REFLEXIVITY REVISITED

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Abstract. We study some aspects of reflexive modules. For example, we search conditions for which reflexive modules are free or close to free modules.

Contents

1. Introduction 1
2. Reflexivity and finiteness 3
3. Homological reflexivity 6
4. Being free 7
5. Reflexivity of multi dual 19
6. Treger’s conjecture 21
7. Remarks on a question by Braun 24
8. Descent from (to) the endomorphism ring and tensor products 27
9. Reflexivity and UFD 31
10. Reflexivity of ideals 33
11. Questions by Holanda and Miranda-Neto 39
References 42

1. Introduction

In this note \( (R, m, k) \) is a commutative noetherian local ring and \( M \) is a finitely generated \( R \)-module, otherwise specializes. The notation \( M \) stands for a general module. For simplicity, the notation \( M^* \) stands for \( \text{Hom}_R(M, R) \). Then \( M \) is called reflexive if the natural map \( \varphi_M : M \to M^{**} \) is bijection. Finitely generated projective modules are reflexive. In his seminal paper [33], Kaplansky proved that projective modules over local rings are free. The local assumption is really important. Indeed, there are a lot of interesting research papers, and even books, on the freeness of projective modules over polynomial rings with coefficients from a field. In general, the class of reflexive modules is extremely big compared to the projective modules.

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As a generalization of Seshadri’s result, Serre observed over 2-dimensional regular local rings that finitely generated reflexive modules are free in 1958. This result has some applications: For instance, in the arithmetical property of Iwasawa algebras (see [51]). It seems freeness of reflexive modules is subtle even over very special rings. For example, in [39] Page 518] Lam says that the only obvious examples of reflexive modules over \( R := k[X,Y]/(X,Y)^2 \) are the free modules \( R^n \). Ramras posed the following:

**Problem 1.1.** (See [23] Page 380]) When are finitely generated reflexive modules free?

Over quasi-reduced rings, Problem 1.1 was completely answered (see Proposition 4.22). Ramras proved any finitely generated reflexive module over BNSI (betti numbers strictly increasing) rings is free. We present some applications of this result. Also, we introduce the class of eventually BNSI rings and we study freeness of reflexive modules over them. We know any nonzero free module decomposes into a direct sum of rank one submodules. Treger conjectured (see [57] Page 462]):

**Conjecture 1.2.** Let \((R, m, k)\) be a complete local (singular and containing a field) normal domain of dimension 2 where \( k = \overline{k} \) and \( \text{char}(k) \neq 2 \). Then \( R \) is a cone such as \( \overline{k}[x,y,z]/(x^2 + y^2 + z^2) \) if and only if every nonzero reflexive module \( M \) decomposes into a direct sum of rank one submodules.

§2 connects reflexivity to the finiteness conditions including the finitely generation. As an application, we study the following:

**Question 1.3.** When are quasi-reflexive modules flat?

§3 collects different notions of reflexivity. It may be worth to mention that there are some relations to the miracle of set theory with application to homological reflexivity. For instance, see the seminal work of Shelah [52].

§4 deals with freeness of reflexive modules over some classes of rings. This section is divided into 5 subsections: First subsection deals with freeness of reflexive modules. The second is about of freeness of totally reflexive modules. In Subsection 4.3 investigates freeness of weakly Gorenstein modules. The forth subsection deals with freeness of \( M^* \) against the freeness of \( M \). Subsection 4.5 is devoted to the freeness of specialized (resp. generalized) reflexive modules.

In §5 we investigate the reflexivity of (multi)-dual modules, i.e, we study some homological aspects of

\[ M^{\ell*} := M^* \cdots \cdots . \]

For instance, see Corollary 5.2 and Proposition 5.5 and 5.6. The new invariant is \( \ell(M^{\ast\ast}) \) and its asymptotic behaviors, namely we study the existence of \( \lim_{n \to \infty} \frac{\ell(M^{\ast\ast})_{\text{type}(R)}}{n} \) and detect some numerical invariants in the case that the limit exists. As a sample we present the following:

**Observation 1.4.** Let \((R, m)\) be such that \( m^2 = 0 \) and \( M \) be a finitely generated module with no free direct summands. Then \( \lim_{n \to \infty} \frac{\ell(M^{\ast\ast})_{\text{type}(R)}}{n} = \beta_0(M) \).

In §6 we settle Conjecture 1.2. In §7 we deal with a question of Braun:
Question 1.5. Let $I \triangleleft R$ be a reflexive ideal of a normal domain with $\text{id}_R(I) < \infty$. Is $I \simeq \omega_R$?

In §8 we descent freeness (resp. reflexivity) from the endomorphism ring to the module. This is inspired by the paper of Auslander-Goldman. Similarly, we descent some data from the higher tensor products to the module. In particular, we slightly extend some results of Vasconcelos, Huneke-Wiegand and the recent work of Česnavičius. For example, see Corollary 8.3.

Grothendieck solved a conjecture of Samuel, see Theorem 9.1. In §9 we try to understand this miracles by looking at the mentioned result of Auslander-Goldman. We do this by using some results of §8. In particular, there is a connection between Problem 1.1 and UFD property of regular (complete-intersection) rings, see Corollary 9.3 as a sample. Samuel remarked that there is no symmetric analogue of Auslander’s theorem on torsion part of tensor powers modules over regular rings. In this regard, we present a tiny remark:

Observation 1.6. Let $(R, \mathfrak{m})$ be a regular local ring and $M$ be of rank one. If $\text{Sym}_n(M)$ is reflexive for some $n \geq \max\{2, \dim R\}$, then $M$ is free.

Also, see Corollary 9.9.

In §10 we investigate the reflexivity of ideals. The motivation is a result of Bass [9, Theorem 6.2] which says that an artinian local ring is Gorenstein iff all of its ideals are reflexive. We remark that reflexivity of maximal ideal does not imply reflexivity of ideals, see Discussion 10.8. In particular, we connect to a recent result of Faber [19]. Then, we extend the mentioned theorem of Bass by showing that an artinian local ring is Gorenstein iff its maximal ideal is reflexive.

In the final section we will give a positive answer to the following question asked by Holanda and Miranda-Neto [30] where they called it as a Gorenstein analogue of the famous conjecture of Zariski-Lipman:

Question 1.7. Let $R$ be a Cohen-Macaulay local ring with canonical module $\omega_R$. Suppose $p.\dim_R(\omega^*_R) < \infty$. Must $R$ be Gorenstein?

It may be worth to note that its proof uses freeness of some reflexive modules. We apply this theme to present more applications to [30]. For more details, see Observation 11.4-11.8.

A number of examples and remarks are given.

2. Reflexivity and finiteness

All rings are noetherian. Following Bass, $M$ is called torsion-less if $\varphi_M$ is injective (this some times is called semi-reflexive). A torsion-less module $M$ is noetherian if and only if $M^*$ is noetherian. Submodules of a torsion-less module are torsion-less. We say $M$ is weakly reflexive if $\varphi_M$ is surjective. In general, neither submodule nor quotient of a weakly reflexive module is weakly reflexive.

Observation 2.1. Let $(R, \mathfrak{m})$ be a zero-dimensional Gorenstein local ring. Then the properties of weakly reflexive and finitely generated are the same. In particular, (weakly) reflexivity is not closed under taking direct limits.
Discussion 2.2. The zero dimensional assumption is important. Indeed, by [16, 4.2.1]. Since \( R \) is zero-dimensional and Gorenstein, any module is Gorenstein-projective (see [16, 4.4.8]). It follows by definition that any module is a submodule of a projective module. Over local rings, and by the celebrated theorem of Kaplansky, any projective module is free. Combining these, any module is a submodule of a free module. It follows by definition that any module is torsion-less. Now, let \( \mathcal{M} \) be weakly reflexive. Thus, \( \mathcal{M} \) is reflexive. It is shown in [50, Corollary 2.4.(4)] that over any commutative artinian ring, every reflexive module is finitely generated. By this, \( \mathcal{M} \) is finitely generated. Conversely, assume that \( \mathcal{M} \) is finitely generated. Since \( R \) is zero-dimensional and Gorenstein, \( \mathcal{M} \) is reflexive. To see the particular case, we remark that any module can be written as a directed limit of finitely generated modules. We use this along with the first part to get the claim. \( \square \)

Lemma 2.3. Let \((R, m)\) be a normal domain of dimension bigger than zero and \( \mathcal{M} \) be quasi-reflexive. Then there is a family of finitely generated and reflexive modules \( \{M_i\}_{i \in I} \) such that \( \mathcal{M} = \lim_{\rightarrow} M_i \). Proof. There is a directed family \( \{N_i\}_{i \in I} \) of finitely generated submodules of \( \mathcal{M} \) such that \( \mathcal{M} = \lim_{\leftarrow} N_i \). Let \( \Spec^1(R) := \{p \in \Spec(R), \text{ht}(p) = 1\} \).

For each \( i \), we set \( M_i := \bigcap_{p \in \Spec^1(R)} (N_i)_p \) where we compute the intersection in \( \mathcal{M}_0 := \mathcal{M} \otimes_R Q(R) \). Yuan proved that any flat module is quasi-reflexive (see [35, Lemma 2]).

There are flat modules that are not reflexive, e.g. the vector space \( \bigoplus_{N} \mathbb{Q} \) and the abelian group \( \mathbb{Q} \). In order to handle this drawback, let \( R \) be a normal domain of dimension bigger than zero with fraction field \( Q(R) \). Following Samuel, \( \mathcal{M} \) is called quasi-reflexive if \( \mathcal{M} = \bigcap_{p \in \Spec(R), \text{ht}(p) = 1} M_p \) where we compute the intersection in \( \mathcal{M}_0 := \mathcal{M} \otimes_R Q(R) \). Yuan proved that any flat module is quasi-reflexive (see [35, Lemma 2]).
Thanks to Proposition 1,
\[ N_i^{**} \cong \bigcap_{p \in \text{Spec}^1(R)} (N_i^{**})_p. \]
We put these together,
\[ N_i^{**} \cong \bigcap_{p \in \text{Spec}^1(R)} (N_i^{**})_p \cong \bigcap_{p \in \text{Spec}^1(R)} (N_i)_p. \]
From this, \( M_i \cong N_i^{**} \). In particular, \( M_i \) is finitely generated and reflexive. This completes the proof.

**Proposition 2.4.** Let \((R, \mathfrak{m})\) be a regular local ring of dimension at most two. There is no difference between flat modules and quasi-reflexive modules. In particular, any direct limit of quasi-reflexive modules is quasi-reflexive.

**Proof.** Recall that any flat module is quasi-reflexive. Conversely, let \( M \) be quasi-reflexive. By the above lemma, there is a family of finitely generated reflexive modules \( \{M_i\} \) such that \( M = \lim_{\rightarrow} M_i \).

By the mentioned result of Serre, each \( M_i \) is free. Clearly, direct limit of free modules is flat. We apply this to observe that \( M \) is flat. To see the particular case, we mention that direct limit of flat modules is again flat.

\[ \square \]

**Corollary 2.5.** Let \((R, \mathfrak{m}, k)\) be a normal domain of dimension bigger than zero. The following are equivalent:

i) there is no difference between flat modules and quasi-reflexive modules,

ii) \( R \) is a regular local ring of dimension at most two.

**Proof.** i) \( \Rightarrow \) ii): The second syzygy of \( k \) is reflexive and so quasi-reflexive. By the assumption it is flat. Finitely generated flat modules over local rings are free. Thus, second syzygy of \( k \) is free. This in turn is equivalent with \( p\dim(k) \leq 2 \). In view of local-global-principle, \( R \) is regular and of dimension at most two.

ii) \( \Rightarrow \) i): This is in Proposition 2.4.

\[ \square \]

In the next section we recall the concept of weakly Gorenstein for finitely generated modules over local rings and in §4 we study their freeness. It seems the origin of this comes back to 1950 when Whitehead posed a problem: Let \( G \) be an abelian group such that \( \text{Ext}^+(G, \mathbb{Z}) = 0 \). Is \( G \) free? Shelah [52] proved that this is undecidable in ZFC. By ZFC we mean the Zermelo–Fraenkel set theory with the axiom of choice included. There are a lot of notions between free abelian groups and reflexive abelian groups if we have no finiteness restrictions:

\[ \text{free} \Rightarrow \text{hereditarily separable} \Rightarrow \text{coseparable} \Rightarrow \text{separable} \Rightarrow \aleph_1\text{-free} \Rightarrow \text{torsionless}, \]

and that

\[ \aleph_1\text{-coseparable} \Rightarrow \text{reflexive} \Rightarrow \text{torsionless}. \]

To see these, we cite the book [18], and record a question:
Question 2.6. Let $G, H$ be reflexive abelian groups. Is $G \otimes \mathbb{Z} H$ reflexive?

Let us connect to [54] and cite the following analytic theme of reflexivity:

Fact 2.7. Let $V$ be a Banach space. Then $\text{Hom}(V, \mathbb{R})$ is a Smith space. If $W$ is a Smith space, then $\text{Hom}(W, \mathbb{R})$ is a Banach space, where in both cases we endow the dual space with the compact-open topology, with the corresponding bi duality maps are isomorphisms.

3. Homological reflexivity

Here, modules are finitely generated. We recall different notions of reflexivity (our reference book on this topic is [16]). A reflexive module $M$ is called totally reflexive if $\text{Ext}^i(M, \mathbb{R}) = \text{Ext}^i(M^*, \mathbb{R}) = 0$ for all $i > 0$. By $\mu(M)$ we mean the minimal number of generators of $M$.

Observation 3.1. Let $(R, \mathfrak{m})$ be a local ring such that $\mathfrak{m}^2 = 0$. Then any finitely generated weakly reflexive module is totally reflexive.

Proof. Let $M$ be weakly reflexive. Suppose first that $\mu(\mathfrak{m}) = 1$. It follows that $R$ is zero-dimensional Gorenstein ring. Over such a ring any finitely generated module is totally reflexive. Suppose now that $\mu(\mathfrak{m}) > 1$. Let $D(-)$ be the Auslander transpose. The cokernel of $(-) \rightarrow (-)^{**}$ is $\text{Ext}^2_R(D(-), R)$. Since $M$ is weakly reflexive, $\text{Ext}^2_R(D(M), R) = 0$. Menzin proved that $\text{Ext}^i_R(D(M), R) = 0$ for some $i > 1$ is equivalent to the freeness (see §4 for more details). □

Recall that a module $M$ is called weakly Gorenstein if $\text{Ext}^i_R(M, \mathbb{R}) = 0$ for all $i > 0$, see e.g. [39].

Definition 3.2. A local ring $(R, \mathfrak{m})$ is called quasi-reduced if it satisfies Serre’s condition $(S_1)$ and be generically Gorenstein.

Remark 3.3. Let $(R, \mathfrak{m})$ be a quasi-reduced local ring. The following are equivalent:

i) any reflexive module is totally reflexive,

ii) any reflexive module is weakly Gorenstein,

iii) $R$ is a Gorenstein ring of dimension at most two.

Proof. i) ⇒ ii): This is clear.

ii) ⇒ iii): Let $M$ be reflexive. By a result of Hartshorne (see Fact 5.14 below) $M^*$ is reflexive. By our assumption, $M$ and $M^*$ are weakly Gorenstein. By definition, $\text{Ext}^+_R(M, \mathbb{R}) = \text{Ext}^+_R(M^*, \mathbb{R}) = 0$. Thus, $M$ is totally reflexive. Over quasi-reduced rings, second syzygy modules are reflexive. From this, $\text{Syz}_2(k)$ is reflexive. We proved that $\text{Syz}_2(k)$ is totally reflexive. By this, $R$ is a Gorenstein ring of dimension at most two.

iii) ⇒ i): This is clear by Auslander-Bridge formula. □

The quasi-reduced assumption is important, see Observation 4.1. Also, we say $M$ is skew Gorenstein if $\text{Ext}^i_R(M^*, \mathbb{R}) = 0$ for all $i > 0$. 


**Definition 3.4.** We say $M$ is homologically reflexive if $\text{Ext}^i(M, R) = \text{Ext}^i(M^*, R) = 0$ for all $i > 0$. Also, $M$ is called strongly reflexive if $\text{Ext}^i_R(D(M), R) = 0$ for all $i > 0$ (in the common terminology: $M$ is $n$-torsionless for all $n$).

**Example 3.5.** Let $(R, \mathfrak{m}, k)$ be zero-dimensional but not Gorenstein. Then $k$ is torsion-less but neither reflexive nor weakly Gorenstein.

**Proof.** Since $R$ is not Gorenstein, $\dim_k(\text{Soc}(R)) > 1$. Also, $0 \neq \text{Soc}(R)$ is equipped with a vector space structure. The same thing holds for its dual. In particular, the natural embedding $k \hookrightarrow \text{Soc}(R)^* = k^{**}$ is not surjective. By definition, $k$ is torsion-less and $k$ is not reflexive. Suppose on the contrary that $k$ is weakly Gorenstein. The condition $\text{Ext}^i_R(k, R) = 0$ for all $i > 0$ implies that $R$ is Gorenstein. This is excluded from the assumption. So, $k$ is not weakly Gorenstein. $\square$

**Definition 3.6.** An $R$-module $M$ is called infinite syzygy if there is an exact sequence

$$0 \rightarrow M \rightarrow F_0 \rightarrow \ldots \rightarrow F_n \rightarrow F_{n+1} \rightarrow \ldots$$

where $F_i$ is free.

We note that any module is torsion-less if and only if any module is infinite syzygy. Over artinian rings we can talk a little more:

**Corollary 3.7.** Let $(R, \mathfrak{m}, k)$ be local and zero-dimensional. The following are equivalent:

i) any finitely generated module is torsion-less,

ii) any finitely generated module is infinite syzygy,

iii) the canonical module $\omega_R$ is torsion-less,

iv) $R$ is a Gorenstein ring.

