We study the convex duality method for robust utility maximization in the presence of a random endowment. When the underlying price process is a locally bounded semimartingale, we show that the fundamental duality relation holds true for a wide class of utility functions on the whole real line and unbounded random endowment. To obtain this duality, we prove a robust version of Rockafellar’s theorem on convex integral functionals and apply Fenchel’s general duality theorem.

1. Introduction

We study the convex duality method for robust utility maximization with a random endowment. Suppose we are given a semimartingale $S$ describing the evolution of the underlying asset prices, a class $\Theta$ of admissible integrands (strategies) for $S$, a utility function $U$, a set $P$ of probability measures, and a random variable (endowment) $B$ which is interpreted as the terminal payoff of a contingent claim. Then the problem is to

\[
\max_\theta \inf_P E_P[U(\theta \cdot S_T + B)] \quad \text{over all } \theta \in \Theta.
\]

The set $P$ is a mathematical formulation of model uncertainty (also called Knightian uncertainty in economics), i.e., each element $P \in P$ is considered as a candidate model of the financial market. We refer to [12] for background information and recent results on robust utility maximization problem.

A way of solving (1.1) is the convex duality method which pass (1.1) to a minimization over a set of local martingale measures for $S$, through the (formal) duality

\[
\sup_\theta \inf_P E_P[U(\theta \cdot S_T + B)]
= \inf_\lambda \inf_Q \inf_P E_P \left[ V \left( \lambda \frac{dQ}{dP} \right) + \lambda \frac{dQ}{dP} B \right],
\]

where $V$ is the conjugate of the utility function $U$ and $\mathcal{M}$ is a set of local martingale measures for $S$. We call the right hand side the dual problem. When $B \equiv 0$ and $\text{dom}(U) = \mathbb{R}_+$, this type of duality is proved by [28] under mild assumptions (see also [12] and [27]), and by [29] for bounded endowment $B$. On the other hand, the
case of utility on the whole real line has only a few references: [11] for $B \equiv 0$, [18] for the exponential utility with a suitably integrable $B$, and its generalization by [19] to the case with utility bounded from above.

The central question of this paper is: to what degree of generality does (1.2) hold true? Under the fundamental assumption that $S$ is locally bounded, we shall prove the duality for a wide class of utility functions $U$ on the whole real line and unbounded random endowments $B$. To the best of our knowledge, this is the most general one among duality results in robust utility maximization with $U$ being finite on the whole $\mathbb{R}$ ([11], [18, 19]). Also, this will allow us to introduce and compute a robust version of utility indifference prices.

It is worth emphasizing that our contribution is not only the slight improvement of the result itself, but also to give a simple unified proof based on a philosophy different from other related works ([28], [27], [29], [11], [18, 19]). A (standard) way of showing the duality in robust utility maximization is to reduce the robust problem to a family of subjective problems with the help of a minimax theorem, i.e., to interchange the order of “inf” and “sup” in the left hand side of (1.2) and then apply the duality for each fixed $P \in \mathcal{P}$. This procedure, however, requires an additional technical assumption that the utility function is bounded from above.

Instead, we follow a functional analytic approach in the spirit of [3] in the subjective case, which appeals to Fenchel’s general duality theorem with the help of Rockafellar’s classical result on convex integral functionals. To apply this approach to our robust problem, we need a Rockafellar-type theorem for a robust version of integral functionals. To this end, we study the functionals of the type

$$X \mapsto \sup_{P \in \mathcal{P}} E_P[f(\cdot, X)],$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is a random convex function, and describe their conjugate functionals. This constitutes the heart of this article, and the duality theorem for the robust utility maximization then follows with the idea of [3] mentioned above.

The rest of this paper is organized as follows. In Section 2, we state the precise assumptions for the robust utility maximization problem (1.1) and the duality theorem (Theorem 2.3) as well as some of its consequences, including a robust version of utility indifference prices. Then we give in Section 2.3 the heuristics behind the proof of the duality theorem and a key lemma (Lemma 2.8), which leads us to the study of functionals of the form (1.3). This is the subject of Section 3, which constitutes the heart of our analysis, giving a Rockafellar-type theorem (Theorem 3.9) for this type of functionals. The proof of the duality theorem is given in Section 4.

2. Main Result

2.1. Setup

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, equipped with a filtration $\mathcal{F} := (\mathcal{F}_t)_{t \in [0, T]}$ satisfying the usual conditions, where $T \in (0, \infty)$ is a fixed time horizon, and $\mathcal{F} = \mathcal{F}_T$. For any probability $P \ll \mathbb{P}$, we write $L^p(P) := L^p(\Omega, \mathcal{F}, P)$, and set $L^p := L^p(\mathbb{P})$. Also, the expectation under $P$ is denoted by $E_P[\cdot]$, and $E[\cdot] := E_\mathbb{P}[\cdot]$. In other words, any probabilistic notations without reference to the probability measure are preserved for $\mathbb{P}$.
The process $S$ modeling the underlying asset prices is supposed to be a $d$-dimensional càdlàg locally bounded semimartingale on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Then it is also a locally bounded semimartingale under all $P \ll \mathbb{P}$. A trading strategy and its gain are modeled respectively by a $d$-dimensional predictable $(S, \mathbb{F})$-integrable process $\theta = (\theta^1, \ldots, \theta^d)$, and its stochastic integral $\theta \cdot S = \int_0^\cdot \theta_s dS_s$. The next class of integrands seems to be a natural choice of admissible strategies:

\[(2.1) \quad \Theta_{bb} := \{ \theta \in L(S) : \theta_0 = 0, \theta \cdot S \text{ is uniformly bounded from below} \},\]

where $L(S) := L(S, \mathbb{P})$ is the set of all predictable $(S, \mathbb{P})$-integrable processes (see [14, 15] for precise definition). Note that if $\theta \in \Theta_{bb}$, the stochastic integral $\theta \cdot S$ is well-defined under all $P \ll \mathbb{P}$, and is then a $P$-supermartingale by Ansel-Stricker's criterion [2] (see also [6, Ch. 7]). However, the class $\Theta_{bb}$ is known to be too small in that it can not admit an optimal strategy even with quite ideal settings (e.g., $\mathcal{P} = \{ \mathbb{P} \}, B \equiv 0$ and $S$ is a geometric Brownian motion) as long as we work with a utility function on the entire real line (see e.g. [26]). Thus the class $\Theta_{bb}$ has to be appropriately enlarged if we want to obtain an optimal strategy, although it is not the purpose of this article. We will introduce a possible choice of such enlargement, after defining some more notations.

We fix also a set $\mathcal{P}$ of probabilities $P \ll \mathbb{P}$, which describes the model uncertainty. Since $P \ll \mathbb{P}$ for all $P \in \mathcal{P}$, we can embed the set $\mathcal{P}$ into $L^1(\mathbb{P})$ by the injection $P \mapsto dP/d\mathbb{P}$. In other words, we identify $\mathcal{P}$ with $\{dP/d\mathbb{P}\}_{P \in \mathcal{P}}$. Then we assume:

(A1) $\mathcal{P}$ is convex and $\sigma(L^1, L^\infty)$-compact in $L^1$.

In this paper, we consider a utility function defined on the entire real line, i.e., $\text{dom}(U) = \{ x \in \mathbb{R} : U(x) \in \mathbb{R} \} = \mathbb{R}$. More specifically, we assume;

(A2) $U : \mathbb{R} \to \mathbb{R}$ is a strictly concave, increasing, and continuously differentiable function satisfying the Inada condition:

$$\lim_{x \to -\infty} U'(x) = +\infty \quad \text{and} \quad \lim_{x \to +\infty} U'(x) = 0.$$ 

The conjugate of $U$ is defined by

$$V(y) := \sup_{x \in \mathbb{R}} (U(x) - xy), \quad y \in \mathbb{R}.$$ 

The assumption (A2) implies that $V$ is a strictly convex differentiable function with $V(y) = +\infty$ for $y < 0$, $V(0) = \sup_x U(x)$, and

\[(2.2) \quad V'(0) := \lim_{y \downarrow 0} V'(y) = -\infty \quad \text{and} \quad V'(\infty) := \lim_{y \uparrow \infty} V'(y) = +\infty.\]

In particular, $V$ is bounded from below. Using the function $V$, we define the $V$-divergence functional by

\[(2.3) \quad V(\nu|P) := \begin{cases} E_P[V(d\nu/dP)] & \text{if } Q \ll P, \\ +\infty & \text{otherwise,} \end{cases}\]

for all positive finite (resp. probability) measures $\nu \ll P$ (resp. $P \ll \mathbb{P}$). We set also

$$V(\nu|P) := \inf_{P \in \mathcal{P}} V(\nu|P),$$

and call this map the robust $V$-divergence. Note that $(\nu, P) \mapsto V(\nu|P)$ is convex and lower semicontinuous (see [11, Lemma 2.7]), hence $\nu \mapsto V(\nu|P)$ is also convex since $\mathcal{P}$ is convex by (A1).
A probability $Q \ll P$ on $(\Omega, \mathcal{F})$ is called an absolutely continuous local martingale measure if $S$ is a $Q$-local martingale. The set of all absolutely continuous martingale measures is denoted by $\mathcal{M}_{loc}$. For the domain of the dual problem, we take a subset of $\mathcal{M}_{loc}$:

\[
\mathcal{M}_V := \{ Q \in \mathcal{M}_{loc} : V(\lambda Q|P) < \infty \text{ for some } \lambda > 0 \}.
\]

Note that this set is convex. To see this, suppose $V(\lambda_i Q_i|P) < \infty$ for $\lambda_i > 0$ and $Q_i \in \mathcal{M}_{loc} (i = 1, 2)$, and let $\alpha \in (0, 1)$. Then taking $\gamma = \lambda_1 \lambda_2/(\alpha \lambda_2 + (1 - \alpha) \lambda_1)$, we have

\[
V(\gamma (\alpha Q_1 + (1 - \alpha) Q_2)|P) = V\left(\frac{\alpha \lambda_2}{\alpha \lambda_2 + (1 - \alpha) \lambda_1} \lambda_1 Q_1 + \frac{(1 - \alpha) \lambda_1}{\alpha \lambda_2 + (1 - \alpha) \lambda_1} \lambda_2 Q_2 \right| P) \\
\leq \frac{\alpha \lambda_2}{\alpha \lambda_2 + (1 - \alpha) \lambda_1} V(\lambda_1 Q_1|P) + \frac{(1 - \alpha) \lambda_1}{\alpha \lambda_2 + (1 - \alpha) \lambda_1} V(\lambda_2 Q_2|P) < \infty
\]

since $\nu \mapsto V(\nu|P)$ is convex. We assume for $\mathcal{M}_V$:

\((A3)\) $\mathcal{M}_V := \{ Q \in \mathcal{M}_V : Q \sim P \} \neq \emptyset$.\)

**Remark 2.1.** This condition implies that there exists a pair $(Q, \bar{P}) \in \mathcal{M}_V \times \mathcal{P}$ as well as $\bar{\lambda} > 0$ such that $\overline{Q} \sim \bar{P} \sim P$ and $V(\bar{\lambda} Q|\bar{P}) < \infty$. In particular, “$\mathcal{P}$ is equivalent to $P$” for any $A \in \mathcal{F}$,

\[
P(A) = 0, \forall P \in \mathcal{P} \iff P(A) = 0.
\]

This yields a kind of no-arbitrage (NA) for $\mathcal{P}$: if $\theta \in \Theta_{bb}$ with $P(\theta \cdot S_T \geq 0) = 1$ for all $P \in \mathcal{P}$, then $P(\theta \cdot S_T > 0) = 0$ for all $P \in \mathcal{P}$. However, each model $P \in \mathcal{P}$ may admit an arbitrage.

