Non-existence of Poisson problem involving regional fractional Laplacian with order in \((0, 1/2]\)

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Abstract

In this paper, some elliptic equation in a bounded open domain in \(\mathbb{R}^N\) \((N \geq 2)\) with \(C^2\) boundary \(\partial \Omega\) is considered. The problem is driven by the regional fractional Laplacian, the infinitesimal generator of the censored symmetric \(2\alpha\)-stable process in \(\Omega\). Probability theory asserts that the censored \(2\alpha\)-stable process can not approach the boundary when \(\alpha \in (0, \frac{1}{2}]\). Our aim of this paper is to obtain non-existence of viscosity super-solutions to Poisson problem

\[
\begin{cases}
(-\Delta)^{\alpha}_\Omega u = f & \text{in } \Omega, \\
u = h & \text{on } \partial \Omega
\end{cases}
\]

for \(\alpha \in (0, \frac{1}{2}]\).

Key Words: Regional Fractional Laplacian, Non-existence, Viscosity solution.

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1 Introduction

In the last few years, there has been an increasing interest in the study of regional fractional Laplacian problems which are now widely used in physics, operation research, queuing theory, mathematical finance, risk estimation, and others (see [1, 4 –7, 14, 20, 24]). Given a bounded open Lipschitz set \(\Omega \subset \mathbb{R}^N\) with \(N \geq 2\), a symmetric \(2\alpha\)-stable process \(X = \{X_t\}\) in \(\mathbb{R}^N\) killed upon leaving the set \(\Omega\) is called a symmetric \(2\alpha\)-stable process in \(\Omega\), denote it by \(X^\Omega\), see [9]. The Dirichlet form of \(X^\Omega\) on \(L^2(\Omega, dx)\) is \((\mathcal{C}, \mathcal{F}_\Omega)\), where

\[
\mathcal{F}_\Omega = \left\{ u \in L^2(\mathbb{R}^N) : u = 0 \text{ q.e. on } \mathbb{R}^N \setminus \Omega, \quad \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x - y|^{N+2\alpha}} \, dx \, dy < +\infty \right\}
\]

and

\[
\mathcal{C}(u, v) = \frac{1}{2} c_{N, \alpha} \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2\alpha}} \, dx \, dy + \int_{\Omega} \kappa_\alpha(x) u(x) v(x) \, dx, \quad \forall u, v \in \mathcal{F}_\Omega.
\]

Here q.e. is the abbreviation for quasi-everywhere (cf. [2], [17]), and the density of the killing measure of \(X^\Omega\)

\[
\kappa_\alpha(x) = c_{N, \alpha} \int_{\mathbb{R}^N \setminus \Omega} |x - y|^{-N-2\alpha} \, dy.
\]

Some properties of killing measure \(\kappa_\alpha\) are given in Appendix.

Elliptic differential equations and diffusion processes play significant roles in the theory of partial differential equations and in probability theory, respectively. There are close relationships between these two subjects. Sometimes for an elliptic operator, there is a diffusion process...
associated with it, so that the elliptic operator is the infinitesimal generator of the diffusion process.

The inhomogenous elliptic problem associated to this kind of process is

\[
\begin{cases}
(-\Delta)^\alpha u = f, & \text{in } \Omega, \\
u = 0, & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

where

\[(-\Delta)^\alpha u(x) = c_{N,\alpha} \text{p.v.} \int_{\mathbb{R}^N} \frac{u(x) - u(z)}{|z - x|^{N+2\alpha}} \, dz\]

is the fractional Laplacian operator and \(c_{N,\alpha} > 0\) is the normalized constant, see [16].

Define \(\tau_\Omega = \inf\{t > 0 : X_t \notin \Omega\}\) and note that \(\lim_{t \to \tau_\Omega} X_t\) exists and typically belongs to \(\Omega\).

Bogdan, Burdzy and Chen [2] (also see Guan and Ma [20]) extended \(X_\Omega\) beyond its lifetime \(\tau_\Omega\) and obtained a version of the strong Markov process, named censored symmetric stable process.

A censored stable process in an open set \(\Omega \subset \mathbb{R}^N\) is obtained from the symmetric stable process by suppressing its jumps from \(\Omega\) to \(\mathbb{R}^N \setminus \Omega\). In fact, the censored stable process is a stable process forced to stay inside \(\Omega\) and the Dirichlet form has no killing term. It is known that the censored stable process has the generator the regional fractional Laplacian defined in \(\Omega\) defined as

\[(-\Delta)^\alpha_{\Omega} u(x) = \lim_{\varepsilon \to 0^+} (-\Delta)^\alpha_{\Omega,\varepsilon} u(x)\]

with

\[(-\Delta)^\alpha_{\Omega,\varepsilon} u(x) = c_{N,\alpha} \int_{\Omega \setminus B_\varepsilon(x)} \frac{u(x) - u(z)}{|z - x|^{N+2\alpha}} \, dz\]

It is believed that censored stable processes deserve to be studied because the classical counterpart, killed Brownian motion, is an important model in both pure mathematics and in applied probability (see [2]). When \(\alpha \in (0, 1)\), Kulczycki [21] establishes the upper and the lower bound estimates of Green function for a bounded open domain. When \(\alpha \in \left(\frac{1}{2}, 1\right)\), [8, 9] give estimates on the heat kernel and Green kernel related to the regional fractional Laplacian. [19] provides a version of integration by formula for regional fractional Laplacian and [10] extends this formula to solve regional fractional problem with kinds of inhomogenous terms. Boundary blowing-up solutions of semilinear regional fractional problem is studied in [11].

It should be pointed out that the censored stable process has some interesting properties, which imply some differences between \((\frac{1}{2}, 1)\) and \((0, \frac{1}{2}]\) (see [21 Theorem 1]):

(i) if \(\alpha \in \left(\frac{1}{2}, 1\right)\), the censored symmetric \(2\alpha\)-stable process in \(\Omega\) has a finite lifetime and will approach \(\partial \Omega\);

(ii) if \(\alpha \in (0, \frac{1}{2}]\), the censored symmetric \(2\alpha\)-stable process in \(\Omega\) is conservative and will never approach \(\partial \Omega\).

In particular, the inhomogenous elliptic problem with regional fractional laplacian

\[
\begin{cases}
(-\Delta)^\alpha_{\Omega} u = f, & \text{in } \Omega, \\
u = h, & \text{on } \partial \Omega
\end{cases}
\]

has the following existence result:
Proposition 1.1 Assume that \( \alpha \in (\frac{1}{2}, 1) \), \( \Omega \) is a bounded \( C^2 \) domain in \( \mathbb{R}^N \) with \( N \geq 2 \) and \( h = 0 \) on \( \partial \Omega \).

