A NEW INVARIANT AND PARAMETRIC CONNECTED SUM OF EMBEDDINGS

A. SKOPENKOV

Abstract. We define an isotopy invariant of embeddings $N \to \mathbb{R}^m$ of manifolds into Euclidean space. This invariant together with the $\alpha$-invariant of Haefliger-Wu is complete in the dimension range where the $\alpha$-invariant could be incomplete. We also define parametric connected sum of certain embeddings (analogous to surgery). This allows to obtain new completeness results for the $\alpha$-invariant and the following estimation of isotopy classes of embeddings. For the piecewise-linear category, a $(3n-2m+2)$-connected $n$-manifold $N$ and $\frac{4n+3}{3} \leq m \leq \frac{4n+2}{2}$ each preimage of $\alpha$-invariant injects into a quotient of $H_{3n-2m+3}(N)$, where the coefficients are $\mathbb{Z}$ for $m-n$ odd and $\mathbb{Z}_2$ for $m-n$ even.

Dedicated to the centennary of K. Borsuk

1. Introduction and main results

This paper is on the classical Knotting Problem: for an $n$-manifold $N$ and a number $m$ describe the set $\text{Emb}^m(N)$ of isotopy classes of embeddings $N \to \mathbb{R}^m$. For recent surveys see [RS99, Sk07]; whenever possible we refer to these surveys not to original papers.

All known complete concrete classification results (except for the Haefliger classification of links and smooth knots and recent results [KS05, Sk06, Sk06', CRS07, CRS]) can be obtained using $\alpha$-invariant of Haefliger-Wu (defined below). For another approaches see [Br68, GW99, CRS04, We].

We define an isotopy invariant of embeddings which, together with the $\alpha$-invariant, is complete in the dimension range where the $\alpha$-invariant could be incomplete (the $\beta$-invariant Theorem of §2). We also define parametric connected sum of certain embeddings (see the end of §2; this is a ‘surgery’ of an embedding preserving the embedded manifold). This allows to obtain new estimations of isotopy classes of embeddings and completeness results for the $\alpha$-invariant (the New Isotopy and Embedding Theorems of §1).

We work in the piecewise linear (PL) category [RS72]. (By [Br72] for $m \geq n+3$ the classification of embeddings of PL manifolds is the same in the PL and the TOP categories. Analogously to [Sk06'] our results give some information for the smooth category.)

1991 Mathematics Subject Classification. Primary: 57Q35, 57Q37; Secondary: 55S15, 55Q91, 57R40.

Key words and phrases. Embedding, deleted product, self-intersection, isotopy, Haefliger-Wu invariant.

The author gratefully acknowledges the support by INTAS Grant No. YSF-2002-393, by the Russian Foundation for Basic Research, Grants No 05-01-00993, 07-01-00648a and 06-01-72551-NCNILa, President of Russian Federation Grants MD-3938.2005.1, MD-4729.2007.1 and NSH-4578.2006.1, and by the Pierre Deligne fund based on his 2004 Balzan prize in mathematics.

1This paper is to appear in Fundamenta Mathematicae 197 (2007).
Let
\[ \tilde{N} = \{(x, y) \in N \times N \mid x \neq y\} \]
be the deleted product of \( N \), i.e. the configuration space of ordered pairs of distinct points of \( N \). For an embedding \( f : N \rightarrow \mathbb{R}^m \) one can define a map \( \tilde{f} : \tilde{N} \rightarrow S^{m-1} \) by the Gauss formula
\[ \tilde{f}(x, y) = \frac{fx - fy}{|fx - fy|}. \]
This map is equivariant with respect to the 'exchange of factors' involution \( t(x, y) = (y, x) \) on \( \tilde{N} \) and the antipodal involution on \( S^{m-1} \).

Define \( \alpha(f) \) to be the equivariant homotopy class of the map \( \tilde{f} \), cf. [Gr86, 2.1.E]. This is clearly an isotopy invariant.

Let \( \pi_{eq}^{m-1}(\tilde{N}) \) be the set of equivariant maps \( \tilde{N} \rightarrow S^{m-1} \) up to equivariant homotopy. Thus the \( \alpha \)-invariant is a map
\[ \alpha : \text{Emb}^m(N) \rightarrow \pi_{eq}^{m-1}(\tilde{N}). \]

It is important that using algebraic topology methods the set \( \pi_{eq}^{m-1}(\tilde{N}) \) can be explicitly calculated in many cases [BG71, Ba75, Ya83, RS99, Sk02, GS06, Sk07, §5]. So it is very interesting to know under which conditions the \( \alpha \)-invariant is bijective.

**Isotopy Theorem.** (a) [Sk07, the Haefliger-Weber Theorem 5.4] The \( \alpha \)-invariant is bijective for embeddings \( N \rightarrow \mathbb{R}^m \) of an \( n \)-polyhedron \( N \), if
\[ 2m \geq 3n + 4. \]

(b) [Sk07, Theorem 5.5] The \( \alpha \)-invariant is bijective for \( m \geq n + 3 \) and embeddings \( N \rightarrow \mathbb{R}^m \) of a closed \( k \)-connected \( n \)-manifold \( N \), if
\[ 2m \geq 3n + 3 - k. \]

These theorems have many specific corollaries [Sk07].

In this paper we study the case one dimension lower than in the Isotopy Theorem (b). By \( \mathbb{Z}_{(d)} \) we denote \( \mathbb{Z} \) for \( d \) even and \( \mathbb{Z}_2 \) for \( d \) odd.

**New Isotopy Theorem.** Let \( N \) be a closed \( k \)-connected orientable \( n \)-manifold,
\[ 2m = 3n + 2 - k \quad \text{and} \quad n \geq 3k + 6 \geq 6. \]

(a) The \( \alpha \)-invariant is surjective and each its preimage maps injectively into certain quotient of \( H_{k+1}(N; \mathbb{Z}_{(m-n-1)}) \).

(b) The \( \alpha \)-invariant is bijective if either \( (n, k, m) = (6, 0, 10) \) or \( N \) is almost parallelizable and \( (n, k, m) = (n, n - 14, n + 8) \), where \( 14 \leq n \leq 18 \).

The new part of the New Isotopy Theorem is estimation of point preimages of the \( \alpha \)-invariant (which is surjective by the Embedding Theorem (b) below). These preimages could apriori be non-trivial by [Sk06', Example 1.6.b] stated below, and could depend on \( n, k, N \) and the element of \( \pi_{eq}^{m-1}(\tilde{N}) \) (of which we take the preimage).
For $m - n$ even the New Isotopy Theorem (a) implies that these preimages are finite; the orientability assumption can be dropped.

