THE PRECISE SHAPE OF THE EIGENVALUE INTENSITY FOR A CLASS OF NON-SELFADJOINT OPERATORS UNDER RANDOM PERTURBATIONS

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Abstract. We consider a non-selfadjoint $h$-differential model operator $P_h$ in the semi-classical limit ($h \to 0$) subject to small random perturbations. Furthermore, we let the coupling constant $\delta$ be $e^{-\frac{1}{Ch}} \leq \delta \ll h^n$ for constants $C, \kappa > 0$ suitably large. Let $\Sigma$ be the closure of the range of the principal symbol. Previous results on the same model by Hager, Bordeaux-Montrieux and Sjöstrand show that if $\delta \gg e^{-\frac{1}{Ch}}$ there is, with a probability close to 1, a Weyl law for the eigenvalues in the interior of the pseudospectrum up to a distance $\gg (-h \ln \delta h)^{\frac{2}{3}}$ to the boundary of $\Sigma$.

We study the intensity measure of the random point process of eigenvalues and prove an $h$-asymptotic formula for the average density of eigenvalues. With this we show that there are three distinct regions of different spectral behavior in $\Sigma$: The interior of the pseudospectrum is solely governed by a Weyl law, close to its boundary there is a strong spectral accumulation given by a tunneling effect followed by a region where the density decays rapidly.

Résumé. Nous considérons un opérateur différentiel non-autoadjoint $P_h$ dans la limite semiclassique ($h \to 0$) soumis à de petites perturbations aléatoires. De plus, nous imposons que la constant couplage $\delta$ vérifie $e^{-\frac{1}{Ch}} \leq \delta \ll h^n$ pour certaines constantes $C, \kappa > 0$ choisies assez grandes. Soit $\Sigma$ l’adhérence de l’image du symbole principal de $P_h$. De précédents résultats par Hager, Bordeaux-Montrieux and Sjöstrand montrent que, pour le même opérateur, si l’on choisit $\delta \gg e^{-\frac{1}{Ch}}$, alors la distribution des valeurs propres est donnée par une loi de Weyl jusqu’à une distance $\gg (-h \ln \delta h)^{\frac{2}{3}}$ du bord de $\Sigma$.

Dans cet article, nous donnons une formule $h$-asymptotique pour la densité moyenne des valeurs propres en étudiant le mesure de comptage aléatoire des valeurs propres. En étudiant cette densité, nous prouvons qu’il y a une loi de Weyl à l’intérieur du pseudospectre, une zone d’accumulation des valeurs propres dû à un effet tunnel près du bord du pseudospectre suivi par une zone où la densité décroit rapidement.

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1. INTRODUCTION

Let \( h > 0 \), we consider the semi-classical operator \( P_h : L^2(S^1) \to L^2(S^1) \) as defined by Hager in [5] given by

\[
P_h := hD_x + g(x), \quad D_x := \frac{d}{dx}, \quad S^1 = \mathbb{R}/2\pi \mathbb{Z}
\]

where \( g \in \mathcal{C}^\infty(S^1; \mathbb{C}) \) is such that \( \text{Im } g \) has exactly two critical points, one minimum and one maximum, say in \( a \) and \( b \), with \( a < b < a + 2\pi \) and \( \text{Im } g(a) < \text{Im } g(b) \). Without loss of generality we may assume that \( \text{Im } g(a) = 0 \). As the natural domain of \( P_h \) we shall take the semiclassical Sobolev space

\[
H^1(S^1) := \left\{ u \in L^2(S^1) : \left( \|u\|^2 + \|hD_x u\|^2 \right)^{\frac{1}{2}} < \infty \right\},
\]

where \( \|\cdot\| \) denotes the \( L^2 \)-norm on \( S^1 \) if nothing else is specified. We will use the standard scalar products on \( L^2(S^1) \) and \( \mathcal{C}^N \) defined by

\[
(f|g) := \int_{S^1} f(x)\overline{g(x)}dx, \quad f, g \in L^2(S^1),
\]

and

\[
(X|Y) := \sum_{i=1}^{N} X_i \overline{Y_i}, \quad X, Y \in \mathcal{C}^N.
\]

Let us denote the semi-classical principal symbol of \( P_h \) by

\[
p(x, \xi) = \xi + g(x), \quad (x, \xi) \in T^* S^1.
\]

(1.2)

The spectrum of \( P_h \) is discrete with simple eigenvalues, given by

\[
\sigma(P_h) = \{ z \in \mathbb{C} : \ z = \langle g \rangle + k\hbar, \ k \in \mathbb{Z} \},
\]

where \( \langle g \rangle := (2\pi)^{-1} \int_{0}^{2\pi} g(y)dy \). Next, consider the equation \( z = p(x, \xi) \). It has precisely two solutions \( \rho_{\pm} := (x_{\pm}, \xi_{\pm}) \) where \( x_{\pm} \) are given by \( \text{Im } g(x_{\pm}) = \text{Im } z, \ \pm \text{Im } g'(x_{\pm}) < 0 \) and \( \xi_{\pm} = \text{Re } z - \text{Re } g(x_{\pm}) \). By the natural projection \( \Pi : \mathbb{R} \to S^1 = \mathbb{R}/2\pi \mathbb{Z} \) and a slight abuse of notation we identify the points \( x_{\pm} \in S^1 \) with points \( x_{\pm} \in \mathbb{R} \) such that \( x_- - 2\pi < x_+ < x_- \). Furthermore, we will identify \( S^1 \) with the interval \( [x_- - 2\pi, x_-] \). The Poisson bracket of \( p \) and \( \overline{p} \) is given by

\[
\{ p, \overline{p} \} = p'_x \cdot \overline{p'_x} - p'_x \cdot \overline{p'_x}.
\]

Adding a random perturbation. Let us consider a random perturbation of \( P_h \) given by

\[
P^\delta_h := P_h + \delta Q_\omega := hD_x + g(x) + \delta Q_\omega,
\]

where \( \delta > 0 \) and \( Q_\omega \) is an integral operator \( L^2(S^1) \to L^2(S^1) \) of the form

\[
Q_\omega u(x) := \sum_{|j|,|k| \leq \left\lfloor \frac{C_1}{\delta} \right\rfloor} \alpha_{j,k}(u|e^k) e^j(x).
\]

Here \( |x| := \max\{ n \in \mathbb{N} : \ x \geq n \} \) for \( x \in \mathbb{R}, \ C_1 > 0 \) is big enough, \( e^k(x) := (2\pi)^{-1/2} e^{ikx} \), \( k \in \mathbb{Z} \), and \( \alpha_{j,k} \) are complex valued independent random variables with complex Gaussian distribution law \( \mathcal{N}_{C}(0, 1) \).

Pseudospectrum. Due to the in general bad resolvent control for non-self-adjoint operators, the spectrum is highly unstable, meaning that the eigenvalues can move a lot even when subject to small perturbations.

A major tool in studying this spectral instability is provided by the so-called \( \varepsilon \)-pseudospectrum which consists of the regions where the resolvent is large and thus indicates how far the
eigenvalues can spread under perturbations. Following the work by Trefethen and Embree [10], it can be defined by

**Definition 1.1.** Let $A$ be an closed linear operator on a Banach space $X$ and let $\varepsilon > 0$ be arbitrary. Then, the $\varepsilon$-pseudospectrum $\sigma_\varepsilon(A)$ of $A$ is defined by

$$\sigma_\varepsilon(A) := \left\{ z \in \mathbb{C} : \| (z - A)^{-1} \| > \frac{1}{\varepsilon} \right\} \cup \sigma(A),$$

or equivalently

$$\sigma_\varepsilon(A) = \bigcup_{B \in \mathcal{B}(X)} \sigma(A + B),$$

or equivalently

$$z \in \sigma_\varepsilon \iff z \in \sigma(A) \text{ or } \exists u \in D(A), \| u \| = 1 \text{ s.t.: } \| (z - A)u \| < \varepsilon.$$

The last condition also implicitly defines the so-called $\varepsilon$-pseudoeigenvectors or quasi-modes. Since in the present work we are in the semiclassical setting, we follow [3] and define

**Definition 1.2.** Let $p(x, \xi)$ be the semiclassical principal symbol of the operator $P_h$ as in (1.2). Then, we define

$$\Sigma := \overline{p(T^*S^1)} \subset \mathbb{C}.$$

In the case of (1.1) and (1.2) $p(T^*S^1)$ is already closed due to the ellipticity of $P_h$.

**Previous results.** The operator $P_h$ and small perturbations of it (deterministic and random) have first been studied by Hager [5] followed by works by Bordeaux-Montrieux [1] and Sjöstrand [8]. In [5] Hager obtained that the eigenvalue distribution of random perturbations of $P_h$ in the interior of $\Sigma$ is given by a Weyl law with a probability close to one.

In [1] Bordeaux-Montrieux extended Hager’s result to strips at a distance $\gg (-h \ln \delta h)^{\frac{2}{3}}$ to the boundary of $\Sigma$. In both cases, the results concern only the interior of the pseudospectrum, thus avoiding an accumulation of eigenvalues effect close to the boundary of the spectrum. This effect can be seen in numerical simulations for the model $P_h$, see Figure 2. It has furthermore been noted in numerical simulations for other models, e.g. in the case of Toeplitz quantization considered in [11].

To the best of the author’s knowledge, there has never been, until now, a precise description of this phenomenon. This leads to the main question treated in this paper: We want to study the distribution of eigenvalues of a random perturbation of the operator $P_h$ in the whole of $\Sigma$. In particular this means studying regions where the norm of the resolvent of the unperturbed operator $P_h$ is much larger, of the same order of magnitude and much smaller than the coupling constant $\delta$.

**Quasimodes.** A major tool will be quasimodes for the unperturbed operator $P_h$. However, we will need to distinguish between the following two cases for $z \in \Sigma$:

- $z \in \Omega \subset \Sigma$ independent of $h$. By [3] we can construct quasimodes for $z$ in the interior of $\Sigma$, i.e. for $z \in \Omega \subset \Sigma$ independent of $h$ with $\operatorname{dist} (\Omega, \partial \Sigma) > 1/C$, there exist $u_h \in D(P_h)$ such that

$$\| (P_h - z)u_h \| < O \left( e^{-\frac{1}{C^2}} \right) \| u_h \|$$

and with the semiclassical wave front set given by $\operatorname{WF}_h(u_h) = \{(x, \xi)\}$, where $p(x, \xi) = z$. To achieve this we shall follow two different approaches (see Section 4): the one by Hager in [5] and the one by Sjöstrand in [8].


constant $C > 0$ we mean implicitly that $d$ denote density of eigenvalues of the randomly perturbed operator.

Remark 1.3

Outline. The principal aim of this work is to give a detailed description of the average eigenvalues of the randomly perturbed operator $P_h$. Section 2 will state our main results: we shall state an equivalent of the perturbed operator $P_h$ and to its link with the symplectic volume of the phase space. Section 7 will state additional results to prove a formula for the first intensity measure of the random point process counting the eigenvalues of $P_h$ which then will be proven in Section 8. Sections 6 to 10 will prove the main results.

- $z$ is close to the boundary of $\Sigma$. Since for all $z_0 \in \partial \Sigma$ and all $\rho \in p^{-1}(z_0)$ the distance of $z_0$ to $p(\rho)$ is greater than zero in Theorem 1.1 given in [7] implies that there exists a constant $C_0 > 0$ such that for every constant $C_1 > 0$ there is a constant $C_2 > 0$ such that for $|z - z_0| < C_1(h \ln \frac{1}{h})^{2/3}$, $h < \frac{1}{C_2}$, the resolvent $(P_h - z)^{-1}$ is well defined and satisfies

$$||(P_h - z)^{-1}|| < C_0 h^{-\frac{2}{3}} \exp \left(\frac{C_0}{h} |z - z_0|^{\frac{3}{2}}\right).$$

This implies:

1. Using the more precise resolvent estimates given in [7], we have for $\delta$ as in Definition 2.5, that

$$\sigma(P_h + \delta Q_\omega) \cap D\left(z_0, C_1 \left(h \ln \frac{1}{h}\right)^{2/3}\right) = \emptyset. \quad (1.5)$$

Thus, there exists a tube of radius $C_1 \left(h \ln \frac{1}{h}\right)^{2/3}$ around $\partial \Sigma$ void of the spectrum of the perturbed operator $P_h + \delta Q_\omega$.

2. It is reasonable to expect that as we approach the boundary of $\Sigma$ the quasi-modes become much less precise. In this case we shall follow the quasi-mode construction introduced by Bordeaux-Montrieux in [1] which generalizes Hager’s construction of quasimodes for $z \in \Omega \subset \Sigma$ with $\text{dist}(\Omega, \partial \Sigma) \gg C h^{2/3}$. Thus, since we are interested in the eigenvalue distribution of $P_h^\delta$, we may from now on assume implicitly that $\text{dist}(\Omega, \partial \Sigma) \gg C h^{2/3}$, for some constant $C > 0$, when considering $\Omega \subset \Sigma$.

Remark 1.3. Throughout this work we shall denote the Lebesgue measure on $\mathcal{C}$ by $L(dz)$; denote $d(z) := \text{dist}(z, \partial \Sigma)$; work with the convention that when we write $O(h)^{-1}$ then we mean implicitly that $0 < O(h) \leq C h$; denote by $f(x) \asymp g(x)$ that there exists a constant $C > 0$ such that $C^{-1} g(x) \leq f(x) \leq C g(x)$; write $\chi_1(x) \asymp \chi_2(x)$, with $\chi_i \in C^\infty_0$, if $\text{supp} \chi_2 \subset \text{supp} (1 - \chi_1)$.

Outline. The principal aim of this work is to give a detailed description of the average density of eigenvalues of the randomly perturbed operator $P_h^\delta$.

Section 2 we shall present our main results: we shall state an $h$-asymptotic formula for the average density of eigenvalues and describe its properties. We will show that the spectrum of $P_h^\delta$ is be distributed, in average, in a band $\subset \Sigma$ which breadth depends on the strength of the coupling constant.

In the interior of this band we will establish a Weyl law for the eigenvalues whereas they exhibit a strong accumulation property close to the boundary of this band. Outside of this band the average density of eigenvalues exhibits a double exponential decay. However, the integral of the average density of eigenvalues with respect to $\text{Im} z$ will still be determined by a Weyl law.

Section 3 will give constructions of quasimodes for $z$ in the interior of $\Sigma$ and close to the the boundary $\partial \Sigma$. In Section 4 we treat the needed Grushin problems for the operator $P_h$. Section 5 is dedicated to a Grushin problem for the perturbed operator $P_h^\delta$ and to its link with the symplectic volume of the phase space. Section 7 will state additional results to prove a formula for the first intensity measure of the random point process counting the eigenvalues of $P_h^\delta$ which then will be proven in Section 8. Sections 6 to 10 will prove the main results.
2. Main results

First, we will establish how to chose the strength of the perturbation.

2.1. The coupling $\delta$. To define the following function will be important:

**Definition 2.1.** Let $\Omega \subset \Sigma$, let $p$ denote the semiclassical principal symbol of $P_h$ in (1.2) and let $\rho_{\pm}(z) = (x_{\pm}(z), \xi_{\pm})$ be as above. Define

$$S := \min \left( \Im \int_{x_{\pm}}^x (z - g(y))dy, \Im \int_{x_{\pm}}^{x-2\pi} (z - g(y))dy \right).$$

**Proposition 2.2.** Let $\Omega \subset \Sigma$ and let $S(z)$ be as in Definition 2.1, then $S(z)$ has the following properties for all $z \in \Omega$:

- $S(z)$ depends only on $\Im z$, is continuous and has the zeros $S(\Im g(a)) = S(\Im g(b)) = 0$;
- $S(z) \geq 0$;
- for $\Im z = \langle \Im g \rangle$ the two integrals defining $S$ are equal; $S$ has its maximum at $\langle \Im g \rangle$ and is strictly monotonously decreasing on $[\langle \Im g \rangle, \Im g(b)]$ and strictly monotonously increasing on $[\Im g(a), \langle \Im g \rangle]$;
- its derivative is piecewise of class $C^\infty$ with the only discontinuity at $\Im z = \langle \Im g \rangle$.

Moreover,

$$S(z) = \int_{\langle \Im g \rangle}^{\Im z} (\partial_{\Im z} S)(t)dt + S(\langle \Im g \rangle),$$

where

$$(\partial_{\Im z} S)(t) := \begin{cases} x_-(t) - x_+(t), & \text{if } \Im z \leq \langle \Im g \rangle, \\ x_-(t) - 2\pi - x_+(t), & \text{if } \Im z > \langle \Im g \rangle. \end{cases} \quad (2.1)$$

- $S$ has the following asymptotic behavior for $z \in \Omega$

$$S(z) \asymp d(z)^{\frac{3}{2}},$$

and

$$|\partial_{\Im z} S(z)| \asymp d(z)^{\frac{1}{2}}.$$

**Remark 2.3.** Note that in (2.1) we chose to define $\partial_{\Im z} S(z) := x_-(z) - x_+(z)$ for $\Im z = \langle \Im g \rangle$. We will keep this definition throughout this text.

With the convention $\| (P_h - z)^{-1} \| = \infty$ for $z \in \sigma(P_h)$ we have the following estimate on the resolvent growth of $P_h$:

**Proposition 2.4.** Let $g(x)$ be as above. For $z \in \mathbb{C}$ and $h > 0$ define,

$$\Phi(z, h) := \begin{cases} -\frac{2\pi i}{h} (z - \langle g \rangle), & \text{if } \Im z < \langle \Im g \rangle, \\ \frac{2\pi i}{h} (z - \langle g \rangle), & \text{if } \Im z > \langle \Im g \rangle, \end{cases}$$

where $\Re \Phi(z, h) \leq 0$. Then, under the assumptions of Definition 2.1 we have for $z \in \Sigma$

$$\| (P_h - z)^{-1} \| = \frac{\sqrt{h} \left| 1 - e^{\Phi(z, h)} \right|^{-1} e^{\frac{\Im(z)}{h}}}{\sqrt{h} \left( \frac{1}{2} \{p, \overline{p} \} (\rho_+) \frac{1}{2} \{\overline{p}, p \} (\rho_-) \right)^\frac{1}{2}} \left( 1 + O(h) \right) \propto \frac{e^{\frac{\Im(z)}{h}}}{\sqrt{h} d(z)^{1/4}},$$

for $|\Im z - \langle \Im g \rangle| > 1/C$, $C \gg 1$,

where $\left| 1 - e^{\Phi(z, h)} \right| = 0$ if and only if $z \in \sigma(P_h)$. Moreover,

$$\left| 1 - e^{\Phi(z, h)} \right| = 1 + O \left( e^{-\frac{\pi}{h} |\Im z - \langle \Im g \rangle|} \right).$$
This proposition will be proven in Section 10.1. The growth of the norm of the resolvent away from the the line \( \text{Im } z = (\text{Im } g) \) is exponential and determined by the function \( S(z) \). Therefore, keeping in mind that we will be comparing the norm of the resolvent with the perturbation strength, it is very useful to write the coupling constant \( \delta \) as follows:

**Definition 2.5.** For \( h > 0 \), define

\[
\delta := \delta(h) := \sqrt{h} e^{-\epsilon_0(h)},
\]

with \((\kappa - \frac{1}{2}) h \ln(h^{-1}) + Ch \leq \epsilon_0(h) < S((\text{Im } g))\) for some \( \kappa > 0 \) and \( C > 0 \) large and where the last inequality is uniform in \( h > 0 \). This is equivalent to the bounds

\[
\sqrt{h} e^{-\epsilon_0(h)} < \delta \ll h^{\kappa}.
\]

**Remark 2.6.** The upper bound has been chosen in order to produce eigenvalues sufficiently far away from the line \( \text{Im } z = (\text{Im } g) \) where we find \( \sigma(P_h) \). The lower bound on \( \epsilon_0(h) \) is needed because we want to consider small random perturbations with respect to \( P_h \).

### 2.2. Auxiliary operator.

To describe the elements of the average density of eigenvalues, it will be very useful to introduce the following operators which have already been used in the study of the spectrum of \( P_h^S \) by Sjöstrand [8]. For the readers convenience, we will give a short overview:

Let \( z \in \Omega \subseteq \Sigma \) and define the \( z \)-dependent elliptic self-adjoint operators

\[
Q(z) := (P_h - z)(P_h - z)^*, \quad \tilde{Q}(z) := (P_h - z)(P_h - z)^*: \quad L^2(S^1) \to L^2(S^1)
\]

with domains \( D(Q(z)), D(\tilde{Q}(z)) = H^2(S^1) \). Since \( S^1 \) is compact and these are elliptic, non-negative, self-adjoint operators their spectra are discrete and contained in the interval \([0, \infty[\). Since

\[
Q(z)u = 0 \Rightarrow (P_h - z)u = 0
\]

it follows that \( \mathcal{N}(Q(z)) = \mathcal{N}(P_h - z) \) and \( \mathcal{N}(\tilde{Q}(z)) = \mathcal{N}((P_h - z)^*) \). Furthermore, if \( \lambda \neq 0 \) is an eigenvalue of \( Q(z) \) with corresponding eigenvector \( e_{\lambda} \) we see that \( f_{\lambda} := (P_h - z)e_{\lambda} \) is an eigenvector of \( \tilde{Q}(z) \) with the eigenvalue \( \lambda \). Similarly, every non-vanishing eigenvalue of \( \tilde{Q}(z) \) is an eigenvalue of \( Q(z) \) and moreover, since \( P_h - z, (P_h - z)^* \) are Fredholm operators of index 0 we see that

\[
\dim \mathcal{N}(P_h - z) = \dim \mathcal{N}((P_h - z)^*).
\]

Hence the spectra of \( Q(z) \) and \( \tilde{Q}(z) \) are equal

\[
\sigma(Q(z)) = \sigma(\tilde{Q}(z)) = \{t_0^2, t_1^2, \ldots \}, \quad 0 \leq t_j \nearrow \infty.
\]

We will show in Proposition 3.7 that

\[
t_0^2(z) \leq O\left(d(z)^{\frac{1}{2}} h e^{-\frac{2\pi}{h}}\right), \quad t_j^2 \geq \frac{d(z)^{\frac{1}{2}} h}{O(1)}.
\]

Now consider the orthonormal basis of \( L^2(S^1) \)

\[
\{e_0, e_1, \ldots\}
\]

consisting of the eigenfunctions of \( Q(z) \). By the previous observations we have

\[
(P_h - z)(P_h - z)^*(P_h - z)e_j = t_j^2(P_h - z)e_j.
\]

Thus defining \( f_0 \) to be the normalized eigenvector of \( \tilde{Q} \) corresponding to the eigenvalue \( t_0^2 \) and the vectors \( f_j \in L^2(S^1) \), for \( j \in \mathbb{N} \), as the normalization of \( (P_h - z)e_j \) such that

\[
(P_h - z)e_j = \alpha_j f_j, \quad (P_h - z)^*f_j = \beta_j e_j \quad \text{with } \alpha_j \beta_j = t_j^2,
\]

yields an orthonormal basis of \( L^2(S^1) \)

\[
\{f_0, f_1, \ldots\}
\]
consisting of the eigenfunctions of $\hat{Q}(z)$. Since
\[
\alpha_j = ((P_h - z)e_j)f_j = (e_j|(P_h - z)^*f_j) = \overline{\beta}_j
\]
we can conclude that $\alpha_j\overline{\alpha}_j = t_j^2$.

We will prove in the Sections 4.2 and 4.4 the following two formulas for the tunnel effect:

**Proposition 2.7.** Let $z \in \Omega \in \Sigma$ and let $e_0$ and $f_0$ be as in (2.4) and in (2.6). Furthermore, let $S$ be as in Definition 2.1 and let $p$ and $\rho_\pm$ be as in Section 1. Let $h^{3/2} \ll d(z)$, then for all $z \in \Omega$ with $|\text{Im}z - \langle \text{Im}g \rangle| > 1/C$, $C \gg 1$,
\[
|(e_0|f_0)| = \left( \frac{\sqrt{2 \{p, \overline{p}\}(\rho_+ \frac{1}{2} \{\overline{p}, p\}(\rho_-)}{\pi h^2} \right) |\partial_{\text{Im}z} S(z)| \left( 1 + \mathcal{O}\left( d(z)^{-3/4} h^{1/2} \right) \right) e^{-\frac{\pi}{2}}.
\]
where for all $(n, m) \in \mathbb{N}^2$
\[
\partial_{\overline{z}}^n \partial_z^m \mathcal{O}\left( d(z)^{-3/4} h^{-1/2} \right) = \mathcal{O}\left( d(z)^{\frac{n+m}{2} - \frac{3}{4} h^{-n-m-1}} \right).
\]

**Proposition 2.8.** Under the same assumptions as in Proposition 2.7, let $\chi \in \mathcal{C}_0^\infty(S^1)$ with $\chi \equiv 1$ in a small open neighborhood of $\{x_+^{-} : z \in \Omega\}$. Then, for $h^{3/2} \ll d(z)$,
\[
|(\langle P_h, \chi | e_0 \rangle f_0)| = \sqrt{h} \left( \frac{\sqrt{2 \{p, \overline{p}\}(\rho_+ \frac{1}{2} \{\overline{p}, p\}(\rho_-)}}{\pi^2} \right) ^{1/2} \left( 1 + \mathcal{O}\left( d(z)^{-3/2} h \right) \right) e^{-\frac{\pi}{2}}.
\]
Furthermore, for all $(n, m) \in \mathbb{N}^2$
\[
\partial_{\overline{z}}^n \partial_z^m \mathcal{O}\left( d(z)^{-3/2} h \right) = \mathcal{O}\left( d(z)^{\frac{n+m}{2} - \frac{3}{2} h^{1-(n+m)}} \right).
\]

### 2.3. Average density of eigenvalues

To calculate an $h$-asymptotic formula for the average density of eigenvalues we shall consider the first intensity measure of the random point process of eigenvalues of $P_h^\delta$ given in the following

**Definition 2.9.** Let $P_h^\delta$ be as in (1.3), then we define the point process
\[
\Xi := \sum_{z \in \sigma(P_h^\delta)} \delta_z,
\]
where the zeros are counted according to their multiplicities and $\delta_z$ denotes the Dirac-measure in $z$.

$\Xi$ is a well-defined random measure (cf. for example [2]) since, for $h > 0$ small enough, $P_h^\delta$ is a random operator with discrete spectrum. In order to state the formula for the density, let us first define the functions giving the first order terms.

The main result giving the average density of eigenvalues of $P_h^\delta$ is the following:

**Theorem 2.10.** Let $\Omega \in \Sigma$ be relatively compact. Let $C > 0$ and let $C_1 > 0$ as in (1.4) such that $C - C_1 > 0$ is large enough. Let $\kappa > 4$ and $\delta > 0$ be as above. Define $N := (2|C_1/h| + 1)^2$ and let $B(0, R) \subset \mathbb{C}^N$ be the ball of radius $R := Ch^{-1}$ centered at zero. Then, there exists a $C_2 > 0$ such that for $h > 0$ small enough and for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$
\[
\mathbb{E}[\Xi(\varphi) \mathbf{1}_{B(0, R)}] = \int \varphi(z) D(z, h, \delta) L(dz) + \mathcal{O}\left( e^{-\frac{C_2}{h}} \right),
\]
with the density
\[
D(z, h, \delta) = \frac{1}{\pi} \mathcal{O}\left( \delta h^{-\frac{3}{4}} d(z)^{-1/4} \right) \Psi(z; h, \delta) \exp\{-\Theta(z; h, \delta)\},
\]
where
which depends smoothly on $z$ and is independent of $\varphi$. Moreover, $\Psi(z; h, \delta) = \Psi_1(z; h) + \Psi_2(z; h, \delta)$ and for $z \in \Omega$ with $d(z) \gg h^{2/3}$

$$\begin{align*}
\Psi_1(z; h) &= \frac{i}{h} \left( \frac{i}{\{p, \overline{p}\}(\rho_+(z))} + \frac{i}{\{\overline{p}, p\}(\rho_-(z))} \right) + O(d(z)^{-2}), \\
\Psi_2(z; h, \delta) &= \frac{|\langle e_0|f_0 \rangle|^2}{\delta^2} \left( 1 + O \left( d(z)^{-3/4}h^{1/2} \right) \right), \\
\Theta(z; h, \delta) &= \frac{|\langle P_h \chi|e_0|f_0 \rangle + O \left( d(z)^{-1/4}h^{-2}d^2(1 + O(h\infty)) \right)|^2}{\delta^2} \left( 1 + O \left( e^{-d(z)/h} \right) \right). \quad (2.10)
\end{align*}$$

Furthermore, in (2.8), $O \left( e^{-\mathcal{O}_{\frac{h}{\pi}}} \right)$ means $\langle T_h, \varphi \rangle$ where $T_h \in \mathcal{D}'(\mathbb{C})$ such that

$$\|T_h \varphi\| \leq C\|\varphi\|_\infty e^{-\mathcal{O}_{\frac{h}{\pi}}}$$

for all $\varphi \in C_0^\infty(\Omega)$ where $C$ is independent of $h$, $\delta$, $\eta$ and $\varphi$.

Let us give some comments on this result. The dominant part of the density of eigenvalues $D$ consists of three parts: the first, $\Psi_1$, is up to a small error the Lebesgue density of the symplectic volume. Indeed, we shall prove in Proposition 6.2 that

$$\frac{1}{h} \left\{ \frac{i}{\{p, \overline{p}\}(\rho_+(z))} + \frac{i}{\{\overline{p}, p\}(\rho_-(z))} \right\} L(dz) = \frac{1}{2h} p_s(d\xi \wedge dx),$$

where $d\xi \wedge dx$ is the symplectic form on $T^*S^1$ and $p$ as in (1.2).

The second part, $\Psi_2$, is given by a tunneling effect. Inside the $\delta$- pseudospectrum $\Psi_2$ contributes vanishes in the error term of $\Psi_1$. However, close to the boundary of the $\delta$- pseudospectrum $\Psi_2$ becomes of order $h^{-2}$ and thus yields a strong accumulation of eigenvalues. This can be seen by comparing the more explicit formula for $\Psi_2$ given in Proposition 2.11 with the expression for the norm of the resolvent of $P_h$ given in Proposition 2.4. More details on the form of $\Psi_2$ in this zone will be given in Proposition 2.15.

The third part, $\exp(-\Theta)$, is also given by a tunneling effect and it plays the role of a cut-off function which exhibits double exponential decay outside the $\delta$- pseudospectrum and is close to 1 inside. This will be made more precise in Section 2.4.

**Proposition 2.11.** Under the assumptions of Definition 2.1 and Theorem 2.10, define for $h > 0$ and $\delta > 0$ the functions

$$\Theta^0(z; h, \delta) := \frac{h}{2} \left( \frac{i}{\{p, \overline{p}\}(\rho_+(z))} + \frac{i}{\{\overline{p}, p\}(\rho_-(z))} \right) \frac{e^{-2\pi z}}{\delta^2}.$$

Then, for $|\text{Im } z - (\text{Im } g)| > 1/C$, $C \gg 1$,

$$\begin{align*}
\Psi_2(z; h, \delta) &= \left( \frac{i}{\{p, \overline{p}\}(\rho_+(z))} + \frac{i}{\{\overline{p}, p\}(\rho_-(z))} \right) \frac{1}{\pi h \delta^2} \left| \partial_{\text{Im } z} S(z) \right|^2 \left( 1 + O \left( d(z)^{-3/4}h^{1/2} \right) \right) e^{-\mathcal{O}_{\frac{h}{\pi}}}, \\
\Theta(z; h, \delta) &= \Theta^0(z; h, \delta) \left( 1 + O \left( d(z)^{-1}h^3 \right) \right) + O \left( d(z)^{1/4}h^{-2}\delta + d(z)^{-1/2}\delta^2h^{-5} \right). \quad (2.11)
\end{align*}$$

Moreover, the estimates in (2.11) are stable under application of $d(z)^{-\frac{n+m}{2}}h^{n+m} \partial_{x}^n \partial_{\overline{z}}^m$.

The growth properties of these functions are as follows:

**Proposition 2.12.** Under the assumptions of Definition 2.1 we have that for $d(z) \gg h^{2/3}$

$$\begin{align*}
i \left\{ \frac{i}{\{p, \overline{p}\}(\rho_+(z))} + \frac{i}{\{\overline{p}, p\}(\rho_-(z))} \right\} \approx d(z), \\
\frac{1}{h} \left\{ \frac{i}{\{p, \overline{p}\}(\rho_+(z))} + \frac{i}{\{\overline{p}, p\}(\rho_-(z))} \right\} \approx \frac{1}{h \sqrt{d(z)}}.
\end{align*}$$
and
\[
\Psi_2(z; h, \delta) \asymp \frac{(d(z))^{3/2}e^{-2Sz}}{h\delta^2}, \quad \Theta_0(z; h, \delta) \asymp h\sqrt{d(z)} \left| 1 - e^{\Phi(z,h)} \right| \frac{e^{-2Sz}}{\delta^2}.
\]

In the next Subsection we will explain the asymptotic properties of the density appearing in (2.8).

2.4. Properties of the average density of eigenvalues and its integral with respect to \( \text{Im} z \). It will be sufficient for our purposes to consider rectangular subsets of \( \Sigma \), defined by
\[
\Sigma_{c,d} := \left\{ z \in \Sigma \left| \min_{x \in S^1} \text{Im} g(x) \leq \text{Im} z \leq \max_{x \in S^1} \text{Im} g(x), \ c < \text{Re} z < d \right. \right\}, \quad c < d \quad (2.12)
\]
Roughly speaking, there exist three regions in \( \Sigma \):

(1) \( z \in \Sigma_w \subset \Sigma \iff \|(P_h - z)^{-1}\| \gg \delta^{-1} \),

(2) \( z \in \Sigma_r \subset \Sigma \iff \|(P_h - z)^{-1}\| \asymp \delta^{-1} \),

(3) \( z \in \Sigma_v \subset \Sigma \iff \|(P_h - z)^{-1}\| \ll \delta^{-1} \),

which depend on the strength of the coupling constant \( \delta > 0 \). In \( \Sigma_w \), the average density is of order \( h^1 \) and is governed by the symplectic volume and thus yielding a Weyl law. In \( \Sigma_r \), the average density spikes and is of order \( h^{-2} \) and is equal to the symplectic volume plus the function \( \Psi_2 \) yielding in total a Poisson-type distribution. In \( \Sigma_v \), the average density is rapidly decaying and is void of eigenvalues with a high probability, since
\[
\Theta \asymp \|(P_h - z)^{-1}\|^{-2}\delta^{-2}
\]
which follows from Proposition 2.8 and Proposition 2.4.

Figure 1. The three zones in \( \Sigma \) with a schematic representation of \( \gamma^h_\pm \). The two boxes indicate zones where the integrated densities are equal up to a small error.

We prove that there exist two smooth curves, \( \Gamma^h_\pm \), along which the average density of eigenvalues obtains its local maxima.