(i) If \( f \) is Hölder continuous, then (1.2) has a classical solution \( u_f \in C_0(\Omega) \);

(ii) If \( f \in L^2(\Omega) \), then (1.2) has a weak solution \( u_f \in H^\alpha_0(\Omega) \), where \( H^\alpha_0(\Omega) \) is the Hilbert space, which is the closure of \( C^\infty_c(\Omega) \) under the norm

\[
||u||_\alpha := \sqrt{\int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))^2}{|x - y|^{N+2\alpha}}dxdy}.
\]

It is worth noting that [22] shows that the space \( H^\alpha_0(\Omega) \) has zero trace when \( \alpha \in (\frac{1}{2}, 1) \) and has no zero trace when \( \alpha \in (0, \frac{1}{2}) \); [3] shows that the optimal constant \( A_{N,\alpha} \) in the fractional Hardy inequality

\[
\int_{\mathbb{R}^N_+} \int_{\mathbb{R}^N_+} \frac{(u(x) - u(y))^2}{|x - y|^{N+2\alpha}}dxdy \geq A_{N,\alpha} \int_{\mathbb{R}^N_+} x^{-2\alpha} u^2(x)dx \quad \text{for any } u \in C_c(\mathbb{R}^N_+),
\]

is positive if \( \alpha \in (\frac{1}{2}, 1) \) and zero if \( \alpha \in (0, \frac{1}{2}) \), where \( \mathbb{R}^N_+ = \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N > 0\} \). These differences related to regional fractional Laplacian indicate that the inhomogenous problem (1.2) with \( \alpha \in (0, \frac{1}{2}) \) would have challenging properties.

In spite of some known results, the theory of elliptic problems driven by regional fractional Laplacian is far from mature and adequate. The purpose of this paper is to investigate non-existence of viscosity super-solutions, which, to some extent, can be seen as an interpretation of the result in [2] and an counterpart of [10] for \( \alpha \in (0, \frac{1}{2}) \). Our main results on problem (1.2) with \( \alpha \in (0, \frac{1}{2}) \) states as follows.

Theorem 1.1 Assume that the integer \( N \geq 1, \alpha \in (0, \frac{1}{2}) \), \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) such that \( \Omega \) is \( C^2 \) if \( N \geq 2 \) and the interior point set of \( \Omega \) is \( \Omega \) if \( N = 1 \).

Let \( h \) be a continuous function on \( \partial \Omega \) if \( N \geq 2 \) and \( h > -\infty \) if \( N = 1 \), \( f \) be a nonnegative function such that

\[
f \geq t_0 \quad \text{in } \Omega_\epsilon := \{x \in \Omega : \rho(x) > \epsilon\}
\]

for some \( t_0 > 0 \) and \( \epsilon \geq 0 \), where \( \rho(x) = \text{dist}(x, \partial \Omega) \).

Then there exists \( \epsilon_0 > 0 \) such that (1.3) holds for \( \epsilon \in [0, \epsilon_0] \), problem (1.2) has no viscosity super solution.

Our idea to prove Theorem 1.1 is based on the non-existence boundary blowing up function satisfying \( (-\Delta)^\alpha_\Omega u \geq 0 \) in \( \Omega \) in the viscosity sense and then the key point is to obtain estimates t of \( (-\Delta)^\alpha_\Omega V_\tau \) for \( \alpha \in (0, \frac{1}{2}) \) near the boundary, where

\[
V_\tau(x) = \begin{cases} 
\rho(x)^\tau & \text{for } x \in A_\delta, \\
l(x) & \text{for } x \in \Omega \setminus A_\delta
\end{cases}
\]

and

\[
V_\tau^*(x) = \begin{cases} 
-\ln \rho(x) & \text{for } x \in A_\delta, \\
l(x) & \text{for } x \in \Omega \setminus A_\delta
\end{cases}
\]
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with $A_\delta = \{ x \in \Omega : \rho(x) < \delta \}$, $\delta \leq \frac{1}{2}$, $\tau \in (-1, 2\alpha)$ and the function $l$ being positive such that $V_\tau$ is $C^2$ in $\Omega$. It is worth noting that when $\tau \in (2\alpha - 1, 0)$ with $\alpha \in (0, \frac{1}{2})$, $(-\Delta)_{\delta}^\alpha V_\tau$ blows up near the boundary positively, which leads to the non-existence of Poisson problem (1.4).

A direct corollary could be stated as follows.

**Corollary 1.1** Assume that $N \geq 1$, $\alpha \in (0, \frac{1}{2}]$ and $\Omega$ is a bounded domain in $\mathbb{R}^N$ such that $\Omega$ is $C^2$ if $N \geq 2$ and the interior point set $\bar{\Omega}$ is $\Omega$ if $N = 1$.

(i) problem

\[
\begin{aligned}
(-\Delta)_{\delta}^\alpha u &= 1 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\]  

(1.6)

has no viscosity solution.

(ii) Lane-Emden equation with regional fractional Laplacian

\[
\begin{aligned}
(-\Delta)_{\delta}^\alpha u &= u^p \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\]  

(1.7)

has no nontrivial nonnegative classical solution if $p > 0$.

The rest of this paper is organized as follows. In Section 2, we recall the definition of viscosity solution and do the estimates of $(-\Delta)_{\delta}^\alpha V_\tau$. In Section 3, we prove the non-existence of Poisson problem (1.2) by contradiction that a boundary blow-up super solution could be constructed if (1.2) has a super solution. Finally, we annex properties of killing measure $\kappa_\alpha$.

2 Preliminary

2.1 Viscosity solution

We start with the definition of viscosity solution, inspired by the definition of viscosity sense for nonlocal problem in [6].

**Definition 2.1** We say that a function $u \in C(\bar{\Omega})$ is a viscosity super solution (sub-solution) of

\[
\begin{aligned}
(-\Delta)_{\delta}^\alpha u &= f \quad \text{in } \Omega, \\
u &= h \quad \text{on } \partial \Omega
\end{aligned}
\]  

(2.1)

if $u \geq h$ (resp. $u \leq h$) on $\partial \Omega$ and for every point $x_0 \in \Omega$ and some neighborhood $V$ of $x_0$ with $V \subset \Omega$ and for any $\varphi \in C^2(\bar{V})$ such that $u(x_0) = \varphi(x_0)$ and $x_0$ is the minimum (resp. maximum) point of $u - \varphi$ in $V$, let

\[
\tilde{u} = \begin{cases}
\varphi & \text{in } V, \\
u & \text{in } \Omega \setminus V,
\end{cases}
\]

we have that

\[-\Delta)_{\delta}^\alpha \tilde{u}(x_0) \geq f(x_0) \quad (\text{resp. } (-\Delta)_{\delta}^\alpha \tilde{u}(x_0) \leq f(x_0)).
\]

