TORSION-FREE $G_{2(2)}^*$-STRUCTURES WITH FULL HOLONOMY ON NILMANIFOLDS

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Abstract. We study the existence of invariant metrics with holonomy $G_{2(2)}^* \subset SO(4,3)$ on compact nilmanifolds, i.e. on compact quotients of nilpotent Lie groups by discrete subgroups. We prove that, up to isomorphism, there exists only one indecomposable nilpotent Lie algebra admitting a torsion-free $G_{2(2)}^*$-structure such that the center is definite with respect to the induced inner product. In particular, we show that the associated compact nilmanifold admits a 3-parameter family of invariant metrics with full holonomy $G_{2(2)}^*$.

1. Introduction

The holonomy group of a pseudo-Riemannian manifold $(M, g)$ at a point $p \in M$ is defined as the group of parallel transports along loops based at $p$. In [2] Berger gave a list of possible holonomy groups of simply connected (pseudo-)Riemannian manifolds under the assumption that the group acts irreducibly on the tangent space at $p$. In the list, the exceptional compact Lie group $G_2$ appears as the holonomy group of a 7-dimensional Riemannian manifold, and its non-compact real form $G_{2(2)}^* \subset SO(4,3)$, as the holonomy group of a manifold with metric of signature $(4,3)$. Bryant proved in [3] that there exist pseudo-Riemannian metrics with exceptional holonomy groups, in particular, with holonomy $G_2$ and $G_{2(2)}^*$. Few explicit examples of signature $(4,3)$-metrics with holonomy group $G_{2(2)}^*$ have been constructed explicitly, see for instance [6], [8] and [13].

Each Riemannian manifold whose holonomy group is contained in $G_2$ is Ricci-flat. In particular, if the holonomy group of a homogeneous space $M$ is contained in $G_2$, then the homogeneous metric has to be flat. In contrast, in [12] it has been shown that there exist indecomposable indefinite symmetric spaces of signature $(4,3)$ whose holonomy is contained in $G_{2(2)}^*$.

In the present paper we show that there exist compact nilmanifolds, i.e. compact quotients of simply-connected nilpotent Lie groups $G$ by uniform discrete subgroups $\Gamma$, admitting invariant metrics of signature $(4,3)$ with
holonomy equal to $G^*_2(2)$. By invariant metric on $G/\Gamma$ we mean a metric induced by a inner product on the Lie algebra of $G$. More precisely, we prove that, up to isomorphism, there exists only one indecomposable 7-dimensional nilpotent Lie algebra admitting a torsion-free $G^*_2(2)$-structure $\varphi$ such that the center is definite with respect to the induced inner product $g_\varphi$. The Lie algebra has structure equations

$$
[e_1, e_2] = -e_4, \quad [e_2, e_3] = -e_5, \quad [e_1, e_3] = e_6,
$$
$$
[e_2, e_6] = -[e_3, e_4] = -[e_1, e_6] = [e_2, e_5] = -2e_7.
$$

For this Lie algebra we exhibit a 3-parameter family of (non-symmetric) metrics with full holonomy $G^*_2(2)$. This Lie algebra gives rise to a compact nilmanifold which inherits a 3-parameter family with holonomy equal to $G^*_2(2)$.

By Nomizu’s theorem [15] the de Rham cohomology of a compact nilmanifold $G/\Gamma$ can be calculated using invariant differential forms and is isomorphic to the Chevalley-Eilenberg cohomology of the Lie algebra $\mathfrak{g}$ of $G$. Moreover, a compact nilmanifold is formal in the sense of Sullivan’s rational homotopy theory [16] if and only if it is a torus. Therefore the compact examples with full holonomy $G^*_2(2)$ that we get are not formal in the sense of Sullivan’s rational homotopy theory [16]. It is still an open problem to see if compact Riemannian manifolds with holonomy $G_2^*$ are formal.

Other examples of torsion-free $G^*_2(2)$-structures with 1-dimensional and 2-dimensional holonomy have been found by M. Freibert on almost Abelian Lie algebras [7], showing that there are examples of calibrated Ricci-flat $G^*_2(2)$-structures on compact nilmanifolds which are not parallel and do not have holonomy contained in $G^*_2(2)$. In addition, it is worth mentioning the recent work [17] where manifolds with 5-dimensional holonomy contained in $G^*_2(2)$ are constructed.

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2. Preliminaries

For more details on the group $G^*_2(2)$ and $G^*_2(2)$-structures as for the proofs not appearing in this section see [11].

Let $M$ be a 7-dimensional manifold, and $L(M)$ its bundle of linear frames.

Definition 2.1. A $G^*_2(2)$-structure on $M$ is a $G^*_2(2)$-reduction of $L(M)$.

A $G^*_2(2)$-structure $P \to M$ on $M$ determines a global 3-form $\varphi$ defined by the equivariant map

$$
\varphi : P \to \bigwedge^3 \mathbb{R}^7, \quad u \mapsto \varphi_0,
$$

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with
\[ \varphi_0 = -e^{127} + e^{135} + e^{146} + e^{236} + e^{245} - e^{347} + e^{567}, \]
where $e^{ijk}$ stands for $e^i \wedge e^j \wedge e^k$. Conversely, a 3-form $\varphi$, i.e. a section of $\bigwedge^3(M) = L(M) \times_{GL(7)} \bigwedge^3(\mathbb{R}^7)$, where $\bigwedge^3(\mathbb{R}^7)$ is the $GL(7)$-orbit of $\varphi_0$, determines a $G_{2(2)}^*$-structure
\[ P = \{ u \in L(M) \mid u^* \varphi_0 = \varphi \}. \]
The inclusion $G_{2(2)}^* \subset SO^+(4,3)$ induces a pseudo-Riemannian metric $g$ of index 4 and a space and time orientation on $M$. For the proof of the following proposition see for instance [3] and [9].

**Proposition 2.2.** Let $P$ be a $G_{2(2)}^*$-structure on $M$, and $\nabla$ the Levi-Civita connection of the associated metric $g$. The following conditions are equivalent:

(a) $\nabla \varphi = 0$;
(b) $d\varphi = 0$ and $d(*\varphi) = 0$, where * is the Hodge star operator of $g$;
(c) The holonomy group of $g$ is isomorphic to a subgroup of $G_{2(2)}^*$.

The $G_{2(2)}^*$-structure $P$ is called torsion-free if one of the previous equivalent conditions hold.

Let $M$ be a manifold endowed with a $G_{2(2)}^*$-structure determined by a $G_{2(2)}^*$-reduction $P$ of $L(M)$ and associated 3-form $\varphi$. Suppose that there is a Lie group $G$ acting transitively on $M$. We say that the $G_{2(2)}^*$-structure is $G$-invariant if $P$ is invariant by the lifted action of $G$ on $L(M)$. By the definition of $\varphi$, this is equivalent to $\varphi$ being $G$-invariant. In addition, the inclusion of $P$ in the frame of orthonormal frames with respect to the associated metric $g_\varphi$ implies that $G$ acts on $M$ by isometries with respect to this metric.

2.1. Almost special $\varepsilon$-Hermitian structures in six dimensions. We recall that a $k$-form on a $n$-dimensional real vector space $V$ is called stable if its $GL(V)$ orbit is an open subset of $\bigwedge^k V^*$. For a collection of basic facts about stable forms and the proofs of the forthcoming results in this section see [6].

