Reaction-rate formula in out of equilibrium quantum field theory

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Abstract

A complete derivation, from first principles, of the reaction-rate formula for a generic reaction taking place in an out of equilibrium quantum-field system is given. It is shown that the formula involves no finite-volume correction. Each term of the reaction-rate formula represents a set of physical processes that contribute to the reaction under consideration.

11.10.Wx, 12.38.Mh, 12.38.Bx
I. INTRODUCTION

Ultrarelativistic heavy-ion-collision experiments at the BNL Relativistic Heavy Ion Collider (RHIC) and at the CERN Large Hadron Collider (LHC) will soon start in anticipation of producing quark-gluon plasma (QGP). Confirmation of the QGP formation is done through analyzing rates of various reactions taking place in a QGP. So far, the reaction-rate formula is derived for reactions taking place in the system in thermal and chemical equilibrium [1–4]. The actual QGP is, however, not in equilibrium but is an expanding nonequilibrium system.

In this paper, as a generalization of [1–4], we present a first-principles derivation of the reaction-probability formula for reactions occurring in a nonequilibrium system. We find that the formula involves no finite-volume corrections. We also find from the procedure of derivation that different contributions to the reaction-probability formula have clear physical interpretation, which is summarized as “out-of-equilibrium cutting rules.”

In Sec. II, we derive from first principles the formula for the transition probability of a generic reaction taking place in a nonequilibrium system. In Sec. III, specializing to quasi-uniform systems near equilibrium or nonequilibrium quasistationary systems, we further deduce the formula, finding that the formula is written in terms of the closed-time-path formalism of real-time thermal field theory [5]. In Sec. IV, we present a calculational procedure of a generic reaction-probability formula obtained in Sec. III.

II. NONEQUILIBRIUM REACTION-PROBABILITY FORMULA

A. Preliminaries

The formalism presented in this paper can be applied to a broad class of theories including QCD (cf. the end of Sec. III), but, for simplicity of presentation, we take a system of self-

\[ \text{Framework for dealing with such systems is comprehensively discussed in [5].} \]
interacting, neutral scalars $\phi$'s with mass $m$ and $\lambda\phi^4$ interaction. The system is inside a cube with volume $V = L^3$. Employing the periodic boundary conditions, we label the single-particle basis by its momentum $p_k = 2\pi k/L$, $k_j = 0, \pm 1, \pm 2, \cdots, \pm \infty$ ($j = 1, 2, 3$).

Physically interesting reactions are of the following generic type,

$$\{A\} + \text{nonequilibrium system} \rightarrow \{B\} + \text{anything}.$$  \hspace{1cm} (2.1)

Here $\{A\}$ and $\{B\}$ designate group of particles, which are different from $\phi$. Examples are highly virtual particles, heavy particles, and particles interacting weakly with $\phi$'s. Generalization to more general process, where among $\{A\}$ and/or $\{B\}$ are $\phi$'s, is straightforward (cf., [4]). For definiteness, let us assume that $\{A\}$ consists of $l\Phi$'s and $\{B\}$ consists of $l'\Phi$'s. Here $\Phi$ is a heavy neutral scalar of mass $M$, so that $\Phi$ is absent in the system. For simplicity of presentation, we assume a $\Phi-\phi$ coupling to be of the form $-g\Phi\phi^n/n!$ ($n \geq 2$).

The transition or reaction probability $P$ of the process (2.1) is written as

$$P = \frac{N}{D}, \hspace{1cm} (2.2a)$$

$$N \equiv \sum_{\{k\}} \sum_{\{n_k\}} \sum_{\{m_k\}} \sum_{\{n'_k\}} \langle \{A\}; \{m_k\} | S^\dagger | \{n'_k\}; \{B\} \rangle$$

$$\times \langle \{B\}; \{n'_k\} | S | \{n_k\}; \{A\} \rangle \langle \{n_k\} | \rho | \{m_k\} \rangle \mathbf{S}, \hspace{1cm} (2.2b)$$

$$D \equiv \sum_{\{k\}} \sum_{\{n_k\}} \sum_{\{m_k\}} \sum_{\{n'_k\}} \langle \{m_k\} | S^\dagger | \{n'_k\} \rangle$$

$$\times \langle \{n'_k\} | S | \{n_k\} \rangle \langle \{n_k\} | \rho | \{m_k\} \rangle \mathbf{S}. \hspace{1cm} (2.2c)$$

Here $\mathbf{S}$ is the symmetry factor [3], $\rho$ is the density matrix, and $\langle \{B\}; \{n'_k\} | S | \{n_k\}; \{A\} \rangle$ is a $S$-matrix element of the vacuum-theory process,

$$\{A\} + \{n_k\} \rightarrow \{B\} + \{n'_k\},$$

where $\{n_k\}$ denotes the group of $\phi$'s, which consists of the number $n_k$ of $\phi_k$ ($\phi$ in a mode $k$).

In Eqs. (2.2), $\sum_{\{k\}}$ denotes summation over momentum/momenta of $\phi/\phi$'s in the final state $|\{n'_k\}; \{B\}\rangle$, and of $\phi/\phi$'s in the "two" initial states $|\{m_k\}; \{A\}\rangle$ and $|\{n_k\}; \{A\}\rangle$. (Among
the final states $|\{n'_k\}; \{B\}\rangle$ is $|0; \{B\}\rangle$. This is also the case for "two" initial states.) Note that the perturbation series for $D$ starts from 1,

$$D = 1 + \cdots .$$

It is to be noted that $\{A\}$ and $\{B\}$ in $\langle S \rangle$, which we write $\{A, B\}_S$, are not necessarily involved in one connected part of $\langle S \rangle$. This is also the case for $\{A, B\}_S^\dagger$. We assume that, in $W \equiv \langle S^\dagger \rangle \langle S \rangle$, $\{A, B\}_S$ and $\{A, B\}_S^\dagger$ are involved in one connected part $W_c (\in W)$. Then, $W$ consists, in general, of $W_c$ and other parts which are disconnected with $W_c$ and include only $\phi$'s. Generalization to other cases is straightforward [4]. It should be remarked on the form of $\rho$ in Eqs. (2.2). Let us recall the following two facts. On the one hand, the statistical ensemble is defined by the density matrix at the very initial time $t_i (\sim -\infty)$. On the other hand, in constructing perturbative framework, an adiabatic switching off of the interaction is required [7,3]. Then, $\rho$ in Eqs. (2.2) is a functional of the in-field $\phi_{in}(t_i, x)$ that constitutes the basis of perturbation theory.

As will be seen below, diagrammatic analysis shows that $N$, Eq. (2.2b), takes the form,

$$N = N_{\text{con}} D ,$$

where $N_{\text{con}}$ corresponds to a connected diagram and $D$ is as in Eq. (2.2c). Then, we have

$$P = N_{\text{con}} .$$

The $S$-matrix element in vacuum theory is obtained through an application of the reduction formula [2,4]:

$$\langle \{B\}; \{n'_k\} | S | \{n_k\}; \{A\}\rangle = \prod_{j=1}^{l'} (iK_{j, \phi_j}) \prod_{m=1}^{l'} (iK^*_{m, \phi_m}) \langle 0 | \prod_k T \left\{ \begin{array}{c} n_k \sum_{i_k=0}^{n_k} \delta(n_k - i_k) \sum_{n'_k=0}^{i_k} n'_k \delta(n'_k - i'_k) N_{i_k, n'_k} \\ \prod_{n'=1}^{i'_k} (iK^*_{k, n'}) \prod_{n=1}^{i_k} (iK_{k, n}) \prod_{n'=1}^{i'_k} \phi_{n'} \prod_{n=1}^{i_k} \phi_n \end{array} \right\} \prod_{j=1}^{l} \Phi_j \prod_{m=1}^{l'} \Phi_m \rangle | 0 \rangle ,$$

where $T$ is the time-ordering symbol and
\[
N_{ik,\,i_k'}^{nk,n_k'} \equiv \left( \frac{\binom{n_k'}{i_{ik}'}}{\binom{n_k}{i_{ik}}} \frac{1}{i'_{ik}! \, i_{ik}!} \right)^{1/2} ,
\]

(2.5)

In Eq. (2.4) \(\delta(\cdots; \cdots)\) denotes the Kronecker’s \(\delta\)-symbol and

\[
K_{k,n} \cdots \phi_n \equiv \frac{1}{\sqrt{Z_\phi}} \int d^4x \, f_{p_k}(x) (\Box + m^2) \cdots \phi(x) ,
\]

\[
K_{j,\, \Phi_j} \cdots \Phi_j \equiv \frac{1}{\sqrt{Z_\Phi}} \int d^4x \, F_j(x)(\Box + M^2) \cdots \Phi(x) ,
\]

\[
K_{m,\, \Phi_m}^* \cdots \Phi_m \equiv \frac{1}{\sqrt{Z_\Phi}} \int d^4x \, G_m^*(x)(\Box + M^2) \cdots \Phi(x) .
\]

