Transport properties of a two-lead Luttinger liquid junction out of equilibrium: fermionic representation

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The electrical current through an arbitrary junction connecting quantum wires of spinless interacting fermions is calculated in fermionic representation. The wires are adiabatically attached to two reservoirs at chemical potentials differing by the applied voltage bias. The relevant scale-dependent contributions in perturbation theory in the interaction up to infinite order are evaluated and summed up. The result allows to construct renormalization group equations for the conductance as a function of voltage (or temperature, wire length). There are two fixed points at which the conductance follows a power law in terms of a scaling variable $\Lambda$, which equals the bias voltage $V$, if $V$ is the largest energy scale compared to temperature $T$ and inverse wire length $L^{-1}$, and interpolates between these quantities in the crossover regimes.

I. INTRODUCTION

In the past few years, exactly one-dimensional quantum wires have become available for experimental investigation in the form of carbon nanotubes, chains of metal atoms or weakly side-coupled molecular chains in solids. The new data emerging from these experiments\textsuperscript{4,5} in particular in non-equilibrium situations, require a more detailed and more general theoretical description than presently available. Electron transport in nanowires has been studied theoretically for more than two decades. In the first papers it was found that electron-electron interaction affects even the conductance of a clean wire\textsuperscript{1–3}. Experimentally, the predictions of theory have been found to be observed, at least qualitatively\textsuperscript{4–12}. In this paper we extend the approach of Yue et al. to transport under stationary non-equilibrium conditions. Following our extensive work on transport in the linear regime through junctions of Luttinger liquids at arbitrary strength of interaction\textsuperscript{13–21} we derive in the following RG-equations for the conductance at finite bias voltage and for any interaction strength. We use the fact that the $\beta$- function of the RG-equation for the conductance can be obtained in very good approximation by summing a class of contributions in perturbation theory in all orders of the interaction\textsuperscript{13}. A comparison of our previous results on the linear response conductances of two\textsuperscript{13} and three-lead junctions\textsuperscript{16,18,21} with or without additional symmetries, or an applied magnetic flux\textsuperscript{27,29}, with the results of the bosonization method, of conformal field theory methods, of Bethe ansatz, where available, are in full agreement provided those results were well-founded. In a few cases where the conformal field theory result was based on an additional assumption we found disagreement, which we interpret as saying that the assumption was not justified.

II. THE MODEL

We consider a system of spinless fermions in one dimension, interacting in the region $a < |x| < L$ (the "wire"), adiabatically connected to reservoirs at $|x| > L$. There is a barrier in the narrow regime $|x| < a$, which scatters the fermions as described by the $S$-matrix (up to overall phase factors in the individual wires)

\begin{equation}
S = \begin{pmatrix} r & t \\ \bar{r} & \bar{t} \end{pmatrix} = \begin{pmatrix} \sin \theta & i \cos \theta e^{-i\varphi} \\ i \cos \theta e^{i\varphi} & \sin \theta \end{pmatrix}
\end{equation}

We choose this parametrization in terms of the transmission and reflection amplitudes $t, r$, since it is readily...
generalizable to the case of multi-wire junctions \((n \text{ wires}, n > 2)\). The above form of the \(S\)-matrix is completely general.

In the continuum limit, linearizing the spectrum at the Fermi energy and adding forward scattering interaction of strength \(g_j\) in wire \(j\), we may write the Hamiltonian \(\mathcal{H}\) in the representation of incoming and outgoing waves as

\[
\mathcal{H} = \int_0^\infty dx \sum_{j=1}^{2} \left[ H_j^0 + H_j^\text{int}(a < x < L) \right],
\]

\[
H_j^0 = v_F \psi_{j,\text{in}}^\dagger \nabla \psi_{j,\text{in}} - v_F \psi_{j,\text{out}}^\dagger \nabla \psi_{j,\text{out}},
\]

\[
H_j^\text{int} = 2\pi v_F g_j \psi_{j,\text{in}}^\dagger \psi_{j,\text{in}} \psi_{j,\text{out}}^\dagger \psi_{j,\text{out}}.
\]  

We are using the chiral representation, labeling electrons in lead \(j\) by \((j, \eta) \equiv j_\eta\) where \(\eta = +1\) for outgoing and \(\eta = -1\) for incoming electrons and all position arguments \(x\) are on the positive semi-axis. The range of the interaction lies within the interval \((a, L)\), where \(a > 0\) serves as an ultraviolet cutoff (at energy scale \(v_F/a\)) and separates the domains of interaction and potential scattering on the junction; non-interacting leads are attached adiabatically at large \(x\) beyond \(L\). In terms of the doublet of incoming fermions \(\Psi_- = (\psi_{1,-}, \psi_{2,-})\) the outgoing fermion operators may be expressed with the aid of the \(S\)-matrix as \(\Psi_+(x) = S \cdot \Psi_-(x)\). For later use we define density operators \(\bar{\rho}_{j,\eta=-1} = \psi_{j,-}^\dagger \psi_{j,-} = \Psi_-^\dagger \rho_j \Psi_-\), and \(\bar{\rho}_{j,\eta=1} = \psi_{j,+}^\dagger \psi_{j,+} = \Psi_+^\dagger \rho_j \Psi_+\), where \(\rho_j = S^+ \cdot \rho_j \cdot S\).

The \(2 \times 2\) matrices are defined by \(\rho_{j,1,\alpha\beta} = \delta_{\alpha\beta} \delta_{\eta j} + (\bar{\rho}_j)_{\alpha\beta} = S_{\alpha j} S_{j\beta}\).

### III. Charge Current of Free Fermions

The net current flowing outward through the point \(z\) in wire \(j\) is composed out of two chiral components, moving towards \((\eta = -1)\) and from \((\eta = 1)\) the junction,

\[
J_j(z) = v_j \left( \langle \bar{\rho}_{j,-}(z) \rangle - \langle \bar{\rho}_{j,+}(z) \rangle \right)
\]  

where \(v_j\) is the group velocity of the fermions. We use units where electrical charge \(e = 1\), also \(\hbar = 1\) and Boltzmann’s constant \(k_B = 1\).

We work with the Green’s functions in this chiral basis and in Keldysh formulation (we denote matrices in Keldysh space by an underbar),

\[
\mathcal{G} = \begin{pmatrix} G^R & G^K \\ 0 & G^A \end{pmatrix}
\]  

Here retarded, advanced and Keldysh components of the Green’s functions, in matrix form in the chiral basis are given by \((2 \times 2)\) matrices in the chiral basis are denoted by a hat, \(\hat{G}_{\eta j}(l, y|j, x) = G(l, \eta y|j, \eta_j, x)\)

\[
\hat{G}_R(l, y|j, x) = -i \frac{1}{\sqrt{v_F v_j^{\ast}}} \theta(\tau) e^{i\omega \tau} \begin{bmatrix} \delta_{lj} & 0 \\ S_{lj} & \delta_{lj} \end{bmatrix}
\]

\[
\hat{G}_A(l, y|j, x) = i \frac{1}{\sqrt{v_F v_j^{\ast}}} \theta(-\tau) e^{i\omega \tau} \begin{bmatrix} \delta_{lj} S_{lj}^* & \delta_{lj} \\ 0 & \delta_{lj} \end{bmatrix}
\]

\[
\hat{G}_K(l, y|j, x) = -i \frac{1}{\sqrt{v_F v_j^{\ast}}} e^{i\omega \tau} \begin{bmatrix} \delta_{lj} h_l & \delta_{lj} h_l^* \\ S_{lj} h_j & S_{lj}^* h_l S_{jm} S_{im} h_m \end{bmatrix}
\]  

\(\tau = \eta y / v_j - (\eta_j - x) / v_j\)

where \(h_j(\omega) = \tanh[(\omega - \mu_j) / 2T]\) is the equilibrium distribution function in the reservoir of lead \(j\), characterized by the chemical potential \(\mu_j\). We shall assume the temperatures \(T\) in the leads to be equal.

The average density of the chiral current at point \(z\), \(\langle \rho_{j,\eta}(z) \rangle\), is represented by the diagram in Fig. 1.

