Formation of Discontinuities in Rectangular Plates as a Result of Residual Stress Relief

I V Menshova\textsuperscript{1,2,*}, A P Kerzhaev\textsuperscript{1}, G Yu\textsuperscript{3,4} and X Zeng\textsuperscript{3,4}

\textsuperscript{1} Institute of Earthquake Prediction Theory and Mathematical Geophysics, Russian Academy of Sciences, Moscow 117997, Russia
\textsuperscript{2} Bauman Moscow State Technical University, Moscow 105005, Russia
\textsuperscript{3} School of Civil Engineering, Qingdao University of Technology, Qingdao 266033, China
\textsuperscript{4} Cooperative Innovation Center of Engineering Construction and Safety in Shandong Blue Economic Zone Qingdao University of Technology, Qingdao 266033, China

\textsuperscript{*}E-mail: menshovairina@yandex.ru

Abstract. The paper deals with the problem of relieving residual stresses in an elastic domain of rectangular shape with free sides as a result of the formation of a discontinuity of particular shape. First, we construct the solution to the problem of residual stresses in an infinite strip with free sides and with a central transverse cut on which a discontinuity of displacements is known. Then, the solution for a rectangle is added to this solution, with the help of which the boundary conditions at the ends are satisfied. The formulas for the residual stresses and for the corresponding displacements are represented in the form of series in Papkovich–Fadle eigenfunctions. The expansion coefficients (Lagrange coefficients) have the form of simple Fourier integrals.

1. Introduction

Accounting for residual stresses in engineering is one of the most difficult scientific problems. We note two books that, in our opinion, deserve the greatest attention from the point of view of practice. The first book [1] is the result of many years of research on various manifestations of residual stresses in the construction and operation of underground structures. In the second book [2], various physical models of residual stresses are constructed.

The determination of residual stresses, particularly in rock samples, is a very difficult problem. Simple initial estimates of the magnitude and distribution laws of residual stresses can be made on the basis of the analysis of the shape and location of cracks formed during residual stress relief, as well as the solution of the corresponding model problems of the type considered in the paper.

Based on the mathematical theory of residual stresses [3–5], in this paper, three types of discontinuities in a rectangular domain were considered, from which preliminary conclusions about residual stresses can be made. Then, these estimates can be refined by solving the corresponding boundary value problems of the theory of elasticity.

The obtained formulas, strictly speaking, describe residual stresses in a rectangular plate and the displacements that occur as a result of their relief with the formation of discontinuities. Therefore, the presentation is conducted in terms of residual stresses. In order to pass to the solutions in the classical formulation (a rectangle with a crack), it is necessary to change the signs in the formulas for displacements or stresses to the opposite ones in a certain way.
The problems of rectangular plates weakened by cracks have been the subject of numerous studies. In the paper [6], the stress intensity factors were investigated for an infinite elastic strip containing a central transverse crack. Reut, Vaysfeld and Zhuravlova [7, 8] investigated the stress state of a semi-strip with a transverse and longitudinal crack. The solution to the problem is reduced to solving a system of three singular integral equations. Their articles also provide an extensive review of publications on this topic. In the papers [9, 10], numerical methods are used to investigate the problems of a central crack in a rectangular plate subjected to external tension. Zhou and Yang [11] used the near crack line analysis method to investigate a central crack loaded by two pairs of point shear forces in a finite plate and obtained the analytical solution.

2. Solution in a strip
Let us consider the infinite free strip \{II: |x| < \infty, |y| \leq 1\} in which there is a field of residual stresses symmetric with respect to the x and y axes. We make the transverse cut \{y: x = 0, |y| \leq \alpha < 1\} and assume that a discontinuity of longitudinal displacements is given on this cut:

\[ U^*(0, y) - U^-(0, y) = 2u(y), \]

where \( U^\pm(0, y) \) denotes the displacements equal to \( u(y) \) on the right and left of the cut, respectively.