**Proof.** i) $\Rightarrow$ ii) $\Rightarrow$ iii): These are trivial.

iii) $\Rightarrow$ i): Recall that $\omega_R = E(k)$. Note that any module can be embedded into injective modules. Any injective module is a direct sum of $E(k)$. By assumption $\omega_R$ is torsion-less. Thus, $\omega_R$ is submodule of a free module. It follows that any module can be embedded into free modules. Therefore, any finitely generated module is torsion-less.

ii) $\Leftrightarrow$ iv): See [42, Proposition 2.9]. Here, we present a shorter argument. Suppose $\omega_R$ is torsionless. So, it is a submodule of a free module $F$. Indeed, $0 \rightarrow \omega_R \rightarrow F \rightarrow \text{coker}(\subseteq) \rightarrow 0$ splits, as $\omega$ is injective. So, $\omega$ is free. Consequently, $R$ is Gorenstein. $\square$

**Question 3.8.** (Iyengar) Suppose $k$ is an infinite syzygy. Is $R$ zero-dimensional and Gorenstein?

4. Being free

This section is divided into 5 subsections: The first deals with freeness of reflexive modules. The second subsection is about of freeness of totally reflexive modules. The third subsection investigates freeness of weakly Gorenstein modules. 4.4 deals with freeness of $M^*$ against the freeness of $M$. The final subsection is devoted to the freeness of specialized (resp. generalized) reflexive modules.
4.1. Freeness of reflexive modules. We present three different arguments for Lam’s prediction. One of them implicitly is in the PhD thesis of Gover [24].

**Observation 4.1.** Let $k$ be an algebraically closed field and let $R := \frac{k[(X,Y)]}{(X,Y)^2}$. Then the only examples of nonzero reflexive modules are the free modules $R^\ell$ for some $\ell \in \mathbb{N}$.

**Proof.** Let $\mathcal{M}$ be reflexive. Recall that over any commutative artinian ring, every reflexive module is finitely generated. Hence, $\mathcal{M}$ is finitely generated. Without loss of the generality we may assume that $\mathcal{M}$ is indecomposable. Several years ago, these modules were classified by Kronecker. We use a presentation given by Hartshorne (see [28, Page 60]). We find that any indecomposable finitely generated is isomorphic to one of the following:

i) $M_n := m^{n-1}/m^{n+1}$,

ii) $N_{n,p} := M_n/(p^n)$ where $p \in P_k^1$ representing a linear form $p = ax + by$ with $a, b \in k$, or

iii) $W_n = M_n/(x^n, y^n)$.

Note that $M_1 = R$ (resp. $M_2 = R$) which is free. Also, $M_2 = m$. We claim that $m$ is not reflexive. Indeed, $m$ is a vector space of dimension 2. It follows that $m^*$ is a vector space of dimension 4 and so $m^{**}$ is a vector space of dimension 8. From this, $m$ is not reflexive. Now, we compute $N_{1,p}^*$. To this end,

\[ N_{1,p}^* = \{ r \in R : rp = 0 \} = m. \]

Hence $N_{1,p}^* \cong k \oplus k$. Since $p \in m$ and $m^2 = 0$, we see $N_{1,p} = R/(p)$ is a vector space and $\dim_k(N_{1,p}) = 2$. From this,

\[ N_{1,p}^{**} = m^* = k^4 \cong \neq N_{1,p}. \]

This implies that $N_{1,p}$ is not reflexive. The same proof shows that $N_{2,p}$ is not reflexive. Recall that $W_1 = k$. Also, $W_{1,p}^* = k^4 \cong k = W_1$ and so $W_1$ is not reflexive. The same proof shows that $W_2$ is not reflexive.

The following is the second proof of Lam’s prediction and is implicitly in Gover’s thesis.

**Discussion 4.2.** Let $k$ be any field and let $R := \frac{k[(X,Y)]}{(X,Y)^2}$. Then the only examples of nonzero reflexive modules are the free modules $R^\ell$ for some $\ell \in \mathbb{N}$.

**Proof.** Let $M$ be reflexive. Since the ring is artinian, we may assume that $M$ is finitely generated. It is easy to show that any finitely generated module such as $M$ can be written $M = M_t \oplus F$ where $F$ is free and $M_t^* = \text{Hom}(M_t, m)$. Since $m^2 = 0$, $m$ is a vector space of dimension 2. We apply this observation to see

\[ M_t^* = \text{Hom}(M_t, m) = \oplus_2 \text{Hom}(M_t, k). \]

Take another duality,

\[ M_t^{**} = \text{Hom}(M_t, m)^* = \oplus_2 \text{Hom}(M_t, k)^* = \oplus_2 \text{Hom}(\text{Hom}(M_t, k), m) =: \oplus_2 \text{Hom}(V, m) \]

His beautiful argument appeared in the proof of the following fact: Let $(S, n)$ be a regular local ring of dimension 2. Then $R := S/n^2$ is not Gorenstein. One may prove this by only saying that type of $R$ is two.
since $V := \text{Hom}(M_t, k)$ does not contain a free direct summand. Hence $M_t^{**} = \oplus \text{Hom}(V, k)$. Following Gover, we take another duality, and this shows that

$$M_t^{***} = \oplus \text{Hom}(\text{Hom}_k(V, k), k).$$

Direct summand and dual of reflexive modules is again reflexive. So, $M_t^{***} \cong M_t^*$. From this we get that

$$2 \dim V = 8 \dim V.$$

In other words $V = 0$, and consequently $M_t^* = 0$. Take another duality, $M_t \cong M_t^{**} = 0$. In particular, $M$ is free. □

Recall that the $i^{th}$ betti number of $M$ is given by $\beta_i(M) := \dim_k \text{Tor}_i R^k(M, k)$. Suppose $x \in m$ is nonzero. Note that $R \xrightarrow{x} R \to R/xR \to 0$ is a part of minimal free resolution. By definition, $\beta_0(R/xR) = \beta_1(R/xR)$. By this reason, a ring is called BNSI if for every non-free module $M$ we have $\beta_i(M) > \beta_{i-1}(M)$ for $i > 1$. We will use the following result several times:

**Fact 4.3.** (See [17, 2.4 and 2.5]) Let $R$ be BNSI. Suppose $\text{Ext}_R^i(M, R) = 0$ for some $i \geq 2$. Then $M$ is free. Also, finitely generated weakly reflexive modules are free.

The following extends Lam’s prediction:

**Corollary 4.4.** Let $(R, m)$ be a local ring such that $\mu(m) > 1$ and suppose $R$ is one of the following types:

i) $R := k[[X_1, \ldots, X_m]]/(X_1^{n_1}, \ldots, X_m^{n_m})$, or

ii) $R$ is such that $m^2 = 0$.

Then the only examples of nonzero reflexive modules are the free modules $R^\ell$ for some $\ell \in \mathbb{N}$. Also, over $k[[X]]/(X)^n$ there is no difference between finitely generated modules and reflexive modules.

**Proof.** Let $M$ be reflexive. Recall that over any commutative artinian ring, every reflexive module is finitely generated. Hence, $M$ is finitely generated. By [17, 3.3], $k[[X_1, \ldots, X_m]]/(X_1^{n_1}, \ldots, X_m^{n_m})$ is BNSI provided that $m > 1$. Also, in the second case, the ring $R$ is BNSI, see the proof of Observation 3.4. In the light of Fact 4.3, $M$ is free and of finite rank. For the last claim its enough to note that $k[[X]]/(X)^n$ is zero-dimensional and Gorenstein (see Observation 2.1). □

The following extends [11, Proposition 2 and 4] and [62, Proposition 2.4] by a new proof.

**Corollary 4.5.** Let $(R, m)$ be a local ring such that $m^2 = 0$. If $\mu(m) \neq 1$, then $R$ is BNSI. In particular,

i) if $\text{Ext}_R^i(M, R) = 0$ for some $i \geq 2$, then $M$ is free,

ii) any finitely generated weakly reflexive module is free,

iii) there is no difference between reflexive modules and free modules of finite rank,

iv) there is no difference between skew Gorenstein modules and free modules of finite rank,

v) there is no difference between strongly reflexive modules and free modules of finite rank.
Proof. Since $m^2 = 0$ and $m \neq 0$, we have $m \subseteq (0 : m) \subseteq m$, i.e., $(0 : m) = m$. We apply this to see
\[
\dim_k \left( \frac{(0 : m) + m^2}{m^2} \right) = \dim_k \left( \frac{m}{m^2} \right) = \mu(m) > 1.
\]

Let $M$ be nonfree. By Auslander-Buchsbaum formula, $p \dim(M) = \infty$ because depth($R$) = 0. In particular, all betti numbers are nonzero. We recall from [20, Proposition 2.4] that
\[
\beta_{i+1}(M) \geq \left( \dim_k \left( \frac{(0 : m) + m^2}{m^2} \right) \right) \beta_i(M) \quad \forall i \geq 1.
\]

Since $\beta_i(M) \neq 0$ and $\dim_k \left( \frac{(0 : m) + m^2}{m^2} \right) > 1$ we have
\[
\left( \dim_k \left( \frac{(0 : m) + m^2}{m^2} \right) \right) \beta_i(M) > \beta_i(M).
\]

We combine these to see
\[
\beta_{i+1}(M) \geq \left( \dim_k \left( \frac{(0 : m) + m^2}{m^2} \right) \right) \beta_i(M) > \beta_i(M)
\]
for all $i \geq 1$. By definition, $R$ is BNSI. Fact 4.3 yields i) and ii). Let $M$ be reflexive. Recall that over any commutative artinian ring, every reflexive module is finitely generated. Hence, $M$ is finitely generated. By Fact 4.3 we get iii). Let $M$ be skew Gorenstein. By i), $M^*$ is free. In general freeness of ($-$) can not follow from ($-$)*. But, if depth of a ring is zero this happens (see [47, Lemma 2.6]). Since depth($R$) = 0, we conclude that $M$ is free. It remains to prove v): This is trivial, because strongly reflexive modules are reflexive and by iii) reflexive modules are free. □

Let $(R, m, k)$ be any Gorenstein ring with $m^3 = 0$ which is not field. Then $k$ is totally reflexive but it is not free. To find nonfree totally reflexive modules over non-Gorenstein ring with $m^3 = 0$ it is enough to look at [62]. The following extends and corrects [31, Examples (3)] where it is shown that weakly Gorenstein modules are free without assuming $m^2 \neq (0 : m)$.

Corollary 4.6. Let $(R, m)$ be a local ring such that $m^3 = 0$, $\mu(m) > 1$ and that $m^2 \neq (0 : m)$. Then $R$ is BNSI. For finitely generated modules we have:

\[
\begin{align*}
\text{weakly Gorenstein} & \iff \text{homologically reflexive} \iff \text{totally reflexive} \iff \text{strongly reflexive} \iff \text{reflexive} \iff \text{weakly reflexive} \iff \text{skew Gorenstein} \iff \text{free}
\end{align*}
\]

Proof. The claim in the case $m^2 = 0$ is in Corollary [13] (here, we used the assumption $\mu(m) > 1$). Without loss of the generality we may assume that $m^2 \neq 0$. Let $M$ be non-free. In view of [36, Lemma 3.9] $\beta_i(M) > \beta_{i-1}(M)$ for $i > 0$. By definition, $R$ is BNSI. In view of Fact [13] we see all of the equivalences except the skew Gorenstein property. Suppose $M$ is skew Gorenstein. In the same vein and similar to Corollary [13] we see that $M$ is free. □

Freeess of weakly Gorenstein modules over $\frac{Q[X,Y]}{(X^2,XY,Y^2)}$ follows from [46, Corollary 4.8]. What can one say on the freeness of reflexive modules? We have:

Example 4.7. Let $R := \frac{Q[X,Y]}{(X^2,XY,Y^2)}$. Then any reflexive module is free.
**Proof.** Let \( M \) be any reflexive. Since \( \dim R = 0 \), \( M \) is finitely generated. We have \( \mathfrak{m} = (x, y) \) and that \( \mathfrak{m}^3 = (x^3, x^2y, xy^2, y^3) = 0 \). Let us compute the socle. By definition,
\[
(0 : \mathfrak{m}) = (0 : x) \cap (0 : y) = (x, y) \cap (x, y^2) = (x, y^2).
\]
Clearly, \( (0 : \mathfrak{m}) \neq \mathfrak{m}^2 \). In view of Corollary 4.4, \( M \) is free. \( \square \)

Freeness of totally reflexive is not enough strong to deduce freeness of reflexive:

**Example 4.8.** Let \( R := \mathbb{Q}[[X,Y,Z]]/(XY,XZ) \). Then totally reflexive are free and there is a reflexive module which is not free.

**Proof.** The ring \( R \) is the fiber product \( \mathbb{Q}[[X]] \times_{\mathbb{Q}} \mathbb{Q}[[Y,Z]] \) (see [15, Ex. 3.9]). Since the ring is not Gorenstein (e.g. it is not equi-dimensional) and is fiber product, any totally reflexive is free, see [16, Corollary 4.8]. Now, we look at \( M := R/(y,z) \). We have,
\[
\text{Hom}(M, R) \simeq \{ r \in R : r(y,z) = 0 \} \simeq xR \simeq R/(0 : x) = R/(y,z) = M.
\]
From this, \( M \simeq M^{**} \). By [16, 1.1.9(b)] a finitely generated module is reflexive if it is isomorphic to its bidual. So, \( M \) is reflexive. Since it annihilated by \( (y,z) \) we see that it is not free. \( \square \)

4.2. **Freeness of totally reflexive modules.**

**Definition 4.9.** We say a local ring is eventually BNSI if there is an \( \ell \geq 1 \) such that for every non-free module \( M \) we have \( \beta_i(M) > \beta_{i-1}(M) \) for \( i > \ell \).

**Example 4.10.** Let \((A, \mathfrak{n})\) be a regular local ring of dimension \( n > 1 \) and \( f \in \mathfrak{n} \) be nonzero. Let \( R := \frac{A}{(y,z)} \). The following assertions hold:

i) Any finitely generated totally reflexive module is free.

ii) For every non-free module \( M \) we have:
\[
\ldots > \beta_{n+2}(M) > \beta_{n+1}(M) > \beta_n(M) \geq \beta_{n-1}(M) \geq \ldots \geq \beta_1(M).
\]

iii) Let \( D(-) \) be the Auslander transpose and let \( M := D(fR) \). Then
\[
\ldots > \beta_{n+2}(M) > \beta_{n+1}(M) > \beta_n(M) \geq \beta_{n-1}(M) \geq \ldots \geq \beta_3(M) = n > \beta_2(M) = \beta_1(M) = 1 < n = \beta_0(M).
\]

In fact, \( fR \) is non-free but reflexive.

In particular, the ring \( R \) is eventually BNSI but is not BNSI.

The fact that \( fR \) is reflexive is due to Ramras by a different argument. Also, part i) extends [55, Example 5.1] in three directions via a new argument.

\*The paper [7, 3.5] claims that \( \beta_i(\cdot) \geq \beta_i(\cdot) \) for all \( i > \text{depth}(R) \) over any Golod ring \( R \) which is not hypersurface and all modules of infinite projective dimension. This is in contradiction with iii). I feel that [7] has a misprint and the mentioned result should be stated that \( \beta_i(\cdot) > \beta_i(\cdot) \) for all \( i > \text{emb}(R) \).
Proof. i) The ring $R$ is Golod. Note that codepth$(R) := \text{emb}(R) - \text{depth } R = n - 0 > 1$. Rings of codepth at most one are called hypersurfaces. Avramov and Martsinkovsky proved, over a Golod local ring that is not hypersurface, that every any module of finite Gdim is of finite p. dim. Also, this is well-known that Gdim is the same as of p. dim provided p. dim is finite. By this, any finitely generated totally reflexive module is free.

ii) Since depth$(R) = 0$ any non-free is of infinite projective dimension. Lescot proved over a Golod ring $R$ which is not hypersurface, $\beta_i(M) > \beta_{i-1}(M)$ for $i > \text{emb}(R)$ and all $M$ of infinite projective dimension (see [37, 6.5]). Also [47] Proposition 3.4] says that the betti sequence is not decreasing. By these,

$$\beta_{n+1}(M) > \ldots > \beta_{n+1}(M) > \beta_n(M) \geq \beta_{n-1}(M) \geq \ldots \geq \beta_1(M).$$

iii) By [47] Proposition 3.5, $R$ is not BNSI. It follows from ii) that there is a non-free module $N$ and some $i < n$ such that $\beta_{i+1}(N) = \beta_i(N)$. Let us find them. Let $n := (X_1, \ldots, X_n)$. Since $(0 : R f) = m$ the minimal presentation of $fR$ is $R^n \rightarrow R \rightarrow fR \rightarrow 0$. Apply Hom(−, $R$) we have

$$0 \rightarrow (fR)^* \rightarrow R^* \rightarrow R^n \simeq (R^n)^* \rightarrow D(fR) \rightarrow 0 \quad (\ast)$$

Then $R \rightarrow R^n$ is the minimal presentation of $D(fR)$ if $(fR)^*$ has no free direct summand. Since depth $R = 0$ and in the light of [47] Lemma 2.6], $(fR)^*$ has no free direct summand. This implies that $R \xrightarrow{\varphi} R^n \rightarrow D(fR) \rightarrow 0$ is the minimal presentation of $D(fR)$, where $\varphi := (x_1, \ldots, x_n)$. Since

$$\ker(\varphi) = \bigcap \text{Ann}(x_i) = (0 : m) = fR \quad (\ast, \ast)$$

we see that $R \rightarrow R \xrightarrow{\varphi} R^n \rightarrow D(fR) \rightarrow 0$ is part of the minimal free resolution of $D(fR)$. That is

$$\beta_2(D(fR)) = \beta_1(D(fR)) = 1 < n = \beta_0(D(fR)).$$

Similarly,

$$\beta_2(D(fR)) = \mu(0 : f) = \mu(m) = n.$$

It follows from $(\ast)$ and $(\ast, \ast)$ that $(fR)^* = fR$. Consequently, $fR$ is reflexive. Since $m$ annihilated $fR$ we deduce that $fR$ is not free.

