A CLASS OF SIMPLE C*-ALGEBRAS WITH STABLE RANK ONE

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ABSTRACT. We study the limits of inductive sequences \((A_i, \phi_i)\) where each \(A_i\) is a direct sum of full matrix algebras over compact metric spaces and each partial map of \(\phi_i\) is diagonal. We give a new characterisation of simplicity for such algebras, and apply it to prove that the said algebras have stable rank one whenever they are simple and unital. Significantly, our results do not require any dimension growth assumption.

1. Introduction

Let \(X\) and \(Y\) be compact Hausdorff spaces. A \(*\)-homomorphism

\[
\phi : M_m(C(X)) \to M_{nm}(C(Y))
\]

is called diagonal if there are \(n\) continuous maps \(\lambda_i : Y \to X\) such that

\[
\phi(f) = \begin{pmatrix}
    f \circ \lambda_1 & 0 & \ldots & 0 \\
    0 & f \circ \lambda_2 & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & f \circ \lambda_n
\end{pmatrix}.
\]

The \(\lambda_i\) are called the eigenvalue maps or simply eigenvalues of \(\phi\). The multiset \(\{\lambda_1, \lambda_2, \ldots, \lambda_n\}\) is called the eigenvalue pattern of \(\phi\) and is denoted by \(ep(\phi)\). This definition can be extended to \(*\)-homomorphisms

\[
\phi : \bigoplus_{i=1}^n M_{n_i}(C(X_i)) \to \bigoplus_{j=1}^m M_{m_j}(C(Y_j))
\]

by requiring, roughly, that each partial map

\[
\phi^{ij} : M_{n_i}(C(X_i)) \to M_{m_j}(C(Y_j))
\]

induced by \(\phi\) be diagonal. (A precise definition can be found in Section 2.)

C*-algebras obtained as limits of inductive systems \((A_i, \phi_i)\) where

\[
A_i = \bigoplus_{j=1}^{n_i} M_{n_{i,j}}(C(X_{i,j}))
\]

and each \(\phi_i\) is diagonal form a rich class. They include AF algebras, simple unital AT algebras (and hence the irrational rotation algebras [10]), Goodearl algebras ([10]), and some interesting examples of Villadsen and the third named author connected to Elliott’s program to classify amenable C*-algebras via K-theory [17].

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The structure of these algebras is only well understood when they satisfy some additional conditions such as (very) slow dimension growth or the combination of real rank zero, stable rank one, and weak unperforation of the $K_0$-group — situations in which the strong form of Elliott’s classification conjecture can be verified ([3], [5], [7], [8], [9]).

In this paper we give a new characterisation of simplicity for AH algebras with diagonal connecting maps. As a consequence we are able to prove that such algebras have stable rank one whenever they are unital and simple. The significance of our result derives from the fact that we make no assumptions on the dimension growth of the algebras; we obtain a general theorem on the structure of algebras heretofore considered “wild”. As suggested by M. Rørdam in his recent ICM address, it is high time we became friends with such algebras, as opposed to treating them simply as a source of pathological examples.

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2. Preliminaries

2.1. Basic notation. We use $M_n$ to denote the set of $n \times n$ complex matrices. Given a closed subset $E$ of a compact metric space $(X, d)$ and $\delta > 0$ we set

$$B_{\delta}(E) = \{x \in X \mid d(E, x) < \delta\},$$

and make the convention that $B_{\delta}(\emptyset) = \emptyset$.

2.2. AH systems with diagonal maps.

**Definition 2.1.** We will say that a unital $*$-homomorphism

$$\phi : \bigoplus_{i=1}^{n} M_{n_i}(C(X_i)) \to M_k(C(Y)) \cong M_k \otimes C(Y)$$

is diagonal if there exist natural numbers $k_1, \ldots, k_n$ such that $\sum_i k_i = k$ and $n_i|k_i$, an embedding

$$\iota : \bigoplus_{i=1}^{n} M_{k_i} \hookrightarrow M_k,$$

and diagonal maps

$$\phi_i : M_{n_i}(C(X_i)) \to M_{k_i} \otimes C(Y)$$

such that

$$\phi = \bigoplus_{i=1}^{n} \phi_i.$$

(Notice that $k_i = 0$ is allowed.) We will say that a unital $*$-homomorphism

$$\phi : \bigoplus_{i=1}^{n} M_{n_i}(C(X_i)) \to \bigoplus_{j=1}^{m} M_{m_j}(C(Y_j))$$

is diagonal if each restriction

$$\phi_j : \bigoplus_{i=1}^{n} M_{n_i}(C(X_i)) \to M_{m_j}(C(Y_j))$$

is diagonal.
Let $A$ be the limit of the inductive sequence $(A_i, \phi_i)$, where

$$A_i = \bigoplus_{t=1}^{k_i} M_{n_{i,t}}(C(X_{i,t})), \quad (1)$$

$X_{i,t}$ is a connected compact metric space, and $n_{i,t}$ and $k_i$ are natural numbers. Define

$$A_{i,t} := M_{n_{i,t}}(C(X_{i,t})), \quad X_i := X_{i,1} \sqcup X_{i,2} \sqcup \cdots \sqcup X_{i,k_i},$$

and

$$\phi_{i,j} := \phi_{j-1} \circ \cdots \circ \phi_i.$$

Let

$$\phi^{t,l}_{i,j} : M_{n_{i,t}}(C(X_{i,t})) \to M_{n_{j,l}}(C(X_{j,l}))$$

and

$$\phi^l_{i,j} : \bigoplus_{t=1}^{k_j} M_{n_{i,t}}(C(X_{i,t})) \to M_{n_{j,l}}(C(Y_{j,l}))$$

denote the appropriate restrictions of $\phi$. If each $\phi_i$ is unital and diagonal, then we refer to $(A_i, \phi_i)$ as an AH system with diagonal maps. The limit algebra $A$ will be called a diagonal AH algebra, and we will refer to $(A_i, \phi_i)$ as a decomposition of $A$.

