OVERREFLECTION AND GENERATION OF GRAVITO-ALFVÉN WAVES IN SOLAR-TYPE STARS

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ABSTRACT

The dynamics of linear perturbations is studied in magnetized plasma shear flows with a constant shearing rate and with gravity-induced stratification. The general set of linearized equations is derived, and the two-dimensional case is considered in detail. The Boussinesq approximation is used in order to examine relatively small scale perturbations of low-frequency modes: gravito-Alfvén waves (GAWs) and entropy-mode (EM) perturbations. It is shown that for flows with arbitrary shearing rate, there exists a finite time interval of nonadiabatic evolution of the perturbations. The nonadiabatic behavior manifests itself in a twofold way, viz., by the overreflection of the GAWs and by the generation of GAWs from EM perturbations. It is shown that these phenomena act as efficient transformers of the equilibrium flow energy into the energy of the perturbations for moderate and high shearing rate solar plasma flows. Efficient generation of GAWs by EM perturbations takes place for shearing rates about an order of magnitude smaller than necessary for development of a shear instability. The latter fact could have important consequences for the problem of angular momentum redistribution within the Sun and solar-type stars.

Subject headings: MHD — stars: rotation — Sun: magnetic fields — Sun: rotation — waves

1. INTRODUCTION

It is well known that the most advanced stellar models, taking into account rotation-induced hydrodynamic processes of meridional circulation and shear mixing, quite adequately describe the structure of massive stars (Maeder & Meynet 2000). For solar-type stars with relatively wide convective regions, however, these models predict large rotation gradients and are in notable contradiction with recent observational helioseismology data (Charbonnel & Talon 2005). This is probably due to the insufficiently high efficiency of the common hydrodynamic instabilities that are characteristic of these models: these instabilities can redistribute the angular momentum between the different layers of the radiative region, but they are not strong enough to ensure the onset of a uniform-rotation regime.

While the radiative interiors of solar-type stars rotate quite uniformly, their convective zones rotate differentially. Mounting observational evidence indicates that the radiative interior of the Sun, for instance, rotates almost as a solid body, with a quasi-flat seismic rotation profile (Chaplin et al. 1999; Couvidat et al. 2003). But the solar convective zone exhibits a strong shearing in the latitudinal direction, with its equatorial layers rotating about four-thirds as fast as the polar regions (Kim & MacGregor 2003). The transition between the differential and the uniform rotation regimes takes place in a relatively thin layer (with thickness \( \lesssim 0.05 R_\odot \)), called the tachocline, where the radial shearing of the rotation is particularly significant (Miesch 2005; Petrovay 2003). It is often argued that some kind of shear instability must be the source of gravity waves at the base of the solar convection zone (Kumar et al. 1999), either in the immediate vicinity or just inside the tachocline. This instability can hardly be of a purely hydrodynamic nature, because from various numerical studies of the buoyant rise of magnetic flux from the bottom of the convection zone to the surface of the Sun, it follows that the tachocline is strongly magnetized, with a poloidal magnetic field of about \( 10^5 - 10^6 \) G (Miesch 2005).

From this perspective, it seems quite plausible to surmise that on timescales on the order of the solar age or less, it is not only purely hydrodynamic internal gravity waves (IGWs) that participate in the process of the angular momentum redistribution within the solar interior but also, or rather, gravito-MHD waves. This idea is not new. In the literature, it has been argued (Zahn et al. 1997; Kumar & Quataert 1997) that gravito-MHD waves could be one of the most promising candidates for the onset of the redistribution processes. It has been argued that a self-consistent model should comprise a large-scale magnetic field in the Sun’s interior and an accurate consideration of the Coriolis effects in the convection zone and in the tachocline (Talon & Charbonnel 1998, 2003, 2004, 2005; Charbonnel & Talon 2005). It has also been argued that turbulent stresses in the convection zone, through the agency of Coriolis effects, induce a meridional circulation, causing the gas from the convection zone to burrow downward and generate both horizontal and vertical velocity shear characterizing the tachocline. However, the interior magnetic field confines the burrowing and, hence, smooths and diminishes the shear, as demanded by the observed structure of the tachocline. According to Charbonnel & Talon (2005), a decisive test of this qualitative scenario, after numerical refinements, would be its quantitative consistency with the observed interior angular velocity structure.

