NON-ABELIAN ANOMALY AND THE LAGRANGIAN FOR RADIATIVE MUON CAPTURE

J. SMEJKAL, E. TRUHLÍK
Institute of Nuclear Physics, Czech Academy of Sciences,
250 68 Rež n. Prague,
Czech Republic
E-mail: smejkal@ujf.cas.cz, truhlik@ujf.cas.cz

F. C. KHANNA
Department of Physics, Theoretical Physics Institute, University of Alberta,
Edmonton, Alberta, Canada, T6G 2J1
and
TRIUMF, 4004 Wesbrook Mall, Vancouver, B.C., Canada, V6T 2A3
E-mail: khanna@phys.ualberta.ca

We discuss an anomalous Lagrangian of the $\pi\rho\omega a_1$ system in the presence of external electroweak fields which is suitable for constructing the amplitude for the radiative muon capture by proton.

1 Introduction

Recent measurement of the elementary reaction

$$\mu^- + p \longrightarrow n + \nu_\mu + \gamma,$$

(1)

at TRIUMF\footnote{TRIUMF, 4004 Wesbrook Mall, Vancouver, B.C., Canada, V6T 2A3} provides a value of the weak induced pseudoscalar constant $g_P$,

$$g_P(q^2 \approx 0.88 m_\mu^2) = (9.8 \pm 0.7 \pm 0.3) g_A(0),$$

(2)

which exceeds its value predicted by PCAC and pion-pole dominance by a factor $\approx 1.5$. The analysis of the data in\footnote{TRIUMF, 4004 Wesbrook Mall, Vancouver, B.C., Canada, V6T 2A3} is based on the radiative muon capture (RMC) amplitude\footnote{TRIUMF, 4004 Wesbrook Mall, Vancouver, B.C., Canada, V6T 2A3} obtained using low energy theorems. We have recently presented\footnote{TRIUMF, 4004 Wesbrook Mall, Vancouver, B.C., Canada, V6T 2A3} the RMC amplitude derived from a Lagrangian of the $\pi\rho\omega a_1$ system which reflects the SU(2)$_L \times$SU(2)$_R$ hidden local symmetry\footnote{TRIUMF, 4004 Wesbrook Mall, Vancouver, B.C., Canada, V6T 2A3}. This amplitude coincides in the leading order with the amplitude obtained from the low energy theorems but these amplitudes differ in the next order in momenta $k$ (photon) and $q$ (weak current).

Here we present our next step in analyzing the structure of the RMC amplitude. We start from the anomalous action for the $\pi\rho\omega a_1 D$ system given by Kaiser and Meissner\footnote{TRIUMF, 4004 Wesbrook Mall, Vancouver, B.C., Canada, V6T 2A3}. The most general Wess-Zumino anomalous action
involving pseudoscalars, vectors, axial vectors and electroweak particles reads

\[ \Gamma_{an}[\xi_L, \xi_R, \xi_M, L, R, \mathcal{L}, \mathcal{R}] = \Gamma_{WZW}^{cov}[U, \mathcal{L}, \mathcal{R}] + \sum_{i=1}^{14} \int_{M^4} \mathcal{L}_i[\xi_L, \xi_R, \xi_M, L, R, \mathcal{L}, \mathcal{R}]. \quad (3) \]

Here \( \Gamma_{WZW}^{cov}[U, \mathcal{L}, \mathcal{R}] \) is the covariant Wess-Zumino-Witten action containing pseudoscalars and the electroweak fields. It already satisfies the anomaly constraints. Generally, the 14 independent terms in the r.h.s. of Eq. (3) are given in Eqs. (3.8) and Eqs. (3.9) of Ref. 6. As the terms \( \mathcal{L}_1-\mathcal{L}_8 \) contain at least 4 particles in each vertex, only the terms \( \mathcal{L}_9-\mathcal{L}_{14} \) are of interest for our purpose. Later on, Kaiser and Meissner drop the weak interaction and also the D meson. In contrast to it, we consider the full electroweak interaction. As a result, our anomalous Lagrangian contains all the natural parity violating vertices which are necessary to construct a contribution to the RMC amplitude in such a way that this particular amplitude satisfies gauge invariance, CVC and PCAC constraints by itself.

