The perturbation $\varphi_{2,1}$ of the $M(p, p + 1)$ models of conformal field theory and related polynomial-character identities

Dedicated to George E. Andrews on his 60th birthday.

by

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Abstract

Using $q$-trinomial coefficients of Andrews and Baxter along with the technique of telescopic expansions, we propose and prove a complete set of polynomial identities of Rogers-Ramanujan type for $M(p, p + 1)$ models of conformal field theory perturbed by the operator $\varphi_{2,1}$. The bosonic form of our polynomials is closely related to corner transfer matrix sums which arise in the computation of the order parameter in the regime $1^+$ of $A_{p-1}$ dilute models. In the limit where the degree of the polynomials tends to infinity our identities provide new companion fermionic representations for all Virasoro characters of unitary minimal series.

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1 Introduction

In 1987, G. Andrews and R. Baxter \cite{1} introduced the very fruitful notion of the $q$ - trinomial coefficients

$$
\left( \begin{array}{c} L, q \\ A \\ \end{array} \right)^n = \begin{cases} \sum_{j \geq 0} q^{j(A - n)} \frac{(q)_L}{(q)_j(q)_j + A(q)_L - 2j - A} & \text{for } |A| \leq L \\ 0 & \text{otherwise} \end{cases} (1.1)
$$

where

$$(a, q)_k = (a)_k = \begin{cases} \prod_{j=0}^{k-1} (1 - a q^j) & \text{for } k \in \mathbb{Z}_{>0}, \\ 1 & \text{for } k = 0, \\ \frac{1}{\prod_{j=1}^{-k} (1 - a q^{-j})} & \text{for } k \in \mathbb{Z}_{<0} \end{cases} (1.2)$$

and $L \in \mathbb{Z}_{\geq 0}; A, n \in \mathbb{Z}$. The rich properties of these objects were extensively studied in \cite{1-4}. Recently, $q$ - trinomial coefficients were used to generalize classical results of Bailey \cite{5}. This generalization \cite{6}, termed “Trinomial Analogue of Bailey’s lemma”, led these authors and P. Pearce \cite{7} to the following $q$ - series identities of Rogers-Ramanujan type (which hold for $p \geq 4, p \in \mathbb{Z}$)

$$
\sum_{m \in \mathbb{Z} \cap (p \equiv_{p-2} 0)} q^{\frac{1}{2} \tilde{C}_{p-1} m} \frac{1}{(q)_m} \prod_{j=1}^{p-2} \left( \frac{1}{2} \tilde{I}_{p-1} m \right)_j = \frac{1}{(q)_\infty} \sum_{j=-\infty}^{\infty} \left\{ q^{p(p+1)j^2 + j} - q^{p(p+1)(p+1)j + 1} \right\} (1.3)
$$

where $\tilde{I}_{p-1}$ is the matrix of dimension $(p-1) \times (p-1)$:

$$
\tilde{I}_{p-1} = -\tilde{I}_{p-1} = -\delta_{p-2} \text{ for } 0 \leq j \leq p-2 \\
\tilde{I}_{p-1} = \delta_{i,j+1} + \delta_{i,j-1} \text{ for } 1 \leq i, j \leq p-2 (1.4)
$$

where

$$
\delta_{a,b} = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise} \end{cases} (1.5)
$$

$\tilde{C}_{p-1} = 2 - \tilde{I}_{p-1}$ and $q$ - binomial coefficients $\binom{n + m}{n}_q$ are defined in standard fashion as

$$
\binom{n + m}{n}_q = \binom{n + m}{m}_q = \begin{cases} \frac{(q)_n (q)_m}{(q)_m} & \text{for } n, m \in \mathbb{Z}_{\geq 0} \\ 0 & \text{otherwise} \end{cases} (1.6)
$$

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Throughout this paper we impose the notation that components of a vector \( \mathbf{a} \) are either denoted by \( (a)_i \) or \( a_i \). By \( \mathbf{a}_i \) we denote a vector labeled by \( i \) and not its \( i^{\text{th}} \) component.

Recalling the Rocha-Caridy (bosonic) form for the Virasoro characters of \( M(p,p + 1) \) minimal models of conformal field theory \[8\]

\[
\chi^{p,p+1}_{r,s}(q) = \frac{1}{(q)_{\infty}} \sum_{j=-\infty}^{\infty} \left\{ q^{pp'j^2 + j(p'r - ps)} - q^{(pj + r)(p'j + s)} \right\}
\] (1.7)

with \( 1 \leq r \leq p - 1; 1 \leq s \leq p \) and \( p' = p + 1 \), one can immediately recognize the right-hand-side of \( (1.3) \) as \( \chi^{p+1}_{1,1}(q) \). Following well-established tradition \[9\]-\[10\] we refer to the left-hand-side of \( (1.3) \) as a fermionic representation for \( \chi^{p+1}_{1,1}(q) \).

It is important to notice that this fermionic representation is quite different from the one previously proposed in \[8\] and proven in \[9\]-\[12\]:

\[
\sum_{m_{p-2} \equiv 0(\text{mod} 2)} q^{\frac{1}{2} m_{p-2} m} \frac{1}{(q)_{m_1} \prod_{j=2}^{n-2} (\frac{1}{2} I_{p-2} m)_{j}} = \chi^{p+1}_{1,1}(q)
\] (1.8)

where \( C_{p-2} = 2 - I_{p-2} \) and \( I_{p-2} \) are Cartan and incidence matrices, respectively, of the simple Lie algebras \( A_{p-2} \), i.e.

\[
(I_{p-2})_{a,b} = \delta_{a,b+1} + \delta_{a,b-1} \quad \text{for} \quad 1 \leq a, b \leq p - 2.
\] (1.9)

To understand the meaning of this new representation let us recall, as was shown by A. Zamolodchikov \[13\], that there exist certain operators \( \{\varphi_{1,3}; \varphi_{1,2}; \varphi_{2,1}\} \) with which one can perturb \( M(p,p + 1) \)-models so that these models will remain exactly integrable even though they will lose conformal invariance. Exact integrability is a very powerful constraint which often enables one to solve model completely in terms of so-called thermodynamic Bethe Ansatz equations \[14\]. We remark that the incidence matrices such as \( I_{p-1}, I_{p-2} \) carry important information about the structure of these equations.

One way to recognize particular integrable perturbation from \( q \)-series identities such as \( (1.3) \), \( (1.8) \) is to find appropriate polynomial identities which reduce to the original ones when the degree of polynomials tends to infinity. This procedure was termed “finitization” in \[15\]. For \( L \in \mathbb{Z}_{\geq 0} \) the finitized version of \( (1.3) \) connected in \[8\] is

\[
\sum_{m_{p-2} \equiv 0(\text{mod} 2)} q^{\frac{1}{2} m_{p-1} m} \prod_{j=0}^{2} \left[ \frac{(\frac{1}{2} I_{p-1} m + L e_0)_{j}}{m_j} \right] = \\
= \sum_{j=-\infty}^{\infty} \left\{ q^{mp'j^2 + j} \left( L, q \right)_{2pj}^0 - q^{(jp + 1)(j'p + 1)} \left( L, q \right)_{2pj + 2}^0 \right\}
\] (1.10)

where the \( p - 1 \) dimensional unit vector in the \( a \)-direction \( \mathbf{e}_a \) is

\[
\mathbf{e}_a = \left\{ \begin{array}{ll}
\delta_{a,b} & \text{for } a, b = 0, 1, 2, \ldots, p - 2 \\
0 & \text{otherwise}.
\end{array} \right.
\] (1.11)

Bosonic polynomials on the right-hand-side of \( (1.10) \) emerged previously in the order parameter calculations \[16\] for regime 1\(^+\) of dilute \( A_{p-1} \)-model. It was shown in \[16\] that

\[
\]
the scaling limit of regime \(1^+A_{p-1}\) - model represents the \(M(p,p+1)\) - model perturbed by \(\varphi_{2,1}\). Thus, the fermionic representation \([1.3]\) for the \(\chi_{1,1}^{p,p+1}(q)\) character is related to the \(M(p,p+1) + \varphi_{2,1}\) model which is to be contrasted with the fermionic representation \([1.8]\) which is related to \(M(p,p+1) + \varphi_{1,3}\).

The object of this paper is to prove \([1.10]\) and its generalizations which would, in particular, provide a complete set of Virasoro characters in the limit \(L \to \infty\). The rest of this article is organized as follows: In Section 2 we will state a complete set of polynomial identities proven in this paper. In Section 3 we will discuss numerous properties of \(q\) - trinomials and derive recursion relations for bosonic polynomials. Section 4 is dedicated to deriving recurrences for fermionic polynomials by means of a telescopic expansion technique and proving identities stated in Section 2. In Section 5 we take the limit \(L \to \infty\) to obtain Virasoro character identities. We will conclude with some closing remarks. Some technical details are relegated to Appendices A and B.

2 Identities

Let us denote by \(f_k^p(L; u; A)\) the fundamental fermionic form defined as

\[
f_k^p(L; u; A) = \sum_{m} q^{m c_{p-1} m - \frac{A m^{p-2}}{2}} \prod_{j=0}^{p-1} \left[ \left( \frac{1}{2} I_{p-1} m + L e_0 + \frac{1}{2} u \right) \right]_q \tag{2.1}
\]

and for \(a, b, i \in \mathbb{Z}_{\geq 0}\) let us set

\[
u_{a,b}^i = \theta(a > 1) e_a + \theta(b > 1) e_b + \tilde{p}(a + \delta_{a,1} + b + \delta_{b,1} + \delta_{i,1}(p-1))(e_1 - e_0) \tag{2.2}
\]

and

\[
A_{a,b}^i = \theta(b > 1) e_b + \tilde{p}(a + \delta_{a,1} + b + \delta_{b,1} + \delta_{i,1}(p-1))e_1 \tag{2.3}
\]

with

\[
\delta_{a,b} = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases} \tag{2.4}
\]

\[
\tilde{p}(a) = \begin{cases} 1 & \text{if } a \equiv 1 \pmod{2} \\ 0 & \text{if } a \equiv 0 \pmod{2} \end{cases} \tag{2.5}
\]

and

\[
\theta \text{ (statement)} = \begin{cases} 1 & \text{if statement is true} \\ 0 & \text{if statement is false} \end{cases} \tag{2.6}
\]

Next we define the objects of most interest to us

\[
F_{a,b}^{p,i}(L) = \begin{cases} f_k^p(L; u_{a,b}^i; A_{a,b}^i) & \text{if } b \neq p - 1 \\ f_{1-i}^p(L; u_{a,b}^i; A_{a,b}^i) & \text{if } b = p - 1 \end{cases} \tag{2.7}
\]
with $i = 0, 1$,

$$B_{a,b}^p(L, s) = \sum_{j=-\infty}^{\infty} \left\{ q^{pp'j^2+j(p'a-ps)} \left( \frac{L, q}{2pj + a - b} \right)^0 
- q^{(pj+a)(pj+s)} \left( \frac{L, q}{2pj + a + b} \right)^0 \right\}, \quad (2.8)$$

and

$$\tilde{B}_{a,b}^p(L, s) = \sum_{j=-\infty}^{\infty} \left\{ q^{pp'j^2+j(p'a-ps)} \left( \frac{L, q}{2pj + a - b} \right)^1 
- q^{pp'j^2+j(p'a+(s-2))+(s-1)-b} \left( \frac{L, q}{2pj + a + b} \right)^1 \right\} \quad (2.9)$$

with $p' = p + 1$. Finally, introducing functions $\varphi(a, b)$ and $\tilde{\varphi}(a, b)$ which are defined by

$$\varphi(a, b) = \frac{(b - a)(b - a + 1)}{4} \quad (2.10)$$

and

$$\tilde{\varphi}(a, b) = \frac{(b - a)(b - a + 3)}{4} \quad (2.11)$$

We are in position to state the main results of this paper.

