FILTERING WITH WAVELET ZEROS AND GAUSSIAN ANALYTIC FUNCTIONS

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Abstract. Following the recent work of Flandrin, we propose a method to filter a signal embedded in Gaussian white noise. Whereas Flandrin’s technique is based on zeros of spectrograms—dubbed “silent points”—we propose a similar method using the zeros of the wavelet transform with window

$$\hat{g}_m(\xi) := \begin{cases} 2^{\frac{\alpha - 1}{2}} \xi^{\frac{\alpha - 1}{2}} e^{-\xi}, & \xi \geq 0, \\ 0, & \xi < 0, \end{cases}$$

since these transforms are analytic functions in the upper half-plane. The main observation is that the zeros of the wavelet transform of complex white noise coincide with those of hyperbolic Gaussian analytic functions (GAFs). The signal is detected by designing a mask in the upper half-plane where the zero set of the wavelet transformed noisy signal is significantly different from the hyperbolic GAF. As a limiting case of wavelet transforms, we consider the Cauchy transform. In this case, white noise maps to a particular Gaussian analytic function which is a determinantal point process.

1. Introduction

Recovering signals embedded in a noisy background is a fundamental problem in signal analysis, since most real world signals are acquired in the presence of noise. Most methods in the literature aim to identify the signal from their large components in some transform domain (corresponding to high energy regions as in (block-)thresholding methods \[14, 38\], curves as in “synchrosqueezing” \[11\] or “reassignment” \[2\] methods \[3\], or ridges \[8, 13\]). In sharp contrast, Flandrin introduced a novel method to identify a signal embedded in Gaussian white noise based on its “silent points”—the zeros of the short-time Fourier transform (STFT) \[20\]. He studied the “loud-silent” dichotomy represented by maxima and minima of Gaussian spectrograms based on observation of their geometric distribution \[21, 23\]. Flandrin’s analysis is based on the fact that, up to a non-vanishing factor, an STFT with a Gaussian window is an entire function and therefore complex analytic methods are available. (This fact is also the basis of the characterization of Gabor frames and Riesz sequences with Gaussian windows \[25, 29, 35\]. See also \[4\] for the structure

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of the phase derivative around the zeros of the STFT and \[1\] for other random point processes associated to the STFT.)

In Flandrin’s method, the spectrogram of the clean signal is singled out as an area of statistical deviation from the pattern expected from noise. The intuition behind the filtering procedure is that, while the zeros of the STFT of pure Gaussian white noise with a Gaussian window are distributed according to a very regular random pattern in the plane, the presence of a deterministic signal perturbs that pattern, and has a repulsive effect on the random point configuration. This insight has been recently revisited in \[5\], by noting that the spectrogram of white noise with respect to a Gaussian window is a symmetric Gaussian entire function—i.e., a random analytic function \( \sum_j a_j z^{\alpha_j} \), where \( a_j \) are i.i.d. real Gaussian random variables—and thus its zero-set obeys well-known statistics \[19\].

In this note we propose a scheme similar to Flandrin’s, based on a continuous wavelet transform \[10\, Ch. 2\] with analyzing wavelets of the form

\[
\hat{g}_\alpha(\xi) := \begin{cases} 
\frac{2^{\alpha-1}}{\sqrt{\Gamma(\alpha-1)}} \xi^{\alpha-1} e^{-\xi}, & \xi \geq 0, \\
0, & \xi < 0,
\end{cases}
\]

with \( \alpha > 1 \). The starting point of our analysis is the observation that these windows lead to wavelet transforms that map into a weighted space of analytic functions that can be identified with a Bergman space in the upper half-plane \[12\]—see also Section 2 and the Appendix. As a consequence, we identify the point process arising from the zeros of the scalograms of (complex) white noise with the zero set of a so-called hyperbolic Gaussian analytic function (GAF), and use this information to propose an adequate filtering procedure for the wavelet transform.

The expected number of the zeros of hyperbolic GAFs within a certain domain is proportional to its hyperbolic area and this information is instrumental in the design of a wavelet filtering mask. Crucially, we define a transformation that accounts for the hyperbolic geometry and uniformizes the distribution of the zeros. After this step, we proceed as in \[20\]: we test for a deviation on the distribution expected from white noise, by selecting significant triangles on a Euclidean Delaunay triangulation of the uniformized set of zeros.

The paper is organized as follows. In Section 2 we introduce the function spaces and transforms underlying the specific wavelet transforms considered in this manuscript. In Section 3 we collect some basic properties of Gaussian analytic functions that will be needed for our filtering approach. White noise and its wavelet transform are introduced and discussed in Section 4. Finally, in Section 5 we present the filtering procedure and an explicit example is given in Section 6.
2. Transforms and Function Spaces

2.1. Analytic wavelets. We will be interested in so-called analytic wavelets, i.e., wavelets $g$ that belong to the Hardy space

$$H^2(\mathbb{R}) = \{ f \in L^2(\mathbb{R}) : \hat{f}(\xi) = 0, \text{ for } \xi < 0 \},$$

and satisfy the usual admissibility condition

$$c_g := \int_{\mathbb{R}} |\hat{g}(\xi)|^2 |\xi|^{-1} d\xi < \infty,$$

where the Fourier transform is normalized by

$$\hat{g}(\xi) = \mathcal{F}g(\xi) = \int_{\mathbb{R}} g(t) e^{-i\xi t} dt.$$ We regard the corresponding wavelet transform of a function $f \in L^2(\mathbb{R})$,

$$W_g f(x, y) := \frac{1}{\sqrt{y}} \int_{\mathbb{R}} f(t) \overline{g(t - x/y)} dt, \quad x \in \mathbb{R}, y > 0,$$

as a function on the upper half-plane

$$\Pi^+ := \{ z = x + iy \in \mathbb{C} : y > 0 \},$$

by encoding the joint time-scale parameter into a complex number $z = x + iy$. Equivalently, the corresponding wavelet transform can also be written in the frequency domain as

$$(2) \quad W_g f(x, y) = \sqrt{y} \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi) e^{\frac{i2\pi}{y} \xi} \overline{\hat{g}(y \xi)} d\xi.$$

