Decay Rates of global weak solutions for the MHD equations in $\dot{H}^s(\mathbb{R}^n)$

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Abstract

We show that $t^{s/2} \| (u, b)(\cdot, t) \|_{\dot{H}^s(\mathbb{R}^n)} \to 0$ as $t \to \infty$ for Leray solutions $(u, b)(\cdot, t)$ of the incompressible MHD equations, where $2 \leq n \leq 4$ and $s \geq 0$. As a corollary of main result described previously we have also that

$$\lim_{t \to \infty} t^{n/4 - n/2} \| (u, b)(\cdot, t) \|_{L^q(\mathbb{R}^n)} = 0, \quad 2 \leq q \leq \infty.$$ 

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1. Introduction

In this work we derive a general decay rate for Leray global weak solutions of incompressible MHD equations (in $\dot{H}^4(\mathbb{R}^n)$, where $n \leq 4$), that is, global solutions $(u, b)(\cdot, t) \in L^\infty([0, \infty), L^2(\mathbb{R}^n)) \cap L^2([0, \infty), \dot{H}^4(\mathbb{R}^n)) \cap C_w([0, \infty), L^2(\mathbb{R}^n))$ of the system

$$u_t + u \cdot \nabla u + \nabla P = \mu \Delta u + b \cdot \nabla b,$$  

(1.1a)

$$b_t + u \cdot \nabla b = \nu \Delta b + b \cdot \nabla u,$$  

(1.1b)

$$\nabla \cdot u(\cdot, t) = 0, \quad \nabla \cdot b(\cdot, t) = 0,$$  

(1.1c)

with initial data $(u_0, b_0) \in L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, that $\|(u, b)(\cdot, t) - (u_0, b_0)\|_{L^2(\mathbb{R}^n)} \to 0$
as \( t \to 0 \) and such that the strong energy inequality
\[
\|(u, b)(\cdot, t)\|_{L^2(\mathbb{R}^n)}^2 + 2\mu \int_r^t \|Du(\cdot, \tau)\|_{L^2(\mathbb{R}^n)}^2 d\tau + 2\nu \int_r^t \|Db(\cdot, \tau)\|_{L^2(\mathbb{R}^n)}^2 d\tau \\
\leq \|(u, b)(\cdot, r)\|_{L^2(\mathbb{R}^n)}^2, \forall t \geq r
\]
holds for a.e. \( r \geq 0 \), including \( r = 0 \). Such solutions were first constructed by Leray ([7]) for the Navier-Stokes system where \( n \leq 3 \) and later by other authors with different methods considering also \( n = 4 \) or even in higher dimensions, see e.g. [3, 4, 6, 8]. All these methods can be adapted for the MHD equations [2, 9, 12]. In (1.1), \( \mu, \nu > 0 \) are given constants, \( u = u(x, t), b = b(x, t) \) and \( P = P(x, t) \) are the unknowns (the flow velocity, magnetic field and total pressure, respectively). As usual, \( L^2_s(\mathbb{R}^n) \) is the space of solenoidal fields \( v = (v_1, ..., v_n) \in L^2(\mathbb{R}^n) \equiv L^2(\mathbb{R}^n)^n \) with \( \nabla \cdot v = 0 \) in the distributional sense, \( \dot{H}^s(\mathbb{R}^n) = \dot{H}^s(\mathbb{R}^n)^n \) where \( \dot{H}^s(\mathbb{R}^n) \) denotes the homogeneous Sobolev space of order \( s \geq 0 \), and \( C_w(I, L^2_s(\mathbb{R}^n)) \) denotes the set of mappings from a given interval \( I \subseteq \mathbb{R} \) to \( L^2_s(\mathbb{R}^n) \) that are \( L^2 \)-weakly continuous at each \( t \in I \). Here, we always assume \( 2 \leq n \leq 4 \). Moreover, similarly to the Navier-Stokes case, there always exists some \( t_\ast \gg 1 \) – depending on the solution \( (u, b) \) – such that one has
\[
(u, b) \in C^\infty([t_\ast, \infty))
\]
and, for each \( m \in \mathbb{Z}_+ \):
\[
(u, b)(\cdot, t) \in L^\infty([t_\ast, T], H^m(\mathbb{R}^n)),
\]
for each \( t_\ast < T < \infty \), that is, \( (u, b)(\cdot, t) \in L^\infty_{loc}([t_\ast, \infty), H^m(\mathbb{R}^n)) \). In [1], Agapito and Schonbek showed that
\[
\|(u, b)(\cdot, t)\|_{L^2(\mathbb{R}^n)} \to 0 \text{ as } t \to \infty,
\]
generalizing the Kato techniques for Navier-Stokes equation (see e.g. [11]) in dimension \( n = 2, 3 \). More recently, in [11] the authors showed the above property for the Navier-Stokes equations in a simple way using Duhamel’s principle and, with the same technique, they provided an \( L^\infty \) decay rate. So, we will adapt this for the MHD equations in a preliminaries section and generalize this argument to obtain (1.4) in \( n = 4 \) dimension. However, it was necessary to prove the following decay property for derivatives
\[
\lim_{t \to \infty} t^{1/2} \|(Du, Db)(\cdot, t)\|_{L^2(\mathbb{R}^n)} = 0, \quad 2 \leq n \leq 4.
\]
\(^1\)For the definition of \( \|(u, b)\|_{L^2(\mathbb{R}^n)} \), \( \|(Du, Db)\|_{L^2(\mathbb{R}^n)} \) and other similar expressions throughout the text, see (1.7e) and (1.7f).
\(^2\)If \( n = 2 \), then \( t_\ast = 0 \).
Studying these problems with this new approach we provide a decay rate for all the derivatives and using interpolation we get the general decay below.

