AXIOMS FOR QUANTUM YANG-MILLS THEORIES - 1. EUCLIDEAN AXIOMS

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Abstract. This paper extends the notion of Schwinger functions to quantum Yang-Mills theories and proposes the axioms they should satisfy. Two main features of this axiom scheme is that we assume existence of gauge-invariant co-located Schwinger functions and impose reflection positivity only on them. This is in accordance with the fundamental principle of gauge theories that only gauge-invariant quantities can be given physical meaning.

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1. Introduction

In this paper we adjust the existing Euclidean axiom scheme [1, 2, 3, 4] for quantum field theory to axiomatize most non-Abelian gauge theories (a.k.a Yang-Mills theories) in current usage. A typical construction program starts with the Euclidean framework and derives the Minkowski theory as a consequence. Therefore, if one seeks to axiomatize gauge theories, proposal of Euclidean axioms is the reasonable starting point.

We discuss two most widely recognized strategies for construction of quantum gauge theories:

- The first approach is the lattice approximation. The continuum infinite volume limit was established for the 2D Abelian Higgs model in a series of papers [5, 6, 7]. The mass generation of the gauge field (Higgs mechanism) is also proven. Furthermore, existing axioms of [1, 2], except for cluster decomposition, are rigorously verified. The Abelian Higgs
model has been rigorously constructed in 3D finite volume by T. Balaban, whose renormalization scheme was later improved by J. Dimock; refer to [8] and references therein. Dimock also has been working on 3D spinor QED, where he established ultraviolet stability and regularity through a series of papers [9, 10, 11]. Regarding lattice approximation of Yang-Mills theories, Balaban wrote a large series of papers, as described in the expository articles [12, 13, 14] and references therein such as [15, 16, 17, 18]. Balaban’s main achievement was the introduction of length scale dependent block averaging. In fact, Dimock took up this set of ideas in analyzing 3D spinor QED.

- The second approach is stochastic quantization. In [19], a functional measure in the axial gauge is constructed as a solution of the 2D Yang-Mills stochastic differential equation. [20] addresses the SPDE for 2D Yang-Mills theory as well, with inclusion of the Higgs field. However, rather than working in a fixed gauge, [20] rigorously constructs the gauge orbit space of physical states and analyzes its topology and establishes a Markov process invariant under the gauge action. Moreover, it conjectures existence of a unique gauge-invariant functional measure associated with this Markov process. In 3D, the Yang-Mills-Higgs model is considered in the paper [21]. Again, the paper rigorously defines the gauge orbit space and obtains a Markov process as the solution of Yang-Mills SPDE. Still, existence of a unique gauge-invariant functional measure is not proven yet.

- There also exist papers such as [22] in which both of the above approaches are addressed. It shows that lattice approximation and stochastic quantization yield the same result in 2D under the complete axial gauge.

What is clear from the above discussion is that there may be multiple paths of constructing gauge theories. Our axioms are focused on the non-Abelian cases and formulated to be independent of which approach is chosen, and designed to apply to the final construction only.

In fact, there exists general consensus within the constructive quantum field theory community on what properties should be required for a sensible gauge theory [23], most of which have been explicitly stated and verified on lattice setting [24]. This paper may be regarded as systematic statement of such requirements in continuum $\mathbb{R}^4$ with fully constructed theories in consideration. This is in accordance with history of development of constructive QFT. The Wightman and Haag-Kastler axioms preceded construction of non-trivial examples in the history of rigorous quantum field theory, as described in the article [25]. The axioms were suggested in early 1960s but first rigorous construction of nontrivial examples was achieved in late 1960s through the $\phi^4$ and Yukawa theories in 1 + 1 spacetime dimensions; more difficult examples such as the $\phi^4$ theory in 3 Euclidean dimensions were constructed during 1970s and 1980s. Therefore, we hope that this paper will serve as a guide post for mathematical construction of quantum Yang-Mills theories.
2. Notations

We closely follow the notations [1, Section 2 and 6] and [26, Chapter 15]. Throughout this paper, the expression

\[ A := B \]

means that \( A \) is defined as \( B \). The terms Yang-Mills theory and non-Abelian gauge theory are used interchangeably.

**Definition 1.** The symbol \( \tilde{x} \) denotes a point in \( \mathbb{R}^4 \) whose coordinates with respect to the standard basis are given by \((x^0, x^1, x^2, x^3)\). We also use notations \( \tilde{y}, \tilde{w} \) and \( \tilde{z} \) for the same purpose. The indices \( i, j \in \{1, 2, 3, 4\} \) label vector components in \( \mathbb{R}^4 \) with respect to the standard basis. Moreover, we assume the Euclidean metric on \( \mathbb{R}^4 \) with respect to the standard basis throughout this paper.

**Definition 2.** For any \( N \in \mathbb{N} \), \( \mathcal{S}(\mathbb{R}^{4N}) \) denotes the space of complex-valued Schwartz functions on \( \mathbb{R}^{4N} \). Denote by \( \{ | \cdot |_k \}_{k \in \mathbb{N} \cup \{0\}} \) a collection of semi-norms giving \( \mathcal{S}(\mathbb{R}^{4N}) \) a nuclear Fréchet topology. An example of such semi-norms is given explicitly in [1, p.86]

Denote by \( \mathcal{S}'(\mathbb{R}^{4N}) \) the space of tempered distributions equipped with the strong dual topology.

**Definition 3.** The gauge group \( G \) denotes a simple compact Lie group, typically \( SU(N) \) for \( N \geq 2 \). We fix a bi-invariant Riemannian metric on \( G \) throughout this paper.

The Lie algebra associated with \( G \) is denoted as \( \mathfrak{g} \) and we write its Lie bracket as \([ , ]\). Following [26, Chapter 15], we fix a set of generators for \( \mathfrak{g} \)

\[ \{ t_\alpha | \alpha = 1, 2, \ldots, \dim \mathfrak{g} \} \]

which are normalized in the sense that \( \text{Tr}(t_\alpha t_\beta) = \delta_{\alpha\beta} \).

