A NOTE ON 3D-1D DIMENSION REDUCTION WITH DIFFERENTIAL CONSTRAINTS

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ABSTRACT. Starting from three-dimensional variational models with energies subject to a general type of PDE constraint, we use Γ-convergence methods to derive reduced limit models for thin strings by letting the diameter of the cross section tend to zero. A combination of dimension reduction with homogenization techniques allows for addressing the case of thin strings with fine heterogeneities in the form of periodically oscillating structures. Finally, applications of the results in the classical gradient case, corresponding to nonlinear elasticity with Cosserat vectors, as well as in micromagnetics are discussed.

1. Introduction. In [1], Acerbi, Buttazzo & Percivale give the first rigorous derivation of a lower dimensional theory for thin (almost one-dimensional) strings using an ansatz-free approach based on variational methods. The work by Le Dret & Raoult [25], which can be viewed as the analogue for objects that are thin in one space dimension, relies on Γ-convergence techniques as well, and has brought up a two-dimensional nonlinear membrane model.

This article focuses on strings and provides an abstract dimension reduction result, which has the potential of covering a number of different applications in continuum mechanics and electromagnetism, like hyperelasticity, micromagnetics or magnetoelasticity. Mathematically, this is achieved by working with energy functionals of integral form whose admissible functions solve a PDE constraint conveyed by a first-order differential operator made precise in (3). For a discussion of the analogous problem in the context of thin films we refer to [19, 20, 22]. The literature on 3d-1d dimension reduction, also with different scalings of the energy leading to models for beams and rods, includes e.g. [24, 29, 28, 34, 33, 11].

For $\varepsilon > 0$, $d > 1$, $L > 0$, and $\omega \subset \mathbb{R}^{d-1}$ a sufficiently smooth (e.g. Lipschitz) domain with $0 \in \omega$, let $\Omega_\varepsilon = \{ x = (x', x_d) \in \mathbb{R}^d : x' \in \varepsilon \omega, x_d \in (0, L) \} \subset \mathbb{R}^d$ model the reference configuration of a string with cross section $\varepsilon \omega$. Without loss of...
Our objective is to analyze the asymptotic behavior as $\varepsilon \to 0$ of variational problems

$$I_{\varepsilon} \to \min \text{ in } X_{\varepsilon},$$

where $X_{\varepsilon} = L^p(\Omega_{\varepsilon}; \mathbb{R}^m)$ with $p \in (1, \infty)$, and $I_{\varepsilon}$ is an integral functional restricted to $A$-free vector fields on $\Omega_{\varepsilon}$. Precisely,

$$I_{\varepsilon}(v) = \frac{1}{\varepsilon^{d-1}} \int_{\Omega_{\varepsilon}} g_{\varepsilon}(y, v(y)) \, dy$$

if $Au = 0$ in $\Omega_{\varepsilon}$ (in the sense of distributions), and $I_{\varepsilon} = \infty$, otherwise in $L^p(\Omega_{\varepsilon}, \mathbb{R}^m)$. Regarding the integrands, we require $g_{\varepsilon} : \Omega_{\varepsilon} \times \mathbb{R}^m \to [0, \infty)$ to be Carathéodory (i.e. $g_{\varepsilon}(\cdot, \xi)$ is measurable for all $\xi \in \mathbb{R}^m$ and $g_{\varepsilon}(y, \cdot)$ is continuous for almost all $y \in \Omega_{\varepsilon}$) with uniform (in $\varepsilon$) $p$-growth and $p$-coercivity. Moreover, let $A$ be a linear first-order differential operator with constant coefficients that can be represented as

$$A = \sum_{k=1}^{d} A^{(k)} \partial_k$$

with matrices $A^{(k)} \in \mathbb{R}^{1 \times m}$ for $k = 1, \ldots, d$, and satisfies Murat’s constant-rank property, which has its origins in the theory of compensated compactness [31, 32, 35], and says that for the symbol $\Lambda$ of $A$,

$$\text{rank} \Lambda(\eta) = \text{rank} \left( \sum_{k=1}^{d} A^{(k)} \eta_k \right) = \text{const.} \quad \text{for all } \eta \in \mathbb{R}^d, \eta \neq 0.$$

If, for example, $A = \text{curl}$ (and $\Omega_{\varepsilon}$ is simply connected, so that curl-free fields on $\Omega_{\varepsilon}$ have a gradient representation), then $I_{\varepsilon}$ in (2) can be interpreted as an integral functional depending on gradients, and hence, as the stored energy of a deformation, such as stretching or shearing, acting on an elastic string. Solenoidal fields are admissible for (1) if one takes $A = \text{div}$. A connection to ferromagnetism becomes apparent by using $A = A^{\text{mag}}$ as defined later in (24).

The variational theory of integral problems with differential constraints as above was essentially established by Fonseca & Müller [14], building amongst others on work by Dacorogna [9]. Further problems in the $A$-free setting, including relaxation and homogenization, are investigated for example in [7, 12, 13, 3].

A suitable rescaling of (1) helps to lose the parameter-dependence of $\Omega_{\varepsilon}$, and thus of $X_{\varepsilon}$. By choosing the change of variables

$$y = (y', y_d) = (\varepsilon x', x_d),$$

with $y' = (y_1, \ldots, y_{d-1})$ and defining $u(x) = v(y)$ for $x \in \Omega_1 = \omega \times (0, 1)$, we obtain

$$E_{\varepsilon}(u) = \left\{ \begin{array}{ll}
\int_{\Omega_1} f_{\varepsilon}(x, u(x)) \, dx & \text{if } Au = 0 \text{ in } \Omega_1, \\
\infty & \text{otherwise,}
\end{array} \right.$$

where the transformed density functions $f_{\varepsilon} : \Omega_1 \times \mathbb{R}^m \to [0, \infty)$ are defined as

$$f_{\varepsilon}(x, \xi) = g_{\varepsilon}(\varepsilon x', x_d, \xi)$$

for $x \in \Omega_1$ and $\xi \in \mathbb{R}^m$, and satisfy

$$c|\xi|\varepsilon - C \leq f_{\varepsilon}(x, \xi) \leq C(1 + |\xi|^p)$$

for all $\xi \in \mathbb{R}^m$, almost all $x \in \Omega_1$, with constants $c, C$ independent of $\varepsilon$. The rescaled differential operator in (6) reads

$$A_{\varepsilon} = \frac{1}{\varepsilon} A' \nabla' + A^{(d)} \partial_d,$$

with $\nabla' u := (\partial_1 u, \partial_2 u, \ldots, \partial_{d-1} u) \in \mathbb{R}^{m(d-1)}$ for any $u \in C^1(\mathbb{R}^d; \mathbb{R}^m)$.
Note that the parameter transformation makes the operator in the PDE constraint dependent (see Figure 1). This amounts to a technical difficulty, since the classical projection onto $A_\varepsilon$-fields may lead to diverging projection errors as $\varepsilon$ gets small.

The main theorem of this paper on a reduced model for homogeneous strings is formulated in the framework of $\Gamma$-convergence, for an introduction the reader is referred to [10, 6]. We remark that the following statement requires the operator $A$ to fulfill the two assumptions $A[1]$ and $A[2]$ which are made precise in Section 2.1 and hold for all the examples mentioned above.

**Theorem 1.1 (Dimension reduction).** Let $A$ be a constant-rank operator according to (4) such that Assumptions $A[1]$ and $A[2]$ are satisfied, and let $f_\varepsilon = f$ for all $\varepsilon > 0$ with a Caratheodory function $f : \Omega_1 \times \mathbb{R}^m \to [0, \infty)$ of $p$-growth and $p$-coercivity.

(i) If $f$ is convex in the second variable, then

$$
\lim_{\varepsilon \to 0} E_\varepsilon(u) = \begin{cases} 
\int_{\Omega_1} f(x, u(x)) \, dx & \text{if } A_0 u = 0 \text{ in } \Omega_1, \\
\infty & \text{else.}
\end{cases}
$$

The $\Gamma$-limit is taken with respect to weak convergence in $L^p(\Omega_1; \mathbb{R}^m)$, and $A_0$ as in (12) is the limit operator of $(A_\varepsilon)$ for $\varepsilon \to 0$.

(ii) If $f$ is only continuous in the second variable, there is the following upper bound: For every $u \in L^p(\Omega_1; \mathbb{R}^m) \cap \ker_{\Omega_1} A_\varepsilon$ there exists a sequence $u_\varepsilon \in L^p(\Omega_1; \mathbb{R}^m) \cap \ker_{\Omega_1} A_\varepsilon$ ($\varepsilon > 0$) such that $u_\varepsilon \to u$ in $L^p(\Omega_1; \mathbb{R}^m)$ with

$$
\limsup_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) \leq \int_{\Omega_1} Q_A f(x, u(x)) \, dx.
$$

Here, $Q_A f$ denotes the $A$-quasiconvex envelope of $f$ regarding the second variable (cf. [13]), i.e.

$$
Q_A f(x, \xi) = \inf_{v \in V_A} \int_{Q^A} f(x, \xi + \nabla v(x)) \, dx, \quad x \in \Omega_1, \xi \in \mathbb{R}^m,
$$
with
\[ \mathcal{V}_A = \{ v \in L^p(T^d; \mathbb{R}^m) \cap \ker \mathcal{T}_d A : \int_{Q^d} v \, dx = 0 \}. \] (8)

Here and in the following, \( T^d \) is the \( d \)-dimensional torus as it results from glueing opposite edges of \( Q^d \), and by \( \mathcal{A} v = 0 \) in \( T^d \), or equivalently \( v \in \ker \mathcal{T}_d A \), we understand that
\[ \int_{Q^d} v \cdot \mathcal{A}^T \varphi \, dx = 0 \quad \text{for all } \varphi \in C^\infty(T^d; \mathbb{R}^l) \] (9)
with \( \mathcal{A}^T := \sum_{k=1}^d (A^{(k)})^T \partial_k \) and \( C^\infty(T^d; \mathbb{R}^l) \) the space of smooth \( Q^d \)-periodic functions. Similarly, by \( v \in \ker \Omega_A \), we mean that (9) holds for all test functions \( \varphi \in C_c^\infty(\Omega_1; \mathbb{R}^l) \).

