Compressive Privatization: Sparse Distribution Estimation under Locally Differential Privacy

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Abstract

We consider the problem of discrete distribution estimation under locally differential privacy. Distribution estimation is one of the most fundamental estimation problems, which is widely studied in both non-private and private settings. In the local model, private mechanisms with provably optimal sample complexity are known. However, they are optimal only in the worst-case sense; their sample complexity is proportional to the size of the entire universe, which could be huge in practice (e.g., all IP addresses). We show that as long as the target distribution is sparse or approximately sparse (e.g. highly skewed), the number of samples needed could be significantly reduced. The sample complexity of our new mechanism is characterized by the sparsity of the target distribution and only weakly depends on the size of the universe. Our mechanism does privatization and dimensionality reduction simultaneously, and the sample complexity will only depend on the reduced dimensionality. The original distribution is then recovered using tools from compressive sensing. To complement our theoretical results, we conduct experimental studies, the results of which clearly demonstrate the advantages of our method and confirm our theoretical findings.

1 Introduction

Discrete distribution estimation (Kamath et al. 2015; Lehmann and Casella 2006; Kairouz, Bonawitz, and Ramage 2016) from samples is a fundamental problem in statistical analysis. In the traditional statistical setting, the primary goal is to achieve best tradeoff between sample complexity and estimation accuracy. In many modern data analytical applications, the raw data often contains sensitive information, e.g. medical data of patients, and it is prohibitive to release them without appropriate privatization. Private data releasing and computation is becoming an increasingly important research topic. Differential privacy is one of the most popular and powerful definitions of privacy (Dwork et al. 2006). Traditional centralized model assumes there is a trusted data collector. In this paper, we consider locally differential privacy (LDP) (Warner 1965; Kasiviswanathan et al. 2011; Beimel, Nissim, and Omri 2008), where users privatize their data before releasing it so as to keep their personal data private even from data collectors. We study the discrete distribution estimation problem under LDP constraint.

The main theme in private distribution estimation is to optimize statistical and computational efficiency under privacy constraints. Given privacy parameter ε and target estimation precision, the primary goal is to minimize the number of samples, i.e., sample complexity. In the local model, the communication cost and computation time are also important complexity parameters. This problem has been widely studied in the local model recently (Warner 1965; Duchi, Jordan, and Wainwright 2013; Erlingsson, Pihur, and Korolova 2014; Kairouz, Oh, and Viswanath 2014; Wang et al. 2016; Pastore and Gastpar 2016; Kairouz, Bonawitz, and Ramage 2016; Acharya, Sun, and Zhang 2018; Ye and Barg 2018; Bassily 2019; Acharya, Sun, and Zhang 2018). Thus far, the worst-case sample complexity, i.e., the minimum number of samples needed to achieve a desired accuracy for the worst-case distribution, has been well-understood (Wang et al. 2016; Ye and Barg 2018; Acharya, Sun, and Zhang 2018). However, worst-case behaviors are often not indicative of their performance in practice; real-world inputs often contain special structures that allow one to bypass such worst-case barriers.

In this paper, we consider sparse or approximately sparse distributions, which are perhaps the most natural structured distributions. As far as we know, there is no non-trivial theoretical results on the sample complexity for sparse distribution in the local differential privacy model. In this paper, using techniques from compressive sensing (CS) theory, we provide privatization schemes with sample complexity only depends on the sparsity of the target distribution rather than on the size of the entire universe as in previous worst-case optimal bounds. In particular, the sample complexity is reduced from $O(k)$ to $O(s^2)$ for $\ell_1$ error; and from $O(k^2)$ to $O(s^2)$ for $\ell_2$ error, where $k$ is the dimensionality of the unknown distribution and $s$ denotes its sparsity. Our main idea is to do privatization and dimensionality compression simultaneously, then perform distribution estimation in the lower dimensional space. This can reduce the sample complexity since the estimation error usually depends on the dimensionality of the distribution. The original distribution is then recovered from the low-dimensional one using tools from compressive sensing. We will call this technique compressive privatization. To complement our theoretical improvements, we also provides empirical studies which demonstrate that the estimation error of our compressive privati-
zation method is lower than previous worst-case optimal approaches as long as the input distribution is relatively sparse.

**Problem Definition and Results**

We consider $k$-ary discrete distribution estimation. W.o.l.g., we assume the target distribution is defined on the universe $\mathcal{X} = [k] := \{1, 2, \ldots, k\}$, which can be viewed as a $k$-dimensional vector $\mathbf{p} \in \mathbb{R}^k$ with $\|\mathbf{p}\|_1 = 1$. Let $\Delta_k$ be the set of all valid $k$-ary discrete distributions. Given $n$ i.i.d. samples, $X_1, X_2, \ldots, X_n$, sampled from the unknown distribution $\mathbf{p}$, the goal is to provide an estimator $\hat{\mathbf{p}}$ such that $d(\mathbf{p}, \hat{\mathbf{p}})$ is minimized, where $d(\cdot)$ is typically the $l_1$ or $l_2$ norm.