**Definition 4.** Let \( V \) be any finite-dimensional real or complex vector space and \( \mathcal{M} \) be the space of smooth \( L(V, V) \)-valued mappings on \( \mathbb{R}^4 \). For any \( k \in \mathbb{N} \), denote by \( [\mathcal{M}]^{\otimes k} \) the \( k \)-fold tensor product space with the topology as in [27, Chapter 43]. Therefore, the mapping

\[ F_1 \otimes \cdots \otimes F_k \in [\mathcal{M}]^{\otimes k} \rightarrow \left( \tilde{x} \rightarrow \text{Tr}(F_1(\tilde{x}) \cdots F_k(\tilde{x})) \right) \in C^\infty(\mathbb{R}^4) \]

may be uniquely extended to a continuous linear mapping from \( [\mathcal{M}]^{\otimes k} \) to \( C^\infty(\mathbb{R}^4) \), which we still denote by \( \text{Tr} \).

*In this paper, we consider only a single neutral gauge particle. Thus, the gauge field \( A_\alpha \) appearing in Section 3 is assumed to be neutral. Restriction of \( G \) to a simple compact Lie group is in accordance with such simplification. Generalization to arbitrary products of compact gauge groups (including \( U(1) \)) and gauge antiparticles is straightforward.

†From Schur’s lemma, such a metric on \( G \) is unique up to multiplication by a positive real number.

‡\( \mathcal{M} \) is a nuclear Fréchet space as it may be identified with a finite direct product of \( C^\infty(\mathbb{R}^4) \)’s. Therefore, the two topologies presented in the reference are isomorphic [27, Theorem 50.1].
Definition 5. Let $g : \mathbb{R}^4 \to G$ be a smooth mapping. Its differential $Dg$ is than a mapping of $x \in \mathbb{R}^4$ such that

$$D_xg \in L(\mathbb{R}^4, T_{g(x)}G)$$

Using the Euclidean metric on $\mathbb{R}^4$ and the assumed bi-invariant Riemannian metric on $T_{g(x)}G$, we may define the operator norm $\|\cdot\|_{\text{op}}$ on $L(\mathbb{R}^4, T_{g(x)}G)$. Note that $\|\cdot\|_{\text{op}}$ does not depend on $x$ by construction. We say that $x \to D_xg$ is rapidly decaying at infinity if

$$\sup_{x \in \mathbb{R}^4}(1 + \|x\|)^n\|D_xg\|_{\text{op}} < \infty$$

for all $n \in \mathbb{N}$. It is straightforward to generalize this notion of rapid decay at infinity to higher-order differentials of $g$.

In order to address the local action of $G$ in the context of tempered distribution, we introduce the following group:

$$\mathcal{G} := \left\{ g : \mathbb{R}^4 \to G \mid g \text{ is smooth with the differential of each order rapidly decaying at infinity} \right\}$$

where the group operation is defined point-wise. That is, $(g_1g_2)(x) := g_1(x)g_2(x)$ and $g^{-1}(x) := [g(x)]^{-1}$.

Since $G$ is a compact Lie group, $\mathcal{G}$ is indeed a group under such operations. Moreover, it is not difficult to observe that components of any $g \in \mathcal{G}$ with respect to any representation of $G$ are smooth bounded functions on $\mathbb{R}^4$ whose partial derivatives are all rapidly decaying at infinity.\footnote{Such $\mathcal{G}$ may be too restrictive, but it is the best we can find for the purpose of defining co-located products of Schwinger functions corresponding to the gauge fields. We do not exclude the possibility of a bigger group than $\mathcal{G}$ which still works well in our formulation of the axioms. In this case, there may be a change in the structure of gauge orbits in the state space arising from the reconstruction theorem later.}

Definition 6. We assume that the matter fields are partitioned into multiplets to furnish representations of the gauge group $G$ in a given theory. More specifically, let $\mathcal{R}$ be the labeling of all multiplets in the theory. Then, there exists a representation of $G$ for each $r \in \mathcal{R}$ such that the $r$-multiplet of matter fields are components with respect to a given (ordered) basis in the representation space. We denote the representation space by $V_r$ and the basis by $\{e_{kr}\}$; here the index $k_r$ takes the value from $\{1, 2, \cdots, \dim V_r\}$.

We assume in addition that the adjoint\footnote{Here the term adjoint refers to the dual spinor representation. For example, it denotes (Euclidean) Dirac adjoint in the case of a (Euclidean) Dirac spinor.} of the matter fields in the $r$-multiplet forms a multiplet in the representation of $G$ dual to that of $r$ with respect to the dual basis of $\{e_{kr}\}$. Such multiplet is labeled by $\bar{r}$. For any $g \in \mathcal{G}$, the notation $g_{(r)}$ is used to make it explicit the representation under which the values of $g$ are expressed.

Note that, by definition, $V_{\bar{r}}$ is the dual space of $V_r$ and $\{e_{kr}\}$ is the dual basis for $\{e_{kr}\}$. We assume further that $\{e_{kr}\}$ is ordered in the same way as $\{e_{kr}\}$ so that $\langle e_{kr}, e_{k\bar{r}} \rangle_{V_{\bar{r}} \times V_r} = \delta_{k_{\bar{r}}, k_r}$ and the adjoint of the matter field component corresponding to $e_{kr}$ is the component in the $\bar{r}$-multiplet.
corresponding to \( e_{k_r} \). We further assume the following completeness relation:

\[
\sum_{k_r,k_r'} \delta_{k_r,k_r'} \langle \phi, e_{k_r} \rangle \langle e_{k_r'}, v \rangle = \langle \phi, v \rangle \quad \text{for all } \phi \in V_r \text{ and } v \in V_r
\]

For notational convenience, we may also use the symbols \( V_r^* \) and \( \{ e^*_{k_r} \} \) to denote the dual space \( V_r \) and the dual basis \( \{ e_{k_r} \} \) respectively. Note that \( e^*_{k_r} = \sum_{k_r} \delta_{k_r,k_r'} e_{k_r'} \).

**Definition 7.** For the spinor indices of matter fields, we modify [1, Section 6]. All fields in each \( r \)-multiplet are assumed to be of the same spinor character. As such, the index \( \nu_r \) describes the (Euclidean) spinor character of the fields in the \( r \)-multiplet. By construction, the index \( \nu_r \) corresponds to the representation of \( SO(4) \) dual to that of \( \nu_r \).

**Definition 8.** \( \Psi_{k_r \nu_r} \) denotes the matter field which is the component of the basis element \( e_{k_r} \) in the \( r \)-multiplet, so that the whole multiplet may be expressed as \( \Psi_{v_r} = \sum_{k_r} \Psi_{k_r \nu_r} e_{k_r} \). With the notations from Definition 6 and Definition 7, the adjoint of the field \( \Psi_{k_r \nu_r} \) is \( \Psi_{k_r \nu_r}^* \) and vice versa.

