Adiabatic formation of bound states in the 1d Bose gas

Rebekka Koch,1 Alvise Bastianello,1,2,3 and Jean-Sébastien Caux1

1Institute for Theoretical Physics, University of Amsterdam, PO Box 94485, 1090 GL Amsterdam, The Netherlands
2Department of Physics and Institute for Advanced Study, Technical University of Munich, 85748 Garching, Germany
3Munich Center for Quantum Science and Technology (MCQST), Schellingstr. 4, D-80799 München, Germany

We consider the 1d interacting Bose gas in the presence of time-dependent and spatially inhomogeneous contact interactions. Within its attractive phase, the gas allows for bound states of an arbitrary number of particles, which are eventually populated if the system is dynamically driven from the repulsive to the attractive regime. Building on the framework of Generalized Hydrodynamics, we analytically determine the formation of bound states in the limit of adiabatic changes in the interactions. Our results are valid for arbitrary initial thermal states and, more generally, Generalized Gibbs Ensembles.

Introduction. — Many-body quantum systems are extremely sensitive to interactions, leading to a wide variety of possible phases of matter. This is particularly evident in low-dimensional systems, where particles are forced to meet, therefore to scatter: hence, the tiniest modification in the interactions can lead to deep physical changes. The 1d world is nowadays routinely probed in the lab, thanks to the astonishing advances in the context of cold atoms [1]: several out-of-equilibrium protocols have been engineered, unveiling new phases of matter [2–4]. Local observables and correlations thereof are measured in great detail thanks to in-situ manipulations [5–8]. As one of the main protagonists in this play the Bose gas with contact interactions, also known as the Lieb-Liniger model (LL) [9, 10], stands out since it naturally emerges when a bosonic gas is confined to an elongated trap [1, 11–20]. The LL model belongs to the class of integrable, or exactly solvable, systems [21, 22] which possess an extensive number of local conserved quantities: this has far-reaching consequences such as hindering of thermalization [23, 24] and ballistic transport [25]. Indeed, homogeneous integrable models relax to a non-equilibrium steady state known as Generalized Gibbs Ensemble (GGE) [26], which is sensitive to the whole set of dynamical constraints.

The interaction of the LL model can be experimentally tuned with great accuracy through Feshbach resonances [27] or through trap squeezing [11, 28], allowing experimentalists to probe an interesting dichotomy in its phase space. Indeed, depending on the sign of the interaction, the LL’s excitation content completely changes: in contrast to the repulsive phase, the attractive one sustains stable bound states of an arbitrary number of particles [29–32]. The dynamical production of bound states due to interaction changes is of primary experimental interest [33], but the highly non-perturbative and strongly correlated nature of the problem makes analytical results scarce. So far, only the sudden interaction quench starting from the non-interacting ground state, i.e. the Bose-Einstein condensate (BEC), has been theoretically understood [34, 35] (although results at special values of the interaction [36] or with a finite number of particles exist [37–39]). In spite of the importance of the result, this protocol has some limitations: first of all, the realization of 1d BECs is difficult due to their instability under thermal fluctuations [40]. Secondly, a realistic experimental setup is intrinsically inhomogeneous due to the presence of a trapping potential. Thirdly, the lack of freedom in choosing the initial state results in a narrow variety of steady-states within the attractive regime. Since the attractive LL is unstable at thermal equilibrium [29], this phase of matter can be studied only in out-of-equilibrium setups, hence controlling a larger variety of nonequilibrium states is of primary importance for state-preparation. Generalized Hydrodynamics (GHD) [41, 42] is a new powerful toolbox to deal with inhomogeneous integrable models and a new hope in analytically controlling the bound states’ production in LL. Originally introduced to study ballistic transport in integrable systems [41–54], diffusive corrections were subsequently included [55–60]. GHD applications and extensions are now far-reaching, ranging from the study of correlation functions [61, 62], quantum fluctuations [63], entanglement spreading [64–66], inhomogeneous potentials [67–70] and integrability-breaking terms [71–74]. Even more, it has been experimentally confirmed [18, 20]. Of particular interest for the problem at hand is the ability of GHD to describe adiabatically slow modifications of the interaction [68], provided the underlying integrability structure smoothly changes while varying the coupling. While in LL this is the case within the repulsive and the attractive regime, this is not true any longer when passing from one phase to the other and the state-of-the-art GHD techniques cannot be applied anymore.

In this work, we analytically solve this problem by matching together the hydrodynamic descriptions within the two phases. Our results allow for a complete characterization of the state preparation obtained starting from arbitrary repulsive thermal states (and more general GGEs) and slowly driving the system into the at-
tractive phase. Our findings can also include the presence of a smooth trapping potential and are feasible for experimental applications, which we briefly discuss. Our result is checked in the low-density limit against ab-initio microscopic calculations. Finally, we generalize our approach to the case where the interaction is modulated in space, connecting together spatial regions with different interaction signs.

The interacting Bose gas and its GHD. — The Hamiltonian of the LL model is

\[ \hat{H} = \int dx \left\{ \frac{\hbar^2}{2m} \partial_x \hat{\psi}^\dagger \partial_x \hat{\psi} + e \hat{\psi}^\dagger \hat{\psi} + V(x) \hat{\psi}^\dagger \hat{\psi} \right\}, \tag{1} \]

where the fields obey standard bosonic commutation relations \[ [\hat{\psi}(x), \hat{\psi}^\dagger(y)] = \delta(x-y), \]
and \( c \) is the interaction strength and \( V(x) \) the external trapping potential. Hereafter, we set our units such that \( \hbar^2/(2m) = 1 \).

In the absence of the trap, the model is integrable [9, 10]: the eigenstates of the Hamiltonian and, more generally, of the whole set of local charges can be understood in terms of quasiparticle excitations labeled by a set of quantum numbers \([\{\lambda_j\}_{j=1}^N]\) known as rapidities or quasimomenta. For a given (quasi-)local charge [75] \( \hat{Q} \), the eigenvalue behaves additively \[ \hat{Q}[\{\lambda_j\}_{j=1}^N] = \sum_{j=1}^N q(\lambda_j)[\{\lambda_j\}_{j=1}^N], \]
where the function \( q(\lambda) \) is called the charge eigenvalue. At finite volume these rapidities are quantized according to the Bethe-Takahashi equations [29], whose solution strongly depends on the sign of the interaction. In the thermodynamic limit and within the repulsive phase \( c > 0 \) the rapidities are real, while for \( c < 0 \) they organize in strings of arbitrary length \( j \) [76, 77]: rapidities belonging to a given string share the same real part, but are shifted along the imaginary direction \( \lambda_a = \lambda - ic(j + 1 - 2a)/2 \) with \( a \in \{1, j\} \) (see Fig. 1). Strings can be viewed as bound states of several particles and a string-dependent charge eigenvalue is constructed summing over the constituents of the string \[ q_j(\lambda) = \sum_{a=1}^j q(\lambda - ic(j + 1 - 2a)/2). \]

The detailed arrangement of rapidities in a given state does not matter in the thermodynamic limit [76, 78] and the eigenstates are described in terms of root densities \( \rho(\lambda) \). In the repulsive case, \( Ld\lambda \rho(\lambda) \) counts how many rapidities in the state are comprised in an interval \([\lambda, \lambda + d\lambda]\). In the attractive case, infinitely many root densities \( \rho_j \) are needed, one for each string species, and describe the occupancy of the real part of the rapidities belonging to the same string. The root densities uniquely identify the thermodynamics of eigenstates and, as such, they are in one-to-one correspondence with the GGEs [79].

