COSMIC-RAY TRANSPORT THEORY IN PARTIALLY TURBULENT SPACE PLASMAS WITH COMPRESSIBLE MAGNETIC TURBULENCE

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ABSTRACT

Recently, a new transport theory of cosmic rays in magnetized space plasmas extending the quasilinear approximation to the particle orbit has been developed for the case of an axisymmetric incompressible magnetic turbulence. Here, we generalize the approach to the important physical case of a compressible plasma. As previously obtained in the case of an incompressible plasma, we allow arbitrary gyrophase deviations from the unperturbed spiral orbits in the uniform magnetic field. For the case of quasi-stationary and spatially homogeneous magnetic turbulence we derive, in the small Larmor radius approximation, gyrophase-averaged cosmic-ray Fokker–Planck coefficients. Upper limits for the perpendicular and pitch-angle Fokker–Planck coefficients and for the perpendicular and parallel spatial diffusion coefficients are presented.

Key words: cosmic rays – magnetic fields – plasmas – turbulence

1. INTRODUCTION

The study of the cosmic-ray transport in turbulent magnetic fields is crucial in many aspects of high-energy astrophysics, such as the efficiency of cosmic-ray diffusive shock acceleration, the modulation and penetration of low-energy cosmic rays in the heliosphere, and their confinement and escape from the Galaxy (Burger & Hattingh 1998; Jaeckel & Schlickeiser 1992).

A new theory of cosmic-ray transport in magnetized plasmas extending the quasilinear approximation to the particle orbit has recently been published by one of us (Schlickeiser 2011, hereafter Paper I). In Paper I the transport parameters of energetic charged particles in turbulent magnetized cosmic plasmas were derived for the case of an incompressible plasma, i.e., plasmas for which the component of the magnetic turbulence, \(\delta B_z = 0\), parallel to the guide magnetic field, \(\vec{B}_0 = B_0 \hat{e}_z\), is set to zero. Here we present the generalization of the theory to the case of compressible magnetic turbulence with \(\delta B_z \neq 0\).

In Section 2, we briefly review the theory developed in Paper I. In Section 3 we obtain the gyrophase-averaged cosmic-ray Fokker–Planck coefficients for a quasi-stationary, spatially homogeneous turbulence under a Corrsin-type assumption on the nature of generalized orbits (Corrsin 1959; Salu & Montgomery 1977; McComb 1990). Simplified formulae for the gyrophase-averaged cosmic-ray Fokker–Planck coefficients are obtained in Section 4 assuming that the magnetic turbulence is asymmetric, while the quasilinear limit of the coefficients is shown in the Appendix. In Section 5, from the Fokker–Planck coefficients, we derive upper and lower limits for the perpendicular and parallel spatial diffusion coefficients. In Section 6 we compare the relative importance of mirror forces and turbulent scattering for the cosmic-ray transport in interstellar plasmas.

2. THE GYROAVERAGED FOKKER–PLANCK EQUATIONS

2.1. Equations of Motion of a Particle in Magnetic Field

For the following treatment we briefly recall the equation of motion of charged particles of mass \(m\), charge \(q\), and Lorentz factor \(\gamma = \sqrt{1 + (p/mc)\^2}\) in a uniform guide magnetic field \(\vec{B}_0 = B_0 \hat{e}_z = (0, 0, B_0)\). A random magnetic field, \(\delta \vec{B}\), is superposed to the guide field

\[
\dot{\vec{p}} = \frac{q}{\gamma mc} \vec{p} \times [\vec{B}_0 + \delta \vec{B}], \quad \dot{x} = \frac{\dot{\vec{p}}}{\gamma m}.
\]

The scalar product of Equation (1) with \(\vec{p}\) readily yields \(p = |\vec{p}| = \text{const.}, v = \text{const.},\) and \(\gamma = \text{const.}\). Introducing the constant relativistic gyrofrequency \(\Omega = q B_0 / \gamma mc\) and scaling the turbulent fields in units of \(B_0\), \(\delta b = \vec{B} / B_0\) we obtain

\[
\frac{d}{dt} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \Omega \begin{pmatrix} v_x (1 + \delta b_x) - v_z \delta b_y \\ -v_y (1 + \delta b_z) + v_z \delta b_x \\ v_z \delta b_y - v_y \delta b_x \end{pmatrix}.
\]

For the time evolution of the particles’ pitch-angle cosine, \(\mu = v_z / v\), and phase, \(\phi = \arctan(v_y / v_x)\), this implies

\[
\frac{d\mu}{dt} = h_\mu(t) = \frac{\Omega}{v} (v_x \delta b_y - v_y \delta b_x) = \Omega \sqrt{1 - \mu^2} (\cos \phi \delta b_x - \sin \phi \delta b_y),
\]

and

\[
\frac{d\phi}{dt} = -\Omega + h_\phi(t), \quad h_\phi(t) = -\Omega \delta b_z + \frac{\Omega \mu}{\sqrt{1 - \mu^2}} (\cos \phi \delta b_x + \sin \phi \delta b_y),
\]

with the two random forces \(h_\mu(t)\) and \(h_\phi(t)\).
In the coordinates of the guiding center
\[ \tilde{X} = (X, Y, Z) = \tilde{x} + \frac{\tilde{v} \times \tilde{z}}{\Omega} = \tilde{x} + \frac{1}{\Omega} \begin{pmatrix} \frac{v_y}{-v_x} \\ 0 \end{pmatrix} \]  

Equation (2) become
\[ \frac{d}{dt} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} v_z \delta b_x - v_x \delta b_z \\ v_z \delta b_y - v_y \delta b_z \\ v_z \end{pmatrix}. \]  

Indicating \( X_i = [X, Y] \) with \( i, j = 1, 2 \), Equation (6) provide the two additional random force terms \( h_i(t) \), proportional to the turbulent magnetic field components
\[ \frac{dX_i}{dt} = h_i(t) = v_i(t)\delta b_i(t) - v_i(t)\delta b_i(t). \]  

2.2. The Ensemble-averaged Particle Distribution Function

The description of the cosmic-ray transport within a large-scale guide magnetic field, which is uniform on the scales of the cosmic-ray particles gyroradii \( R_L = v/|\Omega| \), is given by the solution of the Vlasov (collision-free Boltzmann) equation for the particle distribution function \( F \) (Hall & Sturrock 1968; Schlickeiser 2002). In spherical momentum coordinates \((X, Y, z, p, \mu, \phi)\), the Vlasov equation reads (Hall & Sturrock 1968; Achatz et al. 1991)
\[ \frac{\partial F}{\partial t} + \nu \frac{\partial F}{\partial z} - \Omega \frac{\partial F}{\partial \phi} + F_{\partial p} \partial F_{\partial p}^2 p^{-2} \frac{\partial}{\partial p} [p^2 h_p(t) F] + \frac{\partial}{\partial y_\alpha} [h_0(z, X, Y, p, \mu, \phi, t)] = 0, \]  

where \( y_\alpha \in [X, Y, \mu, \phi] \) and
\[ Q_0(z, X, Y, p, \mu, \phi, t) = S_0(z, X, Y, p, \mu, \phi, t) - N_0 F - R_0 F \]
accounts for sources and sinks \( (S_0) \) and the effects of the mirror force \( (N_0) \) and momentum loss processes \( (R_0) \), where the latter two operate on much longer spatial and timescales than the particle interactions with the stochastic fields. In Equation (8) we use the Einstein sum convention for indices and the short notation \( \delta_\alpha = (\partial/\partial x_\alpha) \). The four phase-space variables \( \mu, p, X, Y \) with non-vanishing stochastic fields \( h_i(t) \) are defined as \( x_{\nu, \sigma} \in [\mu, p, X, Y] \).

