Conserved excitation number and $U(1)$-symmetry operators for the anti-rotating (anti-Jaynes-Cummings) term of the Rabi Hamiltonian

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Introduction

The quantum Rabi model describes the dynamics of a quantized electromagnetic field mode interacting with a two-level atom generated by Hamiltonian [1-5]

$$H_R = \frac{1}{2} \hbar \omega (\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger) + \hbar \omega_0 s_z + \hbar g (\hat{a} + \hat{a}^\dagger)(s_+ + s_-)$$  \hspace{1cm} (1a)

where $\omega$, $\hat{a}$, $\hat{a}^\dagger$ are the quantized field mode angular frequency, annihilation and creation operators, while $\omega_0$, $s_z$, $s_+$, $s_-$ are the atomic state transition angular frequency and operators. We have used $\sigma_z = s_+ + s_-$ and expressed the free field mode Hamiltonian in the symmetrized normal and anti-normal order form $\frac{1}{2} \hbar \omega (\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger)$ for reasons which will become clear below.

Collecting the normal and anti-normal order terms in equation (1a), we express the Rabi Hamiltonian in the symmetrized form

$$H_R = \frac{1}{2} (H + \overline{H})$$  \hspace{1cm} (1b)

where we have identified the normal order rotating component as the Jaynes-Cummings Hamiltonian $H$ obtained as

$$H = \hbar (\omega \hat{a}^\dagger \hat{a} + \omega_0 s_z + 2g(\hat{a}s_+ + \hat{a}^\dagger s_-))$$  \hspace{1cm} (1c)

and the anti-normal order anti-rotating component as the anti-Jaynes-Cummings Hamiltonian $\overline{H}$ obtained as

$$\overline{H} = \hbar (\omega \hat{a}\hat{a}^\dagger + \omega_0 s_z + 2g(\hat{a}s_- + \hat{a}^\dagger s_+))$$  \hspace{1cm} (1d)

We observe that the operator ordering principle which distinguishes the rotating (Jaynes-Cummings) and anti-rotating (anti-Jaynes-Cummings) components $H$, $\overline{H}$ is not arbitrary, but has physical foundation. Noting that an electromagnetic field mode is composed of positive and negative frequency components [6], we provide the physical interpretation that the Jaynes-Cummings interaction represents the coupling of the atomic spin to the rotating positive frequency component of the field mode, such that the algebraic operations which generate the resulting red-sideband state transitions are achieved through normal-operator-ordering, while the anti-Jaynes-Cummings interaction represents the coupling of the atomic spin to the anti-rotating negative frequency component of the field mode, such that the algebraic operations which generate the resulting blue-sideband state transitions are achieved through anti-normal-operator-ordering. We note that blue-sideband effects arising from interactions involving negative frequency radiation have been observed in recent experiments [7, 8].

In [2], the Jaynes-Cummings and ant-Jaynes-Cummings Hamiltonians $H$, $\overline{H}$ have been characterized as the chiral and anti-chiral components, respectively, of the Rabi Hamiltonian. In this respect, we generalize the models of interaction between a single quantized field mode and a single two-level atom to include the asymmetric (anisotropic) Rabi models [3, 4] by introducing a rotation-symmetry or chirality parameter $r$ taking values $-1 \leq r \leq 1$ to express the Rabi Hamiltonian in equation (1b) in general symmetrization form

$$H_R = \frac{1}{2} ( (1 + r)H + (1 - r)\overline{H} ) \hspace{1cm} ; \hspace{1cm} -1 \leq r \leq 1$$  \hspace{1cm} (1e)

such that $r = 1, 0, -1$ specifies that the Rabi Hamiltonian takes respectively the fully rotating (Jaynes-Cummings), symmetric (isotropic) or fully anti-rotating (anti-Jaynes-Cummings) form, while for all other values $r \neq 1, 0, -1$ the Rabi Hamiltonian is asymmetric (anisotropic).
A major challenge, which has remained an outstanding problem in the Rabi model over the years, is that while the Jaynes-Cummings component has a conserved excitation number operator and is invariant under the corresponding $U(1)$-symmetry operation [1, 3], a conserved excitation number and corresponding $U(1)$-symmetry operators for the anti-Jaynes-Cummings component have never been determined, leading to the general belief that the anti-Jaynes-Cummings interaction violates energy conservation principle. We address the long outstanding problem of excitation number and $U(1)$-symmetry operators of the anti-Jaynes-Cummings Hamiltonian in this letter.