Let us now allow the system to be weakly inhomogeneous in space and time, but locally integrable. For example, this is the case when an external trap \( V(x) \) is introduced and the interaction \( c \) becomes space-time dependent. Invoking a separation of scales, one can assume the system locally relaxes to a GGE, which is then slowly evolving: this is the paradigm of GHD [41, 42], which locally describes the system through a space-time dependent root density. The GHD in the presence of a trapping potential and of a space-time dependent interactions is [68]

\[ \partial_t \rho_j + \partial_x (v_j \rho_j) + \partial_\lambda (F_j \rho_j) = 0. \tag{2} \]

Above, we wrote the hydrodynamic equations within the attractive phase: the repulsive case is obtained setting \( c > 0 \) in what follows and keeping only the first string \( \rho(\lambda) = \rho_1(\lambda) \). The effective velocity and force are defined as \( v_j^\text{eff} = (\partial_\lambda \epsilon_j)^{dr}/(\partial_\lambda \rho_j)^{dr} \) and \( F_j^\text{eff} = (\partial_\lambda \epsilon_j)^{dr}/(\partial_\lambda \rho_j)^{dr} - \partial_\lambda V \), while \( \epsilon_j(\lambda) = j\lambda^2 - c^2 j^2/2 \) and \( \rho_j(\lambda) = j\lambda \) are respectively the energy and momentum eigenvalues, and \( f_j(\lambda) = \sum_k \int \frac{d\lambda'}{2\pi} \partial_\lambda \Theta_{j,k}(\lambda - \lambda') \rho_k(\lambda') \). The scattering phase \( \Theta \) takes into account the interacting nature of the model \( \Theta_{j,k}(\lambda) = (1 - \delta_{j,k})\theta_{j-k}(\lambda) + 2\theta_{j-k+1}(\lambda) + \ldots + 2\theta_{j+k-2}(\lambda) + \theta_{j+k}(\lambda) \), and \( \theta_{j}(\lambda) = -2\arctan[2\lambda/(jc)] \). Furthermore, the interactions dress the bare quantities according to the linear integral equations [29]

\[ \tau_{j}^{dr}(\lambda) = \tau_j(\lambda) - \sum_k \int \frac{d\lambda'}{2\pi} \partial_\lambda \Theta_{j,k}(\lambda - \lambda') \theta_{k}(\lambda') \tau_{k}^{dr}(\lambda') \tag{3} \]

where \( \theta_j = 2\pi \rho_j/(\partial_\lambda \rho_j)^{dr} \) is called the filling fraction. Qualitatively, Eq. (2) describes the evolution of noninteracting particles with phase-space density \( \rho_j(t, x, \lambda) \), moving with effective velocity \( v_j^\text{eff} \) and experiencing an effective force \( F_j^\text{eff} \), where the interactions cause a state dependence of the latter.

Crossing \( c = 0 \). — We can finally address our out-of-equilibrium protocol. For the sake of simplicity, let us assume a homogeneous setup and start with a given root density in the repulsive phase, for example a thermal state. The presence of a trap can be included performing the forthcoming analysis at each spatial point. Notice that, in the homogeneous case, one can change variable in the GHD equation \( t \to c(t) \) and parametrize the root density in terms of the value of the interaction.
As $c$ is reduced, the particles are compressed together in the rapidity space and the state keeps on evolving until $c = 0^+$ is reached (see Fig. 2). Here, the endpoint of the adiabatic evolution in the repulsive case must fix the initial conditions for the subsequent evolution in the attractive phase, i.e. determine the set $\{\rho_j\}_{j=1}^\infty$ at $c = 0^-$. Of course, the free point $c = 0$ can be equally seen as the limit from the weakly repulsive or weakly interacting regime; and the expectation of local observables must be continuous at $c = 0$. This is true also for the whole set of (quasi-)local conserved charges, whose expectation value is believed to uniquely determine the root densities [79]. One has $\langle \hat{Q}\rangle$ $\sim \int d\lambda q_{\lambda} f(\lambda)$ $\sim \sum_j \int d\lambda q_{\lambda} \rho_j(\lambda)$, where the system’s size $L$ appears because of extensivity. From the charge conservation, we aim to extract the root densities: in order to do so, we first connect the charge eigenvalues in the attractive case with those in the repulsive phase. The charge eigenvalue is obtained summing over the rapidities belonging to one string, whose imaginary shift vanishes in the $c \to 0^-$ limit $\lim_{c \to 0^-} q_{\lambda}(\lambda) = j \lim_{c \to 0^-} q_{\lambda}(\lambda)$, where the continuity of the charge eigenvalue $q$ is assumed. Finally, it is natural to assume $\lim_{c \to 0^-} q(\lambda) = \lim_{c \to 0^+} q(\lambda)$, because the first string in the attractive case is nothing less than the analytic continuation of the repulsive case. These considerations are easily checked on the energy and momentum eigenvalues and, combined with the assumption of completeness of the charges, lead to the following continuity equation, which we stress is diagonal in the rapidity space

$$\rho_{\lambda}(\lambda) = \sum_j j \rho_j(\lambda).$$

Hence, the charges are unable to fully determine $\{\rho_j\}_{j=1}^\infty$ in the $c \to 0^-$ limit. The interpretation of this equation is extremely simple: at zero interaction, bound states of $j$ particles with real rapidity $\lambda$ are completely indistinguishable from $j$ unbounded particles with the same rapidity (Fig. 1). For example, this is clear in the two-particle sector, where the wavefunction decays exponentially on a length scale $|c|^{-1}$, i.e. $|\phi(x,y)| \propto e^{-|c||x−y|/2}$ (see the Supplementary Material [80]). Precisely, the bound state is indistinguishable from unbound particles when its typical spatial width is much larger than the correlation length of the system. The fact that Eq. (4) is diagonal in the rapidity space can be physically motivated as well with the following argument. The interaction acts locally in real space and it is ramped to negative values in an adiabatic fashion, therefore only particles which remain close to each other for an arbitrary long time can bind together. Excitations with different rapidities necessarily have different effective velocities $v_{eff}(\lambda)$, therefore are eventually dragged far apart before they can form a bound state.

Eq. (4) can be read in two ways: if the interaction is switched from the attractive to the repulsive regime, $\{\rho_j\}_{j=1}^\infty$ are known and Eq. (4) fully settles $\rho$. In the opposite scenario, $\rho$ is fixed and $\rho_j$ must be determined, a task where Eq. (4) does not suffice. In order to do this, we revert to the very definition of GGE, i.e. the state that maximizes the entropy under the constraint of fixing the expectation values of all the local integrals of motion. The Yang-Yang (YY) entropy is [76]

$$S = L \sum_j \int d\lambda \partial_{\lambda} p_j(\lambda) \partial_{\lambda} \log \partial_{\lambda}(1 - \partial_{\lambda}) \log(1 - \partial_{\lambda})$$

and its maximization constrained to Eq. (4) leads to the following equations (see SM [80])

$$\varepsilon_j(\lambda) = j \omega(\lambda) + \sum_k (2 \min(j,k) - \delta_{j,k}) \log(1 + e^{-\varepsilon_k(\lambda)}),$$

where the effective energy parametrizes the filling $\partial_j = (1 + e^{\varepsilon_j})^{-1}$ and $\omega(\lambda)$ is a $\lambda$-dependent Lagrange multiplier to be determined imposing Eq. (4). A similar entropy-maximization strategy has been used in determining the bound-state recombination in the XXZ spin chain affected by a time-dependent magnetic flux [69].

