Two positivity conjectures for Kerov polynomials

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Abstract

Kerov polynomials express the normalized characters of irreducible representations of the symmetric group, evaluated on a cycle, as polynomials in the “free cumulants” of the associated Young diagram. We present two positivity conjectures for their coefficients. The latter are stronger than the positivity conjecture of Kerov-Biane, recently proved by Féray.

1 Kerov polynomials

1.1 Characters

A partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ is a finite weakly decreasing sequence of nonnegative integers, called parts. The number $l(\lambda)$ of positive parts is called the length of $\lambda$, and $|\lambda| = \sum_{i=1}^{r} \lambda_i$ the weight of $\lambda$. For any integer $i \geq 1$, $m_i(\lambda) = \text{card}\{j : \lambda_j = i\}$ is the multiplicity of the part $i$ in $\lambda$.

Let $n$ be a fixed positive integer and $S_n$ the group of permutations of $n$ letters. Each permutation $\sigma \in S_n$ factorizes uniquely as a product of disjoint cycles, whose respective lengths are ordered such as to form a partition $\mu = (\mu_1, \ldots, \mu_r)$ with weight $n$, the so-called cycle-type of $\sigma$.

The irreducible representations of $S_n$ and their corresponding characters are also labelled by partitions $\lambda$ with weight $|\lambda| = n$. We write $\dim \lambda$ for the dimension of the representation $\lambda$ and $\chi^\lambda_\mu$ for the value of the character $\chi^\lambda(\sigma)$ at any permutation $\sigma$ of cycle-type $\mu$.

Let $r \leq n$ be a positive integer and $\mu = (r, 1^{n-r})$ the corresponding $r$-cycle in $S_n$. We write

$$\hat{\chi}^\lambda_r = n(n-1) \cdots (n-r+1) \frac{\chi^\lambda_{r, 1^{n-r}}}{\dim \lambda}$$

for the value at $\mu$ of the normalized character.

It was first observed by Kerov[6] and Biane[2] that $\hat{\chi}^\lambda_r$ may be written as a polynomial in the “free cumulants” of the Young diagram of $\lambda$. 


1.2 Free cumulants

Two increasing sequences \( y = (y_1, \ldots, y_{d-1}) \) and \( x = (x_1, \ldots, x_{d-1}, x_d) \) are said to be interlacing if \( x_1 < y_1 < x_2 < \cdots < x_{d-1} < y_{d-1} < x_d \). The center of the pair is \( c(x, y) = \sum_i x_i - \sum_i y_i \).

To any pair of interlacing sequences with center 0 we associate the rational function

\[
G_{x,y}(z) = \frac{1}{z - x_d} \prod_{i=1}^{d-1} \frac{z - y_i}{z - x_i},
\]

and the formal power series inverse to \( G_{x,y} \) for composition,

\[
G_{x,y}^{(-1)}(z) = z^{-1} + \sum_{k \geq 1} R_k(x, y) z^{k-1}.
\]

Note that \( R_1(x, y) = c(x, y) = 0 \). The quantities \( R_k(x, y), k \geq 2 \) are called the free cumulants of the interlacing pair \((x, y)\).

Being given a partition \( \lambda \), we consider the collection of unit boxes centered on the nodes \( \{(j - 1/2, i - 1/2) : 1 \leq i \leq l(\lambda), 1 \leq j \leq \lambda_i\} \). This defines a compact region in \( \mathbb{R}^2 \), the so-called Young diagram of \( \lambda \). On \( \mathbb{R}^2 \) we define the content function by \( c(u, v) = u - v \).

By convention, the content of a box is the one of its center.

Then it is easily shown that the Young diagram of \( \lambda \) defines a pair of interlacing sequences, formed by the contents \( y_1, \ldots, y_{d-1} \) of its corner boxes, and the contents \( x_1, \ldots, x_{d-1}, x_d \) of the corner boxes of its compliment in \( \mathbb{R}^2 \). We have \( x_1 = -l(\lambda) \), and \( x_d = \lambda_1 \).

Conversely, every pair of interlacing sequences with integer entries and center zero uniquely determines the Young diagram of a partition \( \lambda \).

The free cumulants \( R_k(\lambda), k \geq 2 \) are defined accordingly. These quantities arise in the asymptotic study of representations of symmetric groups [1].

1.3 Known results

The following result was first proved in [2] and attributed to Kerov[6].

**Theorem.** There exist polynomials \( K_r, r \geq 2 \) such that for any partition \( \lambda \) with \( |\lambda| \geq r \), one has

\[
\hat{\chi}_r^\lambda = K_r(R_2(\lambda), R_3(\lambda), \ldots, R_{r+1}(\lambda)).
\]

These polynomials have integer coefficients.

Let \((R_2, \ldots, R_{r+1})\) be the indeterminates of the “Kerov polynomial” \( K_r \) and define \(|\mu|\) as the “weight” of the monomial \( R_\mu = \prod_{i \geq 2} R_{m_i(\mu)}^{m_i(\mu)} \). We may decompose \( K_r \) in its graded components with respect to the weight, writing

\[
K_r = \sum_{s \geq 2} K_{r,s} \quad \text{with} \quad K_{r,s} = \sum_{|\mu| = s} x^{(r)}_{\mu} \prod_{i \geq 2} R_{m_i(\mu)}^{m_i(\mu)}.
\]
Then it may be proved [2] that the term of highest weight is $R_{r+1}$ and that $K_{r,s} = 0$ when $s = r - 2k$.

