Asymptotic stabilization of a system of coupled \( n \)th-order differential equations with potentially unbounded high-frequency oscillating perturbations

R. Vrabel

1 Introduction

Suppose we are interested in assessing and ensuring the robustness of nominal control system \( \dot{x} = f(x, u) \) against modeling errors, system uncertainties, external disturbances, etc., represented by the perturbation term \( w(t, x) \) added to the right side of the nominal system,

\[
\dot{x} = f(x, u) + w(t, x), \quad t \geq 0. \tag{1}
\]

The state vector \( x \), control input \( u \), vector field \( f \) and perturbation term \( w \) are the vectors of suitable dimensions, provisionally let \( x \in \mathbb{R}^d, u \in \mathbb{R}^m, d \geq m \geq 1 \).

We always assume that \( f \) and \( w \) are at least continuous and that \( f(0,0) = 0 \). The perturbations \( w \) are assumed to be potentially unknown but belonging to the class of diminishing functions which also covers the high-frequency oscillating and among them also some unbounded perturbations (Definition 5 and Remark 1). Further, let us assume that the solutions \( x \) of (1) for each admissible control \( u \) are unique to the right, that is, \( x(t; t_0, x_0) \) is uniquely determined by \( (t_0, x_0) \) for \( t \geq t_0 \geq 0 \).

For a motivation, let us consider \( x = 0 \) being an uniform-asymptotically stable equilibrium point of the nominal system \( \dot{x} = f(x, u) \) for an appropriate state-feedback control \( u = g(x) \), \( g(0) = 0 \). What can we say about the stability of any kind of the perturbed system? This question represents one of the fundamental problems in various areas of robust stabilization of the con-
control systems, see, e.g., [4,17,18,32], and in principle, to answer this question, it makes usually a difference whether the origin remains an equilibrium for the perturbed system or not. If \( w(t,0) = 0 \), then the origin is an equilibrium of (1). In this case, then we can analyze the stability behavior of the origin as an equilibrium of the perturbed system. If \( w(t,0) \neq 0 \), then the origin is no longer an equilibrium of (1). In this case, we usually analyze the ultimate boundedness of the solutions of the perturbed system. As is shown in [14, Chapter 9] if for an appropriate choice of the control law \( u(t,x) \), the point \( x = 0 \) becomes an exponentially stable equilibrium point of the nominal system and the perturbation term \( w \) satisfies

\[
|w(t,x)| \leq \gamma(t)|x| + \eta(t), \quad \forall |x| < r, \quad \forall t \geq 0
\]  

(2)

where \( \gamma, \eta : [0, \infty) \rightarrow [0, \infty) \) are continuous, \( \int_0^\infty \gamma(t)dt < \infty \) and \( \eta \) is bounded, then for \( \eta \equiv 0 \), the origin is an exponentially stable equilibrium point of perturbed system, and the solutions of perturbed system are ultimately bounded in the opposite case (that is, if \( \eta \) is not identically zero). In contrast to the case of exponential stability, a nominal system with uniform-asymptotically stable (but not exponentially stable) origin is not robust to the smooth perturbations with arbitrarily small linear growth bounds of the form \( |w(t,x)| \leq \gamma|x|, |x| < r, \quad t \geq 0 \) and \( \gamma > 0 \), see [14] for more details.

These analyses are close to the notion input-to-state stability which has been introduced by E. Sontag in [23], see also [1] and [13], but this theory serves less interesting results for special types of perturbations considered in the present paper—(un)boundedness of solutions versus convergence of the solutions to the origin as \( t \to \infty \).

Definitions of the above concepts are given in the following section.

Summarizing these facts, the general framework for our considerations and analyses is that

1. we will assume the stabilizability of the nominal system \( \dot{x} = f(x,u) \) at \( x = 0 \) by a continuously differentiable state-feedback control \( u = g(x) \), \( g(0) = 0 \). This property is guaranteed as a side effect of the non-singularity assumption of Jacobian matrix of the function \( f \) with respect to the variable \( u \) at the point \( (0,0) \) (Theorem 1 and also Remark 2);

2. we will not assume that \( w(t,x) \) satisfies the inequality constraint of the form (2) and therefore the classical results of Khalil [14, Lemma 9.4, p. 352] based on the Lyapunov’s converse theorem, Coddington and Levinson [8, Theorem 3.1, p. 327], Hartman [11, Chapter X] both based on the state-space model representation, and their various variants (e.g., [5], [6, p. 183], [16,25]) are not applicable here in general. Moreover, for \( \eta(t) \) bounded, but nonvanishing at \( t = \infty \), we obtain the stronger result by considering the subclass of diminishing functions \( w(t,x) \) (Remark 1, Part(P2)), namely vanishing of \( x(t; t_0, x_0) \) at \( t = \infty \) versus boundedness of \( x(t; t_0, x_0) \) only. We will also consider the perturbations \( w(t) \) that are unbounded for \( t \to \infty \) (Example 1). Our approach come out from the impressive results and deep theory developed by Strauss and Yorke [24], whose results are obtained by a thorough and fine analysis of solution behavior.