By ii) the ring is eventually BNSI. By iii) $R$ is not BNSI. \qed

Lemma 4.11. Let $(R, m)$ be a local ring of depth zero. Suppose there is $\ell$ such that the betti sequence $\beta_i(M) > \beta_{i-1}(M)$ for $i > \ell$ and all nonfree totally reflexive module $M$. Then $\beta_i(M) > \beta_{i-1}(M)$ for $i > 0$.

Proof. The idea is taken from [48]. We look at the minimal free resolution

$$\cdots \rightarrow R^{\beta_i(M^*)} \rightarrow \cdots \rightarrow R^{\beta_0(M^*)} \rightarrow M^* \rightarrow 0.$$ 

Since $M$ is totally reflexive Ext$^+(M^*, R) = 0$. By duality, there is an exact sequence

$$0 \rightarrow M^{**} \simeq M \rightarrow R^{\beta_0(M^*)} \xrightarrow{d_0} R^{\beta_1(M^*)} \xrightarrow{d_1} \cdots$$
We set $N := \ker(d_{\ell+1})$. We note $\beta_i(M) = \beta_{i+\ell}(N)$. Since $M$ is totally reflexive and in view of $0 \to M \to R^{\beta_0} \to \ldots \to R^{\beta_\ell} \to N \to 0$, we deduce that $N$ is of finite G-dimension. It follows from Auslander-Bridger-formula that $N$ is totally reflexive. By the assumption, $\beta_i(N) > \beta_{i-1}(M)$ for $i > \ell$. From this we get $\beta_i(M) > \beta_{i-1}(M)$ for $i > 0$. □

Remark 4.12. Depth of any eventually BNSI ring is zero. Indeed, suppose there is a regular element $x$. In view of the exact sequence $0 \to R \to R \to R/xR \to 0$ we see that $\beta_i(M) = \beta_{i-1}(M) = 0$ for $i > 1$, a contradiction.

Problem 1.1 specializes (see [7, Page 402]): When totally reflexive modules are free?

Corollary 4.13. Let $(R, \mathfrak{m})$ be eventually BNSI. Then any finitely generated totally reflexive module is free.

Example 4.10 shows that one can not replace totally-reflexivity with the reflexivity. This can follows from [7]. However, the following proof is so easy:

Proof. There is $\ell$ such that $\beta_i(M) > \beta_{i-1}(M)$ for $i > \ell$ and all non-free $M$. Due to the above remark, depth $R = 0$. In view of Lemma 4.11 we can take $\ell = 0$ provided $M$ is totally reflexive and non-free. Since depth $R = 0$ and in the light of [17] Lemma 2.6], $M^*$ is not free. Since $M^*$ is not free, $D(M)$ is not free. Let $F_1 \to F_0 \to M \to 0$ be a minimal presentation of $M$. We observed that $0 \to M^* \to F_0^* \to F_1^* \to D(M) \to 0$ provides a minimal presentation of $D(M)$. Recall that Gdim$(\cdot) = 0$ if and only if Gdim$(D(\cdot)) = 0$. By this $D(M)$ is totally reflexive. Also,

$$\beta_0(M^*) = \beta_2(D(M)) > \beta_1(D(M)) = \beta_0(M).$$

Another use of the former observation implies that

$$\beta_0(M) = \beta_0(M^{**}) > \beta_0(M^*) > \beta_0(M).$$

This is a contradiction that we searched for it. □

The above argument shows a little more:

Remark 4.14. Let $(R, \mathfrak{m})$ be a local ring. Suppose there is $\ell$ such that $\beta_i(M) > \beta_{i-1}(M)$ for $i > \ell$ and all nonfree totally reflexive module $M$. Then, any finitely generated totally reflexive module is free.

Proof. We assume depth$(R) > 0$. Let $M$ be totally reflexive. Suppose on the contradiction that $M$ is not free. It turns out that $M$ is of infinite projective dimension. Without loss of generality we may assume that $M$ has no free direct summand (any direct summand of weakly reflexive module is weakly reflexive). If $M^*$ has a free direct summand, then its dual $M^{**}$ has a free direct
summand, and in view of $M \simeq M^{**}$ we get a contradiction. We proved that $M^*$ has no free direct summand. Without loss of generality we may assume that $\ell > \text{depth} R$. We look at the minimal free resolution

$$\cdots \rightarrow R^{d_i}(M^*) \rightarrow \cdots \rightarrow R^{d_0}(M^*) \rightarrow M^* \rightarrow 0.$$  

Since $M$ is totally reflexive $\Ext^+(M^*, R) = 0$. By duality, there is an exact sequence

$$0 \rightarrow M^{**} \simeq M \rightarrow R^{d_0}(M^*) \xrightarrow{d_0} R^{d_1}(M^*) \xrightarrow{d_1} \cdots.$$  

We set $N := \ker(d_{i+1})$. Since $M$ is of finite G-dimension and in view of

$$0 \rightarrow M \rightarrow R^{d_0} \rightarrow \cdots \rightarrow R^{d_i} \rightarrow N \rightarrow 0,$$  

we deduce that $N$ is of finite G-dimension. Due to the exact sequence we have $\Ext^i_R(N, R) = \Ext^{\ell-i}(M^*, R)$ for all $0 < i \leq \text{depth} R$, this is zero because $M$ is totally reflexive. Recall that

$$\Gdim(N) = \sup \{i : \Ext^i_R(N, R) \neq 0\}$$  

and that $\Gdim$ is bounded by depth provided it is finite. We combine these to see that $\Gdim(N) = 0$. Also, $p. \dim(N) = \infty$, because $p. \dim(M) = \infty$. By the assumption, $\beta_i(N) > \beta_{i-1}(N)$ for $i > \ell$.

From this we get $\beta_i(M) > \beta_{i-1}(M)$ for $i > \ell$. In particular, we can take $\ell = 0$. Suppose on the contradiction that $p. \dim(M^*) < \infty$. It follows that

$$p. \dim(M^*) = \sup \{i : \Ext^i_R(M^*, R) \neq 0\}.$$  

This is zero because $M$ is totally reflexive. Freeness pass to dual. From this, $M \simeq M^{**}$ is free, a contradiction. Let $F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$ be a minimal free resolution of $M$. Then we have $0 \rightarrow M^* \rightarrow F_0^* \rightarrow F_1^* \rightarrow D(M) \rightarrow 0$. Apply this to see $p. \dim(D(M)) = \infty$. Also, $F_0^* \rightarrow F_1^* \rightarrow D(M) \rightarrow 0$ is the minimal presentation of $D(M)$, because $M^*$ has no free direct summand. Recall that $\Gdim(D(M)) = 0$. We have

$$\beta_0(M^*) = \beta_2(D(M)) > \beta_1(D(M)) = \beta_0(M).$$  

Another use of the former observation implies that $\beta_0(M) = \beta_0(M^{**}) > \beta_0(M^*) > \beta_0(M)$, a contradiction. \qed

One may like a ring for which every non-free module $M$ there is an $\ell(M)$ such that $\beta_i(M) > \beta_{i-1}(M)$ for $i > \ell(M)$. We say a such ring is weakly BNSI. This property is not enough strong to deduce freeness from the totally reflexivity:

**Observation 4.15.** Let $(R, \mathfrak{m})$ be a Gorenstein local ring such that $\mathfrak{m}^3 = 0$ and $\mu(\mathfrak{m}) > 2$. Then $(R, \mathfrak{m})$ is weakly BNSI. Also, there is a nonfree totally reflexive module.

**Proof.** If $\mathfrak{m}^2$ were be zero then we should have $(0 : \mathfrak{m}) = \mathfrak{m}$. Since the ring is Gorenstein, it follows that $\mu(\mathfrak{m}) = 1$. This excluded by the assumption. We assume that $\mathfrak{m}^2 \neq 0$. We set $n := \mu(\mathfrak{m})$. Since $\mathfrak{m}^3 = 0$ we see that $0 \neq \mathfrak{m}^2 \subset (0 : \mathfrak{m})$. Since $\dim(\soc(R)) = 1$, we have $\ell(\mathfrak{m}^2) = 1$. It follows that

$$\ell(R) = 1 + \ell(\mathfrak{m}) = 1 + \ell(\mathfrak{m}/\mathfrak{m}^2) + \ell(\mathfrak{m}^2) = n + 2.$$
Recall from [22, Proposition 2.2] that

Fact A) Let \((A, n)\) be an artinian ring and \(N\) be finitely generated. Let \(h\) be the smallest \(i\) such that \(n^{i+1} = 0\). Then \(\beta_{i+1}(N) \geq (2\mu(n) - \ell(A) + h - 1)\beta_i(N)\) for all \(n \geq \mu(N)\).

Since \(m^3 = 0\) we have \(h = 2\). Also, 

\[2\mu(m) - \ell(R) + h - 1 = 2n - (n + 2) + 2 - 1 = n - 1 \geq 2.\]

Let \(M\) be nonfree. By Auslander-Buchsbaum formula, \(\beta_i(M) \neq 0\) for all \(i\). In view of Fact A) we see

\[\beta_{i+1}(M) \geq 2\beta_i(M) > \beta_i(M)\] for all \(i \geq \mu(M)\).

By definition, \(R\) is weakly BNSI. Every non-free module \(M\) is totally reflexive (e.g. the residue field), because the ring is Gorenstein. \(\square\)

Example 4.16. The ring \(R := \mathbb{Q}[\![X,Y,Z]\!]/(X^2-Y^2, Y^2-Z^2, XY, YZ, ZX)\) is weakly BNSI. 

**Proof.** This is a folklore example of a Gorenstein ring. Also, \(m^3 = 0\) and \(\mu(m) = 3\). By Observation 4.15 \(R\) is weakly BNSI. \(\square\)

4.3. Freeness of weakly Gorenstein modules. Here, modules are finitely generated.

**Observation 4.17.** (After Menzin-Yoshino) Let \((R, m, k)\) be non-Gorenstein Cohen-Macaulay ring of minimal multiplicity and \(k\) be infinite. The following assertions hold:

i) If \(\text{Ext}_R^i(M, R) = 0\) for all \(1 \leq i \leq 2 \dim R + 2\), then \(M\) is free. In particular,

\[\text{strongly reflexive} \iff \text{homologically reflexive} \iff \text{weakly Gorenstein} \iff \text{totally reflexive} \iff \text{free}\]

ii) The modules in part i) are equivalent with skew Gorenstein if and only if \(\dim R = 0\).

iii) Suppose in addition that the ring is complete and quasinormal. Then there is a nonfree reflexive module.

It may be nice to give an example of (iii): For example, any 2-dimensional non-Gorenstein normal local domain with a rational singularity (see [45, Example 4.8]).

**Proof.** i) The first claim is in [15, Proposition 7]. In particular,

\[\text{homologically reflexive} \iff \text{weakly Gorenstein} \iff \text{totally reflexive} \iff \text{free}\]

Suppose \(M\) is strongly reflexive. Since \(\text{Ext}_R^i(M^*, R) = 0\) and in view of the first part we see \(M^*\) is free. This yields the freeness of \(M^{**}\). By the assumption, \(M \simeq M^{**}\). Consequently, \(M\) is free.

ii) If \(R\) is artinian, then \(m^2 = 0\) and desired claim is in Corollary 4.3. Suppose \(R\) is not artinian. Due to the Cohen-Macaulay assumption, \(\text{depth}(R) = \dim R > 0\). We look at \(M := R/m \oplus R\). It follows that \(M^* \simeq R\). Hence, \(\text{Ext}_R^i(M^*, R) = 0\). Thus, \(M\) is skew Gorenstein. Clearly, \(M\) is not free.

iii) Since \(R\) is Cohen-Macaulay and homomorphic image of a Gorenstein ring it poses a canonical module \(\omega_R\). In general, canonical module is not reflexive. However, there is a situation for which canonical module is reflexive. Indeed, Vasconcelos proved that:
Fact A) Over quasi-normal rings, a necessary and sufficient condition for $M$ to be reflexive is that every $R$-sequence of two or less elements be also an $M$-sequence.

The canonical module is maximal Cohen-Macaulay. Due to Fact A) we see $\omega_{R}$ is reflexive. However, $\omega_{R}$ is not free, because the ring is not Gorenstein. □

Following [45], we say $m$ is quasi-decomposable if $m$ contains an $R$-sequence $x$ (the empty set allowed) such that the module $m/xR$ decomposes into nonzero submodules. The following may extend [45, Corollary 6.8] (and [46, 4.7]) by presenting a bound:

**Proposition 4.18.** Let $(R, m, k)$ be a non-Gorenstein local ring such that $m$ is quasi-decomposable. The following holds

i) If $\text{Ext}^{i}_{R}(M, R) = 0$ for all $1 \leq i \leq 6 + 3 \text{depth} R - 2$, then $M$ is free. In particular, strongly reflexive $\iff$ homologically reflexive $\iff$ weakly Gorenstein $\iff$ totally reflexive $\iff$ free

ii) There are situations for which reflexive are free.

iii) Suppose in addition that $R$ is complete and quasi-normal, there are reflexive modules that are not free.

**Proof.** i) Suppose first that $m = I \oplus J$. This translates to $R \simeq R/I \times_k R/J$. In view of [45, Corollary 6.3] we see $\text{p.dim}(M) \leq 1$. If projective dimension a module (−) were be finite, then it should be equal to $\sup\{i : \text{Ext}^{i}_{R}(−, R) \neq 0\}$. We call this property by ($\ast$). From this, $M$ is free. Now, suppose that there is a nonempty set $x = x_{1}, \ldots, x_{n}$ of $R$-sequence such that the module $m/xR$ is decomposable. Let $n := \mu(M)$ and look at $0 \to \text{Syz}(M) \to R^{n} \to M \to 0$. Note that $x_{1}$ is regular over $\text{Syz}(M)$. Let $(-):= - \otimes R/x_{1}R$. By the standard reduction,

$$\text{Ext}^{i}_{R}(\text{Syz}(M), R) = 0 \quad \forall 1 \leq i \leq 6 + 3 \text{depth} R - 2,$$

see [41, Proposition 7] for more details. Following the inductive hypothesis, $\overline{\text{Syz}(M)}$ is projective over $R$. Let $m := \mu(\overline{\text{Syz}(M)})$ and apply ($\ast$) to $0 \to \text{Syz}_{2}(M) \to R^{m} \to \text{Syz}(M) \to 0$. This is in turn imply that $\overline{\text{Syz}_{2}(M)} = 0$. We apply Nakayama’s lemma to conclude that $\overline{\text{Syz}_{2}(M)} = 0$. By definition, $\overline{\text{Syz}(M)}$ is projective, and consequently $\text{p.dim}_{R}(M) \leq 1$. Again, we use ($\ast$) to deduce that $\text{p.dim}_{R}(M) = 0$.

ii) Remark that $R_{1} := k[x, y]/(x^{2}, xy, y^{3})$ is the fiber product $k[x]/(x^{2})$ and $k[y]/(y^{3})$ over $k$. In view of Example 4.7 reflexive modules over $R_{1}$ are free.

iii) It is shown in [15] that the ring presented in Observation 4.17 is quasi-decomposable. In view of Observation 4.17 (iii) we can find a reflexive module which is not free. □

4.4. **Freeness of $M^{*}$ versus freeness of $M$.** Recall from Ramras’ work that over any local ring of depth zero, freeness of $M$ follows from freeness of $M^{*}$.

**Observation 4.19.** Let $(R, m)$ be a local ring such that freeness of each module $M$ follows from freeness of its dual. Then $\text{depth}(R) = 0$. 
Proof. Suppose on the contradiction that \( \text{depth}(R) \geq 1 \). We look at \( M := k \oplus R \). Then \( M^* \) is free and \( M \) is not free. We conclude from this contradiction that \( \text{depth}(R) = 0 \).

The following is converse to [17]:

**Proposition 4.20.** Let \((R, m, k)\) be a local ring. Suppose freeness of each torsion-less module \( M \) follows from freeness of its dual. Then \( \text{depth}(R) < 2 \).

**Proof.** Suppose on the contradiction that \( \text{depth}(R) \geq 2 \). We look at \( 0 \rightarrow m \rightarrow R \rightarrow k \rightarrow 0 \). This induces the following long exact sequence

\[
0 \rightarrow \text{Hom}(k, R) \rightarrow m^* \rightarrow R^* \rightarrow \text{Ext}^1_R(k, R) \rightarrow 0.
\]

Since \( \text{depth}(R) > 1 \), we have \( \text{Ext}^0_R(k, R) = \text{Ext}^1_R(k, R) = 0 \). From this, \( m^* \cong R \) which is free. By the assumption, \( m \) is free. Thus

\[2 \leq \text{depth}(R) \leq \dim(R) \leq \text{gl. dim}(R) = p. \dim(k) \leq p. \dim(m) + 1 = 1.\]

We conclude from this contradiction that \( \text{depth}(R) < 2 \). \(\)

4.5. **Freeness of certain reflexive modules.** Here, modules are finitely generated.

**Proposition 4.21.** Let \((R, m, k)\) be a local ring. Suppose any module of the form \( M^* \) is free. Then \((R, m)\) is a regular ring of dimension at most two.

