SAWTOOTH MODELS AND ASYMPTOTIC INDEPENDENCE IN LARGE COMPOSITIONS

PIERRE TARRAGO

Abstract. In this paper we improve the probabilistic approach of Ehrenborg, Levin and Readdy in [ELR02] by introducing a simpler but more general probabilistic model. As consequence we get some new estimates on the behavior of a uniform random permutation $\sigma$ having a fixed descent set. In particular we find a positive answer to the Conjecture 4 of [BHR03] and we show that independently of the shape of the descent set, $\sigma(i)$ and $\sigma(j)$ are almost independent when $i - j$ becomes large.

1. Introduction

A descent of a permutation $\sigma$ of $n \in \mathbb{N}^*$ is an integer $i$ such that $\sigma(i) > \sigma(i + 1)$. For each permutation $\sigma$, the corresponding descent set $D(\sigma)$ is the set of all the descents of $\sigma$. Since descents can be located everywhere except on $n$, a descent set is just a subset of $\{1, \ldots, n - 1\}$, and for the moment we call a composition of $n$ the data of $n$ and a subset of $\{1, \ldots, n - 1\}$. We can pictorially reformulate this by drawing a composition $D$ as a skew Young diagram $\lambda_D$ of $n$ cells $1, \ldots, n$ with the following rule: cells $i$ and $i + 1$ are neighbors and the cell $i + 1$ is right to $i$ if $i \notin D$, below $i$ otherwise. Therefore the descent set of a permutation $\sigma$ is $D$ if and only if inserting $\sigma(i)$ in each cell $i$ of $\lambda_D$ results in a standard skew-Young tableau. For example the composition $D = \{10, (3, 5, 9)\}$ matches the following skew Young diagram:

![Skew Young diagram](image)

**Figure 1.** Skew Young diagram $\lambda_D$ associated to the composition $D = \{10, (3, 5, 9)\}$

And the permutation $\sigma = (3, 5, 8, 4, 7, 1, 6, 9, 10, 2)$ has the descent set $D$ since the associated filling of $\lambda_D$ results in a skew Young tableau as shown in figure 2.

Conversely for each composition $D$ of $n$, the problem is to count how many permutations of $[1, n]$ have exactly $D$ as descent set; it is equivalent to count the number of standard fillings of the associated skew Young tableau $\lambda_D$.

This latter number,
\( \beta(D) \), is called the descent statistic of \( D \) and has been intensively studied in the last decades (see Viennot [Vie79] and [Vie81], Niven [Niv68], de Bruijn [DB70], ...): the main questions were on one hand to find the compositions of \( n \) having a maximum descent statistic, and on the other hand to find exact or asymptotic formulae for descent statistic of compositions of given shape and large size. For example, Niven and de Bruijn proved in [Niv68] and [DB70] that the two compositions of \( n \) maximizing the descent statistic are \( D_1(n) = \{1,3,5,\ldots\} \cap [1,n] \) and \( D_2(n) = \{2,4,6,\ldots\} \cap [1,n] \), whose associated permutations are called alternating permutations; Désiré Andrée gave long before them in [And81] an asymptotic formula for the number of alternating permutations, showing that \( \beta(D_1(n)) \sim 2(2/\pi)^{n/2} n! \) as \( n \) goes to infinity.

To be able to evaluate the descent statistic of a broad class of compositions, Ehrenborg, Levin and Readdy formalized in [ELR02] a probabilistic approach to the counting problem, by relating each permutation of \([1,n]\) with a particular simplex of \([0,1]^n\). Since the cube \([0,1]^n\) with the Lebesgue measure can be seen as a probability space, it is possible to use probabilistic tools to get interesting results on descent statistics: Ehrenborg obtained in [Ehr02] asymptotic descent statistics for the so-called nearly periodic permutations, which consists essentially in permutations having the same descent pattern repeated several times and with some local perturbations. Once again the asymptotic formula has the shape \( K \lambda^n n! \), with \( K \) and \( \lambda \) some constants depending on the situation. Using this approach together with functional analysis tools, Bender, Helton and Richmond extended in [BHR03] the latter result to a broader class of descent sets, and found asymptotic formulae of the same shape as before. The factorial term of the asymptotic formula is easy to get, since it comes from the cardinality of the set \( \mathfrak{S}_n \) of permutations of \( n \) elements. However the power term is harder to understand. The main point of the article [BHR03] is that the authors identified in this class of descent sets the phenomenon that makes the power term \( \lambda^n \) appear: namely if we consider a large uniformly random permutation with a fixed descent set, the value of \( \sigma(1) \) and \( \sigma(n) \) are nearly independent, which causes a factorization in the asymptotic counting. The natural question is thus to know which compositions induce this phenomenon, and it was conjectured in [BHR03] that every composition have this property as they become large.

In the present article we construct a family of particular statistic models, called sawtooth models, that greatly simplifies the probabilistic approach of Ehrenborg, Readdy and Levin. These models are more general than the ones we need in the combinatoric of descent sets, but the properties we will use thereafter appear more
clearly in this broader case; thus we first study these models in their full generality, before deducing some specific results on descent sets. As a main consequence we derive an affirmative answer to the Conjecture 4 on asymptotic independence from Bender, Helton and Richmond ([BHR03]) and we are able to conclude by the following intuitive result on compositions:

*In the random filling of a composition, the content of two distant cells are almost independent.*

In a forthcoming paper we will use the results of this article to study an analog of the Young lattice that was introduced by Gnedin and Olshanski in [GO06].

2. Preliminaries and results

2.1. Compositions. This paragraph gives definitions and notations concerning compositions.

**Definition 1.** Let \( n \in \mathbb{N} \). A composition \( \lambda \) of \( n \) is a sequence of positive integers \((\lambda_1, \ldots, \lambda_r)\) such that \( \sum \lambda_j = n \).

A unique ribbon Young diagram with \( n \) cells is associated to each composition: each row \( j \) has \( \lambda_j \) cells, and the first cell of the row \( j + 1 \) is just below the last cell of the row \( j \). For example the composition of 10, \((3, 2, 4, 1)\) is represented as in figure 1. This picture shows directly the link between Definition 1 and the definition we stated in the introduction: a composition \( \lambda = (\lambda_1, \ldots, \lambda_r) \) of \( n \) yields a subset \( D_\lambda \) of \( \{1, \ldots, n-1\} \), namely the subset \( \{\lambda_1, \lambda_1 + \lambda_2, \ldots, \lambda_1 + \cdots + \lambda_{r-1}\} \). The latter correspondence is clearly bijective.

The size \( |\lambda| \) of a composition is the sum of the \( \lambda_j \). When nothing is specified, \( \lambda \) will always be assumed to have the size \( n \), and \( n \) will always denote the size of the composition \( \lambda \).

A standard filling of a composition \( \lambda \) of size \( n \) is a standard filling of the associated ribbon Young diagram: this is an assignment of a number between 1 and \( n \) for each cell of the composition, such that every cells have different entries, and the entries are increasing to the right along the rows and decreasing to the bottom along the columns. An example for the composition of figure 1 is shown in figure 2.

In particular, reading the tableau from left to right and from top to bottom gives for each standard filling a permutation \( \sigma \); moreover the descent set of such a \( \sigma \), namely the set of indices \( i \) such that \( \sigma(i + 1) < \sigma(i) \), is exactly the set

\[
D_\lambda = \{\lambda_1, \lambda_1 + \lambda_2, \ldots, \sum_{1}^{r-1} \lambda_i\}.
\]

There is a bijection between the standard fillings of \( \lambda \) and the permutations of \( |\lambda| \) with descent set \( D_\lambda \). For example the filling in figure 2 yields the permutation \((3, 5, 8, 4, 7, 1, 6, 9, 10, 2)\).
2.2. Result on asymptotic independence. We present here the main results that are proven in the present paper.

**Notation 1.** Let \( \lambda \) be a composition. Let \( \Sigma_\lambda \) denote the set of all permutations with descent set \( D_\lambda \). With the uniform counting measure \( \mathbb{P}_\lambda \) it becomes a probability space, and \( \sigma_\lambda \) denotes the random permutation coming from this probability space. As usual \( |\Sigma_\lambda| \) is the cardinal of the set \( \Sigma_\lambda \).

\[ |\Sigma_\lambda| \text{ is thus the descent statistic associated to the composition } \lambda. \]

Denote for each random variable \( X \) by \( \mu(X) \) its law and by \( d_X \) its density, and write \( \mu \otimes \nu \) the independent product of two laws. The goal of the paper is to prove that distant cells in a composition have independent entries, namely:

**Theorem 1.** Let \( \epsilon, r \in \mathbb{N} \). Then there exists \( n \geq 0 \) such that if \( \lambda \) is a composition of \( N \) and \( 0 < i_1 < \cdots < i_r \leq N \) are indices with \( i_{j+1} - i_j \geq n \),

\[
d_\pi \left( \mu \left( \frac{\sigma_\lambda(i_1)}{N} \right) \cdots \frac{\sigma_\lambda(i_r)}{N} \right), \mu \left( \frac{\sigma(i_1)}{N} \right) \otimes \cdots \otimes \mu \left( \frac{\sigma(i_r)}{N} \right) \right) \leq \epsilon,
\]

with \( d_\pi \) denoting the Levy-Prokhorov metric on the set of measures of \([0, 1]^r\).

If the first and last runs of the composition remain bounded, the latter can be improved for the density of the first and last particle. This is the content of the Conjecture 4 of \([BHR03]\) that is proven in this paper and reformulated here in term of permutation:

**Theorem 2.** Let \( \epsilon > 0, A \geq 0 \). There exists \( n \geq 0 \) such that for any composition \( \lambda \) of size larger than \( n \) with first and last run bounded by \( A \),

\[
\|d_{\frac{\sigma_\lambda(1)}{n}}d_{\frac{\sigma_\lambda(n)}{n}} - d_{\frac{\sigma(1)}{n}}d_{\frac{\sigma(n)}{n}}\|_\infty < \epsilon.
\]

2.3. Runs of a composition. Let \( \lambda \) be a composition. We number the cells as we read them, from left to right and from top to bottom. The cells are identified with integers from 1 to \( n \) through this numbering. For example in the standard filling of figure [2], the number 7 is in the cell 5.

We call run any set consisting in all the cells of a given column or row. The set of runs is ordered with the lexicographical order. In the same example as before the runs are

\[
s_1 = (1, 2, 3), s_2 = (3, 4), s_3 = (4, 5), s_4 = (5, 6), s_5 = (6, 7, 8, 9), s_6 = (9, 10),
\]

where we put in the parenthesis the cells of each run.

Note that inside each the cells are ordered by the natural order on integers.

We call extreme cell a cell that is an extremum in a run with respect to this order, and denote by \( \mathcal{E}_\lambda \) the set of extreme cells of \( \lambda \). Apart from the first and last cells of the composition, every extreme belong to two consecutive runs. Let \( P_\lambda \) be the set of extreme cells followed by a column, or preceded by a row and \( V_\lambda \) the set of extreme cells followed by a row or preceded by a column. The elements of \( P_\lambda \) are...
called peaks and the one of $V_\lambda$ valleys. The sets $V_\lambda$ and $P_\lambda$ are also ordered with the natural order:

$$P_\lambda = \{x_1^+ < \cdots < x_r^+\}, V_\lambda = \{x_1^- < \cdots < x_t^-\},$$

with $r - 1 \leq t \leq r + 1$.

The first and last cells are always extreme points. A composition is said being of type ++ (resp. +-,+-,-+) if the first cell is a peak and the last cell is a peak (resp peak-valley, valley-peak, valley-valley).

Finally let $l(s)$, the length of a run $s$, be the cardinal of $s$, and $L(\lambda)$, the amplitude of $\lambda$, be the supremum of all lengths.

2.4. The coupling method. In this paragraph we introduce a probabilistic tool called the coupling method, and set the relative notations for the sequel. We refer to [Lin02] for a review on the subject. We will present the notions in the framework of random variables but we could have done the same with probability laws as well.

