MODULI OF PARABOLIC SHEAVES AND FILTERED KRONECKER MODULES

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Abstract. We give functorial moduli construction of pure parabolic sheaves, in the sense of Álvarez-Cónsul and A. King, using the moduli of filtered Kronecker modules which we introduced in our earlier work. We also use a version of S. G. Langton’s result due to K. Yokogawa to deduce the projectivity of moduli of pure parabolic sheaves of maximal dimension. As an application of functorial moduli construction, we can get the morphisms at the level of moduli stacks.

1. Introduction

C. S. Seshadri first introduced the notion of parabolic structure on vector bundles over a compact Riemann surface, and their moduli constructed by V. B. Mehta and C. S. Seshadri [MS80]. The notion of parabolic bundles and several other related concepts and techniques have been generalized from curves to higher dimensional varieties by M. Maruyama and K. Yokogawa [MY92], and later for pure parabolic sheaves by M. Inaba [In00]. There have been other moduli constructions given by U. Bhosle [Bh92, Bh96] generalizing the notion of the parabolic vector bundles.

In [MY92, In00], the authors consider the moduli of stable parabolic sheaves, which they constructed as an inductive limit of quasi-projective schemes due to lack of strong boundedness result. In the thesis [Sc11], D. Schlüter built the moduli construction for pure parabolic sheaves (with some modification in the parabolic Hilbert polynomial) by following the method of C. Simpson closely.

The Biswas-Seshadri correspondence [Bi97] relates parabolic vector bundles to orbifold bundles up to some choice of Kawamata cover, and this correspondence was later extended by N. Borne and A. Vistoli [BV12]. In [AD15], we have given functorial moduli construction in the sense of [AK07] of Γ-sheaves on higher dimensional varieties, for a finite group Γ, by introducing Kronecker McKay modules.

In this article, we avoid Biswas-Seshadri correspondence and use the filtered object description of parabolic sheaves to give the construction of moduli similar to Álvarez-Cónsul and A. King [AK07]. We introduce the notion of filtered Kronecker module to provide the functorial moduli construction of pure parabolic sheaves. We use the structure of moduli of filtered representations done in our earlier work [AD20] for this purpose.

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The moduli construction of filtered representations was done by adopting the moduli construction of A. King in the quasi-abelian category setup with the help of some results of Y. André [An09]. We will now briefly describe the contents of each Section.

We recall the basic notions of parabolic sheaves and its reinterpretations in terms of filtered sheaves in Section 2 following [MY92, In00]. We also recall the parabolic Hilbert polynomial which is used to give the Gieseker type notion of the stability condition on parabolic sheaves.

We introduce the notion of parabolic filtered Kronecker module and describe the adjoint functors which give the embedding of regular parabolic sheaves in the category of parabolic filtered Kronecker modules. We also extend this embedding to the case of flat families of these objects in Section 3.

The notion of stability on filtered Kronecker modules is introduced in Section 4 so that the embedding given in the Section 3 takes the parabolic semistable sheaves to the semistable filtered Kronecker modules, once we fix the parabolic type.

In Section 5, we consider the notion of moduli functors for both parabolic sheaves of fixed type and moduli functor of filtered Kronecker module. We follow the method of Álvarez-Cónsul and King [AK07] to give the functorial moduli construction in this Section and describe the closed points of moduli space (see Theorem 5.4). More precisely, the closed point of the coarse moduli scheme are in bijection with the $S$-equivalence classes of semistable parabolic sheaves of given parabolic type. This answer the question raised in [Sc11, p. 82] that how strictly semistable orbits in the GIT quotients are identified. We also prove the projectivity, using S. G. Langton’s result, of moduli space of pure parabolic sheaves of fixed type $\tau_p$ such that $\deg(P) = \dim X$.

We extend the functorial relation between moduli functors at the level of moduli stacks of parabolic sheaves and filtered Kronecker modules in the last Section 6.

2. Preliminaries

Let $X$ be a projective scheme over an algebraically closed field $k$ of arbitrary characteristic, let $\mathcal{O}_X(1)$ a very ample invertible sheaf on $X$ and $D$ an effective divisor on $X$. By a sheaf on $X$, we shall mean a coherent $\mathcal{O}_X$-module.

**Definition 2.1.** [MY92, In00] Let $\mathcal{E}$ be a pure sheaf of dimension $d$ such that $\dim(D \cap \text{Supp}(\mathcal{E})) < \dim \text{Supp}(\mathcal{E})$. A quasi-parabolic structure on $\mathcal{E}$ with respect to $D$ is a filtration

$$\mathcal{E} = F_1(\mathcal{E}) \supset F_2(\mathcal{E}) \supset \cdots \supset F_{\ell}(\mathcal{E}) \supset F_{\ell+1}(\mathcal{E}) = \mathcal{E}(-D),$$

where $\ell$ is called the length of the filtration. We denote a quasi-parabolic structure on $\mathcal{E}$ by $(\mathcal{E}, F_\bullet(\mathcal{E}))$ and this pair is referred as a quasi-parabolic sheaf.

A parabolic sheaf on $X$ is a quasi-parabolic sheaf $(\mathcal{E}, F_\bullet(\mathcal{E}))$ together with a sequence of real numbers (called weights) $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_\ell < 1$. 
We shall often denote the parabolic sheaf \((E, F_\bullet(E), \alpha_\bullet)\) by \((E_\bullet, \alpha_\bullet)\) or simply by \(E_\bullet\), when it causes no confusion.

Let \((F_\bullet, \beta_\bullet)\) and \((E_\bullet, \alpha_\bullet)\) be two parabolic sheaves. We say that an \(\mathcal{O}_X\)-module homomorphism \(f: F \to E\) is a parabolic homomorphism if \(f(F_i) \subseteq F_{j+1}(E)\) whenever \(\beta_i > \alpha_j\) (cf. [MS80]). We denote by \(\text{Coh}^{\text{par}}(X, D)\) the category of parabolic sheaves on \(X\) with parabolic structure on \(D\).

A parabolic sheaf \((F_\bullet, \beta_\bullet)\) is called a parabolic subsheaf of \((E_\bullet, \alpha_\bullet)\), if \(F \subseteq E\) and \(F_i(F) \subseteq F_{j+1}(E)\) whenever \(\beta_i > \alpha_j\).

Let \((E_\bullet, \alpha_\bullet)\) be a parabolic sheaf on \(X\) and \(\mathcal{F}\) be a non-zero subsheaf of \(\mathcal{E}\) such that the quotient \(\mathcal{E}/\mathcal{F}\) is pure of dimension \(d\). Put \(F_i(\mathcal{F}) := F_i(\mathcal{E}) \cap \mathcal{F}\). After discarding the equalities and by choosing the maximal index among equalities from the following filtration

\[
\mathcal{F} = F_1(\mathcal{F}) \supseteq F_2(\mathcal{F}) \supseteq \cdots \supseteq F_\ell(\mathcal{F}) \supseteq F_{\ell+1}(\mathcal{F}) = \mathcal{F}(-D),
\]

we get the (strict) filtration on \(\mathcal{F}\). The resulting filtration along with the corresponding weights, say \(\bar{\alpha}_\bullet\), will give the parabolic structure on the subsheaf \(\mathcal{F}\), and we call the parabolic sheaf \((F_\bullet, \bar{\alpha}_\bullet)\) the induced parabolic subsheaf of \((E_\bullet, \alpha_\bullet)\).

Let \((E_\bullet, \alpha_\bullet)\) be a parabolic sheaf. For a real number \(\alpha\), take an integer \(i\) such that \(\alpha_{i-1} < \alpha - \lfloor \alpha \rfloor \leq \alpha_i\), where \(\lfloor \alpha \rfloor\) is the largest integer with \(\alpha - \lfloor \alpha \rfloor \geq 0\), and then put \(E_\alpha = F_i(\mathcal{E})(-\lfloor \alpha \rfloor D)\). This way one obtain an \(\mathbb{R}\)-filtration \(E_\bullet\) of sheaves on \(X\) satisfying the following:

\begin{enumerate}
  \item \(E_0 = \mathcal{E}\),
  \item For all \(\alpha < \beta\), \(E_\beta\) is a subsheaf of \(E_\alpha\),
  \item For sufficiently small \(\epsilon\), \(E_{\alpha-\epsilon} = E_\alpha\) for all \(\alpha\),
  \item For all \(\alpha\), \(E_{\alpha+1} = E_\alpha(-D)\).
\end{enumerate}

Conversely, given an \(\mathbb{R}\)-filtration \(E_\bullet\) of sheaves on \(X\) satisfying the above properties, then the sheaf \(\mathcal{E} := E_0\) has a unique parabolic structure giving the \(\mathbb{R}\)-filtration \(E_\bullet\) [MY92].

**Remark 2.2.** [Yo95, Bo07] Let us regard \(\mathbb{R}\) as an index category, whose objects are real numbers and a unique morphism \(\beta \to \alpha\) exists, by definition, precisely when \(\beta \leq \alpha\). Consider a functor \(E_\bullet: \mathbb{R}^{\text{op}} \to \text{Coh}(X)\), where \(\text{Coh}(X)\) is the category of coherent \(\mathcal{O}_X\)-modules. If \(\alpha \in \mathbb{R}\), then we simply write \(E_\alpha\) for \(E_\bullet(\alpha)\) and \(i_{\alpha, \beta}\) for the morphism \(E_\alpha \to E_\beta\) given by the functor \(E_\bullet\) when \(\alpha \geq \beta\). Given \(E_\bullet\) as above and \(\gamma \in \mathbb{R}\), one can define a new functor \(E[\gamma]_\bullet: \mathbb{R}^{\text{op}} \to \text{Coh}(X)\) as follows:

\[
E[\gamma]_\alpha := E_{\alpha+\gamma},
\]

together with obvious definition on morphisms. If \(\gamma \geq 0\), then there is a natural transformation \(E[\gamma]_\bullet \to E_\bullet\). The functor \(E_\bullet\) is called a \(\mathbb{R}\)-**parabolic sheaf** if it comes with the following:

\begin{enumerate}
  \item each \(E_\alpha\) is pure satisfying \(\dim(D \cap \text{Supp}(E_\alpha)) < \dim \text{Supp}(E_\alpha)\),
\end{enumerate}
(2) there is an isomorphism of functors \( j: E_* \otimes \mathcal{O}_X(-D) \to E[1]_* \) such that the diagram

\[
E_* \otimes \mathcal{O}_X(-D) \xrightarrow{j} E[1]_* \\
\downarrow k_{X,E_*} \downarrow \downarrow \longleftrightarrow j \!
\]

commutes, where \( k_{X,E_*} = 1_{E_*} \otimes \iota_D \).

(3) there is a finite sequence of real numbers \( 0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_\ell < 1 \) such that if \( \alpha \in (\alpha_{i-1}, \alpha_i] \), then \( i_{\alpha_i, \alpha}: E_{\alpha_i} \to E_\alpha \) is the identity map.

A homomorphism \( f_* : F_* \to E_* \) between \( \mathbb{R} \)-parabolic sheaves is nothing but a natural transformation between these two functors. This is also equivalent to \( \mathcal{O}_X \)-module homomorphism \( f: F_0 \to E_0 \) such that \( f(F_\alpha) \subseteq E_\alpha \) for all real numbers \( \alpha \). We shall denote by \( \text{Coh}^{\mathbb{R}\text{-fil}}(X, D) \) the category of \( \mathbb{R} \)-parabolic sheaves on \( X \).

It is clear that Maruyama-Yokogawa definitions [MY92] of parabolic sheaf on \( X \) (Definition 2.1) is equivalent to the notion of \( \mathbb{R} \)-parabolic sheaf.

2.1. The parabolic Hilbert polynomial and semistability. The parabolic Euler characteristic of a parabolic sheaf \( \mathcal{E}_* \) is defined as

\[
\chi_{\text{par}}(\mathcal{E}_*) := \chi(\mathcal{E}(-D)) + \sum_{i=1}^{\ell} \alpha_i \chi(G_i),
\]

where \( G_i := F_i(\mathcal{E})/F_{i+1}(\mathcal{E}) \).

The parabolic Hilbert polynomial of \( \mathcal{E}_* \) is the polynomial with rational coefficients given by

\[
\text{par-}P_{\mathcal{E}_*}(m) := \chi_{\text{par}}(\mathcal{E}_*(m)),
\]

where \( \mathcal{E}_*(m) := \mathcal{E}_* \otimes \mathcal{O}_X(m) \).

