Abstract

We employ Mathematica to find \(Z_N\)-invariant subgroups of \(E_8\) for application in \(M\)-theory. These \(Z_N\)-invariant subgroups are phenomenologically important and in some cases they resemble the gauge groups of our real world. We present a specific example of \(Z_7\)-invariant subgroups of \(E_8\), which turn up in orbifold compactification of \(M\)-theory. However, the procedure can be applied for any \(Z_N\) group that acts by shifts in the root lattice of semi-simple Lie groups.

PACS: 02.20.Rt, 11.25.Mj

Keywords: orbifold compactification, string-theory, \(M\)-theory

1 Introduction

In models where part of spacetime is compactified, the geometry of compact space affects the gauge symmetries of the model. Herein, we consider the Horava-Witten \(M\)-theory \([1]\), where the 11th dimension is compactified to an interval, \(I\), and there are two ten-dimensional planes at the boundaries of \(I\). It is convenient to identify \(I = S^1/Z_2\), acting as \(Z_2 : \phi \rightarrow -\phi\), so that the boundary of \(I\) consists of the fixed points of this \(Z_2\)-action. On each one of these ten-dimensional spacetime planes, \(\mathbb{R}^{1,9}_{L,R}\), there is an independent copy (a principal vector bundle) of \(E_8\) gauge fields. To produce considerably more realistic models with 4-dimensional spacetime, one may proceed as follows:

1. Impose twisted periodicity conditions on six of the ten dimensions of the boundary spacetime planes, passing from \(\mathbb{R}^6 \subset (\mathbb{R}^{1,9})_{L,R}\) to \((T^6/\Delta) = ((\mathbb{R}^6/\Lambda)/\Delta)\), where \(\Lambda\) is a suitable 6-dimensional lattice and \(\Delta\) is a symmetry of \(\Lambda\). We consider \(\Delta = Z_N\).

2. Simultaneously embed the \(\Delta\) action into the \(E_8\) structure group of the gauge fields on each of the two boundary-spacetimes, the structure groups are broken to subgroups of \(E_8\) that are invariant with respect to the \(\Delta\)-action.

This is referred to as “compactifying the Horava-Witten \(M\)-theory on a \(T^6/\Delta\) orbifold”, and \(\Delta\) is the “orbifold group.” Typically, \(\Delta\) acts by rotations on the compact space coordinates, and at the same time by shifts in the \(E_8\) root lattice.

In Ref. \([2]\), we have constructed \(Z_7\)-orbifold models in \(M\)-theory. We used Mathematica to find the \(Z_7\)-invariant subgroups of \(E_8\). In this paper we present the details of the Mathematica computation codes and the procedure that we have used. This procedure may be used for higher order (iterated) orbifolds as well, and also in any situation where one needs to find the \(Z_N\)-invariant subgroups of any of the semi-simple Lie group, where \(Z_N\) acts by shifts in the root lattice. Since each Lie group factorizes as a product of a semi-simple part and its abelian factors, the latter are then treated separately \([2]\).

*Email: mahsan@me.com
†Email: thubsch@howard.edu
2 The Algorithm

Let \( v \) denote a \( \mathbb{Z}_N \)-generating shift vector acting on the root lattice \( \mathcal{P} \) of a group \( G \), where \( G \) is one of the simple lie groups \( A_n, B_n, C_n, D_n, E_6, E_7 \) or \( E_8 \). The root vectors of \( G \) that are invariant with respect to the \( v \)-projection

\[
\rho^{2\pi iv} |p\rangle = |p\rangle, \quad p \in \mathcal{P},
\]

are the root vectors of a subgroup \( H \subset G \) that is invariant with respect to the whole group \( \mathbb{Z}_N \) generated by the element corresponding to \( v \). Different \( \mathbb{Z}_N \)-generating shift vectors define different subgroups; upon identifying those that are equivalent by \( G \)-conjugation, we find the inequivalent \( \mathbb{Z}_N \)-invariant subgroups, \( H_I \), with \( I = 1, 2, \ldots \).

**Note:** The condition (2.1) is trivially satisfied for the root vectors \( p_G = \{0,0,\ldots,0\} \) corresponding to Cartan generators of \( G \). Therefore, \( \text{rank}(H_I) = \text{rank}(G) \), and all \( \mathbb{Z}_N \)-invariant groups are regular.

**Step 1:** Find the set of positive root vectors of \( G \), denoted \( \mathcal{W} \).

**Step 2:** Based on the restrictions implied, we construct all possible \( \mathbb{Z}_N \) shift vectors \( v \).

**Step 3:** Find all the subgroups \( H_I \subset G \).

**Step 4:** For each one of the subgroups \( H_I \subset G \), define the following four variables:

- \( t \) is the set of positive root vectors of \( H_I \subset G \);
- \( p := |t| \) is total number of positive root vectors in \( H_I \subset G \);
- \( r := \text{rank}'(H_I) \), defined as the rank of semi-simple part of \( H_I \subset G \), i.e., without \( U(1) \)-factors;
- \( m \) is number of \( A_1 \) factors, if any, in \( H_I \subset G \).

These three variables can be read off by looking at the subgroup \( H_I \) and can be used as identifiers of the group. If these three variables do not suffice to identify \( H_I \subset G \) unambiguously, define another variable:

\( m_2 \) is the number of \( A_2 \) factors in \( H_I \), if any.

If \( \{p,r,m,m_2\} \) turns out not to suffice to identify \( H_I \subset G \) unambiguously, we look for \( A_3, A_4 \ldots \) factors in \( H_I \), the numbers of which, \( m_3, m_4 \ldots \), will be necessary to identify \( H_I \subset G \) unambiguously.

**Step 5:** Pick the first \( v \) from Step 2.

- **Step 5.a:** Set \( t = \emptyset \). For all \( w_a \in \mathcal{W} \), if \( v \cdot w_a = \mathbb{Z} \), append \( w_a \) into the set \( t \).
- **Step 5.b:** Compute \( \{p,r,m,\ldots\} \) of this \( t \) (see Section 3 for the procedure).
- **Step 5.c:** Identify the subgroup \( H_I \subset G \) by comparing \( \{p,r,m,\ldots\} \) with the list from Step 4.

**Step 6:** Pick the next \( v \) from Step 2, and go to Step 5.a.

Steps 1–4 are preparatory. In particular, Step 4 sets up the string of identifiers \( \{p,r,m,m_2,\ldots\} \) as unambiguous addresses of all the subgroups \( H_I \) of a given simple Lie group \( G \). In our case, with \( G = E_8 \) and \( \mathbb{Z}_N \) acting by shifts in the root lattice, \( \{p,r,m\} \)—in many cases only a subset of this—sufficed.

\(^1\)For all simple Lie groups of rank \( \leq 8 \) and several of higher rank, the maximal subgroups are listed in the literature, such as Ref. 3. Alternatively, these may be obtained via the technique using extended Dynkin diagrams. The non-maximal subgroups are then found by iteratively finding the maximal subgroups of the maximal subgroups, and then dropping some of the factors.

