Uhlenbeck compactness for Yang–Mills flow in higher dimensions

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Abstract
This paper proves a general Uhlenbeck compactness theorem for sequences of solutions of Yang–Mills flow on Riemannian manifolds of dimension $n \geq 4$, including rectifiability of the singular set at finite or infinite time.

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1 Introduction

1.1 Background

This article generalizes the well-known sequential compactness theorems for Yang-Mills connections, due to Uhlenbeck [31] and Nakajima [23], to solutions of the Yang–Mills flow

$$\frac{\partial A}{\partial t} = -D^*_A F_A.$$  \hspace{1cm} (YM)

Here $A(t)$ is a time-dependent family of connections on a vector bundle $E$ over a Riemannian manifold. For background on Yang–Mills flow (YM), we refer the reader to the textbook of Donaldson and Kronheimer [13], §6, or [32], §2. Detailed expository treatments of Uhlenbeck’s compactness theory for gauge fields (connections) have appeared in [13], §2 and 4, and the textbook of Wehrheim [34].

Several notable compactness theorems have been established for geometric flows, building on prior work for the corresponding elliptic equation. The first is due to Brakke [6] in the case of mean curvature flow, generalizing Allard’s compactness theorem [2] for minimal submanifolds with locally bounded area. Hamilton [16] proved a smooth compactness theorem for solutions of the Ricci flow with uniformly bounded Riemann tensor and positive injectivity radius, which has been substantially extended by Chen and Wang [9, 10] (see also Bamler [4, 5]). These are parabolic analogues of the celebrated compactness theory for manifolds with controlled Ricci tensor, due to Cheeger [7], Gromov [14], Anderson [3], Cheeger-Colding-Tian [8], and others. In the case of harmonic maps and harmonic map flow, respectively, the most general compactness theorems are due to Lin [20] and Lin-Wang [21] (see also the textbook [22]).

Compactness results of this kind are primarily used for analyzing individual solutions of the flow—meaning, the structure of finite-time singular sets, infinite-time convergence, and (the ubiquitous) blowup arguments. In the context of Yang–Mills flow on compact Kähler manifolds [27, 28], one typically relies on the work of Hong and Tian [17]. Our main motivation is to strengthen the results of [17] in order to apply them outside the Kähler context. Specifically, our analysis includes the case of finite-time blowup as well as infinite-time blowup in the absence of a holomorphic structure. These results have been used to obtain a partial characterization of the infinite-time singular set for Yang–Mills flow on manifolds of special holonomy [24].

As far as the analysis is concerned, the case of (YM) differs from those discussed above in that certain aspects are simpler—for instance, we are able to prove rectifiability of the singular set using a Laplace-transform trick combined with Preiss’s Theorem—while other aspects are more difficult, especially those related to the gauge freedom of the underlying bundle.

1.2 Statement of results

**Theorem 1.1** Let $M$ be a Riemannian manifold (without boundary) of dimension $n \geq 4$. Fix $0 < \tau < \infty$, and let $\{A_i(x, t)\}$ be a sequence of smooth solutions of (YM) on $M \times [0, \tau)$. Writing $F_i(t) = F_{A_i(t)}$, assume that for any compactly contained open set $U \subset M$, there

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1 Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.
holds
\[
\sup_{i \in \mathbb{N}} \int_U |F_i(t)|^2 \,dV < \infty. \tag{1.1}
\]

For \(\varepsilon_0 > 0\) sufficiently small (depending only on \(n\)), define
\[
\Sigma = \{x \in M \mid \liminf_{R \to 0} \liminf_{i \to \infty} \Phi(A_i; R, x, \tau - R^2) \geq \varepsilon_0\} \tag{1.2}
\]
where \(\Phi\) is the weighted energy functional given by (2.2) below. Then, \(\Sigma\) is closed and of locally finite \((n-4)\)-Hausdorff measure.

Let \(\tau_i \nearrow \tau\). Choose any subsequence, again denoted by \([i]\), such that the limit of measures
\[
\mu = \lim_{i \to \infty} \left( |F_i(x, \tau_i)|^2 \,dV \right)
\]
extists, and redefine \(\Sigma\) according to (1.2). If, for each \(U \Subset M\), there holds
\[
\lim_{\sigma \nearrow \tau} \liminf_{i \to \infty} \int_{\tau \sigma} \int_U |D^*F_i|^2 \,dV \,dt = 0 \tag{1.3}
\]
then \(\Sigma\) is \((n-4)\)-rectifiable, i.e., consists of a countable union of \((n-4)\)-dimensional Lipschitz submanifolds (up to \(\mathcal{H}^{n-4}\)-measure zero).

**Remark 1.2** Afuni [1] has independently obtained the finiteness of \(\mathcal{H}^{n-4}(\Sigma)\).

**Theorem 1.3** Let \([A_i]\) be as in Theorem 1.1, satisfying (1.1). Passing to a further subsequence, there exists a smooth connection \(A_\infty\) on a bundle \(E_\infty \to M \setminus \Sigma\), and an exhaustion of open sets
\[
U_1 \Subset \cdots \Subset U_i \Subset U_{i+1} \Subset \cdots \subset M \setminus \Sigma, \quad \bigcup_{i=1}^{\infty} U_i = M \setminus \Sigma,
\]
Together with bundle maps \(u_i : E|_{U_i} \to E_\infty|_{U_i}\) (independent of time), as follows. For any sequence of times \(\tau_i \nearrow \tau\), we have
\[
u_i (A_i(t_i)) \to A_\infty \text{ in } C^\infty_{\text{loc}}(M \setminus \Sigma). \tag{1.4}
\]
Moreover, if
\[
\int_0^\tau \int_{U_k} |D^*F_i|^2 \,dV \,dt \to 0 \text{ as } i \to \infty \tag{1.5}
\]
for each \(k\), then \(A_\infty\) is a Yang–Mills connection, and
\[
u_i (A_i) \to A_\infty \text{ in } C^\infty_{\text{loc}}((M \setminus \Sigma) \times (0, \tau)), \tag{1.6}
\]
where \(A_\infty\) is the constant solution of (YM) equal to \(A_\infty\).

**Corollary 1.4** (Cf. Hong-Tian [17], Theorem A) Fix an arbitrary sequence \(t_i \nearrow \infty\), and let \(A(t)\) be a smooth solution of (YM) on \(M \times [0, \infty)\), with \(M\) compact. After passing to a subsequence of \(t_i\), there exists a closed, \((n-4)\)-rectifiable set \(\Sigma \subset M\), together with an Uhlenbeck limit \(A_\infty\), which is a smooth Yang–Mills connection on \(E_\infty \to M \setminus \Sigma\), such that
\[
u_i (A(t_i + t)) \to A_\infty \text{ in } C^\infty_{\text{loc}}((M \setminus \Sigma) \times \mathbb{R}).
\]
Remark 1.5 Rescaling arguments at the blowup set of (YM) have been carried out by several authors, including Schlatter [26] in dimension four, and Weinkove [35] for Type-I singularities in higher dimensions. The next theorem follows from the refined blowup analysis due to Lin [20], Lin-Wang [21], and Tian [30], applied to (YM) in the Kähler context by Hong and Tian [17]. The recent paper by Kelleher and Streets [18, 19], §6, also contains an essentially correct proof of the following result.

Theorem 1.6 (Cf. Kelleher-Streets [18], §6, Hong-Tian [17], Theorem C, Lin-Wang [22], §8.5, Tian [30], §3–4) Let \( A_i \) be as in Theorem 1.1, satisfying (1.1) and (1.3). For \( H \nabla^4 \)-a.e. \( x_0 \in \Sigma \), we may pass to a subsequence for which there exist \( x_i \to x_0 \), \( t_i \nearrow \tau \), \( \lambda_i \searrow 0 \), and gauge transformations \( u_i \), such that

\[
\lambda_i u_i (A_i) (x_i + \lambda_i x, t_i + \lambda_i^2 t) \to B(x) \text{ in } C^\infty_{\text{loc}}(\mathbb{R}^{n+1}).
\]

Here, \( B \) is a constant solution of (YM) on \( T_{x_0} M \cong \mathbb{R}^n \), which is the product of a flat connection on \( T_{x_0} \Sigma \) with a nontrivial finite-energy Yang–Mills connection on \( (T_{x_0} \Sigma) \perp \cong \mathbb{R}^4 \).

Remark 1.7 In the case that \( T < \infty \) and \( A_i(t) = A(t) \) is a single smooth solution of (YM) over \( M \times [0, T) \) , the assumption (1.3) is automatically satisfied for \( \tau = T \), and we conclude from Theorem 1.1 that the finite-time singular set is rectifiable. The proof (based on Lemma 2.5) also works for the case of harmonic map flow, where this result has not appeared in the literature. Based on [33], however, it is natural to conjecture that for (YM), the result is vacuously true.

Conjecture 1.8 Let \( A(t) \) be a smooth solution of (YM) on \( M \times [0, T) \) , with \( T < \infty \), satisfying

\[
\sup_{0 \leq t < T} \int_M |F(t)|^2 \, dV < \infty.
\]

Then, at time \( \tau = T \), \( \mathcal{H}^{n-4}(\Sigma) = 0 \) and the defect measure (see (3.9) below) vanishes identically.

2 Technical results

2.1 Hamilton's monotonicity formula

We recall the basic monotonicity formula for (YM) in higher dimensions, due to Hamilton [15].

For \( x, y \in M \) and \( R > 0 \), let

\[
u_{R,x}(y) = \frac{R^{4-n}}{(4\pi)^{n/2}} \exp \left( - \left( \frac{d(x, y)}{2R} \right)^2 \right).
\]

Fix a smooth cutoff function \( \varphi(r) \) supported on the unit interval, with \( \varphi(r) \equiv 1 \) on \([0, 1/2] \), and let

\[
\varphi_{x,\rho}(y) = \varphi \left( \frac{d(x, y)}{\rho} \right)
\]

for \( \rho > 0 \). Also let

\[
\rho_1(x) = \min \left[ \frac{\text{inj}(M, x)}{2}, \sqrt{\tau}, 1 \right]
\]
and
\[ \varphi_x(y) = \varphi_{x, \rho_1(x)}(y). \]

Given a solution \( A = A(t) \) of (YM), define
\[ \Phi(A; R, x, t) = \int_M |F_{A(t)}(y)|^2 u_{R, x}(y) \varphi_x^2(y) \, dV_y \quad (2.2) \]
\[ \Sigma(A; R, x, t_1, t_2) = \int_{t_1}^{t_2} \int_M |D^{*}_{A(t)} F_{A(t)}(y)|^2 u_{R, x}(y) \varphi_x^2(y) \, dV_y \, dt. \quad (2.3) \]

We shall typically suppress \( A \) and write
\[ F(t) = F_{A(t)}, \quad D^* F(t) = D^*_{A(t)} F_{A(t)}, \text{ etc.} \]

Let \( U \subset U_1 \subset M \) be compactly contained open submanifolds such that
\[ B_{\rho_1}(x) \subset U_1 \quad (2.4) \]
for all \( x \in U \). In keeping with (1.1), we shall assume
\[ \sup_{0 \leq t < \tau} \int_{U_1} |F(t)|^2 \, dV \leq E \quad (2.5) \]
for a constant \( E > 0 \).

