The hydrostatic approximation for the primitive equations by the scaled Navier–Stokes equations under the no-slip boundary condition

KEN FURUKAWA, YOSHIKAZU GIGA AND TAKAHITO KASHIWABARA

Dedicated to Professor Matthias Hieber on the occasion of his 60th birthday

Abstract. In this paper, we justify the hydrostatic approximation of the primitive equations in maximal $L^p$-$L^q$-settings in the three-dimensional layer domain $\Omega = \mathbb{T}^2 \times (-1, 1)$ under the no-slip (Dirichlet) boundary condition in any time interval $(0, T)$ for $T > 0$. We show that the solution to the $\epsilon$-scaled Navier–Stokes equations with Besov initial data $u_0 \in B^{s,p}_q(\Omega)$ for $s > 2 - 2/p + 1/q$ converges to the solution to the primitive equations with the same initial data in $E_1(T) = W^{1,p}(0, T; L^q(\Omega)) \cap L^p(0, T; W^{2,q}(\Omega))$ with order $O(\epsilon)$, where $(p, q) \in (1, \infty)^2$ satisfies $\frac{1}{p} \leq \min(1 - 1/q, 3/2 - 2/q)$ and $\epsilon$ has the length scale. The global well-posedness of the scaled Navier–Stokes equations by $\epsilon$ in $E_1(T)$ is also proved for sufficiently small $\epsilon > 0$. Note that $T = \infty$ is included.

1. Introduction

1.1. Problems and main results

We consider the primitive equations of the form

\[
(PE) \begin{cases}
\begin{aligned}
\partial_t u - \Delta u + u \cdot \nabla v + \nabla H \pi &= 0 \quad \text{in} \quad \Omega \times (0, \infty), \\
\partial_z \pi &= 0 \quad \text{in} \quad \Omega \times (0, \infty), \\
\text{div } u &= 0 \quad \text{in} \quad \Omega \times (0, \infty), \\
u &= 0 \quad \text{on} \quad \partial \Omega \times (0, \infty),
\end{aligned}
\end{cases}
\]

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where \( u = (v, w) \in \mathbb{R}^2 \times \mathbb{R} \) and \( \pi \) are the unknown velocity field and pressure field, respectively, \( \nabla_H = (\partial_x, \partial_y)^T \), and \( \Omega = \mathbb{T}^2 \times (-1, 1) \) for \( \mathbb{T} = \mathbb{R}/2\pi \mathbb{Z} \). By the divergence-free condition, \( w \) is given by the formula

\[
 w(x', x_3, t) = -\int_{-1}^{x_3} \text{div}_H v(x', \zeta, t) d\zeta d\zeta = \int_{x_3}^1 \text{div}_H v(x', \zeta, t) d\zeta;
\]

here, we invoked physically reasonable condition \( w(\cdot, \cdot, \pm 1, \cdot) = 0 \). The primitive equations are a fundamental model for geographical flow. Existence of a global weak solution to the primitive equations on the spherical shell with thickness \( a > 0 \) for \( L^2 \)-initial data was proved by Lions, Temam and Wang [29]. Local-in-time well-posedness was proved by Guillén-González, Masmoudi and Rodríguez-Bellido [21]. Although the global well-posedness of the three-dimensional Navier–Stokes equations is the well-known open problem, this problem has been solved by Cao and Titi [3]. Hieber and Kashiwabara [23] extended this result to prove global well-posedness for the primitive equations in \( L^p \)-settings. In these papers, boundary conditions are imposed no-slip (Dirichlet) on the bottom and slip (Neumann) on the top. Recently, the second and last authors together with Gries, Hieber and Hussein [15] obtained global-in-time well-posedness in maximal regularity spaces (mixed Lebesgue–Sobolev spaces) \( W^{1,p}(0, T; L^q(\Omega)) \cap L^p(0, T; W^{2,q}(\Omega)) \) for \( T > 0 \) and appropriate \( 1 < p, q < \infty \) under the Neumann, Dirichlet and Dirichlet–Neumann mixed boundary conditions.

Our aim in this paper is to give a rigorous justification of the derivation of the primitive equations under the Dirichlet boundary condition. We begin by explaining its derivation. Let us consider the anisotropic viscous Navier–Stokes equations in a thin domain of the form

\[
 (ANS) \begin{cases}
 \partial_t u - (\Delta_H + \epsilon^2 \partial_z^2) u + u \cdot \nabla u + \nabla \pi = 0 & \text{in } \Omega_\epsilon \times (0, \infty), \\
 \text{div} u = 0 & \text{in } \Omega_\epsilon \times (0, \infty),
\end{cases}
\]

where \( \Omega_\epsilon = (-\epsilon, \epsilon) \times \mathbb{T}^2 \). If \( \epsilon = 1 \), (ANS) is the usual Navier–Stokes equations.

The equations (ANS) are considered as a good model to describe the motion of an incompressible viscous fluid filled in a thin domain. Actually, if we put the Reynolds number to be equal to one, the apparent viscosity for the vertical direction must be of \( \epsilon^2 \)-order since the vertical length and velocity are of \( \epsilon \)-order. The primitive equations are formally derived from (ANS). We introduce new unknowns from the solution to (ANS) such that

\[
 u_\epsilon := (v_\epsilon, w_\epsilon),
 v_\epsilon(x, y, z, t) := v(x, y, \epsilon z, t),
 w_\epsilon(x, y, z, t) := w(x, y, \epsilon z, t)/\epsilon,
 \pi_\epsilon(x, y, z, t) := \pi(x, y, \epsilon z, t),
\]
where \( x, y \in \mathbb{T}, z \in (-1, 1) \) and \( t > 0 \). Then, \( (u_\epsilon, \pi_\epsilon) \) satisfy the scaled Navier–Stokes equations in the fixed domain

\[
\begin{align*}
\text{(SNS)} \quad & \quad \partial_t u_\epsilon - \Delta u_\epsilon + u_\epsilon \cdot \nabla u_\epsilon + \nabla H \pi_\epsilon = 0 \quad \text{in } \Omega \times (0, \infty), \\
\quad & \quad \epsilon^2 (\partial_t w_\epsilon - \Delta w_\epsilon + u_\epsilon \cdot \nabla w_\epsilon) + \partial_z \pi_\epsilon = 0 \quad \text{in } \Omega \times (0, \infty), \\
\quad & \quad \text{div } u = 0 \quad \text{in } \Omega \times (0, \infty).
\end{align*}
\]

Taking formally \( \epsilon \to 0 \) for the above equations, we get the primitive equations.

The Navier–Stokes equations (SNS) are well studied for \( \epsilon = 1 \) since the work of Leray [27]. In this paper, he considered a global weak solution in \( \Omega = \mathbb{R}^3 \). For a general domain, see Farwig et al. [7]. Fujita and Kato [9] constructed local strong solution for \( H^{1/2} \)-initial data. This result was extended to various domains and various function spaces; see, e.g., Ladyzenskaya [25], Kato [24], Giga and Miyakawa [17] for early developments. The reader is referred to the book of Lemarié-Rieusslet [26] and review articles by Farwig, Kozono and Sohr [8] and Gallagher [13] for recent developments. Many results can be extended for general \( \epsilon > 0 \), but it is not often written explicitly except in a book of Chemin, Desjardins, Gallagher and Grenier [4].

The rigorous justification of the primitive equations from the scaled Navier–Stokes equations was studied by Azérad and Guillén [2]. They obtained weak* convergence in the natural energy space \( L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \) for \( \Omega = \mathbb{T}^2 \times (-1, 1) \) and \( T > 0 \). Recently, Li and Titi [28] improved their result to get strong convergence by energy method with the aid of regularity of the solution to the primitive equations. The authors together with Hieber, Hussein and Wrona [11] extended Li and Titi’s result in maximal regularity spaces \( W^{1,p}(0, T; L^q(\mathbb{T}^3)) \cap L^p(0, T; W^{2,q}(\mathbb{T}^3)) \) with initial trace in the Besov space \( B_{q,p}^{2-2/p}(\mathbb{T}^3) \) for \( T > 0 \) and \( 1/p \leq \min(1-1/q, 3/2-2/q) \) by an operator theoretic approach. The case of \( p = q = 2 \) is corresponding to Li and Titi’s result. Note that the case of the torus corresponds to the Neumann boundary conditions on the top and bottom parts. In the work of Azérad and Guillén, they considered the case of mixed boundary conditions with the Dirichlet boundary condition on the bottom, while Li and Titi treated the case of the Neumann boundary condition only. As we already mentioned, the primitive equations are a model for geophysical flow. Although it is more physically natural to consider the case of Dirichlet–Neumann and Dirichlet boundary conditions, there were no results to justify the derivation of the primitive equations from the Navier–Stokes equations in a strong topology.

Let

\[
\begin{align*}
\mathcal{E}_1(T) &= \{ u \in W^{1,p}(0, T; L^q(\Omega)) \cap L^p(0, T; W^{2,q}(\Omega)); \text{div } u = 0, \ u|_{x_3=\pm 1} = 0 \}, \\
\mathcal{E}_0(T) &= \{ u \in L^p(0, T; L^q(\Omega)); \text{div } u = 0, \ u|_{x_3=\pm 1} = 0 \}, \\
\mathcal{E}_1^T(T) &= \left\{ \pi \in L^p(0, T; W^{1,q}(\Omega)); \int_\Omega \pi \, dx = 0 \right\}, \\
\mathcal{X}_\gamma &= \left\{ u \in B^{2-2/p}_{q,p}(\Omega); \text{div } u = 0, \ u|_{x=\pm 1} = 0 \right\}
\end{align*}
\]
be the initial trace space of $E_1(T)$, where $B_{q,p}^s(\Omega)$ denotes the $L^q$-Besov space of order $s$. In this paper, we frequently use $\| \cdot \|_{E_0(T)}$ as the norm of $L^p(0, T; L^q(\Omega))$ and $\| \cdot \|_{E_1(T)}$ as the norm of $W^{1,p}(0, T; L^q(\Omega)) \cap L^p(0, T; W^{2,q}(\Omega))$ to simplify the notation.

Let us seek the solution $U_\varepsilon = (V_\varepsilon, W_\varepsilon)$ to

$$
\begin{align*}
&\partial_t V_\varepsilon - \Delta V_\varepsilon + \nabla_H P_\varepsilon = F_H \quad \text{in } \Omega \times (0, T), \\
&\partial_t (\varepsilon W_\varepsilon) - \Delta (\varepsilon W_\varepsilon) + \frac{\partial}{\partial t} P_\varepsilon = \varepsilon F_\varepsilon + \varepsilon F \\
&\text{div } U_\varepsilon = 0 \quad \text{in } \Omega \times (0, T), \\
&U_\varepsilon = 0 \quad \text{on } \partial \Omega \times (0, T), \\
&U_\varepsilon(0) = 0 \quad \text{in } \Omega,
\end{align*}
$$

where

$$
\begin{align*}
F_H &= -(U_\varepsilon \cdot \nabla V_\varepsilon + u \cdot \nabla V_\varepsilon + U_\varepsilon \cdot \nabla v), \\
F_\varepsilon &= -(U_\varepsilon \cdot \nabla W_\varepsilon + u \cdot \nabla W_\varepsilon + U_\varepsilon \cdot \nabla w), \\
F &= -(\partial_t w - \Delta w + u \cdot \nabla w).
\end{align*}
$$

The system (1) is the equations of the difference between solutions to (PE) and (SNS).

**Theorem 1.** Let $T > 0$. Suppose $(p, q) \in (1, \infty)^2$ satisfies $\frac{1}{p} \leq \min(1 - 1/q, 3/2 - 2/q)$, $u_0 = (v_0, w_0) \in X_\gamma$ and $v_0 \in B_{q,p}^s(\Omega)$ for $s > 2 - 2/p + 1/q$. Let $u \in E_1(T)$ be a solution of (PE) with initial data $u_0$. Then, there exist positive constants $\epsilon_0 = \epsilon_0(p, q, \|u\|_{E_1(T)})$, $C = C(p, q, \|u\|_{E_1(T)})$ and a unique solution $U_\varepsilon = (V_\varepsilon, W_\varepsilon)$ to (1) such that

$$
\| (V_\varepsilon, \varepsilon W_\varepsilon) \|_{E_1(T)} \leq \epsilon C
$$

for any $\varepsilon \in (0, \epsilon_0)$. Moreover, $u_\varepsilon = (v_\varepsilon, w_\varepsilon) := (v + V_\varepsilon, w + W_\varepsilon)$ is the unique solution to (SNS) in $E_1(T)$.

Note that the case $T = \infty$ is included. We use the assumption $v_0 \in B_{q,p}^s(\Omega)$ in Theorem 1 to estimate $\|u\|_{E_1(T)}$. We give an explicit form of $\epsilon_0$ in Remark 33. This theorem implies the justification of the hydrostatic approximation.

**Corollary 2.** Let $T > 0$ and $0 < \varepsilon \leq 1$. Suppose $(p, q) \in (1, \infty)^2$ satisfies $\frac{1}{p} \leq \min(1 - 1/q, 3/2 - 2/q)$, $u_0 = (v_0, w_0) \in X_\gamma$ and $v_0 \in B_{q,p}^s(\Omega)$ for $s > 2 - 2/p + 1/q$. Let $u$ and $u_\varepsilon$ be a solution of (PE) and (SNS) in $E_1(T)$ under the Dirichlet boundary condition with initial data $u_0$, respectively, such that

$$
\|u\|_{E_1(T)} + \| (v_\varepsilon, \varepsilon w_\varepsilon) \|_{E_1(T)} \leq C_0
$$

for some $C_0 = C_0(u_0, p, q)$. Then, there exists a positive $C = C(p, q, C_0)$ such that

$$
\| (v_\varepsilon - v, \varepsilon (w_\varepsilon - w)) \|_{E_1(T)} \leq \epsilon C.
$$
1.2. Strategy

Our strategy to show Theorem 1 is based on estimates for \((V_\epsilon, \epsilon W_\epsilon)\). The proof consists of two key steps: maximal regularity results of the anisotropic Stokes operator and the improved regularity result for the vertical component of the solution to the primitive equations. We consider the nonlinear terms in (SNS) as an external force \(f\) and set \(u_\epsilon = (v_\epsilon, \epsilon w_\epsilon)\) to get the linear equation

\[
\begin{aligned}
&\partial_t u_\epsilon - \Delta u_\epsilon + \nabla_\epsilon \pi_\epsilon = f & \text{in } \Omega \times (0, T), \\
&\text{div}_\epsilon u_\epsilon = 0 & \text{in } \Omega \times (0, T), \\
&u_\epsilon = 0 & \text{on } \partial \Omega \times (0, T), \\
&u_\epsilon(0) = u_0 & \text{in } \Omega,
\end{aligned}
\]

where \(\nabla_\epsilon = (\partial_1, \partial_2, \partial_3/\epsilon)^T\) and \(\text{div}_\epsilon = \nabla_\epsilon \cdot\). We define the function space \(E_{\epsilon, j}(T)\) for \(j = 0, 1\) similarly as \(E_{j}(T)\) by replacing \(\text{div}\) by \(\text{div}_\epsilon\). Although the space \(E_{\epsilon, j}(T)\) depends on \(\epsilon\), the norm is just the norm in \(W^{1, p}(0, T; L^q(\Omega)) \cap L^p(0, T; W^{2, q}(\Omega))\), so we shall write the norm in \(E_{\epsilon, j}(T)\) simply by \(\|\cdot\|_{E_{\epsilon, j}(T)}\).

We recall some known results on maximal regularity for the Stokes operator, which is corresponding to the case \(\epsilon = 1\). Solonnikov [35] first proved \(L^q\)-\(L^q\) maximal regularity for the Stokes operator by a potential-theoretic approach. The second author [14] established a bound for the pure imaginary powers of the Stokes operator in a bounded domain; see [18] for an exterior domain. This type of property will be simply called bounded imaginary powers, shortly BIP. This BIP implies the maximal regularity \(L^p\)-\(L^q\) regularity via Dore–Venini theory [6]. Indeed, the second author and Sohr [19] established the maximal regularity in an exterior domain by estimating BIP. Further studies on maximal regularity were done by many researchers, for instance, Dore and Venini [6] and Weis [36]. See Denk et al. [5] for further comprehensive research. In our case, we have to clarify \(\epsilon\)-dependence in estimates for maximal regularity, which is a key point. Our key maximal regularity result is

**Lemma 3.** Let \(1 < p, q < \infty, 0 < \epsilon \leq 1\) and \(T > 0\). Let \(f \in E_{\epsilon, 0}(T)\) and \(u_0 \in B^{2(1-1/p)}_{q,p}(\Omega)\) with \(\text{div}_\epsilon u = 0\). Then, there exist constants \(C = C(p, q) > 0\) and \(C' = C'(p, q) > 0\), which are independent of \(\epsilon\), and \((u, \pi)\) satisfying (4) such that

\[
\|\partial_t u\|_{E_0(T)} + \|\nabla_\epsilon^2 u\|_{E_0(T)} + \|\nabla_\epsilon \pi\|_{E_0(T)} \leq C\|f\|_{E_0(T)} + C'\|u_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)}.
\]

Lemma 3 follows from a maximal regularity estimate involving the Stokes operator, which follows from a bound for the pure imaginary powers by Dore–Venini theory. However, we need to clarify that \(C\) and \(C'\) can be taken independent of \(\epsilon\). For \(\epsilon = 1\), a necessary BIP estimate for the Stokes operator has been established by Abels [1], where a resolvent decomposition similar to [14] is used. Unfortunately, the \(\epsilon\)-dependent case is not discussed there. However, the strategy in [1] works for our problem. We construct
the anisotropic Stokes operator by the method in [1] and show the boundedness of its imaginary powers. Note that, in our previous paper [11], maximal regularity of the anisotropic Stokes operator is much easier since the corresponding Stokes operator is essentially the same as the Laplace operator on \( \mathbb{T}^3 \). In the case of the Dirichlet boundary condition, the corresponding Stokes operator becomes to be much more difficult by the effect of boundaries, which is substantially different from the case of the periodic boundary conditions. The maximal regularity was proved in a layer domain for the Stokes operator under various boundary conditions by Saito [33] by proving \( \mathcal{R} \)-boundedness of the resolvent operator when \( \epsilon = 1 \). Unfortunately, it seems very difficult to check the dependence of \( \epsilon \), so we do not take this approach.

The term \( F = \partial_t w - \Delta w + u \cdot \nabla w \) appears in the right-hand side of (1). Thus, we need to improve the regularity of \( w \) and estimate this term in \( L^p(0, T; L^q(\Omega)) \).

