The characterization of aCM line bundles on quintic hypersurfaces in $\mathbb{P}^3$

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Abstract

Let $X$ be a smooth quintic hypersurface in $\mathbb{P}^3$, let $C$ be a smooth hyperplane section of $X$, and let $H = \mathcal{O}_X(C)$. In this paper, we give a necessary and sufficient condition for the line bundle given by a non-zero effective divisor on $X$ to be initialized and aCM with respect to $H$.

1 Introduction

Let $X$ be a projective manifold of dimension $n$ over the complex number field $\mathbb{C}$, and let $H$ be a very ample line bundle on $X$. Let $E$ be a vector bundle on $X$. Then we call $E$ an arithmetically Cohen-Macaulay (aCM for short) bundle with respect to $H$ if $H^i(E \otimes H^l) = 0$ for all integers $l \in \mathbb{Z}$ and $1 \leq i \leq n-1$. Moreover, we say that $E$ is initialized if $E$ satisfies $h^0(E) \neq 0$ and $h^0(E \otimes H^\vee) = 0$. If $X = \mathbb{P}^n$, an aCM bundle is characterized as a vector bundle obtained by a direct sum of line bundles on $X$ ([7]). However, this criterion is not correct for more general polarized manifolds. A vector bundle $E$ on $X$ is aCM with respect to $H$ if and only if $E \otimes H$ is so. Hence, previously, many people have studied indecomposable initialized aCM bundles on $X$ with respect to a given polarization $H$ and families of them. In particular, several results concerning initialized aCM bundles on smooth hypersurfaces in $\mathbb{P}^{n+1}$ are known. For example, if $X$ is a smooth quadric hypersurface in $\mathbb{P}^{n+1}$, then any non-split aCM bundle on $X$ of rank $r \geq 2$ is isomorphic to a spinor bundle up to twisting by the hyperplane class of $X$ ([8]). In particular, if $n = 2$, then any aCM bundle on $X$ splits. If $X$ is a smooth cubic hypersurface in $\mathbb{P}^3$, Casanellas and Hartshorne have studied the families of initialized aCM bundles on $X$ ([2]). Moreover, Faenzi has given a complete classification of indecomposable initialized aCM bundles of rank 2 on $X$ ([6]). If $X$ is a smooth quartic hypersurface in $\mathbb{P}^3$, Coskun and Kulkarni have constructed

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a 14-dimensional family of simple Ulrich bundles on $X$ of rank 2 with $c_1 = H^\otimes 3$ and $c_2 = 14$, in the case where $X$ is a Pfaffian quartic surface ([5, Theorem 1.1]). Casnati has classified indecomposable initialized aCM bundles of rank 2 on $X$, in the case where $X$ is general determinantal ([3]).

In general, if an aCM bundle on $X$ splits into a direct sum of line bundles on $X$, then the line bundles on $X$ which appear in the splitting are also aCM. Conversely, a vector bundle on $X$ which is given by an extension of aCM line bundles on $X$ is aCM. Hence, the classification of aCM line bundles on $X$ is useful for constructing indecomposable aCM bundles on $X$ of higher rank. Indeed, Pons-Llopis and Tonini have classified aCM line bundles on a DelPezzo surface $X$ with respect to the anti-canonical line bundle on $X$, and constructed families of strictly semi-stable aCM bundles on $X$ ([9]). On the other hand, our previous work about the characterization of aCM line bundles on smooth quartic hypersurfaces in $\mathbb{P}^3$ ([11]) is deeply connected with the Casnati’s work ([3]). In addition, recently, several other results (for example [4] and [12]) concerning the classification of aCM line bundles on polarized surfaces are also known. Since a smooth quartic hypersurface in $\mathbb{P}^3$ is a K3 surface and the properties of linear systems on K3 surfaces are well known ([10]), we can obtain the results as in [11] and [12] with comparative ease. On the other hand, if $X$ is a Del Pezzo surface or a ruled surface, any aCM line bundle on $X$ can be precisely denoted by using the generators of the Picard group of $X$. However, in general, it is difficult to investigate initialized aCM line bundles on polarized surfaces. In this paper, we are interested in the characterization of aCM line bundles on smooth quintic hypersurfaces in $\mathbb{P}^3$. Our main theorem is as follows.

**Theorem 1.1** Let $X$ be a smooth quintic hypersurface in $\mathbb{P}^3$, let $H$ be the hyperplane class of $X$, and let $C$ be a smooth member of the linear system $|H|$. Let $D$ be a non-zero effective divisor on $X$ of arithmetic genus $P_a(D)$, and let $k = C.D + 1 - P_a(D)$. Then $\mathcal{O}_X(D)$ is aCM and initialized if and only if the following conditions are satisfied.

(i) $0 \leq k \leq 4$.

(ii) If $0 \leq k \leq 1$, then $C.D = 10 - k$ and $h^0(\mathcal{O}_C(D - C)) = 0$.

(iii) If $k = 2$, then the following conditions are satisfied.

(a) $C.D = 1$ or $4 \leq C.D \leq 8$.

(b) If $C.D = 7$, then $h^0(\mathcal{O}_X(2C - D)) = 1$.

(c) If $C.D = 8$, then $h^0(\mathcal{O}_C(D - C)) = 0$ and $h^0(\mathcal{O}_C(D)) = 3$.

(iv) If $3 \leq k \leq 4$, then the following conditions are satisfied.

(a) $k - 1 \leq C.D \leq 10 - k$.

(b) If $8 - k \leq C.D \leq 10 - k$, then $h^0(\mathcal{O}_C(D)) = 5 - k$. 

Notations and conventions. In this paper, a curve and a surface are smooth and projective. Let $X$ be a curve or a surface. Then we denote the canonical bundle of $X$ by $K_X$. For a divisor or a line bundle $L$ on $X$, we denote $|L|$ the linear system of $L$, and denote the dual of a line bundle $L$ by $L^\vee$. For an irreducible curve $D$ which is not necessarily smooth, we denote by $P_a(D)$ the arithmetic genus of $D$. For irreducible divisors $D_1$ and $D_2$ on a surface $X$, the arithmetic genus of $D_1 + D_2$ is denoted as $P_a(D_1) + P_a(D_2) + D_1 . D_2 - 1$. By induction, the arithmetic genus of a non-zero effective divisor $D$ on $X$ which is not irreducible is also defined. We denote it by the same notation $P_a(D)$. It follows that $2P_a(D) - 2 = D . (K_X + D)$ by the adjunction formula.

Let $C$ be a curve. Then the gonality of $C$ is the minimal degree of pencils on $C$. It is well known that if $C$ is a smooth plane curve of degree $d \geq 5$, the gonality of $C$ is $d - 1$.

Let $X$ be a surface. Then we denote the Picard lattice of $X$ by $\text{Pic}(X)$, and call the rank of it the Picard number of $X$. If the Picard number of $X$ is $\rho$, then by the Hodge index theorem, the signature of $\text{Pic}(X)$ is $(1, \rho - 1)$. Note that this implies that $D_1^2D_2^2 \leq (D_1 . D_2)^2$ for two divisors $D_1$ and $D_2$ on $X$ satisfying $D_1^2 > 0$ and $D_2^2 > 0$.

