Fisher Transformation via Edgeworth Expansion

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Abstract

We show how to calculate individual terms of the Edgeworth series to approximate the distribution of the Pearson correlation coefficient with the help of a simple Mathematica program. We also demonstrate how to eliminate the corresponding skewness, thus making the approximation substantially more accurate. This leads, in a rather natural way, to deriving a superior (in terms of its accuracy) version of Fisher’s $z$ transformation. The code can be easily modified to deal with any sample statistics defined as a function of several sample means, based on a random independent sample from a multivariate distribution.

Keywords: Edgeworth series, Pearson correlation, Fisher transformation, moment generating function

1 Introduction

This article is an extension of the original work of \[1, 2, 3\] (the last one correctly discounting the importance of ‘variance stabilization’), with the aim of explicitly deriving all terms necessary to achieve an $O(n^{-3/2})$ accuracy of the desired approximation. The resulting formula then becomes substantially more accurate than those found in the existing literature.

One of the most natural ways to derive the Fisher transformation of the empirical correlation coefficient $r$ (assuming a sample from a bivariate Normal distribution) is to start by finding the first few terms of the Edgeworth expansion \[4\] of the sampling distribution of an arbitrary function of $r$ say $G(r)$. This is done by standardizing $G(r)$ by an $O(n^{-2})$-accurate (in terms of its error) expected value $m$ and $O(n^{-3})$-accurate variance $V$, and finding
its $O(n^{-3/2})$-accurate skewness $\Gamma_3$ and $O(n^{-2})$-accurate excess (subtracting 3) kurtosis $\Gamma_4$ (where $n$ is the sample size). The approximate probability density function (PDF) of

$$Z := \frac{G(r) - m}{\sqrt{V}}$$

is then given by

$$f_Z(z) := \exp\left(-\frac{z^2}{2}\right) \times \left(1 + \frac{\Gamma_3(z^3 - 3z)}{6} + \frac{\Gamma_4(z^4 - 6z^2 + 3)}{24} + \frac{\Gamma_4^2(z^6 - 15z^4 + 45z^2 - 15)}{72}\right)$$

and has an $O(n^{-3/2})$-proportionate error (compared to an $O(n^{-1/2})$ error of the basic Normal approximation).

It is then possible to find $G$ (based on the resulting differential equation) to make $\Gamma_3$ equal to zero (to the $O(n^{-3/2})$ level of accuracy) for any value of the ‘true’ correlation coefficient $\rho$. This yields the expected ‘arctanh’ transformation, but also suggests a subtle correction to it, making the resulting approximation substantially more accurate.

## 2 Key concepts and formulas

In this section we assume sampling from any specific multivariate distribution; to simplify the notation, our definitions and examples will use a tri-variate case and call the random variables $X_1, X_2$ and $X_3$, (generalizing is easy). The corresponding central moments are then defined by

$$\mu_{i,j,k} := \mathbb{E}[(X_1 - \mu_1)^i(X_2 - \mu_2)^j(X_3 - \mu_3)^k]$$

where $i + j + k$ is the moment’s order. They can be conveniently computed based on the moment generating function (MGF) of the corresponding centralized random variables, namely from

$$M(t_1, t_2, t_3) := \mathbb{E}[\exp(t_1(X_1 - \mu_1) + t_2(X_2 - \mu_2) + t_3(X_3 - \mu_3))]$$

$$\equiv 1 + \sum_{i+j+k \geq 2} \frac{\mu_{i,j,k}}{i!j!k!} t_1^i t_2^j t_3^k$$

by differentiating it with respect to $t_1$, $t_2$ and $t_3$ correspondingly $i, j$ and $k$ times, and then setting each $t_\ell$ equal to zero. If an explicit formula for such an MGF cannot be found (the integration may have no analytic answer), it is sufficient to replace it by the last line of (4), with the summation truncated to exclude terms beyond the fourth order (computationally more feasible).