**Main Theorem.**
For a Leray solution \((u, b)(\cdot, t)\) of (1.1) and \(n \leq 4\), one has

\[
\lim_{t \to \infty} t^{s/2} \|(u, b)(\cdot, t)\|_{H^s(\mathbb{R}^n)} = 0,
\]

for all \(s \geq 0\).

As a consequence, we get the following result.

**Corollary.**
For a Leray solution \((u, b)(\cdot, t)\) of (1.1) and \(n \leq 4\), one has

\[
\lim_{t \to \infty} t^{\frac{n}{2} - \frac{s}{2}} \|(u, b)(\cdot, t)\|_{L^q(\mathbb{R}^n)} = 0,
\]

\(2 \leq q \leq \infty\).

In Section 2, we recall some basic facts and estimates that are needed (or have relevance) for the derivation of Main Theorem in Section 3.

**Notation.** As shown above, boldface letters are used for vector quantities, as in \(u(x, t) = (u_1(x, t), \ldots, u_n(x, t))\). Also, \(\nabla P \equiv \nabla P(\cdot, t)\) denotes the spatial gradient of \(P(\cdot, t)\), \(D_j = \partial / \partial x_j\), \(\nabla \cdot u = D_1 u_1 + \ldots + D_n u_n\) is the (spatial) divergence of \(u(\cdot, t)\).

\(| \cdot |_q\) denotes the Euclidean norm in \(\mathbb{R}^n\), and \(\| \cdot \|_{L^q(\mathbb{R}^n)}\), \(1 \leq q \leq \infty\), are the standard norms of the Lebesgue spaces \(L^q(\mathbb{R}^n)\), with the vector counterparts

\[
\| u(\cdot, t) \|_{L^q(\mathbb{R}^n)} = \left\{ \sum_{i=1}^{n} \int_{\mathbb{R}^n} |u_i(x, t)|^q \, dx \right\}^{1/q},
\]

\[
\| Du(\cdot, t) \|_{L^q(\mathbb{R}^n)} = \left\{ \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} |D_j u_i(x, t)|^q \, dx \right\}^{1/q},
\]

and, in general,

\[
\| D^m u(\cdot, t) \|_{L^q(\mathbb{R}^n)} = \left\{ \sum_{i,j, \ldots, j_m=1}^{n} \int_{\mathbb{R}^n} |D_j \ldots D_{j_m} u_i(x, t)|^q \, dx \right\}^{1/q}
\]

if \(1 \leq q < \infty\); if \(q = \infty\), then \(\| u(\cdot, t) \|_{L^\infty(\mathbb{R}^n)} = \max \{ \| u_i(\cdot, t) \|_{L^\infty(\mathbb{R}^n)} : 1 \leq i \leq n \}\),

\(\| Du(\cdot, t) \|_{L^\infty(\mathbb{R}^n)} = \max \{ \| D_j u_i(\cdot, t) \|_{L^\infty(\mathbb{R}^n)} : 1 \leq i, j \leq n \}\) and, for general \(m \geq 1\):
\[ \| D^m u(\cdot, t) \|_{L^\infty(\mathbb{R}^n)} = \max \left\{ \| D_{i_1} \cdots D_{i_m} u(\cdot, t) \|_{L^\infty(\mathbb{R}^n)} : 1 \leq i, j_1, \ldots, j_m \leq n \right\}. \quad (1.7d) \]

Definitions (1.4) are convenient, but not essential. However, some choice for the vector norms has to be made to fix the values of constants. We define also, for simplicity the following norms for a pair \((u, b)\) as usually made in literature:

\[ \|(u, b)\|_{L^q(\mathbb{R}^n)}^q := \|u\|_{L^q(\mathbb{R}^n)}^q + \|b\|_{L^q(\mathbb{R}^n)}^q \quad (1.7e) \]

and more generally, for all \(m \geq 1\) integer

\[ \|(D^m u, D^m b)\|_{L^q(\mathbb{R}^n)}^q := \|D^m u\|_{L^q(\mathbb{R}^n)}^q + \|D^m b\|_{L^q(\mathbb{R}^n)}^q \quad (1.7f) \]

for all \(1 \leq q \leq \infty\). Similarly, for all \(s \geq 0\),

\[ \|(u, b)\|_{H^s(\mathbb{R}^n)}^2 := \|u\|_{H^s(\mathbb{R}^n)}^2 + \|b\|_{H^s(\mathbb{R}^n)}^2, \quad (1.7g) \]

where,

\[ \|u\|_{H^s(\mathbb{R}^n)} = \left( \sum_{i=1}^n \int_{\mathbb{R}^n} |\xi_i|^{2s} \hat{u}_i(\xi)|^2 d\xi \right)^{1/2} \quad (1.7h) \]

and \(\hat{u}_i\) denote the Fourier transform of \(u_i\). The constants will be represented by the letters \(C, c\) or \(K\). For economy, we will typically use the same symbol to denote constants with different numerical values.