Regarding 2-forms, we recall that a 2-form $\omega$ on a real vector space $V$ of dimension $n = 2m$ is stable if and only if it is non-degenerate, that is, $\omega^m \neq 0$. Let now $V$ be an oriented 6-dimensional vector space. We consider the canonical isomorphism
\[ k : \bigwedge^5 V^* \to V \otimes \bigwedge^6 V^*, \quad \xi \mapsto X \otimes \nu \]
with $i_X \nu = \xi$.

Let $\rho$ be a 3-form on $V$, we define (see [6])

\[ K_\rho(v) = k((i_v \rho) \wedge \rho) \in V \otimes \bigwedge^6 V^* \]
\( \lambda(\rho) = \frac{1}{6} \text{tr}(K_\rho^2) \in \left( \bigwedge^6 V^* \right)^{\otimes 2} \) \\

(3) \[ \phi(\rho) = \sqrt{|\lambda(\rho)|} \in \bigwedge^6 V^* \]

where we have chosen the positive square root with respect to the orientation of \( V \). In the case \( \lambda(\rho) \neq 0 \) we also define

(4) \[ J_\rho = \frac{1}{\phi(\rho)} K_\rho \in \text{End}(V). \]

**Proposition 2.3** ([6]). A 3-form \( \rho \) on \( V \) is stable if and only if \( \lambda(\rho) \neq 0 \). In that case there are two orbits corresponding to \( \lambda(\rho) > 0 \) and \( \lambda(\rho) < 0 \).

The elements in the orbit corresponding to \( \lambda(\rho) > 0 \) have stabilizer \( SL(3, \mathbb{R}) \times SL(3, \mathbb{R}) \) in \( GL^+(V) \) and \( J_\rho \) is an almost para-complex structure, i.e., \( J_\rho^2 = 1 \), \( J_\rho \neq 1 \). The elements in the orbit corresponding to \( \lambda(\rho) < 0 \) have stabilizer \( SL(3, \mathbb{C}) \) in \( GL^+(V) \) and \( J_\rho \) is an almost complex structure, i.e., \( J_\rho^2 = -1 \). In both cases the dual form of \( \rho \) is determined by the formula

\[ \hat{\rho} = J_\rho^* \rho. \]

Finally, it is easy to prove that \( \hat{\rho} = -\rho \) and

(5) \[ J_\rho = -\varepsilon J_\rho. \]

A pair \( (\omega, \rho) \in \bigwedge^2 V^* \times \bigwedge^3 V^* \) of stable forms is called compatible if \( \omega \wedge \rho = 0 \) (or equivalently \( \hat{\rho} \wedge \omega = 0 \)) and normalized if \( \hat{\rho} \wedge \rho = \frac{2}{3} \omega^3 \). Let \( \varepsilon \) be the sign of \( \lambda(\rho) \), every compatible pair \( (\omega, \rho) \) uniquely determines an \( \varepsilon \)-complex structure \( J_\rho \) (i.e., \( J_\rho^2 = \varepsilon \)), an inner product \( g(\omega, \rho) = \varepsilon \omega(\cdot, J_\rho^* \cdot) \) (of signature \( (3, 3) \) for \( \varepsilon = 1 \), and definite or of signature \( (2, 4) \) or \( (4, 2) \) for \( \varepsilon = -1 \)), and a complex volume form \( \Psi = \rho + i_\xi \rho \) of type \( (3, 0) \) with respect to \( J_\rho \) (where \( i_\xi \) is the complex or para-complex imaginary unit accordingly). In addition, the stabilizer of the pair \( (\omega, \rho) \) under \( GL(V) \) is \( SU(p, q) \) for \( \varepsilon = -1 \) and \( SL(3, \mathbb{R}) \subset SO(3, 3) \) for \( \varepsilon = 1 \).

3. **Torsion-free \( G^*_2(2) \)-structures on nilpotent Lie algebras**

Let \( g \) be a 7-dimensional nilpotent Lie algebra with a three form \( \varphi \in \Lambda^3 g^* \) defining a \( G^*_2(2) \)-structure. Let \( \xi \) be an element in the center of \( g \) such that \( g_\varphi(\xi, \xi) \neq 0 \), where \( g_\varphi \) is the inner product of signature \( (4, 3) \) induced by \( \varphi \). The quotient \( \mathfrak{h} = g/\text{span}\{\xi\} \) has a unique Lie algebra structure making \( \mathfrak{h} \) nilpotent and the projection map \( g \rightarrow \mathfrak{h} \) is a Lie algebra epimorphism. Via the pullback we identify basic forms on \( g \) (i.e., forms \( \alpha \) with \( i_\xi \alpha = 0 \)) with forms on \( \mathfrak{h} \). Assume that \( g_\varphi(\xi, \xi) = -\varepsilon \in \{\pm 1\} \). Let \( \eta = -\varepsilon \xi^3 \) so that \( \eta(\xi) = 1 \), in analogy with circle bundles we can think of \( \eta \) as a connection form on the bundle \( g \rightarrow \mathfrak{h} \), and \( d\eta \) as its curvature form. We write

(6) \[ \varphi = \omega \wedge \eta + \psi_+, \]
where $\omega = i_\xi \varphi$ and $\psi_+$ are basic. A simple computation shows that

$$
\ast \varphi = \varepsilon \psi_- \wedge \eta - \frac{1}{2} \varepsilon \omega \wedge \omega,
$$

where $\psi_- = \varepsilon i_\xi (\ast \varphi) = \hat{\psi}_+$ is basic. Since $g_\varphi(\xi, \xi) = \pm 1$, by the stability of $\varphi$, there is always a basis $\{E_1, \ldots, E_7\}$ of $\mathfrak{g}$ such that

$$
\varphi = \varepsilon (E_{127} + E_{347}) + E_{567} + E_{135} + \varepsilon (E_{146} + E_{236} + E_{245})
$$

and $\xi = E_7$. Such basis is orthonormal, so that $\eta = E^7$ and we obtain

$$
\omega = \varepsilon (E_{12} + E_{34}) + E_{56}, \quad \psi_+ = E_{135} + \varepsilon (E_{146} + E_{236} + E_{245}).
$$

A simple computation using the Hodge dual of $\varphi$ shows that with respect to this basis $\psi_- = E_{246} + \varepsilon (E_{235} + E_{145} + E_{136})$. Therefore it is easy to see that $\omega$ and $\psi_+$ are stable and compatible, hence $(\omega, \psi_+)$ defines a special $\varepsilon$-Hermitian structure on $\mathfrak{h}$. Moreover, the pair $(\omega, \psi_-)$ is normalized as $\varepsilon \in \{\pm 1\}$. The same is true for $(\omega, \psi_-)$ (see Proposition 1.14 in [6]).

We now suppose that $\varphi$ is torsion free, i.e., $d\varphi = 0$, $d(\ast \varphi) = 0$. Since $\xi$ is in the center of $\mathfrak{g}$ one has that $\mathcal{L}_\xi \varphi = 0$ and $\mathcal{L}_\xi (\ast \varphi) = 0$, so that

$$
d\omega = d(i_\xi \varphi) = \mathcal{L}_\xi \varphi = 0,
$$

and

$$
d\psi_- = \varepsilon d(i_\xi (\ast \varphi)) = \varepsilon \mathcal{L}_\xi (\ast \varphi) = 0.
$$

This means that the pair $(\omega, \psi_-)$ is a symplectic half-flat structure on $\mathfrak{h}$. Finally taking exterior derivative in (6) and (7) we obtain

$$
0 = d\varphi = \omega \wedge d\eta + d\psi_+, \quad 0 = d(\ast \varphi) = -\varepsilon \psi_- \wedge d\eta.
$$

Therefore, constructing torsion-free $G^*_2(2)$-structures on $\mathfrak{g}$ with $g_\varphi(\xi, \xi) = -\varepsilon \in \{\pm 1\}$ is equivalent to construct symplectic half-flat structures on $\mathfrak{h}$ satisfying

$$
\omega \wedge d\eta + d\psi_+ = 0, \quad \psi_- \wedge d\eta = 0.
$$

**Remark 3.1.** With the previous procedure we can obtain all the torsion-free $G^*_2(2)$-structures for which $\xi$ is unitary. However, let $(\omega, \psi_+)$ be normalized and let $t > 0$. The 3-form $\varphi_t = \frac{t}{2} \omega \wedge \eta + \psi_+$ is a $G^*_2(2)$-structure with associated inner product $g_t$ satisfying $g_t(\xi, \xi) = -\varepsilon t$.