It is a known fact that in the linear case, \( f(x,u) = Ax + Bu \), a necessary and sufficient condition for stabilization (by a linear state-feedback control law \( u = Gx \) and in the sense of the pole placement problem) is that rank of the controllability matrix \( [B, AB, \ldots, A^{d-1}B] \) is equal to the dimension of the state space, \( d \), which in turn is equivalent to the complete controllability in the open loop sense. The situation is quite different in the nonlinear case. The control system with dynamics

\[
f(x_1, x_2, x_3, u_1, u_2) = (u_1, u_2 \cos x_1, u_2 \sin x_1)
\]

is easily seen to completely controllable; however, the system cannot be stabilized to 0 by a \( C^1 \) state-feedback because of the Brockett’s necessary condition for feedback stabilization [7, p. 186]. To see that above example does not satisfy Brockett’s necessary condition, note that the points \((b,0,a)\) with \( a \neq 0 \) are not contained in the image of \( f \) for \( x_1 \) near 0—see also Remark 2 at the end of the paper. This is another demonstration of subtlety of the concept “stabilizability” for nonlinear systems.

This paper is an updated version of the previous preprint [27] and has been written in order to provide theoretical background for extending the existing results regarding the eventual uniform-asymptotic stabilizability of control systems at the origin (Definition 3) by a continuous state-feedback control law \( u = g(x) \) to a wider class of admissible perturbations \( w \), namely when the perturbing term is dimin-
ishing, in Theorem 1. Moreover, as is shown in the mentioned theorem, coming out from the perturbation theory developed for uncontrolled dynamical systems in [24], there is no room for further generalization if \( w(t, x) \) is time-dependent only, \( w(t, x) \equiv w(t) \) in the sense that “\( w = w(t) \) is diminishing” is a necessary and sufficient condition to ensure the respective error dynamics defined in Lemma 1 by a constant matrix \( A_H \) to approach zero as \( t \to \infty \) [24, Theorem B].

There do not seem to be any results for control systems affected by (potentially unbounded) high-frequency oscillating disturbance and so the objective of this paper is to contribute to fill gap that can be observed in the control literature at the theoretical and methodological level. Also the mathematical construction itself of the tracking error \( E(t) \) and the control law in Theorem 1 is not usual in control theory and that could be useful for general tracking problems, not only for the stabilizability ones (Remark 4).

2 Notations and definitions

Let 
\[ \mathbb{R}^d \]

denote the finite-dimensional Euclidean, \( d \)-space and let \( | \cdot | \) denote any \( d \)-dimensional norm, and we will use \( \| \cdot \| \) for the Euclidean norm. For later reference, recall that all norms on \( \mathbb{R}^d \) are equivalent [12, p. 273], so, \( \theta_1 | \cdot | \leq \| \cdot \| \leq \theta_2 | \cdot | \) for some positive constants \( \theta_1 \) and \( \theta_2 \) depending on \( | \cdot | \);
\[ \| \cdot \|_{op} \]
represents the operator norm induced by the norm \( | \cdot | \);
\[ \text{Diag}(A) \]

denotes the column vector of the main diagonal elements of the matrix \( A \);
for \( r > 0 \) and a fixed point \( x_0 \in \mathbb{R}^d \), \( B_r(x_0) \triangleq \{ x \in \mathbb{R}^d : |x - x_0| < r \} \);
the superscript ‘\( T \)’ is used to indicate transpose operator.

We now turn to the definitions of the stabilities stated for the closed loop system (1) with a general (continuous) feedback \( u = g(t, x) \) we will use throughout this paper, and that we have adopted and adapted from [24].

Definition 1 The origin is eventually uniformly stable (EvUS) if for every \( \varepsilon > 0 \), there exists \( \alpha = \alpha(\varepsilon) \geq 0 \) and \( \delta = \delta(\varepsilon) > 0 \) such that
\[ |x(t; t_0, x_0)| < \varepsilon \]
for all \( |x_0| < \delta \) and \( t \geq t_0 \geq \alpha \).

It is uniformly stable (US) if one can choose \( \alpha(\varepsilon) = 0 \).

Definition 2 The origin is eventually uniformly attracting (EvUA) if there exist \( \delta_0 > 0 \) and \( \alpha_0 \geq 0 \) and if for every \( \varepsilon > 0 \) there exists \( T = T(\varepsilon) \geq 0 \) such that
\[ |x(t; t_0, x_0)| < \varepsilon \]
for all \( |x_0| < \delta_0 \), \( t_0 \geq \alpha_0 \), and \( t \geq t_0 + T \).

It is uniformly attracting (UA) if one can choose \( \alpha_0 = 0 \).

Definition 3 The origin is eventually uniformly-asymptotically stable (EvUAS) if it is both EvUS and EvUA. It is uniformly-asymptotically stable (UAS) if it is both US and UA.