Let $X$ be a smooth hypersurface of degree $d$ in $\mathbb{P}^3$. For a hyperplane section $C$ of $X$, we denote the class of it in $\text{Pic}(X)$ by $H$. For an integer $l$, $H^\otimes l$ is often denoted as $\mathcal{O}_X(l)$. By the adjunction formula, $K_X \cong \mathcal{O}_X(d - 4)$. For a vector bundle $E$ on $X$, we will write $E \otimes \mathcal{O}_X(l) = E(l)$.

2 Preliminaries

Let $X$ be a smooth quintic hypersurface in $\mathbb{P}^3$. In this section, we recall several fundamental notions concerning line bundles on $X$, and give some useful propositions about them. Let $D$ be a divisor on $X$ and let $C$ be a smooth hyperplane section of $X$. First of all, since $K_X \cong \mathcal{O}_X(1)$, the Riemann-Roch theorem for $\mathcal{O}_X(D)$ is described as follows.

$$\chi(\mathcal{O}_X(D)) = \frac{1}{2}D . (D - C) + \chi(\mathcal{O}_X),$$

where $\chi(\mathcal{O}_X(D)) = h^0(\mathcal{O}_X(D)) - h^1(\mathcal{O}_X(D)) + h^2(\mathcal{O}_X(D))$. Note that since $h^0(K_X) = h^0(\mathcal{O}_{\mathbb{P}^3}(1)) = 4$, $h^1(\mathcal{O}_X) = 0$, and $h^0(\mathcal{O}_X) = 1$, we have $\chi(\mathcal{O}_X) = 5$.

The Serre duality for $\mathcal{O}_X(D)$ is given by

$$h^i(\mathcal{O}_X(D)) = h^{2-i}(\mathcal{O}_X(C - D)) \quad (0 \leq i \leq 2).$$
By the Riemann-Roch theorem, if \( h^0(\mathcal{O}_X(C - D)) = 0 \), \( h^0(\mathcal{O}_X(D)) \geq \chi(\mathcal{O}_X(D)) \). Hence, the following assertion is useful for estimating the value of \( h^0(\mathcal{O}_X(D)) \).

**Proposition 2.1** Let \( k \) be an integer with \( 0 \leq k \leq 4 \), and let \( D \) be a non-zero effective divisor on \( X \) such that \( P_a(D) = C.D + 1 - k \). If \( C.D \geq 7 - k \), then \( h^0(\mathcal{O}_X(C - D)) = 0 \).

**Remark 2.1.** The linear system \( |\mathcal{O}_C(1)| \) gives an embedding \( C \hookrightarrow \mathbb{P}^2 \). Since \( C \) is a plane quintic, \( C \) is a 4-gonal curve. Therefore, if \( L \) is a line bundle on \( C \) satisfying \( h^0(L) \geq 2 \), then \( \deg(L) \geq h^0(L) + 2 \).

If \( D \) is a non-zero effective divisor, then the arithmetic genus \( P_a(D) \) is given as follows.

\[
P_a(D) = \frac{1}{2} D.(D + C) + 1.
\]

If \( D \) is reduced and irreducible, then \( P_a(D) \geq 0 \). Before the proof of Proposition 2.1, we prepare the following lemmas.

**Lemma 2.1** Let \( D \) be a divisor on \( X \) satisfying \( C.D = 1 \). Then the following conditions are equivalent.

(a) \( h^0(\mathcal{O}_X(D)) > 0 \).
(b) \( h^0(\mathcal{O}_X(D)) = 1 \), \( h^0(\mathcal{O}_X(C - D)) = 2 \), and \( h^1(\mathcal{O}_X(D)) = 0 \).
(c) \( D^2 = -3 \).

**Proof.** (a) \( \implies \) (b). By the hypothesis, we may assume that \( D \) is effective. Since \( C.D = 1 \), by the ampleness of \( C \), \( D \) is reduced and irreducible. Therefore, \( P_a(D) \geq 0 \), and hence, we have \( D^2 \geq -3 \). This means that

\[
2.1 \quad h^0(\mathcal{O}_X(D)) + h^0(\mathcal{O}_X(C - D)) \geq \chi(\mathcal{O}_X(D)) \geq 3.
\]

On the other hand, since \( C.D = 1 \) and \( C.(C - D) = 4 \), by Remark 2.1, we have \( h^0(\mathcal{O}_C(D)) = 1 \) and \( h^0(\mathcal{O}_C(C - D)) \leq 2 \). Since \( C.(D - C) = -4 \), by the ampleness of \( C \), we have \( h^0(\mathcal{O}_X(D - C)) = 0 \). Since \( h^0(\mathcal{O}_X(-D)) = 0 \), by the exact sequence

\[
0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X(C - D) \longrightarrow \mathcal{O}_C(C - D) \longrightarrow 0,
\]
and

\[
0 \longrightarrow \mathcal{O}_X(D - C) \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_C(D) \longrightarrow 0,
\]
we have \( h^0(\mathcal{O}_X(C - D)) \leq 2 \) and \( h^0(\mathcal{O}_X(D)) = 1 \). By the inequality (2.1), we have \( h^0(\mathcal{O}_X(C - D)) = 2 \) and \( \chi(\mathcal{O}_X(D)) = 3 \). Hence, we have \( h^1(\mathcal{O}_X(D)) = 0 \).

(b) \( \implies \) (c). Since \( \chi(\mathcal{O}_X(D)) = 3 \) and \( C.D = 1 \), we have \( D^2 = -3 \).

(c) \( \implies \) (a). Since \( C.D = 1 \), \( h^0(\mathcal{O}_X(D)) + h^0(\mathcal{O}_X(C - D)) \geq \chi(\mathcal{O}_X(D)) = 3 \). If \( h^0(\mathcal{O}_X(D)) = 0 \), we have \( h^0(\mathcal{O}_X(C - D)) \geq 3 \). Since \( C.(C - D) = 4 \), by Remark
Hence, by the Riemann-Roch theorem, we have \( h^0(O_X(C - D)) \leq 2 \). Since \(-D.C = -1\), \( h^0(O_X(-D)) = 0 \). Hence, by the exact sequence

\[
0 \rightarrow O_X(-D) \rightarrow O_X(C - D) \rightarrow O_C(C - D) \rightarrow 0,
\]

we have \( h^0(O_X(C - D)) \leq h^0(O_C(C - D)) \). This is a contradiction. Therefore, we have \( h^0(O_X(D)) > 0 \).

\[\square\]

**Lemma 2.2** Let \( D \) be an effective divisor on \( X \) with \( C.D = 2 \). \( D^2 \leq -6 \) if and only if one of the following cases occurs.

(a) There exists a curve \( D_1 \) on \( X \) with \( D = 2D_1, D_1^2 = -3, \) and \( C.D_1 = 1 \).

(b) There exist curves \( D_1 \) and \( D_2 \) with \( D = D_1 + D_2, D_1, D_2 = 0, D_i^2 = -3, \) and \( C.D_i = 1 \) \((i = 1, 2)\).

**Proof.** Assume that \( D^2 \leq -6 \). If \( D \) is reduced and irreducible, then \( P_a(D) \geq 0 \). This means that \( D^2 \geq -4 \). Hence, there exists a non-trivial effective decomposition \( D = D_1 + D_2 \). Since \( C.D = 2 \), we have \( C.D_1 = C.D_2 = 1 \). Hence, \( D_1 \) and \( D_2 \) are reduced and irreducible and, by Lemma 2.1, \( D_i^2 = -3 \) \((i = 1, 2)\). If \( D_1 = D_2 \), then \( D = 2D_1 \) and \( D^2 = -12 \). If \( D_1 \neq D_2 \), then \( D_1, D_2 \geq 0 \). Hence, we have \( D^2 \geq -6 \). By the assumption, we have \( D^2 = -6 \). Therefore, we have \( D_1, D_2 = 0 \). The converse assertion is clear.