**Definition 2.** Let $(E, \mathcal{E})$ be a probability space and $X, Y$ two random variables on $E$. A coupling of $(X, Y)$ is a random variable $(Z_1, Z_2)$ on $(E \times E, \mathcal{E} \otimes \mathcal{E})$ such that

$$Z_1 \sim_{law} X, Z_2 \sim_{law} Y.$$  

Such a coupling always exists: it suffices to consider two independent random variables $Z_1$ and $Z_2$ with respective law $\mu_X$ and $\mu_Y$. However a coupling is often useful precisely when the resulting random variables $Z_1$ and $Z_2$ are far from being independent. In particular in this article we are mainly interested in the case where $Z_1$ and $Z_2$ respect a certain order on the set $E$. From now on $E$ is a Polish space considered with its borelian $\sigma-$algebra $\mathcal{E}$, and $\prec$ a partial order on $E$ such that the graph $G = \{(x, y), x \prec y\}$ is $\mathcal{E}-$measurable.

**Definition 3.** Let $X, Y$ be two random variables on $E$. $Y$ stochastically dominates $X$ (denoted $Y \succeq X$) if and only if

$$\mathbb{P}(X \in A) \leq \mathbb{P}(Y \in A)$$

for any Borel set $A$ such that

$$x \in A \Rightarrow \{y \in E, x \prec y\} \subset A.$$  

For example if $E = \mathbb{R}$ with the canonical order $\leq$ and $\sigma-$algebra $\mathcal{B}(\mathbb{R})$, then $Y$ stochastically dominates $X$ if and only if for all $x \in \mathbb{R}$,

$$\mathbb{P}(X \in [x, +\infty]) \leq \mathbb{P}(Y \in [x, +\infty])$$

or equivalently, if we denote by $F_X(t)$ and $F_Y(t)$ their respective cumulative distribution function:

$$F_Y(t) \leq F_X(t) \text{ for all } t \in \mathbb{R}.$$  

There are several ways to characterize the stochastic dominance:

**Proposition 1.** The three following statements are equivalent:
\begin{itemize}
\item \( Y \) stochastically dominates \( X \)
\item there exists a coupling \((Z_1, Z_2)\) of \(X, Y\) such that \( Z_1 \prec Z_2 \) almost surely.
\item for any positive measurable bounded function \(f\) that is non-decreasing with respect to \(\prec\),
\[ \mathbb{E}(f(X)) \leq \mathbb{E}(f(Y)) \]
\end{itemize}

The proof is straightforward and can be found in [Lin02]. This yields the following intuitive Lemma:

**Lemme 1.** Let \((X_1, X_2), (Y_1, Y_2)\) be two couples of independent random variables on \(E \times E\) such that \(X_1 \preceq Y_1\) and \(X_2 \preceq Y_2\). Then
\[ \mathbb{P}(X_1 \prec X_2) \geq \mathbb{P}(Y_1 \prec Y_2). \]

**Proof.** Let \(\preccurlyeq\) be the partial order on \(E \times E\) defined by
\[ (x, y) \preccurlyeq (x', y') \iff x \prec x' \text{ and } y' \prec y. \]
If \(Y_1 \succeq X_1\) and \(X_2 \succeq Y_2\), there exists a coupling \((\hat{X}_1, \hat{Y}_1)\) (resp. \((\hat{X}_2, \hat{Y}_2)\)) of \(Y_1\) (resp. \(X_2\)) such that almost surely \(\hat{X}_1 \prec \hat{Y}_1\) (resp \(\hat{X}_2 \succ \hat{Y}_2\)). These two couplings can be chosen independent. Since \((X_1, Y_1)\) and \((X_2, Y_2)\) are also independent, this implies that \((\hat{X}_1 \otimes \hat{X}_2, \hat{Y}_1 \otimes \hat{Y}_2)\) is a coupling of \(((X_1, X_2), (Y_1, Y_2))\) with almost surely
\[ (\hat{X}_1, \hat{X}_2) \preccurlyeq (\hat{Y}_1, \hat{Y}_2). \]
But if \(\hat{Y}_1 \prec \hat{Y}_2\), then \(\hat{X}_1 \prec \hat{Y}_1 \prec \hat{Y}_2 \prec \hat{X}_1\) and thus
\[ \mathbb{P}(Y_1 \prec Y_2) = \mathbb{P}(\hat{Y}_1 \prec \hat{Y}_2) \leq \mathbb{P}(\hat{X}_1 \prec \hat{X}_2) = \mathbb{P}(X_1 \prec X_2). \]

These results will be concretely applied on \(\mathbb{R}^n, n \geq 1\), and thus we need to define a family of partial order on those sets.

**Definition 4.** Let \(n \geq 1\). The partial order \(\leq\) on \(\mathbb{R}^n\) is the natural order on \(\mathbb{R}\) for \(n = 1\), and for \(n \geq 2\) if \((x_i)_{1 \leq i \leq n}, (y_i)_{1 \leq i \leq n} \in \mathbb{R}^n\),
\[ (x_i)_{1 \leq i \leq n} \leq (y_i)_{1 \leq i \leq n} \iff \forall i \in [1; n], x_i \leq y_i. \]
For any word of length \(n\) in \(\{1, 0\}\), the modified partial order \(\leq_\epsilon\) is defined as
\[ (x_i)_{1 \leq i \leq n} \leq_\epsilon (y_i)_{1 \leq i \leq n} \iff \forall i \in [1; n], (-1)^{\epsilon_i} x_i \leq (-1)^{\epsilon_i} y_i. \]

The easiest way to check the stochastical dominance is to look at the cumulative distribution function. The proof of the following Lemma is a direct application of Proposition 1.

**Lemme 2.** Let \((X_i)_{1 \leq i \leq n}\) and \((Y_i)_{1 \leq i \leq n}\) be two random variables of \((\mathbb{R}^n, \leq_\epsilon)\). Then \((Y_i)_{1 \leq i \leq n}\) stochastically dominates \((X_i)_{1 \leq i \leq n}\) if and only if for all \((t_i)_{1 \leq i \leq n} \in \mathbb{R}^n\),
\[ F_{(X_i)}(t_1, \ldots, t_n) \geq_\epsilon F_{(Y_i)}(t_1, \ldots, t_n). \]
The stochastic dominance in the case \((\mathbb{R}^n, \leq_\varepsilon)\) is denoted as \((X_1, \ldots, X_n) \preceq_\varepsilon (Y_1, \ldots, Y_n)\). A consequence of the previous result is that if \((Y_1, \ldots, Y_n)\) stochastically dominates \((X_1, \ldots, X_n)\), then for all subsets \(I = (i_1, \ldots, i_r)\) of \(\{1, \ldots, n\}\), \((Y_{i_1}, \ldots, Y_{i_r})\) also stochastically dominates \((X_{i_1}, \ldots, X_{i_r})\).

Applying Lemma 2 to the case \(n = 2\) yields the following Lemma:

**Lemme 3.** Let \((U_1, V_1), (U_2, V_2)\) be two random variables on \([0, 1]\) such that \(U_2\) and \(V_2\) are independent. Suppose that for all \(0 \leq t \leq 1\),

\[F_{V_1}(t) \leq F_{V_2}(t)\]

and for all \(v \in [0, 1]\),

\[F_{U_1|V_1=v}(t) \leq F_{U_2}(t)\]

There exists a coupling \(((Z_1, \tilde{Z}_1), (Z_2, \tilde{Z}_2))\) of \((U_1, V_1)\) and \((U_2, V_2)\) such that almost surely

\[(Z_1, \tilde{Z}_1) \succeq (Z_2, \tilde{Z}_2)\].

3. **Sawtooth model**

3.1. **Definition of the model.** In this section we introduce a statistical model of particles in a tube, which is a generalization of the probabilistic approach of Ehrenborg, Levin and Readdy in [ELR02]. The model consists in a sequence of particles, each of them moving vertically in an horizontal two-dimensional tube. Each particle has a repulsive action on the two neighbouring particles, and moreover the set of particles splits into two groups: the upper particles and the lower particles. The upper particles are always above the lower ones. The model is depicted in Figure 3.

![Figure 3. Repulsive particles in a tube](image)

Such a system is called a Sawtooth model in the sequel.

**Remark 1.** If there are \(n\) upper-particles, there must be \(m\) lower particles with \(m \in \{n-1, n, n+1\}\), depending on what is the type of the first and the last particles. We define therefore the type \(\epsilon(S)\) of the model \(S\) as the word \(\epsilon_I \epsilon_F\), with \(\epsilon_I = +\) (resp. \(\epsilon_F = +\)) if the first (resp. last) particle is an upper one, and \(\epsilon_I = -\) (resp. \(\epsilon_F = -\)) otherwise.
Unless specified otherwise, the first particle is a lower particle (as in the picture). The particles are ordered from the left, and following this order the upper particles are written \( \{p_1 < p_2 < \cdots < p_n\} \) and the lower particles \( \{q_1 < \cdots < q_m\} \). Since the nature of our results won’t depend of the type of the model, we will also assume that there are \( n+1 \) lower particles, yielding that the last particle is a lower one too. Let \( x_i \) be the position of \( q_i \), \( y_i \) the position of \( p_i \) and denote by \( \xi_i(x_i, y_i) \) (resp. \( \rho_i(y_i, x_{i+1}) \)) the potential of the repulsive force between \( q_i \) and \( p_i \) (resp. \( p_i \) and \( q_{i+1} \)). The probability to get a configuration \( \{x_i, y_i\} \) at the Gibbs equilibrium with a temperature \( T \) is:

\[
(*) \quad d\mathbb{P}_{\text{Gibbs}}(\{x_i, y_i\}) = \frac{1}{Z} \exp\left(-\sum\left(\xi_i(x_i, y_i) + \rho_i(y_i, x_{i+1})\right)\right). 
\]

From now on we assume that the potentials only depend on the relative positions of the particles, namely \( \xi_i(x_i, x_{i+1}) = f_i(|y_i - x_{i+1}|) \) and \( \rho_i(x_i, y_i) = g_i(|x_{i+1} - y_i|) \) for some functions \( f_i, g_i \). Since the forces are repulsive, \( f_i \) and \( g_i \) must be decreasing. Moreover by a rescaling we can assume that \( x_i, y_i \in [0, 1] \).

Aiming the results we stated on compositions, we should answer these questions:

1. As the number of particles goes to infinity, is there some independence between \( X_1 \) and \( X_{n+1} \)?
2. It is possible to estimate the behavior of a particle \( X_r \) by only considering its neighbouring particles?

The probability space at the equilibrium can be simplified:

**Definition 5.** A Sawtooth model \( \mathcal{S} \) is the data of:

- \( \{\mu_i, \nu_i\} \) a collection of finite measures on \( [0, 1] \) with respective density functions \( \{f_i, g_i\}_{1 \leq i \leq n} \), each of them being an increasing \( C^1 \) function on \( [0, 1] \).
- A probability space \( \Omega(\{f_i, g_i\}) = ([0, 1]^{n+1} \times [0, 1]^n, \mathbb{P}) \) with probability density

\[
d\mathbb{P}(\{x_i, y_i\}) = \frac{1}{V} \prod 1_{x_i \leq y_i \geq x_{i+1}} f_i(y_i - x_i) g_i(y_i - x_{i+1}).
\]

The quantity \( V \) is called the volume of \( \mathcal{S} \) and is sometimes denoted \( V(\mathcal{S}) \) to avoid confusion.

- \( 2n+1 \) random variables \( \{X_i\} \) and \( \{Y_i\} \) corresponding to the \( 2n+1 \) coordinates on \( [0, 1]^{n+1} \times [0, 1]^n \).

\( \mathcal{S} \) is said renormalized if each \( \mu_i, \nu_i \) is a probability measure.

If we set \( f_i(r) = \exp(-\tilde{f}_i(r)/(k_B T)) \) and \( g_i(r) = \exp(-\tilde{g}_i(r)/(k_B T)) \), we recover the density of \( (\mathcal{S}) \). The volume has the following expression:

\[
(2) \quad V(\mathcal{S}) = \int_{[0,1]^{2n+1}} \prod 1_{x_i \leq y_i \geq x_{i+1}} f_i(y_i - x_i) g_i(y_i - x_{i+1}) \prod dx_i dy_i.
\]

In particular an appropriate rescaling of the measures \( \mu_i, \nu_i \) can transform any Sawtooth model into a normalized one, without changing the probability space. Thus from now on and unless stated otherwise, the model is assumed normalized. In case
we are considering non-normalized models, we will use the notation \( f_i, g_i, \) etc. for the normalized quantities, and \( \tilde{f}_i, \tilde{g}_i, \) etc. for the non-renormalized one.

