CORON’S PROBLEM FOR THE CRITICAL LANE-EMDEN SYSTEM

SANGDON JIN AND SEUNGHYEOK KIM

Abstract. In this paper, we address the solvability of the critical Lane-Emden system

\[
\begin{aligned}
-\Delta u &= |v|^{p-1} v \quad \text{in } \Omega, \\
-\Delta v &= |u|^{q-1} u \quad \text{in } \Omega, \\
u = v &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \( N \geq 4, \ p \in (1, \frac{N-1}{N-2}), \ \frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}, \) and \( \Omega_\epsilon \) is a smooth bounded domain with a small hole of radius \( \epsilon > 0. \) We prove that the system admits a family of positive solutions that concentrate around the center of the hole as \( \epsilon \to 0, \) obtaining a concrete qualitative description of the solutions as well. To the best of our knowledge, this is the first existence result for the critical Lane-Emden system on a bounded domain, while the non-existence result on star-shaped bounded domains has been known since the early 1990s due to Mitidieri (1993) [30] and van der Vorst (1991) [36].

1. Introduction

Let \( N \geq 3, \ p > 1, \) and \( \Omega \) be a smooth domain that is either bounded or the whole space \( \mathbb{R}^N. \) More than half-century, the Lane-Emden equation

\[
-\Delta u = |u|^{p-1} u \quad \text{in } \Omega \tag{1.1}
\]

has played a role as one of the fundamental equations in the theory of nonlinear partial differential equations, thanks to its profound structural complexity despite its simple appearance. As is well-known, the order relation between \( p \) and the Sobolev exponent \( p_\text{S} := \frac{N+2}{N} \) influences the solution structure of (1.1) in a significant way. Moreover, the solution structure of the critical Lane-Emden equation (i.e., (1.1) with \( p = p_\text{S} \))

\[
\begin{aligned}
-\Delta u &= |u|^4 u \quad \text{in } \Omega, \\
u \in H^1_0(\Omega)
\end{aligned}
\]

(1.2)

subtly depends on the topology and geometry of the domain \( \Omega. \)

In this paper, we are interested in an elliptic system

\[
\begin{aligned}
-\Delta u &= |v|^{p-1} v \quad \text{in } \Omega, \\
-\Delta v &= |u|^{q-1} u \quad \text{in } \Omega, \\
(u, v) &\in X_{p,q}(\Omega),
\end{aligned}
\]

(1.3)

called the Lane-Emden system. Here, \( N \geq 3, \ (p, q) \) is a pair of positive numbers, \( \Omega \) is either a smooth bounded domain or the whole space \( \mathbb{R}^N, \) and \( X_{p,q}(\Omega) \) is a natural energy space compatible to the Dirichlet boundary condition (see [22]). For this, the Sobolev hyperbola

\[
\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N} \tag{1.4}
\]

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plays a similar role to the Sobolev exponent $p$ for equation (1.1), and the critical Lane-Emden system (1.3)–(1.4) serves as the natural counterpart of (1.2) for Hamiltonian-type elliptic systems; refer to a survey paper [5] for detailed account of Hamiltonian elliptic systems.

System (1.3) has received remarkable attention for decades. When $\Omega = \mathbb{R}^N$, Lions [28] found a positive least energy solution to (1.3)–(1.4) in $X_{p,q}(\mathbb{R}^N) = \dot{W}^{2, \frac{2}{1+1/p}}(\mathbb{R}^N) \times \dot{W}^{2, \frac{2}{1+1/q}}(\mathbb{R}^N)$ by using the concentration-compactness principle. Alvino et al. [2] then showed that it is radially symmetric and decreasing in the radial variable, after a suitable translation. After that, Wang [37] and Hulshof and van der Vorst [22] independently deduced that it is unique up to translations and dilations. Under the further assumption that $p \geq 1$, Chen et al. [7] extended their results by showing that a positive solution to (1.6) is unique up to translations and dilations. Recently, Frank, Pistoia, and the second author of this paper [16] established that all least energy solutions are non-degenerate in the sense that their linearized equations have precisely the $(N+1)$-dimensional space of solutions in $X_{p,q}(\mathbb{R}^N)$, originating from the translation and dilation invariance of (1.3). Furthermore, by employing the variational method, Clapp and Saldáñ [12] found finitely many non-radial sign-changing solutions to (1.3)–(1.4) for $N \geq 4$.

On the other hand, there is a longstanding conjecture in the literature, called the Lane-Emden conjecture. It claims that if the pair $(p,q)$ satisfies

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N},$$

then (1.3) with $\Omega = \mathbb{R}^N$ admits no positive solutions. The complete answer is unknown yet, though partial results are available in, e.g., [35, 9] and references therein.

If $\Omega$ is a smooth bounded domain, then the situation changes drastically. Due to the works of Hulshof and van der Vorst [21], Figueiredo and Felmer [14], and Bonheure et al. [4], it is known that (1.3) with (1.5) has a solution provided $pq \neq 1$. Conversely, according to Mitidieri [30] and van der Vorst [36], if $\Omega$ is star-shaped, then (1.3)–(1.4) has no positive solution. If $pq > 1$, then the standard variational method produces a positive least energy solution to (1.3) with (1.5). Guerra [18] and Choi and the second author of this paper [10] investigated the asymptotic behavior of the least energy solutions as $(p,q)$ approaches the Sobolev hyperbola (1.4), showing that the least energy solution blows up at a certain point in $\Omega$. By applying the perturbative argument, the second author and Pistoia [25] built multi-bubble (so higher energy) solutions to (1.3) when $(p,q)$ satisfies (1.5) and is sufficiently close to (1.4), and $\Omega$ is dumbbell-shaped. Also, when $\Omega$ is convex, the second author and Moon [24] classified the asymptotic behavior of all positive solutions to (1.3) with (1.5) as $(p,q)$ tends to (1.4).

However, as far as we know, no existence result for the critical system (1.3)–(1.4) is known in the literature. Therefore, it is natural to ask if the system can possess a positive solution in a bounded domain $\Omega$ under certain conditions on the topology or geometry on $\Omega$. In this paper, we give an affirmative answer for the question by constructing solutions to the system assuming that $\Omega$ has a sufficiently small hole, motivated by Coron’s result [13] for equation (1.2). Unlike Coron who used the variational method, we apply a perturbative argument. As a by-product, we obtain a concrete qualitative description of the solutions.

In what follows, we pay attention to

$$\begin{cases}
-\Delta u = |v|^{p-1}v & \text{in } \Omega_\varepsilon, \\
-\Delta v = |u|^{q-1}u & \text{in } \Omega_\varepsilon, \\
u = v = 0 & \text{on } \partial \Omega_\varepsilon
\end{cases}$$

We remark that the energy functional (defined in (2.3)) of system (1.3)–(1.4) is strongly indefinite in $X_{p,q}(\mathbb{R}^N)$, and the least energy solutions have the Morse index $+\infty$.

The existence theory of linear perturbations of the critical Lane-Emden system (1.3)–(1.4) in a smooth bounded domain $\Omega$, i.e., the Brezis-Nirenberg type problem was studied in [20, 18, 10, 25, 19].
where \( N \geq 4 \), \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^N \) such that \( 0 \in \Omega \), \( \epsilon > 0 \) is a small number, \( \Omega_\epsilon := \Omega \setminus B(0, \epsilon) \), and the pair \((p,q)\) is on the Sobolev hyperbola \((1.3)\).  

Theorem 1.1. There exists a small number \( c_0 > 0 \) such that system \((1.6)\) has a positive solution \( (u_\epsilon, v_\epsilon) \in (C^2(\Omega_\epsilon))^2 \) for each \( \epsilon \in (0, c_0) \). Moreover, the family \( \{(u_\epsilon, v_\epsilon)\}_{\epsilon \in (0, c_0)} \) concentrate around the origin of \( \mathbb{R}^N \) as \( \epsilon \to 0 \) in the sense that

\[
(u_\epsilon, v_\epsilon) = \left( P U_{\mu, \xi} + \psi_{\epsilon, (d, \tau)}, P V_{\mu, \xi} + \phi_{\epsilon, (d, \tau)} \right) \quad \text{in } \Omega_\epsilon.
\]

Here, \( (P U_{\mu, \xi}, P V_{\mu, \xi}) \) is the main part of the solution \( (u_\epsilon, v_\epsilon) \) defined by \((3.5)\) and \((3.9)\),

\[
(\mu, \xi) := \left( \epsilon^\alpha d, \epsilon^\alpha d\tau \right) \in (0, \infty) \times \mathbb{R}^N \quad \text{where } \alpha := \frac{N - 2}{(N - 2)p + N - 4} \in (0, 1) \quad \text{(1.8)}
\]

for some suitably chosen \( d > 0 \) and \( \tau \in \mathbb{R}^N \), and

\[
\| \Delta \psi_{\epsilon, (d, \tau)} \|_{L^\frac{p}{N-2} \left( \Omega_\epsilon \right)} + \| \Delta \phi_{\epsilon, (d, \tau)} \|_{L^\frac{q}{N-2} \left( \Omega_\epsilon \right)} \to 0 \quad \text{as } \epsilon \to 0.
\]

Remark 1.2. We make some comments on the above theorem.

(1) In the proof of Theorem 1.1, we will find several interesting and unique features of system \((1.6)\). They include

- the extremely slow decay of the \( u \)-component \( U_{\mu, \xi} \) of a positive solution \((U_{\mu, \xi}, V_{\mu, \xi})\) in \((2.13)\) to the limit system \((2.3)\) (see Lemma \((2.4)\));
- a strong nonlinear behavior represented by the nonlinear projection \( P U_{\mu, \xi} \) of \( U_{\mu, \xi} \) in \((3.9)\) and the potential analysis in the proof of Propositions \((3.3)\) and \((3.4)\);
- entanglement between the components \( u \) and \( v \) of solutions \((u, v)\) to \((1.6)\), which can be seen in the expansion \((3.11)\) of \( P U_{\mu, \xi} \) and the definition of \( A_{\epsilon, (d, \tau)} \) in \((3.10)\) among others.

(2) We could treat the range \([N - 1, N + 2] \times [N - 2, N - 2] \) of \( p \) as well, but opted to focus on \( p \in \left(1, \frac{N - 2}{N - 2} \right)\) for the following reasons: If \( p \in \left(\frac{N - 1}{N - 2}, \frac{N + 2}{N - 2}\right)\), system \((1.6)\) acts as the scalar equation \((1.2)\), which corresponds to \( p = \frac{N + 2}{N - 2} \) for our system. Hence accompanying analysis is expected to be rather traditional. If \( p \in \left(\frac{N - 1}{N - 2}, \frac{N}{N - 2}\right)\), the system behaves similarly to our case, but an additional technical issue arises involving the regularity of the auxiliary map \( \tilde{H}_y \) in \((2.3)\). It makes the problem extremely complicated and, in our opinion, somewhat hides the features of the system depicted above.

A suitable combination of the argument in this paper and one in \((24)\) Section 5) may produce the existence result analogous to Theorem 1.1 for all \( p \in \left(1, \frac{N - 2}{N - 2} \right)\).

If \( p = 1 \), the system boils down to the biharmonic Lane-Emden equation studied in \((1)\). In our proof, the condition \( p > 1 \) is crucially used in several places - the setting of the linear theory, the energy expansion, and the regularity of the solutions. It seems difficult to deal with the case \( p \in \left(\frac{N - 1}{N - 2}, 1\right) \) with our argument.

(3) We see that the exponent \( \alpha \) in \((1.8)\) tends to \( \frac{N - 2}{N - 2} \) as \( p \to 1^+ \), which matches the exponent in \((1)\) \((2.3)\). Also, a formal computation reveals that the blow-up rate of possible single-bubble solutions to \((1.6)\) must be \( \epsilon^\alpha \) for \( p \in \left[\frac{N - 1}{N - 2}, \frac{N}{N - 2}\right) \), and \( \epsilon^{\frac{1}{2}} \) for \( p \in \left(\frac{N}{N - 2}, \frac{N + 2}{N - 2}\right) \). Our computation for \( p = \frac{N + 2}{N - 2} \) is consistent to \((27)\) Theorem 1.1. Also, \( \alpha \to \frac{1}{2} \) as \( p \to \left(\frac{N}{N - 2}\right)^+ \).

There have been extensive studies on the Coron-type problems, namely, the existence theory of solutions to a critical elliptic equation or system in a smooth bounded domain with single or multiple holes.

The Pohozaev identity tells us that \((1.2)\) has no positive solution if \( \Omega \) is star-shaped. In contrast, if \( \Omega \) is an annulus, \((1.2)\) has a positive radial solution, as shown by Kazdan and Warner \((23)\). Later,
Coron \[13\] revealed that (1.2) has a positive solution whenever the (possibly non-symmetric) smooth bounded domain \(\Omega\) has a sufficiently small hole. Lewandowski \[26\] proved that if the hole is spherical, say \(B(0, \epsilon) := \{x \in \mathbb{R}^N : |x| < \epsilon\}\), Coron’s solutions concentrate around the origin of \(\mathbb{R}^N\) as \(\epsilon \to 0\). Coron’s result was substantially improved by the work \[3\] of Bahri and Coron, which yields that if the singular homology group \(H_d(\Omega; \mathbb{Z}_2)\) with coefficients in \(\mathbb{Z}_2\) is non-trivial for some \(d \in \mathbb{N}\), then (1.2) has a positive solution.

Applying the perturbative argument, researchers attempted to find more solutions to (1.2). Under the presence of one or more spherical holes in \(\Omega\), Li et al. \[27\] proved the existence of single-bubble solutions. Furthermore, Musso and Pistoia \[31\] and Ge et al. \[17\] built sign-changing bubble-tower solutions, where the bubble-tower solution refers to a solution which looks like superpositions of bubbles with different blow-up rates. For further results, we refer to, e.g., \[32, 11, 15\].

In addition, Alarcón and Pistoia \[1\] obtained a positive single-bubble solution for the biharmonic Lane-Emden equation, which corresponds to the critical Lane-Emden system (1.6) with \(p = 1\). For the fractional Lane-Emden equation, Long et al. \[29\] and Chen et al. \[8\] derived the existence of a positive single-bubble solution and sign-changing bubble-tower solutions, respectively.

Lastly, we point out that Coron-type results are available for the critical coupled Schrödinger systems, which are the natural counterparts of (1.2) for gradient-type elliptic systems. Indeed, thanks to the works of Pistoia and Soave \[33\] and Pistoia et al. \[34\], the existence of positive solutions whose component looks like a single-bubble or a bubble-tower is known.\[5\]

Structure of the paper and novelty of our proof. We apply the Lyapunov-Schmidt reduction method to construct the desired solutions to system (1.6). For its successful application, we have to find a very precise ansatz for the solutions that reflects the system’s unique characteristics (see Remark \[12\] (1)) and analyze it carefully, which are the main difficulties in our proof.

In Section 2, we collect some preliminary results such as the definition of the appropriate function spaces and auxiliary maps, the expansion of the regular part of Green’s function of the Laplacian \(-\Delta\) in \(\Omega_\epsilon\), and properties of the solutions to the limit system (2.7).

In Section 3, we define and improve ansatz of the solutions to (1.6). In particular, we set the \(u\)-component of the refined ansatz as the nonlinear projection \(PU_{\mu, \xi}\) of \(U_{\mu, \xi}\) (see (2.13) and (3.9)), and expand it by introducing a suitable function \(A_{\epsilon, (d, \tau)}\) and conducting a delicate potential analysis; refer to Proposition 3.3. Because our solutions will be characterized as saddle points of the reduced energy \(J_\epsilon\) in (4.1), the \(C^1\)-smallness of the remainder term \(R\) in (3.12) is required. Unlike the corresponding situation for (1.2) or other related equations, the presence of the ‘ugly’ function \(A_{\epsilon, (d, \tau)}\) makes such estimate non-trivial. In Proposition 3.4, we tackle this issue through a delicate analysis.

In Section 4, we perform the \(C^1\)-estimate of the reduced energy \(J_\epsilon\). Lots of technical computations appearing in Sections 3 and 4 are postponed to Appendix A.

In Section 5, we carry out the reduction procedure. Several results in this section easily follow from known arguments. For the conciseness of the paper, we concentrate only on the results that do not. Whenever we omit the details, we will leave appropriate references.

In Section 6, we complete the proof of Theorem 1.1.

In Appendix B, we study the regularity of \((\psi_{\epsilon, (d, \tau)}, \phi_{\epsilon, (d, \tau)})\) (see (1.7)) by establishing a general regularity result for linear Hamiltonian-type elliptic systems.\[6\]

Notations. Here, we list some notations used in the paper.

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5 The Lane-Emden system (1.6) is strongly coupled, and only synchronized blowing-up solutions (namely, solutions whose components have the same blow-up point) can exist. On the other hand, the coupled nonlinear Schrödinger system is weakly coupled, and both segregated blowing-up solutions (namely, solutions whose components have different blow-up points) and synchronized blowing-up solutions may exist.

6 There is a gap in the proof of \[25\] Proposition 4.6, since the functions \(Q_1\) and \(Q_2\) in \[25\] (B.5) do not belong to \(L^\infty(\Omega)\). We point out that the argument in Appendix B fills this gap.
- Given $x \in \mathbb{R}^N$ and $r > 0$, $B(x, r)$ stands for an open ball centered at $x$ of radius $r > 0$.
- $S^{N-1}$ is the unit sphere in $\mathbb{R}^N$ centered at the origin.
- Unless otherwise stated, $C > 0$ is a universal constant that may vary from line to line and even in the same line.
- Let $(A)$ be a condition. We set $1_{(A)} = 1$ if $(A)$ holds and $0$ otherwise.
- $O(1)$ denotes a term such that
  \[ |O(1)| \leq C \quad \text{for all } \epsilon > 0 \text{ small (and } x \in \Omega \text{ if it is a function in } \Omega) \]  
  where $C > 0$ is independent of $\epsilon$ (and $x$). Also, $o(1)$ denotes a term such that
  \[ |o(1)| \leq C_\epsilon \quad \text{for all } \epsilon > 0 \text{ small (and } x \in \Omega \text{ if it is a function in } \Omega) \]  
  where $0 < C_\epsilon \to 0$ as $\epsilon \to 0$.

Caution. In the following, we always assume that the pair $(p, q)$ satisfies (1.4).

2. Preliminaries

2.1. Functional setting. Let
  \[ \frac{1}{p^*} = \frac{p}{p+1} - \frac{1}{N} = \frac{1}{q+1} + \frac{1}{N} \quad \text{and} \quad \frac{1}{q^*} = \frac{q}{q+1} - \frac{1}{N} = \frac{1}{p+1} + \frac{1}{N} \]  
  so that $p^*$ and $q^*$ are the Hölder’s conjugates of each other. For any smooth domain $\Omega_\epsilon$ in $\mathbb{R}^N$, we set a Banach space
  \[ X_{p,q}(\Omega_\epsilon) := \left( W^{2, \frac{p+1}{N}}(\Omega_\epsilon) \cap W^{1,p^*}_0(\Omega_\epsilon) \right) \times \left( W^{2, \frac{q+1}{N}}(\Omega_\epsilon) \cap W^{1,q^*}_0(\Omega_\epsilon) \right) \]  
  equipped with the norm
  \[ \| (u, v) \|_{X_{p,q}(\Omega_\epsilon)} := \| \Delta u \|_{L^{\frac{p+1}{N}}(\Omega_\epsilon)} + \| \Delta v \|_{L^{\frac{q+1}{N}}(\Omega_\epsilon)}. \]
  By the Sobolev embedding theorem, we have that $X_{p,q}(\Omega_\epsilon) \subset L^{p+1}(\Omega_\epsilon) \times L^{q+1}(\Omega_\epsilon)$. For small $\epsilon > 0$, we also define an energy functional $I_\epsilon : X_{p,q}(\Omega_\epsilon) \to \mathbb{R}$ by
  \[ I_\epsilon(u, v) = \int_{\Omega_\epsilon} \left( \nabla u \cdot \nabla v - \frac{1}{p^*+1} |v|^{p^*+1} - \frac{1}{q^*+1} |u|^{q^*+1} \right). \]  
  It is of class $C^2$ and its critical point is a weak solution to (1.4).