We say that $u$ is a viscosity solution of (2.1) if it is a viscosity super-solution and also a viscosity sub-solution of (2.1).
Theorem 2.1 Assume that the functions \( f : \Omega \to \mathbb{R}, \ h : \partial \Omega \to \mathbb{R} \) are continuous. Let \( u \) and \( v \) be a viscosity super-solution and sub-solution of (2.1) respectively. Then

\[ v \leq u \quad \text{in} \quad \Omega. \tag{2.2} \]

Proof. Let us define \( w = u - v \), then

\[
\begin{cases}
(-\Delta)_\Omega^\alpha w \geq 0 & \text{in} \ \Omega, \\
w \geq 0 & \text{on} \ \partial \Omega.
\end{cases}
\tag{2.3}
\]

If (2.2) fails, then there exists \( x_0 \in \Omega \) such that

\[ w(x_0) = u(x_0) - v(x_0) = \min_{x \in \Omega} w(x) < 0, \]

then in the viscosity sense,

\[ (-\Delta)_\Omega^\alpha w(x_0) \geq 0. \tag{2.4} \]

Since \( w \) is a viscosity super solution \( x_0 \) is the minimum point in \( \Omega \) and \( w \geq 0 \) on \( \partial \Omega \), then we can take a small neighborhood \( V_0 \) of \( x_0 \) such that \( \tilde{w} = w(x_0) \) in \( V_0 \). From (2.4), we have that

\[ (-\Delta)_\Omega^\alpha \tilde{w}(x_0) \geq 0. \]

But the definition of regional fractional Laplacian implies that

\[
(-\Delta)_\Omega^\alpha \tilde{w}(x_0) = \int_{\Omega \setminus V_0} \frac{w(x_0) - w(y)}{|x_0 - y|^{N+2\alpha}} dy < 0,
\]

which is impossible. \( \square \)

Remark 2.1 Let \( u \) be a continuous function in \( \Omega \) and \( x_0 \) is a minimum point of \( u \), then \( (-\Delta)_\Omega^\alpha u(x_0) > 0 \) in the viscosity sense.

2.2 Estimates for boundary blowing up functions

It is known the derivatives on the boundary blowing-up functions is related the one-dimensional problem and we start this subsection by the analysis of one-dimensional regional fractional problem on the half-line. Let

\[ w_\tau(t) = t^\tau \quad \text{and} \quad w_0^*(t) = -\ln t \quad \text{for} \ t > 0, \tag{2.5} \]

where \( \tau > -1. \)

Lemma 2.1 Assume that \( \alpha \in (0, 1), \ \tau \in (-1, 2\alpha) \) and \( w_\tau, w_0^* \) are defined in (2.5). Let

\[ I_\alpha = \begin{cases} 
(0, 2\alpha - 1) & \text{if} \ \alpha \in \left(\frac{1}{2}, 1\right), \\
(2\alpha - 1, 0) & \text{if} \ \alpha \in \left(0, \frac{1}{2}\right).
\end{cases} \]

Then

\[ (-\Delta)^{\frac{1}{2}}_{\mathbb{R}^+} w_0^* = 0 \quad \text{for} \ t > 0 \tag{2.6} \]
and
\[ (-\Delta)^{\alpha}_{\mathbb{R}^n} w_r = c_\alpha(\tau) t^{\tau-2\alpha} \quad \text{for } t \in \mathbb{R}_+, \]
where
\[ c_\alpha(\tau) = \begin{cases} > 0 & \text{for } \tau \in I_\alpha, \\ = 0 & \text{for } \tau = 2\alpha - 1, \\ < 0 & \text{for } \tau \in (-1, 2\alpha) \setminus I_\alpha. \end{cases} \tag{2.7} \]

**Proof.** Estimates of \( c_\alpha(\tau) \): For \( \epsilon \in (0, \frac{1}{8}) \), by change variable we have that
\[
(-\Delta)^{\alpha}_{\mathbb{R}^n} w_r(t) = c_{1,\alpha} t^{\tau-2\alpha} \lim_{\epsilon \to 0^+} \int_{(0,\infty) \setminus (1-\epsilon,1+\epsilon)} \frac{1 - s^\tau}{|1 - s|^{1+2\alpha}} ds
\]
\[
= c_{1,\alpha} t^{\tau-2\alpha} \lim_{\epsilon \to 0^+} \left( \int_{0}^{1-\epsilon} \frac{1 - s^\tau}{(1-s)^{1+2\alpha}} ds - \int_{1+\epsilon}^{\infty} \frac{1 - s^{-\tau}}{(1-s^{-1})^{1+2\alpha}} s^{-2\alpha-1+\tau} ds \right)
\]
\[
= c_{1,\alpha} t^{\tau-2\alpha} \lim_{\epsilon \to 0^+} \left( \int_{0}^{1-\epsilon} \frac{(1 - s^\tau)(1 - s^{2\alpha-\tau-1})}{(1-s)^{1+2\alpha}} ds + \int_{1-\epsilon}^{1+\epsilon} \frac{(1 - s^\tau)s^{2\alpha-1-\tau}}{(1-s)^{1+2\alpha}} ds \right),
\]
where \( 1 - \epsilon < \frac{1-\epsilon}{1+\epsilon} < 1 - \epsilon + \epsilon^2 \) and
\[
\left| \int_{1-\epsilon}^{1+\epsilon} \frac{(1 - s^\tau)s^{2\alpha-1-\tau}}{(1-s)^{1+2\alpha}} ds \right| \leq c_1 |\tau| \int_{1-\epsilon}^{1+\epsilon} (1-s)^{-2\alpha} ds
\]
\[
\leq c_2 \epsilon^{2-2\alpha} \to 0 \quad \text{as } \epsilon \to 0^+.
\]

As a consequence,
\[ (-\Delta)^{\alpha}_{\mathbb{R}^n} w_r(t) = c_{1,\alpha} \gamma(\alpha, \tau) t^{\tau-2\alpha}, \]
where
\[ \gamma(\alpha, \tau) = \int_{0}^{1} \frac{(1 - s^\tau)(1 - s^{2\alpha-\tau})}{(1-s)^{1+2\alpha}} ds. \]

Note that
\[ \gamma(\alpha, \tau) > 0 \quad \text{if and only if} \quad \tau(2\alpha - 1 - \tau) > 0, \]
\[ \gamma(\alpha, \tau) < 0 \quad \text{if and only if} \quad \tau(2\alpha - 1 - \tau) < 0, \]
\[ \gamma(\alpha, \tau) = 0 \quad \text{if} \quad \tau = 0 \text{ or } \tau = 2\alpha - 1. \]

Therefore, we conclude (2.11) with \( c_\alpha(\tau) = c_{1,\alpha} \gamma(\alpha, \tau). \)