(2.6)

Here

\[
f_{p_k}(x) = \frac{1}{\sqrt{2E_k}} e^{-iP_k \cdot x} , \quad (E_k = \sqrt{P_k^2 + m^2}) ,
\]

with \(P_k^\mu \equiv (E_k, \, p_k)\) and \(F_j(x) \, [G_m^*(x)]\) the wave function of \(j\)th \(\Phi \, (\in \{ A \})\) \(m\)th \(\Phi \, (\in \{ B \})\). \(Z\)'s in Eq. (2.6) are the wave-function renormalization constants. It is to be noted that, in Eq. (2.4), among \(n_k \, (n'_k)\) of \(\phi_k\)'s in the initial (final) state, \(i_k \, (i'_k)\) of \(\phi_k\)'s are absorbed in (emitted from) the \(i_k \, (i'_k)\) vertices in \(S\). Remaining \(n_k - i_k \, (= n'_k - i'_k)\) of \(\phi_k\)'s are merely spectators, which reflects only on the statistical factor in \(\mathcal{F}_i\) in Eq. (3.13) below.

\(\langle S \rangle\) in \(\mathcal{D}\) in Eq. (2.24) is given by a similar expression to Eq. (2.4), where factors related to the \(\Phi\) fields are deleted.

\(\langle S \rangle\) in \(\mathcal{D}\) in Eq. (2.4), we see that the permutation of \(\phi_n \, (n = 1, \cdots, i_k)\) and the permutation of \(\phi_n' \, (n' = 1, \cdots, i'_k)\) give the same Feynman diagram (in vacuum theory), and then \(i_k! \, i'_k!\) same diagrams emerge. Taking this fact into account, we may write (2.4) in the form,

\[
\langle \{ B \}; \{ n'_k \}; S | \{ n_k \}; \{ A \} \rangle = \left( \prod_{j=1}^{\nu} \int d^4x_j F_j(x_j) \right) \left( \prod_{m=1}^{\nu'} \int d^4y_m G_m^*(y_m) \right)
\]

\[
\times \sum_{\{i_k\}} \left[ \prod_{k} N_{ik,\,i_k'}^{nk,n_k'} i_k! i'_k! \left( \prod_{j=1}^{\nu} \int d^4\xi_j \, f_{p_k}(\xi_j) \right) \left( \prod_{j=1}^{\nu'} \int d^4\zeta_j \, f_{p_k}(\zeta_j) \right) \right]
\]

\[
\times A(\{ y \}; \{ \xi \}; \{ \zeta \}; \{ x \}) ,
\]

(2.7)
where \( i'_k = n'_k - n_k + i_k \) and \( \mathcal{A} \) is the truncated Green function in configuration space (in vacuum theory), and, e.g., \( \{ y \} \) collectively denotes \( y_1, y_2, \cdots, y_y \).

Among the Feynman diagrams for \( \mathcal{A} \), are some diagrams, in which some \( \xi \)'s (in \( \xi \)’s) \( \zeta \)'s (in \( \zeta \)’s) coincide with \( x \)'s (in \( x \)’s) and/or \( y \)'s (in \( y \)’s) and/or \( \zeta \)'s (in \( \zeta \)’s) \( \xi \)'s (in \( \xi \)’s). In such cases, \( \mathcal{A} \) is understood to include corresponding \( \delta \)-functions, e.g., \( \delta^4(\xi_{kj} - x_i) \).

The expression for \( \langle S^\dagger \rangle \), the complex conjugate of \( \langle S \rangle \), is obtained by taking the complex conjugate of Eq. (2.4) or Eq. (2.7), where we make the substitution (cf. Eqs. (2.2b) and (2.2c)),

\[
\begin{align*}
   n_k &\rightarrow m_k, \\
   n'_k &\rightarrow m'_k (= n'_k) \\
   i_k &\rightarrow j_k, \\
   i'_k &\rightarrow j'_k.
\end{align*}
\]

This applies also to the expression for \( \langle S^\dagger \rangle \) in Eq. (2.2d).

Substitution of \( W = \langle S^\dagger \rangle \langle S \rangle \) into Eq. (2.2d) yields, with obvious notation,

\[
\mathcal{N} = \left( \prod_{j=1}^{l} \int d^4x_j \right) \left( \prod_{j'=1}^{l'} \int d^4y_{j'} \right) \left( \prod_{m=1}^{p} \int d^4y_m \right) \left( \prod_{m'=1}^{p'} \int d^4y_{m'} \right) \frac{G}{\mathcal{A}^* \mathcal{A}} \\
\times \sum_{\{ k \}} \sum_{\{ i_k \}} \sum_{\{ j_k \}} \sum_{\{ i'_k \}} \sum_{\{ j'_k \}} \left[ \prod_{k} \left( \prod_{j=1}^{i_k} \int d^4\xi_{kj} f_{pk}(\xi_{kj}) \right) \left( \prod_{j'=1}^{i'_k} \int d^4\xi_{kj} f_{pk}'(\xi_{kj}) \right) \right] \\
\times \mathcal{S} \mathcal{W}(\{ x \}, \{ \xi \}, \{ \xi' \}, \{ y \}) \mathcal{S}^* \mathcal{A}^* \mathcal{A}, \quad (2.8)
\]

Here \( i'_k = n'_k - n_k + i_k, \) \( j'_k = n'_k - m_k + j_k \), \( \mathcal{W} = \mathcal{A}^* \mathcal{A} \), and

\[
\mathcal{S} \equiv \sum_{\{ n_k \}} \left( \prod_{k} \mathcal{N} \mathcal{N} \right) \langle n_k \rangle \langle \rho | \{ m_k \} \rangle. \quad (2.9)
\]

### B. Statistical factor \( \mathcal{S} \)

Here, it is convenient to introduce creation and annihilation operators, \( a_{pk}^\dagger \) and \( a_{pk} \), which satisfy \( [a_{pk}, a_{pk}'] = \delta_{k,k'} \) and \( [a_{pk}, a_{pk}'] = 0 \). A Fock space \( \mathcal{F} \) is constructed on \( |0\rangle \), which is defined by \( a_{pk}|0\rangle = 0 \). For the vector \( | \rangle \) (in \( \mathcal{F} \)) that satisfies \( a_{pk}^\dagger a_{pk} | \rangle = n_{pk} | \rangle \) \( (n_{pk} = 0, 1, 2, \cdots) \), we use the same notation as in Eq. (2.3), \( | \{ n_k \} \rangle \), since no confusion
arises. A key observation here is that, using the form (2.5), one can easily show that \( S \), Eq. (2.9), may be represented as

\[
S = \sum_{\{n_k\}} \langle \{m_k\} | \left( \prod_{l=1}^{j} a_{p_l}' \right) \left( \prod_{l=1}^{j'} a_{q_l}' \right) \left( \prod_{l=1}^{i} a_{p_l} \right) | \{n_k\} \rangle \langle \{n_k\} | \rho | \{m_k\} \rangle
\]

\begin{equation}
\equiv \left\langle \left( \prod_{l=1}^{j} a_{p_l}' \right) \left( \prod_{l=1}^{j'} a_{q_l}' \right) \left( \prod_{l=1}^{i} a_{p_l} \right) \right\rangle ,
\end{equation}

(2.10)

where we write

\[
\{P_1, \cdots, P_i\} = \{ \underbrace{\cdots, \underbrace{P_k, \cdots, P_k, \cdots}_{i_k}, \cdots} \}
\]

and then \( i = \sum_k i_k \). Similarly, \( i' = \sum_k i'_k \), \( j = \sum_k j_k \), and \( j' = \sum_k j'_k \). Note that \( \langle \{n_k\} | \rho | \{m_k\} \rangle \), in between which \( \rho \) is sandwiched, are as in Eqs. (2.2) and (2.9).