Goulden and Rattan [5, 10] have given a general formula for $K_{r,r-2k+1}$, expressing it as some coefficient in a formal power series (see also [3]). As a consequence, one has

$$K_{r,r-1} = \frac{1}{4} \binom{r+1}{3} \sum_{|\mu| = r-1} l(\mu)! \prod_{i \geq 2} \frac{((i-1)R_i)^{m_i(\mu)}}{m_i(\mu)!},$$

which had been conjectured by Biane [2] and differently proved by Šniady [11].

The same method provides an explicit form for $K_{r,r-3}$. But as far as $K_{r,r-5}$ (and lower components) are concerned, it seems very difficult to apply. Rattan [10, Theorem 3.5.12] found a messy expression of $K_{r,r-5}$ giving an idea about the complexity of the problem.

The following positivity property had been conjectured by Kerov [6] and Biane [2] and was recently proved by Féray [4].

**Theorem.** The coefficients of $K_r$ are nonnegative integers.

The purpose of this note is to present a stronger conjectural property.

## 2 Conjectures

An algebraic basis of the (abstract) symmetric algebra with real coefficients is formed by the classical symmetric functions, elementary $e_i$, complete $h_i$ or power-sum $p_i$. As usual for any partition $\mu$, denote $e_\mu, h_\mu$ or $p_\mu$ their product over the parts of $\mu$, and $m_\mu$ the monomial symmetric function, sum of all distinct monomials whose exponent is a permutation of $\mu$.

For a clearer display we write

$$R_\mu = \prod_{i \geq 2} ((i-1)R_i)^{m_i(\mu)}/m_i(\mu)!.$$}

Firstly we conjecture that the Kerov components $K_{r,r-2k+1}$ may be described in a unified way, independent of $r$.

**Conjecture 1.** For any $k \geq 1$ there exists an inhomogeneous symmetric function $f_k$, having maximal degree $4(k-1)$, such that

$$K_{r,r-2k+1} = \binom{r+1}{3} \sum_{|\mu| = r-2k+1} (l(\mu) + 2k - 2)! f_k(\mu) R_\mu,$$

where $f_k(\mu)$ denotes the value of $f_k$ at the integral vector $\mu$. This symmetric function is independent of $r$.

The assertion is trivial for $k = 1$ since we have $f_1 = 1/4$. Secondly we conjecture the symmetric function $f_k$ to be positive in the following sense.
Conjecture 2. For $k \geq 2$ the inhomogeneous symmetric function $f_k$ may be written

$$f_k = \sum_{|\rho| \leq 4(k-1)} c^{(k)}_{\rho} m_\rho,$$

where the coefficients $c^{(k)}_{\rho}$ are positive rational numbers.

The positivity of the coefficients of $K_{r,r-2k+1}$ is an obvious consequence. We emphasize that the coefficients of $f_k$ in terms of any other classical basis may be negative.

Conjecture 2 is firstly supported by the case $k = 2$. Using the expression of $K_{r,r-3}$ given in [5], we have the following result, whose proof is postponed to Section 3.

Theorem 1. For $k = 2$, we have

$$5760 f_2 = 3m_4 + 8m_{31} + 10m_{22} + 16m_{212} + 24m_{14} + 20m_3 + 36m_{21} + 48m_{13} + 35m_2 + 40m_{12} + 18m_1.$$

Conjecture 2 is secondly supported by extensive computer calculations, giving the values of the positive numbers $c^{(k)}_{\rho}$ for $k = 3, 4$. The two following conjectures have been checked for any $K_r$ with $r \leq 32$.

Conjecture 3. For $k = 3$, the values of $2.6!8!c^{(3)}_{\rho}$ are given by the table below.

| 8 | 71 | 62 | 61² | 53 | 521 | 51³ | 4² | 431 | 42² | 421² |
|---|----|----|-----|----|-----|-----|----|-----|-----|-----|
| 9 | 48 | 132 | 224 | 240 | 544 | 908 | 294 | 848 | 1132 | 1904 |
| 41¹ | 3² | 3¹² | 3²¹ | 3² | 3¹³ | 3¹ | 2⁴ | 2³¹² | 2²¹⁴ | 2¹⁶ |
| 31⁴ | 1440 | 2440 | 3280 | 5480 | 9040 | 4440 | 7440 | 12360 | 20400 | 33600 |

| 7 | 61 | 52 | 51² | 43 | 421 | 41³ | 3²¹ | 3² | 3²1² | 3²2² | 3²1² | 3²²² | 3²1² | 3²²² |
|---|----|----|-----|----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 216 | 968 | 2296 | 3744 | 3560 | 7704 | 12368 | 9856 | 13072 | 21264 | 33968 |
| 2⁴¹ | 2³¹³ | 2¹⁰ | 1⁶ |
| 28560 | 46080 | 73680 | 117600 |

