THE COVARIOGRAM AND AN EXTENSION OF SIEGEL’S FORMULA

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Abstract. We extend a formula of C. L. Siegel in the geometry of numbers, allowing the body to contain an arbitrary number of interior lattice points. Our extension involves a lattice sum of the covariogram of any compact set $K \subset \mathbb{R}^d$. The Fourier methods herein also allow for more general admissible sets, due to the Poisson summation formula. As one of the consequences of these results, we obtain a new characterization of multi-tilings of Euclidean space by translations, using the lower bound on lattice sums of such covariograms.

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1. Introduction

We extend some of Siegel’s results from the Geometry of numbers by studying the covariogram of a compact set $K$, defined by

\begin{equation}
 g_K(x) := \text{vol}(K \cap (K + x)),
\end{equation}

defined for all $x \in \mathbb{R}^d$. We define a body to be any compact subset of $\mathbb{R}^d$, following the standard convention of convex geometry. Given any full-rank lattice $\mathcal{L} \subset \mathbb{R}^d$, we study the following sum of the covariogram $g_K$ over the lattice $\mathcal{L}$:

\begin{equation}
 \sum_{n \in \mathcal{L}} g_K(n) := \sum_{n \in \mathcal{L}} \text{vol}(K \cap (K + n)),
\end{equation}

arXiv:2204.08606v1 [math.NT] 19 Apr 2022
which is always a finite sum due to the compactness of $K$. The covariogram $g_K$ has been studied intensively in recent years and is sometimes referred to as the set covariance. It follows immediately from basic principles that the covariogram is also equal to the autocorrelation of $1_K$:

$$g_K(x) := 1_K * 1_{-K}(x) := \int_{\mathbb{R}^d} 1_K(t)1_{-K}(x-t)dt,$$

where $1_K$ is the indicator function of the body $K$. Here we first give a formula for $\sum_{n \in \mathcal{L}} g_K(n)$ (Theorem 2, equation (8)), in terms of a dual lattice sum that involves the Fourier transform of $K$. This formula (equation (8) below) extends the classical formula of Carl Ludwig Siegel [13] in the geometry of numbers, and is obtained here by employing the Poisson summation formula.

We then give universal lower bounds for the sum $\sum_{n \in \mathcal{L}} g_K(n)$, and completely characterize the compact sets $K$ that give equality (Corollary 2) in these lower bounds, which turn out to be multi-tiling bodies.

We first recall the classical formula of Siegel, which in itself is also an extension of Minkowski’s first theorem ([11]).

**Theorem (Siegel’s formula).** Let $P \subset \mathbb{R}^d$ be a convex body, and $\mathcal{L} \subset \mathbb{R}^d$ a full-rank lattice. Let $B := \frac{1}{2}P - \frac{1}{2}P$ be its symmetrized body. If the only lattice point of $\mathcal{L}$ in the interior of $B$ is the origin,

$$2^d \det \mathcal{L} = \text{vol} P + 4^d \sum_{\xi \in \mathcal{L}^* - \{0\}} |\hat{1}_{P/2}(\xi)|^2.$$

Taking $Q := \frac{1}{2}P$, we may equivalently write:

$$\det \mathcal{L} = \text{vol} Q + \frac{1}{\text{vol} Q} \sum_{\xi \in \mathcal{L}^* - \{0\}} |\hat{1}_{Q}(\xi)|^2.$$

C. L. Siegel’s original proof of formula 4 used the Parseval identity [13]; our approach here turns out to be slightly different, using the Poisson summation formula. To begin, we first give another variation of the Poisson summation formula, for certain compactly supported functions, which may be of independent interest.

**Theorem 1.** Suppose that $g : \mathbb{R}^d \to \mathbb{C}$ is compactly supported, continuous, and both $g, \hat{g} \in L^1(\mathbb{R}^d)$. Then we have:

$$\sum_{n \in \mathcal{L}} g(n + x) = \frac{1}{\det \mathcal{L}} \sum_{m \in \mathcal{L}^*} \hat{g}(m)e^{2\pi i (m,x)},$$

for any full-rank lattice $\mathcal{L}$, and all $x \in \mathbb{R}^d$. Here the equality holds pointwise, and both series converge absolutely and uniformly to continuous functions.
Given any compact set $K \subset \mathbb{R}^d$, if the Poisson summation formula (6) holds for the function
\begin{equation}
(7) 
    f(x) := (1_K * 1_{-K})(x),
\end{equation}
then we call $K$ an admissible set, and we call $f$ an admissible function. It is very natural to wonder how general the class of such admissible sets can be, because they arise naturally in the study of covariograms. The following result gives a very wide family of admissible sets and admissible functions, and is the first main result of this paper. The other two main results are Theorem 3 and Corollary 2 below.

\textbf{Theorem 2.} Let $Q \subset \mathbb{R}^d$ be a compact set, and $L \subset \mathbb{R}^d$ any full-rank lattice.

Then $Q$ is an admissible set, and $g_Q := 1_Q * 1_{-Q}$ is an admissible function. Consequently, we have:
\begin{equation}
(8) 
    \sum_{n \in L} \text{vol}(Q \cap (Q + n + x)) = \frac{1}{\det L} \sum_{\xi \in L^*} |\hat{1}_Q(\xi)|^2 e^{2\pi i \langle \xi, x \rangle},
\end{equation}
for each $x \in \mathbb{R}^d$.

\textbf{Remark 1.} We may rewrite equation (8) as
\begin{equation}
(9) 
    \det L \sum_{n \in L} \text{vol}(Q \cap (Q + n + x)) = \sum_{\xi \in L^*} |\hat{1}_Q(\xi)|^2 \cos (2\pi \langle \xi, x \rangle) + i \sum_{\xi \in L^*} |\hat{1}_Q(\xi)|^2 \sin (2\pi \langle \xi, x \rangle),
\end{equation}
from which we may conclude that the imaginary part vanishes:
\begin{equation}
(10) 
    \sum_{\xi \in L^*} |\hat{1}_Q(\xi)|^2 \sin (2\pi \langle \xi, x \rangle) = 0,
\end{equation}
for all $x \in \mathbb{R}^d$. It’s easy to see that this identity also follows from first principles. Namely, we first observe that $|\hat{1}_Q(\xi)|^2 = \hat{1}_Q(\xi)\overline{\hat{1}_Q(\xi)} = \hat{1}_Q(\xi)\overline{\hat{1}_Q(-\xi)}$. Now, we may split $\mathbb{R}^d$ into 3 regions defined by the hyperplane orthogonal to $x$, namely $H := \{\xi \in \mathbb{R}^d \mid \langle x, \xi \rangle = 0\}$, and the two half-spaces defined by $H$. Each $\xi \in L^*$ on one side of $H$ gives a summand that cancels with the corresponding $-\xi$ on the other side of $H$, while the $\xi$’s that lie on $H$ yield a vanishing summand.

The above discussion shows that
\begin{equation}
(11) 
    \sum_{n \in L} \text{vol}(Q \cap (Q + n + x)) = \frac{1}{\det L} \sum_{\xi \in L^*} |\hat{1}_Q(\xi)|^2 \cos (2\pi \langle \xi, x \rangle),
\end{equation}
valid for each $x \in \mathbb{R}^d$, is equivalent to equation (8) of Theorem 2. \hfill \Box

We also observe that if a centrally symmetric body $B$ contains exactly $N$ interior lattice points then $N$ must be odd. The reason is easy: for all nonzero $n \in \text{int}(B) \cap L$, we have $-n \in \text{int}(B) \cap L$. Including the origin, we therefore have an odd number of lattice points.
We will relax the hypothesis in Siegel’s formula, namely in eq. (4), so that \( B \) is allowed to contain exactly \( N \) interior lattice points of \( \mathcal{L} \), for a fixed integer \( N \), obtaining the following extension of Siegel’s formula.

