Model Averaging for Generalized Linear Model with Covariates that are Missing completely at Random

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Abstract

In this paper, we consider the estimation of generalized linear models with covariates that are missing completely at random. We propose a model averaging estimation method and prove that the corresponding model averaging estimator is asymptotically optimal under certain assumptions. Simulation results illustrate that this method has better performance than other alternatives under most situations.

Key Words: Missing data completely at random, generalized linear models, asymptotically optimal.
1 Introduction

Our attempt in this paper is at developing an optimal model averaging method for generalized linear models (GLMs) with missing values on some covariates. Model averaging, as an alternative of model selection, is widely adopted for dealing with the uncertainty that the model selection takes. Unlike Bayesian model averaging, the current paper focuses on the method of determining weights from frequencies perspective.

The problem of missing values on some covariates is one of the most common challenge facing empirical researchers, which will bring about a great impact on the subsequent modeling as well as inference process. There is a large collection of literature dealing with missing covariates. See Little (1992) and Toutenburg, Heumann, Nittner, and Scheid (2002) for reviews of this topic. There are three types of missing data: missing completely at random (MCAR), missing at random (MAR) and missing not at random. Referring to Little (1992), data is MCAR if the probability of missing or not for a covariate is independent with any covariates (including itself). Data is MAR if the probability of missing or not for a covariate is independent with any covariates’ missing values and may depend on the covariates’ observed values only. Except for MCAR and MAR, the other types of missing data are all missing not at random. In the current paper, we focus on the case that the covariates are MCAR, which can happen in practice.

A very straightforward and widely used method to handle with missing covariates problem is complete-case (CC) analysis. The CC method estimates a model using observations with complete data and discards other observations that have some missing values of some covariates. Obviously, the discarding of data seems an unnecessary waste of information. Another method that is widely used to deal with missing data is mean imputation. However, it is well known that data imputation generally depends on observed covariates, so it is not appropriate for the MCAR situation. In order to make full use of the sample information and avoid the uncertain influence imputation bring, we propose a model averaging method, which is based on model averaging for GLMs Zhang, Yu, Zou, and Liang (2016) to combine some submodels, including one CC model and some partial data-based models. The partial data-based models can use the sample information sufficiently. Hereafter, we refer to this method as sufficient-sample-information (SSI) method, which will be described in details in section 2.

In this paper, we propose weight choice criterion based on the Kullback-Leibler (KL) loss. We prove that the average estimator is asymptotically optimal in the sense that the corresponding KL loss is asymptotically identical to that of the infeasible best model average estimator. To the best of
our knowledge, the current paper is the first work using model averaging to deal with missing data in GLMs.

The rest of the article is organized as follows. Section 2 describes the model framework and establishes the model averaging process. Section 3 shows the weight selection criteria and methods. Section 4 presents asymptotic optimality. Section 5 reports the results of the simulation studies on logistic and Poisson regression, respectively. Section 6 Summary. The relevant assumptions which are needed for the theorem’s proof and the proof processes are in the appendix.

2 Model framework and model averaging process

We consider the data generating process (DGP)

\[ f(y_i|\theta_i, \phi) = \exp\left\{ \frac{y_i\theta_i - b(\theta_i)}{\phi} + c(y_i, \phi) \right\}, \quad i = 1, \ldots, n. \] (1)

where \( \theta_i \) and \( \phi \) are parameters, and \( b(\cdot) \) and \( c(\cdot, \cdot) \) are known functions. The canonical parameter \( \theta_i \) connects the parameter \( \beta \) and the \( K \)-dimension covariate vector \( x_i \) in the form \( \theta_i = x_i^T \beta \). Here we assume that \( K \) is fixed. Suppose we have \( S \) candidate models and \( S \) is finite. We estimate \( \beta \) under different candidate models by maximum likelihood estimation. Let \( \theta_{0,i} \) be the true value of \( \theta_i \). We do not require that the true value \( \theta_{0,i} \) is indeed a linear combination of \( x_i \). In other words, it is not required that there exist a \( \beta \) so that \( \theta_{0,i} = x_i^T \beta \). Therefore, each of candidate model can be misspecified.

