Risk minimizing of derivatives via dynamic $g$-expectation and related topics*

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Abstract

In this paper, we investigate risk minimization problem of derivatives based on non-tradable underlyings by means of dynamic $g$-expectations which are slight different from conditional $g$-expectations. In this framework, inspired by [1] and [16], we introduce risk indifference price, marginal risk price and derivative hedge and obtain their corresponding explicit expressions. The interesting thing is that their expressions have nothing to do with nonlinear generator $g$, and one deep reason for this is due to the completeness of financial market. By giving three useful special risk minimization problems, we obtain the explicit optimal strategies with initial wealth involved, demonstrate some qualitative analysis among optimal strategies, risk aversion parameter and market price of risk, together with some economic interpretations.

Keywords. dynamic $g$-expectation, risk minimization problem, risk indifferent price, market price of risk, risk aversion parameter.

AMS Mathematics subject classification. 60H10, 91B30, 60H30.

1 Introduction

Recently there are many financial instruments written on non-tradable underlyings such as weather future, catastrophe future, and other financial products. Since they are impossible to perfectly hedge, one has to look for some well correlated tradable assets to cross hedge the risk. As to the pricing and hedging problem for such derivatives, see for example Ankirchner et al ([1], [2]). In this paper, we propose a new framework within which to address the dynamic risk minimization problem of above derivatives.

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As to the risk minimization problem, in the literature there are various forms of works on it, such as Mataramvura and Øksendal [15] in a zero-sum stochastic differential game framework, Øksendal and Sulem [17] via stochastic maximum principle for FBSDEs, Horst and Moreno-Bromberg [12] in a principle agent game framework. In this paper, we consider this problem by a new important representation of dynamic convex risk measures via BSDEs. As we know, g-expectations and conditional g-expectations were introduced by Peng [19] as solutions of a class of nonlinear BSDEs and proved by Rosazza Gianin [21] to provide examples of dynamic coherent or convex risk measures under suitable hypothesis. Therefore, inspired by this, we will use a slight different way to represent dynamic risk measures. More precisely, given $t \in [0, T]$, $x$ being initial wealth at time $t$, $X^{t,x}(T)$ being the terminal wealth, if the following BSDE admits a unique solution $(Y, Z)$ on $[t, T]$,

$$Y(r) = -X^{t,x}(T) + \int_r^T g(s, Z(s))ds - \int_r^T Z(s)dW(s), \quad r \in [t, T],$$  

(1.1)

we then define $\rho(t, X^{t,x}(T)) = Y(t)$ with $t \in [0, T]$ the dynamic $g$-expectation on $[0, T]$, $\rho_1(r, X^{t,x}(T)) = Y(r)$ with $r \in [t, T]$ the conditional $g$-expectation on $[t, T]$. Particularly, if $t = 0$, the above two notions degenerate into the classical $g$-expectations (denoted by $\rho_2(0; X^{0,x}(T))$) and conditional $g$-expectations (denoted by $\rho_3(r; X^{0,x}(T))$ with $r \in [0, T]$) in [19] and [21]. Note that $\rho$, $\rho_1$ and $\rho_3$ above can be regarded as dynamic risk measures via BSDE (1.1).

In this paper, we will make use of dynamic $g$-expectation $\rho$, rather than $\rho_1$ and $\rho_3$, to study the risk minimization problem (see Section 2 below). This is not just because of the natural form of $\rho$, while some deeper meanings are involved. Actually, in order to make the results meaningful from the economic point of view, Björk et al [5] argued that risk aversion parameter (or risk tolerance parameter) should be dependent on initial wealth in the dynamic mean-variance portfolio optimization, see also Hu et al [10], which inspires us to consider the risk measures in more general yet realistic framework. For example, we can represent dynamic convex risk measure by dynamic $g$-expectation above in the form

$$\rho(t, X^{t,x}(T)) = -X^{t,x}(T) + \int_t^T g(s, x, Z(s))ds - \int_t^T Z(s)dW(s),$$  

(1.2)

with $g$ being convex and depending on initial wealth $x$. In particular, when $g(s, x, z) = \frac{1}{2\gamma(x)}|z|^2$,

$$\rho(t, X^{t,x}(T)) = \delta(x) \log E^{F_t} \exp \left( -\frac{X^{t,x}(T)}{\delta(x)} \right), \quad t \in [0, T], \quad x \in \mathbb{R},$$

with $\delta(x)$ being the risk tolerance parameter, is a generalized dynamic entropic risk measure. Similarly ideas also appear in Example 3.1 below. On the other hand, if we similarly define dynamic risk measure $\rho_1$ above on $[t, T]$ in such setting, it will become controversial with one basic axiom of defining dynamic risk measure, that is, independence of the past, since $g$ depends on initial wealth $x$. For example, if $t = 0$, and risk aversion parameter $\gamma$ is a decreasing function in $x$, then it becomes unsuitable to use conditional $g$-expectation in $[0, T]$ to represent dynamic risk measures.

In this paper we mainly consider the case of $g$ being Lipschitz in $z$. If $X^{t,x}(T)$ is the terminal wealth depending on some control variable or investment strategy (see for example
our aim is then to find an optimal strategy to minimize the risk defined in (1.2) in a dynamical manner. For this risk minimization problem, we firstly give one sufficient condition for the existence of optimal strategy by means of comparison theorem for BSDEs. From a simple example, we see that a new minimizing problem, playing a key role in above sufficient condition, is also necessary in some sense. This inspires us to study this minimizing problem more deeply. By using Legendre-Fenchel transform and sub-differential for convex functions, we give one sufficient and necessary condition for this minimization problem, and explain it by two simple examples. Furthermore, we use the procedure here to derive risk indifference price, marginal risk price and derivative hedge, the definitions of which are inspired by the work in [11] and [10]. At last by giving three special and useful cases of $g$ in (1.2), we study the corresponding risk minimization problem and get the explicit form of optimal strategy. As in [5], the strategy here depends on initial wealth and their economic rationality via qualitative analysis are also emphasized. We talk about the tight relations between the existence of optimal solution and two basic financial notions, that are, risk aversion parameter and market price of risk.

The main novelty of this paper contains the following: Firstly we introduce two explicit kinds of risk measures via dynamic $g$-expectation representing two classes of investors with different attitude to risk. By looking for some subjective parameter of investor and studying its connection with market price of risk (objective parameter), we explore the requirements of wellposedness of optimal strategy. Some comparisons between these two cases are also given with some reasonable interpretation. Secondly, inspired by the recent work of [22] and [23], when risk measures are expressed by BSDEs with generators being regarded as continuous time analogue of discrete Gini principle, we derive the explicit optimal strategy for the corresponding risk minimization problem, identify the term representing risk aversion parameter. Note that the form of $\gamma(x) \cdot g$ here, with $\gamma(x)$ being the risk aversion parameter, is just the well-known Huber penalty function in [9]. To our best knowledge, it is the first time to do analysis in above two aspects via dynamic $g$-expectation. Thirdly, when dealing with above special risk minimization problems, we also give some qualitative analysis among optimal strategy, market price of risk, initial wealth, and risk aversion parameter. In other words, we talk about the connections between market price of risk and optimal strategy, compare optimal strategy across different investors and wealth level, which are consistent with the results in [8], [20] and [26]. Fourthly, when $g$ is Lipschitz in $z$, we not only derive the explicit form of risk indifference price, marginal risk price and derivative hedge, but also notice one interesting phenomenon, that is their independence of $g$ representing risk preference of investors. By comparing with [11], we believe that one potential reason for this is the completeness of financial market. Such result obtained via BSDEs is still new to our knowledge.

