ABSTRACT This paper studies the electrical conductivity of a conductive material with structures similar to graphene, which consists of three basic components. The influence of various factors on the electrical conductivity of the fractional order is also analyzed. Moreover, the new fundamentals of the hexagon \( n \) \( RL_\alpha C_\beta \) circuit network in the fractional domain are introduced, and the general laws are summarized. First, we derive three general formulas of the equivalent impedance \( Z_{AD}(n), Z_{BC}(n), \) and \( Z_{AA'}(n) \) of the circuit network by using the matrix transform method and the difference equation model in a special case. In addition, taking \( Z_{AA'}(n) \) as an example, we systematically investigate the effects of the seven system parameters (frequency \( \omega \), the number of circuit units \( n \), fractional order \( \alpha \) for inductance, fractional order \( \beta \) for capacitance, inductance \( L \), capacitance \( C \), and resistance \( R \)) on the impedance characteristics of the 3D fractional order circuit network. Specifically, interesting phenomena and laws are shown by visual figures.

INDEX TERMS 3D fractional order circuit, circuit networks, impedance characteristics.

I. INTRODUCTION

In recent decades, fractional calculus has developed rapidly, and a large number of fractal dimensions exist in nature, science, and technology [1], which are widely employed in biomedical systems [2], chaotic systems [3], signal processing systems [4], control systems [5], and other aspects. From the perspective of nonlinear circuit network systems, many scientific experiments have proven that the mathematical model of capacitance and inductance established by fractional calculus can reflect the characteristics of capacitance and inductance more accurately than those of integer order.

Since the beginning of the 20th century, research on circuit network models has only emerged gradually from the natural science community [6], and the establishment of impedance network models has been studied for more than 100 years. In the early 21st century, the Nobel Society awarded the Nobel Prize in Physics for investigations on the resistance networks of graphene, which fully demonstrates the value of network [7]–[10], and others have made some remarks on fractional order circuits [11]–[14]. However, only a few studies have been conducted on circuit network under fractional order [15]–[19], especially in the field of 3D networks [20]–[24], from which we found that the modeling and analysis of fractional order electrical circuit networks have gained considerable interest in these studies. For example, refs. [17] established an equivalent transformation method to obtain the resistance and impedance of a type of n-step honeycomb RLC circuit network, and [18] proposed a \( 2 \times n \) fractional order electrical circuit network and calculated its equivalent impedances based on the mesh current method. While [19]–[22] introduced impedance characteristics by using the matrix transform method and the difference equation model in a fractional-order sense with different \( RLC \) models \( 2 \times n, \nabla \times n, \) rectangular prism \( n \times n, \) Memristor-LC.

The authors in [19]–[22] constructed the equivalent impedance formulas for arbitrary polygon \( RLC \) networks and made progress. With extensive research on 3D \( RLC \) networks, we naturally expand the polygon network to a 3D fractional order \( RL_\alpha C_\beta \) denoted as \( RL_\alpha C_\beta \), similar to graphene circuit network, as shown in Figure 1, which may lay the groundwork for further research on graphene networks, nonmetallic crystal structures or carbon nanotubes.

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According to the above facts, the following advanced research contents can make our research appealing. First, three general formulas of the equivalent impedance ($Z_{AD}(n)$, $Z_{BC}(n)$ and $Z_{AA}(n)$) for 3D fractional-order $\bigcirc \times n \ RL_{\alpha} C_{\beta}$ circuit network are derived by modeling, which is definitely a significant breakthrough. Second, from the circuit perspective, the third impedance ($Z_{AA}(n)$) characteristics of the special scenario are studied in detail and some key conclusions are acquired by analyzing the results of numerical simulations.

We now focus on the equivalent impedance of the 3D fractional order $\bigcirc \times n \ RL_{\alpha} C_{\beta}$ circuit network (shown in Fig. 1 where $R$ refers to resistor, $Z_{R}$ refers to fractional order inductor and $Z_{C}$ refers to fractional order capacitor). This paper is organized as follows: Section II derives the first general formula of the equivalent impedance ($Z_{AD}(n)$) by modeling in three procedures, then the other two formulas for $Z_{BC}(n)$ and $Z_{AA}(n)$ are just given without derivation by the same approach in Section III. Furthermore, we analyze the impedance characteristics $Z_{AA}(n)$ of the third most complex impedance in Section IV. The last section come to the conclusions about the paper.