**Even Identities**

$$F_{p,0}^{a,0}(L) = \left\{ \begin{array}{ll}
q^{\frac{a(a-1)}{4}}B_{a,1}^p(L, 1) & \text{for } a + \delta_{a,1} \equiv 0 \pmod{2} \\
q^{\frac{(a-1)(a-2)}{4}}B_{a,1}^p(L, 2) & \text{for } a + \delta_{a,1} \equiv 1 \pmod{2}
\end{array} \right\} \quad (2.12)$$

$$F_{p,0}^{b,0}(L) = \left\{ \begin{array}{ll}
q^{\varphi(a,b)}B_{a,b}^p(L, b + 1) & \text{for } b + a + \delta_{a,1} \equiv 0 \pmod{2} \\
q^{\tilde{\varphi}(a,b)}\tilde{B}_{a,b}^p(L, b + 2) & \text{for } b + a + \delta_{a,1} \equiv 1 \pmod{2}
\end{array} \right\} \quad (2.13)$$

**Odd Identities**

$$F_{p,1}^{a,0}(L) = \left\{ \begin{array}{ll}
q^{\varphi(p-a,p-1)}B_{a,1}^p(L, 1) & \text{for } a \equiv 1 \pmod{2} \\
q^{\tilde{\varphi}(p-a,p-1)+1-a}B_{a,1}^p(L, 2) & \text{for } a \equiv 0 \pmod{2}
\end{array} \right\} \quad (2.14)$$

$$F_{p,1}^{b,0}(L) + \delta_{L,0}\delta_{a,b} = \left\{ \begin{array}{ll}
q^{\varphi(p-a,p-1)+\varphi(a,b)-\frac{a-1}{2}}B_{a,b}^p(L, b + 1) & \text{for } a + b \equiv 1 \pmod{2} \\
q^{\tilde{\varphi}(p-a,p-1)+\tilde{\varphi}(a,b)-\frac{a-1}{2}(a+2)}\tilde{B}_{a,b}^p(L, b + 2) & \text{for } a + b \equiv 0 \pmod{2}
\end{array} \right\} \quad (2.15)$$
with \( F_{p-1,b}^{p,1}(L) \equiv F_{1,b}^{p,0}(L) \) for \( 1 \leq b \leq p - 1 \). The identities (2.12) - (2.13) hold for \( L = 0, 1, 2, \ldots; p = 4, 5, 6, \ldots; a = 1, 2, \ldots, p - 2 \) and \( b = 2, 3, \ldots, p - 1 \).

Before we move on, a few comments are in order. The polynomials \( B_{a,b}(L, b + 1) \) and \( \tilde{B}_{a,b}(L, b + 2) \) (with restrictions on \( b \) - parameters which are not important for us here) were introduced in [13]-[17] as configuration sums for the regime 1+ of dilute \( A_{p-1} \) models. In addition, two more polynomials \( B_{p,a,b}(L, b) \) and \( B_{p,a,b}^\prime(L) \)

\[
\sum_{j=-\infty}^{\infty} \left\{ q^{ppj^2 + j(p^a - p)(b+1)} \left( \frac{L, q}{2pj + a - b} \right)^1 - q^{ppj^2 + j(p^a + p(b-1)) + b(a-1)} \left( \frac{L, q}{2pj + a + b} \right)^1 \right\}
\]

(2.16)

were employed in [17] for the reason of completeness. Here we point out that the last two polynomials are trivially related to the first two by

\[
B_{p,a,b}^p(L, p - b + 1) = B_{a,b}^p(L, b)
\]

(2.17)

\[
\tilde{B}_{p,a,b}^p(L, p - b + 2) = q^{b-a}B_{a,b}^p(L).
\]

(2.18)

In what follows we find it convenient to use simplified notations for \( F_{a,b}^{p,i}(L) \), namely

\[
F_{a,b}^{p,i}(L) = \sum_{(i)} q^{\Phi_{a,b}(i)} \left[ \frac{n + m}{n} \right]_{a,b}^{p,L}
\]

(2.19)

where for any \((p - 1)\) - dimensional vectors \( D, E \)

\[
\left[ \frac{n + m + D}{n + E} \right]_{a,b}^{p,L} = \prod_{i=0}^{p-2} \left[ \frac{n_i + m_i + D_i}{n_i + E_i} \right]_q
\]

(2.20)

and \( \sum_{(i)} \) stands for the sum over all \( n, m \) subject to constraints

\[
n + m = \frac{1}{2} I_{p-1} m + L e_0 + \frac{1}{2} u_{a,b}^i
\]

(2.21)

and

\[
m_{p-2} \equiv \begin{cases} i \text{ (mod 2)} & \text{if } b \neq p - 1 \\ 1 - i \text{ (mod 2)} & \text{if } b = p - 1 \end{cases},
\]

(2.22)

and the quadratic form \( \Phi_{a,b}^{p,i}(m) \) is defined as

\[
\Phi_{a,b}^{p,i}(m) = \frac{1}{4} m C_{p-1} m - \frac{1}{2} A_{a,b}^i \cdot m.
\]

(2.23)

We will also use a compact symbol

\[
\{n, m\}_{a,b}^{L,i}
\]

(2.24)
to denote set of all integer solutions to the \( n, m \) - system (2.21) subject to the additional constraint (2.22). (To keep notations tidy we suppressed dependence on \( p \) in (2.24).) It immediately follows from (1.6) and (2.19) that \( n \) and \( m \) are vectors with non-negative integer components, i.e. \( n, m \in \mathbb{Z}_{\geq 0}^p \).

We now list a few easily verifiable consequences of (2.21) which will be frequently used.

\[
L = n_0 + n_1 + 2 \sum_{j=1}^{p-3} j(n - \frac{1}{2} u^i_{a,b})_{j+1} + (p - 2)m_{p-2}, \quad (2.25)
\]

\[
L = n_0 + n_1 + m_0 + m_1, \quad (2.26)
\]

\[
L = n_0 + n_1 + m_2 + 2 \sum_{j=2}^{p-2} (n - \frac{1}{2} u^i_{a,b})_j + m_{p-2}, \quad (2.27)
\]

\[
m_0 = \sum_{j=1}^{p-2} j(n - \frac{1}{2} u^i_{a,b})_j + \frac{1}{2}(p - 1)m_{p-2}, \quad (2.28)
\]

and

\[
m_k = 2 \sum_{j=1}^{p-2-k} j(n - \frac{1}{2} u^i_{a,b})_{k+j} + (p - 1 - k)m_{p-2} \text{ for } 2 \leq k \leq p - 2. \quad (2.29)
\]

In the spirit of Andrews proof of Rogers-Ramanujan identities [18] we will prove (2.12-2.15) by showing that fermionic and bosonic polynomials there satisfy the same recurrences and that our identities hold for \( L = 0 \).

To generate recurrences for fermionic polynomials in (2.12-2.15) we will employ the technique of telescopic expansions which was introduced in [11] and further developed in [19]-[22]. This technique is based on the following elementary recursion relations for \( q \)-binomial coefficients:

\[
\begin{bmatrix} n + m \\ n \end{bmatrix}_q = \begin{bmatrix} n - 1 + m \\ n - 1 \end{bmatrix}_q + q^n \begin{bmatrix} n + m - 1 \\ n \end{bmatrix}_q \quad (2.30)
\]

and

\[
\begin{bmatrix} n + m \\ n \end{bmatrix}_q = \begin{bmatrix} n + m - 1 \\ n \end{bmatrix}_q + q^m \begin{bmatrix} n - 1 + m \\ n - 1 \end{bmatrix}_q \quad (2.31)
\]

Note, however, that (2.30), (2.31) fail for \( n = m = 0 \). In this case one gets the contradiction \( 1 = 0 \). Fortunately, there is a way to modify the definition (1.6) in such a way that (2.30), (2.31) will hold without exceptions. The appropriate definition which we borrow from [23] is

\[
\begin{bmatrix} n + m \\ n \end{bmatrix}_q' = \begin{cases} \frac{(q^{n+1})_m}{(q)_m} & \text{for } m \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases} \quad (2.32)
\]
It is easy to verify that modified $q$-binomials (2.32) can be expressed in terms of standard ones (1.6) as

\[
\binom{n+m}{n}_q = \begin{cases} 
\binom{n+m}{n}_q & \text{if } n, m \geq 0, \\
(-)^m q^{m+1+n}\binom{-n-1}{m}_q & \text{if } n + m < 0 \text{ and } m \geq 0 \\
0 & \text{otherwise}.
\end{cases}
\] (2.33)

Clearly, modified $q$-binomials vanish for negative values of $n$ as long as $n + m \geq 0$. At this point one may wonder if it is possible to replace all standard binomials in (2.19) by modified ones and then use (2.30) and (2.31) universally without hesitation. Equation (2.33) indicates that such a replacement will be legitimate if one can show that $n_i + m_i \geq 0$ for all $i = 0, 1, 2, \ldots, p - 2$. From equation (2.32) we easily infer that all $m$-variables will remain non-negative, i.e.

\[ m_i \geq 0 \text{ for } 0 \leq i \leq p - 2. \] (2.34)

This being the case, it follows from (2.21) that $n_i + m_i \geq 0$ for $i \neq 0$, which in turn implies

\[ n_i \geq 0 \text{ for } 1 \leq i \leq p - 2 \] (2.35)

For $i = 0$, the $n, m$ system (2.21) does not guarantee that $n_0 + m_0 \geq 0$, and, in fact, the sum in (2.19) will receive extra contributions from negative values of $n_0$ if the above mentioned replacement is made. However, in Appendix A we will demonstrate that these negative contributions cancel each other out, unless $L = 0, p - a = b \neq 1, p - 1; i = 1$ in (2.19). In the latter case one has an overall contribution equal to 1. Thus, if we replace all standard binomials in (2.19) by modified ones and denote the resulting polynomials as $\bar{F}_{a,b}^p(L)$, then

\[
\bar{F}_{a,b}^{p,0}(L) = F_{a,b}^{p,0}(L)
\] (2.36)

and

\[
\bar{F}_{a,b}^{p,1}(L) = F_{a,b}^{p,1}(L) + \delta_{L,0}\delta_{a,p-b} \theta(1 < b < p - 1)
\] (2.37)

for $1 \leq a \leq p - 2, 1 \leq b \leq p - 1$.

