Abstract

We consider trigonometric solutions of the KP hierarchy. It is known that their poles move as particles of the Calogero-Moser model with trigonometric potential. We show that this correspondence can be extended to the level of hierarchies: the evolution of the poles with respect to the $k$-th hierarchical time of the KP hierarchy is governed by a Hamiltonian which is a linear combination of the first $k$ higher Hamiltonians of the trigonometric Calogero-Moser hierarchy.

1 Introduction

The Kadomtsev-Petviashvili (KP) hierarchy is an archetypal object in the theory of integrable systems. It is an infinite set of compatible nonlinear partial differential equations involving infinitely many variables $t = \{t_1, t_2, t_3, \ldots\}$ (“hierarchical times”). These equations admit a huge number of solutions of very different nature. Among them, of special interest are singular solutions with a finite number of moving poles. In particular, one can consider solutions for which the dependent variables are trigonometric (or hyperbolic) functions of $t_1 = x$ with poles depending on the times $t_k$ with $k \geq 2$ (“trigonometric solitons”).

Dynamics of poles of singular solutions to nonlinear integrable equations is a rather familiar subject in the theory of integrable systems. These studies were initiated by the seminal paper [1], where elliptic and rational solutions to the Korteweg-de Vries and Boussinesq equations were investigated. A remarkable connection with integrable many-body Calogero-Moser systems [2, 3, 4, 5] was observed. Later in [6, 7] it has been shown that this connection becomes most natural for the more general KP equation: the evolution of poles of rational solutions to the KP equation with respect to the time $t_2$ is the Calogero-Moser dynamics of the many-body system with the rational pairwise

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interaction potential \(\frac{1}{(x_i - x_j)^2}\). In [8] Krichever has extended this connection to elliptic (double periodic) solutions expressed through the Weierstrass \(\wp\)-function. The method suggested by Krichever consists in substituting the singular solution not in the KP equation itself but in the auxiliary linear problem for it, using a suitable pole ansatz for the wave function. This method allows one to obtain the equations of motion together with the Lax representation for them.

The further development is Shiota’s work [9]. Shiota has shown that the correspondence between rational solutions to the KP equation and the Calogero-Moser system with rational potential can be extended to the level of hierarchies: namely, the evolution of poles with respect to the higher times \(t_k\) is governed by the higher Hamiltonians \(H_k\) of the integrable Calogero-Moser system. The higher Hamiltonians are defined as traces of powers of the Lax matrix \(L\): \(H_k = \text{tr} L^k\).

Trigonometric and hyperbolic (single-periodic in the complex plane) solutions to the KP hierarchy are not so well studied up to now. In this paper we extend Shiota’s method to this class of solutions. They have the form

\[
u(x, t) = -\sum_{i=1}^{N} \frac{\gamma^2}{\sinh^2(\gamma(x - x_i(t)))},
\]

where \(\gamma\) is a complex parameter. When \(\gamma\) is purely imaginary (respectively, real), one deals with trigonometric (respectively, hyperbolic) solutions. The limit \(\gamma \to 0\) corresponds to rational solutions. We show that the evolution of the poles \(x_i\) with respect to the higher times \(t_k\) is governed by the Hamiltonians

\[
H_k = \frac{1}{2(k + 1)\gamma} \text{tr} \left( (L + \gamma I)^{k+1} - (L - \gamma I)^{k+1} \right)
\]

which are linear combinations of the Hamiltonians \(H_k = \text{tr} L^k\). Here \(I\) is the identity matrix and \(L\) is the Lax matrix given by

\[
L_{ij} = -p_i \delta_{ij} - \frac{(1 - \delta_{ij})\gamma}{\sinh(\gamma(x_i - x_j))}.
\]

In particular,

\[
H_2 = H_2 + \text{const} = \sum_{i=1}^{N} p_i^2 - \sum_{i \neq j} \frac{\gamma^2}{\sinh^2(\gamma(x_i - x_j))} + \text{const}
\]

is the Hamiltonian of the trigonometric Calogero-Moser system. The quantities \(H_k\) (and \(H_k\)) are integrals of motion because the evolution is an isospectral transformation of the Lax matrix.

Our method consists in a direct solution of the auxiliary linear problems for the wave function and its adjoint using a pole ansatz. A similar result was obtained in [10] within a different approach.

The organization of the paper is as follows. In section 2 we very briefly review the KP hierarchy introducing the main notions of the Lax operator, auxiliary linear problems and tau-function. Section 3 is devoted to the trigonometric solutions. We derive the dynamics of their poles in the time \(t_2\). In section 4 the dynamics with respect to the higher times is
considered and the Hamiltonian equations for the higher flows are derived. In section 5 we consider the Bäcklund transformation of the pole dynamics depending on a parameter. In section 6 we prove the determinant formula for the tau-function for trigonometric solutions which expresses it through the initial values of the coordinates and momenta. In the appendix we give some details on the expansion of the Bäcklund transformation in powers of the parameter which yields equations of motion for the higher times.

2 The KP hierarchy

The KP hierarchy can be understood as an infinite set of evolution equations in the times \( t \) for functions of a variable \( x \). Let

\[
\mathcal{L} = \partial_x + \sum_{k \geq 1} u_k \partial_x^{-k}
\]  

(2.1)

be the pseudo-differential Lax operator, where the coefficient functions \( u_k \) are functions of \( x \) and \( t \). The equations of the KP hierarchy for \( u_k \)'s are encoded in the Lax equations

\[
\partial_{tn} \mathcal{L} = [A_m, \mathcal{L}], \quad A_m = (\mathcal{L}^m)_+,
\]  

(2.2)

where \((\ldots)_+\) means taking the purely differential part of a pseudo-differential operator. In particular, we have \( \partial_{t_1} \mathcal{L} = \partial_x \mathcal{L} \), i.e., \( \partial_{t_1} u_k = \partial_x u_k \) for all \( k \geq 1 \). This means that the evolution in \( t_1 \) is simply a shift of \( x \): \( u_k(x, t) = u_k(x + t_1, t_2, t_3, \ldots) \).

