SU(3)-symmetry breaking effects and mass splitting in scalar and pseudoscalar $D$ mesons from QCD sum rules

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Abstract

Motivated by the similar mass splitting in light-light and heavy-light $J^P = 0^-$ and $J^P = 0^+$ mesons, the SU(3)-symmetry breaking effects splitting the masses in the $0^-$ and $0^+$ channels of the $D$ meson are analyzed in the framework of QCD sum rules with an underlying $c\bar{q}$ structure. We take into account operator mixing to obtain an infrared stable OPE including complete non-perturbative and perturbative $O(m_q)$ corrections to the correlation function. With the same threshold for both channels, the mass splitting arising from the sum-rules has the same behavior as the observed spectrum. In particular, we obtain $m_{D_s} - m_{D_d} \sim 35\text{MeV}$ in the $0^-$ channel and $m_{D_d} - m_{D_s} \sim 12\text{MeV}$ in the $0^+$ channel at a renormalization scale $\mu = 1\text{GeV}$. The splitting can be attributed to the different roles of mass effects and the parity-dependent “force” induced from non-perturbative QCD vacuum. Further analysis shows that due to this “parity-dependent” force it is natural that the mass gap of the two states in the $0^-$ channel is larger than the $0^+$ channel. When we increase the renormalization scale to $\mu = 1.3\text{GeV}$ the splitting remains unchanged which demonstrates a correct scale invariance. Combined with HQET, generalization to other channels of charmed mesons and $b$-systems are briefly discussed.

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I. INTRODUCTION

The SU(3) quark model of hadrons\textsuperscript{1,2} provide an intuitive understanding of hadronic properties. Due to the non-perturbative nature of low energy QCD we have to employ non-perturbative methods in the hadronic sector. The QCD sum rule approach\textsuperscript{3–5} has proven to be a successful non-perturbative method to extract reasonable results in the hadronic sector. Similar success have been achieved with light-cone QCD sum rules\textsuperscript{6, 7} which represent a further-developed version of the original sum-rule approach. We will not dwell on the overall success of the QCD sum rules, but will focus on the light and heavy $J^{P} = 0^{−}$ and $J^{P} = 0^{+}$ channels.

It is observed\textsuperscript{36} that the mass splitting in the $J^{P} = 0^{−}$ channel for the lowest light mesons is in line with their underlying structures from the naive quark model estimates. This splitting of lowest light pseudoscalars with quantum numbers of the $\pi, K, \eta$ and $\eta'$ is well accommodated in QCD sum rules if the instanton effects are appropriately\textsuperscript{8} included since the instanton contributions to the correlation function are different from each member of a multiplet due to its dependence on the isospin and effective mass $m_{q}^{∗1}$. However, the splitting in light scalar meson $J^{P} = 0^{+}$ is the reverse of the naive quark model estimate. If instanton effects are considered, the splitting in the $J^{P} = 0^{+}$ channel above 1GeV (i.e. $f_{0}(1370)$, $a_{0}(1450)$, $K_{0}^{*}(1430)$ and $f_{0}(1500)$) can also be explained within the framework of the QCD sum rule approach\textsuperscript{2}.

If we assume an ordinary light-heavy underlying structure of open-charm systems a similar mass hierarchy as light pseudoscalars can also be observed in the $J^{P} = 0^{−}$ channel of $D$ mesons, the $D_{u}(1869)$ and $D_{s}(1968)$. However, in the $J^{P} = 0^{+}$ channel $D_{s}(2317)$ (which was first discovered by BARBAR Collaboration\textsuperscript{10} and later confirmed by CLEO\textsuperscript{11}) and its isospin partner $D^{*}(2400)^{+}$ (observed by FOCUS Collaboration\textsuperscript{12}) also show similar splitting as light scalars of $J^{P} = 0^{+}$ channel in contradiction to the naive quark model estimate. Among these open-charm systems $D_{s}(2317)$ triggers much attention on its underlying structure. Mass results from Lattice QCD for $D_{s}(2317)$ are larger than the experimental value\textsuperscript{13–15} and the results in Ref.\textsuperscript{13} suggested that $D_{s}(2317)$ might receive a large $DK$ component. The work of Ref.\textsuperscript{16} including this contribution from $DK$ continuum in QCD sum rules based on a $\bar{c}s$ structure

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1 One should not confuse it with the “effective mass” in the constituent quark model.

2 The results show there is large glueball content in $f_{0}(1500)$\textsuperscript{9}.
found that this continuum contribution can significantly lower the mass and decay constant of the $D_s(2317)$. A more complete work on open-charm systems can be found in [17] where the $J^P = 0^-, 0^+, 1^-, 1^+$ channels were studied from a viewpoint of $c\bar{q}$ system where $q = u, d, s$.

On the contrary, in considering the difficulty of ordinary heavy-light structure in decoding the nature of $D_s(2317)$, a four-quark states picture was proposed [19, 20]. Ref. [21] employed the four-quark structure to investigate $D_s(2317)$ using QCD sum rules which suggested that $D_s(2317)$ might be a four-quark states, while the radiative decay of $D_s(2317)$ in light-cone sum rules favors a $\bar{c}s$ structure [22]. All the work both experimentally and theoretically shed some light on the interpretation of $D_s(2317)$.

One might hope that the instanton improved QCD sum rules can also realize the splitting in pseudoscalar and scalar $D$ mesons. But this seems unlikely because there will be rapid suppression of instanton contributions by the $c$-quark effective mass $m_c^*$ and damping exponential factor in the case of the charm quark, and thus the QCD vacuum condensates of various operators are still the dominant non-perturbative corrections. These condensates do not preserve an ideal SU(3) flavor symmetry because of the symmetry-breaking effect of different quark masses.

In the framework of QCD sum rules the SU(3)-symmetry breaking effects present itself mainly from two sides. One is the different values of $\langle \bar{u}u \rangle$, $\langle \bar{d}d \rangle$ and $\langle \bar{s}s \rangle$, and associated mixed condensates. Ref. [23] studied the role of the chiral condensate $\langle \bar{q}q \rangle$ in the mass splitting between the scalar-pseudoscalar $D$ (and $B$) mesons with so-called no-free parameters sum rules in the chiral limit, and recently a parallel analysis was applied to the dependence of heavy-baryons mass splitting on the ratio $\kappa = \langle \bar{s}s \rangle/\langle \bar{d}d \rangle$ [24]. Another symmetry-breaking effect is the perturbative mass correction which is proportional to $m_{q_1}m_{q_2}$ where $q_1$ and $q_2$ labels the quark content of the meson or the current considered. These breaking effects are small for pure-light mesons since the quark condensates, mixing condensates as well as the perturbative mass corrections are always accompanied by the light quark masses and therefore they will be greatly suppressed, especially for $m_u$ and $m_d$. In fact we always use a massless approximation for pure light systems. However, when heavy quarks are involved, the mass effects will be considerable since the large mass of heavy quark takes the place of one of the light ones mentioned above. In other words the large mass of the heavy quark in heavy-light systems results in more significant mass effects than the pure-light system. Thus it is more consistent to take into account the mass corrections to an uniform order. The work [17] went further by including the perturbative
mass corrections, and the splitting in different states was realized by choosing different thresholds. However, the theoretical splitting (given by the central value) in $0^+$ channel, although in agreement with the potential model results [18], still contradicts experiment. Therefore a complete analysis including both operator mixing effects and all the dimension-6 operators is necessary.

