Spatial structure of the modified Coulomb potential in a superstrong magnetic field

S.I. Glazyrin\textsuperscript{1,2,3,4,*} and S.I. Godunov\textsuperscript{2,4,†}

\textsuperscript{1}Dukhov Research Institute of Automatics (VNIIA), ul. Sushchevskaya 22, Moscow, 127055 Russia
\textsuperscript{2}Alikhanov Institute for Theoretical and Experimental Physics, National Research Center Kurchatov Institute, ul. Bol’shaya Cheremushkinskaya 25, Moscow, 117218 Russia
\textsuperscript{3}National Research Nuclear University MEPhI, Kashirskoe sh. 31, Moscow, 115409 Russia
\textsuperscript{4}Novosibirsk State University, Novosibirsk, 630090, Russia

The modification of the Coulomb potential due to the enhancement of loop corrections in a superstrong magnetic field is studied numerically. We calculate the modified potential with high precision and obtain the pattern of equipotential lines. The results confirm the general features known from previous studies, however we emphasize some differences in potential structure that can be important for problems with spatially distributed charges.

I. INTRODUCTION

The Coulomb potential is responsible for the appearance of the bound states of oppositely charged particles. The behaviour \( U \propto 1/r \) originate from a single photon exchange. With the account for radiative corrections the potential will be different from the Coulomb one but the deviations are usually small. In the presence of the external fields these corrections can be enhanced.

It is well known that vacuum polarization in one loop is enhanced by an external magnetic field (proportional to magnetic field \( B \)). It was discovered by Shabad and Usov \cite{1,2} that this enhancement leads to the significant modification (screening) of the Coulomb potential for \( B \gtrsim m^2/e^3 \) where \( m \) and \( e \) are electron mass and charge. They calculated the potential in cylindrical geometry \((\rho, \phi, z)\) numerically for \( z = 0 \) with arbitrary \( \rho \) and for \( \rho = 0 \) with arbitrary \( z \) (the pointlike charge is located at \((\rho, z) = (0,0)\) and magnetic field is directed along \( z \) axis) and found asymptotics analytically.

Interpolation (but still quite accurate) formula for the polarization operator in a superstrong magnetic field was suggested by Vysotsky \cite{3}. With the help of this formula the potential for \( z = 0 \) (arbitrary \( \rho \)) and for \( \rho = 0 \) (arbitrary \( z \)) was calculated analytically \cite{4,5}. It was found \cite{1,2,5} that for \( z \gg 1/m \) the equipotential lines are ellipses, though the potential in the mid-range distances, \( r = \sqrt{\rho^2 + z^2} \lesssim 1/m \), was still unknown. In Fig. 1 in \cite{5} the equipotential lines were approximated by ellipses everywhere.

The aim of the present paper is to find out what happens with the potential at mid-range distances. To do that we evaluate the modified potential numerically. The equipotential lines at distances \( r \sim 1/m \) turn out to be “eye-shaped” (see Fig. 2,3) rather than elliptic. It means that the potential diminishes with \( \rho \) faster than it was expected earlier. Such result should be important for problems with the distributed charges, e.g. for the calculation of energy levels in the field of a nucleus with the finite radius.

In Section II we reproduce basic formulae from \cite{1–5}. In Section III we present the results of numerical calculations. We conclude in Section IV.

II. MODIFIED POTENTIAL

Let us briefly describe current analytical results on the modified potential. In the presence of an external superstrong magnetic field \((B \gg B_0 \equiv m^2/e \approx 4.4 \cdot 10^{13} \text{ G})\) the contribution of vacuum polarization at one loop level to the polarization operator becomes greatly enhanced \cite{6}:

\[
\Pi^{(2)}(k,\mathbf{k}) = -\frac{2e^3B}{\pi} \exp\left(-\frac{k^2}{2eB}\right) T(t),
\]  

\( \text{*} \quad \text{glazyrin@itep.ru} \)
\( \text{§} \quad \text{sgodunov@itep.ru} \)