**Theorem 3.** Let \( Q \) be an admissible set in \( \mathbb{R}^d \), and let \( \mathcal{L} \subset \mathbb{R}^d \) be a full-rank lattice.

(a) If \( Q - Q \) contains exactly \( N \) lattice points of \( \mathcal{L} \) in its interior, then \( N \) is odd and:

\[
\sum_{n \in \text{int}(Q-Q) \cap \mathcal{L}} \text{vol}(Q \cap (Q + n)) = \frac{1}{\det \mathcal{L}} \text{vol}^2 Q + \frac{1}{\det \mathcal{L}} \sum_{\xi \in \mathcal{L}^* - \{0\}} \left| \hat{1}_Q(\xi) \right|^2,
\]

where the sum on the left-hand side contains exactly \( N \) terms.

(b) If \( Q := \frac{1}{2}P \) is also assumed to be centrally symmetric, and convex, then:

\[
\sum_{n \in \text{int}(P) \cap \mathcal{L}} \text{vol}(P \cap (P + 2n)) = \frac{1}{\det \mathcal{L}} \text{vol}^2 P + \frac{2^d}{\det \mathcal{L}} \sum_{\xi \in \mathcal{L}^* - \{0\}} \left| \hat{1}_{\frac{1}{2}P}(\xi) \right|^2.
\]

\( \Box \)

Next, we say that a body \( P \subset \mathbb{R}^d \) \textbf{k-tiles by translations with a lattice} \( \mathcal{L} \subset \mathbb{R}^d \) if

\[
\sum_{n \in \mathcal{L}} 1_{P + n}(x) = k,
\]

for all \( x \in \mathbb{R}^d \), except for \( x \in \partial P + \mathcal{L} \), a measure zero set. When \( k = 1 \), this is the classical definition of tiling by translations, where there are no overlaps between the translated interiors of \( P \). But when \( k > 1 \), we note that for such a \( k \)-tiling the translates of \( P \) will always overlap each other. The field of \( k \)-tiling has recently experienced a renaissance, and one of the early works relating \( k \)-tiling to Fourier analysis was done by M. Kolountzakis [8].

The following somewhat technical result is required for the proof of Theorem 3.

**Corollary 1.** Let \( Q \) be an admissible set in \( \mathbb{R}^d \) and let \( \mathcal{L} \subset \mathbb{R}^d \) be a full rank lattice.

(a) For any \( n \in \mathcal{L} \cap \partial(Q - Q) \), we have

\[
\text{vol}(Q \cap (Q + n)) = 0.
\]

(b) Consequently, we also have:

\[
\sum_{n \in \mathcal{L}} \text{vol}(Q \cap (Q + n)) = \sum_{n \in \mathcal{L} \cap \text{int}(Q - Q)} \text{vol}(Q \cap (Q + n)).
\]

\( \Box \)

The following inequality gives us a best-possible lower bound for the sum of the covariogram over a lattice; interestingly, this lower bound in equation (17) is achieved by an admissible body \( Q \) precisely when \( Q \) is a \( k \)-tiling polytope (as in equation (18)).
Corollary 2. Let $Q \subset \mathbb{R}^d$ be an admissible set and let $\mathcal{L} \subset \mathbb{R}^d$ be a full rank lattice. Then we have:

\begin{equation}
\sum_{n \in \mathcal{L} \cap \text{int}(Q - Q)} \text{vol}(Q \cap (Q + n)) \geq \frac{\text{vol}^2 Q}{\det \mathcal{L}}.
\end{equation}

Moreover, equality occurs in (17) $\iff$ the body $Q$ $k$-tiles $\mathbb{R}^d$ with the lattice $\mathcal{L}$. In the equality case, we necessarily have $k = \frac{\text{vol} Q}{\det \mathcal{L}}$, an integer, and therefore:

\begin{equation}
\sum_{n \in \mathcal{L} \cap \text{int}(Q - Q)} \text{vol}(Q \cap (Q + n)) = k \text{vol} Q.
\end{equation}

Another way to conceptualize Corollary (2) is by recalling a well-known identity, as follows. With $f(x) := (1_Q * 1_{-Q})(x) = \text{vol} \left( Q \cap (Q + x) \right)$, we have $\hat{f}(0) = |\hat{1}_Q(0)|^2 = \text{vol}^2(Q)$. But we also have, via the inverse Fourier transform, $\hat{f}(0) = \int_{\mathbb{R}^d} f(x) dx$, giving us the known identity

\begin{equation}
\int_{\mathbb{R}^d} \text{vol} \left( Q \cap (Q + x) \right) dx = \text{vol}^2 Q.
\end{equation}

Putting this together with the right-hand side of equation (17), we’ve just shown the following reformulation of Corollary (2).

Corollary 3. Let $Q \subset \mathbb{R}^d$ be an admissible set and let $\mathcal{L} \subset \mathbb{R}^d$ be a full rank lattice. Then we have:

\begin{equation}
\sum_{n \in \mathcal{L} \cap \text{int}(Q - Q)} \text{vol}(Q \cap (Q + n)) \geq \frac{1}{\det \mathcal{L}} \int_{\mathbb{R}^d} \text{vol} \left( Q \cap (Q + x) \right) dx.
\end{equation}

Moreover, equality occurs in (20) $\iff$ $Q$ $k$-tiles $\mathbb{R}^d$ with the lattice $\mathcal{L}$. □

Corollary 4. Let $Q \subset \mathbb{R}^d$ be an admissible set, and let $\mathcal{L} \subset \mathbb{R}^d$ be a full rank lattice with the property that $Q$ is disjoint from its nonzero translations by vectors from $\mathcal{L}$. In addition, let $x \in \mathbb{R}^d$ be a nonzero vector with the property that $Q + x$ is also disjoint from its nonzero translations by vectors from $\mathcal{L}$. Then

\begin{equation}
\text{vol}^2(Q) = -\sum_{\xi \in \mathcal{L}^* - \{0\}} |\hat{1}_Q(\xi)|^2 \cos(2\pi \langle \xi, x \rangle).
\end{equation}

□

Another strong motivation for studying the covariogram is the following conjectural characterization of a convex body $K$, posed by G. Matheron [10], and studied intensively over the past 30 years by G. Bianchi, and other researchers [3] [4].

Conjecture 1 (Matheron, 1986). The covariogram $g_K$ determines a convex body $K$, among all convex bodies, up to translations and reflections. □
We note that knowledge of $\hat{1}_K(\xi)$, for all $\xi \in \mathbb{R}^d$, uniquely determines $K$, for any convex body $K$; in other words, the Fourier transform is a complete invariant for any body $K$. Hence Conjecture 1 is equivalent to saying that knowledge of $|\hat{1}_K|$ determines $K$ (up to translations and reflections), because $\hat{g}_K = |\hat{1}_K|^2$.

It is known, for example, that centrally symmetric convex bodies in any dimension are determined by their covariogram, up to translations [4].

2. Preliminaries

We recall that by definition we have $1_K * 1_H(x) := \int_{\mathbb{R}^d} 1_K(t)1_H(x - t)dt$, where $K,H$ are any two measurable sets for which the integral converges. We begin by mentioning an elementary argument that proves the following fact.