We will restrict ourselves to indecomposable 7-dimensional nilpotent Lie algebras. In order to apply the reduction procedure it is reasonable first to consider $G^*_2(2)$-structures for which the center is definite with respect to $g_\varphi$. Although as we shall see this condition is rather strong, it will lead us to examples of metrics with full holonomy $G^*_2(2)$.

Following [3] we have the following obstructions to the existence of a parallel $G^*_2(2)$-structure on $\mathfrak{g}$.
Proposition 3.2. Let \( g \) be a 7-dimensional Lie algebra admitting a \( G^*_2(2) \)-structure with associated 3-form \( \varphi \). Then \( (i_X \varphi)^3 \neq 0 \) for every element \( X \in g \) with \( g_{\varphi}(X, X) \neq 0 \).

Proof. Let \( X \) be an element of \( g \) with \( g_{\varphi}(X, X) \neq 0 \), then without loss of generality we can assume that \( g_{\varphi}(X, X) = -\varepsilon \in \{ \pm 1 \} \). Let \( \eta = -\varepsilon X^3 \), we write \( \varphi = \omega \wedge \eta + \psi_+ \). As seen before there is a basis \( \{ E_1, \ldots, E_7 \} \) of \( g \) such that

\[
\varphi = \varepsilon(E^{127} + E^{347}) + E^{567} + E^{135} + \varepsilon(E^{146} + E^{236} + E^{245}),
\]

with \( E_7 = X \). Therefore

\[
i_X \varphi = \varepsilon(E^{12} + E^{34}) + E^{56},
\]

so that \( (i_X \varphi)^3 \neq 0 \). \( \square \)

Making use of Proposition 3.2 we can see that a Lie algebra \( g \) does not admit a calibrated \( G^*_2(2) \)-structure, i.e. with associated 3-form \( \varphi \) satisfying \( d\varphi = 0 \), with definite center by finding an element \( \xi \) in the center of \( g \) such that \( (i_\xi \varphi)^3 = 0 \) for every closed 3-form \( \varphi \in \wedge^3 g^* \). Note that this is an obstruction to the existence of a \( G^*_2(2) \)-structure with definite center in general, without any assumption about \( d\varphi \) or \( d(\ast \varphi) \).

Proposition 3.3. Let \( g \) be a 7-dimensional nilpotent Lie algebra with a calibrated \( G^*_2(2) \)-structure \( \varphi \) with definite center. If \( \pi : g \to h \) is a Lie algebra epimorphism with kernel contained in the center and \( h \) is 6-dimensional, then \( h \) admits a symplectic form \( \omega \) and the curvature form is in the kernel of

\[
\ast \wedge \omega : H^2(h^*) \to H^4(h^*).
\]

Moreover, if the curvature form is exact, then \( g \) is isomorphic to \( \mathbb{R} \oplus h \) as Lie algebras.

Proof. Let \( \xi \) be in the kernel of \( \pi \), as \( \xi \) is in the center of \( g \) we have that \( g_{\varphi}(\xi, \xi) \neq 0 \), and we can suppose that \( g_{\varphi}(\xi, \xi) = -\varepsilon \in \{ \pm 1 \} \). Let \( \eta = -\varepsilon \xi^3 \), we write \( \varphi = \pi^*\omega \wedge \eta + \pi^*\psi_+ \), where \( \omega \) and \( \psi_+ \) are forms on \( h \). Since \( \pi^*\omega = i_\xi \varphi \) we have that \( \omega \) is non-degenerate. Moreover,

\[
0 = d\varphi = d\pi^*\omega \wedge \eta + \pi^*\omega \wedge d\eta + d\pi^*\psi_+,
\]

where \( d\pi^*\omega, d\eta \) and \( d\pi^*\psi_+ \) are basic. Therefore \( \omega \) is a symplectic form and \( d\eta \) is in the kernel of \( \ast \wedge \omega : H^2(h^*) \to H^4(h^*) \). If the form \( d\eta \) is exact on \( h \) then the epimorphism \( g \to h \) is trivial, hence \( g \) is isomorphic to \( \mathbb{R} \oplus h \) as Lie algebras. \( \square \)

Since a torsion-free \( G^*_2(2) \)-structure is in particular calibrated, making use of Proposition 3.3 we can show that a Lie algebra \( g \) does not admit a torsion-free \( G^*_2(2) \)-structure with definite center by finding an element \( \xi \) in the center of \( g \) contradicting Proposition 3.3 i.e., such that \( h = g/\text{Span}\{\xi\} \) does not admit a symplectic form \( \omega \), or such that \( d\eta \) is not in the kernel of \( \ast \wedge \omega : H^2(h^*) \to H^4(h^*) \).
The aim now is to take Gong’s classification of indecomposable 7-dimensional Lie algebras (see [10]), and eliminate those Lie algebras $\mathfrak{g}$ for which there is an element $\xi \in \mathfrak{g}$ contradicting Proposition 3.2 or Proposition 3.3, because as we have seen those Lie algebras cannot admit a torsion-free $G_2^{(2)}$-structure with definite center. In order to do that we use the work done in [5], where analogous obstructions are used to classify 7-dimensional nilpotent Lie algebras admitting a calibrated $G_2$-structure. More precisely, in that paper the authors start with Gong’s list and eliminate those Lie algebras which do not admit a calibrated $G_2$-structure by finding an element $\xi \in \mathfrak{g}$ contradicting Proposition 1 or Lemma 3 therein, which are the analogues of Proposition 3.2 and Proposition 3.3 for the Riemannian case. This is done in the proof of Lemma 5 and in the Appendix of [5]. In our case, since the obstructions given by Proposition 3.2 and Proposition 3.3 (as well as the obstructions given by Proposition 1 or Lemma 3 of [5]) only depend on the structure of the space of 3-forms and the space of closed 2-forms (and not in the signature of the metric), a simple inspection shows that for every Lie algebra in Gong’s list eliminated in [5], the same element $\xi$ used in [5] also contradicts Proposition 3.2 or Proposition 3.3. This means that any of the Lie algebras eliminated in [5] can admit a torsion-free $G_2^{(2)}$-structure. The only indecomposable Lie algebras in Gong’s list not yielding a contradiction with Proposition 3.2 and Proposition 3.3 are

\[
(0, 0, 12, 0, 0, 13 + 24, 15), \\
(0, 0, 12, 0, 24 + 13, 14, 46 + 34 + 15 + 23), (0, 0, 12, 0, 0, 13, 14 + 25), \\
(0, 0, 12, 0, 13, 24 + 23, 25 + 34 + 16 + 15 - 3 \cdot 26), \\
(0, 0, 0, 12, 23, -13, 2 \cdot 26 - 2 \cdot 34 - 2 \cdot 16 + 2 \cdot 25), \\
(0, 0, 0, 12, 13, 14 + 23, 15), (0, 0, 0, 12, 13, 14, 15) \\
(0, 0, 12, 13, 23, 15 + 24, 16 + 34), \\
(0, 0, 12, 13, 23, 15 + 24, 16 + 25 + 34).
\]

The notation $(0, 0, 0, 12, 23, -13, 2 \cdot 26 - 2 \cdot 34 - 2 \cdot 16 + 2 \cdot 25)$ means that $\mathfrak{g}^*$ has a basis \{${e^1, \ldots, e^7}$\} such that $de^i = 0$, for $i = 1, \ldots, 3$ and $de^4 = e^{12}$, $de^5 = e^{23}$, $de^6 = -e^{13}$, $de^7 = 2e^{26} - 2e^{34} - 2e^{16} + 2e^{25}$. We now show which of these Lie algebras actually admit a torsion-free $G_2^{(2)}$-structure with definite center.

