Systems of conservation laws of Temple class, equations of associativity and linear congruences in $P^4$

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Abstract

We propose a geometric correspondence between (a) linearly degenerate systems of conservation laws with rectilinear rarefaction curves and (b) congruences of lines in projective space whose developable surfaces are planar pencils of lines. We prove that in $P^4$ such congruences are necessarily linear. Based on the results of Castelnuovo, the classification of three-component systems is obtained, revealing a close relationship of the problem with projective geometry of the Veronese variety $V^2 \subset P^5$ and the theory of associativity equations of two-dimensional topological field theory.

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1 Introduction

Hyperbolic systems of conservation laws

$$u_i^t = f^i(u)_x = v_j^i(u)u^j_x, \quad v_j^i = \frac{\partial f^i}{\partial u^j}, \quad i = 1, \ldots, n,$$

(1)

naturally arise in a variety of physical applications and are known to possess a rich mathematical and geometric structure [15], [16], [26], [3], [12]. It was observed recently that many constructions of the theory of systems of conservation laws are, in a sense, parallel to that of the projective theory of congruences. The correspondence proposed in [4] and [8] associates with any system (1) an $n$-parameter family of lines

$$y^i = u^i y^0 - f^i(u), \quad i = 1, \ldots, n$$

(2)

in $(n + 1)$-dimensional projective space $P^{n+1}$ with affine coordinates $y^0, \ldots, y^n$. In the case $n = 2$ we obtain a two-parameter family, or a congruence of lines in $P^3$. In the
19th century the theory of congruences was one of the most popular chapters of classical differential geometry (see, e.g., [14]). We keep the name “congruence” for any \( n \)-parameter family of lines \((\mathbb{P}^{n+1})\) in \( P^{n+1} \).

It turns out that the basic concepts of the theory of systems of conservation laws, such as shock and rarefaction curves, Riemann invariants, reciprocal transformations, linearly degenerate systems and systems of Temple class [29] acquire a clear and simple projective interpretation when reformulated in the language of the theory of congruences. For instance, this correspondence enabled the classification of systems of Temple class to be reduced to a much simpler geometric problem of the classification of congruences with either planar or conical developable surfaces. In particular, some of the results of [29] became intuitive geometric statements about families of lines in projective space. Another application of the correspondence proposed was the construction of the Laplace and Lévy transformations of hydrodynamic type systems in Riemann invariants [9], [11], which, on the geometric level, have been a subject of extensive research in projective differential geometry.

Remark. It should be emphasized that the correspondence between systems (1) and congruences in \( P^{n+1} \) is not one-to-one: some “degenerate” congruences are to be excluded. Indeed, let
\[
y^i = g^i(u)y^0 - f^i(u), \quad u = (u^1, ..., u^n)
\] (3)
be an arbitrary \( n \)-parameter family of lines in \( P^{n+1} \); notice that \( g^i(u) \), as well as \( f^i(u) \), may not happen to be functionally independent. Associated with such a congruence is a system of conservation laws
\[
g^i(u)_t = f^i(u)_x, \quad i = 1, ..., n,
\] (4)
which, for functionally dependent \( g^i(u) \), is not in Cauchy normal form. System (4) can be transformed to the Cauchy normal form provided the characteristic polynomial
\[
det \left( \lambda \frac{\partial g^i}{\partial u^j} - \frac{\partial f^i}{\partial u^j} \right)
\] (5)
is not identically zero which, on the geometric level, is equivalent to the requirement that the lines (3) do not belong to a hypersurface in \( P^{n+1} \). Hypersurfaces in \( P^{n+1} \) carrying \( n \)-parameter families of lines are interesting in their own. For \( n = 2 \) these are planes. In the case \( n = 3 \) these are either one-parameter families of planes or three-dimensional quadrics [24]. For \( n = 4 \) among obvious examples are two-parameter families of planes or one-parameter families of three-dimensional quadrics – see [30], [23] for the classification results. In what follows we consider nondegenerate hyperbolic congruences only, which means that the characteristic polynomial (5) is not zero identically and its roots are real and pairwise distinct. Any such congruence can be parametrized in the form (3).

Let \( \lambda^i(u) \) be the eigenvalues of the matrix \( v^i_j \) of system (1), assumed real and pairwise distinct. Let \( \xi^i(u) \) be the corresponding eigenvectors: \( v^i_j \xi^i = \lambda^i \xi^i \). In this paper we investigate and classify systems of conservation laws which simultaneously satisfy the following two properties:

(a) The integral trajectories of the eigenvectors \( \xi^i \) (called the rarefaction curves of system (1)) are straight lines in coordinates \( u^1, ..., u^n \). This condition was introduced by Temple in [29].
(b) The eigenvalues $\lambda^i$ are constant along rarefaction curves of the i-th family. Such systems are known as linearly degenerate.

Systems (1) satisfying both these conditions will be called **T-systems** for short. In section 2 we include the necessary information about systems of conservation laws and recall the main results of Temple [29] and our recent work [2], [3]. These results imply the correspondence between $n$-component T-systems and congruences in $P^{n+1}$ whose developable surfaces are planar pencils of lines. One can readily establish that for $n = 2$ such congruences are linear (that is, defined by two linear equations in Plücker coordinates) and consist of all lines intersecting two fixed skew lines in $P^3$. Since any two linear congruences in $P^3$ are projectively equivalent, there exists essentially a unique two-component T-system. Recall that projective transformations of congruences (2) correspond to “reciprocal transformations” of systems (1) which preserve the T-property: see section 3 for the details.

**Example 1.** Consider the wave equation

$$f_{tt} - f_{xx} = 0.$$  \hspace{1cm} (6)

Introducing the variables $a = f_{xx}, \ b = f_{xt}$, we readily rewrite (3) as a linear two-component system of conservation laws

$$a_t = b_x, \quad b_t = a_x$$  \hspace{1cm} (7)

which is obviously a T-system (any linear system of conservation laws is a T-system since its eigenvalues and eigenvectors are constant). The corresponding congruence (2)

$$y^1 = ay^0 - b, \quad y^2 = by^0 - a$$  \hspace{1cm} (8)

consists of all lines intersecting the two skew lines $y^0 = 1, \ y^1 = -y^2$ and $y^0 = -1, \ y^1 = y^2$.

**Example 2.** Consider the Monge-Ampère equation

$$f_{xt}^2 - f_{xx}f_{tt} = 1.$$  \hspace{1cm} (9)

Introducing the variables $a = f_{xx}, \ b = f_{xt}$ [20], we readily rewrite (1) as a two-component system of conservation laws

$$a_t = b_x, \quad b_t = \left(\frac{b^2 - 1}{a}\right)_x$$  \hspace{1cm} (10)

which proves to be a T-system. The corresponding congruence

$$y^1 = ay^0 - b, \quad y^2 = by^0 - \frac{b^2 - 1}{a}$$  \hspace{1cm} (11)

consists of all lines intersecting the two skew lines $y^1 = 1, \ y^0 = y^2$ and $y^1 = -1, \ y^0 = -y^2$.

Since congruences (8) and (11) are projectively equivalent, the corresponding systems (2) and (4) are reciprocally related, thus providing a linearization of the nonlinear Monge-Ampère equation (8) (which, of course, is not a new result).

Our main result is the classification of three-component T-systems or, in geometric language, congruences in $P^4$ whose developable surfaces are planar pencils of lines. The main
example which motivated our research comes from the theory of equations of associativity of two-dimensional topological field theory.

**Example 3.** Let us consider the Monge-Ampère type equation

\[ f_{ttt} = f_{xxt}^2 - f_{xxx} f_{xxt}, \]

known as the WDVV or the associativity equation, which was thoroughly investigated by Dubrovin in [7]. Introducing the variables \( a = f_{xxx}, b = f_{xxt}, c = f_{xtt} \) [19], we readily rewrite (12) as a three-component system of conservation laws

\[ a_t = b_x, \quad b_t = c_x, \quad c_t = (b^2 - ac)_x \]

which was observed to be a T-system in [2]. The corresponding congruence in \( P^4 \)

\[ y^1 = ay^0 - b, \quad y^2 = by^0 - c, \quad y^3 = cy^0 - b^2 + ac \]

coincides with the set of trisecant lines of the Veronese variety projected from \( P^5 \) into \( P^4 \) (see section 3). In this sense the projected Veronese variety plays the role of the focal variety of the congruence (14). As follows from the classification result presented below, this example is, in a sense, generic.

We prove that congruences in \( P^4 \) whose developable surfaces are planar pencils of lines are necessarily linear (that is, defined by three linear equations in the Plücker coordinates). In the parametrisation (2) the Plücker coordinates of a congruence in \( P^4 \) are

\[ 1, \quad u^1, u^2, u^3, \quad f^1, f^2, f^3, \quad u^1 f^2 - u^2 f^1, u^1 f^3 - u^3 f^1, u^2 f^3 - u^3 f^2. \]

Linear congruences are characterized by three linear relations among them

\[ \alpha + \alpha_i u^i + \beta_i f^i + \alpha_{ij} (u^i f^j - u^j f^i) = 0, \]

where \( \alpha, \alpha_i, \beta_i, \alpha_{ij} \) are arbitrary constants. Solving these equations for \( f^1, f^2, f^3 \), we arrive at the general formula for the fluxes of three-component T-systems. Notice that the congruence (14) is linear.

**Remark.** Given a congruence in \( P^4 \) whose developable surfaces are planar pencils of lines, its Plücker image is a three-dimensional submanifold of the Grassmanian \( G(1, 4) \) covered by a two-parameter family of lines, the images of planar pencils. Moreover, there are three lines passing through each point of this submanifold. Algebraic threefolds covered by lines were recently classified by Mezzetti and Portelli in [18]. It was demonstrated that threefolds with three lines through each point are intersections of \( G(1, 4) \) with a \( P^6 \), that is, Plücker images of linear congruences. In this paper we approach the classification problem from the point of view of local differential geometry, without imposing any additional algebro-geometric restrictions. It turns out, however, that our local differential-geometric assumptions (namely, that developable surfaces are planar pencils) already imply the algebraizability.