Lemma 1. If a body $Q$ $k$-tiles $\mathbb{R}^d$ we have the identity

$$\sum_{n \in \mathcal{L}} \text{vol}(Q \cap (Q + n + x)) = k \cdot \text{vol} Q$$

for every fixed $x \in \mathbb{R}^d$.

Proof. For a $k$-tiling we must have

$$\sum_{n \in \mathcal{L}} 1_{Q+n}(y) = k$$

for all $y \in \mathbb{R}^d$, except those points $y$ that lie on the boundary of $Q$ or its translates under $\mathcal{L}$. Therefore

$$\sum_{n \in \mathcal{L}} \text{vol}(Q \cap (Q + n + x)) = \sum_{n \in \mathcal{L}} \int_{\mathbb{R}^d} 1_{Q \cap (Q + n + x)}(y)dy$$

$$= \sum_{n \in \mathcal{L}} \int_{\mathbb{R}^d} 1_Q(y) \cdot 1_{Q+n+x}(y)dy$$

$$= \sum_{n \in \mathcal{L}} \int_{\mathbb{R}^d} 1_{Q-x}(y) \cdot 1_{Q+n}(y)dy$$

$$= \int_{\mathbb{R}^d} 1_{Q-x}(y) \cdot \sum_{n \in \mathcal{L}} 1_{Q+n}(y)dy$$

$$= k \cdot \text{vol} Q,$$

where we used the translation-invariance of the integral and exchanged the sum with the integral since the integral is nonzero only for a finite number of $n \in \mathcal{L}$. \hfill \Box

It turns out that the converse of Lemma 1 is also true, and is part of of Corollary 2. A very useful Fourier equivalence for translational tilings is given by the following result, due to M. Kolountzakis [8] (see also [11], Theorem 5.5).
Theorem 4 (Kolountzakis). Let $\mathcal{P}$ be a body in $\mathbb{R}^d$, and let $\mathcal{L} \subseteq \mathbb{R}^d$ be a full-rank lattice. Then the following conditions are equivalent:

1. $\hat{1}_\mathcal{P}(\xi) = 0$, for all nonzero $\xi \in \mathcal{L}^*$, the dual lattice.
2. $\mathcal{P}$ $k$-tiles $\mathbb{R}^d$ by translations with $\mathcal{L}$.

Either of the above conditions already implies that $k = \frac{\text{vol} \mathcal{P}}{\det \mathcal{L}}$, and that $k$ is a positive integer. □

3. A nonconvex polygon that multi-tiles with multiplicity 2

Example 1. Here we give a nonconvex polygon $\mathcal{Q}$ (Figure 1) that multi-tiles with multiplicity $k = 2$. We’ll show how the proof of this nontrivial multi-tiling follows from Corollary 2, and that this polygon does not tile with $k = 1$ for any lattice. In other words, we have a non-trivial 2-tiling. This phenomenon is in sharp contrast with multi-tiling the plane by using convex polygons, because in that convex context the smallest non-trivial multiplicity is $k = 5$ [7].

![Figure 1](image)

Figure 1. Left: A body $\mathcal{Q}$, and a lattice $\mathcal{L}$ with $\det \mathcal{L} = 2$. Right: the difference body $\mathcal{Q} - \mathcal{Q}$, with its 9 interior lattice points of $\mathcal{L}$.

The lattice $\mathcal{L}$ we chose for this multi-tiling is defined by the basis $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$, and has index 2 in $\mathbb{Z}^2$ (also known as the $D_2$ lattice, and drawn with green dots in Figure 1). By Corollary 2, to prove
that the body $Q$ gives a 2-tiling of the plane, it suffices to show the following equality:

\[(24) \quad \sum_{n \in \mathcal{L} \cap \text{int}(Q - Q)} \text{vol}(Q \cap (Q + n)) = 2 \text{vol}Q = 8.\]

A brute-force computation using the software Mathematica [16] reveals that the left-hand side of eq. (24) indeed equals 8, giving us a computational method of verifying that $Q$ 2-tiles the plane with translations by the lattice $\mathcal{L}$.

It is instructive to also see this multi-tiling geometrically, and in Figure 2 we give such a geometric confirmation. We first translate the blue collection of translates of $Q$ by the vector $\binom{1}{1}$, to achieve the green collection. We then translate the same blue collection once again, by the vector $\binom{2}{2}$, to achieve the pink collection. Each of the little squares in the middle of the picture show that their color is the overlap of exactly two translates of the blue collection, and hence every point in their interior gets covered exactly twice. We therefore obtain a 2-tiling.

Finally, to see that $Q$ does not 1-tile with any lattice, we suppose to the contrary that it does. Consulting Figure 3, the translated parallel edge $E_2$, belonging to the purple translate of $Q$, must meet $E_1$, because $E_2$ is the only parallel edge to $E_1$. Similarly the unique edge parallel to $E_3$ must be translated to meet $E_3$, as in the right-hand side of Figure 3. We arrive at a contradiction, because the two translated copies of $Q$, drawn in green and purple, must overlap.
Figure 3. Left: The body $Q$ is translated so that edge $E_1$ meets its translated parallel edge $E_2$. Right: $Q$ is translated again, where edge $E_3$ meets its translated parallel edge $E_4$.

4. A NONCONVEX POLYGON THAT DOES NOT MULTI-TILE

**Example 2.** Here we show how we may use Corollary 2 to prove that a nonconvex polygon does not multi-tile with a given lattice. Consider the non-convex body $Q$ drawn on the left-hand side of Figure 4 below. The difference body of $Q$, namely $Q - Q = \{ p - q : p, q \in Q \}$ is shown on the right-hand side of Figure 4, with the red integer points on its boundary.

We may check that here $Q$ does not multi-tile with the integer lattice $\mathbb{Z}^2$, by using Corollary 2. Namely, we’d like to show that here

$$\sum_{n \in \mathcal{L} \cap \text{int}(Q - Q)} \text{vol}(Q \cap (Q + n)) > \frac{\text{vol}^2 Q}{\det \mathcal{L}}. \quad (25)$$

For this example, $\text{vol} Q = 5$, and it turns out the the left-hand-side of (25) equals 26 (by a brute force computation that used Mathematica), while the right-hand-side equals $\frac{\text{vol}^2 Q}{\det \mathcal{L}} = 25$. So we’ve confirmed that indeed we have $26 > 25$ in our inequality (25), implying by Corollary 2 that $Q$ does not multi-tile with the integer lattice. \qed
Figure 4. Left: the nonconvex body $Q$ of Example 2.
Right: its difference body $Q - Q$, with the (red) integer lattice points on its boundary.

Figure 5. The translated bodies $Q + n$ around $Q$. As we can see here, the intersections $Q \cap (Q + n)$ only contain boundary points of $Q$.

5. Lemmas

The 3 lemmas of this section are well-known, but we also include their proofs here for completeness, and because sometimes the precise relationship between convexity (or lack thereof) of $Q$ and the difference body $Q - Q$ can be subtle.

Lemma 2. Let $Q \subset \mathbb{R}^d$ be a centrally symmetric body.

(a) $Q - Q \supseteq 2Q$.

(b) If we also assume that $Q$ is convex, then $Q - Q = 2Q$. 
Proof. To show (a), let \( x \in 2\mathcal{Q} \). Then \( x = 2y \) with \( y \in \mathcal{Q} \), implying that \( x = y + y = y - (-y) \in \mathcal{Q} - \mathcal{Q} \). Here we used the central symmetry of \( \mathcal{Q} \) by invoking \(-y \in \mathcal{Q} \).