**Definition 9.** For \( U, V \in SU(2) \), we use the same notation \( R(U,V) \) as in [1, p.102] to denote a representation of \( SO(4) \) corresponding to the given spinor index.

**Definition 10.** For any \( m, l \in \mathbb{N} \cup \{0\} \), let \( I, \mathcal{I} \) and \( \mathcal{J} \) denote the following ordered sets of indices:

\[
I := \{(k_r, \nu_r), \ldots, (k_{r_m}, \nu_{r_m}), (\alpha, i_1), \ldots, (\alpha, i_l)\}
\]

\[
\mathcal{I} := \{(\nu_r, \nu_{r'}), \ldots, (\nu_{r_m}, \nu_{r_m'}), (\alpha, i_1, j_1), \ldots, (\alpha, i_l, j_l)\}
\]

\[
\mathcal{J} := \{(\nu_r, \nu_{r'}), \ldots, (\nu_{r_m}, \nu_{r_m'}), (i_1, j_1, i_1', j_1'), \ldots, (i_l, j_l, i_l', j_l')\}
\]

For any permutation \( \sigma \) of \( m + l \) elements, let \( \sigma \cdot I \) be the ordered index set obtained by permuting the elements of Eq. (2) according to \( \sigma \). \( \sigma \cdot I \) and \( \sigma \cdot \mathcal{J} \) are defined similarly.

From now on, summation convention will be assumed between any pair of repeated indices.

**Definition 11.** Adjusting the notations in [4, S4] according to [1, p.103], let

\[
f := \left( f^{m,l}_{\sigma \cdot I} \right)
\]

be a sequence enumerated by \( m, l \in \mathbb{N} \cup \{0\} \) and \( \sigma \cdot I \) as in Definition 10 such that

- \( f^{0,0} \in \mathbb{C} \)
- \( f^{m,l}_{\sigma \cdot I} \in \mathcal{S}(\mathbb{R}^{4(m+l)}) \)
- all but finitely many elements are zero
- For \( (m,l) \neq (0,0) \) the support of \( f^{m,l}_{\sigma \cdot I} \) is contained in \( \{(\mathfrak{x}_1, \cdots, \mathfrak{x}_{m+l}) \mid x^0_1, \cdots, x^0_{m+l} \geq 0\} \)

---

As a concrete example, let us consider (the Euclidean version of) QCD [20, Section 18.7], where quark fields and their Dirac adjoints are the matter fields. In this case, \( \mathfrak{R} = \{u, \bar{c}, \bar{s}, \bar{t}, \bar{d}, \bar{b}, \bar{u}, \bar{c}, \bar{s}, \bar{t}, \bar{d}, \bar{b}\} \) corresponds to the flavors, where \( u \) denotes the \( u \)-quark field while \( \bar{u} \) is its Dirac adjoint, and similarly for other flavors. This is slightly different from the conventional meaning of \( u \) and \( \bar{u} \) as a particle and anti-particle respectively, but makes no essential change. With \( r \) denoting any element of \( \{u, c, t, d, s, b\} \), \( k_r \) takes three values (= colors), furnishing the fundamental representation \( 3 \) of \( SU(3) \). \( k_r \) takes three values as well, furnishing the dual representation \( \mathfrak{F} \).
Also, let
\[
* f := \left( * f^m_l \right)
\]
where
\[
* f^m_l(x_1, \ldots, x_m) := \frac{f^m_l(x_{m+1}, \ldots, x_1)}{I^*}
\]
and \((\sigma \cdot I)^*\) is defined in the same way as \([1\text{, Formula (6.2)}]\). For example,
\[
I^* := \{(\alpha_1, i_1), \ldots, (\alpha_1, i_1), (k_{r_m}, \nu_{r_m}), \ldots, (k_{r_T}, \nu_{r_T})\}.
\]
We may make similar definitions with \(I\) replaced by \(\mathcal{I}\) or \(\mathfrak{I}\). Lastly, the reflection operator \(\Theta\) with respect to the zeroth coordinate is defined as in \([1\text{, p.87]}\).

**Definition 12.** For a tempered distribution \(T \in \mathcal{S}'(\mathbb{R}^{4N})\) and a tensor field \(A\) whose entries are all elements of \(\mathcal{S}'(\mathbb{R}^{4N})\), we understand the notation \(T(A)\) as the entry-wise evaluation. That is, if \(A := (A_{i_1, \ldots, i_k})\) with each entry \(A_{i_1, \ldots, i_k} \in \mathcal{S}'(\mathbb{R}^{4N})\), then \(T(A)\) is a tensor with complex-valued entries defined as
\[
T(A) := (T(A_{i_1, \ldots, i_k}))
\]

**Definition 13.** We denote by \(\{\Delta_n\}_{n \in \mathbb{N}}\) a sequence of elements in \(\mathcal{S}'(\mathbb{R}^{4 \times 3})\) that converges to the restriction onto the thin diagonal in the weak* limit of tempered distributions. That is, for any \(F \in \mathcal{S}'(\mathbb{R}^{4 \times 3})\), we have
\[
\lim_{n \to \infty} \int_{(\mathbb{R}^4)^3} \Delta_n(x, y, z) F(x, y, z) dx dy dz = \int_{\mathbb{R}^4} F(x, x, x) dx.
\]

For \(f \in \mathcal{S}(\mathbb{R}^4)\), we use the notation \(\Delta_n(f)\) to denote an element of \(\mathcal{S}(\mathbb{R}^{4 \times 2})\) defined by
\[
\Delta_n(f) := \int_{\mathbb{R}^4} \Delta_n(\cdot, \cdot, \cdot, z) f(z) dz.
\]
We add the gauge representation indices. More specifically, let \(\{\Delta_{n, k_r, k_{r'}}\} \subset \mathcal{S}'(\mathbb{R}^{4 \times 3})\) be such that
\[
\lim_{n \to \infty} \int_{(\mathbb{R}^4)^3} \Delta_{n, k_r, k_{r'}}(x, y, z) F(x, y, z) dx dy dz = \delta_{k_r, k_{r'}} \int_{\mathbb{R}^4} F(x, x, x) dx.
\]
for any \(F \in \mathcal{S}(\mathbb{R}^{4 \times 3})\). \(\Delta_{n, k_r, k_{r'}}(f)\) is defined in the same ways as \(\Delta_n(f)\).

**3. Motive Heuristics**

With the notations presented in the previous section, we first provide a heuristic outline of the main ideas, which will be stated with full mathematical rigor in the next section.