\(^2\)Since the root lattice shift \( v \) corresponds to a generator \( g(v) \in \mathbb{Z}_N \) so that all elements of \( \mathbb{Z}_N \) are powers of \( g(v) \), it follows that root vectors satisfying \( v \cdot w_a = \mathbb{Z} \) are in fact invariant with respect to all of \( \mathbb{Z}_N \).
3 Roots and Shift Vectors

We take the adjoint representation of the group $G$ and calculate its positive root vectors from the highest root $[3]$. Take for example the group $G = E_8$. The highest root of the irreducible representation $248$ of $E_8$ is $\{0,0,0,0,0,1,0\}$, which can be read off from the Dynkin diagram of $E_8$. The entire root system can be obtained from the highest root by subtracting from it the positive simple root vectors as follows: in any given root vector $w$, the value of the positive $n^{th}$ component indicates how many times the $n^{th}$ positive root may be subtracted $[4]$. For example, $w_1 = \{2,1,0,0,0,0,0,0\}$ is the first positive root and may be subtracted from itself twice, producing (to save space, negative root vector components are denoted by an over-bar: $\overline{1} \overset{\text{def}}{=} -1, 2 \overset{\text{def}}{=} -2$ etc.):

$$\{2,1,0,0,0,0,0,0\} \overset{-w_1}{\longrightarrow} \{0,0,0,0,0,0,0,0\} \overset{-w_1}{\longrightarrow} \{2,1,0,0,0,0,0,0\}. \quad (3.1)$$

All three are indeed in the root system of $E_8$. Starting with $\{0,0,0,0,0,0,1\}$, in this way we end with $\{0,0,0,0,0,0,\overline{1}\}$, having found all 248 roots, spanning $248$, the adjoint representation of $E_8$.

The procedure actually applies generally to all representations of all simple Lie groups; however, general representations are spanned by weight vectors, while the adjoint representation is spanned by roots. By plotting the weights (roots) below those from which they are obtained by subtracting positive roots and connecting them by arrows, we obtain a “spindle shaped” graph called the weight (root) diagram of the representation. In the root diagram, the middle row of the root diagram is populated by eight copies of $\{0,0,0,0,0,0,0\}$, representing the eight Cartan generators. The row immediately above the middle is populated by the 8 positive simple root vectors, and the 120 = $(248-8)/2$ roots above the middle row are the positive root vectors of $E_8$.

The following Mathematica code computes the positive root vectors of $E_8$.

**Input (1)**

```mathematica
a = {{2,1,0,0,0,0,0,0}}, {1,2,1,0,0,0,0,0}, {0,1,2,1,0,0,0,0}, {0,0,1,2,1,0,0,0}, {0,0,0,1,2,1,0,0}, {0,0,0,0,1,2,1,0}, {0,0,0,0,0,1,2,1}, {0,0,0,0,0,0,1,2};
g[0] = {{0,0,0,0,0,0,0,1}};
g[1] = Table[Flatten[g[0]] - a[[Flatten[Position[Flatten[g[0]], 1]]]][], {p, Length[Flatten[Position[Flatten[g[0]], 1]]]]}];
e[x_] := e[x] = Union[Flatten[Table[If[g[x][[j]][[i]] == 1, g[x][[j]] - a[[i]]], {i, 8}, {j, Length[g[x]]}, 1]]];
g[x_] := g[x] = Delete[e[x - 1], 1];
Flatten[Table[g[m], {m, 0, 28}, 1]]
```

**Output (1)**

```
{{0,0,0,0,0,0,1,0}, {0,0,0,0,0,1,0,0}, {0,0,0,0,1,0,0,0}, {0,0,0,1,0,0,0,0}, {0,0,1,0,0,0,0,0}, {0,1,0,0,0,0,0,0}, {1,0,0,0,0,0,0,0}, {1,0,0,0,0,1,0,0}, {1,0,0,0,1,0,0,0}, {1,0,0,1,0,0,0,0}, {1,0,1,0,0,0,0,0}, {1,1,0,0,0,0,0,0}, {1,1,0,0,0,1,0,0}, {1,1,0,0,1,0,0,0}, {1,1,0,1,0,0,0,0}, {1,1,1,0,0,0,0,0}, {1,1,1,0,0,1,0,0}, {1,1,1,0,1,0,0,0}, {1,1,1,1,0,0,0,0}, {1,1,1,1,0,0,1,0}, {1,1,1,1,0,1,0,0}, {1,1,1,1,1,0,0,0}, {1,1,1,1,1,0,0,1}, {1,1,1,1,1,0,1,0}, {1,1,1,1,1,1,0,0}, {1,1,1,1,1,1,0,0,1}, {1,1,1,1,1,1,0,1,0}, {1,1,1,1,1,1,1,0,0}, {1,1,1,1,1,1,1,0,0,1},
{0,0,0,0,0,0,1,0}, {0,0,0,0,0,1,0,0}, {0,0,0,0,1,0,0,0}, {0,0,0,1,0,0,0,0}, {0,0,1,0,0,0,0,0}, {0,1,0,0,0,0,0,0}, {1,0,0,0,0,0,0,0}, {1,0,0,0,0,0,1,0}, {1,0,0,0,0,1,0,0}, {1,0,0,0,1,0,0,0}, {1,0,0,1,0,0,0,0}, {1,0,1,0,0,0,0,0}, {1,0,1,0,0,0,1,0}, {1,0,1,0,0,1,0,0}, {1,0,1,0,1,0,0,0}, {1,0,1,0,1,0,0,1}, {1,0,1,1,0,0,0,0}, {1,0,1,1,0,0,0,1}, {1,0,1,1,0,0,1,0}, {1,0,1,1,0,1,0,0}, {1,0,1,1,1,0,0,0}, {1,0,1,1,1,0,0,1}, {1,0,1,1,1,0,1,0}, {1,0,1,1,1,1,0,0}, {1,0,1,1,1,1,0,0,1}, {1,0,1,1,1,1,0,1,0}, {1,0,1,1,1,1,1,0,0}, {1,0,1,1,1,1,1,0,0,1}, {1,0,1,1,1,1,1,0,1,0}, {1,0,1,1,1,1,1,1,0,0}, {1,0,1,1,1,1,1,1,0,0,1}, {1,0,1,1,1,1,1,1,0,1,0}, {1,0,1,1,1,1,1,1,1,0,0}, {1,0,1,1,1,1,1,1,1,0,0,1}}
```
To compute all the root vectors of $E_8$ (not just the positive ones), add the line

Do[Print[g[m]], {m, 0, 28}]

at the end of Input (1).