**Theorem 2.1** (Hamilton [15]) *For* \( R_0 \geq R_1 \geq R_2 \geq 0 \), \( R_1^2 \leq t < \tau \), *and any* \( x \in U \), *we have*
\[ \Phi(R_2, x, t) \leq e^{C_0(R_1 - R_2)} \Phi(R_1, x, t - R_1^2 + R_2^2) + C_1 R_1^2 (R_1^2 - R_2^2) E. \quad (2.6) \]
*Here* \( C_0 \) *and* \( C_1 \) *depend on the geometry of* \( M \) *near* \( U \). *In particular, for any* \( \epsilon > 0 \), *taking* \( R_0 \) *sufficiently small, we have*
\[ \Phi(R_2, x, t) \leq (1 + \epsilon) \Phi(R_1, x, t - R_1^2 + R_2^2) + \epsilon. \quad (2.7) \]

**Proof** This version of Hamilton’s formula corresponds to the case \( \gamma = 1 \) in Theorem 5.7 of [24]. \[ \square \]

### 2.2 Basic estimates for Yang–Mills flow

This section adapts to general dimension several basic results for Yang–Mills flow in dimension four, from [32], §3 and [33], §3. These estimates have not appeared in the literature in this level of detail.

Let \( U \subset U_1 \subset M \) satisfy (2.4) as above.

**Lemma 2.2** Let \( E_0 > 0 \), *and* \( k \in \mathbb{N} \). *There exists a constant* \( \epsilon_0 > 0 \), *depending on* \( E_0 \) *and* \( n \), *as well as* \( R_0 \), *depending on* \( E_0, R_0 \), *and the geometry of* \( M \) *near* \( U \), *as follows. Let* \( A(t) \) *be solution of* (YM) *on* \( M \times [0, T] \), *satisfying* (2.5). *Assume that for some* \( 0 < R < R_0 \), *and* \( R^2 \leq t_0 < T \), *there hold*
\[ \Phi \left( 2R, x, t_0 - R^2 \right) \leq E_0, \quad \Phi \left( R, x, t_0 - R^2 \right) < \epsilon_0. \quad (2.8) \]

---

\(^2\) We find this formulation of Lemma 2.2 to be more natural than the corresponding version in which \( \epsilon_0 \) depends only on \( n \) (see Theorem 6.4 of [24]). In the proof of Theorems 1.1–1.3, the dependence on \( E_0 \) can easily be removed.
Then, for $t_0 - \frac{R^2}{4} \leq t < t_0$, we have
\[
\|F(t)\|_{C^0(B_{R/2})} \leq \frac{C_n}{R^2}.
\] (2.9)

Letting
\[
\sup_{t_0 - R^2 \leq t < t_0} R^{4-n} \int_{B_R(x)} |F(t)|^2 \, dV = \epsilon, \quad R^{4-n} \int_{t_0 - R^2}^{t_0} \int_{B_R(x)} |D^* F|^2 \, dV \, dt = \delta^2
\] (2.10)
then for $t_0 - \frac{R^2}{4} \leq t < t_0$, we also have
\[
\|\nabla^{(k)} F(t)\|_{C^0(B_{R/4}(x))} \leq C_{k,n} R^{-2-k} \sqrt{\epsilon}, \quad \|\nabla^{(k)} D^* F(t)\|_{C^0(B_{R/4}(x))} \leq C_{k,n} R^{-3-k} \delta.
\] (2.11)

**Proof** The estimate (2.9) follows from (2.8) by the $\epsilon$-regularity theorem (see [11], [29], or Theorem 6.2 of [24] for this version). Upon rescaling $B_{R/2}$ to a unit ball $\tilde{B}_1$ and letting $t_0 = 0$, (2.9) becomes the uniform bound
\[
\sup_{\tilde{B}_1 \times [-1,0]} |F(x,t)| \leq C.
\]
The assumptions (2.10) become
\[
\sup_{-1 \leq t < 0} \|F(t)\|^2_{L^2(\tilde{B}_1)} = \epsilon, \quad \int_{-1}^{0} \int_{\tilde{B}_1} |D^* F|^2 \, dV \, dt = \delta^2.
\]
The scale-invariant estimates (2.11) now follow from the standard Moser iteration and bootstrapping argument of [32], Proposition 3.2. \qed

**Lemma 2.3** For $0 < R < R_0$, $0 \leq t_1 < t_2 < T$, and $x \in U$, assume
\[
\sup_{t_1 \leq t \leq t_2} \Phi(2R, x, t) \leq E_0, \quad \Xi(R, x, t_1, t_2) \leq \xi^2
\] (2.12)
and put
\[
\gamma = \xi \left( \frac{\sqrt{(t_2 - t_1)} E_0}{R} \right).
\]
Then
\[
|\Phi(R, x, t_2) - \Phi(R, x, t_1)| \leq C \gamma.
\] (2.13)
For $0 < \epsilon < \epsilon_0$, if
\[
\Phi(R, x, t_2) + \gamma \leq \epsilon \quad \text{or} \quad \Phi(R, x, t_1) + \gamma \leq \epsilon
\] (2.14)
then, for $t_1 + \frac{3}{4} R^2 \leq t \leq t_2$, there hold
\[
\|\nabla^{(k)} F(t)\|_{C^0(B_{R/4}(x))} \leq C_{k,n} R^{-2-k} \sqrt{\epsilon}, \quad \|\nabla^{(k)} D^* F(t)\|_{C^0(B_{R/4}(x))} \leq C_{k,n} R^{-3-k} \xi.
\] (2.15)

**Proof** Integrating by parts once against $u_{R,x} \varphi_x^2$ in the pointwise energy identity (2.4) of [33], and integrating in time, we obtain
\[
\Phi(R, x, t_2) - \Phi(R, x, t_1) = -2 \Xi(R, x, t_1, t_2)
\]
\[
- 2 \int_{t_1}^{t_2} \left( <D^* F^i, F_{i,j}> \left( \nabla^j u_{R,x} \varphi_x^2 + 2 u_{R,x} \varphi_x \nabla^j \varphi_x \right) \right) \, dV \, dt.
\] (2.16)
\[\Box\] Springer
Letting \( r = d(x, y) \), we have
\[
\left| \nabla u_{R,x} \right| = \frac{R^{4-n}}{(4\pi)^{n/2}} \frac{r}{2R^2} \exp - \left( \frac{r}{2R} \right)^2 = \frac{1}{R} \sqrt{u_{R,x}} \left( \frac{R^{4-n}}{4\pi} \right)^{1/2} \frac{r}{2R} \exp - \frac{r^2}{8R^2} \\
\leq \frac{C}{R} \sqrt{u_{R,x}} \sqrt{u_{2R,x}} \frac{r}{R} \exp - \frac{r^2}{16R^2} \tag{2.17}
\]

We may therefore apply Hölder’s inequality to estimate
\[
\left| \int_{t_1}^{t_2} \langle D^* F^i, F_{ij} \rangle \nabla^j u_{R,x} \varphi_x^2 \, dV \, dt \right| \\
\leq C \sqrt{\mathcal{E}(R, x, t_1, t_2)} \sqrt{\int_{t_1}^{t_2} \Phi(2R, x, t) \, dt} \\
\leq C \frac{\xi}{R} \sqrt{(t_2 - t_1) \mathcal{E}_0}. \tag{2.18}
\]

Next, we estimate
\[
\left| \int_{t_1}^{t_2} \langle D^* F^i, F_{ij} \rangle u_{R,x} \varphi_x \nabla^j \varphi_x \, dV \, dt \right| \\
\leq C \sqrt{\mathcal{E}(R, x, t_1, t_2)} \sqrt{\int_{t_1}^{t_2} |F| u_{R,x} |\nabla \varphi_x|^2 \, dV \, dt} \tag{2.19}
\]
\[
\leq C \xi e^{-\frac{\rho_1(x)^2}{4R^2}} \rho_1(x)^{-1} \sqrt{(t_2 - t_1) \mathcal{E}_0}.
\]

We may assume that \( R_0 \) is sufficiently small that
\[
e^{-\frac{\rho_1(x)^2}{4R_0^2}} \leq \rho_1(x)
\]
for all \( x \in U \). Then, inserting (2.18–2.19) into (2.16) yields (2.13).

Under the assumption (2.14), we conclude from (2.13) that
\[
\Phi(R, x, t) \leq C e
\]
for \( t_1 \leq t \leq t_2 \). The desired bounds (2.15) now follow from Lemma 2.2. \( \square \)

**Lemma 2.4** Fix \( 0 < \tau_0 < \tau \leq T \). Let \( K, \delta > 0 \) and \( \tau_0 \leq t_i < \tau \), for \( i = 1, 2 \), be such that
\[
\delta^2 |t_2 - t_1| \leq K^2. \tag{2.20}
\]

Assume that
\[
\sup_{x \in U_1, 0 \leq t < \tau} |F(x, t)| \leq K \tag{2.21}
\]
and
\[
\int_{U_1} |D^* F|^2 \, dV \, dt \leq \delta^2. \tag{2.22}
\]

Then
\[
\|A(t_2) - A(t_1)\|_{C^0(U)} \leq C_{2.4} \delta \sqrt{|t_2 - t_1|}. \tag{2.23}
\]
Fixing a reference connection $\nabla_{\text{ref}}$ on $E$ and defining the $C^k$ norms accordingly, for $k \in \mathbb{N}$, we have
\[
\|A(t_2) - A(t_1)\|_{C^k(U)} \leq C_{2.4} \delta \sqrt{|t_2 - t_1|} \left( 1 + \|A(t_1)\|_{C^{k-1}(U)}^k \right). \tag{2.24}
\]
The constants $C_{2.4}$ depend on $K$, $k$, $\tau_0$, $\nabla_{\text{ref}}$, and the geometry of $M$ near $U$.

**Proof** In this proof, the constant $C$ will have the dependence of $C_{2.4}$. We shall assume $t_1 \leq t_2$, since the opposite case follows by a similar argument.

