A note on the toric Newton spectrum of a polynomial

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Abstract

We define the toric Newton spectrum of a polynomial and we give some applications. Motivated by mirror symmetry, we show in particular that it has a natural orbifold flavor.

1 Introduction

Let \( f \) be a convenient and Newton nondegenerate polynomial in the sense of Kouchnirenko [11], defined on \( \mathbb{C}^n \) (we recall the definitions in Section 2). In these notes, we study the toric Newton spectrum of \( f \), that is the Hilbert-Poincaré series (we denote it by \( P_{\text{gr}}(\mathcal{N}(\mathfrak{g})) (z) \)) of the ring \( \mathcal{B}/\mathcal{L} := \mathbb{C}[u_1, \ldots, u_n]/(u_1 \frac{\partial f}{\partial u_1}, \ldots, u_n \frac{\partial f}{\partial u_n}) \), graded by the Newton filtration. This toric Newton spectrum fits very well with the Newton filtration and can be easily computed using a Koszul complex provided by [11], see Theorem 3.2. We will see how to use it in order to get alternative proofs of some known (or expected) results. It turns out that in some cases (mirror symmetry, combinatorics), this toric spectrum is the “good one” to consider.

Let us explain more in details our motivations. First, we show that the spectrum at infinity (in the sense of [13]) of a convenient and nondegenerate polynomial can be computed from its toric Newton spectrum: we get in particular in Proposition 4.5 a description that matches with a prior result of Y. Matsui and K. Takeuchi [12, Theorem 5.16].

Second, and this has to do with mirror symmetry, we show that this toric Newton spectrum is related with the orbifold cohomology of a stack naturally produced by the Newton polytope \( P \) of \( f \). More precisely, let \( \Sigma \) be the fan in \( \mathbb{R}^n \) obtained by taking the cones over the proper faces of \( P \) not containing the origin and let \( X_\Sigma \) be the toric variety associated with the fan \( \Sigma \). We denote the vertices of \( P \) different from the origin by \( b_1, \ldots, b_r \). Following [2], we will call the triple \( \Sigma = (\mathbb{Z}^n, \Sigma, \{b_i\}) \) the stacky fan of \( P \). We get in Corollary 5.9 the formula \( P_{\text{gr}}(\mathcal{N}(\mathfrak{g})) (z) = \sum_{\alpha \in \mathbb{Q}} \dim_{\mathbb{Q}} H^{2\alpha}_{\text{orb}}(X, \mathbb{Q}) z^\alpha \) where \( X \) is the stack associated with \( \Sigma \) by [2 Proposition 4.7]. Naturally enough, the polynomial \( f \) and \( X \) are said to be mirror partners if there is an isomorphism of \( \mathbb{Q} \)-graded rings \( H^{2\alpha}_{\text{orb}}(X, \mathbb{Q}) \rightarrow \text{gr}^N(\mathfrak{g}) \): the previous equality gives a positive result at the linear level. It is much more difficult to show the correspondence of the products; in the complete case, and up to to a strictness reason [11 Theorem 4.1 (ii)], this is achieved in [2]. See Section 5.3 for a discussion about the subject.

Third, and this is the combinatorial point of view, the toric Newton spectrum counts weighted lattice points in the Newton polytope \( P \) of \( f \): using [15], it is easy to get the \( \delta \)-vector of \( P \), hence
its Ehrhart polynomial, from its toric Newton spectrum, see Section 6. This is emphasized in [6] when \( f \) is a \textit{Laurent} polynomial.

Last, it should be noticed that the previous results have a straightforward \textit{local} version: this is discussed in Section 5.4.

\textit{Conventions.} Let \( n \) be a positive integer. In this text, \( \mathcal{B} \) will denote the polynomial ring \( \mathbb{C}[u_1, \ldots, u_n] \) and we will write \( u^m := u_1^{m_1} \cdots u_n^{m_n} \) if \( m = (m_1, \ldots, m_n) \in \mathbb{N}^n \).

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\section{Kouchnirenko’s setting}

In this section, we set the framework and the notations. We follow [11]. The \textit{support} of \( g = \sum_{m \in \mathbb{N}^n} a_m u^m \in \mathcal{B} \) is \( \text{supp}(g) = \{ m \in \mathbb{N}^n, a_m \neq 0 \} \) and the \textit{Newton polytope} of \( g \) is the convex hull of \( \{0\} \cup \text{supp}(g) \) in \( \mathbb{R}^n_+ \). The \textit{Newton boundary} of a Newton polytope is the union of its closed faces that do not contain the origin. The polynomial \( f \) is \textit{convenient} if, for each \( i = 1, \ldots, n \), there exists an integer \( n_i \geq 1 \) such that the monomial \( u_i^{n_i} \) appears in \( f \) with a non-zero coefficient. The polynomial \( f \) is \textit{Newton nondegenerate} (for short in this text: nondegenerate) if, for each closed face \( \Delta \) of the Newton boundary, the system

\[
(u_1 \frac{\partial f}{\partial u_1})_\Delta = \cdots = (u_n \frac{\partial f}{\partial u_n})_\Delta = 0
\]

has no solutions on \( (\mathbb{C}^*)^n \) (we define here \( g_\Delta = \sum_{m \in \mathbb{N}^n \cap \Delta} b_m u^m \) if \( g = \sum_{m \in \mathbb{N}^n} b_m u^m \in \mathcal{B} \)).

Let \( f \in \mathcal{B} \) and let \( P \) be its Newton polytope. If \( F \) is a facet (face of dimension \( n-1 \)) of the Newton boundary of \( P \), let \( u_F \in \mathbb{Q}^n \) be such that

\[
F = P \cap \{ n \in \mathbb{R}^n, \, \langle u_F, n \rangle = 1 \}. \tag{1}
\]

The \textit{Newton function} \( \nu : \mathbb{N}^n \to \mathbb{Q} \) of \( P \) is defined by \( \nu(a) := \max_F \langle u_F, a \rangle \), where the maximum is taken over the facets of the Newton boundary of \( P \). We have \( \nu(a + b) \leq \nu(a) + \nu(b) \) with equality if and only if \( a \) and \( b \) belong to the same cone (by a cone, we mean a cone spanned by the faces of the Newton boundary of \( P \)). For \( \alpha \in \mathbb{Q} \), let

\[
\mathcal{B}_\alpha = \{ g \in \mathcal{B}, \, \text{supp}(g) \in \nu^{-1}([-\infty; \alpha]) \}.
\]

We get an increasing filtration \( \mathcal{N}_\bullet \) of the ring \( \mathcal{B} \), indexed by \( \mathbb{Q} \), by setting \( \mathcal{N}_\alpha \mathcal{B} := \mathcal{B}_\alpha \): this is the \textit{Newton filtration} of \( \mathcal{B} \). We put \( \mathcal{B}_{<\alpha} = \bigcup_{\beta < \alpha} \mathcal{B}_\beta \), \( B_\alpha = \frac{\mathcal{B}_{<\alpha}}{\mathcal{B}_{<\alpha}} \) and \( B = \bigoplus_\alpha B_\alpha / \mathcal{B}_{<\alpha} \). The product in \( B \) is described as follows: for \( u^m \in \mathcal{B} \), let

\[
\delta_m = u^m + \mathcal{B}_{\nu(m)} \in \mathcal{B}_{\nu(m)} / \mathcal{B}_{\nu(m)} = B_{\nu(m)};
\]

then,

\[
\delta_{m_1} \cdot \delta_{m_2} = \begin{cases} 
\delta_{m_1+m_2} & \text{if } m_1 \text{ and } m_2 \text{ belong to the same cone,} \\
0 & \text{otherwise.} \end{cases} \tag{2}
\]
We will denote by $P_B(z) = \sum_{\alpha \in Q} \dim B_\alpha z^\alpha$ the Hilbert-Poincaré series of the graded ring $B$.