The case $(n, k, m) = (6, 0, 10)$ of the New Isotopy Theorem (b) shows that [Ba75, Proposition 4] is true in the PL category for 6-manifolds.

Under the assumptions of the New Isotopy Theorem the $\alpha$-invariant is not always injective: For each even $n \not\in \{6, 14\}$ and $2m = 3n + 2$ the $\alpha$-invariant is not injective for embeddings $S^1 \times S^{n-1} \to \mathbb{R}^m$ [Sk06’, Example 1.6.b].²

Some classification results for $(3n - 2m + 2)$-connected manifold $N = S^p \times S^q$ are obtained in [Sk06’, Theorems 1.3 and 1.4, CRS07, CRS]. It is very surprising that something can be proved for general manifolds $N$.

**Conjecture.** (a) If $n \geq 3k + 4$ and $N$ is a closed $k$-connected almost parallelizable $n$-manifold, then there is an exact sequence of sets with an action $w$

$$H_{k+1}(N, \mathbb{Z}_{(m-n-1)}) \xrightarrow{w} \text{Emb}^m(N) \xrightarrow{\alpha} \pi^{m-1}_{eq}(\tilde{N}) \to 0.$$ 

(b) The Isotopy Theorem (a) holds for $(n, k, m) = (7, 1, 11)$ and $N$ spin, as well as for $(n, k, m) = (19, 5, 27)$ and $N$ almost parallelizable.³

The corresponding known and new surjectivity results are as follows.⁴

**Embedding Theorem.** (a) The $\alpha$-invariant is surjective for embeddings $N \to \mathbb{R}^m$ of an $n$-polyhedron $N$, if $2m \geq 3n + 3$ [Sk07, the Haefliger-Weber Theorem 5.4].

(b) The $\alpha$-invariant is surjective for embeddings $N \to \mathbb{R}^m$ of a closed $k$-connected $n$-manifold $N$ when $2m \geq 3n + 2 - k$ and $m \geq n + 3$ [Sk07, Theorem 5.5].

**New Embedding Theorem.** Let $N$ be a closed $k$-connected $n$-manifold. The manifold $N$ embeds into $\mathbb{R}^m$ if there is an equivariant map $\tilde{N} \to S^{m-1}$ and either

- $(n, k, m) = (7, 0, 11)$ and $N$ is orientable, or
- $(n, k, m) = (8, 1, 12)$ and $N$ is spin, or
- $(n, k, m) = (n, n-15, n+8)$ and $N$ is almost parallelizable, where $15 \leq n \leq 20$.

An $n$-manifold is $p$-parallelizable if any embedding $S^p \to N$ can be extended to an embedding $S^p \times D^{n-p} \to N$. Note that 1-parallelizability is equivalent to orientability and 1&2-parallelizability is equivalent to being a spin manifold.⁵ The almost parallelizability condition in the results of §1 can be relaxed to $(k+1)$-parallelizability.

**Acknowledgements.**

These results were presented at the Borsuk Centennary Conference (Bedlewo, 2005). I would like to acknowledge M. Skopenkov and S. Melikhov for useful discussions.

---

²Another examples of non-injectivity of the $\alpha$-invariant are recalled in [RS99, §4, Sk02, §1, Sk07, §5].

³This follows from our proof of the Isotopy Theorem (a) (§2) and an improvement [Sk06’, Standardization Lemma] of the Standardization Lemma of §2.

⁴Examples of non-surjectivity of the $\alpha$-invariant are recalled in [RS99, §4, Sk07, §5].

⁵It would be interesting to reformulate the $(k+1)$-parallelizability condition for $k$-connected manifolds in terms of Stiefel-Whitney classes.
2. Proofs

Almost embeddings and almost concordances.

An embedding $F : N \times I \to \mathbb{R}^m \times I$ is a concordance if $N \times 0 = F^{-1}(\mathbb{R}^m \times 0)$ and $N \times 1 = F^{-1}(\mathbb{R}^m \times 1)$. We tacitly use the facts that in codimension at least 3 concordance implies isotopy [Hu70, Li65], and every concordance or isotopy is ambient [Hu66, Ak69].

Let $N$ be a connected $n$-manifold and $B^n \subset \mathcal{N}$ some $n$-ball. The self-intersection set of a map $F : N \to \mathbb{R}^m$ is

$$
\Sigma(F) := \text{Cl}\{x \in N \mid \#F^{-1}Fx \geq 1\}.
$$

A map $F : N \to \mathbb{R}^m$ is an almost embedding of $(N, B^n)$, if $\Sigma(F) \subset B^n$. Or, equivalently, if $F|_{N-B^n}$ is an embedding and $F(N-B^n) \cap F(B^n) = \emptyset$.

A map $F : N \times I \to \mathbb{R}^m \times I$ is an almost concordance of $(N, B^n)$ if

$$N \times 0 = F^{-1}(\mathbb{R}^m \times 0), \quad N \times 1 = F^{-1}(\mathbb{R}^m \times 1) \quad \text{and} \quad \Sigma(F) \subset B^n \times I.$$

Instead of the pair $(N, B^n)$ we shall always write simply $N$ (no confusion would arise).\footnote{Almost embeddings and almost concordances were called quasi-embeddings and quasi-concordances in [Sk02].}

Almost Embeddings Theorem. Suppose that $N$ is a closed $k$-connected $n$-manifold, $k \geq 0$ and $m \geq n + 2$.

(a) If $f, g : N \to \mathbb{R}^m$ are almost concordant embeddings, then $\alpha(f) = \alpha(g)$ [Sk02, Theorem 5.2.α].

(b) If $2m = 3n + 2 - k$ and $f, g : N \to \mathbb{R}^m$ are embeddings such that $\alpha(f) = \alpha(g)$, then $f$ and $g$ are almost concordant [Sk02, Theorem 2.2.α].

(c) If $2m = 3n + 1 - k$ and $\varphi \in \pi_{eq}^{m-1}(\mathcal{N})$, then there is an almost embedding $F : N \to \mathbb{R}^m$ such that $\alpha(F) = \varphi$ [Sk02, Theorem 2.2.α].

Appendix: some results and conjectures on almost embeddings.

This section is not used in the proof of main results, but is perhaps of independent interest.