The aim of the present paper is to present the results of a detailed study of the linear dynamics of perturbations in gravitationally stratified, magnetized shear flows. Bearing in mind the
highly probable importance of gravito-MHD waves for the structure of the Sun and solar-type stars, we decided to pose and study the following three interrelated issues: (1) how the presence of the nonuniform velocity field affects the propagation of the waves through the stellar plasma; (2) what kind of energy-exchange processes between the different collective modes and between the modes and the ambient flow may happen; and (3) what other astrophysical consequences these processes could have.

Initially, we develop a three-dimensional model, allowing the ambient flow to have velocity shearing in both transverse (prior to the gravity field) directions. Further on, the study is focused by a number of simplifying assumptions, enabling the solution of the equations. First of all, we consider only two spatial dimensions, bearing in mind that it is quite straightforward to extend the analysis to the fully three-dimensional case. Second, we assume a constant steady state (or stationary) shear flow, an assumption that allows us to use the shearing-sheet approximation (Goldreich & Lynden-Bell 1965). And third, we use the Boussinesq approximation, and hence we study the dynamics of relatively small scale\(^2\) perturbations of the low-frequency modes, viz., gravito-Alfvén waves (GAWs) and entropy-mode (EM) perturbations.

Our study leads to the conclusion that for arbitrary shear in the flow, there exists a finite interval of time in which the evolution of the modes is highly nonadiabatic. Outside this interval, the evolution is adiabatic and can be accurately described by a WKB approximation. Therefore, we have an asymptotic problem, quite common in various quantum mechanical applications, where one needs to obtain connection formulae in turning-point problems (Landau & Lifshitz 1977). The relevant analysis is performed in this paper, and it is found that the nonadiabatic behavior of the considered modes manifests itself in the form of two phenomena, viz., the overreflection of the GAWs and the generation of the GAWs by the EM perturbations. We show that for flows with moderate and high shearing rates, both these processes are very effective converters of the equilibrium flow energy into the energy of the waves.

In the context of the astrophysical problems discussed above, our results imply that efficient generation of GAWs by the EM perturbations takes place for shearing rates of about an order of magnitude smaller than necessary for development of shear instability.

The remainder of the paper is organized in the following way: The general mathematical and heuristic formalism is presented in § 2. The overreflection of GAWs is studied in § 3, and the second nonadiabatic process, the generation of the GAWs by EM perturbations, is studied in § 4. Finally, in § 5 we conclude with a summary and a brief discussion of our results in the context of their possible importance for angular momentum redistribution within solar-type stars and the solution of the problem of the uniform rotation of their radiative zones.

2. LINEAR PERTURBATIONS OF SHEARED PLASMA FLOWS

For the sake of generality, we derive the basic set of equations for the case of three-dimensional perturbations in a compressible, gravitationally stratified MHD fluid. This formalism is presented in the next subsection (§ 2.1). Later on (§ 2.2), we restrict the consideration to the somewhat simpler case of two-dimensional perturbations in an incompressible medium. In addition, the Boussinesq approximation will be used and the stationary shear flow will be considered.

2.1. General Theory

In our model, the geometry of the considered problem is simplified in the following way: The equilibrium flow \(U\) is supposed to be plane-parallel, to be directed along the \(x\)-axis, and to have both a horizontal \((A_x)\) and a vertical \((A_z)\) shear:

\[
U \equiv (A_x y + A_z, 0, 0).
\]

The uniform gravity \(g\) is assumed to be directed along the negative direction of the \(x\)-axis:

\[
g \equiv (0, 0, -g).
\]

We consider a simplified model and assume that the equilibrium magnetic field \(B_0\) is toroidal, parallel to \(U\), and that it possesses a gravity-induced vertical stratification:

\[
B_0 \equiv (B(z), 0, 0).
\]

The set of equations of one-fluid ideal MHD, governing the physics of the flow, is

\[
D_t \rho + \rho (\nabla \cdot V) = 0, \tag{4}
\]

\[
\rho D_t V = - \nabla (P + \frac{B^2}{8 \pi}) + \frac{1}{4 \pi} (B \cdot \nabla)B + \rho g. \tag{5}
\]

\[
D_t B = (B \cdot \nabla) V - B (\nabla \cdot V), \tag{6}
\]

\[
D_t S = 0, \tag{7}
\]

while

\[
\nabla \cdot B = 0. \tag{8}
\]

In equations (4)–(7), the total (convective) time derivative operator is denoted by \(D_t \equiv \partial_t + (V \cdot \nabla)\).