\( \text{§} \quad \text{The Gauss system of units is used, } e^2 = \alpha = 1/137.035 \ldots, \text{so } m^2/e^3 = B_0/\alpha \text{ where } B_0 \equiv m^2/e \approx 4.4 \cdot 10^{13} \text{ G.} \)
where \( k = (0, \vec{k}) = \left(0, k_\perp, k_\parallel\right) \) is the momentum of the external photon, \( t \equiv k_\parallel^2/4m^2 \) and
\[
T(t) = 1 - \frac{1}{\sqrt{t(1+t)}} \log \left(\sqrt{1+t} + \sqrt{t}\right).
\]
(2)

Here we took into account only Lowest Landau Level (LLL) contribution into polarization operator which should dominate. Contributions from higher loops are also omitted in what follows. With these approximations we follow \[1\,\text{ff}.\] where the arguments in favour of this approximation can be found.

Since the polarization operator enters the photon propagator, it leads to the modification of the pointlike charge potential (we consider elementary charge \( e \)):
\[
\Phi (\rho, z) = 4\pi e \int d^2k_\perp dk_\parallel \frac{e^{-ik_\perp \cdot \rho} e^{-ik_\parallel z}}{(2\pi)^3 k_\parallel^2 + k_\perp^2 - \Pi^{(2)}(k_\perp, k_\parallel)}.
\]
(3)

After integration over the angle in the plane transverse to the magnetic field the following result was obtained \[1, 2\] (up to units and notations):
\[
\Phi (\rho, z) = e \int_{-\infty}^{\infty} dk_\parallel e^{-ik_\parallel z} \int_0^{\infty} dk_\perp \frac{k_\perp J_0 (k_\perp \rho)}{k_\perp^2 + k_\parallel^2 + 2e^3B/\pi e^3B T(k_\parallel^2/4m)}.
\]
(4)

In \[3\] the interpolation formula for \( T(t) \) was introduced:
\[
T(t) \approx \frac{2t}{2t+3}.
\]
(5)

It is rather simple and allows to perform analytical calculations, but at the same time it provides a very good accuracy (see \[4\] for details).

Using formula (5) the analytical expressions for \( \Phi (0, z) \) and \( \Phi (\rho, 0) \) were found \[4, 5\] (see also asymptotics in \[1, 2\]):
\[
\Phi (0, z) = \frac{e}{|z|} \left(1 - e^{-|z|\sqrt{6m^2}} + e^{-|z|\sqrt{(2/\pi)e^3B+6m^2}}\right),
\]
(6)

and for \( B \gg 3\pi m^2/e^3 \):
\[
\Phi (\rho, 0) = \begin{cases} 
\frac{e}{\rho} \exp \left(-\rho \sqrt{(2/\pi)e^3B}\right), & \rho < l_0, \\
\frac{e}{\rho} \sqrt{\frac{3\pi m^2}{e^3B}}, & \rho > l_0,
\end{cases}
\]
(7)

where \( l_0 \equiv \sqrt{\frac{e^3B}{2\pi e^3B}} \ln \sqrt{\frac{e^3B}{3\pi m^2}} \).

The potential at large distances, \( z \gg 1/m \), was found in \[1, 2, 5\]:
\[
\Phi (\rho, z)|_{z \gg 1/m} = \frac{e}{\sqrt{z^2 + \rho^2 \left(1 + \frac{e^3B}{3\pi m^2}\right)}},
\]
(8)

which means that potential lines at large distances are ellipses.

The equipotential lines were shown in Fig. 1 in \[5\] where they were found with the help of \[6, 7\], and \[8\]. The equipotential lines were supposed to be ellipses everywhere. The aim of this paper is to lift this assumption and to calculate initial integral (4) in all space numerically. This is done in Section \[III\].

III. NUMERICAL RESULTS

We want to calculate the potential numerically with the maximum achievable precision. In order to do that one should calculate (4). But the integral has a singularity at \( k_\perp = 0, k_\parallel = 0 \) and therefore it is not really suitable for numerical evaluation.