To prove (b), it now suffices to show that \( \mathcal{Q} - \mathcal{Q} \subseteq 2\mathcal{Q} \). So letting \( x \in \mathcal{Q} - \mathcal{Q} \), \( x = y - z \) with \( y, z \in \mathcal{Q} \). We can rewrite \( x = y + (-z) = \left( \frac{1}{2}y + \frac{1}{2}(-z) \right) + \left( \frac{1}{2}y + \frac{1}{2}(-z) \right) \). Since \( \mathcal{Q} \) is centrally symmetric \(-z \in \mathcal{Q} \), and now the convexity of \( \mathcal{Q} \) implies that \( \frac{1}{2}y + \frac{1}{2}(-z) \in \mathcal{Q} \). Therefore \( x \in 2\mathcal{Q} \). □

We observe that convexity is essential in part (b), by considering the counterexample \( C := [-2, -1] \cup [1, 2] \), a nonconvex set in \( \mathbb{R} \). Here \( C \) is centrally symmetric, yet \( C - C = [-3, 3] \neq [-4, -2] \cup [2, 4] = 2C \).

**Lemma 3.** Let \( \mathcal{Q} \subset \mathbb{R}^d \) be any set, and fix any \( x \in \mathbb{R}^d \). Then:

\[
Q \cap (Q + x) \neq \emptyset \iff x \in Q - Q.
\]

Consequently, if \( \mathcal{Q} \) is a convex centrally symmetric body, then

\[
Q \cap (Q + x) \neq \emptyset \iff x \in 2\mathcal{Q}.
\]

**Proof.** Let \( y \in Q \cap (Q + x) \). Then \( y = z \) and \( y = w + x \), where \( z, w \in \mathcal{Q} \). This gives us \( z = w + x \), so that \( x = y - w \in Q - Q \), proving (26). The converse follows exactly the same logical steps. The second part follows from the latter condition, together with the fact that \( \mathcal{Q} \) is convex and centrally symmetric implies \( \mathcal{Q} = \frac{1}{2} \mathcal{Q} - \frac{1}{2} \mathcal{Q} \), proving (27). □

For the proof of Theorem 2, we will require the following well-known Plancherel-Polya type inequality (see [12], for example).

**Lemma 4.** Let \( g : \mathbb{R}^d \to \mathbb{C} \) be a compactly supported function, with \( \hat{g} \in L^1(\mathbb{R}^d) \). Then there is a positive constant \( M \) such that

\[
\sum_{m \in \mathcal{L}} |\hat{g}(m)| \leq M \int_{\mathbb{R}^d} |\hat{g}(\xi)|.
\]

**Proof.** Let \( \psi \) be a smooth and compactly supported function such that \( \psi(x) = 1 \) for all \( x \) in the support of \( g \). So we have \( g(x) = \psi(x)g(x) \) for all \( x \in \mathbb{R}^d \). By our assumption that \( \hat{g} \in L^1(\mathbb{R}^d) \) (and
clearly \( \hat{\psi} \in L^1(\mathbb{R}^d) \) we also have \( \hat{g}(\xi) = (\hat{\psi} * \hat{g})(\xi) \). We can now bound the series:

\[
\sum_{m \in \mathcal{L}} |\hat{g}(m)| = \sum_{m \in \mathcal{L}} \left| \int_{\mathbb{R}^d} \hat{\psi}(m - \xi) \hat{g}(\xi) \, d\xi \right|
\]

\[
\leq \sum_{m \in \mathcal{L}} \int_{\mathbb{R}^d} |\hat{\psi}(m - \xi)\hat{g}(\xi)| \, d\xi
\]

\[
= \int_{\mathbb{R}^d} \sum_{m \in \mathcal{L}} |\hat{\psi}(m - \xi)||\hat{g}(\xi)| \, d\xi
\]

\[
\leq \sup_{\xi \in \mathbb{R}^d} \left( \sum_{m \in \mathcal{L}} |\hat{\psi}(m - \xi)| \right) \int_{\mathbb{R}^d} |\hat{g}(\xi)| \, d\xi
\]

\[
= M \int_{\mathbb{R}^d} |\hat{g}(\xi)| \, d\xi < \infty.
\]

The integral-sum interchange in (31) is justified by the monotone convergence Theorem. We observe that \( \sum_{m \in \mathcal{L}} |\hat{\psi}(m - \xi)| := h(\xi) \) is a periodic function which is therefore completely determined on \( \mathbb{R}^d/\mathcal{L} \). Due to the rapid decay of \( \hat{\psi} \), \( h(\xi) \) is uniformly convergent on \( \mathbb{R}^d/\mathcal{L} \), and being a series of continuous functions, \( h(\xi) \) is itself continuous. Since \( \mathbb{R}^d/\mathcal{L} \) is compact, \( h \) attains its supremum there, a finite constant \( M > 0 \).

6. Proofs of Theorem 1 and Theorem 2

Proof. (of Theorem 1) The hypothesis that both \( g \) and \( \hat{g} \in L^1(\mathbb{R}^d) \) implies that we can use Fourier inversion (see [5], Theorem 9.36). So \( g(x) = \mathcal{F}(\hat{g})(-x) \) is the image of an \( L^1 \) function under \( \mathcal{F} \), and therefore uniformly continuous. Similarly \( \hat{g} \) is uniformly continuous. Our goal is to prove that:

\[
\sum_{n \in \mathcal{L}} g(n + x) = \frac{1}{\det \mathcal{L}} \sum_{\xi \in \mathcal{L}^*} \hat{g}(\xi)e^{2\pi i \langle \xi, x \rangle}.
\]

To this end, we first pick an infinitely smooth approximate identity \( \varphi_\varepsilon \) that is supported on the unit ball, with \( \varphi_\varepsilon \geq 0 \) and \( \int_{\mathbb{R}^d} \varphi_\varepsilon(x) \, dx = 1 \) for all \( \varepsilon > 0 \) (i.e. \( \varphi_\varepsilon \) is a bump function [14], page 209).

Since \( \varphi_\varepsilon \) vanishes outside a ball of radius \( \varepsilon \) and \( g \) is compactly supported, we have that \( \varphi_\varepsilon \ast g \) is compactly supported on a set that is independent of \( \varepsilon \). Therefore the sum

\[
\sum_{n \in \mathcal{L}} \varphi_\varepsilon \ast g(n + x) := G(x)
\]
has a finite number of terms and is \( \mathcal{L} \)-periodic. Consequently, for \( m \in \mathcal{L}^* \) we have:

\[
\hat{G}(m) := \int_{\mathbb{R}^d / \mathcal{L}} \left( \sum_{n \in \mathcal{L}} \varphi_{\epsilon} * g(n + x) \right) e^{-2\pi i (m, x)} \, dx
\]

\[
= \sum_{n \in \mathcal{L}} \int_{\mathbb{R}^d / \mathcal{L}} \varphi_{\epsilon} * g(n + x) e^{-2\pi i (m, x)} \, dx
\]

\[
= \sum_{n \in \mathcal{L}} \int_{n + \mathbb{R}^d / \mathcal{L}} \varphi_{\epsilon} * g(y) e^{-2\pi i (m, y)} \, dy
\]

\[
= \sum_{n \in \mathcal{L}} \int_{n + \mathbb{R}^d / \mathcal{L}} \varphi_{\epsilon} * g(y) e^{-2\pi i (m, y)} \, dy
\]

\[
= \int_{\mathbb{R}^d} \varphi_{\epsilon} * g(y) e^{-2\pi i (m, y)} \, dy
\]

\[
= \hat{\varphi}_{\epsilon} * \hat{g}(m)
\]

\[
= \hat{\varphi}(\epsilon m) \hat{g}(m)
\]

\[
= \hat{\varphi}(\epsilon m) \hat{g}(m).
\]