In \([1, 2, 4]\), Schwinger functions are vacuum expectation values of field operators for the given theory. However, a crucial feature of gauge theories is the presence of the local action by a gauge group \(G\), resulting in gauge redundancy. That is, not all state vectors have physical meaning.

**Note** that \([1, 2]\) define Schwinger functions only at non-coinciding arguments while \([4]\) include coinciding arguments as well. Refer to \([28\text{, p.371]}\) for a more detailed comparison. In this paper, we follow the approach of \([4]\) since composite operators at coinciding arguments must be addressed for gauge invariance.
Hence, the notion of vacuum expectation values must be generalized. More specifically, suppose that we are given the following:

- A vector space $H$ of all possible states equipped with a conjugate-linear form $\langle \cdot, \cdot \rangle$ and containing a state vector $\Omega$
- A collection of matter fields $\Psi_{k_i \nu_i}$ and gauge fields $A_{\alpha i}$ which are tempered distributions whose values are operators acting on $H$

Note that $H$ is not necessarily a (pre-)Hilbert space since $\langle \cdot, \cdot \rangle$ is not assumed to be positive (semi-)definite. However, $\Omega$ will be interpreted later as a physical vacuum when restricted to a subspace of $H$, as explained in the Reconstruction Theorems appearing in the subsequent paper.

Even without structure of a Hilbert space, we may still define a Schwinger function by the formula

$$S_I(\xi_1, \ldots, \xi_{m+l}) := \langle \Omega, \Psi_{k_1 \nu_1}(\xi_1) \cdots \Psi_{k_m \nu_m}(\xi_m) A_{\alpha_1 i_1}(\xi_{m+1}) \cdots A_{\alpha_l i_l}(\xi_{m+l}) \Omega \rangle \quad (5)$$

or any permutation of matter fields and gauge fields in the second line with corresponding permutation of $I$ in the first line of Eq. (5); we denote such Schwinger functions by $S_{\sigma, I}$ where $\sigma$ is a permutation of $m + l$ elements. By the condition that field operators $\Psi_{k_i \nu_i}$ and $A_{\alpha i}$ are operator-valued tempered distributions, $S_{\sigma, I}$ is an element of $\mathcal{S}'(\mathbb{R}^{4(m+l)})$ for any $\sigma$. These Schwinger functions are required to satisfy the original axioms in [1, 2] except for reflection positivity.

Let us now consider the local gauge action more in detail. Following [33, Formulas (1.61) and (1.72)] with the (physical) coupling constant absorbed into field operators, such an action of $G$ on the field operators is given by the formulas:

$$\begin{cases} (g \cdot \Psi)_{k_i \nu_i}(f) := \Psi_{k'_i \nu'_i}(f(\cdot)e_{k_i}^* \circ g(\nu_i)(\cdot), e_{k'_i})_{V^*_i \times V_i} \\ t_{\alpha}(g \cdot A)_{\alpha j}(f) := A_{\alpha j}(fgt_{\alpha}g^{-1}) - \int_{\mathbb{R}^4} (\langle \partial_j g^{-1}\rangle f(x)dx \end{cases} \quad (6)$$

These field operators must be interacting. It is well-known that (perturbative) renormalizability of a gauge theory depends on gauge fixing. Here we just assume that renormalized (or interacting) field operators under a certain gauge fixing are given, and present the axioms they should satisfy. At least, all field operators in an equivalence class under the gauge action yield the same gauge-invariant co-located products as in Eq. (17) by construction. A caveat here is that local gauge symmetry may not be transitive at the level of interacting field operators like $\Psi_{k_i \nu_i}$ and $A_{\alpha i}$. In that case, there may exist multiple different spaces of the state vectors and collections of gauge-invariant co-located products corresponding to each gauge equivalence class (= orbit). We conjecture that this non-uniqueness issue is somehow related to existence of multiple superselection sectors [29, p.108] as well as spontaneous symmetry breaking [26, Chapter 19]. Actual construction of field operators and detailed analysis of such properties are the major outstanding issues for the future.

Elitzur’s theorem [30, 31] states that the expectation value of gauge-noninvariant field operators with respect to a gauge-invariant functional measure vanishes identically. This result is originally stated for lattice but clearly holds in continuum limit as well, provided that the limit exists. Such a restriction is bypassed by introducing a gauge-fixing functional into the path integral, as originally proposed by Faddeev and Popov [32] in the heuristic level on continuum and rigorously verified on lattice in [31, 24]. As shown in Section 3, the conjugate-linear form $\langle \cdot, \cdot \rangle$ together with the state space $H$ correspond to path integral for a given gauge theory via the moment problem. Therefore, we are led to the conclusion that they must be constructed under a specific gauge fixing condition in order to avoid Elitzur’s theorem.
where $f \in \mathcal{S}(\mathbb{R}^4)$ and $g \in \mathcal{G}$. In the second formula of Eq. (6), $g$ and $t_\alpha$ are assumed to be in the adjoint representation.

Eq. (5) naturally leads to the local action of $G$ on Schwinger functions as follows:

$$[g \cdot S]_f(x_1, \ldots, x_{m+l}) := \left\langle \Omega, [g \cdot \Psi]_{k_r \nu_r} (x_1) \cdots [g \cdot \Psi]_{k_m \nu_m} (x_m) [g \cdot A]_{\alpha_1,i_1} (x_{m+1}) \cdots [g \cdot A]_{\alpha_l,i_l} (x_{m+l}) \Omega \right\rangle$$

or similarly for any permutation of indices. Using Eq. (6), we may express Eq. (7) without explicit resort to field operators, which will be the formal definition of local gauge action for Schwinger functions in the next section. Here, we give simple examples for the matter fields and gauge fields respectively:

- In the presence of a single matter field and its adjoint, only

$$[g \cdot S]_{s(k_r, \nu_r), (\kappa_r \nu_r)} (f \otimes h) = S_{s(k_r', \nu_r'), (\kappa_r' \nu_r')} \left( f(\cdot) \langle e_{k_r'}^* o g(\cdot), e_{k_r} \rangle_{V_r' \times V_r}, h(\cdot) \langle e_{\kappa_r'}^* o g(\cdot), e_{\kappa_r} \rangle_{V_r' \times V_r} \right)$$

where $f, h \in \mathcal{S}(\mathbb{R}^4)$ and the natural identifications $V_r' = V_r$ and $e_{k_r'} = \delta_{k_r'k_r} e_{k_r}$ are assumed.