An equivalent formulation of the KP hierarchy is through the zero curvature (Zakharov-Shabat) equations

\[
\partial_{t_n} A_m - \partial_{t_m} A_n + [A_m, A_n] = 0.
\]  

(2.3)

The simplest nontrivial equation is obtained at \( m = 2, n = 3 \). It is the famous KP equation for \( u = u_1 \):

\[
3u_{t_2 t_3} = \left( 4u_{t_3} - 12uu_x - u_{xxx} \right)_x.
\]  

(2.4)

The Zakharov-Shabat equations are compatibility conditions for the auxiliary linear problems

\[
\partial_{t_n} \psi = A_m \psi,
\]  

(2.5)

where the wave function \( \psi \) depends on a spectral parameter \( z \): \( \psi = \psi(z; t) \). Together with the wave function \( \psi \) one can introduce the adjoint wave function \( \psi^\dagger \) which satisfies the conjugate linear equations

\[
-\partial_{t_n} \psi^\dagger = A_m^\dagger \psi^\dagger
\]  

(2.6)

The conjugation of a differential operator is performed according to the rule \((f(x) \partial_x^n)^\dagger = (-\partial_x)^n f(x)\). In particular, we have the auxiliary linear problems

\[
\partial_{t_2} \psi = \partial_x^2 \psi + 2u_1 \psi, \quad -\partial_{t_2} \psi^\dagger = \partial_x^2 \psi^\dagger + 2u_1 \psi^\dagger
\]  

(2.7)

which have the form of the non-stationary Schrodinger equation.

A common solution to the KP hierarchy is provided by the tau-function \( \tau = \tau(x, t) \). The whole hierarchy is encoded in the bilinear relation \[11, 12\]

\[
\oint_{\infty} e^{(x-x')z + \xi(t,z) - \xi(t',z')} \tau(x, t - [z^{-1}]) \tau(x', t' + [z^{-1}]) dz = 0
\]  

(2.8)
valid for all \(x, x', t, t'\), where
\[
\xi(t, z) = \sum_{k \geq 1} t_k z^k,
\]
\[
t \pm [z^{-1}] = \{t_1 \pm z^{-1}, t_2 \pm \frac{1}{2} z^{-2}, t_3 \pm \frac{1}{3} z^{-3}, \ldots\}. \quad (2.9)
\]
The integration contour is a big circle around infinity separating the singularities coming from the exponential factor from those coming from the tau-functions. A consequence of the bilinear relation (which is in fact equivalent to the whole hierarchy, see [13]) is the equation
\[
\begin{align*}
\partial_x \tau(x, t + [\lambda^{-1}]) \tau(x, t + [\mu^{-1}]) &- \partial_x \tau(x, t + [\mu^{-1}]) \tau(x, t + [\lambda^{-1}]) \\
&= (\lambda - \mu) \left[ \tau(x, t + [\lambda^{-1}]) \tau(x, t + [\mu^{-1}]) - \tau(x, t + [\lambda^{-1}] + [\mu^{-1}]) \tau(x, t) \right].
\end{align*}
\]
In this form this equation appeared for example in [14, 15, 16]. The differential equations of the hierarchy are obtained by expanding this equation in inverse powers of \(\lambda, \mu\). It is important to note that the tau-functions which differ by an exponential factor of a linear combination of times are equivalent.

The coefficient functions \(u_k\) can be expressed through the tau-function. In particular,
\[
u_1(x, t) = \partial_x^2 \log \tau(x, t). \quad (2.11)
\]
The wave function and its adjoint are expressed through the tau-function according to the formulas
\[
\psi(z; t) = A(z) e^{x z + \xi(t, z)} \frac{\tau(x, t - [z^{-1}])}{\tau(x, t)}, \quad (2.12)
\]
\[
\psi^\dagger(z; t) = A(z) e^{-x z - \xi(t, z)} \frac{\tau(x, t + [z^{-1}])}{\tau(x, t)}, \quad (2.13)
\]
where \(A(z)\) is a normalization factor.

Let us point out another useful corollary of the bilinear relation. Differentiating \((2.8)\) with respect to \(t_m\) and putting \(x = x', t = t'\) after that, we obtain
\[
\frac{1}{2\pi i} \oint z^m \psi^\dagger(z; t) \psi(z; t) dz = \partial_x \partial_x \log \tau(x, t), \quad (2.14)
\]
where the normalization factor is put equal to 1.

3 Trigonometric solutions to the KP equation

For trigonometric solutions the tau-function has the form
\[
\tau = \prod_{i=1}^N (e^{2\gamma x} - e^{2\gamma x_i}), \quad (3.1)
\]
where \(x_i\) depend on \(t\), so that
\[
u_1 = - \sum_i \frac{\gamma^2}{\sinh^2(\gamma(x - x_i))}. \quad (3.2)
\]
These functions have a single period $\pi i/\gamma$ in the complex plane. It is convenient to pass to the exponentiated variables

$$w = e^{2\gamma x}, \quad w_i = e^{2\gamma x_i}, \quad (3.3)$$

then the tau-function becomes a polynomial with the roots $w_i$ and $u_1$ becomes a rational function with double poles at $w_i$:

$$\tau = \prod_i (w - w_i), \quad u_1 = -4\gamma^2 \sum_i \frac{w w_i}{(w - w_i)^2}. \quad (3.4)$$

We begin with investigation of the dynamics of poles in the time $t_2$. According to Krichever’s method, our strategy is to solve the linear problem (2.7) for the $\psi$-function. Equation (2.12) suggests the following ansatz for the wave function:

$$\psi = w^{\frac{1}{2\gamma}} e^{t_1 z + t_2 z^2} \left(1 + \sum_i \frac{2\gamma c_i}{w - w_i}\right), \quad (3.5)$$

where we have put the normalization factor equal to 1 and have put $t_k = 0$ for $k \geq 3$. The coefficients $c_i$ depend on the times (and on $z$). We should substitute expressions (3.4), (3.5) into the linear problem

$$-\partial_{t_2} \psi + 4\gamma^2 w \partial_w w \partial_w \psi + 2u_1 \psi = 0.$$ 