We notice that due to the fact that both \( \varphi_{\epsilon} \) and \( g \) are compactly supported, and that \( \varphi_{\epsilon} \) is infinitely smooth, \( \varphi_{\epsilon} * g \) belongs to the Schwartz class \( S(\mathbb{R}^d) \). Thus, the basic Poisson summation formula for Schwartz functions holds:

\[
\sum_{n \in \mathcal{L}} \varphi_{\epsilon} * g(n + x) = \frac{1}{\det \mathcal{L}} \sum_{m \in \mathcal{L}^*} \hat{\varphi}(\epsilon m) \hat{g}(m) e^{2\pi i (m, x)},
\]

(35)

for every \( x \in \mathbb{R}^d \). Then, since \( g \) is continuous and locally integrable on \( \mathbb{R}^d \) the Lebesgue set of \( g \) is \( \mathbb{R}^d \) and we have

\[
\lim_{\epsilon \to 0^+} \sum_{n \in \mathcal{L}} \varphi_{\epsilon} * g(n + x) = \sum_{n \in \mathcal{L}} \lim_{\epsilon \to 0^+} \varphi_{\epsilon} * g(n + x) = \sum_{n \in \mathcal{L}} g(n + x).
\]

(36)

for every \( x \in \mathbb{R}^d \). Therefore

\[
\sum_{n \in \mathcal{L}} g(n + x) = \lim_{\epsilon \to 0^+} \frac{1}{\det \mathcal{L}} \sum_{m \in \mathcal{L}^*} \hat{\varphi}(\epsilon m) \hat{g}(m) e^{2\pi i (m, x)}.
\]

(37)

Our next goal is to allow the limit to go inside the latter series in (37), which is a subtle point. First, we have

\[
|\hat{\varphi}(\epsilon m)| = \left| \int_{\mathbb{R}^d} \varphi(x) e^{-2\pi i (x, \epsilon m)} \, dx \right| \leq \int_{\mathbb{R}^d} |\varphi(x)| \, dx = 1.
\]

(38)

Summarizing, we have \( |\hat{\varphi}(\epsilon x) \hat{g}(x)| \leq |\hat{g}(x)| \), which is an absolutely summable dominating function because

\[
\sum_{m \in \mathcal{L}^*} |\hat{g}(m)| \leq M \int_{\mathbb{R}^d} |\hat{g}(\xi)| < \infty,
\]

(39)
for some positive constant $M$, by invoking Lemma 4 (a Plancherel-Polya inequality) with $\mathcal{L}$ replaced by $\mathcal{L}^*$. By the Lebesgue dominated convergence theorem (applied to the counting measure on $\mathcal{L}^*$), we have
\[
\lim_{\varepsilon \to 0^+} \sum_{m \in \mathcal{L}^*} \hat{\varphi}(\varepsilon m) \hat{g}(m)e^{2\pi i (m,x)} = \sum_{m \in \mathcal{L}^*} \hat{\varphi}(0) \hat{g}(m)e^{2\pi i (m,x)},
\]
where we’ve used the continuity of $\hat{\varphi}$ in the last equality, and also $\hat{\varphi}(0) = 1$. Finally, putting together (37) with the latter computation, we have
\[
(40) \quad \sum_{n \in \mathcal{L}} g(n + x) = \frac{1}{\det \mathcal{L}} \sum_{m \in \mathcal{L}^*} \hat{g}(m)e^{2\pi i (m,x)}
\]
for all $x \in \mathbb{R}^d$. \hfill \qed

The proof of Theorem 2 below uses the result of Theorem 1 above.

Proof. (of Theorem 2) Since $1_{\mathcal{Q}} \in L^2(\mathbb{R}^d)$, we know that $g_{\mathcal{Q}} := 1_{\mathcal{Q}} * 1_{1-\mathcal{Q}}$ is the convolution of two $L^2$ functions. By Rudin’s Theorem ([11], Theorem 4.11), $g_{\mathcal{Q}} = \hat{h}$, for some $h \in L^1(\mathbb{R}^d)$. But now we see that $g_{\mathcal{Q}}$ must be uniformly continuous because it is the image of the Fourier transform of an $L^1$ function.

Because $1_{\mathcal{Q}} \in L^2(\mathbb{R}^d)$, we also have $\hat{1}_{\mathcal{Q}} \in L^2(\mathbb{R}^d)$. The Cauchy-Schwarz inequality gives us:
\[
(41) \quad \int_{\mathbb{R}^d} |\hat{g}_{\mathcal{Q}}(\xi)| \, d\xi = \int_{\mathbb{R}^d} |\hat{1}_{\mathcal{Q}}(\xi)||1_{1-\mathcal{Q}}(\xi)| \, d\xi \leq \left( \int_{\mathbb{R}^d} |\hat{1}_{\mathcal{Q}}(\xi)|^2 \, d\xi \right)^{1/2} \left( \int_{\mathbb{R}^d} |1_{1-\mathcal{Q}}(\xi)|^2 \, d\xi \right)^{1/2} < \infty,
\]
so that $\hat{g}_{\mathcal{Q}} \in L^1(\mathbb{R}^d)$. Let’s compute the left hand side of equation (40):
\[
\sum_{n \in \mathcal{L}} g_{\mathcal{Q}}(n + x) = \sum_{n \in \mathcal{L}} \int_{\mathbb{R}^d} 1_{\mathcal{Q}}(y)1_{1-\mathcal{Q}}(n + x - y) \, dy
\]
\[
= \sum_{n \in \mathcal{L}} \int_{\mathbb{R}^d} 1_{\mathcal{Q}}(y)1_{\mathcal{Q}}(y - n - x) \, dy
\]
\[
= \sum_{n \in \mathcal{L}} \int_{\mathbb{R}^d} 1_{\mathcal{Q}}(y)1_{\mathcal{Q} + n + x}(y) \, dy
\]
\[
= \sum_{n \in \mathcal{L}} \int_{\mathbb{R}^d} 1_{\mathcal{Q} + n + x}(y) \, dy
\]
\[
= \sum_{n \in \mathcal{L}} \text{vol}(\mathcal{Q} \cap (\mathcal{Q} + n + x)).
\]
On the other hand, the right-hand-side of Poisson summation, namely eq. (40), gives us:

\[ \sum_{\xi \in \mathcal{L}^*} \hat{g}_Q(\xi) e^{2\pi i \langle \xi, x \rangle} = \frac{1}{\det \mathcal{L}} \sum_{\xi \in \mathcal{L}^*} \hat{1}_Q(\xi) \hat{1}_Q(\xi) e^{2\pi i \langle \xi, x \rangle} \]

\[ = \frac{1}{\det \mathcal{L}} \sum_{\xi \in \mathcal{L}^*} \int_{Q} e^{-2\pi i \langle \xi, x \rangle} dx \int_{Q} e^{-2\pi i \langle \xi, x \rangle} dx e^{2\pi i \langle \xi, x \rangle} \]

\[ = \frac{1}{\det \mathcal{L}} \sum_{\xi \in \mathcal{L}^*} \int_{Q} e^{-2\pi i \langle \xi, x \rangle} dx \int_{Q} e^{-2\pi i \langle \xi, -x \rangle} dx e^{2\pi i \langle \xi, x \rangle} \]

\[ = \frac{1}{\det \mathcal{L}} \sum_{\xi \in \mathcal{L}^*} \int_{Q} e^{-2\pi i \langle \xi, x \rangle} dx \int_{Q} e^{-2\pi i \langle \xi, x \rangle} dx e^{2\pi i \langle \xi, x \rangle} \]

\[ = \frac{1}{\det \mathcal{L}} \sum_{\xi \in \mathcal{L}^*} |\hat{1}_Q(\xi)|^2 e^{2\pi i \langle \xi, x \rangle}. \]

