Vector variational problem with knitting boundary conditions

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Abstract  
We consider a variational problem with a polyconvex integrand and nonstandard boundary conditions that can be treated as minimization of the strain energy during the suturing process in the plastic surgery. Existence of minimizers is proved as well as necessary optimality conditions are discussed.

Keywords: Calculus of Variations, polyconvex integrand, coercivity assumptions, trace operator, knitting boundary conditions.

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1 Introduction

Given an open bounded connected domain $\Omega \subset \mathbb{R}^N$ with a sufficiently regular (locally Lipschitz) boundary, $\partial \Omega$, let us consider the integral

$$I(u) := \int_{\Omega} W(\nabla u(x)) \, dx \quad (1)$$

to be minimized on a class of Sobolev functions $u : \Omega \to \mathbb{R}^d$ with a kind of boundary conditions to be described later. All over the paper we assume the integrand $W : \mathbb{R}^{d \times N} \to \mathbb{R} \cup \{+\infty\}$ to be polyconvex. This means that the representation

$$W(\xi) = g(\mathbb{T}(\xi)), \quad \xi \in \mathbb{R}^{d \times N},$$

holds for some convex function $g : \mathbb{R}^{\tau(d,N)} \to \mathbb{R} \cup \{+\infty\}$, 

$$\tau(d,N) := \sum_{s=1}^{d \wedge N} \tau(s), \quad \tau(s) := \binom{s}{d} \binom{s}{N} = \frac{d!N!}{(s!)^2(d-s)!(N-s)!},$$

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where
\[ T(\xi) := (\text{Adj}_1\xi, \text{Adj}_2\xi, \text{Adj}_3\xi, \ldots, \text{Adj}_{d\wedge N}\xi), \quad \xi \in \mathbb{R}^{d\times N}, \]
and \( \text{Adj}_k\xi \) is the vector of all minors of the matrix \( \xi \) of order \( k = 1, 2, \ldots, d\wedge N \), respectively. In particular, \( \text{Adj}_1\xi = \xi \) and \( \text{Adj}_d\xi = \det \xi \) whenever \( d = N \).

It is known that, under strong coercivity assumptions on \( W \) to assure weak convergence of the minors of gradients for the minimizing sequence, the functional \( I \) attains its minimum on \( \bar{u}(\cdot) + W_0^{1,p}(\Omega; \mathbb{R}^d) \), \( p \geq 1 \). We refer to the fundamental work by J. Ball [3] motivated by problems coming from nonlinear elasticity and to [1, 20, 18] for further improvements.

The lower semicontinuity for general polyconvex integrands with respect to the weak convergence in \( W^{1,p}(\Omega; \mathbb{R}^d) \), \( \Omega \subset \mathbb{R}^N \), has been the subject of many investigations. Namely, Marcellini showed in [19] that this property holds whenever \( p < N \). Later, this result was improved by Dacorogna and Marcellini in [9] who proved the lower semicontinuity for \( p > N - 1 \) while Mály in [18] exhibited a counterexample for \( p < N - 1 \). The limit case \( p = N - 1 \) was addressed in [11, 14, 6, 15]. Very recently (see [14] and [11]) the limit case was studied for polyconvex integrands depending on \( x \) and/or on \( u \).

Besides the Dirichlet boundary condition \( u = \bar{u} \) for the displacement one considered, for instance, boundary condition on traction, which somehow depends on the normal derivatives of \( u(\cdot) \) (generalizing the Neumann boundary data). Observe that these conditions (displacement, traction or a mixed one) can be applied either to the whole boundary \( \partial\Omega \), or to some of its subsets of positive Hausdorff measure leaving the rest free. Moreover, some restrictions on the jacobian may be relevant and practically justified. For example, the constraint \( \det \nabla u(x) > 0 \) means that the minimum is searched among the deformations preserving orientation, while \( \det \nabla u(x) = 1 \) refers to the case of incompressible elastic body.

One of the possible applications of the above variational problem is regarded to plastic surgery, namely, in the woman breast reduction, where we deal with a sort of very elastic and soft tissue. Some recent publications (see, e.g., [12, 22, 21, 4]) were devoted to mathematical setting of the related problems and to their numerical simulations. Medical examinations allow to consider the involved tissue as a neo-Hookean compressible material (see [23]). We have a more precise model when the strain energy is defined by the integral (1) with the density \( W : \mathbb{R}^{3\times 3} \rightarrow \mathbb{R} \),

\[ W(\xi) := \mu \left( \text{tr} \left( \xi \cdot \xi^T \right) - 3 - 2 \ln(\det \xi) \right) + \lambda (\det \xi - 1)^2 + \beta \text{tr} \left( \text{Adj} \xi \cdot \text{Adj} \xi^T \right), \quad (2) \]

where "tr" means the trace of a matrix, \( \text{Adj} \xi := \text{Adj}_2\xi \), and the symbol "\( T \)" stands for the matrix transposition. One of the steps of the (breast reduction) surgery is the suturing, which mathematically can be seen as an identification of points of some surface piece \( \Gamma^+ \subset \partial\Omega \) with points of another one \( \Gamma^- \subset \partial\Omega \). Denoting the respective correspondence between the points of \( \Gamma^+ \) and \( \Gamma^- \) by \( \sigma \),
we are led to a new type of constraint

\[ u(x) = (u \circ \sigma)(x), \quad x \in \Gamma^+, \]  

(3)
called the knitting boundary condition. Let us note that the one-to-one mapping \( \sigma \) is not \emph{a priori} given and should be chosen to guarantee the minimum value to the functional \( \mathcal{I} \). In other words, a minimizer of \( \mathcal{I} \) (if any) should be a pair \( (u, \sigma) \) where \( u \in W^{1,p}(\Omega; \mathbb{R}^3) \), \( p \geq 1 \), and \( \sigma : \Gamma^+ \to \Gamma^- \) is sufficiently regular. We set the natural hypothesis that \( \sigma \) and its inverse \( \sigma^{-1} \) are Lipschitz transformations (with the same Lipschitz constant \( L > 0 \)). Practically this means that the sutured tissue can not be extended nor compressed too much.

Motivated by the problem coming from the plastic surgery we will consider just the case \( p = 2 \) and \( d = N = 3 \), although the results remain true in the case \( p > 2 \) and arbitrary \( d = N \geq 2 \) as well.

The paper is organized as follows. In the next section we give the exact setting of the variational problem together with the main hypotheses on the integrand \( W \). For simplicity of references we put here also some important facts regarded with the Sobolev functions. In Section 3 we justify first the well-posedness of the problem by showing that the composed function from the knitting condition (3) belongs to the respective Lebesgue class. Afterwards, we prove existence of a minimizer as an accumulation point of an arbitrary minimizing sequence (the so called \emph{direct method}, see [8]). The paper is concluded with a necessary optimality condition for the given problem (see Section 4) allowing to construct effective numerical algorithms, which can be successively applied in the medical practice.

