CATEGORIZATION OF INTEGRABLE REPRESENTATIONS OF QUANTUM GROUPS

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ABSTRACT. We categorify the highest weight integrable representations and their tensor products of a symmetric quantum Kac-Moody algebra. As byproducts, we obtain a geometric realization of Lusztig’s canonical bases of these representations as well as a new positivity result. The main ingredient in the underlying geometric construction is a class of micro-local perverse sheaves on quiver varieties.

INTRODUCTION

0.1. The purpose of this paper is to categorify integrable representations of quantum groups. More precisely, given a symmetric quantum Kac-Moody algebra $U$ and a tensor product $\Lambda$ of highest weight integrable $U$-modules, we construct the followings. (See 3.3.)

1. An abelian category $\mathcal{C}$ whose Grothendieck group $\mathbf{G}(\mathcal{C})$ has a natural structure of free $\mathbb{Z}[q, q^{-1}]$-module.
2. A set of exact endofunctors of $\mathcal{C}$ each for a generator of $U$.
3. A set of functor isomorphisms each for a defining relation of $U$, so that $\mathbf{G}(\mathcal{C}) \otimes \mathbb{Q}(q)$ is endowed with a structure of $U$-module.
4. An isomorphism of $U$-modules $\mathbf{G}(\mathcal{C}) \otimes \mathbb{Q}(q) \rightarrow \Lambda$ such that the indecomposable projectives of $\mathcal{C}$ give rise to the canonical basis of $\Lambda$.

Among others, we also obtain a positivity result on canonical basis. See Theorem 3.3.6(3) and Remark 3.3.7(3).

0.2. For the simplest quantum group $U_q(sl_2)$, categorification of its representations has been extensively studied in, for instance, [CK07, FK86, La08, Zh07] from various points of view. A categorification of representations of $U_q(sl_n)$ was obtained in [Su07] (see also [Ch06] for a treatment of level two representations). But results for other quantum groups are vacant from the literature (a good reference on this direction is [KL08]).

The present work follows the lines of [Zh07] and is intimately related to the works such as [KS97, Lu90, Lu91, Ma03, Na01, Sai02, Sav05] on the interplay between quiver varieties and quantum groups. Some of the relations will be mentioned in the paper. Especially, many techniques we use here are borrowed from [Lu91, Zh07]. Knowledge on them could be of much help for understanding this paper.

We remark that our categorification can be strengthened in a straightforward manner to yield a set of functor isomorphisms in bijection to the multiplication
rules of a modified quantum enveloping algebra as was done for the modified enveloping algebra of $sl_2$ in [BFK99]. It is however a challenging task to work out a categorification using higher categories as [CR08, La08].

0.3. The main contribution of this paper may be summarized as the construction of suitable categories for categorification task. Below are some underlying ideas hopefully useful for other problems concerning quantum algebras.

First of all, a common feature arising from the papers [Lu91, Zh07] is that one has on hand a certain kind of quiver varieties, each admitting a structure of cotangent bundle, say $T^*M$. Then one uses categories of sheaves on $M$, rather than on the total varieties $T^*M$, to fulfil desired goals.

This is essentially a generalization of the fundamental principle arising at the very beginning of quantum theory in physics: the classical phase space describing the motion of a particle is the symplectic space $\mathbb{R}^3 \times \mathbb{R}^3$ formed by the position space and the momentum space, while the quantum phase space is constituted by functions on a half part of the total classical phase space, either the position space or the momentum space.

The above observation led us to the attempt of similar treatment for the quiver varieties (Nakajima’s quiver varieties [Na94, Na98]) involved in our problem. Unfortunately, these quiver varieties in general do not admit structures of cotangent bundles. So, another idea came into our picture: we find open immersions of these quiver varieties into cotangent bundles, say $T^*M$, and extract our desired categories out by localizing categories of sheaves on $M$.

This is analogous to the definition of the ring of functions on a quasi-affine variety which is a localization of that on the ambient affine variety.

In fact, such application of localization on categories of sheaves is not new. It has already been adopted in the construction of stacks of micro-local perverse sheaves [GMV05, Wa04]. In this sense, our categorification is based on a class of micro-local perverse sheaves on quiver varieties.

Due to some technical reasons, for example characteristic variety has not been defined for an $l$-adic sheaf at present, realization of the above ideas is not straightforward in this paper. We will return to this point in 2.2.

0.4. The paper is organized as follows. In Section 1 we introduce various notions and important facts that will be used in this paper, as well as claim some elementary consequences without proof.

In Section 2 we first construct a triangulated category $\mathcal{D}$, which carries all the information of the highest weight integrable representations and their tensor products of a quantum group $U$. Then we define a set of endofunctors of $\mathcal{D}$ and establish a set of functor isomorphisms categorifying $U$. The main theorem is stated in 2.5.

In Section 3 we extract subcategories from $\mathcal{D}$ each for a tensor product of highest weight integrable $U$-modules. The abelian categorification claimed in 0.1 is fulfilled in the final subsection. It is helpful to have a glance at the main results stated in 3.3 before going into specific details.

\textbf{Contents}

Introduction

1. Preliminaries
1.1. Quantum groups

Assume a finite graph without circle edges is given. Let $I$ be the set of vertices and let $H$ be the set of pairs consisting of an edge and an orientation of it. The pair $(I, H)$ is a quiver in the sense that it also defines a directed graph. The elements of $H$ are referred to as arrows.

We associate to the finite graph a symmetric generalized Cartan matrix $(a_{ij})_{i,j \in I}$ by setting $a_{ii} = 2$ and $-a_{ij} = \text{the number of edges joining } i, j$ if $i \neq j$.

1.1.2. The quantum group $U$ associated to the generalized Cartan matrix $(a_{ij})$ is the $\mathbb{Q}(q)$-algebra defined by the generators $K_i, K_i^{-1}, E_i, F_i, i \in I$ and the relations

$$K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i K_j = K_j K_i,$$
$$K_i E_j = q^{a_{ij}} E_j K_i,$$
$$K_i F_j = q^{-a_{ij}} F_j K_i,$$
$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}},$$
$$\sum_{m=0}^{1-a_{ij}} (-1)^m E_i^{(m)} E_j F_i^{(1-a_{ij} - m)} = 0, \quad i \neq j,$$
$$\sum_{m=0}^{1-a_{ij}} (-1)^m F_i^{(m)} F_j E_i^{(1-a_{ij} - m)} = 0, \quad i \neq j,$$

where

$$E_i^{(n)} := \frac{E_i^n}{[n]_q}, \quad F_i^{(n)} := \frac{F_i^n}{[n]_q},$$
Hence contravariant forms on $U$ satisfying a generalized Cartan matrix ($\Delta U$ travariant form on the $U$ module of $M$) is the quotient of $\Lambda(\omega)$, $\Lambda(\omega_1)$, $\Lambda(\omega_2)$, $\Lambda(\omega')$, and all highest weight integrable $U$-modules are obtained in this way.

Remark 1.1.3. The quantum group $U$ is usually defined a little larger when the generalized Cartan matrix $(a_{ij})$ is not of finite type (cf. [Lu93]). But this makes no difference for our discussion on highest weight integrable representations. So we adopt the present definition.

1.1.4. Let $\varrho : U \to U^{op}$ be the algebra isomorphism defined on the generators by

$$
\varrho(K_i) = K_i, \quad \varrho(E_i) = qK_iF_i, \quad \varrho(F_i) = qK_i^{-1}E_i.
$$

By a contravariant form on a $U$-module $M$ we mean a symmetric bilinear form

$$(\cdot, \cdot) : M \times M \to \mathbb{Q}(q)$$

satisfying

$$(xu, w) = (u, \varrho(x)w) \quad \text{for } x \in U, \ u, w \in M.$$ 

The isomorphism $\varrho$ is compatible with the comultiplication of $U$:

$$(\varrho \otimes \varrho)(\Delta(x)) = \Delta\varrho(x) \quad \text{for } x \in U.$$ 

Hence contravariant forms on $U$-modules $M_1, M_2$ automatically give rise to a contravariant form on the $U$-module $M_1 \otimes M_2$

$$(u_1 \otimes u_2, w_1 \otimes w_2) := (u_1, w_1)(u_2, w_2) \quad \text{for } u_1, w_1 \in M_1, u_2, w_2 \in M_2.$$ 

We always assume tensor product modules are endowed with contravariant form in this way.

1.1.5. There is a $\mathbb{Q}$-algebra involution $- : U \to U$ determined by

$$
\bar{q} = q^{-1}, \quad \bar{K}_i = K_i^{-1}, \quad \bar{E}_i = E_i, \quad \bar{F}_i = F_i.
$$

1.1.6. The highest weight integrable representations of $U$ are described as follows.

Given $\omega = \sum_i \omega_i \cdot i \in \mathbb{N}[I]$, we have a Verma module $M(\omega)$ which by definition is the quotient of $U$ by the left ideal generated by $E_i, K_i - q^{\omega_i}$.

Let $\eta_\omega \in M(\omega)$ be represented by the unit of $U$. There is a unique contravariant form on $M(\omega)$ satisfying $(\eta_\omega, \eta_\omega) = 1$. Its radical $R(\omega)$ is the largest nontrivial submodule of $M(\omega)$. The quotient $\Lambda(\omega) = M(\omega)/R(\omega)$ is a highest weight integrable $U$-module, and all highest weight integrable $U$-modules are obtained in this way.

Also let $\eta_\omega \in \Lambda(\omega)$ be represented by the unit of $U$. We will assume $M(\omega), \Lambda(\omega)$ are both endowed with the unique contravariant form satisfying $(\eta_\omega, \eta_\omega) = 1$.

For a sequence $\omega = (\omega^1, \omega^2, \ldots, \omega^t)$ of elements in $\mathbb{N}[I]$, we set

$$
\Lambda(\omega) := \Lambda(\omega^1) \otimes \Lambda(\omega^2) \otimes \cdots \otimes \Lambda(\omega^t).
$$

In particular, $\Lambda(\emptyset)$ is the trivial $U$-module $\mathbb{Q}(q)$. 

$$
[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q! := [1]_q[2]_q \cdots [n]_q.
$$
1.1.7. The Verma module $M(\omega)$ is universal in the following sense.

Let $M, N$ be $U$-modules and let $\varphi : M \to N$ be a linear homomorphism such that $E_i \varphi(u) = \varphi(E_i u)$ and $K_i \varphi(u) = q^{\alpha_i} \varphi(K_i u)$ for $i \in I$ and $u \in M$. Then there exists a unique $U$-module homomorphism $\tilde{\varphi} : M \otimes M(\omega) \to N$ such that $\tilde{\varphi}(u \otimes \eta) = \varphi(u)$, $u \in M$.

Moreover, if $\varphi$ preserves some preassigned contravariant forms on $M, N$, then so does $\tilde{\varphi}$.

1.2. **Perverse sheaves on algebraic stacks.** The theory of perverse sheaves on schemes [BBD82] has a natural version for algebraic stacks. We refer the readers to [Jo93] for a systematic treatment.

Algebraic stacks arising from this paper are actually very restrictive: each is either (i) a (quasi-projective) algebraic variety or (ii) a quotient stack $[X/G]$ where $X$ is an algebraic variety and $G$ is a product of general linear groups acting on it. In the latter case, the derived category of sheaves on $[X/G]$ is nothing new but the $G$-equivariant derived category described in [BL94].

Adopting the language of algebraic stack allows us to treat usual and equivariant derived categories in a uniform way and sometimes simplify our arguments.

1.2.1. Let $\mathbb{F}$ be an algebraic closure of a finite field and let $X$ be an algebraic stack of finite type over $\mathbb{F}$. We denote by $\mathcal{D}(X) = \mathcal{D}_c^b(X, \mathbb{Q}_l)$ the bounded derived category of constructible $\mathbb{Q}_l$-sheaves on $X$ (cf. [BBD82] 2.2.18, [Jo93] 3.3)), where $l$ is a prime number invertible in $\mathbb{F}$. An object of $\mathcal{D}(X)$ is also referred to as a complex.

The constant sheaf on $X$ is denoted as $\mathbb{Q}_l, X$, or merely $\mathbb{Q}_l$ when $X$ is clear from the context. We choose an isomorphism $\mathbb{Q}_l, pt(1) \cong \mathbb{Q}_l, pt$ once and for all and omit the Tate twist throughout this paper.

1.2.2. Verdier duality and the bifunctors $R\text{Hom}, \otimes$ are defined for $\mathcal{D}(X)$. The Verdier dual of $A \in \mathcal{D}(X)$ is denoted as $DA$.

For a morphism $f : X \to Y$ of algebraic stacks, there is an induced functor $f^* : \mathcal{D}(Y) \to \mathcal{D}(X)$. When $f$ is representable, there are also induced functors $f_!, f_* : \mathcal{D}(X) \to \mathcal{D}(Y)$ and $f^! : \mathcal{D}(Y) \to \mathcal{D}(X)$.

Basic properties (for example, those studies in [KW01] Chapter II,III]) of the functors $f^*, f_!, f_*, f^!, D, R\text{Hom}, \otimes$ for schemes have a natural version for algebraic stacks.

1.2.3. For an embedding of locally closed substack $j : S \to X$, we use $A|_S$ to denote $j^* A$ for a complex $A \in \mathcal{D}(X)$.

1.2.4. Let $\mathcal{M}(X)$ denote the full subcategory of $\mathcal{D}(X)$ consisting of the perverse sheaves. Let $^pH^n : \mathcal{D}(X) \to \mathcal{M}(X)$ denote the $n$-th cohomological functor associated to the perverse t-structure.

A complex $C \in \mathcal{D}(X)$ is said to be semisimple if $C \cong \oplus_n ^pH^n(C)[-n]$ and if $^pH^n(C) \in \mathcal{M}(X)$ is semisimple for all $n$.

1.2.5. **Decomposition theorem.** Let $f : X \to Y$ be a representable proper morphism of algebraic stacks with $X$ smooth. Then $f_! \mathbb{Q}_l, X \in \mathcal{D}(Y)$ is a semisimple complex. Cf. [Jo93] 4.13.
1.2.6. More generally, let \( f : X \to Y \) be a representable morphism of algebraic stacks. Assume there is a stratification \( X = \bigsqcup_n U_n \) by locally closed substacks such that restricting to each stratum \( U_n \) the morphism \( f \) can be factored as

\[
U_n \xrightarrow{f_n'} X_n \xrightarrow{f_n} Y
\]

where \( f_n' \) is a vector bundle of fiber dimension \( d_n \) and \( f_n \) is a representable proper morphism with \( X_n \) smooth. Then \( f!\mathbb{Q}_{l,X} \in \mathcal{D}(Y) \) is a semisimple complex. Moreover,

\[
f!\mathbb{Q}_{l,X} \cong \oplus_n f_n!\mathbb{Q}_{l,X_n}[-2d_n].
\]

The proof is essentially the same as \([Lu85, \text{3.7}]\), which argues that the complexes \((f|_{U_n})!\mathbb{Q}_{l,U_n}\) are induced from pure complexes of the same weight.