Assume that $A$ as above is diagonal. We will view $\phi^{t,l}_{i,j}$ as a diagonal map from $M_{n_{i,t}}(C(X_{i,t}))$ into the cut-down of $M_{n_{j,l}}(C(X_{j,l}))$ by $\phi^{t,l}_{i,j}(1)$. For fixed $i$ and $j$, we will denote by $ep_{ij}$ the multiset which is the union, counting multiplicity, of the eigenvalue patterns of each $\phi^{t,l}_{i,j}$; $ep_{ij}$ is the eigenvalue pattern of $\phi_{i,j}$; an element of $ep_{ij}$ is an eigenvalue map of $\phi_{i,j}$. For fixed $i$, $j$, and $l$, we will denote by $ep^l_{ij}$ the multiset which is the union of the eigenvalue patterns of each $\phi^{t,l}_{i,j}$.

Let us now show that the bonding maps $\phi_i$ may be assumed to be injective. Let $(A_i, \phi_i)$ be a decomposition for a diagonal AH algebra $A$ as above. Fix $i \in \mathbb{N}$ and $1 \leq t < k_i$. For each $j > i$ and $1 \leq l \leq k_j$, let $X_{i,t}^l$ denote the closed subset of $X_{i,t}$ which is the union of the images of the eigenvalue maps of $\phi^{t,l}_{i,j}$. Put $X_{i,t}^j = \sqcup_l X_{i,t}^l$, and $\tilde{X}_{i,t} = \cap_j X_{i,t}^j$. Since $X_{i,t}^l \supseteq X_{i,t}^{l+1}$, we have that $\tilde{X}_{i,t}$ is closed subset of $X_{i,t}$. Define

$$\tilde{A}_{i,t} = M_{n_{i,t}}(C(\tilde{X}_{i,t}))$$

and

$$\tilde{A}_i = \bigoplus_{t=1}^{k_i} \tilde{A}_{i,t}.$$  

Define diagonal maps $\tilde{\phi}^{t,l}_{i,t+1} : \tilde{A}_{i,t} \to \tilde{A}_{i+1,t}$ by replacing the eigenvalue maps of $\phi^{t,l}_{i,t+1}$ with their restrictions to $\tilde{X}_{i,t+1}$. Define $\tilde{\phi}_i : \tilde{A}_i \to \tilde{A}_{i+1}$ in a manner analogous to the definition of $\phi_i$. It follows that $(\tilde{A}_i, \tilde{\phi}_i)$ is a diagonal AH system with limit $A$, and $\tilde{\phi}_i$ is injective by construction. We assume from here on that all bonding maps in diagonal AH systems are injective.

One way to construct a simple diagonal AH algebra is to ensure that for each $i \in \mathbb{N}$ and $x$ in a specified dense subset of $X_i$ there is some $j \geq i$ such that for each $l \in \{1, \ldots, k_j\}$ the diagonal map $\phi^{t,l}_{i,j}$ contains the eigenvalue map $ev_x : X_{j,l} \to X_{i,t}$ given by $ev_x(y) = x$. The next definition gives and approximate version of this situation.
2.4. Paths between permutation matrices. Let \( i \in \{n\} \) be such that
\[
\|u_i^j l f (x) u_i^* - \left( \begin{array}{c} f(x) \\ 0 \end{array} \right) \| < \epsilon
\]
for some appropriately sized \( b_l \). (Note that \( \text{diag}(f(x_0), b_l) \in A_{j,l} \)).

We will prove in the sequel that \( A \) as in Definition 2.2 is simple if and only if it has property \( \mathcal{P} \).

2.3. A characterisation of simplicity. Proposition 2.1 of [4] gives some necessary and sufficient conditions for the simplicity of an AH algebra. We will have occasion to apply these in the proof of our main result, and so restate the said proposition in the particular case of an AH system with diagonal injective maps.

Proposition 2.3. Let \( (A_i, \phi_i) \) be an AH system with diagonal injective maps, and set \( A = \lim_{i \to \infty} (A_i, \phi_i) \). The following conditions are equivalent:

(i) \( A \) is simple;
(ii) For any positive integer \( i \) and any non-empty open subset \( U \) of \( X_i \), there is a \( j_0 \geq i \) such that for every \( j \geq j_0 \) and \( l \in \{1, \ldots, k_j\} \) we have
\[
(\epsilon p_{ij}^{l})^{-1}(U) = X_{j,l},
\]
where \( (\epsilon p_{ij}^{l})^{-1}(U) \) denotes the union of the sets \( \lambda^{-1}(U), \lambda \in \epsilon p_{ij}^{l} \);
(iii) For any non-zero element \( a \) in \( A_i \), there is a \( j_0 \geq i \) such that for every \( j \geq j_0 \), \( \phi_{ij}(a)(x) \) is not zero, for every \( x \) in \( X_j \).

