A CELL FILTRATION OF THE RESTRICTION OF A CELL MODULE

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Abstract. We give a new proof that the restriction of a cell module of the Hecke algebra of the symmetric group $\mathfrak{S}_n$ to the Hecke algebra of $\mathfrak{S}_{n-1}$ has a filtration by cell modules.

1. Introduction

Let $R$ be an integral domain and $q$ be a unit in $R$. The Iwahori–Hecke algebra of the symmetric group, denoted $\mathcal{H}_n = \mathcal{H}_n(q^2)$, is the algebra presented by generators $T_1, \ldots, T_{n-1}$, and relations

\[ T_i T_j = T_j T_i, \quad \text{if } |i - j| > 1, \]
\[ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad \text{for } i = 1, \ldots, n-2, \]
\[ (T_i - q^m T_i + q^{-m}) = 0, \quad \text{for } i = 1, \ldots, n-1. \]

If $v \in \mathfrak{S}_n$, and $v = s_1 s_2 \cdots s_k$ is a reduced expression for $v$ in $\mathfrak{S}_n$, then $T_v = T_{s_1} T_{s_2} \cdots T_{s_k}$ is well defined in $\mathcal{H}_n(q^2)$ and $\{ T_v \mid v \in \mathfrak{S}_n \}$ is an $R$–basis of $\mathcal{H}_n(q^2)$. It follows from this that $\mathcal{H}_n$ imbeds in $\mathcal{H}_{n+1}$ for all $n \geq 0$.

The representation theory of $\mathcal{H}_n(q^2)$ was studied by Dipper and James in [1, 2], generalizing the approach to the representation theory of the symmetric groups via Specht modules in [4]. Murphy developed a new combinatorial approach to the representation theory of the Hecke algebras $\mathcal{H}_n(q^2)$ in [9]. Murphy’s analysis involves certain elements $m_{st}^\lambda$ of the Hecke algebra, indexed by Young diagrams $\lambda$ of size $n$ and a pairs of $s, t$ of standard $\lambda$–tableaux. Murphy shows that his elements satisfy the following properties:

1. The collection of elements $m_{st}^\lambda$, as $\lambda$ varies over Young diagrams of size $n$ and $s, t$ vary over standard $\lambda$–tableaux, is an $R$–basis of $\mathcal{H}_n(q^2)$.
2. For $h \in \mathcal{H}_n(q^2)$, $m_{st}^\lambda h = \sum_r r_v m_{st}^\mu + x$, where the coefficients $r_v \in R$ depend only on $t$ and $h$ (and not on $s$) and $x$ is in the $R$–span of basis elements $m_{st}^\mu$, where $\mu$ is greater than $\lambda$ in dominance order.
3. $(m_{st}^\lambda)^* = m_{st}^{\lambda^\ast}$, where $*$ is the algebra involution of $\mathcal{H}_n(q^2)$ defined by $T_v^* = T_{v^{-1}}$.

Independent of Murphy and more or less simultaneously, Graham and Lehrer [3] introduced a theory of cellular algebras. A cellular algebra is an algebra $A$ over an integral domain $R$ with an $R$–linear involution $\ast$, and auxiliary data consisting of a finite partially ordered set $(\tilde{A}, \unlhd)$ and for each $\lambda \in \tilde{A}$ a finite index set $\tilde{A}^\lambda$. $A$ is required to have an $R$–basis $a_{st}^\lambda$ indexed by elements $\lambda \in \tilde{A}$ and pairs $s, t$ in $\tilde{A}^\lambda$, having properties analogous to properties (2) and (3) of the Murphy basis, listed above. The basis set of $A$ is called a cellular basis. Thus, in the language of Graham and Lehrer, Murphy showed that his collection of elements $m_{st}^\lambda$ is a cellular basis of the Hecke algebra $\mathcal{H}_n(q^2)$. For the Hecke algebra $\mathcal{H}_n = \mathcal{H}_n(q^2)$, the relevant partially ordered set $(\check{H}_n, \supseteq)$ is the set of Young diagrams of size $n$ with dominance order and the index set $\check{H}_n^\lambda$ is the set of standard tableaux of shape $\lambda$.