**Remark 1.**
\( a) \) The essential step in the proof is to handle the parameter dependence in the differential constraint. In particular, this requires the characterization of a suitable limit operator \( A_0 \) of \( (A_\varepsilon^d) \), see Section 2. Then, the remaining relaxation argument follows along the lines of [19].

\( b) \) A lower bound for continuous (in the second argument) \( f \) can be derived in analogy to the case of thin films [19] Theorem 1.1, using a Young measure approach based on localization by blow-up. If \( \mathcal{A} = \text{div} \), then \( Q_{\text{div} \mathcal{f}} = f^{**} \) with \( f^{**} \) the convexification of \( f \) with respect to the second argument, and the matching lower bound is trivial to prove. In general, though, the two bounds will not coincide, so that the characterization of the \( \Gamma \)-limit remains an open problem.

Theorem 1.1 is the basis of further implications regarding thin strings with heterogeneities, see Section 3. In Section 4.1 we apply the abstract results of this paper to the gradient case, discussing applications in elasticity with Cosserat vectors. Finally, Section 4.2 provides a new model for thin ferromagnetic strings.

2. Characterization of the limit operator. This section is devoted to the derivation of the correct limit operator for \( (A_\varepsilon^d) \) as \( \varepsilon \to 0 \), i.e. we are looking for a map \( A_0 : L^p(\Omega_1; \mathbb{R}^m) \to W^{-1,p}(\Omega_1; \mathbb{R}^l) \) whose kernel coincides with the set of weak \( L^p \)-limits of \( A_\varepsilon^d \)-free sequences in \( \Omega_1 \).

2.1. Assumptions on \( A \). Throughout this paper, we always assume without further mentioning that the constant-rank operator \( A \) has the representation
\[ A = A' \nabla' + A^{(d)} \partial_d = A' + A^{(d)} \partial_d = \left( \begin{array}{c} A'_+ \\ 0 \end{array} \right) + \left( \begin{array}{c} A^{(d)}_+ \partial_d \\ A^{(d)}_- \partial_d \end{array} \right), \] (10)
where \( A'_+ = A'_+ \nabla' \) with \( A'_+ \in \mathbb{R}^{s \times m(d-1)} \), \( A^{(d)}_+ \in \mathbb{R}^{(l-s) \times m} \), \( A^{(d)}_- \in \mathbb{R}^{s \times m} \), and \( s = \text{rank } A' \). In other words, the number of nonzero rows in \( A' \) is supposed to equal rank \( A' \). In terms of symbols, (10) says that
\[ h(f) = h'(f') + A^{(d)} \eta_d = \left( \begin{array}{c} h'_+(f') \\ 0 \end{array} \right) + \left( \begin{array}{c} A^{(d)}_+ \eta_d \\ A^{(d)}_- \eta_d \end{array} \right), \quad h = (f', \eta_d) \in \mathbb{R}^d. \]

**Remark 2.**
\( a) \) If rank \( h(f) = r \in \mathbb{N} \) for all \( f \neq 0 \), then \( r = \text{rank } A^{(1)} \leq \text{rank } A' = s \leq \min\{l, m(d-1)\} \).

\( b) \) For \( d = 2 \), postulating (10) is identical with [19] Assumption A2] upon switching the roles of the variables \( x_1 \) and \( x_2 \).

\( c) \) We point out that (10) is not restrictive for the set of \( A \)-free fields in view of Gauss elimination (switching rows and adding rows multiplied by a nonzero factor).
Example 2.1. a) Let \( A = \text{div} \) with \( \text{div} u = \sum_{k=1}^{d} \partial_k u_k = \sum_{k=1}^{d} e_k^T \partial_k u \) for \( u \in C^1(\mathbb{R}^d; \mathbb{R}^d) \), where \( e_k \) is the \( k \)th standard unit vector. Then, \( m = d \) and \( l = s = r = 1 \), so that \( A' = A'_+ \in \mathbb{R}^{1 \times (d-1)} \) and \( A(d) = A_{+}^{(d)} \).

b) For \( A = \text{curl} := \nabla \times \) with \( d = m = l = 3 \), one finds \( r = 2 \) and
\[
A' = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0
\end{pmatrix} \in \mathbb{R}^{3 \times 6}.
\]

Since \( s = \text{rank} A' = 3 \), \( \text{ker} A' \) is realized with \( A' = A'_+ \).

c) For general \( A = \text{curl} \), defined for a function \( U : \mathbb{R}^d \to \mathbb{R}^{n \times d} \cong \mathbb{R}^m \) as
\[
(\text{curl} U)_{hij} = \partial_i U_{hj} - \partial_j U_{hi}, \quad i,j = 1, \ldots, d, \quad i < j, \quad h, i, j, \ldots, n,
\]

one obtains \( m = nd \) and \( l = n(d-1) \). Notice that the requirement \( i < j \) is usually not found in the standard definition of curl. Here it is used to eliminate expressions redundant for the characterization \( A \)-free fields, cf. [19, Section 2.7.2]. A careful analysis of
\[
(A^{(k)})_{hij,qp} = \delta_{kj} \delta_{qh} \delta_{pj} - \delta_{kj} \delta_{qp} \delta_{pi}, \quad i,j,p,k = 1, \ldots, d, \quad i < j, \quad h, q = 1, \ldots, n,
\]

shows that \( r = n(d-1) \) and \( s = \text{rank} A' = l \). Thus, here as well, we find that \( A' = A'_+ \), so that curl as in (11) fits into the framework of (10).

Provided (10) holds, the limit operator of \( (A_x) \) can be defined row-wise, precisely
\[
A_0 = \left( \frac{A'_{+}}{A'_+ \partial_d} \right), \quad \text{or in symbols,} \quad A_0(\eta) = \left( \frac{A_0'_{+}(\eta')}{{A'}_{+} \partial_d \eta_d} \right), \quad \eta \in \mathbb{R}^d.
\]

If (10) is violated, \( A_0 \) may not capture the full asymptotic properties of \( A_x \)-free fields, as [19, Example 2.1] with exchanged roles of \( x_1 \) and \( x_2 \) demonstrates. In the special case \( A' = A'_+ \), as for instance for \( A = \text{div} \) and \( A = \text{curl} \) (cf. Example 2.1), formula (12) reduces to \( A_0 = A' \).

In order to prove that \( A_0 \) is in fact the correct limit operator for \( (A_x) \), two more conditions are needed. The first one regards the extension of \( A_0 \)-free fields from \( \Omega_1 \) to \( \mathbb{T}^d \), while preserving the \( A_0 \)-freeness. Recall that we always assume \( \omega \subset \subset Q^{d-1} \).

Assumption 1 (Extension of \( A_0 \)-free fields). Let \( A \) and \( A_0 \) be as in (10) and (12), respectively. For every \( u \in L^p(\Omega_1; \mathbb{R}^m) \cap \ker \Omega_1 A_0 \) there is a \( \tilde{u} \in L^p(\mathbb{T}^d; \mathbb{R}^m) \cap \ker_{\mathbb{T}^d} A_0 \) such that \( u = \tilde{u} \) in \( \Omega_1 \).

If \( A_0 = A' \), which is the most relevant case for the applications we have in mind, it is sufficient that \( A' \), interpreted as a constant-rank operator in \( (d-1) \) dimensions, admits a suitable extension.

Lemma 2.2 (Extension property of \( A' \) as a sufficient condition). Let \( A_0 = A' \). If there exists a bounded linear operator
\[
T' : L^p(\omega; \mathbb{R}^m) \cap \ker_{\omega} A' \to L^p(\mathbb{T}^{d-1}; \mathbb{R}^m) \cap \ker_{\mathbb{T}^{d-1}} A'
\]
such that \( T'w = w \) in \( \omega \) for all \( w \in L^p(\omega; \mathbb{R}^m) \cap \ker_{\omega} A' \), then Assumption 1 is satisfied.

Proof. Let us identify functions in \( L^p(\Omega_1; \mathbb{R}^m) \cap \ker \Omega_1 A_0 \cong L^p(\Omega_1; \mathbb{R}^m) \cap \ker_{\omega} A' \) and \( L^p(\mathbb{T}^d; \mathbb{R}^m) \cap \ker_{\mathbb{T}^d} A_0 \cong L^p(\mathbb{T}^d; \mathbb{R}^m) \cap \ker_{\mathbb{T}^{d-1}} A' \) without
Assumption A2, if there exist matrices $W$ gives rise to a well-defined linear, bounded operator. Setting $\bar{u} = Tu$ immediately yields Assumption A1.

Remark 3. Lemma 2.2 essentially requires $\omega$ to be an $\mathcal{A}'$-free extension domain in the sense of [17, Definition 4.3]. The example of the differential operator associated with the Cauchy-Riemann equations $(d = m = 2, \mathbb{C} \cong \mathbb{R}^2)$ makes clear that an extension operator as in Lemma 2.2 may not always exist; for more details see [17, Section 4.2].

