The sum-capture problem for abelian groups

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Abstract

Let $G$ be a finite abelian group, let $0 < \alpha < 1$, and let $A \subseteq G$ be a random set of size $|G|^\alpha$. We let

$$
\mu(A) = \max_{B, C: |B| = |C| = |A|} |\{(a, b, c) \in A \times B \times C : a = b + c\}|.
$$

The issue is to determine upper bounds on $\mu(A)$ that hold with high probability over the random choice of $A$. Mennink and Preneel \cite{4} conjecture that $\mu(A)$ should be close to $|A|$ (up to possible logarithmic factors in $|G|$) for $\alpha \leq 1/2$ and that $\mu(A)$ should not much exceed $|A|^{3/2}$ for $\alpha \leq 2/3$. We prove the second half of this conjecture by showing that

$$
\mu(A) \leq |A|^3 / |G| + 4|A|^{3/2} \ln(|G|)^{1/2}
$$

with high probability, for all $0 < \alpha < 1$. We note that $3\alpha - 1 \leq (3/2)\alpha$ for $\alpha \leq 2/3$.

In previous work, Alon et al. have shown that $\mu(A) \leq O(1)|A|^3/|G|$ with high probability for $\alpha \geq 2/3$ while Kiltz, Pietrzak and Szegedy show that $\mu(A) \leq |A|^{3+2\alpha}$ with high probability for $\alpha \leq 1/4$. Current bounds on $\mu(A)$ are essentially sharp for the range $2/3 \leq \alpha \leq 1$. Finding better bounds remains an open problem for the range $0 < \alpha < 2/3$ and especially for the range $1/4 < \alpha < 2/3$ in which the bound of Kiltz et al. doesn’t improve on the bound given in this paper (even if that bound applied). Moreover the conjecture of Mennink and Preneel for $\alpha \leq 1/2$ remains open.

1 Introduction

Let $G$ be a finite abelian group, let $0 < \alpha < 1$, and let $A \subseteq G$ be a random set of set of size $|G|^\alpha$. Define

$$
\mu(A) = \max_{B, C: |B| = |C| = |A|} |\{(a, b, c) \in A \times B \times C : a = b + c\}|.
$$

The main question we consider is to determine upper bounds on $\mu(A)$ that hold with high probability over the random choice of $A$. We are motivated in particular by a conjecture of Preneel and Mennink \cite{4}, who posit the existence of constants $C_1$, $C_2$ such that

$$
\Pr[\mu(A) \geq C_1 |A| \log(|G|)] = o(1)
$$

for $\alpha \leq 1/2$ and such that

$$
\Pr[\mu(A) \geq C_2 |A|^{3/2}] = o(1)
$$

for $\alpha \leq 2/3$. We view $|G|$ as going to infinity, without further structural assumptions on $G$. (The nature of the abelian group, indeed, seems to have little influence.)

Our main result is essentially to prove the second of the two conjectures above. More precisely we show that

$$
\Pr_A \left[ \mu(A) \geq |A|^{3/2}/|G| + 4|A|^{3/2} \ln(|G|)^{1/2} \right] = o(1)
$$

(1)

But these conjectures are originally stated for $G = \mathbb{Z}_2^n$ in \cite{4}.  

1
is negligible as $|G| \to \infty$. Note the first term, $|A|^3/|G|$, is the expected size of the set
\[ \{(a, b, c) \in A \times B \times C : a = b + c\} \]
when $A$, $B$ and $C$ are chosen at random. This term dominates for $\alpha > 2/3$ whereas the second term, $|A|^{3/2} \log(|G|)^{1/2}$, dominates for $\alpha < 2/3$.

More generally, if one defines
\[ \mu(A, B, C) = |\{(a, b, c) \in A \times B \times C : a = b + c\}| \]
then we prove that
\[ \Pr_A \left[ \exists B, C \subseteq G \text{ s.t. } \mu(A, B, C) \geq |A||B||C|/|G| + 4\sqrt{\ln(|G|)|A||B|C|} \right] \geq \Theta(1) \]
(2)
is negligible under the same assumptions as before (i.e. that $|A|$ a fixed power of $|G|$ and that $|G| \to \infty$). The fact that (1) is negligible obviously a direct corollary of the fact that (2) is negligible.