The particle distribution function, \( F(X, Y, z, p, \mu, \phi) \), varies in an irregular way under the influence of the stochastically fluctuating fields, \( h_i(t) \). However, we do not look for the detailed function \( F \), but rather for an ensemble-averaged solution, \( \langle F \rangle \), an expectation value of Equation (8), which results from averaging over different realizations of the fields \( h_i(t) \) with the same statistical averages. In the following treatment we will keep only first-order terms in the fluctuating quantities \( \delta F \)
\[ F = \langle F \rangle + \delta F \]  

and in the turbulent fields, \( h_i(t) \). In other words, we will consider the case of weak turbulence or the quasilinear approximation. Moreover, we will also neglect electric fields \( (h_E = 0) \). As shown in detail in Paper I, under the assumption of weak turbulence the ensemble-averaged solution, \( \langle F \rangle \), can be obtained by solving the kinetic equation
\[ \frac{\partial t}{t} \bigg| \frac{\partial F}{\partial t} + \nu \mu \frac{\partial F}{\partial z} - \Omega \frac{\partial F}{\partial \phi} \bigg| = Q_0(z, X, Y, p, \mu, \phi, t) - \frac{\partial}{\partial x_\alpha} \bigg| P_{a\sigma} \frac{\partial F}{\partial x_\sigma} \bigg| - \frac{\partial}{\partial y_\alpha} \bigg| \frac{\partial F}{\partial y_\alpha} \bigg|, \]  

where \( x_\nu \in [X, Y, \mu] \) and the Fokker–Planck coefficients are given as
\[ P_{a\sigma} = \langle h_a(t) \rangle \int_{t_0}^t ds \ h_a(s) \].

The time integration in Equation (12) is performed over a generalization of the unperturbed gyrocenter orbit in the uniform magnetic field with deviations of the gyrophase given by
\[ X_s = X, \quad Y_s = Y, \quad Z_s = Z + v \mu (s - t), \quad p_s = p, \quad \mu_s = \mu, \quad \phi_s = \phi - \Omega(s - t) + \delta \phi (t - s), \]
that contains the additional arbitrary gyrophase variation \( \delta \phi (t - s) \), with \( \delta \phi = 0 \) for \( s = t \).

Fourier transforming the stochastic force in space, the time integral in Equation (12) becomes
\[ \int_{t_0}^t ds \ h_a(s) = \int d^3k \int_{t_0}^t du \ H_a(k, s) \exp \bigg( i k \cdot \tilde{X} + \nu \mu k_0 (s - t) + i k \cdot v \sqrt{1 - \mu^2} \bigg) \int_0^s dw \cos (\psi - \phi + \Omega (w - t) - \delta \phi (w - t)) \bigg] \]
\[ (14) \]
where the particle position is given as
\[ \tilde{x}(s) = \begin{pmatrix} X + v \sqrt{1 - \mu^2} \int_0^s dw \cos (\phi - \Omega (w - t) + \delta \phi (t - w)) \\ Y + v \sqrt{1 - \mu^2} \int_0^s dw \sin (\phi - \Omega (w - t) + \delta \phi (t - w)) \\ Z + v \mu (s - t) \end{pmatrix} \]
\[ (15) \]
and where we have introduced cylindrical coordinates for the wavenumber vector \( \vec{k} = (k_\perp \cos \psi, k_\perp \sin \psi, k_\parallel) \) and the particle velocity.

As explained in detail in Paper I, in the small Larmor radius approximation (Chew et al. 1956; Kennel & Engelmann 1966) the distribution functions are independent of \( \phi \) to lowest order and can then be expanded as

\[
\langle F \rangle = F_0 + \frac{F_1}{\Omega}
\]  

(16)

and the Larmor phase-averaged equation becomes

\[
\partial_t F_0 + v \mu \partial_z F_0 - Q(z, X, Y, p, \mu, t) = -\frac{\partial}{\partial x_\sigma} D_{\sigma\sigma} \frac{\partial F_0}{\partial x_\sigma}
\]

(17)

with the gyroaveraged source term

\[
Q(z, X, Y, p, \mu, t) = \frac{1}{2\pi} \int_0^{2\pi} d\phi Q_0(z, X, Y, p, \mu, \phi, t),
\]

(18)

and the gyroaveraged Fokker–Planck coefficients

\[
D_{\sigma\sigma} = \Re \left[ \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_{0}^{t}\int_{0}^{t} ds \left( \langle h_\sigma(t)h_\sigma^*(s) \rangle \right) \right],
\]

(19)

where we have replaced \( h_\sigma(t) = h_\sigma^*(t) \) by its complex conjugate because the stochastic forces are real-valued quantities.

The generalization of the time integral in Equation (12) from the unperturbed motion of the gyrocenters in the guide magnetic field to arbitrary gyrophase motions of particles is possible essentially because of the gyrophase averaging in Equation (19). As demonstrated in Paper I the considered general particle gyrophase motion then only modifies the arguments of trigonometric and Bessel functions as compared to the quasilinear approximation of particle orbits.

3. DERIVATION OF FOKKER–PLANCK COEFFICIENTS FOR COMPRESSIBLE MAGNETIC TURBULENCE

Following the approach of Paper I we now derive the Fokker–Planck coefficients for the case of compressible magnetic turbulence. We make the following assumptions on the nature of the turbulence: the turbulence is quasi-stationary, meaning that the correlation function \( \langle h_\sigma(t)h_\sigma(s) \rangle \) depends only on the absolute value of the time difference \( |t - s| = |\tau| \), so that with the substitution \( s = t - \tau \) we find for Equation (19)

\[
D_{\sigma\sigma} = \Re \left[ \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_{0}^{t}\int_{0}^{t} ds \left( \langle h_\sigma(t)h_\sigma^*(s) \rangle \right) \right] = \Re \left[ \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_{0}^{t} ds \left( \langle h_\sigma(t)h_\sigma^*(t - \tau) \rangle \right) \right].
\]

(20)

As a second assumption we use the fact that the turbulent magnetic fields are homogeneously distributed, meaning that the particles are subject to turbulence realizations with equal statistical properties, independently of the actual position of the gyrocenter at time \( t \). This allows us to average the Fokker–Planck coefficients over the spatial position of the guiding center using

\[
\frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} d^3X \ e^{i(k - \vec{k})\vec{X}} = \delta(\vec{k} - \vec{\bar{k}}),
\]

(21)

implying that turbulence fields at different wavevectors are uncorrelated. As explained in detail in Paper I, the Fokker–Planck coefficients then become

\[
D_{\sigma\sigma} = \Re \left[ \frac{1}{2\pi} \int_0^{2\pi} d\phi \int d^3k \int_{0}^{t} d\tau \left( H_{\sigma}(\vec{k}, t)H_{\sigma}^*(\vec{k}, t - \tau) e^{i\mu k_\parallel \tau} \right) \times \exp \left[ -ik_\perp v \sqrt{1 - \mu^2} \left( \int_{t-\tau}^{t} dw \cos(\psi - \phi + \Omega(w - t) - \delta\phi(t - w)) + \frac{\sin(\phi - \psi)}{\Omega} \right) \right] \right].
\]

(22)

A third assumption concerns the nature of the particle orbits, i.e., we consider only orbits where \( \delta\phi(w) \) does not depend upon the fluctuating fields, so that the ensemble averaging in Equation (22) involves only the second-order correlation functions of the stochastic fields. This is generally called the Corrsin independence hypothesis (Corrsin 1959; Salu & Montgomery 1977; McComb 1990).