The paper is organized as follows: In Section 2, we will formulate the risk minimization problem and give some useful notations. In Section 3, we will give one sufficient condition to ensure the existence of optimal strategy. By exploring a minimizing problem especially, we derive the explicit form of risk indifference price, marginal risk price and derivative hedge. Some examples are given to further explain these results. In Section 4, we study three important and inspiring cases by connecting with risk aversion parameter and market price of risk, exploring some qualitative analysis for the expressions and explaining the economic rationality of obtained results.
2 Model formulation and preliminary

Throughout this paper, we let \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) be a complete filtered probability space on which a one-dimensional standard Brownian motion \(W(\cdot)\) is defined with \(\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}\) being its natural filtration augmented by all the \(\mathbb{P}\)-null sets. Suppose that the price of a one-dimensional non-tradable index (such as a stock, temperature or loss index) follows the dynamic
\[
dR_s^{t,r} = b(s, R_s^{t,r})dt + \sigma(s, R_s^{t,r})dW_s, \quad s \in [t, T), \ r_0 \in \mathbb{R},
\]
with \(t \in [0, T], \ b : [t, T] \times \mathbb{R} \to \mathbb{R}\) and \(\sigma : [t, T] \times \mathbb{R} \to \mathbb{R}\) being Lipschitz and linear growth. For simplicity of presentation, we assume that in the correlated financial market there are one non-risky asset used as numeraire, and one risky asset, the price of which evolves according to
\[
dS(r) = S(r)[\alpha(r, R_r^{t,r})dr + \beta(r, R_r^{t,r})dW(r)],
\]
where \(\alpha\) is bounded and \(\beta\) satisfies \(\epsilon \leq |\beta| \leq K\) with \(0 < \epsilon < K\) being two constants. Provided the investor invests \(\pi_t\) in risky asset at time \(t\), the wealth of investor at time \(s\) conditional on the wealth \(x\) at time \(t\) and the index \(R_t = r_0\) is given by
\[
X^{\pi,t,r}(s) = x + \int_t^s \pi(u)[\alpha(u, R_u^{t,r})du + \beta(u, R_u^{t,r})dW(u)]. \tag{2.1}
\]
In the following without special explanation we simplify \(X^{\pi,t,r}(\cdot)\) by \(X^{\pi}(\cdot)\). The investment strategy \(\pi\) is called admissible if \(\mathbb{E} \int_t^T |\pi(s)\sigma(s, R_s^{t,r})|^2 ds < \infty\) and we denote the set of admissible strategies by \(\Pi^{t,r_0}\).

The above model is adapted from the one in [1], where pricing and hedging principles for derivative based on nontradable underlyings are discussed in the framework of exponential utility maximization. In this paper we will study this model from a new view of minimizing the risk of derivatives by using dynamic \(g\)-expectation mentioned in the introduction. More precisely, given \(t \in [0, T]\), \(X^{\pi}(T)\) of \([2, 1]\), and \((\rho(t, X^{\pi}(T)), Z^{\pi}(\cdot))\) satisfying
\[
\rho(t, X^{\pi}(T)) = -X^{\pi}(T) + \int_t^T g(s, x, Z(s))ds - \int_t^T Z(s)dW(s)
\]
with \(x\) being the initial wealth of \(X^{\pi}(T)\), our risk minimization problem can be posed as follows.

**Problem(RM).** Find a \(\pi \in \Pi^{t,r_0}\) such that
\[
\rho_t(X^{\pi}(T)) = \rho(t, X^{\pi}(T)) = \inf_{\pi} \rho(t, X^{\pi}(T)), \ t \in [0, T].
\]
Since the generator depends on \(x\); we are looking for optimal strategy of the form \(\pi = \pi(t; x)\). Besides the optimal strategy, we will introduce three related financial notions. Given a bounded and measurable function \(F : \mathbb{R} \to \mathbb{R}\), suppose an investor receives a derivative of form \(F(R_T^{t,r_0})\) kept in his portfolio until maturity \(T\) where the corresponding optimal strategy, denoted by \(\hat{\pi}(t; x)\) satisfies
\[
\rho_t(X^{\hat{\pi}}(T) + F(R_T^{t,r_0})) = \inf_{\pi} \rho(t; X^{\pi}(T) + F(R_T^{t,r_0})), \ t \in [0; T].
\]
Obviously the appearance of derivative \(F(R_T^{t,r_0})\) leads to a change in the optimal strategy. We then call the difference \(\Delta = \hat{\pi} - \pi\) derivative hedge, which is required to hedge the risk associated with the derivative in the portfolio.
Suppose the investor spends a payment \( q \approx q(t, x) \) for derivative \( F(R_t^{t, ro}) \), we thus give the notion of risk indifferent price.

**Definition 2.1** We call \( q \) the dynamic risk indifference price of the derivative \( F(R_t^{t, ro}) \) at time \( t \), if it is the solution of the equation

\[
\text{essinf}_\pi \rho_t(X_{x-q}^\pi(T) + F(R_t^{t, ro})) = \rho_t(X_{x-q}^\pi(T) + F(R_t^{t, ro})) = \rho_t(X_{x}^\pi(T)) = \text{essinf}_\pi \rho_t(X_{x}^\pi(T));
\]

One important notion related to risk indifference price is marginal risk price while the difference price is nonlinear in the sense that risk indifference price of \( k \cdot F(R_t^{t, ro}) \) does not equal to \( k \) (natural number) times the indifference price of \( F(R_t^{t, ro}) \). We denote marginal risk price by \( p(t, x) \). After paying \( p(t, x) \) for the derivative, the investor is indifferent between buying or not buying an infinitesimal amount of derivative. Similar ideas also appears in Section 6 in [1] where the notion of marginal utility price is introduced.

### 3 Risk minimization problem and related topics

In this section, we will study Problem (RM) and three related financial notions mentioned above. By making use of comparison theorem of BSDEs we will derive one sufficient condition for risk minimization problem, while similar ideas appeared in [8] for the special case of \( t = 0 \) and \( g \) being independent of \( x \). We denote by \( \mathcal{P}^{t,r} \) the set of process

\[
p(\cdot) := p(\cdot, R_t^{t, ro}(\cdot), x) = \pi(\cdot, x) \beta(\cdot, R_t^{t, ro}(\cdot))
\]

with \( \pi(\cdot, x) \in \Pi^{t, ro} \). For given \( t \in [0, T] \), \( r \in [t, T] \), the wealth process in (2.1) can be rewritten as

\[
X^p(r) = x + \int_t^r \theta(u, R_u^{t, ro}) p(u, R_u^{t, ro}, x) du + \int_t^r p(u, R_u^{t, ro}, x) dW(u). \tag{3.1}
\]

For reason of simplicity, we consider the special case of \( F = 0 \). Given \( t \in [0, T] \), from Section 1, the corresponding risk minimization problem is to minimize \( \rho(t; X^p(T)) \), where

\[
\rho(t; X^p(T)) = -X^p(T) + \int_t^T g(s, x, Z^{p,t}(s)) ds - \int_t^T Z^{p,t}(s) dW(s). \tag{3.2}
\]

After observing the fact of

\[
-X^p(T) = -X^p(r) - \int_r^T p(s, R_s^{t, ro}, x) \theta(s, R_s^{t, ro}) ds - \int_r^T p(s, R_s^{t, ro}, x) dW(s),
\]

\[
Y^{p,t}(r) = -X^p(r) + \int_r^T [g(s, x, Z^{p,t}(s)) - p(s, R_s^{t, ro}, x) \theta(s, R_s^{t, ro})] ds
\]

\[
- \int_r^T [Z^{p,t}(s) + p(s, R_s^{t, ro}, x)] dW(s),
\]

we arrive at

\[
\gamma^{p,t}(r) = -x + \int_r^T g(s, x, Z^{p,t}(s), p(s, R_s^{t, ro}, x)) ds - \int_r^T Z^{p,t}(s) dW(s), \quad r \in [t, T],
\]
where

\[
\Psi_{t}^{t}(r) = Y_{t}^{t}(r) + \int_{t}^{r} p(u, R_{u}^{t}, x) \theta(u, R_{u}^{t}) du,
\]

\[
+ \int_{t}^{r} p(u, R_{u}^{t}, x) dW(u),
\]

\[
\mathcal{Z}^{t}(s) = Z^{t}(s) + p(s, R_{s}^{t}, x),
\]

\[
\mathcal{V}(s, x, Z^{t}(s), p(s, R_{s}^{t}, x)) = g(s, x, Z^{t}(s) - p(s, R_{s}^{t}, x)) - p(s, R_{s}^{t}, x) \theta(s, R_{s}^{t}).
\]

\[
\text{It follows from the comparison theorem for BSDEs that}
\]