### II. THE EQUIVALENT IMPEDANCE BETWEEN THE TWO POINTS A AND D ($Z_{AD}(n)$)

#### A. CONSTRUCTION OF THE DIFFERENCE EQUATION MODEL

In this section, we apply basic circuit network theory, matrix transformation method and the difference equation model to study equivalent impedance ($Z_{AD}(n)$).

The sub-network of fractional order $\bigcirc \times n \ RL_{\alpha} C_{\beta}$ circuit network is shown in Fig. 2, where six vertices of a horizontal hexagon respectively are A, B, C, D, E, and F. The constant electricity $I$ flows through the network, which means $I$ flows into the network from A and then flows out the network from D. Moreover, currents in six rows (AA’, BB’, CC’, DD’, EE’, FF’) are $I_{Ak}$, $I_{Bk}$, $I_{Ck}$, $I_{Dk}$, $I_{Ek}$, and $I_{Fk}$ (where $1 \leq k \leq n$), respectively. Similarly, the longitudinal currents are $I_{k}$, $I'_{k}$, and $I''_{k}$ by symmetry ($1 \leq k \leq n+1$).

From Fig. 2, it can be obtained by symmetry that $I_{Ak} = I_{Dk}$, $I_{Bk} = I_{Fk}$, $I_{Ck} = I_{Ek}$, and then according to Kirchhoff’s current law we can obtain the equations of current as

$$I_{Ak} - I_{Ak-1} = -2I_{k}$$

$$I_{Bk} - I_{Bk-1} = I_{k} - I'_{k}$$

$$I_{Ck} - I_{Ck-1} = I_{k} - I''_{k}$$

Then, from Kirchhoff’s voltage law of No. k loop, we have

$$I_{Ck}Z_{c} + I'_{k}R + I_{Bk}Z_{c} + I'_{k+1}R = 0$$

$$I_{Fk}Z_{c} - I_{k}R + I_{Ak}Z_{L} + I_{k+1}R = 0$$

Similarly, the voltage equations of the No. k-1 loop are

$$I_{Ck-1}Z_{c} - I'_{k-1}R + I_{Bk-1}Z_{c} + I'_{k}R = 0$$

$$I_{Fk-1}Z_{c} - I_{k-1}R + I_{Ak-1}Z_{L} + I_{k}R = 0$$

From (4) minus (6), and (5) minus (7), we obtain

$$(I_{Ck} - I_{Ck-1})Z_{c} - (I'_{k} - I'_{k-1})R + (I_{Bk} - I_{Bk-1})Z_{c} + (I'_{k+1} - I'_{k})R = 0$$

$$-(I_{Bk} - I_{Bk-1})Z_{c} + (I_{k} - I_{k-1})R + (I_{Ak} - I_{Ak-1})Z_{L} + (I_{k+1} - I_{k})R = 0$$
According to (1) to (3), (8) and (9) can be rewritten as

\[
(I_{k+1} + I_{k-1}) = (2R + 2Z_c) I_k - Z_c I_k
\]

(10)

\[
(I'_{k+1} + I'_{k-1}) = 2R + Z_c I_k - 2Z_c I_k
\]

(11)

Now, let \( a = \frac{Z_c}{R}, \ b = \frac{Z_c}{R} \), (10) and (11) can be expressed as

\[
\begin{bmatrix}
\lambda_1 & 1 \\
\lambda_2 & 1 \\
\end{bmatrix}
\begin{bmatrix}
I_{k+1} \\
I'_{k+1} \\
\end{bmatrix}
= 
\begin{bmatrix}
2 + 2a + b & -b \\
-2b & 2(1 + b) \\
\end{bmatrix}
\begin{bmatrix}
I_k \\
I'_k \\
\end{bmatrix}
- 
\begin{bmatrix}
\lambda_1 & 1 \\
\lambda_2 & 1 \\
\end{bmatrix}
\begin{bmatrix}
I_{k-1} \\
I'_{k-1} \\
\end{bmatrix}
\]

