Logarithmic corrections to $O(a^2)$ lattice artifacts

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Abstract
We compute logarithmic corrections to the $O(a^2)$ lattice artifacts for a class of lattice actions for the non-linear $O(n)$ sigma-model in two dimensions. The generic leading artifacts are of the form $a^2 \ln(a^2)^{n/(n-2)}$. We also compute the next-to-leading corrections and show that for the case $n = 3$ the resulting expressions describe well the lattice artifacts in the step scaling function, which are in a large range of the cutoff apparently of the form $O(a)$. An analogous computation should, if technically possible, accompany any precision measurements in lattice QCD.
1. Most of our knowledge concerning renormalization of quantum field theories stems
from perturbation theory. Although there are no rigorous proofs in general, many
of the results are structural and hence considered to carry over to non-perturbative
formulations. Indeed there is supporting evidence from various studies, e.g. of
soluble models in 2 dimensions and of $1/n$ expansions of some theories.

The same situation holds concerning cutoff artifacts in lattice regularized the-
ories. It is generally accepted that these artifacts are summarized in Symanzik’s
effective action. In this framework generic lattice artifacts are, in particular for
asymptotically free (or trivial) theories, expected to be integer powers in the lat-
tice spacing $O(a^p), p = 1, 2, \ldots$ up to possible multiplicative logarithmic corrections.
This is an extremely important ansatz in the extrapolation of lattice data to the con-
tinuum limit, especially for present computations of lattice QCD where the lattice
spacings are typically around 0.1fm.

In this letter we will examine in more detail Symanzik’s theory for the 2-
dimensional non-linear $O(n) \sigma$-model. In such a simple bosonic model lattice ar-
tifacts are expected to be of the form $O(a^2)$. It was thus rather surprising that
precision measurements [1], some years ago, of certain observables in the O(3) sigma
model exhibited apparently linearly dependent $O(a)$ artifacts for a rather large range
of computable lattice spacings. The importance of finding the solution of this puzzle
was emphasized by Hasenfratz in his lattice plenary talk in 2001 [2].

To define the measured quantity referred to above, one considers the model
confined to a finite (1-dimensional) box of extension $L$ (with periodic boundary
conditions). The LWW coupling [3] is defined as
\[ u_0 = L m(L), \tag{1} \]
where $m(L)$ is the mass gap of the theory in finite volume. Next one measures $u_1$,
defined similarly with doubled box size. In the continuum limit $u_1$ is a function of
$u_0$, called the step scaling function $u_1 = \sigma(2, u_0)$. For the lattice regularized theory
there are lattice artifacts and
\[ u_1 = 2L m(2L) = \Sigma(2, u_0, a/L). \tag{2} \]
The advantage of this measurement for the purpose of studying lattice artifacts is
that there is no need to know the box size $L$ or the mass gap $m(L)$ in physical units.

The results of the MC measurements are shown in Fig. [1]. One can see that
the lattice artifacts (cutoff effects) are very nearly linear as function of the lattice
spacing $a$ both for the case of the standard lattice action (ST) and for a modified
action (MOD). Although the effects are in this case relatively very small, they seem
not of the theoretically expected form. Note however the encouraging feature that
computations with different lattice actions are consistent with the same continuum
limit, supporting the crucial concept of universality.
The 2-dimensional O(3) model is integrable and the finite volume mass gap (and hence the step scaling function) is exactly calculable using thermodynamic Bethe Ansatz techniques [4]. However, it turned out that the knowledge of the exact continuum limit did not help much in clarifying the problem of artifacts. This is shown in Fig. 2, where the exact continuum values are already subtracted. The “linear” fit assuming the functional form $\delta(a) = c_1 a + c_2 a \ln a + c_3 a^2$, still gives a much better representation ($\chi^2$/dof = 0.9) than a “quadratic” fit of the form $\delta(a) = c_1 a^2 + c_2 a^2 \ln a + c_3 a^4$ which has an unacceptable $\chi^2$/dof = 7.7. Note that although the $L/a \geq 10$ data are used in the fits, the “linear” fit describes well also the three coarsest points.