2.4. Paths between permutation matrices. Given any permutation \( \pi \in S_n \), let \( U[\pi] \) denote the permutation matrix in \( M_n \) corresponding to \( \pi \), that is, \( U[\pi] \) is obtained from the identity of \( M_n \) by moving the \( i^{th} \) row to the \( \pi(i) \)th row, for \( i \in \{1, 2, \ldots, n\} \). Any two permutation matrices are homotopic inside the unitary group \( \mathcal{U}(M_n) \) of \( M_n \), but we want to define some particular homotopies for use in the sequel. Let \( \pi \) and \( \sigma \) be elements of \( S_n \), viewed as permutations of the canonical basis vectors \( e_1, \ldots, e_n \) of \( \mathbb{C}^n \). Let \( R = \{e_{W,1}, \ldots, e_{W, \dim(W)}\} \) be the set of basis vectors upon which \( \pi \) and \( \sigma \) agree, and choose \( \gamma \in S_n \) be such that
\[
\gamma \sigma(v) = \gamma \pi(v) = v, \ \forall v \in R.
\]
Then \( U[\gamma]U[\sigma] \) and \( U[\gamma]U[\pi] \) fix \( e_1, \ldots, e_{|R|} \). Put \( W = \text{span}\{e_1, \ldots, e_{|R|}\} \), and let \( V \) be the orthogonal complement of \( W \). There are a canonical unital embedding of \( M_{\dim(W)} \oplus M_{\dim(V)} \) into \( M_n \) and unitaries \( u, \ v \in \mathcal{U}(M_{\dim(V)}) \) such that
\[
U[\gamma]U[\pi] = 1_{M_{\dim(W)}} \oplus v; \ U[\gamma]U[\sigma] = 1_{M_{\dim(W)}} \oplus u.
\]
Choose a homotopy \( g(t) \) between \( u \) and \( v \) inside \( \mathcal{U}(M_{\dim(V)}) \) — \( g(0) = v \) and \( g(1) = u \) — and put
\[
u(t) = U[\gamma]^{-1}(1_{M_{\dim(V)}} \oplus g(t)).
\]
Then \( u(t) \) is a homotopy of unitaries between \( U[\pi] \) and \( U[\sigma] \) such that
\[
u(t)(v) = U[\pi](v) = U[\sigma](v), \ \forall v \in R, \ \forall t \in [0, 1].
2.5. Applications of Urysohn’s Lemma.

Lemma 2.4. Let \( \sigma \in S_n \) be given. There is a homotopy \( u : [0,1] \to U(M_n) \) between \( u(0) = 1_n \) and \( u(1) = U[\sigma] \) which moreover has the following property: for any complex numbers \( \lambda_1, \lambda_2, \ldots, \lambda_n \) such that \( \lambda_i = \lambda_{\sigma(i)} \) for every \( i = 1, 2, \ldots, n \), we have

\[
\begin{pmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n
\end{pmatrix}
\begin{pmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n
\end{pmatrix} = \begin{pmatrix}
u(t)
\end{pmatrix} = \begin{pmatrix}
u^* (t)
\end{pmatrix}, \forall t \in [0,1].
\]

Proof. Let us first consider the case that \( \sigma \) is a \( k \)-cycle. The hypotheses of the lemma guarantee that the desired conclusion holds already for \( t \in \{0,1\} \). Choose the homotopy \( u(t) \) as in Subsection 2.4 by using our given value of \( \sigma \) and setting \( \pi \) equal to the identity element of \( S_n \). The hypothesis \( \lambda_i = \lambda_{\sigma(i)} \), \( i \in \{1, \ldots, n\} \), implies that there is a \( \lambda \in \mathbb{C} \) such that for each \( i \in \{1, \ldots, n\} \) which is not fixed by \( \sigma \) we have \( \lambda_i = \lambda \). In other words, if one decomposes \( \text{diag}(\lambda_1, \ldots, \lambda_n) \) into a direct sum of two diagonal matrices using the decomposition \( \mathbb{C}^n = W \oplus V \) — then the direct summand corresponding to \( V \) is scalar \( k \times k \) matrix. By construction, \( u(t) = v(t) \otimes 1_{n-k} \), with \( v(t) \in U(M_k) \). It follows that \( u(t) \) commutes with \( \text{diag}(\lambda_1, \ldots, \lambda_n) \) for each \( t \in \{0,1\} \).

Now suppose now that \( \sigma \) is any permutation on \( n \) letters, and write \( \sigma \) as a product of disjoint cycles: \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_l \). For each \( j \in \{1, \ldots, l\} \), let \( u_j(t) \) denote the unitary path between \( U[\text{id}] \) and \( U[\sigma_j] \), constructed as in Subsection 2.4. Now

\[
u(t) := u_1(t) u_2(t) \cdots u_l(t)
\]
is a path of unitaries with \( u(0) = U[\text{id}] \) and \( u(1) = U[\sigma] \), and \( u(t) \) commutes with \( \text{diag}(\lambda_1, \ldots, \lambda_n) \) since each \( u_j(t) \) does. \qed

Lemma 2.5. Let \( \sigma \) be any permutation in \( S_n \), and \( A, B \) disjoint nonempty closed subsets of a metric space \( X \). Let \( \lambda_1, \ldots, \lambda_n : X \to \mathbb{C} \) be continuous. Then, there exists a unitary \( v \in M_n(C(X)) \) such that

(i) \( v(x) = 1_n, \forall x \in A \),
(ii) \( v(x) = U[\sigma], \forall x \in B \), and
(iii) \( v(x) \) commutes with \( \text{diag}(\lambda_1(x), \ldots, \lambda_n(x)) \) whenever \( \lambda_i(x) = \lambda_{\sigma(i)}(x) \) for each \( i \in \{1, \ldots, n\} \).