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a convenient and nondegenerate polynomial. By [11], $f$ has only isolated critical points and its global Milnor number $\mu_f := \dim \mathbb{C}B/(\frac{\partial f}{\partial u_1}, \frac{\partial f}{\partial u_2}, \cdots, \frac{\partial f}{\partial u_n})$ is finite. In this text, a geometric spectrum $\text{Spec}_f$ of $f$ is an ordered sequence of rational numbers $\alpha_1 \leq \cdots \leq \alpha_{\mu_f}$ that we will identify with the generating function $\text{Spec}_f(z) := \sum_{i=1}^{\mu_f} z^{\alpha_i}$. The specifications are the following: the $\alpha_i$’s are positive numbers (positivity) and $\text{Spec}_f(z) = z^n \text{Spec}_f(z^{-1})$ (Poincaré duality: a geometric spectrum is symmetric about $n/2$). We can ask moreover that the multiplicity of $\alpha_1$ in $\text{Spec}_f$ is equal to one (normalization): this normalization is not automatic, see examples 4.7, and appears in the construction of “canonical” Frobenius manifolds in the sense of [9].

3 The toric Newton spectrum of a convenient and nondegenerate polynomial

Let $f$ be a convenient and nondegenerate polynomial on $\mathbb{C}^n$. In what follows, $\mathcal{L}$ will denote the ideal generated by the partial derivative $u_1 \frac{\partial f}{\partial u_1}, \cdots, u_n \frac{\partial f}{\partial u_n}$ of $f$. By projection, the Newton filtration $\mathcal{N}^\bullet$ on $B$ induces the Newton filtration $\mathcal{N}^\bullet$ on $B_L$ and we get the graded ring $\text{gr}^N(B_L) = \oplus_{\alpha \in Q} \text{dim} \text{gr}^N(\alpha_{\mathcal{L}}) z^\alpha$.

**Definition 3.1** The toric Newton spectrum of a convenient and nondegenerate polynomial $f$ is the Hilbert-Poincaré series $P_{\text{gr}^N(B_L)}(z) := \sum_{\alpha \in Q} \text{dim} \text{gr}^N(\alpha_{\mathcal{L}}) z^\alpha$.

We will also see the toric Newton spectrum as the sequence of rational numbers $\alpha$, where each $\alpha$ is counted $\text{dim} \text{gr}^N(\alpha_{\mathcal{L}})$-times.

**Theorem 3.2** Let $f$ be a convenient and nondegenerate polynomial on $\mathbb{C}^n$. Then

$$P_{\text{gr}^N(\mathcal{L})}(z) = (1 - z)^n \sum_{v \in \mathbb{N}^n} z^{\nu(v)}.$$  \hspace{1cm} (3)

In particular, the toric Newton spectrum of $f$ depends only on its Newton polytope.

**Proof.** Let $F_i$ be the class of $x_i \frac{\partial f}{\partial x_i}$ in $B_1 = \frac{B}{B_{>1}}$ and let $F$ be the ideal generated by $F_1, \cdots, F_n$ in $B$. By [11] Théorème 2.8, and because $f$ is nondegenerate, the sequence

$$0 \to B^{(n)} \to B^{(n-1)} \to \cdots \to B^{(1)} \to B \to B/F \to 0$$  \hspace{1cm} (4)

that we get from the Koszul complex of the elements $F_1, \cdots, F_n$ in the ring $B$ is exact. It follows that the map

$$\partial : \mathcal{B}^n \to B$$  \hspace{1cm} (5)

defined by $\partial(b_1, \cdots, b_n) = b_1 u_1 \frac{\partial f}{\partial u_1} + \cdots + b_n u_n \frac{\partial f}{\partial u_n}$ is strict with respect to the Newton filtration, see [11] Théorème 4.1. Hence $\text{gr}^N(\mathcal{L}) \cong \frac{B}{F}$ and

$$P_{\text{gr}^N(\mathcal{L})}(z) = P_{B/F}(z).$$  \hspace{1cm} (6)

Now, by [11], we have also

$$P_{B/F}(z) = (1 - z)^n P_B(z)$$  \hspace{1cm} (7)

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where $P_B(z)$ denotes the Hilbert-Poincaré series of the graded ring $B$. □

For $\Delta$ a closed face of $P$ whose vertices are $b_1, \ldots, b_k$, let us define

$$\square(\Delta) := \{ \sum_{\ell=1}^{k} q_\ell b_\ell, ~ q_\ell \in [0,1[ , ~ \ell = 1, \ldots, k \}.$$  \hspace{1cm} (8)

Let $\mathcal{F}(P)$ be the set of the closed faces of $P$ not contained in the union of the hyperplane coordinates.

**Proposition 3.3** Let $f$ be a convenient and nondegenerate polynomial on $\mathbb{C}^n$ and let $P$ be its Newton polytope. Assume that the faces of $\mathcal{F}(P)$ are simplices. Then

$$P_{\text{gr}N}(\frac{q}{z})(z) = \sum_{q=0}^{n-1} \sum_{\dim \Delta = q} (z-1)^{n-1-q} \sum_{v \in \mathcal{C}(\Delta) \cap \mathbb{N}^n} z^{\nu(v)}$$

where $\Delta \in \mathcal{F}(P)$.