We note these results can be given an interpretation in terms of random Cayley graphs. More precisely, let $H_A$ be the Cayley graph of vertex set $G$ and edge set associated to $A$, i.e., such that a directed edge exists from $g_1 \in V(H_A)$ to $g_2 \in V(H_A)$ if and only if $g_2 - g_1 \in A$. Then $\mu(A, B, C)$ is the number of edges $(u, v)$ such that $v \in B \subseteq V(H_A)$ and $u \in C \subseteq V(H_A)$. Thus our result can loosely be paraphrased as: with high probability over the choice of $A$ (with some fixed size), the size of the largest subgraph induced by two shores of given sizes is not much larger (in some sense) than if those shores were also chosen at random.

We note that
\[ |A||B||C|/|G| \geq \sqrt{|A||B||C|} \iff |A||B||C| \geq |G|^2 \]
so that (2) gives an essentially optimal pseudorandomness result as long as $|A||B||C| \geq |G|^2$.

In previous work (Theorem 4) Alon et al. show that for every $0 < \alpha < 1$, $0 < \beta < 1$ such that $2\alpha + \beta > 2 + 1/\log\log(|G|)$,
\[ \Pr_A \left[ \exists B \subseteq G, |B| \geq |G|^\beta \text{ s.t. } \mu(A, B, B) \geq \Theta(1) \frac{\Theta(1)}{2\alpha + \beta - 2} |A||B|^2/|G| \right] \]
(3)
is negligible, where $\Theta(1)$ denotes some absolute constant, and where $A$ is again chosen uniformly at random from all subsets of $G$ of size $|G|^\alpha$. By comparison, (2) implies that for every $0 < \alpha < 1$, $0 < \beta < 1$ such that $\alpha + 2\beta > 2$ (i.e., such that $|A||B|^2 > |G|^2$),
\[ \Pr_A \left[ \exists B \subseteq G, |B| \geq |G|^\beta \text{ s.t. } \mu(A, B, B) \geq (1 + c)|A||B|^2/|G| \right] \]
is negligible for any constant $c > 0$. (This follows from the fact that $c|A||B|^2/|G| \geq \sqrt{|A||B||B| \log(|G|)^{1/2}}$ when $\alpha + 2\beta > 2$.) Loosely speaking, thus, Alon et al. give an optimal pseudorandomness bound for
\[ \max_B \mu(A, B, B) \]
in the regime $|A|^2|B| \geq |G|^2$ whereas we give an optimal pseudorandomness bound for the same quantity in the regime $|A||B|^2 \geq |G|^2$. The two bounds meet at $|A| = |B| = |G|^{2/3}$.

We also note that (3) implies bounds on $\mu(A)$ for $\alpha > 2/3$. Namely, (3) implies that for all $\alpha > 2/3$ there exists a constant $c_\alpha = O(1/(3\alpha - 2))$ such that
\[ \Pr_A [\mu(A) \geq c_\alpha|A|^3/|G|] \]
is negligible. This result, however, is superseded by our observation that (1) is negligible. (Indeed, the latter implies that $c_\alpha$ can in fact be taken any constant greater than 1, independently of $\alpha$, and moreover supports $\alpha = 2/3$.)
In other, more recent related work, Kiltz et al. [3] show that
\[ \Pr_A \left[ \mu(A) \geq |A|^{1+2\alpha} \right] \]
is negligible for all 0 < \alpha \leq 1/4, where again |A| = |G|^\alpha and \alpha is fixed as |G| grows. This result shows in particular that the exponent 3/2 from [1] can be improved (and indeed made arbitrarily close to 1) when |A| is a small power of |G|. Interestingly, 1 + 2\alpha = 3/2 precisely when \alpha = 1/4, so our result implies the restriction 0 < \alpha \leq 1/4 can be lifted while essentially keeping the same bound. (In fact, while keeping a better bound, since 3/2 < 1 + 2\alpha for \alpha > 1/4.) To summarize, the bound of Kiltz et al. on \mu(A) is the best known for 0 < \alpha \leq 1/4 while ours is the current state of the art for 1/4 < \alpha \leq 1, and sharp bounds are only known for 2/3 \leq \alpha \leq 1 (as given variously by Alon et al.’s or by this paper).