For each subset of particles \( A = (q_{i_1}, \ldots, q_{i_k}, p_{j_1}, \ldots, p_{j_{k'}}) \) and measurable event \( \mathcal{X} \), denote by

\[
d_A|\mathcal{X}(x_{i_1}, \ldots, x_{i_k}, y_{j_1}, \ldots, y_{j_{k'}})
\]

the marginal density of \( A \) conditioned on \( \mathcal{X} \). The subscripts will be dropped when there is no confusion, and we denote by \( X_1 \) the first variable \( X_{i_1} \) and \( X_F \) the last particle \( X_{n+1} \). Finally since the system is fully described by the functions \( \{f_i, g_j\} \), we will refer sometimes to a particular system just by mentioning this set of functions.

The definition of a Sawtooth model yields directly two first facts. The first result stresses the Markovian aspect of a Sawtooth model:

**Lemma 4.** Let \( \mathcal{S} \) be a Sawtooth model of size \( n \), and \( 1 \leq i_1 < i_2, \ldots, i_r \leq n \) be distinct indices. Then for all \( x_{i_1}, \ldots, x_{i_r} \in [0,1] \), and \( i < i_1 \),

\[
d_{X_i}|X_{i_1}=x_{i_1},\ldots,X_{i_r}=x_{i_r} = d_{X_i}|X_{i_1}=x_{i_1}.
\]

The proof is a straightforward rephrasing of the density of the model.

The second one is a generalization of Lemma 3\textsuperscript{−} (a) in [BHR03].:

**Lemma 5.** Let \( 1 \leq r \leq n+1 \), and let \( \mathcal{X} \) be an event depending on the position of all particles except \( X_r \). Then \( d_{X_r}|\mathcal{X}(x_r) \) is decreasing in \( x_r \).

**Proof.** Let \( a \) be in \([0,1]\). By Lemma 4

\[
d_{X_r}|\mathcal{X}(a) = \int_{[0,1]^2} d_{X_r}|\mathcal{X}|Y_{r-1}=z,Y_{r+1}=z'(a) d_{Y_{r-1},Y_{r+1}}|\mathcal{X}(z,z')dzdz'.
\]

Thus it suffices to prove the monotonicity in the case of a conditioning on \( Y_{r-1} = z, Y_{r+1} = z' \). In this case

\[
d_{X_r}|Y_{r-1}=z,Y_{r+1}=z'(a) = 1_{z \geq a, z' \geq a} \frac{1}{R}(g_{r-1}(z - a) f_r(z' - a)),
\]

with \( R \) a renormalizing constant. Since since \( g_{r-1} \) and \( f_r \) are increasing, this concludes the proof. \( \Box \)

The same result holds for upper particles, but in this case the density is increasing.

3.2. The processes \( \mathcal{S}_\lambda \) and \( \Sigma_\lambda \). Let us see how these definitions fit into the framework of compositions. The main idea from [ELR02] is to consider the set of all permutations with a given descent set \( D_\lambda \) as a probability space.

\(|\Sigma_\lambda|\) can indeed be related to the volume of a polytope in \([0,1]^n\) (see for example the survey of Stanley on alternating permutations, [Sta10]). For each sequence of
distincts elements $\tilde{\xi} = (\xi_1, \ldots, \xi_n)$ in $[0, 1]$, let $\text{std}^{-1}(\tilde{\xi})$ (the inverse standardization of $\tilde{\xi}$) be the permutation that assigns to each $j$ the index $i_j$ in the reordering $(\xi_{i_1} < \cdots < \xi_{i_n})$.

**Proposition 2.** Let $\{x_i\}_{1 \leq i \leq n}$ be a collection of independent uniform random variables on $[0, 1]$. Then the law of $\sigma_\lambda$ is the law of $\text{std}^{-1}(x_1, \ldots, x_n)$ conditioned on the fact that $x_i > x_{i+1}$ if and only if $i \in D_\lambda$. In particular the following expression of the number of permutations with descent set $D_\lambda$ holds:

$$|\Sigma_\lambda| = n! \int_{[0,1]^n} \prod_{i \in D_\lambda} 1_{x_i \geq x_{i+1}} \prod_{i \notin D_\lambda} 1_{x_i \leq x_{i+1}} \prod dx_i,$$

with $x_{n+1} = 1$.

The proof of the latter proposition is straightforward as soon as we remark that the volume of the polytope $\{0 \leq x_1, \ldots, x_n \leq 1\}$ is exactly $\frac{1}{n!}$. Since the indicator function in the integrand depends on conditions between neighbouring points, this result can be rephrased in terms of Sawtooth model. Regrouping the inequalities between elements of the same run of $\lambda$ yields:

$$|\Sigma_\lambda| \approx n! \int_{[0,1]^n} 1_{x_1 \leq x_2 \leq \cdots \leq x_n} 1_{x_1 \geq x_{i+1} \geq \cdots \geq x_{i+1} + 1} \prod dx_i,$$

and by integrating over all the coordinates that do not correspond to extreme cells, we get

$$|\Sigma_\lambda| = n! \int_{[0,1]^n} 1_{x_1 \leq x_2 \leq \cdots \leq x_n} \frac{1}{(l(s_1) - 2)!} |x_1^+ - x_1^-|^{|l(s_1)| - 2} \frac{1}{(l(s_2) - 2)!} |x_1^+ - x_2^-|^{|l(s_2)| - 2} \cdots \frac{1}{(l(s_r) - 2)!} |x_r^+ - x_{r+1}^-|^{|l(s_r)| - 2} \prod dx_i^+ \prod dx_i^-.$$

Let $S_\lambda$ be the non-renormalized Sawtooth model with the non-renormalized density functions $\{f_j, g_j\}_{1 \leq j \leq r}$ such that

$$\tilde{f}_j(t) = \frac{1}{(l(s_{j-1}) - 2)!} l(s_{j-1}) - 2, \tilde{g}_j(t) = \frac{1}{(l(s_j) - 2)!} l(s_j) - 2.$$

A comparison between the latter expression of $|\Sigma_\lambda|$ and the expression (2) of the volume of a Sawtooth model gives

$$|\Sigma_\lambda| = |\lambda| V(S_\lambda).$$

To sum up, two processes are constructed from $\lambda$. The first one, $\sigma_\lambda$ comes from the uniform random standard filling of the ribbon Young tableau $\lambda$, and the second one comes from the construction of an associated model $S_\lambda$. They are of course intimately related, even if the first one is discrete and the second one continuous. $\sigma_\lambda$ can be recovered from $S_\lambda$ by the inverse standardization, and when $|\lambda|$ goes to infinity ($\frac{\sigma_\lambda(1)}{n+1}$, $\frac{\sigma_\lambda(n)}{n+1}$) and $(X_I, X_F)$ are approximately the same:
Lemme 6. The following inequality always holds for $0 < \epsilon < 1$, $n \in \mathbb{N}$:

$$\mathbb{P}(\sup(|\sigma_{\lambda}(1) / n + 1 - X_1|, |\sigma_{\lambda}(n + 1) / n - X_F| > \frac{A}{\sqrt{n + 2}}) \leq \frac{2}{A^2}$$

In particular if the densities of $x_I$ and $x_F$ remain bounded by a constant $B$, $
 \|F_{(X_I, X_F)} - F_{\sigma(1), \sigma(n)}\| \rightarrow_{\lambda \rightarrow +\infty} 0.$

Proof. Let us evaluate $\mathbb{P}(|\sigma_{\lambda}(1) / n - X_1| > \frac{A}{n + 2})$. Let condition this on a particular realization $\sigma$ of $\sigma_{\lambda}$, and suppose that $\sigma(1) = k$. In this case, the conditional density of $X_I$ is:

$$d_{X_I|\sigma=\sigma(x_I)} = n! \left( \int_{0 \leq x_{\sigma^{-1}(1)} \leq \cdots \leq x_{\sigma^{-1}(k-1)} \leq x_1 \leq \sigma(i) \leq k-1} \prod dx_i \right) \left( \int_{x_1 \leq x_{\sigma^{-1}(k+1)} \leq \cdots \leq x_{\sigma^{-1}(n)} \leq 1} \prod dx_i \right) = \frac{n!}{(k-1)!(n-k)!} x_1^{k-1}(1-x_1)^{n-k}.$$

Computing the conditional expectation yields $\mathbb{E}(X_I|\sigma) = \frac{k}{n+1}$ and $\mathbb{V}ar(x_I|\sigma) = \left( \frac{k}{n+1} \right) \left( \frac{n+1-k}{n+1} \right) \frac{1}{n+1} \leq \frac{1}{n+2}.$

Thus by the Chebyshev’s inequality,

$$\mathbb{P}_{X_I|\sigma=\sigma(|X_I - \frac{\sigma(1)}{n+1}| > \frac{A}{\sqrt{n+2}})} \leq \frac{1}{A^2}.$$

Integrating this inequality on all the disjoint events $\sigma$ on which $X_I$ can be conditioned yields the first part of the Lemma. The second part is straightforward. □

In the sequel let $\tilde{\gamma}_r$ denote for $r \geq 2$ the function $\tilde{\gamma}_r(t) = \frac{1}{(r-2)!} t^{r-2}$, and $\gamma_r(t) = (r-1)t^{r-2}$ its renormalized density function.

4. Convex Sawtooth Model

4.1. Log-concave densities. To be able to get some results on the behavior of the particles, it is necessary to impose some conditions on the density functions $\{f_i, g_i\}$. Actually the condition we need is quite natural from a physical point of view, since we will require that the repulsive forces in the definition of the Sawtooth model come from a convex potential: the consequence is that the density functions should be log-concave. This motivates the following definition:

Definition 6. A Sawtooth model is called convex if all the functions $(f_i, g_i)_{1 \leq i \leq n}$ are log-concave. This means that for all $1 \leq i \leq n$, $\frac{f_i'(t)}{f_i(t)}$ and $\frac{g_i'(t)}{g_i(t)}$ are decreasing.
The main advantage of the log-concavity is that the behavior of the particles becomes monotone in a certain sense. For $1 \leq s \leq n + 1$ denote by $S_{\rightarrow P_s}$ (resp. $S_{P_s \leftarrow}$) the Sawtooth model obtained by keeping only the particles and interactions between $X_t$ and $P_s$ (resp. $P_s$ and $X_F$).

**Proposition 3.** Let $\{f_i, g_i\}$ be a convex Sawtooth model. Then for $1 \leq s \leq n$, $0 \leq t \leq 1$, $F_{X_s|Y_s=y}(t)$ is decreasing in $y$, and $F_{Y_s|X_{s+1}=x}(t)$ is decreasing in $x$. Moreover

$$F_{X_s|Y_s=y}(t) \geq F_{X_s|S_{\rightarrow X_s}}(t)$$

and

$$F_{Y_s|X_{s+1}=x}(t) \leq F_{Y_s|S_{\rightarrow Y_s}}(t).$$

**Proof.** Let $d(x)$ be the density of $X_s$ in $S_{\rightarrow X_s}$. Then by the definition of the probability density of $S$, the density of $X_s$ in $S$ conditioned on the value of $Y_s$ is $1_{x \leq y} \frac{d(x)f_s(y-x)}{A}$, with $A$ a normalizing constant. Thus the cumulative distribution function $F_y(\cdot)$ of $X_s$ conditioned on $Y_s = y$ is

$$F_y(t) = \int_0^{t/y} \frac{d(x)f_s(y-x)dx}{\int_0^y d(x)f_s(y-x)dx}.$$

For $t > y$ it is clear that $\frac{\partial}{\partial y} F_y(t) = 0$, and from now on we only consider $t \leq y$. Since the logarithm function is increasing, it is enough to show that $\frac{\partial}{\partial y} \log(F_y(t)) \leq 0$. This derivative is equal to

$$\frac{\partial}{\partial y} \log(F_y(t)) = \int_0^t \frac{d(x)f_s'(y-x)dx}{\int_0^y d(x)f_s(y-x)dx} - \int_0^y \frac{d(x)f_s'(y-x)dx}{\int_0^y d(x)f_s(y-x)dx} \int_0^y \frac{d(y)f_s(0)}{\int_0^y d(x)f_s(y-x)dx}.$$

Since $-\int_0^y \frac{d(y)f_s(0)}{\int_0^y d(x)f_s(y-x)dx} \leq 0$, the non-positivity of the remaining part of the sum suffices. Denote

$$\Delta = \int_0^t d(x)f_s'(y-x)dx \int_0^y d(x)f_s(y-x)dx - \int_0^y d(x)f_s'(y-x)dx \int_0^t d(x)f_s(y-x)dx.$$

Thus we have to show that $\Delta \leq 0$. For $t \leq y$,

$$\Delta = \int_0^t d(x)f_s'(y-x)dx \left( \int_0^t d(x)f_s(y-x)dx + \int_y^t d(x)f_s(y-x)dx \right)$$

$$- \left( \int_0^t d(x)f_s'(y-x)dx + \int_y^t d(x)f_s'(y-x)dx \right) \int_0^t d(x)f_s(y-x)dx$$

$$= \int_0^t d(x)f_s'(y-x)dx \int_y^t d(x)f_s(y-x)dx$$

$$- \int_t^y d(x)f_s'(y-x)dx \int_0^t d(x)f_s(y-x)dx.$$
Expressing products of integrals as double integrals yields
\[
\Delta = \int_{0 \leq z_1 \leq t, t \leq z_2 \leq y} d(z_1)d(z_2)f_s'(y - z_1)f_s(y - z_2)dz_1dz_2
\]
\[
- \int_{0 \leq z_1 \leq t, t \leq z_2 \leq y} d(z_1)d(z_2)f_s(y - z_1)f_s'(y - z_2)dz_1dz_2
\]
\[
= \int_{0 \leq z_1 \leq t, t \leq z_2 \leq y} d(z_1)d(z_2)(f_s'(y - z_1)f_s(y - z_2) - f_s(y - z_1)f_s'(y - z_2))dz_1dz_2.
\]

Since \(d(z_1)d(z_2)\) is positive and \(\frac{f'(t)}{f_s(t)}\) is decreasing, \(\Delta \leq 0\) and the first part of the Proposition is proven.