Excitation number operators It follows from the physical interpretation given above that an excitation number operator in the Jaynes-Cummings interaction should be defined in normal-order form, while an excitation number operator in the anti-Jaynes-Cummings interaction should be defined in anti-normal-order form. Taking this operator ordering principle into account, we add and subtract an atomic spin normal order term $\hbar \omega_s s_+$ in equation (1c) and anti-normal order term $\hbar \omega_s s_+$ in equation (1d), then reorganize, noting $s_+ s_- = \frac{1}{2} + s_z$, $s_- s_+ = \frac{1}{2} - s_z$, to obtain the Jaynes-Cummings Hamiltonian in the form

$$H = \hbar \omega (\hat{a}_+ \hat{a} + s_+ s_-) + 2 \hbar g (\alpha s_z + \hat{a} s_+ + \hat{a}^\dagger s_-) - \frac{1}{2} \hbar \omega; \quad \alpha = \frac{\omega_0 - \omega}{2g} \quad (2a)$$

and the anti-Jaynes-Cummings Hamiltonian in the form

$$\overline{H} = \hbar \omega (\hat{a}_+ \hat{a} + s_+ s_-) + 2 \hbar g (\overline{\alpha} s_z + \hat{a} s_- + \hat{a}^\dagger s_+) - \frac{1}{2} \hbar \omega; \quad \overline{\alpha} = \frac{\omega_0 + \omega}{2g} \quad (2b)$$

where we factored out $2g$ and introduced respective dimensionless frequency-detuning parameters $\alpha$, $\overline{\alpha}$ defined as indicated.

In the Jaynes-Cummings Hamiltonian $H$, we identify the normally-ordered Jaynes-Cummings excitation number operator $\hat{N}$, while in the anti-Jaynes-Cummings Hamiltonian $\overline{H}$, we identify the anti-normally ordered anti-Jaynes-Cummings excitation number operator $\overline{\hat{N}}$, respectively defined by

$$\hat{N} = \hat{a}_+ \hat{a} + s_+ s_-; \quad \overline{\hat{N}} = \hat{a}_+ \hat{a} + s_- s_+ \quad (2c)$$

which we introduce in equations (2a), (2b) as appropriate to express the Jaynes-Cummings and anti-Jaynes-Cummings Hamiltonians in the form

$$H = \hbar \omega \hat{N} + 2 \hbar g (\alpha s_z + \hat{a} s_+ + \hat{a}^\dagger s_-) - \frac{1}{2} \hbar \omega; \quad \overline{H} = \hbar \omega \overline{\hat{N}} + 2 \hbar g (\overline{\alpha} s_z + \hat{a} s_- + \hat{a}^\dagger s_+) - \frac{1}{2} \hbar \omega \quad (2d)$$

We observe that the Jaynes-Cummings excitation number operator $\hat{N} = \hat{a}_+ \hat{a} + s_+ s_-$ is a standard conserved operator in the dynamics generated by the rotating component of the Rabi Hamiltonian, while the anti-Jaynes-Cummings excitation number operator $\overline{\hat{N}} = \hat{a}_+ \hat{a} + s_- s_+$, which we establish here as a conserved operator in the dynamics generated by the anti-rotating component of the Rabi Hamiltonian, is a new operator discovered and presented for the first time in the present letter. The discovery of the anti-Jaynes-Cummings excitation number operator, proof of its conservation and specification of the corresponding $U(1)$ and parity symmetry operators in the dynamics generated by the anti-Jaynes-Cummings Hamiltonian are the main results of this letter.