Complete classification of embeddings of a given $n$-manifold $N$ into $S^{n+2}$ up to isotopy (or concordance) seems to be hopeless because it is such for $N = S^n$. So it is interesting to obtain complete classification of embeddings of a given $n$-manifold $N$ into $S^{n+2}$ 'modulo knots $S^n \to S^{n+2}$'. The notion of almost concordance is not only useful to study the initial problem (of classification of embeddings up to concordance) for $m \geq n + 3$, but is a good notion of 'concordance modulo knots $S^n \to S^{n+2}$', because any knot $S^n \to S^{n+2}$ is almost concordant to the trivial knot.\footnote{For $N = S^n \cup \cdots \cup S^n$, the classification of embeddings $N \to \mathbb{R}^{n+2}$ up to link homotopy is motivated by classification of embeddings $N \to \mathbb{R}^{n+2}$ 'modulo knots $S^n \to \mathbb{R}^{n+2}$'.} Cf. [MR05].
We conjecture that almost concordance is equivalent to another natural equivalence relation of 'concordance modulo knots' $S^n \to S^{n+2}$, i.e. that for a closed $n$-manifold $N$ two embeddings $N \to S^{n+2}$ are almost concordant if and only if one can be obtained from an embedding concordant to the other by connected summation with knots $S^n \to S^{n+2}$.

Parts (a) and (b) of the following corollary are implied by the Almost Embeddings Theorem (b) and by [Sk02, Theorem 2.3.q], respectively.

**Corollary.** (a) Let $N$ be a sphere with $g$ handles. Then the set of PL almost embeddings $N \to \mathbb{R}^4$ up to PL almost concordance is in 1–1 correspondence with $\mathbb{Z}_2^{2g} \cong H_1(N; \mathbb{Z}_2)$.

(b) For a closed simply-connected 4-manifold $N$, the set of smooth almost embeddings $N \to \mathbb{R}^6$ up to PL almost concordance is in 1–1 correspondence with $\pi_{eq}^5(N)$.

We conjecture that the set of PL embeddings $S^1 \times S^1 \to \mathbb{R}^4$ up to PL almost concordance consists of exactly 3 elements (i.e. that the almost embedding $S^1 \times S^1 \to \mathbb{R}^4$, corresponding by Corollary (a) to the class $(1,1) \in H_1(S^1 \times S^1; \mathbb{Z}_2)$, is not almost concordant to a PL embedding).\(^9\)

The restriction $k \geq 0$ is essential in the Almost Embeddings Theorem (b).\(^10\)

We conjecture that the Almost Embeddings Theorem (b) holds in the smooth category, and that in [Sk02, Theorem 2.3.q] and in the injectivity of [Sk02, Theorem 2.3.a] we can replace PL category by DIFF (if $N$ is a smooth manifold).\(^11\)

We conjecture that for $n$ even, $m = (3n + 1 - k)/2 \geq n + 3$ and a $k$-connected closed $n$-manifold $N$ such that $H_{k+1}(N)$ is free there is an exact sequence of sets $\text{Emb}^m(N) \xrightarrow{\alpha} \pi_{eq}^{m-1}(\tilde{N}) \xrightarrow{\beta} H_{k+1}(N)$. Cf. the New Embedding Theorem.\(^12\)

**Definition of $\beta$-invariant.**

The Almost Embeddings Theorem (b) suggests the definition of an invariant, required for classification of embeddings when $2m \leq 3n + 2 - k$. For each almost concordance $F$ between embeddings analogously to [Hu69, XI.4.iii, Hu70', p. 408, Ha84, §1] we define an obstruction $\beta(F)$ to modification of $F$ to a concordance. Roughly speaking, $\beta(F)$ measures the linking of $\Sigma(F)$ with $F(N)$.

Analogous invariants are the Sato-Levine invariant of knots, the Hudson-Habegger obstruction to embedding disks and Fenn-Rolfsen-Koschorke-Kirk $\beta$-invariant of link maps, see references in [Sk06']. In the proof of the New Embedding and Isotopy Theorems we do not use the definition but only use the properties of $\beta$-invariant (they are stated in the $\beta$-invariant Theorem of the next subsection).

In this and the next subsections we omit the coefficients $\mathbb{Z}_{(m-n-1)}$ of chain groups in the notation.

---

\(^9\) A related result states that any PL embed $S^2 \sqcup \cdots \sqcup S^2 \to S^4$ is link homotopic to the trivial embedding [BT99, Ba01] (the case of two components was proved earlier by Hosokawa-Suzuki).

\(^10\) Indeed, for $l \notin \{3,7\}$ take a link $f: S^0 \times S^{2l-1} \to \mathbb{R}^{3l}$ such that $\lambda_{12}(f) = \lambda_{21}(f) = [\ell_1, \ell_l]$. Then $\alpha(f) = \Sigma^{\infty} \lambda_{21}(f) = 0$ but $f$ is not almost concordant to the standard embedding.

\(^11\) This could perhaps be proved analogously to the cited results using the relative version of [Sk02, Disjunction Theorem 3.1].

\(^12\) By [Sk02, Theorem 2.3.q] and the $\beta$-invariant Theorem it suffices to prove that for an almost embedding $F: N \to \mathbb{R}^m$ the obstruction $\beta(F)$ does not depend on $F|_{\partial N}$ (only on $F|_{N - \partial N}$). This obstruction is a map $b: \pi_n(M) \to H_{k+1}(N)$. We can prove that this map is constant by checking that $\pi_n(M)$ is finite and for fixed $\varphi_0 \in \pi_n(M)$ the map $\varphi \mapsto b(\varphi) - b(\varphi_0)$ is a homomorphism.
Suppose that \( N \) is a connected orientable \( n \)-manifold (possibly, with boundary) and \( F : N \to B^m \) is a proper general position almost embedding whose restriction to the boundary is an embedding. \((F \text{ could be an almost concordance between embeddings.})\)
Take a triangulation \( T \) of \( N \) such that \( B^n \) is a subcomplex of \( T \) and \( F \) is linear on simplices of \( T \). Then \( \Sigma(F) \) is a subcomplex of \( T \). Denote by \([\Sigma(F)] \in C_{2n-m}(B^n)\) the sum of top-dimensional simplices of \( \Sigma(F) \).