Unlike the case of \( P^3 \), the proof of the linearity of these congruences in \( P^4 \) requires a long computation bringing a certain exterior differential system into involutive form, which is carried over to the Appendix (notice that the linearity does not necessarily hold in \( P^5 \) as simple examples from section 5 show). Once the linearity is established, one can make
use of the results of Castelnuovo [6] who classified linear congruences in $P^4$. He found six projectively different types thereof, thus providing a list of six three-component T-systems which are not reciprocally related. Below we list them as scalar third-order Monge-Ampère type equations. They assume the form (1) in the variables $a = f_{xxx}$, $b = f_{xxt}$, $c = f_{xtt}$. As systems of conservation laws, they differ by a number of Riemann invariants they possess (see section 2 for the definitions). Geometrically, the existence of a Riemann invariant implies the reducibility of the focal variety of the corresponding congruence: if a T-system possesses $k$ Riemann invariants, the focal variety contains $k$ linear subspaces of codimension two.

**Theorem 1** Any strictly hyperbolic three-component T-system can be reduced by a reciprocal transformation to one from the following list.

**I.** T-systems which possess no Riemann invariants,

\[ f_{xxx}f_{ttt} - f_{xxt}f_{ttx} = 1 \]  

and

\[ f_{xxt}^2 + f_{xtt}^2 - f_{xxx}f_{xxt} - f_{ttt}f_{xtt} = 1. \]  

The focal varieties of the corresponding congruences are non-singular projections of the Veronese variety into $P^4$. The congruences consist of the trisecant lines of these projections. Notice that there are two different projections which are not equivalent over the reals.

**II** T-systems which possess one Riemann invariant,

\[ f_{xxx}f_{ttt} - f_{xxt}f_{ttx} = 0 \]  

and

\[ f_{xxt}^2 + f_{xtt}^2 - f_{xxx}f_{xxt} - f_{ttt}f_{xtt} = 0. \]  

The corresponding focal varieties are reducible and consist of a cubic scroll and a plane which intersects the cubic scroll along its directrix. Notice that equations (14) and (15) are related to (14) and (15) by a complex change of variables $x \to (x+t)/\sqrt{2}$, $t \to i(x-t)/\sqrt{2}$.

**III** T-system with two Riemann invariants,

\[ f_{xxt}^2 - f_{xxx}f_{ttx} = 1 \]  

which reduces to the Monge-Ampère equation (9) for $\tilde{f} = f_t$. The corresponding focal variety consists of a two-dimensional quadric and two planes which intersect the quadric along rectilinear generators of different families.

**IV** T-system with three Riemann invariants,

\[ f_{ttt} - f_{xxx} = 0. \]  

The corresponding focal variety consists of three planes.
We discuss the geometry of these examples in some more detail in section 3.

Remark. Equation (15) was discussed by Dubrovin in [7]. As shown in [12], after the transformation \( \tilde{x} = t, \tilde{t} = f_{xx}, \tilde{f}_{\tilde{x} \tilde{x}} = -f_{xt}, \tilde{f}_{\tilde{x} t} = x, \tilde{f}_{\tilde{t} \tilde{t}} = f_{tt}, \) it takes the form (12):

\[
\tilde{f}_{\tilde{t} \tilde{t} \tilde{t}} = \tilde{f}_{\tilde{x} \tilde{x} \tilde{x}} - \tilde{f}_{\tilde{x} \tilde{x}} \tilde{f}_{\tilde{x} \tilde{t}},
\]

Notice that this is not a contact transformation. Geometrically, equations (12) and (15) correspond to projectively equivalent congruences (see section 3).

Equation (17) was discussed in [13] and [27]. The classification of third order equations of Monge-Ampère type was given in [1].

We would like to conclude this introduction by formulating two conjectures about the structure of congruences in \( P^{n+1} \) whose developable surfaces are planar pencils of lines.

1. The focal varieties of such congruences are algebraic (possibly, reducible and singular).
2. The intersection of the focal variety with a developable surface (which is a planar pencil of lines) consists of a point (the vertex of the pencil) and a plane curve of degree \( n-1 \).

For \( n = 2 \) this is obvious. For \( n = 3 \) it follows from the results presented above. Both conjectures are true for general linear congruences in \( P^{n+1} \) (see section 4) and congruences arising from the completely exceptional Monge-Ampère type equations (see section 5). As readily follows from the discussion in section 2, the focal varieties have codimension two and contain \( n \)-parameter families of plane curves (which are conics for \( n = 3 \)). This shows that the problem in question is actually algebro-geometric.

2 T-systems from the point of view of the projective theory of congruences.

In this section we give a brief review of the necessary material from [2], [3] and [29]. Let \( \lambda^i(u) \) be the eigenvalues of the matrix \( v^i_j(u) \) (called the characteristic velocities of system (1)), assumed real and pairwise distinct. Let \( \xi_i = (\xi^1_i, ..., \xi^n_i)^T \) be the corresponding eigenvectors: \( v^i \xi_i = \lambda^i \xi_i \), or, in components, \( v^k_{ij} \xi^k_i = \lambda^i \xi^k_i \). We denote by \( L_i = \xi^k_i \frac{\partial}{\partial u^k} \) the Lie derivative in the direction of \( \xi_i \). It is convenient to introduce the expansions

\[
[L_i, L_j] = c^k_{ij} L_k, \quad c^k_{ij} = -c^k_{ji}.
\]

In the theory of hydrodynamic type systems rarefaction curves play crucial role. Recall that rarefaction curves are integral curves of the eigenvectors \( \xi_i \). Thus, there are \( n \) families of rarefaction curves, and for any point in \( u \)-space there is exactly one rarefaction curve from each family passing through it. Due to the correspondence (2), a curve in \( u \)-space defines a ruled surface, i.e., a one-parameter family of lines in \( P^{n+1} \).

Theorem 2 [2] Ruled surfaces defined by rarefaction curves of the \( i \)-th family are developable, i.e., their rectilinear generators are tangential to a curve. This curve can be parametrized in the form

\[
y^0 = \lambda^i, \quad y^1 = u^1 \lambda^i - f^1(u), ..., y^n = u^n \lambda^i - f^n(u),
\]

where \( u \) varies along the rarefaction curve.

The curve (22) constitutes the singular locus of the developable surface which is called its cuspidal edge. The collection of all cuspidal edges corresponding to rarefaction curves of
the i-th family defines the so-called focal hypersurface $M_i \subset P_n^{n+1}$. In our case parametric equations of $M_i$ coincide with (22), where $u$ is now allowed to take all possible values. By a construction, each line of the congruence (2) is tangential to the focal hypersurface $M_i$. The idea of focal hypersurfaces is obviously borrowed from optics: thinking of the lines of a congruence as the rays of light, one can intuitively imagine focal hypersurfaces as the locus in $P_n^{n+1}$ where the light concentrates (this explains why in German literature focal hypersurfaces are called ‘Brennflächen’, which can be translated as ‘burning surfaces’).

Since the system of conservation laws (1) is strictly hyperbolic, there are precisely $n$ developable surfaces passing through a line of the congruence (2), and each line is tangential to $n$ focal hypersurfaces.

In the theory of weak solutions of systems (1) shock curves play fundamental role. The shock curve with the vertex in $u_0$ is the set of points in the $u$-space such that

$$\sigma(u^i - u^i_0) + f^i(u) - f^i(u_0) = 0, \quad i - 1,...,n,$$

for some function $\sigma(u, u_0)$. For any $u$ on the shock curve the discontinuous function

$$u(x,t) = u_0, \quad x \leq \sigma t,$$
$$u(x,t) = u, \quad x \geq \sigma t,$$

is a weak solution of (1). Shock curves, like rarefaction curves, define special ruled surfaces of the congruence (2), the geometry of which was clarified in [3].

Lax showed that a shock curve with the vertex in a generic point $u_0$ splits into $n$ branches, the $i$th branch being $C^2$–tangent of the associated rarefaction curve of the $i$th family passing through $u_0$.

As pointed out by a number of authors, there are situations when shock curves coincide with their associated rarefaction curves. Systems with coinciding shock and rarefaction curves were studied by Temple [29]. His main result can be formulated as follows.

**Theorem 3** [29] Rarefaction curves of the $i$th family coincide with the associated branches of the shock curve if and only if either

1) every rarefaction curve of the $i$th family is a straight line in the $u$–space

or

2) the characteristic velocity $\lambda^i$ is constant along rarefaction curves of the $i$th family,

$$L_i(\lambda^i) = 0.$$ 

Systems satisfying the condition 2 are known as linearly degenerate. Both these condition have a very natural geometric interpretation.

**Theorem 4** [2] Rarefaction curves of the $i$th family are straight lines if and only if the associated developable surfaces are planar, that is, their cuspidal edges are plane curves.

**Theorem 5** [2] The characteristic velocity $\lambda^i$ is linearly degenerate if and only if the associated developable surfaces are conical, that is, their generators meet in a point. The corresponding focal hypersurface $M_i$ degenerates into a submanifold of codimension two.
As demonstrated in [2] and [3], theorems 4 and 5 provide an elementary geometric proof of Theorem 3.

In what follows, we consider systems (1) which simultaneously satisfy both conditions of Theorem 3, namely, all rarefaction curves are rectilinear (in u-space), and all eigenvalues \( \lambda^i \) are linearly degenerate. Systems of this type naturally arise in the theory of equations of associativity of 2D topological field theory [7] (see examples below). We will call them T-systems for short. In view of Theorems 4 and 5, developable surfaces of the corresponding congruence (2) are planar and conical simultaneously, and hence are planar pencils of lines. The corresponding focal hypersurfaces \( M^i \) degenerate into n submanifolds of codimension 2. In the examples discussed below focal submanifolds \( M^i \) are glued together to form an algebraic variety \( \mathbb{V}^{n-1} \subset \mathbb{P}^{n+1} \) of codimension 2, so that the lines of the congruence (2) can be characterized as \( n \)-secants of \( \mathbb{V}^{n-1} \).

It may happen that \( \mathbb{V}^{n-1} \) contains a linear subspace of codimension 2. This is closely related to the property for system (1) to possess Riemann invariants.

Definition. The Riemann invariant for the \( i \)th characteristic velocity \( \lambda^i \) is a function \( R^i(\mathbf{u}) \) such that

\[
R^i_t = \lambda^i R^i_x
\]

by virtue of (1).

