Poisson metrics and Higgs bundles over noncompact Kähler manifolds

Di Wu\(^1\) · Xi Zhang\(^1\)

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Abstract
In this paper, we study existence, uniqueness of Poisson metrics and \(K\)-analytically stability on flat bundles over noncompact Riemannian manifolds and establish related consequences, specially on concerning generalizations of Corlette–Donaldson–Hitchin–Simpson’s non-abelian Hodge correspondence to noncompact Kähler manifolds setting.

Mathematics Subject Classification 53C07 · 53C21 · 14J60

1 Introduction
1.1 Background

Let \((E, D)\) be a vector bundle over a Riemannian manifold \((M, g)\), we say \((E, D)\) is simple if it admits no proper \(D\)-invariant sub-bundle and \((E, D)\) is semi-simple if it is a direct sum of \((E_i, D_i)\) such that each of them is simple. Moreover, the connection \(D\) is called irreducible if it admits no nontrivial covariantly constant section of \(\text{End}(E)\). Given any metric \(H\) on \(E\), it splits the connection as \(D = D_H + \psi_H\), where \(D_H\) is a connection preserving \(H\) and \(\psi_H \in \Omega^1(\text{End}(E))\) is self-adjoint with respect to \(H\). We call \(H\) a harmonic metric if it satisfies

\[\Delta_H H = 0\]

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\(\mathbf{Xi Zhang}\)
mathzx@ustc.edu.cn

\(\mathbf{Di Wu}\)
wudi123@mail.ustc.edu.cn

\(^1\) School of Mathematics and Statistics, Nanjing University of Science and Technology, Nanjing 210094, People’s Republic of China
where $D^*_H$ is the adjoint of $D_H$. By the Riemann-Hilbert correspondence, suppose $\rho_D : \pi_1(M) \rightarrow GL(r)$ is the representation corresponding to a flat bundle $(E, D)$, then any metric $H$ on $E$ induces a $\rho_D$-equivariant map $f_H$ from $\tilde{M}$ to $GL(r)/U(r)$, where $\tilde{M}$ is the universal of $M$ and $r = \text{rank}(E)$. Then we can see that $H$ being harmonic is equivalent to $f_H$ being a harmonic map or $df_H$ being a harmonic 1-form with values in $f_H^{-1}T(GL(r)/U(r))$.

Let $(M, g)$ be a compact Riemannian manifold, the main existence result is the following

**Theorem 1.1** (Corlette [8], Donaldson [14]) A flat bundle admits a harmonic metric if and only if it arises from a semi-simple representation of the fundamental group $\pi_1(M)$.

In the context of compact Kähler manifolds, the pair $(D^0_0, \psi^1_0)$ will determine a polystable Higgs structure, provided $H$ being harmonic. For this reason, sometimes we then say that $D$ satisfies the Hitchin’s self-duality equation [17] and we call $D$ a Hitchin connection. Combing this with the Donaldson–Uhlenbeck–Yau theorem [13, 42] for Higgs bundles [17, 37], one obtains Corlette–Donaldson–Hitchin–Simpson’s nonabelian Hodge correspondence:

**Theorem 1.2** ([8, 14, 17, 37]) On a compact Kähler manifold, there is an one to one correspondence between the category of (semi-)simple flat complex vector bundles and the category of (poly-)stable Higgs bundles with vanishing Chern classes.

This identification can be traced back to Narasimhan-Seshadri [32] and has yielded serval fascinating topological results, specially on fundamental groups of compact Kähler manifolds, see [10, 39] for more details. In [19, 20], Jost-Zuo studied the representations of fundamental groups for quasi-projective varieties by using the theory of harmonic metrics on quasi-projective varieties. From the view point of harmonic maps, Li [24] considered the case that $M = \overline{M} \setminus D$ equipped with a Poincaré-like metric $g$, where $\overline{M}$ is a compact complex manifold and $D$ is a divisor with normal crossings in $\overline{M}$. Recently, Mochizuki made a significant process towards the generalizations of the nonabelian Hodge correspondence and the existence of pluriharmonic metrics plays an important role, see [27–30]. Readers are refereed to [3, 6, 9, 12, 18, 23, 33, 47] and the references therein for many other important works concerning related topics.

Flat bundles also occur in great abundance since it corresponds local systems and representations of fundamental groups. It was observed by Loftin [25] that for an affine manifold $M$, any flat bundle $(E, D)$ over $M$ corresponds a holomorphic bundle $\tilde{E}$ over $TM$ and the Donaldson–Uhlenbeck–Yau theorem can be generalized to the affine case [5, 25, 36]. We write $(D^*_H\psi)_H^\perp$ for the trace-free part of $D^*_H\psi_H$ and we call $H$ a Poisson metric if $(D^*_H\psi_H)^\perp$ vanishes identically. Collins–Jacob–Yau [7] found that the dimension reduction of Hermitian–Yang–Mills equation on $\tilde{E}$ is given by the Poisson metric equation on $(E, D)$. Motivated by this and the understanding of stable vector bundles on $K3$ surfaces in large complex structure limits, Collins–Jacob–Yau [7] related the existence of Poisson metrics on flat bundles with parabolic structures over punctured Riemannian surfaces to certain notion of slope stability, under some suitable assumptions.

**1.2 Main results**

In the first part of this paper, we obtain the following theorem which can be interpreted as the noncompact version of Corlette–Donaldson’s theorem for harmonic metrics.
Theorem 1.3 Let \((E, D)\) be a vector bundle over a (not necessarily compact) Riemannian manifold \((M, g)\) equipped with a connection \(D\) and a metric \(K\), we have

1. Existence: if the Assumption 2.1 is satisfied, \(D\) is flat and \(K\)-analytically stable such that \(\|(D^*_K \psi_K)^{\perp}\| \leq C|\psi_K|\) for a constant \(C\), then it admits a Poisson metric \(H\) with \(\det h = 1\), \(|h| \in L^\infty\) and \(|Dh| \in L^2\) for \(h = K^{-1}H\).

2. \(K\)-analytically stability: if the Assumption 2.2 is satisfied (or \((M, g)\) being complete with finite volume) and \(D\) is irreducible such that \(\|D^*_K \psi_K\| \in L^1\), then it admits a Poisson metric with the property in (1) only if \(D\) is \(K\)-analytically stable.

3. Uniqueness: if the Assumptions 2.1, 2.2 are satisfied (or \((M, g)\) being complete with finite volume) and \(D\) is \(K\)-analytically stable such that \(\text{tr} D^*_K \psi_K \in L^1\), then any Poisson metric with the property in (1) is unique.

Remark 1.1 The notion of analytically stability will be defined in Section 2.2. If \(M\) is compact, it is known that \(D\) being analytically stable(poly-stable) if and only if it is simple(semi-simple), while the correct substitution should be analytically stability in noncompact setting. If \((M, g)\) is a Kähler manifold and \(D\) being flat, our definition coincides with that in [38].

Remark 1.2 In [8], Corlette’s proof relies on Uhlenbeck’s deep compactness theorem [41]. Even in compact case, our discussion is different and we bypass the Uhlenbeck compactness. Besides that, the existence of harmonic metrics on analytically stable flat complex vector bundles over Kähler manifolds should be well-understood after Simpson’s works [37, 38](also see Mochizuki’s paper [29] for its exposition). However, due to the absence of certain functionals, their methods on deriving a prior estimates are not applicable in Riemannian setting.

Remark 1.3 We stress that the second conclusion indicates \(D\) is analytically stable with respect to the background metric \(K\). This and the uniqueness part both require comparisons of different analytically stabilities, which is a new key ingredient in this paper.

For the existence part, we will take Mochizuki’s exhaustion method in [31], where he proved the existence of exhaustion function \(f\). Denote by \(M_s\) the set where \(f \leq s\), we first consider the parabolic heat flow (3.1) on \(M_s\) and for arbitrary \(D\), we show the Dirichlet boundary problems for harmonic metric and Poisson metric are uniquely solvable, see Propositions 3.3 and 3.4. In order to overcome the difficulty that arises from the zeroth estimates, we observe in Proposition 3.5 the crucial identity (3.21), based on which several core a prior estimates are established. Once the zeroth estimate of a sequence of the solutions to the Dirichlet problems is obtained, the evolved metrics are completely controlled and will converge to a Poisson metric.

For the \(K\)-analytically stability and uniqueness part, we will make a discussion on the \(L^2\) property of the second fundamental form and utilize the noncompact Stokes theorem. It may be mentioned that this property is facilitated by the feature of the stability of vector bundles.

By (2.22), we know that
\[
\text{tr} D^*_H \psi_H - \text{tr} D^*_K \psi_K = -\frac{1}{2} \text{tr} D^*_K (h^{-1} \delta h) = \frac{1}{2} \Delta \log \det h,
\]
where \(h = K^{-1}H\) and \(\delta h = DK - \psi_K\). So by solving a scalar Poisson equation(Theorem 4.3 in [43]) and employing Theorem 1.3, it is easy to conclude

Corollary 1.1 Under the Assumptions 2.1, 2.2 with \(\psi = 1\), and \(|D^*_K \psi_K| \in L^\infty\), \(|\psi_K| \in L^2\), there exists a unique harmonic metric \(H\) such that \(\det h = 1\), \(|h| \in L^\infty\) and \(|Dh| \in L^2\).
provided \(D\) being flat and \(K\)-analytically stable. Conversely, \(D\) is \(K\)-analytically stable if it is irreducible and admits such a harmonic metric.

In the second part of this paper, we shall investigate related applications and consequences after Theorem 1.3. Recall a Higgs bundle \((E, \bar{\nabla}_E, \theta)\) over a complex manifold \(X\) consists of a holomorphic bundle \((E, \bar{\nabla}_E)\) over \(X\) together with holomorphic section \(\theta \in \Omega^{1,0}(\text{End}(E))\) such that \(\theta \wedge \theta = 0\). For a Hermitian metric \(K\), we consider the Hitchin-Simpson connection

\[
D_{\bar{\nabla}_E, \theta, K} = \partial_{\theta, K} + \bar{\nabla}_E \theta, \quad \partial_{\theta, K} = \partial_K + \theta \wedge K, \quad \bar{\nabla}_{E, \theta} = \bar{\nabla}_E + \theta,
\]

where \(D_{\bar{\nabla}_E, \theta, K} = \partial_K + \bar{\nabla}_E\) is the Chern connection. If the curvature \(F_{\bar{\nabla}_E, \theta, K} = D^2_{\bar{\nabla}_E, \theta, K}\) vanishes, we will call \((\bar{\nabla}_E, \theta, K)\) a Higgs flat structure on the underlying vector bundle \(E\). In particular, if the Chern curvature \(F_{\bar{\nabla}_E, K} = D^2_{\bar{\nabla}_E, K}\) vanishes, we then call \((\bar{\nabla}_E, K)\) a flat holomorphic structure.

**Theorem 1.4** Let \((E, D)\) be a \(K\)-analytically stable flat complex vector bundle over a Kähler manifold \((X, \omega)\) satisfying the Assumption 2.1 and \(|(D^*_K \psi_K)^{-1}| \leq C \phi\) for a constant \(C\), we have

1. If \((X, \omega)\) is a complex curve, there is a Higgs flat structure on \(E\).
2. If \((X, \omega)\) is complete with bounded Ricci curvature from below and \(|\psi_K| \in L^2\), there is a Higgs flat structure on \(E\).
3. If \((X, \omega)\) is complete with nonnegative Ricci curvature and \(|\psi_K| \in L^2\), there is a flat holomorphic structure on \(E\).

Especially, the third statement in Theorem 1.4 yields the existence of metric compatible flat connections on \(E\). Finding metric compatible flat connections may be of interest in its own right. In fact, by Corlette’s work in [8], it is easy to see the same conclusion also holds for a flat bundle over a compact Riemannian manifold with nonnegative Ricci curvature and so it follows that the Euler class \(e(E)\) of \(E\) is zero. To prove Theorem 1.4, we need to construct Higgs structures by employing Theorem 1.3, which is a key step to establish Corlette–Donaldson–Hitchin–Simpson’s correspondence, see Proposition 4.3 and Proposition 4.4. If \(X\) is compact and \(H\) being harmonic, this is equivalent to determine whether \(H\) being pluriharmonic, the issue does not appear and it is essentially due to Siu-Sampson [35, 40] and Corlette [8], see also [39]. If \(X\) is noncompact, see [20, 37] for the case that the complement of a divisor in a compact Kähler manifold.