As has been proved in [24], the concepts EvUAS and UAS are equivalent if and only if \( x = 0 \) is a unique-to-the-right solution through \((t_0, 0)\) of (1) with \( u = g(t, x) \) defined on \([t_0, \infty)\). So EvUAS is a natural generalization of uniform-asymptotic stability in which it is not assumed that the zero function is a solution.

These definitions (defined in \( \varepsilon - \delta \) terms) are in fact equivalent to the following statements by using the special comparison functions known as class \( \mathcal{K} \) and class \( \mathcal{K}_\infty \) [14, p. 144 and also Lemma 4.5]:
\[ \text{EvUS} \Leftrightarrow (\exists \alpha \in \mathcal{K})(|x(t; t_0, x_0)| \leq \alpha(|x_0|), \forall |x_0| < \delta, \forall t \geq t_0 \geq \alpha) \]
and
\[ \text{EvUAS} \Leftrightarrow (\exists \beta \in \mathcal{K}_\infty)(|x(t; t_0, x_0)| \leq \beta(|x_0|, t - t_0), \forall |x_0| < \delta_0, \forall t \geq t_0 \geq \alpha_0). \]
The origin is globally EvUAS if and only if the last inequality is satisfied for any initial state \( x_0 (\delta_0 = \infty) \).

A special case of UAS, so-called exponential stability, arises when the class \( \mathcal{K}_\infty \) function \( \beta \) takes the form \( \beta(\tilde{r}, s) = \kappa \tilde{r} e^{-\mu s}, \kappa, \mu > 0 \).

Definition 4 Let \( w : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d \) be continuous. Then, \( w \) is vanishing at \( x = 0 \) if there exists \( t^* \geq 0 \) such that for all \( t \geq t^* \) is \( w(t, 0) = 0 \); and \( w \) is vanishing at \( t = \infty \) if there exists \( r^* > 0 \) such that for all \( |x| \leq r^* \) the function \( w(t, x) \to 0 \) for \( t \to \infty \).

Definition 5 [24] Let \( h : [0, \infty) \to \mathbb{R}^d \) be continuous. Then, \( h \) is diminishing if
\[ \sup_{0 \leq \lambda \leq 1} \left| \int_t^{t+\lambda} h(\tau) d\tau \right| \to 0 \text{ as } t \to \infty. \]
For example, if \( h(t) \to 0 \) as \( t \to \infty \) then \( h \) is diminishing [28]. But, vanishing of \( h(t) \) at \( t = \infty \) is a sufficient condition only, not a necessary one. Indeed, let us consider

\[
h(t) = \left( \cos(e^t), \sin(e^t), 0, \ldots, 0 \right).
\]

Then, \( h \) is diminishing; for any \( \lambda \geq 0 \), by integrating by parts with \( a = e^{-t} \) and \( b = e^t \cos(e^t) \), we get

\[
\left| \int_t^{t+\lambda} \cos(e^\tau) d\tau \right| \leq 2e^{-t}(1 + e^{-\lambda}) \leq 4e^{-t}.
\]

The same inequality holds for the second component of \( h \), and thus,

\[
\left\| \int_t^{t+\lambda} h(\tau) d\tau \right\| \leq \sqrt{32} e^{-t} \to 0,
\]

but \( \|h(t)\| = 1 \). The diminishing function may not be even bounded on \([0, \infty)\). For example,

\[
h(t) = \left( t \cos(t^4), t \sin(t^4), 0, \ldots, 0 \right)
\]

is diminishing as follows from the asymptotic properties of the Fresnel functions for large \( t \), see, e.g., [3, p. 149] or [29], and \( \|h(t)\| = t \to \infty \). In both cases, the functions \( h \) represent high-frequency oscillations, bounded and unbounded, respectively. These functions, depicted in Fig. 1, will be used later in Example 1 to demonstrate the effectiveness of the proposed controller.

**Remark 1**

**P1** For example, if \( h(t) \to 0 \) as \( t \to \infty \) then \( h \) is diminishing [28]. But, vanishing of \( h(t) \) at \( t = \infty \) is a sufficient condition only, not a necessary one. Indeed, let us consider

\[
h(t) = \left( \cos(e^t), \sin(e^t), 0, \ldots, 0 \right).
\]

Then, \( h \) is diminishing; for any \( \lambda \geq 0 \), by integrating by parts with \( a = e^{-t} \) and \( b = e^t \cos(e^t) \), we get

\[
\left| \int_t^{t+\lambda} \cos(e^\tau) d\tau \right| \leq 2e^{-t}(1 + e^{-\lambda}) \leq 4e^{-t}.
\]

The same inequality holds for the second component of \( h \), and thus,

\[
\left\| \int_t^{t+\lambda} h(\tau) d\tau \right\| \leq \sqrt{32} e^{-t} \to 0,
\]

but \( \|h(t)\| = 1 \). The diminishing function may not be even bounded on \([0, \infty)\). For example,

\[
h(t) = \left( t \cos(t^4), t \sin(t^4), 0, \ldots, 0 \right)
\]

is diminishing as follows from the asymptotic properties of the Fresnel functions for large \( t \), see, e.g., [3, p. 149] or [29], and \( \|h(t)\| = t \to \infty \). In both cases, the functions \( h \) represent high-frequency oscillations, bounded and unbounded, respectively. These functions, depicted in Fig. 1, will be used later in Example 1 to demonstrate the effectiveness of the proposed controller.

