Four-Index Equations for Gravitation and the Gravitational Energy-Momentum Tensor *

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Abstract

A new treatment of the gravitational energy on the basis of 4-index gravitational equations is reviewed. The gravitational energy for the Schwarzschild field is considered.

1 Introduction

A covariant physical characteristics of the gravitational field is the Riemann curvature tensor, and it is natural that the problems with the energy-momentum of gravitation can be solved if we can express the gravitational energy in terms of this tensor.

In the papers [1], [2] a new generalized 4-index version of the Einstein equations with the Riemann tensor has been formulated, and the local energy-momentum tensors for the system of gravitation field and matter, linearly depending on the curvature tensor, have been constructed as 4-index tensors.

In the present paper some consequences of this treatment, including the calculation of the gravitational energy for a mass point, will be presented.

2 Four-index equations for the gravitational field

In the standard Einstein-Gilbert gravitational action one can add to the Ricci tensor or to the Riemann tensor arbitrary functions (tensors) with zero contractions:

\[ R = g^{km} R_{km} = \frac{1}{2} (g^{km} g^{il} - g^{im} g^{kl}) (R_{iklm} - \kappa V_{iklm}), \]

where \( \kappa = 8\pi k \), and \( L_m \) is the matter Lagrangian, \( V_{iklm} \) has the same symmetry properties as \( R_{iklm} \), and \( g^{il} V_{iklm} = 0 \).

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So, we can start from the new action function:

\[
S = \frac{1}{2} \int d\Omega \sqrt{-g} \left[ \frac{1}{2} (g^{km} g^{il} - g^{im} g^{kl}) \left( -\frac{1}{\kappa} R_{iklm} + V_{iklm} \right) + L_m \right],
\]

(2)

which is fully equivalent to the Einstein-Gilbert action function Eq. (1). Then we obtain for the variation of the action function [1]:

\[
\delta S = -\frac{1}{2} \int d\Omega \sqrt{-g} \delta g^{km} g^{il} (G_{iklm} - T_{iklm}) = 0,
\]

(3)

where \( T_{iklm} = V_{iklm} + T^{(m)}_{iklm} \), and:

\[
G_{iklm} = \frac{1}{\kappa} \left[ R_{iklm} - \frac{1}{(d-1)(d-2)} (g_{il} g_{km} - g_{im} g_{kl}) R \right],
\]

(4)

\[
T^{(m)}_{iklm} = \frac{1}{(d-2)} (g_{km} T_{il} - g_{kl} T_{im} + g_{il} T_{km} - g_{im} T_{kl}) - \frac{1}{(d-1)(d-2)} (g_{il} g_{km} - g_{im} g_{kl}) T.
\]

(5)

Here \( d \) is the spacetime dimensionality, and \( T_{iklm} \) has the same structure as the Riemann tensor having the representation:

\[
R_{iklm} = C_{iklm} + \frac{1}{(d-2)} (g_{km} R_{il} - g_{kl} R_{im} + g_{il} R_{km} - g_{im} R_{kl}) - \frac{1}{(d-1)(d-2)} (g_{il} g_{km} - g_{im} g_{kl}) R,
\]

(6)

where \( C_{iklm} \) is the Weyl tensor with zero contractions \( g^{il} C_{iklm} = g^{km} C_{iklm} = 0 \).

Thus, we obtain the equations:

\[
g^{il} (G_{iklm} - T_{iklm}) = 0.
\]

(7)

In a general case the expression in the parenthesis is not equal to zero for the arbitrary \( V_{iklm} \) and we can not simply exclude the contractional factor \( g^{il} \). However the tensor \( V_{iklm} \) has 10 independent components which is equal to the number of the Riemann tensor components in the vacuum \( (T^{(m)}_{km} = 0) \) where it is reduced to the Weyl tensor. Therefore, if in this case we choose the \( V_{iklm} \) as equal to:

\[
\frac{1}{\kappa} G_{iklm} = \frac{1}{\kappa} C_{iklm} = V_{iklm},
\]

(8)

the equations hold identically for the solutions of the Einstein equations. Thus, we may write the 4-index equations for the gravitational field as [1]:

\[
G_{iklm} = T_{iklm}.
\]

(9)

We see that \( V_{iklm} \) can be considered as the 4-index energy-momentum density tensor for the gravitational field. Although its 2-index contraction vanish, in the 4-index form it
allows one to determine a nonzero, local and positive defined energy-momentum tensor for the gravitational field.

