Self-similar analytical model of plasma expansion in a magnetic field

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Abstract
The study of hot plasma expansion in a magnetic field is of interest for many astrophysical applications. In order to observe this process in the laboratory, an experiment is proposed in which an ultrashort laser pulse produces a high-temperature plasma by irradiation of a small target. In this paper, an analytical model is proposed for an expanding plasma cloud in an external dipole or homogeneous magnetic field. The model is based on the self-similar solution of a similar problem that deals with the sudden expansion of spherical plasma into a vacuum without ambient magnetic field. The expansion characteristics of the plasma and deceleration caused by the magnetic field are examined analytically. The results obtained can be used to treat experimental and simulation data, and several phenomena of astrophysical and laboratory significance.

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(Some figures in this article are in colour only in the electronic version.)

1. Introduction
The problem of sudden expansion of hot plasma into a vacuum in the presence of an external magnetic field was intensively studied in the mid-1960s in connection with high-altitude nuclear explosions. It has also been discussed in the analysis of many astrophysical and laboratory applications (see e.g. [1, 2] and references therein). Such processes arise in the dynamics of solar flares and the flow of solar wind around the Earth’s magnetosphere, in active experiments with plasma clouds in space, and in the course of interpreting a number of astrophysical observations [1–5]. Research on this problem is of considerable interest in connection with experiments on controlled thermonuclear fusion [6] (a review [1] summarizes research in this area over the past four decades).

The expanding plasma pushes the magnetic field out, which is a consequence of magnetic flux conservation. Due to plasma expansion, the surface of the given magnetic flux tube inside the plasma increases. Conservation of the magnetic flux through the surface of the tube leads to a decrease in the local magnetic field and the formation of a diamagnetic cavity. Thus even if a magnetic field is nonzero initially inside the plasma, it tends to zero during plasma expansion. In addition, the plasma is shielded from penetration of the external field by means of surface currents circulating inside the thin layer on the plasma boundary. Ponderomotive forces resulting from interaction of these currents with the magnetic field would act on the plasma surface as if there were magnetic pressure applied from outside. Thus after some period of accelerated motion, the plasma gets decelerated as a result of this external force acting inward. Plasma has been considered as a highly conducting medium with zero magnetic field inside. From the point of view of electrodynamics, it is similar to the expansion of a superconductor in a magnetic field. An exact self-similar analytic solution for a uniformly expanding, highly conducting plasma sphere in an external uniform and constant magnetic field has been obtained in [7]. The nonrelativistic limit of this theory has been used by Raizer [8] to analyze the energy balance (energy emission and transformation) during plasma expansion. A similar problem has been considered in [9] in a one-dimensional (1D) geometry for a plasma layer.
In our recent papers [10, 11], we obtained an exact self-similar analytic solution for the uniform relativistic expansion of a highly conducting plasma sphere or cylinder in the presence of a dipole or homogeneous magnetic field, respectively.

As mentioned above, the previous treatments [7–11] were obtained assuming a somewhat idealized situation: uniform expansion, zero magnetic field and thermal pressure inside plasma, etc. Obviously such simplified models are not capable of correctly describing plasma expansion dynamics where an essential deceleration (or acceleration) of the plasma boundary may occur. More realistic models for plasma self-similar expansion have been developed, for instance, in [12–14] (see also references therein) for planar (1D) [12] and cylindrical [13, 14] expansions employing ideal magnetohydrodynamic (MHD) equations. It should be noted that the ideal MHD is well justified since the typical experimental parameters are such that the expanding plasma is collisionless and the dissipative effects are negligible [1–5].

In this paper, we study the expansion of a spherical plasma cloud in the presence of a dipole or homogeneous magnetic field, taking into account the thermal effects. We employ the self-similar solution of a similar problem that deals with the expansion of spherical plasma into a vacuum without magnetic field. Unlike previous studies [12–14], we concentrate here on the dynamics (deceleration and/or acceleration) of the sharp plasma boundary. We find an analytical solution that can be used to analyze recent experimental and simulation data (see e.g. [1, 2] and references therein).