From the first part of the Proposition, it suffices to prove the first inequality of the second part only for \(y = 1\). Since \(f_s\) is increasing, there exists a measure \(\mu\) on \([0, 1]\) such that \(f_s(x) = \int_0^x d\mu(u)\). Thus
\[
F_1(t) = \frac{\int_0^t d(x)\left(\int_0^{1-u} d\mu(u)\right)dx}{\int_0^1 d(x)\left(\int_0^{1-u} d\mu(u)\right)dx} = \frac{\int_{[0,1]^2} 1_{x \leq t, u \leq 1-x} d(x)d\mu(u)dx}{\int_{[0,1]^2} 1_{u \leq 1-x} d(x)d\mu(u)dx}.
\]

The main point is to express the latter quantity as the expectation of a random variable almost surely greater than \(\int_0^t d(x)dx\). Interverting the integrals yields
\[
F_1(t) = \frac{\int_0^1 \left(\int_0^{t \wedge (1-u)} d(x)dx\right) d\mu(u)}{\int_0^1 \left(\int_0^{1-u} d(x)dx\right) d\mu(u)}.
\]
Let \(\tilde{U}\) be a random variable absolutely continuous with respect to \(\mu\) and having the density
\[
d_{\tilde{U}}(u) = \frac{\int_0^{1-u} d(x)dx}{\int_0^1 \left(\int_0^{1-u} d(x)dx\right) d\mu(u)}.
\]

Then
\[
F_1(t) = \mathbb{E}_{\tilde{U}}\left(\int_0^{t \wedge (1-\tilde{U})} d(x)dx\right).
\]

Since for each \(u \geq 0\)
\[
\frac{\int_0^{t \wedge 1-u} d(x)dx}{\int_0^{1-u} d(x)dx} \geq \int_0^t d(x)dx,
\]
this concludes the proof.
It is exactly the same for \(F_{Y_s|X_{s+1}=x}(t)\). \(\Box\)
4.2. Alternating pattern of a convex sawtooth model. Proposition 3 yields two main features for the model. The first one is an extension of the previous result.

**Proposition 4.** Let \(1 \leq s \leq r, \ 0 \leq t \leq 1\). Then \(F_{X_s|X_r=x}(t)\) is decreasing in \(x\) and \(F_{X_s|Y_r=y}(t)\) is decreasing in \(y\). Moreover
\[
F_{X_s|S \rightarrow X_r}(t) \leq F_{X_s|Y_r=y}(t)
\]
and
\[
F_{X_s|S \rightarrow Y_r}(t) \geq F_{X_s|X_{r+1}=x}(t).
\]

**Proof.** Let \(s \geq 1\) and let us prove the monotonicity by recurrence on \(r\), starting at \(s = r\). \(F_{X_s|X_r=x}(t)\) is clearly decreasing in \(x\) and from Proposition 3 \(F_{X_s|Y_r=y}(t)\) is decreasing in \(y\). Thus the initialization is done.

Suppose the result proved until \(X_r\). Then
\[
F_{X_s|X_{r+1}=x}(t) = \int_0^1 F_{X_s|Y_r=y,X_{r+1}=x}(t)dyd_{Y_r|X_{r+1}=x}(y)
\]
and by an integration by part, since from Lemma 4 \(F_{X_s|Y_r=y,X_{r+1}=x}(t) = F_{X_s|Y_r=y}(t)\),
\[
F_{X_s|X_{r+1}=x}(t) = F_{X_s|Y_r=1}(t) - \int_0^1 \frac{\partial}{\partial y} F_{X_s|Y_r=y}(t)F_{Y_r|X_{r+1}=x}(y)dy.
\]
Thus
\[
\frac{\partial}{\partial x} F_{X_s|X_{r+1}=x}(t) = - \int_0^1 \frac{\partial}{\partial y} F_{X_s|Y_r=y}(t)\frac{\partial}{\partial x} F_{Y_r|X_{r+1}=x}(y)dy.
\]
By recurrence \(\frac{\partial}{\partial y} F_{X_s|Y_r=y}(t)\) is negative and by Proposition 3 \(\frac{\partial}{\partial x} F_{Y_r|X_{r+1}=x}(x)\) is negative, thus \(\frac{\partial}{\partial x} F_{X_s|X_{r+1}=x}(t)\) is also negative. It is exactly the same for \(F_{X_s|Y_{r+1}=y}(t)\).

Let us prove the second part of the proposition and let \(y \in [0,1]\). Conditioning \(X_s\) on \(X_r\) in \(S \rightarrow X_r\) yields
\[
F_{X_s|S \rightarrow X_r}(t) = \mathbb{E}(F_{X_s|X_r=X_r}(t)),
\]
with \(X_r\) following the law of \(q_r\) in \(S \rightarrow X_r\).

On one hand from the first part of the proposition, \(F_{X_s|X_r=x}(t)\) is decreasing in \(x\).

On the other hand from Proposition 3 \(\hat{X}_r\) stochastically dominates \((X_r|Y_r = y)\).

Thus from Proposition 1
\[
F_{X_s|S \rightarrow X_r}(t) = \mathbb{E}(F_{X_s|X_r=\hat{X}_r}(t)) \leq F_{X_s|Y_r=y}(t).
\]
The same pattern proves the second inequality.

There is an immediate consequence of this Proposition on the behavior of \(F_{X_s|S \rightarrow X_n}(t)\) with \(n \geq s\).

**Corollary 1.** The following inequalities hold for \(n \geq s\):
\[
F_{X_s|S \rightarrow X_s}(t) \leq \cdots \leq F_{X_s|S \rightarrow X_n}(t) \leq \cdots \leq F_{X_s|S \rightarrow Y_n}(t) \cdots \leq F_{X_s|S \rightarrow Y_s}(t).
\]
Proof. The previous Proposition yields directly the following inequalities:

\[ F_{X_s|S \to Y_r}(t) \geq F_{X_s|Y_r=1} \geq F_{X_s|S \to X_r}(t). \]

Moreover

\[
F_{X_s|S \to X_{n+1}}(t) = \int_{[0,1]} F_{X_s|Y_n=y}(t)dY_n|s \to X_{n+1}(y)dy \\
\geq \int_{[0,1]} F_{X_s|S \to X_n}(t)dY_n|s \to X_{n+1}(y)dy \\
\geq F_{X_s|S \to X_n}(t),
\]

the first inequality being due to Proposition 3. By symmetry between \(X_n\) and \(Y_n\) the general result holds. \(\square\)

4.3. Estimates on the behavior of extreme particles. As a second consequence of Proposition 3 we can get a more accurate estimate on the behavior of the first and last particles of \(S\). In particular we can achieve a coupling of \((X_I, X_F)\) with two couples of random variables, which only depend on \(f_1\) and \(g_n\) and give some bounds on \((X_I, X_F)\) in the sense of the stochastic domination.

In this paragraph we will not assume that the first and last particles are lower ones, and deal with model of any type (refer to Remark 4 for the definition of the type of a model). Moreover to describe the bounding random variables we introduce two particular transforms \(\Gamma^+\) and \(\Gamma^-\):

**Definition 7.** Let \(f\) be a positive function on \([0,1]\). Then \(\Gamma^+(f)\) and \(\Gamma^-(f)\) are the functions defined on \([0,1]\) as:

\[
\Gamma^-(f)(t) = \frac{\int_{1-t}^1 f(u)du}{\int_0^1 f(u)du},
\]

and

\[
\Gamma^+(f)(t) = \frac{\int_0^t f(u)du}{\int_0^1 f(u)du}.
\]

Remark that \(\Gamma^-(f)(t)\) (resp. \(\Gamma^+(f)(t)\)) is the cumulative distribution function of the random variable \(1 - Z\) (resp. \(Z\)), \(Z\) being the random variable with density \(\frac{f(x)}{\int_0^1 f(x)dx}\).

**Proposition 5.** Let \(S\) be a convex Sawtooth model of type \(\epsilon\) with density functions \(\{f_i, g_i\}_{1 \leq i \leq n}\) and at least four particles. There exists a probability space and two couples of random variables \((X_+, Y_+), (X_-, Y_-)\) on it, such that:

- \((X_-, Y_-) \preceq \epsilon (X_I, X_F) \preceq \epsilon (X_+, Y_+).\)
- \(X_+\) and \(Y_+\) are independent with distribution function

\[
F_{X_+, Y_+}(s, t) = \Gamma^1(f_1)(s)\Gamma^2(g_n)(t).
\]
Let us first consider
particular that
$16$ Pierre Tarrago
with $\lambda$, a continuous density function on $[0$ 

Proof. We assume without loss of generality that each $f_i, g_i$ is renormalized and, since the type of the Sawtooth model doesn’t change the pattern of the proof, we assume that $S$ is of type $-\cdot$.

On one hand the conditional law of $(X_i, X_F)$ given the value of $Y_1 = y_1, Y_n = y_n$ has for cumulative distribution function :

$$F_{X_i,X_F|Y_1=y_1,Y_n=y_n}(t_1,t_2) = \frac{(\int_0^{t_1 \wedge y_1} f_1(y_1 - x)dx)(\int_0^{t_2 \wedge y_n} g_n(y_n - y)dy)}{(\int_0^{y_1} f_1(x)dx)(\int_0^{y_n} g_n(x)dx)} = F_{X_i|Y_1=y_1}(t)F_{X_F|Y_n=y_n}(t).$$

This together with Proposition 3 gives the bound

$$F_{X_i,X_{n+1}|Y_1=y_1,Y_n=y_n}(t_1,t_2) = F_{X_i|Y_1=y_1}(t)F_{X_F|Y_n=y_n}(t) \geq F_{X_i|Y_1=y_1}(t)F_{X_F|Y_n=y_n}(t).$$

Since

$$F_{X_i|Y_1=1}(s)F_{X_F|Y_n=1}(t) = (1 - F_{f_1}(1 - s))(1 - F_{g_n}(1 - t_2)) = \Gamma^-(f_1)(s)\Gamma^-(g_n)(t),$$

denotes the upper part of the stochastic bound.