Motivated by the important role of heavy quark, in this work we will use QCD sum-rules to investigate the mass effects in splitting the pseudoscalar and scalar $D$ multiplet. To be specific, in our work complete perturbative and no-prerturbative $\mathcal{O}(m_q)$ mass corrections (where $m_q$ is the mass of light quark) are taken into account in the sum rules. Unlike pure-light meson SU(3)-breaking effects mainly introduced by VEVs of renormalization-group invariant mass-dependent operators, in heavy-light system there will be another source introduced by operator mixing mentioned above. Another important point is that in QCD sum rules the threshold also represents SU(3)-symmetry breaking, so different members of the same SU(3) multiplet should have different thresholds. In practice we have no knowledge about SU(3)-symmetry breaking effects a priori; it is only evident from experiments. For instance we even do not know how to set the thresholds for the $0^+$ doublet before we know their exact masses experimentally. Normally if we take the viewpoint of constituent quark, the threshold of $D_s$ should be larger than that of $D_d$. But if the mass of $D_d$ is larger than that of $D_s$ in $0^+$ channel it would be not reasonable to set a larger threshold for $D_s$. For this reason we would like to focus our attention on the splitting trend rather than the exact spectrum of the $D$ mesons. We believe the mass splitting of the doublet $D_s$ and $D_d$ in QCD sum rules under the same “reference” threshold reflects the SU(3)-splitting tendency. Thus for an exact sum rule a suitable modification of threshold due to SU(3)-breaking should enlarge this tendency, otherwise we think it is not natural. To this end in our analysis we first select one member of a multiplet as “benchmark” to fix the suitable threshold and Borel window. Then we apply these parameters to another member with replacing the SU(3)-breaking dependent quantities.

The article is structured as follows. In Section II we first review the necessary results on the operator mixing and cancelation of mass singularities for heavy-light current, then present the sum rules for pseudoscalar and scalar currents of $D^+$ mesons. In Section III the numerical results and discussion will be given. Conclusions are presented in Section IV.
II. THE FORMULAS

A. operator mixing and cancelation of mass singularities

In order to demonstrate the operator mixing and cancelation of mass singularities in heavy-light quark system, we consider the following charmed scalar two-point function:

\[
\Pi(q^2) = i \int d^4x e^{iqx} \langle 0 | T \{ \bar{q}(x)c(x), \bar{c}(0)q(0) \} | 0 \rangle
\]

\[
= \Pi_{\text{pert}}(q^2) + \Pi_{\text{np}}(q^2)
\]

\[
= C_I(q^2)I + \sum_{d \neq 0} C_d(q^2) \langle 0 | O_d | 0 \rangle,
\]

where \(d\) is the dimension of the operator and \(q = u, d, s\). For simplicity the renormalization invariant factor \((\ln(\mu/\Lambda))^{-4/b}\) has been suppressed where \(\mu\) is the normalization point and \(b = (11N_c - 2n_f)/3\). Setting aside the perturbative part until later, the contributions of VEVs of \(d \leq 6\) may be written as [27]:

\[
\Pi_{\text{np}}(q^2) = \bar{C}G^2 \left( \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle \right) + \bar{C}G^3 \left( \left\langle \frac{\alpha_s}{\pi} G^3 \right\rangle \right) + \bar{C}j^2 \left( \left\langle j^2 \right\rangle \right)
\]

\[
+ \bar{C}_{\bar{c}c} \left\{ \left\langle \bar{c}c \right\rangle + \frac{1}{12m_c} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle + \frac{1}{360m_c^3} \left( \left\langle \frac{\alpha_s}{\pi} G^3 \right\rangle + 12 \left\langle j^2 \right\rangle \right) \right\}
\]

\[
+ \bar{C}_{\bar{c}Gc} \left\{ \left\langle \bar{c}Gc \right\rangle - \frac{m_c}{2} \ln \frac{m_c^2}{\mu^2} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle + \frac{1}{12m_c} \left( \left\langle \frac{\alpha_s}{\pi} G^3 \right\rangle + 2 \left\langle j^2 \right\rangle \right) \right\}
\]

\[
+ (c \rightarrow q)
\]

\[
+ \bar{C}_{\bar{q}jq} \left\{ \left\langle \bar{q}jq \right\rangle - \frac{1}{24} \ln \frac{m_q^2}{\mu^2} \left\langle j^2 \right\rangle \right\}
\]

The operators in Eq. (1) are defined as follows:

\[
\left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle = \left\langle \frac{\alpha_s}{\pi} G_{\mu
u}^a G^{a\mu\nu} \right\rangle
\]

\[
\left\langle \frac{\alpha_s}{\pi} G^3 \right\rangle = \left\langle g_s f^{abc} G_{\mu
u}^a G_{\nu\rho}^b G_{\rho\mu}^c \right\rangle
\]

\[
\left\langle j^2 \right\rangle = \left\langle g_s^2 (D_{\mu} G^{a\rho\mu})(D^\rho G_{\rho\mu}^a) \right\rangle = \left\langle g_s^4 \left( \sum_q \bar{q} \gamma^\rho T^a q \right)^2 \right\rangle
\]

\[
\left\langle \bar{q}jq \right\rangle = \left\langle g_s \bar{q} \gamma^\mu (D_\nu G_{\rho\mu}^a) T^a q \right\rangle = \left\langle g_s^2 \left( \bar{q} \gamma^\mu T^a q \right) \sum_q \bar{q} \gamma^\mu T^a q \right\rangle
\]
therefore the contribution from $\langle j^2 \rangle$ is to $O(\alpha_s^2)$, in the following we assume the singularities in $j^2$ are canceled by the mixing and we will omit this term. All the charmed-condensates vanish by virtue of heavy quark expansion\cite{3,27,29}:

$$
\langle \bar{c}c \rangle = \frac{1}{12m_c} \frac{\alpha_s}{\pi} G^2 - \frac{1}{360m_c} \frac{\alpha_s}{\pi} G^3 + ...
$$

$$
\langle \bar{c}Gc \rangle = \frac{m_c}{2} \ln \frac{m_c^2}{\mu^2} \frac{\alpha_s}{\pi} G^2 - \frac{1}{12m_c} \frac{\alpha_s}{\pi} G^3 + ...
$$

thus there are only gluonic and light quark related condensates left. It is clear there are mixing to gluonic operators from $\langle \bar{q}q \rangle$ and $\langle \bar{q}Gq \rangle$ terms, and with the help of this mixing the singular parts in gluonic coefficients in limit $m_q \to 0$ will be well canceled and we are left with an infrared stable expression. The final-form of various non-perturbative coefficients follows Eq. (3):

$$
C_{\bar{q}q} = \bar{C}_{\bar{q}q},
$$

$$
C_{\bar{q}G} = \bar{C}_{\bar{q}G} + \frac{1}{12m_q} \bar{C}_{\bar{q}G} - \frac{m_c}{2} \ln \frac{m_c^2}{\mu^2} \bar{C}_{\bar{q}G},
$$

$$
C_{\bar{q}qG} = \bar{C}_{\bar{q}qG},
$$

$$
C_{\bar{q}qG} = \bar{C}_{\bar{q}qG},
$$

$$
C_{\bar{q}G^3} = \bar{C}_{\bar{q}G^3} + \frac{1}{360m_q^3} \bar{C}_{\bar{q}qG} + \frac{1}{12m_q} \bar{C}_{\bar{q}qG}.
$$