2.2. Green’s functions and related maps. For any smooth domain $\Omega_\epsilon$ in $\mathbb{R}^N$, let $G_{\Omega_\epsilon}$ be the Green’s function of the Laplacian $-\Delta$ in $\Omega_\epsilon$ with the Dirichlet boundary condition and $H_{\Omega_\epsilon} : \Omega_\epsilon \times \Omega_\epsilon \to \mathbb{R}$ its regular part solving
  \[ \begin{cases} 
  -\Delta_x H_{\Omega_\epsilon}(x, y) = 0 & \text{for } x \in \Omega_\epsilon, \\
  H_{\Omega_\epsilon}(x, y) = \frac{\gamma_N}{|x - y|^{N-2}} & \text{for } x \in \partial \Omega_\epsilon,
  \end{cases} \]
  where $\gamma_N := [(N - 2)|S^{N-1}|]^{-1}$. Set
  \[ G = G_{\Omega_\epsilon}, \quad H = H_{\Omega_\epsilon}, \quad G_\epsilon = G_{\Omega_\epsilon}, \quad H_\epsilon = H_{\Omega_\epsilon}, \quad G_{\epsilon,1} = G_{\mathbb{R}^N \setminus \overline{B(0, \epsilon)}}, \quad H_{\epsilon,1} = H_{\mathbb{R}^N \setminus \overline{B(0, \epsilon)}}. \]
  Then $H_{\epsilon,1}$ is simply written as
  \[ H_{\epsilon,1}(x, y) = \gamma_N \epsilon^{N-2} \left| y \right| \left( \frac{\epsilon}{|y|} \right)^2 \left( \frac{\epsilon}{|y|} \right)^{2-N} \]  
  for $x, y \in \mathbb{R}^N \setminus \overline{B(0, \epsilon)}$. As the next result shows, one can decompose $H_\epsilon$ into two parts $H$ and $H_{\epsilon,1}$ modulo a small remainder term.
Lemma 2.1. Let $N \geq 3$ and $\epsilon > 0$ small. For any $x, y \in \Omega_\epsilon$, it holds that
\[
H_\epsilon(x, y) = H(x, y) + H_{\epsilon, 1}(x, y) + O(1)e^{-N} (|x|^{2-N} + |y|^{2-N}).
\]

Proof. Fixing $y \in \Omega_\epsilon$, let $R(x, y) = H_\epsilon(x, y) - H_{\epsilon, 1}(x, y) - H(x, y)$, which solves
\[
\begin{cases}
-\Delta_x R(x, y) = 0 & \text{for } x \in \Omega_\epsilon, \\
R(x, y) = -H_{\epsilon, 1}(x, y) & \text{for } x \in \partial \Omega, \\
R(x, y) = -H(x, y) & \text{for } x \in \partial B(0, \epsilon).
\end{cases}
\]
Since
\[
e^{-N} \left| y \right| \left( x - \frac{c^2 y}{|y|^2} \right)^{2-N} \leq C e^{-N} \frac{1}{|y|^{N-2}} \text{ for } x \in \partial \Omega \text{ and } y \in \Omega_\epsilon
\]
and
\[
H(x, y) \leq C e^{-N} \frac{1}{|x|^{N-2}} \text{ for } x \in \partial B(0, \epsilon) \text{ and } y \in \Omega_\epsilon,
\]
the maximum principle yields
\[
\sup_{x \in \Omega_\epsilon} |R(x, y)| \leq \sup_{x \in \partial \Omega_\epsilon} |R(x, y)| \leq C e^{-N} \left( |x|^{2-N} + |y|^{2-N} \right).
\]

For later use, we introduce two functions: Assume that $N \geq 3$ and $p \in \left( \frac{2}{N-2}, \frac{N}{N-2} \right)$. Given $y \in \Omega$, let $\tilde{G}_y : \Omega \to \mathbb{R}$ be a function such that
\[
\begin{cases}
-\Delta \tilde{G}_y = G^p(\cdot, y) & \text{in } \Omega, \\
\tilde{G}_y = 0 & \text{on } \partial \Omega
\end{cases}
\]
and $\tilde{H}_y : \Omega \to \mathbb{R}$ its regular part given by
\[
\tilde{H}_y(x) = \frac{\tilde{\gamma}_{N, p}}{|x - y|^{(N-2)p-2}} - \tilde{G}_y(x) \text{ for } x \in \Omega
\]
where
\[
\tilde{\gamma}_{N, p} := \frac{\gamma^p_N}{((N-2)p-2)(N-(N-2)p)} > 0.
\]
The maximum principle implies that $\tilde{H}_y(x) > 0$ for all $x, y \in \Omega$. Thanks to the assumption that $p < \frac{N-1}{N-2}$, we have the following regularity result.

Lemma 2.2 (Lemmas 2.1 and 2.11 in [25]). Assume that $p \in \left( \frac{2}{N-2}, \frac{N-1}{N-2} \right)$ and let $\tilde{H}(x, y) = \tilde{H}_y(x)$ for $x, y \in \Omega$. Then
- there exists $\sigma \in (0, 1)$ such that $\| \tilde{H}_y \|_{C^{1,\sigma}(\Omega)}$ is uniformly bounded by $y$ in a compact subset of $\Omega$.
  In particular, for each $y \in \Omega$, the map $x \in \Omega \mapsto (\nabla_x \tilde{H})(x, y)$ is continuous;  
- for each $x \in \Omega$, the map $y \in \Omega \mapsto (\nabla_y \tilde{H})(x, y)$ is continuous.

2.3. Limit system. Let $N \geq 3$ and $(U, V)$ be the unique positive ground state solution to
\[
\begin{cases}
-\Delta u = |u|^{p-1} u & \text{in } \mathbb{R}^N, \\
-\Delta v = |v|^{q-1} v & \text{in } \mathbb{R}^N, \\
(u, v) \in \dot{W}^{2, \frac{2}{N-2}}(\mathbb{R}^N) \times \dot{W}^{2, \frac{2}{N-2}}(\mathbb{R}^N)
\end{cases}
\]
such that $u(0) = \max_{x \in \mathbb{R}^N} u(x) = 1$. According to [2], it is radially symmetric and decreasing in the radial variable.
Lemma 2.3 (Theorem 2 in [22]). If \( p \in \left( \frac{2}{N-2}, \frac{N}{N-2} \right) \), then
\[
\lim_{|x| \to \infty} |x|^{(N-2)p-2} U(x) = a_{N,p} \quad \text{and} \quad \lim_{|x| \to \infty} |x|^{N-2} V(x) = b_{N,p}
\]
where \( a_{N,p} \) and \( b_{N,p} \) are positive constants depending only on \( N \) and \( p \) with
\[
b_{N,p}^2 = a_{N,p} ((N-2)p - 2) (N - (N-2)p) .
\]

Lemma 2.4 (Corollaries 2.6 and 2.7 in [25], Lemma 2.2 in [24]). Suppose that \( N \geq 3 \) and \( p \in \left( \frac{2}{N-2}, \frac{N}{N-2} \right) \). Then there exist some \( \kappa_0, \kappa_1 \in (0,1) \) such that
\[
\left| U(x) - \frac{a_{N,p}}{|x|^{(N-2)p-2}} \right| = O \left( \frac{1}{|x|^{(N-2)p-1}} \right),
\]
\[
\left| V(x) - \frac{b_{N,p}}{|x|^{N-2}} \right| = O \left( \frac{1}{|x|^{N-1}} \right),
\]
\[
\left| \nabla U(x) + ((N-2)p - 2)a_{N,p} \frac{x}{|x|^{(N-2)p}} \right| = O \left( \frac{1}{|x|^{(N-2)p-1+\kappa_0}} \right),
\]
\[
\left| \nabla V(x) + (N-2)b_{N,p} \frac{x}{|x|^{N-1}} \right| = O \left( \frac{1}{|x|^{N-1+\kappa_1}} \right)
\]
for \( |x| \geq 1 \).

Given \( (\mu, \xi) \in (0, \infty) \times \mathbb{R}^N \), we define
\[
(U_{\mu,\xi}(x), V_{\mu,\xi}(x)) = \left( \mu^{-\frac{N}{N+1}} U(\mu^{-1}(x - \xi)), \mu^{-\frac{N}{N+1}} V(\mu^{-1}(x - \xi)) \right)
\]
for \( x \in \mathbb{R}^N \) so that \((U, V) = (U_{1,0}, V_{1,0})\). According to [37, 22], they constitute the entire set of solutions to (2.7). They are also non-degenerate as the next proposition states.

Proposition 2.5 (Theorem 1 in [16]). We define
\[
\begin{align*}
(P_{l,0}(x), \Phi_{l,0}(x)) &= \left( x \cdot \nabla U(x) + \frac{NU(x)}{q+1}, x \cdot \nabla V(x) + \frac{NV(x)}{p+1} \right), \\
(P_{1,0}(x), \Phi_{1,0}(x)) &= (\partial_{x_1} U(x), \partial_{x_1} V(x))
\end{align*}
\]
for \( x \in \mathbb{R}^N \) and \( l = 1, \ldots, N \). Also, given \( \mu > 0 \) and \( \xi \in \mathbb{R}^N \), we set
\[
(U_{\mu,\xi}(x), V_{\mu,\xi}(x)) = \left( \mu^{-\frac{N}{N+1}} P_{l,0}(\mu^{-1}(x - \xi)), \mu^{-\frac{N}{N+1}} \Phi_{l,0}(\mu^{-1}(x - \xi)) \right)
\]
for \( x \in \mathbb{R}^N \) and \( l = 0, 1, \ldots, N \). Then the space of solutions to the linear system
\[
\begin{cases}
-\Delta \Psi = pV_{\mu,\xi}^{-1} \Phi & \text{in } \mathbb{R}^N, \\
-\Delta \Phi = qU_{\mu,\xi}^{-1} \Psi & \text{in } \mathbb{R}^N, \\
(\Psi, \Phi) \in W^{2, \frac{N+1}{2}}(\mathbb{R}^N) \times W^{2, \frac{N+1}{2}}(\mathbb{R}^N)
\end{cases}
\]
is spanned by
\[
\{(\Psi_{0,\xi}(x) \mu_{0,\xi}^0(x)), (\Psi_{1,\xi}(x) \mu_{1,\xi}^1(x), \ldots, (\Psi_{N,\xi}(x) \mu_{N,\xi}^N(x)) \}
\]
3. Approximation of the solution

3.1. Admissible set of parameters. Given a small fixed number \( \delta \in (0, 1) \) to be determined in Section 6 we define the admissible set of parameters

\[
\Lambda_{\delta} = \{(d, \tau) : d \in [\delta, \delta^{-1}], \ \tau = (\tau_1, \ldots, \tau_N) \in \mathcal{B}(0, \delta^{-1})\}.
\]

Also, letting

\[
\mu = \epsilon^\alpha \quad \text{where } \alpha = \frac{N - 2}{(N - 2)p + N - 4} \in (0, 1) \quad \text{for } N \geq 4 \text{ and } p > 1,
\]

we write

\[
\mu = \mu_d \quad \text{and} \quad \xi = \mu \tau = \mu_d \tau.
\]

The following relations will be often useful:

\[
\frac{N(p + 1)}{q + 1} = (N - 2)p - 2 \quad \text{and} \quad \left(\frac{\epsilon}{\mu}\right)^{N-2} = \mu^{\epsilon(N-2)p-2}.
\]

3.2. The first approximation of the solution. We analyze the linear projection of \((U_{\mu, \xi}, V_{\mu, \xi})\) in \([2.13]\) into the space \(W_0^{1\cdot p'}(\Omega) \times W_0^{1\cdot q'}(\Omega)\); see \([2.1]\) for the definition of the pair \((p', q')\). It will serve as the first approximation of the positive solution to \((1.6)\). More precisely, we consider the system

\[
\begin{cases}
-\Delta U_{\mu, \xi} = V_{\mu, \xi}^p & \text{in } \Omega, \\
-\Delta V_{\mu, \xi} = U_{\mu, \xi}^q & \text{in } \Omega, \\
PU_{\mu, \xi} = PV_{\mu, \xi} = 0 & \text{on } \partial \Omega
\end{cases}
\]

Lemma 3.1. Assume \( N \geq 4, \ p \in (1, \frac{N}{N-2}), \) and \( \epsilon > 0 \) small. Let \( \tilde{H} : \Omega \times \Omega \to \mathbb{R} \) be a smooth function such that

\[
\begin{cases}
-\Delta \tilde{H}(x, y) = 0 & \text{for } x \in \Omega, \\
\tilde{H}(x, y) = \frac{1}{|x - y|^{(N-2)p-2}} & \text{for } x \in \partial \Omega.
\end{cases}
\]

Then, for \( x \in \Omega, \)

\[
PU_{\mu, \xi}(x) = U_{\mu, \xi}(x) - a_{N, p} \mu^\frac{Np}{N+p} \tilde{H}(x, \xi) - \mu^\frac{N}{N-2} U(\tau) \frac{\epsilon^{N-2}}{|x|^{N-2}} + R_1(x),
\]

\[
PV_{\mu, \xi}(x) = V_{\mu, \xi}(x) - b_{N, p} \mu^\frac{Np}{N+p} \tilde{H}(x, \xi) - \mu^\frac{N}{N-2} V(\tau) \frac{\epsilon^{N-2}}{|x|^{N-2}} + R_2(x).
\]

Here, \( R_1 \) and \( R_2 \) are the remainder terms such that

\[
|R_1(x)| = O(1) \left[ \mu^\frac{N}{N+p} \mu + \epsilon^{N-2} \mu^\frac{N}{N+p} |x|^{2-N} \left( \mu^{(N-2)p-2} + \frac{\epsilon}{\mu} \right) \right],
\]

\[
|R_2(x)| = O(1) \left[ \mu^\frac{N}{N+p} \left( \mu + \left( \frac{\epsilon}{\mu} \right)^{N-2} \right) + \epsilon^{N-2} \mu^\frac{N}{N+p} |x|^{2-N} \left( \mu^{N-2} + \frac{\epsilon}{\mu} \right) \right]
\]

where \( C > 0 \) in \((1.9)\) is chosen uniformly for \((d, \tau) \in \Lambda_{\delta}\).

Proof. We will treat \((3.7)\) only, because the proof of \((3.8)\) is similar. First, we see that \( R_1 \) is a harmonic function in \( \Omega \). We also have

\[
R_1(x) = \mu^\frac{N}{N+p} \left[ U(\tau) - U \left( \frac{x - \xi}{\mu} \right) \right] + a_{N, p} \mu^\frac{Np}{N+p} \tilde{H}(x, \xi)
\]

\[
= \mu^\frac{N}{N+p} \left[ U(\tau) - U \left( \tau - \frac{\epsilon}{\mu} \frac{x}{|x|} \right) \right] + O(1) \mu^\frac{Np}{N+p} = O(1) \mu^\frac{Np}{N+p} \left( \mu^{(N-2)p-2} + \frac{\epsilon}{\mu} \right).
\]
for $x \in \partial B(0, \epsilon)$, and by (3.10),
\[
R_1(x) = -\mu^{-\frac{N}{N+1}} U(\mu^{-1}(x - \xi)) + \frac{a_{N,p} \mu^{\frac{N}{N+1}}}{|x - \xi|(N-2)p-2} \mu^{-\frac{N}{N+1}} U(\tau) \frac{\tau^{N-2}}{|\tau|^{N-2}}
\]
\[
= \frac{\mu^{\frac{Np}{N+1}}}{|x - \xi|(N-2)p-2} \left[ -U(\mu^{-1}(x - \xi)) |\mu^{-1}(x - \xi)|^{(N-2)p-2} + a_{N,p} \right] + O(1) \mu^{-\frac{N}{N+1}} \epsilon^{N-2}
\]
\[
= O(1) \mu^{1+\frac{Np}{N+1}} + O(1) \mu^{-\frac{N}{N+1}} \epsilon^{N-2} = O(1) \mu^{\frac{Np}{N+1}} \left[ \mu + \left( \frac{\epsilon}{\mu} \right)^{N-2} \mu^{N-(N-2)p} \right] = O(1) \mu^{\frac{Np}{N+1}}
\]
for $x \in \partial \Omega$. Thus the maximum principle gives the upper bound for $|R_1|$.

\[\square\]

**Corollary 3.2.** Under the hypotheses of Lemma 3.1 we have
\[
|\nabla_{(d, \tau)} R_1(x)| = O(1) \left[ \mu^{\frac{Np}{N+1}} \kappa_0 + \epsilon^{N-2} \mu^{-\frac{N}{N+1}} |\mu|^{2-N} \left( \mu^{(N-2)p-2} + \frac{\epsilon}{\mu} \right) \right],
\]
\[
|\nabla_{(d, \tau)} R_2(x)| = O(1) \left[ \mu^{\frac{Np}{N+1}} \kappa_1 + \epsilon^{N-2} \mu^{-\frac{N}{N+1}} |\mu|^{2-N} \left( \mu^{N-2} + \frac{\epsilon}{\mu} \right) \right],
\]
for all $x \in \Omega_\epsilon$, where $C > 0$ in (1.9) is chosen uniformly for $(d, \tau) \in \Lambda_\delta$, and $\kappa_0, \kappa_1 \in (0, 1)$ are the numbers in (2.3).

**Proof.** Using (2.11) and (2.12), one can argue as in the proof of previous lemma. \[\square\]

3.3. **The second approximation of the solution.** It turns out that the $u$-component $PU_{\mu,\xi}$ of the first approximation ($PU_{\mu,\xi}, PV_{\mu,\xi}$) makes a huge error and we must refine it to build an actual solution to (1.6). Our idea is, motivated by Subsection 2.3 in [25], to employ the function $PU_{\mu,\xi}$ solving
\[
\begin{cases}
-\Delta PU_{\mu,\xi} = (PV_{\mu,\xi})^p & \text{in } \Omega_\epsilon, \\
PU_{\mu,\xi} = 0 & \text{on } \partial \Omega_\epsilon
\end{cases}
\]
as the second (i.e., refined) approximation for the $u$-component of the solution.