If a wavelet $g$ is further normalized by $c_g = 1$, it is known that the wavelet transform maps $H^2(\mathbb{R}) \to L^2(\Pi^+, y^{-2}dxdy)$ isometrically [9, Prop. 15.1.1].

The use of analytic wavelets $g$ implies that $W_g f$ discards the negatives frequencies of $f$. However, for real-valued $f$, this does not lead to a loss of information, because $\hat{f}(-\xi) = \overline{\hat{f}(\xi)}$. In particular, $f$ can be recovered from its projection onto $H^2(\mathbb{R})$ by

$$(3) \quad f(x) = P_{H^2} f(x) + P_{H^2} f(-x).$$

In the following, we will choose $g$ given by (1), where $\alpha > 1$. These are special cases of the so-called Morse wavelets. (Note that the limit case $g_1$ is not admissible.) Following [2], the corresponding wavelet transform is given as

$$(4) \quad W_{g_{\alpha}} f(x, y) = \frac{2^{\frac{\alpha+1}{2}} y^{\frac{\alpha}{2}}}{2\pi \sqrt{\Gamma(\alpha - 1)}} \int_{\mathbb{R}^+} \xi^{(\alpha-1)/2} e^{i\xi(x+iy)} \hat{f}(\xi) d\xi.$$
2.2. **Spaces of Analytic Functions.** The Hardy space on the upper half-plane \( H^2(\Pi^+) \) is defined as the space of all analytic functions \( f \) on \( \Pi^+ \) such that the norm

\[
\| f \|_{H^2(\Pi^+)}^2 = \sup_{y > 0} \frac{1}{\pi} \int_{\mathbb{R}} |f(x + iy)|^2 \, dx
\]

is finite. This space is closely related to the one on the real line: any function in \( H^2(\Pi^+) \) has boundary values for almost every point in \( \mathbb{R} \) in the sense of non-tangential limits, and the boundary function then belongs to \( H^2(\mathbb{R}) \). In the other direction, the Cauchy transform of a function \( f \in L^2(\mathbb{R}) \) is defined by

\[
Cf(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{1}{t - z} f(t) \, dt, \quad z \in \Pi^+.
\]

For \( f \in H^2(\mathbb{R}) \), \( Cf \) is a function in \( H^2(\Pi^+) \). Furthermore, for a function \( f \in L^2(\mathbb{R}) \ominus H^2(\mathbb{R}) \), we have \( Cf = 0 \) on \( \Pi^+ \).

Similarly, the Hardy space on the unit disk \( H^2(D) \) is defined as the space of all analytic functions \( f \) on \( D \) such that the norm

\[
\| f \|_{H^2(D)}^2 = \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta
\]

is finite. An orthonormal basis is given by the monomials \( \{ z^n : n \geq 0 \} \). For more background on Hardy spaces, see e.g. [16].

For \( \alpha > 1 \), the Bergman space on the upper half-plane \( A_\alpha^2(\Pi^+) \) is defined as the space of all analytic functions \( f \) on \( \Pi^+ \) such that the norm

\[
\| f \|_{A_\alpha^2(\Pi^+)}^2 = \frac{1}{\pi} \int_{\Pi^+} |f(x + iy)|^2 y^{\alpha - 2} \, dm(z)
\]

is finite. Here, \( dm(z) = dx dy \) is the Lebesgue measure. All the spaces mentioned above are reproducing kernel Hilbert spaces. We will need the kernels of \( H^2(\Pi^+) \) and \( A_\alpha^2(\Pi^+) \), which are given by

\[
K_H(z, w) = \frac{i}{2(z - \bar{w})}
\]

and [18]

\[
K_\alpha(z, w) = \frac{2^{\alpha - 2}(\alpha - 1)}{(-i)^\alpha (z - \bar{w})^\alpha},
\]

respectively. Bergman spaces play an essential role in our analysis as the range of the wavelet transform with a window \( g_\alpha \), \( \alpha > 1 \) consists precisely of functions of the form \( y^{\alpha/2} f \), where \( f \) belongs to the weighted Bergman space \( A_\alpha^2(\Pi^+) \) [33,34].