1.3. Fourier-Deligne transform. We refer the readers to \([KW01]\) as a general reference for this subsection.

Let \( F_p \) be a finite field with \( p \) elements, where \( p \) is the characteristic of \( F \). We fix a nontrivial character \( \chi : F_p \to \mathbb{Q}_l^\times \). The Artin-Schreier covering \( F \to F \) given by \( x \mapsto x^p - x \) has \( F_p \) as a group of covering transformations. Hence the character \( \chi \) gives rise to a \( \mathbb{Q}_l \)-local system \( \mathcal{L} \) of rank one on the affine line \( F \).

1.3.1. Let \( \pi : E \to X, \pi' : E' \to X \) be two vector bundles of constant fiber dimension \( d \) over an algebraic variety \( X \). Assume we are given a bilinear map \( T : E \times_X E' \to F \) which defines a duality between the two vector bundles. We have a diagram \( E \xleftarrow{s} E \times_X E' \xrightarrow{t} E' \) where \( s, t \) are the obvious projections.

The Fourier-Deligne transform is defined to be the functor

\[
\Phi_{E,E'} : \mathcal{D}(E) \to \mathcal{D}(E'), \quad A \mapsto t_!(s^*A \otimes T^*\mathcal{L})[d].
\]

Interchanging the roles of \( E, E' \) we have another Fourier-Deligne transform

\[
\Phi_{E',E} : \mathcal{D}(E') \to \mathcal{D}(E), \quad B \mapsto s_!(t^*B \otimes T^*\mathcal{L})[d].
\]

1.3.2. Fourier inversion formula. There is an isomorphism of functors

\[
\Phi_{E',E} \Phi_{E,E'} \cong \sigma^*
\]

where \( \sigma \) is the multiplication of \(-1\) on each fiber of \( E \).

It is known that \( \Phi_{E,E'} \) is perverse t-exact. In particular, it restricts to an equivalence of categories \( \mathcal{M}(E) \to \mathcal{M}(E') \).

1.3.3. Fourier-Deligne transform almost commutes with Verdier duality

\[
D\Phi_{E,E'} D \cong \Phi_{E,E'} \sigma^*.
\]

In fact, by \([1.3.2]\) the right hand side is inverse to \( \Phi_{E',E} \) hence is right adjoint to \( \Phi_{E'E} \). On the other hand, the left hand side is also right adjoint to \( \Phi_{E,E'} \) as demonstrated by the natural isomorphisms

\[
\text{Hom}_{D(E')}(A, D\Phi_{E,E'} DB) = \text{Hom}_{D(E')}(A, D[d]t_!(s^*DB \otimes T^*\mathcal{L})) = \text{Hom}_{D(E')}(A, [-d]t_i\mathcal{R}\text{Hom}(T^*\mathcal{L}, s^*B)) = \text{Hom}_{D(E)}(s_!(t^*A \otimes T^*\mathcal{L})[d], B) = \text{Hom}_{D(E)}(\Phi_{E',E} A, B).
\]
1.3.4. Let $G$ be an algebraic group. Assume the morphisms $\pi, \pi', T, \sigma$ are $G$-equivariant (letting $G$ acts trivially on $\mathbb{F}$). One may equally well define a Fourier-Deligne transform
$$\Phi_{E', E} : D([E'/G]) \to D([E/G])$$
by using the induced morphisms $[E/G] \to [E \times_X E'/G] \to [E'/G]$ and $T : [E \times_X E'/G] \to \mathbb{F}$. The claims from 1.3.2 and 1.3.3 remain true for this Fourier-Deligne transform, whose proof is the same as the usual version.

1.4. Localization in triangulated categories. A typical example of localization appears in the construction of a derived category, via which all acyclic complexes are forced to zero whence quasi-isomorphisms become isomorphisms.

In some sense, the philosophy of localization is that one kills redundant objects off an ambient category so as to get an appropriate one for usage.

In the present paper, localization is adopted in the same way as in micro-local sheaf theory (cf. [KSc90, 6.1]). Objects being killed are those “supported outside a given place”. We will explain this point later in 2.2.

1.4.1. Let $\mathcal{D}$ be a triangulated category. A thick subcategory of $\mathcal{D}$ is a full triangulated subcategory $\mathcal{N}$ such that if a morphism $A \to B$ in $\mathcal{D}$ factors through an object in $\mathcal{N}$ and can be embedded into an exact triangle $A \to B \to C \to A[1]$ with $C \in \mathcal{N}$ then $A, B \in \mathcal{N}$. (In particular, if two objects in an exact triangle are contained in $\mathcal{N}$, so is the third one.)

Given a thick subcategory $\mathcal{N}$, the class of morphisms $S := \{s : A \to B \mid s \text{ can be embedded into an exact triangle } A \to B \to C \to A[1] \text{ with } C \in \mathcal{N}\}$ is localizing, hence gives rise to a triangulated category $\mathcal{D}[S^{-1}]$ by localization (cf. [GM94, 4.1, 5.1.10]). We denote this category by $\mathcal{D}/\mathcal{N}$.

By definition, the objects of $\mathcal{D}/\mathcal{N}$ are the same as $\mathcal{D}$. A morphism $A \to B$ in $\mathcal{D}/\mathcal{N}$ is an equivalence class of roofs, i.e. an equivalence class of pairs of morphisms $(T \xrightarrow{s} A, T \xrightarrow{t} B)$ in $\mathcal{D}$ with $s \in S$. Two roofs $(s_i, t_i), i = 1, 2$ are equivalent, if there is a third roof $(s, t)$ forming into a commutative diagram

\[
\begin{array}{ccc}
T & \xrightarrow{s} & A \\
\downarrow & & \downarrow \\
T_1 & \xrightarrow{s_1} & A \\
& \searrow & \searrow \\
& T_2 & \xrightarrow{s_2} & A \\
& & \downarrow \\
& & B \\
\end{array}
\]

Remark 1.4.2. (1) It can be shown that an object in $\mathcal{D}/\mathcal{N}$ is isomorphic to zero if and only if it is an object from $\mathcal{N}$.

(2) For localization purpose it suffices to assume $\mathcal{N}$ is a null system (cf. [KSc90, 1.6]). But this does not provide new constructions. In fact, a null system $\mathcal{N}$ can always be completed to a thick subcategory $\tilde{\mathcal{N}}$ (the one generated by $\mathcal{N}$) so that the localized categories $\mathcal{D}/\mathcal{N}, \mathcal{D}/\tilde{\mathcal{N}}$ are identical.

1.4.3. Let $\varphi : \mathcal{D}_1 \to \mathcal{D}_2$ be a functor of triangulated categories and let $\mathcal{N}_2$ be a thick subcategory of $\mathcal{D}_2$. Then the full subcategory $\mathcal{N}_1$ of $\mathcal{D}_1$ whose objects are those sent by $\varphi$ into $\mathcal{N}_2$ is a thick subcategory.
1.4.4. Let \( N_i \) be a thick subcategory of \( D_i, \ i = 1, 2 \) and let \( \varphi : D_1 \to D_2 \) be a functor of triangulated categories such that \( \varphi(N_1) \subset N_2 \). Then the following defines a functor of triangulated categories

\[
\tilde{\varphi} : D_1/N_1 \to D_2/N_2
\]

\[
A \mapsto \varphi(A)
\]

\[
[s,t] \mapsto [\varphi(s), \varphi(t)].
\]

1.4.5. Assume \( \psi : D_1 \to D_2 \) satisfies the same assumptions as \( \varphi \). Then a natural transformation \( \alpha_A : \varphi A \to \psi A, \ A \in D_1 \) induces a natural transformation of the induced functors \( [\text{Id}_A, \alpha_A] : \tilde{\varphi}_A \to \tilde{\psi}_A, \ A \in D_1/N_1 \).

It follows that functor isomorphisms and functor adjunctions are preserved under localization.

1.4.6. Let \( (D^\geq, D^\leq) \) be a t-structure on \( D \) with the core \( M = D^\geq \cap D^\leq \) and cohomological functors \( H^n : D \to M \). We assume

1. the given t-structure is bounded, i.e. for every object \( A \in D \), \( H^n(A) \not\cong 0 \) for only finitely many \( n \) and, moreover, \( A \cong 0 \) if and only if \( H^n(A) \cong 0 \) for all \( n \);
2. every object in \( M \) has finite length.

Let \( \mathcal{N} \) be a full subcategory of \( D \) such that

3. \( A \in \mathcal{N} \) if and only if \( H^n(A) \in \mathcal{N} \) for all \( n \);
4. \( \mathcal{M} \cap \mathcal{N} \) is a Serre subcategory of \( \mathcal{M} \), i.e. \( \mathcal{M} \cap \mathcal{N} \) is stable under extensions and subquotients.

It is easy to check that \( \mathcal{N} \) is the thick subcategory of \( D \) generated by the simple objects from \( \mathcal{M} \cap \mathcal{N} \).

If two thick subcategories \( N_1, N_2 \) of \( D \) satisfy (3)(4), so does the thick subcategory \( \tilde{\mathcal{N}} \) generated by \( N_1, N_2 \). In fact, \( \tilde{\mathcal{N}} \) is generated by the simple objects from \( \mathcal{M} \cap \mathcal{N}_1 \) and \( \mathcal{M} \cap \mathcal{N}_2 \).

1.4.7. With the above assumptions, let \( D/\mathcal{N}^\geq \) (resp. \( D/\mathcal{N}^\leq, \mathcal{M}/\mathcal{N} \)) be the full subcategory of \( D/\mathcal{N} \) consisting of the objects isomorphic to those from \( D^\geq \) (resp. \( D^\leq, \mathcal{M} \)). Some labors on homological algebras will convince the reader that \( (D/\mathcal{N}^\geq, D/\mathcal{N}^\leq) \) defines a t-structure on \( D/\mathcal{N} \) with the core \( \mathcal{M}/\mathcal{N} \).

Moreover, an object \( A \in \mathcal{M} \) is simple in \( \mathcal{M}/\mathcal{N} \) if and only if precisely one simple component of its Jordan-Hölder decomposition in \( \mathcal{M} \) is not contained in \( \mathcal{N} \).

1.4.8. Example. Let \( X \) be an algebraic stack and let \( j : U \to X \) be the inclusion of an open substack. Let \( \mathcal{N} \) be the full subcategory of \( D(X) \) consisting of the objects whose supports are disjoint from \( U \).

Then the triangulated category \( D(X) \) endowed with the perverse t-structure and the full subcategory \( \mathcal{N} \) satisfy the assumptions of 1.4.6. (Condition (2) is stated in [Jo93, 4.3]; the others are clear from definition.) Moreover, the functor \( j^* : D(X) \to D(U) \) and the functor \( j_* : D(U) \to D(X) \) induce (in the sense of 1.4.4) an equivalence of triangulated categories

\[
D(X)/\mathcal{N} \sim D(U).
\]
2. Categories and functors from quiver varieties

Let the quiver \((I, H)\), the generalized Cartan matrix \((a_{ij})\) and the quantum group \(U\) be defined in [1,3,11]

2.1. The category \(\mathcal{D}\).

2.1.1. First, we enlarge \((I, H)\) to a quiver \((\hat{I}, \hat{H})\) by appending to the underlying graph for each vertex \(i \in I\) a new vertex \(\hat{i}\) and an edge joining \(i, \hat{i}\).

\[
\begin{array}{ccc}
1 & 2 & 3 \\
\hline
1 & 2 & 3 \\
\end{array}
\]

\((I, H)\) \hspace{1cm} \((\hat{I}, \hat{H})\) an orientation

2.1.2. Notations. Let \(\hat{\cdot} : \hat{H} \to \hat{H}\) be the involution defined by reversing the orientation. An orientation of \((\hat{I}, \hat{H})\) is a subset \(\Omega \subset \hat{H}\) such that \(\Omega \cup \hat{\Omega} = \hat{H}\) and \(\Omega \cap \hat{\Omega} = \emptyset\). (In particular, \(\Omega\) has the half number of elements as \(\hat{H}\).

For an arrow \(h \in H\), we denote by \(h', h''\) the source vertex and target vertex of \(h\), respectively. A vertex \(i \in \hat{I}\) is called a source (resp. sink) of \(\Omega \subset \hat{H}\), if every arrow \(h \in \Omega\) incident to \(i\) is outgoing (resp. incoming).

2.1.3. Quiver varieties. Assume \(\nu = \sum_{i \in I} \nu_i \cdot i \in N[\hat{I}]\). We set \(V_i = \mathbb{F}^{\nu_i}, i \in \hat{I}\).

For each subset \(\Omega \subset \hat{H}\) we define a variety

\[E_\Omega = E_{\nu, \Omega} := \bigoplus_{h \in \Omega} \text{Hom}(V_{h'}, V_{h''}).\]

Let the algebraic group

\[G_\nu := \prod_{i \in I} GL(V_i)\]

act on \(E_\Omega\) from the right such that \((x \cdot g)_h = g_h^{-1}x_hg_{h'}\) (we define \(g_i = \text{Id}, i \in I\).

2.1.4. Fourier-Deligne transform. Given two orientations \(\Omega, \Omega' \subset \hat{H}\), we regard \(E_\Omega\) and \(E_{\Omega'}\) as vector bundles over \(E_{\Omega \cup \Omega'}\) in the obvious way. The morphism

\[T : E_{\Omega \cup \Omega'} = E_\Omega \times_{E_{\Omega \cup \Omega'}} E_{\Omega'} \to \mathbb{F}, \quad x \mapsto \sum_{h \in \Omega \setminus \Omega'} \text{Tr}(x_h x_h')\]

is clearly \(G_\nu\)-equivariant (letting \(G_\nu\) acts trivially on \(\mathbb{F}\)) and defines a duality of the vector bundles \(E_\Omega, E_{\Omega'}\), hence gives rise to a Fourier-Deligne transform

\[\Phi_{\Omega, \Omega'} : \mathcal{D}(\text{[}E_{\Omega}/G_\nu\text{])} \to \mathcal{D}(\text{[}E_{\Omega'}/G_\nu\text{])}.\]

Proposition 2.1.5. There is an isomorphism of functors \(\Phi_{\Omega_2, \Omega_3} \Phi_{\Omega_1, \Omega_2} \cong \Phi_{\Omega_1, \Omega_3}\) for orientations \(\Omega_1, \Omega_2, \Omega_3 \subset \hat{H}\).