FIG. 2. Top: bound state population for different choices of $\rho(\lambda)$ at $c = 0^-$. The green histogram is obtained using the low-density (incorrect) result $\rho_j = \frac{1}{2\pi e^{\omega_j}}$ (see SM [80]). Middle: evolution of the root densities adiabatically evolving from $c = 4$ to $c = -4$. The system is initialized in a thermal state with density 0.5 and inverse temperature $\beta = 0.1$. Bottom: energy $E$ and $g_2 = \langle (\psi^1)^2 \psi^2 \rangle / \langle \psi^1 \psi \rangle^2$ [80] evolution for the same protocol, but with different initial temperatures and density 0.5. The GHD is numerically solved according to the method of Ref. [68].
In Fig. 2 we study the bound state formation and provide results for physical observables, initializing the gas in thermal states at \( c > 0 \) and driving it in the attractive regime. Continuous quantum systems are notoriously hard to be simulated \([81]\), especially out of equilibrium, therefore alternative checks of the validity of our result are extremely important.

**The first check: Ab-initio analysis of the zero density limit.** — The low-density limit is amenable of explicit calculations. Let \( \{\{\lambda_j\}\}^N_j \) be a non-interacting eigenstate with \( \lambda_j \in [\lambda, \lambda + d\lambda) \). Let \( \{\{\Lambda_j\}\} \) be a state at \( c < 0 \) where \( \Lambda_j \) are the rapidities of the \( j \)-string. The transition probability is \( P = |\langle\{\lambda\}|\langle\{\Lambda\}\rangle|^2 \): since the GGE does not fix the microscopic arrangement of the rapidities, \( P \) must be averaged with respect to the position of the rapidities \( \lambda_j \) and \( \Lambda_j \) in \( [\lambda, \lambda + d\lambda) \). Let \( P(\{n_j\}) \) be the averaged probability of obtaining \( n_j \) bound states in the \( j \)-th string. In the low-density and \( c \to 0^- \) limit one gets \( \bar{P}(\{n_j\}) \propto \prod_j \left[ \frac{(d\lambda_j)}{2\pi} \right]^{n_j} \) \([80]\). For large occupancies \( n_j \), log \( P \) is the low-density approximation of the YY entropy once one identifies \( \Lambda d\rho_j = n_j \) and \( \Lambda d\lambda_j = N \). Therefore, one gets \( \rho_j = \frac{N}{2\pi} e^{\frac{j\pi}{2}} \), which is the low-density limit of Eqs. (4) (5). We leave the details to SM \([80]\).

**The second check: entropy conservation.** — When a large spatial interval is considered, the YY entropy describes the scaling limit of the entanglement entropy \([66]\). Since the entanglement entropy is a property of the state itself, it cannot differ if one looks at it from the \( c = 0^+ \) or \( c = 0^- \) limit, hence the YY entropy is expected to be continuous. Indeed, Eq. (5) ensures the continuity of the YY entropy passing from \( c = 0^+ \) to \( c = 0^- \). Even though the highly non-linear nature of Eq. (5) makes the analytical proof of this statement hard, it can be numerically checked with arbitrary precision \([80]\). This provides a non-trivial check of our results at finite density. Since Eq. (5) is a maximum, it is also the only choice which guarantees the continuity of the YY entropy. We also notice that, since the YY entropy is conserved by the GHD equations \([70, 80]\), it is also conserved during the entire protocol.

**Generalization to spatial inhomogeneities.** — Our result can be promptly extended to spatially inhomogeneous interactions. Let us consider \( c(x \leq 0) \leq 0 \) to be a smoothly inhomogeneous function constant in time. The strings flowing from the attractive to the repulsive region \( (\lambda > 0) \) unbind, while particles can form bound states when traveling in the other direction \( (\lambda < 0) \). In this case, rather than the continuity of the charges one must impose the continuity of the current associated with the latter. This requirement leaves some freedom in choosing the bound-state populations, which can be again determined by maximum entropy considerations. However, in this case, the entropy rate must be maximized: computing \( \partial_t S \) with the help of the GHD equations, one finds that this rate is completely determined by the root densities at \( x = 0 \). A detailed derivation can be found in SM \([80]\), leading to the same equations as before, namely Eqs. (4) and (5).

**Bound states’ detection in experiments.** — We expect our results to be applicable to the state-of-the-art experimental techniques. In Ref. \([33]\) Cesium atoms were trapped in 1d optical traps and the interaction manipulated acting on a Feshbach resonance \([28]\). In particular, by gently tuning the magnetic field, the whole range from weakly repulsive to strongly attractive interactions can be continuously explored. Ref. \([33]\) focused on sudden interaction changes, but slow protocols can be implemented as well. The initial state \( c > 0 \) is expected to be thermal and its temperature can be estimated by measuring the mean kinetic energy trough momentum-space imaging. With the same method, the kinetic energy can be probed at the end of the protocol and compared with the GHD result. Advances in atom-chip setups \([82]\) could lead to even more interesting measurements, given the possibility of real-space density’s profile imaging. Combining the latter with a longitudinal trap release, the rapidity-dependent fillings of the bound states can be reconstructed from the full-counting statistics of the density fluctuations (see SM \([80]\)).

**Conclusions and outlook.** — We analytically predict the bound states’ formation in the 1d interacting Bose gas undergoing adiabatic interaction changes from the repulsive to the attractive regime. Our results are exact also when the correlations are strong and inaccessible to perturbation theory. We considered generic initial thermal states and more generally GGEs: this flexibility allows to greatly control the attractive phase with immediate applications to state-preparation. Our findings are in principle experimentally accessible, but also provide prospects for further developments in inhomogeneous 1d systems. For example, inhomogeneous spin chains can arguably be studied with similar methods and the consequences of bound states’ recombination on transport problem addressed \([83]\). The experimental setup of Ref. \([2]\) represents a major theoretical challenge, with a cold-atom realization of the famous sine Gordon (SG) model, describing the phase interference between two coupled 1d atom tubes. The intrinsic inhomogeneity induced by the experimental setup causes a smooth space dependence on the SG interaction, which strongly affects the local spectrum of the theory, and causes binding and unbinding the topological excitations of the phase. Our findings are a first step towards the solution of this very interesting, but difficult, problem. Future applications to classical systems, such as the 1d Non-Linear Schroedinger equation, are thriving to be addressed as well, in view of the relatively simplicity in numerically checking predictions dragged from integrability \([47, 50–53, 72, 73, 84–88]\).

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Supplementary Material

Adiabatic formation of bound states in the 1d Bose gas

The bound states production from entropy maximization

In this section we present the maximization of the Yang-Yang (YY) entropy at the crossing point between repulsive and attractive interaction \( c \), i.e. when the interaction is \( c = 0^- \). In order to determine the recombination of particles into bound states, we look for a maximum of the YY entropy constrained to particle conservation, i.e. we maximize the following functional \( \mathcal{L} \) with respect to \( \rho_j \)

\[
\mathcal{L} = \sum_j \int \frac{d\lambda}{2\pi} \left[ (\partial_\lambda \rho_j)^{dr}[-\vartheta_j \log \vartheta_j - (1 - \vartheta_j) \log(1 - \vartheta_j)] - \omega(\lambda) \left( \rho - \sum_j j\rho_j \right) \right].
\]  

(S1)

Above, \( \omega(\lambda) \) is a (\( \lambda \)-dependent) Lagrange multiplier.

