Compact binary coalescences: Constraints on waveforms

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To obtain gravitational waveforms, results of analytical approximations for the early phase of compact binary coalescences (CBCs) are ‘stitched’ with –or calibrated against– numerical simulations for the late phase. Each of these calculations requires external inputs and there are additional ambiguities associated with the stitching procedure. Nonetheless, the resulting waveforms have been invaluable for the initial detections by the LIGO-Virgo collaboration. We are now entering an era of abundant detections, requiring greater theoretical precision both for testing general relativity and for estimating source parameters assuming general relativity (GR). The goal of this paper is to show that full non-linear GR imposes an infinite number of sharp constraints on the CBC waveforms which can be used as clear-cut measures (i) to evaluate the accuracy of any waveform in the template bank against exact GR, and (ii) to discriminate between various choices that have to be made to resolve the ambiguities.

PACS numbers: 04.70.Bw, 04.25.dg, 04.20.Cv

I. INTRODUCTION

Foundations of the theory of gravitational waves in full, non-linear General Relativity (GR) were laid down already in the 1960s by Bondi, Sachs, Newman, Penrose and others [1–6]. Sachs’s work, in particular, clarified the structure of outgoing radiation, and Penrose’s conformal completion provided us with future null infinity, \( \mathcal{I}^+ \), that serves as the natural arena to investigate detailed properties of this radiation. By combining Einstein’s equations with suitable boundary conditions and gauge fixing, Sachs [2, 4] showed that the Bondi news –or, in the current terminology, the (retarded-) time derivative of the gravitational waveform– can be freely specified on \( \mathcal{I}^+ \). This feature was re-enforced from several different angles by subsequent investigations: the asymptotic characteristic initial value problem [7], the structure of the covariant phase space of GR [8], and the study of radiative modes of the gravitational field in full GR [9, 10]. Thus, Einstein’s equations by themselves allow all possible gravitational waveforms, without any constraints. This intuition has shaped and continues to underlie much of the mathematical physics research on gravitational waves in full GR.

However, specific physical problems require appropriate asymptotic conditions which can nontrivially restrict the allowed waveforms. This is the case for the class of systems we

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will focus on: Compact Binary Coalescences (CBCs). They have certain features that are not shared by systems considered in the mathematical literature. For example, one is interested in gravitational waves emitted by isolated systems, so there is no incoming radiation. Literature on the characteristic initial value problem and non-linear stability of Minkowski space (see, e.g., [11–13]), on the other hand, generally focuses on source-free solutions of Einstein’s equations (with spacetime topology $\mathbb{R}^4$), and in this case the only solution with no incoming radiation is Minkowski space! Therefore, the general intuition that waveforms at $I^+$ are unconstrained need not extend to systems that are of greatest interest to observers today. More importantly, in CBC one encounters additional asymptotic conditions in the fat past and far future: (i) the Post-Newtonian (PN) literature on CBC assumes that the solution is stationary in the distant past and (ii) Numerical Relativity (NR) shows that spacetime geometry in the asymptotic future is well described by the Kerr metric. We will show that this asymptotic behavior leads to an infinite set of constraints on the gravitational waveforms in exact GR.

What are the implications of these constraints? How could they be used in practice? Recall first that waveforms in the current template banks use diverse ideas, including PN methods and numerical simulations, supplemented by further approximations such as the ‘effective one body’ (EOB) strategy [14–17], phenomenologically motivated models (Phenom) [18–22], as well as methods that extrapolate between numerical simulations [23]. Waveforms in the template bank are constructed using different combinations of these ideas. As we discuss in Appendix A, each step leading to the final hybrid waveform has certain inherent ambiguities, and the procedure to combine them introduces further uncertainties. Therefore, it is desirable to have independent, hard checks on waveforms to determine the precise error bars relative to the predictions of exact GR. However, this goal encounters an immediate obstacle: the exact waveform is not known! Our strategy is to use, instead, certain balance laws at $I^+$, implied by full, non-linear general relativity: they provide us with restrictions that the exact waveform must obey. Therefore, violations of these sharp constraints by any proposed waveform provide a precise measure of the deviation from the exact GR prediction, without knowing what that prediction is. Currently, the extensive work on estimating errors (see e.g. [24]) compares the analytic waveforms with the NR waveforms, which are taken to be exact. Balance laws provide an alternative that is, so to say, ‘outside the PN-NR box’, and can be used as a check on the NR simulation as well. As we discuss in Section III, such checks will become increasingly important as the LIGO-Virgo collaboration continues to provide us with ever richer set of observations, potentially enabling better and better source characterization assuming GR, and more accurate tests of viability of GR itself. Not only will these checks provide clear-cut error-bars for waveforms already in the template bank, but they will also enable us to prefer one choice in the stitching procedure over another, and possibly lead to yet new and more accurate schemes.

The paper is organized as follows. In Section II we obtain the infinite number of constraints on the gravitational waveforms. Section III summarizes the main results and presents three concrete ways in which these constraints could be used by the waveform community. Because the intended audience is quite diverse, we have collected important background material in two appendixes. In Appendix A we summarize the procedure that is currently used to create the CBC waveforms. The practitioners in the PN and NR communities are well aware of this procedure and its limitations. However, mathematical relativists appear to be unaware of the actual steps used, and the associated caveats. Therefore this subsection is addressed to them. This summary provides the background that is necessary
to understand why it is important to have non-trivial checks from full non-linear GR on candidate waveforms, and how the balance laws provide a novel avenue towards this goal. Appendix B, on the other hand, is addressed to the PN and NR communities. The constraints on waveforms arise from a balance law for ‘supermomentum’ at $I^+ \ [10, 25]$. We explain why supertranslations must be included in the asymptotic symmetry group. The mathematical GR community is well aware of this material and can skip it. This material is also needed for the angular momentum considerations discussed in the companion paper [26]. In both these Appendixes we have made an attempt to make the discussion easily accessible to non-experts without undue simplifications.

II. CONSTRAINTS ON WAVEFORMS

This Section is divided into three parts. In the first we introduce the basic framework that will be used to specify fields at $I^+$. In the second, we introduce the notion of the Bondi-Metzner-Sachs (BMS) supermomentum, and the associated balance equations. In the third, we use these balance laws together with the standard asymptotic conditions in the CBC literature to arrive at an infinite family of constraints on waveforms.

A. Underlying Framework

Our conventions are as follows. We work with $(-, +, +, +)$ signature and define the curvature tensors via $2\nabla_a \nabla_b K_c = R_{abc}^d K_d$; $R_{ac} = R_{abc}^b$ and $R = g^{ab} R_{ab}$. We will assume that the physical spacetime $(M, g_{ab})$ is asymptotically Minkowskian in the sense spelled out, e.g., in [27].

As explained in Appendix B, future null infinity, $I^+$, is the natural home for all asymptotic fields. It is a null 3-manifold with topology $S^2 \times \mathbb{R}$, coordinatized by retarded time $u$ and angular coordinates $(\theta, \varphi)$. One can think of $I^+$ either as the future boundary of the conformally completed spacetime $(\hat{M} = M \cup I^+, \hat{g}_{ab} = \Omega^2 g_{ab})$ à la Penrose [5], or as the limiting surface ‘$r = \infty$’ obtained by moving away from sources along $u = \text{const}$ null surfaces à la Bondi and Sachs [1, 2, 4]. Since waveforms are expressed using fields in the physical spacetime without making the conformal completion, in this paper we will do the same.

Since $(M, g_{ab})$ is asymptotically Minkowskian, following Bondi and Sachs let us introduce a foliation of the asymptotic region of $M$ by outgoing null hypersurfaces $u = \text{const}$ and denote its geodesic null normal by $\ell^a$. Introduce an affine parameter $r$ of $\ell^a$ such that each null surface $u = \text{const}$ is foliated by a family of (space-like) 2-spheres $r = \text{const}$. Denote the intrinsic $(+, +)$ metric of these 2-spheres by $q_{ab}$ and the other null-normal to each of these 2-spheres by $n^a$, normalized so that $g_{ab} \ell^a n^b = -1$. Finally, introduce a null complex vector field $m^a$ and its complex conjugate $\bar{m}^a$ such that their real and imaginary parts are tangential to these 2-spheres, and they are normalized such that $g_{ab} m^a \bar{m}^b = 1$. Thus, at each point in the asymptotic region we have a null tetrad $\ell^a, n^a, m^a, \bar{m}^a$ for which the only non-zero contractions are $\ell \cdot n = -1$ and $m \cdot \bar{m} = 1$. Finally, asymptotic conditions imply

\footnote{This notion is weaker than Penrose’s original definition of asymptotic simplicity which requires that every null geodesic in $M$ should have endpoints on $I^\pm$; our conditions refer only to properties of spacetime geometry near infinity.}
that this structure can be set up in such a way that, as \( r \to \infty \), we acquire certain smooth fields on \( \mathcal{I}^+ \), denoted here by a over-circle:

(i) \( \hat{q}_{ab} = \lim_{r \to \infty} r^{-2} q_{ab} \) is an unit, round 2-sphere metric (and thus independent of \( u \));

(ii) \( \hat{n}^a = \lim_{r \to \infty} n^a \) coincides with the null normal \( \partial/\partial u \) to \( \mathcal{I}^+ \);

(iii) \( \hat{\ell}^a = \lim_{r \to \infty} r^2 \ell^a \) is the other null normal to a family of 2-sphere cross-sections \( u = \text{const} \) of \( \mathcal{I}^+ \), and

(iv) \( \hat{m}^a = \lim_{r \to \infty} r \, m^a \) and \( \hat{\bar{m}}^a = \lim_{r \to \infty} r \, \bar{m}^a \) are tangential to these cross-sections.