Proof. Find a unitary path \( u(t) \) connecting \( U[\text{id}] \) to \( U[\sigma] \) using Lemma 2.4 so that \( u(t) \) commutes with \( \text{diag}(\lambda_1(x), \ldots, \lambda_n(x)) \) for each \( t \in (0,1) \) and each \( x \in X \) for which \( \lambda_i(x) = \lambda_{\sigma(i)}(x) \), \( i \in \{1, \ldots, n\} \). By Urysohn’s Lemma, there is a continuous map \( f : X \to [0,1] \) which is equal to zero on \( A \) and equal to one on \( B \). It is straightforward to check that \( v(x) := u(f(x)) \) satisfies the conclusion of the lemma. \qed

Lemma 2.6. Let \( Y \) be a closed subset of a normal space \( X \), and let \( f : Y \to S^n \) be continuous. Then, there is a continuous map \( \tilde{f} \) from \( X \) to the \( n+1 \) disk \( D^{n+1} \) which extends \( f \).

Proof. View \( D^{n+1} \) as the cube \([0,1]^{n+1} \), with \( S^n \) as its boundary. Then

\[
f(y) = (f_1(y), \ldots, f_{n+1}(y)),
\]
where each \( f_i : Y \to [0,1] \) is continuous. Extend each \( f_i \) to a continuous map \( \tilde{f}_i : X \to [0,1] \), and put
\[
\tilde{f}(y) = (\tilde{f}_1(y), \ldots, \tilde{f}_{n+1}(y)).
\]

\[\Box\]

### 3. The Main Theorem

Let \( A \) be a diagonal AH algebra with decomposition \( (A_i, \phi_i) \), and assume that the \( \phi_i \) are injective. In this section we will prove that if \( A \) is simple then it has the property \( \mathcal{P} \) (cf. Section 2). (The converse also holds, but is easier by far.) Let us begin with an outline of our strategy, before plunging headlong into the proof.

Assume first that \( A \) is simple and \( A_i = M_{n_i}(C(X_i)) \), so that there are no partial maps to contend with. Let there be given a natural number \( i \), an element \( f \) of \( A_i \), a point \( x_0 \in X_i \), and some \( \varepsilon > 0 \). Put \( U = B_\varepsilon(x_0) \). By Proposition 2.3 there is a \( j_0 \) with \( j_0 \geq i \) such that for any \( j \geq j_0 \),
\[
X_j = \lambda_1^{-1}(U) \cup \lambda_2^{-1}(U) \cup \ldots \cup \lambda_n^{-1}(U),
\]
where
\[
\phi_{i,j}(f) = \begin{pmatrix}
f \circ \lambda_1 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & f \circ \lambda_n
\end{pmatrix}.
\]

On each closed subset \( \lambda_t^{-1}(U) \), the range of the eigenvalue map \( \lambda_t \) is within \( \varepsilon \) of \( x_0 \). To show that \( A \) has the property \( \mathcal{P} \), we require a unitary \( u \) in \( A_j \) and an element \( b_f \in M_{n_j-n_i}(C(X_j)) \) such that
\[
\left\| u \phi_{i,j}(f) u^* - \begin{pmatrix} f(x_0) & 0 \\ 0 & b_f \end{pmatrix} \right\| < \varepsilon.
\]

We would like \( u(y) \) to exchange the first and \( t \)th diagonal entries of \( \phi_{i,j}(f) \) whenever \( y \in \lambda_t^{-1}(U) \), but this operation is unlikely to be well-defined — the sets \( \lambda_1^{-1}(U), \ldots, \lambda_n^{-1}(U) \) need not be mutually disjoint. The remainder of this section is devoted to overcoming this complication.

Given positive integers \( m \leq n \), let us denote by \( M_m(C(Y)) \oplus 1_{n-m} \) the set of all \( n \times n \) matrices of the form
\[
\begin{pmatrix}
a & 0 \\
0 & 1
\end{pmatrix},
\]
where \( a \in M_m(C(Y)) \) and \( 1_{n-m} \in M_{n-m} \).

**Theorem 3.1.** Let there be given a diagonal *-homomorphism \( \phi : C(X) \to M_n(C(Y)) \) with the eigenvalue pattern \( \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \), a point \( x_0 \) in \( X \), an element \( f \) of \( C(X) \), and a tolerance \( \varepsilon > 0 \). Choose \( \eta > 0 \) such that \( |f(x) - f(y)| < \varepsilon \) whenever \( d(x, y) < 2\eta \) (\( d \) is the metric on \( X \)). Suppose that \( F_1, \ldots, F_m \) are nonempty closed subsets of \( Y \) (\( m \leq n \)) such that \( d(\lambda_i(y), x_0) < \eta \) whenever \( y \in F_i \).
Then, there is a unitary \( u \) in \( M_m(C(Y)) \oplus 1_{n-m} \) and an element \( b \in M_{n-m-1}(C(Y)) \) such that for each \( y \in \bigcup_{i=1}^m F_i \) we have
\[
\left\| u(y)\phi(f)(y)u^*(y) - \begin{pmatrix}
    f(x_0) & 0 & 0 & \ldots & 0 \\
    0 & b(y) & 0 & \ldots & 0 \\
    0 & 0 & \lambda_{m+1}(y) & \ldots & 0 \\
    \vdots & \vdots & \ddots & \ddots & \vdots \\
    0 & 0 & 0 & \ldots & \lambda_n(y)
\end{pmatrix}\right\| < 2\epsilon.
\]
(Note that if \( n - m - 1 = 0 \), then there is no \( b \) in the matrix in (2).)