(12)

To solve \( I_k \) and \( I'_k \) in (12), multiply (12) by a second order matrix \( A = \begin{bmatrix} \lambda_1 & 1 \\ \lambda_2 & 1 \end{bmatrix} \), we obtain

\[
\begin{bmatrix}
\lambda_1 & 1 \\
\lambda_2 & 1 \\
\end{bmatrix}
\begin{bmatrix}
I_{k+1} \\
I'_{k+1} \\
\end{bmatrix}
= 
\begin{bmatrix}
2 + 2a + b & -b \\
-2b & 2(1 + b) \\
\end{bmatrix}
\begin{bmatrix}
I_k \\
I'_k \\
\end{bmatrix}
- 
\begin{bmatrix}
\lambda_1 & 1 \\
\lambda_2 & 1 \\
\end{bmatrix}
\begin{bmatrix}
I_{k-1} \\
I'_{k-1} \\
\end{bmatrix}
\]

(13)

Assume there are two undetermined constants, which are \( t_1 \) and \( t_2 \), respectively. One gets

\[
\begin{bmatrix}
\lambda_1 & 1 \\
\lambda_2 & 1 \\
\end{bmatrix}
\begin{bmatrix}
2 + 2a + b & -b \\
-2b & 2(1 + b) \\
\end{bmatrix}
\begin{bmatrix}
t_1 \\
t_2 \\
\end{bmatrix}
= 
\begin{bmatrix}
\lambda_1 & 1 \\
\lambda_2 & 1 \\
\end{bmatrix}
\begin{bmatrix}
2 + 2a + b & -b \\
-2b & 2(1 + b) \\
\end{bmatrix}
\begin{bmatrix}I_k \\
I'_k \\
\end{bmatrix}
\]

(14)

Therefore,

\[
\begin{align*}
(2a + b) \lambda_i - 2b &= \lambda_i t_i \\
-2b \lambda_i + 2(1 + b) &= t_i
\end{align*}
\]

(15)

Thus,

\[
 b \lambda^2 + (2a - b) \lambda - 2b = 0
\]

(16)

Let \( d = \frac{b}{a} \), that is \( \frac{Z_c}{R} = d \), (16) can be rewritten as

\[
\lambda^2 + (2d - 1) \lambda - 2 = 0
\]

(17)

By solving (17), we obtain

\[
\lambda_1 = \frac{1}{2} \left(1 - 2d + \sqrt{(2d - 1)^2 - 4 \times 1 \times (-2)}\right)
\]

(18)

\[
\lambda_2 = \frac{1}{2} \left(1 - 2d - \sqrt{(2d - 1)^2 - 4 \times 1 \times (-2)}\right)
\]

(19)

From (15), (18) and (19), we obtain

\[
\begin{align*}
t_1 &= 2 + (2 - \lambda_1) b \\
t_2 &= 2 + (2 - \lambda_2) b
\end{align*}
\]

(20)

(21)

Hence, from (12) to (21), a new deference equation can be obtained by matrix transform as

\[
\begin{bmatrix}
\lambda_1 I_{k+1} + I'_{k+1} \\
\lambda_2 I_{k+1} + I'_{k+1}
\end{bmatrix}
= 
\begin{bmatrix}
t_1 & 0 \\
0 & t_2
\end{bmatrix}
\begin{bmatrix}
\lambda_1 I_k + I'_k \\
\lambda_2 I_k + I'_k
\end{bmatrix}
- 
\begin{bmatrix}
\lambda_1 I_{k-1} + I'_{k-1} \\
\lambda_2 I_{k-1} + I'_{k-1}
\end{bmatrix}
\]

(22)

For the second order linear difference equation, the characteristic equation of (22) is

\[
\begin{bmatrix}
x^2 \\
y^2
\end{bmatrix}
= 
\begin{bmatrix}
t_1 & 0 \\
0 & t_2
\end{bmatrix}
\begin{bmatrix}x \\
y
\end{bmatrix}
- 
\begin{bmatrix}1 \\
1
\end{bmatrix}
\]