**Lemma 4.** Let \( T > 0 \) and \( u_0 = (v_0, w_0) \in X_T \) with \( w_0 = - \int_{-1}^{x_3} \text{div}_H v_0 \, d\xi \) and \( v_0 \in B^s_{q,p}(\Omega) \) for \( s > 2 - 2/p + 1/q \) and \( u = (v, w) \) be the solution to (PE). Assume \( v \in \mathbb{E}_1(T) \). Then, there exists a constant \( C > 0 \) such that

\[
\|w\|_{\mathbb{E}_1(T)} \leq C.
\] (6)

Since it is known that \( v \in \mathbb{E}_1(T) \), as we have already mentioned, we see \( w(\cdot, x_3) = - \int_{-1}^{x_3} \text{div}_H v(\cdot, \xi) \, d\xi \in W^{1,p}(0, T; W^{-1,q}(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega)) \). This derivative loss is due to the absence of the equation of time-evolution of \( w \) in the primitive equations. In our previous paper [11], which treats the periodic boundary condition, we recover the regularity of \( w \) by deriving the equation which \( w \) satisfies and applying maximal regularity of the Laplace operator to the equation. However, in the case of the Dirichlet boundary condition, this method is not applicable directly because of the presence of the second-order derivative term at the boundary, which vanishes in the case of periodic boundary condition. Thus, we are forced to impose additional regularity for initial data to get regularity for \( v \). If \( v_0 \in B^s_{q,p}(\Omega) \) for \( s > 2 - 2/p + 1/q \), then we obtain \( v \in L^p(0, T; W^{s+2/p,q}(\Omega)) \) and the trace of the second derivative belongs to \( \mathbb{E}_0(T) \).

Let us explain our strategy to show Theorem 1.

Once Lemma 4 is proved, our main result Theorem 1 can be proved essentially the same way as [11], where the proof we give here is slightly different from that of [11]. Moreover, we give an explicit form of the constant \( C \) in Theorem 1; see the estimate (96). We first show the boundedness of nonlinear terms \( F_H \) and \( F_z \) in (1) in the space \( \mathbb{E}_0(T) \). We know that \( F \) is also bounded in \( \mathbb{E}_0(T) \) by Lemma 3. We next apply Lemma 4 to (1) to get a quadratic inequality, which leads to \( \| (V_\epsilon, \epsilon W_\epsilon) \|_{\mathbb{E}_1(T^*)} \leq C \epsilon \) for some short time \( T^* > 0 \) and \( \epsilon \)-independent constant \( C > 0 \). Since \( C \) depends only on \( p, q, u_0, \| u \|_{\mathbb{E}_1(T)} \) and \( T \), if we take \( \epsilon \) small, we are able to extend the time to all finite time \( T \) by finite step. In the case of \( T = \infty \), we show (2) for sufficiently large \( T' > 0 \). We extend the existence time of \( U_\epsilon \) from \( [0, T') \) to \( [0, \infty) \) by the above procedure and smallness of \( \| u \|_{\mathbb{E}_1(\infty)} - \| u \|_{\mathbb{E}_1(T')} \).
This paper is organized as follows. In Sect. 2, the boundedness of pure imaginary powers is proved. The resolvent operator of the anisotropic Stokes operator is decomposed into three parts, and each part of uniform bound independent of $\epsilon$ is proved. In Sect. 3, improved regularity for $w$ is proved. In Sect. 4, we give a proof of our main theorem by iteration.

1.3. Notation

We introduce our notation. We denote by $\| \cdot \|_{X \to Y}$ the operator norm from a Banach space $X$ to a Banach space $Y$. We denote by $C_0^\infty(\Omega)$ the set of compactly supported smooth functions in $\Omega$. We write $L^q(\Omega)$ to denote the Lebesgue space for $1 \leq q \leq \infty$ equipped with the norm

$$\| f \|_{L^q(\Omega)} = \left( \int_\Omega |f(x)|^q \, dx \right)^{1/q}.$$ 

We use the usual modification when $q = \infty$. For a three-dimensional $L^q$-vector field $f$ on a domain $D$, we write $f \in L^q(D)$ to simplify the notation. For $m \in \mathbb{Z}_{\geq 0}$ and $1 \leq q \leq \infty$ we denote by $W^{m,q}(\Omega)$ the $m$-th-order Sobolev space equipped with the norm

$$\| f \|_{W^{m,q}(\Omega)} = \| \nabla^m f \|_{L^q(\Omega)}.$$ 

We define the fractional Sobolev spaces $W^{s,q}(\Omega)$ by the real interpolation $W^{\lfloor s \rfloor, q}(\Omega), W^{\lfloor s \rfloor + 1, q}(\Omega)_{s-\lfloor s \rfloor, q}$, where $\lfloor \cdot \rfloor$ denotes the Gauss symbol. We define the Bessel potential spaces $H^{s,q}(\Omega)$ by the complex interpolation $[W^{\lfloor s \rfloor, p}, W^{\lfloor s \rfloor + 1, q}]_{s-\lfloor s \rfloor}$ for $s \in \mathbb{R}_{+}\setminus\mathbb{Z}_{\geq 1}$ and $1 < q < \infty$. We define the Fourier transform by

$$\mathcal{F} f(\xi) = \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) \, dx$$

and the Fourier inverse transform by

$$\mathcal{F}^{-1} f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} f(\xi) \, d\xi.$$ 

The Fourier transform on the $d$-dimensional torus $\mathbb{T}^d$ and its inverse transform are defined by $[\mathcal{F}_d f](n) = \hat{f}_n = \int_{\mathbb{T}^d} e^{-in \cdot x} f(x) \, dx$ and $[\mathcal{F}^{-1}_d g](x) = \frac{1}{(2\pi)^d} \sum_n g_n e^{inx}$, respectively. We denote by $\mathcal{F}_{x'}$ the partial Fourier transform with respect to $x' \in \mathbb{R}^2$ and by $\mathcal{F}^{-1}_{\xi'}$ the partial Fourier inverse transform with respect to $\xi'$. We denote by $\mathcal{F}_{d,x'}$ the partial Fourier transform with respect to $x' \in \mathbb{T}^2$ and by $\mathcal{F}^{-1}_{d,n'}$, the partial Fourier inverse transform with respect to $n' \in \mathbb{Z}^2$. We write $\Sigma_{\theta} := \{ \lambda \in \mathbb{C} : |\arg \lambda| < \pi - \theta \}$. For a Fourier multiplier operator $\mathcal{F}^{-1}_{\xi'} m(\xi) \mathcal{F}_{x'}$ in $\mathbb{R}^3$, we denote by $[m]_M$ the Mikhlin constant. $\mathcal{F}^{-1}_{\xi'} m(\xi) \mathcal{F}_{x'}$ is a Fourier multiplier operator in $\mathbb{R}^2$ with Mikhlin constant $[m]_{M'}$. For $0 < \epsilon \leq 1$, $\Delta_\epsilon = \partial_1^2 + \partial_2^2 + \partial_3^2/\epsilon^2$ denotes the anisotropic Laplace
operator. We denote by $E_0$ the zero-extension operator with respect to the vertical variable from $(-1, 1)$ to $\mathbb{R}$, i.e.,

$$[E_0 f](\cdot, \cdot, z) = \begin{cases} f(\cdot, \cdot, z) & \text{if } z \in (-1, 1), \\ 0 & \text{otherwise}, \end{cases}$$

for a function $f$ defined in $\mathbb{R}^2 \times (-1, 1)$ or $T^2 \times (-1, 1)$. We denote by $R_0$ the restriction operator with respect to the vertical variable from $\mathbb{R}$ to $(-1, 1)$. For an integrable function $f$ defined in $\Omega$, we write its vertical and horizontal average by $f = \frac{1}{2} \int_{-1}^{1} f(\cdot, \cdot, \zeta) \, d\zeta$ and $\text{ave}_H(f) = \int_{T^2} f(x', \cdot) \, dx'$, respectively.

2. A uniform bound for pure imaginary powers of the anisotropic Stokes operator and its maximal regularity

In this section, we first establish a uniform bound independent of $\varepsilon$ for the pure imaginary powers to the anisotropic Stokes operator along with [1]. Then, we shall give the proof of Lemma 3.

2.1. Boundedness of Fourier multipliers

Although the case of the infinite layer $\mathbb{R}^2 \times (-1, 1)$ is considered in [1], his method also works in the case of the periodic layer $\Omega = T^2 \times (-1, 1)$ thanks to Fourier multiplier theorem on the torus, e.g., Proposition 4.5 in [22] and Sect. 4 of Grafakos’s book [20].

Proposition 5. [22] Let $1 < p < \infty$ and $m \in C^d(\mathbb{R}^d \setminus \{0\})$ satisfies the Mikhlin condition:

$$[m]_{\mathcal{M}} := \sup_{\alpha \in \{0,1\}^d} \sup_{\xi \in \mathbb{R}^d \setminus \{0\}} \left| \xi^\alpha \partial_\xi^\alpha m(\xi) \right| < \infty. \quad (7)$$

Let $a_k = m(k)$ for $k \in \mathbb{Z}^d \setminus \{0\}$ and $a_0 \in \mathbb{C}$.

For $f(x) = \sum_{n \in \mathbb{Z}^d} \hat{f}_n e^{in \cdot x} \in L^q(T^d)$ and a sequence $a = \{a_n\}_{n \in \mathbb{Z}^d}$, we set the Fourier multiplier operator of discrete type by

$$[Tf](x) := \mathcal{F}_d^{-1} a \mathcal{F}_d f = \sum_{n \in \mathbb{Z}^d} a_n \hat{f}_n e^{in \cdot x}. \quad (8)$$

Then, there exists a constant $C = C(p, d) > 0$ such that

$$\|Tf\|_{L^q(T^d)} \leq C \max([m]_{\mathcal{M}}, a_0) \|f\|_{L^q(T^d)}. \quad (9)$$

Let us consider the resolvent problem to (4);

$$\begin{cases} \lambda u - \Delta u + \nabla_\varepsilon \pi = f & \text{in } \Omega, \\ \text{div}_\varepsilon u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega, \end{cases} \quad (10)$$
for $\lambda \in \Sigma_\theta$ ($0 < \theta < \pi/2$) and $f \in L^q(\Omega)$. As in the case $\epsilon = 1$, there is a topological direct sum decomposition called the (anisotropic) Helmholtz decomposition:

$$L^q(\Omega) := L^q_{\sigma,\epsilon}(\Omega) \oplus G_{\sigma,\epsilon}(\Omega),$$

$$L^q_{\sigma,\epsilon}(\Omega) := \{ u \in L^q(\Omega); \nabla \epsilon u = 0, u|_{x_3=\pm 1} = 0 \}, \quad (1 < p < \infty),$$

for $q \in (1, \infty)$; see Miyakawa [30] for infinite layer domain. The reader is referred to the book of Galdi [12] for fundamentals of the Helmholtz projection. Let $H_\epsilon$ be the projection from $L^q(\Omega)$ to $L^q_{\sigma,\epsilon}(\Omega)$ associated with this decomposition. We shall prove its $L^q$-boundedness in Lemma 16. This projection is called the anisotropic Helmholtz projection. Let $A_\epsilon = H_\epsilon(-\Delta)$ be the Stokes operator with the domain $D(A_\epsilon) = L^q_{\sigma,\epsilon}(\Omega) \cap W^{2,q}(\Omega)$. For $0 < a < 1/2$ and $-a < \Re z < 0$, the fractional power of $A_\epsilon$ is defined via the Dunford calculus

$$A_\epsilon^z = \frac{1}{2\pi i} \int_{\Gamma_\theta} (-\lambda)^z (\lambda + A_\epsilon)^{-1} d\lambda,$$

where $0 < \theta < \pi/2$ and $\Gamma_\theta = \Re e^{i(-\pi+\theta)} \cup \Re e^{i(\pi-\theta)}$. Our aim in this section is to prove

**Lemma 6.** Let $1 < q < \infty$, $0 < \epsilon \leq 1$, $0 < a < 1/2$, $z \in \mathbb{C}$ satisfying $-a < \Re z < 0$ and $0 < \theta < \pi/2$. Then, there exists a constant $C = C(q, a, \theta)$ such that

$$||A_\epsilon^z||_{L^q(\Omega) \to L^q(\Omega)} \leq C e^{\theta|\Im z|}. \quad (11)$$

Once the above lemma is proved, then we obtain the maximal regularity of the anisotropic Stokes operator via the formula

$$\left( \frac{d}{dt} + A_\epsilon \right)^{-1} = \frac{1}{2i} \int_{c-i\infty}^{c+i\infty} \frac{(d/\text{d}t)^z A_\epsilon^{1-z}}{\sin \pi z} \text{d}z \quad (12)$$

for $0 < c < 1$ and the Dore–Venni theory [6].

To show Lemma 6, we decompose the solution $(u, \pi)$ to (10) into three parts:

$$u = R_0 v_1 - v_2 + \nabla_\epsilon \pi_3, \quad (13)$$

$$\nabla_\epsilon \pi = \nabla_\epsilon \pi_1 + \nabla_\epsilon \pi_2, \quad (14)$$

where $v_j$ and $\pi_j$ are solutions to

- (I) $\left\{ \begin{aligned}
\lambda v_1 - \Delta v_1 + \nabla_\epsilon \pi_1 &= E_0 f \quad \text{in } \mathbb{T}^2 \times \mathbb{R}, \\
\nabla_\epsilon v_1 &= 0 \quad \text{in } \mathbb{T}^2 \times \mathbb{R},
\end{aligned} \right.$

- (II) $\left\{ \begin{aligned}
\lambda v_2 - \Delta v_2 + \nabla_\epsilon \pi_2 &= 0 \quad \text{in } \Omega, \\
\nabla_\epsilon v_2 &= 0 \quad \text{in } \Omega, \\
v_2 &= \gamma v_1 - (\gamma v_1 \cdot \nu)v \quad \text{on } \partial\Omega,
\end{aligned} \right.$
and

$$\begin{aligned}
(\text{III}) \quad & \Delta_{\epsilon} \pi_3 = 0 \quad \text{in} \quad \Omega, \\
& \nabla_{\epsilon} \pi_3 \cdot \nu = (\gamma v_1 \cdot \nu) \nu \quad \text{on} \quad \partial \Omega,
\end{aligned}$$

where $\nu$ is the unit outer normal and $\gamma = \gamma_{\pm}$ is the trace operator to the upper and lower boundary, namely,

$$\gamma f(x_1, x_2) = (\gamma_{+} f(x_1, x_2), \gamma_{-} f(x_1, x_2)) = (f(x_1, x_2, 1), f(x_1, x_2, -1))$$

for a vector field $f$ on $\mathbb{T}^2 \times \mathbb{R}$ or $\mathbb{R}^3$. To show Lemma 6, we need to obtain the estimate

$$\left\| \frac{1}{2\pi i} \int_{\Gamma} (-\lambda)^{\frac{\epsilon}{2}} R_{0} v_1 \, d\lambda \right\|_{L^q(\Omega)} + \left\| \frac{1}{2\pi i} \int_{\Gamma} (-\lambda)^{\frac{\epsilon}{2}} v_2 \, d\lambda \right\|_{L^q(\Omega)} + \left\| \frac{1}{2\pi i} \int_{\Gamma} (-\lambda)^{\frac{\epsilon}{2}} \nabla_{\epsilon} \pi_3 \, d\lambda \right\|_{L^q(\Omega)} \leq C e^{\theta \|\text{Im} z\|} \|f\|_{L^q(\Omega)}$$

with some constant $C > 0$ independent of $\epsilon$, $z$ and $f$.

**Remark 7.** For $f \in L^q(\Omega)$ and its horizontal average $\text{ave}_H(f) = \int_{\mathbb{T}^2} f \, dx'$, we solve the resolvent problem (10) with external force $\text{ave}_H(f)$ to get $u = (\lambda - \partial_{3}^{-1})^{-1} \text{ave}_H(f_H, 0)$ and $\pi = \epsilon \int_{\mathbb{R}^3} \text{ave}_H(f_3) \, d\xi / \mathbb{R}$, where $f_H$ is the horizontal component of $f$ and $\mathbb{R}$ means average-free. Since $-\partial_{3}^2$ has BIP and the resolvent operator is linear, by taking the difference between the solution to (10) and $(u, \pi)$, we can always assume without loss of generality that $f$ is horizontal average-free.

We define the space of horizontally average-free $L^q$-vector fields by

$$L^q_{af}(\Omega) := \{ f \in L^q(\Omega) : \text{ave}_H(f) = 0 \}.$$ 

Similarly, we define

$$W^{s,q}_{af}(\Omega) := \{ f \in W^{s,q}(\Omega) : \text{ave}_H(f) = 0 \}.$$ 

Throughout this section, we frequently use partial Fourier transforms to construct solutions and estimate these partial Fourier multipliers.

**Proposition 8.** [1] Let $1 < q < \infty$ and $a, b \in \{-1, 1\}$. Let an integral operator $M$ be

$$Mf(x', x_3) = \int_{-1}^{1} \frac{f(x', \xi)}{|x_3 - a| + |\xi - b|} \, d\xi$$

for $f \in L^q(\Omega)$. Then, there exists a constant $C > 0$ such that

$$||Mf||_{L^q} \leq C ||f||_{L^q}.$$ 

**Proof.** See Lemma 3.3 in [1].
Rescaled $L^q$-Fourier multipliers are also bounded $L^q$ multiplier as a direct consequence of the Mikhlin theorem.

**Proposition 9.** Let $1 < q < \infty$ and $0 < \epsilon \leq 1$. Let $m \in C^d(\mathbb{R}^d \setminus \{0\})$ be a $L^q$-Fourier multiplier with the Mikhlin constant $[m]_{\mathcal{M}} \leq C$ for some $C > 0$. Then, the rescaled one $m_\epsilon(\xi) := m(\epsilon \xi)$ is also bounded from $L^q$ into itself such that

$$[m_\epsilon]_{\mathcal{M}} \leq C. \quad (15)$$

**Proof.** By the Mikhlin theorem, it suffices to prove (15). In fact, we observe that

$$[m]_{\mathcal{M}} = [m_\epsilon]_{\mathcal{M}}$$

since

$$\sup_{\xi \neq 0} |\xi|^{\alpha} \left| \partial^{\alpha}_{\xi} (m(\epsilon \xi)) \right| = \sup_{\xi \neq 0} |\xi|^{\alpha} |\epsilon|^{\alpha} \left| \left( \partial^{\alpha}_{\xi} m \right)(\epsilon \xi) \right| = \sup_{\eta \neq 0} |\eta|^{\alpha} \left| \partial^{\alpha}_{\eta} m(\eta) \right|.$$  

□

The above proposition is frequently used in this section to get $\epsilon$-independent estimate for scaled multipliers. We show boundedness of some Fourier multiplier operators in advance. We set

$$s_\lambda = (\lambda + |\xi'|^2)^{1/2}$$

for $\xi' \in \mathbb{R}^2$. In this paper, we use $s_\lambda$ to denote $(\lambda + |n'|^2)^{1/2}$ for $n' \in \mathbb{Z}^2$ to simplify notation.

