Enriched classification of parafermionic gapped phases with time reversal symmetry

Wen-Tao Xu\(^1\) and Guang-Ming Zhang\(^1,2\)

\(^1\)State Key Laboratory of Low-Dimensional Quantum Physics and Department of Physics, Tsinghua University, Beijing 100084, China.

\(^2\)Collaborative Innovation Center of Quantum Matter, Beijing 100084, China.

(Dated: March 13, 2022)

Based on the recently established parafermionic matrix product states, we study the classification of one-dimensional gapped phases of parafermions with the time reversal (TR) symmetry satisfying \(T^2 = 1\). Without extra symmetry, it has been found that \(\mathbb{Z}_p\) parafermionic gapped phases can be classified as topological phases, spontaneous symmetry breaking (SSB) phases, and a trivial phase, which are uniquely labelled by the divisors \(n\) of \(p\). In the presence of TR symmetry, however, the enriched classification is characterized by three indices \(n, \kappa\) and \(\mu\), where \(\kappa \in \mathbb{Z}_2\) denotes the linear or projective TR actions on the edges, and \(\mu \in \mathbb{Z}_2\) indicates the commutation relations between the TR and (fractionalized) charge operator. For the \(\mathbb{Z}_r\) symmetric parafermionic ground states, where \(r = p\) for trivial or topological phases, and \(r = p/n\) for SSB phases, the original gapped phases with odd \(r\) are divided into two phases, while those phases with even \(r\) are further separated into four phases. The gapped parafermionic phases with the TR symmetry include the symmetry protected topological phases, symmetry enriched topological phases, and the SSB coexisting symmetry protected topological phases. From analyzing the structures and symmetries of their reduced density matrices of those resulting topological phases, we can obtain the topological protected degeneracies of their entanglement spectra.

I. INTRODUCTION

Topological phases of matter and their classification have attracted intensive interests in condensed matter physics. One of the important works among various researches are the classification of the topological insulators and topological superconductors\(^1-4\). However, these topological insulators or superconductors are phases of non-interacting fermions, and the classifications are broken down when the local interactions are included\(^5\). It has been known that one-dimensional interacting fermions without extra symmetry are classified as two phases, and the classification of those with the time reversal (TR) symmetry are given by the \(\mathbb{Z}_8\) group structure\(^6,7\). These pioneer works pave the way for studying the phases of strongly interacting fermions in higher dimensions.

Moreover, those one-dimensional fermionic topological phases possess the Majorana edge zero modes, which have the potentials in fault-tolerant quantum computation\(^8\). In order to perform a more general quantum computation, the exotic parafermion zero modes or fractionalized Majorana zero modes have been proposed\(^9\). The parafermion zero modes can be generated from an effective one-dimensional chain, and the classification of parafermion chains without extra symmetry has been performed based on the symmetry fractionalization on the edges\(^10,11\). Such a classification is beyond the framework for the one-dimensional interacting fermion chains. We have noticed that the classification of parafermion chains with the TR symmetry has been briefly discussed\(^12\), but the complete classification scheme has not been established yet.

Although the possible one-dimensional gapped phases of parafermions can be classified, the structures of their ground state wavefunctions are still unknown. For bosonic/spin systems, their ground state wavefunctions can be expressed by matrix product states (MPS), all relevant information of topological properties are encoded in the local tensors of MPS, and the complete classification of one-dimensional bosonic/spin systems can be implemented within the MPS formalism\(^13-15\). Recently, a framework of fermionic MPS has been proposed, and all possible one-dimensional topological phases of interacting fermions have been classified\(^16,17\). In order to have a deeper understanding of topological phases of parafermions, we have generalized the MPS formalism to the parafermion systems\(^18\) and performed the classification of gapped phases of parafermion chains\(^19\). Without extra symmetry, it has been shown that the \(\mathbb{Z}_p\) parafermionic gapped phases can be classified as topological phases, spontaneous symmetry breaking (SSB) phases, and a trivial phase, each phase is uniquely labelled by the divisor \(n\) of \(p\).

In this paper, using the parafermionic MPS we extend our classification scheme to the one-dimensional gapped phases of parafermions with the time reversal (TR) symmetry. First of all, we carefully consider the experimental realizations of the parafermion chains, and show that the TR transformation on the basis of Fock space of parafermions is just to take complex conjugation, corresponding to the BDI class with \(T^2 = 1\). When the TR symmetry is imposed on the parafermionic MPS, the possible gapped phases are enriched and classified by three indices \(n, \kappa\) and \(\mu\), where \(\kappa \in \mathbb{Z}_2\) corresponds to the linear or projective actions of TR symmetry on the edges, and \(\mu \in \mathbb{Z}_2\) describes the commutation relations between the TR and (fractionalized) charge operator at the virtual degrees of freedom. The resulting gapped parafermionic phases include the symmetry protected
topological (SPT) phases, symmetry enriched topological phases, and the SSB coexisting SPT phases. Furthermore, we systematically analyze the structures and symmetries of reduced density matrices for those resulting topological phases, and derive the topological protected degeneracies of their entanglement spectra (ES).

The paper is organized as follows. In Sec. II, the TR transformation of parafermions is discussed according to the experimental realization setups, and in Sec. III the classification of parafermionic MPS without extra symmetry is briefly reviewed. In Sec. IV, the first two specific examples of $\mathbb{Z}_3$ and $\mathbb{Z}_4$ parafermions with the TR symmetry are considered in detail separately. The general classification of $\mathbb{Z}_p$ parafermions with TR symmetry is presented in Sec. V, and the entanglement spectra of the topological phases are analyzed by using the symmetries of reduced density matrices in Sec. VI. Finally, in Sec. VII we summarize the classification results in terms of group cohomology.

II. TIME REVERSAL SYMMETRY FOR PARAFERMIONS

It is well known that the $\mathbb{Z}_p$ spin operators $\sigma_l$ and $\tau_l$ are the generalization of Pauli matrices $\sigma^x$ and $\sigma^z$. They satisfy the following relations

$$\sigma^p_l = \tau^p_l = 1, \quad \sigma_l \tau_m = \omega_{lp} \tau_m \sigma_l,$$

where $\omega_p = e^{2\pi i/p}$, $\sigma^1 = \sigma^{p-1}$ and $\tau^1 = \tau^{p-1}$. The $\mathbb{Z}_p$ parafermion operators can be introduced via the generalized Jordan-Wigner transformation\textsuperscript{20,21},

$$\chi_{2l-1} = \prod_{k<l} \tau_k \sigma_l, \quad \chi_{2l} = -e^{i\pi/p} \prod_{k<l} \tau_k \sigma_l,$$

which satisfy the generalized Clifford algebra:

$$\chi^p_l = 1, \quad \chi_l \chi_m = \omega_{lp} \chi_m \chi_l, \text{ for } l < m.$$

So it is natural that the TR symmetry for parafermions has to be considered from the $\mathbb{Z}_p$ spin operators\textsuperscript{22,23}. However, the resulting transformation is not meaningful, because the $\mathbb{Z}_p$ spin operators do not directly correspond to any physically realized operators.

