Entanglement spectra of coupled $S = 1/2$ spin chains in a ladder geometry

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We study the entanglement spectrum of spin-1/2 XXZ ladders both analytically and numerically. Our analytical approach is based on perturbation theory starting either from the limit of strong rung coupling, or from the opposite case of dominant coupling along the legs. In the former case we find to leading order that the entanglement Hamiltonian is also of nearest-neighbor XXZ form although with an in general renormalized anisotropy. For the cases of XX and isotropic Heisenberg ladders no such renormalization takes place. In the Heisenberg case the second order correction to the entanglement Hamiltonian consists of a renormalization of the nearest neighbor coupling plus an unfrustrated next nearest neighbor coupling. In the opposite regime of strong coupling along the legs, we point out an interesting connection of the entanglement spectrum to the Lehmann representation of single chain spectral functions of operators appearing in the physical Hamiltonian coupling the two chains.

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I. INTRODUCTION

In the last decade many-body physics has been substantially enriched by the concept of entanglement, whose extensive and systematic study originated in the field of quantum information theory$^1$. In particular, the notion of the entanglement spectrum$^2$ has led to novel insights in the physics of various many-body systems. These include quantum Hall monolayers at fractional filling$^{2-11}$, quantum Hall bilayers at filling factor $\nu = 1^{12}$, spin systems of one$^{13-18}$ and two$^{19-21}$ spatial dimensions, and topological insulators$^{22,23}$. Other topics recently covered encompass rotating Bose-Einstein condensates$^{24}$, coupled Tomonaga-Luttinger liquids$^{25}$, and systems of Bose-Hubbard$^{26}$ and complex paired superfluids$^{27}$.

In Ref. 17 Poilblanc observed that chain-chain entanglement spectra in two-leg spin ladders are remarkably similar to the energy spectrum of a single Heisenberg chain. Furthermore he found that the fitted effective inverse temperature depends on the ratio of the leg to the rung couplings and vanishes in the limit of strong rung coupling. In a parallel study of entanglement spectra of quantum hall bilayers at $\nu = 1$ one of the present authors observed a similarly striking analogy between the entanglement spectrum of a single layer and the energy spectrum of a single physical layer at $\nu = 1/2$.\textsuperscript{12}

In this paper we study the entanglement spectrum of two coupled XXZ chains analytically in two limiting cases a) the case of strong rung coupling and b) the case of weak rung coupling. In case a) we find that the entanglement spectrum is described to leading order by an entanglement Hamiltonian of the XXZ form, albeit with an in general renormalized effective anisotropy (and thus independently reproducing the recently posted result of Ref. 18). For the particular cases of XX and Heisenberg ladders the anisotropy is unaltered, and we arrive at the conclusion that in these cases the entanglement Hamiltonian is indeed the physical Hamiltonian restricted to a single chain. Moreover, we derive explicit results for the next-to-leading order exhibiting deviations of the entanglement Hamiltonian from nearest-neighbors XXZ coupling. In the second case (b) we point out an interesting connection between the entanglement spectrum and the Lehmann representation of single chain spectral functions of the operators contained in the physical Hamiltonian coupling the two chains.

We consider an XXZ spin ladder Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$ with

$$\mathcal{H}_0 = J_{\text{rung}} \sum_m \left[ \frac{1}{2} (S^+_{m,1} S^-_{m,2} + h.c.) + \Delta S^z_{m,1} S^z_{m,2} \right],$$

$$\mathcal{H}_1 = J_{\text{leg}} \sum_{m,\nu} \left[ \frac{1}{2} (S^+_{m,\nu} S^-_{m+1,\nu} + h.c.) + \Delta S^z_{m,\nu} S^z_{m+1,\nu} \right]$$

and $\Delta$ a dimensionless XXZ anisotropy parameter. In the following we will always assume a ladder of length $L$ with periodic boundary conditions if not specified otherwise. In the following we limit our discussion to the antiferromagnetic regime $J_{\text{leg}}, J_{\text{rung}} \geq 0$. Apart from the special case $J_{\text{leg}} = 1, J_{\text{rung}} = 0$ (corresponding to two decoupled critical $S = 1/2$ chains), the system is gapped in this regime, with exponentially decaying correlations.