**Proposition 2.13.** Let \( z \in \Sigma_{c,d} \) be as in (2.12), let \( S(z) \) be as in Definition 2.1 and let \( t^2_0(z) \) be as in (2.3). Let \( \delta > 0 \) and \( \varepsilon_0(h) \) be as in Definition 2.5 with \( \kappa > 4 \). Moreover, let \( D(z, h, \delta) \) be the average density of eigenvalues of the operator of \( P^h_\delta \) given in Theorem 2.10. Then,
• for $0 < h \ll 1$, there exist numbers $y_{\pm}(h)$ such that $\varepsilon_0(h) = S(y_{\pm}(h))$ with
\[
\frac{1}{C} h^\frac{3}{2} < y_{\pm}(h) < \langle \text{Im} g \rangle - c h \ln h^{-1} < \langle \text{Im} g \rangle + c h \ln h^{-1} < y_{\pm}(h) \ll \text{Im} g(b) - C h^\frac{3}{2},
\]
for $c > 1$. Furthermore,
\[
y_{\pm}(h), (\text{Im} g(b) - y_{\pm}(h)) \asymp (\varepsilon_0(h))^{2/3};
\]
• there exists $h_0 > 0$ and a family of smooth curves, indexed by $h \in [h_0, 0[$,
\[
\gamma^h_\pm : [c, d] \rightarrow \mathbb{C} \text{ with } \text{Re} \gamma^h_\pm(t) = t
\]
such that
\[
|t_0(\gamma^h(t))| = \delta.
\]
Moreover,
\[
\|(P_h - \gamma^h_\pm(t))^{-1}\| = \delta^{-1},
\]
and
\[
\text{Im} \gamma^h_\pm(\text{Re} z) = y_{\pm}(\varepsilon_0(h)) \left(1 + \frac{h}{O(1)\varepsilon_0(h)}\right).
\]
Furthermore, there exists a constant $C > 0$ such that
\[
\frac{d}{dt} \text{Im} \gamma^h_\pm(t) = O\left(\exp\left[-\varepsilon_0(h)\right]\right).
\]
• there exists $h_0 > 0$ and a family of smooth curves, indexed by $h \in [h_0, 0[$,
\[
\Gamma^h_\pm : [c, d] \rightarrow \mathbb{C}, \text{ Re} \Gamma^h_\pm(t) = t,
\]
with $\Gamma_- \subset \{\text{Im} z < \langle \text{Im} g \rangle\}$ and $\Gamma_+ \subset \{\text{Im} z > \langle \text{Im} g \rangle\}$, along which $\text{Im} z \mapsto D(z, h)$
takes its local maxima on the vertical line $\text{Re} z = \text{const.}$ and
\[
\frac{d}{dt} \text{Im} \Gamma^h_\pm(t) = O\left(\frac{h^4}{\varepsilon_0(h)^4}\right).
\]
Moreover, for all $c < t < d$
\[
|\Gamma^h_\pm(t) - \gamma^h_\pm(t)| \leq O\left(\frac{h^5}{\varepsilon_0(h)^{13/3}}\right).
\]

With respect to the above described curves we prove the following properties of the average density of eigenvalues:

**Proposition 2.14.** Let $d\xi \wedge dx$ be the symplectic form on $T^*S^1$ and $p$ as in (1.2). Under the assumptions of Theorem 2.10 there exist $\alpha, \beta > 0$ such that

• for $z \in \Sigma_{c,d}$ with
\[
\text{Im} \gamma_-(\text{Re} z) + \alpha h \eta^{-1/2} \ln \frac{\eta^{1/2}}{h} \leq \text{Im} z \leq \text{Im} \gamma_+(\text{Re} z) - \alpha h \eta^{-1/2} \ln \frac{\eta^{1/2}}{h}
\]
we have that
\[
\frac{1}{\pi} \Psi(z; h, \delta) e^{-\Theta(z; h, \delta)} L(dz) = \frac{1}{2\pi h} p_s(d\xi \wedge dx) + O\left(d(z)^{-2}\right) L(dz);
\]
• for
\[
\Omega_1(\alpha) := \left\{ z \in \Sigma_{c,d} \mid \text{Im} \gamma_-(\text{Re} z) - \frac{h}{\varepsilon_0(h)^{1/3}} \ln \beta \ln \frac{\varepsilon_0(h)^{1/3}}{h} \leq \text{Im} z \leq \text{Im} \gamma_+(\text{Re} z) + \frac{h}{\varepsilon_0(h)^{1/3}} \ln \beta \ln \frac{\varepsilon_0(h)^{1/3}}{h} \right\}.
\]
we have that
\[
\int_{z \in \Omega_1(\alpha)} \frac{1}{\pi} \Psi(z; h, \delta) e^{-\Theta(z; h, \delta)} L(dz) = \int_{\Sigma_{c,d}} \frac{1}{2\pi h} p_\ast(d\xi \land dx) + O\left(\varepsilon_0(h)^{-2/3}\right)
\]

- for all \(\varepsilon > 0\)
\[
\int_{z \in \Sigma_{c,d} \setminus \Omega_2(\alpha, \varepsilon)} \frac{1}{\pi} \Psi(z; h, \delta) e^{-\Theta(z; h, \delta)} L(dz) = O\left(\exp\left\{-e^{\varepsilon \pi}\right\}\right),
\]

where
\[
\Omega_2(\alpha, \varepsilon) := \left\{ z \in \Sigma_{c,d} \mid \text{Im} \gamma_-(\text{Re} z) - \frac{h}{\varepsilon_0(h)^{1/3}} \ln \beta \ln \frac{\varepsilon_0(h)^{1/3}}{h} - \varepsilon \leq \text{Im} z \leq \text{Im} \gamma_+(\text{Re} z) + \frac{h}{\varepsilon_0(h)^{1/3}} \ln \beta \ln \frac{\varepsilon_0(h)^{1/3}}{h} + \varepsilon \right\}.
\]

Let us give some comments on this result. First, in the interior of the \(\delta\)-pseudospectrum, up to a distance of order \(h \ln \frac{1}{h}\) to the curves \(\gamma_{\pm}\) (see Figure 1), the density is given by a Weyl law. Second, the eigenvalues accumulate strongly in the close vicinity of these curves. This augmented density can be seen as the accumulated eigenvalues which would be given by a Weyl law in the region from \(\gamma_{\pm}\) up to the boundary \(\partial \Sigma\) (see also Figure 2 for an example).

2.4.1. The density in the zone of spectral accumulation. We give a finer description of the density of eigenvalues close to its local maxima at \(\Gamma_{\pm}^h\):

**Proposition 2.15.** Let \(z \in \Omega @ \Sigma\) and let \(\varepsilon_0(h)\) and \(\delta > 0\) be as in Definition 2.5 with \(\kappa > 4\). Let \(S(z)\) be as in Definition 2.1 and let \(\Psi_2(z, h, \delta)\) and \(\Theta(z; h, \delta)\) be as in Theorem 2.10. Then for \(|\text{Im} z - (\text{Im} g)| > 1/C\) and \(d(z) \gg h^{2/3}\)
\[
\Psi_2 e^{-\Theta} = \left[\frac{\partial_m z S^2}{h^2} S \Theta \left(1 + O\left(d(z)^{-3/4} h^{1/2}\right)\right) + O\left(d(z)^{5/4}\right)\right] e^{-\Theta}.
\]

Let us give some remarks on this result. First, we see that we can approximate the second part of the density of eigenvalues by Poisson distribution scaled by the monotone function \(\partial_m z S(z)\). Second, since \(\Theta \asymp \|(P_h - z)^{-1}\|^{-1} \delta^{-2}\), we see that the effects of the second part of the density vanish in the error term of \(\Psi_1\) as long as \(\|(P_h - z)^{-1}\| \gg \delta^{-1}\). However, for \(\|(P_h - z)^{-1}\| \times \delta^{-1}\) it is of order \(O(d(z) h^{-2})\) and dominates the Weyl term.

2.5. Example: Numerical simulations for \(hD + e^{-iz}\). To illustrate our results we shall look at the discretization of \(P_h = hD + e^{-iz}\) in Fourier space. Thus, for \(N \in \mathbb{N}\), consider the \((2N + 1) \times (2N + 1)\)-matrix \(H = hD + E\), where \(D\) and \(E\) are defined by
\[
(D)_{j,k} := \begin{cases} j & \text{if } j = k, \\ 0 & \text{else} \end{cases} \quad \text{and} \quad (E)_{j,k} := \begin{cases} 1 & \text{if } k = j + 1, \\ 0 & \text{else}, \end{cases}
\]

where \(j, k \in \{-N, -N + 1, \ldots, N\}\). Furthermore, let \(R\) be a \((2N + 1) \times (2N + 1)\) random matrix, where the entries \(R_{j,k}\) are independent and identically distributed complex Gaussian random variables, \(R_{j,k} \sim \mathcal{N}_C(0, 1)\). For \(h > 0\) and \(\delta > 0\) as in Theorem 2.10, we let MATLAB calculate the spectrum \(\sigma(H + \delta R)\). Figure 2 shows sections of the spectrum \(\sigma(H + \delta R)\) for \(N = 6000\), \(h = 2 \cdot 10^{-4}\) and \(\delta = 2 \cdot 10^{-14}\). In this case \(\Sigma\) is given by \(|\text{Im} z| \leq 1\). In the image on the left hand side we can see the three zones described in above:

The first zone, is in the middle of the spectrum. We can see roughly an aequidistribution of points at distance \(\sqrt{h}\) as predicted by Proposition 2.14 which tells us that the zone where we have \(\|(P_h - z)^{-1}\| \gg \delta^{-1}\) is purely governed by a Weyl law.
Figure 2. Sections of the spectrum of the discretization of $hD + \exp(-ix)$ perturbed with a random Gaussian matrix $\delta R$.

Another important property of this zone is that there is an increase in the density of the spectral points as we approach the pseudospectral boundary. This is due to the fact that the density given by the Weyl law becomes more and more singular as we approach the pseudospectral boundary (cf. Proposition 2.12).

We will find the second zone by moving closer to the “edge” of the spectrum. It can be characterized as the zone where $\| (P_h - z)^{-1} \| \approx \delta^{-1}$. We notice an accumulation of the spectrum close to the boundary of the pseudospectrum (cf. as well Figure 5). Furthermore, we see in the image on the right hand side of Figure 2 that in the zone of accumulation the eigenvalues align themselves on roughly a straight line with an average distance of order $12^{-1}$. 

Figure 3. Experimental vs predicted eigenvalue density for $h = 2 \cdot 10^{-3}$ and $\delta = 2 \cdot 10^{-12}$. 
Figure 4. Predicted eigenvalue density for different values $\delta$.

$h$. This is exactly as predicted by Proposition 2.13 and Proposition 2.15 (compare with Figure 3).

The third zone is between the spectral edge and the boundary of $\Sigma$ where we find no spectrum at all. It can be characterized as the zone where $\|(P_h - z)^{-1}\| \ll \delta^{-1}$, a void region as described in Proposition 2.14.

Figure 4 further illustrates the behavior of density of eigenvalues by plotting it for positive imaginary part.

Finally, we remark that when we pick $\delta > 0$ to be exponentially small in $h > 0$ the zone of accumulation moves further into the interior of the pseudospectrum thus diminishing the zone determined by the Weyl law and increasing the zone void of eigenvalues. Figure 5 depicts a simulation corresponding to such a choice of $\delta$. However, at this point we quickly run into numerical difficulties.

Figure 5. Section of the spectrum $\sigma(H + \delta R)$ for $N = 4000$, $h = 0.05$ and $\delta = e^{-1/h}$.
3. Quasimodes

The purpose of this section will be to construct quasimodes for the operator

$$P_h - z$$

for $z \in \Omega \subset \Sigma$ with $\text{dist} (\Omega, \partial \Sigma) > Ch^{2/3}$. Therefore, we shall make the distinction between

- $z$ being in the interior of $\Sigma$, i.e. $z \in \Omega_i \subset \dot{\Sigma}$ such that there exists a constant $C_{\Omega_i} > 0$ such that
  $$\text{dist} (\Omega_i, \partial \Sigma) > \frac{1}{C_{\Omega_i}}.$$ 

In this case, following the approach of Hager [5], we can find quasimodes by a WKB construction for the operator $(P_h - z)$;

- $z$ being close to the boundary of $\Sigma$, i.e. $z \in \Omega \cap (\Omega_a^\eta \cup \Omega_b^\eta)$ where, following the notation used in [1], we define for some constant $C > 0$
  $$\Omega_a^\eta := \left\{ z \in \mathbb{C} : \eta \leq \text{Im } z \leq C \eta \right\},$$ 
  $$\Omega_b^\eta := \left\{ z \in \mathbb{C} : \eta \leq (\text{Im } g(b) - \text{Im } z) \leq C \eta \right\},$$

with $\eta^{2/3} \ll \eta \leq \text{const}$. The precise value of the above constant $C > 0$ is not important for the obtained asymptotic results. We will only consider the case $z \in \Omega_a^\eta$ since $z \in \Omega_b^\eta$ can be treated the same way. We may follow the approach of Bordeaux-Montreux [1] and find quasimodes by a WKB construction for the rescaled operator

$$\tilde{P}_h - \tilde{z} := \frac{h}{\eta^2} D_x + \frac{g(\sqrt{\eta} \tilde{x})}{\eta} - \frac{\tilde{z}}{\eta} := \tilde{h} D_x + \tilde{g}(\tilde{x}) - \tilde{z},$$

with the rescaling

$$S^1 \ni x = \sqrt{\eta} \tilde{x} \quad \text{and} \quad \tilde{h} := \frac{h}{\eta^2}.$$ 

Note that in this case demanding $\tilde{h} \ll 1$ implies the condition $\hbar^{2/3} \ll \eta$. The rescaling is motivated by analyzing the Taylor expansion of $\text{Im } g(x)$ around the critical point $a$ yielding that for $\text{Im } z \to 0$

$$|x_\pm(z) - a| \asymp \sqrt{\eta},$$

where $x_\pm(z)$ are as in Section 1. This shows that the rescaling shifts the problem of constructing quasimodes for $z$ close to the boundary of $\Sigma$ to constructing quasimodes for $z$ well in the interior of the range of the semiclassical principal symbol of the new operator $\tilde{P}_h$.

Remark 3.1. Throughout this work we shall work with the convention that when writing an estimate, e.g. $O(\delta \eta^r h^s)$ or $A \asymp \eta^r h^s$, we implicitly set $\eta = 1$ when $\text{dist} (z, \partial \Sigma) > 1/C$ but keep $\eta$ when $z \in \Omega_a^\eta$.

Let us note, that by Taylor expansion we may deduce that $S = S(z)$ as defined in Definition 2.1 satisfies

$$S(z) \asymp \eta^2$$

(3.3)
3.1. Quasimodes for the interior of $\Sigma$.

**Definition 3.2.** Let $z \in \Omega_t \subset \bar{\Sigma}$ and let $x_-, x_+$ be as in the introduction. Let $\psi \in C_0^\infty(\mathbb{R})$ with $\text{supp} \psi \subset ]0,1[$ and $\int \psi(x)dx = 1$. Define $\chi_e \in C_0^\infty([x_--2\pi, x_-])$ and $\chi_f \in C_0^\infty([x_+, x_+ + 2\pi])$ by

$$
\begin{align*}
\chi_e(x, z; h) &:= \int_{-\infty}^x h^{-\frac{1}{2}} \left\{ \psi \left( \frac{y-x-2\pi}{\sqrt{h}} \right) - \psi \left( \frac{x-y}{\sqrt{h}} \right) \right\} dy, \\
\chi_f(x, z; h) &:= \int_{-\infty}^x h^{-\frac{1}{2}} \left\{ \psi \left( \frac{y-x+\pi}{\sqrt{h}} \right) - \psi \left( \frac{x+y+2\pi}{\sqrt{h}} \right) \right\} dy.
\end{align*}
$$

(3.5)

Furthermore, define for $x \in ]x_--2\pi, x_-[$

$$
\phi_+(x; z) := \int_{x_+}^x (z - g(y)) dy,
$$

and for $x \in ]x_+, x_+ + 2\pi[$

$$
\phi_-(x; z) := \int_{x_-}^x (z - g(y)) dy.
$$

Consider the $L^2(S^1)$-normalized quasimodes

$$
e_{wkb}(x, z; h) := h^{-\frac{1}{2}} a(z; h) \chi_e(x, z; h) e^{i\phi_+(x; z)} \in C_0^\infty([x_--2\pi, x_-])
$$

(3.6)

and

$$
f_{wkb}(x, z; h) := h^{-\frac{1}{2}} b(z; h) \chi_f(x, z; h) e^{i\phi_-(x; z)} \in C_0^\infty([x_+, x_+ + 2\pi])
$$

(3.7)

where $a(h; z)$ and $b(h; z)$ are normalization factors obtained by the stationary phase method. Thus, $a(h; z) \sim a_0(z) + ha_1(z) + \cdots \neq 0$ and $b(h; z) \sim b_0(z) + hb_1(z) + \cdots \neq 0$ depend smoothly on $z$ such that all derivatives with respect to $z$ and $\overline{z}$ are bounded when $h \to 0$.

The quasimodes $e_{wkb}$ and $f_{wkb}$ are WKB approximate null solutions to $(P_h - z^*)$ where $a(z; h)$ and $b(z; h)$ are the asymptotic expansions of the normalization coefficients.

**Lemma 3.3.**

$$
a_0 = \left( \frac{-\text{Im} \ g'(x_+)}{\pi} \right)^{\frac{1}{4}}, \quad \text{and} \quad b_0 = \left( \frac{\text{Im} \ g'(x_-)}{\pi} \right)^{\frac{1}{4}}.
$$

(3.8)

**Proof.** We will show the proof only for $a_0$ since the statement for $b_0$ can be achieved by analogous steps. To gain the asymptotic expansion of the normalization coefficient use the stationary phase method to calculate

$$
I_h := h^{-\frac{1}{2}} \int \chi_e(x, z; h)^2 e^{-\frac{\Phi(x; z)}{h}} dx,
$$

where

$$
i\phi_+(x; z) - i\overline{\phi_+(x; z)} = -2\text{Im} \int_{x_+(z)}^x (z - g(y)) dy = -\Phi(x; z).
$$

On the support of $\chi_e$ the phase $\Phi(x; z)$ has the unique critical point $x = x_+(z)$ which is non-degenerate since $\partial^2_{xx} \Phi(x_+(z); z) = -2\text{Im} \ g'(x_+(z)) > 0$. Thus the Morse Lemma (see e.g.: [4]) guarantees the existence of a local $C^\infty$ diffeomorphism $\kappa : V \to U$, where $V \subset \mathbb{R}$ is a neighborhood of $x_+(z)$ and $U \subset \mathbb{R}$ is a neighborhood of 0, such that

$$
\Phi(\kappa^{-1}(x); z) = \Phi(x_+(z); z) + \frac{x^2}{2},
$$

$\kappa^{-1}(0) = x_+(z)$ and

$$
\frac{d\kappa}{dx}(x_+(z)) = \left| \partial^2_{xx} \Phi(x_+(z); z) \right|^{\frac{1}{2}} = \sqrt{-2\text{Im} \ g'(x_+(z))} \neq 0.
$$

(3.9)
One then gets that
\[ I_h = \sqrt{2\pi} \sum_{n=0}^{N} \frac{1}{n!} \left( \frac{h}{2} \right)^n (\Delta^n u)(0) + O(h^{N+1}) \]
with \( u(y) = \chi_c(\kappa^{-1}(y); z)^2 \chi(\kappa^{-1}(y)) |\kappa'(\kappa^{-1}(y))|^{-1} \). Since \( u(0) = (-2\Im g'(x_+(z)))^{-1/2} \),
\[ I_h = \left( \frac{\pi}{-\Im g'(x_+(z))} \right)^{1/2} + O(h). \]

By the natural projection \( \Pi : \mathbb{R} \to S^1 \) as in Section 1 we can identify
\[ C_0^\infty(\{x_+, x_+ + 2\pi\}) = \{ u \in C^\infty(S^1) : x_+ \notin \text{supp } u \} \]
and
\[ C_0^\infty(\{x_-, x_- - 2\pi\}) = \{ u \in C^\infty(S^1) : x_- \notin \text{supp } u \}, \]
with the slight abuse of notation that on the right hand side \( x_\pm \in \mathbb{R} \) and on the left hand side \( x_\pm \in S^1 \). This identification permits us to define \( e_{wkb}(x, z; h), f_{wkb}(x, z; h) \) on \( C^\infty(S^1) \).

### 3.2. Quasimodes close to the boundary of \( \Sigma \)

Now let \( z \in \Omega^a_\eta \). Following [1], we shall construct quasimodes for the operator \( P_h - \tilde{z} \) by looking at the rescaled operator \( \tilde{P}_h - \tilde{z} \) as defined in (3.2).

Let us first note that \( \frac{i}{\hbar} \phi_+(x; z) \) and \( \frac{i}{\hbar} \phi_-(x; z) \) have the following behavior under the rescaling described at the beginning of this section:
\[ \frac{i}{\hbar} \phi_+(x; z) = \frac{i}{\hbar} \int_{x_+}^{x} (z - g(y)) dy = \frac{i}{\hbar} \int_{\tilde{x}_+}^{\tilde{x}} (\tilde{z} - \tilde{g}(\tilde{y})) d\tilde{y} =: \frac{i}{\hbar} \tilde{\phi}_+(\tilde{x}; \tilde{z}) \] 
(3.10)
and analogously for \( \frac{i}{\hbar} \phi_-(x; z) \). Taylor expansion shows us that the rescaled phase functions \( \tilde{\phi}_\pm(\tilde{x}; \tilde{z}) \) have for \( z \in \Omega^a_\eta \) a non-degenerate critical point \( \tilde{x}_\pm(\tilde{z}) \) which satisfy the relation \( x_\pm(z) = \sqrt{\eta} \tilde{x}_\pm(\tilde{z}) \) and \( x_\pm(z) + 2\pi = \sqrt{\eta}(x_\pm + 2\pi)(\tilde{z}) \).

It is easy to see that locally
\[ (\tilde{P}_h - \tilde{z}) e^{\frac{i}{\hbar} \tilde{\phi}_+(\tilde{x}; \tilde{z})} = 0, \]
Thus, the natural choice of quasimodes for \( z \in \Omega \cap \Omega^a_\eta \) in the rescaled variables is

**Proposition 3.4.** Let \( \Omega \subset \Sigma, z \in \Omega \cap \Omega^a_\eta \) and set \( \tilde{\hbar} := \frac{\hbar}{\eta^{1/2}} \). Then there exist functions
\[ a^n(\tilde{h}; \tilde{z}) \sim a^n_0(\tilde{z}) + \tilde{h} a^n_1(\tilde{z}) + \cdots \neq 0, \quad b^n(\tilde{h}; \tilde{z}) \sim b^n_0(\tilde{z}) + \tilde{h} b^n_1(\tilde{z}) + \cdots \neq 0, \]
depending smoothly on \( \tilde{z} \) such that all \( \tilde{z} \)- and \( \bar{\tilde{z}} \)-derivatives remain bounded as \( h \to 0 \) and \( \tilde{h} \tilde{\hbar} < \eta \to 0 \), such that
\[
eq \begin{align*}
eq e^n_{wkb}(\tilde{x}; \tilde{z}; \tilde{h}) &:= (\tilde{\hbar} \eta)^{-1/2} a^n(\tilde{z}; \tilde{h}) \chi_c(\tilde{x}; \tilde{z}; \tilde{h}) e^{\frac{i}{\hbar} \phi_+(\tilde{x}; \tilde{z})} \quad \text{and} \\
eq f^n_{wkb}(\tilde{x}; \tilde{z}; \tilde{h}) &:= (\tilde{\hbar} \eta)^{-1/2} b^n(\tilde{z}; \tilde{h}) \chi_f(\tilde{x}; \tilde{z}; \tilde{h}) e^{\frac{i}{\hbar} \phi_-(\tilde{x}; \tilde{z})},
\end{align*} \]
where \( \chi_c, f \) are as in Definition 3.2, are \( L^2(S^1/\sqrt{\eta}, \sqrt{\eta}d\tilde{x}) \)-normalized. Furthermore,
\[
eq \begin{align*}
eq a^n_0(\tilde{z}) &\equiv \left( \frac{|\Im g''(a)(\tilde{x}_+(\tilde{z}) - a/\sqrt{\eta})(1 + o(1))|}{\pi} \right)^{1/2}, \quad z \in \Omega^a_\eta, \\
eq b^n_0(\tilde{z}) &\equiv \left( \frac{|\Im g''(a)(\tilde{x}_-(\tilde{z}) - a/\sqrt{\eta})(1 + o(1))|}{\pi} \right)^{1/2}, \quad z \in \Omega^a_\eta.
\end{align*} \]
In particular, the Taylor expansion around the critical point \( k \) yields that
\[
\frac{\chi(x,z)}{\eta} \text{ with } k = \frac{\arctan h}{\eta}.
\]

The stationary phase method yields that (3.11)
\[
\int \chi(x,z) \eta^2 e^{-\frac{\pi}{2} \text{Im} \phi_on(x,z)} dx = \sqrt{\eta} \int \chi(x,z) \eta^2 e^{-\frac{\pi}{2} \text{Im} \phi_on(x,z)} dx.
\]

The stationary phase method yields that (3.11) yields that
\[
\| \chi \|_{L^2(S^1)}^2 \sim h^{\frac{3}{2}}(c_0(z) + hc_1(z) + \ldots)
\]

with
\[
c_0(z) = \left( \frac{\pi}{-\text{Im} g'(x,z)} \right)^{\frac{1}{2}}.
\]

Since \( \chi(x,z) \equiv \chi_0(x,z) \) locally around \( x = x(z) \), we may conclude that for all \( k \in \mathbb{N} \)
\[
\tilde{c}_k = \frac{1}{\eta^{\frac{3}{2}} + \frac{1}{2}} c_k(z).
\]

In particular, the Taylor expansion around the critical point \( \alpha \) yields
\[
\tilde{c}_k = \left( \frac{\pi}{\text{Im} g''(a)(x,z) - a(1 + o(1))} \right)^{\frac{1}{2}}, \quad z \in \Omega^\alpha
\]

Thus, we may conclude the statement of the proposition. \( \square \)

Considering the above describe quasimodes in the original variable \( x \in S^1 \) leads to the following

**Definition 3.6.** Let \( \Omega \in \Sigma \), \( \Omega \in \Omega^\alpha \) and set \( \tilde{h} := \frac{h}{\eta^{1/2}} \). Then define
\[
e_w(x,z) := \left( \frac{\eta}{\eta^{1/2}} \right)^{-\frac{1}{2}} a^0(z) \tilde{h} \chi_0(x,z) e^{\frac{i}{\eta} \phi_0(x,z)}
\]
\[
f_w(x,z) := \left( \frac{\eta}{\eta^{1/2}} \right)^{-\frac{1}{2}} b^0(z) \tilde{h} \chi_1(x,z) e^{\frac{i}{\eta} \phi_0(x,z)},
\]

where \( \chi_{e,f}(x,z) \) is the first order term of the normalization coefficient \( a \) of the quasimode \( \eta \), see Lemma 3.3. Similar for \( b^0 \).

Remark 3.5. In Proposition 3.4, we stated the Taylor expansion of the first order terms of \( a(x) \) and \( b(x) \). However, note that we have
\[
a_0(z) = \left( -\frac{\text{Im} g'(x,z)}{\pi} \right)^{\frac{1}{2}} \eta^\frac{1}{2} a_0^0(z),
\]
where \( a_0 \) is the first order term of the normalization coefficient \( a \) of the quasimode \( e_w \).
3.3. **Approximation of the eigenfunctions of** $Q(z)$ and $\tilde{Q}(z)$. Recall $Q$ and $\tilde{Q}$ given in Section 2.2. We will use the above defined quasimodes to prove estimates on the lowest eigenvalue of $Q$, $\ell_0^Q$. Furthermore, we will give estimates on the approximation of the eigenfunctions $\varepsilon_0$ and $f_0$ by the quasimodes $e_{wkb}$ and $f_{wkb}$. We will prove an extended version of a result in [8, Sec. 7.2 and 7.4].

**Proposition 3.7.** Let $z \in \Omega \Subset \Sigma$ and let $S = S(z)$ be defined as in Definition 2.1. Then, for $h^{3/2} < \eta \leq C$

$$t_0^Q(z) \leq O\left(\eta^2 h e^{-\frac{\eta^2}{h}}\right).$$

Furthermore, there exists a constant $C > 0$, uniform in $z$, such that

$$t_1^Q - t_0^Q \geq \eta^2 h / C$$

for $h > 0$ small enough.

**Remark 3.8.** The case $\text{dist}(z, \partial \Sigma) > 1/C$ has been proven in [8, Sec. 7.1]. Since it will be useful further on we shall give a proof of the statement and indicate how to deduce the statement in the case of $z \in \Omega \cap \Omega^\eta_{1/2}$.

**Proof.** Let us first suppose that $z \in \Omega$. Recall from Section 3 that dist $(\Omega_\eta, \partial \Sigma) > 1/C$. Recall the definition of the self-adjoint operator $Q(z)$ given in (2.2) and define

$$r := r(x, z; h) := Q(z) e_{wkb}(x; z). \quad (3.12)$$

Recall, by (3.6), that $e_{wkb}(x; z) = h^{-1/2} a(z; h) \chi_e(x, z; h) e^{i \phi_+(x; z)}$. Since further on we shall need various derivatives of $\chi_e(x, z; h)$ we shall give them now. By (3.5) we gain

$$\partial_x \chi_e(x, z; h) = h^{-1/2} \left\{ \psi \left( \frac{x - x_0 + 2\pi}{\sqrt{h}} \right) - \psi \left( \frac{x - x_0}{\sqrt{h}} \right) \right\} = O\left(h^{-1/2}\right),$$

$$\partial^2_{xx} \chi_e(x, z; h) = h^{-1} \left\{ \psi' \left( \frac{x - x_0 + 2\pi}{\sqrt{h}} \right) + \psi' \left( \frac{x - x_0}{\sqrt{h}} \right) \right\} = O(h^{-1}),$$

$$\partial_z \chi_e(x, z; h) = -h^{-1/2} \left\{ \psi \left( \frac{x - x_0 + 2\pi}{\sqrt{h}} \right) - \psi \left( \frac{x - x_0}{\sqrt{h}} \right) \right\} \partial_z x_-(z) = O\left(h^{-1/2}\right),$$

$$\partial^2_{xx} \chi_e(x, z; h) = -h^{-1} \left\{ \psi' \left( \frac{x - x_0 + 2\pi}{\sqrt{h}} \right) + \psi' \left( \frac{x - x_0}{\sqrt{h}} \right) \right\} \partial_z x_-(z) = O(h^{-1}),$$

$$\partial^3_{xxx} \chi_e(x, z; h) = -h^{-3/2} \left\{ \psi'' \left( \frac{x - x_0 + 2\pi}{\sqrt{h}} \right) - \psi'' \left( \frac{x - x_0}{\sqrt{h}} \right) \right\} \partial_z x_-(z) = O\left(h^{-3/2}\right). \quad (3.13)$$

Since $x_-(z)$ is smooth in $z$ and all its $z$- and \(\zeta\)-derivatives are independent of $h$, we find that for all $(n, m, l) \in \mathbb{N}^3 \setminus \{0\}$

$$\partial^n_{\zeta} \partial^m_{x} \partial^l_{z} \chi_e(x, z; h) = O\left(h^{-\frac{n+m+l}{2}}\right). \quad (3.14)$$

Note that the derivatives of $\chi_e$ are supported in $|x_0 - 2\pi, x_0 - 2\pi + 2h^{1/2} | x_0 - h^{1/2}, x_0 |$. By definition of $\phi_+(x; z)$

$$(P_h - z)e^{i \phi_+(x; z)} = 0$$

for $x \in [x_0 - 2\pi, x_0 - 2\pi - 2h^{1/2} | x_0 - h^{1/2}, x_0 |$. This implies

$$(P_h - z)e_{wkb}(x; z) = h^{-1/2} a(z; h) [(P_h - z), \chi_e(x, z; h)] e^{i \phi_+(x; z)} = h^{-1/2} a(z; h) \frac{h}{i} \partial_x \chi_e(x, z; h) e^{i \phi_+(x; z)}. \quad (3.15)$$
Continuing, we get
\[(P_h - z)(P_h - z) \psi_{wkb}(x; z) \]
\[= a(z; h) \frac{h^2}{4} \left\{ \frac{h}{2} \partial_{xx} \chi(x, x; h) + \partial_x \chi(x, z; h) \left( \partial_x \phi_+(x; z) + g(x) - z \right) \right\} e^{i \phi_+(x; z)}. \quad (3.16) \]

For \( x \in ]x_-, 2\pi, x_+-2\pi + h^{1/2} | x_--h^{1/2}, x_- [ \)
\[ \partial_x \phi_+(x; z) + g(x) - z = z - g(x) + g(x) - z = -2i \text{Im} (g(x) - z) = O \left( h^{1/2} \right) \quad (3.17) \]
which, together with (3.13), applied to (3.16) yields
\[ r = Q(z) \psi_{wkb}(x; z) = O \left( h^{1/2} \right) e^{i \phi_+(x; z)}. \quad (3.18) \]

Note that \( r \) is supported in \( ]x_-, 2\pi, x_+-2\pi + h^{1/2} | x_--h^{1/2}, x_- [ \)
Thus,
\[ (\psi_{wkb}Q(z)\psi_{wkb}) = \int \mathcal{O} \left( h^{1/2} \right)  \mathbb{I}_{x_--h^{1/2}, x_+-h^{1/2} | x_--h^{1/2}, x_- [ } (x) e^{-\frac{\phi(x; z)}{h}} dx, \quad (3.19) \]
where \( \Phi(x; z) = 2 \int_{x_+}^{x_-} \text{Im} (z - g(y)) dy \)
By Taylor’s formula
\[ \Phi(x; z) = \Phi(x_-; z) + \mathcal{O}(h), \quad \text{for } x \in ]x_-, h^{1/2}, x_- [ \]
\[ \Phi(x; z) = \Phi(x_- - 2\pi; z) + \mathcal{O}(h), \quad \text{for } x \in ]x_--2\pi, x_--2\pi + h^{1/2} [ \]
and thus
\[ e^{-\frac{\phi(x; z)}{h}} \leq \mathcal{O} \left( e^{-\frac{2\pi}{h}} \right), \]
where \( S = \min \left( \text{Im} \int_{x_+}^{x_-} (z - g(y)) dy, \text{Im} \int_{x_--2\pi}^{x_-} (z - g(y)) dy \right) \)
Hence,
\[ (\psi_{wkb}Q(z)\psi_{wkb}) \leq \mathcal{O} \left( h^{1/2} e^{-\frac{2\pi}{h}} \right) \int 1_{x_--h^{1/2}, x_+-h^{1/2} | x_--h^{1/2}, x_- [ } (x) dx \]
\[ = \mathcal{O} \left( h e^{-\frac{2\pi}{h}} \right). \quad (3.20) \]

Since \( Q \) is self-adjoint we can deduce that \( t_0^2(z) = \mathcal{O} \left( h e^{-\frac{2\bar{S}(z)}{h}} \right) \)
Let us remark that similarly
\[ \| r \|^2 \leq \mathcal{O} \left( h^{1/2} e^{-\frac{2\pi}{h}} \right) \int 1_{x_--h^{1/2}, x_+-h^{1/2} | x_--h^{1/2}, x_- [ } (x) dx = \mathcal{O} \left( h^2 e^{-\frac{2\pi}{h}} \right). \quad (3.21) \]
The proof of the desired statement about \( t_0^2(z) - t_0^2(z) \) for \( z \in \Omega_i \) can be found in the proof of Proposition 7.2 in [8, Sec. 7.1].

Suppose now that \( z \in \Omega \cap \Omega_i^\eta \). The desired statement follows by a rescaling argument. Recall (3.2) and, using the quasimodes \( \psi_{wkb}^\eta(x; z) \), note that
\[ t_0^2(Q(z)) = t_0^2(\eta^2 (\tilde{P}_h - \bar{z})^* (\tilde{P}_h - \bar{z})) = \mathcal{O} \left( \eta^2 h e^{-\frac{2\bar{S}(z)}{h}} \right), \]
where \( \bar{S} \) is defined in the obvious way via \( \bar{\phi}_+ \) and
\[ \frac{\bar{S}(z)}{h} = \frac{S(z)}{h}. \quad (3.22) \]
Hence,
\[ t_0^2(z) = \mathcal{O} \left( h \eta^{1/2} e^{-\frac{2\bar{S}(z)}{h}} \right). \quad (3.23) \]
The estimate on \( t_0^2(z) - t_0^2(z) \) in the case \( z \in \Omega \cap \Omega_i^\eta \) can be deduced as well by a rescaling argument: note that \( t_0^2(Q(z)) = t_0^2(\eta^2 (\tilde{P}_h - \bar{z})^* (\tilde{P}_h - \bar{z})) \). The statement then follows by performing the same steps of the proof of Proposition 7.2 in [8, Sec. 7.1] in the rescaled
space $L^2(S^1/\sqrt{\eta}, \sqrt{\eta}d\bar{x})$ and using the quasimode $e_{wkb}^\gamma(x; z)$ together with the estimate given in Proposition 4.3.5 in [1].

**Proposition 3.9.** Let $z \in \Omega \in \Sigma$. Then the eigenvalue $t^2_0(z)$ is a smooth function of $z$ and the eigenfunctions $e_0(z)$ and $f_0(z)$ can be chosen to have the same property.

**Proof.** Let us suppose first that $z \in \Omega$. The operator $Q(z)$ is bounded in $H^2(S^1) \to L^2(S^1)$ and in norm real-analytic in $z$ since for $z_0 \in \Omega$,

$$Q(z) = Q(z_0) - (P - z_0)^* (z - z_0) - (P - z_0)(\bar{z} - \bar{z}_0) + |z - z_0|^2. \quad (3.24)$$

Let $\zeta$ be in the resolvent set $\rho(Q(z))$ of $Q(z)$ and consider the resolvent

$$R(\zeta, Q(z)) := (\zeta - Q(z))^{-1}.$$

By [6, II - §1.3] we know that the resolvent depends locally analytically on the variables $\zeta$ and $z$. More precisely if $\zeta_0 \notin \sigma(Q(z_0))$ for $z_0 \in \Omega$ then $R(\zeta, Q(z))$ is holomorphic in $\zeta$ and real-analytic in $z$ in a small neighborhood of $\zeta_0$ and in a small neighborhood of $z_0$.

**Remark 3.10.** The proof in [6, II - §1.3] is given in the case of finite dimensional spaces. However, it can be extended directly to bounded operators on Banach spaces.