It seems natural to conjecture that
\[ \mu(A) \approx \max(|A|, |A|^3/|G|) \] (4)
with high probability, up lower-order (e.g., polylog(|G|)) factors. If true, this would in particular imply that \mu(A) \approx |A| for |A| \leq |G|^{1/2}, as conjectured by Mennink and Preneel [4]. So far, however, (4) has only been established for 2/3 \leq \alpha \leq 1.

TECHNIQUES. As might be guessed from the uncomplicated form of our bound, our proof is very simple and uses only on basic discrete Fourier analysis. More precisely, we rely on the fact that
\[ \mu(A, B, C) = \langle 1_A, 1_B * 1_C \rangle \]
where 1_Z is the characteristic function of Z \subseteq G, where
\[ \langle f, g \rangle = \sum_{x \in G} f(x)g(x) \]
is the inner product of two functions \( f, g : G \to \mathbb{C} \), and where \( f * g \) is the convolution of functions \( f \) and \( g \), i.e.,
\[ (f * g)(x) = \sum_{y \in G} f(y)g(x - y) \]
for all \( x \in G \). We then use the fact that
\[ \langle 1_A, 1_B * 1_C \rangle = |G| \sum_S \hat{1}_A(S)\hat{1}_B(S)\hat{1}_C(S) \]
where the sum is taken over the characters of \( G \) and where \( \hat{f} \) is the (discrete) Fourier transform of \( f \). The fact that \( A \) is random is used to show that, with high probability, \( |\hat{1}_A(S)| \leq 4\sqrt{|A| \ln(|G|)/|G|} \) for all nontrivial characters \( S \), where we borrow the constant 4 from Hayes [6]. After applying this observation, the result easily follows by separating the trivial character from the rest of the sum, and by an application of Cauchy-Schwarz.

EXTENSIONS. Our main result and its corollaries also hold if \( A \) consists of |G|^\alpha elements sampled uniformly at random with replacement. Indeed, this follows by inspection of the proof of the afore-mentioned result of Hayes (6, Lemma 6.3).

APPLICATIONS. We note that our result implies that the compression function “\( \mathbb{F}_2 \)” from [4] provably achieves preimage resistance of \( \sim 2^{2n/3} \) queries. Thus, of the preimage and collision resistance results in [4], only the collision-resistance of \( \mathbb{F}_2 \) remains conjecture-based.
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VERSION HISTORY AND A MISSING REFERENCE. Shortly after posting this note on the arxiv, József Solymosi, editor at E-JC, pointed out to us that very similar results are already obtained in course notes of Babai [2], a reference which we (as well as the above-mentioned authors, except for Hayes) had overlooked. As our methods are basically the same as those of [2] this note thus offers basically nothing new. Our only merit is editorial: to bring the three groups of references [2, 6], [1, 3] and [4] to the attention of one another, as well as to give a unified discussion of these previous results. While we have left the rest of the paper untouched from the first version, we make no longer make any claims to originality.

2 Proof

Since part of the intended audience for this paper are symmetric-key cryptographers (indeed, [4, 3] are both cryptography papers, and this result seems to have other likely applications in symmetric-key cryptography security proofs) we take the leisure of developing the required Fourier analysis from scratch. For notational convenience we assume that $G = \mathbb{Z}_2^n$. Adapting the argument to an arbitrary group is straightforward (this will be evident from the proof).

Let $G = \mathbb{Z}_2^n$. We identify $G$ with the set $\{0,1\}^n$ of binary strings of length $n$. For $S \subseteq [n] = \{1, \ldots, n\}$ we recall that the character function function $\chi_S : \{0,1\}^n \to \{-1,1\}$ is defined by

$$\chi_S(x) = \prod_{i \in S} (-1)^{x_i} = (-1)^{\sum_{i \in S} x_i},$$

where $x = (x_1, \ldots, x_n) \in G = \{0,1\}^n$. Then $\chi_\phi = 1$ and $\chi_S, \chi_T$ are orthogonal for all $S \neq T$, i.e.,

$$\sum_{x \in \{0,1\}^n} \chi_S(x)\chi_T(x) = 0.$$

Thus, also,

$$E[\chi_S\chi_T] = 0, \quad S \neq T$$

where $E[f]$ is a shorthand for

$$E_x[f(x)] = \frac{1}{|G|} \sum_{x \in \{0,1\}^n} f(x).$$

More precisely,

$$E[\chi_S\chi_T] = \begin{cases} 1 & \text{if } S = T, \\ 0 & \text{if } S \neq T. \end{cases}$$