**P2** The concept of diminishing function can be naturally generalized for the functions depending also on \( x \), see [24, Definition 2.19] and the discussion after it. We restrict ourselves to the diminishing functions of the form \( w(t, x) = D(t)k(x) \), where each column of the \( d \times d \) matrix \( D \) is bounded on \([0, \infty)\) and diminishing in the sense of Definition 5 and vector function \( k : \mathbb{R}^d \to \mathbb{R}^d \) is continuous, that is, \( k(x) \equiv 1 \) is also allowed. The boundedness of the columns of \( D \) is required only if \( k \) is a non-constant function.

### 3 Application to the system of nth-order ODEs

In the framework given by the definitions above, our aim is to prove a new theorem on the eventually uniform-asymptotic stabilizability of the origin 0 for the controlled system of \( m \) \( n \)-th-order ordinary differential equations \((m \geq 1, n > 1)\), which we may think as a special case of the system (1).

\[
Y^{(n)} = F \left( Y, Y^{(1)}, \ldots, Y^{(n-1)}, U \right)
\]

\[
+ W \left( t, Y, Y^{(1)}, \ldots, Y^{(n-1)} \right), \quad t \geq 0,
\]

given that \( Y \in \mathbb{R}^m, U \in \mathbb{R}^m, F = (f_1, \ldots, f_m) \) is \( C^1 \) function from \( \mathbb{R}^{mn+m} \) to \( \mathbb{R}^m \), the perturbation \( W = (w_1, \ldots, w_m) \) is continuous from \([0, \infty) \times \mathbb{R}^{mn} \) to \( \mathbb{R}^m \), and \( Y^{(i)} \) denotes the \( i \)-th derivative with respect to the time \( t \), \( (i = 0, 1, \ldots, n - 1) \), and of course, we identify \( Y^{(0)} \) with \( Y \). For example, the Lagrange’s equations in mechanics produce \( m \) second-order differential
Asymptotic stabilization of a system of coupled ... 1425

 equations for an $m$—degree-of-freedom dynamical system [10, Chapter 1], [15, 19], [20, p. 158], [22, p. 211], [26, p. 435], [30,31].

 Associating $Y$ with $X_1$, and $Y^{(i)}$ with $X_{i+1}, i = 1, \ldots, n - 1$, we get the state variable matrix $X \triangleq [X_1, \ldots, X_n] = [Y, Y^{(1)}, \ldots, Y^{(n-1)}] \in \mathbb{R}^{m \times n}$ and the system (3) can be rewritten into the state-space representation

$$\dot{X}_i = X_{i+1}, \quad i = 1, \ldots, n - 1, \quad \dot{X}_n = F(X, U) + W(t, X).$$

(4)

Our main result is the following theorem.

**Theorem 1** Consider the control system (4). Let $F : \mathbb{R}^{mn+m} \to \mathbb{R}^m$ be a $C^1$ function, $F(0, 0) = 0$ and the corresponding Jacobian matrix with respect to the input variable $U$

$$J_{F, U}(0, 0) \triangleq \frac{\partial (f_1, \ldots, f_m)}{\partial (u_1, \ldots, u_m)}(0, 0)$$

is non-singular (and so bijective on $\mathbb{R}^m$). Let

$$W(t, X) = D(t) K(X),$$

(5)

where each column of an $m \times m$ matrix $D$ is bounded (for non-constant $K(X)$) and diminishing, and the vector function $K = (k_1, \ldots, k_m)^T : \mathbb{R}^m \to \mathbb{R}^m$ is continuous.

Then, there exists a $C^1$ state-feedback control law $U = G(X)$ defined in some open neighborhood of $0 \in \mathbb{R}^{mn}$, $G(0) = 0 \in \mathbb{R}^m$ such that $0 \in \mathbb{R}^{mn}$ is EvUAS for (4) with $U = G(X)$. For $W$ independent of the state variable $X$, the EvUAS of $0$ for (4) implies that $W(t)$ is diminishing. Moreover, if

(a1) for any $X \in \mathbb{R}^{mn}$ the functional $\Phi_X : \mathbb{R}^m \to \mathbb{R}$

$$\Phi_X(U) = \frac{1}{2} |F(X, U)|^2$$

is coercive, that is, $\lim_{|U| \to \infty} \Phi_X(U) = \infty$,

(a2) the Jacobian matrix $J_{F, U}(X^*, U^*)$ is bijective for any $(X^*, U^*) \in \mathbb{R}^{mn+m}$,

then the state-feedback control law $U = G(X)$ is defined globally, that is, for all $X \in \mathbb{R}^{mn}$.

