REALIZABILITY OF UNIVALENCE
MODEST KAN COMPLEXES

WOUTER PIETER STEKELENBURG

Abstract. A modest Kan complex is a modest simplicial set which has a right lifting property with respect to horn inclusions \( \Lambda_k[n] \to \Delta[n] \). This paper shows that there is a univalent universe of modest Kan complexes among simplicial assemblies.

1. Introduction

A PER (Partial Equivalence Relation) is a symmetric transitive relation of the natural numbers. A morphism of PERs \( R \to S \) is a function \( f \) of the equivalence classes, for which there is a partial recursive function \( \phi \) such that \( \phi(x) \in f(y) \) for all \( x \in y \). Together they form a category \( \text{PER} \) which has a lot of interesting properties. PERs provide a semantics for the polymorphic \( \lambda \)-calculus. \cite{AP90, Rum04, Fre89}

The category \( \text{PER} \) is closely related to the category of modest sets, which is a subcategory of the effective topos. \cite{Ros90, HRR90, vO08}

This paper is essentially about simplicial PERs, i.e. the simplicial objects of \( \text{PER} \) and their potential use as a model of homotopy type theory. We study these through the related category of simplicial modest sets.

Concretely, we show that inside the category of assemblies the category of discrete opfibrations over a fixed base category all have their own versions of modest sets and all have a generic modest morphism (theorem 11). We define injective morphisms for arbitrary families of morphisms and show that they too have a generic modest morphism (theorem 22). Finally we introduce the simplicial assemblies. The subcategory of Kan complexes is a model category (theorem 39) and the domain of the generic modest Kan fibration s a Kan complex (theorem 44). Ultimately we show that the generic modest fibration is univalent (theorem 46).

2. Assemblies

This section provides some background on the category of assemblies and the category of modest sets. For general information about the effective topos see \cite{vO08}.

Definition 1 (Assemblies). Let \( \mathbb{N} \) be the set of natural numbers and let \( \mathbb{P}\mathbb{N} \) be its powerset. An assembly is a pair \( (X, \phi) \) where \( X \) is a set and \( \phi : X \to \mathbb{P}\mathbb{N} \) is a function which assigns a non empty set of numbers \( \phi(x) \) to each element of \( X \).

Let \( (X, \phi) \) and \( (Y, \chi) \) be assemblies. A partial recursive function \( f \) tracks \( g : X \to Y \) if \( f : \phi(x) \to \chi(g(x)) \). A morphism \( (X, \phi) \to (Y, \chi) \) is a function \( g : X \to Y \) which is tracked. With composition and identities defined as in the category of sets, \( \text{Asm} \) is the category of assemblies and morphisms of assemblies.

The category of assemblies has a number of useful properties which we will mention without proving them here.

Lemma 2. The category of assemblies...

- is finitely complete and cocomplete;
- is locally Cartesian closed, regular and extensive;
• has a natural number object $\mathbb{N}$.

Proof. See [vO08, Ste13b, Ste13a]. □

The category of modest sets is a subcategory of the category of assemblies, which is complete in a suitable internalized sense and equivalent to an internal category of the category of assemblies.

Definition 3 (Modesty). Let $\nabla 2$ be the assembly $(\{0, 1\}, i \mapsto N)$. A morphism $f : X \to Y$ of assemblies is modest if the following diagram is a pullback:

$$
\begin{array}{ccc}
X & \xrightarrow{id_Y} & X^{\nabla 2} \\
\downarrow f & & \downarrow f^{\nabla 2} \\
Y & \xrightarrow{id_Y} & Y^{\nabla 2}
\end{array}
$$

Here $id_Y$ stands for composition with the unique maps $!_X : X \to 1$ and $!_Y : Y \to 1$ to the terminal object. This is another way of saying that $f$ is right orthogonal to $!: \nabla 2 \to 1$ and to the multiple $W \times !$ for every assembly $W$. A modest set is an assembly $X$ for which $!: X \to 1$ is modest (this means that $id_Y$ is an isomorphism).

Lemma 4. Modest morphisms:

• are closed under composition, pullbacks and products, including dependent products;
• include all monomorphisms and the unique map $!: \mathbb{N} \to 1$;
• are pullbacks of a single generic modest morphism $\mu$.

Proof. See [HRR90, vO08, Ros90]. □

The generic modest morphism $\mu : E \to B$ induces an internal category $\text{PER}$. The object of objects of $\text{PER}$ is $B$. The object of morphisms is $\bigsqcup_{(i,j) \in B \times B} E^E_{ij}$. Since it corresponds to modest sets, it is a complete internal category. Contrary to complete internal categories of $\text{Set}$, which are posets by a theorem of Freyd, $\text{PER}$ is not a poset.

The global sections functor $\Gamma : \text{Asm} \to \text{Set}$ turns $\text{PER}$ into the category with subquotients of $\mathbb{N}$ as objects and tracked functions as morphisms which we described in the introduction of this paper.

3. Modest opfibrations

This section introduces discrete opfibrations, which act like functors from internal categories to $\text{Asm}$. We construct a generic modest morphism for in the category of opfibrations over an arbitrary base category.

3.1. Discrete opfibrations. In order to mimic assembly-valued functors $\mathcal{B} \to \text{Asm}$ we use a kind of functor $\mathcal{E} \to \mathcal{B}$ with the property that the fibres are discrete categories and that a morphism $f : i \to j$ in $\mathcal{B}$ induces a functor $f' : \mathcal{E}_i \to \mathcal{E}_j$ between the fibres. Both of these properties come from the following.

A functor $F : \mathcal{E} \to \mathcal{B}$ is a discrete opfibration if the following square is pullback.

$$
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{dom} & \mathcal{E} \\
\downarrow F & & \downarrow F \\
\mathcal{B} & \xrightarrow{dom} & \mathcal{E}
\end{array}
$$

Here $\mathcal{2}$ is the category with two objects 0, 1 and one non identity arrow $0 \to 1$, so $\mathcal{B}^2 \text{ and } \mathcal{B}^2$ are the arrow categories. In both cases $\text{dom}$ is the projection of the
arrows to their domains. In other words, discrete opfibration are right orthogonal to the functors \( c \mapsto (c, 0) : C \to C \times 2 \).

The category \( \mathcal{B} \) acts on \( \mathcal{E} \) in the following way. For each object \( e \) of \( \mathcal{E} \) and each morphism \( \phi : F(e) \to b \) in \( \mathcal{B} \) there is a unique morphism \( \phi(e) : e \to \phi \cdot e \) such that \( F(\phi(e)) = \phi \).

**Lemma 5.** If \( G : \mathcal{B} \to C \) is a discrete opfibration, then \( F : A \to \mathcal{B} \) is the discrete opfibration if and only of \( GF \) is. Moreover, discrete opfibrations are stable under pullback.

**Proof.** This holds for any class of right orthogonal morphisms, for straightforward reasons. \( \square \)

For each internal category \( \mathcal{B} \) of \( \textbf{Asm} \), \( \textbf{Asm}^\mathcal{B} \) is the category whose objects are discrete opfibrations with codomain \( \mathcal{B} \) and whose morphisms are commutative triangles. The following lemmas serve to demonstrate that this category indeed functions as a category of presheaves.

**Lemma 6.** Let \( F/\mathcal{D} \) be the category that has morphisms \( f : Fc \to d \) as objects and where a morphism \( f \to f' \) is a pair \( (g, g') \) where \( g \) is a morphism of \( \mathcal{C} \), \( g' \) of \( \mathcal{D} \) and \( Fg' \circ f = f' \circ Fg \). For each functor \( F : \mathcal{C} \to \mathcal{D} \) opfibrations are orthogonal to the functor \( I(F) : \mathcal{C} \to \mathcal{F}/\mathcal{D} \) which sends \( c \) to \( \text{Fid}_c \).

**Proof.** Let \( G : \mathcal{D} \to \mathcal{E} \) be an opfibration and let \( D : \mathcal{C} \to \mathcal{D} \) and \( E : F/\mathcal{D} \to \mathcal{E} \) satisfy \( GD = EI \). Define \( H : F/\mathcal{D} \to \mathcal{C} \) by \( H(f : Fc \to d) = E(f, \text{id}_d) \cdot D(c) \) for objects of \( \mathcal{B}^2 \). For \( (g, g') : (f : Fc \to d) \to (f' : Fc' \to d') \) let \( H(g, g') = E(g, g')_{H(f)} \). The functor \( H \) satisfies \( HI(b) = G(b) \) and \( GH(f) = G(E(f, \text{id}_{Fc'}) \cdot D(c)) \). \( \square \)

**Lemma 7.** The category \( \textbf{Asm}^\mathcal{B} \) has all finite limits and colimits and is locally Cartesian closed and regular.

**Proof.** Finite limits is trivial with lemma \( \text{[3]} \) The functor \( \mathcal{C} \to \mathcal{C}^2 \) preserves all coproducts because the category \( \{ \to \} \) is connected. If a discrete opfibration \( F : \mathcal{C} \to \mathcal{D} \) is a regular epimorphism of objects, then so is \( F^2 \), which explains coequalizers and regularity. Since every slice category of \( \textbf{Asm}^\mathcal{B} \) is the category of internal categories and funcors \( \textbf{Asm}^\mathcal{B} \), it suffices to show Cartesian closure.

Let \( F : \mathcal{C} \to \mathcal{B} \) and \( G : \mathcal{D} \to \mathcal{B} \) be opfibrations. The opfibration \( G^F : \mathcal{D}^\mathcal{B} \) is defined as follows. The objects of \( \mathcal{D}^\mathcal{B} \) are pairs \( (b, H) \) where \( b \) is an object of \( \mathcal{B} \) and \( H : b/B \times_B \mathcal{C} \to \mathcal{D} \) is a functor that commutes with \( \text{cod} \times F : b/B \times_B \mathcal{C} \to \mathcal{C} \) and \( G : \mathcal{D} \to \mathcal{B} \). A morphism \( (b, H) \to (b', H') \) is a morphism \( f : b \to b' \) such that \( H(x \circ f, Y) = H'(x, Y) \). This is an opfibration because for each object \( (b, H) \) and each \( f : b \to b' \), \( (b, H \circ f^*) \), where \( f^* \) is composition with \( f \) to the right, is the unique lifting.

Let \( E : \mathcal{E} \to \mathcal{B} \) be another fibration. The opfibration \( G^F : \mathcal{G} \to \mathcal{B} \) is orthogonal to \( \mathcal{E} \times_B \mathcal{C} \to \mathcal{E}/B \times_B \mathcal{C} \) by lemma \( \text{[4]} \) and there is a bijection between functors \( \mathcal{E}/B \times_B \mathcal{C} \to \mathcal{D} \) which commute with \( \text{cod} \times F \) and \( G \) and functors \( \mathcal{E} \to \mathcal{D}^\mathcal{B} \) which commute with \( E \) and \( G^F \) by the definition given above. Note that this also works when \( F \) is not an opfibration. \( \square \)

3.2. **Total category.** Let \( \text{cat}(\textbf{Asm}) \) be the category of internal categories and functors in \( \textbf{Asm} \). There is an obvious functor \( \int : \textbf{Asm}^\mathcal{B} \to \text{cat}(\textbf{Asm}) \) that sends a discrete opfibration \( F : \mathcal{E} \to \mathcal{B} \) to its domain: the category of elements of the discrete opfibration.

**Lemma 8.** The category-of-elements functor \( \int \) has a right adjoint.

**Proof.** For an arbitrary internal category \( \mathcal{C} \) let \( \text{Is} \mathcal{C} \) be the following category. The objects are pairs \( (i, F) \) where \( i \) is an object of \( \mathcal{B} \) and \( F \) is a functor \( i/B \to \mathcal{C} \). A morphism \( (i, F) \to (j, G) \) is a morphism \( \phi : i \to j \) such that \( F \circ \phi^* = G \). Here
\(\phi^*\) means composition with \(\phi\) on the right. The opfibration \(sC : |sC| \rightarrow B\) is the projection to the first variable.

For an arbitrary functor \(H : C \rightarrow D\), let \(sH\) be the functor that satisfies \(sH(i,F) = (i,HF)\) on objects and \(sH(\phi,f) = (\phi,Hf)\) on morphisms.