Three of the $\bar{C}$s have been worked out\cite{30} in expansion in $m_q$. In our notation:

$$
\bar{C}_{\bar{q}q} = -\frac{m_c}{q^2 - m_c^2} + \frac{m_q}{2} \frac{2m_c^2 - q^2}{(q^2 - m_c^2)^2} - \frac{m_c^2 m_q^2}{(q^2 - m_c^2)^3},
$$

$$
\bar{C}_{\bar{q}G^2} = \frac{1}{12(q^2 - m_c^2)} \left\{ \frac{m_c}{m_q} - \frac{q^2}{2(q^2 - m_c^2)} + \frac{m_q q^2}{m_c(q^2 - m_c^2)^2} q^2 + 6m_c^2 
+ 6m_c^2 \left( \frac{1}{2} \ln \frac{m_c^2}{\mu^2} + \ln \frac{\mu m_c}{m_c^2 - q^2} \right) \right\} - \frac{3m_c^2 m_q^2 q^2}{(q^2 - m_c^2)^3} \ln \frac{m_c^2}{\mu^2},
$$

and the singular pieces of $\bar{C}_{\bar{q}G^3}$ as $m_q \to 0$ are\cite{28}:

$$
\bar{C}_{\bar{q}G^3} = \frac{m_c}{360m_q^3(q^2 - m_c^2)} + \frac{q^2 - 2m_c^2}{720m_q^2(q^2 - m_c^2)^2} m_c(15q^2 - m_c^2) \frac{1}{360m_q(q^2 - m_c^2)^3}.
$$

$^3$ The three-gluonic coefficient in \cite{28} was derived from pseudoscalar heavy-light current, here one can obtain the scalar one by replacing $m_c$ by $-m_c$. 

6
Substituting Eq. (4) and Eq. (5) into Eq. (3) all the mass singular parts of $C_G^2$ and $C_G^3$ appearing as $1/m_q$ and $\ln m_q$ are canceled since these terms are remnants of long distance structure of vacuum condensates. For definiteness we write down the explicit form of $C_G^2$ to $O(m_q)$ as:

$$C_G^2 = \frac{1}{12(q^2 - m_c^2)} - \frac{m_q m_c^3}{12(q^2 - m_c^2)^3} + \frac{m_q q^2}{12m_c(q^2 - m_c^2)^3} \left( q^2 + 6m_c^2 + 6m_c^2 \ln \frac{\mu m_c}{m_c^2 - q^2} \right),$$

(6)

Therefore to $O(m_q)$ the operator mixing changes the $C_G^2$ significantly and it is expected there will be new mass effects on the sum rules. Similarly, for $C_G^3$ we have (see the appendix for details):

$$C_G^3 = -\frac{q^2}{720m_c^6} W(-10W^3 + 4W^2 + 3W + 2),$$

(7)

where

$$W = \frac{m_c^2}{m_c^2 - q^2}.$$

Now we have fixed the non-perturbative parts in the OPE of scalar heavy-light current. It is easily to get the non-perturbative parts of pseudoscalar current by replacing $m_c$ by $-m_c$ in Eq. (3). The perturbative part and $C_{\bar{q}jq}$ will be presented in the forthcoming subsection.

### B. the sum rules

The sum rules of scalar and pseudoscalar $D$ are based on the following two-point correlation function:

$$\Pi_\Gamma(q^2) = i \int d^4 x e^{iqx} \langle 0 | T\{\bar{q}(x) \Gamma c(x), \bar{c}(0) \Gamma q(0)\} | 0 \rangle,$$

(8)

where $q$ is the light flavor in the $D$ meson, $\Gamma = \{I, i\gamma_5\}$ for scalar and pseudoscalar $D$ meson respectively. The decay matrix element of scalar $D$ meson is defined as:

$$\langle 0 | \bar{q}c | 0 \rangle = m_D f_D,$$

and following the pseudoscalar one is defined as:

$$m_c \langle 0 | \bar{q}i\gamma_5c | D \rangle = m_D^2 f'_D,$$

where $m_c$ is the c-quark mass and $m_D$ the $D$ mass, and $m_q$ labels the mass of light quark. Including the perturbative mass correction to $O(m_q)$ and to $O(\alpha_s)$ as well as the complete $O(m_q)$
non-perturbative corrections worked out in previous subsection, after Borel transformation the OPE of correlation function in Eq.(8) is given by

\[ \Pi_{\Gamma}^{\text{OPE}}(M^2) = \frac{3}{8\pi^2 M^2} \int_{m_c^2}^{\infty} ds \frac{s - m_c^2}{s} \left[ 1 + \frac{2m_q m_c}{s - m_c^2} + \frac{4\alpha_s}{3\pi} f(s, m_c^2) \right] \exp\left[-\frac{s}{M^2}\right] \]

\[ + \left[ \pm \frac{m_c}{M^2} + \frac{1}{2M^2} \left( 1 + \frac{m_c^2}{M^2} \right) m_q \right] \langle \bar{q}q \rangle \exp\left[-\frac{m_c^2}{M^2}\right] \]

\[ + \left\{ \frac{1}{12M^2} \pm \left[ \frac{m_q m_c^3}{8M^6} + \frac{m_q m_c}{2M^4} \left( 1 - \frac{m_c^2}{2M^2} \right) \left( \gamma_E + \ln \frac{m_c}{m_q} \right) \right] \right\} \langle \alpha_s \pi G_{\mu \nu} G^{\alpha \mu \nu} \rangle \exp\left[-\frac{m_c^2}{M^2}\right] \]

\[ - \frac{m_q}{12m_c M^2} \left( 1 - \frac{2m_c^2}{M^2} \right) \right\} \langle \bar{q}g \sigma_{\mu \nu} \lambda^a G^{a \mu \nu} q \rangle \]

\[ + \left\{ \pm \frac{1}{2M^4} \left( 1 - \frac{m_c^2}{2M^2} \right) m_c \langle \bar{q}g \sigma_{\mu \nu} \lambda^a G^{a \mu \nu} q \rangle \right\} \exp\left[-\frac{m_c^2}{M^2}\right], \]

