increasing the size of the battery. On the other hand, the MBL battery, shows this exponential suppression only at late times, while at early times it follows a $\frac{1}{\sqrt{N}}$ suppression factor, only. The exponential suppression, at early times, of l-SYK battery, makes clear that with this model it is possible to obtain very stable charging protocols, in which the average energy stored in the battery is essentially determined with very high precision, even with relatively small batteries.

It has been shown in [35] (in the similar context of work extraction) that precisely an exponential and a $\frac{1}{\sqrt{N}}$ suppression factors are associated to, respectively, collective processes, i.e. processes in which all the cells are collectively controlled in the protocol, and single cells protocols, in which each cell is individually processed. It is then natural to expect that, due to the non-local nature of its hamiltonian, the l-SYK has a genuine collective dynamics from the very early times, shorter than the optimal charging time $\langle \tau \rangle$, while the MBL battery needs a certain amount of time to start a collective dynamics, with an initial single-body behavior.

This is in perfect agreement with the microscopic description of the fluctuations we have provided in Sec. III and in Sec. IV, where we have shown that the MBL battery needs a large amount of time in order to involve a large portion of the eigenstates of $H_0$ in the expansion

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6.png}
\caption{(a): The early and late time temporal fluctuations, as measured by $\Sigma(t)$, for the MBL spin chain, as a function of the lattice size. Continuous lines correspond to the fit (see text). Fitting parameters are: $a_{\text{early}} = 0.228$, $a_{\text{late}} = 0.146$, $b_{\text{early}} = 0.017$, $b_{\text{late}} = 0.010$. (b): Same quantity for the l-SYK battery. Continuous lines correspond to the fit (see text). Fitting parameters are: $a_{\text{early}} = 0.057$, $a_{\text{late}} = 0.070$, $b_{\text{early}} = 0.004$, $b_{\text{late}} = 0.004$. The results are obtained by averaging over 100 ensemble realizations for the l-SYK battery and over 500 ensemble realizations for the MBL battery.}
\end{figure}
of the evolved state $|\psi(\tau)\rangle$. Moreover, the absence of a crossover in the l-SYK system can be again understood in terms of the microscopic description of Sec. III, where we observed that for the l-SYK system $|\psi(\tau)\rangle$ involves a large portion of the Hilbert space from times which are smaller than the optimal charging time, thus ensuring that all the temporal fluctuations are exponentially suppressed. For completeness, we report in Fig. 7 the analogous plot for the Anderson model. We see that in both the time windows the data are greatly reproduced by a $1/\sqrt{N}$-like function. This result is in line with our expectation, since the Anderson model, as shown in Fig. 2, shows huge temporal fluctuations at all the time scales.

We now discuss the disorder and quantum fluctuations for all these three models. From (5) and (6), we see that both these quantities have to be evaluated at a definite time, $\tau$. Hence, we have chosen two fixed values of the time, one in the early time window and one in the late time window, to evaluate them. Once again, we have defined the following, dimensionless, quantities

$$\Sigma^{(d)}(d, q) \equiv 2 \sqrt{\langle \sigma_N^{(d, q)} \rangle^2 / \Delta_{\hat{H}_0}}. \quad (26)$$

Starting with the disorder fluctuations, the results are reported in Fig. 8. Interestingly, and in agreement with all the previous results, the disorder fluctuations for the l-SYK model are always small, both at early times and at late times. Similarly, by considering the Anderson model, we see that the disorder fluctuations are always large, both at early and late time. Finally, the MBL system shows a crossover when passing from the early time window to the late time window: it shows a behavior similar to the Anderson spin chain at early times and it moves to a behavior very similar to the l-SYK model at late times.