Since $\chi_S\chi_T = \chi_{S\Delta T}$ (where $S\Delta T$ is the symmetric difference of $S$ and $T$), we note this reduces to the fact that

$$E[\chi_S] = \begin{cases} 1 & \text{if } S = \phi, \\ 0 & \text{if } S \neq \phi. \end{cases}$$

\[2\] In a nutshell, this seems to come about as follows: in many cryptographic security proofs an adversary makes “queries” whose answers are randomly drawn from a group $\mathbb{G}$, e.g. $\mathbb{G} = \mathbb{Z}_2^n$; these queries form the set $A$. One must then show that with high probability these queries contain no unexpectedly “helpful structure” for the adversary. The “helpful structure” might be, in certain cases, a high value of $\mu(A)$. 

\[4\] In a nutshell, this seems to come about as follows: in many cryptographic security proofs an adversary makes “queries” whose answers are randomly drawn from a group $\mathbb{G}$, e.g. $\mathbb{G} = \mathbb{Z}_2^n$; these queries form the set $A$. One must then show that with high probability these queries contain no unexpectedly “helpful structure” for the adversary. The “helpful structure” might be, in certain cases, a high value of $\mu(A)$. 

\[5\] In a nutshell, this seems to come about as follows: in many cryptographic security proofs an adversary makes “queries” whose answers are randomly drawn from a group $\mathbb{G}$, e.g. $\mathbb{G} = \mathbb{Z}_2^n$; these queries form the set $A$. One must then show that with high probability these queries contain no unexpectedly “helpful structure” for the adversary. The “helpful structure” might be, in certain cases, a high value of $\mu(A)$.
Every function $f : \{0,1\}^n \to \mathbb{R}$ can be seen as an element of $\mathbb{R}^{|G|}$. Since $\{\chi_S : S \subseteq [n]\}$ is a set of $|G|$ orthogonal functions in $\mathbb{R}^{|G|}$, they form a basis of $\mathbb{R}^{|G|}$. I.e., for every function $f : \{0,1\}^n \to \mathbb{R}$ there exist real numbers $\alpha_S, S \subseteq [n]$ such that

$$f = \sum_{S \subseteq [n]} \alpha_S \chi_S.$$ 

The coefficients $\alpha_S$ are called the fourier coefficients of $f$ and are typically written $\hat{f}(S) := \alpha_S$. Thus

$$f = \sum_{S \subseteq [n]} \hat{f}(S) \chi_S$$

for any $f : \{0,1\}^n \to \mathbb{R}$. One has

$$\hat{f}(S) = E[f \chi_S].$$

More precisely, this can be verified from the fact that

$$E[f \chi_S] = E\left[ \left( \sum_{T \subseteq [n]} \alpha_T \chi_T \right) \chi_S \right] = E[\alpha_S \chi_S \chi_S] = \alpha_S$$

using orthogonality.

We have

$$E[fg] = E\left[ \left( \sum_{T \subseteq [n]} \hat{f}(T) \chi_T \right) \left( \sum_{S \subseteq [n]} \hat{g}(S) \chi_S \right) \right] = E\left[ \sum_{S \subseteq [n]} \hat{f}(S) \hat{g}(S) \right] = \sum_{S \subseteq [n]} \hat{f}(S) \hat{g}(S).$$

for any $f, g : \{0,1\}^n \to \mathbb{R}$. In particular

$$E[f^2] = \sum_{S \subseteq [n]} \hat{f}(S)^2$$

and if $f : \{0,1\}^n \to \{-1,1\}$ then

$$\sum_{S \subseteq [n]} \hat{f}(S)^2 = 1$$

since $E[f^2] = 1$.