**Proposition 3.3.** Assume $N \geq 4$, $p \in (1, \frac{N}{N-2})$, and $\epsilon > 0$ small. Given an auxiliary number $\kappa \in (0, 1)$ small enough (determined by $N$ and $p$)\footnote{Working with $\kappa = 0$ gives the remainder term in the energy expansion in (2.2) and (1.17) too large.} and the parameters $(d, \tau) \in \Lambda_\delta$, we set
\[
A_{\kappa, (d, \tau)}(x) = \int_{B(0,\mu^{-1}) \setminus B(0, \epsilon \kappa^{\frac{1}{p}})} \frac{G_\epsilon(x, \mu y + \xi) V^{p-1}(y)}{|y + \tau|^{N-2}} \, dy \quad \text{for } x \in \Omega_\epsilon
\]
where $(\mu, \xi) \in (0, \infty) \times \mathbb{R}^N$ is defined in (3.3). Then it holds that
\[
\begin{align*}
PU_{\mu,\xi}(x) &= U_{\mu,\xi}(x) - \left( \frac{b_{N,p}}{\gamma N} \right)^p \mu^{\frac{Np}{N+1}} \frac{\mu}{N+1} H_\frac{N-1}{N+1}(x) \\
& \quad - \epsilon^{N-2} \left( \frac{\mu}{N+1} + \frac{N-1}{N+1} \right) \left[ U(\tau) |\tau|^{2-N} + pV(\tau) A_{\kappa, (d, \tau)}(x) \right] + R(x) \quad \text{for } x \in \Omega_\epsilon
\end{align*}
\]
Here, $H_\frac{N-1}{N+1}$ is the function in (2.5) with $y = \xi$ and
\[
R(x) := O(1) c^{(N-2)p} \mu^{-\frac{Np}{N+1}} |\tau|^{2-(N-2)p} + o(1) \left[ \mu^{\frac{Np}{N+1}} + \epsilon^{N-2} \mu^{-\frac{N}{N+1}} \left( |\tau|^{2-N} + A_{\kappa, (d, \tau)}(x) \right) \right]
\]
where $C, C_\epsilon > 0$ in (1.9) and (1.10) are chosen uniformly for $(d, \tau) \in \Lambda_\delta$. Furthermore,
\[
A_{\kappa, (d, \tau)}(x) = O(1) \mu^{(N-2)p-2} |\tau|^{2-(N-2)p} \quad \text{for } x \in \Omega_\epsilon
\]
where $C > 0$ in (1.9) is chosen uniformly for $(d, \tau) \in \Lambda_\delta$.\footnote{Working with $\kappa = 0$ gives the remainder term in the energy expansion in (2.2) and (1.17) too large.}
In contrast to the previous subsection, it is difficult to estimate \( \mathcal{P}U_{\mu, \xi} \) by exploiting the maximum principle. Instead, we conduct the potential analysis. More precisely, we estimate the right-hand side of the formula

\[
(\mathcal{P}U_{\mu, \xi} - PU_{\mu, \xi})(x) = \int_{\Omega_{\epsilon} \cap B(\xi, \mu^{\alpha})} G_\epsilon(x, y) \left( PV^p_{\mu, \xi} - V^p_{\mu, \xi} \right)(y) dy + \int_{\Omega_{\epsilon} \setminus B(\xi, \mu^{\alpha})} G_\epsilon(x, y) \left( PV^p_{\mu, \xi} - V^p_{\mu, \xi} \right)(y) dy,
\]

which holds for all \( x \in \Omega_{\epsilon} \) owing to Green’s representation formula.

In the setting of \( |x| \) and the leading-order term of the second integral is a multiple of \( H_\xi \). In our case, the leading-order term of the first integral is a multiple of \( A_{e,(d, \tau)} \) (so the first integral \textbf{does} contribute to the expansion of \( \mathcal{P}U_{\mu, \xi} - PU_{\mu, \xi} \)), while that of the second integral is a linear combination of \( H_\xi(x) \) and \( |x|^{2-N} \).

On the other hand, by analyzing (3.10), one sees that \( A_{e,(d, \tau)}(x) \ll |x|^{2-N} \) for \( \epsilon < |x| \ll \mu \) (refer to (3.13)) and \( A_{e,(d, \tau)}(x) \gg |x|^{2-N} \) for \( x \in \Omega_{\epsilon} \) away from 0, so \( A_{e,(d, \tau)}(x) \) and \( |x|^{2-N} \) are comparable in the pointwise sense. Nonetheless, they yield the same-order terms in the expansion (3.12) and (3.14) of the reduced energy \( J_\epsilon \) in (1.1). Remarkably, the seemingly ‘ugly’ function \( A_{e,(d, \tau)} \) produces a very near term (3.14) during the process of energy expansion.

\textbf{Proof of Proposition 3.3} Let us estimate the first integral on the right-hand side of (3.14). Using (3.8), (3.10) and the first inequality in (A.1), we evaluate

\[
\int_{\Omega_{\epsilon} \cap B(\xi, \mu^{\alpha})} G_\epsilon(x, y) \left( PV^p_{\mu, \xi} - V^p_{\mu, \xi} \right)(y) dy
\]

\[
= \int_{\Omega_{\epsilon} \cap B(\xi, \mu^{\alpha})} G_\epsilon(x, y) \left[ PV^p_{\mu, \xi}(y) \left\{ \begin{array}{l}
\frac{b_{N,p}}{\gamma_N} \mu^{\frac{N}{\gamma}} H(y, \xi) - \mu^{-\frac{N}{\gamma}} V_0(\tau) e^{\frac{N-2}{\gamma}} + R_2(y) \\
+ O(1) \left( \frac{\mu^{\frac{N}{\gamma}}}{y^{\frac{N}{\gamma}}} + \mu^{-\frac{N}{\gamma}} \frac{e^{(N-2)p}}{|y|^{(N-2)p}} \right) \end{array} \right\} dy
\]

\[
= - (1 + o(1)) e^{N-2} \mu^{-\frac{N}{\gamma}} p V_0(\tau) \int_{\Omega_{\epsilon} \cap B(\xi, \mu^{\alpha})} G_\epsilon(x, y) V^p_{\mu, \xi}(y) \frac{dy}{|y|^{N-2}}
\]

\[
+ O(1) \int_{\Omega_{\epsilon} \cap B(\xi, \mu^{\alpha})} \frac{1}{|x - y|^{N-2}} \left( V^p_{\mu, \xi}(y) \mu^{\frac{N}{\gamma}} + \mu^{-\frac{N}{\gamma}} \frac{e^{(N-2)p}}{|y|^{(N-2)p}} \right) dy
\]

\[
= - (1 + o(1)) e^{N-2} \mu^{-\frac{N}{\gamma}} p V_0(\tau) A_{e,(d, \tau)}(x) + O(1) e^{(N-2)p} \mu^{-\frac{N}{\gamma}} |x|^{-2(N-2)p} + o(1) \mu^{\frac{N}{\gamma}}
\]

for \( x \in \Omega_{\epsilon} \). Here, the third equality is valid because of (A.2),

\[
\mu^{-\frac{N}{\gamma}} \int_{\Omega_{\epsilon} \cap B(\xi, \mu^{\alpha})} \frac{G_\epsilon(x, y) V^p_{\mu, \xi}(y)}{|y|^{N-2}} dy = \mu^{-\frac{N}{\gamma}} \int_{B(0, \mu^{\alpha-1}) \setminus B(-\tau, \frac{\mu^{\alpha}}{2})} \frac{G_\epsilon(x, y + \xi) V^p_{\mu, \xi}(y)}{|y + \tau|^{N-2}} dy
\]

\[
= \mu^{-\frac{N}{\gamma}} A_{e,(d, \tau)}(x)
\]

and

\[
\int_{\Omega_{\epsilon} \setminus B(\xi, \mu^{\alpha})} \frac{dy}{|x - y|^{N-2}|y|^{(N-2)p}} \leq \int_{\mathbb{R}^N} \frac{dy}{|x - y|^{N-2}|y|^{(N-2)p}} = O(1) |x|^{2-N(2)p}.
\]

We next estimate the second integral on the right-hand side of (3.14). For \( y \in \Omega_{\epsilon} \setminus B(\xi, \mu^{\alpha}) \), we have

\[
V_{\mu, \xi}(y) = \mu^{-\frac{N}{\gamma}} V_0(\mu^{-1}(y - \xi))
\]

\[
= \mu^{-\frac{N}{\gamma}} \left[ b_{N,p} |\mu^{-1}(y - \xi)|^{2-N} + O(1) |\mu^{-1}(y - \xi)|^{1-N} \right] \quad \text{(by (2.10))}
\]

\[
= \mu^{-\frac{N}{\gamma}} \left[ b_{N,p} |y - \xi|^{2-N} + O(1) |y - \xi|^{1-N} \right] \quad \text{(by (1.4))}
\]
and so

\[
P V_{\mu,\xi}(y) = V_{\mu,\xi}(y) - \frac{b_{N,p}}{\gamma_N} \frac{\mu^{N}}{\tau^{N/\gamma_N}} H(y, \xi) - \mu^{-\frac{N}{\gamma_N}} V(\tau) \frac{e^{N-2}}{|y|^{N-2}} + R_2(y) \tag{3.18}
\]

\[
(3.17)
\]

\[
onumber
= \frac{b_{N,p}}{\gamma_N} \frac{\mu^{N}}{\tau^{N/\gamma_N}} \left[ \gamma_N |y - \xi|^{2-N} - H(y, \xi) \right] - \mu^{-\frac{N}{\gamma_N}} V(\tau) \frac{e^{N-2}}{|y|^{N-2}} + O(1) \mu^{N-1} \frac{1}{\tau^{N-1}} |y-\xi|^{1-N} + R_2(y)
\]

where

\[
\tilde{R}_2(y) := O(1) \mu^{-\frac{N}{\gamma_N}} |y - \xi|^{1-N} + R_2(y)
\]

Thus, for any \( x \in \Omega_t, \)

\[
\int_{\Omega_t \setminus B(\epsilon, \mu^\gamma)} G_\epsilon(x, y) V_{\mu,\xi}(y) dy
\]

\[
= \frac{b_{N,p}}{\gamma_N} \mu^{\frac{N}{\gamma_N}} \int_{\Omega_t \setminus B(\epsilon, \mu^\gamma)} G_\epsilon(x, y) |y - \xi|^{(2-N)p} \left( 1 + O(1) \mu^{1-N} \right) dy \tag{3.16}
\]

\[
(3.19)
\]

\[
\int_{\Omega_t \setminus B(\epsilon, \mu^\gamma)} G_\epsilon(x, y) |y - \xi|^{(2-N)p} dy
\]

\[
= O(1) \mu^{-\frac{N}{\gamma_N}} \mu^{1-N} \int_{\Omega_t \setminus B(\epsilon, \mu^\gamma)} |x - y|^{2-N} |y|^{(2-N)p} dy
\]

\[
= O(1) \mu^{-\frac{N}{\gamma_N}} \mu^{(1-N) + \kappa(2-(N-2)p)} \tag{A.1}
\]

and by (3.17) and the second inequality in (A.1),

\[
\int_{\Omega_t \setminus B(\epsilon, \mu^\gamma)} G_\epsilon(x, y) PV_{\mu,\xi}(y) dy
\]

\[
= \int_{\Omega_t \setminus B(\epsilon, \mu^\gamma)} G_\epsilon(x, y) \left( \frac{b_{N,p}}{\gamma_N} \frac{\mu^{N}}{\tau^{N/\gamma_N}} G(y, \xi) \right)^p dy
\]

\[
(3.20)
\]

\[
- \int_{\Omega_t \setminus B(\epsilon, \mu^\gamma)} G_\epsilon(x, y) \left| \mu^{-\frac{N}{\gamma_N}} V(\tau) \frac{e^{N-2}}{|y|^{N-2}} - \tilde{R}_2(y) \right|^{p-1} \left( \mu^{-\frac{N}{\gamma_N}} V(\tau) \frac{e^{N-2}}{|y|^{N-2}} - \tilde{R}_2(y) \right) dy
\]

\[
+ O(1) \int_{\Omega_t \setminus B(\epsilon, \mu^\gamma)} G_\epsilon(x, y) \left[ \mu^{-\frac{N}{\gamma_N}} G(y, \xi) \right]^{p-1} \left( \mu^{-\frac{N}{\gamma_N}} V(\tau) \frac{e^{N-2}}{|y|^{N-2}} - \tilde{R}_2(y) \right) dy
\]

\[
= \left( \frac{b_{N,p}}{\gamma_N} \right)^p \frac{\mu^{\frac{N}{\gamma_N}}}{\tau^{N/\gamma_N}} \int_{\Omega_t \setminus B(\epsilon, \mu^\gamma)} G_\epsilon(x, y) G^p(y, \xi) dy - e^{(N-2)p} \mu^{-\frac{N}{\gamma_N}} V^p(\tau) \int_{\Omega_t \setminus B(\epsilon, \mu^\gamma)} G_\epsilon(x, y) dy
\]

\[
+ O(1) R_3(x)
\]

where

\[
R_3(x) := \int_{\Omega_t \setminus B(\epsilon, \mu^\gamma)} G_\epsilon(x, y) \left[ \left( \mu^{-\frac{N}{\gamma_N}} G(y, \xi) \right)^{p-1} \left( \mu^{-\frac{N}{\gamma_N}} V(\tau) \frac{e^{N-2}}{|y|^{N-2}} - \tilde{R}_2(y) \right) \right] dy
\]

\[
+ \left( \mu^{-\frac{N}{\gamma_N}} V(\tau) \frac{e^{N-2}}{|y|^{N-2}} - \tilde{R}_2(y) \right)^{p-1} \tilde{R}_2(y) + |\tilde{R}_2(y)|^{p} dy.
\]
Owing to Lemma 3.5 and 3.6, it holds that

\[
\int_{\Omega \setminus B(\xi, \mu^*)} R_3(x)\, dy = o(1) \mu^{-\frac{N_p}{p+1}},
\]

\[
\int_{\Omega \setminus B(\xi, \mu^*)} \frac{G_e(x, y)}{|y|^{(N-2)p}} \, dy = O(1) \int_{\mathbb{R}^N} \frac{dy}{|x-y|^{N-2}|y|^{(N-2)p}} = O(1)|x|^{2-(N-2)p}
\]  

(3.22)

for \( \kappa \in (0, 1) \) small. Hence, we see from (3.19), (3.20), and (3.22) that

\[
\int_{\Omega \setminus B(\xi, \mu^*)} G_e(x, y) P V_{\mu, \xi}(y) \, dy = \int_{\Omega \setminus B(\xi, \mu^*)} G_e(x, y) |y - \xi|^{(2-N)p} \, dy + o(1) \mu^{-\frac{N_p}{p+1}}
\]

and

\[
\int_{\Omega \setminus B(\xi, \mu^*)} G_e(x, y) P V_{\mu, \xi}(y) \, dy = \left( \frac{b(N)}{\gamma_N} \right)^p \mu^{-\frac{N_p}{p+1}} \int_{\Omega \setminus B(\xi, \mu^*)} G_e(x, y) |y - \xi|^{(2-N)p} \, dy + O(1) \mu^{-\frac{N_p}{p+1}}
\]

(3.23)

for \( \kappa \in (0, 1) \) small. Combining these yields

\[
\int_{\Omega \setminus B(\xi, \mu^*)} G_e(x, y) \left( P V_{\mu, \xi} - V_{\mu, \xi} \right)(y) \, dy
\]

\[
= \left( \frac{b(N)}{\gamma_N} \right)^p \mu^{-\frac{N_p}{p+1}} \int_{\Omega \setminus B(\xi, \mu^*)} G_e(x, y) |y - \xi|^{(2-N)p} \, dy + O(1) \mu^{-\frac{N_p}{p+1}}
\]

(3.24)

We write

\[
\int_{\Omega \setminus B(\xi, \mu^*)} G_e(x, y) \left( |y|^{(2-N)p} - |\gamma_N N^{(2-N)p} - \gamma_N |\right) \, dy
\]

\[
= \int_{B(\xi, \mu^*)} G(x, y) \left( |y|^{(2-N)p} - |\gamma_N N^{(2-N)p} - \gamma_N |\right) \, dy - \int_{\Omega \setminus B(\xi, \mu^*)} G(x, y) \left( |y|^{(2-N)p} - |\gamma_N N^{(2-N)p} - \gamma_N |\right) \, dy
\]

(3.25)

where \( \mathcal{R}(x, y) = H_e(x, y) - H_{e, 1}(x, y) - H(x, y) \). We observe from the first inequality in (A.1) that

\[
\int_{B(\xi, \mu^*)} G(x, y) \left( |y|^{(2-N)p} - |\gamma_N N^{(2-N)p} - \gamma_N |\right) \, dy
\]

\[
= \gamma_N \int_{B(\xi, \mu^*)} G(x, y) |y - \xi|^{(2-N)p} \left[ (1 - \gamma_N^{-1} |y - \xi|^{(2-N)H(y, \xi)})^p - 1 \right] \, dy
\]

\[
= O(1) \mu^{-\frac{N_p}{p+1}} \int_{B(\xi, \mu^*)} \frac{dy}{|x-y|^{N-2}|y|^{(N-2)p+1}} = O(1) \mu^{-\frac{N_p}{p+1}}
\]

(3.25)

and

\[
\int_{\Omega \setminus B(\xi, \mu^*)} \mathcal{R}(x, y) \left( |y|^{(2-N)p} - |\gamma_N N^{(2-N)p} - \gamma_N |\right) \, dy
\]

\[
= O(1) \int_{\Omega \setminus B(\xi, \mu^*)} \left( |y|^{(2-N)p} - |\gamma_N N^{(2-N)p} - \gamma_N |\right) \, dy
\]

\[
= O(1) \mu^{-\frac{N_p}{p+1}} \int_{\Omega \setminus B(\xi, \mu^*)} |y|^{(2-N)p} \, dy = O(1) \mu^{-\frac{N_p}{p+1}}
\]

(3.26)

where the second equation of (3.26) follows from Lemma 2.4 and

\[
|y| \left( \frac{e^2 y}{|y|^2} \right)^{2-N} = |y|^{2-N} |y|^{2-N} \frac{x}{|x|} - \frac{e^2 y}{|x|^2} \right)^{2-N} = O(1) (1 + \epsilon \mu^{-\kappa}) |x|^{2-N} |y|^{2-N}
\]
for \( x \in \Omega \) and \( y \in \Omega \setminus B(\xi, \mu^c) \). Therefore, if we set \( \varphi_\xi \) as the solution to

\[
\begin{align*}
-\Delta \varphi_\xi &= G^p(\cdot, \xi) - \gamma_N^p \cdot |\cdot|^{(2-N)p} \quad \text{in } \Omega, \\
\varphi_\xi &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

then

\[
\int_{\Omega \setminus B(\xi, \mu^c)} G_\epsilon(x, y) \left( G^p(y, \xi) - \gamma_N^p |y - \xi|^{(2-N)p} \right) dy = \varphi_\xi(x) + O(1)\mu^{c(N-(N-2)p)} + O(1)\epsilon^{N-2}|x|^{2-N}. \tag{3.27}
\]

Plugging this into (3.23), we conclude

\[
\int_{\Omega \setminus B(\xi, \mu^c)} G_\epsilon(x, y) \left( PV_{\mu, \xi} - V_{\mu, \xi}(y) \right) dy = \left( \frac{b_N^p}{\gamma_N} \right)^p \mu^{\frac{Np}{N-1}} \varphi_\xi(x) + O(1)\epsilon^{(N-2)p} \mu^{-\frac{Np}{N-1}}|x|^{2-(N-2)p} + o(1)\mu^{\frac{Np}{N-1}} + O(1)\epsilon^{N-2}|x|^{2-N}. \tag{3.28}
\]

As a result, we infer from (3.10), (3.7), (3.15), (3.10), (3.28), (2.8) and the identity

\[
\tilde{H}_\xi(x) = \gamma_N \mu \tilde{H}(x, \xi) - \varphi_\xi(x) \quad \text{(by (3.10), (2.4), (3.10) and (4.2))}
\]

that

\[
P_\mu, \xi(x) = U_\mu, \xi(x) - a_\mu, \xi(x) - \frac{Np}{N-1} \mu^{\frac{Np}{N-1}} \tilde{H}(x, \xi) + \left( \frac{b_N^p}{\gamma_N} \right)^p \mu^{\frac{Np}{N-1}} \varphi_\xi(x) - \mu^{\frac{Np}{N-1}} U(\tau) \epsilon^{N-2} x|\tilde{H}(x, \xi) - \frac{Np}{N-1} \mu^{\frac{Np}{N-1}} \tilde{H}(x, \xi) - \mu^{\frac{Np}{N-1}} U(\tau) \epsilon^{N-2} x
\]

\[
- \mu^{\frac{Np}{N-1}} U(\tau) \epsilon^{N-2} x
\]

\[
- (1 + o(1))\epsilon^{N-2} \mu^{-\frac{Np}{N-1}} \tilde{H}(x, \xi) - \mu^{\frac{Np}{N-1}} U(\tau) \epsilon^{N-2} x
\]

\[
- (1 + o(1))\epsilon^{N-2} \mu^{-\frac{Np}{N-1}} \tilde{H}(x, \xi) - \mu^{\frac{Np}{N-1}} U(\tau) \epsilon^{N-2} x
\]

This concludes the proof of (3.11).