There is a simple criterion for the existence of Riemann invariants in terms of the coefficients \( c_{jk}^i \) defined by (21).

Proposition 1 The characteristic velocity \( \lambda^i \) possesses a Riemann invariant if and only if \( c_{jk}^i = 0 \) for any \((j,k) \neq i\).

Theorem 6 If the characteristic velocity \( \lambda^1 \) of a T-system (4) possesses a Riemann invariant, then the corresponding focal submanifold \( M^i \) is a linear subspace of codimension 2.

Proof: Let us consider, for definiteness, the focal hypersurface \( M_1 \) with the radius vector \( \mathbf{r}_1 = (\lambda^1, u^1 \lambda^1 - f^1, ..., u^n \lambda^1 - f^n) \), corresponding to the characteristic velocity \( \lambda^1 \). We will need the following relations between the densities \( u \) and the fluxes \( f \) of conservation laws of system (1):

\[
L_i(f) = \lambda^i L_i(u) \quad \text{for any} \quad i = 1, ..., n,
\]

\[
L_i L_j(u) = \frac{L_j(\lambda^i)}{\lambda^j - \lambda^i} L_i(u) + \frac{L_i(\lambda^j)}{\lambda^j - \lambda^i} L_j(u) + \frac{\lambda^j - \lambda^k}{\lambda^j - \lambda^i} c_{jk}^i L_k(u), \quad i \neq j,
\]

(see e.g. [24], [28]). In particular, \( f = f^s \) and \( u = u^s \) satisfy (24) and (25) for any \( s = 1, ..., n \). Introducing \( \mathbf{1} = (1, u^1, ..., u^n) \) and applying \( L_1, ..., L_n \) to the radius vector \( \mathbf{r}_1 \), one readily obtains \( L_1(\mathbf{r}_1) = L_1(\lambda^1) \mathbf{1} = 0 \) as \( L_1(\lambda^1) = 0 \) by the linear degeneracy. Thus, the condition \( L_1(\mathbf{r}_1) = 0 \) implies that \( M_1 \) is independent of \( R^1 \), where \( R^1 \) is the Riemann invariant corresponding to \( \lambda^1 \). Since

\[
L_i(\mathbf{r}_1) = L_i(\lambda^1) \mathbf{1} + (\lambda^1 - \lambda^i) L_i(\mathbf{1}),
\]
the tangent space $TM_1$ is spanned by $n - 1$ vectors $L_i(\lambda^1)l + (\lambda^1 - \lambda^i)L_i(l)$, $(i \neq 1)$. This tangent space belongs to the hyperplane $H_1$ spanned by $n$ vectors $l, L_i(l)$, $(i \neq 1)$ which depends on the variable $R^1$ only since $L_k(H_1) \in H_1$ for any $k \neq 1$. The latter follows from the relations

$$L_k(l) \in H_1, \quad L_k^2(l) = A_k L_k(l) \in H_1, \quad L_k L_j H_1 \in H_1 \quad (k, j \neq 1).$$

Here $L_k^2(l) = A_k L_k(l)$ due to the linearity of rarefaction curves, and $L_k L_j H_1 \in H_1$ by virtue of (25) and the condition $c_{kj}^1 = 0$ (Proposition 1). On the other hand, $M_1$ (and hence $TM_1$) does not depend on $R^1$ due to the linear degeneracy. Consequently, the tangent space of $M_1$ is the intersection of any two hyperplanes $H_1$ which correspond to the two different values of $R^1$. Therefore, it is stationary and coincides with the focal submanifold $M_1$. Q.E.D.

Finally, we recall the necessary information about reciprocal transformations of systems of conservation laws. Let $B(u)dx + A(u)dt$ and $N(u)dx + M(u)dt$ be two conservation laws of system (1), understood as the one-forms which are closed by virtue of (1). In the new independent variables $X,T$ defined by

$$dX = B(u)dx + A(u)dt, \quad dT = N(u)dx + M(u)dt,$$

system (1) takes the form

$$u^T_i = V^i_j (u)u^j_X, \quad i = 1, ..., n,$$

where $V = (Bv - AE)(ME - Nv)^{-1}$, $E = id$. The new characteristic velocities $\Lambda^k$ are

$$\Lambda^k = \frac{\lambda^kB - A}{M - \lambda^kN}. \quad (28)$$

Transformations (29) are called reciprocal. Reciprocal transformations are known to preserve the linear degeneracy (see [10]). Moreover, if both integrals (29) are linear combinations of the canonical integrals $u^i dx + f^i dt$ defining the T-system (1),

$$dX = (\alpha_i u^i + \alpha) dx + (\alpha_i f^i + \bar{\alpha}) dt,$$
$$dT = (\beta_i u^i + \beta) dx + (\beta_i f^i + \bar{\beta}) dt,$$

(9) here $\alpha_i, \alpha, \bar{\alpha}, \beta_i, \beta, \bar{\beta}$ are arbitrary constants), then the transformed system will be a T-system, too (2). Furthermore, affine transformations

$$U^i = C^i_j u^j + D^i, \quad C^i_j = const, \quad D^i = const, \quad det C^i_j \neq 0, \quad (30)$$

obviously transform T-systems to T-systems.

**Theorem 7** [2] The transformation group generated by reciprocal transformations (29) and affine transformations (30) is isomorphic to the group of projective transformations of $P^{n+1}$.

Thus, the classification of systems of conservation laws up to transformations (29) and (30) is equivalent to the classification of the corresponding congruences up to projective equivalence. Actually, this observation was the main reason for introducing the geometric correspondence discussed in section 2.
3 Geometry of the examples

Let us first recall some of the well-known properties of the Veronese variety \( V^2 \subset P^5 \) realising \( P^5 \) as the space of \( 3 \times 3 \) symmetric matrices \( Z^{ij}, \ i, j = 0, 1, 2 \). Veronese variety \( V^2 \) is the variety of matrices of rank 1

\[
Z = \begin{pmatrix}
X^0X^0 & X^0X^1 & X^0X^2 \\
X^1X^0 & X^1X^1 & X^1X^2 \\
X^2X^0 & X^2X^1 & X^2X^2
\end{pmatrix}.
\]

It can be viewed as the canonical embedding \( F : P^2 \rightarrow V^2 \subset P^5 \) defined by

\[
Z^{ij} = X^iX^j, \ i = 0, 1, 2,
\]

where \( X^0 : X^1 : X^2 \) are homogeneous coordinates in \( P^2 \). Veronese variety coincides with the singular locus of the cubic symmetroid defined by the equation

\[
\det Z^{ij} = 0,
\]

which also is the bisecant variety of \( V^2 \) consisting of symmetric matrices of rank two. Under the embedding (31) each line in \( P^2 \) is mapped onto a conic on \( V^2 \), therefore, Veronese variety carries a 2-parameter family of conics. The projective automorphism group of \( V^2 \) coincides with the natural action of \( PSL_3 \) on \( P^5 \)

\[
Z \rightarrow g^T Z g, \quad g \in PSL_3,
\]

which obviously preserves \( V^2 \).

Below we discuss in some more detail the geometry of congruences associated with the equations (12), (15) – (20).

3.1 Geometry of the equations with no Riemann invariants

In this subsection we discuss equations (12), (15) and (16). Rewritten as systems of conservation laws, they do not possess Riemann invariants, so that the corresponding focal varieties will be irreducible. We explicitly demonstrate that they coincide with different non-singular projections of the Veronese variety.

**Equation (12).** The focal variety of the corresponding congruence (14) is defined by (22)

\[
y^0 = \lambda, \quad y^1 = a\lambda - b, \quad y^2 = b\lambda - c, \quad y^3 = c\lambda - b^2 + ac
\]

where \( \lambda \) satisfies the characteristic equation

\[
\lambda^3 + a\lambda^2 - 2b\lambda + c = 0.
\]

One can verify that the three focal surfaces corresponding to the three different values of \( \lambda \) are, in fact, "glued" together to form the algebraic variety defined in this affine chart by a system of seven cubics

\[
(y^0)^3 + y^0y^1 - y^2 = 0, \quad (y^2)^2 + y^3(y^0)^2 = 0, \quad y^1y^2y^3 + y^0(y^3)^2 - (y^2)^3 = 0,
\]

\[
y^2(y^0)^2 + y^1y^2 + y^0y^3 = 0, \quad (y^3)^2 - y^1(y^2)^2 + y^0y^2y^3 + y^2(y^1)^2 = 0,
\]

\[
y^0y^1y^3 - y^0(y^2)^2 - y^2y^3 = 0, \quad y^0y^2 + y^1(y^0)^2 + (y^1)^2 + y^3 = 0.
\]
Variety (35) coincides with the projection of the Veronesé variety $V^2 \subset P^5$

$$y^0 = \frac{Z^{02}}{Z^{22}}, \quad y^1 = \frac{Z^{12} - Z^{00}}{Z^{22}}, \quad y^2 = \frac{Z^{01}}{Z^{22}}, \quad y^3 = -\frac{Z^{11}}{Z^{22}}$$

from the point

$$\begin{pmatrix} Z^{00} & 0 & 0 \\ 0 & 0 & Z^{00} \\ 0 & Z^{00} & 0 \end{pmatrix} \quad (36)$$

into $P^4$. Notice that this point does not belong to the bisecant variety $S(V^2)$ and hence the projection is non-singular (we would like to thank A. Oblomkov for clarifying the structure of its ideal). Here we list some of the main properties of this projection which are, of course, well-known.

1. The manifold of trisecant lines of the focal variety (35) is three-dimensional.
2. For a point $p$ on the focal variety the set of trisecants passing through $p$ forms a planar pencil with the vertex $p$.
3. The intersection of the abovementioned planar pencil with the focal variety consists of the point $p$ and a conic. Let us demonstrate this by a direct calculation. Since $(1, \lambda, \lambda^2)^T$ is the eigenvector of the system (13) corresponding to the eigenvalue $\lambda$, the rarefaction curve passing through $p$ is given parametrically by

$$(a, b, c) + s(1, \lambda, \lambda^2) = (a + s, b + s\lambda, c + s\lambda^2),$$

$s$ being the parameter (recall that rarefaction curves are lines). The corresponding pencil of lines

$$y^1 = (a + s)y^0 - (b + s\lambda),$$
$$y^2 = (b + s\lambda)y^0 - (c + s\lambda^2),$$
$$y^3 = (c + s\lambda^2)y^0 - (b + s\lambda)^2 + (a + s)(c + s\lambda^2)$$

belongs to the plane with parametric equations

$$y^0 = X,$$
$$y^1 = aX - b + Y,$$
$$y^2 = bX - c + \lambda Y,$$
$$y^3 = cX - b^2 + ac + \lambda^2 Y.$$ 

It can be readily verified that the intersection of this plane with the focal variety consists of the point $X = \lambda, Y = 0$ and the parabola $Y + X^2 + (a + \lambda)X + \lambda^2 + a\lambda - 2b = 0$.