Theorem 3.1 For given \( t, s, x, z \), suppose that there is one \( \Psi = \Psi(\cdot, R_{t}^{t})(\cdot, x) \in \mathcal{P}^{t,r} \) such that

\[
\mathcal{V}(s, x, z, \Psi(s, R_{t}^{t}, x)) = \text{essinf}_{p} \mathcal{V}(s, x, z, p(s, R_{s}^{t}, x)), \ s \in [t, T], \ x \in \mathbb{R}. \tag{3.3}
\]

Moreover, \( g \) is Lipschitz in \( z \). Then

\[
\Psi_{t}^{t}(r) = \text{essinf}_{p} \Psi_{p}^{t}(r) \text{ with } r \in [t, T].
\]

In particular, \( \Psi_{t}^{t}(t) = \text{essinf}_{p} \Psi_{p}^{t}(t) \). Furthermore, if \( g \) is convex and differentiable in \( z \) such that

\[
\mathcal{V}_{p}(s, x, \Psi_{p}^{t}(s), \Psi(s, R_{s}^{t}, x)) = 0, \ s \in [t, T],
\]

then

\[
\Psi(s, x) = \Psi(s, R_{s}^{t}, x) \beta^{-1}(s, R_{s}^{t}), \ s \in [t, T], \ x \in \mathbb{R},
\]

is an optimal strategy of Problem(RM).

In Section 4 we will give three important examples to illustrate the useful application of Theorem 3.1. Before that let us look at another simple yet interesting example, from which we can see the necessity of condition (3.3) above. Here for \( s \in [t, T], x \in \mathbb{R}, \omega \in \Omega \), we assume that \( p(s, R_{s}^{t}, x) \in \Gamma \equiv \Gamma_{s}(\omega) \) with \( \Gamma_{s}(\omega) \subseteq \mathbb{R} \) being closed and convex. For given \( a \in \mathbb{R} \), we denote by \( \text{dist}_{\Gamma}(a) \) the distance between \( a \) and \( \Gamma \), and \( \Psi_{\Gamma}(a) \) the element of \( \Gamma \) such that

\[
|a - \Psi_{\Gamma}(a)| = \text{dist}_{\Gamma}(a).
\]

Example 3.1 Consider BSDE of

\[
Y^{p,t}(r) = -X^{p,t}(T) + \int_{r}^{T} \frac{\gamma(x)}{2} |Z^{p,t}(s)|^{2} ds - \int_{r}^{T} Z^{p,t}(s) dW(s), \ r \in [t, T],
\]

where \( X^{p,t}(T) \) satisfies \( (3.1) \), \( \gamma \) is a positive and decreasing function of initial wealth \( x \) representing risk aversion parameter. Note that \( Y^{p,t}(\cdot) \) is also a dynamic exponential utility function with \( \gamma(x) \), that is,

\[
Y^{p,t}(r) = \frac{1}{\gamma(x)} \ln \mathbb{E}^{F_{r}}[\exp(-\gamma(x)X^{p,t}(T))], \ r \in [t, T], \ x \in \mathbb{R}.
\]
Our problem is to find an optimal strategy $\pi$ to minimize $Y^{p,t}(t)$ with $t \in [0,T]$. In this case, we have

$$
\pi(s, x, \bar{Z}^{p,t}(s), p(s, R^t_{s-r_0}, x)) = \gamma(x) \left| p(s, R^t_{s-r_0}, x) - \bar{Z}^{p,t}(s) \right|^2 - p(s, R^t_{s-r_0}, x) \theta(s, R^t_{s-r_0})
$$

$$
= \frac{\gamma(x)}{2} \left| p(s, R^t_{s-r_0}, x) - \bar{Z}^{p,t}(s) - \frac{\theta(s, R^t_{s-r_0})^2}{\gamma(x)} \right|
$$

$$
- \bar{Z}^{p,t}(s) \theta(s, R^t_{s-r_0}) - \frac{\theta(s, R^t_{s-r_0})^2}{2 \gamma(x)}
$$

(3.4)

By using the similar ideas as in Hu et al [11] (see also [1]) we deduce that

$$
\pi(s, R^t_{s-r_0}, x) = \Psi \Gamma \left( \bar{Z}^{p,t}(s) + \frac{\theta(s, R^t_{s-r_0})}{\gamma(x)} \right) \beta^{-1}(s, R^t_{s-r_0}),
$$

(3.5)

where

$$
Y^{p,t}(r) = -x + \int_r^T \frac{\gamma(x)}{2} \text{dist}^2 \left( \bar{Z}^{p,t}(s) + \frac{\theta(s, R^t_{s-r_0})}{\gamma(x)}, \Gamma_s(\omega) \right) ds
$$

$$
- \int_r^T \left[ \bar{Z}^{p,t}(s) \theta(s, R^t_{s-r_0}) + \frac{\theta(s, R^t_{s-r_0})^2}{2 \gamma(x)} \right] ds - \int_r^T \bar{Z}^{p,t}(s) dW(s),
$$

with $r \in [t, T]$. By substituting (3.5) into (3.4), we further have for any $p \in \Gamma$,

$$
\gamma(s, x, \bar{Z}^{p,t}(s), p(s, R^t_{s-r_0}, x)) \leq \gamma(s, x, \bar{Z}^{p,t}(s), p(s, R^t_{s-r_0}, x)),
$$

which means

$$
\gamma(s, x, \bar{Z}^{p,t}(s), p(s, R^t_{s-r_0}, x)) = \text{essinf}_p \gamma(s, x, \bar{Z}^{p,t}(s), p(s, R^t_{s-r_0}, x)),
$$

is a necessary condition for $\gamma$ being optimal.

**Remark 3.1** Note that from (3.4) we can tell the influence of initial wealth on optimal strategy. For example, suppose $\alpha$ and $\beta$ are independent of $R^t_{s-r_0}$, $\alpha$ is positive, we are aim to find an optimal strategy with no-shorting. Hence (3.5) becomes

$$
\pi(s, x) = \Pi_{R^+} \left( \frac{\theta(s)}{\gamma(x)} \right) \beta^{-1}(s), \ s \in [t, T], \ x \in \mathbb{R}.
$$

Since $\gamma$ is decreasing in $x$, hence $\pi(s, x)$ is increasing in $x$, which means that the more initial wealth, the more investment for the investor.

**Remark 3.2** Here we can not treat the general case of $g$ being quadratic growth in $z$. One important reason for that lies in the limitation of existed comparison theorem for quadratic BSDEs (see for example Briand and Hu [4]) which is even not applicable on deriving optimal strategy in Example 3.1. For example, some bounded conditions of diffusion term in forward equation are indeed required when studying quadratic Markovian BSDEs, see [4].
From Theorem 3.1 and Example 3.1 we can see the importance and necessity of optimization problem for

$$\text{essinf}_p \overline{\gamma}(s, x, z, p(s, R^{t,ro}_s, x)), \ s \in [t, T], \ x, z \in \mathbb{R}. \quad (3.6)$$

Next we will study problem in (3.6) from some new perspectives. If $g$ is convex, we have

$$\text{essinf}_p \left[ g(s, x, z - p(s, R^{t,ro}_s, x)) + \theta(s, R^{t,ro}_s)(z - p(s, R^{t,ro}_s, x)) \right] = -\text{esssup}_p \left[ -\theta(s, R^{t,ro}_s)(z - p(s, R^{t,ro}_s, x)) - g(s, x, z - p(s, R^{t,ro}_s, x)) \right],$$

therefore

$$\text{essinf}_p \overline{\gamma}(s, x, z, p(s, R^{t,ro}_s, x)) = -G(s, x, \theta(s, R^{t,ro}_s))z, \quad (3.7)$$

where

$$G(s, x, \mu) = \sup_{r \in \text{Dom}(g)} (-\mu \cdot r - g(s, x, r)), \ \mu \in \mathbb{R}, \quad (3.8)$$

with $\text{Dom}(g(s, x, \cdot)) = \{r \in \mathbb{R} : g(s, x, r) < +\infty\}$ being the Legendre-Fenchel transform of $g(s, x, \cdot)$. We call $G(s, x, \mu)$ the polar function. Once we get an optimal point of (3.8), $\overline{\gamma}(s, R^{t,ro}_s, x)$, then

$$\overline{\gamma}(s, R^{t,ro}_s, x) = z - R^{t,ro}_s, \ s \in [t, T], \ x \in \mathbb{R}, \quad (3.9)$$

is the optimal solution of problem (3.6).