With \( \xi = t - w \) and the abbreviation

\[
G(\xi) = \Omega \xi + \delta\phi(\xi),
\]

(23)

the Fokker–Planck coefficients (22) then are

\[
D_{\sigma\sigma} = \Re \left[ \frac{1}{2\pi} \int_0^{2\pi} d\phi \int d^3k \int_{0}^{t} d\tau \langle H_{\sigma}(\vec{k}, t)H_{\sigma}^*(\vec{k}, t - \tau) \rangle \times \exp \left[ \nu \mu k_\parallel \tau + ik_\perp v \sqrt{1 - \mu^2} \left( \int_{t-\tau}^{t} d\xi \cos(\phi - \psi + G(\xi)) - \frac{\sin(\phi - \psi)}{\Omega} \right) \right] \right].
\]

(24)
Later we will also assume that the turbulence has a finite decorrelation time \( t_c \), such that the correlation functions \( \langle h_\nu(t) h_\nu^*(t - \tau) \rangle \to 0 \) fall to a negligible magnitude for \( \tau \to \infty \), so that the upper integration boundary in the \( \tau \)-integral can be replaced by infinity

\[
D_{i\sigma} = \Im \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_0^\infty d\tau \langle h_\nu(t) h_\nu^*(t - \tau) \rangle. \tag{25}
\]

We remark that diffusive transport of cosmic rays happens if the turbulence is quasi-stationary and has a finite decorrelation time \( t_c \), because the resulting gyroaveraged Fokker–Planck coefficients at large times \( t - t_0 \gg t_c \) no longer depend on the time difference \( t - t_0 \).

The equations of motion of the guiding center (Equation (7)) can be written as

\[
\frac{dX_i}{d\tau} = h_i(t) = v\mu \delta b_i(t) - v_i(t) \delta b_2(t), \tag{26}
\]

if

\[
v_i(t) = v\sqrt{1 - \mu^2} \cos \left( (i - 1) \frac{\pi}{2} - \phi \right), \tag{27}
\]

where \( i = [1, 2] \), whereas the pitch-angle random force (Equation (3)) is

\[
\frac{d\mu}{d\tau} = h_\mu(t) = \Omega \sqrt{1 - \mu^2} (\cos \phi \delta b_2(t) - \sin \phi \delta b_1(t)). \tag{28}
\]

### 3.1. Individual Gyroaveraged Fokker–Planck Coefficients

The Fourier transforms of the stochastic fields in Equations (26) and (28) are

\[
H_i(\vec{k}, t) = v\mu \delta b_i(\vec{k}, t) - v_i(t) \delta b_2(\vec{k}, t),
\]

\[
H_i(\vec{k}, t - \tau) = v\mu \delta b_i(\vec{k}, t - \tau) - v_i(t - \tau) \delta b_2(\vec{k}, t - \tau),
\]

\[
H_\mu(\vec{k}, t) = \Omega \sqrt{1 - \mu^2} (\cos \phi \delta b_2(\vec{k}, t) - \sin \phi \delta b_1(\vec{k}, t))
\]

\[
H_\mu(\vec{k}, t - \tau) = \Omega \sqrt{1 - \mu^2} (\cos \phi + G(\tau)) \delta b_2(\vec{k}, t - \tau) - \sin \phi + G(\tau)) \delta b_1(\vec{k}, t - \tau),
\]

where

\[
v_i(t) = v\sqrt{1 - \mu^2} \cos \left( (i - 1) \frac{\pi}{2} - \phi \right), \quad v_i(t - \tau) = v\sqrt{1 - \mu^2} \cos \left( (i - 1) \frac{\pi}{2} - (\phi + G(\tau)) \right).
\]

In terms of the magnetic field correlation tensor

\[
\langle b_i(\vec{k}, t) b_i^*(\vec{k}, t - \tau) \rangle = P_{ij}(\vec{k}, \tau), \tag{30}
\]

we then obtain for the perpendicular Fokker–Planck coefficients

\[
D_{ij} = \Im \frac{v^2}{2\pi} \int d^3k \int_0^{t - t_0} d\tau e^{ik\cdot\vec{x}_\tau} \int_0^{2\pi} d\phi \ H_{ij}(\vec{k}, \tau) \exp \left[ ik_z v \sqrt{1 - \mu^2} \left( \int_0^{\tau} d\xi \ \cos(\phi - \psi + G(\xi)) - \frac{\sin(\phi - \psi)}{\Omega} \right) \right], \tag{31}
\]

where

\[
H_{ij}(\vec{k}, \tau) = \mu^2 P_{ij}(\vec{k}, \tau) - \mu \sqrt{1 - \mu^2} \cos \left( (j - 1) \frac{\pi}{2} - (\phi + G(\tau)) \right) P_{ij}(\vec{k}, \tau) - \mu \sqrt{1 - \mu^2} \cos \left( (i - 1) \frac{\pi}{2} - \phi \right) P_{ij}(\vec{k}, \tau)
+ (1 - \mu^2) \cos \left( (i - 1) \frac{\pi}{2} - \phi \right) \cos \left( (j - 1) \frac{\pi}{2} - (\phi + G(\tau)) \right) P_{ij}(\vec{k}, \tau).
\]

The mixed Fokker–Planck coefficients are instead given as

\[
D_{ij\mu} = \Im \frac{v\Omega \sqrt{1 - \mu^2}}{2\pi} \int d^3k \int_0^{t - t_0} d\tau e^{ik\cdot\vec{x}_\tau} \int_0^{2\pi} d\phi \ H_{ij\mu}(\vec{k}, \tau)
\times \exp \left[ ik_z v \sqrt{1 - \mu^2} \left( \int_0^{\tau} d\xi \ \cos(\phi - \psi + G(\xi)) - \frac{\sin(\phi - \psi)}{\Omega} \right) \right], \tag{33}
\]

where

\[
H_{ij\mu}(\vec{k}, \tau) = \mu \cos(\phi + G(\tau)) P_{ij}(\vec{k}, \tau) - \mu \sin(\phi + G(\tau)) P_{ij}(\vec{k}, \tau) - \sqrt{1 - \mu^2} \cos \left( (i - 1) \frac{\pi}{2} - \phi \right) \cos(\phi + G(\tau)) P_{ij}(\vec{k}, \tau)
+ \sqrt{1 - \mu^2} \cos \left( (i - 1) \frac{\pi}{2} - \phi \right) \sin(\phi + G(\tau)) P_{ij}(\vec{k}, \tau)
\]
where
\[ H_{\mu i}(\vec{k}, \tau) = \mu \cos \phi P_{2z}(\vec{k}, \tau) - \mu \sin \phi P_{1\parallel}(\vec{k}, \tau) - \sqrt{1 - \mu^2} \cos \phi \cos \left( (i - 1) \frac{\pi}{2} - (\phi + G(\tau)) \right) P_{z\parallel}(\vec{k}, \tau) \]
\[ + \sqrt{1 - \mu^2} \sin \phi \cos \left( (i - 1) \frac{\pi}{2} - (\phi + G(\tau)) \right) P_{1\parallel}(\vec{k}, \tau). \]  

