Stochastic Service Placement

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Abstract. Resource allocation for cloud services is a complex task due to the diversity of the services and the dynamic workloads. One way to address this is by overprovisioning which results in high cost due to the unutilized resources. A much more economical approach, relying on the stochastic nature of the demand, is to allocate just the right amount of resources and use additional more expensive mechanisms in case of overflow situations where demand exceeds the capacity. In this paper we study this approach and show both by comprehensive analysis for independent normal distributed demands and simulation on synthetic data that it is significantly better than currently deployed methods.

1 Introduction

The recent rapid development of cloud technology gives rise to “many-and-diverse” services being deployed in datacenters across the world. The allocation of the available resources in the various locations to these services has a critical impact on the ability to provide a ubiquitous cost-effective high quality service. There are many challenges associated with optimal service placement due to the large scale of the problem, the need to obtain state information, and the geographical spreading of the datacenters and users.

One intriguing problem is the fact that the service resource requirement changes over time and is not fully known at the time of placement. A popular way of addressing this important problem is over-provisioning, that is allocating resources for the peak demand. Clearly, this is not a cost effective approach as much of the resources are unused most of the time. An alternative approach is to model the service requirement as a stochastic process with known parameters (these parameters can be inferred from historical data).

Much of the previous work that used stochastic demand modeling \cite{[1],[2],[3],[4]} attempted to minimize the probability of an overflow event and not the cost of this overflow. Their solution was based on solving the Stochastic Bin Packing (SBP) problem, where the goal is to pack a given set of random items to a minimum number of bins such that each bin overflow probability will not exceed a given value.\footnote{\cite{[3],[4]} also looked at the online SBP problem, where the items (services in our case) are assigned to some bin (datacenter) as they arrive.}

Thus, in these works there is no distinction between cases with marginal and substantial overflow of the demand, and under this modeling one again prepares
for the worst and tries to prevent an overflow. However, in reality it is not cost effective to allocate resources according to the worst case scenario. Instead, one often wishes to minimize the cost associated with periods of insufficient resource availability, which is commonly dealt with by either diverting the service request to a remote location or dynamically buying additional resources. In both cases, the cost associated with such events is proportional to the amount of unavailable resources. Thus, our aim is to minimize the expected overflow of the demand over time.

We deviate from previous work in two ways. First, we look at a stochastic packing problem where the number of bins is given, e.g., a company already has two datacenters where services can be placed, and one would like to place services in these two datacenters optimally. Second, as we mentioned before, we look at a more practical optimality criterion where we do not wish to optimize the probability of an overflow, but instead its expected deviation.

To better understand this, assume that the resource we are optimizing is the bandwidth consumption of the service in a datacenter where we have a prebooked bandwidth for each datacenter. Since we are dealing with stochastic bandwidth demand, with some probability the traffic will exceed the prebooked bandwidth and will result with expensive overpayment that depends on the amount of oversubscribed traffic. Obviously, in such a case minimizing the probability of oversubscription may not give the optimal cost, and what we need to minimize is the expected deviation from the prebooked bandwidth, thus we term our problem SP-MED (stochastic packing with minimum expected deviation).

We analyze the case of independent normal distribution and develop algorithms for the optimal partition between two or more datacenters that need not be identical. We prove the correctness of our algorithm by separating the problem into two: one dealing with the continuous characteristics of the stochastic objective function and the other dealing with the discrete nature of the combinatorial algorithm. This approach results in a clean and elegant algorithm and proof.

In fact, our proof technique reveals that the algorithms we developed hold for a large family of optimization criteria, and thus may be applicable for other optimization problems. In particular we study two other natural cost functions that correspond to minimizing overflow probability (rather than expected overflow deviation). We show that these cost functions also fall into our general framework, and therefore the same algorithm works for them as well! As will become clear later the requirements from the cost functions we have are quite natural, and we believe our methods will turn useful for many other applications.

2 Related work

Early work on VM placement (e.g., [5,6,7]) models the problem as a deterministic bin packing problem, namely, for every service there is an estimate of its deterministic demand. Stochastic bin packing was first suggested by Kleinberg, Rabani and Tardos [1] for statistical multiplexing. [1] mostly considered
Bernoulli-type distributions. Goel and Indyk [2] further studied Poisson and exponential distributions. Wang, Meng and Zhang [3] suggested to model real data with the *normal distribution*. Thus, the input to their stochastic packing problem is \( n \) independent services each with demand requirement distributed according to a distribution \( X^{(i)} \) that is normal with mean \( \mu^{(i)} \) and variance \( V^{(i)} \). The output is some partition of the services to bins in a way that minimizes a target function that differs from problem to problem.

A naive approach to such a problem is to reduce it to classical bin packing as follows: for the \( i \)th service define the *effective size* as the number \( e^{(i)} \) such that the probability that \( X^{(i)} \) is larger than \( e^{(i)} \) is small; then solve the classical bin packing problem (or a variant of it) with item sizes \( e^{(1)}, \ldots, e^{(n)} \). However, [3] showed this approach can be quite wasteful, mostly because it adds extra space per service and not per bin. To demonstrate the issue, think about unbiased, independent coin tosses. The probability one coin toss significantly deviates from its mean is 1, while the probability 100 independent coin tosses significantly deviate from the mean is exponentially small. When running independent trials there is a *smoothing* effect that considerably reduces the chance of high deviations. This can also be seen from the fact that the standard deviation of \( n \) independent, identical processes is only \( \sqrt{n} \) times the standard deviation of one process, and so the standard deviation grows much slower than the number of processes.

Breitgand and Epstein [4], building on [3], suggest an algorithm for stochastic bin packing that takes advantage of this smoothing effect. The algorithm assumes all bins have equal capacity. The algorithm first sorts the processes by their variance to mean ratio (VMR), i.e., \( \frac{V^{(1)}}{\mu^{(1)}} \leq \frac{V^{(2)}}{\mu^{(2)}} \leq \ldots \leq \frac{V^{(n)}}{\mu^{(n)}} \). Then the algorithm finds the largest prefix of the sorted list such that allocating that set of services to the first bin makes the probability the first bin overflows at most \( p \). The algorithm then proceeds bin by bin, each time allocating a prefix of the remaining services on the sorted list to the next bin. [4] show that if we allow *fractional solutions*, i.e., we allow splitting services between bins, the algorithm finds an optimal solution, and also show an online, integral version that gives a 2-approximation to the optimum.

### 3 The risk unbalancing principle: A bird’s overview of our technique

In this bird’s overview we focus on the SP-MED problem, and in the next section we generalize it to a much wider class of cost functions. As mentioned above we are given \( k \) bins with capacities \( c_1, \ldots, c_k \). We are looking for a solution that minimizes the expected deviation. As before, we study the case where the stochastic demands are independent and normally distributed. We develop a new general framework to analyze this problem that also sheds light on previous work.

We first observe that an optimal fractional solution to SP-MED is also optimal for any two bins. This is true because the cost function is the sum of the
expected deviation of all the bins, and changing the internal allocation of two bins only affects them and not the other bins. Thus, one possible approach to achieve an optimal solution is by repeatedly improving the internal division of two bins. However, offhand, there is no reason to believe such a sequence of local improvements efficiently converges to the optimal solution. Surprisingly, this is indeed the case. Therefore, we first focus on the two bin case. Later (in Appendix A) we will see that solving the \( k = 2 \) case implies a solution to the general case of arbitrary \( k \).

We recall that if we have \( n \) independent normally distributed services with mean and variance \( (\mu^{(i)}, V^{(i)}) \), and we allocate the services with indices in \( I \subseteq [n] \) to the first bin, and the rest to the second bin, then the first bin is normally distributed with mean \( \mu_1 = \sum_{i \in I} \mu^{(i)} \) and variance \( V_1 = \sum_{i \in I} V^{(i)} \), while the second one is normally distributed with mean \( \mu - \mu_1 \) and variance \( V - V_1 \) where \( \mu = \sum_{i=1}^{n} \mu^{(i)} \) and \( V = \sum_{i=1}^{n} V^{(i)} \). We wish to minimize the total expected deviation of the two bins.

Let us define the function \( \text{Dev} : [0, 1] \times [0, 1] \rightarrow \mathbb{R} \) such that \( \text{Dev}(a, b) \) is the total expected deviation when bin one is distributed with mean \( a \mu \) and variance \( b V \) and bin two with mean \( (1-a) \mu \) and variance \( (1-b) V \). Figure 1 (left) depicts \( \text{Dev}(a, b) \) for two bins with equal capacity. \( \text{Dev} \) has a reflection symmetry around the \( a = b \) line, that corresponds to the fact that we can change the order of the bins as long as they have equal capacity and remain with a solution of the same cost (see Appendix G.2 for a proof of a similar symmetry for the case of different capacity bins). The points \((0, 0)\) and \((1, 1)\) correspond to allocating all the services to a single bin, and indeed the expected deviation there is maximal.

The point \((\frac{1}{2}, \frac{1}{2})\) is a saddle point (see Appendix G.3 for a proof). Figure 1 (middle) shows a zoom in around this saddle point. There are big mountains in the lower left and upper right quarters that fall down to the saddle point, and also valleys going down from the saddle point to the bottom right and top left. Going back to Figure 1 (left) we see that for every fixed \( b \) the function is convex in \( a \), with a single minimum point, and all these minimum points form the two valleys mentioned above.

We now go back and consider the input \( \{(\mu^{(i)}, V^{(i)})\}_{i=1}^{n} \). We can represent service \( i \) with the pair \( (a^{(i)} = \frac{\mu^{(i)}}{\mu}, b^{(i)} = \frac{V^{(i)}}{V}) \). If we split the input to the sets \( I \) and \( [n] \setminus I \), the expected deviation of this partition is \( \text{Dev}(\sum_{i \in I} a^{(i)}, \sum_{i \in I} b^{(i)}) \). Thus, the \( n \) input points induce \( 2^n \) possible solutions \( P_I = (\sum_{i \in I} a^{(i)}, \sum_{i \in I} b^{(i)}) \) for each \( I \subseteq [n] \), and our task is to find the partition \( I \) that minimizes \( \text{Dev} \).

Sorting the services by their VMR, is equivalent to sorting the vectors \( P^{(i)} = (a^{(i)}, b^{(i)}) \) by their angle with the \( a \) axis. If the sorted list is \( P^{(1)}, \ldots, P^{(n)} \) with increasing angles, looking for a solution in a partition that breaks the sorted sequence to two consecutive parts finds a solution among \( (0, 0), P^{(1)}, P^{(1)} + P^{(2)}, \ldots, P^{(1)} + \ldots + P^{(n)} = (1, 1) \). If we connect a line between two consecutive points in the above sequence we get the sorted path, connecting \((0, 0)\) with \((1, 1)\).