Instantaneous values of all physical variables are decomposed into their regular (mean) and perturbational components:

\[
\rho \equiv \rho_0 + \varrho, \quad P \equiv P_0 + p, \quad S \equiv S_0 + s, \tag{9}
\]

\[
V \equiv U + u, \quad B \equiv B_0 + b. \tag{10}
\]

It is straightforward to see that the horizontal background flow does not affect the vertical hydromagnetic equilibrium of the flowing MHD fluid, which is governed by the equation

\[
\partial_z \left( P_0 + \frac{B_0^2}{8 \pi} \right) = - \rho_0 g. \tag{11}
\]

In general, the variation of the density \(d\rho\) is related to variations of the pressure \(dP\) and entropy \(ds\) as

\[
d\rho = \mu ds + (1/C_s^2) dP, \tag{12}
\]

where

\[
\mu \equiv (\partial \rho / \partial s)_p, \quad C_s^2 \equiv (\partial P / \partial \rho)_s. \tag{13}
\]

\(^2\) With the vertical wavelength, \(\ell_v\), much smaller than the stratification length scale \(z_0\).
A linearized set of equations governing the evolution of perturbations within this flow can be written in the following way \(D_t \equiv \partial_t + (A_y + A_z) \partial_z\):

\[
\begin{align*}
D_t \partial_t + (\partial_r \rho_0) v_z + \rho_0 (\nabla \cdot \mathbf{u}) &= 0, \quad (14) \\
D_t u_x &= -A_y u_x - A_z u_z - \frac{1}{\rho_0} \partial_r \rho + \frac{(\partial_z B_0)}{4\pi \rho_0} b_z, \quad (15) \\
D_t u_y &= -\frac{1}{\rho_0} \partial_r \rho - \frac{B_0}{4\pi \rho_0} (\partial_z b_y - \partial_y b_z), \quad (16) \\
D_t u_z &= -\frac{1}{\rho_0} \partial_r \rho + \frac{B_0}{4\pi \rho_0} (\partial_z b_z - \partial_z b_z) - \frac{(\partial_y B_0)}{4\pi \rho_0} b_x - \frac{g}{\rho_0} \theta, \quad (17) \\
D_t s + (\partial_r s) v_z &= 0, \quad (18) \\
D_t b_y &= B_0 \partial_z u_y, \quad D_t b_z = B_0 \partial_z u_z, \quad (19) \\
\partial_z b_x + \partial_y b_y + \partial_z b_z &= 0. \quad (20)
\end{align*}
\]

Because of the stratification, some coefficients on the right-hand sides of equations (14)–(19) depend on \(z\). But when studying the dynamics of small-scale perturbations (with \(l_z \ll z_0\)), one can consider the mean components as constant. In this case, the set of first-order, partial differential equations (14)–(20) can be reduced to a set of ordinary differential equations (ODEs) with time-dependent coefficients if we look for solutions of the following form:

\[
F(r, t) = \tilde{F}(k(t), t)e^{(k \cdot r - \varphi(k, t))} \quad (21)
\]

with

\[
\varphi(k, t) = U \cdot \int k(t) dt. \quad (22)
\]

Here \(k(t)\) is a time-dependent wavenumber vector, determined by the following set of equations:

\[
k^{(1)} + S^T \cdot k = 0 \quad (23)
\]

(Lagnado et al. 1984; Craig & Criminale 1986; Mahajan & Rogava 1999) with \(S^T\) being the transposed shear matrix

\[
S \equiv \begin{pmatrix}
U_{xx} & U_{xy} & U_{xz} \\
U_{yx} & U_{yy} & U_{yz} \\
U_{zx} & U_{zy} & U_{zz}
\end{pmatrix}, \quad (24)
\]

which, in our case has only two nonzero components, \(A_y\) and \(A_z\):

\[
S \equiv \begin{pmatrix}
0 & A_y & A_z \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}. \quad (25)
\]

Equation (23) implies that the transverse components of \(k(t)\) acquire a linear time dependence:

\[
k_x(t) = k_x(0) - A_x t k_x, \quad k_z(t) = k_z(0) - A_z t k_x. \quad (26)
\]

Therefore, one can see that the wavenumber vector \(k(t)\) of the spatial Fourier harmonics (SFHs) varies in time; that is, in the linear approximation there is a “drift” of the SFHs in the \(k\)-space. The physical reason for this drift is related to the fact that in the sheared flow, the perturbations cannot have the form of a simple plane wave, because of the effect of the shearing background on the wave crests.