**Proposition 10.**

1. Let $0 < \theta < \pi/2$, $\lambda \in \Sigma_\theta$, $t > 0$ and $\alpha$ be a positive integer. Then, there exist constants $c > 0$ and $C > 0$ such that

$$[|\xi'|^\alpha e^{-t s_\lambda}]_{\mathcal{M}'} \leq C e^{-c |\lambda|^{1/2} t^\alpha}, \quad \left[ \frac{e^{-s_\lambda}}{s_\lambda} \right]_{\mathcal{M}'} \leq C |\lambda|^{-1/2} e^{-c |\lambda|^{1/2}}. \quad (16)$$

2. Let $-1 \leq x_3 \leq 1$. Then, there exists a constant $C > 0$ which is independent of $\epsilon$, such that

$$\left[ \frac{\sinh(\epsilon |\xi'| x_3)}{\sinh(\epsilon |\xi'|)} \frac{\epsilon |\xi'|}{1 + \epsilon |\xi'|} \right]_{\mathcal{M}'} \leq C, \quad \left[ \frac{\cosh(\epsilon |\xi'| x_3)}{\sinh(\epsilon |\xi'|)} \frac{\epsilon |\xi'|}{1 + \epsilon |\xi'|} \right]_{\mathcal{M}'} \leq C \quad (17)$$

for all $0 < \epsilon \leq 1$.

3. Let $-1 \leq x_3 \leq 1$. Then, there exists a constant $C > 0$, which is independent of $\epsilon$, such that

$$\left[ \frac{\sinh(\epsilon |\xi'| x_3)}{\cosh(\epsilon |\xi'|)} \right]_{\mathcal{M}'} \leq C, \quad \left[ \frac{\sinh(\epsilon |\xi'| x_3)}{\cosh(\epsilon |\xi'|)} \right]_{\mathcal{M}'} \leq C \quad (18)$$

for all $0 < \epsilon \leq 1$. 


Proof. The estimate (16) is a direct consequence of Lemma 3.5 in [1] and the Mikhlin theorem.

By definition of sinh and cosh, we find the formula
\[
\frac{\sinh(\epsilon |\xi'| x_3)}{\sinh(\epsilon |\xi'|)} = \frac{e^{\epsilon |\xi'| x_3} - e^{-\epsilon |\xi'| x_3}}{e^{\epsilon |\xi'|} - e^{-\epsilon |\xi'|}} = \frac{e^{-\epsilon |\xi'| (x_3-1)}}{1 - e^{-2\epsilon |\xi'|}} - \frac{e^{-\epsilon |\xi'| (x_3+1)}}{1 - e^{-2\epsilon |\xi'|}}
\] (19)

and
\[
\frac{\cosh(\epsilon |\xi'| x_3)}{\sinh(\epsilon |\xi'|)} = \frac{e^{-\epsilon |\xi'| (x_3-1)}}{1 - e^{-2\epsilon |\xi'|}} + \frac{e^{-\epsilon |\xi'| (x_3+1)}}{1 - e^{-2\epsilon |\xi'|}}.
\] (20)

Thus, multiplying both sides of (19) by \(\frac{\epsilon |\xi'|}{1 + \epsilon |\xi'|}\), we find from Proposition 9 that
\[
\left[ \frac{\sinh(\epsilon |\xi'| x_3)}{\sinh(\epsilon |\xi'|)} \right] \frac{1 + \epsilon |\xi'|}{\epsilon |\xi'|} \leq C \left[ \frac{e^{-\epsilon |\xi'| (x_3-1)}}{1 - e^{-2\epsilon |\xi'|}} \right] \frac{1}{(1 + \epsilon |\xi'|)} M' + \left[ \frac{e^{-\epsilon |\xi'| (x_3+1)}}{1 - e^{-2\epsilon |\xi'|}} \right] \frac{1}{(1 + \epsilon |\xi'|)} M' \leq C.
\] (21)

The second inequality of (17) is proved by the same way as above using (20). Similarly, by definition of sinh and cosh, the estimate (18) follows. \(\square\)

2.2. Estimate for \(v_1\)

Let us consider the equations (I). For \(a \in \mathbb{R}\) we denote the dilation operator in the last variable by
\[
[\tau_a f](\xi_1, \xi_2, \xi_3) = f(\xi_1, \xi_2, a\xi_3).
\]

The anisotropic Helmholtz projection \(\mathbb{P}_{\epsilon}^{\mathbb{R}^3}\) on \(\mathbb{R}^3\) with symbols
\[
\mathcal{F} \mathbb{P}_{\epsilon}^{\mathbb{R}^3} = I_3 - \xi_\epsilon \otimes \xi_\epsilon, \quad \xi_\epsilon = \left(\xi_1, \xi_2, \frac{\xi_3}{\epsilon}\right) \in \mathbb{R}^3,
\]
is bounded in \(L^q(\mathbb{R}^3)\) by boundedness of the Riesz operator and the formula
\[
\mathcal{F}_{\xi}^{-1}(\tau_a m)\mathcal{F}_x f(\xi) = \tau_{a^{-1}} \left[ \mathcal{F}_{\xi}^{-1} m(\xi) \mathcal{F}_x \tau_a f \right].
\] (22)

Indeed, it follows from (22) that
\[
\|\mathbb{P}_{\epsilon}^{\mathbb{R}^3} f\|_{L^q(\mathbb{R}^3)} = \epsilon^{-1} \|\tau_{1/\epsilon} f\|_{L^q(\mathbb{R}^3)} \leq C \|f\|_{L^q(\mathbb{R}^3)},
\]
since $\mathbb{P}^{\mathbb{R}^3}_1$ is $L^q$-bounded, which follows from the boundedness of the Riesz operator. We define the anisotropic Helmholtz projection $\mathbb{P}^{\mathbb{T}^2 \times \mathbb{R}}_\epsilon$ on $\mathbb{T}^2 \times \mathbb{R}$ with symbols by

$$\mathcal{F}_{x_3} \mathcal{F}_{d, x'} \mathbb{P}^{\mathbb{T}^2 \times \mathbb{R}}_\epsilon = I_3 - \left( \begin{array}{c} n_1 \\ n_2 \\ \xi_3 / \epsilon \end{array} \right) \otimes \left( \begin{array}{c} n_1 \\ n_2 \\ \xi_3 / \epsilon \end{array} \right), \quad n_1, n_2 \in \mathbb{Z}, \ \xi_3 \in \mathbb{R}.$$

We find $\mathbb{P}^{\mathbb{T}^2 \times \mathbb{R}}_\epsilon$ is bounded from $L^q(T^2 \times \mathbb{R})$ into itself by boundedness of $\mathbb{P}^{\mathbb{R}^3}_1$ and Proposition 5 uniformly in $\epsilon \in (-1, 1)$.

**Proposition 11.** Let $1 < q < \infty$, $0 < a < 1/2$, $0 < \epsilon \leq 1$, $z \in \mathbb{C}$ satisfying $-a < \text{Re} z < 0$ and $0 < \theta < \pi/2$. Then, there exists a constant $C = C(q, a, \theta)$ such that

$$\left\| \frac{1}{2\pi i} \mathcal{R}_0 \int_{I_0} (-\lambda)^z (\lambda - \Delta_{\mathbb{T}^2 \times \mathbb{R}})^{-1} \mathbb{P}^{\mathbb{T}^2 \times \mathbb{R}}_\epsilon E_0 f \, d\lambda \right\|_{L^q(T^2 \times \mathbb{R})} \leq C e^{\theta |\text{Im} z|} \| f \|_{L^q(\Omega)} \tag{23}$$

for all $f \in L^q(\Omega)$.

**Proof.** It is easy to prove that the Laplace operator on a cylinder $\mathbb{T}^2 \times \mathbb{R}$ has BIP by calculating the pure imaginary power directly, see, e.g., Appendix A in [19] and Sect. 8 of Nau [31]. For the Laplace operator, BIP was established on a domain with boundary long time ago by Fujiwara [10] and Seeley [34]. Combining these facts with $L^q$-boundedness of $\mathbb{P}^{\mathbb{T}^2 \times \mathbb{R}}_\epsilon$, we have (23). □

**Proposition 12.** Let $1 < q < \infty$, $0 < \epsilon \leq 1$, $0 < \theta < \pi/2$ and $\lambda \in \Sigma_\theta$. Then, there exists a constant $C = C(q) > 0$, which is independent of $\epsilon$, such that

$$\left\| \nabla^2 (\lambda - \Delta_{\mathbb{T}^2 \times \mathbb{R}})^{-1} \mathbb{P}^{\mathbb{T}^2 \times \mathbb{R}}_\epsilon E_0 f \right\|_{L^q(\Omega)} \leq C \| f \|_{L^q(\Omega)}$$

for all $f \in L^q(\Omega)$.

**Proof.** This follows from Propositions 5 and 9 since $\mathbb{P}^{\mathbb{T}^2 \times \mathbb{R}}_\epsilon$ is uniformly bounded from $L^q(T^2 \times \mathbb{R})$ into itself. □

Let us calculate the partial Fourier transform for $v_1$ with respect to the horizontal variable. This is needed to obtain representation formula for $v_2$ later.

Let $g \in L^q(T^2 \times \mathbb{R})$. The solution $\tilde{v}$ to the equation

$$\begin{cases}
\lambda \tilde{v} - \Delta \tilde{v} + \nabla_\epsilon \tilde{\pi} &= g \quad \text{in} \quad \mathbb{T}^2 \times \mathbb{R}, \\
\text{div}_\epsilon \tilde{v} &= 0 \quad \text{in} \quad \mathbb{T}^2 \times \mathbb{R},
\end{cases}$$

is given by

$$\tilde{v} = (\lambda - \Delta_{\mathbb{R}^3})^{-1} \mathbb{P}^{\mathbb{T}^2 \times \mathbb{R}}_\epsilon g.$$
Moreover,
\[
K_{\lambda,e} g := (\lambda - \Delta_{T^2 \times \mathbb{R}})^{-1} \mathbb{R}^2 \times \mathbb{R} g
\]
\[
= \mathcal{F}^{-1}(\lambda + |n'|^2 + \xi_3^2)^{-1} \left( I_3 - \frac{\xi_e \otimes \xi_e}{|\xi_e|^2} \right) \mathcal{F} g
\]
\[
= \mathcal{F}^{-1}_{n'} \int_{\mathbb{R}} k_{\lambda,e}(n', x_3 - \zeta) \mathcal{F} g(n'), \zeta \, d\zeta,
\]
where \( \xi_e = (n', \xi_3/e) \in \mathbb{Z}^2 \times \mathbb{R} \) and
\[
k'_{\lambda,e}(n', x_3)
= \mathcal{F}^{-1}_{\xi_3} \left[ (\lambda + |n'|^2 + \xi_3^2)^{-1} \left( I_3 - \frac{\xi_e \otimes \xi_e}{|\xi_e|^2} \right) \right]
= \frac{e^{-s_\lambda}}{2s_\lambda} \left( \begin{array}{cc} I_2 & 0 \\ 0 & 0 \end{array} \right)
- \left( n' \otimes n' \frac{e^2}{\lambda+(1+e^2)|n'|^2} e^{-|n'|^2|s_\lambda+3e^{-|s_\lambda|}|n'|} \right)
\]
\[
= \frac{e^{-s_\lambda}}{2s_\lambda} \left( I_2 0 \\ 0 0 \right) - \left( n' \otimes n' \eta'_{\lambda,e}(n', x_3) - in' \partial_3 \eta'_{\lambda,e}(n', x_3) \right).
\]
(25)

The kernel function \( k_{\lambda,e}(n', x_3) \) is calculated by the residue theorem. Actually, since poles of \( (\lambda + |n'|^2 + \xi_3^2)^{-1} \) are \( \xi_3 = \pm is_\lambda \), the residue theorem implies the partial

Fourier inverse transform of \( (\lambda + |n'|^2 + \xi_3^2)^{-1} \) with respect to \( \xi_3 \) is given by inserting \( \xi_3 = is_\lambda \) or \( -is_\lambda \) into \( e^{i\lambda \xi_3} \xi_3 \) so that the real part become to be negative. Thus, we have
\[
e'_{\lambda}(n', x_3) := \mathcal{F}^{-1}_{\xi_3} \left( \lambda + |n'|^2 + |x_3|^2 \right)^{-1} = \frac{e^{-|x_3|s_\lambda}}{s_\lambda}.
\]
(26)

Moreover, this formula leads to
\[
\mathcal{F}^{-1}_{\xi_3} \left[ |\xi_e|^2 \right]^{-1} = \mathcal{F}^{-1}_{\xi_3} \left[ \frac{e^2}{e^2 |n'|^2 + \xi_3^2} \right] = \frac{e e^{-|x_3|s_\lambda}}{|n'|}.
\]

Combining the above two calculations and the formula
\[
I_3 - \frac{\xi_e \otimes \xi_e}{|\xi_e|^2} = \left( I_2 0 \\ 0 0 \right) - \left( \frac{n' \otimes n'}{|\xi_e|^2} \frac{\xi_3 n'/e}{|\xi_e|^2} \frac{\xi_3 n'/e}{|\xi_e|^2} \right),
\]
we obtain (25).
2.3. Boundedness of the anisotropic Helmholtz projection

Next, we consider the equation (III) with boundary data $\phi = (\phi_+, \phi_-) \in C_0^\infty(\Omega)$. Applying the partial Fourier transformation to (III), we have

\[
\left\{ \begin{array}{l}
\left( \frac{\alpha^2}{\epsilon^2} - |n'|^2 \right) F_{d,x'} \pi_3(n', x_3) = 0, \\
\frac{\partial}{\partial n} F_{d,x'} \pi_3(n', \pm 1) = F_{d,x'} \phi_{\pm}(n'),
\end{array} \right.
\]

for $n' \in \mathbb{Z}^2 \setminus \{0\}$ and $x_3 \in (-1, 1)$. The solution to (27) is of the form

\[
F_{d,x'} \pi_3(n', x_3) = C_1 e^{\epsilon x_3 |n'|} + C_2 e^{-\epsilon x_3 |n'|}
\]

for some constant $C_1$ and $C_2$. Take the constants so that (27) satisfied, namely

\[
C_1 = \frac{F_{d,x'} \phi_+ + F_{d,x'} \phi_-}{4 |n'| \cosh(\epsilon |n'|)} + \frac{F_{d,x'} \phi_+ - F_{d,x'} \phi_-}{4 |n'| \sinh(\epsilon |n'|)},
\]

\[
C_2 = -\frac{F_{d,x'} \phi_+ + F_{d,x'} \phi_-}{4 |\xi'| \cosh(\epsilon |n'|)} + \frac{F_{d,x'} \phi_+ - F_{d,x'} \phi_-}{4 |\xi'| \sinh(\epsilon |n'|)},
\]

then the solution to (27) is given by

\[
\pi_3(x', x_3) = F_{d,n'}^{-1} \left( \frac{\sinh(\epsilon x_3 |n'|)}{|n'| \cosh(\epsilon |n'|)} \frac{F_{d,x'} \phi_+ + F_{d,x'} \phi_-}{2} \frac{\cosh(\epsilon x_3 |n'|)}{|n'| \sinh(\epsilon |n'|)} \frac{F_{d,x'} \phi_+ - F_{d,x'} \phi_-}{2} \right).
\]

Moreover, its anisotropic gradient given by

\[
\nabla_\pi \pi_3 = F_{d,n'}^{-1} \left( \frac{i n' \sinh(\epsilon x_3 |n'|)}{\cosh(\epsilon x_3 |n'|)} \frac{F_{d,x'} \phi_+ + F_{d,x'} \phi_-}{2} \frac{i n' \cosh(\epsilon x_3 |n'|)}{\sinh(\epsilon x_3 |n'|)} \frac{F_{d,x'} \phi_+ - F_{d,x'} \phi_-}{2} \right)
\]

\[
=: F_{d,n'}^{-1} \alpha e_+ (n', x_3) F_{d,x'} \phi_+ + F_{d,n'}^{-1} \alpha e_- (n', x_3) F_{d,x'} \phi_-.
\]

We apply the trace to (28) to get

\[
\gamma\pm \nabla_\pi \pi_3 = F_{d,n'}^{-1} \left( \frac{\pm i n' \sinh(\epsilon |n'|)}{|n'| \cosh(\epsilon |n'|)} \frac{F_{d,x'} \phi_+ + F_{d,x'} \phi_-}{2} \frac{\pm i n' \cosh(\epsilon |n'|)}{|n'| \sinh(\epsilon |n'|)} \frac{F_{d,x'} \phi_+ - F_{d,x'} \phi_-}{2} \right).
\]

We insert $\phi_{\pm} = \gamma\pm \frac{\mathbb{P}T^2 \times \mathbb{R}}{\epsilon} f$ to (28) for $f \in C_0^\infty(\Omega)$ satisfying $\text{ave}_H(f) = 0$ and set

\[
\Pi_\epsilon f := F_{d,n'}^{-1} \left[ \alpha e_+ (n', x_3) \gamma_+ F_{d,x'} \left( e_3 \cdot \frac{\mathbb{P}T^2 \times \mathbb{R}}{\epsilon} E_0 f \right) \right] + F_{d,n'}^{-1} \left[ \alpha e_- (n', x_3) \gamma_- F_{d,x'} \left( e_3 \cdot \frac{\mathbb{P}T^2 \times \mathbb{R}}{\epsilon} E_0 f \right) \right].
\]

Lemma 13. Let $1 < q < \infty$, $0 < \epsilon \leq 1$ and $s \geq 0$. Then, there exists a constant $C = C(q)$, which is independent of $\epsilon$, the operator $\Pi_\epsilon$ can be extended to a bounded operator from $W^{s,q}_{af}(\Omega)$ into itself such that

\[
||\Pi_\epsilon f||_{W^{s,q}(\Omega)} \leq C ||f||_{W^{s,q}(\Omega)}
\]

for all $f \in W^{s,q}_{af}(\Omega)$. 
Proof. Let \( f \in C_0^\infty(\Omega) \) satisfy \( \text{ave}_H(f) = 0 \). We seek the multiplier of \( \Pi_\epsilon \) by a direct calculation. Recall that the symbol of \( \mathbb{P}^{T^2 \times \mathbb{R}}_\epsilon \) is of the form

\[
\mathcal{F}_{x_3} \mathcal{F}_{d,x^1} \mathbb{P}^{T^2 \times \mathbb{R}}_\epsilon = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} n^2 \otimes n' \\ n_1 \otimes n' \end{pmatrix} \begin{pmatrix} \frac{\epsilon n_1}{n_1^2} & \frac{\epsilon n_1}{n_1^2} \\ \frac{\epsilon n_1}{n_1^2} & \frac{\epsilon n_1}{n_1^2} \end{pmatrix}.
\]