By [6, IV - §3.5] we know that the simple eigenvalue $t^2_0(z)$ depends continuously on $Q(z)$. Thus, by Proposition 3.7 and the continuity of $t^2_0(z)$ there exists, for $h > 0$ small enough, a constant $D > 0$ such that for all $z$ in a neighborhood of a point $z_0 \in \Omega$

$$t^2_0(z) > \frac{h}{D}.$$

Define $\gamma$ to be the positively oriented circle of radius $h/(2D)$ centered at $0$ and consider the spectral projection of $Q(z)$ onto the eigenspace associated with $t^2_0(z)$

$$\Pi_{t^2_0(z)} = \frac{1}{2\pi i} \int_{\gamma} R(\zeta, Q(z))d\zeta.$$

Since the resolvent $R(\zeta, Q(z))$ is smooth in $z$ it follows that $\Pi_{t^2_0(z)}$ is smooth in $z$. Now set $e(x, z)$ to be a smooth quasimode for $P_h - z$ for $z \in \Omega$, as in Section 3 which depends smoothly on $z$. Thus, by setting

$$e_0(x, z, h) = \frac{\Pi_{t^2_0(z)}e_{wkb}(x, z, h)}{\|\Pi_{t^2_0(z)}e_{wkb}(-, z, h)\|},$$

we deduce that also $e_0(x, z)$ depends smoothly on $z$. The statement for $f_0(z)$ follows by performing the same argument for $\bar{Q}(z)$ instead of $Q(z)$ and with the quasimode $f_{wkb}$.

Using that $\Pi_{t^2_0(z)}$ and $Q(z)$ are smooth and that the operator $\Pi_{t^2_0(z)}Q(z)\Pi_{t^2_0(z)}$ has finite rank we see by

$$t^2_0(z) = \text{tr} \left( \Pi_{t^2_0(z)}Q(z)\Pi_{t^2_0(z)} \right)$$

that $t^2_0(z)$ is smooth.

In the case of $z \in \Omega \cap \Omega^a_\eta$ for $h^{2/3} < \eta < \text{const.}$ we follow the exact same steps as above, mutandis mutandis. We take the estimate $t^2_0(z) > \frac{h\sqrt{\eta}}{2D}$ for $z$ in a neighborhood of a fixed $z_0 \in \Omega \cap \Omega^a_\eta$ (following from Proposition 3.7) and thus we pick, as above, $\gamma$ to be the positively oriented circle of radius $h\sqrt{\eta}/(2D)$ centered at $0$. Hence, for $z \in \Omega \cap \Omega^a_{\eta}$

$$\Pi_{t^2_0(z)} = \frac{1}{2\pi i} \int_{\gamma} R(\zeta, Q(z))d\zeta, \quad e_0(x, z, h) = \frac{\Pi_{t^2_0(z)}e_{wkb}^\eta(x, z, h)}{\|\Pi_{t^2_0(z)}e_{wkb}^\eta(-, z, h)\|}.$$
Proposition 3.11. Let \( z \in \Omega \Subset \Sigma \) and let \( e_0 \) and \( f_0 \) be the eigenfunctions of the operators \( Q \) and \( \tilde{Q} \) with respect to their smallest eigenvalue (as in Section 4.1). Let \( S = S(z) \) be defined as in Definition 2.1. Then

- for \( z \in \Omega \) with \( \text{dist}(\Omega, \partial \Sigma) > 1/C \) and for all \((n, m) \in \mathbb{N}^2\)
  \[
  \| \partial_z^n \partial_{\bar{z}}^m (e_0 - e_{wkb}) \|, \| \partial_z^n \partial_{\bar{z}}^m (f_0 - f_{wkb}) \| = \mathcal{O}(h^{-(n+m)} e^{-\frac{z}{h}}). \tag{3.25}
  \]

Furthermore, the various \( z \)- and \( \bar{z} \)-derivatives of \( e_0, f_0, e_{wkb}, \text{and } f_{wkb} \) have at most temperate growth in \( 1/h \), more precisely

- for \( h^{2/3} \ll \eta < \text{const.}, z \in \Omega \cap \Omega_{\eta}^n \) and for all \((n, m) \in \mathbb{N}^2\)
  \[
  \| \partial_z^n \partial_{\bar{z}}^m (e_0 - e_{wkb}) \|, \| \partial_z^n \partial_{\bar{z}}^m (f_0 - f_{wkb}) \| = \mathcal{O}(\eta^{\frac{n+m}{2}} h^{-(n+m)} e^{-\frac{\eta}{h}}). \tag{3.27}
  \]

Furthermore, the various \( z \)- and \( \bar{z} \)-derivatives of \( e_0, f_0, e_{wkb}, \text{and } f_{wkb} \) have at most temperate growth in \( \sqrt{k}/h \), more precisely

- for all \( n, m \in \mathbb{N} \).

Remark 3.12. Let us recall that

- for \( z \in \Omega \Subset \hat{\Sigma} \), in the case where \( \Omega \) is independent of \( h > 0 \) and has a positive distance to the boundary of \( \Sigma \) we have \( 1/C \leq S \leq C \) for some constant \( C > 0 \). Thus, we may formulate the corresponding estimates of Proposition 3.11 uniformly in \( z \);
- for \( h^{2/3} \ll \eta < \text{const.} \) and \( z \in \Omega \cap \Omega_{\eta}^n \), (3.4) implies estimates uniform in \( z \) but \( \eta \) dependent.

This implies the following

**Corollary 3.13. Under the assumptions of Proposition 3.11,**

- for \( z \in \Omega_i \) there exists a constant \( C > 0 \) such that for all \((n, m) \in \mathbb{N}^2\)
  \[
  \| \partial_z^n \partial_{\bar{z}}^m (e_0 - e_{wkb}) \|, \| \partial_z^n \partial_{\bar{z}}^m (f_0 - f_{wkb}) \| = \mathcal{O}(h^{-(n+m)} e^{-\frac{\eta}{h}}). \tag{3.29}
  \]

- for \( h^{2/3} \ll \eta < \text{const.}, z \in \Omega \cap \Omega_{\eta}^{a,b} \) and for all \((n, m) \in \mathbb{N}^2\)
  \[
  \| \partial_z^n \partial_{\bar{z}}^m (e_0 - e_{wkb}) \|, \| \partial_z^n \partial_{\bar{z}}^m (f_0 - f_{wkb}) \| = \mathcal{O}(\eta^{\frac{n+m}{2}} h^{-(n+m)} e^{-\frac{\eta}{h}}). \tag{3.30}
  \]

**Remark 3.14.** The proof of Proposition 3.11 is unfortunately somewhat long and technical. Thus, we have split it into several lemmas.

**Lemma 3.15.** Let \( \Omega \Subset \Sigma \) such that \( \text{dist}(\Omega, \partial \Sigma) > 1/C \). For \( z \in \Omega \) define \( r := r(x, z; h) := Q(z)e_{wkb}(x; z) \) as in (3.12). Then, for all \( n, m \in \mathbb{N}, \supp \partial_z^n \partial_{\bar{z}}^m r \subset [x_- - 2\pi, x_- - 2\pi + h^{1/2}], [x_+ - h^{1/2}, x_+] \) and

- \( \| \partial_z^n \partial_{\bar{z}}^m r \| = \mathcal{O}(h^{1-(n+m)} e^{-\frac{z}{h}}). \)

**Proof.** Using (3.14) and (3.16) we may conclude by the Leibniz rule that for \((n, m) \in \mathbb{N}^2\)

\[
\partial_z^n \partial_{\bar{z}}^m r = \mathcal{O}(h^{\frac{z}{2}-(n+m)}) e^k \phi_z(x;z). \]
which is supported in \( |x_- - 2\pi, x_- - 2\pi + h^{1/2} | \). Thus, by the same reasoning as for the estimate on \( r \) in the proof of Proposition 3.7 above, we conclude
\[
\| \partial_z^n \partial_x^m r \| = O \left( h^{3-2(n+m)e^{-2\varepsilon/\pi}} \right) \int_{|x_- - h^{1/2}, x_- + h^{1/2}|} dx = O \left( h^{2-2(n+m)e^{-2\varepsilon/\pi}} \right).
\]

**Lemma 3.16.** Let \( \Omega \subset \Sigma \) such that dist \((\Omega, \partial \Sigma) > 1/C \) and let \( z \in \Omega \). Moreover, let \( \Pi_{t_0}^2 : L^2(S^1) \to C_0 \) denote the spectral projection of \( Q(z) \) onto the eigenspace associated with \( t_0^2 \). Then,
\[
\| \partial_z^n \partial_x^m \Pi_{t_0}(z) \|_{L^2 \to H^2_{sc}} = O \left( h^{n+m/2} \right).
\]

**Proof.** By virtue of Proposition 3.7 and the continuity of \( t_0^2(z) \) there exists for \( h > 0 \) small enough a constant \( D > 0 \) such that for all \( z \) in a neighborhood of a point \( z_0 \in \Omega \)
\[
t_0^2(z) > h / D.
\]
Define \( \gamma \) to be the positively oriented circle of radius \( h / (2D) \) centered at 0. Note that \( \gamma \) is locally independent of \( z \). Thus, we gain a path such that 0, \( t_0^2(z) \notin \gamma \) and which has length \( |\gamma| = h\pi / D \). For \( \lambda \in \gamma \) we have that
\[
\| (\lambda - Q(z))^{-1} \| = \frac{1}{\text{dist} (\lambda, \sigma(Q(z)))} = O(|\gamma|^{-1}).
\]
By (3.24) and the resolvent identity we see that
\[
\partial_z (\lambda - Q(z))^{-1} = -(\lambda - Q(z))^{-1} (P_h - z) (\lambda - Q(z))^{-1}
\]
as well as
\[
\partial_x (\lambda - Q(z))^{-1} = -(\lambda - Q(z))^{-1} (P_h - z) (\lambda - Q(z))^{-1}.
\]
The higher derivatives \( \partial_z^\alpha \partial_x^\beta (\lambda - Q(z))^{-1} \), for \( (n, m) \in \mathbb{N} \times \mathbb{N} \setminus \{0\} \), are finite linear combinations of terms of the form
\[
(\lambda - Q(z))^{-1} \partial_z^{\alpha_1}(Q(z))(\lambda - Q(z))^{-1} \cdots \partial_x^{\alpha_k}(Q(z))(\lambda - Q(z))^{-1}
\]
with \( \alpha_j = (1, 0), (0, 1), (1, 1) \) and \( \alpha_1 + \cdots + \alpha_k = (n, m) \). Thus it is sufficient to estimate the terms of the form \((P_h - z)(Q(z) - \lambda)^{-1}\) and \((P_h - z)^*(Q(z) - \lambda)^{-1}\). Since \( Q(z) = (P_h - z)^*(P_h - z) \), it follows that
\[
\| (P_h - z) u \|^2 - |\gamma| \| u \|^2 \leq \| (Q(z) - \lambda) u \| \leq \| (Q(z) - \lambda) u \| \| u \|.
\]
Since \( Q(z) > 0 \) is self-adjoint and since dist \((\lambda, \sigma(Q(z))) \approx |\gamma| \) we have the a priori estimate
\[
\| (Q(z) - \lambda) u \| \geq C |\gamma| \| u \|
\]
for all \( u \in H^2_{sc}(S^1) \), where \( C > 0 \) is a constant locally uniform in \( z \). This implies
\[
\| (P_h - z) u \|^2 \leq \| (Q(z) - \lambda) u \| + |\gamma| \| u \| \| u \|
\leq C \| (Q(z) - \lambda) u \| \| u \| \leq \frac{C}{|\gamma|} \| (Q(z) - \lambda) u \|^2;
\]
where \( C > 0 \) is a constant uniform in \( z \). Hence
\[
\| (P_h - z)(Q(z) - \lambda)^{-1} \|_{L^2 \to L^2} = O \left( |\gamma|^{-1/2} \right).
\]
Finally, note that since \( [P_h^*, P_h] = O_{H^2_{sc} \to L^2}(h) \) we can replace \( P_h \) by its adjoint in (3.34) and gain the estimate
\[
\| (P_h - z)^*(Q(z) - \lambda)^{-1} \|_{L^2 \to L^2} = O \left( |\gamma|^{-1/2} \right).
\]
We conclude that for all \( (n, m) \in \mathbb{N} \times \mathbb{N} \setminus \{0\} \)
\[
\| \partial_z^n \partial_x^m (\lambda - Q(z))^{-1} \|_{L^2 \to H^2_{sc}} = O \left( |\gamma|^{-n+m/2} \right).
\]
Similarly, using (3.14), the stationary phase method implies

\[ \| \partial_z^n \partial_\omega^m (\lambda - Q(z))^{-1} \|_{L^2 \rightarrow H^{2n} c} = O\left( h^{-\frac{n+m+2}{2}} \right). \tag{3.35} \]

Since for \( u \in L^2(S^1) \)

\[ \frac{1}{2\pi i} \int_{\gamma} (\lambda - Q(z))^{-1}ud\lambda = \Pi_0 u, \]

(3.35) implies

\[ \| \partial_z^n \partial_\omega^m \Pi_0(z) \|_{L^2 \rightarrow H^{2n} c} = O\left( h^{-\frac{n+m}{2}} \right). \]

**Lemma 3.17.** Under the assumptions of Lemma 3.16 we have

\[ \| \partial_z^n \partial_\omega^m e_{wkb}(\cdot; z) \|, \| \partial_z^n \partial_\omega^m \Pi_0 e_{wkb}(\cdot; z) \| = O(h^{-n-m}). \]

**Proof.** Consider

\[ \partial_z e_{wkb}(x; z) = h^{-\frac{1}{4}} \left\{ \partial_z \chi(x, z; h) a^i(z; h) + \chi(x, z; h)\partial_z a^i(z; h) \right\} e^{\frac{i}{h} \phi_+(x; z)} \]

and use the triangular inequality

\[ \| \partial_z e_{wkb}(-; z) \| \leq h^{-\frac{1}{4}} \| \partial_z \chi(-; z) a^i(z; h) e^{\frac{i}{h} \phi_+(-; z)} \| + h^{-\frac{1}{4}} \| \chi(-; z) \partial_z a^i(z; h) e^{\frac{i}{h} \phi_+(-; z)} \| \]

\[ + h^{-\frac{1}{4}} \| \chi(-; z) a^i(z; h) \| \frac{i}{h} \| \partial_z \phi_+(-; z) e^{\frac{i}{h} \phi_+(-; z)} \|. \]

By (3.13) we see that \( \partial_z \chi(x, z; h) = O(h^{-1/2}) \) and is supported in \( [x_-, 2\pi - x_-, 2\pi + h^{1/2} \cup \] \( x_-, h^{1/2}, x_- \). Thus, by the same argument as we have used for (3.19)

\[ h^{-\frac{1}{4}} \| \partial_z \chi(-; z) a^i(z; h) e^{\frac{i}{h} \phi_+(-; z)} \| = O\left( h^{-\frac{1}{2}} e^{-\frac{h}{2}} \right). \]

Since \( \partial_z a^i(z; h) = O(1) \), the stationary phase method (see the proof of Lemma 3.3) implies

\[ h^{-\frac{1}{4}} \| \chi(-; z) \partial_z a^i(z; h) e^{\frac{i}{h} \phi_+(-; z)} \| = O(1). \]

Furthermore, since

\[ \partial_z \phi_+(x; z) = \int_{x_+(z)}^x dy - \xi_+(z) \partial_z x_+(z) \tag{3.36} \]

it follows by the stationary phase method that

\[ h^{-\frac{1}{4}} \| \chi(-; z) a^i(z; h) \| \frac{i}{h} \| \partial_z \phi_+(-; z) e^{\frac{i}{h} \phi_+(-; z)} \| = \frac{1}{h} |\xi_+(z) \partial_z x_+(z)| + O(1). \]

Hence, by putting all of the above together

\[ \| \partial_z e_{wkb}(-; z) \| = O(h^{-1}). \]

To estimate the higher \( z \)- and \( \omega \)-derivatives of \( e_{wkb} \) recall that \( \partial_z^n \partial_\omega^m a^i(z; h) = O(1) \). Similarly, using (3.14), the stationary phase method implies

\[ \| \partial_z^n \partial_\omega^m e_{wkb}(-; z) \| = O(h^{-n-m}). \]

**Lemma 3.16** then implies by the Leibniz rule that

\[ \| \partial_z^n \partial_\omega^m \Pi_0 e_{wkb} \| = O(h^{-n-m}). \]
Remark 3.18. As in Lemma 3.17, we have for $z \in \Omega$ with $\text{dist} (\Omega, \partial \Sigma) > 1/C$
\[ \| \partial_z^p \partial_{\Sigma}^m f_{\wkb} (\cdot, z) \| = O (h^{-n-m}) \]
and
\[ \| \partial_z^p \partial_{\Sigma}^m \tilde{\Pi}_0 e_{\wkb} \| = O (h^{-n-m}) . \]
where $\tilde{\Pi}_0 : L^2 (S^1) \rightarrow Cf_0$ is the spectral projection of $\tilde{Q} (z)$ onto the eigenspace associated
with the eigenvalue $\lambda_0$.

Proof of Proposition 3.11. Part I - First, suppose that $z \in \Omega$ with $\text{dist} (\Omega, \partial \Sigma) > 1/C$.
Let $r$ be as in Lemma 3.15 and consider for
\[ (\lambda - Q (z)) e_{\wkb} = \lambda e_{\wkb} - r. \]
If $\lambda \notin \sigma (Q (z)) \cup \{0\}$ we have
\[ (\lambda - Q (z))^{-1} e_{\wkb} = \frac{1}{\lambda} e_{\wkb} - \frac{1}{\lambda} (\lambda - Q (z))^{-1} r. \]
As in the proof of Lemma 3.15, define $\gamma$ to be the positively oriented circle of radius $h/(2D)$
centered at 0. $\gamma$ is locally independent of $z$. Thus, we gain a path such that $0, \partial^2 \gamma (z) \notin \gamma$
and which has length $|\gamma| = h\pi / D$. Hence
\[ \frac{1}{2\pi i} \int_{\gamma} (\lambda - Q (z))^{-1} e_{\wkb} d\lambda = e_{\wkb} - \frac{1}{2\pi i} \int_{\gamma} \frac{1}{\lambda} (\lambda - Q (z))^{-1} r d\lambda. \quad (3.37) \]
By Lemma 3.15, (3.23) and (3.31)
\[ \left\| \frac{1}{2\pi i} \int_{\gamma} \frac{1}{\lambda} (\lambda - Q (z))^{-1} r d\lambda \right\| = O \left( e^{-\frac{\gamma}{\pi}} \right). \]
Thus, by (3.37)
\[ \| \Pi_{\tilde{\Pi}_0^2} e_{\wkb} - e_{\wkb} \| = O \left( e^{-\frac{\gamma}{\pi}} \right). \quad (3.38) \]
Recall that $e_{\wkb}$ is normalized. Pythagoras’ theorem then implies
\[ \| \Pi_{\tilde{\Pi}_0^2} e_{\wkb} \|^2 = \| e_{\wkb} \|^2 - \| e_{\wkb} - \Pi_{\tilde{\Pi}_0^2} e_{\wkb} \|^2 = 1 - O \left( e^{-\frac{2G}{\pi}} \right) \]
which yields
\[ e_0 = \frac{1}{\Pi_{\tilde{\Pi}_0^2} e_{\wkb}} = \frac{1}{\Pi_{\tilde{\Pi}_0^2} e_{\wkb}} = \left( 1 + O \left( e^{-\frac{2G}{\pi}} \right) \right) \Pi_{\tilde{\Pi}_0^2} e_{\wkb}. \quad (3.40) \]
Let us now turn to the $z$- and $\Sigma$-derivatives of $e_0 - e_{\wkb}$. By (3.40)
\[ \| \partial_z^p \partial_{\Sigma}^m (e_0 (z) - e_{\wkb} (z)) \| = \| \partial_z^p \partial_{\Sigma}^m \left( \frac{\Pi_{\tilde{\Pi}_0^2} e_{\wkb} (z)}{\| \Pi_{\tilde{\Pi}_0^2} e_{\wkb} (z) \|} - e_{\wkb} (z) \right) \| \]
\[ = \| \partial_z^p \partial_{\Sigma}^m \left( \frac{(\Pi_{\tilde{\Pi}_0^2} - 1) e_{\wkb} + (1 - \| \Pi_{\tilde{\Pi}_0^2} e_{\wkb} \|) e_{\wkb}}{\| \Pi_{\tilde{\Pi}_0^2} e_{\wkb} (z) \|} \right) \| . \]
First, note that Lemma 3.26 together with (3.39) implies
\[ \partial_z^p \partial_{\Sigma}^m \| \Pi_{\tilde{\Pi}_0^2} e_{\wkb} \| = O (h^{-n-m}) . \]
Using this result and (3.39) implies by the Leibniz rule applied to (3.40) that
\[ \| \partial_z^p \partial_{\Sigma}^m e_0 \| = O (h^{-n-m}) . \]
Next, applying Lemma 3.15 and (3.35) to (3.37) yields
\[ \| \partial_z^p \partial_{\Sigma}^m (\Pi_{\tilde{\Pi}_0^2} - 1) e_{\wkb} \| = \| \partial_z^p \partial_{\Sigma}^m \frac{1}{2\pi i} \int_{\gamma} \frac{1}{\lambda} (\lambda - Q (z))^{-1} r d\lambda \| = O \left( h^{-(n+m)} e^{-\frac{\gamma}{\pi}} \right). \]
Thus, Lemma 3.26 and (3.39) together with the Leibniz rule then imply
\[
\|\partial_z^m \partial_\zeta^n (e_0(z) - e_{wkb}(z))\| = O\left(\hbar^{-(n+m)} e^{-\frac{2}{\hbar}}\right).
\]

Part II - Now, let \( z \in \Omega^\eta \) with \( \hbar^2 \ll \eta < \text{const} \). The statements of the proposition follow from a simple rescaling argument. For the rescaling we use the same notation as in the beginning of Section 3. Let \( \tilde{e}_0(\tilde{z}) \) be the \( L^2(S^1/\sqrt{\eta}, d\tilde{x}) \)-normalized eigenfunction of the operator \( \tilde{Q}(\tilde{z}) = (\tilde{P}_h - \tilde{z})^*(\tilde{P}_h - \tilde{z}) \) and note that \( \eta^2 e_{wkb} \) is \( L^2(S^1/\sqrt{\eta}, d\tilde{x}) \)-normalized. Thus,
\[
\left\| \partial_z^m \partial_\zeta^n (\tilde{e}_0(\tilde{z}) - e_{wkb}(\cdot, \tilde{z}, \hbar)) \right\|_{L^2(S^1/\sqrt{\eta}, d\tilde{x})} = O\left(\hbar^{-(n+m)} e^{-\frac{2}{\hbar}}\right),
\]
where \( \tilde{S} \) is as in (3.22). Since \( e_0(z) = \eta^{-1/4} \tilde{e}_0(\tilde{z}) \), it follows by rescaling that
\[
\left\| \partial_z^m \partial_\zeta^n (e_0(z) - e_{wkb}(z)) \right\|_{L^2(S^1, dx)} = O\left(\eta^{\frac{n+m}{2}} \hbar^{-(n+m)} e^{-\frac{2}{\hbar}}\right).
\]
The results on \( \left\| \partial_z^m \partial_\zeta^n e_{wkb} \right\| \) and on \( \left\| \partial_z^m \partial_\zeta^n e_0 \right\| \) can be proven by the same rescaling argument.

Remark 3.19. The analogous result for \( f_{wkb}(z) \), \( f_0(z) \) and \( f_0(z) \) can be proven in the same way with the occasional obvious adaption. \( \square \)

4. GRUSHIN PROBLEM FOR THE UNPERTURBED OPERATOR \( P_h \)

Let us recall from Section 3 that if we write \( z \in \Omega \Subset \Sigma \) we shall always suppose that \( \text{dist}(\Omega, \partial \Sigma) > C \hbar^{2/3} \).

We know by [9] that the eigenvalues of \( P_h^\delta \) are given as the zeros of the function \( E_{\pm}^\delta(z) \) which arises in a Grushin problem for \( P_h^\delta \). The aim of this section is to construct three different Grushin problems for the unperturbed operator \( P_h^\delta \): one that is valid everywhere in \( \Sigma \) but less explicit (here we will follow the construction given in [8, Sec. 7.2 and 7.4]) and two very explicit Grushin problems which are only valid in the interior respectively close to the boundary of \( \Sigma \) (here we will recall the construction given by Hager in [5] respectively Bordeaux-Montrieux in [1]).

4.1. Grushin problem valid in all of \( \Sigma \). Following the ideas of [8], we will use the eigenfunctions \( e_0 \) and \( f_0 \) to set up the Grushin problem.

Proposition 4.1. Let \( z \in \Omega \Subset \Sigma \) and let \( a_0 \) be as in (2.5). Define
\[
R_+: H^1(S^1) \to \mathbb{C}: u \mapsto (u e_0)
\]
\[
R_- : \mathbb{C} \to L^2(S^1): u_- \mapsto u_- f_0.
\]
Then
\[
\mathcal{P}(z) := \begin{pmatrix} P_h - z & R_- \\ R_+ & 0 \end{pmatrix} : H^1(S^1) \times \mathbb{C} \to L^2(S^1) \times \mathbb{C}
\]
is bijective with the bounded inverse
\[
\mathcal{E}(z) = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{++}(z) \end{pmatrix}
\]
where \( E_-(z)v = (v | f_0) \), \( E_+(z)v_+ = v_+ e_0 \) and \( E(z) = (P_h - z)^{-1}|(f_0)_+ \to (e_0)_+ \) and \( E_{++}(z)v_+ = -a_0 v_+ \). Furthermore, we have the estimates
- for \( z \in \Omega \) with \( \text{dist}(\Omega, \partial \Sigma) > 1/C \)
  \[
  \|E_-(z)\|_{L^2 \to \mathbb{C}} \|E_+(z)\|_{\mathbb{C} \to H^1} = O(1),
  \|E(z)\|_{L^2 \to H^1} = O(h^{-1/2}),
  \|E_{++}(z)\| = O\left(\sqrt{h} e^{-\frac{2}{\hbar}}\right) = O\left(e^{\frac{1}{\sqrt{\hbar}}}\right);
  \]

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• for $z \in \Omega \cap \Omega^a_\eta$ with $h^{\frac{2}{3}} \ll \eta < \text{const}$.

\[ \|E_-(z)\|_{L^2 \to C}, \|E_+(z)\|_{C \to H^1} = O(1), \]
\[ \|E(z)\|_{L^2 \to H^1} = O((h \sqrt{\eta})^{-1/2}), \]
\[ |E_{-+}(z)| = O(\sqrt{\eta} \frac{1}{h} e^{-\frac{\pi}{2}}) = O\left(e^{-\frac{\pi}{2h}}\right). \] (4.3)

**Proof.** For a proof of the existence of the bounded inverse as well as the estimate for $\|E(z)\|_{L^2 \to H^1}$ in the case of $\text{dist}(\Omega, \partial \Sigma) > 1/C$ see [8, Section 7.2].

The other estimate for $\|E(z)\|_{L^2 \to H^1}$ can be proven by performing the same steps as in the case of $\|E(z)\|_{L^2 \to H^1}$, mutandis mutandis, together with the estimate given by Bordeaux-Monttrieux in [1, Proposition 4.3.5]. The estimates for $|E_{-+}(z)|$ follow from Proposition 3.7, whereas the estimates on $\|E_-(z)\|_{L^2 \to C}$ and $\|E_+(z)\|_{C \to H^1}$ come from the fact that $e_0$ and $f_0$ are normalized.

Alternatively, one can conclude the result in the case of $z \in \Omega \cap \Omega^a_\eta$ by a rescaling argument similar to the one in the proof of Proposition 3.11.

\[ \square \]

### 4.2. Tunneling

We prove now the following formula for a tunnel effect from which we conclude Proposition 2.7.

**Proposition 4.2.** Let $z \in \Omega \in \Sigma$ and let $e_0$ and $f_0$ be as in (2.4) and in (2.6). Furthermore, let $\Phi(z, h)$ be as in Proposition 2.4, let $S$ be as in Definition 2.1 and let $p$ and $\rho_\pm$ be as in Section 1. Let $h^{\frac{2}{3}} \ll \eta < \text{const}$. Then, for all $z \in \Omega$ with $|\text{Im} z - \langle \text{Im} g \rangle| > 1/C$, $C > 1$,

\[ |(e_0|f_0)| = \left(\frac{\left(\frac{1}{2} \{p, \overline{p}\}\{\rho_+\}^2 \{p, \overline{p}\}\{\rho_-\}^2\}}{\sqrt{\pi h}} \right) \frac{1}{\sqrt{\pi h}} |\partial_{\text{Im} z} S(z)| \left(1 + O\left(\eta^{-\frac{3}{2}} h^{\frac{1}{2}} \right)\right) e^{-\frac{\pi}{2}} \]

where for all $(n, m) \in \mathbb{N}^2$

\[ \partial_n^m \partial_{\overline{z}}^m \partial_{\text{Re} z} \partial_{\text{Im} z} \left(\eta^{-3/4} h^{\frac{1}{2}} \right) = O\left(\eta^{\frac{n+m}{2}} - \frac{3}{4} h^{-(n+m)+\frac{1}{2}} \right). \]

This implies Proposition 2.7. Furthermore, Proposition 4.2 implies by direct calculation the following result:

**Proposition 4.3.** Under the assumptions of Proposition 6.5 we have for $h^{2/3} \ll \eta < \text{const}$.

\[ \partial_{\text{Im} z} |(e_0|f_0)|^2 = \frac{2 \left(\frac{1}{2} \{p, \overline{p}\}\{\rho_+\}^2 \{p, \overline{p}\}\{\rho_-\}^2\}}{\pi h^2} \left|\partial_{\text{Im} z} S(z)\right|^2 \left(-\partial_{\text{Im} z} S(z)\right) e^{-\frac{2\pi}{h}} \]
\[ + O\left(\eta^{5/4} h^{-\frac{3}{2} e^{-\frac{2\pi}{h}}} \right), \]
\[ \partial_{\text{Re} z} |(e_0|f_0)|^2, \partial_{\text{Re} z} \partial_{\text{Im} z} |(e_0|f_0)|^2 = O\left(e^{-\frac{\pi}{h} e^{-\frac{2\pi}{h}}} \right). \]

**Remark 4.4.** Let us point out that we can find an even more detailed formula for $|(e_0|f_0)|$ (cf. (4.8)) valid even for $|\text{Im} z - \langle \text{Im} g \rangle| \geq 1/C$:

\[ |(e_0|f_0)| = \left(\frac{\left(\frac{1}{2} \{p, \overline{p}\}\{\rho_+\}^2 \{p, \overline{p}\}\{\rho_-\}^2\}}{\sqrt{\pi h}} \right) \frac{1}{\sqrt{\pi h}} |\partial_{\text{Im} z} S(z)| \left(1 + \frac{2\pi}{\left|\partial_{\text{Im} z} S(z)\right|} e^{\Phi(z, h)} \right) \]
\[ + O\left(e^{-\frac{\pi}{h}} \right) + O\left(\eta^{3/4} h^{-\frac{3}{2}} e^{-\frac{2\pi}{h} + \Phi(z, h)} \right). \]

However, the result of Proposition 4.2 is sufficient for the purposes of this work.
Proof of Proposition 4.2. First, suppose that $z \in \Omega$ with $\text{dist}(\Omega, \partial \Sigma) > 1/C$. Then, by Proposition 3.11

\[
(e_0|f_0) = (e_0|f_{wkb}) + O\left(e^{-\frac{z}{\pi}}\right) = (e_{wkb}|f_{wkb}) + O\left(e^{-\frac{z}{\pi}}\right). \tag{4.4}
\]

Recall the definition of the quasimodes $e_{wkb}$ and $f_{wkb}$ from Section 3. Moreover, recall from Section 1 that by the natural projection $\Pi : \mathbb{R} \to S^1$ we identify $S^1$ with the interval $[x_-(z) - 2\pi, x_-(z)]$. This choice leads to the fact that $\phi_+$ is given by

\[
\phi_+(x) = \int_{x_+(z)}^x (z - g(y)) \, dy
\]
on this interval, whereas $\phi_-$ is given by

\[
\phi_-(x) = \begin{cases} 
\int_{x_-(z)}^x (z - g(y)) \, dy, & \text{for } x \in [x_+(z), x_-(z)], \\
\int_{x_-(z) - 2\pi}^{x_-(z)} (z - g(y)) \, dy, & \text{for } x \in [x_-(z) - 2\pi, x_+(z)].
\end{cases}
\]

Define

\[
R := \frac{a\tilde{\beta}}{\sqrt{h}} = \frac{\left(\frac{1}{2} \{p, \tilde{p}\}(\rho_+) + \frac{1}{2} \{\tilde{p}, p\}(\rho_-)\right)^{\frac{1}{4}}}{\sqrt{\pi}} + O(\sqrt{h}), \tag{4.5}
\]

where we used Lemma 3.3, Proposition 3.4 and (4.20) to gain the equality. Consider

\[
(e_{wkb}|f_{wkb}) = R \int \chi_e(x)\chi_f(x)e^{\frac{i}{h} \phi_+(x) - \tilde{\beta}_-(x)} \, dx
\]

\[
= R \int_{x_-(z)}^{x_+(z)} \chi_e(x)\chi_f(x)e^{\frac{i}{h} \phi_+(x) - \tilde{\beta}_-(x)} \, dx
\]

\[
+ R \int_{x_-(z) - 2\pi}^{x_-(z)} \chi_e(x)\chi_f(x)e^{\frac{i}{h} \phi_+(x) - \tilde{\beta}_-(x)} \, dx,
\]

and note that

\[
\phi_+(x) - \phi_-(x) = \begin{cases} 
\int_{x_+(z)}^{x_-(z)} (z - g(y)) \, dy, & \text{for } x \in [x_+(z), x_-(z)], \\
\int_{x_+(z)}^{x_-(z) - 2\pi} (z - g(y)) \, dy, & \text{for } x \in [x_-(z) - 2\pi, x_+(z)].
\end{cases}
\]

Thus we may conclude

\[
(e_{wkb}|f_{wkb}) = Re^{\frac{i}{h} \int_{x_+(z)}^{x_-(z) - 2\pi} (z - g(y)) \, dy} \int_{x_-(z) - 2\pi}^{x_+(z)} \chi_e(x)\chi_f(x) \, dx
\]

\[
+ Re^{\frac{i}{h} \int_{x_+(z)}^{x_-(z)} (z - g(y)) \, dy} \int_{x_+(z)}^{x_-(z) - 2\pi} \chi_e(x)\chi_f(x) \, dx. \tag{4.6}
\]
Recall from (3.5) and Definition 3.6 the definitions of the cut-off function $\chi_{e,f}$. Thus
\[
\int_{x_-(z) - 2\pi}^{x_+(z)} \chi_e(x) \chi_f(x) \, dx = x_+(z) - (x_-(z) - 2\pi)
- \int_{x_- - 2\pi + \sqrt{\hbar}}^{x_- - 2\pi} (1 - \chi_e(x)) \, dx - \int_{x_+ + 2\pi - \sqrt{\hbar}}^{x_+ + 2\pi} (1 - \chi_f(x)) \, dx
= x_+(z) - (x_-(z) - 2\pi) + \mathcal{O}(\sqrt{\hbar}),
\]
\[
\int_{x_+(z)}^{x_-(z)} \chi_e(x) \chi_f(x) \, dx = x_-(z) - x_+(z)
- \int_{x_- + \sqrt{\hbar}}^{x_-} (1 - \chi_e(x)) \, dx - \int_{x_+ - \sqrt{\hbar}}^{x_+} (1 - \chi_f(x)) \, dx
= x_-(z) - x_+(z) + \mathcal{O}(\sqrt{\hbar}).
\]

(4.7)

Now let us assume that we are below the spectral line of $P_h$, i.e. $\text{Im} \, z \leq \langle \text{Im} \, g \rangle$. There, we see that
\[
|\langle e_{\text{wkb}} | f_{\text{wkb}} \rangle| = \text{Re} -\frac{1}{\hbar} \text{Im} \int_{x_-}^{x_+} (z - g(y)) \, dy \left| (x_-(z) - x_+(z)) + \mathcal{O}(\sqrt{\hbar}) \right|
+ \left| x_+(z) - (x_-(z) - 2\pi) + \mathcal{O}(\sqrt{\hbar}) \right| e^{-\frac{2\pi}{\hbar}(z(y))}.
\]

Analogously, if we are above the spectral line, i.e. $\text{Im} \, z \geq \langle \text{Im} \, g \rangle$,
\[
|\langle e_{\text{wkb}} | f_{\text{wkb}} \rangle| = \text{Re} -\frac{1}{\hbar} \text{Im} \int_{x_+}^{x_-} (z - g(y)) \, dy \left| (x_+(z) - x_-(z) - 2\pi) + \mathcal{O}(\sqrt{\hbar}) \right|
+ \left| x_- (z) - x_+(z) + \mathcal{O}(\sqrt{\hbar}) \right| e^{-\frac{2\pi}{\hbar}(z(y))}.
\]

Together with (4.4), we conclude that
\[
|\langle e_0 | f_0 \rangle| = \left(\frac{1}{\sqrt{\pi \hbar}} \left\{ p, \eta \right\} (\rho_+) \frac{1}{2} \left\{ p, \eta \right\} (\rho_-) \right)^{\frac{1}{2}} e^{-\frac{\eta}{\hbar}} |\partial \text{Im} \, z S(z)| \left(1 + \frac{2\pi - |\partial \text{Im} \, z S(z)|}{|\partial \text{Im} \, z S(z)|} e^{\Phi(z,h)} \right)
+ \mathcal{O}(e^{-\frac{\eta}{\hbar}}) + \mathcal{O}(\eta^{3/4} h^{-\frac{1}{4}} e^{-\frac{\eta}{\hbar} + \Phi(z,h)})
\]
(4.8)

where $\Phi(z, h)$ is as in Proposition 2.4. Note that $\exp \{ -\Phi(z, h) \}$ is exponentially small for $|\text{Im} \, z - \langle \text{Im} \, g \rangle| > 1/C$. Together with (4.4), we conclude that
\[
|\langle e_0 | f_0 \rangle| = \left(\frac{1}{\sqrt{\pi \hbar}} \left\{ p, \eta \right\} (\rho_+) \frac{1}{2} \left\{ p, \eta \right\} (\rho_-) \right)^{\frac{1}{2}} e^{-\frac{\eta}{\hbar}} |\partial \text{Im} \, z S(z)| \left(1 + \mathcal{O}(\eta^{-3/4} h^{1/2}) \right).
\]
(4.9)

Now let us discuss the $\partial_z^\mu \partial_z^m$-derivatives of the errors. First let us treat the error term $\mathcal{O}(\sqrt{\hbar})$ from the definition of $R$ which is given as a product of the normalization coefficients of the quasimodes $e_{\text{wkb}}$ and $f_{\text{wkb}}$. Thus, it is easy to see that
\[
\partial_z^\nu \partial_z^m \mathcal{O}(\sqrt{\hbar}) = \mathcal{O}(h^{-(n+m-1/2)}).
\]
(4.10)

The $\partial_z^\nu \partial_z^m$-derivatives of the error term in (4.7) can be treated as follows: note that
\[
\partial_z \int_{x_- - 2\pi}^{x_- - 2\pi + \sqrt{\hbar}} (1 - \chi_e(x; z)) \, dx = \partial_z \chi_e(x; z) \left( \chi_e(x_- - 2\pi; z) - \chi_e(x_- - 2\pi + \sqrt{\hbar}; z) \right)
- \int_{x_- - 2\pi}^{x_- - 2\pi + \sqrt{\hbar}} \partial_z \chi_e(x; z) \, dx.
\]
By (3.13)
\[
\int_{x_- - 2\pi}^{x_- - 2\pi + \sqrt{h}} \partial_z \chi_e(x; z) dx = - \int_{x_- - 2\pi}^{x_- - 2\pi + \sqrt{h}} \psi \left( \frac{x - x_- + 2\pi}{\sqrt{h}} \right) \partial_z x_-(z) dx = - \partial_z x_-(z).
\]
Since \( \chi_e(x_- - 2\pi; z) = 0 \) and \( \chi_e(x_- - 2\pi + \sqrt{h}; z) = 1 \),
\[
\partial_z \int_{x_- - 2\pi}^{x_- - 2\pi + \sqrt{h}} (1 - \chi_e(x; z)) dx = 0.
\]
The other integrals in (4.7) as well as the respective \( \tau \)-derivatives can be treated completely analogously. Thus, we may conclude that \( \partial_z^n \partial_{\tau}^m O(\sqrt{h}) = 0 \) for all \( (n, m) \in (\mathbb{N} \times \mathbb{N}) \setminus \{0\} \).