**Lemma 3.4.** With the exception of

\[\mathfrak{g}_1 = (0, 0, 0, 12, 23, -13, 2 \cdot 26 - 2 \cdot 34 - 2 \cdot 16 + 2 \cdot 25),\]

none of the Lie algebras in [9] admits a torsion-free $G_2^{(2)}$-structure with definite center.

**Proof.** We have seen that if a Lie algebra $\mathfrak{g}$ admits a torsion-free $G_2^{(2)}$-structure with definite center, then for every element $\xi$ in the center of $\mathfrak{g}$ we have a reduction $\mathfrak{g} \to \mathfrak{h} = \mathfrak{g}/\text{Span}\{\xi\}$ such that there are stable forms $\omega, \psi_+$ and $\psi_- = \hat{\psi}_+$ satisfying

\[
(10) \quad d\omega = 0, \quad d\psi_- = 0, \quad \omega \wedge d\eta + d\psi_+ = 0, \quad \psi_- \wedge d\eta = 0, \quad \omega \wedge \psi_- = 0.
\]

Moreover, Theorem 3.5. Every Lie algebra $\mathfrak{g}$ admitting a torsion-free structure with holonomy equal to $G$ has a definite center. Note that if for example we reduce by $\xi$, the general form of $\eta = -\varepsilon \xi^3$ is

$$\eta = e^7 + \sum_{i=1}^6 \gamma_i e^i,$$

for some constants $\gamma_i$ depending on $g$. 

- $(0,0,12,0,0,13 + 24,15)$ $\xi = e_7$ $(0,0,12,0,0,13 + 24)$: For any compatible closed forms $\omega$ and $\psi_-$ on $\mathfrak{h}$, imposing $\omega \wedge d\eta + d\psi_+ = 0$, and $\psi_- \wedge d\eta = 0$ we obtain solutions such that $e_6$ is null with respect to the corresponding metric $g_\varphi$ on $\mathfrak{g}$.

- $(0,0,12,0,24 + 13,14,64 + 24 + 15 + 23)$ $\xi = e_7$ $(0,0,12,0,24 + 13,14)$: For any closed forms $\omega$ and $\psi_-$ on $\mathfrak{h}$, equations $\psi_- \wedge d\eta = 0$, $\omega \wedge \psi_- = 0$ and $\omega \wedge d\eta + d\psi_+ = 0$ imply $\omega^3 = 0$.

- $(0,0,12,0,13,14 + 25)$ $\xi = e_7$ $(0,0,12,0,13,14 + 25)$: For any compatible closed forms $\omega$ and $\psi_-$ on $\mathfrak{h}$, equation $\omega \wedge d\eta + d\psi_+ = 0$ implies $\omega^3 = 0$.

- $(0,0,13,16,23,28 + 13,24 + 15 + 1 + 3 - 26)$ $\xi = e_7$ $(0,0,13,16,23,28 + 13,24 + 15 + 1 + 3 - 26)$: For any closed forms $\omega$ and $\psi_-$ on $\mathfrak{h}$, equation $\omega \wedge d\eta + d\psi_+ = 0$ implies $\omega^3 = 0$.

- $(0,0,12,13,14 + 23,15)$ $\xi = e_7$ $(0,0,0,12,13,14 + 23)$: For any compatible closed forms $\omega$ and $\psi_-$ on $\mathfrak{h}$, imposing $\omega \wedge d\eta + d\psi_+ = 0$, and $\psi_- \wedge d\eta = 0$ we obtain solutions such that $e_6$ is null with respect to the corresponding metric $g_\varphi$ on $\mathfrak{g}$.

- $(0,0,0,12,13,14,15)$ $\xi = e_7$ $(0,0,0,12,13,14)$: For any closed forms $\omega$ and $\psi_-$ on $\mathfrak{h}$, equations $\psi_- \wedge d\eta = 0$, $\omega \wedge \psi_- = 0$ and $\omega \wedge d\eta + d\psi_+ = 0$ imply $\omega^3 = 0$ or $\lambda(\psi_-) = 0$.

- $(0,0,12,13,23,15 + 24,16 + 34)$ $\xi = e_7$ $(0,0,12,13,23,15 + 24)$: For any closed forms $\omega$ and $\psi_-$ on $\mathfrak{h}$, equations $\psi_- \wedge d\eta = 0$ and $\omega \wedge d\eta + d\psi_+ = 0$ imply $\omega^3 = 0$.

- $(0,0,12,13,23,15 + 24,16 + 25 + 34)$ $\xi = e_7$ $(0,0,12,13,23,15 + 24)$: For any closed forms $\omega$ and $\psi_-$ on $\mathfrak{h}$, equations $\psi_- \wedge d\eta = 0$, $\omega \wedge \psi_- = 0$ and $\omega \wedge d\eta + d\psi_+ = 0$ imply $\omega^3 = 0$.

Theorem 3.5. The only indecomposable 7-dimensional nilpotent Lie algebra admitting a torsion-free $G_{2(2)}^*$-structure with definite center is

$$\mathfrak{g}_1 = (0,0,0,12,23,-13,2 \cdot 26 - 2 \cdot 34 - 2 \cdot 16 + 2 \cdot 25).$$

Moreover, $\mathfrak{g}_1$ admits a 3-parameter family of torsion-free $G_{2(2)}^*$-structures with holonomy equal to $G_{2(2)}^*$. □
Proof. We first show that \( g_1 \) admits torsion-free \( G^*_{2(2)} \)-structures with definite center. We consider a new basis \( \{e_1, \ldots, e_7\} \) of \( g_1 \) with respect to which \( g_1 \) has structure equations

\[
(0, 0, 0, 12, 13, 23, -2 \cdot 25 - 2 \cdot 34 + 2 \cdot 15 + 2 \cdot 26)
\]

(this can be done by setting \( e_5 = -e'_6, e_6 = e'_5, \) and \( e_i = e'_i \) for \( i \neq 5, 6, \) where by \( \{e'_1, \ldots, e'_7\} \) we denote the old basis of \( g_1 \)). Consider the reduction

\[
g_1 \rightarrow h_1 = \frac{g_1}{\text{span}\{e_7\}} \cong (0, 0, 0, 12, 13, 23).
\]

Let \( \varphi \) be a \( G^*_{2(2)} \)-structure on \( g_1 \). As before we write

\[
\varphi = \omega \wedge \eta + \psi_+, \quad *\varphi = \varepsilon \psi_- \wedge \eta - \frac{1}{2} \varepsilon \omega \wedge \omega.
\]