For \( m-n \) odd the coefficient \( \pm 1 \) of an oriented simplex \( \sigma \subset \Sigma(F) \) is defined as follows.\(^{13}\) Fix in advance any orientation of \( N \) and of \( B^m \). By general position there is a unique simplex \( \sigma' \) of \( T \) such that \( F(\sigma) = F(\sigma') \). The orientation on \( \sigma \) induces an orientation on \( F\sigma \) and then on \( \sigma' \). The orientations on \( \sigma \) and \( \sigma' \) induce orientations on normal spaces in \( N \) to these simplices. These two orientations (in this order) together with the orientation on \( F\sigma \) induce an orientation on \( B^m \). If this orientation agrees with the fixed orientation of \( B^m \), then the coefficient of \( \sigma \) is +1, otherwise -1. Clearly, change of orientation of \( \sigma \) changes the sign of \( \Sigma(F) \), so the sign is well-defined.\(^{14}\)

By \([\text{Hu69, Lemma 11.4, Hu70', Lemma 1}] \ \partial(\Sigma(F)) = 0.\(^{15}\) Hence
\[
[\Sigma(F)] = \partial C \quad \text{for some} \quad C \in C_{2n-m+1}(B^n).
\]
By \([\text{Hu70', Corollary 1.1}] \ \partial FC = 0. \quad \text{Hence}
\[
F(C) = \partial D \quad \text{for some} \quad D \in C_{2n-m+2}(B^m).
\]
By general position
\[
\tilde{D} := (F|_{N-\text{Int}B^n})^{-1}(D) \in C_{3n-2m+2}(N).
\]
Since the support of \( C \) is in \( B^n \), the support of \( F(C) = \partial D \) is in \( F(B^n) \). Hence \( \tilde{D} \) is a cycle and we can define\(^{16}\)
\[
\beta(F) := [\tilde{D}] \in H_{3n-2m+2}(N; \mathbb{Z}_{(m-n-1)}).
\]

Proof that \( \beta(F) \) is well-defined, i.e. is independent of the choices of \( C \) and \( D \). The independence of the choice of \( D \) is standard. Let us prove the independence of the choice of \( C \). For an almost embedding \( F \) if \( \partial C_1 = \partial C_2 = [\Sigma(F)] \) then \( \partial(C_1 - C_2) = 0 \). Hence
\[
C_1 - C_2 = \partial X \quad \text{for some} \quad X \in C_{2n-m+1}(B^n).
\]
Thus \( FC_1 - FC_2 = \partial FX \). Hence we can take chains \( D_1 \) and \( D_2 \) as above and such that \( D_1 - D_2 = FX \). Since the support of \( X \) is in \( B^n \), we have \( \tilde{D}_1 = \tilde{D}_2 \). \( \square \)

---

\(^{13}\) For \( m-n \) even this sign can also be defined but is not used.

\(^{14}\) This definition of sign is equivalent to Hudson’s one given as follows. The orientation on \( \sigma \) induces an orientation on \( F\sigma \) and on \( \sigma' \), hence it induces orientation on their links. Consider the oriented \((2m-2n-1)\)-sphere \( \text{lk}_F \sigma \), that is the link of \( F\sigma \) in certain triangulation of \( B^m \), ‘compatible’ with \( T \). This sphere contains disjoint oriented \((m-n-1)\)-spheres \( F(\text{lk}_T \sigma) \) and \( F(\text{lk}_T \sigma') \). Their linking coefficient \( \text{link}_{lk_F\sigma}(F(\text{lk}_T \sigma), F(\text{lk}_T \sigma')) \in \mathbb{Z}_{(m-n-1)} \) is the coefficient of \( \sigma \) in \([\Sigma(F)]\), which equals \( \pm 1 \).

\(^{15}\) Here we use the fact that coefficients are \( \mathbb{Z}_2 \) for \( m-n \) even.

\(^{16}\) This definition agrees with that for \( N = S^p \times S^q \) [Sk06’, subsection ‘a new embedding invariant’ of §2] when \( m \geq 2p + q + 2 \).
Properties of the $\beta$-invariant and proof of the New Isotopy Theorem (a).

$\beta$-invariant Theorem. Let $N$ be a connected orientable $n$-manifold (possibly with boundary) and $m \geq n + 3$. To each proper general position almost embedding $F : N \to B^m$ whose restriction to the boundary is an embedding there corresponds an element $\beta(F) \in H_{3n-2m+3}(N; \mathbb{Z}_{(m-n-1)})$ such that

(1) If $F$ is an embedding, then $\beta(F) = 0$.

(2) $\beta(F)$ is invariant under almost concordance of $F$ relative to the boundary.

(3) $\beta(F) = 0$ and $N$ is homologically $(3n-2m+1)$-connected, then $F$ is almost concordant relative to $N-B^n$ to an embedding.

(4) Suppose that $N = X \times I$ and $F, F'$ are almost concordances between $f_0$ and $f_1$, $f_1$ and $f_2$, respectively. Denote by $\overline{F}$ the reversed $F$, i.e. $\overline{F}(x, t) = F(x, 1-t)$. Define an almost concordance $F \cup F'$ between $f_0$ and $f_2$ as ‘the union’ of

$$F : X \times [0, 1] \to \mathbb{R}^m \times [0, 1] \quad \text{and} \quad F' : X \times [1, 2] \to \mathbb{R}^m \times [1, 2].$$

Then $\beta(F \cup F') = \beta(F) + \beta(F')$ and $\beta(\overline{F}) = -\beta(F)$.

Here the orientability assumption can be dropped for $m-n$ even.

The obstruction and additivity follow obviously by the definition of $\beta$-invariant.

The invariance is analogous to [Hu70', Lemma 2, cf. Hu69, Lemma 11.6]. The completeness is a non-trivial property, but it is an easy consequence of known results [Hu70', Theorem 2, Ha84, Theorem 4]. See the details below.

Proof of the invariance. Let $F_0$ and $F_1$ be two almost concordances between embeddings $f$ and $g$. Suppose that $\Phi : N \times I \to B^m \times I$ is an almost concordance between $F_0$ and $F_1$.

As in the above definition of $\beta(F_0)$ and $\beta(F_1)$ take chains

$$C_0, C_1 \in C_{2n-m+1}(B^n) \quad \text{such that} \quad \partial C_j = [\Sigma(F_j)] \quad \text{for} \quad j \in \{0, 1\}.$$

Analogously to the above definition of $\beta(F)$ define chain

$$[\Sigma(\Phi)] \in C_{2n-m+1}(B^n \times I).$$

Let $i_j : N \cong N \times j \to N \times I$ be the inclusions. Then

$$\partial [\Sigma(\Phi)] = i_1[\Sigma(F_1)] - i_0[\Sigma(F_0)], \quad \text{so} \quad \partial([\Sigma(\Phi)] + C_0 - C_1) = 0.$$

Therefore there exists

$$C \in C_{2n-m+2}(B^n \times I) \quad \text{such that} \quad \partial C = [\Sigma(\Phi)] + C_1 - C_0.$$

Analogously to [Hu70', Lemma 2] $\partial \Phi C = i_1(F_1C_1) - i_0(F_0C_0)$. Let $\text{pr} : N \times I \to N$ be the projection. Hence $\partial \text{pr} \Phi C = F_1C_1 - F_0C_0$. Thus analogously to the proof of the independence of $\beta$ of the choice of $C$ we obtain $\beta(F_0) = \beta(F_1)$. \hfill \Box

\footnote{Observe that the union of almost concordances is associative up to ambient isotopy.}
Proof of the completeness. Denote
\[ M := B^m - \text{Int} R_{B^m}(F(N - \hat{B}^n), F\partial B^n). \]

Observe that \( F|_{B^n} : B^n \to M \) is a proper map whose restriction to the boundary is an embedding.