For $E_7$ and $E_6$ the input codes are similar. For $E_7$, the highest root of the adjoint representation, $133$, is $\{1,0,0,0,0,0,0\}$. Its $(133-7)/2 = 63$ positive root vectors of $E_7$ are found from the following code:

**Input (2)**

\[
a = \{(2,1,0,0,0,0,0), (1,2,1,0,0,0,0), (0,1,2,1,0,0,0), (1,0,1,2,1,0,0), (0,0,1,2,1,0,0), (0,0,0,1,2,1,0), (0,0,0,0,1,2,0)\};
\]

\[
g[0] = \{(1,0,0,0,0,0,0)\};
\]

\[
g[1] = Table[Flatten[g[0]] - a[[Flatten[Position[Flatten[g[0]], 1]]][[p]]],
\]

\[
\{p, Length[Flatten[Position[Flatten[g[0]], 1]]]}\};
\]

\[
e[x_] := e[x] = Union[Flatten[Table[If[g[x][[j]][[i]] == 1, g[x][[j]] - a[[i]],
\]

\[
\{i, 7\}, {j, Length[g[x]]}]], 1]];\]

\[
g[x_] := g[x] = Delete[e[x - 1], 1];
\]

\[
Flatten[Table[g[m], {m, 0, 16}], 1]
\]

For $E_6$, the highest root (weight of the adjoint representation), $78$ is $\{0,0,0,0,0,0,1\}$. Its $(78-6)/2 = 36$ positive root vectors are found as follows:

**Input (3)**

\[
a = \{(2,1,0,0,0,0,0), (1,2,1,0,0,0,0), (0,1,2,1,0,0,1), (0,0,1,2,1,0,0), (0,0,0,1,2,0,0)\};
\]

\[
g[0] = \{(0,0,0,0,0,1)\};
\]

\[
g[1] = Table[Flatten[g[0]] - a[[Flatten[Position[Flatten[g[0]], 1]]][[p]]]],
\]
For $A_n$, the dimension of the adjoint representation is $n(n+2)$ and the number of positive root vectors is $(n(n+2) - n)/2 = n(n-1)/2$. The Mathematica code computing the positive root vectors of $A_n$, for $n = 5$ for example, is:

**Input (4)**

```mathematica
n = 5;
d = {{2, -1, 0}, {0, -1, 2}, {-1, 2, -1}};
a = If[n > 1, Flatten[{{Flatten[Position[Flatten[g[0]], 1]], 1}}]), Table[Flatten[g[x]], Flatten[g[x]], Flatten[g[x]]] = Delete[e[x - 1], 1];
Flatten[Table[g[m], {m, 0, 10}], 1]
```

Similarly to $B_n$, the dimension of the adjoint representation is $n(2n+1)$ and the number of positive root vectors is $(n(2n+1) - n)/2 = n^2$. The Mathematica code computing the positive root vectors of $B_n$, for $n = 5$ for example, is:

**Input (5)**

```mathematica
n = 5;
d = {{2, -1, 0}, {0, -1, 2}, {-1, 2, -1}};
a = Transpose[Flatten[{{Flatten[Position[Flatten[g[0]], 1]], 1}}]), Table[Flatten[g[x]], Flatten[g[x]], Flatten[g[x]]] = Delete[e[x - 1], 1];
Flatten[Table[g[m], {m, 0, 2 n - 2}], 1]
```

Similarly to $B_n$, the dimension of the adjoint representation of $C_n$ is also $n(2n+1)$ and the number of positive root vectors is also $(n(2n+1) - n)/2 = n^2$. The Mathematica code computing the positive root vectors of $C_n$, for $n = 5$ for example, is:

**Input (6)**

```mathematica
n = 5;
d = {{2, -1, 0}, {0, -1, 2}, {-1, 2, -1}};
a = Transpose[Flatten[{{Flatten[Position[Flatten[g[0]], 1]], 1}}]), Table[Flatten[g[x]], Flatten[g[x]], Flatten[g[x]]] = Delete[e[x - 1], 1];
Flatten[Table[g[m], {m, 0, 2 n - 2}], 1]
```
\[
\text{g}[0] = \{\text{PadRight}[[2], n, 0]\};
\text{g}[1] = \{\text{Flatten}[\text{g}[0]] - a[[\text{Flatten}[\text{Position}[\text{Flatten}[\text{g}[0]], 2]]][[1]]]\};
\]

\[
\text{e}[x_] := \text{e}[x] = \text{Union}[\text{Flatten}\left[\text{Table}\left[\text{If}[\text{g}[x][[j]][[i]] == 1, \text{g}[x][[j]] - a[[i]], \\
\text{If}[\text{g}[x][[j]][[i]] == 2, \text{g}[x][[j]] - a[[i]]], \{i, n\}, \{j, \text{Length}[\text{g}[x]]\}\right], \{1, \text{Length}[\text{g}[x - 1]]\}\right], 2]\};
\]

\[
\text{g}[x_] := \text{g}[x] = \text{Delete}[\text{Union}[\text{Flatten}\left[\text{Table}\left[\text{If}[\text{g}[x - 1][[l]][[k]] == 2, \text{g}[x - 1][[l]] - 2 a[[k]], \{k, n\}, \{l, \text{Length}[\text{g}[x - 1]]\}\right], 2\right], 2]\};
\]

\[
\text{For } D_n \text{, the dimension of the adjoint representation is } n(2n - 1) \text{ and the number of positive root vectors is } (n(2n - 1) - n)/2 = n(n - 1). \text{ The Mathematica code computing the positive root vectors of } B_n, \text{ for } n = 5 \text{ for example, is:}
\]

\[
\text{Input (7)}
\]

\[
\text{n} = 5;
\text{d} = \{\{2, -1, 0, 0\}, \{-1, 2, -1, -1\}, \{0, -1, 2, 0\}, \{0, -1, 0, 2\}, \{-1, 2, -1, 0\}\};
\text{a} = \text{Flatten}\left[\{\text{PadLeft}[\text{d}[[1]], n, 0, n - 4]\}, \{\text{Table}[\text{PadLeft}[\text{d}[[5]], n, 0, n - i - 3], \{i, n - 4\}\} , \{\text{PadRight}[\text{d}[[2]], n, 0, n - 4]\}, \{\text{PadRight}[\text{d}[[3]], n, 0, n - 4]\}\right], 2];
\text{g}[0] = \{\text{PadRight}[[0, 1, 0], n, 0]\};
\text{g}[1] = \text{Table}[\text{Flatten}[\text{g}[0]] - a[[\text{Flatten}[\text{Position}[\text{Flatten}[\text{g}[0]], 1]]][[p]]], \{p, \text{Length}[\text{Flatten}[\text{Position}[\text{Flatten}[\text{g}[0]], 1]]]\}];
\text{g}[x_] := \text{g}[x] = \text{Delete}[\text{Union}[\text{Flatten}\left[\text{Table}\left[\text{If}[\text{g}[x - 1][[l]][[i]] == 1, \text{g}[x - 1][[l]] - a[[i]], \{i, n\}, \{j, \text{Length}[\text{g}[x - 1]]\}\right], 1]\right], 1]\};
\text{Flatten}\left[\text{Table}[\text{g}[m], \{m, 0, 2 n - 2\}\right], 1]\}
\]

The highest root, the level of simple root vectors (i.e., the height of the tower of positive roots) and the dimension of the adjoint representation can be found in Table 8 of [3]. Table 9 of Ref. [3] gives the positive root systems of a few low-rank simple Lie groups. We leave it to the diligent Reader to adapt the above Mathematica codes for the remaining simple Lie groups, $G_2$ and $F_4$.

\[\star\]

In constructing $T^6/Z_N = (\mathbb{R}^6/\Lambda)/Z_N$ orbifolds for superstring theory and its $M$-theory extension, the choices of the $Z_N$ shift vectors (representing the embedding in the gauge group) are restricted. For example, in $M$-theory, the shift vectors must satisfy a supersymmetry condition, and in string theory they satisfy an additional modular invariance condition.

