Pseudo-Poisson Nijenhuis manifolds

Tomoya Nakamura *
email: x-haze@ruri.waseda.jp

March 29, 2017

Abstract

We introduce the notion of pseudo-Poisson Nijenhuis manifolds. These manifolds are generalizations of Poisson Nijenhuis manifolds by Magri and Morosi [11]. We show that any pseudo-Poisson Nijenhuis manifold has an associated quasi-Lie bialgebroid as in the case of Poisson quasi-Nijenhuis manifolds by Stiénon and Xu [13]. Hence, since a quasi-Lie bialgebroid has an associated Courant algebroid, we have new materials to construct Courant algebroids. In the “nondegenerate” case, we show that the conditions of pseudo-Poisson Nijenhuis structures can be reduced. Therefore we can provide lots of non-trivial examples of pseudo-Poisson Nijenhuis manifolds.

1 Introduction

Kosmann-Schwarzbach [5] showed that there is a one-to-one correspondence between the Poisson Nijenhuis manifolds \((M, \pi, N)\) and the Lie bialgebroids \(((TM)_N, (T^*M)_\pi)\), where \((TM)_N\) is a Lie algebroid deformed by the Nijenhuis structure \(N\) and \((T^*M)_\pi\) is the cotangent bundle equipped with the standard Lie algebroid structure induced by the Poisson structure \(\pi\). On the other hand, Stiénon and Xu [13] showed that a Poisson quasi-Nijenhuis manifold \((M, \pi, N, \phi)\) corresponded to a quasi-Lie bialgebroid \(((T^*M)_\pi, d_N, \phi)\) introduced by Roytenberg [12]. A pseudo-Poisson Nijenhuis manifold \((M, \pi, N, \Phi)\) corresponds to “the opposite side” of \(((T^*M)_\pi, d_N, \phi)\), that is, a quasi-Lie bialgebroid \(((TM)_N, d_\pi, \Phi)\). Here \(d_N\) and \(d_\pi\) are operators of \(\Gamma(\Lambda^*TM)\) and \(\Gamma(\Lambda^*TM)\), respectively, and a pseudo-Poisson Nijenhuis structure on \(M\) is a triple consisting of a 2-vector field \(\pi\) which does not need to be

*Department of Mathematics, Waseda University, 3-4-1, Okubo, Shinjuku-ku, Tokyo, Japan
a Poisson structure, a Nijenhuis structure $N$ “compatible” with $\pi$ and a 3-vector field $\Phi$ with conditions

(i) $d_\pi \Phi = 0$,
(ii) $\frac{1}{2} \iota_{\alpha \wedge \beta} [\pi, \pi] = N \iota_{\alpha \wedge \beta} \Phi$,
(iii) $N \iota_{\alpha \wedge \beta} \mathcal{L}_X \Phi - \iota_{\alpha \wedge \beta} \mathcal{L}_{NX} \Phi - \iota_{(\mathcal{L}_X N^*) (\alpha \wedge \beta)} \Phi = 0$

for any $X$ in $\mathfrak{X}(M)$ and $\alpha$ and $\beta$ in $\Omega^1(M)$.

Furthermore, since quasi-Lie bialgebroids (of course, Lie bialgebroids also) construct Courant algebroids, we can obtain a new Courant algebroid from a pseudo-Poisson Nijenhuis manifold similar to a Poisson Nijenhuis and Poisson quasi-Nijenhuis manifold. In other words, we can say that a pseudo-Poisson Nijenhuis structure is a new material for constructing a Courant algebroid structure of $TM \oplus T^*M$. Therefore a pseudo-Poisson Nijenhuis structure on $M$ complements the bottom left of below the correspondence table:

| a Courant algebroid structure [9] of $TM \oplus T^*M$ |
|------------------------------------------------------|
| a quasi-Lie bialgebroid [12]                        |
| $((T^M)_N, d_\pi, \Phi)$                           |
| a Lie bialgebroid [10]                              |
| $((T^M)_N, (T^*M)_\pi)$                            |
| a quasi-Lie bialgebroid [12]                        |
| $(T^*M)_\pi, d_N, \phi$                            |
| a pseudo-Poisson Nijenhuis [11]                     |
| $(\pi, N, \Phi)$                                   |
| $\pi$ : a 2-vector field                            |
| $N$ : a Nijenhuis                                   |
| $\Phi$ : a 3-vector field                           |
| a Poisson Nijenhuis [11]                            |
| $(\pi, N)$                                         |
| $\pi$ : a Poisson                                   |
| $N$ : a Nijenhuis                                   |
| a Poisson quasi-Nijenhuis [13]                      |
| $(\pi, N, \phi)$                                   |
| $\pi$ : a Poisson                                   |
| $\phi$ : a 3-form                                   |
| $N$ : a bundle map                                   |

We also obtain the notion of pseudo-Poisson Nijenhuis Lie algebroids, which is the natural generalization of pseudo-Poisson Nijenhuis manifolds. This is also analogous to Poisson Nijenhuis Lie algebroids [4] and Poisson quasi-Nijenhuis Lie algebroids [2].

This paper is constructed as follows. Section [2] consists of preliminaries to define pseudo-Poisson Nijenhuis manifolds and to describe relations between those manifolds and quasi-Lie bialgebroids or Courant algebroids. In section [3] we define pseudo-Poisson Nijenhuis manifolds and show that there is a one-to-one correspondence between a pseudo-Poisson Nijenhuis manifold $(M, \pi, N, \Phi)$ and a quasi-Lie bialgebroid $((T^M)_N, d_\pi, \Phi)$, which is the main theorem in this paper. Moreover we prove that we can reduce the condition (iii) of pseudo-Poisson Nijenhuis structures under the assumption that a 2-vector field $\pi$ is nondegenerate. It seems very difficult due to the
condition (iii) to find non-trivial pseudo-Poisson Nijenhuis structures but we can construct many “nondegenerate” examples by this theorem. As such examples, we give ones made from symplectic Nijenhuis structures [14] and generalized almost complex structures [3].

2 Preliminaries

We begin by recalling the definitions of Courant algebroids.