Let $A$ be a cellular algebra with data $(\ast, \tilde{A}, \unlhd)$ and cellular basis $\mathcal{A} = \{ a_{st}^\lambda : \lambda \in \tilde{A}^\ast \}$ and $s, t \in \tilde{A}^\lambda$. Fix $\lambda \in \tilde{A}$ and define $\tilde{A}^{\ast\lambda}$ to be the span of basis elements $a_{st}^\mu$ with $\mu \unlhd \lambda$, and likewise $\tilde{A}^{\ast\lambda}$ to be the span of basis elements $a_{st}^\mu$ with $\mu \unlhd \lambda$. It follows that these are two–sided $\ast$–invariant ideals of $A$. Moreover, for any fixed $s \in \tilde{A}^\lambda$, $\Delta^\lambda = \text{span}\{ a_{st}^\lambda + A^{\ast\lambda} : t \in \tilde{A} \}$ is an $A$–submodule of $A^{\ast\lambda} / A^{\ast\lambda}$, which is free as an $R$–module. Up to isomorphism, the cell module $\Delta^\lambda$ is independent of $s$. Graham and Lehrer show that the cell module has a canonical bilinear form. When $R$ is specialized to be a field, the quotient

Key words and phrases. Cellular algebra; Hecke algebra; Murphy basis; cell filtration.

We thank Andrew Mathas for detailed discussions of this problem.

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of the cell module by the radical of the bilinear form is either zero or simple, and all simple modules arise in this way [3, Theorem 3.4].

Let A be a cellular algebra and M an A–module. Say that M has an order preserving cell filtration if M has a filtration by A–submodules:

\[(0) = M_0 \subset M_1 \subset \cdots \subset M_{s-1} \subset M_s = M,\]
such that for each i, there exists \(\lambda^{(i)} \in \mathcal{A}\) such that \(M_i/M_{i-1} \cong \Delta^{\lambda^{(i)}}\), and, moreover, \(\lambda^{(1)} \triangleright \lambda^{(2)} \cdots \triangleright \lambda^{(s)}\).

This note concerns the following theorem regarding restrictions of cell modules of the Hecke algebras \(H_n = H_n(q^2)\):

**Theorem 1.1** (Jost, Murphy). Let \(n \geq 1\) and \(\lambda\) be a Young diagram of size \(n\). Let \(\Delta^\lambda_{H_n}\) be the corresponding cell module of \(H_n\). Then \(\text{Res}^{\mathcal{H}_n}_{\mathcal{H}_{n-1}}(\Delta^\lambda_{H_n})\) has an order preserving filtration by cell modules of \(H_{n-1}\).

Jost [5] has shown, using the Dipper–James description of Specht modules of the Hecke algebras [1], that the restriction of a Specht module has a filtration by Specht modules. Together with Murphy's result that the cell modules of the Hecke algebras can be identified with the Specht modules [9, Theorem 5.3], this shows that the restriction of a cell module has a cell filtration.

It seems that there is no complete proof of Theorem 1.1 in the literature based on Murphy's description of the cellular structure and representation theory of the Hecke algebras. The proof given in [6, Proposition 6.1] has a gap, as it is not evident that the filtration of \(\text{Res}^{\mathcal{H}_n}_{\mathcal{H}_{n-1}}(\Delta^\lambda_{H_n})\) provided there has subquotients isomorphic to cell modules of \(H_{n-1}\). The purpose of this note is to give a detailed proof of this fact.

Our treatment of this problem developed out of a correspondence with Andrew Mathas. In the meanwhile, Mathas has also found a different approach, using the seminormal basis, which extends also to the more general contexts of cyclotomic Hecke algebras and cyclotomic quiver Hecke algebras [8].

We remark that there is a companion theorem regarding cell filtrations of induced modules:

**Theorem 1.2** (Dipper–James, Murphy, Mathas). Let \(\mu\) be a Young diagram of size \(n\) and let \(\Delta^\mu_{H_n}\) be the corresponding cell module of \(H_n\). Then \(\text{Ind}^{\mathcal{H}_{n+1}}_{\mathcal{H}_n}(\Delta^\mu_{H_n})\) has an order preserving filtration by cell modules of \(H_{n+1}\).