The Bergman space on the unit disc, \( A_\alpha^2(D) \) is defined as the space of all analytic functions \( f \) on \( D \) such that the norm

\[
\| f \|_{A_\alpha^2(D)}^2 = \frac{1}{\pi} \int_D |f(z)|^2 (1 - |z|^2)^{\alpha - 2} \, dm(z)
\]
is finite. The reproducing kernel of $A^2_\alpha(\mathbb{D})$ is \([26]\)

$$K^\alpha(z, w) = \frac{\alpha - 1}{(1 - \overline{z}w)^\alpha}.$$ \(8\)

The following normalized monomials form an orthonormal basis of $A^2_\alpha(\mathbb{D})$:

$$\varphi^\alpha_n(z) = \sqrt{\frac{\Gamma(n + \alpha)}{n!\Gamma(\alpha - 1)}} z^n.$$ \(9\)

Based on a conformal map between $\mathbb{D}$ and $\Pi^+$, we define the isometric isomorphism $T^\alpha: A^2_\alpha(\Pi^+) \to A^2_\alpha(\mathbb{D})$ by

$$T^\alpha f(z) = \frac{2}{(1 - z)^\alpha} f \left( \frac{z + 1}{1 - z} \right).$$ \(10\)

The pullback $\Phi^\alpha_n$ of the orthonormal basis $\varphi^\alpha_n$ then satisfies

$$T^\alpha \Phi^\alpha_n(z) = \varphi^\alpha_n(z) = \sqrt{\frac{\Gamma(n + \alpha)}{n!\Gamma(\alpha - 1)}} z^n,$$ \(11\)

and $\Phi^\alpha_n(z)$ has the explicit form

$$\Phi^\alpha_n(z) = 2^{\alpha - 1} \sqrt{\frac{\Gamma(n + \alpha)}{n!\Gamma(\alpha - 1)}} \left( \frac{z - i}{z + i} \right)^n \left( \frac{i}{z + i} \right)^\alpha.$$ \(12\)

Similarly to the Bergman space case, we have an isometric isomorphism between the Hardy spaces $T^H: H^2(\Pi^+) \to H^2(\mathbb{D})$:

$$(T^H f)(z) = \frac{2}{(1 - z)^\alpha} f \left( \frac{z + 1}{1 - z} \right).$$ \(13\)

Consequently, the orthonormal basis of $H^2(\Pi^+)$ that corresponds to the monomial basis on the unit disk has the form

$$\Phi^H_n(z) := \left( \frac{z - i}{z + i} \right)^n \frac{i}{z + i}.$$ \(14\)

3. Gaussian Analytic Functions

In this section, we collect a few facts about Gaussian analytic functions (GAFs) that will be needed for the filtering procedure. A good reference on this topic is \([27]\). The general definition of GAFs involves a reproducing kernel Hilbert space $\mathcal{H}$ of analytic functions on a domain $D$ in the complex plane (for instance, $D = \mathbb{C}$, $D = \Pi^+$, or $D = \mathbb{D}$). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. By a standard complex Gaussian, we mean a complex valued random variable with distribution $\frac{1}{\pi} e^{-|z|^2}$, or equivalently, a variable of the form
$X + iY$ where $X$ and $Y$ are independent real Gaussian random variables with zero mean and variance $1/2$. For an orthonormal basis $e_n$ of $\mathcal{H}$, we can define

\begin{equation}
(15)
f(z) = \sum_n a_n e_n(z),
\end{equation}

with $\{a_n : n \geq 0\}$ a sequence of independent standard complex Gaussians and where the sum is either finite or infinite depending on the dimension of $\mathcal{H}$. The series $\text{(15)}$ converges almost surely and locally uniformly to an analytic function \cite{27}; we think of $f$ as a random analytic function. One can verify that

$$
\mathbb{E}(f(z)f(w)) = K(z, w)
$$

where $K$ is the reproducing kernel of $\mathcal{H}$. More generally, the random vector $(f(z_1), \ldots, f(z_N))$ is a complex Gaussian with a covariance matrix

$$
\det[K(z_j, z_k)]_{1 \leq j, k \leq N}.
$$

For this reason, $K$ is often called the covariance kernel of $f$. In particular, the statistics of the random analytic function $f$ depend on the space $\mathcal{H}$, but not on the particular choice of orthonormal basis.

The central object of interest in the theory of GAFs is their zero set, which is a random point configuration in the plane. We can think of it as a random integer-valued measure, by setting a Dirac mass at each zero. This point process is simple \cite[ Lem. 2.4.1]{27}, which means that singletons have at most measure 1. For $D = \Pi^+$, the first intensity function of the zero set of $f$ is the function $\rho$ satisfying

$$
\mathbb{E}\#\{w \in U : f(w) = 0\} = \int_U \rho(z) dm(z),
$$

for every measurable subset $U \subseteq \Pi^+$. The first intensity of the zero set of GAFs exists and can be computed from the Edelman-Kostlan formula \cite{17, 27, 37} as

$$
\rho(z) = \frac{1}{4\pi} \Delta \log K(z, z).
$$

We will focus on the cases where $\mathcal{H}$ is either one of the Bergman spaces $A^2_\alpha(\Pi^+)$ or the Hardy space $H^2(\Pi^+)$. The corresponding random analytic functions are denoted by $f_\alpha$ and $f_H$. These are called hyperbolic GAFs. The first intensities corresponding to the Bergman and Hardy spaces on $\Pi^+$ are, respectively,

\begin{equation}
(16)\rho_\alpha(z) = \frac{\alpha}{4\pi(\text{Im} z)^2}
\end{equation}

and

\begin{equation}
(17)\rho_H(z) = \frac{1}{4\pi(\text{Im} z)^2}.
\end{equation}
This means that the zeros of GAFs are distributed according to a multiple of the hyperbolic area density on the upper half-plane. Besides this rough description, the zeros of GAFs are known to be quite rigid: the events where the concentration of the zeros deviates significantly from what is prescribed by the first intensity are very unlikely (see, e.g., the large deviations estimates by Sodin [37] and Offord [30]). However, it was recently observed in the Euclidean (non-hyperbolic) setting that strong deviations up to a "forbidden zone" with hardly any zeros can appear, if the distribution is conditioned on a hole event [24].