First note that by covering $U$ with finitely many balls and applying Lemma 2.2, for any $\tau_0 \leq t < \tau$, we may obtain an estimate
\[
\|D^* F(t)\|_{C^0(U)} \leq C \|D^* F\|_{L^2(U_1 \times [t-\tau_0, t])}. \tag{2.25}
\]
To prove (2.23), using (2.25) and Hölder’s inequality, we calculate
\[
\|A(t_2) - A(t_1)\|_{C^0(U)} \leq \int_{t_1}^{t_2} \|D^* F(t)\|_{C^0(U)} \, dt
\leq C \int_{t_1}^{t_2} \|D^* F\|_{L^2(U_1 \times [t-\tau_0, t])} \, dt
\leq C(t_2 - t_1)^{1/2} \left( \int_{t_1}^{t_2} \|D^* F\|^2_{L^2(U_1 \times [t-\tau_0, t])} \, dt \right)^{1/2}
\leq C(t_2 - t_1)^{1/2} \left( \int_{t_1}^{t_2} \int_{t-\tau_0}^t \|D^* F(s)\|^2_{L^2(U_1)} \, ds \, dt \right)^{1/2}.
\]
The domain of integration
\[
t - \tau_0 \leq s \leq t, \quad t_1 \leq t \leq t_2
\]
may be relaxed to
\[
0 \leq s \leq \tau, \quad s \leq t \leq s + \tau_0.
\]
Then (2.26) becomes
\[
\|A(t_2) - A(t_1)\|_{C^0(U)} \leq C(t_2 - t_1)^{1/2} \int_{0}^{\tau} \|D^* F(s)\|^2_{L^2(U_1)} \, ds
\leq C(t_2 - t_1)^{1/2} \delta
\]
which is (2.23).

Next, we calculate as follows:
\[
\partial_t \nabla_{\text{ref}} A = -\nabla_{\text{ref}} D^* F
= -\nabla_A D^* F + A\# D^* F,
\]
\[
\partial_t \nabla_{\text{ref}}^{(2)} A = -\nabla_{\text{ref}}^{(2)} D^* F
= -\nabla_A^{(2)} D^* F + A\# \nabla_A D^* F + \nabla_{\text{ref}} A\# D^* F + A\# A\# D^* F, \text{ etc.}
\]
Continuing in this fashion, we obtain bounds
\[
\|\partial_t \nabla_{\text{ref}}^{(k)} A(t)\|_{C^0(U)} \leq C \left( 1 + \|A(t)\|_{C^{k-1}(U)}^k \right) \sum_{\ell=0}^{k} \|\nabla^{(\ell)} D^* F(t)\|_{C^0(U)} \tag{2.27}
\]
for each $k \in \mathbb{N}$. Integrating (2.27) in time, and applying Lemma 2.3 and Hölder’s inequality as above, we obtain
\begin{equation}
\|A(t_2) - A(t_1)\|_{C^k(U)} \leq C\delta \sqrt{t_2 - t_1} \left( 1 + \sup_{t_1 \leq t \leq t_2} \|A(t)\|_{C^{k-1}(U)}^k \right)
\end{equation}
(2.28)
for $k \in \mathbb{N}$.

To obtain (2.24) from (2.28), we use induction. The base case $k = 0$ is (2.23). Assuming that (2.24) holds for $k - 1$, for any $t_1 \leq t \leq t_2$, we have
\begin{align*}
\|A(t)\|_{C^{k-1}(U)}^k &\leq \left( \|A(t_1)\|_{C^{k-1}(U)}^k + \|A(t) - A(t_1)\|_{C^{k-1}(U)}^{(k-1)t} \right)^k \\
&\leq C \left( 1 + \|A(t_1)\|_{C^{k-1}(U)}^{(k-1)t} \right).
\end{align*}
(2.29)
We have used the induction hypothesis and (2.20) in the second line. Substituting (2.29) into (2.28) gives (2.24) for $k$, completing the induction. \hfill \Box

### 2.3 Weighted density for Preiss’s Theorem

This section contains a lemma that will allow us to appeal directly to Preiss’s Rectifiability Theorem in the parabolic context.

**Lemma 2.5** Let $\mu$ be a locally finite measure on $M^n$, and fix a positive integer $k$. Given a point $x \in M$, define the function
\[ \phi(R) = \frac{1}{R^k} \int_M \exp \left( - \frac{d(x, y)^2}{2R} \right) \varphi^2_\alpha(y) \, d\mu \]
and suppose that $\phi(R)$ is bounded above. Then
\begin{equation}
\lim_{R \searrow 0} \frac{\mu(B_R(x))}{R^k} = \frac{\lim_{R \searrow 0} \phi(R)}{2^k \Gamma \left( \frac{k}{2} + 1 \right)}.
\end{equation}
(2.30)
Here $\Gamma$ is the Euler gamma function.

**Proof** For simplicity, we will suppress the cutoff function $\varphi^2_{\alpha}$ throughout the proof.

Define the increasing function
\[ m(r) = \mu(B_r(x)). \]
(2.31)
Since $\phi(r) \leq E_0$ is bounded above, we clearly have
\[ 0 \leq m(r) \leq e^{1/4} r^{k/2} \phi(r) \leq 2E_0 r^k. \]
(2.32)
For any $C^1$ radial function $f(r)$ with
\[ |f(r)| + r |f'(r)| = O(r^{-k-1}) \text{ as } r \to \infty \]
(2.33)
we have
\[ \int f(r) \, d\mu = - \int_0^\infty f'(r) m(r) \, dr. \]
We first assume that the limit on the LHS of (2.30) exists, so
\[ m(r) \sim L r^k \text{ as } r \to 0 \]
for some \( L > 0 \). Let \( \psi(s) = \exp(-s^2/4) \). Then, by (2.34), we have
\[
\phi(R) = \frac{1}{R^k} \int M \psi \left( \frac{r}{R} \right) d\mu = - \int_0^\infty \psi' \left( \frac{r}{R} \right) \left( \frac{r}{R} \right)^k \frac{m(r)}{r^k} \frac{dr}{R} \]
(2.36)
where \( s = r/R \). Hence
\[
\lim_{R \searrow 0} \phi(R) = -L \int_0^\infty \psi' (s) s^k \, ds
\]
(2.37)
where \( \omega_{k-1} = \text{Vol}(S^{k-1}) \). Applying the formula for \( \omega_{k-1} \) in (2.37), and rearranging, yields (2.30).

Next, we assume that the limit on the RHS of (2.30) exists, and show that the limit on the LHS exists using a Laplace-transform trick.

**Claim 1.** Let \( \chi(x) \) be the characteristic function of the unit interval \([0, 1] \subset \mathbb{R}_{\geq 0}\), and fix an integer \( N_0 > \frac{k+1}{2} \). Given \( \epsilon > 0 \), there exists an integer \( N = N(\epsilon) > N_0 \) and an approximating function \( \tilde{\chi}_\epsilon(x) \), of the form
\[
\tilde{\chi}_\epsilon(x) = \sum_{i=N_0}^{N} a_i \frac{1}{(x+1)^i}
\]
(2.38)
which satisfies
\[
-\epsilon < \tilde{\chi}_\epsilon(x) < 1 + \epsilon \quad (0 \leq x < \infty)
\]
\[
|\tilde{\chi}_\epsilon(x) - \chi(x)| \leq \begin{cases} \epsilon & (0 \leq x \leq 1 - \epsilon) \\ \frac{\epsilon}{\chi^{N_0}} & (1 + \epsilon \leq x < \infty) \end{cases}
\]
(2.39)

**Proof of Claim 1** Let \( u = \frac{1}{x+1} \). We use Weierstrass approximation on the unit interval in the \( u \) variable, as follows. Let \( 0 \leq \sigma_\epsilon \leq 1 \) be a continuous function satisfying
\[
\sigma_\epsilon(u) = \begin{cases} 0 & (0 \leq u \leq \frac{1-\epsilon}{2}) \\ 1 & (\frac{1+\epsilon}{2} \leq u \leq 1) \end{cases}
\]
(2.40)
We may let \( p(u) \) be a polynomial satisfying
\[
|p(u) - u^{-N_0} \sigma_\epsilon(u)| < \epsilon/2 \quad (0 \leq u \leq 1).
\]
(2.41)
Letting 

$$\tilde{\chi}_\epsilon (u) = u^{N_0} p(u)$$

and substituting $u = \frac{1}{x+1}$, we obtain a function $\tilde{\chi}_\epsilon (x)$ of the form (2.38) which satisfies (2.39), as claimed. \hfill \Box

**Claim 2.** Let

$$g_\epsilon (\lambda) = -e^{-1/4\lambda^2} \sum_{i=N_0}^{N} \frac{d_i}{\lambda^2i+1 2^{2i-1}(i-1)!}.$$  \hspace{1cm} (2.42)

Then

$$\tilde{\chi}_\epsilon (x) = \int_0^\infty g_\epsilon (\lambda) \frac{e^{-x/4\lambda^2}}{\lambda^k} \, d\lambda.$$  \hspace{1cm} (2.43)

**Proof of Claim 2** We have the Laplace-transform identity

$$\frac{1}{(x + 1)^i} = \int_0^\infty s^{i-1} e^{-s} \frac{e^{-xs}}{(i-1)!} \, ds.$$  \hspace{1cm} (2.44)

Changing variables $s = \frac{1}{4\lambda^2}$ yields the claim. \hfill \Box

**Claim 3.** Let $x = r^2$, and define

$$\chi_\epsilon (r) = \chi_\epsilon (r^2).$$

The limit

$$L_\epsilon = \lim_{R \to 0} R^{-k} \int \chi_\epsilon \left( \frac{r}{R} \right) \, d\mu$$  \hspace{1cm} (2.45)

exists, and is bounded independently of $\epsilon$.