Dipper and James showed that the induced module of a Specht module of \(H_n\) has a filtration by Specht modules of \(H_{n+1}\). Together with Murphy's theorem [9, Theorem 5.3], this yields Theorem 1.2. A different proof was recently given by Mathas [7]; this proof is based on Murphy's theorem [9, Theorem 7.2] on the existence of a cell filtration of permutation modules of \(H_n\).

## 2. Preliminaries

We will assume familiarity with the usual combinatorial notions related to the representation theory of the symmetric groups and their Hecke algebras. We refer to [6, Ch. 3] for details and notation.

For purposes of this note, for a Young diagram \(\lambda\) of size \(n\), a tableau will mean an assignment of the numbers 1, . . . , \(n\) to the nodes of \(\lambda\). We let \(T(\lambda)\) denote the set of all \(\lambda\)–tableaux and \(T^{\text{Std}}(\lambda)\) the set of standard \(\lambda\)–tableaux. We will let \(t^\lambda\) denote the “superstandard” tableau of shape \(\lambda\), in which the numbers 1 through \(n\) are entered in increasing order from left to right along the rows of \([\lambda]\). Thus when \(n = 6\) and \(\lambda = (3, 2, 1)\),

\[
t^\lambda = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}
\]  

(2.1)

We regard the symmetric group \(S_n\) as acting on the right on the set of numbers 1, . . . , \(n\). The symmetric group \(S_n\) acts on the set of tableaux of size \(n\), by acting on the entries; this action is free and transitive. The Young subgroup \(S_\lambda\) is defined to be the row stabiliser of \(t^\lambda\) in \(S_n\). For each \(t \in T(\lambda)\), let \(w(t)\) denote the unique permutation such that \(t = t^\lambda w(t)\).
If \( \lambda \) is a Young diagram of size \( n \), let \( m_{\lambda} = \sum_{\nu \in \mathfrak{S}_\lambda} q^{l(\nu)} T_{\nu} \), where \( l \) denotes the length function on \( \mathfrak{S}_n \). For standard tableaux \( s, t \) of shape \( \lambda \), let

\[
m_{\lambda}^s = T_{w(s)}^a m_{\lambda} T_{w(t)}^a.
\]

These are Murphy's basis elements, as described in the introduction. Define

\[
m_{\lambda}^t = m_{\lambda} T_{w(t)}^a + \mathcal{H}^{\geq \lambda}.
\]

The set \( \{ m_{\lambda}^t : t \text{ is a standard } \lambda \text{-tableau} \} \) is an \( R \)-basis of the cell module \( \Delta^\lambda \). We will write \( M^\lambda \) for the "permutation module"

\[
M^\lambda = m_{\lambda} \mathcal{H}_n.
\]

Let \( \lambda \) be a Young diagram of size \( n \) and let \( t \in T(\lambda) \). Call a node of \( \lambda \) addable if the addition of the node to \( \lambda \) yields a Young diagram of size \( n + 1 \). Define a removable node similarly. Let \( a \) be an addable node of \( \lambda \). Then we write \( t \cup a \) for the tableau of shape \( \lambda \cup a \) which agrees with \( t \) on the nodes of \( \lambda \) and which has the entry \( n + 1 \) in node \( a \). If \( t \) is a standard \( \lambda \)-tableau and \( 1 \leq k < n \), let \( t_\downarrow_k \) denote the tableau obtained by deleting from \( t \) the nodes containing \( k + 1, \ldots, n \); then \( t_\downarrow_k \) is a standard tableau of size \( k \).