The substitution gives:

$$-\sum_i \frac{\dot{c}_i}{w - w_i} - \sum_i \frac{c_i \dot{w}_i}{(w - w_i)^2} + 8\gamma^2 \sum_i \frac{c_i w^2}{(w - w_i)^3} - 4\gamma z \sum_i \frac{c_i w}{(w - w_i)^2} - 4\gamma^2 \sum_i \frac{c_i w}{(w - w_i)^2} - 8\gamma^2 \left(\sum_i \frac{w w_i}{(w - w_i)^2}\right) \left(\frac{1}{2\gamma} + \sum_k \frac{c_k}{w - w_k}\right) = 0,$$

where dot means the $t_2$-derivative. We should cancel the poles at $w = w_i$. Poles of the third order cancel automatically. The cancellation of the second and first order poles yields the conditions

$$\begin{cases}
\frac{1}{2} \dot{x}_i c_i + (z - \gamma) c_i + 2\gamma \sum_{k \neq i} \frac{w_i c_k}{w_i - w_k} = -w_i \\
\dot{c}_i = 2\gamma \dot{\psi}_i c_i - 8\gamma^2 c_i \sum_{k \neq i} \frac{w_i w_k}{(w_i - w_k)^2} + 8\gamma^2 \sum_{k \neq i} \frac{w_i^2 c_k}{(w_i - w_k)^2} 
\end{cases} \quad (3.6)$$

for $i = 1, \ldots, N$ which are linear equations for the $c_i$’s. In a similar way, the conjugated linear problem (2.7) with the ansatz

$$\psi^\dagger = w^{-\frac{1}{2\gamma}} e^{-t_1 z - t_2 z^2} \left(1 + \sum_i \frac{2\gamma c_i^*}{w - w_i}\right) \quad (3.7)$$

for the adjoint wave function leads to the conditions

$$\begin{cases}
\frac{1}{2} \dot{x}_i c_i^* + (z + \gamma) c_i^* + 2\gamma \sum_{k \neq i} \frac{w_i c_k^*}{w_k - w_i} = w_i \\
\dot{c}_i^* = -2\gamma \dot{x}_i c_i^* - 8\gamma^2 c_i^* \sum_{k \neq i} \frac{w_i w_k}{(w_i - w_k)^2} + 8\gamma^2 \sum_{k \neq i} \frac{w_i^2 c_k^*}{(w_i - w_k)^2}.
\end{cases} \quad (3.8)$$
It is convenient to pass to \( \tilde{c}_i = c_i w_i^{-1/2} \), \( \tilde{c}_i^* = c_i^* w_i^{-1/2} \), then the above conditions can be written in the matrix form

\[
\begin{aligned}
\left((z - \gamma)I - L\right)\tilde{c} &= -W^{1/2}e, \quad \partial_{t_2} \tilde{c} = M\tilde{c}, \\
\tilde{c}^* \left((z + \gamma)I - L\right) &= e^T W^{1/2}, \quad \partial_{t_2} \tilde{c}^* = -\tilde{c}^* \tilde{M},
\end{aligned}
\tag{3.9}
\]

where \( \tilde{c} = (\tilde{c}_1, \ldots, \tilde{c}_N)^T \) is a column vector, \( \tilde{c}^* = (\tilde{c}_1^*, \ldots, \tilde{c}_N^*) \) is a row vector, \( e = (1,1,\ldots,1)^T \) and the matrices \( W, L, M, \tilde{M} \) are

\[
W = \text{diag}(w_1, w_2, \ldots, w_N),
\tag{3.11}
\]

\[
L_{ij} = -\frac{1}{2} \delta_{ij} \dot{x}_i - 2\gamma (1 - \delta_{ij}) \frac{w_i^{1/2} w_j^{1/2}}{w_i - w_j},
\tag{3.12}
\]

\[
M_{ij} = \gamma \dot{x}_i \delta_{ij} - 8\gamma^2 \delta_{ij} \sum_{k \neq i} \frac{w_i w_k}{(w_i - w_k)^2} + 8\gamma^2 (1 - \delta_{ij}) \frac{w_i^{3/2} w_j^{1/2}}{(w_i - w_j)^2},
\tag{3.13}
\]

\[
\tilde{M}_{ij} = -2\gamma \dot{x}_i \delta_{ij} + M_{ji}.
\tag{3.14}
\]

It is straightforward to check the following basic commutation relation:

\[
[L, W] = 2\gamma \left(W^{1/2} E W^{1/2} - W\right),
\tag{3.15}
\]

where \( E = e \otimes e^T \) is the rank 1 matrix with matrix elements \( E_{ij} = 1 \).

The linear system \((3.9)\) is overdetermined. A simple calculation shows that the compatibility condition for this system is

\[
\dot{L} + [L, M] = 0.
\tag{3.16}
\]

We write \( L = -\frac{1}{2} \dot{X} - A, M = \gamma \dot{X} - 2D + 2B \), where the matrices \( A, B, D, X \) are given by

\[
A_{ik} = 2\gamma (1 - \delta_{ik}) \frac{w_i^{1/2} w_k^{1/2}}{w_i - w_k}, \quad B_{ik} = 4\gamma^2 (1 - \delta_{ik}) \frac{w_i^{3/2} w_k^{1/2}}{(w_i - w_k)^2},
\]

\[
D_{ik} = 4\gamma^2 \delta_{ik} \sum_{l \neq i} \frac{w_l w_i}{(w_i - w_l)^2}, \quad X_{ik} = \delta_{ik} x_i.
\]

We have:

\[
\dot{L} + [L, M] = -\frac{1}{2} \dot{X} - \dot{A} - [X, B] + \gamma [\dot{X}, A] + 2[A, D] - 2[A, B].
\]

It can be easily checked that \( \dot{A} + [\dot{X}, B - \gamma A] = 0 \) and \( ([A, B] - [A, D])_{ik} = 0 \) at \( i \neq k \).

Therefore, the compatibility condition reduces to \( \dot{X}_{ii} + 4[A, B]_{ii} = 0 \) or

\[
\ddot{x}_i = -32\gamma^3 \sum_{j \neq i} \frac{w_i w_j (w_i + w_j)}{(w_i - w_j)^3} = -8\gamma^3 \sum_{j \neq i} \frac{\cosh(\gamma (x_i - x_j))}{\sinh^3(\gamma (x_i - x_j))},
\tag{3.17}
\]

These are equations of motion of the trigonometric Calogero-Moser model. Equation \((3.16)\) is their Lax representation. It states that the evolution of the Lax matrix \( L \) is isospectral. Therefore, \( H_k = \text{tr} L^k \), \( k = 1, \ldots, N \), are \( N \) independent integrals of motion. It can be proved (see \cite{17}, section 3.2) that they are in involution, so the system is integrable. Introducing the momenta \( p_i = \frac{1}{2} \dot{x}_i \), we write the Lax matrix \((3.12)\) in the form \((1.2)\), then the Calogero-Moser Hamiltonian \( H_2 \) is given by \((1.3)\).
4 Dynamics of poles in higher times

Our main tool in this section is the relation (2.14) which, after substitution of the wave functions and tau-function in the form specific for trigonometric solutions, acquires the form

\[
\frac{1}{2\pi i} \oint_{\gamma} z^m \left(1 + \sum_i \frac{2\gamma c_i^2}{w - w_i} \right) \left(1 + \sum_k \frac{2\gamma c_k}{w - w_k} \right) \, dz = 4\gamma^2 \sum_i \frac{ww_i \partial_{t_m} x_i}{(w - w_i)^2}.
\]