### 3 Bosonic Recurrences

In the course of this paper we shall require various properties of $q$-trinomial coefficients such as asymptotics, symmetry, recursion relations and some identities which become tautologies when $q = 1$. Below we shall first list necessary formulas and indicate where they first appeared in the literature. Next we shall use the collected results to obtain two new $q$-trinomial identities and to derive recurrences for bosonic polynomials introduced in Section 2.
Limiting Properties

\[
\lim_{L \to \infty} \left( \begin{array}{c}
L \\
q A
\end{array} \right)^0 = \frac{1}{(q)_{\infty}}
\] (3.1)

\[
\lim_{L \to \infty} \left( \begin{array}{c}
L \\
q A
\end{array} \right)^1 = \frac{1 + q^A}{(q)_{\infty}}
\] (3.2)

Equations (3.1) and (3.2) are formulas (2.48) and (2.49) of [1], respectively.

Symmetry Relations

\[
\left( \begin{array}{c}
L \\
q A
\end{array} \right)^0 = \left( \begin{array}{c}
L \\
-A
\end{array} \right)^0
\] (3.3)

\[
\left( \begin{array}{c}
L \\
q A
\end{array} \right)^1 = q^A \left( \begin{array}{c}
L \\
-A
\end{array} \right)^1
\] (3.4)

The above two equations follow immediately from equation (2.15) of [1].

Pascal triangle recurrences

\[
\left( \begin{array}{c}
L \\
q A
\end{array} \right)^0 = \left( \begin{array}{c}
L - 1 \\
q A + 1
\end{array} \right)^0 + q^{L-1-A} \left( \begin{array}{c}
L - 1 \\
q A
\end{array} \right)^1 + q^{L-A} \left( \begin{array}{c}
L - 1 \\
A - 1
\end{array} \right)^0
\] (3.5)

\[
\left( \begin{array}{c}
L \\
q A
\end{array} \right)^1 = \left( \begin{array}{c}
L - 1 \\
q A - 1
\end{array} \right)^0 + q^{L-1} \left( \begin{array}{c}
L - 1 \\
q A
\end{array} \right)^1 + q^A \left( \begin{array}{c}
L - 1 \\
A + 1
\end{array} \right)^0
\] (3.6)

Depth one recurrences (3.5) and (3.6) were first established in [1] where they appeared as formulas (2.26) and (2.25), respectively. In addition, depth two recurrences

\[
\left( \begin{array}{c}
L \\
q A
\end{array} \right)^n = q^{L-A} \left( \begin{array}{c}
L - 1 \\
q A - 1
\end{array} \right)^n + q^{L+A-n} \left( \begin{array}{c}
L - 1 \\
A + 1
\end{array} \right)^n +
\]

\[
+ \left( \begin{array}{c}
L - 1 \\
q A
\end{array} \right)^n + q^{L-1-n} (1 - q^{L-1}) \left( \begin{array}{c}
L - 2 \\
A
\end{array} \right)^n
\] (3.7)

with \( n \in \mathbb{Z} \), have proven to be very useful. For \( n = 0 \), equation (3.7) was first derived in [4] as equation (2.3). Later, the proof given there was further extended for all \( n \in \mathbb{Z} \) in [19] as equation (A.1).
Tautologies

\[
\begin{align*}
\left( \frac{L}{A} \right)^{0} &= \left( \frac{L}{A} \right)^{1} + q^{A} \left( 1 - q^{L} \right) \left( \frac{L - 1}{A + 1} \right)^{0} \\
\left( \frac{L}{A} \right)^{0} - q^{L-A} \left( \frac{L}{A} \right)^{1} &= \left( \frac{L}{A + 1} \right)^{0} - q^{L} \left( \frac{L}{A + 1} \right)^{1} \\
\left( \frac{L}{A} \right)^{0} &= q^{L} \left( \frac{L}{A} \right)^{1} + (1 - q^{L}) \left( \frac{L - 1}{A - 1} \right)^{0} \\
&+ q^{L-1} \left( 1 - q^{L} \right) \left( \frac{L - 1}{A} \right)^{1} \\
\left( \frac{L}{A + 1} \right)^{1} - q^{1-A} \left( \frac{L}{A - 1} \right)^{1} &= q^{A+1} \left( \frac{L - q}{A + 1} \right)^{0} - q^{1-A} \left( \frac{L}{A - 1} \right)^{0}
\end{align*}
\] (3.8)

Identity (3.8) is equation (2.23) of [1] with \( A = B \). Identity (3.9) is equation (2.27) (corrected) of [1]. Identity (3.10) can be obtained from equation (2.13) of [24] with \( p = 2, r = 1 \). Finally, identity (3.11) follows from equation (A.8) of [19] with \( n = 1 \) and \( A \) replaced by \( A + 1 \).

Each of identities (3.5)-(3.11) gives rise to a twin identity if we first replace \( A \) by \(-A\) and then use symmetry relations (3.3), (3.4). In particular, from (3.7) we obtain

\[
\begin{align*}
\left( \frac{L}{A} \right)^{0} &= \left( \frac{L - 1}{A - 1} \right)^{0} + q^{L-1} \left( \frac{L - 1}{A} \right)^{1} + q^{L+A} \left( \frac{L - 1}{A + 1} \right)^{0}
\end{align*}
\] (3.12)

Our object now is to prove two new \( q \)-trinomial coefficients recurrences:

\[
\begin{align*}
\left( \frac{L}{A} \right)^{0} &= q^{L-A} \left( \frac{L - 1}{A - 1} \right)^{1} + \left( \frac{L - 1}{A} \right)^{0} + q^{L-1} \left( \frac{L - 1}{A + 1} \right)^{1} \\
&+ q^{L-1}(q^{L-1} - 1) \left( \frac{L - 2}{A} \right)^{0}
\end{align*}
\] (3.13)

\[
\begin{align*}
\left( \frac{L}{A} \right)^{1} &= \left\{ \left( \frac{L - 1}{A - 1} \right)^{0} + (q^{L-1} - 1) \left( \frac{L - 2}{A - 1} \right)^{0} \right\} + \left( \frac{L - 1}{A} \right)^{1} \\
&+ q^{A} \left\{ \left( \frac{L - 1}{A + 1} \right)^{0} + (q^{L-1} - 1) \left( \frac{L - 2}{A + 1} \right)^{0} \right\} \\
&+ q^{L-2}(q^{L-1} - 1) \left( \frac{L - 2}{A} \right)^{1}.
\end{align*}
\] (3.14)
Both (3.13) and (3.14) are depth two recurrences, but unlike relations (3.7) they mix $q$-trinomials with different superscripts. The proof of (3.13) follows immediately by combining (3.7) with $n = 0$ and (3.8) along with its twin. The proof of (3.14) is just slightly more involved. Here, we first combine (3.10) with the twin of (3.9) to get

$$\left( \begin{array}{c} L, q \\ A - 1 \end{array} \right)^0 = q^{L+1-A} \left( \begin{array}{c} L, q \\ A - 1 \end{array} \right)^1 + (1 - q^L) \left( \begin{array}{c} L - 1, q \\ A - 1 \end{array} \right)^0$$

$$+ q^{L-1}(1 - q^L) \left( \begin{array}{c} L - 1, q \\ A \end{array} \right)^1$$

(3.15)

and its twin

$$\left( \begin{array}{c} L, q \\ A + 1 \end{array} \right)^0 = q^L \left( \begin{array}{c} L, q \\ A + 1 \end{array} \right)^1 + (1 - q^L) \left( \begin{array}{c} L - 1, q \\ A + 1 \end{array} \right)^0$$

$$+ q^{L-1-A}(1 - q^L) \left( \begin{array}{c} L - 1, q \\ A \end{array} \right)^1$$

(3.16)

Next we use (3.7) with $n = 1$ and (3.13), (3.16) with $L$ replaced by $L - 1$ to obtain (3.14).

Now we are ready to establish the following recursion relations for bosonic polynomials introduced in Section 2:

$$B_{a,1}^p(L, 1) = B_{a,1}^p(L - 1, 1) + q^{L+1-a}B_{a,2}^p(L - 1, 3)$$

(3.17)

$$B_{a,1}^p(L, 2) = B_{a,1}^p(L - 1, 2) + q^{L+1-a}B_{a,2}^p(L - 1, 4)$$

(3.18)

$$B_{a,2}^p(L, 3) = B_{a,2}^p(L - 1, 3) + q^{L+2-a}B_{a,1}^p(L - 1, 1)$$

$$+ q^{L-a+2}B_{a,3}^p(L - 1, 5)$$

(3.19)

$$B_{a,2}^p(L, 4) = q^{L-1}B_{a,2}^p(L - 1, 4) + q^{a-2}B_{a,1}^p(L - 1, 2)$$

$$+ B_{a,3}^p(L - 1, 4)$$

(3.20)

$$B_{a,b}^p(L, b + 1) = B_{a,b}^p(L - 1, b + 1) + q^{L-1}(q^{L-1} - 1)B_{a,b}^p(L - 2, b + 1)$$

$$+ q^{L-a+b}B_{a,b+1}^p(L - 1, b + 3) + q^{L-1}B_{a,b-1}^p(L - 1, b + 1)$$

(3.21)

$$B_{a,b}^p(L, b + 2) = B_{a,b+1}^p(L - 1, b + 2) + (q^{L-1} - 1)B_{a,b}^p(L - 2, b + 2)$$

$$+ q^{a-b}(B_{a,b-1}^p(L - 1, b) + (q^{L-1} - 1)B_{a,b-1}^p(L - 2, b))$$

$$+ B_{a,b}^p(L - 1, b + 2) + q^{L-2}(q^{L-1} - 1)B_{a,b}^p(L - 2, b + 2)$$

(3.22)

Because the factor $q^{L-1} - 1$ in (3.21) and (3.22) vanishes for $L = 1$ one needs to specify initial conditions only for $L = 0$ to determine these polynomials uniquely. The validity of (3.17)-(3.19) is readily seen from (3.3) and (3.12). Relation (3.20) follows by use of (3.3) and its twin identity.
Finally (3.13) along with its twin and (3.14) along with its twin imply (3.21) and (3.22), respectively. We close this section by noting some additional properties of bosonic polynomials.

**Reflection properties**

\[
B_{p,a-1}(L,p) = B_{p,a}(L,1) \tag{3.23}
\]

\[
\tilde{B}_{a-1}^{p}(L,p+1) = q^{a-p+1}B_{p,a}(L,2) \tag{3.24}
\]

**Limiting properties**

\[
\lim_{L \to \infty} B_{a,b}^{p}(L,b+1) = \chi_{a,b+1}(q) \tag{3.25}
\]

\[
\lim_{L \to \infty} \tilde{B}_{a,b}^{p}(L,b+2) = \chi_{a,b+2}(q) + q^{a-b} \chi_{a,b}^{p+1}(q) \tag{3.26}
\]

with \(\chi_{a,b}^{p+1}(q)\) defined by (1.7).