Hence, we conclude
\[
\partial_z^n \partial_{\tau}^m O\left(\eta^{-3/4} h^{1/2}\right) = O\left(\eta^{n+m+3/2} h^{-(n+m+1/2)} e^{-\frac{S}{\pi}}\right),
\]
Finally, in the case where \( z \in \Omega \cap \Omega^a \eta \) we can conclude the statement by a rescaling argument similar as in the proof of Proposition 3.11. \( \square \)

**Remark 4.5.** It is a direct consequence of (4.6), (4.4) and Proposition 3.11 that
\[
\partial_z^n \partial_{\tau}^m (e_0|f_0) = O\left(\eta^{n+m+3/2} h^{-(n+m+1/2)} e^{-\frac{S}{\pi}}\right),
\]
where we conclude the case where \( z \in \Omega \cap \Omega^a \eta \) by a rescaling argument similar as in the proof of Proposition 3.11.

**Proof of Proposition 4.3.** The first statement follows directly from Proposition 4.2. The statements regarding the derivatives can be derived by a direct calculation from Proposition 4.2 together with the fact that the \( z \)-respectively the \( \tau \)-derivative of the error term increases its growth at most by a term of order \( \eta^{1/2} h^{-1} \). Moreover, we use that \( e^\Phi \) is exponentially small in \( h \) due to \( |\text{Im} \ z - (\text{Im} \ g)| > 1/C \). Furthermore, we use that the prefactor \( \left( e \frac{1}{2} \{ p, p \} ( p_+ + \frac{1}{2} \{ p, p \} ( p_- ) \right)^{3/2} \) is the first order term of \( R \) (cf. (4.5)). Recall that \( R \) is defined via the normalization coefficients of the quasimodes \( e_{kbb} \) and \( f_{kbb} \). It is thus independent of \( \text{Re} \ z \) and its \( \partial_{m \ z} \) derivative is of order \( O(\eta^{-1/4}) \) which can be seen by the stationary phase method and a rescaling argument similar to the one in the proof of Proposition 3.11. \( \square \)

Now let us give estimates on the derivatives of the effective Hamiltonian \( E_{-+}(z) \).

**Proposition 4.6.** Let \( z \in \Omega \subseteq \Sigma \) and let \( E_{-+}(z) \) be as in Proposition 4.1. Then there exists a \( C > 0 \) such that for \( h > 0 \) small enough and all \( n, m \in \mathbb{N}_0 \)
\[
|\partial_z^n \partial_{\tau}^m E_{-+}(z)| = O\left(\eta^{n+m+1/2} h^{-(n+m)+1/2} e^{-\frac{S}{\pi}}\right).
\]

**Proof.** Take the \( \partial_{\tau} \) derivative and the \( \partial_z \) derivative of the first equation in (2.5) to gain
\[
(P_h - z) \partial_{\tau} e_0 = (\partial_{\tau} a_0) f_0 + a_0 \partial_z f_0 \quad \text{and} \quad (P_h - z) \partial_z e_0 - e_0 = (\partial_z a_0) f_0 + a_0 \partial_\tau f_0.
\]
Now consider the scalar product of these equations with \( f_0 \) and recall from Proposition 4.1 that \( E_{-+}(z) = -a_0(z) \) to conclude
\[
\partial_z E_{-+}(z) = E_{-+}(z) \{ (\partial_z e_0|e_0) - (\partial_{\tau} f_0|f_0) \} \quad \text{and} \quad \partial_{\tau} E_{-+}(z) = E_{-+}(z) \{ (\partial_{\tau} e_0|e_0) - (\partial_z f_0|f_0) \} + (e_0|f_0).
\]
The statement of the Proposition then follows by repeated differentiation of (4.11) and induction using Remark 4.5, the estimate \( |E_{-+}(z)| = O(\eta^{1/4} h^{1/2} e^{-\frac{S}{\pi}}) \) given in (4.2) and (4.3) and the estimates given in Proposition 3.11. \( \square \)

Finally, Proposition 4.2 permits us to prove the following extension of Proposition 3.11:
Proposition 4.7. Let \( z \in \Omega \subseteq \Sigma \) and let \( e_0 \) and \( f_0 \) be the eigenfunctions of the operators \( Q \) and \( \bar{Q} \) with respect to their smallest eigenvalue (as in Section 4.1). Let \( S = S(z) \) be defined as in Definition 2.1. Then

- for \( z \in \Omega \) with \( \text{dist} (\Omega, \partial \Sigma) > 1/C \) and for all \( n, m, k \in \mathbb{N} \)
  \[
  \| \partial_z^n \partial_{\bar{z}}^m D_z^k (e_0 - e_{\text{wkb}}) \|, \| \partial_z^n \partial_{\bar{z}}^m D_z^k (f_0 - f_{\text{wkb}}) \| = O \left( h^{-(n+m+k)} e^{-\frac{z}{\Delta}} \right). 
  \]
  Furthermore, the various \( z \)-, \( \bar{z} \)- and \( x \)-derivatives of \( e_0, f_0, e_{\text{wkb}} \) and \( f_{\text{wkb}} \) have at most temperate growth in \( 1/h \), more precisely
  \[
  \| \partial_z^n \partial_{\bar{z}}^m D_z^k e_{\text{wkb}} \|, \| \partial_z^n \partial_{\bar{z}}^m D_z^k f_{\text{wkb}} \|, \| \partial_z^n \partial_{\bar{z}}^m D_z^k e_0 \|, \| \partial_z^n \partial_{\bar{z}}^m D_z^k f_0 \| = O \left( h^{-(n+m+k)} \right)
  \]
  for all \( n, m, k \in \mathbb{N} \);

- for \( h^{2/3} < \eta < \text{const.} \), \( z \in \Omega \cap \Omega_{\eta} \) and for all \( n, m, k \in \mathbb{N} \)
  \[
  \| \partial_z^n \partial_{\bar{z}}^m D_z^k (e_0 - e_{\text{wkb}}) \|, \| \partial_z^n \partial_{\bar{z}}^m D_z^k (f_0 - f_{\text{wkb}}) \| = O \left( \eta^{\frac{n+m+k}{2}} h^{-(n+m+k)} e^{-\frac{z}{\Delta}} \right). 
  \]
  Furthermore, the various \( z \)-, \( \bar{z} \)- and \( x \)-derivatives of \( e_0, f_0, e_{\text{wkb}} \) and \( f_{\text{wkb}} \) have at most temperate growth in \( \sqrt{h} \), more precisely
  \[
  \| \partial_z^n \partial_{\bar{z}}^m D_z^k e_{\text{wkb}} \|, \| \partial_z^n \partial_{\bar{z}}^m D_z^k f_{\text{wkb}} \|, \| \partial_z^n \partial_{\bar{z}}^m D_z^k e_0 \|, \| \partial_z^n \partial_{\bar{z}}^m D_z^k f_0 \| = O \left( \eta^{\frac{n+m+k}{2}} h^{-(n+m+k)} \right)
  \]
  for all \( n, m, k \in \mathbb{N} \).

Proof. Will show the proof in the case of \( e_0(z) \) since the case of \( f_0(z) \) is similar. Suppose first that \( z \in \Omega \) with \( \text{dist} (\Omega, \partial \Sigma) > 1/C \). Recall from (2.5) that

\[
(P_h - z)e_0 = \alpha_0 f_0 \quad \text{and} \quad (P_h - z)^* f_0 = \overline{\alpha_0} e_0 \quad (4.12)
\]

First consider the \( \partial_z^n \partial_{\bar{z}}^m \) derivatives of (4.12):

\[
(P_h - z)\partial_z^n \partial_{\bar{z}}^m e_0(z) = n\partial_z^{n-1} \partial_{\bar{z}}^m e_0(z) \quad \text{and} \quad \sum_{|a_1 + \beta_1| = n \atop |a_2 + \beta_2| = m} \left( \frac{\eta + \beta}{\beta} \right) (\partial_{\bar{z}}^\beta \alpha_0(z)) (\partial_z^n f_0(z))
\]

\[
(P_h - z)^* \partial_z^n \partial_{\bar{z}}^m f_0(z) = m\partial_z^{n-1} \partial_{\bar{z}}^m f_0(z) \quad \text{and} \quad \sum_{|a_1 + \beta_1| = n \atop |a_2 + \beta_2| = m} \left( \frac{\eta + \beta}{\beta} \right) (\partial_{\bar{z}}^\beta \overline{\alpha_0}(z)) (\partial_z^n e_0(z)) \quad (4.13)
\]

and thus

\[
h\| D_z \partial_z^n \partial_{\bar{z}}^m e_0(z) \| \leq n\| \partial_z^{n-1} \partial_{\bar{z}}^m e_0(z) \| + \sum_{|a_1 + \beta_1| = n \atop |a_2 + \beta_2| = m} \left( \frac{\eta + \beta}{\beta} \right) \| \partial_{\bar{z}}^\beta \alpha_0(z) \| \| \partial_z^n f_0(z) \|
\]

\[
+ \| g - z \|_{L^\infty(S^1)} \cdot \| \partial_z^n \partial_{\bar{z}}^m e_0(z) \|
\]

and

\[
h\| D_z \partial_z^n \partial_{\bar{z}}^m f_0(z) \| \leq m\| \partial_z^{n-1} \partial_{\bar{z}}^m f_0(z) \| + \sum_{|a_1 + \beta_1| = n \atop |a_2 + \beta_2| = m} \left( \frac{\eta + \beta}{\beta} \right) \| \partial_{\bar{z}}^\beta \overline{\alpha_0}(z) \| \| \partial_z^n e_0(z) \|
\]

\[
+ \| g - z \|_{L^\infty(S^1)} \cdot \| \partial_z^n \partial_{\bar{z}}^m f_0(z) \|.
\]

By Proposition 4.6, there exists a constant \( C > 0 \) such that

\[
| \partial_z^k \partial_{\bar{z}}^l \alpha_0(z) | = | \partial_z^k \partial_{\bar{z}}^l E_{\text{wkb}}(z) | = O \left( h^{-(k+j)} e^{-\frac{z}{\Delta}} \right).
\]

By (3.26) we conclude

\[
\| D_z \partial_z^n \partial_{\bar{z}}^m e_0(z) \|, \| D_z \partial_z^n \partial_{\bar{z}}^m f_0(z) \| = O \left( h^{-(n+m+1)} \right).
\]
Repeated differentiation of (4.13) and induction then yield that for all \( l \in \mathbb{N} \)
\[
\|D_x^l \partial_z^m \partial_x^n e_0(z)\|, \|D_x^l \partial_z^m \partial_x^n f_0(z)\| = O\left( h^{-\left(l+n+m\right)} \right).
\]
The estimate
\[
\|D_x^l \partial_z^m \partial_x^n e_{wkb}\|, \|D_x^l \partial_z^m \partial_x^n f_{wkb}\| = O\left( h^{-\left(l+n+m\right)} \right)
\]
follows directly by the stationary phase method (cf. the proofs of Proposition 3.7 and Lemma 3.17). Finally, using (3.15), consider
\[
(P_h - z)(e_0 - e_{wkb}) = \alpha_0 f_0 - h^{-\frac{1}{2}} a(z) \frac{h}{l} \partial_x \chi e^{\frac{1}{i} \phi_+(x)}
\]
which implies for \( k \geq 1 \)
\[
(hD_x)^k \partial_z^m \partial_x^n (e_0 - e_{wkb}) = (hD_x)^{(k-1)} \partial_z^m \partial_x^n (\alpha_0 f_0) - (hD_x)^{(k-1)} \partial_z^m \partial_x^n \left( h^{-\frac{1}{2}} a(z) \frac{h}{l} \partial_x \chi e^{\frac{1}{i} \phi_+(x)} \right).
\]
By induction together with Proposition 3.11 and the improved estimate on \( \partial_z^m \partial_x^n E_{-} \) given in Proposition 3.7, we conclude the first point of the Proposition. The results in the case where \( z \in \Omega \cap \Omega^n \) follow by a rescaling argument similar as in the proof of Proposition 3.11.

4.3. Alternative Grushin problems for the unperturbed operator \( P_h \). In [5] Hager set up a different Grushin problem for \( P_h \) and \( z \in \Omega \), which results in a more explicit effective Hamiltonian \( E^{H+}_-(z) \). To aver confusion, we will mark the elements of Hager’s Grushin problem with an additional “\( H \)”.

Bordeaux-Monttrieux in [1] then extended Hager’s Grushin problem to \( z \in \Omega \cap \Omega^n \). It is very useful for the further discussion to have an explicit effective Hamiltonian. Thus we will briefly introduce Hager’s Grushin problem \( \mathcal{P}^H \) and show that \( E_{-}^{H+}(z) \) and \( E_{-}^{H}(z) \) differ only by an exponentially small error.

**Proposition 4.8** ([5, 1]). For \( z \in \Omega \subseteq \Sigma \), let \( x_{\pm}(z) \in \mathbb{R} \) be as in Section 1.  
- for \( z \in \Omega \) with dist \((\Omega, \partial \Sigma) > 1/C\) let \( I_{\pm} \) be open intervals, independent of \( z \) such that 
  \[
x_{\pm}(z) \in I_{\pm}, \quad x_{\pm}(z) \notin \overline{I_{\pm}} \quad \text{for all } z \in \overline{\Omega}.
\]

Let \( \phi_\pm(x; z) \) be as in Definition 3.2. Then, there exist smooth functions \( c_\pm(z; h) > 0 \) such that
\[
c_\pm(z; h) \sim h^{-\frac{1}{4}} \left( c_{\pm}^0(z) + h c_{\pm}^1(z) + \ldots \right)
\]
and, for \( e_+(z; h) := c_+(z; h) \exp\left( \frac{i \phi_+ (x; z)}{h} \right) \in H^1(I_+) \) and \( e_-(z; h) := c_-(z; h) \exp\left( \frac{i \phi_- (x; z)}{h} \right) \in H^1(I_-) \),
\[
\| e_+ \|_{L^2(I_+)} = 1 = \| e_- \|_{L^2(I_-)}.
\]
Furthermore, we have
\[
c_\pm^0(z) = \left( \frac{-\text{Im} \ g'(x_{\pm}(z))}{\pi} \right)^{\frac{1}{4}}, \quad \text{and } c_\pm^0(z) = \left( \frac{\text{Im} \ g'(x_{\pm}(z))}{\pi} \right)^{\frac{1}{4}}.
\]
- for \( z \in \overline{\Omega} \cap \Omega^n \) with \( h^{2/3} \ll \eta < \text{const.} \) let \( J_{\pm} \) be open intervals, such that
  \[
x_{\pm}(\overline{\Omega^n}) \subset J_{\pm}, \quad \text{dist} (J_+, J_-) > \frac{1}{C} \eta^{1/2}.
\]
Define \( \tilde{I}_{\pm} := S^1 \setminus J_{\pm} \). Let \( \phi_{\pm}(x; z) \) be as in Definition 3.2 and set \( \tilde{h} := h/\eta^{3/2} \). Then, there exist smooth functions \( c_\pm(z; \tilde{h}) > 0 \) such that
\[
c_\pm(z; \tilde{h}) \sim \tilde{h}^{-\frac{1}{4}} \eta^{-1/4} \left( c_{\pm}^0(z) + \tilde{h} c_{\pm}^1(z) + \ldots \right)
\]
and, for $c^\eta_+ (z; h) := c_+ (z; \tilde{h}) \exp (\frac{i \phi \cdot (x, z)}{h}) \in H^1 (\mathcal{I}_+)$ and $c^\eta_- (z; h) := c_- (z; \tilde{h}) \exp (\frac{i \phi \cdot (x, z)}{h}) \in H^1 (\mathcal{I}_-)$,
\[
\| c^\eta_+ \|_{L^2 (\mathcal{I}_+)} = 1 = \| c^\eta_- \|_{L^2 (\mathcal{I}_-)}.
\]

Furthermore, we have
\[
c^{0, \eta}_+ (z) = \left( \frac{\text{Im} \, g'' (a) (\tilde{x}_+ (\tilde{z}) - a / \sqrt{\eta}) (1 + o(1))}{\pi} \right)^{\frac{1}{2}}, \quad z \in \Omega^a_\eta,
\]
\[
c^{0, \eta}_- (z) = \left( \frac{\text{Im} \, g'' (a) (\tilde{x}_- (\tilde{z}) - a / \sqrt{\eta}) (1 + o(1))}{\pi} \right)^{\frac{1}{2}}, \quad z \in \Omega^a_\eta.
\]

Proof. For a proof of the first statement see [5]. The second statement has been proven in [1] with the exception of the representation of $c^{0, \eta}_+ (z)$ which can be achieved by an analogous argument to the one used in the proof of Proposition 3.4.

Note that $(P_h - z) c^\bullet_+ (x; z) = 0$ on $I_+$ and that $(P_h - z) c^\bullet_- (x; z) = 0$ on $I_-$. With these quasimodes Hager and then Bordeaux-Monttrieux set up a Grushin problem $\mathcal{P}^H$ and proved the existence of an inverse $\mathcal{E}^H$.

Proposition 4.9 ([5]). For $z \in \Omega^a_\eta \subseteq \Sigma$ and $x_\pm (z)$ as in Section 1. Let $g \in C^\infty (S^1 : \mathbb{C})$ be as in (1.1) and let $a < b < a + 2\pi$ where $a$ denotes the minimum and $b$ the maximum of $\text{Im} \, g$. Let $J_+ \subset (b, a + 2\pi)$ and $J_- \subset (a, b)$ such that $\{ x_\pm (z) : z \in \Omega^a_\eta \} \subset J_\pm$. Let $\chi_\pm \in C^\infty_c (I_\pm)$ be such that $\chi_\pm \equiv 1$ on $\overline{J_\pm}$ and $\text{supp} (\chi_+) \cap \text{supp} (\chi_-) = \emptyset$. Define
\[
R^H_+ : H^1 (S^1) \rightarrow \mathbb{C} : u \mapsto (u |_{\chi_+} \cdot e_+),
\]
\[
R^H_- : \mathbb{C} \rightarrow L^2 (S^1) : u_- \mapsto u_- \cdot e_-.
\]

Then
\[
\mathcal{P}^H (z) := \begin{pmatrix} P_h - z & R^H_+ \\ R^H_- & 0 \end{pmatrix} : H^1 (S^1) \times \mathbb{C} \rightarrow L^2 (S^1) \times \mathbb{C}
\]
is bijective with the bounded inverse
\[
\mathcal{E}^H (z) = \begin{pmatrix} E^H_+ (z) & E^H_+ (z) \\ E^H_- (z) & E^H_- (z) \end{pmatrix}
\]
where
\[
\| E^H_+ (z) \|_{L^2 \rightarrow H^1} = \mathcal{O} (h^{-1/2}),
\]
\[
\| E^H_- (z) \|_{L^2 \rightarrow \mathbb{C}} = \mathcal{O} (1),
\]
\[
\| E^H_+ (z) \|_{\mathbb{C} \rightarrow H^1} = \mathcal{O} (1),
\]
\[
| E^H_- (z) | = \mathcal{O} \left( e^{- \frac{4}{\pi} h \cdot \eta} \right).
\]

Furthermore,
\[
E^H_- (z) = \left( \frac{i}{2} \left\{ p, \mathcal{P} \right\} (\rho_+) \frac{i}{2} \left\{ \mathcal{P}, p \right\} (\rho_-) \right)^{\frac{1}{2}} \left( \frac{h}{\pi} \right)^{\frac{1}{2}} \left( \frac{h}{\pi} \right)^{\frac{1}{2}} + \mathcal{O} \left( h^2 \right).
\]

Furthermore,
\[
E^H_- (z) = \left( \frac{i}{2} \left\{ p, \mathcal{P} \right\} (\rho_+) \frac{i}{2} \left\{ \mathcal{P}, p \right\} (\rho_-) \right)^{\frac{1}{2}} \left( \frac{h}{\pi} \right)^{\frac{1}{2}} \left( \frac{h}{\pi} \right)^{\frac{1}{2}} + \mathcal{O} \left( h^2 \right).
\]

where the prefactor of the exponentials depends only on $\text{Im} \, z$ and has bounded derivatives of order $\mathcal{O} (\sqrt{h})$.

Proof. See [5].
Proposition 4.10 ([1]). Let \( \Omega \subset \Sigma \). For \( z \in \Omega \cap \Omega^n_{\eta} \) and \( x_{\pm}(z) \) as in Section 1. Let \( g \in C^\infty(S^1) \) be as in (1.1). Let \( J_{\pm} \) and \( I_{\pm} \) be as in the second point of Proposition 4.8. Let \( \chi_{\pm}^n \in C^\infty(I_{\pm}) \) such that \( \chi_{\pm}^n \equiv 1 \) on \( J_{\pm} \) and \( \text{supp}(\chi_{\pm}^n) \cap \text{supp}(\chi^n_{\eta}) = \emptyset \). Define
\[
R^n_+: H^1(S^1) \rightarrow \mathbb{C} : u \mapsto (u|\chi^+_n)
\]
\[
R^n_- : \mathbb{C} \rightarrow L^2(S^1) : u_- \mapsto u_-\chi^-n.
\]
Then
\[
\mathcal{P}^n(z) := \begin{pmatrix} p_n - z & R^n_+ \\ R^n_- & 0 \end{pmatrix} : H^1(S^1) \times \mathbb{C} \rightarrow L^2(S^1) \times \mathbb{C}
\]
is bijective with the bounded inverse
\[
\mathcal{E}^n(z) = \begin{pmatrix} E^n_0(z) & E^n_+(z) \\ E^n_0(z) & E^n_+(z) \end{pmatrix}
\]
where
\[
\|E^n_0(z)\|_{L^2 \rightarrow H^1} = \mathcal{O}((\sqrt{\eta}h)^{-1/2}),
\]
\[
\|E^n_0(z)\|_{L^2 \rightarrow \mathbb{C}} = \mathcal{O}(1),
\]
\[
\|E^n_0(z)\|_{\mathbb{C} \rightarrow H^1} = \mathcal{O}(1),
\]
\[
|E^n_+(z)| = \mathcal{O}\left(\eta^{1/4}h^{1/2}e^{-\frac{z \eta^{3/2}}{\kappa}}\right).
\]
Furthermore,
\[
E^n_-(z) = \left( c_+^0(z)c_-^0(z)(h\sqrt{\eta})^{1/2} + \mathcal{O}\left(h^{3/2}\eta^{-5/4}\right) \right) \left( e^{\frac{3}{4} \int_{x^+_0}^{x^-_0}(z-g(y))dy} - e^{\frac{3}{4} \int_{x^+}^{x^-_0}(z-g(y))dy} \right),
\]
where the prefactor of the exponentials depends only on \( \text{Im} \ z \) and has bounded derivatives of order \( \mathcal{O}(\sqrt{h/\eta}) \).

Proof. See [1]. (4.17) has not been stated in this form on [1]. However, it can easily be deduce from the results in [1] together with Proposition 4.8. \( \square \)

Remark 4.11. The cut-off function \( \chi_{\pm}^n \) in the above proposition can be chosen similarly to \( \chi_{e,f}^n \) in Definition 3.6 (compare also with Definition 3.2).

4.4. Estimates on the effective Hamiltonians. Note that Hager chose to represent \( S^1 \) as an interval between two of the periodically appearing minima of \( \text{Im} \ g \) and thus chose the notation for \( x_{\pm} \) accordingly (this notation was used in (4.15)). In our case however, we chose to represent \( S^1 \) as an interval between two of the periodically appearing maxima of \( \text{Im} \ g \). This results in the following difference between notations:
\[
x_+(z) = x^H_+(z) - 2\pi \quad \text{and} \quad x_-(z) = x^H_-(z).
\]

Thus, in our notation, we have for \( \bullet = H, \eta \)
\[
E_{-+}^\bullet (z) = V^\bullet (z, h) \left( e^{\frac{3}{4} \int_{x^+}^{x^-}(z-g(y))dy} - e^{\frac{3}{4} \int_{x^+}^{x^-}(z-g(y))dy} \right),
\]
where
\[
V^\bullet (z, h) = \begin{cases} \left( \frac{3}{4} \{p, \bar{p}\}(p_+) \frac{1}{2} \{p, p\}(p_-) \right)^{1/2} \left( \frac{3}{4} \right)^{1/2} (1 + \mathcal{O}(h)) & \text{if} \ \bullet = H, \ z \in \Omega_i \\
\left( c_+^0(z)c_-^0(z)(h\sqrt{\eta})^{1/2} + \mathcal{O}\left(\eta^{-5/4}\right) \right) & \text{if} \ \bullet = \eta, \ z \in \Omega^\eta_i,
\end{cases}
\]
Note that Taylor expansion around the point \( a \) yields
\[
\{p, \bar{p}\}(p_+) = -2i \text{Im} \ g'(x_+) = 2i \sqrt{\eta} \left| \text{Im} \ g''(a)(\tilde{x}_\pm(z) - a/\sqrt{\eta})(1 + o_{\sqrt{\eta}}(1)) \right|, \text{ for } z \in \Omega^\eta_i.
\]
Therefore, we may write for all $z \in \Omega \subset \Sigma$
\[
V(z, h) := V^*(z, h) = \left(\frac{i}{2} \{p, \bar{p}\}(\rho_+) \frac{i}{2} \{\bar{p}, p\}(\rho_-)\right)^\frac{1}{2} \left(\frac{h}{\pi}\right)^\frac{1}{2} \left(1 + \mathcal{O}(\eta^{-\frac{3}{2}}h)\right)
\]
(4.21)
where the first order term is $\eta^{1/4}$ for $z \in \Omega \cap \Omega^\eta_\eta$. Note that
\[
\left|e^{\frac{h}{2} \int x_+^{\eta}(z-g(y))dy} - e^{\frac{h}{2} \int x_+^{\eta}(z-g(y))dy}\right| = e^{-\frac{h}{2}x} \left|1 - e^{\Phi(z, h)}\right|,
\]
(4.22)
where $\Phi(z, h)$ is defined already in Proposition 2.4. For the readers convenience:
\[
\Phi(z, h) = \begin{cases} 
-\frac{2\pi i}{h}(z - \langle g \rangle), & \text{if } \Im z < \langle \Im g \rangle, \\
\frac{2\pi i}{h}(z - \langle g \rangle), & \text{if } \Im z > \langle \Im g \rangle,
\end{cases}
\]
Hence
\[
|E^*_{+}(z)| = V(z, h)e^{-\frac{h}{2}} \left|1 - e^{\Phi(z, h)}\right|.
\]
(4.23)
The aim of this section is to prove the following proposition.

**Proposition 4.12.** Let $\Omega \subset \Sigma$, let $\Phi(z, h)$ be as in Proposition 2.4 and let $E_{-+}(z)$ the effective Hamiltonian given in Proposition 4.1. Then, for $h > 0$ small enough, there exists a constant $C > 0$ such that for $h^{\frac{3}{2}} \ll \eta \leq \text{const}$.
\[
|E_{-+}(z)| = V(z, h)e^{-\frac{h}{2}} \left|1 - e^{\Phi(z, h)}\right| \left(1 + \mathcal{O}\left(e^{-\frac{\eta^{3/2}}{h}}\right)\right).
\]
Furthermore, for all $(n, m) \in \mathbb{N}^2$ the $\partial^m \partial^m_\eta$ derivatives of the error terms are bounded and of order
\[
\mathcal{O}\left(\eta^{\frac{n+m}{2}} h^{-(n+m)} e^{-\frac{\eta^{3}}{h}}\right).
\]

**Proof of Proposition 2.8.** Recall that $(P_h - z)e_0 = \alpha_0 f_0$ (cf. (2.5)). Suppose first that $z \in \Omega$ with $\text{dist}(\Omega, \partial \Sigma) > 1/C$. By approximating $f_0$ with the quasimode $f_{wkb}$ (cf. Proposition 3.11) we find
\[
((1 - \chi)(P_h - z)e_0|f_0) = \alpha_0(f_0)(1 - \chi)f_0 = \alpha_0 \left((f_{wkb}(1 - \chi)f_{wkb}) + \mathcal{O}\left(e^{-\frac{1}{\eta}}\right)\right).
\]
Since the phase $f_{wkb}$ has no critical point on the support of $\chi$, it follows that there exists a constant $C > 0$, depending on $\chi$ but uniform in $z \in \Omega$, such that
\[
((1 - \chi)(P_h - z)e_0|f_0) = \mathcal{O}\left(\alpha_0 e^{-\frac{1}{\eta}}\right).
\]
By a similar argument we find that
\[
((P_h - z)\chi e_0|f_0) = \alpha_0 (\chi e_0|e_0) = \mathcal{O}\left(\alpha_0 e^{-\frac{1}{\eta}}\right).
\]
In the case where $z \in \Omega \cap \Omega^\eta_\eta$, we perform a rescaling argument similar to the one in the proof of Proposition 3.11. Thus,
\[
((1 - \chi)(P_h - z)e_0|f_0), ((P_h - z)\chi e_0|f_0) = \mathcal{O}\left(\alpha_0 \exp\left(-\frac{\eta^{3}}{Ch}\right)\right).
\]
Note that Proposition 3.11 implies that each $z$- and $\Sigma$- derivative of the exponentially small error term increases its order of growth at most by factor of order $\mathcal{O}(\eta^{1/2}h^{-1})$. Thus, using (2.5) yields
\[
\alpha_0 = ((1 - \chi + \chi)(P_h - z)e_0|f_0) = ([P_h, \chi]e_0|f_0) + \mathcal{O}\left(\alpha_0 \exp\left(-\frac{\eta^{3}}{Ch}\right)\right)
\]
(4.24)
The statement of the Proposition then follows by the fact that \(|\alpha_0| = |E_{-}(z)|\) (cf. Proposition 4.1) together with Proposition 4.12.

Let us first give estimates on the various elements of the Grushin problems introduced in Section 4.

**Proposition 4.13.** Let \(\Omega \in \Sigma\), let \(E_{\bullet,+}, E_{\bullet,-}, R_{\bullet,\pm}, E^\bullet\) be as in the Propositions 4.1, 4.9 and 4.10, where \(\bullet = -, H, \eta\) with “-” symbolizing no index. Furthermore, let \(S(z)\) as in Definition 2.1. Then we have the following estimates

1. For \(\bullet = -, H\) and for \(z \in \Omega_t \subset \Omega\)
   \[
   \|\partial_z^\alpha \partial_z^\beta R_{\pm}\|, \|\partial_z^\alpha \partial_z^\beta E_{\pm}\| = O\left(h^{-(n+m)}\right),
   \|\partial_z^\alpha \partial_z^\beta E_{-}\| = O\left(h^{-(n+m-\frac{1}{2})}e^{-\frac{|z|}{h}}\right), \|\partial_z^\alpha \partial_z^\beta E^\bullet\| = O\left(h^{-(n+m+1/2)}\right).
   \]

2. For \(\bullet = -, \eta\) and for \(z \in \Omega_{\eta,b} \subset \Omega\)
   \[
   \|\partial_z^\alpha \partial_z^\beta R_{\pm}\|, \|\partial_z^\alpha \partial_z^\beta E_{\pm}\| = O\left(\eta^{\frac{n+m}{2}}h^{-(n+m)}\right),
   \|\partial_z^\alpha \partial_z^\beta E_{-}\| = O\left(\eta^{\frac{n+m+1}{2}}h^{-(n+m-\frac{1}{2})}e^{-\frac{2|\eta|}{h}}\right), \|\partial_z^\alpha \partial_z^\beta E^\bullet\| = O\left(\eta^{\frac{n+m-1}{2}}h^{-(n+m+1/2)}\right).
   \]

**Proof.** Recall the definition of \(R_{\pm}\) and \(E_{\pm}\) given in Proposition 4.1. By the estimates on the \(z\)- and \(\overline{z}\)-derivatives of \(e_0\) and \(f_0\) given in Proposition 3.11, we may conclude for \(z \in \Omega\) that

\[
\|\partial_z^\alpha \partial_z^\beta E_{+}\|_{L^2}, \|\partial_z^\alpha \partial_z^\beta R_{+}\|_{L^1} \leq \|\partial_z^\alpha \partial_z^\beta e_{0}\|_{L^2} = O\left(\eta^{\frac{n+m}{2}}h^{-(n+m)}\right),
\|\partial_z^\alpha \partial_z^\beta E_{-}\|_{L^2} \leq \|\partial_z^\alpha \partial_z^\beta f_{0}\|_{L^2} = O\left(\eta^{\frac{n+m}{2}}h^{-(n+m)}\right),
\]

and thus prove the corresponding “-”-cases in the Proposition. The estimates for the other cases of \(R_{\pm}\) and \(E_{\pm}\) then follow from (4.25), (4.26) and (4.30).

Recall from Proposition 4.1 that \(\mathcal{E}(z)\mathcal{P}(z) = 1\). Thus, note that

\[
\partial_z\mathcal{E}(z) + \mathcal{E}(z)(\partial_z\mathcal{P}(z))\mathcal{E}(z) = 0,
\partial_{\overline{z}}\mathcal{E}(z) + \mathcal{E}(z)(\partial_{\overline{z}}\mathcal{P}(z))\mathcal{E}(z) = 0,
\]

which implies

\[
\partial_z E = -E(\partial_z(P_h - z))E - E(\partial_z R_+)E - E(\partial_z R_-)E = E^2 - E(\partial_z R_+)E - E(\partial_z R_-)E
\]

and

\[
\partial_{\overline{z}} E(z) = -E_+(z)(\partial_{\overline{z}} R_+)E(z) - E(z)(\partial_{\overline{z}} R_-)E_-(z).
\]

Thus, by induction we conclude from this, from (4.25) and from Proposition 4.1 that for \(z \in \Omega\)

\[
\|\partial_z^\alpha \partial_z^\beta E(z)\| = O\left(\eta^{\frac{n+m-1}{2}}h^{-(n+m+\frac{1}{2})}\right).
\]

The estimates on \(\|\partial_z^\alpha \partial_z^\beta E^\bullet(z)\|\), for \(\bullet = \eta, H\), can be conclude by following the same steps and by using the corresponding estimates on \(R_{\pm}^\bullet\) and \(E_{\pm}^\bullet\) and the Propositions 4.9 and 4.10.