Recall that if \( \mathcal{E} \) has Hilbert polynomial \( P \) and \( \mathcal{E}/F_i(\mathcal{E}) \) has Hilbert polynomial \( P_i \) for \( 2 \leq i \leq \ell + 1 \), then we have

\[
\text{par-}P_{\mathcal{E}_*}(m) = P_{\mathcal{E}(-D)}(m) + \sum_{i=1}^{\ell} \alpha_i P_{G_i}(m)
\]

\[= \alpha_{\ell+1} P_{F_{\ell+1}(\mathcal{E})}(m) + \sum_{i=1}^{\ell} \alpha_i [(P_{F_i(\mathcal{E})}(m) - P_{F_{i+1}(\mathcal{E})}(m)]
\]

\[= \alpha_1 P_{F_1(\mathcal{E})}(m) + \sum_{i=2}^{\ell+1} (\alpha_i - \alpha_{i-1}) P_{F_i(\mathcal{E})}(m)
\]

\[= \alpha_{\ell+1} P_{\mathcal{E}}(m) + \sum_{i=2}^{\ell+1} (\alpha_i - \alpha_{i-1}) [P_{F_i(\mathcal{E})}(m) - P_{\mathcal{E}}(m)]
\]

\[= P_{\mathcal{E}}(m) - \sum_{i=2}^{\ell+1} \varepsilon_i [P_{\mathcal{E}}(m) - P_{F_i(\mathcal{E})}(m)]
\]

\[= P(m) - \sum_{i=2}^{\ell+1} \varepsilon_i P_i(m)
\]

where \( \varepsilon_i = \alpha_i - \alpha_{i-1} \) and \( \alpha_{\ell+1} := 1 \).
We also note that
\[
\chi_{\text{par}}(E^\bullet) = \chi(E(-D)) + \sum_{i=1}^\ell \alpha_i [\chi(F_i(E)) - \chi(F_{i+1}(E))]
\]
\[
= \alpha_{\ell+1} \chi(E(-D)) + \sum_{i=1}^\ell \alpha_i [\chi(F_i(E)) - \chi(F_{i+1}(E))]
\]
\[
= \alpha_1 \chi(F_1(E)) + \sum_{i=2}^{\ell+1} (\alpha_i - \alpha_{i-1}) \chi(F_i(E))
\]
\[
= \int_0^1 \chi(E_\alpha) d\alpha
\]

The reduced parabolic Hilbert polynomial par-\(p_{E^\bullet}\) of a parabolic sheaf \(E^\bullet\) of dimension \(d\) is defined by
\[
\text{par-}p_{E^\bullet}(m) := \frac{\text{par-}P_{E^\bullet}(m)}{a_d(E)},
\]
where \(a_d(E)\) is the leading coefficient of the Hilbert polynomial of \(E\).

Recall that the parabolic degree
\[
p-\text{deg}(E^\bullet) = a_{d-1}(E(-D)) + \sum_{i=1}^\ell \alpha_i a_{d-1}(G_i).
\]

The parabolic slope of \(E^\bullet\) is defined as
\[
\mu_{\text{par}}(E^\bullet) := \frac{p-\text{deg}(E^\bullet)}{a_d(E)}.
\]

It is easy to see that
\[
\mu_{\text{par}}(E^\bullet) = \int_0^1 \mu(E_\alpha) d\alpha,
\]
where \(\mu(E_\alpha) = \frac{a_{d-1}(E_\alpha)}{a_d(E_\alpha)}\).

**Definition 2.3.** We say that a parabolic sheaf \(E^\bullet\) is parabolic semistable if for every parabolic subsheaf \(F^\bullet\) of \(E^\bullet\), we have
\[
(2.1) \quad \text{par-}p_{F^\bullet} \leq \text{par-}p_{E^\bullet}.
\]

We say that a parabolic sheaf \(E^\bullet\) is parabolic stable if the inequality \(2.1\) is strict for every proper parabolic subsheaf \(F^\bullet\) of \(E^\bullet\).

**Remark 2.4.** [In00, p. 121] Recall that a subsheaf \(F\) of a pure sheaf \(E\) of dimension \(d\) is called saturated if the quotient sheaf \(E/F\) is pure of dimension \(d\). To check the stability of \(E^\bullet\), it suffices to consider the saturated subsheaves of a parabolic sheaf \(E^\bullet\) with their induced parabolic structures. To see this, let \(F'\) be any parabolic subsheaf of \(E^\bullet\) and \(F\) be the subsheaf of \(E\) containing \(F'\) such that \(\dim(F/F') < d\) and \(E/F\) is pure of dimension \(d\). Let \(\bar{F}\) be the induced parabolic subsheaf of \(\bar{E}\). Then for sufficiently large integer \(m\), we have
\[
\text{par-}p_{\bar{F}^\bullet}(m) = \frac{1}{a_d(F')} \int_0^1 \chi(F'_\alpha(m)) d\alpha
\]
\[
\leq \frac{1}{a_d(F)} \int_0^1 \chi(F_\alpha(m)) d\alpha = \text{par-}p_F(m)
\]
3. A functorial embedding

In this section, we first recall an embedding of the category of regular sheaves into the category of representations of a Kronecker quiver \([AK07]\). Then, we will extend such result to parabolic case.

For integers \(m > n\), let \(T := \mathcal{O}_X(-n) \oplus \mathcal{O}_X(-m)\), and
\[
A := \begin{pmatrix} k & H \\ 0 & k \end{pmatrix}
\]
be the path algebra of the Kronecker quiver \(K : 1 \rightarrow 2\), where \(H := H^0(\mathcal{O}_X(m - n))\) is the multiplicity space for arrows. Recall that a representation \(M\) of \(K\) can be described as the decomposition \(M = M_1 \oplus M_2\) together with a linear map \(\alpha : M_1 \otimes_k H \rightarrow M_2\). Then, \(M\) can also be considered as an \(A\)-module, which will be referred as Kronecker module.

For any coherent sheaf \(E\), we have \(\text{Hom}_X(T, E) = H^0(E(n)) \oplus H^0(E(m))\) together with the multiplication map \(\alpha_E : H^0(E(n)) \otimes H \rightarrow H^0(E(m))\). Thus, we obtained a functor
\[
\Phi := \text{Hom}_X(T, -) : \text{Coh}(X) \rightarrow A\text{-mod}
\]
given by \(E \mapsto \text{Hom}_X(T, E)\). The functor \(\Phi\) has a left adjoint
\[
\Phi^\vee := - \otimes_A T : A\text{-mod} \rightarrow \text{Coh}(X)
\]
Let \(\varepsilon : \Phi^\vee \circ \Phi \rightarrow 1_{\text{Coh}(X)}\) and \(\eta : 1_{A\text{-mod}} \rightarrow \Phi \circ \Phi^\vee\) be the co-unit and unit of the adjunction between \(\Phi\) and \(\Phi^\vee\), respectively. Let \(\text{Coh}^{n\text{-reg}}(X)\) be the full subcategory of \(\text{Coh}(X)\) consisting of \(n\)-regular sheaves on \(X\). Let \(A\text{-mod}^{n\text{-reg}}\) be the full subcategory of \(A\text{-mod}\) consisting of \(A\)-modules \(M\) for which \(\eta_M\) is an isomorphism and \(\Phi^\vee(M)\) is \(n\)-regular.

**Proposition 3.1.** [AK07, Theorem 3.4] If \(\mathcal{O}_X(m - n)\) is regular, then we have an embedding
\[
\Phi : \text{Coh}^{n\text{-reg}}(X) \rightarrow A\text{-mod}
\]
Further, \(\Phi\) gives an equivalence between \(\text{Coh}^{n\text{-reg}}(X)\) and \(A\text{-mod}^{n\text{-reg}}\).

Consider the linear quiver \(A_{\ell+1}^\text{ch} : 1 \leftarrow 2 \leftarrow \cdots \leftarrow \ell \leftarrow \ell + 1\):

Let \(A\text{-mod}^\text{ch}\) (respectively, \(\text{Coh}^\text{ch}(X)\)) denote the category whose objects are representations of a linear quiver \(A_{\ell+1}^\text{ch}\) in the category of Kronecker modules (respectively, coherent sheaves on \(X\)). More precisely, an object \(M_*\) (respectively, \(E_*\)) of \(A\text{-mod}^\text{ch}\) (respectively, \(\text{Coh}^\text{ch}(X)\)) consists of a family of Kronecker modules \(M_i\) (respectively, coherent sheaves \(E_i\)) indexed by the vertices of \(A_{\ell+1}^\text{ch}\) together with a family of morphisms \(M_{i+1} \rightarrow M_i\) (respectively, \(E_{i+1} \rightarrow E_i\)) indexed by the arrows in \(A_{\ell+1}^\text{ch}\). The morphisms are defined in the usual sense.
Let $A\text{-mod}^\text{fl}$ be the full subcategory of $A\text{-mod}^\text{ch}$ consisting of those representations of $A_{t+1}$ for which each morphism $M_{t+1} \to M_i$ is injective. The objects of the category $A\text{-mod}^\text{fl}$ are called filtered Kronecker modules.

**Remark 3.2.** We get a functor $\Phi_{\text{ch}} : \text{Coh}^\text{ch}(X) \to A\text{-mod}^\text{ch}$ defined by $\Phi_{\text{ch}}(E_{\bullet}) := \Phi(E_{\bullet})$. The functor $\Phi_{\text{ch}}$ has a left adjoint

$$\Phi_{\text{ch}}^\vee : A\text{-mod}^\text{ch} \to \text{Coh}^\text{ch}(X).$$

We also denote by $\varepsilon$ and $\eta$ the co-unit and unit of the adjunction between $\Phi_{\text{ch}}$ and $\Phi_{\text{ch}}^\vee$, respectively.

**Definition 3.3.** A quasi-parabolic Kronecker module $M_{\bullet}$ of length $\ell$ is a filtration

$$M_{\bullet} : M = M_1 \supset M_2 \supset \cdots \supset M_\ell \supset M_{\ell+1} = M(-D)$$

in $A\text{-mod}^\text{n-reg}$, where $M(-D) := \Phi(\Phi^\vee(M)(-D))$.

A parabolic filtered Kronecker module is a pair $(M_{\bullet}, \alpha_{\bullet})$ consisting of a quasi-parabolic filtered Kronecker module $M_{\bullet}$ and a sequence of real numbers $\alpha_{\bullet} : 0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_\ell < 1$ (called weights associated to a filtration). We will denote the parabolic filtered Kronecker module $(M_{\bullet}, \alpha_{\bullet})$ simply by $M_{\bullet}$, when it causes no confusion.

Note that any homomorphism $f : N \to M$ of Kronecker modules in $A\text{-mod}^\text{n-reg}$ induces a homomorphism $f(-D) := \Phi(\Phi^\vee(f)(-D)) : N(-D) \to M(-D)$, using the functoriality. A homomorphism $f_{\bullet} : (N_{\bullet}, \beta_{\bullet}) \to (M_{\bullet}, \alpha_{\bullet})$ of parabolic filtered Kronecker modules is defined as a homomorphism $f : N \to M$ of Kronecker modules satisfying $f(N_i) \subset M_j$ whenever $\beta_i > \alpha_j$. We denote by $A\text{-mod}^\text{par}_D$ the category of all parabolic filtered Kronecker modules.

**Remark 3.4.** If weights in the parabolic filtered Kronecker modules $N_{\bullet}$ and $M_{\bullet}$ are same, then a homomorphism $f_{\bullet} : N_{\bullet} \to M_{\bullet}$ is same as a homomorphism $f : N \to M$ which preserves the quasi-parabolic structures, i.e. $f(N_i) \subset M_i$, for $2 \leq i \leq \ell + 1$. In particular, there is a faithful functor from the category of parabolic Kronecker modules of length $\ell$ having fixed weights to the category $A\text{-mod}^\text{ch}$.

A parabolic sheaf $E_{\bullet}$ is called $n$-regular if each $F_i(E)$ is Castelnuovo-Mumford $n$-regular in the usual sense for all $i = 1, 2, \ldots, \ell + 1$. We denote by $\text{Coh}^\text{par}(X, D)^{n\text{-reg}}$ the category of $n$-regular parabolic sheaves on $X$ with the parabolic structure over $D$.

**Now onwards in this section, we will assume that $O_X(m-n)$ is regular.**

Let $E_{\bullet}$ be an $n$-regular parabolic sheaf. Then, we have a filtration

$$E = F_1(E) \supset F_2(E) \supset \cdots \supset F_\ell(E) \supset F_{\ell+1}(E) = E(-D),$$

(3.1) together with the weights $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_\ell < 1$.