Consider the following composition of Alexander and Poincaré duality isomorphisms (with the \( \mathbb{Z} \)-coefficients)
\[ H_i(M) \cong H^{m-i-1}(B^m - M, \partial B^m - M) \cong H^{m-i-1}(N - \hat{B}^n, \partial N) \cong \]
\[ \cong H_{i+n-m+1}(N - \hat{B}^n, \partial B^n) \cong H_{i+n-m+1}(N). \]

Since \( N \) is homologically \((3n - 2m + 1)\)-connected, we obtain that \( M \) is homologically \((2n - m)\)-connected. Since \( M \) is simply-connected, it follows that \( M \) is \((2n - m)\)-connected. Then by [Hu70', Theorem 2, Ha84, Theorem 4] the class \([FC] \in H_{2n-m+1}(M; \mathbb{Z}_{(m-n-1)})\) is the complete obstruction to the existence of a homotopy \( \text{rel} \partial B^n \) from \( F|_{B^n} : B^n \to M \) to an embedding. This class goes to \( \beta(F) = 0 \) under the composition of the above isomorphisms with coefficients \( \mathbb{Z}_{(m-n-1)} \). Hence \( F|_{B^n} \) is homotopic \( \text{rel} \partial B^n \) to an embedding. Extending this embedding over \( N \) by \( F \) we obtain the required embedding \( N \to B^m \). \( \square \)

Proof of the New Isotopy Theorem (a). Fix any \( \varphi \in \pi^{m-1}_{eq}(\tilde{N}) \) and any embedding \( f_0 : N \to \mathbb{R}^m \) such that \( \alpha(f_0) = \varphi \). Define
\[ K := \{ \beta(F_0) \in H_{k+1}(N; \mathbb{Z}_{(m-n-1)}) \mid F_0 \text{ is an almost concordance from } f_0 \text{ to } f_0 \}. \]

By the additivity of \( \beta \)-invariant, \( K \) is a subgroup (depending on \( n, k, N, \varphi \)).

For any embedding \( f : N \to \mathbb{R}^m \) such that \( \alpha(f) = \alpha(f_0) \) by Almost Embeddings Theorem (b) there is an almost concordance \( F \) from \( f \) to \( f_0 \). (This together with the additivity of \( \beta \)-invariant implies that \( K \) does not depend on the choice of \( f_0 \).) So we can define a map
\[ B : \alpha^{-1}(\varphi) \to H_{k+1}(N; \mathbb{Z}_{(m-n-1)})/K \text{ by } B(f) := \beta(F) + K. \]

If \( F \) and \( F' \) are two almost concordances from \( f \) to \( f_0 \), then \( F' \cup \overline{F} \) is an almost concordance from \( f_0 \) to \( f_0 \). Hence the map \( B \) is well-defined by the additivity of \( \beta \)-invariant.

If \( B(f) \in K \), then \( \beta(F) = \beta(F_0) \) for some almost concordance \( F_0 \) from \( f_0 \) to \( f_0 \). Then \( F \cup F_0 \) is an almost concordance from \( f \) to \( f_0 \), and by the additivity of \( \beta \)-invariant \( \beta(F \cup F_0) = 0 \). Hence by the completeness of \( \beta \)-invariant \( f \) is concordant to \( f_0 \). Thus \( B \) is injective. \( \square \)

Parametric connected sum of embeddings.
By \( S^p = D^p_+ \cup_{\partial D^p_+ = S^{p-1} = \partial D^p_-} D^p_- \) we denote the standard decomposition of \( S^p \). Analogously define \( \mathbb{R}^m_{\pm} \) and \( \mathbb{R}^{m-1} \). Identify \( D^p \) with \( D^p_+ \).

For \( m \geq n+2 \) denote by \( i \) the standard embedding which is the composition \( S^p \times S^{n-p} \to \mathbb{R}^{p+1} \times \mathbb{R}^{n-p+1} \subset \mathbb{R}^m \subset S^m \).
Let $N$ be a closed connected $n$-manifold. Denote by $s : S^p \times D^{n-p} \to N$ an embedding. For the ball $B^n \subset N$ from the definition of an almost embedding (concordance) assume that $\text{im} \, s \cap B^n = \emptyset$.

A map $f : N \to S^m$ is called $s$-standardized if $f \circ s : S^p \times D^{n-p} \to D^m_-$ is the restriction of the standard embedding $f$. Roughly speaking, a map $N \to D^m$ is $s$-standardized if its image is put on hyperplane $D^{m-1}$ so that the image intersects the hyperplane in a standardly embedded $S^p \times D^{n-p}$ (indeed, for such a map the set $\text{im} \, s$ can be pulled below the hyperplane to obtain an $s$-standardized embedding in the above sense).

A concordance $F : N \times I \to S^m \times I$ between $s$-standardized maps is called $s$-standardized if

$$F(\text{im} \, s \times I) \subset S^m \times I$$

is the identical concordance and

$$F((N - \text{im} \, s) \times I) \subset \text{Int} \, S^m_+ \times I.$$  

**Standardization Lemma.** (a) If $m \geq n + p + 2$, then any (almost) embedding $g : N \to S^m$ is isotopic to an $s$-standardized (almost) embedding.

(b) If $m \geq n + p + 3$, then any (almost) concordance between $s$-standardized embeddings $N \to S^m$ is isotopic relatively to the ends to an $s$-standardized (almost) concordance.

**Proof of (a).** Fix a point $y \in D^{n-p}_- \subset S^{n-p}$. Since $m \geq n + p + 2 \geq 2p + 2$, it follows that $g|_{S^p \times \{y\}}$ is unknotted in $S^m$. So there is an embedding $\tilde{g} : D^{p+1} \to S^m$ such that

$$\tilde{g}|_{\partial D^{p+1}} = g|_{S^p \times \{y\}} \quad \text{and} \quad \tilde{g} \, \text{Int} \, D^{p+1} \cap \text{Int} \, gN = \emptyset.$$

(The second property holds by general position because $m \geq n + p + 2$.) The regular neighborhood in $S^m$ of $\tilde{g}D^{p+1}$ is homeomorphic to the $m$-ball. Take an isotopy moving this ball to $D^m_-$ and let $f' : N \to S^m$ be the embedding obtained from $g$.