Moreover if $f : \{0,1\}^n \to \{0,1\}$ then $(-1)^f : \{0,1\}^n \to \{-1,1\}$ and $(-1)^f = 1 - 2f$ so

$$1 = \sum_{S \subseteq [n]} (-1)^f(S)^2$$

$$= \sum_{S \subseteq [n]} 1 - 2f(S)^2$$

$$= \sum_{S \subseteq [n]} (\hat{1}(S) - 2\hat{f}(S))^2$$

$$= \sum_{S \subseteq [n]} \hat{1}(S)^2 - 4\hat{1}(S)\hat{f}(S) + 4\hat{f}(S)^2$$

$$= 1 - 4\hat{f}(\phi) + 4 \sum_{S \subseteq [n]} \hat{f}(S)^2$$

from which we deduce:

$$\hat{f}(\phi) = \sum_{S \subseteq [n]} \hat{f}(S)^2, \quad (f : \{0,1\}^n \to \{0,1\}).$$
Define
\[ (f \ast g)(x) = \sum_{y \in \{0,1\}^n} f(y)g(x + y) = |G|E_y[f(y)g(x + y)]. \]
(Note \(x + y = x - y\) in the current group.) Using the fact that \(\chi_S(x + y) = \chi_S(x)\chi_S(y)\) for all \(S, x, y\) we find
\[
\frac{\hat{f} \ast g(S)}{\hat{g}(S)} = E_x[(f \ast g)(x)\chi_S(x)]
\]
\[= E_x \left[ \sum_y f(y)g(x + y)\chi_S(x) \right]\]
\[= |G| \sum_y f(y) \sum_x g(x + y)\chi_S(x)\]
\[= |G| \sum_y f(y) \sum_x g(x)\chi_S(x + y)\]
\[= |G| \left( \sum_y f(y)\chi_S(y) \right) \left( \sum_x g(x)\chi_S(x) \right)\]
\[= |G|\hat{f}(S)\hat{g}(S).\]

We write \(1_Z\) for the characteristic function of a set \(Z \subseteq \{0,1\}^n\). Note that for sets \(A, B, C \subseteq \{0,1\}^n\) we have
\[|\{(z, a, b) \in A \times B \times C : z = a + b\}| = \sum_{x \in \{0,1\}^n} 1_A(x)(1_B \ast 1_C)(x)\]
\[= |G|E[1_A(1_B \ast 1_C)]\]
\[= |G| \sum_{S \subseteq [n]} \hat{1}_A(S)\hat{1}_B(S)\hat{1}_C(S)\]
\[= |G|^2 \sum_{S \subseteq [n]} \hat{1}_A(S)\hat{1}_B(S)\hat{1}_C(S)\]

Now let \(A \subseteq \{0,1\}^n\) consist of \(|G|^\alpha\) elements sampled uniformly at random without replacement, Fix \(S \subseteq [n], S \neq \emptyset\). Let \(\chi_S^+ = \{x \in \{0,1\}^n : \chi_S(x) = 1\}, \chi_S^- = \{x \in \{0,1\}^n : \chi_S(x) = -1\}\) be the supports of the positive and negative supports of \(\chi_S\). Note \(|\chi_S^+| = |\chi_S^-| = |G|/2\) and that
\[|G| \cdot \hat{1}_A(S) = |A \cap \chi_S^+| - |A \cap \chi_S^-|\]

Since the points in \(A\) are uniformly distributed in \(\{0,1\}^n\), \(|G| \cdot \hat{1}_A(S)\) is therefore concentrated around 0. If \(A\) were sampled uniformly with replacement, a Chernoff bound would show
\[\Pr \left[ |G| \cdot \hat{1}_A(S) \geq c\sqrt{|A|} \right] \leq 2e^{-c^2/2},\]
which would imply that, with high probability over the choice of \(A\),
\[|\hat{1}_A(S)| \leq \frac{1}{|G|} \sqrt{(2 + h) \ln(|G|)|A|}\]