Assume \(\Omega_a = (\Omega \setminus \Omega_a') \sqcup \hat{\Omega}_a', a = 1, 2, 3\), where \(\Omega\) is an orientation and \(\Omega_1', \Omega_2', \Omega_3' \subset \Omega\) are disjoint subsets. It is easy to see \(\Phi_{\Omega_a, \Omega_b} \cong \Phi_{\Omega, \Omega_a} \Phi_{\Omega_a, \Omega_b}\) for \(a \neq b\).

Notice that the morphism \(\sigma : [E_{\Omega'/G_\nu}] \to [E_{\Omega'/G_\nu}]\) induced by the multiplication of \(-1\) on a Hom summand of \(E_{\Omega}\) is isomorphic to the identity. From [1,3,2] it follows that \(\Phi_{\Omega_2, \Omega} \Phi_{\Omega_1, \Omega} \cong \text{Id}\).

Therefore, \(\Phi_{\Omega_2, \Omega_3} \Phi_{\Omega_1, \Omega_2} \cong \Phi_{\Omega, \Omega_3} \Phi_{\Omega_2, \Omega} \Phi_{\Omega_1, \Omega} \cong \Phi_{\Omega_1, \Omega_3}\).
2.1.6. We denote by \( x(i) \) the restriction of \( x \in E_\Omega \) to the direct summand
\[
\bigoplus_{h \in \Omega: h' = i} \text{Hom}(V_{h'}, V_{h''}) = \text{Hom}(V_i, V(i))
\]
where
\[
V(i) := \bigoplus_{h \in \Omega: h' = i} V_{h''}.
\]
Define an open subvariety of \( E_\Omega \) for each \( i \in I \)
\[
\hat{E}_{\Omega,i} = \hat{E}_{\nu,\Omega,i} := \{ x \in E_\Omega \mid \text{Ker} \ x(i) = 0 \}.
\]
When \( i \) is a source of \( \Omega \), letting \( \hat{\Omega} \) denote the subset of \( \Omega \) consisting of the arrows not incident to \( i \), we have
\[
\hat{E}_{\Omega,i}/GL(V_i) = \{ (\hat{x}, V) \mid \hat{x} \in E_{\hat{\Omega}i}, V \subset V(i) \text{ is a subspace of dimension } \nu_i \}.
\]

2.1.7. Localization. We choose for each vertex \( i \in I \) an orientation \( \Omega_i \subset \hat{\Omega} \) which has \( i \) as a source. Denote by \( N_\Omega \) the thick subcategory of \( \mathcal{D}([E_{\Omega}/G_\nu]) \) consisting of the complexes whose supports are disjoint from the open substack \( [E_{\Omega,i}/G_\nu] \).

Then for an arbitrary orientation \( \Omega \subset \hat{\Omega} \), we denote by \( N_{\nu,\Omega,i} \) the thick subcategory of \( \mathcal{D}([E_{\Omega}/G_\nu]) \) consisting of the complexes sent by \( \Phi_{\Omega,\Omega_i} \) into \( N_\Omega \) (cf. [1.4.3]). Note that \( N_{\nu,\Omega,i} \) is independent of the choice of \( \Omega_i \).

Define \( N_{\nu,\Omega} \) to be the thick subcategory of \( \mathcal{D}([E_{\Omega}/G_\nu]) \) generated by \( N_{\nu,\Omega,i} \), \( i \in I \).

2.1.8. By Proposition 2.1.5, the categories \( \mathcal{D}([E_{\Omega}/G_\nu]) \) for various orientation \( \Omega \) are related via Fourier-Deligne transforms, thus determine a unique triangulated category up to equivalence which we denote as \( \mathcal{D}([E_\nu/G_\nu]) \). Accordingly, \( N_{\nu,\Omega} \) for various \( \Omega \) give rise to a thick subcategory \( N_\nu \) of \( \mathcal{D}([E_{\nu}/G_\nu]) \).

Now we are in position to define our main category
\[
\mathcal{D} := \bigoplus_{\nu} \mathcal{D}_\nu
\]
where
\[
\mathcal{D}_\nu := \mathcal{D}([E_{\nu}/G_\nu])/N_\nu.
\]

Since Fourier-Deligne transforms are perverse t-exact, perverse t-structure is well defined on \( \mathcal{D}([E_{\nu}/G_\nu]) \), as well as on \( \mathcal{D} \) according to [1.4.6][1.4.8].

According to [1.4.3] and the isomorphism \( \sigma \cong \text{Id} \) claimed in the proof of Proposition 2.1.3 Verdier duality is well defined on \( \mathcal{D}([E_{\nu}/G_\nu]) \). It is clear that \( DN_\nu \subset N_\nu \), hence Verdier duality is well defined on \( \mathcal{D} \), too.

2.2. A micro-local point of view. The category \( \mathcal{D}([E_{\Omega}/G_\nu])/N_{\nu,\Omega} \) actually can be well understood from the viewpoint of micro-local sheaf theory. For simplicity, we assume \((I,H)\) is of finite type.

First, we review in brief Nakajima’s quiver variety \( \text{Na94}, \text{Na98} \) associated to \((I,H)\) and \( \nu \in \mathbb{N}[I] \).

Choose an orientation \( \Omega \subset \hat{\Omega} \) and identify \( E_{\hat{\Omega}} \) with \( T^*E_\Omega \). Then the full quiver variety \( E_{\hat{\Omega}} \) is canonically a symplectic vector space, on which \( G_\nu \) acts by symplectic isomorphisms. So, we have a momentum map \( \mu : E_{\hat{\Omega}} \to \text{Lie}(G_\nu)^* \) which is assumed vanishing at the origin of \( E_{\hat{\Omega}} \).

One can show the algebraic group \( G_\nu \) acts freely on the variety
\[
\mu^{-1}(0)^* := \{ x \in \mu^{-1}(0) \mid \text{Ker} \ x(i) = 0, \ i \in I \},
\]
and Nakajima’s quiver variety is defined to be the quotient

$$\mathcal{M}_\nu := \mu^{-1}(0)^\ast / G_\nu.$$ 

Actually $\mu^{-1}(0)$ and hence $\mathcal{M}_\nu$ are independent of $\Omega$.

Now, observe that $\mu^{-1}(0)$ is precisely the union of the conormal varieties to the $G_\nu$-orbits of $E_{\Omega}$. Intuitively we may take the quotient stack $[\mu^{-1}(0)/G_\nu]$ as the cotangent bundle to $[E_{\Omega}/G_\nu]$. Also observe that $\mathcal{M}_\nu$ is an open substack of $T^*[E_{\Omega}/G_\nu] := [\mu^{-1}(0)/G_\nu]$. (The appropriate “cotangent bundle” should be a cotangent complex [IH71] and there does be a meaningful open immersion of $\mathcal{M}_\nu$ into the cotangent complex to $[E_{\Omega}/G_\nu]$ one can work out.)

Therefore, the category $\mathcal{D}([E_{\Omega}/G_\nu]/\mathcal{N}_{\nu,\Omega})$ is nothing but the localization of $\mathcal{D}([E_{\Omega}/G_\nu])$ by the complexes whose “micro-supports (i.e. characteristic varieties)” are disjoint from $\mathcal{M}_\nu$.

In fact, similar localization has already appeared in the construction of stacks of micro-local perverse sheaves [GKM05]. After the terminology therein, we refer to the perverse sheaves in $\mathcal{D}_\nu$ provisionally (to the extent maybe inappropriately) as micro-local perverse sheaves on the quiver variety $\mathcal{M}_\nu$.

2.3. The functors $\mathcal{R}_i, \mathcal{E}^{(n)}_i, \mathcal{S}^{(n)}_i$.

2.3.1. Assume $\nu' = \nu + n \cdot i$ for a vertex $i \in \mathcal{I}$ and an integer $n \geq 1$. Let $G_{\nu \nu'} := G_\nu \times G_{\nu'}$ act on the variety

$$F_{\nu \nu'} := \{ y \in \bigoplus_{j \in \mathcal{I}} \text{Hom}(V_j, V'_j) \mid \ker y_j = 0, \ y_j = \text{Id}, \ j \in \mathcal{I} \}$$

from the right such that $(y \cdot (g, g'))_j = g'_j^{-1}y_jg_j$.

We associate to every $\Omega \subset \hat{H}$ a variety

$$Z_{\Omega} := \{(x, y) \in E_{\nu', \Omega} \times F_{\nu \nu'} \mid \text{Img} x_h y_{h'} \subset \text{Img} y_{h''} \}$$

and a couple of morphisms

$$p : [Z_{\Omega}/G_{\nu \nu'}] \to [E_{\nu', \Omega}/G_{\nu'}], \quad p(x, y)_h = y_{h''}^{-1}x_h y_{h'},$$

$$p' : [Z_{\Omega}/G_{\nu \nu'}] \to [E_{\nu', \Omega}/G_{\nu'}], \quad p'(x, y)_h = x_h.$$ 

Define the following functors for an orientation $\Omega \subset \hat{H}$

$$K_{\Omega, i} := \text{Id} | \nu_i - \tilde{\nu}_i| : \mathcal{D}([E_{\nu', \Omega}/G_{\nu'}]) \to \mathcal{D}([E_{\nu', \Omega}/G_{\nu'}]),$$

$$F_{\Omega, i}^{(n)} := p| \nu_i^{(n)} | : \mathcal{D}([E_{\nu', \Omega}/G_{\nu'}]) \to \mathcal{D}([E_{\nu', \Omega}/G_{\nu'}]).$$

where

$$\tilde{\nu}_i := \sum_{h \in \hat{H} : h' = i} \nu_{h''} - \nu_i,$$

$$\tilde{\nu}_i(\Omega) := \sum_{h \in \Omega : h' = i} \nu_{h''} - \nu_i.$$ 

When $i$ is a source of $\Omega \subset \hat{H}$, we have a subvariety of $Z_{\Omega}$

$$\tilde{Z}_{\Omega, i} := \{(x, y) \in \hat{E}_{\nu', \Omega, i} \times F_{\nu \nu'} \mid \text{Img} x_h y_{h'} \subset \text{Img} y_{h''} \}.$$ 

Let $\tilde{p}_i, \tilde{p}'_i$ be the restrictions of $p, p'$ to $[\tilde{Z}_{\Omega, i}/G_{\nu \nu'}]$, respectively, and define

$$\mathcal{E}^{(n)}_{\Omega, i} := \tilde{p}_i | \nu_i^{(n)} | : \mathcal{D}([E_{\nu', \Omega}/G_{\nu'}]) \to \mathcal{D}([E_{\nu', \Omega}/G_{\nu'}]).$$
Remark 2.3.2. (1) A $G$-equivariant morphism $X \to Y$ of algebraic varieties automatically induces a representable morphism $[X/G] \to [Y/G]$. If, in addition, $G$ acts trivially on $Y$, there is also an induced (maybe non-representable) morphism $[X/G] \to Y$. As we have done for $p, p'$ above, we usually describe induced morphisms of quotient stacks by their liftings.

(2) The morphisms $p', \tilde{p}, \tilde{p}'$ are representable and smooth, while $p$ may be even not representable. But the functor $p^*$ is well defined.

(3) It is not surprising to see the definitions of $\mathcal{E}, \mathcal{F}$ are less symmetric, for we are in purpose to treat highest weight representations of $U$.

Proposition 2.3.3. There are functor isomorphisms for orientations $\Omega_1, \Omega_2 \subset \hat{H}$ (both having $i$ as a source for the isomorphism of $\mathcal{E}$)

\[
\Phi_{\Omega_1, \Omega_2}^{} : \mathcal{K}_{\Omega_1, i} \cong \mathcal{K}_{\Omega_2, i},
\]

\[
\Phi_{\Omega_1, \Omega_2}^{} : \mathcal{E}_{\Omega_1, i}^{(n)} \cong \mathcal{E}_{\Omega_2, i}^{(n)},
\]

\[
\Phi_{\Omega_1, \Omega_2}^{} : \mathcal{F}_{\Omega_1, i}^{(n)} \cong \mathcal{F}_{\Omega_2, i}^{(n)}.
\]

The isomorphism of $\mathcal{K}$ is obvious. The proof of $\mathcal{E}$ is similar as $\mathcal{F}$. Below we prove the isomorphism of $\mathcal{F}$.

We may assume $\Omega_1, \Omega_2$ differ by a single element. That is, $\Omega_1 \setminus \Omega_2 = \{h\}$. We form the following commutative diagram.

\[
\begin{array}{ccc}
[E_{\nu, \Omega_2}/G_\nu] & \xrightarrow{p} & [E_{\nu', \Omega_2}/G_{\nu'}] \\
\downarrow t & & \downarrow t' \\
[E_{\nu, \Omega_1 \cup \Omega_2}/G_\nu] & \xrightarrow{p} & [E_{\nu', \Omega_1 \cup \Omega_2}/G_{\nu'}] \\
\downarrow s & & \downarrow s' \\
[E_{\nu, \Omega_1}/G_\nu] & \xrightarrow{p} & [E_{\nu', \Omega_1}/G_{\nu'}]
\end{array}
\]

Let $T_Z$ be the morphism

\[
T_Z : [E_{\nu, \Omega_1 \cup \Omega_2}/G_{\nu'}] \to \mathcal{F}, \quad (x, y) \mapsto \text{Tr}(x_{\tilde{h}} x_h).
\]

When $h$ is not incident to $i$, all the squares in the above diagram are cartesian. Otherwise, we assume $h'' = i$. Then the top-left and the bottom-right squares in the above diagram are cartesian. In either case, there are natural isomorphisms for $A \in \mathcal{D}([E_{\nu, \Omega_1}/G_\nu])$

\[
p^* \Phi_{\Omega_1, \Omega_2}^{} A[-\nu_h, \nu_h] = p^* t_1(s^* A \otimes T^* \mathcal{L}) \cong t_{2Z} p^* (s^* A \otimes T^* \mathcal{L})
\]

\[
\cong t_{2Z}(p^* s^* A \otimes T_Z^* \mathcal{L}) = t_{2Z}(s_Z^* p^* A \otimes T_Z^* \mathcal{L}).
\]

Similarly, there are natural isomorphisms for $B \in \mathcal{D}([Z_{\Omega_1}/G_{\nu'}])$

\[
\Phi_{\Omega_1, \Omega_2}^{} B[-\nu_h', \nu_h'] \cong p t_{2Z}(s_Z^* B \otimes T_Z^* \mathcal{L}).
\]

It follows that

\[
p' p^* \Phi_{\Omega_1, \Omega_2}^{} [-\nu_h, \nu_h] \cong \Phi_{\Omega_1, \Omega_2}^{} p' p^* [-\nu_h', \nu_h'].
\]

Therefore, the isomorphism of $\mathcal{F}$ follows.