### FIG. 1: (Color online) Two leg spin ladder consider in this paper. The chains are labeled by $\nu = 1, 2$, while the runs are labeled by $m = 1, \ldots, L$. We consider the entanglement spectrum in the illustrated chain-chain bipartition.
II. EXACT DIAGONALIZATION RESULTS

Before discussing our analytical results in the two limiting cases we present an overview of the full $J_{\text{leg}}/J_{\text{rung}}$ dependence of the entanglement spectrum $\xi := -\log(\lambda)$ obtained from the eigenvalues $\lambda$ of the reduced density matrix of a single chain in a Heisenberg ($\Delta = 1$) spin ladder (cf. bipartition setup shown in Fig. 1). In the region $J_{\text{leg}}/J_{\text{rung}} < 1$ selected numerical spectra have already been presented in Refs. 17,20.

In the upper panel of Fig. 2 the bare entanglement spectrum for a $L = 8$ ladder is shown as a function of $J_{\text{leg}}/J_{\text{rung}}$ (note the logarithmic $x$-axis), where the levels are labeled according to their $S^z$ quantum number$^{28}$. In the limit $J_{\text{leg}}/J_{\text{rung}} \to 0^+$ all the $2^L$ levels of the entanglement spectrum collapse onto a single value $L \ln 2$, and for finite $J_{\text{leg}}/J_{\text{rung}}$ they start to spread. For $J_{\text{leg}}/J_{\text{rung}} \gtrsim 1$ the spectrum rearranges, and a notable feature is that the lowest levels above the singlet ground states are a sizable set of $S = 1$ triplets, displaying a common slope of 2 when plotted as a function of $\ln(J_{\text{leg}}/J_{\text{rung}})$. Note that this is in contrast with the behavior at small $J_{\text{leg}}/J_{\text{rung}}$, where there is only one triplet located below the second singlet in the entanglement spectrum.

In the lower panels of Fig. 2 we highlight the behavior in the two limit cases by appropriately rescaling and shifting the entanglement spectra. In the lower left panel we also show the energy spectrum of a $L = 8$ Heisenberg chain (multiplied by two) to demonstrate the remarkable agreement between the entanglement spectrum and the energy spectrum in this particular limit. As just discussed the structure of the entanglement spectrum is somewhat different in the opposite limit, and the analogy with the energy spectrum of a single chain apparently breaks down.

We now provide an analytical justification of the reported behavior based on perturbation theory around the two limits (a) and (b) described above.

III. STRONG RUNG COUPLING LIMIT

Let us now treat $\mathcal{H}_1$ as a perturbation to $\mathcal{H}_0$ with antiferromagnetic coupling, $J_{\text{rung}} > 0$. The unperturbed ground state reads

$$|0\rangle = \bigotimes_m |s_m\rangle \quad (2)$$

using obvious notation for singlet and triplet states on the rungs,

$$|s_m\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle_{m,1} |\downarrow\rangle_{m,2} - |\downarrow\rangle_{m,1} |\uparrow\rangle_{m,2}) \quad (3)$$

$$|t^+_m\rangle = |\uparrow\rangle_{m,1} |\uparrow\rangle_{m,2} \quad (4)$$

$$|t^0_m\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle_{m,1} |\downarrow\rangle_{m,2} + |\downarrow\rangle_{m,1} |\uparrow\rangle_{m,2}) \quad (5)$$

$$|t^-_m\rangle = |\downarrow\rangle_{m,1} |\downarrow\rangle_{m,2} \quad (6)$$

A. First order

The first-order correction to the ground state can be obtained by elementary calculation,