| 6 | 51 | 42 | 41² | 3² | 3²1² | 3² | 3² | 2³ | 2²¹² | 2¹⁴ | 1⁶ |
|---|----|----|-----|----|-----|-----|----|----|-----|-----|-----|
| 2094 | 7696 | 15450 | 24016 | 19696 | 40592 | 62428 | 53796 | 83848 | 128988 | 198120 |

| 5 | 41 | 32 | 3¹² | 2³ | 2¹³ | 1⁵ |
|---|----|----|-----|----|-----|-----|
| 10588 | 30072 | 51096 | 75232 | 99640 | 146200 | 214040 |

| 4 | 31 | 2² | 2¹² | 1⁴ |
|---|----|----|-----|-----|
| 30109 | 67360 | 87382 | 120912 | 166320 |

| 3 | 2¹ | 1² |
|---|----|-----|
| 48092 | 77684 | 98016 |

| 2 | 1² | 1 |
|---|----|-----|
| 39884 | 43928 | 13200 |
Conjecture 4. For $k = 4$, the values of $2.8!12!c_p^{(4)}$ are given by the table below.

| $k$  | $11.1$ | $10.2$ | $10.1^2$ | $93$   | $921$  | $91^3$  |
|------|--------|--------|----------|--------|--------|---------|
| 495  | 3960   | 16830  | 29040    | 48312  | 113520 | 194392  |
| 84   | 831    | 82^2   | 82^2     | 81^4   | 75     | 741     |
| 99297| 296472 | 403590 | 692912   | 1180248| 150480 | 546480  |
| 732  | 731^2  | 724^2  | 721^3    | 71^6   | 66     | 651     |
| 945120| 1626592| 2219360| 3792480  | 6439200| 172260 | 733920  |
| 642  | 641^2  | 63^2   | 6321     | 631^3  | 64     | 62^4    |
| 1543740| 2654432| 19600992| 4611552 | 7890528| 6305640| 10797440|
| 621^4| 61^9   | 5^42   | 5^41^2   | 543    | 5421   | 541^9   |
| 18388320| 31168800| 1811040 | 3110800 | 2797872| 6566560| 11221392|
| 53^41 | 532^2  | 532^1  | 531^1    | 52^4   | 52^4^3 | 521^5   |
| 8360352| 11420640| 1957392 | 33343968 | 26812800| 45753120| 77733600|
| 51^7 | 4^3    | 4^3^1  | 4^22^2   | 4^21^4 | 4^2^1^4 | 43^2    |
| 131644800| 3402630 | 10157840 | 13861540| 23751840| 40429200| 17629920|
| 43^4^2 | 432^1^4| 432^1^3 | 431^6   | 42^4   | 42^4^1^2 | 42^4^2  |
| 30274720| 41416480| 70724640| 120150240| 56773080| 97050240| 165207840|
| 421^b | 41^8   | 3^4    | 3^21     | 3^1^3  | 3^2^3   | 3^2^1^3  |
| 280274400| 474454400| 22397760 | 52718400 | 90162240| 72246720| 123618880|
| 3^22^1^4| 3^1^9  | 32^1   | 32^1^9   | 32^1^9 | 321^7  | 31^9    |
| 210584640| 357315840| 169727040| 289477440| 492072000| 834301440| 1412328960|
| 2^6  | 2^1^2   | 2^1^4   | 2^1^6    | 2^1^8  | 21^10  | 1^12    |
| 233226000| 398160000| 677678400 | 115063200| 1950278400| 3302208000| 5588352000|