**Equation (15).** Rewritten as a system of conservation laws

$$a_t = b_x, \quad b_t = c_x, \quad c_t = (1 + bc)/a_x, \quad (37)$$

this equation is associated with the congruence

$$y^1 = ay^0 - b, \quad y^2 = by^0 - c, \quad y^3 = cy^0 - (1 + bc)/a, \quad (38)$$

the focal variety of which is defined by (22)

$$y^0 = \lambda, \quad y^1 = a\lambda - b, \quad y^2 = b\lambda - c, \quad y^3 = c\lambda - (1 + bc)/a \quad (39)$$
where \( \lambda \) satisfies the characteristic equation

\[
\lambda^3 - \frac{b}{a} \lambda^2 - \frac{c}{a} \lambda + \frac{1 + bc}{a^2} = 0.
\] (40)

One can verify that the three focal surfaces (39) corresponding to the three different values of \( \lambda \) are glued together to form the algebraic variety defined in this affine chart by a system of cubics

\[
\begin{align*}
1 + y^0(y^1)^2 + y^1y^2 = 0, & \quad y^1(y^0)^2 - y^3 = 0, & \quad (y^0)^3 + y^0y^2y^3 + (y^3)^2 = 0, \\
y^0 + y^0y^1y^2 + y^1y^3 = 0, & \quad y^0y^3 - y^2(y^0)^2 + y^1(y^3)^2 - y^3(y^2)^2 = 0, \\
(y^0)^2 + y^0y^1y^3 + y^2y^3 = 0, & \quad y^0y^1 + (y^1)^2y^3 - y^2 - y^1(y^2)^2 = 0.
\end{align*}
\] (41)

This algebraic variety is the projection of the Veronese variety

\[
y^0 = -\frac{Z^0}{Z^{12}}, \quad y^1 = -\frac{Z^1}{Z^{12}}, \quad y^2 = \frac{Z^2 - Z^0}{Z^{12}}, \quad y^3 = -\frac{Z^0}{Z^{12}}
\]

from the point

\[
\begin{pmatrix}
0 & Z^{01} & 0 \\
Z^{01} & 0 & 0 \\
0 & 0 & Z^{01}
\end{pmatrix}
\] (42)

into \( \mathbb{P}^4 \). Notice that the two points (36) and (42) are equivalent under the action of the group (32) preserving the Veronese variety (indeed, both matrices have the same Lorentzian signature). Hence, both projections and the corresponding congruences of trisecants are projectively equivalent. To be explicit, the projective transformation

\[
y^0 = -\frac{1}{Y^1}, \quad y^1 = \frac{Y^2}{Y^1}, \quad y^2 = \frac{Y^0}{Y^1}, \quad y^3 = -\frac{Y^3}{Y^1}
\] (43)

identifies the systems of cubics (35) and (41). Applying this transformation to the congruence (14) and introducing the new parameters \( A = -1/c, \quad B = b/c, \quad C = a - b^2/c \), we readily rewrite (14) in the form

\[
Y^1 = AY^0 - B, \quad Y^2 = BY^0 - C, \quad Y^3 = CY^0 - (1 + BC)/A
\]

which coincides with (35). This gives geometric explanation of the transformation between equations (12) and (15) mentioned in the introduction. On the level of systems of conservation laws (13) and (37), this transformation is a reciprocal equivalence.

**Equation (16).** Rewritten as a system of conservation laws

\[
a_t = b_x, \quad b_t = c_x, \quad c_t = ((c^2 + b^2 - ac - 1)/b)_x,
\] (44)

this equation is associated with the congruence

\[
y^1 = ay^0 - b, \quad y^2 = by^0 - c, \quad y^3 = cy^0 - ((c^2 + b^2 - ac - 1)/b)
\] (45)
whose focal surfaces are glued together to form the algebraic variety which is the projection of the Veronesé variety

\[ y^0 = -\frac{Z^{02}}{Z^{12}}, \quad y^1 = \frac{Z^{00} - Z^{22}}{Z^{12}}, \quad y^2 = \frac{Z^{01}}{Z^{12}}, \quad y^3 = \frac{Z^{11} - Z^{22}}{Z^{12}}, \]

from the point

\[
\begin{pmatrix}
  Z^{00} & 0 & 0 \\
  0 & Z^{00} & 0 \\
  0 & 0 & Z^{00}
\end{pmatrix}
\]  

(46)

into \( P^4 \). Notice that this point is not equivalent (over the reals) to the points (36) and (42) under the action of the group (32) (indeed, the signature of (46) is Euclidean). Hence, the congruence (45) is not projectively equivalent to any of the congruences (14) or (38). The corresponding systems of conservation laws are not reciprocally related.

We point out that the Veronesé variety \( V^2 \subset P^5 \), being the intersection of quadrics, does not possess trisecant lines. Trisecants appear only after we project \( V^2 \) into \( P^4 \). Indeed, let \( P_0 \) be a point in \( P^5 \) not on the bisecant variety \( S(V^2) \). Viewed as a \( 3 \times 3 \) symmetric matrix, \( P_0 \) defines a non-degenerate conic in \( P^2 \)

\[ \sum_{i,j=0}^{2} (P_0^{-1})^{ij} X^i X^j = 0 \]

(47)

where \( X^0 : X^1 : X^2 \) are homogeneous coordinates. If a plane passes through \( P_0 \) and cuts \( V^2 \) in three points, then pre-images of these points under the embedding (31) are pairwise conjugate with respect to the conic (47). Conversely, \( P_0 \) lies in the plane spanned by the images under (31) of any three points in \( P^2 \) which are pairwise conjugate with respect to (47). Thus, there is a three-parameter family of planes passing through \( P_0 \) and cutting \( V^2 \) in three points. Projecting this family from the point \( P_0 \) into \( P^4 \), we arrive at the congruence of lines in \( P^4 \). By a construction, its lines are trisecants of the projection \( \pi_{P_0}(V^2) \), which is thus the focal surface of our congruence. To see that the developable surfaces of the congruence are planar pencils of lines, we consider a line \( L \) in \( P^2 \) defined by the equation \( L_0 X^0 + L_1 X^1 + L_2 X^2 = 0 \). Under the embedding (31), this line corresponds to a conic on \( V^2 \) lying in the so called conisecant plane of \( V^2 \). In matrix form equations of this plane are \( LZ = 0 \). The three-dimensional subspace \( \Lambda \) spanned by \( P_0 \) and the conisecant plane consists of all \( Z \) such that the vectors \( LZ \) and \( LP_0 \) are collinear. In addition to the conic in the conisecant plane, \( \Lambda \) intersects \( V^2 \) in the point \( P_0^L \) whose pre-image in \( P^2 \) under (31) has homogeneous coordinates \( P_0 L^T \). Consider now the one-parameter family of planes in \( P^5 \) which belong to \( \Lambda \) and pass through the line joining \( P_0 \) and \( P_0^L \). Clearly, each of these planes intersects \( V^2 \) in three points, and the projection of this one-parameter family of planes into \( P^4 \) will be a planar pencil of lines. This gives developable surfaces of our congruence.

### 3.2 Geometry of the equations with one Riemann invariant

In this subsection we discuss equations (13) and (15). Since both equations possess only one Riemann invariant, the corresponding focal varieties will be reducible, consisting of a cubic scroll and a plane intersecting the cubic scroll along its directrix.
Equation (17) can be rewritten as a system of conservation laws

\[ a_t = b_x, \quad b_t = c_x, \quad c_t = (bc/a)_x \]  

(48)

the characteristic velocities of which are \( \lambda^1 = b/a \) and \( \lambda^2, \lambda^3 = \pm \sqrt{c/a} \). The only Riemann invariant \( R^1 = c/a \) corresponds to \( \lambda^1 \). The focal surface corresponding to \( \lambda^1 \) is the plane

\[ y^1 = y^3 = 0, \]  

(49)

while the focal surfaces corresponding to \( \lambda^2 \) and \( \lambda^3 \) are glued together to form the cubic scroll defined by a system of quadrics

\[ y^0 y^1 + y^2 = 0, \quad y^0 y^2 + y^3 = 0, \quad y^1 y^3 - (y^2)^2 = 0. \]  

(50)

The plane (49) intersects the cubic scroll along its directrix

\[ y^1 = y^2 = y^3 = 0. \]  

(51)

The cubic scroll (50) can be obtained by projecting the Veronese variety

\[ y^0 = \frac{Z^{02}}{Z^{12}}, \quad y^1 = \frac{Z^{11}}{Z^{12}}, \quad y^2 = -\frac{Z^{01}}{Z^{12}}, \quad y^3 = \frac{Z^{00}}{Z^{12}}, \]

from the point

\[ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Z^{22} \end{pmatrix}. \]

Notice that the center of this projection lies on the Veronese variety. The directrix (51) is the image of the tangent plane \( Z^{00} = Z^{01} = Z^{11} = 0 \) to the Veronese variety in the centre of projection, and the plane (49) is the projection of the three-dimensional linear subspace in \( P^5 \) spanned by the tangent plane and the point

\[ \begin{pmatrix} 0 & Z^{01} & 0 \\ Z^{01} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]  

(52)

on the bisecant variety. Thus, the focal variety of our congruence is reducible and consists of the plane (49) and the cubic scroll (50). Like in the case of systems without Riemann invariants,

1. the manifold of trisecants of the focal variety is three-dimensional, and

2. for a fixed point \( p \) on the focal variety the set of trisecants passing through \( p \) forms a planar pencil with the vertex \( p \). If \( p \) belongs to the plane (49), the corresponding planar pencil cuts the focal variety in the point \( p \) and a conic. If \( p \) belongs to the cubic scroll, it cuts the focal variety in the point \( p \) and a pair of lines.