In the early 80’s Symanzik was working on the nature of lattice artifacts, in particular with respect to his improvement program [5,6,7,8]. In this letter the general theory is not discussed; we only consider Symanzik’s theory applied to the 2-dimensional O(n) $\sigma$-model. Nevertheless the spirit of the general theory can already be understood by studying this example. We shall see, based on Symanzik’s theory, why in the 2-dimensional $\sigma$-model quadratic artifacts are expected and how the particular logarithmic corrections predicted by this theory solves the puzzle of apparent linear artifacts. Similar conclusions have been reached previously in studies of the O(n) model in the first orders of the $1/n$ expansion [9,10]. We can here only give a brief description; full details will be presented in a separate paper [11].
2. We write the lattice Lagrangian including the source terms symbolically as

$$\mathcal{L}_{\text{latt}} = \frac{1}{2\lambda_0^2} \left( \partial_\mu S \cdot \partial_\mu S \right)_{\text{latt}} - J \cdot S, \quad S^2 = 1,$$

with some lattice regularization of the kinetic term. This is used in the generating functional for bare lattice $S$-field correlation functions, which, after Fourier transformation, become functions of the momenta, the bare lattice coupling $\lambda_0$ and the lattice spacing $a$. Performing, for fixed momenta and coupling, a small $a$ expansion we can write them as

$$G_X^{\text{latt}}(\lambda_0, a) = G_X^{(0)}(\lambda_0, a) + a^2 G_X^{(1)}(\lambda_0, a) + O(a^4),$$

where the upper index $X$ symbolizes any $r$-point correlation function (in $x$-space or in Fourier space), and both the scaling functions $G_X^{(0)}$ and the leading cutoff corrections $G_X^{(1)}$ are still weakly (logarithmically) depending on $a$.

The separation of the full lattice correlation function into a scaling piece and cutoff corrections is unambiguous and straightforward in perturbation theory (PT). In fact, in PT at $\ell$ loop order both terms are finite (order $\ell$) polynomials in $\ln a$.

One of our main assumptions here is that the expansion (4) makes sense also beyond PT. Usual renormalization theory deals with the scaling part $G_X^{(0)}$. Symanzik’s important contribution was to show that the next term $G_X^{(1)}$ can be generated using an effective Lagrangian.
3. Symanzik’s local effective Lagrangian \( \mathcal{L}_{\text{eff}} \) can be specified in the continuum in \( D = 2 - \varepsilon \) dimensions in the framework of dimensional regularization:

\[
- \mathcal{L}_{\text{eff}} = -\mathcal{L} + a^2 \sum_{i=1}^{7} Y_i(g,\varepsilon) U_i ,
\]

where \( \mathcal{L} \) is the continuum Lagrangian with source terms

\[
\mathcal{L} = \frac{1}{2g_0^2} (\partial_\mu S \cdot \partial_\mu S) - \frac{1}{g_0} I \cdot S ,
\]

and the \( U_i \) form a basis of local operators of dimension 4 invariant under the lattice symmetries, discussed below. There are no such operators of dimension 3 and hence no \( O(a) \) terms.

First we recall the continuum part (6). Here \( g_0 \) is the bare coupling and the square of the bare \( O(n) \) field \( S^a(x) \) is normalized to unity which is parameterized, as usual, by \( S^i = g_0 \pi^i, \ i = 1, \ldots, n-1; \ S^n = \sigma = \sqrt{1 - g_0^2 \pi^2} \). The source dependent action \( A = \int d^D x L(x) \) is used in the generating functional

\[
Z[I] = \int (D\pi) \ e^{-A} ,
\]

which can be used to obtain bare correlation functions of the field \( S^a(x) \):

\[
\hat{G}^{a_1 \ldots a_r}(x_1, \ldots, x_r) = g_0^{2r} Z^{-1} (I_0) \frac{\delta}{\delta I^{a_1}(x_1)} \cdots \frac{\delta}{\delta I^{a_r}(x_r)} \bigg|_{I_0} Z[I] ,
\]

where the functional derivative is taken at \( I_0(x) = m_0^2 \delta^{an} \); i.e. a mass term (external magnetic field) is introduced to avoid infrared singularities. As is well known, for \( O(n) \) invariants the limit \( m_0 \to 0 \) can be taken at the end of the calculation.