We will now compute all torsion-free \( G^*_{2(2)} \)-structures with definite center satisfying \( \eta = e^7 \) and \( g_\varphi(e_7, e_7) = -\varepsilon \in \{\pm 1\} \), by solving equations (10) for stable forms \( \omega, \psi_+ \) and \( \psi_- = \psi_+ \) on \( h_1 \). We will also impose the normalization condition \( \psi_- \wedge \psi_+ = \frac{2}{3} \omega^3 \). The closed 2-forms on \( h_1 \) are given by

\[
\omega = r_1 e^{12} + r_2 e^{13} + r_3 e^{14} + r_4 e^{15} + r_5 e^{23} + r_6 e^{24} + r_7 e^{26} + r_8 e^{35}
\]

\[
+ r_9 e^{36} + r_{10}(e^{16} + e^{25}) + r_{11}(e^{16} - e^{34}),
\]

for some parameters \( r_1, \ldots, r_{11} \in \mathbb{R} \). The non-degeneracy condition \( \omega^3 \neq 0 \) is thus

\[
r_3(r_7 r_8 - r_9 r_{10}) + r_4(r_6 r_9 + r_7 r_{11}) - (r_6 r_8 + r_{10} r_{11})(r_{10} + r_{11}) \neq 0.
\]

For the second equation of (10), it is also easy to compute the space of closed 3-forms. In this case imposing also the fourth equation we obtain that \( \psi_- \) must be of the form

\[
\psi_- = m_1(e^{125} - e^{234}) + m_2(e^{126} + e^{134} - e^{125}) + m_3(e^{135} - e^{136})
\]

\[
+ m_4(e^{135} - e^{236}) + m_5(e^{145} - e^{146} - e^{245}) + m_6(e^{145} - e^{246})
\]

\[
+ m_7 e^{123} + m_8 e^{124} + m_9 e^{235} + m_{10} e^{356},
\]

for some parameters \( m_1, \ldots, m_{10} \in \mathbb{R} \). The non-degeneracy condition for \( \psi_- \) is \( \lambda(\psi_-) \neq 0 \) (which is equivalent to \( \lambda(\psi_+) \neq 0 \)) where \( \lambda \) is given by (2). To impose the third equation of (10) we have to compute \( \psi_+ \). Since \( \psi_- = J^*_\psi_+ \psi_+ \), where \( J^*_{\psi_+} \) is given by (4), taking into account \( J^2_{\psi_+} = \varepsilon \) and (5) we obtain \( \psi_+ = -J^*_\psi_+ \psi_- \).

The two remaining conditions are then the vanishing of the forms

\[
\omega \wedge \psi_-,
\]

\[
d\psi_+ + \omega \wedge d\eta.
\]

Solving these two equations and imposing the normalization condition, we obtain after a quite long computation a parametrization of the space of
torsion-free $G^*_2(2)$ structures on $\mathfrak{g}_1$ such that $g_\varphi(e_7,e_7) = -\varepsilon$. Such parametrization is a subset of $\mathbb{R}^{11} \times \mathbb{R}^{10} \times \mathbb{Z}_2$ given by

$$r_3 = 0, \quad r_6 = 0, \quad r_8 = 0, \quad r_9 = 0, \quad r_7 = r_4, \quad r_{11} = -r_4, \quad r_{10} = 0,$$

$$m_1 = \frac{1}{3} r_m r_5 + r_5 r_6 + 2 r_6 r_7 + 2 r_7 r_8 + r_8 r_9 + r_9 r_10,$$

$$m_2 = \frac{1}{3} r_4 r_5 - 2 r_4 r_6 - r_4 r_7 - r_4 r_8.$$

$$m_3 = -\frac{r_4 m_6}{r_4} + m_6 m_10,$$

$$m_7 = -\frac{1}{12 m_4^2 r_4^2 (m_5^2 + m_6^2 + m_5 m_6)} (18 m_5 m_9 m_5 m_5^2 m_8 m_8 4 m_4^2 + 9 m_5^2 m_5^2 m_5^2 m_5 10$$

$$+ 18 m_5 m_9 m_5 m_5^2 m_5^2 m_5 10 - 18 m_5 m_5 m_5^2 m_5^2 m_5 10 + 9 m_5^2 m_5^2 m_5^2 m_5 10$$

$$+ 18 m_5 m_9 m_5 m_5^2 m_5^2 m_5 10 - 18 m_5 m_5 m_5^2 m_5^2 m_5 10 + 9 m_5^2 m_5^2 m_5^2 m_5 10$$

$$+ 18 m_5 m_9 m_5 m_5^2 m_5^2 m_5 10 - 18 m_5 m_5 m_5^2 m_5^2 m_5 10 + 9 m_5^2 m_5^2 m_5^2 m_5 10$$

$$+ 18 m_5 m_9 m_5 m_5^2 m_5^2 m_5 10 - 18 m_5 m_5 m_5^2 m_5^2 m_5 10 + 9 m_5^2 m_5^2 m_5^2 m_5 10$$

$$+ 18 m_5 m_9 m_5 m_5^2 m_5^2 m_5 10 - 18 m_5 m_5 m_5^2 m_5^2 m_5 10 + 9 m_5^2 m_5^2 m_5^2 m_5 10$$

$$+ 18 m_5 m_9 m_5 m_5^2 m_5^2 m_5 10 - 18 m_5 m_5 m_5^2 m_5^2 m_5 10 + 9 m_5^2 m_5^2 m_5^2 m_5 10.$$

$$m_{10} = 2 \varepsilon \frac{r_4^2}{m_5^2 + m_6^2 + m_5 m_6}, \quad \phi = -2 r_4^3,$$

and non-degeneracy conditions $r_4 \neq 0, \quad m_5^2 + m_6^2 + m_5 m_6 \neq 0$. Note that this space has (at least) four connected components due to the value of $(\varepsilon, \text{sign}(r_4))$. The space of free parameters

$$(16) \quad (r_1, r_2, r_4, r_5, m_4, m_5, m_6, m_8, m_9, \varepsilon)$$

is then an open set of $\mathbb{R}^4 \times \mathbb{R}^5 \times \mathbb{Z}_2$ given by the non-degeneracy condition

$$r_4 \neq 0, \quad m_5^2 + m_6^2 + m_5 m_6 \neq 0.$$

The rest of the proof is devoted to exhibit a subfamily of torsion-free $G^*_2(2)$ structures on $\mathfrak{g}_1$ with full holonomy $G^*_2(2)$. Since the non-degeneracy conditions only involve $(r_4, m_5, m_6)$, we choose the values

$$(17) \quad r_1 = 0, \quad r_2 = 0, \quad r_5 = 0, \quad m_4 = 0, \quad m_8 = 0, \quad m_9 = 0.$$

Recall that since the pair $(\omega, \psi_\perp)$ is normalized, the inner product $g_\varphi$ on $\mathfrak{g}_1$ is obtained as $g_\varphi = h - \varepsilon \eta \otimes \eta$, where $h$ is the inner product associated to
\((\omega, \psi_\perp)\). Evaluating at (17) we obtain for \(g_\varphi\) the matrix representation
\[
\begin{pmatrix}
 \frac{m_5}{2} & -\frac{m_5}{2} & 0 & 0 & 0 & 0 \\
 \frac{m_5}{2} & -\frac{m_5}{2} & 0 & 0 & 0 & 0 \\
 0 & 0 & \frac{\varepsilon r_4^2}{(m_5, m_6)} & 0 & 0 & 0 \\
 0 & 0 & 0 & \frac{\varepsilon r_4^2}{(m_5, m_6)} & 0 & 0 \\
 0 & 0 & 0 & 0 & \frac{\varepsilon r_4^2}{(m_5, m_6)} & 0 \\
 0 & 0 & 0 & 0 & 0 & -\varepsilon
\end{pmatrix},
\]
where \((m_5, m_6)\) stands for \(m_5^2 + m_6^2 + m_5 m_6\).