Example 2.3. a) If $\mathcal{A} = \text{div}$, one has $\mathcal{A}'w = \text{div}'w = \sum_{k=1}^{d-1} \partial_k w_k$ for $w \in L^p(\omega; \mathbb{R}^m)$. The existence of an extension operator $T'$ is proven in [10, Proposition 2.1] by solving a Poisson problem with Neumann boundary conditions outside of $\omega$. An alternative approach relies on partitions of unity and reflections.

b) For $\mathcal{A} = \text{curl}$ and $\omega$ simply connected, $T'$ is constructed by suitable extension at the level of potentials. If $W \in L^p(\omega; \mathbb{R}^{n \times d})$ such that $\text{curl}'W = 0$ in $\omega$, i.e.

$$\partial_i W_{hd} = 0 \quad \text{for } i = 1, \ldots, d-1, h = 1, \ldots, n,$$

$$\partial_i W_{hj} - \partial_j W_{hi} = 0 \quad \text{for } i, j = 1, \ldots, d-1, i < j, h = 1, \ldots, n,$$

it follows that the $d$th column of $W$, denoted by $W_d$, is constant, and that

$$W' := (W_1 \ldots W_{d-1}) = \nabla'w,$$

where $w \in W^{1,p}(\omega; \mathbb{R}^n)$. With $S : W^{1,p}(\omega; \mathbb{R}^n) \to W^{1,p}_0(Q^{d-1}; \mathbb{R}^n)$ a bounded linear extension operator for Sobolev spaces, defining

$$T'W = (\nabla'(Sw)|W_d) \in L^p(T^{d-1}; \mathbb{R}^{n \times d}) \cap \ker_{\tau^{d-1}} \text{curl}'$$

for $W \in L^p(\omega; \mathbb{R}^{n \times d}) \cap \ker_{\omega} \text{curl}'$ with $W' = \nabla'w$ gives the sought extension operator.

The second assumption is a structural property on the operator $\mathcal{A}$.

Assumption 2. We say that a differential operator $\mathcal{A} = \sum_{k=1}^d A^{(k)} \partial_k$ satisfies Assumption A2 if there exist matrices $S^{(k)} \in \mathbb{R}^{n \times l}$, $k = 1, \ldots, d-1$, with

$$\sum_{k=1}^{d-1} A^{(k)} S^{(k)} = \begin{pmatrix} I_{d \times l} & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{l \times l},$$

such that $A^{(d)} S^{(d)} A^{(d)} = 0$ for all $k = 1, \ldots, d-1$.

Remark 4. If $s = r = \text{rank } A^{(k)}$ for all $k = 1, \ldots, d-1$, then choosing $S^{(k)}$ as the Moore-Penrose pseudo-inverse of $A^{(k)}$ (denoted by $(A^{(k)})^\dagger$), multiplied with the factor $(d-1)^{-1}$ guarantees (13).

Example 2.4. a) For $\mathcal{A} = \text{div}$, we have $l = s = r = 1$ and $A^{(k)} = e_k^T$ for $k = 1, \ldots, d$. Then, $S^{(k)} = (d-1)^{-1} A^{(k)} \dagger = (d-1)^{-1} e_k$ for $k = 1, \ldots, d-1$ ensures (13) according to Remark 4 and $A^{(d)}(A^{(k)})^\dagger A^{(d)} = (d-1)^{-1} (e_d \cdot e_k) e_d^T = 0$ for $k = 1, \ldots, d-1$.

b) For $\mathcal{A} = \text{curl} = \nabla \times$ in 3d, one has that $A^{(1)} = e_3 \otimes e_2 - e_2 \otimes e_3$, $A^{(2)} = e_1 \otimes e_3 - e_3 \otimes e_1$, and $A^{(3)} = e_2 \otimes e_1 - e_1 \otimes e_2$. Here, $r = 2$ and $s = 3$. It is easy to check that the choice $S^{(1)} = \frac{1}{2} e_2 \otimes e_3 - e_3 \otimes e_2$ and $S^{(2)} = e_3 \otimes e_1 - \frac{1}{2} e_1 \otimes e_3$ entails Assumption A2.
2.2. A characterization result and the proof of Theorem 1.1. The following proposition specifies the statement that \( A_0 \) is the suitable limit operator for \( (A_\varepsilon) \) as \( \varepsilon \to 0 \).

**Proposition 1** (Limit operator \( A_0 \)). Let \( A \) be a constant-rank operator satisfying Assumptions A1 and A2, and let \( A_0 \) be defined in (12).

(i) If \( u_\varepsilon \in L^p(\Omega_1; \mathbb{R}^m) \cap \ker \Omega_1 A_\varepsilon (\varepsilon > 0) \) such that \( u_\varepsilon \rightharpoonup u \) in \( L^p(\Omega_1; \mathbb{R}^m) \) for some \( u \in L^p(\Omega_1; \mathbb{R}^m) \), then \( u \in \ker \Omega_1 A_0 \).

(ii) For every \( u \in L^p(\Omega_1; \mathbb{R}^m) \cap \ker \Omega_1 A_0 \) there are \( u_\varepsilon \in L^p(\Omega_1; \mathbb{R}^m) \cap \ker \Omega_1 A_\varepsilon (\varepsilon > 0) \) such that \( u_\varepsilon \rightharpoonup u \) in \( L^p(\Omega_1; \mathbb{R}^m) \).

Once Proposition 1 is proven, Theorem 1.1 follows essentially as a corollary.

**Proof of Theorem 1.1.** (i) In view of Proposition 1(i), the lower bound becomes trivial and follows immediately from the weak lower semicontinuity of integral functionals with convex integrands.

For a given \( u \in L^p(\Omega_1; \mathbb{R}^m) \cap \ker \Omega_1 A_0 \), the sequence of Proposition 1(ii) is even a recovery sequence, since \( u_\varepsilon \rightharpoonup u \) in \( L^p(\Omega_1; \mathbb{R}^m) \) implies

\[
\lim_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) = \lim_{\varepsilon \to 0} \int_{\Omega_1} f(x, u_\varepsilon) \, dx = \int_{\Omega_1} f(x, u) \, dx
\]

by Lebesgue’s dominated convergence, which is applicable due to the continuity and the \( p \)-growth of \( f \) (in the second argument).

(ii) Let \( u \in L^p(\Omega_1; \mathbb{R}^m) \cap \ker \Omega_1 A_0 \) be given, and \( (u_\varepsilon) \) be the corresponding sequence resulting from Proposition 1(ii). By Theorem 1.1, one finds for every \( \varepsilon > 0 \) an \( A_\varepsilon \)-free sequence \( (u_\varepsilon^{(j)})_j \) such that \( u_\varepsilon^{(j)} \rightharpoonup u_\varepsilon \) in \( L^p(\Omega_1; \mathbb{R}^m) \) as \( j \to \infty \) and

\[
\limsup_{j \to \infty} E_\varepsilon(u_\varepsilon^{(j)}) \leq \int_{\Omega_1} Q_{A_\varepsilon} f(x, u_\varepsilon) \, dx.
\]

We observe that \( Q_{A_\varepsilon} = Q_{A_\varepsilon} f \) for every \( \varepsilon > 0 \), as a consequence of a change of variables in the spirit of [19] Lemma 2.12. Besides, \( Q_{A_\varepsilon} f \) is upper semicontinuous in the second variable by [14] Proposition 3.4. Hence, we may conclude by selecting a diagonal sequence \( (u_\varepsilon^{(j)})_\varepsilon \) with the required properties.

**Remark 5.** Even if \( A \) and its rescaled versions \( A_\varepsilon \) have constant rank, the limit operator \( A_0 : L^p(\Omega_1; \mathbb{R}^m) \to W^{-1,p}(\Omega_1; \mathbb{R}^r) \) as defined in (12) in general does not. In fact, if \( A' = A_\varepsilon \) (or \( s = l \)), then \( A_0 = A' \), and rank \( A'(e_1) = \text{rank } A^{(1)} = r \), while rank \( A'(e_d) = 0 \). This shows that (4) is violated for \( A' \), if \( A \) is non-degenerate (i.e. \( \mathcal{A} \neq 0 \)).

We remark that \( A' \) is of constant rank, though, when interpreted as an operator \( L^p(\omega; \mathbb{R}^m) \to W^{-1,p}(\omega; \mathbb{R}^r) \).

2.3. Proofs. The proof of Proposition 1 is split in two parts, separated by a paragraph on technical tools regarding projections onto \( A_\varepsilon \)-free fields.

**Proof of Proposition 1 [Part I].** (i) As \( u_\varepsilon \in \ker \Omega_1 A_\varepsilon \), for \( \varepsilon > 0 \), one obtains for all test functions \( \varphi \in C_0^\infty(\Omega_1; \mathbb{R}^s) \) that

\[
\frac{1}{\varepsilon} \int_{\Omega_1} u_\varepsilon \cdot (A_\varepsilon')^T \varphi \, dx + \int_{\Omega_1} u_\varepsilon \cdot (A_\varepsilon^{(d)})^T \partial_d \varphi \, dx = 0.
\]

Multiplying with \( 1/\varepsilon \) and letting \( \varepsilon \) tend to zero entails \( A_\varepsilon' u = 0 \) in \( \Omega_1 \). Similarly, we argue that \( A_\varepsilon^{(d)} \partial_d = 0 \).