Let \(D(C) : sC \rightarrow B\) be the functor which sends \((i,F)\) to \(i\) and which is the identity on morphisms. This is a discrete opfibration because for each object \((i,F)\) of \(sC\) and each morphism \(\phi : P(i,F) \rightarrow j\) there is a unique morphism \(\phi : (i,F) \rightarrow (j,F \circ \phi^*)\) over \(\phi\).

This functor \(s\) has the property that there is a bijection between functors \(C \rightarrow |sD|\) which commute with \(F : C \rightarrow B\) and \(sD\) and functors \(F/B \rightarrow D\), because of the definition of \(s\). We can compose \(G : C \rightarrow D\) with \(H : F/C \rightarrow C\) to get the commutative triangle \(F \rightarrow sD\). For \(H : f \rightarrow sD\) we compose the corresponding \(H' : F/B \rightarrow D\) with the functor \(I : C \rightarrow F/B\) of lemma [6].

Hence we get an adjunction \(\int - s\).

3.3. Modest opfibrations. We bring modest morphisms to \(\text{Asm}^s\) by considering morphisms of discrete opfibrations whose underlying functors are modest.

**Definition 9.** A discrete opfibration \(F : X \rightarrow Y\) is *modest* if its object map \(F_0 : X_0 \rightarrow Y_0\) is a modest morphism.

**Lemma 10.** There is a generic modest opfibration.

**Proof.** Let \(\mu : E \rightarrow B\) be the generic modest morphism of \(\text{Asm}\). The category of \(\text{PER}\) was defined with \(B\) as object of objects and with \(\text{PER}(i,j) = E_{ij}\) as homsets. We define the internal category of pointed PERs \(\text{PER}_s\) in a similar way. Its object of objects is \(E\) and \(\text{PER}_s(i,j) = \{f \in E_{\mu(i)} | f(i) = j\}\); the idea is that \(E\) is a set of PERs \(E_i\) paired with a chosen point \(1 \rightarrow E_i\) and that morphisms preserve points.

There is a forgetful functor \(U : \text{PER}_s \rightarrow \text{PER}\); for objects \(U(i) = \mu(i)\) and for morphism \(U\) is the identity. This is a discrete opfibration because for each morphism \(f : E_i \rightarrow E_j\) and each point \(p : 1 \rightarrow E_i\), \(f(p) : 1 \rightarrow E_j\) is the unique point that turns \(f\) into a morphism of pointed PERs. We claim that this is the generic modest opfibration.

Let \(F : C \rightarrow D\) be a modest opfibration. There are morphisms \(e : C_0 \rightarrow E\) and \(b : D_0 \rightarrow B\) such that \(b \circ e = \mu \circ e\) is a pullback square. We turn these into functors in the following way. In \(\text{cat}(\text{Asm})/\mathcal{D}\) the action \(x \mapsto f \cdot x\) for \(f \in D_1\) induces a morphism \(\text{dom}(\mathcal{D}) \rightarrow F^\mathcal{D}\) where \(\text{dom}(\mathcal{D}) : \mathcal{D}^\mathcal{D} \rightarrow \mathcal{D}\) is the domain functor and \(F^\mathcal{D}\) is the exponential as fibration. Because \(F^\mathcal{D} \approx b^*(U^\mathcal{D})\) we get an object map which turns \(b : D_0 \rightarrow B\) into a functor \(\mathcal{D} \rightarrow \text{PER}\).

In \(\text{cat}(\text{Asm})/\mathcal{C}\) for the action \(x \mapsto f \cdot x\) for \(f \in C_1\) induces a morphism \(F\text{dom}(\mathcal{C}) \rightarrow F^\mathcal{C}\). This time we rely on \(F^*(F^\mathcal{C}) \approx (b \circ F)^*(U^\mathcal{C}) = (\mu \circ e)^*(U^\mathcal{C})\) to turn the morphism \(e : C_0 \rightarrow E\) into a functor \(e : \mathcal{C} \rightarrow \text{PER}_s\).

The new functors satisfy \(U \circ e = b \circ F\) and even form a pullback square. For each \(f : i \rightarrow j\) in \(\mathcal{B}\) and each \(g : E_{\mu(k)} \rightarrow E_{\mu(l)}\) in \(\text{PER}_s\), there is an object \(k'\) of \(Y_i\) such that \(F(k') = i\) and \(\epsilon(k') = k\) because the object maps for a pullback square. Hence there is a unique \(g' : k' \rightarrow l'\) in \(\mathcal{B}\) such that \(Fg' = f\). Because \(U(\epsilon(g')) = b(f), \epsilon(g')\) is the unique morphism \(g : k \rightarrow l\) for which \(U(g) = b(f)\). Hence the square \(U \circ e = b \circ F\) is a pullback.

The discrete opfibration \(U : \text{PER}_s \rightarrow \text{PER}\) is clearly modest itself and hence a generic modest morphism.

A morphism \(F : C \rightarrow B\) in \(\text{Asm}^s\) is *modest* precisely when \(f(F)\) is modest. This has the following consequence.

**Theorem 11.** The category \(\text{Asm}^s\) has a generic modest morphism.
Proof. The morphism $sU : s\text{PER}_s \to s\text{PER}$ is the generic modest morphism of $\text{Asm}_{\mathcal{B}}$. If $\int(F) : (\epsilon) \to \int(B)$ is a pullback of $U : \text{PER}_s \to \text{PER}$ along some $X : \int(B) \to \text{PER}$, then $F$ is the pullback of $sU$ along the transpose $X^!$ of $X$ for the following reasons. There is a discrete opfibration orthogonal to coslice categories by lemma [6]. This means that for every point $p \in \text{PER}_s$ and every functor $F : i/\mathcal{B} \to \text{PER}$ such that $U(p) = F(id_i)$, there is a unique functor $F_* : i/\mathcal{B} \to \text{PER}_s$ such that $UF_* = F$ and $F_*(id_i) = p$.

The co-unit $\epsilon_{\text{PER}} : s\text{PER} \to \text{PER}$ is the functor that sends $(i, F)$ in $s\text{PER}$ to $F(id_i)$ and a morphism $f : (i, F) \to (j, G)$ to $Ff : F(id_i) \to F(f) = G(id_j)$. The description for $\epsilon_{\text{PER}}$ is the same. If $(i, F)$ is an object of $s\text{PER}$ and $j$ an object of $\text{PER}$ such that $Uj = \epsilon(i, F) = F(id_i)$, we get a unique functor $F_*$ such that $\epsilon(i, F_*) = j$ and $UF_* = F$ as explained above. Hence the naturality square of the co-unit is a pullback. This implies that $sU$ is modest and that if $\int(F)$ is a pullback of $U$, then $F$ is a pullback of $sU$. □

3.4. Orthogonality and completeness. Just like in $\text{Asm}$, orthogonality characterizes the modest morphisms of $\text{Asm}_{\mathcal{B}}$.

Lemma 12. Let $\nabla^2_{\text{disc}}$ be the discrete category whose object of objects is $\nabla^2$. In $\text{cat}(\text{Asm})$ the modest morphisms are precisely those that are right orthogonal to the constant discrete opfibrations $\nabla^2_{\text{disc}} \times \mathcal{B} \to \mathcal{B}$.

Proof. A discrete opfibration $F : C \to D$ is modest if the underlying object map $F_0 : C_0 \to D_0$ is modest. Due to the adjunction between $X \mapsto X_{\text{disc}}$ and $Y \mapsto Y_0$, $fF_0$ is right orthogonal to $\nabla^2$ when $\int F$ is orthogonal to $\nabla^2_{\text{disc}}$. The constant discrete opfibration is just a multiple and therefore right orthogonality is preserved. □

Proposition 13. For each internal category $\mathcal{B}$ of $\text{Asm}$, $\text{Asm}_{\mathcal{B}}$ has small complete internal categories.

Proof. The class of discrete opfibrations which are right orthogonal to $\nabla^2_{\text{disc}}$, are closed under all existing limits and the fibration is essentially small thanks to the generic modest morphism. □

4. Injectives

The categories $\text{Asm}_{\mathcal{B}}$ are canonically enriched over $\text{Asm}$. Enrichment modifies the lifting properties which ordinarily define Kan fibrations in simplicial sets. This section shows that for any internal family of morphisms in $\text{Asm}_{\mathcal{B}}$, there is a generic modest injective morphism (theorem [22]).

4.1. Enriched injectives.

Definition 14. Let $\text{nat} : (\text{Asm}_{\mathcal{P}})^{\mathcal{P}} \times \text{Asm}_{\mathcal{P}} \to \text{Asm}$ be the functor which sends each pair of opfibration $X, Y$ over $\mathcal{P}$ to the assembly $\text{nat}(X, Y)$ of morphisms between them.

A morphism $f : X \to Y$ has the global right lifting property with respect to a a morphism $g : I \to J$ and $g$ has the global left lifting property with respect to $f$ if the morphism $(f, g^*) = (\text{nat}(id_J, f), \text{nat}(g, id_X))$ in the diagram below is a split epimorphism.
Remark 15 (Lifting power). The ordinary right lifting property only says that composition with \((f_1, g^*)\) induces a surjective function on global sections, while the local right lifting property only requires that \((f_1, g^*)\) is a regular epimorphism. The global version is stronger than either of those.

Example 16. Modest sets have the global right lifting property with respect to \(! : \nabla 2 \to 1\).

Example 17. Discrete opfibrations have the global right lifting property with respect to \(0 : 1 \to \{\to\}\), though they are not the only functors that have it.

Definition 18 (Injective and anodyne). An \(I\)-indexed-family of morphisms in \(\text{Asm}^B\) a morphism \(a : D \to E\) in \(\text{Asm}^{I_{\text{disc}} \times B}\). Let \(I^* : \text{Asm}^B \times \text{Asm}^{I_{\text{disc}} \times B}\) be the functor which sends each opfibration \(F : \mathcal{E} \to \mathcal{B}\) to the opfibration \(I_{\text{disc}} \times \mathcal{E} \to I_{\text{disc}} \times \mathcal{B}\). A morphism \(f : X \to Y\) is \(a\)-injective if \(I^* f\) has the global right lifting property with respect to \(a\). A morphism is \(a\)-anodyne if it has the global left lifting property with respect to all \(a\)-injectives.

4.2. Injectives as algebras. Injectives are a kind of Lambek algebra for a functor \(S : \text{Asm}^B \to \text{Asm}^B\). This allows us to construct a generic modest injective in \(\text{Asm}^B\).

Definition 19 (Algebras). A pointed endofunctor is a pair \((F, \phi)\) where \(F\) is an endofunctor of a category \(\mathcal{C}\) and \(\phi : \text{id}_\mathcal{C} \to F\) is a natural transformation. An algebra for \((F, \phi)\) is a pair \((X, f : FX \to X)\) where \(f \circ \phi_X = \text{id}_X\).

Lemma 20. There is a pointed endofunctor \((S : \text{Asm}^{B \times \to} \to \text{Asm}^{B \times \to}, \sigma)\) such that \(f : X \to Y\) is \(a\)-injective if it has an algebra structure.

Proof. For each morphism \(f : X \to Y\) in \(\text{Asm}^B\) let \(\text{nat}(a, I^*(f))\) be the following object of \(\text{Asm}^{I_{\text{disc}} \times B}\).

\[
\{(x, y) \in \text{nat}(D, I^*(X)) \times \text{nat}(E, I^*(Y)) | f \circ x = y \circ h\}
\]

Alternatively, the following pullback square defines it.

\[
\begin{array}{ccc}
\text{nat}(a, I^*(f)) & \to & \text{nat}(D, I^*(X)) \\
\downarrow & & \downarrow \\
\text{nat}(E, I^*(Y)) & \to & \text{nat}(D, I^*(Y))
\end{array}
\]

The functor \(\text{nat}(E, I^*(-))\) is a composition of three functors which have a left adjoint. The first one \(I^*\) has \(I : \text{Asm}^{I_{\text{disc}} \times B} \to \text{Asm}^B\) which is composition with the opfibration \(I_{\text{disc}} \times \mathcal{B} \to \mathcal{B}\). The functor \(\text{nat}(E, -)\) is the composition of the exponential \(-^E\) and the functor \(\mathcal{B}_* : \text{Asm}^{I_{\text{disc}}} \to \text{Asm}^{I_{\text{disc}} \times B}\) which is right adjoint to the functor \(B^* : \text{Asm}^{I_{\text{disc}}} \to \text{Asm}^{I_{\text{disc}} \times \mathcal{B}}\) which sends each opfibration \(F : \mathcal{E} \to I_{\text{disc}}\) to \(F \times \mathcal{B} : \mathcal{E} \times \mathcal{B} \to I_{\text{disc}} \times \mathcal{B}\). Hence \(I(B^*(-) \times E)\) is left adjoint to \(\text{nat}(E, I^*(-))\).