| $k$  | $10.1$ | $92$  | $91^2$  | $83$   | $821$  | $81^3$  |
|------|--------|-------|---------|--------|--------|---------|
| 25740| 184140 | 708444| 1199440 | 1836252| 4210844| 7075728 |
| 74   | 731    | 72^2  | 721^4   | 71^4   | 65     | 641     |
| 3371544| 9817984| 13294160| 22416768| 37488576| 4518360| 15961880|
| 632  | 631^2  | 621^4  | 621^3   | 61^5   | 51^7   | 542     |
| 27441744| 46345728| 62955728| 105656256| 176236800| 18678880| 38954344|
| 541^2 | 53^2   | 5321   | 531^3   | 52^3   | 52^2^4 | 52^4    |
| 65688480| 49454592| 113453824| 190456128| 154321200| 259588848| 434150016|
| 51^b  | 4^3    | 4^21   | 4^1^4   | 4^3^1  | 43^2   | 432^2   |
| 723211200| 59989160| 137408040| 230425440| 174697600| 237252400| 399329280|
| 431^4 | 421^4  | 421^4  | 421^9   | 41^1^4 | 3^2   | 3^4^2   |
| 667719360| 544244400| 912287040| 1522644480| 2534616000| 301150080| 507776640|
| 3^22^1^4| 3^1^9  | 32^1   | 32^1^9   | 32^1^9 | 321^7  | 321^9   |
| 691290880| 1159603200| 1935373440| 942671520| 1583527680| 2648849280| 4416612480|
| 31^8  | 3^1^8  | 3^1^5  | 3^1^8   | 3^1^7  | 2^1^7  | 1^11    |
| 7350568560| 2163722400| 3624808320| 6054048000| 10089797760| 16795537920| 27941760000|
| n   | 10      | 91      | 82      | 81^2    | 73      | 721     |
|-----|---------|---------|---------|---------|---------|---------|
| 41  | 589545  | 3732696 | 12880197| 21347832| 29796624| 66542608|
| 71^3| 71      | 64      | 631     | 62^2    | 621^2   | 61^4    |
| 109503504 | 48249234 | 136592720 | 184006988 | 304004800 | 498221712 |
| 5^2  | 541     | 532     | 531^2   | 52^21   | 521^3   |         |
| 56379312 | 192905680 | 329380304 | 544358320 | 736055232 | 1210234416 |
| 51^p | 4^2     | 4^11^4  | 43^2    | 4321    | 431^4   |         |
| 1979174160 | 398071454 | 657018384 | 504522128 | 1126245296 | 1850729904 |
| 1523067348 | 2510092224 | 4113855024 | 671963680 | 1427727840 | 1928190880 |
| 3^21^2 | 3^21^4  | 32^51    | 321^3   | 321^5   | 31^1    |         |
| 3180030560 | 5210415840 | 4309828320 | 7080806880 | 11585384160 | 18913547520 |
| 2^6   | 2^4^2   | 2^4^1^4  | 2^4^1^6 | 21^8    | 1^10    |         |
| 5844598200 | 9618960960 | 15768879840 | 25780980960 | 42087911040 | 68660524800 |
| 9    | 7834926 | 43370910 | 132689304 | 214757664 | 270145656 | 585908840 |
| 61^3 | 61      | 54      | 531     | 52^2    | 521^2   | 51^4    |
| 942097728 | 380072484 | 1041283232 | 1395178488 | 2251722880 | 3607638624 |
| 4^11  | 432     | 431^2   | 42^1    | 421^4   | 41^9    |         |
| 1253522292 | 2121348680 | 3421048224 | 4600109272 | 7388286912 | 11813196960 |
| 3^2   | 3^21^2  | 3^21^4  | 32^3    | 321^2   | 321^4   |         |
| 2679266304 | 5805122752 | 9318556608 | 7798935408 | 12559063744 | 20114667264 |
| 31^6  | 2^4^1   | 2^4^1^4  | 2^4^1^6 | 21^8    | 1^10    |         |
| 32123903040 | 16915888080 | 27152536320 | 43428598080 | 69327800640 | 110563004160 |
| 8    | 66992805 | 319460328 | 854070228 | 1345992736 | 1504935432 |
| 62^2 | 521     | 1^3     | 4^2     | 431     | 42^2    |         |
| 562166208 | 4945126296 | 1806665454 | 4760982424 | 6336879340 |
| 421^2 | 41^4    | 3^2     | 3^21^2  | 32^1    |         |         |
| 9953455776 | 15535885752 | 7959879312 | 12492469616 | 16671548080 |
| 321^4 | 31^5    | 2^1     | 2^1^2   | 2^1^4   |         |         |
| 26065233552 | 40592042160 | 22229990472 | 34840460832 | 54337307568 |
| 21^8  | 1^8     |         |         |         |         |         |
| 84517248240 | 131257445760 |
| 7    | 386137224 | 1557181296 | 357220960 | 5460878192 | 5341858632 |
| 51^2 | 421     | 41^3    | 3^2     | 32^2    | 321^2   |         |
| 10769122320 | 16360041456 | 13438992512 | 1772289864 | 26967001248 |
| 31^4 | 2^4^1   | 2^4^3   | 21^5    | 1^7     |         |         |
| 40796325216 | 35619645600 | 53958337440 | 81409500480 | 122509104480 |
| 6    | 1527234687 | 5086528128 | 9789272361 | 14430109232 | 12134469600 | 23282303088 |
| 31^4 | 2^4^2   | 2^4^1^2  | 21^4    | 1^6     |         |         |
| 34060600640 | 30307366254 | 44384647296 | 64583789280 | 93548535360 |
|   |   |   |   |   |
|---|---|---|---|---|
| 5 | 41 | 32 | 31^2 | 2^21 |
| 413557494 | 11019741678 | 17318813292 | 24369700608 | 31165644708 |
| 21^3 | 1^5 |
| 43403668704 | 59946923520 |
| 4 | 31 | 22 | 21^2 | 1^4 |
| 7478442180 | 15298473960 | 19094031000 | 25180566840 | 32685206400 |
| 3 | 21 | 1 |
| 8579601096 | 12733485336 | 15147277200 |
| 2 | 1^2 | 1 |
| 5589321408 | 5773242816 | 1555424640 |

3 Proof of Theorem 1

Following [5, 10] we consider the generating series

\[ C(z) = \sum_{i\geq 0} C_i z^i = \left(1 - \sum_{i\geq 2} (i - 1) R_i z^i\right)^{-1}. \]

By classical methods we have

\[ C_n = \sum_{|\mu|=n} l(\mu)! R_\mu. \]