Now we introduce another natural choice of admissible class enlarging $\Theta_{bb}$:

\[
(2.6) \quad \Theta_V := \{ \theta \in L(S) : \theta_0 = 0, \theta \cdot S \text{ is a } \mathcal{Q} \text{-supermartingale}, \forall \mathcal{Q} \in \mathcal{M}_V \}.
\]

By definition and the comment after (2.1), we have $\Theta_{bb} \subset \Theta_V$. This is a largest natural choice of universally defined admissible strategies from the arbitrage point of view, and in the case without model uncertainty (i.e., $\mathcal{P} = \{P\}$), this class indeed admits an optimizer.

Finally, for the primal/dual problems to be well-defined, we need some restriction on the endowment $B$. In this paper, we do not assume $B$ to be bounded, but instead:

\((A4)\) For some $\varepsilon > 0$, the family $\{ U(-(1 + \varepsilon)B^-)dP/d\mathbb{P} \}_{P \in \mathcal{P}}$ is uniformly integrable, and for every $P \in \mathcal{P}$, there exists some $\varepsilon_P > 0$ such that

\[
E_P[U(\varepsilon_P B^+)] > -\infty.
\]

**Remark 2.2.** (Consequences of (A4) via Young’s inequality). Note that

\[
U(x) \leq V(y) + xy, \quad \forall x, y \in \mathbb{R}.
\]

This direct consequence of the definition of $V$ is called Young’s inequality. From this, we have the following:

1. If we take a triplet $(\lambda, Q, P)$ with $\lambda > 0$ and $V(\lambda Q|P) < \infty$ (then automatically $Q \ll P$), and a positive random variable $D$,

\[
\lambda E_Q[D] \leq V(\lambda Q|P) - E_P[U(-D)].
\]
Taking $D = (1 + \varepsilon)B^-$ and $D = \varepsilon P B^+$, (A4) shows that $B^\pm \in L^1(Q)$, hence $B \in L^1(Q)$, for all $Q \in \mathcal{P} \cup \mathcal{M}_V$ (note that $V(Q|Q) = V(1) < \infty$ for the case of $\mathcal{P}$).

2. In particular, $V(\lambda Q|\mathcal{P}) + \lambda E_Q[B]$ is well-defined with values in $\mathbb{R} \cup \{+\infty\}$ for all $Q \in \mathcal{M}_V$ and $\lambda > 0$ since $V$ is bounded from below. Therefore, the dual problem below of (1.1) is well-defined:

$$\inf_{\lambda > 0, Q \in \mathcal{M}_V} \sup \{V(\lambda Q|\mathcal{P}) + \lambda E_Q[B]\}$$

3. Whenever $X \in L^1(Q)$ for all $Q \in \mathcal{M}_V$, we have, for all $Q \in \mathcal{M}_V$,

$$\inf_{P \in \mathcal{P}} E_p[U(X + B)] \leq E_{P_Q}[U(X + B)]$$

and $\inf_{\lambda > 0, Q \in \mathcal{M}_V} \{V(\lambda Q|\mathcal{P}) + \lambda E_Q[B]\} < \infty$ by (A3).

Furthermore, the infimum in the right hand side is attained by some $(\hat{\lambda}, \hat{Q}, \hat{P})$ in $(0, \infty) \times \mathcal{M}_V \times \mathcal{P}$, i.e., the right hand side is written as $V(\lambda \hat{Q}|\hat{P}) + \hat{\lambda} E_{\hat{Q}}[B]$.

2.2. Duality Theorem and Robust Utility Indifference Valuation

We are now in the position to state the duality theorems in its rigorous form. We start with a basic result, then deduce some of its consequences. In the rest of this section, (A1) – (A4) are always in force as the standing assumptions, and we do not cite them in each statements. The proofs will be given in Section 4.

**Theorem 2.3.** For any $\Theta \subset L(S)$ sandwiched by $\Theta_{bb}$ and $\Theta_V$, i.e., $\Theta_{bb} \subset \Theta \subset \Theta_V$, the duality holds true:

$$\sup_{\theta \in \Theta} \inf_{P \in \mathcal{P}} E_p[U(\theta \cdot S_T + B)] = \inf_{\lambda > 0, Q \in \mathcal{M}_V} \{V(\lambda Q|\mathcal{P}) + \lambda E_Q[B]\}.$$ 

Furthermore, the infimum in the right hand side is attained by some $(\hat{\lambda}, \hat{Q}, \hat{P})$ in $(0, \infty) \times \mathcal{M}_V \times \mathcal{P}$, i.e., the right hand side is written as $V(\lambda \hat{Q}|\hat{P}) + \hat{\lambda} E_{\hat{Q}}[B]$.

When $U(\infty) := \sup_x U(x) < \infty$ (as the exponential utility), the same result is proved in [19, Theorems 2.5 and 2.7], and our Theorem 2.3 slightly generalize that to cover the case with $U(\infty) = \infty$. Although this improvement seems rather minor, we give a unified proof based on a different philosophy, developing a new methodology which is the second and a key contribution of this work. We refer to [19] for a review of other related results.

The reason for stating the robust duality in a “robust form against the choice of $\Theta$” is twofold. The first one concerns the possibility of obtaining an optimal strategy as mentioned above, from which point of view, the larger the class, the better. On the other hand, from the practical point of view, investors living in the real market can use only quite limited strategies (e.g., simple strategies with bounded wealth), thus the smaller the $\Theta$, the better for real investors, provided that the resulting maximal utility (the left hand side of (2.10)) is unchanged. In terms of the first point of view, whether the class $\Theta_V$ can admit an optimizer for the problem (1.1) with some reasonable generality is still open.
Remark 2.4. Whenever $(\hat{\lambda}, \hat{Q}, \hat{P})$ is a dual optimizer, i.e., the minimizer of the right hand side, [19, Theorem 2.7 (a)] shows that $\hat{Q} \sim \hat{P}$. Moreover, we can construct a maximal solution where the term “maximal” is used to mean the maximal support among all solutions to the dual problem. However, even such a maximal solution may fail to be equivalent to the reference measure $\mathbb{P}$. See [27] for the concept of maximal solution and related information.

We now give a couple of variants of Theorem 2.3. In the right hand side of (2.10), the infimum in $Q$ is taken over $\mathcal{M}_V$ which consists of absolutely continuous martingale measures. Then it is natural and important to ask if we can replace $\mathcal{M}_V$ in (2.10) by $\mathcal{M}_V^\prime$, i.e., by elements which are equivalent to $\mathbb{P}$. This point becomes crucial when we want to compute in some concrete setting the dual value function or the indifference prices which we shall discuss below. The answer is positive if we abandon the attainability.

Corollary 2.5. The infimum over $\mathcal{M}_V$ in (2.10) can be replaced by that over $\mathcal{M}_V^\prime$, i.e.,

$$(2.11) \quad \sup_{\theta \in \Theta} \inf_{P \in \mathcal{P}} E_P[U(\theta \cdot S_T + B)] = \inf_{\lambda > 0} \inf_{Q \in \mathcal{M}_V^\prime} (V(\lambda Q|\mathbb{P}) + \lambda E_Q[B]).$$

We proceed to the second variant. The reader might realize that our formulation of the (robust) utility maximization is less general than the following form which seems more common in literature:

$$(2.12) \quad \text{maximize } \inf_{P \in \mathcal{P}} E_P[U(x + \theta \cdot S_T + B)] \quad \text{over all } \theta \in \Theta,$$

where $x \in \mathbb{R}$ is the initial capital. If we consider a utility on the positive half line with $B \equiv 0$, the initial capital is necessary since otherwise the problem is trivial. In our case, however, we can embed the problem (2.12) into (1.1) by replacing $B$ by $x + B$. This is indeed possible since the assumption (A4) is stable under translation by constants: if $B$ satisfies (A4), then so does $x + B$ for any $x \in \mathbb{R}$. We thus obtain:

Corollary 2.6. For any $\Theta$ with $\Theta_{bb} \subset \Theta \subset \Theta_V$ and $x \in \mathbb{R}$,

$$\sup_{\theta \in \Theta} \inf_{P \in \mathcal{P}} E_P[U(x + \theta \cdot S_T + B)]$$

$$(2.13) \quad = \inf_{\lambda > 0} \left\{ \lambda x + \inf_{Q \in \mathcal{M}_V^\prime} (V(\lambda Q|\mathbb{P}) + \lambda E_Q[B]) \right\}.$$

A mathematically straightforward but practically important application of our duality result is the indifference prices of $B$ based on a robust preference. We define buyer’s robust utility indifference price of $B$ by

$$p_b(B) := \sup \left\{ p : \sup_{\theta \in \Theta_{bs}} \inf_{P \in \mathcal{P}} E_P[U(\theta \cdot S_T - p + B)] \geq \sup_{\theta \in \Theta_{bs}} \inf_{P \in \mathcal{P}} E_P[U(\theta \cdot S_T)] \right\}.$$