Finally, we note from (3.10) that

\[
\mathcal{A}_{\epsilon, \mu}(x, \tau) = \gamma_N \mu^{2-N} \int_R dy|\varphi_\xi(x) - \tau^{(2-N)p} x^{2-(N-2)p}|
\]

\[
= O(1)\mu^{2-N} |\varphi_\xi(x) - \tau^{(2-N)p} x^{2-(N-2)p}|
\]

so (3.13) is true. \(\square\)

The papers [25, 24] show that the single- and multi-bubble solutions to slightly subcritical Lane-Emden systems are characterized as local minima of the reduced energy. Hence, to construct them, one only needs the \(C^0\)-estimate (corresponding to our (4.2)) of the reduced energy. In contrast, the solutions to (1.6) found here will be characterized as saddle points of the reduced energy, which forces us to derive its \(C^1\)-estimate. The following corollary, whose proof is not a mere adaptation of the proof of Proposition 3.3, roles as a key result in such estimate; refer to Proposition 4.3 and Lemma 4.6.

**Proposition 3.4.** Under the hypotheses of Proposition 3.3 we have

\[
|\nabla_{\mu, \tau}(x)| = O(1)\epsilon^{(N-2)p} \mu^{-\frac{Np}{N-1}} |x|^{2-(N-2)p}
\]

\[
+ o(1) \left[ \mu^{\frac{Np}{N-1}} + \epsilon^{N-2} \mu^{-\frac{Np}{N-1}} \left( |x|^{2-N} + \mathcal{A}_{\epsilon, \mu}(x) \right) \right] \quad \text{for } x \in \Omega \tag{3.29}
\]
where $C, C_\epsilon > 0$ in (13.9) and (13.10) are chosen uniformly for $(d, \tau) \in \Lambda_\delta$. Furthermore,
\[
|\nabla_{(d,\tau)} A_{\epsilon,(d,\tau)}(x)| = O(1)\mu^{(N-2)p-N} |x|^{2-(N-2)p} \quad \text{for } x \in \Omega,
\]
where $C > 0$ in (13.9) is chosen uniformly for $(d, \tau) \in \Lambda_\delta$.

**Proof.** Here, we only consider the differentiation of the maps $R$ and $A_{\epsilon,(d,\tau)}$ with respect to the $d$-variable, since the differentiation with respect to the $\tau$-variable can be handled in a similar way.

Let us prove (3.29) (where $\nabla_{(d,\tau)}$ is replaced with $\nabla_d$). Thanks to Corollary 3.2 we only have to estimate the map $(d, \tau) \in \Lambda_\delta \mapsto \mathcal{P}U_{\mu,\xi} - \mathcal{P}U_{\mu,\xi}$. We write
\[
\nabla_d (\mathcal{P}U_{\mu,\xi} - \mathcal{P}U_{\mu,\xi})(x)
\]
\[
= p \int_{\Omega_\tau \cap B(\xi,\mu^n)} G_\epsilon(x,y) \left( (\nabla_d V_{\mu,\xi}) \left( (PV_{\mu,\xi}^{p-1} - V_{\mu,\xi}^{p-1}) + PV_{\mu,\xi}^{p-1} \{\nabla_d (PV_{\mu,\xi} - V_{\mu,\xi}) \} \right) \right) \, dy
\]
\[
+ p \int_{\Omega_\tau \cap B(\xi,\mu^n)} G_\epsilon(x,y) \left( PV_{\mu,\xi}^{p-1} (\nabla_d PV_{\mu,\xi}) - V_{\mu,\xi}^{p-1} (\nabla_d V_{\mu,\xi}) \right) \, dy
\]
for $x \in \Omega_\tau$.

Let us estimate the first term in the right-hand side of (3.31). Using the previous computations in (3.15), the fact that $|\nabla_d V_{\mu,\xi}| \leq CV_{\mu,\xi}$ in $\mathbb{R}^N$ for some $C > 0$ depending only on $N$ and $p$, and the last inequality in (3.1.1), we deduce that
\[
\int_{\Omega_\tau \cap B(\xi,\mu^n)} G_\epsilon(x,y) (\nabla_d V_{\mu,\xi})(y) \left( PV_{\mu,\xi}^{p-1} - V_{\mu,\xi}^{p-1} \right) \, dy
\]
\[
= (p - 1) \int_{\Omega_\tau \cap B(\xi,\mu^n)} G_\epsilon(x,y) \left( (\nabla_d V_{\mu,\xi})V_{\mu,\xi}^{p-2} (PV_{\mu,\xi} - V_{\mu,\xi}) + O(1)PV_{\mu,\xi} - V_{\mu,\xi}^p \right) \, dy
\]
\[
= -(1 + o(1)) \epsilon N^{-2} \mu^{-N} V(\tau) \int_{\Omega_\tau \cap B(\xi,\mu^n)} \frac{G_\epsilon(x,y) (\nabla_d V_{\mu,\xi}^{p-1})(y)}{|y|^{N-2}} \, dy
\]
\[
+ O(1) \epsilon^{(N-2)p} \mu^{-N} \frac{N}{p+1} |x|^{-2(N-2)p} + o(1) \mu^{Np}
\]
Applying Corollary 3.2 and the chain of inequalities $0 \leq b^{p-1} - a^{p-1} \leq (b - a)^{p-1}$ for $0 < a \leq b$, we also compute
\[
\int_{\Omega_\tau \cap B(\xi,\mu^n)} G_\epsilon(x,y) PV_{\mu,\xi}^{p-1} \{\nabla_d (PV_{\mu,\xi} - V_{\mu,\xi}) \} \, dy
\]
\[
= (1 + o(1)) \left( \frac{N}{p+1} \right) \epsilon^{-N-2} \mu^{-N} - \frac{N}{p+1} V(\tau) \int_{\Omega_\tau \cap B(\xi,\mu^n)} \frac{G_\epsilon(x,y) V_{\mu,\xi}^{p-1}(y)}{|y|^{N-2}} \, dy
\]
\[
+ O(1) \int_{\Omega_\tau \cap B(\xi,\mu^n)} \frac{1}{|x-y|^{N-2}} \left[ V_{\mu,\xi}^{p-1}(y) \mu^{-N} \right. \left. + \left( \frac{N}{p+1} \right)^{N-(p+1)} \left( \frac{N}{p+1} \left( \frac{N}{p+1} \right)^{-N} \right) \right] \, dy
\]
\[
= -(1 + o(1)) \epsilon N^{-2} \nabla_d \left( \mu^{-N} \right) V(\tau) \int_{\Omega_\tau \cap B(\xi,\mu^n)} \frac{G_\epsilon(x,y) V_{\mu,\xi}^{p-1}(y)}{|y|^{N-2}} \, dy
\]
\[
+ O(1) \epsilon^{(N-2)p} \mu^{-N} |x|^{2-(N-2)p} + o(1) \mu^{Np} + o(1) \mu^{-N} \epsilon^{N-2} |x|^{2-N}.
\]
To derive the second equality in (3.33), we employed the computations in (3.15) and the identities
\[
\frac{N}{p+1} \epsilon^{N-2}(p-1) \int_{\Omega_\tau \cap B(\xi,\mu^n)} \frac{dy}{|x-y|^{-2(N-2)(p-1)}} = O(1) \mu^{Np} \left( \frac{\epsilon}{\mu} \right)^{N-2(p-1)}
\]
\[
= o(1) \mu^{Np} \quad \text{for } p > 1
\]
We next estimate the second term in the right-hand side of (3.31). Since
\[ \nabla_d \left[ \epsilon^{N-2} \mu^{-\frac{N}{p+1}} V(\tau) \int_{\Omega_\epsilon \setminus B(\xi, \mu^\kappa)} G_\epsilon(x, y) \frac{V_{\mu, \xi}^{p-1}(y)}{|y|^{N-2}} dy \right] = o(1) \mu^{\frac{N}{p+1}} \]
for \( \kappa \in (0, 1) \) small, we have
\[ \nabla_d \left[ \epsilon^{N-2} \mu^{-\frac{N}{p+1}} V(\tau) \int_{\Omega_\epsilon \setminus B(\xi, \mu^\kappa)} G_\epsilon(x, y) V_{\mu, \xi}^{p-1}(y) \frac{dy}{|y|^{N-2}} \right] = o(1) \mu^{\frac{N}{p+1}}. \]

As a result, combining (3.32) and (3.33), we obtain that
\[ p \int_{\Omega_\epsilon \setminus B(\xi, \mu^\kappa)} G_\epsilon(x, y) \left[ \nabla_d V_{\mu, \xi} \right] \left( PV_{\mu, \xi}^{p-1} - V_{\mu, \xi}^{p-1} \right) + PV_{\mu, \xi}^{p-1} \{ \nabla_d (PV_{\mu, \xi} - V_{\mu, \xi}) \} (y) dy \]
\[ = -\nabla_d \left[ \epsilon^{N-2} \mu^{-\frac{N}{p+1}} V(\tau) \int_{\Omega_\epsilon \setminus B(\xi, \mu^\kappa)} G_\epsilon(x, y) V_{\mu, \xi}^{p-1}(y) \frac{dy}{|y|^{N-2}} \right] \]
\[ + O(1) \epsilon^{(N-2)p} \mu^{-\frac{N}{p+1}} |x|^{2-(N-2)p} + o(1) \left[ \mu^{\frac{N}{p+1}} + \mu^{-\frac{N}{p+1}} \epsilon^{N-2} \left( |x|^{2-N} + A_{x, (d, \tau)}(x) \right) \right]. \]
for $\kappa \in (0, 1)$ small. Moreover, by (3.37) and (3.38),

$$PV_{\mu, \xi}^{p-1}(y) = \left( \frac{b_{N,p}}{\gamma_N} \mu^{\frac{N}{p+1}} G(y, \xi) \right)^{p-1} + O(1) \left( \frac{\varepsilon^{N-2} y_{\mu}^{\frac{N}{p+1}}}{|y|^{N-2}} \right) + O(1) \mu^{\frac{2(p-1)}{p+1} + \bar{\kappa}}$$

and

$$(\nabla_d PV_{\mu, \xi})(y) = \frac{N d^{-1}}{q+1} \left( \frac{b_{N,p}}{\gamma_N} \mu^{\frac{N}{p+1}} G(y, \xi) \right) - \frac{b_{N,p}}{\gamma_N} \mu^{\frac{N}{p+1}} \nabla_d (H(y, \xi)) + O(1) \left( \mu^{\frac{N}{p+1}} \frac{\varepsilon^{N-2}}{|y|^{N-2}} + \mu^{\frac{N}{p+1} + \bar{\kappa}} \right)$$

for $y \in \Omega_\epsilon \setminus B(\xi, \mu^c)$, where $\bar{\kappa} \in (0, 1)$ is a small number. Hence

$$\int_{\Omega_\epsilon \setminus B(\xi, \mu^c)} G_{\epsilon}(x, y)(\nabla_d PV_{\mu, \xi})(y) PV_{\mu, \xi}^{p-1}(y) dy$$

$$= \left( \frac{b_{N,p}}{\gamma_N} \right)^p \mu^{\frac{N}{p+1}} \int_{\Omega_\epsilon \setminus B(\xi, \mu^c)} G_{\epsilon}(x, y) \left[ \frac{N d^{-1}}{q+1} \left( \frac{b_{N,p}}{\gamma_N} \mu^{\frac{N}{p+1}} G(y, \xi) - G_{\epsilon}^{p-1}(y, \xi) \nabla_d (H(y, \xi)) \right) \right] dy + o(1) \mu^{\frac{N}{p+1}}$$

(3.38)

$$= \left( \frac{b_{N,p}}{\gamma_N} \right)^p \mu^{\frac{N}{p+1}} \int_{\Omega_\epsilon \setminus B(\xi, \mu^c)} G_{\epsilon}(x, y) \left[ \frac{N d^{-1}}{q+1} \left( \frac{b_{N,p}}{\gamma_N} \mu^{\frac{N}{p+1}} G(y, \xi) - G_{\epsilon}^{p-1}(y, \xi) \nabla_d (\gamma_N^{p-1}|y - \xi|^{(2-N)p} - G_{\epsilon}(y, \xi)) \right) \right] dy$$

$$+ o(1) \mu^{\frac{N}{p+1}}$$

for $\kappa \in (0, 1)$ small, where the second equality in (3.38) follows from the estimate

$$\int_{\Omega_\epsilon \setminus B(\xi, \mu^c)} G_{\epsilon}(x, y) G_{\epsilon}^{p-1}(y, \xi) \nabla_d (\gamma_N^{p-1}|y - \xi|^{(2-N)p}) dy$$

$$= \int_{\Omega_\epsilon \setminus B(\xi, \mu^c)} G_{\epsilon}(x, y) \left( \gamma_N^{p-1}|y - \xi|^{(2-N)(p-1)} + O(1) \right) \nabla_d (\gamma_N^{p-1}|y - \xi|^{2-N}) dy$$

$$= \int_{\Omega_\epsilon \setminus B(\xi, \mu^c)} G_{\epsilon}(x, y) \left( \gamma_N^{p-1}|y - \xi|^{(2-N)p} \right) dy + O(1) \int_{\Omega_\epsilon \setminus B(\xi, \mu^c)} |x - y|^{2-N} |y - \xi|^{1-N} dy$$

$$= \int_{\Omega_\epsilon \setminus B(\xi, \mu^c)} G_{\epsilon}(x, y) \nabla_d (\gamma_N^{p-1}|y - \xi|^{(2-N)p}) dy + O(1) \mu^{1-N(N-3)}$$

(3.37)

(3.40)

Summing (3.37) and (3.38) shows

$$\int_{\Omega_\epsilon \setminus B(\xi, \mu^c)} G_{\epsilon}(x, y) \left[ PV_{\mu, \xi}^{p-1}(\nabla_d PV_{\mu, \xi}) - V_{\mu, \xi}^{p-1}(\nabla_d V_{\mu, \xi}) \right] (y) dy$$

$$= \frac{N p}{q+1} \left( \frac{b_{N,p}}{\gamma_N} \right)^p d^{-1} \mu^{\frac{N}{p+1}} \int_{\Omega_\epsilon \setminus B(\xi, \mu^c)} G_{\epsilon}(x, y) \left( G(y, \xi) - \gamma_N^p |y - \xi|^{(2-N)p} \right) dy$$

(3.39)

$$= \frac{N p}{q+1} \left( \frac{b_{N,p}}{\gamma_N} \right)^p d^{-1} \mu^{\frac{N}{p+1}} \int_{\Omega_\epsilon \setminus B(\xi, \mu^c)} G_{\epsilon}(x, y) \nabla_d \left( G(y, \xi) - \gamma_N^p |y - \xi|^{(2-N)p} \right) dy + o(1) \mu^{\frac{N}{p+1}}.$$
Finally, by putting (3.31), (3.35), (3.41) and Corollary 3.2 together, we establish (3.29).

Also, (3.33) and the fact that
\[
\int_{(d,t)} \frac{d\sigma_y}{|x-y|^{N-2}} = \mu^{\kappa(N-1)} \int_{B(0,1)} |\mu y - (x-\xi)|^{N-2}
\]
implies (3.30) (where \(\nabla_{(d,t)}\) is replaced with \(\nabla_d\)).

4. Estimate for the reduced energy

Recall the energy functional \(I_\epsilon\) in (2.3), the set \(\Lambda_\delta\) in (3.1), the relation (3.3) between \((d,\tau)\in \Lambda_\delta\) and \((\mu, \xi)\in (0, \infty)\times \mathbb{R}^N\), and the pair \((\mathcal{P}U_{\mu, \xi}, PV_{\mu, \xi})\) defined by (3.5) and (3.7). Let \(J_\epsilon : \Lambda_\delta \to \mathbb{R}\) be a reduced energy
\[
J_\epsilon(d,\tau) = I_\epsilon(\mathcal{P}U_{\mu, \xi}, PV_{\mu, \xi}) = \int_{\Omega_c} \left[ \nabla \mathcal{P}U_{\mu, \xi} \cdot \nabla PV_{\mu, \xi} - \frac{1}{p+1} (PV_{\mu, \xi})^{p+1} - \frac{1}{q+1} (\mathcal{P}U_{\mu, \xi})^{q+1} \right].
\]

In the next two propositions, we expand \(J_\epsilon\) in the \(C^0\)- and \(C^1\)-sense, respectively.

Proposition 4.1. Assume \(N \geq 4\), \(p \in (1, \frac{N}{N-2})\), and \(\epsilon > 0\) small. Then it holds that
\[
J_\epsilon(d,\tau) = \frac{2}{N} \int_{\mathbb{R}^N} U^{q+1} + \frac{\mu^{(N-2)p-2}}{p+1} \left[ \left( \frac{bN}{\gamma N} \right)^p \int_{\mathbb{R}^N} U^q + (p+1)\gamma N^{1-d^2N}U(\tau)V(\tau) \right] + o(1)\mu^{(N-2)p-2}
\]
where \(\bar{H}_0\) is the function in (2.5) with \(y = 0\) and \(C_c > 0\) in (1.10) is chosen uniformly for \((d,\tau)\in \Lambda_\delta\).

Proof. Because
\[
\int_{\Omega_c} (PV_{\mu, \xi})^{p+1} = \int_{\Omega_c} (-\Delta \mathcal{P}U_{\mu, \xi}) PV_{\mu, \xi} = \int_{\Omega_c} \nabla \mathcal{P}U_{\mu, \xi} \cdot \nabla PV_{\mu, \xi}
\]
we know that
\[
J_\epsilon(d,\tau) = \int_{\Omega_c} \left[ \frac{p}{p+1} (PV_{\mu, \xi}) U_{\mu, \xi}^q - \frac{q}{q+1} (PV_{\mu, \xi})^{q+1} \right]
\]
\[
= \left( \frac{p}{p+1} - \frac{1}{q+1} \right) \int_{\Omega_c} U^{q+1} + \left( \frac{p}{p+1} - 1 \right) \int_{\Omega_c} (\mathcal{P}U_{\mu, \xi} - U_{\mu, \xi}) U_{\mu, \xi}^q
\]
\[
- \frac{1}{q+1} \int_{\Omega_c} [(\mathcal{P}U_{\mu, \xi})^{q+1} - U_{\mu, \xi}^{q+1} - (q+1)(\mathcal{P}U_{\mu, \xi} - U_{\mu, \xi}) U_{\mu, \xi}^q]
\]
(4.3)
\[
= \frac{2}{N} \int_{\Omega_c} U^{q+1} - \frac{q}{q+1} \int_{\Omega_c} (\mathcal{P}U_{\mu, \xi} - U_{\mu, \xi}) U_{\mu, \xi}^q
\]
(4.4)
\[
- \frac{q}{2} \int_{\Omega_c} (t_\epsilon \mathcal{P}U_{\mu, \xi} + (1-t_\epsilon)U_{\mu, \xi})^{q-1} (\mathcal{P}U_{\mu, \xi} - U_{\mu, \xi})^2 dx
\]
where \( t_x \in [0, 1] \).