**Proof of Proposition 2.1.** If \( C.D \geq \max\{6, 7 - k\} \), then \( C.(C - D) \leq -1 \). Hence, \( h^0(O_X(C - D)) = 0 \). We consider the case where \( 2 \leq k \leq 4 \) and \( C.D = 5 \).

Assume that \( |C - D| \neq \emptyset \). Then, by the ampleness of \( C \), we have \( O_X(D) \cong O_X(C) \). This contradicts the assumption that \( P_a(D) = C.D + 1 - k \). Assume that \( 3 \leq k \leq 4 \) and \( C.D = 4 \). Then \( C.(C - D) = 1 \) and \( (C - D)^2 = 1 - 2k \leq -5 \).

By Lemma 2.1, we have \( |C - D| = \emptyset \). We consider the case where \( k = 4 \) and \( C.D = 3 \). Assume that \( |C - D| \neq \emptyset \) and let \( \Gamma \) be a member of \( |C - D| \).

Since \( \Gamma.C = 2 \) and \( \Gamma^2 = -6 \), by Lemma 2.2, there exist curves \( D_1 \) and \( D_2 \) satisfying the conditions that \( \Gamma = D_1 + D_2, D_1.D_2 = 0, D_i^2 = -3, \) and \( C.D_i = 1 \) \((i = 1, 2)\). Since \( |C - \Gamma| = |D| \neq \emptyset \), there exists a hyperplane in \( \mathbb{P}^3 \) containing \( \Gamma \). This is a contradiction. Hence, we have the assertion.

**Proposition 2.2** Let \( D \) be a non-zero effective divisor on \( X \). If \( h^1(O_X(-D)) = 0 \), then \( P_a(D) \geq 0 \).

**Proof.** By the Serre duality, \( h^1(O_X(C + D)) = 0 \), and \( h^2(O_X(C + D)) = 0 \). Hence, by the Riemann-Roch theorem, we have \( h^0(O_X(C + D)) = P_a(D) + 4 \). Since \( h^0(O_X(C + D)) \geq h^0(O_X(C)) = 4 \), we have \( P_a(D) \geq 0 \).

By Proposition 2.2, for any effective divisor \( D \) satisfying the condition (a) or (b) as in Lemma 2.2, we have \( h^1(O_X(-D)) \neq 0 \). In general, the vanishing condition of the cohomology of the sheaf as in Proposition 2.2 can be characterized by the following notion.
**Definition 2.1** Let $m$ be a positive integer. Then a non-zero effective divisor $D$ on $X$ is called $m$-connected if $D_1, D_2 \geq m$, for each effective decomposition $D = D_1 + D_2$.

If a non-zero effective divisor $D$ is 1-connected, then $h^0(\mathcal{O}_D) = 1$. Therefore, by the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0,$$

we have $h^1(\mathcal{O}_X(-D)) = 0$. Hence, by Proposition 2.1, we have $P_a(D) \geq 0$. An effective divisor $D$ satisfying the condition as in Lemma 2.2 (b) is not 1-connected, but reduced. Hence, such a divisor $D$ is not contained by any hyperplane in $\mathbb{P}^3$. On the other hand, since it satisfies $h^0(\mathcal{O}_X(D - C)) = 0$, by the Riemann-Roch theorem and Remark 2.1, the conditions $h^0(\mathcal{O}_X(D)) = 1$ and $h^1(\mathcal{O}_X(D)) = 0$ are also satisfied. Conversely, any non-zero effective divisor $D$ which is not 1-connected is characterized as follows, under the condition that $\mathcal{O}_X(D)$ is initialized and $h^1(\mathcal{O}_X(D)) = 0$.

**Proposition 2.3** Let $D$ be a non-zero effective divisor. If $h^0(\mathcal{O}_X(D - C)) = 0$ and $h^1(\mathcal{O}_X(D)) = 0$, then $h^1(\mathcal{O}_X(-D)) = 0$ or there exist smooth rational curves $D_1$ and $D_2$ on $X$ such that $D = D_1 + D_2$, $D_1, D_2 = 0$, and $C.D_i = 1$ ($i = 1, 2$).

**Proof.** By the ampleness of $C$, $C.D \geq 1$. If $C.D = 1$, then $D$ is reduced and irreducible. Hence, $h^1(\mathcal{O}_X(-D)) = 0$. Assume that $C.D \geq 2$. If $C.D \geq 6$, then $C.(C - D) \leq -1$ and hence, $h^0(\mathcal{O}_C(C - D)) = 0$. Since $h^1(\mathcal{O}_X(C - D)) = 0$, by the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X(C - D) \longrightarrow \mathcal{O}_C(C - D) \longrightarrow 0,$$

we have $h^1(\mathcal{O}_X(-D)) = 0$. Assume that $C.D \leq 5$. Since $C.D \geq 2$, we have $0 \leq C.(C - D) \leq 3$. By Remark 2.1, we have $h^0(\mathcal{O}_C(C - D)) \leq 1$. By the exact sequence (2.2), we have $h^0(\mathcal{O}_X(C - D)) \leq 1$. If $h^0(\mathcal{O}_X(C - D)) = 1$, then $h^0(\mathcal{O}_C(C - D)) = 1$ and $h^1(\mathcal{O}_X(-D)) = 0$. Hence, it is sufficient to consider the case where $h^0(\mathcal{O}_X(C - D)) = 0$. By Remark 2.1, we have $h^0(\mathcal{O}_C(D)) \leq 3$. By the assumption that $h^0(\mathcal{O}_X(D - C)) = 0$ and the exact sequence

$$0 \longrightarrow \mathcal{O}_X(D - C) \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_C(D) \longrightarrow 0,$$

we have $h^0(\mathcal{O}_X(D)) \leq 3$. If $h^0(\mathcal{O}_X(D - C)) = 0$, by the assumption that $h^1(\mathcal{O}_X(D)) = 0$, we have $h^0(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X(D))$, and hence, $-8 \leq D^2 - C.D \leq -4$. Therefore, it is sufficient to show the following lemma.