Now we are done since the embedding $f' \circ s$ is isotopic to the standard embedding by the following result (because $m \geq n + 3$, the pair $(S^p \times D^{n-p}, S^p \times S^{n-p-1})$ is $(n-p-1)$-connected and $n-p-1 \geq 2n-m+1$). □

**Unknotting Theorem Moving the Boundary.** Let $N$ be a compact $n$-dimensional PL manifold and $f, g : N \to D^m$ proper PL embeddings. If $m \geq n + 3$ and $(N, \partial N)$ is $(2n-m+1)$-connected, then $f$ and $g$ are properly isotopic [Hu69, Theorem 10.2, p. 199].

**Proof of (b).** This is a relative version of the proof of (a). Take a concordance $G$ between standardized embeddings $f_0, f_1 : N \to S^m$. There is a level-preserving embedding $\widehat{G} : D^{p+1} \times \{0, 1\} \to S^m \times \{0, 1\}$ whose components satisfy to (*). Since $m+1 \geq p+1+3$, by the Haefliger-Zeeman Unknotting Theorem any concordance $S^p \times I \to S^m \times I$ standard on the boundary is isotopic to the standard concordance. Hence the map $\widehat{G}$ can be extended to an embedding $\widehat{G} : D^{p+1} \times I \subset S^m \times I$ such that

$$\widehat{G}|_{\partial D^{p+1} \times I} = G|_{S^p \times \{y\} \times I} \quad \text{and} \quad \widehat{G} \, \text{Int} \, D^{p+1} \times I \cap G(N \times I) = \emptyset.$$ 

(The second property holds by general position because $m \geq n + p + 3$.) Take a regular neighborhood

$$B^m \times I \quad \text{in} \quad S^m \times I$$ 

of $\widehat{G}D^{p+1}$ such that $(B^m \times I) \cap (S^m \times \{0, 1\}) = D^m_0 \times \{0, 1\}$. 

---

18 Note that standardized in the sense of [Sk06', §2] is $i$-standardized in the sense of this paper.
Take an isotopy of $S^m \times I$ rel $S^m \times \{0,1\}$ moving $B^m \times I$ to $D^m \times I$. Let $F'$ be the concordance obtained from $G$ by this isotopy.

The embedding $F'_{|_{I \times s \times I}} : I \times s \times I \to D^m_\infty \times I$ is isotopic rel $D^m_\infty \times \{0,1\}$ to the identical concordance by the following Unknotting Theorem Moving Part of the Boundary (which is proved analogously to [Hu69, Theorem 10.2 on p. 199]).

Let $N$ be a compact $n$-dimensional PL manifold, $A$ a codimension zero submanifold of $\partial N$ and $f,g : N \to D^m$ proper PL embeddings. If $m \geq n+3$ and $(N,A)$ is $(2n-m+1)$-connected, then $f$ and $g$ are properly isotopic rel $\partial N - A$. □

By $R_k$ we denote the symmetry of $\mathbb{R}^k$ with respect to the plane $x_1 = x_2 = 0$.

**Definition of parametric connected sum.** (a) Take (almost) embeddings $f : N \to \mathbb{R}^m$ and $g : S^p \times S^{n-p} \to \mathbb{R}^m$.

If $m \geq n+p+2$, then by Standardization Lemma (a) we can make isotopies and assume that $f$ and $g$ are s-standardized and i-standardized, respectively. Define an (almost) embedding

$$f \#_sg : N \to \mathbb{R}^m \quad \text{by} \quad (f \#_sg)(a) = \begin{cases} f(a) & a \notin \text{im } s \\ R_m g(x, R_{n-p} y) & a = s(x,y) \end{cases},$$

(b) Take (almost) concordances $F : N \times I \to \mathbb{R}^m \times I$ and $G : S^p \times S^{n-p} \times I \to \mathbb{R}^m \times I$.

If $m \geq n+p+3$, then by the Standardization Lemma (b) we can make isotopies relative to the ends and assume that $F$ and $G$ are s-standardized and i-standardized, respectively. Define a (almost) concordance $F \#_s G : N \times I \to \mathbb{R}^m \times I$ by

$$(F \#_s G)(a,t) = \begin{cases} F(a,t) & a \notin \text{im } s \\ (R_m G(x, R_{n-p} y, t), t) & a = s(x,y) \end{cases}.$$

We do not need parametric connected sum to be independent on the choice of an almost concordance to a *standardized* almost embedding or almost concordance: we denote by $f \#_sg$ or $F \#_s G$ the result for any such choice.

**Proof of the New Isotopy Theorem (b) and the New Embedding Theorem.**

**The Hopf Invariant Lemma.** Take the standard embedding $i : D^{p+1} \times S^q \to S^m$. Represent $\varphi \in \pi_{p+q}(S^{m-q-1})$ by a map (not necessarily an embedding) $\varphi : S^{p+q} \to S^m - i(D^{p+1} \times S^q) \simeq S^{m-q-1}$. If $2m = 3q + 2p + 2$, then $\beta(i \# \varphi) = \pm H \Sigma \varphi$.\(^{19}\)

Proof (analogous to [Ko88, Theorem 4.8]). We may assume that $\varphi$ is a smooth general position framed immersion. Extend $\varphi$ to a smooth general position framed immersion $\hat{\varphi} : B \to \mathbb{R}^m$, where $B := B^{p+q+1}$. Then by [Ko88, Theorem 1.3] and [Ke59, Lemma 5.1] the class $\pm \Sigma \varphi \in \pi_{p+2q+1-m}^S$ is represented by the framed $(p+2q+1-m)$-submanifold

$$\Delta := \{(u,z) \in B \times S^q \mid \hat{\varphi}(u) = i(a,z)\} \quad \text{of} \quad B \times S^q \subset S^{p+2q+1-m}$$

\(^{19}\)Here $i \# \varphi$ is embedded connected sum of linked embeddings but not parametrized connected sum as above; $i \# \varphi = \overline{\varphi}(\varphi)$ in the notation of [Sk06].
(with natural framing). For a 0-chain $X$ with coefficients in $\mathbb{Z}_{(m-p-q-1)}$ in a connected manifold denote by $[X]$ the number of points in $X$ modulo 2 when $m-p-q$ is even (we need only this case for the Non-triviality Lemma) and the algebraic number of points when $m-p-q$ is odd. (I.e. $[X]$ is the 0-dimensional homology class of $X$.) Then by [Ko88, p. 411]

$$\pm H\Sigma \varphi = \{(x,y) \in \Delta \times \Delta \mid \text{pr}_2 x = \text{pr}_2 y \}$$

(this set is finite by general position). Thus

$$\pm H\Sigma \varphi = \{(u,v,z) \in B \times B \times S^q \mid \hat{\varphi}(u) = \hat{\varphi}(v) = i(a,z)\} = [i(a \times S^q) \cap \hat{\varphi} \text{pr}_2 D] = \beta(i\# \varphi),$$

where $D := \{(u,v) \in B \times B \mid \hat{\varphi}(u) = \hat{\varphi}(v)\}$.