In their seminal paper [12] Brézin, Zinn-Justin and Le Guillou prove the renormalizability of the \( O(n) \) model using functional methods. They showed that the generating functional \( Z^{-1} [I_0] Z[I] \) is finite as a function of the renormalized quantities \( j^a(x), g, \mu, m_R \) if we write

\[
I^a(x) = Z_1(g,\varepsilon) Z^{-1/2}(g,\varepsilon) g^2 j^a(x) + I_0^a ,
\]

\[
g_0^2 = \mu^2 Z_1(g,\varepsilon) g^2 ,
\]

\[
m_0^2 = Z_1(g,\varepsilon) Z^{-1/2}(g,\varepsilon) m_R^2 ,
\]

where in the minimal subtraction scheme the renormalization constants contain only pole terms in \( \varepsilon \).

Functional derivation with respect to the source \( j^a(x) \) gives renormalized correlation functions, i.e. correlation functions of the renormalized fields \( S^a_R = Z^{-1/2} S^a \):

\[
\hat{G}^X_{(R)}(g,\mu,\varepsilon) = Z^{-1/2}(g,\varepsilon) \hat{G}^X(g_0,\varepsilon) .
\]
We assume that $X$ is $O(n)$ invariant and that the $m_R \to 0$ limit has been taken. Finiteness means that the limit
\[ G_X^R(g, \mu) = \lim_{\varepsilon \to 0} \tilde{G}_X^R(g, \mu, \varepsilon) \]
exists and defines the renormalized correlation function in two dimensions.

The renormalization group (RG) equations express the fact that the bare correlation functions are independent of the renormalization scale $\mu$. In terms of the renormalized correlation functions this is expressed as
\[ \left\{ D + \frac{r}{2} \gamma(g) \right\} G_X^R(g, \mu) = 0, \]
where the RG differential operator is
\[ D = \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g}, \]
and the RG beta and gamma functions are defined as
\[ \beta(g) = \frac{\varepsilon g}{2} - \frac{\varepsilon g}{2 + g \frac{\partial \ln Z_{1(g, \varepsilon)}}{\partial g}} = -\beta_0 g^3 - \beta_1 g^5 - \beta_2 g^7 + \ldots \]
\[ \gamma(g) = \left\{ \beta(g) - \frac{\varepsilon g}{2} \right\} \frac{\partial \ln Z(g, \varepsilon)}{\partial g} = \gamma_0 g^2 + \gamma_1 g^4 + \ldots . \]

Next, the dimension four operators $U_i$ in (5) are linear combinations of the 5 Lorentz scalar operators considered by Brézin et al [12]:
\begin{align*}
O_1 &= \frac{1}{8} (\partial_\mu S \cdot \partial_\mu S)^2, & O_2 &= \frac{1}{8} (\partial_\mu S \cdot \partial_\nu S) (\partial_\mu S \cdot \partial_\nu S), \\
O_3 &= \frac{1}{2} \Box S \cdot \Box S, & O_4 &= \frac{1}{2} \alpha \partial_\mu S \cdot \partial_\mu S, & O_5 &= \frac{1}{8} \alpha^2,
\end{align*}
\begin{equation}
\text{where} \quad \alpha = \frac{\Box \sigma + I^n(x)}{\sigma},
\end{equation}
and a further two operators which are only invariant under discrete lattice rotations:
\begin{align*}
A &= \sum_{\mu=1}^{D} \hat{t}_{\mu\mu\mu}, & B &= \sum_{\mu=1}^{D} \hat{k}_{\mu\mu\mu},
\end{align*}
\begin{equation}
\text{where} \quad \hat{t} \text{ and } \hat{k} \text{ are the traceless parts of the symmetric tensors } t, k:
\end{equation}
\begin{align*}
t_{\mu\nu\rho\sigma} &= S \cdot \partial_\mu \partial_\nu \partial_\rho \partial_\sigma S, \\
k_{\mu\nu\rho\sigma} &= \frac{1}{3} \{ (\partial_\mu S \cdot \partial_\nu S) (\partial_\rho S \cdot \partial_\sigma S) + 2 \text{ perms} \}.
\end{align*}
5
Although the source dependent operators $O_4$ and $O_5$ look $O(n)$ non-invariant, Brézin et al show that they must be included in the operator renormalization scheme for consistency. The operators renormalize multiplicatively according to

