Chaos of a Markov operator and the fourth moment condition

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fourth-moment theorem

D. Nualart, G. Peccati (2005)

condition for a Wiener chaos to be close to Gaussian
multiple Wiener integrals

simplified (finite-dimensional) model

Wiener (Gaussian) chaos of order \( k \)

\[
F = F(x) = \sum_{i_1,\ldots,i_k=1}^{N} a_{i_1,\ldots,i_k} x_{i_1} \cdots x_{i_k}, \quad x = (x_1, \ldots, x_N) \in \mathbb{R}^N
\]

multi-linear form

\( a_{i_1,\ldots,i_k} \in \mathbb{R} \) symmetric, vanishing on diagonals

\[
\int_{\mathbb{R}^N} F^2 d\gamma_N = 1 \quad \left( \int_{\mathbb{R}^N} F d\gamma_N = 0 \right)
\]

\[
d\gamma_N(x) = e^{-|x|^2/2} \frac{dx}{(2\pi)^n} \quad \text{standard Gaussian measure on } \mathbb{R}^N
\]
multiple Wiener integrals

simplified (finite-dimensional) model

Wiener (Gaussian) chaos of order $k$

\[ F = F(X) = \sum_{i_1,...,i_k=1}^{N} a_{i_1,...,i_k} X_{i_1} \cdots X_{i_k} \]

$X_1, \ldots, X_N$ independent standard normal

$a_{i_1,...,i_k} \in \mathbb{R}$ symmetric, vanishing on diagonals

\[ \mathbb{E}(F(X)^2) = 1 \quad (\mathbb{E}(F(X)) = 0) \]
\[ F = F_n, \quad n \in \mathbb{N} \quad k\text{-chaos} \quad (k \ \text{fixed}) \]

\[ N = N_n \to \infty \]

\[ \int_{\mathbb{R}^N_n} F_n^2 \, d\gamma_{N_n} = 1 \quad \text{(or} \quad \to 1) \]

**Theorem** by D. Nualart, G. Peccati (2005)

distribution of \( F_n \) converges to \( \gamma_1 \) (standard normal on \( \mathbb{R} \))

\[ \int_{\mathbb{R}^N_n} F_n^4 \, d\gamma_{N_n} \to 3 \quad \left( = \int_{\mathbb{R}} x^4 \, d\gamma_1 \right) \]

striking reduction of the moment method
Wiener chaos (multiple Wiener integrals)

\[ F = \int_{[0,1]^k} f(t_1, \ldots, t_k) dB_{t_1} \cdots dB_{t_k} \]

\[ f \in L^2([0,1]^k; \mathbb{R}) \text{ symmetric} \]

\[ F = I_k(f) \]
main tool: multiplication formula

\[ l_k(f) l_\ell(g) = \sum_{r=0}^{k \wedge \ell} r! \binom{k}{r} \binom{\ell}{r} l_{k+\ell-2}(f \tilde{\otimes} rg) \]
main tool: multiplication formula

\[ H_k H_\ell = \sum_{r=0}^{k \wedge \ell} r! \binom{k}{r} \binom{\ell}{r} H_{k+\ell-2r} \]

\(H_k\) Hermite polynomials
main tool: multiplication formula

\[ l_k(f) l_\ell(g) = \sum_{r=0}^{k \wedge \ell} r! \binom{k}{r} \binom{\ell}{r} l_{k+\ell-2}(f \tilde{\otimes}_r g) \]

contraction

\[ f \otimes_r g = \int_{[0,1]^r} f(t_1, \ldots, t_{k-r}, s_1, \ldots, s_r) \times g(t_{k-r+1}, \ldots, t_{k+\ell-2r}, s_1, \ldots, s_r) ds_1 \cdots ds_r \]

\[ f \tilde{\otimes}_r g \quad \text{symmetrized} \]

\[ \int_{\mathbb{R}^N_n} F^4_n \, d\gamma_{N_n} \to 3 \quad \text{implies} \]

\[ \| f_n \tilde{\otimes}_p f_n \|_2 \to 0, \quad p = 1, \ldots, k - 1 \]

combinatorial arguments
D. Nualart, G. Peccati (2005)

\[ I_k(f_n) = W_{T_n} \quad \text{time change} \quad T_n \to 1 \]