2 Main hypotheses and auxiliary results

In what follows we fix a nonempty open bounded and connected set \( \Omega \subset \mathbb{R}^3 \) whose boundary \( \partial \Omega \) is assumed to be locally Lipschitz (see, e.g., [17] p. 354)). By the symbol \( \mathcal{L}^m(dx) \) we denote the Lebesgue measure in the space \( \mathbb{R}^m \), \( m = 2, 3 \), while \( \mathcal{H}^2 \) means the two-dimensional Hausdorff measure (see, e.g., [13]).

Let us divide the surface \( \partial \Omega \) into several parts \( \Gamma_i \), \( i = 1, 2, 3, 4 \), in such a way that \( \mathcal{H}^2(\Gamma_i \cap \Gamma_j) = 0 \) for \( i \neq j \). Moreover, we set \( \Gamma_4 := \Gamma^+ \cup \Gamma^- \) where \( \Gamma^\pm \subset \Gamma \) with \( \mathcal{H}^2(\Gamma^\pm) > 0 \) and \( \mathcal{H}^2(\Gamma^+ \cap \Gamma^-) = 0 \) are also given.

Suppose that \( W : \mathbb{R}^{3 \times 3} \to \mathbb{R} \) is a polyconvex function satisfying the growth assumption:

\[ W(\xi) \geq c_0 + c_1 |\xi|^2 + c_2 |\text{Adj} \xi|^2 + c_3 (\det \xi)^2, \quad \xi \in \mathbb{R}^{3 \times 3}, \]  

(4)
where \( c_0 \in \mathbb{R} \) and \( c_i > 0, i = 1, 2, 3 \), are some given constants. Here and in what follows by \( |\cdot| \) we denote the norm of both a vector in \( \mathbb{R}^n \) and a \( 3 \times 3 \)-matrix.
Taking into account that \( \text{tr}(\xi \cdot \xi^T) = |\xi|^2 \) for each matrix \( \xi \in \mathbb{R}^{3 \times 3} \), we see that the integrand \( W(\xi) \) satisfies the above properties. Indeed, it is convex as a function of \( \xi \) being represented as a sum of three terms, which are convex w.r.t. \( \xi, \det \xi \) and \( \text{Adj} \xi \), respectively. Furthermore,

\[
W(\xi) = -3\mu + \lambda + \mu |\xi|^2 + \beta |\text{Adj} \xi|^2 + f(\det \xi), \quad \xi \in \mathbb{R}^{3 \times 3},
\]

where the function

\[
f(t) := \frac{\lambda}{2} t^2 - 2\lambda t - 2\mu \ln t, \quad t > 0,
\]

is lower bounded by some (negative) constant.

Since on various pieces of the surface \( \partial \Omega \) the boundary conditions are structurally different (some part of \( \partial \Omega \) can be left even free), to set the problem we use the notion of the trace operator, which associates to each \( u \in W^{1,2}(\Omega; \mathbb{R}^3) \) a function \( \text{Tr} u \) defined on the boundary, \( \partial \Omega \), which can be interpreted as the "boundary values" of \( u \). We refer to [17, pp. 465-474], where the existence and uniqueness of the trace operator were proved for scalar Sobolev functions \( u \in W^{1,p}(\Omega) \), \( p > 1 \).

For vector-valued functions \( u : \Omega \rightarrow \mathbb{R}^3 \) instead, we can argue componentwise. So, applying [17, Theorem 15.23], we define the trace as the linear and bounded operator \( \text{Tr} : W^{1,2}(\Omega; \mathbb{R}^3) \rightarrow L^2(\partial \Omega; \mathbb{R}^3) \) satisfying the following properties:

1. \( \text{Tr} u(x) = u(x), \quad x \in \partial \Omega, \) whenever \( u \in W^{1,2}(\Omega; \mathbb{R}^3) \cap C(\overline{\Omega}; \mathbb{R}^3) \);
2. for each \( u \in W^{1,2}(\Omega; \mathbb{R}^3) \) and any test function \( \varphi \in C^1(\overline{\Omega}; \mathbb{R}^3) \) the equalities

\[
\int_{\Omega} u_j \frac{\partial \varphi_j}{\partial x_i} dx = -\int_{\Omega} \varphi_j \frac{\partial u_j}{\partial x_i} dx + \int_{\partial \Omega} \varphi_j \text{Tr}(u_j) \nu_i d\mathcal{H}^2,
\]

hold for each \( i, j = 1, 2, 3 \) where \( \nu := (\nu_1, \nu_2, \nu_3)^T \) means the unit outward normal to \( \partial \Omega \).

In addition to the properties above, observe that the trace operator gives a compact embedding into the space \( L^2(\partial \Omega; \mathbb{R}^3) \) that will be crucial to obtain the main result in Section 3. Namely, the following proposition takes place.

**Proposition 1** Let \( \Omega \subset \mathbb{R}^3 \) be an open bounded set with locally Lipschitz boundary. Then for each \( \{u_n\} \subset W^{1,2}(\Omega; \mathbb{R}^3) \) converging to \( u \) weakly in \( W^{1,2}(\Omega; \mathbb{R}^3) \) the sequence of traces \( \{\text{Tr} u_n\} \) converges to \( \text{Tr} u \) strongly in \( L^2(\partial \Omega; \mathbb{R}^3) \).

The proof is based essentially on the following lemma giving a nice estimate for the surface integral of the trace operator.

**Lemma 1** Let \( \Omega \subset \mathbb{R}^3 \) be as in Proposition 4. Then there exists a constant \( C > 0 \) such that

\[
\int_{\partial \Omega} |\text{Tr} u|^2 d\mathcal{H}^2 \leq C \left( \frac{1}{\varepsilon} \int_{\Omega} |u|^2 dx + \varepsilon \int_{\Omega} |\nabla u|^2 dx \right)
\]

for any \( \varepsilon > 0 \) and any \( u \in W^{1,2}(\Omega; \mathbb{R}^3) \).
Proof. Given $x \in \partial \Omega$, due to the Lipschitz hypothesis there exists a neighborhood $U_x$ of $x$ such that $U_x \cap \partial \Omega$ can be represented as the graph of a Lipschitz function w.r.t. some (local) coordinates. Without loss of generality, we may assume that

$$U_x \cap \Omega = \{(y', y_3): f_x(y') < y_3 \leq f_x(y') + \delta_x, \quad y' \in G_x\}$$

and

$$U_x \cap \partial \Omega = \{(y', f_x(y')): y' \in G_x\},$$

where $\delta_x > 0$, $G_x \subset \mathbb{R}^2$ is an open set in the space of the first two coordinates and $f_x : G_x \to \mathbb{R}$ is a Lipschitz function. By compactness there exists a finite number of points $x^i \in \partial \Omega$, $i = 1, \ldots, q$, such that

$$\partial \Omega = \bigcup_{i=1}^q (U_{x^i} \cap \partial \Omega).$$

Set $\delta_i := \delta_{x^i}$, $U_i := U_{x^i}$, $G_i := G_{x^i}$ and $f_i := f_{x^i}$, $i = 1, \ldots, q$. Denote by $L > 0$ the biggest Lipschitz constant of the functions $f_i$.