**Local Privacy**

In the local model, each $X_i$ is held by a different user. Each user will only send a privatized version of their data to the central server, who will then produce the final estimate. A privatized mechanism is a randomized mapping $Q$ that maps $x \in \mathcal{X}$ to $y \in \mathcal{Y}$ with probability $Q(y|x)$ for some output set $\mathcal{Y}$. The mapping $Q$ is said to be $\varepsilon$-locally differentially private (LDP) (Duchi, Jordan, and Wainwright 2013) if for all $x, x' \in \mathcal{X}$ and $y \in \mathcal{Y}$,

$$Q(y|x) \leq e^\varepsilon Q(y|x').$$  \hspace{1cm} (1)

In this the paper, we will only focus on the **high privacy regime**, i.e., $\varepsilon$ is a small constant (typically $\varepsilon \leq 1$), which is considered as the most interesting regime. The case where $\varepsilon = o(1)$ is left for future work.

**LDP distribution estimation**

Let $Y = (Y_1, Y_2, \cdots, Y_n) \in \mathcal{Y}^n$ be the randomized observations obtained by applying a privatization mechanism $Q$ independently on $X = (X_1, X_2, \cdots, X_n)$. Given privacy parameter $\varepsilon$, the goal of LDP distribution estimation is to design an $\varepsilon$-LDP mapping $Q$ and a corresponding estimator $\hat{Q} : \mathcal{Y}^n \rightarrow \Delta_k$, such that $\mathbb{E}[d(\mathbf{p}, \hat{\mathbf{p}})]$ is minimized. Given $\varepsilon$ and $\alpha$, we are most interested in the number of samples needed (as a function of $\varepsilon$ and $\alpha$) to assure $\varepsilon$-LDP and $\mathbb{E}[d(\mathbf{p}, \hat{\mathbf{p}})] \leq \alpha$. A slightly different definition is also used in the literature: In stead of expected error, it is required that $d(\mathbf{p}, \hat{\mathbf{p}}) \leq \alpha$ with probability at least $0.9$ (Acharya and Zhang 2018). However, up to a constant factor, they are equivalent by Markov's inequalty.

**Sparsity**

A discrete distribution $\mathbf{p} \in \mathcal{Y}$ is called $s$-sparse if the number of non-zeros in $\mathbf{p}$ is at most $s$. Let $[\mathbf{p}]_s$ be the $s$-sparse vector that contains the top-$s$ entries of $\mathbf{p}$. We say $\mathbf{p}$ is approximately $(s, \lambda)$-sparse, if $\|\mathbf{p} - [\mathbf{p}]_s\|_1 \leq \lambda$.

**Our results**

For high privacy regime, existing studies (Warner 1965; Kairouz, Bonawitz, and Ramage 2016; Acharya, Sun, and Zhang 2018, Erlingsson, Pihur, and Korolova 2014; Ye and Barg 2018, Wang et al. 2018; Bassily 2019, Acharya, Sun, and Zhang 2018) have achieved the optimal sample complexity, which is $O\left(\frac{k^2}{\alpha \varepsilon^2}\right)$ for $l_1$ norm and $O\left(\frac{k}{\alpha^2 \varepsilon^2}\right)$ for $l_2$ norm. These worst-case optimal bounds have a dependence on $k$, the size of the entire universe, which could be huge in practice (e.g., all IP addresses). In this paper we propose a new LDP mechanism with sample complexity depends on the sparsity of the target distribution. The main result (informal) is summarized as follows; see Theorem 1.1 for exact bounds and results for approximately sparse distributions.

**Theorem 1.1 (Informal).** For any $0 < \varepsilon < 1$ and $\alpha > 0$, there is an $\varepsilon$-LDP scheme $Q$, which produces an estimator $\hat{\mathbf{p}}$ with error guarantee $d(\mathbf{p}, \hat{\mathbf{p}}) \leq \alpha$. If $\mathbf{p}$ is $s$ sparse, then the sample complexity of $Q$ is $O\left(\frac{s^2 \log(k/s)}{\varepsilon^2 \alpha^2}\right)$ for $l_1$ error and $O\left(\frac{s \log(k/s)}{\varepsilon^2 \alpha^2}\right)$ for $l_2$ error.

Our framework can be easily extended to the regime with $1 \leq \varepsilon \leq \log k$ (see Section 4), which is conceptually simpler than previous methods on this privacy regime. We also note that the sample complexity lower bounds from (Ye and Barg 2018) implies a lower bound of $\Omega\left(\frac{s^2 \log(k/s)}{\varepsilon^2 \alpha^2}\right)$ for $l_1$ error and $\Omega\left(\frac{s \log(k/s)}{\varepsilon^2 \alpha^2}\right)$ for $l_2$ error, which means our sample complexity is optimal up to a $\log(k/s)$ factor.

**Related work**

Differential privacy is arguably the most widely adopted notion of privacy (Dwork et al. 2006), a large body of literature exists (see e.g., (Dwork, Roth et al. 2014) for a comprehensive survey). The local model has become quite popular recently (Warner 1965; Kasiviswanathan et al. 2011; Beimel, Nissim, and Omri 2008). The distribution estimation problem considered in this paper has been studied in (Warner 1965; Duchi, Jordan, and Wainwright 2013; Erlingsson, Pihur, and Korolova 2014; Kairouz, Oh, and Viswanath 2014; Wang et al. 2016; Pastore and Gastpar 2016; Kairouz, Bonawitz, and Ramage 2016; Acharya, Sun, and Zhang 2018). Among them, (Wang et al. 2016; Ye and Barg 2018; Acharya, Sun, and Zhang 2018) have achieved worst-case optimal sample complexity over all privacy regimes.