The transposes of the projections \(\pi_0 : \text{nat}(a, I^*(f)) \to \text{nat}(D, I^*(X))\) and \(\pi_1 : \text{nat}(a, I^*(f)) \to \text{nat}(E, I^*(Y))\) satisfy \(f \circ \pi_{0\text{r}} = \pi_{1\text{r}} \circ I_t(B^*(\text{nat}(a, I^*(f))) \times f)\). The pointed endofunctor comes from the pushout of \(I_t(B^*(\text{nat}(a, I^*(f))) \times f)\) along \(\pi_{0\text{r}}\), as defined below.

\[
\begin{array}{ccc}
I_t(B^*(\text{nat}(a, I^*(f))) \times D) & \xrightarrow{\sigma_0} & X \\
\downarrow & & \downarrow \text{id}_X \\
I_t(B^*(\text{nat}(a, I^*(f))) \times E) & \xrightarrow{\pi_1} & Y \\
\downarrow & & \downarrow S(f) \\
I_t(B^*(\text{nat}(a, I^*(f)))) & \xrightarrow{\sigma(f)} & Y
\end{array}
\]

By definition \(f\) is \(a\)-injective if the canonical morphism \((a^*, f_t) : \text{nat}(E, I^*(X)) \to \text{nat}(a, I^*(f))\) has a section \(g\). The transpose \(g^t\) satisfies \(f \circ g^t = \pi_{1\text{r}}\) and \(g^t \circ
\[ I((B^I)(\text{nat}(a, I^I(f)))) \times f) = \pi^I_0 \] and hence factors through \(S(X)\) giving an algebra structure to \(f\). On the other hand each algebra structure on \(f\) induces an inverse of \((a^I, f_i)\).

\[ \square \]

**Remark 21.** Nothing forces \(S(f)\) to be injective, so the construction above does not necessarily induce a factorization into anodyne and injective morphisms. Small object arguments don’t help because \(\text{Asm}^S\) is not cocomplete. The internal category \(\text{sPER}\) is algebraically complete [Fie91]. Therefore \((S, \sigma)\) generated a free monoid in the monoidal category \(\text{sPER}^{\text{sPER}}\), which is a monad on \(\text{sPER}\) whose algebras are \((S, \sigma)\)-algebras. This means that modest morphisms factorize into modest injective and morphisms which have the global left lifting property with respect to modest injectives. This is not exactly the same thing as anodyne, unfortunately.

**4.3. Generic modest injectives.** We delve deeper into modest injectives.

**Theorem 22.** Every family of morphisms \(a\) has a generic modest \(a\)-injective.

**Proof.** The functor \(S\) is defined by a combination of limits and colimits. Pullbacks preserve all of them because \(\text{Asm}^B\) is locally cartesian closed. The factorization with \(S\) above is therefore stable under pullback, i.e. if \(f : A \to B\) is the pullback of \(g : C \to D\) along some \(h : B \to D\), then \(S(f) : S(A) \to B\) is the pullback of \(S(g) : S(C) \to D\).

\[
\begin{array}{ccc}
A & \xrightarrow{f} & S(A) \xrightarrow{\sigma(f)} B \\
\downarrow & & \downarrow \\
C & \xrightarrow{g} & S(C) \xrightarrow{\sigma(g)} D \\
\end{array}
\]

Let \(\text{sPER}^S\) be the following object of fibrewise left inverses of \(\sigma(sU) : \text{sPER}_* \to S(\text{sPER}_*)\) over \(\text{sPER}\):

\[
\text{sPER}^S_i = \{ f : S(\text{sPER}_*)_i \to (\text{sPER}_*)_i | f \circ \sigma(sU)_i = \text{id}_{(\text{sPER}_*)_i} \}
\]

There is a morphism \(sU^S(\text{sPER}_*) : \text{sPER}^S \to \text{sPER}\) which simply sends each left inverse to its index. When we pull back \(sU\) along \(sU^S(sU)\) we obtain a modest morphism \(sU^S : \text{sPER}^S \to \text{sPER}^S\).

The object \(S(\text{sPER}^S)\) is the fibred product \(\text{sPER}^S \times_{\text{sPER}} S(\text{sPER}_*)\) and by definition of \(\text{sPER}^S\) there is an application morphism \(\epsilon : S(\text{sPER}^S) \to \text{sPER}_*\) which makes \(\epsilon \circ \sigma(sU^S)\) equal to the projection \(\pi : \text{sPER}^S \to \text{sPER}_*\).

\[
\begin{array}{ccc}
sU^S & \xleftarrow{\sigma(sU^S)} & S(\text{sPER}^S) \\
\downarrow & & \downarrow \\
\text{sPER}_* & \xleftarrow{\sigma(sU)} & S(\text{sPER}_*) \\
\text{sPER}_* & \xrightarrow{\pi} & \text{sPER}^S \\
\end{array}
\]

Because \(sU \circ \epsilon = sU^S(sU) \circ S(sU^S)\), \(\epsilon\) factors through \(\text{sPER}^S\) in a unique \(\epsilon' : S(\text{sPER}^S) \to \text{sPER}^S\) which satisfies \(sU^S \circ \epsilon' = S(sU^S)\) and \(\pi \circ \epsilon' = \epsilon\). Because \(sU^S \circ \epsilon' \circ \sigma(sU^S) = sU^S\) and \(\pi \circ \epsilon' \circ \sigma(sU^S) = \pi\), \(\epsilon' \circ \sigma(sU^S)\) is the unique factorization of \((sU^S, \pi)\) through itself, i.e. \(\text{id}_{\text{sPER}^S}\). Hence \(sU^S\) is a modest \(a\)-injective.

Let \(f : X \to Y\) be the pullback of \(sU\) along \(\chi : Y \to \text{sPER}\). Let \(g : S(X) \to X\) satisfy \(f \circ g = S(f)\) and \(g \circ \sigma(f) = \text{id}_X\). The morphism \(S(f) : S(X) \to Y\) is the pullback of \(S(u) : S(\text{sPER}_*) \to \text{sPER}\). We take advantage of fibrewise isomorphisms \(\alpha_y : X_y \to (\text{sPER}_*)_Y\) and \(\beta_y : S(X)_Y \to S(\text{sPER}_*)_Y\) set up by

\[
\begin{array}{ccc}
sU^S & \xrightarrow{\sigma(sU)} & S(sU^S) \\
\downarrow & & \downarrow \\
\text{sPER}^S & \xrightarrow{\sigma(sU^S)} & S(sU^S) \\
\end{array}
\]
these pullbacks. The morphism $\chi$ factors through $s\text{PER}^S$ thanks to the following mapping:

$$y \in Y \mapsto \alpha_y \circ g_y \circ \beta_y^{-1} : s\text{PER}_\gamma) \to (s\text{PER}_\gamma)\chi(y)$$

The following is a diagram chasing way to find that morphism. Work inside $\text{Asm}^B/Y$, where we have actual isomorphisms $\alpha' : f \to \chi^*(sU)$ and $\beta' : S(f) \to \chi^*(S(sU))$. The morphism $\alpha' \circ g \circ (\beta')^{-1} : \chi^*(S(sU)) \to \chi^*(sU)$ has a transpose $\gamma : 1 \to \chi^*(sU)\chi^*(S(sU))$. Local Cartesian closure implies that the canonical morphism $\kappa : \chi^*(sU)^{S(sU)} \to \chi^*(sU)^{S(sU)}$ is an isomorphism. The transpose $(\kappa^{-1} \circ \gamma)^! : \chi \to s\text{PER}^{S(sU)}$ in $\text{Asm}^B/s\text{PER}$ is what we are looking for.

Note that $h \circ \sigma(f) = \text{id}_X$. For this reason $(\kappa^{-1} \circ \gamma)^!$ actually goes to $s\text{PER}^S$, which only contains the right inverses of sections of $\sigma(sU)$.

Because $s\text{PER}^{S(sU)} \circ (\kappa^{-1} \circ \gamma)^! = \chi$ and $f \simeq \chi^*(sU)$, $f \simeq [(\kappa^{-1} \circ \gamma)^!]^*(sU^S)$. Since every modest $a$-injective is a pullback of $sU^S$, $sU^S$ is a generic modest $a$-injective.

\[\square\]

5. Homotopy

This section starts the exploration of simplicial assemblies and their homotopy. If we extend the cofibrations and acyclic Kan fibrations of simplicial sets to simplicial assemblies we still get a factorization system (proposition 28). Not all monomorphisms in $s\text{Asm}$ are cofibrations because the internal logic of $\text{Asm}$ does not satisfy the principle of the excluded middle. Cofibrations are monomorphisms $f : X \to Y$ for which some properties of simplices in $Y$ are nonetheless decidable (proposition 29).

5.1. Simplicial assemblies. The simplex category exists as internal category $\Delta$ of $\text{Asm}$. The category of simplicial assemblies $s\text{Asm}$ is the category of opfibrations $\text{Asm}^{\Delta^{op}}$ over the dual $\Delta^{op}$ of $\Delta$.

5.2. Simplicial assemblies. The category of assemblies has a natural number object $N$. The simplex category $\Delta$ has $N$ as object of objects, but when we refer to its objects, we surround numbers by square brackets: $[n]$. The homset $\Delta([m], [n])$ is the object of nondecreasing morphisms $\{i \mid i \leq m\} \to \{j \leq m\}$. Among the morphisms of $\Delta$ the face maps are regularly used below. The morphism $\delta_i^0 : [n-1] \to [n]$ is the unique injective nondecreasing morphism which skips $i$; it represents the face opposite to the edge $i$.

Of course, there is another kind of simplicial assembly–a presheaf on the external simplex category. Those are more general, because in $s\text{Asm}$ there is a recursive function tracking the action map $(f, x) \mapsto f \cdot x$, while for the presheaves there may only be recursive functions tracking $x \mapsto f \cdot x$ for each $f$ separately.

The simplex $\Delta[n]$ is the opfibration $\Delta/[n]^{op} \to \Delta^{op}$. For each morphism $\phi : [m] \to [n]$ we let $\Delta(\phi) : \Delta[m] \to \Delta[n]$ be the morphism that sends $\xi : [i] \to [m]$ to $\phi \circ \xi : [i] \to [n]$. The cycle $\partial\Delta[n]$ is the subopfibration $C[n] \subseteq \Delta/[n]^{op}$ whose objects are the nondecreasing maps $[m] \to [n]$ that are not surjective. For $n > 0$ the horn $\Lambda_k[n]$ is the subopfibration $H_k[n] \subseteq \Delta/[n]^{op}$ whose objects are the nondecreasing maps $[m] \to [n]$ that are not surjective on the complement of $\{k\} \subseteq [n]$. These seemingly classical definitions work because equality of numbers is recursively decidable.

These simplicial assemblies form a family $\{\Lambda_k[n] \to \Delta[n] \mid n > 0, k \leq n\}$ of horn inclusions and $\{\partial\Delta[n] \to \Delta[n] \mid n \in \mathbb{N}\}$ of cycle inclusions. We introduce the following terms based on these families.

- A Kan fibration is an injective morphism relative to the family of horn inclusions.
• An **acyclic Kan fibration** is a injective morphism relative to the family of cycle inclusions.

• A **Kan complex** or **Kan fibrant object** is a simplicial assembly $X$ for which $!_X : X \to 1$ is a fibration.

• A **cofibration** is an anodyne morphism relative to the family of cycle inclusions.

• A **cofibrant object** is a simplicial assembly $X$ for which $!_X : 0 \to X$ is a fibration.

• An **acyclic cofibration** is an injective morphism relative to the family of horn inclusions.

• A **weak equivalence** is a morphism which factors as an acyclic fibration following an acyclic cofibration.

We often leave out ‘Kan’ and simply talk about fibrations and complexes in the rest of this paper.