**Proof.** Firstly, we assume that \( \text{depth}(R) = 0 \). Let \( M \) be finitely generated. By our assumption, \( M^* \) is free. Recall that Ramras proved that freeness of \( M^* \) implies freeness of \( M \). We deduce from this that any finitely generated module is free. In particular, \( R_m \) is free which implies that \( m = 0 \) and so \( R \) is a field. Now, we assume that \( \text{depth}(R) = 1 \). We look at \( 0 \rightarrow m \rightarrow R \rightarrow k \rightarrow 0 \). This induces the following exact sequence

\[
0 = \text{Hom}(k, R) \rightarrow m^* \rightarrow R^* \rightarrow \text{Ext}^1_R(k, R) \rightarrow 0.
\]

Since \( \text{depth}(R) = 1 \) we know that \( \text{Ext}^1_R(k, R) \cong \oplus k \) is nonzero. In particular, \( p. \dim(k) \leq 1 \). This implies that \( 1 = \text{depth}(R) \leq \text{gl. dim}(R) = 1 \). Thus, \( R \) is a principal ideal domain. Finally, we assume that \( d := \text{depth}(R) \geq 2 \). We look at the exact sequence

\[
0 \rightarrow \text{Syz}_d(k) \rightarrow R^n \rightarrow \text{Syz}_{d-1}(k) \rightarrow 0.
\]

This induces the following long exact sequence

\[
0 \rightarrow \text{Hom}(\text{Syz}_{d-1}(k), R) \rightarrow R^n \rightarrow \text{Hom}(\text{Syz}_d(k), R) \rightarrow \text{Ext}^1_R(\text{Syz}_{d-1}(k), R) \rightarrow 0,
\]

and that \( \text{Ext}^1_R(\text{Syz}_{d-1}(k), R) \cong \oplus k \) is nonzero. By our assumption, \( \text{Syz}_{d-1}(k)^* \) and \( \text{Syz}_d(k)^* \) are free. We deduce that \( p. \dim(k) \leq 2 \). This implies \( R \) is a regular ring of dimension 2. \(\)
The following item suggested by Souvik Dey:

**Second proof of Proposition 4.21.** Replace totally reflexive with free in the proof of Proposition 4.26. We leave the routine modification to the reader. □

Over normal domains, freeness of rank-one reflexive modules implies UFD. Over complete normal rings, freeness of reflexive modules of rank at most two implies regularity, see [57, 2.14].

**Proposition 4.22.** Let \((R, m)\) be a local quasi-reduced ring. Then reflexive modules are free if and only if \(R\) is a regular ring of dimension at most two.

**Proof.** Over quasi-reduced rings, second syzygy modules are reflexive modules (see [39, page 5809]). Trivially, second syzygy modules are free if and only if \(R\) is regular and of dimension at most two. □

**Remark 4.23.** The first item shows that the local assumption is important. The second item shows that quasi-reduced assumption is needed.

i) Consider the ring \(R := \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}\) and \(M := \mathbb{Z}/2\mathbb{Z} \oplus 0\). It is projective and so reflexive. However, \(M\) is not free. In order to find an example which is integral domain, we look at \(R := \mathbb{Z}[\sqrt{5}]\) and \(M := (2, 1 + \sqrt{5}) \triangleleft R\).

ii) See Observation 4.1.

One has \(M^* = \text{Syz}_2(D(M))\). Then \(M\) is reflexive if \(M \cong D_2(D_2(M))\) where \(D_2(M) := \text{Syz}_2(D(M))\). Following Auslander-Bridger, this is equivalent to saying that the map

\[
\text{Ext}_R^1(D_2(D_2(M)), -) \xrightarrow{\cong} \text{Ext}_R^1(M, -)
\]

induced by \(M \to D_2(D_2(M))\) is an isomorphism.

**Definition 4.24.** Set \(D_n(-) := \text{Syz}_n(D(-))\).

i) Following Auslander-Bridger, \(M\) is called \(n\)-reflexive if \(\text{Ext}_R^1(D_n(D_n(M)), -) \xrightarrow{\cong} \text{Ext}_R^1(M, -)\)

induced by \(M \to D_n(D_n(M))\) is an isomorphism.

ii) Following Masek, \(R\) is called \(n\)-Gorenstein if it satisfies \((S_n)\) and is Gorenstein in codimension \(n - 1\).

**Proposition 4.25.** Let \((R, m, k)\) be an \((n-1)\)-Gorenstein local ring. Then \(n\)-reflexive modules are free if and only if \(R\) is a regular ring of dimension at most \(n\).

**Proof.** First, we assume that \(n\)-reflexive modules are free. Let \(M := \text{Syz}_n(k)\). By definition, this is \(n\)-syzygy. Over \((n-1)\)-Gorenstein rings, \(n\)-syzygy modules are \(n\)-torsionless (see [39, Corollary 43]). In view of [3, Theorem 2.17] we observe that any \(n\)-torsionless module is \(n\)-reflexive. So, \(M\) is \(n\)-reflexive. By the assumption, \(M\) is free. From this, \(p. \dim(k) \leq n\). Thus, \(R\) is a regular ring of dimension at most \(n\). To see the converse part, let \(R\) be regular of dimension at most \(n\) and \(M\) be \(n\)-reflexive. Recall from [3, Corollary 4.22] that over Gorenstein rings \(n\)-reflexivity coincides with \(n\)-syzygy. Since \(R\) is regular of dimension at most \(n\) any \(n\)-syzygy module is free. From this, \(M\) is free. □
**Proposition 4.26.** Let \((R, \mathfrak{m}, k)\) be a local ring. Suppose any module of the form \(M^*\) is totally reflexive. Then \(R\) is a Gorenstein ring of dimension at most two.

**Proof.** Recall that \(M^*\) is isomorphic to the second syzygy module of \(D(M)\). Due to the assumption, we know that \(\text{Syz}_2(D(M))\) is totally reflexive for every module \(M\). In particular, \(\text{Syz}_2(D(D(M)))\) is totally reflexive for every \(R\)-module \(M\). Since \(D(D(M)) \approx M\), \(\text{Syz}_2(M)\) is totally reflexive for every \(R\)-module \(M\). So, every \(R\)-module has G-dimension at most 2. Therefore, \(R\) is a Gorenstein ring of dimension at most 2.

\(\square\)

5. **Reflexivity of multi dual**

For simplicity, we set \(M_{\ell}^* := \underbrace{M^* \otimes \cdots \otimes M^*}_{\ell \text{-times}}\) and we put \(M_{\ell}^* := M\) if \(\ell = 0\). Reflexivity of \(M^*\) is subject of [14, 1.3.6]. Over any non-Gorenstein artinian local ring, dual of the residue field is not reflexive. Despite of this, and as a motivation, we recall the following result of Vasconcelos:

**Fact 5.1.** Over quasi-normal rings, \(M^*\) is reflexive. In particular, \(M_{\ell}^*\) is reflexive for all \(\ell \in \mathbb{N}\).

The following extends [35, Page 518] where Lam worked with \(k[\frac{x,y}{(x,y)^2}]\) and \(M := k\).

**Corollary 5.2.** Let \(0 \neq M\) be any nonfree over one of the following local rings:

i) \(R := k[\frac{X_1, \ldots, X_m}{(X_1, \ldots, X_m)}]\) with \(m > 1\), or

ii) \(R\) is such that \(m^2 = 0\) and \(\mu(m) > 1\), or

iii) \(R := \frac{k[X,Y]}{(X^2, X^3, Y)}\).

Then \(M_{\ell}^*\) is not reflexive for all \(\ell \in \mathbb{N}_0\).

**Proof.** We may assume \(\ell \in \mathbb{N}\). We argue by induction on \(\ell\). Without loss of the generality we may assume that \(\ell = 1\).

Claim A) Let \((S,n)\) be an artinian local ring and \(N \neq 0\) be finitely generated. Then \(N^*\) is nonzero. Indeed, since \(N\) is finitely generated and nonzero,

\(\text{Ass}(\text{Hom}(N, S)) = \supp(N) \cap \text{Ass}(S) = \{n\}\).

In particular, \(N^* \neq 0\).

In view of Claim A), we see that \(M^*\) is nonzero. One may find easily that \(M^*\) is finitely generated. Suppose on the contradiction that \(M^*\) is reflexive. The ring \(R\) is BNSI (see §4). By Fact [4.3], \(M^*\) is free. Since \(0 \leq \text{depth}(R) \leq \dim(R) = 0\) we have \(\text{depth}(R) = 0\). If depth of a ring is zero, then freeness descents from \((-)^*\) to \((-)\), see [47, Lemma 2.6]. This immediately implies that \(M\) is free which is excluded from the assumption. This is a contradiction. \(\square\)

The above proof shows:

**Observation 5.3.** Let \((R, \mathfrak{m})\) be artinian BNSI and \(M\) be nonfree. Then \(M_{\ell}^*\) is not reflexive.

It may be natural to ask:
Question 5.4. Let $R$ be artinian non-Gorenstein and $M$ be nonfree. When is $M^{\ell^*}$ (non-)reflexive?

Here, is the answer:

Proposition 5.5. Let $(R, \mathfrak{m})$ be artinian and $M$ be finitely generated. If $M^{\ell^*}$ is reflexive for some $\ell > 1$, then $M^{\ell^*}$ is reflexive for all $n > 0$.

Proof. First, assume $n \geq \ell$. Taking $(n - \ell)^{th}$ times dual from $M^{\ell^*} \xrightarrow{\exists} M^{(\ell+2)^*}$ yields that $M^{\ell^*} \xrightarrow{\exists} M^{(n+2)^*}$. From this, $M^{\ell^*}$ is reflexive. Now, assume that $n < \ell$. We set $N := M^{(\ell-2)^*}$. Then $N^{\ell^*}$ is reflexive. We are going to show that $N^*$ is reflexive. By definition, the natural map $\phi := \phi_{N^{\ell^*}} : N^{\ell^*} \rightarrow N^{\ell^*}$ is an isomorphism. This gives a map $\varphi := N^{\ell^*} \rightarrow N^{\ell^*}$ such that $\varphi \phi = \text{id}_{N^{\ell^*}}$. Since dual modules are torsionless, there is an exact sequence

$$0 \rightarrow N^* \xrightarrow{\phi^*} N^{\ell^*} \rightarrow C \rightarrow 0,$$

where $\phi^* := \varphi_N$. Taking dual, it yields that

$$0 \rightarrow C^* \rightarrow N^{\ell^*} \xrightarrow{\varphi^*} N^{\ell^*}.$$

It is easy to see $\varphi^* = \varphi$. In particular, $C^* = \ker(\varphi^*) = 0$. Thus,

$$0 = \text{Ass}(C^*) = \text{Ass}(\text{Hom}(C, R)) = \text{Supp}(C) \cap \text{Ass}(R) = \text{Supp}(C) \cap \{m\},$$

this is equivalent to saying that $m \notin \text{Supp}(C)$, i.e., $C = 0$. Consequently, $N^* \xrightarrow{\phi^*} N^{\ell^*}$ is an isomorphism. So, $M^{\ell^*} = N^*$ is reflexive, as claimed.

Suppose the ring is artinian and $M$ is nonfree. It is easy to see that $\limsup_{n \rightarrow \infty} \frac{\ell(M^n)^*}{\ell(R)^n} \leq \ell(M)$. Indeed, via induction on $n$, it is enough to show $\ell(M^n) \leq \text{type}(R) \ell(M)$. To show this it is enough to use induction on $\ell(M)$, or see Menzin’s PhD-thesis. If the module is simple or $R$ is Gorenstein, then $\lim_{n \rightarrow \infty} \frac{\ell(M^n)^*}{\ell(R)^n} = \ell(M)$. We ask:

Question 5.6. When does the limit $\lim_{n \rightarrow \infty} \frac{\ell(M^n)^*}{\ell(R)^n}$ exist? when is it equal to $\ell(M)$?

In general, this is not the case:

Example 5.7. Let $(R, \mathfrak{m})$ be artinian non-Gorenstein and $M$ be free. Then $\lim_{n \rightarrow \infty} \frac{\ell(M^n)^*}{\ell(R)^n} = 0$.

Proof. Recall that $M^{\ell^*} = M$ for all $n$, since $M$ is free. Also, $\text{type}(R) > 1$ because $R$ is not Gorenstein. Thus, $\lim_{n \rightarrow \infty} \frac{\ell(M^n)^*}{\ell(R)^n} = \lim_{n \rightarrow \infty} \frac{\ell(M)}{\ell(R)^n} = 0$. 

In particular, we consider to modules with no free direct summands:

Example 5.8. Let $R := \frac{\mathbb{Q}[x, y]}{(x^2, xy, y^2)}$ and $M := \mathfrak{m}$. Then $\lim_{n \rightarrow \infty} \frac{\ell(M^n)^*}{\ell(R)^n} \neq \ell(M)$.

Proof. Note that $R = \mathbb{Q} \oplus \mathbb{Q}x \oplus \mathbb{Q}y \oplus \mathbb{Q}y^2$ and $(0 : \mathfrak{m}) = (x, y^2)$. From this, type of $R$ is 2. Also, $\ell(R) = 3$. Since $xy = 0$, we have

$$m = (x, y) = xR \oplus yR = R/(0 : x) \oplus R/(0 : y) = R/(x, y) \oplus R/(x, y^2) \quad (*).$$
Now, we compute dual of $R/(x, y^2)$:

$$\text{Hom}_R(R/(x, y^2), R) = \{ r : r(x, y^2) = 0 \} = (x, y) = m \quad (\ast, \ast)$$

We combine $(\ast)$ along with $(\ast, \ast)$ to see that $m^* = \mathbb{Q}^* \oplus m$. We take another dual to see

$$m^{**} = \mathbb{Q}^{**} \oplus m^* = \mathbb{Q}^{**} \oplus \mathbb{Q}^* \oplus m.$$

By induction,

$$m^{n*} = \mathbb{Q}^{n*} \oplus \mathbb{Q}^{(n-1)*} \oplus \ldots \oplus \mathbb{Q}^* \oplus m \quad (\ast \ast \ast)$$

Recall that $\mathbb{Q}^* = \oplus_{\text{type}(R)} \mathbb{Q}$ and that $\mathbb{Q}^{n*} = \oplus_{\text{type}(R)} \mathbb{Q} = \mathbb{Q}^{2n}$. We put this along with $(\ast \ast \ast)$ to see

$$\ell(m^{n*}) = \sum_{j=1}^{n} 2^j + \ell(m) = \frac{2^{n+1} + 2}{1 - 2} = 2^{n+1} + 1.$$

Consequently, $\lim_{n \to \infty} \frac{\ell(M^{n*})}{\text{type}(R)^n} = \lim_{n \to \infty} \frac{2^{n+1}}{2^n} = 2 < 3 = \ell(m). \quad \square$

**Example 5.9.** Let $R := \mathbb{Q}[X, Y]_{(X^2, XY, Y^2)}$ and $M := m$. Then $\lim_{n \to \infty} \frac{\ell(M^{n*})}{\text{type}(R)^n} = \ell(M)$.

**Proof.** Note that $R = \mathbb{Q} \oplus \mathbb{Q}x \oplus \mathbb{Q}y$ and $(0 : m) = (x, y)$. From this, type of $R$ is 2. Since $xy = 0$, we have

$$m = (x, y) = xR \oplus yR = R/(0 : x) \oplus R/(0 : y) = R/(x, y) \oplus R/(x, y) = \mathbb{Q} \oplus \mathbb{Q}.$$

By an easy induction, $m^{n*} = \mathbb{Q}^{n*} \oplus \mathbb{Q}^{n*}$. Recall that $\mathbb{Q}^* = \oplus_{\text{type}(R)} \mathbb{Q}$ and that $\mathbb{Q}^{n*} = \mathbb{Q}^{2n}$. This means that $\ell(m^{n*}) = 2^{n+1}$. Consequently,

$$\lim_{n \to \infty} \frac{\ell(M^{n*})}{\text{type}(R)^n} = \lim_{n \to \infty} \frac{2^{n+1}}{2^n} = 2 = \ell(m),$$

as claimed. \square

**Proposition 5.10.** Let $(R, m)$ be such that $m^2 = 0$ and $M$ be a finitely generated module with no free direct summands. Then $\lim_{n \to \infty} \frac{\ell(M^{n*})}{\text{type}(R)^n} = \beta_0(M)$.

**Proof.** We may assume that $m \neq 0$. Due to [H] Proposition 1, $\ell(M^*) = \ell(m) \mu(M)$. We remark that $M^*$ is torsion-less. It is submodule of a free module $F$. Let $f : M^* \to F$. Then $M^*$ is a first syzygy of $\text{coker}(f)$ with respect to a free resolution of $\text{coker}(f)$. Let $\text{Syz}_1(\text{coker}(f))$ be the first syzygy of $\text{coker}(f)$ with respect to the minimal free resolution of $\text{coker}(f)$. There are free modules $R^n$ and $R^m$ ($m \leq n$) such that

$$\text{Syz}_1(\text{coker}(f)) \oplus R^m \cong M^* \oplus R^n.$$

By Krull-Schmidt theorem over complete rings,

$$M^* \cong \text{Syz}_1(\text{coker}(f)) \oplus R^{n-m}.$$
We recall from [17] Lemma 2.6 that $M^*$ has no free direct summands, because $M$ has no free direct summands and depth of the ring is zero. We conclude that $M^* \simeq \text{Syz}_1(\text{coker}(f))$, and so that $M^* \subset \mathfrak{m}F$ for some free module $F$. In particular, $\mathfrak{m}M^* \subset \mathfrak{m}^2F = 0$. Hence

$$\mu(M^*) = \lim_{\mathfrak{m}}(\frac{M^*}{\mathfrak{m}M^*}) = \dim M^* = \ell(M^*).$$

By repeating this, $M'^*$ has no free direct summands for all $n \geq 1$. Also, $\ell(M'^*) = \mu(M'^*)$ for all $n \geq 1$. By the mentioned result of Menzin, $\ell(M^{(n+1)*}) = \ell(\mathfrak{m})\mu(M'^*)$. An easy induction implies that

$$\ell(M^{(n+1)*}) = \ell(\mathfrak{m})\mu(M^*).$$

We use $\mathfrak{m}^2 = 0$ to deduce $\ell(\mathfrak{m}) = \text{type}(R)$. Therefore,

$$\lim_{n \to \infty} \frac{\ell(M^{(n+1)*})}{\text{type}(R)^{n+1}} \mu(M^*) = \lim_{n \to \infty} \frac{\text{type}(R)^n \mu(M^*)}{\text{type}(R)^{n+1}} = \frac{\mu(M^*)}{\text{type}(R)} = \frac{\ell(M^*)}{\ell(\mathfrak{m})} = \mu(M) = \beta_0(M),$$

as claimed. \hfill \Box

**Corollary 5.11.** In addition to Proposition 5.10 assume that $M$ is torsion-less. Then

$$\lim_{n \to \infty} \frac{\ell(M'^*)}{\text{type}(R)^n} = \ell(M).$$

Let $(R, \mathfrak{m})$ be non-Gorenstein such that $\mathfrak{m}^2 = 0$ and $M$ be a module with no free direct summands. Note that $\ell(R) \neq 2$, i.e., $(\ell(R) - 1)^2 - 1 \neq 0$. It follows by [11] Proposition 1] that

$$\lim_{i \to \infty} \frac{\ell(\text{Ext}_R^i(M, R))}{\ell(R)^i} = \lim_{i \to \infty} \frac{(\ell(R) - 1)^{i-2} \beta_i(M)((\ell(R) - 1)^2 - 1)}{\ell(R)^i} = \beta_1(M).$$

**Question 5.12.** Let $(R, \mathfrak{m})$ be an artinian non-Gorenstein ring. When is the limit $\lim_{i \to \infty} \frac{\ell(\text{Ext}_R^i(M, R))}{\ell(R)^i}$ exist?