The fact that we eventually need the \( c \to 0^- \) limit greatly simplifies the dependence of the derivative of the dressed momentum \( (\partial_\lambda \rho_j)^{dr} \) and of the filling fraction \( \vartheta_j \) with respect to the root densities. Since the latter is defined as \( \vartheta_j = \frac{2\pi \rho_j}{(\partial_\lambda \rho_j)^{dr}(\lambda)} \), we focus on the limit of the former. The derivative of the dressed momentum reads as

\[
(\partial_\lambda \rho_j)^{dr}(\lambda) = j - \sum_k \int d\lambda' \partial_\lambda' \Theta_{jk}(\lambda - \lambda') \rho_k(\lambda').
\]  

(S2)

with the scattering phase

\[
\Theta_{j,k}(\lambda) = (1 - \delta_{j,k})\theta_{j,k}(\lambda) + 2\theta_{j,j+k-2}(\lambda) + ... + 2\theta_{j+k-2,j}(\lambda) + \theta_{j+k}(\lambda)
\]  

(S3)

and \( \theta_j(\lambda) = -2\arctan \frac{j\lambda}{c} \). In the limit \( c \to 0^- \) the derivative of the scattering phase becomes a \( \delta \)-function

\[
\lim_{c \to 0^-} \partial_\lambda \vartheta_j(\lambda) = -\lim_{c \to 0^-} \frac{jc}{\left( \frac{jc}{c} \right)^2 + \lambda^2} = -\text{sign}(c)2\pi \delta(\lambda).
\]  

(S4)

Since the same result holds irrespective of \( j \), we get

\[
\lim_{c \to 0^-} \partial_\lambda \Theta_{jk}(\lambda) = -\text{sign}(c)2\pi \delta(\lambda)[2\min(j,k) - \delta_{jk}].
\]  

(S5)

Notice that the \( \delta \)-contribution in the integration kernel makes the relation between \( \rho_j \) and \( (\partial_\lambda \rho_j)^{dr} \) diagonal in the rapidity space, but retains a non-trivial dressing over the string indices. The non-interacting limit of the repulsive regime \( c \to 0^+ \) can be recovered from the previous expression focusing on a single string \( \rho = \rho_{j=1} \) and letting \( c > 0 \). However, we are now interested in the attractive regime and therefore assume \( c < 0 \) in what follows.

While varying \( \mathcal{L} \) with respect to \( \rho_j \) letting \( \rho_j \to \rho_j + \delta\rho_j \), one needs the variation of the \( (\partial_\lambda \rho_j)^{dr} \) and \( \vartheta_j \), which read

\[
\delta[(\partial_\lambda \rho_j)^{dr}(\lambda)] = -2\pi \sum_k \delta[2\min(j,k) - \delta_{jk}] \delta \rho_k(\lambda), \quad \delta \vartheta_j = \delta \left[ \frac{2\pi \rho_j}{(\partial_\lambda \rho_j)^{dr}} \right] = \frac{2\pi}{(\partial_\lambda \rho_j)^{dr}} \left[ \delta \rho_j - \frac{\rho_j}{(\partial_\lambda \rho_j)^{dr}} \delta[(\partial_\lambda \rho_j)^{dr}] \right].
\]  

(S6)

Maximizing the YY entropy we obtain

\[
\sum_k \left[2\min(j,k) - \delta_{jk}\right] \log(1 - \vartheta_k) - \log \left( \frac{\vartheta_j}{1 - \vartheta_j} \right) - j\omega(\lambda) = 0.
\]  

(S7)

These equations together with the conservation of particles \( \rho(\lambda) = \sum_j j\rho_j(\lambda) \) determine the different root densities \( \rho_j \) of the bound states at \( c = 0^- \). Notice that this constraint must hold for any \( \lambda \) so that the Lagrange multiplier is \( \lambda \)-dependent. If we parametrize the filling with the effective energy \( \vartheta_j = (1 + e^{\varepsilon_j})^{-1} \), Eq. (S7) can be rewritten as

\[
\varepsilon_j(\lambda) = j\omega(\lambda) + \sum_k \left[2\min(j,k) - \delta_{jk}\right] \log(1 + e^{-\varepsilon_k(\lambda)})
\]  

(S8)
which is the equation presented in the main text. We notice that, due to the fact that the scattering phase becomes diagonal in the rapidity space (S5), the YY entropy becomes diagonal as well $S = L \int d\lambda s[\{\rho_j(\lambda)\}^N_{j=1}]$ with $s[\{\rho_j(\lambda)\}^N_{j=1}]$ being a functional of the root densities with rapidity $\lambda$. The same considerations hold in the $c \to 0^+$ case as well and it is easily numerically checked to arbitrary precision that Eq. (S8) guarantees the continuity of the functional $s$ and of the YY entropy.

The full protocol

In the main text, we initialize the system in thermal states at $c > 0$ and adiabatically tune the interaction up to negative values. More specifically, we consider an initial thermal state at $c > 0$ determined by the following integral equation [29]

$$\varepsilon(\lambda) = \beta[\varepsilon(\lambda) - \mu] - \frac{d\lambda}{2\pi} \theta_1(\lambda - \lambda') \log \left(1 + e^{-\varepsilon(\lambda')}\right). \quad \text{(S9)}$$

The initial state is then evolved with the GHD equations according to the method of characteristics used in Ref. [68]. The $c = 0^+$ and $c = 0^-$ sectors are connected with Eq. (S8).

During the evolution, we mainly focus on two physically motivated observables, namely the total energy $E$ and the correlated density $\rho_2 = \langle (\psi^\dagger)\psi^2 \rangle / \langle (\psi^\dagger)\psi \rangle^2$. In terms of the root densities and filling fractions, these observables are

$$E = \sum_j \int d\lambda \varepsilon_j(\lambda) \rho_j(\lambda), \quad \langle \psi^\dagger \psi \rangle = \sum_j \int d\lambda \rho_j(\lambda) \quad \text{(S10)}$$

and [34, 35]

$$\langle (\psi^\dagger)\psi^2 \rangle = -\sum_j \int d\lambda \frac{c}{6} j(j^2 - 1) \rho_j(\lambda) + \sum_j \int \frac{d\lambda}{\pi} j \lambda \theta_j(\lambda) f^{\text{dr}}_j(\lambda). \quad \text{(S11)}$$

Both observables are reported in the attractive phase, while the repulsive one is obtained retaining only the first string $j = 1$ and of course $c > 0$.

The bound-state formation in the low density limit

In this section, we derive the ansatz for the formation of the bound states in the zero density limit, using first-principle calculations in the microscopic model. As we will see, the probability of forming a bound state is completely determined from phase-space arguments and therefore from entropy maximization. Focusing on the zero density limit, we will miss effects of the interaction that are important at finite density, such as dressing.