The Fokker–Planck coefficients parallel to the direction of the guide magnetic field are
\[ D_{\mu \mu} = \frac{\Omega^2 (1 - \mu^2)}{2\pi} \int d^3 k \int_0^{\tau_{-0}} d\tau e^{i\nu k_\tau} \int_0^{2\pi} d\phi H_{\mu \mu}(\vec{k}, \tau) \times \exp \left[ i k_\perp v_\perp \sqrt{1 - \mu^2} \left( \int_0^{\tau_\perp} d\xi \cos(\phi - \psi + G(\xi)) - \frac{\sin(\phi - \psi)}{\Omega} \right) \right], \]

where
\[ H_{\mu i}(\vec{k}, \tau) = \cos \phi \cos(\phi + G(\tau)) P_{2z}(\vec{k}, \tau) + \sin \phi \sin(\phi + G(\tau)) P_{1\parallel}(\vec{k}, \tau) - \sin \phi \cos(\phi + G(\tau)) P_{z\parallel}(\vec{k}, \tau) - \cos \phi \sin(\phi + G(\tau)) P_{2\parallel}(\vec{k}, \tau). \]  

The \( \phi \)-integrals are calculated in Appendix A of Paper I and yield
\[ D_{ij} = \frac{\eta v^2}{2} \int d^3 k \int_0^{\tau_{-0}} d\tau e^{i\nu k_\parallel} I_{ij}(\vec{k}, \tau), \]

where
\[ I_{ij}(\vec{k}, \tau) = \left[ 2\mu^2 P_{ij}(\vec{k}, \tau) + (-1)^{|i-j|} (1 - \mu^2) \cos \left( |i - j| \frac{\pi}{2} - G(\tau) \right) P_{z\parallel}(\vec{k}, \tau) \right] J_0(Z) \]
\[ - 2i \sqrt{1 - \mu^2} \left[ (-1)^{j-1} \sin \left( (i - 1) \frac{\pi}{2} - \left( \psi + \arcsin \left( \frac{Z_1}{Z} \right) \right) \right) P_{z\parallel}(\vec{k}, \tau) \right] \]
\[ + (-1)^{i-1} \sin \left( (j - 1) \frac{\pi}{2} - \left( \psi + G(\tau) + \arcsin \left( \frac{Z_1}{Z} \right) \right) \right) P_{1\parallel}(\vec{k}, \tau) \]
\[ + (-1)^{i-1} \left( i - 1 \right) (1 - \mu^2) P_{z\parallel}(\vec{k}, \tau) \cos \left( |i - j| \frac{\pi}{2} - \left( 2\psi + G(\tau) + \arcsin \left( \frac{Z_1}{Z} \right) \right) \right) J_2(Z), \]  

\[ D_{\mu \mu} = \frac{\eta v^2 \sqrt{1 - \mu^2}}{2} \int d^3 k \int_0^{\tau_{-0}} d\tau e^{i\nu k_\parallel} I_{\mu \mu}(\vec{k}, \tau), \]
where

\[ I_{\mu}(k, \tau) = \sqrt{1 - \mu^2} \left[ ( -1 )^{\mu} \sin \left( \left( i - 1 \right) \frac{\pi}{2} - G(\tau) \right) P_{iz}(k, \tau) - \cos \left( \left( i - 1 \right) \frac{\pi}{2} - G(\tau) \right) P_{z}(k, \tau) \right] J_0(Z) \]

\[ + 2 \mu \left[ \sin \left( \psi + \arcsin \left( \frac{Z_1}{Z} \right) \right) P_{2z}(k, \tau) + \cos \left( \psi + \arcsin \left( \frac{Z_1}{Z} \right) \right) P_{z}(k, \tau) \right] J_1(Z) \]

\[ + \sqrt{1 - \mu^2} \left[ ( -1 )^{\mu-1} \sin \left( \left( i - 1 \right) \frac{\pi}{2} - \left( 2 \psi + G(\tau) + \arcsin \left( \frac{Z_1}{Z} \right) \right) \right) \right] P_{iz}(k, \tau) \]

\[ - \cos \left( \left( i - 1 \right) \frac{\pi}{2} - \left( 2 \psi + G(\tau) + \arcsin \left( \frac{Z_1}{Z} \right) \right) \right) \right] P_{z}(k, \tau) \right] J_2(Z), \]

(42)

and

\[ D_{\mu\mu} = \Re \frac{\Omega^2(1 - \mu^2)}{2} \int d^3 k \int_{t_0}^{t_0} d\tau e^{i\psi k \cdot \tau} \]

\[ \times \left[ \left( \cos(G(t)) J_0(Z) - \cos \left( 2 \psi + G(\tau) + 2 \arcsin \left( \frac{Z_1}{Z} \right) \right) J_2(Z) \right) P_{11}(k, \tau) \]

\[ + \left( \cos(G(t)) J_0(Z) + \cos \left( 2 \psi + G(\tau) + 2 \arcsin \left( \frac{Z_1}{Z} \right) \right) J_2(Z) \right) P_{22}(k, \tau) \]

\[ - \left( \sin(G(t)) J_0(Z) + \sin \left( 2 \psi + G(\tau) + 2 \arcsin \left( \frac{Z_1}{Z} \right) \right) J_2(Z) \right) P_{21}(k, \tau) \]

\[ + \left( \sin(G(t)) J_0(Z) - \sin \left( 2 \psi + G(\tau) + 2 \arcsin \left( \frac{Z_1}{Z} \right) \right) J_2(Z) \right) P_{12}(k, \tau) \right], \]

(43)

respectively. In Equations (38)–(43) \( J_n(Z) \) denotes the Bessel function of the first kind and order \( n \),

\[ Z_1 = k_\perp v \sqrt{1 - \mu^2} \int d\xi \cos(G(\xi)) \]

(44)

and

\[ Z = k_\perp v \sqrt{1 - \mu^2} \left[ \left( \int d\xi \cos(G(\xi)) \right)^2 + \left( \frac{1}{\Omega} + \int d\xi \sin(G(\xi)) \right)^2 \right]^{1/2}. \]

(45)

4. AXISYMMETRIC TURBULENCE

Useful formulae can be obtained by assuming that the turbulence is asymmetric, meaning that \( P_{\alpha\beta}(k, \tau) \) are independent of the wave phase \( \psi \)

\[ P_{\alpha\beta}(k, \tau) = P_{\alpha\beta}(k_1, k_\perp, \tau). \]

(46)

The integration over \( \psi \) of the general formulae (38)–(43) then provides

\[ D_{ij} = \Re \pi v^2 \int_{-\infty}^{\infty} dk_i \int_{0}^{\infty} dk_\perp k_\perp \int_{t_0}^{t_0} d\tau e^{i\psi k_\perp \tau} J_0(Z) \]

\[ \times \left[ 2 \mu^2 P_{ij}(k_i, k_\perp, \tau) + ( -1 )^{\mu+j} ( 1 - \mu^2 ) \cos \left( |i-j| \frac{\pi}{2} - G(\tau) \right) P_{z}(k_i, k_\perp, \tau) \right], \]

(47)

\[ D_{ij} = \Re \pi v \Omega(1 - \mu^2) \int_{-\infty}^{\infty} dk_i \int_{0}^{\infty} dk_\perp k_\perp \int_{t_0}^{t_0} d\tau e^{i\psi k_\perp \tau} J_0(Z) \]

\[ \times \left[ \sin \left( ( i - 1 ) \frac{\pi}{2} - G(\tau) \right) P_{z}(k_i, k_\perp, \tau) - \cos \left( ( i - 1 ) \frac{\pi}{2} - G(\tau) \right) P_{z}(k_i, k_\perp, \tau) \right], \]