Since $O_X(m-n)$ is regular, we have $\Phi^\vee\Phi(E) \cong E$, and hence

$$\Phi(E)(-D) := \Phi(\Phi^\vee(\Phi(E))(-D)) = \Phi(E(-D)).$$
We can define a quasi-parabolic structure on $\Phi(\mathcal{E}_\bullet)$ as follows by applying the functor $\Phi$ to (3.1):

\[ \Phi(\mathcal{E}) = \Phi(F_1(\mathcal{E})) \supset \Phi(F_2(\mathcal{E})) \supset \cdots \supset \Phi(F_{n}(\mathcal{E})) \supset \Phi(F_{n+1}(\mathcal{E})) = \Phi(\mathcal{E}(-D)), \]

where $\Phi(\mathcal{E})(-D) = \Phi(\mathcal{E}(-D))$. By assigning the same weights $\alpha_\bullet$ to the filtration (3.2), we get a structure of parabolic filtered Kronecker module on $\Phi(\mathcal{E})$, which we denote by $\Psi(\mathcal{E}_\bullet)$. Now, if $f: \mathcal{F}_\bullet \to \mathcal{E}_\bullet$ is a morphism of parabolic sheaves, then $\Psi(f) = \Phi(f): \Phi(\mathcal{F}) \to \Phi(\mathcal{E})$ is morphism of corresponding parabolic Kronecker modules. To see this, if $\beta_i > \alpha_j$, then $f$ being a morphism of parabolic sheaves, we have $f(F_i(\mathcal{F})) \subseteq F_{j+1}(\mathcal{E})$. Since $\Phi$ preserves monomorphism, we have $\Psi(f)(\Phi(F_i(\mathcal{F}))) \subseteq \Phi(F_{j+1}(\mathcal{E}))$. Therefore, we get a functor $\Psi: \text{Coh}^{\text{par}}(X, D)^{n-\text{reg}} \to A\text{-mod}^\text{par}_D$.

**Proposition 3.5.** The functor $\Psi: \text{Coh}^{\text{par}}(X, D)^{n-\text{reg}} \to A\text{-mod}^\text{par}_D$ is an embedding of categories.

**Proof.** Let $\mathcal{C}$ be a full subcategory of $A\text{-mod}^\text{par}_D$ consisting of those parabolic filtered Kronecker modules $M_\bullet$ for which we have a filtration

\[ \Phi^\vee(M) \supset \Phi^\vee(M_2) \supset \cdots \supset \Phi^\vee(M_\ell) \supset \Phi^\vee(M(-D)). \]

Note that for an object $M$ of $A\text{-mod}^{n-\text{reg}}$, we have

\[ \Phi^\vee(M(-D)) = \Phi^\vee(\Phi((\Phi^\vee(M)(-D))) = (\Phi^\vee M)(-D) \]

For an object $M_\bullet$ of $\mathcal{C}$, by assigning the same weights as in $M_\bullet$ to the filtration (3.3), we get a parabolic structure on $\Phi^\vee(M)$. Let $\Psi^\vee(M_\bullet)$ denote the parabolic sheaf corresponding to an object $M_\bullet$ of $\mathcal{C}$. By definition of $A\text{-mod}^\text{par}_D$, the parabolic sheaf $\Psi^\vee(M_\bullet)$ is $n$-regular. Hence, we get a functor $\Psi^\vee: \mathcal{C} \to \text{Coh}^{\text{par}}(X, D)^{n-\text{reg}}$.

Let $\mathcal{E}_\bullet$ be an $n$-regular parabolic sheaf. Using Proposition 3.1, we can check that $\Psi(\mathcal{E}_\bullet)$ is an object of $\mathcal{C}$ and $\Psi^\vee(\Psi(\mathcal{E}_\bullet)) \cong \mathcal{E}_\bullet$. In other words, the functor $\Psi$ factors through the category $\mathcal{C}$. By the above construction, we have a left adjoint functor $\Psi^\vee : \mathcal{C} \to \text{Coh}^{\text{par}}(X, D)^{n-\text{reg}}$. Since both unit and co-unit of the adjunction are isomorphisms, we get that the functor $\Psi$ factors through an equivalence to a subcategory of $A\text{-mod}^\text{par}_D$. \(\square\)

**Remark 3.6.** We can define the parabolic filtered Kronecker module using $\mathbb{R}$-filtration following the notion of $\mathbb{R}$-parabolic sheaves from Section 2 and the functorial eimbedding. We shall denote this category by $A\text{-mod}^{\mathbb{R}\text{-fil}}_D$ whose objects and morphisms are described as follows:

**Objects:** All functors $M_\bullet: \mathbb{R}^{\text{op}} \to A\text{-mod}^{n-\text{reg}}$ satisfying the following:

(a) there is an isomorphism of functors $j_{A,M_\bullet}: M_\bullet(-D) \to M[1]_\bullet$ such that the diagram

\[
\begin{array}{ccc}
M_\bullet(-D) & \xrightarrow{j_{A,M_\bullet}} & M[1]_\bullet \\
\downarrow{k_{A,M_\bullet}} & & \downarrow{j_{0,1}} \\
M_\bullet & \xrightarrow{[0,1]} & M_\bullet \\
\end{array}
\]


commutes, where $M_*(-D) := \Phi(\Phi^\vee(M_*)((-D)))$ and
\[ k_{A,M_*} = \eta_{M_*} \circ \Phi(k_{X,\Phi^\vee(M_*)}) : M_*(-D) \to M_* . \]

(b) there is a finite sequence of real numbers $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_\ell < 1$ such that if $\alpha \in (\alpha_{i-1}, \alpha_i]$, then $i_{\alpha_i,\alpha} : M_{\alpha_i} \to M_\alpha$ is the identity map.

**Morphisms:** A morphism between two objects $(N_*, j_{A,N_*})$ and $(M_*, j_{A,M_*})$ is a functor $h_* : N_* \to M_*$ making the following diagram
\[
\begin{array}{ccc}
N_*(-D) & \xrightarrow{j_{A,N_*}} & N_*[1] \\
h_*(-D) \downarrow & & \downarrow h_*[1] \\
M_*(-D) & \xrightarrow{j_{A,M_*}} & M_*[1]
\end{array}
\]
commutative, where $h_*(-D) := \Phi(\Phi^\vee(h_*)(-D))$.

The objects of $A\text{-mod}_D^{\mathbb{R}^\text{fil}}$ are called $\mathbb{R}$-parabolic filtered Kronecker modules. Similar to the case of parabolic sheaves, we can check that the category $A\text{-mod}_D^{\mathbb{R}^\text{fil}}$ is equivalent to the category of parabolic filtered Kronecker modules.

### 3.1. Flat families and stratification.

Let $S$ be a scheme. We say that $(\mathcal{E}, F_*(\mathcal{E}), \alpha_\bullet)$ is a flat family over $S$ of parabolic sheaves on $X$ with the parabolic structure over $D$ if $\mathcal{E}$ is a $S$-flat coherent $O_X$-module such that for every geometric point $s$ of $S$, $\mathcal{E}_s$ is of pure dimension $d$, $\dim(D_s \cap \text{Supp}(\mathcal{E}_s)) < \dim \text{Supp}(\mathcal{E}_s)$ and $\mathcal{E} = F_1(\mathcal{E}) \supset \cdots \supset F_{\ell+1}(\mathcal{E}) = \mathcal{E}(-D)$ is a filtration by coherent sheaves such that each $\mathcal{E}/F_i(\mathcal{E})$ is flat over $S$. In other words, we say that the corresponding $\mathbb{R}$-parabolic sheaf $E_\bullet$ is flat if for each $\alpha \in \mathbb{R}$, the sheaf $E_\alpha$ is flat over $S$. We denote by $\mathbf{Coh}_{\text{flat}}^{\text{par}}(X_S, D)^{\text{n-reg}}$ the category of flat family over $S$ of $n$-regular parabolic sheaves on $X$ with the parabolic structure over $D$.

Let $K : 1 \xrightarrow{H} 2$ be a Kronecker quiver, where $H$ is the multiplicity of arrows. Consider the ladder quiver $A_{\ell+1} \times K$ whose set of vertices and arrows can be described as follows:

\[
\begin{array}{cccc}
1_1 & \beta_1^1 & 2_1 & \beta_2^1 \\
H_1 & \beta_2^2 & \cdots & \beta_{\ell-1}^1 \\
1_2 & \beta_1^2 & 2_2 & \beta_2^2 \\
\end{array}
\]

Let $I$ be the ideal in the path algebra $k(A_{\ell+1} \times K)$ generated by $h_i \beta_i^1 - \beta_i^2 h_{i+1}$, where $h_i \in H_i = H, \ i = 1, 2, \ldots, \ell$. Set $B = k(A_{\ell+1} \times K)/I$.

We say that a sheaf $M_\bullet$ of right modules over the sheaf of algebras $O_S \otimes B$ is flat over $S$, if it is locally-free as sheaf of $O_S$-modules. Moreover, we say that a flat family $M_\bullet$ over $S$ is a flat family of filtered Kronecker modules, if for each closed point $s \in S$, the fibre $M_{s,\bullet}$ is a filtered Kronecker module. We denote by $A\text{-mod}_S^{\text{fil}}(S)$ the category of flat families of filtered Kronecker modules over $S$. Furthermore, we say that it is a flat family of parabolic filtered Kronecker modules, if the fibres $M_{s,\bullet}$ are parabolic filtered Kronecker
modules. We denote by \( A\text{-mod}^\text{par}_{\text{flat}}(S, D) \) the category of flat families of parabolic filtered Kronecker modules over \( S \).

Using [AK07, Proposition 4.1] and Proposition 3.5, we have the following:

**Proposition 3.7.** The functor \( \Psi : \text{Coh}^\text{par}_{\text{flat}}(X_S, D)^{n\text{-reg}} \to A\text{-mod}^\text{par}_{\text{flat}}(S, D) \) is an embedding of categories.

**Proof.** Since \( \Psi \) preserves flat families and monomorphism at each fibre, it follows that the functor \( \Psi \) is well defined. We denote by \( \mathcal{C}_S \) the full subcategory of \( A\text{-mod}^\text{par}_{\text{flat}}(S, D) \) consisting of flat families of parabolic filtered Kronecker modules over \( S \) such that the family \( \Psi^\vee(M_\bullet) \) is a flat family of \( n \)-regular parabolic sheaves.

Let \( \mathcal{E}_\bullet \) be a flat family over \( S \) of \( n \)-regular parabolic sheaves on \( X \). Using Proposition 3.5, we can check that \( \Psi(\mathcal{E}_\bullet) \) is an object of \( \mathcal{C}_S \). In other words, the functor \( \Psi \) factors through the category \( \mathcal{C}_S \). By following the construction of the left adjoint functor in the proof of Proposition 3.5, the functor \( \Psi : \text{Coh}^\text{par}_{\text{flat}}(X_S, D)^{n\text{-reg}} \to \mathcal{C}_S \) has a left adjoint functor \( \Psi^\vee : \mathcal{C}_S \to \text{Coh}^\text{par}_{\text{flat}}(X_S, D)^{n\text{-reg}} \). Since both unit and co-unit of the adjunction are isomorphisms, we get that the functor \( \Psi \) factors through an equivalence to a subcategory of \( A\text{-mod}^\text{par}_{\text{flat}}(S, D) \).

Let \( \tau_p \) represent fixed tuple of numerical polynomials \( P, P_1, \ldots, P_\ell \) such that \( \deg P = d \) and \( \deg P_i < d \) for each \( i = 1, 2, \ldots, \ell \), and fixed sequence of real numbers \( 0 < \alpha_1 < \alpha_2 < \cdots < \alpha_\ell < 1 \). We say that a parabolic sheaf \( \mathcal{E}_\bullet = (\mathcal{E}, F_\bullet(\mathcal{E}), \alpha_\bullet) \) is of parabolic type \( \tau_p \) if the following holds:

- \( \chi(\mathcal{E})(k) = P(k), \chi(\mathcal{E}/F_{i+1}(\mathcal{E}))(k) = P_i(k) \) (\( i = 1, \ldots, \ell \)) for all integers \( k \).

**Proposition 3.8.** Assume that \( \mathcal{O}_X(m - n) \) is regular. Let \( M_\bullet \) be a flat family of filtered Kronecker modules of dimension vector \( \mathbf{d} \) over a scheme \( B \). Then, there exist a unique locally closed subscheme \( \iota : B^\text{reg}_{\tau_p} \to B \) such that the following hold.