Let us choose $\varepsilon < \min \{\delta_i : i = 1, \ldots, q\}$ and consider first the function $u \in C^1(\overline{\Omega}) \cap W^{1,2}(\Omega; \mathbb{R}^3)$. Given $i \in \{1, \ldots, q\}$, by the Newton-Leibniz formula, for each $x' \in G_i$ and $x_3 \in \mathbb{R}$ with $f_i(x') \leq x_3 < f_i(x') + \varepsilon$ we have

$$u(x', f_i(x')) = u(x', x_3) - \int_{f_i(x')}^{x_3} \frac{\partial u}{\partial x_3}(x', s) \, ds.$$

In turn, by Cauchy-Schwartz inequality,

$$|u(x', f_i(x'))|^2 \leq 2 \left( |u(x', x_3)|^2 + \left( \int_{f_i(x')}^{x_3} \left| \frac{\partial u}{\partial x_3}(x', s) \right| ds \right)^2 \right) \leq 2 \left( |u(x', x_3)|^2 + \varepsilon \int_{f_i(x')}^{x_3} \left| \frac{\partial u}{\partial x_3}(x', s) \right|^2 ds \right).$$

Hence,

$$|u(x', f_i(x'))|^2 \sqrt{1 + |\nabla f_i(x')|^2} \leq 2 \sqrt{1 + L^2 \left( |u(x', x_3)|^2 + \varepsilon \int_{f_i(x')}^{x_3} \left| \frac{\partial u}{\partial x_3}(x', s) \right|^2 ds \right)}.$$

Integrating both parts of the previous inequality in the cylinder

$$\Omega_i := \{(x', x_3) : f(x') \leq x_3 \leq f(x') + \varepsilon, \quad x' \in G_i\} \subset U_i \cap \Omega_i,$$

gives

$$\varepsilon \int_{U_i \cap \partial \Omega} |u(x)|^2 \, dH^2(x) \leq 2 \sqrt{1 + L^2 \left( \int_{\Omega_i} |u(x)|^2 \, dx + \varepsilon^2 \int_{\Omega_i} \left| \frac{\partial u}{\partial x_3}(x) \right|^2 \, dx \right)} \leq 2 \sqrt{1 + L^2 \left( \int_{\Omega} |u(x)|^2 \, dx + \varepsilon^2 \int_{\Omega} \left| \frac{\partial u}{\partial x_3}(x) \right|^2 \, dx \right)}.$$
Summing in \( i = 1, \ldots, q \) we have
\[
\int_{\partial \Omega} |u(x)|^2 \, dH^2(x) \leq 2q \sqrt{1 + L^2} \left( \frac{1}{\varepsilon} \int_{\Omega} |u(x)|^2 \, dx + \varepsilon \int_{\Omega} \left| \frac{\partial u}{\partial x_3}(x) \right|^2 \, dx \right).
\]

The inequality (5) for each \( u \in W^{1,2}(\Omega; \mathbb{R}^3) \) follows now from (6) by the properties of the trace operator and by the density of smooth functions.

**Proof of Proposition 1.** Given a sequence \( \{u_n\} \subset W^{1,2}(\Omega; \mathbb{R}^3) \) converging weakly to \( u \) in \( W^{1,2}(\Omega; \mathbb{R}^3) \), there exists \( M > 0 \) with
\[
\int_{\Omega} |\nabla u_n(x)|^2 \, dx \leq M
\]
for all \( n \in \mathbb{N} \). Then, by the Rellich-Kondrachov theorem (see \[17\] Theorem 11.21, p. 326), \( u_n \rightharpoonup u \) strongly in \( L^2(\Omega; \mathbb{R}^3) \). Applying Lemma 2 to \( \text{Tr}(u_n - u) = \text{Tr}u_n - \text{Tr}u \), we have
\[
\int_{\partial \Omega} |\text{Tr}u_n - \text{Tr}u|^2 \, dH^2 \leq C \left( \frac{1}{\varepsilon} \int_{\Omega} |u_n - u|^2 \, dx + \varepsilon \int_{\Omega} |\nabla u_n - \nabla u|^2 \, dx \right)
\leq C \left( \frac{1}{\varepsilon} \int_{\Omega} |u_n - u|^2 \, dx + 4\varepsilon M \right).
\]

Hence
\[
\limsup_{n \to \infty} \int_{\partial \Omega} |\text{Tr}u_n - \text{Tr}u|^2 \, dH^2 \leq 4\varepsilon CM.
\]
Letting \( \varepsilon \to 0^+ \) concludes the proof.

We will use also the so called generalized Poincaré inequality (see \[5\] Theorem 6.1-8, p. 281).

**Proposition 2** Given an open bounded domain \( \Omega \subset \mathbb{R}^3 \) with locally Lipschitz boundary and a measurable subset \( \Gamma \subset \partial \Omega \) with \( H^2(\Gamma) > 0 \) there exists a constant \( C > 0 \) such that
\[
\int_{\Omega} |u(x)|^2 \, dx \leq C \left[ \int_{\Omega} |\nabla u(x)|^2 \, dx + \int_{\Gamma} |\text{Tr}u(x)|^2 \, dH^2(x) \right]^2
\]
for each \( u \in W^{1,2}(\Omega; \mathbb{R}^3) \).

Let us formulate now the boundary conditions in terms of the trace operator. Consider first a surface \( S \subset \mathbb{R}^3 \) defined by some continuous function \( h : \mathbb{R}^3 \to \mathbb{R} \),
\[
S := \{ u \in \mathbb{R}^3 : h(u) = 0 \}.
\]
Then, given $L \geq 1$ we denote by $\Sigma_L (\Gamma^+; \Gamma^-)$ the set of all functions $\sigma : \Gamma^+ \to \Gamma^-$ satisfying the inequalities
\[
\frac{1}{L} |x - y| \leq |\sigma (x) - \sigma (y)| \leq L |x - y|
\]
for all $x, y \in \Gamma^+$, and introduce the set $W_L \subset W^{1,2} (\Omega; \mathbb{R}^3) \times \Sigma_L (\Gamma^+; \Gamma^-)$ of all pairs $(u, \sigma)$ satisfying the (boundary) conditions:

- $(C_1)$ $\text{Tr} u (x) = x$ for $\mathcal{H}^2$-a.e. $x \in \Gamma_1$;
- $(C_2)$ $h (\text{Tr} u (x)) = 0$ for $\mathcal{H}^2$-a.e. $x \in \Gamma_2$;
- $(C_3)$ $\text{Tr} u (x) = \text{Tr} u (\sigma (x))$ for $\mathcal{H}^2$-a.e. $x \in \Gamma^+$.