**Definition 1** ([9]). A Courant algebroid is a vector bundle $E \rightarrow M$ equipped with a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ (called the pairing) on the bundle, skew-symmetric bracket $[\cdot, \cdot]$ on $\Gamma(E)$ and a bundle map $\rho : E \rightarrow TM$ such that the following properties are satisfied: for any $e, e_1, e_2, e_3$ in $\Gamma(E)$, any $f$ and $g$ in $C^\infty(M),$

\[ \sum \text{Cycl}(e_1, e_2, e_3) [\langle e_1, e_2 \rangle, e_3] = \frac{1}{3} \sum \text{Cycl}(e_1, e_2, e_3) \mathcal{D}(\langle e_1, e_2 \rangle, e_3), \]

\[ \rho([e_1, e_2]) = [\rho(e_1), \rho(e_2)], \]

\[ [e_1, fe_2] = f[e_1, e_2] + (\rho(e_1)f) e_2 - \langle e_1, e_2 \rangle \mathcal{D} f, \]

\[ \rho \circ \mathcal{D} = 0, \text{ i.e., } \langle \mathcal{D} f, \mathcal{D} g \rangle = 0, \]

\[ \rho(e)\langle e_1, e_2 \rangle = \langle [e, e_1] + \mathcal{D} \langle e, e_1 \rangle, e_2 \rangle + \langle e_1, [e, e_2] + \mathcal{D} \langle e, e_2 \rangle \rangle, \]

where $\mathcal{D} : C^\infty(M) \rightarrow \Gamma(E)$ is the smooth map defined by

\[ \langle \mathcal{D} f, e \rangle = \frac{1}{2} \rho(e) f. \]

The map $\rho$ and the operator $[\cdot, \cdot]$ are called an anchor map and a Courant bracket, respectively.

A Courant algebroid is not a Lie algebroid since the Jacobi identity is not satisfied due to (i). The following example is fundamental.

**Example 1** ([9]). The direct sum $TM \oplus T^*M$ on a manifold $M$ is a Courant algebroid. Here the anchor map, the pairing and the Courant bracket are given by

\[ \rho(X + \xi) = X, \]

\[ \langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\langle \xi, Y \rangle + < \eta, X >), \]

\[ [X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi + \frac{1}{2} d(\langle \xi, Y \rangle - < \eta, X >), \]

where $X$ and $Y$ are in $\mathfrak{X}(M)$ and $\xi$ and $\eta$ are in $\Omega^1(M)$. This is called the standard Courant algebroid.
Next we shall recall the definition of quasi-Lie bialgebroids.

**Definition 2** ([12]). A quasi-Lie bialgebroid is a Lie algebroid \((A, [\cdot, \cdot]_A, a)\) equipped with a degree-one derivation \(d_s\) of the Gerstenhaber algebra \((\Gamma(\Lambda^* A), \wedge, [\cdot, \cdot]_A)\) and a 3-section of \(A, \phi_A\) in \(\Gamma(\Lambda^3 A)\) such that

\[
\begin{align*}
  d_s^2 &= [\phi_A, \cdot]_A, \\
  d_s \phi_A &= 0.
\end{align*}
\]

If the 3-section \(\phi_A\) is equal to 0, the quasi-Lie bialgebroid \((A, d_s, \phi_A)\) is just a Lie bialgebroid.

**Example 2.** Let \((A, d_s, \phi_A)\) be a quasi-Lie bialgebroid, where \(A = (A, [\cdot, \cdot]_A, a)\), and \(d_A : \Gamma(\Lambda^* A^*) \to \Gamma(\Lambda^{s+1} A^*)\) be the Lie algebroid derivative of \(A\). Its double \(E = A \oplus A^*\) has naturally a Courant algebroid structure. Namely, it is equipped with an anchor map \(\rho\), a pairing \(\langle \cdot, \cdot \rangle\) and a Courant bracket \([\cdot, \cdot]\) given by the following: for any \(X, Y\) in \(\Gamma(A)\), any \(\xi, \eta\) in \(\Gamma(\Lambda^3 A)\),

\[
\begin{align*}
  \rho(X + \xi) &= a(X) + a_s(\xi), \\
  \langle X + \xi, Y + \eta \rangle &= \frac{1}{2}(\langle \xi, Y \rangle + \langle \eta, X \rangle), \\
  [X, Y] &= [X, Y]_A \\
  [\xi, \eta] &= [\xi, \eta]_{A^*} + \phi_A(X, Y, \cdot) \\
  [X, \xi] &= (\iota_X d_A \xi + \frac{1}{2} d_A < \xi, X >) \\
  &\quad - (\iota_\xi d_s X + \frac{1}{2} d_s < \xi, X >)
\end{align*}
\]

where the map \(a_s : A^* \to TM\) and the bracket \([\cdot, \cdot]_{A^*}\) are defined by

\[
\begin{align*}
  a_s(\xi) f &= < \xi, d_s f >, \\
  < [\xi, \eta]_{A^*}, X > &= a_s(\xi) < \eta, Y > - a_s(\eta) < \xi, X > - (d_s X)(\xi, \eta),
\end{align*}
\]

for \(\xi, \eta \in \Gamma(A^*),\ X, Y \in \mathfrak{X}(M)\) and \(f \in C^\infty(M)\), respectively.

Taking \(\phi_A = 0\), we obtain the Courant algebroid structure of a double of a Lie bialgebroid in [9].

### 3 Pseudo-Poisson Nijenhuis manifolds

Let \(M\) be a \(C^\infty\)-manifold, \(\pi\) a 2-vector field and \(N : TM \to TM\) a bundle map over \(M\).
Definition 3 ([5]). The 2-vector field \( \pi \) on \( M \) and the bundle map \( N \) over \( M \) are compatible [6] if those satisfy

\[
N \circ \pi^\sharp = \pi^\sharp \circ N^*, 
\]

\[
C^N_\pi = 0, 
\]

where for any \( \alpha \) and \( \beta \) in \( \Omega^1(M) \),

\[
C^N_\pi(\alpha, \beta) := [\alpha, \beta]_{N\pi^\sharp} - [\alpha, \beta]^N_{\pi}, 
\]

\[
[\alpha, \beta]_{N\pi^\sharp} := \mathcal{L}_{N\pi\sharp}\alpha\beta - \mathcal{L}_{N\pi\sharp}\beta\alpha - d < N\pi^\sharp\alpha, \beta >, 
\]

\[
[\alpha, \beta]^{N*}_{\pi} := [N^*\alpha, \beta]_{\pi} + [\alpha, N^*\beta]_{\pi} - N^*[\alpha, \beta]_{\pi} \quad \text{and} 
\]

\[
[\alpha, \beta]_{\pi} := \mathcal{L}_{\pi\sharp}\alpha\beta - \mathcal{L}_{\pi\sharp}\beta\alpha - d < \pi^\sharp\alpha, \beta > . 
\]