The following extends the Third Dual Theorem, see [35] 19.38] (and it is well-known over abelian groups, see [21] Ex. 12.11.(3)):

**Proposition 5.13.** Let $(R, \mathfrak{m})$ be a local ring and $M$ be weakly reflexive. Then $M^{\ell*}$ is reflexive for all $\ell \in \mathbb{N}$.

**Proof.** In view of the Third Dual Theorem, we need to proof the claim only for $\ell := 1$. To this end, we take dual from $M \xrightarrow{\varphi_M} M^* \to 0$ to observe that $0 \to M^{***} \xrightarrow{(\varphi_M)^*} M^*$. We look at $\varphi_{M^*} : M^* \to M^{***}$ and realize that $(\varphi_M)^*\varphi_{M^*} = 1_{M^*}$. That is the monomorphism $\varphi_{M^*} : M^* \to M^{***}$ splits. We have

$$M^{***} \simeq M^* \oplus \text{coker}(\varphi_{M^*}) \simeq M^* \oplus \ker((\varphi_M)^*) \simeq M^* \oplus 0 \simeq M^*.$$

By [14] 1.1.9(b)] $M$ is reflexive if it is isomorphic to its bidual. \hfill \Box

**Fact 5.14.** (Bass 1963, Hartshorne 1992, Masek 2000) Let $(R, \mathfrak{m})$ be a locally quasi-reduced ring. Then $M^{\ell*}$ is reflexive for all $\ell \in \mathbb{N}$. Conversely, if $M^*$ is reflexive for all $M$, then $R$ is quasi-reduced.
Proof. We may assume that $R$ is local. In the sense of Masek, $R$ is 1-Gorenstein. Over a such ring, 2−torsionless is the same as of 2−syzygy, see [39 Corollary 45]. Let $F_1 \to F_0 \to M \to 0$ be a minimal presentation of $M$. In view of

$$0 \to M^* \to F_0^* \to F_1^* \to D(M) \to 0$$

we know $M^*$ is second syzygy. Thus, $M^*$ is 2−torsionless, and so reflexive. From this, $M^{\ell*}$ is reflexive for all $\ell \in \mathbb{N}$.

Conversely, by a result of Bass (see [9 Proposition 6.1]) we know $S^{-1}R$ is a Gorenstein ring where $S$ is the multiplicative closed subset of all regular elements. Note that $S^{-1} = R \cup \{p \in \text{Ass}(R) | p \}$. Since $S^{-1}$ is equi-dimensional, $\text{min}(S^{-1}R) = \text{Ass}(S^{-1}R)$. From this, $\text{min}(R) = \text{Ass}(R)$. This property is just (S1). Let $p \in \text{min}(R)$. Since $(S^{-1}R)_{S^{-1}p} = R_p$ we deduce that $R$ is (G0). Recall that quasi-reduced means (G0)+(S1). This completes the proof. □

Remark 5.15. The quasi-reduced assumption is important, see Observation 4.1. Also, the finitely generated assumption on $M$ is important even over $\mathbb{Z}$:

i) There is an abelian group $G$ such that $G^*$ is not reflexive, see e.g. [18 XI, Theorem 1.13].

ii) There is a non-reflexive abelian group $G$ such that $G \simeq G^{**}$, see [18, Page 355] (by [16, Proposition 1.1.9] such a thing never happens in the setting of noetherian modules). The origin proof uses topological methods.

Recall that there are natural maps $M^{m*} \to (M^{m*})^{**} = M^{(m+2)*}$. We set

$$M_{\text{even}}^{\infty*} := \lim_{\to} \left( M \xrightarrow{\varphi_m} M^{2*} \xrightarrow{\varphi_{M^{2*}}} M^{4*} \xrightarrow{\varphi_{M^{4*}}} M^{6*} \to \cdots \right)$$

and

$$M_{\text{odd}}^{\infty*} := \lim_{\to} \left( M^* \xrightarrow{\varphi_{M^*}} M^{3*} \xrightarrow{\varphi_{M^{3*}}} M^{5*} \xrightarrow{\varphi_{M^{5*}}} M^{7*} \to \cdots \right).$$

Question 5.16. When is $M_{\text{odd}}^{\infty*} \simeq M_{\text{even}}^{\infty*}$?

The question is not true on this generality:

Example 5.17. Let $R := \mathbb{Z}$ and $M := \bigoplus_{\mathbb{N}} \mathbb{Z}$. Recall that $M^* = \prod_{\mathbb{N}} \mathbb{Z}$ and $(\prod_{\mathbb{N}} \mathbb{Z})^* = \bigoplus_{\mathbb{N}} \mathbb{Z}$. From these,

$$M_{\text{even}}^{\infty*} = \lim_{\to} \left( M \to M^{2*} \to M^{4*} \to M^{6*} \to \cdots \right) \cong \bigoplus_{\mathbb{N}} \mathbb{Z}$$

and

$$M_{\text{odd}}^{\infty*} = \lim_{\to} \left( M^* \to M^{3*} \to M^{5*} \to M^{7*} \to \cdots \right) \cong \prod_{\mathbb{N}} \mathbb{Z}.$$ 

So, $M_{\text{odd}}^{\infty*} \not\simeq M_{\text{even}}^{\infty*}$. However, both of $M_{\text{odd}}^{\infty*}$ and $M_{\text{even}}^{\infty*}$ are reflexive.

Remark 5.18. Let $R$ be a local domain and $M$ be finitely generated. Then $M_{\text{odd}}^{\infty*} \simeq M^*$ and $M_{\text{even}}^{\infty*} \simeq M^{**}$ are reflexive. In particular, there are situations for which:

- $M_{\text{odd}}^{\infty*} \not\simeq M_{\text{even}}^{\infty*}$, and also
- $M_{\text{odd}}^{\infty*} \not\simeq M_{\text{even}}^{\infty*}$. 
Here, modules are finitely generated.

Example 6.1. Let \( R := \mathbb{C}[x^4, x^3y, x^2y^2, xy^3, y^4] \). Then \( R \) is a complete normal domain of dimension 2 and every nonzero reflexive module decomposes into a direct sum of rank one submodules.

Proof. Let \( A := \mathbb{C}[x, y] \) and recall that \( R \) is the 4-Veronese subring of \( A \). Recall that \( R \) can be regarded as an invariant ring of cyclic group \( G := \langle g \rangle \) of order 4. Let \( A_i \) be the invariant ring by \( g^i \). We left to the reader to check that each \( A_i \) is reflexive as an \( R \)-module and of rank one. This is well-known that \( A = A_0 \oplus \ldots \oplus A_3 \). Let \( M \) be any indecomposable reflexive \( R \)-module. By [38, Proposition 6.2], any indecomposable reflexive \( R \)-module is a direct summand of \( A \) as an \( R \)-module. From this, there is an \( R \)-module \( N \) such that

\[
M \oplus N \cong A = A_0 \oplus \ldots \oplus A_3.
\]

Krull-Remak-Schmidt property holds over complete rings. It follows that \( M \cong A_i \) for some \( 0 \leq i \leq 3 \). Recall that \( A_i \) is of rank one. In particular, any indecomposable reflexive module is of rank one. □

The following demonstrates the role of 2-dimensional assumption in Treger’s conjecture.

Observation 6.2. Let \( A := \mathbb{C}[[X_1, \ldots, X_n]] \) and let \( (R, m) \) be its \( m \)-Veronese subring. Any nonzero reflexive module decomposes into a direct sum of rank one submodules if and only if \( n \leq 2 \).

Proof. In the case \( n = 1 \), the ring \( R \) is regular and is of dimension one. Also, the case \( m = 1 \) is trivial. The case \( n = 2 \) is a modification of Example 6.1 (it may be worth to note that there is a geometric proof in [20, Lemma 1]). Next we deal with \( n = 3 \) and \( m = 2 \). This is stated in [5, Theorem 4.1] that the canonical module of \( R \) is generated by 3 elements and not less. Recall that \( \omega_R \) is of rank 1. However, if we pass to its first syzygy we get a rank 2 indecomposable maximal Cohen-Macaulay module \( \text{Syz}(\omega_R) \). In particular, \( \text{Syz}(\omega_R) \) is a reflexive module and of rank two. Since \( \text{Syz}(\omega_R) \) indecomposable, it does not decomposable into a direct sum of rank one submodules. Finally we assume either \( n > 3 \) or \( (n = 3 \leq m) \). Again, we use a result of Auslander and Reiten ([5, Theorem 3.1]) to see there are infinity many indecomposable maximal Cohen-Macaulay modules. Hence, there are infinity many indecomposable reflexive modules. The classical group of \( R \) is \( \mathbb{Z}/m\mathbb{Z} \). By definition, there are finitely many rank one reflexive modules. Now, we find an indecomposable reflexive module \( M \) of rank bigger than one. In particular, \( M \) is not direct sum of its rank one submodules. □

Proposition 6.3. Let \( (R, m) \) be a singular standard graded normal hypersurface ring of dimension 2 where \( k \) is algebraically closed with \( \text{char} \ k \neq 2 \). Then \( R = \frac{k[x,y,z]}{(x^2+y^2+z^2)} \) (after a suitable linear change of variables) if and only if every nonzero graded reflexive module decomposes into a direct sum of rank one submodules.
Proof. Suppose every graded reflexive module is decomposable into a direct sum of rank one submodules. Set $C := \text{Proj}(R)$. In view of Serre’s criterion of normality ([27, Page 185]), $C$ is a smooth projective plane curve. According to [27, Proposition I.7.6], $C$ is of degree equal to $d := \text{deg}(f)$. Also, $C$ is reduced, irreducible and connected. Recall that there is no difference between reflexive modules and maximal Cohen-Macaulay modules. Maximal Cohen-Macaulay are locally free over punctured spectrum and their are of constant rank, since $R$ is domain and is regular over punctured spectrum. In this regards, graded reflexive modules correspondence to locally free sheaves. Let us explain a little more. Namely, an $\mathcal{O}_C$-module $F$ is called free if it is isomorphic to a direct sum of copies of $\mathcal{O}_C$. It is called locally free if $C$ can be covered by open (affine) sets $U$ for which $F|_U$ is a free $\mathcal{O}_U$-module. In particular, $F$ is quasi-coherent. We look at the reflexive module

$$M := \bigoplus_{n \in \mathbb{Z}} H^0(C, F \otimes \mathcal{O}_C(n)),$$

see [38, 6.49, Exercise]. According to [27, Proposition II.5.15] $F \cong \tilde{M}$. Here, we use the fact that $R$ is finitely generated by $R_1$ as an $k$-algebra. Recall from [27, Ex. II.5.18(d)] that there is a one-to-one correspondence between isomorphism classes of locally free sheaves and isomorphism classes of vector bundles. In sum, we observed that any vector bundle on $C$ is isomorphic to a direct sum of line bundles. Again, we are going to use the fact that $k = \overline{k}$: Thanks to [8, Theorem 1.1], $C$ is isomorphic to $\mathbb{P}^1$. In the light of [27, V.5.6.1] we know genus is a birational invariant. Since $C$ is smooth,

$$g_C = \frac{(d - 1)(d - 2)}{2}$$

(see [27, Ex. I.7.2]). This is zero, because $g_{\mathbb{P}^1} = 0$. This implies that $d \leq 2$. It $d = 1$ this implies that $R$ is nonsingular which is excluded by the assumption. Then we may assume that $d = 2$. Since $\text{char} \ k \neq 2$ and in view of [27, Ex. I.5.2] and after a suitable linear change of variables, $C$ is defined by $x^2 + y^2 + z^2 = 0$.

Conversely, assume $R = \frac{k[x,y,z]}{(x^2 + y^2 + z^2)}$. In the sense of representation theory, $R$ has singularity of type $A_1$. In particular, there are only two indecomposable maximal Cohen-Macaulay modules. Both of them are of rank one. The proof is now complete. \hfill $\square$

The standard-graded assumption is really important:

Example 6.4. i) Any nonzero reflexive module over $R_n := \frac{C[x,y,z]}{(x^n + y^n + z^n)}$ with $n \geq 2$ decomposable into a direct sum of rank one reflexive submodules (see [38, Example 5.25]). In fact any nonzero indecomposable reflexive $R_n$-module is of rank 1. The same thing works for $\frac{C[x,y,z]}{(x^n + y^n + z^n)}$. ii) Look at $R = \frac{k[x,y,z]}{(x^2 + y^2 + z^2)}$ equipped with graded ring structure given by $G := \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Auslander-Reiten presented an indecomposable $G$-graded reflexive module $M$. For more details, see [3, Page 191]. Also, they remarked that $\tilde{M}$ decomposes into a direct sum of two rank one reflexive submodules.
Remark 6.5. Treger remarked that there are indecomposable reflexive modules over $R = \frac{k[[x,y,z]]}{(x^3+y^5+z^2)}$ of rank bigger than one, see [57, Remark 3.12] (here we adopt some restriction on the characteristic). Let us determine the rank of such modules.

i) Let $(R, \mathfrak{m})$ be a complete 2-dimensional UFD which is not regular and containing a field. Then any indecomposable reflexive module is of rank $1 \leq i \leq 6$.

ii) Let $(R, \mathfrak{m})$ be a complete 2-dimensional ring of Kleinian singularities. Suppose any reflexive module over $R$ decomposes into a direct sum of rank one reflexive submodules. Then $R \simeq \frac{k[[x,y,z]]}{(x^n+y^2+z^2)}$ for some $n > 1$.

iii) If we allow the ring is not local, then we can construct a 2-dimensional UFD with a projective module which is not free.

Proof. Since $R$ is UFD, its classical group is trivial. This implies that any rank one reflexive module is free. According to Remark 4.22 there is a reflexive module which is not free. By this, there is an indecomposable reflexive module of rank bigger than one.

i) Lipman proved over any algebraically closed field $k$ of characteristic $> 5$ the only non-regular normal complete 2-dimensional local ring which is a UFD is $R = \frac{k[[x,y,z]]}{(x^3+y^5+z^2)}$ (see e.g. [27, Ex. V.5.8]). According to [13] there is a complete list of representatives of the isomorphism classes of indecomposable maximal Cohen-Macaulay modules. The ranks are $[1, 6]$.

ii) The proof of this is similar to i) and we leave it to reader.

iii) This item was proved by many authors. For example, the tangent bundle of a 2-dimensional sphere is not trivial, see Swan’s monograph [54].

□

Remark 6.6. Let $R = \frac{k[x_0,\ldots,x_3]}{I}$ be a singular standard graded normal ring of dimension 2 where $k = \overline{k}$ with $\text{char} k \neq 2$. Suppose $\text{deg}(R) \leq 7$. Then $I$ can be generated by degree-two elements if every nonzero graded reflexive module decomposes into a direct sum of rank one submodules.

Proof. Let $C := \text{Proj}(R)$. We may assume that $C$ is genus is zero, see Proposition 6.3. The case $\text{deg}(R) \leq 4$ follows from the following fact:

Fact (Treger-Nagel) Let $X \subset \mathbb{P}^n$ be an arithmetically Cohen-Macaulay variety of degree $d$ and codimension $c > 1$. Then the defining ideal is generated by forms of degree $\lceil \frac{d}{c} \rceil$.

For the simplicity we recall the following classification result: Let $X \subset \mathbb{P}^n$ be an irreducible and reduced normal projective subvariety of dimension one and degree five which is not contained in any hyperplane of $\mathbb{P}^n$. Then $X$ is one of the following four types: i) a curve of genus 6 in $\mathbb{P}^2$, ii) a curve of genus 2 in $\mathbb{P}^3$, iii) an elliptic curve of degree 5 in $\mathbb{P}^4$, or iv) a rational normal curve in $\mathbb{P}^5$. These are not isomorphic with $C$ (we use the presentation of [56, page 4324]). The only projectively normal curve of degree 6 in $\mathbb{P}^3$ not contained in any plane are of genus 3 or 4 (see [27, Ex.V.6.6]). The same citation shows that the only projectively normal curve of degree 7 in $\mathbb{P}^3$ not contained in any plane are of genus 5 or 6. The proof is now complete. □
7. Remarks on a question by Braun

Question 7.1. (Braun, [11 Question 16]) Let \((R, \mathfrak{m})\) be a normal domain and \(I \triangleleft R\) a reflexive ideal with \(\text{id}_R(I) < \infty\). Is \(I\) isomorphic to a canonical module?

By [11 Page 682], the only positive evidence we have is when \(R\) is also Gorenstein.

Proposition 7.2. Let \((R, \mathfrak{m})\) be an analytically normal domain of dimension \(2\) and \(I \triangleleft R\) be reflexive with \(\text{id}_R(I) < \infty\). Then \(I\) isomorphic to a canonical module.