In order to find a well-defined TR symmetry, we consider an experimental setup, which can realize the $\mathbb{Z}_{2m}$ parafermion modes ($m$ is an odd integer) from a fractional topological insulator\textsuperscript{24-26}. This setup can be viewed as two copies of $\nu = \pm 1/m$ fractional quantum Hall states. The edges of this system are gapped out in proximity to ferromagnetic or superconducting regions, and the $\mathbb{Z}_{2m}$ parafermions live in the domain walls between the superconducting and ferromagnetic regions, as shown in Fig. 1. However, this setup can not realize the $\mathbb{Z}_p$ parafermions with odd $p$. But there are many other proposals for the $\mathbb{Z}_p$ parafermions with odd $p$, such as bosonic or $\nu = 2/3$ fractional quantum Hall states\textsuperscript{27,28}. Here we only focus on the parafermions realized at the edges of fractional topological insulator, the other setups can be discussed similarly.

The usual TR transformation for spin-1/2 electrons is defined by $T a^\dagger T^{-1} = a^\dagger$, $T a T^{-1} = -a$, and $T^2 = 1$, where $a$ and $a^\dagger$ are the annihilation operators of spin-up and spin-down electrons, respectively. Since the ferromagnets in proximity to the fractional topological insulator induce backscatterings between the two counter propagating edge modes, we have to include the Zeeman terms $(\lambda a^\dagger a^\dagger + h.c.)$, which explicitly break this usual TR symmetry. Although there is a proposal realizing the Kramers pairs of parafermions with $T^2 = -1$ in the absence of magnetic field\textsuperscript{29}, the original setup is nevertheless invariant under a modified TR symmetry\textsuperscript{30}:

$$T a^\dagger T^{-1} = a^\dagger, \quad T a T^{-1} = -a, \quad T^2 = 1.$$

Under such a TR transformation, the electron charge remains unchanged but the electron spin is flipped.

Based on the bosonization of the edge theory\textsuperscript{24-26}, the $\mathbb{Z}_p$ spin operators have the correspondences

$$\sigma_l \rightarrow e^{i\psi_l}, \quad \tau_l \rightarrow e^{i\pi \rho^l},$$

where $\psi_l$ denotes the non-chiral bosonic field whose derivative is the electron spin density, and $\rho^l_i$ is the electron charge density. When the TR symmetry $T^2 = 1$ is applied, it can be proven that

$$T \psi_l T^{-1} = -\psi_l, \quad T \rho^l_i T^{-1} = \rho^l_i.$$

So the TR transformations for $\mathbb{Z}_p$ spin operators are obtained as

$$T \sigma_l T^{-1} = \sigma_l, \quad T \tau_l T^{-1} = -\tau_l.$$

When writing the wavefunctions for parafermions, we use the Fock space of parafermions with the basis denoted as $|i_1 i_2 \cdots i_L\rangle$, where $i_l \in \mathbb{Z}_p$ are the quantum numbers of $p$-dimensional local Hilbert spaces. With the help of Fock parafermions\textsuperscript{30}, it can be proved that

$$\chi_{2l} |i_1 \cdots i_L \cdots \rangle = -e^{i\pi \frac{1}{p}} \left( \sum_{k<l} \chi^1_k \right) |i_1 \cdots i_l - 1 \cdots \rangle,$$

$$\chi_{2l-1} |i_1 \cdots i_L \cdots \rangle = e^{i\pi \frac{1}{p}} \sum_{k<l} \chi^1_k |i_1 \cdots i_l - 1 \cdots \rangle.$$

Because we can write the $\mathbb{Z}_p$ spin operator in terms of the parafermions as $\tau_l = -\omega_p^{-1/2} \chi_{2l-1} \chi_{2l}$, the basis of...
Fock space become the eigenstates of $\tau_i$, 
$$\tau|i_1 \cdots i_l \cdots i_L\rangle = \omega_p^{i_l} |i_1 \cdots i_l \cdots i_L\rangle. \quad (9)$$

According to the TR transformation of $\tau_i$, the basis of Fock space are invariant under the TR transformation 
$$T|i_1 \cdots i_l \cdots i_L\rangle = |i_1 \cdots i_l \cdots i_L\rangle. \quad (10)$$

Taking into account the expression $\tau \sim e^{i\pi p}$, the physical significance of the Fock basis is the electron charges of quasi-particles modulo $p$ in the SC domains, and it is reasonable that the Fock basis keep invariant under the TR symmetry.

Because the fractional topological insulator has two separate counter propagating edge modes, these two types of edge operators have the definite charges, thus the Hilbert space $\mathbb{H}$ is the direct sum of $\mathbb{H}_p$ subspaces. In $\mathbb{H}_p$, there exist constraints $|i| + |\alpha| - |\beta| = 0$ for all $A^{[i]}_{\alpha\beta}$, where the charge of index $\beta$ is inverted because the opposite charge parafermions in neighboring sites form the charge-0 bonds. Then the local matrices as the components of the local tensors can be expressed as

$$A^{[i]} = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & a^{[i]} & \cdots & 0 \\
0 & 0 & a^{[i]} & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & a^{[i]}_{p-1}
\end{bmatrix}, \quad |i| = 0,$$
$$A^{[i]} = \begin{bmatrix}
0 & a^{[i]} & \cdots & 0 \\
0 & 0 & a^{[i]} & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
a^{[i]}_{p-1} & 0 & 0 & 0 & 0
\end{bmatrix}, \quad |i| = 1,$$
$$A^{[i]} = \begin{bmatrix}
0 & 0 & \cdots & a^{[i]} \\
0 & a^{[i]} & \cdots & 0 \\
0 & 0 & a^{[i]} & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & a^{[i]}_{p-1} & 0
\end{bmatrix}, \quad |i| = p - 1, \quad (15)$$

where $a^{[i]}_r$ with $r \in \mathbb{Z}_p$ are the sub-block matrices.