Altogether, using Theorem 1 (our variation of Poisson summation) gives us the identity:

\[ (42) \sum_{n \in \mathcal{L}} \text{vol} (Q \cap (Q + n + x)) = \frac{1}{\det \mathcal{L}} \sum_{\xi \in \mathcal{L}^*} |\hat{1}_Q(\xi)|^2 e^{2\pi i \langle \xi, x \rangle} \]

for all \( x \in \mathbb{R}^d \). \( \square \)

**Remark 2.** We note that one direction of the known Theorem 4 of Kolountzakis, follows as a Corollary of our Theorem 2. Namely, suppose that \( Q \) multi-tiles \( \mathbb{R}^d \). Then we know by Lemma 1 that the left hand side of (42) is constant for all \( x \) and therefore by uniqueness of Fourier series, all of the Fourier coefficients on the right-hand side of (42) must vanish, except for \( \xi = 0 \). This guarantees that Theorem 4, part 2 \( \Rightarrow \) Theorem 4, part 1, and also that \( k = \frac{\text{vol } Q}{\det \mathcal{L}} \).

### 7. Proof of Corollary 1

**Proof.** We observe that the right-hand side of eq. (42) does not depend on boundary points of \( Q \), which implies that

\[ \sum_{n \in \mathcal{L}} \text{vol} (Q \cap (Q + n)) = \sum_{n \in \mathcal{L}} \text{vol} (\text{int } Q \cap (\text{int } Q + n)). \]

But,

\[ Q \cap (Q + n) \supset (\text{int } Q \cap (\text{int } Q + n)) \]

\[ \Rightarrow \text{vol} (Q \cap (Q + n)) \geq \text{vol} (\text{int } Q \cap (\text{int } Q + n)) \]

\[ \Rightarrow \sum_{n \in \mathcal{L}} \text{vol} (Q \cap (Q + n)) \geq \sum_{n \in \mathcal{L}} \text{vol} (\text{int } Q \cap (\text{int } Q + n)). \]
Since the left hand side of equation (43) is finite, all terms of both sides are non-negative, it follows from the inequality (44), together with the equality (43), that we must have

\[(46) \quad \text{vol} \left( Q \cap (Q + n) \right) = \text{vol} \left( \text{int} Q \cap (\text{int} Q + n) \right)\]

for each \( n \in L \). By Lemma (3), we have

\[(47) \quad \sum_{n \in L} \text{vol} \left( Q \cap (Q + n) \right) = \sum_{n \in L \cap (Q - Q)} \text{vol} \left( Q \cap (Q + n) \right) = \sum_{n \in L \cap \text{int} \left( Q - Q \right)} \text{vol} \left( Q \cap (Q + n) \right) + \sum_{n \in L \cap \partial \left( Q - Q \right)} \text{vol} \left( Q \cap (Q + n) \right) .\]

where in the last step we have an equality since \( Q - Q \) is compact, indeed closed.

On the other hand, again by Lemma (3), we have

\[(48) \quad \sum_{n \in L} \text{vol} \left( \text{int} Q \cap (\text{int} Q + n) \right) = \sum_{n \in L \cap \text{int} \left( Q - Q \right)} \text{vol} \left( \text{int} Q \cap (\text{int} Q + n) \right) \leq \sum_{n \in L \cap \text{int} \left( Q - Q \right)} \text{vol} \left( \text{int} Q \cap (\text{int} Q + n) \right),\]

where the last inequality is justified observing that \( \text{int} Q - \text{int} Q \subset Q - Q \) and that \( \text{int} Q - \text{int} Q \) being open implies \( \text{int} Q - \text{int} Q \subset \text{int}(Q - Q) \). Combining inequalities (47) and (48), and then using identity (43) we get

\[(49) \quad \sum_{n \in L \cap \text{int} \left( Q - Q \right)} \text{vol} \left( \text{int} Q \cap (\text{int} Q + n) \right) \geq \sum_{n \in L \cap \text{int} \left( Q - Q \right)} \text{vol} \left( Q \cap (Q + n) \right) + \sum_{n \in L \cap \partial \left( Q - Q \right)} \text{vol} \left( Q \cap (Q + n) \right) .\]

Inserting eq. (46) into eq. (49) we obtain

\[(50) \quad \sum_{n \in L \cap \partial \left( Q - Q \right)} \text{vol} \left( Q \cap (Q + n) \right) \leq 0 .\]

But, each volume is a non-negative number, thus

\[(51) \quad \text{vol} \left( Q \cap (Q + n) \right) = 0 .\]

for each \( n \in L \cap \partial \left( Q - Q \right) \) as claimed.

Part (b) follows as an immediate consequence, using eq. (47).

\[ \square \]

8. PROOFS OF COROLLARIES 2 AND 4

The following proof of Corollary 2 follows essentially from combining Theorem 2 and Kolountakis’ Theorem 4.
**Proof.** (of Corollary 2) Since \( Q \subset \mathbb{R}^d \) is an admissible set, Theorem 2 gives (with \( x = 0 \)):

\[
\sum_{n \in \mathcal{L}} \text{vol} (Q \cap (Q + n)) = \frac{1}{\text{det} \mathcal{L}} \sum_{\xi \in \mathcal{L}^*} |\hat{1}_Q(\xi)|^2 \geq \frac{1}{\text{det} \mathcal{L}} |\hat{1}_Q(0)|^2 = \frac{\text{vol}^2 Q}{\text{det} \mathcal{L}}.
\]

We may rewrite the left-hand side, using Corollary 1, part (b), as follows:

\[
\sum_{n \in \mathcal{L} \cap \text{int}(Q - Q)} \text{vol} (Q \cap (Q + n)) \geq \frac{\text{vol}^2 Q}{\text{det} \mathcal{L}}.
\]

Now we’ll show that equality holds in (53) if and only if \( Q \) multi-tiles, with the lattice \( \mathcal{L} \). First, let’s assume that \( Q \) multi-tiles \( \mathbb{R}^d \); that is, Theorem 4, part 2 holds. Since Theorem 4, part 2 \( \Rightarrow \) Theorem 4, part 1, we have

\[
\sum_{n \in \mathcal{L} \cap \text{int}(Q - Q)} \text{vol} (Q \cap (Q + n)) = \frac{1}{\text{det} \mathcal{L}} |\hat{1}_Q(0)|^2 = \frac{\text{vol}^2(Q)}{\text{det} \mathcal{L}}.
\]

Conversely, if equality holds in (53), then

\[
\sum_{n \in \mathcal{L} \cap \text{int}(Q - Q)} \text{vol} (Q \cap (Q + n)) = \frac{\text{vol}^2(Q)}{\text{det} \mathcal{L}},
\]

and by Theorem 2 with \( x = 0 \) we therefore must have \( |\hat{1}_Q(\xi)|^2 = 0 \) for all \( \xi \in \mathcal{L}^* \), excluding the origin. This means that Theorem 4, part 1 holds, and its equivalence with Theorem 4, part 2 tells us that \( Q \) multi-tiles \( \mathbb{R}^d \) with the lattice \( \mathcal{L} \).