For a data set with \( K \) covariates and the sample size \( n \) to be fixed, the number of possible covariate missing cases is \( 2^K - 1 \) (we collect covariates whose position of missing are same into one group). Noting that not all such possible cases need be present in a data set, we assume that we have \( S - 1 \leq 2^K - 1 \) covariate missing cases, indexed by \( s = 2, \ldots, S \). For clear illustration, we provide the following example.

Example

Suppose we have a sample with \( K \) covariates and the sample size is \( n \). The covariates are missing completely at random. By sorting the position of missing covariates, the following data matrix can be obtained after a certain rearrangement of rows as well as columns:
\[ X = \begin{pmatrix}
X_{11} & X_{12} & X_{13} & X_{14} & X_{15} \\
X_{21} & missing & missing & X_{24} & X_{25} \\
X_{31} & X_{32} & X_{33} & missing & X_{35} \\
X_{41} & X_{42} & missing & missing & missing 
\end{pmatrix}_{n \times K} \]

where \( X_{i12} (i_1 \in \{1, 2, 3, 4\}, i_2 \in \{1, 2, 3, 4, 5\}) \) can be matrices. There are 5 cases with incomplete covariates:

- \((X_{11} \ X_{21} \ X_{31} \ X_{41})^T, \)
- \((X_{12} \ missing \ X_{32} \ X_{42})^T, \)
- \((X_{13} \ missing \ X_{33} \ missing)^T, \)
- \((X_{14} \ X_{24} \ missing \ missing)^T, \)
- \((X_{15} \ X_{25} \ X_{35} \ missing)^T, \)

so \( S = 6 \) in this example.

We denote the CC model by model 1, under which all the covariates are utilized but the sample size is smaller than \( n \). Alternatively, to utilize sufficient sample size, we may ignore some missing values of covariates. Since we have \( S - 1 \) cases of missing covariates, we can have \( S \) models in all, including one CC model and \( S - 1 \) SSI models. For the \( s^{th} \) model, let \( y^{(s)} \) be the associated dependent variable and \( X^{(s)} \) be the \( n_s \times K_s \) covariate matrix, where \( n_s \) is the sample size, \( K_s \) is the number of covariates. We assume \( X^{(s)} \) to be of full column rank. In example 1, model 1 has covariate matrix \((X_{11} \ X_{12} \ X_{13} \ X_{14} \ X_{15})\). The other five SSI models have following covariate matrices:

- \( X^{(1)} = (X_{11} \ X_{21} \ X_{31} \ X_{41})^T, \)
- \( X^{(2)} = (X_{12} \ X_{32} \ X_{42})^T, \)
- \( X^{(3)} = (X_{13} \ X_{33})^T, \)
- \( X^{(4)} = (X_{14} \ X_{24})^T \) and \( X^{(5)} = (X_{15} \ X_{25} \ X_{35})^T. \)

Let \( \zeta_s \) denote the index set of the columns of \( X \) used in model \( s \). Let \( \pi_s \) denote the projection matrix mapping \( \beta = (\beta_1, \ldots, \beta_K)^T \) to the subvector \( \pi_s \beta = \beta^{(s)} \) of components \( \beta_k, k \in \zeta_s \). Denote the maximum likelihood estimator of \( \beta^{(s)} \) as \( \hat{\beta}^{(s)} \). Then the estimator of \( \beta \) is \( \hat{\beta} = \pi_s^T \hat{\beta}^{(s)} \) for the \( s^{th} \) model. Some components of \( \hat{\beta}^{(s)} \) are zeros.

Let \( w = (w_1, \ldots, w_S)^T \) belonging in set: \( \mathcal{H} = \{ w \in [0,1]^S : \sum_{s=1}^S w_s = 1 \} \). Then the model averaging estimator of \( \beta \) is:

\[
\hat{\beta}(w) = \sum_{s=1}^S w_s \hat{\beta}^{(s)}.
\]
Replace missing values in \( X \) by zeros and denote the resulting matrix by \( \tilde{X} \).