Let us write \( S \), for short, as

\[
S(b_1 b_2 \cdots b_N) = (N = i + j + i' + j').
\]

Let \( l_1, \cdots, l_m \) be a solution in positive integers of

\[
\sum_{j=1}^{m} l_j = N \quad (1 \leq m \leq N).
\]

(2.11)

Pick out \( l_1 \) b’s out of \( b_1, b_2, \cdots, b_N \) and pick out \( l_2 \) b’s out of remaining b’s, and so on, to make \( m \) groups,

\[
\{b_1 \cdots b_{i_1}\} \{b_{i_1+1} \cdots b_{i_1+l_2}\} \cdots \{b_{i_N-l_m+1} \cdots b_N\},
\]

(2.12)

where \( 1 < i_{1+1} < i_{1+l_2+1} < \cdots < i_{N-l_m+1} \leq N \). In Eq. (2.12), let \( b_l \) and \( b_{l'} \) are in between one set of curly brackets. Then, if \( l < l' \), \( b_l \) is located at the left of \( b_{l'} \) and vice versa. We are now in a position to write

\[
S(b_1 \cdots b_N)
= \sum_{m=1}^{N} \sum_{l_1 \cdots} \sum_{l_2 \cdots} S_c(b_1 \cdots b_{i_1})
\times S_c(b_{i_{1+1}} \cdots b_{i_{1+l_2}}) \cdots S_c(b_{i_{N-l_m+1}} \cdots b_N).
\]

(2.13)
Here, the second summation $\sum_{l_s'}$ runs over all solutions in integers of Eq. (2.11) and the third summation $\sum_{gr}$ runs over all ways of making $m$ groups as in Eq. (2.12). From Eq. (2.13), $S_c$ is determined iteratively. For example,

$$S_c(b_1b_2) = S(b_1b_2) - S(b_1)S(b_2)$$

$$S_c(b_1b_2b_3) = S(b_1b_2b_3) - S_c(b_1b_2)S(b_3) - S_c(b_1b_3)S(b_2) - S(b_1)S_c(b_2b_3) - S(b_1b_3)S(b_2).$$

Thus, we have, with obvious notation,

$$S = \sum_{m=1}^{i+j+i'+j'} \sum_{l_s'} \sum_{gr} S_c(\cdots)S_c(\cdots) \cdots S_c(\cdots). \quad (2.14)$$

In the case of equilibrium system, all but $\langle a_p^\dagger a_p^\dagger \rangle$ and $\langle a_q^\dagger a_p \rangle$ vanish. From the definition of $S_c$, it is not difficult to show that, for $N \geq 3$,

$$S_c(b_1b_2 \cdots b_{ji}) = S_c(\vdots b_1b_2 \cdots b_{ji} \vdots), \quad (2.15)$$

where $\vdots \cdots \vdots$ indicates to take the normal ordering with respect to the creation and annihilation operators.

**C. Reaction-probability formula**

Now, $N$ in Eq. (2.8) may be written as

$$N = \left( \prod_{j=1}^{l} \int d^4 x_j d^4 x'_j F_j(x_j) F_j^*(x'_j) \right) \left( \prod_{j=1}^{\nu} \int d^4 y_j d^4 y'_j G_j(y_j) G_j(y'_j) \right)$$

$$\times \sum_{i, j, j', j''} \left( \prod_{j=1}^{i} \int d^4 \xi_j \sum_{p_j} \frac{1}{\sqrt{2E_{p_j}V}} e^{-ip_j\xi_j} \right) \left( \prod_{j=1}^{j'} \int d^4 \xi_j \sum_{q_j} \frac{1}{\sqrt{2E_{q_j}V}} e^{iq_j\xi_j} \right)$$

$$\times \left( \prod_{l=1}^{j} \int d^4 \xi'_{l} \sum_{p'_{l}} \frac{1}{\sqrt{2E_{p'_{l}}V}} e^{ip'_{l}\xi'_{l}} \right) \left( \prod_{l=1}^{j'} \int d^4 \xi'_{l} \sum_{q'_{l}} \frac{1}{\sqrt{2E_{q'_{l}}V}} e^{-iq'_{l}\xi'_{l}} \right)$$

$$\times S \mathcal{W}(\{x'\}, \{\xi'\}; \{\xi''\}, \{y'\}; \{y\}, \{\xi\}; \{\xi\}, \{x\}) S. \quad (2.16)$$

Carrying out the integration over $\xi$'s, $\zeta$'s, $\xi''$'s, $\zeta'$'s and the internal spacetime vertex points, which are included in $\mathcal{W}$, we obtain, with obvious notation,
\[ N = \left( \prod_{j=1}^{\ell} \int d^4 x_j d^4 x'_j F_j(x_j) F_j^*(x'_j) \right) \left( \prod_{j=1}^{\ell'} \int d^4 y_j d^4 y'_j G_j(y_j) G_j(y'_j) \right) \times \sum_{i,j,i',j'} \left( \prod_{j=1}^{i} \sum_{p_j} \frac{1}{\sqrt{2E_{p_j}V}} \right) \left( \prod_{j=1}^{i'} \sum_{q_j} \frac{1}{\sqrt{2E_{q_j}V}} \right) \left( \prod_{l=1}^{j} \sum_{p'_l} \frac{1}{\sqrt{2E_{p'_l}V}} \right) \left( \prod_{l=1}^{j'} \sum_{q'_l} \frac{1}{\sqrt{2E_{q'_l}V}} \right) \times S \mathcal{W}(\{x\}, \{p\}; \{q\}, \{y\}, \{s\}; \{r\}) \] (2.17)

Let us Fourier transform the wave functions \( F_j(x) \), \( G_j(x) \)

\[ F_j(x) = \int \mathcal{D}r_j e^{-iR_j \cdot (x - X_c)} \tilde{F}_j(r_j), \]
\[ G_j(x) = \int \mathcal{D}r_j e^{-iR_j \cdot (x - X_c)} \tilde{G}_j(r_j), \] (2.18)

where \( R_j^\mu = (E_j, r_j) \) with \( E_j = \sqrt{r_j^2 + M^2} \). In Eq. (2.18), \( X_c \) of \( X^\mu_c = (X_c^\alpha, X_c) \) is the space point, around which \( \Phi \)'s are localized and \( X_c^\alpha \) is the time, around which the reaction takes place. In general, \( \tilde{F}_j \) and \( \tilde{G}_j \) also depend on \( X_c \).

Substituting (2.18) into Eq. (2.17) and carrying out the integration over \( x_j \), \( x'_j \), \( y_j \) and \( y'_j \), we obtain

\[ N = \left( \prod_{j=1}^{\ell} \int \mathcal{D}r_j \mathcal{D}r'_j \tilde{F}_j(r_j) \tilde{F}_j^*(r'_j) \right) \left( \prod_{j=1}^{\ell'} \int \mathcal{D}s_j \mathcal{D}s'_j \tilde{G}_j(s_j) \tilde{G}_j(s'_j) \right) \times \sum_{i,j,i',j'} \left( \prod_{j=1}^{i} \sum_{p_j} \frac{1}{\sqrt{2E_{p_j}V}} \right) \left( \prod_{j=1}^{i'} \sum_{q_j} \frac{1}{\sqrt{2E_{q_j}V}} \right) \left( \prod_{l=1}^{j} \sum_{p'_l} \frac{1}{\sqrt{2E_{p'_l}V}} \right) \left( \prod_{l=1}^{j'} \sum_{q'_l} \frac{1}{\sqrt{2E_{q'_l}V}} \right) \times 2\pi\delta[\sum r_{j0} - \sum s_{j0} + \sum p_0 - \sum q_0] \times 2\pi\delta[\sum s_{j0}' - \sum r_{j0}' - \sum p_0' + \sum q_0] \times V\delta(\sum r_j - \sum s_j; \sum q - \sum p) V\delta(\sum s_j' - \sum r_j'; \sum p' - \sum q') \times S \mathcal{W}(\{r\}, \{p\}; \{q\}, \{s\}; \{r\}) \] (2.19)

Note that, when \( \langle S \rangle \) (\( \in W \)) or \( \langle S^\dagger \rangle \) consists of several disconnected parts, corresponding (momentum-conservation) \( \delta \)-function above becomes product of several \( \delta \)-functions.

The form for \( D \), Eq. (2.24), is given by Eq. (2.18) or Eq. (2.19), in which factors related to the \( \Phi \) fields are deleted.

In general, \( N \) consists of several graphically disconnected parts. As assumed in Sec. IIA, all \( \Phi \)'s are included in one connected parts \( N_{con} \). Other parts, which we write \( D \), include
only the constituent particles φ’ of the system. Then, it is obvious that \( N \) takes the form \( N = N_{\text{con}}D \) (cf. Eq. (2.3)). It is also obvious that \( D \) is a contribution to \( D \) in Eqs. (2.2). Then, such contribution does contribute to the reaction-probability \( P \), Eq. (2.2a), as \( N_{\text{con}} \), which has already been dealt with in a lower-order level. Thus, computation of \( N \)’s, which consist of one connected part, is sufficient.