Applying the Ansatz of equation (21) to equations (14)–(19), we reduce the system to the following set of first-order ODEs (hereafter we use the notation \(f^{(\alpha)} \equiv \partial_\alpha f\)):

\[
d^{(1)} = i e v_z + \xi_y + \hat{K}_y(\tau) v_x + \hat{K}_z(\tau) v_y, \quad (27)
\]

\[
e^{(1)} = -\Xi_x v_x - \Xi_y v_y - \sigma^2 \mathcal{P} + \frac{1}{2} i e b_z, \quad (28)
\]

\[
ev_z^{(1)} = -\sigma^2 \hat{K}_x(\tau) \mathcal{P} + b_x - \frac{1}{2} i e b_z + i \alpha d, \quad (29)
\]

\[
e^{(1)} = v_x, \quad h_x^{(1)} = -v_y, \quad h_z^{(1)} = -v_z, \quad (30)
\]

while the Maxwell equation (eq. [20]) reduces to the algebraic relation

\[
b_x + \hat{K}_x(\tau) b_y + \hat{K}_z(\tau) b_z = 0. \quad (32)
\]

In these equations, the following dimensionless notation is used for the constants: \(R_x \equiv A_x / k_y C_y, R_z \equiv A_z / k_z C_z, \hat{K}_y(0) \equiv k_z(0)/k_y, \hat{K}_z(0) \equiv k_y(0)/k_z, \epsilon \equiv (k_z 0^{-1}), \sigma^2 \equiv (C_z / C_y)^2\), and \(N^2 \equiv (N_B V / C_z) k_y^2\). The dimensionless variables appearing in the above set of equations are defined as \(v_i \equiv \hat{u}_i / C_y, b_j \equiv \hat{b}_j / B_0, d \equiv \hat{\theta}/\rho_0, \mathcal{P} \equiv \hat{b}_j / \rho_0 C_z^2, e \equiv -\hat{k}_x \hat{k}_y / (\partial_b s_0), \tau \equiv k_z C_y t, \hat{K}_x(\tau) \equiv \hat{K}_x(0) - R_x \tau, \text{ and } \hat{K}_z(\tau) \equiv \hat{K}_z(0) - R_z \tau\).

Note that this system is not closed, because we have only eight equations for nine variables \((v, b, d, e, \mathcal{P})\). The closure of the set of equations is ensured by the thermodynamic relation that follows from equation (12) \((\sigma \equiv g/k_z C_z)\):

\[
d = \mathcal{P} - i(N^2 / \alpha) e. \quad (33)
\]

This system of equations describes the temporal evolution of gravito-MHD waves modified by the presence of a velocity shear in the two planes transverse to the flow direction. A full analysis of this set of equations is beyond the scope of the present paper. Instead, in the next sections we focus our study on the relatively simple two-dimensional, incompressible case. We will see that even in this simplified case the presence of the velocity shear brings a considerable novelty to the dynamics of perturbations.

2.2. The Incompressible, Two-dimensional Limit

In order to simplify the mathematical aspects of the problem, we make the following assumptions:

1. We consider the two-dimensional case, restricting the problem to the \(x-z\) plane and implying

\[
K_x(0) = R_x = 0. \quad (34)
\]

Hereafter we write \(R \equiv R_x (A \equiv A_x)\). In this case, equation (32) enables us to express the longitudinal (z-) component of the magnetic field perturbation in terms of its transverse (x-) component:

\[
b_x = -\hat{K}_x(\tau) b_z. \quad (35)
\]

2. We drop the \(d^{(1)}\) term in the continuity equation; that is, we adopt the concept of so-called dynamic incompressibility (Landau & Lifshitz 1959). This implies that the velocity of the

\[
\text{The Alfven speed } C_A^2 = B_0^2/\rho_0 \text{ and the Brunt-Vaisalla frequency } N_B^2 = -g \partial_b s_0 / \rho_0 \text{ are defined in the usual way.}
flow is assumed to be small in comparison with the speed of sound. As a consequence, the density perturbations evoked by the pressure perturbations are negligible, and instead of equation (33) we have

\[ d \simeq -i(N^2/\alpha)e. \]  

(36)

However, since the medium is thermally conducting, its density should vary also as a result of the temperature variation, and this effect should be (and is) taken into account.