where

\[ f(s, m_c^2) = \frac{9}{4} + 2 \text{Li}_2 \left( \frac{m_c^2}{s} \right) + \ln \frac{s}{m_c^2} \ln \frac{s}{s - m_c^2} + \frac{3}{2} \ln \frac{m_c^2}{s - m_c^2} \]

\[ + \ln \frac{s}{s - m_c^2} + \frac{m_c^2}{s} \ln \frac{s - m_c^2}{m_c^2} + \frac{m_c^2}{s - m_c^2} \ln \frac{s}{m_c^2}, \]

with

\[ \text{Li}_2(x) = - \int_0^x dt \frac{\ln(1 - t)}{t}, \]

and

\[ \gamma_E = 0.577. \]

is the Euler constant. Contributions from 3-gluonic condensates have been omitted safely since it is \( m_q \)-independent and greatly suppressed by the huge denominator therefore it is not responsible for the splitting as one can see from Eq.(7). The upper and lower signs in Eq.(9) are for the scalar and pseudoscalar channel respectively.

\[ \text{The } m_q \text{-independent term in coefficient of } \langle \alpha_s \pi G^2 \rangle \text{ here is different from } [17] \text{ and } [23], \text{ but this term has no impact on splitting except an uniform shift.} \]
On the other hand the correlation function in Eq.(8) can also be derived from the phenomenological side by the dispersion relation:

\[ \Pi(q^2) = \frac{1}{\pi} \int_{m_c^2}^{\infty} ds \frac{\text{Im}\Pi^\text{ph}(s)}{s - q^2} + \text{subtraction constant}, \]  

(10)

where the spectral density \( \text{Im}\Pi^\text{ph}(s) \) is obtained by inserting a complete set of quantum states \( \Sigma|n\rangle\langle n| \) into Eq.(8) which reads:

\[ \text{Im}\Pi(s) = F_G m_D^2 \pi \delta(s - m_D^2) + \pi \frac{3}{8\pi^2} \frac{(s - m_c^2)^2}{s} \left[ 1 + \frac{2m_q m_c}{s - m_c^2} + \frac{4\alpha_s}{3\pi} f(s, m_c^2) \right] \theta(s - s_0), \]  

(11)

Taking the Borel transformation of Eq.(10) and equating it with Eq.(9), after subtracting the continuum contributions we arrive the desired sum rules:

\[ F_G m_D^2 \exp\left[-\frac{m_D^2}{M^2}\right] = \frac{3}{8\pi^2} \int_{m_c^2}^{\infty} ds \frac{(s - m_c^2)^2}{s} \left[ 1 + \frac{2m_q m_c}{s - m_c^2} + \frac{4\alpha_s}{3\pi} f(s, m_c^2) \right] \exp\left[-\frac{s}{M^2}\right] \]

\[ + \left\{ \pm m_c \langle \bar{q}q \rangle + \frac{1}{2} \left( 1 + \frac{m_c^2}{M^2} \right) m_q \langle \bar{q}q \rangle \right\} \exp\left[-\frac{m_c^2}{M^2}\right] \]

\[ + \left\{ \frac{1}{12} \pm \left[ \frac{m_q m_c^3}{8M^4} + \frac{m_q m_c}{2M^2} \left( 1 - \frac{m_c^2}{2M^2} \right) \left( \gamma_F + \ln \frac{\mu m_c}{M^2} \right) \right] \right. \]

\[ - \frac{m_q}{12m_c} \left( 1 - \frac{2m_c^2}{M^2} \right) \} \left\{ \frac{\alpha_s}{\pi} G^a_{\mu\nu} G^{a\mu\nu} \right\} \exp\left[-\frac{m_c^2}{M^2}\right] \]

\[ + \left\{ \pm \frac{1}{2M^2} \left( 1 - \frac{m_c^2}{2M^2} \right) m_c \langle \bar{q}g\sigma_{\mu\nu} \frac{\lambda^a}{2} G^{a\mu\nu} \rangle \right\} \]

\[ + \frac{m_c^2}{4M^4} \left( 1 - \frac{m_c^2}{3M^2} \right) m_q \langle \bar{q}g\sigma_{\mu\nu} \frac{\lambda^a}{2} G^{a\mu\nu} \rangle \]

\[ - \frac{16\pi\alpha_s}{27M^2} \left( 1 - \frac{m_c^2}{4M^2} - \frac{m_c^4}{12M^4} \right) \langle \bar{q}q \rangle^2 \exp\left[-\frac{m_c^2}{M^2}\right]. \]  

(12)

Now we have completed the sum rules for pseudoscalar and scalar \( D \) mesons. The input parameters in Eq.(12) are as follows \[34,36,40]\:

\[ \Lambda_{\text{QCD}} = 259 \text{MeV}, \quad \alpha_s = 0.517, \quad \langle \alpha_s G^2 \rangle = (0.07 \pm 0.01) \text{GeV}^4, \]

\[ \langle \bar{u}u \rangle = \langle \bar{d}d \rangle = -(0.225 \pm 0.025)^3 \text{GeV}^3, \quad \langle \bar{s}s \rangle = 0.8 \langle \bar{u}u \rangle, \]

\[ m_d = 3 \sim 7 \text{MeV}, \quad m_s = 120 \text{MeV}, \]

\[ \langle \bar{q}\sigma_{\mu\nu} \frac{\lambda^a}{2} G^{a\mu\nu} \rangle = m_0^2 \langle \bar{q}q \rangle, \quad m_0^2 = 0.8 \text{GeV}^2. \]  

(13)
All the values adopted above are given at the scale $\mu = 1\text{GeV}$ and we deduce the QCD scale $\Lambda_{\text{QCD}}$ to one-loop from $\alpha_s(M_Z) = 0.1170 \pm 0.0012$\cite{20}. The renormalization scale dependence is given by\cite{35}:

\begin{align}
  m_q(\mu) &= m_q(\mu_0) \left( \frac{\alpha_s(\mu_0)}{\alpha_s(\mu)} \right)^{-4/b}, \\
  \langle \bar{q}q \rangle(\mu) &= \langle \bar{q}q \rangle(\mu_0) \left( \frac{\alpha_s(\mu_0)}{\alpha_s(\mu)} \right)^{4/b}, \\
  \langle g_s\bar{q}\sigma Gq \rangle(\mu) &= \langle g_s\bar{q}\sigma Gq \rangle(\mu_0) \left( \frac{\alpha_s(\mu_0)}{\alpha_s(\mu)} \right)^{-2/3b}, \\
  \langle \frac{\alpha_s}{\pi} G^2 \rangle(\mu) &= \langle \frac{\alpha_s}{\pi} G^2 \rangle(\mu_0).
\end{align}

(14)

with $b = (11N_c - 2n_f)/3$. We use the following pole mass for the charm quark\footnote{A note in\cite{23} argued that the value $m_c = 1.46\text{GeV}$ used in\cite{17} may be ill-defined. As the sum rule is sensitive to $m_c$ as we can see in the next section, the larger choice $m_c = 1.47 \pm 0.04\text{GeV}$ here might induce new error. But it does not affect the splitting since the thresholds are fixed for same channel in our analysis.}:

$$m_c = 1.47 \pm 0.04\text{GeV},$$

which can be expressed in terms of the running mass through the relation:

$$m_c = \bar{m}_c(\mu) \left[ 1 + \left( \frac{4}{3} + \ln \frac{\mu^2}{m_c^2} \right) \frac{\bar{\alpha}_s(\mu)}{\pi} + \mathcal{O}(\alpha_s^2) \right].$$

(15)

Taking the logarithm of both sides of Eq.(12) and applying the differential operator $M^4\partial/\partial M^2$ to them we can separate the mass from decay constant. Now we have fixed all the ingredients for numerical analysis.