- In the presence of two gauge fields only,

$$(t_\alpha \otimes t_\alpha') [g \cdot S]_{(\alpha, i), (\alpha', i')} (f \otimes h) = S_{(\alpha, i), (\alpha', i')} \left( f g t_\alpha g^{-1} \otimes h g t_\alpha' g^{-1} \right) - S_{\alpha \alpha'} \left( f g t_\alpha g^{-1} \right) \otimes \int_{\mathbb{R}^4} h(\partial_i g) g^{-1}$$

$$- S_{\alpha' \alpha'} \left( f g t_\alpha g^{-1} \right) \otimes \int_{\mathbb{R}^4} f(\partial_i g) g^{-1} + \left( \int_{\mathbb{R}^4} f(\partial_i g) g^{-1} \right) \otimes \left( \int_{\mathbb{R}^4} h(\partial_i g) g^{-1} \right)$$

The fundamental principle of gauge theories is that all physical observables must be gauge-invariant. According to [11, 2], reflection positivity is the key properties of Schwinger functions that lead to positivity of Wightman functions, which in turn makes it possible to reconstruct the underlying Hilbert space, cf. [37, Theorem 3.7]. Therefore, reflection positivity may be regarded as a physical requirement and must be imposed on gauge-invariant quantities only.

For this purpose, we should construct gauge-invariant Schwinger functions starting from Eq. (5). Among local field operators, the simplest gauge-invariant ones for the matter fields is of the following form:

$$\left| \Psi \right|^2_{\nu_r \nu_r'} (x) := \delta_{k_r k_r'} \langle \Psi_{k_r \nu_r}, \Psi_{k_r' \nu_r'} \rangle (x)$$

provided that the co-located product is well-defined.

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@footnote{We retain locality for Schwinger functions themselves. However, non-local quantities such as the Wilson loop will be derived from such Schwinger functions in a later section.}
For the gauge fields $A_{\alpha i}$, situation is more involved. Following [33, Formula (1.78)] or [26, Formula (15.1.13)], the field strength tensor is defined as:

$$t_\alpha F_{\alpha ij}(y) := t_\alpha \left( \partial_i A_{\alpha j}(y) - \partial_j A_{\alpha i}(y) \right) + [t_\beta, t_\gamma] \left( A_{\beta i} A_{\gamma j} \right)(y)$$

(11)

which leads to the gauge-invariant local operator

$$F_{ij'j''}(y) := \left( F_{\alpha ij} F_{\alpha i'j''} \right)(y)$$

(12)

Again, we must assume that the co-located products in Eq. (11) and Eq. (12) are well-defined.

Using Eq. (10) and Eq. (11), we may introduce the following intermediate form of Schwinger functions:

$$\mathcal{S}_I(x_1, \ldots, x_{m+l}) := \Omega, \left| \Psi_{\nu_1, \nu'_1}^2(x_1) \cdots \Psi_{\nu_m, \nu'_m}^2(x_m) F_{\alpha_1 i_1 j_1}(x_{m+1}) \cdots F_{\alpha_l i_l j_l}(x_{m+l}) \Omega \right>$$

(13)

or similarly for any permutation of $I$. Eq. (13) is the non-Abelian version of the formula appearing in [7, p.383 Corollary 4.6].

However, Eq. (13) is still not fully gauge-invariant due to the non-Abelian structure. Rather, one may heuristically compute that

$$\bigotimes_{k=1}^l t_{\alpha_k} \left[ g \cdot \mathcal{S}_I \right] (f_1 \otimes \cdots \otimes f_{m+l})$$

(14)

$$= \left< \Omega, \left| \Psi_{\nu_1, \nu'_1}^2(f_1) \cdots \Psi_{\nu_m, \nu'_m}^2(f_m) F_{\alpha_1 i_1 j_1}(f_{m+1} t_{\alpha_1} g^{-1}) \cdots F_{\alpha_l i_l j_l}(f_{m+l} t_{\alpha_l} g^{-1}) \Omega \right>$$

where $f_1, \ldots, f_{m+l} \in \mathcal{F}(\mathbb{R}^4)$ and the tensor product of matrices with operator-valued entries, where the usual multiplication of scalars is replaced by composition of functions for the entries, is assumed for $F$’s. In fact, Eq. (14) is equivalent to the transformation rule

$$t_\alpha [g \cdot F]_{\alpha ij} = g t_\alpha g^{-1} F_{\alpha ij}$$

for the field strength tensor $F$, which is well-established at the classical level. Therefore, with the transformation rule for a two-fold tensor product of $F_{\alpha ij}$ is given by

$$\left( t_\alpha \otimes t_{\alpha'} \right) [g \cdot F]_{\alpha ij}(\vec{x}) [g \cdot F]_{\alpha' i'j'}(\vec{x}') = \left( F_{\alpha ij}(\vec{x}) g(\vec{x}) t_\alpha g^{-1}(\vec{x}) \right) \otimes \left( F_{\alpha' i'j'}(\vec{x}') g(\vec{x}') t_{\alpha'} g^{-1}(\vec{x}') \right)$$

(15)

Now, let us assume again that the co-located products

$$\left( F_{\alpha ij} F_{\alpha' i'j'} \right)(\vec{x})$$

The sum in the formula of Schwinger functions appearing the cited paper must be replaced by a product, which is a typo confirmed via email correspondence with the authors.
exist. Then, we formally apply the mapping \( \text{Tr} \) in Definition 4 to Eq. (15) to compute that

\[
\text{Tr} \left( [g \cdot F]_{\alpha ij} t_\alpha \otimes [g \cdot F]_{\alpha' i'j'} t_{\alpha'} \right)(x) \\
= (F_{\alpha ij} F_{\alpha' i'j'}) (x) \text{Tr} \left( g(x) t_\alpha g^{-1}(x) g(x) t_{\alpha'} g^{-1}(x) \right) = F^2_{i'j'}(x) \\
= \text{Tr} \left( F_{\alpha ij} t_\alpha \otimes F_{\alpha' i'j'} t_{\alpha'} \right)(x)
\]

which may be understood as gauge invariance of Eq. (12). Motivated by such formulas, let us consider the following Schwinger functions

\[
\mathcal{G}_\gamma (\bar{x}_1, \ldots, \bar{x}_{m+l}) := \left\langle \Omega, |\Psi|^{2}_{\nu_1 i_2 r} (\bar{x}_1) \cdots |\Psi|^{2}_{\nu_m i_{m+l}} (\bar{x}_m) F^2_{i_1j_1} (\bar{x}_{m+1}) \cdots F^2_{i_{m+l}j_{m+l}} (\bar{x}_{m+l}) \Omega \right\rangle
\]

or any permutation of the \( m + l \) index tuples, which we check heuristically to be gauge-invariant. Now, the aforementioned reflection positivity can be imposed on Schwinger functions of the form Eq. (17). This may be the continuum version of the previous works on lattice approximation such as [24, p.448 Theorem 2.1]. With analytic continuation to the Minkowski spacetime as in [1, 2], such positivity of Eq. (17) together with some modification of [34, Theorem 3.7] implies that we may (re)construct an actual Hilbert space completion of a subspace of \( H \) containing \( \Omega \). The Hilbert space may be regarded as a space of physical states, which justifies the previous interpretation of \( \Omega \) as a physical vacuum. Details of such reconstructions will be presented in the next paper.