**Proof** Let us define

- the matrix $X_d(t) \triangleq [X_{1,d}(t), \ldots, X_{n,d}(t)] = [Y_d(t), Y_d^{(1)}(t), \ldots, Y_d^{(n-1)}(t)] \in \mathbb{R}^{m \times n}$ that represents a desired trajectory of the control system (3)—for the purpose of the asymptotic stabilization of system, $X_d(t) \equiv 0$ (see also Remark 4 for a general case);

- the matrix $\Delta = X - X_d(t)$;

\[\square\]

and

- let the tracking error is given as

$$E \triangleq \text{Diag} \left( \begin{bmatrix} \Gamma \end{bmatrix} \right),$$

$$E = (e_1, \ldots, e_m)^T \in \mathbb{R}^m, \quad \begin{bmatrix} \Gamma \end{bmatrix} \in \mathbb{R}^{n \times m},$$

(6)

where each column of $\Gamma$ is such that the polynomials $\gamma_1, j + \gamma_2, j^2 + \cdots + \gamma_{n-1}, j^{n-2} + j^{n-1}, j = 1, \ldots, m$, have the roots which are either negative or pairwise conjugate with negative real parts; therefore, $\Delta(t) \to 0$ if $E(t) \to 0$ for $t \to \infty$; the $n$—tuple $(\gamma_1, j, \ldots, \gamma_{n-1}, j, 1)$ is the $j$th column of $\begin{bmatrix} \Gamma \end{bmatrix}$.

Differentiating (6), we obtain

$$\dot{E} = \text{Diag} \left( \begin{bmatrix} 0 \\ \Gamma \end{bmatrix} \right) + F(X, U) - Y^{(n)}_d(t)$$

$$+ W(t, X) \quad \text{here, } Y^{(n)}_d(t) \equiv 0,$$

and hence,

$$\dot{E} = \Delta H E + \text{Diag} \left( \begin{bmatrix} 0 \\ \Gamma \end{bmatrix} \right) + F(X, U) - A_H E$$

$$\triangleq \hat{F}(X, U) + W(t, X),$$

for provisionally arbitrary $m \times m$ constant matrix $A_H$. Because

$$\text{Diag} \left( \begin{bmatrix} 0 \\ \Gamma \end{bmatrix} \right) - A_H \text{Diag} \left( \begin{bmatrix} \Gamma \\ \Gamma \end{bmatrix} \right), \quad \Delta \equiv X$$

is independent of $U$, the Jacobian matrix $J_{F, U}(0, 0) = \text{Diag} \left( \begin{bmatrix} \Gamma \end{bmatrix} \right)$ and therefore is non-singular, also $\hat{F}(0, 0) = 0$. On the basis of the implicit function theorem, see, e.g., [21, p. 136], there exists a neighborhood $P$ of $0 \in \mathbb{R}^{mn}$, a neighborhood $\mathbb{Q} \subset \mathbb{R}^n$ and a class $C^1$ function $G : P \to Q$ such that $G(0) = 0$ and for all $(X_1, \ldots, X_n) \in P$ is $\hat{F}(X_1, \ldots, X_n, G(X_1, \ldots, X_n)) = 0$.

For a given fixed initial state $(t_0, X(t_0))$, the mapping between the vector’s $E(t)$ individual components
and the rows of $X(t; t_0, X(t_0))$ is one to one; therefore, $X$ can be expressed in terms of $E$ by the variation-of-parameter method, $X = X(E)$. The rest of the proof of the first part of theorem (local stabilizability property) is a consequence of the following lemma.

**Lemma 1** Let us consider the error dynamics

$$\dot{E}(t) = A_H E(t) + \tilde{W}(t, E), \quad t \geq 0, \quad E \in \mathbb{R}^m,$$  \hfill (7)

where all eigenvalues of the matrix $A_H \in \mathbb{R}^{m \times m}$ have negative real parts and $\tilde{W}(t, E) = D(t)K(X(E))$. Then, $0$ is globally $\nu$-UAS for (7).

**Proof** The statement of lemma follows from [24, Corollary 4.5 and 4.6, and Theorem A(i)] applied to (7).

We still need to ensure to be $X(t) \in P$ for all $t \geq t_0$. Let $B_{r\max}(0)$ is the maximal open ball in $P$. From Definition 1, $|E(t; t_0, E_0)| < \varepsilon$ for $t \geq t_0 \geq \alpha$ if

$$|E(t_0)| = \left| \text{Diag} \left( X(t_0) \left[ \begin{array}{c} \Gamma_1 \\ 1 \end{array} \right] \right) \right| < \delta_E(\varepsilon),$$

that is, for $|X(t_0)| < \delta_{E}^*$ for some $\delta_{E}^* = \delta_E^*(\Gamma, \varepsilon) > 0$, which can be calculated from the inequality

$$|E(t_0)| \leq \frac{1}{\theta_1} \left\| \text{Diag} \left( X(t_0) \left[ \begin{array}{c} \Gamma_1 \\ 1 \end{array} \right] \right) \right\| \leq \frac{\gamma^* \sqrt{m}}{\theta_1} \|X(t_0)\| \leq \frac{\gamma^* \theta_2 \sqrt{m}}{\theta_1} |X(t_0)|,$$

where $\gamma^* = \max \{ \text{the absolute value of } \gamma_{i,j}, 1, \ldots, n-1, \quad i = 1, \ldots, n-1, \quad j = 1, \ldots, m \}$, and so, $\delta_{E}^* = \frac{\theta_1}{\gamma^* \theta_2 \sqrt{m}} \delta_E(\varepsilon)$.