We give an example of $Z_7$ vectors. There are 428 eight-component vectors that may be constructed with the components taking values in the set \{0, 1/7, 2/7, 3/7, 4/7, 5/7, 6/7\}. The supersymmetry restriction requires that the components of a $Z_N$ vector add up to an integer. The following is a code that produces all the “supersymmetric” $Z_7$ vectors. We have shown only a sample of the output. Note that in order to find all the possible vectors preserving supersymmetry, we need to take permutations of each one of the vectors shown in this output.

\[
\text{Input (8)}
\]

\[
a = \{1/7, 2/7, 3/7, 4/7, 5/7, 6/7\};
b = \text{Flatten}\left[\text{Table}[\text{Tuples}[a, i], \{i, 2, 8\}, 1]\right];
c = \text{Union}\left[\text{Table}[\text{Sort}[b[[i]]], \{i, \text{Length}[b]\}]\right];
\]
Dynkin diagram in all possible ways, and read off the subgroup represented by the remainder. For example (see Fig 1), if we take out the nodes $\text{SO}$

\[ \begin{array}{c}
\text{Mathematica} \\
\text{Output (8)}
\end{array} \]

One may use a similar code for generating general $\mathbb{Z}_N$ shift vectors in the root lattice.

4 Subgroups of $G$

Our next step is to find all the regular subgroups of $G$. This can be done manually, for example, by using the extended Dynkin diagram technique [4]. The procedure is to remove nodes from the extended Dynkin diagram in all possible ways, and read off the subgroup represented by the remainder. For example

\[ \begin{array}{c}
\text{Input (9)}
\end{array} \]

428

One may use a similar code for generating general $\mathbb{Z}_N$ shift vectors in the root lattice.

4 Subgroups of $G$

Our next step is to find all the regular subgroups of $G$. This can be done manually, for example, by using the extended Dynkin diagram technique [4]. The procedure is to remove nodes from the extended Dynkin diagram in all possible ways, and read off the subgroup represented by the remainder. For example

\[ \begin{array}{c}
\text{Input (9)}
\end{array} \]

428

One may use a similar code for generating general $\mathbb{Z}_N$ shift vectors in the root lattice.

4 Subgroups of $G$

Our next step is to find all the regular subgroups of $G$. This can be done manually, for example, by using the extended Dynkin diagram technique [4]. The procedure is to remove nodes from the extended Dynkin diagram in all possible ways, and read off the subgroup represented by the remainder. For example

\[ \begin{array}{c}
\text{Input (9)}
\end{array} \]

428

One may use a similar code for generating general $\mathbb{Z}_N$ shift vectors in the root lattice.
In the above code, we look for possible combinations of $A_n$ ($1 \leq n \leq 8$), $D_n$ ($4 \leq n \leq 8$) and $E_n$ ($6 \leq n \leq 8$). This code may not work for computers low in processor power and memory. In such a case, one might want to break down the evaluation in $k$ pieces ($k$ is equal to the rank of the group $G$) by changing the variable $i$ in the second line of the input as \{i,k,k\}, instead of \{i,8\} and collecting all the results at the end.

Using the Dynkin Diagram technique or Mathematica, one can find all the regular subgroups of $G$ in general.

Table 1: All the subgroups of $E_8$ and the values of the identifiers.

| Subgroups  | p   | m | r | c | Subgroups  | p   | m | r | c |
|------------|-----|---|---|---|------------|-----|---|---|---|
| $E_8$      | 120 | 0 | 8 |   | $SO_8 \times SU_2^2$ | 14  | 2 | 6 | ✓ |
| $E_7 \times SU_2$ | 64  | 1 | 8 |   | $SU_5 \times SU_3 \times SU_2$ | 14  | 1 | 7 | ✓ |
| $E_7$      | 63  | 0 | 7 | ✓ | $SU_5 \times SU_2^4$ | 14  | 4 | 8 | ✓ |
| $SO_{16}$  | 56  | 0 | 8 |   | $SU_2^2 \times SU_2^2$ | 14  | 2 | 8 | ✓ |
| $SO_{14} \times SU_2$ | 43  | 1 | 8 |   | $SO_8 \times SU_2$ | 13  | 1 | 5 |   |
| $SO_{14}$  | 42  | 0 | 7 | ✓ | $SU_5 \times SU_3$ | 13  | 0 | 6 |   |
| $E_6 \times SU_3$ | 39  | 0 | 8 |   | $SU_5 \times SU_2^3$ | 13  | 3 | 7 |   |
| $E_6 \times SU_2^2$ | 38  | 2 | 8 |   | $SU_4^2 \times SU_2$ | 13  | 1 | 7 |   |
| $E_6 \times SU_2$ | 37  | 1 | 7 | ✓ | $SU_4 \times SU_2^2 \times SU_2$ | 13  | 1 | 8 |   |
| $E_6$      | 36  | 0 | 6 | ✓ | $SO_8$ | 12  | 0 | 4 |   |
| $SU_9$     | 36  | 0 | 8 | ✓ | $SU_5 \times SU_2^2$ | 12  | 2 | 6 |   |
| $SO_{12} \times SU_3$ | 33  | 0 | 8 |   | $SU_2^4$ | 12  | 0 | 6 |   |
| $SO_{12} \times SU_2^2$ | 32  | 2 | 8 |   | $SU_4 \times SU_2^3$ | 12  | 0 | 7 |   |
| $SO_{12} \times SU_2$ | 31  | 1 | 7 |   | $SU_4 \times SU_3 \times SU_2^2$ | 12  | 3 | 8 |   |
| $SO_{12}$  | 30  | 0 | 6 | ✓ | $SU_2^3$ | 12  | 0 | 8 |   |
| $SU_{8} \times SU_2$ | 29  | 1 | 8 |   | $SU_5 \times SU_2$ | 11  | 1 | 5 |   |
| $SU_8$     | 28  | 0 | 7 | ✓ | $SU_4 \times SU_3 \times SU_2^2$ | 11  | 2 | 7 |   |
| $SO_{10} \times SU_4$ | 26  | 0 | 8 |   | $SU_4 \times SU_2^3$ | 11  | 5 | 8 |   |
| $SO_{10} \times SU_3 \times SU_2$ | 24  | 1 | 8 |   | $SU_3^3 \times SU_2^2$ | 11  | 2 | 8 |   |
| $SU_7 \times SU_3$ | 24  | 0 | 8 |   | $SU_5$ | 10  | 0 | 4 |   |
| $SO_{10} \times SU_3$ | 23  | 0 | 7 | ✓ | $SU_4 \times SU_3 \times SU_2$ | 10  | 1 | 6 |   |
| $SO_{10} \times SU_2^3$ | 23  | 3 | 8 | ✓ | $SU_4 \times SU_2^4$ | 10  | 4 | 7 |   |
| $SU_7 \times SU_2^3$ | 23  | 2 | 8 | ✓ | $SU_3^3 \times SU_2$ | 10  | 1 | 7 |   |
| $SO_{10} \times SU_2^4$ | 22  | 2 | 7 | ✓ | $SU_3^2 \times SU_2^4$ | 10  | 4 | 8 |   |
| $SO_{8} \times SU_2^3$ | 22  | 0 | 8 | ✓ | $SU_4 \times SU_3$ | 9   | 0 | 5 |   |