(30)

Since the symbol of \( \mathbb{P}^{T^2 \times \mathbb{R}}_\epsilon \) have poles at \( \xi_3 = \pm i \epsilon \abs{n'} \), we apply \( e_3 \) to (30) by the left-hand side and use the residue theorem so that the power of \( e \) is negative to get

\[
\mathcal{F}_{d,x'}(e_3 \cdot \mathbb{P}^{T^2 \times \mathbb{R}}_\epsilon E_0 f) = -\int_{-1}^{1} \frac{e^{-|x_3 - \zeta| \epsilon \abs{n'}}}{2} \mathcal{F}_{d,x'} f'(n', \zeta) \, d\zeta + \int_{-1}^{1} \frac{e^{-|x_3 - \zeta| \epsilon \abs{n'}}}{2} \mathcal{F}_{d,x'} f_3(n', \zeta) \, d\zeta.
\]

(31)

Note that the integration is due to the relationship between the Fourier transform and convolution. Applying trace operators \( \gamma_\pm \) and \( \alpha_\epsilon, \pm(n', x_3) \), respectively, and taking Fourier inverse transformation with respect to \( n' \), we get

\[
\Pi_\epsilon f(x', x_3)
\]

\[
= -\mathcal{F}_{d,n'}^{-1} \int_{-1}^{1} \alpha_{\epsilon, +}(n', x_3) \frac{e^{-|1 - \zeta| \epsilon \abs{n'}}}{2} \mathcal{F}_{d,x'} f'(n', \zeta) \, d\zeta + \mathcal{F}_{d,n'}^{-1} \int_{-1}^{1} \alpha_{\epsilon, +}(n', x_3) \frac{e^{-|1 - \zeta| \epsilon \abs{n'}}}{2} \mathcal{F}_{d,x'} f_3(n', \zeta) \, d\zeta
\]

\[
= -\mathcal{F}_{d,n'}^{-1} \int_{-1}^{1} \alpha_{\epsilon, -}(n', x_3) \frac{e^{-|1 - \zeta| \epsilon \abs{n'}}}{2} \mathcal{F}_{d,x'} f'(n', \zeta) \, d\zeta + \mathcal{F}_{d,n'}^{-1} \int_{-1}^{1} \alpha_{\epsilon, -}(n', x_3) \frac{e^{-|1 - \zeta| \epsilon \abs{n'}}}{2} \mathcal{F}_{d,x'} f_3(n', \zeta) \, d\zeta
\]

\[
=: I_1 + I_2 + I_3 + I_4.
\]

(32)

By the definition of \( \alpha_{\pm, \epsilon} \),

\[
I_1 = -\mathcal{F}_{d,n'}^{-1} \int_{-1}^{1} \frac{1}{2} \left( \frac{\epsilon n' \sinh(\epsilon x_3 \abs{n'})}{\abs{n'} \cosh(\epsilon \abs{n'})} + \frac{\epsilon n' \cosh(\epsilon x_3 \abs{n'})}{\cosh(\epsilon \abs{n'})} \right) \mathcal{F}_{d,x'} f'(n', \zeta) \, d\zeta.
\]

Symbols in the integral are written by \( A(\epsilon n')(1 + \epsilon \abs{n'})e^{-\epsilon \abs{n'}(|x_3| + \abs{\zeta} + 1)} \) with a symbol \( A \). The Mikhlin constant of \( A(\epsilon n') \) is independent of \( \epsilon \) by Proposition 10. The same argument is valid for \( I_j \) (\( j = 2, 3, 4 \)). Thus, we conclude by Propositions 8 and 9 that
\[ \| \Pi \epsilon f \|_{L^q(\Omega)} \leq C \| f \|_{L^q(\Omega)} + C \left| \int_{-1}^{1} \left\| f(\cdot, \xi) \right\|_{L^q(-1,1)} \, d\xi \right| \leq C \| f \|_{L^q(\Omega)} \]

for all \( f \in C_0^\infty(\Omega) \) satisfying \( \text{ave}_H(f) = 0 \), where the constant \( C \) is independent of \( \epsilon \). Thus, the estimate (29) holds for \( s = 0 \). We find from the formula (32) that \( \partial_j \) commutes with \( \Pi \epsilon \) for \( j = 1, 2 \). Moreover, the equation (27) implies

\[
\partial_3^2 \Pi \epsilon f = -\epsilon^2 (\partial_1^2 + \partial_2^2) \Pi \epsilon f \\
= -\epsilon^2 \Pi \epsilon (\partial_1^2 + \partial_2^2) f.
\]

Thus, we conclude that (29) holds for all positive even numbers. We obtain (29) for all \( s > 0 \) by interpolation.

□

We set the operator

\[ P_{N,\epsilon} := R_0 \left( \mathbb{T}^2 \times \mathbb{R} - \Pi \epsilon \right) f \]

for all \( f \in C_0^\infty(\Omega) \) satisfying \( \text{ave}_H(f) = 0 \). Then, Lemma 13 implies

**Corollary 14.** Let \( 1 < q < \infty \), \( 0 < \epsilon \leq 1 \) and \( s \geq 0 \). Then, there exists a constant \( C > 0 \), which is independent of \( \epsilon \), the operator \( P_{N,\epsilon} \) can be extended to a bounded operator from \( W^{s,q}_a(\Omega) \) into itself such that

\[ \| P_{N,\epsilon} f \|_{W^{s,q}_a(\Omega)} \leq C \| f \|_{W^{s,q}_a(\Omega)} \]

for all \( f \in W^{s,q}_a(\Omega) \).

**Remark 15.** Note that \( P_{N,\epsilon} \) is not the anisotropic Helmholtz projection on \( \Omega \). \( P_{N,\epsilon} \) is the operator which maps from the \( L^q \)-vector fields into \( L^q \)-divergence-free vector fields with tangential trace. However, we find that the anisotropic Helmholtz projection is bounded from \( L^q(\Omega) \) into itself by the same method of Lemma 13.

**Lemma 16.** Let \( 1 < q < \infty \) and \( 0 < \epsilon \leq 1 \). Then, there exists a constant \( C > 0 \), which is independent of \( \epsilon \), such that

\[ \| H_\epsilon f \|_{L^q(\Omega)} \leq C \| f \|_{L^q(\Omega)} \]

for all \( f \in L^q(\Omega) \).

**Proof.** Let \( u \in C_0^\infty(\Omega) \). Then, we obtain the solution \( \pi_\epsilon \) to the Neumann problem

\[
\begin{cases}
\Delta_\epsilon \pi_\epsilon = \text{div}_\epsilon u & \text{in } \Omega, \\
\gamma_\epsilon \frac{\partial_3 \pi_\epsilon}{\epsilon} = u \cdot \nu_+ & \text{on } \partial \Omega.
\end{cases}
\]

(33)

The anisotropic Helmholtz projection \( H_\epsilon \) is represented by

\[ H_\epsilon u = u - \nabla_\epsilon \pi_\epsilon. \]
In the case of the Dirichlet boundary condition, i.e., \( \gamma u = 0 \), the right-hand side of the second equality of (33) is zero. Let us consider the \( L^q \)-boundedness of \( \nabla \epsilon \pi_{\epsilon} \), which implies the boundedness of the anisotropic Helmholtz projection. For the solution \( \pi^0 \) to the equation

\[
\begin{cases}
\partial_3^2 \pi^0(x_3)/\epsilon^2 = \partial_3[\text{ave}_H(u_3)(x_3)]/\epsilon, & x_3 \in (-1, 1), \\
\partial_3 \pi^0(\pm 1)/\epsilon = 0,
\end{cases}
\]

we have

\[
\nabla \epsilon \pi^0 = (0, 0, \text{ave}_H(u_3))^T.
\]

Let \( \pi_{\epsilon}^1 \) and \( \pi_{\epsilon}^2 \) be the solutions to

\[
\Delta \epsilon \pi_{\epsilon}^1 = E_0 \text{div}_\epsilon u \quad \text{in} \quad \mathbb{T}^2 \times \mathbb{R},
\]

and

\[
\begin{cases}
\Delta \epsilon \pi_{\epsilon}^2 = 0 & \text{in} \quad \Omega, \\
\gamma \pm \partial_3 \pi_{\epsilon}^2 /\epsilon = -\gamma \pm v \cdot \nabla \epsilon \Delta_{\epsilon}^{-1} E_0 \text{div}_\epsilon u & \text{on} \quad \partial \Omega,
\end{cases}
\]

respectively, for \( u \in C^\infty_0(\Omega) \) satisfying \( \text{ave}_H(u) = 0 \).

We first consider (35). It follows from integration by parts that

\[
\mathcal{F}_{x_3} \mathcal{F}_{d,x'} E_0 \text{div}_\epsilon u = \mathcal{F}_{d,x'} \int_{-1}^1 e^{-ix_3 \xi_3} \left( \text{div}_H u' \cdot x_3 + \frac{\partial_3 u_3(\cdot, x_3)}{\epsilon} \right) \, dx_3
\]

\[
= i \left( n' \xi_3/\epsilon \right) \cdot \mathcal{F}_{d,x'}(E_0 u)
\]

\[
= \mathcal{F}_{x_3} \mathcal{F}_{d,x'} \text{div}_\epsilon(E_0 u).
\]

This formula, the Mikhlin theorem and Proposition 5 imply

\[
\left\| \nabla \epsilon \pi_{\epsilon}^1 \right\|_{L^q(\Omega)} \leq C \| u \|_{L^q(\Omega)},
\]

where \( C > 0 \) is independent of \( \epsilon \). Moreover, since \( e_3 \cdot \nabla \epsilon \Delta_{\epsilon}^{-1} \text{div}_\epsilon \) is given by the left-hand side of (31), we use the same method as in Lemma 13 to get

\[
\left\| \nabla \epsilon \pi_{\epsilon}^2 \right\|_{L^q(\Omega)} \leq C \| u \|_{L^q(\Omega)},
\]

where \( C > 0 \) is also independent of \( \epsilon \). The formula (34) and estimates (36) and (37) imply \( L^p \)-boundedness of the anisotropic Helmholtz projection on \( \Omega \).

**Proposition 17.** Let \( 0 < \epsilon \leq 1, 1 < q < \infty, 0 < a < 1/2, z \in \mathbb{C} \) satisfying \( -a < \text{Re} z < 0 \) and \( 0 < \theta < \pi/2 \). Then, there exists a constant \( C = C(q, a, \theta) \), which is independent of \( \epsilon \), the solution \( \pi_3 \) to (III) with boundary data \( (\gamma K_{\lambda, \epsilon} f \cdot v) \nu \) satisfies

\[
\left\| \frac{1}{2\pi i} \int_{I_0} (\lambda)^z \nabla \epsilon \pi_3 \, d\lambda \right\|_{L^q(\Omega)} \leq Ce^{\theta |\text{Im} z|} \| f \|_{L^q(\Omega)}
\]

for all \( f \in L^q(\Omega) \).
**Proof.** In view of Remark 7, we may assume \( \text{ave}_H(f) = 0 \) without loss of generality. Since

\[
\nabla \pi_3 = \Pi_\epsilon K_{\lambda, \epsilon} \mathcal{E}_0 f
\]

and the Cauchy integral commutes with \( \Pi_\epsilon \), the conclusion is obtained from Proposition 11 and Lemma 13.

**Proposition 18.** Let \( 0 < \epsilon \leq 1 \), \( 1 < q < \infty \), \( 0 < \theta < \pi / 2 \) and \( \lambda \in \Sigma_\theta \). Then, there exists a constant \( C = C(q, \theta) \), which is independent of \( \epsilon \), the solution \( \pi_3 \) to (III) with boundary data \( (\gamma K_{\lambda, \epsilon} f \cdot \nu) \nu \) satisfies

\[
\| \nabla^2 \nabla \pi_3 \|_{L^q(\Omega)} \leq C \| f \|_{L^q(\Omega)}
\]  

for all \( f \in L^q(\Omega) \).

**Proof.** The estimate (39) is a direct consequence of (24), (38), Lemma 13 and Proposition 12.

2.4. Estimate for \( v_2 \)

Let us consider the equation (II) with tangential boundary data \( g = (g_+, g_-) \). Set

\[
y_{\lambda, \epsilon}'(n') = 2s_\lambda \left( I_2 + \frac{\epsilon |n'| n' \otimes n'}{s_\lambda |n'|^2} \right),
\]

\[
y_{\lambda, \epsilon}(n') = \begin{pmatrix} y_{\lambda, \epsilon}'(n') & 0 \\ 0 & 0 \end{pmatrix}.
\]

Then, \( y_{\lambda, \epsilon} \) satisfies

\[
k_{\lambda, \epsilon}'(n', 0)y_{\lambda, \epsilon}(n') = J_2 := \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix},
\]

where \( k_{\lambda, \epsilon}' \) is defined by (25). We define the multiplier operator \( L_{\lambda, \epsilon} \) as

\[
L_{\lambda, \epsilon} g(n', x_3) = \mathbb{P}_{\epsilon}^{\mathbb{T}^2 \times \mathbb{R}} \mathcal{F}_d^{d,n'}[- \epsilon_{\lambda}'(n', 1 - x_3)y_{\lambda, \epsilon}(n')\mathcal{F}_d x' g_+(n')] + \mathbb{P}_{\epsilon}^{\mathbb{T}^2 \times \mathbb{R}} \mathcal{F}_d^{d,n'}[- \epsilon_{\lambda}'(n', -1 - x_3)y_{\lambda, \epsilon}(n')\mathcal{F}_d x' g_-(n')],
\]

where \( \epsilon_{\lambda}' \) is defined by (26). Let \( p_{\epsilon}'(n', x_3) \) be a partial Fourier transform of the symbol of \( \mathbb{P}_{\epsilon}^{\mathbb{T}^2 \times \mathbb{R}} \) with respect to \( \xi_3 \). Then, we obtain

\[
L_{\lambda, \epsilon} g(n', \cdot) = \mathcal{F}_d^{d,n'}[- \epsilon_{\lambda}'(n', 1 - \cdot)y_{\lambda, \epsilon}(n')\mathcal{F}_d x' g_+(n')] + \mathcal{F}_d^{d,n'}[- \epsilon_{\lambda}'(n', -1 - \cdot)y_{\lambda, \epsilon}(n')\mathcal{F}_d x' g_-(n')],
\]

where \( \cdot *_3 \cdot \) is convolution with respect to \( x_3 \). We set

\[
W_{\lambda, \epsilon} = P_{N, \epsilon} L_{\lambda, \epsilon}.
\]
Then, $W_{\lambda, \epsilon}g$ is a solution to (II) with boundary data $\gamma W_{\lambda, \epsilon}g$. We first get the Fourier multiplier of $\gamma W_{\lambda, \epsilon}$. Next, we show the map $S_{\lambda, \epsilon} : g \mapsto \gamma W_{\lambda, \epsilon}g$ has a bounded inverse for large $\lambda$. Put

$$V_{\lambda, \epsilon}g = W_{\lambda, \epsilon}S_{\lambda, \epsilon}^{-1}g, \quad (45)$$

then, $V_{\lambda, \epsilon}g$ gives the solution to (II) with boundary data $g$.

**Proposition 19.** Let $1 < q < \infty$, $0 < \epsilon \leq 1$, $s \geq 0$ and $0 < \theta < \pi/2$. Then, there exist $r > 0$ and, for $\lambda \in \Sigma_{\theta}$ satisfying $|\lambda| > r$, a bounded operator $R_{\lambda, \epsilon}$ from $W^{s,q}(\mathbb{T}^2)$ into itself satisfying

$$\|R_{\lambda, \epsilon}\|_{W^{s,q}_{af}(\mathbb{T}^2) \to W^{s,q}_{af}(\mathbb{T}^2)} \leq \frac{C}{|\lambda|^{1/2}}, \quad (46)$$

and

$$\|R_{\lambda, \epsilon}\|_{W^{s,q}_{af}(\mathbb{T}^2) \to W^{s+1,q}_{af}(\mathbb{T}^2)} \leq C, \quad (47)$$

where $C > 0$ is independent of $\epsilon$, such that

$$-S_{\lambda, \epsilon}^{-1} = I + R_{\lambda, \epsilon}. \quad (48)$$

**Proof.** Let $g \in C^\infty(\mathbb{T}^2)$ be horizontal average-free. Since $e_{\lambda}'$ is an even function with respect to $x_3$, we find from the change of variable that

$$p'_{\epsilon}(n', \cdot) *_3 e_{\lambda}'(n', 1 - \cdot) = \int_{\mathbb{R}} p'_{\epsilon}(n', \cdot - \zeta) e_{\lambda}'(n', 1 - \zeta) \, d\zeta$$

$$= -\int_{\mathbb{R}} p'_{\epsilon}(n', \eta) e_{\lambda}'(n', -1 + \cdot - \eta) \, d\eta$$

$$= -k'_{\lambda, \epsilon}(n', -1 + \cdot),$$

and similarly

$$p'_{\epsilon}(n', \cdot) *_3 e_{\lambda}'(n', -1 - \cdot) = -k'_{\lambda, \epsilon}(n', 1 + \cdot).$$

Thus, we find from (43) that

$$L_{\lambda, \epsilon}g(n', x_3) = \mathcal{F}^{-1}_{d,n'} \left[ -k'_{\lambda, \epsilon}(n', -1 + x_3) y'_{\lambda, \epsilon}n' \mathcal{F}_{d,x'}g+(n') \right] + \mathcal{F}^{-1}_{d,n'} \left[ -k'_{\lambda, \epsilon}(n', 1 + x_3) y'_{\lambda, \epsilon}(n') \mathcal{F}_{d,x'}g-(n') \right]. \quad (49)$$

We apply $P_{N, \epsilon}$ to (49) to get

$$S_{\lambda, \epsilon}g = \gamma_{\pm} W_{\lambda, \epsilon}g$$

$$= -\mathcal{F}^{-1}_{d,n'} \left[ k'_{\lambda, \epsilon}(n', -1 \pm 1) y_{\lambda, \epsilon}(\xi') \mathcal{F}_{d,x'}g+(n') \right]$$

$$-\mathcal{F}^{-1}_{d,n'} \left[ \alpha_{+, \epsilon}(n', \pm 1) e_3 \cdot k'_{\lambda, \epsilon}(n', 2) y_{\lambda, \epsilon}(n') \mathcal{F}_{d,x'}g-(n') \right]$$

$$-\mathcal{F}^{-1}_{d,n'} \left[ k'_{\lambda, \epsilon}(n', 1 \pm 1) y_{\lambda, \epsilon}(\xi') \mathcal{F}_{d,x'}g-(n') \right]$$

$$-\mathcal{F}^{-1}_{d,n'} \left[ \alpha_{-, \epsilon}(n', \pm 1) e_3 \cdot k'_{\lambda, \epsilon}(n', -2) y_{\lambda, \epsilon}(n') \mathcal{F}_{d,x'}g+(n') \right]$$

$$= I_1 + I_2 + I_3 + I_4. \quad (50)$$
Let us estimate \( I_1 \) and \( I_3 \). The identity (41) implies

\[
\mathcal{F}_{d,n}^{-1} \left[ k_{\lambda,\epsilon}'(n', 0) y_{\lambda,\epsilon}(n') \mathcal{F}_{d,n} g \pm(n') \right] = g \pm.
\] (51)