The remaining crucial issue is how one can actually construct co-located products such as Eq. (10), Eq. (11) and Eq. (12) at least at the level of vacuum expectation values. Motivated by the operator product expansion [26, Chapter 20] and rigorous construction of Wick products [35], we assume existence of certain counter-terms that cancel out ultraviolet singularity of Schwinger functions at coinciding points in order for the limits of point-splitting regularization to exist as in [35, pp.646–674]. As a specific example of such construction for the matter fields, consider

\[
\mathcal{G}_{\nu_r \nu_r} (x) = \left\langle \Omega, |\Psi|^{2}_{\nu_r \nu_r} (x) \Omega \right\rangle
\]

which is the vacuum expectation value of Eq. (10).

Let us start with Schwinger functions of the form

\[
S_{(k_r, \nu_r), (k_{r'}, \nu_{r'})} (\bar{x}, \bar{x}') = \left\langle \Omega, \Psi_{k_r \nu_r} (\bar{x}) \Psi_{k_{r'} \nu_{r'}} (\bar{x}') \Omega \right\rangle
\]

and assume existence of counterterms

\[
C_{(k_r, \nu_r), (k_{r'}, \nu_{r'})} (\bar{x}, \bar{x}')
\]

***If Eq. (10) were just the Wick product of (Euclidean) free fields, then Eq. (18) would be identically zero. However, since interacting fields on \( \mathbb{R}^4 \) are assumed here, we believe that it is unlikely for Eq. (15) to just vanish identically. Again, a detailed analysis of such properties comes with actual construction of a theory, and we only consider Eq. (18) as an illustrative example of the general cases that will be presented in the next section. The same rationale applies to Eq. (22) and Eq. (23).
which are tempered distributions on $(\mathbb{R}^4)^2$ subject to the same local gauge action as Eq. (8) and the following renormalization properties

- The limit
  \[
  \lim_{n \to \infty} \left( S_{(k_r, r)}(\xi^i, \xi^j) - C_{(k_r, r)}(\xi^i, \xi^j) \right) \left( \Delta_n, k_n \right) (f) \tag{20}
  \]
  exists for all $f \in \mathcal{S}(\mathbb{R}^4)$ and any choice of $\{ \Delta_n, k_n \}$.

- Eq. (20) is independent of choice of regularizers in the sense that for each $f$, any choice of $\{ \Delta_n, k_n \}$ gives the same limit.

Note that for any values of the spinor indices, the mapping
\[
\lim_{n \to \infty} \left( S_{(k_r, r)}(\xi^i, \xi^j) - C_{(k_r, r)}(\xi^i, \xi^j) \right) \left( \Delta_n, k_n \right) (f)
\]
is a tempered distribution on $\mathbb{R}^4$ for each $n$. Hence, the weak$^*$ limit Eq. (20) defines a tempered distribution on $\mathbb{R}^4$ [36, Theorem 2.7].

By construction, it is straightforward to check that the limit Eq. (20) is invariant under the gauge action. That is,
\[
\lim_{n \to \infty} \left( g \cdot S_{(k_r, r)}(\xi^i, \xi^j) - g \cdot C_{(k_r, r)}(\xi^i, \xi^j) \right) \left( \Delta_n, k_n \right) (f)
\]
eq \lim_{n \to \infty} \left( S_{(k_r, r)}(\xi^i, \xi^j) - C_{(k_r, r)}(\xi^i, \xi^j) \right) \left( \Delta_n, k_n \right) (f)
\]
for every $f \in \mathcal{S}(\mathbb{R}^4)$, $g \in \mathcal{G}$ and any choice of $\{ \Delta_n, k_n \}$. Comparing Eq. (10) with the Formula (16) or p.383 Corollary 4.6, we may take Eq. (20) as the definition of Eq. (18).

Co-located products for gauge fields are more involved, as we have to go through two steps. For simplicity, we will only present the case of
\[
S_{(\alpha, i, j), (\alpha', i', j')}(\xi, \xi') = \left< \Omega, F_{\alpha ij}(\xi) F_{\alpha' i' j'}(\xi) \Omega \right> \tag{22}
\]
and
\[
\mathcal{G}_{ij ij'}(\xi) = \left< \Omega, F_{ij ij'}^2(\xi) \Omega \right> \tag{23}
\]
in this section. With Eq. (11) in mind, we start with the following formula
\[
t_{\alpha} \otimes t_{\alpha'} \left( \partial_{\alpha'} S_{(\alpha, j), (\alpha', i')} - \partial_{\alpha'} S_{(\alpha, i), (\alpha', j')} - \partial_{\alpha'} S_{(\alpha, i), (\alpha', j')} + \partial_{\alpha} S_{(\alpha, i), (\alpha', j')} \right) (f \otimes h) + t_{\alpha} \otimes [t_{\beta}, t_{\gamma}] \left( \partial_{\beta} S_{(\alpha, i), (\beta, i'), (\gamma, j')} - \partial_{\beta} S_{(\alpha, i), (\beta, i'), (\gamma, j')} \right) (f \otimes \Delta_n'(h))
\]
\[
+ [t_{\beta}, t_{\gamma}] \otimes t_{\alpha} \left( \partial_{\gamma} S_{(\alpha, i), (\beta, j'), (\gamma', j')} - \partial_{\gamma} S_{(\alpha, i), (\beta, j'), (\gamma', j')} \right) (\Delta_n(h) \otimes f) + [t_{\beta}, t_{\gamma}] \otimes [t_{\beta}, t_{\gamma}] \left( \Delta_n(h) \otimes \Delta_n'(h) \right)
\]
for $f, h \in \mathcal{S}(\mathbb{R}^4)$ and any choices of $\Delta_n$ and $\Delta_n'$. Now, assume existence of the following counterterms
\[
C_{(\beta, i), (\gamma, j)} \in \mathcal{S}'(\mathbb{R}^8) \text{ and } C_{(\beta, i), (\gamma, j), (\beta', i'), (\gamma', j')} \in \mathcal{S}'(\mathbb{R}^16)
\]
satisfying the following properties
• Symmetry under the commutation of index pairs in analogy to \([E3]\) in p.103. In fact, we assume the same property for Schwinger functions appearing in Eq. (24) as only vector indices appear here, which are bosonic.