We will seek the solution to the problem in the form of series \((x \geq 0, \Re \lambda_k < 0)\)

\[
U(x, y) = \sum_{k=1}^{\infty} a_k \xi(\lambda_k, y)e^{ikx} + a_k \bar{\xi}(\lambda_k, y)e^{ikx},
\]

\[
V(x, y) = \sum_{k=1}^{\infty} a_k \chi(\lambda_k, y)e^{ikx} + a_k \bar{\chi}(\lambda_k, y)e^{ikx},
\]

\[
\sigma_x(x, y) = \sum_{k=1}^{\infty} a_k s_x(\lambda_k, y)e^{ikx} + a_k \bar{s}_x(\lambda_k, y)e^{ikx},
\]

\[
\sigma_y(x, y) = \sum_{k=1}^{\infty} a_k s_y(\lambda_k, y)e^{ikx} + a_k \bar{s}_y(\lambda_k, y)e^{ikx},
\]

\[
\tau_{xy}(x, y) = \sum_{k=1}^{\infty} a_k t_{xy}(\lambda_k, y)e^{ikx} + a_k \bar{t}_{xy}(\lambda_k, y)e^{ikx},
\]

in the Papkovich–Fadle eigenfunctions

\[
\xi(\lambda_k, y) = \left(1 - \frac{1}{2} \sin \lambda_k - \frac{1 + \nu}{2} \lambda_k \cos \lambda_k \right) \cos \lambda_k y - \frac{1 + \nu}{2} \lambda_k y \sin \lambda_k \sin \lambda_k y,
\]

\[
\chi(\lambda_k, y) = \left(1 + \frac{1 + \nu}{2} \lambda_k \cos \lambda_k + \sin \lambda_k \right) \sin \lambda_k y - \frac{1 + \nu}{2} \lambda_k y \sin \lambda_k \cos \lambda_k y,
\]

\[
s_x(\lambda_k, y) = (1 + \nu) \lambda_k \left[ (\sin \lambda_k - \lambda_k \cos \lambda_k) \cos \lambda_k y - \lambda_k y \sin \lambda_k \sin \lambda_k y \right],
\]

\[
s_y(\lambda_k, y) = (1 + \nu) \lambda_k \left[ (\sin \lambda_k + \lambda_k \cos \lambda_k) \cos \lambda_k y + \lambda_k y \sin \lambda_k \sin \lambda_k y \right],
\]

\[
t_{xy}(\lambda_k, y) = (1 + \nu) \lambda_k^2 \left(\cos \lambda_k \sin \lambda_k y - \sin \lambda_k \cos \lambda_k y \right).
\]

In formulas (2) and (3), the following notation is introduced: \( U(x, y) \) and \( V(x, y) \) are the longitudinal and transverse displacements, respectively, multiplied by the shear modulus \( G \), \( \nu \) is Poisson’s ratio, \( a_k \) are the unknown expansion coefficients, the eigenvalues \( \lambda_k \) are the zeros of the entire function of exponential type \( L(\lambda) = \lambda + \sin \lambda \cos \lambda \).

The expansion coefficients \( a_k \) in formulas (2) are determined from the conditions on the crack as a solution to the problem of conjugation of two functions analytical in the right and left semi-strips. The meaning of their introduction is to extract minimal systems of functions, to which it is possible to construct biorthogonal systems of functions, from the right and left complete systems of the Papkovich–Fadle eigenfunctions involved in the boundary conditions at the junction of the semi-strips, and then, with the help of them, to find the required coefficients of expansions.
To formulate the boundary value problem for an infinite strip with the discontinuity of longitudinal displacements (1), following the papers [3, 4, 12], we introduce two functions:

\[
\Phi(x, y) = -\tau_{yx}(x, y) + i(1 + \nu)\frac{\partial V(x, y)}{\partial y} - \frac{1 - \nu}{2}\sigma_y(x, y),
\]

\[
F(x, y) = -(1 + \nu)\frac{\partial U(x, y)}{\partial y} - \frac{1 - \nu}{2}\tau_{yx}(x, y) - i\sigma_y(x, y).
\]

Substituting here (2), we obtain

\[
\Phi(x, y) = \sum_{k=1}^{\infty} 2\text{Re} \left[ a_k \Phi(\lambda_k, y) e^{\lambda_k x} \right],
\]

\[
F(x, y) = \sum_{k=1}^{\infty} 2\text{Re} \left[ a_k F(\lambda_k, y) e^{\lambda_k x} \right],
\]

where

\[
\Phi(\lambda_k, y) = (1 + \nu)\lambda_k^2 (i\cos \lambda_k + y \sin \lambda_k) e^{\lambda_k y},
\]

\[
F(\lambda_k, y) = (1 + \nu)\lambda_k [i(\lambda_k \cos \lambda_k - \sin \lambda_k) + \lambda_k y \sin \lambda_k] e^{\lambda_k y}.
\]

Denote

\[
\omega_1 = \lambda_1, \omega_2 = -\lambda_1, \omega_3 = \lambda_2, \omega_4 = -\lambda_2, \ldots, a_1^* = A_1, a_2^* = -A_2,
\]

where \( a_1^*, a_2^* \) are the unknown expansion coefficients for the right and left semi-strips, respectively.