\[\square\]

**Proof.** (of Corollary 4) This proof follows from Theorem 2 by observing that the hypotheses guarantee that the left-hand side of equation (11) vanishes. Namely, we have:

\[
0 = \sum_{n \in \mathcal{L}} \text{vol} (Q \cap (Q + n + x)) = \frac{1}{\text{det} \mathcal{L}} \sum_{\xi \in \mathcal{L}^*} |\hat{1}_Q(\xi)|^2 \cos (2\pi \langle \xi, x \rangle).
\]

Since \( \hat{1}_Q(0) = \text{vol} Q \), we’re done.

\[\square\]

9. More examples

**Example 3.** While it is tempting to assert that \( \text{int}(Q - Q) = \text{int} Q - \text{int} Q \), for any compact set \( Q \), we give a rather extreme counterexample to this claim. Consider the usual cantor set \( Q \subset [0, 1] \), whose interior is known to be the empty set. It is known that its difference body satisfies the surprising identity \( Q - Q = [-1, 1] \), implying that \( \text{int}(Q - Q) = (-1, 1) \). However, \( \text{int} Q = \emptyset \), so we find that \( \text{int} Q - \text{int} Q = \emptyset \neq \text{int}(Q - Q) \). This example shows why the analysis in Corollary 1 is necessary. \[\square\]
Example 4. Suppose we have a convex, centrally symmetric body $Q \subset \mathbb{R}^d$ that $k$-tiles $\mathbb{R}^d$ with a lattice $\mathcal{L}$. Consequently, we know that $Q - Q = 2Q$. Here we use Theorem 3 to show that if $Q$ $k$-tiles, and enjoys the property that $Q - Q$ contains exactly 3 lattice points of $\mathcal{L}$ in its interior, say $0, n_0, -n_0$, then either $k = 1$ or $k = 2$.

By Theorem 3, we have:

$$\frac{\text{vol}^2 Q}{\det \mathcal{L}} = \sum_{n \in \text{int}(Q - Q) \cap \mathcal{L}} \text{vol}(Q \cap (Q + n))$$

$$= \text{vol} Q + \text{vol}(Q \cap (Q + n_0)) + \text{vol}(Q \cap (Q - n_0))$$

$$= \text{vol} Q + 2 \text{vol}(Q \cap (Q + n_0)).$$

By Theorem 4, we also know that $k = \frac{\text{vol} Q}{\det \mathcal{L}}$, so that

$$k \text{vol} Q = \text{vol} Q + 2 \text{vol}(Q \cap (Q + n_0)).$$

Since $Q \cap (Q + n_0) \subset Q$ we must have

$$k \text{vol} Q = \text{vol} Q + 2 \text{vol}(Q \cap (Q + n_0)) < 3 \text{vol} Q,$$

giving us

$$k \leq 2,$$

as was claimed. As an aside, together with equation (57), we now also have

$$k = 1 + 2 \frac{\text{vol}(Q \cap (Q + n_0))}{\text{vol} Q} \leq 2.$$ 

and therefore

$$\text{vol}(Q \cap (Q + n_0)) = \frac{1}{2} \text{vol} Q \quad \text{or} \quad \text{vol}(Q \cap (Q + n_0)) = 0.$$  

Example 5. The classical identity of Euler, namely $\zeta(2) := 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$ can be retrieved from Theorem 2 by using the integer lattice $\mathcal{L} = \mathbb{Z}^2$ together with the triangle $Q := \triangle$, defined by the vertices $v_0 := \left(\begin{array}{c}0 \\ 0\end{array}\right), v_1 := \left(\begin{array}{c}1 \\ 0\end{array}\right)$, and $v_2 := \left(\begin{array}{c}0 \\ 1\end{array}\right)$, as shown in Figure 6. On the left-hand side of equation (8), Theorem 2, we have (with $x = 0$):

$$\sum_{n \in \mathbb{Z}^2} \text{vol}(\triangle \cap (\triangle + n)) = \text{vol} \triangle = \frac{1}{2}.$$ 

On the right-hand side of (8) we can compute directly the summands $|\hat{1}_\triangle(\xi)|^2$, using the following known formula.
Lemma 5. For the standard $d$-dimensional simplex $\triangle_d := \{ x \in \mathbb{R}^d_{\geq 0} \mid x_1 + \cdots + x_d \leq 1 \} \subset \mathbb{R}^d$, its Fourier transform vanishes for all integer points $\xi$ with the property that all of their coordinates are nonzero and distinct from each other.

Proof. The Fourier transform of $\triangle_d$ is given by:

$$\hat{1}_{\triangle_d}(\xi) := \int_{\triangle_d} e^{-2\pi i \langle \xi, x \rangle} dx = \frac{1}{(2\pi i)^d} \sum_{i=0}^{d} \frac{e^{-2\pi i \xi_i}}{\prod_{0 \leq j \leq d, j \neq i} (\xi_i - \xi_j)},$$

where we let $\xi_i \neq \xi_j$ for every $i \neq j$, and where we’ve defined $\xi_0 := 0$ ([1], or [9], Lemma 21). If we restrict $\xi$ to be an integer point, then for each coordinate $\xi_i \in \mathbb{Z} (i \geq 1)$, we have $e^{-2\pi i \xi_i} = 1$. By the Lagrange interpolation polynomial applied to the points $(\xi_i, 1) \in \mathbb{R}^2$, we have $\hat{1}_{\triangle_d}(\xi) = 0$. The reason for this vanishing is that these points lie on a horizontal line - the constant polynomial. Consequently, all coefficients in the interpolating formula vanish except the constant coefficient, which equals 1 by uniqueness of the polynomial with degree less than or equal to $d + 1$. □

From the discussion above, it is sufficient to restrict attention to only those integer points $\xi \in \mathbb{Z}^d$ which have at least one vanishing coordinate, or at least two equal coordinates. When $d = 2$, the only integer vectors $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \mathbb{Z}^2$ for which $\hat{1}_{\triangle_2}(\xi)$ possibly does not vanish are those that are orthogonal to the sides of $\triangle_2$, namely the integer points belonging to the family $\mathcal{F} = \left\{ \begin{pmatrix} u \\ k \end{pmatrix} \mid k \in \mathbb{Z} \right\} \cup \left\{ \begin{pmatrix} k \\ u \end{pmatrix} \mid k \in \mathbb{Z} \right\} \cup \left\{ \begin{pmatrix} k \\ k \end{pmatrix} \mid k \in \mathbb{Z} \right\}$, as drawn in Figure 6. By symmetry, $\left| \hat{1}_{\triangle_2}(\begin{pmatrix} u \\ k \end{pmatrix}) \right|^2 = \left| \hat{1}_{\triangle_2}(\begin{pmatrix} k \\ u \end{pmatrix}) \right|^2$, for all $k \in \mathbb{Z}$. We
compute, for each $k \in \mathbb{Z} - \{0\}$:

$$\left| \hat{1}_{\Delta_2}^{(0)} \right|^2 = \left| \int_0^1 \int_0^{-x_2+1} e^{-2\pi ikx_1} dx_1 dx_2 \right|^2 = \frac{1}{4\pi^2 k^2} \left| \int_0^1 (e^{2\pi ikx_2} - 1) dx_2 \right|^2 = \frac{1}{4\pi^2 k^2}.$$ 

A similar computation gives $\left| \hat{1}_{\Delta_2}^{(k)} \right|^2 = \frac{1}{4\pi^2 k^2}$. By Theorem 2, we have:

$$\frac{1}{2} = \text{vol} \Delta_2 = \frac{1}{\det \mathbb{Z}^2} \sum_{\xi \in \mathcal{F}} \left| \hat{1}_{\Delta_2}(\xi) \right|^2 = \left| \hat{1}_{\Delta_2}(0) \right|^2 + 3 \sum_{k \in \mathbb{Z} - \{0\}} \frac{1}{4\pi^2 k^2} = (\text{vol} \Delta_2)^2 + 3 \sum_{k \in \mathbb{Z} - \{0\}} \frac{1}{k^2} = \frac{1}{4} + \frac{3}{2\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2},$$

giving the required classical identity $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$. \hfill \square

**Example 6.** Here we show that it is possible to recover the values of the Riemann zeta function at all positive even integers, namely $\zeta(2d)$, by using Theorem 2. This approach extends Example 5, and we begin by considering the integer lattice, together with the standard simplex $Q := \Delta_d$. We will compute $\zeta(4) := \sum_{n=1}^{\infty} \frac{1}{n^4}$.