The interpretation of this “price” is same as in the subjective case (i.e., $\mathcal{P}$ is a singleton, see e.g. [25, 16, 5]): the quantity $\sup_{\theta \in \Theta_{bs}} \inf_{P \in \mathcal{P}} E_P[U(\theta \cdot S_T - p + B)]$ represents the maximal admissible robust utility with buying the claim $B$ at the price $p$, while $\sup_{\theta \in \Theta_{bs}} \inf_{P \in \mathcal{P}} E_P[U(\theta \cdot S_T)]$ is that without buying the claim; thus $p_b(B)$ is understood the maximal acceptable price of $B$ for the buyer whose preference is determined by the robust utility functional $X \mapsto \inf_{P \in \mathcal{P}} E_P[U(X)]$. Seller’s price is defined in a symmetric way. Given Corollary 2.6, the following expression is immediate.
Corollary 2.7. \textit{Buyer’s robust utility indifference price} $p_0(B)$ \textit{of} $B$ \textit{is expressed as:}

(2.14) 
\[ p_0(B) = \inf_{Q \in \mathcal{M}_p^b} (E_Q[B] + \gamma(Q)), \]

where

(2.15) \[ \gamma(Q) := \inf_{\lambda > 0} \frac{1}{\lambda} \left( V(\lambda Q|P) - \inf_{\lambda' > 0} \inf_{Q' \in \mathcal{M}_p^b} V(\lambda' Q'|P) \right). \]

\textbf{Proof of Corollary 2.7.} By Corollary 2.6, we have

\begin{align*}
\sup_{\theta \in \Theta_{ad}} \inf_{P \in \mathcal{P}} E_P[U(\theta \cdot S_T - p + B)] &= \inf_{\lambda > 0} Q \in \mathcal{M}_p^b (V(\lambda Q|P) - \lambda p + \lambda E_Q[B]),
\end{align*}

\begin{align*}
\sup_{\theta \in \Theta_{ad}} \inf_{P \in \mathcal{P}} E_P[U(\theta \cdot S_T)] &= \inf_{\lambda > 0} Q \in \mathcal{M}_p^b V(\lambda Q|P) =: v_0.
\end{align*}

Hence,

\begin{align*}
\sup_{\theta \in \Theta_{ad}} \inf_{P \in \mathcal{P}} E_P[U(\theta \cdot S_T - p + B)] &\geq \sup_{\theta \in \Theta_{ad}} \inf_{P \in \mathcal{P}} E_P[U(\theta \cdot S_T)]
\end{align*}

\begin{align*}
\iff \inf_{Q \in \mathcal{M}_p^b} (V(\lambda Q|P) - \lambda p + \lambda E_Q[B]) \geq v_0
\end{align*}

\begin{align*}
\iff \inf_{Q \in \mathcal{M}_p^b} (E_Q[B] + \gamma(Q))
\end{align*}

\begin{align*}
= \inf_{Q \in \mathcal{M}_p^b} \left( E_Q[B] + \frac{1}{\lambda} (V(\lambda Q|P) - v_0) \right) \geq p, \quad \forall \lambda > 0.
\end{align*}

Therefore, we have (2.14). \hfill \Box

2.3. \textbf{Heuristics and a key Lemma}

To motivate ourselves, and to make it clear what the difficulty and our contribution are, we give here a functional analytic view on the duality theorem. We begin by recalling a classical theorem in functional analysis, called Fenchel’s duality theorem. Let $f : E \to \mathbb{R} \cup \{+\infty\}$ (resp. $g : E \to \{-\infty\}$) be a proper convex (resp. concave) function defined on a topological vector space $E$. Then Fenchel’s theorem (Rockafellar’s version [20]) states that \textit{if either} $f$ \textit{or} $g$ \textit{is continuous at some} $x_0 \in \text{dom} f \cap \text{dom} g$, \textit{then}

(2.16) \[ \sup_{x \in E} (g(x) - f(x)) = \min_{y \in E^*} (f^*(x) - g_*(y)), \]

where $E^*$ denotes the (topological) dual of $E$, and $f^*$ (resp. $g_*$) is the convex (resp. concave) conjugate of $f$ (resp. $g$) defined by

\begin{align*}
f^*(x) &:= \sup_{x \in E} ((x, y) - f(x)), \quad \forall y \in E^* \\
g_*(y) &:= \inf_{x \in E} ((x, y) - g(x)), \quad \forall y \in E^*.
\end{align*}

If we take a convex cone $C \subset E$, and set $f(x) = \delta_C(x) := 0$ (resp. $\infty$) if $x \in C$ (resp. $x \notin C$), which is obviously convex, then its conjugate is given by

\[ \delta_C^*(-y) = \sup_{x \in C} (x, y) = \delta_{C^c}(y), \]

where $C^c$ is the polar cone defined by

\[ C^c := \{ y \in E^* : \langle x, y \rangle \leq 0, \forall x \in C \}. \]
Then if $g$ is continuous at some point $x_0 \in C$, we have
\begin{equation}
\sup_{x \in C} g(x) = \sup_{x \in E} (g(x) - \delta_C(x)) = \min_{y \in C^*} -g_y(y).
\end{equation}

This is a well-established duality argument in abstract functional analysis. We now translate our specific problem into this framework. We take $E = L^\infty$, then its dual is $ba$ with $(X, \nu) = \int_{\Omega} X d\nu =: \nu(X)$ (see Definition 3.2 and Lemma 3.3). Also, define
\begin{equation}
C := \{ X \in L^\infty : X \leq \theta : \cal{S}_T, \exists \theta \in \Theta_{ba} \}.
\end{equation}

This is the set of all super-replicable claims with zero initial costs, and it is well-known (see e.g. [6]) in mathematical finance that $C$ is a convex cone containing $L^\infty$, and every $\sigma$-additive element of $C^o$ is a positive multiple of some local martingale measure, that is,
\begin{equation}
C^o \cap ba^\sigma = \{ \lambda Q : \lambda \geq 0, Q \in \cal{M}_{loc} \} =: \text{cone}_{\cal{M}_{loc}}.
\end{equation}

Here $ba^\sigma$ denotes the set of all $\sigma$-additive elements of $ba$ (i.e., finite signed measures $\nu \ll \check{\nu}$).

For the concave function $g$ and its conjugate, we take
\begin{equation}
u_{B,P}(X) := \inf_{P \in P} E_P[U(X + B)], \quad X \in L^\infty,
\end{equation}
\begin{equation}
u_{B,P}(\nu) := \sup_{X \in L^\infty} (\nu_{B,P}(X) - \nu(X)) = -\nu_{B,P}^*(\nu), \quad \nu \in ba.
\end{equation}

Note that $\nu_{B,P}$ is well-defined proper concave function on $L^\infty$ due to Item 3 of Remark 2.2 above. Now if $\nu_{B,P}$ is continuous, the above argument shows:
\begin{equation}
\sup_{X \in C} \nu_{B,P}(X) = \min_{\nu \in C^o} \nu_{B,P}(\nu) = \min_{\nu \in \text{cone}_{\nu_{B,P}}} \nu_{B,P}(\nu).
\end{equation}

Thus, if we have the continuity of $\nu_{B,P}$ and the complete description of $\nu_{B,P}$, everything else is cleared by the classical duality arguments via Fenchel’s theorem. The key is therefore:

**Lemma 2.8 (Key Lemma).** Assume $(A1) - (A4)$. Then $\nu_{B,P}$ is finite and norm continuous on the whole $L^\infty$, and
\begin{equation}
\nu_{B,P}(\nu) = \begin{cases} V(\nu|\cal{P}) + \nu(B) & \text{if } \nu \in ba^\sigma \text{ and } V(\nu|\cal{P}) < \infty, \\ +\infty & \text{otherwise}, \end{cases}
\end{equation}
where $\nu(B) := \int_{\Omega} B \check{\nu} d\nu$.

As soon as we obtain this key lemma, the above arguments shows the following abstract duality:
\begin{equation}
\sup_{X \in C} \nu_{B,P}(X) = \min_{\lambda \geq 0, Q \in \cal{M}_V} (V(\lambda Q|\cal{P}) + \lambda E_Q[B]).
\end{equation}

Indeed, the representation (2.23) and (2.19) imply $C^o \cap \text{dom} \nu_{B,P} \subset \text{cone}_{\nu_{B,P}}$, and (2.22) shows (2.24). Then some slight adjustments will finish the proof of the duality (2.10).

If we look at this lemma from more abstract point of view, we come up to the convex functionals of the type:
\begin{equation}
\cal{I}_{f,P}(X) := \sup_{P \in \cal{P}} E_P[f(\cdot, X)], \quad X \in L^\infty,
\end{equation}
where $f : \Omega \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is a random convex function. In fact, if we take $f(\omega, x) = -U(-x + B(\omega))$, then $\nu_{B,P}(X) = -\cal{I}_{f,P}(-X)$, and $\nu_{B,P}(\nu) = \cal{I}_{f,P}(\nu)$. 
When \( P \) consists of a single element (say \( \{P\} \)), the corresponding functional \( X \rightarrow E[f(\cdot, X)] \) is called the convex integral functional, and its basic properties including the expression of the conjugate functional are obtained by R. T. Rockafellar in [21, 22, 23]. The core of our analysis is thus to extend these classical results to the robust version of integral functionals of the form (2.25).

3. Analysis of Robust Convex Functionals

We now proceed to the key part of the paper, namely the analysis of the robust convex functionals formally defined by (2.25). We start by introducing some terminologies.

Definition 3.1 (Normal Convex Integrands). A map \( f : \Omega \times \mathbb{R} \mapsto \mathbb{R} \cup \{+\infty\} \) is said to be a normal convex integrand if:

1. \( f \) is \( F \otimes B(\mathbb{R}) \)-measurable;
2. \( x \mapsto f(\omega, x) \) is a lower semicontinuous proper convex function for a.e. \( \omega \).

Several equivalent definitions of normality are found in [24]. An immediate but important consequence of the normality in this sense is that the map \( \omega \mapsto f(\omega, X(\omega)) \) is \( F \)-measurable, hence the following definition makes sense:

\[
(3.1) \quad I_f(X) := E[f(\cdot, X)], \quad X \in L^\infty.
\]

Note that the normality of \( f \) already implies that the “\( \omega \)-wise” conjugate defined below is also normal:

\[
(3.2) \quad f^*(\omega, y) := \sup_{x \in \mathbb{R}} (xy - f(\omega, x)), \quad \forall y \in \mathbb{R},
\]

Therefore, the integral functional \( I_f^* \) on \( L^1 \) also makes sense (provided some integrability condition).

As we are considering the integral functionals on \( L^\infty \), we need the description of the dual of \( L^\infty \), that is the space \( ba \), which already appeared in Section 2.3. For the article to be self-contained, we recall the definition and some of basic properties. For the proofs and more details, we refer the reader to [9], [13] and [4].

