Parametric instabilities in a two ion species plasma as a driver of super Alfvénic waves

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Abstract. Linear dispersion relation of parametric instabilities of a left hand circularly polarized wave in a three species plasma consisting of electrons, protons, and α particles is derived and the obtained dispersion relation is numerically solved. It is shown that the so-called super Alfvénic waves, which are thought to play important roles in preferential acceleration of heavy ions, can get excited through a parametric decay instability when the parent wave is on the proton cyclotron branch and the plasma beta is sufficiently low. A variety of other parametric instabilities are also generated.

1. Introduction

A preferential acceleration of heavy ions is often observed in many space and astrophysical circumstances [1, 2, 3]. While the mechanism of heavy ion acceleration is still under debate, [4] proposed a mechanism that some α particles are preferentially accelerated through the interaction simultaneously with two left hand circularly polarized waves in a plasma consisting of electrons, protons, and α particles. According to them, an efficient acceleration of α particles occurs when one of the two waves is a high phase velocity super Alfvénic (SPA) wave, which is on the high frequency proton cyclotron branch of left hand circularly polarized waves in the three species plasma. Although they assume the presence of an SPA wave, the generation mechanism of the SPA wave was not proposed in [4]. Recently, [5] showed that the acceleration process proposed by [4] occurs in a self-consistent particle-in-cell (PIC) simulation. In the simulation a proton temperature anisotropy is given as an initial condition, while other species, electrons and α particles, are isotropic and all the species are homogeneously distributed in space. [5] explained that the SPA waves are nonlinearly generated through the parametric decay instability of large amplitude proton cyclotron waves which are generated due to the so-called proton EMIC (electromagnetic ion cyclotron) instability.

In this paper we discuss the linear dispersion relation of parametric instabilities in a three species plasma to show that the proton cyclotron waves generated by an EMIC
instability can be the source of generation of the SPA waves. The linear analysis of parametric instabilities of a circularly polarized wave in a three species plasma were studied by a number of authors, e.g., [6, 8, 7], although they investigated the case when the relative drift between protons and α particles is present. We discuss here a more general situation that there is no relative drift among the species.

The paper is organized as follows. In section 2 the dispersion relation is derived. Its numerical solutions are shown in section 3. Then, summary is given in section 4.

2. Linear Analysis

2.1. Formulations and Basic Assumptions

We consider a plasma, composed of electrons, protons, and alpha particles, governed by the following equations.

\[
\left( \frac{\partial}{\partial t} + v_j \cdot \nabla \right) v_j = \frac{q_j}{m_j} \left( E + \frac{v_j \times B}{c} \right) - \frac{v_{ij}^2}{n_{j0}} \nabla n_j
\]

(1)

\[
\frac{\partial n_j}{\partial t} = -\nabla \cdot (n_j v_j)
\]

(2)

\[
\nabla \cdot E = 4\pi \rho
\]

(3)

\[
\nabla \times E = -\frac{1}{c} \frac{\partial B}{\partial t}
\]

(4)

\[
\nabla \times B = \frac{4\pi c}{c} J + \frac{1}{c} \frac{\partial E}{\partial t}
\]

(5)

\[
J = \sum_j q_j n_j v_j
\]

(6)

\[
\rho = \sum_j q_j n_j
\]

(7)

Here, \(v_j = (u_j, v_\perp j)\) is the velocity, \(v_{ij} = (T_j/m_j)^{1/2}\) the thermal speed, \(n_j\) the density, \(q_j\) the particle charge, \(m_j\) the particle mass, \(c\) the speed of light, and \(E\) and \(B\) are the electric and magnetic fields, respectively. The subscript \(j\) denotes particle species.

It is assumed that a circularly polarized parent wave propagates parallel to the constant magnetic field, \(B_0\), which is along the x-axis. Allowing both electrostatic and electromagnetic perturbations, we write,

\[
n_j = n_{j0} + \delta n_j,
\]

(8)

\[
u_j = \delta u_j,
\]

(9)

\[
B = B_0 + B_p + \delta B,
\]

(10)

\[
v_\perp j = v_{pj} + \delta v_j,
\]