For the first and last terms on the rightmost side of (4.3), we have

\[
\int_{\Omega} U_{\mu, \xi}^{q+1} = \int_{\Omega} U_{\mu, \xi}^{q+1} + \int_{B(-\tau, \phi)} U_{\mu, \xi}^{q+1} + \int_{\Omega \setminus \mu^{-1}(\Omega - \xi)} U_{\mu, \xi}^{q+1}
\]

\[
= \int_{\Omega} U_{\mu, \xi}^{q+1} + O(1) \left[ \left( \frac{\mu}{\mu} \right)^N + \mu^N \right] = \int_{\Omega} U_{\mu, \xi}^{q+1} + o(1)\mu \rho (N^2 - 2) p - 2 \quad \text{(by (3.4))}
\]

and

\[
0 \leq \int_{\Omega_x} (t_x PU_{\mu, \xi} + (1 - t_x) U_{\mu, \xi})^{q-1} (PU_{\mu, \xi} - U_{\mu, \xi})^2 dx \leq \int_{\Omega_x} U_{\mu, \xi}^{q-1} (PU_{\mu, \xi} - U_{\mu, \xi})^2,
\]

for \( 0 \leq PU_{\mu, \xi} \leq U_{\mu, \xi} \) in \( \Omega_x \). If we define

\[
Q_{\mu, \xi}(x) = \int_{\Omega_x} U_{\mu, \xi}^{\mu^N} \left[ \frac{2N + 1}{\mu^N} + \left( \frac{\mu}{\mu} \right)^N \right] \mu^N \nabla^2 \mu_{\xi}(x) + O(1) \mu \rho (N - 2)
\]

\[
\mu \rho \nabla^2 \mu_{\xi}(x) + O(1) \mu \rho (N - 2) \mu^N \nabla^2 \mu_{\xi}(x)
\]

for \( x \in \Omega_x \), (4.6)

For the second term in the rightmost side of (4.3), we see from (3.11) that

\[
\int_{\Omega_x} (t_x PU_{\mu, \xi} + (1 - t_x) U_{\mu, \xi})^{q-1} (PU_{\mu, \xi} - U_{\mu, \xi})^2 dx = O(1)Q_{\mu, \xi} = o(1)\mu \rho (N^2 - 2) p - 2.
\]

Applying Lemma 2.2, 2.7, 2.9, and Green’s representation formula, we compute

\[
\mu \rho \nabla^2 \mu_{\xi}(x) = \mu \rho \nabla^2 \mu_{\xi}(x) + O(1)\rho \mu \rho (N - 2) \mu^N \nabla^2 \mu_{\xi}(x)
\]

\[
\mu \rho \nabla^2 \mu_{\xi}(x) + O(1) \mu \rho (N - 2) \mu^N \nabla^2 \mu_{\xi}(x)
\]

and

\[
\epsilon^{N-2} \mu \rho \nabla^2 \mu_{\xi}(x) = \left( \frac{\epsilon}{\mu} \right)^{N-2} \int_{\Omega_x} \frac{(-\Delta \mu)(x)}{|x + \tau|^N} dx
\]

\[
= \left( \frac{\epsilon}{\mu} \right)^{N-2} \int_{\Omega_x} \frac{\gamma_1 \mu_1(\tau)}{|x + \tau|^{N-2}} dx + O(1) \left( \frac{\epsilon}{\mu} \right)^2
\]

\[
= \left( \frac{\epsilon}{\mu} \right)^{N-2} \left( \gamma_1 \mu_1(\tau) + o(1) \right).
\]

Moreover, Lemma A.3 and 4 imply that

\[
\epsilon^{N-2} \mu \rho \nabla^2 \mu_{\xi}(x) = \epsilon^{N-2} \mu \rho \nabla^2 \mu_{\xi}(x) = O(1) \left( \frac{\epsilon}{\mu} \right)^{N-2} = o(1)\mu \rho (N^2 - 2) p - 2.
\]
Next, using (3.10), we observe
\[
\epsilon^{N-2} \frac{-N}{N+1} \int_{\Omega_{\epsilon}} U(x) \mathcal{A}_{\epsilon}(d,\tau) \, dx
\]
and
\[
\int_{\Omega_{\epsilon}} U(x) \mathcal{A}_{\epsilon}(d,\tau) \, dx
\]
Then Lemma A.7 allows us to deduce
\[
(II) = o(1) \mu_{(N-2)p-2}.
\]
Let us estimate (I). By virtue of Fubini’s theorem and (4.12),
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{U(x)Vp^{-1}(y)}{|x-y|^{N-2}|y+\tau|^{N-2}} \, dx \, dy = \int_{\mathbb{R}^N} \frac{Vp^{-1}(y)}{|y+\tau|^{N-2}} \int_{\mathbb{R}^N} \frac{U(x)}{|x-y|^{N-2}} \, dx \, dy
\]
\[
= \gamma_N^{-1} \int_{\mathbb{R}^N} \frac{Vp(y)}{|y+\tau|^{N-2}} \, dy = \gamma_N^{-2} U(\tau),
\]
and
\[
= O(1) \int_{\mathbb{R}^N} \frac{Vp^{-1}(y)}{|y+\tau|^{N-2}} \left[ \int_{\mathbb{R}^N \setminus B(0,|C\mu|^{-1})} \frac{dx}{|x-y|^{N-2}|x-\tau|^{N-2}} + \int_{B(0,|C\mu|^{-1})} \frac{dx}{|x-y|^{N-2}|x-\tau|^{N-2}} \right] \, dy
\]
\[
= O(1) \mu_{(N-2)p-2} \int_{\mathbb{R}^N} \frac{Vp^{-1}(y)}{|y+\tau|^{N-2}} \min \left\{ 1, |\mu y|^{2-N} \right\} \, dy
\]
and
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N \setminus B(0,\mu^{-1})} \frac{U(x)Vp^{-1}(y)}{|x-y|^{N-2}|y+\tau|^{N-2}} \, dx \, dy
\]
\[
= \int_{\mathbb{R}^N \setminus B(0,\mu^{-1})} \frac{Vp^{-1}(y)}{|y+\tau|^{N-2}} \int_{\mathbb{R}^N} \frac{U(x)}{|x-y|^{N-2}} \, dx \, dy
\]
\[
= \gamma_N^{-1} \int_{\mathbb{R}^N \setminus B(0,\mu^{-1})} \frac{Vp(y)}{|y+\tau|^{N-2}} \, dy
\]
Proposition 4.2. Thus, from (4.3), (4.4), (4.5), (4.7) and (4.16), we arrive at (4.2).

Consequently,

\[ \text{(I)} = (1 + o(1)) \left( \frac{\epsilon}{\mu} \right)^{N-2} \gamma_N^{-1} U(\tau). \quad (4.14) \]

In view of (4.12)–(4.14), we conclude that

\[ \epsilon^{N-2} \mu^{-\frac{N}{N-2}} V(\tau) \int_{\Omega} U_{\mu,\xi}^q \frac{\nabla_{(d,\tau)} \left( b_{N,p} \gamma_N \right)^p \tau \tilde{H}_0(0) \int_{\mathbb{R}^N} U^q + (p + 1)\gamma_N^{-1} d^{-N} U(\tau)V(\tau)}{d^{(N-2)p-2}}. \quad (4.15) \]

Combining (4.8)–(4.11) and (4.15), we establish

\[ \int_{\Omega} (\mathcal{P}U_{\mu,\xi} - U_{\mu,\xi}) U_{\mu,\xi}^{q-1} \mu e^{-\frac{N}{N-2}} U(\tau) \int_{\mathbb{R}^N} U^q + (p + 1)\gamma_N^{-1} d^{-N} U(\tau)V(\tau) + o(1) \mu e^{-\frac{N}{N-2}}. \quad (4.16) \]

Hence, from (4.8), (4.9), (4.10), (4.11) and (4.16), we arrive at (4.2).

**Proposition 4.2.** Under the hypotheses of Proposition 4.1, we have

\[ \nabla_{(d,\tau)} J_{\epsilon}(d, \tau) = \left( \frac{\mu e^{-\frac{N}{N-2}}}{p + 1} \right) \nabla_{(d,\tau)} \left[ \left( \frac{b_{N,p}}{\gamma_N} \right)^p \tau \tilde{H}_0(0) \int_{\mathbb{R}^N} U^q + (p + 1)\gamma_N^{-1} d^{-N} U(\tau)V(\tau) \right] + o(1) \mu e^{-\frac{N}{N-2}} \]

where \( C_{\epsilon} > 0 \) in (4.10) is chosen uniformly for \((d, \tau) \in \Lambda_{\delta} \).

**Proof.** Here, we only consider the differentiation of the map \( J_{\epsilon} \) with respect to the \( d \)-variable.

By (4.1), (3.15), (3.9), (3.11)–(3.13) and (3.29)–(3.30), we have

\[ \nabla_{(d,\tau)} J_{\epsilon}(d, \tau) = \int_{\Omega} \left[ U_{\mu,\xi} - (\mathcal{P}U_{\mu,\xi})^q \right] \nabla_{(d,\tau)} U_{\mu,\xi}^{q-1} \nabla_{(d,\tau)} \left( b_{N,p} \gamma_N \right)^p \tau \tilde{H}_0(0) \int_{\mathbb{R}^N} U^q + (p + 1)\gamma_N^{-1} d^{-N} U(\tau)V(\tau) + o(1) \mu e^{-\frac{N}{N-2}} \]

where \( C_{\epsilon} > 0 \) in (4.10) is chosen uniformly for \((d, \tau) \in \Lambda_{\delta} \).

Firstly, it holds that

\[ \int_{\Omega} \left( U_{\mu,\xi}^{q-1} \nabla_{(d,\tau)} \left( b_{N,p} \gamma_N \right)^p \tau \tilde{H}_0(0) \int_{\mathbb{R}^N} U^q + (p + 1)\gamma_N^{-1} d^{-N} U(\tau)V(\tau) + o(1) \mu e^{-\frac{N}{N-2}} \]

Combining this and the equality

\[ q \int_0^r r^{N-1}(q)(\partial_r U)(r)\,dr = -\int_0^r r^{N-1}U^q(\tau)\,dr \quad \text{where } r = |x| \text{ and } U(r) = U(x), \]

we find

\[ (\text{III}) = \mu e^{-\frac{N}{N-2}} \left( \frac{b_{N,p}}{\gamma_N} \right)^p \left( \frac{N}{q + 1} \right) d^{(N-2)p-3} \tilde{H}_0(0) \int_{\mathbb{R}^N} U^q + o(1) \mu e^{-\frac{N}{N-2}}. \quad (4.19) \]
Secondly, arguing similarly to (4.14), we see
\[
\int_{\Omega_\epsilon} \left( qU^{-1}_\mu \nabla dU_{\mu, \xi} \right)(x) dx = -\gamma^{-1}_N \mu^{2-\frac{2N}{p+1}} d^{-1} \left[ \Phi_0^0(-\tau) + \tau \cdot (\nabla V)(-\tau) + o(1) \right] 
\]
\[
= - \left( \frac{N}{p+1} \right) \gamma^{-1}_N \mu^{2-\frac{2N}{p+1}} d^{-1} (V(\tau) + o(1)) \quad \text{(by (2.14)),}
\]
from which we deduce
\[
(IV) = - \left( \frac{N}{p+1} \right) \gamma^{-1}_N \left( \frac{\epsilon}{\mu} \right)^{N-2} d^{-1} U(\tau)V(\tau) + o(1)\mu^p_{\epsilon} (N-2) p^{-2}.
\]
Lastly, as in (4.15), we obtain
\[
\int_{\Omega_\epsilon} \left( qU^{-1}_\mu \nabla dU_{\mu, \xi} \right)(x) A_{e,(d, \tau)}(x) dx = - \frac{1}{p} \gamma^{-1}_N \mu^{2-\frac{2N}{p+1}} d^{-1} \left[ \Phi_0^0(-\tau) + \tau \cdot (\nabla U)(-\tau) + o(1) \right] 
\]
\[
= - \frac{1}{p} \left( \frac{N}{q+1} \right) \gamma^{-1}_N \mu^{2-\frac{2N}{p+1}} d^{-1} (U(\tau) + o(1)) \quad \text{(by (2.14)).}
\]
Hence
\[
(V) = - \left( \frac{N}{q+1} \right) \gamma^{-1}_N \left( \frac{\epsilon}{\mu} \right)^{N-2} d^{-1} U(\tau)V(\tau) + o(1)\mu^p_{\epsilon} (N-2) p^{-2}.
\]
Putting (4.18), (4.21) and (1.4) together, we establish (4.17) (where \(\nabla_{(d, \tau)}\) is replaced with \(\nabla_d\)). \(\square\)

5. Reduction process

In this section, we outline the main steps of the Lyapunov-Schmidt reduction.

5.1. Reformulation of the problem. We consider a map \(I_\epsilon : L^{\frac{p+1}{p}}(\Omega_\epsilon) \times L^{\frac{q+1}{q}}(\Omega_\epsilon) \to X_{p,q}(\Omega_\epsilon)\) given by \(I_\epsilon(f, g) = (u, v)\), where
\[
\begin{cases}
-\Delta u = f \quad & \text{in } \Omega_\epsilon, \\
-\Delta v = g \quad & \text{in } \Omega_\epsilon, \\
u = v = 0 \quad & \text{on } \partial \Omega_\epsilon, \\
\end{cases}
\quad \begin{cases}
u(x) = \int_{\Omega_\epsilon} G_\epsilon(x, y) f(y) dy, \\
v(x) = \int_{\Omega_\epsilon} G_\epsilon(x, y) g(y) dy, \\
\end{cases}
\]
for \(x \in \Omega_\epsilon\).

Then the operator norm of \(I_\epsilon\) is uniformly bounded in \(\epsilon > 0\) small, and system (1.6) is rewritten as
\[
(u, v) = I_\epsilon \left( |v|^{p-1} v, |u|^{q-1} u \right).
\]

Given a pair \((p, q)\) and a small \(\epsilon > 0\) fixed, we write \(X_\epsilon = X_{p,q}(\Omega_\epsilon)\) for brevity. Also, we recall the functions \((U_{\mu, \xi}, V_{\mu, \xi})\) and \((\Phi^l_{\mu, \xi}, \Phi^l_{\mu, \xi})\) in (2.13) and (2.15). For \((d, \tau) \in \Lambda_\delta\), let \(Y_{e,(d, \tau)}\) and \(Z_{e,(d, \tau)}\) be the subspaces of \(X_\epsilon\) defined as
\[
Y_{e,(d, \tau)} = \text{span} \left\{ (P\Phi^l_{\mu, \xi}, P\Phi^l_{\mu, \xi}) : l = 0, \ldots, N \right\}
\]
and
\[
Z_{e,(d, \tau)} = \left\{ (\psi, \phi) \in X_\epsilon : \int_{\Omega_\epsilon} (pV^{-1}_{\mu, \xi} \Phi^l_{\mu, \xi} \phi + qU^{-1}_{\mu, \xi} \Phi^l_{\mu, \xi} \psi) = 0 \right\}
\]
for \(l = 0, \ldots, N\).

where \((P\Phi^l_{\mu, \xi}, P\Phi^l_{\mu, \xi})\) is the unique smooth solution to the system
\[
\begin{align*}
-\Delta P\Phi^l_{\mu, \xi} &= pV^{-1}_{\mu, \xi} \Phi^l_{\mu, \xi} \quad \text{in } \Omega_\epsilon, \\
-\Delta P\Phi^l_{\mu, \xi} &= qU^{-1}_{\mu, \xi} \Phi^l_{\mu, \xi} \quad \text{in } \Omega_\epsilon, \\
P\Phi^l_{\mu, \xi} &= 0 \quad \text{on } \partial \Omega_\epsilon.
\end{align*}
\]
Then, arguing as in [25] Lemma 3.1, one sees that \( Y_{\epsilon}(d, \tau) \) and \( Z_{\epsilon}(d, \tau) \) are topological complements of each other. Besides, if \( \Pi_{\epsilon}(d, \tau) : X_{\epsilon} \to Y_{\epsilon}(d, \tau) \) is the linear operator defined by

\[
\Pi_{\epsilon}(d, \tau)(\psi, \phi) = \sum_{l=0}^{N} c_{\epsilon,l} \left( P\Psi_{\mu, \xi}^{l}, P\Phi_{\mu, \xi}^{l} \right)
\]

where the coefficients \( c_{\epsilon,l} \)'s are defined by the relation

\[
\sum_{m=0}^{N} c_{\epsilon,m} \int_{\Omega_{\epsilon}} \left( pV^{p-1}_{\mu, \xi} \Phi_{\mu, \xi}^{m} P\Phi_{\mu, \xi}^{m} + qU^{q-1}_{\mu, \xi} \Phi_{\mu, \xi}^{l} P\Psi_{\mu, \xi}^{l} \right) = \int_{\Omega_{\epsilon}} \left( pV^{p-1}_{\mu, \xi} \Phi_{\mu, \xi}^{l} \phi + qU^{q-1}_{\mu, \xi} \Psi_{\mu, \xi}^{l} \psi \right)
\]

for \( l = 0, \ldots, N \),

then a slight modification of the proof of [25] Corollary 3.2 shows that \( \Pi_{\epsilon}(d, \tau) \) is well-defined and its operator norm is uniformly bounded in \( \epsilon > 0 \) small and \((d, \tau) \in \Lambda_{\delta}\).

To build a solution to (1.6), we shall look for \((d, \tau) \in \Lambda_{\delta} \) such that \((\psi, \phi) = (\psi_{\epsilon}(d, \tau), \phi_{\epsilon}(d, \tau)) \in Z_{\epsilon}(d, \tau) \) satisfies

1. the auxiliary equation:

\[
(\text{Id}_{X_{\epsilon}} - \Pi_{\epsilon}(d, \tau)) \left( (PU_{\mu, \xi} + \psi, PV_{\mu, \xi} + \phi) - \mathcal{I}_{\epsilon} \left( (PV_{\mu, \xi} + \phi)^{p-1} (PV_{\mu, \xi} + \phi), (PU_{\mu, \xi} + \psi)^{q-1} (PU_{\mu, \xi} + \psi) \right) \right) = 0;
\]

2. the bifurcation equation:

\[
\Pi_{\epsilon}(d, \tau) \left( (PU_{\mu, \xi} + \psi, PV_{\mu, \xi} + \phi) - \mathcal{I}_{\epsilon} \left( (PV_{\mu, \xi} + \phi)^{p-1} (PV_{\mu, \xi} + \phi), (PU_{\mu, \xi} + \psi)^{q-1} (PU_{\mu, \xi} + \psi) \right) \right) = 0
\]

where \( \text{Id}_{X_{\epsilon}} \) is the identity operator on \( X_{\epsilon} \).

Let \( L_{\epsilon}(d, \tau) : Z_{\epsilon}(d, \tau) \to Z_{\epsilon}(d, \tau) \) be a bounded linear operator

\[
L_{\epsilon}(d, \tau)(\psi, \phi) = (\psi, \phi) - (\text{Id}_{X_{\epsilon}} - \Pi_{\epsilon}(d, \tau)) \left[ \mathcal{I}_{\epsilon} \left( (PV_{\mu, \xi} + \phi)^{p-1} \phi, q(\Phi_{\mu, \xi})^{q-1} \phi \right) \right]
\]

for all \((\psi, \phi) \in Z_{\epsilon}(d, \tau) \). We also define an error term

\[
\mathcal{E}_{\epsilon}(d, \tau) = (PU_{\mu, \xi} + \psi, PV_{\mu, \xi} + \phi) - \mathcal{I}_{\epsilon} \left( (PV_{\mu, \xi} + \phi)^{p-1} (PV_{\mu, \xi} + \phi), (PU_{\mu, \xi} + \psi)^{q-1} (PU_{\mu, \xi} + \psi) \right)
\]

and a nonlinear operator \( N_{\epsilon}(d, \tau) : Z_{\epsilon}(d, \tau) \to L^{\frac{Np}{N-2}}(\Omega_{\epsilon}) \times L^{\frac{Nq}{N-2}}(\Omega_{\epsilon}) \) by

\[
N_{\epsilon}(d, \tau)(\psi, \phi) = \left( (PV_{\mu, \xi} + \phi)^{p-1} (PV_{\mu, \xi} + \phi), (PU_{\mu, \xi} + \psi)^{q-1} (PU_{\mu, \xi} + \psi) \right) - (PU_{\mu, \xi} + \psi)^{q-1} (PU_{\mu, \xi} + \psi) - q(\Phi_{\mu, \xi})^{q-1} \phi
\]

Then the auxiliary equation is rewritten as

\[
(\text{Id}_{X_{\epsilon}} - \Pi_{\epsilon}(d, \tau)) \left[ L_{\epsilon}(d, \tau)(\psi, \phi) + \mathcal{E}_{\epsilon}(d, \tau) - \mathcal{I}_{\epsilon} \left( N_{\epsilon}(d, \tau)(\psi, \phi) \right) \right] = 0.
\]

5.2. Error estimates. This subsection is devoted to estimating the error term \( \mathcal{E}_{\epsilon}(d, \tau) \) in (5.6).