The both sides are rational functions of \(w\) with poles at \(w = w_i\) vanishing at infinity. Identifying the coefficients in front of the second order poles at \(w = w_i\), we obtain

\[
\partial_{t_m} x_i = \frac{1}{2\pi i} \oint_{\gamma} z^m \bar{c}_i^* w_i^{-1} \bar{c}_i \, dz.
\]

(Comparison of the first order poles leads to the same relation.) From (3.9), (3.10) we conclude that

\[
\bar{c} = -(zI - (L + \gamma I))^{-1} W^{1/2} e, \quad \bar{c}^* = e^T W^{1/2} (zI - (L - \gamma I))^{-1},
\]

and, therefore, (4.1) reads

\[
\partial_{t_m} x_i = - \operatorname{res}_\infty \sum_{k,k'} \left( z^m w_k^{1/2} \left( \frac{1}{zI - (L - \gamma I)} \right) w_i^{-1} \left( \frac{1}{zI - (L + \gamma I)} \right) \right) \bar{c}_i \bar{c}_k \bar{c}_{k'} \bar{c}_i \bar{c}_{k'}.
\]

where we imply the convention \(\operatorname{res}_\infty (z^{-n}) = \delta_{n1}\) and \(E_i\) is the diagonal matrix with matrix elements \((E_i)_{jk} = \delta_{ij}\delta_{ik}\). Obviously, \(E_i = -\partial L / \partial p_i\). Using the commutation relation (3.13), we write:

\[
\partial_{t_m} x_i = \frac{1}{2\gamma} \operatorname{res}_\infty \left( z^m (LW - WL) + 2\gamma W \right) \left( \frac{1}{zI - (L - \gamma I)} \right) \left( \frac{1}{zI - (L + \gamma I)} \right)
\]

Let us calculate

\[
\begin{align*}
\operatorname{tr} \left( (LW - WL) \frac{1}{zI - (L - \gamma I)} \right) \frac{1}{zI - (L + \gamma I)} W^{-1} \frac{\partial L}{\partial p_i} \frac{1}{zI - (L + \gamma I)}
\end{align*}
\]

\[
= \operatorname{tr} \left( \frac{1}{zI - (L - \gamma I)} W^{-1} \frac{\partial L}{\partial p_i} \frac{L}{zI - (L + \gamma I)} \right) - \operatorname{tr} \left( \frac{1}{zI - (L - \gamma I)} W^{-1} \frac{\partial L}{\partial p_i} \frac{1}{zI - (L + \gamma I)} \right)
\]

\[
= -\operatorname{tr} \left( \frac{1}{zI - (L - \gamma I)} W^{-1} \frac{\partial L}{\partial p_i} \frac{1}{zI - (L + \gamma I)} \right) \frac{1}{zI - (L + \gamma I)}
\]

\[
+(z - \gamma) \operatorname{tr} \left( \frac{1}{zI - (L - \gamma I)} W^{-1} \frac{\partial L}{\partial p_i} \frac{1}{zI - (L + \gamma I)} \right)
\]

\[
-(z + \gamma) \operatorname{tr} \left( \frac{1}{zI - (L - \gamma I)} W^{-1} \frac{\partial L}{\partial p_i} \frac{1}{zI - (L + \gamma I)} \right)
\]

\]

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and use equations (3.9), (3.10). In this way we get:

\[-2\gamma \text{tr} \left( W zI - (L - \gamma I) W^{-1} \frac{\partial L}{\partial p_i} zI - (L + \gamma I) \right)\]

(it is taken into account that the diagonal matrices \( W \) and \( \partial L/\partial p_i \) commute). Therefore, we get from (4.3):

\[
\partial_{t_m} x_i = \frac{1}{2 \gamma} \text{res tr} \left[ z^m \frac{\partial L}{\partial p_i} \left( \frac{1}{zI - (L + \gamma I)} - \frac{1}{zI - (L - \gamma I)} \right) \right]
\]

\[
= \frac{1}{2 \gamma} \text{tr} \left( \frac{\partial L}{\partial p_i} (L + \gamma I)^m - \frac{\partial L}{\partial p_i} (L - \gamma I)^m \right) \quad \text{(4.4)}
\]

\[
= \frac{1}{2(m+1) \gamma} \frac{\partial}{\partial p_i} \text{tr} \left( (L + \gamma I)^{m+1} - (L - \gamma I)^{m+1} \right) = \frac{\partial H_m}{\partial p_i},
\]

where

\[
H_m = \frac{1}{2(m+1) \gamma} \text{tr} \left( (L + \gamma I)^{m+1} - (L - \gamma I)^{m+1} \right)
\]

\[
\quad = H_m + \sum_{k \geq 1} \frac{m!}{(m-2k)! (2k+1)!} \gamma^{2k} H_{m-2k}.
\]

In this way we have obtained one part of the Hamiltonian equations for the higher time flows. We see that the Hamiltonian corresponding to the \( m \)-th flow is a linear combination of the Calogero-Moser Hamiltonians \( H_m, H_{m-2}, H_{m-4}, \ldots \). For example, \( H_3 = H_3 + \gamma^2 H_1, H_4 = H_4 + 2\gamma^2 H_2 + \frac{\gamma^4}{5}, H_5 = H_5 + \frac{10}{3} \gamma^2 H_3 + \gamma^4 H_1 \) and so on. In the case \( \gamma \to 0 \) (rational solutions) Shiota’s result is reproduced.

In order to obtain the remaining part of the Hamiltonian equations we differentiate (4.1) with respect to \( t_2 \):

\[
\partial_{t_m} \dot{x}_i = -2\gamma \text{res} \left( z^m \bar{c}_i^* \bar{c}_j w_i^{-1} \bar{c}_i \right) + \text{res} \left( z^m (\bar{c}_i^* w_i^{-1} \partial_{t_2} \bar{c}_i + \partial_{t_2} \bar{c}_i^* w_i^{-1} \bar{c}_i) \right)
\]

and use equations (3.9), (3.10). In this way we get:

\[
\partial_{t_m} p_i = \frac{1}{2} \partial_{t_m} \dot{x}_i = \frac{1}{2} \sum_k \text{res} \left( z^m \left( \bar{c}_i^* w_i^{-1} M_{ik} \bar{c}_k - \bar{c}_k^* M^T_{ki} w_i^{-1} \bar{c}_i \right) \right)
\]

\[
= - \text{res} \left[ z^m \text{tr} \left( W^{1/2} E W^{1/2} \frac{1}{zI - (L - \gamma I)} \frac{1}{zI - (L + \gamma I)} \right) \right]
\]

\[
= - \frac{1}{2\gamma} \text{res tr} \left( z^m (LW - WL + 2\gamma W) \frac{1}{zI - (L - \gamma I)} G^{(i)} \frac{1}{zI - (L + \gamma I)} \right),
\]

where the matrix \( G^{(i)} \) is \( G^{(i)} = \frac{1}{2} (W^{-1} E_i M - M^T W^{-1} E_i) \). Its matrix elements are

\[
G^{(i)}_{jk} = 4\gamma^2 w_j^{1/2} w_k^{1/2} \left( \delta_{ij} - \delta_{ik} \right).
\]