(11)

where the subscript 0 represents the zeroth order constant quantities, the subscript \(p\) denotes the zeroth order parent wave, and \(\delta\) represents a small perturbation, respectively.
These parent and perturbed quantities, assumed to propagate parallel or anti-parallel to \( B_0 \), are further expressed as follows,

\[
B_p = \frac{B_p}{\sqrt{2}} \exp(i\phi_0) \hat{e} + \text{c.c.,} \tag{12}
\]

\[
\delta B = \frac{1}{\sqrt{2}} (\delta B_+ \exp(i\phi_+) + \delta B_- \exp(i\phi_-)) \hat{e} + \text{c.c.,} \tag{13}
\]

\[
\delta n_j = \delta n_j \exp(i\phi) + \text{c.c.,} \tag{14}
\]

where \( B_p \) is real, \( \phi_0 = k_0 x - \omega_0 t, \phi_\pm = k_\pm x - \omega_\pm t, (k_\pm, \omega_\pm) = (k_0 \pm k, \omega_0 \pm \omega), (k_0, \omega_0) \) and \((k, \omega)\) are the wavenumber and frequency of a parent wave and a density perturbation, \( \hat{e} = (\hat{y} - i\hat{z})/\sqrt{2} \) (\( \hat{y} \) and \( \hat{z} \) are the unit vectors along respective directions), and c.c. denotes complex conjugate.

The parallel and perpendicular velocities are also written in a same manner. It is convenient to note the following relations.

\[
\hat{e} \cdot \hat{e} = 0, \quad \hat{e} \cdot \hat{e}^* = 1
\]

\[
\hat{e} \times \hat{e} = \hat{e}^* \times \hat{e}^* = 0
\]

\[
\hat{e} \times \hat{e}^* = -\hat{e}^* \times \hat{e} = i\hat{x}
\]

\[
\hat{x} \times \hat{e} = i\hat{e}
\]

For the zeroth order, the transverse velocity of each species is obtained as

\[
\frac{v_{pj}}{c} = \frac{\omega_0}{c k_0} \frac{\Omega_j \eta}{\omega_0 - \Omega_j}, \tag{15}
\]

where \( \eta = B_p/B_0 \) and \( \Omega_j = q_j B_0/m_j c \) is the cyclotron frequency of species \( j \). The frequency of the parent wave, \( \omega_0 \), is given by the zeroth order dispersion relation,

\[
\frac{c^2 k_0^2}{\omega_0^2} = 1 - \sum_j \frac{\omega_{pj}^2}{\omega_0(\omega_0 - \Omega_j)}, \tag{16}
\]

where \( \omega_p \) is the plasma frequency.

Let us assume that the electron mass is negligible \((m_e = 0)\) and the two ion species are cold \((T_p = T_\alpha = 0)\). Furthermore, the speed of light is assumed to be much faster than phase velocities of any type of waves considered here. The following linear eigenvalue equations are then obtained from eqs.(1) - (14).

\[
\delta v_{e+} - \frac{\eta}{2} \delta u_e = -\frac{\omega_+}{k_+} \frac{\delta B_+}{B_0}
\]

\[
\delta v_{e-} - \frac{\eta}{2} \delta u_e = -\frac{\omega_-}{k_-} \frac{\delta B_-}{B_0}
\]

\[
(\omega_+ - \Omega_p) \delta v_{p+} - \frac{1}{2} (k_0 v_p - \Omega_p \eta) \delta u_p = \frac{\omega_+ e \delta B_+}{k_+ m_p c}
\]