Definition 3.2. \( ba := ba(\Omega, F, P) \) is the set of all bounded finitely additive signed measures absolutely continuous w.r.t. \( P \), i.e., \( \nu \in ba \) if and only if \( \nu \) is a real valued function on \( F \) such that (1) \( \sup_{A \in F} |\nu(A)| < \infty \), (2) for every \( A \in F \), \( P(A) = 0 \) implies \( \nu(A) = 0 \), (3) if \( A, B \in F \) and \( A \cap B = \emptyset \), then \( \nu(A \cup B) = \nu(A) + \nu(B) \). Also, \( ba^+ \) (resp. \( ba^- \)) denotes the set of positive (resp. \( \sigma \)-additive) elements of \( ba \), and set \( ba := ba^+ \cap ba^- \).

Lemma 3.3. The following assertions hold:

1. (Jordan decomposition). Every \( \nu \in ba \) admits a unique decomposition \( \nu = \nu_+ - \nu_- \), where \( \nu_+ \in ba^+ \). Thus, \( |\nu| = \nu_+ + \nu_- \in ba^+ \) is well-defined.
2. \( ba \) is a Banach space equipped with the total variation norm \( ||\nu|| = |\nu|(\Omega) \), and \( ba \simeq (L^\infty)^* \) with \( \langle X, \nu \rangle = \int_{\Omega} Xd\nu =: \nu(X) \). Also, \( L^1 \simeq ba^\sigma \) is a closed subspace of \( ba \).
3. (Yosida-Hewitt decomposition) Every \( \nu \in ba \) admits a unique decomposition \( \nu = \nu_r + \nu_s \), where \( \nu_r \in ba^\sigma \) and \( \nu_s \) is purely finitely additive, i.e., for every \( \mu \in ba^\sigma \), \( 0 \leq \mu \leq |\nu_s| \) implies \( \mu = 0 \).
4. \( \nu \in ba \) is purely finitely additive if and only if there exists an increasing sequence \( (A_n) \subset F \) such that \( P(A_n) \nearrow 1 \) and \( |\nu|(A_n) = 0 \) for all \( n \).
With these preparations, we now recall the classical Rockafellar theorem.

**Theorem 3.4** ([22, Theorem 1, Corollary 2A]). Let $f$ and $f^*$ be as above, and assume:

\[(3.3)\] there exists some $X \in L^\infty$ such that $f(\cdot, X)^+ \in L^1$;

\[(3.4)\] there exists some $Y \in L^1$ such that $f^*(\cdot, Y)^+ \in L^1$.

Then $I_f$ (resp. $I_{f^*}$) is well-defined as a lower semicontinuous proper convex functional on $L^\infty$ (resp. $L^1$), and for any $\nu \in ba$,

\[(3.5)\] $\sup_{X \in L^\infty} (\nu(X) - I_f(X)) = I_f^* (d\nu/dP) + \sup_{X \in \text{dom}(I_f)} \nu_s(X),$

where $\nu_s$ (resp. $\nu_c$) is the regular (resp. singular) part of $\nu$ in the Yosida-Hewitt decomposition. In particular, if $\text{dom}(I_f) = L^\infty$, then $I_f$ is norm continuous on the whole $L^\infty$ and

\[(3.6)\] $\sup_{X \in L^\infty} (\nu(X) - I_f(X)) = \begin{cases} I_f^* (d\nu/dP) & \text{if } \nu \text{ is } \sigma\text{-additive,} \\ +\infty & \text{otherwise.} \end{cases}$

**Remark 3.5.** All the assertions of the theorem remain valid if we replace $P$ by any $P \sim P$ with some obvious modifications, e.g., replacing $L^1$ by $L^1(P)$, $I_f$ by $I_{f,P}(X) := E_P[f(\cdot, X)]$, and $d\nu/dP$ by $d\nu/dP$.

3.1. Robust Version of Integral Functionals

We now proceed to the robust situation. Let $P$ be as in Section 2.1, satisfying the assumption (A1) of convexity and weak compactness, and $f$ a normal convex integrand. From a technical reason, we assume that $f$ is finite valued, i.e.,

\[(3.7)\] $P(f(\cdot, x) < \infty, \forall x \in \mathbb{R}) = 1.$

In particular, $x \mapsto f(\omega, x)$ is continuous. We then define the robust version of integral functional associated to $f$:

\[(3.8)\] $\mathcal{I}_{f,P}(X) := \sup_{P \in \mathcal{P}} E_P[f(\cdot, X)] = \sup_{P \in \mathcal{P}} \int_{\Omega} f(\omega, X(\omega))P(d\omega), \ \forall X \in L^\infty,$

and its conjugate defined on $ba$:

\[(3.9)\] $(\mathcal{I}_{f,P})^*(\nu) := \sup_{X \in L^\infty} (\nu(X) - \mathcal{I}_{f,P}(X)), \ \forall \nu \in ba = ba(\Omega, \mathcal{F}, P).$

In what follows, we investigate the regularity properties of $\mathcal{I}_{f,P}$ and the description of the conjugate $(\mathcal{I}_{f,P})^*$ in the spirit of Rockafellar’s theorem.

The functional $\mathcal{I}_{f,P}$ is the pointwise supremum over $P \in \mathcal{P}$ of the $P$-integral functionals $X \mapsto I_{f,P}(X) := E_P[f(\cdot, X)]$ restricted to $L^\infty = L^\infty(P)$. But there is an alternative point of view that $\mathcal{I}_{f,P}$ is a pointwise supremum over $\mathcal{P}$ of “$P$-integral functionals” associated to the integrands $(dP/dP)f$. Note that these new integrands are normal whenever $f$ is, and we can naturally deal with all $I_{f,P}$ on the same domain $L^\infty$. This simple point of view leads us to another type of conjugate: $\tilde{f}(\omega, y, z) := (z f)^*(\omega, y)$, that is,

\[(3.10)\] $\tilde{f}(\omega, y, z) := \sup_{x \in \mathbb{R}} (xy - z f(\omega, x)), \ \omega \in \Omega, y \in \mathbb{R}, z \geq 0.$

By definition, we have

\[(3.11)\] $xy \leq z f(\omega, x) + \tilde{f}(\omega, y, z), \ \forall \omega \in \Omega, \forall x, y \in \mathbb{R}, \forall z \geq 0.$
The next lemma is elementary.

**Lemma 3.6.** $\hat{f}$ is a normal convex integrand on $\Omega \times \mathbb{R} \times \mathbb{R}_+$, i.e., $\hat{f}$ is $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable, and $(y, z) \mapsto \hat{f}(\omega, y, z)$ is a lower semicontinuous proper convex functional on $\mathbb{R} \times \mathbb{R}_+$. Moreover, $\hat{f}$ is explicitly written as:

$$
\hat{f}(y, z) = \begin{cases} 
0 & \text{if } y = z = 0, \\
+\infty & \text{if } y \neq 0, z = 0, \\
zf^*(y/z) & \text{if } z > 0.
\end{cases}
$$ (3.12)

**Proof.** Since $\hat{f}(\cdot, y, z) \geq -zf(\cdot, 0) > -\infty$ and $\hat{f}(\cdot, 0, 0) = 0$, $\hat{f}$ is proper. The lower semicontinuity and the convexity are consequences of the fact that $\tilde{f}$ is a point-wise supremum of linear functions $(y, z) \mapsto xy - zf(x)$ when $x$ runs through all reals. As $x \mapsto f(\cdot, x)$ is lower semicontinuous (actually continuous), the supremum over reals can be replaced by that over rationals, which shows the measurability of $\hat{f}$. Finally, (3.12) is verified by direct computation. \qed

Using $\hat{f}$, we introduce another type of integral functional which plays the role of $I_f$ in the classical case:

$$
\mathcal{J}_{\hat{f}, P}(Y) := \inf_{f \in \mathcal{P}} E[\hat{f}(:, Y, dP/d\mathbb{P})], \quad Y \in L^1.
$$ (3.13)

We first verify that the functionals $\mathcal{I}_{f, P}$ and $\mathcal{J}_{\hat{f}, P}$ are indeed well-defined, under natural integrability conditions corresponding to (3.3) and (3.4).

**Proposition 3.7.** Assume (3.7) and that

(3.14) for some $X \in L^\infty$, $\sup_{P \in \mathcal{P}} E_P[f(:, X)\uparrow] < \infty$;

(3.15) for any $P \in \mathcal{P}$, there exists $Y \in L^1$ such that $\hat{f}(\cdot, Y, dP/d\mathbb{P})\uparrow \in L^1$.

Then we have the following.

(a) $\mathcal{I}_{f, P}$ is well-defined as a lower semicontinuous proper convex functional on $L^\infty$;

(b) $\mathcal{J}_{\hat{f}, P}$ is well-defined as a proper convex functional on $L^1$;

(c) for all $X \in L^\infty$ and $Y \in L^1$,

$$
E[XY] \leq \mathcal{I}_{f, P}(X) + \mathcal{J}_{\hat{f}, P}(Y).
$$ (3.16)

**Proof.** (a) Fix $P \in \mathcal{P}$, and take $Y \in L^1$ as in (3.15). By (3.11), we have

$$
\frac{dP}{d\mathbb{P}} f(\cdot, X) \geq XY - \hat{f}(:, Y, dP/d\mathbb{P})\uparrow, \quad \forall X \in L^\infty.
$$ (3.17)

Since the right hand side is integrable, we see that $E_P[f(\cdot, X)] > -\infty$ for all $X \in L^\infty$, while $E_P[f(\cdot, X)] < \infty$ for some $X \in L^\infty$ by (3.14). Thus, the functional $X \mapsto E_P[f(\cdot, X)]$ is a proper convex functional on $L^\infty$, where the convexity is clear from the convexity of $f$. Also, if $(X_n) \subset L^\infty$ is any norm convergent sequence with the limit $X$, then $X^n Y \rightarrow XY$, a.s. and in $L^1$. Then (3.17) allows us to use Fatou’s lemma to conclude

$$
E_P[f(\cdot, X)] \leq \liminf_{n \rightarrow \infty} E_P[f(\cdot, X_n)],
$$

hence we have that $X \mapsto E_P[f(\cdot, X)]$ is a lower semicontinuous proper convex functional on $L^\infty$. Since this holds for all $P \in \mathcal{P}$, $X \mapsto \sup_{P \in \mathcal{P}} E_P[f(\cdot, X)]$ is again a lower semicontinuous convex functional as a point-wise supremum of such
functionals, and \( I_{f,p}(X) > -\infty \) for all \( X \in L^\infty \). Finally, \( I_{f,p}(X) < \infty \) for some \( X \in L^\infty \) again by (3.14), hence \( I_{f,p} \) is proper.