1. \( \Psi^\vee(\iota^* M_\bullet) \) is a flat family over \( B^\text{reg}_{\tau_p} \) of \( n \)-regular parabolic sheaves on \( X \) with parabolic type \( \tau_p \) and the unit map

\[
\eta_{\iota^* M_\bullet} : \iota^* M_\bullet \to \Psi \circ \Psi^\vee(\iota^* M_\bullet)
\]

is an isomorphism.

2. If \( \sigma : S \to B \) is such that \( \sigma^* M_\bullet \cong \Psi(E_\bullet) \) for a flat family \( E_\bullet \) over \( S \) of \( n \)-regular parabolic sheaves on \( X \) with parabolic type \( \tau_p \), then \( \sigma \) factors through \( \iota : B^\text{reg}_{\tau_p} \to B \) and \( E_\bullet \cong \Psi^\vee(\sigma^* M_\bullet) \).

**Proof.** Using the flattening stratification for \( \Psi^\vee M_\bullet \) over the projection \( X \times B \to B \), we can conclude that, there exists a locally closed subscheme \( j : B_{\tau_p} \to B \) such that \( j^* \Psi^\vee M_\bullet \) is a flat family over \( B_{\tau_p} \) of chain of coherent sheaves on \( X \), and the closed points of \( B_{\tau_p} \) are precisely those \( b \in B \) for which the fibers \( M_{b, \bullet} \) have following properties:

- \( \Psi^\vee M_{b_{\iota + 1}} = \Psi^\vee M_{b_1}(-D) \).
- for each \( i \), the natural map \( \Psi^\vee M_{b_{\iota + 1}} \to \Psi^\vee M_{b_i} \) is injective.
the Hilbert polynomial of \(\Psi^\vee M_{b_i}\) is \(P\) and the Hilbert polynomial of \(\Psi^\vee M_{b_i}/\Psi^\vee M_{b_{i+1}}\) is \(P_i\) for each \(i = 1, 2, \ldots, \ell\).

It is clear that \(B_{\tau_p}\) contains an open set \(B_{\tau_p}^{\text{reg}}\) of points \(b\) for which the sheaf \(\Psi^\vee M_{b_i}\) is \(n\)-regular and the morphism \(\eta_{b_i} : M_{b_i} \to \Psi(\Psi^\vee (M_{b_i}))\) is an isomorphism for each \(i\) (as both are open conditions). Now, using the proof of Proposition 3.7, it follows that \(\Psi(\Psi^\vee M_{\bullet})\) is a flat family of parabolic filtered Kronecker modules over \(B_{\tau_p}^{\text{reg}}\) and the unit map \(\eta_{\bullet} : \iota^* M_{\bullet} \to \Psi(\Psi^\vee (\iota^* M_{\bullet}))\) is an isomorphism, where \(\iota : B_{\tau_p}^{\text{reg}} \to B\). Therefore, we get a locally closed subscheme \(\iota : B_{\tau_p}^{\text{reg}} \to B\) satisfying (1). The proof of (2) follows the similar line of arguments as in [AK07, Proposition 4.2].

4. Preservation of semi-stability

In this section, we will now study the semistability of parabolic sheaves and determine the stability parameter for the filtered Kronecker modules. This will be useful to reduce the problem of constructing the moduli space to GIT quotient by choosing an appropriate parameter space into the representation space of filtered Kronecker modules.

Recall that \(\tau_p\) represent fixed tuple of numerical polynomials \(P, P_1, \ldots, P_{\ell}\) such that \(\deg P = d\) and \(\deg P_i < d\) for any \(i\), and fixed sequence of real numbers \(\alpha_{\bullet} = (\alpha_1, \ldots, \alpha_{\ell})\) such that \(0 < \alpha_1 < \alpha_2 < \cdots < \alpha_{\ell} < 1\).

In [In00], the collection of \(e\)-stable parabolic sheaves of given type \(\tau_p\) is shown to be bounded following the boundedness result of [MY92]. In [Sc11], using the results of [Si94, La04], the author has extended the boundedness result to the semistable parabolic sheaves of given type \(\tau_p\).

**Theorem 4.1.** [Sc11, Theorem 4.5.2] Let \(\mathcal{F}^{\text{ss}}_X(\tau_p)\) be the family of parabolic semistable sheaves on \(X\) of parabolic type \(\tau_p\). Then, the family \(\mathcal{F}^{\text{ss}}_X(\tau_p)\) is bounded.

The above result is very crucial to give a characterization of semistability of parabolic sheaves of a given type as in the following Proposition 4.2. This, in turn, helps to determine the right stability parameter for the filtered Kronecker modules to give a functorial construction of the coarse moduli space of semistable parabolic sheaves having a parabolic type \(\tau_p\).

**Proposition 4.2.** There exists an integer \(N\) such that for any pure \(d\)-dimensional parabolic sheaf \(\mathcal{E}_{\bullet}\) on \(X\) of parabolic type \(\tau_p\), the following are equivalent:

1. \(\mathcal{E}_{\bullet}\) is parabolic semistable.

2. For all \(n \geq N\), \(\mathcal{E}_{\bullet}\) is \(n\)-regular, and for all proper subsheaf \(\mathcal{E}' \subset \mathcal{E}\) the inequality of polynomials

   \[
   \left( \sum_{i=1}^{\ell+1} \varepsilon_i h^0(F_i(\mathcal{E}')(n)) \right)a_d(\mathcal{E}) \leq \left( \sum_{i=1}^{\ell+1} \varepsilon_i h^0(F_i(\mathcal{E})(n)) \right)a_d(\mathcal{E}')
   \]

   holds, where \(\varepsilon_i = \alpha_i - \alpha_{i-1}\), \(\alpha_0 := 0\), \(\alpha_{\ell+1} := 1\).
Moreover, if $\mathcal{E}_\bullet$ is parabolic semistable, and $\mathcal{E}'$ is a proper subsheaf of $\mathcal{E}$, then equality holds in (4.1) if and only if $\text{par-}p_{\mathcal{E}_\bullet} = \text{par-}p_{\mathcal{E}'_\bullet}$.

Proof. If $\mathcal{E}_\bullet$ is parabolic semistable, then by [MY92, Proposition 2.5] together with Theorem 4.1, for any proper subsheaf $\mathcal{E}'$ of $\mathcal{E}$, we have

$$\frac{1}{a_d(\mathcal{E}')} \int_0^1 h^0(\mathcal{E}'_\alpha(n))d\alpha \leq \frac{1}{a_d(\mathcal{E})} \int_0^1 h^0(\mathcal{E}_\alpha(n))d\alpha$$

for sufficiently large $n$. This is equivalent to (4.1).

Conversely, to check that $\mathcal{E}_\bullet$ is parabolic semistable, it is enough to show that for any saturated subsheaf $\mathcal{E}'$ of $\mathcal{E}$, we have $\text{par-}p_{\mathcal{E}'_\bullet} \leq \text{par-}p_{\mathcal{E}_\bullet}$. Let $\mathcal{E}'$ be a saturated subsheaf of $\mathcal{E}$. If $\mu_{\text{par}}(\mathcal{E}'_\bullet) < \mu_{\text{par}}(\mathcal{E}_\bullet)$, then it can not destabilize the parabolic sheaf $\mathcal{E}_\bullet$. So, assume that $\mu_{\text{par}}(\mathcal{E}'_\bullet) \geq \mu_{\text{par}}(\mathcal{E}_\bullet)$. Then for some $\alpha \in \mathbb{R}$, we have $\mu(\mathcal{E}'_\alpha) \geq \mu_{\text{par}}(\mathcal{E}_\bullet)$. Using [HL97, Lemma 1.7.9], we can conclude that the family of saturated subsheaves of $\mathcal{E}$ with the property that $\mu(\mathcal{E}'_\alpha) \geq \mu := \mu_{\text{par}}(\mathcal{E}_\bullet)$ for some $\alpha$, is bounded. In particular, we can conclude that $\mathcal{E}'_\bullet$ is $n$-regular for sufficiently large $n$. Since the inequality (4.1) holds for $\mathcal{E}'$, we have

$$\text{par-}p_{\mathcal{E}'_\bullet}(n) = \frac{1}{a_d(\mathcal{E}')} \left( \sum_{i=1}^{\ell+1} \varepsilon_i h^0(F_i(\mathcal{E}')(n)) \right) \leq \frac{1}{a_d(\mathcal{E})} \left( \sum_{i=1}^{\ell+1} \varepsilon_i h^0(F_i(\mathcal{E})(n)) \right) = \text{par-}p_{\mathcal{E}_\bullet}(n)$$

for sufficiently large enough $n$. This implies that $\text{par-}p_{\mathcal{E}'_\bullet} \leq \text{par-}p_{\mathcal{E}_\bullet}$. \hfill $\square$

Proposition 4.3. There exists an integer $P_{LS}$ such that, for all $n \geq P_{LS}$ the following are equivalent for any pure $d$-dimensional parabolic sheaf $\mathcal{E}_\bullet$ of parabolic type $\tau_p$:

1. $\mathcal{E}_\bullet$ is parabolic semistable.
2. For all $n \geq P_{LS}$, $\mathcal{E}_\bullet$ is $n$-regular, and for all proper subsheaf $\mathcal{E}' \subset \mathcal{E}$ the inequality of polynomials

$$(4.2) \quad \sum_{i=1}^{\ell+1} \varepsilon_i h^0(F_i(\mathcal{E}')(n)) P_\mathcal{E} \leq \text{par-}p_{\mathcal{E}'_\bullet}(n) P_{\mathcal{E}'}$$

holds.

Moreover, if $\mathcal{E}_\bullet$ is parabolic semistable, and $\mathcal{E}'$ is a proper subsheaf of $\mathcal{E}$, then equality holds in (4.2) if and only if $\text{par-}p_{\mathcal{E}_\bullet} = \text{par-}p_{\mathcal{E}'_\bullet}$.

Proof. This immediately follows from the Proposition 4.2 by noting that $a_d(\mathcal{E}')$ and $a_d(\mathcal{E})$ are leading terms in the Hilbert polynomials $P_{\mathcal{E}'}$ and $P_{\mathcal{E}}$, respectively. \hfill $\square$

Proposition 4.4. [Sc11, Proposition 4.4.3] For a fixed parabolic type $\tau_p$, the full subcategory of category of all parabolic sheaves on $X$ consisting of semistable parabolic sheaves having parabolic type $\tau_p$ is an abelian, Noetherian and Artinian category. Moreover, its simple objects are precisely the stable parabolic sheaves having parabolic type $\tau_p$. 

From Proposition 4.4, it follows that a parabolic semistable sheaf $E$ having parabolic type $\tau_p$ admits a Jordan-Hölder filtration
\[ \{0\} = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_k = E, \]
such that the successive quotients $E_{i+1}/E_i$ are stable parabolic sheaves having parabolic type $\tau_p$. The associated graded object is
\[ \text{gr}(E) = \bigoplus_{i=0}^{k-1} E_{i+1}/E_i \]
which does not depend on the choice of the filtration up to an isomorphism.

We say that two parabolic semistable sheaves $E$ and $F$ are $S$-equivalent if the associated graded objects $\text{gr}(E)$ and $\text{gr}(F)$ are isomorphic.

Now onwards, in this section we choose $m \gg n \gg 0$ such that the following holds:

(C1) All parabolic semistable sheaves of parabolic type $\tau_p$ are $n$-regular.
(C2) The Le Potier-Simpson estimates hold, i.e., $n \geq P_{LS}$, for $P_{LS}$ as in Proposition 4.3.
(C3) $\mathcal{O}_X(m - n)$ is regular.

Let $E$ be any parabolic sheaf of parabolic type $\tau_p$ which is $n$-regular. For each $j$, let $ev_j : H^0(F_j(E)(n)) \otimes \mathcal{O}_X(-n) \to E$ be the natural evaluation map. For subspaces $V'_j \subseteq H^0(F_j(E)(n))$, let $E'_j$ and $F'_j$ be the image and kernel of restriction of $ev_j$ to $V'_j \otimes \mathcal{O}_X(-n)$. Let $S$ be the set of all sheaves $E'_j$, $F'_j$ that arises in this way, and all saturated subsheaves $E' \subset E$, where $E$ is $n$-regular parabolic sheaf having parabolic type $\tau_p$ and $\mu_{\text{par}}(E') \geq \mu_{\text{par}}(E)$. Then by Grothendieck lemma, the family $S$ is bounded.