Here the last equality holds because

$$C := \{(u,v) \in \partial B \times B \mid \varphi(u) = \varphi(v)\}, \quad D \quad \text{and} \quad \tilde{D} := \hat{\varphi} \text{pr}_2 D$$

are as in the definition of $\beta$-invariant ($C$ has natural orientation for $m-p-q$ odd); the groups $H_p(S^p \times S^q \times \mathbb{Z}_{(m-p-q-1)})$ and $\mathbb{Z}_{(m-p-q-1)}$ are identified by the isomorphism $\gamma \mapsto [a \times S^q]$. The definition of $\beta$-invariant does make sense in the piecewise-smooth category (and hence in the smooth category). Recall that piecewise-smooth category is equivalent to the PL category, i.e. the forgetful map from the set of PL embeddings up to PL isotopy to the set of piecewise differentiable embeddings up to piecewise differentiable isotopy is a 1–1 correspondence [Ha67]. □

**Non-triviality Lemma.** For $1 \leq p < l \leq \{3,7\}$ there exists an almost embedding $G : S^p \times S^{2l} \to \mathbb{R}^{3l+p+1}$ such that $\beta(G) = 1$.

**Proof.** Since $p \geq 1$, the group $\pi_{2l+p}(S^{l+p})$ is either stable or metastable, so the stable suspension $\Sigma^\infty$ is epimorphic. Since $l \in \{3,7\}$, the Hopf invariant $H$ is epimorphic. Hence there exists $\varphi \in \pi_{2l+p}(S^{l+p})$ whose stable Hopf invariant $H\Sigma^\infty(\varphi) = 1 \in \mathbb{Z}_2$. By the Hopf Invariant Lemma for $q = 2l$ and $m = 3l + p + 1$ we obtain $\beta(i\# \varphi) = H\Sigma^\infty(\varphi) = 1$. □

**Realization Lemma.** Let $N$ be an orientable $(p-1)$-connected $p$-parallelizable closed $n$-manifold and $n \geq 2p + 1$. Then any homology class $x \in H_p(N; \mathbb{Z}$ or $\mathbb{Z}_2)$ is realizable by an embedding $\varphi : S^p \times D^{n-p} \to N$.

**Proof.** Since $N$ is $(p-1)$-connected, any homology class in $H_p(N; \mathbb{Z})$ can be realized by a map $S^p \to N$. Hence the same holds for $\mathbb{Z}_2$-coefficients. Since $n \geq 2p + 1$ every such map is homotopic to an embedding $S^p \to N$. Since $N$ is $p$-parallelizable, it follows that this embedding can be extended to an embedding $\varphi : S^p \times D^{n-p} \to N$. □

**#-additivity Lemma.** If $p = 3n - 2m + 3 \geq 0$, $m \geq n + p + 2$ and $s : S^p \times D^{n-p} \to N$ is an embedding, then $\beta(f \# s) = \beta(f) + \beta(g)[s]$ for almost embeddings $f : N \to \mathbb{R}^m$ and $g : S^p \times S^{n-p} \to \mathbb{R}^m$, where $\beta(g)$ is considered as an element of $\mathbb{Z}_{(m-n-1)}$.

**Proof.** Since $m \geq n + p + 2$, by the Standardization Lemma (a) we may assume that $f$ and $g$ are standardized. Since $R_m$ and $R_{n-p}$ are isotopic to the identity maps of $\mathbb{R}^m$ and of $S^{n-p}$, they do not change orientations. Hence

$$[\Sigma(f \# [s]) = [\Sigma(f)] + s_*(id S^p \times R_{n-p})_*[\Sigma(g)],$$

so we can take

$$C_{f \# s} := C_f + s_*(id S^p \times R_{n-p})_*C_g \quad \text{and} \quad D_{f \# s} := D_f + R_m D_g.$$
We may assume that the supports of $D_f$ and $D_g$ are in $\mathbb{R}^m_+$. Identify $S^p \times D^{n-p}$ with $S^p \times D_+^{n-p}$ so that it would contain the support of $\tilde{D}_g$. Then

$$
\beta(f \# s g) = [\tilde{D}_{f \# s g}] = [(F \# s G|_{N-\text{Int } B^n})^{-1} D_f] + [(F \# s G|_{N-\text{Int } B^n})^{-1} R_m D_g] = [\tilde{D}_f] + s [\tilde{D}_g] = \beta(f) + [s] \beta(g). \quad \square
$$

**Proof of the New Embedding Theorem.** Take any $\varphi \in \pi^{m-1}_{eq}(\tilde{N})$. Since $2m = 3n+1-k$, by the Almost Embedding Theorem (c) there is an almost embedding $f : N \rightarrow \mathbb{R}^m$ such that $\alpha(f) = \varphi$. By the Realization Lemma there is an embedding $s : S^{k+1} \times D^{n-k-1} \rightarrow N$ such that $[s] = -\beta(f)$. By the Non-triviality Lemma there is an almost embedding $G : S^{k+1} \times S^{n-k-1} \rightarrow \mathbb{R}^m$ such that $\beta(G) = 1$. By the #-additivity Lemma $\beta(f \# s G) = \beta(f) + [s] = 0$. By the completeness of $\beta$-invariant $f \# s G$ is almost concordant to an embedding $N \rightarrow \mathbb{R}^m$. (Note that possibly $\alpha(f) \neq \alpha(f \# s G)$.) \square

**Proof of the New Isotopy Theorem (b).** Since $2m = 3n+2-k$, the surjectivity follows by the Embedding Theorem (b). The following proof of the injectivity is a relative version of the proof of the New Embedding Theorem.

Take two embeddings $f, f' : N \rightarrow \mathbb{R}^m$ such that $\alpha(f) = \alpha(f')$. By the Almost Embeddings Theorem (b) there is an almost concordance $F$ from $f$ to $f'$. By the Realization Lemma there is an embedding $s : S^{k+1} \times D^{n-k-1} \rightarrow N$ such that $[s] = -\beta(F)$. Then $\alpha(f) = \alpha(f')$.