Proof. Due to the Serre’s criterion of normality, the ring is \((S_2)\). Since \(\dim R = 2\), we see that \(R\) is Cohen-Macaulay. In the light of Fact A) in Observation 4.17, \(\text{depth} I \geq 2\). Since \(\dim R = 2\) we deduce that \(I\) is maximal Cohen-Macaulay. We may assume that the ring is complete: Let

\[ K_R := \text{Hom}_R(H^2_{\mathfrak{m}}(R), E(R/\mathfrak{m})). \]

Thanks to [61 Definition 5.6], \(K_R \otimes \hat{R} \simeq K_{\hat{R}}\). Also, recall that if \(M \otimes \hat{R} \simeq N \otimes \hat{R}\), then \(M\) is also isomorphic to \(N\) (see [61 Lemma 5.8]). Since \(R\) is complete and Cohen-Macaulay, \(K_R = \omega_R\), in the sense that it is maximal Cohen-Macaulay module of type one and of finite injective dimension. For the simplicity, we bring the following from [38 Proposition 11.7]:

Fact A) Let \(A\) be a Cohen-Macaulay local ring with canonical module \(\omega_A\). If a module \(M\) is both maximal Cohen-Macaulay and of finite injective dimension, then \(M \simeq \oplus_n \omega_R\) for some \(n\).

We apply the above fact to see \(I \simeq \oplus_n \omega_R\) for some \(n\). Since the ring is domain, \(I_1I_2 \neq 0\) where \(0 \neq I_i \subset I\) are ideals of \(R\). Thus, \(n = 1\) and that \(I \simeq \omega_R\).

Proposition 7.3. Let \((R, \mathfrak{m})\) be an analytically normal domain and \(I \triangleleft R\) be totally reflexive with \(\text{id}_R(I) < \infty\). Then \(I\) isomorphic to a canonical module.

Proof. Due to Bass’ conjecture (which is a theorem), \(R\) is Cohen-Macaulay because there is a finitely generated module of finite injective dimension. Due to the above argument, we may assume \(R\) is complete. By definition, totally reflexive are of zero \(G\)-dimension. According to Auslander-Bridger formula,

\[ \text{depth}(I) = \text{Gdim}(I) + \text{depth}(I) = \text{depth}(R) = \dim(R). \]

In other words, \(I\) is maximal Cohen-Macaulay. We apply Fact A) in Proposition 7.2 to see \(I \simeq \oplus_n \omega_R\) for some \(n\). Since the ring is domain, \(I \simeq \omega_R\).

8. Descent from (to) the endomorphism ring and tensor products

A module \(M\) satisfied \((S_n)\) if \(M_p\) is either maximal Cohen-Macaulay or \(0\) when \(\text{ht}(p) \leq n\), and \(\text{depth}(M_p) \geq n\) if \(\text{ht}(p) > n\) with the convenience that \(\text{depth}(0) = \infty\). In [4 Proposition 4.6], there is a criterion of reflexivity over normal local domains. Here, we prove this for Gorenstein rings:

Proposition 8.1. Let \((R, \mathfrak{m})\) be a Gorenstein ring. The following holds:

i) Assume \(\text{Ext}_R^1(M, M) = 0\) and \(\text{Hom}_R(M, M)\) is reflexive. Then \(M\) is reflexive.
ii) If $M$ is reflexive, then $\text{Hom}_R(M, M)$ is reflexive.

Proof. One can assume that the ring is local and $M$ is nonzero.

i): We argue by induction on $d := \dim R$. When $d = 0$, any module is totally reflexive. Suppose, inductively, that $d > 0$ and that the result has been proved for Gorenstein rings of smaller dimensions. By the inductive assumption, $M$ is reflexive over the punctured spectrum. Since $\text{Hom}_R(M, M)$ is torsion-less,

$$m \notin \text{Ass}(\text{Hom}_R(M, M)) = \text{Supp}(M) \cap \text{Ass}(M) = \text{Ass}(M).$$

If $d = 1$, this means that $M$ is maximal Cohen-Macaulay. Since these modules over Gorenstein rings are reflexive, we can assume that $d > 1$. It follows from definition that $M$ is $(S_1)$, i.e., $M$ is torsion-less. We conclude that there is an exact sequence $0 \rightarrow M \rightarrow M^{**} \rightarrow L \rightarrow 0$. Since $M$ is reflexive over the punctured spectrum, $L$ is of finite length. In particular,

$$\text{Ass}(L) \subset \text{Supp}(L) \subset \{m\} \subseteq \text{Supp}(M).$$

We are going to show that $L = 0$. Keep in mind that a module $(-)$ is zero if $\text{Ass}(-) = \emptyset$. Since

$$\text{Ass}(\text{Hom}(M, L)) = \text{Supp}(M) \cap \text{Ass}(L) = \text{Ass}(L),$$

we see $L = 0$ if and only if $\text{Hom}(M, L) = 0$. There is an exact sequence

$$0 \rightarrow \text{Hom}_R(M, M) \rightarrow \text{Hom}_R(M, M^{**}) \rightarrow \text{Hom}(M, L) \rightarrow \text{Ext}^1_R(M, M) = 0.$$

Since $\text{Hom}_R(M, M)$ is reflexive, $\text{depth}(\text{Hom}_R(M, M)) \geq 2$. In the same vein, $\text{depth}(M^{**}) \geq 2$. In view of [4, Proposition 4.7] we observe that

$$\text{depth}(\text{Hom}_R(M, M^{**})) \geq 2.$$

Suppose on the way of contradiction that $\text{Hom}(M, L) \neq 0$. Since $\text{Hom}(M, L)$ is artinian, $\text{depth}(\text{Hom}(M, L)) = 0$. We use the depth lemma to see that

$$\text{depth}(\text{Hom}(M, L)) \geq \left\{\text{depth}(\text{Hom}_R(M, M)) - 1, \text{depth}(\text{Hom}_R(M, M^{**}))\right\} \geq 1.$$

This contradiction shows that $M$ is reflexive.

ii): Recall that $M$ is $(S_2)$. Let $p \in \text{Supp}(\text{Hom}_R(M, M))$ of height at most two. Then $p \in \text{Supp}(M)$. It follows from $(S_2)$ that $M_p$ is maximal Cohen-Macaulay. Due to [4, Proposition 4.7],

$$\text{depth}(\text{Hom}_{R_p}(M_p, M_p)) \geq \text{ht}(p).$$

This means that $\text{Hom}_R(M, M)_p$ is maximal Cohen-Macaulay. Let $p$ be of height greater than two. It follows from $(S_2)$ that $\text{depth}(M_p) \geq 2$. Another use of [4, Proposition 4.7] implies that

$$\text{depth}(\text{Hom}_{R_p}(M_p, M_p)) \geq 2.$$ This means that $\text{Hom}_R(M, M)$ is $(S_2)$. This condition over Gorenstein rings implies the reflexivity. □
Example 8.2. Proposition 8.1 (ii) is true over rings that are Gorenstein in codimension one. The Gorenstein assumption in Proposition 8.1 (i) is important: Let $R := k[[X^3, X^4, X^5]]$. Recall that for any maximal Cohen-Macaulay module $M$, $\Ext_R^0(M, \omega_R) = 0$. Applying this for the canonical module, $\Ext_R^1(\omega_R, \omega_R) = 0$. Also, $\Hom_R(\omega_R, \omega_R) \simeq R$ is reflexive. By [32, 4.8], $\omega_R$ is not reflexive.

The ring in the next result is more general than [4, Proposition 4.9]:

Lemma 8.3. Let $(R, \mathfrak{m})$ be local, $M$ be locally free on the punctured spectrum of depth at least two such that $\text{depth}(\Hom_R(M, M)) \geq 3$. Then $\Ext_R^1(M, M) = 0$.

Proof. We have $\Ext_R^1(M, M)$ is of finite length. Suppose on the contradiction that $\Ext_R^1(M, M) \neq 0$. Then $\text{depth}(\Ext_R^1(M, M)) = 0$. Let $x$ be $M$-regular and let

$$C := \ker \left( x : \Ext_R^1(M, M) \rightarrow \Ext_R^1(M, M) \right).$$

We look at the exact sequence

$$0 \rightarrow \Hom_R(M, M)/x \Hom_R(M, M) \rightarrow \Hom_R(M, M/xM) \rightarrow C \rightarrow 0.$$

Note that

- $\text{depth} \left( \frac{\Hom_R(M, M)}{x \Hom_R(M, M)} \right) \geq 2$, and
- $\text{depth} \left( \Hom_R(M, \frac{M}{xM}) \right) > 0$.

We use depth lemma to see

$$\text{depth}(C) \geq \left\{ \text{depth} \left( \frac{\Hom_R(M, M)}{x \Hom_R(M, M)} \right) - 1, \text{depth} \left( \Hom_R(M, \frac{M}{xM}) \right) \right\} \geq 1,$$

a contradiction. \qed

The following deals with the reflexivity assumption of [15, Theorem 2.3] and may regard as a generalization of [4, Theorem 4.4]:

Corollary 8.4. Let $(R, \mathfrak{m})$ be an abstract complete intersection of dimension at least 4. Suppose $M$ is locally free on the punctured spectrum and $\text{depth}(\Hom_R(M, M)) \geq 4$. The following are equivalent:

i) $M$ is free,
ii) $M$ is reflexive,
iii) $\Ext_R^1(M, M) = 0$.

Proof. i) $\Rightarrow$ ii) is trivial and ii) $\Rightarrow$ iii) is a special case of Lemma 8.3.

iii) $\Rightarrow$ i) Since $\Hom_R(M, M)$ is $(S_4)$, it is particularly $(S_2)$. This condition over Gorenstein rings implies the reflexivity. We are in a position to apply Proposition 8.1 (i). Due to Proposition 8.1 (i) $M$ is reflexive. This allow us to use [15, Theorem 2.3] to show $M$ is free. \qed

Fact 8.5. (See [60, Theorem 3.1]) Let $(R, \mathfrak{m})$ be a one-dimensional Gorenstein ring and $M$ a finitely generated $R$-module. Then $M$ is projective provided $\Hom_R(M, M)$ is projective.
Due to an example of Vasconcelos, Fact 8.5 can’t be extended to higher-dimensional Gorenstein rings. However, we show:

**Proposition 8.6.** Let \((R, \mathfrak{m})\) be a d-dimensional Gorenstein ring. Assume the following conditions:

i) \(\text{Ext}^i_R(M, M) = 0\) for all \(0 < i < d\)

ii) \(\text{Hom}_R(M, M)\) is projective.

Then \(M\) is projective.

**Proof.** One can assume that the ring is local. We argue by induction on \(d\). When \(d \leq 1\), the claim is in Fact 8.5. Suppose, inductively, that \(d > 1\) and that the result has been proved for Gorenstein rings of smaller dimensions. Recall that

\[
\mathfrak{m} \notin \text{Ass}(\text{Hom}_R(M, M)) = \text{Supp}(M) \cap \text{Ass}(M) = \text{Ass}(M).
\]

Let \(x\) be a regular element both on \(R\) and on \(M\). We set \(\omega := (\omega - x) / x\) and we look at

\[
0 \rightarrow M \xrightarrow{x} M \rightarrow \overline{M} \rightarrow 0 \quad (*).
\]

Since \(\text{Ext}^1_R(M, M) = 0\), \(\text{Hom}_R(M, \overline{M}) \simeq \text{Hom}(M, M)\). We combine this along with \(\text{Hom}_R(M, \overline{M}) \simeq \text{Hom}_R(M, \overline{M})\) to deduce that \(\text{Hom}(M, \overline{M})\) is free over \(\overline{R}\). First, we may assume \(d = 2\). Thanks to Fact 8.5, \(M\) is free over \(R\). The exact sequence

\[
0 \rightarrow \text{Syz}(M) \rightarrow F \rightarrow M \rightarrow 0,
\]

induces

\[
\text{Tor}^1_R(M, \overline{R}) \rightarrow \text{Syz}(\overline{M}) \rightarrow F \rightarrow \overline{M} \rightarrow 0.
\]

Since \(\text{Tor}^1_R(M, \overline{R}) = \ker(x : M \rightarrow M) = 0\) and \(F \simeq \overline{M}\), we see \(\text{Syz}(M) = 0\). Nakayama’s lemma says that \(\text{Syz}(M) = 0\). By definition, \(M\) is free. For simplicity, we may assume \(d = 3\). Again, (*) induces the exact sequence

\[
0 = \text{Ext}^1_R(M, M) \rightarrow \text{Ext}^1_R(M, \overline{M}) \rightarrow \text{Ext}^2_R(M, M) = 0.
\]

Hence \(\text{Ext}^1_R(M, \overline{M}) = 0\). On the other hand, we have \(\text{Ext}^1_R(M, \overline{M}) \simeq \text{Ext}^1_R(M, \overline{M})\), and consequently, \(\text{Ext}^1_R(M, \overline{M}) = 0\). Due to the 2-dimensional case, \(\overline{M}\) is free over \(\overline{R}\), and so \(M\) is free. \(\square\)

**Example 8.7.** The Gorenstein assumption in Proposition 8.6 is important. Indeed, let \((R, \mathfrak{m})\) be any local complete Cohen-Macaulay ring which is not Gorenstein. We look at the canonical module. Recall that \(\text{Ext}^1_R(\omega_R, \omega_R) = 0\) and \(\text{Hom}_R(\omega_R, \omega_R) \simeq \overline{R}\). But, \(\omega_R\) is not projective.

The following may be extend [52, Theorem 3.1] by weakening of \((S_n)\) to \((S_{n-1})\).

**Proposition 8.8.** Let \((R, \mathfrak{m})\) be a hypersurface of dimension \(n > 2\) and \(0 \neq M\) be of constant rank. If \(M \otimes M\) is \((S_{n-1})\), then \(M\) is free.

*there is nothing when \(d = 1\).
Proof. Note that \( \operatorname{Supp}(M) = \operatorname{Spec}(R) \). From this, \( \operatorname{Supp}(M \otimes M) = \operatorname{Spec}(R) \). In the light of [32, Theorem 3.1] we need to show \( \operatorname{depth}(M \otimes M) = n \). Suppose on the contradiction that \( \operatorname{depth}(M \otimes M) \neq n \). Since \( M \otimes M \) is \( (S_{n-1}) \), we deduce that \( \operatorname{depth}(M \otimes M) = n - 1 \). Since \( n > 2 \), \( M \otimes M \) is \( (S_2) \). This condition over complete-intersection rings implies the reflexivity. Due to the reflexivity we are in a position to use Second Rigidity Theorem [32, Theorem 2.7] to see that \( \operatorname{Tor}^+_{(M, M)} = 0 \). This vanishing result allow us to apply the depth formula ([32, Proposition 2.5]):

\[
\operatorname{depth}(M) + \operatorname{depth}(M) = \operatorname{depth}(R) + \operatorname{depth}(M \otimes M) = n + (n - 1).
\]

The left hand side is an even number and the right hand side is odd. This is a contradiction. □

Remark 8.9. The first item shows that having the constant rank in Proposition 8.8 is really needed. The second item shows that the assumption \( (S_{n-1}) \) can’t be weekend to \( (S_{n-2}) \):

i) We look at \( R := \mathbb{Q}[X, Y, Z, W]/(XY) \) and \( M := R/xR \). It is easy to see that \( M^{\ell \otimes} \) is \( (S_2) \) (resp. reflexive) for all \( \ell > 0 \) but \( M \) is not free. Also, this shows that [43, Theorem 3.1] needs the extra assumption: the module \( M \) has constant rank.

ii) Let \((R, m, k)\) be a 3-dimensional regular local ring and let \( M := \text{Syz}_2^k \). We left to the reader to check that \( M \otimes R \) satisfies \( (S_1) \). Clearly, \( M \) is not free.

9. Reflexivity and UFD

As an application to Corollary 8.4, we recover the following:

Theorem 9.1. (Grothendieck 1961) Let \((R, m)\) be a local complete intersection domain. If \( R_P \) is UFD for all \( P \) of height \( \leq 3 \), then \( R \) is UFD.

Proof. The proof is by induction on \( d := \dim R \). We may assume that \( d > 3 \). First, we deal with the case \( d = 4 \). Let \( p \) be any height one prime ideal. Since \( \dim R = 4 \) and \( R_P \) is UFD for every prime ideal \( P \) of height \( \leq 3 \) we deduce that \( p \) is locally free on the punctured spectrum. One dimensional normal rings are regular. Thus, \( R \) is \( (S_2) \) and \( (R_1) \). According to Serre, \( R \) is normal.

From this, \( \operatorname{Hom}_R(p, p) = R \). In particular, \( \operatorname{depth}_R(\operatorname{Hom}_R(p, p)) \geq 4 \). Then,

\[
p^* = \bigcap_{q \in \operatorname{Spec}^1(R)} pR_q = pR_p \cap \left( \bigcap_{q \in \operatorname{Spec}^1(R) \setminus \{p\}} pR_q \right) = pR_p \cap R = p.
\]

In view of Corollary 8.3, \( p \) is free. Thus, \( p \) is principal. Since any height-one prime ideal is principal, it follows that \( R \) is UFD. Suppose inductively that the claim holds for \( d - 1 \). This implies that \( p \) is locally free. Similar to the case \( d = 4 \), we know \( \operatorname{depth}_R(\operatorname{Hom}_R(p, p)) \geq 4 \) and also \( p \) is reflexive. In view of Corollary 8.3, \( p \) is free and so principal. In sum, \( R \) is UFD. □

Fact 9.2. (Samuel) Let \( R = \bigoplus_{n=0}^{\infty} R_n \) be an integral domain and \( Q \) be the fraction field of \( R_0 \). Then \( R \) is UFD if and only if \( R_0 \) is UFD, each \( R_n \) is reflexive and \( R \otimes_{R_0} Q \) is UFD.