For the convenience of discussion, we introduce two $n \times n$ block matrices

$$\mathbb{Y}_n = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
1 & 0 & 0 & 0 & 0
\end{bmatrix},$$
$$\mathbb{Q}_n = \text{diag}(1, \omega_p, \omega_p^2, \ldots, \omega_p^{n-1}), \quad (16)$$

where the dimensions of the sub-block identities $\mathbb{1}$ should be coincided with that of the sub-block matrices $a^{[i]}_r$. The matrix $\mathbb{Q}_n$ measures the $\mathbb{Z}_n$ charge, and the matrix $\mathbb{Y}_n$ flips the $\mathbb{Z}_n$ charge. When the identity matrices $\mathbb{1}$ are reduced to the number 1, we will denote $\mathbb{Y}_n$ by $Y_n$ and $\mathbb{Q}_n$ by $Q_n$. With these definitions, Eq. (15) can be written into a more concise form:

$$A^{[i]} = Y^{[i]}_p \otimes a^{[i]}_r, \quad (17)$$

Supposing that all $a^{[i]}_r$ cannot become equal under any gauge transformations, the MPS generated by the matrices of Eq. (17) belong to the trivial phase. If all sub-blocks can be equal under a gauge transformation, i.e., $a^{[i]}_r = a^{[i]}$, the local matrices of MPS can be written as

$$A^{[i]} = Y_p^{[i]} \otimes a^{[i]}, \quad (18)$$

The generated MPS represent the ground states of a topological phase with unpaired $\mathbb{Z}_p$ parafermion zero
edge modes. Moreover, if \( a^{[i]}_r = a^{[i]}_{(r+p)/n} \) mod \( p \) under a gauge transformation, where \( n \) is a divisor of \( p \), there are \( p/n \) unequal sub-block matrices \( a^{[i]}_r \) with \( r \in \mathbb{Z}_{p/n} \), and there are two different situations. If \( n \) and \( p/n \) are mutually prime, they are the topological phases with diagonal phase. Otherwise, they are SSB phases. 

The generated MPS correspond to a \( \mathbb{Z}_p \) symmetric topological phase with unpaired \( \mathbb{Z}_n \) parafermion zero edge modes. Such a topological phase is characterized by the \( n \)-fold degenerate ES.

In the case that \( n \) and \( p/n \) are not mutually prime, it is impossible that the local matrices can be gauge transformed into the form of Eq. (19), because \( Y_p \sim Y_n \otimes Y_{p/n} \) only if \( n \) and \( p/n \) are mutually prime. With \( Q_n = \text{diag}(1, \omega_n^0, \omega_n^1, \ldots, \omega_n^{n-1}) \), we can write \( Y_p \sim Q_n \otimes Y_{p/n} \), and then the local matrices can be gauge transformed into

\[
A^{[i]} = \tilde{Q}_n^{[i]} \otimes d^{[i]},
\]

where the gauge transformation breaks the \( \mathbb{Z}_p \) charge symmetry but preserves the \( \mathbb{Z}_{p/n} \) charge symmetry. The block diagonal forms of \( A^{[i]} \) represent a SSB phase, where the \( \mathbb{Z}_p \) symmetry is spontaneously broken down to \( \mathbb{Z}_{p/n} \) symmetry.

So the number of gapped phases of \( \mathbb{Z}_p \) parafermions is the same as the number of divisors of \( p \), and each divisor \( n \) uniquely labels a gapped phase. If \( n \) and \( p/n \) are mutually prime, they are the topological phases with \( \mathbb{Z}_n \) parafermion zero edge modes except the \( n = 1 \) trivial phase. Otherwise, they are SSB phases.

**IV. \( \mathbb{Z}_3 \) AND \( \mathbb{Z}_4 \) PARAFERMI ON PHASES ENRICHED BY TIME REVERSAL SYMMETRY**

**A. \( \mathbb{Z}_3 \) parafermionic phases**

In Sec. II, we have shown that the basis of Fock space are invariant under TR transformation. The matrix form of TR symmetry operator can be expressed as \( T_{ij} = \delta_{ij}K \), where \( K \) is the complex conjugation operator. Because a TR symmetric MPS are invariant under TR transformation up to a gauge transformation \( T \), the projective representation of \( T \), the MPS local matrices satisfy

\[
\sum_j \delta_{ij} K A^{[j]} = \bar{A}^{[i]} = T^{-1} A^{[i]} T,
\]

where \( \bar{A}^{[i]} \) are the complex conjugate of \( A^{[i]} \). Since \( T^2 = 1 \), we return back to the original matrices via twice TR transformation

\[
A^{[i]} = T T A^{[i]} T^{-1} T^{-1}.
\]

Because there are two distinct gapped phases for \( \mathbb{Z}_3 \) parafermion chains, their enriched classification with TR symmetry should be discussed separately.

For the original trivial phase, the irreducible local matrices \( A^{[i]} \) are injective, so the only way to fulfill Eq. (22) is

\[
TT = a_0 1.
\]

Without loss of generality, one can assume \( a_0 = (-1)^\kappa \), where \( \kappa \in \mathbb{Z}_2 \) labels two distinct classes of projective representations. In addition, the systems have the intrinsic \( \mathbb{Z}_3 \) charge symmetry, the MPS are invariant under the action of the \( \mathbb{Z}_3 \) charge operator up to the a gauge transformation \( Q_3 \):

\[
Q_3^{-1} A^{[i]} Q_3 = \sum_j (Q_3)_{ij} A^{[j]} = \omega_3^{0} A^{[i]}.
\]

Accordingly the MPS should also be invariant under the combined action of \( T \) and \( Q_3 \), which are not commute each other. Comparing the transformations of \( T \) and \( Q_3 \) with different orders, we find that the injectivity of \( A^{[i]} \) requires \( T \) with a definite \( \mathbb{Z}_3 \) charge, i.e.,

\[
Q_3^{-1} T Q_3 = \omega_3^{3} T, \quad |T| = 0, 1, 2.
\]

Obviously \( T \) and \( \bar{T} \) have the same charge, and Eq. (22) uniquely determines that \( T \) has charge-zero, i.e., \( |T| = 0 \) and \( Q_3 T = T Q_3 \). Unlike the fermionic MPS, the commutation relation between \( Q_3 \) and \( T \) does not lead to a topological invariant. By imposing the TR symmetry, the trivial phase of \( \mathbb{Z}_3 \) parafermions is thus split into two TR symmetric phases labelled by \( \kappa = 0, 1 \). They are the trivial phase (\( \kappa = 0 \)) and SPT phase (\( \kappa = 1 \)) with Kramers doublets at the edges.