In Penrose’s conformal completion, one can always make the null normal \( \hat{n}^a \) to \( \mathcal{I}^+ \) divergence-free by an appropriate choice of \( \Omega \), and for each such choice, we acquire a pair \((\hat{q}_{ab}, \hat{n}^a)\) of fields defined intrinsically on \( \mathcal{I}^+ \), with \( \hat{q}_{ab} \) the (degenerate) metric of signature \((+,+)\) and \( \hat{n}^a \) the null normal to \( \mathcal{I}^+ \), both induced by the conformal metric \( \hat{g}_{ab} \). However, we can further restrict our conformal factor such that \( \hat{q}_{ab} \) is a unit 2-sphere metric. The restricted pair \((\hat{q}_{ab}, \hat{n}^a)\) is said to provide a ‘Bondi (conformal) frame’. In terms of structures introduced in the last para in physical spacetime, we have \( q_{ab} = \hat{q}_{ab} \) and \( n^a = \hat{n}^a \) at \( \mathcal{I}^+ \). The structure introduced is depicted in Fig. 1.

We will therefore refer to the pair \((\hat{q}_{ab}, \hat{n}^a)\) as a Bondi-frame at \( \mathcal{I}^+ \). A change in the initial set up of the last paragraph can lead to a change in the Bondi frame. Physically, each Bondi-frame selects an asymptotic Lorentz frame because the vector field \( \hat{n}^a \) provided by the Bondi-frame is an asymptotic time translation, i.e., limit to \( \mathcal{I}^+ \) of an asymptotic time translation vector field in spacetime. Recall that we have a 3-parameter family of time-translation Killing fields –or Lorentz frames– in Minkowski space. Therefore, one would expect that there would be 3-parameter family of Bondi-frames. This is indeed the case: Bondi-frames are related to one another by (asymptotic) boosts. If we perform a boost by 3-velocity \( \vec{v} \), the initial Bondi-frame \((\hat{q}_{ab}, \hat{n}^a)\) transforms to \((\hat{q}_{ab}, \hat{n}^a)\) given by

\[
\hat{\ell}^a = \omega^2 \hat{q}_{ab}, \quad \hat{\bar{m}}^a = \omega^{-1} \hat{n}^a \quad \text{with} \quad \omega = \gamma \left( 1 - \frac{\vec{v} \cdot \hat{x}}{c} \right),
\]

where \( \gamma = (1 - (v/c)^2)^{-\frac{1}{2}} \) is the standard Lorentz factor and \( \hat{x} \) the unit radial vector with components \((\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)\). We will use the terms ‘Bondi-frame’ and ‘asymptotic Lorentz-frame’ interchangeably.

Recall from special relativity that the Lorentz frame in which the 4-momentum \( P_a \) of the system is purely time-like –i.e., in which the 3-momentum \( \vec{P} \) vanishes is referred to as the rest frame of the system. In most of this paper we will work with the past rest frame at \( \mathcal{I}^+ \), i.e., the frame in which the Bondi 3-momentum, defined in Section II B below, vanishes in the asymptotic past, as \( u \to -\infty \). Finally, at one intermediate step we will also have to use the Bondi-frame in which the Bondi 3-momentum vanishes in the distant future –i.e., \( u \to \infty \). In a CBC this is the asymptotic Lorentz frame in which the final black hole is at rest, and represents the future rest frame.

In terms of the null-tetrads we have: \( g_{ab} = -2\ell_{(a}m_{b)} + 2m_{(a}\bar{m}_{b)} = -2\ell_{(a}n_{b)} + q_{ab} \) on the physical spacetime \( M \), and \( \hat{q}_{ab} = 2\hat{m}_{(a}\hat{\bar{m}}_{b)} \) on \( \mathcal{I}^+ \). Of special interest is the shear \( \sigma = -m^a m^b \nabla_a \ell_b \) of \( \ell^a \) because its asymptotic value \( \sigma^a \) is directly related to the gravitational
FIG. 1: A depiction of future null infinity, $\mathcal{I}^+$ (in light blue) that constitutes the future boundary of space-time and has topology $S^2 \times \mathbb{R}$. The (yellow) outgoing null surface $u = u_1$ in the asymptotic region of space-time intersects $\mathcal{I}^+$ in a 2-sphere cross-section. The null tetrad $(\ell^a, n^a, m^a, \bar{m}^a)$ and the metric $q_{ab}$ on $r = \text{const}$ 2-spheres in the interior have limits $\hat{\ell}^a, \hat{n}^a, \hat{m}^a, \bar{\hat{m}}^a$ and $\hat{q}_{ab}$ respectively on $\mathcal{I}^+$. The figure also depicts the typical waveform produced by a CBC.

waveforms. In Minkowski spacetime we can choose the $u = \text{const}$ surfaces to be the light cones emanating from points of a time-like geodesic. Then $\ell^a$ is shear-free. In asymptotically Minkowskian spacetimes, shear $\sigma$ of $\ell^a$ falls-off as $1/r^2$ and

$$\sigma^\circ(u, \theta, \varphi) := - \lim_{r \to \infty} r^2 (\ell^a m^b \nabla_a \ell_b) \equiv (h^\circ_+ + i h^\circ_\times)(u, \theta, \varphi)$$

(2.2)

is well-defined on $\mathcal{I}^+$, where, in the last step we have expressed $\sigma^\circ$ in terms of the commonly used waveforms $h^\circ_+ \equiv r h_+$ and $h^\circ_\times \equiv r h_\times$. (Because of the $'m^a m^b'$ factor, $\sigma^\circ$ has spin weight $+2$, while the more commonly used combination $h^\circ_+ - i h^\circ_\times = \bar{\sigma}^\circ$ has spin-weight $-2$.) The shear tensor $\sigma^\circ_{ab}$ at $\mathcal{I}^+$, given by

$$\sigma^\circ_{ab}(u, \theta, \varphi) = - (\sigma^\circ \bar{m}_a \hat{m}_b + \bar{\sigma}^\circ \hat{m}_a m_b)(u, \theta, \varphi),$$

(2.3)

is a symmetric, traceless tensor field, transversal to the null normal to $\mathcal{I}^+$ - i.e. satisfies $\sigma^\circ_{ab} \hat{n}^a = 0$ and has spin weight zero. It captures the two Transverse-Traceless or radiative modes of the gravitational field in full, non-linear GR. Its time derivative is the Bondi news
tensor:  
\[ N_{ab} := 2\mathcal{L}_{\dot{\gamma}}\sigma_{ab}^\circ \equiv 2\dot{\sigma}_{ab}^\circ. \]  
(2.4)

One often introduces a ‘news function’ \( N \) via
\[ N_{ab} = 2\left( \nabla_{\dot{m}_a}\dot{m}_b + \nabla_{\dot{\bar{m}}_a}\bar{m}_b \right), \]
(2.5)
so that \( N = -\partial_u\sigma^\circ \equiv -\dot{\sigma}^\circ \) has spin weight -2.