A crucial observation is that all the \( 2^n \) possible solutions \( P_I \) lie on or above that sorted path. See Lemma 1 for a rigorous statement and proof and Figure 1 (right) for an example. The fact that the sorted path always lies beneath all
**Fig. 1.** The left figure depicts $Dev(a, b)$ when $\mu = 160$, $V = 6400$ and $c_1 = c_2 = 100$. The middle figure is a zoom in around the saddle point $\left( \frac{1}{2}, \frac{1}{2} \right)$. The white lines represent the zero level sets of the partial derivatives. The right figure depicts $Dev(a, b)$ with the $2^{10}$ possible partitions represented in orange. The bottom sorted path (that sorts services by their VMR in increasing order) and the upper sorted path (that sorts services by their VMR in decreasing order), together with their integral points, are in black. Notice that all partition points are confined by both sorted paths.

possible solutions is independent of the target function, and only depends on the fact that we deal with the normal distribution. Actually, this is true for any distribution where allocating a subset of services to a bin amounts to adding the corresponding means and variances.

A second crucial observation, proved in Theorem 1, is that the optimal fractional solution lies on the sorted path. In fact, the proof shows this is a general phenomenon that holds for many possible target functions. What we need from the cost function $Dev$ we are trying to minimize is:

- $Dev$ has a symmetry of reflection around a line that passes through a saddle point $(a', \frac{1}{2})$.
- For every fixed $b$, $Dev$ is strictly uni-modal in $a$ (i.e., has a unique minimum in $a$). We call the set of solutions $\{ (m(b), b) \}$ the valley. The saddle point lies on the valley.
- $Dev$ restricted to the valley is strictly monotone for $b \leq \frac{1}{2}$ and $b \geq \frac{1}{2}$ with a maximum at the saddle point that is obtained when $b = \frac{1}{2}$.

To see why these conditions suffice, consider an arbitrary fractional solution $(a, b)$. By the symmetry property we may assume without loss of generality that $b \leq \frac{1}{2}$. If $(a, b)$ is left to the valley, we can improve the solution by keeping $b$ and moving $a$ towards the valley, until we either hit the sorted path or the valley. If we hit the valley first, we can improve the expected deviation, by going down the valley until we hit the sorted path. A similar argument applies when $(a, b)$ is right to the valley. The conclusion is that we can always find another fractional solution that is better than $(a, b)$ and lies on the sorted path.

Thus, quite surprisingly, we manage to decouple the question to two separate and almost orthogonal questions. The first question is what is the behavior of the function $Dev$ over its entire domain. Notice that $Dev$ is a function of only
two variables, independent of $n$. This question does not depend on the input \( \{ (\mu^{(i)}, V^{(i)}) \} \) at all. We study $Dev$ using analytic tools. The second question is what is the geometric structure of the set of feasible points, and here we investigate the input, completely ignoring the target function $Dev$. For this question we use geometric intuition and combinatorial tools.

In Appendix B we prove our three cost functions fall into the above framework. The discussion above also gives an intuitive geometric interpretation of the results in [13]. We believe the framework is also applicable to many other target functions.

There is an intuitive explanation why the optimal fractional solution lies on the sorted path. Sorting services by their VMR essentially sorts them by their risk, where risk is the amount of variance per one unit of expectation. Partitioning the sorted list corresponds to putting all the low risk services in one bin, and all the high risk services in the other. We call this the risk unbalancing principle. Intuitively, we would like to give the high risk services as much spare capacity as possible, and we achieve that by grouping all low risk services together and giving them less spare capacity. In contrast, balancing risk amounts to taking the point \( (\frac{1}{2}, \frac{1}{2}) \) (for the case where the two bins have equal capacities). Having the geometric picture in mind, we immediately see that this saddle point is not optimal, and can be improved by taking the point on the valley that intersects the sorted path.

The technique also applies to bins having different capacities that previous work did not analyze, and reveals that we should allocate low risk services to the bin with the lower capacity. This follows from a neat geometric argument that when $c_1 < c_2$ the optimal fractional solution lies on the sorted path (that sorts services by their VMR in increasing order) and not the upper sorted path (that sorts services by their VMR in decreasing order). See Section 5.2 and Theorem 1 for more details. If we have $k$ bins, we should sort the bins by their capacity and the services by their VMR, and then allocate consecutive segments of the sorted list of services to the bins, with lower risk services allocated to smaller capacity bins. This double-sorting again intuitively follows from the risk unbalancing principle, trying to preserve as much spare capacity to the high risk services. We prove the double-sorting algorithm in Appendix A.

Finally, we are left with the question of finding the right partition of the sorted list. With two bins one can simply try all $n$ partition points. However, with $k$ bins there are \( \binom{n}{k-1} = \Theta(n^{k-1}) \) possible partition points. We show a dynamic programming algorithm that finds the best partition in poly$(n)$ time. The optimal solution can probably be found much faster and we have candidate

\[2 \text{ In a similar vein if we have total capacity } c \text{ to split between two bins, it is always better to make the bin capacities unbalanced as much as possible, i.e., the minimum expected deviation decreases as } c_2 - c_1 \text{ increases. In particular the best choice is having a single bin with capacity } c \text{ and the worst choice is splitting the capacities evenly between the two bins. Obviously, in practice there might be other reasons to have several bins, but if there is tolerance in each bin capacity it is always better to minimize the number of bins. We give a precise statement and proof in Appendix J.} \]
algorithms that work well in practice and we are currently working on formally proving their correctness.

4 Formal treatment

The input to the problem consists of \( k \) and \( n \), specifying the number of bins and services, integers \( \{c_i\}_{i=1}^k \), specifying the bin capacities, and values \( \{(\mu^{(i)}, V^{(i)})\}_{i=1}^n \) where the demand distribution \( X^{(i)} \) of service \( i \) is normal with mean \( \mu^{(i)} \) and variance \( V^{(i)} \). \( X^{(i)} \) are independent. The output is a partition of \([n]\) to \( k \) disjoint sets \( S_1, \ldots, S_k \subseteq [n] \), where \( S_j \) includes indices of services that needs to be allocated to bin \( j \). Our goal is to find a partition minimizing a given cost function \( D(S_1, \ldots, S_k) \). The optimal (integral) partition is the partition with minimum cost among all possible partitions.

We let \( c \) denote the total capacity, \( \mu \) the total demand and \( V \) the total variance, i.e., \( c = \sum_{j=1}^k c_j \), \( \mu = \sum_{i=1}^n \mu^{(i)} \) and \( V = \sum_{i=1}^n V^{(i)} \). The value \( c - \mu \) represents the total spare capacity we have. The (total) standard deviation is \( \sigma = \sqrt{V} \) which represents the standard deviation of the input had all services been put into a single bin. We let \( \Delta \) denote the spare capacity in units of the standard deviation, i.e., \( \Delta = \frac{c - \mu}{\sigma} \). We assume that \( c \geq \mu \).

We remind the reader that the sum of independent normal distributions with mean \( \mu^{(i)} \) and variance \( V^{(i)} \) is normal with mean \( \sum \mu^{(i)} \) and variance \( \sum V^{(i)} \). Consider a partition \( S_1, \ldots, S_k \). Then, the demand distribution of bin \( j \) is normal with mean \( \mu_j = \sum_{i \in S_j} \mu^{(i)} \) and variance \( V_j = \sum_{i \in S_j} V^{(i)} \). The standard deviation of bin \( j \) is \( \sigma_j = \sqrt{V_j} \) and its spare capacity in units of its standard deviation is \( \Delta_j = \frac{c_j - \mu_j}{\sigma_j} \). Following previous work, we assume that for every \( i \), \( \mu^{(i)} \geq 0 \) and \( V^{(i)} \) is small enough compared with \( (\mu^{(i)})^2 \) so that the probability of getting negative demand in service \( i \) is negligible.

Next we normalize everything with regard to the total mean \( \mu \) and the total variance \( V \). The input to the function \( D \) are two vectors \( a_1, \ldots, a_{k-1}, b_1, \ldots, b_{k-1} \) s.t., \( \sum_{j=1}^{k-1} a_j \leq 1 \) and \( \sum_{j=1}^{k-1} b_j \leq 1 \). We think of the vectors \( a_1, \ldots, a_{k-1} \) and \( b_1, \ldots, b_{k-1} \) as representing the fraction of mean and variance each bin takes. We let \( a_k = 1 - \sum_{j=1}^{k-1} a_j \) and \( b_k = 1 - \sum_{j=1}^{k-1} b_j \) be the fraction of the last bin (that takes all the remaining mean and variance). We let \( \mu_j = a_j \mu \) and \( V_j = b_j V \) for \( j = 1, \ldots, k \) and we define \( \sigma_j = \sqrt{V_j} = \sqrt{b_j} \sigma \) (where \( \sigma = \sqrt{V} \)) and \( \Delta_j = \frac{c_j - \mu_j}{\sigma_j} \).

In this notation the cost function is \( D(a_1, \ldots, a_{k-1}; b_1, \ldots, b_{k-1}) \).

The function \( D \) is defined over all tuples \( a_1, \ldots, a_{k-1}, b_1, \ldots, b_{k-1} \in [0,1] \) s.t., \( \sum_{j=1}^{k-1} a_j \leq 1 \) and \( \sum_{j=1}^{k-1} b_j \leq 1 \). There are \( k^n \) possible partitions that correspond to the \( k^n \) (possibly) different tuple values. We call a tuple integral if it is induced by some partition. Our goal is to approximate the integral solution minimizing \( D \) over all integral inputs.

4.1 What do we require form a cost function?

We can handle a wide variety of cost functions. Specifically, we require the following from the cost function:
1. In any solution to a \( k \)-bin problem, the allocation for any two bins is also optimal. Formally, if \( S_1, \ldots, S_k \) is optimal for a \( k \)-bin problem, then for any \( j \) and \( j' \), the partition \( S_j, S_{j'} \) is optimal for the two-bin problem defined by the services in \( S_j \cup S_{j'} \) and capacities \( c_j, c_{j'} \). We remark that this is a natural condition that is true for almost any cost function we know.

For \( k = 2 \) we require that:

2. \( D(a, b) = D(1 - a - \frac{c_2}{c_1}, 1 - b) \). When \( c_1 = c_2 \) this simply translates to \( D(a, b) = D(1 - a, 1 - b) \) and there is no difference between allocating the set \( S_1 \) to the first bin or to the second one.