(ii) The following reasoning is in analogy to the proof of [19] Proposition 4.1.
Step 1: Extension and mollifications. By Assumption $A^2$ there exists $\tilde{u} \in L^p(\mathbb{T}^d; \mathbb{R}^m) \cap \ker T \cap \mathcal{A}_0$ such that $\tilde{u} = u$ in $\Omega_1$. Mollifying $\tilde{u}$ with standard unit kernels is an operation compatible with the $\mathcal{A}_0$-free constraint and hence, gives rise to a smooth approximating sequence $\{\tilde{u}_j\} \subset C^\infty(\mathbb{T}^d; \mathbb{R}^m) \cap \ker T \cap \mathcal{A}_0$. In view of a diagonalization procedure, it is sufficient to prove $(ii)$ with $\Omega_1 = Q^d$ for each $\tilde{u}_j$. In the following, we may therefore assume that $u$ is smooth and that $\mathcal{A}_0 u = 0$ in $\mathbb{T}^d$.

Step 2: Special case $u = u(x_d)$. In this case, an explicit construction gives a suitable $\mathcal{A}_c$-free approximation of $u$. Let $u \in C^\infty(\mathbb{T}^d; \mathbb{R}^m) \cap \ker T \cap \mathcal{A}_0$ with $\nabla' u = 0$ in $Q^d$. We assert that

$$u_\varepsilon(x) := u(x_d) - \varepsilon \sum_{k=1}^{d-1} x_k S^{(k)} A^{(d)} \partial_d u(x_d), \quad x \in Q^d,$$

where $S^{(k)}$ are the matrices resulting from Assumption $A^2$ has the desired properties. By construction, $u_\varepsilon \in C^\infty(Q^d; \mathbb{R}^m)$ and it is immediate to see that $u_\varepsilon \to u$ in $L^p(Q^d; \mathbb{R}^m)$. Next, we show that

$$\mathcal{A}_c u_\varepsilon = 0 \quad \text{in } Q^d.$$

Indeed, since $u_\varepsilon$ is smooth, we compute for every $x \in Q^d$,

$$\mathcal{A}_c u_\varepsilon(x) = A^{(d)} \partial_d u(x_d) - \mathcal{A}^{(d)} \left( \sum_{k=1}^{d-1} x_k S^{(k)} A^{(d)} \partial_d u(x_d) \right)$$

$$- \varepsilon A^{(d)} \partial_d \left( \sum_{k=1}^{d-1} x_k S^{(k)} A^{(d)} \partial_d u(x_d) \right)$$

$$= \left( \frac{A^{(d)}_{+}}{A^{(d)}_{-}} \right) \partial_d u(x_d) - \sum_{k=1}^{d-1} A^{(k)} S^{(k)} A^{(d)} \partial_d u(x_d)$$

$$- \varepsilon \sum_{k=1}^{d-1} x_k A^{(d)} S^{(k)} A^{(d)} \partial_d u(x_d) = 0.$$

For the last equality, we exploited Assumption $A^2$ as well as $A^{(d)} \partial_d u(x_d) = 0$ in view of $\mathcal{A}_0 u = 0$ in $Q^d$. \hfill \square

For the remaining steps of the proof of Proposition $1(ii)$, projections play a decisive role. The heuristic idea is to define $u_\varepsilon := \mathcal{P}_{\mathcal{A}_c} u$ for all $u \in \ker \mathcal{A}_0$ not covered by Step 2 of the previous proof, where $\mathcal{P}_{\mathcal{A}_c}$ is a suitable projection operator onto $\mathcal{A}_c$-free fields. Then,

$$"u_\varepsilon = \mathcal{P}_{\mathcal{A}_c} u \to \mathcal{P}_{\mathcal{A}_c} u = u" \tag{14}$$

in a sense to be made precise later.

To make these considerations rigorous, we take a new viewpoint by switching to the Fourier space setting. There, projections can conveniently be defined algebraically. For $\eta \in \mathbb{R}^d \setminus \{0\}$ and a differential operator $\mathcal{A}$ of the form (3) we consider the orthogonal projection onto $\ker \mathcal{A}(\eta) \subset \mathbb{R}^m$, calling it $\mathcal{P}_\mathcal{A}(\eta) \in \text{Lin}(\mathbb{R}^m; \mathbb{R}^m)$. Let us recall that a map $m : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ is called a $L^p$-Fourier multiplier, if the operator $T_m$ defined by $T_m := F^{-1}(m F)$ on the Schwartz space $S(\mathbb{R}^d)$, where $F$ denotes the Fourier transform and $F^{-1}$ its inverse, extends to a bounded linear operator on $L^p(\mathbb{R}^d)$. The space of $L^p$-Fourier multipliers will be denoted by $\mathcal{M}_p(\mathbb{R}^d)$, and a norm is defined by $\|m\|_{\mathcal{M}_p(\mathbb{R}^d)} = \|T_m\|_{\text{Lin}(L^p(\mathbb{R}^d); L^p(\mathbb{R}^d))}$. 

\[62\text{ CAROLIN KREISBECK} \]
Proof. The proof of (i) can be found in [13, Lemma 2.14], where it is employed that \( P_A \) is 0-homogeneous and smooth owing to the constant-rank property of \( A \). To prove (ii), one possibility is to invoke the Mihlin multiplier theorem along with a scaling argument for Fourier multipliers in the spirit of [19, Lemma 2.7], leading to

\[ \|P_A \|_{M_p([0,r])} \leq c L \] \hspace{1cm} (15)

for every \( r > 0 \).

(ii) There is a constant \( C > 0 \) (independent of \( \varepsilon \)) such that

\[ \|P_A \|_{M_p([0,\varepsilon])} \leq C \]

for every \( \varepsilon > 0 \).

Proof. The proof of (i) can be found in [13, Lemma 2.14], where it is employed that \( P_A \) is 0-homogeneous and smooth owing to the constant-rank property of \( A \). To prove (ii), one possibility is to invoke the Mihlin multiplier theorem along with a scaling argument for Fourier multipliers in the spirit of [19, Lemma 2.7], leading to

\[ \|P_A \|_{M_p([0,\varepsilon])} \leq c L \] \hspace{1cm} (15)

for every \( \varepsilon > 0 \).

\[ \|P_A \|_{M_p([0,\varepsilon])} \leq C \]

for every \( \varepsilon > 0 \).

Remark 6. As an immediate consequence of the \( \varepsilon \)-independence of \( C_L(\varepsilon, \varepsilon) \), we observe that \( \|P_A \|_{L_p([0,\varepsilon])} \) is uniformly bounded with respect to \( \varepsilon \).

In terms of the discrete Fourier multiplier operators associated with \( P_A \), one obtains the following:

Lemma 2.6 (Projection onto \( A \)-free fields). Let \( A \) be a constant-rank operator. Then there are linear, uniformly \( (\varepsilon > 0) \) bounded operators \( P_{A_0} : L_p(\mathbb{T}^d; \mathbb{R}^m) \rightarrow L_p(\mathbb{T}^d; \mathbb{R}^m) \), such that \( A_0(P_{A_0} \mathcal{u}) = 0 \) in \( \mathbb{T}^d \) and

\[ \|P_{A_0} \mathcal{u} - \mathcal{u}\|_{L_p(\mathbb{T}^d; \mathbb{R}^m)} \leq C \|A_0 \mathcal{u}\|_{W^{-1,p}(\mathbb{T}^d; \mathbb{R}^m)} \]

for all \( u \in L_p(\mathbb{T}^d; \mathbb{R}^m) \) with a constant \( C > 0 \) independent of \( \varepsilon \).

Proof. Let \( P_{A_0} \) be the discrete multiplier operator corresponding to \( (P_{A_0}(\varepsilon))_{\varepsilon \in \mathbb{Z} \setminus \{0\}} \), precisely, for \( u \in L_p(\mathbb{T}^d; \mathbb{R}^m) \),

\[ P_{A_0} \mathcal{u}(x) := \hat{\mathcal{u}}(0) + \sum_{\varepsilon \in \mathbb{Z} \setminus \{0\}} P_{A_0}(\varepsilon) \hat{\mathcal{u}}(\varepsilon) e^{2\pi i x \cdot \varepsilon}, \quad x \in \mathbb{T}^d, \]

with \( \hat{\mathcal{u}} \) the discrete Fourier coefficients of \( \mathcal{u} \). In view of (15), there is \( c > 0 \) such that

\[ \|P_{A_0} \mathcal{u}\|_{L_p(\mathbb{T}^d; \mathbb{R}^m)} \leq c \|\mathcal{u}\|_{L_p(\mathbb{T}^d; \mathbb{R}^m)} \]

for all \( u \in L_p(\mathbb{T}^d; \mathbb{R}^m) \).

Proving the estimate of the projection error follows closely along the lines of [19, Theorem 2.8], which in turn is a modification of [14, Lemma 2.14]. Here, again, the Lizorkin multiplier theorem is needed to get \( C > 0 \) independent of \( \varepsilon \).
Remark 7. Notice that the constant-rank property of \( \mathcal{A} \) is essential here. In particular, a projection operator \( \mathcal{P}_{\mathcal{A}_0} \) (cf. [14]) with the properties of Lemma 2.6 does not exist in general, as the following counterexample (slightly adapted from an idea by Krömer [21]) illustrates: Let \( \mathcal{A} = \text{div} \) with \( d = m = 2, l = 1 \), and consider the corresponding limit operator \( \text{div}_0 \) with \( \text{div}_0 u = \text{div'} u = \partial_1 u_1 \). We observe that the orthogonal projection onto \( \text{div}_0 \)-free fields is given by
\[
\mathcal{P}u(x) := \left( \int_0^1 u_1(t, x_2) \, dt \right) e_1 + u_2(x)e_2, \quad x \in Q^2.
\]
For the sequence \((u_j) \subset L^2(T^d; \mathbb{R}^2)\) defined by
\[
u_j(x) = \sin(2\pi x_1) \sin(2\pi j x_2)e_1, \quad x \in Q^2;
\]
one finds that
\[
\| \text{div}_0 u_j \|_{W^{-1,2}(\mathbb{T}^2)} \to 0 \quad \text{for } j \to \infty,
\]
since \( \text{div}_0 u_j = 2\pi \cos(2\pi x_1) \sin(2\pi j x_2) \to 0 \) in \( L^2(T^2) \), but
\[
\| \mathcal{P}u_j - u_j \|_{L^2(\mathbb{T}^2; \mathbb{R}^2)} = \| u_j \|_{L^2(\mathbb{T}^2; \mathbb{R}^2)} = \int_0^1 \sin^2(2\pi t) \, dt = \frac{1}{2} > 0.
\]

The next lemma deals with the convergence of the kernels of the symbols \( \mathbb{A}_\varepsilon \), formulated in terms of the corresponding projections \( \mathbb{P}_{\mathcal{A}_\varepsilon} \).