It remains to prove the estimates on \(\partial_z^\alpha \partial_z^\beta E_{-}\) and \(\partial_z^\alpha \partial_z^\beta E_{+}\): let us first consider the case where \(z \in \Omega_t \subset \Omega\). Recall (4.18) and recall from Proposition 4.9 that the prefactor \(V^H(z)\) has bounded \(z\)- and \(\overline{z}\)-derivatives of order \(O(\sqrt{h})\). Thus, the statement...
follows immediately.

In the case where \( z \in \Omega_{\eta}^{b} \subset \Omega \), recall again (4.18) note from Proposition 4.10 that the prefactor \( V_{n}^{\eta}(z) \) has bounded \( z- \) and \( \pi- \)derivatives of order \( O(\sqrt{h}/\sqrt{\eta}) \). Furthermore, note that

\[ e^{\frac{i}{h} \int_{z_{+}}^{z} (z-g(y))dy} - e^{\frac{i}{h} \int_{z_{+}}^{z} (z-g(y))dy} = e^{\frac{i}{h} \int_{z}^{z_{+} - 2\pi i} (z-g(y))dy} - e^{\frac{i}{h} \int_{z}^{z_{+} - 2\pi i} (z-g(y))dy}. \]

Thus, by (3.4)

\[ |\partial_{z}^{m} \partial_{\pi}^{n} \mathcal{E}_{+}(z)| = \eta^{-(n+m)} |\partial_{z}^{m} \partial_{\pi}^{n} \mathcal{E}_{+}(z)| \leq O \left( \frac{n+m+1/2}{h} e^{\frac{-\eta}{h}} \right). \]

Proof of Proposition 4.12. Let \( \bullet = H, \eta \) denote the quasimodes and elements of the Grushin problems corresponding to the different zones of \( z \).

Since \( \mathcal{P} \mathcal{E}^{\bullet} : L^{2}(S^{1}) \times C \rightarrow L^{2}(S^{1}) \times C \) let us introduce the following norm for an operator-valued matrix \( A : L^{2}(S^{1}) \times C \rightarrow L^{2}(S^{1}) \times C \):

\[ ||A||_{\infty} := \max_{1 \leq i \leq 2} \sum_{j=1}^{2} ||A_{ij}||, \]

where \( ||A_{ij}|| \) denotes the respective operator norm for \( A_{ij} \). Next, note that

\[ \mathcal{P} \mathcal{E}^{\bullet} = (\mathcal{P} + (\mathcal{P} - \mathcal{P}^{\bullet})) \mathcal{E}^{\bullet} = 1 + (\mathcal{P} - \mathcal{P}^{\bullet}) \mathcal{E}^{\bullet}. \]

Estimates for \( (\mathcal{P} - \mathcal{P}^{\bullet}) \) Recall the definition of \( \mathcal{P} \) and of \( \mathcal{P}^{\bullet} \) from the Propositions 4.1, 4.9 and 4.10 and note that

\[ \mathcal{P} - \mathcal{P}^{\bullet} = \begin{pmatrix} 0 & R_{-} - R_{+}^{\bullet} \\ R_{+} - R_{+}^{\bullet} & 0 \end{pmatrix}. \]

We will now prove that for all \( (n, m) \in \mathbb{N}^{2} \)

\[ ||\partial_{z}^{m} \partial_{\pi}^{n} (R_{+} - R_{+}^{\bullet})||_{H^{1}(S^{1}) \rightarrow C} \leq ||\partial_{z}^{m} \partial_{\pi}^{n} (e_{0} - \chi_{+}^{\bullet}e_{+}^{\bullet})|| \]

\[ = \begin{cases} O \left( h^{-(n+m)} e^{-\frac{\eta}{2h}} \right), & \text{for } z \in \Omega, \text{ dist } (\Omega, \partial \Sigma) > 1/C, \\ O \left( \eta^{\frac{n+m+1/2}{2}} h^{-(n+m)} e^{-\frac{\eta}{h}} \right), & \text{for } z \in \Omega_{\eta}, \end{cases} \]

where the first estimate follows from the Cauchy-Schwartz inequality. Note that

\[ ||\partial_{z}^{m} \partial_{\pi}^{n} (e_{0} - \chi_{+}^{\bullet}e_{+}^{\bullet})|| \leq ||\partial_{z}^{m} \partial_{\pi}^{n} (e_{wkb}^{\bullet} - \chi_{+}^{\bullet}e_{+}^{\bullet})|| + ||\partial_{z}^{m} \partial_{\pi}^{n} (e_{0} - e_{wkb}^{\bullet})||. \]

By Proposition 3.11 it remains to prove the desired estimate on \( ||\partial_{z}^{m} \partial_{\pi}^{n} (e_{wkb}^{\bullet} - \chi_{+}^{\bullet}e_{+}^{\bullet})|| \). Recall the definition of the quasimodes \( e_{wkb}^{\bullet} \) and \( e_{+}^{\bullet} \) from Section 3 and from Proposition 4.8.

Let us first consider the case of \( z \in \Omega \) with dist \( (\Omega, \partial \Sigma) > 1/C \): recall from Proposition 4.9 that all \( z- \) and \( \pi- \)derivatives of \( \chi_{+} \) are bounded independently of \( h > 0 \), whereas for the derivatives of \( \chi_{e} \) we have (3.14). Thus

\[ \partial_{z}^{m} \partial_{\pi}^{n} \chi_{+}, \partial_{z}^{m} \partial_{\pi}^{n} \chi_{e} = O(h^{-(n+m)/2}). \]
Thus, since $\chi_e(-; z) \geq \chi_+$ for all $z \in \overline{\Omega}_i$, which implies that $x_+(z) \notin \text{supp}(\chi_e(-; z) - \chi_+)$ for all $z \in \overline{\Omega}_i$, the Leibniz rule then implies

$$
\left\| \partial_z^n \partial_\xi^m \left( (\chi_e(-; z) - \chi_+) e^{\frac{i}{h} \phi_+(-z)} \right) \right\| = \left( \int \left| \partial_z^n \partial_\xi^m \left( (\chi_e(-; z) - \chi_+) e^{\frac{i}{h} \phi_+(-z)} \right) \right|^2 \, dx \right)^{\frac{1}{2}} \leq O \left( h^{-(n+m)} e^{\frac{F}{\pi}} \right). \tag{4.28}
$$

where $F > 0$ is given by the infimum of $\text{Im} \, \phi(x; z)$ over all $z \in \overline{\Omega}$ and all $x \in (\bigcup_{z \in \overline{\Omega}} \text{supp}(\chi_e(-; z))) \setminus \{ x \in I_+ : \chi_+ \equiv 1 \}$. Note that $F > 0$ is strictly positive because $x_-(z) \notin T_+$ for all $z \in \overline{\Omega}$ and $\chi_+ \in C_0^\infty(I_+)$ (cf. Propositions 4.9 and 4.8).

Recall that $h^{-1/4} a(z; h)$ and $c_+(z; h)$ are the normalization factors of $e_{\text{wkb}}$ and $e_+$ (cf. (3.6) and Proposition 4.8). Hence, for $z \in \Omega_i$,

$$
h^{-\frac{1}{4}} \partial_z^n \partial_\xi^m a(z; h), \partial_z^n \partial_\xi^m c_+(z; h) = O \left( h^{-(n+m+1/2)} \right).
$$

Thus the Leibniz rule implies

$$
\left| \partial_z^n \partial_\xi^m c_+(z; h) - \partial_z^n \partial_\xi^m h^{-1/4} a(z; h) \right| = \left\| \partial_z^n \partial_\xi^m \left( (\chi_e(-; z) e^{\frac{i}{h} \phi_+(-z)} ) - \left( (\chi_+ e^{\frac{i}{h} \phi_+(-z)} \right) \right\| \leq O \left( h^{-(n+m+1/2)} e^{-\frac{F}{\pi}} \right).
$$

Since $h^{-\frac{1}{4}} a(z; h), c_+(z; h) = O(h^{-\frac{1}{4}})$, the Leibniz rule and the above imply that for $z \in \Omega_i$

$$
\left\| \partial_z^n \partial_\xi^m \left( e_{\text{wkb}} - \chi_+ e_+ \right) \right\| = \left\| \partial_z^n \partial_\xi^m \left( h^{-1/4} a(z; h)(\chi_e(-; z) - \chi_+) e^{\frac{i}{h} \phi_+(-z)} + (h^{-1/4} a(z; h) - c_+(z; h)) \chi_+ e^{\frac{i}{h} \phi_+(-z)} \right) \right\| \leq O \left( h^{-(n+m+1/2)} e^{-\frac{F}{\pi}} \right).
$$

Thus there exists a constant $C > 0$, for $h > 0$ small enough, such that for $z \in \Omega_i$

$$
\left\| \partial_z^n \partial_\xi^m \left( e_{\text{wkb}} - \chi_+ e_+ \right) \right\| = O \left( h^{-(n+m)} e^{-\frac{F}{\pi}} \right). \tag{4.29}
$$

Now let us consider the case $z \in \Omega \cap \Omega_0^b$: recall the quasimodes $e_{\text{wkb}}^\eta$ and $e_+^\eta$ as given in Definition 3.6 and Proposition 4.8. A rescaling argument similar to the one in the proof of Proposition 3.11 then implies

$$
\left\| \partial_z^n \partial_\xi^m \left( e_{\text{wkb}}^\eta - \chi_+^\eta e_+^\eta \right) \right\| = \eta^{-(n+m)} O \left( h^{-(n+m+1/2)} e^{\frac{-2}{n} \frac{3}{2}} \right) = O \left( \eta^{n+m+3/2} h^{-(n+m+1/2)} e^{\frac{-2}{n} \frac{3}{2}} \right).
$$

Absorbing the factor $\eta^{3/4} h^{-1/2}$ into $e^{\frac{-2}{n} \frac{3}{2}}$ then yields the desired estimate.
It is possible to achieve an analogous estimate for $R_- - R^\bullet_-$, namely that for all $z \in \Omega$ and for all $(n, m) \in \mathbb{N}^2$\n\n$$\|\partial_\zeta^m \partial_\zeta^m (R_- - R^\bullet_-)\|_{C^0 \rightarrow H^1(S^1)} = \|\partial_\zeta^m \partial_\zeta^m (f_0 - \chi^\bullet e^\bullet)\|$$
\n\begin{align*}
&= \begin{cases}
\mathcal{O} \left( h^{-(n+m)} e^{-\frac{1}{h^n}} \right), & \text{for } z \in \Omega, \text{ dist } (\Omega, \partial \Sigma) > 1/C, \\
\mathcal{O} \left( \eta^{n+m} h^{-(n+m)} e^{-\frac{2}{h^n}} \right), & \text{for } z \in \Omega^\eta,
\end{cases} \\
&\quad (4.30)
\end{align*}

This can be achieved by analogous reasoning as for the estimate on $R_+ - R^\bullet_+$.\n
**A formula for $E_{-+}$**. It is easy to see, that for $h > 0$ small enough
\n$$\|(\mathcal{P} - \mathcal{P}^\bullet)\mathcal{E}^\bullet\|_{\infty} \ll 1.$$\n
Thus, $1 + (\mathcal{P} - \mathcal{P}^\bullet)\mathcal{E}^\bullet$ is invertible by the Neumann series, wherefore
\n$$\mathcal{P}\mathcal{E}^\bullet [1 + (\mathcal{P} - \mathcal{P}^\bullet)\mathcal{E}^\bullet]^{-1} = 1.$$\n
We conclude that
\n$$\mathcal{E} = \mathcal{E}^\bullet \sum_{n \geq 0} (-1)^n [(\mathcal{P} - \mathcal{P}^\bullet)\mathcal{E}^\bullet]^n.$$\n
Define $g_- := R_- - R^\bullet_-$ and $g_+ := R_+ - R^\bullet_+$. Hence, by Propositions 4.9 and 4.10 as well as by (4.27) and (4.26), there exists a constant $C > 0$ such that
\n$$(\mathcal{P} - \mathcal{P}^\bullet)\mathcal{E}^\bullet = \begin{pmatrix} g_- E_-^\bullet & g_- E_+^\bullet \\ g_+ E_-^\bullet & g_+ E_+^\bullet \end{pmatrix} = \begin{pmatrix} \mathcal{O} \left( e^{-\frac{1}{h^n}} \right) & E_-^\bullet + \mathcal{O} \left( e^{-\frac{1}{h^n}} \right) \\ \mathcal{O} \left( e^{-\frac{1}{h^n}} \right) & \mathcal{O} \left( e^{-\frac{1}{h^n}} \right) \end{pmatrix}.$$\n
By induction it follows that for $n \in \mathbb{N}$
\n$$[(\mathcal{P} - \mathcal{P}^\bullet)\mathcal{E}^\bullet]^n = \begin{pmatrix} \mathcal{O} \left( e^{-\frac{n}{h^n}} \right) & E_-^\bullet + \mathcal{O} \left( e^{-\frac{n}{h^n}} \right) \\ \mathcal{O} \left( e^{-\frac{n}{h^n}} \right) & \mathcal{O} \left( e^{-\frac{n}{h^n}} \right) \end{pmatrix}.$$\n
We conclude that
\n$$E_{-+}(z) = E_{-+}^\bullet \left( 1 + \sum_{n \geq 1} \mathcal{O} \left( e^{-\frac{n}{h^n}} \right) \right) = E_{-+}^\bullet \left( 1 + \mathcal{O} \left( e^{-\frac{1}{h^n}} \right) \right).$$\n
Finally, by the estimates on $g_+$ and $g_+$ obtained above and by the estimates given in Proposition 4.13 we conclude the desired estimates on the $z$- and $\zeta$-derivatives of the error term. \hfill $\square$

5. **Grushin problem for the perturbed operator $P_h^\delta$**

For $\delta > 0$ small enough, we can use the Grushin problem for the unperturbed operator $P_h$ to gain a well-posed Grushin problem for the perturbed operator $P_h^\delta$. However, before we can do this, let us introduce a cut-off in our probability space. We have the following result of Bordeaux-Montrieux, see [8]:

**Lemma 5.1.** Let $h > 0$. If $C > 0$ is large enough, then
\n$$\|Q_\omega\|_{HS} \leq \frac{C}{h}$$ \quad with probability $\geq 1 - e^{-\frac{C}{h^{1/2}}}.$\n
**Remark 5.2.** Since $\|Q_\omega\|_{HS} = \sum |a_{j,k}(\omega)|^2$, we can also view the bound on $\|Q_\omega\|_{HS}$ in the preceding Lemma as restricting the probability space to a ball of radius $C/h$. Hence, from now on we shall work in the restricted probability space and we shall assume that
for $z \in \Omega$, 

\[ \delta \ll h^{\frac{3}{2}}, \]

since this implies that $\| \delta Q_\Omega \|_{HS} \ll h^{1/2}$ which is natural since we want the perturbation to be small;

- for $h^{2/3} \ll \eta \leq \text{const. and } z \in \Omega_{a,b}$ 

\[ \delta \ll h^{\frac{3}{2}} \eta^{\frac{1}{4}}, \]

since this implies that $\| \delta Q_\Omega \|_{HS} \ll h^{1/2} \eta^{\frac{1}{4}}$.

**Proposition 5.3.** Let $z \in \Omega \in \Sigma$, let $h^{2/3} \ll \eta \leq \text{const. and let } R_-, R_+ \text{ be as in Proposition 4.1. Then}

\[ \mathcal{P}_\delta(z) := \begin{pmatrix} P_h^d - z & R_- \\ R_+ & 0 \end{pmatrix} : H^1(S^1) \times C \rightarrow L^2(S^1) \times C \]

is bijective with the bounded inverse

\[ \mathcal{E}_\delta(z) = \begin{pmatrix} E^\delta(z) & E^\delta_+(z) \\ E^-_\delta(z) & E^\delta_+(z) \end{pmatrix} \]

where

\[ E^\delta(z) = E(z) + O(\eta^{-1/2} \delta h^{-2}) = O(\eta^{-1/4} h^{-1/2}) \]

\[ E^\delta_-(z) = E_- (z) + O(\delta \eta^{-1/4} h^{-3/2}) = O(1) \]

\[ E^\delta_+(z) = E_+ (z) + O(\delta \eta^{-1/4} h^{-3/2}) = O(1) \]

and

\[ E^\delta_- (z) = E_- (z) - \delta \left( E_- Q_\Omega E_+ + \sum_{n=1}^{\infty} (-\delta)^n E_- Q_\Omega (E Q_\Omega)^n E_+ \right) \]

\[ = E_- (z) - \delta \left( E_- Q_\Omega E_+ + O(\delta \eta^{-1/4} h^{-5/2}) \right) \]

(5.1)

**Proof.** The statement follows immediately from Proposition 4.1 by use of the Neumann series.

By (4.2) we get

\[ E_- Q_\Omega E_+ = \sum_{|j|, |k| \leq \left\lceil \frac{C_1}{h} \right\rceil} \alpha_{j,k} (e_0 | e^j) \cdot (e^j | f_0) = \sum_{|j|, |k| \leq \left\lceil \frac{C_1}{h} \right\rceil} \alpha_{j,k} \hat{e}_0(k) \hat{f}_0(j). \]

For a more convenient notation we make the following definition:

**Definition 5.4.** For $x \in \mathbb{R}$ we shall denote the Gauss brackets by $[x] := \max \{ k \in \mathbb{Z} : k \leq x \}$. Let $C_1 \geq 0$ be big enough as above and define $N := (2 \left\lceil \frac{C_1}{h} \right\rceil + 1)^2$. For $z \in \Omega \in \Sigma$ let $X(z) = (X_{j,k}(z))_{|j|, |k| \leq \left\lceil \frac{C_1}{h} \right\rceil} \in \mathbb{C}^N$ be given by

\[ X_{j,k}(z) = \hat{e}_0(z; k) \hat{f}_0(z; j), \text{ for } |j|, |k| \leq \left\lceil \frac{C_1}{h} \right\rceil. \]

Let us introduce at this point a more convenient notation for $E^\delta_- (z)$. Define

\[ T(z; \alpha) := \sum_{n=1}^{\infty} (-\delta)^n E_- Q_\Omega (E Q_\Omega)^n E_+ = O(\delta \eta^{-1/4} h^{-5/2}), \]

(5.2)
where the estimate comes from Proposition 5.3. Note that $T(z; \alpha)$ is $C^\infty$ in $z$ and holomorphic in $\alpha$ in a ball of radius $C/h$, $B(0, C/h) \subset \mathbb{C}^N$, by Lemma 5.1. Thus, for $z \in \Omega \Subset \Sigma$ and $\alpha \in B(0, C/h) \subset \mathbb{C}^N$

$$E_{\delta_+}^\delta(z) = E_{-\delta}(z) - \delta [X(z) \cdot \alpha + T(z; \alpha)],$$

where the dot-product $X(z) \cdot \alpha$ is the bilinear one.

**Proposition 5.5.** Let $z \in \Omega \Subset \Sigma$, let $X(z)$ be as in Definition 5.4. Let $h|k| \geq C$ for $C > 0$ large enough, then the Fourier coefficients satisfy

$$\hat{e}_0(z; k), \hat{f}_0(z; k) = \mathcal{O}\left(|k|^{-M} \text{dist} (\Omega, \partial \Sigma)^{-\frac{M}{2}}\right), \quad \text{dist} (\Omega, \partial \Sigma) \gg h^\frac{5}{2}$$

for all $M \in \mathbb{N}$. In particular

$$\|X(z)\| = 1 + \mathcal{O}(h^\infty).$$

**Proof. Estimates on the Fourier coefficients** Will show the proof in the case of $e_0(z)$ since the case of $f_0(z)$ is similar. Let us first suppose that $z \in \Omega$ with $\text{dist} (\Omega, \partial \Sigma) > 1/C$. Recall the definition of the quasimode $e_{wkb}$ given in (3.6). By Proposition 4.7

$$\hat{e}_0(z; k) = \int \left(e_{wkb}(z; x) + \mathcal{O}_{C^\infty} \left(e^{-\frac{x}{\sqrt{\eta}}}ight)\right) e^{-ikx} dx.$$  

For $k \in \mathbb{Z}\setminus\{0\}$, repeated integration by parts using the operator

$$t_L := \frac{i}{\kappa} \frac{d}{dx}$$

applied to the error term yields by Proposition 4.7 that for all $n \in \mathbb{N}$

$$\hat{e}_0(z; k) = \int e_{wkb}(z; x)e^{-ikx} dx + \mathcal{O}(|k|^{-n} h^\infty).$$

Define the phase function $\Phi(x, z) := (\phi_+(x, z)h^{-1} - kx)$. Since $h|k| \geq C$ is large enough and since $\Omega$ is relatively compact, it follows that

$$|\partial_x \Phi(x, z)| = |\partial_x \phi_+(x, z)h^{-1} - k| \geq C_1 |k| > 0.$$  

Repeated integration by parts using the operator

$$t_L' := \frac{1}{\partial_x \Phi(x, z)} D_x$$

yields that for all $n \in \mathbb{N}$

$$\int e_{wkb}(z; x)e^{-ikx} dx = \mathcal{O}(|k|^{-n}).$$

Thus, for all $n \in \mathbb{N}$

$$\hat{e}_0(z; k) = \mathcal{O}(|k|^{-n}).$$

For $z \in \Omega \cap \Omega' \subset \mathbb{C}^N$ one performs a similar rescaling argument as in the proof of Proposition 3.11. Since in the rescaled coordinates $\tilde{k} = \sqrt{\eta} k$, we conclude that for all $n \in \mathbb{N}$

$$|\hat{e}_0(k)| \leq \mathcal{O} \left(\eta^{-\frac{5}{2}} |k|^{-n}\right).$$

**Application of the estimates** Note that

$$\|X(z)\|^2 = \sum_{|j|, |k| \leq N} |\hat{e}_{0}(z; j)|^2 |\hat{f}_{0}(z; k)|^2$$

$$= \left(\sum_{j \in \mathbb{Z}} |\hat{e}_{0}(z; j)|^2 - \sum_{|j| \geq \frac{|k|}{N}} |\hat{e}_{0}(z; j)|^2\right) \left(\sum_{j \in \mathbb{Z}} |\hat{f}_{0}(z; k)|^2 - \sum_{|j| \geq \frac{|k|}{N}} |\hat{f}_{0}(z; k)|^2\right).$$
The Parseval identity and the estimates on the Fourier coefficients above then imply
\[ |X(z)|^2 = (e_0(z)|e_0(z))(f_0(z)|f_0(z)) + O(h^\infty). \]
Note that in the above we also used the Parseval identity together with the fact that \( \|e_0\|, \|f_0\| = 1 \) to conclude that
\[ \sum_{|k| > \left(\frac{1}{T}\right)} |\hat{e}_0(z; j)|^2 < O(h^\infty), \quad \sum_{|k| > \left(\frac{1}{T}\right)} |\hat{f}_0(z; k)|^2 < O(h^\infty). \]
Using once more \( \|e_0\|, \|f_0\| = 1 \) lets us conclude the second statement of the Proposition. \( \square \)

The following is an extension of Proposition 5.5.

**Proposition 5.6.** Let \( z \in \Omega \subseteq \Sigma \), let \( X(z) \) be as in Definition 5.4. Let \( h|k| \geq C \) for \( C > 0 \) large enough, then for dist \((\Omega, \partial \Sigma) \gg h^{\frac{3}{2}} \) and for all \( n, m \in \mathbb{N}_0 \)
\[ \partial_x^n \partial_\xi^m \hat{e}_0(z; k), \partial_x^n \partial_\xi^m \hat{f}_0(z; k) = \left( |k|^{-M} \text{dist} \((\Omega, \partial \Sigma) - \frac{M}{2}\right). \]
Furthermore,
\[ \|\partial_x^n \partial_\xi^m X(z)\| = O\left(\text{dist} \((\Omega, \partial \Sigma)\frac{n+m}{2} h^{-(n+m)}\right). \]

**Proof.** Since
\[ \partial_x^n \partial_\xi^m \hat{e}_0(z; k) = \int \partial_x^n \partial_\xi^m e_0(z; x)e^{-ikx}dx. \]
We then conclude similar to the proof of Proposition 5.5 that for all \( N \in \mathbb{N} \)
\[ |\partial_x^n \partial_\xi^m \hat{e}_0(z; k)| = O\left(\frac{1}{\eta^\frac{N}{2}} |k|^{-N}\right). \]
The second statement of the Proposition is a direct consequence of Parseval’s identity and Proposition 3.11. \( \square \)

6. Connections with symplectic volume and tunneling effects

Recall the effective Hamiltonian \( E_{(1)}^{\delta, +} \) for the perturbed operator \( P_0^\delta \) given in (5.1). The terms up to the second order are of vital importance for the proof of Theorem 2.10. In particular, Proposition 6.1 will reveal a relation to the symplectic volume and Proposition 6.5 will show a link to the tunneling effects described in Section 4.2.

6.1. Link with the symplectic volume

**Proposition 6.1.** Let \( z \in \Omega \subseteq \Sigma \) and let \( p \) and \( \rho_\pm \) be as in Section 1. Let \( X(z) \) be as in Definition 5.4. Then we have for \( h > 0 \) small enough and \( h^{2/3} \ll \eta \ll \text{const.} \)
\[ (\partial_x X | \partial_x X) - \frac{|(\partial_x X | X)|^2}{\|X\|^2} = \frac{1}{h} \left( \frac{i}{\{p, \bar{p}\}(\rho_+(z))} - \frac{i}{\{p, \bar{p}\}(\rho_-(z))} \right) + O(\eta^{-2}), \]
where
\[ |\{p, \bar{p}\}(\rho_\pm)| \asymp \sqrt{\eta}. \]
The \( \partial_x^n \partial_\xi^m \) derivatives of the error term \( O(\eta^{-2}) \) are of order \( O\left(\frac{n+m}{2} h^{-(n+m)}\right). \)

**Proposition 6.2.** Let \( z \in \Omega \subseteq \Sigma \), let \( p \) and \( \rho_\pm \) be as in Section 1 and let \( d\xi \wedge dx \) be the symplectic form on \( T^* S^1 \). Then,
\[ \frac{1}{h} \left( \frac{i}{\{p, \bar{p}\}(\rho_+(z))} - \frac{i}{\{p, \bar{p}\}(\rho_-(z))} \right) L(dz) = \frac{1}{2h} (d\xi_- \wedge dx_- - d\xi_+ \wedge dx_+) = \frac{1}{2h} p_4 (d\xi \wedge dx) \]
To prove Proposition 6.1 we first prove the following result.

Lemma 6.3. Let $\Omega \Subset \Sigma$ such that dist$(\Omega, \partial \Sigma) > 1/C$ and let $g \in C^\infty(\Sigma)$ and $\rho_{wkb}$ as in Section 1. Let $e_{wkb}$ and $f_{wkb}$ be as in (3.6) and (3.7). Let $\Pi_{e_{wkb}} : L^2(S^1) \to L^2(S^1)$ and $\Pi_{f_{wkb}} : L^2(S^1) \to L^2(S^1)$ denote the orthogonal projections onto the subspaces spanned by $e_{wkb}$ and $f_{wkb}$ respectively. Then,

$$
\|(1 - \Pi_{e_{wkb}})\partial_ze_{wkb}(\cdot; z)\|^2 = \frac{-1}{2h\Im g'(x_+(z))} + O(1),
$$

$$
\|(1 - \Pi_{f_{wkb}})\partial_zf_{wkb}(\cdot; z)\|^2 = \frac{1}{2h\Im g'(x_-(z))} + O(1).
$$

Remark 6.4. In the following, we shall regard $z$ as a fixed parameter. Hence, by the support of functions depending on both $x$ and $z$ we mean the support with respect to the variable $x$.

Proof. We will consider only the case of $e_{wkb}$ since the case of $f_{wkb}$ is similar. One calculates

$$
\partial_ze_{wkb}(x; z) = h^{-\frac{1}{2}} \left\{ \partial_ze(x; z)a(z; h) + \chi_e(x; z)\partial_za(z; h) + \chi_e(x; z)a(z; h)\frac{i}{h}\partial_z\phi_+(x; z) \right\} e^{i\phi_+(x; z)}. \tag{6.1}
$$

Thus

$$
(\partial_ze_{wkb}|e_{wkb}) = h^{-\frac{1}{2}} \int \left( (\partial_ze(x; z))|a(z; h)|^2 + (\partial_za(z; h))\overline{a(z; h)}\chi_e(x; z) \right. \\
+ |a(z; h)|^2 \chi_e(x; z)\frac{i}{h}\partial_\phi(x; z) \left. \right) \chi_e(x; z)e^{\frac{i}{h}(\phi_+(x; z) - \phi_+(x; z))} \, dx. \tag{6.2}
$$

The phase can be rewritten as follows

$$
\frac{i}{h}(\phi_+(x; z) - \phi_+(x; z)) = -\Im \frac{2}{h} \int_{x_+(z)}^{x} (z - g(y)) dy = -\frac{\Phi(x; z)}{h}. \tag{6.3}
$$

First, we will compute

$$
h^{-\frac{1}{2}} \int (\partial_ze(x; z))\chi_e(x; z)|a(z; h)|^2 e^{-\frac{\Phi(x; z)}{h}} \, dx. \tag{6.4}
$$

By (3.13)

$$
\partial_ze(x; z) = -h^{-\frac{1}{2}} \left\{ \psi \left( \frac{x - x_+}{\sqrt{h}} \right) + 2\pi \right\} \partial_z\chi(x_-(z) = O(h^{-1}) \tag{6.5}
$$

and has support in $]x_- - 2\pi, x_- - 2\pi + h^{1/2}[\cup]x_--h^{1/2}, x_-[$. By Taylor’s formula we can expand the phase on this domain as

$$
\begin{cases}
\Phi(x; z) = \Phi(x_-(z); z) + O(h), \text{ for } x \in ]x_- - h^{1/2}, x_-[ \\
\Phi(x; z) = \Phi(x_-(z) - 2\pi; z) + O(h), \text{ for } x \in ]x_- - 2\pi, x_- - 2\pi + h^{1/2}].
\end{cases}
$$

Thus

$$
\frac{-\Phi(x; z)}{h} \leq O\left( e^{-2\pi \frac{n}{h}} \right),
$$

where $S$ is as in Definition 2.1. Now, applying this, together with (6.5), to (6.4), yields

$$
h^{-\frac{1}{2}}|a(z; h)|^2 \int \partial_ze(x; z)\chi_e(x; z)e^{-\frac{\Phi(x; z)}{h}} \, dx = O\left( h^{-\frac{1}{2}} e^{-2\pi \frac{n}{h}} \right). \tag{6.6}
$$

Next, we will treat the other two contributions to (6.2). First, consider

$$
h^{-\frac{1}{2}}\left(\partial_za(z; h)\overline{a(z; h)}\right) \int \chi_e(x; z)^2 e^{-\frac{\Phi(x; z)}{h}} \, dx.
$$
Since $h^{-\frac{1}{2}} |a(z; h)|^2$ is the normalization factor of $\|e_{wkb}\|^2$ we see that
\[
 h^{-\frac{1}{2}} \partial_z a(z; h) \overline{a(z; h)} \int \chi_e(x; z) e^{-\frac{\Phi(x; z)}{h}} \, dx = \frac{\partial_z a(z; h)}{a(z; h)}.
\] (6.7)

Let us now turn to the third contribution to (6.2)
\[
 I_h := h^{-\frac{1}{2}} |a(z; h)|^2 \int \frac{i}{h} \partial_z \phi_+(x; z) \chi_e(x; z) e^{-\frac{\Phi(x; z)}{h}} \, dx.
\]
The stationary phase method (analogous to the proof of Lemma 3.3) then implies together with (3.8)
\[
 I_h = i \frac{h}{h} \partial_z \phi_+(x_+(z); z) + O(1).
\] (6.8)

Thus, by combining (6.6), (6.7) and (6.8)
\[
 (\partial_z e_{wkb}|e_{wkb}) = i \frac{h}{h} \partial_z \phi_+(x_+(z); z) + O(1)
\]
and thus
\[
 (\partial_z e_{wkb}|e_{wkb}) e_{wkb}(x; z) = h^{-\frac{1}{2}} \left\{ a(z; h) i \frac{h}{h} \partial_z \phi_+(x_+(z); z) + O(1) \right\} \chi_e(x; z) e^{i \phi_+(x; z)}. \] (6.9)

Subtract (6.9) from (6.1) and note that the term $a(z; h) \partial_z \chi_e(x; z) e^{i \phi_+(x; z)}$ is exponentially small in $h$ like in (6.5). Thus
\[
 (1 - \Pi_{e_{wkb}}) \partial_z e_{wkb}(x; z)
 = h^{-\frac{1}{2}} \left\{ a(z; h) \chi_e(x; z) i \frac{h}{h} \left( \partial_z \phi_+(x; z) - \partial_z \phi_+(x_+(z); z) \right) \right\} e^{i \phi_+(x; z)} + O_L^2(1). \] (6.10)

It remains to treat
\[
 I_h := \left\| a(z; h) \chi_e(x; z) \frac{i}{h} \left( \partial_z \phi_+(x; z) - \partial_z \phi_+(x_+(z); z) \right) e^{i \phi_+(x; z)} \right\|^2
 = h^{-\frac{1}{2}} \int \chi_e(x; z)^2 |a(z; h)|^2 \left| i \frac{h}{h} \left( \partial_z \phi_+(x; z) - \partial_z \phi_+(x_+(z); z) \right) \right|^2 e^{-\frac{\Phi(x; z)}{h}} \, dx, \] (6.11)

where $\Phi(x; z) = 2 \text{Im} \int_{x_+}^x (z - g(y)) dy$. This can be done by the stationary phase method, as in the proof of Lemma 3.3. Thus
\[
 I_h = \sqrt{2\pi} \sum_{n=0}^N \frac{1}{n!} \left( \frac{\hbar}{2} \right)^n \left( \Delta_y u \right)(0) + O(h^{N+1}),
\]
where
\[
 u(y) = \chi_e(\kappa^{-1}(y); z)^2 \frac{|a(z; h)|^2}{|\kappa'(\kappa^{-1}(y))|} \left| i \frac{h}{h} \left( \partial_z \phi_+(\kappa^{-1}(y); z) - \partial_z \phi_+(x_+(z); z) \right) \right|^2.
\]
Recall that $\kappa^{-1}(0) = x_+(z)$. This implies that $u(0) = 0$ and thus we have to calculate the second order term in the above asymptotics. Thus

$$\Delta_y u(y) = \Delta_y \left( \chi_{\kappa^{-1}(y)} \frac{|a(z;h)|^2}{|\kappa'(\kappa^{-1}(y))|} \right) \left| \frac{i}{h} \left( \partial_z \phi_+ (\kappa^{-1}(y); z) - \partial_z \phi_+ (x_+(z); z) \right) \right|^2$$

$$+ \frac{d}{dy} \left( \chi_{\kappa^{-1}(y)} \frac{|a(z;h)|^2}{|\kappa'(\kappa^{-1}(y))|} \right) \frac{d}{dy} \left( \left| \frac{i}{h} \left( \partial_z \phi_+ (\kappa^{-1}(y); z) - \partial_z \phi_+ (x_+(z); z) \right) \right|^2 \right)$$

$$+ \chi_{\kappa^{-1}(y)} \frac{|a(z;h)|^2}{|\kappa'(\kappa^{-1}(y))|} \Delta_y \left( \left| \frac{i}{h} \left( \partial_z \phi_+ (\kappa^{-1}(y); z) - \partial_z \phi_+ (x_+(z); z) \right) \right|^2 \right).$$

Note that at $y = 0$ the first and the second term of the right hand side vanish. By (3.36)

$$\Delta_y \left( \left| \frac{i}{h} \left( \partial_z \phi_+ (\kappa^{-1}(y); z) - \partial_z \phi_+ (x_+(z); z) \right) \right|^2 \right) = 2 \left| \frac{d}{dy} \kappa^{-1}(y) \right|^2.$$

Thus, since $\chi_{\kappa^{-1}(0); z} = \chi_{x_+(z); z} = 1$

$$(\Delta_y u)(0) = \frac{2|a(z;h)|^2}{h^2|\kappa'(x_+(z))|^3}. $$

Recall from (3.9) that $\kappa'(x_+(z)) = \sqrt{-2\text{Im} g'(x_+(z))} \neq 0$. Thus by (3.8)

$$(\Delta_y u)(0) = \frac{1}{\sqrt{2\pi h^2}} \left( -\text{Im} g'(x_+(z)) \right)^{-1} + O(h^{-1})$$

which yields

$$I_h = \frac{-1}{2h\text{Im} g'(x_+(z))} + O(1).$$

This, together with (6.10), yields

$$\|(1 - \Pi_{e_w k}) \partial_z e_{w k} (-; z)\|^2 = \frac{1}{2h\text{Im} g'(x_+(z))} + O(1). \quad \blacksquare$$