(23)

Now let \( \alpha \) and \( \beta \) be the two solutions of the equation about \( x \), and \( \gamma \) and \( \delta \) be the two solutions of the equation about \( y \). From (23), the solutions are

\[
\begin{bmatrix}
\alpha \\
\beta
\end{bmatrix}
= \frac{1}{2} \left(2 + (2 - \lambda_1) b + \sqrt{(2 - \lambda_1) b + 4}\right)
\]

\[
\begin{bmatrix}
\gamma \\
\delta
\end{bmatrix}
= \frac{1}{2} \left(2 + (2 - \lambda_2) b + \sqrt{(2 - \lambda_2) b + 4}\right)
\]

(24)

(25)

Therefore, by solving the second order difference equations, we get

\[
\begin{bmatrix}x_k \\
y_k
\end{bmatrix}
= \frac{1}{\alpha - \beta} \left(\frac{y_k - \lambda_2 I_k + I'_k}{y_k - \lambda_1 I_k + I'_k}\right)
\]

(26)

where

\[
\begin{bmatrix}x_k \\
y_k
\end{bmatrix}
= \frac{1}{\alpha - \beta} \left(\frac{y_k - \lambda_2 I_k + I'_k}{y_k - \lambda_1 I_k + I'_k}\right)
\]

(27)

For (26), we sum k from 1 to n+1. With (27), there are

\[
(x_k - \beta x_k) \frac{1 - \alpha^{n+1}}{1 - \alpha} - (x_k - \alpha x_k) \frac{1 - \beta^{n+1}}{1 - \beta} = (\alpha - \beta) I
\]

(28)

\[
(y_k - \delta y_k) \frac{1 - \gamma^{n+1}}{1 - \gamma} - (y_k - \gamma y_k) \frac{1 - \gamma^{n+1}}{1 - \delta} = -\frac{1}{2} (\gamma - \delta) I
\]

(29)

Similarly, according to the network analysis, the equations of the current of left margin are derived

\[
I_{A1} = I - 2I_1
\]

(30)
Then, from Fig. 3, the voltage equation of the first loop is given by

\[ I_{C1}Z_c - I_1' + I_{B1}Z_c = I_2' + I_2 = 0 \]  
\[ -I_{B1}Z_c - I_1 - I_{A1}Z_L + I_2 = 0 \]

According to (30) to (32), (33) and (34) can be rewritten as

\[
\begin{bmatrix}
I_2 \\
I_2'
\end{bmatrix} = \begin{bmatrix}
2a + b + 1 & -b \\
-2b & 2b + 1
\end{bmatrix} \begin{bmatrix}
I_1 \\
I_1'
\end{bmatrix} - \begin{bmatrix}
aI \\
0
\end{bmatrix}
\]  

(35)

Multiply (35) by matrix \( A \), it becomes

\[
\begin{bmatrix}
x_2 \\
y_2
\end{bmatrix} = \begin{bmatrix}
(\alpha + \beta - 1)x_1 - \lambda_1a \\
(\delta + \gamma - 1)y_1 - \lambda_2a
\end{bmatrix} I
\]  

(36)

And with

\[
\alpha + \beta = t_1 = 2 + (2 - \lambda_1)b \\
\gamma + \delta = t_2 = 2 + (2 - \lambda_2)b
\]  

(37)

\[ a = db \left( \frac{a}{b} \right) 
\]  

(39)

According to (37) to (39), one gets

\[
\lambda_1a = \frac{\lambda_1d}{2 - \lambda_1} (\alpha + \beta - 2) \\
\lambda_2a = \frac{\lambda_2d}{2 - \lambda_2} (\gamma + \delta - 2)
\]  

(40)

(41)

So (36) can be rewritten as

\[
\begin{bmatrix}
x_2 \\
y_2
\end{bmatrix} = \begin{bmatrix}
(\alpha + \beta - 1)x_1 - \frac{\lambda_1d}{2 - \lambda_1} (\alpha + \beta - 2) I \\
(\delta + \gamma - 1)y_1 - \frac{\lambda_2d}{2 - \lambda_2} (\gamma + \delta - 2) I
\end{bmatrix}
\]  