$$|1\rangle = \frac{J_{\text{leg}}}{4J_{\text{rung}}} \sum_m \left[ \frac{2}{1+\Delta} (\cdots |t^+_m\rangle |t^-_{m+1}\rangle \cdots) + \frac{2}{1+\Delta} (\cdots |t^0_m\rangle |t^0_{m+1}\rangle \cdots) - \Delta (\cdots |t^0_m\rangle |t^0_{m+1}\rangle \cdots) \right] \quad (7)$$
where the dots denote singlet states on each rung not explicitly specified. The reduced density operator is obtained by tracing out one of the legs from \( \rho = (|0\rangle + |1\rangle)(|0\rangle + |1\rangle) \) and is given within first order in \( J_{\text{leg}} / J_{\text{rung}} \) by

\[
\rho_{\text{red}}^{(1)} = \frac{1}{2\beta} \left( 1 - \frac{4J_{\text{leg}}}{J_{\text{rung}}(1 + \Delta)} \sum_m \left( S_m^+ S_{m+1}^- + h.c. \right) + \frac{1}{2}(\Delta + \Delta^2) S_m^z S_{m+1}^z \right),
\]

(8)

with \( L \) being the number of rungs. Again within first order perturbation theory, this result can be formulated as

\[
\rho_{\text{red}}^{(1)} = \frac{1}{Z} \exp \left( -\mathcal{H}_{\text{ent}}^{(1)} \right)
\]

(9)

with \( Z = \text{Tr} \exp(-\mathcal{H}_{\text{ent}}^{(1)}) \) being a partition function. The entanglement Hamiltonian \( \mathcal{H}_{\text{ent}}^{(1)} \) is given as:

\[
\mathcal{H}_{\text{ent}}^{(1)} = \frac{4}{1 + \Delta} \times \frac{J_{\text{leg}}}{J_{\text{rung}}} \sum_m \left[ \frac{1}{2}(S_m^+ S_{m+1}^- + h.c.) + \Delta S_m^z S_{m+1}^z \right],
\]

(10)

and is of nearest neighbor XXZ form, with a renormalized anisotropy parameter,

\[
\tilde{\Delta} = \frac{1}{2}(\Delta + \Delta^2).
\]

(11)

Note that \( \Delta = 0 \) (XX case) and \( \Delta = 1 \) (Heisenberg interactions) are invariant, i.e. the entanglement Hamiltonian is proportional to the physical Hamiltonian restricted to the block, as observed numerically for \( \Delta = 1 \) earlier in Ref. 17. Since in these two specific cases the physical Hamiltonian on a chain and the entanglement Hamiltonian are simply proportional to each other (to first order), one can define an ad hoc inverse temperature

\[
\beta = \frac{4}{1 + \Delta} \times \frac{J_{\text{leg}}}{J_{\text{rung}}},
\]

(12)

such that \( \mathcal{H}_{\text{ent}}^{(1)} = \beta \mathcal{H}_{XXZ} \), with a \( J_{\text{leg}} / J_{\text{rung}} \) dependence solely in \( \beta \).

### B. Second order

The second-order contribution to the ground state is somewhat lengthy and given in appendix A. For general anisotropy the result does not seem to be amenable to a simple interpretation. In the isotropic case \( \Delta = 1 \), however, one finds up to second order

\[
\rho_{\text{red}}^{(2)} = \frac{1}{2L} \left( 1 - \frac{2J_{\text{leg}}}{J_{\text{rung}}} \sum_m \bar{S}_m \bar{S}_{m+1} \right)
\]

(13)

\[
+ \frac{1}{2} \left( \frac{2J_{\text{leg}}}{J_{\text{rung}}} \right)^2 \left( \left[ \sum_m \bar{S}_m \bar{S}_{m+1} \right]^2 - \frac{3}{16} L \right)
\]

\[
+ \frac{1}{4} \sum_m \left[ \bar{S}_m \bar{S}_{m+2} - \bar{S}_m \bar{S}_{m+1} \right] \right),
\]