**Proof of** \( (-\Delta)^{1/2}_{\mathbb{R}^n} w^*_0 = 0. \) By change variable we have that
\[
(-\Delta)^{1/2}_{\mathbb{R}^n} w^*_0(t) = c_{1,1/2} t^{-1} \lim_{\epsilon \to 0^+} \int_{(0,\infty) \setminus (1-\epsilon,1+\epsilon)} \frac{\ln s}{|1 - s|^2} ds
\]
\[
= c_{1,1/2} t^{-1} \lim_{\epsilon \to 0^+} \left( \int_{0}^{1-\epsilon} \frac{\ln s}{(1-s)^2} ds + \int_{1+\epsilon}^{\infty} \frac{-\ln s}{(1-s)^2} ds \right)
\]
\[
= c_{1,1/2} t^{-1} \lim_{\epsilon \to 0^+} \int_{1-\epsilon}^{1+\epsilon} \frac{-\ln s}{(1-s)^2} ds,
\]
where \( 1 - \epsilon < \frac{1}{1+\epsilon} < 1 - \epsilon + \epsilon^2 \) and for \( c_3, c_4 > 0 \),

\[
\left| \int_{1-\epsilon}^{1} \frac{-\ln s}{(1-s)^2} ds \right| \leq c_3 \int_{1-\epsilon}^{1} (1-s)^{-1} ds \\
\leq c_4 \epsilon \to 0 \quad \text{as} \quad \epsilon \to 0^+.
\]

Therefore, \((-\Delta)^{\frac{1}{2}}_{\mathbb{R}_+} w_0^* = 0 \) in \( \mathbb{R}_+ \). We complete the proof. \( \square \)

**Proposition 2.1** Assume that \( \alpha \in (0, 1) \), \( I_{\alpha} = (-1, 2\alpha) \), \( \Omega \) is a bounded \( C^2 \) domain in \( \mathbb{R}^N \) with \( N \geq 1 \) and \( V_\tau \) is given in \((1.4)\) with \( \tau \in I_{\alpha} \). Let

\[
I_{\alpha}^* = \begin{cases} (2\alpha - 1, 0) & \text{if } \alpha \in (0, \frac{1}{2}), \\
(0, 2\alpha - 1) & \text{if } \alpha \in (\frac{1}{2}, 1), \quad \text{and} \quad I_{\alpha}^# = \begin{cases} I_{\alpha}^* & \text{if } \alpha \neq \frac{1}{2}, \\
\emptyset & \text{if } \alpha = \frac{1}{2}. \end{cases} \end{cases} \tag{2.8}
\]

Then there exist \( \delta_1 \in (0, \delta) \) and constant \( c_5 > 1 \) such that:

(i) If \( \tau \in I_{\alpha}^* \), then

\[
\frac{1}{c_5} \rho(x)^{\alpha - 2\alpha} \leq (-\Delta)^{\alpha}_{\Omega} V_\tau(x) \leq c_5 \rho(x)^{\alpha - 2\alpha} \quad \text{for all} \quad x \in A_{\delta_1}.
\]

(ii) If \( \tau \in I_{\alpha} \setminus I_{\alpha}^# \), then

\[
\frac{1}{c_5} \rho(x)^{\alpha - 2\alpha} \leq (-\Delta)^{\alpha}_{\Omega} V_\tau(x) \leq c_5 \rho(x)^{\alpha - 2\alpha} \quad \text{for all} \quad x \in A_{\delta_1}.
\]

(iii) If \( \tau = 2\alpha - 1 \) and \( \alpha \neq \frac{1}{2} \), then

\[
\left| (-\Delta)^{\alpha}_{\Omega} V_\tau(x) \right| \leq c_5 (\rho(x)^{\alpha - 2\alpha + 1} + \rho(x)^{\alpha} + 1) \quad \text{for all} \quad x \in A_{\delta_1}.
\]

**Proof.** When \( N = 1 \), we assume that \((0, \delta_0) \subset \Omega \), for \( x_1 \in (0, \frac{\alpha}{2}) \), we have that

\[
(-\Delta)^{\alpha}_{\Omega} V_\tau(x) = c_{1,\alpha} \text{p.v.} \int_0^{\delta_0} \frac{x^\tau - y^\tau}{|x-y|^{1+2\alpha}} dy + \int_{\Omega \setminus (0,\delta_0)} \frac{x^\tau - y^\tau}{|x-y|^{1+2\alpha}} dy
\]

\[
= c_{1,\alpha} x^{\tau - 2\alpha} \int_0^{\delta_0} \frac{1 - t^\tau}{|1-t|^{1+2\alpha}} dt + O(1)(x^{\tau} + 1)
\]

\[
= x^{\tau - 2\alpha} \left( e_{\alpha}(\tau) - \left( \frac{\delta_0}{x} \right)^{\tau - 2\alpha} \right) + O(1)(x^{\tau} + 1)
\]

\[
= c_{\alpha}(\tau) x^{\tau - 2\alpha} + O(1)(x^{\tau} + 1).
\]

Now we concentrate on the case that \( N \geq 2 \). By compactness we prove that the corresponding inequality holds in a neighborhood of any point \( \bar{x} \in \partial \Omega \) and without loss of generality we may assume that \( \bar{x} = 0 \) and we further assume that \(-e_1\) is the outer normal vector of \( \Omega \) at 0, where \( e_1 = (1, 0, \cdots) \in \mathbb{R}^N \).
For a given $0 < \eta \leq \delta$, we define
\[
Q_\eta = \{ y = (y_1, y') \in \mathbb{R} \times \mathbb{R}^{N-1}, 0 < y_1 < \eta, |y'| < \eta \}
\]
and
\[
\tilde{Q}_\eta = \{ y = (y_1, y') \in \mathbb{R} \times \mathbb{R}^{N-1}, |y_1| < \eta, |y'| < \eta \}.
\]

Since $\partial \Omega$ is $C^2$, then there is a $C^2$ function $\varphi : \mathbb{R}^{N-1} \to \mathbb{R}$ such that $(z_1, z') \in \Omega \cap \tilde{Q}_2\eta$ if and only if $z_1 = \varphi(z')$ for $|z'| < 2\eta$ for $\eta > 0$ small enough. In fact, we take $\eta > 0$ small enough, then for any $z \in A_t := \{ x \in \Omega : \rho(x) = t \}$ with $t \in (0, \eta]$, there exists a unique point $z_0 \in \partial \Omega$ such that $|z - z_0| = t$, $z - z_0$ is a normal vector at $z_0$ and $A_t$ is $C^2$.