4. White Noise, Zeros of Scalograms, and GAFs

4.1. White Noise. We adopt a Gaussian Hilbert space approach to white noise [28]. Let \((\Omega, \mathcal{F}, P)\) be a probability space. Heuristically, one thinks of white noise on \(\mathbb{R}\) as a
\[
N = \sum_{n=0}^{\infty} a_n e_n
\]
where \(a_n\) are independent standard (real or complex) Gaussians and \(\{e_n : n \geq 0\}\) is an orthonormal basis of \(L^2(\mathbb{R})\). With probability 1, this sum does not converge in \(L^2(\mathbb{R})\). However, for any \(f \in L^2(\mathbb{R})\), the sum
\[
N(f) := \sum_{n=0}^{\infty} a_n \langle f, e_n \rangle
\]
converges in \(L^2(\Omega, \mathcal{F}, P)\) to a complex Gaussian variable with mean zero and variance \(\|f\|^2\). A precise definition of white noise is then as the collection of random variables \(G:=\{N(f) : f \in L^2(\mathbb{R})\}\). The space \(G\) is a Gaussian Hilbert space, that is, a Hilbert space consisting of Gaussian random variables. Its inner product is induced by
\[
\|N(f)\|^2_G := \|f\|^2.
\]
We will call the white noise real or complex depending on whether the variables \(a_n\) are real or complex standard Gaussians. The definition is independent of the choice of orthonormal basis.

Alternatively, white noise could be defined as a random element in some larger class, for example tempered distributions, but, with such definition, \(N(f)\) would not be defined for all \(f \in L^2(\mathbb{R})\).

4.2. Wavelet Transform of White Noise. Let \(\{e_n : n \geq 0\}\) be an orthonormal basis of \(L^2(\mathbb{R})\). The wavelet transform with respect to the windows \(g_\alpha\) of (real or complex) white noise are formally defined by
\[
W_{g_\alpha}(\mathcal{N})(z) = \langle \mathcal{N}, y^{-1/2}g_\alpha((\cdot - x)/y) \rangle = \sum_{n=0}^{\infty} a_n W_{g_\alpha} e_n(z),
\]
where \(z = x + iy\). By the isometry property of the wavelet transform, \(\{W_{g_\alpha}(e_n) : n \geq 0\}\) is an orthonormal set in \(L^2(\Pi^+, y^{-2}dx dy)\). Furthermore, the functions \(\{y^{\alpha/2}W_{g_\alpha} e_n(z) : \)
$n \geq 0}\} form an orthonormal basis of the Bergman space $A^2_\alpha$. Therefore, as discussed in Section 4.1, the sum in (18) converges almost surely locally uniformly and, when $a_j$ are standard complex Gaussians, it is a realization of $y^{\alpha/2} f_\alpha(z)$, where $f_\alpha$ is the hyperbolic GAF. We conclude the following.

**Proposition 4.1.** For $\alpha > 1$, the set of zeros of $W_{g_\alpha}(N)$, where $N$ is complex white noise, has the same distribution as those of $f_\alpha$, the hyperbolic GAF associated with the Bergman space $A^2_\alpha$.

To make the connection to [6,7,31] more explicit, we can choose the following orthonormal basis in $H^2(R)$ (this basis also appears in the analysis of time-frequency localisation operators [10,12]):

$$
\Psi^\alpha_n(t) := 2\pi F^{-1} l^\alpha_n(t),
$$

where $F^{-1}$ is the inverse Fourier transform and $l^\alpha_n$ are the Laguerre functions

$$
l^\alpha_n(\xi) := \begin{cases} 
\sqrt{n!} \pi^{-\alpha/2} \frac{\Gamma(n+\alpha)}{2^{n+\alpha}} (2\xi)^{\alpha-1} e^{-\xi} L_{n-1}^{\alpha-1}(2\xi), & \xi > 0, \\
0, & \xi < 0.
\end{cases}
$$

Here, $L^\alpha_n$ denotes the associated Laguerre polynomial. We will show in Appendix A that the wavelet transform maps this basis to the basis $\Phi^\alpha_n$ of $A^2_\alpha(\Pi^+)$ given in (12), up to a multiplicative factor. More precisely, we have

$$
\sqrt{\pi y^{-\alpha/2}} W_{g_\alpha} \Psi^\alpha_n(z) = \Phi^\alpha_n(z).
$$

Consequently, we obtain

$$
\sqrt{\pi y^{-\alpha/2}} W_{g_\alpha}(N)(z) = \sum_{n=0}^{\infty} a_n \sqrt{\pi y^{-\alpha/2}} W_{g_\alpha} \Psi^\alpha_n(z)
$$

(19)

$$
= \sum_{n=0}^{\infty} a_n 2^{\alpha-1} \sqrt{\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha-1)}} \left(\frac{z-i}{z+i}\right)^{n} \left(\frac{i}{z+i}\right)^{\alpha}.
$$

Finally, using (11), we obtain the following connection to the well-known hyperbolic GAF on the disk

$$
T^\alpha \left(\sqrt{\pi y^{-\alpha/2}} W_{g_\alpha}(N)\right)(z) = \sum_{n=0}^{\infty} a_n \sqrt{\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha-1)}} z^n.
$$