On the other hand, the irreducible \( \mathbb{Z}_3 \) parafermionic MPS for the topological phase do not have the injective property. Since the structures of local matrices \( A^{[i]} = Y_3^{[i]} \otimes a^{[i]} \) are featured by the matrix \( Y_3 \), the regular representation of the \( \mathbb{Z}_3 \) generator, the MPS of the topological phase are \( \mathbb{Z}_3 \) injective. Considering the special structures of \( A^{[i]} \), we can without loss of generality assume that \( T \) has a well-defined charge, i.e., one of the three matrices \( T_0, T_1, T_2 \) with charge-0, charge-1 and charge-2 fulfills the TR transformation. Because \( Y_3 \) commutes with all \( A^{[i]} \), \( T_0, T_1 \) and \( T_2 \) are not independent, they are connected by \( Y_3 \):

\[
T_0 = T_0 Y_3, \quad T_1 = T_1 Y_3^2, \quad T_2 = T_2 Y_3.
\]

Moreover, the possible way satisfying Eq. (22) is given by

\[
T_0 T_0 = a_0 1, \quad T_1 T_1 = a_1 Y_3^2, \quad T_2 T_2 = a_2 Y_3.
\]
TABLE I: The classification of gapped phases of $Z_3$ parafermions with TR symmetry. Different phases are labelled by the indices $n$ and $\kappa$, where $\kappa$ describes whether there exist Kramers doublets at the edges.

| Phase | Trivial | Non-trivial |
|-------|---------|-------------|
| $n$   | 0       | 1           |
| $\kappa$ | 0       | 1           |

An analogy to the trivial phase, we generally have $\alpha_0 = (-1)^\kappa$ with $\kappa = 0, 1$. From Eq. (26) and Eq. (27), we can determine $\alpha_1 = \alpha_2 = \alpha_0$ as well as $\alpha_1 = \alpha_0 \alpha_2$, and another relation between $\alpha_0$ and $\alpha_1$ can be obtained by the commutation relation between $T_0$ and $Y_3$:

$$Y_3 T_0 = \omega^{\mu}_3 T_0 Y_3, \quad \omega^{\mu}_3 = \alpha_0 \alpha_1,$$  

where $\mu = 0, 1, 2$. In the classification of $Z_2$ fermion chains $\mathbb{Z}_2$, $\mu$ distinguishes different phases. So we might expect that different values of $\mu$ also label different parafermion phases, but there are some redundancies.

In general, the MPS of topological phase intrinsically have $\mathbb{Z}_p$ charge symmetry, which is implemented by a gauge transformation $Q_p$ as $A[i] = Q_p^{r} A[i] Q_p^{r}$, where $r \in \mathbb{Z}_p$, and the ground state wavefunctions generated by $A[i]$ and $A[i]'$ are the same. We then denote that $T_0'$ is the projective charge-0 representation of $T$ associated with the local matrices $A[i]'$, i.e., $A[i]' = T_0' A[i] T_0'$. Compare to Eq. (21), the $Z_3$-injectivity of $A[i]'$ gives rise to

$$T_0' = Q_p^{-2r} T_0, \quad Y_p T_0' = \omega^{\mu-2r} Y_p T_0,$$  

indicating that $\mu$ and $(\mu - 2r) \mod p$ should characterize the same phase. Therefore, for even $p$, only the parity of $\mu$ can distinguish different topological phases, while for odd $p$, all phases with different $\mu$ are equivalent.

As a result, only two non-trivial topological phases with the TR symmetry labelled by $\kappa = 0, 1$ are obtained from the topological phase of $Z_3$ parafermions, and they are referred to as the symmetry enriched topological phases. By including the one SPT phase and the trivial phase, there exist four phases labelled by the indices $n$ and $\kappa$, which are summarized in the Tab. I.

B. $Z_4$ parafermionic phases

For the trivial phase of $Z_4$ parafermions, we still have

$$T \bar{T} = (-1)^\kappa \mathbb{1},$$  

where $\kappa = 0$ and 1 characterize two different gapped phases. Similar to Eq. (28), $Q_4^{-1} T Q_4 = \omega_4^{T} T$ with $|T| = 0, 1, 2, 3$ for the $Z_4$ parafermions. $T$ is further required to have a definite $Z_4$ charge. Since $T$ and $\bar{T}$ have the same charge, it is only possible that $T$ has charge-0 or charge-2, determined by

$$Q_4 T = (-1)^\mu T Q_4$$  

with $\mu = 0, 1$. Unlike the $Z_3$ case, the charge of $T$ can take two different values and $\mu$ is thus a topological invariant. So from the trivial phase of $Z_4$ parafermions, there emerge four different gapped phases labelled by $\kappa = 0, 1$ and $\mu = 0, 1$. Among them, there are three SPT phases with the TR symmetry.

According to the structures of local matrices $A[i] = Y_4^{[i]} \otimes a[i]$, the MPS for the non-trivial topological phase of $Z_4$ parafermions are $Z_4$-injective. Similar to the $Z_3$ case, the projective representation $T$ can be restricted to the charge-$q$ matrices $T_q$ satisfying $T_q T_0 = \alpha_q Y^{2q}$ with $q = 0, 1, 2, 3$. Among them, $T_0$ is used to define two different topological invariants like the $Z_3$ classification:

$$T_0 \bar{T}_0 = (-1)^\kappa \mathbb{1}, \quad Y_4 T_0 = \omega^{\mu} T_0 Y_4,$$  

where $\kappa = 0, 1$ and $\mu = 0, 1, 2, 3$. From the Eq. (29), $\mu$ and $(\mu - 2r) \mod 4$ label the same phases. Therefore, there exist four different symmetry enriched topological phases labelled by $\kappa = 0, 1$ and $\mu = 0, 1$. $\kappa = 1$ implies the existence of the Kramers degeneracy, and $\mu$ classifies the actions of the TR transformation on parafermion zero edge modes.

In addition, for the $Z_4$ parafermions, there also exists a SSB phase. According to Eq. (20), the local matrices can be expressed as

$$A[i] = \begin{bmatrix} d[i] & 0 \\ 0 & \omega_4^{[i]} d[i] \end{bmatrix},$$

$$d[i] = \begin{bmatrix} a_0[i] & 0 \\ a_1[i] & 0 \end{bmatrix}, \quad |i| = 0, 2,$n

$$d[i] = \begin{bmatrix} 0 & a_0[i] \\ a_1[i] & 0 \end{bmatrix}, \quad |i| = 1, 3.$$  

To fulfill the unity requirement of twice TR transformation, the projective TR representation $T$ should have the following block diagonal form

$$T = \begin{bmatrix} T_{0,0} & 0 \\ 0 & T_{1,1} \end{bmatrix}.$$  

And the TR transformations for the sub-blocks $d[i]$ and $\omega_4^{[i]} d[i]$ yield

$$T_{0,0} T_{0,0} = (-1)^\kappa \mathbb{1}, \quad T_{1,1} T_{1,1} = \alpha_1 \mathbb{1}.$$  

Actually, $T_{0,0}$ and $T_{1,1}$ are not independent. Considering that the MPS generated by $d[i]$ are injective and $Z_2$ parity symmetric, i.e., $Q_2 d[i] Q_2 = \omega_2^{[i]} d[i]$, we then have

$$T_{1,1} = Q_2 T_{0,0}, \quad T_{0,0} Q_2 = (-1)^\mu Q_2 T_{0,0}.$$  

The relation between $T_{0,0}$ and $T_{1,1}$ leads to $\alpha_1 = (-1)^{\mu+\kappa}$, and $\mu$ just indicates the $Z_2$ parity of $T_{0,0}$. So we can obtain four different SSB coexisting SPT phases, labelled by $\kappa = 0, 1$ and $\mu = 0, 1$.