**Proof of Claim 3** In the $r$ variable, the bounds (2.39) become

$$-\epsilon < \chi_\epsilon (r) < 1 + \epsilon \quad (0 \leq r < \infty)$$

$$|\chi_\epsilon (r) - \chi (r)| \leq \begin{cases} \epsilon & (0 \leq r \leq 1 - \epsilon) \\ \frac{\epsilon}{r^{2N_0}} & (1 + \epsilon \leq r < \infty) \end{cases}.$$  \hspace{1cm} (2.46)

Since $2N_0 > k + 1$, the boundedness follows from (2.32). To show that the limit exists for a given $\epsilon > 0$, from Claim 2, we have

$$R^{-k} \int \chi_\epsilon \left( \frac{r}{R} \right) \, d\mu = R^{-k} \int_0^\infty g_\epsilon (\lambda) \frac{e^{-r^2/4R^2\lambda^2}}{\lambda^k} \, d\lambda \, d\mu$$

$$= \int_0^\infty g_\epsilon (\lambda) \int_0^{R^2} e^{-r^2/4R^2\lambda^2} \frac{d\mu}{(R\lambda)^k} \, d\lambda$$

$$= \int g_\epsilon (\lambda) \phi (\lambda R) \, d\lambda.$$  \hspace{1cm} (2.47)

By assumption, $\phi (\cdot)$ is continuous at zero and bounded, while $g_\epsilon (\lambda)$ is absolutely integrable. Hence (2.46) yields

$$\lim_{R \to 0} R^{-k} \int \chi_\epsilon \left( \frac{r}{R} \right) \, d\mu = \left( \lim_{s \to 0} \phi (s) \right) \int_0^\infty g_\epsilon (\lambda) \, d\lambda.$$
This proves the claim. \hfill \square

**Claim 4.** We have
\[
\int \chi_\epsilon \left( \frac{r}{(1 - \epsilon) R} \right) d\mu - C \epsilon R^k \leq \mu(B_R) \leq \int \chi_\epsilon \left( \frac{r}{(1 + \epsilon) R} \right) d\mu + C \epsilon R^k. \tag{2.47}
\]

**Proof of Claim 4** From \((2.45)\), we have
\[
\chi_\epsilon \left( \frac{r}{(1 - \epsilon) R} \right) \leq \chi \left( \frac{r}{R} \right) + \epsilon \left( \frac{2R}{R + r} \right)^{2N_0} \tag{2.48}
\]
and
\[
\chi \left( \frac{r}{R} \right) \leq \chi_\epsilon \left( \frac{r}{(1 + \epsilon) R} \right) + \epsilon \left( \frac{2R}{R + r} \right)^{2N_0}. \tag{2.49}
\]
From \((2.32)\) and \((2.34)\), we have
\[
\int \epsilon \left( \frac{2R}{R + r} \right)^{2N_0} d\mu \leq C \epsilon \int_0^\infty \frac{(2R)^{2N_0 + k}}{(R + r)^{2N_0 + 1}} dr \leq C \epsilon R^k. \tag{2.50}
\]
Integrating \((2.48)\) and \((2.49)\) in \(r\) and applying \((2.50)\) yields the claim. \hfill \square

To complete the proof of Lemma 2.5, let \(L_\epsilon\) be the limit in Claim 3. Dividing \((2.47)\) by \(R^k\) and taking the limit as \(R \downarrow 0\), we obtain
\[
(1 - \epsilon)^k L_\epsilon - C \epsilon \leq \liminf_{R \downarrow 0} \frac{\mu(B_R)}{R^k} \tag{2.51}
\]
and
\[
\limsup_{R \downarrow 0} \frac{\mu(B_R)}{R^k} \leq (1 + \epsilon)^k L_\epsilon + C \epsilon. \tag{2.52}
\]
Subtracting \((2.51)\) from \((2.52)\) yields
\[
\limsup_{R \downarrow 0} \frac{\mu(B_R)}{R^k} - \liminf_{R \downarrow 0} \frac{\mu(B_R)}{R^k} \leq C \left( 1 + |L_\epsilon| \right) \epsilon.
\]
Since \(\epsilon > 0\) was arbitrary and \(L_\epsilon\) is bounded, we conclude that the limit on the LHS of \((2.30)\) exists.

By the first part of the proof, the two limits must again satisfy \((2.30)\). \hfill \square

**Lemma 2.6** Let \(\mu\) and \(\phi(r)\) be as in Lemma 2.5, and assume \(\phi(r) \leq E_0\) for all \(r > 0\). If, for some \(R > 0\) and \(\epsilon \leq 1\), we have
\[
m(R) = \mu(B_R) \leq \epsilon R^k \tag{2.53}
\]
then
\[
\phi \left( \epsilon^{1/2k} R \right) \leq C_{k,n} \left( \sqrt{\epsilon} + E_0 \exp \left( -\epsilon^{-1/2k} \right) \right). \tag{2.54}
\]
Proof

Let \( 0 \leq \alpha \leq 1 \), and put \( R_1 = \alpha R \). Notice from (2.53) that for \( R_1 \leq r \leq R \), we have

\[
\frac{m(r)}{r^k} = \left( \frac{R}{r} \right)^k R^{-k} m(r) \leq \left( \frac{R}{R_1} \right)^k R^{-k} m(R) \leq \alpha^{-k} \epsilon. \tag{2.55}
\]

Let \( \psi(x) = \exp -x^2/4 \) as above. Then, by (2.34), we have

\[
\phi(R_1) \leq \frac{1}{R_1^k} \int_M \psi \left( \frac{r}{R_1} \right) \, d\mu - \psi(R_1) m(R_1) \frac{r^k}{R_1^k} - \left( \int_{R_1}^{R} + \int_{R}^{\infty} \right) \psi' \left( \frac{r}{R_1} \right) \frac{r^k}{R_1^k} \frac{m(r)}{r^k} \int R_1 \frac{m(R)}{R_1^k} \, dr \leq C \alpha^{-k} \epsilon \left( 1 + \int_1^{\alpha^{-1}} \| \psi'(s) \| s^k ds \right) + CE_0 \int_1^{\alpha^{-1}} \| \psi'(s) \| s^k ds \tag{2.56}
\]

where we have let \( s = r/R_1 \). This gives

\[
\phi(R_1) \leq C \left( \alpha^{-k} \epsilon + E_0 \exp -\left( \alpha^{-2}/S \right) \right). \tag{2.57}
\]

Letting \( \alpha = \epsilon^{1/2k} \) yields the claim. \( \square \)

2.4 Hausdorff-measure estimates

We collect here the Hausdorff estimates which will be used in the proof of Theorems 1.1–1.3.

For a sequence of solutions \( \{ A_i \} \) as in Theorem 1.1, let \( \Phi_i \) and \( \Xi_i \) denote the quantities (2.2–2.3) corresponding to \( A_i \). Write

\[
\Phi = \liminf_{i \to \infty} \Phi_i, \quad \Xi = \liminf_{i \to \infty} \Xi_i. \tag{2.58}
\]

Proposition 2.7

For \( \Sigma \) as in Theorem 1.1, we have \( \mathcal{H}^{n-4}(\Sigma \cap U) < \infty \) for any \( U \subset M \).

Proof

Without loss of generality, we replace \( \Sigma \) by \( \Sigma \cap U \) in the proof. By (1.1), we may assume (2.5).

Notice, from Hamilton’s monotonicity formula (2.7), that there exists \( R_0 > 0 \) such that for any \( 0 < R_1 < R_0 \), and for every \( x \in \Sigma \), we have

\[
\Phi \left( R_1, x, \tau - R_1^2 \right) \geq \frac{\epsilon_0}{2}. \tag{2.59}
\]

Let \( \epsilon > 0 \) be such that the RHS of (2.54) is less than \( \epsilon_0/2 \), and let \( R = \frac{R_1}{\epsilon^{1/2(n-4)}} \). \( \tag{2.60} \)

The contrapositive of Lemma 2.6 and (2.59) imply

\[
\liminf_{i \to \infty} \int_{B_R(x)} \left| F_i \left( \tau - R_1^2 \right) \right|^2 \, dV \geq \epsilon R^{n-4} \tag{2.61}
\]

for any \( x \in \Sigma \).
We now estimate the Hausdorff measure by the argument of Nakajima. By the Vitali covering lemma, we may let \( x_k \in \Sigma \) be such that \( \Sigma \subset \bigcup B_{S_k} (x_k) \) and \( B_{R_k} (x_k) \cap B_R (x_\ell) = \emptyset \) for \( k \neq \ell \). We then have

\[
\mathcal{H}^{n-4}_S (\Sigma) \leq \sum_k (5R)^{n-4} \leq \frac{5^{n-4}}{\epsilon} \sum_k \liminf_{i \to \infty} \int_{B_R (x_k)} |F_i (\tau - R_1^2)|^2 \, dV
\]

\[
\leq \frac{5^{n-4}}{\epsilon} \liminf_{i \to \infty} \sum_k \int_{B_R (x_k)} |F_i (\tau - R_1^2)|^2 \, dV
\]

\[
\leq \frac{5^{n-4}}{\epsilon} \liminf_{i \to \infty} \int_M |F_i (\tau - R_1^2)|^2 \, dV
\]

\[
\leq \frac{5^{n-4} E}{\epsilon}.
\]

Since \( R \) tends to zero with \( R_1 \) (by (2.60)), we are done. \( \square \)

**Proposition 2.8** For \( \tau_i \not\sim \tau \), let \( f(x) = \lim inf_{i \to \infty} |F_i (x, \tau_i)|^2 \). Then

\[
\lim_{R \to 0} \int_M f(y) \mu_{R, x}(y) \varphi_x^2(y) \, dV_y = 0 \quad (2.62)
\]

for all \( x \in M \setminus \Sigma \) and for \( \mathcal{H}^{n-4} \)-a.e. \( x \in \Sigma \).

**Proof** First, note from the \( \epsilon \)-regularity theorem that \( f(x) \) is locally bounded on \( M \setminus \Sigma \). Hence the limit (2.62) is zero if \( x \in M \setminus \Sigma \).

We may again replace \( \Sigma \) by \( \Sigma \cup U \) for an open subset \( U \subset M \). Let \( S_j \subset \Sigma \) be the set of \( x \) such that

\[
\limsup_{R_k \to 0} \int_M f(y) \mu_{R, x}(y) \varphi_x^2(y) \, dV_y \geq \frac{1}{j} \quad (2.63)
\]

We will show that \( \mathcal{H}^{n-4} (S_j) = 0 \).

Let \( \delta > 0 \). Define \( \epsilon_j > 0 \) such that the RHS of (2.54), with \( \epsilon = \epsilon_j \), is equal to \( 1/2j \). By the contrapositive of Lemma 2.6, (2.62) implies that for each \( x \in S_j \), there exists \( 0 < R_x < \delta \) such that

\[
R_x^{4-n} \int_{B_{R_x} (x)} f(y) \, dV_y \geq \epsilon_j \quad (2.64)
\]

Let \( \{ x_k \} \subset \Sigma \) be such that \( \Sigma \subset \bigcup B_{S_k} (x_k) \) and \( B_{R_k} (x_k) \cap B_{R_\ell} (x_\ell) = \emptyset \) for \( k \neq \ell \). By (2.64), we have

\[
\mathcal{H}^{n-4}_S (S_j) \leq S^{n-4} \sum_k R_x^{n-4} \leq \frac{5^{n-4}}{\epsilon_j} \sum_k \int_{B_{R_k} (x_k)} f(y) \, dV_y
\]

\[
\leq \frac{5^{n-4}}{\epsilon_j} \int_{\Sigma_j} f(y) \, dV_y = \frac{5^{n-4}}{\epsilon_j} \int_M \chi_{\Sigma_j^c} f(y) \, dV_y.
\]

Since \( \Sigma \) is closed and of Lebesgue measure zero, \( \chi_{\Sigma_j^c} f(y) \to 0 \) pointwise almost everywhere as \( \delta \to 0 \). Hence, the last integral tends to zero by the dominated convergence theorem. This completes the proof that \( \mathcal{H}^{n-4} (S_j) = 0 \).