For \( 1 \leq i, j \leq n \), let

\[
T_{i,j} = \begin{cases} T_i T_{i+1} \cdots T_{j-1} = T_{(j,j-1,\ldots,i)}, & \text{if } j \geq i, \\ T_{j-1} T_{j-2} \cdots T_j = T_{(j,j+1,\ldots,i)}, & \text{if } i > j. \end{cases}
\]

Lemma 2.1. Let \( \lambda \) be a Young diagram of size \( n \), let \( a \) be a removable node of \( \lambda \), and let \( \mu = \lambda \setminus \alpha \). Let \( \alpha \) be the entry of \( \alpha^\lambda \) in the node \( a \). Let \( s \in T(\mu) \) be a \( \mu \)-tableau. Then

\[
w(s \cup \alpha) = (n, n - 1, \ldots, a) w(s),
\]

and

\[
T_{w(s \cup \alpha)} = T_{(n,n-1,\ldots,a)} T_{w(s)} = T_{a,n} T_{w(s)}.
\]

Proof. We have

\[
s \cup \alpha = (w \cup \alpha) w(s) = \alpha^\lambda (n, n - 1, \ldots, a) w(s).
\]

Therefore,

\[
w(s \cup \alpha) = (n, n - 1, \ldots, a) w(s).
\]

Now one can check that \( (n, n - 1, \ldots, a) \) is a distinguished left coset representative of \( \mathfrak{S}_{n-1} \) in \( \mathfrak{S}_n \). Therefore,

\[
T_{w(s \cup \alpha)} = T_{(n,n-1,\ldots,a)} T_{w(s)} = T_{a,n} T_{w(s)}.
\]

Lemma 2.2. Let \( \lambda \) be a Young diagram of size \( n \), let \( a \) be a removabale node of \( \lambda \), and let \( \mu = \lambda \setminus \alpha \). Let \( r \) be the row index of \( a \) and let \( b \) and \( a \) be the first and last entry in the \( r \)-th row of the standard tableaux \( \alpha^\lambda \). Write

\[
D(\alpha) = 1 + q T_{a-1} + q^2 T_{a-1} T_{a-2} + \cdots + q^{a-b} T_{a-1} T_{a-2} \cdots T_b.
\]

Then

\[
D(\alpha) T_{a,n} m_{\mu} = m_{\lambda} T_{a,n}.
\]

(2.2)

Proof. Let \( \lambda' = (\mu_1, \ldots, \mu_r, 1, \mu_{r+1}, \ldots, \mu_l) \). One has \( T_{n,a}^{-1} T_j T_{n,a} = T_{j+1} \) if \( a \leq j \leq n - 1 \). This follows from the identity in the braid group:

\[
(\sigma_1^{a-1} \cdots \sigma_{a-1}^{n-1}) \sigma_j (\sigma_{n-1} \cdots \sigma_a) = \sigma_{j+1},
\]

for \( a \leq j \leq n - 1 \), where the elements \( \sigma_i \) are the Artin generators of the braid group. From this, we obtain:

\[
m_{\lambda'} = T_{n,a}^{-1} m_{\mu} T_{n,a}.
\]

Note that \( \mathfrak{S}_{\lambda'} \subset \mathfrak{S}_{\lambda} \) and \( D(\alpha) = \sum q^{l(\alpha)} T_{\alpha} \), as where the sum is over the distinguished right coset representatives of \( \mathfrak{S}_{\lambda'} \) in \( \mathfrak{S}_{\lambda} \). Hence \( m_{\lambda} = m_{\lambda'} D(\alpha) \), and the result follows.
3. Proof of Theorem 1.1

In this section we give a proof of Theorem 1.1. The proof is based on Murphy’s fundamental paper [9], but we refer specifically to Mathas’ reworking of Murphy’s theory in [6, Ch. 3].

Let \( \lambda \) be a Young diagram of size \( n \). We recall the definition of a Garnir tableau of shape \( \lambda \), see [6, page 33]. Suppose both \((i, j)\) and \((i+1, j)\) are nodes of \( \lambda \). The \((i, j)\)-Garnir strip consists of all nodes of \( \lambda \) in row \( i \) weakly to the right of \((i, j)\) together with all nodes in row \((i+1)\) weakly to the left of \((i+1, j)\). Let \( a \) be the entry of \( t^k \) in the node \((i, j)\) and \( b \) the entry of \( t^k \) in the node \((i+1, j)\). The \((i, j)\)-Garnir tableau \( g = g_{(i,j)} \) is the (row standard) tableau which agrees with \( t^k \) outside the Garnir strip, and in which the numbers \( a, a+1, \ldots, b \) are entered from left to right in the Garnir strip, first in row \( i+1 \) and then in row \( i \).