I. Nourdin, G. Peccati (2009)

moments \[ \mathbb{E}(I_k(f_n)^q) \to \int_{\mathbb{R}} x^q d\gamma_1 \]

D. Nualart, S. Ortiz-Latorre (2008)

stochastic calculus (Malliavin)

\[ \text{further equivalence} \]

\[ \text{Var}_{\gamma_{N_n}}(\|\nabla F_n\|^2) \to 0 \]

I. Nourdin, J. Rosinski (2012)

covariance criterion
first objectives

Gaussian $k$-chaos

\[ F = F(x) = \sum_{i_1, \ldots, i_k = 1}^{N} a_{i_1, \ldots, i_k} x_{i_1} \cdots x_{i_k} \]

understand what is used on $F$

why a fourth moment condition

\[ \int_{\mathbb{R}^N} F^4 d\gamma_N \sim 3 \]

connection with $\text{Var}_{\gamma_N}(|\nabla F|^2)$
Gaussian $k$-chaos

\[ F = F(x) = \sum_{i_1, \ldots, i_k=1}^{N} a_{i_1, \ldots, i_k} x_{i_1} \cdots x_{i_k} \]

first feature: **eigenfunction**, eigenvalue $k$

\[ -LF = kF \]

\[ L = \Delta - x \cdot \nabla \quad \text{Ornstein-Uhlenbeck operator on } \mathbb{R}^N \]

\[ L x_1 = -x_1, \quad L x_1 x_2 = -2 x_1 x_2 \]

invariant (reversible) measure $\gamma_N = \gamma$

integration by parts

\[ \int_{\mathbb{R}^N} f(-Lg) d\gamma = \int_{\mathbb{R}^N} \nabla f \cdot \nabla g \, d\gamma \]
connection with \( \text{Var}_{\gamma} (|\nabla F|^2) \)

\[-LF = k F\]

integration by parts

\[k \int_{\mathbb{R}^N} F^4 d\gamma = \int_{\mathbb{R}^N} F^3 (-LF) d\gamma = 3 \int_{\mathbb{R}^N} F^2 |\nabla F|^2 d\gamma\]

similarly

\[k \int_{\mathbb{R}^N} F^2 d\gamma = \int_{\mathbb{R}^N} F (-LF) d\gamma = \int_{\mathbb{R}^N} |\nabla F|^2 d\gamma\]

normalization \( \int_{\mathbb{R}^N} F^2 d\gamma = 1 \)

\[\int_{\mathbb{R}^N} |\nabla F|^2 d\gamma = k\]

\[\text{Var}_{\gamma} (|\nabla F|^2) = \int_{\mathbb{R}^N} (|\nabla F|^2 - k)^2 d\gamma\]
\[ k \int_{\mathbb{R}^N} F^4 d\gamma = \int_{\mathbb{R}^N} F^3(-LF) d\gamma = 3 \int_{\mathbb{R}^N} F^2 |\nabla F|^2 d\gamma \]

\[ \int_{\mathbb{R}^N} |\nabla F|^2 d\gamma = k \]

\[ k \left( \frac{1}{3} \int_{\mathbb{R}^N} F^4 d\gamma - 1 \right) = \int_{\mathbb{R}^N} F^2 (|\nabla F|^2 - k) d\gamma \]

\[ \int_{\mathbb{R}^N} F^4 d\gamma \sim 3 \implies |\nabla F|^2 \sim k \]

\[ \text{Var}_\gamma(|\nabla F|^2) = \int_{\mathbb{R}^N} (|\nabla F|^2 - k)^2 d\gamma \]

\[ \text{technical task} \]

\[ \text{from } \int_{\mathbb{R}^N} F^2(|\nabla F|^2 - k) d\gamma \text{ to } \int_{\mathbb{R}^N} (|\nabla F|^2 - k)^2 d\gamma \]
main step