The general problem we want to address is the following: Let us consider a non-interacting eigenstate labeled by $N$ particles $|\{\lambda_i\}^N_{i=1}\rangle$. Then, we consider another state in the weakly attractive phase featuring some bound states. Let us label it as $|\{\lambda_a\}^N_{a=1}\rangle$, where $j$ labels the species of the bound state and $a$ is an internal label running on the rapidities $\Lambda$ of the bound states of the same species. The transition probability $P$ from one state to the other is the overlap squared

$$P = |\langle\{\lambda_a\}^N_{a=1}|\{\lambda_i\}^N_{i=1}\rangle|^2. \quad \text{(S12)}$$

Our ultimate goal is to compare our calculations with GHD predictions: since the GGE fixes the average occupancy in each rapidity cell $[\lambda, \lambda + d\lambda]$, but it is not sensitive to the microscopic arrangements of the rapidities, the probability $P$ must be averaged accordingly. We already know from the charge eigenvalues that quasiparticles with distinct rapidities cannot bind together. Therefore, let us focus on the case where all the incoming rapidities belong to the same interval $\lambda_i \in [\lambda, \lambda + d\lambda]$. The overlap will vanish if the rapidities of the bound states do not belongs to the same interval (as we will explicitly see) and, similarly, we are not interested in their exact location on the rapidity axis, but only in their number. Let $\{n_j\}^N_{j=1}$ be the number of bound states of each species at $c = 0^-$, then we are interested in the following averaged probability

$$\bar{P}(\{n_j\}) = \left(\frac{d\lambda L}{2\pi}\right)^{-N} \sum_{\lambda_i \in [\lambda, \lambda + d\lambda]} \sum_{\Lambda_a^j} |\langle\{\lambda_i\}^N_{i=1}|\{\lambda_a\}^N_{a=1}\rangle|^2 \{n_j\} \text{ fixed}. \quad \text{(S13)}$$
Above, the \( \lambda_i \) rapidities are independently summed over the interval \([\lambda, \lambda + d\lambda]\). The prefactor is just the phase space of the rapidities which are quantized as integer multiples of \( 2\pi/L \), with \( L \) the system size.

This object can be explicitly computed in the low density limit in view of very simple considerations. However, as a warm up, it is useful to study first a simple case in detail, namely the probability for \( N \) particles to form the largest possible bound state.

The probability of maximal binding

For finite \( c < 0 \), the bound state of \( N \) particles with rapidity \( \Lambda \) and at finite volume in the coordinate representation reads \[77\]

\[
|\Lambda\rangle = e^{iN-1\Lambda \sum_{i=1}^N x_i} \sqrt{\frac{|c|^{N-1}}{N!}} \Phi_N(|c|x_1, ..., |c|x_N). \tag{S14}
\]

The wavefunction \( \Phi \) within the string hypothesis explicitly reads

\[
\Phi_N(x_1, ..., x_N) = \frac{1}{\sqrt{N!}} \exp \left[ \sum_{a=1}^N x_a (N + 1 - 2a)/2 \right], \quad x_1 \leq ... \leq x_N, \tag{S15}
\]

and the symmetric extension for other ordering of the coordinates is assumed. However, we will never use the explicit form of \( \Phi \), but only that it is translational invariant and fast decaying when the coordinates are stretched far apart. The incoming state is

\[
|\{\lambda_i\}_{i=1}^N\rangle = \frac{1}{\sqrt{N!L^N}} \sum_{\Lambda} e^{i \sum_{a} \lambda_{F(a)} x_a} \tag{S16}
\]

where the sum is over the possible permutations \( P \) of \( N \) elements. Strictly speaking, we assume \( \{\lambda_i\} \) to be all different: coinciding rapidities will change the normalization constant, but they are not important after the averaging. We now examine the overlap

\[
\langle \Lambda|\{\lambda_i\}\rangle = \sqrt{\frac{|c|^{N-1}}{N!}} \frac{1}{\sqrt{N!L^N}} \sum_{\Lambda} \int d^N x \Phi(|c|x_1, ..., |c|x_N) e^{i \sum_{a} \lambda_{F(a)} x_a - iN^{-1} \Lambda \sum_a x_a}. \tag{S17}
\]

Above, the symmetry of the wavefunction \( \Phi \) under global translations allows one to integrate the center of mass \( N^{-1} \sum_a x_a \) and the oscillating phases impose the momentum constraint \( \Lambda = N^{-1} \sum_\lambda \lambda_a \). Importantly, we work at large but finite volume, hence conservation laws are not enforced through Dirac deltas, but Kronecker deltas and, of course, \( L \) factors. Rather than aiming for a brute-force computation of the above integral, it is more convenient to keep the formal integral representation and plug it directly into Eq. (S13)

\[
\tilde{P} = \left( \frac{d\Lambda L}{2\pi} \right)^{-N} \frac{|c|^{N-1}}{L} \frac{1}{N!L^N} \sum_{\Lambda, \lambda} \sum_{\Lambda'} \int d^N x \int d^N x' \Phi(|c|x_1, ..., |c|x_N) \Phi(|c|x_1', ..., |c|x_N') e^{i \sum_{a} \lambda_{F(a)} x_a - iN^{-1} \Lambda \sum_a x_a - iN^{-1} \Lambda' \sum_a x_a'} \tag{S18}
\]

The summation over the \( \Lambda \) states is trivially performed, since the only non-zero contribution is when \( \Lambda = N^{-1} \sum_\lambda \lambda_a \). Then, we use the fact that since we are summing over all the rapidities and coordinates, the result is invariant under permutations. Therefore, we can pick a single arrangement of coordinates and rapidities and introduce a prefactor \( (N!)^2 \) to keep into account the double summation over the rapidities, which gives

\[
\tilde{P} = \left( \frac{d\Lambda L}{2\pi} \right)^{-N} \frac{N!|c|^{N-1}}{L^{N+1}} \sum_{\Lambda, \lambda} \int d^N x \int d^N x' \Phi(|c|x_1, ..., |c|x_N) \Phi(|c|x_1', ..., |c|x_N') e^{i \sum_{a} (\lambda_a - N^{-1} \sum_{a'} \lambda_{a'}) (x_a - x_a')} \tag{S19}
\]
Next, we notice that the integrand is invariant under translations $x_a \to x_a + \text{const.}$ and similarly under translations $x'_a \to x'_a + \text{const.}$, so that we get a factor $L$ for each of the two translational symmetries and we can fix $x_1 = x'_1 = 0$ in the integrand. Furthermore, we take the thermodynamic limit $\sum_\lambda \to \frac{L}{2\pi} \int d\lambda$

$$P = \left( \frac{d\lambda L}{2\pi} \right)^{-N} \frac{N!|\lambda|^{N-1}N!}{(2\pi)^N} \times \int d^N \lambda \int d^{N-1} x \int d^{N-1} x' \Phi(0, |c|x_2, ..., |c|x_N)\Phi(0, |c|x'_2, ..., |c|x'_N)e^{i \sum_{\alpha=2}^N (\lambda_{\alpha} - N-1) \lambda_{\alpha'}}(x_{\alpha} - x'_{\alpha}).$$  \hspace{1cm} (S20)

Now, we notice that the integrand is invariant under global rapidity shifts $\lambda_a \to \lambda_a + \text{const.}$, hence it is convenient to change variables as $X_1 = -N^{-1} \sum_a \lambda_a$ and $X_{a>1} = \lambda_a - N^{-1} \sum_a \lambda_a$. The change of coordinate $M_{a,a'}X_{a'} = \lambda_a$ has a non-trivial Jacobian which must be taken into account when changing variable. In particular $\det M = -N^{-1}$. After the change of variables one gets

$$P = \left( \frac{d\lambda L}{2\pi} \right)^{-N} \frac{Ld\lambda N!}{(2\pi)^N} \int d^{N-1} x \int d^{N-1} x' \int d^{N-1} y \Phi(0, |c|x_2, ..., |c|x_N)\Phi(0, |c|x'_2, ..., |c|x'_N)e^{i \sum_{\alpha=2}^N \lambda_{\alpha}(x_{\alpha} - x'_{\alpha})}.$$  \hspace{1cm} (S21)