Secondly, we will focus on generalizing the nonabelian Hodge correspondence to non-compact Kähler manifolds setting. Following Simpson [37], given a Higgs structure \((\bar{\nabla}_E, \theta)\) on \(E\), we set

\[
\deg_\omega(E, K) = \int_X \sqrt{-1} \text{tr} \Lambda_{\omega} F_{\bar{\nabla}_E, \theta, K} \text{dvol}_\omega,
\]

where \(\text{dvol}_\omega\) is the volume form of the base Hermitian manifold. Suppose \(V \subseteq \mathcal{O}_X(E)\) is a sub-sheaf which comes from a sub-bundle \(V\) outside a singular set \(\Sigma_V\) of codimension at least two. Let \(\pi_K\) denotes the projection onto \(V\) and \(K\) restricts to a metric on \(V\), so that one defines \(\deg_\omega(V, K)\) by integrating outside \(\Sigma_V\). Then \((\bar{\nabla}_E, \theta)\) is called \(K\)-analytically stable if for any proper sub-Higgs sheaf \(V \subset \mathcal{O}_X(E)\), it holds

\[
\frac{\deg_\omega(V, K)}{\text{rank}(V)} < \frac{\deg_\omega(E, K)}{\text{rank}(E)}.
\]

We fix a Hermitian metric \(K\) on a complex vector bundle \(E\) and define
• $\mathcal{M}_{\text{Flat},K}$: the space of isomorphic classes of $K$-analytically stable and irreducible flat connections with $|(D^*_K \psi_K)^\perp| \in L^1$.
• $\mathcal{M}_{\text{Higgs},K}$: the space of isomorphic classes of $K$-analytically stable and irreducible Higgs structures with $\text{tr} \, F_{\tilde{\partial}_E,\theta,K} = 0$ and $|\Lambda_\omega F_{\tilde{\partial}_E,\theta,K}| \in L^1$.
• $\mathcal{M}_{\text{Flat},K,\phi}$: the space of isomorphic classes of $K$-analytically stable and irreducible flat connections with $|(D^*_K \psi_K)^\perp| \leq C \phi$ for a constant $C$.
• $\mathcal{M}_{\text{Higgs},K,\phi}$: the space of isomorphic classes of $K$-analytically stable and irreducible Higgs structures with $\text{tr} \, F_{\tilde{\partial}_E,\theta,K} = 0$ and $|\Lambda_\omega F_{\tilde{\partial}_E,\theta,K}| \leq C \phi$ for a constant $C$.

In the above, isomorphic means that they are in the the same $G$-orbit, where $G$ denotes the set of $L^\infty$-sections of $\text{Aut}(E)$. We say a Higgs structure $(\tilde{\partial}_E, \theta)$ is irreducible if $D_{\tilde{\partial}_E,\theta,H}$ is irreducible, for any Hermitian-Einstein metric $H$ with $\det h = 1$, $|h| \in L^\infty$ and $h = K^{-1} H$.

Our third theorem can be stated as the follows.

**Theorem 1.5** On a noncompact complex curve satisfying the Assumptions 2.1, 2.2, we have

1. There exists a correspondence from $\mathcal{M}_{\text{Flat},K,\phi}$ to $\mathcal{M}_{\text{Higgs},K}$.
2. There exists a correspondence from $\mathcal{M}_{\text{Higgs},K,\phi}$ to $\mathcal{M}_{\text{Flat},K}$.

Notice the correspondences in Theorem 1.5 preserve $K$-analytically stabilities. Moreover, it can be seen that $\mathbb{C}$ and punctured Riemannian surfaces satisfy the assumptions in Theorem 1.5. For higher dimensional case, under some extra technical assumptions (see [31, 37]), the direction that from $K$-analytically stable and irreducible Higgs structures to $K$-analytically stable and irreducible flat connections can be finished by the results of Simpson [37], Mochizuki [31] and the discussion as that in (2) of Theorem 1.5. Conversely, start with a $K$-analytically stable and irreducible flat connection, using Proposition 4.4 (or Proposition 4.5) instead, we can obtain an irreducible Higgs structure. In the process, the analytic estimates in Theorem 1.5 remain valid, but currently it is not clear that when the resulting Higgs structure is stable with respect to $K$.

Theorem 1.5 is established based on Theorem 1.3 and the results on Hermitian-Einstein metrics in [31, 37, 45], while analytic estimates and comparisons of different analytically stabilities play the significant roles. It is relevant to point out that the nonabelian Hodge correspondence has been generalised to parabolic and noncompact context, specially on the case of a projective variety equipped with normal crossing divisors, in a series of works including those of Simpson [38] and Biquard-Boalch [2] in complex dimension one. The higher dimensional results can be found in the papers of Biquard and Mochizuki, see [1, 27, 29, 30] for details.

### 1.3 Organization

The rest of this paper is organized as follows. In Sect. 2, we discuss the assumptions on the base manifolds, introduce the notation of stability of vector bundles, and present some useful calculations. In Sect. 3, we solve the Dirichlet boundary problems for harmonic metric equation and Poisson metric equation, then give the proof of Theorem 1.3. In Sect. 4, we first investigate the uniqueness of Poisson metrics and then prove Theorems 1.4, 1.5. As a byproduct, also a vanishing theorem of Kamber–Tondeur classes is obtained.
2 Preliminaries

2.1 Assumptions on base spaces

Let \((M, g)\) be a Riemannian manifold, we list the following assumptions that needed in this paper.

**Assumption 2.1** There is a function \(\phi : M \to \mathbb{R}_{\geq 0}\) with \(\phi \in L^1\), such that if \(f\) is a nonnegative bounded function on \(M\) satisfying \(\Delta f \geq -B\phi\) for a positive constant \(B\) in distribution sense, we have

\[
\sup_M f \leq C(B)(1 + \int_M f\phi \, d\text{vol}_g). \tag{2.1}
\]

Furthermore, if \(f\) satisfies \(\Delta f \geq 0\), we have \(\Delta f = 0\).

**Assumption 2.2** It admits an exhaustion function \(\rho : M \to \mathbb{R}_{\geq 0}\) with \(|\Delta \rho| \in L^1\).

The above assumptions are introduced for Kähler manifolds in [31, 37].

**Example 2.1** Compact Riemannian manifolds satisfy Assumptions 2.1, 2.2.

**Example 2.2** Let \(M\) be a Zariski open subset of a compact Kähler manifold \((X, g)\), \(g\) is the restriction of \(g\), then it satisfies Assumptions 2.1, 2.2.

**Example 2.3** Consider \((\mathbb{R}^2, g_{\mathbb{R}^2})\) with the Euclidean metric \(g_{\mathbb{R}^2}\) and \(\phi_{\mathbb{R}^2} = (1 + |r|^2)^{-1-\delta}\), \(\delta > 0\), then it satisfies Assumption 2.1. Moreover, \((\mathbb{R}^2, \phi_{\mathbb{R}^2} g_{\mathbb{R}^2})\) satisfies Assumptions 2.1, 2.2.

**Example 2.4** Any product manifold \((\mathbb{R}^2, g_{\mathbb{R}^2}) \times (N, g_N)\) satisfies Assumption 2.1, where \((N, g_N)\) is a compact Riemannian manifold.

2.2 Stability of vector bundles

Let \((E, D)\) be a vector bundle over a Riemannian manifold \((M, g)\), equipped with a metric \(K\) on \(E\).

**Definition 2.1** The \(K\)-analytically degree of \((E, D)\) is defined by

\[
\deg_g(E, D, K) = -\int_M \text{tr} D^*_K \psi_K \, d\text{vol}_g. \tag{2.2}
\]

For any \(D\)-invariant sub-bundle \(S \subseteq E\), we define

\[
\deg_g(S, D, K) = -\int_M \text{tr} D^*_K \psi_K \, d\text{vol}_g, \tag{2.3}
\]

where \(K_S\) is the restricted metric on \(S\). Then \((E, D)\) is said to be \(K\)-analytically stable if for any nontrivial \(D\)-invariant sub-bundle \(S\), it holds that

\[
\frac{\deg_g(S, D, K)}{\text{rank}(S)} < \frac{\deg_g(E, D, K)}{\text{rank}(E)}. \tag{2.4}
\]
Remark 2.1 If $(M, \omega)$ is a Kähler manifold and $D$ being a flat connection, we have
\[
\deg_E(E, D, K) = \deg_\omega(E, D_0^1, K) = -\deg_\omega(E, \delta^{0,1}_K, K),
\]
where $\delta^{0,1}_K = D_0^1 - \psi^{0,1}_K$, $\deg_\omega(E, D_0^1, K)$, $\deg_\omega(E, \delta^{0,1}_K, K)$ are the $K$-analytically degrees of the holomorphic bundles $(E, D_0^1)$ and $(E, \delta^{0,1}_K)$ respectively.

For any $D$-invariant sub-bundle $S$, we have the Chern-Weil formula written in terms of $D_K^*\psi_K$,
\[
\deg_S(S, D, K) = -\int_M \left( \text{tr}(\pi_K \circ D_K^*\psi_K) + \frac{1}{2} |D\pi_K|^2_K \right) \text{dvol}_g,
\](2.6)
where $\pi_K$ is the orthogonal projection onto $S$.

Proposition 2.1 If a vector bundle $(E, D)$ admits a Poisson metric $K$ with $\text{tr} D_K^*\psi_K \in L^1$, it is an orthogonal direct sum of $K$-analytically stable bundles.

Proof For any nontrivial $D$-invariant sub-bundle $S \subset E$, the Chern-Weil formula yields
\[
\frac{\deg_S(S, D, K)}{\text{rank}(S)} = \frac{-\int_M \left( \text{tr}(\pi_K \circ D_K^*\psi_K) + \frac{1}{2} |D\pi_K|^2_K \right) \text{dvol}_g}{\text{rank}(S)}
= \frac{-\int_M \left( \frac{\text{rank}(S)}{\text{rank}(E)} \text{tr} D_K^*\psi_K + \frac{1}{2} |D\pi_K|^2_K \right) \text{dvol}_g}{\text{rank}(S)}
\leq \frac{\deg_S(E, D, K)}{\text{rank}(E)}.
\](2.7)

If the equality holds, we obtain $D\pi_K = 0$. Let $S^\perp$ denote the orthogonal complement of $S$ in $E$ with respect to $K$, then it is also a $D$-invariant sub-bundle such that $(E, D) \cong (S, D_S) \oplus (S^\perp, D_{S^\perp})$, where $D_S$ and $D_{S^\perp}$ are the induced connections. Moreover, the induced metrics $K_S$ and $K_{S^\perp}$ both are Poisson metrics and we complete the proof by an easy induction. \hfill \Box

2.3 Some useful calculations

We present some important calculations, it is emphasized that we are not working under the local flat frame and hence most of them need not the assumption that the underlying connection is flat.

Given a vector bundle $E$ over a Riemannian manifold $(M, g)$, the exterior differential operator $D : \Omega^p(E) \to \Omega^{p+1}(E)$ relative to a connection is given by
\[
D\omega(e_0, \ldots, e_p) = \sum_{k=0}^p (-1)^k D_{e_k} (\omega(e_0, e_1, \ldots, \hat{e}_k, \ldots, e_p))
+ \sum_{k<l} (-1)^{k+l} \omega([e_k, e_l], e_1, \ldots, \hat{e}_k, \ldots, \hat{e}_l, \ldots, e_p).
\](2.8)

Since the Levi-Civita connection on $TM$ is torsion-free, we also have
\[
D\omega(e_0, \ldots, e_p) = \sum_{k=0}^p (-1)^k (\nabla_{e_k} \omega)(e_0, \ldots, \hat{e}_k, \ldots, e_p),
\](2.9)
where $\tilde{\nabla}$ is the induced connection on $\Omega^*(E)$. For a metric $K$ on $E$, we define the inner-product

$$K(\omega, \theta)(x) = \sum_{i_1 < \ldots < i_p} (\omega(e_{i_1}, \ldots, e_{i_p}), \theta(e_{i_1}, \ldots, e_{i_p}))_K, \quad (2.10)$$

where $\omega, \theta \in \Omega^p(E)$ and $\{e_i\}_{i=1}^{\dim M}$ is the orthogonal unit basis of $T_xM$. Then the connection decomposition is achieved by choosing $\psi_K$ such that

$$K(\psi_K(X), Y) = \frac{1}{2}(K(DX, Y) + K(X, DY) - dK(X, Y)), \quad (2.11)$$

for any $X, Y \in \Omega^0(E)$. With respect to the Riemannian structures of $E$ and $TM$, the co-differential operator $D^*_K : \Omega^p(E) \to \Omega^{p-1}(E)$ is characterized via the formula

$$\int_M K(D_K\omega, \theta) \, dvol_g = \int_M K(\omega, D^*_K\theta) \, dvol_g, \quad (2.12)$$

for any $\omega \in \Omega^{p-1}(E), \theta \in \Omega^p(E)$ and either $\omega$ or $\theta$ is compactly supported. Then it holds

$$D^*_K\theta(e_1, \ldots, e_{p-1}) = -\sum_i (\nabla_{e_i, \theta}(e_i), e_1, \ldots, e_{p-1}), \quad (2.13)$$

where $\nabla_K$ is the connection on $\Omega^*(E)$, induced by the Levi-Civita connection $\nabla$ and $D_K$.