**Closing properties**

\[
B_{a,p}(L,p+1) = \tilde{B}_{a,p}^{p}(L,p+2) = 0 \tag{3.27}
\]

Equations (3.23) and (3.27) can be easily verified by inspection. Equations (3.25)-(3.26) are simply consequences of (3.1) and (3.2). Identity (3.24) follows from (3.11). Tautology (3.11) can be also used to derive a further interesting identity

\[
B_{a,1}^{p}(L,2) = \tilde{B}_{a,1}^{p}(L,2) \tag{3.28}
\]

**4 Fermionic Recursion Relations**

We now turn our attention to the proof of the following fermionic recurrences:

\[
F_{a,1}^{p,0}(L) = F_{a,1}^{p,0}(L-1) + q^{L-\frac{1}{2}}F_{a,2}^{p,0}(L-1) \quad \text{for } a + \delta_{a,1} \equiv 0 \pmod{2} \tag{4.1}
\]

\[
F_{a,1}^{p,0}(L) = F_{a,1}^{p,0}(L-1) + q^{L-1}F_{a,2}^{p,0}(L-1) \quad \text{for } a + \delta_{a,1} \equiv 1 \pmod{2} \tag{4.2}
\]

\[
F_{a,2}^{p,0}(L) = q^{L-1}F_{a,2}^{p,0}(L-1) + q^{L-1}F_{a,1}^{p,0}(L-1) + F_{a,2}^{p,0}(L-1) \quad \text{for } a + \delta_{a,1} \equiv 0 \pmod{2} \tag{4.3}
\]
\[ F_{a,2}^p(L) = \frac{1}{\sqrt{q}} F_{a,3}^p(L - 1) + F_{a,1}^p(L - 1) + q^{L-1} F_{a,2}^p(L - 1) \]

for \( a + \delta_{a,1} \equiv 1 \pmod{2} \) \hspace{1cm} (4.4)

\[ F_{a,b}^p(L) = q^{L-1} F_{a,b+1}^p(L - 1) + q^{L-1} F_{a,b-1}^p(L - 1) + F_{a,b}^p(L - 2) \]

for \( b + a + \delta_{a,1} \equiv 0 \pmod{2} \), \( 2 < b \leq p - 1 \) \hspace{1cm} (4.5)

\[ F_{a,b}^p(L) = \frac{1}{\sqrt{q}} \left\{ F_{a,b+1}^p(L - 1) + (q^{L-1} - 1) F_{a,b+1}^p(L - 2) \right\} \]

\[ + \left\{ F_{a,b-1}^p(L - 1) + (q^{L-1} - 1) F_{a,b-1}^p(L - 2) \right\} \]

\[ + \left\{ F_{a,b}^p(L - 1) + q^{L-2}(q^{L-1} - 1) F_{a,b}^p(L - 2) \right\} \]

for \( b + a + \delta_{a,1} \equiv 1 \pmod{2} \), \( 2 < b \leq p - 1 \) \hspace{1cm} (4.6)

with \( F_{a,p}^p(L) \equiv 0 \). In contrast with many identities relating \( q \)-trinomial coefficients used in the proof of the bosonic recurrences (3.17-22) the only identities we shall require here are the elementary recursion relations for \( q \)-binomials (2.30-2.31). Recurrences (4.1-5) will be proven in this section. The proof of equation (4.6) will be relegated to Appendix B.

### 4.1 Proof of (4.1) and (4.2)

Using equation (2.30) we can expand \( F_{a,1}^p(L) \) as

\[ F_{a,1}^p(L) = \sum_{(0)} q^{(0)} \Phi_{a,1}^p(m) \left[ \frac{n + m - e_0}{n - e_0} \right]_{a,1}^{p,L} + \]

\[ + \sum_{(0)} q^{(0)} \Phi_{a,1}^p(m) + n_0 \left[ \frac{n + m - e_0}{n} \right]_{a,1}^{p,L} \]. \hspace{1cm} (4.7)

Now equations (2.25) and (2.28-29) imply that

\[ \{n, m\}_{a,1}^{L,0} - \{e_0, 0\} = \{n, m\}_{a,1}^{L-1,0} \]

and

\[ \{n, m\}_{a,1}^{L,0} - \{0, e_0\} = \{n, m\}_{a,1}^{L-1,0} \]

The first equation of (2.21)

\[ n_0 + m_0 = L - \frac{m_2}{2} - \frac{1}{2} \tilde{p}(a + \delta_{a,1} + b + \delta_{b,1}) \]

with \( b = 1 \) gives

\[ \Phi_{a,1}^p(m) + n_0 = L - \frac{1}{2} - \frac{1}{2} \tilde{p}(a + \delta_{a,1}) + \Phi_{a,2}^p(m - e_0) \]. \hspace{1cm} (4.10)
Hence, we can identify the first and second terms in the right-hand-side of (4.7) as
\[ F_{a,1}^p(L - 1) \text{ and } q^{-\frac{1}{2} - \frac{1}{2} (a + \delta_{a,1})} F_{a,2}^p(L - 1), \]
respectively. Thus (4.1) and (4.2) are established.

### 4.2 Proof of (4.3) and (4.4)

To prove (4.3), we start by expanding \( F_{a,2}^p(L) \) with \( a + \delta_{a,1} \equiv 0 \pmod{2} \) in telescopic fashion as
\[
F_{a,2}^p(L) = \sum \left(0 \right) q^{\Phi_{a,2}^p(m)+n_0} \left[ n + m - e_0 - e_2 \right]_{a,2}^{p,L} + \sum \left(0 \right) q^{\Phi_{a,2}^p(m)+n_0+m_2} \left[ n + m - e_0 - e_2 \right]_{a,2}^{p,L} + \sum \left(0 \right) q^{\Phi_{a,2}^p(m)} \left[ n + m - e_0 \right]_{a,2}^{p,L},
\]
where (2.31) was used to combine first and second terms in the right-hand-side of (4.12) and (2.30) was used in the last step.

Next with the help of (2.25) and (2.28-29) one shows that
\[
\{n, m\}_{a,2}^{L,0} - \{0, e_0 + e_2\} = \{n, m\}_{a,3}^{L-1,0},
\]
(4.13)
\[
\{n, m\}_{a,2}^{L,0} - \{e_2, e_0\} = \{n, m\}_{a,3}^{L-1,0},
\]
(4.14)
and
\[
\{n, m\}_{a,2}^{L,0} - \{e_0, 0\} = \{n, m\}_{a,3}^{L-1,0}.
\]
(4.15)
Using equation (4.10) with \( b = 2 \) and \( a + \delta_{a,1} \equiv 0 \pmod{2} \) one readily checks that
\[
\Phi_{a,2}^p(m) + n_0 = L - 1 + \Phi_{a,3}^p(m - e_0 - e_2)
\]
(4.16)
and
\[
\Phi_{a,2}^p(m) + n_0 + m_2 = L - \frac{1}{2} + \Phi_{a,1}^p(m - e_0).
\]
(4.17)
Equations (4.13-17) enable one to recognize first, second, and third terms in the right-hand-side of (4.12) as \( q^{L-1} F_{a,3}^p(L - 1), q^{L-\frac{1}{2}} F_{a,1}^p(L - 1), \text{ and } F_{a,2}^p(L - 1), \) respectively. Thus, (4.3) is established.

The proof of (4.4) is however considerably more involved then the simple demonstration of (4.3). Again, we start by expanding \( F_{a,2}^p(L) \) with \( a + \delta_{a,1} \equiv 1 \pmod{2} \) in telescopic fashion as
\[
F_{a,2}^p(L) = \sum \left(0 \right) q^{\Phi_{a,2}^p(m)+n_0} \left[ n + m - e_1 - e_2 \right]_{a,b}^{p,L} + \sum \left(0 \right) q^{\Phi_{a,2}^p(m)+n_0+m_2} \left[ n + m - e_1 - e_2 \right]_{a,b}^{p,L} + \sum \left(0 \right) q^{\Phi_{a,2}^p(m)+m_1} \left[ n + m - e_1 \right]_{a,b}^{p,L},
\]
(4.18)
where \((2.31)\) was used twice to recombine all terms in the right-hand-side. In the same manner as before, it is possible to show that

\[
\frac{1}{\sqrt{q}} F_{a,3}^{p}(L-1) = \sum \phi_{a,2}^{(0)}(m) \left[ \frac{n + m - e_1 - e_2}{n} \right]_{a,b}^{p,L},
\]

(4.19)

\[
F_{a,1}^{p}(L-1) = \sum \phi_{a,2}^{(0)}(m) \left[ \frac{n + m - e_0 - e_2}{n - e_2} \right]_{a,2}^{p,L},
\]

(4.20)

and

\[
F_{a,2}^{p}(L-1) = \sum \phi_{a,2}^{(0)}(m) \left[ \frac{n + m - e_0}{n - e_0} \right]_{a,2}^{p,L}.
\]

(4.21)

Employing \((2.31)\) one can expand \((4.20)\) further as

\[
F_{a,1}^{p}(L-1) = \sum \phi_{a,2}^{(0)}(m) \left[ \frac{n + m - E_{0,2}}{n - E_{1,2}} \right]_{a,2}^{p,L} + \sum \phi_{a,2}^{(0)}(m) \left[ \frac{n + m - E_{0,2}}{n - e_2} \right]_{a,2}^{p,L},
\]

(4.22)

where for \(b \geq a\) \((p - 1)\)-dimensional vector \(E_{a,b}\) is defined as

\[
E_{a,b} = \sum_{i=a}^{b} e_i.
\]

(4.23)

Next making use of \((2.30)\) we expand the second term in the right-hand-side of \((4.18)\) as

\[
\sum \phi_{a,2}^{(0)}(m) + m_2 \left[ \frac{n + m - E_{0,2}}{n - e_2} \right]_{a,2}^{p,L} = \sum \phi_{a,2}^{(0)}(m) + m_2 \left[ \frac{n + m - E_{0,2}}{n - e_0 - e_2} \right]_{a,2}^{p,L} + \sum \phi_{a,2}^{(0)}(m) + m_2 + n_0 \left[ \frac{n + m - E_{0,2}}{n - e_2} \right]_{a,2}^{p,L},
\]

(4.24)

and then perform the change of summation variables

\[
n \rightarrow n + e_0 - e_1 + e_2; m \rightarrow m - e_0 + e_1
\]

(4.25)

in the first sum appearing in the right-hand-side of \((4.24)\) to obtain

\[
\sum \phi_{a,2}^{(0)}(m) + m_2 \left[ \frac{n + m - E_{0,2}}{n - e_0 - e_2} \right]_{a,2}^{p,L} = \sum \phi_{a,2}^{(0)}(m) + m_0 + m_1 + \frac{m_2}{2} \left[ \frac{n + m - E_{0,2}}{n - E_{1,2}} \right]_{a,2}^{p,L},
\]

(4.26)

Combining \((4.22)\), \((4.24)\), \((4.26)\) and making use of \((4.10)\) with \(b = 2, a + \delta_{a,1} \equiv 1(\text{mod}\ 2)\), we have

\[
\sum \phi_{a,2}^{(0)}(m) + m_2 \left[ \frac{n + m - E_{0,2}}{n - e_2} \right]_{a,2}^{p,L} = F_{1}(L - 1) + (q^{L-1} - 1) \sum \phi_{a,2}^{(0)}(m) - m_0 + \frac{m_2 + 1}{2} \left[ \frac{n + m - E_{0,2}}{n - e_2} \right]_{a,2}^{p,L}.
\]