**Lemma 2.3** Let $D$ be a non-zero effective divisor with $2 \leq C.D \leq 5$ satisfying the condition that $h^1(\mathcal{O}_X(D)) = 0$, $h^0(\mathcal{O}_X(D - C)) = 0$, and $h^0(\mathcal{O}_X(C - D)) = 0$. Moreover, we assume that there exists an integer $k$ with $2 \leq k \leq 4$ satisfying $D^2 = C.D - 2k$. If $D$ is not 1-connected, then there exist smooth rational curves $D_1$ and $D_2$ on $X$ such that $D = D_1 + D_2$, $D_1, D_2 = 0$, and $C.D_i = 1$ ($i = 1, 2$).
Proof. Assume that $D$ is not 1-connected, and let $D = D_1 + D_2$ be a non-trivial effective decomposition with $D_1.D_2 \leq 0$. Since $D^2 = D.D_1 + D.D_2$, we may assume that $D.D_1 = D^2 - D.D_2 \leq \frac{D^2}{2}$. Then we have

\begin{equation}
\frac{D^2}{2} \leq D.D_2 = D_2^2 + D_1.D_2 \leq D_2^2.
\end{equation}

By the assumption that $D^2 = C.D - 2k$ and $h^0(\mathcal{O}_X(C - D)) = 0$ and the inequality (2.3), we have

\begin{equation}
\chi(\mathcal{O}_X(D_2)) \geq h^0(\mathcal{O}_X(D)) + \frac{1}{4}(C.D_1 - C.D_2 + 2k).
\end{equation}

Assume that $C.D_1 < C.D_2$. Since $C.D_1 \geq 1$, we have $C.D_2 \geq 2$. Hence, we have $C.(C - D_2) \leq 3$. By Remark 2.1, we have $h^0(\mathcal{O}_X(C - D_2)) \leq 1$. Assume that $h^0(\mathcal{O}_X(C - D_2)) = 0$. Since $C.D \leq 5$ and $2 \leq k \leq 4$, we have

$$C.D_1 - C.D_2 + 2k = 2C.D_1 - C.D + 2k \geq 2k + 2 - C.D \geq 2k - 3 \geq 1.$$ 

Hence, by the inequality (2.4), we have $h^0(\mathcal{O}_X(D_2)) > h^0(\mathcal{O}_X(D))$. This contradicts the fact that $\mathcal{O}_X(D_2) \subset \mathcal{O}_X(D)$.

Assume that $h^0(\mathcal{O}_X(C - D_2)) = 1$. Since $h^0(\mathcal{O}_X(D - C)) = 0$, we have $C.(C - D_2) \geq 1$. Since $C.D_1 \geq 1$ and $C.D_2 \leq 4$, if $k = 4$, then $C.D_1 - C.D_2 + 2k + 4 \geq 1$. By the inequality (2.4), we have $h^0(\mathcal{O}_X(D_2)) > h^0(\mathcal{O}_X(D))$. However, by the same reason as above, this is a contradiction. If $k = 2$ or $3$, $C.D_1 - C.D_2 + 2k - 4 \geq -3$. By the inequality (2.4), we have

$$h^0(\mathcal{O}_X(D_2)) \geq h^0(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X(D)) = 5 - k \geq 2.$$ 

By the assumption that $h^0(\mathcal{O}_X(D - C)) = 0$, we have $h^0(\mathcal{O}_X(D_2 - C)) = 0$. By Remark 2.1, we have $h^0(\mathcal{O}_C(D_2)) \leq 2$. Hence, by the exact sequence

\begin{equation}
0 \longrightarrow \mathcal{O}_X(D_2 - C) \longrightarrow \mathcal{O}_X(D_2) \longrightarrow \mathcal{O}_C(D_2) \longrightarrow 0,
\end{equation}

we have $k = 3$, $h^0(\mathcal{O}_C(D_2)) = 2$, and $C.D_2 = 4$. Since $C.(C - D_2) = 1$, by Lemma 2.1, we have $(C - D_2)^2 = -3$ and hence, $D_2^2 = 0$. Since $C.D \leq 5$, we have $C.D_1 = 1$. By the same reason, $D_1^2 = -3$. Since $D^2 = C.D - 6 = -1$, we have $D_1.D_2 = 1$. This contradicts the hypothesis that $D_1.D_2 \leq 0$. By the above argument, we have $C.D_1 \geq C.D_2$.

Since $C.(C - D_2) \leq 4$, by Remark 2.1, we have $h^0(\mathcal{O}_X(C - D_2)) \leq 2$. If $h^0(\mathcal{O}_X(C - D_2)) = 0$, by the inequality (2.4), $h^0(\mathcal{O}_X(D_2)) > h^0(\mathcal{O}_X(D))$. Hence, by the same reason as above, we have a contradiction.

Assume that $h^0(\mathcal{O}_X(C - D_2)) = 1$. Since $C.D_1 \geq C.D_2$, if $3 \leq k \leq 4$, then, by the inequality (2.4), we have $h^0(\mathcal{O}_X(D_2)) > h^0(\mathcal{O}_X(D))$. By the same reason as above, we have a contradiction. Assume that $k = 2$. Since $h^0(\mathcal{O}_X(D)) = 3$,
by the inequality (2.4), we have $h^0(O_X(D_2)) \geq 3$. Since $|C - D_2| \neq \emptyset$ and $h^0(O_X(D - C)) = 0$, we have $C.(C - D_2) \geq 1$. Since $C.D_2 \leq 4$, by Remark 2.1 and the exact sequence (2.5), we have a contradiction.

Assume that $h^0(O_X(C - D_2)) = 2$. Then, we have $C.(C - D_2) = 4$. Hence, we have $C.D_2 = 1$. By Lemma 2.1, we have $D_2^2 = -3$. Hence, by the inequality (2.3), we have $C.D \leq 2k - 6$. Since $C.D \geq 2$, by the assumption that $2 \leq k \leq 4$, we have $k = 4$, $C.D = 2$, $D_1.D_2 = 0$ and $D^2 = -6$. Since $C.D_1 = C.D_2 = 1$, by Lemma 2.1, we have $P_a(D_1) = P_a(D_2) = 0$. Therefore, we have the assertion of Lemma 2.3.

We can construct an example of an effective divisor $D$ on a smooth quintic hypersurface $X$ satisfying the condition as in Lemma 2.2 (b).

**Example 2.1.** Let $X$ be the quintic hypersurface in $\mathbb{P}^3$ defined by the equation $x_0^5 + x_1^5 + x_2^5 + x_3^5 = 0$ for a suitable homogeneous coordinate $(x_0 : x_1 : x_2 : x_3)$ on $\mathbb{P}^3$. Let $D_1$ and $D_2$ be lines on $X$ which are defined by the equations $x_0 + x_1 = x_2 + x_3 = 0$ and $x_0 + x_2 = x_1 + \omega x_3 = 0$, respectively. Here, $\omega$ is a primitive 5th root of 1. If we let $C$ be the hyperplane section of $X$ defined by the equation $x_1^5 + x_2^5 + x_3^5 = 0$, then $C.D_1 = C.D_2 = 1$. Obviously, we have $D_1.D_2 = 0$. Hence, the divisor $D = D_1 + D_2$ on $X$ is not 1-connected.

By Proposition 2.3, we have the following assertion.

**Corollary 2.1.** Let $D$ be a non-zero effective divisor satisfying the condition that $h^1(O_X(D)) = 0$ and $h^0(O_X(D - C)) = 0$. If $D^2 > -6$, then we have $h^1(O_X(-D)) = 0$.

### 3 ACM bundles on quintic hypersurfaces in $\mathbb{P}^3$

Let $X$ be as in section 2. In this section, we recall a well known fact about aCM bundles on $X$ and prepare a proposition to prove our main theorem.

**Definition 3.1** We call a vector bundle $E$ on $X$ an arithmetically Cohen-Macaulay (aCM for short) bundle if $H^1(E(l)) = 0$ for all integers $l \in \mathbb{Z}$.

**Definition 3.2** We say that a vector bundle $E$ on $X$ is initialized if it satisfies the conditions $H^0(E) \neq 0$ and $H^0(E(-1)) = 0$.