The bundle map \( N \) is a Nijenhuis structure on \( M \) if the Nijenhuis torsion \( T_N(X, Y) := [NX, NY] - N[X, Y]_N \) vanishes for any \( X \) and \( Y \) in \( \mathfrak{X}(M) \), where

\[
[X, Y]_N := [NX, NY] + [X, NY] - N[X, Y]. 
\]

Remark 1. (i) \((T^*M)_\pi = (T^*M, [\cdot, \cdot]_{\pi^\sharp}, \pi^\sharp)\) is a Lie algebroid if and only if a 2-vector field \( \pi \) on \( M \) is Poisson. We define the derivation \( d_\pi \) by

\[
(d_\pi \alpha)(X_0, \ldots, X_n) = \sum_{i=0}^{n} (-1)^i a(X_i)(\alpha(X_0, \ldots, \hat{X}_i, \ldots, X_n)) 
\]

\[
+ \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j]_A, X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_n), 
\]

where \( \alpha \) is in \( \Gamma(\Lambda^*A^*) \) and \( X_i \) is in \( \Gamma(A) \). The operator satisfies \( d_\pi D = [\pi, D] \) for any \( D \) in \( \Gamma(\Lambda^*TM) \).

(ii) \((TM)_N = (TM, [\cdot, \cdot]_N, N)\) is a Lie algebroid if and only if the map \( N \) is a Nijenhuis structure. We define the derivation \( d_N \) by the same formula as (12).

Let \((\pi, N)\) be a compatible pair and set \( \pi_N(\alpha, \beta) := < N\pi^\sharp\alpha, \beta > \). Then it follows from (6) that \( \pi_N \) is a 2-vector field on \( M \). We denote the bracket \([\cdot, \cdot]_{N\pi^\sharp}\) by \([\cdot, \cdot]_{\pi_N}\). If \( \pi \) is a Poisson structure on \( M \), so is \( \pi_N \).

As an analogy to the definition of Poisson quasi-Nijenhuis manifolds [13], we have the following.
Definition 4. Let $M$ be a $C^\infty$-manifold, $\pi$ a 2-vector field on $M$, a bundle map $N : TM \to TM$ over $M$ a Nijenhuis structure compatible with $\pi$, and $\Phi$ a 3-vector field on $M$. Then a triple $(\pi, N, \Phi)$ is a pseudo-Poisson Nijenhuis structure on $M$ if the following conditions hold:

(i) $[\pi, \Phi] = 0$,  \hspace{1cm} (13)
(ii) $\frac{1}{2} \iota_{\alpha \wedge \beta} [\pi, \pi] = N \iota_{\alpha \wedge \beta} \Phi$,  \hspace{1cm} (14)
(iii) $N \iota_{\alpha \wedge \beta} \mathcal{L}_X \Phi - \iota_{\alpha \wedge \beta} \mathcal{L}_{NX} \Phi - \iota_{(\mathcal{L}_X N^*)}(\alpha \wedge \beta) \Phi = 0$, \hspace{1cm} (15)

where $X$ is in $\mathfrak{x}(M)$ and $\alpha$ and $\beta$ are in $\Omega^1(M)$. The quadruple $(M, \pi, N, \Phi)$ is called a pseudo-Poisson Nijenhuis manifold.

Remark 2. The reason why we use not “quasi-” but “pseudo-” is to avoid confusion with another notion quasi-Poisson manifold in [1] [7].

Now we describe the main theorem in this paper. This is one of the fundamental properties of pseudo-Poisson Nijenhuis manifolds. A similar result for Poisson quasi-Nijenhuis manifolds is also known [13].

Theorem 3.1. Let $M$ be a $C^\infty$-manifold, $\pi$ a 2-vector field on $M$, $N$ a Nijenhuis structure on $M$ compatible with $\pi$ and $\Phi$ a 3-vector field on $M$. Then a quadruple $(M, \pi, N, \Phi)$ is a pseudo-Poisson Nijenhuis manifold if and only if $((TM)_N, d_\pi, \Phi)$ is a quasi-Lie bialgebroid.

Proof. We assume that a quadruple $(M, \pi, N, \Phi)$ is a pseudo-Poisson Nijenhuis manifold. To prove that $((TM)_N, d_\pi, \Phi)$ is a quasi-Lie bialgebroid, we need to confirm the following three conditions: i) $d_\pi$ is a degree-one derivation of the Gerstenhaber algebra $(\Gamma(A^*TM), \wedge, [\cdot, \cdot]_N)$, ii) $d^2_\pi = [\Phi, \cdot]_N$ and iii) $d_\pi \Phi = 0$.

i) is equivalent to the following: for any $X$ and $Y$ in $\mathfrak{x}(M)$,

$$d_\pi [X, Y]_N = [d_\pi X, Y]_N + (-1)^{\deg X + 1}[X, d_\pi Y]_N$$ \hspace{1cm} (16)

holds. On the other hand, we see that for any $\alpha$ and $\beta$ in $\Omega^1(M)$,

$$d_N [\alpha, \beta]_\pi = [d_N \alpha, \beta]_\pi + (-1)^{\deg \alpha + 1}[\alpha, d_N \beta]_\pi$$ \hspace{1cm} (17)

holds due to the proof of (i) $\Rightarrow$ (iii) of Proposition 3.2 in [5]. In addition, Theorem 3.10 in [9] essentially means that (16) holds if and only if (17) holds. In fact, Proposition 3.6, Proposition 3.8 and Lemma 3.11 in [9] are used to prove this theorem. However it is only used that $\pi$ is a 2-vector field in the proofs of these propositions. The induced bundle map $\tilde{\pi}_P^\sharp$ in Proposition
3.6 in [9] is $\pi^2_N := N \circ \pi^2$ in our case and we use only the skew-symmetry of $\pi^2_N$ to prove (16). Since the skew-symmetry of $\pi^2_N$ holds because of the compatibility of $\pi$ and $N$, we obtain (16). Therefore i) holds.