\[
(\omega_- - \Omega_p) \delta v_{p-} - \frac{1}{2} (k_0 v_p - \Omega_p \eta) \delta u_p = \frac{\omega_- e \delta B_-}{k_+ m_p c}
\]
\[
(\omega - \Omega_p)\delta v^*_{p_--} - \frac{1}{2} (k_0 v'_p - \Omega_p \eta) \delta u_p = \frac{\omega e \delta B^*}{k_- m_p c} \tag{20}
\]
\[
(\omega - \Omega_\alpha)\delta v^*_{\alpha+} - \frac{1}{2} (k_0 v'_\alpha - \Omega_\alpha \eta) \delta u_\alpha = \frac{\omega_+ e_\alpha \delta B_+}{k_+ m_\alpha c} \tag{21}
\]
\[
(\omega - \Omega_\alpha)\delta v^*_{\alpha-} - \frac{1}{2} (k_0 v'_\alpha - \Omega_\alpha \eta) \delta u_\alpha = \frac{\omega_- e_\alpha \delta B^*_+}{k_- m_\alpha c} \tag{22}
\]
\[
k^2 T_e \delta u_e = 4\pi e \sum_j q_j n_j 0 \delta u_j + \frac{e}{c} v_e (\delta B^*_+ - \delta B^*_-) - \omega \frac{e}{c} B_p (\delta v^*_e + \delta v^*_e) \tag{23}
\]
\[
\omega^2 \delta u_p = \frac{e}{m_p} \Sigma_j 4\pi q_j n_j 0 \delta u_j + \omega v_p \frac{e}{m_p c} (\delta B^*_+ - \delta B^*_-) - \omega \Omega_p \eta (\delta v^*_p - \delta v^*_p) \tag{24}
\]
\[
\omega^2 \delta u_\alpha = \frac{e_\alpha}{m_\alpha} \Sigma_j 4\pi q_j n_j 0 \delta u_j + \omega v_\alpha \frac{e_\alpha}{m_\alpha c} (\delta B^*_+ - \delta B^*_-) - \omega \Omega_\alpha \eta (\delta v^*_\alpha - \delta v^*_\alpha) \tag{25}
\]
\[
\delta B_+ = - \sum_j 4\pi q_j n_j 0 \frac{k v_j}{k_+ c} \left( \delta v^*_j + \frac{k v_j}{2 \omega} \delta u_j \right) \tag{26}
\]
\[
\delta B^*_+ = - \sum_j 4\pi q_j n_j 0 \frac{k v_j}{k_- c} \left( \delta v^*_j + \frac{k v_j}{2 \omega} \delta u_j \right) \tag{27}
\]

Here, the asterisk (*) implies complex conjugate. Substituting eqs.(17) and (18) into eq.(23) yields
\[
\left( 1 + k^2 \lambda_{De}^2 \right) \delta u_e - \frac{n_{p0}}{n_{e0}} \delta u_p - \frac{2 n_{e0}}{n_{e0}} \delta u_\alpha = \frac{\omega \Omega_p}{\omega_p^2} \left[ \left( v_e + \frac{\omega_+}{k_+} \eta \right) \frac{\delta B^*_+}{B_0} - \left( v_e + \frac{\omega_-}{k_-} \eta \right) \frac{\delta B^*_+}{B_0} \right] \tag{28}
\]

where \( \lambda_{De} = T_e / 4 \pi n_{e0} e^2 \) is the Debye length. Similarly,
\[
\omega_p^2 \delta u_e + \left( \omega^2 - \omega_{pp}^2 - \frac{\Omega_p \eta \omega^2 (k_0 v'_p - \Omega_p \eta)}{(\omega_+ - \Omega_p)(\omega_- - \Omega_p)} \right) \delta u_p = -2 \omega_{p0} \delta u_\alpha \tag{29}
\]

is obtained from eqs.(19), (20), and (24), and eqs.(21), (22), and (25) lead to
\[
\frac{1}{2} \omega_p^2 \delta u_e - \frac{1}{2} \omega_{pp}^2 \delta u_p + \left( \omega^2 - \omega_{ps}^2 - \frac{\Omega_\alpha \eta \omega^2 (k_0 v'_\alpha - \Omega_\alpha)}{(\omega_+ - \Omega_\alpha)(\omega_- - \Omega_\alpha)} \right) \delta u_\alpha \tag{30}
\]

Eqs.(26) and (27) read
\[
\left[ 1 + \frac{\omega_+}{k_+ c^2} \left( \frac{\omega_p^2}{\Omega_p^2} + \frac{\omega_{pp}^2}{\omega_+ - \Omega_p} + \frac{\omega_{ps}^2}{\omega_+ - \Omega_p} \right) \right] \delta B_+ = \frac{4 \pi e_{n0}}{k_- c} \left[ \frac{\eta + k v_e}{\omega} \right] \delta u_e - \frac{e n_{p0}}{2} \left( \frac{k_0 v'_p - \Omega_p \eta + k v'_p}{\omega_+ - \Omega_p} \right) \delta u_p - \frac{e n_{p0}}{2} \left( \frac{k_0 v'_\alpha - \Omega_\alpha \eta + k v'_\alpha}{\omega_+ - \Omega_\alpha} \right) \delta u_\alpha \tag{31}
\]
Substituting these two equations into eqs. (28), (29), and (30) results in

\[
\left\{ 1 + \frac{\omega^2_p}{k_c^2} \left( \frac{\omega^2_p}{\Omega_p} + \frac{\omega^2_{pp}}{\omega_+ - \Omega_p} + \frac{\omega^2_{pa}}{\omega_- - \Omega_a} \right) \right\} \delta B^* = \frac{4\pi}{k_c} \left[ \frac{en_{p0}}{2} \left( \eta + \frac{k v_e}{\Omega} \right) \delta u_e - \frac{1}{2} \left( \frac{k_0v_p - \Omega_p \eta}{\omega_+ - \Omega_p} + \frac{k v_p}{\omega} \right) \right] \]