![FIG. 1: The diagram showing the zeroth order contribution to the current.](image)

In terms of the Green’s function matrix and defining the external vertex by the Keldysh matrix \(\gamma_{\text{ext}}\)

\[
\gamma_{\text{ext}} = \frac{i}{2} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}
\]  

we have

\[
\langle \rho_{j,\eta}(z) \rangle = \int \frac{d\Omega}{2\pi} \text{Tr}_K \left[ \gamma_{\text{ext}} \cdot \mathcal{G}_\Omega(j, \eta |z|j, \eta, z) \right]
\]  

with the trace \(\text{Tr}_K\) taken over the Keldysh indices. Using the expressions \(\hat{G}\), we obtain

\[
v_j \langle \rho_{j,-}(z) \rangle = \frac{1}{2} \int \frac{d\Omega}{2\pi} \left( 1 - h_j(\Omega) \right),
\]

\[
v_j \langle \rho_{j,+}(z) \rangle = \frac{1}{2} \int \frac{d\Omega}{2\pi} \left( 1 - \sum_m |S_{jm}|^2 h_m(\Omega) \right)
\]  

Notice here that the incoming current in the \(j\)th wire is characterized by the distribution function referring to the same wire. The outgoing current in the \(j\)th wire is characterized by the distribution functions referring to the remaining wires. The dependence on \(z\) vanishes in the d.c. limit considered here.

Using the unitarity property (i.e. charge conservation), \(\sum_m |S_{jm}|^2 = 1\), we may represent the net current in the form

\[
J_j^{(0)}(z) = \frac{1}{2} \int \frac{d\Omega}{2\pi} \sum_m |S_{jm}|^2 (h_j(\Omega) - h_m(\Omega))
\]
which is a well-known expression. For the above choice of the weight function $h_j(\Omega)$, the remaining integration can be easily done with the result

$$J_j^{(0)}(z) = \frac{1}{2\pi} \sum m |S_{jm}|^2 (\mu_m - \mu_j) \tag{10}$$

The conductance (in units of the conductance quantum $e^2/2\pi \hbar$) of a two-lead junction is in lowest order given by

$$G = J/V = |S_{12}|^2 = t^2 \tag{11}$$

where $V = \mu_1 - \mu_2$ is the applied bias voltage. In the following we will find it convenient to introduce the quantity $Y = 1 - 2G$ characterizing the conductance.

IV. CURRENT TO FIRST ORDER IN THE INTERACTION

The first order correction to the current in the non-equilibrium case is represented as the diagram depicted in Fig. 2. Here the wavy line stands for the electronic contribution to the current of chirality $\eta_n$ in the $n$–$th$ wire can be expressed as

$$J_j^{(1)}(z) = v_j \int d\Omega d\omega \int dx dy \sum_{\mu=1,2} \sum_{l,\eta} \text{Tr}_{\hat{K}}[\gamma_{ext}^{-1} \hat{G}_\Omega(j_n, z)[l,\eta, x] \gamma^\mu \hat{G}_{\Omega + \omega}(l,\eta, x)[l,\eta, y] \gamma^{\mu} \gamma_{ext}^{-1} \hat{G}_\Omega(l,\eta, x)[l,\eta, z])] \tag{12}$$

The trace $\text{Tr}_{\hat{K}}$ is over the lower (fermionic) Keldysh indices; the fermion-boson vertices, $\gamma^\mu_{ij} \to 2 \gamma^\mu_{ij}$, $\gamma^\mu_{ij} \to \tilde{\gamma}^\mu$, tensors of rank 3 defined in Keldysh space, are given by

$$\gamma^1_{ij} = \tilde{\gamma}^2_{ij} = \frac{1}{\sqrt{2}} \delta_{ij}, \quad \gamma^2_{ij} = \tilde{\gamma}^1_{ij} = \frac{1}{\sqrt{2}} \tau^1_{ij} \tag{13}$$

with $\tau^1$ the first Pauli matrix.

Notice that, similarly to the case of zeroth order in the interaction, the factor $v_j$ at the external point $z$ is compensated by the prefactor coming from the Green’s function, Eq. (5). If the point of the observation $z$ lies outside the interacting region, $z > L$, then the dependence on $z$ disappears in the outgoing current, $J_j^{(1)}(z > L) = J_j^{(1)}$, whereas the corrections to the incoming current are altogether absent, $J_j^{(1)}(z > L) = 0$. In what follows we discuss the corrections to the outgoing current. In view of the later generalization involving an infinite summation of higher order terms it is useful to represent the above first order expression as

$$J_j^{(1)} = i \int \frac{d\omega}{2\pi} \int dx dy \sum_{l,\eta, m, \eta} \text{Tr}_{\hat{K}}[\gamma_{ext}^{-1} \hat{G}_\Omega(j_n, y)[l,\eta, x; j, +, z] \hat{L}_{\eta, \omega}(l,\eta, x)[m,\eta, y)] \tag{14}$$

where we recall the definitions

$$l_\eta = (l, \eta)$$

etc. Here we defined a ”boson propagator” representing the interaction line

$$\hat{L}_{\eta, \omega}(l,\eta, x|m,\eta, y) = (2\pi g \lambda_1) \delta(x - y) \delta_{lm} \tau^1_{\eta, \eta} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{15}$$

and the quantity $T$ representing the triangle of Green’s functions in Fig. 2

$$T^\mu_{\omega}(m,\eta, x|l,\eta, y; j_n, z) = v_j \int \frac{d\Omega}{2\pi} \text{Tr}_{\hat{K}}[\gamma_{ext}^{-1} \hat{G}_\Omega(j_n, z|m,\eta, y) \gamma^\mu \hat{G}_{\Omega + \omega}(m,\eta, x)[l,\eta, x] \gamma^\mu \hat{G}_\Omega(l,\eta, x)[m,\eta, z]) \tag{16}$$

The triangle diagram is characterized by two Keldysh indices and thus is subdivided into four blocks. Symbolically, we write

$$\text{Tr}_{\hat{K}}[TL] = T^{11} L^R + T^{22} L^A$$

anticipating that $T^{21} = 0, L^{21} = 0$ (to be shown later).

When integrating over $\Omega$ in (16) we find two generic integrals. One of them is

$$\int d\Omega (h_j(\Omega + \omega) - h_j(\Omega)) = 2\omega$$

and the other is

$$\int d\Omega \left[ 1 - h_j(\Omega + \omega) h_m(\Omega) \right] = 2F(\omega + \mu_m - \mu_j). \tag{17}$$

For the above form of $h_j(\Omega)$, we have $F(x) = x \coth(x/2T)$.  

FIG. 2: The diagram providing first-order correction to the current due to interaction.
As mentioned above there are no corrections to the incoming currents. In addition to this observation we should recall Kirchhoff’s law, stating the conservation of charge. Given that the total incoming current is equal to the total outgoing current, we should have $J_1 + J_2 = 0$, which is indeed confirmed by direct calculation (see Appendix C).

Taking these facts into account, only the difference of the currents, $J^{(1)} = \frac{1}{2}(J_2^{(1)} - J_1^{(1)})$, is of interest. This involves the difference of the components of $T$ belonging to different leads. Accordingly, for the case of two leads, we define the weighted difference (denoted by the same symbol, $T$, but dependent on fewer variables),

$$T^{\mu\nu}(m_{\eta}, y|l_{\eta}, x; 1, +, z > L)$$

$$- T^{\mu\nu}(m_{\eta}, y|l_{\eta}, x; 2, +, z > L) \quad (18)$$

The complete expressions for $T^{\mu\nu}(m_{\eta}, y|l_{\eta}, x; j, +, z)$ are given in Appendix C. Here we show its odd-in-$V$ part, $T^{\mu\nu}_{\text{odd}}$, which is only of importance below. This is because the current should be odd in $V$, whereas in lowest order $L_0(\omega)$ is independent of $V$ and is an even function of $V$ in higher orders of perturbation theory.