The sufficiently small $\varepsilon$ is chosen such that $|X(t)| < r_{\max}$ (for $t \geq t_0 \geq \alpha$) by estimating solutions to the system

$$\left\{ \begin{array}{l}
\text{Diag} \left( X \left[ \begin{array}{c} \Gamma_1 \\ 1 \end{array} \right] \right) = \}
\quad \text{Diag} \left( \left[ Y, Y^{(1)}, \ldots, Y^{(n-1)} \right] \left[ \begin{array}{c} \Gamma_1 \\ 1 \end{array} \right] \right) = E(t),
\quad |E(t)| < \varepsilon,
\end{array} \right.$$  \hfill (8)

which is for $E(t) \equiv 0$ globally exponentially stable. Thus, for the suitable constants $\kappa \geq 1$, $\mu_{\Gamma} > 0$ and $t \geq t_0$, coming out from the state-space representation of (8), we obtain the inequalities

$$|X(t)| \leq \kappa |X(t_0)| e^{-\mu_{\Gamma} (t-t_0)} + \int_{t_0}^{t} e^{-\mu_{\Gamma} (t-\tau)} |E(\tau)| d\tau$$

$$\leq \kappa |X(t_0)| e^{-\mu_{\Gamma} (t-t_0)} + \frac{\varepsilon}{\mu_{\Gamma}} \left( 1 - e^{-\mu_{\Gamma} (t-t_0)} \right)$$

Hence, $|X(t)| < r_{\max}$ if $|X(t_0)| < \frac{1}{\kappa} (r_{\max} - \varepsilon / \mu_{\Gamma}) \triangleq \delta_{X}^*(\Gamma, \varepsilon)$ and $\varepsilon < r_{\max} / \mu_{\Gamma}$. So, $X(t) \in P$ for $t \geq t_0 \geq \alpha$ if $|X(t_0)| < \min \{ \delta_{E}^*, \delta_{X}^* \}$.

The second part of theorem, the global stabilizability property, is the consequence of [9, Theorem 1] and the fact that a linear mapping on $\mathbb{R}^m$ given by the matrix $A_H$ is a globally Lipschitz function in the sense of definition in [24, Sect. 4] with the Lipschitz constant $L = \| A_H \|_{op}$ on the whole $\mathbb{R}^m$ and $W(t, X)$ of the form (5) is globally diminishing with regard to the variable $X$ [24, Definition 2.19], namely the mentioned definitions hold for the open balls $B_r(0)$ with center $E = 0 \in \mathbb{R}^m$ and $X = 0 \in \mathbb{R}^{mn}$ and each radius $r > 0$, respectively. The proof of Theorem 1 is complete. \hfill \Box

**Example 1** As an illustrative example, let us consider the error dynamics (7) in the form

$$\dot{E} = \left[ \begin{array}{cc} -1 & 2 \\ 0 & -1.5 \end{array} \right] E + \tilde{W}(t, E), \quad t \geq 0$$

with the diminishing perturbation term

$$\tilde{W}(t) = \left( 0.5t \sin(t^4), -t \cos(t^4) \right)^T$$

and

$$\tilde{W}(t, e_1, e_2) = \left( -e_2 \sin(e'), 2(e_1^{1/3} + e_2 + 1) \cos(e') \right)^T,$$

respectively. The time evolutions of error $E(t) = (e_1(t), e_2(t))^T$ with an initial value $E(0) = (-1, 1.5)^T$ for both cases are displayed in Fig. 2. Recall, that these perturbations do not satisfy the inequality (2), $\tilde{W}$ is unbounded in the first case and does not meet the inequality $|\tilde{W}(t, E)| \leq \gamma |E| + \eta$ for any $\gamma, \eta > 0$ in the neighborhood of $E = 0$ due to the "$e_1^{1/3}$" in the second one.

The paper will end with three remarks.

**Remark 2** It is now a classical result that there exists a linear and continuous stabilizing control law for $\dot{x} = f(x, u)$ with $f(0, 0) = 0$ provided the unstable modes of the linearized system are controllable and there exists
Asymptotic stabilization of a system of coupled ... 1427

\[
\dot{\mathbf{E}} = A_H \mathbf{E} + \tilde{W}(t, \mathbf{E})
\]

with the matrix 
\[
A_H = \begin{bmatrix}
-1 & 2 \\
0 & -1.5
\end{bmatrix}
\]

and the perturbing term
\[
\tilde{W}(t) = \begin{pmatrix}
0.5t \sin(t^4), -t \cos(t^4)
\end{pmatrix}^T
\]

(the top row) and
\[
\tilde{W}(t, e_1, e_2) = \begin{pmatrix}
-e_2 \sin(e') \\
2(e_1^{1/3} + e_2 + 1) \cos(e')
\end{pmatrix}^T
\]

(the bottom row), respectively.