(4.27)
Finally, let us perform the change of summation variables

\[ n \rightarrow n - e_0 - e_1 + e_2 \ ; \ m \rightarrow m + e_0 + e_1 \]  \hspace{1cm} (4.28)

in the second sum in (4.27) to derive for the second term in the right-hand-side of (4.18)

\[
\sum_{a_2}^{} q^{p,0}_{\Phi,2}(m) + m_2 \left[ \frac{n + m - E_{1,2}}{n - e_2} \right]^{p, L}_{a_2} = \\
F_1 (L - 1) + (q^{L-1} - 1) \sum_{a_2}^{} q^{p,0}_{\Phi,2}(m) + m_1 \left[ \frac{n + m - E_{0,1}}{n - E_{0,1}} \right]^{p, L}_{a_2} . \]  \hspace{1cm} (4.29)

We now turn to the last term in the right-hand-side of (4.18). We start by expanding it with the aid of (2.30) as

\[
\sum_{a_2}^{} q^{p,0}_{\Phi,2}(m) + m_1 \left[ \frac{n + m - e_1}{n - e_1} \right]^{p, L}_{a_2} = \sum_{a_2}^{} q^{p,0}_{\Phi,2}(m) + m_1 + n_0 \left[ \frac{n + m - E_{0,1}}{n - e_1} \right]^{p, L}_{a_2} \\
+ \sum_{a_2}^{} q^{p,0}_{\Phi,2}(m) + m_1 \left[ \frac{n + m - E_{0,1}}{n - E_{0,1}} \right]^{p, L}_{a_2} . \]  \hspace{1cm} (4.30)

Next we perform the charge of summation variables

\[ n \rightarrow n - e_0 + e_1 \ ; \ m \rightarrow m + e_0 - e_1 \]  \hspace{1cm} (4.31)

in the first term in the right-hand-side of (4.30) to get

\[
\sum_{a_2}^{} q^{p,0}_{\Phi,2}(m) + m_1 \left[ \frac{n + m - e_1}{n - e_1} \right]^{p, L}_{a_2} = \sum_{a_2}^{} q^{p,0}_{\Phi,2}(m) + m_1 + n_0 \left[ \frac{n + m - E_{0,1}}{n - e_1} \right]^{p, L}_{a_2} \\
+ \sum_{a_2}^{} q^{p,0}_{\Phi,2}(m) + m_1 \left[ \frac{n + m - E_{0,1}}{n - E_{0,1}} \right]^{p, L}_{a_2} . \]  \hspace{1cm} (4.32)

where we also used (4.10) with \( b = 2, a + \delta_{a,1} \equiv 1(\text{mod} \ 2) \). If we now add and subtract from the right-hand-side of (4.32) the following sum

\[
\sum_{a_2}^{} q^{p,0}_{\Phi,2}(m) + m_1 + (L-1) \left[ \frac{n + m - E_{0,1}}{n - E_{0,1}} \right]^{p, L}_{a_2} 
\]

and then use (2.31) to recombine two terms, the result is

\[
\sum_{a_2}^{} q^{p,0}_{\Phi,2}(m) + m_1 \left[ \frac{n + m - e_1}{n - e_1} \right]^{p, L}_{a_2} = \\
= \sum_{a_2}^{} q^{p,0}_{\Phi,2}(m) + m_1 + n_0 \left[ \frac{n + m - e_0}{n - e_0} \right]^{p, L}_{a_2} \\
+ (1 - q^{L-1}) \sum_{a_2}^{} q^{p,0}_{\Phi,2}(m) + m_1 \left[ \frac{n + m - E_{0,1}}{n - E_{0,1}} \right]^{p, L}_{a_2} . \]  \hspace{1cm} (4.33)
Recalling (4.21), we can rewrite (4.33) as

\[ \sum_{(0)} q^{\Phi_{a,2}(m)+m_1} \left[ \frac{n+m-e_1}{n-e_1} \right]_{a,2}^{p,L} = \]
\[ = q^{L-1} F_{a,2}^{p,0}(L-1) + (1 - q^{L-1}) \sum_{(0)} q^{\Phi_{a,2}(m)+m_1} \left[ \frac{n+m-E_{0,1}}{n-E_{0,1}} \right]_{a,2}^{p,L}. \] (4.34)

Finally, combining (4.18), (4.19), (4.29) and (4.34) we arrive at the desired result (4.4). Thus (4.4) is established.

### 4.3 Proof of (4.5)

Once again, we start by using (2.30), (2.31) to expand \( F_{a,0}^{p,0}(L) \) with \( a + \delta_{a,1} + b + \delta_{b,1} \equiv 0 \pmod{2} \), \( 2 < b \leq p - 1 \) in telescopic fashion as

\[ F_{a,b}^{p,0}(L) = \theta(b \neq p - 1) \sum_{(0)} q^{\Phi_{a,b}(m)+n_0} \left[ \frac{n+m-e_0-E_{2,b}}{n} \right]_{a,b}^{p,L} + \]
\[ + \sum_{l=3}^b \sum_{(0)} q^{\Phi_{a,b}(m)+n_0+m_l} \left[ \frac{n+m-e_0-E_{2,l}}{n-e_L} \right]_{a,b}^{p,L} + \]
\[ + \sum_{(0)} q^{\Phi_{a,b}(m)+n_0+m_2} \left[ \frac{n+m-e_0-e_2}{n-e_2} \right]_{a,b}^{p,L} + \]
\[ + \sum_{(0)} q^{\Phi_{a,b}(m)} \left[ \frac{n+m-e_0}{n-e_0} \right]_{a,b}^{p,L}. \] (4.35)

Employing (2.25), (2.28), (2.29) along with (4.10) it is straightforward (through tedious) to verify the following:

\[ q^{L-1} F_{a,b+1}^{p,0}(L-1) = \theta(b \neq p - 1) \sum_{(0)} q^{\Phi_{a,b}(m)+n_0} \left[ \frac{n+m-e_0-E_{2,b}}{n} \right]_{a,b}^{p,L} \] (4.36)
\[ F_{a,b}^{p,0}(L-1) = \sum_{(0)} q^{\Phi_{a,b}(m)} \left[ \frac{n+m-e_0}{n-e_0} \right]_{a,b}^{p,L} \] (4.37)
\[ q^{L-1} F_{a,b-1}^{p,0}(L-1) = \sum_{(0)} q^{\Phi_{a,b}(m)+n_0+1-m_{b-1}+m_b} \left[ \frac{n+m-e_0-E_{2,b-1}+e_{b-1}-e_b}{n+e_{b-1}-e_b} \right]_{a,b}^{p,L} \] (4.38)
\[ q^{L-1} F_{a,b}^{p,0}(L-2) = \sum_{(0)} q^{\Phi_{a,b}(m)+n_0+1-m_{b-1}+m_b} \left[ \frac{n+m-E_{0,1}-2E_{2,b-1}+e_{b-1}-e_b}{n+e_{b-1}-e_b} \right]_{a,b}^{p,L} \] (4.39)
\[ q^{2(L-1)}F_{a,b}^{p,0}(L-2) = \sum_{q} (0) \sum \phi_{a,b}^{p,0}(m_{+} + 2n_{0} + m_{2} + n + m + 2e_{0} - e_{2} - e_{2})_{a,b} \] (4.40)

with \( m_{p-1} \equiv 0 \) and \( e_{p-1} \equiv 0 \).

To proceed further, it’s expedient to expand (4.38) with the aid of (2.31) in telescopic fashion as follows

\[ q^{L-\frac{1}{2}}F_{a,b}^{p,0}(L-1) = \sum_{q} (0) \sum \phi_{a,b}^{p,0}(m)_{n_0 + 1 - m_{b-1} + m_b} \sum_{l=2}^{b-1} \sum_{n} (0) \sum \phi_{a,b}^{p,0}(m)_{n_0 + 1 - m_{b-1} + m_b} \left( \begin{array}{c} n + m + 2e_{0} - e_{0} - e_{b-1} - e_{b} \\ n + e_{b-1} - e_{b} \end{array} \right)_{a,b} \] (4.41)

Using (2.31) again, we split the first sum in (4.41) into two pieces as

\[ \sum_{q} (0) \sum \phi_{a,b}^{p,0}(m)_{n_0 + 1 - m_{b-1} + m_b} \left( \begin{array}{c} n + m - e_{0} - 2E_{2,b-1} + e_{b-1} - e_{b} \\ n + e_{b-1} - e_{b} \end{array} \right)_{a,b} = \] (4.42)

Recalling (4.39), we identify the first place as \( q^{L-1}F_{a,b}^{p,0}(L-2) \). Next we perform the change of summation variables

\[ n \rightarrow n - e_{0} + e_{1} - e_{2} - e_{b-1} + e_{b} ; m \rightarrow m + 2E_{2,b-1} \] (4.43)

in the second piece to obtain

\[ \sum_{q} (0) \sum \phi_{a,b}^{p,0}(m)_{n_0 + 1 - m_{b-1} + m_b + 1} \left( \begin{array}{c} n + m - e_{0} - 2e_{0} - e_{2} - e_{b-1} - e_{b} \\ n - e_{1} + e_{b-1} - e_{b} \end{array} \right)_{a,b} = \] (4.44)

Hence by (4.39), (4.41), (4.42) and (4.44),

\[ q^{L-\frac{1}{2}}F_{a,b}^{p,0}(L-1) - q^{L-1}F_{a,b}^{p,0}(L-2) = \sum_{q} (0) \sum \phi_{a,b}^{p,0}(m)_{n_0 + 2} \left( \begin{array}{c} n + m - 2e_{0} - e_{2} \\ n - e_{0} - e_{2} \end{array} \right)_{a,b} = \] (4.45)

\[ \sum_{l=2}^{b-1} \sum_{q} (0) \sum \phi_{a,b}^{p,0}(m)_{n_0 + 1 - m_{b-1} + m_b + (m - 1)} \left( \begin{array}{c} n + m - e_{0} - 2E_{2,b-1} + e_{b-1} - e_{b} - E_{l,b-1} \\ n + e_{b-1} - e_{b} - e_{l} \end{array} \right)_{a,b} \]
Let us now perform the \( l \)-dependent change of summation variables
\[
\begin{align*}
n &\rightarrow n - e_{l-1} + e_l + e_{b-1} - e_b \\
m &\rightarrow m - 2E_{l,b-1}
\end{align*}
\] (4.46)
in the double sum featured in equation (4.35) to derive
\[
\sum_{l=3}^{b} \sum_{m=2}^{b} \Phi_{a,b}^{p,0}(m + n_0 + m_l) \begin{bmatrix} n + m - e_0 - E_{2,l} \\ n - e_l \end{bmatrix}^{p,L}_{a,b} = \]
\[
= b-1 \sum_{l=2}^{b-1} \Phi_{a,b}^{p,0}(m + n_0 + m_l - m_{l-1} + m_{l-1}) \begin{bmatrix} n + m - e_0 - E_{2,b-1} + e_{b-1} - e_b - E_{l,b-1} \\ n + e_{b-1} - e_b - e_l \end{bmatrix}^{p,L}_{a,b}. \] (4.47)