**Proof of Proposition 6.1.** Recall that $e_0(z)$ (respectively $f_0(z)$) denotes an eigenfunction of the $z$-dependent operator $Q(z)$ (respectively $Q(z)$). We will denote the $j$-th Fourier coefficient of these eigenfunctions by $\hat{e}_0(z; j)$ and $\hat{f}_0(z; j)$ respectively. By Definition 5.4 we have the following identity

$$\left( \partial_z X \right)^{\|X\|^2} - \frac{|(\partial_z X | X)^2}{\|X\|^2} = \sum_{|j|, |k| \leq \left[ \frac{Q}{h} \right]} \left( \partial_z \hat{e}_0(z; j) \overline{\hat{f}_0(z; k)} + \hat{e}_0(z; j) \overline{\partial_z \hat{f}_0(z; k)} \right) \left( \partial_z \hat{e}_0(z; j) \overline{\hat{f}_0(z; k)} + \hat{e}_0(z; j) \overline{\partial_z \hat{f}_0(z; k)} \right) - \frac{1}{\|X\|^2} \sum_{|j|, |k| \leq \left[ \frac{Q}{h} \right]} \left( \partial_z \hat{e}_0(z; j) \overline{\hat{f}_0(z; k)} + \hat{e}_0(z; j) \overline{\partial_z \hat{f}_0(z; k)} \right)^2. \quad (6.12)$$

Proposition 5.5, Corollary 5.6 and the Parseval identity then imply

$$\left( \partial_z X \right)^{\|X\|^2} - \frac{|(\partial_z X | X)^2}{\|X\|^2} =$$

$$= (\partial_z e_0 | \partial_z e_0) + (\partial_z e_0 | \partial_z f_0 | f_0) + (e_0 | \partial_z e_0 | f_0) + (\partial_z f_0 | \partial_z f_0) - |(\partial_z e_0 | e_0)|^2 - |(f_0 | \partial_z f_0 | f_0)|^2 - (\partial_z e_0 | e_0)(\partial_z f_0 | f_0) - (e_0 | \partial_z e_0 | f_0)(\partial_z f_0 | f_0) + O(h^{\infty})$$

$$= (\partial_z e_0 | \partial_z e_0) - |(\partial_z e_0 | e_0)|^2 + (\partial_z f_0 | \partial_z f_0) - |(f_0 | \partial_z f_0 | f_0)|^2 + O(h^{\infty}).$$

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Suppose that $z \in \Omega$ with $\text{dist}(\Omega, \partial \Omega) > 1/C$. By Corollary 3.13 it then follows that

$$(\partial_2 e_0 | \partial_2 e_0) - |(\partial_2 e_0 | e_0)|^2 = (\partial_2 e_{\text{wkb}} | \partial_2 e_{\text{wkb}}) - |(\partial_2 e_{\text{wkb}} | e_{\text{wkb}})|^2 + O\left(h^{-1} e^{-\frac{1}{Ch}}\right).$$

Let $\Pi_{e_{\text{wkb}}}$ and $\Pi_{f_{\text{wkb}}}$ be as in Lemma 6.3 and note that

$$\|(1 - \Pi_{e_{\text{wkb}}}) \partial_2 e_{\text{wkb}}\|^2 = \|(1 - \Pi_{e_{\text{wkb}}}) \partial_2 e_{\text{wkb}}\|^2$$

and

$$\|(1 - \Pi_{f_{\text{wkb}}}) \partial_2 f_{\text{wkb}}\|^2 = \|(1 - \Pi_{f_{\text{wkb}}}) \partial_2 f_{\text{wkb}}\|^2.$$  \hspace{1cm} (6.13)

Hence

$$(\partial_2 X | \partial_2 X) - \frac{|(\partial_2 X | X)|^2}{\|X\|^2} = (1 - \Pi_{e_{\text{wkb}}}) \partial_2 e_{\text{wkb}}\|^2 + \|(1 - \Pi_{f_{\text{wkb}}}) \partial_2 f_{\text{wkb}}\|^2 + O\left(h^{-1} e^{-\frac{1}{Ch}} + h^\infty\right).$$ \hspace{1cm} (6.14)

Since $\{p, \overline{p}\}(\rho_\pm) = g'(x_\pm) - g'(x_\pm)$, it follows by Lemma 6.3 and by (6.14) that

$$(\partial_2 X | \partial_2 X) - \frac{|(\partial_2 X | X)|^2}{\|X\|^2} = \frac{1}{\hbar} \left( i \frac{\{p, \overline{p}\}(\rho_+ (z))}{\{p, \overline{p}\}(\rho_-(z))} - i \frac{\{p, \overline{p}\}(\rho_+ (z))}{\{p, \overline{p}\}(\rho_-(z))} \right) + O(1)$$ \hspace{1cm} (6.15)

Now let us consider the case where $z \in \Omega \cap \Omega_{\eta}^n$. Similar to Lemma 6.3 we get that

$$\|(1 - \Pi_{e_{\text{wkb}}}^n) \partial_2 e_{\text{wkb}}(z)\|^2_{L^2(S^1/\sqrt{\eta}, \sqrt{\eta} \xi dz)} = \frac{-1}{2h \text{Im} g'(x_+(z))} + O(1),$$

$$\|(1 - \Pi_{f_{\text{wkb}}}^n) \partial_2 f_{\text{wkb}}(z)\|^2_{L^2(S^1/\sqrt{\eta}, \sqrt{\eta} \xi dz)} = \frac{1}{2h \text{Im} g'(x_-(z))} + O(1),$$

where $|\text{Im} g'(x_+(z))| \approx \sqrt{\eta}$. A rescaling argument similar to the one in the proof of Proposition 3.11 and Corollary 3.13 then imply

$$|(\partial_2 e_0 | \partial_2 e_0) - |(\partial_2 e_0 | e_0)|^2 = \frac{-1}{2h \text{Im} g'(x_+(z))} + O(\eta^{-2})$$

and similar for $(\partial_2 f_0 | \partial_2 f_0) - |(f_0 | \partial_2 f_0)|^2$. Hence,

$$(\partial_2 X | \partial_2 X) - \frac{|(\partial_2 X | X)|^2}{\|X\|^2} = \frac{1}{\hbar} \left( i \frac{\{p, \overline{p}\}(\rho_+ (z))}{\{p, \overline{p}\}(\rho_-(z))} - i \frac{\{p, \overline{p}\}(\rho_+ (z))}{\{p, \overline{p}\}(\rho_-(z))} \right) + O(\eta^{-2})$$

with $|\{p, \overline{p}\}(\rho_\pm (z))| \approx \sqrt{\eta}$. The statement on the derivatives of the error estimates follow by the Stationary phase method and the usual rescaling argument. \hfill \Box

**Proof of Proposition 6.2.** In the following we will conform to ideas from [5, 1, 8]: Since $p(x_\pm, \xi_\pm) = z$, we find the following system of linear equations

$$\begin{cases}
\{ p'_{+} \cdot \partial_2 x_+ + p'_{-} \cdot \partial_2 \xi_+ &= 1 \\
\{ p'_{+} \cdot \partial_2 x_+ + p'_{-} \cdot \partial_2 \xi_+ &= 0 
\end{cases}$$

and since $x_\pm, \xi_\pm \in \mathbb{R}$

$$\begin{cases}
\{ p'_{+} \cdot \partial_2 x_+ + p'_{-} \cdot \partial_2 \xi_+ &= 1 \\
\{ p'_{+} \cdot \partial_2 x_+ + p'_{-} \cdot \partial_2 \xi_+ &= 0 
\end{cases}$$

This system can be solved and we find

$$\partial_2 x_+ = \frac{\{-p'_+\}}{\{p, \overline{p}\}(\rho_\pm)}, \quad \partial_2 \xi_+ = \frac{\{-p'_-\}}{\{p, \overline{p}\}(\rho_\pm)}$$ \hspace{1cm} (6.16)

and

$$\partial_2 \xi_+ = \frac{\{p'_{+}\}}{\{p, \overline{p}\}(\rho_\pm)}, \quad \partial_2 x_+ = \frac{\{p'_{-}\}}{\{p, \overline{p}\}(\rho_\pm)}.$$
Hence we have
\[ d\xi_\pm \wedge dx_\pm = (\partial_\xi_\pm \partial_\eta x_\pm - \partial_\xi_\pm \partial_\eta x_\pm) d\zeta \wedge d\bar{\zeta} = \left( \frac{1}{\{p, p\}}(\rho_\pm) \right) d\zeta \wedge d\bar{\zeta}. \]

Since the Lebesgue measure with the standard orientation of \( \mathbb{C} \) can be represented as
\[ L(d\zeta) = \frac{i}{2} d\zeta \wedge d\bar{\zeta}, \]
the statement of the Proposition follows from (6.15). \( \square \)

6.2. Link with the tunneling effects. We will prove the following result in the light of Proposition 4.2.

**Proposition 6.5.** Let \( z \in \Omega \in \Sigma \), let \( X(z) \) be as in Definition 5.4 and let \( E_{-+}(z) \) be as in Proposition 4.1. Let \( S \) be as in Definition 2.1. Then,
\[ \left| \partial_\eta E_{-+}(z) - E_{-+}(z) \left( \frac{\partial_\eta X(z)|X(z)}{\|X(z)\|^2} \right) - (e_0|f_0) \right| \leq \mathcal{O}\left( h^{\infty} e^{-\frac{s}{2}} \right). \]

**Proof of Proposition 6.5.** Recall the relations between \( e_0 \) and \( f_0 \) given in (2.5):
\[ (P_h - z)e_0 = \alpha_0 f_0 \quad \text{and} \quad (P_h - z)^* f_0 = \beta_0 e_0 \tag{6.17} \]
with \( \alpha_0 = \overline{\beta_0} \). Apply the \( \partial_\eta \) derivative to the first equation in (6.17),
\[ (P_h - z)\partial_\eta e_0 - e_0 = \partial_\eta \alpha_0 \cdot f_0 + \alpha_0 \partial_\eta f_0. \]
Taking the scalar product with \( f_0 \) (which is \( L^2 \)-normalized) then yields
\[ (\partial_\eta e_0)(P_h - z)^* f_0) - (e_0|f_0) = \partial_\eta \alpha_0 + \alpha_0 (\partial_\eta f_0|f_0). \]
Recall from Proposition 4.1 that \( E_{-+}(z) = -\alpha_0(z) \) and use the second equation in (6.17) to see
\[ \partial_\eta E_{-+}(z) - E_{-+}(z)((\partial_\eta e_0|e_0) - (\partial_\eta f_0|f_0)) - (e_0|f_0) = 0. \tag{6.18} \]
By Definition 5.4 we have the following identity
\[ (\partial_\eta X|X) = \sum_{|j|,|k| \leq \lfloor \frac{C_1}{\hbar} \rfloor} \left( \partial_\eta \widehat{e}_0(z;j)\overline{f_0(z;k)} + \widehat{e}_0(z;j)\overline{\partial_\eta f_0(z;k)} \right). \]
Proposition 5.5, Corollary 5.6 and the Parseval identity then imply
\[ \frac{(\partial_\eta X|X)}{\|X\|^2} = (\partial_\eta e_0|e_0) + (f_0|\partial_\eta f_0) + \mathcal{O}(h^{\infty}). \tag{6.19} \]
Note that in the above we also used that \( e_0 \) and \( f_0 \) are normalized. Since \( (f_0|\partial_\eta f_0) = -\overline{(f_0|\partial_\eta f_0)} \) we conclude by the triangular inequality
\[ \left| \partial_\eta E_{-+}(z) - E_{-+}(z) \left( \frac{\partial_\eta X(z)|X(z)}{\|X(z)\|^2} \right) - (e_0|f_0) \right| \leq \mathcal{O}(h^{\infty})|E_{-+}(z)|. \]
The statement of the proposition then follows by the estimate \( |E_{-+}(z)| = \mathcal{O}\left( \eta^{\frac{1}{2}} h^{\frac{1}{2}} e^{-\frac{s}{2}} \right) \) given in Proposition 4.6. \( \square \)
7. Preparations for the distribution of eigenvalues of $P_h^\delta$

7.1. Intensity measure of the point process of eigenvalues of $P_h^\delta$. By [8] we know that $E_{-+}^\delta(z)$ can be rendered holomorphic by multiplying with a smooth non-vanishing function, since it satisfies the $\bar{\partial}$-equation

$$\bar{\partial}z E_{-+}^\delta(z) + f^\delta(z) E_{-+}^\delta(z) = 0$$

which follows from the differentiation of $E_\delta(z)P_\delta(z) = 1$. Hence the function

$$e^{F^\delta(z)}E_{-+}^\delta(z)$$

with $\bar{\partial}z F^\delta(z) = f^\delta(z)$ (7.1) is holomorphic in $z$ and has the same zeros as $E_{-+}^\delta(z)$. In particular, the zeros of $E_{-+}^\delta(z)$ form a discrete set of locally finite multiplicity where the notion of multiplicity of a zero $z_0$ of $E_{-+}^\delta(z)$ is defined to be the multiplicity of $z_0$ as a zero of $e^{F^\delta(z)}E_{-+}^\delta(z)$.

As is discussed in [9], the key property of a Grushin problem is that $P_h^\delta - z$ is invertible if and only if $E_{-+}^\delta(z)$ is invertible. This permits us to study the distribution of the eigenvalues of $P_h^\delta - z$ by studying the distribution of the zeros of $E_{-+}^\delta(z)$.

The point process $\Xi$ defined in Definition 2.7 can be written as a point process corresponding to the zeros of $E_{-+}^\delta(z)$:

$$\Xi = \sum_{z \in (E_{-+}^\delta)^{-1}(0)} \delta_z,$$

where the zeros are counted according to their multiplicities and $\delta_z$ denotes the Dirac-measure in $z$. To prove Theorem 2.10 we will calculate for $\varphi \in C_0^\infty(\mathbb{C})$

$$E[\Xi(\varphi)] = E \left[ \sum_{z \in (E_{-+}^\delta)^{-1}(0)} \varphi(z) \right],$$

which is also sometimes called 1-point intensity measure of $\Xi$ or simply intensity measure of $\Xi$.

7.2. Counting zeros. To calculate the intensity measure of $\Xi$ we shall make use of the following observations:

Lemma 7.1. Let $\Omega \subset \mathbb{C}$ be open and convex and let $g, F : \Omega \rightarrow \mathbb{C}$ be $C^\infty$ such that $g \neq 0$ and

$$\bar{\partial}z g(z) + \bar{\partial}z F(z) \cdot g(z) = 0$$

(7.2) holds for all $z \in \Omega$. The zeros of $g$ form a discrete set of locally finite multiplicity. The notion of multiplicity here is the same as for holomorphic functions, more details can be found in the proof. Furthermore, for all $\varphi \in C_0^\infty(\Omega)$

$$\left\langle \chi \left(\frac{g}{\varepsilon}\right) \frac{1}{\varepsilon^2} |\bar{\partial}z g|^2, \varphi \right\rangle \rightarrow \sum_{z \in g^{-1}(0)} \varphi(z), \quad \varepsilon \rightarrow 0,$$

where $\chi \in C_0^\infty(\Omega)$ such that $\chi \geq 0$ and $\int \chi(w)L(dw) = 1$ and the zeros are counted according to their multiplicities.

Proof. (7.2) implies that

$$e^{F(z)}g(z)$$

is holomorphic in $\Omega$. $g$ has the same zeros as the holomorphic function (7.3). Thus, the zeros of $g$ in $\Omega$ form a discrete set and the notion of the multiplicity of the zeros of $g$ is well-defined since we can view the zeros as those of a holomorphic function.
Let \( z_0 \in g^{-1}(0) \) have multiplicity \( n \). Therefore, there exists a neighborhood \( W \subset \Omega \) of \( z_0 \) such that \( W \cap g^{-1}(0) = z_0 \). Since \( e^{F(z)}g(z) \) is holomorphic, there exists a neighborhood \( U \subset \Omega \) of \( z_0 \) and a holomorphic function \( f : U \to \mathbb{C} \) such that for all \( z \in U \)
\[
f(z) \neq 0, \quad \text{and} \quad e^{F(z)}g(z) = f(z)(z - z_0)^n.
\]

Choose a \( \lambda > 0 \) such that \( |e^{-F(z)}f(z) - e^{-F(z_0)}f(z_0)| < |e^{-F(z_0)}f(z_0)| \) for \( |z - z_0| < \lambda \). In this disk we can define a single-valued branch of \( \sqrt[n]{e^{-F(z)}f(z)} \).

We take a test function \( \varphi \in C_0(\Omega) \) with
\[
supp \varphi \subset (U \cap W \cap \{ z : |z - z_0| < \lambda \}) =: N \quad \text{(7.4)}
\]
and consider for \( \varepsilon > 0 \)
\[
\left\langle \chi \left( \frac{g}{\varepsilon} \right) \frac{1}{\varepsilon^2} |\partial_z g|^2, \varphi \right\rangle = \frac{1}{\varepsilon^2} \int_N \chi \left( \frac{g(z)}{\varepsilon} \right) |\partial_z g(z)|^2 \varphi(z)L(dz).
\]

Let us perform a change of variables. Define
\[
w := g(z) = (z - z_0)^n e^{-F(z)}f(z), \quad \text{(7.5)}
\]
Since
\[
\partial_z w(z) = (z - z_0)^{n-1} e^{-F(z)} (nf(z) + (z - z_0)(\partial_z f(z) - \partial_z F(z)f(z))), \\
\partial_z w(z_0) = 0,
\]
the implicit function theorem implies that we can invert equation (7.5) for \( z \) in a small neighborhood of \( z_0 \) without \( \{ z_0 \} \), say the disk \( D(0, r) \setminus \{ z_0 \} \) for some radius \( r > 0 \), and \( w \) in the \( n \)-fold covering surface of \( w(D(0, r) \setminus \{ z_0 \}) \). Thus, if we denote the domain on each leaf of the covering by \( B_k \), for \( k = 1, \ldots, n \), as a subset of \( \mathbb{C} \), and the respective branch of \( g \) by \( g_k \) we get for \( \varepsilon > 0 \) small enough
\[
\left\langle \chi \left( \frac{g}{\varepsilon} \right) \frac{1}{\varepsilon^2} |\partial_z g|^2, \varphi \right\rangle = \sum_{k=1}^n \frac{1}{\varepsilon^2} \int_{B_k} \varphi(g_k^{-1}(w)) \chi \left( \frac{w}{\varepsilon} \right) (1 + \mathcal{O}(w^2))L(dw),
\]
with \( g_k^{-1}(0) = z_0 \). In the above we used that
\[
L(dw) = (|\partial_z g(z)|^2 - |\partial_w g(z)|^2) Ld(z)
\]
and the \( \bar{\partial} \)-equation (7.2) which implies
\[
|\partial_w g(z)|^2 = |\partial_w F(z)g(z)|^2 \asymp w^2.
\]

Thus we can conclude
\[
\left\langle \chi \left( \frac{g}{\varepsilon} \right) \frac{1}{\varepsilon^2} |\partial_z g|^2, \varphi \right\rangle \rightarrow \sum_{k=1}^n \varphi(z(0)) = n\varphi(z_0), \quad \text{for } \varepsilon \to 0. \quad \text{(7.6)}
\]

Since \( g \) has at most countably many zeros in \( \Omega \), there exists some index set \( I \subset \mathbb{N} \) such that we can denote the set of zeros of \( g \) in \( \Omega \) by \( \{ z_i \}_{i \in I} := g^{-1}(0) \cap \Omega \). Furthermore, let \( m(i) \) for all \( i \in I \) denote the multiplicity of the respective zero \( z_i \).

For each zero \( z_i \) we can construct a neighborhood \( N_i \), as above, such that for a test function with support in \( N_i \) we have the convergence as in (7.6). By potentially shrinking the \( N_i \) we can gain \( N_i \cap N_j = \emptyset \) for \( i \neq j \). Consider the following locally finite open covering of \( \Omega \)
\[
\Omega = \left( \bigcup_{i \in I} N_i \right) \cup (\Omega \setminus \{ z_i : i \in I \}) \quad \text{(7.7)}
\]

Let \( \{ \chi_i \}_{i \in I \cup \{ 0 \}} \) be a partition of unity subordinate to this open covering such that
\[
1 = \sum_{i \in I} \chi_i + \chi_0.
\]

Let \( \{ \chi_i \}_{i \in I \cup \{ 0 \}} \) be a partition of unity subordinate to this open covering such that
\[
1 = \sum_{i \in I} \chi_i + \chi_0.
\]
Here \( \chi_i \in C^\infty_0(N_i) \) and \( \chi_i \equiv 1 \) in a neighborhood of \( z_i \) for all \( i \in I \). Furthermore, \( \chi_0 \in C^\infty_0(\Omega) \) and \( z_i \notin \text{supp} \chi_0 \) for all \( i \in I \). Let \( \varphi \in C_0(\Omega) \) be an arbitrary test function. By (7.6) we have for \( \varepsilon \to 0 \)

\[
\left\langle \chi \left( \frac{g}{\varepsilon} \right) \frac{1}{\varepsilon^2} |\partial_z g|^2, \varphi \right\rangle = \sum_{i \in I} \left\langle \chi \left( \frac{g}{\varepsilon} \right) \frac{1}{\varepsilon^2} |\partial_z g|^2, \chi_i \varphi \right\rangle \longrightarrow \sum_{i \in I} m(i) \chi_i(z_i) \varphi(z_i).
\]

Since \( g(z) \neq 0 \) for all \( z \in \text{supp} \chi_0 \) we have for \( \varepsilon > 0 \) small enough

\[
\left\langle \chi \left( \frac{g}{\varepsilon} \right) \frac{1}{\varepsilon^2} |\partial_z g|^2, \chi_0 \varphi \right\rangle = 0
\]

and thus we can conclude the statement of the Lemma.

\[ \square \]

### 7.3. An implicit function theorem.

**Lemma 7.2.** Let \( R > 0 \) and \( a > c \geq 0 \) be constants. Let \( D(0, R) \subset \mathbb{C} \) be the open disk of radius \( R \) centered at 0 and let \( g, f : D(0, R) \rightarrow \mathbb{C} \) be holomorphic such that

\[
\|g\|_\infty \leq c, \quad \text{and for all } z \in D(0, R) : \partial_z f(z) = a + g(z).
\]

(7.7)

Assume that

\[
\xi \in D(f(0), (a-c)R) \subset \mathbb{C}.
\]

Then the equation

\[
f(z) = \xi
\]

has exactly one solution \( z = z(\xi) \in D(0, R) \) and it depends holomorphically on \( \xi \).

**Proof.** For \( z \in D(0, R) \)

\[
f(z) = \int_0^z (a + g(w)) \, dw + f(0) = az + f(0) + G(z),
\]

where \( G(z) := \int_0^z g(w) \, dw \). Now let us consider the equation

\[
az + f(0) - \xi = 0.
\]

It has a unique solution in the disk \( D(0, R) \) since

\[
\frac{|\xi - f(0)|}{a} < \frac{|a-c|}{a} R < R.
\]

Now consider for \( \varepsilon > 0 \) and for \( z \in D(0, R - \varepsilon) \) the equation

\[
f(z) - \xi = az + f(0) - \xi + G(z) = 0.
\]

Recall that \( \xi \in D(f(0), (a-c)R) \) which implies that there exists a \( \varepsilon(\xi) > 0 \) such that

\[
|\xi - f(0)| \leq (a-c)(R - \varepsilon(\xi)).
\]

Thus for all \( \varepsilon < \varepsilon(\xi) \)

\[
|az + f(0) - \xi| \geq |az| - |f(0) - \xi| > a|z| - (a-c)(R - \varepsilon)
\]

and, using that \( |G(z)| \leq c|z| \), we may conclude that for \( |z| = R - \varepsilon \)

\[
|G(z)| < |az + f(0) - \xi|.
\]

By Rouché’s theorem we have that \( az + f(0) - \xi \) and \( f(z) - \xi \) have the same number of zeros in the disk \( D(0, R - \varepsilon) \). We also see that \( f(z) - \xi \) has no zero in \( D(0, R) \setminus D(0, R - \varepsilon) \) and the result follows.

\[ \square \]

**Proposition 7.3.** Let \( a > c \geq 0 \) be constants, \( n \in \mathbb{N} \), let \( \Omega \subset \mathbb{C}^n \) be open, bounded and of the form

\[
\Omega = \{ z = (z', z_n) \in \mathbb{C}^n : z' \in \Omega', \ |z_n| < R_{z'} \}
\]

where \( R_{z'} > 0 \) is continuous in \( z' \). Furthermore, assume that

- \( g, F : \Omega \rightarrow \mathbb{C} \) are holomorphic such that

\[
\|g\|_\infty \leq c, \quad \text{and for all } z \in \Omega : \partial_{z_n} F(z) = a + g(z),
\]

(7.8)
Lemma 7.2 implies the existence of the solutions

Proof. Let \( \eta \) be as in (2.4) and (2.6). There exist functions \( \Phi(z) \) and \( \Theta(z) \) such that for all \( \eta \), \( \Psi(z) \) and \( \Theta(z) \) depend holomorphically on \( \xi \) and on \( z' \).

8. A Formula for the Intensity Measure of the Point Process of Eigenvalues of \( P_h^\delta \)

We shall prove the following formula for the intensity measure of \( \Xi \):

Proposition 8.1. Let \( h^{2/3} \ll \eta < \text{const.} \) and let \( \Omega := \Omega_h^\eta \subset \Sigma \). Let \( C > 0 \) and let \( C_1 > 0 \) be as in (1.4) such that \( C - C_1 > 0 \) is large enough. Let \( \delta \) be as in Definition 2.5, define \( N := (2[C_1/h] + 1)^2 \) and let \( B(0, R) \subset C^N \) be the ball of radius \( R := C^{-1} \) centered at zero. For \( z \in \Omega \) let \( X(z) \) be as in Definition 5.4, let \( E_{-\pm}(z) \) be as in Proposition 4.1 and let \( e_0 \) and \( f_0 \) be as in (2.4) and (2.6). There exist functions

\[
\Psi(z; h, \delta) = (\partial_z X)(\partial_z X) - \frac{1}{\|X\|^2} \|\partial_z X \|X\|^2 \\
+ \delta^{-2} \left| (e_0[f_0] + \mathcal{O}(h^\infty)) + \mathcal{O}(\eta^{1/4} \delta^2 h^{-7/2}) \right|^2 + \mathcal{O}(\delta^3 h^{-3}),
\]

(8.1)

\[
\Theta(z; h, \delta) = \frac{|E_{-\pm}(z) + \mathcal{O}(\delta^2 \eta^{-1/4} h^{-5/2})|^2}{\delta^2 \|X(z)\|^2},
\]

(8.2)

and \( D > 0 \) and \( \tilde{C} > 0 \) such that for all \( \varphi \in C_0^\infty(\Omega) \) and for \( h > 0 \) small enough

\[
\mathbb{E} \left[ \Xi(\varphi) \mathbf{1}_{B(0,R)} \right] = \int \varphi(z) \frac{1 + \mathcal{O}(\delta^2 \eta^{-1/4} h^{-3/2})}{\pi \|X\|^2} \Psi(z; h, \delta) e^{-\Theta(z; h, \delta)} L(dz) \\
+ \mathcal{O}(e^{-\frac{D}{h^2}}).
\]

Here, \( \mathcal{O}(\eta^{-1/4} \delta^3 h^{-3/2}) \) is independent of \( \varphi \) and \( \mathcal{O}(e^{-\frac{D}{h^2}}) \) means \( (T_h, \varphi) \) such that \( |(T_h, \varphi)| \leq C \|\varphi\|_\infty e^{-\frac{D}{h^2}} \) for all \( \varphi \in C_0^\infty(\Omega) \) where \( C \) and \( D \) is independent of \( h, \delta, \eta \) and \( \varphi \). Moreover, the estimates in (8.1) and (8.2) are stable under application of \( \eta^{-\frac{\alpha - m}{2}} h^{n+m} \partial_\nu^\alpha \).

Proof. Step I We will approximate the Dirac-delta as in Lemma 7.1 with

\[
w \mapsto \varepsilon^{-2} \chi \left( \frac{w}{\varepsilon} \right), \quad \text{where } 0 \leq \chi \in C_0^\infty(\Sigma), \quad \text{and } \int \chi(w) L(dw) = 1.
\]
By Lemma 7.1, Fubini’s theorem and the Lebesgue dominated convergence theorem we have

\[
E \left[ \sum_{z \in (E_{-+}^\delta)^{-1}(0)} \varphi(z) \mathbf{1}_{B(0,R)} \right] \\
= \lim_{\epsilon \to 0} \int_{B(0,R)} \varphi(z) E \left[ \chi \left( \frac{E_{-+}^\delta(z; \alpha)}{\epsilon} \right) \frac{1}{\epsilon^2} \left| \partial_z E_{-+}^\delta(z; \alpha) \right|^2 \mathbf{1}_{B(0,R)} \right] L(dz) \\
= \lim_{\epsilon \to 0} \int \varphi(z) \left\{ \pi^{-N} \int_{B(0,R)} \chi \left( \frac{E_{-+}^\delta(z; \alpha)}{\epsilon} \right) \frac{1}{\epsilon^2} \left| \partial_z E_{-+}^\delta(z; \alpha) \right|^2 e^{-\alpha L}(d\alpha) \right\} L(dz). \quad (8.3)
\]

**Step II** Let us give an estimate on \( \partial_z E_{-+}^\delta(z) \). By (5.3)

\[
\partial_z E_{-+}^\delta(z) = \partial_z E_{-+}(z) - \delta (\partial_z X(z) \cdot \alpha + \partial_z T(z; \alpha)), \quad (8.4)
\]

where the derivative \( \partial_z \) acts on \( X(z) \) component wise and the dot-product \( \partial_z X(z) \cdot \alpha \) is bilinear. To estimate \( \partial_z T(z; \alpha) \), recall (5.2) and consider the derivative

\[
\partial_z E_-(Q_{\omega}(EQ_{\omega})^n E_+ \left[ \sum_{j=1}^n (EQ_{\omega})^{j-1} (\partial_z E)Q_{\omega}(EQ_{\omega})^{n-j} \right] E_+ + E_- Q_{\omega}(EQ_{\omega})^n(\partial_z E_+),
\]

where we use the convention \( (EQ_{\omega})^0 = 1 \). By considering the Grushin problem for the unperturbed operator \( P_h \) and by taking the derivative with respect to \( z \) of the relation \( \mathcal{E}(z)P(z) = 1 \) we gain

\[
\partial_z \mathcal{E}(z) + \mathcal{E}(z)(\partial_z P(z))\mathcal{E}(z) = 0.
\]

A direct calculation yields

\[
\partial_z E = -E(\partial_z (P_h - z))E - E_+(\partial_z R_+)E - E(\partial_z R_-)E_-
\]

\[
= E^2 - E_+ (\partial_z R_+) E - E (\partial_z R_-) E_-
\]

Recall the definition of \( R_+ \) and \( R_- \) given in (4.1). By the estimates on the \( \zeta \)- and \( \tau \)-derivatives of \( e_0 \) and \( f_0 \) given in Lemma 3.11, we may conclude

\[
\| \partial_z R_+ \|_{H^1 \to C} \leq \| \partial_\zeta e_0 \|_{L^2} = \mathcal{O} \left( \eta^{1/2} h^{-1} \right),
\]

\[
\| \partial_z R_- \|_{C \to L^2} = \| \partial_\zeta f_0 \|_{L^2} = \mathcal{O} \left( \eta^{1/2} h^{-1} \right).
\]

Note, furthermore, that we have the same estimates on \( \| \partial_z E_+ \|_{C \to L^2} \) and \( \| \partial_z E_- \|_{H^1 \to C} \). Thus, since \( \| E(z) \|_{L^2 \to H^1} = \mathcal{O} \left( (h\sqrt{\eta})^{-1/2} \right) \) and \( \| E_\pm \| = \mathcal{O}(1) \), we gain the following estimate

\[
\| \partial_z E \|_{L^2 \to H^1} = \mathcal{O} \left( \eta^{1/4} h^{-3/2} \right).
\]

If we put all of this together, we get that the series of \( \partial_z T(z; \alpha) \) converges again geometrically and we gain the estimate

\[
\partial_z T(z; \alpha) = \mathcal{O} \left( \eta^{1/4} \delta h^{-7/2} \right). \quad (8.5)
\]

Analogously, we may conclude

\[
\eta^{-\frac{n+m}{2}} h^{n+m} \partial_z \partial_\zeta \partial_\tau T(z; \alpha) = \mathcal{O} \left( \eta^{-1/4} \delta h^{-5/2} \right).
\]

Thus,

\[
\partial_z E_{-+}^\delta(z) = \partial_z E_{-+}(z) - \delta \partial_z X(z) \cdot \alpha + \mathcal{O} \left( \eta^{1/4} \delta h^{-7/2} \right).
\]
**Step III** Let us again consider the integral (8.3) in the light of (5.3). We choose as a basis of the \( \alpha \)-space vectors \( e_1, e_2, \ldots \in \mathbb{C}^N \) such that \( e_1 = \bar{X}/\|X\| \) and such that \( e_1, e_2 \) and \( \|X\| \), \( \partial_\alpha X \) span the same space. Therefore, we perform a unitary transformation in the \( \alpha \)-space such that with a slight abuse of notation

\[
\alpha = \alpha_1 \frac{X(z)}{\|X(z)\|} + \alpha_2 b \left( \frac{\partial_z X(z)}{\|\partial_z X(z)\|} - \frac{(\partial_z X(z)|X(z))X(z)}{\|\partial_z X(z)\| \|X(z)\|^2} \right) + \alpha^\perp,
\]

where \( \alpha_1, \alpha_2 \in \mathbb{C} \) and \( \alpha^\perp \in \mathbb{C}^{N-3} \) and \( b > 0 \) is a factor of normalization,

\[
b = \frac{\|\partial_z X(z)\| \|X(z)\|}{\sqrt{\|\partial_z X(z)\|^2 \|X(z)\|^2 - \|\partial_z X(z)(X(z))\|^2}}.
\]

The change of variables is well defined thanks to Lemma 6.1 below.