Fig. 2 Time evolution of the error dynamics

\[
\dot{\mathbf{E}} = A_H \mathbf{E} + \tilde{W}(t, \mathbf{E})
\]

for the problem considered here, for \( n > 1 \) the number of state variables \((mn)\) is greater than the control inputs \((m)\). But from the specific form of nominal part of the system (4), \( \dot{x}_i = \tilde{F}_i(x_1, \ldots, x_{mn}, u_1, \ldots, u_m) \), \( i = 1, \ldots, mn \), the Jacobian matrix \( J_{\tilde{F}_i,x}(0, 0) \) is directly, after an appropriate rearranging of the rows, in the canonical controllability form [2, p. 283]. This fact together with a non-singularity of \( J_{F,u}(0, 0) \), allowing the transformation of input matrix to the required canonical form, ensures the controllability of the linear part of above system. Therefore, does not matter how are distributed the eigenvalues of \( J_{\tilde{F}_i,x}(0, 0) \) in the complex plane, all eigenvalues are controllable. These findings point to an alternative approach to the local asymptotic stabilization of nominal system by a linear state-feedback control law \( u = Gx \), where \( G \) is a suitable \( m \times mn \) constant matrix ensuring the asymptotic stability of linear part of closed loop system.

Remark 3 For the practical computations, especially for the large matrices, here can be useful the sufficient condition to be the Jacobian matrix \( J_{F,u}(0, 0) \) non-singular, given by the implication: If the matrix \( B = (b_{ij}) \in \mathbb{R}^{m \times m} \) is strictly diagonally dominant, that is,
\[ |b_{ii}| > \sum_{j=1}^{m} |b_{ij}| \text{ for all } i = 1, \ldots, m, \text{ then } B \text{ is non-singular.} \] This result is known as the Levy–Desplanques theorem [12, p. 349].

**Remark 4** As indicated in the first lines of the proof of Theorem 1, its basic idea can be used also for a general state-trajectory tracking problem, meaning that \( X_d(t) \) is not identically equal to \( 0 \in \mathbb{R}^{m \times n} \) on \([t_0, \infty)\), under the assumption that \( F(X_d(t), 0) = 0 \) for each \( t \geq t_0 \). In this non-stationary case, the control law \( U \) is calculated separately for every \( t \in [t_0, \infty) \) fixed. The function

\[
\tilde{F}(t, \Delta, U) = \text{Diag} \left( \Delta \begin{bmatrix} 0 \\ \Gamma \end{bmatrix} + F(\Delta + X_d(t), U) \right)
- \gamma_d^{(n)}(t) - A_H \text{Diag} \left( \Delta \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)
\]

and \( U = G(t, \Delta), \Delta = X - X_d(t), t \geq t_0 \). Globally defined stabilizing control law \( U \), that is, defined for all \( \Delta \in \mathbb{R}^{mn}(= P_1) \), eliminates the problems caused by possibly too-fast-shrinking \( \Delta \)-domain \( P = P_t \) of \( G \) as \( t \to \infty \).

### 4 Concluding remarks

In this paper, we discussed and analyzed the stabilizability problem of the control systems consisting of the coupled \( n \)-th-order differential equations, affected by the high-frequency oscillating perturbations (often interpreted as an exogenous disturbance) \( w(t, x) \) belonging to the class of diminishing functions, which are not necessarily bounded and vanishing at \( t = \infty \) or/and at \( x = 0 \) as is traditionally considered in the literature. Under easily verifiable assumptions given in Theorem 1, we have shown that there exists a \( C^1 \) state-feedback control law preserving (generalized) uniform-asymptotic stability of the closed loop equilibrium point \( x = 0 \) of the nominal (unperturbed) system.

### Compliance with ethical standards

**Conflicts of interest** The author declares that he has no conflict of interest.

### References

1. Agrachev, A.A., Morse, A.S., Sontag, E.D., Sussmann, H.J., Utkin, V.I.: Nonlinear and Optimal Control Theory: Lectures

given at the C.I.M.E. Summer School held in Cetraro, Italy, June 19–29, 2004 (Lecture Notes in Mathematics/C.I.M.E. Foundation Subseries), Springer (2008)

2. Antsaklis, P.J., Michel, A.N.: Linear Systems. Birkhauser, Boston (2nd Corrected Printing, Originally published by McGraw-Hill, Englewood Cliffs, NJ, 1997) (2006)

3. Bateman, H., Erdelyi, A., et al. (eds.): Higher Transcendental Functions, 2. Bessel Functions, Parabolic Cylinder Functions, Orthogonal Polynomials. McGraw-Hill, New York (1953)

4. Bagherzadeh, M.A., Askari, J., Ghaisari, J., Mojiri, M.: Robust asymptotic stability of parametric switched linear systems with dwell time. IET Control Theory Appl. 12(4), 477–483 (2018)