For every pair of morphisms $f, g : X \to Y$ a homotopy between them is a map $h : \Delta[1] \times X \to Y$ such that $h \circ (\delta^1_0 \times \text{id}_X) = f$ and $h \circ (\delta^1_1 \times \text{id}_X) = g$ where $\delta^1_0$ and $\delta^1_1$ are the face maps mentioned above. The morphisms $f$ and $g$ are homotopic if there is a homotopy between them. A morphism $f : X \to Y$ is a homotopy inverse of $g : Y \to X$ if $f \circ g$ is homotopic to $\text{id}_Y$ and $g \circ f$ is homotopic to $\text{id}_X$. If $f$ has a homotopy inverse, then $f$ is a homotopy equivalence.

5.3. **Decidability.** Not all monomorphisms of simplicial assemblies are cofibrations apparently because not all subobjects in the category of assemblies have complements. The next few subsections explain the connection between decidability and cofibrancy.

**Definition 23** (Locally decidable). A monomorphism in $\text{Asm}$ or $\text{sAsm}$ is **decidable** if it is isomorphic to a coproduct inclusion. A monomorphism $m : X \to Y$ is locally decidable if the object map of $\int m : \int X \to \int Y$ is decidable in $\text{Asm}$.

The terminology is slightly misleading. Decidable monomorphisms in $\text{sAsm}$ cover all locally decidable monomorphisms, but also some monomorphisms which are not locally decidable.

**Example 24.** All cycle inclusions $\partial \Delta[n] \to \Delta[n]$ are locally decidable, because equality of numbers is decidable.

Not all locally decidable monomorphisms are cofibrations, but many are. The distinction is degeneracy.

**Definition 25.** Let $X$ be a simplicial assembly. A simplex $x$ of $X$ is degenerate if there is an epimorphism $e$ in $\Delta$ and a simplex $y$ in $X$ such that $x = e \cdot y$. A face is a nondegenerate simplex.

**Lemma 26.** Let $f : X \to Y$ be a monomorphism in $\text{sAsm}$. If $f$ is locally decidable and degeneracy is decidable for simplices in the complement of $X$, then $f$ is a cofibration.

**Proof.** Let $Y_j$ be the union of $X$ with all $j$-dimensional faces of $Y$. There is an assembly $S_j$ of $j$-dimensional faces of $Y$ which are not simplices of $X$ thanks to decidability. Let $\Delta^* : \text{Asm} \to \text{sAsm}$ be the constant simplicial assembly functor. The inclusion $X \to Y_0 = X + \Delta^*(S_0)$ is a pushout of $S_0$ copies of the cofibrations $0 \to 1$ and hence a cofibration. For $j > 0$, if $y \in S_j$ then $Y_j \cap y$ is the boundary of $y$, hence $Y_{j-1} \to Y_j$ is a pushout of $S_j$ copies of the cofibration $\partial \Delta[j] \to \Delta[j]$ and hence a cofibration. Because $f : X \to Y$ is the colimit of the inclusions $X \to Y_j$ and those inclusions are compositions of cofibrations, $f$ is a cofibration. □
Proposition 29 below shows that these locally decidable monomorphisms are all cofibration in $\mathsf{sAsm}$. The locally decidable monomorphisms include all pullbacks of cofibrations as the following lemma shows.

**Lemma 27.** Locally decidable monomorphisms are stable under pullback and there is a classifier for locally decidable monomorphism in $\mathsf{sAsm}$.

**Proof.** The category of assemblies $\mathsf{Asm}$ is extensive, so pullbacks preserve coproduct inclusions. Either coproduct inclusion $1 \to 1 + 1$ is a generic decidable monomorphism. For every locally decidable monomorphism $f : X \to Y$ in $\mathsf{sAsm}$ the opfibration $\int f$ is a pullback of $1 : 1 \to 2$. The underlying map of $\int f$ is decidable and therefore a pullback of the underlying map of $1 : 1 \to 2$. Because $f$ is an opfibration the arrow map is a pullback too. So a morphism in $\mathsf{sAsm}$ is locally decidable if and only if it is a pullback of $s1 : s1 \to s2$. □

5.4. Factorization. The category $\mathsf{sAsm}$ has to few colimits for the small object argument to work for arbitrary families of morphism, but we don’t need it for cofibrations and acyclic fibrations.

**Proposition 28.** Every morphism $f : X \to Y$ factors as an acyclic fibration following a cofibration.

**Proof.** Let $Z$ be the following simplicial assembly. First let the assemblies $Z'[n]$ of $n \in \mathbb{N}$ consist of quadruples $(a,b,c,d)$ where $a : [n] \to [p]$ is an epimorphism of $\Delta$, $b : \Delta/[p]^{\mathsf{op}} \to 2$ is a functor, $c : \{b\} \to \int X$ is a functor on the full subcategory $\{b\}$ of $\Delta/[p]^{\mathsf{op}}$ on the objects $\xi$ for which $b(\xi) = 1$ and $d \in Y[p]$.

Let $\top$ stand for the functor $\Delta/[n]^{\mathsf{op}} \to 2$ which sends everything to $1$. Equivalently, $\top$ is the unique functor $\Delta/[n]^{\mathsf{op}} \to 2$ satisfying $(\text{id}_{[n]}) = 1$. If a simplex $(a,b,c,d)$ of $Z'$ has $b = \top$, then it represents a simplex of $X$. Let $Z[n] \subseteq Z'[n]$ consists of those quadruples $(a,b,c,d)$, where $a = \text{id}$ whenever $b = \top$. This subobject exists because $b = \top$ is equivalent to $b(\text{id}_{[n]}) = 1$, which is a decidable property of $b$.

In order to define the action of morphisms in the simplicial assembly, we use a case distinction for $b = \top$ and use the fact that in $\Delta$ every morphism $\phi : [m] \to [n]$ factors uniquely as a monomorphism $m(\phi)$ following an epimorphism $e(\phi)$. For every $\phi : [m] \to [n]$ and $(a,b,c,d) \in Z[n]$ let

$$
\phi \cdot (a,b,c,d) = \begin{cases} 
\text{id}, & \top, a \circ \phi, d \\
(c(a \circ \phi), b \circ m(a \circ \phi), c \circ m(a \circ \phi)), m(a \circ \phi) \cdot d & b(\phi) = 1 \\
\phi \cdot 0 & b(\phi) = 0
\end{cases}
$$

Here $\phi_1 : \Delta/[m]^{\mathsf{op}} \to \Delta/[n]^{\mathsf{op}}$ is defined as composition with $\phi$ and $m(a \circ \phi)_1$ is defined similarly.

There is a morphism $g : X \to Z$ satisfying $g[n](x) = (\text{id}_{[n]}, \top, x', f[n](x))$ where $x' : \Delta/[n]^{\mathsf{op}} \to x$ is the functor which sends $\xi$ to $\xi \cdot x$. This morphism is a cofibration by lemma 26.

There is a morphism $h : Z \to Y$ satisfying $h[n](a,b,c,d) = a \cdot d$ and this map is an acyclic fibration for the following reasons. Let $z : \partial Z[n] \to Z$, $y : Z[n] \to Y$ and $h \circ z = z \circ k_n$ where $k_n : \partial Z[n] \to Z[n]$ is the inclusion of the boundary. The functor $f_z : C[n] \to \int Z$ is a system of simplices $z(\xi) = (a(\xi), b(\xi), c(\xi), d(\xi))$ for $\xi \in C[n]$. A filler $w : \Delta[n] \to Z$ corresponds to a simplex $z = (a,b,c,d) \in Z[n]$ for which $a \cdot d \in Y[n]$ corresponds with $y$ and where $\xi \cdot z = z(\xi)$ for all $\xi \in C[n]$.

We make a case distinction.

- If $a(\xi) = \text{id}$ for all monomorphisms $\xi$, then $a = \text{id}$. Let the functor $b : \Delta/[n] \to 2$ send all epimorphisms to $0$. For every other morphism $\phi : [i] \to [j]$, we let $b(\phi) = b(m(\phi))(e(\phi))$. If $b(\phi) = 1$, then $c(\phi) = c(m(\phi))(e(\phi))$.

Finally $d$ is the simplex in $Y[n]$ which correspond with the morphism $y$. [26]
• If $a(\xi) \neq \text{id}$ for some monomorphisms $\xi$, then there is a greatest monomorphism $\mu : [m] \to [n]$ such that $\mu \neq \text{id}$ but $a(\mu) = \text{id}$. The reason is that there are finitely many monomorphisms to $[n]$ and the subcategory of those monomorphisms $\xi$ for which $a(\xi) = \text{id}$ is closed under pushouts. In this case $a$ should be the unique inverse of $\mu$ for which $e(a \circ \xi) = a(\xi)$ for all monomorphisms. We then let $z = a \cdot z(\mu)$.

These two cases cover all possible commutative squares with the cycle inclusion $k_n$, because there are finitely many monomorphisms $\xi : [m] \to [n]$ and $a(\xi) = \text{id}$ is a decidable property. For this reason, $h$ has the global right lifting property which respect to the family of all cycle inclusions. Therefore $h$ is an acyclic cofibration. □

The factorization in the proof above turns the implication in lemma 26 into an equivalence.

**Proposition 29.** A monomorphism $f : X \to Y$ is a cofibration if and only if it is decidable whether simplices of $Y$ are in $X$ and whether simplices outside of $X$ are degenerate.

**Proof.** Lemma 26 shows the ‘if’ direction. For ‘only if’ factor $f : X \to Y$ as in the proof of lemma 28 to get (another) cofibration $g : X \to Z$ and an acyclic fibration $h : Z \to Y$. Because $\text{id}_Y \circ f = h \circ g$ there is a $k : Y \to Z$ such that $h \circ k = \text{id}_Y$ and $k \circ f = g$ by the global lifting property. For each simplex $y$ of $Y$, $k(y) = (a, b, c, d)$. The simplex $y$ is in the image of $x$ if and only if $b = \top$ and this is decidable. A simplex $y$ for which $b \neq \top$ is nondegenerate if and only if $a = \text{id}$ for the following reasons. Since $k$ is a section of $h$ it is a monomorphism, and monomorphisms preserve nondegeneracy. The morphism $k$ commutes with the actions of morphisms in $\Delta$ and degenerates of $Z$ outside of the image of $g$ have $a \neq \text{id}$. □

## 6. Kan complexes

Because acyclic cofibrations are less stable under pullback than general cofibrations, it is not clear we can factorize arbitrary morphisms as fibrations following acyclic cofibrations by a similar construction as the one we used in proposition 28. We retreat to the category $\text{Asm}_f$ of fibrant objects and use the extra structure to get model structure here (theorem 39).

### 6.1. Pushout products.

The proof for a model structure on simplicial assemblies relies on the pushout-product construction, which preserves (acyclic) cofibrations.

**Definition 30.** For each pair of morphisms $f : W \to X$ and $g : Y \to Z$ the pushout product $f \otimes g$ is the unique factorization of the cospan $(f \times \text{id}_Z, \text{id}_X \times g)$ though the pushout of $f \times \text{id}_Y$ and $\text{id}_W \times g$.

$$
\begin{array}{c}
W \times Y \xrightarrow{f \times \text{id}} X \times Y \\
W \times g \\
W \times Z \xrightarrow{f \times \text{id}} X \times Z
\end{array}
$$

Because $\text{sAsm}$ is Cartesian closed, the pushout product with a fixed morphism has a right adjoint.

**Definition 31.** For each pair of morphisms $f : W \to X$ and $g : Y \to Z$ the pullback exponential $g^f$ is the unique factorization of the span of the span $(Y^f, g^X)$ though
the pullback of $Z_f$ and $g_W$.

\[
\begin{array}{c}
\text{Y}^X \\
\downarrow g^X \quad \downarrow g_f \\
\quad \downarrow g \\
Z^X \\
\end{array}
\]

\[
\begin{array}{c}
\text{Y}^W \\
\downarrow g^W \quad \downarrow g_W \\
\quad \downarrow g \\
Z^W \\
\end{array}
\]

**Lemma 32** (Pushout product). If $f$ and $g$ are cofibrations, then so is $f \otimes g$. Moreover, if either $f$ or $g$ is acyclic then so is $f \otimes g$.

**Proof.** A simplex $(x, z)$ of $X \times Z$ is outside of the image $f \otimes g$ and nondegenerate if and only if both $x$ is outside of the image of $f$ and nondegenerate and $z$ is outside of the image of $g$ and nondegenerate. This is a decidable property of simplices of $X \times Z$ and hence $f \otimes g$ is a cofibration by proposition 29.