### III. OUT-OF-EQUILIBRIUM REACTION-PROBABILITY FORMULA

#### A. Preliminaries

In this section, we restrict our concern to quasiuniform systems near equilibrium and nonequilibrium quasistationary systems, which we simply refer to as out-of-equilibrium systems. Such systems are characterized \(^3\) by weak dependence of the reaction probabilities on \( X_c \) (cf. above after Eq. (2.18)). More precisely, there exists a spacetime scale \( L^\mu \), such that the reaction probabilities do not appreciably depend on \( X_c \), when \( X_c \) is in the spacetime region \( |X_c^\mu - X_{c0}^\mu| \lesssim L^\mu \) with \( X_{c0}^\mu \) an arbitrary spacetime point. For such systems, the reactions are regarded as taking place in the region \( |X_c^\mu - X_{c0}^\mu| \lesssim L^\mu \). Going to the momentum space, this means that the contribution (to the reaction probability \( N \)) from the state that includes “very soft” momentum \( |P^\mu| \lesssim 1/L^\mu \) should be small. More precisely, the contribution from the summation-region in Eq. (2.16), in which at least one momentum (out of \( \{ p_j, q_j, p'_j, q'_j \} \)) is “very soft” is negligibly small. \(^2\)

\(^2\)This is the case for most practical cases, which can be seen as follows. Let \( \mathcal{T} \) be a typical scale(s) of the system under consideration. In the case of thermal-equilibrium system, \( \mathcal{T} \) is the temperature of the system. Due to interactions, an effective mass is induced and the vacuum-theory mass \( m \) turns out to the effective mass \( M_{\text{eff}}(X_c) \). In the case of \( m >> \sqrt{xT} \), \( M_{\text{eff}}(X_c) \) is not much different from \( m \) and, for \( m \lesssim \sqrt{xT} \), a tadpole diagram induces mass of \( O(\sqrt{xT}) \), so that \( M_{\text{eff}}(X_c) = O(\sqrt{xT}) \). \( \sqrt{xT} \) (or even \( \lambda T \)) is the scale that characterizes reactions. We assume
Let us pick out $\langle a_p \rangle$ from $\mathcal{S}$ in (2.14), which appears in $\mathcal{N}$, Eq. (2.16), in the form

$$\sum_p \frac{1}{\sqrt{2E_p V}} \langle a_p \rangle e^{-iP_\omega},$$

where $\omega$ stands for $\xi_j$ or $\zeta_j'$. The above observation shows that the quantity (3.1) does not appreciably depend on $\omega^\mu$, when $|\omega^\mu - X^\mu_0| \lesssim L^\mu$. This means that $\langle a_p \rangle \simeq 0$ for $|p^i| \gtrsim 1/L^i$ and $p^0 = E_p \gtrsim 1/L^0$. Then, the argument at the end of the above paragraph shows that the contribution to $\mathcal{N}$ that include $\langle a \rangle$ can be ignored. Same reasoning shows that the contribution including $\langle a^\dagger \rangle$ and/or $\mathcal{S}_c(aa \cdots a)$ and/or $\mathcal{S}_c(a^\dagger a^\dagger \cdots a^\dagger)$ may also be ignored.

Recalling the identity (2.15), we pick out from Eq. (2.14) one $\mathcal{S}_c(a^\dagger_{p_1} \cdots a^\dagger_{p_j} a_{p_{j+1}} \cdots a_{p_n})$ ($n \geq 3$). In $\mathcal{N}$ in Eq. (2.16), this factor appears in the form

$$\sum_{\{p\}} \left( \prod_{l=1}^{n} \frac{1}{\sqrt{2E_{p_l} V}} \right) \mathcal{S}_c \left( \left( \prod_{l=1}^{j} a^\dagger_{p_l} \right) \left( \prod_{l'=j+1}^{n} a_{p_{l'}} \right) \right) \times \exp \left[ i \left( \sum_{l=1}^{j} P_l \cdot z_l - \sum_{l'=j+1}^{n} P_{l'} \cdot z_{l'} \right) \right],$$

where $p_{l0} = E_p$ ($l = 1, \cdots, n$). It is not difficult to show that among the contributions to $\mathcal{N}$, there are contributions, whose counterparts of Eq. (3.2), together with Eq. (3.2), can be united into the form

$$\mathcal{C}(\{z\}) \equiv i^{n-1} \mathcal{S}_c (: \phi(z_1) \cdots \phi(z_n) :) .$$

Here

$$\phi(z) = \sum_p \frac{1}{\sqrt{2E_p V}} \left[ a_p e^{-iP \cdot z} + a^\dagger_p e^{iP \cdot z} \right],$$

that this scale is much larger than the “very soft” momentum scale, $1/L^\mu \ll \sqrt{\lambda T}$ (or $\lambda T$). Most amplitudes, when computed in perturbation theory (to be deduced below), are insensitive to the region $|P^\mu| \lesssim O(\sqrt{\lambda T})$. Then, the contribution from the region $|P^\mu| \lesssim 1/L^\mu$ is small, since the phase-space volume is small. Incidentally, in the case of equilibrium thermal QED or QCD ($m = 0$), there are some quantities that diverge at infrared limits to leading order in hard-thermal-loop resummation scheme [8,9]. For such cases, more elaborate analysis is required.
where \( p_0 = E_p \) and ‘: \( \cdot \cdot \cdot \) ’ in Eq. (3.3) indicates to take the normal ordering. As discussed at the beginning of this subsection, for the system under consideration, the function (3.2) does not change appreciably in the region \(|\Delta Z^\mu| \lesssim L^\mu \) \((Z = \sum_{l=1}^n z_l/n)\). This leads to an approximate momentum conservation for the function (3.2):

\[
\left| \sum_{l=1}^j P_l^\mu - \sum_{l'=j+1}^n P_{l'}^\mu \right| \lesssim 1/L^\mu.
\]

(3.4)

This is also the case for \( C(\{z\}) \) in Eq. (3.3). The conditions under which the initial correlations may be ignored are discussed in [10]. In the following, we ignore the initial correlations, inclusion of which into the formula obtained below is straightforward.

After all this, in \( S \) in Eq. (2.16), we keep only \( \langle a^\dagger a \rangle \)'s:

\[
S = \sum_{m,n} \sum_{gr} \langle a^\dagger_{p^j_{i'}} a_{q^j_{i'}} \rangle \cdots \langle a^\dagger_{p^j_{i-m+1}} a_{q^j_{i-m+1}} \rangle \\
\times \left( \delta_{q_{k'_{i}}, q'_{l_{m}}} + \langle a_{q_{k'}_{i}} a_{q'_{l_{m}}} \rangle \right) \cdots \left( \delta_{q_{k_{l-n+1}}, q'_{l'l_{1}}} + \langle a_{q_{k_{l-n+1}}}, a_{q'_{l'l_{1}}} \rangle \right) \\
\times \langle a_{q_{i-n}} a_{p_{i}} \rangle \cdots \langle a_{q_{i1}} a_{p_{i1}} \rangle \langle a_{p'_{i1}} a_{p_{i1}} \rangle \cdots \langle a_{p'_{i'}} a_{p_{i'}} \rangle,
\]

(3.5)

where \( j - n = j' - m \) and \( i' - m = i - n \), which leads to \( i + j' = j + i' \).

Referring to (2.16), we use the following set-symbols throughout in the sequel:

\[
\mathcal{V}_{\Phi} = \mathcal{V}_{\Phi}^S \cup \mathcal{V}_{\Phi}^{St} ; \quad \mathcal{V}_{\Phi}^S = \{x\} \cup \{y\} ; \quad \mathcal{V}_{\Phi}^{St} = \{x'\} \cup \{y'\},
\]

\[
\mathcal{V}_{e} = \mathcal{V}_{e}^S \cup \mathcal{V}_{e}^{St} ; \quad \mathcal{V}_{e}^S = \{\xi\} \cup \{\zeta\} ; \quad \mathcal{V}_{e}^{St} = \{\xi'\} \cup \{\zeta'\},
\]

and \( \mathcal{V}_{i} = \mathcal{V}_{i}^S \cup \mathcal{V}_{i}^{St} \) with \( \mathcal{V}_{i}^S \) [\( \mathcal{V}_{i}^{St} \)] the set of internal-vertex points in \( \langle S \rangle [\langle S^\dagger \rangle] \) \((\in \mathcal{W})\). When the vertex point \( \xi_{j} \) (\( \xi'_{j} \)) or \( \zeta_{i} \) (\( \zeta'_{i} \)) coincides with one of the vertex points in \( \mathcal{V}_{\Phi}^S \) (\( \mathcal{V}_{\Phi}^{St} \)), we include it in \( \mathcal{V}_{\Phi}^S \) (\( \mathcal{V}_{\Phi}^{St} \)). At the final stage, \( \mathcal{V}_{\Phi} \) (\( \mathcal{V}_{e} \cup \mathcal{V}_{i} \)) turns out to the set of external-vertex (internal-vertex) points of the out-of-equilibrium amplitude (3.19) representing \( \mathcal{P} \).