**Preliminaries on Compressive Sensing**

Let $x$ be an unknown $k$-dimensional vector. The goal of compressive sensing (CS) is to reconstruct $x$ from only a few linear measurements (Candes and Tao 2005; Candes, Romberg, and Tao 2006; Donoho 2006). To be precise, let $B \in \mathbb{R}^{m \times k}$ be the measurement matrix with $m \ll k$ and $e \in \mathbb{R}^m$ be an unknown noise vector, given $y = Bx + e$, CS aims to recover a sparse approximation $\hat{x}$ of $x$ from $y$. This
Lemma 1.2. If $B$ satisfies $(s, \delta)$-RIP, then for every $s$-sparse vectors $x$,
\[(1 - \delta) \|x\|_2 \leq \|Bx\|_2 \leq (1 + \delta) \|x\|_2.
\]
We will use the following results from (Cai and Zhang 2013)

Lemma 1.2. If $B$ satisfies $(2s, 1/\sqrt{2})$-RIP. Given $y = Bx + e$, there is a polynomial time algorithm, which outputs $\hat{x}$ that satisfies $\|x - \hat{x}\|_2 \leq \frac{C}{\sqrt{2}}\|x - [x]_s\|_1 + D\|e\|_2$ for some constant $C, D$.

The measurement matrix we use in our implementation is $B = \frac{1}{\sqrt{n}} A$, where the entries of $A \in \{-1, +1\}^{m \times n}$ are i.i.d. Rademacher random variables, i.e., takes +1 or −1 with equal probability. It is known that if $m \geq O(s \log \frac{n}{s})$, $B$ satisfies $(s, 1/\sqrt{2})$-RIP with probability $1 - e^{-m}$ (Baraniuk et al. 2008).

2 Our Method for high privacy

An overview of our method To estimate an unknown distribution $p \in \mathbb{R}^k$, all previous LDP mechanisms essentially apply a probability transition matrix $Q$ mapping $p$ to $q$, where $q$ is the distribution of the privatized samples. The central server gets $n$ independent samples from $q$, from which it computes an empirical estimate of $q$, denoted as $\hat{q}$, and then computes an estimator of $p$ from $\hat{q}$ by solving $Qp = \hat{q}$.

The key problem is to design an appropriate $Q$ such that it satisfies privacy guarantees and achieves lower recovery error. To ensure its invertibility, the dimension of $q$ is the same as that of $p$ in all previous works. The error of $\hat{p} = Q^{-1}\hat{q}$ is dictated by the estimation error of $\hat{q}$ and the spectral norm of $Q^{-1}$. The main difference of our method is that, we map $p$ to a much lower dimensional $q$, and then $\hat{q}$ with similar estimation error can be obtained with much less number of samples. However, now $Q$ is not invertible; to reconstruct $\hat{p}$ from $\hat{q}$, we use tools from sparse recovery (Candes and Tao 2005) and compressive sampling (Candes, Romberg, and Tao 2006). The overall structure of our method is illustrated in Figure 1. We remark that, with privacy constraints, recovery results from CS cannot be applied directly. In this paper, we provide a sufficient condition on $Q$ for efficient estimation of distributions with arbitrary sparsity. In fact, for the case that $p$ is dense ($s \approx k$), the Hadamard matrix arises naturally from our sufficient condition. Thus, Hadamard Response of (Acharya, Sun, and Zhang 2018) can be viewed as a special case of our framework.

Compressive Privatization Scheme

Privatization. Our mechanism $Q$ is a mapping from $[k]$ to $[m]$ for each $x \in [k]$, we pick a set $C_x \subseteq [m]$, which will be specified later, and let $n_x = |C_x|$. Our privatization scheme $Q$ is given by the conditional probability of $y$ given $x$:

\[Q(y|x) = \begin{cases} 
\frac{e^y}{n_x e^y + m-n_x} & \text{if } y \in C_x \\
\frac{e^y}{n_x e^y + m-n_x} & \text{if } y \in [m]\setminus C_x
\end{cases} \quad (2)
\]

The key problem is to design an appropriate $Q$ such that it satisfies privacy guarantees and achieves lower recovery error. To ensure its invertibility, the dimension of $q$ is the same as that of $p$ in all previous works. The error of $\hat{p} = Q^{-1}\hat{q}$ is dictated by the estimation error of $\hat{q}$ and the spectral norm of $Q^{-1}$. The main difference of our method is that, we map $p$ to a much lower dimensional $q$, and then $\hat{q}$ with similar estimation error can be obtained with much less number of samples. However, now $Q$ is not invertible; to reconstruct $\hat{p}$ from $\hat{q}$, we use tools from sparse recovery (Candes and Tao 2005) and compressive sampling (Candes, Romberg, and Tao 2006). The overall structure of our method is illustrated in Figure 1. We remark that, with privacy constraints, recovery results from CS cannot be applied directly. In this paper, we provide a sufficient condition on $Q$ for efficient estimation of distributions with arbitrary sparsity. In fact, for the case that $p$ is dense ($s \approx k$), the Hadamard matrix arises naturally from our sufficient condition. Thus, Hadamard Response of (Acharya, Sun, and Zhang 2018) can be viewed as a special case of our framework.