### 4.3. Cauchy Transform of White Noise

The Cauchy transform corresponds formally to the wavelet transform with $g_1$:

$$
W_{g_1} f(z) = 2\pi \sqrt{y} C f(z).
$$

Since the window $g_1$ is not admissible, we cannot speak of the wavelet transform in the normal sense. Instead, we consider the Cauchy transform as an isometric isomorphism from $H^2(R)$ to $H^2(\Pi^+)$. Recall also that $C f = 0$ on $\Pi^+$ for $f \in L^2(R) \ominus H^2(R)$. We
can therefore conclude, as we did for the wavelet transform, that complex white noise is mapped to the random analytic function

\[ f_H := \sum_n a_n e_{H,n}(z) \]

where \( e_{H,n} \) is an orthonormal basis of the Hardy space \( H^2(\Pi^+) \). To give an explicit form, we can use the basis in \([14]\):

\[ \Phi_n^H(z) = \left( \frac{z - i}{z + i} \right)^n \left( \frac{i}{z + i} \right). \]

The GAF \( f_H \) is especially interesting because, due to a result of Shirai \([36]\), based on a result of Peres and Virag \([31]\), it is a determinantal point process (DPPs) \([27]\), and, therefore, all statistical quantities have very explicit forms.

5. Filtering

5.1. Wavelet Masking. We consider a square-integrable real-valued function \( f : \mathbb{R} \to \mathbb{R} \) contaminated with white noise \( \tilde{f} = f + N \). For our theoretical analysis we need to assume that \( N \) is complex white noise. Although in practice the difference between real and complex noise seems to be minor, let us notice that if \( f \) is contaminated with real white noise, we can artificially add an imaginary part with simulated white noise, without altering the amount of available information.

The key task is to design a wavelet mask \( m : \mathbb{R} \times (0, \infty) \to \mathbb{R} \) that discards those parts of the scalogram of \( \tilde{f} \) where the statistics of white noise can be recognized. The goal is that

\[ mW_{g_\alpha} \tilde{f}(t) \approx W_{g_\alpha} f. \]

Once this is achieved, the wavelet multiplier

\[ M_m \tilde{f}(t) = W^*_{g_\alpha} mW_{g_\alpha} \tilde{f}(t) = y^{-1/2} \int_0^\infty \int_{\mathbb{R}} m(x, y) W_{g_\alpha} \tilde{f}(x, y) g_\alpha \left( \frac{t-x}{y} \right) dx \frac{dy}{y^2} \]

provides the approximation

\[ (20) \quad M_m \tilde{f} \approx W^*_{g_\alpha} W_{g_\alpha} f = c_{g_\alpha} P_{H^2} f. \]

Here, \( P_{H^2(\mathbb{R})} \) is the orthogonal projection onto \( H^2(\mathbb{R}) \) and \( W^*_{g_\alpha} \) is the adjoint of \( W_{g_\alpha} \) considered as a mapping from \( L^2(\mathbb{R}) \) to \( L^2(\Pi^+, y^{-2} dx dy) \); see, e.g., \([9\text{, Ch. 15]}\) and \([12]\).

As a second step, we can approximately recover \( f \) using \([3]\), because \( f \) is real-valued.
5.2. Design of the Filtering Mask. We start by computing the zeros of the scalogram,

\[ \mathcal{Z} = \{ z \in \Pi^+ : W_{g_0} \tilde{f}(z) = 0 \}. \]

This requires some care because a simple thresholding can be very unstable. We use the fact that the function \( y^{\alpha/2} \cdot W_{g_0} f(x, y) \) is analytic, and, therefore, every local minimum of \( y^{\alpha/2}|W_{g_0} f(x, y)| \) is a zero. Indeed, for an analytic function \( F \), the function \( \log |F| \) is harmonic outside the set of zeros of \( F \) and therefore satisfies the minimum principle. In practice, this computation is carried out on a grid by comparing each value of \( |F| \) to its immediate neighbors, see Section 6.

The mask \( m \) is designed as follows. Due to Proposition 4.1 and the Edelman-Kostlan formula (16), in the absence of signal \( f \), the set \( \mathcal{Z} \) is distributed according to a multiple of the hyperbolic area measure \( y^{-2} dxdy \). To account for this geometry, we apply the map

\[ M: (x, y) \mapsto \left( x, \frac{1}{y} \right) \]

to \( \mathcal{Z} \).

**Proposition 5.1.** In the absence of a signal (\( f = 0 \)), the expected number of points of \( M \mathcal{Z} \) inside a measurable domain of the upper half-plane is proportional to its Lebesgue measure.

In more technical terms, if we denote by \( \nu \) the counting measure of the zero set of a hyperbolic GAF, the pushforward point process \( \tilde{\nu} := \nu \circ M^{-1} \) has constant one-point intensity.

**Proof.** Let us denote the one-point intensities of the point processes \( \nu \) and \( \tilde{\nu} \) by \( \rho_{\nu} \) and \( \rho_{\tilde{\nu}} \) respectively. Based on (16) and (17), we can write \( \rho_{\nu}(x, y) = c y^{-2} dxdy \), with \( c > 0 \). We note that the absolute value of the Jacobian matrix of the map \( M^{-1} \) is \( |\det DM^{-1}(u, v)| = v^{-2} \). We also have \( \rho_{\nu}(M^{-1}(u, v)) = cv^2 \). Hence, for a measurable set \( S \subseteq \Pi^+ \),

\[
\int_S \rho_{\tilde{\nu}}(u, v)dudv = \mathbb{E} \int_S d\tilde{\nu} = \mathbb{E} \int_{M^{-1}(S)} d\nu = \int_{M^{-1}(S)} \rho_{\nu}(x, y)dxdy = \int_S \rho_{\nu}(M^{-1}(u, v))|\det DM^{-1}(u, v)|dudv = c \int_S dudv.
\]

As this holds for all measurable subsets \( S \), we conclude that \( \rho_{\tilde{\nu}} = c \).
In Figure 1, we show a realization of the zeros of white noise in the hyperbolic setting for $\alpha = 5$. The effect of Proposition 5.1 is illustrated in Figure 2, where the zeros of the same realization of white noise are shown with the $y$-axis inverted.