2. Theoretical model

Usually, the motion of the expanding plasma boundary is approximated as the motion with constant velocity (uniform expansion). In the present study, a quantitative analysis of plasma dynamics is developed on the basis of a 1D spherical radial model. Within the scope of this analysis, the initial stage of plasma acceleration, the later stage of deceleration and the process of stopping at the point of maximum expansion are examined.

We consider a collisionless magnetized plasma expanding into vacuum. The relevant equations governing the expansion are those of ideal MHD [15], assuming that the characteristic length scales for plasma flow are much larger than the Debye length and Larmor radius of the ions. Thus

\[
\frac{\partial P}{\partial t} + \nabla \cdot (P \nabla v) = 0,
\]

\[
\rho \left[ \frac{\partial v}{\partial t} + (v \cdot \nabla) v \right] = \frac{1}{4 \pi} [\nabla \times H] \times H - \nabla P,
\]

\[
\frac{\partial H}{\partial t} = \nabla \times [v \times H], \quad \nabla \cdot H = 0,
\]

where \(\rho\), \(v\), \(H\) and \(P\) are mass density, velocity, magnetic field and pressure, respectively. These equations must be accompanied by the adiabatic equation of state \(P \sim \rho^\gamma\) (\(\gamma = C_p/C_V > 1\) is the adiabatic index; \(C_p\) and \(C_V\) are the heat capacities at constant pressure and constant volume, respectively) and the equation of conservation of entropy, which expresses the fact that plasma dynamics is adiabatic in the absence of dissipation. Using the thermodynamic relation between entropy, pressure and internal energy as well as equation (1), the equation for pressure reads [16]

\[
\frac{\partial P}{\partial t} + (v \cdot \nabla) P + \gamma P (\nabla \cdot v) = 0.
\] (2)

The equation of conservation of energy is derived from the system of equations (1) and is given by [15]

\[
\frac{\partial \varepsilon}{\partial t} + \nabla \cdot \Phi = 0,
\] (3)

where \(\varepsilon\) and \(\Phi\) are energy and energy flux densities, respectively, with

\[
\varepsilon = \frac{\rho v^2}{2} + \frac{P}{\gamma - 1} - \frac{H^2}{8\pi},
\]

\[
\Phi = v \left( \frac{\rho v^2}{2} + \frac{P}{\gamma - 1} \right) + \frac{1}{4\pi} [H \times [v \times H]].
\] (4)

Here the energy \(\varepsilon\) consists of kinetic (first term), internal (second term) and magnetic field (third term) energies. The last term in the energy flux density \(\Phi\) represents the Poynting vector (let us recall that in ideal MHD the electric field is given by \(E = -\frac{1}{c} [v \times H]\)).

With the theoretical basis presented so far, we now take up the main topic of this paper. This is to study the dynamics of the plasma boundary expanding in an ambient magnetic field. Consider the magnetic dipole \(p\) and a plasma spherical cloud with radius \(a(t)\) located at the origin of the coordinate system. The dipole is placed in the position \(r_0\) from the center of the plasma cloud \((a(t) < r_0)\). The orientation of the dipole is given by the angle \(\theta_0\) between the vectors \(p\) and \(r_0\). We denote the strength of the magnetic field of the dipole by \(H_0(r)\). The energy, which is transferred from plasma to electromagnetic field, is the mechanical work performed by the plasma on the external magnetic pressure \(H_0^2(r)/8\pi\).

Taking into account this effect and the energy conservation (3) integrated over the spherical plasma volume \(\Omega_p\) (with \(0 \leq r \leq a(t)\)), the equation of balance of plasma energy is as follows:

\[
\frac{4\pi}{\gamma} \int_0^{a(t)} P(r, t) r^2 dr + 2\pi \int_0^{a(t)} \rho(r, t) v^2(r, t) r^2 dr + \int_0^{H_0^2(r)/8\pi} d\Omega = W_0,
\] (5)

where \(\Omega\) is the volume of the spherical shell \(a_0 \leq r \leq a(t)\), \(a_0 = a(0)\) \((a(t) \geq a_0)\) and \(W_0\) are the initial radius and energy of the plasma. When the plasma cloud is introduced into a background magnetic field, the plasma expands and excludes the background magnetic field to form a magnetic cavity. The magnetic energy of the dipole in the excluded volume is represented by the last term in equation (5). Initial plasma velocity is supposed to be \(v(r, 0) = v_m(r/a_0)\) at \(r \leq a_0\) and \(v(r, 0) = 0\) at \(r > a_0\), where \(v_m\) is the initial velocity of the plasma boundary \((v_m = a(0))\).