By including the SPT phases split from the trivial phase and four symmetry enriched topological phases, we
have obtained three different families of $Z_4$ parafermionic gapped phases, each of them consists of four different phases labelled by the indices $\kappa$ and $\mu$. The results are summarized in the Table II.

V. ENRICHED CLASSIFICATION OF $Z_p$ PARAFERMIONIC PHASES WITH TR SYMMETRY

The general cases are more complicated than $Z_3$ and $Z_4$ cases, because there exist various topological phases and SSB phases. We will classify all these phases with TR symmetry in the following, and the obtained results are summarized in Tab. II.

A. From the trivial phase

The local matrices generating the MPS for the trivial phase of $Z_p$ parafermions have been given by Eq. (17). The injectivity of irreducible local matrices $A[i]$ and Eq. (22) determine

$$T_0 = (-1)^\kappa \mathbb{1}, \quad \kappa = 0, 1.$$  

(37)

Eq. (25) is still valid even when generalizing to $Z_p$ parafermions,

$$Q^{-1}_{p/n} Q_p = \omega_p^{j|T|} T, \quad |T| \in \mathbb{Z}_p,$$  

(38)

which enforces that $T$ has a definite $Z_p$ charge. Since $T$ and $T$ have the same charge, $T$ must be charge-0 for odd $p$, and charge-0 or charge-$p/2$ for even $p$, corresponding to $Q_p = (-1)^\mu Q_p T$ with $\mu = 0, 1$, respectively. So the trivial phase of $Z_p$ parafermions with odd $p$ is split into two different phases labelled by $\kappa = 0, 1$, while the trivial phase with even $p$ is split into four phases labelled by $\kappa = 0, 1$ and $\mu = 0, 1$. $\kappa = 1$ signifies that there is the Kramers degeneracy at the each end of the chains, while $\mu = 1$ indicates that $T$ shifts the $Z_p$ charges of the edge states by $p/2$.

| phase   | Trivial | SSB   | Non-trivial |
|---------|---------|-------|-------------|
| $n$     | 1       | 2     | 4           |
| $\kappa$| 0       | 0     | 0           |
| $\mu$   | 1       | 0     | 0           |

B. From the non-trivial phases

There might be many non-trivial topological phases, and each phase is labelled by $n$ ($n \neq 1$), satisfying $n$ and $p/n$ are mutually prime. The local matrices of the non-trivial phases have the structures $A[i] = Y_{p/n}^i \otimes d[i]$, where $d[i]$ can generate injective MPS. Since $Y_{p/n}^i$ commute with all $A[i]$ which are $Z_n$ injective, we can restrict $T$ to the matrices $T_p$ with arbitrary $Z_n$ charges $r \in \mathbb{Z}_n$. They satisfy $T_{\phi} T = \alpha_{\phi} Y_{p/n}^{2r}$ and are transformed into each other by multiplying $Y_{p/n}$ several times. Analogue to the previous examples, we have

$$T_0 T_0 = (-1)^\kappa \mathbb{1},$$  

(39)

and the relations among $\alpha_{\phi}$ are determined by

$$Y_{p/n} T = \omega_{p/n} T Y_{p/n},$$  

(40)

where $\mu$’ and $(\mu - 2r) \mod n$ represent the same phases as explained before. Therefore, if $n$ is odd, different $\mu$’ are equivalent. But if $n$ is even, the parities of $\mu$’ define the equivalent classes.

It should be noticed that the MPS generated by $d[i]$ have the $Z_{p/n}$ symmetry. Eq. (40) implies that $T_0$ has the structure $T_0 = Q_{p/n}^i \otimes T_0$. Similar to Eq. (20), $T_{0,0}$ should have a definite $Z_{p/n}$ charge,

$$Q_{p/n}^{-1} T_{0,0} Q_{p/n} = \omega_{p/n}^{j|T_0|} T_{0,0}.$$  

(41)

Then for odd $p/n$, we have $|T_{0,0}| = 0$, while for even $p/n$, we have $|T_{0,0}| = 0$ or $p/(2n)$, corresponding to $T_{0,p/n} Q_{p/n} = (-1)^\mu Q_{p/n} T_{0,p/n}$ with topological invariant $\mu = 0, 1$, respectively.

Now we have three topological indices $\kappa = 0, 1$, $\mu = 0, 1$, and $\mu = 0, 1$. The MPS matrices generally have both a trivial part and a non-trivial part, and $\mu$ comes from the trivial part and $\mu$’ originates from the non-trivial part. Since they are mutually prime for the non-trivial phases, $n$ and $p/n$ can not be even number simultaneously. When $p$ is odd, both $\mu$ and $\mu$’ are equal to zero. If $p$ is even and odd $p/n$, $\mu = 0, 1$ and $\mu = 0$; while for both even $p$ and $p/n$, $\mu = 0, 1$ and $\mu = 0$. So in Tab. III and the following discussion, we do not distinguish $\mu$ from $\mu$’ and both are denoted as $\mu$. When we count the degeneracy of ES, however, we must remember which part of $\mu$ stems from. Therefore, for odd $p$, there are two phases labelled by $\kappa = 0, 1$, while for even $p$ there are four phases simply labelled by $\kappa = 0, 1$ and $\mu = 0, 1$.

C. From the SSB phases

We now consider the SSB phases, where $Z_p$ symmetry is broken down to $Z_{p/n}$ and $n$ and $p/n$ are not mutually prime in this case. Applying the $Z_p$ charge matrix to all physical degrees of freedom will shift one ground state to another degenerate ground state, and it will go back...
to the original ground state after the $\mathbb{Z}_p$ charge operator acts $n$ times. Therefore, the local matrices can be generally written as Eq. (20), where the sub-blocks $\omega_p^{[i]} d^{[i]}$ generate a $\mathbb{Z}_{p/n}$ symmetric ground state, which is the same as the MPS of $\mathbb{Z}_{p/n}$ trivial phase.