**B. Two-point function**

\[ \langle i \tilde{\Delta}(\rho, \sigma) \rangle \equiv \sum_{p,p'} \frac{1}{\sqrt{2E_p \sqrt{2E_{p'}}}} \langle a^\dagger_{p'} a_p \rangle e^{-i(p \cdot \rho - p' \cdot \sigma)}, \]

\[ 1 \]
where \( p_0 = E_p, \ p'_0 = E_p' \), and \( p \in \{p\} \cup \{q'\} \), \( p' \in \{p'\} \cup \{q\} \), \( \rho \in \{\xi\} \cup \{\xi'\} \) and \( \sigma \in \{\xi'\} \cup \{\zeta\} \). Changing \( p \) and \( p' \) to

\[
\begin{align*}
p_+ &= (p + p')/2, \\
p_- &= p - p',
\end{align*}
\]

we get

\[
i\tilde{\Delta}(\rho, \sigma) = \sum_{p, p'} \frac{1}{\sqrt{E_+ V} \sqrt{E_- V}} e^{-i(p,(\rho - \sigma))} \\
& \quad \times \tilde{N}(X; p_+), \tag{3.6}
\]

\[
\tilde{N}(X; p_+) = \sum_{p_-} e^{-i(E_+ - E_-)X_0 e^{i\rho} \cdot X} \\
& \quad \times \langle a_{p_-}^+ a_{p_+ + p_-/2} \rangle, \tag{3.7}
\]

where \( X = (\rho + \sigma)/2 \), \( E_\pm = E_{|p_+ + i\nabla X/2|} \) and \( p_+^0 = (E_+ + E_-)/2 \). It is worth mentioning in passing that one can easily derive from Eq. (3.7) \( P \cdot \partial X \tilde{N} = 0 \).

Now, Eq. (3.8) may be written as

\[
i\tilde{\Delta}(\rho, \sigma) = \sum_{p} D^4 P e^{-iP,(\rho - \sigma)} \frac{p_0}{\sqrt{E_+ E_-}} \frac{2\pi\theta(p_0)}{2\pi} \\
& \quad \times \delta \left( p_0^2 - \frac{(E_+ + E_-)}{2} \right) \tilde{N}(X; p), \tag{3.8}
\]

where

\[
\sum_{p} D^4 P \equiv \int \frac{dp_0}{2\pi} \sum_p \frac{1}{V}.
\]

As usual, we rewrite \( p = |p| \) in terms of \( p_0 \) by using \( \delta_+ (p_0^2 - \cdots) \) in Eq. (3.8). In doing so we obtain

\[
\frac{p_0}{\sqrt{E_+ E_-}} \tilde{N}(X; p) \rightarrow \left[ 1 + \frac{(p \cdot \nabla X)^2}{4p_0^4} \right]^{-1/2} \tilde{N}(X; p_0, \hat{p}),
\]

where \( \hat{p} \equiv p/p \). Carrying out the derivative expansion (expansion with respect to \( \partial X_\mu \)) and keeping up to the second-order \( X \)-derivative terms, we obtain

\[
i\tilde{\Delta}(\rho, \sigma) = \sum_{p} D^4 P e^{-iP,(\rho - \sigma)} \\
& \quad \times 2\pi\delta_+ (P^2 - m^2) N(X; p_0, \hat{p}), \tag{3.9}
\]
where $\delta_+(P^2 - m^2) = \theta(p_0)\delta(P^2 - m^2)$ and

$$N(X; p_0, \hat{p}) = \left[ 1 - \frac{1}{4\partial m^2} \left( \nabla_x^2 - (v \cdot \nabla x)^2 \right) \right.$$  

$$. \left. \frac{(v \cdot \nabla x)^2}{\delta p_0^2} + \cdots \right] \tilde{N}(X; p_0, \hat{p}).$$

(3.10)

Here $\frac{\partial}{\partial m^2}$ acts on $\delta_+(P^2 - m^2)$ in Eq. (3.9) and $v = \frac{p}{p_0}$.

In case of the system in which translation invariance holds, $\langle a_{\hat{p}}^\dagger a_q \rangle \propto \delta_{p, q}$. Eq. (3.7) tells us that $N(p_0, \hat{p}) = \tilde{N}(p_0, \hat{p})$ is the number density of a particle with momentum $p$. This allows us to interpret $N(X; p_0, \hat{p})$ as the “bare” number density of a quasiparticle with $p$ at the spacetime point $X^\mu$. (For more details, see [10].)

C. Construction of out-of-equilibrium propagators

So far, for simplicity of presentation, we have dealt with real-scalar-field systems. Physical meaning of the propagators to be deduced below can be determined more transparent manner by employing a complex-scalar-field systems, which we deal with in the sequel of this section. Let $a_p$ ($a_{\hat{p}}^\dagger$) be an annihilation [a creation] operator for a particle of momentum $p$. The antiparticle counterpart of $a_p$ ($a_{\hat{p}}^\dagger$) is $b_p$ ($b_{\hat{p}}^\dagger$). For simplicity, we assume that the density-matrix operator $\rho$ commutes with charge operator $Q$, $[\rho, Q] = 0$. Then, all but $\langle a_{\hat{p}}^\dagger a_q \rangle$, $\langle b_{\hat{p}}^\dagger b_q \rangle$, $\langle a_{\hat{p}}^\dagger b_q \rangle$, $\langle a_p b_q \rangle$ vanish. Same reasoning as at the beginning of this section shows that $\langle a_{\hat{p}}^\dagger b_q \rangle$ and $\langle a_p b_q \rangle$ are negligibly small. Thus, we are left with $\langle a_{\hat{p}}^\dagger a_q \rangle$’s and $\langle b_{\hat{p}}^\dagger b_q \rangle$’s.

A) Let us take a Feynman diagram $\mathcal{F}$ for $\mathcal{N}$ (cf. Eq. (2.16)), and pick out from $\mathcal{F}$ a vacuum-theory propagator $i\Delta^{(0)}(z_1 - z_2) = \langle 0 | T \phi(z_1)\phi(z_2) | 0 \rangle \in \langle S \rangle (\in \mathcal{N})$. Then, we pick up the following two diagrams for $\mathcal{N}$. The first one is the same as $\mathcal{F}$, except that $i\Delta^{(0)}(z_1 - z_2)$ is replaced by

$$\sum_p \frac{1}{\sqrt{2EpV}} e^{-iP \cdot z_1} \sum_q \frac{1}{\sqrt{2EqV}} e^{iQ \cdot z_2} \langle a_{\hat{p}}^\dagger a_q \rangle,$$
which is involved in Eq. (2.16). The second one is the same as \( F \), except that \( i \Delta^{(0)}(z_1 - z_2) \) is replaced by

\[
\sum_q \frac{1}{\sqrt{2E_qV}} e^{-iQz_2} \sum_p \frac{1}{\sqrt{2E_pV}} e^{iPz_1} \langle b_{p_q} b_{-q} \rangle,
\]

with \( P \equiv (E_p, -p) \), etc. Adding the above two contributions to the original contribution, and Fourier transforming on \( z_1 - z_2 \), we obtain for the relevant part,

\[
i \Delta_{11} \left( \frac{z_1 + z_2}{2}; P \right) \equiv \frac{i}{P^2 - m^2 + i0^+} + 2\pi\delta(P^2 - m^2)N \left( \frac{z_1 + z_2}{2}; p_0, \hat{p} \right).
\]  

(3.11)

Here, \( N \) with \( p_0 > 0 \) is as in Eq. (3.10) with (3.7), while, for \( p_0 < 0 \), \( N \) takes the same form (3.10) where \( \tilde{N} \) is defined, with obvious notation, as

\[
\tilde{N} \left( \frac{z_1 + z_2}{2}; p_0, \hat{p} \right) = \sum_{p_+} e^{i(E_+ + E_-)(z_{10} + z_{20})/2} \times e^{-i\hat{p}_- \cdot (z_1 + z_2)/2} \langle b_{p_+ + \hat{p}_-/2} b_{-\hat{p}_-/2} \rangle
\]

with, as before, \( E_{\pm} = E_{|p_+ \mp i\nabla_X/2|} \).