Finally let us introduce the Newman-Penrose components of the Weyl tensor that also feature prominently in the discussion of gravitational waves produced by CBCs. Since the stress energy tensor (if any) of the system is of compact spatial support, curvature in the neighborhood of infinity is fully captured by the Weyl tensor. Because \((M, g_{ab})\) is asymptotically Minkowskian, components of the Weyl tensor in the null tetrad have a specific fall-off, known as ‘peeling properties’ \([2, 6]\). The leading order components of the Weyl tensor in the null tetrad –which carry a superscript \( \circ \)– can be regarded as fields on \( \mathcal{I}^+ \):

\[ \Psi_4^\circ(u, \theta, \varphi) = \lim_{r \to \infty} r^4 C_{abcd}^\circ a^m b^c \ell^d; \quad \Psi_3^\circ(u, \theta, \varphi) = \lim_{r \to \infty} r^2 C_{abcd}^\circ a^m b^c \ell^d; \]
\[ 2\Re \Psi_2^\circ(u, \theta, \varphi) = \lim_{r \to \infty} r^3 C_{abcd}^\circ a^m b^c \ell^d; \quad 2\Im \Psi_2^\circ(u, \theta, \varphi) = -i \lim_{r \to \infty} r^3 C_{abcd}^\circ a^m b^c \ell^d; \]
\[ \Psi_1^\circ(u, \theta, \varphi) = \lim_{r \to \infty} r^4 C_{abcd}^\circ a^m b^c \ell^d; \quad \Psi_0^\circ(u, \theta, \varphi) = \lim_{r \to \infty} r^5 C_{abcd}^\circ a^m b^c \ell^d. \]  
(2.6)

The spin-weight -2 component \( \Psi_4^\circ \) is referred to as the radiation field because: (i) it is the leading order coefficient of curvature component with asymptotic fall-off \( 1/r \); and, (ii) it is directly related to the waveforms, namely \( \Psi_4^\circ = \dot{\sigma}^\circ \equiv \partial_u \dot{\sigma}^\circ \). By contrast, the real part of the spin-weight zero component \( \Psi_2^\circ \) encodes the ‘Coulombic’ information contained in the Bondi mass. The angular momentum information resides in the spin-weight 1 component \( \Psi_1^\circ \) \([28, 29]\). This may seem surprising at first because one is accustomed to the statement that in Kerr spacetime the only non-zero component of the Weyl tensor is \( \Psi_2 \). However, that statement refers to components in principal null directions which do not agree with the pair of null vector fields \((\dot{\gamma}^a, \ell^a)\) at \( \mathcal{I}^+ \). In terms of the null tetrad at \( \mathcal{I}^+ \), in a Kerr spacetime with mass \( M \) and angular momentum \( J \), \( \Psi_2^\circ \) is a real constant with value \(-GM\) and \( \Psi_1^\circ = (3i/2)\sin \theta GJ \). Finally, in this paper we will not need the component \( \Psi_0^\circ \); it was introduced just for completeness.

**Remark:** The fall-off in \( 1/r \) of fields constructed from \( g_{ab} \) in the physical spacetime translates to degree of differentiability –or smoothness– of the conformally rescaled metric \( \hat{g}_{ab} \) in the Penrose completion. There has been considerable debate in the literature on whether these assumptions are physically reasonable (for a summary, see e.g. \([30]\)). Indeed, using the PN perspective it was argued in the early days that they may not be appropriate for CBC \([31]\). However, as noted in Appendix B, the current consensus in the PN literature is that the asymptotic form of the PN metric is completely consistent with the Bondi-Sachs-Penrose framework \([32]\). Similarly, the key notions underlying this framework –such as \( \Psi_4^\circ \), \( N \), and \( \sigma^\circ \)– and relations between them are heavily used in NR.

\(^2\) It has a clear-cut geometric meaning in the conformally completed spacetime: \( N_{ab} \) is the conformally invariant part of the curvature of the intrinsic connection on \( \mathcal{I}^+ \) \([9]\).
B. The supermomentum balance law

Much of the previous work on supermomentum is carried out in the conformally completed spacetime, without introducing auxiliary structures such as the tetrad vectors $\ell^a, m^a, \bar{m}^a$ that are not canonically defined at $\mathcal{I}^+$ [10, 25, 33]. Similarly, one does not require that the intrinsic metric $\tilde{q}^{ab}$ on $\mathcal{I}^+$ is the round, unit 2-sphere metric. This generality made it manifest that the results have all the required invariance and covariance properties. However, since the waveform community does not use conformal completion, we will now present the material using only those notions that are introduced in Section II A (and in Appendix B). On the other hand, to check transformation properties, e.g., from one Bondi frame to another, it is easiest to use the original invariant framework.

As explained in Appendix B, one of the major surprises to emerge from the Bondi-Sachs work was that, although spacetimes under consideration are asymptotically Minkowskian, the asymptotic symmetry group is not the Poincaré group $\mathcal{P}$, but an infinite dimensional enlargement thereof, the BMS group $\mathfrak{B}$. While $\mathfrak{B}$ does admit a canonical 4-dimensional normal subgroup $\mathcal{T}$ of translations [3], it also contains an infinite dimensional Abelian subgroup $\mathcal{S}$ of supertranslations, which can be thought of as ‘angle dependent translations’. There is a direct relation between this enlargement and the presence of gravitational waves: In absence of gravitational radiation the group naturally reduces back to the Poincaré group [9, 33, 34]. Intuitively, the enlargement occurs because the ripples of curvature propagate out all the way to infinity, introducing an ambiguity in the choice of the ‘background Minkowski metric’ that the physical metric $g_{ab}$ is approaching as $1/r$. In any Bondi-frame at $\mathcal{I}^+$, we have the following properties:

(i) translations are represented by vector fields $\xi^a_{\alpha} := \alpha(\theta, \phi)\hat{n}^a$ at $\mathcal{I}^+$, where $\alpha$ is a (real-valued) linear combination of the first 4 spherical harmonics:

$$\alpha(\theta, \varphi) = \alpha_0 Y_{0,0} + \sum_{m=-1}^{m=1} \alpha_m Y_{1,m}(\theta, \varphi)$$  \hspace{1cm} (2.7)

for some constants $\alpha_0, \alpha_m$;

(ii) the vector field $\xi^a_{(0)} = \hat{n}^a$ is the time-translation in the chosen Bondi frame and the vector fields $\xi^a_{(\alpha)} = \sum_{m=-1}^{m=1} \alpha_m Y_{1,m}(\theta, \varphi) \hat{n}^a$ are spatial translations in that frame;

(iii) the more general vector-fields $\xi^a_{(f)} = f(\theta, \varphi) \hat{n}^a$ where $f$ is any smooth function on a 2-sphere are called supertranslations; so translations are just special cases of supertranslations. If we fix a Bondi-frame –as we will in most of the paper, taking it to be the rest-frame in the asymptotic past– we can speak of ‘pure’ supertranslations: These correspond to

$$f(\theta, \varphi) = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell,m} Y_{\ell,m}(\theta, \varphi).$$  \hspace{1cm} (2.8)

In what follows, we will use the symbol $\alpha(\theta, \varphi)$ to refer to BMS translation and $f(\theta, \varphi)$ to refer to generic BMS supertranslations. 3

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3 If we change the Bondi-frame, the vector field $\hat{n}^a$ is rescaled as in Eq. (2.1). Since the supertranslation
Recall that for fields in Minkowski space, energy-momentum arises as the Hamiltonian generating canonical transformations corresponding to spacetime translations. In the same spirit, in GR one can construct a phase space $\Gamma_{\text{Rad}}$ of radiative modes at $\mathcal{I}^+$ [8], show that the action of BMS supertranslations $\xi^a_f$ on $\mathcal{I}^+$ induces canonical transformations on $\Gamma_{\text{Rad}}$, and calculate the corresponding Hamiltonians [10]. They are given by an integral over $\mathcal{I}^+$ which has the interpretation of the total flux $F_f$ across $\mathcal{I}^+$ of the component of supermomentum defined by $f(\theta, \varphi)$:

$$F_f = \frac{1}{32\pi G} \int_{\mathcal{I}^+} du d^2V f(\theta, \varphi) \left[ N_{ab}N^{ab} + 2\dot{D}_a\dot{D}_b N^{ab} \right] (u, \theta, \varphi)$$

Here, in the first step we have raised the indices of $N_{ab}$ using the unit 2-sphere metric $\delta_{ab}$ on the $u = \text{const}$ 2-spheres, and $\dot{D}_a$ is the derivative operator compatible with $\dot{q}_{ab}$. In the second step we have used the Newman-Penrose angular derivative operator $\bar{\sigma}$ whose action on a spin-weight $s$ scalar $A$, given by

$$\bar{\sigma} A = \frac{1}{\sqrt{2}}(\sin \theta)^s \left( \frac{\partial}{\partial \theta} + \frac{i}{\sin \theta} \frac{\partial}{\partial \varphi} \right) \sin \theta^{-s} A,$$

yields a scalar with spin-weight $s + 1$. (In our case, $\dot{\sigma}$ has spin-weight $-2$, whence $\bar{\sigma}^2 \dot{\sigma}$ has spin-weight zero.) The second equality in (2.9) provides us with supermomentum fluxes directly in terms of the waveform $\sigma^o(u, \theta, \varphi) = (h^o_+ + ih^o_\times)(u, \theta, \varphi)$.