Thus, we can write the knitting variational problem in the form
\[
\min \left\{ \int_{\Omega} W (\nabla u (x)) \, dx : (u, \sigma) \in W_L \right\}.
\] (9)

Figure 1: General scheme of plastic surgery.

Let us emphasize the difference between the boundary conditions $(C_1) − (C_3)$ above (see Fig. 1). The first condition $(C_1)$ means that the surface $\Gamma_1$ is a part of the elastic body (breast) that remains fixed. The condition $(C_2)$, instead, can be interpreted as the knitting of a part of the incised breast (surface $\Gamma_2$) to the fixed surface $\mathcal{S}$ of the woman’s chest, while $(C_3)$ means the knitting of the cut breast surface $\Gamma_4 := \Gamma^+ \cup \Gamma^-$, of $\Gamma^+$ into $\Gamma^-$. Finally, the piece $\Gamma_3$ of the boundary $\partial \Omega$ remains free and admits an arbitrary configuration depending on the knitting process.

3 Existence of minimizers

Before proving the main existence theorem let us justify that for each $\sigma \in \Sigma_L (\Gamma^+; \Gamma^-)$ the boundary condition $(C_3)$ makes sense.
Lemma 2 Let $\Omega \subset \mathbb{R}^3$ be an open bounded connected domain with locally Lipschitz boundary and $\Gamma^\pm \subset \partial \Omega$ be $\mathcal{H}^2$-measurable sets with $\mathcal{H}^2(\Gamma^+ \cap \Gamma^-) = 0$. Then for each $u \in W^{1,2}(\Omega; \mathbb{R}^3)$ and $\sigma \in \Sigma_{L}(\Gamma^+; \Gamma^-)$, $L \geq 1$, the composed function $\text{Tr} u \circ \sigma : \Gamma^+ \rightarrow \mathbb{R}^3$ is measurable w.r.t. the measure $\mathcal{H}^2$ on $\Gamma^+$.

Proof. Given $u \in W^{1,2}(\Omega; \mathbb{R}^3)$ by the density argument there exists a sequence of continuous functions $v_n : \Omega \rightarrow \mathbb{R}^3$ converging to $u$ in the space $W^{1,2}(\Omega; \mathbb{R}^3)$. Consequently (see the properties 1 and 2 of traces), $v_n = \text{Tr} v_n \rightarrow \text{Tr} u$, as $n \rightarrow \infty$, in $L^2(\partial \Omega; \mathbb{R}^3)$. Then, up to a subsequence, we have

$$v_n(y) \rightarrow \text{Tr} u (y) \quad \forall y \in \Gamma^- \setminus E^-_0$$

(10)

where $E^-_0 \subset \Gamma^-$ is some set with null Hausdorff measure. So, it remains to prove that

$$\mathcal{H}^2(\sigma^{-1}(E^-_0)) = 0$$

(11)

because in such case we deduce from (10) that the sequence of (continuous) functions $\{v_n(\sigma(x))\}$ converges to $\text{Tr} u (\sigma(x))$ for all $x \in \Gamma^+$ up to the negligible set of points $E^+_0 := \sigma^{-1}(E^-_0)$.

On the other hand, (11) follows easily from the left inequality in (8) and from the definition of Hausdorff measure (see [13, p. 171]):

$$\mathcal{H}^2(E) := \pi \lim_{\varepsilon \to 0} \inf \sum_{i=1}^{\infty} (\text{diam } A_i)^2$$

where the infimum is taken over all coverings $\{A_i\}_{i=1}^{\infty}$ of $E$ with the diameters $\text{diam } A_i \leq \varepsilon$. In fact, given $\eta > 0$ we find $\varepsilon > 0$ and a family $\{A_i\}_{i=1}^{\infty}$ with $\bigcup_{i=1}^{\infty} A_i \supset E^-_0$, $\text{diam } A_i \leq \varepsilon/L$ and

$$\sum_{i=1}^{\infty} (\text{diam } A_i)^2 \leq \frac{\eta}{L^2}.$$ 

Since due to (8) obviously $\text{diam } (\sigma^{-1}(A_i)) \leq L \text{ diam } A_i \leq \varepsilon, i = 1, 2, \ldots$; the family $\{\sigma^{-1}(A_i)\}_{i=1}^{\infty}$ is a covering of $E^+_0$ and

$$\sum_{i=1}^{\infty} (\text{diam } (\sigma^{-1}(A_i)))^2 \leq L^2 \sum_{i=1}^{\infty} (\text{diam } A_i)^2 \leq \eta,$$

we conclude that $\mathcal{H}^2(E^+_0) = 0$, and the $\mathcal{H}^2$-measurability of $x \mapsto \text{Tr} u (\sigma(x))$ on $\Gamma^+$ follows.

Let us give now an a priori estimate for the "weighted" integral

$$\int_{\Gamma^+} |\text{Tr} u (\sigma(x))|^2 \, d\mathcal{H}^2 (x),$$

implying, in particular, that the composed function $\text{Tr} u \circ \sigma$ belongs to the class $L^2(\partial \Omega; \mathbb{R}^3)$. 

8
Lemma 3 Let $\Omega \subset \mathbb{R}^3$ and $\Gamma^\pm \subset \partial \Omega$ be such as in Lemma 2. Then given $L \geq 1$ there exists a constant $\mathcal{L}_L > 0$ such that the inequality

$$
\int_{\Gamma_+} |\text{Tr} u (\sigma (x))|^2 \, d\mathcal{H}^2 (x) \leq \mathcal{L}_L \int_{\Gamma_-} |\text{Tr} u (y)|^2 \, d\mathcal{H}^2 (y)
$$

(12)

holds whenever $u \in W^{1,2} (\Omega; \mathbb{R}^3)$ and $\sigma \in \Sigma_L (\Gamma^+; \Gamma^-)$.

Proof. To prove the estimate (12) we employ the local lipschitzianity of the surfaces $\Gamma^\pm$. Namely, given $y \in \Gamma^-$ we choose an (open) ball $B (y, \varepsilon_y)$, $\varepsilon_y > 0$, such that $\Gamma^- \cap B (y, \varepsilon_y)$ can be represented as the graph of a Lipschitz continuous function with respect to some (local) coordinates. Without loss of generality we can suppose that this function (say $f_y^-$) is defined on an open set $D_y^-$ from the cartesian product of the first two components $x' := (x_1, x_2)$ and admits as values the component $x_3$, i.e.,

$$
\Gamma^- \cap B (y, \varepsilon_y) = \{(z', f_y^- (z')) : z' \in D_y^-\}. 
$$

(13)

By the compactness of $\Gamma^-$ one can find a finite number of points $y^1, \ldots, y^q \in \Gamma^-$ with

$$
\Gamma^- = \Gamma^- \cap \bigcup_{j=1}^q B \left( y^j, \frac{\varepsilon_y}{2} \right).
$$

(14)

Set $\varepsilon_j := \varepsilon_{y^j}$, $D_j^- := D_{y^j}$ and $f_j^- (z') := f_{y^j}^- (z')$, $z' \in D_{y^j}^-$, $j = 1, \ldots, q$.