**Proposition 5.1.** Let \( N \geq 4, \ p \in (1, \frac{N}{N-2}) \), and \( \epsilon > 0 \) small. Then it holds that

\[
\| \mathcal{E}_{\epsilon}(d, \tau) \|_{X_{\epsilon}} = O(1) \left[ \mu^{(N-2)p-2} \| \ln \mu \|_{\frac{N}{N-2}} + \left( \frac{\epsilon}{\mu} \right)^{\frac{Np}{N-2}} \left( \ln \left( \frac{\epsilon}{\mu} \right) \right)^{\frac{Nq}{N-2}} \right]
\]

uniformly in \((d, \tau) \in \Lambda_{\delta}\).

**Proof.** By (5.5), (3.9) and (3.3), we have

\[
\| \mathcal{E}_{\epsilon}(d, \tau) \|_{X_{\epsilon}} = \| \mathcal{I}_{\epsilon} \left( 0, U_{\mu, \xi}^{q} - (PU_{\mu, \xi})^{q} \right) \|_{X_{\epsilon}} = O(1) \left\| U_{\mu, \xi}^{q} - (PU_{\mu, \xi})^{q} \right\|_{L^{\frac{Nq}{N-2}}(\Omega_{\epsilon})}
\]
\[ e^{N-2} \mu^{-\frac{\mu}{q+1}} A_{c,d}(x) = O(1) e^{N-2} \mu^{-\frac{\mu}{q+1}} |x|^{2-(N-2)p} \]

\[ = \left( \frac{\epsilon}{\mu} \right)^{N-2} \mu^{\frac{\mu}{q+1}} |x|^{2-(N-2)p} \]  

(5.9)

for \( x \in \Omega \). We observe from (3.11), (5.9) and the estimate \( (N-2)p \mu^{-\frac{\mu}{q+1}} = o(1) (\mu)^{N-2} \mu^{\frac{\mu}{q+1}} \) that

\[ \left| U_{\mu,\xi}^q(x) - \mathcal{P} U_{\mu,\xi}^q(x) \right| \leq \frac{O(1)}{\mu^{\frac{\mu}{q+1}}} \left( |x|^{2-N} + A_{c,d}(x) \right) + \frac{e^{N-2} \mu^{-\frac{\mu}{q+1}} |x|^{2-(N-2)p} \epsilon_{\mu,\xi}}{\mu^{\frac{\mu}{q+1}}} \]

(\( \mu^{\frac{\mu}{q+1}} + e^{N-2} \mu^{-\frac{\mu}{q+1}} |x|^{2-N} + A_{c,d}(x) \)) + \frac{e^{N-2} \mu^{-\frac{\mu}{q+1}} |x|^{2-(N-2)p} \epsilon_{\mu,\xi}}{\mu^{\frac{\mu}{q+1}}}

\[ \leq O(1) \left[ \mu^N + (N-2)(q+1) \mu^{-N} |x|^{(2-N)(q+1)} \right] + \left( \frac{\epsilon}{\mu} \right)^{(N-2)(q+1)} \mu^N |x|^{-N(p+1)} \]

for \( x \in \Omega \).

On the other hand, applying Lemma (3.3), \((N-2)q - 2 = Np = (q+1)(N - (N-2)p)\) and

\[ ((N-2)p - 2) q^2 - 1 \quad \frac{(N-2)(q+1)}{q} > ((N-2)p - 2) q^2 - 1 \quad \frac{(N-2)(q+1)}{q} > N, \]

we deduce

\[ \mu^{\frac{\mu}{q+1}} \int_{\Omega} \frac{U_{\mu,\xi}^{2-1}(x)dx}{\mu^{\frac{\mu}{q+1}}} = O(1) \mu^{\frac{\mu}{q+1}} \left[ \frac{\mu^N}{\mu^{\frac{\mu}{q+1}}} \ln \mu + \mu^{\frac{\mu}{q+1}} \right] = O(1) \left[ \frac{\mu^{\frac{\mu}{q+1}}}{\mu^{\frac{\mu}{q+1}}} \ln \mu + \mu^{N} \right], \]

and

\[ \left( \frac{\epsilon}{\mu} \right)^{(N-2)(q+1)} \mu^{\frac{\mu}{q+1}} \int_{\Omega} U_{\mu,\xi}^{2-1}(x) |x|^{(2-N)(q+1)} \mu^{\frac{\mu}{q+1}} dx = O(1) \left[ \frac{\epsilon}{\mu} \right]^N \left[ \ln \left( \frac{\epsilon}{\mu} \right) + \left( \frac{\epsilon}{\mu} \right)^{(N-2)(q+1)} \mu^{N} \right] \]

We also note that

\[ \int_{\Omega} \left[ \mu^N + (N-2)(q+1) \mu^{-N} |x|^{(2-N)(q+1)} \right] dx = O(1) \left[ \mu^N + \left( \frac{\epsilon}{\mu} \right)^N \right]. \]

Therefore, we arrive at

\[ \left\| U_{\mu,\xi}^q - \mathcal{P} U_{\mu,\xi}^q \right\|_{L^{\frac{\mu}{q+1}}(\Omega)} = O(1) \left[ \mu^{\frac{\mu}{q+1}} \ln \mu + \mu^N + \left( \frac{\epsilon}{\mu} \right)^N \ln \left( \frac{\epsilon}{\mu} \right) + \left( \frac{\epsilon}{\mu} \right)^{(N-2)(q+1)} \right] \]

(by (1.8) and (3.4)),

which together with (5.8) yields (5.7).
5.3. Linear theory. Recall the operator $L_{\epsilon,(d,\tau)}$ in (5.1). Here, we examine the unique solvability of the linear equation

$$L_{\epsilon,(d,\tau)}(\psi, \phi) = (h_1, h_2)$$

(5.10)

for any given $(h_1, h_2) \in Z_{\epsilon,(d,\tau)}$.

Employing Proposition 5.3, we obtain the following result. Its proof is similar to that of [25, Proposition 4.2], so we omit it.

**Proposition 5.2.** Assume $N \geq 4$, $p \in (1, \frac{N}{N-2})$, $\epsilon > 0$ small, and $(d, \tau) \in \Lambda_\delta$. Then there exists a constant $C > 0$ independent of $\epsilon > 0$ and $(d, \tau) \in \Lambda_\delta$ such that

$$\|L_{\epsilon,(d,\tau)}(\psi, \phi)\|_{X_\epsilon} \geq C\|\psi, \phi\|_{X_\epsilon}$$

for all $(\psi, \phi) \in Z_{\epsilon,(d,\tau)}$.

Because of the assumption $p > 1$, one can write the operator $L_{\epsilon,(d,\tau)}$ as the sum of the identity operator on $X_\epsilon$ and a compact operator; see [25, Remark 4.1]. As a consequence, Proposition 5.2 and the Fredholm alternative readily imply the following result.

**Corollary 5.3.** Under the hypotheses of Proposition 5.2, let $(h_1, h_2) \in Z_{\epsilon,(d,\tau)}$. Then (5.10) admits a unique solution $(\psi, \phi) \in Z_{\epsilon,(d,\tau)}$. Moreover,

$$\|(h_1, h_2)\|_{X_\epsilon} \geq C\|\psi, \phi\|_{X_\epsilon}$$

where $C > 0$ is the constant in Proposition 5.2. In particular, the operator $L_{\epsilon,(d,\tau)}^{-1}: Z_{\epsilon,(d,\tau)} \rightarrow Z_{\epsilon,(d,\tau)}$ given as $L_{\epsilon,(d,\tau)}^{-1}(h_1, h_2) = (\psi, \phi)$ is well-defined and uniformly bounded in $\epsilon > 0$ and $(d, \tau) \in \Lambda_\delta$.

5.4. Nonlinear problems. Owing to Corollary 5.3, the auxiliary equation (5.6) is further reduced to

$$(\psi, \phi) = L_{\epsilon,(d,\tau)}^{-1}(\text{Id}_{X_\epsilon} - \Pi_{\epsilon,(d,\tau)}) \left[ T_{\epsilon}(N_{\epsilon,(d,\tau)}(\psi, \phi)) - E_{\epsilon,(d,\tau)} \right].$$

(5.11)

The next proposition concerns its unique solvability.

**Proposition 5.4.** Assume $N \geq 4$, $p \in (1, \frac{N}{N-2})$, $\epsilon > 0$ small, and $(d, \tau) \in \Lambda_\delta$. Then (5.11) has a unique solution $(\psi_{\epsilon,(d,\tau)}, \phi_{\epsilon,(d,\tau)}) \in Z_{\epsilon,(d,\tau)}$ satisfying

$$\left\|(\psi_{\epsilon,(d,\tau)}, \phi_{\epsilon,(d,\tau)})\right\|_{X_\epsilon} \leq C\|E_{\epsilon,(d,\tau)}\|_{X_\epsilon}$$

(5.12)

where $C > 0$ is independent of $\epsilon > 0$ and $(d, \tau) \in \Lambda_\delta$. Furthermore, $(\psi_{\epsilon,(d,\tau)}, \phi_{\epsilon,(d,\tau)}) \in (L^\infty(\Omega_\epsilon))^2$ and the map $(d, \tau) \in \Lambda_\delta \mapsto (\psi_{\epsilon,(d,\tau)}, \phi_{\epsilon,(d,\tau)}) \in X_\epsilon$ is of $C^1$-class.

**Proof.** By employing the Banach fixed-point theorem, we deduce the unique existence of the solution $(\psi_{\epsilon,(d,\tau)}, \phi_{\epsilon,(d,\tau)})$ to (5.11) satisfying (5.12). The fact that $(\psi_{\epsilon,(d,\tau)}, \phi_{\epsilon,(d,\tau)}) \in (L^\infty(\Omega_\epsilon))^2$ will be proved in Appendix 12. The $C^1$-regularity of the map $(d, \tau) \mapsto (\psi_{\epsilon,(d,\tau)}, \phi_{\epsilon,(d,\tau)})$ is the consequence of the implicit function theorem, the Fredholm alternative and $(\psi_{\epsilon,(d,\tau)}, \phi_{\epsilon,(d,\tau)}) \in (L^\infty(\Omega_\epsilon))^2$. 

5.5. Lyapunov-Schmidt reduction. For $\epsilon > 0$ small, we set a $C^1$-functional $J_\Omega: \Lambda_\delta \rightarrow \mathbb{R}$ by

$$J_\Omega(d, \tau) = I_{\epsilon} \left( PU_{\mu,\xi} + \psi_{\epsilon,(d,\tau)}, PV_{\mu,\xi} + \phi_{\epsilon,(d,\tau)} \right)$$

where $I_{\epsilon}$ is the energy functional in (2.2) and $(\psi_{\epsilon,(d,\tau)}, \phi_{\epsilon,(d,\tau)})$ is the solution to (5.11) described in Proposition 5.3. Let $\text{Int}(\Lambda_\delta)$ be the interior of $\Lambda_\delta$.

**Proposition 5.5.** Assume $N \geq 4$, $p \in (1, \frac{N-1}{N-2})$, and $\epsilon > 0$ small. Suppose that $(d, \tau) \in \text{Int}(\Lambda_\delta)$ is a critical point of $J_\Omega$. Then $(PU_{\mu,\xi} + \psi_{\epsilon,(d,\tau)}, PV_{\mu,\xi} + \phi_{\epsilon,(d,\tau)})$ is a critical point of $I_{\epsilon}$ which belongs to $(C^2(\Omega_\epsilon))^2$, and hence a classical solution to (1.6) having the form (1.7). Moreover, one may assume that its components are positive in $\Omega_\epsilon$. 
Lemma 5.6. Assume \( N \geq 4, p \in \left( 1, \frac{N-2}{2} \right), \) and \( \epsilon > 0 \) small. Let

\[
c_0 = \frac{2}{N} \int_{\mathbb{R}^N} U^{q+1}, \quad c_1 = \frac{1}{p+1} \left( \frac{b_N \varrho}{\gamma_N} \right)^p \int_{\mathbb{R}^N} U^q > 0, \quad c_2 = \gamma_N^{-1} > 0,
\]

and

\[
\Theta(d, \tau) = c_1 d^{(N-2)p-2} + c_2 d^2 N U(\tau) V(\tau) \quad \text{for} \quad (d, \tau) \in A_\delta.
\]

Then

\[
J_{0\epsilon}(d, \tau) = J_{\epsilon}(d, \tau) + o(1) \mu\epsilon^{(N-2)p-2} = c_0 + \mu\epsilon^{(N-2)p-2} \Theta(d, \tau) + o(1) \mu\epsilon^{(N-2)p-2}
\]

and

\[
\nabla_{(d, \tau)} J_{0\epsilon}(d, \tau) = \nabla_{(d, \tau)} J_{\epsilon}(d, \tau) + o(1) \mu\epsilon^{(N-2)p-2} - \mu\epsilon^{(N-2)p-2} \nabla_{(d, \tau)} \Theta(d, \tau) + o(1) \mu\epsilon^{(N-2)p-2}
\]

where \( C_\epsilon > 0 \) in (4.10) is chosen uniformly for \( (d, \tau) \in A_\delta \).

Proof. Let \( q^* \in (1, 2) \) be the number in (2.1). Estimate (5.7) and the inequality that \( N q q^* > (N-2)(q+1) \) imply

\[
\| \xi_{(d, \tau)} \|_{X_{\epsilon}}^\prime = o(1) \mu\epsilon^{(N-2)p-2}.
\]

Employing (5.17), one can check the first inequality in (5.16) as in the proof of [25] Lemma 5.4.

We verify the first equality in (5.16). Let \( s \) be one of the parameters \( d, \tau_1, \ldots, \tau_N \). We have

\[
\nabla_s (J_{0\epsilon} - J_{\epsilon})(d, \tau) = \left[ I_1 \left( \mathcal{P}U_{\mu, \xi} + \psi_{(d, \tau)}, PV_{\mu, \xi} + \phi_{(d, \tau)} \right) - I_1 \left( \mathcal{P}U_{\mu, \xi}, PV_{\mu, \xi} \right) \right] \nabla_s \mathcal{P}U_{\mu, \xi}
\]

\[
+ I_1' \left( \mathcal{P}U_{\mu, \xi} + \psi_{(d, \tau)}, PV_{\mu, \xi} + \phi_{(d, \tau)} \right) \left( \nabla_s \psi_{(d, \tau)}, \nabla_s \phi_{(d, \tau)} \right)
\]

=: (VI) + (VII)

By differentiating (3.5) and (3.9) with respect to \( s \), we get the equation of \( \nabla_s \mathcal{P}U_{\mu, \xi}, \nabla_s PV_{\mu, \xi} \). Using it, the third equality in (5.11), Lemma 5.8, Hölder’s inequality, the Sobolev inequality, the inequalities that \( 0 \leq \mathcal{P}U_{\mu, \xi} \leq U_{\mu, \xi} \) and \( 0 \leq PV_{\mu, \xi} \leq V_{\mu, \xi} \) in \( \Omega \), (5.12) and (5.17), we compute

\[
(VI) = - \int_\Omega \left[ \mathcal{P}V_{\mu, \xi} + \phi_{(d, \tau)} \right] q^{-1} \left( \mathcal{P}U_{\mu, \xi} + \psi_{(d, \tau)} \right) - (\mathcal{P}U_{\mu, \xi})^q - q(\mathcal{P}U_{\mu, \xi})^{q-1} \psi_{(d, \tau)} \nabla_s \mathcal{P}U_{\mu, \xi}
\]

\[
- \int_\Omega \left[ \mathcal{P}V_{\mu, \xi} + \phi_{(d, \tau)} \right] q^{-1} \left( \mathcal{P}V_{\mu, \xi} + \phi_{(d, \tau)} \right) - (\mathcal{P}V_{\mu, \xi})^q - p(\mathcal{P}V_{\mu, \xi})^{q-1} \phi_{(d, \tau)} \nabla_s \mathcal{P}V_{\mu, \xi}
\]

\[
- q \int_\Omega \left[ (\mathcal{P}U_{\mu, \xi})^{q-1} \nabla_s (\mathcal{P}U_{\mu, \xi} - U_{\mu, \xi}) + \nabla_s U_{\mu, \xi} \left( (\mathcal{P}U_{\mu, \xi})^{q-1} - U_{\mu, \xi}^{q-1} \right) \phi_{(d, \tau)} \right] \psi_{(d, \tau)}
\]

\[
= O(1) \int_\Omega \left[ (\mathcal{P}U_{\mu, \xi})^{q-2} \psi_{(d, \tau)}^2 + |\psi_{(d, \tau)}| q > 2 \right] \nabla_s \mathcal{P}U_{\mu, \xi} + (\mathcal{P}V_{\mu, \xi})^{q-2} \phi_{(d, \tau)} \nabla_s \mathcal{P}V_{\mu, \xi}
\]

\[
+ O(1) \int_\Omega \left[ (\mathcal{P}U_{\mu, \xi})^{q-1} |\nabla_s (\mathcal{P}U_{\mu, \xi} - U_{\mu, \xi})| + U_{\mu, \xi}^{q-1} |\mathcal{P}U_{\mu, \xi} - U_{\mu, \xi}| \right] \psi_{(d, \tau)}
\]

\[
= O(1) \left[ (\psi_{(d, \tau)}, \phi_{(d, \tau)}) \right]_{X_{\epsilon}}^2
\]

\[
+ \left[ (\nabla_s (\mathcal{P}U_{\mu, \xi} - U_{\mu, \xi}))_{L^{q+1}(\Omega)} + \left( \mathcal{P}U_{\mu, \xi} - U_{\mu, \xi} \right)_{L^{q+1}(\Omega)} \right] |\psi_{(d, \tau)}|_{L^{q+1}(\Omega)}
\]

\[
= o(1) \mu\epsilon^{(N-2)p-2} + O(1) \left[ \left( \nabla_s (\mathcal{P}U_{\mu, \xi} - U_{\mu, \xi}))_{L^{q+1}(\Omega)} + \left( \mathcal{P}U_{\mu, \xi} - U_{\mu, \xi} \right)_{L^{q+1}(\Omega)} \right] |\xi_{(d, \tau)}|_{X_{\epsilon}}.
\]
Besides, (5.11), (5.29) and (5.30) yield
\[ \left\| \nabla_s (PU_{\mu, \xi} - U_{\mu, \xi}) \right\|_{L^{t+1}(\Omega, \tau)} + \left\| PU_{\mu, \xi} - U_{\mu, \xi} \right\|_{L^{t+1}(\Omega, \tau)} = O(1) \left[ \frac{\mu}{\mu + \frac{1}{2}} + \left( \frac{\epsilon}{\mu} \right) \frac{\mu}{\mu + 1} \right]. \] (5.18)

By (5.4) and (5.7),
\[ \left( \left\| \nabla_s (PU_{\mu, \xi} - U_{\mu, \xi}) \right\|_{L^{t+1}(\Omega, \tau)} + \left\| PU_{\mu, \xi} - U_{\mu, \xi} \right\|_{L^{t+1}(\Omega, \tau)} \right) \left\| \xi_{e,(d, \tau)} \right\|_{\mathbb{L}_{e,(d, \tau)}} = o(1) \mu^{(N-2)p-2}. \]

Thus it holds that (VI) = o(1)\mu^{(N-2)p-2}.