To detect the signal, we scan for deviations of the stochastics prescribed by Proposition 5.1. Following [20], we compute a Delaunay triangulation of $MZ$, and select those triangles whose perimeter exceeds a certain threshold. The mask $m$ is defined as the characteristic function of the union of the selected triangles. A detailed example is provided in the following section.

6. Numerical Experiments

We consider a clean signal $f$ represented by $L = 128 \cdot 1024$ time samples and a sampling rate of $r_s = 1024$. The clean signal at time $t = n/1024$ is given as

$$f(t) = \sin \left(2\pi(5 + (1/32 \cdot t)^2)t\right)e^{-(t-48)^2/16^2}$$

We model noise by attaching to each sample independently a standard complex Gaussian random variable. Adding this sequence of i.i.d. noise random variables to the clean signal,
we obtain the noisy signal \( f_{no} \). The variance of the noise is chosen such that the signal to noise ratio is 0.2.

Recall that we want to simulate the wavelet transform

\[
W_{g_{\alpha}} f(x, y) = \frac{1}{\sqrt{y}} \int_{0}^{\infty} f(t) \overline{g_{\alpha}} \left( \frac{t - x}{y} \right) dt.
\]

We consider the case \( \alpha = 5 \) where we can explicitly calculate

\[
g_5(t) = \frac{4}{\sqrt{6\pi}} \frac{1}{(1 - it)^{3/2}}.
\]

One can check that about 90% of the energy of \( g_5 \) is contained in the interval \([-2, 2]\). We approximate:

\[
W_{g_5} f(x, y) \approx \frac{1}{\sqrt{y}} \int_{0}^{L/r_s} f(t) g_5 \left( \frac{x - t}{y} \right) dt
\]
for $x \in [2y, \frac{L}{rs} - 2y]$. We can rewrite (21) as

\[
\frac{1}{\sqrt{y}} \int_0^{L/rs} f(t)g_5 \left( \frac{x-t}{y} \right) dt = \frac{\sqrt{y}}{2\pi} \int_0^{\infty} \hat{g}_5(s'y)e^{is'x}\hat{f}(s')ds'.
\]

We further approximate:

\[
\hat{f}(s') = \int_0^{L/rs} f(t)e^{-its'} dt \approx \sum_{n=1}^{L} e^{-i\frac{n}{rs}s'} \int_{(n-1)/rs}^{n/r} f(t) dt.
\]

We denote the emerging discrete representation of the signal as $f[n] = \int_{(n-1)/rs}^{n/r} f(t) dt$. Using the discrete Fourier transform, we can thus approximate $\hat{f}(s')$ at particular values of $s'$ as

\[
\hat{f}(s') \approx \sum_{n=1}^{L} f[n]e^{-2\pi i \frac{n}{rs}s'} =: \hat{f}[m]
\]

where $m = \frac{s'L}{2\pi rs}$ and $m = 1, \ldots, L$. Based on this discretization, we can approximate (22) as

\[
\sqrt{y} \int_0^{\infty} \hat{g}_5(s'y)e^{is'x}\hat{f}(s')ds' \approx \frac{\sqrt{y}}{2\pi} \sum_{m=1}^{L} \hat{g}_5 \left( \frac{2\pi rsy}{L} \right) e^{2\pi i \frac{rsy}{L}x}\hat{f}[m]
\]

\[
= \frac{\sqrt{y}}{2\pi} \sum_{m=1}^{L} \hat{g}_5 \left( \frac{2\pi rsy}{L} \right) \hat{f}[m]e^{2\pi i \frac{rsy}{L}x}
\]

which is the inverse discrete Fourier transform of $m \mapsto \sqrt{y}g_5 \left( \frac{2\pi rsy}{L} \right) \hat{f}[m]$ at $n = xr_s$. Summarizing, we take a DFT of the discrete signal representation, multiply pointwise with $\hat{g}_5$ for various scales $y$ and take the IDFT of the resulting vector. As, according to Proposition 5.1, we expect the zeros of pure noise to have a constant one-point intensity if we use a reciprocal scale for the $y$-axis. We choose 1000 equidistant values for $1/y$ on the interval $[\pi, 41\pi]$.

We next estimate local minima of this discretized continuous wavelet transform by comparing each value with the four neighboring points. A point is classified as a zero if it is smaller than any of its 4 neighbors, cf. Section 5.2. A Delaunay triangulation is used to connect the resulting zeros and a threshold of $3\alpha = 15$ on the perimeter of the triangles is used to find the region where the signal has significant components. Based on these triangles a mask of the spectrogram is calculated and the signal reconstructed following (20) and (3). This procedure is illustrated in Figure 4 and a comparison of the clean signal, the noisy signal, and the reconstructed signal is given in Figure 3. The signal-to-noise ratio after filtering is over several realizations of white noise approximately 5.
Figure 3. Comparison of the clean signal, noisy signal, and filtered signal.