On the other hand, the density of $(Y_1, Y_n)$ conditioned on the value of $(X_2, X_n)$ is

$$d_{Y_1,Y_n|X_2=x_2,X_n=x_n}(y_1, y_n) = 1_{y_1 \geq x_2, y_n \geq x_n} \frac{\int_0^{y_1} f_1(x)dx g_1(y_1 - x)}{\int_2^{y_2} F_{f_1}(y_1 - x)g_1(z - x)} \frac{\int_0^{y_n} g_n(x)dx f_n(y_n - x)}{\int_2^{y_n} F_{g_n}(y_n - x)g_1(z - x)}$$

$$= 1_{y_1 \geq x_2, y_n \geq x_n} \frac{F_{f_1}(y_1)g_1(y_1 - x_2)}{\int_2^{y_2} F_{f_1}(z)g_1(z - x_2)dz} \frac{F_{g_n}(y_n)g_1(z - x_n)}{\int_2^{y_n} F_{g_n}(z)g_1(z - x_n)dz}.$$

Factorizing the latter density yields

$$d_{Y_1,Y_n|X_2=x_2,X_n=x_n}(y_1, y_n) = d_{Y_1|X_2=x_2}(y_1)d_{Y_n|X_n=x_n}(y_n).$$

Let us first consider $Y_1$. Recall that $g_1$ is an increasing $C^1$ function. This means in particular that

$$g_1(x) = \frac{1}{K} \int_0^x d\lambda(u),$$

with $\lambda$ a probability measure on $[0, 1]$ having eventually a dirac mass at 0 and then a continuous density function on $[0, 1]$. Thus the density of $Y_1$ conditioned on the
value of \(X_2\) is
\[
d_{y_1|x_2=x_2}(y_1) = \frac{1}{A} 1_{y_1 \geq x_2} F_{f_1}(y_1) \int_{x_2}^{y_1} d\lambda(u - x_2),
\]
with \(A\) a normalizing constant. Let \(d_u\) be the density function defined for \(0 \leq u \leq 1\) by
\[
d_u(y) = \frac{1}{A_u} 1_{y \geq u} F_{f_1}(y),
\]
with \(A_u\) a normalizing constant depending on \(u\) and let \(F_u(t)\) be the associated cumulative distribution function. On one hand
\[
F_{y_1|x_2=x_2}(t) = \frac{\int_0^t 1_{y_1 \geq x_2} F_{f_1}(y_1) \int_{x_2}^{y_1} d\lambda(u - x_2) dy_1}{\int_0^t 1_{y_1 \geq x_2} F_{f_1}(y_1) \int_{x_2}^{y_1} d\lambda(u - x_2) dy_1} = \frac{\int_0^t \int_{x_2}^{y_1} 1_{y_1 \geq u} F_{f_1}(y_1) d\lambda(u - x_2) dy_1}{\int_0^t \int_{x_2}^{y_1} 1_{y_1 \geq u} F_{f_1}(y_1) d\lambda(u - x_2) dy_1},
\]
and after interverting the integrals, since \(F_u(1) = 1\),
\[
F_{y_1|x_2=x_2}(t) = \frac{\int_{x_2}^{t} \int_{u}^{1} A_u F_u(t) d\lambda(u - x_2)}{\int_{x_2}^{t} A_u d\lambda(u - x_2)} = \mathbb{E}_{\tilde{U}}(F_{\tilde{U}}(t)),
\]
with \(\tilde{U}\) a random variable with law \(d\tilde{U}(u) = 1_{u \geq x_2} \frac{A_u d\lambda(u - x_2)}{\int_{x_2}^{1} A_u d\lambda(u - x_2)}\).

On the other hand
\[
F_u(t) = 1_{t \geq u} \frac{\int_{u}^{t} F_{f_1}(u) du}{\int_{u}^{t} F_{f_1}(u) du} = 1_{t \geq u} \frac{F_{f_1}(t) - F_{f_1}(u)}{F_{f_1}(1) - F_{f_1}(u)},
\]
with \(F_{f_1}\) being the primitive of \(F_{f_1}\) taking the value 0 at 0. This yields
\[
\frac{\partial}{\partial u} F_u(t) = \frac{\partial}{\partial u} \left( 1_{u \leq t} \frac{F_{f_1}(t) - F_{f_1}(u)}{F_{f_1}(1) - F_{f_1}(u)} \right)
= 1_{u \leq t} \frac{\partial}{\partial u} \left( (F_{f_1}(t) - F_{f_1}(1)) \frac{1}{F_{f_1}(1) - F_{f_1}(u)} + 1 \right)
= 1_{u \leq t} (F_{f_1}(t) - F_{f_1}(1)) \frac{\partial}{\partial u} \left( \frac{1}{F_{f_1}(1) - F_{f_1}(u)} \right)
= 1_{u \leq t} (F_{f_1}(t) - F_{f_1}(1)) \frac{F_{f_1}(u)}{(F_{f_1}(1) - F_{f_1}(u))^2} \leq 0,
\]
and thus
\[ F_u(t) \leq F_0(t) = \frac{\mathcal{F}_{f_1}(t)}{\mathcal{F}_{f_1}(1)}. \]

Integrating with respect to \( \tilde{U} \) yields
\[ F_{Y_1|X_2=x_2}(t) = \mathbb{E}_{\tilde{U}}(F_{\tilde{U}}(t)) \leq \mathbb{E}_{\tilde{U}}(F_0(t)), \]
and finally \( F_{Y_1|X_2=x_2}(t) \leq \frac{\mathcal{F}_{f_1}(t)}{\mathcal{F}_{f_1}(1)} \). We can now integrate this inequality to get a bound on the cumulative distribution function of \( X_I \) conditioned on \( X_2 : \)
\[
F_{X_I|X_2=x_2}(t) = \int_0^1 F_{X_I|Y_1=y(t)}dy_{Y_1|X_2=x_2(y)}dy \\
= F_{X_I|Y_1=y(t)} - \int_0^1 \frac{\partial}{\partial y} F_{X_I|Y_1=y(t)}F_{Y_1|X_2=x_2(y)}dy \\
\leq F_{X_I|Y_1=y(t)} - \int_0^1 \frac{\partial}{\partial y} F_{X_I|Y_1=y(t)}\frac{\mathcal{F}_{f_1}(y)}{\mathcal{F}_{f_1}(1)}dy \\
\leq \int_0^1 F_{X_I|Y_1=y(t)}\frac{\mathcal{F}_{f_1}(y)}{\mathcal{F}_{f_1}(1)}dy.
\]

Note that the sense of the inequality on the third line is due to the negative sign of \( \frac{\partial}{\partial y} F_{X_I|Y_1=y(t)} \). Since
\[
\int_0^1 F_{X_I|Y_1=y(t)}\frac{\mathcal{F}_{f_1}(y)}{\mathcal{F}_{f_1}(1)}dy = \int_0^1 \int_0^t fl(y-u)du F_{f_1}(y) \frac{\mathcal{F}_{f_1}(y)}{\mathcal{F}_{f_1}(1)}dy \\
= \int_0^1 \int_u^t fl(y-u)du \frac{\mathcal{F}_{f_1}(y)}{\mathcal{F}_{f_1}(1)}dy \\
= \int_u^t fl(1-u)du \frac{\mathcal{F}_{f_1}(1)}{\mathcal{F}_{f_1}(1)} = \Gamma-(f_1)(t),
\]
this yields the inequality
\[ F_{X_I|X_2=x_2}(t) \leq \Gamma-(\circ f_1)(t). \]

Note that the latter inequality is valid even if the model has only three particles (see the next Corollary). Finally since in our case there are at least four particles, \( X_F \neq X_2 \), and thus \( F_{X_I|X_2=x_2,X_F=y(t)} = F_{X_I|X_2=x_2}(t) \). Therefore
\[ F_{X_I|X_F=y(t)} \leq \Gamma-(\circ f_1)(t), \]
and by averaging on \( y, \)
\[ F_{X_I}(t) \leq \Gamma-(\circ f_1)(t). \]

Doing the same with \( X_F \) gives the bound:
\[ F_{X_F}(t) \leq \Gamma-(\circ g_n)(t). \]
The result follows from Lemma 3.

In particular as a corollary of the latter proposition (and as a corollary of the proof in the case \( n = 2 \)), the following result holds:

**Corollary 2.** Let \( \mathcal{S} \) be a convex Sawtooth model of type \( \epsilon \) with density functions \( \{f_i, g_i\}_{1 \leq i \leq n} \). There exists a couple of random variables \((Z^{(1)}, Z^{(2)})\) such that for \( y \in [0, 1] \),

- \( Z^{(1)} \leq_{\mathcal{S}} (X_I | X_F = y) \leq_{\mathcal{S}} Z^{(2)} \),
- The cumulative distribution function of \( Z^{(2)} \) is:
  \[
  F_{Z^{(2)}}(t) = \Gamma_{\epsilon}^{(1)}(f_1)(t).
  \]
- The cumulative distribution function of \( Z^{(1)} \) is
  \[
  F_{Z^{(1)}}(t) = \Gamma_{\epsilon}^{(1)}(f_1)^{\ast}(t).
  \]

**Proof.** For \( n \geq 3 \), the result is deduced from the latter Proposition. In the case \( n = 2 \), the proof is exactly the same as in the latter Proposition, except that we only deal with the left case, and thus we don’t need anymore the fact that \( X_2 \neq X_F \). \( \square \)

5. **The independence theorem in a bounded Sawtooth Model**

5.1. **Decorrelation principle and bounding Lemmas.** This section is devoted to the proof of the independence of \( X_I \) and \( X_F \) when the number of particles grows whereas the repulsion forces remain bounded.

**Definition 8.** Let \( A > 0 \). A Sawtooth model \( \mathcal{S} \) with density functions \( \{f_i, g_i\} \) is bounded by \( A \) if

\[
\sup(\|f_i\|_{[0, 1]}, \|g_i\|_{[0, 1]}) \leq A.
\]

The purpose is to prove the following Theorem:

**Theorem 3.** Let \( A > 0 \). For all \( \epsilon > 0 \) there exists \( N_A \geq 0 \) such that for all Sawtooth model \( \mathcal{S} \) bounded by \( A \) and with \( 2n \geq N_A \) particles we have:

\[
\|d_{X_I, X_F}(x, y) - d_{X_I}(x)d_{X_F}(y)\|_{\infty} \leq \epsilon
\]

The pattern of the proof is the following: conditioned on the fact that a particle \( P \) - from now on called a splitting particle - is closed to the boundary of the domain, the left part \( \mathcal{S}_{-P} \) and the right part \( \mathcal{S}_{-P} \) of the system are almost not correlated anymore (see Figure 4).

However we may still not have independence if the law of \( X_I \) and \( X_F \) depends on which particle splits the system. Thus we have to find a set of particles that is large enough, so that with probability close to one an element of this set is close to the boundary, and such that nonetheless conditioning on having any particle from this set closed to the boundary yields the same law on \((X_I, X_F)\).

Let us first begin by bounding the density of the \((X_I, X_F)\).
Figure 4. Decorrelation of the process

**Lemma 7.** Suppose that $\| f_1 \|_\infty \leq A$ and let $\mathcal{S}$ be a Sawtooth model larger than 2. Then there exist $K_A$ only depending on $A$ such that for all event $\mathcal{X}$ depending on $\{ X_i, Y_i \}_{i \geq 2}$:

$$\| d_{X_1|\mathcal{X}} \|_\infty \leq K_A.$$ 

More precisely $K_A = 4A^2$ fits.

This Lemma was already mentioned in the specific context of compositions in [BHR03]. We provide here a different proof.

**Proof.** By Lemma 4, it suffices to prove it for a conditioning on $\{ X_2 = x_2 \}$. From Lemma 5, $d_{X_1|X_2=x_2}(x)$ is decreasing in $x$ and thus it is enough to bound $d_{X_1|X_2=x_2}(0)$. We have

$$d_{X_1|X_2=x_2}(0) = \frac{\int_{x_2}^1 f_1(z) g_1(z - x_2) dz}{\int_{x_2}^1 F_{f_1}(z) g_1(z - x_2) dz} \leq A \frac{\int_{x_2}^1 g_1(z - x_2) dz}{\int_{x_2}^1 F_{f_1}(z) g_1(z - x_2) dz}.$$ 

Remark that

$$\frac{\int_{x_2}^1 g_1(z - x_2) dz}{\int_{x_2}^1 F_{f_1}(z) g_1(z - x_2) dz} = \frac{1}{\mathbb{E}_{\tilde{Z}}(F_{f_1}(\tilde{Z}))},$$ 

with $\tilde{Z}$ being a random variable with density $1_{z > x_2} g_1(z - x_2)$. Since $\| F_{f_1}' \| \leq A$ and $F_{f_1}(1) = 1$, $F_{f_1}(t) \geq 1/2$ on $[1 - 1/(2A)]$; moreover $z \mapsto g_1(z - x_2)$ is increasing, thus $\mathbb{P}(\tilde{Z} \in [1 - 1/(2A), 1]) \geq \frac{1}{2A}$ and by Markov’s inequality $\mathbb{E}_{\tilde{Z}}(F_{f_1}(\tilde{Z})) \geq 1/4A$. Finally

$$d_{X_1|X_2=x_2}(0) \leq 4A^2.$$ 

□

The next step is to get a bound on the first derivative of $d_{X_1}$. This is possible only if $g_1$ is also bounded by $A$ and the model is large enough.