Continued...
Table 1: All the subgroups of $E_8$ and the values of the identifiers. (Continued)

| Subgroups | p | m | r | c | Subgroups | p | m | r | c |
|-----------|---|---|---|---|-----------|---|---|---|---|
| $SU_7 \times SU_2$ | 22 | 1 | 7 | ✓ | 76 | $SU_4 \times SU_3^3$ | 9 | 3 | 6 |
| $SO_{10} \times SU_2$ | 21 | 1 | 6 | ✓ | 77 | $SU_3^3$ | 9 | 0 | 6 |
| $SU_7$ | 21 | 0 | 6 | ✓ | 78 | $SU_3^2 \times SU_2^3$ | 9 | 3 | 7 |
| $SU_6 \times SU_4$ | 21 | 0 | 8 | ✓ | 79 | $SU_3 \times SU_2^6$ | 9 | 6 | 8 |
| $SO_{10}$ | 20 | 0 | 5 | ✓ | 80 | $SU_4 \times SU_2^3$ | 8 | 2 | 5 |
| $SU_5^2$ | 20 | 0 | 8 | ✓ | 81 | $SU_3^2 \times SU_2^3$ | 8 | 2 | 6 |
| $SO_8 \times SU_4 \times SU_2$ | 19 | 1 | 8 | ✓ | 82 | $SU_3 \times SU_2^5$ | 8 | 5 | 7 |
| $SU_6 \times SU_3 \times SU_2$ | 19 | 1 | 8 | ✓ | 83 | $SU_5^8$ | 8 | 8 | 8 |
| $SO_8 \times SU_4$ | 18 | 0 | 7 | ✓ | 84 | $SU_4 \times SU_2$ | 7 | 1 | 4 |
| $SO_8 \times SU_2^3$ | 18 | 0 | 8 | ✓ | 85 | $SU_2^3 \times SU_2$ | 7 | 1 | 5 |
| $SU_6 \times SU_3$ | 18 | 0 | 7 | ✓ | 86 | $SU_3 \times SU_2^4$ | 7 | 4 | 6 |
| $SU_6 \times SU_2^3$ | 18 | 3 | 8 | ✓ | 87 | $SU_2^3$ | 7 | 7 | 7 |
| $SO_8 \times SU_3 \times SU_2^2$ | 17 | 2 | 8 | ✓ | 88 | $SU_4$ | 6 | 0 | 3 |
| $SU_6 \times SU_2^3$ | 17 | 2 | 7 | ✓ | 89 | $SU_3^8$ | 6 | 0 | 4 |
| $SU_5 \times SU_4 \times SU_2$ | 17 | 1 | 8 | ✓ | 90 | $SU_3 \times SU_2^3$ | 6 | 3 | 5 |
| $SO_8 \times SU_3 \times SU_2$ | 16 | 1 | 7 | ✓ | 91 | $SU_2^3$ | 6 | 5 | 5 |
| $SO_8 \times SU_2^3$ | 16 | 4 | 8 | ✓ | 92 | $SU_3 \times SU_2^2$ | 5 | 2 | 4 |
| $SU_6 \times SU_2$ | 16 | 1 | 6 | ✓ | 93 | $SU_3^5$ | 5 | 5 | 5 |
| $SU_5 \times SU_4$ | 16 | 0 | 7 | ✓ | 94 | $SU_3 \times SU_2$ | 4 | 1 | 3 |
| $SU_5 \times SU_2^2$ | 16 | 0 | 8 | ✓ | 95 | $SU_2^4$ | 4 | 4 | 4 |
| $SO_8 \times SU_3$ | 15 | 0 | 6 | ✓ | 96 | $SU_3$ | 3 | 0 | 2 |
| $SO_8 \times SU_2^3$ | 15 | 3 | 7 | ✓ | 97 | $SU_2^3$ | 3 | 3 | 3 |
| $SU_6$ | 15 | 0 | 5 | ✓ | 98 | $SU_2^2$ | 2 | 2 | 2 |
| $SU_5 \times SU_3 \times SU_2^2$ | 15 | 2 | 8 | ✓ | 99 | $SU_2$ | 1 | 1 | 1 |
| $SU_2^4 \times SU_3$ | 15 | 0 | 8 | ✓ | 100 | — | 0 | 0 | 0 |

The $U(1)$ factors are not shown; regular subgroups have $(8-r)$ of them.

Once we get all the subgroups of $E_8$, we calculated the number of positive root vectors for each subgroup and list them in column $p$. The values of the other identifiers ($r$, $m$ and possibly $m_2, m_3, \ldots$) turned out not to be necessary in most cases for our purposes. Before using them, we found the possible candidates which are $\mathbb{Z}_7$ invariant subgroups of $E_8$ through a procedure given in Input/Output (10). This greatly reduced the complexity of the codes in the next section and saves in the Mathematica evaluation time.

We have 428 $\mathbb{Z}_7$ shift vectors in Output \cite{5} and once we take their permutations, this gives a total of 823,542 shift vectors. We take the first $\mathbb{Z}_7$ vector $\{\frac{1}{7}, \frac{6}{7}, 0, 0, 0, 0, 0, 0\}$ from the previous section and

\footnote{The identifiers $r$ and $m$ are shown in Table \ref{table1} for completeness, and for the benefit of possible generalizations to $\mathbb{Z}_N$-actions where the supersymmetry condition is relaxed. The additional identifiers, $m_i$ in \textbf{Step 4}, are easily added.}
calculate the number of positive root vectors that satisfy the condition $p \cdot v = \mathbb{Z}$ using the following code:

**Input (10)**

```math
q = \text{(not shown here: 428 \(\mathbb{Z}_7\) vectors from Output (8))}
```

```math
w = \text{(not shown here: 120 positive root vectors of \(E_8\) from Output (1))}
```

```math
p = Flatten[Table[Permutations[q[[i]]], {i, 1, 1}], 1];
a = Length[w];
l = Length[p];
```

```math
u = Table[Table[w[[i]].p[[j]], {i, a}], {j, l}];
```

```math
v = Table[Table[IntegerQ[u[[j, i]]], {i, a}], {j, l}];
r = Table[Count[v[[j]], True], {j, l}];
```

```math
\text{Union}[r] \quad \text{\(\rightarrow\) \(\mathbb{Z}_7\)Roots;}
```

**Output (10)**

\{37, 42\}

The Output of this evaluation (\{37, 42\}) is written in an external file \texttt{\(\mathbb{Z}_7\)Roots}. We do this evaluation for the other \(\mathbb{Z}_7\) vectors in Output (8) and the results are collected from the text file \texttt{\(\mathbb{Z}_7\)Roots}. This gives the possible values of \(p\) for \(\mathbb{Z}_7\) vectors as

\[p : \{14, 15, 16, 21, 22, 23, 28, 30, 36, 37, 42, 63\}\]  \hspace{1cm} (4.1)

This narrows down our choices to 30 subgroups of \(E_8\), marked by a check in column \(c\) of Table 1. Now we use the values of \(m\) (number of \(A_1\) factors in \(H_I \subset E_8\)) to identify the possible subgroups \(H_I \subset E_8\). When \(p\) and \(m\) do not specify \(H_I \subset E_8\) unambiguously, we use the values of \(r\) (rank of the semi-simple part of \(H_I\)). The values of \(p\), \(m\) and \(r\) are also calculated from the root vectors that survive a \(\mathbb{Z}_7\) shift. This is shown in the next section.