$$U_i(R) = K_{ij}(g, \varepsilon) U_j,$$  

(20)

where the matrix of renormalization constants is block diagonal, consisting of a $2 \times 2$ block for $i, j = 6, 7$ ($\propto A, B$), and a $5 \times 5$ block for $i, j = 1, 2, 3, 4, 5$. The operator renormalization matrix is of the form

$$\mathcal{K}_{ij}(g, \varepsilon) = \delta_{ij} - \frac{g^2}{\varepsilon} k_{ij} + \frac{g^4}{2 \varepsilon} \nu_{ij}^{(2)} + \frac{g^4}{2 \varepsilon^2} (k_{is}k_{sj} + 2\beta_0 k_{ij}) + \ldots.$$  

(21)

Our one loop result $k_{ij}$ for the $5 \times 5$ sub-block agrees with that which is obtained from the computation of Brézin et al [12] in the basis (16).

Renormalized correlation functions with one operator insertion are given by

$$G^X_{i(R)}(g, \mu) = \lim_{\varepsilon \to 0} Z^{-r/2}(g, \varepsilon) K_{ij}(g, \varepsilon) G^X_j(g_0, \varepsilon).$$  

(22)

They satisfy the RG equation

$$\left\{ D + \frac{r}{2} \gamma(g) \right\} G^X_{i(R)}(g, \mu) + \nu_{ij}(g) G^X_{j(R)}(g, \mu) = 0,$$  

(23)

where the anomalous dimension matrix is defined by

$$\nu_{ij}(g) = K_{is}(g, \varepsilon) \left( \beta(g) - \frac{\varepsilon g}{2} \right) \frac{\partial (K^{-1})_{sj}(g, \varepsilon)}{\partial g} = -k_{ij}g^2 + \nu_{ij}^{(2)} g^4 + \nu_{ij}^{(3)} g^6 + \ldots.$$  

(24)

We have chosen the basis $U_i$ such that the one-loop anomalous dimension matrix $k_{ij}$ is diagonal of the form

$$k_{ij} = 2\beta_0 \Delta_i \delta_{ij},$$  

(25)

with eigenvalues corresponding to

$$\Delta_i = \left\{ \frac{n}{n-2} : -1 ; \frac{1-n}{n-2} ; \frac{1}{n-2} ; 0 ; -1 \right\}.$$  

(26)

Now defining

$$\tilde{c}_j(g, \varepsilon) = \sum_{i=1}^{7} Y_i(g, \varepsilon) K_{ij}^{-1}(g, \varepsilon)$$  

(27)

we can rewrite the second term in (5) as

$$\sum_{i=1}^{7} Y_i U_i = \sum_{i=1}^{7} \tilde{c}_i U_i(R).$$  

(28)
The limit $\hat{c}_i(g, 0)$ must therefore exist:

$$c_i(g) = \hat{c}_i(g, 0) = \sum_{\ell=0}^{\infty} c_i^{(\ell)} g^{2\ell}.$$  \hspace{1cm} (29)

The coefficients $c_i$ depend on the particular lattice action. In our investigations we only considered actions quadratic in the spins:

$$A = \frac{\beta}{2} \sum_{x,y} \sum_a S^a(x) K(x-y) S^a(y) ,$$  \hspace{1cm} (30)

with $\beta = 1/\lambda_0^2$. Here $K$ is short range, satisfying $\sum_x K(x) = 0$, and $K(z) = K(Rz)$, where $R$ is a lattice rotation or reflection. For the standard action $K(z) = \sum_{\mu} [2\delta_{z,0} - \delta_{z,\mu} - \delta_{z,-\mu}]$.

The tree level effective action for this class of lattice actions involves only the operators $A$ and $O_3$:

$$- \mathcal{L}^{(0)}_{\text{eff}} = - \frac{1}{2\lambda_0^2} (\partial_{\mu} S \cdot \partial_{\mu} S) + \frac{a^2}{\lambda_0^2} \left\{ \frac{e_4}{24} A + \frac{e_0}{16} O_3 \right\} + O(a^4) ,$$  \hspace{1cm} (31)

where $e_0 = e_4 = 1$ for the standard action, and off-shell improved actions are such that $e_0 = e_4 = 0$. From this we obtain directly the leading coefficients $c_i^{(0)}$ in (29).