Proof of conservation: state transition operators Using standard atomic spin and field mode operator algebraic relations

$$[s_+, s_-] = 2s_z; \quad [s_z, s_-] = -s_-; \quad [s_z, s_+] = s_+; \quad s_+ s_- = \frac{1}{2} + s_z; \quad s_- s_+ = \frac{1}{2} - s_z$$

we easily prove that the excitation number operators $\hat{N}$, $\overline{\hat{N}}$ in equation (2c) commute with the respective Jaynes-Cummings and anti-Jaynes-Cummings Hamiltonians $H$, $\overline{H}$ in equation (2d) according to

$$[\hat{N}, H] = 0; \quad [\overline{\hat{N}}, \overline{H}] = 0 \quad (3b)$$

which proves the standard dynamical property that the excitation number operator $\hat{N} = \hat{a}_+ \hat{a} + s_+ s_-$ is conserved in the dynamics generated by the Jaynes-Cummings Hamiltonian $H$ and the new dynamical
property that the excitation number operator \( \hat{N} = \hat{a} \hat{a}^\dagger + s_- s_+ \) is conserved in the dynamics generated by the anti-Jaynes-Cummings Hamiltonian \( \hat{H} \).

To make the proof even more transparent, we introduce two new conserved dynamical operators, namely, Jaynes-Cummings state transition operator \( \hat{A} \) and anti-Jaynes-Cummings state transition operator \( \hat{A} \), respectively defined by
\[
\hat{A} = \alpha s_z + \tilde{\alpha} s_+ + \hat{a}^\dagger s_- ; \quad \hat{A} = \tilde{\alpha} s_z + \tilde{\alpha} s_- + \hat{a}^\dagger s_+ \quad (3c)
\]
which on squaring and applying standard atomic spin and field mode operator algebraic relations
\[
\hat{a}^2 = \frac{1}{4} ; \quad \hat{s}_z = \hat{s}^2 = 0 ; \quad s_+ s_- - s_- s_+ = 1 ; \quad s_z s_+ + s_+ s_z = 0 ; \quad s_z s_- + s_- s_z = 0 ; \quad \hat{a} \hat{a}^\dagger = \hat{a}^\dagger \hat{a} + 1 \quad (3d)
\]
provide the respective Jaynes-Cummings and anti-Jaynes-Cummings excitation number operators \( \hat{N} \), \( \hat{N} \) defined in equation (2c) in the form
\[
\hat{A}^2 = \hat{a}^\dagger \hat{a} + s_+ s_- + \frac{1}{4} \alpha^2 = \hat{N} + \frac{1}{4} \alpha^2 \quad ; \quad \hat{A}^2 = \hat{a}^\dagger \hat{a} + s_- s_+ + \frac{1}{4} \tilde{\alpha}^2 - 1 = \hat{N} + \frac{1}{4} \tilde{\alpha}^2 - 1 \quad (3e)
\]
Substituting \( \hat{A} \), \( \hat{A} \) from equation (3c) and \( \hat{N} = \hat{A}^2 - \frac{1}{4} \alpha^2 \), \( \hat{N} = \hat{A}^2 - \frac{1}{4} \tilde{\alpha}^2 + 1 \) from equation (3e) into equation (2d), we express the Jaynes-Cummings and anti-Jaynes-Cummings Hamiltonians in terms of the respective state transition operators in the form
\[
\hat{H} = \hbar (\omega \hat{A}^2 + 2g \hat{A}) - \frac{1}{4} \hbar \omega \alpha^2 - \frac{1}{2} \hbar \omega \quad ; \quad \hat{\Pi} = \hbar (\omega \hat{A}^2 + 2g \hat{A}) - \frac{1}{4} \hbar \omega \pi^2 + \frac{1}{2} \hbar \omega \quad (3f)
\]
Using equations (3e) and (3f) easily confirms the commutation relations in equation (3b). In addition, it is easy to establish that the Jaynes-Cummings excitation number operator is not conserved in the dynamics generated by the anti-Jaynes-Cummings Hamiltonian \( \hat{H} \) and likewise, the anti-Jaynes-Cummings excitation number operator is not conserved in the dynamics generated by the Jaynes-Cummings Hamiltonian \( \hat{H} \) according to the commutation relations
\[
[ \hat{N} , \hat{\Pi} ] \neq 0 \quad ; \quad [ \hat{\Pi} , \hat{H} ] \neq 0 \quad (3g)
\]
Similarly, the state transition operators \( \hat{A} \), \( \hat{A} \) are conserved in the dynamics generated by the respective Hamiltonians \( \hat{H} \), \( \hat{H} \), but not in the dynamics generated by the other component Hamiltonian according to the commutation relations
\[
[ \hat{A} , \hat{H} ] = 0 \quad ; \quad [ \hat{A} , \hat{\Pi} ] \neq 0 \quad ; \quad [ \hat{A} , \hat{H} ] = 0 \quad ; \quad [ \hat{A} , \hat{H} ] \neq 0 \quad (3h)
\]
We have thus proved the desired conservation of the anti-Jaynes-Cummings excitation number operator and the state transition operators. We have established in another paper [9] that the Jaynes-Cummings state transition operator \( \hat{A} \) generates red-sideband transitions between polariton qubit states arising in the rotating Jaynes-Cummings interaction \( 2\hbar g(\hat{a} \hat{a}^\dagger s_- + \hat{a} \hat{a}^\dagger) \), while the anti-Jaynes-Cummings state transition operator \( \hat{A} \) generates blue-sideband transitions between anti-polariton qubit states arising in the anti-rotating anti-Jaynes-Cummings interaction \( 2\hbar g(\hat{a} \hat{a}^\dagger s_+ + \hat{a} \hat{a}^\dagger) \). Eigenvectors and eigenvalues of the respective Jaynes-Cummings and anti-Jaynes-Cummings Hamiltonians \( \hat{H} \), \( \hat{\Pi} \) in equation (3f), interpreted as polariton and anti-polariton qubit Hamiltonians, have been determined easily in [9]. The coupling of the atomic spin to the anti-rotating negative frequency component of the field mode leading to blue-sideband transitions accounts for the excitation number and energy conservation in the anti-Jaynes-Cummings interaction.