(13)

which can, within the same order, be reformulated as

\[
\rho_{\text{red}}^{(2)} = \frac{1}{Z} \exp \left( -\mathcal{H}_{\text{ent}}^{(2)} \right)
\]

(14)

with \( Z = \text{Tr} \exp(-\mathcal{H}_{\text{ent}}^{(2)}) \) and

\[
\mathcal{H}_{\text{ent}}^{(2)} = \frac{2J_{\text{leg}}}{J_{\text{rung}}} \sum_m \bar{S}_m \bar{S}_{m+1}
\]

(15)

\[
+ \frac{1}{2} \left( \frac{J_{\text{leg}}}{J_{\text{rung}}} \right)^2 \sum_m \left[ \bar{S}_m \bar{S}_{m+1} - \bar{S}_m \bar{S}_{m+2} \right].
\]

Thus, also within second order \( J_{\text{leg}} / J_{\text{rung}} \) the entanglement Hamiltonian is quite similar to the physical Hamiltonian on a single chain. The second order correction contains a renormalization of the amplitude of the nearest neighbor coupling and the appearance of an frustrated ferromagnetic next nearest neighbor Heisenberg coupling. The latter observation is in accordance with the numerical findings in Refs. 18, 20. We expect this picture to hold also at higher order. According to general linked cluster ideas, sites at distance \( r \) will only be able to couple starting at order \( r \) with a leading amplitude proportional to \( \left( \frac{J_{\text{leg}}}{J_{\text{rung}}} \right)^r \). Given the Heisenberg nature of the leg interactions, to leading order a Heisenberg interaction \( \sum_m \bar{S}_m \bar{S}_{m+r} \) is the only symmetry-allowed interaction.

In Fig. 3 we present a comparison between the rescaled numerical entanglement spectra for \( L = 8 \) and the perturbative result in the form of the energy spectrum of Eq. (15) for \( J_{\text{leg}} / J_{\text{rung}} \leq 1 \). The second order corrections only seem to become visible for this system size for \( J_{\text{leg}} / J_{\text{rung}} \gtrsim 0.1 \), and the initial deviations from a straight line in the numerics are correctly described by the second order corrections. However for larger \( J_{\text{leg}} / J_{\text{rung}} \) ratios further neighbor Heisenberg couplings and multi-spin interactions become non-negligible, as observed in Ref. 20.

### IV. LIMIT OF WEAKLY COUPLED CHAINS

We now discuss the opposite limit of weakly coupled chains. At the starting point \( J_{\text{rung}} / J_{\text{leg}} = 0 \) the ground state is the product of the ground states of the individual chains.

\[
|\psi_0\rangle = |0\rangle_A |0\rangle_B
\]

(16)

In this limit the entanglement spectrum is trivial, composed of only a single value \( \xi = 0 \).
functions of the two isolated chains, while single chain excitation energies $E$ over the spin components $\{S^z\}$ where $\sum_j S_j^z = S^z$. The expression for the first order wavefunction to $\psi_1$ is the Hamiltonian coupling the two chains [cf. Eq. (15)]. After a straightforward calculation one obtains the following expression for the first order wavefunction $\psi_1$ in the product basis of chain eigenfunctions:

$$\psi_1 = \langle 0 \rangle_A |0\rangle_B + \sum_{n,n',\neq 0} \frac{(n|S^z_0)|n'\rangle S^z_0|0\rangle_B}{\Delta_n + \Delta_n'} |n\rangle_A |n'\rangle_B$$

where $\Delta_k = 1/2$. After straightforward calculation one obtains the following expression for the first order wavefunction $\psi_1$ in the product basis of chain eigenfunctions:

$$\psi_1 = \langle 0 \rangle_A |0\rangle_B + \sum_{n,n',\neq 0} \frac{(n|S^0_0)|n'\rangle S^0_0|0\rangle_B}{\Delta_n + \Delta_n'} |n\rangle_A |n'\rangle_B$$