Substituting these two equations into eqs. (28), (29), and (30) results in

\[
\left\{ 1 + \frac{\omega^2_p}{k_c^2} \left( \frac{\omega^2_p}{\Omega_p} + \frac{\omega^2_{pp}}{\omega_+ - \Omega_p} + \frac{\omega^2_{pa}}{\omega_- - \Omega_a} \right) \right\} \delta u_e - \frac{n_{p0}}{n_{e0}} \left\{ 1 - \omega \left[ \frac{1}{\Omega_p} \left( \frac{v_e}{c} + \omega_+ \eta \right) \left( \frac{k_0v_p - \Omega_p \eta}{\omega_+ - \Omega_p} + \frac{k v_p}{\omega} \right) \right] \right\} \delta u_p \\
- \frac{2n_{e0}}{n_{p0}} \left\{ 1 - \omega \left[ \frac{1}{\Omega_p} \left( \frac{v_e}{c} + \omega_+ \eta \right) \left( \frac{k_0v_p - \Omega_p \eta}{\omega_+ - \Omega_p} + \frac{k v_p}{\omega} \right) \right] \right\} \delta u_a = 0 \tag{31}
\]

\[
\left\{ \frac{\omega^2_p}{2} \left( \eta + \frac{k v_e}{\omega} \right) \right\} \left\{ \frac{\omega^2_p}{\Omega_p} \left( v_p - \frac{\omega_+ \Omega_p \eta}{k_+ \omega_+ - \Omega_p} \right) \right\} \delta u_e + \left\{ \omega^2 - \omega^2_{pp} \right\} \left\{ \frac{\omega^2_p}{\Omega_p} \left( v_p - \frac{\omega_+ \Omega_p \eta}{k_+ \omega_+ - \Omega_p} \right) \right\} \delta u_p \\
+ \frac{\omega^2_{pp}}{2} \left( v_p - \frac{\omega_+ \Omega_p \eta}{k_+ \omega_+ - \Omega_p} \right) \left( \frac{k_0v_p - \Omega_p \eta}{\omega_+ - \Omega_p} + \frac{k v_p}{\omega} \right) \\
+ \frac{\omega^2_{pa}}{2} \left( v_p - \frac{\omega_- \Omega_p \eta}{k_- \omega_- - \Omega_p} \right) \left( \frac{k_0v_p - \Omega_p \eta}{\omega_- - \Omega_p} + \frac{k v_p}{\omega} \right) \right\} \delta u_a = 0 \tag{32}
\]
\[
\left\{ \frac{\omega_{pi}^2}{2} \frac{\omega}{2} \left( \eta + \frac{k v_e}{\omega} \right) \left[ \frac{\omega_{pi}^2/2}{T_+ k_+ c^2} \left( v_\alpha - \frac{\omega_+ - \Omega_\alpha}{k_+ \omega_+ - \Omega_\alpha} \right) \right] \right.
- \frac{\omega_{pp}^2/2}{T_+ k_+ c^2} \left( v_\alpha - \frac{\omega_+ - \Omega_\alpha}{k_+ \omega_+ - \Omega_\alpha} \right) \delta u_e \\
+ \left\{ \frac{\omega_{pp}^2}{2} \frac{\omega}{2} \left( v_\alpha - \frac{\omega_+ - \Omega_\alpha}{w_+ - \Omega_\alpha} \right) \left( \frac{k_0 v_p - \Omega_\alpha}{\omega_+ - \Omega_\alpha} \right) \right. \\
+ \frac{\omega_{pa}^2}{2} \omega \left( v_\alpha - \frac{\omega_+ - \Omega_\alpha}{w_+ - \Omega_\alpha} \right) \left( \frac{k_0 v_\alpha - \Omega_\alpha}{\omega_+ - \Omega_\alpha} \right) \right\} \delta u_p \\
\left. + \left\{ \frac{\omega_{pi}^2}{2} \frac{\omega}{2} \left( \eta + \frac{k v_e}{\omega} \right) \left[ \frac{\omega_{pi}^2/2}{T_+ k_+ c^2} \left( v_\alpha - \frac{\omega_+ - \Omega_\alpha}{k_+ \omega_+ - \Omega_\alpha} \right) \right] \right. \right. \\
- \frac{\omega_{pp}^2/2}{T_+ k_+ c^2} \left( v_\alpha - \frac{\omega_+ - \Omega_\alpha}{k_+ \omega_+ - \Omega_\alpha} \right) \left( \frac{k_0 v_p - \Omega_\alpha}{\omega_+ - \Omega_\alpha} \right) \right\} \delta u_\alpha = 0 
\] (33)