\( U(1) \)-symmetry operators The Jaynes-Cummings excitation number operator \( \hat{N} = \hat{a}^\dagger \hat{a} + s_- s_+ \) generates a free time evolution operator \( U_0(t) \) obtained as
\[
U_0(t) = e^{-i\omega t \hat{N}} \quad \Rightarrow \quad U_0^\dagger(t) = e^{i\omega t \hat{N}} \quad (4a)
\]
which provides field mode and atomic spin operator time evolution in the form
\[
U_0^\dagger(t) \hat{a} U_0(t) = e^{-i\omega t} \hat{a} ; \quad U_0^\dagger(t) \hat{a}^\dagger U_0(t) = e^{i\omega t} \hat{a}^\dagger ; \quad U_0^\dagger(t) s_- U_0(t) = e^{-i\omega t} s_- ; \quad U_0^\dagger(t) s_+ U_0(t) = e^{i\omega t} s_+ \quad (4b)
\]
The operator $U_0(t)$ thus generates symmetry operations on the Jaynes-Cummings and anti-Jaynes-Cummings Hamiltonians in equation (2d) in the form

$$U_0^\dagger(t)HU_0(t) = H \quad ; \quad U_0^\dagger(t)\overline{H}U_0(t) = \hbar \omega \hat{N} + 2\hbar g (s_+ + e^{-2i\omega t}\hat{a}s_- + e^{2i\omega t}\hat{a}^\dagger s_+) - \frac{1}{2}\hbar \omega \quad (4c)$$

which shows that the Jaynes-Cummings excitation number operator generated free time evolution operator $U_0(t) = e^{-i\omega t\hat{N}}$ is a $U(1)$-symmetry operator of the Jaynes-Cummings Hamiltonian $H$, but not a symmetry operator of the anti-Jaynes-Cummings Hamiltonian $\overline{H}$.