3. For every fixed \( b \in [0, 1] \), \( D(a, b) \) has a unique minimum on \( a \in [0, 1] \), at some point \( a = m(b) \), i.e. \( D \) is decreasing at \( a < m(b) \) and increasing at \( a > m(b) \). We call the points on the curve \( \{(m(b), b)\} \) the valley.

4. \( D \) has a unique maximum over the valley at the point \( (m(\frac{1}{2}), \frac{1}{2}) \). In fact by the symmetry above this point is \( (\frac{1}{2} - \frac{c_2 - c_1}{2c_1}, \frac{1}{2}) \). This means that \( D(m(b), b) \) is increasing for \( b \leq \frac{1}{2} \) and decreasing for \( b \geq \frac{1}{2} \).

4.2 Three cost functions

We have already seen SP-MED, which minimizes the expected deviation. Previous work on SBP (stochastic bin packing) studied the worst overflow probability of a bin. We define SP-MWOP to be the problem that minimizes the worst overflow probability of a bin. Another natural problem is what we call SP-MOP that minimizes the probability that some bin overflows. The three cost functions are related but behave differently. For example, SP-MWOP is not differentiable but the other two are. It is relatively simple to find the valley in SP-MED and SP-MWOP but we are not aware of any explicit description of the valley in SP-MOP.

Nevertheless, it is remarkable that all three cost functions fall into our framework and we prove that in Appendix B. The proof is not always simple, and we have used Lagrange multipliers and the log concavity of the cumulative distribution function of the normal distribution for proving this for SP-MOP. It follows that all the machinery we develop in the paper holds for these cost functions.

5 The two bin case

We first consider the two bin case \( (k = 2) \). We shall later see (in Appendix A) that solving the \( k = 2 \) case is key for solving the general problem for any \( k > 2 \).

As explained before we decouple the question to one about the function \( D \) and another about the structure of the integral points.

5.1 The sorted path

We now consider the input \( (\mu^{(1)}, V^{(1)}), \ldots, (\mu^{(n)}, V^{(n)}) \). We represent service \( i \) with the pair \( (a^{(i)} = \frac{\mu^{(i)}}{\mu}, b^{(i)} = \frac{V^{(i)}}{V}) \). If we split the input to the sets \( I \)
We keep all notation as before. The algorithm works as follows:

**5.2 The sorting algorithm**

We now present the algorithm for the general case of arbitrary $k$. The algorithm for the general case of arbitrary $k$ is presented in Appendix A. The algorithm gets as input $n$, $\{(\mu^{(i)}, V^{(i)})\}_{i=1}^n, c_1, c_2$ and outputs a partition $S_1, S_2$ of $[n]$ minimizing cost.

We keep all notation as before. The algorithm works as follows:

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**The sorting algorithm: 2 bins**

1. Sort the bins by their capacity such that $c_1 \leq c_2$.
2. Sort the services by their VMR such that $\mu^{(1)} \leq \mu^{(2)} \leq \cdots \leq \mu^{(n)}$.
3. Calculate the points $P^{[0]} = (0,0), P^{[1]}, \ldots, P^{[n]} = (1,1)$ on the sorted path.
4. Calculate $D(P^{[i]})$ for each $0 \leq i \leq n$ and find the index $i^*$ such that the point $P^{[i^*]}$ achieves the minimal cost among all points $P^{[i]}$.
5. Let $S_1 = \{1, \ldots, i^*\}$ and $S_2 = [n] \setminus S_1$. Output $(S_1, S_2)$.
The optimal fractional point is the fractional point that minimizes cost. In Section 3, we said, and soon will prove, that the optimal solution allocates low risk services to one bin and the rest to the other. However, offhand, it is not clear whether to allocate the smaller risk services to the lower capacity bin or the higher capacity bin. In fact, offhand, it is not clear whether the optimal solution is on the lower sorted path or the upper sorted path, and it might even depend on the input. Remarkably, we prove that it always lies on the bottom sorted path, meaning that it is always better to allocate low risk services to the smaller capacity bin and high risk services to the higher capacity bin. We gave an intuitive explanation to this phenomenon in Section 3. We prove Theorem 1 in Appendix E.

**Theorem 1.** The optimal fractional point lies on the bottom sorted path. The optimal fractional solution splits at most one service between two bins.

### 6 Simulation results

In this section we present our simulation results for the two bin and k bin cases. We run our simulation on bins with equal capacity. We compare the sorting algorithms for SP-MED, SP-MWOP and SP-MOP to an algorithm we call BM (Balanced Mean). The algorithm goes through the list, item by item, and allocates each item to the bin which is less occupied, which is a natural benchmark and also much better than other naive solutions like first-fit and first-fit decreasing.

We show simulation results on synthetic data. We first generate our stochastic input \(\{\bar{\mu}^{(i)}, \bar{\sigma}^{(i)}\}\) for \(n = 100\) and for \(n = 500\). Our sample space is a mixture of three populations: all items have the same mean (we fixed it at \(\mu^{(i)} = 500\)) but 50% had standard deviation picked uniformly from \([0, 0.1 \cdot \bar{\mu}^{(i)}]\), 25% had standard deviation picked uniformly from \([0.1 \cdot \bar{\mu}^{(i)}, 0.5 \cdot \bar{\mu}^{(i)}]\) and 25% had standard deviation picked uniformly from \([0.5 \cdot \bar{\mu}^{(i)}, 0.9 \cdot \bar{\mu}^{(i)}]\).

We then randomly generated 500 sample values \(x_i^{(l)}\) for each \(1 \leq i \leq n\) and \(1 \leq l \leq 500\) using the normal distribution \(N[\bar{\mu}^{(i)}, \bar{\sigma}^{(i)}]\) and from this we inferred parameters \(\mu^{(i)}, V^{(i)}\), best explaining the sample as a normal distribution. Both

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3 At first, we also wanted to compare our algorithm with variants of the algorithms considered in [31] for the SBP problem. In both papers, the authors consider the algorithms First Fit and First Fit Decreasing [3] with item size equal to the effective size, which is the mean value of the item plus an extra value that guarantees an overflow probability be at most some given value \(p\). Their algorithm chooses an existing bin when possible, and otherwise opens a new bin. However, when the number of bins is fixed in advance, taking effective size rather than size does not change much. For a new item (regardless of its size or effective size) we keep choosing the bin that is less occupied, but this time we measure occupancy with respect to effective size rather than size. Thus, if elements come in a random order, the net outcome of this is that the two bins are almost balanced and a new item is placed in each bin with almost equal probability.
the sorting algorithm and the BM algorithm got as input \( \{ (\mu^{(i)}, V^{(i)}) \}_{i=1}^{n} \), as well as \( c_1 = \ldots = c_k = \frac{c}{\mu} \) and output their partition.

To check the suggested partitions, we viewed each sample \( x_{l}^{(i)} \) as representing an item instantiation in a different time slot. We then computed the cost function. For example, for SP-MED, the deviation value for bin \( j \) at time slot \( l \) is: \( \max \left\{ 0, 100 \frac{\sum_{i \in S_{j}} x_{l}^{(i)} - c_j}{\sum_{i=1}^{n} \mu^{(i)}} \right\} \), i.e., the deviation is measured as a percent of the total mean value \( \mu \). We generated 20 such lists and calculated the average cost for these 20 input lists for each algorithm. We run the experiment for different values of \( c \).

Figure 2 shows the 2 bins average cost of both algorithms for SP-MED, SP-MWOP and SP-MOP as a function of \( \frac{c}{\mu} \). As expected, the average cost decreases as the value \( \frac{c}{\mu} \) increases, i.e. as the total spare capacity increases. We also see that the sorting algorithm out-performs BM. Figure 3 shows the average cost of the BM algorithm divided by the average cost of the sorting algorithm for the three problems, again as a function of \( \frac{c}{\mu} \).

Figure 4 shows the 4 bins average cost and cost ratio of both algorithms for SP-MED as a function of \( \frac{c}{\mu} \) for \( n = 100 \) and \( n = 500 \). As we saw in the two bins case, the average cost decreases as the value \( \frac{c}{\mu} \) increases. We can also see that the average deviation for \( n = 100 \) is higher than the deviation for \( n = 500 \). However, the ratio between the BM algorithm average cost and the sorting algorithms average cost for \( n = 100 \) is lower than the ratio we get for \( n = 500 \). Still, in both \( n \) values, the sorting algorithm out-performs the BM.
Fig. 4. The left figure depicts the average cost of the sorting algorithm and the BM algorithm for SP-MED with four bins, both for $n = 100$ and for $n = 500$. The right figure depicts the average cost of the BM algorithm divided by the average cost of the sorting algorithm for $n = 100$ and for $n = 500$. The $x$ axis measures $\frac{c}{\mu}$.

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A The general case

We now analyze the general $k$ bin case using the results we have obtained for the two-bin case. We first show the optimal factional solution is obtained by first sorting the bins by their capacities and the services by their VMR ratio, and then partitioning the sorted list to $k$ consecutive parts.

Assume the bins are sorted by their capacity, $c_1 \leq c_2 \leq \ldots \leq c_k$. A fractional solution is called sorted if for every $j < j'$ and every two services $i$ and $i'$ such that service $i$ (resp. $i'$) is allocated to bin $j$ (resp. $j'$) it holds that the VMR of service $i$ is at most that of service $i'$. 
**Theorem 2.** The optimal fractional solution is sorted.

**Proof.** Assume there is an optimal solution $S_1, \ldots, S_k$ that is not sorted. I.e., there exist $j < j'$ and $i, i'$ such that service $i$ (resp. $i'$) is allocated to bin $j$ (resp. $j'$) and the VMR of service $i$ is strictly larger than that of service $i'$. By Theorem 1 the sorted fractional solution for the problem defined by bins $j$ and $j'$ is strictly better than the one offered by the solution $S_1, \ldots, S_k$. This means the solution is not optimal for bins $j$ and $j'$ - in contradiction to item (1) in Section 4.1.

The fact that with two bins and different capacities $c_1 < c_2$, low risk services should be allocated to the smaller capacity bin (See Theorem 1 and the discussion before it) implies also that for $k$ bins lower risk services should be allocated to lower capacity bins.

Next we present an algorithmic framework for the problem:

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**The double sorting algorithm framework**

- Sort the bins by their capacity $c_1 \leq c_2 \leq \ldots \leq c_k$.
- Sort the services by their VMR $\frac{v^{(1)}}{\mu^{(1)}} \leq \frac{v^{(2)}}{\mu^{(2)}} \leq \ldots \leq \frac{v^{(n)}}{\mu^{(n)}}$.
- Use a partitioning algorithm to find $k-1$ partition points $\ell_0 = 0 \leq \ell_1 < \ldots \ell_{k-1} \leq \ell_k = n$ and output $S_1, \ldots, S_k$ where $S_j = \{\ell_{j-1} + 1, \ldots, \ell_j\}$.
- Allocate the services in $S_j$ to bin number $j$ (with capacity $c_j$).