As discussed at the end of the last subsection, \( N(X; p_0, \hat{p}) \) with \( p_0 > 0 \) is the “bare” number density of a quasiparticle with momentum \( p \) at the point \( X^\mu \). Similarly, \( N(X; p_0, \hat{p}) \) with \( p_0 < 0 \) is the “bare” number density of an anti-quasiparticle with momentum \( -p \) at \( X^\mu \).

B) Starting from \( \langle S^\dagger \rangle (\in \mathcal{N}) \) that includes a vacuum-theory propagator \( [i \Delta^{(0)}(z_1 - z_2)]^\ast \) and proceeding as above A), we obtain

\[
i \Delta_{22}(X; P) \equiv [i \Delta_{11}(X; P)]^\ast
\]

\[
= \frac{-i}{P^2 - m^2 - i0^+} + 2\pi\delta(P^2 - m^2)N \left( \frac{z_1 + z_2}{2}; p_0, \hat{p} \right).
\]  

(3.12)
C) Let us take a set of Feynman diagrams $\mathcal{F}_1$ and $\mathcal{F}_2$. $\mathcal{F}_1$ contains (cf. Eq. (3.3))

$$
\sum_{p_j} \frac{1}{\sqrt{2E_{p_j}V}} e^{-iP_{j} \cdot z_1} \sum_{q_k} \frac{1}{\sqrt{2E_{q_k}V}} e^{iQ_{k} \cdot z_2} \\
\times \left( \delta_{q_k, p_j} + \langle a_{q_k}^\dagger a_{p_j} \rangle \right)
$$

(3.13)

with $z_1 \in \{\zeta\}' (\in \langle S^\dagger \rangle)$ and $z_2 \in \{\zeta\} (\in \langle S \rangle)$. $\mathcal{F}_2$ is same as $\mathcal{F}_1$ except that (3.13) is replaced by

$$
\sum_{q_k} \frac{1}{\sqrt{2E_{q_k}V}} e^{-iQ_{k} \cdot z_2} \sum_{p_j} \frac{1}{\sqrt{2E_{p_j}V}} e^{iP_{j} \cdot z_1} \langle b_{-p_j}^\dagger b_{-q_k} \rangle
$$

with $z_1 \in \{\zeta\}' (\in \langle S^\dagger \rangle)$ and $z_2 \in \{\zeta\} (\in \langle S \rangle)$. Adding the contributions from $\mathcal{F}_1$ and from $\mathcal{F}_2$, we extract the relevant part, of which the Fourier transformation on $z_1 - z_2$ is

$$
i \Delta_{21}(X; P) \equiv 2\pi \delta(P^2 - m^2) \\
\times \left[ \theta(p_0) + N \left( \frac{z_1 + z_2}{2}; p_0, \hat{P} \right) \right].
$$

(3.14)

D) Let us take a set of Feynman diagrams $\mathcal{F}'_1$ and $\mathcal{F}'_2$. $\mathcal{F}'_1$ contains

$$
\sum_{p_j} \frac{1}{\sqrt{2E_{p_j}V}} e^{-iP_{j} \cdot z_1} \sum_{q_k} \frac{1}{\sqrt{2E_{q_k}V}} e^{iQ_{k} \cdot z_2} \langle a_{q_k}^\dagger a_{p_j} \rangle
$$

(3.15)

with $z_1 \in \{\zeta\} (\in \langle S \rangle)$ and $z_2 \in \{\zeta\}' (\in \langle S^\dagger \rangle)$. $\mathcal{F}'_2$ is same as $\mathcal{F}'_1$ except that (3.13) is replaced by

$$
\sum_{q_k} \frac{1}{\sqrt{2E_{q_k}V}} e^{-iQ_{k} \cdot z_2} \sum_{p_j} \frac{1}{\sqrt{2E_{p_j}V}} e^{iP_{j} \cdot z_1} \\
\times \left( \delta_{p_j, q_k} + \langle b_{-p_j}^\dagger b_{-q_k} \rangle \right)
$$

with $z_1 \in \{\zeta\} (\in \langle S \rangle)$ and $z_2 \in \{\zeta\}' (\in \langle S^\dagger \rangle)$. Adding the contributions from $\mathcal{F}'_1$ and from $\mathcal{F}'_2$, we extract the relevant part, of which the Fourier transformation on $z_1 - z_2$ is

$$
i \Delta_{12}(X; P) \equiv 2\pi \delta(P^2 - m^2) \\
\times \left[ \theta(-p_0) + N \left( \frac{z_1 + z_2}{2}; p_0, \hat{P} \right) \right].
$$

(3.16)

Above derivation of $i \Delta_{ij}$ ($i, j = 1, 2$) is self explanatory for their physical meaning or interpretation. The physical interpretation is summarized as generalized cutting rules, which is a generalization of Cutkosky’s cutting rules in vacuum theory. (For more details, see [4].)
D. Closed-time-path formalism

\[ i\Delta_{ij} \ (i, j = 1, 2) \] obtained above are nothing but the propagators in the closed-time-path (CTP) formalism of out-of-equilibrium quantum field theory. The CTP formalism is constructed on the directed time-path \( C = C_1 \oplus C_2 \) in a complex-time plane, where \( C_1 = (-\infty \rightarrow +\infty) \) and \( C_2 = (+\infty \rightarrow -\infty) \). A field \( \phi(x_0, x) \) with \( x_0 \in C_1 \ [x_0 \in C_2] \) is denoted by \( \phi_1(x_0, x) [\phi_2(x_0, x)] \) and is called a type-1 [type-2] field. The interaction Lagrangian density is of the form,

\[ L_{\text{int}} = L^{(1)}_{\text{int}} - L^{(2)}_{\text{int}}, \]

\[ L^{(i)}_{\text{int}} = -\frac{\lambda}{4} (\phi_i^\dagger \phi_i)^2 - \frac{g}{(n!)^2} \Phi_i (\phi_i^\dagger \phi_i)^n, \quad (i = 1, 2). \]

Then, the vertex factor for the “type-1 vertex” that comes from \( L^{(1)}_{\text{int}} \) is the same as in vacuum theory, while the vertex factor for the “type-2 vertex” is minus the corresponding “type-1 vertex factor.” The CTP propagators are defined by the statistical average of the time-path-ordered product of fields, which are written as

\[ i\Delta_{11}(x, y) = \langle T\phi_1(x)\phi_1^\dagger(y) \rangle_c, \]

\[ i\Delta_{22}(x, y) = \langle T\phi_2(x)\phi_2^\dagger(y) \rangle_c = [i\Delta_{11}(y, x)]^*, \]

\[ i\Delta_{12}(x, y) = \langle \phi_2^\dagger(y)\phi_1(x) \rangle_c, \]

\[ i\Delta_{21}(x, y) = \langle \phi_2(x)\phi_1^\dagger(y) \rangle_c, \]

(3.17)

where \( T \ (\overline{T}) \) is the time-ordering (anti-time-ordering) symbol. In computing (3.17), one identifies \( \phi_2 \) with \( \phi_1 \). Comparing Eq. (3.17) with the above deduction of \( \Delta_{ij} \ (i, j = 1, 2) \), Eqs. (3.11), (3.12), (3.14), and (3.16), we see that \( x \) of \( \phi_1(x) \) in Eq. (3.17) corresponds to a vertex-point in \( \langle S \rangle \ (\in W) \) and \( x \) of \( \phi_2(x) \) corresponds to a vertex-point in \( \langle S^\dagger \rangle \). The vertex factors in \( \langle S \rangle \ (\in W) \) are \(-i\lambda \) for \(-\lambda (\phi^\dagger \phi)^2/4 \) interaction and \(-ig \) for \(-g\Phi (\phi^\dagger \phi)^n/(n!)^2 \) interaction. Then, the vertex factors in \( \langle S^\dagger \rangle \ (\in W) \) are, in corresponding order to the above, \( i\lambda \) and \( ig \). This is in accord with the above-mentioned vertex factors in the CTP formalism.
E. Reaction-probability formula

Observation made so far shows that $N$ in Eq. (2.16) with Eq. (3.5) corresponds to an amplitude in the CTP formalism of the “process,”

$$\sum_{j=1}^{l} \Phi_{1j} + \sum_{j=1}^{l'} \Phi_{2j} \rightarrow \sum_{j=1}^{l} \Phi_{2j} + \sum_{j=1}^{l'} \Phi_{1j} .$$  \hspace{1cm} (3.18)