One can show that the $F_f$ is in fact an integral of an exact 3-form, whence the integral on $\mathcal{I}^+$ can be expressed as the difference between two 2-sphere integrals, performed at $i^0$ (i.e. on the ‘$u = -\infty$’ 2-sphere) and $i^+$ (i.e. on the ‘$u = \infty$’ 2-sphere) [10]:

$$F_f = \lim_{u_0 \to -\infty} P_f|_{u = u_0} - \lim_{u_0 \to \infty} P_f|_{u = u_0}$$

where

$$P_f|_{u = u_0} := -\frac{1}{4\pi G} \oint_{u = u_0} d^2V f(\theta, \varphi) \Re \left[ \Psi^2_2 + \bar{\sigma} \dot{\sigma} \right] (\theta, \varphi)$$

is the $f$-component of the supermomentum of the system at the retarded time $u = u_0$. Eq. (2.11) constitutes an infinite family of balance laws –one for each choice of a supertranslation, i.e., of smooth $f(\theta, \varphi)$ on a 2-sphere– that will provide us with an infinite family of constraints on waveforms. Finally, references [10, 25] provide more general expression of supermomentum that hold on any 2-sphere cross-section of $\mathcal{I}^+$ (not necessarily $u = \text{const}$) and formula of supermomentum flux through any patch $\Delta \mathcal{I}^+$ of $\mathcal{I}^+$. Eqs (2.9) and (2.12) are just special cases of those expressions to the setup introduced in Section II A.

Since the balance law holds for any supertranslation $f(\theta, \varphi)\dot{n}^a$, it holds in particular for translations $\alpha(\theta, \varphi)\dot{n}^a$. In this case $P_{(\alpha)}$ is simply the Bondi-Sachs energy-momentum of the

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is given by $\xi^a = f\dot{n}^a = \dot{f}\dot{n}^a$, the labels $f$ and $\dot{f}$ in the two frames are related by $\dot{f} = \omega f$. In the Penrose conformal picture, $f$ is not a scalar but carries a conformal weight $1$. It turns out that the notion of a BMS translation is invariant with respect to this change of the Bondi-frame, but the notion of a ‘pure supertranslation’ is not.
system at the retarded time \( u = u_\circ \). Since the functions \( \alpha(\theta, \varphi) \) are linear combinations of the first four spherical harmonics, the flux formula simplifies. For, \( \hat{q}^{ab}D_a\alpha \) is a conformal Killing field of the 2-sphere metric \( \hat{q}_{ab} \), whence \( \hat{D}_a\hat{D}_b\alpha \propto \hat{q}_{ab} \). Therefore, integrating the first equation in (2.9) by parts and using the fact that \( N_{ab} \) is trace-free, one obtains:

\[
\mathcal{F}_{(\alpha)} = \frac{1}{4\pi G} \int_{J^+} \mathrm{d}u \mathrm{d}^2\Omega \alpha(\theta, \varphi) |\sigma^\circ|^2(u, \theta, \varphi). \tag{2.13}
\]

In particular, for any time-translation, \( \alpha(\theta, \varphi) \) is positive, whence the flux of energy is necessarily positive. This is the celebrated Bondi-Sachs result. To calculate the final black hole kick, one computes the 3-momentum flux \( \mathcal{F}_{(\vec{\alpha})} \), where the function \( \alpha \) is a general linear combination of the three \( Y_1, m \).

**Remark:** In the literature between 1960s and early 1980s there was considerable confusion about supermomentum because the flux expressions were not obtained from a physically well-motivated procedure. For example, expressions of supermomentum given in [35–38] lead to a non-zero flux between general cross-sections of \( J^+ \) in Minkowski space [39–41]! The expression (2.9) by contrast vanishes anytime the news tensor vanishes, in particular in any stationary spacetime (relation between these flux expressions is discussed in [40]). Even when Hamiltonian considerations were used to provide a physical basis for the derivation, the flux expression in the early literature was incorrect (see, e.g. second article in [4] and [42]) because of a subtle fact that the radiative phase space is an affine space, rather than a vector space (for details, see [10]). These examples serve to bring out the fact that considerable care is needed to arrive at viable balance laws.

### C. Implications for hybrid waveforms

We can now apply the balance laws to isolated systems undergoing CBC. Appendix A summarizes the assumptions, approximations and phenomenological inputs that go in the construction of waveforms used in the template banks. Since the balance laws (Eq. (2.11)) refer to exact GR, we will not need to make any approximations. However, we need to incorporate the fact that we are now restricting ourselves to CBC. We will do so through two physically motivated assumptions.

1. **We assume that the Bondi news tensor \( N_{ab} \) on \( J^+ \) goes to zero as \( u \to \pm \infty \) as \( 1/|u|^{1+\epsilon} \) for some \( \epsilon > 0 \).**

This condition is necessary and sufficient to ensure that the flux of the BMS angular momentum across \( J^+ \) is finite [10, 26]. For vacuum solutions, the Christodoulou-Klainerman results, for example, ensure that this condition holds with \( \epsilon = 1/2 \). Hybrid waveforms constructed so far satisfy this assumption because the Bondi news goes to zero rapidly at early and late times; within the approximations made and accuracy achieved, \( N_{ab} \) is indistinguishable from zero outside some finite \( u \)-interval. In terms of the waveform \( \sigma^0 = h_+^c + ih_x^c \), our assumption will be:

\[
\sigma^0(u, \theta, \varphi) = \sigma_\pm(\theta, \varphi) + |u|^{-\epsilon} \sigma_\pm^{(1)}(\theta, \varphi) + O(|u|^{-\epsilon-1}) \quad \text{as} \quad u \to \pm \infty, \tag{2.14}
\]

so that \( \sigma_\pm(\theta, \varphi) \) are the limits of \( \sigma^0(u, \theta, \varphi) \), and \( \pm \varepsilon \sigma_\pm^{(1)}(\theta, \varphi) \) are the limits of \( |u|^{1+\epsilon} \tilde{N}(u, \theta, \varphi) \) as \( u \to \pm \infty \).
Numerical simulations show that at late times the spacetime metric approaches the Kerr solution, and thus becomes stationary. As explained in Appendix A, PN inspired waveforms are obtained assuming that system is stationary in the past, i.e., for times \( t < -\tau \) \[32\]. We will need a much weaker assumption to capture the physical behavior of the compact binary in the asymptotic future and past. Since \( N_{ab} = O(1/|u|^{1+\epsilon}) \) as \( u \to \pm \infty \), Einstein’s equations together with Bianchi identities imply that \( \partial_u \Psi_4, \partial_u \Psi_3 \) and \( \partial_u \Psi_2 \) go to zero as \( u^{-(3+\epsilon)}, u^{-(2+\epsilon)} \) and \( u^{-(1+\epsilon)} \) respectively. In this precise sense these three complex components of the asymptotic curvature become ‘time-independent’ or ‘stationary’ in the limit \( u \to \pm \infty \). Because \( N_{ab} \) is invariant under the change of Bondi-frame, this fall-off behavior holds in any Bondi-frame. Furthermore, \( \Psi_4 \) and \( \Psi_3 \) themselves vanish in these limits, and for \( \Psi_2 \) we have

\[
\Psi_2^o(u, \theta, \varphi) = \psi_\pm(\theta, \varphi) + |u|^{-2} \psi_\pm^{(1)}(\theta, \varphi) + O(|u|^{-1}) \quad \text{as} \quad u \to \pm \infty ,
\]

again in any Bondi-frame, although the limiting values \( \psi_\pm(\theta, \varphi) \) and \( \psi_\pm^{(1)}(\theta, \varphi) \) depend on the choice of the frame.

However, assumption (1) does not imply that \( \Psi_1^o \) (or \( \Psi_0^o \)) become stationary as \( u \to \pm \infty \). Furthermore, the limiting values of \( \partial_u \Psi_1^o \) depend on the choice of the Bondi-frame and in general the rest-frame of the final black hole is different from that of the system in the distant past because of black hole kicks \[43, 44\]. We will now make our second assumption:

\[
(2) \quad \partial_u \Psi_1^o \to 0 \text{ in the past Bondi-frame as } u \to -\infty , \text{ and in the future Bondi-frame as } u \to \infty .
\]

Recall that the past Bondi-frame is the one in which the Bondi 3-momentum vanishes in the distant past (i.e., \( u \to -\infty \)), and the future Bondi-frame is the one in which the Bondi 3-momentum vanishes in the distant future (i.e., \( u \to \infty \)). Thus, we are not requiring that the system should become stationary in the past and the future in the same rest-frame. As we discuss below, that requirement would have been too restrictive. Ours is quite weak, much weaker than what is assumed in PN waveforms and found to hold in the NR waveforms.

