Limited profit in predictable stock markets

Roland Rothenstein and Klaus Pawelzik
Institute for Theoretical Physics,
University of Bremen, FB 1
Otto-Hahn-Allee, 28334 Bremen

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It has been assumed that arbitrage profits are not possible in efficient markets, because future prices are not predictable. Here we show that predictability alone is not a sufficient measure of market efficiency. We instead propose to measure inefficiencies of markets in terms of the maximal profit an ideal trader can take out from a market. In a stock market model with an evolutionary selection of agents this method reveals that the mean relative amount of realizable profits is very limited and we find that it decays with rising number of agents in the markets. Our results show that markets may self-organize their collective dynamics such that it becomes very sensitive to profit attacks which demonstrates that a high degree of market efficiency can coexist with predictability.

Many complex systems in nature and society consist of rather simple elements whose interactions self-organize in a way that the available dynamics of the system achieves a desired performance. In brains, for instance, individual neurons can realize only rather simple functions while their interactions are permanently re-adapted to ultimately yield successful behavior in non-stationary environments. Further examples are found in the metabolic and genetic networks in cells and in the brain. It has been suspected, that also collective human behavior can sometimes be described in a similar way. A particularly prominent example for social interactions is a market in which each trader has only limited access to relevant information, while the dynamics of the whole system is considered to exploit the overall information quite effectively. In economy it is believed that the collective behavior of traders leads to an efficient market such that in real markets no profit can be made from an analysis of price time series. However, since high frequency data are available research has revealed that some temporal correlations in price time series do in fact exist. But it is still an open question whether these correlations imply inefficiencies (see for an overview). For a solution of this problem it would be highly desirable to have a quantitative measure of inefficiency, for which, however, no generally accepted definition has been proposed until now.

Former studies of efficiency were challenged by possible irrationalities of traders in the market. They either postulated a market of rational traders or tried to measure efficiency of real markets. The drawback of these approaches is that in the first a comparison to real markets in principle cannot be complete, while the second lacked a clear definition of irrelevant noise and meaningful information.

In this paper we propose a novel measure of inefficiency and apply it to a model which combines the approaches of evolutionary games and agent-based models. The method we propose for measuring the inefficiency of a dynamical system modeling a market is general and can be applied to other models and in principle also to real markets. In our view our approach is the first step towards an objective measure of efficiency of stock markets.

Our starting point is to consider a market as an open dynamical system \( \mu \). The future return \( r_{t+1} = (p_{t+1} - p_t)/p_t \) is then given by

\[
r_{t+1} = \mu(\Phi_t),
\]

where \( p_t \) is the price at time \( t \). \( \Phi_t \) denotes all parameters determining the next price including the actions of the traders participating in the market at time \( t \).

This formulation of a market shows why it is not straightforward to use the notion of predictability for defining efficiency: a dynamical system can in principle be perfectly predicted if all of its constituents are known. Making profit, however, for every trader requires to interact with the dynamics of a market. If an external additional trader enters the market at time \( t \) her interaction \( W \) will influence the resulting returns

\[
\tilde{r}_{t+1} = \mu'(\Phi_t, W)
\]

and will thereby also modify the profit the trader can achieve. Note that \( \mu'(\Phi_t, 0) = \mu(\Phi_t) \).

For the purpose of analyzing a given market model we can formally close a given system with respect to capital and information by using a procedure analogous to the notion of baths in thermodynamics. This is done by assuming that the system is coupled to an ideal trader \( T \) who has perfect knowledge of the system and infinite resources. Furthermore, we assume \( T \) always chooses her interaction with the market such that she achieves the maximal profit possible. Clearly, the ideal situation of complete information is very artificial and in reality only a small part of the information determining the dynamics \( \mu \) is available to a trader. However, the profit of this ideal trader per definition provides a strict upper bound on the profit achievable by any trader.
In a simple stock market with one risk less asset $M$ e.g. cash and a single stock $S$ the optimal interaction of the ideal trader is characterized by the amount of $M$ and a price limit $p$ for buying or selling, i.e. $W = (M, p)$. In the following we assume that $T$ always chooses the optimal value for $p$ such that the strength of interaction is characterized solely by the magnitude of risk less assets $M$ she is willing to exchange.

The ideal trader $T$ would buy stocks with $M > 0$, if the return $\hat{r}_{t+1}$ resulting from her interaction would be positive. If in contrast the return $\hat{r}_{t+1}$ resulting from an interaction would be negative the ideal trader sell her stocks with $M < 0$. The profit is calculated according to the difference between buying and selling at time $t+1$:

$$P_t(M) = \frac{M}{M_{tot}} \hat{r}_{t+1}(M).$$  \hspace{1cm} (3)

$M = pS$ denotes the value of the stocks at time $t$ the trader buys or sells measured in risk less assets. $M_{tot}$ denotes the total value of risk less assets present in the market. The profit is equivalent to the amount of risk less assets $\Delta M = \frac{M_{tot} + M}{M_{tot}}$ the ideal trader could virtually win in one time step. This can be seen as the relative loss of money of the participants in the market potentially transferred to the ideal trader.