The set of \( x \) satisfying (2.62) is the complement of the union of \( S_j \), for \( j = 1, \ldots, \infty \), and therefore has full \( \mathcal{H}^{n-4} \)-measure in \( \Sigma \). \( \square \)
Proposition 2.9 Assuming (1.3), we have
\[ \lim_{\sigma \to \tau} \limsup_{R \to 0} \Xi(R, x, \sigma, \tau) = 0 \] (2.66)
for all \( x \in M \setminus \Sigma \) and for \( \mathcal{H}^{n-4} \)-a.e. \( x \in \Sigma \). Here \( \Xi \) is defined by (2.58).

Proof The proof is similar to that of the previous Proposition, with an extra step. First, note from Lemma 2.2 that the limit (2.66) is zero if \( x \in M \setminus \Sigma \).

Let \( S_j \) be the set of points \( x \in \Sigma \) such that
\[ \limsup_{\sigma \to \tau} \limsup_{R \to 0} \Xi(R, x, \sigma, \tau) \geq \frac{1}{j} \] (2.67)

We will show that \( \mathcal{H}^{n-4}(S_{j,k}) \to 0 \) as \( k \to \infty \), for each fixed \( j \). Since \( S_j \subset S_{j,k} \), this will imply that \( \mathcal{H}^{n-4}(S_j) = 0 \).

Let \( \delta > 0 \). As above, define \( \epsilon_j > 0 \) such that the RHS of (2.54), with \( \epsilon = \epsilon_j \), is equal to \( 1/2j \). By the contrapositive of Lemma 2.6, (2.67) implies that for each \( x \in S_{j,k} \), there exists \( 0 < R_x < \delta \) such that
\[ R_x^{4-n} \liminf_{i \to \infty} \int_{\sigma_i}^{\tau} \int_{B_{R_x}(x)} |D^* F_i|^2 dV dt \geq \epsilon_j. \] (2.68)

Let \( \{x_\ell\} \subset S_{j,k} \) be such that \( S_{j,k} \subset \bigcup B_{5R_{x_\ell}}(x_\ell) \) and \( B_{R_{x_\ell}}(x_\ell) \cap B_{R_{x_m}}(x_m) = \emptyset \) for \( \ell \neq m \). By (2.68), we have
\[ \mathcal{H}^{n-4}_{\mathcal{S}}(S_{j,k}) \leq \sum_\ell \langle 5R_{x_\ell} \rangle^{n-4} \leq \frac{5^{n-4}}{\epsilon_j} \sum_\ell \liminf_{i \to \infty} \int_{\sigma_i}^{\tau} \int_{B_{R_{x_\ell}}(x_\ell)} |D^* F_i|^2 dV dt \]
\[ \leq \frac{5^{n-4}}{\epsilon_j} \liminf_{i \to \infty} \int_{\sigma_i}^{\tau} \int_{B_{R_{x_\ell}}(x_\ell)} |D^* F_i|^2 dV dt \]
\[ \leq \frac{5^{n-4}}{\epsilon_j} \int_{\sigma_i}^{\tau} \int_{B_{R_{x_\ell}}(x_\ell)} |D^* F_i|^2 dV dt \]
\[ \leq \frac{5^{n-4}}{k \epsilon_j}. \]

We may let \( \delta \to 0 \), then let \( k \to \infty \), to conclude that
\[ \lim_{k \to \infty} \mathcal{H}^{n-4}(S_{j,k}) = 0 \]
and therefore \( \mathcal{H}^{n-4}(S_j) = 0 \).

As before, the set of \( x \) satisfying (2.62) is the complement of the union of \( S_j \), for \( j = 1, 2, \ldots \), and therefore has full \( \mathcal{H}^{n-4} \)-measure in \( \Sigma \). \( \square \)
Corollary 2.10 Assuming (1.3), for any $L > 0$, we have

$$\lim_{R \downarrow 0} \mathbb{X} \left( R, x, \tau - LR^2, \tau \right) = 0$$

(2.69)

for all $x \in M \setminus \Sigma$ and for $\mathcal{H}^{n-4}$-a.e. $x \in \Sigma$.

**Proof** For a given $x$, the condition (2.69) is implied by (2.66). $\square$

### 2.5 Gauge-patching lemmas

This section carries out a minor correction to Lemmas 4.4.5–4.4.7 and Corollary 4.4.8 of Donaldson and Kronheimer [13], originally due to Uhlenbeck [31] in a different form.³

**Remark 2.11** As in Lemma 2.4, we fix a reference connection $\nabla_{ref}$ on the bundle $E$, which we use to define the $C^k$ norms on bundle-valued differential forms. By definition, the $C^k$ norm of a connection is equal to $\|A\|_{C^k}$, where $A$ is the unique (global) 1-form such that $\nabla_A = \nabla_{ref} + A$.

**Lemma 2.12** (Cf. [13], Lemma 4.4.5) Suppose that $A_i$ is a sequence of connections on a bundle $E$ over a base manifold $\Omega$, and $\hat{\Omega} \subseteq \Omega$ is an interior domain. Suppose that there are gauge transformations $u_i \in \text{Aut } E$ and $\tilde{u}_i \in \text{Aut } E|_{\hat{\Omega}}$ such that $u_i(A_i)$ converges in $C^\infty_{loc}(\Omega)$ and $\tilde{u}_i(A_i)$ converges over $C^\infty_{loc}(\hat{\Omega})$. Then for any compact set $K \subset \hat{\Omega}$, there exists a subsequence $(j) \subset (i)$, with $j \geq j_0$, and gauge transformations $w_j$ such that

$$w_j =\begin{cases} \tilde{u}_j^{-1}u_j & \text{on } K \\ u_j^{-1}u_j & \text{on } \Omega \setminus \hat{\Omega} \end{cases}$$

(2.70)

and the connections $w_j(A_j)$ converge in $C^\infty_{loc}(\Omega)$.

**Proof** Define the gauge transformations over $\hat{\Omega}$

$$v_i = \tilde{u}_i u_i^{-1}.$$  

These satisfy

$$v_i(u_i(A_i)) = \tilde{u}_i(A_i).$$

(2.71)

Choose an open set $N$ with

$$K \subset N \subseteq \hat{\Omega}.$$  

Since both $u_i(A_i)$ and $\tilde{u}_i(A_i)$ are smoothly convergent, we conclude from the usual bootstrapping argument (e.g. [13], p. 64) applied to (2.71), that $v_i$ are bounded in $C^k(N)$ for all $k$. By the Arzela-Ascoli Theorem, we may extract a convergent subsequence $v_j$. Choosing $j_0$ sufficiently large, we may assume

$$z_j = v_j^{-1}v_j$$

³ Note the phrase “we extend $\tilde{u}|_N$ arbitrarily over $\Omega$...” in the middle of p. 159 of [13]. To see that it is not always possible to extend gauge transformations, consider the trivial $SU(2)$ bundle on $B_1 \subset \mathbb{R}^4$ and the gauge transformation defined over $B_1 \setminus B_1/2$ corresponding to the identity map of $SU(2) \cong S^3$.  

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is arbitrarily close to the identity on $N$ for all $j \geq j_0$. Letting $\xi_j = \log z_j$, and choosing a cutoff $\psi$ for $K \subset N$, we may extend $z_j$ over $\Omega$ by the formula

$$z_j = \exp(\psi \xi_j).$$

Defining

$$w_j = u_{j_0}^{-1} z_j u_j$$

yields the desired sequence of gauge transformations over $\Omega$. \hfill \Box

Lemma 2.13 (Cf. [13], Lemma 4.4.7) Suppose $\Omega$ is a union of domains $\Omega = \Omega_1 \cup \Omega_2$, and $A_i$ is a sequence of connections on a bundle $E$ over $\Omega$. Choose a compactly contained subdomain $\Omega' \subset \Omega$. If there are sequences of gauge transformations $v_i \in \text{Aut } E|_{\Omega_1}$ and $w_i \in \text{Aut } E|_{\Omega_2}$ such that $v_i(A_i)$ and $w_i(A_i)$ converge over $\Omega_1$ and $\Omega_2$, respectively, then there is a subsequence $\{j\}$ and gauge transformations $u_j$ over $\Omega'$ such that $u_j(A_j)$ converges over $\Omega'$.

**Proof** We may assume without loss that $\Omega' \subset \Omega_1' \cup \Omega_2'$, for subdomains $\Omega_1' \subset \Omega_1$ and $\Omega_2' \subset \Omega_2$. Applying Lemma 2.12 with $\Omega = \Omega_1$ and $K = \Omega_1' \cap \Omega_2'$, we obtain gauge transformations $v'_j$ over $\Omega_1$ such that $v'_j = w_{j_0}^{-1} w_j$ over $\Omega_1'$. Then each $v'_j$ glues together with $w_{j_0}^{-1} w_j$ to define gauge transformations $u_j$ over $\Omega'$ such that $u_j(A_j)$ converges, as desired. \hfill \Box

Corollary 2.14 (Cf. [13], Corollary 4.4.8) Suppose $A_i$ is a sequence of connections on $E \to \Omega$ with the following property. For each point $x \in \Omega$, and each subsequence $\{j\} \subset \{i\}$, there is a neighborhood $D$ of $x$, a further subsequence $\{k\} \subset \{j\}$, and gauge transformations $v_k$ defined over $D$ such that $v_k(A_k)$ converges over $D$. Then for any subdomain $\Omega' \subset \Omega$, there is a single subsequence $\{j\}$ and gauge transformations $u_j$ defined over $\Omega'$, such that $u_j(A_j)$ converges over all of $\Omega'$.

**Proof** Choose a finite cover of $\Omega'$ by balls $D_\ell \subset \Omega$ with the stated property. Applying Lemma 2.13 with $\Omega_1 = D_1 \cup \cdots \cup D_{m-1}$ and $\Omega_2 = D_m$, we may cobble together the gauges on $D_\ell$, in finitely many steps, to obtain a sequence of global gauges on $\Omega'$ with the desired property. \hfill \Box

Corollary 2.15 (Cf. [13], Lemma 4.4.6) Suppose that $\Omega$ is exhausted by an increasing sequence of precompact open sets

$$U_1 \subset U_2 \subset \cdots \subset \Omega, \quad \bigcup_{m=1}^\infty U_m = \Omega. \quad (2.72)$$

Let $A_i$ be a sequence of connections on $E \to \Omega$ with the property stated in Corollary 2.14. Then there is a subsequence $\{j\}$, a bundle $E_\infty \to \Omega$, and bundle maps

$$u_j : E|_{U_j} \to E_\infty|_{U_j} \quad (2.73)$$

such that $u_j(A_j)$ converges in $C^\infty_{\text{loc}}(\Omega)$ to a connection $A_\infty$ on $E_\infty \to \Omega$.