We show the other terms are \( O(1/|\lambda|^{1/2}) \). By (25) and (40), we have

\[
k_{\lambda,\epsilon}'(\xi', \pm 2) y_{\lambda,\epsilon}(n')
= e^{-s_{\lambda}} \left( J_2 + \frac{\epsilon |n'|}{s_{\lambda}} J_2 n \otimes J_2 n \right)
\]

\[
- J_2 n \otimes J_2 n \frac{e^2}{\lambda + (1 - \epsilon^2) |n'|^2} e^{-2s_{\lambda}} \left( J_2 + \frac{\epsilon |n'|}{s_{\lambda}} J_2 n \otimes J_2 n \right)
\]

\[
- J_2 n \otimes J_2 n \frac{e^2}{\lambda + (1 - \epsilon^2) |n'|^2} e^{-2\epsilon |n'|} \frac{e^{-2s_{\lambda}}}{s_{\lambda}} \left( J_2 + \frac{\epsilon |n'|}{s_{\lambda}} J_2 n \otimes J_2 n \right)
\]

\[=: II_1 + II_2 + II_3.
\]

We find from (16) and (17) in Proposition 10 and the estimate

\[
\left[ \frac{|\xi'|}{s_{\lambda}} \right]_{\mathcal{M}'} + \left[ \frac{J_2 \xi \otimes J_2 \xi}{|\xi'|^2} \right]_{\mathcal{M}'} \leq C, \quad \xi = (\xi', \xi_3) \in \mathbb{R}^3,
\] (52)

that

\[
[II_1]_{\mathcal{M}'} \leq C e^{-c|\lambda|^2}, \quad \left[ |\xi'| \right]_{II_1} \leq C e^{-c|\lambda|^2},
\] (53)

where we interpret that the multiplier \( II_1 \) is extended from \( \mathbb{Z}^2 \) to \( \mathbb{R}^2 \) by the canonical way. Since

\[
\left[ \frac{1}{\lambda + (1 - \epsilon^2) |\xi'|^2} \right]_{\mathcal{M}'} \leq \frac{C}{|\lambda|},
\] (54)

by the same way as above, we have

\[
[II_2]_{\mathcal{M}'} \leq \frac{C e^{-c|\lambda|^{1/2}}}{|\lambda|}, \quad \left[ |\xi'| \right]_{II_2} \leq C e^{-c|\lambda|^{1/2}},
\] (55)

for \( \lambda \in \Sigma_{\theta} \), where the constants \( c \) and \( C > 0 \) are independent of \( \epsilon \). Note that \( II_3 \) has a little bit problem near \( \epsilon = 0 \) since we cannot use the decay of \( e^{-2\epsilon |\xi'|} \) to obtain uniform boundedness of the Mikhlin constant at this point. However, we can use the decay of \( 1/(\lambda + (1 - \epsilon^2) |\xi'|^2) \) around \( \epsilon = 0 \). On the other hand, when \( \epsilon \) is away from 0, we have no problem to use decay of \( e^{-2\epsilon |\xi'|} \). Thus, combining this observation with Proposition 9, (52) and (54), we conclude that

\[
[II_3]_{\mathcal{M}'} \leq \frac{C}{|\lambda|^2}, \quad \left[ |\xi'| \right]_{II_3} \leq C,
\] (56)
where $C > 0$ is independent of $\epsilon$. Thus, we find from (53), (55) and (56) that

$$
||I_1 + I_3 - g+ - g-||_{L^q(\mathbb{T}^2)}
\leq \left|\left|\mathcal{F}_{d,n}^{-1} \left[ k_{\lambda,\epsilon}^{(n')}(n', 2)y_{\lambda,\epsilon}(n') \mathcal{F}_{d,x'} g(n') \right] \right|_{L^q(\Omega)} \right|
+ \left|\left|\mathcal{F}_{d,n}^{-1} \left[ k_{\lambda,\epsilon}^{(n')}(-2)y_{\lambda,\epsilon}(n') \mathcal{F}_{d,x'} g(n') \right] \right|_{L^q(\Omega)} \right|
\leq \frac{C}{|\lambda|^2} ||g||_{L^q(\mathbb{T}^2)}. \quad (57)
$$

Next, we estimate $I_2$ and $I_4$. It follows from (25) that

$$
e_3 \cdot k_{\lambda,\epsilon}^{(n')} (n', \pm 2) y_{\lambda,\epsilon}(n') \mathcal{F}_{d,x'} g_+(n')
= e_3 \cdot \left[ \frac{e^{-s_\lambda}}{2s_\lambda} y_{\lambda,\epsilon}(n') \mathcal{F}_{d,x'} g_+ - \left( \eta^{(n')}_{\lambda,\epsilon}(n', \pm 2) n' \otimes n' y^{(n')}_{\lambda,\epsilon}(n') 0 \right) \mathcal{F}_{d,x'} g_+ \right]
= - \left( -\partial_3 \eta^{(n')}_{\lambda,\epsilon}(n', \pm 2) n' \otimes n' y^{(n')}_{\lambda,\epsilon}(n') 0 \right) \mathcal{F}_{d,x'} g_+. \quad (58)
$$

Recall $\partial_3 \eta_{\lambda,\epsilon}(n', \pm 2) = \frac{e^2}{\lambda + (1 - e^2) |n'|^2} \frac{e^{-2s_\lambda} - e^{-2s_\lambda}}{2}$. Then, we find from the first inequality of (16) and (54) that

$$
\left[ (1 + \epsilon \left| \xi' \right|) \partial_3 \eta_{\lambda,\epsilon} (\xi', \pm 2) y^{(\xi')}_{\lambda} \right]_{\mathcal{M}'}
= \left[ \frac{e^2}{\lambda + (1 - e^2) |\xi'|^2} (1 + \epsilon \left| \xi' \right|) s_\lambda \left( e^{-2s_\lambda} - e^{-2s_\lambda} \right) \left( I_2 + \frac{\epsilon \left| \xi' \right|}{s_\lambda} \frac{\xi' \otimes \xi'}{|\xi'|^2} \right) \right]_{\mathcal{M}'}
\leq \frac{C}{|\lambda|^2},
$$

and

$$
\left[ \left| \xi' \right| (1 + \epsilon \left| \xi' \right|) \partial_3 \eta_{\lambda,\epsilon} (\xi', \pm 2) y^{(\xi')}_{\lambda} \right]_{\mathcal{M}'} \leq C, \quad (59)
$$

where $C > 0$ is independent of $\epsilon$. The formula (28), estimates (17) and (18) lead to

$$
\sup_{0 < \epsilon < 1} \left[ \frac{\alpha_{\pm,\epsilon} (\xi', \pm 1) \epsilon \left| \xi' \right|}{1 + \epsilon \left| \xi' \right|} \right]_{\mathcal{M}'} < \infty.
$$

We conclude from Proposition 5 that

$$
||I_2 + I_4||_{L^q(\mathbb{T}^2)}
\leq \left|\left|\mathcal{F}_{d,n}^{-1} \alpha_{+,\epsilon}^{(n')} (n', \pm 1) e_3 \cdot k_{\lambda,\epsilon}^{(n')} (n', 2) y_{\lambda,\epsilon}(n') \mathcal{F}_{d,x'} g - \right|_{L^q(\mathbb{T}^2)} \right|
+ \left|\left|\mathcal{F}_{d,n}^{-1} \alpha_{-,\epsilon}^{(n')} (n', \pm 1) e_3 \cdot k_{\lambda,\epsilon}^{(n')} (n', -2) y_{\lambda,\epsilon}(n') \mathcal{F}_{d,x'} g_+ \right|_{L^q(\mathbb{T}^2)} \right|
\leq \frac{C}{|\lambda|^2} ||g||_{L^q(\mathbb{T}^2)}, \quad (60)
$$
where $C$ is independent of $\epsilon$. Thus, taking $|\lambda|$ sufficiently large; clearly, the choice of $\lambda$ is also independent of $\epsilon$; we conclude by (50), (51), (57) and (60) that

$$-S_{\lambda, \epsilon} = I + O(|\lambda|^{-1/2}).$$

By the Neumann series argument, we obtain (46) for $s = 0$. Moreover, we find from (53), (55), (56) and (59) that (47) holds for $s = 0$. Since $\partial_j$ ($j = 1, 2$) commutes Fouier multiplier operators, we obtain (46) and (47) for $s > 0$.

**Proposition 20.** Let $1 < q < \infty$, $0 < \epsilon \leq 1$, $0 < \theta < \pi/2$ and $\lambda \in \Sigma_\theta$. Then, there exist $r > 0$ and a constant $C > 0$, which is independent of $\epsilon$ and $\lambda$, if $|\lambda| \geq r$, $V_{\lambda, \epsilon}$ defined by (45) satisfies

$$\left| \left| V_{\lambda, \epsilon} g \right| \right|_{L^q(\partial \Omega)} \leq C |\lambda|^{-1/2q} \left| \left| g \right| \right|_{L^q(\partial \Omega)}$$

for all $g \in L^q(\partial \Omega)$ satisfying $\text{ave}_H(g) = 0$.

**Proof.** We take $r > 0$ so that $R_{\lambda, \epsilon}$ exists. Then, $S_{\lambda, \epsilon}^{-1}$ is bounded on $L^q(\Omega)$. We find from the resolvent estimate for the Dirichlet Laplacian on $\Omega$, see Lemma 5.3 in [1] and Proposition 19 that

$$\left| \left| L_{\lambda, \epsilon} S_{\lambda, \epsilon}^{-1} g \right| \right|_{L^q(\Omega)} \leq C |\lambda|^{-1/2q} \left| \left| g \right| \right|_{L^q(\Omega)},$$

where $C > 0$ is independent of $\epsilon$. By Corollary 14, we obtain (61).

**Proposition 21.** Let $1 < q < \infty$ and $0 < \epsilon \leq 1$. Then, there exists a constant $C > 0$, which is independent of $\epsilon$, such that

$$\left| \left| \mathcal{F}_{\delta, \epsilon, \delta} \frac{1 + \epsilon}{|n'|} \left( e_3 \cdot \mathcal{F}_{\delta, \epsilon} \left( \mathbb{P}_\epsilon \mathbb{R}_0 f \right) \right) \right| \right|_{L^q(\Omega)} \leq C \left| \left| f \right| \right|_{L^q(\Omega)}$$

for all $f \in L^q(\Omega)$.

**Proof.** Since the symbol has poles at $\xi_3 = \pm i \epsilon |n'|$, we calculate its partial Fourier transform with respect to $\xi_3$ by the residue theorem and obtain

$$\mathcal{F}_{\delta, \epsilon} \frac{1}{|n'|} \left( e_3 \cdot \mathcal{F}_{\delta, \epsilon} \left( \mathbb{P}_\epsilon \mathbb{R}_0 f \right) \right) = \frac{1}{2} \mathcal{F}_{\delta, \epsilon} \int_{-1}^{1} \frac{1}{|n'|} \left[ e^{-|\xi_3 - \zeta|} i \epsilon n' \cdot \mathcal{F}_{\delta, \epsilon} f'(n', \zeta) + e^{-|\xi_3 - \zeta|} \epsilon n' \cdot \mathcal{F}_{\delta, \epsilon} f_3(n', \zeta) \right] d\zeta.$$

This formula and Proposition 5 imply

$$\left| \left| \mathcal{F}_{\delta, \epsilon} \frac{1}{|n'|} \left( e_3 \cdot \mathcal{F}_{\delta, \epsilon} \left( \mathbb{P}_\epsilon \mathbb{R}_0 f \right) \right) \right| \right|_{L^q(\mathbb{R}^2)} \leq C \left| \left| f \right| \right|_{L^q(\mathbb{R}^2)},$$

where $C$ is independent of $\epsilon$. Combining this estimate with the boundedness of $\mathbb{P}_\epsilon \mathbb{R}_0$, we obtain (62).
Let us show BIP for the solution operator for the equation (II).

**Proposition 22.** Let $1 < q < \infty$, $0 < \epsilon \leq 1$, $0 < \theta < \pi/2$, $\lambda \in \Sigma_\theta$, $0 < a < 1/2$, and $z$ satisfying $-a < \text{Re} z < 0$. Then, there exists a constant $C = C(q, a, \theta)$, it holds that

$$
\left\| \frac{1}{2\pi i} \int_{\gamma} (-\lambda)^z V_{\lambda, \epsilon} \left[ \gamma v_1 - (\gamma v_1 \cdot v)v \right] \, d\lambda \right\|_{L^q(\Omega)} \leq C e^{\left|\text{Im} z\right| \theta} ||f||_{L^q(\Omega)}
$$

(63)

for all $f \in L^q(\Omega)$, where $v_1 = K_{\lambda, \epsilon} f$.

**Proof.** In view of Remark 7, we may assume $\text{ave}_H (f) = 0$ without loss of generality. It holds by (24) that

$$
\gamma v_1 - (\gamma v_1 \cdot v)v = \gamma K_{\lambda, \epsilon} E_0 f - \gamma \Pi_\epsilon K_{\lambda, \epsilon} E_0 f.
$$

We find from this formula, (28), (42), (44), (45) and (48) that the integrand of the left-hand side of (63) can be essentially written as

$$
P_{N, \epsilon} \mathbb{P}_{\mathbb{T}^2 \times \mathbb{R}}^{(d, n')_\epsilon} \int_{-1}^{1} e'(\xi', \pm 1 - x_3) y_{\lambda, \epsilon}(n')^* e'(\xi', \pm 1 - \xi)
$$

$$
\times \mathcal{F}_{d, x'} \left( \mathbb{P}_{\mathbb{T}^2 \times \mathbb{R}}^{(d, n')_\epsilon} E_0 f \right) (n', \xi) \, d\xi
$$

$$
+ P_{N, \epsilon} \mathbb{P}_{\mathbb{T}^2 \times \mathbb{R}}^{(d, n')_\epsilon} \int_{-1}^{1} e'(n', \pm 1 - x_3) y_{\lambda, \epsilon}(n')^* e'(n', \pm 1 + x_3)
$$

$$
\times \frac{e|n'|}{1 + e|n'|} \mathcal{F}_{d, x'} \left( \mathbb{P}_{\mathbb{T}^2 \times \mathbb{R}}^{(d, n')_\epsilon} E_0 f \right) (n', \xi) \, d\xi
$$

$$
+ W_{\lambda, \epsilon} R_{\lambda, \epsilon} \left[ \gamma K_{\lambda, \epsilon} E_0 f - \gamma \Pi_\epsilon K_{\lambda, \epsilon} E_0 f \right] = I_1 + I_2 + I_3,
$$

(64)

where $\pm$ should be take properly.

It follows from (16) and (52) that

$$
\left[ e'(\xi', \pm 1 - x_3) y_{\lambda, \epsilon}(\xi') e'(\xi', \pm 1 - \xi) \right]_{\mathcal{M}'}
$$

$$
= 2 \left[ e^{-|\pm 1 - x_3| \xi} \left( I_2 + \frac{e |\xi'|}{s_{\lambda}} \frac{\xi' \otimes \xi'}{|\xi'|^2} e^{-|\pm 1 - \xi| s_{\lambda}} \right) \right]_{\mathcal{M}'}
$$

$$
\leq C e^{-c|\lambda|^{1/2}} \frac{|\lambda|^{1/2}}{|\lambda|^{1/2}}
$$

(65)

Let $R > 0$ be large enough so that $S_{\lambda, \epsilon}^{-1}$ in Proposition 19 exists. Then, we find from the change of integral curve around the origin to ensure $|\lambda| > R$ and Proposition 5 that
Lemma 5.3 in [1], and Lemma 13 imply $\delta > 0$ for some small $\delta > 0$. Applying Proposition 8, we obtain

$$
\left| \frac{1}{2\pi i} \int_{\Gamma} (-\lambda)^{\frac{m}{2}} I_1 \, d\lambda \right|_{L^q(\Omega)} \leq C \left| \int_{-1}^{1} \int_{R} \frac{e^{-c|\lambda|^{1/2}(|x_3-a|+|\zeta-b|)} e^{-c|\lambda|^{1/2}(|x_3-a|+|\zeta-b|)}}{\lambda^{1/2}} \left| \mathbb{P}_\epsilon^{L^2_{\Omega}} \mathcal{E}_0 f (\cdot, \zeta) \right|_{L^q(\Omega)} \, d\lambda \, d\zeta \right|_{L^q(-1,1)}
$$

Thus, we find from Proposition 21 that

$$
\left| \frac{1}{2\pi i} \int_{\Gamma} (-\lambda)^{\frac{m}{2}} I_2 \, d\lambda \right|_{L^q(\Omega)} \leq C e^{\theta \text{Im} \lambda} \left\| f \right\|_{L^q(\Omega)}.
$$

It follows from (26), (28), (40) and Proposition 10 that

$$
\left| \mathcal{E}_0 f (\cdot, \zeta) \right|_{L^q(\Omega)} \leq C \frac{e^{-c|\lambda|^{1/2}(|x_3-a|+|\zeta-b|)}}{|\lambda|^{1/2}} \mathcal{M}(\xi).
$$

Thus, we find from Proposition 21 that

$$
\left| \frac{1}{2\pi i} \int_{\Gamma} (-\lambda)^{\frac{m}{2}} I_2 \, d\lambda \right|_{L^q(\Omega)} \leq C e^{\theta \text{Im} \lambda} \left\| f \right\|_{L^q(\Omega)}.
$$

By Proposition 19, the trace theorem, Lemma 13 and the resolvent estimate for the Laplace operator on $\mathbb{T}^2 \times \mathbb{R}$, we have

$$
\left| \mathcal{E}_0 f (\cdot, \zeta) \right|_{L^q(\Omega)} \leq C \frac{e^{-c|\lambda|^{1/2}(|x_3-a|+|\zeta-b|)}}{|\lambda|^{1/2}} \mathcal{M}(\xi).
$$

for some small $\delta > 0$. The resolvent estimate for the Dirichlet Laplacian on $\Omega$, see Lemma 5.3 in [1], and Lemma 13 imply

$$
\left| P_{N_{\epsilon}} L_{\lambda, \epsilon} \right|_{L^q(\partial \Omega) \rightarrow L^q(\Omega)} \leq C \frac{1}{|\lambda|^{1/2}}
$$
for some small $\delta > 0$. We find from the above two inequalities

$$||I_3||_{L^q(\Omega)} \leq C |\lambda|^{-3/2+\delta} ||f||_{L^q(\Omega)}.$$  \hspace{1cm} (68)

Thus, we find from the change of integral line around the origin that

$$\left| \frac{1}{2\pi i} \int_{\Gamma} (-\lambda)^\gamma I_3 \, d\lambda \right|_{L^q(\Omega)} \leq C ||f||_{L^q(\Omega)},$$

where $C > 0$ is independent of $\epsilon$. \hspace{1cm} $\Box$

**Proof of Lemma 6.** Lemma 6 is a direct consequence of Propositions 11, 17 and 22. \hspace{1cm} $\Box$

We next prove Lemma 3 from Lemma 6. For this purpose we need further uniform estimate for the resolvent to compare $\|\nabla^2 u\|_{L^q(\Omega)}$ and $\|A_\epsilon u\|_{L^q(\Omega)}$. For resolvent estimates we begin with Proposition 23.