**Proof.** For $\Delta$ a closed face of $P$, let $\text{Cone}(\Delta)$ be the union of the half straight lines starting from the origin and passing through $\Delta$ and $B_\Delta = \{ g \in B, \supp(g) \in \mathbb{N}^n \cap \text{Cone}(\Delta) \}$. Because the polynomial $f$ is convenient, there is by [11, Proposition 2.6] an exact sequence of graded $B$-modules

$$0 \to B \to C_{n-1} \to C_{n-2} \to \cdots \to C_0 \to 0$$

where $C_q = \oplus_{\dim \Delta = q} B_\Delta$ for $\Delta \in \mathcal{F}(P)$. It follows that

$$P_B(z) = \sum_{q=0}^{n-1} \sum_{\dim \Delta = q} (-1)^{n-1-q} P_{B_\Delta}(z)$$

where $\Delta \in \mathcal{F}(P)$. Using (7), we get

$$P_{B/F}(z) = \sum_{q=0}^{n-1} \sum_{\dim \Delta = q} (1-z)^{n-1-q} (1-z)^{n-1-q} (1-z)^{q+1} P_{B_\Delta}(z).$$

Last,

$$(1-z)^{q+1} P_{B_\Delta}(z) = (1-z)^{q+1} \sum_{v \in \mathcal{C}(\Delta) \cap \mathbb{N}^n} z^{\nu(v)} = \sum_{v \in \mathcal{C}(\Delta) \cap \mathbb{N}^n} z^{\nu(v)}$$

if $\Delta \in \mathcal{F}(P)$ is a simplex and $\dim \Delta = q$. Now the assertion follows from (6). □

**Remark 3.4** Define $\mu_P := n! \text{vol}(P)$ where the volume $\text{vol}(P)$ of the Newton polytope $P$ is normalized such that the volume of the cube is equal to one. Then $\dim \mathcal{C} \frac{R}{\mathbb{Z}} = \mu_P$, see [17].

Last, we give some expected consequences of the previous results (in particular we will get a complete description of the toric Newton spectrum for $n = 2$, compare with [9] Example 4.17 and [15, Lemma 3.13]). If $I$ is an interval of $\mathbb{R}$ and $P_C(z) = \sum_{\alpha \in \mathbb{Q}} \dim C_\alpha z^\alpha$ we define $P_C^I(z) = \sum_{\alpha \in I} \dim C_\alpha z^\alpha$. 

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Corollary 3.5 Let \( f \) be a convenient and nondegenerate polynomial on \( \mathbb{C}^n \) and let \( P \) be its Newton polytope. Then,

1. \( P_{\operatorname{gr}^N(\frac{f}{z})}(z) = P_{\operatorname{gr}^N(\frac{f}{z})}^{[0,n]}(z) \) if \( P \) is simplicial,
2. \( P_{\operatorname{gr}^N(\frac{f}{z})}^{[0,1]}(z) = \sum_{v \in \mathbb{N}, \nu(v) < 1} z^{\nu(v)}, \)
3. the multiplicity \( \mu_1 \) of 1 in \( P_{\operatorname{gr}^N(\frac{f}{z})}(z) \) is \( \mu_1 = \operatorname{Card}(N \cap \mathbb{N}^n) - n \) where \( N \) denotes the Newton boundary of \( P \).

Proof. The first assertion follows from Proposition 3.3 because \( \nu(v) < q + 1 \) if \( v \) belongs to a \( q \)-dimensional face of \( P \). Because the map (5) is strict with respect to the Newton filtration, we have \( \mathcal{L}(f) \cap \mathcal{N}_\alpha \mathcal{B} = \operatorname{im} \partial \cap \mathcal{N}_\alpha \mathcal{B} = \partial(N_{\alpha-1}(\mathcal{B}^n)) \) and because \( N_\beta \mathcal{B} = 0 \) if \( \beta < 0 \), we get

\[
\frac{\mathcal{N}_\alpha \mathcal{B}}{\mathcal{L}(f) \cap \mathcal{N}_\alpha \mathcal{B} + \mathcal{N}_{< \alpha} \mathcal{B}} = \frac{\mathcal{N}_\alpha \mathcal{B}}{\mathcal{N}_{< \alpha} \mathcal{B}}
\]

for \( \alpha < 1 \). Thus, \( P_{\operatorname{gr}^N(\frac{f}{z})}^{[0,1]}(z) = \sum_{\alpha < 1} \dim \mathcal{B}_\alpha z^\alpha \) and this shows the second assertion. The third one follows from the computation of the coefficient of \( z \) in the formula of Theorem 3.2.

Remark 3.6 By Corollary 3.5 and because \( \nu(v) = 0 \) if and only if \( v \) is the origin, the value 0 appears in the toric Newton spectrum with multiplicity one. It follows that the toric Newton spectrum is not symmetric about \( \frac{n}{2} \). In particular, the toric Newton spectrum is not a geometric spectrum in the sense of Section 2.

4 Toric Newton spectrum and spectrum at infinity

Let \( f \) be a convenient and nondegenerate polynomial on \( \mathbb{C}^n \) with Newton polytope \( P \), \( \mathcal{I} \) is the ideal generated by the partial derivative \( \frac{\partial f}{\partial u_1}, \ldots, \frac{\partial f}{\partial u_n} \) of \( f \) and \( \mu_f := \dim_{\mathbb{C}} \mathcal{B}_P \) is the Milnor number of \( f \). We define an increasing filtration, indexed by \( \mathbb{Q} \), \( W \) on \( \mathcal{B} := \mathbb{C}[u_1, \ldots, u_n] \) by

\[
W_\alpha \mathcal{B} := \{ g \in \mathcal{B}, \operatorname{supp}(gu_1 \cdots u_n) \in \nu^{-1}(| - \infty; \alpha|) \}.
\]

By projection, we get a filtration on \( \mathcal{B}_P \). We set \( \mathcal{B}_\alpha := \operatorname{gr}_\alpha W \mathcal{B} \) and \( \mathcal{B}^* := \oplus \mathcal{B}_\alpha^* \).

Definition 4.1 The Newton spectrum of the convenient and nondegenerate polynomial \( f \) is the Hilbert-Poincaré series \( P_{\mathcal{B}^*}(z) = \sum_{\alpha \in \mathbb{Q}} \dim \mathcal{B}_\alpha z^\alpha \).

The Newton spectrum can be seen as a sequence of rational numbers with the following property: the frequency of \( \alpha \) in the sequence is equal to \( \dim \mathcal{B}_\alpha^* \).

In order to justify this definition (in particular to explain the twist by \( u_1 \cdots u_n \)), we recall the definition of the spectrum at infinity of a convenient and nondegenerate polynomial \( f \) defined on \( U = \mathbb{C}^n \) (we use the notations of [9, 2.e]). Let \( G \) be the Fourier-Laplace transform of the Gauss-Manin system of \( f \) and let \( G_0 \) be its Brieskorn lattice. Because \( f \) is convenient and nondegenerate, \( G_0 \) is indeed a free \( \mathbb{C}[\theta] \)-module of rank \( \mu_f \) and \( G = \mathbb{C}[\theta, \theta^{-1}] \otimes G_0 \); this follows for instance from the strictness of the map (5), see [9, Remark 4.8]. Last, let \( V \) be the \( V \)-filtration of \( G \) at infinity, that is along \( \theta^{-1} = 0 \). It provides, by projection, a \( V \)-filtration on the \( \mu_f \)-dimensional vector space \( \Omega_f := \Omega^n(U)/df \wedge \Omega^{n-1}(U) = G_0/\theta G_0 \).
**Definition 4.2** \[13\] Section 1 and Corollary 10.2] The spectrum at infinity of a convenient and nondegenerate polynomial \( f \) is the sequence \( \alpha_1, \alpha_2, \cdots, \alpha_{\mu_f} \) of rational numbers with the following property: the frequency of \( \alpha \) in the sequence is equal to \( \dim \text{gr}_\alpha \Omega_f \).