In the following we will also use the notation \( (\alpha_1, \alpha_2, \alpha^\perp) = (\alpha_1, \alpha') \). This choice of basis yields by (5.2) and (5.3)

\[
E_{\delta}^{\alpha_1}(z) = E_{\alpha}(z) - \delta \|X(z)\| \alpha_1 + O\left( \eta^{-1/4} \delta^2 h^{-5/2} \right)
\]

and by (8.4)

\[
\partial_\alpha E_{\delta}^{\alpha_1}(z) = \partial_\alpha E_{\alpha}(z) - \delta \frac{(\partial_\alpha X(z)|X(z))X(z)}{\|X(z)\|^2} \alpha_1
\]

\[
- \delta b \left( \frac{\|\partial_\alpha X(z)\| - \frac{|(\partial_\alpha X(z)|X(z))|^2}{\|\partial_\alpha X(z)\| \|X(z)\|^2}} \right) \alpha_2 + O\left( \eta^{-1/4} \delta^2 h^{-7/2} \right)
\]

\[
= \partial_\alpha E_{\alpha}(z) - \delta \frac{(\partial_\alpha X(z)|X(z))X(z)}{\|X(z)\|^2} \alpha_1
\]

\[
- \delta \left( \frac{\|\partial_\alpha X(z)\|^2 - \frac{|(\partial_\alpha X(z)|X(z))|^2}{\|X(z)\|^2}}{\|X(z)\|^2} \right)^{1/2} \alpha_2 + O\left( \eta^{-1/4} \delta^2 h^{-7/2} \right).
\]

Now let us split the ball \( B(0, R), R = Ch^{-1} \), into two pieces: pick \( C_0 > 0 \) such that \( 0 < C_1 < C_0 < C \) and define \( R_0 := C_0 h^{-1} \). Then we shall consider one piece such that \( \|\alpha'\|_{\mathbb{C}^{N-1}} < R_0 \) and the other such that \( \|\alpha'\|_{\mathbb{C}^{N-1}} > R_0 \). Hence, by (8.3)

\[
\mathbb{E} \left[ \sum_{z \in (E_{\delta}^{\alpha_1})^{-1}(0)} \phi(z) 1_{B(0,R)} \right]
\]

\[
= \lim_{\varepsilon \to 0} \int \phi(z) \left\{ \chi \left( \frac{E_{\delta}^{\alpha_1}(z; \alpha)}{\varepsilon} \right) \frac{1}{\varepsilon^2} \left| \partial_\alpha E_{\delta}^{\alpha_1}(z; \alpha) \right|^2 e^{-a\pi L(d\alpha)} \right\} L(dz)
\]

\[
+ \lim_{\varepsilon \to 0} \int \phi(z) \left\{ \chi \left( \frac{E_{\delta}^{\alpha_1}(z; \alpha)}{\varepsilon} \right) \frac{1}{\varepsilon^2} \left| \partial_\alpha E_{\delta}^{\alpha_1}(z; \alpha) \right|^2 e^{-a\pi L(d\alpha)} \right\} L(dz)
\]

\[
=: I_1(\phi) + I_2(\phi).
\]

**Step IV** In this step we will calculate \( I_1(\phi) \) of equation (8.9). There we can perform a change of variables such that \( \beta := E_{\delta}^{\alpha_1}(z; \alpha) \) is one of them. Due to (8.7) it is natural to
express $\alpha_1$ as a function of $\beta$ and $\alpha'$. To this purpose we will apply Proposition 7.3 to the function $E^\delta_{-+}(z; \alpha)$:

$E^\delta_{-+}(z; \alpha_1, \alpha')$ is holomorphic in $\alpha$ in ball of radius $R = Ch^{-1}$ centered at 0. Here, $\alpha$ plays the role of $z$ in the Proposition, in particular $\alpha_1$ plays the role of $z_n$. Recall (5.3) and note that since $T(z; \alpha) = O(\eta^{-1/4} \delta h^{-5/2})$ (cf. (5.2)) we can conclude by the Cauchy inequalities that

$$\partial_{\alpha_1} \delta T(z; \alpha) = O(\eta^{-1/4} \delta^2 h^{-3/2})$$

which implies

$$\partial_{\alpha_1} E^\delta_{-+}(z; \alpha_1, \alpha') = -\delta \|X(z)\| + O(\eta^{-1/4} \delta^2 h^{-3/2}) .$$

By Proposition 5.5 we have that $\|X(z)\| = 1 + O(h^\infty)$ which implies that

$$\partial_{\alpha_1} E^\delta_{-+}(z; \alpha_1, \alpha') = -\delta \left(1 + O\left(h^\infty + \eta^{-1/4} \delta h^{-3/2}\right)\right).$$

Hence, $E^\delta_{-+}(z; \alpha)$ satisfies the assumptions of Proposition 7.3. Since we restricted $\alpha'$ to $\|\alpha'\|_{C^{N-1}} < R_0$ and since

$$|\alpha_1| < R^2 - \|\alpha'\|_{C^{N-1}} =: R_0,$$

it follows by Proposition 7.3 that for

$$\beta \in \bigcap_{|\alpha'|_{C^{N-1}} < R_0} D \left(E^\delta_{-+}(z; 0, \alpha'), r_{\alpha'}\right)$$

with

$$r_{\alpha'} \geq \delta \left(1 + O\left(h^\infty + \eta^{-1/4} \delta h^{-3/2}\right)\right) \frac{\sqrt{C^2 - C_0^2}}{h} \geq \frac{\delta h^{-1}}{O(1)} > 0 .$$

and $h > 0$ small enough, $\beta = E^\delta_{-+}(z; \alpha_1, \alpha')$ has exactly one solution $\alpha_1(\beta, \alpha')$ in the disk $D(0, R_{\alpha'})$ and it depends holomorphically on $\beta$ and $\alpha'$. More precisely,

$$\alpha_1(\beta, \alpha') = \frac{-\beta + E_{-+}(z) + O(\eta^{-1/4} \delta^2 h^{-5/2})}{\delta \|X(z)\|} .$$

Furthermore,

$$L(\,d\alpha) = |\partial_{\alpha_1} E^\delta_{-+}|^{-2} L(\,d\beta) L(\,d\alpha').$$

Note that, since the support of $\chi$ is compact, we can restrict our attention to $\beta$ and $E^\delta_{-+}(z; 0, \alpha')$ in a small disk of radius $\varepsilon > 0$ centered at 0. By choosing $\varepsilon < \delta h^{-1}/C$, $C > 0$ large enough, as in (8.12) we see that $\beta, E^\delta_{-+}(z; 0, \alpha') \in D(0, \varepsilon)$ implies (8.11). Thus, by performing this change of variables and by picking $\varepsilon > 0$ small enough as above, we get

$$I_1(\varphi) = \lim_{\varepsilon \to 0} \int \varphi(z) \left\{ \int \chi \left(\frac{\beta}{\varepsilon}\right) \frac{1}{\varepsilon^2} A(\beta; z) L(\,d\beta) \right\} L(\,dz),$$

where $A(\beta; z)$ depends smoothly on $z$ and on $\beta$ and, using (8.8), is given by

$$A(\beta, z) := \pi^{-N} \int_{|\alpha'|_{C^{N-1}} < R_0} \mathbb{1}_{B(0,R)}(\alpha_1(\beta, \alpha', z), \alpha') \left| \partial_{\alpha_1} E^\delta_{-+}(\alpha_1(\beta, \alpha', z), \alpha'; z) \right|^{-2}$$

$$\cdot \left| A(\alpha, z) - \beta \frac{\partial_{\alpha} X(z) X(z)}{\|X(z)\|^2} - B(z) \alpha_2 + O\left(\eta^{1/4} \delta^2 h^{-7/2}\right) \right|^2$$

$$\cdot \exp \left\{ -\alpha' \overline{\alpha'} - \frac{-\beta + E_{-+}(z) + O(\eta^{-1/4} \delta^2 h^{-5/2})}{\delta \|X(z)\|} \right\} L(\,d\alpha'),$$

(8.15)
where \( A(\alpha, z) \) and \( B(z) \) are defined as follows:

\[
A(\alpha, z) := \partial_2 E_{-+}(z) - \frac{(\partial_2 X(z)|X(z))}{\|X(z)\|^2} (E_{-+}(z) + O\left(\eta^{-1/4} \delta^2 h^{-5/2}\right)) + O\left(\eta^1 \delta^2 h^{-7/2}\right)
= (\varepsilon_0 f_0) (1 + O(h^\infty)) + O\left(\eta^{1/4} \delta^2 h^{-7/2}\right)
= O\left(\eta^{3/4} h^{-\frac{3}{2}} e^{-\frac{\alpha z^2}{h}}\right) + O\left(\eta^{1/4} \delta^2 h^{-7/2}\right).
\] (8.16)

The second identity for \( A \) is due to Proposition 6.5 and the following estimate

\[
\left| \frac{(\partial_2 X(z)|X(z))}{\|X(z)\|^2} \right| \leq \|\partial_2 X(z)\| = (1 + O(h^\infty)) O\left(\eta^{1/2} h^{-1}\right) = O\left(\eta^{1/2} h^{-1}\right)
\]

which follows from Proposition 5.5 and Proposition 5.6. In the last line we used Proposition 4.6 together with (3.4). Furthermore, recall by Step II and Step III that \( A(\alpha, z) \) is holomorphic in \( \alpha \).

Similarly, we define

\[
B(z) := \delta \left( \|\partial_2 X(z)\|^2 - \frac{|(\partial_2 X(z)|X(z))|^2}{\|X(z)\|^2} \right)^\frac{1}{2}
= O\left(\eta^{-1/4} \delta^{2} h^{-\frac{3}{2}}\right).
\] (8.17)

**Remark 8.2.** It follows from Proposition 6.5, Proposition 4.6, Proposition 5.6 and from (8.6) that

\[
\eta^{-\frac{n+m}{2}} h^{n+m} \partial_n^m \partial_2^m A(z) = O\left(\eta^{3/4} h^{-\frac{3}{2}} e^{-\frac{\alpha z^2}{h}}\right) + O\left(\eta^{1/4} \delta^2 h^{-7/2}\right),
\]

\[
\eta^{-\frac{n+m}{2}} h^{n+m} \partial_n^m \partial_2^m B(z) = O\left(\eta^{-1/4} \delta^{2} h^{-\frac{3}{2}}\right).
\] (8.18)

Since \( A(\beta, z) \) is continuous in \( \beta \), the Lebesgue dominated convergence theorem shows that the limit (8.14) is equal to

\[
\int \varphi(z) A(0, z) L(dz).
\]

Next, let us look at the indicator function \( I_{B(0,R)}(\alpha_1(\beta, \alpha', z), \alpha') \) for \( \|\alpha'\| < R_0 \): By (8.13) we have

\[
|\alpha_1(0, \alpha')| = \frac{|E_{-+}(z) + O(\delta^2 h^{-5/2})|}{\delta \|X(z)\|}.
\]

Thus,

\[
I_{B(0,R)}(\alpha_1(0, \alpha'; z), \alpha') = 1, \text{ if } \left\{ \begin{array}{l}
|\alpha_1(0, \alpha')| \leq R^2 - R_0^2 = C_2^2, \|\alpha'\| < R_0^2,
R^2 - R_0^2 < |\alpha_1(0, \alpha')|^2 < R^2, \|\alpha'\| < R_0^2 - |\alpha_1(0, \alpha')|^2,
\end{array} \right.
0, \text{ if } R^2 \leq |\alpha_1(0, \alpha')|^2,
\] (8.19)

with \( C_2 := C^2 - C_0^2 \).

**Remark 8.3.** Note that this restriction amounts to

\[
I_{B(0,R)}(\alpha_1(0, \alpha'; z), \alpha') = 1, \text{ if } \delta \gg h|E_{-+}(z)| \text{ or } \delta \approx h|E_{-+}(z)|,
0, \text{ if } \delta \ll h|E_{-+}(z)|,
\]
Hence, we split $\Lambda(0, z)$ into

$$\Lambda(0, z) = \Lambda(0, z) \left( I_{\{\sqrt{\Theta(z; h, \delta)} \leq \frac{C}{h} \}}(z) + I_{\{\frac{C}{h} < \sqrt{\Theta(z; h, \delta)} < R \}}(z) \right)$$

$$=: \Lambda_1(0, z) + \Lambda_2(0, z),$$

where

$$\Theta(z; h, \delta) := \frac{|E_{-\pi}(z) + \mathcal{O}(\delta^2 \eta^{-1/4} h^{-5/2})|^2}{\delta^2 \|X(z)\|^2}.$$  

In the following we will treat the first term of the above as the main part, whereas we will treat the second term as an error term which we shall estimate at the end of this step.

Note that the function

$$\{\|\alpha'\|_{CN-1} < R_0 \} \ni \alpha' \mapsto \exp \left\{-|\alpha_1(0, \alpha'; z)|^2 \right\} \in [0, 1]$$

is continuous, bounded and recall that (8.13) holds for all $\alpha' \in \{\|\alpha'\|_{CN-1} < R_0 \}$. Furthermore, note that all factors in the integral (8.15) are positive. Since the ball $\{\|\alpha'\|_{CN-1} < R_0 \}$ is simply connected the intermediate value theorem yields

$$\Lambda_1(0, z) = \pi^{-N} I_{\{\sqrt{\Theta(z; h, \delta)} \leq \frac{C}{h} \}}(z) \left| \delta \|X(z)\| + \mathcal{O}\left(\eta^{-1/4} \delta^2 h^{-3/2}\right) \right|^{-2}$$

$$\cdot \exp\{ -\Theta(z; h, \delta) \} \int_{\|\alpha'\|_{CN-1} < R_0} |A(\alpha, z) - \delta B(\alpha) \alpha_2^2 \exp \{ -\alpha' \alpha \} L(d\alpha').$$

(8.20)

Here we also applied (8.10). Before we can further simplify (8.20), let us consider the following technical Lemma:

**Lemma 8.4.** Let $h > 0$ and let $C_0 > 0$. Let $n \in \mathbb{N}^{-1}, m \in \mathbb{N}^{N-1}$, let $R_0 = C_0/h$ and let $\alpha \in \mathcal{C}^N$ where $N := (2 \lfloor \frac{C}{r} \rfloor + 1)^2$. If $C_0 > C_1 > 0$ are large enough and such that

$$\ln \left( 2 + \frac{e R_0^2}{N-2} \right) < \frac{R_0^2}{2(N-2)},$$

then, for $h > 0$ small enough, there exists a constant $D_{n,m} =: D > 0$ such that

$$\int_{\|\alpha'\|_{CN-1} \geq R_0} \alpha^n \alpha^m e^{-\alpha' \alpha} L(d\alpha') = O\left( e^{-D^2} \right).$$

**Proof.** Define

$$2u := \begin{cases} |n| + |m|, & \text{if it is even} \\
|n| + |m| + 1, & \text{else} \end{cases}$$

and notice

$$\int_{\|\alpha'\|_{CN-1} \geq R_0} \alpha^n \alpha^m e^{-\alpha' \alpha} L(d\alpha') \leq \pi^{1-N} \left| S^{2N-3} \right| \int_{R_0}^{\infty} r^{2u+2N-3} e^{-r^2} dr$$

$$= \frac{2}{(N-2)!} \int_{R_0}^{\infty} \tau^{u+N-2} e^{-\tau^2} d\tau.$$  

Repeated partial integration then yields

$$\frac{2}{(N-2)!} e^{-R_0^2} \sum_{i=0}^{u+N-2} \binom{u+N-2}{i} (u+N-2-i)! R_0^{2i}. \tag{8.21}$$

Stirling’s formula

$$\sqrt{2\pi n} \left( \frac{n}{e} \right)^n \leq n! \leq e \sqrt{n} \left( \frac{n}{e} \right)^n \tag{55}.$$
The quantity (8.21) is 

\[(u + N - 2 - i)! \leq e^{\sqrt{u + N - 2}} \left(\frac{u + N - 2}{e}\right)^{u + N - 2 - i}\]

and 

\[(N - 2)! \geq 2\pi(N - 2) \left(\frac{N - 2}{e}\right)^{N - 2}.

The quantity (8.21) is ≤

\[
\frac{e^{\sqrt{u + N - 2}}}{(N - 2)!} e^{-R_0^2} \sum_{i=0}^{u + N - 2} \left(\frac{u + N - 2}{e}\right)^i R_0^{2i}
\]

\[
\leq \frac{e^{\sqrt{u + N - 2}}}{2\pi(N - 2)} e^{-R_0^2} \left(\frac{e}{N - 2}\right)^{N - 2} \left(R_0^2 + \frac{u + N - 2}{e}\right)^{u + N - 2}
\]

\[
= e^{-R_0^2} e^{\frac{u}{2\pi}} \sqrt{1 + \frac{u}{N - 2}} \left(\frac{R_0^2 e}{N - 2} + 1 + \frac{u}{N - 2}\right)^{N - 2} \left(R_0^2 + \frac{u + N - 2}{e}\right)^u.
\]

Since \(u/(N - 2)\) is bounded for \(h > 0\) small, it remains to consider

\[
\exp \left\{ -R_0^2 + (N - 2) \ln \left(\frac{R_0^2 e}{N - 2} + 1 + \frac{u}{N - 2}\right) + u \ln \left(R_0^2 + \frac{u + N - 2}{e}\right) \right\}. \tag{8.22}
\]

However, by the assumption in the statement, there exists a \(1 > \kappa > 0\) such that

\[-R_0^2 + (N - 2) \ln \left(\frac{R_0^2 e}{N - 2} + 1 + \frac{u}{N - 2}\right) \leq -R_0^2 \kappa = -\frac{C_0^2}{h^2},
\]

which implies that (8.22) is dominated by

\[
\exp \left\{ -\frac{C_0^2}{h^2} \left(\kappa - \frac{h^2}{1\ln(h)}\right) \right\}.
\]

Thus, we may conclude the statement of the Lemma for \(h > 0\) small enough. \(\square\)

Now let us return to (8.20). We turn to the integral from (8.20)

\[
\pi^{-N} \int_{\|\alpha\|_{C^{N-1}} < R_0} |A - B\alpha_2|^2 \exp \{-\alpha'\alpha'\} L(d\alpha'). \tag{8.23}
\]

Since \(B\) is constant in \(\alpha\) and since \(\int |\alpha|^2 \exp(\alpha'\alpha') L(d\alpha') = \pi^{N-1}\), we can conclude, by Lemma 8.4 for \(C_0 > C_1 > 0\) large enough and \(h > 0\) small enough, that there exists a constant \(D > 0\) such that

\[
\pi^{-N} \int_{\|\alpha\|_{C^{N-1}} < R_0} |B\alpha_2|^2 \exp \{-\alpha'\alpha'\} L(d\alpha') = \pi^{-1}|B|^2 + \mathcal{O}\left(\eta^{-1/2} h^2 e^{-\frac{D}{\sqrt{h}}}\right).
\]

The mean value theorem, (8.16) and Lemma 8.4 imply that there exists a constant \(D > 0\) (not necessarily the same as above) such that

\[
\pi^{-N} \int_{\|\alpha\|_{C^{N-1}} < R_0} |A|^2 \exp \{-\alpha'\alpha'\} L(d\alpha') = \pi^{-1}|A|^2 + \mathcal{O}\left(e^{-\frac{D}{\sqrt{h}}}\right).
\]

Note that after the equality sign we have \(A = A(\alpha', z)\) for an \(\alpha' \in B(0, R_0)\) given by the mean value theorem. Now consider

\[
\pi^{-N} \int_{\|\alpha\|_{C^{N-1}} < R_0} |\tilde{A}B\alpha_2| \exp \{-\alpha'\alpha'\} L(d\alpha') = \pi^{-N}B \int_{\|\alpha\|_{C^{N-1}} < R_0} |\tilde{A}\alpha_2| \exp \{-\alpha'\alpha'\} L(d\alpha').
\]

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Since $A(\alpha, z)$ is holomorphic in $\alpha$ we gain from (8.16) by the Cauchy inequalities

$$|\partial_{\alpha_2} A| = \mathcal{O}\left(\eta^{1/4} \delta^2 h^{-5/2}\right).$$

Here we used that the first term in (8.16) is independent of $\alpha$. Extend $A$ to a function on $\mathbb{C}^{N-1}$ such that the above estimate still holds. Then, by Lemma 8.4 there exists a constant $D > 0$

$$\pi^{-N} B \int_{\|\alpha\|_{\mathbb{C}^{N-1}} \geq R_0} \mathcal{A}_{\alpha_2} \exp \{-\alpha' \alpha\} L(d\alpha') = \mathcal{O}\left(\eta^{1/2} h^{-1} \delta e^{-\frac{\pi^2 \delta^2}{h}} + \delta^3 h^{-4}\right) e^{-\frac{D}{\pi}}.$$  

Here we used (8.16) and (8.17). Stokes' theorem implies

$$\pi^{-N} B \int_{\mathbb{C}^{N-1}} \mathcal{A}_{\alpha_2} \exp \{-\alpha' \alpha\} L(d\alpha') = \pi^{-N} B \int_{\mathbb{C}^{N-1}} (\partial_{\alpha_2} A) \exp \{-\alpha' \alpha\} L(d\alpha') \leq \mathcal{O}(\delta^3 h^{-3}).$$

Plugging the above into (8.23), we gather that there exist a constant $D > 0$ such that

$$\pi^{-N} \int_{\|\alpha\|_{\mathbb{C}^{N-1}} < R_0} |A - B_{\alpha_2}|^2 \exp \{-\alpha' \alpha\} L(d\alpha') = \pi^{-1} \left(|A(z)|^2 + |B(z)|^2\right) + \mathcal{O}\left(\delta^3 h^{-3} + e^{-\frac{D}{\pi}}\right)$$

$$=: \delta^2 \Psi(z, h, \delta)$$  \hspace{1cm} (8.24)

with

$$\pi^{-1} \left(|A(z)|^2 + |B(z)|^2\right)$$

$$= \frac{\delta^2}{\pi} \left((\partial_z X|\partial_z X) - \frac{1}{\|Z\|^2} |(\partial_z X|Z)|^2 + \delta^{-2} \left|\left(c_0|f_0\right)/(1 + \mathcal{O}(1)) + \mathcal{O}\left(\eta^{1/4} \delta^2 h^{-7/2}\right)^2\right|\right).$$

Note that for the identity we used (8.16) and (8.17). Plug the above into (8.20) and use

$$\left|\delta\|X(z)\| + \mathcal{O}\left(\eta^{1/4} \delta^2 h^{-3/2}\right)^{-2} \right| = \left(1 + \mathcal{O}(\eta^{1/4} \delta h^{-3/2})\right)/\delta^2 \pi \|X(z)\|^2,$$

to conclude that for $h > 0$ small enough, there exists a constant $D > 0$ such that

$$A_1(0, z) := \frac{1 + \mathcal{O}(\eta^{1/4} \delta h^{-3/2})}{\pi \|Z\|^2} \mathbb{1}_{\mathcal{C}_{\mathbb{C}^{N-1}}}(z) \Psi(z, h, \delta) \exp \{-\Theta(z; h, \delta)\}. \hspace{1cm} (8.25)$$

Finally, let us estimate $A_2(0, z)$: applying (8.16), (8.17) and Lemma 8.4 to (8.15) yields

$$A_2(0, z) \leq e^{-\frac{D}{\pi^2}} \mathcal{O}\left(\delta^4 \eta^{1/2} h^{-7} + \eta^{1/2} h^{-5/2} \delta e^{-\frac{\pi^2 \delta^2}{h}}\right) = \mathcal{O}\left(e^{-\frac{D}{\pi^2}}\right),$$

for some $D > 0$. Thus, up to an error of order $\mathcal{O}(e^{-\frac{D}{\pi^2}})$, we can substitute $\mathbb{1}_{\mathcal{C}_{\mathbb{C}^{N-1}}}(z)$ with 1 in (8.25).
Step V: In this step we will estimate the second integral of equation (8.9), \( I_2(\varphi) \). Therefore, we will increase the space of integration

\[
\int_{B(0,R) \cap \|\alpha\|_{C^{N-1}} > R_0} \chi \left( \frac{E^{\beta}_{\alpha}(z; \alpha)}{\varepsilon} \right) \frac{1}{\varepsilon^2} \left| \partial_{z \alpha} E^{\beta}_{\alpha}(z; \alpha) \right|^2 e^{-\alpha z} L(\alpha) \, d\alpha
\]

where

\[
\lim_{\varepsilon \to 0} \int_{R_0 < \|\alpha\|_{C^{N-1}} < 2R_0} \mathbb{I}_{B(0,2R)}(\alpha(0, \alpha'; z), \alpha') \left| \partial_{z \alpha} \beta(\alpha_1, \alpha'; z) \right|^{-2} |A(\alpha; z) - B(z)\alpha_2|^2 \exp \left\{ -\alpha' \alpha - \Lambda(z, h, \delta)^2 \right\} L(\alpha').
\]

By (8.16), (8.17) and Lemma 8.4 we see that there exists a constant \( D > 0 \) such that

\[
\pi^{-N} \int_{R_0 < \|\alpha\|_{C^{N-1}} < 2R_0} |A - B\alpha_2|^2 e^{-\alpha' \alpha} L(\alpha') \leq e^{-\frac{D}{\pi^2}} O \left( \delta \eta^{1/2} h^{-7} + \eta^{1/2} h^{-1} \delta e^{-\frac{\eta^{3/2}}{h}} \right) = O \left( e^{-\frac{D}{\pi^2}} \right).
\]

Finally note that the statement about the derivatives of the error estimates follows from (8.18) and (8.6).

9. Average Density of Eigenvalues

First, we will give the proof the main result of this work:

Proof of Theorem 2.10. By (1.5) we know that there exists a \( C > 0 \) such that the tube of radius \( C^{-1} \left( h \ln \frac{1}{\pi} \right)^{2/3} \) around \( \partial \Sigma \) is void of the spectrum of \( P_h + \delta Q_\omega \). Thus, we may consider without loss of generality \( \Omega \in \Sigma \) with \( \text{dist}(\Omega, \partial \Sigma) > C^{-1} \left( h \ln \frac{1}{\pi} \right)^{2/3} \).

Let \( \left( h \ln \frac{1}{\pi} \right)^{2/3} \ll \eta \leq C \), where \( C > 0 \) is some constant, as above. Recall the definition of \( \Omega^a_\eta \) given in (3.1):

\[
\Omega^a_\eta = \left\{ z \in \Omega : \frac{\eta}{C} \leq \Im z \leq C \eta \right\}
\]

for some constant \( C > 0 \). Define

\[
\tilde{\Omega}^a_\eta := \left\{ z \in \Omega : \frac{\eta}{2C} \leq \Im z \leq 2C \eta \right\}.
\]

Define \( \eta_j := C^{-j}, j \in \mathbb{N}_0 \), and consider the open covering of \( \Omega \)

\[
\Omega \subset \bigcup_{j \in \mathbb{N}_0} \tilde{\Omega}^a_{\eta_j} \cup \left( \Omega \setminus \bigcup_{j \in \mathbb{N}_0} \Omega^a_{\eta_j} \right),
\]

where \( \text{dist}(\Omega \setminus \bigcup_{j \in \mathbb{N}_0} (\Omega^a_{\eta_j}, \partial \Sigma)) > 1/C \), thus, conforming with the previous notation, we may define

\[
\Omega_i := \Omega \setminus \bigcup_{j \in \mathbb{N}_0} \Omega^a_{\eta_j},
\]

Thus, we have

\[
\left| \mathcal{I}(\varphi) \right| \leq \int_{B(0,R) \cap \|\alpha\|_{C^{N-1}} > R_0} \chi \left( \frac{E^{\beta}_{\alpha}(z; \alpha)}{\varepsilon} \right) \frac{1}{\varepsilon^2} \left| \partial_{z \alpha} E^{\beta}_{\alpha}(z; \alpha) \right|^2 e^{-\alpha z} L(\alpha) \, d\alpha.
\]
Let \( \{ \chi_{\eta_j} \}_{j \in \mathbb{N}_0} \) be a partition of unity subordinate to this locally finite open subcovering such that
\[
1 = \sum_{j \in \mathbb{N}} \chi_{\eta_j} + \chi_{\eta_0},
\]
in a neighborhood of \( \Omega \). Here, for \( j \in \mathbb{N} \), \( \chi_{\eta_j} \in C_0^\infty(\tilde{\Omega}_\eta) \), supported in either \( \tilde{\Omega}_\eta \). Furthermore, \( \chi_{\eta_0} \in C^\infty(\Omega_\eta) \). This partition of unity together with Proposition 8.1 yields
\[
\mathbb{E} \left[ \mathbb{E}(\varphi) \mathbb{1}_{B(0, R)} \right] = \sum_{j \in \mathbb{N}} \mathbb{E} \left[ \mathbb{E}(\varphi \chi_{\eta_j}) \mathbb{1}_{B(0, R)} \right] + \mathbb{E} \left[ \mathbb{E}(\varphi \chi_{\eta_0}) \mathbb{1}_{B(0, R)} \right]
\]
Furthermore, Proposition 8.1 and Proposition 6.1 yield that
\[
\text{Analysis of the density } \Psi \text{ Recall the formula for the density of eigenvalues given in Proposition 8.1. Define }
\]
\[
\Psi_1(z; h, \delta) = (\partial_z X) |(\partial_z X)| - \frac{1}{\|X\|^2} |(\partial_z X)|^2 + \mathcal{O}(\delta^3 h^{-3})
\]
Since the error above is of order \( \mathcal{O}(1) \), it follows from Proposition 6.1 that
\[
\Psi_1(z, h, \delta) = \frac{1}{h} \left\{ \frac{i}{\{p, \bar{p}\}(\rho_+(z))} - \frac{i}{\{p, \bar{p}\}(\rho_-(z))} \right\} + \mathcal{O}(\text{dist } (z, \partial \Sigma)^{-2})
\]
where we used that \( \text{Im } z \approx \eta_j \) for \( z \in \Omega_{\eta_j} \). Proposition 6.2 implies
\[
\Psi_1(z, h, \delta) L(dz) = \frac{1}{2h} p_\nu(d\xi \wedge dx) + \mathcal{O}(\text{dist } (z, \partial \Sigma)^{-2}) L(dz).
\]
Furthermore, Proposition 8.1 and Proposition 6.1 yield that
\[
\eta_j^{-2} - \mathcal{O}(\eta_j^{-2})
\]
Next, let us turn to the second part of \( \Psi \):
\[
\delta^{-2} |(e_0, f_0)(1 + \mathcal{O}(h^\infty)) + \mathcal{O}(\eta_j^{1/4} \delta^2 h^{-7/2})|^2
\]
\[
= \delta^{-2} |(e_0, f_0)|^2 (1 + \mathcal{O}(h^\infty)) + \mathcal{O}(\eta_j^{1/2} \delta^2 h^{-7}) + \mathcal{O}(\eta_j^{1/4} h^{-7/2} |(e_0, f_0)|)
\]
\[
= \delta^{-2} |(e_0, f_0)|^2 (1 + \mathcal{O}(h^\infty)) + \mathcal{O}(\eta_j h^{-4} e^{-\frac{\delta}{\eta}}) + \mathcal{O}(\eta_j^{1/2} \delta^2 h^{-7}).
\]
In the last line, we applied an estimate on \( |(e_0, f_0)| \) which follows from Proposition 4.2. The error term \( \mathcal{O}(\eta_j h^{-4} e^{-\frac{\delta}{\eta}}) \) is bounded by \( \mathcal{O}(\eta_j) \) because \( \eta \gg (-h \ln h)^{2/3} \). We then absorb
As in (8.18), the error estimates don’t change if we apply $\Im\frac{\omega_j}{z}$ since

$$\left\langle \frac{\omega_j}{\Im} \right\rangle$$

where we used that

$$\chi$$

Analysis of the exponential $\Theta$

Recall from Proposition 4.1 that $-\alpha_0 = E_+$ and use (4.24) to find that

$$E_+(z) = ([P_h, \chi]e_0 | f_0) \left( 1 + \mathcal{O} \left( \eta_j^{-3/4} h^{1/2} \right) \right),$$

where $\chi \in C^\infty_0(S^1)$ with $\chi \equiv 1$ in a small open neighborhood of $\{x_-(z) : z \in \Omega\}$. Thus, using $\|X\| = (1 + \mathcal{O}(h^\infty))$ (cf. Proposition 5.5), we have the following equation for $\Theta$ given in Proposition 8.1

$$\Theta(z, h, \delta) = \frac{\left| E_+(z) + \mathcal{O} \left( \eta_j^{-1/4} \delta^2 h^{-5/2} \right) \right|}{\delta^2 \|X\|^2} = \frac{\left| ([P_h, \chi]e_0 | f_0) + \mathcal{O} \left( \eta_j^{-1/4} \delta^2 h^{-5/2} \right) \right|^2}{\delta^2 (1 + \mathcal{O}(h^\infty))} \left( 1 + \mathcal{O} \left( \frac{\eta_j^{3/2}}{h} \right) \right).$$

As in (8.18), the error estimates stay invariant under the action of $\eta_j^{\frac{n+m}{2}} h^{n+m} \partial_z^a \partial_{\overline{z}}^m$.

Finally, to conclude the density given in the Theorem, note that

$$\frac{1 + \mathcal{O} \left( \eta_j^{-1/4} \delta h^{-3/2} \right)}{\pi \|X\|^2} = \frac{1 + \mathcal{O} \left( \Delta \left( z, \partial \Sigma \right)^{-1/4} \delta h^{-3/2} \right)}{\pi} \quad \Box$$

In the case of the operator $P_{h}^\delta$, it is possible to state more explicit formulas for the different parts of the density of eigenvalues given in Theorem 2.10:

It follows by Propositions 6.1 and 6.2 that

$$\frac{1}{2h} p_+ (d\xi \wedge dx) = \frac{1}{\hbar} \left\{ \frac{i}{\{p, \overline{p}\} (\rho_+(z))} + \frac{i}{\{p, \overline{p}\} (\rho_-(z))} \right\} L(dz) = \frac{1}{h\sqrt{\text{dist}(z, \partial \Sigma)}} L(dz)$$

where we used that $\Im z \simeq \eta_j$ for $z \in \Omega_{\eta_j}^\delta$. For our purposes we can assume that $|\Im g| > 1/C, C \gg 1$, since inside this tube $\Psi_2$ and $\Theta$ are exponentially small in $h > 0$. In the case of $\Psi_2$, this follows from the assumptions on $\delta$ (cf. Definition 2.5) and from Remark 4.4. In the case of $\Theta$, this follows from the assumptions on $\delta$ and Proposition 4.12 and (9.3). Thus, applying Proposition 4.2 to (9.2) yields

$$\Psi_2(z; h, \delta) = \frac{1}{\pi h^2} \left\{ \frac{i}{\overline{p, p}} (\rho_+(z)) + \frac{i}{p, p} (\rho_-(z)) \right\} L(dz) \approx \frac{1}{h \sqrt{\text{dist}(z, \partial \Sigma)}} L(dz)$$

As in (8.18), the error estimates don’t change if we apply $\eta_j^{\frac{n+m}{2}} h^{n+m} \partial_z^a \partial_{\overline{z}}^m$. Moreover, since $\Im z \simeq \eta_j$ for $z \in \Omega_{\eta_j}^\delta$,

$$\Psi_2^0(z; h, \delta) \approx \frac{\left( \text{dist}(z, \partial \Sigma)^{\frac{3}{2}} e^{-\frac{2\pi}{\hbar}} \right)}{h^2 \delta^2}.$$
Apply Proposition 4.12 to (9.3) gives that
\[
\Theta(z, h, \delta) = V(z, h)^2 e^{-\frac{2\pi}{\delta^2}} \left(1 + O(h^\infty) + O\left(e^{-\frac{\eta^{3/2}}{h}}\right)\right)
+ O\left(\eta_j^{-1/2} \delta^2 h^{-5}\right) + O\left(Vh^{-5/2} e^{-\frac{\eta}{\delta}}\right).
\] (9.5)

Since \(0 \leq V = O\left(\eta_j^{1/4} h^{1/2}\right)\) by (4.21), it follows that
\[
\Theta(z, h, \delta) = \frac{h}{\pi} \left(\frac{1}{2} \{p, \overline{p}\} (\rho_+)^{\frac{1}{2}} \{\overline{p}, p\} (\rho_-)^{\frac{1}{2}} e^{-\frac{2\pi}{\delta^2}} \left(1 + O\left(\eta_j^{-1/4} h^{\frac{1}{2}}\right)\right)\right)
+ O\left(\eta_j^{-1/2} \delta^2 h^{-5}\right) + O\left(\eta_j^{1/4} h^{-2} e^{-\frac{\eta}{\delta}}\right).
\]

Furthermore, for \(e^{-\frac{2\pi}{\delta^2}} \delta^{-2} < 1\), the error term \(O\left(\eta_j^{1/4} h^{-2} e^{-\frac{\eta}{\delta}}\right)\) is bounded by \(O(\eta_j^{1/4} h^{-2} \delta)\) since there we have that \(e^{-\frac{\eta}{\delta}} \leq \delta\). For \(e^{-\frac{2\pi}{\delta^2}} \delta^{-2} < 1\), we have that
\[
O\left(\eta_j^{1/4} h^{-2} e^{-\frac{\eta}{\delta}}\right) \leq O\left(\eta_j^{1/4} h^{-2} \delta e^{-\frac{2\pi}{\delta^2}}\right) \leq O\left(\eta_j^{1/4} h^{-2} e^{-\frac{2\pi}{\delta^2}}\right).
\]

Thus,
\[
\Theta(z, h, \delta) = \frac{h}{\pi} \left(\frac{1}{2} \{p, \overline{p}\} (\rho_+)^{\frac{1}{2}} \{\overline{p}, p\} (\rho_-)^{\frac{1}{2}} e^{-\frac{2\pi}{\delta^2}} \left(1 + O\left(\eta_j^{-1/4} h^{\frac{1}{2}}\right)\right)\right)
+ O\left(\eta_j^{1/4} h^{-2} \delta + \eta_j^{-1/2} \delta^2 h^{-5}\right).
\] (9.6)

Analogous to (8.18), the error estimates stay invariant under the action of \(\eta_j^{-\frac{n+m}{2}} h^{n+m} \partial^n \partial^m\).

Moreover,
\[
\Theta^0(z; h, \delta) \asymp h \sqrt{\text{dist}(z, \partial \Sigma)} e^{-\frac{2\pi}{\delta^2}}.
\]

We have thus proven Proposition 2.12 and Proposition 2.11. Since we will need it later on we will state the following formulas:

**Lemma 9.1.** Under the assumptions of Theorem 2.10 and for \(h^{2/3} \ll \eta < \text{const.}\) we have
\[
\partial_{\text{Im} z} \Psi_1 = -\frac{1}{4h} \left(\frac{\text{Im} g''(x_-)}{(\text{Im} g'(x_-))^3} - \frac{\text{Im} g''(x_+)}{(\text{Im} g'(x_+))^3}\right) + O(\eta^{-2}) = O\left(\eta^{-3/2} h^{-1}\right)
\]
and for \(|\text{Im} z - (\text{Im} g)| > 1/C, C > 0\) large enough,
\[
\partial_{\text{Im} z} \Psi_2(z, h) = \frac{2}{\pi h^2} \left(\frac{1}{2} \{p, \overline{p}\} (\rho_+)^{\frac{1}{2}} \{\overline{p}, p\} (\rho_-)^{\frac{1}{2}}\right) |\partial_{\text{Im} z} S(z)|^2 (-\partial_{\text{Im} z} S(z)) e^{-\frac{2\pi}{\delta^2}}
+ O(\eta^{-3/4} h^{-3} \delta + \delta^2 h^{-6}).
\]

**Proof.** Let us first treat \(\Psi_1\): Recall from the proof of Proposition 6.5 that \(\Psi_1\) was given by an oscillatory integral where the phase vanishes at the critical point. Thus, the \(\partial_{\text{Im} z}\) derivative of the error term \(O(\eta^{-2})\) grows at most by \(\eta^{-1}\). Thus, taking the derivative of (9.1) yields
\[
\partial_{\text{Im} z} \Psi_1 = -\frac{1}{4h} \left(\frac{\text{Im} g''(x_-)}{(\text{Im} g'(x_-))^3} - \frac{\text{Im} g''(x_+)}{(\text{Im} g'(x_+))^3}\right) + O(\eta^{-3}) = O\left(\eta^{-3/2} h^{-1}\right),
\]
where the last estimate follows from $|2\text{Im }g'(x_{\pm})| = |\{p,\bar{p}\}(\rho_{\pm})| \asymp \sqrt{\eta}$ (cf. Proposition 6.1) and from the fact that the $z$- and $\bar{z}$-derivative of the error term grow at most by a factor of $O(\eta^{1/2}h^{-1})$.