The $4 \times 4$ matrices appearing here are now direct products of $2 \times 2$ matrices in chiral space (outer block structure) and $2 \times 2$ matrices in lead space (inner block structure, see Table II).

$$T^{11}_{\text{odd}} = \frac{r^2 t^2}{8\pi} [F(\omega + V) - F(\omega - V)]$$

$$\times \Phi_\omega(y)$$

$$\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & -1 & 1 & 0 \\
1 & -1 & 0 & 0
\end{bmatrix} \Phi^*_\omega(x),$$

$$T^{22}_{\text{odd}} = -(T^{11}_{\text{odd}})^T |_{x \leftrightarrow y}, \quad T^{21}_{\text{odd}} = 0.$$  

$$\Phi_\omega(x) = \text{diag} \left[ e^{-i\omega x/v_1}, e^{-i\omega x/v_2}, e^{i\omega x/v_1}, e^{i\omega x/v_2} \right]$$

$$\quad (19)$$

The vanishing of $T^{21}_{\text{odd}}$ implies that the Keldysh component of the renormalized interaction, $L^K$, does not enter. Inserting the components of $T^{\mu\nu}_{\text{odd}}$ and $L_0$ into the expression (14) for the current we find

$$J^{(1)} = -g t^2 r^2 \int_0^{\omega_c} \frac{d\omega}{\omega} [F(\omega + V) - F(\omega - V)] \sin^2 \frac{\omega L}{v}.$$  

Here we apply an upper cut-off $\omega_c$ given in the microscopic model either as $\omega_c = v/\alpha$ as mentioned above or $\omega_c = W$, the band width. The conductance as a function of voltage $V$, temperature $T$, wire length $L$, is found from there as

$$G^{(1)} = -2g G(1 - G) \Lambda(V, T, L). \quad (20)$$

Here we introduced the scaling variable $\Lambda$

### Table I: Convention for the indices

| $\alpha$ | 1 | 2 | 3 | 4 |
|----------|---|---|---|---|
| before/after wire # | B | A | 1 | 2 |

$$\Lambda(V, T, L) = \int_0^{\omega_c} \frac{d\omega}{\omega} \frac{F(\omega + V) - F(\omega - V)}{V} \sin^2 \frac{\omega L}{v}. \quad (21)$$

The factor $\sin^2(\omega L/v)$ guarantees convergence of the integral at $\omega < 1/t_0 = \pi v/L$. At $\omega > 1/t_0$ we may average this rapidly oscillating function and approximate $\sin^2(\omega L/v) \approx 1/2$. Employing this and analogous procedures for the cases of small $V, L^{-1}$ or small $V, T$ we may approximate $\Lambda$ as

$$\Lambda(V, T, L) \approx \ln \left( \frac{\omega_c}{\max\{V, T, v/L\}} \right). \quad (22)$$

### V. SCALE-DEPENDENT PART OF THE CURRENT: SUMMATION TO INFINITE ORDER IN THE INTERACTION

As shown in our previous work, the perturbative treatment may be extended into the strong coupling regime by summing up an infinite series of relevant scale-dependent contributions to the conductance in all orders ("ladder approximation"). These represent a self-energy renormalization of the "boson propagator" $L_0$ introduced above. They thus technically constitute a renormalized one-loop contribution to the RG equation. This series can still be represented by the generic diagram of Fig. 2 but the wavy line of electronic interaction should be dressed by screening effects, as discussed below.

As a result of this summation the interaction line, $g$, acquires non-locality and retardation effects. Moreover, if we have initially only diagonal components in Keldysh space, after the summation we generate a Keldysh component and in general a rather complicated structure. Schematically, we replace $L_0$ by $L$ in Eq. (14):

**FIG. 3**: A series of diagrams showing the screening. The negative sign of the coordinate corresponds to the incoming (B) electrons.
\[
L_0 \rightarrow L = \begin{pmatrix}
L^R_\omega(l_\eta, x|m_\eta, y) & L^K_\omega(l_\eta, x|m_\eta, y) \\
0 & L^A_\omega(l_\eta, x|m_\eta, y)
\end{pmatrix}
\] (23)

We now embark on the calculation of \( L \). Introducing compact notation, we express the lowest order result \( L_0 \) in the form \( L^{K\nu}_\omega(l_\eta, x|m_\eta, y) = \delta_{\nu \tau} \tau^\dagger_\eta_\eta \delta_\eta_\eta \delta(x - y) = 1_K \otimes \tau^\dagger_\omega \otimes 1_\omega \delta(x - y). \)

The integral equation describing the summation of the relevant infinite class of diagrams (see Fig. 3) takes the form

\[
L = L_0 + L_0 \ast \Pi \ast L_0 + L_0 \ast \Pi \ast L_0 \ast \Pi \ast L_0 + \ldots
\] (24)

where \( \Pi \) represents a fermion bubble

\[
\Pi^{\mu\nu}_\omega(l_\eta, j_\eta, y) = \int \frac{d\Omega}{2\pi} \text{Tr} \left[ \gamma^\mu G_\omega + \omega \gamma^\nu \right] \sqrt{\gamma^\omega} C(l_\eta, j_\eta, y|l_\eta, x)
\] (25)

The multiplication \( \ast \) is defined as

\[
(\Pi \ast L)^{\mu\nu}_\omega(j_\eta, y|n_\eta, x) = \int dz \sum_{l_\eta} \sum_{l_\eta=1,2} \Pi^{\mu\lambda}(j_\eta, y|l_\eta, z) \times L^{\nu\lambda}_\omega(l_\eta, z|n_\eta, x)
\]

At the level of Keldysh structure we have

\[
\begin{pmatrix}
L^R & L^K \\
0 & L^A
\end{pmatrix}
= \begin{pmatrix}
L_0 & 0 \\
0 & L_0
\end{pmatrix} + \begin{pmatrix}
L_0 & 0 \\
0 & L_0
\end{pmatrix} \ast \begin{pmatrix}
\Pi^R & \Pi^K \\
0 & \Pi^A
\end{pmatrix} \ast \begin{pmatrix}
L^R & L^K \\
0 & L^A
\end{pmatrix}
\]

which means that we can solve the integral equation in three steps.

First, we solve the coupled integral equations in the retarded sector

\[
L^R = L_0 + L_0 \ast \Pi^R \ast L^R
\] (26)

Second, considering that

\[
L^A = L_0 + L_0 \ast \Pi^A \ast L^A
\]

if we are using the relation between \( \Pi^A \) and \( \Pi^R \), we need not solve this equation separately. Third, we notice for completeness that

\[
L^K = L_0 \ast \Pi^R \ast L^K + L_0 \ast \Pi^K \ast L^A
\]

and hence,

\[
L^K = (1 - L_0 \ast \Pi^R)^{-1} \ast L_0 \ast \Pi^K \ast L^A = L^R \ast \Pi^K \ast L^A
\]

where we used (26) to obtain the second equality. This means that once we have \( L^R \), we can easily find the two remaining components, \( L^A \) and \( L^K \). We recall, however, that as pointed out above the component \( L^K \) does not enter the calculation of the current.

The solution for \( L^R \) in the linear response case was presented previously in our work\(^{19}\). We follow that derivation but reformulate it somewhat for the present purposes. First we define the integral (scalar) kernel

\[
P_\omega(j, x|l, z) = \left( 2\pi v_1 v_2 \right)^{-1} \left( \delta(\tau) + i\omega \eta(\tau)e^{i\omega \tau} \right)
\]

and the matrix quantity

\[
\hat{\Pi}^R = \begin{pmatrix}
\Pi(-x - z), & Y^T \Pi(-x|z) \\
Y \Pi(x|z), & \Pi(x|z)
\end{pmatrix}
\] (28)

where \( \Pi(j|x) = \delta_{\nu} P_\omega(j, x|l, z), Y \Pi(x|z) = Y_{jl} P_\omega(j, x|l, z) \) and \( Y^T \Pi(x|z) = Y_{jl} P_\omega(j, x|l, z) \) with \( Y_{jl} = |S_{jl}|^2 \). In the case of two leads we have \( Y = Y^T \).

Notice that \( Y^T \Pi(-x|z) = 0 \) for \( x, z > 0 \), and we use the full form (28) for future reference.