The above three equations are reduced with a matrix form as follows.

\[
\begin{pmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{pmatrix} \begin{pmatrix} \delta u_e \\ \delta u_p \\ \delta u_\alpha \end{pmatrix} = 0 
\] (34)

Here,

\[
D_{11} = S_e - \frac{U_e V_{e+}}{T_+} + \frac{U_e V_{e-}}{T_-}, \\
D_{12} = -\frac{n_{p0}}{n_{e0}} \left( 1 - \frac{U_{p+} V_{e+}}{T_+} + \frac{U_p - V_{e-}}{T_-} \right), \\
D_{13} = -\frac{2n_{\alpha0}}{n_{e0}} \left( 1 - \frac{U_{\alpha+} V_{e+}}{T_+} + \frac{U_\alpha - V_{e-}}{T_-} \right), \\
D_{21} = \frac{\omega_{pi}^2}{2} \left( 1 - \frac{U_e V_{p+}}{T_+} + \frac{U_e V_{p-}}{T_-} \right), \\
D_{22} = S_p - W_p + \omega_{pp}^2 \frac{U_{p+} V_{p+}}{T_+} - \frac{\omega_{pp}^2}{2} \frac{U_p - V_{p-}}{T_-}, \\
D_{23} = -2 \omega_{pa}^2 \left( 1 - \frac{U_{\alpha+} V_{p+}}{T_+} + \frac{U_{\alpha-} V_{p-}}{T_-} \right), \\
D_{31} = \frac{\omega_{pi}^2}{2} \left( 1 - \frac{U_e V_{\alpha+}}{T_+} + \frac{U_e V_{\alpha-}}{T_-} \right), \\
D_{32} = -\frac{\omega_{pp}^2}{2} \left( 1 - \frac{U_{p+} V_{\alpha+}}{T_+} + \frac{U_{p-} V_{\alpha-}}{T_-} \right), \\
D_{33} = S_\alpha - W_\alpha + \omega_{pa}^2 \frac{U_{\alpha+} V_{\alpha+}}{T_+} - \frac{\omega_{pa}^2}{2} \frac{U_{\alpha-} V_{\alpha-}}{T_-},
\]

where \( \omega_{pi}^2 = 4 \pi n_{e0} e^2 / m_p \),

\[
T_+ = 1 + \frac{\omega_+}{k_+ c^2} \left( \frac{\omega_{pi}^2}{\Omega_p} + \frac{\omega_{pp}^2}{\omega_+ - \Omega_p} + \frac{\omega_{pa}^2}{\omega_+ - \Omega_\alpha} \right),
\] (35)
The dispersion relation is obtained as a zero determinant of the coefficient matrix in eq. (34). Let us assume here that the amplitude of a parent wave is rather small so that $\eta$ and $v_j$ are small parameters. Then, neglecting $O(\eta^4, v_j^4)$ terms results in