Now let us turn to $\Psi_2$: one calculates from (9.4) that for $|\text{Im }z - \langle \text{Im }g \rangle| > 1/C$

$$\partial_{\text{Im }z} \Psi_2(z, h) = \frac{2(\frac{i}{2} \{p, \bar{p}\}(\rho_+) + \frac{i}{2} \{p, \bar{p}\}(\rho_-))}{\pi h^2} |\partial_{\text{Im }z} S(z)|^2 (-\partial_{\text{Im }z} S(z)) \frac{e^{-\frac{2z}{\bar{z}}}}{\partial^2} \cdot \left(1 + O\left(\eta^{-3/4}h^{1/2}\right)\right).$$

Here we used that the $z$- and $\bar{z}$-derivative of the error terms grow at most by a factor of $O(\eta^{1/2}h^{-1})$.

Finally, let us turn to $\Theta$: as in the proof of Proposition 4.3 one calculates the formula for $\partial_{\text{Im }z} \Theta$ from (9.6).  \hfill \Box

10. Properties of the density

In this section we will discuss and prove the results stated in Section 2.4.

10.1. Local maximum of the average density. First, we prove the resolvent estimate given in Proposition 2.4.

Proof of Proposition 2.4. Recall that the operator $Q(z)$ is self-adjoint and that $|t_0(z)| = |\alpha_0(z)|$; see Section 2.2. It follows that

$$\| (P_h - z)^{-1} \| = |t_0(z)|^{-1} = |\alpha_0(z)|^{-1}.$$ 

Recall the Grushin problem posed in Proposition 4.1. Since $E_{-+}^{-1} = -\alpha_0$, it follows by Proposition 4.12 that

$$\| (P_h - z)^{-1} \| = \frac{\exp\left\{ \frac{\sqrt{2}}{h} \right\}}{V(z)|1 - e^{\Phi(z)}| \left(1 + O\left(e^{-\frac{\sqrt{2}h^{1/2}}{h}}\right)\right)^{1/2}}. \tag{10.1}$$

For the readers convenience: $V$ as given in (4.21):

$$V(z, h) = \left(\frac{i}{2} \{p, \bar{p}\}(\rho_+) + \frac{i}{2} \{p, \bar{p}\}(\rho_-)\right) \left(\frac{h}{\pi}\right)^{1/2} \left(1 + O\left(\eta^{-3/2}h\right)\right).$$

The result about the asymptotic behavior of the resolvent follows from the above together with the fact that $|\{p, \bar{p}\}(\rho_{\pm})| \asymp \sqrt{\eta}$ (cf. Proposition 6.1).  \hfill \Box

We have cut the proof of Proposition 2.13 into the following two Lemmata:

Lemma 10.1. Let $z \in \Sigma_{\kappa,d}$ be as in (2.12) and let $S(z)$ be as in Definition 2.1. Let $\delta > 0$ and $\varepsilon(h)$ be as in Definition 2.5 with $\kappa > 7/2$. Moreover, let $E_{-+}(z)$ be as in Proposition 4.1. Then,

- for $0 < h \ll 1$, there exist numbers $y_{\pm}(h)$ such that $\varepsilon_0 = S(y_{\pm}(h))$ with

$$\frac{1}{C} h^{\frac{5}{2}} \ll y_{-}(h) < \langle \text{Im }g \rangle - ch \ln h^{-1} < \langle \text{Im }g \rangle + ch \ln h^{-1} < y_{+}(h) \ll \text{Im }g(b) - Ch^{\frac{5}{2}},$$

for $c > 1$. Furthermore,

$$y_{-}(h), (\text{Im }g(b) - y_{+}(h)) \asymp (\varepsilon_0(h))^{2/3};$$
that there exists $h_0 > 0$ and a family of smooth curves, indexed by $h \in ]h_0, 0[$,
\[ \gamma_{\pm}^h : ]c, d[ \to \mathbb{C} \text{ with } \text{Re} \gamma_{\pm}^h(t) = t \]
such that
\[ |E_{-+}(\gamma_{\pm}^h(t))| = \delta, \]
and
\[ \text{Im} \gamma_{\pm}^h(\text{Re } z) = y_{\pm}(\varepsilon_0(h)) \left( 1 \mp \frac{h}{\mathcal{O}(1)\varepsilon_0(h)} \right). \]

Furthermore, there exists a constant $C > 0$ such that
\[ \frac{d\text{Im } \gamma_{\pm}^h}{dt}(t) = \mathcal{O}\left( \exp \left[ \frac{-\varepsilon_0(h)}{Ch} \right] \right). \]

Lemma 10.2. Assume the same hypothesis as in Lemma 10.1 and let
\[ D(z, h) := \frac{1 + \mathcal{O}(\delta h^{-\frac{3}{2}} \text{dist}(z, \partial \Sigma)^{-1/4})}{\pi} \Psi(z; h, \delta) \exp\{-\Theta(z; h, \delta)\} \]
be the average density of eigenvalues of the operator of $P_h^\delta$ given in Theorem 2.10. Then, there exists $h_0 > 0$ and a family of smooth curves, indexed by $h \in ]h_0, 0[$,
\[ \Gamma_{-+}^h : ]c, d[ \to \mathbb{C}, \text{ Re } \Gamma_{-+}^h(t) = t, \]
with $\Gamma_+ \subset \{ \text{Im } z < \langle \text{Im } g \rangle \}$ and $\Gamma_- \subset \{ \text{Im } z > \langle \text{Im } g \rangle \}$, along which $\text{Im } z \mapsto D(z, h)$ takes its local maxima on the vertical line $\text{Re } z = \text{const.}$ and
\[ \frac{d}{dt} \text{Im } \Gamma_{-+}^h(t) = \mathcal{O}\left( \frac{h^4}{\varepsilon_0(h)^{13/3}} \right). \]
Moreover, for all $c < t < d$
\[ |\Gamma_{-+}^h(t) - \gamma_{-+}^h(t)| \leq \mathcal{O}\left( \frac{h^5}{\varepsilon_0(h)^{13/3}} \right). \]

Proof of Proposition 2.13. The first two points of the proposition follow from Lemma 10.1 together with the observations that $|E_{-+}(z)| = |a_0| = |t_0(z)|$ (cf. Proposition 4.1) and that by (10.1)
\[ \| (P_h - \gamma_{-+}^h)^{-1} \| = \delta^{-1}. \]
The third point has been proven with Lemma 10.2.

Proof of Lemma 10.1. Recall from Proposition 2.2 that $S$ is strictly monotonous above and below the spectral line, i.e. $\text{Im } z = \langle \text{Im } g \rangle$. Furthermore, recall from Definition 2.5 that $-(\kappa - \frac{1}{2}) h \ln h + Ch \leq \varepsilon_0(h) < S(\langle \text{Im } g \rangle)$. Thus, the implicit function theorem implies that there exist $y_{\pm}(\varepsilon_0(h)) \in \mathbb{R}$ such that $S(y_{\pm}(\varepsilon_0(h))) = \varepsilon_0(h)$. Note that in the case where $\varepsilon_0(h)$ is independent of $h$, the same holds true for $y_{\pm}(\varepsilon_0(h))$. For the rest of the proof we will only treat the case where $\text{Im } z \leq \langle \text{Im } g \rangle$ (corresponding to $y_-$) since the other case is similar.

Consider $z \in \Sigma_{c,d}$ with $\text{Re } z = \text{const.}$ First, let us prove some a priori estimates: assume that there exists a $\zeta_-$ with $0 = \langle \text{Im } g(a) \rangle \leq \zeta_- \leq \langle \text{Im } g \rangle$ such that $|E_{-+}(\text{Re } z + i\zeta_-)|\delta^{-1} = 1$. Recall Proposition 2.2 and note that
\[ S(z) - \varepsilon_0(h) = \int_{\langle \text{Im } g \rangle}^{\text{Im } z} (\partial_{\text{Im } z} S)(t) \, dt + S(\langle \text{Im } g \rangle) - \varepsilon_0(h) \]
\[ = \int_{y_-(\varepsilon_0(h))}^{\text{Im } z} (\partial_{\text{Im } z} S)(t) \, dt + S(y_-(\varepsilon_0(h))) - \varepsilon_0(h). \]
Recall Proposition 4.12 and Definition 2.5. It follows by (10.2), that if $|\zeta_0 - (\text{Im } g)| \leq \frac{1}{C}$, $C > 0$ large enough, then $|E_{-+}(\text{Re } z + i\zeta_{0})| \delta^{-1} \leq O\left(\eta^{1/4}e^{-\frac{y}{\delta}}\right)$ for some $D > 0$ large. Thus, we may assume that, in case it exists,

$$|\zeta_{-} - (\text{Im } g)| > \frac{1}{C}. \quad (10.3)$$

We conclude from (10.2) that

$$y_-(h) \asymp (\varepsilon_0(h))^{2/3} \quad (10.4)$$

and that for $C > 0$ large enough

$$|(\text{Im } g) - y_-(\varepsilon)| > \frac{1}{C}. \quad (10.5)$$

Now let us prove the existence of the points $\zeta_-$. More precisely, we will prove that for $z \in \Sigma_{c,d}$ with $\text{Im } z < (\text{Im } g) - 1/C$ (cf. (10.3)) and fixed $\text{Re } z$ there exist exactly one $\zeta_-$ such that $|E_{-+}(\text{Re } z, \zeta_{-})| \delta^{-1} = 1$. For $z \in \Omega \cap \Omega_{\eta}^c \in \Sigma_{c,d}$ one calculates from by Proposition 4.12 that

$$\partial_{\text{Im } z}|E_{-+}(z)| = \left\{-V(z)\frac{\partial_{\text{Im } z}S(z)}{h}[1 - e^{\Phi(z)}]\left(1 + O\left(\frac{\eta^{2/3}}{h}\right)\right)\right\} + \partial_{\text{Im } z}V(z)[1 - e^{\Phi(z)}] \left(1 + O\left(\frac{\eta^{2/3}}{h}\right)\right) e^{-\frac{S(z)}{h}}. \quad (10.6)$$

Recall that $V$ is the product of the normalization factors of the quasimodes $e_{\text{wkb}}$ and $f_{\text{wkb}}$ when $z \in \Omega$ with $\text{dist } (\Omega, \partial \Sigma) > 1/C$ and the product of the normalization factors of the quasimodes $e_{\text{wkb}}^\eta$ and $f_{\text{wkb}}^\eta$ when $z \in \Omega \cap \Omega_{\eta}^c$ (cf. (4.19)). Since the derivative with respect to $\text{Im } z$ of the imaginary part of the phase function $\text{Im } \phi_{\pm}$ is equal to zero at $x_{\pm}$, it follows that

$$|\partial_{\text{Im } z}V(z)| = O(h^{1/2}\eta^{-3/4}). \quad (10.7)$$

The a priori bound (10.3) implies that there exists a constant $C > 1$ such that

$$|1 - e^{\Phi(z)}| = 1 + O\left(\frac{\eta}{\delta}\right), \text{ and } |\partial_{\text{Im } z}|1 - e^{\Phi(z)}| = O\left(\frac{\eta}{\delta}\right). \quad (10.8)$$

The fact that $\partial_{\text{Im } z}S(z) > 0$ (cf. (2.2)) implies that $\partial_{\text{Im } z}|E_{-+}(z)| < 0$. Note that in the case where $\text{dist } (\Omega, \partial \Sigma) > 1/C$ one sets in the above $\eta = 1$. Recall from Propositions 4.9 and 4.10 that $V$ is independent of $\text{Re } z$. Using

$$\partial_{\text{Re } z}|1 - e^{\Phi(z)}| = O\left(\frac{\eta}{\delta}\right),$$

we conclude that

$$\partial_{\text{Re } z}|E_{-+}(z)| = \partial_{\text{Re } z} \left\{V(z)[1 - e^{\Phi(z)}] \left(1 + O\left(\frac{\eta^{2/3}}{h}\right)\right) e^{-\frac{S(z)}{h}}\right\} = O\left(\frac{\eta^{2/3}}{h}\right) e^{-\frac{S(z)}{h}}. \quad (10.9)$$

This implies that the gradient $|E_{-+}(z)|$ is non-zero for all $z$ with $|\text{Im } z - (\text{Im } g)| > 1/C$ (cf. (10.3)) and thus we may conclude by the implicit function theorem, that for $\delta$ as above there exist locally smooth curves $\gamma_{\delta}^{\pm}(\text{Re } z) := (\text{Re } z, \zeta_{-}(\varepsilon_0(h), \text{Re } z))$ such that $|E_{-+}(\gamma_{\delta}^{\pm}(\text{Re } z))| = \delta$. Furthermore, we may extend $\gamma_{-}(\text{Re } z)$ smoothly for $c < \text{Re } z < d$. By the mean value theorem applied to $|E_{-+}(z)|$, there exists a $\xi$ between $y_-(h)$ and $\text{Im } \gamma_{\delta}^{\pm}(\text{Re } z)$ such that

$$|E_{-+}(\text{Re } z + iy_-(h))| - |E_{-+}(\gamma_{\delta}^{\pm}(\text{Re } z))| = |(\partial_{\text{Im } z}|E_{-+}(z))|(\text{Re } z + i\zeta_{-}(\varepsilon_0(h), \text{Re } z) - \text{Im } \gamma_{\delta}^{\pm}(\text{Re } z)).$$
Here we used that the Proof of Lemma 10.2. (cf. (10.6)), it follows that
\[ |y_-(h) - \text{Im} \gamma^h_\varepsilon(\text{Re} z)| = \mathcal{O} \left( \eta^{-1/2} h \right). \]
\[ \eta \asymp y_-(h) \asymp (\varepsilon_0(h))^{2/3} \] implies that also \( \text{Im} \gamma^h_\varepsilon(\text{Re} z) \asymp \eta \asymp (\varepsilon_0(h))^{2/3} \). The fact that \( \text{Im} z \mapsto |E_+(z)| \) is strictly monotonously decreasing, implies more precisely that
\[ \text{Im} \gamma^h_\varepsilon(\text{Re} z) = y_-(\varepsilon_0(h)) \left( 1 + \frac{h}{\mathcal{O}(1) \varepsilon_0(h)} \right). \]
Finally, by
\[ 0 = \frac{d}{d\text{Re} z} |E_+((\gamma^h_\varepsilon)(\text{Re} z))| \]
\[ = \partial_{\text{Re} z} |E_+((\gamma^h_\varepsilon)(\text{Re} z))| + \partial_{\text{Im} z} |E_+((\gamma^h_\varepsilon)(\text{Re} z))| \frac{d\text{Im} \gamma^h_\varepsilon(\text{Re} z)}{d\text{Re} z}, \]
and by (10.6) and (10.9) we may then conclude
\[ \frac{d\text{Im} \gamma^h_\varepsilon(\text{Re} z)}{d\text{Re} z} = \mathcal{O} \left( e^{-\frac{z^2}{\pi}} \right) \quad (10.10) \]
which, using \( \eta \asymp y_-(h) \asymp (\varepsilon_0(h))^{2/3} \), yields the last statement of the Lemma.

Proof of Lemma 10.2. The idea of this proof is to search for the critical points of the average density of eigenvalues via the Banach fix point theorem. We shall only consider the case where \( \text{Im} z \leq \langle \text{Im} g \rangle \) since the other case is similar.

Recall from Proposition 2.11 the explicit form the density given in Theorem 2.10. Proposition 6.1 and the fact that \( \text{Im} g \) has exactly two critical points imply that \( \Psi_1 \) is strictly monotonously decreasing. Thus, we may assume similar to (10.3) that for \( C > 0 \) large enough
\[ |\text{Im} z - \langle \text{Im} g \rangle| > \frac{1}{C}. \quad (10.11) \]
since else \( \Psi_2 = \mathcal{O}(e^{-\frac{1}{\pi h}}) \) with \( D > 0 \) large. Now, to find the critical points of the density of eigenvalues consider
\[ \pi \partial_{\text{Im} z} D(z, h) = (\partial_{\text{Im} z} \Psi(z; h, \delta) \exp\{-\Theta(z; h, \delta)\}) \left( 1 + \mathcal{O} \left( \delta \eta^{-1/4} h^{-3/2} \right) \right) \]
\[ + \Psi(z; h, \delta) \exp\{-\Theta(z; h, \delta)\} \mathcal{O} \left( \delta \eta^{-1/4} h^{-5/2} \right) \]
\[ = 0. \quad (10.12) \]
Here we used that the \( z \)- and \( \pi \)-derivative of the error term \( \mathcal{O}(\delta \eta^{-1/4} h^{-3/2}) \) increases its order of growth at most by a term of order \( \mathcal{O}(\eta^{1/2} h^{-1}) \) (cf. Theorem 2.10). By
\[ \partial_{\text{Im} z} \Psi(z; h, \delta) e^{-\Theta(z; h, \delta)} = (\partial_{\text{Im} z} \Psi_1 + \partial_{\text{Im} z} \Psi_2 - (\Psi_1 + \Psi_2) \partial_{\text{Im} z} \Theta) e^{-\Theta(z; h, \delta)}, \]
and by Lemma 9.1 and Proposition 2.11, we can write (10.12) as
\[ h^{-3} F(z, h, \delta) + \frac{2 e^{-\frac{2z}{\pi}}}{\delta^2} |\partial_{\text{Im} z} S(z)|^2 (-\partial_{\text{Im} z} S(z)) \left( \frac{1}{2} \langle p, \overline{p} \rangle (\rho_+) + \frac{1}{2} \langle \overline{p}, p \rangle (\rho_-) \right) \]
\[ \cdot \left( 1 + \mathcal{O} \left( \eta^{-3/4} h^{1/2} \right) \right) \left( 1 + \mathcal{O}(\eta^{-3/2} h^{-1/2}) \right) = 0, \quad (10.13) \]
where \( F(z, h, \delta) \) is a function depending smoothly on \( z \), satisfying the bound
\[
F(z, h, \delta) \asymp -\frac{h^2}{\eta^{3/2}}.
\]

Here we used \( \partial_{\text{Im} z} \Psi_1 \asymp - (\eta^{3/2} h)^{-1} \) which follows from Lemma 9.1 using the fact that \( \text{Im} \, g \) has only two critical points: a minimum at \( a \) and a maximum at \( b \).

**Remark 10.3.** In the case \( \text{Im} \, z > \langle \text{Im} \, g \rangle \) we find similarly that \( F(z, h, \delta) \asymp \frac{h^2}{\eta^{3/2}} \).

Furthermore, the functions in (10.13) are smooth in \( z \) and the \( z \)- and \( \bar{z} \)-derivate increase their order of growth at most by \( O(\eta^{1/2} h^{-1}) \). Recall \( |E_-(z)| \) as given in Proposition 4.12 and define
\[
l(z) := |E_{-\delta}(z)| = \frac{h}{\pi} \left( \frac{3}{4} \{ p, \bar{p} \} \rho_+ \right) \left( \frac{3}{4} \{ p, \bar{p} \} \rho_- \right)^{3/2} e^{-\frac{2}{3} \eta} \left( \frac{1 + O(\eta^{-3/2} h)}{\delta^2} \right).
\]

Thus, (10.13) is equal to zero if and only if
\[
G(z, h, \delta) + l (1 - l) = 0,
\]
where \( G(z, h, \delta) \) is a function depending smoothly on \( z \), satisfying
\[
G(z, h, \delta) = \frac{F(z, h, \delta)}{2 \| \partial_{\text{Im} z} S(z) \|^2 (-\partial_{\text{Im} z} S(z))} \left( 1 + O(\eta^{-3/4} h^{1/2}) \right) \asymp \frac{h^2}{\eta^2}.
\]
The \( z \)- and \( \bar{z} \)-derivate increase the order of growth of \( G \) at most by \( O(\eta^{1/2} h^{-1}) \). For \( l \geq 0 \) to be a solution to (10.14), it is necessary that
\[
l = 1 + \frac{h^2}{O(1) \eta^2}.
\]
Thus, \( l \approx 1 \). Define the smooth function
\[
z \mapsto t(z) := \frac{\eta^3}{h^2} (l(z) - 1),
\]
with \(-c_0 \leq t \leq C_0 \) and \( c_0, C_0 > 0 \) large enough. As in (10.6) on calculates
\[
\frac{h^2}{\eta^2} \partial_{\text{Im} z} t = - \frac{2 \partial_{\text{Im} z} S}{h} \left( 1 + O(\eta^{-3/2} h) \right) l(\text{Im} z) \asymp -\frac{\eta^{1/2} h}{2},
\]
where we used that \( \partial_{\text{Im} z} S \asymp \sqrt{\eta} \) (cf. Proposition 2.2) and that the \( \partial_{\text{Im} z} \) derivative of \( \left( \frac{3}{4} \{ p, \bar{p} \} \rho_+ \right) \left( \frac{3}{4} \{ p, \bar{p} \} \rho_- \right)^{3/2} \) is of order \( O(\eta^{-1/2}) \) due to the scaling \( \bar{z} = z \eta \) as in the proof of Proposition 3.11. The implicit function theorem then implies that we may locally invert and that \( t \mapsto (\text{Im} z)(t) \) is smooth. Since \(-c_0 \leq t \leq C_0 \) we may continue \( (\text{Im} z)(t) \) smoothly to all open subsets of the domain of \( t \). Furthermore, we conclude that
\[
\frac{d(\text{Im} z)}{dt} \asymp - \eta^{-7/2} h^3
\]
(10.15)

Substitute \( \text{Im} z = \text{Im} z(t) \) in (10.14). To find the critical points, it is then enough to consider
\[
t - \tilde{G}(t, \text{Re} \, z, h, \delta) = 0, \quad \tilde{G}(t, \text{Re} \, z, h, \delta) := \frac{G(\text{Im} z(t), \text{Re} \, z, h, \delta)}{\eta^{-3} h^2 (1 + \eta^{-3} h^2 t)}
\]
and one finds
\[
\frac{d}{dt} \tilde{G}(t, \text{Re} \, z, h, \delta) = \mathcal{O}(h^2 \eta^{-3}).
\]
Thus, using \( t(\gamma^h) = 0 \) as starting point, which corresponds to \( t(\gamma^h) = 1 \), the Banach fixed-point theorem implies that for each \( \text{Re} \, z \) there exist a unique zero, \( t^*_z(\text{Re} \, z) \), of (10.13), it depends smoothly on \( \text{Re} \, z \) and satisfies
\[
|t^*_z(\text{Re} \, z) - t(\gamma^h)| \leq \mathcal{O}(h^2 \eta^{-3}).
\]
Thus, one calculates
\[
\frac{dt^*_t(Re z)}{dRe z} = \frac{1}{1 - \left(\frac{\partial_{Re z} \widetilde{G}}{G}(t^*_t, Re z, h, \delta)\right)}(\partial_{Re z} \widetilde{G})(t^*_t, Re z, h, \delta)
\]
which implies the result given in Proposition 2.4. □

Taylor's formula applied to \((\text{Im } z)(t)\) yields that
\[
(\text{Im } z)(t^*_t(Re z)) = \text{Im } z(t(\text{Im } \gamma^h_t(Re z))) + \int_{t(\text{Im } \gamma^h_t(Re z))}^{t^*_t(Re z)} \frac{d\text{Im } z}{dt} (\tau) d\tau.
\]
By (10.16) and (10.15) we conclude that
\[
(\text{Im } z)(t^*_t(Re z)) = \text{Im } \gamma^h_{\pm}(Re z) + \mathcal{O}(\eta^{-13/2}h^5) \tag{10.17}
\]
and using (10.10) that
\[
\frac{d}{dRe z} (\text{Im } z)(t^*_t(Re z)) = \mathcal{O}(\eta^{-6}h^4).
\]
It follows by Proposition 2.15 that the density has local maxima along the curves \(\Gamma^h_{\pm}(Re z) := (Re z, \text{Im } z(t^*_t(Re z)))\). Applying this definition to (10.17) yields that
\[
|\text{Im } \Gamma^h_{\pm}(Re z) - \text{Im } \gamma^h_{\pm}(Re z)| \leq \mathcal{O}(\eta^{-13/2}h^5)
\]
for all \(z \in \Sigma_{c,d}\). By Lemma 10.1 we have that \(\text{Im } \gamma^h_{\pm}(Re z) \asymp \varepsilon_0(h)^{2/3}\). Thus,
\[
\text{Im } \Gamma^h_{\pm}(Re z) = \text{Im } \gamma^h_{\pm}(Re z) \left(1 + \mathcal{O}(\varepsilon_0(h)^{-5}h^5)\right),
\]
which in particular implies that \(\text{Im } \Gamma^h_{\pm}(Re z) \asymp \varepsilon_0(h)^{2/3}\). This concludes the proof of the lemma. □

Proof of Proposition 2.15. Proposition 2.11 implies that for \(|\text{Im } z - \langle \text{Im } y \rangle| > 1/C\)
\[
\Psi_2(z, h, \delta) = \left(\frac{1}{2} \{p, p\}(\rho_+)\frac{1}{2} \{p, p\}(\rho_-)\right)^\frac{1}{2} \frac{e^{2\pi}}{2\pi h \delta^2} |\partial_{im z} S(z)|^2 \left(1 + \mathcal{O}(\eta^{-3/4}h^{1/2})\right) \tag{10.18}
\]
and
\[
\Theta(z; h, \delta) = \frac{h \left(\frac{1}{2} \{p, p\}(\rho_+)\frac{1}{2} \{p, p\}(\rho_-)\right)^\frac{1}{2} \frac{e^{2\pi}}{2\pi \delta^2}}{(1 + \mathcal{O}(\eta^{-1/4}h^{3/2}))}
+ \mathcal{O}(\eta^{1/4}h^{-2}\delta + \eta^{-1/2}\delta^2h^{-5}).
\]
Thus, one calculates
\[
\left|\Psi_2 - \frac{|\partial_{im z} S|^2}{h^2} \Theta\right| \leq \left(\frac{1}{2} \{p, p\}(\rho_+)\frac{1}{2} \{p, p\}(\rho_-)\right)^\frac{1}{2} \frac{e^{2\pi}}{2\pi h \delta^2} |\partial_{im z} S(z)|^2 \mathcal{O}(\eta^{-3/4}h^{1/2})
+ \mathcal{O}(\eta^{5/4}h^{-4}\delta + \eta^{1/2}\delta^2h^{-7}),
\]
which implies the result given in Proposition 2.4. □
Proof of Proposition 2.14. We will only consider the case $z \in \Sigma_{c,d}$ with $\text{Im } z \leq \langle \text{Im } g \rangle$.

A priori restrictions on the domain of integration Let $y_-(h)$ and $\gamma_-(\text{Re } z)$ be as in Lemma 10.1 and note that similarly to (10.2), we have

$$S(\text{Im } z) - \varepsilon_0(h) = \int_{y_-(h)}^{\text{Im } \gamma_-(h)} (\partial_{\text{Im } z} S)(t) dt + \int_{\text{Im } \gamma_-(h)}^{\text{Im } z} (\partial_{\text{Im } z} S)(t) dt. \quad (10.19)$$

Recall from Lemma 10.1 that $(\text{Im } \gamma_-(h) - y_-(h)) \asymp h \varepsilon_0^{-1/3}$. Then, one calculates using the mean value theorem and Proposition 2.2, similar as in the proof of Lemma 10.1 (cf. (10.4)) that

$$\int_{y_-(h)}^{\text{Im } \gamma_-(h)} (\partial_{\text{Im } z} S)(t) dt \asymp (\text{Im } z - \text{Im } \gamma_-(h)) \eta^{1/2},$$

where $\eta$ should be set to 1 in case of $\text{dist } (z, \partial \Sigma_{c,d}) > 1/C$. Next, (10.19) and Proposition 2.11 implies that

$$\Theta(z; h, \delta) = \frac{\eta^{1/2}}{O(1)} \exp \left\{ - \left( \frac{\text{Im } z - \text{Im } \gamma_-(h)}{h} \right) \eta^{1/2} \right\} + O\left( \eta^{1/4} h^{-2} \delta + \eta^{-1/2} \delta^2 h^{-5} \right).$$

Here, we used that $\delta = \sqrt{h} \exp\{-\varepsilon_0(h)/h\}$; see Definition 2.5. Thus, for $\text{Im } \gamma_-(h) < \text{Im } z < \langle \text{Im } g \rangle$

$$\exp\{-\Theta(z; h, \delta)\} = \left( 1 + O\left( \eta^{1/2} \exp \left\{ - \left( \frac{\text{Im } z - \text{Im } \gamma_-(h)}{Ch} \right) \eta^{1/2} \right\} + \eta^{1/4} h^2 \right) \right) \quad (10.20)$$

and for $\text{Im } z \leq \text{Im } \gamma_-(h)$

$$\frac{1}{C} \exp \left\{ -C \eta^{1/2} \exp \left\{ - \left( \frac{\text{Im } z - \text{Im } \gamma_-(h)}{Ch} \right) \eta^{1/2} \right\} \right\} \leq \exp\{-\Theta(z; h, \delta)\} \leq C \exp \left\{ - \frac{\eta^{1/2}}{C} \exp \left\{ - C \left( \frac{\text{Im } z - \text{Im } \gamma_-(h)}{h} \right) \eta^{1/2} \right\} \right\}. \quad (10.21)$$

Similarly, by Proposition 2.11

$$\Psi_2(z; h, \delta) \leq \frac{\eta^{3/2}}{O(1)h^2} \left( 1 + O(\eta^{-1}) \exp\{-\Phi(z; h)\} \right) \exp \left\{ - \left( \frac{\text{Im } z - \text{Im } \gamma_-(h)}{Ch} \right) \eta^{1/2} \right\}. \quad (10.22)$$

Thus, for $\text{Im } \gamma_-(\text{Re } z) + ah \eta^{-1/2} \ln \frac{\eta^{1/2}}{h} \leq \text{Im } z \leq \langle \text{Im } g \rangle$ with $a > 0$ large enough, we see that the average density of eigenvalues (cf. Theorem 2.10)

$$D(z, h, \delta)L(dz) = \frac{1}{2h} P_\delta(d\xi \wedge dx) + O(\eta^{-2})L(dz). \quad (10.22)$$

We then conclude the first statement of the proposition.

Next, recall from Lemma 5.1 that restricting the probability space to the ball $B(0, R)$ of radius $R = Ch^{-1}$ implies that $\|Q_\omega\| \leq C/h$ with probability $\geq \left( 1 - e^{-\frac{1}{C\delta^2}} \right)$. It follows from

$$\|(P_h^\delta - z)^{-1}\| = \left\| (P_h - z)^{-1} \sum_{n \geq 1} (-\delta)^n (Q_\omega(P_h - z)^{-1})^n \right\|_{68}$$
that for \( z \notin \sigma(P_h) \) such that \( \delta\|Q_\omega\|(P_h - z)^{-1} \| < 1 \), \( z \notin \sigma(P_h^\beta) \) with probability \( \geq 1 - e^{-\frac{1}{\|z\|^2}} \). Proposition 2.4 implies that with probability \( \geq 1 - e^{-\frac{1}{\|z\|^2}} \)

\[
\delta\|Q_\omega\|(P_h - z)^{-1} \| \leq \frac{C}{h^{3/2}} \frac{1}{\pi} \frac{1}{\|z\|^2} \ln \left( \frac{\|z\|^2}{\epsilon_0(h)^{1/3}} \right)
\]

Since \( S(z) \approx \eta^{3/2} \), it follows that \( \eta \approx \epsilon_0(h)^{2/3} \). Using the mean value theorem together with Proposition 10.1 implies that with probability \( \geq 1 - e^{-\frac{1}{\|z\|^2}} \) there are no eigenvalues of \( P_h^\beta \) with

\[
\text{Im} \ z \leq \beta_1 := \text{Im} \gamma_+ - C \frac{h}{\epsilon_0(h)^{1/3}} \ln \left( \frac{\epsilon_0(h)^{1/3}}{h} \right), \quad C \gg 1.
\]

Thus, to count eigenvalues, it is sufficient to integrate the density given in Theorem 2.10 over subsets of

\[
\Sigma_{c,d}' = \{ z \in \Sigma_{c,d} \mid \beta_1 \leq \text{Im} \ z \leq \langle \text{Im} \ g \rangle, \ c < \text{Re} \ z < d \}\.
\]

Similarly, for an \( \alpha \) large enough as above, define

\[
\alpha_1 := \text{Im} \gamma_-(\text{Re} \ z) + \frac{h}{\epsilon_0^{1/3}} \ln \left( \frac{\epsilon_0(h)^{1/3}}{h} \right)
\]

and note that (10.22) implies the second statement of the proposition for \( \text{Im} \ z \geq \alpha_1 \).

**Approximate Primitive** Define \( d(z) := \text{dist}(z, \partial \Sigma) \) and recall from (3.1) that \( \eta \approx d(z) \).

Recall that the density of eigenvalues given in Theorem 2.10 is given by \( \Psi_1, \Psi_2 \) and \( \Theta \) which are expressed explicitly in Proposition 2.11 and Theorem 2.10. Since \( \text{Im} \ g(x_{\pm}) = \text{Im} \ z \) and \( \xi_{\pm} = \text{Re} \ z - \text{Re} \ g(x_{\pm}) \) (cf. Section 1), we conclude together with Proposition 6.2 that for \( \beta_1 \leq \text{Im} \ z \leq \alpha_1 \)

\[
\Psi_1(z; h) = \frac{1}{2h} \partial_{\text{Im} \ z} (x_-(z) - x_+(z)) + \mathcal{O}(d(z)^{-2}) = \frac{1}{2h} \partial_{\text{Im} \ z} S(z) + \mathcal{O}(d(z)^{-2}).
\]

Next, it follows by (10.18) and Lemma 9.1 that

\[
|2h \Psi_2 - (\partial_{\text{Im} \ z} S)(-\partial_{\text{Im} \ z} \Theta)| = \mathcal{O} \left( d(z)^{3/4} h^{1/2} e^{-\frac{2\pi}{\delta}} \right) + \mathcal{O}(d(z)^{3/4} h^{-3} \delta).
\]

Thus,

\[
\frac{1 + \mathcal{O}(\delta d(z)^{-1/2} h^{-3/2})}{\pi} \{ \Psi_1(z; h) + \Psi_2(z; h, \delta) \} e^{-\Theta(z; h, \delta)}
\]

\[
= \frac{1}{2\pi h} \partial_{\text{Im} \ z} \left[ (\partial_{\text{Im} \ z} S(z)) e^{-\Theta(z; h, \delta)} \right] + R(z; h, \delta) e^{-\Theta(z; h, \delta)}, \quad (10.23)
\]

where

\[
R(z; h, \delta) := \mathcal{O} \left( d(z)^{-2} + d(z)^{3/4} h^{-1/2} e^{-\frac{2\pi}{\delta}} \right).
\]

Let \( \beta_1 \leq \beta_2 \leq \alpha_1 \). Let us first treat the error term \( R \). Similar as for (10.20), it follows that

\[
R(z; h, \delta) = \mathcal{O} \left( d(z)^{-2} + d(z)^{-3/4} h^{-1/2} \exp \left\{ -\frac{(\text{Im} \ z - \text{Im} \gamma_+^h) d(z)^{1/2}}{Ch} \right\} \right).
\]

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Hence,
\[
\left| \int_{\beta_1}^{\alpha_1} R(z; h, \delta) e^{-\Theta(z; h, \delta)} d(\text{Im} z) \right| \leq \left[ \frac{d(z)^{-1}}{\mathcal{O}(1)} \exp\{ -\Theta(z, \alpha_1; h, \delta) \} + \frac{d(z)^{1/4} h^{1/2}}{\mathcal{O}(1)} \exp\left\{ -\frac{(\text{Im} z - \text{Im} \gamma_h)(d(z)^{1/2})}{Ch} \right\} \right]_{\beta_1}^{\alpha_1} \\
= \frac{\beta_1^{-1}}{\mathcal{O}(1)} \exp\left\{ -\frac{(\alpha_1 - \text{Im} \gamma_h)\alpha^{1/2}}{Ch} \right\} \right|_{\beta_1}^{\alpha_1} \\
= \frac{\epsilon_0(h)^{-2/3}}{\mathcal{O}(1)}.
\]

Next,
\[
\frac{1}{2\pi h} \int_{\beta_2}^{\alpha_1} \partial_{\text{Im} z} \left[ (\partial_{\text{Im} z} S(z)) e^{-\Theta(z; h, \delta)} \right] L(\text{Im} z) \left. \right|_{\beta_2}^{\alpha_1} = \frac{1}{2\pi h} (x_-(\text{Im} z) - x_+(\text{Im} z)) e^{-\Theta(z; h, \delta)} \left. \right|_{\beta_2}^{\alpha_1}. \tag{10.25}
\]
Since,
\[
\int_{\Sigma_{c,d}}^{0 \leq \text{Im} z \leq \alpha_1} p_x(d\xi \wedge dx)(dz) = \frac{1}{2\pi h} (x_-(\alpha_1) - x_+(\alpha_1)) \int_c^d d\text{Re} z
\]
we conclude by (10.21) the second statement of the proposition for
\[
\beta_2 = \text{Im} \gamma_-(\text{Re} z) - \frac{h}{\epsilon_0(h)^{1/3}} \ln \beta \ln \frac{\epsilon_0(h)^{1/3}}{h}
\]
with \( \beta > 0 \) large enough. The last statement of the proposition can be deduced similarly from (10.21), (10.25) and (10.24).

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