**Proof** By Corollary 2.14 and a diagonal argument, we can pass to a subsequence of $A_i$ such that for each $m$, there exist gauge transformations $v_i^m$ defined over $U_m$, for $i \geq m$, such that $v_i^m(A_i)$ converges in $C^\infty(U_m)$ as $i \to \infty$. © Springer
From the $v^m_i$, we construct new sequences $w^m_i$, for $j \geq j_m$, as follows. Let $w^1_i = v^1_i$, for $i \geq 1$. Assuming that $w^m_j$ has been constructed, by applying Lemma 2.12 to the sequence $w^m_j$ and $v^m_{j+1}$, we may obtain a sequence $w^{m+1}_j$ on $U_{m+1}$, for $j \geq j_{m+1}$, such that

$$w^{m+1}_j|_{U_m} = (w^m_{j_{m+1}})^{-1} w^m_j$$  (2.74)

and $w^{m+1}_j(A_j)$ converges on $U_{m+1}$.

We now define the bundle $E_\infty$ as having transition functions $g_{(m+1)m} = w^m_{j_{m+1}}$ from $U_{m+1}$ to $U_m$. We may then define $u_m$ to equal $w^m_{j_m}$ on $U_m$, which gives a well-defined bundle map of the form (2.73). By (2.74), the images $u_m(A_m)$ converge on $E_\infty|_{U_m}$ for each $n$. □

**Remark 2.16** In the case that $U_i$ is a deformation retract of $\Omega$ for sufficiently large $i$, we may take $E_\infty = E|_\Omega$, and the $u_i$ may be assumed to be defined over all of $\Omega$ (see Wehrheim [34]). Such is the case in dimension four, where $\Sigma$ is a finite set of points, and in the Kähler situation in higher dimensions, where $\Sigma$ is a holomorphic subvariety [27].

### 3 Proof of Theorem 1.1

**Proof of Theorem 1.1** As above, we let $\Phi_i$ and $\Xi_i$ denote the quantities (2.2–2.3) corresponding to $A = A_i$, and define $\Phi$ and $\Xi$ by (2.58). Fix $U \Subset U_1 \Subset M$ satisfying (2.4), and let

$$\rho_0 = \inf_{x \in U} \rho_1(x).$$

Then, it suffices to prove closedness and rectifiability of $\Sigma \cap U \subset U$. We replace $\Sigma$ by $\Sigma \cap U$ for the remainder of the proof.

Note that (1.1) implies a bound of the form (2.5) for all $A_i$, with a uniform $E > 0$. Then, by Hamilton’s monotonicity formula (2.7), we have a uniform bound

$$\Phi_i(R, t, x) \leq CE =: E_0$$  (3.1)

for all $0 < R < \rho_0$, $x \in U$, $R^2 \leq t < \tau$, and all $i$.

Closedness of $\Sigma$ follows by adapting the argument of Nakajima [23], as follows. Suppose $\{x_j\} \subset \Sigma$ is a sequence converging to $x \in U$. Let $\epsilon > 0$, and choose $R > 0$ sufficiently small that

$$R^{4-n} \exp - \left( \frac{\rho_0}{4R} \right)^2 < \epsilon.$$  (3.2)

We may fix $j$ sufficiently large that

$$\exp - \left( \frac{d(x, y)}{2R} \right)^2 \varphi_x(y) \geq (1 - \epsilon) \exp - \left( \frac{d(x, y)}{2R} \right)^2 \varphi_{x_j}(y) - C \exp - \left( \frac{\rho_0}{4R} \right)^2.$$  (3.3)

Integrating against $R^{4-n}|F_i(\tau - R^2)|^2$ yields

$$\Phi_i(R, x, \tau - R^2) \geq (1 - \epsilon) \Phi_i(R, x_j, \tau - R^2) - C \left( R^{4-n} \exp - \left( \frac{\rho_0}{4R} \right)^2 \right) E.$$  (3.4)

Now, because $x_j \in \Sigma$, there exists $0 < R' \leq R$ such that for all sufficiently large $i$, we have

$$\Phi_i(R', x_j, \tau - R^2) \geq \epsilon_0 - \epsilon.$$
Applying the monotonicity formula (2.7) yields
\[ \Phi_i(R, x_j, \tau - R^2) \geq \epsilon_0 - 2\epsilon \] (3.5)
provided that \( R < R_0 \). Applying \( \lim \inf \) to both sides of (3.4), and inserting (3.2) and (3.5) yields
\[ \Phi(R, x, \tau - R^2) \geq (1 - \epsilon) (\epsilon_0 - 2\epsilon) - C\epsilon E \]
\[ \geq \epsilon_0 - C\epsilon (\epsilon_0 + E). \] (3.6)
Since \( \epsilon \) was arbitrary, we conclude
\[ \lim \inf_{R \to 0} \Phi(R, x, \tau - R^2) \geq \epsilon_0 \] (3.7)
as desired.

This completes the proof that \( \Sigma \) is closed. Local finiteness of the \( \mathcal{H}^{n-4} \)-measure is shown in Proposition 2.7.

Next, by weak compactness of locally uniformly bounded measures (1.1), we may pass to a subsequence such that the limit of measures
\[ \mu = \lim_{i \to \infty} \left( |F_i(\tau_i)|^2 \right) dV \] (3.8)
exists. By Fatou’s Lemma, we may then write
\[ \mu = \left( \lim \inf_{i \to \infty} |F_i(\tau_i)|^2 \right) dV + \nu \] (3.9)
where \( \nu \) is a nonnegative measure supported on \( \Sigma \).

To show rectifiability assuming (1.3), we claim that
\[ \phi(x) = \lim_{R \to 0} \int_M \exp \left( -\frac{d(x, y)^2}{2R} \right) \varphi^2_\tau(y) \, d\nu \] (3.10)
exists and is nonzero for \( \mathcal{H}^{n-4} \)-a.e. \( x \in \Sigma \).

Let \( \epsilon > 0 \). By (3.9) and Proposition 2.8, we may replace \( d\nu \) with \( d\mu \) in the limit (3.10) for \( \mathcal{H}^{n-4} \)-a.e. \( x \in \Sigma \). Then (3.10) becomes
\[ \phi(x) = \lim_{R \to 0} \lim_{i \to \infty} \Phi_i(R, x, \tau_i) \] (3.11)
where the inner limit exists by (3.8). Given (1.3), by Corollary 2.10, we may assume that \( x \) is a point such that
\[ \lim_{R \to 0} \Xi(R, x, \tau - R^2, \tau) = 0 \] (3.12)
where \( \Xi \) is defined by (2.58).

Then, for \( R > 0 \) be sufficiently small, there exists an infinite subsequence of integers \( \{j\} \) such that
\[ \Xi_j(R, x, \tau - R^2, \tau) < \epsilon^2. \]
By Lemma 2.3, this implies that
\[ \left| \Phi_j(R, x, \tau_j) - \Phi_j(R, x, t) \right| \leq C\epsilon \] (3.13)
for any \( \tau_j - R^2 \leq t \leq \tau_j \).

\( \Xi \) Springer
For any $0 < R' < R$, we may apply the monotonicity formula (2.7) with $R_1 = R$ and $R_2 = R'$, to obtain

$$
\Phi_j (R', x, \tau_j) \leq (1 + \epsilon) \Phi_j (R, x, \tau_j - R^2 + R'^2) + C\epsilon.
$$

Inserting (3.1) and (3.13), we have

$$
\Phi_j (R', x, \tau_j) \leq \Phi_j (R, x, \tau_j) + (C + E_0) \epsilon.
$$

This demonstrates that for $R$ sufficiently small, and any $0 < R' < R$, we in fact have

$$
\lim_{i \to \infty} \Phi_i (R', x, \tau_i) \leq \lim_{i \to \infty} \Phi_i (R, x, \tau_i) + C\epsilon.
$$

Therefore

$$
\limsup_{R \searrow 0} \lim_{i \to \infty} \Phi_i (R, x, \tau_i) \leq \liminf_{R \searrow 0} \lim_{i \to \infty} \Phi_i (R, x, \tau_i) + C\epsilon.
$$

Since $\epsilon > 0$ was arbitrary, this implies that the limit $R \searrow 0$ in (3.11) exists, as claimed.

We may conclude from (3.10), Lemma 2.5, and Preiss’s Theorem [25] (stated as Theorem 1.1 of [12]) that the measure $\nu$ is $(n - 4)$-rectifiable. By (3.9) and Proposition 2.8, $\Sigma = \text{supp } \nu$ up to measure zero, hence the same is true of $\Sigma$.

4 Proof of Theorem 1.3 and Corollary 1.4

**Proof of Theorem 1.3** To construct the required subsequence and exhaustion, we argue as follows.

Given an open subset $U_* \subset (M \setminus \Sigma) \times \mathbb{N}$ (with the box topology), write

$$
\mathcal{I}(U_*) = \pi_2(U_*).
$$

Consider the collection of open subsets

$$
\mathcal{S} \subset \{ U_* \subset (M \setminus \Sigma) \times \mathbb{N} \text{ open} \}
$$

which satisfy the following conditions: for all $i, j \in \mathcal{I}(U_*)$, there hold

$$
\begin{align*}
\sup_{x \in U_i} |F_{A_j}(x, t)| &\leq i \quad (i \leq j) \\
(1 - \frac{1}{j}) \tau \leq t \leq \tau \\
U_i \subseteq U_j \quad (i < j).
\end{align*}
$$

The collection $\mathcal{S}$ is nonempty. For, we may let $x \in M \setminus \Sigma$ and $0 < R < R_0$ be such that a subsequence $\{j\}$ satisfies

$$
\Phi_j (R, x, \tau - R^2) < \epsilon_0
$$

for all $j$. By the $\epsilon$-regularity Theorem 6.4 of [24], there exists $\delta > 0$ such that

$$
\sup_{B_{\delta R}(x) \times [(1 - \delta)\tau, \tau]} |F_{A_j}| \leq \frac{C}{(\delta R)^2}.
$$

We may then let $U_j = B_{(1 - 1/j)\delta R}(x)$ and choose $i = \lceil \frac{C}{(\delta R)^2} \rceil$, so that (4.3) implies (4.2).
Define a partial ordering on $S$ by

$$U_* \leq V_* \text{ if } \bigcup_i U_i \subset \bigcup_i V_i.$$  

Letting

$$U^1_* \leq \cdots \leq U^k_* \leq U^{k+1}_* \leq \cdots$$  

be a chain in $S$, we may construct an upper bound $V_* \in S$ by the following “diagonal” argument. We will construct an increasing sequence of elements $V^k_* \in S$, and then let $V_* = \bigcup_k V^k_*$. 