In hydrodynamics, the second assumption is known as the Boussinesq approximation: in the conservation equations, the terms proportional to the density perturbation are retained if, and only if, they produce a buoyancy force.\(^4\) The Boussinesq approximation automatically implies (Landau & Lifshitz 1959; Gill 1982) that the vertical length scale of perturbations, \(l_z \sim k_z^{-1}\), is much smaller than the characteristic stratification length scale \(z_0\). This condition was already used for the derivation of equations (27)–(31). In terms of our dimensionless notation, this condition transcribes into

\[ \epsilon \ll 1. \]  

(37)

When this condition is satisfied, the above approach enables us to study the dynamics of small-scale, low-frequency perturbations.\(^5\)

These assumptions make the basic set of equations much simpler. Taking into account equations (35) and (36), we get

\[ v_z + K_\gamma(\gamma)v_z = 0, \]  

(38)

\[ v_z^{(1)} = -Re_z - \sigma^2\beta, \]  

(39)

\[ v_z^{(2)} = -\sigma^2K_\gamma(\gamma)\beta + K_\gamma(\gamma)b_z + N^2e, \]  

(40)

\[ e^{(1)} = v_z, \quad b_z^{(1)} = -v_z. \]  

(41)

From the last two equations, it is apparent that there is an algebraic relation between the entropy perturbations and the magnetic field perturbations, viz.,

\[ e + b_z = I = \text{const}(\tau). \]  

(42)

Another, less obvious, relation between the perturbed quantities may be found if one takes the time derivative of equation (38) and takes into account equations (39) and (40). The result can be used to express the pressure perturbation as a function of the other variables, that is,

\[ \sigma^2\beta = -\frac{2R}{K^2(\gamma)}v_y + j\beta e - \frac{W^2}{K^2(\gamma)}ay. \]  

(43)

Having three algebraic relations, equations (38), (42), and (43), for five variables, we can reduce the system to one second-order ODE. For instance, for the variable \(b(\tau)\), defined as \(b(\tau) \equiv K_\gamma(\gamma)b_z\), we can derive the following explicit, second-order, inhomogeneous ODE:

\[ \frac{d^2b}{d\tau^2} + \left[ 1 + \frac{N^2}{K^2(\gamma)} - \frac{R^2}{K^4(\gamma)} \right] b = \frac{N^2}{K(\gamma)}I. \]  

(44)

This equation describes the dynamics of the so-called continuous eigenspectrum (see, e.g., Balmforth & Morrison 2002 and references therein). In order to take into consideration the perturbations belonging to the discrete eigenspectrum, one should specify some boundary conditions with respect to the vertical spatial variable \(z\). In the absence of an external magnetic field, it is known (Drazin & Reid 1981) that all the discrete eigenmodes are stable if the Richardson number, defined as \(Ri \equiv N^2/AR^2\), is greater than \(\frac{1}{4}\).

Equation (44) describes two important physical phenomena related to the dynamics of the continuous eigenspectrum, which will be considered in detail in the two subsequent sections, viz., (1) the (over-) reflection of the GAWs and (2) the conversion of the EM perturbations into GAWs. It will be shown that the EM perturbations can be a very efficient source of GAWs, even for relatively small shearing rates (\(Ri > \frac{1}{4}\)).

3. OVERREFLECTION OF GRAVITO-ALFVÉN WAVES

In order to describe the phenomenon of the overreflection of the GAWs in its pure form, let us study equation (44) with \(I = 0\), yielding

\[ \frac{d^2b}{d\tau^2} + \left[ 1 + \frac{N^2}{K^2(\gamma)} - \frac{R^2}{K^4(\gamma)} \right] b = 0. \]  

(45)

The standard method for the analysis of similar equations (Olver 1974) is well known in quantum mechanics (Landau & Lifshitz 1977, p. 304). The behavior of the solution is fully determined by the shear-modified frequency:

\[ \omega^2(\gamma) \equiv 1 + \frac{N^2}{K^2(\gamma)} - \frac{R^2}{K^4(\gamma)}. \]  

(46)

From the mathematical point of view, the (over-) reflection is caused by the cavity of \(\omega^2(\gamma)\) that appears in the vicinity of the point \(\tau^*\), where \(K_\gamma(\tau^*) = 0\).