**Proposition 23.** Let $1 < q < \infty$, $0 < \epsilon \leq 1$, $0 < \theta < \pi/2$. Let $\lambda \in \Sigma_\theta$ be sufficiently large so that $S_{\lambda, \epsilon}^{-1}$ exists in Proposition 19. Then, there exists a constant $C = C(q, \theta)$, it holds that

$$\left\| \nabla^2 V_{\lambda, \epsilon} \left[ (\gamma v_1 - (\gamma v_1 \cdot v))v \right] \right\|_{L^q(\Omega)} \leq C ||f||_{L^q(\Omega)}$$

for all $f \in L^q(\Omega)$, where $v_1 = K_{\lambda, \epsilon} f$.

**Proof.** In view of Remark 7, we may assume $\ave_H(f) = 0$ without loss of generality. It is enough to estimate the second derivative of the left-hand side of (64) in $L^q(\Omega)$. We find from (16) and (65) that

$$\left[ |n'|^2 e_\lambda' (\xi', \pm 1 - x_3) y_{\lambda, \epsilon} (\xi') e_\lambda' (\xi', \pm 1 - \zeta) \right]_{M'} = 2 \left| \xi' \right|^2 e^{-|\pm 1 - x_3| s_\lambda} \left( I_2 + \frac{\epsilon \left| \xi' \right|}{s_\lambda} \xi' \otimes \xi' \right) \frac{e^{-|\pm 1 - \zeta| s_\lambda}}{s_\lambda} \right]_{M'} \leq \frac{C}{|\pm 1 - x_3| + |\pm 1 - \zeta|},$$

where $C > 0$ is independent of $\epsilon$. Similarly, it follows from (40), (28) and Proposition 10 that

$$\left[ |\xi'|^2 e_\lambda' (\xi', \pm 1 - x_3) y_{\lambda, \epsilon} (\xi') \omega_{\epsilon, \pm (\xi', \pm 1)} e_\lambda' (\xi', \pm 1 - \zeta) \right] \frac{\epsilon \left| \xi' \right|}{1 + \epsilon \left| \xi' \right|} \right]_{M'} \leq \frac{C}{|\pm 1 - x_3| + |\pm 1 - \zeta|}.$$  

Thus, we find from Corollary 14 and Proposition 8 that

$$\left\| \nabla H \otimes \nabla H I_j \right\|_{L^q(\Omega)} \leq C ||f||_{L^q(\Omega)}, \quad j = 1, 2,$$
where $\nabla_H = (\partial_1, \partial_2)^T$, $I_j$ is defined in (64) and $C > 0$ is independent of $\epsilon$. Since
\[
\partial_3 e'_\lambda(n', \pm 1 - x_3) = \frac{\pm e^{-i x_3} x_3}{2},
\]
\[
\partial_2^2 e'_\lambda(n', \pm 1 - x_3) = \frac{s_\lambda e^{-i x_3} x_3}{2},
\]
we use the same way as above to get
\[
\left\| \nabla H \partial_3 I_j \right\|_{L^q(\Omega)} + \left\| \partial_2^2 I_j \right\|_{L^q(\Omega)} \leq C \| f \|_{L^q(\Omega)}, \quad j = 1, 2,
\]
where $I_j$ is defined in (64) and $C > 0$ is independent of $\epsilon$. Propositions 12 and 19, Lemma 13 and the trace theorem imply
\[
R_{\lambda,\epsilon} \left[ \gamma K_{\lambda,\epsilon} E_0 f - \gamma E_0 f \right] \in W^{3-1/q, q}(\mathbb{T}^2)
\]
and its norm is bounded uniformly on $\epsilon$. By the definition of the operator $L_{\lambda,\epsilon}$, see (42), we have
\[
L_{\lambda,\epsilon} R_{\lambda,\epsilon} \left[ \gamma K_{\lambda,\epsilon} E_0 f - \gamma E_0 f \right]
\]
solves the elliptic equations $\lambda u - \Delta u = 0$. Moreover, the boundary data belong to $W^{3-1/q, q}(\mathbb{T}^2)$ by (26), (40) and Proposition 5. Thus, we find from (45), Corollary 14 and smoothing effect of the solution operator to the elliptic equation that
\[
\left\| W_{\lambda,\epsilon} R_{\lambda,\epsilon} \left[ \gamma K_{\lambda,\epsilon} E_0 f - \gamma E_0 f \right] \right\|_{W^{2,q}(\Omega)} \leq C \| f \|_{L^q(\Omega)}
\]
\[
\leq C \| L_{\lambda,\epsilon} R_{\lambda,\epsilon} \left[ \gamma K_{\lambda,\epsilon} E_0 f - \gamma E_0 f \right] \|_{W^{2,q}(\Omega)}
\]
\[
\leq C \| R_{\lambda,\epsilon} \left[ \gamma K_{\lambda,\epsilon} E_0 f - \gamma E_0 f \right] \|_{W^{2-1/q+\delta,q}(\mathbb{T}^2)}
\]
\[
\leq C \| f \|_{L^q(\Omega)},
\]
where $\delta > 0$ is small and $C$ is independent of $\epsilon$. \hfill \Box

**Lemma 24.** Let $1 < q < \infty$, $0 < \epsilon \leq 1$, $0 < \theta < \pi/2$ and $\lambda \in \Sigma_\theta$ satisfying $|\lambda| > R$ for sufficiently large $R > 0$. Then, there exists a constant $C = C(q, \theta)$ such that
\[
\left\| \nabla^2 (\lambda + A_\epsilon)^{-1} f \right\|_{L^q(\Omega)} \leq C \| f \|_{L^q(\Omega)}
\]
for all $f \in L^q(\Omega)$.

**Proof.** This is a direct consequence of Propositions 12, 18 and 23. \hfill \Box

**Lemma 25.** Let $1 < q < \infty$, $0 < \epsilon \leq 1$. Then, there exists a constant $C = C(q)$ such that
\[
\| \nabla^2 u \|_{L^q(\Omega)} \leq C \| A_\epsilon u \|_{L^q(\Omega)}
\]
for all $u \in D(A_\epsilon)$. \hfill \Box
Proof of Lemma 3. Let $u$ be a solution of (4). Our uniform BIP yields
\[
\| \partial_t u \|_{E_0(T)} + \| A_\epsilon u \|_{E_0(T)} \leq C \left( \| f \|_{E_0(T)} + \| u_0 \|_{B^2_{q,p}(\Omega)} \right)\]
by the Dore–Venni theory, where $C > 0$ is independent of $\epsilon$ and $T$. Applying an a priori estimate Lemma 25, we can replace $\| A_\epsilon u \|_{E_0(T)}$ by $\| \nabla^2 u \|_{E_0(T)}$. Since $(u, \pi)$ solves (4) and $\partial_t u$ and $\nabla^2 u$ are controlled, we are able to estimate $\| \nabla \pi \|_{E_0(T)}$. This completes the proof of Lemma 3. □

It remains to prove Lemma 25. We first observe an a priori estimate slightly weaker than Lemma 25, which is proved by using the resolvent estimate Lemma 24.

Proposition 26. Let $1 < q < \infty$ and $0 < \epsilon \leq 1$. There exists a unique solution $(u, \pi) \in D(A_\epsilon) \times L^q(\Omega)/\mathbb{R}$ to
\[
-\Delta u + \nabla \epsilon \pi = f \quad \text{in} \quad \Omega,
\]
\[
\text{div}_\epsilon u = 0 \quad \text{in} \quad \Omega,
\]
\[
u \quad u = 0 \quad \text{on} \quad \partial \Omega,
\]
for $f \in L^q(\Omega)$, such that
\[
\| \nabla^2 u \|_{L^q(\Omega)} + \| \nabla \epsilon \pi \|_{L^q(\Omega)} \leq C \| f \|_{L^q(\Omega)} + C \| u \|_{L^q(\Omega)},
\]
where $C > 0$ is independent of $\epsilon$ and $f$.

Proof. The equations are equivalent to
\[
\lambda_0 u - \Delta u + \nabla \epsilon \pi = f + \lambda_0 u \quad \text{in} \quad \Omega,
\]
\[
\text{div}_\epsilon u = 0 \quad \text{in} \quad \Omega,
\]
\[
u \quad u = 0 \quad \text{on} \quad \partial \Omega,
\]
for sufficiently large $\lambda_0 > 0$. We find from Lemma 24 that
\[
\| \nabla^2 u \|_{L^q(\Omega)} \leq C \| \nabla^2 (\lambda_0 + A_\epsilon)^{-1} \|_{L^q(\Omega)\rightarrow L^q(\Omega)} \| f + \lambda_0 u \|_{L^q(\Omega)}
\]
\[
\leq C \left( \| f \|_{L^q(\Omega)} + \lambda_0 \| u \|_{L^q(\Omega)} \right)
\]
for some constant $C > 0$, which is independent of $\epsilon$. The first equation in (69) implies
\[
\| \nabla \epsilon \pi \|_{L^q(\Omega)} \leq \| \nabla^2 u \|_{L^q(\Omega)} + \| f \|_{L^q(\Omega)}
\]
\[
\leq C \left( \| f \|_{L^q(\Omega)} + \lambda_0 \| u \|_{L^q(\Omega)} \right).
\]

For uniqueness we multiply the first equation by $u$ and integrating by parts yields $\nabla u = 0$. By the Poincaré inequality, it implies $u = 0$. This argument works for $q \geq 2$ since $\Omega$ is bounded. Since $(\lambda_0 + A_\epsilon)^{-1}$ is compact in $L^q(\Omega)$, the Riesz–Schauder theorem implies that $0$ is in resolvent since $\ker A_\epsilon = \{0\}$. In particular, (69) is uniquely solvable for any $f \in L^q(\Omega)$ for $q \geq 2$. By duality argument the solvability of $q \geq 2$ implies the uniqueness of (69) for $1 < q < 2$. Again by compactness of $(\lambda_0 + A_\epsilon)^{-1}$ the solvability for (69) follows. □
Proof of Lemma 25. Assume that if the statement were false, then there would exist a sequence \( \{ \epsilon_k \} \in \mathbb{Z}_{>1}, (0 < \epsilon_k \leq 1) \) and \( u_k \in D(A_\varepsilon) \) such that
\[
\| \nabla^2 u_k \|_{L^q(\Omega)} > \frac{k}{k} \| f_k \|_{L^q(\Omega)}, \quad f_k = A_\epsilon u_k.
\]
Since the problem is linear we may assume that
\[
\| \nabla^2 u_k \|_{L^q(\Omega)} \equiv 1, \quad \| f_k \|_{L^q(\Omega)} \leq \frac{1}{k} \to 0, \ (k \to 0).
\]
By \( A_\epsilon u_k = f_k \) and Proposition 26, we have
\[
1 \leq C \left( \| f_k \|_{L^q(\Omega)} + \| u_k \|_{L^q(\Omega)} \right)
\]
for some constant \( C > 0 \), which is independent of \( \epsilon_k \). Letting \( k \to \infty \) implies
\[
\frac{1}{C} \leq \liminf_{k \to \infty} \| u_k \|_{L^q(\Omega)}. \tag{70}
\]
By the Poincaré inequality for \( u_k \), our bound \( \| \nabla u_k \|_{L^q(\Omega)} \) implies that \( u_k \) and \( \nabla u_k \) are bounded in \( L^q(\Omega) \). By Rellich’s compactness theorem, we observe that \( u_k \to u \) for some \( u \in L^q(\Omega) \) strongly in \( L^q(\Omega) \) by taking a subsequence. The estimate (70) implies that
\[
\| u \|_{L^q(\Omega)} \geq \frac{1}{C}.
\]
We may assume \( \epsilon_k \to \epsilon_* \in [0, 1] \) and \( u_k \to u \) as \( k \to \infty \) by taking a subsequence.

The situation is divided into two cases, i.e., \( \epsilon_* = 0 \) or \( \epsilon_* > 0 \). By definition,
\[
\begin{align*}
-\Delta u_k + \nabla_\epsilon \pi_k &= f_k \quad \text{in } \Omega, \\
\text{div}_\epsilon u_k &= 0 \quad \text{in } \Omega, \\
u_k &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]
with some function \( \pi_k \) satisfying \( \int_{\Omega} \pi_k \, dx = 0 \). Since \( \| \nabla^2 u_k \|_{L^q(\Omega)} \leq 1 \), we see that
\[
\| \nabla_\epsilon \pi_k \|_{L^q(\Omega)} \leq \| f \|_{L^q(\Omega)} + 1.
\]
By the Poincaré inequality, \( \{ \pi_k \} \) is bounded in \( L^q(\Omega) \). By Rellich’s compactness theorem we may assume \( \pi_k \to \pi \) in \( L^q(\Omega) \) for some \( \pi \in L^q(\Omega) \) strongly by taking a subsequence. If \( \epsilon_* = 0 \), this implies \( \pi \) is independent of \( z \). Since \( \text{div}_\epsilon u_k = 0 \) and the vertical component \( \omega_k = 0 \) on \( x_3 = \pm 1 \), integration vertically on \( (-1, 1) \) yields that the horizontal limit \( v \) satisfies
\[
\text{div}_H \vec{v} = 0,
\]
where \( \text{div}_H = \nabla_\epsilon \cdot \). Thus, the horizontal component \( v \) satisfies the hydrostatic Stokes equations
\[
\begin{align*}
-\Delta u + \nabla_\epsilon \pi &= 0 \quad \text{in } \Omega, \\
\text{div}_H \vec{v} &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]
Since we know the only possible $W^{2,q}$-solution is zero, so we conclude that $v = 0$. Since $\|\nabla u_k\|_{L^q(\Omega)}$ is bounded, $\text{div} \varepsilon_k$-free condition implies that the horizontal limit $w$ is independent of the vertical variable. By the boundary condition $w = 0$ at $x_3 = \pm 1$, this implies $w$ must be zero. We thus observe that $u_k \rightarrow 0$ strongly in $L^q(\Omega)$, this contradicts $\|u\|_{L^q(\Omega)} \geq 1/C > 0$. The case $\epsilon_\ast$ is easier since the limit satisfies the anisotropic Stokes equations

\[-\Delta u + \nabla \epsilon_\ast x = 0 \quad \text{in} \quad \Omega,\]
\[\text{div} \epsilon_\ast u = 0 \quad \text{in} \quad \Omega,\]
\[u = 0 \quad \text{on} \quad \partial \Omega.\]

By the uniqueness $u \equiv 0$ in $\Omega$. This again contradicts $\|u\|_{L^q(\Omega)} \geq 1/C > 0$. The proof of Lemma 25 is now complete. □

As an application of Lemma 3, we obtain

**Corollary 27.** Let $p, q \in (1, \infty)$, $T > 0$, $F = (f_H, f_z) \in \mathbb{E}_0(T)$, $U_0 \in X_\gamma$ and $0 < \epsilon \leq 1$. Then, there is a unique solution $(U_\epsilon, P_\epsilon) \in \mathbb{E}_1(T) \times \mathbb{E}_0(T)$ to the equations

\[
\begin{aligned}
\partial_t V - \Delta V + \nabla H P &= f_H \quad \text{in} \quad \Omega \times (0, T), \\
\partial_t (\epsilon W) - \Delta (\epsilon W) + \frac{\partial \epsilon P}{\epsilon} &= f_z \quad \text{in} \quad \Omega \times (0, T), \\
\text{div} H V + \frac{\partial \epsilon}{\epsilon} (\epsilon W) &= 0 \quad \text{in} \quad \Omega \times (0, T), \\
U &= 0 \quad \text{on} \quad \partial \Omega \times (0, T), \\
U(0) &= U_0 \quad \text{in} \quad \Omega,
\end{aligned}
\]

(71)

where $P$ is unique up to a constant. Moreover, there exist constants $C > 0$ and $C_T > 0$, which is independent of $\epsilon$, such that

\[
\| (V, \epsilon W) \|_{\mathbb{E}_1(T)} + \| \nabla \epsilon P \|_{\mathbb{E}_0(T)} \leq C \| F \|_{\mathbb{E}_0(T)} + C_T \| (V_0, \epsilon W_0) \|_{B_{q,p}^{2(1-1/p)}(\Omega)}.
\]

(72)

**Proof.** Lemma 3 implies there exists a solution $(\tilde{U}, \tilde{P})$ to (4) with initial data $U_0$ such that

\[
\| \tilde{U} \|_{\mathbb{E}_1(T)} + \| \nabla \epsilon \tilde{P} \|_{\mathbb{E}_0(T)} \leq C \| F \|_{\mathbb{E}_0(T)} + C_T \| U_0 \|_{X_\gamma}.
\]

Set

\[
V = \tilde{V}, \quad W = \epsilon \tilde{W}, \quad P = \tilde{P}.
\]

Then, $(U, P)$ is the desired solution satisfying (72). Note that the spectrum of the anisotropic Stokes operator is positive. This implies exponential decay of $\|(V, \epsilon W)\|_{B_{q,p}^{2(1-1/p)}(\Omega)}$. Thus, the constant $C_T$ is uniformly bounded on $T$. □

3. Nonlinear estimates and regularity for $w$

In this section, we begin by recalling some estimates of products of functions. Then, we estimate terms $F_H, F_z, F$ and derive necessary regularity of $w$. Although the following propositions have been already proved in [11], we restate them to explain our restriction for $p$ and $q$ and for the reader’s convenience.
Proposition 28. Lemma 4.3 in [11] Let $T > 0$, $p, q \in (1, \infty)$ such that $2/3p + 1/q \leq 1$. Then, there exists a constant $C = C(p, q) > 0$ such that

$$||v_1 \partial_x v_2||_{E_0(T)} \leq C ||v_1||_{E_1(T)} ||v_2||_{E_1(T)}$$

for all $v_1, v_2 \in E_1(T)$.