2. Preliminaries

First, we will obtain the derivatives monotonicity in \(L^2(\mathbb{R}^n)\),

\[ \|(Du, Db)(\cdot, t)\|_{L^2(\mathbb{R}^n)} \leq \|(Du, Db)(\cdot, t_0)\|_{L^2(\mathbb{R}^n)}. \]

Staring with \(n = 3\). This next argument is adapted from [5]. Using (1.1) and (1.3), we get,

\[ \|(Du, Db)(\cdot, t)\|_{L^2(\mathbb{R}^3)}^2 + 2 \min\{\mu, \nu\} \int_{t_0}^t \| (D^2 u, D^2 b)(\cdot, \tau) \|_{L^2(\mathbb{R}^3)}^2 \, d\tau \]

\[ \leq \|(Du, Db)(\cdot, t_0)\|_{L^2(\mathbb{R}^3)}^2 \]

\[ + C \int_{t_0}^t \| (u, b)(\cdot, \tau) \|_{L^\infty(\mathbb{R}^3)} \| (Du, Db)(\cdot, \tau) \|_{L^2(\mathbb{R}^3)} \| (D^2 u, D^2 b)(\cdot, \tau) \|_{L^2(\mathbb{R}^3)} \, d\tau \]

\[ \leq \|(Du, Db)(\cdot, t_0)\|_{L^2(\mathbb{R}^3)}^2 \]

\[ + C \int_{t_0}^t \| (u, b)(\cdot, \tau) \|_{L^2(\mathbb{R}^3)}^{1/2} \| (Du, Db)(\cdot, \tau) \|_{L^2(\mathbb{R}^3)}^{1/2} \| (D^2 u, D^2 b)(\cdot, \tau) \|_{L^2(\mathbb{R}^3)} \, d\tau, \quad (2.1) \]
where we have used the Sobolev-Nirenberg-Gagliardo (SNG) inequalities (see (2.10a)). By (1.2), we can choose 
\[ t \]
where we have used the Sobolev-Nirenberg-Gagliardo (SNG) inequalities (see (2.10a)).
so that (2.1) gives 
\[ \| (Du, Db)(\cdot, t) \|_{L^2(\mathbb{R}^3)} \leq \| (Du, Db)(\cdot, t_0) \|_{L^2(\mathbb{R}^3)} \]
for all \( t \) near \( t_0 \) by continuity. Actually, with this choice, it follows from ((2.1) again) that 
\[ C^2 \| (u_0, b_0) \|_{L^2(\mathbb{R}^3)} \| (Du, Db)(\cdot, s) \|_{L^2(\mathbb{R}^3)} < (\min\{\mu, \nu\})^2, \quad \forall s \geq t_0. \tag{2.2} \]
Recalling (2.1), (2.2) implies that 
\[ \| (Du, Db)(\cdot, t) \|_{L^2(\mathbb{R}^3)} \leq \| (Du, Db)(\cdot, t_0) \|_{L^2(\mathbb{R}^3)}, \tag{2.3} \]
for all \( t \geq t_0 \). Since, by (1.2), \( \| (Du, Db)(\cdot, t) \|_{L^2(\mathbb{R}^3)}^2 \) is integrable in \((0, \infty)\) one has, by (2.3), that 
\[ \lim_{t \to \infty} \int_0^t (Du, Db)(\cdot, s) \, ds = 0. \tag{2.4} \]
A similar argument hold for \( n = 2 \) (with \( t_* = 0 \)). For \( n = 4 \), we proceed as before,
\[
\| (Du, Db)(\cdot, t) \|_{L^2(\mathbb{R}^4)}^2 + 2 \min\{\mu, \nu\} \int_0^t \| (D^2u, D^2b)(\cdot, \tau) \|_{L^2(\mathbb{R}^4)}^2 \, d\tau \\
\leq \| (Du, Db)(\cdot, t_0) \|_{L^2(\mathbb{R}^4)}^2 \\
+ C \int_0^t \| (u, b)(\cdot, \tau) \|_{L^2(\mathbb{R}^4)} \| (Du, Db)(\cdot, \tau) \|_{L^2(\mathbb{R}^4)} \| (D^2u, D^2b)(\cdot, \tau) \|_{L^2(\mathbb{R}^4)} \, d\tau \\
\leq \| (Du, Db)(\cdot, t_0) \|_{L^2(\mathbb{R}^4)}^2 \\
+ C \int_0^t \| (Du, Db)(\cdot, \tau) \|_{L^2(\mathbb{R}^4)} \| (D^2u, D^2b)(\cdot, \tau) \|_{L^2(\mathbb{R}^4)} \, d\tau,
\]
where we have used the Sobolev-Nirenberg-Gagliardo (SNG) inequalities (see (2.12)). Now, proceeding as in the 3D case we get 
\[ \| (Du, Db)(\cdot, t) \|_{L^2(\mathbb{R}^4)} \leq \| (Du, Db)(\cdot, t_0) \|_{L^2(\mathbb{R}^4)}. \]
and consequently as in (2.4) one has 
\[
\lim_{t \to \infty} t^{1/2} \| (Du, Db)(\cdot, t) \|_{L^2(\mathbb{R}^n)} = 0, \quad 2 \leq n \leq 4. \tag{2.5}
\]
In order to derive some Sobolev inequalities, we observe, by (1.7(a)), that 
\[ \|u\|_{L^q(\mathbb{R}^n)} \leq \|(u, b)\|_{L^q(\mathbb{R}^n)}, \tag{2.6} \]
\[ ^3 \text{Because a monotonic function } f \in C^0((a, \infty)) \cap L^1((a, \infty)) \text{ has to satisfy } f(t) = o(1/t) \text{ as } t \to \infty \text{ (see e.g. [5], p. 236).} \]
for $1 \leq q \leq \infty$. The study of Leray solutions in dimension $n \leq 4$ is facilitated by the fact that they are necessarily smooth for large $t$. A further simplification for $n = 2, 3$ is that pointwise values of functions can be estimated in terms of $H^2$ norms and so we begin with this case. One has