The integrand does not depend on $\chi_1$, hence one can explicitly integrate over $\chi_1$, getting a $d\lambda$ overall factor, i.e. the length of the interval on which we are averaging. Lastly, we change variables rescaling $y_a = |c|x_a$, $y'_a = |c|x'_a$ and $\mu_a = \chi_a/|c|$. Notice that, since $\lambda_a$ lived in an interval of width $d\lambda$, $\mu_a$ belongs on an interval of length $d\lambda/|c|$, which diverges with $|c| \to 0$. Hence, in the $|c| \to 0$ limit one gets

$$P = \left( \frac{d\lambda L}{2\pi} \right)^{-N} \frac{Ld\lambda N!}{(2\pi)^N} \int d^{N-1} x \int d^{N-1} y \Phi(0, y_2, ..., y_N)\Phi(0, y'_2, ..., y'_N)e^{i \sum_{\alpha=2}^N \mu_{\alpha}(y_{\alpha} - y'_{\alpha})}.$$  \hspace{1cm} (S22)

Now, we could first integrate in the coordinates $y_a y'_a$ and then in the the variables $\mu_a$. If one proceeds in this way, a decaying function of the $\mu_a$ variables is found. In terms of the original rapidities $\lambda_a$, this means the function decays as $\lambda_a$ are dragged apart from their center of mass on a typical length scale $\sim |c|$. In other words, the function is very peaked in the $\lambda$–space: this will be used in the forthcoming section. However, for the time being it is better to integrate first in the $\chi_a$ coordinates: this results in $N - 1$ Dirac deltas that enforce $y_a = y'_a$

$$P = \left( \frac{d\lambda L}{2\pi} \right)^{-N} \frac{Ld\lambda N!}{(2\pi)^N} N(2\pi)^{N-1} \int d^{N-1} y |\Phi(0, y_2, ..., y_N)|^2.$$  \hspace{1cm} (S23)

Finally, one notices that $\int d^{N-1} y |\Phi(0, y_2, ..., y_N)|^2 = 1$ because of the normalization of the bound state wavefunction. The final simple result is the following

$$P = \left[ \frac{1}{N!} \left( \frac{d\lambda L}{2\pi} \right)^N \right]^{-1} \frac{d\lambda N}{2\pi}.$$  \hspace{1cm} (S24)

This result can be now interpreted in terms of phase-space densities. The quantity $d\lambda N/(2\pi)$ is the number of possible rearrangements of the momentum of the bound state of $N$ particles in the interval $[\lambda, \lambda + d\lambda]$. Indeed, since $\Lambda = N^{-1} \sum_a \lambda_a$ and $\lambda_a$ are quantized in units of $2\pi L^{-1}$, $\Lambda$ is therefore quantized in units of $2\pi |c| L^{-1}$. On the other hand, $\frac{1}{N!} \left( \frac{d\lambda L}{2\pi} \right)^N$ is the phase space of the incoming states (the $N!$ term keeps into account the fact that particles are indistinguishable). The phase-space density result suggests that the probability in the general case can be determined on the basis of simple arguments. This is indeed the case.

The probability of the generic transition

We can now go back to the problem of computing the transition probability to an arbitrary state, in the $c \to 0$ limit. In order to do this, we use some assumptions which, based on our previous calculations, are expected to be valid in the zero-density limit. Let us consider the generic overlap $\langle \{\lambda_1\}|\{\Lambda'_a\} \rangle$: this overlap can be divided into a product of Kronecker deltas enforcing the conservation laws of the momenta times a smooth part $F$

$$\langle \{\lambda_1\}|\{\Lambda'_a\} \rangle = [\text{conservation law}] \times F(\lambda_1, ..., \lambda_N).$$  \hspace{1cm} (S25)
In the maximal binding calculation the conservation law was extremely simple \(\delta_{\lambda,N-1}\sum \lambda_i\). Instead, in the general case, the constraint is a complicated product of Kronecker deltas (and sum of this product over rapidity permutations). The set of rapidities \(\{\lambda_i\}\) is partitioned into groups of rapidities, where the center of mass of each group is enforced to be equal to the rapidity of a certain bound state. However, there is no need to make this constraint explicit. As we commented in the explicit computation in the maximal binding case, in the \(c \to 0^-\) limit the smooth part of the overlap becomes extremely peaked around the rapidity of the bound state: we assume it is the case in the generic overlap as well, therefore \(F\) is peaked for the rapidities \(\lambda_i\) close to the rapidity of the associated bound state. This observation allows one to write

\[
\sum_{\lambda_i \in \{\lambda, \lambda+d\lambda\}} |\{\lambda_i\}\{\Lambda^j_i\}|^2 \rightarrow \sum_{\lambda_i} |\text{conservation law}| \times F(\lambda_1, \ldots, \lambda_N) = N!.
\] (S26)

The second sum is unconstrained. Since the rapidities of the bound state belong to the interval \([\lambda, \lambda + d\lambda]\), the function \(F\) is zero whenever one of the rapidities \(\lambda_i\) lays outside of the interval. The unconstrained sum is then equal to \(N!\), because of the completeness of the \(|\{\lambda_i\}\) states

\[
\sum_{\lambda_i \in \{\lambda, \lambda+d\lambda\}} \sum_{\lambda_i \in \{\lambda, \lambda+d\lambda\}} |\{\lambda_i\}\{\Lambda^j_i\}|^2 \rightarrow N!|\{\Lambda^j_i\}|(\sum_{\lambda_i < \ldots < \lambda_N} |\{\lambda_i\}\{\Lambda^j_i\}|) = N!|\{\Lambda^j_i\}| = N! (S27)
\]

In order to get the averaged probability \(\bar{P}\), we need now to sum over the possible positions of the bound states within the interval \([\lambda, \lambda + d\lambda]\). The rapidity of a bound state of species \(j\) is quantized in units of \(2\pi/(Lj)\), therefore we get

\[
\sum_{\lambda_i \in \{\lambda, \lambda+d\lambda\}} \sum_{\lambda_i \in \{\lambda, \lambda+d\lambda\}} |\{\lambda_i\}\{\Lambda^j_i\}|^2 = \sum_{\lambda_i \in \{\lambda, \lambda+d\lambda\}} N! \prod_j \left[ \frac{1}{n_j!} \left( \frac{d\lambda Lj}{2\pi} \right)^{n_j} \right] .
\] (S28)

Above, the \(n_j!\) terms account for the indistinguishability of the bound states and, in the zero density limit, we quantize the rapidities of the bound states independently. This simple analysis gives us the following simple averaged probability

\[
\bar{P}(\{n_j\}) = \left[ \frac{1}{N!} \left( \frac{d\lambda L}{2\pi} \right)^N \right]^{-1} \prod_j \left[ \frac{1}{n_j!} \left( \frac{d\lambda Lj}{2\pi} \right)^{n_j} \right] + \ldots,
\] (S29)

which is of course consistent with the maximal binding probability previously derived. We notice that the averaged probability is completely determined in terms of the phase space. Above, further corrections are present: indeed, the probability \(\bar{P}\) is not correctly normalized. One can simply observe, for example, that the configuration \(n_1 = N\) and \(n_j > 1 = 0\) already saturates the probability \(\bar{P}(n_1 = N, n_2 = 0, \ldots) = 1\). The reason is the following: the argument we provided only captures the leading behavior in the thermodynamic limit of a given configuration \(\{n_j\}\). Indeed, if one considers the power counting in \(L\) factors finds \(L^{\sum n_j - N}\). Using the constraint \(N = \sum_j jn_j\), the power counting can be rewritten as \(L^{-1/2 + (j-1)n_j} \); therefore, strictly speaking, in the \(L \to \infty\) limit only the configuration \(n_1 = N\) survives, while other configurations vanish and the probability is correctly normalized. Hence, at finite \(L\), the probability of each configuration has non-trivial corrections to subleading orders in \(L\).