if \( \int_{\mathbb{R}^N} F^4 d\gamma \sim 3 \)

then 
\[ \text{Var}_\gamma (|\nabla F|^2) = \int_{\mathbb{R}^N} (|\nabla F|^2 - k)^2 d\gamma \sim 0 \]

if \( |\nabla F|^2 \sim k \quad (\text{Var}_\gamma (|\nabla F|^2) \sim 0) \)

then the distribution of \( F \) is approximatively Gaussian
second (main) step

if \[ \int_{\mathbb{R}^N} F^4 d\gamma \sim 3 \]

then \[ \text{Var}_\gamma(|\nabla F|^2) = \int_{\mathbb{R}^N} (|\nabla F|^2 - k)^2 d\gamma \sim 0 \]

first step

if \[ |\nabla F|^2 \sim k \quad (\text{Var}_\gamma(|\nabla F|^2) \sim 0) \]

then the distribution of \( F \)

is approximatively Gaussian
first step

Ornstein-Uhlenbeck operator on $\mathbb{R}^N$

$$L = \Delta - x \cdot \nabla$$

$F : \mathbb{R}^N \to \mathbb{R}$ Gaussian chaos

$F$ eigenfunction of $L$

$$-LF = \lambda F \quad (\lambda > 0)$$

$\varphi : \mathbb{R} \to \mathbb{R}$ smooth

chain rule formula for $L$ (Laplacian)

$$L(\varphi \circ F) = \varphi'(F) LF + \varphi''(F) |\nabla F|^2 = -\lambda F \varphi'(F) + \varphi''(F) |\nabla F|^2$$
\[ L(\varphi \circ F) = -\lambda F\varphi'(F) + \varphi''(F) |\nabla F|^2 \]

if \( |\nabla F|^2 = \lambda \) then

\[ L(\varphi \circ F) = -\lambda F\varphi'(F) + \lambda \varphi''(F) \]

\[ L(\varphi \circ F) = \lambda (L\varphi)(F) \]

\( L\psi = \psi'' - x\psi' \) on \( \mathbb{R} \) (one-dimensional O-U operator)

\( \gamma \#_F \) distribution of \( F : \mathbb{R}^N \to \mathbb{R} \) under \( \gamma \)

\[ 0 = \int_{\mathbb{R}^N} L(\varphi \circ F) d\gamma = \lambda \int_{\mathbb{R}} L\varphi d\gamma \#_F \]

\( \gamma \#_F \) invariant measure of \( L \)

\( \gamma \#_F = \gamma_1 \)
Stein’s method

quantify the preceding

I. Nourdin, G. Peccati (2009)

\[-L F = \lambda F\]

\(\gamma\#_F\) distribution of \(F : \mathbb{R}^N \rightarrow \mathbb{R}\) under \(\gamma\)

\[
\left| \int_{\mathbb{R}} \varphi \, d\gamma\#_F - \int_{\mathbb{R}} \varphi \, d\gamma_1 \right| \leq \frac{C\varphi}{\lambda} \Var_{\gamma}(\|\nabla F\|^2)^{1/2}
\]

sufficiently many smooth \(\varphi : \mathbb{R} \rightarrow \mathbb{R}\)

if \(\Var_{\gamma}(\|\nabla F\|^2) \sim 0\)

then distribution \(\gamma\#_F\) close to Gaussian \(\gamma_1\)
second (main) step

when does

$$\int_{\mathbb{R}^N} F^4 d\gamma \sim 3$$

imply that

$$\text{Var}_\gamma(|\nabla F|^2) \sim 0 ?$$

is it enough to use

$$-L F = k F ?$$

more information is needed
convenient framework
calculus
Markov operator \( L \) on state space \( E \)
\( \mu \) invariant symmetric probability measure
\( \Gamma \) (bilinear) operator
\[
\Gamma(f, g) = \frac{1}{2} \left[ L(fg) - fLg - gLf \right]
\]
\( f, g : E \to \mathbb{R} \) in some nice algebra \( \mathcal{A} \)
integration by parts
\[
\int_E f(-Lg) \, d\mu = \int_E \Gamma(f, g) \, d\mu
\]
Ornstein-Uhlenbeck operator on $E = \mathbb{R}^N$