5. Brauer, F.: Nonlinear differential equations with forcing terms. Proc. Am. Math. Soc. 15, 758–765 (1964)

6. Brauer, F., Nohel, J.A.: The Qualitative Theory of Ordinary Differential Equations: An Introduction. Dover Publications Inc, New York (1969)

7. Brockett, R.W.: Asymptotic stability and feedback stabilization. In: Brockett, R.W., Millman, R.S., Sussmann, H.J. (eds.) Differential Geometric Control Theory, pp. 181–191. Birkhauser, Boston (1983)

8. Coddington, E.A., Levinson, N.: Theory of Ordinary Differential Equations. McGraw-Hill, New York (1955)

9. Galewski, M., Radulescu, M.: On a global implicit function theorem for locally Lipschitz maps via non-smooth critical point theory. Quaest. Math. 41(4), 515–528 (2018)

10. Goldstein, H., Poole, C., Safko, J.: Classical Mechanics, 3rd edn. Addison Wesley, Boston (2002)

11. Hartman, P.: Ordinary Differential Equations, 2nd edn. SIAM, Philadelphia (2002)

12. Horn, R.A., Johnson, C.R.: Matrix Analysis. Cambridge University Press, Cambridge (1990)

13. Kameneva, T., Nesic, D.: Input-to-state stabilization of nonlinear systems with quantized feedback. In: Proceedings of the 17th World Congress, The International Federation of Automatic Control Seoul, Korea, July 6–11, pp. 12480–12485, (2008)

14. Khalil, H.K.: Nonlinear Systems, 3rd edn. Prentice-Hall, Englewood Cliffs, NJ (2002)

15. Lacarbonara, W., Balachandran, B., Ma, J., Machado, J.A.T., Stepan, G.: Nonlinear dynamics and control. In: Proceedings of the First International Nonlinear Dynamics Conference (NODYCON 2019), Volume II, Springer, (2020)

16. Ladde, G.S.: Variational comparison theorem and perturbations of nonlinear systems. Proc. Am. Math. Soc. 52, 181–187 (1975)

17. Li, H., Wang, Y.: Robust stability and stabilisation of Boolean networks with disturbance inputs. Int. J. Syst. Sci. 48(4), 750–756 (2017)

18. Ma, R., Zhao, J., Dimirovski, G.M.: Backstepping design for global robust stabilisation of switched nonlinear systems in lower triangular form. Int. J. Syst. Sci. 44(4), 615–624 (2013)

19. Meehan, P.A.: Investigation of chaotic instabilities in railway wheel squeal. Nonlinear Dyn. (2020). https://doi.org/10.1007/s11071-020-05493-x

20. Murray, R.M., Li, Z., Sastry, S.S.: A Mathematical Introduction to Robotic Manipulation. CRC Press, Boca Raton (1994)
21. Shirali, S., Vasudeva, H.L.: Multivariable Analysis. Springer, London (2011). https://doi.org/10.1007/978-0-85729-192-9
22. Slotine, J.-J.E., Li, W.: Applied Nonlinear Control. Prentice Hall, Englewood Cliffs (1991)
23. Sontag, E.D.: Smooth stabilization implies coprime factorization. IEEE Trans. Automat. Contr. 34(4), 435–443 (1989)
24. Strauss, A., Yorke, J.A.: Perturbing uniform asymptotically stable nonlinear systems. J. Differ. Equ. 6, 452–483 (1969). https://doi.org/10.1016/0022-0396(69)90004-7
25. Struble, R.A.: A note on damped oscillations in nonlinear systems. SIAM Rev. 6(3), 257–259 (1964)
26. Vidyasagar, M.: Nonlinear System Analysis, 2nd edn. Prentice Hall, Englewood Cliffs (1993)
27. Vrabel, R.: Asymptotic stabilization of a system of coupled nth-order differential equations with potentially unbounded high-frequency oscillating perturbations, arXiv:1907.06720 [math.OC], (2019)
28. Vrabel, R.: Criterion for robustness of global asymptotic stability to perturbations of linear time-varying systems, (2019). arXiv:1903.00873v2 [math.DS]
29. Weisstein, E.W.: Fresnel Integrals. From MathWorld: A Wolfram Web Resource; http://mathworld.wolfram.com/FresnelIntegrals.html. Accessed 2 Feb 2020
30. Yan, Y., Zeng, J., Mu, J.: Complex vibration analysis of railway vehicle with tread conicity variation. Nonlinear Dyn. (2020). https://doi.org/10.1007/s11071-020-05498-6
31. Zhang, M., Zhang, Y., Ouyang, H., et al.: Adaptive integral sliding mode control with payload sway reduction for 4-DOF tower crane systems. Nonlinear Dyn. (2020). https://doi.org/10.1007/s11071-020-05471-3
32. Zhu, B., Ma, J., Zhang, Z., Feng, H., Li, S.: Robust stability analysis and stabilisation of uncertain impulsive positive systems with time delay. Int. J. Syst. Sci. 49(14), 2940–2956 (2018)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.