Let $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ be a $C^2$ diffeomorphism such that
\[
\Phi(z) = (z', z_1 + \varphi(z')) \quad \text{for} \quad z = (z_1, z') \in Q_\eta.
\]

Note that
\[
\Phi(te_1) = te_1 \quad \text{for} \quad t \in (0, \eta), \quad \Phi(0, y') \in \partial \Omega \cap \tilde{Q}_\eta \quad \text{for} \quad |y'| < \eta
\]
and
\[
\tilde{Q}_{\frac{\eta}{2}} \cap \Omega \subset \Phi(Q_\eta) \subset \tilde{Q}_{2\eta} \cap \Omega.
\]

Moreover, $\rho(\Phi(y)) = y_1 + O(|y'|^2)$ for $y \in Q_\eta$ and there exists $c_6 > 0$ such that
\[
\Phi(0) = 0, \quad |D\Phi(0) - E_1| \leq c_6 |z'| \quad \text{and} \quad |\Phi(y) - y| \leq c_6 |y'|^2 \quad \text{for} \quad y = (y_1, y') \in Q_\eta,
\]
where $E_1 = (d_{ij})_{N \times N}$ is the unit matrix, i.e. $d_{ii} = 1$ and $d_{ij} = 0$ if $i \neq j$.

Let $x_1 \in (0, \eta/4)$, $x = (x_1, 0)$ be a generic point in $A_{\eta/4}$. We observe that $|x - \tilde{x}| = \rho(x) = x_1$. By definition we have
\[
(-\Delta)^{\alpha} V_\tau(x) = c_{N, \alpha} \text{p.v.} \int_{\Phi(Q_\eta)} \frac{x_1 - \rho^\tau(z)}{|x - z|^{N+2\alpha}} dz + c_{N, \alpha} \int_{\Omega \setminus \Phi(Q_\eta)} \frac{V_\tau(x) - V_\tau(z)}{|x - z|^{N+2\alpha}} dz
\]
and we see that
\[
|\int_{\Omega \setminus \Phi(Q_\eta)} \frac{V_\tau(x) - V_\tau(z)}{|x - z|^{N+2\alpha}} dz| \leq c_7 (x_1^{\alpha} + 1),
\]
where the constant $c_7$ is independent of $x$. Thus we only need to study the asymptotic behavior of the first integral, that from now on, we denote by
\[
E(x_1) = \text{p.v.} \int_{\Phi(Q_\eta)} \frac{x_1 - \rho^\tau(z)}{|x - z|^{N+2\alpha}} dz.
\]

Let $z = \Phi(y)$, then we have that
\[
E(x_1) = \text{p.v.} \int_{Q_\eta} \frac{x_1^\tau - y_1^\tau}{|x - \Phi(y)|^{N+2\alpha}} |D\Phi(y)| dy + \int_{Q_\eta} \frac{O(|y'|^2)}{|x - \Phi(y)|^{N+2\alpha}} |D\Phi(y)| dy.
\]

Observe that
\[
|D\Phi(y)| = 1 + O(|y'|),
\]
\[
|x - \Phi(y)|^2 = |x - y + y - \Phi(y)|^2 = |x_1 - y_1|^2 + |y'|^2 + O(|y'|^4)
\]
and
\[ | \int_{Q_R} \frac{O(|y'|^2)}{|x - \Phi(y)|^{N+2\alpha}} |D\Phi(y)|dy | \leq C_8, \quad (2.13) \]
where we use the fact that $|y - \Phi(y)| = O(|y'|^2)$. Then we have that
\[
\tilde{E}(x_1) := \text{p.v.} \int_{Q_R} \frac{x_1^T - y_1^T}{(|x_1 - y_1|^2 + |y'|^2 + O(|y'|^4))^{N+2\alpha}} (1 + O(|y'|^2))dy
\]
\[= \text{p.v.} \int_0^\eta (x_1^T - y_1^T) \int_{B_1^{N-1}(0)} \frac{(1 + O(|y'|))}{(|x_1 - y_1|^2 + |y'|^2)^{N+2\alpha}} dy' dy_1
\]
\[= \text{p.v.} \int_0^\eta \frac{x_1^T - y_1^T}{|x_1 - y_1|^{1+2\alpha}} \int_{B_1^{N-1}(0)} \frac{(1 + O(|x_1 - y_1|z'))}{(1 + |z'|^2)^{N+2\alpha}} dz' dy_1
\]
\[= \text{p.v.} \int_0^\eta \frac{x_1^T - y_1^T}{|x_1 - y_1|^{1+2\alpha}} \mathcal{F}_\alpha(|x_1 - y_1|) dy_1,
\]
where
\[\mathcal{F}_\alpha(|x_1 - y_1|) = \int_{B_1^{N-1}(0)} \frac{(1 + O(|x_1 - y_1|z'))}{(1 + |z'|^2)^{N+2\alpha}} dz'
\]
\[= \int_{B_1^{N-1}(0)} \frac{1}{(1 + |z'|^2)^{N+2\alpha}} dz' + O(|x_1 - y_1|) \int_{B_1^{N-1}(0)} \frac{|z'|}{(1 + |z'|^2)^{N+2\alpha}} dz'
\]
\[= \frac{1}{(1 + |z'|^2)^{N+2\alpha}} dz' + O(|x_1 - y_1|^{1+2\alpha}) + \int_{\mathbb{R}^{N-1}} \frac{|z'|}{(1 + |z'|^2)^{N+2\alpha}} dz'O(|x_1 - y_1|)
\]
\[= d_\alpha + O(|x_1 - y_1|),
\]
with $d_\alpha = \int_{\mathbb{R}^{N-1}} \frac{1}{(1 + |z'|^2)^{N+2\alpha}} dz' > 0$ and $B_1^{N-1}(0)$ being the ball with center at the origin and radius $r$ in $\mathbb{R}^{N-1}$. Thus we have that
\[E(x_1) = d_\alpha \text{p.v.} \int_0^\eta \frac{x_1^T - y_1^T}{|x_1 - y_1|^{1+2\alpha}} dy_1 + O(\text{p.v.} \int_0^\eta \frac{x_1^T - y_1^T}{|x_1 - y_1|^{2\alpha}} dy_1) + O(1)
\]
\[= d_\alpha \left( x_1^{\tau - 2\alpha} \text{p.v.} \int_0^{\frac{\eta}{x_1}} \frac{1 - t^{\tau + 2\alpha}}{|1 - t|^{1+2\alpha}} dt + O(x_1^{\tau - 2\alpha + 1} \text{p.v.} \int_0^{\frac{\eta}{x_1}} \frac{1 - t^{\tau + 2\alpha}}{|1 - t|^{2\alpha}} dt) \right) + O(1)
\]
\[= d_\alpha \left[ x_1^{\tau - 2\alpha} \left( c_\alpha(\tau) - \frac{\eta}{x_1} \right) + O(x_1^{\tau - 2\alpha + 1}) \right] + O(1)
\]
\[= d_\alpha c_\alpha(\tau) x_1^{\tau - 2\alpha} + O(x_1^{\tau - 2\alpha + 1}) + O(1)
\]
Therefore, we conclude from (2.10) and (2.16) that
\[(-\Delta)_1^{\tau} V_\tau(x) = d_\alpha c_\alpha(\tau) x_1^{\tau - 2\alpha} + O(x_1^\tau) + O(x_1^{\tau - 2\alpha + 1}) + O(1) \quad \text{as} \quad x_1 \to 0^+. \quad (2.17)\]
Combining with Lemma 2.1 Proposition 2.1 follows and the proof is complete.