In the areas where \(K_\gamma(\tau) \gg R\), the evolution of \(b(\tau)\) is adiabatic \([db(\tau)/d\tau \ll \omega^2(\tau)]\), and therefore, the approximate solution has the form

\[ b(\tau) \approx A_+b_+(\tau) + A_-b_-(\tau), \]  

(47)

where

\[ b_\pm(\tau) = \frac{1}{\sqrt{\omega(\tau)}}e^{\pm i\omega(\tau)d\tau}. \]  

(48)

are WKB solutions having positive and negative phase velocities along the \(x\)-axis, respectively, while the \(A_\pm\) can be treated as the intensities of the corresponding types of perturbations.

Let us assume that \(A_+^2\) and \(A_-^2\) are the WKB amplitudes far on the left- and right-hand sides from the point \(\tau^*\), respectively. It is well known (Olver 1974) that the conservation of the Wronskian leads to the following invariant:

\[ |A_+^2| - |A_-^2| = |A_+^8| - |A_-^8|, \]  

(49)

which physically corresponds to the conservation of the wave action (see, e.g., Gogoberidze et al. 2004 and references therein). If initially \(A_+^8 \equiv 0\), the refraction and transmission coefficients may be defined in the usual way:

\[ \operatorname{Re} = \frac{|A_+^8|^2}{|A_+^8|}, \quad D = \frac{|A_-^8|^2}{|A_+^8|}, \]  

(50)

\(^4\) The last term in eq. (30).

\(^5\) For a detailed analysis of the Boussinesq approximation in stratified shear flows, see Didebulidze (1997) and Didebulidze et al. (2004).
while from equation (49) we obviously have

\[ 1 + \text{Re} = D. \]  

(51)

The dependence of the transmission coefficient \( D(R) \) on \( R \), obtained through the numerical solution of equation (45), is presented in Figure 1 for different values of the Brunt-Väisälä frequency, viz., \( N = 0 \) (solid line), \( N = 1 \) (dashed line), and \( N = 2 \) (dash-dotted line). The initial conditions for the numerical solution are chosen from the WKB solutions (eq. [48]) far on the left-hand side of the resonant area \([k_\perp(0)]\). The analysis leads to the conclusion that the amplification of the energy density of the perturbations is always finite, but it can become arbitrarily large with a proper increase of the shearing rate. It can be seen that the velocity shear is able to cause a strong amplification of the GAWs, even for moderate values of the normalized shearing rate.

According to the numerical study of equation (45), the amplitude of the reflected wave exceeds the amplitude of the incident wave if

\[ R^2 > 2. \]  

(52)

This phenomenon, originally discovered by Miles (1957) for acoustic waves, is called overreflection.

Bearing in mind that the energy of the mode is proportional to the square of its amplitude, we can easily surmise that in the course of its nonadiabatic evolution, the perturbation energy is increasing. According to equations (49) and (51), the ratio of the total “postreflection” energy to the initial one equals \( \text{Re} + D > 1 \). The energy increases at the expense of the mean (shear) flow energy. Consequently, overreflection represents a quite efficient mechanism for transferring the mean flow energy to perturbations.

In the limit \( R \to 0 \), the reflection coefficient becomes exponentially small with respect to the parameter \(-1/R\), that is, \( \text{Re} \sim \exp(-1/R) \) (Gogoberidze et al. 2004).

4. THE GENERATION OF GRAVITO-ALFVÉN WAVES

Let us return to the analysis of the full, inhomogeneous equation (44). When the condition for adiabatic evolution,

\[ d\omega(\tau)/d\tau \ll \omega^2(\tau), \]  

holds, then the approximate solution of equation (51) has the following form:

\[ b(\tau) \approx A_+ b_+(\tau) + A_- b_-(\tau) + b_E(\tau) \]  

(53)

(Chagelishvili et al. 1994), where the nonperiodic solution \( b_E(\tau) \) is given by

\[ b_E(\tau) \approx \frac{N^2}{K^2(\tau)} \left[ 1 + \frac{N^2}{K^2(\tau)} - \frac{R^2}{K^4(\tau)} \right]^{-1} I, \]  

(54)

which can be readily identified as the shear-modified entropy-mode perturbation.

It turns out that another physically important phenomenon that takes place in the system under consideration for moderate and high shearing rates is the generation of GAWs by the EM perturbations. An illustrative example is given by Figure 2, where the numerical solution of equation (44) for the case in which \( N = 1 \) and \( R = 0.7 \) for the variable \( b(\tau) \) is shown. The initial conditions were chosen in such a way that at the beginning [at
$K_2(0) = 35$, there exists only the EM perturbation; namely, the initial conditions were found by means of equations (53) and (54) with $A_\pm = 0$ and $I = 1$. The figure clearly demonstrates the generation of GAWs while the EM passes through the interval of nonadiabatic evolution, situated in the vicinity of the point $K_2(\tau) = 0$.