In Example 1:

\[
\tilde{X} = \begin{pmatrix}
X_{11} & X_{12} & X_{13} & X_{14} & X_{15} \\
X_{21} & 0 & 0 & X_{24} & X_{25} \\
X_{31} & X_{32} & X_{33} & 0 & X_{35} \\
X_{41} & X_{42} & 0 & 0 & 0
\end{pmatrix}_{n \times K}
\]

Suppose \( \tilde{X} \) to be of full column rank, \( y = (y_1, \ldots, y_n)^T \), \( \theta = (\theta_1, \ldots, \theta_n)^T \). \( \theta_0 \) is the true value of \( \theta \). Then a model averaging estimator of \( \theta_0 \) is:

\[
\theta \{ \hat{\beta}(w) \} = [\theta_1 \{ \hat{\beta}(w) \}, \ldots, \theta_n \{ \hat{\beta}(w) \}]^T = \tilde{X} \hat{\beta}(w).
\]

Note that \( \tilde{X} \), not \( X \), appears in \( \theta \{ \hat{\beta}(w) \} \), because the use of \( X \) is infeasible.

### 3 Weight Choice

Our weight choice criterion is:

\[
\mathcal{G}(w) = 2\phi^{-1}B\{\hat{\beta}(w)\} - 2\phi^{-1}y^T \theta \{ \hat{\beta}(w) \} + \lambda_n w^T k,
\]

where \( \lambda_n = 2 \) like that of the penalty term of AIC, \( K = (k_1, \ldots, k_S)^T \), \( k_s \) is the number of columns of \( X \) used in the \( s^{th} \) candidate model.

\( \mathcal{G}(w) \) comes from the Kullback-Leibler (KL) loss which is defined as follows. Let \( \mu = Ey, B_0 = \sum_{i=1}^n b(\theta_{0_i}), B\{\hat{\beta}(w)\} = \sum_{i=1}^n b[\theta_i \{ \hat{\beta}(w) \}], \) and

\[
J(w) = \phi^{-1}B\{\hat{\beta}(w)\} - \phi^{-1}\mu^T \theta \{ \hat{\beta}(w) \}.
\]

The KL loss of \( \theta \{ \hat{\beta}(w) \} \) is

\[
KL(w) = 2 \sum_{i=1}^n E_{y^*} \{ log \{ f(y^* | \theta_0, \phi) \} - log \{ f(y^* | \theta_0 \hat{\beta}(w), \phi) \} \}
\]

\[
= 2\phi^{-1}B\{\hat{\beta}(w)\} - 2\phi^{-1}\mu^T \theta (\hat{\beta}(w)) - 2\phi^{-1}B_0 + 2\phi^{-1}\mu^T \theta_0
\]

\[
= 2J(w) - 2\phi^{-1}B_0 + 2\phi^{-1}\mu^T \theta_0.
\]

where \( y^* \) is another realization from \( f(\cdot | \theta_0, \phi) \) and independent of \( y \). Assume \( \phi \) is known. Typically, in logistic and Poisson regressions, \( \phi = 1 \). If \( \mu \) was known, we could obtain a weight vector by minimizing \( J(w) \) given the relationship between \( J(w) \) and \( KL(w) \) in (3). In practice, the minimization of \( J(w) \) is infeasible owing to the unknown parameter \( \mu \). An intuitive solution
is to estimate \( J(w) \). That is, we may use \( y \) to estimate \( \mu \) directly, i.e., we plug \( y \) into \( J(w) \). Then we can obtain weights by minimizing \( \phi^{-1}B\{\hat{\beta}(w)\} - \phi^{-1}y^T \theta\{\hat{\beta}(w)\} \). Unfortunately, this intuitive procedure leads to overfitting. To avoid the overfitting, we use (2) as our weight choice criterion.