By \[13\], the spectrum at infinity is a geometric spectrum. Notice however that it does not satisfy the normalization condition in general.

In order to define a Newton filtration on \( \Omega^n(U) \), we decrete that the Newton order of the volume form \( \frac{du_1 \wedge \cdots \wedge du_n}{u_1} \) is zero: the Newton filtration \( N_\bullet \) on \( \Omega^n(U) \) is thus the increasing filtration \( N_\bullet \) indexed by \( \mathbb{Q} \) such that

\[
N_\alpha \Omega^n(U) := \{ \omega \in \Omega^n(U), \text{ supp}(\omega) \in \nu^{-1}([-\infty; \alpha]) \}
\]

where \( \text{supp}(\omega) = \{ m + (1, \cdots, 1) \in \mathbb{N}^n, a_m \neq 0 \} \) if \( \omega = \sum_{m \in \mathbb{Z}^n} a_m u^m du_1 \wedge \cdots \wedge du_n \). This filtration induces a filtration on \( \Omega_f \) and the Newton spectrum is equal to the spectrum of this Newton filtration on \( \Omega_f \). This normalization is explained by the following result (in the local case, that is if \( f \) is the germ of a holomorphic function with an isolated critical point at the origin, a similar result is given in \[10\] and \[14\]):

**Theorem 4.3** \[13\] Theorem 12.1] The spectrum at infinity of a convenient and nondegenerate polynomial \( f \) is equal to its Newton spectrum.

It happens that the Newton spectrum (hence the spectrum at infinity) can be computed from the toric Newton spectrum. The first step is given by the following counterpart of \[3\] Proposition B.1.2.3):

**Lemma 4.4** Let \( f \) be a convenient and nondegenerate polynomial on \( \mathbb{C}^n \). Then the multiplication by \( u_1 \cdots u_n \) induces isomorphisms

\[
\lambda_\alpha : \text{gr}_\alpha^W \frac{B}{I} \to \text{gr}_\alpha^N \frac{Bu_1 \cdots u_n + L(f)}{L(f)} \subset \text{gr}_\alpha^N \frac{B + L(f)}{L(f)}
\]

for \( \alpha \in \mathbb{Q} \).

**Proof.** Let \( g \in B \) such that \( gu_1 \cdots u_n \in L(f) \cap N_\alpha B \cap Bu_1 \cdots u_n \). Then, as in \[3\] Appendix 1], there exists \( g_1, \cdots, g_n \in B \) such that

\[
gu_1 \cdots u_n = u_1 \frac{\partial f}{\partial u_1} u_2 \cdots u_ng_1 + \cdots + u_n \frac{\partial f}{\partial u_n} u_1 \cdots u_{n-1}g_n
\]

and \( \nu(u_2 \cdots u_ng_1) \leq \alpha - 1, \cdots, \nu(u_1 \cdots u_{n-1}g_n) \leq \alpha - 1 \). Thus, the multiplication by \( u_1 \cdots u_n \) induces \( \lambda : \frac{B}{I} \to \frac{B_{u_1 \cdots u_n}}{L(f)B_{u_1 \cdots u_n}} \) and injective maps

\[
\lambda_\alpha : \frac{W_\alpha B}{I \cap W_\alpha B + W_{<\alpha} B} \to \frac{N_\alpha B \cap Bu_1 \cdots u_n}{L(f) \cap N_\alpha B \cap Bu_1 \cdots u_n + N_{<\alpha} B \cap Bu_1 \cdots u_n}.
\]

For \( 1 \leq p \leq n \), let \( I_p = \{(i_1, \cdots, i_p), 1 \leq i_1 < \cdots < i_p \leq n\} \) and let \( P^{(i_1, \cdots, i_p)}(\frac{f}{\text{gr}_\alpha^N(\frac{\Theta}{\omega})}) \) be the toric Newton spectrum of the restriction \( f_{(i_1, \cdots, i_p)} \) of \( f \) at \( u_1 = \cdots = u_p = 0 \), with the convention \( P^{(1, \cdots, n)}(\frac{f}{\text{gr}_\alpha^N(\frac{\Theta}{\omega})}) = 1 \). The restrictions \( f_{(i_1, \cdots, i_p)} \) are convenient and nondegenerate if \( f \) is so.

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Proposition 4.5 Let \( f \) be a convenient and nondegenerate polynomial with Newton polytope \( P \). Then

\[
P_{B^*}(z) = P_{\text{gr}^N(\mathbb{F}_2)}(z) + \sum_{p=1}^{n-1} (-1)^p \sum_{(i_1,\ldots,i_p) \in I_p} P_{\text{gr}^N(\mathbb{F}_2)}^{(i_1,\ldots,i_p)}(z) + (-1)^n.
\]

Proof. Assume that \( g u_1 \cdots u_{n-1} \in \mathcal{L}(f) \cap N_\alpha \mathbb{C}[u_1, \ldots, u_n]u_1 \cdots u_{n-1} \). Then, by \([3\text{ Appendix 1}]\), there exists \( g_1, \ldots, g_n \in \mathcal{B} \) such that

\[
g u_1 \cdots u_{n-1} = u_1 \frac{\partial f}{\partial u_1} u_2 \cdot \cdots \cdot u_{n-1} g_1 + \cdots + u_{n-1} \frac{\partial f}{\partial u_{n-1}} u_2 \cdot \cdots \cdot u_{n-1} g_n + u_n \frac{\partial f}{\partial u_n} u_1 \cdots u_{n-1} g_n
\]

Hence \( g = \frac{\partial f}{\partial u_1} g_1 + \cdots + \frac{\partial f}{\partial u_{n-1}} g_{n-1} + u_n \frac{\partial f}{\partial u_n} g_n \). We deduce the isomorphisms (put \( u_0 = 0 \) in the previous formula), induced by the multiplication by \( u_1 \cdots u_{n-1} \) as in Lemma 4.4.

\[
\text{gr}_\alpha^W \mathbb{C}[u] \cong \text{gr}_\alpha^N \mathbb{C}[u_1, \ldots, u_{n-1}]u_1 \cdots u_{n-1} \mathcal{L}(f) \cap \mathbb{C}[u_1, \ldots, u_{n-1}]u_1 \cdots u_{n-1}
\]

where \( u' = (u_1, \ldots, u_{n-1}) \). We also have \( P_{B^*}(1) = \mu_f \) because the filtration \( W_\bullet \) is exhaustive and finite on \( \mathbb{F}_2 \). By \([11 \text{ Théorème 1.15}]\), it follows that

\[
P_{B^*}(1) = n! V_n - (n-1)! V_{n-1} + \cdots + (-1)^{n-1} V_1 + (-1)^n
\]

where, for \( 1 \leq q \leq n-1 \), \( V_q \) is the sum of the \( q \)-dimensional volumes of the intersection of \( P \) with the hyperplane coordinates of dimension \( q \). Because \( P_{\text{gr}^N(\mathbb{F}_2)}(1) = n! V_n \), a dimension argument, Lemma 4.4 and equation (10) yield

\[
P_{\text{gr}^N(\mathbb{F}_2)}(z) = P_{B^*}(z) + \sum_{p=1}^{n} \sum_{(i_1,\ldots,i_p) \in I_p} P_{B^*}^{(i_1,\ldots,i_p)}(z)
\]

where \( P_{B^*}^{(i_1,\ldots,i_p)}(z) \) is the Newton spectrum of \( f^{(i_1,\ldots,i_p)} \), and \( P_{B^*}^{(1,\ldots,n)}(z) = 1 \). Now we are done by induction because \( P_{B^*}(z) = P_{\text{gr}^N(\mathbb{F}_2)}(z) - 1 \) if \( n = 1 \) by Proposition 3.3 and Lemma 4.4. \( \square \)