Equation (18) can be rewritten as a system of conservation laws

\[ a_t = b_x, \quad b_t = c_x, \quad c_t = ((c^2 + b^2 - ac)/b)_x \]  

(53)
the characteristic velocities of which are \( \lambda^1 = c/b \) and \( \lambda^2, \lambda^3 = (c-a \pm \sqrt{4b^2 + (c-a)^2})/2b \). The only Riemann invariant \( R^1 = (c-a)/b \) corresponds to \( \lambda^1 \). The focal surface corresponding to \( \lambda^1 \) is the plane

\[
y^1 = y^3, \quad y^2 = 0, \tag{54}
\]

while the focal surfaces corresponding to \( \lambda^2 \) and \( \lambda^3 \) are glued together to form the cubic scroll defined by a system of quadrics

\[
y^0 y^3 + y^2 = 0, \quad y^0 y^2 + y^1 = 0, \quad y^1 y^3 - (y^2)^2 = 0. \tag{55}
\]

The plane (54) intersects the cubic scroll (55) along its directrix

\[
y^1 = y^2 = y^3 = 0. \tag{56}
\]

The cubic scroll (55) can be obtained by projecting \( V^2 \)

\[
y^0 = \frac{Z^{02}}{Z^{12}}, \quad y^1 = \frac{Z^{00}}{Z^{12}}, \quad y^2 = -\frac{Z^{01}}{Z^{12}}, \quad y^3 = \frac{Z^{11}}{Z^{12}},
\]

from the point

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & Z^{22}
\end{pmatrix}
\]

on \( V^2 \). The directrix (54) is the image of the tangent plane \( Z^{00} = Z^{01} = Z^{11} = 0 \) to \( V^2 \) in the centre of projection, and the plane (54) is the image of the three-dimensional linear subspace spanned by the tangent plane and the point

\[
\begin{pmatrix}
Z^{00} & 0 & 0 \\
0 & Z^{00} & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

on the bisecant variety.

A coordinate-free construction of the congruences discussed above can be described as follows. Take a point \( P_0 \in S(V^2) \) which is represented by a symmetric matrix of rank two. Then there is a nonzero vector \( X \in P^2 \) such that \( P_0 X = 0 \). Consider the tangent plane to \( V^2 \) at the point \( F(X) \). The projection of \( V^2 \) into \( P^4 \) from the point \( F(x) \) is a cubic scroll. The projection of the tangent plane is the directrix. The projection of the three-dimensional space spanned by the tangent plane and \( P_0 \) is the plane intersecting the cubic scroll along its directrix.

Although the last two examples look pretty similar, they are not projectively equivalent, indeed, the points (52) and (57) have different signatures.

### 3.3 Geometry of the equation with two Riemann invariants.

In this subsection we discuss equation (19). Due to the existence of two Riemann invariants, the corresponding focal variety will be reducible consisting of two planes and a two-dimensional quadric.
Equation (19) can be rewritten as a system of conservation laws

\[ a_t = b_x, \quad b_t = c_x, \quad c_t = \left( (c^2 - 1)/b \right)_x \]  \tag{58}

with the characteristic velocities \( \lambda^1 = 0 \) and \( \lambda^2, \lambda^3 = (c \mp 1)/b \). The system has 2 Riemann invariants \((c \pm 1)/b \) corresponding to \( \lambda^2 \) and \( \lambda^3 \), respectively. The focal surfaces of the associated congruence corresponding to \( \lambda^2 \) and \( \lambda^3 \) are the planes

\[ y^2 = \mp 1, \quad y^0 = \mp y^3, \]  \tag{59}

while the third focal surface, corresponding to \( \lambda^1 \), is the quadric

\[ y^0 = 0, \quad y^1 y^3 - (y^2)^2 + 1 = 0. \]  \tag{60}

The planes (58) intersect the quadric (60) along the rectilinear generators

\[ y^0 = 0, \quad y^2 = \mp 1, \quad y^3 = 0 \]

which belong to different families and meet at infinity.

One can describe this congruence in a coordinate-free form as follows. Consider a quadric \( Q \) in a hyperplane \( \Lambda \subset P^4 \). Choose a point \( p \in Q \) and draw two rectilinear generators \( l_1, l_2 \) of \( Q \) through \( p \). Choose two planes \( \pi_1 \) and \( \pi_2 \) which are not in \( \Lambda \) such that \( l_i \subset \pi_i \) and \( \pi_1 \cap \pi_2 = p \). The union of \( \pi_1, \pi_2 \) and \( Q \) is the focal variety in question. Its trisecants define a congruence in \( P^4 \).

### 3.4 Geometry of the equation with three Riemann invariants.

As follows from Theorem 6, the focal varieties of congruences corresponding to diagonalizable \( n \)-component T-systems are collections of \( n \) linear subspaces of codimension two in \( P^{n+1} \). For \( n = 4 \) we have 3 planes in \( P^4 \). To ensure the nondegeneracy, we require that the points of their pairwise intersections are distinct.

Equation (20) can be rewritten as a linear system of conservation laws

\[ a_t = b_x, \quad b_t = c_x, \quad c_t = b_x \]  \tag{61}

with the characteristic velocities \( \lambda^1 = 0, \lambda^2 = 1, \lambda^3 = -1 \). Being linear, this system has 3 Riemann invariants. The focal surfaces of the associated congruence are the planes

\[ y^1 = y^3, \quad y^0 = 0 \quad \text{for} \quad \lambda^1 = 0, \]

\[ y^3 = -y^2, \quad y^0 = 1 \quad \text{for} \quad \lambda^2 = 1 \]

and

\[ y^3 = y^2, \quad y^0 = -1 \quad \text{for} \quad \lambda^1 = 0, \]

respectively.
4 Linear congruences.

A congruence \((\mathbf{2})\) is called linear (or general linear) if its Plücker coordinates

\[1, u^i, f^i, u^i f^j - u^j f^i\]

satisfy \(n\) linear equations of the form

\[
\alpha + \alpha_i u^i + \beta_i f^i + \alpha_{ij} (u^i f^j - u^j f^i) = 0 \tag{62}
\]

where \(\alpha, \alpha_i, \beta_i, \alpha_{ij}\) are arbitrary constants (notice that equations (62), being linear in \(f\), define \(f^i\) as explicit functions of \(u\)). We emphasize that all examples discussed above belong to this class.

**Theorem 8** Congruences corresponding to three-component T-systems are linear.

The **Proof** is technical and relegated to the Appendix.

Let \(q\) be a fixed point in \(P^{n+1}\). For the lines of the congruence \((\mathbf{2})\) passing through \(q\) we have \(f^i = u^i q^0 - q^i\), which, upon the substitution into (62), implies a linear system for \(u\). In general, this system possesses a unique solution, so that there exists a unique line of our congruence passing through \(q\) (such congruences are said to be of order one). The focal variety \(V\) (also called the jump locus) consists of those \(q\) for which the corresponding linear system is not uniquely solvable for \(u\). One can show that \(V\) has codimension at least two, and in the case it equals two, the developable surfaces are planar pencils of lines. Moreover, the intersection of any of these planes with the focal variety \(V\) consists of a point and a plane curve of degree \(n - 1\).

There are at least two different ways one could approach the classification of three-component T-systems or, equivalently, the line congruences in \(P^4\) whose developable surfaces are planar pencils of lines. The first way is to establish their linearity. In the parametrization \((\mathbf{3})\) this means that the three-dimensional surface with the radius-vector \((1, u, f, u \wedge f)\) representing our congruence in the Grassmanian \(G(1, 4)\) lies in a linear subspace of codimension three. After the linearity is established, the results of Castelnuovo [6] (who demonstrated that the corresponding focal varieties are projections of the Veronese variety into \(P^4\)) complete the classification and imply Theorem 1.

Another way makes use of the Theorem of Segre [25] saying that a surface in projective space carrying a two-parameter family of plane curves (not lines) is either a cone or the surface of Veronese \(V^2\) or its projection into \(P^4\). Moreover, the corresponding plane curves are conics. This theorem is intimately related to our problem. Indeed, let \(M_1, M_2, M_3\) be three focal surfaces of our congruence in \(P^4\). Take a point \(p \in M_1\) and consider the planar pencil of lines passing through it. This plane intersects \(M_2\) and \(M_3\) in the curves \(\gamma_2\) and \(\gamma_3\), respectively. Varying \(p\), we conclude that both \(M_2\) and \(M_3\) contain two-parameter families of plane curves and hence are projections of the Veronese variety (the case of a cone can be easily ruled out). Moreover, the curves \(\gamma_2\) and \(\gamma_3\) are conics. To show that both \(M_2\) and \(M_3\) are actually parts of one and the same Veronese variety, it is sufficient to demonstrate that \(\gamma_2\) and \(\gamma_3\) are parts of one and the same plane conic. This can be done as follows: intersect \(\gamma_2\) and \(\gamma_3\) by a line passing through \(p\) and construct the tangent lines to \(\gamma_2\) and \(\gamma_3\) in the points of intersection. These tangent lines meet in a point \(q\) lying in
the same plane. Doing this for all lines of the pencil with vertex \( p \) we arrive at the curve \( q \), which clearly must be a line (called the polar of \( p \)) in case \( \gamma_2 \) and \( \gamma_3 \) are parts of one and the same conic.

Unfortunately, both proofs require differential identities which do not immediately follow from the geometric data given. Thus, it proves necessary to directly investigate the exterior differential system governing three-component T-systems, transforming it into the involutive form. Once it is done, the verification of both properties mentioned above reduces to a straightforward calculation.

In the case \( n = 4 \) the geometry of focal varieties of general linear congruences, known as the Palatini scrolls, was investigated in [22] (see also [17] and [21] for further properties of the Palatini scrolls) (we thank F. Zak for providing these references). T-systems of conservation laws corresponding to general linear congruences will be discussed elsewhere.