Then, from the conjugation conditions for \( x = 0 \), we obtain two equations:

\[
\sum_{k=1}^{\infty} 2\text{Re} [A_k \Phi(\omega_k, y)] = 0, \quad \sum_{k=1}^{\infty} 2\text{Re} [A_k F(\omega_k, y)] = 2 \frac{du(x)}{dy}.
\]

The solution to the system of equations (8) is constructed in the same way as in [3], namely with the help of the functions \( \Phi_k(y) \) and \( F_k(y) \) biorthogonal to the functions \( \Phi(\lambda_k, y) \) and \( F(\lambda_k, y) \). The biorthogonal functions are determined from the equations [3]

\[
\int_{-\infty}^{\infty} \Phi(\lambda, y) \Phi_k(y) dy = \frac{\lambda L(\lambda)}{\lambda - \lambda_k}, \quad \int_{-\infty}^{\infty} F(\lambda, y) F_k(y) dy = \frac{L(\lambda)}{\lambda - \lambda_k}.
\]

Using the biorthogonality relations following from equations (9), from (8) we obtain the system of two algebraic equations

\[
\begin{align*}
A_k \omega_k M_k + A_k \omega_2 M_k &= 0; \\
A_k M_k + A_k M_k &= f_k + f_k^*.
\end{align*}
\]

Here

\[
f_k = \frac{u_k}{\lambda_k}, \quad u_k = \int_{-\infty}^{\infty} u(y) u_k(y) dy, \quad u_k(y) = \frac{\lambda_k \cos \lambda_k y}{(1 + \nu) \sin \lambda_k},
\]

The numbers \( u_k \) are the Lagrange coefficients of the expanded function \( u(y) \), the functions \( u_k(y) \) are the finite parts of the functions biorthogonal to the Papkovich–Fadle eigenfunctions \( \xi(\lambda_k, y) \), \( M_k = \cos^2 \lambda_k \) are normalizing factors.

Solving the system of equations (10) and taking into account the notation (7), we find

\[
a_k^* = \frac{\lambda_k (f_k + f_k^*)}{(\lambda_k - \lambda_k) M_k}.
\]

We substitute (12) into (2) and separate the zero-series, following [3, 4], for example. As a result, we obtain formulas that describe the solution to the problem under consideration:

\[
U'(x, y) = -\text{sgn}(x) \sum_{k=1}^{\infty} 2\text{Re} \left[ u_k \frac{\xi(\lambda_k, y) \text{Im}(\lambda_k e^{\lambda_k y})}{\lambda_k M_k} \right],
\]

\[
V'(x, y) = -\sum_{k=1}^{\infty} 2\text{Re} \left[ u_k \frac{\xi(\lambda_k, y) \text{Im}(\lambda_k e^{\lambda_k y})}{\lambda_k M_k} \right].
\]
\[ \sigma_i'(x,y) = -\sum_{k=1}^{\infty} 2\text{Re} \left\{ \frac{u_k s_i(\lambda_k, y) \text{Im}(\lambda_k e^{k|x|})}{\lambda_k M_k} \right\}, \]

\[ \sigma_j'(x,y) = -\sum_{k=1}^{\infty} 2\text{Re} \left\{ \frac{u_k s_j(\lambda_k, y) \overline{\lambda_k M_k}}{\lambda_k M_k} \right\}, \]

\[ \tau_{ij}'(x,y) = -\text{sgn}(x) \sum_{k=1}^{\infty} 2\text{Re} \left\{ \frac{u_k u_k}{\lambda_k M_k} \overline{\lambda_k M_k} \right\}. \]

Formulas (13) describe the residual stresses in the infinite strip \( \Pi \) and the displacements that occur during their relief with the formation of the discontinuity (1).