**Remark 4.6** Combining Proposition 3.3 and Proposition 4.5 we get \([12 \text{ Theorem 5.16}]\).

**Example 4.7** In the following examples, the toric Newton spectrum is computed using Proposition 3.3 and the Newton spectrum is computed using Proposition 4.5.

1. Let \( f(u,v) = u^2 + u^2 v^2 + v^2 \). Then \( \mu_P = 8 \) and \( \mu_f = 5 \). We get

\[
P_{\text{gr}^N(\mathbb{F}_2)}(z) = 1 + 3z + 3z^{1/2} + z^{3/2}
\]

and

\[
P_{B^*}(z) = P_{\text{gr}^N(\mathbb{F}_2)}(z) - (1 + z^{1/2}) - (1 + z^{1/2}) + 1 = z^{1/2} + 3z + z^{3/2}.
\]
2. Let \( g(u, v) = u + v + u^2v^4 \) on \( \mathbb{C}^2 \). Then \( \mu_g = 5 \) and \( \mu_P = 6 \). We get

\[
P_{B^*}(z) = z^{1/2} + z^{3/4} + z + z^{3/2} + z^{5/4}
\]

and

\[
P_{grN}(\frac{g}{z})(z) = 1 + z^{1/2} + z^{3/4} + z + z^{3/2} + z^{5/4}.
\]

Because the linear forms supporting the facets of the Newton boundary of \( g \) have equations

\[u - \frac{1}{4}v \quad \text{and} \quad -\frac{3}{2}u + v,
\]

the Newton order of the volume form \( du \wedge dv \) is \( 3/4 \) and the Newton order of the form \( vdu \wedge dv \) is \( 1/2 \): the Newton order of the standard volume form \( du \wedge dv \) is not necessarily the smallest element in the Newton spectrum.

3. Let \( f(u, v, w) = u + v + w + u^2v^2w^2 + v^2w^2 \). Then \( \mu_P = 12 \) and \( \mu_f = 8 \). We first analyze the contributions of \( v \in \Delta(\Delta), \Delta \in \mathcal{F}(P) \):

- the contribution of \( v = (0, 0, 0) \) is \( 4 + 4(z - 1) + (z - 1)^2 = z^2 + 2z + 1 \),
- the contribution of \( v = (1, 1, 1) \) is \( (4 + 4(z - 1) + (z - 1)^2)z^{1/2} = (z^2 + 2z + 1)z^{1/2} \),
- the contribution of \( v = (1, 2, 2) \) is \( (2 + (z - 1))z = (z + 1)z = z + z^2 \),
- the contribution of \( v = (0, 1, 1) \) is \( (2 + (z - 1))z^{1/2} = (z + 1)z^{1/2} = z^{1/2} + z^{3/2} \).

Thus,

\[
P_{grN}(\frac{f}{z})(z) = 1 + 2z^{1/2} + 3z^{3/2} + 3z + 2z^2 + z^{5/2}
\]

and

\[
P_{B^*}(z) = P_{grN}(\frac{f}{z})(z) - (3 + z^{1/2} + z + z^{3/2}) + 3 - 1 = z^{1/2} + 2z^{3/2} + 2z + 2z^2 + z^{5/2}.
\]

5. Toric Newton spectrum and orbifold cohomology

5.1 Technical interlude: Hodge-Deligne polynomials

In this section, \( f \) is a convenient and nondegenerate polynomial defined on \( \mathbb{C}^n \) with Newton polytope \( P \). Let \( \Sigma \) be the fan in \( \mathbb{R}^n \) obtained by taking the cones over the faces of the Newton boundary of \( P \) and let \( X_\Sigma \) be the toric variety associated with the fan \( \Sigma \). We will call \( \Sigma \) the fan of \( P \) and \( X_\Sigma \) the toric variety of \( P \). We will denote by \( \Sigma^* \) the set of the cones of \( \Sigma \) not contained in the hyperplane coordinates. We first gather some results about Hodge-Deligne polynomials.
Definition 5.1 Let $v \in \mathbb{N}^n$ and let $\sigma(v)$ be the smallest cone of $\Sigma$ containing $v$. Then,

1. $E_v(z) := \sum_{\tau \in \Sigma, \sigma(v) \subseteq \tau} (z - 1)^{n - \dim \tau}$ is the Hodge-Deligne polynomial of $v$,

2. $E^*_v(z) := \sum_{\tau \in \Sigma^*, \sigma(v) \subseteq \tau} (z - 1)^{n - \dim \tau}$ is the relative Hodge-Deligne polynomial of $v$.

The case $v = 0$ deserves a special attention: we will call $E_0(z)$ the Hodge-Deligne polynomial of the toric variety $X_\Sigma$ and $E^*_0(z)$ the relative Hodge-Deligne polynomial of $X_\Sigma$.

Remark 5.2 Assume that $P$ is simplicial. Then $E^*_0(z) - 1 = E_0(z) - z^n$. Indeed, and thanks to the simpliciality assumption, we have

$$E_0(z) = E^*_0(z) + \sum_{i=0}^{n-1} \binom{n}{i} (z - 1)^{n-i} = E^*_0(z) + z^n - 1.$$ 

Example 5.3 Assume that $n = 2$. Then $E_0(z) = z^2 + cz$ and $E^*_0(z) = cz + 1$ where $c$ is the number of vertices of $P$ not contained in the hyperplane coordinates.

Proposition 5.4 Let $f$ be a convenient and nondegenerate polynomial on $\mathbb{C}^n$ and let $P$ be its Newton polytope. Assume that $\Sigma$ is simplicial. Then:

1. $X_\Sigma$ has no odd cohomology and $E_0(z) = \sum_{k=0}^{n} \dim H^k_c(X_\Sigma, \mathbb{Q}) z^k$,

2. $z^n E_0(z^{-1}) = E^*_0(z)$.