Then we express the integral equation for \( L^R \) as a \( 2 \times 2 \) matrix equation in the chiral space

\[
\hat{L}^R(x|y) = 2\pi \delta(x - y) \begin{pmatrix} g & 0 \\ g & 0 \end{pmatrix}
- 2\pi \int_a^L dz \begin{pmatrix} gY \Pi(x|z) & g \Pi(x|z) \\ g \Pi(-x|z) & 0 \end{pmatrix} \hat{L}^R(z|y).
\]

Here \( \hat{L}^R \) is a \( 4 \times 4 \) (in the general case of \( n \) leads \( 2n \times 2n \) ) matrix. The elements of the \( 2 \times 2 \) matrices in chiral space (denoted by a hat) are \( 2 \times 2 \) matrices in the space of the 2 leads (denoted by bold letters). The matrix of interaction constants is then given by \( g = \text{diag}[g_1 v_1, g_2 v_2] \). The scattering properties of the junction are encoded in the \( 2 \times 2 \) matrix \( Y \). The equation for \( L^A \) is similar to the above, but

\[
\hat{\Pi}^A = \left( \hat{\Pi}^R \right)^{-1}|_{x\rightarrow z = 0} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \hat{\Pi}^R \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}|_{x\rightarrow -\omega, \nu\rightarrow Y^T}
\]

Because \( L_0 \) does not contain \( \omega, Y \), it follows that

\[
\hat{L}^A = \left( \hat{L}^R \right)^{-1}|_{x\rightarrow y = 0}
\] (30)

The Keldysh part of the kernel takes the form presented in the appendix\(^{12}\). We show there, that \( \Pi^K \) is an even function of \( V \), and therefore \( L^K \) does not contribute to the current.

Following the method of solution of the integral equation described in\(^{12}\) we first solve the equation for the case \( Y_{jl} = 0 \), to give a partial summation resulting in an auxiliary quantity \( C \)
\[ \hat{C}(x|y) = 2\pi \delta(x-y) \begin{pmatrix} 0 & g \\ g & 0 \end{pmatrix} \]

\[ -2\pi \int_\alpha^L dz \left( \begin{array}{cc} 0 & g\Pi(x|z) \\ g\Pi(-x|-z) & 0 \end{array} \right) \hat{C}(z|y), \quad (31) \]

In terms of \( \hat{C} \) the integral equation for \( \hat{L}^R \) may be expressed as

\[ \hat{L}^R(x|y) = \hat{C}(x|y) - 2\pi \int_\alpha^L dz_1 dz_2 \hat{C}(x|z_1) \times \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) \hat{L}^R(z_2|y), \quad (32) \]

The solution of the integral equation for \( \hat{C} \), which is of the Wiener-Hopf type, may be found by an appropriate ansatz described in [12]. By construction, \( \hat{C}(x|y) \) is diagonal in wire space, \( \hat{C}_{j}(x|y) = \delta_{j} \hat{C}_{j}(x|y) \). The explicit expressions for \( \hat{C}_{j}(x|y) \) are presented in the Appendix E.

Returning to the integral equation (32) for \( \hat{L}^R \) in terms of \( \hat{C} \) we observe that its kernel is separable and thus the solution may be readily obtained. The explicit expressions and some details of the derivation of this result are given in Appendix E.

An inspection of Eqs. (14), (19) shows that the \( x, y \) dependence of \( T_{\alpha\beta}^{\nu\nu'} \) comes only from the matrices \( \Phi^*_\omega(x), \Phi_\omega(y) \). We combine these matrices with \( \hat{L}^R \) and integrate over the position

\[ \hat{L}^R_{\text{simple}} = \int_\alpha^L dx dy \Phi^*_\omega(x) \hat{L}^R(x|y) \Phi_\omega(y). \quad (33) \]

Making now use of relations (19), (30), we arrive at a much simpler algebraic expression for the current. Instead of (14) we have

\[ J^{(L)} = -2 \text{Im} \frac{1}{2\pi} \text{Tr} \left[ \hat{T}^{11}_{\text{core}} \hat{L}^R_{\text{simple}} \right], \quad (34) \]

with \( \hat{T}^{\nu\nu'}_{\text{core}} \) obtained by putting \( \Phi^*_\omega(x) = \Phi_\omega(y) = 1 \) in Eq. (19). We show the algebraic relation (34) diagrammatically in Fig. 4.

Introducing the quantities \( d_j \) and \( q_j \)

\[ d_j^2 = 1 - g_j^2, \quad q_j^{-1} = \frac{g_j}{1 + id_j \cot \left( \frac{\pi L}{V_j d_j} \right)}. \quad (35) \]

we present the simpler expressions of \( \hat{C}, \hat{L} \) integrated over position.

\[ \hat{L}^R_{j,k,\text{simple}} = \frac{2\pi i}{\omega} \left( \delta_{jk} \hat{C}_{j,\text{simple}} + Y_{jk} \begin{pmatrix} V_{1,j}V_{2,k} & V_{1,j}V_{1,k} \\ V_{2,j}V_{2,k} & V_{2,j}V_{1,k} \end{pmatrix} \right), \quad (36) \]

Again the \( \frac{1}{\omega} \)-singularity of the integrand leads to a logarithmically divergent contribution, which we identify as a scaling contribution. The singularity is controlled by the largest of the three energy scales, (i) energy scale \( \omega_L = v/L \) controlled by the length \( L \), (ii) temperature \( \omega_T = T \), (iii) bias voltage \( \omega_V = V \). In the limit \( V \to 0, T \to 0 \) we have \( F(\omega+V) - F(\omega-V) = 2V \text{sign}(\omega) \).

Combining the above results, (19), (34), (36), (37), we find the current for two equal wires, \( g_j = g, V_j = v \), as

\[ J^{(L)} = \frac{g}{8\pi} \int_0^{\omega_L} d\omega F(\omega + V) - F(\omega - V) \times \text{Re} \left\{ \frac{2(1-Y^2)}{1 - gY + i\omega L/dv} \right\}. \quad (38) \]

with

\[ \hat{C}_{j,\text{simple}} = \begin{pmatrix} -1, 0 \\ 0, -1 \end{pmatrix}, \quad Y_{jk} = V_{1,j}(1 - Q_j^{-1}Y_{jl}) \quad (37) \]

\[ V_{1,j} = (\hat{C}_{j,\text{simple}})_{12}, \quad V_{2,j} = (\hat{C}_{j,\text{simple}})_{11}, \quad Y_{jk} = Y_{j}(1 - Q_j^{-1}Y_{jl}) \quad (38) \]

The \( \frac{1}{\omega} \)-singularity is in this case cut off at the scale \( \omega_L = dv/L \) by the cot \( \left( \frac{\pi L}{V_d} \right) \) term in the denominator. Above this scale we may average the rapidly oscillating function in the curly brackets in (38) over one oscillation period, \( \omega_0 < \omega < \omega_0 + (\pi/t_0) \), with \( t_0 = L/dv \):

\[ \frac{t_0}{\pi} \int_{\omega_1}^{\omega_1 + \pi/t_0} d\omega \frac{d\omega}{1 - gY \pm id\cot \omega t_0} = (1 - gY + d)^{-1}, \]

such that the correction to the conductance is obtained as

\[ G^{(L)} = -g \frac{(1 - Y^2)}{1 - gY + d} \ln \frac{L}{a}, \]

FIG. 4: The schematic diagram, with already algebraic quantity \( T(\omega) \) and dressed interaction line \( L(\omega) \).
in agreement with\textsuperscript{15}. In the general case we find accordingly

\[ G^{(L)} = -g \frac{1 - Y^2}{1 - gY + d} \Lambda \]  \hspace{1cm} (39)

where \( \Lambda = \ln(\omega_c/\max\{V,T,v/L\}) \). In the limit of long wires, \( L \to \infty \), a closed expression is found in Appendix G in the form

\[ \Lambda = \ln \left( \frac{\omega_c}{2\pi T} \right) - \text{Re} \left[ \psi \left( 1 + \frac{iV}{2\pi T} \right) \right] , \]  \hspace{1cm} (40)

with \( \psi(x) \) digamma function. This function shows a smooth interpolation between the regimes with \( \ln \left( \frac{\omega_c}{2\pi T} \right) + 0.577 \) at \( V \ll T \) and \( \ln(\omega_c/V) \) at \( V \gg T \).