Proposition 29. Lemma 4.5 in [11] Let $T > 0$ and $x_3 \in (-1, 1)$. Let $p, q \in (1, \infty)$ such that $1/p + 1/q \leq 1$. Then, there exists a constant $C = C(p, q) > 0$ such that

$$||w_1 \partial_3 v_2||_{E_0(T)} \leq C ||v_1||_{E_1(T)} ||v_2||_{E_1(T)}$$

for all $v_1, v_2 \in E_1(T)$ and $w_1 := - \int_{-1}^{x_3} \text{div} H v_1 \, d\zeta$.

Proposition 30. Let $T > 0$ and $x_3 \in (-1, 1)$. Let $p, q \in (1, \infty)$ such that

$$\frac{1}{p} + \frac{1}{q} \leq 1, \quad \frac{1}{p} + \frac{2}{q} \leq \frac{3}{2}.$$  

Then, there exists a constant $C = C(p, q) > 0$ such that

$$\left\| \int_{-1}^{x_3} \partial_j v_1 \partial_k v_2 \, d\zeta \right\|_{E_0(T)} \leq C \left\| \partial_j v_1 \right\|_{E_0(T)} \left\| \partial_k v_2 \right\|_{E_0(T)}$$  

(73)

for all $v_1, v_2 \in E_1(T)$, $-1 \leq x_3 \leq 1$, $j = 1, 2$ and $k = 1, 2, 3$.

Proof. We find from the Hölder inequality that

$$\left\| \int_{-1}^{x_3} \partial_j v_1 \partial_k v_2 \, d\zeta \right\|_{L^q(\Omega)} \leq C \left\| \partial_j v_1 \partial_k v_2 \right\|_{L^q(\mathbb{R}^2; L^1(-1, 1))} \leq \begin{cases} C \|v_1\|_{W^{1,2q}(\mathbb{R}^2; L^2(-1, 1))} \|v_2\|_{W^{1,2q}(\mathbb{R}^2; L^2(-1, 1))}, & (k = 1, 2), \\ C \|v_1\|_{W^{1,2q}(\mathbb{R}^2; L^2(-1, 1))} \|v_2\|_{L^{2q}(\mathbb{R}^2; W^{1,2}(-1, 1))}, & (k = 3). \end{cases}$$

Applying $L^p$-norm for the time variable and using the Hölder inequality again, we have

$$\left\| \int_{-1}^{x_3} \partial_j v_1 \partial_k v_2 \, d\zeta \right\|_{E_0(T)} \leq \begin{cases} C \|v_1\|_{L^{2p}(0,T; W^{1,2q}(\mathbb{R}^2; L^2(-1, 1)))} \|v_2\|_{L^{2p}(0,T; W^{1,2q}(\mathbb{R}^2; L^2(-1, 1)))}, & (k = 1, 2), \\ C \|v_1\|_{L^{2p}(0,T; W^{1,2q}(\mathbb{R}^2; L^2(-1, 1)))} \|v_2\|_{L^{2p}(0,T; L^{2q}(\mathbb{R}^2; W^{1,2}(-1, 1)))}, & (k = 3). \end{cases}$$

By the Sobolev inequality and the mixed-derivative theorem (interpolation inequality)

$$\|v_1\|_{W^{\theta,p}(0,T; W^{2(1-\theta),q}(\Omega))} \leq C \|v_1\|_{E_1(T)}, \quad \theta \in (0, 1),$$
we have
\[
\|v_i\|_{L^2_p(0,T; W^{1,2q}(T^2; L^2(-1,1)))} \\
\leq C \|v_i\|_{W^{1,2p,p}(0,T; W^{1+1/q,q}(T^2; W^{1/q-1/2,q}(-1,1)))} \\
\leq C \|v_i\|_{W^{1,2p,p}(0,T; W^{1/2+1/q,q}()} \\
\leq C \|v_i\|_{E_1(T)}
\]
for \(i = 1, 2\) and \(p, q\) satisfying \(1/2p + 1/q \leq 3/4\) and \(q \leq 2\). For the mixed derivative theorem, the reader is referred to the book [32]. In the case \(q \geq 2\), we find from the Sobolev inequality and the mixed-derivative theorem that
\[
\|v_i\|_{L^2_p(0,T; W^{1,2q}(T^2; L^2(-1,1)))} \\
\leq C \|v_i\|_{W^{1,2p,p}(0,T; W^{1+1/q,q}(T^2; L^q(-1,1)))} \\
\leq C \|v_i\|_{W^{1,2p,p}(0,T; W^{1+1/q,q}(\Omega))} \\
\leq C \|v_i\|_{E_1(T)}
\]
for \(i = 1, 2\) and \(p, q\) satisfying \(1/p + 1/q \leq 1\). By the Sobolev inequality and the mixed-derivative theorem, we get
\[
\|v_2\|_{L^2_p(0,T; L^{2q}(T^2; W^{1,2}(-1,1)))} \\
\leq C \|v_2\|_{W^{1,2p,p}(0,T; W^{1+1/q,q}(T^2; W^{1/2+1/q,q}(-1,1)))} \\
\leq C \|v_2\|_{W^{1,2p,p}(0,T; W^{1/2+1/q,q}(\Omega))} \\
\leq C \|v_2\|_{E_1(T)}
\]
for \(p, q\) satisfying \(1/2p + 1/q \leq 3/4\) and \(q \leq 2\). We next consider the case \(q \geq 2\). We argue in same way as above to get
\[
\|v_2\|_{L^2_p(0,T; L^{2q}(T^2; W^{1,2}(-1,1)))} \\
\leq C \|v_2\|_{W^{1,2p,p}(0,T; W^{1/q,q}(T^2; W^{1/q}(-1,1)))} \\
\leq C \|v_2\|_{W^{1,2p,p}(0,T; W^{1+1/q,q}(\Omega))} \\
\leq C \|v_2\|_{E_1(T)}
\]
for \(p, q\) satisfying \(1/p + 1/q \leq 1\). Thus, we obtain (73).  

\begin{proposition}
Let \(T > 0, 1 < p, q < \infty\) and \(s > 1/q\). Let \(w_0 \in B^{2(1-1/p)}_{q,p}(\Omega)\) and \(v \in E_1(T)\). Let \(f\) be a quadratic nonlinear function and \(g\) be a bi-linear function satisfying
\[
\|f(v_1)\|_{E_0(T)} \leq C \|v_1\|_{W^{1/p}(0,T; W^{1,q}(\Omega)) \cap L^p(0,T; W^{2+1/s,q}(\Omega))} \\
+ C \|v_1\|_{E_1(T)}^2, \\
\|g(v_1, v_2)\|_{E_0(T)} \leq C \|v_1\|_{E_1(T)} \|v_2\|_{E_1(T)},
\]
\end{proposition}
for $v_1 \in W^{1,p}(0, T; W^{s,q}(\Omega)) \cap L^p(0, T; W^{2+s,q}(\Omega))$ and $v_2 \in E_1(T)$. Then, there is a constant $\delta > 0$ depending only on $p, q, v$ and $f$ such that if $T$ is taken so that $\|v\|_{E_1(T)} \leq \delta$, the solution $w$ to

$$\begin{align*}
  \partial_t w - \Delta w &= f(v) + g(v, w) \quad \text{in} \quad \Omega \times (0, T), \\
  w &= 0 \quad \text{on} \quad \partial \Omega \times (0, T), \\
  w(0) &= w_0 \quad \text{in} \quad \Omega
\end{align*}$$

exists and satisfies the estimate

$$\|w\|_{E_0(T)} \leq C \left( 1 + \|v\|_{W^{1,p}(0, T; W^{s,q}(\Omega)) \cap L^p(0, T; W^{2+s,q}(\Omega))} + \|w_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \right)^2$$

with some constant $C > 0$.

**Proof.** Let $u = S(h, w_0)$ solve

$$\begin{align*}
  \partial_t u - \Delta u &= h \quad \text{in} \quad \Omega \times (0, T), \\
  u &= 0 \quad \text{on} \quad \partial \Omega \times (0, T), \\
  u(0) &= w_0 \quad \text{in} \quad \Omega
\end{align*}$$

for an external force $h \in E_0(T)$ and initial data $w_0 \in B_{q,p}^{2(1-1/p)}(\Omega)$. The operator $S$ has the maximal regularity of the form

$$\|S(h, w_0)\|_{E_0(T)} \leq C \left( \|h\|_{E_1(T)} + \|w_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \right).$$

(75)

We define the linear bounded operator $H : E_0(T) \to E_0(T)$ by

$$H(h) = h - g(v, S(h, 0))$$

for $h \in E_0(T)$. By the definition of $g$ and (75), we get

$$\|g(v, S(h, 0))\|_{E_0(T)} \leq C \|v\|_{E_1(T)} \|h\|_{E_0(T)}.$$

If we take $T$ sufficiently small so that

$$\|v\|_{E_0(T)} \leq \frac{1}{2C},$$

we have

$$\|g(v, S(h, 0))\|_{E_0(T)} \leq \frac{1}{2} \|h\|_{E_1(T)}.$$

Using the Neumann series argument, we find that for small $T$ the inverse operator of $H$ exists such that

$$\|H^{-1}\|_{E_0(T) \to E_0(T)} = \|(I - g(v, S(\cdot, 0)))^{-1}\|_{E_0(T) \to E_0(T)} \leq 1.$$

(76)
Put
\[ F := f(v) + g(v, S(0, w_0)), \]
\[ w := S\left( H^{-1}(F), w_0 \right) = S\left( H^{-1}(F), 0 \right) + S(0, w_0). \]

By the definition and (76), we obtain
\[ \|w\|_{E_1(T)} \leq C \left( \|H^{-1}(F)\|_{E_1(T)} + \|w_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)} \right) \]
\[ \leq C \left( \|F\|_{E_1(T)} + \|w_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)} \right) \]
\[ \leq C \left( 1 + \|v_1\|_{W^{1,p}(0,T;W^{s,q}(\Omega))} + \|w_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)} \right)^2. \]

We now check that \( w \) is the solution to (74). It follows from the definition of \( w \) that
\[ \partial_t w - \Delta w - g(v, w) \]
\[ = H^{-1}(F) - \left[ g \left( v, S\left( H^{-1}(F), 0 \right) \right) + g(v, S(0, w_0)) \right] \]
\[ = H\left( H^{-1}(F) \right) - g(v, S(0, w_0)) \]
\[ = F - g(v, S(0, w_0)) \]
\[ = f(v). \]

Let us show \( w \in E_1(T) \). In our previous paper [11], we first derive the equation which \( w \) satisfies by applying \( \int_1^3 \text{div}_H \cdot d\zeta \) to the equations \( v \) satisfies. Then, estimating the corresponding nonlinear terms and applying the maximal regularity principle, we obtain \( w \in E_1(T) \). Note that, in the present paper, we invoke additional regularity for \( v \) to deal with the trace of the second derivative.

Although, in [15], the authors treat higher-order regularity of the solution to the primitive equations, they do not explicitly write the maximal regularity in fractional Sobolev spaces. However, it is easy to modify their proof to get the maximal regularity in the fractional Sobolev spaces. In [16], the argument to get \( H^\infty \)-calculus of hydrostatic Stokes operator is based on \( H^\infty \)-calculus for the Laplace operator and perturbations arguments. Since the Laplace operator admits \( H^\infty \)-calculus in fractional Sobolev spaces, it is not difficult to establish \( H^\infty \)-calculus of the hydrostatic Stokes operator in fractional Sobolev spaces. We also find local well-posedness of the primitive equations in fractional maximal regularity space \( W^{1,p}(0,T;W^{s,q}(\Omega)) \cap L^p(0,T;W^{2+s,q}(\Omega)) \) for \( s > 1/q \) in the same way [15] to get local well-posedness, namely, using Lemma 6.1, Corollary 6.2 and Theorem 5.1 in [15].

**Remark 32.** It is already known that \( v \in E_1(T) \) for initial data \( v_0 \in X_T \) by Giga et al. [15,16].
**Proof of Lemma 4.** Integrating (PE) both sides over \((-1, 1),\) we find \((\bar{\nu}, \bar{\pi})\) satisfy

\[
\begin{align*}
\partial_t \bar{\nu} - \Delta \bar{\nu} + \nabla_H \bar{\pi} &= - \int_{-1}^1 \nu \cdot \nabla_H \nu + w \partial_3 \nu \, d\zeta + \left(\partial_3 \nu\right)|_{x_3 = -1} \quad \text{in } \Omega \times (0, T), \\
\div_H \bar{\nu} &= 0 \quad \text{in } \Omega \times (0, T), \\
\bar{\nu} &= 0 \quad \text{on } \partial \Omega \times (0, T), \\
\bar{\nu}(0) &= \bar{\nu}_0 \quad \text{in } \Omega.
\end{align*}
\]

(77)

It is clear that \(\tilde{\nu} := \nu - \bar{\nu}\) satisfies

\[
\|\tilde{\nu}\|_{\text{E}_1(T)} \leq 2\|\nu\|_{\text{E}_1(T)}.
\]

Then, \(\tilde{u} = (\tilde{\nu}, w)\) solves

\[
\begin{align*}
\partial_t \tilde{\nu} - \Delta \tilde{\nu} &= -\tilde{\nu} \cdot \nabla_H \tilde{\nu} - w \partial_3 \tilde{\nu} \\
&\quad - \bar{\nu} \cdot \nabla_H \bar{\nu} - \tilde{\nu} \cdot \nabla_H \bar{\nu} \\
&\quad - \frac{1}{2} \int_{-1}^1 \tilde{\nu} \cdot \nabla_H \tilde{\nu} - (\div_H \tilde{\nu})^2 \, d\zeta \\
&\quad + \frac{1}{2} \left(\partial_3 \nu\right)|_{x_3 = -1} \quad \text{in } \Omega \times (0, T), \\
\div_H \tilde{\nu} + \partial_3 w &= 0 \quad \text{in } \Omega \times (0, T), \\
\tilde{\nu} &= 0 \quad \text{on } \partial \Omega \times (0, T), \\
\tilde{\nu}(0) &= \nu(0) - \bar{\nu}_0 \quad \text{in } \Omega.
\end{align*}
\]

(78)

Note that the pressure term no longer appears in the above equations and

\[
\div_H \tilde{\nu} + \partial_3 w = 0.
\]

Applying \(-\div_H\) to (78) and integrating over \((-1, x_3)\) with respect to vertical variable, we find

\[
\begin{align*}
\partial_t w - \Delta w &= \partial_3 \div_H \tilde{\nu}|_{x_3 = -1} - \int_{-1}^{x_3} \frac{1}{2} \div_H \left[(\partial_3 \nu)|_{x_3 = -1}\right] d\zeta \\
&\quad + \int_{-1}^{x_3} \div_H (\tilde{\nu} \cdot \nabla_H \tilde{\nu} - w \partial_3 \tilde{\nu} - \bar{\nu} \cdot \nabla_H \bar{\nu} - \tilde{\nu} \cdot \nabla_H \bar{\nu}) d\zeta \\
&\quad - \frac{1}{2} \int_{-1}^{x_3} \div_H \int_{-1}^{1} \tilde{\nu} \cdot \nabla_H \tilde{\nu} - (\div_H \tilde{\nu})\tilde{\nu} \, d\zeta \, d\eta \\
&=: I_1 + I_2 + I_3,
\end{align*}
\]

with initial data \(w_0.\) Since \(v_0 \in B^s_{q,p}(\Omega)\) for \(s > 2 - 2/p + 1/q,\) we have \(v \in \text{E}_1(T) \cap L^p(0, T; W^{2+1/q+\delta,q}(\Omega))\) for some \(\delta > 0\) by [15] and [16], and thus, \(\|I_1\| \leq \|I_2\| \leq \|I_3\| \leq \}

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$C$ for some $C > 0$. We use integration by parts to get
\[
I_2 = \tilde{v} \cdot \nabla_H w - w \text{div}_H \tilde{v} + \tilde{v} \cdot \nabla_H w \\
+ \int_{-1}^{x_3} \partial_j \tilde{v} \cdot \nabla_H \tilde{v}_j - (\partial_\xi \tilde{v} \cdot \nabla_H w) + \nabla_H w \cdot \partial_\xi \tilde{v} - \partial_\xi w \text{div}_H \tilde{v} \, d\xi \\
+ \int_{-1}^{x_3} \partial_j \tilde{v} \cdot \nabla_H \tilde{v}_j + \partial_j \tilde{v} \cdot \nabla_H \tilde{v}_j \, d\xi \\
:= I_{21}(v, w) + I_{22}(v),
\]
where $j = 1, 2$ and Einstein’s summation convention is used. We find from Propositions 28, 29 and 30 that
\[
||I_{21}(v, w)||_{E_0(T)} \leq C ||w||_{E_1(T)} ||v||_{E_1(T)}
\]
and
\[
||I_{22}(v)||_{E_0(T)} \leq C ||v||_{E_1(T)}^2.
\]
Similarly, $I_3$ is decomposed into $w$-depend part $I_{31}(v, w)$ and $w$-independent part $I_{32}(v)$ and estimated as
\[
||I_{31}(v, w)||_{E_0(T)} \leq C ||w||_{E_1(T)} ||v||_{E_1(T)}
\]
and
\[
||I_{32}(v)||_{E_0(T)} \leq C ||v||_{E_1(T)}^2.
\]
Thus, we find from Proposition 31 that
\[
||w||_{E_1(T')} \leq C
\]
for some constant $C > 0$ and small $T'$. The solution to the primitive equations $v$ is smooth in the time interval $[T', T)$. The formula $w = \int_{-1}^{x_3} \text{div}_H \tilde{v} \, d\xi$ implies that $w$ is also smooth in on $[T', T)$. Thus, we conclude
\[
||w||_{E_1(T)} \leq C
\]
\[\square\]

4. Justification of the hydrostatic approximation and global-well-posedness of the anisotropic Navier–Stokes equations

Let us prove our main theorem. Recall that $u = (v, w)$ is the solution to the primitive equations and $U_\epsilon = (V_\epsilon, W_\epsilon)$ is the solution to (1). We construct the solution $u_\epsilon = (v_\epsilon, w_\epsilon)$ to (SNS) of the form $u_\epsilon = u + U_\epsilon$. The key is construction of $U_\epsilon$ by iteration. Note that our idea to construct the solution is based on the principle which small data implies the global well-posedness.
Proof of Theorem 1. Let $C_1$ be the maximum of constants $C$ in Propositions 28 and 29, (72) and the constant in the trace theorem. Let us construct a solution $(V_\varepsilon, \epsilon W_\varepsilon)$ to (1) with zero initial data on $[0, T]$. Set $(u_\varepsilon, p_\varepsilon) := (v + V_\varepsilon, w + W_\varepsilon, p + P_\varepsilon)$, then this is the desired solution to (SNS). We denote by $\| \cdot \|_{E_1(a,b)}$ and $\| \cdot \|_{E_0(a,b)}$ the $E_1$-norm and $E_0$-norm on the time interval $[a, b]$, respectively. We choose $0 < T \leq 1$ so small that

$$T = NT$$

for sufficiently large integer $N$ and

$$\| u \|_{E_1(mT, (m+1)T)} \leq \frac{1}{10C_1},$$

for all integer $m \in [1, N]$. The choice of $T$ depends on $T$ and $u$ but is independent of $\epsilon$. We divide the time interval $[0, T]$ into $\bigcup_{m=0}^{N}[mT, (m+1)T]$. We denote the left-hand side of (1) by

$$F(U_\varepsilon, u) = F(V_\varepsilon, W_\varepsilon, u) := (F_H(V_\varepsilon, W_\varepsilon, u), F_z(V_\varepsilon, W_\varepsilon, u)).$$