Suppose that there is a short-range correlated ground state which is invariant under only a subgroup of the whole symmetry group. It is possible to have the gapped phases in which long-range order and SPT order coexist. To classify these phases, we can combine the symmetry breaking and symmetry fractionalization using two sets of data: the subgroup and the SPT order under the subgroup. Considering Eq. (21) and Eq. (22) together, $\mathcal{T}$ should have the form

$$\mathcal{T} = \text{diag}(\mathcal{T}_{0,0}, \mathcal{T}_{1,1}, \ldots, \mathcal{T}_{n-1,n-1}) \times (P \otimes 1),$$

where $P$ is the $n \times n$ permutation matrix occurring in classification for SSB phases. When the TR symmetry is imposed, we can in general assume that the matrices of the zeroth ground state satisfy $d^{[i]} = \mathcal{T}_{0,0}^{-1} d^{[i]} \mathcal{T}_{0,0}$.

However, the matrices $\omega_p^{[i]} d^{[i]}$ of the $r$-th ground state fulfill the TR transformation via

$$\omega_p^{r[i]} d^{[i]} = \mathcal{T}_{0,0}^{-1} Q_{p/n} (\omega_p^{(n-r)[i]} d^{[i]} ) Q_{p/n}^{-1} \mathcal{T}_{0,0},$$

where the phases $\omega_p^{[i]}$ have been canceled due to $Q_{p/n} d^{[i]} Q_{p/n}^{-1} = \omega_p^{-[i]} d^{[i]}$. Since the matrices $\omega_p^{(n-r)[i]} d^{[i]}$ in the right hand side generate the $(n-r)$-th ground state, the TR symmetry transforms the $r$-th ground state to the $(n-r)$-th ground state. Thus $\mathcal{T}_{r,r} = Q_{p/n}^{-1} \mathcal{T}_{0,0}$ for $r = 1, 2, \ldots, n-1$ and the permutation matrix is $P_{\delta} = \delta_{i,n-j}$.

Finally, we need to discuss the properties of $\mathcal{T}_{0,0}$, which satisfies

$$\mathcal{T}_{0,0} \mathcal{T}_{0,0} = (-1)^{n} 1$$

with $\kappa = 0, 1$. Since $\mathcal{T}_{0,0}$ also has a definite $\mathbb{Z}_{p/n}$ charge, we have

$$\mathcal{T}_{0,0} Q_{p/n} = (-1)^{\mu} Q_{p/n} \mathcal{T}_{0,0},$$

For odd $p/n$, the charge of $\mathcal{T}_{0,0}$ can only be zero, and there are two different phases labelled by $\kappa = 0, 1$. While for even $p/n$, the charge of $\mathcal{T}_{0,0}$ can be 0 or $p/(2n)$, and there are four phases labelled by $\kappa = 0, 1$ and $\mu = 0, 1$.

Different from the previous classification, we find that the phases of $\mathbb{Z}_p$ parafermions with even $p$ do not always split into 4 phases. For example, the SSB phase of $\mathbb{Z}_{18}$ parafermions labelled by $n = 6$ just splits into two different gapped phases.

### VI. ENTANGLEMENT SPECTRA OF THE TR ENRICHED $\mathbb{Z}_p$ PARAFERMIONIC PHASES

It has been known that different topological phases can be featured by the necessary degeneracies of the their $\mathbb{E}$ eigenstates. Here we consider the degeneracy of $\mathbb{E}$ via a left-right bipartition of an infinite long chain, so there is only one boundary in the reduced system. The topological protected degeneracy of $\mathbb{E}$ is determined by the structure and symmetries of the reduced density matrix. According to the holographic principle, the reduced density matrix obtained via the left-right bipartition of an infinite long chain can be derived from the dominant eigenvectors of the transfer operator $\mathbb{E} = \sum A^{[i]} \otimes A^{[i]}$. We will divide our discussion into three different cases. The corresponding results have been summarized in Tab. III.

#### A. For SPT phases

These SPT phases are enriched from the trivial phase. According to the properties of injective MPS, the dominant eigenvalue of $\mathbb{E}$ is non-degenerate. We denote its left dominant eigenvector by $\sigma_L$, and the right dominant eigenvector by $\sigma_R$, as shown in Fig. 2(a) and (b), respectively, and the dominant eigenvectors can be reshaped into matrices. Because of the TR transformation satisfied by $A^{[i]}$ shown in Eq. (21), the transfer operator $\mathbb{E}$ is TR symmetric, i.e.,

$$(\mathcal{T} \otimes \mathcal{T}^{-1})\mathbb{E}(\mathcal{T}^{-1} \otimes \mathcal{T}) = \mathbb{E},$$

where $\mathcal{T}$ effectively plays a role of TR transformation on virtual degrees of freedom. Thus it takes the complex conjugation $\mathcal{E}$ in the left hand side of Eq. (46). By acting the TR on the eigen-equations, we can demonstrate that $\mathcal{T}^{-1} \sigma_L^T \mathcal{T}$ and $\mathcal{T} \sigma_R^T \mathcal{T}^{-1}$ are also left and right dominant eigenvectors of $\mathbb{E}$, as shown in Fig. 2(c) and (d). Moreover, because $\sigma_L$ and $\sigma_R$ are Hermitian operators, i.e.,

| $p$ | odd | even |
|-----|-----|------|
| phase | trivial | non-trivial | SSB | trivial | non-trivial | SSB |
| divisor $n$ | 1 | $n$ | 1 | $n$ | 1 | $n$ |
| $\mu$ | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 |
| $\kappa$ | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| top. deg. | 1 | 2 | $n$ | 2$n$ | 1 | 2 | 2 | 2 | 2 |

| $\text{gcd}(p/n, 2)n$ | 2$n$ | 1(only for even $p/n$) |

The last two columns do not exist for odd $p/n$. The last row shows the topological protected degeneracy, where “gcd” denotes the greatest common divisor and $\text{gcd}(p/n, 2)n$ means the degeneracy is $n$ for odd $p/n$ and $2n$ for even $p/n$. 

Suppose that there is a short-range correlated ground state which is invariant under only a subgroup of the whole symmetry group. It is possible to have the gapped phases in which long-range order and SPT order coexist. To classify these phases, we can combine the symmetry breaking and symmetry fractionalization using two sets of data: the subgroup and the SPT order under the subgroup. Considering Eq. (21) and Eq. (22) together, $\mathcal{T}$ should have the form

$$\mathcal{T} = \text{diag}(\mathcal{T}_{0,0}, \mathcal{T}_{1,1}, \ldots, \mathcal{T}_{n-1,n-1}) \times (P \otimes 1),$$

where $P$ is the $n \times n$ permutation matrix occurring in classification for SSB phases. When the TR symmetry is imposed, we can in general assume that the matrices of the zeroth ground state satisfy $d^{[i]} = \mathcal{T}_{0,0}^{-1} d^{[i]} \mathcal{T}_{0,0}$.