Lemma 2.7 (Convergence of the projection operators in Fourier space). Suppose \( \mathcal{A} \) is a constant-rank operator. If \( \eta' \in \mathbb{R}^d \) is such that \( \eta' \neq 0 \), then \( \mathbb{P}_{\mathcal{A}_\varepsilon}(\eta) \to \mathbb{P}_{\mathcal{A}_\varepsilon}(\eta') \) pointwise.

Remark 8. Notice that \( \eta' \neq 0 \) is necessary. To see this, let \( \eta \neq 0 \) with \( \eta' = 0 \), and observe that \( \ker \mathbb{A}_\varepsilon(\eta) = \ker \mathcal{A}^{(d)} \) for all \( \varepsilon > 0 \), while \( \ker \mathbb{A}_0(\eta) = \ker \mathcal{A}_-^{(d)} \). If \( \ker \mathcal{A}^{(d)} \neq \ker \mathcal{A}_-^{(d)} \), which is the case in all the examples discussed in this work, the statement of Lemma 2.7 cannot be true.

Proof of Lemma 2.7. The following arguments are adapted from [19] Lemma 2.3 and 4.3. Let \( \eta' \in \mathbb{R}^d \) with \( \eta' \neq 0 \) be fixed.

Step 1: Conservation of rank. To start with, we prove that
\[
\text{rank} \mathbb{A}_0(\eta) = \text{rank} \mathbb{A}_\varepsilon(\eta) = r.
\]
or equivalently, that
\[
\dim \ker \mathbb{A}_0(\eta) = \dim \ker \mathbb{A}_\varepsilon(\eta) = m - r.
\]
Recall from (12) that
\[
\mathbb{A}_0(\eta) = \frac{\mathbb{A}_+(\eta')}{\mathcal{A}_-^{(d)} \eta_d}.
\]
As \( \mathcal{A}' = \sum_{k=1}^{d-1} \mathcal{A}^{(k)} \partial_k \) is a constant-rank operator in \((d-1)\) dimensions and \( \eta' \neq 0 \), it follows that \( \text{rank} \mathbb{A}'_+ (\eta') = \text{rank} \mathbb{A}'_- (\eta') = r \), which trivially yields \( \text{rank} \mathbb{A}_0(\eta) \geq r \), and even (16) in the case \( s = l \).

To prove (16) for \( s < l \), we start by setting
\[
\mathbb{B}_\varepsilon(\eta) = \left( \frac{\varepsilon \mathbb{A}_\varepsilon(\eta) + \mathbb{A}_-^{(d)} \eta_d}{\mathbb{A}_+^{(d)} \eta_d} \right) \quad \text{with} \quad \mathbb{A}_\varepsilon(\eta)_+ = \frac{1}{\varepsilon} \mathbb{A}_-^{(d)}(\eta') + \mathcal{A}_+^{(d)} \eta_d.
\]
It is immediate to see that \( \mathbb{B}_\varepsilon(\eta) \to \mathbb{A}_0(\eta) \) as \( \varepsilon \to 0 \). Moreover, we assert that \( \text{rank} \mathbb{B}_\varepsilon(\eta) = r \) for small \( \varepsilon \), which follows from
\[
\text{rank} \mathbb{A}_\varepsilon(\eta)_+ = r \quad \text{for small } \varepsilon.
\]
To verify the latter, observe that \( \varepsilon \mathcal{A}_z(\eta)_+ \to \mathcal{A}_+^\prime(\eta') \) as \( \varepsilon \to 0 \). Analyzing the convergence of \( r \times r \)-subdeterminants of \( \varepsilon \mathcal{A}_z(\eta)_+ \) to those of \( \mathcal{A}_+^\prime(\eta') \) reveals that \( \text{rank } \mathcal{A}_z(\eta)_+ = \text{rank } \mathcal{A}_z(\eta) \geq r \) for \( \varepsilon \) sufficiently small. With the reverse inequality being trivial, (17) is proven.

Now, for any \((r+1) \times (r+1)-\)submatrix of \( \mathcal{B}_z(\eta) \) and \( \mathcal{A}_0(\eta) \) (denoted by \( M(\mathcal{B}_z(\eta)) \) and \( M(\mathcal{A}_0(\eta)) \)) that contains exactly one row of \( \mathcal{A}_+^d(\eta) \), it follows from the continuity of the determinant that \( M(\mathcal{A}_0(\eta)) = \lim_{\varepsilon \to 0} M(\mathcal{B}_z(\eta)) = 0 \). Consequently, the rows of \( \mathcal{A}_+^d(\eta) \) are linear combinations of the rows of \( \mathcal{A}_+^\prime(\eta') \), so that \( \text{rank } \mathcal{A}_0(\eta) = \text{rank } \mathcal{A}_+^\prime(\eta') = r \). This finishes the proof of (16).

Step 2: Convergence of rows of \( \mathcal{A}_z \). The kernel of the symbol of \( \mathcal{A}_0 \) can be expressed in terms of \( r \) linearly independent rows of \( \mathcal{A}_0(\eta) \), i.e.

\[
\ker \mathcal{A}_0(\eta) = \text{span}\{ e_i^T \mathcal{A}_0(\eta) : i \in I_0 \}^\perp
\]

with \( I_0 \subset \{1, \ldots, l\} \) such that \( \mathbb{I} I_0 = r \), and \( U^\perp \) denoting the orthogonal complement of a subspace \( U \) of \( \mathbb{R}^m \). Since the rows of \( \mathcal{A}_+^d(\eta) \) can be written as linear combinations of the rows of \( \mathcal{A}_+^\prime(\eta') \) by Step 1, we may actually take \( I_0 \subset \{1, \ldots, s\} \). Suppose that \( \varepsilon > 0 \) is small enough, so that \( e_i^T \mathcal{A}_z(\eta) \neq 0 \) for all \( i \in I_0 \). Then,

\[
\lim_{\varepsilon \to 0} \frac{e_i^T \mathcal{A}_z(\eta)}{|e_i^T \mathcal{A}_z(\eta)|} = \lim_{\varepsilon \to 0} \frac{e_i^T \mathcal{A}_z(\eta)_+}{|e_i^T \mathcal{A}_z(\eta)_+|} = \frac{e_i^T \mathcal{A}_+^\prime(\eta')}{|e_i^T \mathcal{A}_+^\prime(\eta')|} = \frac{e_i^T \mathcal{A}_0(\eta)}{|e_i^T \mathcal{A}_0(\eta)|}, \quad i \in I_0.
\]

As a consequence, the \( r \) vectors \( e_i^T \mathcal{A}_z(\eta) \) with \( i \in I_0 \) are linearly independent, and

\[
\ker \mathcal{A}_z(\eta) = \text{span}\{ e_i^T \mathcal{A}_z(\eta) : i \in I_0 \}^\perp
\]

for \( \varepsilon \) sufficiently small.

Step 3: Orthogonalization and representation of the projections. Performing Gram-Schmidt orthogonalization in parallel on both sets of vectors \( \{e_i^T \mathcal{A}_z(\eta)\}_{i \in I_0} \) and \( \{e_i^T \mathcal{A}_0(\eta)\}_{i \in I_0} \) yields orthonormal systems \( \{w_i^z\}_{i=1,\ldots,r} \) and \( \{w_i^0\}_{i=1,\ldots,r} \) in \( \mathbb{R}^m \) such that

\[
\text{span}\{w_i^z : i = 1, \ldots, r\} = \ker \mathcal{A}_z(\eta)^\perp, \quad \text{span}\{w_i^0 : i = 1, \ldots, r\} = \ker \mathcal{A}_0(\eta)^\perp,
\]

and

\[
w_i^z \to w_i^0 \quad \text{in } \mathbb{R}^m \text{ as } \varepsilon \to 0.
\]

Then the orthogonal projections \( \mathbb{P}_{\mathcal{A}_z}(\eta) \) and \( \mathbb{P}_{\mathcal{A}_0}(\eta) \) can be represented as

\[
\mathbb{P}_{\mathcal{A}_z}(\eta)v = v - \sum_{i=1}^r (v \cdot w_i^z) w_i^z \quad \text{and} \quad \mathbb{P}_{\mathcal{A}_0}(\eta)v = v - \sum_{i=1}^r (v \cdot w_i^0) w_i^0, \quad v \in \mathbb{R}^m.
\]

Finally, (18) implies \( \mathbb{P}_{\mathcal{A}_z}(\eta)v \to \mathbb{P}_{\mathcal{A}_0}(\eta)v \) in \( \mathbb{R}^m \) as \( \varepsilon \to 0 \). \( \square \)

Having collected these tools about projections in Fourier space, we continue the proof of the characterization of \( \mathcal{A}_0 \) as the limit operator for \( (\mathcal{A}_z) \).