$$\| u \|_{L^\infty(\mathbb{R}^2)} \leq \| u \|_{L^2(\mathbb{R}^2)}^{1/2} \| D^2 u \|_{L^2(\mathbb{R}^2)}^{1/2} \quad (2.7a)$$

for arbitrary $u \in H^2(\mathbb{R}^2)$; likewise,

$$\| u \|_{L^\infty(\mathbb{R}^3)} \leq \| u \|_{L^2(\mathbb{R}^3)}^{1/4} \| D^2 u \|_{L^2(\mathbb{R}^3)}^{3/4} \quad (2.7b)$$

for $u \in H^2(\mathbb{R}^3)$. These are easily shown by Fourier transform and Parseval’s identity (see e.g. [10], where the optimal versions of (2.7) and their higher dimensional analogues are obtained. By Fourier transform, we also get (for any $n$):

$$\| Du \|_{L^2(\mathbb{R}^n)} \leq \| u \|_{L^2(\mathbb{R}^n)}^{1/2} \| D^2 u \|_{L^2(\mathbb{R}^n)}^{1/2} \quad (2.8a)$$

or, more generally,

$$\| D^\ell u \|_{L^2(\mathbb{R}^n)} \leq \| u \|_{L^2(\mathbb{R}^n)}^{1-\theta} \| D^m u \|_{L^2(\mathbb{R}^n)}^{\theta}, \quad \theta = \frac{\ell}{m} \quad (2.8b)$$

Combining (2.6), (2.7) and (2.8), we get the following basic inequalities.

**Lemma 1.** For $n = 2$, one has

$$\| (u, b) \|_{L^\infty(\mathbb{R}^2)} \| (Du, Db) \|_{L^2(\mathbb{R}^2)} \leq C \| (u, b) \|_{L^2(\mathbb{R}^2)} \| (D^2 u, D^2 b) \|_{L^2(\mathbb{R}^2)} \quad (2.9a)$$

$$\| (u, b) \|_{L^\infty(\mathbb{R}^2)} \| (D^2 u, D^2 b) \|_{L^2(\mathbb{R}^2)} \leq C \| (u, b) \|_{L^2(\mathbb{R}^2)} \| (D^3 u, D^3 b) \|_{L^2(\mathbb{R}^2)} \quad (2.9b)$$

$$\| (Du, Db) \|_{L^\infty(\mathbb{R}^2)} \| (Du, Db) \|_{L^2(\mathbb{R}^2)} \leq C \| (u, b) \|_{L^2(\mathbb{R}^2)} \| (D^3 u, D^3 b) \|_{L^2(\mathbb{R}^2)} \quad (2.9c)$$

and, for general $m \geq 2, 0 \leq \ell \leq m - 2$:

$$\| (D^\ell u, D^\ell b) \|_{L^\infty(\mathbb{R}^2)} \| (D^{m-\ell} u, D^{m-\ell} b) \|_{L^2(\mathbb{R}^2)} \leq C \| (u, b) \|_{L^2(\mathbb{R}^2)} \| (D^{m+1} u, D^{m+1} b) \|_{L^2(\mathbb{R}^2)} \quad (2.9d)$$

for some $C > 0$. 
Similarly, in dimension $n = 3$.

**Lemma 2.** For $n = 3$, one has

\[
\| (u, b) \|_{L^\infty(\mathbb{R}^3)} \| (Du, Db) \|_{L^2(\mathbb{R}^3)} \\
\leq C \| (u, b) \|_{L^2(\mathbb{R}^3)}^{1/2} \| (Du, Db) \|_{L^2(\mathbb{R}^3)}^{1/2} \| (D^2u, D^2b) \|_{L^2(\mathbb{R}^3)} \tag{2.10a}
\]

\[
\leq C \| (u, b) \|_{L^\infty(\mathbb{R}^3)} \| (D^2u, D^2b) \|_{L^2(\mathbb{R}^3)} \tag{2.10b}
\]

\[
\leq C \| (u, b) \|_{L^2(\mathbb{R}^3)}^{1/2} \| (Du, Db) \|_{L^2(\mathbb{R}^3)}^{1/2} \| (D^3u, D^3b) \|_{L^2(\mathbb{R}^3)} \tag{2.10c}
\]

\[
\leq C \| (u, b) \|_{L^\infty(\mathbb{R}^3)} \| (D^2u, D^2b) \|_{L^2(\mathbb{R}^3)} \tag{2.10d}
\]

\[
\leq C \| (u, b) \|_{L^2(\mathbb{R}^3)}^{3/4} \| (D^2u, D^2b) \|_{L^2(\mathbb{R}^3)}^{3/4} \| (D^3u, D^3b) \|_{L^2(\mathbb{R}^3)} \tag{2.10e}
\]

and, for general $m \geq 3$, $0 \leq \ell \leq m - 3$:

\[
\| (D^\ell u, D^\ell b) \|_{L^\infty(\mathbb{R}^3)} \| (D^{m-\ell}u, D^{m-\ell}b) \|_{L^2(\mathbb{R}^3)} \leq C \| (u, b) \|_{L^2(\mathbb{R}^3)}^{\ell+3/2} \| (D^{\ell+2}u, D^{\ell+2}b) \|_{L^2(\mathbb{R}^3)}^{1/2} \| (D^{m+1}u, D^{m+1}b) \|_{L^2(\mathbb{R}^3)} \tag{2.10f}
\]

for some $C > 0$.