If $H$ is another metric on $E$, we define the positive endomorphism $h = K^{-1}H$ by setting $H(\cdot, \cdot) = K(h(\cdot, \cdot))$. A straightforward calculation, using (2.11), shows that

$$D_H = D_K + \frac{1}{2}h^{-1}\delta_K h = \frac{1}{2}(D_K + h^{-1} \circ D_K \circ h + \psi_K - h^{-1} \circ \psi_K \circ h), \quad (2.14)$$

$$\psi_H = \psi_K - \frac{1}{2}h^{-1}\delta_K h = \frac{1}{2}(\psi_K + h^{-1} \circ \psi_K \circ h + D_K - h^{-1} \circ D_K \circ h), \quad (2.15)$$

$$\delta_H = \delta_K + h^{-1}\delta_K h = h^{-1} \circ \delta_K \circ h. \quad (2.16)$$

**Lemma 2.1** If $H(t)$ is a family of fiber metrics on the vector bundle $(E, D)$, then we have

$$\frac{\partial D_{H(t)}}{\partial t} = \frac{1}{2}D_{H(t)} \left( h^{-1}(t) \frac{\partial h(t)}{\partial t} \right) - \frac{1}{2} \left[ \psi_{H(t)}, h^{-1}(t) \frac{\partial h(t)}{\partial t} \right], \quad (2.17)$$

$$\frac{\partial \psi_{H(t)}}{\partial t} = -\frac{1}{2}D_{H(t)} \left( h^{-1}(t) \frac{\partial h(t)}{\partial t} \right) + \frac{1}{2} \left[ \psi_{H(t)}, h^{-1}(t) \frac{\partial h(t)}{\partial t} \right], \quad (2.18)$$

$$\frac{\partial D^*_{H(t)} \psi_{H(t)}}{\partial t} = -\frac{1}{2}D^*_{H(t)}D_{H(t)} \left( h^{-1}(t) \frac{\partial h(t)}{\partial t} \right) + \frac{1}{2} \left[ D^*_{H(t)} \psi_{H(t)}, h^{-1}(t) \frac{\partial h(t)}{\partial t} \right]$$

$$-\frac{1}{2}g^{ij} \left[ \psi_{H(t)} \left( \frac{\partial }{\partial x^i} \right), \psi_{H(t)} \left( \frac{\partial }{\partial x^j} \right), h^{-1}(t) \frac{\partial h(t)}{\partial t} \right], \quad (2.19)$$

where $h(t) = K^{-1}H(t)$.  

\[ Springer \]
Proof It suffices to prove the (2.17) and (2.19). By making use of (2.14), (2.15), we deduce

\[
\frac{\partial D_H}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} \left( h^{-1} D_K h - h^{-1} \circ \psi_K \circ h \right)
\]

\[
= -\frac{1}{2} h^{-1} \frac{\partial h}{\partial t} h^{-1} D_K h + \frac{1}{2} h^{-1} D_K \frac{\partial h}{\partial t} + \frac{1}{2} h^{-1} \frac{\partial h}{\partial t} h^{-1} \psi_K h - \frac{1}{2} h^{-1} \psi_K \frac{\partial h}{\partial t} 
\]

\[
= -\frac{1}{2} h^{-1} \frac{\partial h}{\partial t} h^{-1} D_K h + \frac{1}{2} h^{-1} D_K \frac{\partial h}{\partial t} - \frac{1}{2} \left[ \psi_H, h^{-1} \frac{\partial h}{\partial t} \right] 
\]

\[
+ \frac{1}{2} h^{-1} \frac{\partial h}{\partial t} \left( \psi_H - \psi_K - D_K + h^{-1} \circ D_K \circ h \right) 
\]

\[
- \frac{1}{2} \left( \psi_H - \psi_K - D_K + h^{-1} \circ D_K \circ h \right) h^{-1} \frac{\partial h}{\partial t} 
\]

\[
= \frac{1}{2} D_H \left( h^{-1} \frac{\partial h}{\partial t} \right) - \frac{1}{2} \left[ \psi_H, h^{-1} \frac{\partial h}{\partial t} \right] - \frac{1}{2} h^{-1} \frac{\partial h}{\partial t} h^{-1} D_K h 
\]

\[
+ \frac{1}{2} h^{-1} D_K \frac{\partial h}{\partial t} + \frac{1}{2} \left[ h^{-1} \frac{\partial h}{\partial t}, h^{-1} \circ D_K \circ h \right] 
\]

\[
= \frac{1}{2} D_H \left( h^{-1} \frac{\partial h}{\partial t} \right) - \frac{1}{2} \left[ \psi_H, h^{-1} \frac{\partial h}{\partial t} \right] . 
\]

On the other hand, let’s choose the local normal coordinate \((x^1, \ldots, x^n)\) at the considered point, using (2.17) and (2.18), we have

\[
\frac{\partial D_H \psi_H}{\partial t} = -\frac{\partial}{\partial t} \left[ D_H, \frac{\partial \psi_H}{\partial x^i} \right] \psi_H \left( \frac{\partial}{\partial x^i} \right) \right] 
\]

\[
= -\frac{1}{2} \left[ D_H \frac{\partial}{\partial x^i} \psi_H \left( \frac{\partial}{\partial x^i} \right) \right] - \frac{1}{2} \left[ D_H \frac{\partial}{\partial x^i} \psi_H \left( \frac{\partial}{\partial x^i} \right) \right] 
\]

\[
+ \frac{1}{2} \left[ D_H \frac{\partial}{\partial x^i} D_H \frac{\partial}{\partial x^i} \psi_H \left( \frac{\partial}{\partial x^i} \right) \right] - \frac{1}{2} \left[ \psi_H \left( \frac{\partial}{\partial x^i} \right), \psi_H \left( \frac{\partial}{\partial x^i} \right), h^{-1} \frac{\partial h}{\partial t} \right] 
\]

\[
= \frac{1}{2} D_H \frac{\partial}{\partial x^i} D_H \frac{\partial}{\partial x^i} \left( h^{-1} \frac{\partial h}{\partial t} \right) - \frac{1}{2} \left[ \psi_H \left( \frac{\partial}{\partial x^i} \right), \psi_H \left( \frac{\partial}{\partial x^i} \right), h^{-1} \frac{\partial h}{\partial t} \right] 
\]

\[
- \frac{1}{2} \left[ D_H \frac{\partial}{\partial x^i} \psi_H \left( \frac{\partial}{\partial x^i} \right) \right] - \frac{1}{2} \left[ D_H \frac{\partial}{\partial x^i} \psi_H \left( \frac{\partial}{\partial x^i} \right), h^{-1} \frac{\partial h}{\partial t} \right] 
\]

\[
= -\frac{1}{2} D_H D_H \left( h^{-1} \frac{\partial h}{\partial t} \right) + \frac{1}{2} \left[ D_H \frac{\partial}{\partial x^i} \left( \frac{\partial}{\partial x^i} \right), h^{-1} \frac{\partial h}{\partial t} \right] 
\]

\[
- \frac{1}{2} \left[ \psi_H \left( \frac{\partial}{\partial x^i} \right), \psi_H \left( \frac{\partial}{\partial x^i} \right), h^{-1} \frac{\partial h}{\partial t} \right] . 
\]

\[
\square
\]

Under the local coordinate \((x^1, \ldots, x^n)\), by using (2.14) and (2.15), it follows

\[
D_H^* \psi_H = D_K^* \psi_K - \frac{1}{2} D_K^* \left( h^{-1} \delta_K h \right) - \frac{1}{2} g^{ij} \left[ h^{-1} \delta_K, \frac{\partial}{\partial x^i} \right] h \psi_K \left( \frac{\partial}{\partial x^j} \right) 
\]

\[
= D_K^* \psi_K + \frac{1}{2} g^{ij} D_{\frac{\partial}{\partial x^i}} \left( h^{-1} \delta_K, \frac{\partial}{\partial x^j} \right) h - \frac{1}{2} g^{ij} \Gamma_{ij}^k h^{-1} \delta_K, \frac{\partial}{\partial x^k} h . 
\]
Unless indicated explicitly, below the norms and inner products are taken with respect to $K$.

**Lemma 2.2** Let $H$ and $K$ be two fiber metrics on the vector bundle $(E, D)$, it holds

\[
(D^* \psi_H - D^* \psi_K, h) = \frac{1}{2} \Delta h - \frac{1}{2} |h^{-\frac{1}{2}} \delta_K h|^2,
\]

\[
(D^* \psi_K - D^* \psi_H, h^{-1})_H = \frac{1}{2} \Delta h^{-1} - \frac{1}{2} |h^{\frac{1}{2}} \delta_H h^{-1}|^2_H,
\]

where $h = K^{-1}H$.

**Proof** We only proof (2.23) and (2.24) follows from (2.23). Choose the local normal coordinate $(x^1, \ldots, x^n)$ at the considered point, based on (2.22), we have

\[
(D^* \psi_H - D^* \psi_K, h)
\]

\[
= \frac{1}{2} \left( D^*_K \left( \frac{\partial}{\partial x^i} h^{-1} \delta_K h \right), h \right)_H - \frac{1}{2} \left( h^{-1} \delta_K \frac{\partial}{\partial x^i} h, D_K \left( \frac{\partial}{\partial x^i} \right) h \right)_H
\]

\[
= \frac{1}{2} \Delta h - \frac{1}{2} |h^{-\frac{1}{2}} \delta_K h|^2.
\]

\[
\square
\]

Following [13], the Donaldson’s distance on the space of metrics is defined by

\[
\sigma(K, H) = \text{tr}(K^{-1}H) + \text{tr}(H^{-1}K) - 2 \text{rank}(E).
\]

(2.26)

It is obvious that $\sigma(H, K) \geq 0$ with equality if and only if $H = K$.

By Lemma 2.2, we get

**Corollary 2.1** Let $H$ and $K$ be two harmonic metrics on the vector bundle $(E, D)$, then we have

\[
\Delta \sigma(H, K) = |h^{-\frac{1}{2}} \delta_K h|^2 + |h^{\frac{1}{2}} \delta_H h^{-1}|^2_H,
\]

(2.27)

where $h = K^{-1}H$. In particular, if $(E, D)$ is simple and $(M, g)$ satisfies the Assumption 2.1, the mutually bounded harmonic metrics are unique up to scaling.

Now we consider the evolution equation

\[
H^{-1}(t) \frac{\partial H(t)}{\partial t} = 2 D^*_{H(t)} \psi_{H(t)},
\]

(2.28)

a direct calculation yields
Lemma 2.3 Let \((E, D)\) be a vector bundle over \((M, g)\), assume \(H(t)\) and \(K(t)\) are two solutions of the heat flow (2.28), then we have
\[
\left( \frac{\partial}{\partial t} - \Delta \right) \sigma (H(t), K(t)) = -|\tilde{h}(t)|^{-\frac{1}{2}} \delta_{K(t)} \tilde{h}(t) |K(t)|^{2} + |\tilde{h}^{1}(t) |H(t)||^{2},
\]
(2.29)
where \(\tilde{h}(t) = K^{-1}(t)H(t)\).

Lemma 2.4 Let \((E, D)\) be a vector bundle over \((M, g)\), along the heat flow (2.28) it holds
\[
\frac{d}{dt} ||\psi_H||_{H, L^2}^2 = -2 ||D_H^* \psi_H||_{H, L^2}^2 - 2 \int_M d\eta_H,
\]
(2.30)
\[
\frac{d}{dt} ||D_H^* \psi_H||_{H, L^2}^2 = -2 ||D_H D_H^* \psi_H||_{H, L^2}^2 - 2 ||[\psi_H, D_H^* \psi_H]||_{H, L^2}^2 + 2 \int_M d\tilde{\eta}_H,
\]
(2.31)
\[
(\frac{\partial}{\partial t} - \Delta) ||\psi_H||_{H}^2 = -2 ||D_H D_H^* \psi_H||_{H}^2 - 2 ||[\psi_H, D_H^* \psi_H]||_{H}^2 + 2 \int_M d\eta_H,
\]
(2.32)
where \(\eta_H = H(D_H^* \psi_H, \psi_H), \tilde{\eta}_H = H(D_H^* \psi_H, *D_H D_H^* \psi_H)\). Furthermore, if \(D\) is a flat connection, we have
\[
\left( \frac{\partial}{\partial t} - \Delta \right) ||\psi_H||_{H}^2 = -2 ||[\psi_H, D_H^* \psi_H]||_{H}^2 - 2(\psi_H \circ \text{Ric}, \psi_H)_H - 2|\nabla_H \psi_H||_{H}^2,
\]
(2.33)
where \(\text{Ric}\) is the Ricci transformation of \((M, g)\).