III. RESULTS AND DISCUSSION

Firstly we present the criteria followed in our analysis:

1. To specify the appropriate threshold and Borel window, we demand that the continuum contribution [i.e., the part in the dispersive integral from $s_0$ to $\infty$ which has been subtracted from both sides of Eq.(12)] should not be too large (less than 30% of the total dispersive integral). This criterion give us an upper limit on the Borel momentum $M^2$. Furthermore, the non-perturbative dimension-six operators corrections should be less than 10% which establishes a lower limit.
2. In order to check the mass effects on the splitting of the same $J^P$ channel, states of that channel will be analyzed under same threshold while the flavor-dependent parameters such as $m_q$, $\langle \bar{q}q \rangle$ as well as $\langle \bar{q}\sigma Gq \rangle$ change correspondingly. This will supply us with an appropriate comparison in the same channel with different light content.

3. As mentioned above our primary concern is a correct splitting trend, not the whole spectrum in same channel. Thus we select one state as our “benchmark” to determine the threshold and Borel window according to criterion 1. For definiteness we select $D_d$ and $D_s$ as our sample in $0^-$ and in $0^+$ channel respectively. After this we turn on another state following criterion 2. If it is a natural sum rule, it should produce a correct splitting trend that agrees with the experiment.

With these criteria in mind we plot the mass curves of the two states in the same channel against the Borel momentum $M^2$ in a diagram for different threshold and charm mass $m_c$ since it is convenient to observe the splitting. The working windows which satisfy the criterion 1 are marked by two short lines(or one short line which labels the upper limit only) while the narrow ranges from which we read our numerical value are marked by shaded bars. If there is no an obvious extremum within the window we determined, the central value will be adopted. Under these criteria we find for fixed threshold and charm mass $m_c$ as well as scale parameter $\mu$ the working windows for $D_d$ and $D_s$ in each channel are very close. The upper limit of our working windows decrease as the thresholds decrease, while it seems that the lower limit is nearly invariant which can be seen from the following graphs. When we scale up to $\mu = 1.3$GeV the upper limit increases compared with $\mu = 1$GeV while there is no obvious impact on the lower limit.

First, we study the $0^-$ channel from the pseudoscalar sum rule given in Eq.(12) at the scale $\mu = 1$GeV. The numerical results of the two states of $J^P = 0^-$ channel are shown in figure. We can see from figure that the results following the pseudoscalar sum rules of Eq.(12) accurately reproduce the mass of $D_d$ with a large charm mass which is very close to the experimental $D_d(1869)$. When the parameters of $D_s$ turn on and with the threshold fixed, the resulting mass from the sum rule Eq.(12) is still lower compared with the experimental one $D_{s0}(1668)$ but it is always larger than $D_d$. The mass splitting from Eq.(12) is $\sim 35$MeV, a value much lower than the observed splitting $\sim 100$MeV. At a first glance it appears that the SU(3)-breaking effects can not supply a realistic splitting. But it is important that the sum rule does present a
FIG. 1: Mass of $D_d$(dashed line) and $D_s$ meson (solid line) of $0^-$ channel from pseudoscalar sum rule of Eq.(12) as function of Borel momentum $M^2$ at scale $\mu = 1\, \text{GeV}$.

reasonable mass splitting trend in the $0^-$ channel which is in line with the observed spectrum. The failure of pseudoscalar sum rules of Eq.(12) in providing the entire mass gap in $0^-$ channel is understandable: there is a large gap between the two $0^-$ states thus a threshold appropriate to $D_d(1869)$ is too low to produce $D_s(1986)$. On the other hand in QCD sum rules we notice the difference of threshold between different members also reflects SU(3)-symmetry breaking. Therefore it is expected that if we determine the threshold of each member separately, at some lager threshold than the one for $D_d$ the mass of $D_s$ will be well produced from the sum rules. In addition, the theoretical results are very sensitive to charm mass: as we can see from figure\textsuperscript{1}
FIG. 2: Mass of $D_d$ (dashed line) and $D_s$ meson (solid line) from scalar sum rule of Eq. (12) as function of Borel momentum $M^2$ at $\mu = 1$GeV.

at the lowest value adopted in our work, even at a much higher $s_0$ it is still difficult to produce $D_d$. We can read from figure 1 that it seems the pole mass $m_c = 1.47$GeV is more appropriate than the other two choices. The results are summarized in Table I.

Now we turn to the analysis of $0^+$ channel from scalar sum rule of Eq. (12). The observed spectrum of $0^+$ channel is reversed from the estimate from naive quark model. The numerical results following scalar sum rules of Eq. (12) are shown in figure 2. We can see that under the threshold determined from $D_s$, a mass gap $M_{D_d} - M_{D_s} \sim 15$MeV between $D_d$ and $D_s$ can be realized which is also lower than the experimental one $\sim 35$MeV. However, it indeed
TABLE I: Mass of pseudoscalar $D_d$ and $D_s$ read from shaded areas marked by short bars in figure\[1\] for different threshold $s_0$ and $m_c$ at scale $\mu = 1\text{GeV}$.

| $s_0$ (GeV$^2$) | $m_c$ (GeV) | $D_d$ (MeV) | $D_s$ (MeV) | $M_{D_s} - M_{D_d}$ (MeV) |
|-----------------|-------------|-------------|-------------|-----------------------------|
| 7.0             | 1.43        | 1812        | 1846        | 34                          |
| 6.8             | 1.47        | 1856        | 1888        | 32                          |
| 5.8             | 1.51        | 1873        | 1911        | 38                          |

TABLE II: Masses of scalar $D_d$ and $D_s$ read from figure\[2\] for different threshold $s_0$ and $m_c$ at scale $\mu = 1\text{GeV}$. The first two values are read from the shaded area marked by short bars while the third is the central value between the two short lines in the third graph.

| $s_0$ (GeV$^2$) | $m_c$ (GeV) | $D_d$ (MeV) | $D_s$ (MeV) | $M_{D_d} - M_{D_s}$ (MeV) |
|-----------------|-------------|-------------|-------------|-----------------------------|
| 7.9             | 1.43        | 2356        | 2340        | 16                          |
| 7.1             | 1.47        | 2341        | 2328        | 13                          |
| 6.5             | 1.51        | 2354        | 2340        | 14                          |

gives a correct splitting trend which agrees with experiment. The results in the $0^+$ channel are summarized in Table\[3\]