4. With these preparations the precise relation between the lattice correlation functions and those obtained by using the effective action can now be specified as

$$G_{X(0)}(\lambda_0, a) = y^x(g) G_{(R)}^X \left( g, \frac{1}{a} \right) ,$$  \hspace{1cm} (32)

$$G_{X(1)}(\lambda_0, a) = y^x(g) \sum_{i=1}^{7} c_i(g) G_{i(R)}^X \left( g, \frac{1}{a} \right) ,$$

where we have identified (for simplicity) the scale parameter $\mu$ of dimensional regularization with the inverse of the lattice spacing $a$. The finite wave function renormalization constant $y(g)$ comes from the relation $j^a(x) = y(g) J^a(x)$ between the lattice source $J$ and the (renormalized) dimensional regularization source $j$.

The result (32) can also be written as

$$G_{\text{tatt}}^X(\lambda_0, a) = G_{X(0)}(\lambda_0, a) \{ 1 + a^2 \delta^X(\lambda_0, a) \} + O(a^4) ,$$  \hspace{1cm} (33)

where

$$\delta^X(\lambda_0, a) = \sum_{i=1}^{7} c_i(g) \delta^X_i(g, a)$$  \hspace{1cm} (34)
\[ \delta_i^X(g,a) = \frac{\mathcal{G}^X_{i(R)}(g,\frac{1}{a})}{\mathcal{G}^X_{(R)}(g,\frac{1}{a})}. \] (35)

It is easy to see that the functions \( \delta_i^X \) satisfy the RG equation
\[ \left\{ -a \frac{\partial}{\partial a} + \beta(g) \frac{\partial}{\partial g} \right\} \delta_i^X(g,a) = -\nu_{ij}(g) \delta_j^X(g,a). \] (36)

To derive an explicit expression for \( \delta^X \) we must solve this partial differential equation. For this purpose we introduce the matrix \( U_{ij}(g) \), which solves the ordinary differential equation
\[ U'_{ij}(g) = -\rho_{is}(g) U_{sj}(g), \] (37)
where
\[ \rho_{ij}(g) := \frac{\nu_{ij}(g)}{\beta(g)} = \frac{2\Delta_i}{g} \delta_{ij} + \sum_{\ell=2}^{\infty} \rho_{ij}^{(\ell)} g^{2\ell-3}. \] (38)

If we find the solution of (37) we can write the general solution of (36) as
\[ \delta_i^X(g,a) = U_{ij}(g) D_j^X(\Lambda), \] (39)
and the lattice artifacts are of the form
\[ \delta^X(\lambda_0,a) = \sum_{i=1}^{7} \hat{v}_i(g) D_i^X(\Lambda), \quad \hat{v}_i(g) = \sum_{s=1}^{7} c_s(g) U_{si}(g). \] (40)

The functions \( D_j^X \) depend only on \( \Lambda \), the RG invariant combination of \( g \) and \( a \). These functions are non-perturbative and depend on the quantity \( X \) we are considering. On the other hand, the coefficients \( \hat{v}_i(g) \) are perturbative and they remain the same for all physical quantities (but depend on the lattice action we started with).

We take the following ansatz:
\[ U_{ij}(g) = \left\{ \delta_{ij} + \sum_{\ell=2}^{\infty} k_{ij}^{(\ell)} g^{2\ell-2} \right\} g^{-2\Delta_i}, \] (41)

Here the coefficients \( k_{ij}^{(\ell)} \) may still weakly (logarithmically) depend on the coupling. This can arise if the difference between two eigenvalues \( \Delta_i - \Delta_j \) is a non-zero integer, which is possible in our case, i.e. for \( n = 3 \). We will however ignore this subtlety, because we have verified that for the quantities we need here it plays no role.