Moreover, by using (5.3) and the fact that \((\psi_{e,(d, \tau)}, \phi_{e,(d, \tau)}) \in Z_{e,(d, \tau)}\), we find
\[ \text{(VII)} = \sum_{l=0}^{N} \tilde{c}_{e,l} \int_{\Omega} \left[ dV_{\mu, \xi} \Phi_{\mu, \xi} \nabla_s \phi_{e,(d, \tau)} + qU_{\mu, \xi} \Psi_{\mu, \xi} \nabla_s \psi_{e,(d, \tau)} \right] \]
\[ = -\sum_{l=0}^{N} \tilde{c}_{e,l} \int_{\Omega} \left[ dV_{\mu, \xi} \Phi_{\mu, \xi} \phi_{e,(d, \tau)} + \nabla_s \left( qU_{\mu, \xi} \Phi_{\mu, \xi} \right) \psi_{e,(d, \tau)} \right] \] (5.19)
where the coefficients \(\tilde{c}_{e,l}\)'s are defined by the relation
\[ \sum_{l=0}^{N} \tilde{c}_{e,l} \left( PU_{\mu, \xi}, PV_{\mu, \xi} \right) = \left( PU_{\mu, \xi} + \psi_{e,(d, \tau)}, PV_{\mu, \xi} + \phi_{e,(d, \tau)} \right) \]
\[ = \mathcal{T}_{e}^{*} \left( \left( PU_{\mu, \xi} + \phi_{e,(d, \tau)} \right) U_{\mu, \xi}, \left( PV_{\mu, \xi} + \phi_{e,(d, \tau)} \right) \right) \]

see (5.2). Since
\[ p \int_{\Omega} V_{\mu, \xi}^{p-1} \Phi_{\mu, \xi}^{p-1} PV_{\mu, \xi} = \int_{\Omega} \nabla P_{\mu, \xi} \cdot \nabla PV_{\mu, \xi} = \int_{\Omega} P_{\mu, \xi} U_{\mu, \xi} \] (by (5.1) and (3.5)),
\[ q \int_{\Omega} U_{\mu, \xi}^{q-1} \Phi_{\mu, \xi}^{q-1} PU_{\mu, \xi} = \int_{\Omega} \nabla P_{\mu, \xi} \cdot \nabla PU_{\mu, \xi} = \int_{\Omega} P_{\mu, \xi} \left( PU_{\mu, \xi} \right)^{p} \] (by (5.1) and (3.9)),
if we define a number
\[ M_{e,(d, \tau)}^{m} = \int_{\Omega} \left[ pV_{\mu, \xi}^{p-1} \Phi_{\mu, \xi}^{p-1} PV_{\mu, \xi} - \left| PV_{\mu, \xi} + \phi_{e,(d, \tau)} \right| U_{\mu, \xi} \right] \]
\[ + \int_{\Omega} \left[ qU_{\mu, \xi}^{q-1} \Phi_{\mu, \xi}^{q-1} PU_{\mu, \xi} - \left| PU_{\mu, \xi} + \psi_{e,(d, \tau)} \right| U_{\mu, \xi} \right] \]
for \(m = 0, \ldots, N\), then
\[ M_{e,(d, \tau)}^{m} = -\int_{\Omega} \left[ PV_{\mu, \xi} + \phi_{e,(d, \tau)} \right] U_{\mu, \xi} \]
\[ - \int_{\Omega} \left[ PU_{\mu, \xi} + \psi_{e,(d, \tau)} \right] U_{\mu, \xi} \]
\[ - \int_{\Omega} \left[ (PU_{\mu, \xi})^{q} - U_{\mu, \xi}^{q} \right] U_{\mu, \xi} \] (5.20)
By (5.20), (5.7), (5.12), (5.18), the inequalities that \( |P_{\mu, \xi}^{m}| \leq C\mu^{-1} U_{\mu, \xi} \) and \( |P_{\mu, \xi}^{m}| \leq C\mu^{-1} V_{\mu, \xi} \) in \( \Omega_e \), it follows that
\[ M_{e,(d, \tau)}^{m} = O(1) \left\| \Phi_{\mu, \xi} \right\|_{L^{t+1}(\Omega, \tau)} \left\| \phi_{e,(d, \tau)} \right\|_{L^{t+1}(\Omega, \tau)} \]
\[ + O(1) \left\| \Phi_{\mu, \xi} \right\|_{L^{t+1}(\Omega, \tau)} \left( \left\| \phi_{e,(d, \tau)} \right\|_{L^{t+1}(\Omega, \tau)} + \left\| PU_{\mu, \xi} - U_{\mu, \xi} \right\|_{L^{t+1}(\Omega, \tau)} \right) \]
\[ = O(1) \mu^{-1} \left[ \frac{\mu}{\mu + \frac{1}{2}} + \left( \frac{\epsilon}{\mu} \right) \frac{\mu}{\mu + 1} \right]. \] (5.21)
We arrive at (VII) = 0.

Taking \( r \) where a simple degree argument guarantees the existence of a critical point of \( \tilde{\Theta} \) in \( \text{Int}(\Lambda) \), we have a non-degenerate saddle point \( \tilde{\Theta} \).

By combining (5.19), (5.22), (5.7) and (5.21), we deduce

\[
\delta_{c,m} = \mu^2 \sum_{l=0}^{N} (1_{m=l} + o(1)) M_{l,m}^{\mu} \left[ \int_{\mathbb{R}^N} \left( p V^{p-1}(\Phi_{l,m}^\mu)^2 + q U^{q-1}(\Phi_{l,m}^\mu)^2 \right) \right]^{-1} = O(1) \mu \left[ \frac{\mu N}{\nu + 1} + \left( \frac{\epsilon}{\mu} \right)^{\frac{N}{\nu + 1}} \right].
\]

By combining (5.19), (5.22), (5.7) and (5.21), we deduce

\[
\int_{\Omega_x} \left[ \nabla \left( p V^{p-1}(\Phi_{l,m}^\mu) \right) \psi_{c,l,m,r}(d,r) + \nabla \left( q U^{q-1}(\Phi_{l,m}^\mu) \right) \psi_{c,l,m,r}(d,r) \right] d\mathbf{x} = O(1) \mu \left[ \left\| \Phi_{l,m}^\mu \right\|_{L^p(\Omega_x)} \left\| \psi_{c,l,m,r}(d,r) \right\|_{L^{p+1}(\Omega_x)} \right]
\]

we arrive at (VII) = o(1) \mu^{(N-2)p-2}.

Consequently, the first equality in (5.16) is true.

The second equalities in (5.16) and (5.17) follow from (12) and (4.17), respectively.

6. Completion of the proof of Theorem 1.1

We are now ready to conclude the proof of our main theorem. Below, we show the function \( \Theta \) in (5.14) has a non-degenerate saddle point \( (\tilde{d}, \tilde{\tau}) \) in \( \text{Int}(\Lambda_\delta) \) with a suitable choice of \( \delta > 0 \). Then Lemma 5.6 and a simple degree argument guarantee the existence of a critical point of \( J_{0,\epsilon} \) in \( \text{Int}(\Lambda_\delta) \) for \( \epsilon > 0 \) small.

Proposition 5.3 implies the existence of a small number \( \epsilon_0 > 0 \) and a family of solutions \( \{ (u_\epsilon, v_\epsilon) \} \in (0, \epsilon_0) \) to system (1.6) depicted in the statement of Theorem 1.1.

Let

\[
d = \left[ \frac{c_2 (N - 2)V(0)}{c_1 (N - 2)p - 2} \right]^{(N-2)p-2} > 0 \quad \text{and} \quad \tilde{\tau} = 0 \in \mathbb{R}^N
\]

where \( c_1, c_2 > 0 \) are the numbers in (5.13). Then, using \( U(0) = 1 \) and \( \nabla U(0) = \nabla V(0) = 0 \), we easily check that \( \nabla_{(d,r)} \Theta(d, \tilde{\tau}) = 0 \). Furthermore, it holds that

\[
\partial^2_{d^l} \Theta(d, \tilde{\tau}) = 0 \quad \text{for each} \quad l = 1, \ldots, N,
\]

\[
\partial^2_{d^l} \Theta(d, \tilde{\tau}) = \tilde{d}^{l-N} \left[ c_1 ((N - 2)(N - 2) - 2) ((N - 2)p - 3) \tilde{d}^{(N-2)p-2} + c_2 (N - 2)(N - 1)(UV)(0) \right]
\]

\[
= \tilde{d}^{l-N} c_2 (N - 2)((N - 2)p + N - 4)V(0) > 0,
\]

and

\[
\partial^2_{d^l d^m} \Theta(d, \tilde{\tau}) = c_2 \tilde{d}^{2-N} \left[ \left( \partial_{d^l}^2 U \right)(0)V(0) + \left( \partial_{d^l}^2 V \right)(0)U(0) \right]
\]

\[
= \begin{cases} c_2 \tilde{d}^{2-N} \left[ \left( \partial_{d^l}^2 U \right)(0)V(0) + \left( \partial_{d^l}^2 V \right)(0) \right] = -c_2 N^{-1} \tilde{d}^{2-N} (V^{p+1}(0) + 1) < 0 & \text{if} \ l = m, \\
0 & \text{if} \ l \neq m
\end{cases}
\]

for \( l, m = 1, \ldots, N \). Consequently, the Hessian matrix \( D^2_{d,r} \Theta(d, \tilde{\tau}) \) of \( \Theta \) at \( (\tilde{d}, \tilde{\tau}) \) has one positive eigenvalue and \( N \) negative eigenvalues. Taking \( \delta = \tilde{d} + \tilde{d}^{-1} \), we see that \( (\tilde{d}, \tilde{\tau}) \) is a non-degenerate saddle point of \( \Theta \) in \( \text{Int}(\Lambda_\delta) \). The proof of Theorem 1.1 is now finished.

---

\(^8\)Let \( r = |x| \). Abusing notation, we write \( U(r) = U(x) \) and \( V(r) = V(x) \). Then system (1.6) is rewritten as

\[-(\partial_{x}^2 U)(r) - \frac{N - 1}{r} (\partial_r U)(r) = V^p(r) \quad \text{and} \quad -(\partial_{x}^2 V)(r) - \frac{N - 1}{r} (\partial_r V)(r) = U^q(r)
\]

for \( r \in (0, \infty) \).

Taking \( r \to 0 \) on the both equations, we find \( -N(\partial_{x}^2 U)(0) = V^p(0) \) and \( -N(\partial_{x}^2 V)(0) = U^q(0) = 1 \).
Appendix A. Technical computations

An elementary calculus yields the following inequalities.

**Lemma A.1.** Assume that $1 < p < 2$. Then there exists a constant $C > 0$ depending only on $p$ such that

$$
\begin{align*}
\|a + b|^{p-1}(a + b) - |a|^{p-1}a - |b|^{p-1}b\| &\leq C \left(|a|^{p-1}|b| + |b|^p\right),
\|a + b|^{p-1}(a + b) - |a|^{p-1}a - |b|^{p-1}b\| &\leq C|a|^{p-1}|b|, \\
|a + b| - |a|^{p-1}a - |b|^{p-1}b &\leq C\min \left\{|a|^{p-2}b^2, |b|^p\right\},
\end{align*}
$$

(A.1)

for any $a, b \in \mathbb{R}$.

**Lemma A.2.** Let $N \geq 3$, $a < 2$, $\lambda \in (0, 1)$ and $\xi \in \mathbb{R}^N$. Then there exists a constant $C > 0$ depending only on $N$ and $a$ such that

$$
\int_{B(\xi, \lambda)} \frac{dy}{|x - y|^{N-2}|y - \xi|^a} \leq \frac{C\lambda^{2-a}}{1 + |\lambda^{-1}(x - \xi)|^{N-2}} \quad \text{for } x \in \mathbb{R}^N.
$$

(A.2)

**Proof.** Applying a change of variable $y - \xi \mapsto \lambda y$, we observe

$$
\int_{B(\xi, \lambda)} \frac{dy}{|x - y|^{N-2}|y - \xi|^a} = \lambda^{2-a} \int_{B(0, 1)} \frac{dy}{|z - y|^{N-2}|y|^a}
$$

where $z := \lambda^{-1}(x - \xi)$.

Since

$$
\int_{B(0, 1)} \frac{dy}{|z - y|^{N-2}|y|^a} \leq C|z|^{2-N} \int_{B(0, 1)} |y|^{-a} dy \leq C|z|^{2-N} \quad \text{for } |z| \geq 2,
$$

we have

$$
\int_{B(\xi, \lambda)} \frac{dy}{|x - y|^{N-2}|y - \xi|^a} \leq C\lambda^{2-a} \left|\lambda^{-1}(x - \xi)\right|^{2-N} \quad \text{provided } |x - \xi| \geq 2\lambda.
$$

Besides, it holds that

$$
-\Delta_z \int_{B(0, 2)} \frac{dy}{|z - y|^{N-2}|y|^a} = \gamma_N^{-1}|z|^{-a} \in L^q(B(0, 2)) \quad \text{for some } q > \frac{N}{2}.
$$

By elliptic regularity, we conclude that

$$
\int_{B(\xi, \lambda)} \frac{dy}{|x - y|^{N-2}|y - \xi|^a} \leq C\lambda^{2-a} \quad \text{provided } |x - \xi| < 2\lambda.
$$

\hfill \Box

In the rest of the appendix, we assume that $(d, \tau) \in \Lambda_\delta$ (see (3.1)) and (3.2)–(3.3) holds.

**Lemma A.3.** Let $N \geq 3$, $p \in (1, \frac{N}{N-1})$, $a > 0$ and $b \in \mathbb{R}$. Then we have

$$
\int_{\Omega} \left|U_{\mu, \xi}^\alpha(x)|x|^{-b}dx = \begin{cases} O(1)\mu^{-\frac{N+1}{\alpha}+N-b}\left(\frac{\xi}{\mu}\right)^N & \text{for } b > N, \\
O(1)\mu^{-\frac{N}{\alpha+1}+N-b}\ln\left(\frac{\mu}{\alpha}\right) & \text{for } b = N. \end{cases}
$$

(A.3)

If we further assume that $b < N$, then

$$
\int_{\Omega} \left|U_{\mu, \xi}^\alpha(x)|x|^{-b}dx = \begin{cases} O(1)\mu^{-\frac{N}{\alpha+1}+N-b} & \text{for } ((N - 2)p - 2)a + b > N, \\
O(1)\mu^{-\frac{N}{\alpha+1}+N-b}\ln\mu & \text{for } ((N - 2)p - 2)a + b = N, \\
O(1)\mu^{-\frac{N}{\alpha+1}+((N-2)p-2)a} & \text{for } ((N - 2)p - 2)a + b < N. \end{cases}
$$

(A.4)
Proof. By (A.6), we have
\[
\int_{\Omega_N} U_{\mu, \varepsilon}^{a}(x)|x|^{-b}\,dx = O(1)\mu^{-\frac{a}{N-2} + N - b} \int_{\mu^{-1}\Omega_N} \frac{dx}{(1 + |x|)((N-2)p-2)a)|x|^{b}}.
\]
If \(b \geq N\), then \(((N-2)p-2)a + b > N\), and hence
\[
\int_{\mu^{-1}\Omega_N} \frac{dx}{(1 + |x|)((N-2)p-2)a)|x|^{b}} = \begin{cases} O(1) & \text{for } b > N, \\ O(1)\ln\mu & \text{for } (N-2)p-2)a + b = N, \\ O(1)\mu^{-N+(N-2)p-2)a+b} & \text{for } (N-2)p-2)a + b < N. \end{cases}
\]
On the other hand, if \(b < N\), then
\[
\int_{\mu^{-1}\Omega_N} \frac{dx}{(1 + |x|)((N-2)p-2)a)|x|^{b}} = \begin{cases} O(1) & \text{for } (N-2)p-2)a + b > N, \\ O(1)\ln\mu & \text{for } ((N-2)p-2)a + b = N, \\ O(1)\mu^{-N+(N-2)p-2)a+b} & \text{for } (N-2)p-2)a + b < N. \end{cases}
\]
Combining all the computations, we derive (A.3) and (A.4).

Lemma A.4. Let \(N \geq 3\) and \(\kappa \in (0, 1)\) small. It holds that
\[
\int_{\mathbb{R}^N \setminus B(0, 1)} \frac{dy}{|x-y|^{N-2}|y|^b} = \begin{cases} O(1) & \text{if } a < 2, \\ O(1)\ln\mu & \text{if } 2 = a, \\ O(1)\mu^{(2-b)} & \text{if } 2 < a < 0, \\ O(1)\ln|x| & \text{if } b < 2, \\ O(1)|x|^{2-b} & \text{if } 2 < b < N, \end{cases}
\]
for \(x \in \mathbb{R}^N\). In addition,
\[
\int_{\Omega \setminus B(0, \mu^\kappa)} \frac{dy}{|x-y|^{N-2}|y|^b} = \begin{cases} O(1) & \text{if } a < 2, \\ O(1)\ln|x| & \text{if } 2 = a, \\ O(1)\mu^{(2-b)} & \text{if } 2 < a < 0, \\ O(1)\ln|x| & \text{if } b < 2, \\ O(1)|x|^{2-b} & \text{if } 2 < b < N, \end{cases}
\]
for \(x \in \Omega\).

Proof. We will only verify (A.6). Estimate (A.5) is well-known, and the proof of (A.6) essentially covers that of (A.7). We write \(\Omega_{\mu^\kappa} = \Omega \setminus B(0, \mu^\kappa)\).

If \(|x| \leq \frac{\mu^\kappa}{2}\), then \(|x-y| \geq \frac{|y|}{2}\) for \(y \in \Omega_{\mu^\kappa}\). Thus
\[
\int_{\Omega_{\mu^\kappa}} \frac{dy}{|x-y|^{N-2}|y|^b} \leq C \int_{\Omega_{\mu^\kappa}} \frac{dy}{|y|^{N+b-2}} \leq \begin{cases} O(1) & \text{if } b < 2, \\ O(1)\ln|x| & \text{if } 2 = b, \\ O(1)\mu^{(2-b)} & \text{if } b > 2, \end{cases}
\]
for \(|x| > \frac{\mu^\kappa}{2}\). By applying the inequalities
\[
\begin{align*}
|y| &\geq |x| - |x-y| > \frac{|x|}{2} & \text{for } y \in B(x, \frac{|x|}{2}), \\
|x-y| &\geq |x| - |y| > \frac{|x|}{2} & \text{for } y \in B(0, \frac{|x|}{2}), \\
|y| &\leq |y-x| + |x| \leq 3|x-y| & \text{for } y \in \mathbb{R}^N \setminus \left[ B(x, \frac{|x|}{2}) \cup B(0, \frac{|x|}{2}) \right],
\end{align*}
\]
we derive
\[
\int_{\Omega_{\mu^\kappa} \cap B(x, \frac{|x|}{2})} \frac{dy}{|x-y|^{N-2}|y|^b} = O(1)|x|^{-b} \int_{B(x, \frac{|x|}{2})} \frac{dy}{|x-y|^{N-2}} = O(1)|x|^{2-b},
\]
Proof. A straightforward computation using (2.9) shows that
\[ O(1) |x|^{2-N} \int_{B(0, \frac{|x|}{2})} \frac{dy}{|y|^b} = O(1) |x|^{2-b} \quad (\text{since } b < N), \]
\[ O(1) \int_{\Omega \setminus B(x, \frac{|x|}{4})} \frac{dy}{|y|^{N-2+b}} = \begin{cases} O(1) & \text{if } b < 2, \\ O(1) |\ln |x|| & \text{if } b = 2, \\ O(1) |x|^{2-b} & \text{if } b > 2. \end{cases} \]
This completes the verification of (A.6). \( \square \)

Lemma A.5. Let \( N \geq 4, p \in (1, \frac{N}{N-2}) \) and \( \kappa \in (0, \frac{1}{N-2}) \). Let \( R_3 \) be the function in (2.21). It holds that
\[ R_3(x) = O(1) \mu^{-\frac{N+2}{2}} \left[ \mu^{1+\kappa(1-N)} + \left( \frac{\epsilon}{\mu} \right)^{N-2} \mu^{\kappa(2-(N-2)p)} + \epsilon^{N-2} \mu^{\kappa(2-(N-2)p)} \right] \tag{A.8} \]
for \( x \in \Omega_\epsilon \).