As mentioned at the end of Sec. II, only connected $N$’s contribute to the reaction-probability $\mathcal{P}$. Thus, we finally obtain

$$\mathcal{P} = \left( \prod_{j=1}^{l} \int d^{4}x_{j} d^{4}x'_{j} F_{j}(x_{j}) F^{*}(x'_{j}) \right) \times \left( \prod_{j=1}^{l'} \int d^{4}y_{j} d^{4}y'_{j} G^{*}_{j}(y_{j}) G_{j}(y'_{j}) \right) \times \sum_{\text{diagrams}} \int d^{4}\omega_{1} \cdots \omega_{N_{d}} \mathcal{F}_{i}(X; \{ (\omega_{k} - \omega_{k'}) \}) ,$$  \hspace{1cm} (3.19)

where $\mathcal{F}_{i}$ is a connected amplitude in the CTP formalism which includes all $\Phi$’s. In Eq. (3.19), we have used $\{ \omega \}$ for collectively denoting all the (external and internal) vertex-points and the summation runs over diagrams. A pair of $\omega$’s, $\omega_{k}$ and $\omega_{k'}$, in a pair of brackets $(\cdots)$ in $\mathcal{F}_{i}$ denotes the vertex-points that are connected by $i \Delta_{kl}(\omega_{k(k')}, \omega_{k'(k)})$.

Here some remarks are in order.

1) As mentioned at the beginning of section, inclusion of the initial correlations (3.2) or (3.3) is straightforward.

2) Taking the infinite-volume limit $V \rightarrow \infty$ goes as follows:

$$\sum_{\mathbf{p}} \rightarrow \frac{V}{(2\pi)^{3}} \int d^{3}p ,$$

$$a_{\mathbf{p}} \rightarrow \sqrt{\frac{(2\pi)^{3}}{V}} a(\mathbf{p}) ,$$  \hspace{1cm} etc.

Above deduction shows that there is no finite-volume correction, in the sense that there do not exist extra contributions to $\mathcal{N}$, which disappear in the limit $V \rightarrow \infty$. It should be stressed that this statement holds for periodic boundary conditions.
3) It is clear from the above deduction (cf. Subsecs. B and C) that the CTP formalism here is formulated in terms of the “bare” number density of quasiparticles. A canonical CTP formalism is formulated in terms of the physical or observed number density of quasiparticles. How to translate the former into the latter is discussed in [10].

Finally, we make a comment on gauge theories. If we choose a physical gauge like the Coulomb gauge or the Landshoff-Rebhan variant [11] of a covariant gauge, the gauge boson may be dealt with in a similar manner to the above scalar-field case. If we adopt a traditional covariant gauge, a straightforward modification is necessary.

IV. COMPUTATIONAL PROCEDURE

In this section, we present a concrete procedure of computing the reaction probability $P$ up to $n$th-order terms with respect to the $X_\mu$ derivatives.

1) From $\mathcal{F}_i$ in Eq. (3.19), we pick out $i\Delta_{ij}(\rho, \sigma)$,

$$\Delta_{ij}(\rho, \sigma) = \sum_{\rho, \sigma} \mathcal{D}^4P e^{-iP \cdot (\rho - \sigma)} \Delta_{ij} \left( \frac{\rho + \sigma}{2}; P \right).$$

Since $\mathcal{F}_i$ includes $\Phi$’s, the vertex-point $\rho$ [$\sigma$] is connected with a vertex-point $v$ [$v'$] $\in \mathcal{V}_\Phi$ (cf. Fig. 1):

$$\frac{\rho + \sigma}{2} = \frac{1}{2} \left[ -\sum_{j=0}^k (\omega_{j+1} - \omega_j) + \sum_{j=0}^{k'} (\omega_j' - \omega_{j+1}') + v + v' \right],$$

where $\omega_0 = \rho$, $\omega_0' = \sigma$, $\omega_{k+1} = v$, $\omega_{k'+1} = v'$, with $v, v' \in \mathcal{V}_\Phi$. In Eq. (4.2), each pair of spacetime points in a pair of brackets, $\omega_{j+1}$ and $\omega_j$ [$\omega_j'$ and $\omega_{j+1}'$], is connected by one or

3Note that, in general, the vertex-points $v$ and $v'$ are not uniquely singled out. ($v$ can coincides with $v'$.) However, different choices of $v$ and $v'$ leads to the same reaction probability $P$ within the accuracy under consideration.
several $i\Delta_{kl}(\omega_{j+1}, \omega_j)$ [$i\Delta_{k'l'}(\omega_{j'}, \omega_{j'+1})$] in $F_i$ (cf. Fig. 1). Here, we note that $v$ and $v'$ may be written as

$$v = X + \tilde{v}, \quad v' = X + \tilde{v}'$$

where $X$ is the mid-point of the external-vertex points, around which the reaction is taking place:

$$X = \frac{1}{2(l + l')} \left[ \sum_{j=1}^{l}(x_j + x'_j) + \sum_{j=1}^{l'}(y_j + y'_j) \right].$$

2) Using Eqs. (4.2) and (4.3), we expand $\Delta_{ij}((\rho + \sigma)/2; P)$ in Eq. (4.1) as

$$\Delta_{ij} \left( \frac{\rho + \sigma}{2}; P \right) = \Delta_{ij}(X; P) + \frac{1}{2} \sum_{j=0}^{k} (\omega_{j+1} - \omega_j) + \sum_{j=0}^{k'} (\omega'_{j+1} - \omega'_j) \cdot \frac{\partial}{\partial X} \Delta_{ij}(X; P) + \cdots,$$  

where '⋯' stands for terms with higher-order derivative with respect to $X$. The series (4.4) is truncated at the $n$th-order terms with respect to the $X^\mu$ derivatives. The approximation in which '⋯' is ignored is called the gradient approximation.

3) Let us deal with the term with $(\omega_{j+1} - \omega_j)$ in Eq. (4.4). It can easily be shown that

$$(\omega_{j+1} - \omega_j) i\Delta_{kl}(\omega_{j+1}, \omega_j)$$

becomes

$$\begin{align*}
(\omega_{j+1} - \omega_j)^\mu \sum \mathcal{D}^4 P^' e^{-i(P+P') \cdot (\omega_{j+1}-\omega_j)} & i\Delta_{kl} \left( \frac{\omega_{j+1} + \omega_j}{2} ; P + P' \right) \\
= \sum \mathcal{D}^4 P^' e^{-i(P+P') \cdot (\omega_{j+1}-\omega_j)} & \frac{\partial}{\partial P^\mu} i\Delta_{kl} \left( \frac{\omega_j + \omega_{j+1}}{2} ; P + P' \right).
\end{align*}$$

Other terms and higher $X^\mu$-derivative terms '⋯' in Eq. (4.4) may be dealt with similarly.

All other parts of $F_i$, Eq. (3.19), than the one (4.1) may be dealt with similarly.

---

4 As in the case of some self-energy-type subdiagram, there are several $i\Delta_{kl}(\omega_{j+1}, \omega_j)$'s [$i\Delta_{k'l'}(\omega_{j'}, \omega_{j'+1})$'s] (cf. Fig. 1). In such a case, one chooses any one of them.
4) Carrying out the integrations over all vertex-points except those in $\mathcal{V}_\Phi$, we have momentum-conservation $\delta$-functions at each internal vertex point.