First, we have $\sum_{n \in \mathbb{Z}^3} \text{vol} (\Delta_3 \cap (\Delta_3 + n)) = \text{vol} \Delta_3 = \frac{1}{6}$. The crucial ingredient on the right-hand side of equation (8) of Theorem 2 is the transform $\left| \hat{1}_{\Delta_3}(\xi) \right|^2 = \left| \int_{\Delta_3} e^{-2\pi i \xi \cdot x} dx \right|^2$. The only $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{Z}^3$ for which the latter integral does not vanish are those integer points lying on 6 particular hyperplanes, by Lemma 5. More precisely, these are exactly the integer points in the family

$$\mathcal{F} = \{(0, k, k), (k, 0, k), (k, k, 0), (k, k, l), (k, l, k), (l, k, k)\}_{k,l \in \mathbb{Z}}.$$

After some computations we obtain

$$\left| \hat{1}_{\Delta_3}(0, k, l) \right|^2 = \frac{1}{16\pi^4} \cdot \frac{1}{k^2 l^2} \text{ for } k, l \neq 0 \text{ and } k \neq l,$$

$$\left| \hat{1}_{\Delta_3}(0, k, k) \right|^2 = \frac{1}{16\pi^4} \cdot \frac{4}{k^2} \text{ for } k \neq 0,$$

$$\left| \hat{1}_{\Delta_3}(k, k, l) \right|^2 = \frac{1}{16\pi^4} \cdot \frac{1}{k^2 (k - l)^2} \text{ for } k, l \neq 0 \text{ and } k \neq l,$$

$$\left| \hat{1}_{\Delta_3}(0, 0, k) \right|^2 = \left| \hat{1}_{\Delta_3}(k, k, k) \right|^2 = \frac{1}{16\pi^4} \cdot \left( \frac{1}{k^2} + \frac{\pi^2}{k^2} \right) \text{ for } k \neq 0.$$
Figure 7. The Tetrahedron $\triangle_3$, the integer lattice $\mathbb{Z}^3$, and the hyperplanes where $\hat{1}_{\triangle_3}(\xi) \neq 0$. There are four more analogous hyperplanes with this property.

By symmetry, all permutations of the coordinates yield the same result for each computation above.

By Theorem 2 we have

$$\frac{1}{6} = \text{vol} \triangle_3 = \frac{1}{\det \mathbb{Z}^3} \sum_{\xi \in \mathbb{F}} |\hat{1}_{\triangle_3}(\xi)|^2 = |\hat{1}_{\triangle_3}(0,0,0)|^2 + 3 \sum_{k,l \neq 0, k \neq l} |\hat{1}_{\triangle_3}(0,k,l)|^2 + 3 \sum_{k,l \neq 0, k \neq l} |\hat{1}_{\triangle_3}(k,k,l)|^2 + 3 \sum_{k \neq 0} |\hat{1}_{\triangle_3}(0,0,k)|^2$$

$$+ 3 \sum_{k \neq 0} |\hat{1}_{\triangle_3}(0,k,0)|^2 + \sum_{k \neq 0} |\hat{1}_{\triangle_3}(0,k,k)|^2 + 3 \sum_{k \neq 0} |\hat{1}_{\triangle_3}(0,0,0)|^2$$

$$= \left(\frac{\text{vol} \triangle_3}{6}\right)^2 + \frac{1}{16\pi^4} \left(3 \sum_{k,l \neq 0, k \neq l} \frac{1}{k^2l^2} + 3 \sum_{k,l \neq 0, k \neq l} \frac{1}{k^2(k-l)^2} + 3 \sum_{k \neq 0} \left(\frac{1}{k^4} + \frac{\pi^2}{k^2}\right) + \sum_{k \neq 0} \left(\frac{1}{k^4} + \frac{\pi^2}{k^2}\right) + 3 \sum_{k \neq 0} \frac{4}{k^4}\right)$$

$$= \frac{1}{36} + \frac{1}{16\pi^4} \left(3 \sum_{k,l \neq 0, k \neq l} \frac{1}{k^2l^2} - 3 \sum_{k \neq 0} \frac{1}{k^4} + 3 \sum_{k \neq 0, k \neq l} \frac{1}{k^2(k-l)^2} - 3 \sum_{k \neq 0} \frac{1}{k^4} + 4 \sum_{k \neq 0} \left(\frac{1}{k^4} + \frac{\pi^2}{k^2}\right) + 3 \sum_{k \neq 0} \frac{4}{k^4}\right)$$

$$= \frac{1}{36} + \frac{1}{16\pi^4} \left(24 \sum_{k,l>0} \frac{1}{k^2l^2} + 20 \sum_{k>0} \frac{1}{k^4} + 8 \sum_{k>0} \frac{\pi^2}{k^2}\right)$$

$$= \frac{1}{36} + \frac{1}{16\pi^4} \left(24 \left(\frac{\pi^2}{6}\right)^2 + 20 \sum_{k>0} \frac{1}{k^4} + 8\pi^2 \cdot \frac{\pi^2}{6}\right)$$

$$= \frac{1}{36} + \frac{1}{16\pi^4} \left(2\pi^4 + 20 \sum_{k>0} \frac{1}{k^4}\right)$$

$$= \frac{11}{72} + \frac{5}{4\pi^4} \sum_{k>0} \frac{1}{k^4}.$$
giving us the well-known identity \( \sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90} \).

10. Further remarks

Here we mention some directions for future research.

**Problem 1.** Extend the notion of admissible sets to include unbounded sets, and classify them.

An answer to this problem would enable a generalization of the current results to functions whose support is unbounded. Regarding multi-tiling bodies, the following simple-sounding question is still open.

**Problem 2.** If \( Q \) is a body that \( k \)-tiles \( \mathbb{R}^d \) (nontrivially), then \( k \neq 2 \).

Problem 2 is known to be true for \( d = 2 \) and \( d = 3 \) [7].

**Problem 3.** Apply the methods and results herein to study Matheron’s Conjecture 1.

**Problem 4.** Using Theorems 2 and 3, with some appropriate admissible set \( Q \) (possibly nonconvex), find a geometric interpretation for \( \zeta(3), \zeta(5) \), etc.

We note that there are already some fascinating geometric interpretations for the odd special values \( \zeta(2n + 1) \), following the work of Ed Witten on volumes of certain moduli spaces ([15], eq. 4.93). In Example 6, we picked simplices, and we retrieved the even values of \( \zeta(s) \) from a spectral representation of their volumes. Perhaps more can be done for Problem 4 by picking more complex sets and computing their volumes in Theorem 2.

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