5 Completely exceptional Monge-Ampère type equations.

Another important class of examples of T-systems is provided by completely exceptional Monge-Ampère equations studied in [3]. Equations of this type are defined as follows. Introduce the Hankel matrix

\[
\begin{pmatrix}
\frac{\partial^2 m_{i,j}}{\partial x^m \partial y} & \frac{\partial^2 m_{i,j}}{\partial x^{m-1} \partial y^2} & \frac{\partial^2 m_{i,j}}{\partial x^{m-2} \partial y^3} & \cdots & \frac{\partial^2 m_{i,j}}{\partial x \partial y^m} \\
\frac{\partial^2 m_{i,j}}{\partial x^{m-1} \partial y} & \frac{\partial^2 m_{i,j}}{\partial x^{m-2} \partial y^2} & \frac{\partial^2 m_{i,j}}{\partial x \partial y^m} & \cdots & \frac{\partial^2 m_{i,j}}{\partial y^{m-1} \partial x} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{\partial^2 m_{i,j}}{\partial x \partial y^m} & \frac{\partial^2 m_{i,j}}{\partial y^{m-1} \partial x} & \cdots & \cdots & \cdots
\end{pmatrix}
\] (63)

and denote by \( M_{J,K}(u) \) its minor of order \( l \) whose rows and columns are encoded in the multiindices \( J = (j_1,...,j_l) \) and \( K = (k_1,...,k_l) \), respectively. PDE’s in question are defined by linear combinations of these minors,

\[
\sum A_{J,K}^I M_{J,K} = 0
\] (64)

where the summation is over all possible \( l, J, K \), and \( A_{J,K}^I \) are arbitrary constants. Any such equation can be rewritten as \( \frac{\partial^2 m_{i,j}}{\partial x^m \partial y} = f(\frac{\partial^2 m_{i,j}}{\partial x^{m-1} \partial y},...,\frac{\partial^2 m_{i,j}}{\partial y^{m-1} \partial x}) \), and after the introduction of \( a_1 = \frac{\partial^2 m_{i,j}}{\partial x \partial y}, a_2 = \frac{\partial^2 m_{i,j}}{\partial x^m \partial y},..., a_m = \frac{\partial^2 m_{i,j}}{\partial y^m \partial x}, \) assumes the conservative form

\[
a_1 = a_2, \quad a_2 = a_3, \quad..., \quad a_m = f(a_1,...,a_{m-1}).
\] (65)

One can show that this is always a T-system (in fact, its linear degeneracy was demonstrated in [3]), and the corresponding congruence [3] has the following properties:
- its developable surfaces are planar pencils of lines,
- its focal variety has codimension at least 2,
- each developable surface intersects the focal variety in a point, which is the vertex of the pencil, and a plane curve of degree \( n - 1 \).

To obtain systems of this type for odd \( n \), one should consider equations (64) which are independent of \( \frac{\partial^2 m_{i,j}}{\partial x^m \partial y} \). Introducing \( v = \frac{\partial a_1}{\partial x} \) and rewriting the resulting equation for \( v \) (which
is of the order $2m - 1$) as a system of conservation laws, one arrives at the congruence with the properties as formulated above. We are planning to investigate the geometry of these examples elsewhere. When $n \geq 4$ these congruences are not necessarily linear. In this case the focal varieties must be singular, as follows from [17].

6 Appendix

6.1 The exterior representation of hydrodynamic type systems

Investigating nondiagonalizable systems

$$u^i_t = v^i_j(u)u^j_x, \quad i = 1, 2, 3,$$

it is convenient to use the following exterior notation: let \( l^i_j = (l^i_j(u)) \) be left eigenvectors of the matrix \( v^i_j \) corresponding to the eigenvalues \( \lambda^i \), i.e., \( l^i_j v^j_k = \lambda^i l^i_k \). With the eigenforms \( \omega^i = l^i_j du^j \), the system (66) is rewritten in the following exterior form:

$$\omega^i \wedge (dx + \lambda^i dt) = 0, \quad i = 1, 2, 3.$$  \hspace{1cm} (67)

Differentiation of \( \omega^i \) and \( \lambda^i \) gives the structure equations

$$d\omega^i = -c^i_{jk} \omega^j \wedge \omega^k, \quad (c^i_{jk} = -c^i_{kj}), \quad d\lambda^i = \lambda^j \omega^i,$$  \hspace{1cm} (68)

containing all the necessary information about the system under study. Notice that if \( \omega^i(\xi_j) = \delta^i_j \) then the coefficients \( c^i_{jk} \) are the same as that appearing in (21).

6.2 The structure equations

The three theorems formulated below show that the structure equations of three-component T-systems take surprisingly simple forms. We give the detailed proof of the first theorem and only sketch the proofs of two others. Notice that \( \omega^i \) are defined up to a nonzero normalization \( \omega^i \rightarrow p^i \omega^i, \quad p^i \neq 0 \).

Theorem 9 The eigenforms of a three-component T-system with no Riemann invariants can be normalized so that the structure equations take the form

$$d\omega^1 = \omega^2 \wedge \omega^3, \quad d\omega^2 = \epsilon \omega^3 \wedge \omega^1, \quad d\omega^3 = \omega^1 \wedge \omega^2,$$  \hspace{1cm} (69)

where \( \epsilon = \pm 1 \).

Proof: For a system with no Riemann invariants the forms \( \omega^i \) can be normalized in such a way that the structure equations take the form

$$d\omega^1 = a \omega^1 \wedge \omega^2 + b \omega^1 \wedge \omega^3 + \omega^2 \wedge \omega^3, \quad d\omega^2 = p \omega^2 \wedge \omega^1 + q \omega^2 \wedge \omega^3 + \epsilon \omega^3 \wedge \omega^1, \quad d\omega^3 = r \omega^3 \wedge \omega^1 + s \omega^3 \wedge \omega^2 + \omega^1 \wedge \omega^2,$$  \hspace{1cm} (70)

where \( \epsilon = \pm 1 \). Below we assume \( \epsilon = 1 \), since the complex normalization

$$\omega^1 \rightarrow i \omega^1, \quad \omega^2 \rightarrow \omega^2, \quad \omega^3 \rightarrow i \omega^3$$
reduces the case $\epsilon = -1$ to the case $\epsilon = 1$ which allows to treat both cases on equal footing. Since the systems under consideration are strictly hyperbolic, the eigenforms $\omega^i$ constitute a basis so that the differential of a function $u$ can be decomposed as $du = u_i \omega^i$ where $u_i = L_i(u)$. Differentiating the relations $du = u_i \omega^i$, $df = \lambda^i u_i \omega^i$ (compare with \[24\]) and equating to zero coefficients at $\omega^i \wedge \omega^j$, one obtains

$$
\begin{align*}
    u_{12} &= u_{2} \frac{\lambda_2^2}{\lambda_2^2 - \lambda_1^2} + u_{1} \frac{\lambda_1^2}{\lambda_2^2 - \lambda_1^2} + u_{1} a + u_{3} \frac{\lambda^3 - \lambda_2^2}{\lambda_1^2 - \lambda_2^2}, \\
    u_{21} &= u_{2} \frac{\lambda_2^2}{\lambda_2^2 - \lambda_1^2} + u_{1} \frac{\lambda_1^2}{\lambda_2^2 - \lambda_1^2} + u_{2} p + u_{3} \frac{\lambda^3 - \lambda_1^2}{\lambda_1^2 - \lambda_2^2}, \\
    u_{31} &= u_{3} \frac{\lambda_3^2}{\lambda_3^2 - \lambda_1^2} + u_{1} \frac{\lambda_1^2}{\lambda_3^2 - \lambda_1^2} + u_{3} r + u_{2} \frac{\lambda^3 - \lambda_1^2}{\lambda_1^2 - \lambda_2^2}, \\
    u_{13} &= u_{3} \frac{\lambda_3^2}{\lambda_3^2 - \lambda_1^2} + u_{1} \frac{\lambda_1^2}{\lambda_3^2 - \lambda_1^2} + u_{1} b + u_{2} \frac{\lambda^3 - \lambda_2^2}{\lambda_1^2 - \lambda_2^2}, \\
    u_{32} &= u_{3} \frac{\lambda_3^2}{\lambda_3^2 - \lambda_1^2} + u_{2} \frac{\lambda_2^2}{\lambda_3^2 - \lambda_2^2} + u_{3} s + u_{1} \frac{\lambda^3 - \lambda_2^2}{\lambda_1^2 - \lambda_2^2}, \\
    u_{23} &= u_{3} \frac{\lambda_3^2}{\lambda_3^2 - \lambda_1^2} + u_{2} \frac{\lambda_2^2}{\lambda_3^2 - \lambda_2^2} + u_{2} q + u_{1} \frac{\lambda^3 - \lambda_1^2}{\lambda_1^2 - \lambda_2^2},
\end{align*}
$$

where $u_{ij}$ are defined as $du_i = u_{ij} \omega^j$. Let us introduce the vector $\vec{u} = (u^1, u^2, u^3)$. The condition of the linearity of rarefaction curves in the coordinates $u^1, u^2, u^3$ can be written in the form

$$
\vec{u}_{11} = A_1 \vec{u}_1, \quad \vec{u}_{22} = A_2 \vec{u}_2, \quad \vec{u}_{33} = A_3 \vec{u}_3
$$

where $A_1, A_2$ and $A_3$ are certain proportionality factors. Taking into account the known expressions for $\vec{u}_{ij}$ when $i \neq j$, one obtains the following expressions for $d \vec{u}_i$:

$$
\begin{align*}
    d \vec{u}_1 &= A_1 \vec{u}_1 \omega^1 + \left( \vec{u}_1 \left( a + \frac{\lambda_1^2}{\lambda_2^2 - \lambda_1^2} \right) + \frac{u_{2}}{\lambda_2^2 - \lambda_1^2} \frac{\lambda_1^2}{\lambda_2^2 - \lambda_1^2} + \frac{u_{3}}{\lambda_2^2 - \lambda_1^2} \left( \lambda^3 - \lambda_2^2 \right) \right) \omega^2 \\
    &+ \left( \vec{u}_1 \left( b + \frac{\lambda_1^2}{\lambda_3^2 - \lambda_1^2} \right) + \frac{u_{2}}{\lambda_3^2 - \lambda_1^2} \frac{\lambda_1^2}{\lambda_3^2 - \lambda_1^2} + \frac{u_{3}}{\lambda_3^2 - \lambda_1^2} \right) \omega^3,
\end{align*}
$$