It is instructive to study the scale dependence of our results since physical quantities are scale-independent thus it will supply a natural check on our results. Therefore the theoretical splitting should be unchanged when calculated with another scale. To this end we evolve the related parameters according to Eq.(14) to a higher scale $\mu = 1.3\text{GeV}$ which is still lower than charm mass. The results for $0^-$ and $0^+$ are shown in figure\[3\] and figure\[4\] respectively. We can see that when the scale increases, the results of $0^-$ channel still keep a well behavior as $\mu = 1\text{GeV}$. We can read from figure\[3\] the splitting $M_{D_s} - M_{D_d} \sim 35\text{MeV}$ which agrees well compared with $\mu = 1\text{GeV}$. But the situation is not so good in the $0^+$ channel when we scale up. It is obvious from figure\[4\] that at $\mu = 1.3\text{GeV}$ the calculated mass is monotonically decreasing within the window satisfying the criterion 1 in $0^+$ channel. But fortunately the calculated mass of $D_d$ is always larger than $D_s$ within the selected Borel windows. For example, if we take the central values of in figure\[4\] we find the mass gap $M_{D_d} - M_{D_s} \sim 15\text{MeV}$ which is also consistent
with the splitting obtained at $\mu = 1$GeV. Therefore the splitting in both channels are invariant which shows a correct scale invariance. Combined the results at $\mu = 1$GeV we conjecture the reason why it is difficult to develop an extremum value at large charm mass in $0^+$ channel maybe, as pointed out in [23], is that a large charm mass will induce large error. Our results imply that it is more appropriate to take a lower pole mass for charm. In fact as the scale increases, we approach to the asymptotic free side further thus the non-perturbative effects will have reduced impact. We can see obviously from figure 4 that at the high energy side in the $0^+$ channel the mass gap decreases.

FIG. 3: Mass of $D_d$ (dashed line) and $D_s$ meson (solid line) of $0^-$ channel from pseudoscalar sum rule of Eq. (12) as function of Borel momentum $M^2$ at scale $\mu = 1.3$GeV.
FIG. 4: Mass of $D_d$ (dashed line) and $D_s$ meson (solid line) from scalar sum rule of Eq. (12) as function of Borel momentum $M^2$ at $\mu = 1.3$ GeV. It is obvious with the charm mass increasing, it is difficult to develop an reasonable extremum value in the allowed Borel window.

The different importance of mass effects in realizing the splitting in $0^-$ and $0^+$ channel of $D$ meson is not surprising. One should notice the “force” induced by the QCD vacuum or equivalently, the non-perturbative effects is parity-dependent which is well indicated by the contribution of the dominant condensates $m_c\langle \bar{q}q \rangle$ and $m_c\langle \bar{q}\sigma Gq \rangle$ as well as the parts in $\langle \alpha_s G^2 / \pi \rangle$ introduced by their mixing. In $0^-$ channel the entire effect of these two terms give positive contributions to the correlation function thus an attractive force is induced by the QCD vacuum. While in $0^+$ channel these two terms supply negative contributions to the correlation
TABLE III: Mass of pseudoscalar $D_d$ and $D_s$ read from shaded area marked by short bars in figure.\textsuperscript{3}

for different threshold $s_0$ and $m_c$ at scale $\mu = 1.3$GeV.

| $s_0$(GeV$^2$) | $m_c$(GeV) | $D_d$(MeV) | $D_s$(MeV) | $M_{D_s} - M_{D_d}$(MeV) |
|---------------|-------------|-------------|-------------|-----------------|
| 7.6           | 1.43        | 1807        | 1845        | 38              |
| 7.2           | 1.47        | 1851        | 1886        | 35              |
| 6.6           | 1.51        | 1885        | 1922        | 37              |

function, then a repulsive force is induced by the QCD vacuum. Since $-\langle \bar{d}d \rangle > -\langle \bar{s}s \rangle$ we could expect $M_{D_s} > M_{D_d}$ in $0^-$ and $M_{D_s} < M_{D_d}$ in $0^+$ channel. Furthermore the “force” is scale dependent which implies that the larger the scale is, the farther we leave from the confinement sector, therefore the importance of non-perturbative effects will be discounted compared with a lower scale. In the $0^+$ channel the effect of the quark condensate $\langle \bar{q}q \rangle$ overpowers other mass effects, so we find $M_{D_s} < M_{D_d}$. On the contrary if we set $\langle \bar{d}d \rangle = \langle \bar{s}s \rangle$ then the mass difference in $0^+$ channel is produced by $m_q$-dependent terms only; thus it is expected there will be mass-flipping. The mass curves are shown in figure.\textsuperscript{5} It is obvious that the mass gap of the two scalars are very sensitive to the ratio $\kappa = \langle \bar{s}s \rangle / \langle \bar{u}u \rangle$. However, in $0^-$ channel the sign of the splitting remains unchanged, the mass gap of the two states is not sensitive to this ratio.

In fact we can categorize corrections into two parts: one is parity-dependent and mainly proportional to light quark masses, $\langle \bar{q}q \rangle$ and $\langle \bar{q}\sigma Gq \rangle$, another is parity-independent such as $m_q \langle \bar{q}q \rangle$, $m_q \langle \bar{q}\sigma Gq \rangle$ and $\langle \bar{q}q \rangle^2$. It seems the former overpower the latter because the latter are doubly suppressed by the $m_q$ and the $\langle \bar{q}q \rangle$, however, their magnitudes for SU(3)-breaking are comparable. The former change their signs when we alter from $0^+$ to $0^-$ channel and vice versa while the latter do not. We can learn from pseudoscalar sum rule in Eq.(12) that in the $0^-$ channel these two parts provide a consistent response to the splitting since there is flipping in their signs. For instance the mass gap from quark condensates is:

$$m_c(\langle \bar{s}s \rangle - \langle \bar{d}d \rangle) > 0,$$

$$m_d\langle \bar{d}d \rangle - m_s\langle \bar{s}s \rangle > 0,$$

and from the other condensates all contribute positive differences, all these positive difference are inclined to broaden the mass gap between $D_s$ and $D_d$ in $0^-$ channel. However, in the $0^+$
FIG. 5: Mass curves of $D_d$ (dashed line) and $D_s$ (solid line) vs. Borel momentum $M^2$ in $0^+$ channel at $\mu = 1\text{GeV}$, where we set $\langle \bar{d}d \rangle = \langle \bar{s}s \rangle$. It is clear the splitting of the two states reverse.

channel all the non-perturbative corrections keep the same positive parity, and consequently a competitive role to the splitting of $D_d$ and $D_s$ developed:

$$m_c(\langle \bar{d}d \rangle - \langle \bar{s}s \rangle) < 0,$$
$$m_d\langle \bar{d}d \rangle - m_s\langle \bar{s}s \rangle > 0,$$

so does the $\langle \bar{q}\sigma Gq \rangle$. This means there are some compensations to the splitting from the parity-independent terms which prefer to weaken the splitting between $D_d$ and $D_s$ in $0^+$ channel. Therefore the mass gap is broader in the $0^-$ channel than in $0^+$ of $D$ meson. It is worthy to mention that this phenomenon is partly noticed in [23]. In fact these results can be generalized to the $0^-$ and $0^+$ channels of pure-light mesons, but the mass difference induced by various condensates is greatly suppressed by small light quark mass thus it is expected the splitting is tiny by this mechanism, so to realize a realistic splitting in QCD sum rules based on a naive quark model should take into account instanton effects [8, 9].