We can now write the lattice artifacts as
\[ \delta^X(\lambda_0,a) = \sum_{i=1}^{7} v_i(g) g^{-2\Delta_i} D_i^X(\Lambda), \] (42)
where
\[ v_i(g) = c_i(g) + \sum_s c_s(g) \sum_{\ell=2}^{\infty} k^{(\ell)}_{s_i} g^{2\ell-2} = \sum_{\ell=0}^{\infty} v_i^{(\ell)} g^{2\ell}. \] (43)

The spectrum of one-loop eigenvalues given by (26) plays a crucial role in our considerations. The leading term corresponds to
\[ \Delta_1 = \frac{n}{n-2} = n\chi = 1 + 2\chi, \] (44)
(where \( \chi := 1/(n-2) \)) and the subleading one to
\[ \Delta_5 = \frac{1}{n-2} = \chi. \] (45)

We thus have the leading expansion
\[
\delta^X(\lambda_0, a) = v_1 D_1^X (g^{-2})^{1+2\chi} + v_5 D_5^X (g^{-2})^\chi + \ldots,
\]
\[ = D_1^X \left\{ (g^{-2})^{1+2\chi} v^{(0)}_1 + v^{(1)}_1 (g^{-2})^2 \chi \right\} + O ((g^{-2})^\chi). \] (46)

It turns out that for the \( n=3 \) case \( v^{(0)}_5 = c^{(0)}_5 = 0 \). This means that in this case the corrections start one power later and we have the leading expansion
\[
\delta^X(\lambda_0, a) = D_1^X \left\{ v^{(0)}_1 g^{-6} + v^{(1)}_1 g^{-4} + v^{(2)}_1 g^{-2} \right\} + O(1). \] (47)

The first expansion coefficients are
\[
v^{(0)}_1 = c^{(0)}_1 = \frac{e_0}{4(n-1)}, \quad v^{(1)}_1 = c^{(1)}_1 + \sum_s c^{(0)}_s k^{(2)}_{s_1}. \] (48)

Concretely we have
\[ k^{(2)}_{s_1} = \frac{1}{2(\Delta_1 - 1 - \Delta_s) \rho^{(2)}_{s_1}}, \] (49)

which is different from zero for \( s = 1, 2 \) only.

This and the connection between the lattice coupling \( \lambda_0 \) and \( g \) is all we need to write down the final result:
\[
\delta^X(\lambda_0, a) = e_0 \left| D_1^X (\Lambda) \left\{ \left( \frac{\tilde\beta}{\beta} \right)^{1+2\chi} + r^{(2)} \left( \frac{\tilde\beta}{\beta} \right)^2 \right\} + O (\tilde\beta^\chi) \right| \] (50)
for \( n \geq 4 \), and
\[
\delta^X(\lambda_0, a) = e_0 \left| D_1^X (\Lambda) \left\{ \tilde\beta^3 + r^{(2)} \tilde\beta^2 + r^{(3)} \tilde\beta \right\} + O(1) \right|, \] (51)
for \( n = 3 \), where we have introduced the inverse coupling \( \tilde{\beta} = 2\pi/\lambda_0^2 \). We expect this form of artifacts to be generically present for all observables. We note that our final result, eq. (50), is completely consistent with the large \( n \) results of ref. [9].

We have computed the coefficient \( r^{(2)} \) which is composed of three contributions:

\[
 r^{(2)} = r^{(2)}_I + r^{(2)}_{II} + r^{(2)}_{III} .
\]  

(52)

For \( n = 3 \) it would be nice to know the 3-loop coefficient \( r^{(3)} \); its computation would however be a major undertaking\(^1\).

For the computation of the first term in (52)

\[
 r^{(2)}_I = \frac{8\pi(n - 1)}{\epsilon_0} c_1^{(1)} ,
\]  

(53)

we need the 1-loop coefficients of the effective action. We obtained these by calculating the 2- and 4-point functions in both regularizations. More precisely we need the scaling part and the \( O(a^2) \) piece of the lattice correlation functions, and in the continuum we need the original correlation functions as well as the ones where those dimension four operators that appear in the tree level effective action are inserted. This is a long computation\(^2\), the details of which will be published in ref. [11].