**Proof.** Let \( \rho \) denote the metric on \( Y \). Choose \( \delta > 0 \) such that \( d(\lambda_i(x), \lambda_i(y)) < \eta \) whenever \( \rho(x, y) \leq \delta \), \( i \in \{1, \ldots, n\} \). For each \( 1 \leq i \leq m \),
\[
|f \circ \lambda_i(y) - f(x_0)| < \epsilon, \text{ for all } y \in B_{\delta}(F_i).
\]
Set \( \delta_i(y) = f \circ \lambda_i(y) - f(x_0) \) for all \( y \) in \( B_{\delta}(F_i) \). Then, \( \delta_i \) is a continuous map from \( B_{\delta}(F_i) \) to the disk of radius \( \epsilon \) in the complex plane. By Lemma 2.6, \( \delta_i \) can be extended to a continuous function from \( Y \) to the complex plane such that \( ||\delta_i|| \leq \epsilon \) (let us also denote this extension map by \( \delta_i \)). For \( m < i \leq n \), set \( \delta_i = 0 \).
For each \( i \in \{1, \ldots, n\} \), put \( g_i = f \circ \lambda_i - \delta_i \), so that \( g_i \in C(X) \). Set
\[
g = \text{diag}(g_1, g_2, \ldots, g_n).
\]
Then, for each \( i \in \{1, \ldots, m\} \) and \( y \in B_{\delta}(F_i) \), we have
\[
g_i(y) = f(x_0);
\]
if \( i \in \{m+1, \ldots, n\} \), then \( g_i = f \circ \lambda_i \). For any unitary \( u \in M_n(C(Y)) \) we have
\[
||u\phi(f)u^* - ngu^*|| = ||\text{diag}(\delta_1, \ldots, \delta_n)|| < 2\epsilon.
\]
We have therefore reduced our problem to proving the following claim:

**Claim:** There is a unitary \( u \) in \( M_m(C(Y)) \oplus 1_{n-m} \) such that
\[
uu^* = \begin{pmatrix}
    f(x_0) & 0 & 0 & \ldots & 0 \\
    0 & b(x) & 0 & \ldots & 0 \\
    0 & 0 & g_{m+1}(x) & \ldots & 0 \\
    \vdots & \vdots & \ddots & \ddots & \vdots \\
    0 & 0 & 0 & \ldots & g_n(x)
\end{pmatrix}, \forall x \in \bigcup_{i=1}^m F_i,
\]
where \( b \in M_{n-m-1}(C(Y)) \).

**Proof of claim:** We will assume that for some \( 1 \leq k < m \) there is a unitary \( u_k \in M_k(C(Y)) \oplus 1_{n-k} \) such that
\[
u_ku_k^* = \begin{pmatrix}
    f(x_0) & 0 & 0 & \ldots & 0 \\
    0 & b(x) & 0 & \ldots & 0 \\
    0 & 0 & g_{k+1}(x) & \ldots & 0 \\
    \vdots & \vdots & \ddots & \ddots & \vdots \\
    0 & 0 & 0 & \ldots & g_n(x)
\end{pmatrix}, \forall x \in \bigcup_{i=1}^k F_i,
\]
and then prove that the same statement holds with \( k \) replaced by \( k + 1 \). Since (3) clearly holds when \( k = 1 \) — just take \( u_1 \) to be the identity matrix of \( M_m(C(Y)) \) — this recursive argument will prove our claim.
Assume that \( 3 \) holds for some \( k < n \). Put \( B = F_{k+1} \) and \( A = Y \setminus B_3(B) \). Apply Lemma 2.5 with these choices of \( A \) and \( B \) and with \( \sigma = (1 \, k+1) \) to obtain a unitary \( v \in M_n(C(Y)) \). We then have that \( v = 1_n \) on \( A \) and \( v = U[(1 \, k+1)] \) on \( B \). Inspecting the construction of \( v \), we find that it has the following form:

\[
(4) \quad v(y) = U[(2 \, k+1)] \begin{pmatrix} v'(y) & 0 \\ 0 & 1 \end{pmatrix} U[(2 \, k+1)], \forall y \in Y,
\]

where \( v'(y) \) is a unitary matrix in \( M_2(C(Y)) \) equal to \( 1_2 \) on \( A \) and equal to \( U[(12)] \) on \( B \). Define \( u_{k+1} := vu_k \).

Let us show that \( u_{k+1} \) satisfies the requirements of the claim. It is clear that \( u_{k+1} \) is an element of \( M_{k+1}(C(Y)) \oplus 1_{n-k-1} \). First suppose that \( y \in B \), so that \( g_{k+1}(y) = f(x_0) \). Since \( u_{k+1} = vu_k \) we have

\[
u_{k+1}(y)g(y)u^*_k(y) = U[(1 \, k+1)] \begin{pmatrix} c(y) & 0 & \ldots & 0 \\ 0 & f(x_0) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & g_n(y) \end{pmatrix} U[(1 \, k+1)] = \\
= \begin{pmatrix} f(x_0) & 0 & 0 & \ldots & 0 \\ 0 & b(y) & 0 & \ldots & 0 \\ 0 & 0 & g_{k+2}(y) & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & g_n(y) \end{pmatrix},
\]

for some \( c(y), b(y) \in M_k \).

Now suppose that \( y \in \bigcup_{i=1}^m F_i \setminus B_3(B) \subseteq Y \setminus B_3(B) \). In this case \( v(y) = 1_n \) and \( u_{k+1}(y) = u_k(y) \) and there is nothing to prove.