Assume without loss that $U^1_* \neq \emptyset$, and let $V^1_* = U^1_\ell$ for any $\ell \in I(U^1_*)$. Assuming that $V^k_*$ has been chosen, we may choose $\ell_{k+1} \in I(U^k_{\ell+1})$ with $\ell_{k+1} > \ell_k$, such that

$$\bigcup_{m \leq k \atop i \leq \ell_k} U_i^m \subseteq U^{k+1}_{\ell_{k+1}}. \quad (4.5)$$

We then let

$$V^{k+1}_* = V^k_* \cup U^{k+1}_{\ell_{k+1}}.$$ 

By construction, the resulting set

$$V_* = \bigcup_k V^k_*$$

is an element of $S$. Since $\ell_k \to \infty$ as $k \to \infty$, from (4.5), we have

$$\bigcup_{m,i} U_i^m \subset \bigcup_{k,i} V_i^k = \bigcup_i V_i.$$ 

Therefore, $V_*$ is an upper bound in $S$ for the given chain (4.4), as required.

We conclude from Zorn’s lemma that $S$ contains a maximal element, $W_*$. Defining the singular set $\Sigma W \supset \Sigma$ for the subsequence $I(W_*)$ via (1.2), we claim that $W_*$ is an exhaustion of $M \setminus \Sigma$. For, if there existed $x \in M \setminus (\bigcup_i W_i \cup \Sigma)$, we could choose $R > 0$ and a further subsequence $J \subset I(W_*)$ such that $\sup_{B_R(x), j \in J} |F_{A_j}| \leq i$, for some $i$. But then, defining $V_* \in S$ by

$$V_j = W_j \cup B_{(1-1/j)}(x) \quad (4.6)$$

for all $j \in J$ with $j \geq i$, we conclude that $W_*$ was not maximal, which is a contradiction. Therefore $\bigcup_i W_i = M \setminus \Sigma$, as claimed.

Finally, we pass entirely to the subsequence $I(W_*)$, which we relabel as $\{i\}_i=1^\infty$, and replace $\Sigma$ by $\Sigma$. The assumption (4.2) then becomes, for each $i \in \mathbb{N}$ and a certain $\tau_i < \tau$, the crucial bound

$$\sup_{x \in U_i, \quad \tau_i \leq t < \tau} \quad j \geq i \quad |F_j(x, t)| < \infty. \quad (4.7)$$

By Lemma 2.3, (4.7) may be improved to the derivative estimates

$$\sup_{x \in U_i, \quad \tau_i \leq t < \tau} \quad j \geq i \quad |\nabla^{(k)} F_j(x, t)| < \infty. \quad (4.8)$$
for each \( i, k \in \mathbb{N} \).

With (4.8) now in hand, the construction of the bundle maps \( u_i \) and Uhlenbeck limit \( A_\infty \) follows the standard argument. By the Theorem of Uhlenbeck [31], for each \( x \in M \setminus \Sigma \), there exists a ball \( D \ni x \) and a gauge transformation \( v_j \) on \( D \) such that \( v_j(A_j(\tau_j)) \) is in Coulomb gauge on \( E|_D \), for each \( j \geq i \). From (4.8), the estimates of Donaldson-Kronheimer [13], Lemma 2.3.11, or Wehrheim [34], Theorem 5.5, give uniform \( C^k \) estimates on \( v_j(A_j(\tau_j)) \) over \( D \), for each \( k \in \mathbb{N} \) (where the \( C^k \) norms are defined with respect to a fixed reference connection \( \nabla_{\text{ref}} \), see Remark 2.11). By the Arzela-Ascoli Theorem, there exists a subsequence which is smoothly convergent over \( D \). We claim that given any \( \tau \), there exists a ball \( D \ni \tau \) for \( \exists k \) uniform bounded in \( \tau \), for each \( k \in \mathbb{N} \).

We now turn to the proofs of (1.4) and (1.6). Fix \( k \in \mathbb{N} \); for \( i \geq k \), we have

\[
\| u_i(A_i(t)) - A_\infty \|_{C^k(U_k)} 
\leq \| u_i(A_i(t)) - u_i(A_i(\tau_i)) \|_{C^k(U_k)} + \| u_i(A_i(\tau_i)) - A_\infty \|_{C^k(U_k)}. \tag{4.10}
\]

The second term on the RHS tends to zero with \( i \), by (4.9). Let

\[
\delta_i = \sqrt{\int_{\Omega} \int_{U_{k+1}} |D^* F_i|^2 \, dV \, dt}
\]

which, by the assumption (1.1) and the local energy inequality, are uniformly bounded. By (4.7), for \( i \) sufficiently large and \( \tau_{k+1} \leq t < \tau \), we have

\[
\| F_i(t) \|_{C^0(U_{k+1})} \leq K \tag{4.11}
\]

for some \( K > 0 \).

To prove (1.4), note that \( u_i(A_i(t)) \) are smooth solutions of (YM) on \( E_\infty|_{U_{k+1}} \). Applying Lemma 2.4, we obtain

\[
\| u_i(A_i(t)) - u_i(A_i(\tau_i)) \|_{C^k(U_k)} \leq C \delta_i \sqrt{|\tau_i - t|} \left( 1 + \| u_i(A_i(\tau_i)) \|_{C^{k-1}(U_k)} \right)
\]

for \( \tau_{k+1} \leq t \leq \tau \). We have absorbed the last factor because \( u_i(A_i(\tau_i)) \) is convergent, hence uniformly bounded in \( C^{k-1}(U_k) \). Since the \( \delta_i \) are bounded and \( |\tau_i - t_i| \to 0 \), (4.12) gives

\[
\| u_i(A_i(t_i)) - u_i(A_i(\tau_i)) \|_{C^k(U_k)} \to 0.
\]

Returning to (4.10), we have

\[
\| u_i(A_i(t_i)) - A_\infty \|_{C^k(U_k)} \to 0 \quad \text{as} \quad i \to \infty.
\]

Since \( k \) was arbitrary, this proves (1.4).

To prove (1.6), we now assume (per (1.5)) that

\[
\delta_i \to 0 \quad \text{as} \quad i \to \infty. \tag{4.13}
\]

We claim that given any \( \tau_0 > 0 \), for \( i \) sufficiently large, a bound of the form (4.11) will hold for all \( 0 < \tau_0 \leq t < \tau \). This is easily seen from Lemma 2.3, which may be applied on a cover.
of $U_{k+1}$, in view of (3.1) and (4.13). It also follows from (2.15) that $A_\infty$ is a Yang–Mills connection, since

$$
\| D^* F_{u_i A_i(\tau_i)} \|_{C^0(U_k)} \leq C \| D^* F_{u_i A_i} \|_{L^2(M \times (0,\tau))} \leq C \delta_i \to 0 \text{ as } i \to \infty
$$

and $u_i A_i(\tau_i) \to A_\infty$ in $C^\infty(U_k)$.

We now apply Lemma 2.4, to again obtain the bound (4.12) for $0 < \tau_0 \leq t < \tau$. Since $\delta_i(\tau_i - t) \to 0$, we again obtain

$$
\| u_i(A_i(t)) - u_i(\tau_i) \|_{C^k(U_k)} \to 0
$$

and (4.10) becomes

$$
\| u_i(A_i(t)) - A_\infty \|_{C^k(U_k)} \to 0 \text{ as } i \to \infty.
$$

Since $\tau_0$ and $k$ were arbitrary, this implies (1.6), completing the proof.

**Proof of Corollary 1.4** Since $M$ is compact, we have

$$
\int_M |F(t)|^2 \, dV + 2 \int_0^t \int_M |D^* F(s)|^2 \, dV \, ds = \int_M |F(0)|^2 \, dV
$$

for any $0 \leq t < \infty$. Therefore, for any $\tau > 0$ and $t_i \not\to \infty$, we have

$$
\int_{t_i}^{t_i + \tau} \int_M |D^* F(t)|^2 \, dV \, dt \to 0.
$$

We may therefore apply Theorem 1.3 to the sequence of solutions

$$
A_i(t) = A(t_i + t)
$$

to obtain $A_\infty$ and bundle maps $u_i$ satisfying (1.6) over $[t_i + 1, t_i + 2]$, say. By the argument just used in proving (4.14) above, the limit extends over arbitrary time intervals. \(\square\)

**5 Proof of Theorem 1.6**

We first note the following straightforward variant of Theorems 1.1–1.3: assume that $M_i \subset M_\infty$ are open submanifolds which exhaust $M_\infty$, and $g_i$ are metrics on $M_i$ such that

$$
g_i \to g_\infty \text{ as } i \to \infty
$$

in $C^\infty_{\text{loc}}(M_\infty, g_\infty)$. Then, letting $A_i$ be solutions of (YM) with respect to the metrics $g_i$ on bundles $E_i \to M_i$ of fixed structure group, we obtain a rectifiable singular set $\Sigma \subset M_\infty$ and Uhlenbeck limit $A_\infty$ on a bundle $E_\infty \to M_\infty \setminus \Sigma$. Given a fixed manifold $(M, g)$ and $x_0 \in M$, this version can be used for blowup analysis: let $(M_\infty, g_\infty) = (\mathbb{R}^n, g_{\text{Euclidean}})$ and

$$
(M_i, g_i) = \left( B_{\rho_i(x_0)}(x_i), \lambda_i^{-2} g(x_i + \lambda_i x) \right)
$$

for sequences $x_i \to x_0 \in M$ and $\lambda_i \searrow 0$.

**Proof of Theorem 1.6** Let $\tau_i \not\to \tau$, and define the measures $\mu$ and $\nu$ by (3.9), as in the proof of Theorem 1.1.
Without loss of generality, we may take \( x_0 \) to be a point where the tangent measure \( T_{x_0}v = T_{x_0} \nu \) is equal to a measure of constant density along an \((n-4)\)-plane \( V \subset T_{x_0}M \). Let \( \rho_i \searrow 0 \) be a sequence such that the rescaled curvature measures converge

\[
\rho_i^2 |F_i(x_0 + \rho_i x, \tau_i)| \to T_{x_0}v
\]

(5.1)

weakly on \( \mathbb{R}^n \). By Proposition 2.9, we may also assume that \( x_0 \) is such that

\[
\lim_{\sigma \nearrow \tau} \limsup_{R \searrow 0} \frac{\rho_i}{\Phi_{i}^{1}(R, x, \sigma, \tau)} = 0.
\]

(5.2)

Before proceeding with the proof, we replace the original sequence \( A_i \) with the blown-up sequence of solutions \( \rho_i A_i(x_0 + \rho_i x, \tau_i + \rho_i^2 t) \), which solve \((YM)\) on an exhaustion of \( T_{x_0}M \times (-\infty, 0) \) with respect to the rescaled metrics \( \rho_i^{-2} g(x_0 + \rho_i x) \) (converging locally uniformly to the Euclidean metric).