(4.6)
A calculation similar to the one done above in this section shows that
\[
\partial_t m p_i = -\frac{1}{2\gamma} \text{res}_\infty \left[ z^m \text{tr} \left( W G^{(i)} \frac{1}{z I - (L + \gamma I)} - G^{(i)} W \frac{1}{z I - (L - \gamma I)} \right) \right]
\]
\[
= -\frac{1}{4\gamma} \text{res}_\infty \left[ z^m \text{tr} \left( (W G^{(i)} + G^{(i)} W) \left( \frac{1}{z I - (L + \gamma I)} - \frac{1}{z I - (L - \gamma I)} \right) \right) \right]
\]
\[
- \frac{1}{4\gamma} \text{res}_\infty \left[ z^m \text{tr} \left( (W G^{(i)} - G^{(i)} W) \left( \frac{1}{z I - (L + \gamma I)} + \frac{1}{z I - (L - \gamma I)} \right) \right) \right]
\]
\[(4.7)\]

The last line here is actually equal to zero because
\[
(W G^{(i)} - G^{(i)} W)_{jk} = -2\gamma L_{jk}(\delta_{ij} - \delta_{ik})
\]
and so
\[
\text{tr}((W G^{(i)} - G^{(i)} W)L^m) = -2\gamma \sum_{j,l} L_{jl}(\delta_{ij} - \delta_{il})(L^m)_{lj}
\]
\[
= -2\gamma \sum_i L_{ii}(L^m)_{ii} + 2\gamma \sum_j (L^m)_{ij} L_{ji} = 2\gamma((L^{m+1})_{ii} - (L^{m+1})_{ii}) = 0
\]
for any integer \( m \). Next, it is not difficult to prove the identity
\[
W G^{(i)} + G^{(i)} W = 2 \frac{\partial L}{\partial x_i}.
\]
\[(4.8)\]

Therefore, we obtain:
\[
\partial_t m p_i = -\frac{1}{2\gamma} \text{res}_\infty \left[ z^m \text{tr} \left( \frac{\partial L}{\partial x_i} \left( \frac{1}{z I - (L + \gamma I)} - \frac{1}{z I - (L - \gamma I)} \right) \right) \right]
\]
\[
= -\frac{1}{2\gamma} \text{tr} \left( \frac{\partial L}{\partial x_i} (L + \gamma I)^m \right) - \frac{\partial L}{\partial x_i} (L - \gamma I)^m \]
\[
= -\frac{\partial H_m}{\partial x_i}
\]
\[(4.9)\]

which is the remaining part of the Hamiltonian equations.

5 Bäcklund transformation and equations of motion in the higher times

Instead of dynamics of poles of the wave function one can consider equations which connect the dynamics of poles of the wave function with dynamics of its zeros. This yields a transformation of the Calogero-Moser system of the Bäcklund type.

According to (2.12) we have \( \psi(\mu, t) = e^{\tau + \xi(t, \mu)} \hat{\tau}(t)/\tau(t) \), where \( \hat{\tau}(t) = \tau(t - [\mu^{-1}]) \). In terms of \( \tau \) and \( \hat{\tau} \) the auxiliary linear problem (2.7) becomes
\[
\partial_{t_2} \log \frac{\hat{\tau}}{\tau} = (\partial_{x_2} \log \frac{\hat{\tau}}{\tau})^2 + 2\mu \partial_{x_2} \log \frac{\hat{\tau}}{\tau}.
\]
\[(5.1)\]
For trigonometric solutions

\[ \tau = \prod_i \left( e^{2\gamma x} - e^{2\gamma y} \right), \quad \hat{\tau} = \prod_i \left( e^{2\gamma x} - e^{2\gamma y} \right) \]  

(5.2)

and in terms of the variables \( w = e^{2\gamma x}, \, w_i = e^{2\gamma x_i}, \, v_i = e^{2\gamma y_i} \) equation (5.1) acquires the form

\[
\sum_i \frac{(\dot{y}_i + 2\mu - 2\gamma)v_i}{w - v_i} - \sum_i \frac{(\dot{x}_i + 2\mu + 2\gamma)v_i}{w - w_i} - 2\gamma \sum_i \frac{w_i^2}{(w - w_i)^2} - 2\gamma \sum_i \frac{v_i^2}{(w - v_i)^2} + 2\gamma \left( \sum_i \frac{v_i}{w - v_i} - \sum_i \frac{w_i}{w - w_i} \right)^2 = 0,
\]

where dot means the \( t_2 \)-derivative. The second order poles cancel identically. Equating residues at the poles at \( w = w_i \) and \( w = v_i \) to zero, we get the system of equations

\[
\begin{align*}
p_i &= -\mu + \gamma \sum_{k \neq i} \frac{w_i + w_k}{w_i - w_k} - \gamma \sum_k \frac{w_i + v_k}{w_i - v_k} \\
\tilde{p}_i &= -\mu - \gamma \sum_{k \neq i} \frac{v_i + v_k}{v_i - v_k} + \gamma \sum_k \frac{v_i + w_k}{v_i - w_k}
\end{align*}
\]

(here \( p_i = \frac{1}{2} \dot{x}_i, \, \tilde{p}_i = \frac{1}{2} \dot{y}_i \)) or

\[
\begin{align*}
p_i &= -\mu + \gamma \sum_{k \neq i} \coth(\gamma(x_i - x_k)) - \gamma \sum_k \coth(\gamma(x_i - y_k)) \\
\tilde{p}_i &= -\mu - \gamma \sum_{k \neq i} \coth(\gamma(y_i - y_k)) + \gamma \sum_k \coth(\gamma(y_i - x_k)).
\end{align*}
\]

(5.3)

Note that this is a canonical transformation \((p_i, x_i) \to (\tilde{p}_i, y_i)\) with the generating function

\[
F = \sum_{i < j} \log \left[ \sinh(\gamma(x_i - x_j)) \sinh(\gamma(y_i - y_j)) \right] - \sum_{i, j} \log \sinh(\gamma(x_i - y_j)) - \mu \sum_i (x_i - y_i)
\]

(5.4)

since we have \( p_i = \partial F/\partial x_i, \, \tilde{p}_i = -\partial F/\partial y_i \). Equations (5.3) appeared in [18] under the name of Bäcklund transformation and in [19] under the name of self-dual equations of motion.