Example 3.1. Let \( \lambda = (3, 2, 1) \) then

\[
\begin{align*}
\varnothing(1,1) &= \begin{array}{c} \text{ } \end{array}, & \varnothing(1,2) &= \begin{array}{c} \text{ } \end{array}, & \varnothing(2,1) &= \begin{array}{c} \text{ } \end{array}.
\end{align*}
\]

Fix a Young diagram of size \( n \) and a Garnir tableau \( g = g_{(i,j)} \) of shape \( \lambda \). All of the row standard \( \lambda \)-tableaux which agree with \( t^k \) outside the \((i, j)\)-Garnir strip, apart from \( g \), are in fact standard. Moreover, a standard \( \lambda \)-tableau \( \tau \) agrees with \( t^k \) outside the Garnir strip if and only if \( \tau \triangleright g \). Define

\[
h_{g} = m_{\lambda} T_{w(g)} + \sum_{\tau \triangleright g} m_{\lambda} T_{w(\tau)},
\]

where the sum is over standard \( \lambda \)-tableaux \( \tau \triangleright g \). It follows from [6, Lemma 3.13] and the cellularity of the Murphy basis that \( h_{g} \) is an element of \( M^{\lambda} \cap H_{n}^{\triangleright \lambda} \). Let \( M_{0}^{\lambda} \) be the right \( H_{n} \)-module generated by the elements \( h_{g} \), as \( g \) varies over all Garnir tableaux of shape \( \lambda \). Then we have \( M_{0}^{\lambda} \leq M^{\lambda} \cap H_{n}^{\triangleright \lambda} \).

Lemma 3.2. \( M_{0}^{\lambda} = M^{\lambda} \cap H_{n}^{\triangleright \lambda} \).

Proof. By the proof of [6, Lemma 3.15], if \( t \) is a row standard \( \lambda \)-tableau that is not standard, then \( m_{\lambda} T_{w(t)} = x + h \), where \( x \) is a linear combination of Murphy basis elements \( m_{\lambda} T_{w(v)} \) (with \( v \in \mathcal{T}^{\text{Std}}(\lambda) \)) and \( h \in M_{0}^{\lambda} \). Thus we have

\[
M^{\lambda} = \text{span}\{m_{\lambda} T_{w(v)} : v \in \mathcal{T}^{\text{Std}}(\lambda)\} + M_{0}^{\lambda}.
\]

It follows from this that \( M^{\lambda} \cap H_{n}^{\triangleright \lambda} \leq M_{0}^{\lambda} \).

Lemma 3.3. Let \( \lambda \) be a Young diagram of size \( n \). Let \( S \) be a subset of the set of row standard \( \lambda \)-tableaux and let \( I \) be a subset of \( \{1, 2, \ldots, n-1\} \), with the following properties:

1. If \( s \in S \) and \( t \) is a row standard \( \lambda \)-tableau with \( t \trianglerighteq s \), then \( t \in S \).
2. If \( s \in S \) and \( i \in I \), with \( i \) and \( i+1 \) in different rows of \( s \), then \( ss_{i} \in S \).

Let \( \mathcal{H}_{I} \) be the unital subalgebra of \( H_{n} \) generated by \( \{T_{i} : i \in I\} \). Then

\[
M = \text{span}\{m_{\lambda} T_{w(t)} : t \in S \cap \mathcal{T}^{\text{Std}}(\lambda)\} + (M^{\lambda} \cap H_{n}^{\triangleright \lambda})
\]

is a right \( \mathcal{H}_{I} \)-submodule of \( M^{\lambda} \).