\[
T_+ T_- \omega^2 [S_e \omega^2 - k^2 \lambda_D^2 (\omega_{pp}^2 + \omega_{pa}^2)] \\
+ (S_e S_p + \omega_{pp}^2) (-T_+ T_- W_\alpha + \omega_{pa}^2 T_+ U_\alpha + V_\alpha - \omega_{pa}^2 T_+ U_\alpha - V_\alpha) \\
+ (S_e S_\alpha - \omega_{pp}^2) (-T_+ T_- W_p + \omega_{pp}^2 T_+ U_p + V_p - \omega_{pp}^2 T_+ U_p - V_p) \\
+ (S_p S_\alpha - \omega_{pp}^2) U_e (-T_- V_\alpha + T_+ V_e) \\
+ \omega_{pa}^2 (T_+ U_\alpha V_\alpha + T_+ U_\alpha V_e - T_+ U_\alpha V_\alpha + T_+ U_\alpha V_e) \\
+ \omega^2 (T_+ U_\alpha V_\alpha + T_+ U_\alpha V_e - T_+ U_\alpha V_\alpha + T_+ U_\alpha V_e) \\
- k^2 \lambda_D^2 \omega_{pp}^2 \omega_{pa}^2 (-T_- U_\alpha V_\alpha + T_+ U_\alpha V_\alpha - T_- U_\alpha V_\alpha + T_+ U_\alpha V_\alpha) = 0.
\]

Furthermore, when $k^2 \lambda_D^2 << 1$ so that $O(\eta^2 k^2 \lambda_D^2)$ terms can also be neglected,

\[
T_+ T_- [\omega^2 - k^2 \lambda_D^2 (\omega_{pp}^2 + \omega_{pa}^2)] \\
- T_+ T_- W_\alpha + \omega_{pa}^2 T_+ U_\alpha + V_\alpha - \omega_{pa}^2 T_+ U_\alpha - V_\alpha \\
- T_+ T_- W_p + \omega_{pp}^2 T_+ U_p + V_p - \omega_{pp}^2 T_+ U_p - V_p \\
+ (\omega^2 - \omega_{pp}^2) U_e (-T_- V_\alpha + T_+ V_e) \\
+ \omega_{pa}^2 (-T_- U_\alpha V_\alpha + T_+ U_\alpha V_e - T_- U_\alpha V_\alpha + T_+ U_\alpha V_e) \\
+ \omega_{pp}^2 (-T_- U_\alpha V_\alpha + T_+ U_\alpha V_e - T_- U_\alpha V_\alpha + T_+ U_\alpha V_e) = 0.
\]

Eq. (46) is numerically solved in the following.
3. Numerical solutions

We first discuss a specific solution for $\eta = 0$. The parent wave assumed here is a left hand polarized wave with $(k_0v_A/\Omega_p, \omega_0/\Omega_p) = (1.6, 0.81966)$ which is a solution of eq.(16) under the appropriate conditions $(c^2k_0^2/\omega_0^2 \gg 1, |\omega_0| \ll |\Omega_\parallel|$, and $\alpha_n(n = n_{e0}/n_{e0}) = 0.1$). Fig.1 shows the solution for $\beta = 0.02$, where $\beta = 8\pi n_{e0}T_e/B_0^2$. The lines labeled with $s\pm$ denote sound waves, while other lines are the solutions of $T_{\pm} = 0$. When $\eta \neq 0$, a number of parametric instabilities set in around the intersections of the different lines. Some examples are shown in Fig.2 - Fig.5. Since the dispersion diagram is symmetric with respect to the origin, we only show the first and the fourth quadrants in the following. Hereafter, space and time are normalized to $v_A/\Omega_p$ and $\Omega_p^{-1}$, where $v_A^2 = B_0^2/4\pi n_{e0}m_p$.

Figs.2 to 5 correspond to the cases where a parent wave is on a proton cyclotron branch ($\omega_0 > 0.5$). When $\beta = 0.02$ and $\omega_0 = 0.81966$, the maximum growth occurs around $k \approx 2$ as in Fig.2. This instability is a decay instability excited through an interaction between $s+$ and $lp-$ branches. This type of decay instability is not present in an electron-proton plasma, since the excited transverse daughter waves are the SPA waves which are on the high frequency branch with large phase velocity possible to exist only in a multi-ion species plasma. The SPA waves observed in the Fig.2 of [5] is considered to be generated via this decay instability. The growth rate increases when $\eta$ becomes large. In Fig.2 a number of other parametric instabilities are seen. An another decay instability occurs around the intersection between $s+$ and $l\alpha-$ branches. This is also confirmed in Fig.2 of [5]. The so-called modulational instability is found...
at small $k < 1$. The instability at the intersection between $lp^+$ and $lp^-$ is known as a beat instability. There is also an instability around the intersection between $s^-$ and $l\alpha^+$ branches. The instabilities associated with $l\alpha^\pm$ branches may not grow in practice because of strong damping due to kinetic effects of $\alpha$ particles which have been neglected in our analysis.