Next, we show ii). For any $f$ in $C^\infty(M)$ and any $\alpha$ and $\beta$ in $\Omega^1(M)$, we compute

$$(d^2_\pi f)(\alpha, \beta) = [[\pi, f]](\alpha, \beta) = \frac{1}{2}[[\pi, \pi], f](\alpha, \beta)$$

$$= \frac{1}{2}t_{\pi f}[\pi, \pi](\alpha, \beta) = \frac{1}{2}[\pi, \pi](df, \alpha, \beta)$$

$$= \frac{1}{2}[\pi, \pi](\alpha, \beta, df) = \frac{1}{2}t_{\alpha \wedge \beta}[\pi, \pi](df),$$

where the second equality follows from the graded Jacobi identity and the fact is used that $[D, f] = (-1)^{k+1}t_{\pi f}D$ for any $D$ in $\Gamma(\Lambda^k TM)$ in the third equality. On the other hand, we have

$$[\Phi, f]_N(\alpha, \beta) = t_{N^* df}\Phi(\alpha, \beta) + \Phi(N^* df, \alpha, \beta)$$

$$= \Phi(\alpha, \beta, N^* df) = \rho_{\alpha \wedge \beta}\Phi(N^* df),$$

where we use the fact that $[D, f]_N = (-1)^{k+1}t_{N^* df}D$ for any $D$ in $\Gamma(\Lambda^k TM)$ in the first step. Therefore it follows that $d^2_\pi = [\Phi, \cdot]_N$ on $C^\infty(M)$ if and only if the equality (14) holds as a linear map on the exact 1-forms. By $C^\infty(M)$-linearity of (14) and the fact that the exact 1-forms generate locally the 1-forms as a $C^\infty(M)$-module, the equality (14) holds on $\Omega^1(M)$ if and only if $d^2_\pi = [\Phi, \cdot]_N$ holds on $C^\infty(M)$.

Next, for any $X$ in $\mathfrak{X}(M)$, any $\alpha$, $\beta$ and $\gamma$ in $\Omega^1(M)$, we obtain

$$(d^2_\pi X)(\alpha, \beta, \gamma) = [[\pi, X]](\alpha, \beta, \gamma) = \frac{1}{2}[[\pi, \pi], X](\alpha, \beta, \gamma)$$

$$= \frac{1}{2}[X, [\pi, \pi]](\alpha, \beta, \gamma) - \frac{1}{2}(\mathcal{L}_X[\pi, \pi])(\alpha, \beta, \gamma)$$

$$= \frac{1}{2}\{\mathcal{L}_X([[\pi, \pi]](\alpha, \beta, \gamma)) - [\pi, \pi](\mathcal{L}_X\alpha, \beta, \gamma)$$

$$- [\pi, \pi](\alpha, \mathcal{L}_X\beta, \gamma) - [\pi, \pi](\alpha, \beta, \mathcal{L}_X \gamma)\}$$

$$= -\mathcal{L}_X\left(\frac{1}{2}t_{\alpha \wedge \beta}[\pi, \pi](\gamma) + \frac{1}{2}t_{\mathcal{L}_X\alpha \wedge \beta}[\pi, \pi](\gamma)$$

$$+ \frac{1}{2}t_{\alpha \wedge \mathcal{L}_X \beta}[\pi, \pi](\gamma)$$

$$+ \frac{1}{2}t_{\alpha \wedge \beta}[\pi, \pi](\mathcal{L}_X \gamma) + \frac{1}{2}t_{\alpha \wedge \beta}[\pi, \pi](\mathcal{L}_X \gamma)\}$$

$$= -\mathcal{L}_X((N\rho_{\alpha \wedge \beta}\Phi)(\gamma)) + (N\rho_{\mathcal{L}_X\alpha \wedge \beta}\Phi)(\gamma)$$

7
where the second equality follows from the graded Jacobi identity and we use the equality \([14]\) in the seventh equality. On the other hand, we obtain

\[
\left[ \Phi, X \right]_N (\alpha, \beta, \gamma) = -\left[ X, \Phi \right]_N (\alpha, \beta, \gamma) = -\left( \mathfrak{L}^N \Phi \right)(\alpha, \beta, \gamma)
\]

\[
= -\mathfrak{L}^N_X (\Phi(\alpha, \beta, \gamma)) + \Phi(\mathfrak{L}^N_X \alpha, \beta, \gamma)
\]

\[
+ \Phi(\alpha, \mathfrak{L}^N_X \beta, \gamma) + \Phi(\alpha, \beta, \mathfrak{L}^N_X \gamma)
\]

\[
= -\mathfrak{L}^N_{\mathfrak{L}X}(\Phi(\alpha, \beta, \gamma))
\]

\[
+ \Phi(\mathfrak{L}^N_{\mathfrak{L}X} \alpha - (\mathfrak{L} X^*) \alpha, \beta, \gamma)
\]

\[
+ \Phi(\alpha, \mathfrak{L}^N_{\mathfrak{L}X} \beta - (\mathfrak{L} X^*) \beta, \gamma)
\]

\[
+ \Phi(\alpha, \beta, \mathfrak{L}^N_{\mathfrak{L}X} \gamma - (\mathfrak{L} X^*) \gamma)
\]

\[
= -\mathfrak{L}^N_{\mathfrak{L}X}(\Phi(\alpha, \beta, \gamma)) + \Phi(\mathfrak{L}^N_{\mathfrak{L}X} \alpha, \beta, \gamma)
\]

\[
+ \Phi(\alpha, \mathfrak{L}^N_{\mathfrak{L}X} \beta, \gamma) + \Phi(\alpha, \beta, \mathfrak{L}^N_{\mathfrak{L}X} \gamma)
\]

\[
- \Phi((\mathfrak{L} X^*) \alpha, \beta, \gamma) - \Phi(\alpha, (\mathfrak{L} X^*) \beta, \gamma)
\]

\[
- \Phi(\alpha, \beta, (\mathfrak{L} X^*) \gamma)
\]