(b), (c). Take \( X \) as in (3.14). Then \( f(\cdot, X) \in L^1(P) \) for all \( P \), and by (3.11),
\[
\tilde{f}(\cdot, Y, dP/d\tilde{P}) \geq XY - \frac{dP}{d\tilde{P}} f(\cdot, X), \quad \forall Y \in L^1, \forall P \in \mathcal{P}.
\]
Therefore, \( J_{f,p} \) is proper since
\[
\inf_{P \in \mathcal{P}} E[\tilde{f}(\cdot, Y, dP/d\tilde{P})] \geq \inf_{P \in \mathcal{P}} (E[XY] - E_P[f(\cdot, X)])
= E[XY] - \sup_{P \in \mathcal{P}} E_P[f(\cdot, X)] > \infty, \forall Y \in L^1.
\]

The convexity of \( J_{f,p} \) follows from that of \( (y, z) \mapsto \tilde{f}(\cdot, y, z) \) (Lemma 3.6) and of \( \mathcal{P} \), and we have (b). As for (c), we may assume \( I_{f,p}(X) < \infty \) and \( J_{f,p}(Y) < \infty \), since otherwise the assertion is trivial. But this case is already proved by (3.18). \( \square \)

**Remark 3.8.** The condition (3.15) is equivalent to:
\[
\forall P \in \mathcal{P}, \exists \tilde{Y} \in L^1(P) \text{ such that } f^*(\cdot, \tilde{Y}) \in L^1(P).
\]
Indeed, if \( \tilde{f}(\cdot, Y, dP/d\tilde{P})+ \in L^1 \), then \( Y = 0 \) on \( \{ \frac{dP}{d\tilde{P}} = 0 \} \) by (3.12) and \( \tilde{f}(\cdot, Y, dP/d\tilde{P})+ = 1_{\{ \frac{dP}{d\tilde{P}} > 0 \}} \frac{dP}{d\tilde{P}} f^*(\cdot, Y/(dP/d\tilde{P})) + \). Therefore, \( \tilde{Y} := 1_{\{ \frac{dP}{d\tilde{P}} > 0 \}} Y/(dP/d\tilde{P}) \), which is \( P \)-integrable since \( Y \) is \( P \)-integrable, satisfies the condition. Conversely, if \( \tilde{Y} \in L^1(P) \) satisfies \( f^*(\cdot, \tilde{Y})+ \in L^1(P) \), we set \( Y = \tilde{Y}dP/d\tilde{P} \). Then we have by (3.12) that \( \tilde{f}(\cdot, Y, dP/d\tilde{P})+ = \frac{dP}{d\tilde{P}} f^*(\cdot, \tilde{Y})+ \in L^1 \). In particular, (3.15) coincides with (3.4) of Theorem 3.4 when \( \mathcal{P} \) is a singleton.

3.2. A Robust Version of the Rockafellar Theorem

We have arrived at the main theorem of this section, which is also the heart of the whole paper. Recall that \( I_{f,p} \) and \( J_{f,p} \) are defined respectively by (3.9) and (3.13), and the normal convex integrand \( f \) is assumed to be finite valued, thus in particular, \( x \mapsto f(\omega, x) \) is continuous for a.e. \( \omega \).

To obtain a nice description of the conjugate \((I_{f,p})^*\), the integrability assumption (3.14) is not enough, and we need a slightly stronger assumption. Let
\[
\mathcal{D} := \{ X \in L^\infty : \{ f(\cdot, X)^+ dP/d\tilde{P} \}_{P \in \mathcal{P}} \text{ is uniformly integrable} \}.
\]
If \( X \in \mathcal{D} \), then \( \sup_{P \in \mathcal{P}} E_P[f(\cdot, X)] \leq \sup_{P \in \mathcal{P}} E_P[f(\cdot, X)^+] < \infty \), hence \( X \in \text{dom}(I_{f,p}) \), i.e., \( \mathcal{D} \subset \text{dom}(I_{f,p}) \). The proof of the next theorem will be given in Section 3.3.

**Theorem 3.9.** Assume (3.7), (3.15), (A1) (i.e., \( \mathcal{P} \) is convex and weakly compact) and that \( \mathcal{D} \neq \emptyset \). Then for all \( \nu \in ba \) with the Yosida-Hewitt decomposition \( \nu = \nu_r + \nu_s \),
\[
J_{f,p} \left( \frac{d\nu_r}{d\tilde{P}} \right) + \sup_{X \in \mathcal{D}} \nu_s(X) \leq (I_{f,p})^*(\nu)
\]
(3.21)
\[
\leq J_{f,p} \left( \frac{d\nu_r}{d\tilde{P}} \right) + \sup_{X \in \text{dom}(I_{f,p})} \nu_s(X).
\]

**Corollary 3.10** (Restriction to \( ba^\infty \simeq L^1 \)). For any \( \nu \in ba^\infty \), we have
\[
(I_{f,p})^*(\nu) = J_{f,p}(d\nu/d\tilde{P}) = \inf_{P \in \mathcal{P}} E[\tilde{f}(\cdot, d\nu/d\tilde{P}, P)]
\]
(3.22)
In particular, \( J_{f,p} \) is also lower semicontinuous.
Proof. The first assertion is immediate from (3.21) since the second terms in the left and right hand sides are zero if $\nu_s = 0$. The lower semicontinuity follows from a general fact: if $(E, E')$ is a dual pair (see [1] for definition), and $\phi$ is a convex function on $E$, the conjugate $\phi^*$ on $E'$ is $\sigma(E', E)$-lower semicontinuous. Here $E'$ does not have to be the topological dual of $E$ under the original topology. □

When $\mathcal{P}$ is a singleton, two sets $\mathcal{D}$ and $\text{dom}(\mathcal{I}_{f, \mathcal{P}})$ coincide, hence the two inequalities in (3.21) actually holds as a single equality, which is nothing other than the Rockafellar theorem (Theorem 3.4). In the general case, however, the inclusion $\mathcal{D} \subset \text{dom}(\mathcal{I}_{f, \mathcal{P}})$ can be strict (see Examples 3.12 and 3.13).

To illustrate the situation, we give an alternative form of integrability conditions. Define
\[ L^1(\mathcal{P}) := \left\{ X \in L^0 : \|X\|_{1, \mathcal{P}} := \sup_{P \in \mathcal{P}} E_P[|X|] < \infty \right\}. \]
This is an “$L^1$-type space” under the sublinear expectation $X \mapsto \sup_{P \in \mathcal{P}} E_P[X]$. If we introduce an equivalence relation by $X \sim_{\mathcal{P}} Y$ if $X = Y$, $\mathcal{P}$-quasi surely ($\Leftrightarrow$ $P$-a.s. for all $P \in \mathcal{P}$), the resulting quotient space $L^1(\mathcal{P}) := L^1(\mathcal{P})/\sim_{\mathcal{P}}$ is indeed a Banach space with the norm $\| \cdot \|_{1, \mathcal{P}}$. Then the integrability condition (3.14) is equivalent to saying that there exists some $X \in L^\infty$ such that $f(\cdot, X)^+ \in L^1(\mathcal{P})$.

If $\mathcal{P}$ is a singleton, the $L^1$-type space $L^1(\mathcal{P})$ is nothing but the usual $L^1$ space, and the condition (3.14) coincides with (3.3) in the Rockafellar theorem.

Under the sublinear expectation, there is another natural $L^1$-type space:
\[ L^1_a(\mathcal{P}) := \left\{ X \in L^0 : \lim_{N \to \infty} \sup_{P \in \mathcal{P}} E_P[|X|1_{\{|X| \geq N\}}] = 0 \right\}, \]
and we set $L^1_a(\mathcal{P}) := L^1_a(\mathcal{P})/\sim_{\mathcal{P}}$. It is immediate to show that $L^1_a(\mathcal{P}) \subset L^1(\mathcal{P})$, and $L^1(\mathcal{P}) = L^1_0(\mathcal{P}) = L^1(\mathcal{P})$. $L^1_a(\mathcal{P})$ is also a Banach space endowed with the norm $\| \cdot \|_{1, \mathcal{P}}$, hence $L^1_a(\mathcal{P})$ is closed in $L^1(\mathcal{P})$. Moreover, we can show also that
\[ (3.23) \quad X \in L^1_a(\mathcal{P}) \iff \{XdP/d\mathbb{P}\}_{P \in \mathcal{P}} \text{ is uniformly integrable.} \]
In particular, the set $\mathcal{D}$ is equivalently written as:
\[ (3.24) \quad \mathcal{D} = \{ X \in L^\infty : f(\cdot, X)^+ \in L^1_a(\mathcal{P}) \}. \]

Remark 3.11. The spaces $L^1(\mathcal{P})$ and $L^1_a(\mathcal{P})$ correspond respectively to $L^1$ and $L^1_0$ in [8, Section 2.2] which provide some basic properties including a counter part of (3.23) in a slightly different setting. In [8] $\Omega$ is a complete metric space with $\mathcal{F}$ being its Borel $\sigma$-field but $\mathcal{P}$ is not necessarily dominated by a single probability $\mathbb{P}$, while we assume $\mathcal{P}$ is dominated but impose no topological restriction to $\Omega$ and $\mathcal{F}$.

Finally, we used the subscript “$u$” to keep the uniform integrability in mind, while the subscript “$b$” in [8] comes from the fact that $\mathbb{L}_b$ is the completion under $\| \cdot \|_{1, \mathcal{P}}$ of the space of bounded functions.

Example 3.12 ($\mathcal{D} = \emptyset$ but $\text{dom}(\mathcal{I}_{f, \mathcal{P}}) = L^\infty$). We first give an extreme example following [8, Example 20]. We take $\Omega = \mathbb{N}$ with $\mathcal{F} = 2^{\mathbb{N}}$. In this case, every probability measure is absolutely continuous w.r.t. $\mathbb{P}$, given by $\mathbb{P}(\{n\}) = 2^{-n}$. For each $n \in \mathbb{N}$, we define $P_n$ by
\[ (3.25) \quad P_n(\{1\}) = 1 - 1/n, P_n(\{n\}) = 1/n, P_n(\{k\}) = 0 \text{ if } k \notin \{1, n\}. \]
Then we set $\mathcal{P} = \overline{\text{conv}}(P_n; n \in \mathbb{N})$. This $\mathcal{P}$ is weakly compact. To see this, it suffices to show the uniform integrability of $\{P_n\}_n$ since closed convex hull of
uniformly integrable family is again uniformly integrable (see [7, Th. II, 20]). Noting that \(dP_n/dP = 2(1 - 1/n)1_{[1]} + 2^n/n1_{[n]}\), \(E[(dP_n/dP)1_{(dP_n/dP \geq N)}] = (1/n)1_{[1]} + 2^n/n1_{[n]}\) for every \(N \geq 2\), hence \(\sup_n E[(dP_n/dP)1_{(dP_n/dP \geq N)}] = 1/n\), where \(n_N := \min(n : 2^n/n \geq N) \to \infty\) as \(N \to \infty\).