**Lemma 8.** Suppose that $\sup(\| f_1 \|_\infty, \| g_1 \|_\infty) \leq A$ and that $\mathcal{S}$ is a Sawtooth model with at least four particles. Then there exists a constant $R_A$ only depending on $A$ such that for any event $\mathcal{X}$ depending on $\{ X_{i+1}, Y_i \}_{i \geq 2}$,

$$\| (d_{X_1|\mathcal{X}})' \|_\infty \leq R_A.$$
Proof. For exactly the same reasons as in the previous proof, it suffices to bound the derivative of the density conditioned on \( X = \{ Y_2 = y_2 \} \). The expression of the density probability yields

\[
d_{X_1|Y_2=y_2}(x) = \frac{\int_0^1 f_1(y_1 - x)dy_1|_{Y_2=y_2}(y_1)dy_1}{\int_0^1 \left( \int_0^1 f_1(y_1 - x)dy_1|_{Y_2=y_2}(y_1)dy_1 \right)dx}.
\]

Let \( \Delta = \int_0^1 \left( \int_0^1 f_1(y_1 - x)dy_1|_{Y_2=y_2}(y_1)dy_1 \right)dx \), which is independent of \( x \). Then

\[
\left| \frac{\partial}{\partial x} d_{X_1|Y_2=y_2}(x) \right| = \frac{1}{\Delta} \left| \frac{\partial}{\partial x} \int_x^1 f_1(y_1 - x)dy_1|_{Y_2=y_2}(y_1)dy_1 \right|
\]

\[
= \frac{1}{\Delta} \left| \int_x^1 \left( \frac{\partial}{\partial x} f_1(y_1 - x) \right)dy_1|_{Y_2=y_2}(y_1)dy_1 - f_1(0)dy_1|_{Y_2=y_2}(y_1)dy_1 \right|
\]

\[
\leq \frac{1}{\Delta} \left( \int_x^1 -\left( \frac{\partial}{\partial x} f_1 \right)(y_1 - x)dy_1|_{Y_2=y_2}(y_1)dy_1 + |f_1(0)|dy_1|_{Y_2=y_2}(y_1)dy_1 \right).
\]

Let us first bound the numerator. By the expression of the density of \( Y_1 \) conditioned on \( Y_2 = y_2 \),

\[
d_{Y_1|Y_2=y_2}(y_1) = \frac{F_{Y_1}(y_1)dy_1|_{Y_2=y_2}(y_1)}{\mathbb{E}_{\tilde{Y}_1}(F_{Y_1}(\tilde{Y}_1))},
\]

with \( \tilde{Y}_1 \) having the density \( dy_1|_{Y_2=y_2} \). Since \( g_1 \) is bounded by \( A \), from Lemma \( \ref{lem:7} \) \( |dy_1|_{Y_2=y_2} \leq K_A \). From Lemma \( \ref{lem:5} \) \( dy_1|_{Y_2=y_2}(y) \) is increasing in \( y \), and \( |F_{f_1}'| \leq A \), thus \( \mathbb{E}_{\tilde{Y}_1}(F_{f_1}(\tilde{Y}_1)) \geq \frac{1}{4A} \) and

\[
|f_1(0)|dy_1|_{Y_2=y_2}(y_1)dy_1 \leq 4A^2K_A^2.
\]

Let us bound also the first term of the sum: \( f_1 \) being increasing, \( \frac{\partial}{\partial x} f_1(y_1 - x) \leq 0 \) and we can thus remove the absolute value in this first term. An other application of Lemma \( \ref{lem:7} \) yields:

\[
\int_x^1 -\left( \frac{\partial}{\partial x} f_1 \right)(y_2 - x)dy_2|_{Y_2=y_2}(y_1)dy_1 \leq K_A \left( \int_x^1 \frac{\partial}{\partial x} f_1(y_2 - x)dy_2 \right) 
\]

\[
\leq K_A((f_1(1 - x) - f_1(0)) \leq A \times K_A.
\]

The numerator is thus bounded by \( AK_A + 4A^2K_A^2 \).

Interverting the integrals in \( \Delta \) yields:

\[
\Delta = \int_0^1 F_{f_1}(y_1)dy_1|_{Y_2=y_2}(y_1)dy_1.
\]

Since \( F_{f_1}' \) is bounded by \( A \) and \( F_{f_1}(1) = 1 \), we can conclude as in the previous proof that \( F_{f_1}(t) \geq \frac{1}{3A} \) on \([1 - 1/(2A), 1] \). Moreover \( Y_1 \) is an upper particule and thus by
Lemma 5. \(d_{Y_1|Y_2=y_2}(y_1)\) is increasing in \(y_2\). Since \(\int_{[0,1]} d_{Y_1|Y_2=y_2} = 1\), this implies that
\[
\int_{1-1/(2A)}^{1} d_{Y_1|Y_2=y_2}(y_1)dy_1 \geq \frac{1}{2A},
\]
and yields \(\Delta \geq \frac{1}{4A}\). The bounds on the numerator and on \(\Delta\) yield:
\[
\left| \frac{\partial}{\partial x} d_{X_1|Y_2=y_2}(x) \right| \leq 4A^3(K_A + 4AK_A^2).
\]

As an application of the latter Lemma, we can also prove that \(y \mapsto F_{X_1|X_F=y}(t)\) is Lipschitz:

**Proposition 6.** Let \(S\) be a Sawtooth model with \(n \geq 3\) lower particles. Suppose that \(\{f_1, g_1, f_n, g_n\}\) are bounded by \(A > 0\). Let \(R_A\) be the constant of Lemma 5 (with \(R_A \geq 1\)). Then on a neighbourhood \([0, 1/R_A]\) of 0,
\[
F : \left\{ [0, 1/R_A] \rightarrow (C([0, 1], \mathbb{R}), \|\|. \right\}
\]
is Lipschitz with a Lipschitz constant \(B_A\) only depending on \(A\).

**Proof.** It suffices to prove that for \(x \in [0, 1]\), \(y \mapsto d_{X_1|X_F=y}(x)\) is Lipschitz on \([0, 1/R_A]\) with a Lipschitz constant independent of \(x\).

From Lemma 5, \(d_{X_F}\) is decreasing and thus on \([1/R_A, 1]\), \(d_{X_F} \leq d_{X_F}(1/R_A)\). From Lemma 8, \(\left| \frac{\partial}{\partial y} d_{X_F}(y) \right| \leq R_A\) and thus on \([0, 1/R_A]\), \(d_{X_F}(y) \leq d_{X_F}(1/R_A) + R_A(1/R_A - y)\). This implies that
\[
\int_{[0,1]} d_{X_F}(y)dy \leq \int_{0}^{1/R_A} d_{X_F}(1/R_A) + R_A(1/R_A - y)dy + \int_{1/R_A}^{1} d_{X_F}(1/R_A)
\]
\[
\leq d_{X_F}(1/R_A) + \frac{1}{2R_A}.
\]

Since \(\int_{[0,1]} d_{X_F} = 1\), this implies that \(d_{X_F}(1/R_A) \geq 1 - \frac{1}{2R_A}\), and thus that \(d_{X_F} \geq 1 - \frac{1}{2R_A}\) on \([0, 1/R_A]\).

From Lemma 8, \(\left\| \frac{\partial}{\partial y} d_{X_F|x=x} \right\| \leq R_A\). Thus since \(\|f_1\| \leq A\) this yields by applying Lemma 7 on \(d_{X_1,X_F}(x, y) = d_{X_F|x=x}(y)d_{X_1}(x)\):
\[
\left| \frac{\partial}{\partial y} d_{X_1,X_F}(x, y) \right| \leq K_AR_A.
\]

Thus on \([0, 1/R_A]\),
\[
\left| \frac{\partial}{\partial y} d_{X_1|X_F=y}(x) \right| = \frac{1}{d_{X_F}(y)} \left| \frac{\partial}{\partial y} d_{X_1,X_F}(x, y) \right| - \frac{d_{X_1,X_F}(x, y)}{d_{X_F}(y)} \frac{\partial}{\partial y} d_{X_F}(y)
\]
\[
\leq \frac{1}{1 - 1/(2R_A)}(K_AR_A + \frac{R_AK_A^2}{1 - 1/(2R_A)})
\]
Set $B_A = \frac{1}{1-1/(2R_A)}(K_A R_A + \frac{R_A K^2}{1-1/(2R_A)})$. Then $\mathcal{F}$ is $B_A$-Lipschitz on $[0,1/R_A]$. □

5.2. Behavior of $\{X_i\}$ for large models. The purpose of this subsection is to find for a model $\mathcal{S}$ a large set of intermediate particles $\{X_r\}$ for which almost surely one of these particles is close to 0 and such that $F_{X_l|X_r=0}$ is essentially the same for all particles of this set. The first part is a essentially probability computation:

**Proposition 7.** Let $\eta > 0, \epsilon > 0$. There exists $N_0$ such that for any model $\mathcal{S}$ of size $N$ larger than $N_0 + 4$ and for any $2 \leq r \leq N - N_0$, $y_{r+N_0} \in [0,1], \mathbb{P}(\bigcup_{r \leq r+N_0} \{X_i < \eta\}) | Y_{r+N_0} = y_{r+N_0}) \geq 1 - \epsilon.$

**Proof.** Let $N_0$ be an integer to specify later and $\mathcal{S}, r$ as in the statement of the Proposition. Let $\tilde{P} = \mathbb{P}(\bigcap_{r \leq r+N_0} \{X_i \geq \eta\}) | Y_{r+N_0} = y_{r+N_0}).$ Condition this probability on the value of $Y_{r-1} = y_{r-1}$ and denote by $P$ this quantity. Then we have

$$P = \frac{\int_{[0,1]}^{N_0 + 1} \int_{[0,1]}^{N_0} \prod_{r \leq i \leq r+N_0} 1_{x_i \leq y, y_{i-1}} f_i(y_i - x_i) g_i(y_i - x_i) \prod dx_i dy_i}{\int_{[0,1]}^{N_0 + 1} \int_{[0,1]}^{N_0} \prod_{r \leq i \leq r+N_0} 1_{x_i \leq y, y_{i-1}} f_i(y_i - x_i) g_i(y_i - x_i) \prod dx_i \prod dy_i}.$$ 

We can operate the linear change of variable

$$\begin{cases} x_i \to u_i = \frac{1}{1-\eta}(x_i - \eta) \\ y_i \to v_i = \frac{1}{1-\eta}(y_i - \eta) \end{cases}$$

on the numerator. This yields

$$P = \int \frac{(1-\eta)^{2N_0+1} \prod_{r \leq i \leq r+N_0} 1_{u_i \geq u, u_{i+1}} f_i((-1-\eta)(v_i - u_i)) g_i((-1-\eta)(v_i - u_{i+1}))}{\prod_{r \leq i \leq r+N_0} 1_{x_i \leq y, y_{i-1}} f_i(y_i - x_i) g_i(y_i - x_i) \prod dx_i \prod dy_i} \times g_r(y_{r-1} - ((1-\eta)u_{r-1} - \eta)) f_{r+N_0}(y_{r+N_0} - ((1-\eta)u_{r+N_0} + \eta)) \prod du_i dv_i.$$ 

Now recall that each $f_i, g_i$ is increasing. Moreover $(1-\eta)(u - v) \leq (u - v)$, and $y - ((1-\eta)u + \eta) \leq y - u$. Thus

$$P \leq (1-\eta)^{2N_0+1}.$$ 

Let $N_0$ be such that $(1-\eta)^{2N_0+1} \leq \epsilon$. Then by averaging on $y_{r-1},$

$$\tilde{P} \leq \epsilon,$$

and this concludes the proof. □

As said before, it is also necessary that $F_{X_l|X_r=0}$ remains almost constant among this subset of particles. This is possible for large Sawtooth models, thank to the monotony results of Proposition 4:

**Proposition 8.** Let $A, \epsilon > 0, M \in \mathbb{N}^*$. There exists $N_{e,A,M}$ such that for any Sawtooth model bounded by $A$ and of size $N \geq N_{e,A,M}$, there exists $1 \leq r \leq N - M$ such that for $r \leq i, j \leq r + M$,

$$\|F_{X_l|X_r=0} - F_{X_l|X_j=0}\| \leq \epsilon.$$
Proof. Let $S$ be a Sawtooth model bounded by $A$ and of size $N$. Denote by $F_i$ the function $t \mapsto F_{X_i|X_{i-1}=0}(t)$ for $2 \leq i \leq N$. By Lemma 7 all the $F_i$ are $K_{A}-Lipschitz$. Let $K = \lceil \frac{2K}{\epsilon} \rceil$. It suffices to find $r \geq 2$ such that for all $r \leq i, j \leq r + M$, and all $0 \leq k \leq K$,

$$|F_i\left(\frac{k}{K}\right) - F_j\left(\frac{k}{K}\right)| \leq \epsilon \frac{\epsilon}{3}.$$

Denote by $v_i \in [0, 1]^{K+1}$ the vector $(F_i(\frac{k}{K}))_{0 \leq k \leq K}$. Let $N_{e,A,M} = M(\lceil \frac{2K}{\epsilon} \rceil + 1)^{K+1}$. Then if $N \geq N_{e,A,M}$, by the Dirichlet principle on $[0, 1]^{K+1}$, there exists an hypercube of size $\frac{\epsilon}{3}$ that contains at least $M$ distinct points $v_{i_1}, \ldots, v_{i_M}$ (with $i_1 < \cdots < i_M$). Moreover by Proposition 11 for all fixed $0 \leq k \leq K$, $v_i(k) = F_i(\frac{k}{K})$ is decreasing and thus for all $i_1 \leq i \leq i_M$, $v_i(k) \leq v_j(k) \leq v_i(k)$. This yields for all $i_1 \leq i, j \leq i_M$,

$$\|v_i - v_j\|_\infty \leq \epsilon.$$  

\[\square\]

5.3. Proof of Theorem 8. Theorem 8 is a consequence of the following proposition :

**Proposition 9.** Let $A > 0$. For all $\epsilon > 0$, there exists a number $N_{A,\epsilon} \geq 0$ such that for all Sawtooth model $S$ bounded by $A$ and with $2n \geq N_{A,\epsilon}$ particles we have :

$$|F_{X_i|X_{i-1}=y}(t) - F_{X_{i-1}}(t)| \leq \epsilon.$$  

for all $t, y \in [0, 1]$.

**Proof.** Set $\eta = \inf(\frac{1}{R_A}, \frac{\epsilon}{A})$ with $R_A, B_A$ the constants given respectively by Lemma 8 and Proposition 6. Let $N_{0}$ be the constant given for $\eta$ and $\epsilon$ by Proposition 7. And finally set $N_{A,\epsilon} = N_{\epsilon/4,A,N_0} + 4$ given by Proposition 8.

Let $S$ be a Sawtooth model bounded by $A$ of size larger than $N_{A,\epsilon}$. Then by Proposition 8 there exists $2 \leq r \leq N_{A,\epsilon} - 2 - N_{0}$ such that for all $r \leq i, j \leq r + N_{0}$, 

$$\|F_{X_i|X_{i-1}=0} - F_{X_j|X_{j-1}=0}\|_\infty \leq \epsilon.$$  

Denote $t = r + N_{0}$ and let $y_t \in [0, 1]$. For $r \leq i \leq r + N_{0}$, set $L_i = \{X_i \leq \eta \cap \{\forall s > i, X_s > \eta\}\}$. Note that $L_i \cap L_j = \emptyset$ for all $i \neq j$ and $\bigcup L_i = L$ with $L = \bigcup_{r \leq i \leq r + N_{0}} \{X_i \leq \eta\}$. Moreover since $L_i$ is $(X_s, Y_s)_{s \geq t}-$measurable, by Lemma 11 conditioning $X_t$ on $\{X_i = u, Y_t = y_t\} \cap L_i$ is the same as conditioning $X_t$ on $\{X_i = u\}$. Thus

$$\|F_{X_i|L_i, Y_t = y_t} - F_{X_{i-1}}(t)| \leq 2\epsilon,$$  

where $F_{X_i|L_i, Y_t = y_t} = \int_0^\eta (F_{X_i|X_i = u} - F_{X_{i-1}}(t))dX_i|L_i, Y_t = y_t(u)du$.
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by the choice of \( \eta \). Recall that if \( A = \bigcup A_i \), with \( A_i \) disjoint events, then for any event \( C \),

\[
P(C|A) = \sum P(C|A_i)P(A_i|A)
\]

In particular for \( L = \bigcup L_i \) this yields

\[
\|F_{X|L,Y_t=y_t} - F_{X|X_r=0}\| = \sum_i \|F_{X|L_i,Y_t=y_t} - F_{X|X_r=0}\| \leq \sum_i \|F_{X|L_i,Y_t=y_t} - F_{X|X_r=0}\| \leq 2\epsilon.
\]

By Proposition \( \square \) and the choice of \( N_0 \), \( P(L|Y_t=y_t) \geq 1 - \epsilon \), and thus

\[
\|F_{X|Y_t=y_t} - F_{X|X_r=0}\|_\infty \leq 3\epsilon.
\]

By averaging on \( y_t \) with the density \( d_{Y_t|X_F=y} \) we get

\[
\|F_{X|X_F=y} - F_{X_i}\|_\infty \leq 4\epsilon.
\]

Let us end the proof of the Theorem \( \boxplus \), which consists essentially in a rewriting in terms of densities of the latter Proposition.

**Proof.** Let \( A > 0, \epsilon > 0 \). Set \( \epsilon_1 = \left( \frac{\epsilon}{K_A} \right) \) and let \( S \) be a Sawtooth model bounded by \( A \) of size larger than \( N_{A,\epsilon_1} \) (\( N_{A,\epsilon_1} \) being given by Proposition \( \boxplus \)). Then from Proposition \( \boxplus \) for \( y \in [0,1] \),

\[
(4) \quad \|F_{X|X_F=y} - F_{X_i}\|_\infty \leq \frac{(\epsilon/K_A)^2}{2R_A}.
\]

Moreover the following result holds for \( C^1 \)-functions on \([0,1]\):

**Lemme 9.** Let \( f, g : [0,1] \rightarrow [0,1] \) be two \( C^1 \)-functions, such that \( \|f'\|_\infty, \|g'\|_\infty \leq M \). Then for \( \epsilon > 0 \), if \( F, G \) are two primitive of \( f, g \) and

\[
\|F - G\|_\infty \leq \frac{\epsilon^2}{2M},
\]

then \( \|f - g\|_\infty \leq \epsilon \).

Applying this Lemma to (4) yields for \( y \in [0,1] \),

\[
\|d_{X|X_F=y} - d_{X_i}\|_\infty \leq \epsilon/K_A.
\]

And finally,

\[
|d_{X_i,X_F}(x,y) - d_{X_i}(x)d_{X_F}(y)| = |d_{X_F}(y)||d_{X|X_F=y}(x) - d_{X_i}(x)| \leq K_A \frac{\epsilon}{K_A} \leq \epsilon.
\]

\( \square \)
6. Application to compositions

Theorem 3 can be applied to the framework of compositions:

**Corollary 3.** Let $A \geq 0$, $\epsilon > 0$. There exists $n \geq 0$ such that for any composition $\lambda$ of size larger than $n$ with every runs bounded by $A$,

$$ \|d_{S_\lambda}(x, y) - d_{S_\lambda}(x)d_{S_\lambda}(y)\| < \epsilon $$

**Proof.** Each run of $\lambda$ of length $l$ yields a density function $\gamma_l$ in $S_\lambda$, and $\|\gamma_l\|_\infty = l - 1$. Thus if any run of $\lambda$ is bounded by $A$, then all the density functions $\{f_i, g_i\}$ in $S_\lambda$ are bounded by $A - 1$. It suffices then to apply Theorem 3. □

The purpose of this section is to strengthen Corollary 3 and to prove the following Theorem:

**Theorem 4.** Let $\epsilon > 0$, $A \geq 0$. There exists $n \geq 0$ such that for any composition $\lambda$ of size larger than $n$ with first and last run bounded by $A$,

$$ \|d_{S_\lambda}(x, y) - d_{S_\lambda}(x)d_{S_\lambda}(y)\| < \epsilon $$

This Theorem was actually Conjecture 4 in [BHR03]. By Lemma 6, the latter Theorem is equivalent to Theorem 2. The proof of Theorem 4 is followed by some applications.

6.1. Effect of a large run on the law of $(X_I, X_F)$. From Corollary 3 it is enough to prove that the presence of a large run inside the composition disconnects the behaviors of $X_I$ and $X_F$. The main reason for this is the Lemma below: for each composition $\lambda$, denote by $\lambda^+$ the composition $\lambda$ with a cell added on the last run, and by $\lambda^-$ the composition $\lambda$ with a cell removed on the last run.

**Lemme 10.** Let $A > 0$ and $\lambda$ a composition with more than three runs and with the first run smaller than $A$. If the last run of $\lambda$ is of size $R$,

$$ \|d_{S_\lambda} - d_{S_{\lambda^+}}\|_\infty \leq \frac{K_A}{R - 1}, $$

where $K_A$ is the bound on the density of $X_I$ as defined in Lemma 7.

**Proof.** Let us prove it in the case where the first run of $\lambda$ is increasing and the last run decreasing, the other cases having the same proofs. The expression (3) yields

$$ d_{X_I, X_F}^{S_{\lambda^+}}(x, y) = \frac{\int_y^1 d_{X_I, X_F}^{S_\lambda}(x, z)dz}{\int_{[0,1]^2} (\int_y^1 d_{X_I, X_F}^{S_\lambda}(x, z))dxdy}. $$
Thus by integrating on $y$ and then interverting the integrals, this yields
\[
d_{\lambda_1}^{\mathbf{S}_\lambda}(x) = \frac{\int_0^1 \int_0^1 d_{\lambda_1}^{\mathbf{S}_\lambda}(x) \mathbf{1}_{y \leq z} dy \, dz}{\int_0^1 \int_0^1 d_{\lambda_1}^{\mathbf{S}_\lambda}(x) \mathbf{1}_{y \leq z} dy \, dz}.
\]

Factorizing by $d_{\lambda_1}^{\mathbf{S}_\lambda}(x)$ makes a conditional expectation appear and thus
\[
d_{\lambda_1}^{\mathbf{S}_\lambda}(x) = d_{\lambda_1}^{\mathbf{S}_\lambda}(x) \frac{\mathbb{E}_{\lambda}(X_F | X_I = x)}{\mathbb{E}_{\lambda}(X_F)}.
\]

Moreover Proposition 5 yields
\[
F_{Z_1} \leq F_{X_F | X_I = x} \leq F_{Z_2},
\]
with $F_{Z_1} = \Gamma^-(\gamma_R)$ and $F_{Z_2} = \Gamma^-(\gamma_R)$. Since $\Gamma^-(\gamma_R)(t) = 1 - (1 - t)^R$ and $\Gamma^-(\gamma_R)(t) = 1 - (1 - t)^{R-1}$, by stochastic dominance applying Proposition 1 gives
\[
\frac{1}{R} \leq \mathbb{E}_{\lambda}(X_F | X_I = x) \leq \frac{1}{R-1}.
\]

Integrating the latter result on $x$ yields $\frac{1}{R} \leq \mathbb{E}_{\lambda}(X_F) \leq \frac{1}{R-1}$, and thus
\[
\frac{R-1}{R} \leq \frac{\mathbb{E}_{\lambda}(X_F | X_I = x)}{\mathbb{E}_{\lambda}(X_F)} \leq \frac{R}{R-1}.
\]

This yields
\[
|d_{\lambda_1}^{\mathbf{S}_\lambda}(x) - d_{\lambda_1}^{\mathbf{S}_\lambda}(x)| \leq |d_{\lambda_1}^{\mathbf{S}_\lambda}(x)| \frac{1}{R-1} \leq \frac{K_A}{R-1}.
\]

In particular the previous Lemma can used to bound the conditional law of the first particle with respect to the last one. For each composition $\lambda$, and each cell $i \in \lambda$, denote by $\lambda_{\rightarrow i}$ the composition $\lambda$ truncated just after the cell $i$. Moreover denote by $R_{\text{int}}(\lambda)$ the set of all runs of $\lambda$ except the first and last ones.