Finally, suppose that \( y \in (B_3(B) \setminus B) \cap (\bigcup_{i=1}^m F_i) \). As in the case \( y \in B \), we have \( g_{k+1}(y) = f(x_0) \). From \( 3 \) and this last fact we have

\[
U[(2 \, k+1)]u_k(y)g(y)u^*_k(y)U[(2 \, k+1)] = \\
\begin{pmatrix} f(x_0) & 0 & 0 & 0 & \ldots & 0 \\ 0 & f(x_0) & 0 & 0 & \ldots & 0 \\ 0 & 0 & d(y) & 0 & \ldots & 0 \\ 0 & 0 & 0 & g_{k+2}(y) & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & g_n(y) \end{pmatrix}
\]

for some \( d(y) \in M_{k-1} \). Since the upper left \( 2 \times 2 \) corner of the matrix above is scalar, the entire matrix commutes with \( v(y) \oplus 1_{n-2} \). It follows that \( u_{k+1}(y)g(y)u^*_k(y) \) is equal to

\[
U[(2 \, k+1)] \\
\begin{pmatrix} f(x_0) & 0 & 0 & 0 & \ldots & 0 \\ 0 & f(x_0) & 0 & 0 & \ldots & 0 \\ 0 & 0 & b_1(y) & 0 & \ldots & 0 \\ 0 & 0 & 0 & g_{k+2}(y) & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & g_n(y) \end{pmatrix} U[(2 \, k+1)].
\]
Computing this product yields a matrix of the form
\[
\begin{pmatrix}
  f(x_0) & 0 & 0 & \ldots & 0 \\
  0 & b(y) & 0 & \ldots & 0 \\
  0 & 0 & g_{k+2}(y) & \ldots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \ldots & g_n(y)
\end{pmatrix},
\]
as required. \( \square \)

For any C*-algebra \( B \) there is an isomorphism
\[ \pi : B \otimes M_n \rightarrow M_n(B) \]
given by
\[ \pi(b \otimes (a_{ij})) = \begin{pmatrix} ba_{11} & \cdots & ba_{1n} \\ \vdots & \ddots & \vdots \\ ba_{n1} & \cdots & ba_{nn} \end{pmatrix}. \] (5)

**Proposition 3.2.** Suppose that \( \phi : M_m(C(X)) \rightarrow M_{nm}(C(Y)) \) is a diagonal *-homomorphism with \( \epsilon p(\phi) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \). Let \( \tilde{\phi} : C(X) \rightarrow M_n(C(Y)) \) be the diagonal *-homomorphism given by
\[ \tilde{\phi}(f) = \begin{pmatrix}
  f \circ \lambda_1 & 0 & \ldots & 0 \\
  0 & f \circ \lambda_2 & \ldots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \ldots & f \circ \lambda_n
\end{pmatrix}. \]

Then, \( \tilde{\phi} \otimes \text{id}_{M_m} : C(X) \otimes M_m \rightarrow M_n(C(Y)) \otimes M_m \) is unitarily equivalent to \( \phi \).

**Proof.** On the one hand we have
\[ \tilde{\phi} \otimes \text{id}_{M_m}(f \otimes (c_{ij})) = \begin{pmatrix} f \circ \lambda_1 & 0 & \ldots & 0 \\
  0 & f \circ \lambda_2 & \ldots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \ldots & f \circ \lambda_n \end{pmatrix} \otimes (c_{ij}), \]
while on the other we have
\[ \phi(f \otimes (c_{ij})) = \begin{pmatrix}
  (f \circ \lambda_1) \otimes (c_{ij}) & 0 & \ldots & 0 \\
  0 & (f \circ \lambda_2) \otimes (c_{ij}) & \ldots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \ldots & (f \circ \lambda_n) \otimes (c_{ij})
\end{pmatrix}. \]

With the identifications \( C(X) \otimes M_n \cong M_n(C(X)) \) and \( M_n(C(Y)) \otimes M_m \cong M_{nm}(C(Y)) \) given by \[ \text{in mind, one sees that} \]
\[ \text{Ad}(U[\pi]) \circ (\tilde{\phi} \otimes \text{id}_{M_m}) = \phi, \]
where \( \pi \) is the permutation in \( S_{nm} \) and given by \( \pi(kn + i) = (i - 1)m + k + 1 \) for \( k = 0, 1, 2, \ldots, m-1 \) and \( i = 1, 2, \ldots, n \). \( \square \)
Corollary 3.3. Let $\phi : M_n(C(X)) \to M_m(C(Y))$ be a diagonal $*$-homomorphism with eigenvalue pattern $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, and let $\epsilon > 0$ be given. Let $f$ be any element of $M_m(C(X))$ and choose $\eta > 0$ such that

$$\|f(x) - f(y)\| < \epsilon \text{ whenever } d(x, y) < 2\eta$$

($d$ is the metric on $X$). Let $U$ be the open ball centered at $x_0 \in X$ with radius $\eta$, and suppose that

$$Y = \lambda_1^{-1}(U) \cup \lambda_2^{-1}(U) \cup \ldots \cup \lambda_n^{-1}(U).$$

Then there is a unitary $u$ in $M_n(C(Y))$ and element $b \in M_{nm-m}(C(Y))$ such that

$$\left\| u\phi(f)u^* - \begin{pmatrix} f(x_0) & 0 \\ 0 & b \end{pmatrix} \right\| < \epsilon.$$

Proof. By Proposition 3.2 we can assume that $m = 1$. Let $F_i$ be the closure of $\lambda_i^{-1}(U)$ for each $i$. Now, apply Theorem 3.1. Since $Y = \bigcup_{i=1}^n F_i$, we are done. \(\square\)

Now, we are ready to prove the main theorem of this section.

Theorem 3.4. Let $A = \lim(A_i, \phi_i)$ be a unital diagonal AH algebra. Then, $A$ is simple if and only if $A$ has the property $\mathcal{P}$ of Definition 2.2.