Similarly, for any $x \in \Gamma^+$ there exist $\delta_x > 0$, an open domain $D_x^+ \subset \mathbb{R}^2$ and a Lipschitz function $f_x^+: D_x^+ \to \mathbb{R}$ such that

$$
\Gamma^+ \cap B (x, \delta_x) = \{(z', f_x^+ (z')) : z' \in D_x^+\},
$$

We do not loose generality assuming that the value of $f_x^+$ is the last component of the vector $z \in \Gamma^+$ (as in (13)). Again due to the compactness argument there exists a finite number of points $x^1, \ldots, x^r \in \Gamma^+$ such that

$$
\Gamma^+ = \Gamma^+ \cap \bigcup_{i=1}^r B \left( x^i, \delta_i \right)
$$

(15)

where

$$
\delta_i := \min_{1 \leq j \leq q} \left( \delta_{x^i, \frac{\varepsilon_j}{2}} \right), \quad i = 1, \ldots, r.
$$

(16)

Set also $D_i^+: = D_{x^i}^+$ and $f_i^+ (z') := f_{x^i}^+ (z')$, $z' \in D_{x^i}^+$, $i = 1, \ldots, r$, and denote by

$L_i$ the maximal Lipschitz constant of the functions $f_j^- : D_j^- \to \mathbb{R}$, $j = 1, \ldots, q$, and $f_i^+ : D_i^+ \to \mathbb{R}$, $i = 1, \ldots, r$.

We claim that given $i = 1, \ldots, r$ and $\sigma \in \Sigma_L$, the image $\sigma \left( \Gamma^+ \cap B (x^i, \delta_i) \right)$ is contained in $\Gamma^- \cap B (y^j, \varepsilon_j)$ for some $j = 1, \ldots, q$. Indeed, let us choose $j$ such that $\sigma (x^i) \in B (y^j, \varepsilon_j/2)$ (see (14)). Taking an arbitrary $z \in \Gamma^+ \cap B (x^i, \delta_i)$ we, in particular, have $|z - x^i| < \frac{\varepsilon_j}{2}$ (see (15)) and by (16) $|\sigma (z) - \sigma (x^i)| < \varepsilon_j/2$. 
However, assuming that \( \sigma (z) \notin \Gamma^- \cap B (y^j, \varepsilon_j) \) we have \(|\sigma (z) - y^j| \geq \varepsilon_j \) and hence
\[
|\sigma (z) - \sigma (x^i)| \geq |\sigma (z) - y^j| - |\sigma (x^i) - y^j| \geq \varepsilon_j - \frac{\varepsilon_j}{2} = \frac{\varepsilon_j}{2},
\]
which is a contradiction. In what follows we associate to each \( \sigma \in \Sigma_L \) and to each \( i = 1, \ldots, r \) an index \( j = j (\sigma, i) \in \{ 1, \ldots, q \} \) such that
\[
\sigma \left( \Gamma^+ \cap B \left( x^i, \delta_i \right) \right) \subset \Gamma^- \cap B \left( y^j, \varepsilon_j \right).
\]

The latter inclusion allows us to define correctly the (injective) mapping \( \psi_i^\sigma : D_i^+ \rightarrow D_j^+(\sigma, i) \) such that
\[
\sigma (x) = \sigma (x', f_i^+ (x')) = (\psi_i^\sigma (x'), f_j^+ (\psi_i^\sigma (x'))), x' \in D_i^+.
\] (16)

From (8) it follows immediately that \( \psi_i^\sigma \) is Lipschitz:
\[
|\psi_i^\sigma (x') - \psi_i^\sigma (z')| \leq |\sigma (x', f_i^+ (x')) - \sigma (z', f_i^+ (z'))| \\
\leq L \left( |x' - z'|^2 + (f_i^+ (x') - f_i^+ (z'))^2 \right)^{1/2} \\
\leq L \sqrt{1 + L^2_i} |x' - z'|, \ x', z' \in D_i^+.
\] (17)

So, by Rademacher’s Theorem, \( \psi_i^\sigma \) is \( \mathcal{L}^2 \)-a.e. differentiable on \( D_i^+ \) with \( \mathcal{L}^2 \)-measurable gradient, and the inequality
\[
|\nabla \psi_i^\sigma (x')| \leq M := L \sqrt{1 + L^2_i}
\]
holds for a.e. \( x' \in D_i^+ \). Notice that the inverse mapping \( (\psi_i^\sigma)^{-1} \) is well defined on \( G_i^\sigma := \psi_i^\sigma (D_i^+) \subset D_j^+(\sigma, i) \) by the formula similar to (16), namely,
\[
\sigma^{-1} \left( y', f_j^- (y') \right) = \left( (\psi_i^\sigma)^{-1} (y'), f_i^+ \left( (\psi_i^\sigma)^{-1} (y') \right) \right), y' \in G_i^\sigma.
\]

In the same way as (17) we deduce, from (8), that \( (\psi_i^\sigma)^{-1} \) is Lipschitz on the (open) set \( G_i^\sigma \) and, the estimate
\[
|\nabla (\psi_i^\sigma)^{-1} (y')| \leq M
\] (18)
holds for \( \mathcal{L}^2 \)-a.e. \( y' \in G_i^\sigma \).