It may be shown (see a proof in Section 7 below) that if \( \phi \) is a polynomial in \( i \), there exists a symmetric function \( \hat{\phi} \) such that

\[ \sum_{(i,j,k)\in\mathbb{N}^3 \atop i+j+k=n} \phi(i) C_i C_j C_k = \sum_{|\mu|=n} (l(\mu) + 2)! \hat{\phi}(\mu) R_\mu, \]

where \( \hat{\phi}(\mu) \) denotes the value of \( \hat{\phi} \) at the integral vector \( \mu \). For \( \phi(i) = a + bi + ci^2 \), we have

\[ \hat{\phi} = a/2 + bn/6 + c(n^2 + p_2)/12. \]

The following explicit form of \( K_{r,r-3} \) was given in [5, Theorem 3.3]

\[ K_{r,r-3} = \binom{r+1}{3} \sum_{(i,j,k)\in\mathbb{N}^3 \atop i+j+k=r-3} (a(r) + b(r)i^2) C_i C_j C_k, \]

with

\[ a(r) = -\frac{1}{2880}(r - 1)(r - 3)(r^2 - 4r - 6), \quad b(r) = \frac{1}{480}(2r^2 - 3). \]

As a straightforward consequence, we have

\[ K_{r,r-3} = \binom{r+1}{3} \sum_{|\mu|=r-3} (l(\mu) + 2)! f_2(\mu) R_\mu, \]
with

\[ f_2(\mu) = \frac{1}{2}a(r) + \frac{1}{12}b(r)((r - 3)^2 + p_2(\mu)). \]

But since \(|\mu| = p_1(\mu) = r - 3\), this can be rewritten

\[ f_2 = \frac{1}{5760} \left( 2p_2p_1^2 + p_1^4 + 12p_2p_1 + 8p_1^3 + 15p_2 + 20p_1^2 + 18p_1 \right). \]

Using for instance ACE [12] we easily obtain

\[ f_2 = \frac{1}{5760} \left( 3m_4 + 8m_{31} + 10m_{22} + 16m_{21^2} + 24m_4 + 20m_3 + 36m_{21} + 48m_{13} + 35m_2 + 40m_{12} + 18m_1 \right). \]

Observe that in this particular situation, the coefficients of \( f_2 \) in terms of power sums are nonnegative. This property is not true for \( K_{r,r-5} \) and lower components.

Starting from [10, Theorem 3.5.12], Conjecture 3 may probably be proved along the same line.

4 C-expansion

Goulden and Rattan [5, 10] have considered the expansion of Kerov polynomials in terms of the indeterminates \( C_i \). They have given the following positivity conjecture, proved for \( k = 1, 2 \), which is stronger than the one of Kerov and Biane.

**Conjecture.** For \( k \geq 1 \) the coefficients of \( K_{r,r-2k+1} \) in terms of the \( C_i \)'s are nonnegative rational numbers.

In analogy with Section 2 we conjecture that for any \( k \geq 1 \) one has

\[ K_{r,r-2k+1} = \binom{r + 1}{3} \sum_{\nu \in \mathbb{N}^{2k-1}} F_k(\nu) \prod_{i=1}^{2k-1} C_{\nu_i}, \]

where \( F_k \) is an inhomogeneous symmetric function, having maximal degree \( 4(k - 1) \) and independent of \( r \).

This is clear for \( k = 1 \) since

\[ K_{r,r-1} = \frac{1}{4} \binom{r + 1}{3} C_{r-1}, \]

hence \( F_1 = 1/4 \). For \( k = 2 \) we have seen in Section 3 that

\[ F_2(\nu) = a(r) + \frac{1}{3}b(r)p_2(\nu) \]
with \( \nu = (i, j, k) \). Since \(|\nu| = p_1(\nu) = r - 3 \), we obtain

\[
F_2 = \frac{1}{2880} \left( 4p_2p_1^2 - p_1^4 + 24p_2p_1 - 4p_1^3 + 30p_2 + 5p_1^2 + 18p_1 \right).
\]

However we emphasize that, unlike those of \( f_2 \), the coefficients of \( F_2 \) in terms of monomial symmetric functions are not positive. One has

\[
F_2 = \frac{1}{2880} \left( 3m_4 + 4m_{31} + 2m_{22} - 4m_{212} + 20m_3 + 12m_{21} - 24m_{13} + 35m_2 + 10m_{12} + 18m_1 \right).
\]

Therefore it seems that \( C \)-positivity and \( R \)-positivity are of a different nature.

5 New expansion

For a better understanding of the difference between the \( C \) and \( R \) expansions, it is useful to introduce new polynomials \( Q_i \) in the free cumulants. Define \( Q_0 = 1 \), \( Q_1 = 0 \) and for any \( n \geq 2 \),

\[
Q_n = \sum_{|\mu|=n} (l(\mu) - 1)! R_\mu.
\]

Writing for short

\[
Q_\mu = \prod_{i \geq 2} Q_i^{m_i(\mu)/m_i(\mu)!}, \quad C_\mu = \prod_{i \geq 2} C_i^{m_i(\mu)/m_i(\mu)!},
\]

the correspondence between these three families is given by

\[
Q_n = \sum_{|\mu|=n} (-1)^{l(\mu)} (l(\mu) - 1)! C_\mu,
\]

\[
C_n = \sum_{|\mu|=n} l(\mu)! R_\mu = \sum_{|\mu|=n} Q_\mu,
\]

\[
(1 - n) R_n = \sum_{|\mu|=n} (-1)^{l(\mu)} Q_\mu = \sum_{|\mu|=n} (-1)^{l(\mu)} l(\mu)! C_\mu.
\]