In dimension $n = 4$, we start with the fundamental Sobolev inequality,

\[
\| u \|_{L^4(\mathbb{R}^4)} \leq \| Du \|_{L^2(\mathbb{R}^4)}. \tag{2.11}
\]

Hence, using (2.6), (2.8) and (2.11), we have the result below.

**Lemma 3.** For all $m \geq 1$, $0 \leq \ell \leq m - 1$, one actually has

\[
\| (D^\ell u, D^\ell b) \|_{L^4(\mathbb{R}^4)} \| (D^{m-\ell}u, D^{m-\ell}b) \|_{L^4(\mathbb{R}^4)} \leq C \| (Du, Db) \|_{L^2(\mathbb{R}^4)} \| (D^{m+1}u, D^{m+1}b) \|_{L^2(\mathbb{R}^4)}, \tag{2.12}
\]
for some $C > 0$.

When we derive energy inequalities for higher order derivatives of Leray solutions to MHD equations, the importance of lemmas above becomes clear. In euclidean plane $\mathbb{R}^2$, it turns out that all solutions of MHD system (1.1) are the same, i.e., the uniqueness is well established; the solutions are also to be smooth, in other words, $(u, b) \in C^\infty(\mathbb{R}^2 \times (0, \infty))$ and moreover $(u, b)(\cdot, t) \in C((0, \infty), H^m(\mathbb{R}^2))$ for all $m \geq 0$. When $n > 2$, the absence of smoothness previously cited complicates the study of Leray solutions; in particular, their uniqueness and precise regularity properties are still unresolved as in the Navier-Stokes system case.

Now, we will generalize the argument in [11] for the MHD system (1.1) in dimension $n = 4$. Since $u(\cdot, t)$ is smooth for large $t$, it can be written as

$$u(\cdot, t) = e^{\mu \Delta (t-t_0)} u(\cdot, t_0) - \int_{t_0}^t e^{\mu \Delta (t-\tau)} Q_1(\cdot, \tau) \, d\tau, \quad t > t_0$$

(2.13)

for $t_0$ large enough, where $Q_1 = u \cdot \nabla u + \nabla P - b \cdot \nabla b$, and $e^{\mu \Delta t}$ denotes the heat semigroup. From (2.13), we get

$$\|u(\cdot, t)\|_{L^2(\mathbb{R}^4)} \leq \|v_0(\cdot, t)\|_{L^2(\mathbb{R}^4)} + \int_{t_0}^t \|u(\cdot, \tau) \cdot \nabla u(\cdot, \tau)\|_{L^2(\mathbb{R}^4)} \, d\tau$$

$$+ \int_{t_0}^t \|b(\cdot, \tau) \cdot \nabla b(\cdot, \tau)\|_{L^2(\mathbb{R}^4)} \, d\tau$$

$$\leq \|v_0(\cdot, t)\|_{L^2(\mathbb{R}^4)} + \sqrt{2} \int_{t_0}^t \|u(\cdot, \tau)\|_{L^4(\mathbb{R}^4)} \|D u(\cdot, \tau)\|_{L^4(\mathbb{R}^4)} \, d\tau$$

$$+ \sqrt{2} \int_{t_0}^t \|b(\cdot, \tau)\|_{L^4(\mathbb{R}^4)} \|D b(\cdot, \tau)\|_{L^4(\mathbb{R}^4)} \, d\tau$$

$$\leq \|v_0(\cdot, t)\|_{L^2(\mathbb{R}^4)} + 2\sqrt{2} \int_{t_0}^t \|(D u, D b)(\cdot, \tau)\|_{L^2(\mathbb{R}^4)} \|D^2 u, D^2 b)(\cdot, \tau)\|_{L^2(\mathbb{R}^4)} \, d\tau$$

by (2.11), where $v_0(\cdot, t) := e^{\mu \Delta (t-t_0)} u(\cdot, t_0)$, and using that (by Helmholtz projection or directly by Fourier transform [5]): $\|Q_1(\cdot, \tau)\|_{L^2(\mathbb{R}^n)} \leq \|u(\cdot, \tau) \cdot \nabla u(\cdot, \tau)\|_{L^2(\mathbb{R}^n)}$.

This shows that, given $\epsilon > 0$, taking $t_0 \gg 1$ we get $\|u(\cdot, t)\|_{L^2(\mathbb{R}^4)} < \epsilon$ for all $t$ large enough, since the integrand on the right hand side above is in $L^1((t_*, \infty))$. One can repeat the previous analysis for $b(\cdot, t)$ using $Q_2 = u \cdot \nabla b - b \cdot \nabla u$ and obtain

$$\|(u, b)(\cdot, t)\|_{L^2(\mathbb{R}^4)} \to 0, \quad \text{as } t \to \infty,$$

which implies (with (1.4)) that

$$\|(u, b)(\cdot, t)\|_{L^2(\mathbb{R}^n)} \to 0, \quad \text{as } t \to \infty,$$

(2.14)
for $n = 2, 3, 4$.