Proof. Let \( t \in S \cap \mathcal{T}^{\text{Std}}(\lambda) \) and let \( i \in I \). We have to show that \( m_{\lambda} T_{w(t)} T_{i} \in M \). If \( i \) and \( i+1 \) are in the same row of \( t \), then \( m_{\lambda} T_{w(t)} T_{i} = q^{2} m_{\lambda} T_{w(t)} \). If \( i \) and \( i+1 \) are in different rows and different columns, then \( t' = ts_{i} \in S \cap \mathcal{T}^{\text{Std}}(\lambda) \) by hypothesis, and \( m_{\lambda} T_{w(t)} T_{i} \) is a linear combination of \( m_{\lambda} T_{w(t)} \) and \( m_{\lambda} T_{w(t')} \). Finally, if \( i \) and \( i+1 \) are in the same column of \( t \), then \( t' = ts_{i} \) is a row standard but not standard, \( t' \in S \) by hypothesis, and \( m_{\lambda} T_{w(t')} T_{i} = m_{\lambda} T_{w(t')} \). By [6, Lemma 3.15] and cellularity of the Murphy basis, \( m_{\lambda} T_{w(t')} = x + h \), where \( x \) is a linear combination of elements \( m_{\lambda} T_{w(v)} \) with \( v \) standard and \( v \trianglerighteq t' \), and \( h \in M^{\lambda} \cap H_{n}^{\triangleright \lambda} \). By hypothesis, each such \( v \) is in \( S \), so \( m_{\lambda} T_{w(t')} T_{i} = m_{\lambda} T_{w(t')} \in M \).

Let \( \lambda \) be a Young diagram of size \( n \) and let \( t \in \mathcal{T}^{\text{Std}}(\lambda) \). For any \( 1 \leq i \leq n \), let \( \text{row}(i) \) denote the row in which \( i \) appears in \( t \).

Corollary 3.4. Let \( \lambda \) be a Young diagram of size \( n \) and let \( r \geq 1 \). Then

\[
\text{span}\{m_{\lambda} T_{w(t)} : t \in \mathcal{T}^{\text{Std}}(\lambda) \text{ and } \text{row}(i) \geq r\} + (M^{\lambda} \cap H_{n}^{\triangleright \lambda})
\]

is a right \( \mathcal{H}_{n-1} \)-submodule of \( M^{\lambda} \).
Proof. In Lemma 3.3, take $S$ to be the set of row standard tableaux $s$ such that row$_s(n) \geq r$ and take $I = \{1, 2, \ldots, n - 2\}$.  

In the following discussion, $[s]$ denotes the shape of a standard tableau $s$.

**Corollary 3.5.** Let $\lambda$ be a Young diagram of size $n$ and let $\gamma$ be a node of $\lambda$. Let $m$ denote the entry of $\lambda^j$ in node $\gamma$. Let $H_{m,n}$ be the unital subalgebra of $H_n$ generated by $\{T_m, \ldots, T_{n-1}\}$. Then

$$\text{span}\{m_l T_{w(i)} : t \in T^\text{Std}(\lambda) \text{ and } [t \downarrow_{m-1}] = [\lambda^j \downarrow_{m-1}] + (M^j \cap H_n^{\text{ std}})\}$$

is a right $H_{m,n}$-submodule of $M^j$.

**Proof.** In Lemma 3.3, take $S$ to be the set of row standard tableaux $s$ such that $[s] \downarrow_{m-1} = [\lambda^j \downarrow_{m-1}]$ and take $I = \{m, \ldots, n - 1\}$.  

For the remainder of this section, we fix $n \geq 1$ and a Young diagram $\lambda$ of size $n$. Let $\alpha_1, \ldots, \alpha_p$ be the list of removable nodes of $\lambda$, listed from bottom to top, and let $\mu^{(j)} = \lambda \setminus \alpha_j$. Let $N_0 = (0)$ and for $1 \leq j \leq p$, let $N_j$ be the $R$-submodule of $\Delta_n^\lambda$, spanned by the basis elements $m^\lambda_t$ such that node$_{\alpha_j}(n) = [\alpha_1, \ldots, \alpha_j]$. Then we have

$$(0) = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_p = \text{Res}_{H_n}^{H_{n-1}}(\Delta_n^\lambda).$$

The explicit form of the assertion of Theorem 1.1 is that the $N_j$ are $H_{n-1}$-submodules of $\text{Res}_{H_n}^{H_{n-1}}(\Delta_n^\lambda)$ and $N_j/N_{j-1} \cong \Delta_n^\mu$ for $1 \leq j \leq p$. The isomorphism is determined by

$$m^\mu$$

(3.3)

**Corollary 3.6.** For each $j$, $N_j$ is a right $H_{n-1}$-submodule of $\Delta_n^\lambda$.

**Proof.** Immediate from Corollary 3.4.

Our goal is to show that $N_j/N_{j-1} \cong \Delta_n^\mu$ as $H_{n-1}$-modules, for each $j \geq 1$.