The obtained energy balance equation can be effectively used if profiles of velocity \(v(r, t)\), pressure \(P(r, t)\) and mass density \(\rho(r, t)\) are known functions of the plasma radius \(a(t)\).
We will take these dependencies from the solution of a similar problem that deals with the sudden expansion of spherical plasma into a vacuum without ambient magnetic field [16]. The simplest class of solutions available in this case are so-called self-similar solutions. They are realized under specified initial conditions. We will set the initial conditions with a parabolic distribution of pressure and mass density, which describe a hot and dense initial plasma state with a sharp boundary localized at $r = a_0$. The self-similar solutions are characterized by a velocity distribution linearly dependent on $r$ (see e.g. [16]). At $r \leq a(t)$

$$v(r, t) = r \frac{\dot{a}(t)}{a(t)}, \quad (6)$$

where unknown $a(t)$ is the radius of the sharp plasma boundary while $\dot{a}(t)$ is the velocity of the boundary. The specification of the mass density profile at $r \leq a(t)$ is given by

$$\rho(r, t) = \frac{\Gamma}{\pi^{\frac{3}{2}} \Gamma(1 + q)} \frac{M}{a^3(t)} \left[1 - \frac{r^2}{a^2(t)}\right]^q, \quad (7)$$

and equation (6) for the velocity automatically satisfies the continuity equation in (1) for an arbitrary function $a(t)$ and for an arbitrary parameter $q$. Here $M = \text{const}$ is the total mass of the plasma cloud and $\Gamma(z)$ is the Euler function. Substitution of $v(r, t)$ into equation (2) gives at $r \leq a(t)$ the following solution for the pressure:

$$P(r, t) = p_{\text{max}} \left[\frac{a_0}{a(t)}\right]^3 \left[1 - \frac{r^2}{a^2(t)}\right]^q, \quad (8)$$

where $s$ is an arbitrary parameter and $p_{\text{max}}$ is the thermal pressure at the center of the spherical plasma cloud at $t = 0$. In addition, the quantities $v(r, t)$, $\rho(r, t)$ and $P(r, t)$ vanish at $r > a(t)$, $v(r, t) = \rho(r, t) = P(r, t) = 0$. Substituting (7) and (8) into the fluid equation of motion (1) and neglecting the term involving the magnetic field $H$ yields a second-order differential equation governing the motion of the plasma boundary $a(t)$. The problem considered is not isentropic in general except for the case when $s = q'$. In the latter case of the isentropic expansion, the equation of state is given by $P \rho^{-q'} = \text{const}$. Throughout this paper we will assume that $q > 0$ and $s \geq 0$. The self-similar solution given by equations (6)–(8) has been considered by many authors. We refer to [16] (see also references therein) for details and for some applications of this solution.

There are several limitations and shortcomings of the self-similar model considered above. It should be emphasized that in general the plasma expansion process with and without ambient magnetic field is not self-similar since there is a characteristic length scale, that is, the initial radius of the plasma $a_0$. However, during the later stage (when $a(t) \gg a_0$) the plasma expansion asymptotically approaches the self-similar regime since the role of the radius $a_0$ becomes less important. This property is supported by the asymptotic solution of the system of partial differential equations (1) and (2) [16, 17] as well as by the numerical simulations [16] for free expansion (i.e. at $H = 0$). In addition, it has been shown that at an early stage of the expansion the deviation of the solution (6)–(8) from a nonsimilar one is small and is additionally reduced due to the spatial average in the equation of energy balance (5).