5 \(Z_N\) Invariant Subgroups of \(G\)

To illustrate the procedure of calculating the values of \(m\) and \(r\) from the surviving root vectors we give the same example as in Section 3 of our previous paper [2]. Take the shift vector \(v = \{1, 1, 0, 0, 0, 0, 0, 0\}\), which is one of the permutations of \(\{1, 1, 1, 1, 1, 1, 1, 0\}\). The \(E_8\) root vectors that survive the shift are

\[
\begin{align*}
&\{0,0,0,0,1,\bar{1},0,0\}, \{1,\bar{1},0,0,0,0,0,1\}, \{1,\bar{1},1,0,0,0,0,\bar{1}\}, \\
&\{\bar{1},1,0,0,\bar{1},1,0,0\}, \{0,0,\bar{1},0,1,0,0,1\}, \{0,0,0,0,1,0,0,\bar{1}\}, \\
&\{1,\bar{1},0,0,0,1,0,0\}, \{0,0,0,0,0,\bar{1},0,1\}, \{0,0,1,0,0,\bar{1},0,\bar{1}\}, \\
&\{0,1,\bar{1},1,1,1,1,0\}, \{\bar{1},0,0,\bar{1},0,0,1,0\}, \{1,\bar{1},1,0,\bar{1},0,0,0\}, \\
&\{\bar{1},1,\bar{1},0,1,\bar{1},0,1\}, \{\bar{1},1,0,0,1,\bar{1},0,\bar{1}\}, \{0,0,\bar{1},0,0,0,0,2\},
\end{align*}
\]

\hspace{1cm} (5.1)

and are thus invariant under the action of the whole \(\mathbb{Z}_7\) generated by this shift. Call these root vectors \(t[i], i = 1, 2, \ldots 15\), and set \(p = 15\).

Next, we need to identify which subgroup \(H_I \subset E_8\)—from among those listed in Table 1—do these roots (together with their negatives and the Cartan root vectors) generate. We look for possible relations
in the form $t[i] + t[j] = t[k]$ and find the following:

$$
\begin{align*}
\mathcal{t}[2] + \mathcal{t}[14] &= \mathcal{t}[1], & \mathcal{t}[3] + \mathcal{t}[13] &= \mathcal{t}[1], & \mathcal{t}[3] + \mathcal{t}[15] &= \mathcal{t}[2], \\
\mathcal{t}[5] + \mathcal{t}[9] &= \mathcal{t}[1], & \mathcal{t}[5] + \mathcal{t}[12] &= \mathcal{t}[2], & \mathcal{t}[6] + \mathcal{t}[8] &= \mathcal{t}[1], \\
\mathcal{t}[6] + \mathcal{t}[12] &= \mathcal{t}[3], & \mathcal{t}[6] + \mathcal{t}[15] &= \mathcal{t}[5], & \mathcal{t}[7] + \mathcal{t}[8] &= \mathcal{t}[2], \\
\mathcal{t}[7] + \mathcal{t}[9] &= \mathcal{t}[3], & \mathcal{t}[7] + \mathcal{t}[13] &= \mathcal{t}[5], & \mathcal{t}[7] + \mathcal{t}[14] &= \mathcal{t}[6], \\
\mathcal{t}[9] + \mathcal{t}[15] &= \mathcal{t}[8], & \mathcal{t}[10] + \mathcal{t}[11] &= \mathcal{t}[4], & \mathcal{t}[12] + \mathcal{t}[13] &= \mathcal{t}[8], \\
\mathcal{t}[12] + \mathcal{t}[14] &= \mathcal{t}[9], & \mathcal{t}[14] + \mathcal{t}[15] &= \mathcal{t}[13].
\end{align*}
$$

Note that the root vectors $t[7], t[10], t[11], t[12], t[14]$ and $t[15]$ cannot be expressed as a sum of any other root vectors, which means that they must correspond to 6 positive, simple root vectors in $H_I$, whence the rank of the semi-simple part of $H_I$ must be 6, and the remaining two Cartan roots correspond to a $U(1)^2$ factor. Also, all the 15 root vectors appear in (5.2), meaning there are no $A_1$ factors in $H_I$, each of which would have had to have a single, isolated, positive root vector. So looking at the surviving root vectors we can define the variables $m$ and $r$ as:

- $m$ is the number of root vectors that do not appear in the equation of the form $t[i] + t[j] = t[k]$ and so must be single, isolated, positive root vectors; here, $m = 0$.
- $r$ is the number of root vectors that do not appear on the right side of the equation of the form $t[i] + t[j] = t[k]$ and so must be simple; here, $r = 6$.

Using $\{p, m, r\} = \{15, 0, 6\}$, we identify unambiguously the subgroup from Table 1 as $SO_8 \times SU_3$.

We employ this analysis in the construction of the Mathematica codes below and find the subgroups of $E_8$ that are invariant under a $\mathbb{Z}_7$ shift listed in Table 2, and so the whole $\mathbb{Z}_7$ group action generated by that shift. This reduces our choice to only 14 subgroups.

**Table 2: $\mathbb{Z}_7$-invariant subgroups of $E_8$.**

| Group | Group | Group | Group |
|-------|-------|-------|-------|
| 1     | $E_7$ | 5     | $SO_{12}$ | 9 | $SU_8$ | 13 | $SU_5 \times SU_4$ |
| 2     | $E_6 \times SU_2$ | 6 | $SO_{10} \times SU_3$ | 10 | $SU_7 \times SU_2$ | 14 | $SU_5 \times SU_3 \times SU_2$ |
| 3     | $E_6$ | 7 | $SO_{10} \times SU_2$ | 11 | $SU_7$ | |
| 4     | $SO_{14}$ | 8 | $SO_5 \times SU_3$ | 12 | $SU_6 \times SU_2$ | |