$$
\begin{align*}
    d \vec{u}_2 &= \left( \vec{u}_2 \left( p + \frac{\lambda_2^2}{\lambda_2^2 - \lambda_1^2} \right) + \frac{u_{1}}{\lambda_2^2 - \lambda_1^2} \frac{\lambda_1^2}{\lambda_2^2 - \lambda_1^2} + \frac{u_{3}}{\lambda_2^2 - \lambda_1^2} \left( \lambda^3 - \lambda_1^2 \right) \right) \omega^1 + A_2 \vec{u}_2 \omega^2 \\
    &+ \left( \vec{u}_2 \left( q + \frac{\lambda_2^2}{\lambda_3^2 - \lambda_2^2} \right) + \frac{u_{1}}{\lambda_3^2 - \lambda_2^2} \frac{\lambda_1^2}{\lambda_3^2 - \lambda_2^2} + \frac{u_{3}}{\lambda_3^2 - \lambda_2^2} \right) \omega^3,
\end{align*}
$$

$$
\begin{align*}
    d \vec{u}_3 &= \left( \vec{u}_3 \left( r + \frac{\lambda_3^2}{\lambda_3^2 - \lambda_1^2} \right) + \frac{u_{1}}{\lambda_3^2 - \lambda_1^2} \frac{\lambda_1^2}{\lambda_3^2 - \lambda_1^2} + \frac{u_{2}}{\lambda_3^2 - \lambda_1^2} \left( \lambda^3 - \lambda_2^2 \right) \right) \omega^1 \\
    &+ \left( \vec{u}_3 \left( s + \frac{\lambda_3^2}{\lambda_3^2 - \lambda_2^2} \right) + \frac{u_{1}}{\lambda_3^2 - \lambda_2^2} \frac{\lambda_1^2}{\lambda_3^2 - \lambda_2^2} + \frac{u_{2}}{\lambda_3^2 - \lambda_2^2} \right) \omega^2 + A_3 \vec{u}_3 \omega^3,
\end{align*}
$$

where $\vec{u}_i = (u^1_i, u^2_i, u^3_i)$.

Differentiating these equations and equating to zero coefficients at $\omega^i \wedge \omega^j$, one obtains 9 equations which are linear in $u_{ij}$. Since $u^1, u^2, u^3$ are functionally independent, these equations split with respect to $u_{ij}$, thus providing 27 equations for the derivatives of $\lambda^k$, the factors $A_1, A_2, A_3$ and the coefficients of the structure equations $a, b, p, q, r, s$. The coefficients at $\vec{u}_3 \omega^1 \wedge \omega^2$ and $\vec{u}_2 \omega^3 \wedge \omega^1$ of the result of differentiation of the first equation \[71\] allow to find $A_1$ and $\lambda_1^1$. Similarly, the coefficients at $\vec{u}_3 \omega^1 \wedge \omega^2$ and $\vec{u}_1 \omega^2 \wedge \omega^3$ of the result of differentiation of the second equation \[72\] give $A_2$ and $\lambda_2^2$. Finally, the coefficients at $\vec{u}_1 \omega^2 \wedge \omega^3$ and $\vec{u}_2 \omega^3 \wedge \omega^1$ of the result of differentiation of the third equation \[73\] give $A_3$ and $\lambda_3^3$,

$$
\lambda_1^1 = \frac{2p - 2r}{\lambda_2^2 - \lambda_1^2} \left( \lambda^3 - \lambda_2^2 \right) \left( \lambda_2^2 - \lambda_1^2 \right),
$$

\[72\]
respectively. (We do not write down these intermediate expressions for nonzero $a$)

to find all second derivatives of $\lambda$ to their complexity.) Substituting these derivatives back into the coefficients at $\lambda$ and $u$, one obtains a closed system for $(71)$, one obtains a closed system for $A_1 = \frac{(r-p) (\lambda^2 - 2 \lambda^1 + \lambda^3)}{\lambda^2 - \lambda^3} + \frac{2 \lambda_1^3 (\lambda^2 - \lambda^1)}{(\lambda^1 - \lambda^3) (\lambda^2 - \lambda^3)} + \frac{2 \lambda_1^2 (\lambda^3 - \lambda^1)}{(\lambda^2 - \lambda^3) (\lambda^2 - \lambda^1)}, \quad (75)

$A_2 = \frac{(s-a) (\lambda^1 - 2 \lambda^2 + \lambda^3)}{\lambda^1 - \lambda^3} + \frac{2 \lambda_2^3 (\lambda^1 - \lambda^2)}{(\lambda^1 - \lambda^3) (\lambda^2 - \lambda^3)} + \frac{2 \lambda_2^2 (\lambda^3 - \lambda^2)}{(\lambda^1 - \lambda^3) (\lambda^1 - \lambda^2)}, \quad (76)

$A_3 = \frac{(q-b) (\lambda^1 - 2 \lambda^3 + \lambda^2)}{\lambda^1 - \lambda^2} + \frac{2 \lambda_3^3 (\lambda^2 - \lambda^3)}{(\lambda^1 - \lambda^3) (\lambda^1 - \lambda^2)} + \frac{2 \lambda_3^2 (\lambda^3 - \lambda^1)}{(\lambda^2 - \lambda^3) (\lambda^1 - \lambda^2)}. \quad (77)$

For linearly degenerate systems the first three of these equations imply

$$r = p, \ s = a, \ q = b.$$ \quad (78)

For the eigenvalues of linearly degenerate systems and the first derivatives thereof we have

$$d\lambda^1 = \lambda_1^1 \omega^2 + \lambda_1^3 \omega^3$$

$$d\lambda^2 = \lambda_2^2 \omega^1 + \lambda_2^3 \omega^3$$

$$d\lambda^3 = \lambda_3^3 \omega^1 + \lambda_3^2 \omega^2$$

$$d\lambda_1^1 = (p \lambda_2^1 - \lambda_3^1) \omega^1 + \lambda_3^2 \omega^2 + \lambda_3^3 \omega^3$$

$$d\lambda_1^2 = (r \lambda_1^1 + \lambda_2^1) \omega^1 + (\lambda_2^2 - q \lambda_2^1 + s \lambda_1^1) \omega^2 + \lambda_3^3 \omega^3$$

$$d\lambda_1^2 = \lambda_1^2 \omega^1 + (a \lambda_2^2 + \lambda_3^2) \omega^2 + (\lambda_3^1 + b \lambda_1^2 \omega^2 + \lambda_3^2 \omega^3$$

$$d\lambda_2^2 = \lambda_2^3 \omega^1 + (s \lambda_3^2 - \lambda_1^2) \omega^2 + \lambda_2^3 \omega^3$$

$$d\lambda_3^2 = \lambda_3^1 \omega^1 + \lambda_3^2 \omega^2 + (b \lambda_3^1 - \lambda_3^2) \omega^3$$

$$d\lambda_2^3 = (\lambda_3^1 + a \lambda_1^3 + p \lambda_2^3) \omega^1 + \lambda_3^2 \omega^2 + (q \lambda_3^1 + \lambda_3^2) \omega^3$$

Taking into account $(78)$ and substituting the expressions for $A_1, A_2$ and $A_3$ back into $(71)$, one obtains a closed system for $\vec{u}_1, \vec{u}_2, \vec{u}_3$. Differentiating this system once again and collecting coefficients at $\vec{u}_1 \omega^2 \land \omega^k$, one arrives at the 27 compatibility conditions involving the second derivatives of $\lambda$ and the first derivatives of $a, b$ and $p$. These conditions allow to find all second derivatives of $\lambda$. Moreover, they imply $a = b = p = 0$.

Indeed, differentiating the first equation $(71)$ and collecting coefficients at $\vec{u}_1 \omega^1 \land \omega^2$, $\vec{u}_2 \omega^1 \land \omega^2$, $\vec{u}_1 \omega^2 \land \omega^3$, $\vec{u}_2 \omega^2 \land \omega^3$ and $\vec{u}_3 \omega^3 \land \omega^1$, we calculate $\lambda_1^2, \lambda_1^1, \lambda_2^3, \lambda_3^1$ and $\lambda_3^2$, respectively. (We do not write down these intermediate expression for nonzero $a, b, p$ due to their complexity.) Substituting these derivatives back into the coefficients at $\vec{u}_3 \omega^2 \land \omega^3$ and $\vec{u}_1 \omega^3 \land \omega^1$, one arrives at the equations

$$pa - 5b - a_1 = 0$$

$$bp + 5a - b_1 = 0$$

which determine $a_1$ and $b_1$. It is remarkable that these equations for the structure coefficients do not include the eigenvalues and the derivatives thereof.
Similarly, the differentiation of the second equation (71) defines \( \lambda_{12}^1, \lambda_{22}^3 \), by calculating the coefficients at \( \vec{u}_1 \omega^1 \wedge \omega^2 \) and \( \vec{u}_3 \omega^2 \wedge \omega^3 \), respectively, along with the equations

\[
\begin{aligned}
pa + 5b - p_2 &= 0 \\
ab - 5p - b_2 &= 0 \\
bp + 3a - p_3 &= 0 \\
ab - 2a_3 + b_2 - p &= 0
\end{aligned}
\]

which result from the coefficients at \( \vec{u}_2 \omega^1 \wedge \omega^2, \vec{u}_1 \omega^3 \wedge \omega^1, \vec{u}_2 \omega^3 \wedge \omega^1 \) and \( \vec{u}_2 \omega^2 \wedge \omega^3 \), respectively.

Finally, differentiation of the third equation (71) defines \( \lambda_{33}^3 \) and \( \lambda_{33}^3 \) from the coefficients at \( \vec{u}_2 \omega^2 \wedge \omega^3 \) and \( \vec{u}_1 \omega^3 \wedge \omega^1 \), and, eventually, the equations \( b = 0, p = 0 \) and \( a = 0 \) from the coefficients at \( \vec{u}_3 \omega^1 \wedge \omega^2, \vec{u}_3 \omega^2 \wedge \omega^3 \) and \( \vec{u}_3 \omega^3 \wedge \omega^1 \).

For \( a = b = p = 0 \) not only the system (71), but also the system (80) is in involution.