$$L = \Delta - x \cdot \nabla$$

invariant measure $\mu = \gamma$

$\gamma$ standard Gaussian measure on $\mathbb{R}^N$

$$\Gamma(f, g) = \nabla f \cdot \nabla g$$

$$\int_{\mathbb{R}^N} f(-Lg) d\gamma = \int_{\mathbb{R}^N} \nabla f \cdot \nabla g d\gamma$$
iterated gradients

\[ \Gamma_m, \quad m \geq 2 \]

\[ \Gamma(f, g) = \frac{1}{2} [L(fg) - fLg - gLf] \]

\[ \Gamma_2(f, g) = \frac{1}{2} [L \Gamma(f, g) - \Gamma(f, Lg) - \Gamma(g, Lf)] \]

D. Bakry, M. Émery (1985)

\( \Gamma_2 \) operator (criterion) : Bochner’s formula

\[ \Gamma_m(f, g) = \frac{1}{2} [L \Gamma_{m-1}(f, g) - \Gamma_{m-1}(f, Lg) - \Gamma_{m-1}(g, Lf)] \]

\[ \Gamma_0(f, f) = f^2, \quad \Gamma_1 = \Gamma \]

\[ \Gamma_m(f) = \Gamma_m(f, f) \]
example

Ornstein-Uhlenbeck operator on $E = \mathbb{R}^N$

$L = \Delta - x \cdot \nabla$

\[
\Gamma(f, f) = \Gamma(f) = \Gamma_1(f) = |\nabla f|^2
\]

$\Gamma_2(f) = |\nabla \otimes^2 f|^2 + |\nabla f|^2$

$\Gamma_3(f) = |\nabla \otimes^3 f|^2 + 3 |\nabla \otimes^2 f|^2 + |\nabla f|^2$

$|\nabla \otimes^2 f|^2 = \Gamma_2(f) - \Gamma_1(f)$

$|\nabla \otimes^3 f|^2 = \Gamma_3(f) - 3 \Gamma_2(f) + 2 \Gamma_1(f)$
Gaussian chaos

\[ F = F(x) = \sum_{i_1, \ldots, i_k=1}^{N} a_{i_1, \ldots, i_k} x_{i_1} \cdots x_{i_k} \]

\[-L F = k F\]

\[ |\nabla^{\otimes k+1} f|^2 = 0 \]

\[ (|\nabla^{\otimes k} f|^2 = \text{constant}) \]

intrinsic description in terms of the \( \Gamma_m \)
L (pure) point spectrum

\[ S = \{ 0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots \} \]

polynomials

\[ Q_k(X) = \prod_{i=0}^{k-1} (X - \lambda_i) = \sum_{i=1}^{k} q_i X^i \]

\[ Q_k(\Gamma) = \sum_{i=1}^{k} q_i \Gamma_i \]

bilinear forms on \( A \times A \)

\[ Q_k(\Gamma)(f, f) = Q_k(\Gamma)(f) \]
example

Ornstein-Uhlenbeck operator on \( E = \mathbb{R}^N \)

\[
L = \Delta - x \cdot \nabla
\]

spectrum \( S = \mathbb{N} \)

\[
Q_2(X) = X(X - \lambda_1) = X^2 - X
\]

\[
|\nabla^\otimes 2 f|^2 = \Gamma_2(f) - \Gamma_1(f) = Q_2(\Gamma)(f)
\]

\[
Q_3(X) = X(X - \lambda_1)(X - \lambda_2) = X^3 - 3X^2 + 2X
\]

\[
|\nabla^\otimes 3 f|^2 = \Gamma_3(f) - 3\Gamma_2(f) + 2\Gamma_1(f) = Q_3(\Gamma)(f)
\]

\[
|\nabla^\otimes^k f|^2 = Q_k(\Gamma)(f)
\]
**Definition** An eigenfunction $F$ of $-L$ with eigenvalue $\lambda_k$

$$-LF = \lambda_k F$$

is said to be a *chaos of degree* $k \geq 1$ relative to $S = (\lambda_n)_{n \in \mathbb{N}}$ if

$$Q_{k+1}(\Gamma)(F) = 0$$

($\mu$-almost everywhere)