Proposition 2.3.4. Let $\Omega \subset \hat{H}$ be an orientation (having $i$ as a source for the statement of $\mathcal{E}$). Then $K_{\Omega, i}(N_{\nu}, \Omega) \subset N_{\nu}, \mathcal{E}_{\Omega, i}^{(n)}(N_{\nu}, \Omega) \subset N_{\nu}, \mathcal{F}_{\Omega, i}^{(n)}(N_{\nu}, \Omega) \subset N_{\nu}, \Omega$. 

If \( k \in I \) is a source of \( \Omega \), we have clearly \( K_{\Omega,i}(N_{\nu^\prime,i}, k) \subset N_{\nu^\prime,i}, k \), \( E_{\Omega,i}^{(n)}(N_{\nu^\prime,i}, k) \subset N_{\nu^\prime,i}, k, \) \( F_{\Omega,i}^{(n)}(N_{\nu^\prime,i}, k) \subset N_{\nu^\prime,i}, k \). Hence from the above proposition the claim for \( K, F \) follows.

We are left to show \( E_{\Omega,i}^{(n)}(N_{\nu^\prime,i}, k) \subset N_{\nu^\prime,i} \) where \( k \in I \) is a vertex joined by some edge(s) to \( i \). Below we assume \( n = 1 \), proof of general case postponed to the next subsection.

Let \( \Omega^\prime \) be obtained from \( \Omega \) by reversing the arrows joining \( k, i \). We may assume \( k \) is a source of \( \Omega^\prime \). We have to show

\[
\hat{p}_i \hat{p}_i^* A \in N_{\nu, \Omega} \quad \text{for} \quad A \in N_{\nu^\prime, i}. \tag{2.3.5}
\]

Let \( H_0 \subset H \) be the set of the arrows going from \( k \) to \( i \). Note that \( \Omega \cup \Omega^\prime = \Omega \cup H_0 = \Omega^\prime \cup H_0 \). Define for every subset \( \hat{\Omega} \subset H \) a variety

\[
W_{\hat{\Omega}} := \{(x, x', y, y') \in E_{\nu, \hat{\Omega}} \setminus H_0 \times E_{\nu^\prime, \hat{\Omega} \setminus H_0} \times F_{\nu^\prime} \times \bigoplus_{j \in i} \text{Hom}(V_j, V_j) \mid y_j y_j = \text{Id}_{V_j} \}
\]

and a couple of morphisms

\[
q : [W_{\hat{\Omega}}/G_{\nu^\prime}] \to [E_{\nu, \hat{\Omega}}/G_{\nu}], \quad q(x, x', y, y')_h = x_h \text{ or } y_h x_h, \quad y_h \text{ or } x_h,
\]

\[
q' : [W_{\hat{\Omega}}/G_{\nu^\prime}] \to [E_{\nu^\prime, \hat{\Omega}}/G_{\nu}], \quad q'(x, x', y, y')_h = y_h x_h, \quad y_h x_h, \text{ or } x_h.
\]

Assume \( A \cong \Phi_{\nu^\prime, \Omega} B \) where \( B \in N_{\nu^\prime, k} \). By the same reason as Proposition 2.3.3 we have \( \Phi_{\nu^\prime, \Omega} p^*_i B \cong p^*_i \Phi_{\nu^\prime, \Omega} B[n_1] \) and \( \Phi_{\nu^\prime, \Omega} q^*_i B \cong q^*_i \Phi_{\nu^\prime, \Omega} B[n_2] \) for some integers \( n_1, n_2 \). Clearly \( p p^*_i B, q q^*_i B \in N_{\nu^\prime, k} \). Hence

\[
p p^*_i A, \quad q q^*_i A \in N_{\nu, \Omega} \tag{2.3.6}
\]

We stratify \( Z_\Omega, W_\Omega \) as follows

\[
Z_1 := \{(x, y) \in Z_\Omega \mid p(x, y) \notin \hat{E}_{\nu, \Omega, i} \},
\]

\[
Z_2 := Z_3 \setminus Z_1,
\]

\[
Z_3 := Z_\Omega \setminus Z_4,
\]

\[
Z_4 := \{(x, y) \in Z_\Omega \mid x \in \hat{E}_{\nu^\prime, \Omega, i} \},
\]

\[
W_1 := \{(x, y, y') \in W_\Omega \mid q(x, y, y') \notin \hat{E}_{\nu^\prime, \Omega, i} \},
\]

\[
W_2 := W_\Omega \setminus W_1,
\]

and denote by \( z_a, w_a \) the inclusions of locally closed substacks

\[
z_a : [Z_a/G_{\nu^\prime}] \to [Z_\Omega/G_{\nu^\prime}],
\]

\[
w_a : [W_a/G_{\nu^\prime}] \to [W_\Omega/G_{\nu^\prime}].
\]

By definition,

\[
p z_a^*_i z^*_a p^*_i A, \quad q w_a^*_i q^*_i A \in N_{\nu, \Omega, i} \tag{2.3.7}
\]

Moreover, our assumption \( n = 1 \) implies the morphism

\[
[W_2/G_{\nu^\prime}] \to [Z_2/G_{\nu^\prime}], \quad (x, y, y') \mapsto (x y', y)
\]
is an isomorphism which forms into the following commutative diagram.

\[
\begin{array}{ccc}
Z_2/G_{\nu'} & \xrightarrow{p_2} & E_{\nu',\Omega}/G_{\nu} \\
\downarrow{p_2} & \sim & \downarrow{q_w} \\
[E_{\nu',\Omega}/G_{\nu}] & \xleftarrow{q_w} & [W_2/G_{\nu'}]
\end{array}
\]

Thus

\[
(2.3.8) \quad q_w w_2^* q'\pi^* A \cong p_2 z_2^* p'^* A.
\]

From (2.3.6)-(2.3.8), it follows that all the complexes in the following adjunction triangles are sent by \( p_q \) or \( q_t \) into \( N_{\nu',\Omega} \).

\[
\begin{align*}
& z_{\Omega} z_{\nu} p'^* A \rightarrow p'^* A \rightarrow z_{\Omega} z_{\nu} p'^* A \quad [1] \\
& z_{\Omega} z_{\nu} p'^* A \rightarrow z_{\Omega} z_{\nu} p'^* A \rightarrow z_{\Omega} z_{\nu} p'^* A \quad [1] \\
& w_2^* w_2^* q'\pi^* A \rightarrow q'\pi^* A \rightarrow w_1^* w_1^* q'\pi^* A \quad [1]
\end{align*}
\]

Finally, observe that \( \tilde{p}_t s^* \pi^* A \cong p_2 z_2^* p'^* A. \) Our claim (2.3.5) follows.

**Remark 2.3.9.** We have cheated in the above proof: the functors \( p_t, q_t \) are not well defined since the morphisms \( p, q \) are not representable.

We can overcome this gap as follows. Choose a connected smooth principal \( GL(V') \)-bundle \( P \) and replace the functors \( \tilde{p}_t, p_t, q_t \) by the compositions of the following well defined ones (defined in the obvious way)

\[
\begin{align*}
& \mathcal{D}(\tilde{Z}_{\Omega,i}/G_{\nu'}) \xrightarrow{s^*} \mathcal{D}(\tilde{Z}_i \times P/G_{\nu'}) \xrightarrow{(\tilde{p}_t s^*)} \mathcal{D}(E_{\nu,\Omega}/G_{\nu}) \\
& \mathcal{D}(Z_\Omega/G_{\nu'}) \xrightarrow{s^*} \mathcal{D}(Z_\Omega \times P/G_{\nu'}) \xrightarrow{(p_t s^*)} \mathcal{D}(E_{\nu,\Omega}/G_{\nu}) \\
& \mathcal{D}(W_\Omega/G_{\nu'}) \xrightarrow{s^*} \mathcal{D}(W_\Omega \times P/G_{\nu'}) \xrightarrow{(q_t s^*)} \mathcal{D}(E_{\nu,\Omega}/G_{\nu})
\end{align*}
\]

Then the above proof goes through to yield

\[
\tilde{p}_t s^* \pi^* A \in N_{\nu,\Omega} \quad \text{for} \quad A \in N_{\nu',\Omega}. \]

Now, given a perverse sheaf \( A \in N_{\nu',\Omega,k} \), let \( m \) be such that \( p_{\tau \leq m} \tilde{p}_t \tilde{p}_t s^* \pi^* A \cong \tilde{p}_t s^* \pi^* A \) where \( p_{\tau \leq m}, p_{\tau \geq m} \) are the truncation functors associated to the perverse t-structure. Choosing \( P \) such that \( H^n(P, \mathbb{Q}_l) = 0 \) for \( 1 \leq n \leq 2m \), we have

\[
p_{\tau \geq -m} (\tilde{p}_t s^* \pi^* A) \cong p_{\tau \geq -m} \tilde{p}_t s_{\geq -m} (\tilde{p}_t s^* \pi^* A) \cong \tilde{p}_t s^* \pi^* A.
\]

Since both thick subcategories \( N_{\nu',\Omega,k}, N_{\nu,\Omega} \) are generated by simple perverse sheaves therein (cf. 1.4.16 1.4.8), it follows that \( \tilde{p}_t s^* \pi^* A \in N_{\nu,\Omega} \) holds for arbitrary \( A \in N_{\nu',\Omega,k} \).

**2.3.10.** By the above propositions, the functors \( \mathcal{K}_{\Omega,i}, \mathcal{C}_{\Omega,i}, \mathcal{P}_{\Omega,i} \) pass to localized categories and induce the following functors unambiguously

\[
\begin{align*}
& \mathfrak{K}_{\nu,i} : \mathcal{D}_{\nu} \rightarrow \mathcal{D}_{\nu}, \\
& \mathfrak{C}_{\nu,i} : \mathcal{D}_{\nu+n} \rightarrow \mathcal{D}_{\nu}, \\
& \mathfrak{P}_{\nu,i} : \mathcal{D}_{\nu} \rightarrow \mathcal{D}_{\nu+n}.
\end{align*}
\]
Assemble these functors to endofunctors of $\mathcal{D}$

$$\mathcal{F}_i := \bigoplus_{\nu} \mathcal{F}_{\nu,i},$$

$$\mathcal{E}^{(n)}_i := \bigoplus_{\nu} \mathcal{E}^{(n)}_{\nu,i},$$

$$\mathcal{G}^{(n)}_i := \bigoplus_{\nu} \mathcal{G}^{(n)}_{\nu,i}.$$

Below we will drop the superscript $(n)$ off when $n = 1$.

2.4. **An alternative approach to $\mathcal{E}^{(n)}_i, \mathcal{G}^{(n)}_i$.** Recall that for every vertex $k \in I$ there is an open subvariety $\bar{E}_{\nu,\Omega,k} \subset E_{\nu,\Omega}$. Let $j_{\Omega,k} : [\bar{E}_{\nu,\Omega,k}/G_{\nu}] \to [E_{\nu,\Omega}/G_{\nu}]$ be the induced open immersion.

2.4.1. Let $\Omega \subset \bar{H}$ be an orientation having a vertex $k \in I$ as a source. There are a couple of morphisms obtained from $p, p'$ by restriction

$$p_k : [\bar{Z}_{\nu,\Omega,k}/G_{\nu}] \to [\bar{E}_{\nu,\Omega,k}/G_{\nu}], \quad p_k(x, y) = y^{-1}x y,$$

$$p'_k : [\bar{Z}_{\nu,\Omega,k}/G_{\nu}] \to [\bar{E}_{\nu',\Omega,k}/G_{\nu'}], \quad p'_k(x, y) = x.$$

We define a functor

$$\tilde{\mathcal{F}}^{(n)}_{i,k,i} := p'_k p'^*_k [n\nu'_i(\Omega)] : \mathcal{D}([\bar{E}_{\nu',\Omega,i}/G_{\nu'}]) \to \mathcal{D}([\bar{E}_{\nu',\Omega,i}/G_{\nu'}]).$$

Assume $k \neq i$. In the notations from 2.4.1 we can identify

$$\bar{Z}_{\nu,\Omega,k}/GL(V_k) \times GL(V'_k) = \{(\hat{x}, V, \hat{y}) \mid \hat{x} \in E_{\nu',\Omega}, \, \hat{y} \in \bar{E}_{\nu',\Omega}, \, \text{Img} \hat{x} \hat{y} \subset \text{Img} \hat{y} \hat{y}', \, V \subset V(k), \, \dim V = \nu_k\}$$

where

$$\bar{E}_{\nu',\Omega} := \{y \in \bigoplus_{j \in I \setminus \{k\}} \text{Hom}(V_j, V'_j) \mid \text{Ker} y_j = 0, \, y_j = \text{Id}, \, j \in I\}$$

so that $p_k(x, y) = (y^{-1}x y, V)$, $p'_k(x, y) = (x, y(V)).$

2.4.2. Let $\Omega \subset \bar{H}$ be an orientation having $i$ as a source. Define functors

$$\tilde{\mathcal{E}}^{(n)}_{i,i,i} := p_i p^*_i [n\nu_i] : \mathcal{D}([\bar{E}_{\nu,\Omega,i}/G_{\nu}]) \to \mathcal{D}([\bar{E}_{\nu,\Omega,i}/G_{\nu}]),$$

$$\tilde{\mathcal{F}}^{(n)}_{i,i} := p'_i p'^*_i [n\nu'_i] : \mathcal{D}([\bar{E}_{\nu,\Omega,i}/G_{\nu}]) \to \mathcal{D}([\bar{E}_{\nu,\Omega,i}/G_{\nu}]).$$

Note that the functors $\tilde{\mathcal{E}}^{(n)}_{i,i,i}, \tilde{\mathcal{F}}^{(n)}_{i,i}$ coincide.