**Input (11)**

```mathematica
q = (not shown here: 428 $\mathbb{Z}_7$ vectors from Output (5) )
w = (not shown here: 120 positive root vectors of $E_8$ from Output (7) )
CleanSlate[];
p = Flatten[Table[Permutations[q[[i]]],{i,1,1}],1];
u = Table[Table[w[[i]].p[[j]],[i,Length[w]],[j,Length[p]]];
v = Table[Table[IntegerQ[u[[j,i]]],[i,Length[w]],[j,Length[p]]];
s = Table[Flatten[Position[v[[k]],True]],[k,Length[v]]];
t = Table[Table[w[[s[[j]]][[i]]],[i,Length[s[[j]]]],[j,Length[v]]];
```
\( \Phi[k] := \Phi[k] = \text{Evaluate}[b = \text{Table}[\text{Table}[t[[k]][[i]] + t[[k]][[j]],
\{i, \text{Length}[t[[k]]]\}], \{j, \text{Length}[t[[k]]]\}];
\]
c = \text{Table}[\text{MemberQ}[t[[k]], b[[i, j]]],
\{i, \text{Length}[t[[k]]]\}], \{j, \text{Length}[t[[k]]]\}];
f = \text{Position}[c, \text{True}];
g = \text{Union}[\text{Table}[\text{Sort}[f[[i]]],
\{i, \text{Length}[f]\}]]; 
x = \text{Table}[g[[i]][[1]],
\{i, \text{Length}[g]\}];
y = \text{Table}[g[[i]][[2]],
\{i, \text{Length}[g]\}];
h = \text{Table}[t[[k]][[x[[i]]]] + t[[k]][[y[[i]]]]],
\{i, \text{Length}[x]\}];
z = \text{Flatten}[\text{Table}[\text{Position}[t[[k]], h[[i]]],
\{i, \text{Length}[h]\}]]; 
o = \text{Table}[l,
\{l, \text{Length}[t[[k]]]\}];
m = \text{Length}[\text{Complement}[o, \text{Union}[x, y, z]]];
n = \text{Length}[\text{Complement}[o, z]]];
\text{Table}[\text{If}[\text{Length}[t[[k]]] == 14, \text{Evaluate}[\Phi[k];
\text{If}[m == 1, a[1] a[2] a[4], \text{If}[m == 2, 
\text{If}[n == 6, a[1] a[1] d[4], \text{If}[n == 8, a[1] a[3] a[5]],
\text{If}[m == 4, a[1] a[1] a[1] a[1] a[1] a[1] a[1] a[4]]]]]]];
\text{If}[\text{Length}[t[[k]]] == 15, \text{Evaluate}[\Phi[k];
\text{If}[m == 0, \text{If}[n == 5, a[1] a[2] d[4],
\text{If}[n == 6, a[1] a[2] a[4]]],
\text{If}[n == 8, a[1] a[3] a[3]],
\text{If}[m == 2, a[1] a[1] a[1] a[4], \text{If}[m == 3, a[1] a[1] a[1] d[4]]]]]]];
\text{If}[\text{Length}[t[[k]]] == 16, \text{Evaluate}[\Phi[k];
\text{If}[m == 0, \text{If}[n == 7, a[3] a[4], \text{If}[n == 8, a[2] a[2] a[4]]],
\text{If}[m == 1, \text{If}[n == 6, a[1] a[5], \text{If}[n == 7, a[1] a[2] d[4]]],
\text{If}[m == 4, a[1] a[1] a[1] a[1] a[1] d[4]]]]]]];
\text{If}[\text{Length}[t[[k]]] == 21, \text{Evaluate}[\Phi[k];
\text{If}[m == 0, \text{If}[n == 6, a[6], \text{If}[n == 8, a[3] a[5]]],
\text{If}[m == 1, a[1] d[5]]]]];
\text{If}[\text{Length}[t[[k]]] == 22, \text{Evaluate}[\Phi[k];
\text{If}[m == 0, a[1] a[6], \text{If}[m == 2, a[1] a[1] d[5]]]]];
\text{If}[\text{Length}[t[[k]]] == 23, \text{Evaluate}[\Phi[k];
\text{If}[m == 0, a[2] a[5], \text{If}[m == 2, a[1] a[1] a[6], \text{If}[m == 3, a[1] a[1] a[1] d[5]]]]]]];
\text{If}[\text{Length}[t[[k]]] == 28, a[7], \text{If}[\text{Length}[t[[k]]] == 30, d[6],
\text{If}[\text{Length}[t[[k]]] == 36, \text{Evaluate}[\Phi[k];
\text{If}[n == 6, e[6], \text{If}[n == 8, a[8]]],
\text{If}[\text{Length}[t[[k]]] == 37, a[1] e[6], \text{If}[\text{Length}[t[[k]]] == 42, d[7],
\text{If}[\text{Length}[t[[k]]] == 63, e[7]]]]]]], \{k, \text{Length}[t]\}];
Union[\%]>>>Z7Groups;

\text{Output (11)}$
\{d[7], a[1] e[6]\}$

In the above code \text{Length}[t[[k]]] is the number of surviving root vectors \((p)\), \text{m} is the number of \text{A}_1 factors \((m)\) and \text{n} is the rank \((r)\) of a group. These three variables are calculated from the surviving root vectors as explained in the example, \(E_8 \rightarrow SO(8) \times SU(3)\) branching. The output of this evaluation is written in an external file \textit{Z7Groups} where a group \text{A}_n is identified as \text{a}[n], \text{D}_n as \text{d}[n] and \text{E}_n as \text{e}[n].

For other orbifolds there are situations where \(p, m\) and \(r\) do not suffice to specify the group unambiguously. In those cases we we look for \text{A}_2, \text{A}_3, \ldots\ factors in \(H_I\) by looking at root vector relations. For example, an \text{A}_2 factor would have to be spanned by three root vectors \\{t[i], t[j], t[k]\} that satisfy an equation of the form \(t[i] + t[j] = t[k]\) and occur in no equation involving any other root vectors. Equivalently, we can look for root vectors that do not appear in any equation of the form \(t[i] + t[j] + t[k] = t[l]\).
6 Automation

Due to limitations of computer’s processor speed and memory, it may be necessary to partition the computation. The following shows how it may be done for the $\mathbb{Z}_7$ orbifold example in $M$-theory.

(i) Collect all the $\mathbb{Z}_7$ vectors $q$ in Output (8) and all the $E_8$ root vectors $w$ in Output (1) of section 3 and put them in a notebook, say, $\textit{NB}_0$. Use the package ‘CleanSlate’ and put this in one of Mathematica’s home directory. This package helps in clearing the Mathematica kernel memory so that successive evaluations can use the maximum possible memory. The input of $\textit{NB}_0$ are as follows:

\textbf{Input (12)}

\begin{verbatim}
q = ; (no output shown here: 428 $\mathbb{Z}_7$ vectors from Output (8))
w = ; (no output shown here: 120 positive root vectors of $E_8$ from Output (1))
\end{verbatim}

\begin{verbatim}
<< CleanSlate.m;
orbifold = EvaluationNotebook[];
NotebookSave[orbifold]
NotebookOpen["NB 1.nb"]
\end{verbatim}

(ii) We create a notebook $\mathbb{Z}_7$\_Generic which contains the code of Input (10) with some added lines of codes to make use of the automation process:

\textbf{Input (13)}

\begin{verbatim}
NotebookClose[orbifold]
CleanSlate[];
p = Flatten[Table[Permutations[q[[i]]], {i, a, a}], 1];
a = Length[w];
l = Length[p];
u = Table[Table[w[[i]].p[[j]], {i, a}], {j, l}];
v = Table[Table[IntegerQ[u[[j, i]]], {i, a}], {j, l}];
r = Table[Count[v[[j]], True], {j, l}];
Union[r] >>> Z7\_Roots;
orbifold = EvaluationNotebook[];
NotebookSave[orbifold]
\gamma = a + 1;
"NB " <> ToString[\gamma] <> ".nb"