Take an almost embedding $G : S^{k+1} \times S^{n-k} \rightarrow \mathbb{R}^{m+1}$ given by the Non-triviality Lemma. Analogously to the Standardization Lemma we may assume that $G(S^{k+1} \times D^{n-k}_-) \subset D^{m+1}_-$ is the standard embedding,

$G(S^{k+1} \times \frac{1}{2} D^{n-k}_+) \subset \frac{1}{2} D^{m+1}_+$ is the standard embedding and

$$
G(S^{k+1} \times (D^{n-k}_- - \frac{1}{2} D^{n-k}_+)) \subset D^{m+1}_+ - \frac{1}{2} D^{m+1}_+.
$$

The latter inclusion $(\ast)$ gives an almost $G_0$ concordance between standard embeddings such that $\beta(G_0) = 1$. Then $F \# s G_0$ is an almost concordance from $f$ to $f'$.

Analogously to the #-additivity Lemma one proves the following.

If $p = 3n-2m+2 \geq 0$, $m \geq n + p + 3$ and $s : S^p \times D^n-p \rightarrow N$ is an embedding, then $\beta(F \# s G_0) = \beta(F) + \beta(G_0)[s]$ for almost concordances $F : N \times I \rightarrow \mathbb{R}^m \times I$ and $G_0 : S^p \times S^{n-p} \times I \rightarrow \mathbb{R}^m \times I$, where $\beta(G_0)$ is considered as an element of $\mathbb{Z}_{(m-n-1)}$.

So $\beta(F \# s G_0) = \beta(F) + [s] = 0$. Hence by the completeness of $\beta$-invariant $f$ is isotopic to $f'$. \square

**References**

[Ak69] E. Akin, *Manifold phenomena in the theory of polyhedra*, Trans. Amer. Math. Soc. **143** (1969), 413–473.

[Ba75] D. R. Bausum, *Embeddings and immersions of manifolds in Euclidean space*, Trans. AMS **213** (1975), 263–303.

[Ba01] A. Bartels, *Higher dimensional links are singular slice*, Math. Ann. **320** (2001), 547–576.

[BG71] J. C. Becker and H. H. Glover, *Note on the Embedding of Manifolds in Euclidean Space*, Proc. of the Amer. Math. Soc. **27**:2 (1971), 405-410; doi:10.2307/2036329.

[Br68] W. Browder, *Embedding smooth manifolds*, Proc. Int. Congr. Math. Moscow 1966 (1968), 712–719.

[Br72] J. L. Bryant, *Approximating embeddings of polyhedra in codimension 3*, Trans. Amer. Math. Soc. **170** (1972), 85–95.
A NEW INVARIANT AND PARAMETRIC CONNECTED SUM OF EMBEDDINGS

[BT99] A. Bartels and P. Teichner, *All two-dimensional links are null-homotopic*, Geom. Topol. 3 (1999), 235–252.

[CRS04] M. Cencelj, D. Repovš and A. Skopenkov, *On the Browder-Levine-Novikov embedding theorems*, Trudy Mat. Inst. Steklova 247 (2004), 280–290; English translation: Proc. of the Steklov Inst. of Math. 247 (2004).

[CRS07] M. Cencelj, D. Repovš and M. Skopenkov, *Homotopy type of the complement of an immersion and classification of embeddings of tori*, Uspekhi Mat. Nauk 62:5 (2007), 165-166; English transl: Russian Math. Surveys 62:5 (2007).

[CRS] M. Cencelj, D. Repovš and M. Skopenkov, *Knotted tori and the β-invariant*, preprint (2005).

[Gr86] M. Gromov, *Partial differential relations*, Ergeb. Math. Grenzgeb., Springer, Berlin, 1986.

[GS06] D. Gonçalves and A. Skopenkov, *Embeddings of homology equivalent manifolds with boundary*, Topol. Appl. 153:12 (2006), 2026-2034.

[GW99] T. Goodwillie and M. Weiss, *Embeddings from the point of view of immersion theory, II*, Geometry and Topology 3 (1999), 103–118.

[Ha67] A. Haefliger, *Lissage des immersions-I*, Topology 6 (1967), 221–240.

[Ha84] N. Habegger, *Obstruction to embedding disks II: a proof of a conjecture by Hudson*, Topol. Appl. 17 (1984).

[Hu66] J. F. P. Hudson, *Extending piecewise linear isotopies*, Proc. London Math. Soc. (3) 16 (1966), 651–668.

[Hu69] J. F. P. Hudson, *Piecewise-Linear Topology*, Benjamin, New York, Amsterdam, 1969.

[Hu70] J. F. P. Hudson, *Concordance, isotopy and diffeotopy*, Ann. of Math. 91:3 (1970), 425–448.

[Hu70'] J. F. P. Hudson, *Obstruction to embedding disks*, In: Topology of manifolds (1970), 407–415.

[Ka59] M. Kervaire, *An interpretation of G. Whitehead’s generalization of H. Hopf’s invariant*, Ann. of Math. 69 (1959), 345–362.

[Ke88] U. Koschorke, *Link maps and the geometry of their invariants*, Manuscripta Math. 61:4 (1988), 383–415.

[KS05] M. Kreck and A. Skopenkov, *A classification of smooth embeddings of 4-manifolds in 7-space*, submitted; [arXiv:math/0512594].

[Li65] W. B. R. Lickorish, *The piecewise linear unknotting of cones*, Topology 4 (1965), 67–91.

[MR05] S. Melikhov and D. Repovš, *n-Quasi-isotopy: I. Questions of nilpotence*, J. Knot Theory Ramif. 14 (2005), 571–602; [arXiv:math/0103113].

[RS72] C. P. Rourke and B. J. Sanderson, *Introduction to Piecewise-Linear Topology*, Ergeb. Math. Grenzgeb. 69, Springer, Berlin, 1972.

[RS99] D. Repovš and A. Skopenkov, *New results on embeddings of polyhedra and manifolds into Euclidean spaces*, Uspekhi Mat. Nauk 54:6 (1999), 61–109 (in Russian); English transl., Russ. Math. Surv. 54:6 (1999), 1149–1196.

[Sk02] A. Skopenkov, *On the Haefliger-Hirsch-Wu invariants for embeddings and immersions*, Comment. Math. Helv. (2002), 78–124.

[Sk06] A. Skopenkov, *A classification of smooth embeddings of 3-manifolds in 6-space*, Math. Zeitschrift, to appear; [arXiv:math/0603429].

[Sk06'] A. Skopenkov, *Classification of embeddings below the metastable dimension*; [arXiv:math/0607422].

[Sk07] A. Skopenkov, *Embedding and knotting of manifolds in Euclidean spaces*, in: *Surveys in Contemporary Mathematics, Ed. N. Young and Y. Choi*, London Math. Soc. Lect. Notes 347 (2007), 248–342; [arXiv:math/0604041].

[We] M. Weiss, *Second and third layers in the calculus of embeddings*, preprint.

[Ya83] T. Yasui, *On the map defined by regarding embeddings as immersions*, Hiroshima Math. J. 13 (1983), 457–476.