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The algorithmic framework is not complete since it does not specify how to find the partition points on the sorted path. With two bins there is only one partition point, and we can check all possible $n - 1$ integral partition points. With $k$ bins there are $\binom{n}{k-1}$ possible partition points, and checking all partition points may be infeasible.

We now describe a dynamic programming algorithm that solves the problem, and works under very general conditions.
Finding an optimal integral partition

For each $1 \leq t \leq \log k$, $1 \leq i < i' \leq n$ and $j = 1, \ldots, k$ keep the two values $\text{Partition}(2^t, i, i', j)$ and $D(2^t, i, i', j)$ that are defined as follows:

- **Base case** $t = 1$: $\text{Partition}(2^1, i, i', j)$ is the best integral partition point for the two bin problem with inputs $X_i, \ldots, X_{i'}$ and capacities $c_j, c_{j+1}$. $D(2, i, i', j)$ is its corresponding expected deviation.

- **Induction step.** Suppose we have built the tables for $t$, we show how to build the tables for $t + 1$. We let $\text{Dev}(2^{t+1}, i, i', j)$ be

  $$\min_{i \leq i'' \leq i'} \{D(2^t, i, i'' - 1, j) + D(2^t, i'', i' + 2^t)\}$$

  and we let $\text{Partition}(2^{t+1}, i, i', j)$ be the partition point that obtains the minimum in the above equation.

The optimal partition to $k$ bins (assuming $k$ is a power of two) can be recovered from the tables. For example for 4 bins $\text{Partition}(4, 1, n, 1)$ returns the middle point $\ell_2$ of the partition, and the partition points within each half are obtained by $\ell_1 = \text{Partition}(2, 1, \ell_2 - 1, 1)$ and $\ell_3 = \text{Partition}(2, \ell_2, n, 3)$.

It can be seen (by a simple induction) that $\text{Partition}(2^t, i, i', j)$ gives the middle point of the optimal integral solution on the sorted path to the problem of dividing $X_i, \ldots, X_{i'}$ to $2^t$ bins with capacities $c_j, c_{j+1}, \ldots, c_{j+2^t-1}$. It can also be verified that the running time of the algorithm is $O(n^3k \log k)$.

We analyze the error the integral solution we find has compared with the optimal fractional solution. The error analysis builds upon the error analysis of the two bin case. Start with an optimal fractional solution $\pi$. By Theorem 2 we know $\pi$ is sorted. $\pi$ defines some sorted fractional solution to the problem defined by the first two bins. By the remark after Theorem 3 we know that one of the integral points to the left or to the right of the fractional solution has a very close deviation to that of $\pi$, and we fix the first partition point accordingly to make it integral. Now assume after fixing $j$ stages we hold some fractional solution $\pi^{(j)}$ that is integral on the first $j$ partition points and possibly fractional on the rest. We look at bins $j + 1$ and $j + 2$ and find an integral partition point (to the left or to the right of the next fractional partition point) that is almost as good as the fractional one. Doing the above for $j = 1, \ldots, k - 1$ we end up with an integral solution whose deviation is close to that of the optimal fractional solution, where the error bound in Theorem 3 is multiplied by a factor of $k - 1$. We remark that this also shows an efficient way of converting an optimal fractional solution to an integral solution close to it.

The algorithm is quite general and works whenever we can solve the $k = 2$ case, and therefore may be seen as a reduction from arbitrary $k$ to the $k =$
The main point we want to stress is that even in this generality the complexity is polynomial. The main disadvantage of the algorithm is that it runs in about \( n^3 \) time. For specific cases (such as SP-MED and SP-MOP and probably other natural variants) the optimal solution may be probably found much faster, and we are currently working on proving the correctness of an almost linear algorithm.

B Three cost functions

B.1 Stochastic Packing with Minimum Expected Deviation (SP-MED)

In Appendix G.1 we prove the expected deviation of bin \( j (j = 1, \ldots, k) \), denoted by \( \text{Dev}_{S_j} \), is

\[
\text{Dev}_{S_j} = \sigma_j [\phi(\Delta_j) - \Delta_j (1 - \Phi(\Delta_j))],
\]

where \( \phi \) is the probability density function (pdf) of the standard normal distribution and \( \Phi \) is its cumulative distribution function (CDF). Denoting \( g(\Delta) = \phi(\Delta) - \Delta (1 - \Phi(\Delta)) \) we see that \( \text{Dev}_{S_j} = \sigma_j g(\Delta_j) \). With two bins \( \text{Dev} \) is a function from \([0, 1]^2\) to \( \mathbb{R} \) and \( \text{Dev}(a, b) = \sigma_1 g(\Delta_1) + \sigma_2 g(\Delta_2) \) where the first bin has mean \( a\mu \) and variance \( bV \), the second bin has mean \( (1 - a)\mu \) and variance \( (1 - b)V \) and \( \sigma_j, \Delta_j \) are defined as above.

Lemma 3. \( \text{Dev} \) respects conditions (1)-(4) of Section 4.1.

Proof. We go over the conditions one by one.

item (1): Let \( S_1, \ldots, S_k \) be an optimal solution. Suppose there are \( j \) and \( j' \) for which \( S_j, S_{j'} \) are not the optimal solution for the two-bin problem defined by the services in \( S_j \cup S_{j'} \) and capacities \( c_j, c_{j'} \). Change the allocation of the solution \( S_1, \ldots, S_k \) on bins \( j \) and \( j' \) to an optimal solution for the two-bin problem. This change improves the expected total deviation of the two bins while not affecting the expected deviation of any other bin. In total we get a better solution, in contradiction to the optimality of \( S_1, \ldots, S_k \).

item (2): In Appendix G.2 we prove \( \text{Dev}(a, b) = \text{Dev}(1 - a - \frac{c_1 - a\mu}{\mu}, 1 - b) \). We remark that for \( c_1 = c_2 \) this simply says we can switch the names of the first and second bin.

item (3): In Appendix G.3 we calculate \( \frac{\partial^2 \text{Dev}}{\partial b^2} = \mu^2 \left[ \frac{\phi(\Delta_2)}{\Delta_2} + \frac{\phi(\Delta_1)}{\Delta_1} \right] \geq 0 \). It follows that for any \( 0 < b < 1 \), \( \text{Dev}(a) \) is convex and has a unique minimum.

The unique point \((m(b), b)\) on the valley is the one where \( \Delta_1 = \Delta_2 \).

item (4): We first explicitly determine what \( \text{Dev} \) restricted to the valley is as a function \( D(b) = \text{Dev}(m(b), b) \) of \( b \). As \( \text{Dev}(a, b) = \sigma_1 g(\Delta_1) + \sigma_2 g(\Delta_2) \) and on the valley \( \Delta_1 = \Delta_2 \) we see that on the valley \( \text{Dev}(a, b) = (\sigma_1 + \sigma_2)g(\Delta_1) \).

However, \( \sigma_1 + \sigma_2 \) also simplifies to \( \frac{c_1 - a\mu}{\Delta_1} + \frac{c_2 - (1-a)\mu}{\Delta_2} = \frac{c - \mu}{\Delta_1} \). Altogether,

---

4 A quick background on the normal distribution is given in Appendix F.
we conclude that on the valley $Dev(a, b) = (c - \mu) \frac{\sigma(\Delta_2)}{\Delta_2}$ is a function of $\Delta_1$ alone.

Now it is a straightforward calculation that $\frac{\partial Dev(\Delta_1)}{\partial \Delta_1} = -(c - \mu) \frac{\sigma(\Delta_1)}{\Delta_1} < 0$. Also $\frac{\partial \Delta_1}{\partial b}$ is negative when $b \leq \frac{1}{2}$ and positive when $b \geq \frac{1}{2}$ (see Appendix G.3 and G.3). As $\frac{\partial D}{\partial b} = \frac{\partial Dev}{\partial \Delta_1} \cdot \frac{\partial \Delta_1}{\partial b}$, we see that $D(b)$ is increasing for $b \leq \frac{1}{2}$ and decreasing for $b \geq \frac{1}{2}$ as claimed. The saddle point $(a = \frac{1}{2} - \frac{c_2 - c_1}{2\sigma}, b = \frac{1}{2})$ lies on the valley and is a maximum point for $Dev$ restricted to the valley.

We remark that we could simplify the proof by using Lagrange multipliers (as we do in Lemma 8). However, since here it is easy to explicitly find $Dev$ restricted to the valley we prefer the explicit solution. Later, we will not be able to explicitly find the restriction to the valley and we use instead Lagrange multipliers that solves the problem with an implicit description of the valley.

**Lemma 4.** The difference between the expected deviation in the integral point found by the sorting algorithm and the optimal integral (or fractional) point for SP-MED is at most $\frac{1}{\sqrt{2\pi e}} \cdot \frac{1}{a} \cdot \frac{\sigma}{b} \cdot \mu$. In particular, when $L = o(n)$ and $\alpha$ is a constant, the error is $o(\mu)$.

**Proof.** We know from Theorem 3 that the difference is at most

$$\min \{|\nabla D(\xi_1)|, |\nabla D(\xi_2)|\} \frac{L}{n},$$

where $\xi_1 \in [O_1, OPT]$, $\xi_2 \in [OPT, O_2]$ and $O_1$ and $O_2$ are the two points on the bottom sorted path between which $OPT$ lies. Using the partial derivatives calculated in Appendix G.3 we see that

$$|\nabla (\sigma g)(\Delta_2)| \leq |\mu (1 - \Phi(\Delta_2))| + \frac{\sigma}{2\sqrt{1 - b}} \phi(\Delta_2) \leq \mu + \frac{\sigma}{2\sqrt{1 - b}} \phi(\Delta_2).$$

Moreover, $\frac{\sigma}{2\sqrt{1 - b}} \phi(\Delta_2) = \frac{\sigma}{2\sqrt{1 - b}} \frac{1}{\Delta_2} \Delta_2 \phi(\Delta_2)$ and a simple calculation shows that the function $\Delta \phi(\Delta)$ maximizes at $\Delta = 1$ with value at most $\frac{1}{\sqrt{2\pi e}}$. By our assumption that $\Delta_j \geq 0$ for every $j$, we get that

$$\frac{\sigma}{2\sqrt{1 - b}} \phi(\Delta_2) \leq \frac{\sigma}{2\sqrt{1 - b}} \frac{\sigma\sqrt{1 - b}}{c_2 - (1 - a)\mu} \frac{1}{\sqrt{2\pi e}} \leq \frac{V}{2\sqrt{2\pi e}} \frac{1}{c_2 - (1 - a)\mu}.$$

Applying the same argument on $O_2$ shows the error can also be bounded by $\frac{V}{2\sqrt{2\pi e}} \frac{1}{c_2 - (1 - a)\mu}$.