In the notations from 2.1.6 we can identify

$$\bar{Z}_{\nu,\Omega,i}/GL(V_i') = \{(\hat{x}, V \subset V') \mid \hat{x} \in E_{\Omega}, V, V' \subset V(i), \, \dim V = \nu_i, \, \dim V' = \nu'_i\}$$

so that $p_i(x, V, V') = (\hat{x}, V)$, $p'_i(x, V, V') = (\hat{x}, V')$.

It follows that $p_i, p'_i$ are fibred in the Grassmannians $Gr(\nu'_i, \nu'_i)$, $Gr(\nu_i, \nu'_i)$ of $\nu'_i, \nu_i$-dimensional subspaces in a $\nu'_i, \nu_i$-dimensional $F$-vector space, respectively. In particular, the morphisms $p_i, p'_i$ are proper and smooth of relative dimension $n\nu'_i, n\nu_i$, respectively.

**Proposition 2.4.3.** We have functor isomorphisms

$$\tilde{\mathcal{E}}^{(n)}_{i,i,i} \cong j_{\Omega,i}^{-1}\tilde{\mathcal{E}}^{(n)}_{\Omega,i} j_{\Omega,i}^* \quad \text{and} \quad j_{\Omega,k}^{-1}\tilde{\mathcal{F}}^{(n)}_{i,i} \cong \tilde{\mathcal{F}}^{(n)}_{\Omega,k} j_{\Omega,k}^*.$$
The first one follows readily from definition. To see the second one, we form the following commutative diagram

\[
\begin{array}{ccc}
[\tilde{E}_{\nu,\Omega,k}/G_{\nu}] & \xrightarrow{\tilde{p}_k} & [\tilde{Z}_{\Omega,k}/G_{\nu}] \\
\downarrow j & & \downarrow j_{\Omega,k} \\
[E_{\nu,\Omega}/G_{\nu}] & \xrightarrow{p} & [Z_{\Omega}/G_{\nu}] & \xrightarrow{p'} & [E_{\nu',\Omega}/G_{\nu'}]
\end{array}
\]

in which the right square is cartesian. Then

\[j_{\Omega,k}^* p^* \cong p'_k j_{\Omega,k}^* \Rightarrow p'_k j_{\Omega,k}^* = j_{\Omega,k} j_{\Omega,k}^*.\]

Hence

\[j_{\Omega,k}^* F_{\Omega,i} = j_{\Omega,k}^* p'^*_k [n\tilde{\nu}_i(\Omega)] = p'_k p^*_k j_{\Omega,k}^* [n\tilde{\nu}_i(\Omega)] = \tilde{F}_{\Omega,i}^{(n)} j_{\Omega,k}^*.\]

**Proposition 2.4.4.** We have functor isomorphisms

\[
\tilde{\xi}_{\Omega,i}^{(n-1)} \cong \bigoplus_{0 \leq m < n} \tilde{\xi}_{\Omega,i}^{(n)}[n - 1 - 2m],
\]

\[
\tilde{\mathcal{F}}_{\Omega,i}^{(n-1)} \cong \bigoplus_{0 \leq m < n} \tilde{\mathcal{F}}_{\Omega,i}^{(n)}[n - 1 - 2m].
\]

We prove the second isomorphism. Assume \(\nu' = \nu + i\), \(\nu'' = \nu + ni\). Consider the following commutative diagram which contains a cartesian square at the bottom-right corner.

\[
\begin{array}{ccc}
[\tilde{Z}_{\Omega,i}^{1}/G_{\nu''}] & \xrightarrow{p_1} & [\tilde{E}_{\nu''},\Omega,i/G_{\nu''}] \\
\downarrow p_3 & & \downarrow p_2 \\
[Y/G_{\nu''}'] & \xrightarrow{q_2} & [\tilde{Z}_{\Omega,i}^{2}/G_{\nu''}] \\
\downarrow q_1 & & \downarrow q_1 \\
[\tilde{E}_{\nu,\Omega,i}/G_{\nu}] & \xleftarrow{p_3} & [\tilde{Z}_{\Omega,i}^{2}/G_{\nu}] & \xleftarrow{p_1} & [\tilde{E}_{\nu''},\Omega,i/G_{\nu''}]
\end{array}
\]

where (1) \(\tilde{Z}_{\Omega,i}^{a}, p_a, p'_a, a = 1, 2, 3\) are those defining the functors \(\tilde{\mathcal{F}}_{\Omega,i}, \tilde{\mathcal{F}}_{\Omega,i}^{(n-1)}, \tilde{\mathcal{F}}_{\Omega,i}^{(n)}\), respectively, (2) \(Y := \tilde{Z}_{\Omega,i}^{1} \times E_{\nu,\Omega,i}, \tilde{Z}_{\Omega,i}^{2}\) and \(q_3(\hat{x}, V, V', V'') := (\hat{x}, V, V'')\) under the identification

\[
Y/G_{\nu'''} \times GL(V''') = \{(\hat{x}, V \subset V' \subset V'') \mid \hat{x} \in E_{\Omega}, V, V', V'' \subset V(i), \dim V = \nu, \dim V' = \nu', \dim V'' = \nu''\}.
\]

We have

\[
\tilde{\mathcal{F}}_{\Omega,i}^{(n-1)} \cong p'_2 p'_3 p'_1 [t] \cong p'_2 q_2 q'_1 [t] = p'_2 q_2 q_3 q' [t]
\]

where

\[
t := \tilde{\nu}' + (n - 1)\tilde{\nu}'' = n\tilde{\nu}'' + n - 1.
\]

Note that \(q_3\) is a \(\mathbb{P}^{n-1}\)-bundle. By the decomposition theorem,

\[
q_3|\bar{\mathcal{Q}}_{\Omega,i}[Y/G_{\nu'''}] \cong \bigoplus_{0 \leq m < n} \bar{\mathcal{Q}}_{\Omega,i}[\tilde{Z}_{\Omega,i}^{2}/G_{\nu}''][-2m].
\]
Hence
\[ q_m q_{1} \cong q_m [\mathcal{Q}_{\mathcal{L}[Y/G_{\nu^*,\nu}]}) \otimes = \bigoplus_{0 \leq m < n} [-2m]. \]

It follows that
\[ \mathcal{F}_{\Omega, i}^{(n-1)} \mathcal{F}_{\Omega, i} = \bigoplus_{0 \leq m < n} p_m p_n^*[t - 2m] = \bigoplus_{0 \leq m < n} \mathcal{F}_{\Omega, i}^{(n)}[n - 1 - 2m]. \]

2.4.5. **Proof of Proposition 2.3.4 (continue).** Since \( j_{\Omega, i}^* \mathcal{F}_{\Omega, i} \cong \mathrm{Id} \), it follows from the above propositions that
\[ \mathcal{E}_{\Omega, i}^{(n-1)} \mathcal{E}_{\Omega, i} = \bigoplus_{0 \leq m < n} \mathcal{E}_{\Omega, i}^{(n)}[n - 1 - 2m]. \]

Thus by induction on \( n \), the claim \( \mathcal{E}_{\Omega, i}^{(n)}(\mathcal{N}_{\nu', \Omega}) \subset \mathcal{N}_{\nu, \Omega} \) is reduced to the case \( n = 1 \), which has already been proved.

2.4.6. Let \( \mathcal{N}_{\nu, \Omega, k} \) be the thick subcategory of \( \mathcal{D}(\mathcal{E}_{\nu, \Omega, k}/G_{\nu}) \) consisting of those complexes sent by the functor \( j_{\Omega, k} \) into the thick subcategory \( \mathcal{N}_{\nu, \Omega} \) of \( \mathcal{D}(\mathcal{E}_{\nu, \Omega}/G_{\nu}) \) (cf. 1.4.3). Then the functors \( j_{\Omega, k} \) and \( j_{\Omega, k}^* \) induce an equivalence of categories (cf. 1.4.8)
\[ \mathcal{D}(\mathcal{E}_{\nu, \Omega, k}/G_{\nu})/\mathcal{N}_{\nu, \Omega, k} \sim \mathcal{D}(\mathcal{E}_{\nu, \Omega}/G_{\nu})/\mathcal{N}_{\nu, \Omega}. \]

Therefore, \( \mathcal{D}(\mathcal{E}_{\nu, \Omega, k}/G_{\nu})/\mathcal{N}_{\nu, \Omega, k} \) provides an alternative realization of \( \mathcal{D}_{\nu} \).

Moreover, by Proposition 2.4.3 \( \mathcal{E}_{\Omega, i}^{(n)}(\mathcal{N}_{\nu', \Omega, i}) \subset \mathcal{N}_{\nu, \Omega, i}, \mathcal{F}_{\Omega, k, i}^{(n)}(\mathcal{N}_{\nu, \Omega, k}) \subset \mathcal{N}_{\nu, \Omega, k} \) so that \( \mathcal{E}_{\Omega, i}^{(n)}, \mathcal{F}_{\Omega, k, i}^{(n)} \) pass to localized categories and realize the functors \( \mathcal{E}_{\nu, i}^{(n)} \) and \( \mathcal{F}_{\nu, i}^{(n)} \), respectively.

2.5. **Functor isomorphisms.** We categorify in this subsection the defining relations of \( U \).

**Proposition 2.5.1.** The endofunctors \( R_i, \mathcal{E}_{\nu, i}^{(n)}, \mathcal{F}_{\nu, i}^{(n)} \) of \( \mathcal{D} \) have the functors
\[ R_i^{-1}, \mathcal{R}_i^{\nu} \mathcal{E}_{\nu, i}^{(n)}[-n^2], \mathcal{R}_i^{\nu} \mathcal{E}_{\nu, i}^{(n)}[-n^2] \]
as left adjoints and have the functors
\[ R_i^{-1}, \mathcal{R}_i^{\nu} \mathcal{E}_{\nu, i}^{(n)}[n^2], \mathcal{R}_i^{\nu} \mathcal{E}_{\nu, i}^{(n)}[n^2] \]
as right adjoints, respectively.

Recall from 2.4.2 that the morphisms \( p_i, p'_i \) are proper and smooth of relative dimension \( n\nu'_i, n\nu_i \), respectively. Hence \( p^*_i = p_i^* \) and \( p'_i[-n\nu'_i] = p'_i[n\nu_i] \). We have natural isomorphisms for \( A \in \mathcal{D}(\mathcal{E}_{\nu, \Omega, l}/G_{\nu}), B \in \mathcal{D}(\mathcal{E}_{\nu', \Omega, i}/G_{\nu'}) \)
\[ \text{Hom}_{\mathcal{D}(\mathcal{E}_{\nu, \Omega, l}/G_{\nu})}(\mathcal{F}_{\Omega, i}^{(n)}A, B) = \text{Hom}_{\mathcal{D}(\mathcal{E}_{\nu', \Omega, i}/G_{\nu'})}(p_i' p_i^*[n\nu'_i]A, B) = \text{Hom}_{\mathcal{D}(\mathcal{E}_{\nu, \Omega, l}/G_{\nu})}(A, p_i p_i^*[-n\nu_i]B) = \text{Hom}_{\mathcal{D}(\mathcal{E}_{\nu', \Omega, i}/G_{\nu'})}(A, p_i p_i^*[2n\nu_i - n\nu_i]B) = \text{Hom}_{\mathcal{D}(\mathcal{E}_{\nu, \Omega, l}/G_{\nu})}(A, [n\nu_i - n\nu_i] \mathcal{E}_{\Omega, i}^{(n)}[n^2]B). \]
It follows that $\mathfrak{E}_i^{(n)}$ is left adjoint to $R^n_\ast E_i^{(n)}[n^2]$ (cf. 1.2.4). Similarly for the others.

**Theorem 2.5.2.** There are isomorphisms of endofunctors of $\mathcal{D}$.

1. $\mathcal{R}_i \mathcal{R}_j = \mathcal{R}_j \mathcal{R}_i$;
2. $\mathcal{R}_i E_j^{(n)} = E_j^{(n)} \mathcal{R}_i [-na_{ij}]$;
3. $\mathcal{R}_i \mathfrak{E}_j^{(n)} = \mathfrak{E}_j^{(n)} \mathcal{R}_i [na_{ij}]$;
4. $E_i^{(n-1)} E_i \cong \bigoplus_{0 \leq m < n} E_i^{(n)} [n - 1 - 2m]$;
5. $\mathfrak{E}_i^{(n)} \mathfrak{E}_j \cong \bigoplus_{0 \leq m < n} \mathfrak{E}_i^{(n)} [n - 1 - 2m]$;
6. $E_i \mathcal{E}_j \oplus \bigoplus_{\nu} \bigoplus_{0 \leq m < \nu - \bar{\nu}_i} \text{Id}_{\mathcal{D}_\nu} [\nu_i - \bar{\nu}_i - 1 - 2m]$;
7. $E_i \mathfrak{E}_j \cong \mathfrak{E}_j E_i$, $i \neq j$;
8. $\bigoplus_{0 \leq m \leq 1 - a_{ij}} E_i^{(m)} E_j^{(1-a_{ij}-m)} \cong \bigoplus_{0 \leq m \leq 1 - a_{ij}} E_i^{(m)} E_j^{(1-a_{ij}-m)}$, $i \neq j$;
9. $\bigoplus_{0 \leq m \leq 1 - a_{ij}} \mathfrak{E}_i^{(m)} \mathfrak{E}_j^{(1-a_{ij}-m)} \cong \bigoplus_{0 \leq m \leq 1 - a_{ij}} \mathfrak{E}_i^{(m)} \mathfrak{E}_j^{(1-a_{ij}-m)}$, $i \neq j$.

(1)(2)(3) are obvious. (4)(5) are immediate from Proposition 2.4.4. (8) follows from the above proposition and (3)(9). The remaining three are proved below.