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Appendix A. Wavelet Transform of an Explicit Basis

We first argue that the functions

$$\Psi_n^\alpha(t) = \int_0^\infty \sqrt{\frac{n!}{\pi\Gamma(n+\alpha)}} (2\xi)^{\frac{\alpha-1}{2}} e^{-\xi L_n^{\alpha-1}(2\xi)} e^{it\xi} dt$$

are indeed orthonormal. To this end, we first note that

$$\hat{\Psi}_n^\alpha(\xi) = \sqrt{\frac{4\pi n!}{\Gamma(n+\alpha)}} (2\xi)^{\frac{\alpha-1}{2}} e^{-\xi L_n^{\alpha-1}(2\xi)}.$$

Due to the orthogonality property of the Laguerre functions, we have (see \[\text{[15, eq. (6)]}\])

$$\int_0^\infty e^{-\xi L_n^{\alpha-1}(\xi)} L_m^{\alpha-1}(\xi) d\xi = \frac{\Gamma(n+\alpha)}{n!} \delta_{nm}.$$
**Figure 4.** The clean signal (scalogram top left) is corrupted by noise (scalogram top right). The zeros of the scalogram are calculated and a Delaunay triangulation (mid left) captures the signal region with large triangles, found by a thresholding of the perimeters (mid right). Finally, the scalogram is filtered using the chosen triangles as mask (bottom) and the signal reconstructed.
and, in turn,

\[
\int_0^\infty \widehat{\Psi}_n^\alpha(\xi) \overline{\widehat{\Psi}_m^\alpha(\xi)} d\xi = 2\pi \sqrt{\frac{n!m!}{\Gamma(n+\alpha)\Gamma(m+\alpha)}} \int_0^\infty \xi^{\alpha-1} e^{-\xi} L_n^{\alpha-1}(\xi)L_m^{\alpha-1}(\xi) d\xi = 2\pi \delta_{nm}.
\]

Recalling that we defined the Fourier transform not unitary, we easily see that this results in

\[
\int_\mathbb{R} \Psi_n^\alpha(t) \overline{\Psi_m^\alpha(t)} dt = \delta_{nm}.
\]

We next want to show that \(\sqrt{\pi}y^{-\alpha/2}W_{g_{\alpha}} \Psi_n^\alpha(z) = \Phi_n^\alpha(z)\). Starting from (4), we obtain

\[
W_{g_{\alpha}} \Psi_n^\alpha(x,y) = \frac{2^{\alpha-\frac{1}{2}} y^{\alpha}}{2\pi \Gamma(\alpha-1)^{\frac{\alpha}{2}}} \int_{\mathbb{R}^+} \xi^{\alpha-1} e^{i\xi(x+iy)} \sqrt{\frac{4\pi n!}{\Gamma(n+\alpha)}} (2\xi)^{\alpha-1} e^{-\xi} L_n^{\alpha-1}(2\xi) d\xi
\]

\[
= \sqrt{\frac{y^n n!}{\pi \Gamma(\alpha-1) \Gamma(n+\alpha)}} \int_{\mathbb{R}^+} (2\xi)^{\alpha-1} e^{i\xi(x+iy)} e^{-\xi} L_n^{\alpha-1}(2\xi) d\xi
\]

\[
= 2^{\alpha-1} \sqrt{\frac{y^{n} \Gamma(n+\alpha)}{\pi n! \Gamma(\alpha-1)}} \left(\frac{z-i}{z+i}\right)^n \left(\frac{i}{z+i}\right)^\alpha
\]

\[
= y^{\frac{\alpha}{2}} \Phi_n^\alpha(z),
\]

where we used in (28) the Fourier transform of the Laguerre functions given in [15, Lem. 1].

**References**

[1] L. D. Abreu, J. M. Pereira, J. L. Romero, and S. Torquato. The Weyl-Heisenberg ensemble: hyperuniformity and higher Landau levels. *J. Stat. Mech. Theory Exp.*, 2017(4), 2017.

[2] F. Auger and P. Flandrin. Improving the readability of time-frequency and time-scale representations by the reassignment method. *IEEE Trans. Signal Process.*, 43(5):1068–1089, May 1995.

[3] F. Auger, P. Flandrin, Y.-T. Lin, S. Mclaughlin, S. Meignen, T. Oberlin, and H.-T. Wu. Time-frequency reassignment and synchrosqueezing: An overview. *IEEE Signal Process. Mag.*, 30(6):32–41, Nov. 2013.

[4] P. Balazs, D. Bayer, F. Jaillet, and P. Soendergaard. The pole behaviour of the phase derivative of the short-time Fourier transform. *Appl. Comput. Harmon. Anal.*, 30(3):610–621, May 2016.

[5] R. Bardenet, J. Flamant, and P. Chainais. On the zeros of the spectrogram of white noise. *arXiv preprint:1708.00082*, 2017.

[6] J. Buckley. Fluctuations in the zero set of the hyperbolic Gaussian analytic function. *Int. Math. Res. Not. IMRN*, 2015(6):1666–1687, Jan. 2015.

[7] J. Buckley, A. Nishry, R. Peled, and M. Sodin. Hole probability for zeroes of Gaussian Taylor series with finite radii of convergence. *Probab. Theory Related Fields*, 171(1–2):377–430, June 2018.

[8] R. A. Carmona, W. L. Hwang, and B. Torresani. Characterization of signals by the ridges of their wavelet transforms. *IEEE Trans. Signal Process.*, 45(10):2586–2590, Oct. 1997.