• The limits

\[
\begin{align*}
\text{dd} \\
\text{dd} \\
\end{align*}
\]

and

\[
\begin{align*}
\text{dd} \\
\end{align*}
\]

• The local gauge action is defined by

4. **The Axioms for Schwinger functions of a Gauge theory**

In this section, we formally state the OS axioms adjusted to encompass non-Abelian gauge symmetry, whose motivations and simple examples are presented in the previous section:

**Definition 14.** The following five collections of tempered distributions

\[
\begin{align*}
\bigcup_{m,l \in \mathbb{N}\cup\{0\}} \left\{ S_{\sigma, l} \left( x_{\pi(1)}, \ldots, x_{\pi(m+l)} \right) \mid S_I \in \mathcal{S}'(\mathbb{R}^{4(m+l)}) \text{ and } \pi \text{ is any permutation of } m+l \text{ elements} \right\} \\
\quad (S1)
\end{align*}
\]

\[
\begin{align*}
\bigcup_{m,l \in \mathbb{N}\cup\{0\}} \left\{ S_{\sigma, l} \left( x_{\pi(1)}, \ldots, x_{\pi(m+l)} \right) \mid S_I \in \mathcal{S}'(\mathbb{R}^{4(m+l)}) \text{ and } \pi \text{ is any permutation of } m+l \text{ elements} \right\} \\
\quad (S2)
\end{align*}
\]

\[
\begin{align*}
\bigcup_{m,l \in \mathbb{N}\cup\{0\}} \left\{ \mathcal{G}_{\pi, l} \left( x_{\pi(1)}, \ldots, x_{\pi(m+l)} \right) \mid \mathcal{G}_I \in \mathcal{S}'(\mathbb{R}^{4(m+l)}) \text{ and } \pi \text{ is any permutation of } m+l \text{ elements} \right\} \\
\quad (S3)
\end{align*}
\]

with \(S_0 = \mathcal{G}_0 = 1\) are called the Schwinger functions for a quantum Yang-Mills theory with the gauge group \(G\) under a gauge fixing and the following two collections of tempered distributions

\[
\begin{align*}
\bigcup_{m,l \in \mathbb{N}\cup\{0\}} \left\{ C_{\sigma, l} \left( x_{\pi(1)}, \ldots, x_{\pi(m+l)} \right) \mid C_I \in \mathcal{S}'(\mathbb{R}^{4(m+l)}) \text{ and } \pi \text{ is any permutation of } m+l \text{ elements} \right\} \\
\quad (C1)
\end{align*}
\]

\[
\begin{align*}
\bigcup_{m,l \in \mathbb{N}\cup\{0\}} \left\{ C_{\sigma, l} \left( x_{\pi(1)}, \ldots, x_{\pi(m+l)} \right) \mid C_I \in \mathcal{S}'(\mathbb{R}^{4(m+l)}) \text{ and } \pi \text{ is any permutation of } m+l \text{ elements} \right\} \\
\quad (C2)
\end{align*}
\]

are called the corresponding counterterms for above Schwinger functions if they satisfy the following axioms:

- For all of Eq. (S1), (S2), (S3), (C1) and (C2),

  \(\rightarrow\) **Euclidean Covariance**

  Eq. (S1) satisfies the following transformation law:

\[
S_{\sigma, l} \left( x_{\pi(1)}, \ldots, x_{\pi(m+l)} \right) = R(U, V)_{\pi(1)}^{\nu_1} \cdots R(U, V)_{\pi(m)}^{\nu_m} R(U, V)_{\pi(1)}^{\mu_1} \cdots R(U, V)_{\pi(m+l)}^{\mu_l} S_{\sigma, l'} \left( R x_{\pi(1)} + a, \ldots, R x_{\pi(m+l)} + a \right)
\]

\(\text{(EC)}\)
where $R = R(U, V)$, $a \in \mathbb{R}^4$ and $I' := \{(k_1, \nu_1'), \cdots , (k_m, \nu_m'), (\alpha_1, \iota_1'), \cdots , (\alpha_l, \iota_l')\}$. Eq. (S2), (S3), (C1) and (C2) satisfy transformation properties according to their own index sets in a way analogous to Eq. (E3).

\[ S_{\sigma' I'}(\bar{x}_{\pi(1)}, \cdots \bar{x}_{\pi(m+l)}) = \pm S_I(\bar{x}_1, \cdots \bar{x}_{m+l}) \quad \text{(AC)} \]

and

\[ C_{\sigma' I'}(\bar{x}_{\pi(1)}, \cdots \bar{x}_{\pi(m+l)}) = \pm C_I(\bar{x}_1, \cdots \bar{x}_{m+l}) \quad \text{(AC')} \]

where $+$ is for the cases in which $\sigma$ carries out an even number of transpositions of fermionic indices, and $-$ is for an odd number of such transpositions. Note that Eq. (AC) and (AC') correspond to [11] Formula (E3). On the other hand, Eq. (S2), (S3) and (C2) are invariant under permutation of $m + l$ elements in the sense that

\[ S_{\sigma I'}(\bar{x}_{\pi(1)}, \cdots \bar{x}_{\pi(m+l)}) = S_I(\bar{x}_1, \cdots \bar{x}_{m+l}) \]

\[ C_{\sigma I'}(\bar{x}_{\pi(1)}, \cdots \bar{x}_{\pi(m+l)}) = C_I(\bar{x}_1, \cdots \bar{x}_{m+l}) \]

and

\[ \mathcal{S}_{\sigma I'}(\bar{x}_{\pi(1)}, \cdots \bar{x}_{\pi(m+l)}) = \mathcal{S}_I(\bar{x}_1, \cdots \bar{x}_{m+l}). \]

However, they satisfy the (anti)commutation property for permutation of the spinor indices within each tuple, which corresponds to a single argument in the tempered distribution. That is,

(a) In both $\mathcal{I}$ and $\mathcal{J}$, transposition of $\nu_{\alpha}$ and $\nu'_{\alpha}$ yields a minus sign if they are fermionic, for any $\alpha \in \{1, \cdots , m\}$. Transposition of $i_b$ and $j_b$ in $(\alpha_b, i_b, j_b)$ for any $b \in \{1, \cdots , l\}$ yields a minus sign.