These relations are better understood by using the theory of symmetric functions. Actually let \( A \) be the (formal) alphabet defined by

\[
(i - 1) R_i = -h_i(A), \quad Q_i = -p_i(A)/i, \quad C_i = (-1)^{i} e_i(A).
\]

Writing

\[
u_{\mu} = l(\mu)! \prod_{i \geq 1} m_i(\mu)!, \quad \epsilon_{\mu} = (-1)^{n-l(\mu)}, \quad z_{\mu} = \prod_{i \geq 1} i^{m_i(\mu)} m_i(\mu)!,
\]

\[
u_{\mu} = \prod_{i \geq 1} i^{m_i(\mu)} m_i(\mu)!, \quad \epsilon_{\mu} = (-1)^{n-l(\mu)}.
\]

9
the previous relations are merely the classical properties [9, pp. 25 and 33]

\[ p_n = -n \sum_{|\mu|=n} (-1)^{l(\mu)} u_\mu h_\mu / l(\mu) = -n \sum_{|\mu|=n} \epsilon_\mu u_\mu e_\mu / l(\mu), \]

\[ e_n = \sum_{|\mu|=n} \epsilon_\mu u_\mu h_\mu = \sum_{|\mu|=n} \epsilon_\mu z_\mu^{-1} p_\mu, \]

\[ h_n = \sum_{|\mu|=n} z_\mu^{-1} p_\mu = \sum_{|\mu|=n} \epsilon_\mu u_\mu e_\mu. \]

From these relations, it is clear that \( C \)-positivity implies \( Q \)-positivity, which itself implies \( R \)-positivity. In particular the following conjecture is \textit{a priori} stronger than the one of Kerov-Biane and weaker than the one of Goulden-Rattan.

\textbf{Conjecture 5.} For \( k \geq 1 \) the coefficients of \( K_{r,r-2k+1} \) in terms of the \( Q_i \)'s are nonnegative rational numbers.

The assertion is trivial for \( k = 1 \) since

\[ K_{r,r-1} = \frac{1}{4} \left( \frac{r+1}{3} \right) C_{r-1} = \frac{1}{4} \left( \frac{r+1}{3} \right) \sum_{|\mu|=r-1} Q_\mu. \]

This leads us to the following conjecture (with obviously \( g_1 = 1/4 \)).

\textbf{Conjecture 6.} For any \( k \geq 1 \) there exists an inhomogeneous symmetric function \( g_k \), having maximal degree \( 4(k-1) \), such that

\[ K_{r,r-2k+1} = \left( \frac{r+1}{3} \right) \sum_{|\mu|=r-2k+1} (2k-1)^{l(\mu)} g_k(\mu) Q_\mu, \]

where \( g_k(\mu) \) denotes the value of \( g_k \) at the integral vector \( \mu \). This symmetric function is independent of \( r \).

It is a highly remarkable fact that, in contrast with the \( C \)-expansion, the \( Q \)-positivity is completely analogous to the \( R \)-positivity (and possibly equivalent).

\textbf{Conjecture 7.} For \( k \geq 2 \) the inhomogeneous symmetric function \( g_k \) may be written

\[ g_k = \sum_{|\rho| \leq 4(k-1)} a^{(k)}_\rho m_\rho, \]

where the coefficients \( a^{(k)}_\rho \) are positive rational numbers.

The assertion of Conjecture 5 is a direct consequence. Conjecture 7 is supported by the following result for \( k = 2 \), which will be proved in Section 6.
Theorem 2. For $k = 2$, we have

$$8640g_2 = 9m_4 + 20m_{31} + 22m_{22} + 28m_{212} + 24m_{4} + 60m_3 + 84m_{21} + 72m_{13} + 105m_2 + 90m_{12} + 54m_1.$$ 

Conjecture 7 is also supported by computer calculations, giving the positive numbers $a^{(k)}_\rho$ for $k = 3, 4$.

Conjecture 8. For $k = 3$, the values of $500.5!7!\ a^{(3)}_\rho$ are given by the table below.

|     | 8   | 71  | 62  | 61^2 | 53  | 521 | 51^4 | 4^2  | 43   | 421 | 41^4 | 3^2 | 3^2 | 32^2 |
|-----|-----|-----|-----|------|-----|-----|------|------|------|-----|------|-----|-----|------|
| 1125| 5400| 13500| 21480| 23400| 46200| 69072| 28350| 69000| 84900 |
| 421^2| 41^4 | 3^2 | 3^2 | 32^2 | 321^4 | 31^4 | 2^4 | 2^1^2 | 2^1^4 |
| 126168| 174864| 104400| 157152| 190704| 265632 | 338880| 233208| 322128| 414432 |
| 21^6 | 1^8 | 486720 | 524160 |