3. Solution in a rectangle

The solution for the free rectangle \( \{P: |x| \leq d, |y| \leq 1\} \) with the discontinuity (1) is obtained simply. To the solution (13) for the strip with the discontinuity, we need to add the solution for the rectangle that relieves the normal and tangential stresses

\[ \sigma_i'(\pm d, y) = -\sum_{k=1}^{\infty} 2\text{Re} \left\{ \frac{u_k s_i(\lambda_k, y) \text{Im}(\lambda_k e^{k|x|})}{\lambda_k M_k} \right\}, \]

\[ \tau_{ij}'(\pm d, y) = -\text{sgn}(\pm d) \sum_{k=1}^{\infty} 2\text{Re} \left\{ \frac{u_k u_k}{\lambda_k M_k} \overline{\lambda_k M_k} \right\}, \]

in the cross sections \( x = \pm d \) of the strip.

The final formulas for the displacements and stresses in the rectangle when normal and tangential stresses are given at its ends in the case of its even symmetric deformation with respect to the central axes are as follows:

(a) if only normal stresses are given at the ends of the rectangle, then

\[ U(x,y) = \sum_{k=1}^{\infty} 2\text{Re} \left( \frac{\lambda_k}{\lambda_k M_k} \right) \text{Im}(\lambda_k \sinh \lambda_k d \sinh \lambda_k x), \]

\[ V(x,y) = \sum_{k=1}^{\infty} 2\text{Re} \left( \frac{\lambda_k}{\lambda_k M_k} \right) \text{Im}(\lambda_k \sinh \lambda_k d \cosh \lambda_k x), \]

\[ \sigma_i(x,y) = \sum_{k=1}^{\infty} 2\text{Re} \left( \frac{s_i(\lambda_k, y)}{M_k} \right) \text{Im}(\lambda_k \sinh \lambda_k d \cosh \lambda_k x), \]

\[ \sigma_j(x,y) = \sum_{k=1}^{\infty} 2\text{Re} \left( \frac{s_j(\lambda_k, y)}{M_k} \right) \text{Im}(\lambda_k \sinh \lambda_k d \cosh \lambda_k x), \]

\[ \tau_{ij}(x,y) = \sum_{k=1}^{\infty} 2\text{Re} \left( \frac{t_{ij}(\lambda_k, y)}{\lambda_k M_k} \right) \text{Im}(\lambda_k \sinh \lambda_k d \cosh \lambda_k x), \]

(b) if only tangential stresses are given at the ends of the rectangle, then

\[ U(x,y) = -\sum_{k=1}^{\infty} 2\text{Re} \left( \frac{\lambda_k}{\lambda_k M_k} \right) \text{Im}(\lambda_k \cosh \lambda_k d \sinh \lambda_k x), \]

\[ V(x,y) = -\sum_{k=1}^{\infty} 2\text{Re} \left( \frac{\lambda_k}{\lambda_k M_k} \right) \text{Im}(\lambda_k \cosh \lambda_k d \cosh \lambda_k x), \]

\[ \sigma_i(x,y) = -\sum_{k=1}^{\infty} 2\text{Re} \left( \frac{s_i(\lambda_k, y)}{M_k} \right) \text{Im}(\lambda_k \cosh \lambda_k d \cosh \lambda_k x), \]

\[ \sigma_j(x,y) = -\sum_{k=1}^{\infty} 2\text{Re} \left( \frac{s_j(\lambda_k, y)}{M_k} \right) \text{Im}(\lambda_k \cosh \lambda_k d \cosh \lambda_k x). \]
\[ \tau_{xy}(x, y) = -\sum_{k=1}^{\infty} 2\text{Re}\left( \frac{t_n(x, y)}{M_k} \text{Im}(\lambda_k \cosh \lambda_k d \sinh \lambda_k x) \right) \frac{\text{Im}(\lambda_k \sinh \lambda_k d \cosh \lambda_k d)}{\lambda_k}. \]

The numbers \( \sigma_k \) and \( \tau_k \) should be taken equal to
\[ \sigma_k = \frac{u_k}{\lambda_k}, \quad \text{Im}(\lambda_k e^{k|d|}), \quad \tau_k = -u_k \frac{\text{Im}(e^{k|d|})}{\lambda_k}. \] (17)

The complete solution to the problem for the rectangle with the discontinuity of longitudinal displacements (1) is obtained by adding the solutions (13), (15) and (16).