Proof of Proposition [Part II]. After discussing another special case in Step 3, the general situation \( u \in C^\infty(T^d; \mathbb{R}^m) \cap \ker \tau_1 \mathcal{A}_0 \) (see Step 1) is addressed in Step 4.

Step 3: Special case \( u \) with \( \int_{Q^d} u(y', x_d) \, dy' = 0 \). As \( u \in C^\infty(T^d; \mathbb{R}^m) \),

\[
u(x) = \sum_{\eta \in \mathbb{R}^d} \hat{u}(\eta) e^{2\pi i x \cdot \eta}, \quad x \in Q^d,
\]
can be applied. Precisely, and employing that the matrices \( \varepsilon \) as well-defined for every \( \varepsilon \in \mathbb{Z}^d \) with \( \eta' = 0 \).

By Step 1, may assume that \( A_0 u = 0 \) in \( \mathbb{T}^d \), hence, \( A_0(\eta) \hat{u}(\eta) = 0 \) for all \( \eta \in \mathbb{Z}^d \).

According to Lemma 2.5 (and a transference argument), the function

\[
A_\varepsilon(x) := \sum_{\eta \in \mathbb{Z}^d \setminus \{0\}} P_{A_\varepsilon}(\eta) \hat{u}(\eta) e^{2\pi i \varepsilon \cdot \eta}, \quad x \in Q^d,
\]

is well-defined for every \( \varepsilon > 0 \) and lies in \( L^p(Q^d; \mathbb{R}^m) \). Thus,

\[
\|u_\varepsilon - u\|_{L^p(Q^d; \mathbb{R}^m)} = \int_{Q^d} \left| \sum_{\eta \in \mathbb{Z}^d, \eta' \neq 0} (P_{A_\varepsilon}(\eta) \hat{u}(\eta) - \hat{u}(\eta)) e^{2\pi i \varepsilon \cdot \eta}\right|^p \, dx \\
\leq \left( \sum_{\eta \in \mathbb{Z}^d, \eta' \neq 0} |(P_{A_\varepsilon}(\eta) - P_{A_0}(\eta)) \hat{u}(\eta)| \right)^p \rightarrow 0
\]
as \( \varepsilon \to 0 \). To show the convergence in the previous line we argue with Lebesgue’s convergence theorem for series, making use of the pointwise convergence of the projections \( P_{A_\varepsilon} \) to \( P_{A_0} \) outside the linear subspace \( \{ \eta \in \mathbb{R}^d : \eta' = 0 \} \) by Lemma 2.7 and employing that the matrices \( P_{A_\varepsilon}(\eta) \) are uniformly bounded due to Remark 6.

**Step 4: General case.** We split \( u \) into two parts to which the results of Step 2 and 3 can be applied. Precisely,

\[
u(x) = u^{(1)}(x) + u^{(2)}(x) := (u - \int_{Q^{d-1}} u(y', x_d) \, dy') + \int_{Q^{d-1}} u(y', x_d) \, dy'
\]

for \( x \in Q^d \). Note that this decomposition preserves \( A_0 \)-freeness, i.e., \( A_0 u^{(1)} = A_0 u^{(2)} = 0 \) in \( Q^d \) in view of \( A_0 u = 0 \) in \( Q^d \). This follows immediately from \( A_{\varepsilon}^{(2)} u^{(2)} = 0 \) in \( Q^d \) and

\[
A_{\varepsilon}^{(d)} \partial_d u^{(2)}(x_d) = A_{\varepsilon}^{(d)} \partial_d \int_{Q^{d-1}} u(y', x_d) \, dy' = \int_{Q^{d-1}} A_{\varepsilon}^{(d)} \partial_d u(y', x_d) \, dy' = 0
\]

for \( x_d \in (0, 1) \), which entails \( A_0^{(2)} u = 0 \) in \( Q^d \). To conclude, we define \( u_\varepsilon := u_\varepsilon^{(1)} + u_\varepsilon^{(2)} \), where \( u_\varepsilon^{(1)} \) and \( u_\varepsilon^{(2)} \) are the \( A_\varepsilon \)-free corresponding to \( u^{(1)} \) and \( u^{(2)} \) by Step 3 and 2, respectively.

**3. Strings with heterogeneities.** We continue by investigating the asymptotic behavior of \( 0 \) with

\[
f_\varepsilon(x, \xi) = f\left(\frac{x'}{\varepsilon^{\alpha-1}}, \frac{x_d}{\varepsilon^\alpha} \xi\right), \quad x \in \Omega_1, \ \xi \in \mathbb{R}^m, \quad (19)
\]

where \( \alpha > 0 \), and \( f : \mathbb{R}^d \times \mathbb{R}^m \to [0, \infty) \) is supposed to be measurable and \( Q^d \)-periodic in its first variable, convex and continuously differentiable in the second argument \( \xi \). Moreover, \( f \) has \( p \)-growth and \( p \)-coercivity, so that \( 7 \) is fulfilled.

Studying this problem contributes to a better understanding of thin strings with periodically oscillating structures such as they arise from heterogeneities in material composition. The parameter \( \alpha \) describes the relative magnitude between the thickness of the string and the characteristic length scale of the fine substructures, see Figure 2.

The results of this section are closely related to the analogous statements for thin films as presented in detail in \( 20 \). Essentially, once Proposition \( 1 \) is proven, that
Figure 2. Examples of heterogeneous strings. a) Fibered heterogeneities \((f \text{ constant in } y_d)\); b) Fine material layers \((f \text{ constant in } y')\).

is the limit operator \(A_0\) is characterized, and the projection result Lemma 2.6 is available, the difference between films and strings comes down to interchanging the roles of the variables \(x'\) and \(x_d\). Therefore, we will be content with sketching the ideas of the proofs and pointing out the necessary modifications.

From now on, let us focus on the situation where the heterogeneities are fine compared to the diameter of the cross section, i.e. \(\alpha > 1\) (see Remark 9 for a comment on \(\alpha \leq 1\)).

**Theorem 3.1** (Local \(\Gamma\)-limit for \(\alpha > 1\)). Let \(A\) be a constant-rank operator satisfying Assumptions \(A_1\) and \(A_2\). Suppose that \(f_\varepsilon : \Omega_1 \times \mathbb{R}^m \rightarrow [0, \infty)\) as in (19).

Then,

\[
\Gamma \lim_{\varepsilon \to 0} E_\varepsilon(u) = \begin{cases} 
\int_{\Omega_1} f_{\text{hom}}^A(u) \, dx & \text{if } A_0u = 0 \text{ in } \Omega_1, \\
\infty & \text{otherwise},
\end{cases}
\]

regarding weak convergence in \(L^p(\Omega_1; \mathbb{R}^m)\). The homogenized energy density is defined as

\[
f_{\text{hom}}^A(\xi) = \inf_{v \in \mathcal{V}_A} \int_{Q_d} f(y, \xi + v(y)) \, dy,
\]

see (8) for the definition of \(\mathcal{V}_A\).

Along with this \(\Gamma\)-convergence result comes the compactness of sequences of bounded energy, i.e. a sequence \((u_\varepsilon) \subset L^p(\Omega_1; \mathbb{R}^m)\) with \(\sup_{\varepsilon > 0} E_\varepsilon(u_\varepsilon) < \infty\) is relatively sequentially compact with respect to the weak \(L^p(\Omega_1; \mathbb{R}^m)\)-topology. This is an immediate consequence of the \(p\)-coercivity of \(f\).

The proof of the lower bound is based on multiscale convergence methods, precisely, on Proposition 2 regarding necessary conditions for a multiscale limit (suitably adapted to the context under consideration, see the definition below) of \(A_\varepsilon\)-free fields.

**Definition 3.2** (A notion of reduced weak multiscale convergence). A sequence \((u_\varepsilon) \subset L^p(\Omega_1; \mathbb{R}^m)\) is said to converge weakly to \(w \in L^p(\Omega_1 \times Q_d; \mathbb{R}^m)\) in the
reduced three-scale sense if
\[ \int_{\Omega_1} u_\varepsilon(x) \cdot \varphi \left( \frac{x}{\varepsilon^{\alpha - 1}}, \frac{x_d}{\varepsilon} \right) \, dx \rightarrow \int_{\Omega_1} \int_{Q^d} w(x,y) \cdot \varphi(x,y) \, dy \, dx \]
for all test functions \( \varphi \in L^p(\Omega_1; C^\infty(T^d; \mathbb{R}^m)) \). In formulas, \( u_\varepsilon \overset{m-s^\alpha}{\rightarrow} w \) in \( L^p(\Omega_1; \mathbb{R}^m) \).

By minor modifications of [2, Theorem 1.2] it follows that uniformly bounded sequences in \( L^p(\Omega_1; \mathbb{R}^m) \) are compact with respect to the weak \( m-s^\alpha \)-convergence.