Proof. For the first point we follow closely [15, Lemma 4.1]: let us denote by $\rho$ the ray in $-\Sigma$ whose primitive generator is $v_\rho = (-1, \cdots, -1) \in \mathbb{Z}^n$ and let $\Sigma'$ (resp. $\Sigma'_\rho$) be the fan given by the cones of $\Sigma$ and the cones generated by $\rho$ and the cones of $\Sigma$ contained in the hyperplane coordinates (resp. be the quotient fan $\Sigma'/\rho$). By the proof of loc. cit., we have an exact sequence of cohomology with compact support

$$0 \rightarrow H^2_c(X_\Sigma, \mathbb{Q}) \rightarrow H^2_c(X_{\Sigma'}, \mathbb{Q}) \rightarrow H^2_c(X_{\Sigma'_\rho}, \mathbb{Q}) \rightarrow 0.$$ 

Because $X_{\Sigma'}$ and $X_{\Sigma'_\rho}$ are complete and simplicial, the first formula holds for $X_{\Sigma'}$ and $X_{\Sigma'_\rho}$ (see for instance [5, section 14]) and we get

$$\sum_{k=0}^{n} \dim H^k_c(X_{\Sigma'}, \mathbb{Q}) z^k = E^\Sigma_0'(z) - E^\Sigma'_{0,\rho}(z)$$

$$= \sum_{\tau \in \Sigma'} (z - 1)^{n - \dim \tau} - \sum_{\rho \subseteq \tau} (z - 1)^{n - \dim \tau} = \sum_{\tau \in \Sigma} (z - 1)^{n - \dim \tau} = E_0(z)$$

where $E^\Sigma_0'(z)$ (resp. $E^\Sigma'_{0,\rho}(z)$) denotes the Hodge-Deligne polynomial of $X_{\Sigma'}$ (resp. $X_{\Sigma'_\rho}$). In order to get the second equality, we have used two facts: first, the orbit closure $V(\rho) = \cup_{\rho \subseteq \tau} O(\tau)$ of the orbit $O(\rho)$ and the toric variety $X_{\Sigma'/\rho}$ are isomorphic, see for instance [4, Proposition 3.2.7]; second, the Hodge-Deligne polynomial of $\mathbb{C}^*$ is $z - 1$ and $O(\tau) \cong (\mathbb{C}^*)^{n - \dim \tau}$, see [4, Section 3.2]. This shows the first point.
By Poincaré duality for the complete and simplicial toric varieties $X_{\Sigma'}$ and $X_{\Sigma''}$, we have
\[ z^n E_0(\Sigma')(z^{-1}) = E_0(\Sigma')(z) \quad \text{and} \quad z^{n-1} E_0(\Sigma''(z^{-1}) = E_0(\Sigma''(z) \]
Using (11), we get
\[ z^n E_0(z^{-1}) - E_0(z) = (1 - z)E_0(\Sigma')(z) \]
thus
\[ z^n E_0(z^{-1}) = E_0(z) - (z - 1)E_0(\Sigma')(z) = E_0(\Sigma'') \]
This gives the second point.  

**Corollary 5.5** Assume that $\Sigma$ is simplicial. Then
\[ E_0^*(z) = \sum_{\nu \in \mathbb{N}^n} \dim H^{2i}(X_{\Sigma'}, \mathbb{Q}) z^i. \]

**Proof.** Follows from Proposition 5.4 and Poincaré duality for the simplicial toric variety $X_{\Sigma}$.  

### 5.2 Toric Newton spectrum and cohomology

We now express the toric Newton spectrum of $f$ in terms of Hilbert-Poincaré series of toric varieties. We will denote by $b_1, \ldots, b_r$ the vertices of $P$ different from the origin. Let us define, for $\Delta$ a closed face of $P$ whose vertices are $b_{i_1}, \ldots, b_{i_k}$,

- $\square(\Delta) := \{ \sum_{\ell=1}^k q_{\ell} b_{i_{\ell}} \mid q_{\ell} \in [0,1], \ell = 1, \ldots, k \}$,
- $\square(P) = \cup_{\Delta \in F(P)} \square(\Delta)$

(recall that $F(P)$ denotes the set of the closed faces of $P$ not contained in the union of the hyperplane coordinates).

**Theorem 5.6** Assume that $\Sigma$ is simplicial. Then
\[ P_{\text{gr}N_{\Sigma}(\mathbb{Q})}(z) = \sum_{v \in \square(P) \cap \mathbb{N}^n} E_{\nu}(z) z^{\nu(v)}. \]

**Proof.** This is Proposition 3.3.  

For $\sigma$ a cone in the fan $\Sigma$, let $\Sigma/\sigma$ be the quotient fan $\Sigma/\sigma := \{ \overline{\tau} = \tau + (L_{\sigma})_{\mathbb{R}}, \sigma \subseteq \tau, \tau \in \Sigma \}$ in $L(\sigma)_{\mathbb{R}}$, where $L_{\sigma}$ is the sublattice of $L := \mathbb{Z}^n$ generated by the points of $\sigma \cap L$ and $L(\sigma) := L/L_{\sigma}$. For $v \in \mathbb{N}^n$, let $\sigma(v)$ be the smallest cone of $\Sigma$ containing $v$ and let $X_{\Sigma/\sigma(v)}$ be the toric variety associated with the quotient fan $\Sigma/\sigma(v)$. Last, for $\alpha \in \mathbb{Q}$, we define
\[ H^{2\alpha}_{K,\text{tor}}(\Sigma, \mathbb{Q}) := \oplus_{v \in \square(P) \cap \mathbb{N}^n} H^{2(\alpha - \nu(v))}(X_{\Sigma/\sigma(v)}, \mathbb{Q}). \]

We get:

**Corollary 5.7** Let $f$ be a convenient and nondegenerate polynomial on $\mathbb{C}^n$. Assume that $\Sigma$ is simplicial. Then
\[ P_{\text{gr}N_{\Sigma}(\mathbb{Q})}(z) = \sum_{\alpha \in \mathbb{Q}} \dim Q H^{2\alpha}_{K,\text{tor}}(\Sigma, \mathbb{Q}) z^\alpha. \]

**Proof.** Follows from Theorem 5.6 and Proposition 5.4.
Remark 5.8 Corollary 5.7 explains integral shifts in the toric Newton spectrum: if \( v \in \Box(P) \cap N \) and \( v \) doesn’t belong to the hyperplane coordinates, the sequence \( \nu(v), \nu(v)+1, \cdots, \nu(v)+n-\dim \sigma(v) \) is a part of the toric Newton spectrum.

If \( X_\Sigma \) is simplicial, one associates to the triple (a stacky fan) \( \Sigma := (\mathbb{Z}^n, \Sigma, \{ b_i \}) \) a Deligne-Mumford stack \( X \) whose coarse moduli space is \( X_\Sigma \), see [2]. We get a geometric interpretation of the toric Newton spectrum in terms of the orbifold cohomology of \( X \) (see for instance [1] Definition 4.8):

Corollary 5.9 Let \( f \) be a convenient and nondegenerate polynomial on \( \mathbb{C}^n \). Assume that \( X_\Sigma \) is simplicial. Then

\[
P_{\text{gr}^\Sigma} (\frac{f}{z}) (z) = \sum_{\alpha \in \mathbb{Q}} \dim_{\mathbb{Q}} H^{2\alpha}_\text{orb}(\mathcal{X}, \mathbb{Q}) z^\alpha.
\]

Proof. By [2] Proposition 4.7, 
\[
H^{2\alpha}_\text{orb}(\mathcal{X}, \mathbb{Q}) = \oplus_{v \in \Box(P) \cap \mathbb{Z}^n} H^2(\alpha - \nu(v))(X_\Sigma/\sigma(v), \mathbb{Q}).
\]

Remark 5.10 By Theorem 5.2 the Hilbert-Poincaré series \( P_{\text{gr}^\Sigma} (\frac{f}{z}) (z) \) is equal to the weighted \( \delta \)-vector denoted by \( \delta^\Sigma(t) \) in [13] and Corollary 5.9 could also be deduced from [13] Theorem 4.3.