Proof Firstly, applying the equality (2.18) yields
\[
\frac{d}{dt} ||\psi_H||_{H, L^2}^2 = 2 \int_M \left( \frac{1}{2} D_H \left( H^{-1} \frac{\partial H}{\partial t} \right) + \frac{1}{2} \left[ \psi_H, \left( H^{-1} \frac{\partial H}{\partial t} \right) \right] \right), \psi_H)_H dvol_g
\]
\[
= -2 \int_M (D_H D_H^* \psi_H, \psi_H)_H dvol_g
\]
(2.34)
\[
= -2 ||D_H^* \psi_H||_{H, L^2}^2 - 2 \int_M d\eta_H.
\]
Next, (2.19) indicates that
\[
\frac{\partial D_H^* \psi_H}{\partial t} = -D_H D_H D_H^* \psi_H - g^{ij} \left[ \psi_H \left( \frac{\partial}{\partial x^j} \right), \left[ \psi_H \left( \frac{\partial}{\partial x^j} \right), D_H^* \psi_H \right] \right],
\]
(2.35)
as it is easy to check \(D_H^* \psi_H\) is self-adjoint with respect to \(H\), it then follows
\[
\frac{d}{dt} ||D_H^* \psi_H||_{H, L^2}^2 = -2 \int_M (D_H^* D_H D_H^* \psi_H, D_H^* \psi_H)_H dvol_g
\]
\[
- 2 \int_M g^{ij} \left[ \psi_H \left( \frac{\partial}{\partial x^j} \right), \left[ \psi_H \left( \frac{\partial}{\partial x^j} \right), D_H^* \psi_H \right] \right], D_H^* \psi_H)_H dvol_g
\]
\[
= -2 ||D_H D_H^* \psi_H||_{H, L^2}^2 - 2 ||[\psi_H, D_H^* \psi_H]||_{H, L^2}^2 + 2 \int_M d\tilde{\eta}_H,
\]
(2.36)
and
\[
\left( \frac{\partial}{\partial t} - \Delta \right) |D^*_H \psi_H|^2_H = -2(D^*_H D_H D^*_H \psi_H, D^*_H \psi_H)_H
\]
\[
- 2g^{ij} \left( \left[ \psi_H \left( \frac{\partial}{\partial x^i} \right) \right], \left[ \psi_H \left( \frac{\partial}{\partial x^j} \right) \right], D^*_H \psi_H \right)_H
\]
\[
+ 2 \left( D^*_H D_H D^*_H \psi_H, D^*_H \psi_H \right)_H - 2|D_H D^*_H \psi_H|^2_H
\]
\[
= -2|D_H D^*_H \psi_H|^2_H - 2||\psi_H, D^*_H \psi_H||^2_H. \tag{2.37}
\]

On the other hand, we have
\[
\left( \frac{\partial}{\partial t} - \Delta \right) |\psi_H|^2_H = 2 \left( -\frac{1}{2} D_H \left( H^{-1} \frac{\partial H}{\partial t} \right) + \frac{1}{2} [\psi_H, H^{-1} \frac{\partial H}{\partial t}], \psi_H \right)_H
\]
\[
+ 2 (\nabla^* H \nabla H \psi_H \psi_H)_H - 2|\nabla H \psi_H|^2_H
\]
\[
= -2 (D_H D^*_H \psi_H - \nabla^*_H \nabla H \psi_H, \psi_H)_H - 2|\nabla H \psi_H|^2_H. \tag{2.38}
\]

Now using the flatness of $D$: $D_H \psi_H = 0$ and $D^2_H = -\frac{1}{2} [\psi_H, \psi_H]$, it is equal to
\[
2g^{ij} \left( D^2_H \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \psi_H \right)_H - \psi_H \left( R\nabla \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \psi_H \right)_H - 2|\nabla H \psi_H|^2_H
\]
\[
= -\left( g^{ij} [\psi_H, \psi_H] \frac{\partial}{\partial x^i}, \psi_H \frac{\partial}{\partial x^j} \right)_H + 2\psi_H \circ \text{Ric}, \psi_H)_H - 2|\nabla H \psi_H|^2_H \tag{2.39}
\]
\[
= -||\psi_H, \psi_H||^2_H - 2 (\psi_H \circ \text{Ric}, \psi_H)_H - 2|\nabla H \psi_H|^2_H.
\]

\[\square\]

3 Analytically stability and existence of Poisson metrics

Given a vector bundle $(E, D)$ over a compact Riemannian manifold $(X, g)$, consider the following heat flow
\[
\begin{cases}
H^{-1} \frac{\partial H}{\partial t} = 2D^*_H \psi_H, \\
H(0) = K.
\end{cases} \tag{3.1}
\]

And if $X$ has nonempty smooth boundary $\partial X$, for any given compatible data $\tilde{H}$, we impose the boundary condition $H(t)|_{\partial X} = \tilde{H}$ in the above system.

With the calculations in previous section at hand and following [13], we have

**Proposition 3.1** Let $(E, D)$ be a vector bundle over a compact Riemannian manifold $(X, g)$ (with possibly nonempty boundary), then heat flow (3.1) admits unique solution defined on $[0, \infty)$.

**Proposition 3.2** Assume $(E, D)$ is a vector bundle over a compact Riemannian manifold $(M, g)$, then for any $\epsilon > 0$, there exists a fiber metric $H_\epsilon$ such that $||D^*_H \psi_{H_\epsilon}||_{L^2} \leq \epsilon$.

**Proof** Let $H = H(t)$ be the solution to (3.1), defined on $[0, +\infty)$. Since (3.1) is the negative gradient flow of the energy $||\psi_H||_{L^2}$, by (2.31) we know $||D^*_H \psi_H||^2_{H, L^2} \rightarrow 0$, as $t$ is going to infinity. Then the proof can be finished by using (2.32) and parabolic Moser’s iteration. \[\square\]
Proposition 3.3 Let \((E, D)\) be a vector bundle over a compact Riemannian manifold \((X, g)\) with smooth boundary \(\partial X\). Then for any given data \(K\), there exists a unique harmonic metric \(H\) on \((E, D)\) such that \(H = K\) on \(\partial X\).

**Proof** Due to Corollary 2.1, it remain to prove the existence statement. By the standard elliptic equation theory, let \(u\) be the solution to the following Dirichlet problem
\[
\Delta u = -|D^*_K \psi_K|, u|_{\partial X} = 0. \tag{3.2}
\]
Let \(H(t)\) be a long-time solution of (3.1) with \(H(t) = K\) on \(\partial X\) and set
\[
v(x, t) = \int_0^t |D^*_H(s)\psi_{H(s)}|_{H(s)} ds - u, \tag{3.3}
\]
by (2.32) it is easy to see
\[
\begin{align*}
\left\{ \left( \frac{\partial}{\partial t} - \Delta \right) v(x, t) \leq 0, \\
v(x, 0) = -u(x), \\
v(x, t)|_{\partial X} = 0.
\end{align*} \tag{3.4}
\]
Then the maximum principle gives
\[
\int_0^t |D^*_H(s)\psi_{H(s)}|_{H(s)} ds \leq \max_X u. \tag{3.5}
\]
Now for \(t_1 \leq t \leq t_2\) and \(\tilde{h}(t) = H^{-1}(t_1) H(t), \dot{h}(t) = H^{-1}(t) H(t_1)\), it is straightforward to check
\[
\left| \frac{\partial}{\partial t} \log \text{tr} \tilde{h}(t) \right| \leq 2|D^*_H(t)\psi_{H(t)}|_{H(t)}, \left| \frac{\partial}{\partial t} \log \text{tr} \dot{h}(t) \right| \leq 2|D^*_H(t)\psi_{H(t)}|_{H(t)}. \tag{3.6}
\]
A simple integration yields
\[
\sigma(H(t_1), H(t_2)) \leq 2 \text{rank}(E) \left( e^{2 \int_{t_1}^{t_2} |D^*_H(s)\psi_{H(s)}|_{H(s)} ds} - 1 \right). \tag{3.7}
\]
So by combining (3.5) and (3.7), we know that \(H(t)\) converge in \(C^0\)-topology to \(H_\infty\) as \(t \to \infty\). Using (3.5) and arguing as Lemma 3.3 in [46], the elliptic regularity theory implies that there exists a subsequence \(H(t_1)\) converging to \(H_\infty\) in \(C^\infty\)-topology and \(H_\infty\) is a harmonic metric satisfying the desired boundary condition. \(\square\)

Proposition 3.4 Let \((E, D)\) be a vector bundle over a compact Riemannian manifold \((X, g)\) with smooth boundary \(\partial X\). Then for any given data \(K\), there exists a unique Poisson metric \(H\) on \((E, D)\) such that \(H = K\) on \(\partial X\) and \(\det H = \det K\).

**Proof** Due to Proposition 3.3, there exists a unique harmonic metric \(\tilde{H}\) such that \(\tilde{H} = K\) on \(\partial X\). Set \(H = \tilde{H} e^f\), it follows
\[
D^*_H \psi_H = -\frac{1}{2} D^*_H (df \otimes \text{id}_E) = \frac{\text{tr} D^*_H \psi_H}{\text{rank}(E)} \text{id}_E. \tag{3.8}
\]
Now we set \(f = \frac{\log \det(\tilde{H}^{-1} K)}{\text{rank}(E)}\), then we have \(H = K\) on \(\partial X\) and \(\det H = \det K\). \(\square\)

On each \(M_s\), the following Dirichlet problem is solvable,
\[
\begin{align*}
D^*_H \psi_{H_s} &= c_s \text{id}_E, \\
H|_{\partial M_s} &= K, \\
\det h_s &= 1, h_s &= K^{-1} H_s. \tag{3.9}
\end{align*}
\]
Due to the inequality \( \log \left( \frac{\text{tr} h_s}{\text{rank}(E)} \right) \geq \log \det \frac{h_s}{\text{rank}(E)} = 0 \) and the boundary condition, we have

\[
\frac{\partial \log \left( \frac{\text{tr} h_s}{\text{rank}(E)} \right)}{\partial \bar{n}} \geq 0, \tag{3.10}
\]

where \( \bar{n} \) is the inward unit normal vector field at \( \partial M_s \). We extend \( \log \left( \frac{\text{tr} h_s}{\text{rank}(E)} \right) \) to the function \( \tilde{\log} \left( \frac{\text{tr} h_s}{\text{rank}(E)} \right) \) on whole \( M \) by setting 0 outside \( M_s \). For any non-negative compactly supported function \( \phi \), the Green’s formula yields

\[
\int_M \log \left( \frac{\text{tr} h_s}{\text{rank}(E)} \right) \Delta \phi \, d\nu_g = \int_{M_s} \log \left( \frac{\text{tr} h_s}{\text{rank}(E)} \right) \Delta \phi \, d\nu_g
\]

\[
= \int_{M_s} \phi \Delta \left( \frac{\text{tr} h_s}{\text{rank}(E)} \right) \, d\nu_g
\]

\[
+ \int_{\partial M_s} \phi \frac{\partial}{\partial \bar{n}} \left( \frac{\text{tr} h_s}{\text{rank}(E)} \right) \, d\nu_{\partial M_s}
\]

\[
- \int_{\partial M_s} \left( \frac{\text{tr} h_s}{\text{rank}(E)} \right) \frac{\partial \phi}{\partial \bar{n}} \, d\nu_{\partial M_s}
\]

\[
\geq \int_{M_s} \phi \Delta \left( \frac{\text{tr} h_s}{\text{rank}(E)} \right) \, d\nu_g. \tag{3.11}
\]

Now by Lemma 2.2, we see

\[
\Delta \left( \frac{\text{tr} h_s}{\text{rank}(E)} \right) = \frac{2(D^* H_s \psi H_s - D^* K \psi_K, h_s) + |h_s^{-1/2} \delta_K h_s|^2}{\text{tr} h_s} \tag{3.12}
\]

\[
\geq -2 \left( |D^* \psi_K | h_s \right) \frac{\text{tr} h_s}{\text{rank}(E)} \text{tr} h_s
\]

\[
\geq -2 C \phi,
\]

where we have used the following simple inequality

\[
\frac{|\text{tr} \delta_K h_s|^2}{\text{tr} h_s} \leq |h_s^{-1/2} \delta_K h_s|^2. \tag{3.13}
\]

And therefore

\[
\int_M \left( \frac{\text{tr} h_s}{\text{rank}(E)} \right) \Delta \phi \, d\nu_g \geq -2 C \int_{M_s} \phi \phi \, d\nu_g \geq -2 C \int_{M} \phi \phi \, d\nu_g. \tag{3.14}
\]

That is, \( \tilde{\Delta} \log \left( \frac{\text{tr} h_s}{\text{rank}(E)} \right) \geq -2 C \phi \) as a distribution. So by Assumption 2.1 we get

\[
\max_{M_s} \left( \frac{\text{tr} h_s}{\text{rank}(E)} \right) \leq C_1 (1 + \int_{M_s} \left( \frac{\text{tr} h_s}{\text{rank}(E)} \right) \phi \, d\nu_g), \tag{3.15}
\]
where \( C_1 \) depends on \( C \). Combing this with \( \det h_s = 1 \) yields
\[
\max_{M_s} |\log h_s| \leq C_2 \left( 1 + \int_{M_s} |\log h_s| \phi \text{dvol}_g \right),
\]
(3.16)
for a constant \( C_2 \) depending only on \( C_1 \) and \( \text{rank}(E) \), where we have used that
\[
\left( \frac{\text{tr} h_s}{\text{rank}(E)} \right) \leq |\log h_s| \leq \text{rank}(E)^2 \text{tr} \log h_s.
\]
(3.17)
Our goal is to show the quantity
\[
\int_{M_s} |\log h_s| \phi \text{dvol}_g
\]
(3.18)
is uniformly bounded, under the assumption that \( D \) is \( K \)-analytically stable.

For further consideration, we recall some notations. Given \( \rho \in C^\infty(\mathbb{R}, \mathbb{R}), \Theta \in C^\infty(\mathbb{R} \times \mathbb{R}, \mathbb{R}), \chi \in \Omega^* (\text{End}(E)) \) and a self-adjoint bundle endomorphism \( \sigma \), we define \( \rho[\sigma] \) and \( \Theta[\sigma](\chi) \) as follows. For every \( x \in M \), we set
\[
\rho[\sigma] = \rho(\lambda_\alpha)e^\alpha \otimes e_\alpha, \Theta[\sigma](\chi) = \Theta(\lambda_\alpha, \lambda_\beta)\chi_\alpha^\beta e^\alpha \otimes e_\beta.
\]
(3.19)
where \( \{e_\alpha\}_{\alpha=1}^{\text{rank}(E)} \) is an orthogonal basis with respect to \( K \), such that
\[
\sigma(e_\alpha) = \lambda_\alpha e_\alpha, \chi = \chi_\alpha^\beta e^\alpha \otimes e_\beta.
\]
(3.20)
Next we prove the following proposition.