Consider a random variable \(W\) defined by \(W(n) = n\). Then since \(E_P[W] = 1 \cdot (1 - 1/n) + n \cdot (1/n) = 2 - 1/n\), we see \(\sup_{P \in \mathcal{P}} E_P[W] = \sup_n E_P[n] = 2\). But,

\[
E_P[nW1_{\{W \geq N\}}] = \begin{cases} 0 & \text{if } n < N, \\ 1 & \text{if } n \geq N. \end{cases}
\]

Hence \(\sup_n E_P[nW1_{\{W \geq N\}}] = 1 \neq 0\).

Let \(f(\omega, x) = W(\omega)^{-}\). For any constant random variable \(a\), \(f(\omega, a)^+ = e^a W\), hence \(\sup_{P \in \mathcal{P}} E_P[f(\omega, X)^+] = \sup_n E_P[f(\omega, X)^+] = e^a \sup_n E_P[W] = 2e^a\), but \(\sup_{P \in \mathcal{P}} E_P[f(\omega, X)^+1_{\{f(\omega, X) \geq N\}}] = e^a \sup_n E_P[W1_{\{e^a \geq W \geq n\}}] = e^a\), for all \(N \in \mathbb{N}\). This shows that \(a \in \text{dom}(I_{f,P}) \setminus \mathcal{D}\). Moreover, we have \(\emptyset = \mathcal{D} \neq \text{dom}(I_{f,P}) = L^\infty\).

Therefore, \(X \in \text{dom}(I_{f,P})\) for all \(X \in L^\infty\), but \(X \not\in \mathcal{D}\) for all \(X \in L^\infty\).

**Example 3.13.** Let \(W\) be same as the previous example, and set \(f(\omega, x) := W(\omega)^{-}\). Since \(W \geq 1\), this \(f\) is well-defined. For \(X \equiv 1\), we have \(f(\cdot, X) \in L^1(\mathcal{P}) \setminus L_1^s(\mathcal{P})\). On the other hand, if \(\|X\|_\infty \leq \gamma < 1\),

\[
\sup_{P \in \mathcal{P}} E_P[f(\cdot, X)1_{\{|f(\cdot, X)| \geq n\}}] \leq \sup_{P \in \mathcal{P}} E_P[W1_{\{W \geq n\}}] \leq \sup_{P \in \mathcal{P}} E_P[W^\gamma] \cdot \sup_{P' \in \mathcal{P}} P'(W^\gamma \geq n)^1 - \gamma - n^{-\gamma} = 0.
\]

The last convergence comes from the uniform integrability of \(\mathcal{P}\). Therefore, we have \(X \in \mathcal{D}\), and consequently, \(\emptyset \neq \mathcal{D} \subset \text{dom}(I_{f,P})\).

Although the inequalities in (3.21) can generally be strict, we can still have an exact estimate in the case corresponding to the second assertion of Theorem 3.4.

**Corollary 3.14.** If \(\mathcal{D} = L^\infty\), \(I_{f,P}\) is norm continuous on the whole \(L^\infty\), and

\[
\sup_{X \in L^\infty} (\nu(X) - I_{f,P}(X)) = \begin{cases} J_{f,P}(d\nu/dP) & \text{if } \nu \text{ is } \sigma\text{-additive}, \\ +\infty & \text{otherwise}. \end{cases}
\]

**Proof.** The equality (3.26) is again a direct consequence of (3.21) since

\[
\sup_{X \in L^\infty} \nu_s(X) = \begin{cases} 0 & \text{if } \nu_s = 0, \\ +\infty & \text{if } \nu_s \neq 0. \end{cases}
\]

The assumption \(\mathcal{D} = L^\infty\) implies also that \(I_{f,P}\) is finite on the whole \(L^\infty\), thus the continuity follows from the general fact that a lower semicontinuous convex function on a Banach space is continuous on the interior of its effective domain, which is equal to \(L^\infty\) in this case. See e.g., [10]. \(\square\)
An important question is when the condition $D = L^\infty$ holds. A trivial case is that $f$ is "deterministic", i.e., $f$ does not depend on $\omega$. Indeed, for any $X \in L^\infty$, the random variable $f(X)$ is again bounded, since we are assuming $\text{dom}(f) = \mathbb{R}$, hence $f$ is continuous. In the general case with $f$ being "random", $x \mapsto f(\omega, x)$ is still continuous, but the bound $\sup_{|x| \leq \|X\|_\infty} f(\omega, x)$ depends on $\omega$. The next criterion is easy, but will turn out to be useful.

**Corollary 3.15.** Suppose that there exists a continuous function $g : \mathbb{R} \to \mathbb{R}$ and a random variable $W$ such that $\{W dP/d\mathbb{P}\}_{P \in \mathcal{P}}$ is uniformly integrable, and

\[
(3.27) \quad f(\omega, x) \leq g(x) + W(\omega).
\]

Then $D = L^\infty$ and hence (3.26) holds.

**Proof.** It suffices to note that a family $\mathcal{X}$ in $L^1$ is uniformly integrable if it is dominated by a uniformly integrable family $\mathcal{Y}$ in the sense that for every $X \in \mathcal{X}$, there exists $Y \in \mathcal{Y}$ such that $|X| \leq |Y|$. In the present case, we take $\mathcal{Y} = \{(g(X) + |W|) dP/d\mathbb{P}\}_{P \in \mathcal{P}}$ which is uniformly integrable by the assumptions, and dominates \(f(\cdot, X)^+ dP/d\mathbb{P}\) for large $\gamma_0$. \hfill \Box

### 3.3. **Proof of Theorem 3.9**

**Lemma 3.16.** Let $D$ be a random variable with $D^- \in L^1$, and $\alpha$ be a constant with $\alpha < E[D]$. Then we can take some $Z \in L^1$ such that $Z < D$, a.s., and $E[Z] > \alpha$.

**Proof.** Take $\varepsilon > 0$ with $E[D] - \varepsilon > \alpha$. Then for any $\gamma \in \mathbb{R}$, $(D - \varepsilon) \wedge \gamma < D$, $(D - \varepsilon) \wedge \gamma \in L^1$ since $D^- \in L^1$, and $E[(D - \varepsilon) \wedge \gamma] = E[D] - \varepsilon > \alpha$ by the monotone convergence theorem. Therefore, $Z := (D - \varepsilon) \wedge \gamma_0$ satisfies the condition of the statement for a large $\gamma_0$. \hfill \Box

**Lemma 3.17** ([21, Lemma 6]). Let $g$ be a normal convex integrand, and $Z$ be a random variable such that

\[
(3.28) \quad \inf_{x \in \mathbb{R}} g(\cdot, x) < Z, \text{ a.s.}
\]

There then exists some finite valued random variable $X$ such that

\[
(3.29) \quad g(\cdot, X) \leq Z, \text{ a.s.}
\]

**Lemma 3.18.** For any $X \in \mathcal{D}$, the map $P \mapsto E_P[f(\cdot, X)]$ is weakly upper semi-continuous on $\mathcal{P}$.

**Proof.** Let $A_\alpha := \{P \in \mathcal{P} : E_P[f(\cdot, X)] \geq \alpha\}$, which is convex, hence is weakly closed if and only if strongly closed. Thus, it suffices to show that $A_\alpha$ is strongly closed for all $\alpha \in \mathbb{R}$. Let $(P_n)$ be a convergent sequence in $A_\alpha$, i.e., $dP_n/d\mathbb{P} \to dP/d\mathbb{P}$ in $L^1$ for some probability measure $P \in \mathcal{P}$, and we show that $P \in A_\alpha$. Taking a subsequence if necessary, we may assume the a.s. convergence. For each $n$,

\[
\frac{dP_n}{d\mathbb{P}} f(\cdot, X) \leq \frac{dP_n}{d\mathbb{P}} f(\cdot, X)^+,
\]

and the family $\{f(\cdot, X)^+ dP_n/d\mathbb{P}\}_{n}$ is uniformly integrable and a.s. convergent. Therefore, we can apply (reverse) Fatou’s lemma to get:

\[
E_P[f(\cdot, X)] \leq \lim \sup_n E_{P_n}[f(\cdot, X)] \leq E_P[f(\cdot, X)].
\]
Hence $P \in A_\alpha$ and the proof is complete. \hfill \Box

**Proof of Theorem 3.9.** We start from the second inequality. Noting that $\nu(X) = \nu_r(X) + \nu_s(X) = E[Xd\nu_r/\mathbb{d}P] + \nu_s(X)$, the inequality (3.16) in Proposition 3.7 shows that

$$\sup_{X \in L^\infty} (\nu(X) - \mathcal{I}_{f,P}(X)) = \sup_{X \in \text{dom}(\mathcal{I}_{f,P})} (\nu(X) - \mathcal{I}_{f,P}(X))$$

$$= \sup_{X \in \text{dom}(\mathcal{I}_{f,P})} (E[Xd\nu_r/\mathbb{d}P] - \mathcal{I}_{f,P}(X) + \nu_s(X))$$

$$\leq \mathcal{J}_{f,P}(d\nu_r/\mathbb{d}P) + \sup_{X \in \text{dom}(\mathcal{I}_{f,P})} \nu_s(X).$$

The first inequality is more subtle. Observe that $X \mapsto \nu(X) - E_P[f(\cdot,X)]$ is concave on $\mathcal{D}$, and $P \mapsto \nu(X) - E_P[f(\cdot,X)]$ is convex and lower semicontinuous on the weakly compact set $\mathcal{P}$ for all $X \in \mathcal{D}$ by Lemma 3.18. Thus a minimax theorem shows:

$$\sup_{X \in L^\infty} (\nu(X) - \mathcal{I}_{f,P}(X)) = \sup_{X \in L^\infty} \inf_{P \in \mathcal{P}} (\nu(X) - E_P[f(\cdot,X)])$$

$$\geq \sup_{X \in \mathcal{D}} \inf_{P \in \mathcal{P}} (\nu(X) - E_P[f(\cdot,X)])$$

$$= \inf_{P \in \mathcal{P}} \sup_{X \in \mathcal{D}} (\nu(X) - E_P[f(\cdot,X)]).$$

We shall show:

**Claim.** For any $\alpha < \mathcal{J}_{f,P}(d\nu_r/\mathbb{d}P)$ and $\beta < \sup_{X \in \mathcal{D}} \nu_s(X)$, we have

$$\sup_{X \in \mathcal{D}} (\nu(X) - E_P[f(\cdot,X)]) > \alpha + \beta, \quad \forall P \in \mathcal{P}. \quad (3.30)$$

**Proof of Claim.** Note first that there exists by definition an element $X_s \in \mathcal{D}$ with $\nu_s(X_s) > \beta$. Also, there exists an increasing sequence $(A_n)$ in $\mathcal{F}$ such that $P(A_n) \nearrow 1$ and $|\nu_s|(A_n) = 0$ for each $n$, by the singularity of $\nu_s$. In particular, for any $X \in L^\infty$, $\nu_s(X_1A_n + X_s1_{A_n}) = \nu_s(X_s) > \beta$.