*The averaged population in the low-density limit*

We finally use the probability \(\bar{P}\) (S29) to compute the average bound-state population. Since \(\bar{P}\) captures only the leading order in \(L^{-1}\), the resulting expectation values \(\langle n_j \rangle\) will be valid at the leading order as well, i.e. the leading order in the zero density limit.

Let us consider \(\bar{P}\) (S29) in the limit of large occupation numbers \(n_j\). In this case, one can use the Stirling approximation and write

\[
\bar{P} \propto \exp \left[ \sum_j n_j \log \left( \frac{d\lambda Lj}{2\pi} \right) - n_j \log n_j + n_j \right].
\] (S30)
This expression can be immediately compared with the low-density regime of the YY entropy. Indeed, if one identifies $Ld\lambda \rho_j = n_j$ and $Ld\lambda = N$, $\bar{P}$ can be expressed as

$$
\bar{P} \propto \exp \left[ Ld\lambda \sum_j \rho_j \log (\partial_\lambda \rho_j) - \rho_j \log(\rho_j) + \rho_j \right], 
$$
where we used that $\partial_\lambda \rho_j = j/(2\pi)$. The argument in the exponential is nothing else than the leading order in the $\rho_j \to 0$ limit of the YY entropy. In the $L \to \infty$ limit, the probability is peaked around its maximum and therefore the expectation values are determined by the saddle point, namely entropy maximization. While doing so, one should take into account the constraint $\rho = \sum_j j \rho_j$ that is trivially derived from $N = \sum_j j n_j$. The entropy maximization gives the following simple result

$$
\rho_j = \frac{j}{2\pi} e^{-\omega_j},
$$
with $\omega$ a Lagrange multiplier. Then, $\omega$ is fixed by $\rho = \sum_j j \rho_j$. As expected, this is the low-density limit of the GHD result.

**Bound-states formation in spatially-inhomogeneous setups**

In this section, we generalize our approach to study spatially inhomogeneous interactions. Let us consider a 1d Bose gas with a time-independent but smoothly spatially modulated interaction $c \to c(x)$. Without loss of generality, we assume $c(x = 0) = 0$ and $c(x \leq 0) \leq 0$: generalizing to multiple zeroes is trivial. Within the $x > 0$ and $x < 0$ regions, the Eulerian dynamics is entirely governed by the GHD equations and, similarly to the time-dependent protocol, one has to find the proper boundary conditions at the transition point. First, we find the analogue of charge-conservation, which can be easily understood to be the continuity of currents. Indeed, any discontinuity of a current, would imply a divergent growth of the associated charge density, that is of course unphysical. The exact expression for currents has already been proposed in the original papers on GHD \[41, 42\] and it reads

$$
\langle \tilde{J} \rangle = \sum_j \int d\lambda \, v^{\text{eff}}(\lambda) q_j(\lambda) \rho_j(\lambda),
$$
where $q_j$ is the charge eigenvalue of the charge $\hat{Q}$ associated with the current.

Assuming the analiticity of the charge eigenvalues and their completeness (together with $\lim_{\lambda \to 0-} q_j(\lambda) = j \lim_{\lambda \to 0+} q(\lambda)$), one gets a continuity equation. Thanks to the fact that, at $c \to 0$, the dressing acts diagonally in the rapidity space, from the very definition of the effective velocity one can easily show that, both in the repulsive and attractive regime, it holds $v^{\text{eff}}(\lambda) = v^{\text{eff}}(\lambda) = 2\lambda$. This further simplifies the continuity equation obtained from the currents which, in the end, is identical to the time-dependent case

$$
\rho(\lambda) = \sum_j j \rho_j(\lambda).
$$

Similarly to the charge conservation in the time-dependent case, Eq. (S34) does not completely fix the boundary conditions, since it allows a possible rearrangement of the bound states. More specifically, the current flowing into the junction is of course fixed by the left and right bulks, while the out-going current must be found. The notion of in-going and out-going is determined by the sign of $v^{\text{eff}}(\lambda) = v^{\text{eff}}(\lambda) = 2\lambda$. In order to unambiguously determine the bound state recombination, we consider the entropy once again.

Within the inhomogeneous setup, rather than considering the YY entropy, one should focus on its growth. Let us consider $\partial_t S = \partial_t S_{x<0} + \partial_t S_{x>0}$, where $S_{x<0}$ is the Yang-Yang entropy in the left and right halves of the system, respectively. The GHD equations have been proved to conserve the entropy \[70\], however this is true only in the absence of boundary terms (see also Ref. \[69\]). Indeed, let us consider the Yang-Yang entropy within the attractive regime (i.e. in the region $x < 0$) and compute its time derivative, using the GHD equations one straightforwardly obtains

$$
\partial_t S_{x<0} = \partial_t \left( \int_{-\infty}^0 dx \int d\lambda \sum_j \rho_j^2(\lambda) \eta(\dot{\varphi}_j) \right) = -\sum_j \int_{-\infty}^0 dx \int d\lambda \partial_x (v^{\text{eff}}_j \rho_j^2 \eta(\dot{\varphi}_j)) + \partial_\lambda \left( F^{\text{eff}}_j \rho_j \eta(\dot{\varphi}_j) \right),
$$
where $\eta(x) = -x \log x - (1 - x) \log(1 - x)$. Since we integrate exact differentials, only boundary terms matter. In the hypothesis that the filling vanishes at large rapidities and for $x \to -\infty$, we get a non-trivial contribution only from the boundary at $x = 0$

$$\partial_t S_{x<0} = -\sum_j \int d\lambda \, v_j^v \rho_j^\alpha \eta(\partial_j) \bigg|_{x=0}. \quad (S36)$$

A similar conclusion holds for $S_{x>0}$. Now, we are left with the problem of maximizing the entropy rate with the constrain $(S34)$. Of course, we are considering the $c \to 0$ limit, hence the dressing is diagonal in the rapidity space and $v_j^v = 2\lambda$: using this identity in $(S36)$, we obtain that the integrand in the entropy growth is, apart from the factor $2\lambda$, exactly the YY entropy we maximized in the time-dependent case. Since the dressing acts diagonally in the rapidity space, the $2\lambda$ prefactor is ineffective in the entropy maximization. Besides, the continuity equation $(S34)$ is formally the same as what we had in the time-dependent case. Hence, the entropy maximization leads to exactly the same non-linear equations $(4-5)$.

Detecting bound states through trap release

Our result is an exact analytical expression for the production of bound states after a slow interactions change. In this section, we describe how bound states can be experimentally detected. Let us imagine a $1d$ interacting Bose gas confined in an elongated trap. In Ref. [70] it has been pointed out that, within the repulsive phase, the gas expansion following a longitudinal trap release (but maintaining the transverse confinement), allows one to reconstruct the rapidity-dependent filling function, integrated in space. The same method can be used to detect the population of the bound states within the attractive phase through correlated density measurements of the expanding cloud. First, we quickly recap the measurement proposed in Ref. [70], then we move to discuss the attractive regime.