**Example 3.2** VaR (or AVaR) is currently one of most widely used financial risk measures. It is shown that the solution $Y(\cdot)$ of BSDE with simple generator

$$g(s, x, z) = \begin{cases} |z|, & \alpha < \frac{1}{2}, \\ -\frac{1}{\alpha} |z|, & \frac{1}{2} \leq \alpha \leq 1, \\ \end{cases}$$

is the limit of discrete AVaR under suitable conditions, see Section 7 in [23]. If we consider problem (3.8), we then have

$$G(s, x, \mu) = \sup_{r \in \mathbb{R}} \left( -\mu \cdot r - |r| I_{\alpha < \frac{1}{2}} + \frac{(\alpha - 1)}{\alpha} |r| I_{\alpha \in [\frac{1}{2}, 1]} \right), \ \mu \in \mathbb{R}.$$ 

After some calculations,

$$G(s, x, \mu) = \begin{cases} 0, & \mu \in [-1, 0], \\ +\infty, & \text{others}, \end{cases}$$

and $\overline{\gamma} = 0$ is the optimal point of problem (3.8).

**Remark 3.3** If $g(s, x, z)$ is Lipschitz in $z$, mathematically $\theta(s; R^{t,ro}_s)$ should be less than the Lipschitz constant so as to ensure $G$ is finite, see p.35 in [21]. Moreover, such property can have some more deeper and interesting financial interpretation in some special cases, see Section 4 next.

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Before studying (3.10), for any \((s, x) \in [0, T] \times \mathbb{R}\), we need the notion of sub-differential of a convex function \(f\) defined on \(\mathbb{R}\). Given \(a\), it is a set and we denote it by \(\partial f(s, x, a)\); where
\[
\partial f(s, x, a) = \{ \theta; f(s, x, b) - f(s, x, a) \geq -\theta(b - a), \forall b \in \text{Dom}(f(s, x, \cdot)) \},
\]
and the elements are called sub-gradient of \(f\) at \(a\). By Section E in [7] we can derive the following result.

**Theorem 3.2** If \(g\) is a continuous and convex function in \(r\), and \(G(s, x, \mu)\) is its polar function, then \(\hat{r} \in \partial G(s, x, \mu)\) if and only if \(\hat{r}\) is optimal for the following problem, that is,
\[
-G(s, x, \mu) = \inf_{r} g(s, x, r) + \mu \cdot r = g(s, x, \hat{r}) + \mu \cdot \hat{r}.
\]

Given \(\theta(s, R_s^t, r)\), if \(\partial G(s, x, \theta(s, R_s^t, r))\) is non-empty, then the optimal solution exists. According to Section E in [7], if \(\mu\) is in the interior of \(\text{Dom}(G(s, x, \cdot))\), then \(\partial G(s, x, \mu)\) is non-empty. In particular, if \(G\) is finite with \(\mu \in \mathbb{R}\), then \(\partial G(s, x, \mu)\) is nonempty for any \(\mu\). Note that in Example 3.2, if \(\mu \in (-1, 0)\), we can find at least one point \(0 \in \partial G(s, x, \mu)\). Next we will give another example when \(G\) is finite.

**Example 3.3** Given \(s\) and \(x\), \(g(s, x, r) = r^2\) with \(r \in \mathbb{R}\); then
\[
G(s, x, \mu) = \sup_{r \in \mathbb{R}} (-\mu r - r^2), \text{ with } \mu \in \mathbb{R}.
\]

After some calculations, \(G(s, x, \mu) = \frac{\mu^2}{4}\) and \(\tau = -\frac{\mu}{2}\) is the optimal point. In this case, \(G(s, x, \mu)\) is differential, hence \(\partial G(s, x, \mu)\) is nonempty for any \(\mu \in \mathbb{R}\).

Next we will make use of (3.7) above to derive the explicit form of risk indifferent price, derivative hedge and marginal risk price. Before it, we need a furthermore assumption.

**\((H1)\)** \(F\) is bounded and differentiable with bounded derivative, \(\frac{\partial \alpha(s, r)}{\partial r}\) and \(\frac{\partial \beta(s, r)}{\partial r}\) are bounded.

Given \(t \in [0, T]\), the risk indifference price \(q, \tilde{p}(\cdot, R^{t,r_0}(\cdot), x) = \tilde{\pi}(\cdot, x)\beta(\cdot, R^{t,r_0}(\cdot)), \tilde{\pi}\) and \(\pi\) being defined respectively in Section 2, we derive that \(X^\tilde{\pi}_{x-q}(T)\) is terminal wealth with initial wealth being \(x - q\), therefore, by (3.1) and (3.2),
\[
\rho_t(X^\tilde{\pi}_{x-q}(T) + F(R_t^{t,r_0})) = \rho_t(X^\pi_{x-q}(T) + F(R_t^{t,r_0}))
\]
\[
= -X^\tilde{p}_{x-q}(T) - F(R_t^{t,r_0}) + \int_t^T g(s, x, Z^{\tilde{p},t}(s))ds - \int_t^T Z^{\tilde{p},t}(s)dW(s).
\]
Using similar ideas as above, from (3.1) we have
\[
\rho_t(X^\tilde{\pi}_{x-q}(T) + F(R_t^{t,r_0}))
\]
\[
= -x + q - F(R_T^{t,r_0}) - \int_t^T G(s, x, \theta(s, R_s^{t,r_0}))ds
\]
\[
- \int_t^T Z^{\tilde{\pi},t}(s)\theta(s, R_s^{t,r_0})ds - \int_t^T Z^{\tilde{\pi},t}(s)dW(s).
\]
Similarly
\[
\rho_t(X_z^T(T)) = -x - \int_t^T G(s, x, \theta(s, R_s^{t, r_0}))ds \\
- \int_t^T \Delta \pi_t(s, \theta(s, R_s^{t, r_0}))ds - \int_t^T \Delta \pi_t(s)dW(s).
\]

As a result, the risk indifferent price \( q \) of \( q(t, x) \) can be expressed as
\[
q_t = \mathbb{E}^F_t F(R_t^{t, r_0}) = \frac{\mathbb{E}^P_t A(T) F(R_t^{t, r_0})}{A(t)}, \tag{3.10}
\]
where \( A(\cdot) \) is defined in \([t, T]\),
\[
\frac{dQ}{dP}\bigg|_{F_t} = A(s) = e^{-\int_t^s \theta(v, R_v^{v, r_0})dv - \frac{1}{2} \int_t^s \theta^2(v, R_v^{v, r_0})dv}, \quad s \in [t, T]. \tag{3.11}
\]

Note that the indifference price \( q \) is linear in \( F(R_t^{t, r_0}) \) and independent of \( x \), therefore the marginal risk price of derivative is still \( q_t \) itself, i.e., \( p(t, x) = q(t, x) \). As to the derivative hedge, it follows from (3.9) that
\[
\Delta(s, R_s^{t, r}) = \frac{[\hat{\rho}(s, R_s^{t, r_0}, x) - \rho(s, R_s^{t, r_0}, x)]}{\beta(s, R_s^{t, r_0})} = \frac{[\hat{\pi}_t(s) - \pi_t(s)]}{\beta(s, R_s^{t, r_0})},
\]
with \( s \in [t, T] \). If we denote
\[
\Delta \hat{Y}^t(r) = \hat{Y}^{t, \hat{t}}(r) - \hat{Y}^{t, t}(r), \quad \Delta \hat{Z}^t(r) = \hat{Z}^{t, \hat{t}}(r) - \hat{Z}^{t, t}(r), \quad r \in [t, T],
\]
where for example
\[
\hat{Y}^{t, \hat{t}}(r) = -x - F(R_t^{t, r_0}) - \int_t^T G(s, x, \theta(s, R_s^{t, r_0}))ds \\
- \int_t^T \Delta \hat{\pi}_t(s, \theta(s, R_s^{t, r_0}))ds - \int_t^T \Delta \hat{\pi}_t(s)dW(s),
\]
then
\[
\Delta \hat{Y}^t(r) = -F(R_t^{t, r_0}) - \int_t^r \Delta \hat{\pi}_t(s, \theta(s, R_s^{t, r_0}))ds - \int_r^T \Delta \hat{\pi}_t(s)dW(s) \\
= -F(R_t^{t, r_0}) - \int_t^r \Delta \hat{\pi}_t(s)d\hat{W}(s),
\]
where
\[
\hat{W}(s) = \int_s^r \theta(v, R_v^{v, r_0})dv + W(s), \quad s \in [t, T].
\]