Using all information, the ideal trader will also take into account her own influence on the market to optimize her profit $P_t$ in every time step $t$:

$$\hat{P}_t = P_t(\hat{M}_t) = \max_M P_t(M).$$ \hspace{1cm} (4)

Assuming that $P_t(M)$ is differentiable we obtain for the optimal invested capital

$$\hat{M}_t = - \frac{\partial r_{t+1}(\hat{M}_t)}{\partial M \hat{r}_{t+1}(\hat{M}_t)}. \hspace{1cm} (5)$$

For the optimal profit then follows:

$$\hat{P}_t(\hat{M}_t) = - \frac{\partial^2 r_{t+1}(\hat{M}_t)}{\partial M^2 \hat{r}_{t+1}(\hat{M}_t)}.$$ \hspace{1cm} (6)

This equation shows that the optimal profit consists of two contributions: The nominator represents the volatility of the return, resulting from the interaction with $T$. The denominator characterizes the responsiveness of the market to increasing $\hat{M}$, and reflects the sensitivity of the system.

The main problem of determining the inefficiency of a market is to specify $\hat{r}(M)$. One can in principle estimate this function in experiments or measure it in model markets (see below) or one can postulate generic properties of this function and analyze the effect on the inefficiency $\delta_t(M) = \frac{\hat{r}_{t+1}}{\hat{r}_{t+1}}(\hat{r}_{t+1}(M) - r_{t+1})$. One might expect that for an optimal order $|\hat{M}| > 0$, the market reaction will be defensive and $\delta$ will be negative (see in Fig. 1 the dashed line). While for large orders any realistic market, as e.g. the example below, will ensure this, for intermediate orders and irrational traders a market can be non-defensive ($\delta > 0$) (see in Fig. 1 the dotted line). In the latter case, the ideal trader will maximally exploit these irrationalities of a market which leads to the counterintuitive result that a feedback of knowledge into a market can in principle increase its overall volatility.

We illustrate the determination of $\hat{r}$ and how one can measure the efficiency using a simple order-based stock market model. Our model reproduces the statistics of empirical market returns to an astonishing large degree, including characteristic scaling of return distributions and strong temporal correlations of volatility (volatility clusters). Because this model and it’s properties have been described in detail before we here outline only its main constituents.

In our model $N$ agents trade by swapping stocks into cash and vice versa. Initially every agent $i$ receives $S_i$ stocks and an amount $M_0$ of cash. The decision of each agent to buy or to sell stocks is determined by a set of $n$ parameters $\alpha_i^\Delta t = 0, ..., (n - 1))$ and is based on the history of the returns $r_{t-1}, r_{t-2}, ..., r_{t-n+1}$. The parameters define simple linear prediction models for each agent and are randomly drawn from a normal distribution with mean 0 and variance 1.

Every iteration of the model yields the next return. It can be divided in three parts: First, the agents make a prognosis of two successive future returns $\hat{r}_t, \hat{r}_{t+1}$ based on their individual parameters of the linear prediction model.
the maximum turnover is the minimum of both functions at price \( a \):

\[
r_t^i = a_t^i + \sum_{\Delta t=1}^{\tau} a_t^i r(t - \Delta t)
\]

\[
r_{t+1}^i = a_{t+1}^i + a_{t+1}^i r_{t+1} + \sum_{\Delta t=2}^{\tau} a_t^i r(t - \Delta t + 1).
\]

Then the agents decide to buy/sell stocks if they predict that the second return \( r_{t+1} \) is positive/negative. In the second part each agent \( i \) places an order, which is stored in an order book. The overall demand \( D(p) \) of the stocks is calculated according to a limit price given by the agents and the size of the orders. \( S^i \) is the total number of stocks agent \( i \) placed in its order respectively \( \Delta S^i = \text{int}(\fr{N}{p_t^i}) \) the number of stocks agent \( i \) is willing to buy with his money \( M^i \):

\[
O(p) = \sum_{i \in \text{onbook}, p < 0} S^i \Theta(p - \hat{p}_t^i)
\]

\[
D(p) = \sum_{i \in \text{onbook}, p > 0} \Delta S^i \Theta(\hat{p}_t^i - p),
\]

where \( \Theta(x) \) is the Heaviside step function. After the orders are placed, a turnover function is calculated. That is the minimum of both functions at price \( p \):

\[
Z(p) = \min\{O(p), D(p)\}
\]

In order to determine the new price the minimum and the maximum argument of \( Z(p) \) at the interval of the maximum turnover \( p_{\text{min}} = \min(\text{argmax} Z(p)) \) and \( p_{\text{max}} = \max(\text{argmax} Z(p)) \) are computed and the new price is then defined by the weighted mean between these two points:

\[
p(t) = \frac{p_{\text{min}} O(p_{\text{min}}) + p_{\text{max}} D(p_{\text{max}})}{O(p_{\text{min}}) + D(p_{\text{max}})}
\]

In the last step of each iteration one agent is replaced by a new random one. In case of an evolutionary update the poorest agent is replaced. To investigate the effects of evolution on efficiency we also consider the case where a randomly chosen agent becomes replaced in each step. Previously we have shown that for large \( N \) the return distributions have a stationary and characteristic shape. While the shape of the distributions varies for the different evolutionary mechanisms, the variances \((r_t^2)\) for both cases are nearly equal and enter a constant value for sufficiently many traders \( N > 1000 \) (see dashed line in Fig. \( \ref{fig:2} \)).