Trap release in the repulsive phase

Let us consider the $1d$ repulsive Bose gas in a trap, with homogeneous and time-independent interaction. Within the GHD description, the state at time $t$ is fully characterized by an inhomogeneous filling function $\vartheta(t, \lambda, x)$. Let us imagine that the longitudinal confinement is suddenly removed and the gas let free to expand. Hence, the evolution will be described by the GHD equation without force terms $\partial_t \vartheta + v^\text{eff} \partial_x \vartheta = 0$. One can write an implicit solution to this continuity equation in the form

$$\vartheta(t, \lambda, x) = \int dy \, \delta \left(y - x + \int_0^t dt' v^\text{eff}(t') \right) \vartheta(0, \lambda, y), \quad (S37)$$

where we set $t = 0$ to be the time when the trap is released. This is only an implicit solution, since the effective velocity depends on the state itself. However, in the large time limit the cloud expansion reduces the local density and the dressing effects on the bare velocity become negligible. Within this assumption, we can replace the effective velocity with the bare value $v^\text{eff} \to 2\lambda$ since the particles travel with their bare velocity for most of the trap expansion

$$\vartheta(t, \lambda, x) \approx \int dy \, \delta \left(y - x + 2\lambda t \right) \vartheta(0, \lambda, y). \quad (S38)$$

Now, let us consider the density profile and compute $\langle \hat{\psi}^\dagger(x) \hat{\psi}(x) \rangle = \int d\lambda \, \rho(t, \lambda, x)$. Again, in the large time limit one can neglect the dressing and approximate $\rho \approx \frac{1}{2\pi} \vartheta$, where we can use Eq. $(S38)$ and reach the following expression for the density profile

$$\langle \hat{\psi}^\dagger(x) \hat{\psi}(x) \rangle = \frac{1}{4\pi t} \int dy \, \vartheta \left(0, \frac{x - y}{2t}, y \right), \quad (S39)$$

with the final approximation that the filling is smooth in the rapidities and the time is much larger than the trap size at $t = 0$ divided by the typical velocity of the particles (in terms of adimensional quantities $\frac{x - y}{2t} \times [\partial_\lambda \vartheta / \vartheta]_\lambda = \frac{x - y}{v} \ll 1$) one can finally approximate

$$\langle \hat{\psi}^\dagger(x) \hat{\psi}(x) \rangle = \frac{1}{4\pi t} \int dy \, \vartheta \left(0, \frac{x}{2t}, y \right). \quad (S40)$$
Hence, measuring the density profile of the expanding cloud, one can measure $\int dy \vartheta(0, \lambda, y)$ as a function of the rapidity $\lambda$.

**The attractive phase**

A similar reasoning can be applied if the gas is in the attractive phase, although measurements of the density profile do not allow to discern among the bound states. Indeed, similarly to the repulsive phase, one has

$$\vartheta_j(t, \lambda, x) \simeq \int dy \, \delta(y - x + 2\lambda) \vartheta_j(0, \lambda, y),$$

which eventually leads to the density profile below

$$\langle \hat{\psi}^\dagger(x) \hat{\psi}(x) \rangle = \frac{1}{4\pi t} \sum_j \int dy \, j \vartheta_j \left(0, \frac{x}{2t}, y\right).$$

This single observable cannot distinguish the bound states and further measurements are needed. Physically, bound states are cluster of correlated particles which travel together. Hence, the correlated densities $O_{\ell}(x) = \langle \hat{\psi}^\dagger(x) \rangle [\hat{\psi}(x)]^\ell$ are the natural candidates to probe the bound states’ population. Within the GHD perspective, these observables can be expressed in terms of a functional of the root densities at the same position $\langle O_{\ell}(x) \rangle = O_{\ell}[\rho_{j}(x)]$. At finite density, such a functional is complicated and in general not known. However, we can use the fact that in the large time limit after the trap release, the local density of particles is small and attempt a linear expansion

$$\langle O_{\ell}(x) \rangle \simeq \sum_j \int d\lambda \, C_j^\ell(\lambda) \rho_{j}(t, x, \lambda).$$

Since the coefficients $C_j^\ell$ describe the low density limit of the expectation values $\langle O_{\ell}(x) \rangle$, they can be explicitly computed in the finite-particle sector. More precisely, let $|j, \lambda\rangle$ be the bound state of $j$ constituents and rapidity $\lambda$, with $L$ the system’s length

$$|j, \lambda\rangle = \sqrt{\frac{c^{j-1}}{j!L^{j}}} \int d^L x \, e^{i \frac{\lambda}{2} \sum_{k=1} x_k} \Phi_j(|c|x_1, |c|x_2, ..., |c|x_j\rangle \hat{\psi}^\dagger(x_1) ... \hat{\psi}^\dagger(x_j)|0\rangle.$$

The wavefunction $\Phi$ is reported in Eq. (S15). The normalization $N_j$ in Eq. (S15) can be fixed imposing

$$\int x_1=0 \, d^{j-1} x \, |\Phi_j(x_1, ..., x_j)|^2 = 1$$

Let us now focus on computing the expectation value of $O_{\ell}(x)$ on this state. Thanks to translational invariance, we can consider $O_{\ell}(x = 0)$ and one readily finds

$$\langle j, \lambda|O_{\ell}(0)|j, \lambda\rangle = \begin{cases} \frac{1}{j!|\ell|!} \int d^{j-\ell} x \, |\Phi_j(x_1, x_2, ..., x_{j-\ell}, 0, 0)|^2 & j \geq \ell \\ 0 & j < \ell \end{cases}$$

Comparing the above with Eq. (S43) and using that the state $|j, \lambda\rangle$ is represented by a root density $\rho_{j'}(\lambda') = L^{-1} \delta_{j,j'} \delta(\lambda - \lambda')$, one can determine the coefficients $C_j^\ell(\lambda)$:

$$C_j^\ell(\lambda) = \begin{cases} \frac{1}{j!|\ell|!} \int d^{j-\ell} x \, |\Phi_j(x_1, x_2, ..., x_{j-\ell}, 0, 0)|^2 & j \geq \ell \\ 0 & j < \ell \end{cases}$$

Notice that, due to Galilean invariance, $C_j^\ell(\lambda)$ is actually $\lambda$–independent. Once these coefficients have been computed, one can analyze the expanding cloud by mean of Eqs. (S41) and (S43), lastly determining $\int dx \, \vartheta_j(t, \lambda, x)$ as a function of $\lambda$.

The computation of $C_j^\ell$ requires performing a finite-dimensional integral in at most $j$ coordinates of a simple wavefunction (S15). Albeit tedious, this task can be straightforwardly performed, especially in the case where only the first strings are populated and one can truncate the sum (S43).
The local correlated density $[\psi^\dagger(x)]^\ell[\psi(x)]^\ell$ can be difficult to be directly measured in experiment. However, the same strategy can be used on the moments of the particle numbers in an interval. Let us define $\hat{N}_\Delta(x) = \int_{x-\Delta/2}^{x+\Delta/2} dx' \hat{\psi}^\dagger(x')\hat{\psi}(x')$, i.e. the particle number operator in an interval of length $\Delta$ centered around $x$. In the long time limit after the trap release, one can write

$$\langle [\hat{N}_\Delta(x)]^\ell \rangle = \frac{1}{4\pi t} \sum_j \int dy \tilde{C}_j^\ell \vartheta_j \left( 0, \frac{x}{2t}, y \right), \quad (S48)$$

where the computation of the coefficients $\tilde{C}_j^\ell$ is a trivial generalization of the strategy that led to determine $C_j^\ell$. 