Equations (6)–(8) are an exact solution in the case of expansion into a vacuum without magnetic field. However, equation (8) does not satisfy the boundary condition $P(a(t), t) = H_\text{mag}^2/8\pi$, which is imposed in the case of expansion into an ambient magnetic field. Therefore, in the present case with nonzero magnetic field for the validity of equations (6)–(8) the domain close to the boundary of the plasma is most critical, $r \approx a(t)$. On the other hand, if the magnetic pressure is smaller than the plasma average pressure $\bar{P}$, $P_{\text{mag}} \ll \bar{P}$, the difference between the exact solution in the magnetic field and free expansion model is small and is localized in a narrow area near the surface of the cloud. These deviations are additionally reduced due to the integration in the equation of energy balance. Estimating the accuracy of the free expansion model, one should take into account that the long stage of plasma deceleration corresponds to a high expansion ratio, $a(t)/a_0 \gg 1$. The average plasma pressure drops significantly and plays no role in energy balance equation (5) during this stage. In accordance with the above boundary condition, the local pressure near the plasma edge must be equal to the magnetic pressure outside. It causes deviation from the profile equation (7) and accumulation of plasma in this area. This is confirmed independently by the numerical simulations [18]. In the limiting case when all plasma is localized near the front, one can expect an increase in the kinetic energy and longer stage of plasma deceleration as compared with the self-similar expansion model.

Another critical domain for the violation of the self-similar solution (6)–(8) is the final stage of expansion when plasma is fully stopped by the magnetic field. As mentioned above, the average plasma pressure is strongly reduced compared to the magnetic pressure and equations (6)–(8) clearly become invalid in this case. However, if the critical time interval $\Delta t$, where $\bar{P} < P_{\text{mag}}$ is much smaller than the typical time scale of the plasma flow (up to the full stop), the contribution of this interval to the overall plasma dynamics is negligible and use of the self-similar solution is justified.

In the case of dipole magnetic field, the volume integral in the last term of equation (5) has been evaluated in [10]. The result reads

$$\int_{\Omega} \frac{H_\text{mag}^2(x)}{8\pi} \text{d}x = \frac{\bar{P}^2}{32 r_0^3} \left[Q(\eta x(t)) - Q(\eta)\right], \quad (9)$$

where $\eta = a_0/r_0 < 1$, $x(t) = a(t)/a_0$ (note that $a(t)/r_0 = \eta x(t) < 1$) and

$$Q(\eta) = \frac{1}{(1 - \eta^2)^3} \left[\eta \left(1 - \eta^4\right) \left(3 \cos^2 \theta - 1\right) - 8 \eta^3 \left(1 + \cos^2 \theta\right)\right] = \frac{3 \cos^2 \theta - 1}{2} \ln \frac{1 + \eta}{1 - \eta}, \quad (10)$$

Substituting equations (6)–(8) into (5) and integrating over $r$ yields a first-order differential equation for $a(t)$, which
already satisfies initial condition $\dot{a}(0) = v_n$,
\[ \dot{x}^2(\tau) + \frac{\beta}{x^{3(\gamma-1)}} + \alpha \left[ \frac{Q}{\eta x(\tau)} - Q(\eta) \right] = 1. \]  
(11)

Here two dimensionless quantities are introduced, 
\[ \alpha = \frac{p^2}{32W_0 \eta}, \quad \beta = \frac{\pi^{3/2} P_{\text{max}} a_0^3}{\gamma(\gamma-1) W_0^3} \Gamma(\frac{1}{\gamma} + x), \]  
which determine the magnetic and thermal energies, respectively, in terms of the total initial energy $W_0$. The latter is easily obtained from equation (5) and reads
\[ W_0 = \frac{3M v_m^2}{2(5 + 2q)} + \frac{\pi^{3/2} P_{\text{max}} a_0^3}{\gamma(\gamma-1)} \Gamma(\frac{1}{\gamma} + x). \]  
(13)

From equations (12) and (13) it is seen that $\beta < 1$. New dimensionless variables are introduced as follows:
\[ x(\tau) = a(t)/a_0, \quad \tau = t/t_0, \quad t_0 = a_0/v_m, \]  
where $v_m = [2(5 + 2q)W_0/3M]^{1/2}$ is the velocity of plasma expansion, achieved asymptotically at $t \to \infty$ in the case of expansion into a vacuum without magnetic field (i.e. at $a = 0$).