Let us introduce the differential operator

\[
D(\mu) = \sum_{k \geq 1} \frac{\mu^{-k}}{k^2} \partial_{x_k},
\]

(5.5)

then \( \hat{\tau} = e^{-D(\mu)} \tau \) and \( y_i = e^{-D(\mu)} x_i, \, \tilde{p}_i = e^{-D(\mu)} p_i \) and equations (5.3) can be written in the form

\[
\begin{align*}
p_i &= -\mu + \gamma \sum_{k \neq i} \coth(\gamma(x_i - x_k)) - \gamma \sum_k \coth(\gamma(x_i - e^{-D(\mu)} x_k)) \\
p_i &= -\mu - \gamma \sum_{k \neq i} \coth(\gamma(x_i - x_k)) + \gamma \sum_k \coth(\gamma(x_i - e^{D(\mu)} x_k))
\end{align*}
\]

(5.6)
(the second equation here is obtained from the second one in (5.3) by an overall shift of times). Subtracting the two equations in (5.6), we get
\[ \sum_k \coth(\gamma(x_i - e^{-D(\mu)}x_k)) + \sum_k \coth(\gamma(x_i - e^{D(\mu)}x_k)) - 2 \sum_{k \neq i} \coth(\gamma(x_i - x_k)) = 0. \]
(5.7)
These equations comprise the generating form of all equations of motion of the Calogero-Moser system for all higher flows with Hamiltonians \( \mathcal{H}_k \). Namely, the equations of motion are obtained by expansion of (5.6) and (5.7) in powers of \( \mu \). Some details of this expansion are given in the appendix.

6 The tau-function

In this section we prove that the tau-function for the trigonometric solutions to the KP hierarchy is given by the determinant formula
\[
\tau(t) = \det_{N \times N} \left( wI - \exp\left( - \sum_{k \geq 1} t_k \mathcal{L}_k \right) W_0 \right),
\]
(6.1)
where \( W_0 = W(0) \),
\[
\mathcal{L}_k = (L_0 + \gamma I)^k - (L_0 - \gamma I)^k, \quad L_0 = L(0).
\]
For the proof that the eigenvalues of the matrix \( \exp\left( - \sum_{k \geq 1} t_k \mathcal{L}_k \right) W_0 \) are \( e^{2\gamma x_i} \), where \( x_i \) are coordinates of the Calogero-Moser particles (as functions of the \( t_k \)'s under the Hamiltonian flows with the Hamiltonians \( \mathcal{H}_k \)), see [20].

Let \( V \) be a diagonalizing matrix for \( \exp\left( - \sum_{k \geq 1} t_k \mathcal{L}_k \right) W_0 \):
\[
V \exp\left( - \sum_{k \geq 1} t_k \mathcal{L}_k \right) W_0 V^{-1} = W,
\]
where \( W \) is diagonal. It is defined up to the left multiplication by a diagonal matrix. We fix this freedom by the condition
\[
V W_0^{-1/2} e = W^{-1/2} e.
\]
We know that the matrices \( W_0, L_0 \) satisfy the commutation relation (3.15) which we write here in the form
\[
W_0^{-1/2} L_0 W_0^{1/2} - W_0^{1/2} L_0 W_0^{-1/2} = 2\gamma (E - I).
\]
(6.2)
Let us prove that the matrices \( W \) and \( L = VL_0V^{-1} \) satisfy the same commutation relation. We have, following [20]:
\[
W^{-1/2} L W^{-1/2} - W^{1/2} L W^{-1/2} = W^{1/2} (W^{-1} L W - L) W^{-1/2}
\]
\[
= W^{1/2} V W_0^{-1/2} (W_0^{-1/2} L_0 W_0^{1/2} - W_0^{1/2} L_0 W_0^{-1/2}) W_0^{1/2} V^{-1} W^{-1/2}
\]
\[
= 2\gamma W^{1/2} V W_0^{-1/2} (e \otimes e^T - I) W_0^{1/2} V^{-1/2} W^{-1/2} \\
= 2\gamma \left( W^{1/2} V W_0^{-1/2} e \otimes e^T W_0^{1/2} V^{-1/2} W^{-1/2} - I \right).
\]

Since the diagonal elements of the matrix in the left hand side are equal to 0, we conclude that \(e^T W^{1/2} V^{-1} = e^T W^{1/2}\) and obtain \(2\gamma (e \otimes e^T - I)\) in the right hand side.

Below we will prove that the function (6.1) satisfies the bilinear relation (2.10) which differs from (2.10) by a shift of the time variables.

Substituting everything into (6.3), we write the left hand side of (6.3) (divided by \(\det(I)\))

\[
= \frac{\partial \tau(t + [\lambda^{-1}] - [\mu^{-1}])}{\tau(t)} - \frac{\partial \tau(t)}{\tau(t)} \left( \frac{\tau(t + [\lambda^{-1}] - [\mu^{-1}])}{\tau(t)} - \frac{\tau(t)}{\tau(t)} \left( \frac{\tau(t)}{\tau(t)} - \frac{\tau(t - [\mu^{-1}])}{\tau(t)} \right) \right) = 0
\]

which differs from (2.10) by a shift of the time variables.