The two assumptions lead to a key simplification in the surface terms representing asymptotic values of supermomentum (2.12) used in the balance laws (2.11). First, in any Bondi-frame, Einstein’s equations and Bianchi identities imply that the following relations hold on all of \( \mathcal{I}^+ \) (see, e.g., \[6\]):

\[
\partial_u \Psi_1^o = m^a \dot D_a \Psi_2^o + 2 \sigma^\theta \Psi_3^o \equiv \partial \Psi_2^o - 2 \sigma^\theta \partial \dot \sigma^\theta , \quad \text{and,} \quad \text{Im } \Psi_2^o = \text{Im} \left( \partial^2 \sigma^\theta + \sigma^\theta N \right) .
\]

They have two important consequences. Let us first consider the past Bondi-frame. Then, in the limit \( u \to -\infty \) the left side of (2.16) vanishes by assumption (2), while the second term on the right hand side vanishes by assumption (1). Therefore \( m^a \dot D_a \Psi_2^o \) vanishes i.e., \( \Psi_2^o \) becomes spherically symmetric in the limit \( u \to -\infty \). Next, thanks to assumption (1), Eq. (2.17) implies \( \lim_{u \to -\infty} \text{Im } \Psi_3^o = \partial^2 \text{Im } \sigma_\theta \). Integrating both sides over a 2-sphere, and using spherical symmetry of \( \text{Im } \Psi_2^o \) we conclude that \( \text{Im } \Psi_2^o = 0 \) in the limit. To evaluate the limit of \( \text{Re } \Psi_2^o \) we will use the expression (2.12) to evaluate the past limit of the Bondi 4-momentum. This limit is purely time-like in the rest-frame at \( i^0 \), and the limiting Bondi energy is precisely the initial (or the total) mass \( M_{i^0} \). Therefore, setting \( f = 1 \) in Eq. (2.12), and taking the limit \( u_o \to -\infty \) we obtain: \( \lim_{u \to -\infty} \text{Re } \Psi_2^o = -GM_{i^0} \). Thus, we conclude:

\[
\lim_{u \to -\infty} \text{Re } \Psi_2^o(u, \theta, \varphi) = -GM_{i^0} , \quad \text{and} \quad \lim_{u \to -\infty} \text{Im } \Psi_2^o(u, \theta, \varphi) = 0 .
\]

\[2.18\]
Now, the asymptotic rest frame of the system at $u = \infty$ is generically different from that in the past. To work out the implication of condition (2) on the behavior of $\Psi_2$ in the distant future, let us for a moment switch to the future Bondi-frame—in which the final black hole is at rest—and denote fields in that frame with a prime. Then the reasoning we used also implies that $\lim_{u \to \infty} \Psi_2' = -GM_{i+}$.

Since the balance law is formulated in a general but fixed Bondi-frame, and we have chosen to work in the past Bondi-frame, we need to transform $\Psi_2'$ to that frame. The two frames are related to each other by a boost defined by the velocity $\vec{v}$ of the final black hole (in the past rest-frame). For definiteness let us choose our $z$ axis in the direction of this velocity, so that $\vec{v} = v \hat{z}$. To calculate $v$ we note that, since initially the Bondi 3-momentum is zero, the momentum $P_z$ of the final black hole in the past rest-frame is just the negative of the Bondi 3-momentum flux $F(\alpha = \cos \theta)$ carried by gravitational waves, and hence completely determined by the waveform. From Eq. (2.13) we obtain

$$\gamma (M_{i+}) v \equiv P_z = -\frac{1}{4\pi G} \int du d^2 \hat{V} \cos \theta |\dot{\sigma}^\circ(u, \theta, \varphi)|^2,$$

where $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$ is the standard Lorentz factor. In terms of this velocity $v$, the transformation property of $\Psi_2$ under the change of Bondi-frames implies that the future limit of $\Psi_2$ in the past Bondi-frame is given by

$$\Psi_2^\circ|_{u=\infty} = \frac{\Psi_2'}{\gamma^3 (1 - \frac{v}{c} \cos \theta)^3} = -\frac{GM_{i+}}{\gamma^3 (1 - \frac{v}{c} \cos \theta)^3}.$$  

Thus, using assumptions (1) and (2) we have now expressed $\Psi_2$ in both asymptotic limits $u \to \pm \infty$ using the the past Bondi-frame through Eqs. (2.18) and (2.20). We can now use the balance law (2.11) for a general supertranslation $f(\theta, \varphi)$ for CBC to conclude that the waveform $\sigma^\circ = h^\circ_\perp + i h^\circ_\times$ must satisfy

$$F(\theta, \varphi) := \int du [|\dot{\sigma}^\circ|^2 - \Re(\bar{\partial}^2 \dot{\sigma}^\circ)](u, \theta, \varphi) = GM_{i\circ} - \frac{GM_{i+}}{\gamma^3 (1 - \frac{v}{c} \cos \theta)^3}$$

where we have used the fact that $f(\theta, \varphi)$ in (2.9) and (2.12) is an arbitrary function of $(\theta, \varphi)$. $F(\theta, \varphi)$ can be thought of as the ‘total flux of supermomentum in the $(\theta, \varphi)$ direction’. Since, Eq.(2.21) holds for each $(\theta, \varphi)$ we have an infinite set of conditions that constrains the waveform.

Now, every calculation leading to a waveform in the current bank provides us with:

(i) the waveform $\sigma^\circ(u, \theta, \varphi) = h^\circ_\perp + i h^\circ_\times$ that appears on the left side of (2.21);

(ii) $M_{i\circ}$—the total mass of the system—from the PN part of the calculation;

(iii) the mass $M_{i+}$ of the final black hole, from NR.

---

4 This transformation property follows from the conformal rescaling of $\hat{n}^a$ given in Eq. (2.1), the normalization condition $g_{ab}n^a n^b = -1$ in physical spacetime, the definition of $\Psi_2^\circ(u, \theta, \varphi)$ given in Eq. (2.6) and the fact that the radial coordinate changes via $r' = \gamma (1 - \frac{v}{c} \cdot \hat{x}) r + O(1)$ under a boost.
If the global procedure used to create the waveform is to be consistent with exact GR, then the outcome must satisfy the infinite set of constraints (2.21). As explained in Appendix B, the ingredients come from very different quarters, each with its own approximations and additional inputs. The mass $M_{i^{+}}$ of the final black hole is typically deduced from the final isolated horizon – rather than from fields on $3^{+}$ – to minimize NR errors. Construction of the hybrid waveform $\sigma^o$ involves several inputs: For example, in the NR part, waveform is extracted at some large but finite radius [45]; one uses various ‘approximants’ in addition to the choice of the PN order at which the calculation is truncated [32, 46]; and the stitching procedure requires additional inputs while matching the PN parameters with the NR ones [47]. Finally, the initial mass $M_i^{o}$ is calculated using PN methods assuming exact stationarity at early times. While a few thousand numerical simulations are available, the template bank has a few hundred thousand waveforms, obtained by introducing additional assumptions and interpolations mentioned in Section I. Eq. (2.21) provides sharp, clearcut tests to check if the end-products of this entire procedure is consistent with fully non-linear GR and to calculate ‘objective’ error-bars that, e.g., do not involve comparisons with NR simulations.

To get an intuitive feel for these constraints, let us consider the special case where there is no kick, i.e., where the velocity $v$ of the final black hole is zero. Then the right side of (2.21) is a constant, while the left side $F(\theta, \varphi)$, constructed from the waveform, is a function of $(\theta, \varphi)$. The infinite set of constraints now imply that $F(\theta, \varphi)$ must be a constant. However, zero kick represents a very special CBC. Numerical simulations show that typical kick velocities are a few hundred km/s. For definiteness, let us take it be $\sim 300$ km/s, so $v/c \sim 10^{-2}$. One can then Taylor expand the second term in the right side of (2.21) in $v/c$,

$$GM_i^{o} - \frac{GM_{i^{+}}}{\gamma^3 (1 - \frac{v}{c} \cos \theta)^3} = GM_i^{o} - GM_{i^{+}} \left(1 + 3 \cos \theta \frac{v}{c} - \left(\frac{3}{2} - 6 \cos^2 \theta \right) \frac{v^2}{c^2} + \ldots\right)$$

(2.22)

and keep terms to the desired accuracy. To test if a given waveform is accurate to $\sim 0.3\%$, we only need to retain the first order term in $v/c$, namely, $(v/c) GM_{i^{+}} \cos \theta$. Then the right side has only linear combinations of $\ell = 0, 1$ spherical harmonics, whence, in the expansion of $F(\theta, \varphi)$ in terms of spherical harmonics, all coefficients of the flux must vanish for $\ell \geq 2$. This means that the waveform must be such that the total flux of the infinitely many components of pure supermomentum across $3^{+}$ must vanish to $\sim 0.3\%$ accuracy. If we are interested in a 0.01% accuracy, we would keep terms up to $O(v^2/c^2)$ and find that the coefficients of the total flux must vanish for $\ell > 2$, and so on. To summarize, (2.21) provides an infinite number of conditions that any proposed waveform must satisfy if it is to approximate the outcome of the exact GR calculation to a given desired accuracy. Furthermore, the resulting error-bars are relative to exact GR and are not contaminated by other inputs.