**Proposition 2** (Multiscale limit of \( A_\varepsilon \)-free fields). Let \( \alpha > 1 \), \( A \) as in \([10]\), \( u_\varepsilon \in L^p(\Omega_1; \mathbb{R}^m) \cap \ker_{\Omega_1} A_\varepsilon \) (\( \varepsilon > 0 \)) with \( u_\varepsilon \rightharpoonup u \) in \( L^p(\Omega_1; \mathbb{R}^m) \), and \( w \in L^p(\Omega_1 \times Q^d; \mathbb{R}^m) \). If \( u_\varepsilon \overset{m-s^\alpha}{\rightarrow} w \) in \( L^p(\Omega_1; \mathbb{R}^m) \), then \( \int_{Q^d} w(\cdot, y) \, dy = u \) with \( A_0 u = 0 \) in \( \Omega_1 \), and
\[ A[w(\cdot, \cdot)] = 0 \quad \text{in } T^d \text{ for almost every } x \in \Omega_1. \]

With this characterization at hand the liminf-inequality for Theorem 3.1 can be derived as follows: Let \( u_\varepsilon \in L^p(\Omega_1; \mathbb{R}^m) \cap \ker_{\Omega_1} A_\varepsilon \) (\( \varepsilon > 0 \)) with \( u_\varepsilon \rightharpoonup u \) in \( L^p(\Omega_1; \mathbb{R}^m) \). By the compactness of the reduced weak three-scale convergence, there exists a subsequence of \( (u_\varepsilon) \) (not relabeled) and \( w \in L^p(\Omega_1 \times Q^d; \mathbb{R}^m) \) with \( u_\varepsilon \overset{m-s^\alpha}{\rightarrow} w \) in \( L^p(\Omega_1; \mathbb{R}^m) \). An adapted version of the lower semicontinuity result for multiscale convergence \([13]\) Lemma 4.4] finally yields
\[ \liminf_{\varepsilon \to 0} \int_{\Omega_1} f \left( \frac{x'}{\varepsilon^{\alpha - 1}}, \frac{x_d}{\varepsilon} u_\varepsilon(x) \right) \, dx \geq \int_{\Omega_1} \int_{Q^d} f(y, w(x,y)) \, dy \, dx \]
\[ \geq \int_{\Omega_1} f_A^\text{hom}(u(x)) \, dx, \]
in view of Proposition 2.

Concerning the construction of a recovery sequence for Theorem 3.1 a sequence of asymptotically optimal structures realizing constant functions \( u \) (\( u = \xi, \xi \in \mathbb{R}^m \)) in the limit results from a perturbation with suitable oscillations. Precisely, we select a diagonal sequence \( (u_\varepsilon) = (u_\varepsilon^\delta(\xi)) \) of
\[ u_\varepsilon^\delta(x) = \xi + v(\delta) \left( \frac{x'}{\varepsilon^{\alpha - 1}}, \frac{x_d}{\varepsilon} \right), \quad x \in \Omega_1, \varepsilon, \delta > 0, \]
where \( v(\delta) \in \mathcal{V}_A \) is such that
\[ \int_{Q^d} f(y, \xi + v(\delta)(y)) \, dy \leq f_A^\text{hom}(\xi) + \delta. \]
The extension of this construction to find recovery sequences for general \( u \in L^p(\Omega_1; \mathbb{R}^m) \cap \ker_{\Omega_1} A_0 \) requires an approximation by piecewise affine functions, as well as a localization argument that allows to deal with the latter. Technical difficulties in gluing recovery sequences for constant functions are due to the fact that a cut-off in the direction of the cross section, i.e. in \( x' \), is not easily possible. Indeed, for a cut-off function \( \psi \in C^1_\varepsilon(Q^d; \mathbb{R}^m) \) and \( u_\varepsilon \in L^p(Q^d; \mathbb{R}^m) \cap \ker_{Q^d} A_\varepsilon \) (\( \varepsilon > 0 \)) with \( u_\varepsilon \rightharpoonup 0 \) in \( L^p(Q^d; \mathbb{R}^m) \), one observes that
\[ A_\varepsilon(\psi u_\varepsilon) = \frac{1}{\varepsilon} u_\varepsilon A' \psi + u_\varepsilon A^{(d)} \partial_d \psi. \] (21)
Hence, in the attempt to recover \( A_\varepsilon \)-freeness by projection (compare Lemma 2.6), we will lose control of the projection error, if the first term on the right-hand side in \([21]\) diverges. The main idea of the localization step is the observation that
this effect can be compensated, if \( u_\varepsilon \) features sufficiently fast oscillations along the string, i.e. in \( x_d \)-direction. The analogue of \cite{20} Lemma 3.9 for thin films (with interchanged roles of \( x' \) and \( x_d \)) is the following:

**Lemma 3.3 (Localization in \( x' \)).** Let \((u^\varepsilon)\subset L^p(Q^d; \mathbb{R}^m)\) with \( u^\varepsilon \in \ker(Q^{4-1} \times \mathbb{T}^1 \Lambda)\) be a uniformly bounded sequence with fast oscillations in \( x_d \), i.e. for every \( \varepsilon > 0 \) the function \( u^\varepsilon \) is \( \tau_\varepsilon \)-periodic in the \( x_d \)-variable, where \( (\tau_\varepsilon) \subset \mathbb{R}_+ \) with \( \tau_\varepsilon \varepsilon^{-1} \to 0 \) as \( \varepsilon \to 0 \).

Then, for every cut-off function \( \psi : Q^d \to \mathbb{R} \) with \( \psi(x) = \tilde{\psi}(x') \) for \( x \in Q^d \) and \( \tilde{\psi} \in C^1_c(Q^{d-1}; [0,1]) \),

\[
\lim_{\varepsilon \to 0} \left\| A_\varepsilon [\psi(u^\varepsilon - \bar{u}_\varepsilon)] \right\|_{W^{-1,p}(\mathbb{T}^d; \mathbb{R}^m)} = 0,
\]

where \( \bar{u}_\varepsilon \) is defined as

\[
\bar{u}_\varepsilon(x) := \int_0^1 u^\varepsilon(\gamma, \tau_\varepsilon \gamma d) d\gamma_d, \quad x \in Q^d.
\]

In the proof of Theorem \cite{3.1} this lemma is applied with \( \alpha = \varepsilon^n \). Notice that only for \( \alpha > 1 \), the necessary requirement \( \tau_\varepsilon \varepsilon^{-1} \to 0 \) is satisfied.

**Remark 9.** The previous reasoning does not carry over to the case \( \alpha \leq 1 \), meaning to strings with coarse heterogeneities. In fact, the counterexample of \cite{20} Section 6.2 (by exchanging the roles of the variables \( x' \) and \( x_d \)) indicates that the \( \Gamma \)-limit of \((E_\varepsilon)\) (if existent) may even fail to be local in the sense that it need not have an integral representation with respect to the Lebesgue measure.

4. **Applications.** In this last section we implement the statements of Theorems \cite{1.1} and \cite{3.1} for two relevant applications and give a brief interpretation of the resulting limit models for strings in \( \mathbb{R}^3 \).

4.1. **Elastic strings and Cosserat vectors.** Let \( \Omega_1 \subset \mathbb{R}^3 \) be a bounded and simply connected Lipschitz domain, modeling the rescaled (see \cite{3}) reference configuration of a thin string with cross section \( \varepsilon \omega \) made of hyperelastic (possibly heterogeneous) material. Let \( u : \Omega_1 \to \mathbb{R}^3 \) stand for the rescaled deformation. Since energies in finite elasticity commonly depend on the deformation gradient, the admissible vector fields (in the rescaled setting) have the form

\[
U = (U'|U_d) = \left( \frac{1}{\varepsilon} \nabla' u|\partial_3 u \right).
\]

The functional \( E^{\text{el}}_\varepsilon : L^p(\Omega_1; \mathbb{R}^{3 \times 3}) \to [0, \infty] \) given by

\[
E^{\text{el}}_\varepsilon(U) = \begin{cases} 
\int_{\Omega_1} f_\varepsilon(x, U) \, dx & \text{if } U = \left( \frac{1}{\varepsilon} \nabla' u, \partial_3 u \right) = : \nabla z u, u \in W^{1,p}(\Omega_1; \mathbb{R}^3), \\
\infty & \text{otherwise,}
\end{cases}
\]

with density function \( f_\varepsilon : \Omega_1 \times \mathbb{R}^{3 \times 3} \to [0, \infty) \) represents a typical (rescaled) elastic energy.

We observe that choosing \( A = \text{curl} \) as in Example \cite{2.1}(c) with \( n = d = 3 \) leads to the equality \( E^{\text{el}}_\varepsilon = E_\varepsilon \) with \( E_\varepsilon \) defined in \cite{6}. Hence, the following \( \Gamma \)-convergence result is an immediate consequence of Theorems \cite{1.1} and \cite{3.1} considering that for \( U \in L^p(\Omega_1; \mathbb{R}^{3 \times 3}) \),

\[
\text{curl}_0 U = 0 \text{ in } \Omega_1,
\]
if and only if, for almost every \( x \in \Omega_1 \),
\[
U(x) = (\nabla'w(x', x_3))
\]  
(22)
with \( z \in W^{1,p}(0, 1; \mathbb{R}^3) \), \( w \in L^p(0, 1; W^{1,p}(\omega; \mathbb{R}^3)) \). Note that \( z' \in L^p(0, 1; \mathbb{R}^3) \)
denotes the weak derivative of \( z \).

**Corollary 1.** Let the assumptions of either Theorem 1.1 (homogeneous case) or Theorem 3.1 (heterogeneous case) hold for the density function \( f_\varepsilon : \Omega_1 \times \mathbb{R}^{3\times 3} \rightarrow [0, \infty) \). Then,
\[
\Gamma\text{-lim}_{\varepsilon \to 0} E^\varepsilon(U) = \begin{cases} 
\int_{\Omega_1} h(x, U) \, dx & \text{if } U = (U'|U_d) \text{ as in (22),} \\
\infty & \text{otherwise,}
\end{cases}
\]
where the \( \Gamma \)-limit is taken with respect to weak \( L^p \)-convergence.