In order to show that this family of metrics has full holonomy \(G_{2(2)}^*\) we show that the Lie algebra spanned by the curvature endomorphisms \(R_{XY} : g_1 \to g_1\), \(X, Y \in g_1\), and the endomorphisms \((\nabla_Z R)_{XY} : g_1 \to g_1\), \(Z, X, Y \in g_1\), has dimension equal to 14. As it is well known (see [1]), the holonomy algebra \(\mathfrak{hol}\) of \(g_\varphi\) contains this Lie algebra. Therefore since \(\mathfrak{hol} \subset g_{2(2)}^*\) and \(\dim(g_{2(2)}) = 14\) we have that \(\mathfrak{hol} = g_{2(2)}^*\), so that \(g_\varphi\) has full holonomy \(G_{2(2)}^*\). A set of linearly independent endomorphisms spanning \(\mathfrak{hol}\) can be found in the Appendix at the end of the manuscript.

The three parameter family of metrics with full holonomy \(G_{2(2)}^*\) constructed on \(g_1\) provides a family of left-invariant metrics with full holonomy \(G_{2(2)}^*\) on the simply-connected nilpotent Lie group \(G\) associated to \(g_1\). It is worth noting the difference with the Riemannian counterpart, where any homogeneous Ricci-flat metric must be flat. Moreover, since the structure equations of \(g_1\) are rational, \(G\) admits a co-compact lattice \(\Gamma\) (see [14]). Therefore the family of left-invariant metrics on \(G\) induce a family of metrics with full holonomy \(G_{2(2)}^*\) on the compact nilmanifold \(\Gamma \setminus G\).

**Remark 3.6.** So far, the authors have not found any value of the parameters (10) such that \(g_\varphi\) has not holonomy equal to \(G_{2(2)}^*\).

**Remark 3.7.** Note that we have parameterized all torsion-free \(G_{2(2)}^*\)-structures with definite center on \(g_1\) but we have not studied their isomorphism classes. There are choices for the parameters that give isometric structures and choices that give non-isometric structures. For example setting (17), the choices \(\{r_4 = -1, m_5 = 1, m_6 = -1\}\) and \(\{r_4 = -1, m_5 = -1, m_6 = 1\}\) for \(\varepsilon = \pm 1\) give isometric metrics. On the other hand, the choices \(\{r_4 = -1, m_5 = 1, m_6 = 0\}\) and \(\{r_4 = -2, m_5 = 1, m_6 = 0\}\) give non-isometric metrics.

As it has been pointed out, the condition that a nilpotent Lie algebra has definite center with respect to \(g_\varphi\) is rather strong. New examples of torsion-free \(G_{2(2)}^*\)-structures can be obtained relaxing this condition and just asking the existence of a non-null element in the center of the Lie algebra. This is done for instance in the proof of Lemma 3.4 for the Lie algebras \((0, 0, 12, 0, 13 + 24, 15)\) and \((0, 0, 0, 12, 13, 14 + 23, 15)\).
Example 3.8. Consider the Lie algebra \( g = (0, 0, 12, 13, 15, 14 + 23) \) and the reduction
\[
(0, 0, 12, 13, 15, 14 + 23) \overset{e_7}{\rightarrow} h = (0, 0, 12, 13, 15).
\]
We construct a pair \( (\omega, \psi_-) \) of stable forms on \( h \) satisfying
\[
(18) \quad d\omega = 0, \quad d\psi_- = 0, \quad \omega \wedge d\eta + d\psi_+ = 0, \quad \psi_- \wedge d\eta = 0, \quad \omega \wedge \psi_+ = 0,
\]
and the normalization condition
\[
\psi_- \wedge \psi_+ = \frac{2}{3} \omega^{3}.
\]
For the sake of simplicity we shall restrict ourselves to the case \( \eta = e^7 \). The closed 2-forms on \( h \) are given by
\[
\omega = r_{12} e^{12} + r_{13} e^{13} + r_{14} e^{14} + r_{15} e^{15} + r_{16} e^{16} + r_{23} e^{23} + r_{24} e^{24} + r_{34} (e^{25} + e^{34}) + r_{35} e^{35},
\]
for some parameters \( r_{ij} \in \mathbb{R} \). The non-degeneracy condition \( \omega^3 \neq 0 \) is thus
\[
-r_{16} r_{24} r_{35} + r_{16} r_{34}^2 \neq 0.
\]
A closed 3-form \( \psi_- \) is of the form
\[
\psi_- = m_{123} e^{123} + m_{124} e^{124} + m_{125} e^{125} + m_{126} e^{126} + m_{134} e^{134} + m_{135} e^{135} + m_{136} e^{136} + m_{145} e^{145} + m_{146} e^{146} + m_{156} e^{156} + m_{234} e^{234} + m_{235} e^{235} + m_{236} e^{236} + m_{345} (e^{236} + e^{345}) + m_{345} e^{356},
\]
for some parameters \( m_{ijk} \in \mathbb{R} \). The non-degeneracy condition for \( \psi_- \) is \( \lambda(\psi_-) \neq 0 \). Solving \( (18) \) together with the normalization condition we obtain
\[
r_{14} = -r_{23}, \quad r_{34} = 0, \quad r_{16} = r_{35},
\]
\[
m_{145} + m_{235} = 0, \quad m_{345} + m_{146} = 0, \quad m_{356} = m_{156} = 0,
\]
\[
m_{124} = -\frac{1}{r_{35}} (2m_{235} r_{23} + m_{135} r_{24} + m_{234} r_{15} - m_{345} r_{12}),
\]
\[
m_{126} = \frac{1}{r_{35}} (m_{235} r_{35} - m_{345} r_{15}),
\]
\[
m_{136} = -\frac{1}{r_{24}} (m_{234} r_{35} + 2m_{345} r_{23}),
\]
\[
m_{125} = \frac{1}{8m_{345} r_{35}^2} (6m_{345}^5 m_{235} r_{23}^2 - 4m_{345} m_{235} r_{35} r_{15} - 2m_{345}^7 r_{15}^2 + r_{35}^3 \phi^3),
\]
\[
r_{24} = -\varepsilon \frac{|m_{345}|^3}{|r_{35}|^3}, \quad \phi = 2\varepsilon \frac{|m_{345}|^3}{|r_{35}|^3},
\]
with non-degeneracy condition \( r_{35} \neq 0 \) and \( m_{345} \neq 0 \). Note that this parametrization has at least eight connected components due to the value of the triple \((\varepsilon, \text{sign}(m_{345}), \text{sign}(r_{35}))\). The space of free parameters

\[
(r_{12}, r_{13}, r_{15}, r_{23}, r_{35}, m_{123}, m_{134}, m_{135}, m_{234}, m_{235}, m_{345}, \varepsilon)
\]

is thus an open set of \( \mathbb{R}^5 \times \mathbb{R}^6 \times \mathbb{Z}_2 \) given by equations \( m_{345} \neq 0 \) and \( r_{35} \neq 0 \).