For an aCM bundle $E$ on $X$, we consider the graded module $H^0(E) := \bigoplus_{l \in \mathbb{Z}} H^0(E(l))$ over the homogeneous coordinate ring of $X$. First of all, we mention the following result concerning the minimal number of generators of it.

**Proposition 3.1** ([2, Theorem 3.1 and Corollary 3.5]). Let $E$ be an aCM bundle of rank $r$ on $X$, and let $\mu(E)$ be the minimal number of generators of $H^0(E)$. Then we get $\mu(E) \leq 5r$. Moreover, if $E$ is initialized, then $h^0(E) \leq 5r$. 

An initialized aCM bundle $E$ of rank $r$ with $h^0(E) = 5r$ is called an Ulrich bundle. An Ulrich bundle of rank $r$ is characterized as an initialized aCM bundle whose Hilbert polynomial is equal to $5r\binom{r+2}{2}$. In particular, the line bundle $\mathcal{O}_X(D)$ defined by an effective divisor $D$ on $X$ with $D^2 = D.K_X = 10$ as in Theorem 1.1 (ii) is Ulrich. If there exists such a line bundle $\mathcal{O}_X(D)$ on $X$, by taking the minimal free resolution of $H^0_*(\mathcal{O}_X(D))$ as a module over the homogeneous coordinate ring of $\mathbb{P}^3$, it follows that $X$ is linear determinantal (i.e., $X$ is defined as the zero locus of the determinant of a $5 \times 5$-matrix of linear forms).

Remark 3.1. A line bundle on $X$ is aCM if and only if the dual of it is aCM.

Proposition 3.2 Let $D$ be a non-zero effective divisor on $X$, let $C$ be a smooth hyperplane section of $X$, and let $k$ be a positive integer satisfying $C.D + 5 < 5k$. If $h^1(\mathcal{O}_X(lC - D)) = 0$ for $0 \leq l \leq k$, then $\mathcal{O}_X(D)$ is aCM.

Proof. Since $K_C = \mathcal{O}_C(2)$, by the Serre duality, for $n \geq 1$, we have

$$h^1(\mathcal{O}_C((n + 1)C - D)) = h^0(\mathcal{O}_C(D - (n - 1)C)).$$

If $n \geq k$, by the assumption, $\deg(\mathcal{O}_C(D - (n - 1)C)) = C.D - 5(n - 1) < 0$, and hence, we have $h^1(\mathcal{O}_C((n + 1)C - D)) = 0$. Since $h^1(\mathcal{O}_X(lC - D)) = 0$ for $1 \leq l \leq k$, by the exact sequence

$$0 \longrightarrow \mathcal{O}_X(nC - D) \longrightarrow \mathcal{O}_X((n + 1)C - D) \longrightarrow \mathcal{O}_C((n + 1)C - D) \longrightarrow 0,$$

for any integer $n \geq 1$, we have $h^1(\mathcal{O}_X(nC - D)) = 0$ by induction.

On the other hand, if $m \geq 0$, we have $h^0(\mathcal{O}_C(-mC - D)) = 0$. Since $h^1(\mathcal{O}_X(-D)) = 0$, by the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-(m + 1)C - D) \longrightarrow \mathcal{O}_X(-mC - D) \longrightarrow \mathcal{O}_C(-mC - D) \longrightarrow 0,$$

for any integer $m \geq 0$, we have $h^1(\mathcal{O}_X(-mC - D)) = 0$ by induction. Hence, $\mathcal{O}_X(-D)$ is aCM. By Remark 3.1, $\mathcal{O}_X(D)$ is also aCM. □

4 Proof of Theorem 1.1

Let $X$ be a smooth quintic hypersurface in $\mathbb{P}^3$, and let $C$ be a smooth hyperplane section of $X$. Let $D$ be a non-zero effective divisor of arithmetic genus $P_a(D)$. In this section, we give a proof of our main theorem. First of all, we have the following assertion.

Proposition 4.1 If $\mathcal{O}_X(D)$ is aCM and initialized, then the following conditions are satisfied.
(a) $C.D - 3 \leq P_a(D) \leq C.D + 1$.
(b) If $P_a(D) = C.D - 3$, then $3 \leq C.D \leq 6$.
(c) If $P_a(D) = C.D - 2$, then $2 \leq C.D \leq 7$.
(d) If $P_a(D) = C.D - 1$, then $C.D = 1$ or $4 \leq C.D \leq 8$.
(e) If $P_a(D) = C.D$, then $6 \leq C.D \leq 9$.
(f) If $P_a(D) = C.D + 1$, then $7 \leq C.D \leq 10$.

By the ampleness of $C$, we have $C.(C - D) \leq 4$. Hence, $h^0(O_\mathcal{C}(C - D)) \leq 2$, by Remark 2.1. By the exact sequence

$$0 \rightarrow O_X(-D) \rightarrow O_X(C - D) \rightarrow O_\mathcal{C}(C - D) \rightarrow 0,$$

we have $h^0(O_X(C - D)) \leq 2$. Hence, we divide the assertion of Proposition 4.1 into the following three lemmas depending on the value of $h^0(O_X(C - D))$.

**Lemma 4.1** Assume that $h^0(O_X(C - D)) = 2$. Then $P_a(D) = 0$ and $C.D = 1$.

*Proof.* By the exact sequence (4.1), we have $h^0(O_X(C - D)) = 2$. Hence, we have $C.D = 1$. By Lemma 2.1, we have $D^2 = -3$ and hence, $P_a(D) = 0$.  

**Lemma 4.2** Assume that $h^0(O_X(C - D)) = 1$. If $O_X(D)$ is $aCM$ and initialized, then one of the following cases occurs.

(i) $P_a(D) = 0$ and $C.D = 2$.
(ii) $P_a(D) = 1$ and $C.D = 3$.
(iii) $P_a(D) = 3$ and $C.D = 4$.

*Proof.* Since $h^1(O_X(D)) = 0$, by the Riemann-Roch theorem, we have

$$h^0(O_X(D)) = P_a(D) + 3 - C.D.$$

Since $|C - D| \neq \emptyset$, $C.(C - D) \geq 0$. Hence, we have $C.D \leq 5$. If $C.D = 5$, $O_X(D) \cong O_X(C)$. This contradicts the assumption that $O_X(D)$ is initialized. Hence, we have $C.D \leq 4$. Since $O_X(D)$ is $aCM$ and initialized, by the exact sequence

$$0 \rightarrow O_X(D - C) \rightarrow O_X(D) \rightarrow O_\mathcal{C}(D) \rightarrow 0,$$

we have $h^0(O_\mathcal{C}(D)) = h^0(O_X(D))$. By Remark 2.1, $h^0(O_X(D)) \leq 2$. Assume that $h^0(O_X(D)) = 2$. Since we have $C.D = 4$, by the equality (4.2), $P_a(D) = 3$. Assume that $h^0(O_X(D)) = 1$. Then $P_a(D) = C.D - 2 \leq 2$. If $P_a(D) = 2$, then $C.D = 4$. Hence, $C.(C - D) = 1$ and $(C - D)^2 = -5$. However, since $|C - D| \neq \emptyset$, this contradicts Lemma 2.1. Since $h^1(O_X(-D)) = 0$, by Proposition 2.2, $P_a(D) \geq 0$. Hence, we have the assertion.  