However, $(c_1 - a\mu) + (c_2 - (1 - a)\mu) = c - \mu$ which is the total spare capacity, and at least one of the bins takes spare capacity that is at least half of that, namely $\frac{c - \mu}{2}$. Since the error is bounded by either term, we can choose the one where the spare capacity is at least $\frac{c - \mu}{2}$ and we therefore see that the error is at most $\frac{V}{2\sqrt{2\pi e}} \frac{2}{c - \mu}$. Since we assume $c - \mu \geq 2\mu$ for some constant $\alpha > 0$, the error is at most $\frac{V}{2\sqrt{2\pi e}} \frac{1}{\alpha \mu}$. As we assume $V \leq \mu^2$, $\frac{V}{\alpha} \leq \mu$ which completes the proof.
This shows the approximation factor goes to 1 and linearly (in the number of services) fast. Thus, from a practical point of view the theorem is very satisfying.

**B.2 Stochastic Packing with Min Worst Overflow Probability (SP-MWOP)**

The SP-MWOP problem gets as input integers \( k \) and \( n \), specifying the number of bins and services, integers \( c_1, \ldots, c_k \), specifying the bin capacities and values \( \{ (\mu^{(i)}, V^{(i)}) \}_{i=1}^n \), specifying that the demand distribution \( X^{(i)} \) of service \( i \) is normal with mean \( \mu^{(i)} \) and variance \( V^{(i)} \). A solution to the problem is a partition of \([n]\) to \( k \) disjoint sets \( S_1, \ldots, S_k \subseteq [n] \) that minimizes the worst overflow probability.

The SP-MWOP problem is a natural variant of SBP. For a given partition let \( OFP_j \) (for \( j = 1, \ldots, k \)) denote the overflow probability of bin \( j \). Let \( WOFP \) denote the worst overflow probability, i.e., \( WOFP = \max_{j=1}^{k} \{ OFP_j \} \). In the SBP problem we are given \( n \) normal distributions and wish to pack them into few bins such that the \( OFP \leq k \) for some given parameter \( p \). Suppose we solve the SBP problem for a given \( p \) and know that \( k \) bins suffice. We now ask ourselves what is the minimal \( WOFP \) achieved with the \( k \) bins (this probability is clearly at most \( p \) but can also be significantly smaller). We also ask what is the partition that achieves this minimal worst overflow probability. The problem SP-MOP does exactly that.

In Appendix H.1 we prove the overflow probability of bin \( j \) (for \( j = 1, \ldots, k \)), is \( OFP_j = 1 - \Phi(\Delta_j) \) where \( \Delta_j = \frac{c_j - \mu_j}{\sigma_j} \). Thus,

\[
WOFP = \max_{j=1}^{k} \{ 1 - \Phi(\Delta_j) \}.
\]

With two bins \( WOFP \) is a function from \([0,1]^2\) to \( \mathbb{R} \) and \( WOFP(a,b) = \max \{ 1 - \Phi(\Delta_1), 1 - \Phi(\Delta_2) \} \) where the first bin has mean \( a\mu \) and variance \( bV \), the second bin has mean \( (1-a)\mu \) and variance \( (1-b)V \) and \( \sigma_j, \Delta_j \) are defined as above.

**Lemma 5.** \( WOFP \) respects conditions (1)-(4) of Section 4.1

**Proof.** We go over the conditions one by one.

**item (1):** Let \( S_1, \ldots, S_k \) be an optimal solution. Suppose there are \( j \) and \( j' \) for which \( S_j, S_j' \) are not the optimal solution for the two-bin problem defined by the services in \( S_j \cup S_j' \) and capacities \( c_j, c_{j'} \). Change the allocation of the solution \( S_1, \ldots, S_k \) on bins \( j \) and \( j' \) to an optimal solution for the two-bin problem. This change improves the worst overflow probability of the two bins while not affecting the overflow probability of any other bin. In total we get a better solution, in contradiction to the optimality of \( S_1, \ldots, S_k \).

**item (2):** The same proof as in Appendix G.2 shows \( WOFP(a,b) = WOFP(1-a, 1-b) \).
item (3): Fix \(b\). Denote \(OFP_1(a, b) = OFP_{S_1}(a\mu, bV) = 1 - \Phi(\Delta_1)\). It is a simple calculation that \(\frac{\partial OFP_1}{\partial a}(a, b) = \frac{\mu}{b\sigma} \cdot \phi(\Delta_1) > 0\). Similarly, if \(OFP_2(a, b)\) denotes the overflow probability in the second bin when the first bin has total mean \(a\mu\) and total variance \(bV\), then \(\frac{\partial OFP_2}{\partial a} = \frac{-\mu}{\sqrt{1-b^2}} \cdot \phi(\Delta_2) < 0\). Thus, \(OFP_1\) is monotonically increasing in \(a\) and \(OFP_2\) is monotonically decreasing in \(a\), and therefore there is a unique minimum for \(OFP(a, b)\) (when \(b\) is fixed and \(a\) is free) that is obtained when \(OFP_1(a, b) = OFP_2(a, b)\), i.e., when \(\Delta_1 = \Delta_2\).

item (4): We first explicitly determine what \(WOFP\) restricted to the valley is as a function \(D(b) = WOFP(m(b), b)\) of \(b\). From before we know that on the valley \(\Delta_1 = \Delta\). Therefore, following the same reasoning as in the SP-MED case,

\[
D(b) = \frac{c - \mu}{\sigma} \frac{1}{\sqrt{b + \sqrt{1 - b}}}. 
\]

It follows that \(D(b)\) is monotonically decreasing in \(b\) for \(b \leq \frac{1}{2}\) and increasing otherwise. The maximal point is obtained in the saddle point that is the center of the symmetry.

**Lemma 6.** The difference between minimal worst overflow probability in the integral point found by the sorting algorithm and the optimal integral (or fractional) point for SP-MWOP is at most \(O(\frac{\log L}{\alpha n})\). In particular, when \(L = o(n)\) and \(\alpha\) is a constant, the difference is \(o(1)\).

**Proof.** We know from Theorem 3 that the difference is at most

\[
\min \{ |\nabla D(\xi_1)|, |\nabla D(\xi_2)| \} \frac{L}{n},
\]

where \(\xi_1 = (a_1, b_1) \in [O_1, OPT_j]\), \(\xi_2 = (a_2, b_2) \in [OPT_j, O_2]\) and \(O_1\) and \(O_2\) are the two points on the bottom sorted path between which \(OPT_j\) lies, and notice that even though \(WOFP\) is not differentiable when \(\Delta_1 = \Delta_2\), it is differentiable everywhere else. We now use the partial derivatives calculated in Appendix G.3. We also replace \(\frac{\partial \phi(\Delta_2)}{\partial c_2} \frac{\Delta_2 \phi(\Delta_2)}{c_2 - (1 - \alpha) \mu}\) and similarly for the other term. We get:

\[
\min \left\{ \left| \Delta_2 \phi(\Delta_2) \right| \cdot \left| \left( \frac{c_2 - (1 - a_1) \mu}{2(1 - b_1)} \right) \cdot \frac{1}{2b_2} \right|, \left| \Delta_1 \phi(\Delta_1) \right| \cdot \left| \left( \frac{c_1 - a_2 \mu}{c_1 - a_2 \mu} \right) \cdot \frac{1}{2b_2} \right| \right\} \frac{L}{n}.
\]

\(\Delta \phi(\Delta)\) maximizes at \(\Delta = 1\) with value at most \(\frac{1}{\sqrt{2\pi\sigma}}\). Also, \((c_1 - a_2 \mu) + (c_2 - (1 - a_1) \mu) = c - \mu - (a_2 - a_1) \mu \geq c - \mu \frac{1}{2} \geq \frac{\alpha}{2} \mu\), where \(\alpha\) is the total space capacity, and a constant by our assumption. Hence, at least one of the terms \(\frac{\mu}{c_2 - (1 - a_1) \mu}, \frac{\mu}{c_1 - a_2 \mu}\) is at most \(\frac{4}{\alpha}\). Also, for that term, the spare capacity is maximal, and therefore it takes at least half of the variance. Altogether, the difference is at most \(O(\frac{\log L}{\alpha n})\) which completes the proof.
B.3 Stochastic Packing with Minimum Overflow Probability (SP-MOP)

The SP-MOP problem gets as input integers \(k\) and \(n\), specifying the number of bins and services, integers \(c_1, \ldots, c_k\), specifying the bin capacities and values \(\{(\mu^{(i)}, V^{(i)})\}_{i=1}^n\), specifying that the demand distribution \(X^{(i)}\) of service \(i\) is normal with mean \(\mu^{(i)}\) and variance \(V^{(i)}\). A solution to the problem is a partition of \([n]\) to \(k\) disjoint sets \(S_1, \ldots, S_k \subseteq [n]\) that minimizes the overflow probability.

The total overflow probability is \(OFP = 1 - \prod_{j=1}^k (1 - OFP_j)\) where as we computed before (in Appendix H.1) \(OFP_j = 1 - \Phi(\Delta_j)\). With two bins \(OFP\) is a function from \([0, 1]^2\) to \(\mathbb{R}\) and \(OFP(a, b) = 1 - \Phi(\Delta_1)\Phi(\Delta_2)\) where the first bin has mean \(a\mu\) and variance \(bV\), the second bin has mean \((1 - a)\mu\) and variance \((1 - b)V\) and \(\sigma_j, \Delta_j\) are defined as above.

**Lemma 7.** \(OFP\) respects conditions (1)-(3) of Section 4.1.

**Proof.** We go over the conditions one by one.

**item (1):** Let \(S_1, \ldots, S_k\) be an optimal solution. Suppose there are \(j\) and \(j'\) for which \(S_j, S_{j'}\) are not the optimal solution for the two-bin problem defined by the services in \(S_j \cup S_{j'}\) and capacities \(c_j, c_{j'}\). Change the allocation of the solution \(S_1, \ldots, S_k\) on bins \(j\) and \(j'\) to an optimal solution for the two bin problem. This change improves \((1 - OFP_j)(1 - OFP_{j'})\) while not affecting the overflow probability of any other bin. In total we get a better solution, in contradiction to the optimality of \(S_1, \ldots, S_k\).