**Proposition 3.5** Let \((E, D)\) be a vector bundle over a compact Riemannian manifold \((X, g)\) with non-empty boundary, and \( H, K \) be two fiber metrics with \( H|_{\partial X} = K|_{\partial X} \). Then we have
\[
\int_X (D_H^* \psi_H - D_K^* \psi_K, s) \text{dvol}_g = -\frac{1}{2} \int_X (\Theta[s](Ds), Ds) \text{dvol}_g,
\]
(3.21)
in the sense of (3.19), where \( s = \log(K^{-1}H), \Theta(x, y) = \frac{e^y - e^x}{y - x} \) for \( x \neq y \) and \( \Theta(x, x) = 1 \).

**Proof** Consider the local normal coordinate at the considered point and for \( h = K^{-1}H \), noting that \( (h^{-1}\delta_K, \frac{\partial}{\partial x^i}, h, s) = \text{tr}(s\delta_K, \frac{\partial}{\partial x^i}, s) \), through computing, we have
\[
(D_H^* \psi_H - D_K^* \psi_K, s) = -\frac{1}{2} \left( D_K^* \left( h^{-1}\delta_K h \right), s \right) - \frac{1}{2} \left( h^{-1}\delta_K \frac{\partial}{\partial x^i} h \psi_K \left( \frac{\partial}{\partial x^i} \right), s \right)
\]
\[
= \frac{1}{2} \left( D_K \frac{\partial}{\partial x^i} \left( h^{-1}\delta_K h \right), s \right) + \frac{1}{2} \left( h^{-1}\delta_K h \left( \frac{\partial}{\partial x^i} \right), s \right)
\]
\[
= \frac{1}{2} \frac{\partial}{\partial x^i} \left( h^{-1}\delta_K h \right), s \right) + \frac{1}{2} \left( h^{-1}\delta_K h \left( \frac{\partial}{\partial x^i} \right), s \right)
\]
\[
= \frac{1}{2} \frac{\partial}{\partial x^i} \left( s \delta_K \frac{\partial}{\partial x^i} s \right) - \frac{1}{2} \left( h^{-1}\delta_K h \left( \frac{\partial}{\partial x^i} \right), s \right)
\]
\[
= \frac{1}{2} \frac{\partial}{\partial x^i} \left( s \frac{\partial}{\partial x^i} s \right) - \frac{1}{2} \left( h^{-1}\delta_K h, \delta_K s \right).
\]
(3.22)
On the other hand, according to Section 7.4 in [26], there is an open dense subset \( W \subset X \) and for every \( x \in W \), we may choose an orthogonal basis \( \{ e_\alpha \}_{\alpha=1}^{\text{rank}(E)} \) for \( E \) with respect to \( K \), defined over a neighborhood of \( x \), such that

\[
h = \sum e^{\lambda_{\alpha}} e_\alpha \otimes e^\alpha, \quad s = \sum \lambda_{\alpha} e_\alpha \otimes e^\alpha.
\]

(3.23)

We set \( De_\alpha = A_\beta^\alpha e_\beta \), then it follows

\[
Dh \circ h^{-1} = \Theta[s](Ds) = \sum_{\alpha} d\lambda_{\alpha} e_\alpha \otimes e^\alpha + \sum_{\alpha \neq \beta} (1 - e^{\lambda_{\alpha} - \lambda_{\beta}}) A_\alpha^\beta e_\alpha \otimes e^\beta,
\]

and therefore

\[
(h^{-1} \delta_K h, \delta_K s) = (Ds, Dh \circ h^{-1}) = (\Theta[s](Ds), Ds).
\]

(3.25)

Finally, by (3.22) and (3.25), we have

\[
\int_X (D^*_H \psi_H - D^*_K \psi_K, s) \, d\text{vol}_g = \frac{1}{4} \int_X d(\ast d|s|^2) - \frac{1}{2} \int_X (\Theta[s](Ds), Ds) \, d\text{vol}_g = -\frac{1}{2} \int_X (\Theta[s](Ds), Ds) \, d\text{vol}_g.
\]

(3.26)

We shall prove (3.18) by contradictory argument, the method is a combination of Proposition 3.5 and Simpson’s trick. We may assume that there exists a sequence \( s \to \infty \), such that

\[
ls = \int_{M_s} |\log h_s| \phi \, d\text{vol}_g \to +\infty.
\]

(3.27)

Let’s set \( u_s = \log h_s/ls \) with \( \text{tr} u_s = 0 \) and \( \int_{M_s} |u_s| \phi \, d\text{vol}_g = 1 \). By (3.16) we see

\[
\sup_{M_s} |u_s| \leq \frac{C_2}{ls} (1 + ls) \leq C_3,
\]

(3.28)

for a constant \( C_3 \) doesn’t depend on \( s \). Using Proposition 3.5 and the fact that \( H_s \) is a Poisson metric, one has

\[
\int_{M_s} l_s (\Theta[l_s u_s](Du_s), Du_s) \, d\text{vol}_g = 2 \int_{M_s} (D^*_K \psi_K - D^*_H \psi_H, u_s) \, d\text{vol}_g = 2 \int_{M_s} ((D^*_K \psi_K)^\perp, u_s) \, d\text{vol}_g.
\]

(3.29)

Notice that

\[
l\Theta(lx, ly) \to \begin{cases} \frac{1}{x-y}, & x > y, \\ +\infty, & x \leq y. \end{cases}
\]

(3.30)

increases monotonically as \( l \to +\infty \). It holds that

\[
\int_{M_s} (\rho[u_s](Du_s), Du_s) \, d\text{vol}_g \leq 2 \int_{M_s} ((D^*_K \psi_K)^\perp, u_s) \, d\text{vol}_g.
\]

(3.31)

for any \( \rho : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) with \( \rho(x, y) < \frac{1}{x-y} \) whenever \( x > y \) and \( s \) large enough.
Due to the zeroth order estimate of $u_s$, and $|(D^+_K \psi_K)^ot| \leq C \phi$, by taking $\rho$ small enough, it follows that $Du_s$ is bounded in $L^2$-norm on any compact subset of $M$. We may assume that $u_s \to u_\infty$ weakly in $L^2_{1, \text{loc}}$-topology with $\|u_\infty\|_{L^\infty} \leq C_3$. Since $\phi \in L^1$, for any $\epsilon > 0$ there exists a $s_\epsilon$ such that for any $s \geq s_\epsilon$, it holds

$$\int_{M - M_s} \phi \, d\text{vol}_g \leq \epsilon. \quad (3.32)$$

Hence for $s_\epsilon \leq s_1 \leq s_2$, (3.28) indicates

$$1 - C_3 \epsilon \leq \int_{M_{s_2}} |u_{s_2}| \phi \, d\text{vol}_g - \int_{M_{s_2} - M_{s_1}} |u_{s_2}| \phi \, d\text{vol}_g = \int_{M_{s_1}} |u_{s_2}| \phi \, d\text{vol}_g \leq 1. \quad (3.33)$$

Fixing $s_1$ and by the weak compactness of $\{u_s\}$ in $L^2_{1, \text{loc}}$-topology, we get

$$1 - C_3 \epsilon \leq \int_{M_{s_1}} |u_\infty| \phi \, d\text{vol}_g \leq 1, \quad (3.34)$$

and then taking $s_1 \to \infty$ and $\epsilon \to 0$ yields $\int_M |u_\infty| \phi \, d\text{vol}_g = 1$, which implies $u_\infty \neq 0$.

Now it follows from (3.28), (3.31), (3.32) and $|(D^+_K \psi_K)^ot| \leq C \phi$ that

$$\int_{M_{s_1}} (\rho[u_{s_2}](Du_{s_2}), Du_{s_2}) \, d\text{vol}_g \leq \int_{M_{s_2}} (\rho[u_{s_2}](Du_{s_2}), Du_{s_2}) \, d\text{vol}_g \leq 2 \left( \int_{M_{s_2} - M_{s_1}} + \int_{M_{s_1}} \right) ((D^+_K \psi_K)^ot, u_{s_2}) \, d\text{vol}_g \leq 2CC_3 \epsilon + 2 \int_{M_{s_1}} ((D^+_K \psi_K)^ot, u_{s_2}) \, d\text{vol}_g. \quad (3.35)$$

Let’s take $s_2 \to \infty$, $s_1 \to \infty$ and $\epsilon \to 0$ successively, it yields

$$\int_M (\rho[u_\infty](Du_\infty), Du_\infty) \, d\text{vol}_g \leq 2 \int_M ((D^+_K \psi_K)^ot, u_\infty) \, d\text{vol}_g \leq 2CC_3 \int_M \phi \, d\text{vol}_g. \quad (3.36)$$

As long as (3.36) is established, by using Simpson’s discussion in [37, p 886–888] and Loftin’s observation in [25], it’s easy to conclude (3.18) and therefore we obtain

**Proposition 3.6** If $(M, g)$ satisfies the Assumption 2.1 and the flat bundle $(E, D)$ is $K$-analytically stable with $|(D^+_K \psi_K)^ot| \leq C \phi$ for a constant $C$, it holds

$$\max_{M_s} |\log h_s| \leq \tilde{C}_1, \quad (3.37)$$

for a constant $\tilde{C}_1$ that doesn’t depend on $s$. 
Using Lemma 2.2 again shows
\[
|h_s^{-\frac{1}{2}} \delta_K h_s|_{L^2(M_s)}^2 = \int_{M_s} \left( \Delta \text{tr} h_s - 2(D^*_H \psi_{H_s} - D^*_K \psi_K, h_s) \right) \, d\text{vol}_g
\]
\[
= \int_{M_s} \left( \Delta \text{tr} h_s + 2((D^*_K \psi_K)^\perp, h_s) \right) \, d\text{vol}_g
\]
\[
= - \int_{\partial M_s} \frac{\partial \text{tr} h_s}{\partial n} \, d\text{vol}_{\partial M_s} + 2 \int_{M_s} ((D^*_K \psi_K)^\perp, h_s) \, d\text{vol}_g
\]
\[
\leq 2 \int_{M_s} ((D^*_K \psi_K)^\perp, h_s) \, d\text{vol}_g
\]
\[
\leq CC_4 \int_M \phi \, d\text{vol}_g,
\]
where $C_4$ depends only on $\tilde{C}_1$ and rank($E$). So we obtain

**Proposition 3.7** If $(M, g)$ satisfies the Assumption 2.1 and the flat bundle $(E, D)$ is $K$-analytically stable with $|(D^*_K \psi_K)^\perp| \leq C\phi$ for a constant $C$, we have
\[
||Dh_s||_{L^2(M_s)} \leq \tilde{C}_2 ||\phi||_{L^1},
\] (3.39)
for a constant $\tilde{C}_2$ that depends only on $C$, $\tilde{C}_1$ and rank($E$).

**Proposition 3.8** If $(M, g)$ satisfies the Assumption 2.1 and the flat bundle $(E, D)$ is $K$-analytically stable with $|(D^*_K \psi_K)^\perp| \leq C\phi$ for a constant $C$, then there exists a Poisson metric $H$ such that $\det h = 1$, $|h| \in L^\infty$ and $|Dh| \in L^2$ for $h = K^{-1}H$.

**Proof** Using the flatness of $D$, we deduce
\[
\Delta |\psi_{H_s}|_{H_s}^2 \geq -2g^{ij} \left( D^2_{H_s} \left( \frac{\partial}{\partial x^i} \left( \psi_{H_s} \left( \frac{\partial}{\partial x^j} \right) \right), \psi_{H_s} \right) \right)_{H_s}
\]
\[
+ 2g^{ij} \left( \psi_{H_s} \left( R \left( \frac{\partial}{\partial x^i} \right), \frac{\partial}{\partial x^j} \right) \right)_{H_s} - 2 \left( D_{H_s} D^*_H \psi_{H_s}, \psi_{H_s} \right)_{H_s}
\]
\[
\geq -C_5 |\psi_{H_s}|_{H_s}^2 - \frac{2}{\text{rank}(E)} \left( d \text{tr} D^*_H \psi_{H_s} \otimes \text{id}_E, \psi_{H_s} \right)_{H_s}
\]
\[
\geq -C_5 |\psi_{H_s}|_{H_s}^2 - \frac{2}{\text{rank}(E)} \left( d \text{tr} D^*_K \psi_K \otimes \text{id}_E, \psi_{H_s} \right)_{H_s}
\]
\[
\geq -C_6 |\psi_{H_s}|_{H_s}^2 - C_7,
\]
where $C_6$ depends only on the lower bounded of the Ricci curvature at the considered point, $C_7$ depends only on $|d \text{tr} D^*_K \psi_K|$ and rank($E$).

On the other hand, (2.15) implies there exists two constants $C_8, C_9$ that depend only on $\tilde{C}_1$ and rank($E$), such that
\[
|\psi_{H_s}|_{H_s}^2 \leq C_8 (|\psi_K|^2 + |Dh_s|^2),
\] (3.41)
\[
|Dh_s|^2 \leq C_9 (|\psi_K|^2 + |\psi_{H_s}|_{H_s}^2).
\] (3.42)
So combing (3.41), (3.42) with Proposition 3.7, (3.40) and Theorem 9.20 in [16], we get the uniform local boundedness of $|DH_S|$. Now on each $M_j$, by the Poisson metric equation, we have

$$D^*_S (h^{-1}_S \delta_K h_s) = 2(D_K^* \psi_K)^\perp - g^{ij} \left[ h^{-1}_S \delta_{K, \frac{\partial}{\partial x_j}} h_s, \psi_K \left( \frac{\partial}{\partial x_j} \right) \right],$$

(3.43)

the standard bootstrapping procedure implies the uniform local higher order estimates of $\{h_s\}$ and by diagonal subsequence argument, $H_s$ converge to $H_\infty$ on whole $M$ in $C^\infty_{k\psi}$-topology. It obvious that $H_\infty$ is a Poisson metric with det $h = 1, |h| \in L^\infty$ and $|DH| \in L^2$.