(42)

And then submission of (42) into (28) and (29) gives

\[
x_1 = \frac{\alpha - \beta}{\alpha^{n+1} - \beta^{n+1}} + \frac{\lambda_1d}{2 - \lambda_1} \left( 1 - \frac{\alpha^n - \beta^n + \alpha - \beta}{\alpha^{n+1} - \beta^{n+1}} \right) I
\]  

(43)

\[
y_1 = \frac{-\frac{1}{2} (\gamma - \delta)}{\gamma^{n+1} - \delta^{n+1}} + \frac{\lambda_2d}{2 - \lambda_2} \left( 1 - \frac{\gamma^n - \delta^n + \gamma - \delta}{\gamma^{n+1} - \delta^{n+1}} \right) I
\]  

(44)

Finally, (43) and (44) are the final current formulas with the boundary condition.

**C. DERIVATION OF EQUIVALENT IMPEDANCE**

In order to calculate the equivalent impedance, according to Ohm’s law \( Z_{AD} = \frac{U_{AD}}{I_{AD}} \), the voltage drop between two parts of a and d is needed. From Fig. 2, \( U_{AD} = (2I_1 + I_1')R \), and according to \( x_1 = \lambda_1I_1 + I_1' \) and \( y_1 = \lambda_2I_1 + I_1' \), we obtain

\[
U_{AD} = \frac{2 - \lambda_2}{\lambda_1 - \lambda_2} x_1 + \frac{\lambda_1 - 2}{\lambda_1 - \lambda_2} y_1 \]

(45)

According to Ohm’s law, the equivalent impedance can be obtained by

\[
Z_{AD} = \left( \frac{2 - \lambda_2}{\lambda_1 - \lambda_2} \left[ \frac{\alpha - \beta}{\alpha^{n+1} - \beta^{n+1}} + \frac{\lambda_1d}{2 - \lambda_1} f(\alpha, \beta) \right] + \frac{\lambda_1 - 2}{\lambda_1 - \lambda_2} \left[ \frac{-\frac{1}{2} (\gamma - \delta)}{\gamma^{n+1} - \delta^{n+1}} + \frac{\lambda_2d}{2 - \lambda_2} f(\gamma, \delta) \right] \right) \]

(46)

where \( f(\alpha, \beta) = \left( 1 - \frac{\alpha^n - \beta^n + \alpha - \beta}{\alpha^{n+1} - \beta^{n+1}} \right) \), \( f(\gamma, \delta) = \left( 1 - \frac{\gamma^n - \delta^n + \gamma - \delta}{\gamma^{n+1} - \delta^{n+1}} \right) \).

Equation (46) is the fractional-order equivalent impedance between the two points A and D for \( n = 1, 2, 3, \ldots \).

**D. THE EQUIVALENT IMPEDANCE (Z_{BC}(N) AND Z_{AA‘}(N))**

Similarly, the other two formulas of the equivalent impedance of the fractional-order circuit network between the two points B and C (Z_{BC}(n)) and A and A’ (Z_{AA’}(n)) are just given without derivation by the above approach.

\[
Z_{BC} = \left[ \frac{1}{2} \left( 1 - \frac{\mu_1^n - \mu_2^n}{\mu_1^{n+1} - \mu_2^{n+1}} \right) + \frac{p + q}{4q} \right] \left( 1 - \frac{\gamma_1^n - \gamma_2^n}{\gamma_1^{n+1} - \gamma_2^{n+1}} \right) \]

\[
Z_{AA'} = \left[ \frac{\lambda_1}{2p + 4} Z_L + \left( \frac{p + 1}{p + 2} \right)^2 \left( \frac{2 - p + q}{p + q + 2} f(\lambda_1, \lambda_2) \right) + \frac{\lambda_2}{2p + 4} \left( \frac{p + 1}{p + 2} \right)^2 \left( \frac{2 - p + q}{p + q + 2} f(\mu_1, \mu_2) \right) \right] \]

(47)

(48)

where \( f(\lambda_1, \lambda_2) = 1 - \frac{\lambda_1^n - \lambda_2^n + \lambda_2 - \lambda_1}{\lambda_1^{n+1} - \lambda_2^{n+1}} \), \( f(\mu_1, \mu_2) = 1 - \frac{\mu_1^n - \mu_2^n + \mu_2 - \mu_1}{\mu_1^{n+1} - \mu_2^{n+1}} \).