In the homogeneous case, one has \( h = f \), and in the heterogeneous case, \( h(x, \xi) = h(\xi) = f_{\text{hom}}(\xi) \) for \( x \in \Omega_1 \) and \( \xi \in \mathbb{R}^{3\times 3} \) with
\[
f_{\text{hom}}(\xi) := \inf_{v \in W^{1,p}_0(Q; \mathbb{R}^3)} \int_Q f(y, \xi + \nabla v(y)) \, dy.
\]  
(23)

**Remark 10.** a) The homogenized energy density \( f_{\text{hom}} \) in (23) is the classical cell formula, which naturally emerges in homogenization problems with convex integrands in the gradient setting [27, 30], and coincides with \( f_{\text{hom}}^{\text{curl}} \) as defined in [20].

b) For a closely related result on the asymptotics of (heterogeneous) thin films in hyperelasticity, we refer to [20, Section 1].

The function \( z \) in (22), with one independent real variable, describes the deformation of the mid-fiber, while the two columns of \( U' = (U_1|U_2) \) capture the deformation of the cross section in the limit problem and can be interpreted as Cosserat vector fields.

Indeed, let \((U_\varepsilon)\) be a bounded energy sequence for \( (E^\varepsilon_\varepsilon) \), i.e. \( E^\varepsilon_\varepsilon(U_\varepsilon) < C \) for all \( \varepsilon > 0 \). Then, (after passing to a subsequence) the weak \( L^p \)-limit of \((U_\varepsilon)\), called \( U \), has exactly the form (22). In particular, \( U' \) keeps track of the expression \( (\frac{1}{2} \nabla' u_\varepsilon) \) in the weak limit as \( \varepsilon \to 0 \), and therefore contains valuable information about the limit behavior of \((U'_\varepsilon)\), which will be nontrivial in general, even though \( (\nabla' u_\varepsilon) \) has to vanish asymptotically regarding weak convergence.

As a consequence, the \( \Gamma \)-limit of Corollary 1 is not purely one-dimensional as in [1]. A similar effect appears in the context of membrane theory. In contrast to [25], the model studied by Bouchitté, Fonseca & Mascarenhas in [5, 4] involves an additional Cosserat vector or bending moment, leading to a limit energy that depends nontrivially on the direction orthogonal to the film.

**4.2. Thin strings in micromagnetics.** In analogy to [18] for thin films, we establish here a model for ferromagnetic strings suitable for small samples in three space dimensions. With \( \omega \subset \mathbb{R}^2 \) smooth and simply connected, let \( \bar{m} : \Omega_\varepsilon \rightarrow \mathbb{R}^3 \) be the magnetization (uniformly saturated with \( |\bar{m}| = 1 \)), and \( \bar{h} : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) the induced magnetic field. To encode the relation between \( \bar{m} \) and \( \bar{h} \), which is governed by the static Maxwell equations, we use the operator
\[
A^\text{mag} \left( \begin{array}{c}
\bar{m} \\
\bar{h}
\end{array} \right) = \left( \begin{array}{c}
\text{div}(\bar{m} + \bar{h}) \\
\text{curl} \bar{h}
\end{array} \right).
\]  
(24)
As pointed out in [14], \( A^\text{mag} \) has the constant-rank property [4].
Let $m$ and $h$ denote the rescaled versions (cf. (5)) of $\bar{m}$ and $\bar{h}$, respectively, and identify $m$ with its trivial extension to $\mathbb{R}^3$ by zero. The micromagnetic energy comprises an exchange, an anisotropy and an induced energy contribution, see e.g. [8, 23]. After rescaling one obtains

$$E^\text{mag}_\varepsilon(m, h) = \int_{\Omega_1} \alpha |\nabla_3 m|^2 + \varphi(m) \, dx + \frac{1}{2} \int_{\mathbb{R}^3} |h|^2 \, dx,$$

for $(m, h) \in V^\text{mag}_\varepsilon := \{(m, h) \in [W^{1,2}(\Omega_1; \mathbb{R}^3) \times L^2(\mathbb{R}^3; \mathbb{R}^3)] \cap \ker_{\mathbb{R}^3} A^\text{mag} : |m| = 1 \text{ in } \Omega_1\}$, and $E^\text{mag} = \infty$ otherwise in $W^{1,2}(\Omega_1; \mathbb{R}^m) \times L^2(\mathbb{R}^3; \mathbb{R}^3)$. Notice that $(m, h) \in \ker_{\mathbb{R}^3} A^\text{mag}$ if the equations $\text{div}_\varepsilon(m+h) = 0$ and $\text{curl}_\varepsilon h = 0$, or equivalently,

$$\frac{1}{\varepsilon} \text{div}(m + h) + \partial_3(m_3 + h_3) = 0$$

and

$$\frac{1}{\varepsilon} \partial_2 h_3 - \partial_3 h_2 = 0, \quad \partial_3 h_1 - \frac{1}{\varepsilon} \partial_1 h_3 = 0, \quad \partial_1 h_2 - \partial_2 h_1 = 0,$$

hold in $\mathbb{R}^3$ in the sense of distributions. In (25), $\alpha > 0$ is a material constant and the continuous function $\varphi : \mathbb{R}^m \to \mathbb{R}$ favors the easy axes of magnetization.

In view of Proposition 1 and (12), the limit operator $A^\text{mag}_0$ for $(A^\text{mag}_\varepsilon)$ is represented by

$$A^\text{mag}_0 \left( \begin{array}{c} m \\ h \end{array} \right) = \left( \begin{array}{c} \text{div}_0(m + h) \\ \text{curl}_0 h \end{array} \right) = \left( \begin{array}{c} \text{div}(m + h) \\ \text{curl } h \end{array} \right)$$

$$= \left( \begin{array}{c} \partial_1(m_1 + h_1) + \partial_2(m_2 + h_2) \\ \partial_2 h_3 - \partial_3 h_2 \\ -\partial_1 h_3 \\ \partial_1 h_2 - \partial_2 h_1 \end{array} \right),$$

which leads to the following result.

**Corollary 2.** The $\Gamma$-limit of $(E^\text{mag}_\varepsilon)$ with respect to weak convergence in $W^{1,2}(\Omega_1; \mathbb{R}^3) \times L^2(\mathbb{R}^3; \mathbb{R}^3)$ is characterized by

$$\Gamma:\text{-lim}_{\varepsilon \to 0} E^\text{mag}_\varepsilon(m, h) = \begin{cases} \int_{\Omega_1} \alpha |\partial_3 m|^2 + \varphi(m) \, dx + \frac{1}{2} \int_{\mathbb{R}^3} |h|^2 \, dx & \text{if } (m, h) \in \gamma^\text{mag}_0, \\ \infty & \text{otherwise}, \end{cases}$$

where $\gamma^\text{mag}_0 = \{(m, h) \in [W^{1,2}(\Omega_1; \mathbb{R}^3) \times L^2(\mathbb{R}^3; \mathbb{R}^3)] \cap \ker_{\mathbb{R}^3} A^\text{mag}_0 : \nabla' m = 0 \text{ and } |m| = 1 \text{ in } \Omega_1\}$.

**Proof.** The lower bound is immediate, since for any bounded energy sequence $(m_\varepsilon, h_\varepsilon)$ of $(E^\text{mag}_\varepsilon)$ it follows that $\nabla' m_\varepsilon \to 0$ in $L^2(\Omega_1; \mathbb{R}^{3\times 2})$. Moreover, after passing to a subsequence, $m_\varepsilon \to m$ in $L^2(\Omega_1; \mathbb{R}^3)$ for some $m \in W^{1,2}(\Omega_1; \mathbb{R}^3)$, ensuring that $m$ meets the nonconvex constraint $|m| = 1$. Exploiting Proposition 1(ii) and the lower semicontinuity of the norm concludes the proof of the lower bound.

The construction of a recovery sequence is based on Proposition 1(ii) in a version on the whole space $\mathbb{R}^3$. This modification simply requires to replace Fourier series by Fourier transforms in the proof, cf. [18] Section 4. Hence, for every $(m, h) \in \gamma^\text{mag}_0$ there exists $(\tilde{m}_\varepsilon, \tilde{h}_\varepsilon) \in [L^2(\mathbb{R}^3; \mathbb{R}^3) \times L^2(\mathbb{R}^3; \mathbb{R}^3)] \cap \ker_{\mathbb{R}^3} A^\text{mag}_0$ such that $\tilde{m}_\varepsilon \to m$ and $\tilde{h}_\varepsilon \to h$ both in $L^2(\mathbb{R}^3; \mathbb{R}^3)$. Notice, though, that $\tilde{m}_\varepsilon$ cannot be expected to
admissible for $E^\text{mag}_\varepsilon$, as, in particular, the nonconvex constraint may be violated. Therefore, in analogy to [18, Proposition 4.7], we set
\[ m_\varepsilon = m \]
\[ h_\varepsilon = P_{\text{curl}_\varepsilon}(\hat{h}_\varepsilon - m + \hat{m}_\varepsilon) \]
(27) to obtain the sought recovery sequence. Indeed, the properties of the projection operator $P_{\text{curl}_\varepsilon}$ imply that $h_\varepsilon \in L^2(\mathbb{R}^3; \mathbb{R}^3) \cap \ker_{\mathbb{R}^3} \text{curl}_\varepsilon$ for all $\varepsilon > 0$ and $h_\varepsilon \to h$ in $L^2(\mathbb{R}^3; \mathbb{R}^3)$. Let us point out that, with $F$ the notation for the Fourier transform, $P_{\text{curl}_\varepsilon}$ in (27) is to be understood as $P_{\text{curl}_\varepsilon} := F^{-1}(P_{\text{curl}_\varepsilon} F)$.

**Remark 11.** We observe that $m$ in (26) describes the magnetization of the one-dimensional center line of the string.

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