where $k$ runs over the lattice momenta of a single chain, $\alpha$ runs over the spin components $\{x,y\}$, $n$ and $n'$ label the eigenfunctions of the two isolated chains, while $\Delta_n$ denotes the single chain excitation energies $E_n - E_0$. One recognizes that the nontrivial part of the wave function is composed of contributions which also enter the Lehmann representation of the $S^{\alpha\alpha}(k,\omega)$ spectral functions of a single chain,

$$S^\alpha(k,\omega) = \sum_n |c_{n}^{k,\alpha}|^2 \times \delta(\omega - \Delta_n),$$

$$c_{n}^{k,\alpha} = \langle n|S^\alpha_k|0\rangle,$$

To first order the wave function reads:

$$|\psi_1\rangle = [1 - RQ_0 H_0]|\psi_0\rangle,$$  \(\text{Eq. (17)}\)

where $H_0 = 0$ is the Hamiltonian coupling the two chains [cf. Eq. (1)], $Q_0 = 0$ is the projector onto the subspace orthogonal to $|\psi_0\rangle$, and $R$ denotes the resolvent operator $R = (H_1 - E_0)^{-1}$. After a straightforward calculation one obtains the following expression for the first order wavefunction $|\psi_1\rangle$ in the product basis of chain eigenfunctions:

$$\psi_1 = \langle 0 \rangle_A |0\rangle_B + \sum_{n,n',\neq 0} \frac{(n|S^z_0)|n'\rangle S^z_0|0\rangle_B}{\Delta_n + \Delta_n'} |n\rangle_A |n'\rangle_B$$

where $k$ runs over the lattice momenta of a single chain, $\alpha$ runs over the spin components $\{x,y\}$, $n$ and $n'$ label the eigenfunctions of the two isolated chains, while $\Delta_n(n')$ denotes the single chain excitation energies $E_n(n') - E_0$. One recognizes that the nontrivial part of the wave function is composed of contributions which also enter the Lehmann representation of the $S^{\alpha\alpha}(k,\omega)$ spectral functions of a single chain,

$$S^\alpha(k,\omega) = \sum_n |c_{n}^{k,\alpha}|^2 \times \delta(\omega - \Delta_n),$$

$$c_{n}^{k,\alpha} = \langle n|S^\alpha_k|0\rangle,$$
resent all the predicted entanglement levels appearing at this order. The numerical results match perfectly the analytical prediction up to the finite precision threshold of the numerics.

V. DISCUSSION AND CONCLUSION

In summary, we have investigated the entanglement spectra of spin-1/2 XXZ ladders using both exact numerical diagonalizations and perturbation theory approaches. In the limit of strong rung coupling perturbation theory predicts in leading order an entanglement Hamiltonian being also of nearest-neighbor XXZ form with a renormalized anisotropy parameter. For XX and isotropic Heisenberg ladders no such renormalization takes place and in this case one can define an ad hoc effective temperature proportional to the $J_{\text{rung}}/J_{\text{leg}}$ ratio. Moreover, in next-to-leading order the entanglement Hamiltonian exhibits spin couplings of longer range, and all the above findings are in perfect agreement with our numerical exact diagonalization results. The first order perturbation theory result has also been obtained in parallel work by Peschel and Chung. Similar numerical observations have already made for quantum Hall bilayers at filling factor $\nu = 1$. It remains an open task to devise analogous perturbational arguments for such systems with long-ranged interactions.

In the opposite regime of strong coupling along the legs, we point out an interesting connection of the entanglement spectrum to the Lehmann representation of single chain spectral functions of the operators contained in the physical Hamiltonian coupling the two chains. This aspect also holds true for other ladder systems, such as e.g. Hubbard ladders, where the entanglement spectrum will be determined by elements of the single particle addition and removal spectral functions in the weakly coupled chain regime.