(48)

\[ D_{\mu\mu} = \Re \pi v \Omega(1 - \mu^2) \int_{-\infty}^{\infty} dk_i \int_{0}^{\infty} dk_\perp k_\perp \int_{t_0}^{t_0} d\tau e^{i\psi k_\perp \tau} J_0(Z) \]

\[ \times \left[ \cos(G(\tau)) P_{11}(k_i, k_\perp, \tau) + P_{z}(k_i, k_\perp, \tau) \right] + \sin(G(\tau))( P_{12}(k_i, k_\perp, \tau) - P_{21}(k_i, k_\perp, \tau) \right). \]

(49)

and

\[ D_{\mu\mu} = \Re \pi v^2 \Omega^2(1 - \mu^2) \int_{-\infty}^{\infty} dk_i \int_{0}^{\infty} dk_\perp k_\perp \int_{t_0}^{t_0} d\tau e^{i\psi k_\perp \tau} J_0(Z) \]

\[ \times \left[ \cos(G(\tau)) P_{11}(k_i, k_\perp, \tau) + P_{z}(k_i, k_\perp, \tau) \right] + \sin(G(\tau))( P_{12}(k_i, k_\perp, \tau) - P_{21}(k_i, k_\perp, \tau) \right). \]

(50)
Introducing the left-handed and right-handed polarized stochastic magnetic field components

\[ \delta b_{L,R} = \frac{1}{\sqrt{2}} [\delta b_1 \pm i \delta b_2], \]

so that

\[ 2P_{LL} = P_{11} + P_{22} + i P_{21} - i P_{12}, \quad 2P_{RR} = P_{11} + P_{22} + i P_{12} - i P_{21}, \]

we obtain for the pitch-angle Fokker–Planck coefficient (50)

\[
D_{\mu\mu} = \Re \Omega^2 (1 - \mu^2) \int_{-\infty}^{\infty} dk_\parallel \int_{-\infty}^{\infty} dk_\perp k_\parallel \int_{0}^{T} d\tau J_0(Z) [e^{i(v\mu k_\parallel + G(\tau))} P_{LL}(k_\parallel, k_\perp, \tau) + e^{i(v\mu k_\parallel - G(\tau))} P_{RR}(k_\parallel, k_\perp, \tau)].
\]

5. UPPER AND LOWER LIMITS OF THE GENERAL FOKKER–PLANCK COEFFICIENTS IN THE DIFFUSION LIMIT

If we now consider a magnetic field fluctuation decorrelation time \( t_c = \gamma^{-1} \) (Schlickeiser & Achatz 1993; Bieber et al. 1994)

\[ P_{ij}(\vec{k}, \tau) = P_{ij0}(\vec{k}) e^{-\gamma \tau}, \]

then in the diffusion limit \( t - t_0 \gg t_c \), the general Fokker–Planck coefficients (47)–(53) in asymmetric turbulence become

\[
D_{\mu\mu} = \Re \Omega^2 (1 - \mu^2) \int_{-\infty}^{\infty} dk_\parallel \int_{-\infty}^{\infty} dk_\perp k_\parallel \int_{0}^{T} d\tau J_0(Z) e^{-\gamma \tau} \times [\cos(v\mu k_\parallel + G(\tau)) P_{LL0}(k_\parallel, k_\perp) + \cos(v\mu k_\parallel - G(\tau)) P_{RR0}(k_\parallel, k_\perp)],
\]

and

\[
D_{ij} = \pi v^2 \int_{-\infty}^{\infty} dk_\parallel \int_{-\infty}^{\infty} dk_\perp k_\parallel \int_{0}^{T} d\tau J_0(Z) e^{i(v\mu k_\parallel - \gamma \tau)} \times \left[ 2 \mu^2 P_{ij0}(k_\parallel, k_\perp) + (1 - \mu^2) \frac{1}{2} (e^{iG(\tau)} + e^{-iG(\tau)}) P_{\perp\perp0}(k_\parallel, k_\perp) \right],
\]

Note that in Equations (55)–(58) we consider only the real part of the integral.

Because of the existence of the finite turbulence decorrelation time \( \gamma^{-1} \), the correlation functions \( P_{ij}(k_\parallel, k_\perp, \tau) \) fall to a negligible magnitude for \( \tau \rightarrow \infty \), allowing us to replace the upper integration boundary in the \( \tau \)-integrals in Equations (55)–(58) by infinity. We recover the diffusion limit which is valid for times \( t - t_0 \gg \gamma^{-1} \).

If \( J_0(A) \leq 1 \) and \( \cos(x) \leq 1 \), Equations (55)–(58) become

\[
D_{ij} < D_{ij}^{\text{max}} = \frac{v^2 \mu^2}{\gamma} 2\pi \int_{-\infty}^{\infty} dk_\parallel \int_{-\infty}^{\infty} dk_\perp k_\parallel \int_{0}^{\infty} d\tau J_0(Z) \cos(v\mu k_\parallel) e^{-\gamma \tau} \times \frac{v^2(1 - \mu^2)}{2} 2\pi \int_{-\infty}^{\infty} dk_\parallel \int_{0}^{\infty} dk_\perp k_\parallel P_{\perp\perp0}(k_\parallel, k_\perp),
\]

\[
\int_{0}^{\infty} d\tau J_0(Z) e^{-\gamma \tau} \times \frac{v^2(1 - \mu^2) \delta_{ij}}{2\gamma} + \frac{v^2(1 - \mu^2) \delta_{ij}}{2\gamma},
\]

and

\[
D_{\mu\mu} < D_{\mu\mu}^{\text{max}} = \frac{\pi \Omega(1 - \mu^2)}{2} 2\pi \int_{-\infty}^{\infty} dk_\parallel \int_{0}^{\infty} dk_\perp k_\parallel \int_{0}^{\infty} d\tau J_0(Z) e^{-\gamma \tau} \times \left[ \cos(v\mu k_\parallel + G(\tau)) + \cos(v\mu k_\parallel - G(\tau)) \delta_{ij} P_{\perp\perp0}(k_\parallel, k_\perp) \right] - \frac{\pi \Omega(1 - \mu^2)}{2\gamma} \left[ \delta_{ij} \epsilon \delta_{ij} - \delta_{ij} \epsilon \delta_{ij} \right],
\]
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spatial diffusion coefficients are given by the derivatives of the cosmic-ray Larmor radius. In particular, considering the case of a magnetic power spectrum of Alfvénic forces, \( k_\perp \frac{\mu_\perp}{\mu_\parallel} \), the ratio of the perpendicular mirror spatial diffusion coefficient to the parallel turbulent spatial diffusion coefficient is

\[
\frac{D_{\mu \mu}}{D_{\mu \mu}} \propto \delta b^2_{1T} - \delta b^2_{1z}.
\] (61)

According to the diffusion approximation (Schlickeiser 2002), neglecting the influence of the mirror force contribution \( \mathcal{N}_0 \) in Equation (8), the perpendicular spatial diffusion coefficients for the isotropic part of the cosmic-ray phase-space density are given by the pitch-angle average

\[
\kappa_{ij} = \frac{1}{2} \int_{-1}^{1} d\mu D_{ij}(\mu).
\] (64)

From Equation (59) we find the upper limits

\[
\kappa_{ij} < \kappa_{ij}^{\max} = \frac{v^2}{3\gamma} (\delta b^2_{ij} + \delta b^2_{zz} \delta_{ij}).
\] (65)