**Proof of Theorem 1.3** The existence is given by Proposition 3.8 and the uniqueness will be proved in Proposition 4.1. Conversely, we assume $D$ is irreducible and $H$ is a Poisson metric which shares the same properties with $H_\infty$, we shall demonstrate that it is also analytically stable with respect to $K$. For any $D$-invariant sub-bundle $S \subset E$, $K$ and $H$ restrict to the metrics $K_S$ and $H_S$ on $S$. Taking the orthogonal complement of $S$ in $E$ with respect to $K$, we have the orthogonal decomposition $E = S \oplus S^\perp$ and the projection $\pi_K$ onto $S$.

If $|D\pi_K| \notin L^2$, by (2.6) and $|D^*_K \psi_K| \in L^1$, we know

$$\deg_g (S, D, K) = -\infty.$$  

(3.44)

If $|D\pi_K| \in L^2$, we know $\deg_g (S, D, K)$ is finite. Moreover, we have

$$h_S = \pi_K \circ h \circ \pi_K,$$

(3.45)

$$D_S h_S = \pi_K \circ Dh \circ \pi_K + D\pi_K \circ (id_E - \pi_K) \circ h \circ \pi_K,$$

(3.46)

where $h_S = K^{-1}_S H_S$ and $D_S$ is the induced connection on $S$. It then follows

$$\int_M |\text{tr}(h_S^{-1} \delta_{DS, K_S} h_S)|^2 \text{dvol}_g \leq \int_M |h_S^{-1}|^2 |D_S h_S|^2 \text{dvol}_g < \infty,$$

(3.47)

where $\delta_{DS, K_S} = D_{S, K} - \psi_{DS, K_S}$. So Simpson’s Lemma 5.2 in [37] (or Yau’s Lemma in [44]) implies

$$\lim_{j \to \infty} \int_{M_j} \frac{1}{2} d(*d \log \det h_S) = \lim_{j \to \infty} \int_{M_j} \frac{1}{2} d(*\text{tr}(h_S^{-1} \delta_{DS, K_S} h_S)) = 0,$$

(3.48)

where each $M_j$ is an exhaustion subset. It then follows from the convergence theorem that

$$\lim_{j \to \infty} \int_{M_j} \text{tr} D^*_S \psi_{HS} \text{dvol}_g = \lim_{j \to \infty} \int_{M_j} \text{tr} D^*_K \psi_{KS} \text{dvol}_g$$

$$+ \lim_{j \to \infty} \int_{M_j} \frac{1}{2} \Delta \log \det h_S \text{dvol}_g$$

$$= - \deg_g (S, D, K),$$

(3.49)

and hence

$$\lim_{j \to \infty} \int_{M_j} \text{tr} D^*_H \psi_{HS} \text{dvol}_g = \int_M \text{tr} D^*_H \psi_{HS} \text{dvol}_g = - \deg_g (S, D, H).$$

(3.50)

In summary, we conclude either

$$\deg_g (S, D, K) = -\infty$$

(3.51)

or

$$\deg_g (S, D, K) = \deg_g (S, D, H).$$

(3.52)
On the other hand, it follows from Proposition 2.1 that \((E, D)\) is \(H\)-analytically stable and hence the proof is completed. \(\square\)

## 4 Some applications and consequences

### 4.1 Uniqueness of Poisson metrics

Firstly, we get the following uniqueness result and this implies the third conclusion in Theorem 1.3 under weaker assumption.

**Proposition 4.1** Let \((E, D)\) be a \(K\)-analytically stable vector bundle over a Riemannian manifold \((M, g)\) satisfying the Assumptions 2.1, 2.2 (or complete with finite volume). Suppose \(\text{tr} D_K^* \psi_K \in L^1\) and \(H\) is a Poisson metric such that \(\det h = 1\), \(|h| \in L^\infty\) and \(|Dh| \in L^2\), where \(h = K^{-1}H\). If \(\tilde{H}\) is another Poisson metric which is mutually bounded with \(H\) and \(\det H = \det \tilde{H}\), we have \(H = \tilde{H}\).

**Proof** Let \(\tilde{h} = H^{-1} \tilde{H}\) and we have
\[
\Delta \text{tr} \tilde{h} = 2(D_h^* \psi_{\tilde{H}} - D_{\tilde{H}}^* \psi_H, \tilde{h})_H + |\tilde{h}^{-\frac{1}{2}} \delta_H \tilde{h}|_H^2.
\]

(4.1)

So the assumptions imply \(\tilde{h}\) is \(D\)-parallel and let \(E = \bigoplus_{j=1}^m E_j\) denote the eigendecomposition of \(\tilde{h}\), this decomposition is orthogonal with respect \(H\) and \(\tilde{H}\). Moreover, we have \(\tilde{H}|_{E_j} = c_j H|_{E_j}\) for some constants \(c_j\) and
\[
- \int_M \text{tr} D_H^* \psi_H \text{dvol}_g = - \sum_{j=1}^m \int_M \text{tr} D_{H|_{E_j}}^* \psi_{H|_{E_j}} \text{dvol}_g.
\]

(4.2)

Then running the same argument in the proof of Theorem 1.3 shows \(D\) being \(H\)-analytically stable. As a consequence,
\[
\frac{\deg(E, D, H)}{\text{rank}(E)} = - \sum_{j=1}^m \int_M \text{tr} D_{H|_{E_j}}^* \psi_{H|_{E_j}} \text{dvol}_g \frac{\text{rank}(E_j)}{\text{rank}(E)}
\]

(4.3)

the strict inequality occurs if and only if \(m > 1\) and therefore we complete the proof. \(\square\)

**Proposition 4.2** Let \((E, D)\) be a \(K\)-analytically stable flat bundle over a Riemannian manifold \((M, g)\) satisfying the Assumption 2.1 (or complete with finite volume) and there is a positive exhaustion function \(\rho\) with \(|d \log \rho| \in L^1\). Assume that \(|\psi_K| \in L^2\) and \(H_1\) is a Poisson metric such that \(|h_1|, |h_1^{-1}| \in L^\infty\) and \(|Dh_1| \in L^2\), where \(h_1 = K^{-1}H_1\). If \(H_2\) is another Poisson metric which is mutually bounded with \(H_1\) and \(\det H_1 = \det H_2\), then \(H_1 = H_2\).
Proof It is a flat bundle analogy of Mochizuki’s uniqueness result on Hermitian-Einstein metrics in [31]. Similar, let \( E = \bigoplus_{j=1}^{m} E_j \) denote the eigendecomposition of \( h_{12} = H_1^{-1} H_2 \) and \( H_1|_{E_j} = c_i H_2|_{E_j} \) for some constants \( c_j \). Let \( \pi_j \) denotes the projection onto \( E_j \) and \( h_i = K^{-1} H_i \), then \( \pi_j \) are bounded with respect to \( K \), as \( K \) and \( H_i \) are mutually bounded.

Now \( \delta_K \pi_j = \delta_{H_i} \pi_j - [h_{1}^{-1} \delta_k h_1, \pi_j] = -[h_{1}^{-1} \delta_k h_1, \pi_j] \) implies \( \delta_K \pi_j \) and \( D \pi_j \) are \( L^2 \). We consider the fiber metric \( H \) determined by the direct sum of \( K|_{E_j} \), then \( H \) and \( K \) are mutually bounded. By definition, we have \( h \triangleq K^{-1} H = \sum_{j=1}^{m} \pi_j \circ \pi_j \) and it follows

\[
h^{-1} D h = h^{-1} \sum_{j=1}^{m} D \pi_j \circ \pi_j, \quad h^{-1} \delta_K h = h^{-1} \sum_{j=1}^{m} \pi_j \circ \delta_K \pi_j,
\]

(4.4)

which implies \( h^{-1} D h \) and \( h^{-1} \delta_K h \) are \( L^2 \).

Next we compute \( D(h^{-1} \delta_K h) = -h^{-1} D h h^{-1} \delta_K h + h^{-1} D(\delta_K h) \)

\[
D(\delta_K h) = D \left( \sum_{j=1}^{m} \pi_j \circ \delta_K \pi_j \right) = \sum_{j=1}^{m} \left( D \pi_j \circ \delta_K \pi_j + \pi_j \circ D(\delta_K \pi_j) \right)
\]

(4.5)

\[
= \sum_{j=1}^{m} \left( D \pi_j \circ \delta_K \pi_j + 4 \pi_j \circ [D^2_{\pi_j}, \pi_j] \right)
\]

\[
= \sum_{j=1}^{m} \left( D \pi_j \circ \delta_K \pi_j - 2 \pi_j \circ [\psi_\pi, \psi_\pi, \pi_j] \right).
\]

where we have used the flatness of \( D \). As a consequence, we find \( \Delta \log \det h \) is \( L^1 \).

Finally we set \( \chi_N = \eta \left( \frac{\rho}{N} \right) \), where \( \eta \) is a nonnegative function with \( \eta(t) = 0 \) if \( t \geq 2 \) and \( \eta(t) = 1 \) if \( t \leq 1 \). We then deduce

\[
\int_M \chi_N \Delta \log \det h \ dvol_g = - \int_M \nabla \chi_N \cdot \nabla \log \det h \ dvol_g
\]

\[
= - \int_{\{N \leq \rho \leq 2N\}} \frac{\eta'}{N} \nabla \rho \cdot \nabla \log \det h \ dvol_g
\]

(4.6)

\[
\leq \int_{\{N \leq \rho \leq 2N\}} \frac{2|\eta'|}{\rho} |\nabla \rho||\nabla \log \det h| \ dvol_g
\]

\[
= \int_{\{N \leq \rho \leq 2N\}} 2|\eta'| \left( \frac{\rho}{N} \right)|\nabla \log \rho||\nabla (h^{-1} \delta_K h)| \ dvol_g.
\]

Combing this and using the fact that \( \Delta \log \det h \) lies in \( L^1 \), it follows

\[
- \int_M \text{tr} \ D^*_K \psi_K \ dvol_g = - \int_M \text{tr} \ D^*_H \psi_H \ dvol_g + \frac{1}{2} \int_M \Delta \log \det h \ dvol_g
\]

(4.7)

\[
= - \sum_{j=1}^{m} \int_M \text{tr} \ D^*_K \psi_K|_{E_j} \ dvol_g.
\]
Since $D$ is $K$-analytically stable, we obtain $m = 1$ and the proof is completed.  

\[ \square \]

### 4.2 Applications on nonabelian Hodge correspondence

**Proposition 4.3** Let $(X, \omega)$ be a noncompact complex curve satisfying the Assumption 2.1, and $(E, D)$ be a $K$-analytically stable flat complex vector bundle over $X$ with $|(D_K^* \psi_K)^\perp| \leq C\phi$ for a constant $C$, then there exists a Higgs structure on $E$.

**Proof** According to Theorem 1.3, there is a Poisson metric $H$ on $(E, D)$ and we have

\[ D_H \psi_H^\perp = D_H \left( \psi_H^{1,0} \right) = 0, \tag{4.8} \]

\[ D_H^* \psi_H^\perp = \sqrt{-1} D_H \left( \psi_H^{1,0} - \psi_H^{1,0} \right) = 0, \tag{4.9} \]

where $\psi_H^\perp$ is the trace-free part of $\psi_H$. Then we easily get $D_H^0 \psi_H^{1,0} = 0$ and the pair $(D_H^0, \psi_H^{1,0})$ gives rise a Higgs structure on $E$.  

For higher dimensional case, we have

**Proposition 4.4** Let $(X, \omega)$ be a complete Kähler manifold with bounded Ricci curvature from below and satisfying the Assumption 2.1. Assume $(E, D)$ is a $K$-analytically stable flat complex vector bundle over $X$ with $|(D_K^* \psi_K)^\perp| \leq C\phi$ for a constant $C$ and $|\psi_K| \in L^2$. Then there exists a Higgs structure on $E$.