**Proposition 10.** Let $A \geq 0$ and $\lambda$ a composition with first run bounded by $A$. Then
\[
\|F_{X_I | X_F = x} - F_{X_I}\|_\infty \leq \frac{K_A}{\sup_{s \in R_{\text{int}}(\lambda)} l(s) - 2}.
\]

**Proof.** Let $t \in [0, 1]$. Let $s_0$ be the run with maximal length $R$ in $R_{\text{int}}$ and let $i_0$ be the rightest cell of this run. This cell corresponds to a particle $X_i$ of $Y_i$ in $\mathbf{S}_\lambda$. Let us assume without loss of generality that this particle is a lower one. From Proposition
Lemma 11. Let $\lambda_b$ be the composition with three runs of respective length 2, $b$ and 2, and $d_b(x, y) = d_{X_t|Y_2=y}(x)$. Then the following convergence holds:

$$
\lim_{b \to \infty} \sup_{[0,1]^2} (d_b(x, y) - (1 - x^b)) = 0.
$$

In particular the asymptotic independence:

$$
\lim_{b \to \infty} \sup_{x,y,y'} (d_b(x, y) - d_b(x, y')) = 0.
$$

is valid.

Proof. After integrating in (3) the coordinates of the particles inside the composition:

$$
d_b(x, y) = \frac{1 - x^b - (1 - y)^b + ((x - y) \wedge 0)^b}{(1 - 1/(b + 1))(1 - (1 - y)^b) + y/(b + 1)(1 - y)^b}.
$$
Let us show that \( \lim_{b \to \infty} d_b(x, y) - (1 - x^{b+1}) = 0 \) uniformly in \( x \) and \( y \). In the denominator of (7), letting \( b \) go to \( +\infty \) yields
\[
(1 - \frac{1}{b+1})(1 - (1 - y)^b) + y/(b + 1)(1 - y)^b \sim_{b \to \infty} 1 - (1 - y)^b,
\]
with the equivalent being uniform in \( x \) and \( y \). Indeed
\[
\frac{y/(b + 1)(1 - y)^b}{1 - (1 - y)^b} = \frac{1}{b + 1} \sum_{k=0}^{b-1} (1 - y)^k \leq \frac{1}{b + 1}.
\]
Since for \( x \in [0, 1/2], y \in [1/2, 1], d_b(x, y) \) converges uniformly to 1, it suffices to consider in the sequel that \( x \in [1/2, 1] \) and \( y \in [0, 1/2] \). Let \( \Delta \) be defined as
\[
\Delta(x, y) = \frac{1 - x^b - (1 - y)^b + (x - y)^b}{1 - (1 - y)^b} - (1 - x^b)
\]
\[
= (1 - x^b - (x - y)^b) \frac{1 - (1 - y)^b}{1 - (1 - y)^b} - (1 - x^b) = \frac{(x - y)^b - (1 - y)^b x^b}{1 - (1 - y)^b}.
\]
A derivative computation shows that \( \Delta(x, y) \leq \frac{1}{7} \), which proves the uniform convergence. Since \( \lim_{b \to \infty} \|d_b(x, y) - (1 - x^{b+1})\|_{\infty, [0,1]^2} = 0 \),
\[
\lim_{b \to \infty} \sup_{y,y',x} (d_b(x, y) - d_b(x, y')) = 0.
\]

From the latter result can be deduced the asymptotic independence with a large second run:

**Lemma 12.** Let \( A, \epsilon > 0 \). There exist \( B_A \in \mathbb{N} \) such that if \( \lambda \) is a composition with at least three runs, the extreme runs bounded by \( A \) and the second run larger than \( B_A \), then
\[
\|d_{X_I, X_F} - d_{X_I} d_{X_F}\|_{\infty} \leq \epsilon
\]

**Proof.** Let \( \lambda \) be a composition with first run of length \( a \) and second run of length \( b \). From the definition of the density \( d_{X_I, X_F} \) in (3), conditioning the law of \( X_I \) on the position \( x_P \) of the particle \( P = a + b \) yields
\[
d_{X_I|x_P=y}(x) = \frac{\int_x^1 (\int_0^{z_1} y^{a-2} (z_1 - a)^b - 2 d z_2) d z_1}{\mathcal{Z}}.
\]
Let \( 2 \leq a \leq A \). Then
\[
d_{X_I|x_P=y}(x) = \frac{\int_x^1 (u - x)^{a-3} d_b(u, y) du}{\frac{1}{a-2} \int_0^1 u^a d_b(u, y) du}
\]
From the first part of Lemma [11] \( |d_b(u, y) - (1 - u^b)| \to_{b \to \infty} 0 \) uniformly in \( u \) and \( y \), and thus
\[
\frac{1}{a-2} \int_0^1 u^{a-2} d_b(u, y) du \to_{b \to \infty} \frac{1}{(a-2)(a-1)}.
\]
uniformly in $y$. Since $a$ is bounded by $A$, and from the second part of Lemma [11],
\[
\|d_{X_I}|_{x_p=y} - d_{X_I}|_{x_p=y'}\|_\infty \leq A^2 \sup_{y,y',x} (d_b(x, y) - d_b(x, y')) \to 0
\]
uniformly in $y$. Thus for $b$ large enough, \(\|d_{X_I}|_{x_p=y} - d_{X_I}|_{x_p=y'}\| < \epsilon/A\) for all $y, y'$; then averaging on the law of $x_p$ conditioned on $X_F = y$ yields \(|d_{X_I}|_{X_F=y} - d_{X_I}|_{X_F=y'}| < \epsilon/A\) for all $y, y'$. And finally this implies that
\[
\|d_{X_I,X_F} - d_{X_I}d_{X_F}\|_\infty \leq \epsilon.
\]

\[
\square
\]

The proof of Theorem [4] is just a gathering of all the previous results:

**Proof.** Let $A, \epsilon > 0$. Since the first and last runs are bounded by $A$, any composition large enough has at least three runs. Let $B_A$ be given by Lemma [12], $R$ be the associate constant given by Lemma [8] for $B_A$, and set $C = \frac{2K_AR}{(\epsilon/A)^2}$. Finally, let $n$ be the integer given by Corollary [3] for compositions of runs bounded by $C$. Suppose that $\lambda$ is a composition larger than $n$. By Lemma [12] if the second run is larger than $B_A$, (5) is verified. Thus we can suppose that the second run is bounded by $B_A$.

If $\lambda$ has a run larger than $C$, then from Proposition [10]
\[
\|F_{X_I}|_{X_F=x} - F_{X_I}\|_\infty \leq \frac{KA}{C-1} \leq \frac{(\epsilon/A)^2}{2R}.
\]
But from Lemma [8], $d'_{X_I}$ is bounded by $R$, thus the latter inequality yields with Lemma [9]:
\[
\|d_{X_I}|_{X_F=y} - d_{X_I}\| \leq \epsilon/A.
\]
And $d_{X_I}$ being bounded by $A$, this yields (5).
Thus we can assume that all the runs of $\lambda$ are bounded by $C$. Once again by the choice of $n$ and Corollary [3], (5) is verified. \[\square\]

Note that we actually proved something stronger than Theorem [4] namely:

**Corollary 4.** Let $A, \epsilon > 0$. There exists $n$ such that for every composition $\lambda$ of size larger than $n$ and first run bounded by $A$, and for all $y, y' \in [0,1]$,
\[
\|d_{X_I}|_{X_F=y} - d_{X_I}|_{X_F=y'}\| \leq \epsilon.
\]

6.3. **Consequences and proof of Theorem [4]**. Here are some interesting consequences of Theorem [4]. Let us first remove the constraints on the extreme runs.

**Lemme 13.** Let $\epsilon > 0$. There exists $n \geq 0$ such that for all compositions larger than $n$ with at least two runs,
\[
\sup_{(y,y')\in[0,1]^2} (\|F_{X_I}|_{X_F=y} - F_{X_I}|_{X_F=y'}\|_\infty) \leq \epsilon.
\]
on the neighbouring particles: for any composition \( \lambda \) larger than \( n \), from (3), the law of Proposition 5 applied to \( S_\lambda \),

\[
1 - (1 - t)^R \leq F_{X_1 | X_F = y}(t) \leq 1 - (1 - t)^{R-1}.
\]

Since \( \sup_{[0,1]}(u^{R-1} - u^R) \to_{R \to \infty} 0 \), there exists \( A \) such that for any composition with first run larger than \( A \),

\[
\sup_{[0,1]^2} \| F_{X_1 | X_F = y} - F_{X_1 | X_F = y'} \| \leq \epsilon.
\]

Applying Corollary 4 to \( A, \epsilon \) yields that there exists \( n \) such that for any composition larger than \( n \),

\[
\sup_{[0,1]^2} \| F_{X_1 | X_F = y} - F_{X_1 | X_F = y'} \| \leq \epsilon.
\]

This result can be adapted to show that the law of the first particle depends only on the neighbouring particles: for any composition \( \lambda \) of size \( N \), and \( n \leq N \), denote by \( \lambda(n) \) the composition \( \lambda \) containing only the \( n \) first cells.

**Proposition 11.** Let \( \epsilon > 0 \). There exists \( n_0 \geq 1 \) such that for any \( n \geq n_0 \) and any composition \( \lambda \) of size larger than \( n \) with first run smaller than \( n \),

\[
\| F_{S_\lambda} - F_{S_{\lambda(n)}} \|_\infty \leq \epsilon.
\]

The proof consists only in an averaging of the inequality of the previous Lemma. We will close this paper by proving Theorem 1:

**Proof.** By iteration it is enough to prove the result in the case \( r = 2 \). Let \( 1 \leq i_1 < i_2 \leq N \). Denote by \( \nu_1 \) (resp. \( \nu_2 \), resp. \( \nu_3 \)) the composition containing all the cells \( i \) of \( \lambda \) such that \( i \leq i_1 \) (resp. \( i_1 \leq i \leq i_2 \), resp \( i_2 \leq i \)).

Let \( a \) be the length of the last run of \( \nu_1 \) and \( b \) the length of the first run of \( \nu_2 \). We suppose that these two runs are increasing (the proof is the same in other cases).

From (3), conditioning on the position \( x_{i_1-a} = z \) and \( x_{i_1+b} = z' \) yields the density of \( x_{i_1} \)

\[
\tilde{d}_{z,z'}(x) = \frac{(\int_0^{z \wedge x}(x-z)^a dz)(\int_0^{z' \wedge x}(z'-x)^b dz')}{\int_0^1 (\int_0^{z \wedge x}(x-z)^a dz)(\int_0^{z' \wedge x}(z'-x)^b dz') dx}.
\]

From the latter expression, as \( \min(a, b) \to +\infty \), \( d_\pi(\mu(x_{i_1}), \delta_{x_{i_1}}) \) goes to zero, implying the independence. Thus we assume that \( a \) and \( b \) are bounded by some constant \( R \) and that the same holds for the first run of \( \nu_3 \) and the last run of \( \nu_2 \).

From (3), the law of \( x_{i_1} \) and \( x_{i_2} \) is

\[
d_{x_{i_1}, x_{i_2}}(x, y) = \frac{d_{X_{i_1}, X_{i_2}}(x, y)dx_{i_1, i_2}(y)}{\mathbb{E}_{X_{i_1}, X_{i_2}}(d_{X_{i_1}, i_2}(X_{i_1}, i_2)(X_F))}.
\]

From the boundedness on the extreme runs and Proposition 5 there exists \( K \) such that \( \mathbb{E}_{X_{i_1}, X_{i_2}}(d_{X_{i_1}, i_2}(X_{i_1}, X_{i_2}(X_F))) \geq K \), and \( \| d_{X_{i_1}, i_2} \|, \| d_{X_{i_1}, i_3} \| \leq K \). Thus as
\( \nu_2 \) becomes larger,
\[
\|d_{x_{i_1}x_{i_2}} - d_{x_{i_1}d_{x_{i_2}}}\|_\infty \to 0,
\]
independently of the shape of \( \mu_2 \). Finally by Lemma 6 there exists \( n \) such that if \( i_2 - i_1 \geq n \),
\[
d_x(\mu(\frac{\sigma_\lambda(i_1)}{n}, \frac{\sigma_\lambda(i_r)}{n}), \mu(\frac{\sigma(i_1)}{n}) \otimes \mu(\frac{\sigma(i_r)}{N})) \leq \epsilon.
\]

\[\square\]

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