(C4) All the sheaves in $S$ are $m$-regular.
(C5) For a parabolic sheaf $E$ of parabolic type $\tau_p$, let $P_j(k) = \chi(F_j(E)(k))$. Then for any integers $c_j \in \{0, 1, \ldots, P_j(n)\}$ and sheaves $E' \in S$, the polynomial relation $P(\sum_j \varepsilon_j c_j) \sim P_E\text{par} - P_E(n)$ is equivalent to the relation $P_E(m) \sum_j \varepsilon_j c_j \sim P_E(m)\text{par} - P_E$, where $\sim$ is any of $\leq$ or $< \text{or } =$.

4.1. Semistability of filtered Kronecker modules. In [AD20], we gave the moduli construction of filtered quiver representation with the appropriate notion of semistability using rank and degree functions. In the following, we will consider a specific rank and degree, which corresponds to the stability of parabolic sheaves.

Let
\[ M = M_1 \supset M_2 \supset \cdots \supset M_\ell \supset M_{\ell+1} \]
be a filtered Kronecker module, where $M_i$’s are Kronecker modules. The category of all filtered Kronecker modules will be denoted as $A^\text{fil}\text{-mod}$. We define a slope of $M$ as
\[ \mu(M) := \frac{\sum_{i=1}^{\ell+1} \varepsilon_i \dim M_i}{\dim M_{\ell+1}}, \]
which takes values in $[0, \infty]$. 


Let $d = (d_{11}, d_{12}, \ldots, d_{(\ell+1)1}, d_{(\ell+1)2})$ be a dimension vector of $A_{\ell+1} \times K$. If $M_\bullet$ is an object of $A\text{-mod}^{\text{fil}}$ having dimension vector $d$, we set $\text{rk}(d) = d_{12}$.

Let $M'_\bullet$ be a non-zero subrepresentation of $M_\bullet$ in the category of filtered Kronecker modules. We say that $M'_\bullet$ is degenerate if $M'_{12} = 0$.

Let $\Theta_{j1} = \varepsilon_j$ and $\Theta_{j2} = 0$. Then, we get degree function

$$\Theta(d(M_\bullet)) := \sum_{w \in (A_{\ell+1} \times K)_0} \Theta_w d(M)_w.$$  

(4.4) $\theta(M'_\bullet) := (\sum_{j=1}^{\ell+1} (\Theta_{j1} \text{ dim } M'_{j1}))d_{12} - (\sum_{j=1}^{\ell+1} (\Theta_{j1} d_{j1})) \text{ dim } M'_{12}$

where $\Theta_{j1} = \varepsilon_j$ and $\Theta_{j2} = 0$ for each $j$.

We say that $M_\bullet$ is $\theta$-semistable if $\theta(M'_\bullet) \leq 0$ for all subrepresentations $M'_\bullet$ of $M_\bullet$. Note that the slope $\mu(M'_\bullet)$ is well-defined for all non-degenerate subrepresentations of $M_\bullet$.

**Lemma 4.5.** Let $E_\bullet$ be an $n$-regular parabolic sheaf having parabolic type $\tau_p$, and let $M_\bullet := \Psi(E_\bullet)$ be a corresponding filtered Kronecker module. Then, $M_\bullet$ does not have any degenerate subrepresentation.

**Proof.** This follows from the observation in the proof of [GRT, Lemma 8.8]. □

We can reformulate the $\theta$-semistability in terms of slope semistability.

**Proposition 4.6.** Assume that a filtered Kronecker module $M_\bullet$ having dimension vector $d$ does not have any degenerate subrepresentation. Then, $M_\bullet$ is $\theta$-semistable if and only if for all non-zero subrepresentations $M'_\bullet$ of $M_\bullet$, we have $\mu(M'_\bullet) \leq \mu(M_\bullet)$.

**Proof.** Note that

$$\theta(M'_\bullet) := (\sum_{j=1}^{\ell+1} (\varepsilon_j \text{ dim } M'_{j1}))d_{12} - (\sum_{j=1}^{\ell+1} (\varepsilon_j d_{j1})) \text{ dim } M'_{12}$$

Hence, we have

$$\theta(M'_\bullet) = d_{12} \text{ dim } M'_{12}(\mu(M'_\bullet) - \mu(M_\bullet)).$$

From this the assertion follows. □

**Definition 4.7.** Let $M'_\bullet$ and $M''_\bullet$ be two subobjects of $M_\bullet$. We say that $M'_\bullet$ is subordinate to $M''_\bullet$ if

$$M'_{j1} \subseteq M''_{j1} \text{ for each } j, \text{ and } M'_{12} \subseteq M''_{12}.$$  

We say that $M'_\bullet$ is tight if whenever it is subordinate to a subobject $M''_\bullet$ of $M_\bullet$, we have $M'_{j1} = M''_{j1}$ for each $j$, and $M'_{12} = M''_{12}$. In this case, we have $\mu(M'_\bullet) = \mu(M''_\bullet)$. 


Lemma 4.8. Let \( \tilde{M}_j \) be a subobject of \( M_* \). Then \( \tilde{M}_j \) is subordinate to some tight subobject \( M'_* \) of \( M_* \).

**Proof.** Let \( M_* \) be a filtered Kronecker module. In other words, we have linear maps

\[ \rho_j : M_{j1} \otimes H \to M_{j2} \]

with injective linear maps

\[ f_j : M_{(j+1)1} \to M_{j1} \quad \text{and} \quad f_{j2} : M_{(j+1)2} \to M_{j2} \]

such that \( f_{j2} \circ \rho_{j+1} = \rho_j \circ f_j \) for each \( j = 1, 2, \ldots, \ell \).

Let \( M'_{12} := \rho_1(\tilde{M}_{11} \otimes H) \) and \( M'_{11} := \{ v \in M_{11} \mid \alpha_j(v \otimes h) \in M'_{12} \text{ for all } h \in H \} \). For \( j \geq 2 \), we define \( M'_{j1} := (f_{11} \circ f_{21} \circ \cdots \circ f_{(j-1)1})^{-1}(M'_{11}) \) and \( M'_{j2} := \rho_j(M'_{j1} \otimes H) \). Since each \( f_j \) is injective, we have \( M'_{j1} = \tilde{M}_{1j} \cap M_j \) for each \( j \).

Let \( M'_* \) be a subobject of \( M_* \) given by \( \{ M'_{j1}, M'_{j2} \} \) with the induced maps from \( \rho' \)'s and \( f' \)'s. From the definition, it follows that \( \tilde{M}_{11} \subseteq M'_{11} \) and \( \tilde{M}_{12} \supseteq M'_{12} \). Since \( f_{j1}(\tilde{M}_{(j+1)1}) \subseteq \tilde{M}_{1j} \) and \( \tilde{M}_{11} \subseteq M'_{11} \), by induction, we can conclude that \( \tilde{M}_{1j} \subseteq M'_{1j} \) for all \( j \geq 2 \). This proves that \( M_* \) is subordinate to \( M'_* \).

To see that \( M'_* \) is a tight subobject of \( M_* \), let \( M''_* \) be a subobject of \( M_* \) such that \( M'_* \) is subordinate to \( M''_* \). That is, we have \( M'_{j1} \subseteq M''_{j1} \) and \( M'_{j2} \subseteq M''_{j2} \). Note that

\[ M''_{12} = \rho_1(\tilde{M}_{11} \otimes H) \subseteq \rho_1(M'_{11} \otimes H) \subseteq \rho_1(M''_{11} \otimes H) \subseteq M''_{12} \]

This proves that \( M''_{12} = M''_{12} \), and hence by definition of \( M''_{11} \), we have \( M''_{11} = M''_{11} \). Observe that \( M''_{j1} \subseteq M''_{11} = M'_{11} \), and hence \( M''_{j1} \subseteq (M'_{11} \cap M_j) = M'_{j1} \) for \( j \geq 2 \). This completes the proof that \( M'_* \) is a tight subobject of \( M_* \). \( \square \)

Lemma 4.9. Let \( E_* \) be an \( n \)-regular parabolic sheaf with parabolic type \( \tau_p \), and let \( E'_* \) be a subobject of \( E_* \). Then there exists a subsheaf \( E' \) of \( E \) such that \( E'_* \) is subordinate to \( E'_* \). In particular, if \( M'_* \) is tight and non-degenerate, then \( \mu(M'_*) = \mu(\Psi(E'_*)) \).

**Proof.** Let \( E_* : E_1 \supset E_2 \supset \cdots \supset E_L \supset E(-D) \), and let \( M_* : M_1 \supset M_2 \supset \cdots \supset M_L \supset M(-D) \). Let \( E'_* \) be a subobject of \( M_* \). Let \( E'_j := \text{Im}(M'_{j1} \otimes \mathcal{O}_X(-n) \to E) \), and \( E' := E'_1 \). Then, the \( E' \) being a subsheaf of \( E \), we get an induced parabolic structure on \( E' \), which we shall denote by \( E'_* \). Note that

\[ M'_{j1} \subset H^0(E'_j(n)) \subset H^0(F'_j(E'(n))). \]

For each \( j \), there is a short exact sequence \( 0 \to F_j \to M'_{j1} \otimes \mathcal{O}_X(-n) \to E'_j \to 0 \). By \( C4 \), \( F_j \) is \( m \)-regular, and hence the map \( M'_{j1} \otimes H^0(\mathcal{O}_X(m-n)) \to H^0(E'_j(m)) \) is surjective. Since \( M'_* \) is a subobject of \( M_* \), it follows that

\[ H^0(F'_j(E'(m))) = H^0(E'(m)) \subset M'_{12}. \]

This proves that \( M'_* \) is subordinate to \( \Psi(E'_*) \). \( \square \)
Remark 4.10. Let \( \text{sk}(A\text{-mod}^{\text{fil}}) \) be the skeleton of \( A\text{-mod}^{\text{fil}} \), that is, the set of isomorphism classes of objects of \( A\text{-mod}^{\text{fil}} \). From the above Lemma 4.5, we get a rank function 
\[
\text{rk}: \text{sk}(A\text{-mod}^{\text{fil}}) \to \mathbb{N}
\]
given by \( \text{rk}(d) = d_{12} \).

We can consider a slope function \( \mu: \text{sk}(A\text{-mod}^{\text{fil}}) - \{0\} \to \mathbb{Q} \) as follows:
\[
\mu_{\Theta,\text{rk}}(M) := \frac{\Theta(d(M))}{\text{rk}(d(M))}.
\]
Note that the slope defined in (4.3) agree with \( \mu_{\Theta,\text{rk}} \) on \( A\text{-mod}^{\text{fil}} \).

Theorem 4.11. A parabolic sheaf \( \mathcal{E}_\bullet \) of parabolic type \( \tau_p \) is parabolic semistable if and only if it is pure, \( n \)-regular and \( \Psi(\mathcal{E}_\bullet) \) is \( \theta \)-semistable.

Proof. Suppose that \( \mathcal{E}_\bullet \) is parabolic semistable. Then, by definition, it is pure and \( n \)-regular by (C1). By Lemma 4.5 and Proposition 4.6, we need to check that \( \mu(M'_\bullet) \leq \mu(M_\bullet) \) for all tight subobjects \( M'_\bullet \) of \( M_\bullet \). Let \( M'_\bullet \) be a tight subobject of \( M_\bullet \). By Lemma 4.9, we get a parabolic subsheaf \( \mathcal{E}'_\bullet \) of \( \mathcal{E}_\bullet \) such that \( \mu(M'_\bullet) = \mu(\Psi(\mathcal{E}'_\bullet)) \). Since \( \mathcal{E}_\bullet \) is parabolic semistable, by Proposition 4.3, we have
\[
\sum_{i=1}^{\ell+1} \varepsilon_i h^0(F_i(\mathcal{E}')(n)) P_{\mathcal{E}} \leq \text{par}-P_{\mathcal{E}_\bullet}(n) P_{\mathcal{E}'}.
\]
This polynomial inequality implies the following numerical inequality:
\[
\sum_{i=1}^{\ell+1} \varepsilon_i h^0(F_i(\mathcal{E}')(n)) P_{\mathcal{E}}(m) \leq \text{par}-P_{\mathcal{E}_\bullet}(n) P_{\mathcal{E}'}(m).
\]
In other words,
\[
\mu(M'_\bullet) = \frac{\sum_{i=1}^{\ell+1} \varepsilon_i h^0(F_i(\mathcal{E}')(n))}{P_{\mathcal{E}'}(m)} \leq \frac{\sum_{i=1}^{\ell+1} \varepsilon_i h^0(F_i(\mathcal{E}')(n))}{P_{\mathcal{E}}(m)} = \mu(M_\bullet).
\]
This proves that \( \Psi(\mathcal{E}'_\bullet) \) is \( \theta \)-semistable.