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Appendix A: Second-order correction to the ground state and the reduced density matrix

The second-order correction to the ground state can be obtained as

$$|2\rangle = \left( \frac{J_{\text{leg}}}{J_{\text{rung}}} \right)^2 \sum_{|m-n|>1} \left[ \frac{1}{4(1+\Delta)^2} \left( |t_{m,n}^+, t_{m+1,n}^-, t_{n+1}^-; t_{m+1,n}^+, t_{n+1}^-; t_{n+1}^-; t_{m+1,n}^+; t_{n+1}^+ \rangle \right) - \frac{\Delta}{4(3+\Delta)} \left( \frac{1}{2} + \frac{1}{1+\Delta} \right) \left( |t_{m,n}^+, t_{m+1,n}^-, t_{0,n+1}^0; t_{m+1,n}^+, t_{0,n+1}^0; t_{0,n+1}^0; t_{m+1,n}^+; t_{0,n+1}^+ \rangle \right) + \frac{\Delta^2}{32} \left( |t_{m,n}^0; t_{m+1,n}^0; t_{n+1}^0 \rangle \right) \right] + \left( \frac{J_{\text{leg}}}{J_{\text{rung}}} \right)^2 \sum_m \left[ \frac{1}{2(1+\Delta)} \left( |t_{m,n}^+, t_{m+2,n}^0+; t_{m,n}^0; t_{m+2,n}^+ \rangle + \frac{\Delta^2}{8} |t_{m,n}^0; t_{m+2,n}^0 \rangle \right) + \frac{\Delta}{8(1+\Delta)} \left( \frac{1}{2} + \frac{1}{1+\Delta} \right) \left( |t_{m,n}^+, t_{m+1,n}^-, t_{m+1,n}^+; t_{m,n}^0; t_{m+1,n}^0 \rangle \right) - \frac{1}{4(1+\Delta)} \left( |t_{m,n}^0; t_{m+1,n}^0 \rangle \right) \right] \right) \right). \quad (A1)$$

Note that no terms with an odd number of triplets occur. In the isotropic Heisenberg case $\Delta = 1$, the second-order contribution to the reduced density matrix is the sum of the following expressions,

$$\text{tr}_{\text{leg}} (|0\rangle \langle 2| + |2\rangle \langle 0|)$$

$$= \left( \frac{J_{\text{leg}}}{J_{\text{rung}}} \right)^2 \frac{1}{2T} \left( \sum_{|m-n|>1} (\tilde{S}_m \tilde{S}_{m+1}) \left( \tilde{S}_n \tilde{S}_{n+1} \right) \right) + \sum_m \left[ \tilde{S}_m \tilde{S}_{m+2} - \tilde{S}_m \tilde{S}_{m+1} - \frac{3}{16} L \right], \quad (A2)$$

$$\text{tr}_{\text{leg}} (|1\rangle \langle 1|)$$

$$= \left( \frac{J_{\text{leg}}}{J_{\text{rung}}} \right)^2 \frac{1}{2T} \left( \sum_{|m-n|>1} (\tilde{S}_m \tilde{S}_{m+1}) \left( \tilde{S}_n \tilde{S}_{n+1} \right) \right) + \frac{1}{2} \sum_m \left[ \tilde{S}_m \tilde{S}_{m+2} - \tilde{S}_m \tilde{S}_{m+1} + \frac{3}{16} L \right]. \quad (A3)$$
Now using the identities

\[ \frac{1}{2} \vec{S}_1 \vec{S}_3 = (\vec{S}_1 \vec{S}_2) (\vec{S}_2 \vec{S}_3) + (\vec{S}_2 \vec{S}_3) (\vec{S}_1 \vec{S}_2) \]  \hspace{1cm} (A4)

and

\[ \frac{3}{16} - \frac{1}{2} \vec{S}_1 \vec{S}_2 = (\vec{S}_1 \vec{S}_2)^2 \]  \hspace{1cm} (A5)

for spin-1/2-operators, one derives the result (13).