By using (H1), we can verify the results of (see the appendix for detailed proof),
\[
\mathbb{E}_Q|F(R_t^{t, r_0})| < \infty, \quad \mathbb{E}_Q \left[ \int_t^T |D_s F(R_s^{t, r_0})|^2 ds \right]^{\frac{1}{2}} < \infty,
\]
\[
\mathbb{E}_Q \left[ |F(R_t^{t, r_0})| \cdot \left( \int_t^T h(r) dr \right)^{\frac{1}{2}} \right] < \infty. \tag{3.12}
\]
with $D_s M$ being the Malliavin derivative of random variable $M$, and

$$h(r) = \left( \int_t^T D_r \theta(s, R_s^{t, r_0}) dW(s) + \int_t^T D_s \theta(s, R_s^{t, r_0}) \cdot \theta(s, R_s^{t, r_0}) ds \right)^2,$$

therefore we have

$$\Delta Z^I(r) = -\mathbb{E}_Q^F \left[ D_r F(R_t^{t, r_0}) - F(R_t^{t, r_0}) \int_r^T D_s \theta(s, R_s^{t, r_0}) d\tilde{W}(s) \right], \quad (3.13)$$

therefore the derivative hedge $\Delta(s, R_s^{t, r_0})$ with $s \in [t, T]$ can be given by

$$\Delta(s, R_s^{t, r_0}) = \frac{-\mathbb{E}_F^s \left[ A(T) F_s(R_t^{t, r_0}) \cdot D_s R_t^{t, r_0} \right]}{\beta(s, R_s^{t, r_0}) A(s)} + \frac{\mathbb{E}_F^s \left[ A(T) F(R_s^{t, r_0}) \int_s^T \theta(u, R_u^{t, r_0}) D_s R_u^{t, r_0} d\tilde{W}(u) \right]}{\beta(s, R_s^{t, r_0}) A(s)}.$$

To sum up, we have

**Theorem 3.3** Suppose (H1) hold, $g$ is Lipschitz in $z$, then the indifference price $q(t, x)$ and the derivative hedge $\Delta(s, R_s^{t, r_0})$ can be expressed in (3.10) and (3.14) respectively. In addition, $p(t, x) = u(t, x)$.

**Remark 3.4** Mathematically the expression of $p, q$ and $\Delta$ just depends on some parameters of financial market. Some other subjective parameter represented by $g$, such as risk aversion parameter, will not affect them. Since all the variable are supposed to be one dimensional, $\beta$ is invertible, and admissible strategy is non-constrained, one important reason for this phenomenon, we believe, is the completeness of financial market. Otherwise, for example, if the admissible strategy is restricted to take value in subset of $\mathbb{R}$, then $q(t, x)$ will depend on $g$ and some delicate analysis will be needed to get the explicit expression, see [1].

**Remark 3.5** As we know, in the real world there are lots of deals that do not go through the exchange trading and people called them over-the-counter (OTC for short). Hence how to find a fair and suitable price for the agreements between the sellers and buyers is faced by certain investment institutions who are offering the OTC deals. Of course, from the view of minimizing risk for financial position, risk indifference price seems to be a reasonable choice. Generally speaking, this price should be nonlinear on the amount of derivatives (see [1]) which prompt us to introduce marginal risk price to characterize the price for each one derivative. However, here we are lucky to get the linear form, which is to say, the price of $k \times F(R_T^{t, r_0})$ is just $k$ times the price of $F(R_T^{t, r_0})$. This seems a little too ideal to have ”discount” in this procedure. We believe that the requirement of complete financial market is also one of main reasons for that, and we hope to show more results of incomplete financial market case in future.

4 Some special risk minimization problems with explicit solutions

In this section, we will give several special cases to illustrate the well application of Theorem 3.1 above, where there are two basic financial notions, market price of risk (see for example
and risk aversion parameter involved. The former one is objective and determined by the financial market while the latter one is subjective and mainly depends on the investor. We will consider their relations with the optimal strategy of our risk minimization problem above.

### 4.1 Case 1

In this subsection, we consider one risk minimization problem by supposing

$$g(s; z, x) = k(s; R^{t, r_0}_s; x) \left( \sqrt{1 + |z|^2} - 1 \right), \quad s \in [t, T], \ x, z \in \mathbb{R},$$

where $k(s; r; x)$: $[0; T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is positive, decreasing in $x$; and $k(s; r; x) > |\theta(s; r)|$ with $s \in [0; T]$ and $r \in \mathbb{R}$; $x \in \mathbb{R}$. Suppose that

We claim that $k$ above can be regarded as a subjective parameter representing investors’ attitude towards risk, that is, the larger $k$ becomes, the more risk averse the investor is. Actually, suppose that two investors $A$ and $B$ are facing one contingent claim $\xi$, and they will measure its risk in two different way, such as two BSDEs with generator $g_1$ and $g_2$ respectively,

$$g_i(s; z, x) = k_i(s; R^{t, r_0}_s; x) \left( \sqrt{1 + |z|^2} - 1 \right), \quad s \in [t, T], \ x, z \in \mathbb{R}, \ i = 1, 2.$$

In addition, $k_1 \leq k_2$. It follows from the comparison theorem for BSDEs that $\rho^A(\xi) \leq \rho^B(\xi)$ which means that the risk considered by investor $A$ is less than that by $B$. In other words, investor $A$ is less sensitive to risk than $B$, or is more risk tolerance (less risk averse) than $B$ when facing the same uncertainty. This can be identified as one important role of $k$. Next we will derive the explicit expression of optimal strategy according to Theorem 3.1. First we arrive at,

$$\bar{g}(s, x; \mathcal{Z}^{p,t}_s(s), p(s, R^{t, r_0}_s, x)) = k(s, R^{t, r_0}_s, x) \left( \sqrt{1 + \left| \mathcal{Z}^{p,t}_s(s) - p(s, R^{t, r_0}_s, x) \right|^2} - 1 \right) - p(s, R^{t, r_0}_s, x)\theta(s, R^{t, r_0}_s),$$

and

$$\frac{\partial \bar{g}(s, x; \mathcal{Z}^{p,t}_s(s), p(s, R^{t, r_0}_s, x))}{\partial p} = k(s, R^{t, r_0}_s, x) \frac{p(s, R^{t, r_0}_s, x) - \mathcal{Z}^{p,t}_s(s)}{\sqrt{1 + \left| p(s, R^{t, r_0}_s, x) - \mathcal{Z}^{p,t}_s(s) \right|^2}} - \theta(s, R^{t, r_0}_s).$$

Therefore, if $-1 < \frac{\theta(s, R^{t, r_0}_s)}{k(s, R^{t, r_0}_s, x)} < 0$, by Theorem 3.1, there exists a unique optimal strategy $\mathcal{P}(s, R^{t, r_0}_s, x)$, together with $\mathcal{Z}^{p,t}_s(s)$ such that $\frac{\partial \bar{g}(s, x; \mathcal{Z}^{p,t}_s(s), \mathcal{P}(s, R^{t, r_0}_s, x))}{\partial p} = 0$, where

$$\mathcal{P}(s, R^{t, r_0}_s, x) = \frac{\theta(s, R^{t, r_0}_s)}{k^2(s, R^{t, r_0}_s, x) - \theta^2(s, R^{t, r_0}_s)} + \mathcal{Z}^{p,t}_s(s), \quad (4.1)$$

and

$$\mathcal{Z}^{p,t}_s(r) = -x + \int_r^T \left[ \frac{k^2(s, R^{t, r_0}_s, x) - \theta^2(s, R^{t, r_0}_s) - k(s, R^{t, r_0}_s, x)}{k^2(s, R^{t, r_0}_s, x) - \theta^2(s, R^{t, r_0}_s)} \right] ds - \int_r^T \mathcal{Z}^{p,t}_s(s) dW(s),$$

(4.2)
is the so-called minimal risk value at time $r \geq t$. By using Girsanov theorem, we can rewrite (4.2) into

$$Y^{p,t}(r) = -x + \mathbb{E}^T_r \left[ \int_r^T \left( \sqrt{k^2(s, R^t_{s, x})} - \theta^2(s, R^t_{s, x}) - k(s, R^t_{s, x}) \right) ds \right],$$

where $A(\cdot)$ is defined in (3.11) and $dQ = A(T) dP$, $s \in [t, T]$.