Remark 5.11 Using Proposition 4.5 we get an analogous description for the Newton spectrum of a convenient and nondegenerate polynomial \( f \) on \( \mathbb{C}^n \) with Newton polytope \( P \). For a face \( \Delta \) of \( P \) with vertices \( b_{i_1}, \cdots, b_{i_k} \), let \( \text{Box}(\Delta) := \{ \sum_{\ell=1}^k q \ell b_{i_{\ell}}, \ q \in [0,1], \ \ell = 1, \cdots, k \} \) and let \( \text{Box}(P) := \bigcup_{\Delta \in \mathcal{F}(P)} \text{Box}(\Delta) \). For \( \alpha \in \mathbb{Q} \), we define (we denote by \( H \) the union of the hyperplane coordinates)

\[
H^{2\alpha}_K(\Sigma, \mathbb{Q}) := \oplus_{v \in \text{Box}(P) \cap \mathbb{Z}^n} H^2(\alpha - \nu(v))(X_\Sigma/\sigma(v), \mathbb{Q}) \bigoplus \oplus_{v \in P \cap \mathbb{Z}^n} H^2(\alpha - \nu(v))(X_\Sigma/\sigma(v), \mathbb{Q})
\]

where \( H^2_\Sigma(X_\Sigma/\sigma(v), \mathbb{Q}) = H^2(\Sigma, \mathbb{Q}) \) if \( \beta = 0 \) and \( H^0(\Sigma, \mathbb{Q}) = 0 \). Then \( P_B^\Sigma(z) = \sum_{\alpha \in \mathbb{Q}} \dim_{\mathbb{Q}} H^{2\alpha}_K(\Sigma, \mathbb{Q}) z^\alpha \) if \( P \) is simplicial.

5.3 Toric Newton spectrum and mirror symmetry

Let \( f \) be a convenient and nondegenerate polynomial defined on \( \mathbb{C}^n \) with Newton polytope \( P \) and let \( X \) be the Deligne-Mumford stack associated with the stacky fan \( \Sigma := (\mathbb{Z}^n, \Sigma, \{ b_i \}) \) of \( P \) (we assume here that \( X_\Sigma \) is simplicial) as in Section 5.2. Then (and this can be justified by Corollary 5.9), \( f \) and \( X \) are said to be mirror partners if there is an isomorphism of \( \mathbb{Q} \)-graded rings

\[
H^{2\alpha}_\text{orb}(\mathcal{X}, \mathbb{Q}) \rightarrow \text{Gr}(\frac{B}{L}),
\]

(12)

Because the map (5) is strict with respect to the Newton filtration, the graded ring \( \text{Gr}(\frac{B}{L}) \) looks like the "Stanley-Reisner presentation" of \( X \) given by the right hand side of [2] Theorem 1.1 (but we can’t use directly this result because \( X_\Sigma \) is not complete). The product on \( \text{Gr}(\frac{B}{L}) \) is in principle easy to compute and should give, with the help of such an isomorphism, a concrete description of the orbifold product on \( H^{2\alpha}_\text{orb}(\mathcal{X}, \mathbb{Q}) \). On the other hand, Poincaré duality on \( H^{2\alpha}_\text{orb}(\mathcal{X}, \mathbb{Q}) \) (see for instance [1] Proposition 4.11) should be helpful in order to understand the pairing on \( \text{Gr}(\frac{B}{L}) \) and possible symmetries of the toric Newton spectrum. Notice that if \( f \) is a convenient and
nondegenerate \textit{Laurent} polynomial then \(X_{\Sigma}\) is complete and the map \([12]\) is an isomorphism of graded rings by \([2, \text{Theorem 1.1}]\); in this case, \(f\) and \(\mathcal{X}\) are mirror partners (see also \([7]\)).

In the same way, a product \(\ast\) on \(B^* := \text{gr}^W \mathcal{B}\) is defined by

\[
g \ast h := \lambda^{-1}(\lambda(g) \bullet \lambda(h))
\]

for \(g, h \in B^*\), where \(\lambda : \text{gr}^W \mathcal{B} \rightarrow \text{gr}^N \mathcal{B}_{u_1 \cdots u_n}\) is the isomorphism of Lemma \([4, 3]\). Equipped with the (commutative and associative, but without identity) product \(\ast\), \(B^*\) becomes a graded ring and this would give an "orbifold cup-product" on the vector space \(H^2_{orb}(\Sigma, \mathbb{Q})\) defined in Remark \([5.11]\).

\textbf{Example 5.12} Let us return to the example \(f(u, v) = u^2 + 2u^2v^2 + v^2\). A basis of \(\text{gr}^N \mathcal{B}_{\mathcal{L}(f)}\) is given by \(1, uv, u^2v^2, u^3v^3, u, v, u^2v, \) \(uv^2\), with respective grading \(0, \frac{1}{2}, 1, \frac{3}{2}, \frac{5}{2}, 1, 1, 1\), and we get the following table for the product \(\ast\) in \(\text{gr}^N \mathcal{B}_{\mathcal{L}(f)}\):

\[
\begin{array}{cccccccc}
& 1 & uv & u^2v^2 & u^3v^3 & u & v & u^2v & uv^2 \\
1 & 1 & uv & u^2v^2 & u^3v^3 & u & v & 0 & 0 \\
v & uv & uv & 0 & 0 & 0 & 0 & 0 & 0 \\
u & uv & u^2v & u^3v^3 & 0 & 0 & 0 & 0 & 0 \\
u & uv & u^2v & u^3v & 0 & 0 & 0 & 0 & 0 \\
u & uv & u^2v & u & 0 & 0 & 0 & 0 & 0 \\
u & uv & uv & 0 & 0 & -u^2v^2 & 0 & -u^3v^3 & 0 \\
u & uv & u^2v & 0 & 0 & 0 & -u^2v^2 & 0 & -u^3v^3 \\
u & uv & uv & 0 & 0 & 0 & 0 & 0 & 0 \\
u & uv & u & 0 & 0 & 0 & 0 & 0 & 0 \\
u & uv & uv & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