Moving to the quantum fluctuations, the results are reported in Fig. 9. Again, we find that the MBL model shows a crossover when moving from the early time window to the late time window, becoming more similar to the l-SYK behavior only at late times. We also see that the quantum fluctuations for the l-SYK model are larger than for all the other models we have considered. Furthermore, it is worth to notice that for all the three models, the values of the temporal and disorder fluctuations are quite similar, while the quantum fluctuations are always larger than the other sources of fluctuations, reaching the largest value of $\sim 0.50$ for the l-SYK battery. This large value of quantum fluctuations for the l-SYK model
As a byproduct, we have found evidences that a new interesting time scale can be uncovered during the charging of a quantum battery; namely the time scale at which the charging protocol turns to be collective, which corresponds to the time at which one can observe a transition in the strength of the temporal fluctuations as a function of the size of the system. We have also provided a microscopic understanding of this new time scale, as the time scale at which an initially localized state (in the eigenbasis of the constant hamiltonian) has spread to cover a large portion of the eigenbasis of the constant hamiltonian.

By making use of this last point of view, and using also the temporal evolution of the ergotropy as a further probe, we then conjecture that the high stability of the charging protocol based on the SYK model is just another manifestation of the fastest scrambling (and fast thermalizing) property of the SYK hamiltonian, thus suggesting that the stability reached by the SYK quench puts an upper bound on the level of stability that a QB can show.

Of course, there are many open points which would be worth to explore. One major open point, that we hope to address in the near future, is a precise characterization of the crossover time, by introducing a good observable to characterize with high precision the crossover time for a given model. It would be also highly desirable to find further evidences for the conjecture that the charging stability of the SYK QBs is an upper bound for the charging stability of a generic QB. Another promising line of research would be to study the charging protocol described in this paper from the holographic point of view, perhaps along the lines of [67]. Such an approach could be also relevant both to confirm the presence of an upper bound on the possible charging stability of a quantum battery and also to find its possible implications in the physics of the black holes.

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VI. SUMMARY AND OUTLOOK

In this paper, we have introduced a new class of quantum batteries, in which the unitary charging protocol is realized via a sudden quench with a SYK-like hamiltonian. We have argued, and shown via extensive numerical computations, that such a charging protocol is able to dramatically suppress the strength of the erratic temporal fluctuations.

could be put in relation with its high charging power, [65], since it has been recently observed in Ref. [66] that high levels of quantum fluctuations are necessary in order to increase the charging power of a QB.
Appendix A: The role of non-locality in the stability of a QB

We have seen that the l-SYK battery is extremely efficient in reducing the temporal fluctuations of the charging protocol. A natural question to raise is the role of non-locality on reducing these fluctuations. More explicitly, one may wonder if integrable models with non-local interactions can efficiently suppress temporal fluctuations, as well as the l-SYK, without relying on the chaotic properties of the latter.

This result confirms that, to achieve stability in the charging protocol, non-local, interacting and chaotic hamiltonians are necessary, and that the temporal fluctuations are linked with the physics of thermalization, thus confirming the prominence of the l-SYK model as a perfect candidate.

Appendix B: The role of the local hamiltonian

Another interesting check to perform is to investigate the role of the local term in the hamiltonian, $\hat{H}_0$, on the charging performance of a given QB. To this end, we can study a slightly different version of the l-SYK model, usually called “mass-deformed” SYK model (m-SYK), studied in [68–70]. In this model, the quench hamiltonian, $\hat{H}_1$, is the usual quartic hamiltonian of the l-SYK model, $h\hat{H}_4$, as defined in (11), while the constant term is given by the non-local random mass term, $h\hat{H}_2$, defined in (A2).

We have compared, for the same realizations of the disorder couplings $J_{ijkl}$ in both the models and for a realization of $K_{ij}$, the function $R(\tau)$ for both the l-SYK and the m-SYK batteries. We have renormalized the bandwidth of $\hat{H}_2$ such that the constraint $\Delta h\hat{H}_{loc} = \Delta h\hat{H}_2$ was satisfied.

From Fig. 11 we clearly see that the two performances are almost the same, both in terms of the maximal value reached by $R(\tau)$, and in terms of the strength of the fluctuations, with a small advantage for the l-SYK model. This shows that the role of the particular $\hat{H}_0$ term on the charging performance is very limited, and that only the quench hamiltonian really matters in the unitary charging protocol.
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