We now use (4.36-37), (4.45) and (4.47) to simplify (4.35) as follows
\[
F_{a,b}^{p,0}(L) = q^{L-1} F_{a,b+1}^{p,0}(L - 1) + F_{a,b}^{p,0}(L - 1) + q^{L-1} F_{a,b-1}^{p,0}(L - 1) \\
- q^{L-1} F_{a,b}^{p,0}(L - 2) - \sum_{l=2}^{b-1} \Phi_{a,b}^{p,0}(m + n_0 + m_l) \begin{bmatrix} n + m - 2e_0 - e_2 \\ n - e_0 - e_2 \end{bmatrix}^{p,L}_{a,b} \\
+ \sum_{l=2}^{b-1} \Phi_{a,b}^{p,0}(m + n_0 + m_l) \begin{bmatrix} n + m - e_0 - e_2 \\ n - e_2 \end{bmatrix}^{p,L}_{a,b}. \] (4.48)

The last two terms in the right-hand-side of (4.48) can be combined together with the help of (2.30) as
\[
\sum_{l=2}^{b-1} \Phi_{a,b}^{p,0}(m + n_0 + m_l) \begin{bmatrix} n + m - 2e_0 - e_2 \\ n - e_2 \end{bmatrix}^{p,L}_{a,b} \] (4.49)
which according to (4.40) is nothing else but \( q^{2(L-1)} F_{a,b}^{p,0}(L - 2) \). Hence,
\[
F_{a,b}^{p,0}(L) = q^{L-1} F_{a,b+1}^{p,0}(L - 1) + F_{a,b}^{p,0}(L - 1) + q^{L-1} F_{a,b-1}^{p,0}(L - 1) \\
+ q^{L-1}(q^{L-1} - 1) F_{a,b}^{p,0}(L - 2). \] (4.50)

Thus, (4.5) is established.

### 4.4 Proof of even identities (2.12-13)

Comparing (3.17)-(3.22) and (4.1)-(4.6) one readily checks that polynomials in the left-hand-side and right-hand-side of (2.12-13) obey exactly the same recurrences, which specifies these polynomials uniquely. Therefore, to complete the proof one needs to show that identities (2.12-13) hold for \( L = 0 \).

Keeping in mind that all \( n \) and \( m \) variables in (2.19) are non-negative and taking advantage of (2.25) and (2.28-29) one concludes after a bit of analysis that the \( n - m \) system (2.21) with \( L = 0 \) has solutions if and only if \( a = b \). In the latter case the solution is
\[
n = e_a; m = 0 \text{ for } L = 0, a = b \] (4.51)
Substituting (4.51) into (2.19), yields
\[ F_{a,b}^{p,0}(0) = \delta_{a,b}. \] (4.52)

On the other hand, using
\[ \left( \begin{array}{c} 0, q \\ A \end{array} \right)^n = \delta_{A,0} \] (4.53)
one immediately gets
\[ B_{a,b}(0, s) = \tilde{B}_{a,b}(0, s) = \delta_{a,b}. \] (4.54)

From (2.10) one has
\[ \varphi(a, a) = 0. \] (4.55)

Hence by (4.52), (4.54) and (4.55), it is clear that (2.12-13) hold for \( L = 0 \). Therefore, we completed the proof of even identities (2.12-13).

**4.5 Proof of odd identities (2.14-15)**

Close examination of the proof of fermionic recurrences (4.1)-(4.6) given above shows that the parity of the \( m_{p-2} \) variable played a minor role in it. One can literally repeat all the steps in the proof at (4.1)-(4.6) with parity of \( m_{p-2} \) reversed in all the earlier formulas to find “odd” analogues of the even recurrences. They are listed as follows:

\[ \tilde{F}_{p-a,1}^{p,1}(L) = \tilde{F}_{p-a,1}^{p,1}(L-1) + q^{L-\frac{1}{2}} \tilde{F}_{p-a,2}^{p,1}(L-1) \text{ for } a \equiv 1 \pmod{2} \] (4.56)

\[ \tilde{F}_{p-a,1}^{p,1}(L) = \tilde{F}_{p-a,1}^{p,1}(L-1) + q^{L-1} \tilde{F}_{p-a,2}^{p,1}(L-1) \text{ for } a \equiv 0 \pmod{2} \] (4.57)

\[ \tilde{F}_{p-a,2}^{p,1}(L) = q^{L-1} \tilde{F}_{p-a,3}^{p,1}(L-1) + q^{L-\frac{1}{2}} \tilde{F}_{p-a,1}^{p,1}(L-1) + \tilde{F}_{p-a,2}^{p,1}(L-1) \]
for \( a \equiv 1 \pmod{2} \) (4.58)

\[ \tilde{F}_{p-a,2}^{p,1}(L) = \frac{1}{\sqrt{q}} \tilde{F}_{p-a,3}^{p,1}(L-1) + \tilde{F}_{p-a,1}^{p,1}(L-1) + q^{L-1} \tilde{F}_{p-a,2}^{p,1}(L-1) \]
for \( a \equiv 0 \pmod{2} \) (4.59)

\[ F_{p-a,b}^{p,1}(L) = q^{L-1} F_{p-a,b+1}^{p,1}(L-1) + q^{L-\frac{1}{2}} F_{p-a,b-1}^{p,1}(L-1) + F_{p-a,b}^{p,1}(L-1) \]
\[ + q^{L-1}(q^{L-1} - 1) \tilde{F}_{p-a,b}^{p,1}(L-2) \]
for \( a + b \equiv 1 \pmod{2}, 2 < b \leq p - 1 \) (4.60)

\[ \tilde{F}_{p-a,b}^{p,1}(L) = \frac{1}{\sqrt{q}} \left\{ \tilde{F}_{p-a,b+1}^{p,1}(L-1) + (q^{L-1} - 1) \tilde{F}_{p-a,b+1}^{p,1}(L-2) \right\} \]
\[ + \left\{ \tilde{F}_{p-a,b-1}^{p,1}(L-1) + (q^{L-1} - 1) \tilde{F}_{p-a,b-1}^{p,1}(L-2) \right\} \]
\[ + \left\{ \tilde{F}_{p-a,b}^{p,1}(L-1) + q^{L-2}(q^{L-1} - 1) \tilde{F}_{p-a,b}^{p,1}(L-2) \right\} \]
for \( a + b \equiv 0 \pmod{2}, 2 < b \leq p - 1 \) (4.61)
with $2 \leq a \leq p - 2$, $\bar{F}^{1}_{p-a,b}(L) \equiv 0$. For $2 \leq a \leq p - 2$, $1 \leq b \leq p - 1$ polynomials $\bar{F}^{p,1}_{p-a,b}(L)$ were defined as

$$\bar{F}^{p,1}_{p-a,b}(L) = \sum_{m} \left( \prod_{j=0}^{p-2} \frac{n_j + m_j}{n_j} \right)'(L)$$

with $n, m$ variables being constrained by (2.21), (2.22) with $a$ replaced by $p - a$ and $i = 1$. We would like to remind the reader that as in the discussion given at the end of Section 2

$$\bar{F}^{p,1}_{p-a,b}(L) = \tilde{F}^{p,1}_{p-a,b}(L) + \delta_{L,0} \cdot \delta_{a,b} \cdot \theta(1 < b < p - 1).$$

Once again, comparing (4.56)-(4.61) with (3.17)-(3.22) shows that bosonic and fermionic polynomials in (2.14), (2.15) obey exactly the same recursion relations. Thus, to complete the proof it is sufficient to demonstrate that (2.14) holds for all $L \geq 0$. This is easily done as follows. First, from definition (2.7), it is obvious that

$$\bar{F}^{p,1}_{p-a,b}(L) = \tilde{F}^{p,1}_{p-a,b}(L).$$

Next, replacing $a$ by $p - a$ and setting $b = p - 1$ in the identity (2.13), we get

$$\bar{F}^{p,1}_{p-a,b}(L) = \tilde{F}^{p,1}_{p-a,b}(L) + \delta_{L,0} \cdot \delta_{a,b} \cdot \theta(1 < b < p - 1).$$

with $2 \leq a \leq p - 2$. Finally, making use of reflection properties (3.23), (3.24) we derive (2.14). Thus the odd identities (2.14) and (2.15) are established for $2 \leq a \leq p - 2$, $1 \leq b \leq p - 1$. Observing that

$$\bar{F}^{p,1}_{p-a,b}(L) \equiv \tilde{F}^{p,1}_{p-a,b}(L)$$

and

$$\varphi(p - 1, p - 1) = \hat{\varphi}(p - 1, p - 1) = 0,$$

we conclude that odd identities with $a = 1, 1 \leq b \leq p - 1$ reduce to the even ones proven earlier in Section 4.4.

## 5 Infinite L. Virasoro character identities

In this section we will let $L \to \infty$ in the formulas (2.12)-(2.15). To this end it’s convenient to define fermionic objects

$$F^{1}_{a,b}(q) = \lim_{L \to \infty} F^{p,1}_{a,b}(L)$$

with $1 \leq a \leq p - 2; 1 \leq b \leq p - 1; i = 0, 1$. Employing the well-known formula describing limiting behavior of $q$-binomials

$$\lim_{L \to \infty} \left[ \begin{array}{c} L \\ k \end{array} \right] = \frac{1}{(q)_k} \text{ with } L, k \in \mathbb{Z}_{\geq 0}.$$
it is easy to derive explicit expressions for $F e r_{a,b}^{p,i}(q)$

$$F e r_{a,b}^{p,i}(q) = \sum_{m} \frac{1}{q^i} \prod_{j=1}^{p-2} \left( \frac{f_{p-j}^{m} + u_{a,b}^{j} m_j}{m_j} \right)_q$$

where

$$d(i) = \begin{cases} i \text{ for } b \neq p - 1 \\ 1 - i \text{ for } b = p - 1 \end{cases}$$

and the rest of notation is the same as in Section 1 and Section 2. Equipped with definitions (5.1),(5.3) and using limiting properties (3.25),(3.26) we can obtain character and extended character identities.

**Character identities**

$$F e r_{a,1}^{p,0}(q) = \begin{cases} q^{\varphi(a-1)} \chi_{a,1}^{p,0}(q) & \text{for } a + \delta_{a,1} \equiv 0 \pmod{2} \\ q^{\varphi(a-1)(a-2)} \chi_{a,2}^{p,0}(q) & \text{for } a + \delta_{a,1} \equiv 1 \pmod{2} \end{cases}$$

(5.5)

$$F e r_{p-a,1}^{p,1}(q) = \begin{cases} q^{\varphi(p-a,p-1)} \chi_{a,1}^{p,1}(q) & \text{for } a \equiv 1 \pmod{2} \\ q^{\varphi(p-a,p-1)} \chi_{a,2}^{p,1}(q) & \text{for } a \equiv 0 \pmod{2} \end{cases}$$

(5.6)

$$F e r_{a,b}^{p,0}(q) = q^{\varphi(a,b)} \chi_{a,b+1}^{p,0}(q)$$

for $b + a + \delta_{a,1} \equiv 0 \pmod{2}$

(5.7)

$$F e r_{p-a,b}^{p,1}(q) = q^{\varphi(p-a,p-1)+\varphi(a,b)-a+1} \chi_{a,b+1}^{p,1}(q)$$

for $a + b \equiv 0 \pmod{2}$, $2 \leq b \leq p - 1$

(5.8)

with $1 \leq a \leq p - 2$, $1 \leq b \leq p - 1$

and

$$F e r_{p-1,b}^{p,1}(q) = F e r_{1,b}^{p,0}(q)$$

for $1 \leq b \leq p - 1$.