\[
= -\left( \mathfrak{L}^N \Phi \right)(\alpha, \beta, \gamma) - \Phi((\mathfrak{L} X^*) \alpha, \beta, \gamma)
\]

\[
- \Phi(\alpha, \mathfrak{L}^N X^* \beta, \gamma) - \Phi(\alpha, \beta, (\mathfrak{L} X^*) \gamma)
\]

where \(\mathfrak{L}^N\) is the Lie derivative of \((TM, [\cdot , \cdot ]_N, N)\) defined by the Cartan formula on \(\Omega^*(M)\) and we use the property that \(\mathfrak{L}^N_X \alpha = \mathfrak{L}^N_{\mathfrak{L}X} \alpha - (\mathfrak{L} X^*) \alpha\) for any \(X\) in \(\mathfrak{X}(M)\) and any \(\alpha\) in \(\Omega^1(M)\). Therefore, we obtain

\[
(d^2 - [\Phi, X])_N (\alpha, \beta, \gamma) = -\left( \mathfrak{L}^N_X \Phi \right)(\alpha, \beta, N^* \gamma) - \Phi(\alpha, \beta, (\mathfrak{L} X^*) \gamma)
\]

\[
+ \Phi(\alpha, \mathfrak{L}^N X^* \beta, \gamma) + \Phi(\alpha, \beta, (\mathfrak{L} X^*) \gamma)
\]

\[
+ \Phi(\alpha, \mathfrak{L}^N X^* \beta, \gamma) + \Phi(\alpha, \beta, (\mathfrak{L} X^*) \gamma)
\]
\[\begin{align*}
&= - (\mathcal{L}_X \Phi)(\alpha, \beta, N^* \gamma) + (\mathcal{L}_{NX} \Phi)(\alpha, \beta, \gamma) \\
&\quad + \Phi((\mathcal{L}_X N^*) \alpha, \beta, \gamma) + \Phi(\alpha, (\mathcal{L}_X N^*) \beta, \gamma) \\
&= - (N \iota_{\alpha \wedge \beta} \mathcal{L}_X \Phi - \iota_{\alpha \wedge \beta} \mathcal{L}_{NX} \Phi - \iota_{(\mathcal{L}_X N^*) \alpha \wedge \beta} \Phi)(\gamma).
\end{align*}\]

Hence it follows that \(d^2_\pi = [\Phi, \cdot]_N\) on \(\mathcal{X}(M)\) if and only if the equality (15) holds.

Since \(d^2_\pi\) and \([\Phi, \cdot]_N\) are derivatives on \((\Gamma(\Lambda^* T M), \wedge)\), it follows that \(d^2_\pi = [\Phi, \cdot]_N\) on \(C^\infty(M) \oplus \mathcal{X}(M)\) if and only if \(d^2_\pi = [\Phi, \cdot]_N\) on \(\Gamma(\Lambda^* T M)\).

Finally, iii) is equivalent to (13) because of Remark 1. Therefore the proof has been completed.

By the theorem, we have the following result of Kosmann-Schwarzbach [5].

**Corollary 3.2.** Under the same assumption as Theorem 3.1, the triple \((M, \pi, N)\) is a Poisson Nijenhuis manifold if and only if \(((TM)_N, d_\pi)\) is a Lie bialgebroid.

As in the case of Poisson quasi-Nijenhuis Lie algebroids [2], we can consider a straightforward generalization of pseudo-Poisson Nijenhuis manifolds.

**Definition 5.** A pseudo-Poisson Nijenhuis Lie algebroid \((A, \pi, N, \Phi)\) is a Lie algebroid \(A\) equipped with a 2-section \(\pi\) in \(\Gamma(\Lambda^2 A)\), a Nijenhuis structure \(N : A \to A\) compatible with \(\pi\) in the sense of Definition 3 and a 3-section \(\Phi\) in \(\Gamma(\Lambda^3 A)\) satisfying the conditions (13), (14) and (15) replaced \([\cdot, \cdot]\) and \(\mathcal{L}\) with \([\cdot, \cdot]_A\) and \(\mathcal{L}^A\), respectively.

**Theorem 3.3.** If a quadruple \((A, \pi, N, \Phi)\) is a pseudo-Poisson Nijenhuis Lie algebroid, then \((A_N, d_\pi, \Phi)\) is a quasi-Lie bialgebroid, where \(A_N\) is a Lie algebroid deformed by the Nijenhuis structure \(N\).

Now we show three simple and important examples of pseudo-Poisson Nijenhuis manifolds.

**Example 3.** A triple \((\pi, N, \Phi)\), where \(\Phi = 0\), is a pseudo-Poisson Nijenhuis structure if \((\pi, N)\) is a Poisson-Nijenhuis structure.

**Example 4.** Let \((M, \pi)\) be a Poisson manifold and set \(N = 0\). For any \(d_\pi\)-closed 3-vector field \(\Phi\), the triple \((\pi, N, \Phi)\) is a pseudo-Poisson Nijenhuis structure. Therefore, by Theorem 3.1 and Example 2 \(((TM)_N, d_\pi, \Phi)\) is...
a quasi-Lie bialgebroid and \(((TM)_N \oplus (T^*M)_\pi, \langle \cdot, \cdot \rangle, [[\cdot, \cdot]]^\Phi, \rho)\) is a Courant algebroid, where

\[
[[X, Y]]^\Phi = [X, Y]_0 = 0,
\]

\[
[\xi, \eta]^\Phi = [\xi, \eta]_\pi + \Phi(\xi, \eta, \cdot),
\]

\[
[[X, \xi]]^\Phi = (\iota_X d\pi \xi + \frac{1}{2} d\pi < \xi, X >) - (\iota_\xi d_\pi X + \frac{1}{2} d_\pi < \xi, X >) = -\iota_\xi d_\pi X - \frac{1}{2} d_\pi < \xi, X >,
\]

the anchor map satisfies \(\rho(X + \xi) = N X + \pi^\sharp \xi = \pi^\sharp \xi\) and the pairing is given by \([\cdot, \cdot]\) for any \(X, Y\) in \(\mathfrak{X}(M)\), any \(\xi\) and \(\eta\) in \(\Omega^1(M)\).