|     | 7   | 61  | 52  | 51^2 | 43  | 421 | 41^4 | 3^2 | 31^3 | 3^2 | 32^2 |
|-----|-----|-----|-----|------|-----|-----|------|-----|------|-----|------|
| 27000| 107400| 231000| 345360| 345000| 630840 | 874320| 785760| 953520 |
| 321^4 | 31^4 | 2^1 | 2^1 | 2^1 | 31^4 | 1^4 | |
| 1328160| 1694400| 1610640| 2072160| 2433600| 2620800 |

|     | 6   | 51  | 42  | 41^2 | 3^2 | 321 | 31^3 | 3^2 | 2^3 | 2^2 | 1^2 |
|-----|-----|-----|-----|------|-----|-----|------|-----|-----|-----|-----|
| 261750| 840300| 1532250| 2121660| 1907400| 3217080 | 4095696| 3896460| 5001672 |
| 21^4 | 1^0 | 5853744 | 6274080 |

|     | 5   | 41  | 32  | 31^2 | 2^1 | 31^3 | 1^3 |
|-----|-----|-----|-----|------|-----|------|-----|
| 1323500| 3322300| 5017400| 6358480| 7740360| 8988720 | 9530400 |

|     | 4   | 31  | 2^2 | 21^2 | 1^4 |
|-----|-----|-----|-----|------|-----|
| 3763625| 7093100| 8590950| 9830340| 10212600 |
| 3   | 21  | 1^3 |
| 6011500| 8045700| 8043000 |

|     | 2   | 1^2 | 1   |
|-----|-----|-----|-----|
| 4985500| 4595000| 1650000 |

This conjecture has been checked for any $K_r$ with $r \leq 32$. Starting from [10, Theorem 3.5.12], it may probably be proved by the method given in the next section.

We have also obtained the values of the positive numbers $a^{(4)}_\rho$. Listing them here would be tedious, but they are available upon request to the author.
6 Proof of Theorem 2

We start from the following lemma of symmetric function theory. It is better understood in the language of \( \lambda \)-rings. This method allows to handle symmetric functions acting on “sums”, “products” or “multiples” of alphabets. Here we shall not enter into details, and refer the reader to [7, Chapter 2] or [8, Section 3] for a short survey.

If \( f \) is a symmetric function, we denote \( f[\mathbf{A}] \) its \( \lambda \)-ring action on the alphabet \( \mathbf{A} \), which should not be confused with its evaluation \( f(\mathbf{A}) \). For instance \( p_n[-z+2]=-z^n+2 \) and \( p_n(-z+2)=(-z+2)^n \).

**Lemma 1.** On any alphabet \( \mathbf{A} \) and for any positive integer \( n \), we have

\[
\sum_{(i,j,k)\in\mathbb{N}^3} e_i e_j e_k = \sum_{|\mu|=n} (-1)^{n-l(\mu)} 3^{l(\mu)} z^{-1} p_\mu,
\]

\[
\sum_{(i,j,k)\in\mathbb{N}^3} i^2 e_i e_j e_k = \sum_{|\mu|=n} (-1)^{n-l(\mu)} 3^{l(\mu)-2} \left( n^2 + 2p_2(\mu) \right) z^{-1} p_\mu.
\]

**Sketch of proof.** Recall the “Cauchy formula” [7, (1.6.6)], or [9, (4.1) p. 62-65] or [8, p. 222],

\[
e_n[\mathbf{A}\mathbf{B}] = \sum_{|\mu|=n} (-1)^{n-l(\mu)} z^{-1} p_\mu[\mathbf{A}] p_\mu[\mathbf{B}].
\]

The first relation evaluates \( e_n[3\mathbf{A}] \) by using this formula together with the identity \( p_\mu[p] = p^{l(\mu)} \) valid for any real number \( p \).

For the second relation, we evaluate similarly \( e_n[(z+2)\mathbf{A}] \). Then we differentiate two times and fix \( z=1 \). At the left-hand side we get \( \sum_{i+j+k=n} i(i-1)e_i e_j e_k[\mathbf{A}] \). At the right-hand side, we compute

\[
\partial^2_z(p_\mu[z+2])\bigg|_{z=1} = \partial^2_z \left( \prod_{i\geq 1} (z^i+2)^{m_i(\mu)} \right)\bigg|_{z=1} = 3^{l(\mu)-2} \left( n^2 - 3n + 2p_2(\mu) \right).
\]

Observe that by differentiating \( r \) times, we might similarly get \( \sum_{i+j+k=n} \binom{i}{r} e_i e_j e_k. \)

Specializing the alphabet \( \mathbf{A} \) as in Section 5, so that

\[
Q_i = -p_i(\mathbf{A})/i, \quad C_i = (-1)^i e_i(\mathbf{A}),
\]

we obtain

\[
\sum_{(i,j,k)\in\mathbb{N}^3} (a+bi+ci^2) C_i C_j C_k = \sum_{|\mu|=n} 3^{l(\mu)} \left( a + \frac{b}{3} n + \frac{c}{9} (n^2 + 2p_2(\mu)) \right) Q_\mu.
\]

By insertion in the expression [5, Theorem 3.3]

\[
K_{r,r-3} = \binom{r+1}{3} \sum_{(i,j,k)\in\mathbb{N}^3} (a(r)+b(r)i^2) C_i C_j C_k,
\]