(b) For each $(i_b, j_b, i'_b, j'_b)$ in $\mathcal{J}$, transposition of $i_b \& j_b$ or $i'_b \& j'_b$ yields a minus sign.

However, transposition of the two pairs does not cause any change in the sign.

The two rules apply the same way for any $\sigma \cdot \mathcal{I}$ or $\sigma \cdot \mathcal{J}$.

\[ \text{For the Schwinger functions Eq. (S1), (S2) and (S3)} \]

\[ \rightarrow \text{Cluster Decomposition} \]

Let $f := (f_{\sigma' I'}^{m,l})$ and $g := (g_{\sigma' I'}^{m',l'})$ be as in Definition [11]. Then, for any 4-vector $a := (0, \bar{a})$ with nonzero $\bar{a} \in \mathbb{R}^3$

\[ \lim_{\lambda \rightarrow \infty} \sum_{m,l,m',l'} \left\{ S_{\sigma I', \sigma' I'} \left( \Theta [f_{\sigma' I'}^{m,l}] \otimes g_{\sigma' I'}^{m',l'} (\lambda a) \right) - S_{\sigma' I'} \left( \Theta [f_{\sigma'I'}^{m,l}] \right) S_{\sigma' I'} \left( g_{\sigma' I'}^{m',l'} \right) \right\} = 0 \quad \text{(CD)} \]

where $g_{\sigma' I', (\lambda a)}$ is the translation of all arguments of $g_{\sigma' I'}^{m',l'}$ by $\lambda a$ as defined in [1] p.87.

Eq. (S2) and (S3) satisfy decomposition properties analogous to Eq. (CD) with respect to their own index sets.

\[ \rightarrow \text{Growth Bounds} \]
There exist constants $A, B > 0$ and some fixed $k \in \mathbb{N} \cup \{0\}$ such that

$$\max\left\{ |S(f_1 \otimes \cdots \otimes f_{m+l})|, |\mathcal{S}(f_1 \otimes \cdots \otimes f_{m+l})|, |\mathcal{S}_2(f_1 \otimes \cdots \otimes f_{m+l})| \right\} \leq A^{m+l}(m+l)! \prod_{s=1}^{m+l} |f_s|^k$$

for all $m, l \in \mathbb{N} \cup \{0\}$, $f_1, \cdots, f_{m+l} \in \mathcal{H} (\mathbb{R}^4)$ and any choice of the index sets.

- For Eq. (S2) with respect to Eq. (S1) and (C1)
  \[\rightarrow \text{Renormalized Co-location}\]
  \[1 + 1 = 2\] \hspace{1cm} (S2-1)

- For Eq. (S3) with respect to Eq. (S2) and (C2)
  \[\rightarrow \text{Renormalized Co-location}\]
  \[1 + 1 = 2\] \hspace{1cm} (S3-1)

- For Eq. (S3)
  \[\rightarrow \text{Reflection Positivity}\]
  \[1 + 1 = 2\] \hspace{1cm} (S3-1)

It is not difficult to check that Euclidean covariance and (anti)commutation of Eq. (S2) and Eq. (S3) can be deduced from However, it is not obvious that cluster decomposition for Eq. (S2) and Eq. (S3) follows from above construction, so we impose the property separately.

Such Schwinger functions and counterterms are designed to admit a local gauge structure, defined as follows:

**Definition 15.** The local action of the gauge group $G$ on Schwinger functions in Eq. (S1) is defined on

It is not difficult to check via density argument that this action is indeed a group action:

**Proposition 1.**

**Proof.** dd

With respect to the local gauge symmetry given by Definition 15 we may establish gauge invariance of Eq. (S3) in the following sense, which justifies the requirement that Eq. (S3) must satisfy reflection positivity:

**Proposition 2.**

**Proof.** dd
5. **Sanity Check-1: 2D Pure Yang-Mills Theories**

The main reason we only considered $\mathbb{R}^4$ so far is that failure of the spin-statistics theorem in lower dimensions makes the (anti)commutation axiom complicated.

However, for a pure Yang-Mills theory in 2 Euclidean dimension, we may still check if the above axioms scheme works properly. In fact, complete axial gauge makes the issue of gauge fixing quite simple. Moreover, the field strength tensor $F$ is a Gaussian field on the Lie group $G$, which implies that their co-located products may still be defined as the (Euclidean) Wick product.

6. **Sanity Check-2: 3D Yang-Mills Higgs Model**

Stochastic quantization (SQ) aims to construct the functional measure for a theory directly. My axiom scheme is related to SQ via the *moment problem*. That is, the Schwinger functions are moments of the functional measure in question, as presented in [37, 38].

In this section, we reconstruct the 3D Yang-Mills-Higgs theory described in [21], starting from Schwinger functions satisfying the axioms of Sec. 4 as well as suitable dynamics for the theory. A caveat in the moment problem is that a collection of moments does NOT necessarily lead to a unique functional measure, which is pointed out in [37]. We need additional properties, such as ergodicity, to ensure uniqueness [38].

7. **Rigorous Derivation of Chiral Anomaly**

8. **Relation to non-local objects**

In [39, Ch.8], it is pointed out that non-local objects, such as Wilson loops, provide a more natural framework for gauge theories as they are better suited for describing behaviors of particles in such theories.

In fact, the list of Euclidean axioms for Schwinger functions defined in terms of Wilson loops is presented in [39, p.164]. We aim to show that such “non-local” Schwinger functions can be constructed from “local” ones satisfying the axioms stated in Sec. 4.

9. **Comparison with existing axiom schemes**

10. **Conclusion**

The next natural step is to proceed toward Wightman functions on the Minkowski spacetime via analytic continuation in analogy to [1, 2]. The Wightman Reconstruction Theorem [34, p.117 Theorem 3.7] will be modified for gauge theories as well. Lastly, we aim to establish connection between our extended Wightman axioms and the existing AQFT formalism for gauge theories.

We are deeply grateful for Professors Arthur Jaffe, Jürg Fröhlich, Erhard Seiler, Klaus Fredenhagen, Martin Hairer, Abdelmalek Abdesselam, and Iosif Pinelis for their valuable helps and insights.
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