On the cohomology side, we define an isomorphism

\[
H^2_{orb}(\mathcal{X}, \mathbb{Q}) \longrightarrow \text{Gr}(\mathcal{B}_{\mathcal{L}})
\]

by the assignments

\[
1_{v_0} \mapsto 1, 1_{v_1} \mapsto u, 1_{v_2} \mapsto u^2v, 1_{v_3} \mapsto uv, 1_{v_4} \mapsto uv^2, 1_{v_5} \mapsto v, 1_{v_0} p \mapsto u^2v^2, 1_{v_3} p \mapsto u^3v^3.
\]

A basis of the orbifold cohomology is \(1_{v_0}, 1_{v_0} p, 1_{v_1}, 1_{v_1} p, 1_{v_2}, 1_{v_3}, 1_{v_3} p, 1_{v_4}, 1_{v_5}\), with respective grading \(0, 1, \frac{1}{2}, 1, \frac{3}{2}, 1, \frac{5}{2}, \frac{7}{2}\) and we get the following table for an "orbifold" cup-product in \(H^2_{orb}(\mathcal{X}, \mathbb{Q})\):

\[
\begin{array}{cccccccc}
\cup_{\text{orb}} & 1_{v_0} & 1_{v_1} & 1_{v_2} & 1_{v_3} & 1_{v_4} & 1_{v_5} & 1_{v_0} p & 1_{v_3} p \\
1_{v_0} & 1_{v_0} & 1_{v_1} & 1_{v_2} & 1_{v_3} & 1_{v_4} & 1_{v_5} & 1_{v_0} p & 1_{v_3} p \\
1_{v_1} & 1_{v_1} & 1_{v_2} & 1_{v_3} & 1_{v_4} & 1_{v_5} & 1_{v_0} p & 1_{v_3} p & 1_{v_3} p \\
1_{v_2} & 1_{v_2} & 1_{v_3} & 1_{v_4} & 1_{v_5} & 1_{v_0} p & 1_{v_3} p & 1_{v_3} p & 1_{v_3} p \\
1_{v_3} & 1_{v_3} & 1_{v_4} & 1_{v_5} & 1_{v_0} p & 1_{v_3} p & 1_{v_3} p & 1_{v_3} p & 1_{v_3} p \\
1_{v_4} & 1_{v_4} & 1_{v_5} & 1_{v_0} p & 1_{v_3} p & 1_{v_3} p & 1_{v_3} p & 1_{v_3} p & 1_{v_3} p \\
1_{v_5} & 1_{v_5} & 1_{v_0} p & 1_{v_3} p & 1_{v_3} p & 1_{v_3} p & 1_{v_3} p & 1_{v_3} p & 1_{v_3} p \\
1_{v_0} p & 1_{v_0} p & 1_{v_3} p & 1_{v_3} p & 1_{v_3} p & 1_{v_3} p & 1_{v_3} p & 1_{v_3} p & 1_{v_3} p \\
1_{v_3} p & 1_{v_3} p & 1_{v_3} p & 1_{v_3} p & 1_{v_3} p & 1_{v_3} p & 1_{v_3} p & 1_{v_3} p & 1_{v_3} p \\
1_{v_3} p & 1_{v_3} p & 1_{v_3} p & 1_{v_3} p & 1_{v_3} p & 1_{v_3} p & 1_{v_3} p & 1_{v_3} p & 1_{v_3} p
\end{array}
\]
5.4 A small digression: the local case

Analogous results hold in the local case: let now $f$ be a power series with an isolated critical point at the origin, and let $\mu_0 := \dim \mathbb{C} \{u_1, \ldots, u_n\}/(\partial_{u_1} f, \ldots, \partial_{u_n} f)$ be its Milnor number at the origin. Let $\Gamma(f)$ be the convex hull of $\cup_{a \in \supp(f) - \{0\}} (a + \mathbb{R}_+^n)$ in $\mathbb{R}_+^n$, let $\Gamma(f)$ be the union of the compact faces of $\Gamma(f)$ and let $P$ be the union of all the segments starting from the origin and ending on $\Gamma(f)$: this is the Newton polyhedron of $f$.

A Newton function $\nu_0$ is defined as in Section 2 putting $\nu_0(a) = \min F < u_F, a >$ for $a \in \mathbb{N}^n$, see [11 Section 2.1]. It now satisfies $\nu_0(a+b) \geq \nu_0(a) + \nu_0(b)$ for $a, b \in \mathbb{N}^n$, with equality if $a$ and $b$ belong to the same cone. This provides a decreasing Newton filtration $\mathcal{N}_0$ on $\mathcal{A} := \mathbb{C}\{u_1, \ldots, u_n\}$, a local toric Newton spectrum $P_{\mathcal{N}_0}(\nu)(z)$ (where $\mathcal{L}$ still denotes the ideal generated by the partial derivative $u_1 \partial_{u_1}, \ldots, u_n \partial_{u_n}$ of $f$) and a local Newton spectrum. The previous results still hold with minor modifications. In particular, let $\Sigma$ be the fan built over the faces of the Newton boundary of $P$ (the union of the closed faces of $P$ that do not contain the origin), and let $X_{\Sigma}$ be the toric variety associated with the fan $\Sigma$. Assume that $X_{\Sigma}$ is simplicial and let $\mathcal{X}$ be the Deligne-Mumford stack associated with the stacky fan $\Sigma = (\mathbb{Z}^n, \Sigma, \{b_i\})$ where the $b_i$'s are the vertices of $P$ different from the origin (see [2]).

**Theorem 5.13** Let $f \in \mathcal{A}$ be a convenient and nondegenerate power series. Assume that the fan $\Sigma$ of its Newton polyhedron is simplicial. Then $P_{\mathcal{N}_0}(\nu)(z) = \sum_{\alpha \in \mathbb{Q}} \dim \mathbb{Q} H_{\mathcal{O}_F}^2(\mathcal{X}, \mathbb{Q}) z^\alpha$. □

6 Toric Newton spectrum and Ehrhart polynomials

By Theorem 3.2, the toric Newton spectrum is equal to the weighted $\delta$-vector defined in [15]: in particular, it counts weighted lattice points in its Newton polytope. Theorem 4.2 and Proposition 4.5 of [6] still hold, with the same proofs, for the toric Newton spectrum of the Newton polytope of a convenient and nondegenerate polynomials. This gives a recipe in order to calculate Ehrhart polynomials of Newton polytopes of polynomials from their toric Newton spectrum, see [6] Section 4.2 and [6] Section 4.3. For instance, let us consider the simplest example possible, namely the polynomial $f(u_1, \ldots, u_n) = u_1 + \cdots + u_n$ on $\mathbb{C}^n$. Its Newton polytope is

$$P := \text{conv}((0, \ldots, 0), (1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), (0, \cdots, 0, 1)),$$

the standard simplex in $\mathbb{R}^n$. The toric Newton spectrum of $f$ is 1 and it follows from [6] Proposition 4.5 that the $\delta$-vector of $P$ is given by $\delta_0 = 1$, $\delta_2 = \cdots = \delta_n = 0$: its Ehrhart polynomial is $L_P(z) = \binom{z + n}{n}$, see loc. cit. for details and more involved examples.

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