The second derivatives \( \lambda_{jk}^1 \) and \( \lambda_{jj}^3 \) can be obtained by cyclic permutations from

\[
\begin{aligned}
\lambda_{12}^3 &= \lambda_{23}^1 \left( \frac{1}{\lambda - \lambda^1} + \frac{1}{\lambda - \lambda^2} \right) + \lambda_{13}^2 \left( \frac{1}{\lambda - \lambda^1} + \frac{1}{\lambda - \lambda^3} \right) - \lambda_{12}^1 \left( \frac{1}{\lambda - \lambda^1} + \frac{1}{\lambda - \lambda^2} \right) + \\
&+ \lambda_{33}^1 \left( \frac{\lambda^1 - \lambda^3}{\lambda - \lambda^3} \right)^2 - \lambda_{33}^3 \left( \frac{\lambda^2 - \lambda^3}{\lambda - \lambda^3} \right)^2
\end{aligned}
\]

(81)

\[
\lambda_{11}^3 = -2\lambda_{11}^3 \left( \frac{1}{\lambda - \lambda^1} + \frac{1}{\lambda - \lambda^2} \right) + 2 \frac{(\lambda_1^2)^2}{\lambda - \lambda^1} + 2 \frac{(\lambda_2^2 - \lambda^1)(\lambda^2 - \lambda^1)}{\lambda - \lambda^1}
\]

(82)

Q.E.D.

The differential form \( \omega^i \) is proportional to the differential of a function \( \omega^i = p^i dR^i \) if and only if \( d\omega^i \wedge \omega^i = 0 \) (this is a special case of the Frobenius theorem). Recall that the function \( R^i \) is called the Riemann invariant of the system (80).

**Theorem 10** The eigenforms of a three-component T-system with one Riemann invariant can be normalized so that the structure equations take the form

\[
d\omega^1 = \epsilon \omega^2 \wedge \omega^3, \quad d\omega^2 = \omega^3 \wedge \omega^1, \quad \omega^3 = dR^3, \quad \text{where } \epsilon = \pm 1.
\]

(83)

**Proof:** Let the system (80) have only one Riemann invariant \( R^3 : dR^3 = \omega^3 \). Then the structure equations (80) can be normalized as follows:

\[
\begin{aligned}
d\omega^1 &= a \omega^1 \wedge \omega^2 + b \omega^1 \wedge dR^2 + c \omega^2 \wedge dR^3, \\
d\omega^2 &= p \omega^2 \wedge \omega^1 + q \omega^2 \wedge dR^3 + dR^3 \wedge \omega^1.
\end{aligned}
\]

(84)

As in the case with no Riemann invariants, we can find \( \vec{u}_{ij} = \vec{U}_{ij}(\lambda^1, \lambda^2, \vec{u}, a, b, c, p, q), \quad i \neq j \). Now, the linear degeneracy and the compatibility conditions for

\[
\begin{aligned}
d\vec{u}_1 &= A_1 \vec{u}_1 \omega^1 + \vec{U}_{12} \omega^2 + \vec{U}_{13} \omega^3, \\
d\vec{u}_2 &= \vec{U}_{21} \omega^1 + A_2 \vec{u}_2 \omega^2 + \vec{U}_{23} \omega^3, \\
d\vec{u}_3 &= \vec{U}_{31} \omega^1 + \vec{U}_{32} \omega^2 + A_3 \vec{u}_3 \omega^3
\end{aligned}
\]

(85)

imply

\[
\begin{aligned}
a_1 &= p_2, \\
b_1 &= b_1, \\
c_2 &= b_2, \\
a_3 - b_2 &= pc + aq, \\
b_1 - p_3 &= a - pb, \\
c_1 &= 0, \\
c_2 &= 0, \\
c_3 &= 2c(q - b).
\end{aligned}
\]

(86)
The first five formulae (86) are equivalent to $d(p \omega^1 + a \omega^2 + b dR^3) = 0$ and $d((q - b) dR^3) = 0$. Define the functions $\phi$ and $\psi$ by

$$-\frac{d\phi}{\phi} = p \omega^1 + a \omega^2 + b dR^3, \quad \frac{d\psi}{\psi} = (q - b)dR^3.$$

It is clear that $\psi$ depends on $R^3$ only. Renormalize the forms and introduce the new Riemann invariant $\hat{R}^3$ as follows:

$$\hat{\omega}^1 = \frac{\omega^1}{\phi}, \quad \hat{\omega}^2 = \frac{\psi}{\phi} \omega^2, \quad d\hat{R}^3 = \psi(R^3) dR^3.$$

Thus renormalized forms satisfy

$$d\hat{\omega}^1 = \frac{c}{\psi^2} \omega^2 \wedge d\hat{R}^3, \quad d\hat{\omega}^2 = d\hat{R}^3 \wedge \hat{\omega}^1.$$

The last three equations (86) give $d(c/\psi^2) = 0$, so $c/\psi^2$ is constant. One can always choose $\psi$ to guarantee $c/\psi^2 = \pm 1$. With the structure equation (83), the system (85) is in involution. Q.E.D.

**Theorem 11** The eigenforms of a three-component T-system with two Riemann invariants can be normalized so that the structure equations take the form

$$d\omega^1 = \omega^2 \wedge \omega^3, \quad \omega^2 = dR^2, \quad \omega^3 = dR^3. \quad (87)$$

**Proof:** One can normalize $\omega^1$ so that

$$d\omega^1 = a \omega^1 \wedge dR^2 + b \omega^1 \wedge dR^3 + dR^2 \wedge dR^3.$$

Now, the compatibility conditions for (85) imply

$$a_1 = a_3 = b_1 = b_2 = 0,$$

which means that $a$ is a function of $R^2$ and $b$ is a function of $R^3$ only. Define

$$d\hat{R}^2 = exp \left( \int a(R^2) dR^2 \right) dR^2, \quad d\hat{R}^3 = exp \left( \int b(R^3) dR^3 \right) dR^3,$$

$$\tilde{\omega}^1 = exp \left( \int a(R^2) dR^2 + \int b(R^3) dR^3 \right) \omega^1.$$

The so renormalized forms and the so defined Riemann invariants satisfy $d\tilde{\omega}^1 = d\hat{R}^2 \wedge d\hat{R}^3$. As before, with the structure equation (87), the corresponding system (85) is in involution. Q.E.D.

**Remark.** Equations (15) and (16) have the structure equations (69) with $\epsilon = -1$ and $\epsilon = 1$, respectively. Equations (17) and (18) have the structure equations (83) with $\epsilon = -1$ and $\epsilon = 1$, respectively. Equation (19) has the structure equations (87).
6.3 Proof of the Theorem

In the parametrization ([2]) the linearity of the congruence means that the three-dimensional submanifold with the radius-vector \( \mathbf{q} = (1, \mathbf{u}, f, \mathbf{u} \wedge f) \) representing the congruence in the Grassmanian \( G(1, 4) \) lies in a linear subspace of codimension three (here \( \mathbf{u} = (u^1, u^2, u^3) \) and \( f = (f^1, f^2, f^3) \)). The osculating space of this submanifold is spanned by the vectors \( \mathbf{q}_i = L_i(\mathbf{q}) \), \( \mathbf{q}_{ii} = L_i^2(\mathbf{q}) \) and \( \mathbf{q}_{ij} = L_jL_i(\mathbf{q}) \). The conditions \( \lambda_i^j = 0 \), \( \mathbf{u}_{ii} = A_i \mathbf{u}_i \) and \( \mathbf{f}_i = \lambda^i \mathbf{u}_i \) imply \( \mathbf{q}_i = (0, \mathbf{u}_i, \lambda^i \mathbf{u}_i, \mathbf{u}_i \wedge (f - \lambda^i \mathbf{u})) \) and \( \mathbf{q}_{ii} = A_i \mathbf{q}_i \equiv 0 \mod(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) \). Using the first equation of (71), one has

\[
\mathbf{q}_{12} = L_2(\mathbf{q}_1) = \frac{\lambda_2^1}{\lambda_1^2-\lambda_2^2} \mathbf{q}_2 + \lambda_2^1(0, 0, \mathbf{u}_2, \mathbf{u} \wedge \mathbf{u}_2) + \frac{\lambda_1^1}{\lambda_1^2-\lambda_2^2} \mathbf{q}_1 + \lambda_1^1(0, 0, \mathbf{u}_1, \mathbf{u} \wedge \mathbf{u}_1) + \frac{\lambda_3^1}{\lambda_1^2-\lambda_2^2} (\mathbf{q}_3 + (0, 0, \mathbf{u}_3, \mathbf{u} \wedge \mathbf{u}_3)) + (\lambda_2^2 - \lambda_1^2)(0, 0, 0, \mathbf{u}_1 \wedge \mathbf{u}_2),
\]

which, \( \mod(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) \), equals

\[
\tilde{\mathbf{q}}_{12} = \lambda_2^1(0, 0, \mathbf{u}_2, \mathbf{u} \wedge \mathbf{u}_2) + \lambda_2^1(0, 0, \mathbf{u}_1, \mathbf{u} \wedge \mathbf{u}_1) + \frac{\lambda_3^1(\lambda_2^3-\lambda_3^3)}{\lambda_1^2-\lambda_2^2}(0, 0, \mathbf{u}_3, \mathbf{u} \wedge \mathbf{u}_3) + (\lambda_2^2 - \lambda_1^2)(0, 0, 0, \mathbf{u}_1 \wedge \mathbf{u}_2).
\]

Define \( \tilde{\mathbf{q}}_{23} \) and \( \tilde{\mathbf{q}}_{31} \) in a similar way. Thus, the osculating space is spanned by six vectors \( \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \tilde{\mathbf{q}}_{12}, \tilde{\mathbf{q}}_{23}, \tilde{\mathbf{q}}_{31} \). Using the formula (31) (and the ones obtained from it by cyclic permutations) one shows by a direct computation that \( L_3(\tilde{\mathbf{q}}_{12}) \equiv 0 \mod(\tilde{\mathbf{q}}_{12}, \tilde{\mathbf{q}}_{23}, \tilde{\mathbf{q}}_{31}) \). This relation along with \( \mathbf{q}_{ii} \equiv 0 \mod(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) \) implies that the osculating space is stationary, so the three-dimensional submanifold in question lies in six-dimensional linear subspace. In the cases with one or two Riemann invariants the proofs are essentially the same. Q.E.D.

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