Since the degeneracy condition only involves \( r_{35} \) and \( m_{345} \), we select the subfamily of \( G_{2(2)}^* \)-structures given by the choice

\[
\begin{align*}
    r_{12} &= 0, r_{13} = 0, r_{15} = 0, r_{23} = 0, m_{123} = 0, m_{134} = 0, m_{135} = 0, \\
    m_{234} &= 0, m_{235} = 0.
\end{align*}
\]

For the sake of simplicity we also suppose \( r_{35} > 0 \) and \( m_{345} > 0 \) (the other cases are analogous). Evaluating at these values, we obtain

\[
\varphi = \varepsilon \frac{m_{345}^2}{r_{35}} (-e^{145} + e^{126} + e^{235}) - \varepsilon \frac{m_{345}^3}{r_{35}^2} e^{247} + r_{35} (e^{346} + e^{167} + e^{357}),
\]

whence \( g_\varphi \) has matrix representation

\[
\begin{pmatrix}
    0 & 0 & 0 & 0 & m_{345} & 0 \\
    0 & \varepsilon m_{345}^4 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & \varepsilon^2 r_{35} m_{345} \\
    0 & 0 & m_{345} & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & -\varepsilon
\end{pmatrix}.
\]

Note that \( e_6 \) is null as expected. A straightforward computation shows that the space spanned by the curvature endomorphisms of \( g_\varphi \) and its first derivatives has dimension 6. In addition, the second order derivatives of the curvature are linearly dependent with the curvature and its first derivatives. This implies that the holonomy of \( g_\varphi \) is 6-dimensional. So far, the authors have not found any value of the parameters \((19)\) such that \( g_\varphi \) has not 6-dimensional holonomy.

**Remark 3.9.** As in the previous example, all the torsion-free \( G_{2(2)}^* \)-structures on the Lie algebra \((0, 0, 0, 12, 13, 14 + 23, 15)\) obtained by the authors have 6-dimensional holonomy. In this case, in order to obtain solutions to \((10)\) one must consider \( \eta = e^T + \sum_{i=1}^6 \gamma_i e^i \), with \( \gamma_6 \neq 0 \).

**Remark 3.10.** The reduction procedure shown at the beginning of this section gives a simple method for constructing torsion-free \( G_{2(2)}^* \)-structures on decomposable 7-dimensional Lie algebras of the form \( g = h \oplus \mathbb{R} \). Recall that by \((10)\) the only decomposable 7-dimensional nilpotent Lie algebra which is not of this form is \((0, 0, 0, 0, 12, 34, 36)\).

Let \( g = h \oplus \mathbb{R} \) be a 7-dimensional nilpotent Lie algebra. Let \( \varphi \) be a torsion-free \( G_{2(2)}^* \)-structure on \( g \) such that the summand \( \mathbb{R} \) is not null and
orthogonal to $\mathfrak{h}$ with respect to $g_{\varphi}$. The reduction $\mathfrak{g} \rightarrow \mathfrak{h}$ has curvature form $d\eta = 0$, so that writing again (6) and (7), equations (8) transform into $d\psi_+ = 0$. Therefore, constructing a torsion-free $G^*_2(\mathbb{R})$-structure on $\mathfrak{g}$ with non-null summand $\mathbb{R}$ orthogonal to $\mathfrak{h}$ is equivalent to construct a torsion-free almost $\varepsilon$-special Hermitian structure on $\mathfrak{h}$, i.e., a compatible pair $(\omega, \psi_+)$ with $d\omega = 0$, $d\psi_+ = 0$, $d\psi_- = 0$ (8). Note that with this procedure the holonomy of $g_{\varphi}$ will be equal to the holonomy of $g_{\varphi|\mathfrak{h}}$, and in particular it will be contained in $SL(3, \mathbb{R})$ or $SU(2, 1)$.

**Appendix**

We denote by $e^i_\varepsilon$ the endomorphism $e_i \otimes e^i$ of $\mathfrak{g}_1$ and $(m_5, m_6) = m_2^2 + m_6^2 + m_5 m_6$, where \{e_1, \ldots, e_7\} is the basis such that $\mathfrak{g}_1$ has structure equations $(0, 0, 0, 12, 13, 25, -2 - 25 - 2 - 34 + 2 \cdot 15 + 2 \cdot 26)$. By direct calculation, a set of linearly independent endomorphisms of $\mathfrak{g}_1$ spanning $\mathfrak{sof}$ is

\[
R_{e_{1}e_{2}} = -\frac{3(m_5 + m_6)}{2r^2_4} e_1 - \frac{3m_6}{2r^2_4} e_1^2 + \frac{3m_5}{2r^2_4} e_1^2 + \frac{3(m_5 + m_6)}{2r^2_4} e_2 - \frac{1}{2} \varepsilon^7_3
\]

\[
+ \frac{m_6}{r^2_4} e_5^5 - \frac{m_5 m_6}{r^2_4} e_5^6 - \frac{m_5 + m_6}{r^2_4} e_6^6 - \frac{m_6}{r^2_4} e_6^6 - \frac{(m_5, m_6)}{2r^2_4} e_5^7,
\]

\[
R_{e_{1}e_{3}} = -\frac{3m_5}{2r^2_4} e_1^3 + \frac{1}{2} \varepsilon^7_1 - \frac{2m_6}{2r^2_4} e_2^2 - \frac{1}{2} \varepsilon^7_2 + \frac{3\varepsilon^r_4}{(m_5, m_6)} e_1^3 + \frac{3\varepsilon^r_4}{(m_5, m_6)} e_3^3
\]

\[
+ \frac{m_5 + m_6}{2r^2_4} e_4^5 + \frac{m_6}{2r^2_4} e_4^6 - \frac{2\varepsilon^r_4}{(m_5, m_6)} e_4^6 - \frac{\varepsilon m_6}{(m_5, m_6)} e_7^1 + \frac{\varepsilon m_5}{(m_5, m_6)} e_7^2,
\]

\[
R_{e_{1}e_{4}} = \frac{m_6}{2r^2_4} e_1^4 - \frac{m_5 + m_6}{2r^2_4} e_2^4 - \frac{m_5 + m_6}{2r^2_4} e_3^4 - \frac{m_6}{2r^2_4} e_6^4 - \frac{(m_5, m_6)}{r^2_4} e_4^1
\]

\[
- \frac{(m_5, m_6)}{r^2_4} e_5^5 - \frac{m_5}{r^2_4} e_5^6 + \frac{m_6}{r^2_4} e_6^6 + \frac{(m_5, m_6)}{2r^2_4} e_7^5,
\]

\[
R_{e_{1}e_{5}} = \frac{4m_5 + 3m_6}{2r^2_4} e_1^5 + \frac{3m_6}{2r^2_4} e_1^6 - \frac{3m_5}{2r^2_4} e_2^5 - \frac{2m_5 + 3m_6}{2r^2_4} e_2^6 - \frac{2\varepsilon^r_4}{(m_5, m_6)} e_3^4
\]

\[
- \frac{3(m_5, m_6)}{r^2_4} e_4^5 - \frac{m_5}{r^2_4} e_4^6 + \frac{2\varepsilon(m_5 - 3m_6)}{(m_5, m_6)} e_5^1 + \frac{8\varepsilon m_5}{(m_5, m_6)} e_5^2
\]

\[
+ \frac{2\varepsilon(2m_5 + 3m_6)}{(m_5, m_6)} e_6^1 + \frac{6\varepsilon m_6}{(m_5, m_6)} e_6^2 - \frac{\varepsilon m_5}{(m_5, m_6)} e_6^4,
\]

\[
R_{e_{1}e_{6}} = \frac{m_6}{2r^2_4} e_1^6 + \frac{m_6}{2r^2_4} e_1^6 + \frac{m_6}{2r^2_4} e_1^6 + \frac{2\varepsilon m_6}{(m_5, m_6)} e_5^1 + \frac{2\varepsilon m_6}{(m_5, m_6)} e_5^2 - \frac{2\varepsilon m_6}{(m_5, m_6)} e_6^1
\]