12
we obtain
\[ g_2(\mu) = a(r) + \frac{1}{9} b(r)((r-3)^2 + 2p_2(\mu)). \]
Since \(|\mu| = p_1(\mu) = r - 3\), this can be rewritten
\[ g_2 = \frac{1}{8640} \left( 8p_2p_1^2 + p_1^4 + 48p_2p_1 + 12p_1^3 + 60p_2 + 45p_1^2 + 54p_1 \right). \]
Using ACE \([12]\) the conversion to monomial functions is performed immediately. \(\square\)

7 Theorem 1 revisited

In Section 3 (proof of Theorem 1) we used the property
\[ \sum_{(i,j,k) \in \mathbb{N}^3 \atop i+j+k=n} (a + bi + c i^2) C_i C_j C_k = \frac{1}{2} \sum_{|\mu| = n} (l(\mu) + 2)! \left( a + \frac{b}{3} n + \frac{c}{6} (n^2 + p_2(\mu)) \right) R_\mu, \]
which may also be proved by \(\lambda\)-rings method. It is obtained by specialization of the following lemma.

Lemma 2. On any alphabet \(A\) and for any positive integer \(n\), we have
\[ \sum_{(i,j,k) \in \mathbb{N}^3 \atop i+j+k=n} e_i e_j e_k = \frac{1}{2} \sum_{|\mu| = n} (-1)^{n-l(\mu)} \frac{(l(\mu) + 2)!}{\prod_i m_i(\mu)!} h_\mu, \]
\[ \sum_{(i,j,k) \in \mathbb{N}^3 \atop i+j+k=n} i^2 e_i e_j e_k = \frac{1}{12} \sum_{|\mu| = n} (-1)^{n-l(\mu)} \frac{(l(\mu) + 2)!}{\prod_i m_i(\mu)!} \left( n^2 + p_2(\mu) \right) h_\mu. \]

Sketch of proof. Recall the “Cauchy formula” \([7, (1.6.3)]\), or \([9, (4.2)\ p. 62-65]\) or \([8, p. 222]\),
\[ (-1)^n e_n[A B] = \sum_{|\mu| = n} m_\mu[ -B ] h_\mu[A]. \]
The first relation evaluates \(e_n[3A]\) by using this formula and the identity \([7, (2.2.2)]\) valid for any real number \(p\),
\[ m_\mu[p] = p(p-1) \cdots (p-l(\mu)+1)/\prod_i m_i(\mu)! \].

For the second relation, we evaluate similarly \(e_n[(z+2)A]\). Then we differentiate two times and fix \(z = 1\), obtaining \(\sum_{i+j+k=n} i(i-1)e_i e_j e_k[A]\) at the left-hand side. At the right-hand side, we compute
\[ \prod_{i \geq 1} m_i(\mu)! \frac{\partial^2}{\partial z^2} (m_\mu[-z-2]) \bigg|_{z=1} = (-1)^{l(\mu)} (l(\mu) + 2)! \left( n^2 - 2n + p_2(\mu) \right) / 12. \]
\(\square\)
8 Final remark

In this note, we have considered three conjectural developments of the Kerov component $K_{r, r-2k+1}$, namely up to $\binom{r+1}{3}$,

$$
\sum_{\nu \in \mathbb{N}^{2k-1} \atop |\nu|=r-2k+1} F_k(\nu) \prod_{i=1}^{2k-1} C_{\nu_i} = \sum_{|\mu|=r-2k+1} (l(\mu) + 2k - 2)! f_k(\mu) \mathcal{R}_\mu
$$

$$
= \sum_{|\mu|=r-2k+1} (2k - 1)^{|\mu|} g_k(\mu) \mathcal{Q}_\mu.
$$

As indicated above, these relations are better understood in the framework of symmetric functions. Choosing

$$(i - 1)R_i = -h_i(A), \quad Q_i = -p_i(A)/i, \quad C_i = (-1)^i e_i(A),$$

they are the specializations at $A$ of the abstract identities

$$
\sum_{\nu \in \mathbb{N}^{2k-1} \atop |\nu|=n} F_k(\nu) e_\nu = \sum_{|\mu|=n} (-1)^{n-l(\mu)} f_k(\mu) \frac{(l(\mu) + 2k - 2)!}{\prod_i m_i(\mu)!} h_\mu
$$

$$
= \sum_{|\mu|=n} (-1)^{n-l(\mu)} g_k(\mu) (2k - 1)^{|\mu|} z_\mu^{-1} p_\mu.
$$

Moreover these identities are themselves related with the classical Cauchy formulas. Using the values of $p_\mu[p]$ and $m_\mu[p]$ given above, they may be written

$$
(-1)^n \sum_{\nu \in \mathbb{N}^{2k-1} \atop |\nu|=n} F_k(\nu) e_\nu = \sum_{|\mu|=n} f_k(\mu) m_\mu[-2k+1] h_\mu
$$

$$
= \sum_{|\mu|=n} g_k(\mu) p_\mu[-2k+1] z_\mu^{-1} p_\mu.
$$

Therefore it seems plausible that the conjectured positivity properties of $f_k$ and $g_k$ are equivalent, and reflect some abstract pattern of the theory of symmetric functions.

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