We pick one removable node $\alpha = \alpha_k$ of $\lambda$, and write $\mu = \lambda \setminus \alpha$. Let $D(\alpha)$ be defined as in the statement of Lemma 2.2. For a $H_n$ module $M$, we will write $\text{Res}(M)$ for $\text{Res}_{H_n}^{H_{n-1}}(M)$. Because of Equation (2.2), we have an $H_{n-1}$-module homomorphism from $M^\mu$ to $\text{Res}(M^\lambda)$ defined by

$$\varphi_0 : m_\mu h \mapsto m^\lambda h_{\alpha_1, \ldots, \alpha_\mu}h.$$

If $s \in T(\mu)$ is a $\mu$-tableau, then we have

$$\varphi_0(m_\mu T_{w(s)}) = m^\lambda T_{w(s)} = m^\lambda T_{w(s \alpha_\mu)}$$

by Lemma 2.1.

Let $\varphi$ be the composite homomorphism

$$\varphi : M^\mu \to \text{Res}(M^\lambda) \to \text{Res}(\Delta_n^\lambda)/(\text{Res}(\Delta_n^\mu)/N_{k-1})$$

where the latter two maps are canonical quotients maps. We claim that $\varphi$ factors through $\Delta_n^\mu$. Because of Lemma 3.2, it suffices to show that if $g_0$ is a Garnir tableau of shape $\mu$, then $\varphi(h_{g_0}) = 0$.

Let $g_0$ be the $(i, j)$-Garnir tableau of shape $\mu$ and let $g$ be the $(i, j)$-Garnir tableau of shape $\lambda$. There are two cases to consider:

- **Case 1.** The node $\alpha$ is not in the Garnir strip of $g$. In this case, there is a one to one correspondence between row standard tableaux $\tau_0$ of shape $\mu$ such that $\tau_0 \supseteq g_0$, and row standard tableaux $\tau$ of shape $\lambda$ such that $\tau \supseteq \alpha$, given by

$$\tau(n, n-1, \ldots, m) = \tau_0 \cup \alpha,$$

where $m = g(\alpha) = \lambda^j(\alpha)$. We claim that (when $\tau$ and $\tau_0$ are so related)

$$T_{w(\tau)} = T_{w(\tau_0)}T_{m,n}.$$  

(3.5)
and (3.5) follows. Now we have
\[
\varphi_0(h_{\omega_0}) = m_\lambda \left( \sum_{\tau \in \mathcal{G}_0} T_{w(\tau),\omega} \right) = m_\lambda \left( \sum_{\tau \in \mathcal{A}} T_{w(\tau)} \right) T_{m,n} = h_T T_{m,n}.
\]
(3.6)
Thus \( \varphi_0(h_{\omega_0}) \in M^\lambda \cap \mathcal{H}_n^{\lambda\lambda}, \) so \( \varphi(h_{\omega_0}) = 0. \)