2 Contribution from the Wess-Zumino-Witten anomalous action

The covariant Wess-Zumino-Witten anomalous action of pseudoscalars reads

\[ \Gamma_{WZW}^{cov}[U, \mathcal{L}, \mathcal{R}] = -i \frac{N_c}{240 \pi^2} \int_{M^5} \text{Tr}[\alpha^5]_{\text{covariantized}}, \quad (4) \]

where \( N_c \) is the number of colours and \( \alpha \) is a differential one-form

\[ \alpha = (\partial_\mu U)U^\dagger dx_\mu, \quad U(x) = exp[-i\Pi^a(x)\tau^a/f_\pi] \equiv \xi^2. \quad (5) \]

The integral is over a five-dimensional manifold \( M^5 \) whose boundary is the ordinary Minkowski space \( M^4 \). In the process of covariantization, one adds terms to the non-covariantized anomalous action which contain external gauge fields \( \mathcal{R}_\mu \) and \( \mathcal{L}_\mu \) in such a way that the covariantized form would satisfy the Wess-Zumino anomaly equation

\[ \delta \Gamma_{WZW}^{cov}[U, \mathcal{L}, \mathcal{R}] = -\frac{N_c}{24\pi^2} \int_{M^4} \text{Tr}[\epsilon_L \{(d\mathcal{L})^2 + \frac{1}{2}i(d\mathcal{L})^3\}] - (L \leftrightarrow R). \quad (6) \]

The external fields \( \mathcal{L}_\mu \) and \( \mathcal{R}_\mu \) are directly related to the external gauge bosons of the Standard Model.
The contribution from the action to the 3-point Lagrangian of interest is
\[ [L_{WZW}]^3 = \frac{e^2}{8\pi^2 f_\pi} \epsilon_{\kappa\lambda\mu} (\partial_\kappa \tilde{B}_\lambda) (\partial_\mu \tilde{V}_\nu \cdot \tilde{\Pi}), \tag{7} \]
where \( \tilde{B}_\lambda \) is an electroweak neutral field and \( \tilde{V}_\nu \) is a weak vector field (see below).

3 Contribution from the homogenous terms

As we have already mentioned, we include both the electromagnetic and the weak interactions and put \( D = 0 \). Then our analogue of Eqs. (3.10) of Ref. is
\[ \omega_L = \frac{1}{2} g \omega + \frac{1}{2} (\beta_A - \beta_V) - \frac{1}{6} e \tilde{B}, \tag{8} \]
\[ \omega_R = \frac{1}{2} g \omega - \frac{1}{2} (\beta_A + \beta_V) - \frac{1}{6} e \tilde{B}, \tag{9} \]
\[ \omega_M = \beta_M, \tag{10} \]
\[ F_L + \xi_M F_R \xi_M^\dagger = g d\omega + F_V, \tag{11} \]
\[ \xi_L \tilde{F}_L \xi_L^\dagger + \xi_M \xi_R F_R \xi_M^\dagger = \frac{1}{3} e d\tilde{B} + \tilde{F}_V, \tag{12} \]
\[ F_L - \xi_M F_R \xi_M^\dagger = -F_A, \tag{13} \]
\[ \xi_L \tilde{F}_L \xi_L^\dagger - \xi_M \xi_R F_R \xi_M^\dagger = -\tilde{F}_A. \tag{14} \]

The isoscalar and isovector components are separated as
\[ \beta_A = -g \tilde{\tau} \cdot \tilde{a} + e \tilde{\tau} \cdot \tilde{A} \equiv -g a + e A, \tag{15} \]
\[ \beta_V = -g \tilde{\tau} \cdot \tilde{\rho} + e \tilde{\tau} \cdot \tilde{V} \equiv -g \rho + e V, \tag{16} \]
\[ \beta_M = -g \tilde{\tau} \cdot \tilde{a}, \tag{17} \]
\[ F_V = g \tilde{\tau} \cdot d\tilde{\rho} + \frac{1}{2} ig^2 [ (\tilde{\tau} \cdot \tilde{\rho})(\tilde{\tau} \cdot \tilde{\rho}) + (\tilde{\tau} \cdot \tilde{a})(\tilde{\tau} \cdot \tilde{a}) ], \tag{18} \]
\[ F_A = g \tilde{\tau} \cdot d\tilde{a} + \frac{1}{2} ig^2 [ (\tilde{\tau} \cdot \tilde{\rho})(\tilde{\tau} \cdot \tilde{a}) + (\tilde{\tau} \cdot \tilde{\rho})(\tilde{\tau} \cdot \tilde{a}) ]. \tag{19} \]