The resultant weight vector is defined as

\[
\hat{w} = \arg\min_{w \in \mathcal{W}} \mathcal{B}(w)
\]  

(4)

4 Asymptotic Optimality

Let \( \beta^*_s \) be the parameter vector which minimizes the KL divergence between the true model (1) and the \( s^{th} \) candidate model. From Theorem 3.2 of White (1982), we know that, under certain regularity conditions, for \( s \in \{1, \ldots, S\} \),

\[
\hat{\beta}(s) - \beta^*_s = O_p(n^{-1/2}).
\]

Furthermore, if \( \frac{n_1}{n} c \), where \( n_1 \) is the sample size of CC model and \( c \) is a positive constant, since \( n > n_s > n_1 \), we have \( n_s = c^* n \) for some positive constant, then

\[
\hat{\beta}(s) - \beta^*_s = O_p(n^{-1/2}),
\]

(5)

for \( s \in \{1, \ldots, S\} \).

In order to study the optimality of the model averaging estimator, we need the following conditions.

**Condition(C.1)** \( ||\hat{X}^T \mu|| = O(n), ||\hat{X}^T \epsilon|| = O_p(n^{1/2}), \) and uniformly for \( w \in \mathcal{W} \),

\[
||\partial B(\beta)/\partial \beta^T|_{\beta=\hat{\beta}(w)}|| = O_p(n)
\]

for every \( \hat{\beta}(w) \) between \( \hat{\beta}(w) \) and \( \beta^*(w) \).

**Condition(C.2)** Uniformly for \( s \in \{1, \ldots, S\} \), \( n^{-1/2} ||\theta(\beta^*_s)||^2 = O(1) \).

**Condition(C.3)** \( n \xi_n^{-2} = o(1) \).

The following theorem establishes the asymptotic optimality of the model averaging estimator \( \theta\{\hat{\beta}(w)\} \).

**Theorem 1** If equation (5) and Conditions (C.1)-(C.3) are satisfied, and \( n^{-1/2} \lambda_n = O(1) \), then

\[
\frac{KL(\hat{w})}{\inf_{w \in \mathcal{W}} KL(w)} \rightarrow 1
\]

in probability as \( n \rightarrow \infty \).
Proof 1 See appendix.

Theorem 1 tell us, based on the estimated weight \(\hat{w}\), the model averaging estimate achieves the infimum of the KL loss.

5 Simulation study

In this section, we conduct two simulation experiments: logistic regression and Poisson regression, to demonstrate the finite sample performance of our model averaging method. Because our method can achieve asymptotic optimality for missing data, we denote it by MOPT. In the simulations, we compare the MOPT method with the CC model, model after mean imputation (MIM), model averaging after mean imputation (MIMA). We set sample size \(n \in \{100, 200\}\) and use KL loss (divided by \(n\)) for assessment. For each setting, we generate 1000 simulated data. To mimic the situation that all candidate models are misspecified, we pretend that the last covariate missed in all candidate models.

The simulation design is a logistic model. \(y_i\) is generated from \(\text{Binomial}(1, p_{0i})\) with

\[
p_{0i} = \frac{\exp(x_i^T \beta)}{1 + \exp(x_i^T \beta)},
\]

where \(\beta = (1, 0.2, -1.2, -1, 0.1)^T\) and \(x_i = (x_{i1}, x_{i2}, x_{i3}, x_{i4}, x_{i5})\) follow normal distribution with mean zeros, variance ones and the correlations between different components of \(x_i\) being 0.75. In order to simulate covariates which are missing completely at random, following Zhang (2013), we construct the missing data matrix as follows. Data missing only occurs with \(x_{i3}\) and \(x_{i4}\) for some \(i\). In order to control the missing structures, we generate \(\epsilon_i = (\epsilon_{i1}, \epsilon_{i2}) \sim \mathcal{N}(0, I_2)\), which is independent with \(x_i\). When \(\epsilon_{ik} < a, x_{i(k+2)}\) is missed \((k = 1, 2)\). We set the parameter \(a \in \{-0.3, 0, 0.5\}\) to control the ratio of missing observations. For MIMA, we consider all possible submodels.