$F$ is a chaos eigenfunction (with eigenvalue $\lambda_k$)

Wiener (Gaussian) chaos

$$F = F(x) = \sum_{i_1,\ldots,i_k=1}^{N} a_{i_1,\ldots,i_k} x_{i_1} \cdots x_{i_k}$$

$F$ is a $k$-chaos eigenfunction
\[ Q_{k+1}(X) = \prod_{i=0}^{k} (X - \lambda_i) = \sum_{i=1}^{k+1} q_i X^i \]

Further polynomials

\[ R_{k+1}(X) = \frac{1}{X^2} \left[ Q_{k+1}(X) - q_1 X \right] = \sum_{i=2}^{k+1} q_i X^{i-2} \]

\[ T_{k+1}(X) = R_{k+1}(X + \lambda_k) - R_{k+1}(\lambda_k) \]

For example

\[ Q_2(X) = X^2 - \lambda_1 X, \quad R_2 \equiv 1, \quad T_2 \equiv 0 \]

\[ Q_3(X) = X^3 - (\lambda_1 + \lambda_2)X^2 + \lambda_1 \lambda_2 X \]

\[ R_3(X) = X - (\lambda_1 + \lambda_2), \quad T_3(X) = X \]
The main statement is
\[ R_{k+1}(X) = \frac{1}{X^2} \left[ Q_{k+1}(X) - q_1 X \right] = \sum_{i=2}^{k+1} q_i X^{i-2} \]

\[ T_{k+1}(X) = R_{k+1}(X + \lambda_k) - R_{k+1}(\lambda_k) \]

\[ \pi_k = \lambda_1 \cdots \lambda_k, \quad k \geq 1 \quad (\pi_0 = 1) \]

**Theorem** Let \( F \) be a \( k \)-chaos eigenfunction with eigenvalue \( \lambda_k \). Set \( \Gamma = \Gamma(F) \). Then

\[ \pi_{k-1} \int_E \Gamma^2 d\mu = \pi_k \int_E F^2 \Gamma d\mu + (-1)^k \int_E \Gamma T_{k+1}(\frac{\Gamma}{2}) \Gamma d\mu \]
\[ \pi_{k-1} \int_E \Gamma^2 d\mu = \pi_k \int_E F^2 \Gamma d\mu + (-1)^k \int_E \Gamma T_{k+1}(\frac{1}{2}) \Gamma d\mu \]

**Corollary**  
*Recall the spectrum* \( S = (\lambda_n)_{n \in \mathbb{N}} \) *of* \(-L\). If

\[ (-1)^k T_{k+1}(-\frac{\lambda_n}{2}) \leq 0 \quad \text{for every} \quad n \in \mathbb{N} \]

*then*

\[ \int_E \Gamma^2 d\mu \leq \lambda_k \int_E F^2 \Gamma d\mu \]

\( F \) *normalized in* \( L^2(\mu) \)

\[ \int_E \Gamma d\mu = \int_E F(-LF)d\mu = \lambda_k \int_E F^2 d\mu = \lambda_k \]

\[ \text{Var}_\mu(\Gamma) \leq \lambda_k \left( \int_E F^2 \Gamma d\mu - \lambda_k \right) \]
\[ k \int_{\mathbb{R}^N} F^4 d\gamma = \int_{\mathbb{R}^N} F^3(-LF) d\gamma = 3 \int_{\mathbb{R}^N} F^2 |\nabla F|^2 d\gamma \]

\[ \int_{\mathbb{R}^N} |\nabla F|^2 d\gamma = k \]

\[ k \left( \frac{1}{3} \int_{\mathbb{R}^N} F^4 d\gamma - 1 \right) = \int_{\mathbb{R}^N} F^2 (|\nabla F|^2 - k) d\gamma \]

\[ \int_{\mathbb{R}^N} F^4 d\gamma \sim 3 \implies |\nabla F|^2 \sim k \]

\[ \text{Var}_\gamma (|\nabla F|^2) = \int_{\mathbb{R}^N} (|\nabla F|^2 - k)^2 d\gamma \]