Performing the similarity transformation with the diagonalizing matrix \(V\) under the determinant (6.1), we have:

\[
\tau(t) = \det(w - W), \\
\tau(t + [\lambda^{-1}]) = \det\left( wI - \frac{\lambda - \gamma}{\lambda + \gamma} I - L \right)
\]

\[
= \frac{\det\left( ((\lambda + \gamma) I - L)(wI - W) + 2\gamma W \right)}{\det((\lambda + \gamma)I - L)} = \frac{\det\left( (wI - W)((\lambda + \gamma) I - L) + 2\gamma \tilde{E} \right)}{\det((\lambda + \gamma)I - L)},
\]

where \(\tilde{E} = W^{1/2} EW^{1/2}\) and we used the commutation relation (3.15). Using the formula \(\det(I + A) = 1 + \text{tr} A\) valid for any rank 1 matrix \(A\), we get:

\[
\tau(t + [\lambda^{-1}]) = \tau(t) \left( 1 + 2\gamma \text{tr}\left[ ((\lambda + \gamma) I - L)^{-1} (wI - W)^{-1} \tilde{E} \right] \right).
\]

In a similar way, we obtain:

\[
\tau(t - [\mu^{-1}]) = \tau(t) \left( 1 - 2\gamma \text{tr}\left[ (wI - W)^{-1} ((\mu - \gamma) I - L)^{-1} \tilde{E} \right] \right).
\]

At last, for the tau-function \(\tau(t + [\lambda^{-1}] - [\mu^{-1}])\) we have:

\[
\tau(t + [\lambda^{-1}] - [\mu^{-1}]) = \det\left( wI - \frac{\lambda - \gamma}{\lambda + \gamma} I - L \left( \frac{\mu + \gamma}{\mu - \gamma} I - L \right) W \right)
\]

\[
= \det\left( wI - \frac{\lambda - \gamma}{\lambda + \gamma} I - L \left( \frac{\mu + \gamma}{\mu - \gamma} I - L \right) W \right)
\]

\[
= \frac{\det\left( ((\mu - \gamma) I - L)(wI - W)((\lambda + \gamma) I - L) - 2\gamma(\lambda - \mu) \tilde{E} \right)}{\det((\lambda + \gamma)I - L)\det((\mu - \gamma)I - L)}
\]

\[
= \tau(t) \left( 1 - 2\gamma(\lambda - \mu) \text{tr}\left[ ((\lambda + \gamma) I - L)^{-1} (wI - W)^{-1} ((\mu - \gamma) I - L)^{-1} \tilde{E} \right] \right).
\]

Substituting everything into (6.3), we write the left hand side of (6.3) (divided by \(\lambda - \mu\)) in the form

\[
\text{LHS of (6.3)} \propto 2\gamma \text{tr}\left( \frac{1}{(\lambda + \gamma) I - L} \frac{w}{(wI - W)^2} \frac{1}{(\mu - \gamma) I - L} \tilde{E} \right)
\]

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\[(\lambda - \mu) \text{tr} \left( \frac{1}{(\lambda + \gamma)I - L} \frac{1}{wI - W} \frac{1}{(\mu - \gamma)I - \tilde{E}} \right) \]
\[\begin{array}{c}
- \text{tr} \left( \frac{1}{wI - W} \frac{1}{(\mu - \gamma)I - L} \tilde{E} \right) + \text{tr} \left( \frac{1}{(\lambda + \gamma)I - L} \frac{1}{wI - W} \tilde{E} \right) \\
- 2\gamma \text{tr} \left( \frac{1}{wI - W} \frac{1}{(\mu - \gamma)I - L} \tilde{E} \right) \text{tr} \left( \frac{1}{(\lambda + \gamma)I - L} \frac{1}{wI - W} \tilde{E} \right).
\end{array}\]

It is a rational function of \(w\) with second and first order poles at \(w = w_i\) vanishing at infinity. It is easy to see that poles of the second order cancel. The analysis of the first order poles is more complicated. The residue at the pole at \(w = w_i\) is equal to

\[(\lambda - \mu + 2\gamma) \sum_{j,k} \left( \frac{1}{(\lambda + \gamma)I - L} \right)_{ij} \left( \frac{1}{(\mu - \gamma)I - L} \right)_{ik} w_i^{1/2} w_j^{1/2} \]
\[\begin{array}{c}
- \sum_j \left( \frac{1}{(\mu - \gamma)I - L} \right)_{ij} w_i^{1/2} w_j^{1/2} + \sum_j \left( \frac{1}{(\lambda + \gamma)I - L} \right)_{ji} w_i^{1/2} w_j^{1/2} \\
- 2\gamma \sum_{j \neq i} \sum_k \sum_{k'} \left( w_i^{1/2} w_j^{1/2} w_i^{1/2} w_j^{1/2} \right) \left[ \left( \frac{1}{(\mu - \gamma)I - L} \right)_{ik} \left( \frac{1}{(\lambda + \gamma)I - L} \right)_{k'j} \right] \\
+ \left( \frac{1}{(\mu - \gamma)I - L} \right)_{jk} \left( \frac{1}{(\lambda + \gamma)I - L} \right)_{k'i} \right].
\end{array}\]

Since \(L_{ij}\) are given by (3.12), the last triple sum is equal to

\[-2\gamma \sum_{k,k'} \left( \frac{1}{(\mu - \gamma)I - L} \right)_{ik} \left( \frac{L}{(\lambda + \gamma)I - L} \right)_{k'i} \]
\[\begin{array}{c}
- \left( \frac{L}{(\mu - \gamma)I - L} \right)_{ik} \left( \frac{1}{(\lambda + \gamma)I - L} \right)_{k'i} \right] w_k^{1/2} w_{k'}^{1/2} \\
= \sum_k \left( w_i^{1/2} w_k^{1/2} \right) \left( \frac{1}{(\mu - \gamma)I - L} \right)_{ik} - \sum_k \left( w_i^{1/2} w_k^{1/2} \right) \left( \frac{1}{(\lambda + \gamma)I - L} \right)_{ki} \\
- (\lambda - \mu + 2\gamma) \sum_{k,k'} \left( \frac{1}{(\mu - \gamma)I - L} \right)_{ik} \left( \frac{1}{(\lambda + \gamma)I - L} \right)_{k'i} w_k^{1/2} w_{k'}^{1/2}
\end{array}\]

and thus the residue is equal to 0. We have proved that the left hand side of (6.3) vanishes and, therefore, the function (6.1) is indeed the tau-function of the KP hierarchy.

In fact this also follows from the result of Kasman and Gekhtman [21] (see also [10]): for any square matrices \(X, Y, Z\) such that the matrix \(XZ - YX\) has rank 1 the function

\[
\tau = \det \left( X \exp \left( \sum_{k \geq 1} t_k Z^k \right) + \exp \left( \sum_{k \geq 1} t_k Y^k \right) \right) \quad (6.6)
\]

is a tau-function of the KP hierarchy. In our case \(X = -W_0, Z = L_0 - \gamma I, Y = L_0 + \gamma I\) and the condition that \(XZ - YX\) has rank 1 is equivalent to the commutation relation (6.2). Nevertheless, we found it instructive to give an independent direct proof.
7 Conclusion

In this paper we have shown, using basically the method developed by Shiota for rational solutions, that the hierarchy of the KP equations for solutions with trigonometric dependence on $t_1$ generates the hierarchy of the trigonometric Calogero-Moser dynamical equations for poles of the solutions. The KP hierarchical flow $t_k$ gives rise to the Hamiltonian flow with the Hamiltonian $H_k$ which is an explicitly known linear combination of the first $k$ Hamiltonians $H_m = \text{tr} L^m$ of the Calogero-Moser system. Therefore, there is an important difference with the rational case considered by Shiota, where the Hamiltonian for the $k$-th flow is $H_k$ itself.