□
Lemma 4.3. Assume that $h^0(\mathcal{O}_X(C-D)) = 0$. If $\mathcal{O}_X(D)$ is aCM and initialized, then one of the following cases occurs.

(i) $P_a(D) = C.D - 3$ and $3 \leq C.D \leq 6$.
(ii) $P_a(D) = C.D - 2$ and $4 \leq C.D \leq 7$.
(iii) $P_a(D) = C.D - 1$ and $5 \leq C.D \leq 8$.
(iv) $P_a(D) = C.D$ and $6 \leq C.D \leq 9$.
(v) $P_a(D) = C.D + 1$ and $7 \leq C.D \leq 10$.

Proof. First of all, by Proposition 3.1, we have $h^0(\mathcal{O}_X(D)) \leq 5$. Let $k$ be an integer with $0 \leq k \leq 4$ such that $h^0(\mathcal{O}_X(D)) = 5 - k$. Since $h^1(\mathcal{O}_X(D)) = 0$, by the Riemann-Roch theorem, we have

\[(4.3) \quad P_a(D) = C.D + 1 - k.\]

On the other hand, we have $h^0(\mathcal{O}_X(D)) + 5 - C.D = \chi(\mathcal{O}_X(2C-D))$. Since $h^1(\mathcal{O}_X(2C-D)) = 0$, we have

\[(4.4) \quad C.D \leq h^0(\mathcal{O}_X(D)) + 5.\]

If $h^0(\mathcal{O}_X(D)) = 1$, then we have $P_a(D) = C.D - 3$ by the equality (4.3). Since $h^1(\mathcal{O}_X(-D)) = 0$, by Proposition 2.2, we have $P_a(D) \geq 0$. Hence, we have $C.D \geq 3$. By the inequality (4.4), we have $C.D \leq 6$.

Assume that $h^0(\mathcal{O}_X(D)) \geq 2$. Then $0 \leq k \leq 3$. By the same argument as in the proof of Lemma 4.2, we have $h^0(\mathcal{O}_C(D)) = 5 - k$. By Remark 2.1, we have $C.D \geq 7 - k$. By the inequality (4.4), we have $C.D \leq 10 - k$. Hence, we have the assertion. $\square$

By Proposition 4.1, we have the following necessary condition for $\mathcal{O}_X(D)$ to be aCM and initialized.

Proposition 4.2. Let $k = C.D + 1 - P_a(D)$. If $\mathcal{O}_X(D)$ is aCM and initialized, then the following conditions are satisfied.

(i) $0 \leq k \leq 4$.
(ii) If $0 \leq k \leq 1$, then $C.D = 10 - k$ and $h^0(\mathcal{O}_C(D-C)) = 0$.
(iii) If $k = 2$, then the following conditions are satisfied.
   (a) $C.D = 1$ or $4 \leq C.D \leq 8$.
   (b) If $C.D = 7$, then $h^0(\mathcal{O}_X(2C-D)) = 1$.
   (c) If $C.D = 8$, then $h^0(\mathcal{O}_C(D-C)) = 0$ and $h^0(\mathcal{O}_C(D)) = 3$.
(iv) If $3 \leq k \leq 4$, then the following conditions are satisfied.
   (a) $k - 1 \leq C.D \leq 10 - k$.
   (b) If $8 - k \leq C.D \leq 10 - k$, then $h^0(\mathcal{O}_C(D)) = 5 - k$. 


Proof. First of all, by Proposition 4.1, the assertion of (i) is clear. Assume that \(C.D \geq 8 - k\). By Proposition 2.1, we have \(|C - D| = \emptyset\). Since \(h^1(\mathcal{O}_X(D)) = 0\), by the Riemann-Roch theorem, we have \(h^0(\mathcal{O}_X(D)) = 5 - k\). Since \(\mathcal{O}_X(D)\) is aCM and initialized, by the exact sequences

\[
0 \to \mathcal{O}_X(D - C) \to \mathcal{O}_X(D) \to \mathcal{O}_C(D) \to 0,
\]

and

\[
0 \to \mathcal{O}_X(D - 2C) \to \mathcal{O}_X(D - C) \to \mathcal{O}_C(D - C) \to 0,
\]

we have \(h^0(\mathcal{O}_C(D)) = 5 - k\) and \(h^0(\mathcal{O}_C(D - C)) = 0\). Therefore, by Proposition 4.1, we get the assertion of (iv).

On the other hand, we have

\[
(h.5) \quad h^0(\mathcal{O}_X(2C - D)) = \chi(\mathcal{O}_X(2C - D)) = 10 - k - C.D.
\]

(ii) We consider the case where \(0 \leq k \leq 1\). By the above argument, it is sufficient to show that \(C.D = 10 - k\). By Proposition 4.1, \(7 - k \leq C.D \leq 10 - k\). Assume that \(7 - k \leq C.D \leq 8 - k\). Then, by the equality (4.5), we have \(h^0(\mathcal{O}_X(2C - D)) \geq 2\). On the other hand, since \(|C - D| = \emptyset\) and \(h^1(\mathcal{O}_X(C - D)) = 0\), by the equality (4.5) and the exact sequence

\[
0 \to \mathcal{O}_X(C - D) \to \mathcal{O}_X(2C - D) \to \mathcal{O}_C(2C - D) \to 0,
\]

we have \(h^0(\mathcal{O}_C(2C - D)) = 10 - k - C.D\). By Remark 2.1, we have \(C.(2C - D) \geq 12 - k - C.D\). This contradicts the assumption that \(0 \leq k \leq 1\).

Assume that \(C.D = 9 - k\). By the equality (4.5), \(h^0(\mathcal{O}_X(2C - D)) = 1\). If \(k = 0\), we have \(C.(2C - D) = 1\) and \((2C - D)^2 = -7\). However, this contradicts Lemma 2.1. Assume that \(k = 1\) and let \(\Gamma\) be the member of \(|2C - D|\). Then we have \(P_a(\Gamma) = -1\). By Proposition 2.2, this means that \(h^1(\mathcal{O}_X(D - 2C)) \neq 0\). This contradicts the assumption that \(\mathcal{O}_X(D)\) is aCM. Hence, we have the assertion.

(iii) If \(k = 2\) and \(C.D = 7\), then, by the equality (4.5), we have the assertion of (b). By Proposition 4.1 and the above argument, we have the assertion. □

From now on we show that each condition from (ii) to (iv) as in Proposition 4.2 is a sufficient condition for \(\mathcal{O}_X(D)\) to be aCM and initialized. Since the proof is long and complex, we divide the converse assertion of Proposition 4.2 into several propositions.

**Proposition 4.3** Assume that \(P_a(D) = C.D - 3\). If the following conditions are satisfied, then \(\mathcal{O}_X(D)\) is aCM and initialized.

(a) \(3 \leq C.D \leq 6\).

(b) If \(4 \leq C.D \leq 6\), then \(h^0(\mathcal{O}_C(D)) = 1\).
Proof. First of all, \( \mathcal{O}_X(D) \) is initialized. Indeed, if \( C.D \leq 4 \), then we have \( C.(D - C) \leq -1 \), and hence, \( |D - C| = 0 \). Since \( h^0(\mathcal{O}_C(1)) = 3 \), if \( 5 \leq C.D \leq 6 \), by the assumption (b), \( h^0(\mathcal{O}_C(D - C)) = 0 \). Moreover, since \( C.(D - 2C) < 0 \), we have \( h^0(\mathcal{O}_X(D - 2C)) = 0 \). By the exact sequence

\[
(4.6) \quad 0 \rightarrow \mathcal{O}_X(D - 2C) \rightarrow \mathcal{O}_X(D - C) \rightarrow \mathcal{O}_C(D - C) \rightarrow 0,
\]

we have \( h^0(\mathcal{O}_X(D - C)) = 0 \).