**technical task**

from \[ \int_{\mathbb{R}^N} F^2 (|\nabla F|^2 - k) d\gamma \] to \[ \int_{\mathbb{R}^N} (|\nabla F|^2 - k)^2 d\gamma \]
\[ \text{Var}_\mu(\Gamma) \leq \lambda_k \left( \int_E F^2 \Gamma \, d\mu - \lambda_k \right) \]

recall: integration by parts \( \int_E F^2 \, d\mu = 1 \)

\[ \lambda_k \left( \frac{1}{3} \int_E F^4 \, d\mu - 1 \right) = \int_E F^2 \Gamma \, d\mu - \lambda_k \]

\[ \text{Var}_\mu(\Gamma) \leq \lambda_k^2 \left( \frac{1}{3} \int_E F^4 \, d\mu - 1 \right) \]

if \( \int_E F^4 \, d\mu = 3 \) then \( \Gamma = \Gamma(F) \) constant

Stein’s method (first step)

distribution of \( F \) is Gaussian

Nualart-Peccati theorem
main statement

\[ R_{k+1}(X) = \frac{1}{X^2} \left[ Q_{k+1}(X) - q_1 X \right] = \sum_{i=2}^{k+1} q_i X^{i-2} \]

\[ T_{k+1}(X) = R_{k+1}(X + \lambda_k) - R_{k+1}(\lambda_k) \]

\[ \pi_k = \lambda_1 \cdots \lambda_k, \quad k \geq 1 \quad (\pi_0 = 1) \]

**Theorem**  Let \( F \) be a \( k \)-chaos eigenfunction with eigenvalue \( \lambda_k \). Set \( \Gamma = \Gamma(F) \). Then

\[ \pi_{k-1} \int_E \Gamma^2 d\mu = \pi_k \int_E F^2 \Gamma d\mu + (-1)^k \int_E \Gamma T_{k+1}(\frac{1}{2}) \Gamma d\mu \]
key argument of the proof

$F$ eigenfunction of $-L$ eigenvalue $\lambda$, $-LF = \lambda F$

$\Gamma_m = \Gamma_m(F), \quad m \geq 1$

$\Gamma_m(F) = \frac{1}{2} L \Gamma_{m-1}(F) - \Gamma_{m-1}(F, L F)$

$\Gamma_m = \frac{1}{2} L \Gamma_{m-1} + \lambda \Gamma_{m-1}$

consequences

$\Gamma_m = \left(\frac{1}{2} L + \lambda \text{Id}\right)^{m-1} \Gamma$

$\int_E F^2 \Gamma_m d\mu = \int_E \Gamma_0 \Gamma_m d\mu = \int_E \Gamma \Gamma_{m-1} d\mu$
**proof when** \( k = 1 \)

\[
Q_2(X) = X^2 - \lambda_1 X, \quad R_2 \equiv 1, \quad T_2 \equiv 0, \quad \pi_1 = \lambda_1
\]

\[
F \text{ 1-chaos}
\]

\[-LF = \lambda_1 F
\]

\[
Q_2(\Gamma) = \Gamma_2 - \lambda_1 \Gamma = 0
\]

**multiply by** \( F^2 \), **integrate**

\[
\int_E F^2 \Gamma_2 \, d\mu = \lambda_1 \int_E F^2 \Gamma \, d\mu
\]
key argument of the proof