Note that $I_\alpha = \{0\}$ when $\alpha = \frac{1}{2}$ and in this case, we need do the estimate for the following function

$$V_\epsilon(x) = \begin{cases} -\ln \rho(x) & \text{for } x \in A_\delta, \\ l(x) & \text{for } x \in \Omega \setminus A_\delta, \end{cases}$$

(2.18)

where $\delta \in (0, \frac{1}{4})$ and the function $l$ is positive such that $V_\epsilon$ is $C^2$ in $\Omega$.

**Remark 2.2** For $\tau \in (-1, 2\alpha) \setminus \{2\alpha - 1, 0\}$, then

$$\lim_{\rho(x) \to 0^+ \rho^2} \rho^{2\alpha - \tau}(x) (-\Delta)_\Omega^\alpha V_\tau(x) = d_\alpha c_\alpha(\tau).$$

**Proposition 2.2** Assume that $\Omega$ is a bounded $C^2$ domain in $\mathbb{R}^N$ with $N \geq 1$ and $V_\epsilon$ is given in (2.15).

Then there exists $c_9 > 0$ such that

$$| (-\Delta)^{1/2}_\Omega V_\epsilon(x) | \leq c_9 V_\epsilon(x), \quad \forall x \in \Omega.$$

**Proof.** We omit the proof for the case $N = 1$. When $N \geq 2$ and under the same setting in the proof of Proposition 2.1, we have

$$(-\Delta)^{1/2}_\Omega V_\epsilon(x) = c_{N, \alpha} \text{p.v.} \int_{\Phi(Q_\eta)} -\ln x_1 + \ln \rho(z) |x - z|^{-N+1} dz + c_{N, \alpha} \int_{\Omega \setminus \Phi(Q_\eta)} V_\epsilon(x) - V_\epsilon(z) |x - z|^{-N+1} dz$$

(2.19)

and we see that

$$\left| \int_{\Omega \setminus \Phi(Q_\eta)} \frac{V_\epsilon(x) - V_\epsilon(z)}{|x - z|^{-N+2\alpha}} dz \right| \leq c_{10} V_\epsilon(x),$$

(2.20)

where the constant $c_{10}$ is independent of $x$. Thus we only need to study the asymptotic behavior of the first integral, from now on, we denote by

$$\mathbb{E}(x_1) = \text{p.v.} \int_{\Phi(Q_\eta)} -\ln x_1 + \ln \rho(z) |x - z|^{-N+1} dz.$$

Let $z = \Phi(y)$ and $0 < x_1 < 1$, it follows from (2.11), (2.12) and (2.13) that

$$\begin{align*}
\mathbb{E}(x_1) &= \text{p.v.} \int_{Q_\eta} -\ln x_1 + \ln y_1 |x - \Phi(y)|^{-N+1} |D\Phi(y)| dy + \int_{Q_\eta} \frac{O(|y'|^2)}{|x - \Phi(y)|^{-N+1}} |D\Phi(y)| dy, \\
&= \text{p.v.} \int_{Q_\eta} -\ln x_1 + \ln y_1 \left( |x_1 - y_1|^2 + |y'|^2 + O(|y'|^4) \right)^{\frac{N+1}{2}} (1 + O(|y'|)) dy, \\
&= \text{p.v.} \int_0^\eta (-\ln x_1 + \ln y_1) \int_{B_{N-1}^\eta(0)} \left( |x_1 - y_1|^2 + |y'|^2 \right)^{\frac{N+1}{2}} (1 + O(|y'|)) dy' dy_1, \\
&= \text{p.v.} \int_0^\eta -\ln x_1 + \ln y_1 \int_{|x_1 - y_1|^{1+1}}^{B_{N-1}^\eta_{|x_1 - y_1|}(0)} \left( |x_1 - y_1|^2 + |y'|^2 \right)^{\frac{N+1}{2}} (1 + O(|y'|)) dy' dy_1, \\
&= \text{p.v.} \int_0^\eta -\ln x_1 + \ln y_1 \int_{|x_1 - y_1|^2}^{|x_1 - y_1|} \left( |x_1 - y_1|^2 + |y'|^2 \right)^{\frac{N+1}{2}} (1 + |y'|^2) dy' dy_1, \\
&= \text{p.v.} \int_0^\eta -\ln x_1 + \ln y_1 \int_{|x_1 - y_1|^2}^{|x_1 - y_1|} \frac{F_2(1 + |y'|^2)}{|x_1 - y_1|^2} dy_1.
\end{align*}$$
where $F_1(|x_1 - y_1|) = d_{\frac{1}{2}} + O(|x_1 - y_1|)$, $d_{\frac{1}{2}} = \int_{R^N} \frac{1}{(1+|x'|^2)^{\frac{1}{2}} - \frac{1}{2}} d\tilde{z}' > 0$.

Thus we have that

$$E(x_1) = d_{\frac{1}{2}} \text{p.v.} \int_0^\eta - \frac{\ln x_1 + \ln y_1}{|x_1 - y_1|^2} dy_1 + O(\text{p.v.} \int_0^\eta - \frac{\ln x_1 + \ln y_1}{|x_1 - y_1|} dy_1) + O(1)$$

$$= \frac{d_{\frac{1}{2}}}{2} \int_{x_1}^{1} \frac{\ln t}{|1-t|^2} dt + O\left(\int_{x_1}^{1} \frac{\ln t}{1-t} dt\right) + O(1)$$

$$= d_{\frac{1}{2}} \left[ -\left(\frac{\eta}{x_1}\right)^{-1} \right] + O\left(\ln \frac{\eta}{x_1} + O(1)\right)$$

$$= O(\ln \frac{1}{x_1}) + O(1), \quad (2.21)$$

where we use (2.6) and p.v. $\int_0^\eta \frac{\ln t}{|1-t|^2} dt = -\int_{x_1}^{+\infty} \frac{\ln t}{|1-t|^2} dt$,

$$\int_{x_1}^{+\infty} \frac{\ln t}{|1-t|^2} dt \leq 2 \int_{x_1}^{+\infty} \frac{\ln t}{t^2} dt$$

$$= -\ln t \mid_{t=x_1}^{+\infty} + \frac{1}{t} \mid_{t=x_1}^{+\infty} \leq -c_0 x_1 \ln x_1 + c_0 x_1.$$

Then we conclude from (2.20) and (2.21) that

$$|(-\Delta)^{1/2}_{\Omega} V_*(x)| \leq c_{11} V_*(x) \quad \text{for all } x \in \Omega,$$  \quad (2.22)

which ends the proof. □

3 Non-existence of positive solution

Lemma 3.1 Let

$$w_1(x) = V_*(x) - V_{\tau_0}(x) \quad \text{for } x \in \Omega,$$

where $\tau_0 = \frac{1}{4}$.