The total energy of the plasma cloud at time $t$ is obtained from equation (11):
\[ W(t) = W_0 - \frac{p^2}{32\tau} \left[ Q(\eta x(t)) - Q(\eta) \right]. \]  
(14)

Note that the function $Q(\eta)$ monotonically increases with the argument and the plasma cloud energy decreases with time.

Consider also the case of a uniform magnetic field when $H_0 =$ const. In this case, the volume integral in equation (9) is replaced by $(H_0^2/6)(a^3(t) - a_0^3)$ and the differential equation (11) for the plasma boundary reads
\[ \dot{x}^2(\tau) + \frac{\beta}{x^{3(\gamma-1)}} + \alpha [x^3(\tau) - 1] = 1, \]  
(15)

where $\sigma = W_{\text{mag}}/W_0$, $W_{\text{mag}} = 4\pi a_0^3/3)P_{\text{mag}}$ is the initial magnetic energy in the plasma volume and $P_{\text{mag}} = H_0^2/8\pi$ is the magnetic field pressure.

Equations (11) and (15) coincide with the equation of the 1D motion of the point-like particle in the potential $U(x)$, which is determined by the second and third terms of equations (11) and (15). The distance $x_\sigma$ of plasma cloud motion up to the full stop (the stopping length) at the turning point is determined by $U(x_\sigma) = 1$. In particular, it is easier to obtain stopping length in the case of homogeneous magnetic field and at vanishing thermal pressure ($\beta = 0$). Then from equation (15) one obtains the equation of motion
\[ \dot{x}^2 = 1 + \sigma - \sigma x^3. \]  
(16)

It is seen that in this case the stopping length is given by $x_\sigma = (1 + 1/\sigma)^{1/3}$. The solution of equation (16) can be represented in the form
\[ t = \frac{t_0}{\sqrt{\sigma + 1}} \left[ F \left( \frac{x(t)}{x_\sigma} \right) - F \left( \frac{1}{x_\sigma} \right) \right], \]  
(17)

where $F(z) = F(\frac{1}{\alpha}; 1, \frac{1}{\alpha}; z)$ and the latter is the hypergeometric function. Substituting in equation (17) $x(t) = x_\sigma$, we obtain the corresponding stopping time as a function of magnetic field and plasma kinetic energy:
\[ t_s = \frac{t_0}{2} \left[ \sqrt{\frac{1}{\sigma + 1}} \left( C \left( \frac{\sigma + 1}{\sigma} \right) - F \left( \frac{\sigma}{\sigma + 1} \right) \right) \right]. \]  
(18)

Here $C = F(1) = \sqrt{\pi} \Gamma(\frac{1}{\alpha})/\Gamma(\frac{4}{\alpha}) \approx 1.4$ is a constant. At vanishing ($\sigma \ll 1$) and very strong ($\sigma \gg 1$) magnetic fields, the stopping time becomes $t_s \simeq C R_m/v_m = C(a_0/v_m)\sigma$, and $t_s \simeq 2R_m^2/3v_m a_0^2 = (2a_0/3v_m \sigma)^2$, respectively, where the radius $R_m = (6W_0/H_0^2)^{1/3}$ is obtained by equating the initial kinetic energy $W_0$ of an initially spherical plasma cloud to the energy of the magnetic field that it pursues out in expanding to the radius $R_m$. It is worth mentioning that in the case of weak magnetic fields, $\sigma \ll 1$, and at vanishing thermal pressure ($\beta = 0$), the stopping time does not depend on the initial plasma radius, $t_s \sim (M/v_m P_{\text{mag}})^{1/3}$.