**Example 5.** Let \(M\) be a \(C^\infty\)-manifold and set \(N = a \cdot \text{id}_{TM}\), where \(a\) is a non-zero real number. For any 2-vector field \(\pi\) in \(\Gamma(A^2TM)\), the triple \((\pi, N, \Phi)\), where \(\Phi = \frac{1}{2a} [\pi, \pi]\), is a pseudo-Poisson Nijenhuis structure. Therefore \(((TM)_N, d_\pi, \Phi)\) is a quasi-Lie bialgebroid and \(((TM)_N \oplus (T^*M)_\pi, \langle \cdot, \cdot \rangle, [[\cdot, \cdot]]^\Phi, \rho)\) is a Courant algebroid, where

\[
[[X, Y]]^\Phi = [X, Y]_{a \cdot \text{id}_{TM}} = a[X, Y],
\]

\[
[\xi, \eta]^\Phi = [\xi, \eta]_\pi + \frac{1}{2a} [\pi, \pi] (\xi, \eta, \cdot),
\]

\[
[[X, \xi]]^\Phi = (\iota_X d_{a \cdot \text{id}_{TM}} \xi + \frac{1}{2} d_{a \cdot \text{id}_{TM}} < \xi, X >) - (\iota_\xi d_\pi X + \frac{1}{2} d_\pi < \xi, X >) = a(\iota_X d\pi \xi + \frac{1}{2} d < \xi, X >) - (\iota_\xi d_\pi X + \frac{1}{2} d_\pi < \xi, X >),
\]

the anchor map satisfies \(\rho(X + \xi) = aX + \pi^\sharp \xi\) and the pairing is given by \([\cdot, \cdot]\) for any \(X, Y\) in \(\mathfrak{X}(M)\), \(\xi\) and \(\eta\) in \(\Omega^1(M)\).

Example 5 is an example of not a Poisson Nijenhuis manifold but a pseudo-Poisson Nijenhuis manifold.

The following proposition means that two given pseudo-Poisson Nijenhuis manifolds generate a new one.

**Proposition 3.4.** Let \((M_i, \pi_i, N_i, \Phi_i), i = 1, 2\), be a pseudo-Poisson Nijenhuis manifolds. Then the product \((M_1 \times M_2, \pi_1 + \pi_2, N_1 \oplus N_2, \Phi_1 + \Phi_2)\) is a pseudo-Poisson Nijenhuis manifold.

**Proof.** Using the fact that \([X_1, X_2] = 0\) for any \(X_i\) in \(\mathfrak{X}(M_i), i = 1, 2\), etc., we can see that the triple \((\pi_1 + \pi_2, N_1 \oplus N_2, \Phi_1 + \Phi_2)\) satisfies that \(N_1 \oplus N_2\) is a Nijenhuis structure on \(M_1 \times M_2\), the compatibility of \((\pi_1 + \pi_2, N_1 \oplus N_2)\) and the conditions \([\cdot, \cdot]\), \([\cdot, \cdot]\) and \([\cdot, \cdot]\) of Definition 4. \qed
From now on, we shall assume that a 2-vector field $\pi$ is nondegenerate. Then we can reduce one condition of a pseudo-Poisson Nijenhuis structure. This fact is important in the sense to be able to find pseudo-Poisson Nijenhuis structures easily.

**Theorem 3.5.** Let $\pi$ be a nondegenerate 2-vector field, $N$ a Nijenhuis structure and $\Phi$ a 3-vector field. If a triple $(\pi, N, \Phi)$ satisfies the conditions (13) and (14) in Definition 4, then $(\pi, N, \Phi)$ is a pseudo-Poisson Nijenhuis structure, i.e., $(\pi, N, \Phi)$ satisfies the condition (15).

**Proof.** We shall prove (15). By the nondegeneracy of $\pi$, the map $\pi^\sharp : T^*M \to TM$ is a bundle isomorphism. Therefore a set $\{\pi^\sharp df | f \in C^\infty(M)\}$ generates the vector fields $\mathfrak{X}(M)$ as a $C^\infty(M)$-module. We have proved in Theorem 3.1 that the equality (14) holds if and only if $d^2_\pi = [\Phi, \cdot]$ holds on $C^\infty(M)$. Thus we compute, for any $f$ in $C^\infty(M)$,

\[
\begin{align*}
d^2_\pi(\pi^\sharp df) &= d^2_\pi(-d_\pi f) = -d_\pi (d^2_\pi f) = -d_\pi [\Phi, f]_N \\
&= -((d_\pi \Phi, f)_N + [\Phi, d_\pi f]_N) \\
&= -[\Phi, d_\pi f]_N = [\Phi, \pi^\sharp df]_N,
\end{align*}
\]

where we use $\pi^\sharp df = -d_\pi f$ in the first and the last step, the fourth equality follows from (16) and the fifth equality does from (13). Therefore $d^2_\pi = [\Phi, \cdot]$ holds on the set $\{\pi^\sharp df | f \in C^\infty(M)\}$. Since $d^2_\pi = [\Phi, \cdot]$ holds on $C^\infty(M) \oplus \{\pi^\sharp df | f \in C^\infty(M)\}$ and since both $d^2_\pi$ and $[\Phi, \cdot]_N$ are derivatives on $(\Gamma(\Lambda^*TM), \wedge)$, we obtain that $d^2_\pi = [\Phi, \cdot]$ holds on $\mathfrak{X}(M)$. This is equivalent to the condition (15), so that the proof has been completed.

Using this theorem, we obtain the following example of a pseudo-Poisson Nijenhuis manifold.

**Proposition 3.6.** Let $(M, \omega, N)$ be a symplectic Nijenhuis manifold, i.e., for the nondegenerate Poisson structure $\pi$ corresponding to $\omega$, the pair $(\pi, N)$ is a Poisson Nijenhuis structure, and $\phi$ a closed 3-form such that $\iota_{\pi X} \phi = 0$ for any $X$ in $\mathfrak{X}(M)$. Then $(M, \pi, N, \Phi)$ is a pseudo-Poisson Nijenhuis manifold, where $\Phi = \pi^\sharp \phi$.