It is well known (and easy to prove) that, when replacing the individual random variables by their respective sample means, the MGF of the new set, namely of $\overline{X}_1 - \mu_1, \overline{X}_2 - \mu_2,$
\( \overline{X_3} - \mu_3 \), is given by

\[
M \left( \frac{t_1}{n}, \frac{t_2}{n}, \frac{t_3}{n} \right)^n \quad (5)
\]

This implies that finding

\[
\bar{\mu}_{i,j,k} := \mathbb{E} \left[ (\overline{X_1} - \mu_1)^i (\overline{X_2} - \mu_2)^j (\overline{X_3} - \mu_3)^k \right] \quad (6)
\]

can be achieved by the same \( i, j, k \)-fold differentiation of (5) and subsequent \( t_\ell = 0 \) substitution as before. Thus, for example, we get

\[
\bar{\mu}_{2,1,1} = \frac{\mu_{2,0,0}\mu_{0,1,1} + 2\mu_{1,1,0}\mu_{1,0,1} + \mu_{2,1,1} - \mu_{2,0,0}\mu_{0,1,1} - 2\mu_{1,1,0}\mu_{1,0,1}}{n^2} \quad (7)
\]

e.tc. Not surprisingly, the complexity of these formulas increases ‘exponentially’ with the moment’s order.

To find an \( O(n^{-3/2}) \)-accurate approximation to the PDF of any function of sample means, say \( H(\overline{X_1}, \overline{X_2}, \overline{X_3}) \), we need to re-write this function as

\[
H(\mu_1 + \varepsilon(\overline{X_1} - \mu_1), \mu_2 + \varepsilon(\overline{X_2} - \mu_2), \mu_3 + \varepsilon(\overline{X_3} - \mu_3)) \quad (8)
\]

and expand it in \( \varepsilon \), up to and including \( \varepsilon^3 \) terms (with the understanding that \( \varepsilon \) will be set to 1 eventually); note that the first (constant) term of this expansion is \( H(\mu_1, \mu_2, \mu_3) \).

We then compute the expected value of the first four powers of the result, after subtracting \( H(\mu_1, \mu_2, \mu_3) \) from it (to simplify the corresponding algebra). This requires further expanding of these powers in \( \varepsilon \), up to and including \( \varepsilon^2, \varepsilon^4, \varepsilon^4 \) and \( \varepsilon^6 \) terms respectively, before applying (6) to the individual terms.

The resulting four moments of \( H(\overline{X_1}, \overline{X_2}, \overline{X_3}) - H(\mu_1, \mu_2, \mu_3) \) are then easily converted to the corresponding \( m, V, \Gamma_3 \) and \( \Gamma_4 \); these should be further simplified by keeping only the leading terms of their \( \frac{1}{n} \) expansion (with the exception of \( V \), where both \( \frac{1}{n} \) and \( \frac{1}{n^2} \) proportional terms are needed). This ensures that all terms contributing to the final \( O(n^{-3/2}) \) accuracy of the final answer are included, while the rest of them (often incorrect, since incomplete) have been eliminated.

### 3 Examples

Pearson’s correlation coefficient \( r \) is defined, using the \( \varepsilon \) notation of the previous section, as

\[
r := \frac{\rho + \varepsilon(\overline{XY} - \rho) - \varepsilon^2\overline{X} \cdot \overline{Y}}{\sqrt{\left(1 + \varepsilon(\overline{X^2} - 1) - \varepsilon^2\overline{X}^2\right) \left(1 + \varepsilon(\overline{Y^2} - 1) - \varepsilon^2\overline{Y}^2\right)}} \quad (9)
\]

Note that \( r \) is a function of five different sample means. To investigate its sampling distribution, we assume a random independent sample of size \( n \) from a bivariate Normal distribution
of $X$ and $Y$, with both means equal to 0 and both variances equal to 1, while the true (or ‘population’) correlation coefficient is $\rho$ (the $r$ distribution is the same whatever means and variances we use; we have thus made the simplest choice).