As for the regular part, since $\alpha < \mathcal{J}_{f,P}(d\nu_r/\mathbb{d}P) = \inf_{P \in \mathcal{P}} E[f(\cdot,d\nu_r/\mathbb{d}P)]$, Lemma 3.16 shows the existence, for each $P \in \mathcal{P}$, of an integrable random variable $Z_P$ such that

$$E[Z_P] > \alpha$$

and

$$Z_P < \hat{f} \left( \left( \frac{d\nu_r}{\mathbb{d}P}, \frac{dP}{\mathbb{d}P} \right) = \sup_{x \in \mathbb{R}} \left( x \frac{d\nu_r}{\mathbb{d}P} - \frac{dP}{\mathbb{d}P} f(\cdot,x) \right) \right), \text{ a.s.} \quad (3.31)$$

The latter condition and Lemma 3.17 applied to the normal integrand $(\omega,x) \mapsto f(\omega,x)(dP/d\mathbb{P})(\omega) - x(d\nu_r/d\mathbb{P})(\omega)$ yields a measurable selection $X^0_P \in L^0$ with

$$X^0_P d\nu_r/d\mathbb{P} - f(\cdot,X^0_P)dP/d\mathbb{P} \geq Z_P. \quad (3.32)$$

Note that $X^0_P$ is not an element of $\mathcal{D}$ (not even in $L^\infty$) in general. Thus we need to **approximate** $X^0_P$ with elements of $\mathcal{D}$. Recall that we are assuming $f$ to be finite valued. Let $B_n := \{|X^0_P| \leq n\} \cap \{|f(\cdot,X^0_P)| \leq n\}$, and $C_n := A_n \cap B_n$, for each $n$. Then $P(C_n) \not\nearrow 1$, $|\nu_s|(C_n) = 0$, and

$$X^0_P := X^0_P1_{C_n} + X_s1_{C^c_n} \in \mathcal{D}, \quad \text{for each } n.$$

Indeed, $X^0_P1_{C_n}$ and $1_{C_n}f(\cdot,X^0_P)$ are bounded by the construction, $f(\cdot,X^0_P) = 1_{C_n}f(\cdot,X^0_P) + 1_{C^c_n}f(\cdot,X_s)$, and hence the family $\{f(\cdot,X^0_P)^+dP/d\mathbb{P}\}_{P \in \mathcal{P}}$ is uniformly integrable by the weak compactness of $\mathcal{P}$ and the uniform integrability of
\[ \{ f(\cdot,X_n)^+dP/d\bar{P}\}_{P \in \mathcal{P}}. \] We have
\[
E[X^*_Pd\nu_r/d\bar{P}] - E_P[f(\cdot,X^*_P)]
\]
\[
= E \left[ 1_{C_\alpha} \left( X^*_P \frac{d\nu_r}{d\bar{P}} - \frac{dP}{d\bar{P}} f(\cdot,X^*_P) \right) \right] + E \left[ 1_{C_\beta} \left( X^*_P \frac{d\nu_r}{d\bar{P}} - \frac{dP}{d\bar{P}} f(\cdot,X_n) \right) \right]
\]
\[
\geq E[1_{C_\alpha} Z_P] + E \left[ 1_{C_\beta} \left( X^*_P \frac{d\nu_r}{d\bar{P}} - \frac{dP}{d\bar{P}} f(\cdot,X_n) \right) \right]
\]
\[
= E[Z_P] + E[1_{C_\alpha} \Xi_P],
\]
where \( \Xi_P := X^*_P \frac{d\nu_r}{d\bar{P}} - f(\cdot,X_n) dP/d\bar{P} - Z_P \in L^1. \) Since \( \nu_s(X^*_P) = \nu_s(X_n) > \beta, \)
\[
\sup_{X \in D} (\nu(X) - E_P[f(\cdot,X)]) \geq E[X^*_Pd\nu_r/d\bar{P}] - E_P[f(\cdot,X^*_P)] + \nu_s(X^*_P)
\]
\[
\geq E[Z_P] + \nu_s(X_n) + E[1_{C_\alpha} \Xi_P],
\]
for each \( n. \) Since \( \lim_{n} E[1_{C_\alpha} \Xi_P] = 0, \) we have
\[
\sup_{X \in D} (\nu(X) - E_P[f(\cdot,X)]) \geq E[Z_P] + \nu_s(X_n) > \alpha + \beta,
\]
and the claim is proved.

We now complete the proof of the first inequality in (3.21). By taking \( \alpha = \mathcal{J}_{f,P}(d\nu_r/d\bar{P}) - \varepsilon/2 \) and \( \beta = \sup_{X \in D} \nu_s(X) - \varepsilon/2, \) we have
\[
\inf_{P \in \mathcal{P}} \sup_{X \in D} (\nu(X) - E_P[f(\cdot,X)])
\]
\[
\geq (\mathcal{J}_{f,P}(d\nu_r/d\bar{P}) - \varepsilon/2) + (\sup_{X \in D} \nu_s(X) - \varepsilon/2)
\]
\[
= \mathcal{J}_{f,P}(d\nu_r/d\bar{P}) + \sup_{X \in D} \nu_s(X) - \varepsilon,
\]
for all \( \varepsilon > 0, \) and the proof is complete. \( \square \)

4. PROOF OF THE DUALITY

We now complete our program outlined in Section 2.3. We start by translating the context of robust utility maximization into the language of Section 3. Set
\[
f(\omega,x) := -U(-x + B(\omega)), \quad \forall (\omega,x) \in \Omega \times \mathbb{R},
\]
which is \( \mathcal{F} \otimes \mathcal{B}(\mathbb{R}) \)-measurable, and \( x \mapsto f(\omega,x) \) is convex and continuous, hence is a finite valued normal convex integrand in the sense of Definition 3.1. The conjugate of \( f \) is given by \( f^*(\cdot,y) = V(y) + yB, \) and
\[
\tilde{f}(\omega,y,z) = \begin{cases} 0 & \text{if } y = z = 0, \\ +\infty & \text{if } y \neq 0, z = 0, \\ zV(y/z) + yB(\omega) & \text{if } z > 0. \end{cases}
\]
Noting that \( u_{B,P}(X) = -\mathcal{I}_{f,P}(-X), \) and \( v_{B,P} = (\mathcal{I}_{f,P})^*, \) we now arrive at the position to prove the key lemma.

Proof of Lemma 2.8. We shall apply Corollary 3.15 to \( (f,\tilde{f}) \) given by (4.1) and (4.2). Since \( f^*(\cdot,1) = V(1) + B, \) Remark 2.2.1, and Remark 3.8.2 guarantee that (3.15) is satisfied. On the other hand, the concavity of \( U \) implies that
\[
f(\cdot,x) \leq -\frac{\varepsilon}{1 + \varepsilon} U \left( \frac{1 + \varepsilon}{\varepsilon} x \right) - \frac{1}{1 + \varepsilon} U(-(1 + \varepsilon)B^-) =: g(x) + W.
\]
Here \( g \) is continuous and finite on \( \mathbb{R}, \) while \( \{WdP/d\bar{P}\}_{P \in \mathcal{P}} \) is uniformly integrable by (A4). Hence we can apply Corollary 3.15, and \( u_{B,P} \) is continuous in particular.
It remains to compute the explicit form of $J_{f,P}$ on $ba^+_\kappa$. We first show:

\begin{equation}
E\left[ \tilde{f}\left( \cdot, \frac{d\nu}{d\mathbb{P}}, \frac{dP}{d\mathbb{P}} \right) \right] = \begin{cases} V(\nu|P) + \nu(B) & \text{if } V(\nu|P) < \infty \\ +\infty & \text{otherwise.} \end{cases}
\end{equation}

On the set $\{d\nu/d\mathbb{P} > 0, dP/d\mathbb{P} = 0\}$, we have $\tilde{f}(\cdot, d\nu/d\mathbb{P}, dP/d\mathbb{P}) = +\infty$, while on $\{dP/d\mathbb{P} > 0\}$, [17, Lemma 3.4] shows

\begin{equation}
\frac{\varepsilon}{1 + \varepsilon} \frac{d\mathbb{P}}{d\mathbb{P}} \left( V\left( \frac{d\nu}{d\mathbb{P}} \right) - V(1) \right) + \frac{d\mathbb{P}}{d\mathbb{P}} U(- (1 + \varepsilon)B^{-}) \\
\leq \tilde{f}\left( \frac{d\nu}{d\mathbb{P}}, \frac{dP}{d\mathbb{P}} \right) \leq \frac{1 + \varepsilon}{\varepsilon} \frac{d\mathbb{P}}{d\mathbb{P}} V\left( \frac{d\nu}{d\mathbb{P}} \right) - \frac{1}{\varepsilon} \frac{d\mathbb{P}}{d\mathbb{P}} U(- \varepsilon B^+). \tag{4.4}
\end{equation}

Recalling that $V$ is bounded from below, the integrability assumption (A4) implies that $\tilde{f}(\cdot, d\nu/d\mathbb{P}, dP/d\mathbb{P})^{-} \in L^1$ for all $\nu \in ba^+_\kappa$, and $\tilde{f}(\cdot, d\nu/d\mathbb{P}, dP/d\mathbb{P}) \in L^1$ if and only if $V(\nu|P) < \infty$ (which implies $\nu \ll P$). Further, when the latter condition is satisfied, we have $B \in L^1(\nu)$, hence $E[\tilde{f}(\cdot, d\nu/d\mathbb{P}, dP/d\mathbb{P})] = E[\{d\nu/dP\}(V(d\nu/dP) + (d\nu/dP)B)] = V(\nu|P) + \nu(B)$. We thus obtain (4.3), and taking the infimum over $\mathcal{P}$, we have for all $\nu \in ba^+_\kappa$,

\begin{equation}
\mathcal{J}_{f,P}(d\nu/d\mathbb{P}) = \begin{cases} V(\nu|P) + \nu(B) & \text{if } V(\nu|P) < \infty \\ +\infty & \text{otherwise.} \end{cases} \tag{4.5}
\end{equation}