Assume $f$ is acyclic. Let $e$ be an arbitrary fibration. The problem of filling a commutative square with $f \otimes g$ opposite to $e$ reduces to the problem of filling commutative squares with the pushout product $h \otimes k$ of a cycle inclusion $k : \partial \Delta[m] \to \Delta[m]$ and a horn inclusion $h : \Lambda_k[n] \to \Delta[n]$ opposite to $e$. By standard simplicial homotopy, $h \otimes k$ is an acyclic cofibration (see [Hov99] section 3.3, [GJ99] section I.5).

The reduction uses the pullback exponential. For each horn inclusion $h : \Lambda_k[n] \to \Delta[n]$, $e^h$ is an acyclic fibration, because the problem of lifting a cycle inclusion $k : \partial \Delta[m] \to \Delta[m]$ reduces to the problem of lifting the acyclic cofibration $h \otimes k$. Since $h \otimes g$ has the global left lifting property with respect to all fibrations, it is an acyclic fibration. For the fibration $e$ this implies that $e^g$ is a fibration, because the problem of lifting a horn $h$ against $e^g$ reduces the the problem of lifting $g$ against the acyclic fibration $e^h$. This means that $f$ has the left lifting property with respect to $e^g$. By generalization $f \otimes g$ has the left lifting property with respect to all fibrations, which means it is an acyclic cofibration.

Lemma 32 has a counterpart for pullback exponentials.

**Corollary 33.** If $e : U \to V$ is a fibration and $f : W \to X$ is a cofibration, then $e^f$ is a fibration. Moreover, if either $f$ or $e$ are acyclic, then $e^f$ is acyclic.

6.2. Deformation retracts. In sAsm weak equivalences are morphisms which factor as acyclic fibrations following acyclic cofibration. For the model structure on Kan complexes a specific type of weak equivalences plays an important role.

**Definition 34.** A deformation retract is a morphism $f : X \to Y$ with a left inverse $g : Y \to X$ and a homotopy $h : \Delta[1] \times Y \to Y$ between $\text{id}_Y$ and $f \circ g$, i.e. $h \circ (\Delta(\delta_0^1) \times \text{id}_Y) = \text{id}_Y$ and $h \circ (\Delta(\delta_1^1) \times \text{id}_Y) = f \circ g$, where $\Delta(\delta_i^1) : 1 \to \Delta[1]$ are the points of $\Delta[1]$.

**Lemma 35.** Let $f : X \to Y$ be a deformation retract with left inverse $g$ and homotopy $h$. If $f$ is a cofibration, it is an acyclic cofibration. If $g$ is a fibration, then it is an acyclic fibration.
Lemma 36. Let \( f : X \to Y \) be a deformation retract with left inverse \( g : Y \to X \) and homotopy \( h \). The split monic \( f \) factors as an acyclic fibration following an acyclic cofibration.

**Proof.** By lemma [28] \( f \) factors as an acyclic fibration \( a : Z \to Y \) and cofibration \( b : X \to Z \). The morphism \( b' = g \circ a \) is a left inverse of \( b' \), and \( a \circ b \circ b' = f \circ g \circ a \) is homotopic to \( a \) thanks to \( h \circ (\text{id}_{\Delta[1]} \times a) \). The morphism \( a \) reflects this homotopy because it is an acyclic fibration.

Let \( c_1 \) be the cycle \( 1 + 1 \to \Delta[1] \). The pushout product \( c_1 \otimes b \) is a cofibration by lemma [22]. The domain of \( c_1 \otimes b \) is \( W = \Delta[1] \times X + X + X (Z + Z) \), and there is a morphism \( d = (b \circ \pi_1, (\text{id}_Z, b \circ b')) : W \to Z \) such that \( a \circ d = h \circ (\text{id}_{\Delta[1]} \times a) \circ c_1 \otimes b \).

The filler \( \Delta[1] \times Z \to Z \) is a homotopy between \( \text{id}_Z \) and \( b \circ b' \). Lemma [35] now tells us that \( b \) is an acyclic cofibration.

There is a conventional method for factoring morphisms between Kan complexes as fibrations following deformation retracts, which allows us to prove the following proposition, which brings a model structure on Kan complexes much closer.
Proposition 37. Every morphism \( f : X \to Y \) between Kan complexes factors as a fibration following an acyclic cofibration.

Proof. Because \( !_Y : Y \to 1 \) is a fibration, the morphism \((d_0, d_1) : Y^\Delta[1] \to Y \times Y\) defined by composition with the cycle inclusion \(1 + 1 \to \Delta[1]\) is a fibration and the components \(d_i : Y^\Delta[1] \to Y\) are acyclic fibrations by corollary 39. Pulling back \((d_0, d_1)\) along \(f \times \text{id}_Y\) produces the homotopy graph \(Y/f\) of \(f\) together with projections \(f_0 : Y/f \to X\) and \(f_1 : Y/f \to Y\) where \(f_0\) is an acyclic fibration because it is the pullback of \(d_0\) and \(f_1\) is a fibration because it is the composition of the fibrations \((f_0, f_1)\) and \(X \times Y \to Y\).

There is a deformation retract \(r = (\text{id}_X, Y' \circ f) : X \to Y/f\) with \(f_1\) as left inverse. By lemma 39 \(r = g \circ h\) for some acyclic fibration \(g\) and some acyclic cofibration \(h\). This means \(f = (f_0 \circ g) \circ h\) where \(f_0 \circ g\) is a fibration and \(h\) is a cofibration. \(\Box\)

6.3. Weak equivalences. To show that weak equivalences, fibrations and cofibrations form a model structure, we now only need to show that weak equivalences satisfy 2-out-of-3, if we want to get a model structure.

Lemma 38 (2-out-of-3). Let \(f : X \to Y\) and \(g : Y \to Z\) be morphisms of sAsm. If any two of \(f, g\) or \(g \circ f\) are weak equivalences, then all three are.

Proof. Compositions of acyclic fibrations are acyclic fibrations and the same holds for acyclic cofibrations. To show that all compositions of weak equivalences are weak equivalences, we just have to show that \(g \circ f\) factors as an acyclic fibration following an acyclic cofibration when \(g\) is an acyclic cofibration and \(f\) is an acyclic fibration.

By proposition 28, \(g \circ f = h \circ k\) for some acyclic fibration \(h : W \to Z\) and a cofibration \(k : X \to W\). Since \(Y\) is fibrant, \(g\) has a left inverse \(g'\) by the lifting property. Since \(f \circ \text{id} = g' \circ g \circ f = (g' \circ h) \circ k\) there is a morphism \(k'\) such that \(f \circ k' = g' \circ h\) and \(k' \circ k = \text{id}\), so \(k\) has its own left inverse.

\[
\begin{array}{ccc}
X & \xrightarrow{k} & W \\
\downarrow{f} & & \downarrow{h} \\
Y & \xrightarrow{g} & Z \\
\downarrow{\text{id}} & & \downarrow{g'} \\
X & \xrightarrow{k'} & X \\
\end{array}
\]

The morphism \(Z^{c_1} : Z^\Delta[1] \to Z \times Z\) defined by composition with the cycle \(c_1 : 1 + 1 \to \Delta[1]\), is a fibration because \(Z\) is fibrant. The transpose of the filler of the commutative square \((\text{id}_Z, g \circ g') \circ g = Z^{c_1} \circ (Z' \circ g)\) is a homotopy \(\chi\) between \(\text{id}_Z\) and \(g \circ g'\). We lift \(\chi\) to a homotopy of \(\Delta[1] \times W \to W\) between \(\text{id}_W\) and \(k \circ k'\) using the pushout product \(c_1 \circ k\) which is a cofibration by lemma 42. The domain of \(c_1 \circ k\) is \(V = \Delta[1] \times X + X + X(W + W)\) and there is a map \(l = (k \circ \pi, \text{id}_W, k \circ k') : V \to W\) such that \(h \circ l = \chi \circ (\text{id}_\Delta[1] \times h) \circ c_1 \circ k\). The lifting property induces the homotopy
Lemma 35 tells us that $k$ is acyclic because it is a deformation retract. So at this point we know that weak equivalences are closed under composition.

The second case we tackle is where $g$ and $g \circ f$ are weak equivalences.

First assume that $g$ and $g \circ f$ are acyclic fibrations. By proposition 35, $f$ factors as an acyclic fibration $h : W \to Y$ following a cofibration $k : X \to W$. Because $(g \circ f) \circ \text{id} = (g \circ h) \circ k$ and $g \circ h$ is an acyclic fibration, $k$ has a left inverse $k' : X \to W$ which satisfies $g \circ f \circ k' = g \circ h$. The pushout product of the cycle $c_1 : 1 + 1 \to \Delta[1]$ with $k$ is a cofibration by corollary 32. The domain of $c_1 \otimes k$ is the pushout $V = \Delta[1] \times X + X + X (W + W)$, and there is a morphism $a = (k \circ \pi_1, \text{id}_W, k \circ k') : V \to W$ such that $h \circ a = h \circ \pi_1 \circ (c_1 \otimes k)$. The filler $\Delta[1] \times W \to W$ is a homotopy between $\text{id}_W$ and $k \circ k'$. By lemma 35, $k$ is an acyclic cofibration and $f$ is a weak equivalence.

Next assume that $g$ and $g \circ f$ are acyclic cofibrations. The monomorphism $g$ reduces decisions on membership and degeneracy of a simplex $y$ in $Y$ to the same questions about $g(y)$ in $Z$. Therefore $f$ is a cofibration. Because $X$, $Y$, and $Z$ are fibrant, $g$ and $g \circ f$ are deformation retracts. Construction of left inverses and homotopies of $f$ from those of $g$ and $g \circ f$ is easy. They prove that $f$ is a deformation retract and acyclic by lemma 35.

In the general case where $g$ and $g \circ f$ are general weak equivalences, we factor $f$ as an acyclic fibration $h : W \to Y$ following a cofibration $k : X \to W$. Because weak equivalences are closed under composition, $g \circ h$ is a weak equivalence. We only need to show that $k$ is acyclic. Factor both $g \circ f$ and $g \circ h$ as acyclic fibrations following acyclic cofibrations, so $g \circ f = a \circ b$ and $g \circ h = c \circ d$. The lifting properties induce a morphism $l$ such that $l \circ b = d \circ k$ and $c \circ l = a$. The morphism $l$ is a weak equivalence because $a$ and $c$ are acyclic fibrations. Because of closure under composition, the morphism $l \circ b = c \circ k$ is both a weak equivalence and a cofibration and hence an acyclic cofibration. Since $c$ is an acyclic cofibration, so is $k$.

The last case is where $f$ and $g \circ f$ are weak equivalences.

First assume that $f$ and $g \circ f$ are acyclic cofibrations. The morphism $g$ factors as an fibration $h : W \to Z$ following an acyclic cofibration $k : Y \to W$ by lemma 37. The fibration $h$ has a right inverse $h'$ which satisfies $h' \circ g \circ f = k \circ f$ because $h \circ (k \circ f) = \text{id} \circ (g \circ f)$. Because $k \circ f$ is an acyclic cofibration and $h' \circ h \circ k \circ f = k \circ f$, there is a homotopy between $h \circ h'$ and $\text{id}_W$ for the following reasons. The morphism $W^{c_1} : W^{\Delta[1]} \to W \times W$ defined by composition with the cycle $c_1 : 1 + 1 \to \Delta[1]$, is a fibration for standard reasons. There is a commutative square $W^{c_1} \circ W^{c_2} \circ k \circ f = (\text{id}_W, h' \circ h) \circ (k \circ f)$ and the transpose of the filler $W \to W^{\Delta[1]}$ is the homotopy. Since $h : W \to Z$ is the left inverse of a deformation retract, lemma 35 tells us it is an acyclic fibration.

Next assume that $f$ and $g \circ f$ are acyclic fibrations. Let $k_n$ be the cycle $\partial \Delta[n] \to \Delta[n]$. Let $a : \partial \Delta[n] \to Y$ and $b : \Delta[n] \to Y$ satisfy $b \circ k_n = g \circ a$. Because $\partial \Delta[n] \circ k$ is cofibrant, there is an $a' : \partial \Delta[n] \to X$ such that $f \circ a' = a$ and hence $(g \circ f) \circ a' = b \circ c$. There is a filler $d : \Delta[n] \to X$ for this commutative square. The morphism $f \circ d$ is a filler for the square $b \circ c_n = g \circ a$. This proves $g$ is an acyclic fibration.
In the general case where \( f \) and \( g \circ f \) are weak equivalences is now dual to the case where \( g \) and \( q \circ f \) are weak equivalences. This means that weak equivalences indeed satisfy 2-out-of-3. \( \square \)

This lead us to the next theorem of this paper.