A natural unsolved problem is to extend these results to elliptic (double periodic in the complex plane) solutions to the KP hierarchy whose poles are known to move as Calogero-Moser particles with elliptic interaction potential. The $t_2$ and $t_3$ flows are studied in [8, 22].

8 Appendix: expansion of equations (5.6)

Let us give some details of the expansion of equations (5.6) in powers of $\mu$. For brevity, denote

$$c(x) = \gamma \coth(\gamma x), \quad c(x) = \frac{1}{x} + \frac{\gamma^2}{3} x + O(x^3) \quad \text{as} \ x \to 0.$$  

We also expand

$$e^{D(\mu)} - 1 = \sum_{k \geq 1} h_k(\tilde{\partial}) \mu^{-k},$$

where $h_k(t)$ are Schur polynomials and $\tilde{\partial} = \{\partial_{t_1}, \frac{1}{2} \partial_{t_2}, \frac{1}{3} \partial_{t_3}, \ldots\}$. The first few operators $h_k(\tilde{\partial})$ are:

$$h_1(\tilde{\partial}) = \partial_{t_1},$$
$$h_2(\tilde{\partial}) = \frac{1}{2} (\partial_{t_2} + \partial_{t_1}^2),$$
$$h_3(\tilde{\partial}) = \frac{1}{6} (2 \partial_{t_3} + 3 \partial_{t_2} \partial_{t_1} + \partial_{t_1}^3),$$
$$h_4(\tilde{\partial}) = \frac{1}{24} (6 \partial_{t_4} + 8 \partial_{t_3} \partial_{t_1} + 3 \partial_{t_2}^2 + 6 \partial_{t_2} \partial_{t_1}^2 + \partial_{t_1}^4).$$

Their action to the $x_i$’s is as follows: $h_1(\tilde{\partial}) x_i = -1$ (because the solution essentially depends only on $x + t_1$), $h_2(\tilde{\partial}) x_i = \frac{1}{2} \dot{x}_i$, $h_3(\tilde{\partial}) x_i = \frac{1}{3} \partial_{t_3} x_i$, $h_4(\tilde{\partial}) x_i = \frac{1}{4} \partial_{t_4} x_i + \frac{1}{3} \ddot{x}_i$.

Expanding the second equation in (5.6), we get:

$$p_i = -c\left(e^{D(\mu)} - 1\right) x_i - \mu + \sum_{j \neq i} c(x_{ij} - (e^{D(\mu)} - 1) x_j) - c(x_{ij})$$

$$= c\left(\mu^{-1} - \sum_{k \geq 2} h_k(\tilde{\partial}) x_i \mu^{-k}\right) - \mu$$

$$- \sum_{j \neq i} c'(x_{ij}) \left(\sum_{k \geq 1} h_k(\tilde{\partial}) x_j \mu^{-k}\right) - \frac{1}{2} c''(x_{ij}) \left(\sum_{k \geq 1} h_k(\tilde{\partial}) x_j \mu^{-k}\right)^2 + \ldots$$

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\[ h_2(\ddot{\vartheta}) x_i + \left( h_3(\ddot{\vartheta}) x_i + (h_2(\ddot{\vartheta}) x_i)^2 + \sum_{j \neq i} c'(x_{ij}) + \frac{\gamma^2}{3} \right) \mu^{-1} \]

\[ + \left( h_4(\ddot{\vartheta}) x_i + 2h_2(\ddot{\vartheta}) x_i h_3(\ddot{\vartheta}) x_i + (h_2(\ddot{\vartheta}) x_i)^3 \right) \]

\[ - \sum_{j \neq i} \left( (h_2(\ddot{\vartheta}) x_j c'(x_{ij}) - \frac{1}{2} c''(x_{ij})) - \frac{\gamma^2}{6} h_2(\ddot{\vartheta}) x_i \right) \mu^{-2} + O(\mu^{-3}) \]

(here \( x_{ij} = x_i - x_j \)). The similar expansion of the first equation in (5.6) is

\[ p_i = -h_2(-\ddot{\vartheta}) x_i + \left( \gamma^2 \right) \mu^{-1} \]

\[ + \left( -h_4(-\ddot{\vartheta}) x_i + 2h_2(-\ddot{\vartheta}) x_i h_3(-\ddot{\vartheta}) x_i - (h_2(-\ddot{\vartheta}) x_i)^3 \right) \]

\[ + \sum_{j \neq i} \left( (h_2(-\ddot{\vartheta}) x_j c'(x_{ij}) - \frac{1}{2} c''(x_{ij})) + \frac{\gamma^2}{6} h_2(-\ddot{\vartheta}) x_i \right) \mu^{-2} + O(\mu^{-3}). \]

Matching the coefficients in front of powers of \( \mu \), we get the relation \( p_i = \frac{1}{2} \dot{x}_i \), the equations of motion \( \ddot{x}_i = -4 \sum_{j \neq i} c'(x_{ij}) \) and the Hamiltonian equations

\[ \partial_{t_i} x_i = -3p_i^2 - 3 \sum_{j \neq i} c'(x_{ij}) - \gamma^2 = \frac{\partial}{\partial p_i} (H_3 + \gamma^2 H_1), \]

\[ \partial_{t_i} x_i = 4p_i^3 + 4 \sum_{j \neq i} (2p_i + p_j)c'(x_{ij}) + 4\gamma^2 p_i = \frac{\partial}{\partial p_i} (H_4 + 2\gamma^2 H_2), \]

where

\[ H_1 = - \sum_i p_i, \]

\[ H_2 = \sum_i p_i^2 + \sum_{i \neq j} c'(x_{ij}), \]

\[ H_3 = - \sum_i p_i^3 - 3 \sum_{i \neq j} p_i c'(x_{ij}), \]

\[ H_4 = \sum_i p_i^4 + \sum_{i \neq j} \left( 4p_i^2 + 2p_i p_j \right)c'(x_{ij}) + 2 \sum_{i \neq j \neq k} c'(x_{ij}) c'(x_{jk}) + \sum_{i \neq j} \left( c'(x_{ij}) \right)^2. \]

This is in full agreement with the result of section 4.

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