Conversely, suppose that \( \mathcal{E}_\bullet \) is pure, \( n \)-regular and \( \Psi(\mathcal{E}_\bullet) \) is \( \theta \)-semistable. Let \( \mathcal{E}'_\bullet \) be a saturated subsheaf of \( \mathcal{E} \) such that \( \mu_{\text{par}}(\mathcal{E}'_\bullet) \geq \mu_{\text{par}}(\mathcal{E}_\bullet) \). Since \( \Psi(\mathcal{E}_\bullet) \) is \( \theta \)-semistable, we have \( \theta(\Psi(\mathcal{E}'_\bullet)) \leq 0 \). Note that
\[
\theta(\Psi(\mathcal{E}'_\bullet)) = \left( \sum_{i=1}^{\ell+1} \varepsilon_i h^0(F_i(\mathcal{E}')(n)) \right) P_{\mathcal{E}}(m) - \left( \sum_{i=1}^{\ell+1} \varepsilon_i h^0(F_i(\mathcal{E}')(n)) \right) P_{\mathcal{E}'}(m)
\]
Hence, we have
\[
\left( \sum_{i=1}^{\ell+1} \varepsilon_i h^0(F_i(\mathcal{E}')(n)) \right) P_{\mathcal{E}}(m) \leq \left( \sum_{i=1}^{\ell+1} \varepsilon_i h^0(F_i(\mathcal{E}')(n)) \right) P_{\mathcal{E}'}(m)
\]
By condition (C5), this numerical inequality is equivalent to the polynomial inequality
\[
\left( \sum_{i=1}^{\ell+1} \varepsilon_i h^0(F_i(\mathcal{E}')(n)) \right) P_{\mathcal{E}} \leq \left( \sum_{i=1}^{\ell+1} \varepsilon_i h^0(F_i(\mathcal{E}')(n)) \right) P_{\mathcal{E}'}.
\]
Now, using Proposition 4.3, we can conclude that $E_\bullet$ is parabolic semistable.

From the Theorem 4.11 and Proposition 3.5, we can deduce that the functor $\Psi$ induces an equivalence between the category $\text{Coh}^\text{par}(X, D)^{ss}(\tau_p)$ consisting of semistable parabolic sheaves on $X$ having parabolic type $\tau_p$ and the full subcategory category $C^{ss}(\mu)$ of $C$ consisting of semistable parabolic filtered Kronecker module having fixed slope $\mu$, which is determined by the parabolic type $\tau_p$. Since $\text{Coh}^\text{par}(X, D)^{ss}(\tau_p)$ is an abelian category, it follows that Jordan-Hölder theorem holds for $C^{ss}(\mu)$. In other words, if $M_\bullet$ is an object of $C^{ss}(\mu)$, then there exists a filtration

$$\{0\} = M_{0_\bullet} \subset M_{1_\bullet} \subset M_{2_\bullet} \subset \cdots \subset M_{k_\bullet} = M_\bullet$$

such that the successive quotients $M_{(i+1)_\bullet}/M_{i_\bullet}$ are stable objects in $C^{ss}(\mu)$. The associated graded object is

$$\text{gr}^{JH}(M_\bullet) = \bigoplus_{i=0}^k M_{(i+1)_\bullet}/M_{i_\bullet}$$

which does not depend on the choice of the filtration up to an isomorphism.

We say that two objects $M_\bullet$ and $N_\bullet$ in $C^{ss}(\mu)$ are $S^{JH}$-equivalent if the associated graded objects $\text{gr}^{JH}(M_\bullet)$ and $\text{gr}^{JH}(N_\bullet)$ are isomorphic.

**Proposition 4.12.** Let $\mathcal{E}_\bullet$ be a semistable parabolic sheaf having parabolic type $\tau_p$, and let $M_\bullet := \Psi(\mathcal{E}_\bullet)$ be the corresponding filtered Kronecker module. If $\mathcal{E}_{1_\bullet} \subset \mathcal{E}_{2_\bullet}$ are parabolic subsheaves of $\mathcal{E}_\bullet$ having parabolic type $\tau_p$, then

$$\Psi(\mathcal{E}_{2_\bullet})/\Psi(\mathcal{E}_{1_\bullet}) \cong \Psi(\mathcal{E}_{2_\bullet}/\mathcal{E}_{1_\bullet}).$$

Moreover, we have

$$\text{gr}^{JH}(M_\bullet) \cong \Psi(\text{gr}(\mathcal{E}_\bullet)). \text{ and } \text{gr}(\mathcal{E}_\bullet) \cong \Psi' \text{ gr}^{JH}(M_\bullet).$$

**Proof.** First note that if $\mathcal{E}_\bullet'$ is a parabolic subsheaf of $\mathcal{E}_\bullet$ having parabolic type $\tau_p$, then $\mathcal{E}_\bullet'$ is also semistable, and hence $n$-regular by the condition (C1). Consider the short exact sequence

$$(4.5) \quad 0 \rightarrow \mathcal{E}_{1_\bullet} \rightarrow \mathcal{E}_{2_\bullet} \rightarrow \mathcal{E}_{2_\bullet}/\mathcal{E}_{1_\bullet} \rightarrow 0.$$ 

For each $\alpha$, we have a short exact sequence

$$0 \rightarrow \mathcal{E}_{1_\alpha} \rightarrow \mathcal{E}_{2_\alpha} \rightarrow \mathcal{E}_{2_\alpha}/\mathcal{E}_{1_\alpha} \rightarrow 0.$$ 

Since $\mathcal{E}_{1_\alpha}$ is $n$-regular, by applying the functor $\Psi$ to the short exact sequence (4.5), we get the isomorphism

$$\Psi(\mathcal{E}_{2_\bullet})/\Psi(\mathcal{E}_{1_\bullet}) \cong \Psi(\mathcal{E}_{2_\bullet}/\mathcal{E}_{1_\bullet}).$$

The rest of the proof follows in the same line as in [AK07, Corollary 5.11].
5. Moduli functors and construction

We are now ready to give a functorial construction of the following moduli problem. Consider the moduli functor
\[ M(\tau_p)_{X, \text{ss}} : (\mathbf{Sch}/k)^{\circ} \to \mathbf{Set} \]
which assigns to a k-scheme S the set of isomorphism classes of flat families over S of semistable parabolic sheaves on X having parabolic type \( \tau_p \). Similarly, there is a moduli functor \( M(\tau_p)_{X, \text{ss}} \) for stable sheaves.

Let us fix the dimension vector \( d \) for the ladder quiver
\[ A_{\ell+1} \times K = ((A_{\ell+1} \times K)_0, (A_{\ell+1} \times K)_1), \]
where \((A_{\ell+1} \times K)_0\) is the set of vertices and \((A_{\ell+1} \times K)_1\) is the set of arrows as described in (3.4). Let
\[ R := \bigoplus_{a \in (A_{\ell+1} \times K)_1} \text{Hom}_k(k^{d_{i(a)}}, k^{d_{t(a)}}) \]
be the space of representations of \( A_{\ell+1} \times K \) having dimension vector \( d \). Let \( R_{\text{rel}} \) be the closed subset of \( R \) consisting of \( (\alpha_a) \) satisfying the relation in I (see, (3.4)). Let \( R_{\text{fil}} \) be an open subset of \( R_{\text{rel}} \) consisting of \( (\alpha_a) \in R_{\text{rel}} \) such that for any \( a \in A_{\ell+1} \times Q_0 \), we have \( \alpha_a \) injective map. Let
\[ G(A_{\ell+1} \times K) := \prod_{w \in (A_{\ell+1} \times K)_0} \text{GL}(k^{d_w}). \]
There is a natural action of the group \( G(A_{\ell+1} \times K) \) on \( R \) such that the isomorphism classes in \( \text{Rep}(A_{\ell+1} \times K) \) correspond to the orbits in \( R \) with respect to this action. Let \( G := G(A_{\ell+1} \times K)/\Delta \), where \( \Delta := \{(t1_{k^{d_w}})_{w \in (A_{\ell+1} \times K)_0} | t \in k^{*}\} \) and there is an action of this group \( G \) on \( R \).

Note that \( R_{\text{fil}} \) is a locally closed subscheme of \( R \), and is invariant under this action. Given a weight \( \sigma \in \mathbb{R}^{(A_{\ell+1} \times K)_0} \), let
\[ R_{\text{fil}}^{\sigma, \text{ss}} = \{ x \in R_{\text{fil}} | \text{the corresponding representation } M_x \text{ is } \sigma\text{-semistable} \} \]
be the subset of \( R_{\text{fil}} \), which is \( \sigma\)-semistable locus in \( R_{\text{fil}} \). Similarly, let \( R_{\text{fil}}^{\sigma, \text{ss}} \) denote the \( \sigma\)-stable locus in \( R_{\text{fil}} \). Recall that if \( \sigma \) is an integral weight, then \( R_{\text{fil}}^{\sigma, \text{ss}} \) and \( R_{\text{fil}}^{\sigma, s} \) are open subset of \( R_{\text{fil}} \). By [Ch08, Corollary 2.3], it follows that they are open subsets of \( R_{\text{fil}} \) for arbitrary weights.

Now, for a fixed parabolic type \( \tau_p \), we have a fixed parabolic weights \( \alpha_\ast \). From this, we get a weight \( \theta \in \mathbb{R}^{(A_{\ell+1} \times K)_0} \) as in (4.4) so that the functor \( \Psi \) preserve the semistability (see, Section 4). More precisely, for a vertex \( w \in (A_{\ell+1} \times K)_0 \), let
\[ \theta_w = \begin{cases} 
\varepsilon_j d_{i_2} & \text{if } w = j_1, j = 1, 2, \ldots, \ell + 1 \\
-\sum_{i=1}^{\ell+1} \varepsilon_i d_{i_1} & \text{if } w = l_2 \\
0 & \text{otherwise,} 
\end{cases} \]
where \( d = (d_{11}, d_{12}, \ldots, d_{(\ell+1)1}, d_{(\ell+1)2}) \) is the dimension vector of \( A_{\ell+1} \times K \) determined by the fixed parabolic type \( \tau_p \).

By [Ch08, Corollary 2.3], there exist an integral weight \( \sigma \in \mathbb{Z}^{(A_{\ell+1} \times K)}_0 \) such that
\[
\mathcal{R}^\sigma_{\text{ss}} = \mathcal{R}^{\theta_{\text{ss}}}_{\text{fil}} \quad \text{and} \quad \mathcal{R}^{\sigma_{s}}_{\text{fil}} = \mathcal{R}^{\theta_{s}}_{\text{fil}}.
\]

Let \( \chi: G \to k^* \) be the character determined by the integral weight \( \sigma \). Then, we have
\[
\mathcal{R}^{\chi_{\text{ss}}}_{\text{fil}} = \mathcal{R}^{\theta_{\text{ss}}}_{\text{fil}} \quad \text{and} \quad \mathcal{R}^{\chi_{s}}_{\text{fil}} = \mathcal{R}^{\theta_{s}}_{\text{fil}}.
\]

where \( \mathcal{R}^{\chi_{\text{ss}}}_{\text{fil}} \) denote the open subset of \( \mathcal{R}_{\text{fil}} \) which is \( \chi_{\text{ss}} \)-semistable locus in \( \mathcal{R}_{\text{fil}} \).

For simplicity, we shall denote \( \mathcal{R}^{\theta_{\text{ss}}}_{\text{fil}} \) and \( \mathcal{R}^{\theta_{s}}_{\text{fil}} \) simply by \( \mathcal{R}^{\text{ss}}_{\text{fil}} \) and \( \mathcal{R}^{s}_{\text{fil}} \), respectively.

Recall [AD20, Section 4.2], the moduli functor
\[
\mathcal{M}_{\text{ss}}(d): \left( \text{Sch}/K \right)^{\circ} \to \text{Set}
\]
which assigns to a \( k \)-scheme \( S \) the set of isomorphism classes of flat families over \( S \) of semistable filtered Kronecker modules of given dimension vector \( d \).

**Remark 5.1.** Let \( M_\bullet \) be a tautological family of filtered Kronecker modules on \( \mathcal{R}_{\text{fil}} \). It will be a trivial vector bundle for each vertex of the ladder quiver, as it is a restriction of the tautological family over the affine space \( \mathcal{R} \). We can use this fact to see that if two families are related by an action of group element over a base scheme \( S \), then they are isomorphic as families.