**Theorem 39.** The category \( \text{sAsm}_f \) of Kan complexes is a model category.

### 7. Universe

This section shows that \( \text{sPER}^S \) is a complex (theorem 41) and that \( \text{sU}^S \) is univalent in \( \text{sAsm} \) (theorem 40).

#### 7.1. Fibrancy

Morphisms \( \Delta[n] \to \text{sPER} \) are transposes of functors \( \Delta/[n]^{op} \to \text{PER} \). Similarly, for each horn \( \Lambda_k[n] \) morphisms \( \Lambda_k[n] \to \text{sPER} \) are transposes of functors \( H_k[n]^{op} \to \text{PER} \). The problem is to show that any \( f : H_k[n]^{op} \to \text{PER} \) has an extension \( g \) to \( \Delta/[n]^{op} \), such that the transpose of \( g \) factors through \( \text{sPER}^S \), the category of algebras.

The lowest dimensional case where \( n = 1 \) is special. The horns \( \Delta(\delta_0^1), \Delta(\delta_1^1) : 1 \to \Delta[1] \) are split monomorphisms, because they are sections of the map \( ! : \Delta[1] \to 1 \). The map \( ! \) corresponds to the forgetful functor \( \text{dom} : \Delta/[1] \to \Delta \). We let \( \text{dom}^*(f) \) be the extension of each functor \( f : \Delta^{op} \to \text{PER} \) along either \( \delta_1^1 \). This construction corresponds to sending a modest complex \( X \) to the projection \( \Delta[1] \times X \to \Delta[1] \). This map is trivially a fibration.

We present a construction which works for all \( n > 1 \) below. Note that this construction does not always produce fibrations for \( n = 1 \), so we still need the construction above.

Let \( n > 1 \). Let \( H : H_k[n] \to \Delta/[n] \) be the inclusion. Composition determines a functor \( H^* : \text{PER}\Delta/[n]^{op} \to \text{PER}(H_k[n])^{op} \) and because \( \text{PER} \) is complete and \( H^* \) preserves all limits, this functor has a right adjoint \( H_* : \text{PER}(H_k[n])^{op} \to \text{PER}\Delta/[n]^{op} \). More importantly, \( H_* \) can be defined in such a way that it is a strict inverse of \( H^* \):

\[
H_*(f)(\phi) = \begin{cases} 
  f(\phi) & \phi \in (H_k[n])_0 \\
  \lim_{\alpha \to \phi, \alpha \in (H_k[n])_0} f(\alpha) & \phi \notin (H_k[n])_0
\end{cases}
\]

This is useful, because we are looking for an extension \( g \) of \( f \) such that \( H^*(g) = f \). Sadly, \( H_* \) corresponds to the dependent product along \( h : \Lambda_k[n] \to \Delta[n] \), which does not preserve fibrations.

The solution is that \( g(\delta_k^i) \), where \( \delta_k^i \) is the face opposite to the point \( k \), equals \( H_* f(id) \) i.e. the problematic simplices get a supporting edge over \( k \). We extend \( g \) to other objects \( \xi : [m] \to [n] \) by adding more of these supporting edges.

Define the distance of \( \xi \) to \( H_k[n] \) as follows.

\[
||\xi|| = \# \left( \prod_{i \in [n], i \neq k} p \in [m]|\xi(p) = i \right)
\]

Here, \( \# \) stands for the number of elements in this finite set. The distance \( ||\xi|| \) is the number of ways \( \delta_k^i \) factors through \( \xi \).

Next we define a functor \( K : \Delta/[n] \to \Delta/[n] \). For each object \( \xi : [m] \to [n] \) we let \( K\xi : [m + ||\xi||] \to [n] \) satisfy:

\[
K\xi(i) = \begin{cases} 
  \xi(i) & \xi(i) < k \\
  k & \xi(i) \geq k, \xi(i - ||\xi||) < k \\
  \xi(i - ||\xi||) & \xi(i - ||\xi||) \geq k
\end{cases}
\]

If we view \( \xi \) as a finite nondecreasing sequence then this functor simply adds \( ||\xi|| \) \( k \)'s to the sequence in such a way that the new sequence is still nondecreasing.
In order to define $K$ for morphisms, we introduce some extra notation. For $\xi : [m] \to [n]$ and $i \in [n]$, let $\xi_i$ be the partial ordered set $\{p \in [m] | \xi(p) = i\}$. Using ordinal arithmetic, we get the following isomorphism:

$$|m + \|\xi\|| \simeq \sum_{i < k} \xi_i + \prod_{i \neq k} \xi_i + \sum_{i \geq k} \xi_i$$

Of course, $i \in [n]$. A morphism $\phi : \xi \to \xi'$ of $\Delta/[n]$ is a sequence of $n + 1$ nondecreasing maps $\phi_i : \xi_i \to \xi'_i$ to which we apply the same construction:

$$K\phi = \sum_{i < k} \phi_i + \prod_{i \neq k} \phi_i + \sum_{i \geq k} \phi_i$$

Composition to the right defines a functor $K^* : \text{PER}^{\Delta/[n]} \to \text{PER}^{\Delta/[n]}$, which has a left adjoint $K_!$ because $\text{PER}$ has all finite colimits. To show that $K^*H_*$ preserves fibrations, we show that $K_!H^*$ preserves acyclic cofibrations.

7.2. Preservation of acyclic cofibrations. Suppose we have a horn inclusion $j : L\alpha[m] \to \Delta[m]$ and a morphism $\Delta(\xi) : \Delta[m] \to \Delta[n]$. We will first show that $K_!$ sends this horn to a monomorphism.

**Lemma 40.** Let $\delta^m_{pq} : [m - 2] \to [m]$ for $p, q \in [m], p \leq q$, be the unique nondecreasing map that only skips $p$ and $q$. Seen as subobject of $K(\xi)$, $K(\xi \circ \delta^m_{pq})$ is the intersection of $K(\xi \circ \delta^m_p)$ and $K(\xi \circ \delta^m_q)$.

**Proof.** Ordinal sums and products preserve pullbacks and therefore so does $K$. We start with the pullback square $\delta^m_q \circ \delta^m_{p-1} = \delta^m_p \circ \delta^m_{q-1}$ where both sides compose to $\delta^m_{pq}$. □

The morphism $\Delta(\xi) : \Delta[m] \to \Delta[n]$ and $\Delta(\xi) \circ j : L\alpha[n] \to \Delta[n]$ are modest, which allows us to apply $K_!$ to them.

**Corollary 41.** The domain $K_!(L\alpha[n])$ of $K_!(\Delta(\xi) \circ j)$ is a subobject of $\Delta[m + \|\xi\|]$, which is the domain of $K_!(\xi)$.

**Proof.** The effect of $K_!$ on any map $\Delta(\chi) : \Delta[m] \to \Delta[n]$ is straightforward: $K_!(\Delta(\chi)) = \Delta(K(\chi))$. Because $K_!$ is a left adjoint, it preserves colimits. The morphism $\Delta(\xi) \circ j : L\alpha[n] \to \Delta[n]$ is a colimit of the diagram which consists of the objects $\Delta(\xi \circ \delta^m_p)$ for $p \neq l$ and their intersections $\Delta(\xi \circ \delta^m_{pq})$. Since $K_!$ preserves these intersections by lemma [40], $K_!(\Delta(\xi) \circ j)$ is the union of $K_!(\Delta(\xi \circ \delta^m_{pq}))$ for $p \neq l$. □

We introduce some notation in order to describe the effect of $H^*K_!$.

1. We want to keep track of the elements which $K$ adds to the domain of $\xi : [m] \to [n]$. For this we use a nondecreasing injection $\kappa$ which sends the product $\prod_{i \neq k} \xi_i$ to the interval in $[m + \|\xi\|]$ which starts at the least $i$ such that $K\xi(i) = k$.

2. There is another injection $\lambda : [m] \to [m + \|\xi\|]$ which skips the image of $\kappa$: $\lambda(i) = i$ if $\xi(i) < k$ and $\lambda(i) = i + \|\xi\|$ if $\xi(i) \geq k$.

3. We extend the face notation. For each $U \subseteq [m + \|\xi\|]$ let $\delta(U)$ be the face of $\Delta[m + \|\xi\|]$ which is opposite to all points in $U$.

4. To apply $\kappa$ and $\lambda$ to all elements of a subset of their domains we use $\exists_\kappa$ and $\exists_\lambda$.

We can describe the action of $K_!$ using unions of faces. Let $A = \exists_\kappa(\prod_{i \neq k} \xi_i)$ and for each $q \in [m]$ let:

$$A_q = \{ \kappa(p) \in A | \xi(q) \neq k, p_{\xi(q)} = q \}$$

The set $A_q$ contains the supporting edges of $\xi$ which are not supporting edges for $\xi \circ \delta^m_q$. Therefore, the functor $K$ sends the face $\delta^m_p : \xi \circ \delta^m_p \to \xi$ to $\delta(A_p \cup \{ \lambda(p) \})$. 

REALIZABILITY OF UNIVALENCE MODEST KAN COMPLEXES 17
Preserved unions imply
\[ K_i(\Lambda_k[m]) = \bigcup_{p \neq i} \delta(A_p \cup \{\lambda(p)\}) \]

The effect of \( H^* \) is also easy to describe in terms of unions of faces.

\[
\begin{align*}
H^* K_i(\Delta[m]) &= \bigcup_{i \neq k} \delta(\exists_{\lambda}(\xi_i)) \\
H^* K_i(\Lambda_i[m]) &= \bigcup_{i \neq k \atop p \neq l} \delta(\exists_{\lambda}(\xi_i) \cup A_p \cup \{\lambda(p)\})
\end{align*}
\]

We first proof a technical lemma about acyclic cofibrations.

**Lemma 42** (Face completion). Let \( F \) be an inhabited decidable set of faces of \( \Delta[p] \) which all have an edge \( e \) in common. The inclusion \( \bigcup F \to \Delta[p] \) is an acyclic cofibration.

**Proof.** For all \( j \in [p] \) let \( F_j \) be the union of \( F \) with the set of \( j \)-dimensional faces of \( \Delta[p] \) which contain the edge \( e \). Because \( F \) is inhabited, \( e \in \bigcup F \) and hence \( F_0 = F \). Because \( \Delta[p] \) is a \( p \)-dimensional face of \( \Delta[p] \) which contains \( e \), \( \bigcup F_p = \Delta[p] \). For \( j > 0 \) let \( S_j \) be the set of \( j \)-dimensional faces of \( \bigcup F_j \) which are not already contained in \( \bigcup F_{j-1} \). If a \( j \)-dimensional face \( \delta(\Sigma) \) of \( \bigcup F_j \) opposes \( e \) it is part of a higher dimensional face which is a member of \( F \). Therefore each face \( \delta(\Sigma) \in S_j \) contains \( e \). For this reason \( \delta(\Sigma) \cap \bigcup F_{j-1} \) is the horn whose central edge is \( e \). The inclusion \( \bigcup F_{j-1} \to \bigcup F_j \) is therefore the pushout of a coproduct of horn inclusions indexed over \( S_j \) and therefore an acyclic cofibration. Because acyclic cofibrations are closed under composition, \( \bigcup F = F_0 \to F_p = \Delta[p] \) is an acyclic cofibration. \( \square \)

**Lemma 43** (Descent). The inclusion \( H^* K_i(\Lambda_i[m]) \to H^* K_i(\Delta[m]) \) is an acyclic cofibration.

**Proof.** If \( ||\xi|| = 0 \) and hence \( \prod_{i \neq k} \xi_i = 0 \), then neither \( K_i \) nor \( H^* \) change anything about the horn \( \Lambda_i[m] \to \Delta[m] \), so we only need to worry about the cases where \( ||\xi|| > 0 \).

We will add the faces \( U_i = \delta(\exists_{\lambda}(\xi_i)) \) of \( H^* K_i(\Delta[m]) = \bigcup_{i \neq k} \delta(\exists_{\lambda}(\xi_i)) \), saving the most difficult case \( i = \xi(l) \) for last.