The MGF of the required five centralized variables, namely $X$, $Y$, $X^2 - 1$, $Y^2 - 1$ and $XY - \rho$, is the result of the following (rather routine) double integration

$$\int_{-\infty}^{\infty} \frac{\exp \left( -\frac{x^2 + y^2 - 2\rho xy}{2(1 - \rho^2)} + t_1 x + t_2 y + t_3 (x^2 - 1) + t_4 (y^2 - 1) + t_5 (xy - \rho) \right)}{2\pi \sqrt{1 - \rho^2}} \, dx \, dy$$

(10)

quoted in the Mathematica code of Figure 1 (which calls it $M$). This expression is then easily converted into the MGF of $\bar{X}$, $\bar{Y}$, $\bar{X}^2 - 1$, $\bar{Y}^2 - 1$ and $\bar{XY} - \rho$, and the first four moments of $r - \rho$ (no further transformation is applied to $r$ in this example) are found; the corresponding $m$, $V$, $\Gamma_3$ and $\Gamma_4$, properly truncated in their $\frac{1}{n}$ expansions, then easily follow. The complete program and its output are displayed in Figure 1.

The resulting approximation is than given by (1) and (2); the PDF of $Z$ can be easily converted to an approximate PDF of $r$ by

$$f(r) \simeq f_Z \left( \frac{r - m}{\sqrt{V}} \right)$$

(11)

Using $n = 35$ and $\rho = -0.85$, we show (in Figure 2 - the exact PDF is red, the approximate one is blue) how the approximation compares to the exact answer (which, rather atypically for a sample statistic of this complexity, has an analytic form - see [2]). The maximum error of this approximation, realized when computing $Pr(-0.9685 < r < -0.9133)$, is less than 0.8%.

3.1 Fisher transformation

Re-running the same program with the ‘$G[x_] := x$’ line removed yields, for $\Gamma_3$, the following expression

$$3G''[\rho] \left( (1 - \rho^2)G''[\rho] - 2\rho \, G'[\rho] \right) / \sqrt{n}$$

(12)

To make it zero (for any value of $\rho$), $G$ needs to be either a constant (which is clearly inadmissible) or a solution to $(1 - \rho^2)G''(\rho) - 2\rho \, G'(\rho) = 0$, namely $G(\rho) = \text{arctanh}(\rho)$, having chosen its simplest form (all other possibilities would yield the same $Z$).
\[ M = \text{Exp} \left[ \left( t_1^2 + t_2^2 + 2 \rho t_1 t_2 - \frac{1}{2} \left( 1 - \rho^2 \right) t_3 t_4 + t_1 t_4 - t_1 t_2 - t_3 t_2 \right) \right] / 2 / \left( 1 - 2 t_2 - 2 t_4 + 4 (1 - \rho^2) t_3 t_4 - 2 \rho t_3 t_4 - t_3 - t_4 - \rho t_3 \right) / \text{Sqrt} \left[ 1 - 2 t_2 - 2 t_4 + 4 (1 - \rho^2) t_3 t_4 - 2 \rho t_3 t_4 - t_3 - t_4 - \rho t_3 \right] \];

\[ \text{MGF} = (M \cdot t \cdot t / n)^{n} ; \]

\[ \mu \_i, j, k, l, m, n = \text{Collect} \left[ \text{D} \{ \text{MGF}, \{ t_1, l \}, \{ t_2, j \}, \{ t_3, k \}, \{ t_4, m \} \} / \text{t} \rightarrow 0, n, \text{Expand} \right] \]

\[ \rho = \frac{\mu \_i, j, k, l, m, n}{\sqrt{\left( 1 + \rho u_3 - \rho^2 u_3 \right) \cdot \left( 1 + \rho u_4 - \rho^2 u_4 \right)}} ; \]

\[ \text{EV} \left[ \sigma, \alpha \right] := \text{Expand} \left[ \mu \_i, j, k, l, m, n, \text{Collect} \left[ \text{Series} \left[ \sigma, \{ \alpha, 0, \alpha \} \right] \right. \right. // \text{Normal}, \alpha, \text{Factor} \right] / \mu \_i, j, k, l, m, n / \alpha \rightarrow 1 / \alpha \rightarrow \mu ; \]

\[ q \left[ \left\{ \text{EV} \left[ \left( \text{G} \left[ \text{G} \right] - \text{G} \left[ \rho \right] \right), 2 \right] ; \text{EV} \left[ \left( \text{G} \left[ \text{G} \right] - \text{G} \left[ \rho \right] \right)^2, 4 \right] ; \text{EV} \left[ \left( \text{G} \left[ \text{G} \right] - \text{G} \left[ \rho \right] \right)^3, 4 \right] ; \text{EV} \left[ \left( \text{G} \left[ \text{G} \right] - \text{G} \left[ \rho \right] \right)^4, 6 \right] \right\} \right] ; \]