Simulation results are as follows (\(\times 10^{-1}\)):
Table 1: Figure 1. Binomial

| n    | MOPT     | CC       | MIM      | MIMA     |
|------|----------|----------|----------|----------|
|      | mean     | median   | SD       | mean     | median   | SD       | mean     | median   | SD       |
| a=-0.3 | 100 | 0.887 | 0.867 | 0.380 | 0.899 | 0.899 | 0.959 | 200 | 0.732 | 0.739 | 0.510 | 0.722 | 0.795 | 3.114 | 14.688 | 0.549 | 3.834 |
| a=0   | 100 | 1.068 | 0.994 | 1.755 | 1.162 | 1.092 | 1.169 | 200 | 0.906 | 0.892 | 0.797 | 0.940 | 0.974 | 3.700 | 65.761 | 7.864 | 4.420 |
| a=0.5 | 100 | 1.321 | 1.244 | 0.532 | 1.791 | 0.484 | 1.785 | 200 | 1.256 | 1.226 | 3.040 | 1.430 | 1.355 | 3.290 | 12859.850 | 1.471 | 0.573 |

Table 1 shows that when sample size n increases or the proportion of missing observations decreases, the mean and median values of the KL loss of the MOPT decrease. The mean and median values of the other four methods also decrease as the sample size n increases or the proportion of the missing observations decreases.

Except for the case $a = -0.3, n = 200$, the mean and median values of MOPT are both lower than all others. This pattern is also almost true regarding standard deviation (SD) values except two cases of $a = -0.3, n = 200$, in which MOPT yields larger SD than MIM, and $a = 0.5, n = 100$, in which MOPT yields larger SD than MIMA.

Moreover, the standard SD values of CC model are very huge comparing with other models in most cases. This phenomenon shows that the KL loss is not stable for each replication of simulation for CC model. This is because that each replication may have different data missing pattern that leads to significant difference in sample size for the CC model.
6 Concluding Remarks

In this paper, we propose a model averaging method (MOPT) to combine the
generalized linear models for the case when covariates are missing com pletely
at random to utilize the largest set of available cases. The asymptotic opti-
mality of our method has been proved. Developing model averaging method
to the generalized additive models (GAMs) with missing data and the non-
linear models is also worthy of study in the future.

7 Appendix

All the limiting properties here and throughout the text hold under $n \to \infty$.
Let $\epsilon = (\epsilon_1, \ldots, \epsilon_n)^T = y - \mu$, $\sigma^2 = \max_{i \in \{1, \ldots, n\}} \text{var}(\epsilon_i)$, $\beta^*(w) = \sum_{s=1}^{S} w_s \beta^*_s$, $KL^*(w) = 2\phi^{-1}B\{\beta^*(w)\} - 2\phi^{-1}B_0 - 2\phi^{-1}\mu^T[\theta\{\beta^*(w)\} - \theta_0]$, and $\xi_n = \inf_{w \in \mathcal{W}} KL^*(w)$.

Proof of Theorem 1

Let $\hat{\mathcal{G}}(w) = \mathcal{G}(w) - 2\phi^{-1}B_0 + 2\phi^{-1}\mu^T\theta_0$. Obviously, $\hat{w} = \text{argmin}_{w \in \mathcal{W}} \hat{\mathcal{G}}(w)$.

According to the proof of Theorem 1' in Wan et al.(2010), Theorem 1 is valid if the following hold:

\[
\sup_{w \in \mathcal{W}} \frac{|KL(w) - KL^*(w)|}{KL^*(w)} = o_p(1) \tag{A.1}
\]

and

\[
\sup_{w \in \mathcal{W}} \frac{|\hat{\mathcal{G}}(w) - KL^*(w)|}{KL^*(w)} = o_p(1) \tag{A.2}
\]

By (5), we know that uniformly for $w \in \mathcal{W}$,

\[
\hat{\beta}(w) - \beta^*(w) = \sum_{s=1}^{S} w_s(\hat{\beta}_s - \beta^*_s) = O_p(n^{-1/2}) \tag{A.3}
\]