**Proof.** Since the Nijenhuis structure $N$ is compatible with the nondegenerate Poisson structure $\pi$, we need to prove that $(\pi, N, \Phi)$ satisfies (13) and (14). In this case, the condition (13) is

$$N_{\iota_{\alpha \wedge \beta}} \Phi = 0 \ (\alpha, \beta \in \Omega^1(M))$$
because of $[\pi, \pi] = 0$. By computing that, for any $\gamma$ in $\Omega^1(M)$,
\[
<N_{1\alpha}^\wedge\beta\Phi, \gamma> = \Phi(\alpha, \beta, N^*\gamma) = (\pi^2\phi)(\alpha, \beta, N^*\gamma) = -\phi(\pi^2\alpha, \pi^2\beta, N\pi^2\gamma) = -\phi(\pi^2\alpha, \pi^2\beta, N\pi^2\gamma) = -\phi(\pi^2\alpha, \pi^2\beta, N\pi^2\gamma) = 0,
\]
where we use $N\pi^2 = \pi^2N^*$ and $\iota_{NX}\phi = 0$, we conclude that (14) holds. By a straightforward computation using the property of the Poisson structure $\pi$ and the fact that $\phi$ is closed, we have
\[
(d\pi\Phi)(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (d\phi)(\pi^2\alpha_1, \pi^2\alpha_2, \pi^2\alpha_3, \pi^2\alpha_4) = 0
\]
for any $\alpha_i$ in $\Omega^1(M)$.

**Example 6.** On the 6-torus $T^6$ with angle coordinates $(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6)$, we consider the standard symplectic structure $\omega := d\theta_1 \wedge d\theta_2 + d\theta_3 \wedge d\theta_4 + d\theta_5 \wedge d\theta_6$ and a regular Poisson structure with rank 2, $\pi^\lambda := \partial/\partial\theta_a \wedge (\partial/\partial\theta_b + \lambda \partial/\partial\theta_c)$, where $\lambda$ is in $\mathbb{R}$ and $a, b$ and $c$ are three distinct numbers (see [8]). Setting $N^\lambda := \pi^\lambda \circ \omega^\phi$, we obtain a symplectic Nijenhuis structure $(\omega, N^\lambda)$ on $T^6$. Since the rank of $N^\lambda$ is 2 at each points, the kernel of $N^\lambda$ is a subbundle with rank 4 of the cotangent bundle of $T^6$. Hence for any closed 3-form $\phi$ in $\Gamma(\Lambda^3\text{Ker}N^\lambda)$, a triple $(\pi^\lambda, N^\lambda, \pi^\lambda \circ \omega^\phi)$ is a pseudo-Poisson Nijenhuis structure on $T^6$, where $\pi^\lambda$ is the Poisson structure corresponding to $\omega$.

Finally, we describe a relation between pseudo-Poisson Nijenhuis manifolds and generalized almost complex structures [3].

**Definition 6.** A generalized almost complex structure on a $C^\infty$-manifold $M$ is a bundle map over $M$,
\[
J : TM \oplus T^*M \longrightarrow TM \oplus T^*M
\]
satisfying the conditions
\[
J^2 = -id_{TM \oplus T^*M} \quad \text{and} \quad \langle Jv, Jw \rangle = \langle v, w \rangle
\]
for any $v$ and $w$ in $\Gamma(TM \oplus T^*M)$, where the pairing $\langle \cdot, \cdot \rangle$ is given by (2) in Example [1]. The Courant-Nijenhuis torsion $\mathcal{T}_J$ of $J$ is given by
\[
\mathcal{T}_J(v, w) := [Jv, Jw] - J[Jv, w] - J[v, w], \quad \langle v, w \rangle_J := J[Jv, w] + [v, Jw] - J[v, w]
\]
for any $v$ and $w$ in $\Gamma(TM \oplus T^*M)$. 

12
The conditions (18) imply that $J$ is of the form

$$J = \begin{pmatrix} N & \pi^\sharp \\ \sigma & -N^* \end{pmatrix},$$

where $\pi$ in $\Gamma(\Lambda^2 TM)$ is a 2-vector field, $\sigma$ in $\Omega^2(M)$ is a 2-form and $N : TM \to TM$ is a bundle map over $M$. We have the corresponding result to Proposition 7.5 in [13].

**Proposition 3.7.** Let $J : TM \oplus T^* M \to TM \oplus T^* M$ be a bundle map satisfying (18) and of the form (21). Then $(M, \pi, N, \pi^\sharp d\sigma_N)$ is a pseudo-Poisson Nijenhuis manifold if a generalized almost complex structure $J$ satisfies the following conditions:

(i) $\pi$ is nondegenerate,

(ii) $\sigma$ is closed,

(iii) $T_J|_{T^* M} : (2,1)$-tensor,

(iv) $T_J|_{TM} : (1,2)$-tensor and

(v) $T_J(\alpha, \beta) + \pi_N^\sharp (T_J(\pi^\sharp \alpha, \pi^\sharp \beta)) = 0$ $(\alpha, \beta \in \Omega^1(M), X, Y \in \mathfrak{X}(M))$,

where $\sigma_N$ is the 2-form given by $\sigma_N(X, Y) := \sigma(NX, Y)$.

**Proof.** As observed in [2], $\pi$ and $N$ are compatible by the conditions $T_J|_{T^* M} \equiv 0$ and $J^2 = -id_{TM \oplus T^* M}$, and $N$ is Nijenhuis by the conditions $T_J(X, Y)|_{TM} \equiv 0$ and $d\sigma = 0$. Moreover, by computing $T_J(\alpha, \beta)|_{TM}$ and $T_J(X, Y)|_{T^* M}$, we obtain

$$[\pi^\sharp \alpha, \pi^\sharp \beta] - \pi^\sharp [\alpha, \beta] = T_J(\alpha, \beta),$$

$$\iota_X \wedge \iota_Y d\sigma_N = -T_J(X, Y).$$

Since $\pi$ is nondegenerate, we only need to prove that $(\pi, N, \pi^\sharp d\sigma_N)$ satisfies (13) and (14). First we compute

$$\left\langle \frac{1}{2} \iota_{\alpha \wedge \beta} [\pi, \pi], \gamma \right\rangle = \left\langle \pi^\sharp [\alpha, \pi^\sharp \beta] - \pi^\sharp [\alpha, \beta], \gamma \right\rangle = \left\langle T_J(\alpha, \beta), \gamma \right\rangle$$