We also introduce the following quantities,
\[ \tilde{F}_V = e (\tilde{\tau} \cdot d\tilde{\rho}) + \frac{1}{2} ie^2 [ (\tilde{\tau} \cdot \tilde{\rho})(\tilde{\tau} \cdot \tilde{\rho}) + (\tilde{\tau} \cdot \tilde{a})(\tilde{\tau} \cdot \tilde{a}) ], \tag{20} \]
\[ \tilde{F}_A = e (\tilde{\tau} \cdot d\tilde{a}) + \frac{1}{2} ie^2 [ (\tilde{\tau} \cdot \tilde{\rho})(\tilde{\tau} \cdot \tilde{a}) + (\tilde{\tau} \cdot \tilde{\rho})(\tilde{\tau} \cdot \tilde{a}) ]. \tag{21} \]
where the fields $\tilde{V}$ and $\tilde{A}$ are defined by the prescription

\begin{align}
(\tilde{V} + \tilde{A}) \cdot \tilde{\tau} &= \xi \left[ (\tilde{V} + \tilde{A}) \cdot \tilde{\tau} \right] \xi^\dagger + \frac{2}{e} i (d\xi) \xi^\dagger, \\
(\tilde{V} - \tilde{A}) \cdot \tilde{\tau} &= \xi^\dagger \left[ (\tilde{V} - \tilde{A}) \cdot \tilde{\tau} \right] \xi - \frac{2}{e} i \xi^\dagger (d\xi). 
\end{align}

Due to the properties of the forms $\tilde{F}_V$ and $\tilde{F}_A$ the following equations hold

\begin{align}
(\tilde{F}_V + \tilde{F}_A) \cdot \tilde{\tau} &= \xi \left[ (\tilde{F}_V + \tilde{F}_A) \cdot \tilde{\tau} \right] \xi^\dagger, \\
(\tilde{F}_V - \tilde{F}_A) \cdot \tilde{\tau} &= \xi^\dagger \left[ (\tilde{F}_V - \tilde{F}_A) \cdot \tilde{\tau} \right] \xi,
\end{align}

where the forms $\tilde{F}_V$ and $\tilde{F}_A$ correspond to the fields $\tilde{V}$ and $\tilde{A}$

\begin{align}
F_V &= e (\tilde{\tau} \cdot d\tilde{V}) + \frac{1}{2} ie^2 \left[ (\tilde{\tau} \cdot \tilde{V})(\tilde{\tau} \cdot \tilde{V}) + (\tilde{\tau} \cdot \tilde{A})(\tilde{\tau} \cdot \tilde{A}) \right], \\
F_A &= e (\tilde{\tau} \cdot d\tilde{A}) + \frac{1}{2} ie^2 \left[ (\tilde{\tau} \cdot \tilde{A})(\tilde{\tau} \cdot \tilde{V}) + (\tilde{\tau} \cdot \tilde{V})(\tilde{\tau} \cdot \tilde{A}) \right].
\end{align}

External fields $\tilde{V}$ and $\tilde{A}$ correspond to the gauge fields of the Standard Model

\begin{align}
V_{\mu}^\pm &= -A_{\mu}^\pm = \frac{1}{\sin \Theta_w} W_{\mu}^\pm \cos \Theta_e, \\
V_{\mu}^3 &= B_{\mu} + \cot \theta_w Z_{\mu} = \tilde{B}_{\mu} + \frac{1}{\sin (2 \Theta_w)} Z_{\mu}, \\
A_{\mu}^3 &= -\frac{1}{\sin (2 \Theta_w)} Z_{\mu}. 
\end{align}

Let us now use the 'unitary gauge'. Employing our Eqs. (8)-(14) we get analogues of Eqs. (3.12) for the Lagrangians $L_9$-$L_{14}$

\begin{align}
L_9 &= -g d\omega \; Tr[\beta V \beta A] + (g \omega - \frac{1}{3} e \tilde{B}) Tr[F_V \beta A], \\
L_{10} &= -g d\omega \; Tr[-2 \beta V \beta M] - 2 (g \omega - \frac{1}{3} e \tilde{B}) Tr[F_V \beta M], \\
L_{12} &= -\frac{1}{3} e d\tilde{B} \; Tr[\beta V \beta A] + (g \omega - \frac{1}{3} e \tilde{B}) Tr[\tilde{F}_V \beta A], \\
L_{13} &= -\frac{1}{3} e d\tilde{B} \; Tr[-2 \beta V \beta M] - 2 (g \omega - \frac{1}{3} e \tilde{B}) Tr[\tilde{F}_V \beta M], \\
L_{11} &= L_{14} = 0.
\end{align}
In order to be compatible with the final result of Ref. 6 given in Eqs. (A.1) of appendix A we define new linear combinations

\[ \tilde{\mathcal{L}}_7 = \mathcal{L}_9 + \frac{1}{2} \mathcal{L}_{10}, \quad (36) \]

\[ \tilde{\mathcal{L}}_8 = \frac{1}{2} \mathcal{L}_9 - \frac{1}{4} \mathcal{L}_{10}, \quad (37) \]

\[ \tilde{\mathcal{L}}_9 = \mathcal{L}_{12} + \frac{1}{2} \mathcal{L}_{13}, \quad (38) \]

\[ \tilde{\mathcal{L}}_{10} = \frac{1}{2} \mathcal{L}_{12} - \frac{1}{4} \mathcal{L}_{13}. \quad (39) \]