2 Let us note that there are contributions to the potential other than from vacuum polarization at one loop with electrons at LLL. So generally speaking, it is pointless to infinitely hunt for the precision in one particular contribution. But it is important for us to check that we can achieve high precision so in future we can take into account other effects as well.
This expression is finite for any \( \rho \) and \( z \) though for numerical integration some additional regularizations are required. The expression for \( T(t) \), Eq. (2), should be expanded into series for small \( t \). \( \Delta \Phi (0, z) \) between numerical and analytical results for any \( \rho \) and \( z \), in Fig. 1a, where both numerical and analytical results are shown for \( \Phi (0, z) \). Evaluating this exact integral we can check the precision of analytical estimations. With (6) we obtain the analytical estimate for \( \Delta \Phi (0, 0) \):

\[
\Delta \Phi_{\text{analyt}} (0, 0) = \lim_{z \to 0} \frac{e^{-z \sqrt{6m^2}} - e^{-z \sqrt{(2/\pi)eB + 6m^2}}}{z} = \sqrt{(2/\pi)eB + 6m^2} - \sqrt{6m^2}. \tag{10}
\]

For \( B = 10^4 B_0 \) we get \( \Delta \Phi_{\text{analyt}} (0, 0) / m \approx 4.793 \). The numerical value of exact integration is 4.41692858. We see that at this point the analytical solution is quiet close to the numerical one. This agreement is further confirmed in Fig. 1a where both numerical and analytical results are shown for \( \Phi (0, z) \). One can see a very good agreement between numerical and analytical results for any \( z \).

In Fig. 1b both numerical and analytical results are shown for \( \Phi (\rho, 0) \). The agreement is not so good as for \( \Delta \Phi (0, z) \).

![Figure 1](image_url)

Figure 1: Numerical and analytical results for \( B = 10^4 B_0 \) and \( B = 10^5 B_0 \). The lowest dashed line corresponds to the Coulomb potential; two blue lines (solid and dashed) above the Coulomb potential correspond to numerical and analytical results for \( B = 10^5 B_0 \); two upper lines (solid and dashed green) correspond to numerical and analytical results for \( B = 10^5 B_0 \).

Using numerical results we can obtain the correct spatial structure of the potential. The equipotential lines for \( B = 10^4 B_0 \) and \( B = 10^5 B_0 \) are shown in Fig. 2 and Fig. 3 correspondingly. The central part of the pattern shown
in Fig. 3a is magnified in Fig. 3b. We see that outer equipotential line has elliptic shape while inner lines are “eye-shaped” rather than elliptic. It means that the modified potential of the pointlike charge diminishes with $\rho$ faster than it was expected from previous studies. If we consider the nucleus with finite radius instead of the pointlike charge, we obtain that the potential magnitude along $\rho = 0$ line will be smaller for mid-range distances, $z \lesssim 1/m$, than in the case of elliptic lines (since the contribution to the potential from the charges dislocated from $\rho = 0$ will be smaller). It can be easily seen in the following configuration: let us consider the charge distributed along the ring...
\( \rho = \rho_0 = \text{const} \) in the \( z = 0 \) plane. In this case the potential along \( z \) axis (\( \rho = 0 \)) will be the same as the pointlike charge potential along \( \rho = \rho_0 \) line. Since the pointlike charge potential diminishes with \( \rho \) faster than in the case of elliptic lines, it proves the claim from above.

Let us consider an electron in the field of a nucleus with a finite size in the presence of a superstrong magnetic field. The electron is localized in the direction transverse to the magnetic field at distances \( \sim a_H \equiv 1/\sqrt{eB} \). When the magnetic field is so strong that \( a_H \) gets smaller than the nucleus size, the charge distribution becomes important. As we have shown above the potential is weaker than it was expected. It means that energy levels in the field of such a nucleus will be higher. The detailed investigation of this problem is a subject for a separate study.

IV. CONCLUSIONS

The paper considers the Coulomb potential in a superstrong external magnetic field. Due to the enhancement of loop corrections the potential is modified (screened). The results for these corrections were known from previous studies, but not in the whole space. In this paper we calculated numerically the modified potential in all space. With the help of numerical results we estimated the precision of analytical results and obtained some new features. It turned out that the equipotential lines are not ellipses in the mid-range distances, \( z \lesssim 1/m \), see Fig. 2-3. It means that potential diminishes with \( \rho \) faster than it was expected from previous studies. This feature may be important for some problems, e.g. with spatially distributed charges (like energy levels calculation for atoms and ions with high-Z nuclei).

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