\[
- \frac{\varepsilon m_6}{(m_5, m_6)} e_6^2,
\]

\[
R_{e_{1}e_{7}} = \frac{(m_5, m_6)}{2r^2_4} e_1^7 - \frac{m_5 + m_6}{2r^2_4} e_2^7 + \frac{m_5}{2r^2_4} e_2^7 + \frac{\varepsilon m_5}{(m_5, m_6)} e_3^1 + \frac{\varepsilon(m_5 + m_6)}{(m_5, m_6)} e_3^2
\]

\[
+ \frac{(m_5, m_6)}{r^2_4} e_4^5 - \frac{2\varepsilon m_5}{(m_5, m_6)} e_4^6 - \frac{2\varepsilon m_6}{(m_5, m_6)} e_6^1 - \frac{\varepsilon}{r^2_4} e_7^7,
\]
$$R_{e2e3} = \frac{-3m_5}{2r_4^2} e_1^3 + \frac{3(m_5 + m_6)}{2r_4^2} e_2^3 + \frac{1}{2} e_7^2 - \frac{3\varepsilon r_4}{(m_5, m_6)} e_3^1 - \frac{m_5}{r_4^2} e_4^1 - \frac{m_5 + m_6}{r_4^2} e_6^1,$$

$$+ \frac{2\varepsilon r_4}{(m_5, m_6)} e_5^1 - \frac{2\varepsilon}{(m_5, m_6)} e_6^2 + \frac{\varepsilon (m_5 + m_6)}{(m_5, m_6)} e_7^1 + \frac{\varepsilon m_6}{(m_5, m_6)} e_7^2,$$

$$R_{e2e4} = -\frac{m_5 + m_6}{2r_4^2} e_1^4 + \frac{m_5}{2r_4^2} e_2^4 + \frac{m_5}{2r_4^2} e_3^5 + \frac{m_5 + m_6}{2r_4^2} e_3^6 - \frac{(m_5, m_6)^2}{r_4^2},$$

$$+ \frac{(m_5, m_6)}{r_4^2} e_3^7 - \frac{m_6}{r_4^2} e_3^6 + \frac{m_5 + m_6}{r_4^2} e_7^6 + \frac{(m_5, m_6)}{r_4^2} e_7^7,$$

$$R_{e2e5} = \frac{(m_5, m_6)}{2r_4^2} e_1^5 + \frac{m_5}{2r_4^2} e_1^7 - \frac{(m_5, m_6)}{2r_4^2} e_2^5 + \frac{m_6}{2r_4^2} e_2^7 + \frac{\varepsilon m_6}{(m_5, m_6)} e_3^1,$$

$$- \frac{\varepsilon m_5}{(m_5, m_6)} e_3^3 + \frac{(m_5, m_6)}{r_4^2} e_4^6 - \frac{2\varepsilon m_6}{(m_5, m_6)} e_5^4 - \frac{2\varepsilon (m_5 + m_6)}{(m_5, m_6)} e_5^6 + \frac{\varepsilon e_1^7}{r_4^2} + \frac{\varepsilon e_2^7}{r_4^2},$$

$$R_{e2e7} = \frac{(m_5 + m_6)}{(m_5, m_6)} e_1^1 + \frac{\varepsilon m_5}{(m_5, m_6)} e_2^1 - \frac{\varepsilon m_6}{(m_5, m_6)} e_2^2 - \frac{\varepsilon (m_5 + m_6)}{(m_5, m_6)} e_2^2 - \frac{\varepsilon e_3^7}{r_4^2},$$

$$- \frac{2\varepsilon m_6}{(m_5, m_6)} e_5^5 + \frac{2\varepsilon m_5}{(m_5, m_6)} e_5^6 + \frac{2\varepsilon (m_5 + m_6)}{(m_5, m_6)} e_5^7 + \frac{2\varepsilon m_6}{(m_5, m_6)} e_6^6 - \frac{\varepsilon e_7^7}{r_4^2},$$

$$(\nabla e_1 R)_{e1e2} = \frac{m_6}{r_4^2} e_1^4 - \frac{m_5 + m_6}{r_4^2} e_2^4 - \frac{m_5}{2r_4^2} e_3^5 - \frac{m_5 + m_6}{2r_4^2} e_3^6 - \frac{2(m_5, m_6)}{r_4^2} e_4^1,$$

$$- \frac{(m_5, m_6)}{r_4^2} e_3^7 - \frac{m_5 + m_6}{r_4^2} e_7^5 + \frac{(m_5, m_6)}{r_4^2} e_7^6 + \frac{(m_5, m_6)}{r_4^2} e_7^7,$$

$$+ \frac{(m_5, m_6)}{2r_4^2} e_7^7,$$

$$(\nabla e_1 R)_{e1e3} = \frac{2m_5 + m_6}{r_4^2} e_1^5 + \frac{m_6}{r_4^2} e_1^6 - \frac{m_5 - 2m_6}{2r_4^2} e_2^5 - \frac{m_5 + m_6}{2r_4^2} e_2^6 - \frac{4\varepsilon r_4}{(m_5, m_6)^4} e_3^4,$$

$$- \frac{4(m_5, m_6)}{r_4^2} e_3^7 + \frac{2\varepsilon (3m_5 - m_6)}{(m_5, m_6)} e_5^1 + \frac{2\varepsilon (4m_5 + m_6)}{(m_5, m_6)} e_5^2,$$

$$+ \frac{\varepsilon (m_5 + 3m_6)}{(m_5, m_6)} e_6^1 + \frac{6\varepsilon m_6}{(m_5, m_6)} e_6^2 + \frac{\varepsilon m_6}{(m_5, m_6)} e_7^3,$$

$$(\nabla e_1 R)_{e1e7} = -\frac{2m_5m_6 + m_5^2 + m_6^2}{r_4^2} e_1^6 - \frac{m_6(m_5 + m_6)}{r_4^2} e_6^1 + \frac{3m_5m_6 + 3m_5^2 + m_6^2}{2r_4^2} e_2^5,$$

$$+ \frac{3m_5m_6 + m_5^2 + m_6^2}{2r_4^2} e_2^6 + \frac{\varepsilon m_6}{(m_5, m_6)} e_3^4 + \frac{m_6(m_5, m_6)}{r_4^2} e_3^4 - \frac{2(m_5, m_6)}{r_4^2} e_4^1,$$

$$+ \frac{2\varepsilon (2m_5m_6 + m_5^2 + m_6^2)}{r_4^2(m_5, m_6)} e_5^1 - \frac{2\varepsilon (m_5m_6 + 2m_5^2 + m_6^2)}{r_4^2(m_5, m_6)} e_5^2,$$

$$- \frac{2\varepsilon m_5(m_5 + m_6)}{r_4^2(m_5, m_6)} e_1^1 - \frac{2\varepsilon m_5m_6}{r_4^2(m_5, m_6)} e_6^2 + \frac{2\varepsilon}{r_4^2} e_7^4,$$
\[
(\nabla_e R)_{e_1 e_2} = - \frac{m_5 + m_6}{r^2_4} e_1^4 + \frac{m_5}{r^2_4} e_2^4 - \frac{m_6}{2r^2_4} e_3^5 + \frac{m_5}{2r^2_4} e_4^3 - \frac{2(m_5, m_6)}{r^4_4} e_4^2
+ \frac{m_5}{4} e_5^7 - \frac{(m_5, m_6)}{r^4_4} e_6^3 + \frac{m_6}{2r^4_4} e_7^5.
\]

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