As discussed at the beginning of Sec. III, the wave functions of $\Phi$’s should be localized within the space region $\lesssim L^i$ ($i = 1, 2, 3$). However, for simplicity, we assume in the sequel that the wave functions of $\Phi$’s are of plane-wave form:

$$F_j(x) = e^{-iR_j \cdot x} / \left(2V \sqrt{r_j^2 + M^2}\right)^{1/2},$$
$$G_j(y) = e^{-iR_j' \cdot y} / \left(2V \sqrt{r_j'^2 + M^2}\right)^{1/2}. \tag{4.5}$$

5) We carry out the integrations over all vertex-points in $\mathcal{V}_\Phi$ to yield momentum-conservation $\delta$-functions at those vertex points and we are left with integrations over the independent or loop momenta. Keeping the terms up to the $n$th-order terms with respect to the $X_\mu$ derivatives, we obtain the final formula, which may be written in the form,

$$\mathcal{P} = \int d^4X A(X; R'_1, \cdots, R'_l; R_1, \cdots, R_l). \tag{4.6}$$

Note that $A$ depends weakly on $X$ through $N(X; Q_k)$’s. From Eq. (4.6), we see that $A$ is the reaction rate per unit volume. Incidentally, were it not for this $X$-dependence, integration over $X$ in Eq. (4.6) would yield $VT$, where $V$ is the volume of the system and $T = t_f - t_i$ is the time interval during which the reaction takes place. In the limit $V, T \to \infty$, the $VT$ becomes

$$\lim_{V, T \to \infty} V T = (2\pi)^4 \delta^4(0).$$

Example

Here, for the purpose of illustration, we deal with the heavy-$\Phi$ production process,

---

5It is to be noted that, if we use the the plane-wave form \[4.5\] in Eq. (2.13), $X$-dependence disappears. In the procedure presented here, $X$-dependence of $\mathcal{N}$ is already (partially) taken into account before arriving at 4).
The system is composed of real scalar $\phi$ with $L_{\text{int}} = -\lambda \phi^3 / 3!$, and $\Phi$ interacts with $\phi$ through $L_{\phi\Phi} = -g \Phi \phi^2 / 2$. We analyze the contribution from Fig. 2 for $P$ in Eqs. (2.2). Using Eq. (2.16), we have

$$
N = g^2 \lambda^2 \int d^4x' G^*(x') \int d^4y' G(y') \int d^4\xi \int d^4\xi' \sum_{\mathbf{p}_1} \frac{1}{\sqrt{2E_{\mathbf{p}_1}V}} e^{-i\mathbf{P}_1 \cdot \mathbf{x}'} \sum_{\mathbf{p}_2} \frac{1}{\sqrt{2E_{\mathbf{p}_2}V}} e^{-i\mathbf{P}_2 \cdot \xi} \sum_{\mathbf{q}} \frac{1}{\sqrt{2E_{\mathbf{q}'}V}} e^{i\mathbf{P}_2' \cdot \mathbf{x}'} \sum_{\mathbf{p}_1'} \frac{1}{\sqrt{2E_{\mathbf{p}_1'}V}} e^{i\mathbf{P}_1' \cdot \mathbf{y}'} \sum_{\mathbf{q}'} \frac{1}{\sqrt{2E_{\mathbf{q}'}V}} e^{-i\mathbf{P}_2' \cdot \xi'} \sum_{\mathbf{p}_1''} \frac{1}{\sqrt{2E_{\mathbf{p}_1''}V}} e^{i\mathbf{P}_1'' \cdot \mathbf{x}'} \sum_{\mathbf{q}''} \frac{1}{\sqrt{2E_{\mathbf{q}''}V}} e^{i\mathbf{P}_2'' \cdot \mathbf{y}'} \sum_{\mathbf{q}'''} \frac{1}{\sqrt{2E_{\mathbf{q'''}}V}} e^{-i\mathbf{P}_2'' \cdot \xi'} S \Delta^{(0)} (\xi - x') (i\Delta^{(0)} (\xi' - y'))^*,
$$

where $\Delta^{(0)}$ is the vacuum-theory propagator of $\phi$ and $S$ (cf. Eq. (2.10)) takes the form

$$
S = \langle a_{\mathbf{p}_1}^{\dagger} a_{\mathbf{p}_2}^{\dagger} a_{\mathbf{q}} a_{\mathbf{q}'} a_{\mathbf{p}_2} a_{\mathbf{p}_1} \rangle = \langle a_{\mathbf{p}_1}^{\dagger} a_{\mathbf{p}_2}^{\dagger} (\delta_{\mathbf{q}', \mathbf{q}} + a_{\mathbf{q}'}^{\dagger} a_{\mathbf{q}}) a_{\mathbf{p}_1} a_{\mathbf{p}_2} \rangle.
$$

We compute the contributions that include only two-point functions. If necessary, the contributions including initial correlations may be written down in a straightforward manner. Keeping the terms that do not vanish kinematically, we have

$$
S = S_1 + S_2,
$$

$$
S_1 = \langle a_{\mathbf{p}_1}^{\dagger} a_{\mathbf{p}_2}^{\dagger} a_{\mathbf{q}} a_{\mathbf{q}'} a_{\mathbf{p}_2} a_{\mathbf{p}_1} \rangle [\delta_{\mathbf{q}', \mathbf{q}} + \langle a_{\mathbf{q}'}^{\dagger} a_{\mathbf{q}} \rangle],
$$

$$
S_2 = S_1 \bigg|_{\mathbf{p}_1 \leftrightarrow \mathbf{p}_2}.
$$

We compute the contribution $N_1$ from $S_1$. The contribution from $S_2$ may be computed similarly. Following the procedure presented above, we obtain

$$
N_1 = g^2 \lambda^2 \int d^4x' G^*(x') \int d^4y' G(y') \int d^4\xi \int d^4\xi' \sum_{\mathbf{p}_1} D^4P_1 e^{-i\mathbf{P}_1 \cdot (x' - y')} 2\pi \delta_+(P_1^2 - m^2) N \left( \frac{x' + y'}{2} ; P_1 \right)
$$

22
\[
\times \sum \int D^4 P_2 e^{-iP_2 \cdot (\xi - \xi')} 2\pi \delta_+(P_2^2 - m^2) N\left(\frac{\xi + \xi'}{2}; P_2\right) \\
\times \sum \int D^4 Q e^{-iQ \cdot (\xi' - \xi)} 2\pi \delta_+(Q^2 - m^2) \left\{1 + N\left(\frac{\xi + \xi'}{2}; Q\right)\right\} \\
\times \sum \int D^4 P' e^{-iP' \cdot (\xi - x')} \frac{i}{P'^2 - m^2 + i0^+} \sum \int D^4 Q' e^{-iQ' \cdot (y' - \xi')} \frac{-i}{Q'^2 - m^2 - i0^+}.
\]

Here we observe that
\[
\frac{\xi + \xi'}{2} - \frac{x' + y'}{2} = \frac{1}{2} [\xi - x' + \xi' - y'] \rightarrow -\frac{i}{2} \left(\frac{\partial}{\partial P'} - \frac{\partial}{\partial Q'}\right),
\]
where the partial derivatives are understood to act on the “propagators” in momentum representation.

Making the plane-wave approximation for \(G(x)\),
\[
G(x) = \frac{e^{-iR \cdot x}}{\sqrt{2E_\Phi V}} \quad (E_\Phi = \sqrt{r^2 + M^2}),
\]
we finally obtain, within the gradient approximation,
\[
\mathcal{N}_1 \simeq \frac{g^2 \lambda^2}{2E_\Phi V} \int d^4 X \sum \int D^4 P_1 \sum \int D^4 P_2 \left[2\pi \delta_+(P_1^2 - m^2) \tilde{N}(X; P_1)\right] \\
\times \left[2\pi \delta_+(P_2^2 - m^2) \tilde{N}(X_2; P_2)\right] \left[2\pi \delta_+(Q^2 - m^2)\{1 + \tilde{N}(X_1; Q)\}\right] \\
\times \left[1 - \frac{i}{2} \left(\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_1}\right) \cdot \left(\frac{\partial}{\partial P'} - \frac{\partial}{\partial Q'}\right)\right] \frac{1}{P'^2 - m^2 + i0^+} \\
\times \frac{1}{Q'^2 - m^2 - i0^+} \bigg|_{X_1 = x_2 = x, Q' = P'},
\]
where \(X = (x' + y')/2\) and \(P' = Q' = P_1 - R\) and \(Q = P_2 + P_1 - R\).

Eq. (4.8) corresponds to a contribution to the amplitude in the CTP formalism of the “process” (cf. Eq. (3.18)), \(\Phi_2(R) \rightarrow \Phi_1(R)\), and constitutes a part of the diagram as depicted in Fig. 3 in the CTP formalism. As a matter of fact, Eq. (4.8) represents Fig. 3 with \((p_{10} > 0, p_{20} > 0, q_0 > 0)\) plus Fig. 3 with \((p_{10} > 0, p_{20} < 0, q_0 < 0)\).
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FIGURES

FIG. 1. A diagram for $F_i$ in Eq. (3.19). $i$, $j$, $k$, and $l$ are the vertex-type. Each $\Phi$ is either type-1 or type-2.

FIG. 2. A diagram representing $N$, Eq. (2.24), for the process (4.7). The spacetime points $\xi$ and $x'$ ($\xi'$ and $y'$) are connected by a vacuum-theory propagator. The dot-dashed line stands for the final-state-cut line. The group of particles on top of the figure represents the spectator particles.

FIG. 3. An amplitude for the “process” $\Phi_2(R) \to \Phi_1(R)$ in the CTP formalism, a part of which represents the contribution (4.8).
FIG. 1
FIG. 2
FIG. 3