Proof. Suppose that $A$ has property $\mathcal{P}$. Let $f \in A_i$ be nonzero, so that there is a point $x_0$ in $X_i$ such that $f(x_0) \neq 0$. By the definition of property $\mathcal{P}$, there are an integer $j > i$ and unitaries $u_l \in A_{j,l}$, $l \in \{1, \ldots, k_j\}$ such that

$$\left\| u_l\phi_{i,j}(f)u_l^* - \begin{pmatrix} f(x_0) & 0 \\ 0 & b_l \end{pmatrix} \right\| < \epsilon$$

for some appropriately sized $b_l$. We may assume that $\epsilon < \|f(x_0)\|$, so that $\phi_{i,j}(f)$ is nowhere zero. This implies that the ideal of $A_j$ generated by $\phi_{i,j}(f)$ is all of $A_j$, and that the ideal of $A$ generated by the image of $f$ is all of $A$. Since $f$ was arbitrary, $A$ is simple.

Now assume that $A$ is simple, and let $f \in A_{i,t}$ be nonzero. Recall that the $\phi_i$ may be taken to be injective. By Proposition 2.3 there exists, for each $x_0 \in X_{i,t}$ and $\epsilon > 0$, a $j_0 > i$ with the following property: for every $j \geq j_0$ and every $l \in \{1, \ldots, k_j\}$ we have

$$\epsilon p^{-1}(\phi_{i,j}(f), (B_\delta(x_0))) = X_{j,t},$$

where $\delta$ is some positive number such that

$$d(f(x), f(y)) < \epsilon \text{ whenever } d(x, y) < 2\delta.$$

As pointed out in Section 2, the map $\phi_{i,j}^{t,l}$ may be viewed as a diagonal map from $A_{i,t}$ into the cut-down of $A_{j,l}$ by the projection $\phi_{i,j}^{t,l}(1)$; any unitary $u$ in this corner of $A_{j,l}$ gives rise to a unitary $\tilde{u}$ in $A_{j,t}$ by setting $\tilde{u} = u \oplus (1_{A_{j,t}} - \phi_{i,j}^{t,l}(1))$. Combining this observation with Corollary 3.3, we see that there exists, for each $l \in \{1, \ldots, k_j\}$, a unitary $u_l \in A_{j,t}$ such that

$$\left\| u_l\phi_{i,j}^{t,l}(f)u_l^* - \begin{pmatrix} f(x_0) & 0 \\ 0 & b_l \end{pmatrix} \right\| < \epsilon.$$

Thus, $A$ has property $\mathcal{P}$, as desired. \(\square\)
4. Stable Rank

**Theorem 4.1.** Let \( A = \lim_{i \to \infty} (A_i, \phi_i) \) be a simple unital diagonal AH algebra. Then, \( A \) has stable rank one.

Before proving Theorem 4.1, we situate it relative to other results on the stable rank of general approximately homogeneous (AH) algebras. Recall that an AH algebra is an inductive limit \( C^* \)-algebra \( A = \lim_{i \to \infty} (A_i, \phi_i) \), where

\[
A_i = \bigoplus_{l=1}^{n_i} p_{i,l}(C(X_{i,l}) \otimes \mathcal{K}) p_{i,l}
\]

for compact connected Hausdorff spaces \( X_{i,l} \), projections \( p_{i,l} \in C(X_{i,l}) \otimes \mathcal{K} \), and natural numbers \( n_i \). If \( A \) is separable, then one may assume that the \( X_{i,l} \) are finite CW-complexes (1, 11). The inductive system \( (A_i, \phi_i) \) is referred to as a decomposition for \( A \). All AH algebras in this paper are assumed to be separable.

If an AH algebra \( A \) admits a decomposition as in (6) for which

\[
\max_{1 \leq l \leq n_i} \left\{ \frac{\dim(X_{i,1})}{\text{rank}(p_{i,1})}, \ldots, \frac{\dim(X_{i,n_i})}{\text{rank}(p_{i,n_i})} \right\} \to 0
\]

then we say that \( A \) has slow dimension growth. Theorem 1 of [2] states that every simple unital AH algebra with slow dimension growth has stable rank one. Villadsen in [17] constructed simple diagonal AH algebras which do not have slow dimension growth, but which do have stable rank one; the converse of [2] Theorem 1 does not hold. There are in fact a wealth of simple diagonal AH algebras without slow dimension growth which exhibit all sorts of interesting behaviour (cf. 14, 15, 16), whence Theorem 4.1 is widely applicable.

Simple AH algebras may have stable rank strictly greater than one, and there is reason to believe that Theorem 4.1 is quite close to being best possible. One might be able to generalise our result to the setting of AH algebras where the projections \( \phi_{i,j}(p_{i,l}) \) appearing in (6) can be decomposed into a direct sum of a trivial projection \( \theta_j \) and a second projection \( q_j \) such that \( \tau(q_j) \to 0 \) as \( j \to \infty \) for any trace \( \tau \). Otherwise, one finds oneself in a situation very similar to the construction of 18, where the stable rank is always strictly greater than one.

Let us now prepare for the proof of Theorem 4.1.

**Lemma 4.2.** Let \( a \in M_m(C(X)) \) be block diagonal, i.e.,

\[
a = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & a_1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & a_n
\end{pmatrix},
\]

where \( a_i \in M_{k_i}(C(X)) \) for natural numbers \( k_1, \ldots, k_n \). If the size of the matrix 0 in the upper left-hand corner of \( a \) is strictly greater than \( \max_{1 \leq i \leq n} k_i \), then \( a \) can be approximated arbitrarily closely by invertible elements in \( M_m(C(X)) \).