We now turn to the general equations determined by (11) and (15). At the initial stage of plasma expansion ($t \ll t_0$), from these equations we obtain
\[ x(t) \approx 1 + \frac{v_m t}{a_0} + \frac{3}{4} h \left( \frac{t}{t_0} \right)^2, \]  
(19)

where $h = \beta(\gamma - 1) - \kappa, \kappa = \frac{\pi}{4} \eta Q(\eta)$ and $\sigma = \kappa$ for the dipole and homogeneous magnetic fields, respectively. Here the prime indicates the derivative with respect to the argument. Thus at the initial stage, the plasma cloud may get accelerated or decelerated depending on the sign of the quantity $h$ (in other words, on the relation between thermal and magnetic pressures). For instance, in the homogeneous magnetic field acceleration occurs when $p_{\text{max}} > p_c$, where
\[ p_c = \frac{4}{3\pi} \sigma \Gamma(\frac{5+s}{3}) P_{\text{mag}} / a_0. \]  
(20)

(i.e. at $h > 0$) and continues until $x(t)$ reaches some value $x_\sigma > 1$ given by $x_\sigma = (p_{\text{max}}/p_c)^{1/3}$. The time interval $0 \leq t < t_s$ of the acceleration is determined from the equation of motion (15). The critical radius $x_\sigma$ and time $t_s$ correspond to the beginning of plasma deceleration. Further plasma motion at $t > t_s$ is an expansion with slowing-down velocity. It ends up at the turning point which corresponds to the maximum of expansion, $U(x_\sigma) = 1$. However, in the opposite case of low thermal pressure with $p_{\text{max}} < p_c$, the plasma systematically gets decelerated in the whole time interval of its dynamics.

A characteristic stopping time of plasma motion up to the full stop at the turning point is given by the integral of equations (11) and (15):
\[ t_s = \int_{x_1}^{x_\sigma} \frac{dy}{\sqrt{1 - U(y)}} \approx 2 \sqrt{x_\sigma - 1} / U(x_1). \]  
(21)

Calculating the time $t_s$ needed for plasma to reach this point, one can simplify the integrand, taking into account that the main contribution comes from the vicinity of the upper limit of integration. This approximation is expressed by the second part of equation (21). In the case of weak and homogeneous magnetic fields this yields universal expressions, $t_s \sim (M/v_m P_{\text{mag}})^{1/3}$ and $a_\sigma \sim a_{\text{mag}}$. It is worth mentioning that in the case of weak magnetic fields the
It is seen that stopping length and time decrease with some values of the normalized plasma thermal pressure \( \gamma \), \( \eta \) and \( \theta_0 \). From the right panel of figure 1 it is seen that at \( h > 0 \) (solid and dashed lines) there is a short initial period of acceleration, \( 0 < t \lesssim t_0 \), when the plasma boundary is accelerated according to equation (19). During this period (which is only weakly sensitive to magnetic field strength), the dimensionless radius \( a(t)/a_0 \) increases up to 2–3, and at \( t_0 \lesssim t < t_c \) almost all initial total energy \( W_0 \) is transferred into kinetic energy of free radial expansion at constant velocity \( \sim u_{m} \). As expected (see above), the time \( t_c \) is reduced on increasing the strength of the magnetic field and the free expansion period is shorter for larger \( \alpha \). Further increasing the strength of the magnetic field (figure 1, dotted line) results in a plasma dynamics with systematically slowing-down velocity.

For the same set of parameters \( \gamma \), \( \eta \) and \( \theta_0 \) in figure 2, the normalized stopping length (left panel) and stopping time (right panel) of the plasma cloud as a function of the dimensionless strength \( \alpha \) of the dipole magnetic field for some values of the normalized plasma thermal pressure \( \beta \) are shown. It is seen that stopping length and time decrease with

\[
\frac{dx(t)}{dt} = \alpha \frac{dx}{dt} + \beta \frac{dx}{dt}
\]

\[
x(t) = \frac{\alpha}{\beta} \ln \left( 1 + \frac{\beta}{\alpha} t \right)
\]

\[
\tau_s(\alpha) = \frac{\alpha}{\beta} \ln \left( 1 + \frac{\beta}{\alpha} \right)
\]

\[
\frac{dx}{dt} = \frac{\alpha}{\beta} \left( 1 + \frac{\beta}{\alpha} t \right)^{-1}
\]