If $0 < \frac{\theta(s, R^t_{s, x})}{k(s, R^t_{s, x})} < 1$, we can also prove that the above $p$ is optimal. Particularly, when $\theta$ and $k$ are deterministic, i.e., they are independent of $R^t_{s, x}$, then

$$Z^{p,t}(s) = 0, \quad p(s, x) = \frac{\theta(s)}{\sqrt{k^2(s, x) - \theta^2(s)}} \text{ with } s \in [t, T],$$

hence the optimal strategy

$$\pi(s, x) = \frac{\theta(s)}{\sqrt{k^2(s, x) - \theta^2(s)}} \beta^{-1}(s), \quad (4.3)$$

and minimal risk value at time $t$ is described by

$$Y^{p}(t) = -x + \int_t^T [\sqrt{k^2(s, x) - \theta^2(s)} - k(s, x)] ds.$$

If $\beta(\cdot)$ is positive, then from (4.3) the optimal strategy $\pi$ is increasing in market price of risk. Similar point also appeared by Pirvu and Zhang in [20] where optimal investment, consumption and life insurance acquisition for wage earners with constant relative risk aversion (CRRA) preference is carried out. On the other hand, (4.3) also indicates that $\pi$ is decreasing in $k$. By combining this to the role of $k$ claimed above, it is easy to say, the larger $k$ becomes, the more risk averse the investor is, and the less investment in the risky asset, which is consistent with results in [25].

**Remark 4.1** Let us give two points on parameter $k$ here. Inspired by arguments in [12] and [16], we suppose that risk aversion parameter $k$ is dependent on $x$. Actually, if $k$ is independent of $x$, from (4.7) and (4.2) $p$ is also independent of $x$. This means that optimal strategy is independent of the initial wealth, which seems unreasonable according to the arguments in Section 3 of [15]. On the other hand, (4.3) also indicates that $\pi$ is decreasing in $k$. By combining this to the role of $k$ claimed above, it is easy to say, the larger $k$ becomes, the more risk averse the investor is, and the less investment in the risky asset, which is consistent with results in [25].

**Remark 4.2** As we can see from (4.3), $\pi$ is increasing (decreasing) in $\theta$ when $\beta$ is positive (negative). However, in order to make the risk be minimized, there should be a threshold for $\theta$,
which for example means people can not keep on investing more all the time when $\theta$ increases and $\beta$ is positive. This indicates that for a given investor with subjective parameter $k$, his/her investment strategy $\theta$ is controlled by $k$. From the view of mathematics, the case of $|\frac{\theta(s, R_{s}^{t, r_0})}{k(s, R_{s}^{t, r_0})}| > 1$ will cause the illposedness of optimal strategy. On the other hand, given two investors $A$ (with $k_1$) and $B$ (with $k_2$) and $k_1 \leq k_2$, $|\theta(s, R_{s}^{t, r_0})| \leq k_1$ implies that $|\theta(s, R_{s}^{t, r_0})| \leq k_2$, which means investor $B$ has larger boundary to keep market price of risk. It also indicates that people who are more risk aversion have larger probability to find the optimal strategy to minimize the risk. Such relations among subjective parameter (like $k$), objective parameter (like $\theta$), and wellposedness of risk minimization problem are new to our best knowledge.

**Remark 4.3** Here the parameter $k$ is assumed to keep unchanged as the time goes by. However, in real world, people’s attitude to risk and uncertainty is always different at different time period. For example, as is shown in [20], when the wage earner becomes older he/she has a higher demand for life insurance/pension annuities due to the higher hazard rate. That is to say, a more general but applicable case for $k$ should be $k(t; s; r; x)$. In this case, the risk is described by

\[
\begin{align*}
\bar{p}^{\pi,t}(t, r) &= -x + \int_r^T k(t, s, R_{s}^{t, r_0}, x) \left( \sqrt{1 + |\bar{Z}^{\pi,t}(t, s) - p(s, R_{s}^{t, r_0}, x)|^2} - 1 \right) ds \\
&\quad - \int_r^T \left[ p(s, R_{s}^{t, r_0}, x) \theta(s, R_{s}^{t, r_0}) - \bar{Z}^{\pi,t}(t, s) \theta(s, R_{s}^{t, r_0}) \right] ds - \int_r^T \bar{Z}^{\pi,t}(t, s) dW(s),
\end{align*}
\]

with $r \in [t, T]$, and $\bar{p}$, depending on $t$, should be

\[
\bar{p}(t, s, R_{s}^{t, r_0}, x) = \frac{\theta(s, R_{s}^{t, r_0})}{\sqrt{k^2(t, s, R_{s}^{t, r_0}, x) - \theta^2(s, R_{s}^{t, r_0})}} + \bar{Z}^{\pi,t}(t, s), \tag{4.4}
\]

with $s \in [t, T]$, where $\bar{Z}^{\pi,t}(t, s)$ satisfies

\[
\begin{align*}
\bar{Z}^{\pi,t}(t, r) &= -x + \int_r^T \left( \sqrt{k^2(t, s, R_{s}^{t, r_0}, x) - \theta^2(s, R_{s}^{t, r_0})} - k(t, s, R_{s}^{t, r_0}, x) \right) ds \\
&\quad - \int_r^T \bar{Z}^{\pi,t}(t, s) \theta(s, R_{s}^{t, r_0}) ds - \int_r^T \bar{Z}^{\pi,t}(t, s) dW(s).
\end{align*}
\]

If $r = t$, $(\varphi^{\pi,t}(\cdot), \bar{Z}^{\pi,t}(\cdot, \cdot))$ satisfies a linear backward stochastic Volterra integral equation (BSVIE for short) studied for example in [20], with $\varphi^{\pi,t}(t) = \bar{Y}^{\pi,t}(t, t)$. From (4.4) $\pi$ is not time consistent strategy due to dependence on time $t$. At the moment, it is not clear to us how one can study the time consistent solution in such case.