\( F \) eigenfunction of \(-L\) eigenvalue \( \lambda \), \(-LF = \lambda F\)

\[ \Gamma_m = \Gamma_m(F), \quad m \geq 1 \]

\[ \Gamma_m(F) = \frac{1}{2} L \Gamma_{m-1}(F) - \Gamma_{m-1}(F, LF) \]

\[ \Gamma_m = \frac{1}{2} L \Gamma_{m-1} + \lambda \Gamma_{m-1} \]

consequences

\[ \Gamma_m = \left( \frac{1}{2} L + \lambda \text{Id} \right)^{m-1} \Gamma \]

\[ \int_E F^2 \Gamma_m d\mu = \int_E \Gamma_0 \Gamma_m d\mu = \int_E \Gamma \Gamma_{m-1} d\mu \]
proof when $k = 1$

\[ Q_2(X) = X^2 - \lambda_1 X, \quad R_2 \equiv 1, \quad T_2 \equiv 0, \quad \pi_1 = \lambda_1 \]

\[ F \text{-1-chaos} \]

\[ -LF = \lambda_1 F \]

\[ Q_2(\Gamma) = \Gamma_2 - \lambda_1 \Gamma = 0 \]

multiply by $F^2$, integrate

\[ \int_E F^2 \Gamma_2 \, d\mu = \lambda_1 \int_E F^2 \Gamma \, d\mu \]

\[ \pi_{k-1} \int_E \Gamma^2 \, d\mu = \pi_k \int_E F^2 \Gamma \, d\mu + (-1)^k \int_E \Gamma \, T_{k+1}(\frac{1}{2}) \, d\mu \]
proof when \( k = 1 \)

\[
Q_2(X) = X^2 - \lambda_1 X, \quad R_2 \equiv 1, \quad T_2 \equiv 0, \quad \pi_1 = \lambda_1
\]

\[
F \quad \text{1-chaos}
\]

\[
-LF = \lambda_1 F
\]

\[
Q_2(\Gamma) = \Gamma_2 - \lambda_1 \Gamma = 0
\]

multiply by \( F^2 \), integrate

\[
\int_E \Gamma^2 \, d\mu = \lambda_1 \int_E F^2 \Gamma \, d\mu
\]

\[
\pi_{k-1} \int_E \Gamma^2 \, d\mu = \pi_k \int_E F^2 \Gamma \, d\mu + (-1)^k \int_E \Gamma T_{k+1}(\frac{1}{2}) \Gamma \, d\mu
\]
proof when \( k = 2 \)

\[
Q_3(X) = X^3 - (\lambda_1 + \lambda_2)X^2 + \lambda_1\lambda_2X
\]

\[
R_3(X) = X - (\lambda_1 + \lambda_2), \quad T_3(X) = X
\]

\[
\pi_2 = \lambda_2\lambda_1, \quad \pi_1 = \lambda_1
\]

\[
F \quad \text{2-chaos}
\]

\[
-LF = \lambda_2 F
\]

\[
Q_3(\Gamma) = \Gamma_3 - (\lambda_1 + \lambda_2)\Gamma_2 + \lambda_1\lambda_2 \Gamma = 0
\]

multiply by \( F^2 \), integrate

\[
\int_E \Gamma \Gamma_2 d\mu - (\lambda_1 + \lambda_2) \int_E \Gamma^2 d\mu + \lambda_1\lambda_2 \int_E F^2 \Gamma d\mu = 0
\]
\[
\int_E \Gamma \Gamma_2 d\mu - (\lambda_1 + \lambda_2) \int_E \Gamma^2 d\mu + \lambda_1 \lambda_2 \int_E F^2 \Gamma d\mu = 0
\]

\[
\Gamma_2 = \frac{1}{2} L \Gamma + \lambda_2 \Gamma
\]

\[
\frac{1}{2} \int_E \Gamma L \Gamma d\mu - \lambda_1 \int_E \Gamma^2 d\mu + \lambda_1 \lambda_2 \int_E F^2 \Gamma d\mu = 0
\]

\[
T_3(X) = X
\]

\[
\pi_{k-1} \int_E \Gamma^2 d\mu = \pi_k \int_E F^2 \Gamma d\mu + (-1)^k \int_E \Gamma T_{k+1}(\frac{1}{2}) \Gamma d\mu
\]
\[
\pi_{k-1} \int_E \Gamma^2 d\mu = \pi_k \int_E F^2 \Gamma d\mu + (-1)^k \int_E \Gamma T_{k+1} \left( \frac{1}{2} \right) \Gamma d\mu
\]