Sufficient conditions for $Q$ The difference between our mechanism, RR (Warner 1965) and HR (Acharya, Sun, and Zhang 2018) is the choice of each $C_x$, or equivalently the matrix $A$. In RR, $C_x = \{x\}$ for all $x \in [k]$, while in HR, $A$ is the Hadamard matrix. In our privatization scheme, any matrix $A$ whose column sums are close to 0 and that satisfies RIP will suffice. More formally, given target error $\alpha$, privacy parameter $\varepsilon$ and target sparsity $s$, we require $A$ to have the following 2 properties:

- **P1**: $(1 - \beta) \frac{m}{s} \leq n_x \leq (1 + \beta) \frac{m}{s}$ for all $i \in [k]$, with $\beta \leq \frac{s}{2}$ and $\beta \leq c\alpha$ for some $c$ depending on the error norm.
- **P2**: $\frac{1}{\sqrt{m}} A$ satisfies $(s, \delta)$-RIP, where $s$ is the target sparsity and $\delta \leq 1/\sqrt{2}$.

It is known that a Rademacher random matrix satisfies RIP with high probability if $m \geq s \log \frac{n}{s}$, but how to deterministically construct optimal RIP matrices remains a major open problem (Gamarnik 2019). Therefore, in principle, without running time constraints, our mechanism only needs private randomness, but in practice, the matrix $A$ will be sampled using public randomness.

In (Acharya, Sun, and Zhang 2018), Hadamard matrix is used to specify each set $C_x$. The proportion of +1 entries is exactly half in each column of $H$ (except for the first column), and thus P1 is automatically satisfied with $\beta = 0$. Since Hadamard matrix is orthonormal, it is $(k, 0)$-RIP.

Privacy Guarantee

Lemma 2.1. If $(1 - \beta) \frac{m}{s} \leq n_x \leq (1 + \beta) \frac{m}{s}$ for all $i \in [k]$ and $0 < \beta < 1$, then the privacy mechanism $Q$ from (2)
satisfies $(\varepsilon + 2\beta)$-LDP.

**Proof.** Observe that for any $x_1, x_2 \in [k]$, we have

$$\max_{y \in [n]} \frac{Q(y|x_1)}{Q(y|x_2)} \leq \frac{n_{x_2} e^{-n_{x_2} e^\varepsilon} \beta}{n_{x_1} e^{-n_{x_1} e^\varepsilon} \beta}. \text{ By assumption,}$$

$$\frac{n_{x_2} e^\varepsilon + m - n_{x_2} e^\varepsilon}{n_{x_1} e^\varepsilon + m - n_{x_1} e^\varepsilon} \leq \frac{(1 + \beta) m n_{x_2} e^\varepsilon + m}{(1 + \beta) m n_{x_1} e^\varepsilon + m - (1 - \beta) m n_{x_1} e^\varepsilon} \leq 1 + \beta/2$$

$$\leq 1 + 2\beta.$$  

where the second inequality is from $\frac{e^{\varepsilon_1}}{\varepsilon_1 + 1} \leq \frac{1}{2}$ for $\varepsilon \in (0, 1)$ and the last inequality is from 0 $\leq \beta \leq 1$. It follows that

$$\max_{x_1, x_2 \in [k]} \frac{Q(y|x_1)}{Q(y|x_2)} \leq \max_{x_1, x_2 \in [k]} \frac{n_{x_2} e^\varepsilon + m - n_{x_2} e^\varepsilon}{n_{x_1} e^\varepsilon + m - n_{x_1} e^\varepsilon} \leq 1 + 2\beta e^{\varepsilon+2\beta}.$$ 

The last inequality is from $\ln(1 + x) \leq x$ for $x > -1$. □

When the matrix $A$ used in (2) satisfies

$$1 - (1-\varepsilon)^m \leq n_i \leq 1 - (1+\varepsilon)^m/2 \text{ for all } i \in [k],$$

then $Q$ is $3\varepsilon$-LDP. We can rescale $\varepsilon$ in the beginning by a constant to ensure $\varepsilon$-LDP, which will only affect the sample complexity by a constant factor.

**Estimation Algorithm**

Recall $n_i$ is the number of $+1$’s in the $i$-th column of $A$. Let $d_i = \frac{1}{n_i e^{-2m-n_i}}$ and $D' = \frac{m(e^{\varepsilon+1})}{2} \cdot \text{diag}(d_1, d_2, \cdots, d_k)$. The estimation algorithm is provided in Algorithm 1 which will be explained in more details next.

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**Algorithm 1: Estimation**

**Result:** Estimate of $p$

**Input:** Privatized samples $Y_1, Y_2, \cdots, Y_n$, privacy parameter $\varepsilon$, $A \in \mathbb{R}^{m \times k}$, sparsity $s$

$q = (0, 0, \cdots, 0) \in \mathbb{R}^m$

$1 = (1, 1, \cdots, 1) \in \mathbb{R}^m$

for $i \leftarrow 1$ to $m$

$q[i] = \frac{\sum_{j=1}^n Y_j(i) \cdot e^\varepsilon}{n}$

end

Apply Lemma 1.2 with

$y = \frac{(\varepsilon + 1)}{(\varepsilon - 1)} \left( \sqrt{mq} - 1/\sqrt{m} \right)$, $B = \frac{1}{\sqrt{m}} A$ and sparsity $s$; let $f$ be the output