**item (2):** The same proof as in Appendix G.2 shows \(OFP(a, b) = OFP(1 - a - \frac{\epsilon_2 - \epsilon_1}{a\mu}, 1 - b)\).

**item (3):** Fix \(b\).

\[
\frac{\partial^2 OFP}{\partial^2 a} = \mu^2 \left[ \frac{\Delta_1}{\sigma_1^2} \phi(\Delta_1) \Phi(\Delta_2) + \frac{\Delta_2}{\sigma_2^2} \phi(\Delta_2) \Phi(\Delta_1) + \frac{2}{\sigma_1 \sigma_2} \phi(\Delta_1) \phi(\Delta_2) \right].
\]

In particular \(\frac{\partial^2 OFP}{\partial a^2} > 0\) and for every fixed \(b\), \(OFP(a, b)\) is convex over \(a \in [0..1]\) and has a unique minimum \(a = m(b)\).

Proving there exists a unique maximum over the valley is more challenging. We wish to find all extremum points of the cost function \(D\) (OFP in our case) over the valley \(\{(m(b), b)\}\). Define \(m(a, b) = a - m(b)\). Then we wish to maximize \(D(a, b)\) subject to \(m(a, b) = 0\). Before, we computed the restriction \(D(b)\) of the cost function over the valley and found its extremum points. However, here we do not know how to explicitly find \(D(b)\). Instead, we use Lagrange multipliers that allow working with the implicit form \(m(a, b) = 0\) without explicitly finding \(D(b)\). We prove a general result:

**Lemma 8.** If a cost function \(D\) is differentiable twice over \([0, 1] \times [0, 1]\) and respects conditions (1)-(3) of Section 4.1 and if for every \(b \in [0, 1]\), \(D(a, b)\) is strictly convex for \(a \in [0, 1]\), then any extremum point of \(D\) over the valley must have zero gradient at \(Q\), i.e., \(\nabla(D)(Q) = 0\).
Proof. The valley is defined by the equation \( m(a, b) = \frac{\partial D}{\partial a}(a, b) \). Using Lagrange multipliers we find that at any extremum point \( Q \) of \( D \) over the valley,

\[
\nabla(D)(Q) = \lambda \nabla m(Q).
\]

For some real value \( \lambda \). However,

\[
\nabla(D)(Q) = (\frac{\partial D}{\partial a}(Q), \frac{\partial D}{\partial b}(Q)) = (0, \frac{\partial D}{\partial b}(Q)),
\]

because \( Q \) is on the valley. H

Also, as for every fixed \( b \), \( D(a, b) \) is strictly convex, we have that

\[
\frac{\partial m}{\partial a}(Q) = \frac{\partial^2 D}{\partial^2 a}(Q) > 0.
\]

Therefore, we conclude that \( \lambda = 0 \). This implies that \( \frac{\partial D}{\partial b}(Q) = 0 \). Hence,

\[
\nabla(D)(Q) = 0.
\]

With that we prove:

Lemma 9. OFP respects condition \([4]\) of Section \([4.1]\).

Proof. Let \( Q = (a, b) \) be an extremum point of OFP over the valley. We look at the range \( b \in [0..\frac{1}{2}) \), \( b \geq \frac{1}{2} \) is obtained by the symmetry. Then, by Lemma \([8]\)

\[
\phi(\Delta_1) \Phi(\Delta_2) \frac{\partial \Delta_1}{\partial a} = -\phi(\Delta_2) \Phi(\Delta_1) \frac{\partial \Delta_2}{\partial a}, \text{ and}
\]

\[
\phi(\Delta_1) \Phi(\Delta_2) \frac{\partial \Delta_1}{\partial b} = -\phi(\Delta_2) \Phi(\Delta_1) \frac{\partial \Delta_2}{\partial b}.
\]

Dividing the two equations we get

\[
\frac{\partial \Delta_1}{\partial a} \frac{\partial \Delta_2}{\partial b} = \frac{\partial \Delta_2}{\partial a} \frac{\partial \Delta_1}{\partial b}.
\]

Plugging the partial derivatives of \( \Delta_i \) by \( a \) and \( b \), we get the equation

\[
\frac{\Delta_1}{\Delta_2} = \sqrt{\frac{b}{1-b}}.
\]

As \( b \leq \frac{1}{2}, b < 1 - b \) and we conclude that at \( Q \) \( \Delta_1 < \Delta_2 \). However, using the log-concavity of the normal c.d.f function \( \Phi \) we prove in Appendix \([1]\) that:

Lemma 10. \( \frac{\partial OFP}{\partial a} = 0 \) at a point \( Q = (a, b) \) with \( b \leq \frac{1}{2} \) implies \( \Delta_1 \geq \Delta_2 \).
Together, this implies that the only extremum point of \( OFP \) over the valley is at \( b = \frac{1}{2} \). However, at \( b = 0 \), the best is to fill the largest bin to full capacity with variance 0, and thus, \( OFP(m(0),0) = 1 - \Phi(\Delta) \) where \( \Delta = \frac{c - \mu}{\sigma} \). On the other hand, at \( b = \frac{1}{2} \), \( OFP(a = m(\frac{1}{2}), \frac{1}{2}) = 1 - \Phi(\frac{c - \mu}{\sqrt{2} \sigma})\Phi(\frac{c - (1 - a)\mu}{\sqrt{2} \sigma}) \). As \((c_1 - a\mu) + (c_2 - (1 - a)\mu) = c - \mu\), either \( c_1 - a\mu \) or \( c_2 - (1 - a)\mu \) is at most \( \frac{c - \mu}{2} \) and therefore \( \Phi(\frac{c - \mu}{\sigma})\Phi(\frac{c - (1 - a)\mu}{\sigma}) \leq \Phi(\frac{\sqrt{2}(c - \mu)}{2\sigma}) = \Phi(\frac{c - \mu}{\sqrt{2}\sigma}) \leq \Phi(\frac{c - \mu}{\sigma}) \).

We conclude that \( OFP(a, \frac{1}{2}) \geq OFP(m(0),0) \) and there is a unique maximum point on the valley and it is obtained at \( b = \frac{1}{2} \).

### C Bounding the approximation error of the sorting algorithm

We assume that no input service is too dominant. Recall that we represent service \( i \) with the point \( P^{(i)} = (a^{(i)}, b^{(i)}) \) and \( P^{(1)} + P^{(2)} + \ldots + P^{(n)} = (1, 1) \). Thus, \( \sum_i |P^{(i)}| \geq |(1, 1)| = \sqrt{2} \) (by the triangle inequality) and \( \sum_i |P^{(i)}| \leq 2 \) (because the length of the longest increasing path from \((0, 0)\) to \((1, 1)\), is obtained by the path going from \((0, 0)\) to \((1, 0)\) and then to \((1, 1)\)). Hence, the average length of an input point \( P^{(i)} \) is somewhere between \( \frac{\sqrt{2}}{n} \) and \( \frac{2}{n} \). Our assumption states that no element takes more than \( L \) times its "fair" share, i.e., that for every \( i \), \(|P^{(i)}| \leq \frac{L}{n}\). With that we prove:

**Theorem 3.** Let \( OPT_f \) be the fractional optimal solution. If \( D \) is differentiable, the difference between the cost on the integral point found by the sorting algorithm and the cost on the optimal integral (or fractional) point is at most \( \min \{|\nabla D(\xi_1)|, |\nabla D(\xi_2)|\} \cdot \frac{\epsilon}{n} \), where \( \xi_1 \in [O_1, OPT_f] \), \( \xi_2 \in [OPT_f, O_2] \) and \( O_1 \) and \( O_2 \) are the two points on the bottom sorted path between which \( OPT_f \) lies.

**Proof.** Suppose we run the sorting algorithm on some input. Let \( OPT_{int} \) be the integral optimal solution, \( OPT_f \) the fractional optimal solution and \( OPT_{sort} \) the integral point the sorting algorithm finds on the bottom sorted path. We wish to bound \( D(OPT_{sort}) - D(OPT_{int}) \) and clearly it is at most \( D(OPT_{sort}) - D(OPT_f) \). We now look at the two points \( O_1 \) and \( O_2 \) on the bottom sorted path between which \( OPT_f \) lies (and notice that as far as we know it is possible that \( OPT_{sort} \) is none of these points). Since \( D(OPT_f) \leq D(OPT_{sort}) \leq D(O_1) \) and \( D(OPT_f) \leq D(OPT_{sort}) \leq D(O_2) \) the error the sorting algorithm makes is at most \( \min \{D(O_1) - D(OPT_f), D(O_2) - D(OPT_f)\} \).

We now apply the mean value theorem and use our assumption that for every \( i \), \(|P^{(i)}| \leq \frac{L}{n}\).

We define a new system constant, relative spare capacity, denoted by \( \alpha \) which equals \( \frac{\epsilon\mu}{n} \), i.e., it expresses the spare capacity as a fraction of the total mean. We assume that the system has some constant (possibly small) relative spare capacity. Also, we only consider solutions where each bin is allocated services with
total mean not exceeding its capacity. Equivalently, we only consider solutions where \( \Delta_j \geq 0 \) for every \( 1 \leq j \leq k \). We will later see that under these conditions the sorting algorithm solves all three cost functions with a small error going fast to zero with \( n \).

We remark that in fact the proof shows something stronger: the deviation of any (not necessarily optimal) fractional solution on the bottom sorted path, is close to the deviation of the integral solution to the left or to the right of it on the bottom sorted path.

### D Proof of Lemma 1

**Proof.** We introduce some notation. Let \( \tau = \tau_1, \ldots, \tau_n \) be a sequence of \( n \) elements that is a reordering of \( \{1, \ldots, n\} \). We associate with \( \tau \) the \( n \) partial sums \( P_\tau^{[1]}, \ldots, P_\tau^{[n]} \) where \( P_\tau^{[i]} = \sum_{j=1}^{i} P(\tau_j) \), i.e., \( P_\tau^{[i]} \) is the integral point that is the sum of the first \( i \) points according to the sequence \( \tau \). We also define \( P_\tau^{[0]} = (0, 0) \) and \( P_\tau^{[n]} = (1, 1) \). The curve connecting \( \tau \) is the curve that is formed by connecting \( P_\tau^{[i]} \) and \( P_\tau^{[i+1]} \) with a line, for \( i = 0, \ldots, n-1 \).