(5.9)

**Extended characters identities**

$$F e r_{a,b}^{p,0}(q) = q^{\varphi(a,b)} \left\{ \theta(b \neq p - 1) \cdot \chi_{a,b+2}^{p,0}(q) \\ + q^{a-b} \chi_{a,b}^{p,0}(q) \right\}$$

for $b + a + \delta_{a,1} \equiv 1 \pmod{2}$, $2 \leq b \leq p - 1$, $1 \leq a \leq p - 2$

(5.10)

$$F e r_{p-a,b}^{p,1}(q) = q^{\varphi(p-a,p-1)+\varphi(a,b)-a+1} \left\{ \theta(b \neq p - 1) \chi_{a,b+2}^{p,1}(q) \\ + q^{a-b} \chi_{a,b}^{p,1}(q) \right\}$$

for $a + b \equiv 0 \pmod{2}$, $2 \leq b \leq p - 1$

and $1 \leq a \leq p - 2$

(5.11)
with \( \chi_{p,p+1}^{a,b}(q) \) defined by (1.7).

S.O. Warnaar pointed out to us that the variable \( m_1 \) in (2.1) and (2.7) can be summed out by means of the q-analogue of the Chu-Vandermonde formula [(3.3.10) of \cite{25}]. Having done so and taking the limits \( L \to \infty \) we obtain from (2.13) with \( 2 \leq b \leq p-2 \), \( 2 \leq a \leq p-2 \) and \( a+b \equiv 1 \pmod{2} \) the following character identity

\[
\sum_{m_{p-2} \equiv 0 \pmod{2}} q^{mc_{p-2,m+1}\frac{m_1-m_k}{m}} \frac{1}{(q)_{m_1}} \prod_{j=2}^{p-2} \left[ \left( \frac{1}{2} I_{p-2,m} \right)_j + \frac{1}{2} \left( \delta_{a,j} + \delta_{b,j} \right) \right] q^{\varphi(a,b) \left\{ \chi_{a,b+2}^{a,b+1}(q) + q^{a-b} \chi_{a,b}^{a,b+1}(q) \right\}},
\]

which was not noticed previously.

6 Closing remarks

Here we would like to point out some salient features of the identities (2.12)-(2.15) proven in this paper. First, the need to take into account certain negative solutions to the \( n, m \) system (2.21) as discussed at the end of Section 2 makes our identities somewhat similar to those for non-unitary minimal models \( M(p,p') \neq p+1 \) perturbed by the operator \( \varphi_{1,3} \) \cite{22}, \cite{26}. Second, the bosonic forms of our polynomials required two types of \( q \)-trinomial coefficients. The analogous feature was previously seen in the study of \( N = 1 \) superconformal models \( SM(p,p') \) where two types of \( q \)-trinomials were used to describe Neveu-Schwarz and Ramond sector identities \cite{19}, \cite{20}, \cite{24}, \cite{27}. This similarity is not very surprising because both dilute and \( N = 1 \) superconformal models share common spin-1 properties.

It is rather intriguing to notice, that despite the fact that we have a complete set of Virasoro character identities (5.5)-(5.8), we find in addition extended character identities (5.9),(5.10). In this regard, we want to mention that linear combinations of Virasoro characters in (5.10),(5.11) were interpreted by K. Seaton and L. Scott in \cite{17} as arising from configuration sums for excited states. Thus, it is conceivable that extended character identities provide a hint for the existence of some superselection rules which break up the space of our models into a number of orthogonal subsectors.

Identities of this paper are associated with \( A \)-modular invariant models of conformal field theory in the classification of \cite{28}. The only known result for \( D \)-modular invariant models perturbed by operator \( \varphi_{2,1} \) is for three-state Potts model \( M_D(5,6) \). In this case the identities are of a parafermionic form \( Z_3 \) introduced by J. Lepowsky and M. Primc \cite{29} and are quite different from the ones studied here. In particular, the para-fermionic form has two particles instead of the four particles found here for \( M(5,6) \). Thus, one may speculate that not only different integrable perturbations lead to distinct Rogers-Ramanujan type identities but also different modular invariance specifications may yield distinct new identities. Finally, it is highly desirable to find a Partition Theoretical interpretation of polynomials \( B^p_{a,b}(L,s), B^p_{a,b}(L,s) \). Such an interpretation may lead to the \( q \)-trinomial generalization of the Burge iterative process \cite{30} which was recently used by O. Foda, et al. \cite{31} to give another proof of Rogers-Ramanujan type identities for \( M(p,p') \) models perturbed by operator \( \varphi_{1,3} \).
Note added After the completion of this paper S.O.Warnaar informed us that he has obtained an independent proof of identity (1.10).

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Appendix A

Here we will show that under replacement

\[
\begin{bmatrix} n_0 + m_0 \\ n_0 \end{bmatrix}_q \rightarrow \begin{bmatrix} n_0 + m_0 \\ n_0 \end{bmatrix}'_q \quad (A.1)
\]

in (2.19), the combined contribution due to the negative values of \(n_0\) vanishes, unless \(L = 0; p - a = b \neq 1, p - 1; i = 1\) in (2.19). Since the main features of our treatment remain essentially the same for all \(1 \leq a \leq p - 2; 1 \leq b \leq p - 1; i = 0, 1\) in (2.19), we first limit our attention to the case \(a = b = 1; i = 0\) in order to simplify a bit the discussion below. Note, that in this case equation (2.27) becomes

\[
L = (n_0 + n_1) + m_2 + 2 \sum_{j=2}^{p-2} n_j + m_{p-2} . \quad (A.2)
\]

For \(n_0\) to be negative

\[
m_2 > 2L \quad (A.3)
\]

should hold. Evaluating (2.19) with replacement (A.1) for \(m_2 > 2L\), one encounters the sum

\[
\sum_{m_1=0}^{C_0} (-)^{m_1} q^{m_1^2 - m_1} \left[ \frac{m_1^2}{2} - Lm_1 \right] \left[ \frac{m_1}{2} - L - 1 + m_0 \right] \left[ \frac{m_2}{2} - L - 1 \right]_q \left[ \frac{m_2}{m_1} \right]_q \quad (A.4)
\]

with variables \(m_0, m_1\) being constrained by \(m_0 + m_1 = C_0\). Below we will show that this sum becomes zero, unless

\[
0 \leq C_0 = m_0 + m_1 \leq L . \quad (A.5)
\]

However, condition (A.5) is incompatible with (A.2), (2.34), (2.35). Indeed, from (A.5) and (2.26) it is readily seen that

\[
n_0 + n_1 \geq 0 . \quad (A.6)
\]

Now combining (A.2), (A.3), (A.6), (2.34) and (2.35) we arrive at contradiction

\[
L > 2L \quad (A.7)
\]

which proves our claim.
It remains to demonstrate that sum (A.4) will vanish for $C_0 > L$. To this end let

$$P(L, C_0, B) = \sum_{m_1=0}^{\infty} (-1)^{m_1} q^{\frac{m_1^2 - m_1}{2} - Lm_1} \frac{B + C_0 - m_1}{B} \left[ \begin{array}{c} B + 1 + L \\ m_1 \end{array} \right]_q. \quad (A.8)$$

Using the q-Chu-Vandermonde sum (equation II.6 in [23]) the expression (A.8) is evaluated as

$$P(L, C_0, B) = (-1)^{C_0} q^{\frac{C_0(C_0-1)}{2} - C_0L} \left[ \begin{array}{c} L \\ C_0 \end{array} \right]_q. \quad (A.9)$$

Clearly $P(L, C_0, B)$ vanishes for $C_0 > L$. Finally, observing that (A.4) is nothing else but $P(L, C_0, m_2 - L - 1)$, we are done.

Proceeding as above in general case: $1 \leq a \leq p - 2; 1 \leq b \leq p - 1; i = 0, 1$, we conclude that contributions due to negative values of $n_0$ must vanish unless (A.6) holds and $m_2 \geq \begin{cases} 2L + 1 & \text{if } m_2 \text{ is odd} \\ 2L + 2 & \text{if } m_2 \text{ is even} \end{cases}. \quad (A.10)$

Again, combining (2.27), (A.6), (A.10), (2.34) and (2.35) we arrive at a contradiction, unless $L \leq 1$. Detailed analysis of the remaining cases with $L = 0, 1$ shows that one can have an overall contribution due to negative values of $n_0$ if and only if $L = 0; p - a = b \neq 1, p - 1$ and $i = 1$ in (2.19). In the latter case one needs to take into account the following solution to (2.21) with negative $n_0$:

$$n_0 = -n_1 = -1 \quad (A.11)$$

$$n_i = 0 \text{ for } 2 \leq i \leq p - 2 \quad (A.12)$$

$$m_0 = m_1 = 0 \quad (A.13)$$

and for $i = 2, 3, \ldots, p - 2$

$$m_i = \begin{cases} (i - 1) & \text{for } 2 \leq i \leq \min \{b, p - b\} \\ (p - 1 - i) & \text{for } \max \{b, p - b\} \leq i \leq p - 2 \\ \min \{p - b, b\} - 1 & \text{otherwise} \end{cases}. \quad (A.14)$$

It can be easily checked that this solution contributes as 1.