2.5.3. **Proof of (6).** Let $\Omega \subset \hat{H}$ be an orientation having $i$ as a source. It suffices to show there is an isomorphism of endofunctors of $\mathcal{D}([E_{\nu}, \Omega, i/G_{\nu}])$

$$\mathcal{E}_{\Omega, i} \mathfrak{F}_{\Omega, i} \oplus \bigoplus_{0 \leq m < \nu_i - \bar{\nu}_i} \text{Id} [\nu_i - \bar{\nu}_i - 1 - 2m]$$

(2.5.4)

$$\cong \mathfrak{F}_{\Omega, i} \mathcal{E}_{\Omega, i} \oplus \bigoplus_{0 \leq m < \nu_i - \bar{\nu}_i} \text{Id} [\bar{\nu}_i - \nu_i - 1 - 2m].$$

Assume $\nu^1 = \nu + i$, $\nu^2 = \nu - i$, $\nu' = \nu$. Consider the following commutative diagrams, both having a cartesian square at the bottom-right corner

$$\begin{array}{ccc}
[Y/G_{\nu \nu'}] & \xrightarrow{\pi'} & [E_{\nu'}, \Omega, i/G_{\nu'}] \\
\pi & \downarrow & \downarrow p_1 \\
[Y^1/G_{\nu \nu'}, \nu'] & \xrightarrow{p} & [Z_{\Omega, i}^1/G_{\nu \nu' \nu'}] \\
& \downarrow & \downarrow p_1' \\
[\mathcal{E}_{\nu, \Omega, i}/G_{\nu}] & \xrightarrow{p_1} & [Z_{\Omega, i}^1/G_{\nu \nu' \nu'}] \\
& & \xrightarrow{p_1'} [E_{\nu', \Omega, i}/G_{\nu'}]
\end{array}$$
where (1) \( \tilde{Z}_{\Omega,i}^a, p_a, p'_a, a = 1, 2 \) are those defining the functors \( \hat{E}_{\Omega,i}, \hat{F}_{\Omega,i} \), (2) \( Y \) is such that
\[
Y/G_{\nu} \times GL(V_i) = \{ (\dot{x}, V, V') | \dot{x} \in E_{\Omega,i}, V, V' \subset V(i), \dim V = \dim V' = \nu_i \}
\]
and \( \pi(\dot{x}, V, V') := (\dot{x}, V), \pi'(\dot{x}, V, V') := (\dot{x}, V') \), (3) \( Y^a := \tilde{Z}_{\Omega,i}^a \times E_{\nu,a,i} \tilde{Z}_{\Omega,i}^a \) and \( r_a(\dot{x}, V, V_a, V') := (\dot{x}, V, V') \) under the identification
\[
Y^1/G_{\nu,\nu'} \times GL(V_i) = \{ (\dot{x}, V \subset V^1 \supset V') | \dot{x} \in E_{\Omega,i}, V, V^1, V' \subset V(i), \dim V = \dim V' = \nu_i, \dim V^1 = \nu_i + 1 \},
\]
\[
Y^2/G_{\nu,\nu'} \times GL(V_i) = \{ (\dot{x}, V \supset V^2 \subset V') | \dot{x} \in E_{\Omega,i}, V, V^2, V' \subset V(i), \dim V = \dim V' = \nu_i, \dim V^2 = \nu_i - 1 \}.
\]

Set \( A_a := r_a^!(\mathbb{Q}^i_{[Y^a/G_{\nu,a,\nu}, [\nu_i + \nu_i - 1]} \cdot \text{Identify } [\hat{E}_{\nu,\Omega,i}/G_{\nu}] \) with the diagonal part \( \Delta \) of \([Y/G_{\nu,\nu'}]\) and let \( \iota : \Delta \to [Y/G_{\nu,\nu'}] \) be the inclusion. We have
\[
\text{Id}_{\mathbb{P}(\hat{E}_{\nu,\Omega,i}/G_{\nu})} \cong \pi'_{\Omega,i}^! \mu^* \cong \pi'_{\Omega,i}!(u \mathbb{Q}_i^\Delta \otimes \pi'^*),
\]
(2.5.5) \( \hat{E}_{\nu,\Omega,i} \hat{F}_{\Omega,i} = p_1! p_1^! p_1'! [\nu_i + \nu_i] \cong \pi'_{\Omega,i}! r_1! \pi^* \pi'^*[\nu_i + \nu_i - 1] \cong \pi'_{\Omega,i}(A_1 \otimes \pi'^*),
\]
\( \hat{F}_{\nu,\Omega,i} \hat{E}_{\nu,\Omega,i} = p_2! p_2^! p_2'! [\nu_i + \nu_i] \cong \pi'_{\Omega,i}! r_2! \pi^* \pi'^*[\nu_i + \nu_i - 1] \cong \pi'_{\Omega,i}(A_2 \otimes \pi'^*). \)

Note that \( r_1, r_2 \) are fibred respectively in \( \mathbb{P}^{\nu_i - 1} \) and \( \mathbb{P}^{\nu_i - 1} \) over \( \Delta \). Away from \( \Delta \), both \( r_1, r_2 \) restrict to an isomorphism onto the closed substack \([U/G_{\nu,\nu'}] \subset \[Y/G_{\nu,\nu'}\] \setminus \Delta \) where \( U \subset Y \) is such that
\[
U/G_{\nu} \times GL(V_i) = \{ (\dot{x}, V, V') | \dim(V + V') = \nu_i + 1 \}
\]
\[
= \{ (\dot{x}, V, V') | \dim(V \cap V') = \nu_i - 1 \}.
\]

It follows that
\[
A_1|_{\Delta} \cong \bigoplus_{0 \leq m < \nu_i} \mathbb{Q}_{i,\Delta}[\nu_i + \nu_i - 1 - 2m],
\]
\[
A_2|_{\Delta} \cong \bigoplus_{0 \leq m < \nu_i} \mathbb{Q}_{i,\Delta}[\nu_i + \nu_i - 1 - 2m],
\]
\[
A_1|_{[Y/G_{\nu,\nu'}]\setminus \Delta} \cong A_2|_{[Y/G_{\nu,\nu'}]\setminus \Delta}.
\]

On the other hand, \([Y^a/G_{\nu,a,\nu'}]\) is a tower of Grassmannian bundles over \( [\hat{E}_{\nu,\Omega,i}/G_{\nu}] \), hence is a smooth algebraic stack and is proper over \([Y/G_{\nu,\nu'}]\). By the decomposition theorem, \( A_1, A_2 \) are semisimple complexes. Therefore, by the classification of
simple perverse sheaves,
\[
A_1 \oplus \bigoplus_{\nu_i \leq m < \nu_i} \nu! \mathcal{Q}_t[\nu_i] \oplus \nu_i [\nu_i + 1 - 2m] \]
\[= A_2 \oplus \bigoplus_{\nu_i \leq m < \nu_i} \nu! \mathcal{Q}_t[\nu_i] \oplus \nu_i [\nu_i + 1 - 2m].\]
(2.5.6)

From (2.5.5), (2.5.6) the isomorphism (2.5.4) follows.

2.5.7. Proof of (7). Let $\Omega \subset \tilde{H}$ be an orientation having $i$ as a source. Assume $\nu' = \nu - i + j$. It suffices to show there is an isomorphism of functors from $D([E_{\nu,\Omega,i}/G_{\nu}])$ to $D([E_{\nu',\Omega,i}/G_{\nu'}])$

\[\tilde{E}_{\Omega,i} \tilde{E}_{\Omega,i,j} \cong \tilde{E}_{\Omega,i,j} \tilde{E}_{\Omega,i}.\]

Let $\nu^1 = \nu + j, \nu^2 = \nu - i$. In a similar way as the proof of (6), we reduce the problem to the consideration of the diagram, $a = 1, 2$,

\[
\begin{array}{c}
[Y^a / G_{\nu,\nu'}] \\
\downarrow q_a \\
[\tilde{E}_{\nu,\Omega,i}/G_{\nu}] \\
\downarrow q_a' \\
[\tilde{E}_{\nu',\Omega,i}/G_{\nu'}]
\end{array}
\]

where $Y^a$ is the fibred product of some $\tilde{Z}_{\Omega,i}$ over $\tilde{E}_{\nu,\nu,\Omega,i}$ and

\[q_a(\hat{x}, V, V', \hat{y}) := (y^{-1} \hat{y} \hat{x}, V),\]
\[q_a'(\hat{x}, V, V', \hat{y}) := (\hat{x}, V').\]

under the identification
\[
Y^1 / G_{\nu'} \times GL(V_i) \times GL(V_i) = \{ (\hat{x}, V, V', \hat{y}) | \hat{x} \in E_{\nu,\Omega,i}, \hat{y} \in \tilde{E}_{\nu,\Omega,i}, \}
\]
\[Img \hat{x} h_y h_v \subset \text{Img} \hat{y} h_v, V \subset V(i), V' \subset V^1 = \hat{y}(V),\]
\[\text{dim} V = \nu_i, \text{dim} V' = \nu_i - 1,\]
\[
Y^2 / G_{\nu'} \times GL(V_i') \times GL(V_i') = \{ (\hat{x}, V, V', \hat{y}) | \hat{x} \in E_{\nu,\Omega,i}, \hat{y} \in \tilde{E}_{\nu,\Omega,i}, \}
\]
\[Img \hat{x} h_y h_v \subset \text{Img} \hat{y} h_v, V^2 \subset V \subset V(i), V' = \hat{y}(V^2),\]
\[\text{dim} V = \nu_i, \text{dim} V' = \nu_i - 1.\]

(\text{Note that } E_{\nu,\Omega} = E_{\nu,2,\Omega}, E_{\nu,3,\Omega} = E_{\nu,\Omega} \text{ and } \tilde{F}_{\nu,\nu} = \tilde{F}_{\nu,\nu}.)

We have
\[\tilde{E}_{\Omega,i} \tilde{E}_{\Omega,i,j} \cong q_{a1} q_{a1}^t [\nu_j^2 (\Omega) + \nu_j^t],\]
\[\tilde{E}_{\Omega,i,j} \tilde{E}_{\Omega,i,j} \cong q_{a2} q_{a2}^t [\nu_j^2 + \nu_j^t (\Omega)] = q_{a2} q_{a2}^t [\nu_i + \nu_j (\Omega) - 2].\]

Notice the isomorphism
\[s : [Y^2 / G_{\nu,\nu'}] \to [Y^1 / G_{\nu,\nu'}], s(\hat{x}, V, V', \hat{y}) = (\hat{x}, V, V', \hat{y}).\]

We have $q_2 = q_1 s$ and $q'_2 = q'_1 s$. Hence $q_{a1} q_{a1}^t \cong q_{a2} q_{a2}^t$. Hence $\tilde{E}_{\Omega,i,j} \tilde{E}_{\Omega,i,j} \cong \tilde{E}_{\Omega,i,j} \tilde{E}_{\Omega,i,j}.\]
2.5.8. Proof of (9). Let $\Omega \subset \hat{H}$ be an orientation having $i$ as a source. Assume $\nu' = \nu + (1 - a_{ij})i + j$. It suffices to show there is an isomorphism of functors from $\mathcal{D}([E_{\nu,\Omega,i}/G_{\nu}])$ to $\mathcal{D}([E_{\nu',\Omega,i}/G_{\nu'}])$

$$\bigoplus_{0 \leq m \leq n - a_{ij} \atop m \text{ odd}} \tilde{F}_{\Omega,i,j}^{(1-a_{ij}-m)} \cong \bigoplus_{0 \leq m \leq n - a_{ij} \atop m \text{ even}} \tilde{F}_{\Omega,i,j}^{(1-a_{ij}-m)} \tilde{F}_{\Omega,i}^{(m)}.$$

Let $\nu^m = \nu + mi$, $\nu'^m = \nu + mi + j$. In a similar way as the proof of (6), we reduce the problem to the consideration of the diagram

$$\begin{array}{ccc}
Y^m/G_{\nu^m,\nu'^m} & \cong & \tilde{E}_{\nu,\Omega,i}/G_{\nu}' \\
\downarrow r_m & & \downarrow \pi' \downarrow \pi\
[Y/G_{\nu''}] & & \tilde{E}_{\nu',\Omega,i}/G_{\nu''}
\end{array}$$

where $Y, Y^m$ are algebraic varieties such that

$$Y/GL(V_i) \times GL(V'_i) = \{(\hat{x}, V, V', \hat{y}) \mid \hat{x} \in E_{\nu',\hat{\Omega}}, \hat{y} \in \hat{E}_{\nu''}, \dim V = \nu_i, \dim V' = \nu'_i\},$$

$Y^m/G_{\nu^m,\nu'^m} \times GL(V_i) \times GL(V'_i) = \{(\hat{x}, V, V^m, V', \hat{y}) \mid \hat{x} \in E_{\nu',\hat{\Omega}}, \hat{y} \in \hat{E}_{\nu''}, \dim V = \nu_i, \dim V^m = \nu^m_i, \dim V' = \nu'_i\},$

and

$$\pi(\hat{x}, V, V', \hat{y}) := (\hat{y}^{-1} \hat{x} \hat{y}, V),$$

$$\pi'(\hat{x}, V, V', \hat{y}) := (\hat{x}, V'),$$

$$r_m(\hat{x}, V, V^m, V', \hat{y}) := (\hat{x}, V, V', \hat{y}).$$

Set $A_m := r_m[\tilde{E}_{\nu,i}/[Y^m/G_{\nu^m,\nu'^m}]]$. We have

$$\tilde{F}_{\Omega,i,j}^{(1-a_{ij}-m)} \tilde{F}_{\Omega,i}^{(m)} \cong \pi'_m r_m^* \pi^* [t_m] \cong \pi'_m (A_m [t_m] \otimes \pi^* -)$$

where

$$t_m := m \tilde{v}_m^p + \tilde{v}_j^p (\Omega) + (1 - a_{ij} - m) \tilde{v}_j' = \tilde{v}_j' (\Omega) + (1 - a_{ij}) \tilde{v}_j' + m(1 - m).$$

So, we are left to show

$$(2.5.9) \bigoplus_{0 \leq m \leq n - a_{ij} \atop m \text{ odd}} A_m [m(1 - m)] \cong \bigoplus_{0 \leq m \leq n - a_{ij} \atop m \text{ even}} A_m [m(1 - m)].$$

Consider the stratification $Y = \bigsqcup_{n=1}^{1-a_{ij}} U_n$ where $U_n$ is such that

$$U_n/GL(V_i) \times GL(V'_i) = \{(\hat{x}, V, V', \hat{y}) \mid \dim(V' \cap \hat{y}(V(i))) = \nu_i + n\}.$$ 

Note that the morphism $r_m$ is fibred in the Grassmannian $Gr(m, n)$ of $m$-dimensional subspaces in an $n$-dimensional $F$-vector space over $[U_n/G_{\nu''}]$. So, it is a routine matter to verify that the isomorphism holds on each stratum $[U_n/G_{\nu''}]$.

On the other hand, the variety $Y^m/G_{\nu^m,\nu'^m} \times GL(V_i) \times GL(V'_i)$ is a tower of Grassmannian bundles over

$$\{(\hat{x}, \hat{y}) \in E_{\nu',\hat{\Omega}} \times \hat{E}_{\nu''} \mid \text{Img} \hat{x} \hat{y} \subset \text{Img} \hat{y}''\}.$$
and the latter is a vector bundle over $\hat{F}_{\nu'}$. Hence $[Y^m/G_{\nu m,\nu' m}]$ is a smooth algebraic stack and is proper over $[Y/G_{\nu'}]$. By the decomposition theorem, each $A_m$ is a semisimple complex. Therefore, the validity of the isomorphism \(2.5.9\) follows from those on the strata $[U_n/G_{\nu'}]$.