Assume that in the sequence \( \tau = \tau_1, \ldots, \tau_n \) there is some index \( i \) such that the VMR of \( P(\tau_i) \) is larger than the VMR of \( P(\tau_{i+1}) \). Consider the sequence \( \tau' = \tau_1, \ldots, \tau_{i-1}, \tau_{i+1}, \tau_i, \tau_{i+2}, \ldots, \tau_n \). We claim that the curve connecting \( \tau' \) lies beneath the curve connecting \( \tau \). To see that notice that both curves are the same up to the point \( P_\tau^{[i-1]} \). There, the two paths split. \( \tau \) adds \( P(\tau_i) \) and then \( P(\tau_{i+1}) \) while \( \tau' \) first adds \( P(\tau_{i+1}) \) and then \( P(\tau_i) \). Then the two curves coincide and overlap all the way to \((1, 1)\). In the section where the two paths differ, the two different paths form a parallelogram with \( P(\tau_i) \) and \( P(\tau_{i+1}) \) as two neighboring edges of the parallelogram. As the angle \( P(\tau_{i+1}) \) has with the \( a \) axis is smaller than the angle \( P(\tau_i) \) has with the \( a \) axis, the curve connecting \( \tau' \) goes beneath that of \( \tau \).

To finish the argument, let \( P_I \) be an arbitrary integral point for some \( I \subseteq [n] \). Look at the sequence \( \tau \) that starts with the elements of \( I \) followed by the elements of \([n] \setminus I\) in an arbitrary order. Notice that \( P_I \) lies on the curve connecting \( \tau \). Now run a bubble sort on \( \tau \), each time ordering a pair of elements by their VMR. Notice that the process terminates with the sequence that sorts the elements by their VMR and the curve connecting the final sequence is the bottom sorted path. Thus, we see that the bottom sorted path lies beneath the curve connecting \( \tau \), and in particular \( P_I \) lies above the bottom sorted path. A similar argument shows \( P_I \) lies underneath the upper sorted path.

### E Proof of Theorem 1: The optimal solution is on the bottom sorted path

**Proof.** Consider an arbitrary fractional point \((a_0, b_0)\) lying strictly inside the polygon confined by the upper and bottom sorted paths. If \( b_0 \leq \frac{1}{2} \), then by
keeping $b = b_0$ constant and changing $a$ till it reaches the valley we strictly decrease cost (because $D$ is strictly monotone in this range). Now, when changing $a$ we either hit the bottom sorted path or the valley. If we hit the bottom sorted path, we found a point on the bottom sorted path with less cost and we are done. If we hit the valley, we can go down the valley until we hit the bottom sorted path and again we are done (as $D$ is strictly monotone on the valley).

If $b_0 \geq \frac{1}{2}$ we recall two facts that we already know:

- The point $(1 - a_0, 1 - b_0) = \varphi(a_0, b_0)$ is fractional (since $(a_0, b_0)$ is fractional and $\varphi$ maps fractional points to fractional points), and,
- By the reflection symmetry we know that $D(a_0, b_0) = D(1 - a_0 - \zeta, 1 - b_0)$ where $\zeta = \frac{c_2 - c_1}{\mu} \geq 0$.

Now, $(1 - a_0 - \zeta, 1 - b_0)$ has $b$ coordinate that is at most $\frac{1}{2}$. Also $(1 - a_0 - \zeta, 1 - b_0)$ lies to the left of the fractional point $(1 - a_0, 1 - b_0)$ (since $\zeta > 0$) and therefore it lies above the bottom sorted path. We therefore see that the point $(a_0, b_0)$ has a corresponding fractional point with the same cost and with $b$ coordinate at most $\frac{1}{2}$. Applying the argument that appears in the first paragraph of the proof we conclude that there exists some point on the bottom sorted path with less cost, and conclude the proof.

### F Standard Normal Distribution

The probability density function of the standard normal distribution (that has mean 0 and variance 1) is $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ and the cumulative distribution function is $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$. Clearly, $\Phi'(x) = \phi(x)$. Also, $\Phi'(x) = -\frac{x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = -x\phi(x)$. The second derivative is $\Phi''(x) = -\phi(x) + x^2\phi(x) = (x^2 - 1)\phi(x)$.

### G SP-MED

**G.1 Expected Deviation of a single bin**

By definition the expected deviation of a single bin is $Dev_{S_j} = \frac{1}{\sigma_j \sqrt{2\pi}} \int_{c_j}^{\infty} (x - c_j) e^{-\frac{(x-c_j)^2}{2\sigma_j^2}} dx$. Doing the variable change $t = \frac{x - \mu_j}{\sigma_j}$ and then the variable change $y = \frac{-t^2}{2}$ we get:

$$Dev_{S_j} = (\mu_j - c_j)[1 - \phi(\frac{c_j - \mu_j}{\sigma_j})] - \frac{\sigma_j}{\sqrt{2\pi}} \int_{\frac{-\infty}{2(c_j - \mu_j)^2}}^{-\infty} e^{y} dy$$

$$= -\sigma_j \Delta_j [1 - \phi(\Delta_j)] + \frac{\sigma_j}{\sqrt{2\pi}} e^{-\Delta_j^2} = \sigma_j [\phi(\Delta_j) - \Delta_j (1 - \Phi(\Delta_j))].$$

Denoting $g(\Delta) = \phi(\Delta) - \Delta [1 - \Phi(\Delta)]$ we see that $Dev_{S_j} = \sigma_j g(\Delta_j)$. 
G.2 Symmetry

Claim. \( \text{Dev}(a, b) = \text{Dev}(1 - a - \frac{c_2 - c_1}{\mu}, 1 - b). \)

Proof. Let us define \( \sigma_1(b) = \sqrt{b} \sigma, \sigma_2(b) = \sqrt{\frac{1}{b}} \sigma, \Delta_1(a, b) = \frac{c_1 - \mu}{\sigma_1(b)} \) and \( \Delta_2(a, b) = \frac{c_2 - (1 - \omega - \phi)}{\sigma_2(b)} \). We know that \( \text{Dev}(a, b) = \sigma_1(b)g(\Delta_1(a, b) + \sigma_2(b)g(\Delta_2(a, b)). \)

To prove the claim it is enough to show that the following four equations hold:

\[
\sigma_1(b) = \sigma_2(1 - b), \quad \sigma_2(b) = \sigma_1(1 - b), \quad \Delta_1(a, b) = \Delta_2(1 - a + \frac{c_1 - c_2}{\mu}, 1 - b) \quad \text{and} \quad \Delta_2(a, b) = \Delta_1(1 - a + \frac{c_2 - c_1}{\mu}, 1 - b).
\]

Indeed, \( \sigma_1(1 - b) = \sqrt{1 - b} \sigma = \sigma_2(b) \) and similarly \( \sigma_2(1 - b) = \sigma_1(b) \). Also,

\[
\Delta_2(1 - a - \frac{c_2 - c_1}{\mu}, 1 - b) = \frac{c_2 - (1 - a + \frac{c_1 - c_2}{\mu})\mu}{\sigma_2(1 - b)}
\]

A similar check shows that \( \Delta_1(1 - a - \frac{c_2 - c_1}{\mu}, 1 - b) = \Delta_2(a, b) \).

G.3 The partial derivatives of \( \text{Dev} \)

The system has absolute constants \( \mu \) and \( V, \sigma = \sqrt{V}. \) We let \( \sigma_1 = \sqrt{b} \sigma \) and \( \sigma_2 = \sqrt{\frac{1}{b}} \sigma. \) We let \( \Delta_1 = \frac{c_1 - \mu}{\sigma_1} = \frac{c_1 - \phi}{\sigma_1} \) and \( \Delta_2 = \frac{c_2 - \mu}{\sigma_2} = \frac{c_2 - (1 - \omega - \phi)}{\sigma_2}. \) In this notation, \( \text{Dev}(a, b) = \sigma_1 g(\Delta_1) + \sigma_2 g(\Delta_2). \)

We calculate the \( g \) derivatives. Since \( g(\Delta) = \phi(\Delta) + \Delta \Phi(\Delta) - \Delta \) we have that

\[
g'(\Delta) = \phi'(\Delta) + \phi(\Delta) + \Delta \phi(\Delta) - 1 = -\Delta \phi(\Delta) + \phi(\Delta) + \Delta \phi(\Delta) - 1 = -1 - \Phi(\Delta)
\]

and \( g''(\Delta) = \phi'(\Delta). \) As \( \Phi(\Delta) \leq 1, g'(\Delta) \leq 0 \) and \( g \) is monotonically decreasing. As \( g''(\Delta) > 0, g \) is convex.

Deriving by \( a \) We have: \( \frac{\partial \Delta_1}{\partial a} = -\frac{\mu}{\sigma_1} \) and \( \frac{\partial \Delta_2}{\partial a} = \frac{\mu}{\sigma_2}. \) Also,

\[
\frac{\partial \text{Dev}}{\partial a}(a, b) = \sigma_1 \frac{\partial g(\Delta_1)}{\partial \Delta_1} \frac{\partial \Delta_1}{\partial a} + \sigma_2 \frac{\partial g(\Delta_2)}{\partial \Delta_2} \frac{\partial \Delta_2}{\partial a} = -\sigma_1 \frac{\mu}{\sigma_1} [\phi(\Delta_1) - 1] + \sigma_2 \frac{\mu}{\sigma_2} [\phi(\Delta_2) - 1] = -\sigma_1 [\phi(\Delta_1) - 1] + \sigma_2 [\phi(\Delta_2) - 1] = \mu [\phi(\Delta_2) - \phi(\Delta_1)]
\]

Since \( \Phi \) is monotonic, a zero value is achieved when \( \Delta_1 = \Delta_2. \) Deriving again by \( a: \)

\[
\frac{\partial^2 \text{Dev}}{\partial a^2} = \mu \left[ \frac{\partial^2 \Phi(\Delta_2)}{\partial \Delta_2^2} \frac{\partial \Delta_2}{\partial a} - \frac{\partial^2 \Phi(\Delta_1)}{\partial \Delta_1^2} \frac{\partial \Delta_1}{\partial a} \right] = \mu^2 \left[ \frac{\phi(\Delta_2)}{\sigma_2} + \phi(\Delta_1) \right]
\]