Corollary 9.3. Let \((R, m)\) be a 2-dimensional regular local ring and \( M \) be finitely generated. Then \( \text{Sym}_R(M) \) is regular if and only if \( \text{Sym}_R(M) \) is UFD.
Proof. Assume that $\text{Sym}_R(M)$ is UFD. By Fact \ref{criterion}, $M$ is reflexive. Since $R$ is 2-dimensional and regular, $M$ is free. Let $n := \text{rank } M$. Then $\text{Sym}_R(M) = R[X_1, \ldots, X_n]$ which is regular. \hfill \Box

The 2-dimensional assumption is important:

Example 9.4. Let $(R, \mathfrak{m}, k)$ be a 3-dimensional regular local ring and $M := \text{Syz}_2(k)$. Then $\text{depth}(\text{Sym}_n(M)) = 2$ for all $n > 0$. In particular, $\text{Sym}_R(M)$ is nonregular UFD.

Proof. Let $\{x, y, z\}$ be a regular parameter sequence of $R$. In view of \cite[Proposition 5]{[19]} we know that $\text{Sym}(M) \cong \frac{R[X, Y, Z]}{(xX + yY + zZ)}$ is UFD. Due to Fact \ref{criterion} we deduce that $\text{Sym}_n(M)$ is reflexive. Since the following relation

$$(xX + yY + zZ)_R = (xX + yY + zZ)$$

is nontrivial, we deduce that $\text{sym}_n(M)$ is free. Thus $\text{depth}(\text{Sym}_n(M)) = 2$. We set

$$n := \mathfrak{m} \oplus \bigoplus_{n > 0} \text{Sym}_n(M).$$

As $xX + yY + zZ \in n^2$, we see $\text{Sym}_R(M)$ is not regular. \hfill \Box

Samuel constructed a reflexive module $M$ such that $\text{Sym}_2(M)$ is not reflexive. Here, there is another one:

Example 9.5. Let $(R, \mathfrak{m}, k)$ be a 4-dimensional regular local ring. Then $M := \text{Syz}_2(k)$ is reflexive. In view of \cite[Proposition 3.1.11]{[59]}, $\text{p.dim}(\text{Sym}_2(M)) = 4$. By Auslander-Buchsbaum formula, $\text{depth}(\text{Sym}_2(M)) = 0$. In particular, $\text{Sym}_2(M)$ is not reflexive.

Conjecture 9.6. (See \cite[Conjecture 6.1.4]{[59]}) Let $R$ be a regular local ring. If $\text{Sym}_R(M)$ is UFD, then $\text{p.dim}(M) \leq 1$.

Samuel remarked that there is no symmetric analogue of Auslander’s theorem on torsion part of tensor powers. We present a tiny remark:

Proposition 9.7. Let $(R, \mathfrak{m})$ be a regular local ring and $M$ be of rank one. If $\text{Sym}_n(M)$ is reflexive for some $n \geq \max\{2, \text{dim } R\}$, then $M$ is free.

Proof. We proceed by induction on $d := \text{dim } R$. Suppose $d = 1$. Due to the structure theorem for finitely generated modules over DVR, $M = F \oplus T$ where $F$ is free and $T := \bigoplus_{i=1}^t \frac{R}{\mathfrak{m}^i}$ is the torsion part. Suppose on the contradiction that $T \neq 0$, and recall that

$$\text{Sym}_n(M) = \bigoplus_{i+j=n} \text{Sym}_i(F) \otimes_R \text{Sym}_j(T) \cong L_1 \oplus (L_2 \otimes \text{Sym}_1(\frac{R}{\mathfrak{m}^n})) = L_1 \oplus (L_2 \otimes \frac{R}{\mathfrak{m}^n}),$$

where $L_2 \neq 0$. Hence, $\text{Sym}_n(M)$ is not reflexive. This contradiction says that $T = 0$ and consequently, $M$ is free. Suppose now that $d > 1$ and the claim holds for all regular rings of smaller dimension. Let $p \in (\text{Spec}(R) \setminus \text{max}(R))$, and recall that $\text{Sym}_n(M)_p \cong \text{Sym}_n(M_p)$ is reflexive over $R_p$. Inductively, $M_p$ is free over $R_p$. In particular, we may assume that $M$ is locally free over the punctured spectrum.
There is a surjective map $M^\otimes n \rightarrow \pi \rightarrow \text{Sym}_n(M) \rightarrow 0$. Set $K := \ker(\pi)$. Again, let $p$ be a prime ideal which is not maximal. Then, we have

$$0 \rightarrow K_p \rightarrow (M^\otimes n)_p \rightarrow \text{Sym}_n(M)_p \rightarrow 0 \quad (*)$$

Since $M$ is locally free of rank 1, we have

$$\text{Sym}_n(M)_p \cong \text{Sym}_n(M_p) \cong (R_p[X])_n.$$ 

This is free over $R_p$ and is of rank 1. Also, $(M^\otimes n)_p$ is free over $R_p$ and is of rank 1. In particular, the sequence $(*)$ splits, i.e., $K_p \oplus R_p \cong R_p$. It follows that $K_p = 0$. Hence, $K$ is supported in $m$. In particular, it is of finite length. According to Grothendieck’s vanishing theorem, we know $H^2_m(K) = 0$. This yields the following exact sequence

$$0 = H^1_m(K) \rightarrow H^1_m(M^\otimes n) \rightarrow H^1_m(\text{Sym}_n(M)) \rightarrow H^2_m(K) = 0 \quad (+)$$

Another use of Fact 9.2 implies that $\text{Sym}_n(M)$ is reflexive. Since $\dim R > 1$, we deduce that $\text{depth}(\text{Sym}_n(M)) \geq 2$. Due to the cohomological characterization of depth, $H^1_m(\text{Sym}_n(M)) = 0$. In view of $(+)$ we see $H^1_m(M^\otimes n) = 0$. Now, we recall the following recent result:

**Fact A)** (See [2]) Let $(A, n)$ be a local complete intersection ring and let $N$ be rigid and locally free. Let $m \geq \max\{2, \dim A\}$. If $H^i_m(N^\otimes n) = 0$ for some $0 \leq i \leq \text{depth}_R(N)$, then $N$ is free. Over regular rings any module is rigid. Therefore, Fact A) implies that $M$ is free.

In the case of ideals the following fact is due to Micali and Samuel:

**Fact 9.8.** Let $M$ be a module of rank one. If $\text{Sym}_R(M)$ is UFD, then $M$ is projective.

**Proof.** We may assume that $R$ is local. By Fact 9.2 $M$ is reflexive and $R$ is UFD. Since $R$ is normal, $\text{Cl}(R) = \text{Pic}(R)$. Recall that $\text{Pic}(R)$ is the isomorphism classes of rank 1 reflexive modules. Since $\text{Cl}(R) = 0$, $M$ is free.

**Corollary 9.9.** Let $(R, m)$ be a 3-dimensional regular local ring, $M$ be torsion-free and of rank one. If $\text{Sym}_n(M)$ is reflexive for some $n > 0$, then $M$ is free.

**Proof.** The case $n = 1$ (resp. $n \geq 3$ ) is in Fact 9.2 (resp. Proposition 9.7). Without loss of the generality we assume that $n = 2$. By the proof of Proposition 9.7, we assume that $M$ is locally free and that $H^1_m(M^\otimes 2) = 0$. Since $M$ is torsion-free, it follows that $M$ is free.

The rank 1 condition is important, see Example 9.4.

10. **Reflexivity of ideals**

In this section we study some aspects of the following problem and also its assumption, e.g., the artinian and et cetera:

**Problem 10.1.** Let $(R, m)$ be artinian and suppose $m^n \neq 0$ is reflexive. Find $n$ such that $R$ becomes Gorenstein.
Proposition 10.2. Let \( (R, m, k) \) be a local artinian ring. Assume that there exists a non-negative integer \( n \) such that \( m^n \neq 0 \) and \( m^{n+1} = 0 \). If \( m^n \) is reflexive, then \( R \) is Gorenstein.

The proof shows \((m^n)^*\) is cyclic.

**Proof.** Consider the minimal presentation of \( \text{Hom}_R(m^n, R) \)

\[
R^{m_1} \rightarrow R^{m_0} \rightarrow (m^n)^* \rightarrow 0 \quad (+)
\]

This gives the exact sequence \( 0 \rightarrow (m^n) \rightarrow R^{m_0} \xrightarrow{f} R^{m_1}, \) where \( f := (a_{ij})_{m_1 \times n_0} \) with \( a_{ij} \in m \) and so \( f(Soc(R^{m^n})) = 0 \). Now,

\[
\text{Soc}(R)^{m_0} = \text{Soc}(R^{m_0}) \subseteq \text{ker}(f) \cong m^n.
\]

As \( m^{n+1} = 0 \), one has \( m^n \subseteq \text{Soc}(R) \). Consequently, \( \text{Soc}(R)^{m_0} \subseteq m^n \subseteq \text{Soc}(R) \). This implies that \( n_0 = 1 \) and \( m^n = \text{Soc}(R) \). We plug the first one into \((+)\) and observe that \((m^n)^*\) is cyclic. Applying \((-)^*\) to it, yields that \( \text{Soc}(R)^* \) is cyclic. Recall from its definition, \( \text{Soc}(R) \) is an \( R/m \)-vector space, and by denoting its dimension with \( t := \dim_k(\text{Soc}(R)) \) we have:

\[
\text{Soc}(R)^* = \text{Hom}(\oplus_k R, R) = \oplus_t \text{Hom}(R/m, R) = \oplus_t \{ r \in R | rm = 0 \} = \oplus_t \text{Soc}(R).
\]

Since \( \text{Soc}(R)^* \) is cyclic we know \( t = 1 \) and also \( \text{Soc}(R) \) is cyclic. In other words, type of \( R \) is one. It remains to note that Cohen-Macaulay rings of type one are Gorenstein. \( \square \)

**Second proof of Proposition [10.2].** Since \( m \) annihilates \( m^n \), we observe that \( m^n \) is a \( k \)-vector space. Since any directed summand of a reflexive module is again reflexive, we deduce that \( R/m \) is reflexive. By the first argument, we have

\[
R/m \cong \bigoplus_{\text{Soc}(R)} \bigoplus_{\text{Soc}(R)} R/m,
\]

i.e., \( \text{type}(R) = 1 \), and so \( R \) is Gorenstein. \( \square \)

**Example 10.3.** Let \( R := k[x,y]_{(x,y)} \). Then \( m^3 \) is not reflexive.

**Proof.** Since \( \{ xy, y^3 \} \subseteq \text{Soc}(R) \), \( R \) is not Gorenstein. Recall that \( m^4 = 0 \) and \( m^3 = y^3R \neq 0 \). Proposition [10.2] shows that \( m^3 \) is not reflexive. \( \square \)

**Fact 10.4.** Let \( (R, m) \) be a local ring. If \( \text{depth } R \geq 2 \), then \( m^n \) is not reflexive for all \( n \geq 0 \).

**Proof.** On the contrary, suppose that \( m^n \) is reflexive for some natural number \( n \). As \( m^n \) is a reflexive \( R \)-module and \( \text{depth } R \geq 2 \), by [12] Proposition 1.4.1, we conclude that \( \text{depth}_R m^n \geq 2 \). The short exact sequence \( 0 \rightarrow m^n \rightarrow R \rightarrow R/m^n \rightarrow 0 \) yields the following long exact sequence of local cohomology modules:

\[
0 \rightarrow \text{H}^0_m(m^n) \rightarrow \text{H}^0_m(R) \rightarrow \text{H}^0_m(R/m^n) \rightarrow \text{H}^1_m(m^n) \rightarrow \text{H}^1_m(R) \rightarrow \text{H}^1_m(R/m^n) \rightarrow \cdots.
\]

As \( \text{depth } R \geq 2 \), we have \( \text{H}^0_m(R) = \text{H}^0_m(R) = 0 \), and so \( \text{H}^0_m(m^n) \cong \text{H}^0_m(R/m^n) = R/m^n \neq 0 \). This implies that \( \text{depth}_R (m^n) = \inf \{ i \in \mathbb{N}_0 | \text{H}^i_m(m^n) \neq 0 \} \leq 1 \). This is a contradiction. \( \square \)
Recall that Bass \cite[Theorem 6.2]{Bass} proved that an artinian local ring is Gorenstein iff all of its ideals are reflexive.

**Proposition 10.5.** Let \((R, \mathfrak{m})\) be a local ring. The following assertions are true:

(i) If \(\dim R = 0\) and \(\mathfrak{m}\) is reflexive, then \(R\) is Gorenstein.

(ii) If \(\text{depth} R \geq 2\), then \(\mathfrak{m} \not\cong \mathfrak{m}^{**}\). In fact, \(\mathfrak{m}^n\) is not reflexive for all \(n > 0\).

(iii) If \(R\) is quasi-reduced and \(\text{depth} R = 1\), then \(\mathfrak{m} \cong \mathfrak{m}^{**}\).

**Proof.** i) Apply \((-)^*\) to the short exact sequence \(0 \rightarrow \mathfrak{m} \xrightarrow{i} R \rightarrow R/\mathfrak{m} \rightarrow 0\) implies the exact sequence fits in the following diagram:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \text{Hom}(R/\mathfrak{m}, R) & \longrightarrow & R^* & \longrightarrow & \mathfrak{m}^* & \longrightarrow & \text{Ext}^1_R(R/\mathfrak{m}, R) & \longrightarrow & 0 \\
& & \uparrow & \cong & \uparrow & \cong & \uparrow & \cong & \uparrow & \cong \\
0 & \longrightarrow & \bigoplus_{i=1}^t R/\mathfrak{m} & \longrightarrow & R & \longrightarrow & \mathfrak{m}^* & \longrightarrow & \bigoplus_{i=1}^{t_1} R/\mathfrak{m} & \longrightarrow & 0,
\end{array}
\]

where \(t\) is the type of \(R\) and \(t_1\) is an integer. Let us break down the bottom sequence into the following short exact sequences:

1. \(0 \rightarrow \bigoplus_{i=1}^t R/\mathfrak{m} \rightarrow R \rightarrow L \rightarrow 0\),
2. \(0 \rightarrow L \rightarrow \mathfrak{m}^* \rightarrow \bigoplus_{i=1}^{t_1} R/\mathfrak{m} \rightarrow 0\).

Taking duality from 2) yields

\[
0 \longrightarrow \bigoplus_{i=1}^t \bigoplus_{i=1}^{t_1} R/\mathfrak{m} \longrightarrow \mathfrak{m} \cong \mathfrak{m}^{**}.
\]

In particular, \(0 \rightarrow \bigoplus_{i=1}^t \bigoplus_{i=1}^{t_1} R/\mathfrak{m} \subseteq R\). Taking socle and computing the length yields that \(tt_1 \leq t\). Since \(t > 0\) we deduce that \(t_1 \leq 1\). We have two possibilities:

(a) \(t_1 = 0\),
(b) \(t_1 \neq 0\).

(a) Assume \(t_1 = 0\). Recall that \(\mu_i := \dim \text{Ext}^i_R(R/\mathfrak{m}, R)\) is the \(i\)-th Bass’ number and \(t_1 = \mu_1\). Suppose on the way of contradiction that \(\text{id}(R) = \infty\). In this case Bass \cite[Lemma 3.5]{Bass} proved that \(\mu_i > 0\) for all \(i > \dim R\). Since \(t_1 = 0\) we get a contradiction. So, \(\text{id}(R) < \infty\), i.e., \(R\) is Gorenstein.

(b) Assume \(t_1 \neq 0\). This yields that \(t_1 = 1\). Then, we have \(0 \rightarrow R/\text{Soc}(R) \rightarrow \mathfrak{m}^* \rightarrow R/\mathfrak{m} \rightarrow 0\).

Recall that

\[
\text{Hom}(R/\text{Soc}(R), R) = \{ r \in R : r \text{Soc}(R) = 0 \} = \mathfrak{m}.
\]

First, we take another star and then apply the last observation to deduce the following exact sequence

\[
0 \longrightarrow \bigoplus_{i=1}^t R/\mathfrak{m} \longrightarrow \mathfrak{m}^{**} \longrightarrow \mathfrak{m} \xrightarrow{\phi} \text{Ext}^1_R(R/\mathfrak{m}, R) \cong \bigoplus_{i=1}^{t_1} R/\mathfrak{m} = R/\mathfrak{m}.
\]
In other words:

$$0 \to \bigoplus_{i=1}^{t} R/m \to m^{**} \to m \to \text{im} \phi \to 0.$$ 

Recall that \(\text{im} \phi \subset R/m\). This says that the length of \(\text{im} \phi\) is at most one. Let us take the length and use its additivity. This yields

$$0 = \ell\left(\bigoplus_{i=1}^{t} R/m\right) = \ell(m^{**}) + \ell(m) - \ell(\text{im} \phi) = t - \ell(\text{im} \phi),$$

here, we used the fact that \(m\) is reflexive. So,

$$0 < t = \ell(\text{im} \phi) \leq 1,$$

i.e., type of \(R\) is one and so \(R\) is Gorenstein.

The proof is now complete.

ii) By Fact \[10.3\] depth\((m) = 1\). Since depth\((m^{**}) \geq 2\) we have \(m \not\cong m^{**}\).

iii) Having Fact \[5.14\] in mind, the argument is a slight modification of \[19\] Proposition 4.1. We leave the routine details to the reader. \(\square\)

Suppose depth \(R = 0\) and \(\dim R > 0\). When \(m \not\cong m^{**}\)? Here, we present a sample:

**Example** 10.6. Let \(S\) be any regular local ring of dimension \(n > 0\) and \(p\) be a prime ideal of \(S\). Let \(R := S/pm\). Then \(m \cong m^{**}\) if and only if \(n = 1\).

For example, \(R := k[[X_1, \ldots, X_n]]/p(X_1, \ldots, X_n)\).