3. Proof of Main Theorem

Let $(u, b)(\cdot, t)$ be any given Leray solution of the system (1.1). Observe that, by (2.14) and (2.5), the result is true for $s = 0$ and $s = 1$. Our strategy will be to show that the main theorem is valid for $s > 0$ integer, i.e.,

$$
\lim_{t \to \infty} t^{m/2} \|(D^m u, D^m b)(\cdot, t)\|_{L^2(\mathbb{R}^n)} = 0,
$$

(3.1)

for all $m \geq 0$ integer.

By (2.5) and (2.14), given $\epsilon > 0$, there exist $t_0 > t_\ast$ (see (1.3)) sufficiently large such as

$$
\|(u, b)(\cdot, t)\|_{L^2(\mathbb{R}^n)} \leq \epsilon
$$

((3.2a))

and

$$
t^{1/2} \|(Du, Db)(\cdot, t)\|_{L^2(\mathbb{R}^n)} \leq \epsilon,
$$

(3.2b)

for all $t \geq t_0$.

Starting with $n = 3$, let $t_\ast \geq 0$ be chosen so that (1.3) holds. Now, Differentiating (1.1a) and (1.1b) with respect to $x_\ell$, taking the dot product of (1.1a) and (1.1b) by $(t - t_0)D_\ell u$ and $(t - t_0)D_\ell b$, respectively, and integrating the result on $\mathbb{R}^3 \times [t_0, t]$, we get summing over $1 \leq \ell \leq 3$,

$$(t - t_0) \|(Du, Db)(\cdot, t)\|_{L^2(\mathbb{R}^3)}^2
+ 2 \min\{\mu, \nu\} \int_{t_0}^{t} (\tau - t_0) \|(D^2 u, D^2 b)(\cdot, \tau)\|_{L^2(\mathbb{R}^3)}^2 d\tau
\leq \int_{t_0}^{t} \|(Du, Db)(\cdot, \tau)\|_{L^2(\mathbb{R}^3)}^2 d\tau
+ C \int_{t_0}^{t} (\tau - t_0) \|(u, b)(\cdot, \tau)\|_{L^\infty(\mathbb{R}^3)} \|(Du, Db)(\cdot, \tau)\|_{L^2(\mathbb{R}^3)} \|(D^2 u, D^2 b)(\cdot, \tau)\|_{L^2(\mathbb{R}^3)} d\tau
\leq \int_{t_0}^{t} \|(Du, Db)(\cdot, \tau)\|_{L^2(\mathbb{R}^3)}^2 d\tau
+ C \int_{t_0}^{t} (\tau - t_0) \|(u, b)(\cdot, \tau)\|_{L^2(\mathbb{R}^3)} \|(Du, Db)(\cdot, \tau)\|_{L^2(\mathbb{R}^3)} \|(D^2 u, D^2 b)(\cdot, \tau)\|_{L^2(\mathbb{R}^3)}^2 d\tau,
$$

where we have used integration by parts, (1.1c) and (2.10a). Therefore by (2.5) and
For some constant $C > 0$, by (2.14), and summing over $1 + \ell_2 > t_0$ sufficiently large, we have,

\[
(t - t_0) \| (D^2 u, D^2 b)(\cdot, t) \|_{L^2(\mathbb{R}^3)}^2 \leq \int_{t_0}^t \| (D u, D b)(\cdot, \tau) \|_{L^2(\mathbb{R}^3)}^2 d\tau.
\]

For some constant $C > 0$.

Now, we go to the next step similarly: differentiating (1.1a) and (1.1b) twice (with respect to $x_{\ell_1}, x_{\ell_2}$, for example), multiplying (1.1a) and (1.1b) by $(t - t_0)^2 \ell_{\ell_1} \ell_{\ell_2} u(x, t)$ and by $\ell_{\ell_1} \ell_{\ell_2} b(x, t)$, respectively, we get, integrating the result on $\mathbb{R}^3 \times [t_0, t]$, $t \geq t_0$ and summing over $1 \leq \ell_1, \ell_2 \leq 3,$

\[
(t - t_0)^2 \| (D^2 u, D^2 b)(\cdot, t) \|_{L^2(\mathbb{R}^3)}^2
\]

\[
+ 2 \min\{\mu, \nu\} \int_{t_0}^t (\tau - t_0)^2 \| (D^3 u, D^3 b)(\cdot, \tau) \|_{L^2(\mathbb{R}^3)}^2 d\tau
\]

\[
\leq 2 \int_{t_0}^t (\tau - t_0) \| (D^2 u, D^2 b)(\cdot, \tau) \|_{L^2(\mathbb{R}^3)}^2 d\tau
\]

\[
+ C \int_{t_0}^t (\tau - t_0)^2 \left\{ \| (u, b)(\cdot, \tau) \|_{L^\infty(\mathbb{R}^3)} \| (D^2 u, D^2 b)(\cdot, \tau) \|_{L^2(\mathbb{R}^3)} \right\} \| (D^3 u, D^3 b)(\cdot, \tau) \|_{L^2(\mathbb{R}^3)}^2 d\tau.
\]