If \( n \geq 3 \), we have \( C.D + 5 < 5n \). Hence, by Proposition 3.2, it is sufficient to show that \( h^1(\mathcal{O}_X(lC - D)) = 0 \) for \( 0 \leq l \leq 3 \). By Proposition 2.1, we have \( |C - D| = 0 \). If \( C.D = 3 \), by Remark 2.1, we have \( h^0(\mathcal{O}_C(D)) = 1 \). Hence, by the assumption (b) and the exact sequence

\[
(4.7) \quad 0 \rightarrow \mathcal{O}_X(D - C) \rightarrow \mathcal{O}_X(D) \rightarrow \mathcal{O}_C(D) \rightarrow 0,
\]

we have \( h^0(\mathcal{O}_X(D)) = 1 \). Since \( \chi(\mathcal{O}_X(D)) = 1 \), we have

\[
h^1(\mathcal{O}_X(C - D)) = h^1(\mathcal{O}_X(D)) = 0.
\]

By the exact sequence (4.6) and (4.7), we have \( h^1(\mathcal{O}_X(lC - D)) = 0 \) for \( 2 \leq l \leq 3 \). Since \( \mathcal{O}_X(D) \) is initialized, and satisfies \( h^1(\mathcal{O}_X(D)) = 0 \) and \( P_a(D) \geq 0 \), by Corollary 2.1, we have \( h^1(\mathcal{O}_X(-D)) = 0 \). Hence, we have the assertion. \( \square \)

Proposition 4.4 Assume that \( P_a(D) = C.D - 2 \). If the following conditions are satisfied, then \( \mathcal{O}_X(D) \) is aCM and initialized.

(a) \( 2 \leq C.D \leq 7 \).
(b) If \( 5 \leq C.D \leq 7 \), then \( h^0(\mathcal{O}_C(D)) = 2 \).

Proof. First of all, \( \mathcal{O}_X(D) \) is initialized. Indeed, if \( C.D \leq 4 \), \( C.(D - C) \leq -1 \) and hence, we have \( |D - C| = 0 \). Since \( h^0(\mathcal{O}_C(1)) = 3 \), if \( 5 \leq C.D \leq 7 \), then the assumption (b) implies \( h^0(\mathcal{O}_C(D - C)) = 0 \). Moreover, since \( C.(D - 2C) < 0 \), we have \( h^0(\mathcal{O}_X(D - 2C)) = 0 \). Hence, by the exact sequence (4.6), we have \( h^0(\mathcal{O}_X(D - C)) = 0 \). If \( n \geq 3 \), then we have \( C.D + 5 < 5n \). Hence, by Proposition 3.2, it is sufficient to show that \( h^1(\mathcal{O}_X(lC - D)) = 0 \) for \( 0 \leq l \leq 3 \).

Assume that \( 2 \leq C.D \leq 3 \). By Remark 2.1 and the exact sequence (4.7), we have \( h^0(\mathcal{O}_X(D)) = h^0(\mathcal{O}_C(D)) = 1 \). Since \( \chi(\mathcal{O}_X(D)) = 2 \), \( h^0(\mathcal{O}_X(C - D)) \geq 1 \). Since \( 2 \leq C.(C - D) \leq 3 \), we have \( h^0(\mathcal{O}_C(C - D)) \leq 1 \). Hence, by the exact sequence

\[
(4.8) \quad 0 \rightarrow \mathcal{O}_X(-D) \rightarrow \mathcal{O}_X(C - D) \rightarrow \mathcal{O}_C(C - D) \rightarrow 0,
\]

we have \( h^0(\mathcal{O}_X(C - D)) = h^0(\mathcal{O}_C(C - D)) = 1 \). Hence, we have

\[
h^1(\mathcal{O}_X(C - D)) = h^1(\mathcal{O}_X(D)) = 0.
\]
Assume that $4 \leq C.D \leq 7$. Since, by Proposition 2.1, $|C - D| = \emptyset$, we have $h^0(\mathcal{O}_X(D)) \geq \chi(\mathcal{O}_X(D)) = 2$. If $C.D = 4$, by Remark 2.1, $h^0(\mathcal{O}_C(D)) \leq 2$. By the assumption (b) and the exact sequence (4.7), we have $h^0(\mathcal{O}_X(D)) = h^0(\mathcal{O}_C(D)) = 2$. Hence, we have $h^1(\mathcal{O}_X(C - D)) = h^1(\mathcal{O}_X(D)) = 0$. Hence, by the exact sequence (4.7), we have $h^1(\mathcal{O}_X(2C - D)) = h^1(\mathcal{O}_X(D - C)) = 0$. Since $h^0(\mathcal{O}_C(D - C)) = 0$, by the exact sequence (4.6), we have

$$h^1(\mathcal{O}_X(3C - D)) = h^1(\mathcal{O}_X(D - 2C)) = 0.$$ 

Since $\mathcal{O}_X(D)$ is initialized and satisfies $h^1(\mathcal{O}_X(D)) = 0$ and $P_\alpha(D) \geq 0$, by Corollary 2.1, we have $h^1(\mathcal{O}_X(-D)) = 0$. Therefore, the assertion follows. 

\textbf{Proposition 4.5} Assume that $P_\alpha(D) = C.D - 1$. If the following conditions are satisfied, then $\mathcal{O}_X(D)$ is aCM and initialized.

(a) $C.D = 1$ or $4 \leq C.D \leq 8$.
(b) If $C.D = 7$, then $h^0(\mathcal{O}_X(2C - D)) = 1$.
(c) If $C.D = 8$, then $h^0(\mathcal{O}_C(D - C)) = 0$ and $h^0(\mathcal{O}_C(D)) = 3$.

\textit{Proof.} First of all, we show that $\mathcal{O}_X(D)$ is initialized. If $C.D = 1$ or $4$, then $C.(D - C) < 0$, and hence, we have $|D - C| = \emptyset$. We consider the case where $5 \leq C.D \leq 7$. Assume that $|D - C| \neq \emptyset$. If $C.D = 5$, then $\mathcal{O}_X(D) \cong \mathcal{O}_X(C)$. This contradicts the assumption that $P_\alpha(D) = 4$. If $C.D = 6$, then $(D - C)^2 = -5$ and $C.(D - C) = 1$. This contradicts Lemma 2.1. If $C.D = 7$, then by the assumption (b), $|2C - D| \neq \emptyset$. Hence, if we let $\Gamma$ be the member of $|D - C|$, $\Gamma$ is contained by a hyperplane in $\mathbb{P}^3$. Since $\Gamma.C = 2$ and $\Gamma^2 = -6$, by Lemma 2.2, this is a contradiction. Hence, $|D - C| = \emptyset$. If $C.D = 8$, then $C.(D - 2C) = -2$, and hence, $|D - 2C| = \emptyset$. By the assumption (c) and the exact sequence (4.6), we have $h^0(\mathcal{O}_X(D - C)) = 0$. If $n \geq 3$, then $C.D + 5 < 5n$. Hence, it is sufficient to show that $h^1(\mathcal{O}_X(lC - D)) = 0$ for $0 \leq l \leq 3$.