**Proof.** Without loss of generality, we may assume \(p \neq 0\). First, assume that \(\text{ht}(p) \geq 2\). According to a result of Ramras, \(R\) is BNSI. In particular, any reflexive module is free. This yields that \(m \cong m^{**}\) if and only if \(n = 1\). Now, we consider to the case \(\text{ht}(p) = 1\). Since \(S\) is UFD, \(p = (x)\) for some \(x\). Without loss of generality we assume that \(n > 1\), and we are going to show \(m\) is not reflexive. Suppose on the way of contradiction, \(m\) is reflexive and search a contradiction. Recall that

\[
\text{Hom}_R(k, R) = \{ r : rm = 0 \} = xR \quad (*)
\]

Apply \((-)^*\) to short exact sequence

$$0 \to m \xrightarrow{i} R \to R/m \to 0,$$

gives us:

$$0 \to \text{Hom}(R/m, R) \to R^* \to m^* \to \text{Ext}_R^1(R/m, R) \to 0$$

$$= \uparrow \quad \cong \uparrow \quad \uparrow \quad \uparrow \quad \uparrow$$

$$0 \to xR \to R \to m^* \to \bigoplus_{i=1}^{t_1} R/m \to 0,$$
where \( t_1 \) is an integer. In other words, the sequence

\[
0 \rightarrow R/xR \rightarrow m^* \rightarrow \bigoplus_{i=1}^{t_1} k \rightarrow 0
\]
is exact. We take another dual and use

\[
\text{Hom}_R(R/xR, R) = \{ r : rx = 0 \} = m.
\]

Let us put things into the following diagrams:

\[
\begin{array}{cccccccccccc}
0 & \rightarrow & \bigoplus_{i=1}^{t_1} \text{Hom}(R/m, R) & \rightarrow & m^* & \rightarrow & (R/xR)^* & \rightarrow & \bigoplus_{i=1}^{t_1} \text{Ext}_R^1(R/m, R) \\
& & & & \cong & & = & & = \\
0 & \rightarrow & \bigoplus_{i=1}^{t_1} xR & \rightarrow & m & \rightarrow & m & \rightarrow & \bigoplus_{i=1}^{t_1} \bigoplus_{i=1}^{t_1} R/m.
\end{array}
\]

On the one hand, there is an equality \((0 : x) = m\). From this,

\[
\text{Soc}(xR) = \text{Soc}(R/(0 : x)) = \text{Soc}(k) = k,
\]

and consequently

\[
\dim(\text{Soc}(xR)) = t_1 \dim(\text{Soc}(xR)) = t_1.
\]

On the other hand, \( \bigoplus_{i=1}^{t_1} xR \subseteq R \). By taking length from \( \bigoplus_{i=1}^{t_1} xR \subseteq \text{Soc}(R) = xR \) we deduce \( t_1 \leq 1 \). In view of [12, Page 373] we observe

\[
1 \geq t_1 = \mu_1(m, R) \geq \dim R = n - 1 \geq 1.
\]

This implies \( n = 2 \). In this case \( \dim R = 1 \). In sum, we proved that

\[
\mu_{\dim R}(m, R) = 1.
\]

This allows us to apply a beautiful result of Roberts (see [12, 9.6.3]) to deduce that \( R \) is Cohen-Macaulay, i.e.,

\[
0 = \text{depth}(R) = \dim R = 1.
\]

This is a contradiction that we searched for it. \( \square \)

**Corollary 10.7.** Let \( R := k[[X, Y]]/X(X, Y) \). Then \( m^n \) is not reflexive for all \( n > 0 \).

**Proof.** In view of Example 10.6 we know \( m \) is not reflexive. Now, let \( n > 1 \) and suppose on the way of contradiction that \( m^n = y^n R \) is reflexive. In particular,

\[
(m^n)^* \cong (m^n)^{**} \quad (z)
\]

Recall that

\[
(m^n)^* = \text{Hom}(R/(0 : y^n), R) = \text{Hom}(R/xR, R) \cong \{ r \in R : rx = 0 \} = m \quad (\dagger)
\]

Taking two times dual from (\( \dagger \)) yields

\[
(m^n)^{**} \cong m^* \quad (\ddagger)
\]
Combining these together
\[ \mathfrak{m} \overset{\dagger}{=} (\mathfrak{m}^n)^* \overset{\natural}{=} (\mathfrak{m}^n)^{**} \overset{\sharp}{=} \mathfrak{m}^{**} \quad (+) \]
Then (+) shows that \( \phi_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{m}^{**} \) is an isomorphism. By definition, \( \mathfrak{m} \) is reflexive. This is in contradiction with Example \( \text{HOC} \). So, \( \mathfrak{m}^n \) is not reflexive. \( \square \)

**Discussion 10.8.** Reflexivity of \( \mathfrak{m} \) does not imply reflexivity of ideals:

i) Let \((R, \mathfrak{m})\) be a 1-dimensional reduced ring which is not Gorenstein. Then \( \mathfrak{m} \) is reflexive and there is an ideal which is not reflexive. Indeed, thanks to Proposition \( \text{HOC}(iii) \) we know \( \mathfrak{m} \) is reflexive. Recall that 1-dimensional reduced rings are Cohen-Macaulay. It remains to recall from \( \text{[R]} \) (6.2) Theorem] that any one-dimensional Cohen-Macaulay ring is Gorenstein if and only if every ideal is reflexive.

ii) Concerning to the first item, canonical module is the proposed ideal. To see an explicit example, let \( R := k[[x^3, x^4, x^5]] \). Then \( \omega_R = (x^3, x^4) \) is not reflexive.

The Gorenstein property of a 1-dimensional ring may follow by observing that \( \mathfrak{m} \) is totally reflexive. This involves to checking infinitely many vanishing of Ext-modules. Vanishing of some initial Ext-modules may yield the Gorenstein property:

**Remark 10.9.** Let \((R, \mathfrak{m})\) be a 1-dimensional Cohen-Macaulay local ring. Then \( R \) is Gorenstein if and only if \( \text{Ext}_R^1(\mathfrak{m}, R) = \text{Ext}_R^1((\mathfrak{m}^2)^*, R) = 0 \).

**Proof.** First, assume that \( R \) is Gorenstein. For any nonzero submodule \( L \) of a finite rank free \( R \)-module, we have \( \text{depth}_R L \geq 1 \), and so the Auslander-Bridger formula implies that

\[ \text{sup}\{i \in \mathbb{N}_0 \mid \text{Ext}_R^i(L, R) \neq 0\} = \text{Gdim}_R L = 0. \]

Conversely, we assume \( \text{Ext}_R^1(\mathfrak{m}, R) = \text{Ext}_R^1((\mathfrak{m}^2)^*, R) = 0 \), and we are going to show \( R \) is Gorenstein. By \( \text{[R]} \) Proposition 4.1, we know that \( \mathfrak{m}^{**} \cong \mathfrak{m} \). Set \( \mu := \text{vdim}_k(\mathfrak{m}/\mathfrak{m}^2) \) and \( t := \text{vdim}_k(\text{Ext}_R^1(R/\mathfrak{m}, R)) \). The short exact sequence

\[ 0 \longrightarrow \mathfrak{m}^2 \overset{i}{\longrightarrow} \mathfrak{m} \longrightarrow \mathfrak{m}/\mathfrak{m}^2 \longrightarrow 0 \]

implies the exact sequence

\[ 0 \longrightarrow \text{Hom}(\mathfrak{m}/\mathfrak{m}^2, R) \longrightarrow \mathfrak{m}^* \longrightarrow (\mathfrak{m}^2)^* \longrightarrow \text{Ext}_R^1(\mathfrak{m}/\mathfrak{m}^2, R) \longrightarrow \text{Ext}_R^1(\mathfrak{m}, R). \]

By definition of \( \mu \) and \( t \), one has

\[ \text{Ext}_R^1(\mathfrak{m}/\mathfrak{m}^2, R) \cong \bigoplus_{i=1}^{\mu} R/\mathfrak{m}. \]

Since \( \text{depth}_R(R) > 0 \), we can easily see that \( \text{Hom}(\mathfrak{m}/\mathfrak{m}^2, R) = 0 \). We combine this along with \( \text{Ext}_R^1(\mathfrak{m}, R) = 0 \) to deduce the following short exact sequence

\[ 0 \longrightarrow \mathfrak{m}^* \overset{i^*}{\longrightarrow} (\mathfrak{m}^2)^* \longrightarrow \bigoplus_{i=1}^{\mu} R/\mathfrak{m} \longrightarrow 0. \]
Since
\[(R/m)^* = 0 = \text{Ext}^1_R((m^2)^*, R),\]
and via applying the functor \((-)^*\) to the later exact sequence, it yields the following exact sequence
\[0 \to (m^2)^{**} \to m^{**} \to \bigoplus_{i=1}^{\mu} R/m \to 0.\]

Now recall that \(m^{**} \cong m\). In other words, \(m/(m^2)^{**} \cong \bigoplus_{i=1}^{\mu} R/m\). By definition, \(m/m^2 = \bigoplus_{i=1}^{\mu} R/m\). Let us recall from \(m^2 \subseteq (m^2)^{**} \subseteq m\) that \(m/m^2 \to m/(m^2)^{**} \to 0\). We put all things together and we get to the following diagram:

\[
\begin{array}{ccc}
m/m^2 & \longrightarrow & m/(m^2)^{**} \\
\downarrow & & \downarrow \\
\bigoplus_{i=1}^{\mu} R/m & \longrightarrow & \bigoplus_{i=1}^{\mu} R/m.
\end{array}
\]

We deduce from this diagram that \(f\) is surjective. This yields the validity of \(\mu \geq \mu^t\). Consequently, \(0 < t \leq 1\). Therefore, \(R\) is Gorenstein.

\[\Box\]

11. Questions by Holanda and Miranda-Neto

The following question was asked in [30, Question 5.24]:

**Question 11.1.** Let \(R\) be a Cohen-Macaulay local ring with canonical module \(\omega_R\). If \(p \cdot \dim_R(\omega_R^*) < \infty\), must \(R\) be Gorenstein?

First, we deal with low-dimensional cases:

**Proposition 11.2.** Suppose \(\dim(R) \leq 3\). Then Question 11.1 is satisfied.

**Proof.** Suppose first that \(d := \dim R = 0\). Recall that \(p \cdot \dim_R(\omega_R^*) < \infty\). This allows us to apply Auslander-Buchsbaum formula, to observe that
\[p \cdot \dim_R(\omega_R^*) = \text{depth}(R) - \text{depth}(\omega_R^*) = 0.\]

By the mentioned result of Kaplansky, \(\omega_R^*\) is free. Since \(\text{depth}(R) = 0\), and in view of Ramras’ result from subsection 4.4, \(\omega_R\) is free. Thus, \(R\) is Gorenstein.

Suppose \(d = 1\). By a formula of Auslander-Goldman,
\[\text{depth}(\text{Hom}(\omega_R, R)) \geq \min\{2, \text{depth}(\omega_R)\} = 1.\]

By another use of Auslander-Buchsbaum formula, \(\omega_R^*\) is free. Now, recall

Fact i): (See [17, Lemma 3.9]) Suppose \(R\) is Cohen-Macaulay with dimension 1, and suppose \(M \in CM(R)\). Then \(M^*\) free implies \(M\) is free.
So, $\omega_R$ is free. Consequently, $R$ is Gorenstein.

Suppose $d = 2$. The assumptions behave well with respect to the localization. According to the previous cases, we may and do assume that $R$ is $(G_1)$. So, $R$ is quasi-normal. Over any quasi-normal ring, and by the mentioned result of Vasconcelos, a module $(-)$ is reflexive iff any $R$-regular sequence of length at most two is $(-)$-regular. So, $\omega_R$ is reflexive. By a formula of Auslander-Goldman,

$$\text{depth}(\text{Hom}(\omega_R, R)) \geq \min\{2, \text{depth}(\omega_R)\} = 2.$$ 

By another use of Auslander-Buchsbaum formula, $\omega_R^*$ is free. So, $\omega_R \cong \omega_R^{**}$ is free. Consequently, $R$ is Gorenstein.

Suppose $d = 3$. It is easy to see that $\omega^*_R$ is locally free over punctured spectrum, of projective dimension at most one, and reflexive. So, there is an exact sequence

$$0 \longrightarrow R^n \longrightarrow R^m \longrightarrow \omega^*_R \longrightarrow 0.$$ 

Applying $- \otimes R \omega_R$ to it gives us

$$0 = \text{Tor}^R_1(\omega_R, R^m) \longrightarrow \text{Tor}^R_1(\omega_R, \omega^*_R) \longrightarrow R^m \otimes R \omega_R \longrightarrow \omega_R \otimes \omega^*_R \longrightarrow 0.$$ 

Since $\text{Tor}^R_1(\omega_R, \omega^*_R)$ is of finite length and $\omega^*_R$ is of positive depth, then $\text{Tor}^R_1(\omega_R, \omega^*_R) = 0$. The long exact sequence of local cohomology module shows that

$$0 = H^1_\text{m}(\omega^*_R) \longrightarrow H^1_\text{m}(\omega_R \otimes \omega^*_R) \longrightarrow H^2_\text{m}(\omega_R^n) = 0,$$

because $\text{depth}(\omega^*_R) = \text{depth}(\omega_R^n) = \dim(R) > 2$.

Now, we recall the following:

**Fact ii):** (See [2, Corollary 8.2]) Suppose $\text{depth}(R) > 0$ and $M$ is locally free on punctured spectrum. If $H^1_\text{m}(M \otimes M^*) = 0$, then $M$ is free.

So, $\omega_R$ is free. Consequently, $R$ is Gorenstein. \hfill $\Box$

**Theorem 11.3.** The desired property of Question 11.1 is true.

**Proof.** We proceed by induction on $d := \dim R$ which is finite by the local assumption. The case $d = 0$ is in Proposition 11.2. The assumptions behave well with respect to the localization. In particular, we are able to apply induction hypothesis to deduce that $R$ is Gorenstein on the punctured spectrum.

**Fact:** (See [1, Proposition 3.4]) Let $(R, m, k)$ be a local ring with an ideal $\mathfrak{a}$, $M$ and $N$ be such that $\dim(M) < \infty$ and one of them is locally free over $\text{Spec}(R) \setminus V(\mathfrak{a})$. Let $0 \leq r < d := \dim R$ be such that $\text{grade}_R(\mathfrak{a}, M) + \text{grade}_R(\mathfrak{a}, N) \geq d + r + 1$. Then $H^0_\mathfrak{m}(M \otimes_R N) = \ldots = H^1_\mathfrak{m}(M \otimes_R N) = 0$.

According to the previous result, we may assume that $d > 3$. In particular, $\text{depth}(\omega^*_R) \geq 2$. In the previous fact, apply $\mathfrak{a} := m$, $M := \omega^*_R$, $N := \omega_R$, and $r := 1$. Recall that $\text{grade}_R(\mathfrak{a}, M) + \text{grade}_R(\mathfrak{a}, N) \geq d + 1 + 1$. So, the above fact can be applied. It gives $H^1_\mathfrak{m}(\omega_R \otimes \omega^*_R) = 0$. In the light of Fact ii) from Proposition 11.2, $\omega_R$ is free. Consequently, $R$ is Gorenstein. \hfill $\Box$
The following question was asked in [30, Question 3.17]:

**Question 11.4.** Let \( R \) be a Cohen-Macaulay local ring with canonical module \( \omega_R \) and dimension \( d \). Let \( M \) be a finite \( R \)-module satisfying \((S_k)\) with either \( k = 1 \) or \( 3 \leq k < d \), and such that \( \text{Ext}_R^i(M, \omega_R) = 0 \) for all \( i = 1, \ldots, \max\{1, d - k\} \). Is it true that \( M \) is maximal Cohen-Macaulay?

**Observation 11.5.** The answer to Question 11.4 is yes.

**Proof.** We may assume the ring is complete. There is nothing to prove if \( \text{depth}(M) = d \). So, let \( \text{depth}(M) \leq i < d \). By local duality, \( H^i_m(M) = \text{Ext}_R^{d-i}(M, \omega_R^e) \) which is zero by the assumption. By homological characterization of depth, we know \( H^i_m(M) = 0 \) for all \( i < \text{depth}(M) \). Combining these, \( H^i_m(M) = 0 \) for all \( i < d \). So, \( M \) is maximal Cohen-Macaulay. \( \square \)

The following question was asked in [30, Question 4.7]:

**Question 11.6.** Let \( R \) be a Cohen-Macaulay local ring with canonical module \( \omega_R \) and dimension \( d > 0 \). If \( \text{Ext}_R^+(\omega_R^*, R) = 0 \), must \( R \) be Gorenstein?

We support it with the following two results:

**Observation 11.7.** Suppose \( R \) is quasi-normal. The desired property of Question 11.6 is valid.

**Proof.** It is easy to see \( \omega \) is reflexive. Let \( F_\bullet : \cdots \to F_0 \to \omega_R^* \to 0 \) be a free resolution. Since \( \text{Ext}_R^+(\omega_R^*, R) = 0 \), the sequence \( 0 \to \omega_R^* \to F_0^* \to \cdots \) is exact. Since \( \omega \) is reflexive, and by pinching \( F_\bullet \) to \( F_\bullet^* \), we see \( \omega \) is totally acyclic. This property behaves well with respect to reduction via regular sequences. Let \( \underline{z} \) be a regular sequence of length \( d \). Let \( \underline{R} := R/\underline{z}R \). Then there is \( \omega_{\underline{R}} \cong \omega_R/\underline{z}\omega_R \subseteq \underline{F} \) where \( \underline{F} \) is free. So, \( \omega_{\underline{R}} \) is torsionless. By Corollary 3.7, \( \underline{R} \) is Gorenstein. Consequently, \( R \) is as well. \( \square \)

**Observation 11.8.** Suppose \( R \) is of minimal multiplicity with infinite residue field. The answer to Question 11.6 is yes.

**Proof.** According to the standard reduction by a regular sequence, we may and do assume that \( \text{dim} R = 1 \). Suppose on the way of contradiction that \( R \) is not Gorenstein. This allows us to apply Observation 11.7 to deduce that \( \omega_R^* \) is free. By Fact ii) from Proposition 11.2, \( \omega_R \) is free. Consequently, \( R \) is Gorenstein. \( \square \)

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*By our terminology, \( \omega_R \) is skew Gorenstein.
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