**Proof.** Let \( \epsilon > 0 \) be given, and let \( k \) denote the size of the matrix 0 in the upper left-hand corner of \( a \). Since \( k > k_i, i \in \{1, \ldots, n\} \), there is a permutation matrix
$U$ such that

$$Ua = \begin{pmatrix} 0 & * & \cdots & * \\ 0 & 0 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

is nilpotent. As was proved in [13], every nilpotent element in a unital $C^*$-algebra can be approximated arbitrarily closely by invertible elements. We may thus find an invertible element $b \in M_n(C(X))$ such that $||Ua - b|| < \epsilon$, and

$$||a - U^{-1}b|| = ||U^{-1}Ua - U^{-1}b|| \leq ||U^{-1}|| \cdot ||Ua - b|| < \epsilon.$$  

The lemma now follows from the fact that $U^{-1}b$ is invertible.  

It easy to prove that $a \in M_n(C(X))$ is invertible if and only if $a(x)$ is invertible for each $x \in X$. The proof of the next lemma is also straightforward.

**Lemma 4.3.** Let $p, q$ be orthogonal projections in a $C^*$-algebra $A$, and let $\epsilon > 0$ be given. If elements $a$ and $b$ in $A$ can be approximated to within $\epsilon$ by invertible elements in $pAp$ and $qAq$, respectively, then $a + b$ can be approximated to within $\epsilon$ by an invertible element in $(p + q)A(p + q)$.

Now, we are ready to prove Theorem 4.1.

**Proof of Theorem 4.1.** Since every element in $A$ can be approximated arbitrarily closely by elements in $\bigcup_{i=1}^{\infty} A_i$, it will suffice to prove for any $\epsilon > 0$ and any $a \in A_i$, there is an invertible element in $A$ whose distance to $a$ is less than $\epsilon$. (Note that we are using the injectivity of the $\phi_i$ to identify $A_i$ with its image in $A$.)

By Lemma 4.3 we may assume that $A_i = M_{n_i}(C(X_i))$. We also assume that $a$ is not invertible. By the comment preceding Lemma 4.3 there is a point $x_0 \in X_i$ such that $\det(a(x_0)) = 0$. There are permutation matrices $u, v \in M_{n_i}$ and a matrix $c \in M_{n_i-1}$ such that

$$ua(x_0)v = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix},$$

Let $b$ denote the element $uv$. Following the lines of the proof of Lemma 4.2, it will suffice to prove that $b$ can be approximated to within $\epsilon$ by an invertible element of $A$.

For each $j > i$, $\phi_{i,j}(b)$ is a tuple of $k_j$ elements. If each coordinate of $\phi_{i,j}(b)$ can be approximated to within $\epsilon$ by an invertible element in the corner of $A$ generated by the unit of $A_j$, then $\phi_{i,j}(b)$ can be approximated to within $\epsilon$ by an invertible element of $A$. We may therefore assume that $A_j = M_{n_j}(C(Y_j))$, and concern ourselves with proving that $\phi_{i,j}(b)$ is approximated to within $\epsilon$ by an invertible element in $A$.

By Theorem 3.4 there exist an integer $j > i$, a unitary $w \in A_j$ and an element $b'$ such that

$$\left|\left|w\phi_{i,j}(b)w^* - \begin{pmatrix} b(x_0) & 0 \\ 0 & b' \end{pmatrix}\right|\right| < \epsilon/2.$$  

We have

$$\begin{pmatrix} b(x_0) & 0 \\ 0 & b' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & b'' \end{pmatrix},$$
where $b'' = \text{diag}(c, b')$. Put $d = w\phi_{i,j}(b)w^*$, and note that it will suffice to prove that $\phi_{j,m}(d)$ is approximated to within $\epsilon/2$ by an invertible element in $A_m$ for some $m > j$.

Since $A$ is simple, there is an integer $m > j$ large enough so that, for each $t \in \{1, \ldots, k_m\}$, either the number of the eigenvalue maps of $\phi_{j,m}^{1,t}$ counted with multiplicity is strictly larger than the size of the matrix $b''$, or the image of $\phi_{j,m}^{1,t}$ is finite-dimensional. In the latter case, $\phi_{j,m}^{1,t}(d)$ is approximated to within $\epsilon$ by an invertible element in the image of $\phi_{j,m}^{1,t}$ since finite-dimensional C$^*$-algebras have stable rank one, so we may assume that the number of eigenvalue maps of $\phi_{j,m}^{1,t}$, counted with multiplicity, is strictly larger than the size of the matrix $b''$. Then, $\phi_{j,m}^{1,t}(d)$ is unitarily equivalent to

$$
\begin{pmatrix}
0 & 0 & \ldots & 0 \\
0 & b_1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & b_l
\end{pmatrix}
$$

inside $A_{m,t}$, where $b_k$ is the composition of $b''$ and the $k^{\text{th}}$ eigenvalue map of $\phi_{j,m}^{1,t}$, and the size of the matrix 0 at the upper left-hand corner of the above matrix is strictly bigger than the size of the matrix $b_k$ for every $k$. By Lemma 4.2, the matrix above can be approximated to within $\epsilon/2$ by an invertible in $A_{m,t}$, as required. □

**Corollary 4.4.** Let $A$ be a simple unital diagonal AH algebra. If $A$ has real rank zero and weakly unperforated K$_0$-group, then $A$ is tracially AF.

**Proof.** By Theorem 4.1 $A$ has stable rank one. The corollary then follows from a result of Lin ([12, Theorem 2.1]). □

Corollary 4.4 applies, for instance, to simple unital diagonal AH algebras for which the spaces $X_i$ in some diagonal decomposition for $A$ are all contractible. This contractibility hypothesis may seem strong, but it does not substantially restrict the complexity of $A$; if one wants to classify all such $A$ via K-theory and traces, then the additional assumption of very slow dimension growth and the full force of [8] and [9] are required; the collection of all such cannot cannot be classified by topological K-theory and traces alone ([12]).

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