Compute $\hat{p}' = D'^{-1} f$, and then compute the projection of $\hat{p}'$ onto the set $\Delta_k$, denoted as $\hat{p}$

return $\hat{p}$

---

We will first show how to model the estimation of $p$ as a standard compressive sensing problem. Let $q \in \Delta_m$ be the real distribution of a privatized sample, given that the input sample is distributed according to $p \in \Delta_k$. Then, for each $j \in [m]$, we have

$$q[j] = \sum_{i=1}^k p[i] \cdot Q(Y = j | X = i)$$

$$= \sum_{i,j : j \in C_i} \frac{e^\varepsilon \cdot p[i]}{n_i e^\varepsilon + m - n_i} + \sum_{i,j \in [k] \setminus C_i} \frac{p[i]}{n_i e^\varepsilon + m - n_i}.$$  

By writing above formula in the matrix form, we get

$$q = \left( \begin{array}{c} e^\varepsilon - 1 \\ e^\varepsilon + 1 \end{array} \right) \left( \begin{array}{c} \sqrt{mq} - 1/\sqrt{m} \\ 1/\sqrt{m} \end{array} \right) D p.$$  

where $J \in \mathbb{R}^{m \times k}$ is the all-one matrix and $D$ is the diagonal matrix with $d_i = \frac{1}{n_i e^{-2m-n_i}}$ in the $i$-th column on the diagonal. As mentioned above, the matrix $\frac{1}{\sqrt{m}} A$ to be used will satisfy RIP. We then rewrite (3) to the form of a standard noisy compressive sensing problem

$$\begin{align*}
\begin{bmatrix}
(e^\varepsilon + 1) \\
(e^\varepsilon - 1)
\end{bmatrix} \begin{bmatrix}
\sqrt{mq} - 1/\sqrt{m} \\
1/\sqrt{m}
\end{bmatrix}
&= \begin{bmatrix}
1/\sqrt{m} A (D'pr) + e^\varepsilon + 1/\sqrt{m} (e^\varepsilon - 1) J (D'p - I)p
\end{bmatrix}
&= \begin{bmatrix}
1/\sqrt{m} A (D'pr) + e^\varepsilon + 1/\sqrt{m} (e^\varepsilon - 1) J (D'p - I)p
&+ e^\varepsilon - 1/\sqrt{m} (q - q).
\end{bmatrix}
\end{align*}$$

where $1 \in \mathbb{R}^m$ is the all-one vector, $I \in \mathbb{R}^{k \times k}$ is the identity matrix and $D' = m(e^{\varepsilon+1}) I$.

However, in our problem, the exact $q$ is also unknown, and we can only get an empirical estimate $\hat{q}$ from the privatized samples. So, we need to add a new noise term that corresponds the estimation error of $\hat{q}$, and the actual underdetermined linear system is

$$\begin{align*}
\begin{bmatrix}
(e^\varepsilon + 1) \\
(e^\varepsilon - 1)
\end{bmatrix} &\begin{bmatrix}
\sqrt{mq} - 1/\sqrt{m} \\
1/\sqrt{m}
\end{bmatrix}
&= \begin{bmatrix}
1/\sqrt{m} A (D'pr) + e^\varepsilon + 1/\sqrt{m} (e^\varepsilon - 1) J (D'p - I)p
\end{bmatrix}
&= \begin{bmatrix}
1/\sqrt{m} A (D'pr) + e^\varepsilon + 1/\sqrt{m} (e^\varepsilon - 1) J (D'p - I)p
&+ \sqrt{m} (e^\varepsilon + 1) (\hat{q} - q).
\end{bmatrix}
\end{align*}$$

Given the LHS of (5), $\frac{1}{\sqrt{m}} A$ and a target sparsity $s$, we reconstruct $\hat{D'} pr$ by applying Lemma 1.2. Then compute $\hat{p}' = D'^{-1} \hat{D'} pr$ ($D'$ is known) and project it to the probability simplex. This is what Algorithm 1 does.

### 3 Estimation Error and Sample Complexity

By Lemma 1.2, the reconstruction error depends on the $l_2$ norm of $e_1$ and $e_2$ in (5). Next we first bound of norm of the noise terms in Lemma 3.1 and Lemma 3.2.

**Lemma 3.1.** For a fixed $\beta \in (0, 0.5)$, if $n_i \in (1 \pm \beta)\frac{m}{2}$ for all $i \in [k]$, then $\|e_1\|_2 \leq \frac{\beta}{1 - \beta} \leq 2\beta$. 

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Proof. By definition

\[ \|e_1\|_2 = \left\| \frac{e^* + 1}{\sqrt{m(e^* - 1)}} J(D' - I)p \right\|_2 \]

\[ = \frac{e^* + 1}{\sqrt{m(e^* - 1)}} \| J(D' - I)p \|_2 \]

\[ = \frac{e^* + 1}{\sqrt{m(e^* - 1)}} \sqrt{\| (D' - I)p \| \cdot J^T \cdot (D' - I)p } \]

\[ \leq \frac{e^* + 1}{e^* - 1} \sum_i |d'_i - 1| p_i, \]

where \( d'_i = \frac{m(e^*+1)/n^2}{n_i e^* + m - n_i} \). By the assumption \( n_i \in (1 + \beta) \frac{m}{2} \) for all \( i \in [k] \), we have

\[ |d'_i - 1| = \frac{|m/2 - n_i|(e^* - 1)}{n_i e^* + m - n_i} \leq \frac{\beta m(e^* - 1)/2}{n_i e^* + m - n_i} \]

\[ \leq \frac{(1 - \beta)\frac{m}{2}e^* + \frac{m}{2} - \beta m}{(1 - \beta)(e^* - 1)} \]

\[ = \frac{\beta(e^* - 1)}{(1 - \beta)(e^* + 1)}. \]