The parallel spatial diffusion coefficient for the isotropic part of the cosmic-ray phase-space density is given by the pitch-angle average

\[
\kappa_{\parallel} = \frac{v^2}{8} \int_{-1}^{1} d\mu \frac{(1 - \mu^2)^2}{D_{\mu \mu}(\mu)},
\] (66)

and its lower limit does not change with respect to the incompressible case examined in Paper I

\[
\kappa_{\parallel} > \kappa_{\parallel}^{\min} = \frac{\gamma' v^2}{3\Omega^2 [\delta b^2_{xx} + \delta b^2_{yy}]}.\] (67)

6. MIRROR FORCES AND TURBULENT SCATTERING IN THE SOLAR WIND PLASMA

Mirror forces are produced by large-scale spatial variations of the guide magnetic field. The perpendicular component of the mirror force generates gradients and curvature drifts of the cosmic-ray guiding center (Boyd & Sanderson 1969). In the presence of mirror forces the diffusion coefficients are given by the sum of the turbulent contribution, \( k^{(T)}_{\mu \nu} \), and of the contribution due to the mirror forces, \( k^{(M)}_{\mu \nu} \),

\[
k_{\mu \nu} = k^{(T)}_{\mu \nu} + k^{(M)}_{\mu \nu}.
\] (68)

We will now compare the effect of mirror forces and of the turbulent contribution, \( k^{(T)}_{\mu \nu} \), calculated in Section 5, on the properties of cosmic-ray transport in the solar wind plasma. For mirror forces, Schlickeiser & Jenko (2010) showed that in the case of a symmetric choice of the pitch-angle Fokker–Planck coefficients the ratio of the perpendicular mirror spatial diffusion coefficient to the parallel turbulent spatial diffusion coefficient is given by the derivatives of the cosmic-ray Larmor radius. In particular, considering the case of a magnetic power spectrum of Alfvénic slab turbulence \( P(k) \propto k^{-s} \) with \( s < 2 \), Schlickeiser & Jenko (2010) found that the ratios of the perpendicular to parallel non-zero spatial diffusion coefficients are

\[
\begin{align*}
\frac{k^{(M)}_{XX}}{k^{(T)}_{XX}} & = \frac{2 - s}{6 - s} \left( \frac{R_L}{3L_2} \right)^2, \\
\frac{k^{(M)}_{YZ}}{k^{(T)}_{YZ}} & = \frac{2 - s}{6 - s} \left( \frac{R_L}{3L_2} \right)^2, \\
\frac{k^{(M)}_{YY}}{k^{(T)}_{YY}} & = \frac{2 - s}{6 - s} \left( \frac{R_L}{3L_1} \right)^2.
\end{align*}
\] (69)
where the perpendicular magnetic field scale lengths are (Schlickeiser & Jenko 2010)

\[ L_1^{-1} = -B^{-1} \frac{\partial B}{\partial x}, \]
\[ L_2^{-1} = -B^{-1} \frac{\partial B}{\partial y}. \]  \( \text{(70)} \)

Summing the diagonal terms of the diffusion matrix we obtain

\[ \frac{\kappa_{XX}^{(M)} + \kappa_{YY}^{(M)}}{\kappa_{ZZ}^{(T)}} = \frac{2 - s}{6 - s} \left( \frac{R_L}{3} \right)^2 \left( \frac{1}{L_1^2} + \frac{1}{L_2^2} \right), \]  \( \text{(71)} \)

where the cosmic-ray gyroradius \( R_L \) is defined as

\[ R_L = \frac{v}{\Omega} = \frac{p_c}{ZeB_0}. \]  \( \text{(72)} \)

From Equation (71) we have

\[ \kappa_{\parallel}^{\min} \left( \kappa_{XX}^{(M)} + \kappa_{YY}^{(M)} \right) > v^2 \left( \frac{R_L}{3} \right)^4 \frac{2 - s}{6 - s} \left( \frac{1}{L_1^2} + \frac{1}{L_2^2} \right) \]  \( \text{(73)} \)

since

\[ \kappa_{\parallel}^{\min} = \frac{vR_L}{3}. \]  \( \text{(74)} \)

The relevant magnetic field random irregularities for the cosmic-ray transport properties are the fast magnetosonic waves (Lee & Völk 1975; Cho & Lazarian 2003). If we consider isotropic magnetosonic waves (Schlickeiser 2002)

\[ P_{xx} = \frac{g(k)}{8\pi k^2} \cos^2 \Theta, \]
\[ P_{yy} = \frac{g(k)}{8\pi k^2}, \]
\[ P_{zz} = \frac{g(k)}{8\pi k^2} \sin^2 \Theta, \]  \( \text{(75)} \)

then the random contributions to the field irregularities are

\[ \delta b_{xx}^2 = \frac{1}{4} \int_{-1}^{1} d\mu \mu^2 \int dk \, g(k) = \frac{1}{6} \int dk \, g(k), \]
\[ \delta b_{yy}^2 = \frac{1}{4} \int_{-1}^{1} d\mu \int dk \, g(k) = \frac{1}{2} \int dk \, g(k), \]
\[ \delta b_{zz}^2 = \frac{1}{4} \int_{-1}^{1} d\mu (1-\mu^2) \int dk \, g(k) = \frac{1}{3} \int dk \, g(k). \]  \( \text{(76)} \)

Using the upper and lower limits for the diffusion coefficients from random turbulent forces in Equations (65) and (67) and applying them for the case of fast magnetosonic waves, we have

\[ \kappa_{\parallel}^{(T)\min} \left( \kappa_{XX}^{(M)} + \kappa_{YY}^{(M)} \right) \geq \left( \frac{vR_L}{3} \right)^2 \left[ 1 + \frac{2\delta b_{zz}^2}{\delta b_{xx}^2 + \delta b_{yy}^2} \right] = 2 \left( \frac{vR_L}{3} \right)^2. \]  \( \text{(77)} \)

Taking the ratio of Equation (73) with Equation (77), we obtain a relation for the product of perpendicular diffusion coefficients independent of \( \kappa_{ZZ} \)

\[ \frac{\kappa_{XX}^{(M)} + \kappa_{YY}^{(M)}}{\kappa_{XX}^{(T)\max} + \kappa_{YY}^{(T)\max}} \geq \left( \frac{R_L}{3} \right)^2 \left( \frac{L_1^2 + L_2^2}{L_1^2 L_2^2} \right)^{2(6-s)} \frac{2 - s}{18(6-s)} \left( \frac{R_L}{\min[L_1, L_2]} \right)^2. \]  \( \text{(78)} \)

Perpendicular spatial diffusion is thus dominated by turbulent forces at low particle momenta, where the gyroradius is less than the minimum of the perpendicular magnetic field focusing lengths. Alternatively, at high momenta, where the gyroradius is larger than the minimum of the perpendicular magnetic field focusing lengths, perpendicular diffusion is dominated by the mirror force contribution.
7. SUMMARY AND CONCLUSIONS

In a large-scale magnetized plasma the description of cosmic-ray transport is given by the solution of the Vlasov equation for the particle distribution function. The influence of stochastically fluctuating fields on the particle distribution function can be studied by looking for an ensemble-averaged solution of the Vlasov equation, which results from averaging over different realizations of turbulent fields with the same statistical properties.