Since \( \frac{\partial^2 \text{Dev}}{\partial a^2} \geq 0, \) it follows that for any \( 0 < b < 1, \) \( \text{Dev}(a) \) is convex and hence \( \Delta_1 = \Delta_2 \) is a minimum point in this range.
Deriving by $b$ We have: $\frac{\partial \sigma_1}{\partial b} = \frac{\sigma_1}{2\sqrt{b}}$ and $\frac{\partial \sigma_2}{\partial b} = -\frac{\sigma_2}{2\sqrt{1-b}}$. Also, $\frac{\partial \Delta_1}{\partial b} = \frac{\Delta_1}{2(1-b)}$. Now,

$$\frac{\partial \text{Dev}}{\partial b} = \frac{\partial \sigma_1}{\partial b} g(\Delta_1) + \sigma_1 \frac{\partial g(\Delta_1)}{\partial \Delta_1} \frac{\partial \Delta_1}{\partial b} + \frac{\partial \sigma_2}{\partial b} g(\Delta_2) + \sigma_2 \frac{\partial g(\Delta_2)}{\partial \Delta_2} \frac{\partial \Delta_2}{\partial b}$$

$$= \frac{\sigma_1 \phi(\Delta_1)}{2\sqrt{b}} - \frac{\phi(\Delta_2)}{\sqrt{1-b}} = V \left[ \frac{\phi(\Delta_1)}{\sigma_1} - \frac{\phi(\Delta_2)}{\sigma_2} \right]$$

Deriving a second time we get:

$$\frac{\partial^2 \text{Dev}}{\partial b^2} = \frac{\sigma_1}{2} \left[ \frac{1}{\sqrt{b}} \frac{\partial^2 \phi(\Delta_1)}{\partial \Delta_1^2} \frac{\partial \Delta_1}{\partial b} - \frac{1}{2b} \phi(\Delta_1) \right] - \frac{1}{\sqrt{1-b}} \frac{1}{2b} \phi(\Delta_2)$$

$$= \frac{\sigma_1}{4} \left[ \frac{\phi(\Delta_1)}{b \sqrt{b}} (\Delta_1^2 - 1) + \frac{\phi(\Delta_2)}{(1-b) \sqrt{1-b}} (\Delta_2^2 - 1) \right]$$

The Mixed Derivative

$$\frac{\partial^2 \text{Dev}}{\partial b \partial a} = \frac{\partial}{\partial b} \frac{\partial \text{Dev}}{\partial a} = \frac{\partial}{\partial b} \left[ \mu [\Phi(\Delta_2) - \Phi(\Delta_1)] \right] = \mu \left[ \frac{\partial \Phi(\Delta_2)}{\partial \Delta_2} \frac{\partial \Delta_2}{\partial b} - \frac{\partial \Phi(\Delta_1)}{\partial \Delta_1} \frac{\partial \Delta_1}{\partial b} \right]$$

$$= \mu \left[ \frac{1}{2b} \Delta_2 \phi(\Delta_1) + \phi(\Delta_1) \frac{\Delta_1}{2b} \right] = \mu \left[ \frac{1}{2b} \Delta_1 \phi(\Delta_1) + \frac{1}{2(1-b)} \Delta_2 \phi(\Delta_2) \right]$$

The saddle point The Hessian of the function $\text{Dev}$ contains the second order partial derivatives of $\text{Dev}$, i.e., it is a $2 \times 2$ matrix $H$,

$$H(a,b) = \begin{pmatrix} \frac{\partial^2 \text{Dev}}{\partial b^2} (a,b) & \frac{\partial^2 \text{Dev}}{\partial b \partial a} (a,b) \\ \frac{\partial^2 \text{Dev}}{\partial a \partial b} (a,b) & \frac{\partial^2 \text{Dev}}{\partial a^2} (a,b) \end{pmatrix}$$

$H$ is symmetric and therefore at any point $(a,b)$ the matrix $H(a,b)$ has two real eigenvalues. Now, assume the first order derivatives of $\text{Dev}$ vanish at some point $(a_0,b_0)$. If the Hessian at that point $(a_0,b_0)$ has negative determinant, then the product of these two eigenvalues is negative, and hence one is positive and the other negative. It then follows that the point $(a_0,b_0)$ is a saddle point of $\text{Dev}$.

We now compute the Hessian at the point $(a_0,b_0) = \left( \frac{1}{2} - \frac{\sigma_1 - \sigma_2}{2}, \frac{1}{2} \right)$ which is the center of the reflection symmetry. We know that at this point $\Delta_1 = \Delta_2$ and $\sigma_1 = \sigma_2$. If then follows that:

$$H(a_0,b_0) = \begin{pmatrix} 2\sqrt{2} \mu \frac{\sigma_1}{2} \frac{\phi(\Delta_1)}{\Delta_1} & 2 \mu \frac{\Delta_1}{\Delta_1} \phi(\Delta_1) \\ 2 \mu \frac{\Delta_1}{\Delta_1} \phi(\Delta_1) & 2 \sqrt{2} \mu \phi(\Delta_1) [\Delta_1^2 - 1] \end{pmatrix}$$

We can now compute the determinant:
\[ \text{det}[H(a_0, b_0)] = \frac{\partial^2 \text{Dev}}{\partial a^2}(a_0, b_0) \frac{\partial^2 \text{Dev}}{\partial b^2}(a_0, b_0) - \left[ \frac{\partial^2 \text{Dev}}{\partial a \partial b}(a_0, b_0) \right]^2 \]
\[ = 2 \sqrt{2} \mu^2 \frac{\phi(\Delta_1)}{\sigma} \sqrt{2} \sigma \phi(\Delta_1) [\Delta^2_1 - 1] - 4 \mu^2 \Delta^2_1 [\phi(\Delta_1)]^2 \]
\[ = -4 \mu^2 [\phi(\Delta_1)]^2 < 0. \]

As \( \text{det}[H(a_0, b_0)] < 0 \) and the partial derivatives of the first order vanish at \((a_0, b_0)\) we conclude that \((a_0, b_0)\) is a saddle point.

**H  SP-MWOP**

**H.1  Expected overflow probability of a single bin**

The overflow probability of bin \( j \), denoted by \( \text{OFP}_{S_j} \), is
\[ \text{OFP}_{S_j}(\mu_j, V_j) = \frac{1}{\sigma_j \sqrt{2 \pi}} \int_\infty^{c_j} e^{-\frac{(x-\mu_j)^2}{2 \sigma_j^2}} \, dx. \]
Substituting \( t = \frac{x-\mu_j}{\sigma_j} \) we get
\[ \text{OFP}_{S_j}(\mu_j, V_j) = \frac{1}{\sqrt{2 \pi}} \int_{c_j-\mu_j}^{\infty} e^{-\frac{t^2}{2}} \, dt = 1 - \Phi\left( \frac{c_j-\mu_j}{\sigma_j} \right) = 1 - \Phi(\Delta_j). \]

**I  SP-MOP**

**I.1  Proof of Lemma 10**

*Proof.* The condition \( \frac{\partial \text{OFP}}{\partial a} = 0 \) is equivalent to
\[ \phi(\Delta_1) \Phi(\Delta_1) = \frac{\sigma_1}{\sigma_2} \cdot \phi(\Delta_2) \Phi(\Delta_2). \]
As \( b < \frac{1}{2}, \) \( b < 1 - b \) and \( \sigma_1 < \sigma_2. \) Hence,
\[ \frac{\phi(\Delta_1)}{\Phi(\Delta_1)} < \frac{\phi(\Delta_2)}{\Phi(\Delta_2)}. \]
Denote \( h(\Delta) = \frac{\phi(\Delta)}{\Phi(\Delta)} \). We will prove that \( h \) is monotone decreasing, and this implies that \( \Delta_1 > \Delta_2 \).
To see that \( h \) is monotone decreasing define \( H(\Delta) = \ln(\Phi(\Delta)) \). Then \( h = H' \). Therefore, \( h' = H'' \). However, \( \Phi \) is log-concave, hence \( H'' < 0 \). We conclude that \( h' < 0 \) and \( h \) is monotone decreasing.
J Unbalancing bin capacities is always better

Suppose we are given a capacity budget $c$ and we have the freedom to choose capacities $c_1, c_2$ that sum up to $c$ for two bins. Which choice is the best? Off-hand, it is possible that for each input there is a different choice of $c_1$ and $c_2$ that minimizes the expected deviation. In contrast, we show that the minimum expected deviation always decreases as the difference $c_2 - c_1$ increases.

Lemma 11. Given a capacity budget $c$, the minimum expected deviation decreases as $c_2 - c_1$ increases. In particular the best choice is having a single bin with capacity $c$ and the worst choice is splitting the capacities evenly between the two bins.

Proof. Recall that $\Delta_1(a, b) = \frac{c_1 - a\mu}{\sigma\sqrt{b}}$ and $\Delta_2(a, b) = \frac{c_2 - (1-a)\mu}{\sigma\sqrt{1-b}}$. Therefore, if we reduce $c_1$ by $\tilde{c}$ and increase $c_2$ by $\tilde{c}$, we get

$$\tilde{\Delta}_1(a, b) = \frac{c_1 - \tilde{c} - a\mu}{\sigma\sqrt{b}} = \frac{c_1 - (a - \frac{\tilde{c}}{\mu})\mu}{\sigma\sqrt{b}} = \Delta_1(a - \frac{\tilde{c}}{\mu}, b).$$

Similarly, $\tilde{\Delta}_2(a, b) = \Delta_2(a - \frac{\tilde{c}}{\mu}, b)$. Let $Dev_{c_1, c_2}(a, b)$ denote the expected deviation with bin capacities $c_1, c_2$. As $Dev(a, b) = \sigma_1(b)g(\Delta_1(a, b)) + \sigma_2(b)g(\Delta_2(a, b))$ we see that

$$Dev_{c_1, c_2}(a, b) = Dev_{c_1, c_2}(a - \frac{\tilde{c}}{\mu}, b),$$

i.e., the graph is shifted left by $\frac{\tilde{c}}{\mu}$.

Notice that the bottom sorted path does not depend on the bin capacities and is the same in both cases. Let $(a, b)$ be the optimal fractional solution for bin capacities $c_1, c_2$. We know that $(a, b)$ is on the bottom sorted path. Let $\tilde{a} = a - \frac{\tilde{c}}{\mu}$. We know that $Dev_{c_1 - \tilde{c}, c_2 + \tilde{c}}(\tilde{a}, \tilde{b}) = Dev_{c_1, c_2}(a, b)$. The point $(\tilde{a}, \tilde{b})$ lies to the left of the bottom sorted path and therefore above it. As the optimal solution for bin capacities $c_1 - \tilde{c}, c_2 + \tilde{c}$ is also on the bottom sorted path and is strictly better than any internal point, we conclude that the expected deviation for bin capacities $c_1 - \tilde{c}, c_2 + \tilde{c}$ is strictly smaller than the expected deviation for bin capacities $c_1, c_2$.

An immediate corollary is the trivial fact that putting all the capacity budget in one bin is best. Obviously, this is not always possible nor desirable, but if there is tolerance in each bin capacity, we recommend minimizing the number of bins.