for all \(S \neq \emptyset\), where \(h > 0\) can be any fixed value. Unfortunately \(A\) is sampled without replacement so Chernoff bounds must be eschewed in favor of Martingales and of Azuma-type inequalities. Such results, in fact, have already been obtained by Hayes [5], who among others proves the following:
Theorem 1 (Hayes, Theorem 1.13). Let $\varepsilon > 0$. Let $G$ be a finite abelian group, and let $0 \leq m \leq |G|$. For all but an $O(|G|^{-\varepsilon})$ fraction of subsets $A \subseteq G$ such that $|A| = m$, the maximum non-principal fourier coefficient of $1_A$ is upper bounded by

$$\frac{2}{|G|} \sqrt{2(1 + \varepsilon) \ln(|G|)} m'$$

in absolute value, where $m' = \min(m, |G| - m)$.

In particular, returning to $G = \mathbb{Z}_p^n$ (although this choice of $G$ will play in an increasingly small role in the remainder), and setting (say) $\varepsilon = 1$ is Hayes’s theorem, we have

$$|\hat{1}_A(S)| \leq \frac{4}{|G|} \sqrt{\ln(|G|)|A|}$$

for all $S \subseteq [n], S \neq \phi$, with overwhelming probability over the choice of $A$, $|A| = |G|^\alpha$. For what follows, we assume such a “generic” $A$. Then for all $B, C \subseteq G$ we have

$$|\{(a, b, c) \in A \times B \times C : a = b + c\}| = |G|^2 \sum_{S \subseteq [n]} \hat{1}_A(S) \hat{1}_B(S) \hat{1}_C(S)$$

$$= |G|^2 \left|\frac{|A||B||C|}{|G|^2} + \sum_{S \neq \phi} \hat{1}_A(S) \hat{1}_B(S) \hat{1}_C(S)\right|$$

$$\leq \frac{|A||B||C|}{|G|} + |G|^2 \sum_{S \neq \phi} |\hat{1}_A(S)| |\hat{1}_B(S)| |\hat{1}_C(S)|.$$  

Note that

$$\sum_{S \neq \phi} |\hat{1}_B(S)\hat{1}_C(S)| \leq \sqrt{\sum_{S \subseteq [n]} |\hat{1}_B(S)|^2} \sqrt{\sum_{S \subseteq [n]} |\hat{1}_C(S)|^2} = \sqrt{\hat{1}_B(\phi)\hat{1}_C(\phi)} = \frac{1}{|G|} \sqrt{|B||C|}$$

by Cauchy-Schwarz. So, by (5),

$$\sum_{S \neq \phi} |\hat{1}_A(S)| |\hat{1}_B(S)| |\hat{1}_C(S)| \leq \frac{4}{|G|^2} \sqrt{\ln(|G|)|A||B||C|}$$

and, altogether,

$$|\{(a, b, c) \in A \times B \times C : a = b + c\}| \leq \frac{|A||B||C|}{|G|} + 4\sqrt{\ln(|G|)|A||B||C|}$$

for all sets $B, C \subseteq G$. (Looking back on the proof, we note that the constant 4 can be replaced with $2\sqrt{2} + h$ for any $h > 0$.)

References

[1] N. Alon, T. Kaufman, M. Krivelevich, D. Ron: Testing triangle-freeness in general graphs. SIAM J. Discrete Math. 22(2), 786819 (2008)

[2] László Babai, The Fourier Transform and Equations over Finite Abelian Groups: An introduction to the method of trigonometric sums (lecture notes), Version 1.3, Section 4. http://people.cs.uchicago.edu/~laci/reu02/fourier.pdf.

[3] Eike Kiltz, Krzysztof Pietrzak, and Mario Szegedy. Digital Signatures with Minimal Overhead from Indifferentiable Random Invertible Functions. CRYPTO 2013, LNCS 8042, pp. 571–588, Springer-Verlag, 2013.
[4] Bart Mennink, Bart Preneel. Hash Functions Based on Three Permutations: A Generic Security Analysis. CRYPTO 2012, LNCS 7417, pp. 330–347, Springer-Verlag, 2012.

[5] Jooyoung Lee, Yannick Seurin, personal communication, 2013.

[6] Thomas P. Hayes, A large-deviation inequality for vector-valued martingales, http://www.cs.unm.edu/~hayes/papers.