In the small Larmor approximation it was shown in Paper I that one can obtain the solution of the Vlasov equation for arbitrary gyrophase motions of the particles, extending the quasilinear approximation to the particle orbit. In Paper I the transport parameters of energetic charged particles in turbulent magnetized cosmic plasmas were derived for the case of an incompressible plasma, i.e., plasmas for which the component of the magnetic turbulence, \( \delta B_z = 0 \), parallel to the guide magnetic field, \( \vec{B}_0 = B_0 \vec{e}_z \), is set to zero. Here we have presented the generalization of the theory to the case of compressible magnetic turbulence with \( \delta B_x \neq 0 \). Under the assumption that the turbulence is quasi-stationary and homogeneous we have obtained the gyroaveraged Fokker–Planck coefficients for a Corrsin type of generalized orbits. For an axisymmetric turbulence we have derived upper and lower limits for the perpendicular and pitch-angle Fokker–Planck coefficients in the diffusion limit. We have shown upper and lower limits for the perpendicular and parallel spatial diffusion coefficients, respectively, describing the spatial diffusion of the isotropic part of the cosmic-ray phase-space density. Finally, using the upper and lower limits for the turbulent motion we have compared the effects on the transport of cosmic-ray particles of the turbulent and of mirror forces, if the latter cannot be neglected.

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APPENDIX

QUASILINEAR LIMIT

Following the approach in Paper I and assuming \( \delta \phi = 0 \) for the particle orbit in Equation (23)

\[
G(\xi) = \Omega \xi + \delta \phi(\xi),
\]

we obtain the quasilinear approximation to the particle orbits (Shalchi & Schlickeiser 2004). The argument of the Bessel functions of first kind

\[
Z = k_\perp v \sqrt{1 - \mu^2} \left[ \left( \int \xi d\xi \cos(G(\xi)) \right)^2 + \left( \frac{1}{\Omega} + \int \xi d\xi \sin(G(\xi)) \right)^2 \right]^{1/2}
\]

at order \( n = 0 \) becomes

\[
Z = Z_0 = k_\perp v \sqrt{1 - \mu^2} \frac{\sin^2(\Omega \tau) + (1 - \cos^2(\Omega \tau))^2}{\Omega} = \frac{k_\perp v \sqrt{1 - \mu^2}}{\Omega} \left[ 2 (1 - \cos(\Omega \tau)) \right]^{1/2} = \frac{2 k_\perp v \sqrt{1 - \mu^2}}{\Omega} \sin \left( \frac{\Omega \tau}{2} \right).
\]

The perpendicular Fokker–Planck coefficients (47) then become

\[
D_{ij}^{QL} = \Re \pi v^2 \int_0^\infty d\tau \int_0^\infty dk_\perp d k_\parallel e^{i k_\perp \tau \tau} J_0 \left( \frac{2 k_\perp v_\perp}{\Omega} \sin \left( \frac{\Omega \tau}{2} \right) \right) \times \left[ 2 \mu^2 P_i(k_1, k_\perp, \tau) + (1 - \mu^2) \cos \left( i - j \frac{\pi}{2} - \Omega \tau \right) P_{zz}(k_\parallel, k_\perp, \tau) \right].
\]

The mixed Fokker–Planck coefficients (48 and 49) are

\[
D_{ij}^{QL} = \Re \pi v \Omega (1 - \mu^2) \int_0^\infty dk_\perp \int_0^\infty dk_\parallel \int_0^{t - \Delta \Omega} d\tau e^{i \nu k_\perp \tau} J_0 \left( \frac{2 k_\perp v_\perp}{\Omega} \sin \left( \frac{\Omega \tau}{2} \right) \right) \times \left[ \sin \left( i - j \frac{\pi}{2} - \Omega \tau \right) P_{zz}(k_\parallel, k_\perp, \tau) \right]
\]

and

\[
D_{ii}^{QL} = \Re \pi v \Omega (1 - \mu^2) \int_0^\infty dk_\perp \int_0^\infty dk_\parallel \int_0^{t - \Delta \Omega} d\tau e^{i \nu k_\perp \tau} J_0 \left( \frac{2 k_\perp v_\perp}{\Omega} \sin \left( \frac{\Omega \tau}{2} \right) \right) \times \left[ (1 - \mu^2) \cos \left( i - j \frac{\pi}{2} - \Omega \tau \right) P_{zz}(k_\parallel, k_\perp, \tau) \right].
\]

whereas the Fokker–Planck coefficient (53) reduces to

\[
D_{ii}^{QL} = \Re \pi \Omega^2 (1 - \mu^2) \int_0^{t - \Delta \Omega} d\tau \int_0^\infty dk_\perp \int_0^\infty dk_\parallel J_0 \left( \frac{2 k_\perp v_\perp}{\Omega} \sin \left( \frac{\Omega \tau}{2} \right) \right) \times \left[ e^{i \nu k_\perp \Omega} P_{LL}(k_\parallel, k_\perp, \tau) + e^{i \nu k_\perp \Omega} P_{RR}(k_\parallel, k_\perp, \tau) \right].
\]
Note that the quasilinear Fokker–Planck coefficients in axisymmetric turbulence no longer involve infinite sums of products of Bessel functions which enormously aids their numerical computation for specified turbulence field correlation tensors.

We now explicitly calculate the different Fokker–Planck coefficients:

\[
D_{x\tau}^{QL} = 3\pi v^2 \int_0^\infty dt \int_{-\infty}^\infty dk_1 \int_{-\infty}^\infty dk_{\perp} k_{\perp} e^{i k_{\perp} \tau} J_0 \left( \frac{2 k_{\perp} v_1}{\Omega} \right) \sin \left( \frac{\Omega \tau}{2} \right) \times \left[ 2 \mu^2 P_{xx}(k_1, k_{\perp}, \tau) + (1 - \mu^2) \cos(\Omega \tau) P_{zz}(k_1, k_{\perp}, \tau) \right],
\] (A8)

\[
D_{y\tau}^{QL} = 3\pi v^2 \int_0^\infty dt \int_{-\infty}^\infty dk_1 \int_{-\infty}^\infty dk_{\perp} k_{\perp} e^{i k_{\perp} \tau} J_0 \left( \frac{2 k_{\perp} v_1}{\Omega} \right) \sin \left( \frac{\Omega \tau}{2} \right) \times \left[ 2 \mu^2 P_{xy}(k_1, k_{\perp}, \tau) + (1 - \mu^2) \sin(\Omega \tau) P_{zz}(k_1, k_{\perp}, \tau) \right],
\] (A9)

\[
D_{x\mu}^{QL} = 3\pi v^2 \int_0^\infty dt \int_{-\infty}^\infty dk_1 \int_{-\infty}^\infty dk_{\perp} k_{\perp} e^{i k_{\perp} \tau} J_0 \left( \frac{2 k_{\perp} v_1}{\Omega} \right) \sin \left( \frac{\Omega \tau}{2} \right) \times \left[ 2 \mu^2 P_{x\mu}(k_1, k_{\perp}, \tau) - (1 - \mu^2) \sin(\Omega \tau) P_{zz}(k_1, k_{\perp}, \tau) \right],
\] (A10)

\[
D_{y\mu}^{QL} = 3\pi v^2 \int_0^\infty dt \int_{-\infty}^\infty dk_1 \int_{-\infty}^\infty dk_{\perp} k_{\perp} e^{i k_{\perp} \tau} J_0 \left( \frac{2 k_{\perp} v_1}{\Omega} \right) \sin \left( \frac{\Omega \tau}{2} \right) \times \left[ 2 \mu^2 P_{y\mu}(k_1, k_{\perp}, \tau) + (1 - \mu^2) \cos(\Omega \tau) P_{zz}(k_1, k_{\perp}, \tau) \right],
\] (A11)