\[
x(t) = \frac{\alpha}{\beta} \ln \left( 1 + \frac{\beta}{\alpha} t \right)
\]

\[
\tau_s(\alpha) = \frac{\alpha}{\beta} \ln \left( 1 + \frac{\beta}{\alpha} \right)
\]

\[
\frac{dx}{dt} = \frac{\alpha}{\beta} \left( 1 + \frac{\beta}{\alpha} t \right)^{-1}
\]

\[
x(t) = \frac{\alpha}{\beta} \ln \left( 1 + \frac{\beta}{\alpha} t \right)
\]

\[
\tau_s(\alpha) = \frac{\alpha}{\beta} \ln \left( 1 + \frac{\beta}{\alpha} \right)
\]
the strength of the magnetic field and are not sensitive to the variation of plasma thermal pressure.

Note that at otherwise unchanged parameters the strength of the dipole magnetic field is maximal at the orientation \( \theta_p = 0 \) and monotonically decreases with \( \theta_p \). For instance, the strength \( H_0(0) \) of the dipole magnetic field at the center of the plasma cloud is reduced by a factor of 2 by varying the dipole orientation from \( \theta_p = 0 \) to \( \theta_p = \pi / 2 \). Therefore, the effect of the magnetic field shown in figures 1 and 2 is weakened at the orientation \( \theta_p = \pi / 2 \) of the dipole. In particular, this results in larger stopping lengths and times than those shown in figure 2.

In this paper, we have assumed that the self-similar solutions (6)–(8) and hence the shape of the plasma remain isotropic during plasma expansion. Such an assumption is not evident since the distribution of magnetic pressure on the plasma surface is anisotropic in general. Thus one can expect strong deformation of the initially spherical plasma surface. Consider briefly this effect assuming constant magnetic field \( H_0 \). Taking into account the effect of the induced magnetic field, the total magnetic pressure on the plasma surface is \( P_{\text{mag}}(\theta) = (9 / 4) P_{\text{mag}} \sin^2 \theta \), where \( \theta \) is the angle between the radius vector \( r \) and \( H_0 \) [8, 10]. The magnetic pressure vanishes at \( \theta = 0, \pi \) and is maximal at \( \theta = \pi / 2 \). Therefore, one can expect that the spherical shape of the plasma is deformed into an ellipsoidal one, which is elongated along the magnetic field lines. This effect is supported by numerical simulations [4, 5]. Thus, the isotropic self-similar model considered here is valid as long as the deformation of the plasma shape is much smaller than the typical length scale for plasma flow. However, while solutions (6)–(8) are violated in the case of strong anisotropy plasma dynamics can be described (at least qualitatively) by equations (15)–(21) replacing the isotropic magnetic pressure \( P_{\text{mag}} \) by the angular-dependent one \( P_{\text{mag}}(\theta) \). The expansion radius \( a(t, \theta) \) as well as the stopping length \( a_i(\theta) \) and time \( t_i(\theta) \) should also be anisotropic in this case.

3. Conclusions

An analytical self-similar solution of the radial expansion of a spherical plasma cloud in the presence of a dipole or homogeneous magnetic field has been obtained. The analysis of plasma expansion into ambient magnetic field shows that there are processes of acceleration, retardation and stopping at the point of maximum expansion that are distinct and separated in space and time. The scaling laws obtained are, in general, functions of two dimensionless parameters, \( \alpha \) (or \( \sigma \) for constant magnetic field) and \( \beta \), which can be varied by means of the choice of the external magnetic field, the thermal pressure and the initial energy of the plasma. This allows one to test the different regimes of plasma dynamics over a wide range of external conditions.

We expect that our theoretical findings are useful in experimental investigations as well as in numerical simulations of plasma expansion into an ambient magnetic field (either uniform or nonuniform). One of the improvements of our present model will be the derivation of the dynamical equation for plasma surface deformation. In this case, it is evident that the problem is not isotropic with respect to the center of the plasma cloud \( (r = 0) \), and a full 3D analysis is required. A study of this and other aspects will be reported elsewhere.

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