On the other hand, the anti-Jaynes-Cummings excitation number operator $\overline{N} = \hat{a}\hat{a}^\dagger + s_-s_+$ generates a free time evolution operator $\overline{U}_0(t)$ obtained as

$$\overline{U}_0(t) = e^{-i\omega t\overline{N}} \quad \Rightarrow \quad \overline{U}_0^\dagger(t) = e^{i\omega t\overline{N}} \quad (4d)$$

which provides field mode and atomic spin operator time evolution in the form

$$\overline{U}_0^\dagger(t)\hat{a}\overline{U}_0(t) = e^{-i\omega t}\hat{a} \quad ; \quad \overline{U}_0^\dagger(t)\hat{a}^\dagger\overline{U}_0(t) = e^{i\omega t}\hat{a}^\dagger \quad ; \quad \overline{U}_0^\dagger(t)s_-\overline{U}_0(t) = e^{i\omega t}s_- \quad ; \quad \overline{U}_0^\dagger(t)s_+\overline{U}_0(t) = e^{-i\omega t}s_+ \quad (4e)$$

The operator $\overline{U}_0(t)$ thus generates symmetry operations on the Jaynes-Cummings and anti-Jaynes-Cummings Hamiltonians $H, \overline{H}$ in equation (2d) in the form

$$\overline{U}_0^\dagger(t)H\overline{U}_0(t) = \overline{H} \quad ; \quad \overline{U}_0^\dagger(t)\overline{H}\overline{U}_0(t) = \hbar \omega \overline{N} + 2\hbar g (\alpha s_- + e^{-2i\omega t}\hat{a}s_+ + e^{2i\omega t}\hat{a}^\dagger s_-) - \frac{1}{2}\hbar \omega \quad (4f)$$

which shows that the anti-Jaynes-Cummings excitation number operator generated free time evolution operator $\overline{U}_0(t) = e^{-i\omega t\overline{N}}$ is a $U(1)$-symmetry operator of the anti-Jaynes-Cummings Hamiltonian $\overline{H}$, but not a symmetry operator of the Jaynes-Cummings Hamiltonian $H$.

**Parity-symmetry operator** It follows from equations (4c) and (4f) that we can determine a common symmetry operator of both Jaynes-Cummings and anti-Jaynes-Cummings Hamiltonians $H, \overline{H}$ by imposing the free evolution symmetry condition

$$e^{-2i\omega t} = e^{2i\omega t} = 1 \quad \Rightarrow \quad 2\omega t = 2n\pi \quad ; \quad \omega t = n\pi \quad ; \quad n = 1, 2, 3, ... \quad (5a)$$

where $n = 0$ defines the identity operator. Substituting $\omega t = n\pi$ into equations (4a), (4d), we obtain the common Jaynes-Cummings and anti-Jaynes-Cummings symmetry operator $\hat{\Pi}_n(\pi)$ in the form

$$\hat{\Pi}_n(\pi) = U_0(n\pi) = e^{-i\pi n\hat{N}} = \overline{U}_0(n\pi) = e^{-i\pi n\overline{N}} \quad ; \quad n = 1, 2, 3, ... \quad (5b)$$

which we express in the form

$$\hat{\Pi}_n(\pi) = (e^{-i\pi \hat{N}})^n = (e^{-i\pi \overline{N}})^n = (\hat{\Pi})^n \quad (5c)$$

from which we identify the standard Jaynes-Cummins and anti-Jaynes-Cummings parity-symmetry operator $\hat{\Pi}$ defined here by

$$\hat{\Pi} = e^{-i\pi \hat{N}} = e^{-i\pi \overline{N}} \quad (5d)$$

Substituting $\hat{N} = \hat{a}\hat{a} + s_+s_-,$ $\overline{N} = \hat{a}\hat{a}^\dagger + s_-s_+$ and using algebraic relations

$$\hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a} + 1 \quad ; \quad s_-s_+ = s_+s_- - 2s_z \quad ; \quad \hat{N} = \hat{N} + 2s_-s_+ \quad (5e)$$

we obtain

$$e^{-i\pi \hat{N}} = e^{-i\pi \hat{N}} e^{-2\pi s_-s_+} \quad ; \quad e^{-2\pi s_-s_+} = I \quad \Rightarrow \quad e^{-i\pi \hat{N}} = e^{-i\pi \hat{N}} \quad (5f)$$

which establishes the common Jaynes-Cummings and anti-Jaynes-Cummings parity-symmetry operator relation in equation (5d).