We now

1. put Eqs. (31)-(34) into Eqs. (36)-(39),
2. introduce Eqs. (15)-(17) into the new equations for \( \tilde{\mathcal{L}}_7 - \tilde{\mathcal{L}}_{10} \),
3. because we need only the 3-particle terms of the invariants \( \tilde{\mathcal{L}}_7 - \tilde{\mathcal{L}}_{10} \), we take into account only the linear parts of \( \tilde{\mathcal{V}}, \tilde{\mathcal{A}}, \) and \( \tilde{\mathcal{F}}_V \) using Eqs. (22)-(25)

\[ [\tilde{\mathcal{F}}_V]^l = \mathcal{F}_V, \quad (40) \]

\[ [\tilde{\mathcal{V}}]^l = \mathcal{V}, \quad (41) \]

\[ [\tilde{\mathcal{A}}]^l = \mathcal{A} + \frac{i}{e} [(d\xi)^l \xi^l + \xi^l (d\xi)]^l = \mathcal{A} + \frac{1}{e f_\pi} d\Pi, \quad (42) \]

Here \( Q = \vec{Q} \cdot \vec{\tau} \) for \( Q = \mathcal{V}, \mathcal{A}, \Pi. \)

4. remove the non-physical \( \pi - a_1 \) mixing by the redefinition of the axial meson field

\[ \tilde{a} = a - \frac{1}{2 f_\pi g} d\Pi. \quad (43) \]

As a result, we have the following set of equations

\begin{align*}
[\tilde{\mathcal{L}}_7][^3] = g d\omega \ Tr[(g\rho - e\mathcal{V})(\frac{1}{f_\pi} d\Pi + e\mathcal{A})] \\
+ (g\omega - \frac{1}{3} e \vec{B}) Tr[F_\mathcal{V}(\frac{1}{f_\pi} d\Pi + e\mathcal{A})], \quad (44) \\

[\tilde{\mathcal{L}}_8][^3] = -g d\omega \ Tr[(g\rho - e\mathcal{V})(\tilde{a} - \frac{1}{2} e\mathcal{A})] \\
- (g\omega - \frac{1}{3} e \vec{B}) Tr[F_\mathcal{V}(\tilde{a} - \frac{1}{2} e\mathcal{A})], \quad (45)
\end{align*}
\[ [\tilde{\mathcal{L}}_9]^{[3]} = + \frac{1}{3} e dB \text{Tr}[(g\rho - e\nu)(\frac{1}{f_\pi} d\Pi + eA)] \\
+ (g\omega - \frac{1}{3} e \tilde{B}) \text{Tr}[F\nu(\frac{1}{f_\pi} d\Pi + eA)], \quad (46) \]

\[ [\tilde{\mathcal{L}}_{10}]^{[3]} = - \frac{1}{3} e dB \text{Tr}[(g\rho - e\nu)(g\tilde{a} - \frac{1}{2} eA)] \\
- (g\omega - \frac{1}{3} e \tilde{B}) \text{Tr}[F\nu(g\tilde{a} - \frac{1}{2} eA)]. \quad (47) \]

We can now compare our Eqs. (44)-(47) with Eqs. (A.1) of Ref.\(^6\). It is easy to see that our \( \tilde{\mathcal{L}}_7 \) and \( \tilde{\mathcal{L}}_8 \) coincide with the same Lagrangians derived in Ref.\(^6\), if we put the external fields to zero. In this approximation, the Lagrangians \( \tilde{\mathcal{L}}_9 \) and \( \tilde{\mathcal{L}}_{10} \) are also zero. Naturally they are absent in Ref.\(^6\).