$$= \left\langle -\pi_N^\sharp (T_J(\pi^\sharp \alpha, \pi^\sharp \beta)), \gamma \right\rangle = \left\langle T_J(\pi^\sharp \alpha, \pi^\sharp \beta), \pi_N^\sharp \gamma \right\rangle$$

$$= \left\langle \iota_{\pi^\sharp \alpha \wedge \pi^\sharp \beta} d\sigma_N, \pi^\sharp N^* \gamma \right\rangle$$

$$= -d\sigma_N(\pi^\sharp \alpha, \pi^\sharp \beta, \pi^\sharp N^* \gamma) = (\pi^\sharp d\sigma_N)(\alpha, \beta, N^* \gamma)$$

$$= \iota_{\alpha \wedge \beta} (\pi^\sharp d\sigma_N, \pi^\sharp N^* \gamma) = \left\langle N \iota_{\alpha \wedge \beta} (\pi^\sharp d\sigma_N), \gamma \right\rangle.$$
for any $\gamma$ in $\Omega^1(M)$, where we use the conditions (22), (23) and a property of 2-vector fields, $\frac{1}{2}k_{\alpha \wedge \beta} [\pi, \pi] = [\pi \xi \alpha, \pi \xi \beta] - \pi \xi [\alpha, \beta]$ (see [14]). Therefore the condition (13) holds. Next, we calculate

$$(d_\pi (\pi \xi ds_N)) (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$$

$$= \sum_{i=0}^{4} (-1)^{i+1} (\pi \xi \alpha_i) ((\pi \xi ds_N)(\alpha_1, \ldots, \hat{\alpha_i}, \ldots, \alpha_4))$$

$$+ \sum_{i<j} (-1)^{i+j} (\pi \xi ds_N)([\alpha_i, \alpha_j], \alpha_1, \ldots, \hat{\alpha_i}, \ldots, \hat{\alpha_j}, \ldots, \alpha_4)$$

$$= -d(ds_N)(\pi \xi \alpha_1, \pi \xi \alpha_2, \pi \xi \alpha_3, \pi \xi \alpha_4)$$

$$+ (ds_N)(\pi \xi (T_J(\pi \xi \alpha_1, \pi \xi \alpha_2)), \pi \xi \alpha_3, \pi \xi \alpha_4))$$

$$- (ds_N)(\pi \xi (T_J(\pi \xi \alpha_1, \pi \xi \alpha_3)), \pi \xi \alpha_2, \pi \xi \alpha_4))$$

$$+ (ds_N)(\pi \xi (T_J(\pi \xi \alpha_1, \pi \xi \alpha_4)), \pi \xi \alpha_2, \pi \xi \alpha_3))$$

$$+ (ds_N)(\pi \xi (T_J(\pi \xi \alpha_2, \pi \xi \alpha_3)), \pi \xi \alpha_1, \pi \xi \alpha_4))$$

$$- (ds_N)(\pi \xi (T_J(\pi \xi \alpha_2, \pi \xi \alpha_4)), \pi \xi \alpha_1, \pi \xi \alpha_3))$$

$$+ (ds_N)(\pi \xi (T_J(\pi \xi \alpha_3, \pi \xi \alpha_4)), \pi \xi \alpha_1, \pi \xi \alpha_2)) = 0$$

for any $\alpha_i$ in $\Omega^1(M)$, where we use the definitions of $d$, $d_\pi$ and the condition (22). Hence the condition (13) holds. We conclude that $(M, \pi, N, \pi \xi ds_N)$ is a pseudo-Poisson Nijenhuis manifold. \qed

References

[1] A. Alekseev and Y. Kosmann-Schwarzbach. Manin pairs and moment maps. J. Diff. Geom. 56 (2000) 133–165.

[2] R. Caseiro, A. de Nicola and J. M. Nunes da Costa. On Poisson quasi-Nijenhuis Lie algebroids. arXiv:0806.2467v1. (2008).

[3] M. Gualtieri. Generalized complex geometry. Ann. of Math.(2) 174 (2011), no.1, 75–123.

[4] J. Grabowski and P. Urbanski. Lie algebroids and Poisson-Nijenhuis structures. Rep. Math. Phys. 40 (1997), 195–208.

[5] Y. Kosmann-Schwarzbach. The Lie bialgebroid of a Poisson-Nijenhuis manifold. Lett. Math. Phys. 38 (1996), no. 4, 421–428.
[6] Y. Kosmann-Schwarzbach and F. Magri. Poisson-Nijenhuis structures. *Ann. Inst. H. Poincaré Phys. Théor.* **53** (1990) no. 1, 35–81.

[7] Y. Kosmann-Schwarzbach and E. Meinrenken. Quasi-Poisson manifolds. *Canad. J. Math.* **54**, no.1 (2000) 3–29.

[8] R. Loja Fernandes and I. Mărcut. Lectures on Poisson Geometry. (Springer, 2015).

[9] Z-J. Liu, A. Weinstein and P. Xu. Manin triple for Lie bialgebroids. *J. Differential Geom.* **45** (1997), no. 3 547–574.

[10] K. Mackenzie and P. Xu. Lie bialgebroids and Poisson groupoids. *Duke Math. J.* **73** (1994), no.2, 415–452.

[11] F. Magri and C. Morosi. On the reduction theory of the Nijenhuis operators and its applications to Gel’fand-Dikii equations. *Proceedings of the IUTAM-ISIMM symposium on modern developments in analytical mechanics.* **117** (1983), Vol. II, 559–626.

[12] D. Roytenberg. Quasi-Lie bialgebroids and twisted Poisson manifolds. *Lett. Math. Phys.* **61** (2002) 123–137.

[13] M. Stiénon and P. Xu. Poisson Quasi-Nijenhuis manifolds. *Comm. Math. Phys.* **50** (2007), 709–725.

[14] I. Vaisman. Complementary 2-forms of Poisson structures. *Compositio math.* **101**(1996), no.1, 55–75.

[15] P. Xu. Gerstenhaber algebras and BV-algebras in Poisson geometry. *Comm. math. Phys.* **200**(1999) 545–560.