This concludes the proof of the corollary.

\begin{corollary}
For every $Q \in \mathcal{M}_V$, we have
\begin{equation}
v_{B,P}(\lambda Q) = V(\lambda Q|\mathcal{P}) + \lambda E_Q[B], \quad \forall \lambda \geq 0. \tag{4.6}
\end{equation}
\end{corollary}

\begin{proof}
Let $Q \in \mathcal{M}_V$, then $B \in L^1(Q)$ by Remark 2.2, and hence the right hand side is well-defined for all $\lambda \geq 0$. Since $v_{B,P}(\lambda Q) < \infty$ if and only if $V(\lambda Q|\mathcal{P}) < \infty$, (4.6) is true in this case, and otherwise, both sides are $+\infty$.
\end{proof}

Recall that the convex cone $\mathcal{C}$ is defined by (2.18). We need the following lemma, which guarantees that the element $0 \in ba^+_\kappa$ never contributes to the dual problem.

\begin{lemma}
We have
\begin{equation}
\inf_{\lambda > 0} \inf_{Q \in \mathcal{M}_V} (V(\lambda Q|\mathcal{P}) + \lambda E_Q[B]) < V(0). \tag{4.7}
\end{equation}
\end{lemma}

\begin{proof}
This is trivial if $V(0) = +\infty$, thus we assume $V(0) < \infty$. Taking a pair $(Q,P) \in \mathcal{M}_V \times \mathcal{P}$ as well as $\lambda > 0$ with $Q \sim \lambda Q \sim \mathbb{P}$ and $V(\lambda Q|\mathcal{P}) < \infty$ by Remark 2.1, the result will follow if we can show $V(0) > E^P[V(\lambda dQ/d\mathbb{P}) + \lambda dQ/d\mathbb{P})B]$ for some $\lambda > 0$. Set $\varphi(\lambda) := E^P[V(\lambda dQ/d\mathbb{P}) + \lambda dQ/d\mathbb{P})B]$, which is convex and finite on $\lambda \in [0,\bar{\lambda}]$ since $V(0) < \infty$. It is easy (cf. the proof of [17, Lemma 3.7]) to show that $\varphi$ is differentiable, and $\varphi'(\lambda) = E^P[V'(\lambda dQ/d\mathbb{P}) + B]$, hence $\varphi(0) = V(0) + E^Q[B] = -\infty$ by (2.2). Thus, we have $\varphi(\lambda) < V(0)$ for some $\lambda > 0$.
\end{proof}

The next one is an abstract version of the duality.

\begin{proposition}
Under the assumptions of Theorem 2.3, we have
\begin{equation}
\sup_{X \in \mathcal{C}} u_{B,P}(X) = \min_{\lambda > 0, Q \in \mathcal{M}_V} (V(\lambda Q|\mathcal{P}) + \lambda E_Q[B]) < \infty. \tag{4.8}
\end{equation}
\end{proposition}
Proof. Since $u_{B,P}$ is continuous on the whole $L^\infty$, Fenchel’s duality theorem ([20, Theorem 1]) shows that

$$\sup_{X \in C} u_{B,P}(X) = \min_{\nu \in C^o} v_{B,P}(\nu) = \min_{\nu \in C^{\sup} \cap \text{dom}(v_{B,P})} v_{B,P}(\nu).$$  

(4.9)

Here “min” means of course that it is attained by some $\hat{\nu} \in C^o \cap \text{dom}(v_{B,P})$. By Lemma 2.8, $\nu \in \text{dom}(v_{B,P})$ if and only if $\nu$ is $\sigma$-additive and $V(\nu|P) < \infty$, while every $\sigma$-additive element $\nu \in C^o$ is a positive multiple of some local martingale measure, i.e., $\nu = \lambda Q$ with $\lambda \geq 0$ and $Q \in \mathcal{M}_{\text{loc}}$. Thus we can rewrite the right hand side of (4.9) using (4.6):

$$\min_{\nu \in C^{\sup} \cap \text{dom}(v_{B,P})} v_{B,P}(\nu) = \min_{\lambda \geq 0, Q \in \mathcal{M}_V} v_{B,P}(\lambda Q) = \min_{\lambda \geq 0, Q \in \mathcal{M}_V} (V(\lambda Q|P) + \lambda E_Q[B]).$$

But Lemma 4.2 implies that $\lambda = 0$ never contributes to the minimum, hence we obtain (4.8), and the finiteness of the right hand side follows from (A3). □

The next step is to replace the “inf$_{X \in C}$” by the infimum over stochastic integrals $\theta \cdot S_T$. By the definition of $C$, the inclusion $\Theta_{bb} \subset \Theta_V$ (see (2.1) and (2.6) for definitions) as well as the monotonicity of utility function, we have

$$\sup_{X \in C} u_{B,P}(X) \leq \sup_{\theta \in \Theta_V} \inf_{P \in \mathcal{P}} E_P[U(\theta \cdot S_T + B)] \leq \sup_{\theta \in \Theta_{V}} \inf_{P \in \mathcal{P}} E_P[U(\theta \cdot S_T + B)].$$  

(4.10)

Therefore, it suffices to show:

**Lemma 4.4.** We have

$$\sup_{\theta \in \Theta_{V}} \inf_{P \in \mathcal{P}} E_P[U(\theta \cdot S_T + B)] \leq \inf_{\lambda > 0} \inf_{Q \in \mathcal{M}_V} (V(\lambda Q|P) + \lambda E_Q[B]).$$  

(4.11)

Proof. Let $\theta \in \Theta_{V}$ be fixed and take any pair $(Q, P) \in \mathcal{M}_V \times \mathcal{P}$ as well as $\lambda > 0$ with $V(\lambda Q|P) < \infty$ ($\Rightarrow Q \ll P$). Then by Young’s inequality,

$$U(\theta \cdot S_T + B) \leq V(\lambda dQ|dP) + \lambda dQ|dP(\theta \cdot S_T + B).$$

Since $\theta \cdot S$ is a $Q$-supermartingale (hence $\theta \cdot S_T$ is $Q$-integrable in particular), and $B \in L^1(Q)$, we obtain by taking $P$-expectation

$$E_P[U(\theta \cdot S_T + B)] \leq E_P\left[V(\lambda dQ|dP) + \lambda dQ|dP B + E_Q[\theta \cdot S_T]\right] \leq V(\lambda Q|P) + \lambda E_Q[B].$$

When $(Q, P) \in \mathcal{M}_V \times \mathcal{P}$ but $V(\lambda Q|P) = \infty$, the right hand side is $+\infty$, hence this inequality is valid for all $\theta \in \Theta_{V}$, $\lambda > 0$ and $(Q, P) \in \mathcal{M}_V \times \mathcal{P}$. Taking the infimum over $P \in \mathcal{P}$, we have $\inf_{P \in \mathcal{P}} E_P[U(\theta \cdot S_T + B)] \leq V(\lambda Q|P) + \lambda E_Q[B]$. Now taking the supremum over $\theta \in \Theta_{V}$ and infimum over $(\lambda, Q) \in (0, \infty) \times \mathcal{M}_V$, we obtain (4.11). □

**Proof of Theorem 2.3.** The equality (2.10) as well as the existence of a dual minimizer follow from (4.8) and the inequality (4.11). Let $(\hat{\lambda}, \hat{Q}) \in (0, \infty) \times \mathcal{M}_V$ be a dual minimizer. Then since $\mathcal{P}$ is weakly compact, and $(\nu, P) \mapsto V(\nu|P)$ (hence $P \mapsto V(\hat{\lambda} \hat{Q}|P)$) is weakly lower semicontinuous ([11, Lemma 2.7]), there exists a $\hat{P}$.
with $V(\hat{\lambda}\hat{Q}|\hat{P}) = V(\hat{\lambda}\hat{Q}|\hat{P})$. This triplet is a desired solution to the dual problem, i.e.,
\[
V(\hat{\lambda}\hat{Q}|\hat{P}) + \hat{\lambda}E_Q[B] = \inf_{\lambda > 0} \inf_{(Q,P) \in \mathcal{M}_V \times \mathcal{P}} (V(\lambda Q|P) + \lambda E_Q[B]),
\]
and we finish the proof. □

Proof of Corollary 2.5. Again we take a triplet $(\bar{\lambda}, \bar{Q}, \bar{P}) \in (0, \infty) \times \mathcal{M}_V \times \mathcal{P}$ satisfying $V(\bar{\lambda}\bar{Q}|\bar{P}) < \infty$ and $\bar{Q} \sim \bar{P} \sim P$ by Remark 2.1, then set $\nu_\alpha := \alpha \lambda \bar{Q} + (1 - \alpha)\hat{\lambda}\hat{Q}$ and $P_\alpha := \alpha \hat{P} + (1 - \alpha)\hat{P}$. Note that $\nu_\alpha \sim P_\alpha \sim P$ for all $\alpha \in (0, 1]$, and the function $\alpha \mapsto V(\nu_\alpha|P_\alpha) + \nu_\alpha(B)$ is convex and finite valued on $[0, 1]$. Therefore, this function is upper semicontinuous on $[0, 1]$, which implies $\inf_{\alpha \in (0,1]} V(\nu_\alpha|P_\alpha) + \nu_\alpha(B) \leq \limsup_{\alpha \searrow 0} V(\nu_\alpha|P_\alpha) + \nu_\alpha(B) \leq V(\hat{\lambda}\hat{Q}|\hat{P}) + \hat{\lambda}E_Q[B]$. This implies that
\[
\inf_{\lambda > 0} \inf_{Q \in \mathcal{M}_V} (V(\lambda Q|P) + \lambda E_Q[B]) \leq \inf_{\alpha \in (0,1]} (V(\nu_\alpha|P_\alpha) + \nu_\alpha(B)) \leq V(\hat{\lambda}\hat{Q}|\hat{P}) + \hat{\lambda}E_Q[B] = \inf_{\lambda > 0} \inf_{Q \in \mathcal{M}_V} (V(\lambda Q|P) + \lambda E_Q[B]).
\]
This concludes the proof. □

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