Integrating the function \(|\text{Tr} u (\sigma (x))|\) on the surface piece \( \Gamma^+ \cap B \left( x^i, \delta_i \right) \) we pass first to the double integral
\[
\int_{\Gamma^+ \cap B (x^i, \delta_i)} |\text{Tr} u (\sigma (x))|^2 d\mathcal{H}^2 (x)
\]
\[
= \int_{D_i^+} \int_{D_i^+} |\text{Tr} u (\sigma (x', f_i^+ (x')))|^2 \sqrt{1 + |\nabla f_i^+ (x')|^2} dx' \\quad (19)
\]
\[
\leq \sqrt{1 + L_i^2} \int_{D_i^+} \int_{D_i^+} |\text{Tr} u (\sigma (x', f_i^+ (x')))|^2 dx'.
\]
Due to the representation (16) we can make the change of variable \( y_i = \psi_i(x') \), \( x' \in D_i^+ \), in the integral (19), and returning then to the surface integral on a piece of \( \Gamma^- \), we have

\[
\int_\Gamma^- \left| \text{Tr} \left( \sigma \left( x', f_i^+ (x') \right) \right) \right|^2 dH^2 (x) \leq 6M^3 \int_\Gamma^- \left| \text{Tr} \left( \sigma \left( y, f_j^- (y') \right) \right) \right|^2 \frac{1}{\sqrt{1 + \left| \nabla f_j^- (y') \right|^2}} dy' + 1 
\]

(20)

\[
\leq 6M^3 \int_\Gamma^- \left| \text{Tr} \left( \sigma \right) \right|^2 dH^2 (y).
\]

Here we used the estimate (18) and the obvious inequality \( |\det A| \leq 6 \| A \|^3 \) (\( A \) is an arbitrary \( 3 \times 3 \)-matrix). Since the sets \( \Gamma^+ \cap B (x^i, \delta_i), i = 1, \ldots, r \), cover the surface \( \Gamma^+ \), taking into account (19) and (20) we conclude that

\[
\int_{\Gamma^+} \left| \text{Tr} \left( \sigma \left( x \right) \right) \right|^2 dH^2 (x) \leq \sum_{i=1}^{r} \int_{\Gamma^+ \cap B (x^i, \delta_i)} \left| \text{Tr} \left( \sigma \left( x \right) \right) \right|^2 dH^2 (x)
\]

\[
\leq \mathcal{L} \int_{\Gamma^-} \left| \text{Tr} \left( \sigma \left( y \right) \right) \right|^2 dH^2 (y)
\]

where \( \mathcal{L} := 6rM^3 \sqrt{1 + L_\Gamma^2} > 0 \) depends just on the Lipschitz constant \( L \geq 1 \) and on the properties of the domain \( \Omega \) (namely, of its boundary).  

Proving the existence theorem we pay the main attention to the validity of the boundary condition (C3) where Lemma 3 is crucial.

**Theorem 1** Let \( W : \mathbb{R}^{3 \times 3} \to \mathbb{R} \cup \{ + \infty \} \) be a polyconvex function satisfying the growth assumption (4). Then problem (4) admits a minimizer whenever there exists at least one pair \( \omega := (u, \sigma) \in W_L \) with

\[
I (\omega) := \int_{\Omega} W \left( \nabla u \left( x \right) \right) \, dx < + \infty.
\]

**Proof.** Let us consider a minimizing sequence \( \{(u_n, \sigma_n)\} \subset W_L \) of the functional (1), e.g., such as

\[
\int_{\Omega} W \left( \nabla u_n \left( x \right) \right) \, dx \leq \inf \left\{ \int_{\Omega} W \left( \nabla u \left( x \right) \right) \, dx : (u, \sigma) \in W_L \right\} + \frac{1}{n} < + \infty. \quad (21)
\]
Taking into account the estimate \(4\) we deduce from (21) that the sequences \(\{\nabla u_n\}, \{\det \nabla u_n\}\) and \(\det \nabla u_n\) are bounded in \(L^2(\Omega; \mathbb{R}^{3 \times 3})\) and in \(L^2(\Omega; \mathbb{R})\), respectively. Applying Proposition 2 and the boundary condition (C1) we find a constant \(C > 0\) such that the inequality
\[
\int_{\Omega} |u_n(x)|^2 \, dx \leq C \left[ \int_{\Omega} |\nabla u_n(x)|^2 \, dx + \int_{\Gamma_1} x \, d\mathcal{H}^2(x) \right]
\]
holds for each \(n \geq 1\). So, the sequence \(\{u_n\}\) is bounded in \(W^{1,2}(\Omega; \mathbb{R}^3)\) and by the Banach-Alaoglu theorem, up to a subsequence, converges weakly to some function \(\bar{u} \in W^{1,2}(\Omega; \mathbb{R}^3)\). Without loss of generality, we can also assume that \(\{\det \nabla u_n\}\) and \(\det \nabla u_n\) converge weakly to some functions \(\xi \in L^2(\Omega; \mathbb{R}^{3 \times 3})\) and \(\eta \in L^2(\Omega; \mathbb{R})\), respectively. Now, by Theorem 8.20 \([3, \text{pp. 395-396}]\) due to the uniqueness of the limit we deduce that \(\xi(x) = \det \nabla u(x)\) and \(\eta(x) = \det \nabla u(x)\) for almost all \(x \in \Omega\). Thus we have the weak convergence of the sequence \(\{\nabla u_n(x)\}\) to the vector-function \(\nabla u\) in the space \(L^2(\Omega; \mathbb{R}^{3 \times 3})\).

On the other hand, recalling that \(\{\sigma_n\} \subset \Sigma_L(\Gamma^+; \Gamma^-)\) (see (3)) by Ascoli’s theorem up to a subsequence, not relabeled, \(\{\sigma_n\}\) converges uniformly to \(\sigma \in \Sigma_L(\Gamma^+; \Gamma^-)\).

Since the integrand \(W\) is polyconvex, it can be represented as \(W(\xi) = g(\nabla u)\), \(\xi \in \mathbb{R}^{3 \times 3}\), with some convex function \(g : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}\), and, therefore,
\[
\int_{\Omega} W(\nabla u(x)) \, dx = \int_{\Omega} g(\nabla u(x)) \, dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} g(\nabla u_n(x)) \, dx
\]
\[
\leq \inf \left\{ \int_{\Omega} W(\nabla u(x)) \, dx : (u, \sigma) \in \mathcal{W}_L \right\}.
\]
Thus, it remains just to prove that \(\bar{\omega} := (\nabla u, \sigma) \in \mathcal{W}_L\) (i.e., that the Sobolev function \(\bar{u}\) satisfies the boundary conditions (C1)–(C3) above with the transformation \(\bar{\sigma}\)). The validity of (C1) and (C2) follows immediately from Proposition 4. In fact, the weak convergence of \(\{u_n\}\) in \(W^{1,2}(\Omega; \mathbb{R}^3)\) implies the strong convergence of traces \(\{\text{Tr} u_n\}\) in \(L^2(\partial \Omega; \mathbb{R}^3)\). So, up to a subsequence, \(\text{Tr} u_n(x) \rightarrow \text{Tr} \bar{u}(x)\) for \(\mathcal{H}^2\)-a.e. \(x \in \partial \Omega\). In particular, \(\text{Tr} \bar{u}(x) = x\) and \(h(\text{Tr} \bar{u}(x)) = 0\) almost everywhere on \(\Gamma_1\) and on \(\Gamma_2\), respectively (w.r.t. the Hausdorff measure).