VI. RENORMALIZATION GROUP EQUATION FOR THE CONDUCTANCE

The above calculation of the leading scale-dependent contribution to the current allows us to derive a renormalization group (RG) equation for the conductance \( G = I/V \) as a function of the scaling variable \( \Lambda = \ln(\omega_c/\max\{V,T,v/L\}) \), \( G = G(\Lambda) \). We thereby use the scaling property of \( G(V,T,v/L,G_0;g) = G(\Lambda;g) \). In our previous work\textsuperscript{11,12} we explicitly checked this property in the equilibrium situation. We directly calculated all the contributions to the conductance up to third order in the interaction, which involves about \( 10^4 \) diagrams. It was shown that the principal contribution near the fixed points (FPs) of this equation is obtained in one-loop order, with the interaction being dressed as described above, \( g \to L \). The scaling exponents obtained this way are identical to those found earlier by the method of bosonization.

Away from the FPs one finds in general additional non-universal contributions, appearing first in the third order. These determine the prefactor in the scaling law near the FP and also fine details of the conductance in the intermediate regime. In the present study focused on the transport far out of equilibrium it would be too costly to perform a similar direct computation of all contributions up to third order. Instead we assume that even out of equilibrium we have the scaling property and the scaling exponents are fully determined by the contribution provided by the approximation of fully dressing the interaction line of the one-loop calculation.

We now briefly review the logic by which the RG-equation is derived from the perturbative result. We start from the result for the renormalized conductance \( G \) as a power series expansion in the interaction, and dependent on the scattering properties of the junction (encapsulated in the conductance \( G_0 \) in the absence of interaction) obtained above, which takes the general form,

\[ G = G_0 - gf(g,G_0)\Lambda + \mathcal{O}(g^2\Lambda^2) \]  \hspace{1cm} (41)

In the approximation of summing up the leading terms in each order, considered above, a very good approximation \( f_{\text{app}} \) of the function \( f(g,G) \) has been obtained, see Eq. (29). We do not calculate the terms of order \( g^2\Lambda^2 \) and higher in this paper. The relation Eq. (41) is valid in the asymptotic regime \( g\Lambda \to 0 \). With the aid of the scaling property we may find the analytic continuation to finite values of \( g\Lambda \). To this end we first invert the relation (making use of \( G = G_0 + \mathcal{O}(g\Lambda) \) ) and write

\[ G_0(g,G;\Lambda) = G + gf(g,G)\Lambda + \mathcal{O}(g^2\Lambda^2) \]

Formally \( G_0 \) here is a function of \( G, g \) and \( \Lambda \). We now employ the crucial property that the value of the bare conductance, \( G_0 \), should not depend on the scaling variable \( \Lambda \) , which means

\[ 0 = \frac{\partial G_0}{\partial \Lambda} + \frac{\partial G_0}{\partial G} \frac{dG}{d\Lambda} \]  \hspace{1cm} (42)

and hence

\[ \frac{dG}{d\Lambda} = -\frac{gf(g,G) + \mathcal{O}(g^2\Lambda)}{1 + g\Lambda[\partial f(g,G)/\partial G] + \mathcal{O}(g^2\Lambda^2)} \]  \hspace{1cm} (43)

The scaling property of \( G \) implies that the explicit \( \Lambda \) -dependence in (43) cancels. This leads to the definition of the RG \( \beta \)-function

\[ \frac{dG}{d\Lambda} = \beta(g,G) = -gf(g,G) . \]  \hspace{1cm} (44)

Our earlier direct third order calculation in\textsuperscript{15,17} showed that the above ratio was indeed independent of \( \Lambda \) to the considered accuracy \( \Lambda \). The function \( gf(g,G) \) has been calculated beyond the ladder approximation in\textsuperscript{15} for the present case of a two-lead junction with the result

\[ \frac{dG}{d\Lambda} = -gf_{\text{app}}(g,G) + c_3 g^3 G^2 (1 - G)^2 + \mathcal{O}(g^4) \]  \hspace{1cm} (45)

The second term here of order \( g^3 \) originates from terms not contained in the perturbation series for \( L \) considered above. This term is subleading in the sense that it vanishes more rapidly on approach to the fixed points at \( G = 0,1 \) than the first term and does therefore not influence the critical properties. There are indications that this is also the case with the higher order contributions not captured by the ladder summation.

A similar conclusion regarding the relative unimportance of corrections beyond the ladder summation \( g \to L \) was reached in\textsuperscript{22} for the more general case of the three-lead Y-junction. In the symmetrical setup the Y-junction was characterized by two conductances, and after extensive computer analysis of perturbative corrections we arrived at a set of two coupled RG equations. We found that the three-loop corrections, not contained in the ladder series of diagrams, did not contribute to the scaling exponents.

We expect that non-universal contributions to the \( \beta \)-function will also exist in the case of non-equilibrium, but those terms will again be unimportant when it comes
to determine the critical behavior at the fixed points. We will therefore approximate the exact function \( f(g,G) \) by the one determined in the ladder approximation and given through eq. (41), which gives rise to the \( \beta \)-function

\[
\frac{d}{d\Lambda} G = -4g \frac{G(1-G)}{1 - g(1-2G)} + d \tag{46}
\]

Introducing the Luttinger parameter \( K = \frac{\sqrt{(1-g)}/(1+g)}{\frac{1}{2}} \), Eq. (46) may be re-expressed as

\[
\frac{d}{d\Lambda} G = -2(1-K) \frac{G(1-G)}{K + (1-K)G}, \tag{47}
\]

which is explicitly solved in the next section.

VII. SOLUTION OF RG EQUATION

Inverting Eq. (47) we write

\[
2(1-K)d\Lambda = -dG \left( \frac{K}{G} + \frac{1}{1-G} \right), \tag{48}
\]

which is integrated with the result

\[
\frac{1-G}{G^K} = \frac{1-G_0}{G_0^K} e^{2(1-K)\Lambda}. \tag{49}
\]

It is more instructive to exclude here the bare conductance, \( G_0 \), and to represent our result as (cf.\(^2\))

\[
\frac{(1-G)/G^K|_{V_1,T_1}}{(1-G)/G^K|_{V_2,T_2}} = e^{2(1-K)\Lambda(V_1,T_1)}/e^{2(1-K)\Lambda(V_2,T_2)}. \tag{50}
\]

The latter exponential can be written as

\[
e^{2(1-K)\Lambda} = \left( \frac{\omega_c}{\max\{V, T, v/L\}} \right)^{(1-K)}.
\]

We see that near the two fixed points of the RG equation, \( G = 0 \), \( G = 1 \), we have the well-established scaling behavior\(^3\)

\[
G \simeq (V/\omega_0)^{2(1-K)} , \quad G \to 0 ,
\]

\[
1 - G \simeq c_*(\omega_0/V)^{2(1-K)}, \quad G \to 1.
\]

with an appropriate \( \omega_0 \) and where \( V \) should be replaced by \( \exp(A(V,T,v_{PE}/l)) \) in the more general situation. At the same time, (49) provides a smooth crossover between the fixed points, i.e. for those values of \( G \) which, strictly speaking, are inaccessible by the bosonization approach.

We notice further, that if the overall energy scale \( \omega_0 \) is fixed near one fixed point, then the constant \( c_* \) is entirely defined by the three-loop and higher-loop terms in the RG equation. In the approximation of neglecting the three-loop terms, as in Eqs. (47), (40), the coefficient \( c_* = 1 \). Keeping the three-loop terms, Eq. (49) is approximately given by\(^1\)

\[
\frac{G^K}{1-G} (1 + G^{1-K})^{c_3(1-K)} = \left[ \frac{\max\{V, T, v/L\}}{\omega_0} \right]^{2(1-K)}, \tag{52}
\]

which implies \( c_* = K^{-c_3(1-K)} \).

