A Macroscopic Multifractal Analysis of Parabolic Stochastic PDEs

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Abstract: It is generally argued that the solution to a stochastic PDE with multiplicative noise—such as \( \dot{u} = \frac{1}{2} u'' + u \xi \), where \( \xi \) denotes space-time white noise—routinely produces exceptionally-large peaks that are “macroscopically multifractal.” See, for example, Gibbon and Doering (Arch Ration Mech Anal 177:115–150, 2005), Gibbon and Titi (Proc R Soc A 461:3089–3097, 2005), and Zimmermann et al. (Phys Rev Lett 85(17):3612–3615, 2000). A few years ago, we proved that the spatial peaks of the solution to the mentioned stochastic PDE indeed form a random multifractal in the macroscopic sense of Barlow and Taylor (J Phys A 22(13):2621–2626, 1989; Proc Lond Math Soc (3) 64:125–152, 1992). The main result of the present paper is a proof of a rigorous formulation of the assertion that the spatio-temporal peaks of the solution form infinitely-many different multifractals on infinitely-many different scales, which we sometimes refer to as “stretch factors.” A simpler, though still complex, such structure is shown to also exist for the constant-coefficient version of the said stochastic PDE.

1. Introduction

1.1. The main result. Let \( \xi \) denote space-time white noise, normalized so that

\[
\text{Cov}[\xi(t, x), \xi(s, y)] = \delta_0(t-s) \cdot \delta_0(x-y) \quad \text{for all } s, t \geq 0 \text{ and } x, y \in \mathbb{R},
\]

and consider, throughout the paper, the stochastic heat equation

\[
\dot{u}(t, x) = \frac{1}{2} u''(t, x) + \sigma(u(t, x))\xi(t, x), \tag{1.1}
\]

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defined on \((t, x) \in (0, \infty) \times \mathbb{R}\) with initial datum \(u(0) = u_0 \in L^\infty(\mathbb{R})\). We always will assume that \(u_0\) and \(\sigma\) are nonrandom real-valued functions on \(\mathbb{R}\), that \(\sigma\) is Lipschitz continuous and satisfies \(\sigma(0) = 0\), and that \(\inf_{x \in \mathbb{R}} u_0(x) > 0\). Among many other things, these conditions ensure that (1.1) has a unique continuous and strictly-positive solution \(u(t, x)\) that has finite moments of all order, uniformly in \(x\) and locally uniformly in \(t\) [11–13,21,25,27].

The main objective of this paper is to study the intermittency properties of the solution to the stochastic PDE (1.1). In order to recall the meaning of this phrase, let us first define

\[
\gamma(k) := \liminf_{t \to \infty} t^{-1} \log \inf_{x \in \mathbb{R}} E \left( |u(t, x)|^k \right) \quad \text{and} \\
\overline{\gamma}(k) := \limsup_{t \to \infty} t^{-1} \log \sup_{x \in \mathbb{R}} E \left( |u(t, x)|^k \right),
\]

for all \(k \in [2, \infty)\). The functions \(\gamma\) and \(\overline{\gamma}\) are known respectively as the lower and the upper moment Lyapunov exponents of the solution \(u\) to (1.1). According to Jensen’s inequality, both \(\gamma\) and \(\overline{\gamma}\) are nondecreasing functions on \([2, \infty)\). The solution \(u\) to (1.1) is said to be intermittent if \(\gamma\) and \(\overline{\gamma}\) are both strictly increasing on \([2, \infty)\); see [15] and [5,6,16,23,24,28] for earlier variations.

It is known that the solution to (1.1) is intermittent [15] under the additional constraint that \(\sigma\) satisfies the following condition:

\[
\inf_{|z| > 0} |\sigma(z)/z| > 0. \quad (1.2)
\]

For this reason, we might refer to Condition (1.2) as an “intermittency condition.” The intermittency condition (1.2) also has quantitative consequences. For example, it implies—see [5,7,8,15,18]—that there exist finite and positive constants \(M_0 < N_0\) and \(M < N\) such that

\[
M_0^k e^{Mk^3 t} \leq E \left( |u(t, x)|^k \right) \leq N_0^k e^{Nk^3 t}, \quad (1.3)
\]

uniformly for all real numbers \(x \in \mathbb{R}, t > 0, \) and \(k \geq 2\). It follows from these bounds that

\[
Mk^3 \leq \gamma(k) \leq \overline{\gamma}(k) \leq Nk^3 \quad \text{for all } k \in [2, \infty).
\]

Also, (1.3) suggests that the tall spatio-temporal peaks of the stochastic process \(u\) might grow exponentially with time. For a heuristic argument, see the Introductions of Bertini and Cancrini [5] and Camona and Molchanov [6], together with Chapter 7 of Khoshnevisan [21]. With this connection to spatio-temporal peaks in mind, let us consider the random space-time set

\[
\mathcal{P}(\beta) := \left\{ (x, t) \in \mathbb{R} \times (e, \infty) : u(t, x) > e^{\beta t} \right\},
\]

of peaks of height profile \(t \mapsto \exp(\beta t)\) for every \(\beta > 0\). When \(t \gg 1\) and \((x, t) \in \mathcal{P}(\beta)\), we have \(u(t, x) > e^{\beta t} > 1\). The present work, in a sense, implies that a “typical” such pair \((x, t)\) in fact satisfies \(u(t, x) \approx e^{\beta t}\). Thus, we see that if \(\mathcal{P}(\beta) \neq \emptyset\) a.s. for infinitely-many distinct \(\beta > 0\), then there are infinitely-many natural length scales in which one can measure the tall peaks of the solution to (1.1). This will verify, quantitatively, a property that is believed to hold for a large class of “complex systems”; see Gibbon and Titi [17] for an argument.