Then there exist $\delta_2 > 0$ and $c_{12} > 1$ such that

$$\frac{1}{c_{12}} \rho^{\tau_0-1}(x) \leq (-\Delta)^{1/2}_{\Omega} w_1(x) \leq c_{12} \rho^{\tau_0-1}(x), \quad \forall x \in A_{\delta_2}.$$

Proof. From Proposition 2.1 for $\delta_1 > 0$ and $c_5 > 1$,

$$\frac{1}{c_5} \rho^{-3/4}(x) \leq -(-\Delta)^{1/2}_{\Omega} V_{\tau_0}(x) \leq c_5 \rho^{-3/4}(x), \quad \forall x \in A_{\delta_1}$$

and from Proposition 2.2,

$$|(-\Delta)^{1/2}_{\Omega} V_*(x)| \leq c_9 V_*(x), \quad \forall x \in A_{\delta_1}.$$
Then there exist $\delta_2 > 0$ and $c_{12} > 1$ such that
\[
\frac{1}{c_{12}} \rho^{\alpha-1}(x) \leq (-\Delta)^{1/2}_\Omega w_1(x) \leq c_{12} \rho^{\alpha-1}(x), \quad \forall x \in A_{\delta_2}.
\]
The proof ends.

**Proof of Theorem 1.1.** Case 1: $\alpha \in (0, \frac{1}{2})$. Take $\tau_0 = \alpha - \frac{1}{2} \in (2\alpha - 1, 0)$ and from Proposition 2.1 there exist $\delta_1 > 0$ and $c_{12} > 0$ such that
\[
(-\Delta)^{\alpha}_{\Omega} V_{\tau_0}(x) \geq c_{12} \rho^{\alpha-2\alpha}, \quad x \in A_{\delta_1}.
\]
Since $V_{\tau_0}$ is $C^2$ in $\Omega$, then there exists $T_0 > 0$ such that
\[
|(-\Delta)^{\alpha}_{\Omega} V_{\tau_0}(x)| \leq T_0 \quad \text{for } x \in \Omega \setminus A_{\delta_1}.
\]

Let $f$ be a nonnegative function satisfying (1.3) with $\epsilon \leq \delta_1$. By contradiction, we assume that problem (1.2) has a positive solution $u_f$. Let
\[
w = V_{\tau_0} + \frac{T_0}{t_{\delta_0}} u_f \quad \text{in } \Omega,
\]
then in the viscosity sense, we have that
\[
(-\Delta)^{\alpha}_{\Omega} w(x) \geq 0 \quad \text{for } x \in \Omega.
\]
that is, $w$ is a viscosity super-solution of $(-\Delta)^{\alpha}_{\Omega} u = 0$ and blows up at boundary of $\Omega$, since $u_f = g$ on the boundary.

Since $w$ blows up at boundary of $\Omega$, then there exists a point $x_0$ such that
\[
w(x_0) = \min_{x \in \Omega} w(x),
\]
and then there exists $\delta_3 > 0$ such that $w \geq 2w(x_0)$ in $A_{\delta_3}$. Therefore, taking a small neighborhood $V_0$ of $x_0$ such that $\tilde{w} = w(x_0)$ in $V_0$ and
\[
(-\Delta)^{\alpha}_{\Omega} w(x_0) \leq (-\Delta)^{\alpha}_{\Omega} \tilde{w}(x_0) = c_{N,\alpha} \int_{\Omega \setminus V_0} \frac{w(x_0) - w(y)}{|x_0 - y|^{N+2\alpha}} dy < 0,
\]
which contradicts (3.1).

**Case 2:** $\alpha = \frac{1}{2}$. From Lemma 3.1 we have that
\[
(-\Delta)^{\alpha}_{\Omega} w_1(x) \geq c_{12} \rho^{\alpha-2\alpha}, \quad x \in A_{\delta_1}.
\]
Since $w_1$ is $C^2$ in $\Omega$, then there exists $T_1 > 0$ such that
\[
|(-\Delta)^{\alpha}_{\Omega} V_{\tau_0}(x)| \leq T_1 \quad \text{for } x \in \Omega \setminus A_{\delta_1}.
\]
The rest proof is the same as Case 1.

**Proof of Corollary 1.1.** Part (i) follows directly by Theorem 1.1 with $f \equiv 1$.

**Part (ii):** Assume that $u_p \geq 0$ is a nontrivial classical solution of (1.7), if there exists $x_0 \in \Omega$ such that $u_p(x_0) = 0$, then it is the minimum point in $\Omega$ and the definition of fractional laplacian implies that
\[
0 > (-\Delta)^{\alpha}_{\Omega} u_p(x_0) = u_p^p(x_0) = 0,
\]
which is impossible. Thus, $u_p$ is positive in $\Omega$ and $f = u_p^p$ verifies condition (1.3) and it contradicts Theorem 1.1.
Appendix: properties of killing measure

The killing measure $\kappa_\alpha$ is the connection between global fractional Laplacian and regional fractional Laplacian. For a regular function $w$ such that $w = 0$ in $\mathbb{R}^N \setminus \Omega$, we observe that

$$(-\Delta)_\Omega^\alpha w(x) = (-\Delta)^\alpha w(x) - w(x)\kappa_\alpha(x), \quad \forall x \in \Omega,$$

where

$$\kappa_\alpha(x) = c_{N,\alpha} \int_{\mathbb{R}^N \setminus \Omega} \frac{1}{|x - z|^{N + 2\alpha}} dz.$$

**Proposition A.1** Let $N \geq 2$, $\Omega$ be a $C^2$ domain and $\kappa_\alpha$ be defined in (1.3) and $\rho(x) = \text{dist}(x, \partial \Omega)$, then $\kappa_\alpha \in C^1_{\text{loc}}(\Omega)$ and

$$\lim_{\rho(x) \to 0^+} \kappa_\alpha(x)\rho(x)^{2\alpha} = d_\alpha c_{N,\alpha},$$

where

$$d_\alpha = \int_{\mathbb{R}^{N-1}} \frac{1}{(1 + |z'|^2)^{\frac{N + 2\alpha}{2}}} dz'.$$

*Proof.* **Lipchitz continuity of $\kappa_\alpha$.** For $x_1, x_2 \in \Omega$ and any $z \in \mathbb{R}^N \setminus \Omega$, we have that

$$|z - x_1| \geq \rho(x_1) + \rho(z), \quad |z - x_2| \geq \rho(x_2) + \rho(z)$$

and

$$||z - x_1|^{N + 2\alpha} - |z - x_2|^{N + 2\alpha}| \leq c_{13}(|x_1 - x_2|(|z - x_1|^{N + 2\alpha} - 1|z - x_2|^{N + 2\alpha} - 1),$$

for some $c_2 > 0$ independent of $x_1$ and $x_2$. Then

$$|\kappa_\alpha(x_1) - \kappa_\alpha(x_2)| \leq c_{N,\alpha} \int_{\mathbb{R}^N \setminus \Omega} \left[ \frac{|z - x_2|^{N + 2\alpha} - |z - x_1|^{N + 2\alpha}}{|z - x_1|^{N + 2\alpha} - 1|z - x_2|^{N + 2\alpha}} dz \right].$$