\[
D_{\mu\tau}^{QL} = 3\pi v^2 \int_0^\infty dt \int_{-\infty}^\infty dk_1 \int_{-\infty}^\infty dk_{\perp} k_{\perp} e^{i k_{\perp} \tau} J_0 \left( \frac{2 k_{\perp} v_1}{\Omega} \right) \sin \left( \frac{\Omega \tau}{2} \right) \times \left[ - \sin(\Omega \tau) P_{x\tau}(k_1, k_{\perp}, \tau) + \cos(\Omega \tau) P_{y\tau}(k_1, k_{\perp}, \tau) \right],
\] (A12)

\[
D_{\mu\mu}^{QL} = 3\pi v^2 \int_0^\infty dt \int_{-\infty}^\infty dk_1 \int_{-\infty}^\infty dk_{\perp} k_{\perp} e^{i k_{\perp} \tau} J_0 \left( \frac{2 k_{\perp} v_1}{\Omega} \right) \sin \left( \frac{\Omega \tau}{2} \right) \times \left[ \cos(\Omega \tau) P_{x\mu}(k_1, k_{\perp}, \tau) + \sin(\Omega \tau) P_{y\mu}(k_1, k_{\perp}, \tau) \right],
\] (A13)

\[
D_{\tau\tau}^{QL} = 3\pi v^2 \int_0^\infty dt \int_{-\infty}^\infty dk_1 \int_{-\infty}^\infty dk_{\perp} k_{\perp} e^{i k_{\perp} \tau} J_0 \left( \frac{2 k_{\perp} v_1}{\Omega} \right) \sin \left( \frac{\Omega \tau}{2} \right) \times \left[ \sin(\Omega \tau) P_{x\tau}(k_1, k_{\perp}, \tau) - \cos(\Omega \tau) P_{y\tau}(k_1, k_{\perp}, \tau) \right],
\] (A14)

\[
D_{\tau\mu}^{QL} = 3\pi v^2 \int_0^\infty dt \int_{-\infty}^\infty dk_1 \int_{-\infty}^\infty dk_{\perp} k_{\perp} e^{i k_{\perp} \tau} J_0 \left( \frac{2 k_{\perp} v_1}{\Omega} \right) \sin \left( \frac{\Omega \tau}{2} \right) \times \left[ \cos(\Omega \tau) P_{x\tau}(k_1, k_{\perp}, \tau) - \sin(\Omega \tau) P_{y\tau}(k_1, k_{\perp}, \tau) \right].
\] (A15)

Using the Bessel function addition theorem (see Appendix in Paper I) with \( r_1 = r_2 = 1, \lambda = k_{\perp} v_1 / \Omega, \) and \( \theta = \Omega \tau \) then

\[
J_0(Z_\theta) = J_0 \left( \frac{k_{\perp} v_1}{\Omega} \right) \frac{2(1 - \cos(\Omega \tau))^{1/2}}{\theta} = \sum_{n=-\infty}^{\infty} J_n^2 \left( \frac{k_{\perp} v_1}{\Omega} \right) e^{i \Omega \tau},
\] (A16)

so that the perpendicular Fokker–Planck coefficients can be written as

\[
D_{x\tau}^{QL} = 3\pi v^2 \int_0^{r_{-b}} dt \int_{-\infty}^\infty dk_1 \int_{-\infty}^\infty dk_{\perp} k_{\perp} \left[ 2 \mu^2 P_{xx}(k_1, k_{\perp}, \tau) \sum_{n=-\infty}^{\infty} e^{i(k_{\perp} v_1 - n \Omega \tau)} J_n^2 \left( \frac{k_{\perp} v_1}{\Omega} \right) + \frac{1}{2}(1 - \mu^2) \right]
\]
\times \sum_{n=-\infty}^{\infty} e^{i(k_{\perp} v_1 - n \Omega \tau)} \left( J_{n-1}^2 \left( \frac{k_{\perp} v_1}{\Omega} \right) + J_{n+1}^2 \left( \frac{k_{\perp} v_1}{\Omega} \right) \right) P_{zz}(k_1, k_{\perp}, \tau),
\] (A17)

\[
D_{y\tau}^{QL} = 3\pi v^2 \int_0^{r_{-b}} dt \int_{-\infty}^\infty dk_1 \int_{-\infty}^\infty dk_{\perp} k_{\perp} \left[ 2 \mu^2 P_{xy}(k_1, k_{\perp}, \tau) \sum_{n=-\infty}^{\infty} e^{i(k_{\perp} v_1 - n \Omega \tau)} J_n^2 \left( \frac{k_{\perp} v_1}{\Omega} \right) + \frac{1}{2i}(1 - \mu^2) \right]
\]
\times \sum_{n=-\infty}^{\infty} e^{i(k_{\perp} v_1 - n \Omega \tau)} \left( J_{n-1}^2 \left( \frac{k_{\perp} v_1}{\Omega} \right) - J_{n+1}^2 \left( \frac{k_{\perp} v_1}{\Omega} \right) \right) P_{zz}(k_1, k_{\perp}, \tau),
\] (A18)
Using the addition theorem (A16) the pitch-angle Fokker–Planck coefficient (A7) becomes

\[ D^{QL}_{\mu \mu} = \Re \pi v^2 \int_0^{t-t_0} d\tau \int_0^{\infty} dk_\parallel \int_0^{\infty} dk_\perp k_\perp \left[ 2\mu^2 P_{yy}(k_\parallel, k_\perp, \tau) \sum_{n=-\infty}^{\infty} e^{i(k_\parallel - n\Omega)\tau} J_n^2 \left( \frac{k_\perp v_\perp}{\Omega} \right) + \frac{1}{2} (1 - \mu^2) \right. \]

\[ \times \sum_{n=-\infty}^{\infty} e^{i(k_\parallel - n\Omega)\tau} \left( J_{n-1}^2 \left( \frac{k_\perp v_\perp}{\Omega} \right) - J_{n+1}^2 \left( \frac{k_\perp v_\perp}{\Omega} \right) \right) P_{zz}(k_\parallel, k_\perp, \tau) \].

(A19)

\[ D^{QL}_{\mu \mu} = \Re \pi v^2 \int_0^{t-t_0} d\tau \int_0^{\infty} dk_\parallel \int_0^{\infty} dk_\perp k_\perp \left[ 2\mu^2 P_{yy}(k_\parallel, k_\perp, \tau) \sum_{n=-\infty}^{\infty} e^{i(k_\parallel - n\Omega)\tau} J_n^2 \left( \frac{k_\perp v_\perp}{\Omega} \right) + \frac{1}{2} (1 - \mu^2) \right. \]

\[ \times \sum_{n=-\infty}^{\infty} e^{i(k_\parallel - n\Omega)\tau} \left( J_{n-1}^2 \left( \frac{k_\perp v_\perp}{\Omega} \right) - J_{n+1}^2 \left( \frac{k_\perp v_\perp}{\Omega} \right) \right) P_{zz}(k_\parallel, k_\perp, \tau) \].

(A20)

The pitch-angle coefficient in (A21) does not change with respect to the case of incompressible plasma treated in Paper I. As remarked in Paper I the pitch-angle coefficient agrees exactly with Equation (12.2.5) of Schlickeiser (2002) for incompressible, axisymmetric turbulence.

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