It is easy to establish that the Jaynes-Cummings and anti-Jaynes-Cummings parity-symmetry operator $\hat{\Pi}$ is a symmetry operator of the Rabi Hamiltonian $H_R = \frac{1}{2}(H + \overline{H})$ in equation (1b) according to the symmetry transformation operations

$$\hat{\Pi}^\dagger H \hat{\Pi} = H \quad ; \quad \hat{\Pi}^\dagger \overline{H} \hat{\Pi} = \overline{H} \quad ; \quad \hat{\Pi}^\dagger H_R \hat{\Pi} = H_R \quad (5g)$$
Finally, we observe that an important dynamical feature emerges from the Jaynes-Cummings-anti-Jaynes-Cummings common parity-symmetry relation in equation (5d). Substituting $\hat{N} = \hat{A}^2 - \frac{i}{4} \alpha^2$, $\hat{\mathcal{N}} = \hat{A}^2 - \frac{i}{4} \alpha^2 + 1$ from equation (3e) into equation (5d) and reorganizing, we obtain the common parity-symmetry relation in the form

$$e^{-i\pi \hat{A}^2} = e^{-i\pi \hat{b}^2} e^{i\pi (\frac{i}{4} \alpha^2 - \frac{i}{4} \alpha^2 - 1)}$$

(6a)

which on using $\alpha = \frac{\omega_0 - \omega}{2g}$, $\alpha = \frac{\omega_0 + \omega}{2g}$ from equations (2a), (2b) to evaluate

$$\frac{1}{4} \alpha^2 - \frac{1}{4} \alpha^2 = \frac{\omega_0\omega}{4g^2} = \beta^2$$

(6b)

takes the form

$$e^{-i\pi \hat{A}^2} = e^{-i\pi \hat{b}^2} e^{i\pi (\beta^2 - 1)} ; \quad \beta^2 = \frac{\omega_0\omega}{4g^2}$$

(6c)

which suggests that there exists a critical coupling constant $g_c$ at which the global phase factor $e^{i\pi (\beta^2 - 1)}$ equals unity obtained as

$$g = g_c : \quad e^{i\pi (\beta^2 - 1)} = 1 ; \quad \beta_c^2 = \frac{\omega_0\omega}{4g_c^2} = 1 \quad \Rightarrow \quad g_c = \frac{1}{2} \sqrt{\omega_0\omega}$$

(6d)

giving common parity-symmetry relation at the critical coupling $g_c$ in the form

$$g = g_c : \quad \alpha_c = \hat{\alpha}_c = \frac{\omega_0 - \omega}{2g_c} ; \quad \beta_c^2 = \frac{\omega_0\omega}{4g_c^2} ; \quad \hat{\Pi}_c = e^{-i\pi \hat{A}_c^2} = e^{-i\pi \hat{b}_c^2}$$

(6e)

We identify $g_c = \frac{i}{2} \sqrt{\omega_0\omega}$ to be exactly the critical coupling constant at which the Rabi interaction undergoes quantum phase transition as determined in a recent study [5]. It follows that parity-symmetry breaking may occur at a quantum phase transition. We have presented quantum phase transition phenomena in the Rabi and the more general Dicke models in another paper.

Conclusion We have applied operator-ordering as the fundamental algebraic property to determine the conserved excitation number and $U(1)$-symmetry operators for the rotating (Jaynes-Cummings) and anti-rotating (anti-Jaynes-Cummings) components of the Rabi Hamiltonian. The specification of the anti-Jaynes-Cummings excitation number operator means that the eigenvalue spectrum of the anti-Jaynes-Cummings Hamiltonian can now be determined explicitly. The Rabi Hamiltonian is thus composed of two algebraically complete Jaynes-Cummings and anti-Jaynes-Cummings components, each specified by its characteristic excitation number, state transitions, $U(1)$-symmetry and red or blue sideband eigenvalue spectrum. We have determined the parity-symmetry operator as the common symmetry operator for both Jaynes-Cummings and anti-Jaynes-Cummings components, leading to the standard algebraic property that parity operator is a symmetry operator of the Rabi Hamiltonian. The parity-symmetry may break at a critical coupling constant $g_c$ where quantum phase transition occurs.

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