2.6. Verdier duality and the bifunctor $\text{Ext}$. The propositions below show that Verdier duality and the bifunctor $\text{Ext}^\bullet_{\mathcal{D}}(-, D-) \text{ categorify the bar involution of } U$ and the contravariant form of $U$-module, respectively.

**Proposition 2.6.1.** We have isomorphisms of endofunctors of $\mathcal{D}$

$$D[-1] = [1]D, \quad D\hat{\mathcal{R}}_i = \hat{\mathcal{R}}_i^{-1}D, \quad D\mathcal{E}^{(n)}_i = \mathcal{E}^{(n)}_iD, \quad D\mathcal{G}^{(n)}_i = \mathcal{G}^{(n)}_iD.$$

The first two isomorphisms are obvious.

Recall from 2.4.2 that the morphisms $p_i, p'_i$ are proper and smooth of relative dimension $n\nu_i, n\nu'_i$, respectively. Hence $Dp_{nu} = p_iD$ and $Dp'_i[n\nu_i] = p'_i[n\nu_i]D$. It follows that $D\hat{\mathcal{R}}_{n\nu_i} = \hat{\mathcal{R}}_{n\nu_i}D$. Similarly, $D\mathcal{F}_{n\nu_i} = \mathcal{F}_{n\nu_i}D$. These conclude the last two isomorphisms (cf. 1.4.1).

**Proposition 2.6.2.** There are natural isomorphisms for $A, B \in \mathcal{D}$

\[
\begin{align*}
\text{Ext}^\bullet_{\mathcal{D}}(A, DB) &= \text{Ext}^\bullet_{\mathcal{D}}(B, DA), \\
\text{Ext}^\bullet_{\mathcal{D}}(\mathcal{R}_i A, DB) &= \text{Ext}^\bullet_{\mathcal{D}}(A, D\mathcal{R}_i B), \\
\text{Ext}^\bullet_{\mathcal{D}}(\mathcal{E}^{(n)}_i A, DB) &= \text{Ext}^\bullet_{\mathcal{D}}(A, D\mathcal{R}_i \mathcal{E}^{(n)}_i [-n^2]B), \\
\text{Ext}^\bullet_{\mathcal{D}}(\mathcal{G}^{(n)}_i A, DB) &= \text{Ext}^\bullet_{\mathcal{D}}(A, D\mathcal{R}_i \mathcal{G}^{(n)}_i [-n^2]B).
\end{align*}
\]

The first one actually states that Verdier duality $D$ is self adjoint, which is an obvious fact.

The others follow from Proposition 2.5.1 and Proposition 2.6.1.

3. Categorification of integrable representations

Let the quiver $(I, H)$, the generalized Cartan matrix $(a_{ij})$ and the quantum group $U$ be defined in 1.1.

3.0.1. Notations. An element $\mu = \sum_{i \in I} \mu_i \cdot i \in \mathbb{N}[I]$ is called of

1. type I, if $\mu = n\iota$ for some $n \geq 1$ and $i \in I$;
2. type II, if $\mu \in \mathbb{N}[I \setminus I]$, i.e. $\mu_i = 0$ for $i \in I$.

In this section, we use $\omega$ (resp. $\omega, \hat{\mu}$) to denote a type II element (resp. a sequence of type II elements, a sequence of type I,II elements) in $\mathbb{N}[\hat{I}]$. We write $\hat{\mu} \triangleright \omega$ for the fact that the subsequence of $\hat{\mu}$ formed by the type II terms is identical to $\omega$.

Let $M(\omega), R(\omega), L(\omega), L(\hat{\omega})$ be the $U$-modules defined in 1.1.6 by regarding $\mathbb{N}[I \setminus I]$ as $\mathbb{N}[\hat{I}]$.

For a sequence $\nu = (\nu^1, \nu^2, \ldots, \nu^s)$ we set $|\nu| = \sum_{a=1}^{s} \nu^a$ and write $\bar{\nu}\mu$ for the sequence $(\nu^1, \nu^2, \ldots, \nu^s, \mu)$.

3.1. The functor $\mathcal{F}_\mu$. First, we fix a full flag

$$\mathbb{F}^0 \subset \mathbb{F}^1 \subset \mathbb{F}^2 \subset \mathbb{F}^3 \subset \cdots$$

so that there is an unambiguous inclusion $\mathbb{F}^m \subset \mathbb{F}^n$ for $m \leq n$. 

3.1.1. Suppose \( \mu = \sum_{i \in I} \mu_i \cdot i \in \mathbb{N}[\hat{I}] \) has discreet support, i.e. \( \mu_i \cdot \mu_{h''} \) vanishes for all \( h \in \hat{H} \). In particular, \( \mu \) can be any type I or type II element.

Assume \( \nu' = \nu + \mu \). Using the variety

\[
F_{\nu'} := \{ y \in \bigoplus_{j \in I} \text{Hom}(V_j, V'_j) \mid \ker y_j = 0, \ y_j \text{ is the inclusion, } j \in I \}
\]

we generalize the functor \( F_{\Omega,i}^{(n)} \) defined in 2.3.1 to the following

\[
F_{\Omega,\mu} := \tilde{p}^1 \widetilde{p}^* [\widetilde{\nu}_i' (\Omega)] : D([E_{\nu,\Omega}/G_{\nu}]) \to D([E_{\nu', \Omega'}/G_{\nu'}]).
\]

where

\[
\tilde{\nu}_i (\Omega) := \sum_{i \in I} \mu_i \tilde{v}_i (\Omega) + \sum_{i \in I \setminus I} \mu_i \sum_{h \in \Omega; h'' = i} \nu_{h''}. \]

Note that \( F_{\Omega,i}^{(n)} = F_{\Omega,ni} \).

**Proposition 3.1.2.** We have \( \Phi_{\Omega_1,\Omega_2} F_{\Omega_1,\mu} \cong F_{\Omega_2,\mu} \Phi_{\Omega_1,\Omega_2} \) and \( F_{\Omega,\mu}(N_{\nu,\Omega}) \subset N_{\nu',\Omega} \) for orientations \( \Omega, \Omega_1, \Omega_2 \subset \hat{H} \).

The proof is the same as Proposition 2.3.3 and the easy part of Proposition 2.3.4.

3.1.3. It follows that \( F_{\Omega,\mu} \) induces a well defined functor

\[
\mathfrak{F}_{\nu,\mu} : \mathcal{D}_\nu \to \mathcal{D}_{\nu+\mu}.
\]

Define an endofunctor of \( \mathcal{D} \) as usual

\[
\mathfrak{F}_\mu := \bigoplus_{\nu} \mathfrak{F}_{\nu,\mu}.
\]

3.1.4. Suppose \( \mu \) has discreet support. Let \( \Omega \subset \hat{H} \) be an orientation having the support of \( \mu \) as sinks, i.e. \( i \in \hat{I} \) is a sink of \( \Omega \) for all nonvanishing \( \mu_i \).

Assume \( \nu = \nu' + \mu \). We have an unambiguous inclusion \( E_{\nu',\Omega} \subset E_{\nu,\Omega} \). Moreover, regarding \( G_{\nu'} \) as a subgroup of \( G_\nu \) so that the inclusion \( E_{\nu',\Omega} \subset E_{\nu,\Omega} \) is \( G_{\nu'} \)-equivariant, we get a representable morphisms \( \iota_\mu : [E_{\nu',\Omega}/G_{\nu}] \to [E_{\nu,\Omega}/G_\nu] \) which is unique up to isomorphism.

**Proposition 3.1.5.** Let \( \Omega \) be an orientation and let \( \mu \) be of type II.

(1) \( R_\mu \mathfrak{F}_\mu = \mathfrak{F}_\mu R_\mu [-\mu] \) and \( \mathcal{C}_\mu^{(n)} \mathfrak{F}_\mu \cong \mathfrak{F}_\mu \mathcal{C}_\mu^{(n)} \);

(2) \( D\mathfrak{F}_\Omega,\mu \cong \mathfrak{F}_\Omega,\mu D \), \( D\mathfrak{F}_\mu \cong \mathfrak{F}_\mu D \);

(3) \( \mathfrak{F}_\Omega,\mu \) and \( \mathfrak{F}_\mu \) are fully faithful;

(4) \( \mathfrak{F}_\Omega,\mu \) and \( \mathfrak{F}_\mu \) send simple perverse sheaves to simple perverse sheaves.

The rest of this subsection is dedicated to the proof of the proposition.

Assume \( \Omega \) has the vertices \( \hat{I} \setminus I \) as sinks and \( \nu = \nu' + \mu \). Note that \( G_\nu = G_{\nu'} \) thus \( \iota_\mu : [E_{\nu',\Omega}/G_{\nu}] \to [E_{\nu,\Omega}/G_\nu] \) is an inclusion of closed substack.

**Lemma 3.1.6.** \( \mathfrak{F}_{\Omega,\mu} = \iota_\mu \).

Our assumption on \( \Omega \) implies \( \tilde{\nu}_i' (\Omega) = 0 \). Note that \( p : [Z_{\Omega}/G_{\nu}] \to [E_{\nu,\Omega}] \) is an isomorphism and \( p' = \iota_\mu p \). Therefore, \( \mathfrak{F}_{\Omega,\mu} = \tilde{p}^1 \iota_\mu p^* = \iota_\mu \).

**Lemma 3.1.7.** Let \( \Omega, \Omega_1, \Omega_2 \subset \hat{H} \) be orientations all having the vertices \( \hat{I} \setminus I \) as sinks. We have \( \Phi_{\Omega_1,\Omega_2} \iota_\mu^* \Phi_{\Omega_1,\Omega_2} \cong \iota_\mu^* \Phi_{\Omega_1,\Omega_2} \) and \( \iota_\mu^* (N_{\nu,\Omega}) \subset N_{\nu',\Omega} \).

The proof is the same as Proposition 2.3.3 and the easy part of Proposition 2.3.4.
3.1.8. Proof of Proposition 3.2.2. Assume $\Omega$ has the vertices $\hat{I} \setminus I$ as sinks.

(1) The first isomorphism is obvious. To see the second one, we assume $i$ is a source of $\Omega$ and form the following cartesian squares for $\bar{\nu} = \nu + ni$, $\bar{\nu}' = \nu' + ni$

$$
\begin{array}{ccc}
[E_{\nu,\Omega}/G_{\nu}] & \xrightarrow{\tilde{\beta}_i} & [\tilde{Z}_{\Omega,i}/G_{\nu}] \\
\downarrow{\iota_{\nu}} & & \downarrow{\iota_{\nu}} \\
[E_{\nu',\Omega}/G_{\nu'}] & \xrightarrow{\tilde{\beta}_i'} & [\tilde{Z}_{\Omega,i}/G_{\nu'}]
\end{array}
$$

By proper base change

$$
\mathcal{E}_{\Omega,\mu}^{(n)} \mathcal{F}_{\Omega,\mu} = \tilde{\varphi}_i \tilde{\varphi}_i' [n\nu_i][n\nu_i'] \cong \iota_{\mu!} \tilde{\varphi}_i \tilde{\varphi}_i'[n\nu_i] = \mathcal{F}_{\Omega,\mu} \mathcal{E}_{\Omega,\mu}^{(n)}.
$$

The second isomorphism follows.

(2) is immediate from the isomorphism $D_{\mu!} \cong \iota_{\mu!} D$.

(3) Clearly $\mathcal{F}_{\Omega,\mu} = \iota_{\mu!}$ is fully faithful. By Lemma 3.1.1, $\iota_{\mu!}$ gives rise to an endofunctor $\mathfrak{Res}_\mu$ of $\mathcal{D}$. Since $\iota_{\mu!}$ is left adjoint to $\iota_{\mu!}$ and since the adjunction morphism $\iota_{\mu!} \iota_{\mu*} \cong \text{Id}$ is an isomorphism, $\mathfrak{Res}_\mu$ is left adjoint to $\mathfrak{Res}_\mu$ and the adjunction morphism $\mathfrak{Res}_\mu \mathfrak{Res}_\mu \rightarrow \text{Id}$ is an isomorphism. Therefore, the functor $\mathfrak{Res}_\mu$ is fully faithful.

(4) Clearly $\mathcal{F}_{\Omega,\mu} = \iota_{\mu!}$ sends simple perverse sheaves to simple perverse sheaves. Thus $\mathfrak{Res}_\mu$ sends a simple perverse sheaf either to a simple perverse sheaf or to zero (cf. 1.4.4). But the latter case may not happen, for $\mathfrak{Res}_\mu$ is fully faithful.

3.2. The subcategory $\Omega_{\mathcal{S}}$ of $\mathcal{D}$.

3.2.1. Note that the variety $E_{0,\Omega}$ is a single point and $G_0$ is a trivial group. Thus $\mathcal{D}_0 = D([E_{0,\Omega}/G_0]) / N_{0,\Omega} = D([E_{0,\Omega}/G_0]) = D(\text{pt})$ for every orientation $\Omega \subset \hat{I}$.

So, for a sequence $\bar{\mu} = (\mu^1, \mu^2, \ldots, \mu^s)$ of type I,II elements in $\mathbb{N}[\hat{I}]$, we have a well defined complex

$$
\mathcal{L}_{\bar{\mu}} := \mathfrak{S}_{\mu^1} \cdots \mathfrak{S}_{\mu^s} \mathfrak{S}_{\mu^1} \hat{Q}_{\mathcal{S},\text{pt}} \in \mathcal{D}.
$$

By definition $\mathcal{L}_{\bar{\mu}}$ is represented by

$$
\mathcal{L}_{\Omega,\bar{\mu}} = \mathcal{F}_{\Omega,\mu^1} \cdots \mathcal{F}_{\Omega,\mu^s} \mathcal{F}_{\Omega,\mu^1} \hat{Q}_{\mathcal{S},\text{pt}}.
$$

Proposition 3.2.2. $\mathcal{L}_{\Omega,\bar{\mu}}$ and hence $\mathcal{L}_{\bar{\mu}}$ are semisimple complexes.