**Corollary**  Recall the spectrum \( S = (\lambda_n)_{n \in \mathbb{N}} \) of \( -L \). If

\[
(-1)^k T_{k+1} \left( - \frac{\lambda_n}{2} \right) \leq 0 \quad \text{for every} \quad n \in \mathbb{N}
\]

then

\[
\int_E \Gamma^2 d\mu \leq \lambda_k \int_E F^2 \Gamma d\mu
\]

\[
\text{Var}_\mu(\Gamma) \leq \lambda_k \left( \int_E F^2 \Gamma d\mu - \lambda_k \right)
\]
spectral condition

\((-1)^k T_{k+1}( - \frac{\lambda_n}{2}) \leq 0, \quad n \in \mathbb{N}\)

**Theorem**  The *spectral condition*

\((-1)^k T_{k+1}( - \frac{\lambda_n}{2}) \leq 0, \quad n \in \mathbb{N}\)

is *satisfied* when

\[
S = (\lambda_n)_{n \in \mathbb{N}} = \mathbb{N}
\]

Wiener (Gaussian) chaos

**Nualart-Peccati** theorem
spectral condition

\[ (-1)^k T_{k+1}( - \frac{\lambda_n}{2} ) \leq 0, \quad n \in \mathbb{N} \]

\[ S = (\lambda_n)_{n \in \mathbb{N}} = \mathbb{N} \]

\[ Q_{k+1}(X) = \prod_{i=0}^{k} (X - i) \]

\[ \pi_k = k! \]

\[ \left( \frac{n}{2} - k \right)^{-2} \left[ \prod_{i=0}^{k} \left( \frac{n}{2} - i \right) - k! \left( \frac{n}{2} - k \right) \right] \geq (k - 1)! \]

elementary exercise
extensions

infinite dimensional Wiener chaos

abstract Markov chaos

continuous and discrete (cube, Poisson $S = (\lambda_n)_{n \in \mathbb{N}} = \mathbb{N}$)

convergence to other distributions (gamma)

I. Nourdin, G. Peccati (2009)

Wigner chaos (free probability) ?

T. Kemp, I. Nourdin, G. Peccati, R. Speicher (2012)
convergence to gamma distributions

**Theorem**  
F-k-chaos with eigenvalue \( \lambda_k \) such that  
\[ \int_E F^2 \, d\mu = p > 0. \]  
Set  \( \Gamma = \Gamma(F) \). Under the spectral condition

\[ (-1)^k T_{k+1}( - \frac{\lambda_n}{2} ) \leq 0, \quad n \in \mathbb{N} \]

it holds

\[
\text{Var}_\mu(\Gamma - \lambda_k F) \leq \lambda_k \int_E F^2 \, \Gamma \, d\mu + A_k \int_E F \, \Gamma \, d\mu - pB_k - p^2 \lambda_k^2
\]

where

\[
A_k = \frac{2(-1)^k \lambda_k}{\pi_{k-1}} R_{k+1}(\frac{\lambda_k}{2}) \quad \text{and} \quad B_k = \frac{(-1)^k \lambda_k^2}{\pi_{k-1}} R_{k+1}(\frac{\lambda_k}{2})
\]
\[ S = \mathbb{N} \]

\( k \) even

\[ \frac{3}{k^2} \text{Var}_\mu(\Gamma - k F) \leq \int_E F^4 d\mu - 6 \int_E F^3 d\mu + 6p - 3p^2 \]

\[ (F_n)_{n \in \mathbb{N}} \quad \text{sequence of } k\text{-chaos} \quad \int_E F_n^2 d\mu = p \]

if \[ \int_E F_n^4 d\mu - 6 \int_E F_n^3 d\mu + 6p - 3p^2 \rightarrow 0 \]

then \[ (F_n + p)_{n \in \mathbb{N}} \quad \text{converges in distribution} \]

\[ \text{to gamma distribution } p \]

I. Nourdin, G. Peccati (2009)