**III. THE DETAILED ANALYSIS OF IMPEDANCE CHARACTERISTICS (Z_{AA’}(N))**

Due to the similarity of the simulation analysis of various types of impedance networks, in this segment, our research will focus on the third impedance between the two points A and A’ (Z_{AA’}(n)), which is the most complicated and its detailed analysis.

The equivalent impedance is given by

\[
Z_{AA'} = \frac{n}{2p + 4} Z_L + \left( \frac{p + 1}{p + 2} \right)^2 \left( \frac{2 - p + q}{p + q + 2} f(\lambda_1, \lambda_2) \right) + \frac{n}{2p + 4} Z_L + \left( \frac{p + 1}{p + 2} \right)^2 \left( \frac{2 - p + q}{p + q + 2} f(\mu_1, \mu_2) \right) \]

(49)

where \( p = 2d - 1, q = \sqrt{(2d - 1)^2 + 8} \),

\[
\begin{bmatrix}
\lambda_1 \\
\lambda_2
\end{bmatrix} = \left[ \frac{1}{2} \left( 2 + 2a + \frac{1}{2} (p + q) b + \sqrt{2 + 2a + \frac{1}{2} (p + q) b} \right) \right]^{-1} \]

(50)

\[
\begin{bmatrix}
\gamma_1 \\
\gamma_2
\end{bmatrix} = \left[ \frac{1}{2} \left( 2 + 2a + b + \sqrt{2 + 2a + b} \right) \right]^{-1}
\]  

(51)
FIGURE 4. The absolute value of the impedance varies with frequency $\omega$ when the number of circuit units $n$ takes different values and other parameters have fixed values.

FIGURE 5. The absolute value of the impedance varies with frequency $\omega$ when the fractional order $\alpha$ takes different values and other parameters have fixed values.

\[
\begin{bmatrix}
\mu_1 \\
\mu_2
\end{bmatrix} = \begin{bmatrix}
\frac{1}{2} \\
\frac{1}{2}
\end{bmatrix} \begin{bmatrix}
2+2a+\left[1 - \frac{1}{2} (p - q)\right] b+ \\
2+2a+\left[1 - \frac{1}{2} (p - q)\right] b-
\end{bmatrix}
\right]
\end{bmatrix}
\]

where $a = \frac{Z_L}{R}$, $b = \frac{Z_C}{R}$, $\frac{Z_L}{Z_C} = d$, with $Z_L$ and $Z_C$ satisfying

\[
\begin{align*}
\textbf{(52)}
\end{align*}
\]

\[
Z_C = \frac{\cos\left(\frac{\alpha \pi}{2}\right)}{\omega^\alpha C} - j \frac{\sin\left(\frac{\alpha \pi}{2}\right)}{\omega^\alpha C}
\]

\[
Z_L = \omega^\beta L \cos\left(\frac{\beta \pi}{2}\right) + j \omega^\beta L \sin\left(\frac{\beta \pi}{2}\right)
\]

A. TWO-DIMENSIONAL CHARACTERISTICS OF EQUIVALENT IMPEDANCE BETWEEN THE TWO POINTS $A$ AND $A'$

In this subsection, we study the two-dimensional characteristics of the equivalent impedance $Z_{AA'}(n)$ on fractional-order

FIGURE 6. (a), (b), (c), and (d) is respectively corresponding to the absolute value of the impedance varies with frequency $\omega$, when fractional order $\beta$, resistance, inductance or capacitance takes different values and other parameters have fixed values.
circuit network in detail, and new regulations will be presented in the following contents.