As operator mixing changes the coefficient of two-gluonic condensates significantly, we can see the $m_q$-dependent parts in $C_{G2}$ are parity-dependent and have a complicated form but also
an obvious scale-dependence. It is instructive to investigate the impact on the calculated mass of both channels. For this purpose we turn off \( m_q \)-dependent parts in \( C_G^2 \) in sum rules. The mass curves are shown in figure 6. It is obvious that in this case in both channels \( M_{D_s} \) is larger than \( M_{D_d} \). We find in \( 0^- \) channel the mass gap is broader than there is \( m_q \)-dependent corrections in gluonic condensate \( \sim 10 \text{MeV} \) so our results show this correction is negative in the mass gap for the \( 0^- \) channel. In \( 0^+ \) channel the situation is just opposite to the case when \( m_q \)-dependent corrections turn on, thus the results show this contribution is positive to realize a reasonable splitting in \( 0^+ \) channel.

A natural idea is to generalize these arguments to the \( 1^- \) and \( 1^+ \) channel of \( D \). Unfortunately it does not work which can be well understood from heavy quark effective theory\[37\]. When there is no orbital excitation of light content, the c-quark with spin \( s_c = 1/2 \) and the light degrees of freedom with spin \( s_{l} = 1/2 \) forming a multiplet of hadrons with spin:

\[
j = \frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1,
\]

thus it is clear there is a unique \( J^P = 1^- \) multiplet. While if there is orbital excitation of light degrees of freedom, the spin of light content will be:

\[
s_{l} = l \pm \frac{1}{2} = \frac{1}{2} \quad \text{or} \quad \frac{3}{2},
\]

FIG. 6: Mass curves of \( 0^- \) (left) and \( 0^+ \) (right) channels vs. Borel momentum \( M^2 \) at \( \mu = 1 \text{GeV} \), where we have turned off the \( m_q \)-dependent corrections in \( C_G^2 \) in sum rules. The dashed and solid line in both graphs denote \( D_d \) and \( D_s \) respectively.
combining with the spin of c-quark $s_c = 1/2$:

$$j = \frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1,$$

$$j = \frac{1}{2} \otimes \frac{3}{2} = 1 \oplus 2.$$

so there will be two $1^+$ multiplets, the $D_1^*$ and the $D_1$ states experimentally. These two states are very close, thus the single resonance approximation in QCD sum rules is not viable. If we let the charm mass go to infinity these two $1^+$ states can be separated in the formalism of the heavy quark effective theory\[38\]. However, the $1/m_c$ corrections which are the same order of SU(3)-breaking effects make these two states mix again. So we still cannot get a simple correspondence between $1^-$ and $1^+$. We can resort to the experimental data directly\[36\]. The mass splitting between $D_s$ and $D_d$ in $1^-$ is about 100MeV, while it is only about 40MeV in the lower $1^+$ (which corresponds to $1^-$ in the heavy quark mass limit). A similar effect still appears.

Finally let us briefly mention the $B$ case although there is not enough experimental evidence. Since the $b$-quark mass is so large the SU(3)-breaking effects are smeared in the formalism Eq.(12). We can hope the similar effects will be recovered in the formalism of the heavy quark effective theory. Certainly, SU(3)-breaking effects also appear in $1/m_h$ corrections which might not be small and could cause some differences between $D$ mesons and $B$ mesons.

IV. CONCLUSIONS

In this work, based on the pseudoscalar and scalar sum rule from $c\bar{q}$ structure we investigate the SU(3)-breaking effects enhanced by the large charm mass on the splitting of the pseudoscalar and scalar $D$ multiplet. Since the quark condensates $\langle \bar{q}q \rangle$ and $\langle \bar{q}\sigma Gq \rangle$ are greatly enhanced by the heavy quark mass, they play more important roles in the SU(3)-breaking in the heavy-light mesons. The sign of the $\langle \bar{q}q \rangle$ and $\langle \bar{q}\sigma Gq \rangle$ contributions is different in $0^-$ and $0^+$ channels thus resulting a “parity- dependent” force which can explain the relatively-lower mass of $D_s(2317)$. Furthermore, these parity-dependent corrections broaden the mass gap in the $0^-$ channel while weakening it in $0^+$ channel. The results show that the $O(m_q)$ corrections in two-gluonic condensates introduced by operator mixing are noticeable in $0^+$ channel. Such a “parity-dependent” force should exist generally in the heavy-light mesons, but its magnitude should be dependent on the specific system. This force is also energy-dependent, so its effect
should be weaker in high excited states. We also analyze the cases $1^-$ and $1^+$. The experimental data hints such a force also exists but is weaker than that in $0^-$ and $0^+$. 

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Appendix A: derivation of Eq.(7)

It is a little effort to work out $C_{G3}$ in scalar current expansion from a vector current expansion. To this end, let us consider vector current two-point function:

$$\Pi_{\mu\nu}(q^2) = i \int d^4x e^{iqx} \langle 0|T\{\bar{q}(x)\gamma_\mu c(x), \bar{c}(0)\gamma_\nu q(0)\}|0\rangle$$

$= (-g_{\mu\nu}q^2 + q_\mu q_\nu)\Pi^V(q^2) + q_\mu q_\nu \Pi^S(q^2), \quad (A1)$

To single out the scalar part we contract Eq.(A1) with $q_\mu q_\nu$:

$$q_\mu q_\nu \Pi_{\mu\nu}(q^2) = q^4\Pi^S(q^2), \quad (A2)$$

Then it is convenient to consider the following two-point function based on the four-divergence of vector current:

$$\Pi^S(q^2) = i \int d^4x e^{iqx} \langle 0|T\{\partial_\mu \bar{q}(x)\gamma_\mu c(x)\partial_\nu \bar{c}(0)\gamma_\nu q(0)\}|0\rangle$$

$= i \int d^4x e^{iqx} \langle 0|\left\{\bar{q}(x)\not\partial c(x) + \bar{q}(x)\not\partial c(x)\right\} \times \left\{\bar{c}(0)\not\partial q(0) + \bar{c}(0)\not\partial q(0)\right\}|0\rangle$