Proof. Let \( \Omega_{\mu^\kappa} = \Omega_\epsilon \setminus B(0, \mu^\kappa) \). A direct computation shows that
\[ R_3(x) = O(1) \int_{\Omega_{\mu^\kappa}} \frac{1}{|x-y|^{N-2}} \left[ \frac{\mu^{N+2-N} \epsilon^{N-2}}{|y|^{(N-2)p}} + |\tilde{R}_2(y)|^p \right. \\
&\left. + \left\{ 1 + \left( \frac{\epsilon}{\mu} \right)^{(N-2)(p-1)} \right\} \frac{\mu^{N(p-1)}}{|y|^{(N-2)(p-1)}} |\tilde{R}_2(y)| \right] dy. \]
By (A.6) and (3.18), we discover
\[ \frac{\mu^{N+2-N} \epsilon^{N-2}}{|x-y|^{N-2}} \int_{\Omega_{\mu^\kappa}} \frac{dy}{|y|^{(N-2)p}} = O(1) \mu^{-\frac{N+2}{2}} \left( \frac{\epsilon}{\mu} \right)^{N-2} \mu^{\kappa(2-(N-2)p)}, \]
\[ \int_{\Omega_{\mu^\kappa}} \frac{1}{|x-y|^{N-2}} |\tilde{R}_2(y)|^p dy = O(1) \mu^{-\frac{N+2}{2}} \left[ \mu^{1+\kappa(1-N)} + \left( \frac{\epsilon}{\mu} \right)^{N-2} \mu^{\kappa(2-(N-2)p)} + \left( \frac{\epsilon}{\mu} \right)^{(N-1)p} \mu^{\kappa(2-(N-2)p)} \right], \]
and
\[ \frac{\mu^{N(p-1)}}{|x-y|^{N-2}} \int_{\Omega_{\mu^\kappa}} |\tilde{R}_2(y)| dy = O(1) \mu^{-\frac{N+2}{2}} \left[ \mu^{1+\kappa(1-N)} + \left( \frac{\epsilon}{\mu} \right)^{N-2} \mu^{\kappa(2-(N-2)p)} + \left( \frac{\epsilon}{\mu} \right)^{N-1} \mu^{\kappa(2-(N-2)p)} \right]. \]
Combining these, we deduce (A.8). \( \square \)

Lemma A.6. Let \( N \geq 4 \) and \( p \in (1, \frac{N}{N-2}) \). If \( Q_{\mu, \xi} \) is the integral in (4.6), then
\[ Q_{\mu, \xi} = o(1) \mu^{(N-2)p-2}. \tag{A.9} \]

Proof. A straightforward computation using (2.9) shows that
\[ \mu^{-\frac{N+2}{2}} \int_{\Omega_\epsilon} U_{\mu, \xi}^{q-1} = o(1) \mu^{(N-2)p-2}. \]
Also, in light of Lemma A.3 and (3.19), we have
\[ \left( \epsilon^{N-2} - \frac{N+2}{2} \right)^2 \int_{\Omega_\epsilon} U_{\mu, \xi}^{q-1} |x|^{2(2-N)} dx = O(1) \left( \frac{\epsilon}{\mu} \right)^N \left( 1 + \frac{\ln \epsilon}{\mu} \right) 1_{N=4}, \]
\[ \left( \epsilon^{(N-2)p} - \frac{N+2}{2} \right)^2 \int_{\Omega_\epsilon} U_{\mu, \xi}^{q-1} |x|^{2-(N-2)p} dx = o(1) \left( \frac{\epsilon}{\mu} \right)^{N-2}. \]
Lemma A.7. Let \( N \geq 4, \kappa \in (0, 1) \) and \( p \in (1, \frac{N}{N-2}) \). Then it holds that
\[
\epsilon^{N-2} \int_{(\Omega \setminus \Omega_{\epsilon})} \int_{B(0, \mu^{n-1}) \setminus B(-\tau, \frac{\tau}{\mu})} H_\kappa(\mu x + \xi, \mu y + \xi) U^q(x) V^{p-1}(y) \frac{dy}{|y + \tau|^{N-2}} dx = o(1) \left( \frac{\epsilon}{\mu} \right)^{N-2}.
\] (A.10)

Proof. An application of Fubini’s theorem, (2.4) and (2.10) yields
\[
\epsilon^{N-2} \int_{(\Omega \setminus \Omega_{\epsilon})} \int_{B(0, \mu^{n-1}) \setminus B(-\tau, \frac{\tau}{\mu})} H_\kappa(\mu x + \xi, \mu y + \xi) U^q(x) V^{p-1}(y) \frac{dy}{|y + \tau|^{N-2}} dx
= O(1) \epsilon^{N-2} \int_{\mathbb{R}^N} U^q(x) dx \int_{B(0, \mu^{n-1}) \setminus B(-\tau, \frac{\tau}{\mu})} V^{p-1}(y) \frac{dy}{|y + \tau|^{N-2}}
= O(1) \epsilon^{N-2} \int_{B(0, \mu^{n-1}) \setminus B(-\tau, \frac{\tau}{\mu})} \frac{1}{1 + |y|^{(N-2)(p-1)}} \frac{dy}{|y|^{N-2}}
= O(1) \left( \frac{\epsilon}{\mu} \right)^{N-2} \mu^{(N-2)p-2} = o(1) \left( \frac{\epsilon}{\mu} \right)^{N-2},
\]
and
\[
\epsilon^{2(N-2)} \frac{\mu^{N-2} \int_{(\Omega \setminus \Omega_{\epsilon})} \int_{B(0, \mu^{n-1}) \setminus B(-\tau, \frac{\tau}{\mu})} \frac{|x + \tau|^{2N-2} + |y + \tau|^{2N-2}}{|y + \tau|^{N-2} \frac{dy}{x U^q(x) V^{p-1}(y)}} dx}{|y + \tau|^{N-2}}
= O(1) \epsilon^{2(N-2)} \frac{\mu^{N-2} \int_{B(0, \mu^{n-1}) \setminus B(-\tau, \frac{\tau}{\mu})} \frac{U^q(x) V^{p-1}(y)}{|y + \tau|^{N-2} + |y + \tau|^{2N-2}} dy}{|y + \tau|^{N-2}}
= O(1) \epsilon^{2(N-2)} \frac{\mu^{N-2} \int_{B(\tau, \mu^{n-1}) \setminus B(0, \frac{\tau}{\mu})} \left( \frac{1}{|y|^{N-2}} + \frac{1}{|y|^{2(N-2)}(p-1)} \right) \frac{dy}{1 + |y|^{N-2}(p-1)}} = o(1) \left( \frac{\epsilon}{\mu} \right)^{N-2}.
\]

Appealing the above computations and Lemma 2.1, we establish (A.10). □

Lemma A.8. Given \((d, \tau) \in \Lambda_\delta\), let \( s \) be one of the parameters \( d, \tau_1, \ldots, \tau_N \). Then there exists a constant \( C > 0 \) depending only on \( N, p \) and \( \delta \) such that
\[
|\nabla_s P U_{\mu, \xi}(x)| \leq CP_{\mu, \xi}(x) \quad \text{and} \quad |\nabla_s PV_{\mu, \xi}(x)| \leq CPV_{\mu, \xi}(x) \quad \text{for} \ x \in \Omega.
\] (A.11)

Proof. In light of (2.13), (2.15) and Lemma 2.4, we know
\[
\begin{cases}
-\Delta (CPV_{\mu, \xi} \pm \nabla_s PV_{\mu, \xi}) = U^{q-1}_{\mu, \xi} (CU_{\mu, \xi} \pm q \nabla_s U_{\mu, \xi}) \geq 0 \quad \text{in} \ \Omega, \\
CPV_{\mu, \xi} \pm \nabla_s PV_{\mu, \xi} = 0 \quad \text{on} \ \partial \Omega.
\end{cases}
\]
for some $C > 0$ depending only on $N$, $p$ and $\delta$. The maximum principle shows that the second inequality in (A.11) holds. Also, applying an analogous argument to (3.9), we see that the first inequality in (A.11) is true. □

**APPENDIX B. PROOF OF \((\psi, (d, \tau), \phi, (d, \tau)) \in (L^\infty(\Omega_*))^2\)**

Fix $\epsilon > 0$ small and \((d, \tau) \in \Lambda_\delta\). In this appendix, we prove that \((\psi, (d, \tau), \phi, (d, \tau)) \in (L^\infty(\Omega_*))^2\) as stated in Proposition 5.4.

Equation (A.11) reads

\[
\begin{cases}
-\Delta \psi = |\phi + PV_{\mu, \xi}|^{p-1} (\phi + PV_{\mu, \xi}) + Q_{11} & \text{in } \Omega, \\
-\Delta \phi = |\psi + PU_{\mu, \xi}|^{q-1} (\psi + PU_{\mu, \xi}) + Q_{21} & \text{in } \Omega, \\
\psi = \phi = 0 & \text{on } \partial \Omega,
\end{cases}
\tag{B.1}
\]

where $Q_{11}, Q_{21} \in C^\infty(\Omega_*)$.

From the third inequality in (A.11), we find

\[|\phi + PV_{\mu, \xi}|^{p-1} (\phi + PV_{\mu, \xi}) = |\phi|^{p-1} (\phi + P_1) + Q_{12}\]

where $P_1 := p PV_{\mu, \xi} \in C^\infty(\Omega_*)$ and $Q_{12} \in L^\infty(\Omega_*)$. On the other hand, $q$ may be greater than 2, so we have

\[|\psi + PU_{\mu, \xi}|^{q-1} (\psi + PU_{\mu, \xi}) = |\psi|^{q-1} (\psi + P_2) + O \left( |\psi|^{q-2} (PU_{\mu, \xi})^2 \right) + Q_{22}\]

where $P_2 := q PU_{\mu, \xi} \in C^\infty(\Omega_*)$ and $Q_{22} \in L^\infty(\Omega_*)$.

Consequently, (B.1) is reduced to

\[
\begin{cases}
-\Delta \psi = |\phi|^{p-1} (\phi + P_1) + Q_{13} & \text{in } \Omega, \\
-\Delta \phi = |\psi|^{q-1} (\psi + P_2) + O \left( |\psi|^{q-2} \right) + Q_{23} & \text{in } \Omega, \\
\psi = \phi = 0 & \text{on } \partial \Omega,
\end{cases}
\tag{B.2}
\]

where $(P_1, P_2, Q_{13}, Q_{23}) \in (L^\infty(\Omega_*))^2$.

At this moment, we need a regularity result which holds for any linear Hamiltonian-type elliptic system (B.4). It is a modified version of [25] Lemma B.1 and its proof is inspired by that of [6] Theorem 1.3.

**Lemma B.1.** Suppose that $N \geq 4$, $p \in (1, \frac{N}{N-2})$, and $\Omega_*$ is a smooth bounded domain in $\mathbb{R}^N$. Given sufficiently small numbers $\zeta_1, \zeta_2 > 0$ satisfying $\zeta_1 (q + 1) = \zeta_2 (p + 1)$, we set

\[\sigma_1 = \frac{p + 1}{p - \zeta_1} \quad \text{and} \quad \sigma_2 = \frac{q + 1}{q - \zeta_2}.\]

Suppose that $(\psi, \phi) \in X_{p,q}(\Omega_*)$ and $(Q_1, Q_2) \in L^{p_1}(\Omega_*) \times L^{q_2}(\Omega_*)$. There is a small constant $\delta > 0$ depending only on $N$, $p$, $\Omega_*$, $\zeta_1$, $\zeta_2$ such that if

\[||F_1||_{L^\frac{p+1}{p-\zeta_1}(\Omega_*)} + ||F_2||_{L^\frac{q+1}{q-\zeta_2}(\Omega_*)} < \delta\]

and

\[
\begin{cases}
-\Delta \psi = F_1 \phi + Q_1 & \text{in } \Omega_*, \\
-\Delta \phi = F_2 \psi + Q_2 & \text{in } \Omega_*, \\
\psi = \phi = 0 & \text{on } \partial \Omega_*,
\end{cases}
\tag{B.4}
\]

then $(\psi, \phi) \in L^{\frac{N\sigma_1}{N-2}}(\Omega_*) \times L^{\frac{N\sigma_2}{N-2}}(\Omega_*)$. 


Proof. Let \((r, s) = (\frac{N\sigma_1}{N-2\sigma_1}, \frac{N\sigma_2}{N-2\sigma_2})\). Then
\[
    r > \frac{N(p+1)/p}{N-2(p+1)/p} = q + 1, \quad s > \frac{N(q+1)/q}{N-2(q+1)/q} = p + 1 > \frac{N}{N-2},
\]
and
\[
    \frac{1}{r} + \frac{1}{p+1} - \frac{1}{q+1} = \frac{1}{\sigma_1} - \frac{2}{N} + \frac{1}{p+1} - \frac{1}{q+1}
    = \frac{p}{p+1} - \frac{2}{N} + \frac{1}{p+1} - \frac{1}{q+1} - \frac{\zeta_1}{p+1} = \frac{1 - \zeta_1}{p+1}
    = \frac{1}{p+1} - \frac{\zeta_2}{q+1} = \frac{q - \zeta_2}{q+1} - \frac{2}{N} = \frac{1}{\sigma_2} - \frac{2}{N} = \frac{1}{s}.
\]

Let \(T_1\) and \(T_2\) be the operators given as
\[
    (T_1 g)(x) = \int_{\Omega} G(x, y) (F_1 g)(y) dy \quad \text{and} \quad (T_2 f)(x) = \int_{\Omega} G(x, y) (F_2 f)(y) dy
\]
for \(x \in \Omega_\ast\), where \(G\) is the Green’s function of the Dirichlet Laplacian \(-\Delta\) in \(\Omega_\ast\). Applying the Hardy-Littlewood-Sobolev inequality, Hölder’s inequality, \((B.3)\) and \((B.4)\), we obtain
\[
    \|T_1 g\|_{L^r(\Omega_\ast)} \leq C \|F_1 g\|_{L^{\frac{nr}{nq+r}}(\Omega_\ast)} \leq C \|F_1\|_{L^{\frac{n+1}{\sigma_1}}(\Omega_\ast)} \|g\|_{L^r(\Omega_\ast)},
\]
and similarly,
\[
    \|T_2 f\|_{L^s(\Omega_\ast)} \leq C \|F_2 f\|_{L^{\frac{ns}{nq+s}}(\Omega_\ast)} \leq C \|F_2\|_{L^{\frac{n+1}{\sigma_2}}(\Omega_\ast)} \|f\|_{L^s(\Omega_\ast)}.
\]
Therefore, if we define the operator \(T\) by \(T(f, g) = (T_1 g, T_2 f)\), then it maps \(L^r(\Omega_\ast) \times L^s(\Omega_\ast)\) into itself. In fact, \((B.3)\) indicates that \(T\) is a contraction mapping on \(L^r(\Omega_\ast) \times L^s(\Omega_\ast)\) provided \(\delta > 0\) small enough.

Set
\[
    Q_1(x) = \int_{\Omega} G(x, y) Q_1(y) dy \quad \text{and} \quad Q_2(x) = \int_{\Omega} G(x, y) Q_2(y) dy
\]
for \(x \in \Omega_\ast\), which belong to \(L^r(\Omega_\ast) \times L^s(\Omega_\ast)\) thanks to the condition \((Q_1, Q_2) \in L^{\sigma_1}(\Omega_\ast) \times L^{\sigma_2}(\Omega_\ast)\). We also write \((B.4)\) in the operator form
\[
    (\psi, \phi) = T(\psi, \phi) + (Q_1, Q_2).
\]

Then, by invoking the Banach fixed-point theorem and the uniqueness of solutions to \((B.7)\), we deduce that \((\psi, \phi) \in L^r(\Omega_\ast) \times L^s(\Omega_\ast)\). The proof is concluded.

By \((5.7)\), we have that \(\|E_{\epsilon, \eta, \frac{d}{(r, s)}}\|_{X_\epsilon} \to 0\) as \(\epsilon \to 0\). Thus, in view of \((5.12)\), all the hypotheses in Lemma \((B.1)\) are fulfilled for system \((B.2)\) with the choice
\[
\begin{align*}
    \Omega_\ast &= \Omega_\epsilon, \quad (F_1, F_2) = (|\phi|^{p-1}, |\psi|^{q-1}), \\
    Q_1 &= |\phi|^{p-1} P_1 + Q_{13} \in L^{\frac{n+1}{\sigma_1}}(\Omega_\ast), \quad Q_2 = |\psi|^{q-1} P_2 + O(|\psi|^{q-2} 1_{q>2}) + Q_{23} \in L^{\frac{n+1}{\sigma_2}}(\Omega_\ast).
\end{align*}
\]

Consequently, there exists a small number \(\zeta > 0\) such that
\[
    (\psi, \phi) \in L^{q+1+\zeta}(\Omega_\ast) \times L^{p+1+\zeta}(\Omega_\ast).
\]

In particular, we see from \((B.2)\) that
\[
    -\Delta \psi \in L^{\frac{p+1+\zeta}{p}}(\Omega_\ast) \quad \text{and} \quad -\Delta \phi \in L^{\frac{q+1+\zeta}{q}}(\Omega_\ast).
\]

Combined with the Calderón-Zygmund estimate and the Sobolev embedding theorem, this yields
\[
    \psi \in W^{2, \frac{n(p+1+\zeta)}{np-2(p+1+\zeta)}}(\Omega_\ast) \quad \text{and} \quad \phi \in W^{2, \frac{n(q+1+\zeta)}{nq-2(q+1+\zeta)}}(\Omega_\ast)
\]
provided \(np > 2(p+1+\zeta)\) and \(nq > 2(q+1+\zeta)\)\(\text{\footnote{If } np \leq 2(p+1+\zeta), \text{ then } \psi \in L^a(\Omega_\ast) \text{ for all } a \geq 1. \text{ Similarly, if } nq \leq 2(q+1+\zeta), \text{ then } \phi \in L^b(\Omega_\ast) \text{ for all } b \geq 1.}\)}\).

Besides, an elementary computation shows
\[
    \frac{n(p+1+\zeta)}{np-2(p+1+\zeta)} \geq q + 1 + (1 + \eta) \zeta \quad \text{and} \quad \frac{n(q+1+\zeta)}{nq-2(q+1+\zeta)} \geq p + 1 + (1 + \eta) \zeta.
\]
for some small number \( \eta > 0 \) independent of \( \zeta \). Hence
\[ \psi \in L^{q+1+(1+\eta)\zeta}(\Omega_\epsilon) \quad \text{and} \quad \psi \in L^{p+1+(1+\eta)\zeta}(\Omega_\epsilon). \]
Iterating the above process finitely many times, we deduce
\[ (\psi, \phi) \in L^a(\Omega_\epsilon) \times L^b(\Omega_\epsilon) \quad \text{for all} \quad a, b \geq 1. \]
Finally, by feeding this information back to (B.2), we arrive at \((\psi, \phi) \in (L^\infty(\Omega_\epsilon))^2\) as desired.

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(Sangdon Jin) DEPARTMENT OF MATHEMATICS, CHUNG-ANG UNIVERSITY, SEOUL 06974, KOREA
Email address: sdjin@cau.ac.kr

(Seunghyeok Kim) DEPARTMENT OF MATHEMATICS AND RESEARCH INSTITUTE FOR NATURAL SCIENCES, COLLEGE OF NATURAL SCIENCES, HANYANG UNIVERSITY, 222 WANGSIMNI-RO SEONGDONG-GU, SEOUL 04763, REPUBLIC OF KOREA
Email address: shkim0401@hanyang.ac.kr shkim0401@gmail.com