However, the matrices $\omega_p^{[i]} d^{[i]}$ of the $r$-th ground state fulfill the TR transformation via

$$\omega_p^{r[i]} d^{[i]} = \mathcal{T}_{0,0}^{-1} Q_{p/n} (\omega_p^{(n-r)[i]} d^{[i]} ) Q_{p/n}^{-1} \mathcal{T}_{0,0},$$

where the phases $\omega_p^{[i]}$ have been canceled due to $Q_{p/n} d^{[i]} Q_{p/n}^{-1} = \omega_p^{-[i]} d^{[i]}$. Since the matrices $\omega_p^{(n-r)[i]} d^{[i]}$ in the right hand side generate the $(n-r)$-th ground state, the TR symmetry transforms the $r$-th ground state to the $(n-r)$-th ground state. Thus $\mathcal{T}_{r,r} = Q_{p/n}^{-1} \mathcal{T}_{0,0}$ for $r = 1, 2, \ldots, n-1$ and the permutation matrix is $P_{\delta} = \delta_{i,n-j}$.

Finally, we need to discuss the properties of $\mathcal{T}_{0,0}$, which satisfies

$$\mathcal{T}_{0,0} \mathcal{T}_{0,0} = (-1)^{n} 1$$

with $\kappa = 0, 1$. Since $\mathcal{T}_{0,0}$ also has a definite $\mathbb{Z}_{p/n}$ charge, we have

$$\mathcal{T}_{0,0} Q_{p/n} = (-1)^{\mu} Q_{p/n} \mathcal{T}_{0,0},$$

For odd $p/n$, the charge of $\mathcal{T}_{0,0}$ can only be zero, and there are two different phases labelled by $\kappa = 0, 1$. While for even $p/n$, the charge of $\mathcal{T}_{0,0}$ can be 0 or $p/(2n)$, and there are four phases labelled by $\kappa = 0, 1$ and $\mu = 0, 1$. Different from the previous classification, we find that the phases of $\mathbb{Z}_p$ parafermions with even $p$ do not always split into 4 phases. For example, the SSB phase of $\mathbb{Z}_{18}$ parafermions labelled by $n = 6$ just splits into two different gapped phases.
\[ T^{-1} \sigma_L T = \sigma_L, \quad T \sigma_R T^{-1} = \sigma_R, \quad (47) \]

which manifest that \( \sigma_L \) and \( \sigma_R \) are also \( T \) symmetric.

Via an isometry map \( U \), the entanglement Hamiltonian \( H_E \) is given by \( e^{H_E} = U^{\dagger} \sqrt{\sigma_L \sigma_R} \sqrt{\sigma_L \sigma_R} U \) for an infinite long chain.\(^{17,32} \) Since the reduced density matrix \( e^{H_E} \) and \( \rho = \sigma_L \sigma_R \) share the same eigenvalue spectrum, we study the spectrum properties \( \rho \) for convenience. The \( T \) symmetric \( \sigma_L \) and \( \sigma_R \) give rise to the \( T \) symmetric \( \rho \):

\[ T \rho T^{-1} = \rho, \quad (48) \]

where \( T \) is effectively anti-unitary. Then the different behaviors of \( T \) give rise to the different topological protected degeneracies.

We suppose that \( v_j \) are the eigenstates of \( \rho \) with eigenvalues \( \epsilon_j \), i.e., \( \rho v_j = \epsilon_j v_j \). Since \( \rho \) is \( T \) symmetric, it can be easily derived that \( T \bar{v}_j \) are also the eigenstates of \( \rho \) with the same eigenvalues: \( \rho T \bar{v}_j = \epsilon_j T \bar{v}_j \). Then we just need to determine whether \( v_j \) and \( T \bar{v}_j \) describe the same states or not. Let us first consider the effects of \( \kappa \), which comes from \( T \bar{T} = (-1)^n \mathbb{1} \). For \( \kappa = 0 \), we need not to distinguish the eigenstates \( v_j \) and \( T \bar{v}_j \). However, for \( \kappa = 1 \), i.e., \( T \bar{T} = -1 \), \( v_j \) and \( T \bar{v}_j \) are certainly different, guaranteed by the Kramers theorem. So the eigenstates of \( \rho \) form the Kramers pairs \( \{(v_j, T \bar{v}_j)\} \) and the ES is two-fold degenerate.

Then we consider the influences of \( \mu \) defined by \( T Q_p = (-1)^{p/n} Q_p \). When \( Q_p \) and \( T \) commute, \( T \) doesn’t change the charges of eigenstates of \( \rho \), namely \( v_j \) and \( T \bar{v}_j \) have the same charges. When \( Q_p \) and \( T \) anti-commute, the charges of \( v_j \) are shifted by \( p/2 \) under the action of \( T \). Thus \( v_j \) and \( T \bar{v}_j \) have different charges. They must correspond to different states with the same eigenvalues \( \epsilon_j \), and two-fold degeneracy is produced in the ES.

Integrating the effects of both \( \kappa \) and \( \mu \), we conclude that as long as one of \( \kappa \) and \( \mu \) is not zero, the ES must be at least two-fold degenerate. In the situation that \( \kappa = \mu = 1 \), the Kramers pairs \( \{(v_j, T \bar{v}_j)\} \) consist of different charge states, so the necessary degeneracy is not enlarged. So the ES is not necessary degenerate for \( \kappa = \mu = 0 \), otherwise it is at least two-fold degenerate.

**B. For symmetry enriched topological phases**

Because the local matrices of the non-trivial topological phases have the peculiar structures: \( A^i = Y^i_{p/n} \otimes d^i \), we introduce the sub-block transfer operator \( \mathbb{E}' = \sum_i d^i \otimes d^i \). The dominant left (right) eigenvector \( \sigma_L (\sigma_R) \) of \( \mathbb{E}' \) is non-degenerate, because the MPS generated by \( d^i \) are injective. Moreover, there are \( n \)-fold degenerate left and right dominant eigenvectors of \( \mathbb{E} \), which are given by \( \sigma_L = Y_n^i \otimes \sigma_L \) and \( \sigma_R = Y_n^i \otimes \sigma_R \) with \( r \in \mathbb{Z}_n \). It can be proved that the spectrum of the whole reduced density matrix is only determined by the charge-0 dominant eigenvectors:\(^{17,19} \)

\[ \rho = \sigma_{L,0} \sigma_{R,0} = 1_n \otimes \sigma_L^0 \sigma_R^0 = 1_n \otimes \rho'. \quad (49) \]

So the ES has \( n \)-fold degeneracy without imposing the TR symmetry.