There is a natural functor
\[
h: \mathcal{R}^{\text{ss}}_{\text{fil}} \to \mathcal{M}^{\text{ss}}_{\text{fil}}(d),
\]
which induces a local isomorphism
\[
\tilde{h}: \mathcal{R}^{\text{ss}}_{\text{fil}}/G \to \mathcal{M}^{\text{ss}}_{\text{fil}}(d) \quad \text{[AD20, Proposition 4.11].}
\]

**Theorem 5.2.** There exist moduli spaces \( M^{\text{ss}}_{\text{fil}}(d) \) (respectively, \( M^{s}_{\text{fil}}(d) \)) of \( \theta_{\text{ss}} \)-semistable (respectively, \( \theta_{\text{s}} \)-stable) filtered Kronecker modules having dimension vector \( d \), where \( M^{\text{ss}}_{\text{fil}}(d) \) is a quasi-projective scheme. Further, the closed points of the moduli space \( M^{\text{ss}}_{\text{fil}}(d) \) (respectively, \( M^{s}_{\text{fil}}(d) \)) correspond to the \( S \)-equivalence classes of \( \theta_{\text{ss}} \)-semistable (respectively, \( \theta_{\text{s}} \)-stable) objects of \( A_{\text{-mod}}^{\text{fil}} \) which have dimension vector \( d \).

**Proof.** In view of Proposition 4.6, this is a special case of [AD20, Theorem 4.12]. \( \square \)

Let \( Q = (\mathcal{R}_{\text{fil}})^{\text{reg}}_{\tau_p} \) be the locally closed subscheme of \( \mathcal{R}_{\text{fil}} \) corresponding to the family \( M_\bullet \) as in the Proposition 3.8.

Let \( \mathcal{M}_1, \mathcal{M}_2: \text{Sch}^{\text{op}} \to \text{Set} \) be two functors. Recall that a morphism of functors \( g: \mathcal{M}_1 \to \mathcal{M}_2 \) is said to be a local isomorphism, if it induces an isomorphism of sheafification in the Zariski topology.

**Lemma 5.3.** For a fixed parabolic type \( \tau_p \), the moduli functor \( \mathcal{M}(\tau_p)^{\text{ss}}_X \) is locally isomorphic to the quotient functor \( \overline{Q^{[\text{ss}]}/G} \).

**Proof.** Let \( Q^{[\text{ss}]}/Q \) be an open loci, where the fibres of the tautological flat family \( \mathcal{F}_\bullet = \Psi^v(i^*M_\bullet) \) over \( Q \) are semistable parabolic sheaves. Then, we get a natural transformation
\[
g^{\text{ss}}: \overline{Q^{[\text{ss}]}} \to \mathcal{M}(\tau_p)_X^{\text{ss}}.
\]
which is restriction of the natural transformation $g: Q \rightarrow \mathcal{M}(\tau_p)_{X}^{\text{reg}}$ defined by

$$g^{\text{ss}}(\sigma: S \rightarrow Q^{[\text{ss}]}) \mapsto [\Psi(\sigma^* \iota^* M_\bullet)]$$

If we change the family $\sigma^* \iota^* M_\bullet$ by the action of the group $G(S)$, then as we noticed in Remark 5.1, the tautological family comes with the trivial vector bundle over each vertex of the ladder quiver, and hence gives the isomorphic family. Since an isomorphism of families is preserved after applying the functor $\Psi^\vee$, we get the well defined natural transformation $\tilde{g}^{\text{ss}}: Q^{[\text{ss}]} / G \rightarrow \mathcal{M}(\tau_p)_{X}^{\text{reg}}$. By Proposition 3.8(1), we have the following commutative diagram of natural transformations

$$
\begin{array}{ccc}
Q^{[\text{ss}]} & \xrightarrow{\iota} & R^{\text{ss}}_{\text{fil}} \\
\downarrow g^{\text{ss}} & & \downarrow h \\
\mathcal{M}^{\text{ss}}(\tau_p)_X & \xrightarrow{j} & \mathcal{M}^{\text{ss}}_{\text{fil}}(d)
\end{array}
$$

For any $k$-scheme $S$, the map $Q^{[\text{ss}]} \rightarrow \mathcal{M}(\tau_p)_{X}^{\text{ss}} \times_{\mathcal{M}^{\text{ss}}_{\text{fil}}(d)} R^{\text{ss}}_{\text{fil}}$ defined by

$$(\sigma: S \rightarrow Q^{[\text{ss}]}) \mapsto (g_S(\sigma), \iota \circ \sigma)$$

is a bijection, by Proposition 3.8(2). In other words, we have the following Cartesian diagram

$$
\begin{array}{ccc}
Q^{[\text{ss}]} / G & \longrightarrow & R^{\text{ss}}_{\text{fil}} / G \\
\downarrow g^{\text{ss}} & & \downarrow \tilde{h} \\
\mathcal{M}^{\text{ss}}(\tau_p)_X & \xrightarrow{j} & \mathcal{M}^{\text{ss}}_{\text{fil}}(d)
\end{array}
$$

Since $\tilde{h}$ is a local isomorphism, it follows that $\tilde{g}^{\text{ss}}: Q^{[\text{ss}]} / G \rightarrow \mathcal{M}(\tau_p)_{X}^{\text{ss}}$ is a local isomorphism.

Theorem 5.4. There exists a quasi-projective scheme $\mathbf{M}_{X}^{\text{ss}}(\tau_p)$ which corepresents the moduli functor $\mathcal{M}^{\text{ss}}(\tau_p)_X$. The closed points of $\mathbf{M}_{X}^{\text{ss}}(\tau_p)$ correspond to the $S$-equivalence classes of semistable parabolic sheaves having parabolic type $\tau_p$.

Proof. First we show that the good quotient $\pi_X: Q^{[\text{ss}]} \rightarrow Q^{[\text{ss}]} / G =: \mathbf{M}_{X}^{\text{ss}}(\tau_p)$ exists. Set $Z := Q^{[\text{ss}]} \cap R^{\text{ss}}_{\text{fil}}$. Let $O(M_\bullet)$ be an orbit of a point in $Q^{[\text{ss}]}$ corresponding to $M_\bullet = \Psi(E_\bullet)$, where $E_\bullet$ is a semistable parabolic sheaf on $X$ having parabolic type $\tau_p$. The closed orbit in $Z$ contained in the closure of the orbit $O(M_\bullet)$ in $Z$ is the orbit corresponding to the associated graded object $\text{gr}^{\text{JH}}(M_\bullet)$ (see, [AD20, Proposition 4.9]). By Proposition 4.12, $\text{gr}^{\text{JH}}(M_\bullet) \cong \Psi(\text{gr}(E_\bullet))$, and $\text{gr}(E_\bullet)$ is semistable. Hence, this closed orbit is also in $Q^{[\text{ss}]}$. Now following the arguments as in the proof of [AK07, Proposition 6.3], we conclude that the good quotient $\pi_X: Q^{[\text{ss}]} \rightarrow \mathbf{M}_{X}^{\text{ss}}(\tau_p)$ exists.

Since $\tilde{g}^{\text{ss}}: Q^{[\text{ss}]} / G \rightarrow \mathcal{M}(\tau_p)_{X}^{\text{ss}}$ is a local isomorphism, it follows that $\mathbf{M}_{X}^{\text{ss}}(\tau_p)$ corepresents the moduli functor $\mathcal{M}(\tau_p)_{X}^{\text{ss}}$. 

□
Moreover, we have the following commutative diagram

\[
\begin{array}{ccc}
Q^{ss} & \xrightarrow{i} & R_{fil}^{ss} \\
\pi_X \downarrow & & \downarrow \pi \\
M_X^{ss}(\tau_p) & \xrightarrow{\varphi} & M_{fil}^{ss}(d)
\end{array}
\]

where the morphism \( \varphi : M_X^{ss}(\tau_p) \to M_{fil}^{ss}(d) \) is a set-theoretic injection of the closed points [AK07, Proposition 6.3]. In other words, it induces a bijection

\[
\varphi : M_X^{ss}(\tau_p)(k) \to \pi(Q^{ss})(k)
\]

Hence, the closed points of \( M_X^{ss} \) correspond to the \( S \)-equivalence classes of semistable filtered Kronecker modules that are of the form \( \Psi(E_\bullet) \) for semistable parabolic sheaves \( E_\bullet \) having parabolic type \( \tau_p \). By Proposition 4.12, the assertion for closed points follows.

**Remark 5.5.** The image of the map \( \varphi : M_X^{ss}(\tau_p)(k) \to M_{fil}^{ss}(d)(k) \) is contained in the set of all closed points of \( M_{fil}^{ss}(d) \) which correspond to the \( \mathcal{SH} \)-equivalence classes of filtered Kronecker modules (see the remark after [AD20, Theorem 4.12]). We have not considered the case of parabolic weights with \( \alpha_1 = 0 \), as we do not know the validity of [Sc11, Proposition 4.4.3]. But in these cases, it is possible to use the more general notion of \( S \)-equivalence as in [AD20] to get the description of two types of points of \( M_X^{ss}(\tau_p) \).

The projectivity of moduli space \( M_X^{ss}(\tau_p) \) follows essentially from [Yo93, §5] using Langton’s criterion. In the following, we provide some essential steps with appropriate references.

Let \( R \) be a discrete valuation ring over \( k \) with quotient filed \( L \). Let \( i : X_k \to X_R \) be a closed immersion, and \( j : X_L \to X_R \) an open immersion.

**Proposition 5.6.** [Yo93, Theorem 5.7] Let \( \mathcal{E}_\bullet \) be a flat family of parabolic sheaves having parabolic type \( \tau_p \) on \( X \) over \( \text{Spec}(R) \) such that \( j^*\mathcal{E}_\bullet \) is a flat family of semistable parabolic sheaves on \( X \) over \( \text{Spec}(L) \). Then, there exists an \( \mathcal{O}_{X_R} \)-submodule \( \mathcal{E}' \) of \( \mathcal{E} \) such that \( j^*\mathcal{E}' = j^*\mathcal{E} \) and \( \mathcal{E}'_\bullet \) is a flat family of parabolic sheaves on \( X \) over \( \text{Spec}(R) \) and \( i^*\mathcal{E}'_\bullet \) is semistable; where \( \mathcal{E}'_\alpha = \mathcal{E}' \cap \mathcal{E}_\alpha \) for all \( \alpha \).

**Proposition 5.7.** Let \( \tau_p \) be a fixed parabolic type with \( \deg(P) := \dim X \). The moduli space \( M_X^{ss}(\tau_p) \) is projective over \( \text{Spec}(k) \).

**Proof.** Since \( M_X^{ss}(\tau_p) \) is quasi-projective, we only need to show that it is proper over \( \text{Spec}(k) \). For this, let \( R \) be a discrete valuation ring with quotient field \( L \) and residue field \( k \). Let \( x : \text{Spec}(L) \to X^{ss}(\tau_p) \). Then, there exist a field extension \( L' \) of \( L \) and \( L' \)-valued point \( x' \) of \( Q^{ss} \) such that the following diagram

\[
\begin{array}{ccc}
\text{Spec}(L') & \xrightarrow{x'} & Q^{ss} \\
\downarrow & & \downarrow \pi \\
\text{Spec}(L) & \xrightarrow{x} & M_X^{ss}(\tau_p)
\end{array}
\]
commutes. Let $E' = x^*F'$, where $F' = \Psi'(\iota^*M)$ is the tautological family of parabolic semistable sheaves having parabolic type $\tau_p$ on $Q^{[ss]}$. Let $R'$ be a discrete valuation ring dominating $R$ with quotient filed $L'$. Let $F$ be a coherent $O_{X_{R'}}$-submodule of $j_*E'$ such that $j^*F = E'$, where $j : X_L \to X_{R'}$ (cf. [La75, Proposition 6]). By [HL97, Exercise 2.B.2], there is a subsheaf $F' \subset F$ such that $j^*F' = j^*F = E'$ and $i^*F'$ is pure, where $i : X_k \to X_{R'}$ is a closed immersion. By properness of the Quot scheme, there exists a unique subsheaf $F'_\alpha$ of $F'$ such that $j^*F'_\alpha = E'_\alpha$ and $F'/F'_\alpha$ is flat over $R'$ for each $0 \leq \alpha \leq 1$. Since $i^*F'$ is pure and hence using the assumption on $\tau_p$, we get $\dim(Supp(i^*(F')) \cap D) < \dim(Supp(i^*(F'))).$ Therefore, the map $i^*(F' \otimes_{O_{X_{R'}}} O_{X_{R'}}(-D_{R'})) \to i^*F'$ is injective, and hence $F'/\langle F' \otimes_{O_{X_{R'}}} O_{X_{R'}}(-D_{R'}) \rangle$ is flat over $R'$.