We can see in Fig. 4 that when the frequency $\omega < 1$ Hz, with the increase in the number of circuit units $n$, the absolute value of the impedance of the network also increases and is accompanied by the appearance of the wave crest. In addition, with the increase in units $n$, the wave crest appears gradually backward. When $n$ is smaller (such as $n = 10$), the crest is not obvious and the impedance gradually increases with $w$. When $n$ is gradually increased, the impedance curves exhibit obvious peaks with increasing $w$ (such as $n = 100, n = 300$). When $n$ is further increased, the wave crest of the impedance curve will gradually disappear, and the impedance curves decrease with increasing $w$ (such as $n = 500$).

As seen in Fig. 5, when $\omega < 10^{-2}$ Hz and $\omega > 10^4$ Hz, the impedance values are basically stable and very small, which is close to 0. If $\omega < 10^{-2}$ Hz, the impedance values will be reduced with the increasing fractional order $\alpha$. Interestingly, an “impulsive impedance” phenomenon is observed when $10^0$ Hz < $\omega$ < $10^2$ Hz. In other words, a large impedance value can also be obtained in the low frequency region, and such a pulse phenomenon becomes increasingly obvious as $\alpha$ increases.

As seen in Fig. 6, the similarity is the coherent trend of the four curves along with the change of the fractional-order $\beta$, resistance $R$, inductance $L$ or capacitance $C$, the impedance increases first and then decreases, finally reaching a stable value with the growth of $\omega$. The difference is that the peaks become sharper as $\beta$ increases in Fig. 6(a), and the point of steady state gradually moves forward as $\beta$ increases. For different $R$, the curve configuration displays a high degree of consistency, but when $\omega$ is given, the impedance always

**FIGURE 7.** The absolute value of the impedance varies with the fractional-order $\alpha$ and fractional-order $\beta$. 

![Graphs showing the absolute value of the impedance varying with $\alpha$ and $\beta$.](image-url)
increases with increasing $R$, as shown in Fig. 6(b). When $L$ becomes larger, the impedance increases rapidly, and the peaks of the impedance curves successively move to the low-frequency direction (see Fig. 6(c)). For the low- and high-frequency regions, $C$ has almost no effect on the values of the impedance, while for $10^{-3}\text{Hz} < \omega < 10^5$, the impedance exhibits a peak, which decreases with increasing $C$, as shown in Fig. 6(d).

Similarly, according to the above analysis, another important conclusion is that the absolute value of the impedance varies with other variables ($n, \alpha, \beta, L, C, R$) is summarized in Table 1 with variable $\omega$.

**B. THE GRAPHIC MODELS OF CHARACTERISTICS OF EQUIVALENT IMPEDANCE BETWEEN THE TWO POINTS A AND A'**

In order to describe the influence of two of the variables ($\omega, n, \alpha, \beta, L, C, R$) while other parameters have fixed values on the characteristics of equivalent impedance between the two points A and A’ simultaneously and figure out which parameter is the main effect acting on the impedance values, the following graphic models are given.

It is seen in Fig. 7(a)(b) that when $\omega = 0.01$ Hz, the impedance values are mainly determined by the fractional order $\beta$. As $\omega$ increases gradually, the dominant role of $\beta$ weakens, while the dominant role of the fractional order $\alpha$ increases. When $\omega$ continues to rise (such as $\omega = 100$ Hz in Fig. 7(e)(f)), $\beta$ slowly plays a secondary role, and $\alpha$ plays a significant role. This means that the change in $\omega$ alters the primary and secondary positions of the fractional order. As $\omega$ increases, the dominant role $\beta$ becomes $\alpha$. Then, it can be observed in Fig. 7 that the leading role of $\alpha$ or $\beta$ has not been changed while units $n$ move from 1 to 100 in the diagram of (a)(b), (c)(d), and (e)(f). That is, the change of $n$ does not affect the primary and secondary status of $\alpha$ and $\beta$. 