Using (2.10b) and (2.10c), we have,

\[
(t - t_0)^2 \| (D^2 u, D^2 b)(\cdot, t) \|_{L^2(\mathbb{R}^3)}^2
\]

\[
+ 2 \min\{\mu, \nu\} \int_{t_0}^t (\tau - t_0)^2 \| (D^3 u, D^3 b)(\cdot, \tau) \|_{L^2(\mathbb{R}^3)}^2 d\tau
\]

\[
\leq 2 \int_{t_0}^t (\tau - t_0) \| D^2 (u, b)(\cdot, \tau) \|_{L^2(\mathbb{R}^3)}^2 d\tau
\]

\[
+ C \int_{t_0}^t (\tau - t_0)^2 \| (u, b)(\cdot, \tau) \|_{L^2(\mathbb{R}^3)} \| (D^2 u, D^2 b)(\cdot, \tau) \|_{L^2(\mathbb{R}^3)} \| (D^3 u, D^3 b)(\cdot, \tau) \|_{L^2(\mathbb{R}^3)}^2 d\tau,
\]

for some constant $C > 0$ (whose value need not concern us). Hence, by (2.5) and (2.14), there exist $t_0 > t_*$ sufficiently large such as,

\[
(t - t_0)^2 \| (D^2 u, D^2 b)(\cdot, t) \|_{L^2(\mathbb{R}^3)}^2 + C \int_{t_0}^t (\tau - t_0)^2 \| (D^3 u, D^3 b)(\cdot, \tau) \|_{L^2(\mathbb{R}^3)}^2 d\tau
\]

\[
\leq 2 \int_{t_0}^t (\tau - t_0) \| (D^2 u, D^2 b)(\cdot, \tau) \|_{L^2(\mathbb{R}^3)}^2 d\tau.
\]

(3.4)
Given $\epsilon > 0$ arbitrary, there exist $t_0 > t_*$ large enough so that,

$$\int_{t_0}^{t} \| (Du, Db)(\cdot, \tau) \|^2_{L^2(\mathbb{R}^3)} d\tau \leq \epsilon,$$

by the fundamental energy inequality (1.2) and (3.2). Hence, by (3.3), one has,

$$\int_{t_0}^{t} (\tau - t_0) \| (D^2u, D^2b)(\cdot, \tau) \|^2_{L^2(\mathbb{R}^3)} d\tau \leq \epsilon.$$

Using (3.4), we conclude that,

$$t^2 \left( \frac{t - t_0}{t} \right)^2 \| (D^2u, D^2b)(\cdot, t) \|^2_{L^2(\mathbb{R}^3)} = (t - t_0)^2 \| (D^2u, D^2b)(\cdot, t) \|^2_{L^2(\mathbb{R}^3)} \leq \epsilon.$$

Consequently

$$t \| (D^2u, D^2b)(\cdot, t) \|_{L^2(\mathbb{R}^3)} \rightarrow 0, \text{ as } t \rightarrow \infty$$

(3.5)

and (3.5) solves (3.1) for $m = 2$. Similarly, we go to the next step and use the previous decay (3.5) and the Sobolev inequalities (2.10e) and (2.10f) to obtain the 3rd order decay. Now, by induction, we have,

$$(t - t_0)^m \| (D^m u, D^m b)(\cdot, t) \|^2_{L^2(\mathbb{R}^3)}$$

$$+ 2 \min\{\mu, \nu\} \int_{t_0}^{t} (\tau - t_0)^m \| (D^{m+1} u, D^{m+1} b)(\cdot, \tau) \|^2_{L^2(\mathbb{R}^3)} d\tau$$

$$\leq m \int_{t_0}^{t} (\tau - t_0)^{m-1} \| (D^m u, D^m b)(\cdot, \tau) \|^2_{L^2(\mathbb{R}^3)} d\tau$$

$$+ C \int_{t_0}^{t} (\tau - t_0)^m \| (D^{m+1} u, D^{m+1} b)(\cdot, \tau) \|_{L^2(\mathbb{R}^3)} \sum_{\ell = 0}^{[m/2]} \| (D^\ell u, D^\ell b)(\cdot, \tau) \|_{L^\infty(\mathbb{R}^3)} d\tau,$$

for general $m \geq 3$, where $[m]$ is the integer part of $m$. By (2.10f),

$$(t - t_0)^m \| (D^m u, D^m b)(\cdot, t) \|^2_{L^2(\mathbb{R}^3)} + C \int_{t_0}^{t} (\tau - t_0)^m \| (D^{m+1} u, D^{m+1} b)(\cdot, \tau) \|^2_{L^2(\mathbb{R}^3)} d\tau$$

$$\leq m \int_{t_0}^{t} (\tau - t_0)^{m-1} \| (D^m u, D^m b)(\cdot, \tau) \|^2_{L^2(\mathbb{R}^3)} d\tau.$$

By the same previous argument, it follows that

$$t^{m/2} \| (D^m u, D^m b)(\cdot, t) \|_{L^2(\mathbb{R}^3)} \rightarrow 0 \text{ as } t \rightarrow \infty,$$
which completes the proof of (3.1) for \( n = 3 \). The proof of (3.1) for \( n = 2 \) is similar, using the inequalities of Lemma 1 (2.9) instead of Lemma 2.