**Proof** Due to Theorem 1.3, there is a Poisson metric $H$ and we compute

\[ \Delta |\psi_H^\perp|^2_H = -2 \left( \nabla_H^\perp \nabla_H \psi_H^\perp, \psi_H^\perp \right)_H + 2|\nabla_H \psi_H^\perp|_H^2 \]

\[ = -2g^{ij} \left( D_H^2 \left( \frac{\partial}{\partial x^i} \psi_H^\perp \right), \frac{\partial}{\partial x^j} \psi_H^\perp \right)_H + 2g^{ij} \left( \nabla_H \left( \psi_H^\perp \right), \frac{\partial}{\partial x^j} \psi_H^\perp \right)_H + 2|\nabla_H \psi_H^\perp|_H^2 \]

\[ = \left( g^{ij} \left[ \left[ \psi_H^\perp, \psi_H^\perp \right] \left( \frac{\partial}{\partial x^i} \right), \frac{\partial}{\partial x^j} \psi_H^\perp \right] \right)_H + 2 \psi_H^\perp \circ \text{Ric}, \psi_H^\perp \]

\[ \geq -2C_s |\psi_H^\perp|^2_H + 2|\nabla_H \psi_H^\perp|_H^2, \]  

where $-C_s$ the lower bound of Ricci curvature. Let $R$ be any positive constant and we fix point $x_0$, choose a cut-off function $\eta$ satisfying

\[
\begin{cases}
\eta(x) = 1, & x \in B_{x_0}(R), \\
\eta(x) = 0, & x \in X \setminus B_{x_0}(2R), \\
0 \leq \eta \leq 1, & |\nabla \eta| \leq C_1 R,
\end{cases}
\tag{4.11}
\]

where $C_1$ is a positive constant and $B_{x_0}(R)$ is the geodesics ball centered at $x_0$ with radius $R$. Then we have

\[ 2 \int_X \eta^2 |\nabla_H \psi_H^\perp|_H^2 d\omega \leq \int_X \eta^2 \left( \Delta |\psi_H^\perp|_H^2 + 2C_s |\psi_H^\perp|^2_H \right) d\omega \]

\[ \leq \int_X 4\eta |\nabla \eta||\psi_H^\perp|_H|\nabla_H \psi_H^\perp|_H d\omega + 2C_s \int_X \eta^2 |\psi_H^\perp|^2_H d\omega \]

\[ \square \]
\[
\leq \int_X \eta^2 |\nabla H \psi_H^1|^2_{H} \text{dvol}_\omega + \int_X \left(2C_*\eta^2 + 4|\nabla H \eta|^2\right) |\psi_H^1|^2_{H} \text{dvol}_\omega
\]
\[
\leq \int_X \eta^2 |\nabla H \psi_H^1|^2_{H} \text{dvol}_\omega + \int_X \left(2C_*\eta^2 + \frac{4C^2}{R^2}\right) |\psi_H^1|^2_{H} \text{dvol}_\omega.
\]

(4.12)

And it also holds
\[
\int_X |\psi_H^1|^2_{H} \text{dvol}_\omega = \int_X |\psi_H^1| - \frac{1}{2} h^{-1} D h + h^{-1}[\psi_H^1, h]^2 \text{dvol}_\omega < \infty,
\]

(4.13)

where \( h = K^{-1} H \). So by letting \( R \) goes infinity, we see
\[
\int_X |\nabla H \psi_H^1|^2_{H} \text{dvol}_\omega \leq 2C_1 \int_X |\psi_H^1|^2_{H} \text{dvol}_\omega < \infty.
\]

(4.14)

Now let’s consider the pseudo-curvature \( G_H \triangleq (D_H^{0,1} + \psi_H^{1,0})^2 \). The flatness of \( D \) implies
\[
D_H^{1,0} \psi_H^{0,1} = -D_H^{0,1} \psi_H^{1,0},\ D_H^{1,0} D_H^{0,1} = -\frac{1}{2}[\psi_H^{1,0}, \psi_H^{1,0}],\ D_H^{0,1} D_H^{1,0} = -\frac{1}{2}[\psi_H^{0,1}, \psi_H^{1,0}],
\]

(4.15)

and it follows that
\[
G_H^* = \left((D_H^{0,1})^2 + D_H^{0,1} \psi_H^{1,0} + \psi_H^{1,0} \wedge \psi_H^{1,0}\right)^* = -D_H^{1,0} D_H^{0,1} + D_H^{1,0} \psi_H^{0,1} - \psi_H^{0,1} \wedge \psi_H^{0,1}
\]

(4.16)

By \( \text{Kähler identity} \) we know \( \sqrt{-1} \Lambda \omega G_H^\perp = \frac{1}{2} (D_H^* \psi_H^1)^\perp = 0 \) and
\[
\text{tr}(G_H^\perp \wedge G_H^\perp) \wedge \omega^{n-2} = \text{tr} \left(G_H^\perp \wedge \ast(G_H^\perp)^*\right) = |G_H^\perp|^2_{H} \text{dvol}_\omega,
\]

(4.17)

where \( n \) is the complex dimension of \( X \). On the other hand, a direct calculation shows
\[
\text{tr}(G_H^\perp \wedge G_H^\perp) \wedge \omega^{n-2} = -2 \text{tr} \left(\psi_H^{0,1} \wedge \psi_H^{1,0} \wedge \psi_H^{1,0} \wedge \psi_H^{0,1}\right) \wedge \omega^{n-2}
\]
\[
- \text{tr} \left(D_H^{0,1} \psi_H^{1,0} \wedge D_H^{0,1} \psi_H^{0,1}\right) \wedge \omega^{n-2}
\]
\[
- \frac{\text{tr} \left(D_H^{0,1} \psi_H^{1,0}\right) \wedge \text{tr} \left(D_H^{0,1} \psi_H^{0,1}\right)}{\text{rank}(E)} \wedge \omega^{n-2}
\]
\[
= -\overline{\text{tr}} \left(\psi_H^{1,0} \wedge D_H^{1,0} \psi_H^{0,1}\right) \wedge \omega^{n-2}
\]
\[
+ \overline{\text{tr}} \left(\psi_H^{1,0} \wedge D_H^{1,0} \psi_H^{0,1}\right) \wedge \omega^{n-2}
\]

(4.18)

where we have used (4.15), \( D_H^{0,1} \psi_H^{0,1} = 0 \) and \( [D_H^{1,0}, D_H^{0,1}] = -[\psi_H^{1,0}, \psi_H^{0,1}] \).
Now as it holds $D^{1,0}_H \psi^\perp_H = \sum_{i=1}^n dz^i \wedge \nabla_H, a \frac{\partial}{\partial z^i} \psi^\perp_H$, it follows from (4.14) that
\[
\int_X |\text{tr}(\psi^{1,0}_H \wedge D^{1,0}_H \psi^\perp_H) \wedge \omega^{n-2}| \, d\text{vol}_\omega < \infty.
\] (4.19)
Therefore, by (4.17), (4.18), (4.19) and Yau’s Lemma in [44], we conclude $G^\perp_H$ vanishes identically and the pair $(D^{0,1}_H, \psi^{1,0}_H)$ is a Higgs structure on $E$. \hfill \Box

**Proof of Theorem 1.4** Proposition 4.3 and Proposition 4.4 indicate $(D^{0,1}_H, \psi^{1,0}_H)$ determines a Higgs structure on $E$. Moreover, we see the corresponding Hitchin-Simpson curvature
\[
F_{D^{0,1}_H, \psi^{1,0}_H, H} = D^2_H + D_H \psi^\perp_H + \frac{1}{2} \left[ \psi^\perp_H, \psi^\perp_H \right] = 0,
\] (4.20)
where we have used the flatness of $D$. Hence we complete of proof of (1) and (2). For (3), if $(X, \omega)$ has nonnegative Ricci curvature, we have for any $\epsilon > 0$ that
\[
\Delta \left( \frac{1}{2} |\psi^\perp_H|^2 + \epsilon \right) \leq \frac{1}{4} \Delta |\psi^\perp_H|^2 - \frac{1}{4} |\frac{1}{2} d |\psi^\perp_H|^2 |^2 \leq \frac{1}{2} \left( \frac{1}{2} |\psi^\perp_H|^2 + \epsilon \right) \left( 1 - \frac{1}{2} |\psi^\perp_H|^2 \right)
\] (4.21)
and by letting $\epsilon$ goes zero, it follows that $|\psi^\perp_H|$ is subharmonic. On the other hand, as $|\psi^\perp_H| \in L^2$, Yau’s Liouville theorem in [44] applies and we conclude $|\psi^\perp_H|$ is constant. But the volume of $(X, \omega)$ is infinite, so we know $\psi^\perp_H = 0$ and then (3) follows easily. \hfill \Box

In a similar way, using Corollary 1.1 instead, we can obtain the following proposition.

**Proposition 4.5** Let $(X, \omega)$ be a Kähler manifold satisfying the Assumptions 2.1, 2.2 with $\phi = 1$ and $(E, D)$ be a K-analytically stable flat complex vector bundle over $X$ such that $|D^*_K \psi_K| \in L^\infty$ and $|\psi_K| \in L^2$. Assume either $\dim_{\mathbb{C}} X = 1$ or $(X, \omega)$ being complete with bounded Ricci curvature from below, then $(E, D)$ comes from a Higgs bundle.

If $(E, D)$ is a flat bundle coming from a Higgs bundle $(E, \bar{\partial}_E, \theta)$ via a harmonic metric $H$, then we call $(E, D, H)$ is a harmonic bundle. Denote by $H^*_D(E, X)$ and $H^*_{\text{Dol}}(E, X)$ the set of $L^\infty$-elements in the cohomology groups of the complexes $(\Omega^*(E), D)$ and $(\Omega^*(E), \bar{\partial}_{E, \theta})$ respectively.

**Lemma 4.1** Assume $(X, \omega)$ is a Kähler manifold satisfying the Assumption 2.1, we have

1. If $(E, D)$ is a flat bundle and $H$ being a harmonic metric, then any $D$-parallel $L^\infty$-section is also $D^{0,1}_H + \psi^{1,0}_H$-parallel.
2. If $(E, \bar{\partial}_E, \theta)$ is a Higgs bundle and $H$ is a Hermitian metric with $\Lambda_\omega F_{\bar{\partial}_E, \theta, \omega} = 0$, then any $\bar{\partial}_{E, \theta}$-parallel $L^\infty$-section is also $D^{1,0}_{\bar{\partial}_{E, \theta}}$-parallel.
3. If $(E, D, H)$ is a harmonic bundle, then $H^0_{DR}(E, X) = H^0_{\text{Dol}}(E, X)$. 

\[\text{Springer}\]
Proof The statement (3) follows from (1) and (2). Firstly suppose \( f \) is \( D \)-parallel, we have
\[
\Delta|f|^2_H = \sqrt{-1}\Lambda_\omega \left( D^{0,1} f, f \right)_H + \sqrt{-1}\Lambda_\omega \left( D^{1,0} f, D^{1,0} f \right)_H \\
- \sqrt{-1}\Lambda_\omega \left( D^{0,1} f, D^{1,0} f \right)_H + \sqrt{-1}\Lambda_\omega \left( f, D^{0,1} D^{1,0} f \right)_H \\
= |D^{0,1} f|^2_H + \psi^1_0 |f|^2_H - \sqrt{-1}\Lambda_\omega \left( D^{1,0} \psi^1_0 f, f \right)_H - \sqrt{-1}\Lambda_\omega \left( f, D^{0,1} \psi^1_0 f \right)_H \\
= |D^{0,1} f|^2_H + \psi^1_0 |f|^2_H + \sqrt{-1}\Lambda_\omega \left( \psi^1_0 D^{1,0} f, f \right)_H + \sqrt{-1}\Lambda_\omega \left( f, \psi^1_0 D^{0,1} f \right)_H \\
= |D^{0,1} f|^2_H + \psi^1_0 |f|^2_H - \sqrt{-1}\Lambda_\omega \left( D^{1,0} f, \psi^1_0 f \right)_H + \sqrt{-1}\Lambda_\omega \left( \psi^1_0 f, D^{1,0} f \right)_H \\
= 2\left( D^{0,1} f, \psi^1_0 f \right)_H. \quad (4.22)
\]

By assumption we have \( f \) is \( D^{0,1} + \psi^1_0 \)-parallel.

On the other hand, if \( f \) is \( \overline{\sigma}_{E,\theta} \)-parallel, we deduce
\[
\Delta|f|^2_H = \sqrt{-1}\Lambda_\omega (\partial f, \partial f)_H + \sqrt{-1}\Lambda_\omega (f, \overline{\sigma}_{E,\theta} \partial f)_H \\
= |\partial f|^2_H + \sqrt{-1}\Lambda_\omega (f, \overline{\sigma}_{E,\theta} f)_H - \sqrt{-1}\Lambda_\omega (f, [\theta, \theta^* H]) f)_H \\
= |\partial f, H f|^2_H, \quad (4.23)
\]
and then we see \( f \) is \( D_{\overline{\sigma}_{E,\theta}, H} \)-parallel. \( \square \)

Consider a complex vector bundle \( E \) over a complex manifold \((X, \omega)\), equipped with a Hermitian metric \( H \) and a connection \( D \). The contraction of \( G_H \) can be written as
\[
\sqrt{-1}\Lambda_\omega G_H = \frac{\sqrt{-1}}{4} \Lambda_\omega \left( D^{0,1} + \delta^{0,1}_H, D^{1,0} - \delta^{1,0}_H \right) \\
= \frac{\sqrt{-1}}{4} \Lambda_\omega \left( D^2 - (\delta^{1,0}_H + D^{0,1})^2 + (D^{1,0} + \delta^{0,1}_H)^2 - \delta^2_H \right), \quad (4.24)
\]
If \( D \) is flat, we see
\[
\sqrt{-1}\Lambda_\omega G_H = \frac{\sqrt{-1}}{4} \Lambda_\omega \left( - (\delta^{1,0}_H + D^{0,1})^2 + (D^{1,0} + \delta^{0,1}_H)^2 \right) \\
= \frac{\sqrt{-1}}{4} \Lambda_\omega \left( - \left[ D^{0,1}, \delta^{1,0}_K + h^{-1} \delta^{1,0}_K h \right] + \left[ \delta^{0,1}_K + h^{-1} \delta^{0,1}_K h, D^{1,0} \right] \right) \\
= \sqrt{-1}\Lambda_\omega G_K + \frac{\sqrt{-1}}{4} \Lambda_\omega D(h^{-1} D^*_K h), \quad (4.25)
\]
where \( h = K^{-1} H \) and \( D^*_K = \delta^{0,1}_K - \delta^{1,0}_K \).