InputForm[%]
NotebookOpen[%];
\end{verbatim}

Next we create a notebook $\mathbb{Z}_7$\_Generator with the following set of codes,

\textbf{Input (14)}

\begin{verbatim}
Do[NotebookPut[NotebookGet[First[Notebooks["Z7 Generic.nb"]]]/."α"->β];
NotebookSave[SelectedNotebook[],"NB "<>ToString[β]<>".nb"];
Pause[2];
NotebookClose[SelectedNotebook[]],β,1,428]
\end{verbatim}

Once the Input (14) is run, it creates 428 notebooks with the contents of Input (13) where the value of $\alpha = 1, 2, 3, \cdots, 428$, respectively, for each notebook. The files are created in the $\$\text{HomeDirectory}$.

\footnote{This package is available on-line at: http://library.wolfram.com/infocenter/MathSource/4718/}
Our next step is to evaluate these 428 notebooks in a way such that when we open \texttt{NB\_0}, it automatically evaluates it’s content and the contents of notebooks \texttt{NB\_1, NB\_2} and so on so forth. The \texttt{NotebookClose[orbifold]} input line closes the previous notebook that has been evaluated. In this way the screen is not cluttered with open \textit{Mathematica} notebooks, improving the performance of the computer’s memory. The memory is also managed by the input line \texttt{CleanSlate[]}. Note that the ‘CleanSlate’ package is called in after \textit{Mathematica} stores the values of \(q\) and \(w\) in its memory which is necessary for the whole evaluation process. The end result is collected from the text file \texttt{Z7\_Roots} created in \$\textit{HomeDirectory} and is given in Eq. (4.1).

We apply a similar procedure for the evaluation of the \(Z_7\) invariant groups, \textit{Input (11)}. The results are collected from the text file \texttt{Z7\_Groups} and are summarized in Table 2.

In order for the automation process to work we need to make the following changes to \textit{Mathematica} preferences,

1. \texttt{Notebook Options \rightarrow File Options \rightarrow Notebook Autosave (False \rightarrow True)}
2. \texttt{Notebook Options \rightarrow File Options \rightarrow ClosingAutosave (False \rightarrow True)}
3. \texttt{Notebook Options \rightarrow File Options \rightarrow AutogeneratedPackage (Manual \rightarrow None)}
4. \texttt{Notebook Options \rightarrow Evaluation Options \rightarrow Initialization CellEvaluation (Automatic \rightarrow True)}
5. \texttt{Notebook Options \rightarrow Evaluation Options \rightarrow Initialization CellWarning (True \rightarrow False)}
6. \texttt{Cell Options \rightarrow Evaluation Options \rightarrow Initialization Cell (False \rightarrow True)}

This automation process has been tested in \textit{Mathematica} version 5.2, 6.0.1 and 7.0. There appears to be a problem in the latter two versions, which prevents the evaluation of a notebook when it is opened by another notebook, even though the \texttt{Initialization CellEvaluation} and \texttt{Initialization Cell} are changed to \texttt{True} (globally). In those versions of \textit{Mathematica}, the automation process (iii) can be performed using a code such as:

\textit{Input (15)}

\begin{verbatim}
nb = NotebookOpen["notebook.nb"]; SelectionMove[nb, All, Notebook]; SelectionEvaluate[nb];
orbifold = EvaluationNotebook[];
NotebookSave[orbifold];
NotebookClose[orbifold];
\end{verbatim}

Corresponding changes need to be made also in \textit{Input (11)} and \textit{Input (13)} for the automation process to work.

7 Conclusion

We have shown in detail how to find the \(Z_7\)-invariant subgroups of \(E_8\) using \textit{Mathematica}. These groups, obtained in orbifold \(M\)-theory, turn out to be closely related to string theory compactification down to four dimensions: In the limit \(x^{11} \rightarrow 0\), the two \(Z_7\)-invariant subgroups of \(E_8\) (one on each of the two boundaries of \(x^{11}\)) coalesce into \(H_{I,L} \times H_{I',R}\), which turn out to coincide with the gauge groups found in \(Z_7\)-orbifold models in string theory [5]. We have tested our codes also for \(Z_2, Z_3, Z_4\) and \(Z_6\) orbifolds. The so-obtained
subgroups upon the limit $x^{11} \to 0$ coincide with those found in string theory compactification. This would imply that our codes can also be used for $\mathbb{Z}_8$ and $\mathbb{Z}_{12}$ orbifolds.

In the presence of gauge background fields (Wilson lines) the four-dimensional gauge group breaks down to some smaller groups. Since these Wilson lines provide additional shifts in the group lattice, it should be possible to employ our procedure also in those types of models.

For the simple Lie groups $A_n, B_n, C_n, D_n, E_6$, and $E_7$, our procedure can be applied in finding the unbroken gauge symmetry under any $\mathbb{Z}_N$ rotations. In section 3 we provided the root vectors for these groups. As semi-simple Lie groups are products of simple Lie groups, the procedure presented herein merely needs to be applied to each factor separately. Finally, the remaining abelian factors in a Lie group are then treated separately [2].

Finally, our present goal was the demonstration that Mathematica can be used to compute $\Delta$-invariant subgroups of semi-simple Lie groups. Having been motivated by applications in $M$-theory, we have restricted ourselves to “supersymmetric” $\Delta$-actions, and for simplicity to $\Delta = \mathbb{Z}_N$. Generalizations in both respects seem to be desirable, but are beyond our present scope. Similarly, it would seem desirable to restructure, optimize and package the computations presented herein into a single, user-friendly interactive Mathematica package, but that too is deferred to a subsequent effort.

Acknowledgment: We are indebted to the Department of Energy for the generous support under the grant DE-FG02-94ER-40854. TH wishes to thank for the recurring hospitality and resources provided by the Physics Department of the University of Central Florida, Orlando, and the Physics Department of the Faculty of Natural Sciences of the University of Novi Sad, Serbia, where part of this work was completed.

References

[1] P. Hořava and E. Witten, \textit{Heterotic and Type I String Dynamics from Eleven Dimensions}, Nucl. Phys. B460 (1996) 506–524, \texttt{hep-th/9510209}.

[2] T. Hübisch and M. Ahsan, $\mathbb{Z}_7$ orbifold models in $M$-theory, J. Phys. A 42 (2009) 355209, \texttt{arXiv.org:0810.4543}.

[3] R. Slansky, \textit{Group Theory for Unified Model Building}, Phys. Rept. 79 (1981) 1–128.

[4] B. G. Wybourne, \textit{Classical Groups for Physicists}. John Wiley & Sons Inc., 1974.

[5] Y. Katsuki, Y. Kawamura, T. Kobayashi, and N. Ohtsubo, $\mathbb{Z}_7$ Orbifold Models, Phys. Lett. B212 (1988) 339.