In order to verify the condition (C3) we observe first that
\[
\text{Tr} u_n(x) = \text{Tr} u_n(\sigma_n(x)), n = 1, 2, \ldots, \tag{22}
\]
for \(\mathcal{H}^2\)-a.e. \(x \in \Gamma^+\) and consider the surface integral
\[
\mathcal{J} := \int_{\Gamma^+} |\text{Tr} \bar{u}(x) - \text{Tr} \bar{u}(\sigma(x))|^2 \, d\mathcal{H}^2(x).
\]
By the Minkowski’s inequality we have
\[
\mathcal{J}^{1/2} \leq (\mathcal{J}_1^n)^{1/2} + (\mathcal{J}_2^n)^{1/2} + (\mathcal{J}_3^n)^{1/2} \tag{23}
\]
where
\[
\begin{align*}
J_1^n & := \int_{\Gamma^+} |\text{Tr } \bar{u} (x) - \text{Tr } u_n (\sigma_n (x))|^2 \, d\mathcal{H}^2 (x); \\
J_2^n & := \int_{\Gamma^+} |\text{Tr } u_n (\sigma_n (x)) - \text{Tr } \bar{u} (\sigma (x))|^2 \, d\mathcal{H}^2 (x); \\
J_3^n & := \int_{\Gamma^+} |\text{Tr } \bar{u} (\sigma (x)) - \text{Tr } \bar{u} (\tilde{\sigma} (x))|^2 \, d\mathcal{H}^2 (x).
\end{align*}
\]

Taking into account the equalities (22) by using Proposition 1 we immediately obtain that \(J_1^n \to 0\) as \(n \to \infty\).

Due to the linearity of the trace operator, applying Lemma 3 and again Proposition 1 we arrive at
\[
J_2^n \leq L \int_{\Gamma^-} |\text{Tr } (u_n - \bar{u}) (x)|^2 \, d\mathcal{H}^2 (x) \to 0 \text{ as } n \to \infty.
\]

Let us approximate now \(\bar{u} \in W^{1,2} (\Omega; \mathbb{R}^3)\) by a sequence of continuous functions \(v_k : \bar{\Omega} \to \mathbb{R}^3, k = 1, 2, \ldots\) (with respect to the norm of \(W^{1,2} (\Omega; \mathbb{R}^3)\)). Then (see Proposition 1) \(v_k = \text{Tr } v_k \to \text{Tr } \bar{u}\) as \(k \to \infty\) in \(L^2 (\partial \Omega; \mathbb{R}^3)\). In particular, given \(\varepsilon > 0\) there exists an index \(k^* \geq 1\) such that
\[
\int_{\Gamma^+} |\text{Tr } u (y) - v_k (y)|^2 \, d\mathcal{H}^2 (y) \leq \varepsilon.
\]

By using Lemma 3 similarly as was done to estimate the integral \(J_2^n\) we have
\[
\int_{\Gamma^+} |\text{Tr } u_n (\sigma_n (x)) - v_k (\sigma_n (x))|^2 \, d\mathcal{H}^2 (x) \leq L \int_{\Gamma^-} |\text{Tr } \bar{u} (y) - v_k^* (y)|^2 \, d\mathcal{H}^2 (y) \leq L \varepsilon, \quad n = 1, 2, \ldots, \quad (24)
\]
and similarly
\[
\int_{\Gamma^+} |\text{Tr } \bar{u} (\tilde{\sigma} (x)) - v_k^* (\tilde{\sigma} (x))|^2 \, d\mathcal{H}^2 (x) \leq L \varepsilon. \quad (25)
\]

On the other hand, by the uniform continuity of \(v_k\) and the uniform convergence \(\sigma_n \to \bar{\sigma}\) as \(n \to \infty\), we find a number \(n^* \geq 1\) such that
\[
|v_k^* (\sigma_n (x)) - v_k^* (\bar{\sigma} (x))| \leq \varepsilon
\]
for all \(n \geq n^*\) and all \(x \in \Gamma^+\), and, consequently,
\[
\int_{\Gamma^+} |v_k^* (\sigma_n (x)) - v_k^* (\tilde{\sigma} (x))|^2 \, d\mathcal{H}^2 (x) \leq \mathcal{H}^2 (\Gamma^+) \varepsilon^2, n \geq n^*. \quad (26)
\]

Joining together the inequalities (24), (25) and (26) we obtain that
\[
(J_2^n)^{1/2} \leq (L \varepsilon)^{1/2} + (L \varepsilon)^{1/2} + (\mathcal{H}^2 (\Gamma^+) \varepsilon^2)^{1/2}, n \geq n^*.
\]

Since \(\varepsilon > 0\) is arbitrary and the constant \(L \varepsilon\) does not depend on \(n = 1, 2, \ldots\), we conclude that all the three integrals in the right-hand side of (23) tend to zero as \(n \to \infty\). Thus \(J = 0\), or, in other words, \(\text{Tr } \bar{u} (x) - \text{Tr } \bar{u} (\tilde{\sigma} (x)) = 0\) for \(\mathcal{H}^2\)-a.e. \(x \in \Gamma^+\), and the theorem is proved. \(\blacksquare\)
4 Necessary conditions of optimality

In this section, under some additional hypotheses, we deduce necessary conditions of optimality for problem (9).

To simplify, assume that the function $W$ is twice continuously differentiable and $h$ is continuously differentiable. Moreover, suppose that the surfaces $\Gamma_1, \Gamma_2, \Gamma^+, \Gamma^-$ are sufficiently smooth.

Given $\Gamma \subset \partial \Omega$, with $H^2(\Gamma) > 0$, in what follows we denote by $C^1(\Gamma^+; \mathbb{R}^3)$ the family of restrictions to $\Gamma$ of all functions $u : \Omega \to \mathbb{R}^3$, whose gradients are continuous up to the boundary. Let us supply $C^1(\Omega; \mathbb{R}^3)$ with the natural sup-norm.

We consider the problem (9) defined in the space $C^1(\Omega; \mathbb{R}^3) \times C^1(\Gamma^+; \mathbb{R}^3)$.