A. Comparison with the exact solution at \( K = 1/2 \)

To understand better the limitations of our formula (49), we compare it with the exact result at \( K = 1/2 \). Explicitly, our expression in this case reads as

\[
G = 1 - \frac{\sqrt{1 + 4\pi^2} - 1}{2\pi^2}, \quad x = \frac{T}{T^*} \exp \text{Re} [\psi [1 + \frac{iV}{4\pi T}]], \tag{53}
\]

The exact formula, obtained with the aid of the Bethe ansatz\(^2\) is

\[
G_{1/2} = 1 - \frac{4\pi T^*}{V} \text{Im} \psi \left[ \frac{1}{2} + \frac{T^*}{T} + i \frac{V}{4\pi T} \right], \tag{54}
\]

with \( T^* \) depending on the impurity backscattering amplitude and the ultraviolet cutoff. In two important limiting cases we have for the linear conductance

\[
G_{1/2}(T) = G_{1/2}(V \to 0, T),
\]

\[
= 1 - \frac{T^*}{T} \text{Im} \psi \left( \frac{1}{2} + \frac{T^*}{T} \right),
\]

\[
\simeq \frac{1}{12} \left( \frac{T}{T^*} \right)^2, \quad T \ll T^*, \tag{55}
\]

\[
\simeq 1 - \frac{\pi^2 T^*}{2 T}, \quad T \gg T^*.
\]

And for the nonlinear conductance

\[
G_{1/2}(V) = G_{1/2}(V, T \to 0),
\]

\[
= 1 - \frac{4\pi T^*}{V} \arctan \frac{V}{4\pi T^*},
\]

\[
\simeq \frac{1}{12} \left( \frac{V}{2\pi T^*} \right)^2, \quad V \ll T^*, \tag{56}
\]

\[
\simeq 1 - \frac{\pi^2 T^*}{V}, \quad V \gg T^*.
\]

These expressions indicate the existence of non-universal three-loop terms in the RG \( \beta \)-function. Indeed, fixing the overall scale at small \( T \) by \( G_{1/2}(T) = (T/T^*_1)^2 \) with \( T^*_1 = \sqrt{12} T^* \) gives the above constant \( c_* = \pi^2/(4\sqrt{3}) \approx 1.424 \). At the same time, fixing the scale at small \( V \) by \( G_{1/2}(V) = (V/T^*_2)^2 \) with \( T^*_2 = 2\pi \sqrt{12} T^* \) produces \( c_* = \pi/(2\sqrt{3}) \approx 0.911 \).

This means, firstly, that the three-loop term \( c_3 g^3 G^2 (1-G)^2 \) in the RG equation (45) has a different prefactor \( c_3 \), depending on whether the choice of low-energy cutoff is \( T \) or \( V \). This fact was noted in\(^\text{19} \) on the
basis of direct computation of perturbative corrections. From the above estimate \( c_4 = K^{-4\alpha(1-K)} \) we retrieve \( c_3 \simeq 0.255 \) and \( c_4 \simeq -0.070 \) for \( G_{1/2}(T) \) and \( G_{1/2}(V) \), respectively.

Secondly, in the absence of three-loop RG terms \( (c_n = 1) \) the ratio \( G^K/(1 - G) \), appearing in [50], should be a linear function of \( V \), \( T \) at \( K = 1/2 \). Plotting this ratio for the functions [55], [56], we compare it with the straight line corresponding to Eq. (53). We confirm much better agreement with the straight line in the case of the non-linear conductance \( G_{1/2}(V, T \to 0) \), see Fig. 5.

In practical terms these observations mean the following. When fitting experimental data with one universal curve for the whole range of conductances, one should use slightly different expressions for \( G(V = 0, T) \) and \( G(V, T = 0) \). The generic formula is [52], where the value \( c_3 = 1/4 \) is appropriate for \( G(V = 0, T) \), while \( c_3 \simeq -0.07 \) is better suited for \( G(V, T = 0) \).

**VIII. CONCLUSION**

Electron transport through one-dimensional quantum wires of various types has been studied experimentally in several recent works. In a typical set-up stationary charge transport is measured in a two-point geometry of a system of one or several wires connected by a junction. The quantum wires are adiabatically connected to reservoirs kept at a fixed chemical potential and temperature. These systems are described by modeling the quantum wires as Luttinger liquids (spinless, or spinful) of fermions with linear dispersion subject to point-like interaction and treating the reservoirs as non-interacting. A useful picture of the transport process is to think of individual electrons entering the interaction region (quantum wires plus junction) from an initial reservoir and leaving as individual electrons into the final reservoir. If we model the reservoirs as non-interacting systems there is no room for collective excitations such as fractional quasiparticles or multiple quasiparticles in the final state.

Conventionally this problem has been addressed by the bosonization method, which takes advantage of the fact that the exact excitations of a clean Luttinger wire are bosons, at least in the infinite wire. The problem of including the transformation of incoming electrons into bosons has been addressed for the clean wire, and is believed to be solved. For the case of semi-wires connected by a junction there is no convincing calculation of the above transformation available. In order to avoid this difficulty we are using a fermionic representation.

Our approach starts with determining the leading scale-dependent contributions to the conductances in all orders of perturbation theory. We have demonstrated in the linear response case that by summing up these terms one arrives at a description of the critical properties near the fixed points (i.e. the location of the FPs and the critical exponents describing the power laws followed by the conductances). For this it is necessary to establish the scaling property of the conductances (or else to assume its validity, which is usually done), allowing to derive a set of renormalization group equations out of the perturbative result.

In the present paper we followed this approach for the case of stationary non-equilibrium transport. We first derived a general result for the scale-dependent terms in the conductances of an \( n \)-lead junction in first order of the interaction. Then we presented the infinite order summation for the dressed interaction. At this point we specialized our considerations to the case of two symmetric semi-wires. We derived the corresponding RG-equation for the conductance. In general the scaling is dependent on three energy scales, bias voltage \( V \), temperature \( T \), and infrared cut-off provided by the wire length \( v/L \). Whenever one of these energy scales dominates, the scaling variable is varying logarithmically, \( \Lambda = \ln (\max \{V/T, v/L\}) \). In the case \( V, T \gg v/L \) we were able to determine the form of the scaling variable describing the crossover from the regime characterized by \( V \gg T \) to \( V \ll T \), as well.

The intermediate results presented for the general case of an \( n \)-lead junction should be a good starting point for analyzing the behavior of non-equilibrium transport through \( Y \)-junctions or even four-lead junctions. Work in this direction is in progress.

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Appendix A: Normalization of wave functions

The usual summation over the quantum states in the infinite medium is done as an integration over the momentum $\int dk/(2\pi)$ or the summation, $\sum_n$ over the quasi-momentum $k = 2\pi n/L$ in a ring geometry with a finite length $L$. In our situation with a broken translational symmetry we should resort to the integration over the energy, then the correct normalization factor is given by the density of states, which is the inverse Fermi velocity $v_F$.

Thus in the formula for the retarded Green’s function,

$$\tilde G_E^R(l,y|x) = \int dE \frac{\phi_{E,l}(y)\phi_{E,j}(x)}{\omega - E + i0}, \quad (A1)$$

we adopt the wave functions in the $j$th wire in the form

$$\phi_{E,j}(y) = e^{iEy/v_j}/\sqrt{2\pi v_j} \quad (A2)$$

and come to the formula [5]. Notice also that the integration in [A1] should be restricted by the electronic bandwidth $|E| < W = E_F$, which can be modeled by introducing the density of states function, $N_j(E)$, with the property $N_j(0) = v_j^{-1}$. So strictly speaking the formulas [5] are defined at $|\omega| \ll W$, which justifies the upper cutoff in the calculation of logarithmic corrections and the RG procedure.

Appendix B: Keldysh structure of the triangle $T$

The straightforward calculation shows that only a few terms in the complicated expression for $T$ contribute to the final result. Let us sketch here the derivation and present arguments showing the selection of the relevant terms.

To condense our writing, we use the position dependent notation, $G^R \to R$, and position in the product denotes the position in the initial expression, (10). So that $G^R_0(z|x)G^L_0(x|y)G^A_0(y|z) \leftrightarrow RLA$ etc. up to a numerical factor we have

$$T_{11} = RAA + (K + A)(RK - RR + KA), \quad T_{22} = RRA + (RK + KA + AA)(K - R), \quad T_{21} = RKA + KAA + RRK - RRR + AAA. \quad (B1)$$

The combinations $RRR$ and $AAA$ are necessarily zero for the point $z$ outside the interacting region. We may suggest (and it is confirmed by the direct calculation), that the contributing terms in (B1) are those which contain two Keldysh components, $K$. In this sense, we may keep only the terms

$$T_{11} \simeq KRR + KKA, \quad T_{22} \simeq RKK + KAK, \quad T_{21} \simeq 0 \quad (B2)$$

Note that the notation “$\simeq$” here also means that the combination $KK$ should be regularized at $\Omega \to \pm \infty$ by subtracting 1 from the product of distribution factors $h_1(\omega + \Omega)h_1(\Omega)$. This regularization is suggested by inspection of the corresponding expressions in the direct calculation. A closer inspection shows that the combinations $h_1(\Omega)h_1(\Omega)$, not containing $\omega$ do not contribute to the corrections, when multiplied by $L(\omega)$.