### 4.2 Case 2

In this subsection, we consider one special case of Problem(RM) by supposing

\[
g(s, x, z) = l(s; R_{s}^{t, r_0}, x) \cdot \ln \frac{1}{2}(1 + e^{-z}), \ s \in [t, T], \ x, z \in \mathbb{R},
\]
where \(l(s; r; x) : [0; T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) is bounded and positive. By the similar analysis as above, \(l\) is also a subjective parameter for investors, that is, the larger \(l\) becomes, the more risk averse the investor is. In order to derive explicit strategy in such setting, firstly we have

\[
\tilde{y}(s, x, \tilde{Z}^{p,t}(s), p(s, R^{t,ro}_s, x)) = l(s, R^{t,ro}_s, x) \ln \frac{1 + \exp \left\{ p(s, R^{t,ro}_s) - \tilde{Z}^{p,t}(s) \right\}}{2} - p(s, R^{t,ro}_s, x) \theta(s, R^{t,ro}_s),
\]

and

\[
\frac{\partial \tilde{y}(s, x, \tilde{Z}^{p,t}(s), p(s, R^{t,ro}_s, x))}{\partial p} = l(s, R^{t,ro}_s, x) \frac{\exp \left\{ p(s, R^{t,ro}_s, x) - \tilde{Z}^{p,t}(s) \right\}}{1 + \exp \left\{ p(s, R^{t,ro}_s, x) - \tilde{Z}^{p,t}(s) \right\}} - \theta(s, R^{t,ro}_s).
\]

If \(\frac{\theta(s, R^{t,ro}_s)}{l(s, R^{t,ro}_s, x)} < 0\), i.e., \(\theta(s, R^{t,ro}_s)\) is negative, then \(\frac{\partial \tilde{y}(s, x, \tilde{Z}^{p,t}(s), p(s, R^{t,ro}_s, x))}{\partial p} > 0\), therefore the optimal strategy does not exist. If \(\frac{\theta(s, R^{t,ro}_s)}{l(s, R^{t,ro}_s, x)} \geq 1\), then \(\frac{\partial \tilde{y}(s, x, \tilde{Z}^{p,t}(s), p(s, R^{t,ro}_s, x))}{\partial p} < 0\), hence the optimal strategy also does not exist.

As to the case of \(0 \leq \frac{\theta(s, R^{t,ro}_s)}{l(s, R^{t,ro}_s, x)} \leq 1\), by Theorem 3.1 there exists only one \(\tilde{p}(s, R^{t,ro}_s, x)\), such that \(\frac{\partial \tilde{y}(s, x, \tilde{Z}^{p,t}(s), p(s, R^{t,ro}_s, x))}{\partial p} = 0\), where

\[
\tilde{p}(s, R^{t,ro}_s, x) = \ln \left[ \frac{\theta(s, R^{t,ro}_s)}{l(s, R^{t,ro}_s, x) - \theta(s, R^{t,ro}_s)} \right] + \tilde{Z}^{p,t}(s),
\]

and \(\tilde{Z}^{p,t}(s)\) satisfies

\[
\tilde{Z}^{p,t}(s) = -x + \int_t^T l(s, R^{t,ro}_s, x) \ln \frac{l(s, R^{t,ro}_s, x)}{2[l(s, R^{t,ro}_s, x) - \theta(s, R^{t,ro}_s)]} ds
\]

\[- \int_t^T \theta(s, R^{t,ro}_s) \ln \frac{\theta(s, R^{t,ro}_s)}{l(s, R^{t,ro}_s, x) + \theta(s, R^{t,ro}_s)} ds
\]

\[- \int_t^T \tilde{Z}^{p,t}(s) \theta(s, R^{t,ro}_s) ds - \int_t^T \tilde{Z}^{p,t}(s) dW(s).
\]

Note that when \(\theta\) and \(l\) are deterministic,

\[
\tilde{Z}^{p,t}(s) = 0, \quad \tilde{p}(s, x) = \ln \left[ \frac{\theta(s)}{l(s, x) - \theta(s)} \right] \text{ with } s \in [t, T],
\]

hence the optimal strategy

\[
\tilde{p}(s) = \ln \left[ \frac{\theta(s)}{l(s, x) - \theta(s)} \right] \beta^{-1}(s), \quad (4.5)
\]

and minimal risk value at time \(t\) is described by

\[
\tilde{Y}^{p}(t) = -x + \int_t^T \left( l(s, x) \ln \frac{l(s, x)}{2[l(s, x) - \theta(s)]} - \theta(s) \ln \frac{\theta(s)}{l(s, x) - \theta(s)} \right) ds.
\]
Suppose $\alpha$ is non-negative, $\theta$ is positive, then $\beta$ is also positive. Comparing with Case 1, (4.5) also indicates three points which are consistent with the one in [3], [20] and [25]. Firstly the optimal strategy $\pi$ is increasing in market price of risk $\theta$. Secondly, $\pi$ is decreasing in $l$, i.e., the more risk averse investor is, the less investment he/she will spend in risky assets. Thirdly, since $l$ is decreasing in initial wealth $x$, by combining the second point above, $\pi$ is increasing in $x$ which reflects the relations between optimal strategy and initial wealth.

**Remark 4.4** Speaking of the comparison of results between Case 1 and here, on the one hand, similar ideas in the remarks there are also applicable here. On the other hand, in order to ensure the existence of optimal strategy, the investor with such risk preference has more requirements (or restriction) of financial market. For example, market price of risk here should be assumed to be non-negative, otherwise the more investment in the risky asset, the more risk he/she will take, which means there is no best choice for him/her. Since it is reasonable to have $\alpha > 0$, then positive $\pi$ implies positive volatility rate of risky asset $\beta$ which, in the literature, is not an unacceptable condition on financial market. Also negative $\beta$ in some sense reflects certain negative (or unexpected) influence of random factors on the price of risky asset, which reasonably implies the increasing risk for investor.

### 4.3 Case 3

In this case, we will consider one special case of Problem(RM) by supposing generator to be

$$g(x, z) = \begin{cases} 
|z| - \frac{1}{2\gamma(x)}, & |z| \geq \frac{1}{\gamma(x)}, \\
\frac{1}{2\gamma(x)}|z|^2, & |z| < \frac{1}{\gamma(x)},
\end{cases} \quad (4.6)$$

with $\gamma(x)$ being a decreasing positive function. Note that $h(x, z) = \gamma(x)g(x, z)$ is the Huber penalty function (see [9]) which plays an important role in robust statistics. As proved in [22] and [23], such BSDE is regarded as the continuous time analogue of the discrete Gini principle of form

$$V^\gamma(t_i, \xi) = \operatorname{essinf}_Q \left( \mathbb{E}_Q^{t_i} \xi + \frac{1}{\gamma(x)} C_{t_i}(Q|P) \right), \quad t_i \in [0, T],$$

with $C_{t_i}(Q|P)$ being the Gini index (we refer the reader to see the meaning of notations there). From (4.6), $g$ is increasing in $\gamma$, thus $\gamma$, like $k$ and $l$ before, can also represent investor’s attitude to risk. On other hand, according to Theorem 24 in [14], under certain conditions imposed on $\xi$, Gini principle of $V^\gamma(0, \xi)$ agrees with mean-variance functional of the form

$$J(0, \xi) = \mathbb{E}\xi - \frac{1}{2\gamma(x)} \operatorname{Var}(\xi), \quad (4.7)$$

where $\gamma$ in the mean-variance functional (4.7) is a risk aversion parameter, see for example [5]. Therefore, roughly speaking, it follows from above two points that $\gamma$ in (4.6) is the risk aversion parameter in our setting.

Next we will derive the expression of optimal strategy by means of Theorem 3.1. When
\[ |\theta(s, R_s)| \leq 1; \text{ we have} \]

\[
\bar{\gamma}(x, z, p) = \begin{cases} 
  z - p - p\theta - \frac{1}{2\gamma(x)}, & z - p \geq \frac{1}{\gamma(x)}, \\
  p - z - p\theta - \frac{1}{2\gamma(x)}, & p - z \geq \frac{1}{\gamma(x)}, \\
  \frac{\gamma(x)}{2} |p - z|^2 - p\theta, & -\frac{1}{\gamma(x)} \leq p - z \leq \frac{1}{\gamma(x)} \\
  z - p, & z - p \geq 1 \\
  p - z, & p - z \geq 1 \\
  -\frac{1}{\gamma(x)} \leq p - z \leq \frac{1}{\gamma(x)} \\
  -\theta + \frac{1}{2\gamma(x)} + \frac{\theta}{\gamma(x)} \gamma(x), & z - p \geq \frac{1}{\gamma(x)} \\
  -\theta + \frac{1}{2\gamma(x)} - \frac{\theta}{\gamma(x)} \gamma(x), & p - z \geq \frac{1}{\gamma(x)} \\
  -\theta - \frac{\theta^2}{2\gamma(x)}, & -\frac{1}{\gamma(x)} \leq p - z \leq \frac{1}{\gamma(x)}. 
\end{cases}
\]

After some basic calculations, by using Theorem 3.1, we arrive at

\[
\bar{\pi}(s, R^{r,0}_s, x) = \bar{Z}^{\pi,t}(s) + \frac{\theta(s, R^{r,0}_s)}{\gamma(x)}, \tag{4.8}
\]

where

\[
\bar{Y}^{\pi}(r) = -x - \int_r^T \left( \bar{Z}^{\pi,t}(s)\theta(s, R^{r,0}_s) + \frac{\theta^2(s, R^{r,0}_s)}{2\gamma(x)} \right) ds - \int_t^T \bar{Z}^{\pi,t}(s)dW(s),
\]

with \( r \in [t, T] \). As to the case of \( |\theta(s, R^{r,0}_s)| > 1 \), we have \( \inf_p \bar{\gamma}(x, z, p) = -\infty \), which means optimal strategy does not exist.