So far we have focused on the ‘global balance law’ (2.11) that involves all of $3^{+}$. However, we also have a local balance law for any portion $\Delta 3^{+}$ that is is bounded by two cross-sections $C_1$ and $C_2$ [10, 25]. For waveform considerations it suffices to take $C_1$ and $C_2$ to be $u = u_1(\theta, \varphi)$ and $u = u_2(\theta, \varphi)$ cross-sections in any Bondi-frame. Then we have:

$$F(\theta, \varphi)|_{u_2} = P_f|_{u=u_2} - P_f|_{u=u_1}$$

(2.23)

where the supermomentum $P_f|_{u}$ is again given by (2.12) and the flux on the left side is given by

$$F(\theta, \varphi)|_{u_1} = \frac{1}{4\pi G} \int d^2V f(\theta, \varphi) \int_{u_1}^{u_2} du \left[|\sigma^o|^2 - \text{Re}(\sigma^o \sigma^o)\right](u, \theta, \varphi).$$

(2.24)
One can again ‘peel off’ the 2-sphere integral using the fact that \( f(\theta, \varphi) \) is arbitrary and arrive at the local analog of (2.21):

\[
F(\theta, \varphi) \bigg|_{u_1}^{u_2} := - \int_{u_1}^{u_2} \! du \left[ |\dot{\sigma}^o|^2 - \text{Re}(\bar{\sigma}^o \ddot{\sigma}^o) \right](u, \theta, \varphi) = \text{Re}\left[ \Psi_2^o + \bar{\sigma}^o \dot{\sigma}^o \right](u_1, \theta, \varphi) - \text{Re}\left[ \Psi_2^o + \bar{\sigma}^o \dot{\sigma}^o \right](u_2, \theta, \varphi). \tag{2.25}
\]

This local balance law can also be used to measure on how close a proposed waveform is to that of exact GR. In particular, in the case when the interval \((u_1, u_2)\) is in the PN regime, one can use the asymptotic form of the analytical metric to calculate \( \text{Re}\Psi_2^o \) at the two retarded instants of time, \( u = u_1(\theta, \varphi) \) and \( u = u_2(\theta, \varphi) \), and the waveform \( \sigma^0 = h^o_x + ih^o_y \) between \( u_1 \) and \( u_2 \) to find the extent to which the local constraint (2.24) is violated. This can provide an independent resource to decide between candidate ‘approximants’ and help choose the order at which the expansion is truncated.

We will conclude with a few remarks:

1. Had spacetime been exactly stationary for \( t < -\tau \), as is generally assumed in the PN literature [31, 32, 48, 49], then we could have used the multipolar expansion of the metric that is shown to hold rigorously for stationary spacetimes outside some spatially compact region [50]. The past Bondi-frame is of course the one in which \( \ddot{n}^a \) is the limit to \( \mathcal{I}^+ \) of the stationary Killing field. Using this expansion, one can calculate the asymptotic Weyl tensor and show that \( \Psi_2^o \) is real and spherically symmetric on \( \mathcal{I}^+ \) in the past Bondi-frame on an infinite interval \( u < -u_0 \) for some \( u_0 \). Since our notion of asymptotic stationary in the past is extremely weak in comparison, we can only conclude that \( \Psi_2^o \) becomes spherically symmetric in the limit \( u \to -\infty \).

2. The other analytical approximations used to construct waveforms, and discussed in the Introduction, agree with the PN one in the distant past. Our weak assumption (2), being inspired by the PN stationarity condition, therefore includes such models as well.

3. In the above discussion we assumed that the velocity \( \vec{v} \) of the final black hole is in the \( z \)-direction just for simplicity of presentation. In the general case, \( \vec{v} = v_x \hat{x} + v_y \hat{y} + v_z \hat{z} \), we only have to replace \( (v/c) \cos \theta \) in (2.21) with \( (1/c) \vec{v} \cdot \hat{x} \equiv (1/c)(v_x \sin \theta \cos \varphi + v_y \sin \theta \sin \varphi + v_z \cos \theta) \).

4. Since the Lorentz transformation of energy only involves the first power of \( \gamma(\vec{v} \cdot \hat{x}/c) \) in the denominator, the appearance of the cube of this factor in (2.21) may seem surprising. The additional factor of \( (\gamma(\vec{v} \cdot \hat{x}/c))^{-2} \) comes from the transformation property of the 2-sphere area-element: We have ‘peeled-off’ a 2-sphere integral in the passage from (2.11) to (2.21).

5. If there is no kick, i.e., if \( v = 0 \) identically, then the past and future Bondi frames coincide. This is of course a very special case realized, e.g., in an equal mass head-on collision. In this case, Eq. (2.20) simplifies to \( \Psi_2^o|_{u=\infty} = -GM_i^+ \). Thus \( \Psi_2^o \) becomes spherically symmetric in the same Bondi-frame in both asymptotic limits, \( u \to \pm \infty \). Therefore, in the terminology used in the literature, the ordinary memory vanishes. Recently, this implication of asymptotic stationarity of the system in the same Bondi
frame as $u \to \pm \infty$ was arrived at using detailed calculations in the physical spacetime
frame as $u \to \pm \infty$ was arrived at using detailed calculations in the physical spacetime
[51]. Analysis at $I^+$ summarized above brings out the minimal setup needed to obtain
this result:

(i) One only needs that the past and future rest frames are the same, and $\partial_u \Psi_1 \to 0$
as $u \to \pm \infty$ along $I^+$ in this Bondi frame. There is no explicit assumption on
the underlying metric and, at the level of curvature, $\Psi_0$ does not have to become
stationary;

(ii) In full GR, the result follows in a couple of steps if one uses Eqs (2.16) and (2.17)
at $I^+$, which are immediate consequences of Bianchi identities and Einstein’s
equations at $I^+$ [6].

Finally, note that we do allow kicks in our main analysis; our assumption (2) does not
require that the past and the future rest frame is the same and the ordinary memory
does not vanish.

6. The PN assumption of ‘past stationarity’ [31, 32]—i.e., that the system is stationary
before some time $t = -\tau$—is extremely strong. Sometimes one models the behavior
of the system in distant past by assuming that it is represented by a binary in a
quasi-circular orbit, rather than by a stationary system. In this case, velocities of the
individual bodies would go to zero asymptotically and our two assumptions would still
be satisfied. In practice, one may want to avoid the limit $u \to -\infty$ altogether and
simply start at some finite time $u = u_1$ sufficiently in the past when the binary is well
within the PN regime, and use (2.25) in place of (2.21) with $u_2 = \infty$.

III. DISCUSSION

In this paper we showed that the balance laws of supermomentum that follow from ex-
act GR have interesting implications for gravitational waveforms from CBCs. As Appendix
A explains, waveforms are obtained by combining inputs from PN methods, NR simula-
tions, EOB strategy, phenomenological models and interpolation methods. Each step in the
procedure involves approximations and choices, and the procedure used to stitch together
analytical and numerical results has several ambiguities. Nonetheless, thanks to the ingenu-
ity and the combined technical expertise of the community, the matched filtering procedure
based on these waveforms has led to the dramatic discovery of several CBCs.

The pace of detection has gone up considerably already during the current LIGO-Virgo
observation run and, with a global network of gravitational wave observatories, the detection
rate could be as high as 1000 binary black holes coalescences a year, with masses below
100$M_\odot$. LISA and the third generation ground based observatories will further increase the
total rate and widen the parameter range significantly. This abundance of data could be
used to arrive at much more accurate source characterization, as well as more stringent tests
of GR. Therefore, it is important to have external criteria to ensure that the waveforms
used in the analysis do represent predictions of GR to a very high degree of accuracy. As
we remarked in the Introduction, the conceptual difficulty is that we do not have waveforms
from exact GR with which to compare the hybrid waveforms in the template bank! Therefore
errors are generally estimated by comparing the PN and/or other analytic waveforms with
the NR ones, which are taken to be the ‘practical substitutes’ for the exact waveforms (see e.g. [24]). In Section II we showed that the supermomentum balance laws provide an avenue to overcome this limitation: One can test the accuracy of a proposed waveform vis a vis exact GR using Eq. (2.21) or, more generally, Eq. (2.25). These equations impose an infinite number of constraints on the full waveform that must be satisfied in exact GR. Therefore violations of these constraints provides sharp error-bars on any waveform candidate.

One can use (2.21) in three ways. First, one can test individual steps in the procedure used. For example, we can consider the more local version (2.25) of the balance law just for the PN part of the waveform to test its accuracy. Similarly, one can test the accuracy of the NR estimate of the final parameters — generally calculated from the final horizon geometry. For this, one can use the full hybrid waveform, together with the PN values of the initial total mass $M^+_{\text{in}}$, to calculate the final mass $M^+_{\text{f}}$ using global balance laws (2.21) for a large number of values of $(\theta, \varphi)$. Second, as discussed above, one can test the accuracy of the entire procedure using (2.21) for multiple values of $(\theta, \varphi)$. In particular, as explained in Appendix A, the NR waveforms are given only for the leading ‘2-2’ mode and a few subsequent modes. What is the error one makes by considering just the 2-2 mode versus the full waveform? One can use the 2-2 mode waveform in (2.21) and calculate the resulting violation of constraints coming from full GR. Finally, one can use the global or the local balance laws to discriminate between various choices that have been made to resolve ambiguities – the PN ‘approximants, the PN order of truncation, the radius at which waveforms are extracted in numerical simulations, truncation errors in NR, the time or frequency interval over which the two waveforms are to be matched, the way PN parameters and NR parameters are matched, phenomenological choices made in [18], conceptual simplifications made in [14], . . . . Making the choices that minimize the violation of the infinite set of constraints imposed by GR could significantly enhance the accuracy of waveforms admitted to the template bank. Finally, since each element that goes in the balance law (2.11) is readily available for each waveform, it should be straightforward to write a single program with those elements as inputs to test the accuracy of any number of waveforms.