We will now consider the \( n = 4 \) case. Basically, we will use the inequalities (2.11) and (2.12). However, the energy estimates will suffer some changes. So, let \((u, b)(\cdot, t)\) be any given Leray solution to (1.1). Differentiating (1.1a) and (1.1b) with respect to \( x_\ell \), taking the dot product of (1.1a) and (1.1b) by \((t - t_0)D_\ell u\) and \((t - t_0)D_\ell b\), respectively, and integrating the result on \( \mathbb{R}^4 \times [t_0, t] \), the energy estimate, summing over \( 1 \leq \ell \leq 4 \), is now

\[
(t - t_0) \| (D u, D b)(\cdot, t) \|^2_{L^2(\mathbb{R}^4)} + 2 \min\{\mu, \nu\} \int_{t_0}^t (\tau - t_0) \| (D^2 u, D^2 b)(\cdot, \tau) \|^2_{L^2(\mathbb{R}^4)} d\tau 
\leq \int_{t_0}^t \| (D u, D b)(\cdot, \tau) \|^2_{L^2(\mathbb{R}^4)} d\tau
\]

\[
+ C \int_{t_0}^t (\tau - t_0) \| (u, b)(\cdot, \tau) \|^2_{L^4(\mathbb{R}^4)} \| (D u, D b)(\cdot, \tau) \|^2_{L^4(\mathbb{R}^4)} \| (D^2 u, D^2 b)(\cdot, \tau) \|^2_{L^2(\mathbb{R}^4)} d\tau
\]

\[
\leq \int_{t_0}^t \| (D u, D b)(\cdot, \tau) \|^2_{L^2(\mathbb{R}^4)} d\tau
\]

\[
+ C \int_{t_0}^t (\tau - t_0) \| (D u, D b)(\cdot, \tau) \|^2_{L^2(\mathbb{R}^4)} \| (D^2 u, D^2 b)(\cdot, \tau) \|^2_{L^2(\mathbb{R}^4)} d\tau,
\]

by the Hölder inequality and (2.11) for vector field \((u, b)\). Using (2.5), one has,

\[
(t - t_0) \| (D u, D b)(\cdot, t) \|^2_{L^2(\mathbb{R}^4)} + C \int_{t_0}^t (\tau - t_0) \| (D^2 u, D^2 b)(\cdot, \tau) \|^2_{L^2(\mathbb{R}^4)} d\tau \]

\[
\leq \int_{t_0}^t \| (D u, D b)(\cdot, \tau) \|^2_{L^2(\mathbb{R}^4)} d\tau,
\]

for some constant \( C > 0 \). We proceed for general \( m \geq 2 \) by induction similarly as
in the case $n = 3$,
\[
(t - t_0)^m \| (D^m u, D^m b)(\cdot, t) \|^2_{L^2(\mathbb{R}^4)}
\]
\[
+ 2 \min\{\mu, \nu\} \int_{t_0}^t (\tau - t_0)^m \| (D^{m+1} u, D^{m+1} b)(\cdot, \tau) \|^2_{L^2(\mathbb{R}^4)} d\tau
\]
\[
\leq m \int_{t_0}^t (\tau - t_0)^{m-1} \| (D^m u, D^m b)(\cdot, \tau) \|^2_{L^2(\mathbb{R}^4)} d\tau
\]
\[
+ C \int_{t_0}^t (\tau - t_0)^m \| (D^{m+1} u, D^{m+1} b)(\cdot, \tau) \|_{L^2(\mathbb{R}^4)}
\]
\[
\sum_{\ell = 0}^{[m/2]} \| (D^\ell u, D^\ell b)(\cdot, \tau) \|_{L^4(\mathbb{R}^4)} \| (D^{m-\ell} u, D^{m-\ell} b)(\cdot, \tau) \|_{L^4} d\tau.
\]

Using (2.12) and (2.5), we have
\[
(t - t_0)^m \| (D^m u, D^m b)(\cdot, t) \|^2_{L^2(\mathbb{R}^4)} + C \int_{t_0}^t (\tau - t_0)^m \| (D^{m+1} u, D^{m+1} b)(\cdot, \tau) \|^2_{L^2(\mathbb{R}^4)} d\tau
\]
\[
\leq m \int_{t_0}^t (\tau - t_0)^{m-1} \| (D^m u, D^m b)(\cdot, \tau) \|^2_{L^2(\mathbb{R}^4)} d\tau.
\]

By the same argument in the $n = 3$ case, we conclude the proof of (3.1). Now, we just have to apply a simple interpolation and the proof of Theorem I turns out.

### 4. Proof of (1.6)

We begin with $n = 2$. Using the Sobolev inequality (2.7a) for the pair $(u, b)$, we have
\[
\| (u, b) \|_{L^\infty(\mathbb{R}^2)} \leq C \| (u, b) \|_{L^2(\mathbb{R}^2)}^{1/2} \| (D^2 u, D^2 b) \|_{L^2(\mathbb{R}^2)}^{1/2}.
\]
By Main Theorem, we get
\[
t^{1/2} \|(u, b)\|_{L^\infty(\mathbb{R}^2)} \to 0, \text{ as } t \to \infty.
\]
Using the same idea and (2.7b) for a pair $(u, b)$, we conclude that
\[
t^{n/4} \|(u, b)\|_{L^\infty(\mathbb{R}^n)} \to 0, \text{ as } t \to \infty,
\]
for $n = 2, 3$. A particular case of the fundamental Gagliardo-Nirenberg inequality ensures that
\[
\| (u, b) \|_{L^\infty(\mathbb{R}^4)} \leq C \|(D^2 u, D^2 b)\|_{L^2(\mathbb{R}^4)}
\]

and using the Main Theorem again one has the same property above in dimension \( n = 4 \). Now, we just have to apply a simple \( L^2 \leftrightarrow L^\infty \) interpolation to obtain

\[
\lim_{t \to \infty} t^{\frac{4-n}{2q}} \| (u, b)(\cdot, t) \|_{L^q(\mathbb{R}^n)} = 0,
\]

where \( 2 \leq q \leq \infty \) and \( 2 \leq n \leq 4 \).

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