Thus, \( \|e_1\|_2 \leq \frac{\beta}{1 - \beta} \), which completes the proof. \( \square \)

Lemma 3.2. Let \( n \) be the number of samples, then

\[ \mathbb{E}[\|e_2\|_2] \leq \frac{s + 2}{n} \sqrt{\frac{m}{m - 1}}. \]

Proof. Let \( Y_1, Y_2, \cdots, Y_N \) be the privatized samples received by the server. We have, for each \( i \in [m] \),

\[ \hat{q}_i = \frac{\sum_{j=1}^N \mathbb{I}(Y_j = i)}{n}, \]

where I is the indicator function. Thus \( \mathbb{E}[\hat{q}_i] = q_i \) and \( \text{Var}[q_i] = \frac{q_i(1 - q_i)}{n^2} \leq \frac{1}{n} \). It follows that

\[ \mathbb{E}[\|\hat{q} - q\|_2] \leq \sqrt{\mathbb{E}[\|\hat{q} - q\|^2]} \]

\[ = \sqrt{\sum_i \text{Var}(q_i)} \leq \sqrt{\frac{1}{n}} \]

where the first inequality is from Jensen’s inequality. Multiplying \( \frac{s + 2}{n} \sqrt{\frac{m}{m - 1}} \) on both sides of the inequality will conclude the proof. \( \square \)

Let \( p' = D'p \), \( f \) be the recovery result of \( p' \), and \( \hat{p} \) be the projection of \( D'^{-1}f \) onto \( \Delta_k \).

Lemma 3.3. If \( \forall i \in [k], n_i \leq (1 + \beta) \cdot \frac{m}{2} \) for \( 0 \leq \beta \leq 1 \), then \( \|p - \hat{p}\|_2 \leq \left(1 + \frac{\beta}{2}\right) \|p' - f\|_2 \).

Proof. Since \( \Delta_k \) is convex, \( \|p - \hat{p}\|_2 \leq \|p - D'^{-1}f\|_2 \). Since \( p = D'^{-1}p' \), it follows that

\[ \|p - \hat{p}\|_2 \leq \|D'^{-1}(p' - f)\|_2 \leq \max_i \frac{1}{d'_i} \cdot \|p' - f\|_2. \]

where \( d'_i = \frac{m(e^*+1)/n^2}{n_i e^* + m - n_i} \). Since \( \forall i \in [k], n_i \leq (1 + \beta) \cdot \frac{m}{2} \) for \( 0 \leq \beta \leq 1 \), then we have

\[ \|p - \hat{p}\|_2 \leq \max_i \frac{1}{d'_i} \cdot \|p' - f\|_2 \]

\[ \leq \left(1 + \frac{\beta}{2}\right) \|p' - f\|_2 \leq \left(1 + \frac{\beta}{2}\right) \|p' - f\|_2. \]

Theorem 3.4. If A satisfies two properties in Section 2 for some constant C,

\[ \mathbb{E}[\|p - \hat{p}\|_2] \leq \left(1 + \frac{\beta}{2}\right) \left(2D\beta + D(e^* + 1) \frac{m}{e^* - 1} \frac{1}{n}\right). \]

Proof. By Lemma 3.2 we have

\[ \|p' - f\|_2 \leq \frac{C}{\sqrt{s}} \|p' - [p]\|_1 + D \|e_1 + e_2\|_2. \]

\[ \leq \max_i \frac{1}{d'_i} \cdot \frac{C}{\sqrt{s}} \|p - [p]\|_1 + D (\|e_1\|_2 + \|e_2\|_2). \]

Note max, \( \frac{1}{d'_i} \leq (1 + \beta/2) \) as \( n_i \leq (1 + \beta) \cdot \frac{m}{2} \). By Lemma 3.1 and 3.3, we get

\[ \mathbb{E}[\|p - \hat{p}\|_2] \leq \left(1 + \frac{\beta}{2}\right) \left(2D\beta + D(e^* + 1) \frac{m}{e^* - 1} \frac{1}{n}\right). \]

The sample complexity to achieve an error \( \alpha \) and \( \epsilon \)-LDP for \( 0 \leq \epsilon \leq 1 \) is summarized as follows.

Corollary 3.4.1. If A satisfies two properties in Section 2 with \( \beta \leq \frac{c}{4D} \), then \( \ell_2 \) error and \( p \) is \( s \) sparse, the sample complexity of our method is \( O(\frac{m}{\epsilon^2 s^2}) \) for \( \ell_2 \) error and \( O(\frac{m}{\epsilon^2 s^2}) \) for \( \ell_1 \) error. The \( \ell_2 \) result also holds for \( (s, \sqrt{s\alpha}) \)-sparse p and the \( \ell_1 \) result also holds for \( (s, \alpha) \)-sparse p. The communication cost for each user is \( \log m \) bits.