Next, let us discuss the two assumptions on which the main results of sections II rest. The first has direct physical motivation: We assume that the Bondi news –the time derivative of the waveform– decays sufficiently rapidly so that the total flux of supermomentum across $\mathcal{I}^+$ is well defined. (This is also the necessary and sufficient condition for the flux of Bondi angular momentum across $\mathcal{I}^+$ to be finite. The condition is satisfied in, e.g., the analysis of non-linear stability of Minkowski space given in [11].) This condition guarantees that the system becomes asymptotically stationary in the weak sense that time derivatives of the components $\Psi_4^o, \Psi_3^o$ and $\Psi_2^o$ of the asymptotic Weyl curvature go to zero in any Bondi frame as $u \to \pm \infty$. (If these conditions hold in one Bondi frame, they hold in all.) The second condition we imposed is a slight strengthening of this notion of asymptotic stationarity: We require that the time derivative of $\Psi_1^o$ also goes to zero as $u \to \pm \infty$. But now the limits depend on the choice of the Bondi-frame, whence the requirement in the past is imposed only in the Bondi frame in which the system is at rest in distant past – i.e. the limit of the Bondi 3-momentum vanishes as $u \to -\infty$ – and similarly the requirement in the future is imposed only in the Bondi frame in which the system is at rest in distant future. We saw that, had we demanded the time derivative of $\Psi_1^o$ vanishes as $u \to \pm \infty$ in the same Bondi-frame, the condition would have been physically too restrictive – in particular it would have ruled out black hole kicks [43, 44]! By contrast, our condition requires asymptotic stationarity in an extremely weak sense. In particular, PN analysis used in the early phase of coalescence
assumes a much stronger notion and NR simulations bear out that our assumption is satisfied in the distant future. One can even avoid having to take limits $u \to \pm \infty$ altogether by using (2.25) in place of (2.21).

Finally we would like to clarify a potential confusion about the field $\Psi_1^0$ on $\mathcal{I}^+$. Global results by Christodoulou and Klainermann \cite{11} on non-linear stability of Minkowski space showed that $\Psi_4, \Psi_3$ and $\Psi_2$ ‘peel’ as $1/r$, $1/r^2$ and $1/r^3$ respectively, as implied by the Newman-Penrose asymptotic conditions \cite{6}. However, in that analysis, $\Psi_1$ is guaranteed to fall-off only as $1/r^{7/2}$, rather than as $1/r^4$. Therefore, it follows from Eq. (2.6) that $\Psi_1^0$ need not exist on $\mathcal{I}^+$ for the class of initial data considered in Ref. \cite{11}. Note that this is not a statement about limits $u \to \pm \infty$ along $\mathcal{I}^+$ considered in this paper; rather $\Psi_1^0$ need not exist anywhere on $\mathcal{I}^+$. In particular, this means that the angular momentum of the system would be infinite at any retarded time $u = u_o$ \cite{28, 29}. From a physical viewpoint, therefore, this analysis caters to too broad a class of systems. In CBC, in particular, angular momentum is well-defined and constitutes an important parameter in the system characterization. Indeed, even in the vacuum case to which Ref. \cite{11} restricts itself to, Chrusciel and Delay \cite{12} have shown that there is a non-linear neighborhood of Minkowski initial data that evolves to a unique global solution in which the Newman-Penrose peeling holds, i.e. $\Psi_1^0$ has a well-defined $C^k$ limit to $\mathcal{I}^+$. Finally, note that all these global results refer to vacuum (or electrovac) situations in which there is outgoing as well as incoming gravitational radiation. Physically we are much more interested in gravitational waves produced by sources --such as compact binaries— with no incoming radiation on $\mathcal{I}^-$. Implications to such systems of the global existence and uniqueness results for vacuum solutions are unclear because the only vacuum solution with no incoming radiation is Minkowski spacetime!

\section*{Acknowledgments}

We would like to thank K. G. Arun, Anuradha Gupta, and B. Sathyaprakash for comments on the first draft, Bala Iyer for extensive correspondence on the PN approximation and participants of the APS meeting in Denver for questions and suggestions. This work was supported in part by the NSF grant PHY-1806356, grant UN2017-92945 from the Urania Stott Fund of Pittsburgh Foundation and the Eberly research funds of Penn State.

\section*{Appendix A: Waveforms}

In this Appendix we summarize the procedure used to create waveforms through analytical methods and numerical simulations, emphasizing the conceptual aspects and key assumptions and approximations. This is only a bird’s eye view addressed to mathematical physicists. Knowledge of this material is necessary to appreciate why non-trivial checks on waveforms are needed and how this purpose is served by results of Section II.

The main focus of the waveform community has been on the part of CBC that is directly relevant to the sensitivity band of the current gravitational wave detectors. For quasi-circular motion this translates to $\sim 100$ orbits where dynamics is expected to be well-modeled by the slow motion approximation of PN expansions, and the last $\sim 10 - 15$ orbits for which dynamics must incorporate strong field effects of full GR where NR simulations are necessary. In principle, one could use NR for the entire process. However, the required computational time and effort would be too large, given that we need to cover an 8 (or greater) dimensional
parameter space associated with the binary. That is why a stitching procedure is used, where the early waveform comes from the PN analysis and the late waveform from numerical simulations. The result is often referred to as the \textit{hybrid} waveform. In addition, a number of strategies –the effective one body (EOB) method \cite{14–17}, phenomenological interpolation \cite{18–22}, grid-based parameter estimation method \cite{52}, NR-surrogate models \cite{23}, etc– have been used to enhance the reach of analytical waveforms, and/or to interpolate between parameters used in numerical simulations to create a large bank of waveforms. Thus, while currently there are a few thousand CBC numerical simulations, the template banks contain $\sim 100$ times as many waveforms. The full bank is currently used by the LIGO-Virgo collaboration for detection, parameter estimation, and testing GR. (For further details, see e.g., the review articles \cite{32, 53, 54} and references therein.)

Various steps in this process involve approximations, guesses based on intuition, and choices that are necessary to resolve ambiguities. Let us begin with the PN expansion. This is an expansion of physical quantities in small velocity –truncated to various $v/c$ orders– which however is not convergent; it is at best an asymptotic series. For example for luminosity of gravitational waves in the extreme mass limit, the PN expansion starts to deviate significantly from the exact result for $v/c \gtrsim 0.2$, and the contributions up to $(v/c)^4$ and $(v/c)^5$ terms do so in opposite directions \cite{46}. Consequently, even when one can carry out calculations to a high order, it is not easy to systematically control the truncation errors. A second issue goes under the name of \textit{Taylor approximants}. The post-Newtonian waveforms are obtained starting from the PN expansions of the energy of the system $E(v/c)$ and the flux of radiated energy $F(v/c)$. However because the procedure involves rational –rather than polynomial– functionals of $E(v/c)$ and $F(v/c)$, there is some freedom in expanding out these quantities to obtain the waveform to a given PN order. Because of this freedom, several different PN waveforms arise at a given order; this is the so-called ‘ambiguity in the choice of Taylor approximants.’ For unequal masses, this is generally the largest source of errors in the PN waveforms (see, e.g., \cite{24, 46}). Finally, in the PN literature, there is a fixed background Minkowski space at all orders and the PN solution is assumed to be \textit{stationary} in the past, before some time $t < -\tau$ \cite{31, 32, 48, 49}. This assumption would seem unreasonably strong to mathematical relativists since for sources for which the initial value problem is well posed in full general relativity, if a solution is stationary in the past in this strong sense, then it is stationary everywhere. However, in the PN strategy the system is non-stationary in the future due to radiation reaction effects and the assumption of past-stationary primarily serves to make various tail terms finite. \vspace{0.5em}

In NR we encounter different types of errors. First, there are the truncation errors that are common to all numerical simulations. Second, the waveform is extracted at a large but finite radius, whereas the radiation field becomes truly gauge invariant and unambiguous only at infinity. Therefore the results inherit error-bars associated with the choice of extraction radius \cite{45}. Third, the waveform is obtained by integrating twice with respect to time the radiation field encoded in the component $\Psi_4$ of the Weyl tensor. This requires introduction of coordinate systems and null tetrads which also become unambiguous only at infinite distance from sources. Finally, although one does have tools to calculate full $\Psi_4$ (modulo the ambiguities inherent in working at a finite radius), there are numerical errors due to

\footnote{In Section II C, we have used a \textit{much} weaker condition, where past stationarity holds in a limiting sense as one goes to past infinity \textit{along} $I^+$, and that too only for a certain field. The assumption is mild perfectly compatible with non-stationary solutions in full GR.}
high frequency oscillations which are suppressed if one calculates only the first few (spin-weighted) spherical harmonics because of the ‘averaging’ involved. Therefore, only the most dominant modes are generally reported in the NR results, rather than the full waveform.