Clearly, the choice of $T$ is independent of $F$. We denote the solution $(U, P)$ to (71) with initial data $U_0$ and external force $F$ by

$$(U, P) = (\mathcal{R}^u(F, U_0), \mathcal{R}^p(F, U_0)) = \mathcal{R}(F, U_0).$$

We inductively set

$$U_{\varepsilon,1} = \mathcal{R}^u(F(0, u), 0), \quad P_{\varepsilon,1} = \mathcal{R}^p(F(0, u), 0),$$

$$U_{\varepsilon,j+1} = \mathcal{R}^u(F(U_j, u), 0), \quad P_{\varepsilon,j+1} = \mathcal{R}^p(F(U_j, u), 0).$$

Propositions 28, 29 and Corollary 27 lead to

$$\begin{align*}
\| (V_{\varepsilon,j+1}, \epsilon W_{\varepsilon,j+1}) \|_{E_1(T)} + \| \nabla_\varepsilon P_{\varepsilon,j+1} \|_{E_0(T)} \\
\leq C_1 \left( \| u \|_{E_1(T)} \| (V_{\varepsilon,j}, \epsilon W_{\varepsilon,j}) \|_{E_1(T)} + \| (V_{\varepsilon,j}, \epsilon W_{\varepsilon,j}) \|_{E_1(T)}^2 \right) + \epsilon C_1 \left( \| u \|_{E_1(T)} + \| u \|_{E_1(T)}^2 \right).
\end{align*}$$

(85)

This quadratic inequality and (84) imply

$$\begin{align*}
\| (V_{\varepsilon,j}, \epsilon W_{\varepsilon,j}) \|_{E_1(T)} + \| \nabla_\varepsilon P_{\varepsilon,j} \|_{E_0(T)} \leq 2\epsilon C^* \\
\quad \text{for } C^* = (1/10 + 1/100C_1) \quad \text{and small } \epsilon > 0.
\end{align*}$$

(86)

We set the differences

$$\tilde{U}_{\varepsilon,j} = U_{\varepsilon,j+1} - U_{\varepsilon,j}, \quad \tilde{U}_{\varepsilon,0} = U_{\varepsilon,1},$$

$$\tilde{P}_{\varepsilon,j} = P_{\varepsilon,j+1} - P_{\varepsilon,j}, \quad \tilde{P}_{\varepsilon,0} = P_{\varepsilon,1}.$$
Then, seeking the equation which \((\tilde{U}_{\epsilon,j}, \tilde{P}_{\epsilon,j})\) satisfies and applying Propositions 28, 29 and Corollary 27, we have

\[
\begin{aligned}
&\| (\tilde{V}_{\epsilon,j+1}, \epsilon \tilde{W}_{\epsilon,j+1}) \|_{E_1(T)} + \| \nabla \epsilon \tilde{P}_{\epsilon,j+1} \|_{E_0(T)} \\
&\leq C_1 (\| (V_{\epsilon,j}, \epsilon W_{\epsilon,j}) \|_{E_1(T)} + \| (V_{\epsilon,j+1}, \epsilon W_{\epsilon,j+1}) \|_{E_1(T)} \\
&\quad + 2\| u \|_{E_1(T)} ) (\| \tilde{V}_{\epsilon,j}, \epsilon \tilde{W}_{\epsilon,j} \|_{E_1(T)}) \\
&\leq \frac{7}{10} (\| (\tilde{V}_{\epsilon,j}, \epsilon \tilde{W}_{\epsilon,j}) \|_{E_1(T)} + \| \nabla \epsilon \tilde{P}_{\epsilon,j} \|_{E_0(T)}). \tag{87}
\end{aligned}
\]

Thus, \((U_{\epsilon}, P_{\epsilon}) := (\lim_{j \to \infty} U_j, \lim_{j \to \infty} P_j) = (\sum_{j=0}^\infty U_{\epsilon,j}, \sum_{j=0}^\infty \tilde{P}_{\epsilon,j})\) exists in \(E_1(T)\) and satisfies

\[
\| (V_{\epsilon}, \epsilon W_{\epsilon}) \|_{E_1(T)} + \| \nabla \epsilon P_{\epsilon} \|_{E_0(T)} \leq 2\epsilon C^*. \tag{88}
\]

By construction \((U_{\epsilon}, P_{\epsilon})\) satisfies (1) on \([0, T]\). Moreover, by trace theorem there exists a constant \(C_{tr} > 0\) such that

\[
\| (V_{\epsilon}(T), \epsilon W_{\epsilon}(T)) \|_{B^{2(1-1/p)}_{\rho,p}(\Omega)} \leq C_{tr} \| (V_{\epsilon}, \epsilon W_{\epsilon}) \|_{E_1(0,T)} \leq 2\epsilon C^* C_{tr}. \tag{89}
\]

We next construct the solution to (1) on \([T, 2T]\) with initial data

\[
U_{\epsilon}(T) = (V_{\epsilon}(T), W_{\epsilon}(T)).
\]

We set

\[
a_{\epsilon,1} = (b_{\epsilon,1}, c_{\epsilon,1}) = \mathcal{R}^u(0, U_{\epsilon}(T)),
\]

\[
\pi_{\epsilon,1} = \mathcal{R}^p(0, U_{\epsilon}(T)).
\]

Then, \((a_{\epsilon,1}, \pi_{\epsilon,1})\) solves

\[
\begin{aligned}
&\partial_t b_{\epsilon,1} - \Delta b_{\epsilon,1} + \nabla_H \pi_{\epsilon,1} = 0, \\
&\partial_t (\epsilon c_{\epsilon,1}) - \Delta (\epsilon c_{\epsilon,1}) + \frac{\partial}{\epsilon} \pi_{\epsilon,1} = 0, \\
&\text{div}_\epsilon (b_{\epsilon,1}, \epsilon c_{\epsilon,1}) = 0, \\
&(b_{\epsilon}(T), \epsilon c_{\epsilon}(T)) = (V_{\epsilon}(T), \epsilon W_{\epsilon}(T)).
\end{aligned}
\]

Corollary 27 implies

\[
\| (b_{\epsilon,1}, \epsilon c_{\epsilon,1}) \|_{E_1(T,2T)} + \| \nabla \epsilon \pi_{\epsilon,1} \|_{E_0(T,2T)} \leq 2\epsilon C^* C_{tr} C_T. \tag{90}
\]

Let the vector field \(a_{\epsilon} = (b_{\epsilon}, c_{\epsilon})\) be the solution to

\[
\begin{aligned}
&\partial_t b_{\epsilon} - \Delta b_{\epsilon} + \nabla_H \pi_{\epsilon} = F_H(b_{1,\epsilon} + b_{\epsilon}, c_{\epsilon,1} + c_{\epsilon}, u), \\
&\partial_t (\epsilon c_{\epsilon}) - \Delta (\epsilon c_{\epsilon}) + \frac{\partial}{\epsilon} \pi_{\epsilon} = \epsilon F_c(b_{1,\epsilon} + b_{\epsilon}, c_{\epsilon,1} + c_{\epsilon}, u), \\
&\text{div}_\epsilon (b_{\epsilon}, \epsilon c_{\epsilon}) = 0, \\
a_{\epsilon}(T) = 0. \tag{91}
\end{aligned}
\]
If we put $U_\epsilon := a_{\epsilon,1} + a_\epsilon$ and $P_\epsilon := \pi_{\epsilon,1} + \pi_\epsilon$, then $(U_\epsilon, P_\epsilon)$ is the solution to (1) with initial data $U_\epsilon(T)$. We inductively set

$$a_{\epsilon,j+1} = a_{\epsilon,1} + \mathcal{R}^u(F(b_{1,\epsilon} + b_{\epsilon,j} + c_{\epsilon,1} + c_{\epsilon,j}, u, 0), 0),$$

$$\pi_{\epsilon,j+1} = \mathcal{R}^p(F(b_{1,\epsilon} + b_{\epsilon,j} + c_{\epsilon,1} + c_{\epsilon,j}, u, 0), 0),$$

for $j \geq 1$. Applying Propositions 28, 29 and Corollary 27 to (91), we find

\[
\begin{align*}
&\| (b_{\epsilon,j+1}, \epsilon c_{\epsilon,j+1}) \|_{E_1(T,2T)} + \| \nabla_\epsilon \pi_{\epsilon,j+1} \|_{E_0(T,2T)} \\
&\leq C_1 \| u \|_{E_1(T,2T)} \| (b_{\epsilon,1} + b_{\epsilon,j}, \epsilon (c_{\epsilon,1} + c_{\epsilon,j})) \|_{E_1(T,2T)} \\
&\quad + C_1 \| (b_{\epsilon,1} + b_{\epsilon,j}, \epsilon (c_{\epsilon,1} + c_{\epsilon,j})) \|_{E_1(T,2T)}^2 \\
&\quad + \epsilon C_1 \left[ \| u \|_{E_1(T,2T)} + \| u \|_{E_1(T,2T)}^2 \right] \\
&\leq C_1 \| (b_{\epsilon,j}, \epsilon c_{\epsilon,j}) \|_{E_1(T,2T)}^2 \\
&\quad + C_1 \| u \|_{E_1(T,2T)}^2 + 2 \| (b_{\epsilon,1}, \epsilon c_{\epsilon,1}) \|_{E_1(T,2T)} \| (b_{\epsilon,j}, \epsilon c_{\epsilon,j}) \|_{E_1(T,2T)} \\
&\quad + \epsilon C_1 \left[ \| u \|_{E_1(T,2T)} \| (b_{\epsilon,1}, \epsilon c_{\epsilon,1}) \|_{E_1(T,2T)} + \| (b_{\epsilon,1}, \epsilon c_{\epsilon,1}) \|_{E_1(T,2T)}^2 \right] \\
&\quad + \epsilon C_1 \left[ \| u \|_{E_1(T,2T)} + \| u \|_{E_1(T,2T)}^2 \right].
\end{align*}
\]

If we take $\epsilon$ so small that

$$\| (b_{\epsilon,1}, \epsilon c_{\epsilon,1}) \|_{E_1(T,2T)} \leq 2\epsilon C^* C_T C_T \leq \frac{1}{8C_1},$$

(92)

we have

\[
\begin{align*}
&\| (b_{\epsilon,j+1}, \epsilon c_{\epsilon,j+1}) \|_{E_1(T,2T)} + \| \nabla_\epsilon \pi_{\epsilon,j+1} \|_{E_0(T,2T)} \\
&\leq C_1 \| (b_{\epsilon,j}, \epsilon c_{\epsilon,j}) \|_{E_1(T,2T)}^2 \\
&\quad + \frac{1}{2} \| (b_{\epsilon,j}, \epsilon c_{\epsilon,j}) \|_{E_1(T,2T)} + \epsilon C^* C_T C_T + \epsilon C^* \\
&\leq C_1 \| (b_{\epsilon,j}, \epsilon c_{\epsilon,j}) \|_{E_1(T,2T)}^2 + \epsilon C^* (1 + C_T C_T)
\end{align*}
\]

(93)

Thus, we inductively obtain

$$\| (b_{\epsilon,j}, \epsilon c_{\epsilon,j}) \|_{E_1(T,2T)} + \| \nabla_\epsilon \pi_{\epsilon,j} \|_{E_0(T,2T)} \leq 2\epsilon C^* (1 + C_T C_T)$$

for all $j \geq 1$. Set

$$\tilde{a}_{\epsilon,j} = a_{\epsilon,j+1} - a_{\epsilon,j} \quad (j \geq 1), \quad \tilde{a}_{\epsilon,0} = a_{\epsilon,0},$$

$$\tilde{\pi}_{\epsilon,j} = \pi_{\epsilon,j+1} - \pi_{\epsilon,j} \quad (j \geq 1), \quad \tilde{\pi}_{\epsilon,0} = \pi_{\epsilon,0}.$$

Applying Propositions 28, 29 and Corollary 27 to the equations that

$$(\tilde{a}_{\epsilon,j+1}, \tilde{\pi}_{\epsilon,j+1})$$
satisfies, we find
\[
\| (\tilde{b}_{e,j+1}, e\tilde{c}_{e,j+1}) \|_{E_1(T,2T)} + \| \nabla_\epsilon \tilde{\pi}_{e,j+1} \|_{E_0(T,2T)} \\
\leq C_1 \left( \| (b_{e,j}, e\epsilon c_{e,j}) \|_{E_1(T,2T)} + \| (b_{e,j+1}, e\epsilon c_{e,j+1}) \|_{E_1(T,2T)} + 2\|u\|_{E_1(T,2T)} \right) \\
\times \| (\tilde{b}_{e,j}, e\tilde{c}_{e,j}) \|_{E_1(T,2T)} \\
\leq \left[ C_1 \left( \| (b_{e,j}, e\epsilon c_{e,j}) \|_{E_1(T,2T)} + \| (b_{e,j+1}, e\epsilon c_{e,j+1}) \|_{E_1(T,2T)} + \frac{1}{5} \right) \\
\times \| (\tilde{b}_{e,j}, e\tilde{c}_{e,j}) \|_{E_1(T,2T)} \right] \\
\leq \frac{7}{10} \| (\tilde{b}_{e,j}, e\tilde{c}_{e,j}) \|_{E_1(T,2T)}. \quad (94)
\]

The last inequality holds if \( \epsilon \) is sufficiently small. Thus,
\[
(a_\epsilon, \pi_\epsilon) := \left( \lim_{j \to \infty} a_{e,j}, \lim_{j \to \infty} \pi_{e,j} \right) = \left( \sum_{j=0}^{\infty} \tilde{a}_{e,j}, \sum_{j=0}^{\infty} \tilde{\pi}_{e,j} \right)
\]
exists in \( E_1(T,2T) \) and satisfies (91) such that
\[
\| (b_{e}, e\epsilon c_{e}) \|_{E_1(T,2T)} + \| \nabla_\epsilon \pi_{e} \|_{E_0(T,2T)} \leq 2\epsilon C^* (1 + C_T C_{tr}).
\]

The functions \((U_\epsilon, P_\epsilon)\) solves (1) on the time interval \([T,2T]\) with initial data \(U_\epsilon(T)\) such that
\[
\| (V_\epsilon, \epsilon W_\epsilon) \|_{E_1(T)} + \| \nabla_\epsilon P_\epsilon \|_{E_0(T)} \\
\leq \| (b_{e,1}, \epsilon\epsilon c_{e,1}) \|_{E_1(T,2T)} + \| (b_{e}, \epsilon\epsilon c_{e}) \|_{E_1(T,2T)} \\
+ \| \nabla_\epsilon \pi_{e,1} \|_{E_0(T,2T)} + \| \nabla_\epsilon \pi_{e} \|_{E_0(T,2T)} \\
\leq 2\epsilon C^* C_{tr} C_T + 2\epsilon C^*(1 + C_{tr} C_T) \leq 2\epsilon C^*(1 + 2C_{tr} C_T),
\]
where we used (90) in the last inequality. By induction, the solution \((U_\epsilon, P_\epsilon)\) constructed by the same way on the time interval \([mT,(m+1)T]\) inductively satisfies
\[
\| (V_\epsilon, \epsilon W_\epsilon) \|_{E_1(mT,(m+1)T)} + \| \nabla_\epsilon P_\epsilon \|_{E_0(mT,(m+1)T)} \\
\leq 2\epsilon C^* \beta_m, \quad (95)
\]
where \(\beta_m\) is defined by \(\beta_0 = 1 + 2C_{tr} C_T\) and \(\beta_m = 1 + 2C_{tr} C_T \beta_{m-1}\) for \(m = 1, 2, \ldots, N-1\). Since \(T\) is finite, the induction ends in finite steps. Thus, we conclude
\[
\| (V_\epsilon, \epsilon W_\epsilon) \|_{E_1(T)} + \| \nabla_\epsilon P_\epsilon \|_{E_0(T)} \leq 2\epsilon \sum_{1 \leq j \leq N} \beta_j. \quad (96)
\]

In the case of \(T = \infty\), we first show (96) for a sufficiently large \(T' > 0\). Since \(\|u\|_{E_1(T',\infty)}\) is small, we extend the existence time of \((U_\epsilon, P_\epsilon)\) from \([0,T')\) to \([0,\infty)\) by one step. \(\square\)
Remark 33. It is worth pointing out an upper bound of $\epsilon$. We assume the estimate (95) holds for some $m$. In the proof of Theorem 1, we chose $\epsilon$ so that the estimates (85), (87), (92), (93) and (94) hold. In order to extend the existence time of the solution $(U_\epsilon, \Pi_\epsilon, \cdot)$ from $[0, (m+1)T]$ to $[0, (m+2)T]$, it suffices to choose $\epsilon$ such that

$$
\epsilon = \min \left( \frac{1}{16C^* C_1 C_{tr} C_T}, \frac{1}{10C^2_1 \beta_m C_{tr} C_T}, \frac{125C_1}{20 + C_1 + 5C^2_1 + 50C^3_1}, \frac{1}{8C_1 \beta_m} \right).
$$

Since $\beta_m$ is increasing in $m$, we have an example of upper bound of $\epsilon$ as

$$
\epsilon = \min \left( \frac{1}{16C^* C_1 C_{tr} C_T}, \frac{1}{10C^2_1 \beta_{N-1} C_{tr} C_T}, \frac{125C_1}{20 + C_1 + 5C^2_1 + 50C^3_1}, \frac{1}{8C_1 \beta_{N-1}} \right),
$$

where $N$ is the integer in (83). The contribution of the initial data $v_0$ in Theorem 1 is implicitly included in the choice of $T$.

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Ken Furukawa, Yoshikazu Giga and Takahito Kashiwabara
Graduate School of Mathematical Sciences
The University of Tokyo
3-8-1, Komaba, Meguro-ku
Tokyo 153-8914
Japan
E-mail: ken.furukawa@riken.jp

Yoshikazu Giga
E-mail: labgiga@ms.u-tokyo.ac.jp

Takahito Kashiwabara
E-mail: tkashiwa@ms.u-tokyo.ac.jp

Present Address
Ken Furukawa
RIKEN
7-1-26, Minatojima-minami-machi, Chuo-ku
Kobe Hyogo 650-0047
Japan

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