Note that because \( n > 1 \), there are always \( i \in [n] \) such that \( i \neq k \), \( i \neq \xi(l) \). The intersection of \( H^* K_i(\Lambda_i[m]) \) with \( \delta(\exists_{\lambda}(\xi_i)) \) is:

\[
H^* K_i(\Lambda_i[m]) \cap U_i = \bigcup_{p \neq l} \delta(\exists_{\lambda}(\xi_i) \cup A_p \cup \{\lambda(p)\})
\]

The set \( F = \{\delta(\exists_{\lambda}(\xi) \cup A_p \cup \{\lambda(p)\})|p \neq l\} \) is inhabited and decidable. Each face in \( F \) contains the edge \( \lambda(l) \). Hence the inclusion \( H^* K_i(\Lambda_i[m]) \cap U_i \to U_i \) is an acyclic cofibration by lemma \[42\]

Let \( L \) be the union of \( H^* K_i(\Lambda_i[m]) \) with \( U_i \) for all \( i \in [n] - \{k, \xi(l)\} \). The inclusion \( H^* K_i(\Lambda_i[m]) \to L \) is a pushout of a coproduct of acyclic cofibrations indexed over \( [n] - \{k, \xi(l)\} \) and hence is an acyclic cofibration. If \( \xi(l) = k \), then \( L = H^* K_i(\Lambda_i[m]) \) and we are done. Otherwise we still have to deal with the face \( \delta(\exists_{\lambda}(\xi(l)) \cup \{p\} \cup A_p) \). If \( p \neq l \) these are faces of \( U_{\xi(l)} \) which are part of \( L \). Hence \( L \) is the following union of faces.

\[
L = \left( \bigcup_{i \in [n] - \{k, \xi(l)\}} U_i \right) \cup \left( \bigcup_{p \in [m] - \{l\}} V_p \right)
\]

For each \( \tilde{p} \in \prod_{i \in [n] - k, \xi(l)} \xi_i \) let \( B_{\tilde{p}} = \{ q \in [m] | \xi(q) \in [n] - k, \xi(l), q > p_{\xi(q)} \} \) and let \( W_{\tilde{p}} = \delta(\exists_{\lambda}(\xi(l)) \cup B_{\tilde{p}}) \). For all \( j \in \mathbb{N} \), let \( L_j = L \cup \bigcup_{(\tilde{p} | p_j < j} W_{\tilde{p}} \). By this definition
\( L_0 = L \) and \( L_{n+\|\xi\|+1} = H^* K_\xi \Delta[m] \) because \( B_{\vec{p}} = \emptyset \) and therefore \( W_{\vec{p}} = U_{\xi | l} \) if \( p_i \) are the maximal elements of \( \xi \) for each \( i \in [n] - \{ k \} \).

For every \( j \in \mathbb{N} \) the inclusion \( L_j \to L_{j+1} \) is an acyclic cofibration for the following reasons.

As long as \( K_\xi(j) < k \), \( L_j = L \) because \( \bigcup_{\kappa(\vec{p}) < j} W_{\vec{p}} \) is empty. If \( K_\xi(j + 1) < k \) too, then \( L_j \to L_{j+1} \) is an acyclic cofibration because it is an identity.

If \( K_\xi(j + 1) = k \) then \( j + 1 = \kappa(\vec{p}) \) or \( j + 1 = \lambda(\vec{p}) \) for \( p \in \xi_k \). We first consider the case that \( j + 1 = \kappa(\vec{p}) \). If \( \vec{p}, \vec{q} \in \prod_{i \neq k} \xi_j \) and \( p_i \leq q_i \) for all \( i \in [n] - k \), then \( \vec{p} \leq \vec{q} \) in the lexicographical order of the ordinal product and hence \( \kappa(\vec{p}) \leq \kappa(\vec{q}) \).

Therefore \( W_{\vec{p}} \subseteq L_{\kappa(\vec{q})} \). For that reason, the intersection \( L_j \cap W_{\vec{p}} \) is the union of the following families of faces.

\[
\begin{align*}
U_i \cap W_{\vec{p}} &= \delta(\exists \xi(\xi_i \cup \xi(l) \cup B_{\vec{p}})) \quad \text{for } i \in [n] - \{ k \} \{ \xi(l) \} \\
V_q \cap W_{\vec{p}} &= \delta(\exists \xi(\xi_i \cup \{ p \} \cup B_{\vec{p}})) \quad \text{for } q \in [m] - \{ l \} \\
W_{\vec{p}} \cap W_{\vec{q}} &= \delta(\exists \xi(\xi_i \cup B_{\vec{p}} \cup B_{\vec{q}})) \quad \text{for } \kappa(\vec{q}) < \kappa(\vec{p})
\end{align*}
\]

Let \( \vec{p}[l] \in \prod_{i \neq k} \xi_l \) satisfy \( \vec{p}[l]_i = l \) if \( i = \xi(l) \) and \( \vec{p}[l]_i = p_i \) if \( i \neq \xi(l) \). The intersection \( L_j \cap W_{\vec{p}} \) is a union of faces which contain the supporting point \( \kappa(\vec{p}[l]) \) for the following reasons. The supporting edge \( \kappa(\vec{p}[l]) \) is a member of \( U_i \) and \( W_{\vec{p}} \) because those faces are only opposite to edges of the images of \( \lambda \). The faces \( V_q \) contains \( \kappa(\vec{p}[l]) \) if \( \kappa(\vec{p}) \notin A_q \). If \( \xi(q) = k \), then \( A_q = \emptyset \) and if \( \xi(q) = \xi(l) \), then \( \kappa(\vec{p}[l]) \notin A_q \) because \( \kappa(\vec{p}[l]) \in A_l \) and \( A_l \) and \( A_q \) are disjoint because \( q \neq l \). Otherwise, \( q = p_{\xi(l)} \) by definition of \( A_q \).

Either \( \xi(q - 1) = \xi(q) \) or \( q \) is the least member of \( \xi(q) \). If \( \xi(q - 1) = \xi(q) \) let \( \vec{p}[q - 1] \in \prod_{i \neq k} \xi_l \) satisfy \( \vec{p}[q - 1]_l = q = \xi(l) \) and \( \vec{p}[q - 1]_i = p_i \) if \( i \neq \xi(l) \) for some \( p \in \xi_k \). By definition, \( B_{\vec{p}[q - 1]} = B_{\vec{p}} \cup \{ q \} \) and therefore \( V_q \cap W_{\vec{p}} \subseteq W_{\vec{p}[q - 1]} \). As noted before \( \kappa(\vec{p}[q - 1]) < \kappa(\vec{p}) \), so \( W_{\vec{p}[q - 1]} \subseteq L_j \). If \( q \) is the least member of \( \xi(q) \) then \( \xi(q) \subseteq B_{\vec{p}} \cup \{ q \} \) and therefore \( V_q \cap W_{\vec{p}} \subseteq U_{\xi(q)} \).

In all cases where \( V_q \) opposes \( \kappa(\vec{p}[l]) \), some other face of \( L_j \cap W_{\vec{p}} \) contains both \( V_q \) and \( W_{\vec{p}} \). Therefore \( L_j \cap W_{\vec{p}} \) is a union of faces which contain the supporting edge \( \kappa(\vec{p}[l]) \). By lemma \( \square \) \( L_j \cap W_{\vec{p}} \to W_{\vec{p}} \) is an acyclic cofibration and so is \( L_j \to L_{j+1} \).

As \( \kappa(\vec{p}) \) grows, \( B_{\vec{p}} \) shrinks to an empty set. By the time \( j + 1 = \lambda(p) \) for some \( p \in \xi_k \), \( L_j = H^* K_\xi \Delta[m] \) and \( L_j \to L_{j+1} \) is the identity. The same holds for all \( L_j \to L_{j+1} \) where \( K_\xi(j + 1) > k \) and where \( j + 1 > m + \| \xi \| \).

Since acyclic cofibrations are closed under composition, \( L \to H^* K_\xi \Delta[m] \) and \( H^* K_\xi \Delta[m] \to H^* K_\xi \Delta[m] \) are acyclic cofibrations.

The seemingly classical reasoning above is actually constructive because we are working with finite and decidable sets of number–or equivalently with functions \( \mathbb{N} \to \{ 0, 1 \} \)–in every case. This makes it work in \textit{sAsm}.

Theorem 44. The simplicial assembly \( \text{sPER}^S \) is a complex.

Proof. The two lemmas above show that each map \( f : \Lambda_k [n] \to \text{sPER}^S \) has an extension \( K^* H f : \Delta[n] \to \text{sPER}^S \). The construction is sufficiently constructive to turn into an algebra structure \( S(\text{sPER}^S) \to \text{sPER}^S \).

7.3. Univalence. A fibration \( f : X \to Y \) is univalent if weakly equivalent pullbacks of \( f \) are homotopic. For the generic modest fibration \( \sigma : \bar{U} \to U \) this means we can turn a weak equivalence of modest complexes \( w : X \to Y \) into a modest fibration \( W \to \Delta[1] \) such that the pullbacks \( \Delta(\delta_0^1)^* (W) = Y \) and \( \Delta(\delta_1^1)^* (W) = X \)–such a fibration is a correspondence between \( X \) and \( Y \). More precisely, for each pair of functors \( f,g : \Delta^{op} \to \text{PER} \) which correspond to modest complexes and each natural transformation \( w : f \to g \) which corresponds to a weak equivalence there should be a functor \( h : \Delta/[1]^{op} \to \text{PER} \) such that the composition with the constant inclusions \( (\delta_1^1) : \Delta^{op} \to \Delta/[1]^{op} \) are equal to \( f \) and \( g \).
The objects of $\Delta/[1]$ are morphisms $\chi : [n] \to [1]$. We can think of these morphisms as pairs $([n], [i])$ for $i \geq -1$, $i \leq n$, where $[i] \subseteq [n]$ is an initial segment and $[-1]$ stands for the empty initial segment. A morphism $\phi : \chi \to \chi'$ corresponds to a pair of morphism $(\phi : [n] \to [n'], \psi : [i] \to [i'])$ where $\psi$ is a restriction of $\phi$.

For each pair $f, g : \Delta^{op} \to \text{PER}$ and each natural transformation $\iota : f \to g$, we define the \textit{homotopy cograph} $\text{hcg}(\iota) : \Delta/[1]^{op} \to \text{PER}$ as follows. On objects:

$$\text{hcg}(\iota)([n], [i]) = \begin{cases} 
\{ (x, y) \in g(n) \times f(i) | x \cdot [i] = \iota(y) \} & i = n \\
\left\{ f([n]) \right\} & i = -1 \\
g([n]) & \text{otherwise}
\end{cases}$$

Here $x \cdot [i]$ is the restriction of $x$ along the inclusion $[i] \to [n]$ of $[i]$ as an initial segment of $[n]$.

Let $(\phi, \psi) : ([m], [i]) \to ([n], [j])$ be any morphism. Note that the following restrictions apply: if $j = -1$, then $i = -1$; if $j = n$, then $i = n$. The reason is that $(\delta^1_0, \delta^1_1) : 1 + 1 \to \Delta[1]$ induces a sieve on $\Delta/[1]$. With those in mind, let:

$$\text{hcg}(\iota)(\phi, \psi) = \begin{cases} 
\left\{ f(\psi) \right\} & j = n \\
\left\{ f(\psi) \circ \pi_1 \right\} & -1 < j < n, i = m \\
g(\phi) \times f(\psi) & -1 < i < m \\
g(\phi) \circ \pi_0 & -1 < j < n, i = 0 \\
g(\phi) & j = 0
\end{cases}$$

The graph has $f$ gradually fading out as $i$ counts down to 0. The functor $\text{hcg}(\iota) : \Delta/[1]^{op} \to \text{PER}$ satisfies $\text{hcg}(\iota)(\delta^1_0) = f$ and $\chi(\iota)(\delta^1_1) = g$. Therefore it induces a modest fibration $\text{hcg}(\iota)^*(\bar{\mu}) : Z \to \Delta[1]$ such that $f^*(\bar{\mu})$ and $g^*(\bar{\mu})$ are pullbacks along $\Delta(\delta^1_1)$.

\textbf{Lemma 45.} If $\iota : f \to g$ induces a weak equivalence of modest fibrations, then $\text{hcg}(\iota)$ factors through $\text{sPER}^S$.