The parametrization of the interactions in the \( \pi\rho\omega a_1 \) system in the form \( \tilde{\mathcal{L}}_7 - \tilde{\mathcal{L}}_{10} \) has the advantage that each of these Lagrangians leads to the currents satisfying PCAC by itself. But in this case, the field of the \( a_1 \) meson enters isolated from the pion field, while the external gauge field \( \mathcal{A} \) enters simultaneously with the fields of both mesons. It will be more suitable for our purpose to separate the \( a_1 \) and \( \mathcal{A} \) fields. This is accomplished in the new combinations of the Lagrangians \( \tilde{\mathcal{L}}_7 - \tilde{\mathcal{L}}_{10} \)

\[ \tilde{\mathcal{L}}_i = \tilde{\mathcal{L}}_i, \quad i = 7, 9, \quad (48) \]

\[ \tilde{\mathcal{L}}_j = \tilde{\mathcal{L}}_j - \frac{1}{2} \tilde{\mathcal{L}}_{j-1}, \quad j = 8, 10. \quad (49) \]

We now present the final result written in the form convenient for construction of the amplitude for the radiative muon capture. With the help of Eqs. (A.2)\(^6\) we get

\[ [\tilde{\mathcal{L}}_7]^{[3]} = 2ig \varepsilon_{\kappa\lambda\mu\nu} \{ \partial_\kappa \omega_\lambda [(g\tilde{\rho}_\mu - e\nu_{\mu})(\frac{1}{f_\pi} \partial_\nu \Pi + e\tilde{A}_\nu)] \\
+ (g\omega_\kappa - \frac{1}{3} e \tilde{B}_\kappa) [(\partial_\lambda \tilde{\rho}_\mu)(\frac{1}{f_\pi} \partial_\nu \Pi + e\tilde{A}_\nu)], \quad (50) \]

\[ [\tilde{\mathcal{L}}_8]^{[3]} = -2ig \varepsilon_{\kappa\lambda\mu\nu} \{ \partial_\kappa \omega_\lambda [(g\tilde{\rho}_\mu - e\nu_{\mu})(g\tilde{a}_\nu + \frac{1}{2f_\pi} \partial_\nu \Pi)] \\
+ (g\omega_\kappa - \frac{1}{3} e \tilde{B}_\kappa) [(\partial_\lambda \tilde{\rho}_\mu)(g\tilde{a}_\nu + \frac{1}{2f_\pi} \partial_\nu \Pi)], \quad (51) \]
\[
\mathcal{L}_3^{[3]} = 2ie\varepsilon_{\kappa\lambda\mu\nu} \left\{ \frac{1}{3} \partial_\kappa \tilde{B}_\lambda \left[ (g\tilde{\rho}_\mu - e\tilde{V}_\mu) \cdot \left( \frac{1}{f_\pi} \partial_\nu \Pi + e\tilde{A}_\nu \right) \right] \\
+ (g\omega_\kappa - \frac{1}{3} e\tilde{B}_\kappa) \left[ (\partial_\lambda \tilde{V}_\mu) \left( \frac{1}{f_\pi} \partial_\nu \Pi + e\tilde{A}_\nu \right) \right] \right\}, \tag{52}
\]

\[
\mathcal{L}_{10}^{[3]} = -2ie\varepsilon_{\kappa\lambda\mu\nu} \left\{ \frac{1}{3} \partial_\kappa \tilde{B}_\lambda \left[ (g\tilde{\rho}_\mu - e\tilde{V}_\mu) \cdot (g\tilde{a}_\nu + \frac{1}{2f_\pi} \partial_\nu \Pi) \right] \\
+ (g\omega_\kappa - \frac{1}{3} e\tilde{B}_\kappa) \left[ (\partial_\lambda \tilde{V}_\mu) (g\tilde{a}_\nu + \frac{1}{2f_\pi} \partial_\nu \Pi) \right] \right\}. \tag{53}
\]

The strong vertices \(\rho\omega\pi\) and \(\rho\omega a_1\) are already known from the appendix A of Ref.\(\text{[6]}\). Then Eqs.(50)-(53) provide all other terms due to the presence of the external electroweak interactions which violate the natural parity and which should be used when constructing the amplitude for the radiative muon capture by proton. Their presence guarantees that such an amplitude will satisfy gauge invariance and CVC and PCAC constraints at the tree level exactly. The construction of the amplitude is in progress.

Acknowledgments

This work was supported in part by the grant GA ČR 202/97/0447. The research of F.C.K. is supported in part by NSERC.

References

1. G. Jonkmans et al., Phys. Rev. Lett. 77, 4512 (1996).
2. D. S. Beder and H. W. Fearing, Phys. Rev. D 39, 3493 (1989).
   H. W. Fearing, Phys. Rev. C 21, 1951 (1980).
3. J. Smejkal et al., Few–Body Systems 26, 175 (1999).
4. M. Bando et al., Phys. Rep. 164, 217 (1988).
5. U. - G. Meissner, Phys. Rep. 161, 213 (1988).
6. N. Kaiser and U. - G. Meissner, Nucl. Phys. A 519, 671 (1990).
7. J. Smejkal et al., Nucl. Phys. A 624, 655 (1997).
8. E. Witten, Nucl. Phys. B 223, 422 (1983).