**Theorem 2** Let $(\bar{u}, \bar{\sigma}) \in C^1(\Omega; \mathbb{R}^3) \times C^1(\Gamma^+; \mathbb{R}^3)$ be a minimizer of problem (9). Assume that $\nabla h(\bar{u}(x)) \neq 0$, $x \in \Gamma_2$ and $\det \nabla \bar{u}(\bar{\sigma}(x)) \neq 0$, $x \in \Omega$. Then the following conditions are satisfied:

\[
\begin{align*}
\text{Div}(\nabla W)(\nabla \bar{u}(x)) &= 0, \quad x \in \Omega; \\
\nabla W(\nabla \bar{u}(x))\nu(x) &= 0, \quad x \in \Gamma_3; \\
\nabla W(\nabla \bar{u}(x))\nu(x) \times \nabla h(\bar{u}(x)) &= 0, \quad x \in \Gamma_2; \\
\nabla W(\nabla \bar{u}(x))\nu(x) &= 0, \quad x \in \Gamma^\pm. 
\end{align*}
\]

**Proof.** Let us write the constraints in the minimization problem (9) as $F(u, \sigma) = 0$ where the map $F : C^1(\Omega; \mathbb{R}^3) \times C^1(\Gamma^+; \mathbb{R}^3) \to C^1(\Gamma_1; \mathbb{R}^3) \times C^1(\Gamma_2; \mathbb{R}) \times C^1(\Gamma^+; \mathbb{R}^3)$ is given by

\[
F(u, \sigma) := (u(x) - x, h(u(x)), u(x) - u(\sigma(x))).
\]

Under our assumptions the map $F$ and the functional $I$ are both Fréchet differentiable. In particular, for the (Fréchet) derivative of $F$ at the point $(\bar{u}, \bar{\sigma})$ we have

\[
DF(\bar{u}, \bar{\sigma})(\tilde{u}, \tilde{\sigma}) = \begin{pmatrix}
\tilde{u}(x) \\
\langle \nabla h(\bar{u}(x)), \tilde{u}(x) \rangle \\
\langle \nabla \bar{u}(x), \tilde{\sigma}(x) \rangle - \nabla \bar{u}(\tilde{\sigma}(x))\tilde{\sigma}(x)
\end{pmatrix}.
\]

Here, and in what follows, $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^3$. Taking into account that $\nabla h(\bar{u}(x)) \neq 0$ on $\Gamma_2$ and that the jacobian matrix $\nabla \bar{u}(\bar{\sigma}(x))$ is not degenerated, we have that the linear operator $DF(\bar{u}, \bar{\sigma})$ is onto the space $C^1(\Gamma_1; \mathbb{R}^3) \times C^1(\Gamma_2; \mathbb{R}) \times C^1(\Gamma^+; \mathbb{R}^3)$. By the Lagrange multipliers rule (see, e.g., [10]) there exist linear continuous functionals $\lambda_1, \lambda_2, \lambda^+$ on $C^1(\Gamma_1; \mathbb{R}^3)$, $C^1(\Gamma_2; \mathbb{R})$ and $C^1(\Gamma^+; \mathbb{R}^3)$, respectively, such that

\[
\int_{\Omega} \sum_{i,j=1}^3 \frac{\partial W(\nabla \bar{u}(x))}{\partial x_j} \frac{\partial \tilde{u}_i(x)}{\partial x_j} dx = \lambda_1(\tilde{u}) + \lambda_2(\langle \nabla h(\bar{u}), \tilde{u} \rangle) + \lambda^+(\tilde{u} - \bar{u}(\bar{\sigma}) - \nabla \bar{u}(\bar{\sigma})\bar{\sigma}) = 0.
\]

Applying the Divergence theorem we get

\[
- \int_{\Omega} \langle \text{Div}(\nabla W(\nabla \bar{u}(x))), \tilde{u}(x) \rangle dx + \int_{\partial \Omega} \langle \nabla W(\nabla \bar{u}(x))\nu(x), \tilde{u}(x) \rangle dH^2(x)
+ \lambda_1(\tilde{u}) + \lambda_2(\langle \nabla h(\bar{u}), \tilde{u} \rangle) + \lambda^+(\tilde{u} - \bar{u}(\bar{\sigma}) - \nabla \bar{u}(\bar{\sigma})\bar{\sigma}) = 0.
\]
Here $\nu(x)$ is the unit outer normal to the boundary. Varying $\tilde{u}$ in (31) such that $\tilde{u}(x) = 0$ on $\partial \Omega$, we obtain (27). Taking then $\tilde{u} \in C^1(\Omega; \mathbb{R}^3)$ with $\tilde{u}(x) = 0$ on $\partial \Omega \setminus \Gamma_3$ we arrive at (28). Furthermore, choosing appropriate functions $\tilde{u}$ in (30) we obtain

\[
\int_{\Gamma_2} \langle \nabla W(\nabla \tilde{u}(x))\nu(x), \tilde{u}(x) \rangle \, d\mathcal{H}^2(x) + \lambda_2(\langle \nabla h(\tilde{u}), \tilde{u} \rangle) = 0 \quad (32)
\]

whenever $\tilde{u}(x) = 0$, $x \in \partial \Omega \setminus \Gamma_2$, and

\[
\int_{\Gamma_+ \cup \Gamma_-} \langle \nabla W(\nabla \tilde{u}(x))\nu(x), \tilde{u}(x) \rangle \, d\mathcal{H}^2(x) + \lambda^+(\tilde{u} - \tilde{u}(\bar{\sigma}) - \nabla \tilde{u}(\bar{\sigma})\bar{\sigma}) = 0 \quad (33)
\]

whenever $\tilde{u}(x) = 0$, $x \in \partial \Omega \setminus (\Gamma_+ \cup \Gamma_-)$.

Denote by $\Gamma_0^2$ the part of $\Gamma_2$ where the vectors $a(x) := \nabla W(\nabla \tilde{u}(x))\nu(x)$ and $b(x) := \nabla h(\tilde{u}(x))$ are co-linear. Taking an arbitrary $c \in C^1(\Gamma_2; \mathbb{R})$ such that $c(x) = 0$ and $\nabla c(x) = 0$ on $\Gamma_0^2$, let us define

\[
\hat{u}(x) := \begin{cases} 
\frac{a(x)(a(x),b(x)) - b(x)(a(x))^2}{||a(x)||^2 - ||a(x)||^2||b(x)||^2}c(x), & x \in \Gamma_2 \setminus \Gamma_0^2, \\
0, & x \in \Gamma_0^2.
\end{cases}
\]

Obviously, $\hat{u} \in C^1(\Gamma_2; \mathbb{R}^3)$, $\langle \hat{u}(x), a(x) \rangle = 0$ and $\langle \hat{u}(x), b(x) \rangle = c(x)$ for $x \in \Gamma_2$. From (32) we get $\lambda_2(c) = \lambda_2(\langle b, \hat{u} \rangle) = 0$. Hence, varying $\hat{u} \in C^1(\Gamma_2; \mathbb{R}^3)$ in (32) in a suitable way (in particular, setting $\hat{u}(x) = 0$ on $\Gamma_0^2$) we get $a(x) = 0$ in $\Gamma_2 \setminus \Gamma_0^2$. Thus the equality (29) follows.

Finally, taking $\hat{u} = 0$, from (33) we get $\lambda^+ = 0$, and, as a consequence,

\[
\int_{\Gamma_-} \langle \nabla W(\nabla \tilde{u}(x))\nu(x), \tilde{u}(x) \rangle \, d\mathcal{H}^2(x) + \int_{\Gamma_+} \langle \nabla W(\nabla \tilde{u}(x))\nu(x), \tilde{u}(x) \rangle \, d\mathcal{H}^2(x) = 0,
\]

which implies (30). □

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