Thus the expressions for $T_{ij}$ can be simplified even further:

$$T_{11} \simeq KKA, \quad T_{22} \simeq RKK, \quad T_{21} \simeq 0 \quad (B3)$$

The last expression means that the corrections to the incoming current are absent, because $G^A_0(z|x) = 0$ and $G^R_0(z|x) = 0$ in this case, due to the step functions in [5].

Appendix C: Explicit expressions for $T$.

For completeness and future reference we provide here the explicit form of those components of $T_{\omega}^{\mu\nu}(m_\eta, y|l_\eta, x)$ which did not enter our calculation of the current corrections.

Given that the total incoming current is equal to the total outgoing current, we should have $\delta J_1 + \delta J_2 = 0$. We have for the corresponding quantity

$$T_0 = \frac{1}{2}[T_{\omega}^{\mu\nu}(\ldots; 1, +, z > L) + T_{\omega}^{\mu\nu}(\ldots; 2, +, z > L)]$$

the following expressions

$$T_0^{11} = \frac{1}{8\pi} \Phi_\omega(y) \left\{ F(0) \begin{bmatrix} \theta_{xy} & 0 & 0 & 0 \\ 0 & \theta_{xy} & 0 & 0 \\ r^2 & t^2 & \theta_{yz} & 0 \\ t^2 & r^2 & 0 & \theta_{yx} \end{bmatrix} - F(\omega) \begin{bmatrix} 0 & 0 & r^2 & t^2 \\ 0 & 0 & t^2 & r^2 \\ t^2 & r^2 & 0 & 0 \\ F_2(\omega, V) r^2 t^2 & 0 & 0 & 0 \end{bmatrix} \right\} \Phi^*_\omega(x), \quad (C1)$$

$$T_0^{22} = -(T_0^{11})^\dagger \quad (C2)$$

$$T_0^{21} = \frac{\omega}{8\pi} \Phi_\omega(y) \begin{bmatrix} 1 & 0 & r^2 & t^2 \\ 0 & 1 & t^2 & r^2 \\ r^2 & t^2 & 1 & 0 \\ t^2 & r^2 & 0 & 1 \end{bmatrix} \Phi^*_\omega(x),$$

with $\theta_{yx} = \theta(x - y)$ and

$$F_2(\omega, V) = F(\omega + V) + F(\omega - V) - 2F(\omega). \quad (C2)$$

We see that the quantity $T_0$ is an even function of the voltage bias $V$, whereas the current is odd in $V$. Therefore the contribution of $T_0$ to the current vanishes, as confirmed by direct calculation.
The part of the current $\delta J_1 - \delta J_2$ corresponds to Eq. 18. The odd-in-$V$ part of $T^{1\mu}(m_n, y | l_n, x)$ is given by Eq. (19). For completeness, we also show the irrelevant even-in-$V$ part

$$T^{11} = \frac{1}{8\pi} \Phi_\omega(y) \left\{ F(0) Y \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\theta_{xy} & 0 & 0 \\ -\theta_{xy} Y_{yx} & 0 & 0 & 0 \\ t^2 & -t^2 & 0 & -t^2 \end{bmatrix} + F(\omega) \begin{bmatrix} -Y & 0 & 0 & 0 \\ 0 & -Y & -2t^2 & 0 \\ -t^2 & 0 & -1 & 0 \\ t^2 & -t^2 & 0 & 1 \end{bmatrix} - \frac{2}{r} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 2\theta_{xy} & 0 \\ -1 & -1 & 0 & -2\theta_{yx} \end{bmatrix} + 2F(\mu) r^2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \right\} \Phi_\omega^*(x),$$

$$T^{22} = -(T^{11})^\dagger_{x \rightarrow y},$$

$$T^{21} = \frac{\omega}{8\pi} \Phi_\omega(y) \begin{bmatrix} 1 & 0 & r^2 & t^2 \\ 0 & 1 & t^2 & 0 \\ r^2 & t^2 & 0 & 1 \\ t^2 & r^2 & 0 & 1 \end{bmatrix} \Phi_\omega^*(x),$$

### Appendix D: Keldysh kernel of integral equation

$$\Pi^K = \frac{i}{2\pi} \Phi_\omega^*(x) \begin{bmatrix} F(\omega) & YF(\omega) \\ YF(\omega) & K(\omega) \end{bmatrix} \Phi_\omega(y)$$

with

$$K_{jl}(\omega) = \int_{-\infty}^{\infty} \frac{d\Omega}{2} \sum_{m,n} S'_{jm} S_{lm} S'_{ln} S_{jn}(1-h_m(\Omega)h_n(\Omega+\omega))$$

The latter quantity may be cast in the form

$$K(\omega) = F(\omega) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + r^2 t^2 F_2(\omega, V) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

with $F(\omega)$ and $F_2(\omega, V)$ defined in (17) and (C2), respectively. Importantly, $K(\omega)$ is an even function of $\omega$.

### Appendix E: Full form of the solution for $\hat{L}^0_j(x|y)$

The solution of (31) can be found as follows. We iterate the right-hand side of the equation once, to arrive at the diagonal kernel with components of the form

$$g^2 \delta\left(\frac{y-z}{\nu}\right) + i\frac{\gamma}{\nu} (e^{-i\omega|x+z|/\nu} + e^{-i\omega|x-z|/\nu})$$

and another component obtained from here by changing $x \rightarrow L-x$, $y \rightarrow L-y$. We pick first the easier part of this iterated kernel, $\propto \delta(x-z)$, and arrive at the equation for $\tilde{C}_j(x|y)$ with more complicated inhomogeneity instead of $L_0$ and non-singular kernel. This latter kernel shows a jump in its derivative at $x = z$, which we use by twice differentiating $\tilde{C}_j(x|y)$ with respect to $x$. We thus arrive at a second-order differential equation, similarly to what was done in (31). The difference now is that we deal with a $2 \times 2$ matrix for $\tilde{C}_j(x|y)$ for each wire $j$. We determine the solution to this differential equation dependent on the $x$ variable up to $x$ terms proportional to $e^{\pm i\omega x/(\nu_j \nu)}$ which are multiplied by as yet unknown matrices $\hat{A}(y)$, $\hat{B}(y)$, respectively. Considering the initial Eq. (31) for $\tilde{C}_j(x|y)$ in the simpler cases $x = 0$, $x = L$, we form a set of two coupled (matrix) equations for $\hat{A}(y)$, $\hat{B}(y)$, which is eventually solved. As a result we obtain the quantity $\tilde{C}$ diagonal in wire space, with its diagonal elements $\tilde{C}_j$ of the form

$$\tilde{C}_j(x|y) = \frac{2\pi v_j g_j}{d_j^2} \delta(x-y) \begin{bmatrix} -g_j, & 1 \\ -1, & -g_j \end{bmatrix} + i\frac{\omega \nu g_j^2}{d_j^2} \times e^{i\omega|x-y|/\nu_j \nu_j} \begin{bmatrix} d_j sgn(y-x) - 1, & g_j \\ g_j, & d_j sgn(x-y) - 1 \end{bmatrix} + i\frac{\pi \nu g_j^2}{d_j^2} \begin{bmatrix} e^{i\omega x/\nu_j \nu} \hat{A}_j(y) + d_j e^{i\omega (2L-x)/\nu_j \nu_j} \hat{B}_j(y) \end{bmatrix}$$

with $d_j$, $q_j$ defined in (35) and

$$\hat{A}_j(y) = \begin{bmatrix} g_j (q_j^{-1} - g_j), & g_j (1 - g_j q_j^{-1}) \\ (d_j - 1)(q_j^{-1} - g_j), & (d_j - 1)(1 - g_j q_j^{-1}) \end{bmatrix} \times \cos\left(\frac{\omega y}{v_j d_j}\right) + i d_j \sin\left(\frac{\omega y}{v_j d_j}\right) \begin{bmatrix} -g_j^{-1}, & 1 \\ -1, & -g_j^{-1} \end{bmatrix},$$

$$\hat{B}_j(y) = i \cos\left(\frac{\omega y}{v_j d_j}\right) \begin{bmatrix} (d_j - 1) g_j^{-1}, & (1 - d_j) \end{bmatrix} + d_j \sin\left(\frac{\omega y}{v_j d_j}\right) \begin{bmatrix} d_j^{-1}, & 0 \\ 1, & 0 \end{bmatrix}$$