[9] O. Christensen. *An introduction to frames and Riesz bases*. Applied and Numerical Harmonic Analysis. Birkhäuser, second edition, 2016.
[10] I. Daubechies. *Ten Lectures on Wavelets*. SIAM, 1992.
[11] I. Daubechies and S. Maes. A nonlinear squeezing of the continuous wavelet transform based on auditory nerve models. In A. Aldroubi and M. Unser, editors, *Wavelets in Medicine and Biology*, pages 527–546. CRC Press, Boca Raton, FL, 1996.
[12] I. Daubechies and T. Paul. Time-frequency localisation operators—a geometric phase space approach: II. The use of dilations. *Inverse Probl.*, 4(3):661–680, Aug. 1988.
[13] N. Delprat, B. Escudie, P. Guillemin, R. Kronland-Martinet, P. Tchamitchian, and B. Torresani. Asymptotic wavelet and Gabor analysis: Extraction of instantaneous frequencies. *IEEE Trans. Inf. Theory*, 38(2):644–664, Mar. 1992.
[14] D. L. Donoho and I. M. Johnstone. Ideal spatial adaptation by wavelet shrinkage. *Biometrika*, 81(3):425–455, Sept. 1994.
[15] P. Duren, E. A. Gallardo-Gutierrez, and A. Montes-Rodriguez. A Paley-Wiener theorem for Bergman spaces with application to invariant subspaces. *Bull. London Math. Soc.*, 39(3):459–466, 2007.
[16] P. L. Duren. *Theory of $H^p$ spaces*. Pure and Applied Mathematics, Vol. 38. Academic Press, New York, 1970.
[17] A. Edelman and E. Kostlan. How many zeros of a random polynomial are real? *Bulletin of the American Mathematical Society*, 32(1):1–37, 1995.
[18] S. J. Elliott and A. Wynn. Composition operators on weighted Bergman spaces of a half-plane. *Proc. Edinb. Math. Soc. (2)*, 54(2):373–379, 2011.
[19] N. D. Feldheim. Zeroes of Gaussian analytic functions with translation-invariant distribution. *Israel J. Math.*, 195(1):317–345, 2013.
[20] P. Flandrin. Time-frequency filtering based on spectrogram zeros. *IEEE Signal Process. Lett.*, 22(11):2137–2141, 2015.
[21] P. Flandrin. The sound of silence: Recovering signals from time-frequency zeros. In *Proc. Asilomar Conf. Signals Syst. Comput.*, Pacific Grove, CA, Nov. 2016.
[22] P. Flandrin. On spectrogram local maxima. In *Proc. IEEE Int. Conf. Acoust. Speech Signal Process.*, pages 3979–3983, New Orleans, LA, Mar. 2017.
[23] T. J. Gardner and M. O. Magnasco. Sparse time-frequency representations. *Proc. Natl. Acad. Sci. U.S.A.*, 103(16):6094–6099, Apr. 2006.
[24] S. Ghosh and A. Nishry. Gaussian complex zeros on the hole event: the emergence of a forbidden region. *arXiv preprint:1609.00084*, 2016. Comm. Pure Appl. Math., to appear.
[25] K. Gröchenig. *Foundations of Time-Frequency Analysis*. Appl. Numer. Harmon. Anal. Birkhäuser, 2001.
[26] H. Hedenmalm, B. Korenblum, and K. Zhu. *The Theory of Bergman Spaces*. Springer, New York, NY, 2000.
[27] J. B. Hough, M. Krishnapur, Y. Peres, and B. Virág. Gaussian complex zeros on the hole event: the emergence of a forbidden region. *arXiv preprint:1609.00084*, 2016. Comm. Pure Appl. Math., to appear.
[28] S. Janson. *Gaussian Hilbert spaces*, volume 129 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1997.
[29] Y. Lyubarskii. Frames in the Bargmann space of entire functions. In B. Y. Levin, editor, *Entire and subharmonic functions*, volume 11 of *Adv. Soviet Math.*, pages 167–180. Amer. Math. Soc., Providence, RI, 1992.
[30] A. C. Offord. The distribution of zeros of power series whose coefficients are independent random variables. *Indian J. Math.*, 9:175–196, 1967.

[31] Y. Peres and B. Virág. Zeros of the i.i.d. Gaussian power series: a conformally invariant determinantal process. *Acta Math.*, 194(1):1–35, 2005.

[32] Z. Průša, P. L. Søndergaard, N. Holighaus, C. Wiesmeyr, and P. Balazs. The Large Time-Frequency Analysis Toolbox 2.0. In M. Aramaki, O. Derrien, R. Kronland-Martinet, and S. Ystad, editors, *Sound, Music, and Motion*, Lecture Notes in Computer Science, pages 419–442. Springer, 2014.

[33] K. Seip. Regular sets of sampling and interpolation for weighted Bergman spaces. *Proc. Amer. Math. Soc.*, 117(1):213–220, 1993.

[34] K. Seip. *Interpolation and sampling in spaces of analytic functions*, volume 33 of *University Lecture Series*. Amer. Math. Soc., Providence, RI, 2004.

[35] K. Seip and R. Wallstén. Density theorems for sampling and interpolation in the Bargmann-Fock space II. *J. Reine Angew. Math.*, 429:107–114, 1992.

[36] T. Shirai. *Limit theorems for random analytic functions and their zeros*. RIMS Kôkyûroku Bessatsu, B34. Res. Inst. Math. Sci. (RIMS), Kyoto, 2012.

[37] M. Sodin. Zeros of Gaussian analytic functions. *Math. Res. Lett.*, 7(4):371–381, 2000.

[38] G. Yu, S. Mallat, and E. Bacry. Audio denoising by time-frequency block thresholding. *IEEE Trans. Signal Process.*, 56(5):1830–1839, May 2008.

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