The second term in (52)

\[
 r^{(2)}_{II} = \frac{n}{n - 2} (1 - \psi) ,
\]  

(54)

involves the 1-loop relation between the lattice coupling \( \lambda_0 \) and the renormalized coupling of dimensional regularization, \( g \):

\[
g^2 = \lambda_0^2 + \frac{\psi}{2\pi} \lambda_0^4 + \ldots
\]  

(55)

This has been known for a wide class of actions for a long time. e.g. for the case of ST at \( n = 3 \) one gets \( r^{(2)}_{II} = -3.98028 \).

Finally the third term can be obtained from the \( 5 \times 5 \) two-loop anomalous dimension matrix of the dimensionally regularized scalar operators

\[
r^{(2)}_{III} = (2\pi)^2 \left[ \frac{1}{n - 2} \nu_{11}^{(2)} + \frac{1}{n} \nu_{21}^{(2)} \right].
\]  

(56)

\(^1\)It is built from (among other things) the two-loop coefficient \( c_1^{(2)} \) appearing in the effective action and the three-loop anomalous dimension matrix elements \( \nu_{ij}^{(3)} \).

\(^2\)In the case of operator insertions there are many Feynman diagrams to be calculated. In the 4-point function case for example there are 11 non-trivial diagrams, which cannot be reduced to simpler ones like renormalization of tree diagrams or insertion of 2-point function subgraphs with operator insertion. Each of these is a complicated function of the four momenta and since they are topologically distinct, it is difficult to automate the calculation.
This is again a lengthy computation which leads to the simple result
\[ r_{III}^{(2)} = -2 - \frac{9}{2(n-2)}. \] (57)

5. For the standard action we finally obtain
\[ r^{(2)} = -0.7625 - \frac{5.6416}{n-2} \frac{n}{4}, \] (58)
giving a very large negative coefficient, \(-7.1541\), for \(n = 3\). Using the result (51), we can write in this case the leading terms of the asymptotic series describing the lattice artifacts in terms of \(a\) and the inverse coupling:
\[ \text{const.} \, a^2 \left[ \beta^3 - 1.1386\beta^2 + O(\beta) \right]. \] (59)

Due to the big negative value of the subleading term, the coupling dependent function in the square bracket is a very rapidly growing function, which can compensate the decreasing of one power of the lattice spacing, resulting in a behavior approximately linear in \(a\) in this limited range of coupling between \(\beta \sim 1.6\) and \(\beta \sim 2.0\) as observed in [1]. To demonstrate this, we multiplied the measured lattice artifacts by \(L^2\) to remove the \(a^2\) factor and fitted a polynomial cubic in \(\beta\) to the resulting function. We first made a two-parameter fit, with the result
\[ 3.60 \left[ \beta^3 - 1.52\beta^2 \right]. \] We also tried a 2-parameter fit, where the relative coefficient of the subleading term was fixed at the value we obtained from our analysis; we found
\[ 3.06 \left[ \beta^3 - 1.14\beta^2 - 0.60\beta \right]. \] This fit is shown in Fig. 3. For both fits the two smallest lattices were omitted. Both fits are good, indicating that the MC measurements are consistent with the functional form given by our result (59).

For the modified action MOD, where in addition to the nearest neighbor coupling there is also coupling between spins in the diagonal direction (with equal strength), the results are similar. For this lattice \(e_0 = 5/3\) and
\[ r^{(2)} = -3.0153 - \frac{7.2625}{n-2} + 3n \frac{20}{20}, \] (60)
giving \(-9.8278\) for \(n = 3\). The artifacts are given by a formula similar to (59), where the overall constant is bigger by a factor \(5/3\). For the relative (MOD/ST) artifacts the overall non-perturbative constant cancels and we find the following parameter-free 1-loop prediction (for \(n = 3\)):
\[ \text{MOD/ST} = \frac{5}{3} \left[ 1 + \frac{0.224}{\beta} + O(1/\beta^2) \right], \] (61)
which varies between 1.85 and 1.89 in our limited range of coupling. This is in very good agreement with the ratio of artifacts found in [1], as can be seen in Fig. 1.
6. We conclude that Symanzik’s theory describes well lattice artifacts in the O(3) sigma-model. By looking into the details of this problem we had the opportunity to further gain confidence in Symanzik’s theory of artifacts and improvement. Similar computations should, in our opinion, accompany precision lattice studies of QCD in order to control and better estimate systematic errors arising from lattice artifacts for extrapolations to the continuum limit.

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