Case 2. The node \( \alpha \) is in the Garnir strip of \( g \). Let \( m = g(\alpha) \), the largest entry in the Garnir strip of \( g \). The row standard tableaux \( \tau \) such that \( \tau \geq g \) either have node \( (m) = \alpha \) or node \( (m) = (i + 1, j) \). Let \( \mathcal{A} \) be the set of \( \tau \) such that node \( (m) = \alpha \) and let \( \mathcal{B} \) be the set of \( \tau \) such that node \( (m) = (i + 1, j) \). The set \( \mathcal{A} \) is in one to one correspondence with the set of row standard tableaux \( \tau_0 \) of shape \( \mu \) with \( \tau_0 \geq \omega_0 \); the correspondence is given by
\[
\tau(n, n-1, \ldots, m) = \tau_0 \cup \alpha.
\]
For \( \tau \) and \( \tau_0 \) so related we have
\[
T_{w(\tau),\omega} = T_{w(\tau)} T_{m,n}.
\]
Thus we have
\[
\varphi_0(h_{\omega_0}) = m_\lambda \left( \sum_{\tau \in \mathcal{G}_0} T_{w(\tau),\omega} \right) = m_\lambda \left( \sum_{\tau \in \mathcal{A}} T_{w(\tau)} \right) T_{m,n}
\]
\[
= h_T T_{m,n} - m_\lambda \left( \sum_{\tau \in \mathcal{B}} T_{w(\tau)} \right) T_{m,n}
\]
(3.7)
If \( \tau \in B \), then \([\tau \downarrow_{m-1}] = [\lambda^\downarrow_{m-1}] \). By Corollary 3.5, for \( \tau \in B \), \( m_\lambda T_{w(\tau)} T_{m,n} = x + h \), where \( h \in M^\lambda \cap \mathcal{H}_n^{\lambda\lambda} \) and \( x \) is a linear combination of Murphy basis elements \( m_\lambda T_{w(\varnothing)} \) with \( v \in \mathcal{T}^{\text{Std}}(\lambda) \) and \([\nu \downarrow_{m-1}] = [\lambda^\downarrow_{m-1}] \). For such \( v \), row\( v(n) \geq i + 1 \), so the node of \( n \) in \( v \) is one of \( \alpha_1, \ldots, \alpha_{k-1} \). Hence, \( \varphi_0(h_{\omega_0}) \) is contained in
\[
M^\lambda \cap \mathcal{H}_n^{\lambda\lambda} + \text{span}\{ m_\lambda T_{w(\varnothing)} : v \in \mathcal{T}^{\text{Std}}(\lambda) \text{ and } \text{node}_v(n) \in [\alpha_1, \ldots, \alpha_{k-1}] \}.
\]
It follows that \( \varphi(h_{\omega_0}) = 0. \) This completes the proof that \( \varphi \) factors through \( \Delta^{\mu}_{\mathcal{H}_{n-1}} \).

The map \( \varphi : \Delta^{\mu}_{\mathcal{H}_{n-1}} \to \text{Res}(\Delta^\lambda_{\mathcal{H}_n})/N_{k-1} \) determined by \( \varphi \) satisfies
\[
\varphi(m^\mu_{\lambda,\mu}) = m^\lambda_{\mu,\mu} + N_{k-1},
\]
so has range \( N_k/N_{k-1} \), and is an isomorphism onto its range. This completes the proof of Theorem 1.1.

References

[1] Richard Dipper and Gordon James, *Representations of Hecke algebras of general linear groups*, Proc. London Math. Soc. (3) 52 (1986), no. 1, 20–52. MR812444
[2] ______, *Blocks and idempotents of Hecke algebras of general linear groups*, Proc. London Math. Soc. (3) 54 (1987), no. 1, 57–82. MR872250 (89m:20084)
[3] J. I. Graham and G. I. Lehrer, *Cellular algebras*, Invent. Math. 123 (1996), no. 1, 1–34. MR1376244
[4] G. D. James, *The representation theory of the symmetric groups*, Lecture Notes in Mathematics, vol. 682, Springer, Berlin, 1978. MR513828 (80g:20019)
[5] Thomas Jost, *Morita equivalence for blocks of Hecke algebras of symmetric groups*, J. Algebra 194 (1997), no. 1, 201–223. MR1461487
[6] Andrew Mathas, *Iwahori-Hecke algebras and Schur algebras of the symmetric group*, University Lecture Series, vol. 15, American Mathematical Society, Providence, RI, 1999. MR1711316
[7] ______, *A Specht filtration of an induced Specht module*, J. Algebra 322 (2009), no. 3, 893–902. MR2531227
[8] ______, private communication, (2015).
[9] G. E. Murphy, *The representations of Hecke algebras of type \( A_n \)*, J. Algebra 173 (1995), no. 1, 97–121. MR1327362

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