$= i(m_c - m_q)^2 \int d^4x e^{iqx} \langle 0|\bar{q}(x)c(x)\bar{c}(0)q(0)|0\rangle, \quad (A3)$

where the Dirac equations:

$$\not\partial \psi(x) = -im\psi(x),$$

and

$$\not\partial \bar{\psi}(x)\not\partial = im\bar{\psi}(x).$$
have been used. In this way we can associate OPE of scalar current expansion with the vector one:

\[ i \int d^4x e^{iqx} \langle 0 | \bar{q}(x)c(x)\bar{c}(0)q(0) | 0 \rangle = \frac{\Pi^S(q^2)}{(m_c - m_q)^2}, \quad (A4) \]

Combined with Eq. (A2) it is straightforward to obtain:

\[ i \int d^4x e^{iqx} \langle 0 | \bar{q}(x)c(x)\bar{c}(0)q(0) | 0 \rangle = \frac{q^4 \Pi^S(q^2)}{(m_c - m_q)^2}, \quad (A5) \]

And \( C_{G^3} \) has been worked out\[39\] in heavy-light vector current expansion, here we write it in a standard form:

\[
C_{G^3}(q^2) = (-g_{\mu\nu}q^2 + q_\mu q_\nu)A + g_{\mu\nu}(m_c - m_q)^2
= (-g_{\mu\nu}q^2 + q_\mu q_\nu) \left( A - \frac{(m_c - m_q)^2}{q^2} \right) B + q_\mu q_\nu \left( -\frac{(m_c - m_q)^2}{q^2} \right) B. \quad (A6)\]

where

\[
A = \frac{W^2}{4320\pi^2 m_c^6}(-30W^2 - 8W - 3),
\]

\[
B = \frac{W}{2880\pi^2 m_c^6}(-10W^3 + 4W^2 + 3W + 2),
\]

\[
W = \frac{m_c^2}{m_c^2 - q^2}.
\]

 Combined with Eq. (A5) we can obtain \( C_{G^3} \) in a scalar heavy-light expansion:

\[
C_{G^3} = -\frac{q^2}{720m_c^6} W(-10W^3 + 4W + 3W + 2). \quad (A7)\]

This is Eq. (7) where we have suppressed the factor \( \frac{\alpha_s}{\pi} \).

[1] M. Gell-Mann, Phys. Lett. 8, 214(1964).
[2] G. Zweig, CERN preprint 8419/Th412, 8182/Th401 (unpublished).
[3] M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B147, 385, 448(1979).
[4] L. J. Reinders, H. Rubinstein and S. Yazaki, Phys. Rep. 127, 1(1985).
[5] V. A. Novikov, M. A. Shifman A. I. Vainshtein et al, Fortschr. Phys. 32, 585(1984).
[6] I. I. Balitsky, V. M. Braun and A. V. Kolesnichenko, Nucl. Phys. B312, 509(1989).
[7] V. L. Chernyak, I. R. Zhitnisky, Nucl. Phys. B345, 137(1990).
[8] E. V. Shuryak, Nucl Phys, B214, 237(1983); Nucl. Phys. B203, 93, 116, 140(1982).

[9] J. Zhang, H. Y. Jin, Z. F. Zhang, T. G Steele, D. H. Lu, Phys. Rev. D79, 114033(2009).

[10] BARBAR Collaboration, B. Aubert et al., Phys. Rev. Lett. 90, 242001(2003).

[11] CLEO Collaboration, D. Besson et al., Phys. Rev. D68, 032002(2003).

[12] FOCUS Collaboration, J. M. Link, et al., Phys. Lett. B586, 11(2004).

[13] G. S Bali, Phys. Rev. D68, 071501(2003).

[14] A. Dougall, R. D Kenway, C. M. Maynard, C. McNeile, Phys. Lett. B569, 41(2003).

[15] H. W. Lin, S. Ohta, A. Soni, N. Yamada, Phys. Rev. D74, 114506(2006).

[16] Y-B Dai, X-Q. Li, S-L. Zhu, Y-B. Zuo, Eur. Phys. J. C. 55, 249(2008).

[17] A. Hayashigaki, K Terasaki, arXiv: hep-ph/0411285.

[18] M. Di. Pierro and E. Eichten, Phys. Rev. D64, 114004(2001).

[19] H-Y. Cheng, W-S. Hou, Phys. Lett. B566, 193(2003).

[20] T. Barnes, F. E. Close, H. J. Lipkin, Phys. Rev. D68, 054006(2003).

[21] M. E. Bracco, A. Lozea, R. D Matheus, F. S Navarra, M. Nielsen, Phys. Lett. B624, 217(2005).

[22] P. Colangelo, F. De Fazio and A. Ozpineci, Phys. Rev. D72, 074004(2005).

[23] S. Narison, Phys. Lett. B605, 319(2005).

[24] R. M. Albuquerque, S. Narison, M. Nielsen, arXiv: hep-ph/0904.3717.

[25] S. Narison, Phys. Lett. B520, 115(2001).

[26] Q. Mason, et al., Phys. Rev. Lett, 95, 052002(2005).

[27] S. C Generalis and D. J. Broadhurst, Phys. Lett. B139, 85(1984).

[28] E. Bagan, J. I. Latorre, P. Pascual and R. Tarrach, Nucl. Phys, B254, 555(1985).

[29] E. Bagan, H. G Dosch, P. Gosdzinsky, S. Narison, J.-M. Richard, Z. Phys. C64, 57(1994).

[30] M. Jamin, M. Münz, Z. Phys. C60, 569(1993), arXiv: hep/ph-9208201.

[31] L. J. Reinders, H. R. Rubinstein and S. Yazaki, Phys. Lett. B97, 257(1980)

[32] T. M. Aliev and V. L. Eletsky, Sov. J. Nucl. Phys. 38, 936(1983).

[33] A. Khodjamirian and R. Rückl, in: Heavy Flavours, 2nd edition, eds. A. J. Buras and M. Lindner (World Scientific, Singapore), arXiv: hep-ph/9801443

[34] A. Khodjamirian, T. Mannel and M. Melcher, Phys. Rev. D70, 094002(2004).

[35] H-Y Cheng, C-K Chua and K-C Yang, Phys. Rev. D73, 014017(2006).
[36] Particle Data Group. C. Amsler et al., Phys. Lett. B667, 1(2008).

[37] N. Isgur, M. B. Wise, Phys. Lett. B232, 113(1989).
   E. Eichten, B. Hill, Phys. Lett. B234, 511(1990).
   H. Georgi, Phys. Lett. B240, 447(1990).
   And the comprehensive ones:
   M. Neubert, Phys. Rept. 245, 259(1994), and the reference therein.
   A. V. Manohar and M. B. Wise, Heavy Quark Physics, Cambridge University Press, 2000, and
   the reference therein.

[38] Y. B. Dai, C. S. Huang, C. Liu and S. L. Zhu, Phys. Rev. D68, 114011(2003)

[39] S. C. Generalis, J. Phys. G: Nucl. Part. Phys. 16, 367(1990).

[40] S. Narison, Phys. Rev. D74, 034031(2006).