We show that $h^1(\mathcal{O}_X(C - D)) = 0$ and $h^0(\mathcal{O}_X(D)) = h^0(\mathcal{O}_C(D))$. If $C.D = 1$, by the exact sequence (4.7), we have $h^0(\mathcal{O}_X(D)) = h^0(\mathcal{O}_C(D)) = 1$. By Lemma 2.1, we have $h^1(\mathcal{O}_X(C - D)) = h^1(\mathcal{O}_X(D)) = 0$.

Assume that $C.D = 4$. Since $C.(C - D) = 1$, we have $h^0(\mathcal{O}_C(C - D)) \leq 1$. By the exact sequence (4.8), we have $h^0(\mathcal{O}_X(C - D)) \leq 1$. Since $\chi(\mathcal{O}_X(D)) = 3$, we have $h^0(\mathcal{O}_X(D)) \geq 2$. By Remark 2.1 and the exact sequence (4.7), we have $h^0(\mathcal{O}_X(D)) = h^0(\mathcal{O}_C(D)) = 2$. Hence, we have $h^0(\mathcal{O}_X(C - D)) = 1$ and $h^1(\mathcal{O}_X(C - D)) = h^1(\mathcal{O}_X(D)) = 0$.

We consider the case where $5 \leq C.D \leq 8$. By Proposition 2.1, we have $|C - D| = \emptyset$. Hence, $h^0(\mathcal{O}_X(D)) \geq 3$. We have $h^0(\mathcal{O}_X(D)) = 3$. Indeed, if $C.D = 5$, then $h^0(\mathcal{O}_X(D)) = h^0(\mathcal{O}_C(D)) = 3$, by Remark 2.1 and the exact sequence (4.7). Assume that $6 \leq C.D \leq 7$. Since $\mathcal{O}_X(D)$ is initialized, $h^0(\mathcal{O}_X(2C - D)) \geq \chi(\mathcal{O}_X(2C - D)) = 8 - C.D$. By the exact sequence

$$0 \longrightarrow \mathcal{O}_X(C - D) \longrightarrow \mathcal{O}_X(2C - D) \longrightarrow \mathcal{O}_C(2C - D) \longrightarrow 0,$$
we have $h^0(\mathcal{O}_C(2C - D)) \geq 8 - C.D$. Since $C.(2C - D) = 10 - C.D$, by Remark 2.1, we have $h^0(\mathcal{O}_C(2C - D)) = 8 - C.D$. Since $K_C \cong \mathcal{O}_C(2)$, by the Riemann-Roch theorem, we have $h^0(\mathcal{O}_C(D)) = 3$. By the exact sequence (4.7), we have $h^0(\mathcal{O}_X(D)) = 3$. If $C.D = 8$, by the assumption (c) and the exact sequence (4.7), we have $h^0(\mathcal{O}_X(D)) = 3$. Hence, we have $h^1(\mathcal{O}_X(C - D)) = h^1(\mathcal{O}_X(D)) = 0$. For each case as above, by using the exact sequence (4.7), we have $h^1(\mathcal{O}_X(2C - D)) = h^1(\mathcal{O}_X(D - C)) = 0$. Since $P_a(D) \geq 0$, by Corollary 2.1, we have $h^1(\mathcal{O}_X(-D)) = 0$.

We show that $h^1(\mathcal{O}_X(3C - D)) = 0$. If $C.D = 1, 4,$ or $8$, then we have $h^0(\mathcal{O}_C(D - C)) = 0$. Hence, by the exact sequence (4.6), we have

$$h^1(\mathcal{O}_X(3C - D)) = h^1(\mathcal{O}_X(D - 2C)) = 0.$$ 

Assume that $5 \leq C.D \leq 7$. Then $\chi(\mathcal{O}_X(2C - D)) = 8 - C.D > 0$. Since $\mathcal{O}_X(D)$ is initialized and $h^1(\mathcal{O}_X(2C - D)) = 0$, we have $h^0(\mathcal{O}_X(2C - D)) > 0$. Since $|C - D| = 0$ and $(2C - D)^2 \geq -5$, if we apply Corollary 2.1 to $\mathcal{O}_X(2C - D)$, we have $h^1(\mathcal{O}_X(3C - D)) = h^1(\mathcal{O}_X(D - 2C)) = 0$. Hence, the assertion follows. □

**Proposition 4.6** Let $k = C.D + 1 - P_a(D)$. Assume that $0 \leq k \leq 1$ and $C.D = 10 - k$. If $h^0(\mathcal{O}_C(D - C)) = 0$, then $\mathcal{O}_X(D)$ is aCM and initialized.

**Proof.** First of all, we show that $\mathcal{O}_X(D)$ is initialized. First of all, we have $|D - 2C| = \emptyset$. In fact, since $C.(D - 2C) = -k$, if $|D - 2C| \neq \emptyset$, then we have $k = 0$ and $\mathcal{O}_X(D) \cong \mathcal{O}_X(2C)$. This contradicts the assumption that $P_a(D) = 11$. By the exact sequence (4.6) and the assumption that $h^0(\mathcal{O}_C(D - C)) = 0$, we have $h^0(\mathcal{O}_X(D - C)) = 0$. If $n \geq 4$, then we have $C.D + 5 < 5n$. Hence, it is sufficient to show that $h^1(\mathcal{O}_X(lC - D)) = 0$ for $0 \leq l \leq 4$.

We show that $h^1(\mathcal{O}_X(C - D)) = 0$. Since $h^0(\mathcal{O}_C(D - C)) = 0$ and $0 \leq k \leq 1$, we have $h^0(\mathcal{O}_C(2C - D)) = 0$. By the Riemann-Roch theorem, we have $h^0(\mathcal{O}_C(D)) = 5 - k$. On the other hand, since $C.(C - D) = k - 5 < 0$, we have $|C - D| = \emptyset$. Hence, we have $h^0(\mathcal{O}_X(D)) \geq \chi(\mathcal{O}_X(D)) = 5 - k$. By the exact sequence (4.7), we have $h^0(\mathcal{O}_X(D)) = 5 - k$. Therefore, we have $h^1(\mathcal{O}_X(C - D)) = h^1(\mathcal{O}_X(D)) = 0$. By the exact sequence (4.8), we have $h^1(\mathcal{O}_X(-D)) = 0$. By the exact sequence (4.6) and (4.7), we have $h^1(\mathcal{O}_X(lC - D)) = 0$ for $2 \leq l \leq 3$. If $k = 1$, then since $C.D + 5 < 15$, by Proposition 3.2, the assertion is clear. Assume that $k = 0$. Since $h^0(\mathcal{O}_C(D - C)) = 0$, we have $h^0(\mathcal{O}_C(D - 2C)) = 0$ and hence, by the exact sequence

$$0 \longrightarrow \mathcal{O}_X(D - 3C) \longrightarrow \mathcal{O}_X(D - 2C) \longrightarrow \mathcal{O}_C(D - 2C) \longrightarrow 0,$$

we have $h^1(\mathcal{O}_X(4C - D)) = h^1(\mathcal{O}_X(D - 3C)) = 0$. Hence, we have the assertion. □

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