We next use these expressions in Eq. (32), which can be schematically represented as

$$L = C + C \ast Y \ast L = C + C \ast Y \ast C,$$

$$\mathcal{Y} = Y + Y \ast C \ast Y + \ldots = Y \ast (1 - C \ast Y)^{-1}$$
and obtain finally
\[ L_{jk}^R(x|y) = \delta_{jk} \tilde{C}_j(x|y) - i\omega \frac{2\pi g_j g_k}{d_j d_k} \gamma_{jk} \]
\[ \times \left( V_{1,j}(x)V_{2,k}(y) - V_{1,j}(y)V_{2,k}(x) \right) \]
\[ V_{1,j}(x) = (1 - g_j q_j^{-1}) \cos(\frac{\omega y}{v_j d_j}) + i d_j \sin(\frac{\omega y}{v_j d_j}) \]
\[ V_{2,j}(x) = (q_j^{-1} - g_j) \cos(\frac{\omega y}{v_j d_j}) - i d_j q_j^{-1} \sin(\frac{\omega y}{v_j d_j}) \]
with \( \gamma \) given in Eq. \[37\]

Appendix F: Analytic properties of \( L(\omega) \).

In contrast to our previous studies\[15,19\], we see now the appearance of poles in the \( \omega \)-plane of the quantities \( q_j \), Eq. \[35\], which we further integrate over \( \omega \). Given the arbitrariness of the S-matrix, reflected in \( Y_{ik} = |S_{ik}|^2 \), we check here the absence of singularities in \( L^R(\omega) \) in the upper semiplane of complex \( \omega \).

The poles of \( q_j \) correspond to the solution of
\[ \tan \tilde{\omega} = -i d_j, \]
where we introduced \( \tilde{\omega} = \omega L/v_j d_j \). The last equation means that we have an infinite sequence of roots
\[ \tilde{\omega} = -i \arctanh d_j + \pi n, \quad n = 0, \pm 1, \pm 2, \ldots \]
hence the poles of \( q_j^{-1} \) are always in the lower semiplane of complex \( \omega \), as it should be for a retarded function.

Less trivial is the question about the position of the poles of the above expression \((1 - Q^{-1} \cdot Y)^{-1}\). We consider it for a simpler situation with identical wires, \( g_j = g \), \( d_j = d = \sqrt{1 - g^2} \), \( v_j = v \).

The poles are defined by
\[ \det(1 - Q^{-1} \cdot Y) = \det \left[ 1 - \frac{g Y}{1 + i d \cot \tilde{\omega}} \right] = 0. \]
Since the denominator in the last expression cannot modify the location of poles, and \( \sin \tilde{\omega} = 0 \) is not a solution, we can rewrite
\[ \det [i d \cot \tilde{\omega} + 1 - g Y] = 0. \]
Defining the eigenvalues of \( Y \) as \( \gamma_j \), with \( j = 1, \ldots, N \) (for a junction connecting \( N \) wires), the poles are defined by conditions (cf. above) \( \tan \tilde{\omega} = -i \frac{d}{1 - g \gamma_j} \), or
\[ \tilde{\omega} = -i \arctanh \frac{d}{1 - g \gamma_j} + \pi n, \quad j = 1, \ldots, N. \]

Generally we have \( |\gamma_j| \leq 1 \) for all \( j \). This is evident for \( N = 2 \), and can be easily extended to any \( N \). The proof of this statement is as follows:\[12\] Consider a set of diagonal \( N \times N \) matrices \( \lambda_j \) with values 1 in the \( j \)th row and 0 otherwise. This is a set of \( N \) generators of a Cartan subalgebra of the algebra \( U(N) \), normalized according to \( \text{Tr}(\lambda_j \lambda_k) = \delta_{jk} \). A rotation of these generators is defined as \( \tilde{\lambda}_j = S^j \lambda_j S \) where \( S \) is the unitary matrix. Obviously the new set \( \tilde{\lambda}_j \) remains orthonormal \( \text{Tr}(\tilde{\lambda}_j \tilde{\lambda}_k) = \delta_{jk} \). The operator \( P \), defined by \( PA = \sum \tilde{\lambda}_j \text{Tr}(\tilde{\lambda}_j A) \), is a projection operator, \( P^2 = P \).

We have \( P \lambda_k = \sum \tilde{\lambda}_j Y_{jk} \) because \( Y \) can be written as \( Y_{jk} = |S_{jk}|^2 = \text{Tr}(\tilde{\lambda}_j \tilde{\lambda}_k) \). Let \( \{\epsilon_j\} \) be an eigenvector of \( Y \), i.e. \( \sum \tilde{\lambda}_j \epsilon_j c_j = y \epsilon_j \). Introducing the diagonal matrix \( \lambda_* = \sum \epsilon_j \lambda_j \), we obtain \( P \lambda_* = y \lambda_* \), with \( \lambda_* \) is the rotated vector \( \lambda_* \). Since \( ||P \lambda_*|| \leq ||\lambda_*|| \text{ and } ||\tilde{\lambda}_j|| = ||\lambda_*|| \), we conclude that \( |y| \leq 1 \).

It follows that the above ratio \( \frac{d}{1 - g \gamma_j} \) is always positive. As a result, the poles of \( (1 - Q^{-1} \cdot Y)^{-1} \) lie in the lower semiplane of \( \omega \).

Appendix G: Scaling variable \( \Lambda \) in the crossover regime.

In the limit \( L \to \infty \) we have to evaluate the integral
\[ P(T, V) = \int_{-W}^{W} \frac{d\omega}{\omega} [F(\omega + V) - F(\omega - V)], \] (G1)
with the upper cutoff being \( W = v_f/a \) or the bandwidth.

After the rescaling \( \omega = 2Tx \), \( V = 2Ty \), \( W = 2Tw \) we have
\[ P(T, V) = 2T \int_{-w}^{w} \frac{dx}{x} \left[ \frac{x + y}{\tanh(x + y)} - \frac{x - y}{\tanh(x - y)} \right]. \]
We can make a shift \( x \to x \pm y \) in two terms here; the contributions from the limits \( \pm w \) do not cancel upon this shift, but add a constant in the limit \( w \to \infty \). After simple calculation we get in this limit
\[ P(T, V) = 4Ty \left( 2 + \int_{-w}^{w} \frac{dx}{x^2 - y^2} \right), \] (G2)
where the principal value of the integral should be taken.

Next we close the contour of integration in the upper semiplane of complex \( x \), by adding a semicircle of radius \( w \). The contribution from this semicircle vanishes as \( O(w^{-2}) \) and we reduce the remaining integral to the sum over the residues of \( \coth x \) at \( x = i\pi n \) as follows:
\[ \int_{-w}^{w} \frac{dx}{x^2 - y^2} \to \sum_{n=1}^{w/\pi} \frac{2n}{n^2 + (y/\pi)^2}, \] (G3)
The last sum is easily evaluated with the final result of the form \( P(T, V) = 4V \Lambda \) with \( \Lambda \) given by
\[ \Lambda = \ln \left( \frac{W \pi}{2 \pi e} \right) - \frac{1}{2} \left[ \psi \left( 1 + \frac{V}{2 \pi e} \right) + \psi \left( 1 - \frac{V}{2 \pi e} \right) \right]. \] (G4)
with \( e = 2.718 \ldots \) and \( \psi(x) \) digamma function.
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