The tensors $G_{iklm}$ and $T_{iklm}$ have the symmetry properties of the Riemann tensor and, therefore, we have 20 equations. The tensor $G_{iklm}$ is a function of the metric tensor $g_{ik}$ which has 6 independent components. The tensor $T_{iklm}^{(m)}$ has been combined from the ordinary energy-momentum tensor of the matter $T_{ik}$ and it has 4 independent functions (the energy density $\epsilon$ and 3 components of the velocity). These 10 functions obey to 10 Einstein equations. The tensor $V_{iklm}$ gives additional 10 independent components.

So, we have 20 equations for 20 independent functions. If we take solutions of the Einstein equations for the metrics and $T_{ik}$, then we have the additional 10 equations for 10 components of $V_{iklm}$. Therefore, the solutions of the Einstein equations exactly define all components of $V_{iklm}$ and in the paper we can find $V_{iklm}$ for the known standard metrics.

In the vacuum $T_{ik} = T = 0$, $R_{il} = R = 0$ and we have the equations Eq.(8). We see that in the vacuum the tensor $V_{iklm}$ plays the role of the source for the empty spacetime curvature $C_{iklm}$.

The covariant derivatives of the 4-index tensors are also related as:

$$G^i_{klm;i} = T^{(m)}_{klm;i}.$$  \hfill (10)

In the case $d = 4$ we have:

$$G^i_{klm;i} = T_{km;l} - T_{kl;m} - \frac{1}{3}(g_{km}T_{l;m} - g_{kl}T_{m;l}),$$ \hfill (11)

$$T^{(m)}_{klm;j} = \frac{1}{2} \left[ T_{km;j} - T_{kl;m} - \frac{1}{3}(g_{km}T_{l;m} - g_{kl}T_{m;l}) \right] = \frac{1}{2} G^i_{klm;i}.$$ \hfill (12)

Then we obtain the relationship:

$$V^j_{klm;j} = G^j_{klm;j} - T^{(m)}_{klm;j} = \frac{1}{2} G^i_{klm;i}.$$ \hfill (13)

and, therefore,

$$V^i_{klm;j} = T^{(m)}_{klm;j}.$$ \hfill (14)

In the vacuum, therefore, there are local conservation laws:

$$G^i_{klm;j} = V^i_{klm;j} = 0.$$ \hfill (15)

The integral energy-momentum tensor for the system of matter and gravitational field can be defined as:

$$P^i_{lm} = \int d^3 x T^i_{klm}.$$ \hfill (16)

On the hypersurface $x^0 = \text{const}$ we have:

$$P^k_{lm} = \int d^3 x \sqrt{-g} T^{(m)0k}_{..lm} = \int d^3 x \sqrt{-g} (T^{(m)0k}_{..lm} + V^{0k}_{..lm}).$$ \hfill (17)

The energy-momentum vector for matter can be obtained as: $P^i = P^i_{ik}$.