**Remark 4.5** Since \( \gamma \) in (4.6) can be seen as risk aversion parameter, it is natural to be decreasing in initial wealth from a obvious intuition. To study the relation between initial wealth and optimal strategy, we look at a special case, i.e., \( \theta \) is independent of \( R^{r,0}_s \). By (4.8), we have

\[
\pi(s; x) = \frac{\alpha(s)}{\beta^2(s)\gamma(x)}, \quad s \in [t, T], \ x \in \mathbb{R}. \tag{4.9}
\]

By reasonably assuming \( \alpha \) is non-negative, we deduce that \( \pi(s; x) \) is increasing in initial wealth since \( \gamma \) is decreasing in \( x \). This is consistent with the results in Example 3.1, the above two cases and the utility maximization problem with \( U \) exhibiting decreasing absolute risk aversion in [3].

**Remark 4.6** By looking at (4.10), and supposing \( \alpha \) to be non-negative, \( \pi \) is decreasing in risk aversion parameter \( \gamma \), which is a comparative result among investors consistent with the one in Section 4 of [25].

**Remark 4.7** By assuming \( \beta \) to be positive, from (4.10), optimal strategy \( \pi \) is also increasing in market price of risk \( \theta \) which is consistent with Case 1 and 2 above. The difference between there and here is the boundary of \( \theta \) in such setting is a constant independent of certain subjective parameter.

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Appendix

In this section, we will verify the conditions in (3.12) in order to use the generalized Clark representation formula in [18]. For readers’ convenience, we give several basic notations required in the sequel. For given $t \in [0, T]$, $S[t, T]$ is the class of smooth random variable $F$ of

$$ F = f \left( \int_t^T h_1(s) dW(s), \ldots, \int_t^T h_n(s) dW(s) \right), \quad f \in C_\infty(R^n), \quad h_i \in L^2[t, T], $$

with $n \geq 1$, $L^2[t, T]$ being the set of deterministic square integral functions, $D_r H$ being the Malliavin derivative of $H$ at time $r$. We denote by $D_1, 2[t, T]$ the closure of the class of smooth random variables $S[t, T]$ with respect to the norm

$$ \|H\|_{1, 2} = \left( E|H|^2 + E \int_t^T |D_r H|^2 dr \right)^{\frac{1}{2}}. $$

We denote by $L^{1, 2}[t, T]$ the class of square integral, $\mathbb{F}$-adapted process $x$ such that $x(r) \in D^{1, 2}$ for almost $r \in [t, T]$, and

$$ E \int_t^T \int_t^T |D_s x(t)|^2 dt ds < \infty, $$

with $D_s x$ being a measurable version. For given $r, s \in [t, T]$, since $\frac{\partial h_i(s, r)}{\partial r}, \frac{\partial \sigma_i(s, r)}{\partial r}$ exists and are continuous, by Theorem 2.2.1 in [24], we have $D_r R^{t, r_0}_s \in D^{1, 2}$,

$$ D_r R^{t, r_0}_s = \sigma(r, R^{t, r_0}_r) + \int_r^s b_r(u, R^{t, r_0}_u) D_r R^{t, r_0}_u du + \int_r^s \sigma_r(u, R^{t, r_0}_u) D_r R^{t, r_0}_u dW(u), $$

for $r \leq s$, a.e., $D_r R^{t, r_0}_s = 0$ for $r > s$, and

$$ \sup_{r \in [t, T]} E \left( \sup_{s \in [r, T]} |D_r R^{t, r_0}_s(s)|^k \right) < \infty, \quad \text{with} \quad k \geq 2. \quad (4.10) $$
For \( r \leq s, s \in [t, T] \), we can express \( D_s R_t^{r, \rho_0} \) by
\[
D_s R_t^{r, \rho_0} = \sigma(r, R_t^{r, \rho_0}) e^t \left[ b_s(u, R_t^{r, \rho_0})du + \sigma_s(u, R_t^{r, \rho_0})dW(u) - \frac{1}{2} \sigma^2_s(u, R_t^{r, \rho_0})du \right].
\]
Since \( \frac{\partial \alpha(t, r)}{\partial r} \) and \( \frac{\partial \beta(t, r)}{\partial r} \) are bounded, then
\[
\frac{\partial \theta(t, r)}{\partial r} = \frac{1}{\beta(t, r)} \frac{\partial \alpha(t, r)}{\partial r} - \frac{\alpha(t, r)}{\beta^2(t, r)} \frac{\partial \beta(t, r)}{\partial r}
\]
is bounded, thus by combining (11), we have
\[
\mathbb{E} \int_t^T \int_t^T |D_s \theta(s, R_t^{r, \rho_0})|^2 dr ds < \infty,
\]
and we get that \( \theta(s, \cdot) \in L^{1,2}[t, T] \). Since \( F \) has bounded derivative, then we get \( F(R_t^{r, \rho_0}) \in L^{1,2}[t, T] \). Next we will verify the condition in (3.12). Obviously the first inequality is easy to see. As to the second one, we have
\[
\mathbb{E} Q \left[ \int_t^T |D_s F(R_t^{r, \rho_0})|^2 ds \right]^\frac{1}{2} = \mathbb{E} A(T) \left[ \int_t^T |D_s F(R_t^{r, \rho_0})|^2 ds \right]^{\frac{1}{2}} \leq C \left( \mathbb{E} A^2(T) \right)^{\frac{1}{2}} \left( \mathbb{E} \int_t^T |D_s F(R_t^{r, \rho_0})|^2 ds \right)^{\frac{1}{2}}.
\]
Since \( \theta \) is bounded, thus \( \mathbb{E} A^2(T) < \infty \), and therefore
\[
\mathbb{E} \int_t^T |D_s F(R_t)|^2 ds = \mathbb{E} \int_t^T |F_s(R_t)|^2 |D_s R_t|^2 ds < \infty,
\]
due to the fact that \( F \) has bounded derivative. As to the last one, we have
\[
\mathbb{E} A(T) \left[ \int_t^T \left( \int_t^T \left| D_u \theta(s, R_s^{r, \rho_0}) dW(s) \right|^2 du \right) \right]^{\frac{1}{2}} \leq C \left( \mathbb{E} A^2(T) \right)^{\frac{1}{2}} \left[ \mathbb{E} \int_t^T \left( \int_t^T |D_u \theta(s, R_s^{r, \rho_0})|^2 ds \right) du \right]^{\frac{1}{2}} \leq C \left[ \mathbb{E} \int_t^T \left( \int_t^T \left| \frac{\partial \theta(s, R_s^{r, \rho_0})}{\partial r} \right|^2 |D_u R_s^{r, \rho_0}|^2 ds \right) du \right]^{\frac{1}{2}} < \infty,
\]
and
\[
\mathbb{E} A(T) \left[ \int_t^T \left( \int_t^T \left| D_u \theta(s, R_s^{r, \rho_0}) \cdot \theta(s, R_s^{r, \rho_0}) ds \right|^2 du \right) \right]^{\frac{1}{2}} \leq C \left[ \mathbb{E} \int_t^T \int_t^T \left| \frac{\partial \theta(s, R_s^{r, \rho_0})}{\partial r} \right|^2 \left| \theta(s, R_s^{r, \rho_0}) \right|^2 |D_u R_s^{r, \rho_0}|^2 ds du \right]^{\frac{1}{2}} < \infty,
\]
then the third inequality holds.