It follows from (A.3), Condition (C.1) and Taylor expansion that uniformly for $w \in \mathcal{W}$,

\[
|B\{\hat{\beta}(w)\} - B\{\beta^*(w)\}| \leq \left|\frac{\partial B(\beta)}{\partial \beta^T}|_{\beta = \hat{\beta}(w)}\right|||\hat{\beta}(w) - \beta^*(w)|| = O_p(n^{1/2}),
\]

\[
\mu^T[\theta\{\hat{\beta}(w)\} - \theta\{\beta^*(w)\}] \leq ||\mu^T X||||\hat{\beta}(w) - \beta^*(w)|| = O_p(n^{1/2}),
\]
and
\[ \epsilon^T[\theta\{\hat{\beta}(w)\} - \theta\{\beta^*(w)\}] \leq ||\epsilon^T X|| ||\hat{\beta}(w) - \beta^*(w)|| = O_p(1), \]

where \( \hat{\beta}(w) \) is a vector between \( \hat{\beta}(w) \) and \( \beta^*(w) \).

In addition, using the central limit theorem and Condition (C.2), we know that uniformly for \( w \in \mathcal{W} \),
\[ \epsilon^T \theta\{\beta^*(w)\} = \sum_{s=1}^{S} w_s \epsilon^T \theta(\beta^*_s) = O_p(n^{1/2}). \tag{A.4} \]
These arguments indicate that
\[ \sup_{w \in \mathcal{W}} |KL(w) - KL^*(w)| \]
\[ = \sup_{w \in \mathcal{W}} |2\phi^{-1}(B\{\hat{\beta}(w)\} - B\{\beta^*(w)\}) - 2\phi^{-1} \mu^T(\theta\{\hat{\beta}(w)\} - \theta\{\beta^*(w)\})| \]
\[ \leq 2\phi^{-1} \sup_{w \in \mathcal{W}} |B\{\hat{\beta}(w)\} - B\{\beta^*(w)\}| + 2\phi^{-1} \sup_{w \in \mathcal{W}} |\mu^T(\theta\{\hat{\beta}(w)\} - \theta\{\beta^*(w)\})| \]
\[ = O_p(n^{1/2}) \tag{A.5} \]
and
\[ \sup_{w \in \mathcal{W}} |\hat{G}(w) - KL_*(w)| \]
\[ = \sup_{w \in \mathcal{W}} |2\phi^{-1}(B\{\hat{\beta}(w)\} - B\{\beta^*(w)\}) - 2\phi^{-1}(y^T \theta\{\hat{\beta}(w)\} - \mu^T \theta\{\beta^*(w)\}) + \lambda_n w^T k| \]
\[ \leq 2\phi^{-1} \sup_{w \in \mathcal{W}} |B\{\hat{\beta}(w)\} - B\{\beta^*(w)\}| \]
\[ + 2\phi^{-1} \sup_{w \in \mathcal{W}} |y^T \theta\{\hat{\beta}(w)\} - \mu^T \theta\{\beta^*(w)\}| + \lambda_n w^T k \]
\[ \leq 2\phi^{-1} \sup_{w \in \mathcal{W}} |B\{\hat{\beta}(w)\} - B\{\beta^*(w)\}| \]
\[ + 2\phi^{-1} \sup_{w \in \mathcal{W}} |\mu^T(\theta\{\hat{\beta}(w)\} - \theta\{\beta^*(w)\})| \]
\[ + 2\phi^{-1} \sup_{w \in \mathcal{W}} |\epsilon^T \theta\{\beta^*(w)\} - \theta\{\beta^*(w)\}| + \lambda_n w^T k \]
\[ = O_p(n^{1/2}) + \lambda_n w^T k \tag{A.6} \]

From (A.5), (A.6), Condition (C.3) and \( n^{-1/2} \lambda_n = O(1) \), we can obtain (A.1) and (A.2). This complete the proof.
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