To quantitatively describe of the wave generation process, analogously to the reflection and transmission coefficients defined in the previous section, we define the wave generation coefficient $G_\pm$ in the following way:

$$G_\pm = \frac{|A_\pm|^2}{|I|^2 N^2}. \quad (55)$$

With this definition, $G_\pm$ is proportional to the ratio of the generated wave energy to the energy of incident EM perturbation. From the symmetry of the problem, it is obvious that $G_\pm = G_\mp \equiv G$. In Figure 3, the dependence of the generation coefficient $G(R)$ on the normalized shear parameter $R$ is presented for different values of the Brunt-Väisälä frequency: $N = 1$ (solid line), $N = 2$ (dashed line), and $N = 3$ (dash-dotted line).

5. DISCUSSION

In order to explain the near-uniformity of the rotation profile in the radiative region of the Sun and similar regions in solar-type stars, usually one of two different mechanisms is proposed. The first implies the presence of traveling IGWs generated at the base of the convective region (Charbonnel & Talon 2005), while the second is based on the presumed (and for the Sun, observationally justified) existence of a magnetic field in the radiative zone (Gough & McIntyre 1998).

It seems reasonable to assume that these two mechanisms are not necessarily alternatives but could be envisaged as complementary. In particular, we may argue that if magnetic fields are indeed present inside the radiative zone and the gravito-MHD waves do pass through the zone, this will lead to intense energy-exchange processes between the waves and the stellar plasma. This circumstance, in turn, might lead to a more efficient redistribution of the angular momentum within these stars, establishing quasi-flat rotation profiles. The presence of a magnetic field implies that it is more reasonable to speak not about purely hydrodynamic IGWs, but about gravito-MHD waves in general, and gravito-Alfvén waves in particular.

The evolution of the latter mode, present only in the magnetized case and absent in a purely hydrodynamic medium, is studied in the present paper. The theoretical background for studying the evolution of the full set of gravito-MHD waves is also developed.

From the general theory of IGWs, we know that all the discrete eigenmodes are stable under the condition $Ri > \frac{1}{2}$ (Drazin & Reid 1981). Our study shows that the generation of GAWs by entropy-mode perturbations could provide an efficient mechanism for creating the required flux of GAWs, even for relatively low shearing rates, when the discrete eigenmodes are stable. Note in this context that for the case presented in Figure 2, for instance, the Richardson number was about $Ri \approx 1.02$.

In the magnetized case, we studied the dependence of the generation coefficient $G(Ri)$ on the Richardson number $Ri$ for a relatively weak external magnetic field ($N = 4$). The results of this study are presented in Figure 4, which clearly indicates that even though the generation coefficient falls rapidly with increasing $Ri$, efficient generation of GAWs by the EM perturbations still takes place for $Ri < 1.25$.

Obviously, our study is of a simplified nature and only indicates the possibly important role of nonmodal phenomena related to GAWs in the redistribution of angular momentum and the flattening of the rotation curves of solar-type stars. To make this claim more convincing, we have to consider the compressible, three-dimensional analog of this problem, and we have to verify whether the full set of gravito-MHD waves is as efficient (or even more efficient) in exchanging energy with the equilibrium flow. We know from previous studies that in the absence of gravity-induced stratification among the shear-modified MHD waves, the fast magnetosonic waves are the most efficient in exchanging energy with the equilibrium flow (Poedts et al. 1998, 1999; Rogava et al. 2000; Rogava 2004).

It will be of great interest to check whether the fast gravito-magnetosonic waves have the same quality and to determine what their contribution to the angular momentum redistribution could be. The study of the linear dynamics of all gravito-MHD waves is currently under way, and the results will be published in a subsequent paper.

Finally, one has to remember that the nonmodal approach that has been applied in the present paper provides no information about the spatial aspects of the shear-induced processes, because the study of the spatial Fourier harmonics is confined to the phase space of the wavenumber vectors $k(t)$. In order to have a clear idea about the spatial appearance of these processes, one has to study them numerically (similarly to the uniform, nonstratified flow study by Bodo et al. 2001) and to check how the nonmodal phenomena couple with the traditional rotational instabilities.

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