By Proposition 5.6, we get a flat family $E''_\eta$ of semistable parabolic sheaves on $X$ over $\text{Spec}(R')$ such that $j^*E''_\eta = E'_\eta$. The corresponding classifying map $x'_{R'} : \text{Spec}(R') \to M^\text{reg}(\tau_p)$ is the lift of $x'$. The rest of the proof follows the identical arguments as in the proof of [AK07, Proposition 6.6].

6. Moduli stacks and embedding

Recall that a quasi-parabolic sheaf on a scheme $X$ is called a sheaf of fixed type if each terms of the associated graded sheaf has fixed Hilbert polynomial. It is called $n$-regular if each of the sub-sheaves in the filtration is $n$-regular. We can check that the quasi-parabolic sheaf is $n$-regular if each subquotients are $n$-regular. We will say that the parabolic sheaf is $n$-regular, if the underlying quasi-parabolic sheaf is $n$-regular.

**Definition 6.1.** The moduli stack of $n$-regular parabolic sheaves is a category fibered in groupoids defined as,

$$P_X : \mathcal{M}_X^{\text{reg}}(\tau_p) \to \text{Sch}/\mathbb{K}; \mathcal{M}_X^{\text{reg}}(\tau_p)(T) \mapsto T$$

where $\mathcal{M}_X^{\text{reg}}(\tau_p)(T)$ is the groupoid of all flat families of $n$-regular parabolic sheaves of fixed type parametrised by the scheme $T$ and morphism is given by isomorphism of families of quasi-parabolic sheaves. The morphisms of the category $\mathcal{M}_X^{\text{reg}}(\tau_p)$ are pairs $(f : T' \to T, \phi)$, similar to the example described in [Go01, Example 2.16].

There are also two open moduli sub-stacks of the moduli stack $\mathcal{M}_X^{\text{reg}}$ determined by the notion of parabolic stability condition. The moduli stack of semistable (respectively, stable) parabolic sheaves will be denoted as $\mathcal{M}_X^{\text{ss}}(\tau_p)$ (respectively, $\mathcal{M}_X^{\text{st}}(\tau_p)$).

**Remark 6.2.** Using the boundedness result of Schlüter [Sc11, Theorem 4.5.2], originally due to Maruyama and Yokogawa, and its extension by Inaba in some special cases, we can conclude that semistable and stable sub-stacks are algebraic stacks with respect to fppf topology on $\text{Sch}$. Moreover, we get a canonical maps $Q^{[ss]} \to \mathcal{M}_X^{\text{ss}}(\tau_p)$ and $Q^{[s]} \to \mathcal{M}_X^{\text{st}}(\tau_p)$.

We will denote by $M^\text{reg}_X(\tau_p)$ (respectively, $M^\text{ss}_X(\tau_p)$) the coarse moduli space of semistable (respectively, stable) parabolic moduli stack (see Theorem 5.4). Similar to the parabolic case, we can also define the moduli stack of representations of fixed dimension vector.
**Definition 6.3.** The moduli stack of filtered representations is a category fibered in groupoids defined as,

\[ P_A : \mathcal{M}_{\text{fil}}(d) \to \text{Sch}/K; \mathcal{M}_{\text{fil}}(d)(T) \mapsto T \]

where \( \mathcal{M}_{\text{fil}}(d)(T) \) is the groupoid of all flat families of filtered modules of fixed dimension vector parametrised by the scheme \( T \) and morphism is given by isomorphism of families of modules. The morphism in this groupoid can be described as pair \( (f : T' \to T, \phi : M' \congto M) \), where \( \phi \) is an isomorphism of two flat families of filtered modules \( M \) and \( M' \).

Now, using the notion of stability via slope function, we get two open sub-stacks of \( \mathcal{M}_{\text{fil}}(d) \), denoted as \( \mathcal{M}_{\text{ss}}^{\text{fil}}(d) \) and \( \mathcal{M}_{\text{s}}^{\text{fil}}(d) \). We will also denote by \( \mathcal{M}_{\text{ss}}^{\text{fil}}(d) \) (respectively, \( \mathcal{M}_{\text{s}}^{\text{fil}}(d) \)) the coarse moduli space of semistable (respectively, stable) filtered moduli stack.

Now, we can lift the embedding of \( \mathcal{M}_{\text{ss}}^{\text{fil}}(\tau_p) \) in \( \mathcal{M}_{\text{fil}}^{\text{reg}}(d) \) as 1-morphisms at the level of stacks \([\text{Stacks20, Definition 02XS}]\).

**Proposition 6.4.** There exists a morphism of moduli stacks \( \Psi : \mathcal{M}^{\text{reg}}_X(\tau_p) \to \mathcal{M}_{\text{fil}}(d) \), i.e. \( P_A \circ \Psi = P_X \). Moreover, \( \Psi \) is fully-faithful functor.

*Proof.* By Proposition 3.7, we have an embedding of the category of flat family of \( n \)-regular parabolic sheaves having parabolic type \( \tau_p \) into the category of flat family of filtered Kronecker modules having dimension vector \( d \), where \( d \) is determined by the fixed parabolic type \( \tau_p \).

If we have any flat family of parabolic sheaves on \( X \times S \) over a scheme \( S \), then we get a filtered locally free sheaf on \( S \), using regularity, which has fiberwise \( A \)-module structure. This defines a functor at the object level, which will be denoted by the same symbol \( \Psi \), similar to \([\text{AK07}]\).

If we have an isomorphism of flat families of parabolic sheaves over a fixed base scheme, then it will also induce an isomorphism of flat families of filtered Kronecker modules over the same base scheme. Hence, using the functoriality, we can extend the functor to an embedding of the groupoid

\[ \Psi_S : \mathcal{M}^{\text{reg}}_X(\tau_p)(S) \to \mathcal{M}_{\text{fil}}(d)(S); E \mapsto \Psi_S(E). \]

If we have a morphism between the base schemes \( f : S \to S' \), then we can get a functor \( (f \times 1)^* : \mathcal{M}^{\text{reg}}_X(\tau_p)(S') \to \mathcal{M}^{\text{reg}}_X(\tau_p)(S) \). Similar map exists between the fiber groupoids at module side.

Using the fact that cohomology commutes with flat base change, we can get the morphism between moduli stacks \( \Psi : \mathcal{M}^{\text{reg}}_X(\tau_p) \to \mathcal{M}_{\text{fil}}(d) \). Now, the result \([\text{Stacks20, Lemma 003Z}]\) proves that the functor \( \Psi \) is fully faithful. \( \square \)
Lemma 6.5. There is a commutative diagram with horizontal maps being locally closed immersion on topological space of closed points

\[
\begin{array}{ccc}
\mathcal{M}^\text{ss}_X(\tau_P) & \longrightarrow & \mathcal{M}^\text{ss}_{\text{fil}}(d) \\
\downarrow & & \downarrow \\
M^\text{ss}_X(\tau_P) & \longrightarrow & M^\text{ss}_{\text{fil}}(d)
\end{array}
\]

and there is a similar diagram with stable in place of semistable loci and lower horizontal arrow is locally closed scheme theoretic embedding.

Proof. The morphism of stack on top is obtained by restricting the functor \(\Psi\). We will use the same notations for the category fibered in sets given by the moduli functors of last section. Hence, we get the following canonical commutative diagram

\[
\begin{array}{ccc}
\mathcal{M}^\text{ss}_X(\tau_P) & \longrightarrow & \mathcal{M}^\text{ss}_{\text{fil}}(d) \\
\downarrow & & \downarrow \\
\mathcal{M}(\tau_P)^\text{ss}_X & \longrightarrow & \mathcal{M}^\text{ss}_{\text{fil}}(d)
\end{array}
\]

Now, using the commutative square coming from the Theorem 5.4, we get the required commutative square. Since both \(Q^\text{ss} \rightarrow M^\text{ss}_X(\tau_P)\) and \(R^\text{ss}_{\text{fil}}(d) \rightarrow M^\text{ss}_{\text{fil}}(d)\) are good quotients with the same \(S\)-equivlalence classes, the last assertion on closed points follows.

We can get the commutative square on stable locus by similar argument. Since \(Q^\text{ss}\) is closed equivariant subscheme of \(R^\text{ss}_{\text{fil}}(d)\) which is a principal bundle over \(M^\text{ss}_{\text{fil}}(d)\), we get the required scheme theoretic embedding (see [AK07, Proposition 6.7]). □

Proposition 6.6. There is a commutative diagram of 1-morphisms between the stacks i.e. following diagram which commutes up to natural isomorphism (or 2-morphism)

\[
\begin{array}{ccc}
[Q^\text{ss}/G(\mathbb{A}_{\ell+1} \times K)] & \xrightarrow{i} & [R^\text{ss}_{\text{fil}}/G(\mathbb{A}_{\ell+1} \times K)] \\
\downarrow \tilde{\alpha} & & \downarrow \tilde{\beta} \\
\mathcal{M}^\text{ss}_X(\tau_P) & \xrightarrow{\Psi} & \mathcal{M}^\text{ss}_{\text{fil}}(d)
\end{array}
\]

such that the vertical morphisms are isomorphisms. A similar assertion holds for the stable in place of semistable.

Proof. Using the canonical inclusion of \(Q^\text{ss} \hookrightarrow R^\text{ss}\), we get the horizontal 1-morphism of stacks. The bottom horizontal map is defined using the previous Proposition 6.4. The vertical morphisms are defined using the local universal family of modules, exactly similar to the maps \(\tilde{\alpha}\) and \(\tilde{\beta}\), say \(\tilde{\alpha}\) and \(\tilde{\beta}\), respectively. The vertical maps are isomorphism follow from [Go01, Proposition 3.3].

We can check that the square commutes up to natural isomorphism given by the unit of the adjunction. The assertion for the stable case follows using similar arguments. □

Now we get the following assertion from the description given in [Go01, Example 2.29] and properties of GIT quotient from the Mumford’s GIT.
Lemma 6.7. There exists a pullback diagram with vertical maps being canonical map to coarse moduli spaces with horizontal maps being (set-theoretic) immersions between stacks and schemes respectively.

\[
\begin{array}{ccc}
[Q^{ss}/G(A_{\ell+1} \times K)] & \longrightarrow & [R_{\text{fil}}^{ss}/G(A_{\ell+1} \times K)] \\
\downarrow & & \downarrow \\
Q^{ss}/\!/G & \longrightarrow & R_{\text{fil}}^{ss}/\!/G \\
\end{array}
\]

and there is a similar diagram for stable locus with the lower horizontal map being scheme theoretic immersion.

Proposition 6.8. There is a commutative diagram, up to natural isomorphism, where vertical maps are canonical map to coarse moduli spaces

\[
\begin{array}{ccc}
\mathcal{M}_X^{ss}(\tau_P) & \longrightarrow & \mathcal{M}_X^{ss}(d) \\
\sim & & \sim \\
[Q^{ss}/G(A_{\ell+1} \times K)] & \longrightarrow & [R_{\text{fil}}^{ss}/G(A_{\ell+1} \times K)] \\
\downarrow & & \downarrow \\
M_X^{ss}(\tau_P) & \longrightarrow & M_{\text{fil}}^{ss}(d) \\
\downarrow & & \downarrow \\
Q^{ss}/\!/G & \longrightarrow & R_{\text{fil}}^{ss}/\!/G \\
\end{array}
\]

We get the similar diagram for stable locus with the vertical maps in bottom square being isomorphisms.

Proof. The two vertical squares come from the canonical maps coming from the definition of coarse moduli spaces of respective stacks. Now, using previous Proposition 6.6, we get that the top horizontal square will commute up to natural isomorphism and the bottom horizontal square commutes using the Theorem 5.4.

Next, the front vertical square commutes using the Lemma 6.7 and the back vertical square commutes using the Lemma 6.5. The commutative diagram for the stable in place of semistable follows from similar arguments.

Remark 6.9. In the stable case, the front square in the above diagram becomes isomorphic to the square in the background, and hence we can deduce properties of the morphism between moduli stacks and coarse moduli spaces in the background square using the properties of morphism between quotient stacks diagram.

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