\[ m = \text{Collect} \left[ \rho + q \left[ 1 \right], n, \text{Simplify} \right] \]

\[ \rho = \frac{\rho \left( -1 + \rho^2 \right)}{2 n} \]

\[ V = \text{Collect} \left[ \text{Series} \left[ \left( q \left[ 2 \right] - q \left[ 1 \right] \right)^2, \{ n, \text{Infinity}, 2 \} \right) // \text{Normal}, n, \text{Simplify} \right] \]

\[ V = \left( \frac{-1 + \rho^2}{n} \right)^2 + \left( \frac{-1 + \rho^2}{n} \right)^2 \left( 2 + 11 \rho^2 \right) / 2 n^2 \]

\[ \Gamma_3 = \text{Collect} \left[ \text{Series} \left[ \left( q \left[ 3 \right] - 3 q \left[ 2 \right] \cdot q \left[ 1 \right] + 2 q \left[ 1 \right] \right)^2 / \left( 1 - \rho^2 \right)^3, \{ n, \text{Infinity}, 1 \} \right) // \text{Normal}, n, \text{PowerExpand} \right] \]

\[ \Gamma_4 = \text{Collect} \left[ \text{Series} \left[ \left( q \left[ 4 \right] - 4 q \left[ 3 \right] \cdot q \left[ 1 \right] + 6 q \left[ 2 \right] q \left[ 1 \right]^2 - 3 q \left[ 1 \right]^4 \right) / V^2 - 3, \{ n, \text{Infinity}, 1 \} \right) // \text{Normal}, n, \text{Expand} \right] \]

\[ \frac{-6 - 72 \rho^2}{n} \]

Figure 1: Mathematica code
Running the same program one more time with ‘$G[x_] := \text{ArcTanh}[x]$’ then results in

$$m = \text{arctanh}(\rho) + \frac{\rho}{2n}$$
$$V = \frac{1}{n} + \frac{6 - \rho^2}{2n^2}$$
$$\Gamma_3 = 0 \quad \text{(by design)}$$
$$\Gamma_4 = \frac{2}{n}$$

This time

$$f(r) \simeq \frac{f_Z\left(\frac{\text{arctanh}(r) - m}{\sqrt{V}}\right)}{(1 - r^2)\sqrt{V}}$$

(13)

which, when using the previous choice of $n$ and $\rho$, yields a PDF visually indistinguishable from the exact answer; its maximum error (now involving a large interval of values - thus not the fairest of comparisons) is 0.36%. This accuracy is maintained (actually, rather fortuitously reduced to 0.05%) after dropping the $\Gamma_4$ term, whose contribution is (for this transformation of $r$) practically negligible, due to its relatively small size.

Note that the ‘basic’ Fisher transformation similarly ignores $\Gamma_4$, but it also drops the $\frac{1}{n}$-proportional correction to $m$ and uses $V = \frac{1}{n - 3}$ in place of our result. This affects, quite adversely, its accuracy, as seen in Figure 3.
Figure 3: Exact (red) and basic-Fisher (blue) PDF of $r$

The largest error of this approximation is over 3.5% - clearly inacceptable!

These two examples should suffice to illustrate using the Edgeworth series in situations going well beyond its original formulation and intended purpose.

4 Conclusion

We have delineated a procedure for constructing an accurate approximation for a PDF of any function of several sample means, when sampling a specific univariate or multivariate distribution. It is based on finding, to a specific accuracy (in terms of their $\frac{1}{n}$ expansions), the corresponding mean, variance, skewness and excess kurtosis; in the case of a single-parameter distribution, it is usually possible to find a transformation of the sample statistics which eliminates skewness, thus making the approximation both simpler and more accurate. We have tested the technique against the 'classical' example of Fisher transformation (suggesting a minor modification leading to a significant improvement), but its main applicability is to situations with no exact solution (constructing an approximate PDF is then the best we can do). We should mention that we have not attempted to optimize the algorithm (this would require introducing cumulants), but since our program takes only a few seconds to execute, this would not appear necessary in most cases of interest.
References

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