Let us consider three types of discontinuities of longitudinal displacements:
\[ u_1(y) = \sqrt{\alpha^2 - y^2}, \quad |y| < \alpha < 1; \quad u_2(y) = 2\left(\alpha^2 - y^2\right), \quad |y| < \alpha < 1; \quad u_3(y) = \frac{8}{9}\left(\alpha^2 - y^2\right)^2, \quad |y| < \alpha < 1; \] (18)

Accept \( \alpha = 0.5 \). Using formulas (11) and (17), we find the Lagrange coefficients \( u_k, \sigma_k \) and \( \tau_k \) for each of the functions (18), and then substitute them into (13), (15) and (16). Figures 1 and 2 show the comparison of the three solutions depending on the shape of the discontinuity. The solid lines correspond to the solution for the discontinuity \( u_1(y) \), the dashed lines for \( u_2(y) \), and the dotted lines for \( u_3(y) \).

**Figure 1.** Distribution of the normal stresses \( \sigma_y(0.01, y) \) depending on the shape of the discontinuity.

**Figure 2.** Distribution of the normal stresses \( \sigma_y(0.01, y) \) depending on the shape of the discontinuity.

### 4. Conclusions

The paper proposes a new method for solving the problem of relieving residual stresses in an elastic rectangular domain with a transverse crack on which a displacement discontinuity is given. It is based on the theory of expansions in Papkovich–Fadle eigenfunctions. Three types of discontinuities in the rectangular domain were considered. The comparison of the three solutions was given depending on the shape of the discontinuity. The solutions obtained make it possible to reconstruct, from the shape of the discontinuity, the fields of residual stresses that led to their formation.

The problem is somewhat more complicated when stresses are given on the crack. Using the same scheme, one can obtain similar solutions for a rectangle with discontinuities of displacements or with stresses given at the discontinuity with other boundary conditions on its sides, particularly with stiffeners. Using the simple superposition method, one can construct solutions for a set of discontinuities if their shapes and locations are known. In the future, the authors intend to construct similar exact solutions for finite bodies of canonical shape in oblique and polar coordinate systems, in particular, for a rectangle with an oblique crack. This problem has an important application in the problem of increasing the recovery of oil-bearing layers.
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References
[1] Tazhibaev K T 2016 Stresses, Processes of Deformation and Dynamic Destruction of Rocks In two volumes (Bishkek: Altyn Print) (in Russian)
[2] Moroz A I 2004 Self-Stressed State of Rocks (Moscow: MGGU) (in Russian)
[3] Kovalenko M D, Menshova I V and Shulyakovskaya T D 2013 Expansions in Fadle–Papkovich functions: examples of solutions in a half-strip Mech. Solids 48(5) 584–602
[4] Kovalenko M D, Menshova I V and Kerzhaev A P 2018 On the exact solutions of the biharmonic problem of the theory of elasticity in a half-strip Z. Angew. Math. Phys. 69 121
[5] Kovalenko M D, Abrukov D A, Menshova I V, Kerzhaev A P and Yu G 2019 Exact solutions of boundary value problems in the theory of plate bending in a half-strip: basics of the theory Z. Angew. Math. Phys. 70 98
[6] Goldstein R V, Ryskov I N and Salganik R L 1970 Central transverse crack in an infinite strip Int. J. Fract. 6(1) 104–5
[7] Reut V, Vaysfeld N and Zhuravlova Z 2019 Investigation of the stress state of the elastic semi-stripe with a transverse crack Theor. Appl. Fract. Mech. 100 105–9
[8] Vaysfeld N and Zhuravlova Z 2020 The investigation of semi-stripe’s stress state with a longitudinal crack Z. Angew. Math. Mech. 100(3) e201900289
[9] Mohsin N R 2015 Static and dynamic analysis of center cracked finite plate subjected to uniform tensile stress using finite element method Int. J. Mech. Eng. Technol. 6(1) 56–70
[10] Li X and You X 2006 Boundary collocation method for a cracked rectangular plate with double external tension Appl. Anal. 85(9) 1103–11
[11] Zhou X P and Yang H Q 2009 Elastoplastic solution for an eccentric crack loaded by two pairs of point tensile forces Theor. Appl. Fract. Mech. 51(1) 62–72
[12] Kovalenko M D, Menshova I V, Kerzhaev A P and Yu G 2018 Mixed boundary value problems in the theory of elasticity in an infinite strip Acta Mech. 229(11) 4339–56