\textit{Proof.} We start with an analysis of what happens to horns in this construction. Let $n > 0$ and $i, k \leq n$. These are the indices for the horn $\Lambda_k[n] \to \Delta[n]$ together with a morphism $\Delta(([n], [i])) : \Delta[n] \to \Delta[1]$.

If we pullback the horn along $\Delta(\delta^1_1) : 1 \to \Delta[1]$, then we get a monomorphism which we express as an inclusion unions of faces of $\Delta([n])$ to get the following formula:

$$\bigcup_{q \leq n, q \neq k} \delta(\{ p \in [n] | i < p \}) \cup \{ q \} \to \delta \{ p \in [n] | i < p \}$$

(1) For $i < n - 1$, both sides are the face $\delta \{ p \in [n] | i < p \}$, because both $\{ p \in [n] | i < p \} \cup \{ n \}$ and $\{ p \in [n] | i < p \} \cup \{ n - 1 \}$ equal $\{ p \in [n] | i < p \}$ and $k$ is unequal to either $n - 1$ or $n$. Hence the pullback of the horn is the acyclic cofibration $\text{id}_{\Delta[q]} : \Delta[i] \to \Delta[i]$, or, if $i = -1$ the acyclic cofibration $\text{id}_0 : 0 \to 0$.

(2) For $i = n - 1$ and $k < n$, both sides are the face $\delta \{ n \}$ and the pullback of the horn is the acyclic cofibration $\text{id}_{\Delta[n-1]} : \Delta[n-1] \to \Delta[n-1]$. For $k = n$ however, we find that the pullback is the cycle $\partial \Delta[n-1] \to \Delta[n-1]$.

(3) For $i = n$ the pullback is all of $\Lambda_k[n] \to \Delta[n]$.

With this analysis we can tackle the problem of lifting horns along $\text{hcg}(\iota)^*(\bar{\mu}) : Z \to \Delta[1]$ and this hinges on the morphism $([n], [i]) : \Delta[n] \to \Delta[i]$ just like the definition of $\text{hcg}(\iota)$ does.

(1) if $i = -1$, then filling $\Lambda_k[n] \to \Delta[n]$ in $\text{hcg}(\iota)^*(\bar{\mu})$ reduces to filling it in $g^*(\bar{\mu})$.

(2) if $i = n$, then filling $\Lambda_k[n] \to \Delta[n]$ in $\text{hcg}(\iota)^*(\bar{\mu})$ reduces to filling it in $f^*(\bar{\mu})$. 
(3) if $-1 < i < n$, then filling $\Lambda_k[n] \to \Delta[n]$ in $\text{hcg}(\iota)^*(\overline{\mu})$ reduces to filling it in $g^*(\overline{\mu})$ and filling its pullback along $\Delta([n], [i])$ in $f^*(\overline{\mu})$ in a commutative way.

We can do 1 and 2 because $f, g$ are complexes. The third 3 is mostly not a problem, because the pullbacks are identities. There is just one exception: $k = n$ and $\Delta([n], [n-1])$. To fill this cycle in $f$, in a way that is consistent with the filler of the horn in $g$, we use the fact that $\iota$ factors an acyclic fibration following an acyclic fibration. For these two types of maps, we have an elegant solution.

Firstly assume that $\iota$ is an acyclic fibration. When we fill $\Lambda_n[n]$ in $g^* (\vec{\mu})$, we get an $n-1$ simplex opposite to the edge $n$. The cycle $\partial \Delta[n-1] \to \Delta[n-1]$ commutes with the acyclic fibration $\iota^*(\overline{\mu})$, which means there is a filler for is. That way we fill $\partial \Delta[n-1]$ in $f^*(\overline{\mu})$ in a way that commutes with the filler of $\Lambda_n[n]$ in $Y$.

Secondly assume that $\iota$ is an acyclic cofibration. Because $f^*(\overline{\mu})$ is a complex, $\iota$ has a right inverse $\iota'$. After filling the horn $\Lambda_n[n]$ in $Y$ we use this right inverse to find a suitable filler in $X$.

Since every weak equivalence $\iota$ factors as an acyclic fibration following an acyclic cofibration, $\text{hcg}(\iota)$ is a Kan complex. □

**Theorem 46.** The generic modest fibration $U^S$ is univalent.

_Proof._ For every pair $f, g : X \to \text{sPER}^S$ and each homotopy equivalence $h : f^*(U^S) \to g^*(U^S)$ we get a homotopy $k : \Delta/[1] \times X \to \text{sPER}^S$ between $f$ and $g$ by applying the $\text{hcg}$ construction pointwise for the various points of $X$. □

### 8. Conclusion

We have established that there is a univalent generic modest Kan fibration in the model category of Kan complexes $\text{sAsm}_f$. This may be a useful model of homotopy type theory.

Before ending this paper, want want to add a few interesting observations.

8.1. **Dependent types.** The following lemma show that we can interpret dependent types in the category of Kan complexes $\text{sAsm}_f$.

**Proposition 47.** Fibrations are closed under dependent products up to weak equivalence.

_Proof._ Acyclic cofibration are stable under pullback along fibrations with cofibrant domains for the following reasons. Cofibrant objects are simplicial assemblies where degeneracy is decidable. Cofibrations are locally decidable monomorphisms and locally decidable monomorphisms with cofibrant domains are cofibrations by proposition 29. Therefore the pullback of a cofibration along a morphism with a cofibrant domain is a cofibration. Since $\text{sAsm}_f$ is proper, pullbacks along fibrations preserve
weak equivalences and pullbacks along fibrations with cofibrant domains preserve acyclic cofibrations. This implies that dependent products of fibrations along fibrations with cofibrant domains are fibrations.

For each morphism \( f : X \to Y \) splits as a fibration \( f_1 : Y/f \to Y \) following a deformation retract \( r : X \to Y/f \) with inverse \( f_1 \) and \( Y/f \) has a cofibrant replacement \( c : Z \to Y/f \). The homotopical dependent product of a fibration \( g : W \to X \) along \( f \) is \( \prod_{f_1} (g) = \prod_{f_1 \circ c}(f_1 \circ c)(g) \). □

8.2. Exact completions. The constructions in this paper allow us to show that toposes can have complete internal categories in a strong sense.

**Proposition 48.** The exact completion \( \text{Asm}_{\text{ex}} \) of \( \text{Asm} \) . . . 

1. It is a topos.
2. It has a generic modest morphism.

**Proof.** For 1. see [HvO03]. In order to show 2., we a category of truncated simplicial assemblies.

Let \( \Delta_0 \) be the full subcategory of \( \Delta \) on the objects \([0]\) and \([1]\). The category \( \text{Asm}^{\Delta_0} \) is the category of 0-truncated \( n \)-simplices.

Let \( Z : \Delta_0 \to \Delta \) be the inclusion functor. It induces a functor \( Z^* : \text{sAsm} \to \text{Asm}^{\Delta_0} \). If we apply \( Z^* \) to cycle and horn inclusions we find that the higher dimensional ones become isomorphisms. The only cycle inclusions are which don’t turn into isomorphisms are \( 0 \to 1, 1 + 1 \to \Delta[1] \). For horn inclusions we are left with \( \Delta(\delta^1_0) : 1 \to \Delta[1] \) and \( \Lambda_1[2] \to \Delta[2] \).

The injective objects, i.e. the complexes, for these 5 horn inclusions are precisely the pseudoequivalence relations of [CV98]. The category \( \text{Asm}_{\text{ex}} \) therefore has the same objects as the full subcategory of complexes \( \text{Asm}^{\Delta_0}_f \subseteq \text{Asm}^{\Delta_0} \). Morphisms of \( \text{Asm}_{\text{ex}} \) are equivalence classes of morphisms in \( \text{Asm}^{\Delta_0}_f \) for the relation of being homotopic.

We have a generic modest injective in \( \text{Asm}^{\Delta_0}_f \) by theorem [22] Pullbacks of fibrations are preserved by the quotient functor \( Q : \text{Asm}^{\Delta_0}_f \to \text{Asm}_{\text{ex}} \) because they are examples of homotopy pullbacks [GJ99]. Therefore \( Q(\text{sU}^S) \) is a generic family of modest sets in \( \text{Asm}_{\text{ex}} \).

**Remark 49** (Lifting power revisited). If we use the local rather than the global lifting property (see remark [15]) in the definition of fibrations in \( \text{Asm}^{\Delta_0} \), then the homotopy category is equivalent to the effective topos. The effective topos has no generic modest morphism [HRR90].

8.3. Conjectures of independence. The only thing really peculiar to \( \text{Asm} \) that makes the proofs in this paper work are the modest morphisms. A similar class of morphisms exists in the categories of assemblies for other partial combinatory algebras [vO08]. Further generalizations to relative realizability seem unproblematic and I am not sure if order partial combinatory algebras cause any trouble. Moreover, the category of sets way in the background could be any locally Cartesian closed pretopos.

The model structure does not require

8.4. Building factorization systems. A nice way to get an anodyne-injective factorization system is to generate a free monoid from the functor \( S \) in subsection [4.2] in the monoidal category of functors \( \text{sAsm} \to \text{sAsm} \). Garner shows that this is possible in the category of simplicial sets in [Gar12], which is a reflective subcategory of \( \text{sAsm} \). The algebraic completeness of \( \text{sPER} \) makes it easy to find the free monoid of \( S \) there (see remark [21]). I could not figure out how to combine these two constructions into one.
Acknowledgments. I am grateful to the Warsaw Center of Mathematics and Computer Science for the opportunity to write this paper. I am also grateful for discussions with Marek Zawadowski and the seminars on simplicial homotopy theory he organized during my stay at Warsaw University. Richard Garner, Peter LeFanu Lumsdaine and Thomas Streicher made invaluable comments on a early draft of this paper.

References

[AP90] M. Abadi and G. D. Plotkin. A per model of polymorphism and recursive types. In Fifth Annual IEEE Symposium on Logic in Computer Science (Philadelphia, PA, 1990), pages 355–365. IEEE Comput. Soc. Press, Los Alamitos, CA, 1990.

[CV98] Aurelio Carboni and E. M. Vitale. Regular and exact completions. J. Pure Appl. Algebra, 125(1-3):79–116, 1998.

[Fre89] Peter J. Freyd. POLYNAT in PER. In Categories in computer science and logic (Boulder, CO, 1987), volume 92 of Contemp. Math., pages 67–68. Amer. Math. Soc., Providence, RI, 1989.

[Fre91] Peter J. Freyd. Algebraically complete categories. Lecture notes in mathematics, 1488:95–104, 1991.

[Gar12] Richard Garner. Understanding the small object argument. Appl. Categ. Structures, 20(2):103–141, 2012.

[Gar99] Paul G. Goerss and John F. Jardine. Simplicial Homotopy Theory, volume 174 of Progress in Mathematics. Birkhäuser, Basel, 1999.

[Hov99] Mark Hovey. Model Categories, volume 63 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, 1999.

[HRR90] J. M. E. Hyland, E. P. Robinson, and Guiseppe Rosolini. The discrete objects in the effective topos. Proc. London Math. Soc. (3), 60(1):1–36, 1990.

[HvO03] Pieter J. W. Hofstra and Jaap van Oosten. Ordered partial combinatory algebras. Math. Proc. Cambridge Philos. Soc., 134(3):445–463, 2003.

[Ros90] Giuseppe Rosolini. About modest sets. Internat. J. Found. Comput. Sci., 1(3):341–353, 1990. Third Italian Conference on Theoretical Computer Science (Mantova, 1990).

[Rum04] Ivar Rummelhoff. Polynat in PER models. Theoret. Comput. Sci., 316(1-3):215–224, 2004.

[Ste13a] Wouter Pieter Stekelenburg. Realizability Categories. PhD thesis, Utrecht University, 2013.

[Ste13b] Wouter Pieter Stekelenburg. Regular functors and relative realisability categories. Mathematical Structures in Computer Science, FirstView:1–29, 5 2013.

[vO08] Jaap van Oosten. Realizability: an introduction to its categorical side, volume 152 of Studies in Logic and the Foundations of Mathematics. Elsevier B. V., Amsterdam, 2008.

FACULTY OF MATHEMATICS, INFORMATICS AND MECHANICS, UNIVERSITY OF WARSAW, BANacha 2, 02-097 Warszawa, Poland
E-mail address: w.p.stekelenburg@gmail.com