Proof. When \( p \) is \( s \) sparse, the second error term in Theorem 3.4 is 0. The first term is bounded by \( 2(\frac{1}{n} + \frac{1}{\sqrt{n}}) \). Thus, when \( n \geq \frac{6m}{\epsilon^2 s^2} \) for some large enough constant b, the expected \( \ell_2 \) error is at most \( \alpha \). Given sparsity parameter \( s \), \( p \) returned by a typical recovery algorithm is also \( s \)-sparse, so \( p - \hat{p} \) is \( 2s \)-sparse. By Cauchy-Schwarz,

\[ \|p - \hat{p}\|_1 \leq \frac{\sqrt{2}}{\sqrt{s}} \|p - \hat{p}\|_2. \]

Thus, to achieve an \( \ell_1 \) error of \( \alpha \), it is sufficient to get an estimate with \( \alpha' = \alpha / \sqrt{s} \) for \( \ell_2 \). The sample complexity is \( O(\frac{m s}{\epsilon^2 s^2}) \); also note that in this case, we should require \( \beta \leq \frac{c}{4D \sqrt{s}} \). For \( (s, \sqrt{s\alpha}) \)-sparse \( p \), the second error term in Theorem 3.4 is bounded by \( O(\alpha) \); thus the result still holds for \( \ell_2 \) error (up to a constant). For \( \ell_1 \) error with \( p \) being \( (s, \alpha) \)-sparse, now \( p - \hat{p} \) is \( (2s, \alpha) \)-sparse. We have

\[ \|p - \hat{p}\|_1 \leq \|p - \hat{p}\|_2 + \|p - \hat{p}\|_1 \]

\[ \leq \sqrt{2s} \|p - \hat{p}\|_2 + \alpha. \]
Then, the $\ell_1$ result follows by a similar argument as for the exact sparse case.

Guarantees on Random Matrices

The measurement matrix we use is $B = \frac{1}{\sqrt{m}} A$, where the entries of $A \in \{-1, +1\}^{m \times n}$ are i.i.d Rademacher random variables, i.e., takes +1 or -1 with equal probability. It is known that for $m \geq O(s \log \frac{k}{\varepsilon})$, $B$ satisfies $(s, 1/\sqrt{2})$-RIP with probability $1 - e^{-m}$ [Baraniuk et al. 2008]. By standard concentration inequalities, it is not difficult to show that the first property from Section 2, we require for $A$, i.e., $n_i \in (1 \pm \beta) \frac{m}{2}$, is satisfied by a random matrix with high probability.

**Lemma 3.5.** For $0 \leq \beta \leq 1$, with probability at least $1 - 2ke^{-\frac{c^2 m}{2}}$, $(1 - \beta) \frac{m}{2} \leq n_i \leq (1 + \beta) \frac{m}{2}$ holds simultaneously for all $i \in [k]$.

The proof of the above Lemma is provided in supplementary material. As stated in Section 2, we need $\beta \leq \varepsilon/2$ and $\beta \leq c_0$ for some $c_0$ (with $c = O(1)$ for $\ell_2$ error and $c = O(1/\sqrt{s})$ for $\ell_1$ error). From the above lemma, this hold with probability $1 - \delta$ as long as $m = \Omega \left( \max \left( \frac{\log \frac{k}{\varepsilon}}{c^2}, \frac{\log \frac{k}{\varepsilon}}{\varepsilon^2}, \frac{\log \frac{k}{\varepsilon}}{s \log \frac{k}{s}} \right) \right)$

with $c = O(1)$ for $\ell_2$ error and $c = O\left(\frac{1}{\sqrt{s}}\right)$ for $\ell_1$ error, if the entries of $A \in \{-1, +1\}^{m \times k}$ are i.i.d Rademacher random variables, then with probability at least $1 - \delta - e^{-m}$, our method has sample complexity $O \left( \frac{m}{c^2} \right)$ for $\ell_2$ error, and $O \left( \frac{c^2 m}{\sqrt{s}} \right)$ for $\ell_1$ error. The $\ell_2$ result holds for $(s, \sqrt{5c})$-sparse $p$ and the $\ell_1$ result holds for $(s, \alpha)$-sparse $p$. The communication cost for each user is $\log m$ bits.

4 Medium privacy regimes

Our method for high privacy regimes can be extended to medium privacy regime easily. The only difference here is to a different choice of the RIP matrix $A$. Recall that, in high privacy regime, the number of +1’s in each column of $A$ is close to $\frac{m}{2}$. In medium privacy, to get better sample complexity, $n_i$’s are required to be close to $\frac{m}{2}$. To achieve this, we set the entries of $A$ to be i.i.d. random variables that follows the following distribution:

$$A_{ij} = \begin{cases} +1 & \text{P} = \frac{1}{2} \\ -1 & \text{P} = 1 - \frac{1}{2} \end{cases}$$

(6)

One can verify that $A$ satisfies above property with high probability by concentration inequalities.

**Lemma 4.1.** If $(1 - \beta) \frac{m}{2} \leq n_i \leq (1 + \beta) \frac{m}{2}$ for all $i \in [k]$ and $0 \leq \beta \leq 1$, then the privacy mechanism $Q$ from $\mathbb{C}$ satisfies $\varepsilon - 2\beta$-LDP.