![Figure 8](image-url)
Fig. 8(a)(b) shows that when $\omega = 0.01$ Hz, the impedance values are mainly determined by inductance $L$. When $\omega$ increases, the dominant effect of $L$ is weakened gradually, while the determining effect of resistance $R$ increases gradually (such as $\omega = 1$ Hz in Fig. 8(c)(d)). When $\omega$ continues to increase (such as $\omega = 100$ Hz in Fig. 8(e)(f)), $L$ plays a secondary role, but $R$ plays a primary role instead. That is, the main factor of the influence of the impedance mode is $L$ in the first place and then becomes $R$ with the increase in $\omega$. Similarly, it can also be deduced from the figures that the dominant status of $L$ and $R$ does not change while the units $n$ change from 10 to 100, namely, the change in $n$ does not affect the primary and secondary position of $L$ and $R$ on the impedance values.

In Fig. 9(a), (c), (e) or (b), (d), (f), we observe that impedance is mainly determined by inductance $L$ when $\omega$ equals 0.01 Hz, then increases to 1 Hz, and finally to 100 Hz. It is exhibited that the units $n$ in Fig. 9(a), (b) or (c), (d) or (e), (f) change from 1 to 100, but the leading role of capacitance $C$ and $R$ does not change. That is, the change in $n$ and $\omega$ does not make a difference in the primary and secondary status of $C$ and $R$ on the impedance values, where $R$ plays an important role.

According to the above comparative analysis, important conclusions are obtained in Table 2.
FIGURE 10. The absolute value of the impedance varies with the capacitance and inductance.

TABLE 1. Effects of the system variables on the impedance values.

| Variable | Z | ω | Whether Linear |
|----------|---|---|----------------|
| ω       | increases first, then decreases and finally stabilizes when n is small, the peak gradually disappears as n gets larger | when n is small, first larger and then closer to 0 as values increase, then first larger and then smaller as values decrease; otherwise, first larger and then smaller as n gets larger | NO |
| n       | increases gradually and finally stabilizes | increases almost linearly | NO |
| α       | increases gradually and finally stabilizes | getting smaller as values increase | NO |
| β       | decreases gradually and finally stabilizes | first larger and then smaller as values decrease | NO |
| L       | increases gradually and finally stabilizes | first larger and then smaller as values decrease | NO |
| C       | increases gradually and finally stabilizes | first larger and then smaller as values increase | NO |
| R       | increases gradually and finally stabilizes | first larger and then smaller as values increase | NO |

TABLE 2. Decisive parameter for impedance values.

| Variables | Decisive factor with increasing of ω |
|-----------|-------------------------------------|
| α and β   | From β to α                          |
| L and R   | From L to R                          |
| C and R   | mainly by R (don't affected by ω)    |
| C and L   | mainly by L (don't affected by ω)    |

Fig. 11 shows that impedance increases rapidly with the increase in units n when the network circuit is fixed in fractional order (the fractional-order α of inductance and β of capacitance have a fixed value). Units n has more influence on the absolute value of the impedance of the network than ω.

As shown in Fig. 12, the impedance increases rapidly with increasing L, and L has more influence on the absolute value of the impedance than the fractional-order α of L. Especially, when α is small, as L increases, the impedance increases rapidly. In contrast, with the increase in β, the impedance decreases rapidly, then approaches 0 and remains steady.

The impedance increases rapidly with the increase in β (less than 5) in Fig. 13, and the fractional-order β has more influence on the impedance than C. Extraordinarily, when β is more than 5, the impedance will remain constant no matter how β and C change.
This paper derives three general formulas of impedance for the system parameters ($\omega$, $n$, $\alpha$, $\beta$, $L$, $C$, $R$) on the impedance characteristics of the special scenario are systematically analyzed (summarized in Table 1). In addition, graphic models are presented to describe the effects of two parameters on the impedance and determine which parameter is the main effect acting on the impedance values (summarized in Table 2). Finally, new fundamentals are depicted with numerical simulations, the relevant analysis of this special kind of impedance is presented, and some important conclusions are acquired. The results show that these characteristics are relevant since they have more flexibility in the design of electrical circuit networks. Therefore, it is possible to use the derived formulas to design and analyze electrical circuit networks in practical applications.

Investigating the characteristics of the 3D fractional order RLC circuit network by phase characteristics and sensitivity analysis is the work to be developed in the follow-up of this study.

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