The stitching procedure is inherently ambiguous because it involves several choices (see, e.g., [47]). First, one has to decide at what stage in the CBC evolution one stitches the PN and the NR waveforms. Second, one has to decide which PN order and which T-approximant to use. Third, the PN and the NR waveforms are generally computed using different coordinate systems and therefore one has to introduce additional inputs for a meaningful matching. These choices are driven by intuition and guided by past experience rather than clear-cut, unambiguous mathematical physics procedures. Next, because the PN expansion and NR simulation are based on quite different conceptual frameworks, there are several ad-hoc elements involved. In PN calculations, the sources are taken to be point particles in Minkowski space. In NR, there is no background Minkowski space and black holes are represented by dynamical horizons (and neutron stars by suitable fluids). In the case of black holes, the individual masses and spins are determined by the horizon geometry. Therefore, for the stitching procedure, one starts with a controlled set of NR initial data (given by the Bowen-York [55] or the Brandt-Brügmann [56] strategy) satisfying constraints of exact GR and evolves. Now, these data contain some ‘spurious radiation’ which escapes the grid quickly. After this occurs, one re-evaluates the source parameters in the numerical solution and matches them with the source parameters of the PN solution. One then chooses an interval in the time or the frequency domain and evolves both the PN and NR solutions and compares their waveforms. There are several ways to ‘measure’ the difference between the two waveforms and one minimizes it by tweaking the time or frequency of matching, the interval over which the matching is done, and the choice of source parameters in the two schemes. Conceptually, it is important to note that the matching is done only for the waveform – i.e., for the two asymptotic forms of the metric that capture the radiative modes in the two schemes. In the interior, there is no obvious correspondence between the PN and NR solutions. In particular, there is no simple relation between the ‘particle trajectories’ representing the black holes, determined by the PN equations, and the dynamical horizons determined by numerical simulations.

These considerations make it clear that even for the fraction of waveforms in the template banks that are obtained just from PN and NR, we do not yet have a systematic way to measure how well they agree with the predictions of exact GR. Inputs that go into the construction of the rest of the template bank are even less driven by fundamental considerations. It is a tribute to the physical intuition and technical ingenuity behind these hybrid waveforms, that the matched-filtering procedure could lead to detections of coalescing binaries.

Appendix B: Null infinity

In Section II we used several known facts about the structure of null infinity and properties of fields thereon. Since some in the PN and NR communities may not be familiar with them, in this Appendix we present a brief summary, focusing only on those features that we need.

Even though Einstein, Eddington and others explored the properties of gravitational waves in the weak field approximation around Minkowski space soon after the discovery of general relativity, there was considerable confusion about the reality of gravitational waves in full, nonlinear GR for several subsequent decades largely because of the coordinate freedom:
What appeared to be a wave-like behavior in one coordinate system could appear stationary in another. This confusion was resolved only in the 1960s when Bondi, Sachs and others showed that one can unambiguously disentangle gravitational waves by moving away from isolated sources in retarded null directions, i.e., in the usual terminology, by taking the limit \( r \to \infty \) keeping the retarded time constant \( u = \text{const} \). The asymptotic boundary conditions introduced by Bondi and Sachs \([1, 2]\) were geometrized by Penrose \([5]\) through the notion of a conformal completion of spacetime, i.e., by attaching to spacetime a 3-dimensional boundary \( \mathcal{I}^+ \), representing ‘future null infinity’.

These frameworks provided a definitive, coordinate invariant characterization of gravitational radiation in asymptotically flat spacetimes and introduced techniques to analyze its properties in exact, non-linear general relativity. However, initially there were concerns as to whether the underlying assumptions are too strong to be satisfied by realistic isolated systems such as compact binaries (see, e.g., \([31]\)). The current consensus is that they are not too strong. In particular, the asymptotic form of the PN metric is completely consistent with the Bondi-Sachs-Penrose framework, as shown for instance by Theorem 4 in \([32]\). Similarly, the key notions of this framework –such as the radiation field \( \Psi_4^0 \), the Bondi news \( N \), and the asymptotic shear \( \sigma^0 \)– and their properties are heavily used in numerical simulations of waveforms and calculations of energy and momentum flux in NR. These notions and properties are summarized in Section II.

The detailed analysis of gravitational radiation at null infinity brought to forefront an unforeseen result that plays a key role in Section II: Even though spacetimes representing isolated gravitating systems are asymptotically Minkowskian, the asymptotic symmetry group is \textit{not} the Poincaré group \( \mathcal{P} \), but rather an infinite dimensional generalization thereof, the BMS group \( \mathfrak{B} \). This is a consequence of the fact that gravitational radiation is on a really different footing from, say, the electromagnetic one, in one important respect. It introduces ripples in spacetime curvature that extend all the way to infinity, i.e., \( \mathcal{I}^+ \), making it impossible to single out a \textit{preferred} \( \mathcal{P} \) using asymptotic Killing vectors. This difficulty can be seen in concrete terms as follows. Suppose we have a metric \( g_{ab} \) that is asymptotically flat in the sense of Bondi and Sachs. So it approaches a Minkowski metric \( \eta_{ab} \) as \( r \to \infty \) keeping \( u = t - r \) constant. Therefore, Poincaré transformations of \( \eta_{ab} \) provide us with asymptotic Killing fields for \( g_{ab} \). Now consider a diffeomorphism \( t \to t' = t + f(\theta, \varphi), \quad \vec{x} \to \vec{x}' = \vec{x} \) where \( t, \vec{x} \) are Cartesian coordinates of \( \eta_{ab} \). This is an angle dependent translation, whence the metric \( \eta_{ab} \) is sent to a \textit{distinct} flat metric \( \eta'_{ab} \).\(^6\) One can verify that since \( g_{ab} \) approaches \( \eta_{ab} \) as \( 1/r \) à la Bondi-Sachs, it also approaches \( \eta'_{ab} \) as \( 1/r' \) à la Bondi-Sachs! Therefore, the Poincaré transformations of \( \eta'_{ab} \) are also asymptotic Killing fields of our physical \( g_{ab} \). But since the two Minkowski metrics are distinct, their isometry groups \( \mathcal{P} \) and \( \mathcal{P}' \) are also distinct. The BMS group can be interpreted as a ‘consistent union’ of Poincaré groups associated with all these Minkowski metrics, related to one another by ‘angle dependent translations.’ These are known as \textit{supertranslations}. Detailed examination brought out another subtlety: All these Poincaré groups define the same translation subgroup asymptotically, whence the BMS group does admit a canonical, 4-dimensional translation subgroup \( \mathcal{T} \).\(^3\) However, the Lorentz subgroups \( \mathfrak{L} \) of various Poincaré groups are different even asymptotically. Recall that the Poincaré group \( \mathcal{P} \) admits a 4-parameter family of Lorentz subgroups –each of which defines rotations and boosts about one specific origin in Minkowski space– related to

\(^6\) We chose a time-translation just for definiteness: the argument continues to hold if the ‘angle dependent translation’ is space-like or null.
one another by a translation. By contrast, the BMS group $\mathfrak{B}$ admits an infinite parameter family of Lorenz subgroups that are related to one another by supertranslations. This gives rise to the well-known ‘supertranslation ambiguity’ in the notion of angular momentum at null infinity. We discuss this issue in [26], again in the context of CBC.

In this paper we focus on supertranslations. Just as the translational symmetries of the Minkowski metric lead to the notion of energy-momentum for fields in Minkowskian physics, supertranslation symmetries on $\mathcal{I}^+$ lead to the notion of supermomenta. In the case of energy-momentum, we have two different quantities available at $\mathcal{I}^+$. The first is the Bondi 4-momentum—a 2-sphere integral on a cross-section $C$ at $\mathcal{I}^+$, representing the energy momentum of the system, left over at the retarded instant of time $u = u_0$ defined by $C$. The second is the notion of flux of energy-momentum carried away by gravitational waves through a ‘patch’ $\Delta \mathcal{I}^+$ of $\mathcal{I}^+$. As a consequence we have a balance law: The difference between the Bondi 4-momentum evaluated on two different cross-sections $C_1$ and $C_2$ of $\mathcal{I}^+$ is the flux of energy-momentum across the patch $\Delta \mathcal{I}^+$ bounded by them. It turns out that the same is true for supermomentum. Thus we have an infinite number of balance laws, Eq. (2.23) in the main text, each characterized by a function on a 2-sphere defining the supertranslation. As we discuss in Section II, these lead to an infinite set of constraints—imposed by full, non-linear GR— that any waveform must satisfy in a CBC.

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