The decay instability through the SPA waves is suppressed when $\beta$ becomes large ($\beta = 0.1$) as in Fig.3. In this case the two corresponding branches, $s^+$ and $lp^-$, do not intersect anymore. Furthermore, in this particular case, the modulational instability becomes also stable. The dominant instability is the decay instability associated with
Figure 4. Solution of eq.(46) with $\alpha_n = 0.1$ and $\beta = 0.02$ for a parent wave on a proton cyclotron branch with $(k_0, \omega_0) = (1.0, 0.71903)$. The left and the right panels correspond to $\eta = 0.01$ and 0.05, respectively.

Figure 5. Solution of eq.(46) with $\alpha_n = 0.1$ and $\beta = 0.1$ for a parent wave on a proton cyclotron branch with $(k_0, \omega_0) = (1.0, 0.71903)$. The left and the right panels correspond to $\eta = 0.01$ and 0.05, respectively.

Channels of possible instabilities are variable also when choosing a different set of $k_0$ and $\omega_0$. For instance, Fig.4 and 5 represent dispersion diagrams for $(k_0, \omega_0) = (1.0, 0.71903)$. When $\beta$ is small, Fig.4 ($\beta = 0.02$), the number of instabilities increases to 6 for $\eta = 0.01$. For $\beta = 0.1$ (Fig.5), five instabilities become possible, while it is only three in Fig.3. In this case too the dominant instability is the decay instability based on the SPA daughter waves when $\beta$ is small.

For a parent wave on an $\alpha$ cyclotron branch ($\omega_0 < 0.5$), instabilities associated with the $lp$– branch are not allowed for. The dispersion diagrams for $(k_0, \omega_0) = (0.8, 0.41491)$
are shown in Fig.6 and Fig.7. The most unstable instability for $\beta \leq 0.1$ is the decay instability in which $s+$ and $l\alpha-$ branches intersect. The modulational instability with $k < 0.5$, the beat instability due to an interaction between $l\alpha-$ and $lp+$, and between $lp-$ and $l\alpha+$ also get excited.

4. Summary

We derived the dispersion relation of the parametric instabilities of a left hand circularly polarized wave in a three species plasma consisting electrons, protons, and $\alpha$ particles,
where all the parent and daughter waves are assumed to propagate along the ambient magnetic field. It is found that the SPA waves are generated through a decay instability when a parent wave is on the proton cyclotron branch and the plasma beta is low.

Besides the SPA waves a variety of daughter waves are generated in the three species plasma. A decay instability generating $s^+$ and $l\alpha^-$ daughter waves, a modulational instability for small $k < 1$, a beat instability generating $lp^+$ and $lp^-$ waves, an another decay instability generating $s^-$ and $l\alpha^+$ waves, and so on.

Some daughter waves have frequencies close to the cyclotron frequencies of protons or $\alpha$ particles. However, these daughter waves are probably easily damped due to the cyclotron damping through the kinetic effect of corresponding ion species, which have neglected in our analysis. Therefore, the corresponding instabilities may not occur or be strongly suppressed in the actual system.

[1] Reames, V. R., 2017, *Solar Phys.*, 291, 156
[2] Klecker, B., Möbius, E. & Popecki, M. A., 2007, *Space Sci. Rev.*, 130, 273
[3] Kronberg, E. A., Ashour-Abdalla, M., Dandouras, I., Deleurt, D. C., Grigorenko, E. E., Kistler, L. M., Kuzichev, I. V., Liao, J., Maggiolo, R., Molova, H. V., Orlova, K. G., Peroomian, V., Shklyar, D. R., Shprits, Y. Y., Welling, D. T. & Zelenyi, L. M., 2007, *Space Sci. Rev.*, 184, 173
[4] Mizuta, T., & Hoshino, M., 2001, *Geophys. Res. Lett.*, 28, 3099
[5] Matsukiyo, S., Alamizu, T., & Hada, T., 2019, *Astrophys. J. Lett.*, 887, L2
[6] Hollweg, Joseph, V., Esser, R., & Jayanti, V., 1993, *J. Geophys. Res.*, 98, 3491
[7] Jayanti, V., & Hollweg, Joseph, V., 1994, *J. Geophys. Res.*, 99, 23449
[8] Gomberoff, L., Gratton, F., & Gnivi, G., 1994, *J. Geophys. Res.*, 99, 14717