By direct computation, we have that

$$\int_{\mathbb{R}^N \setminus \Omega} \frac{1}{|z - x_1|^{N + 2\alpha} - 1|z - x_2|^{N + 2\alpha}} dz \leq \int_{\mathbb{R}^N \setminus B_{\rho(x_1)}(x_1)} \frac{1}{|z - x_1|^{N + 2\alpha + 1}} dz$$

$$+ \int_{\mathbb{R}^N \setminus B_{\rho(x_2)}(x_2)} \frac{1}{|z - x_2|^{N + 2\alpha + 1}} dz \leq c_{14}[\rho(x_1)^{-1 - 2\alpha} + \rho(x_2)^{-1 - 2\alpha}]$$

and similar to obtain that

$$\int_{\mathbb{R}^N \setminus \Omega} \frac{1}{|z - x_1|^{N + 2\alpha} - 1|z - x_2|^{N + 2\alpha}} dz \leq c_{15}[\rho(x_1)^{-1 - 2\alpha} + \rho(x_2)^{-1 - 2\alpha}],$$
where \( c_{15} > 0 \) is independent of \( x_1, x_2 \). Then
\[
|\kappa_\alpha(x_1) - \kappa_\alpha(x_2)| \leq c_{14} c_{15} [\rho(x_1)^{-1 + 2\alpha} + \rho(x_2)^{-1 + 2\alpha}] |x_1 - x_2|,
\]
that is, \( \kappa_\alpha \) is \( C^{0,1} \) locally in \( \Omega \).

**Proof of (3.2).** By compactness we prove that the corresponding inequality holds in a neighborhood of any point \( \bar{x} \in \partial \Omega \) and without loss of generality we may assume that \( \bar{x} = 0 \) and we further assume that \( -e_1 \) is the outer normal vector of \( \Omega \) at 0, where \( e_1 = (1, 0, \cdots) \in \mathbb{R}^N \).

For a given \( 0 < \eta \leq \delta \), we define
\[
Q^\eta_\eta = \{ y = (y_1, y') \in \mathbb{R} \times \mathbb{R}^{N-1}, -\eta < y_1 \leq 0, |y'| < \eta \}
\]
and
\[
\tilde{Q}^\eta_\eta = \{ y = (y_1, y') \in \mathbb{R} \times \mathbb{R}^{N-1}, |y_1| < \eta, |y'| < \eta \}.
\]

For \( C^2 \) domain \( \Omega \), we recall \( C^2 \) diffeomorphism mapping \( \Phi: \mathbb{R}^N \to \mathbb{R}^N \) that
\[
\Phi(z) = (z', z_1 + \phi(z')) \quad \text{for} \quad z = (z_1, z') \in Q^\eta_\eta.
\]

More properties of \( \Phi \) is given in the proof of Proposition 2.7.

Let \( x_1 \in (0, \eta/4), x = (x_1, 0) \) be a generic point in \( A_{\eta/4} \). By the definition of \( \kappa_\alpha \), we have that
\[
\frac{1}{c_{N,\alpha}} \kappa_\alpha(x) = \int_{\Phi(Q^\eta_\eta)} \frac{1}{|x - y|^{N+2\alpha}} dy + \int_{\Omega^c \setminus \Phi(Q^\eta_\eta)} \frac{1}{|x - y|^{N+2\alpha}} dy,
\]
where \( \Omega^c = \mathbb{R}^N \setminus \Omega \).

On the one hand, we have that \( |x - z| \geq \frac{1}{4} (\eta + |z|) \) for any \( z \in \Omega^c \setminus \Phi(Q^\eta_\eta) \) and then
\[
\int_{\Omega^c \setminus \Phi(Q^\eta_\eta)} \frac{1}{|x - z|^{N+2\alpha}} dz \leq \int_{\mathbb{R}^N} \frac{4^{N+2\alpha}}{(\eta + |z|)^{N+2\alpha}} dy \leq c_{15},
\]
where \( c_{15} > 0 \) depends on \( \eta \).

On the other hand, by change variable \( z = \Phi(y) \), we have that
\[
\int_{\Phi(Q^\eta_\eta)} \frac{1}{|x - z|^{N+2\alpha}} dz = \int_{Q^\eta_\eta} \frac{|D\Phi(y)|}{|x - \Phi(y)|^{N+2\alpha}} dy.
\]

Observe that
\[
|D\Phi(z)| = 1 + O(|z'|),
\]
\[
|x - \Phi(y)|^2 = |x - z + z - \Phi(y)|^2 = |x_1 - y_1|^2 + |y'|^2 + O(|y'|^4),
\]
then
\[
\int_{\Phi(Q)} \frac{1}{|x - z|^{N+2\alpha}} \, dx = \int_{Q} \frac{1 + O(|y'|)}{|x - y|^2 + |y'|^2}^{N+2\alpha} \, dz
\]
\[
= \int_{-\eta}^{0} \frac{1}{|x_1 - y_1|^{1+2\alpha}} \int_{B_{N-1}}^{B_{N-1}(0)} \frac{1 + O(|x_1 - y_1||\zeta'|)}{(1 + |\zeta'|^2)^{N+2\alpha}} \, d\zeta' \, dz_1
\]
\[
= \frac{1}{2\alpha} \int_{R^{N-1}} \frac{1}{(1 + |\zeta'|^2)^{\frac{N+2\alpha}{2}}} d\zeta' \left( x_1^{-2\alpha} - (x_1 + \eta_1)^{-2\alpha} \right)
\]
\[+ O\left( \int_{R^{N-1}} \frac{|\zeta'|}{(1 + |\zeta'|^2)^{\frac{N+2\alpha}{2}}} d\zeta' \varphi(x_1) \right),
\]

where \(B_{N-1}(0)\) is the ball center at origin with radius \(r\) in \(R^{N-1}\), \(\varphi_\alpha(t) = t^{1-2\alpha}\) if \(2\alpha \neq 1\) and \(\varphi_\frac{1}{2}(t) = -\log t\). The proof ends.

\[\square\]

**Remark A.1.** When the domain has the property of uniformly interior tangent ball, i.e. then it has the rough estimate:
\[
\frac{1}{c_{16}} \rho(x)^{-2\alpha} \leq \kappa_\alpha(x) \leq c_{16} \rho(x)^{-2\alpha}, \quad \forall x \in \Omega,
\]

where \(c_{16} > 1\).

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