Notice that the assumption (1.1) is preserved by parabolic rescaling, due to the monotonicity formula. Applying the variant of Theorem 1.1 discussed above to the rescaled sequence, we have

\[
\liminf_{i \to \infty} \int_{\sigma}^{0} \int_{A_i} |D^* F_i(x, t)|^2 u_{0, R}(x) \phi_{0, \rho_i(x_0)}^2(x) dV_x dt = 0
\]

(5.3)

for any \( R > 0 \) and \(-\infty < \sigma < 0\). We may pass to a subsequence such that for \( R = 1 \), \( \lim \inf \) may be replaced by \( \lim \) in (5.3), so that (1.5) is satisfied. By Theorem 1.3, the assumption (5.1) implies that \( F_{A_i} \equiv 0 \), so we have

\[
F_{A_i}(x, t) \to 0 \text{ as } i \to \infty
\]

(5.4)

locally uniformly on \( (T_{x_0}M \setminus V) \times (-\infty, 0) \). Also, by Lemma 2.3 and (5.3), we have

\[
\liminf_{R \to 0} \Phi(R, x, t) \geq \epsilon_0
\]

(5.5)

for all \( x \in V \) and \( t < 0 \).

For \( x \in T_{x_0}M \), write

\[
x = (y, z)
\]

where \( y \in V \cong \mathbb{R}^{n-4} \) and \( z \in V^\perp \cong \mathbb{R}^4 \), and choose coordinates such that \( V \) is spanned by \( e_1, \ldots, e_{n-4} \). Define

\[
f_i(y) = \int_{B_{1}^{n-4}(y_0)} \int_{B_{1}^{n-4}(0)} |D^* F_i(y, z, t)|^2 dV_z dt
\]

\[
g_i(y, t) = \int_{B_{1}^{n-4}(y_0)} \sum_{\alpha=1}^{n-4} \sum_{\beta=1}^{n} |(F_i)_{\alpha\beta}(y, z, t)|^2 dV_z
\]

\[
h_i(y, t) = \int_{B_{1}^{n-4}(y_0)} |F_i(y, z, t)|^2 dV_z.
\]

(5.6)

By (5.3), for any \( y_0 \in V \), we have

\[
\int_{B_{1}^{n-4}(y_0)} f_i(y) dV_y \to 0 \text{ as } i \to \infty.
\]

(5.7)
Given (5.7), the monotonicity-formula trick of Lin-Wang [22], p. 211, implies
\[
\int_{-1}^{-1/8} \int_{B^n_{t_i}(0)} g_i(y, t) \, dV_y \, dt \to 0 \text{ as } i \to \infty.
\] (5.8)

The monotonicity formula also implies
\[
\int_{B^n_{t_i}(0)} h_i(y, t) \, dV_y \leq CE
\] (5.9)
for all \(-1 \leq t < 0\).

Define the Hardy-Littlewood maximal functions
\[
M(f_i)(y) = \sup_{0 < R \leq 1} R^{4-n} \int_{B^n_R(0)} f_i(w) \, dV_w
\]
\[
M(g_i)(y, t) = \sup_{0 < R \leq 1} R^{2-n} \int_{t-R^2}^{t} \int_{B^n_R(0)} g_i(w, s) \, dV_w \, ds
\] (5.10)
\[
M(h_i)(y, t) = \sup_{0 < R \leq 1} R^{4-n} \int_{B^n_R(0)} h_i(w, t) \, dV_w.
\]

Letting \(L_i = \int_{B^n_{t_i}(y_0)} f_i(y) \, dV_y\), by the weak \(L^1\)-estimate for the maximal function, we have
\[
\mu_{\text{Leb}} \{ M(f_i) \geq \epsilon \} \leq C \frac{L_i}{\epsilon}
\] (5.11)
for any \(\epsilon > 0\). Since \(L_i \to 0\) (5.7), we may pass to a subsequence for which there exist points \(y_i \in B^n_{1/2}(0)\) such that
\[
M(f_i)(y_i) \to 0.
\] (5.12)

Arguing similarly, by (5.8), we may choose \(t_i \in [-1/2, -1/4]\) such that
\[
M(g_i)(y_i, t_i) \to 0.
\] (5.13)

Also, by (5.9), we may assume that \(y_i, t_i\) are such that
\[
M(h_i)(y_i, t_i - \delta_i^2) \leq CE.
\] (5.14)

For \(y, s \in \mathbb{R}^{n-4}\) and \(R > 0\), let
\[
v_{R, y}(s) = \frac{R^{4-n}}{(4\pi)^{n/2}} \exp\left(-\frac{|y - s|^2}{4R^2}\right).
\] (5.15)

Then, since
\[
R^{4-n} \exp\left(-\frac{r^2}{4R^2}\right) \leq C (r + R)^{4-n} \exp\left(-\frac{r^2}{2R^2}\right)
\]
it is easily seen from (5.12–5.13) that for any \(y \in \mathbb{R}^{n-4}\), we have
\[
\sup_{0 < R \leq 1} \int_{B^n_{t_i}(0)} f_i(s)v_{R, y_i + Ry}(s) \, dV_s \to 0
\]
\[
\sup_{0 < R \leq 1} R^{-2} \int_{t_i - R^2}^{t_i} \int_{B^n_{t_i}(0)} g_i(s)v_{R, y_i + Ry}(s) \, dV_s \, dt \to 0
\] as \(i \to \infty\) (5.16)
locally uniformly in $y$. Since $|F_i|$ are continuous, in light of (5.4), for $i$ sufficiently large it is possible to choose $\delta_i > 0$ and $z_i \in B_{1/4}^4(0)$ such that

$$
\Phi_{0,1}(A_i; \delta_i, (y_i, z_i), t_i) = \int |F_i(x, t_i)|^2 u(x, z_i) dV_x
$$

$$= \frac{\epsilon_0}{2} = \max_{z \in B_{1/2}^4} \Phi_{0,1}(A_i; \delta_i, (y_i, z), t_i).$$

(5.17)

It follows from (5.4) and (5.5) that $\delta_i \to 0$ and $z_i \to 0 \in \mathbb{R}^4$ as $i \to \infty$.

Finally, we rescale further to define the sequence

$$\bar{A}_i ((y, z), t) = \delta_i A_i ((y_i + \delta_i y, z_i + \delta_i z), t_i + \delta_i^2 (t + 1)).$$

(5.18)

Passing again to a subsequence, let $B(x)$ be the Uhlenbeck limit of $A_i$ on $\mathbb{R}^n \cong T_i M$, with $\tau = 0$, per Theorems 1.1–1.3. For the rescaled sequence, (5.16) with $R = \delta_i$ becomes

$$\int_{B_{1/4}^4(0)} |F_{\bar{A}_i}(x, -1)|^2 dV_x \leq CE.$$  

(5.21)

The statement (5.17) becomes

$$\Phi_i(1, 0, -1) = \frac{\epsilon_0}{2} = \max_{z \in \mathbb{R}^4} \Phi_i(1, (0, z), -1).$$

(5.22)

The cutoff function $\varphi_{0,1}$ of (5.17) is negligible after the rescaling, and we will suppress it in the following calculations.

We shall argue that the convergence $\bar{A}_i \to B$ is smooth over all compact subsets of $\mathbb{R}^n$, and $B(x)$ splits as a product. To this end, we compute

$$\frac{\partial}{\partial x^\alpha}|F|^2 = \langle \nabla_\alpha F_{\beta\gamma}, F_{\beta\gamma} \rangle$$

$$= \langle (\nabla_\beta F_{\alpha\gamma} + \nabla_\gamma F_{\beta\alpha}), F_{\beta\gamma} \rangle$$

$$= 2 \langle \nabla_\beta F_{\alpha\gamma}, F_{\beta\gamma} \rangle$$

(5.23)

$$= 2 \langle \nabla_\beta (F_{\alpha\gamma} F_{\beta\gamma}) - \langle F_{\alpha\gamma}, \nabla_\beta F_{\beta\gamma} \rangle \rangle$$

$$= 2 \langle \nabla_\beta (F_{\alpha\gamma} F_{\beta\gamma}) + \langle F_{\alpha\gamma}, D^\alpha F_{\beta\gamma} \rangle \rangle.$$
where we have used (5.23) in the third line. Define
\[ \Psi_i(R, x) = \int_{-1}^{0} \Phi_i(R, x, t) \, dt. \]
By (2.17), we have
\[ |\nabla u_{1,x}(w)| \leq C \sqrt{u_{1,x}u_{2,x}(w)}. \]
Letting \( \alpha \in \{1, \ldots, n - 4\} \), we may integrate (5.24) in time and apply Hölder’s inequality, to obtain
\[ \left| \frac{\partial}{\partial y^\alpha} \Psi_i(1, x) \right| \leq C \eta_i(x) (\Psi_i(2, x) + \xi_i(x)). \quad (5.25) \]
By (5.19–5.20), the RHS of (5.25) tends to zero locally uniformly in \( x \) as \( i \to \infty \). Then for any \( y \in \mathbb{R}^{n-4} \) and \( z \in \mathbb{R}^4 \), (5.25) implies
\[ |\Psi_i(1, (y, z)) - \Psi_i(1, (0, z))| \to 0 \text{ as } i \to \infty. \]
From (5.22), we have
\[ \limsup_{i \to \infty} \Psi_i(1, (y, z)) = \limsup_{i \to \infty} \Psi_i(1, (0, z)) \leq \frac{\epsilon_0}{2}. \quad (5.26) \]
But from Lemma 2.3 and (5.19), we have
\[ \lim_{i \to \infty} \Psi_i(1, x) = \limsup_{i \to \infty} \Phi_i(1, x, -1) = \limsup_{i \to \infty} \Phi_i(1, x, t) \]
for any \( t \in \mathbb{R} \). Therefore (5.26) yields
\[ \limsup_{i \to \infty} \Phi_i(1, x, t) \leq \frac{\epsilon_0}{2} \quad (5.27) \]
for all \( x \in \mathbb{R}^n \) and \( t \in \mathbb{R} \). Therefore \( \Sigma = \emptyset \), and the limit \( \tilde{A}_i \to B \) takes place smoothly on \( \mathbb{R}^{n+1} \).

Since the convergence is strong on compact sets and the energy is locally bounded, (5.21) and (5.22) are preserved in the limit. Therefore \( B \) is not flat, and has finite energy on the strip \( B_1^{n-4}(0) \times \mathbb{R}^4 \). By (5.20), \( V \sqcup F_B = 0 \), hence \( B(y, z) \) reduces to a finite-energy Yang–Mills connection on \( \mathbb{R}^4 \), as claimed. \( \square \)

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