Finally, the 3-index integral energy-momentum of the gravitational field can be defined as:

$$P^{(g)i}_{lm} = \int d^3 x \sqrt{-g} V^i_{..lm}.$$ \hfill (18)
3 The gravitational energy for the Schwarzschild field

Let us consider the energy of the Schwarzschild field with the line element:

\[ ds^2 = \left(1 - \frac{r_g}{r}\right)dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2(d\theta^2 + \sin^2 \vartheta d\varphi^2), \]  \hspace{2cm} (19)

where \( r_g = 2Gm \) is the gravitational radius, and the components of the metric are: \( g_{22} = -r^2 \), \( g_{33} = -r^2 \sin^2 \vartheta \), and:

\[ g_{00} = g_{11} = 1 - \frac{r_g}{r}. \]  \hspace{2cm} (20)

We calculate the energy-momentum tensor:

\[ V_{iklm} = \frac{1}{\kappa} R_{iklm} \]  \hspace{2cm} (21)

for this solution of the Einstein equations. Nonzero components of the \( V_{iklm} = R_{iklm}/\kappa \) with this metric are:

\[ V_{0101} = \frac{r_g}{\kappa r^3} = -V(r)g_{00}g_{11}, \]  \hspace{2cm} (22)

\[ V_{0202} = -\frac{r_g(r - r_g)}{2\kappa r^2} = \frac{1}{2} V(r)g_{00}g_{22}, \]  \hspace{2cm} (23)

\[ V_{0303} = -\frac{r_g(r - r_g)}{2\kappa r^2} \sin^2 \vartheta = \frac{1}{2} V(r)g_{00}g_{33}, \]  \hspace{2cm} (24)

\[ V_{1212} = \frac{r_g}{\kappa(r - r_g)} = \frac{1}{2} V(r)g_{11}g_{22}, \]  \hspace{2cm} (25)

\[ V_{1313} = \frac{r_g \sin^2 \vartheta}{\kappa(r - r_g)} = \frac{1}{2} V(r)g_{11}g_{33}, \]  \hspace{2cm} (26)

\[ V_{2323} = -\frac{r_g r}{\kappa} \sin^2 \vartheta = -V(r)g_{22}g_{33}, \]  \hspace{2cm} (27)

where:

\[ V(r) = \frac{r_g}{\kappa r^3} = \frac{m}{4\pi r^3} = -\frac{m}{8\pi} \frac{\partial}{\partial r}(r^{-2}). \]  \hspace{2cm} (28)

We see, that the 2-index contraction of this tensor vanishes:

\[ V_{il} = g^{km}V_{iklm} = g_{il}[-V(r) + \frac{1}{2} V(r) + \frac{1}{2} V(r)] = 0. \]

The physical components of the gravitational energy-momentum tensor \( V_{.,lim} = g^{ip}g^{jq}V_{pqlm} \) are:

\[ V_{.,01} = V_{.,10} = V_{.,23} = V_{.,32} = -V(r), \]  \hspace{2cm} (29)

\[ V_{.,02} = V_{.,20} = V_{.,03} = V_{.,30} = V_{.,12} = -V(r), \]  \hspace{2cm} (30)

\[ V_{.,21} = V_{.,13} = V_{.,31} = \frac{1}{2} V(r). \]  \hspace{2cm} (31)

They allow us to calculate one of components of integral gravitational energy-momentum of the static mass point as:
The spatial volume integral can be represented as a spatial surface integral, and we obtain:

\[ cP_{01}^{(g)} = \int dS_0 \sqrt{-g} V_{01}^{(g)} = \int dS_0 \sqrt{-g} [-V(r)]. \]  

(32)

where \( dS_0 = n_r r^2 \) is a 2-dimensional surface element with the normal vector \( n_r \) along \( r \).

References

[1] Zakir Z. (1999) *New Equations for Gravitation with the Riemann Tensor and 4-Index Energy-Momentum Tensors for Gravitation and Matter*. (gr-qc/9905009), in "Zakir Z. (2003) Structure of Space-Time and Matter. CTPA, Tashkent".

[2] Zakir Z. (1999) *Four-Index Energy-Momentum Tensors for the Gravitation and the Matter*. (gr-qc/9905036), in "Zakir Z. (2003) Structure of Space-Time and Matter. CTPA, Tashkent".