**Appendix B**

Here we will prove recurrences (4.6). Throughout this appendix it is assumed that $b + a + \delta_{a,1} \equiv 1 \pmod{2}$ and $1 \leq a \leq p - 2, 2 < b \leq p - 1$. As usual, we start with telescopic expansion for $F^{p,0}_{a,b}(L)$.

$$F^{p,0}_{a,b}(L) = \theta(b \neq p - 1) \sum_{m=0}^{\infty} q^{\phi^{p,0}_{a,b}(m)} \left[ \begin{array}{c} n + m - E_{1,b} \\ n \end{array} \right]_{a,b}^{p,L} + \sum_{l=3}^{b} \sum_{m=0}^{\infty} q^{\phi^{p,0}_{a,b}(m)+m_l} \left[ \begin{array}{c} n + m - E_{1,t} \\ n - e_l \end{array} \right]_{a,b}^{p,L} + \sum_{l=1}^{2} \sum_{m=0}^{\infty} q^{\phi^{p,0}_{a,b}(m)+m_l} \left[ \begin{array}{c} n + m - E_{1,t} \\ n - e_l \end{array} \right]_{a,b}^{p,L} \quad (B.1)$$
with \( m_{p-1} \equiv 0 \). Next we perform the \( l \) - dependent change of summation variables

\[
  n \rightarrow n - e_{l-1} + e_l + e_{b-1} - e_b; \quad m \rightarrow m - 2E_{l,b-1}
\]  \hspace{1cm} \text{(B.2)}

in the first double sum in (B.1) to find after replacing \( l - 1 \) by \( l \) that

\[
  \sum_{l=3}^{b} \sum_{l=3}^{b} q^{p_{a,b}(m)+ml} \left[ \frac{n + m - E_{1,l}}{n - e_l} \right]^{p,L}_{a,b} =
\]

\[
  = \sum_{l=2}^{b-1} \sum_{l=2}^{b-1} q^{p_{a,b}(m)+mb_{b-1}+ml} \left[ \frac{n + m - E_{1,b-1} - E_{l,b-1} + e_{b-1} - e_b}{n - e_l + e_{b-1} - e_b} \right]^{p,L}_{a,b}.
\]  \hspace{1cm} \text{(B.3)}

To proceed further we will use

\[
  F_{a,b+1}^{p,0}(L - 1) = \sqrt{q} \sum_{l=3}^{b} q^{p_{a,b}(m)} \left[ \frac{n + m - E_{1,b}}{n} \right]^{p,L}_{a,b}, \hspace{1cm} \text{(B.4)}
\]

\[
  F_{a,b-1}^{p,0}(L - 1) = \sum_{l=2}^{b} q^{p_{a,b}(m)+mb_{b-1}+ml+1} \left[ \frac{n + m - E_{1,b-1} + e_{b-1} - e_b}{n + e_{b-1} - e_b} \right]^{p,L}_{a,b}, \hspace{1cm} \text{(B.5)}
\]

\[
  q^{L-2}F_{a,b}^{p,0}(L - 2) = \sum_{l=1}^{b-2} q^{p_{a,b}(m)+mb_{b-1}+ml+1+n_0} \left[ \frac{n + m - E_{0,b-1} - E_{2,b-1} + e_{b-1} - e_b}{n + e_{b-1} - e_b} \right]^{p,L}_{a,b}, \hspace{1cm} \text{(B.6)}
\]

\[
  q^{L-1}F_{a,b}^{p,0}(L - 1) = \sum_{l=1}^{b-1} q^{p_{a,b}(m)+n_0+ml} \left[ \frac{n + m - e_0}{n - e_l} \right]^{p,L}_{a,b}. \hspace{1cm} \text{(B.7)}
\]

The telescopic expansion of (B.5) yields

\[
  F_{a,b-1}^{p,0}(L - 1) = \sum_{l=2}^{b-1} q^{p_{a,b}(m)+mb_{b-1}+ml+1} \left[ \frac{n + m - e_1 - 2E_{2,b-1} + e_{b-1} - e_b}{n + e_{b-1} - e_b} \right]^{p,L}_{a,b} +
\]

\[
  \sum_{l=2}^{b-1} q^{p_{a,b}(m)+mb_{b-1}+ml+1} \left[ \frac{n + m - E_{1,b-1} - E_{l,b-1} + e_{b-1} - e_b}{n - e_l + e_{b-1} - e_b} \right]^{p,L}_{a,b}.
\]  \hspace{1cm} \text{(B.8)}

Combining (B.1), (B.4), (B.3) and (B.8) one readily derives

\[
  F_{a,b}^{p,0}(L) = \frac{1}{\sqrt{q}} F_{a,b+1}^{p,0}(L - 1) + F_{a,b-1}^{p,0}(L - 1)
\]

\[
  - \sum_{l=2}^{b} q^{p_{a,b}(m)+mb_{b-1}+ml+1} \left[ \frac{n + m - e_1 - 2E_{2,b-1} + e_{b-1} - e_b}{n + e_{b-1} - e_b} \right]^{p,L}_{a,b}
\]

\[
  + \sum_{l=1}^{2} q^{p_{a,b}(m)+ml} \left[ \frac{n - E_{1,l}}{n - e_l} \right]^{p,L}_{a,b}.
\]  \hspace{1cm} \text{(B.9)}

If we now replace \( L \) by \( L - 1 \) in (B.9) and use

\[
  \{n, m\}^{L,0}_{a,b} = \{n, m\}^{L,0}_{a,b} - \{e_0, 0\}, \hspace{1cm} \text{(B.10)}
\]
we obtain

\[ F_{a,b}^{p,0}(L - 1) = \frac{1}{\sqrt{q}} F_{a,b+1}^{p,0}(L - 2) + F_{a,b-1}^{p,0}(L - 2) \]

\[- \sum_{m=0}^{(0)} q^{p,a,b}(m) + m_{b-1} + 1 \left[ n + m - E_{0,1} - 2E_{2,b-1} + e_{b-1} - e_b \right]_{a,b} \]

\[ + \sum_{l=1}^{2} \sum_{m} q^{p,a,b}(m) + m_l \left[ n - e_0 - E_{1,l} \right]_{a,b} \]

\[ \text{(B.11)} \]

Next subtract (B.11) from (B.9) and use (2.30) and (B.6) to get

\[ F_{a,b}^{p,0}(L) - F_{a,b}^{p,0}(L - 1) = \frac{1}{\sqrt{q}} \left\{ F_{a,b+1}^{p,0}(L - 1) - F_{a,b+1}^{p,0}(L - 2) \right\} + \]

\[ + \left\{ F_{a,b-1}^{p,0}(L - 1) - F_{a,b-1}^{p,0}(L - 2) \right\} - q^{L-2} F_{a,b-1}^{p,0}(L - 2) + \]

\[ + \sum_{l=1}^{2} \sum_{m} q^{p,a,b}(m) + n_{0} + m_{l} \left[ n - E_{0,l} \right]_{a,b} \]

\[ \text{(B.12)} \]

It remains to process the double sum in (B.12). To this end we expand (B.7) with the help of (2.31) as

\[ q^{L-1} F_{a,b}^{p,0}(L - 1) = \sum_{m=0}^{(0)} q^{p,a,b}(m) + n_{0} + m_{1} \left[ n + m - E_{0,1} \right]_{a,b} \]

\[ + \sum_{m=0}^{(0)} q^{p,a,b}(m) + n_{0} + m_{1} + 1 \left[ n + m - E_{0,1} \right]_{a,b} \]

\[ \text{(B.13)} \]

and perform a change of summation variables

\[ n \rightarrow n - e_0 + e_1 ; \quad m \rightarrow m + e_0 - e_1 \]

(B.14)

in the last sum in (B.13) to find

\[ \sum_{m=0}^{(0)} q^{p,a,b}(m) + n_{0} + m_{1} + 1 \left[ n + m - E_{0,1} \right]_{a,b} = \]

\[ q^{L-1} \sum_{m=0}^{(0)} q^{p,a,b}(m) + m_{1} \left[ n + m - E_{0,1} \right]_{a,b} \]

\[ \text{(B.15)} \]

Making use of (B.11), (B.13) and (B.15), it is readily seen that

\[ \sum_{m=0}^{(0)} q^{p,a,b}(m) + n_{0} + m_{1} \left[ n + m - E_{0,1} \right]_{a,b} = \]

\[ q^{L-1} \left\{ \frac{1}{\sqrt{q}} F_{a,b+1}^{p,0}(L - 2) + F_{a,b-1}^{p,0}(L - 2) \right\} - \sum_{m=b-1}^{(0)} q^{p,a,b}(m) + m_{b-1} + 1 \left[ n + m - E_{0,1} - 2E_{2,b-1} + e_{b-1} - e_b \right]_{a,b} \]

\[ + \sum_{m=2}^{(0)} q^{p,a,b}(m) + m_{2} \left[ n + m - E_{0,2} \right]_{a,b} \}

\[ \text{(B.16)} \]
The last equation together with (B.12) yields

\[
F_{a,b}^{p,0}(L) = \frac{1}{\sqrt{q}} \left\{ F_{a,b+1}^{p,0}(L-1) + (q^{L-1} - 1)F_{a,b}^{p,0}(L-2) \right\} + \\
+ \left\{ F_{a,b-1}^{p,0}(L-1) + (q^{L-1} - 1)F_{a,b}^{p,0}(L-2) \right\} + \\
+ F_{a,b}^{p,0}(L-1) + q^{L-2}(q^{L-1} - 1)F_{a,b}^{p,0}(L-2) + Z(L,a,b) 
\]

(B.17)

where

\[
Z(L,a,b) = -q^{L-1}X(L,a,b) + Y(L,a,b) 
\]

(B.18)

with

\[
X(L,a,b) = q^{L-2}F_{a,b}^{p,0}(L-2) + \sum (q^{p^{i+1}}(m+mb_{i-1}+1) \left\{ n + m - E_{0,1} - 2E_{2,b-1} + e_{b-1} - e_b \right\}^{p,L}_{a,b} 
\]

(B.19)

and

\[
Y(L,a,b) = \sum (q^{p^{i+1}}(m) \left\{ n + m - E_{0,2} \right\}^{p,L}_{a,b} + \sum (q^{p^{i+1}}(m+L+1+m2) \left\{ n + m - E_{0,2} \right\}^{p,L}_{a,b} 
\]

(B.20)

Making use of (B.6) and (2.30) we can simplify (B.19) to

\[
X(L,a,b) = \sum (q^{p^{i+1}}(m) + mb_{i-1}+1) \left\{ n + m - E_{0,1} - 2E_{2,b-1} + e_{b-1} - e_b \right\}^{p,L}_{a,b} 
\]

(B.21)

Next we perform the change of summation variables

\[
n \rightarrow n + e_b - e_{b-1} - e_2 ; \quad m \rightarrow m + 2E_{2,b-1} - e_0 + e_1 
\]

in (B.21) to get

\[
X(L,a,b) = \sum (q^{p^{i+1}}(m) - m_0 + m_{i-1} + 1) \left\{ n + m - e_0 - e_2 \right\}^{p,L}_{a,b} 
\]

(B.23)

Finally, multiplying both sides in (B.23) by \(q^{L-1}\) and using (4.10) gives

\[
q^{L-1}X(L,a,b) = \sum (q^{p^{i+1}}(m) + m_0 + m_2) \left\{ n + m - e_0 - e_2 \right\}^{p,L}_{a,b} 
\]

(B.24)

It remains to process (B.20). To this end we perform the change of summation variables

\[
n \rightarrow n + e_0 - e_1 ; \quad m \rightarrow m - e_0 + e_1 
\]

(B.25)
in the second sum in (B.20) to find
\[ \sum_{q} \Phi_{a,b}^{p,0}(m) \left[ n + m - E_{0,2} \right]_{a,b}^{p,L} = \]
\[ \sum_{q} \Phi_{a,b}^{p,0}(m) n_{0} + m_{2} + m_{1}\left[ n + m - E_{1,2} \right]_{a,b}^{p,L}. \quad (B.26) \]

We can now combine the two terms in (B.20) with the help of (2.31) as
\[ Y(L,a,b) = \sum_{q} \Phi_{a,b}^{p,0}(m) n_{0} + m_{2} + m_{1}\left[ n + m - e_{0} - e_{2} \right]_{a,b}^{p,L}. \quad (B.27) \]

The desired result (4.6) now easily follows from (B.17), (B.18), (B.24), (B.27). Thus, recurrences (4.6) are established.

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