The ideal trader applied to our model maximizes the amount of money she virtually gets from the system without giving stocks away. The only free variable the ideal trader can optimize is the money she uses to buy or sell shares. Following Eqn. \( \ref{eq:4} \) she maximizes her profit \( P_t(M_t) \). To do this, the trader has to place an order in the order book. For every price \( p \) one can calculate a spread \( \sigma(p) \) between offered shares \( O(p) \) and demanded shares \( D(p) \):

\[
\sigma(p) = O(p) - D(p)
\]

If at time \( t \) \( T \) would buy \( S = \text{int}(\fr{N}{p_t}) \) shares, she placed an order at the lowest \( p \) with \( S \geq \sigma(p) \). This leads to a new demand function and therefore to a new price \( \hat{p}_{t+1} \). The procedure for selling is analogous. To calculate the optimal profit \( \hat{P} \) (see Eqn. \( \ref{eq:3} \) the ideal trader repeats this procedure for all possible values of \( M_t \), and finally invests the optimal amount of capital \( \hat{M}_t \). Per construction \( T \) does not further interact with the market in the subsequent time step, which makes the profit \( P_t \) in fact virtual. After estimating \( P_t \) the influence of \( T \) on market is canceled and the method is applied de novo in the next time step. For our market model this approach corresponds to an interaction which does not change the nature of the system. It provides a strict upper bound on the profit achievable by any realistic trader who in contrast would need to interact with the market at least twice (e.g. first buy and then sell). Our method avoids this more realistic scenario because such an interaction would effectively remove stocks (or money) from the market which can change the dynamics of the system substantially and thereby will prevent a reliable characterization of the market’s properties.

In Fig. \( \ref{fig:2} \) one can see that especially in the case of small \( N \) the volatility becomes strongly amplified, while for larger \( N \) the market becomes much more defensive. Interestingly, there is no significant difference in the amplification of the cases with and without evolution (not shown).

Fig. \( \ref{fig:3} \) shows that the mean of the optimal investment in units of the total money in the market drops with a
power law, i.e. $\frac{\langle \hat{M}_t \rangle}{M_{tot}} \propto N^{-1/2}$. We find that the ideal invested capital in a market without evolution always lies above the market with evolution and on average is a factor of $1.39 \pm 0.35$ greater (not shown). Both results clearly have an influence on the mean profit $\hat{P}$ an ideal trader can achieve in our market (see Fig. 4). We find that the possible profit $\hat{P}$ in a market with evolution is for all $N$ smaller and is on average only 44\% $\pm$ 29\% of the average profit $\hat{P}$ of the case of random agent replacement. Eqn. 6 indicates that this effect is mainly caused by the sensitivity $\hat{r}(M)$ of the system because the variances do not differ in both markets. Furthermore, we see that the highest achievable profit is decreasing for a rising number of agents. In the regime $N > 1000$ the decrease of $\hat{P}$ is mainly due to the fact that $\hat{M}$ is getting smaller. Most striking are the absolute values of $\hat{P}$. Taking into account that the agents of our market model are not very smart, the fraction of capital the ideal trader can get in one time step on average is below 1\% in the case of 1000 agents and below 0.1\% for 20000 agents. Our results furthermore indicate that the profit $\hat{T}$ an ideal trader can achieve in the market might fall with a power law $\propto N^{-1/2}$ which would imply a vanishing profit opportunity for any trader in the thermodynamic limit.

Prediction of financial time series is an interesting and challenging topic of research on its own. Our analysis, however, shows that predictability alone is not sufficient to characterize the efficiency of a market. Markets can have different degrees of defensiveness and sensitivity, which limits the profit possible for any trader. Our model not only illustrates the notion of the ideal trader, but shows that the profit option for a trader might vanish if the market is large. Furthermore, the comparison of the evolutionary market with the case of random replacements of agents underlines the long standing hypothesis that efficiency of markets might emerge from Darwinian mechanisms. Finally, our measure of inefficiency is general enough to be applicable to every well defined market model and can be used to analyze simple models [11]. We believe our novel method to measure market inefficiencies paves the way towards a systematic understanding of the influences of irrationalities of traders and we expect it will help to better understand the dynamics of real markets.

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