When discussing the degeneracy of ES with TR symmetry, we must know which part of \( \mu \) produces. For even \( n, \mu \) comes from the non-trivial part of the MPS. Considering the relation previously derived: \( T_0 = Q_{p/n} \otimes T_{0,0} \), the sub-block matrices \( d^i \) obey the transformation \( d^i = \omega_{p/n}^i \sigma L_{0,0}^{-1} d^i \sigma L_{0,0} \), from which \( \mathbb{E}' = T_{0,0} \sigma_L \sigma_R \sigma_L^{-1} \sigma_R^{-1} = \mathbb{E} \).

\[ (T_{0,0} \otimes T_{0,0}^{-1}) \mathbb{E}' (T_{0,0}^{-1} \otimes T_{0,0}) = \mathbb{E}'. \quad (50) \]

So \( T_{0,0} \) is irrelevant to the definition of \( \mu \), the symmetry of \( \mathbb{E}' \) as well as that of \( \sigma_L \) and \( \sigma_R \), have the same properties for \( \mu = 0 \), and we can predict that \( \mu \) will not change the degeneracy of ES for even \( n \). Then the discussion for \( T_{0,0} \) of \( \rho' \) is the same as that of the last subsection, because \( \rho' \) is also \( T_{0,0} \) symmetric:

\[ T_{0,0} \rho' T_{0,0}^{-1} = \rho'. \quad (51) \]

Notice that \( p/n \) must be odd for even \( n \). Therefore, if \( \kappa = 0 \), the parafermion zero modes produce the \( n \)-fold degeneracy. If \( \kappa = 1 \), the eigenstates \( v'_j \) of \( \rho' \) form the Kramers pairs \( \{(v'_j, T_{0,0} v'_j)\} \), so the total degeneracy of ES is \( 2n \).

For the odd \( n \) case, \( \mu \) comes from the trivial part and is defined by \( T_{0,0} \sigma_{p/n} = (-1)^{p/n} \sigma_{p/n} T_{0,0} \), and Eq.(50) is still satisfied. Because \( \mu \) must be 0 for odd \( p/n \), the ES is \( n \)-fold degenerate for \( \kappa = 0 \) and \( 2n \)-fold degenerate for \( \kappa = 1 \). For even \( p/n \), the \( n \)-fold degenerate ES is contributed by the parafermion zero edge modes for \( \kappa = 0 \) and \( \mu = 0 \), otherwise there is a \( 2n \)-fold degenerate ES protected by both parafermion zero edge modes and SPT order.

\[ \text{FIG. 2: The transfer operator and its left dominate eigenvector (a) and right dominate eigenvector (b). (c) Applying the TR on the eigen-equation. (d) Because } T^{-1} \sigma_R T \text{ is the eigenvector of } \mathbb{E}, \sigma_R^{\sigma R} T^{-1} \text{ is also the eigenvector of } \mathbb{E}. \]
C. For SSB coexisting SPT phases

In this case, all sub-blocks have the same sub-block transfer operator $E' = \sum_{i} d[i] \otimes \bar{d}[i]$, and the reduced density matrix for whole ground state subspace is thus a direct sum of the reduced density matrices for individual ground states,

$$\rho = \bigoplus_{r=0}^{n-1} \rho' = \bigoplus_{r=0}^{n-1} \sigma'_L \sigma'_R,$$

where $\sigma'_L$ and $\sigma'_R$ are the left and right dominant eigenvectors of the sub-block transfer operator $E'$. Because the system just picks one of the degenerate ground states, we just consider the ES of $\rho'$. The corresponding analysis is the same as that of SPT phases. So there is no topological degeneracy. $\kappa$ and $\mu$ have the same interpretations as the SPT phases.

VII. DISCUSSION AND CONCLUSION

Actually, for the SPT phases and the SSB coexisting SPT phases, the MPS have the same properties as those of bosonic MPS in one dimension. So their classification can be fitted into the framework of second group cohomology classifying bosonic SPT phases. In fact, the SPT phases enriched from the trivial phase are classified by the second group cohomology $H^2(\mathbb{Z}_p \times \mathbb{Z}_n^T, U(1)) = \mathbb{Z}_2 \times \mathbb{Z}_{gcd(p,n)}$, where $\mathbb{Z}_n^T$ is the TR symmetry group and “gcd” denotes the greatest common divisor. The SSB coexisting SPT phases are also classified by the subgroup $\mathbb{Z}_p/n \times \mathbb{Z}_n^T$ and SPT order $H^2(\mathbb{Z}_p/n \times \mathbb{Z}_n^T, U(1)) = \mathbb{Z}_2 \times \mathbb{Z}_{gcd(p,n)}$ under this subgroup.

However, the classification of those symmetry enriched topological phases is different from those bosonic MPS. Employing the recent classification for the one-dimensional interacting fermions with on-site symmetries, we can also integrate our results for the one-dimensional parafermion systems into this generalized framework. Therefore, the $\mathbb{Z}_p$ symmetric non-trivial topological phases labelled by $n$ with the TR symmetry can be classified by both $H^2(\mathbb{Z}_p \times \mathbb{Z}_n^T, U(1)) = \mathbb{Z}_2 \times \mathbb{Z}_{gcd(p,n)}$ and $H^1(\mathbb{Z}_n^T, \mathbb{Z}_n) = \mathbb{Z}_{gcd(n,2)}$, where $\mathbb{Z}_2$, $\mathbb{Z}_{gcd(p,n)}$, and $\mathbb{Z}_{gcd(n,2)}$ correspond to the indices $\kappa$, $\mu$, and $\nu$, respectively. Here the second group cohomology $H^2(\mathbb{Z}_p/n \times \mathbb{Z}_n^T, U(1))$ labels the SPT order under the symmetry group $\mathbb{Z}_p/n \times \mathbb{Z}_n^T$, and the first group cohomology $H^1(\mathbb{Z}_n^T, \mathbb{Z}_n)$ just classifies the actions of TR symmetry on the edge modes. Moreover, the non-trivial SPT order given by the second group cohomology double the degeneracy of the ES.

In conclusion, using the parafermionic MPS, we have established the complete classification of one-dimensional gapped phases of $\mathbb{Z}_p$ parafermions with the TR symmetry satisfying $T^2 = 1$. The possible gapped phases are enriched and classified by both $H^1(\mathbb{Z}_n^T, \mathbb{Z}_n)$ and $H^2(\mathbb{Z}_p/n \times \mathbb{Z}_n^T, U(1))$.

Acknowledgment.- The authors would like to thank Guo-Yi Zhu and Zi-Qi Wang for their stimulating discussion and acknowledges the support of National Key Research and Development Program of China (No.2017YFA0302902).
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