Let $B = \frac{1}{\sqrt{m}} A - \frac{\mathbb{E}[A]}{\sigma(A)}$, where $\sigma(A)$ is the standard deviation of $A_{ij}$. It also satisfies $(s, 1/\sqrt{2})$-RIP with probability $1 - e^{-m}$ for $m \geq O(s \log \frac{k}{\varepsilon})$ [Baraniuk et al. 2008]. Then, Theorem 3.6 in section 3 can be reformulated as the following under-determined linear system:

$$\frac{2\varepsilon \varepsilon}{(e^{\varepsilon} - 1)^2} (\sqrt{m} \hat{q} - \frac{1}{\sqrt{m}}) = BDp$$

$$+ \frac{2\varepsilon - 1}{\sqrt{m}(e^{\varepsilon} - 1)^2} I(D' - I)p + \sqrt{m}(2e^{\varepsilon} - 1) (\hat{q} - q)$$

where $D' = m(2 - \frac{1}{2})D$. Note that the estimation algorithm remains the same as before, with only the definition of $y$ and $D'$ changed. By a similar analysis of the norm of $e_1$ and $e_2$, we get the following result on the estimation error.

**Theorem 4.2.** If $A$ consists of i.i.d. random variables that follow the distribution in $\mathbb{E}$, then with probability at least $1 - 2ke^{-\frac{c^2 m}{2}}$, for some constant $C$,

$$E[(\|p - \hat{p}\|_2) \leq D \left(1 + \beta \frac{m}{2} \right) \left( \frac{2}{3} \beta + \sqrt{\frac{m}{n}} \right) \left( 2e^{\varepsilon} - 1 \right) \left( \frac{e^{\varepsilon} - 1}{(e^{\varepsilon} - 1)^2} \right) + \left(1 + \beta \frac{m}{2} \right) \left( \frac{C}{\sqrt{8}} \right) \left( \|p - [p]_s\|_1 \right)$$

The sample complexity to achieve an error $\alpha$ and $\varepsilon$-LDP for $1 \leq \varepsilon \leq \log m$ is summarized as follows

**Theorem 4.3.** For $\ell_2$ error and $c = O(1/\sqrt{s})$ for $\ell_1$ error, if the entries of $A \in \{-1, +1\}^{m \times k}$ are i.i.d random variables that follow the distribution in $\mathbb{E}$, then with probability at least $1 - \delta - e^{-m}$, our method has sample complexity $O \left( \frac{m}{c^2} \right)$ for $\ell_2$ error, and $O \left( \frac{c^2 m}{\sqrt{s}} \right)$ for $\ell_1$ error. The $\ell_2$ result holds for $(s, \sqrt{5c})$-sparse $p$ and the $\ell_1$ result holds for $(s, \alpha)$-sparse $p$. The communication cost for each user is $\log m$ bits.

5 Experiments

In this section, we conduct experiments comparing our method with RR (Warner [1965], RAPPOR [Erlingsson, Piher, and Korolova 2014], SS (Ye and Barg [2018]) and HR (Acharya, Sun, and Zhang [2018]). It’s known that HR has achieved the optimal sample and communication complexity at the same time for dense distributions. The comparison between HR and our method for sparse distributions is shown in Table 1. To implement our recovery process, we use the orthogonal matching pursuit (OMP) algorithm [Tropp 2004]. We test the performances on two types of (approximately) sparse distributions: 1) geometric distributions $Geo(\lambda)$ with $p(i) \propto (1 - \lambda)^i / \lambda$; and 2) sparse uniform distributions $Unif(s)$ where $|\text{supp}(p)| = s$ and $p(i) = \frac{1}{s}$ for $i \in \text{supp}(p)$.

In our experiments, the dimensionality of the unknown distribution is $k = 10000$, and the value of $m$ in our method is set to 500. The default value of the privacy parameter is $\varepsilon = 0.5$. Here we provide the results of different algorithms on Geo(0.8), Geo(0.6), Unif(10) and Unif(25). We record
For baseline methods, we use the code provided by the authors in (Acharya, Sun, and Zhang 2018) with the default settings in their code. For skewed distributions, Kairouz, Bonawitz, and Ramage (2016) propose a heuristic called projected decoder, which has been proved very effective experimentally. We also implement the baseline methods with projected decoders. The results are shown in Figure 2. It can be seen from the numerical results that the performance of our compressive privatization approach are almost always better than the baselines whether with or without projected decoders. In the default setting, with the same number of samples, the error of our method is 2-10 times lower than all baseline methods. With the projected decoder heuristic, the performance of all baselines expect for RR improves significantly. However, this decoder is computationally more expensive. For our method, simple normalization decoder has already achieved similar accuracy.

Since our estimation algorithm employs the projection decoder in its last step, to eliminate the influence of the projection decoder, we have changed the the last step to a normalized decoder (Kairouz, Bonawitz, and Ramage 2016), which is the default setting in the code provided by the authors in (Acharya, Sun, and Zhang 2018), and then conducted experiments to test the performance. The results are shown in Figure 3. It can be seen from the experimental results that the type of decoder has little effect on the performance of our estimation algorithm. Our method can always achieve good performance for different $\varepsilon$ regardless of the decoder. The improvement of our method on sparse distribution mainly comes from our compressive privatization technique, rather than the choice of decoder.

### 6 Conclusion

We study sparse distribution estimation in the local differential privacy model. We propose compressive privatization, which is the first LDP mechanism with provable guarantee on sparse or approximately sparse distributions. In particular, the sample complexity of our method depends on the sparsity rather than the size of the entire universe. Empirical results are provided which confirm our theoretical improvements.
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