Finally, we are in the position to prove Theorem 1.5.

Proof of Theorem 1.5 Given \(|D| \in \mathcal{M}_{Flat, K, \phi} \), by Theorem 1.3, we yield a Poisson metric \( H \) with \( \det h = 1, |h| \in L^\infty \) and \(|Dh| \in L^2 \), where \( h = K^{-1} H \). Then Proposition 4.3 implies that \((\overline{\sigma}_{E, \theta}) = (D^{0,1}_H, \psi^{1,0}_H)\) determines a Higgs structure with \( \text{tr} F_{\overline{\sigma}_{E, \theta}, K} = 0 \). Moreover, it can be seen \( H \) is harmonic with respect to \( D_H + \psi^1_H \) and due to Lemma 4.1, the assignment \([D] \mapsto ([\overline{\sigma}_{E, \theta}]) \) is well-defined. Then we compute
\[
\sqrt{-1} \Lambda_\omega F_{\bar{\sigma},E,\theta,K} = \sqrt{-1} \Lambda_\omega F_{\bar{\sigma},E,\theta,H} + \sqrt{-1} \Lambda_\omega \bar{\sigma}_{E,\theta,H}(h \partial_{\theta,H} h^{-1}) \\
= \sqrt{-1} \Lambda_\omega D_{\bar{\sigma},E,\theta,H}(h \partial_{\theta,H} h^{-1}) - \sqrt{-1} \Lambda_\omega \bar{\sigma}_{E,\theta,H}(h \partial_{\theta,H} h^{-1}) \\
= -\sqrt{-1} \Lambda_\omega D_{\bar{\sigma},E,\theta,H}(h \partial_{\theta,H} h^{-1}) + \sqrt{-1} \Lambda_\omega D_{\bar{\sigma},E,\theta,H}(\bar{\sigma}_{E,\theta,H} h^{-1}) \\
- \sqrt{-1} \Lambda_\omega \partial_{\theta,H}(h \partial_{\theta,H} h^{-1}) \\
= 4\sqrt{-1} \Lambda_\omega \bar{\sigma}_{E,\theta,H}(h \partial_{\theta,H} h^{-1}) + \sqrt{-1} \Lambda_\omega D_{\bar{\sigma},E,\theta,H}(\bar{\sigma}_{E,\theta,H} h^{-1}) \\
- \sqrt{-1} \Lambda_\omega \partial_{\theta,H}(h \partial_{\theta,H} h^{-1}) \\
= -2(D^*_K \psi_K) + \sqrt{-1} \Lambda_\omega \partial_{\theta,H}(h \partial_{\theta,H} h^{-1}) \\
+ \sqrt{-1} \Lambda_\omega \bar{\sigma}_{E,\theta,H}(h \partial_{\theta,H} h^{-1}) - \sqrt{-1} \Lambda_\omega \partial_{\theta,H}(h \partial_{\theta,H} h^{-1}),
\]

(4.26)

and

\[
\sqrt{-1} \Lambda_\omega \partial_{\theta,H}(h \partial_{E,\theta,H} h^{-1}) = \sqrt{-1} \Lambda_\omega (\partial_{\theta,H} h \wedge \bar{\sigma}_{E,\theta,H} h^{-1}) + \sqrt{-1} \Lambda_\omega (\bar{\sigma}_{E,\theta,H} h \wedge \partial_{\theta,H} h^{-1}) \\
- \sqrt{-1} \Lambda_\omega (\bar{\sigma}_{E,\theta,H} h \wedge \partial_{\theta,H} h^{-1}) - \sqrt{-1} \Lambda_\omega (h \partial_{E,\theta,H} h^{-1}) \\
= \sqrt{-1} \Lambda_\omega (\partial_{\theta,H} h \wedge \bar{\sigma}_{E,\theta,H} h^{-1}) + \sqrt{-1} \Lambda_\omega (\bar{\sigma}_{E,\theta,H} h \wedge \partial_{\theta,H} h^{-1}) \\
- \sqrt{-1} \Lambda_\omega F_{\bar{\sigma},E,\theta,K},
\]

(4.27)

where we have used (4.25). We obtain from above two equalities that

\[
\sqrt{-1} \Lambda_\omega F_{\bar{\sigma},E,\theta,K} = -(D^*_K \psi_K) + \frac{\sqrt{-1}}{2} \Lambda_\omega (\partial_{\theta,H} h \wedge \bar{\sigma}_{E,\theta,H} h^{-1}) \\
+ \frac{\sqrt{-1}}{2} \Lambda_\omega (\bar{\sigma}_{E,\theta,H} h \wedge \partial_{\theta,H} h^{-1}) + \frac{\sqrt{-1}}{2} \Lambda_\omega \bar{\sigma}_{E,\theta,H}(h \partial_{E,\theta,H} h^{-1}) \\
- \frac{\sqrt{-1}}{2} \Lambda_\omega \partial_{\theta,H}(h \partial_{\theta,H} h^{-1}),
\]

(4.28)

from which we conclude \(|\Lambda_\omega F_{\bar{\sigma},E,\theta,K}| \in L^1\).

For any \(\theta\)-invariant sub-holomorphic bundle \(V \subset E\), we have the orthogonal decomposition \(E = V \oplus V^\perp\) with respect to \(K\) and the projection \(\pi_K\) onto \(V\). We denote by \(K_V\), \(H_V\) the metrics on \(V\) induced by \(K\) and \(H\). If \(|\bar{\sigma}_{E,\theta,\pi_K}| \notin L^2\), by Chern-Weil formula

\[
\deg_{\omega}(V, K) = \int_X \left( \sqrt{-1} \text{tr}(\pi_K \circ \Lambda_\omega F_{\bar{\sigma},E,\theta,K}) - |\bar{\sigma}_{E,\theta,\pi_K}|^2 |K\right) d\text{vol}_{\omega}.
\]

(4.29)

and \(|\Lambda_\omega F_{\bar{\sigma},E,\theta,K}| \in L^1\), we know

\[
\deg_{\omega}(V, K) = -\infty.
\]

(4.30)

If \(|\bar{\sigma}_{E,\theta,\pi_K}| \in L^2\), the same reason indicates \(\deg_{\omega}(V, K)\) is finite. It also holds

\[
h_V = \pi_K \circ \bar{\sigma}_{E,\theta,\pi_K},
\]

(4.31)

\[
\bar{\sigma}_{V,\theta h_V} = \pi_K \circ \bar{\sigma}_{E,\theta h} \circ \pi_K + \bar{\sigma}_{E,\theta \pi_K} \circ (\text{id}_E - \pi_K) \circ h \circ \pi_K,
\]

(4.32)

where \(h_V = K_V^{-1} H_V\) and \(\bar{\sigma}_{V,\theta h_V}\) is the induced Higgs structure on \(V\). Therefore

\[
||\text{tr}(h^{-1}_V \partial_{\theta h_V} h V) ||_{L^2} \leq ||h_V^{-1} ||_{L^\infty} ||\bar{\sigma}_{V,\theta h_V} h V ||_{L^2} < \infty.
\]

(4.33)
So Simpson’s Lemma 5.2 in [37] applies,

$$\lim_{j \to \infty} \int_{X_j} \sqrt{-1} \partial \bar{\partial} (h_v^{-1} \partial_{\theta_V} K_H V) = 0, \quad (4.34)$$

where \( \{X_j\} \) is a sequence of exhaustion subsets. And

$$\lim_{j \to \infty} \int_{X_j} \sqrt{-1} \partial \bar{\partial} (h_v^{-1} \partial_{\theta_V} K_H V) = \lim_{j \to \infty} \int_{X_j} \sqrt{-1} \partial \bar{\partial} (h_v^{-1} \partial_{\theta_V} K_H V)$$

$$+ \lim_{j \to \infty} \int_{X_j} \sqrt{-1} \partial \bar{\partial} (h_v^{-1} \partial_{\theta_V} (h_v^{-1} \partial_{\theta_V} K_H V)) = \deg_{\omega}(V, K), \quad (4.35)$$

then we have

$$\lim_{j \to \infty} \int_{X_j} \sqrt{-1} \partial \bar{\partial} (h_v^{-1} \partial_{\theta_V} K_H V) = \int_X \sqrt{-1} \partial \bar{\partial} (h_v^{-1} \partial_{\theta_V} K_H V) = \deg_{\omega}(V, H). \quad (4.36)$$

Since \((\bar{\partial}_E, \theta)\) is \(H\)-analytically stable, we conclude by (4.30), (4.35) and (4.36) that \((\bar{\partial}_E, \theta)\) is \(K\)-analytically stable.

Next assume \(H\) is another Hermitian-Einstein metric such that \(\det \hat{h} = 1\) and \(|\hat{h}| \in L^\infty\) for \(\hat{h} = K^{-1} H\). Set \(\hat{h} = H^{-1} \bar{H}\), we have \(\sqrt{-1} \Lambda_\omega \bar{\partial} \partial \hat{h} = |\hat{h}^{-\frac{1}{2}} \partial_{\theta_H} h_H|^2\), and \(\partial_{\theta_H} h_H = 0\). But \(D_{\bar{\partial}_E, \theta, H}\) is irreducible, we see \(H = \bar{H}\) and therefore \(|(\bar{\partial}_E, \theta)\) \(\in \mathcal{M}_{Higgs, K}\).

Conversely, given \([(\bar{\partial}_E, \theta)] \in \mathcal{M}_{Higgs, K, \phi}\), by the result in [45] we know that there exists a Hermitian-Einstein metric \(H\) with \(\det h = 1\), \(|h| \in L^\infty\) and \(|\bar{\partial}_E, \theta, h| \in L^2\), where \(h = K^{-1} H\). Moreover, the Hitchin-Simpson connection \(D_{\bar{\partial}_E, \theta, H}\) is flat and irreducible. Due to Lemma 4.1, the map \([(\bar{\partial}_E, \theta)] \mapsto [D_{\bar{\partial}_E, \theta, H}]\) is well-defined. Next as \(H\) is harmonic with respect to \(D \triangleq D_{\bar{\partial}_E, \theta, H}\), a similar discussion as that in (4.28) shows

\[
D_{\psi}^+_K \psi_K = -\sqrt{-1} \Lambda_\omega \bar{\partial} \partial h - \frac{1}{2} \Lambda_\omega \left( \partial_{\theta_H} \partial h \wedge \bar{\partial}_E, \theta, h^{-1} \right) + \frac{1}{2} \Lambda_\omega (\partial_{\theta_H} (\bar{\partial}_E, \theta, h^{-1})) + \frac{1}{2} \Lambda_\omega \bar{\partial}_E, \theta, (\bar{\partial}_E, \theta, h^{-1}) \quad (4.37)
\]

and therefore \(|(D_{\psi}^+_K \psi_K)| = |D_{\psi}^+_K \psi_K| \in L^1\). Then proceeding the argument in the proof of the Theorem 1.3, we conclude \(D_{\bar{\partial}_E, \theta, H}\) is analytically stable with respect to the background metric \(K\) and hence \([D_{\bar{\partial}_E, \theta, H}] \in \mathcal{M}_{Flat, K}\).

\[\square\]

### 4.3 A vanishing theorem of characteristic classes

Associated with a vector bundle \((E, D)\), up to some normalization coefficients, the Kamber-Tondeur classes are defined by

$$\alpha_{2k+1}(E, D) = [\mathrm{tr} \psi^k_H] \in H^{2k+1}_{DR}(M, \mathbb{R}), \quad (4.38)$$

and the class is independent of the choice of fiber metrics, see [4, 11, 15, 21, 33] for the related study of these characteristic classes.
By Proposition 4.4, we know $\psi_{H}^{1,0} \wedge \psi_{H}^{1,0} = 0$ and following [22], for any $k \geq 1$ we deduce

$$\text{tr } \psi_{H}^{2k+1} = \text{tr} \left( (\psi_{H}^{1,0} \wedge \psi_{H}^{0,1})^{2k} \wedge \psi_{H}^{1,0} \right) + \text{tr} \left( (\psi_{H}^{0,1} \wedge \psi_{H}^{1,0})^{2k} \wedge \psi_{H}^{0,1} \right)$$

$$= \text{tr} \left( \psi_{H}^{1,0} \wedge (\psi_{H}^{0,1} \wedge \psi_{H}^{1,0})^{2k} \right) + \text{tr} \left( \psi_{H}^{0,1} \wedge (\psi_{H}^{1,0} \wedge \psi_{H}^{0,1})^{2k} \right) \quad (4.39)$$

$$= 0.$$

So we conclude the following vanishing theorem.

**Theorem 4.1** Let $(X, \omega)$ be a complete Kähler manifold with bounded Ricci curvature from below and satisfying the Assumption 2.1. Assume $(E, D)$ is a $K$-analytically stable flat bundle over $X$ such that $|(D^{*}_{K} \psi_{K})^{\perp}| \leq C \phi$ for a constant $C$ and $|\psi_{K}| \in L^{2}$. Then the characteristic classes $\alpha_{2k+1}(E, D)$ vanish for all $k \geq 1$.

We mention that in the compact setting, this result was first proved in [34].

**Data Availability** No additional data are available.

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