Assume $\bar{\mu} = (\mu^1, \mu^2, \ldots, \mu^s)$. Set $\nu := |\bar{\mu}|, |\bar{\mu}| := \sum_{1 \leq a} \mu^a, F_{\bar{\mu}} := \prod_{a=1}^s F_{|\bar{\mu}|^{a-1}|\bar{\mu}|^a}$, $G_{\bar{\mu}} := \prod_{a=1}^s G_{|\bar{\mu}|^a}$ and

$$
Z_{\bar{\mu}} := \{(x, y) \in E_{\nu,\Omega} \times F_{\bar{\mu}} \mid \text{Img} xh(y, y_{s-1} \cdots y_a)_{h'} \subset \text{Img}(y, y_{s-1} \cdots y_a)_{h''}\}.
$$

Let $\pi_{\bar{\mu}} : [Z_{\bar{\mu}}/G_{\bar{\mu}}] \rightarrow [E_{\nu,\Omega}/G_{\nu}]$ be the obvious projection. An easy proper base change argument shows that

$$
\mathcal{L}_{\Omega,\bar{\mu}} \cong \pi_{\bar{\mu}} \hat{Q}_{\mathcal{S},[Z_{\bar{\mu}}/G_{\bar{\mu}}]}[\hat{\bar{\mu}}(\Omega)]
$$

where

$$
\hat{\bar{\mu}}(\Omega) := \sum_{a < b} (\sum_{h \in \Omega} \mu^a_{h^b} \mu^b_{h^a} - \sum_{i \in I} \mu^a_{i^b} \mu^b_{i^a}).
$$

Let $G := \prod_{a=1}^s G_{|\bar{\mu}|^a}$. Note that $Z_{\bar{\mu}}/G$ is a vector bundle over the partial flag variety $F_{\bar{\mu}}/G$ and is proper over $E_{\nu,\Omega}$. Thus $[Z_{\bar{\mu}}/G_{\bar{\mu}}]$ is smooth and $\pi_{\bar{\mu}}$ is representable and proper. By the decomposition theorem $\mathcal{L}_{\Omega,\bar{\mu}}$ is a semisimple complex.
3.2.3. Let $\mathcal{Q}_\mathcal{D}$ be the full subcategory of $\mathcal{D}$ formed by the finite direct sums $\oplus_r A_r [n_r]$ where $n_r \in \mathbb{Z}$ and $A_r$ is a direct summand of some $L_{\mu}^n$ with $\mu \triangleright \varnothing$.

Similarly define a full subcategory $\mathcal{Q}_{\Omega,\mathcal{D}}$ of $\bigoplus_\nu \mathcal{D}([E_{\nu,\Omega}/G_{\nu}])$ for an orientation $\Omega$.

**Proposition 3.2.4.** $\mathcal{Q}_\mathcal{D}$ is stable under the functors $\mathfrak{R}_i^{\pm 1}$, $\mathfrak{S}_i^{(n)}$, $\mathfrak{T}_i^{(n)}$.

The claim is clear for $\mathfrak{R}_i^{\pm 1}$, $\mathfrak{S}_i^{(n)}$.

By Theorem 2.5.2(4) $E_i^{(n)} A$ is a direct summand of $E_i^n A[n_{n-1}^2]$ for $A \in \mathcal{D}$. So it suffices to show $E_i L_{\mu}^n \in \mathcal{Q}_\mathcal{D}$ for $\mu \triangleright \varnothing$. By a similar argument, we may assume the type I terms in $\mu$ are in the form $j$ for various $j \in I$.

We prove by induction on the length of $\mu$.

If $\mu = \emptyset$, then $E_i L_{\mu} = E_i Q_{I, \text{pt}} = 0$; we are done.

If $\mu = \mu' \omega$, by Proposition 3.1.5.1 $E_i L_{\mu} = E_i \mathfrak{S}_j L_{\mu'} = \mathfrak{S}_j E_i L_{\mu'}$, which by the inductive hypothesis is contained in $\mathcal{Q}_\mathcal{D}$.

If $\mu = \mu' j$ for some $j \in I$, by Theorem 2.5.2(6) $E_i L_{\mu} = E_i \mathfrak{S}_j L_{\mu'}$ is either (i) isomorphic to $\mathfrak{S}_j E_i L_{\mu'}$ or (ii) isomorphic to a direct summand of $\mathfrak{S}_j E_i L_{\mu'}$ or (iii) isomorphic to the direct sum of $\mathfrak{S}_j E_i L_{\mu'}$ and several $L_{\mu'}[n_r]$. In any case, $E_i L_{\mu}^n$ is contained in $\mathcal{Q}_\mathcal{D}$ by the inductive hypothesis.

**Proposition 3.2.5.** $\sum_n \dim \text{Ext}^n(A, B) \cdot q^{-n} \in \mathbb{N}[q, q^{-1}]$ for $A, B \in \mathcal{Q}_\mathcal{D}$.

Assume $A, B \in \mathcal{Q}_{\Omega} \cap \mathcal{D}_\nu$ and assume both are in the form $L_{\mu}^n$ with $\mu \triangleright \varnothing$. We prove by induction on the norm $\|\nu\| := \sum_{i \in I} v_i$ and the length of $\varnothing$.

If $\nu = 0$, then $A = B = Q_{I, \text{pt}}$; we are done.

If either of $A, B$, say $A$, has the form $L_{\mu}$ with $\mu \triangleright \varnothing$, $n \geq 1$, $i \in I$, then by Proposition 2.5.1 $E_i^\ast(A, B) = E_i^\ast(\mathfrak{S}_i^{(n)} L_{\mu}, B) = \text{Ext}^\ast(D, E_i^\ast(\mathfrak{S}_i^{(n)}[n^2] B))$.

Thus by the inductive hypothesis, the proposition is true for $A, B$.

If $A = L_{\mu}^n$, then $B = L_{\mu'}^n$ such that $\mu, \mu' \triangleright \varnothing$, then by Proposition 3.1.5.3 $E_i^\ast(A, B) = \text{Ext}^\ast(D, \mathfrak{S}_i L_{\mu}, \mathfrak{S}_i L_{\mu'}) = \text{Ext}^\ast(D, L_{\mu}, L_{\mu'})$.

Thus by the inductive hypothesis, the proposition is also true for $A, B$.

**Proposition 3.2.6.** Let $A \in \mathcal{Q}_{\Omega, \mathcal{D}}$ be a simple perverse sheaf. Then

1. $A$ is self dual, i.e. $DA \cong A$ (therefore, both $\mathcal{Q}_{\Omega, \mathcal{D}}$ and $\mathcal{D}_\nu$ are stable under Verdier duality); and
2. there is an isomorphism with $\mu_r, \mu'_r \triangleright \varnothing$

$A \oplus L_{\Omega, \mu_1}[n_1] \oplus L_{\Omega, \mu_2}[n_2] \oplus \cdots \oplus L_{\Omega, \mu_s}[n_s] \cong L_{\Omega, \mu'_1}[n'_1] \oplus L_{\Omega, \mu'_2}[n'_2] \oplus \cdots \oplus L_{\Omega, \mu'_t}[n'_t]$.

We prove the proposition in the rest of this subsection.

3.2.7. Let $\Omega \subset \mathcal{H}$ be an orientation having a vertex $i \in I$ as a sink. We denote by $\bar{x}(i)$ the restriction of $x \in E_{\nu, \Omega}$ to the direct summand

$$\bigoplus_{h \in \Omega, \ k' = i} \text{Hom}(V_{h'}, V_{k'}) = \text{Hom}(\bigoplus_{h \in \Omega, \ k' = i} V_{h'}, V_i).$$

There is a filtration of closed substacks

$$[E_{\nu, \Omega}/G_{\nu}] = U_{\geq 0} \supset U_{\geq 1} \supset \cdots \supset U_{\geq \nu} \supset U_{\geq \nu+1} = \emptyset$$
where

\[ U_{\geq n} := \{ x \in E_{v,\Omega} \mid \dim \text{Coker} \bar{x}(i) \geq n \}/G_v. \]

Let \( U_n \) be the stratum

\[ U_n := U_{\geq n} \setminus U_{\geq n+1} = \{ x \in E_{v,\Omega} \mid \dim \text{Coker} \bar{x}(i) = n \}/G_v. \]

**Lemma 3.2.8.** Assume \( \nu' = \nu + ni \) for some \( i \in I \). Then \( \text{Supp}(F_{\Omega,ni}A) \subset U'_{\geq n} \) for \( A \in \mathcal{D}([E_{v,\Omega}/G_v]). \)

This is clear from the inclusion \( q'([Z_{\Omega}/G_{\nu\nu'}]) \subset U'_{\geq n}. \)

**Lemma 3.2.9.** Assume \( \nu = \nu' + ni \). Let \( \iota_n : [E_{v',\Omega}/G_{\nu}] \to [E_{v,\Omega}/G_v] \) be the representable morphism induced by the inclusion \( E_{v',\Omega} \subset E_{v,\Omega} \) (cf. 3.1.4).

Note that \( \iota_n \) restricts to a representable morphism \( \iota_n|_{U'_0} : U'_0 \to U_n \) which is smooth with connected nonempty fibers of dimension \( n\nu \). Thus the functor \( (\iota_n|_{U'_0})^*[nu_i] \) is perverse t-exact, commutes with Verdier duality and induces a fully faithful functor \( \mathcal{M}(U_n) \to \mathcal{M}(U'_0). \)

**Lemma 3.2.10.** Let \( A \in \mathcal{D}([E_{v,\Omega}/G_v]) \) be a simple perverse sheaf. Assume \( \text{Supp}(A) \subset U_{\geq n} \) and \( \text{Supp}(A) \cap U_n \neq \emptyset \). Then

1. \( \iota^*_n[nu_i]A|_{U'_0} \) is a simple perverse sheaf.

Moreover, let \( B \in \mathcal{D}([E_{v',\Omega}/G_{\nu}]) \) be the intermediate extension of \( \iota^*_n[nu_i]A|_{U'_0}. \)

2. If \( B \) is self dual, so is \( A. \)

3. \( A|_{U_n} \cong F_{\Omega,ni}B|_{U_n}. \)

First, our assumptions on \( A \) imply that \( A \) is the intermediate extension of the simple perverse sheaf \( A|_{U_n}. \) It follows that \( \iota^*_n[nu_i]A|_{U'_0} = (\iota_n|_{U'_0})^*[nu_i](A|_{U_n}) \) is a simple perverse sheaf. This proves (1).

Moreover, if \( B \) is self dual, so is \( B|_{U'_0} = \iota^*_n[nu_i]A|_{U'_0} = (\iota_n|_{U'_0})^*[nu_i](A|_{U_n}) \), hence so is \( A|_{U_n}, \) hence so is \( A \). This proves (2).

Next, we form the following commutative diagram

\[
\begin{array}{ccc}
Z_{\Omega}\setminus/G_{\nu\nu'} & \xrightarrow{f} & E_{v',\Omega} \setminus/G_{\nu
u'} \\
p \downarrow & & \downarrow \pi \\
E_{v',\Omega}\setminus/G_{\nu} & \xrightarrow{\iota_n} & E_{v,\Omega}\setminus/G_{\nu}
\end{array}
\]

where \( Z_{\Omega}, p, p' \) are those defining \( F_{\Omega,ni}, \pi \) is the presentation of the quotient stack and \( f(x) := (iex^{-1}, i), \ \iota \in F_{v',v} \) being the inclusion. \( (f \) is well defined since \( i \) is a sink of \( \Omega). \)

The commutative diagram restricts to the following one

\[
\begin{array}{ccc}
U_Z & \xleftarrow{f} & \pi^{-1}(U'_0) \\
\downarrow p|_{U_Z} & & \downarrow \pi \\
U'_0 & \xrightarrow{\iota_n|_{U'_0}} & U_n
\end{array}
\]

where \( U_Z := \{ (x,y) \in Z_{\Omega} \mid \dim \text{Coker} \bar{x}(i) = n \}/G_{\nu\nu'}. \)
Note that $p'_lu_z : U_Z \to U_n$ is an isomorphism. We have
\[
\pi^*(\tau_n|U'_u)^*|nu_i|(\mathcal{F}_{\Omega, m} B|U_n) = \pi^*(\tau_n|U'_u)^*(p'_l p^* B|U_n)
= f^*(p'_lu_z)^*(p'_lu_z)(p|lu_z)^*(B|U'_u)
\cong \pi^*(B|U'_u) = \pi^*(\tau_n|U'_u)^*|nu_i|(A|U_n).
\]
Since the functors $\pi^*(\dim G_{\nu'})$ and $(\tau_n|U'_u)^*|nu_i|$ are perverse $t$-exact and induce fully faithful functors $\mathcal{M}(U_n) \to \mathcal{M}(U'_u) \to \mathcal{M}(\pi^{-1}(U'_u))$, it follows that $A|U_n \cong \mathcal{F}_{\Omega, m} B|U_n$. This proves (3).

**Lemma 3.2.11.** Assume $\tilde{\mu} = (\mu^1, \mu^2, \ldots, \mu^s)$ and $\nu = \nu' + ni = |\tilde{\mu}|$. We have
\[
i'_n \mathcal{L}_{\Omega, \tilde{\mu}} \cong \bigoplus_{\tilde{\mu}' \in M} \mathcal{L}_{\Omega, \tilde{\mu}'} \boxtimes u_1(\tilde{\mu}_1, \mu_{1}'|n_i')
\]
for some integers $n_i'$, where
\[
M := \{\tilde{\mu}' | \mu_i'^a \leq \mu_i^a, |\tilde{\mu}'| = |\tilde{\mu}| - ni\},
\]
and $u : P_{\tilde{\mu}'} \to pt$ is the point map of the partial flag variety
\[
P_{\tilde{\mu}'} := \{(0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_s) | \dim V_a/V_{a-1} = \mu_i^a - \mu_i'^a\}.
\]
Keep the notations of Proposition 3.2.2. We set
\[
Z := \{(x, y) \in Z|n_i' | x \in E_{\nu', \Omega}\},
\]
\[
G := \prod_{a=1}^{s-1} G|_{\tilde{\mu}'|n_i'} \times G_{\nu'}.
\]
Let $\pi : [Z/G] \to [E_{\nu', \Omega}/G_{\nu'}]$ be the obvious projection. By proper base change
\[
i'_n \mathcal{L}_{\Omega, \tilde{\mu}} \cong \pi_! \mathcal{Q}_l|Z/G|.
\]

We have a stratification $[Z/G] = \sqcup_{\tilde{\mu}' \in M} [U_{\tilde{\mu}'} / G]$ where
\[
U_{\tilde{\mu}'} := \{(x, y) \in Z | \dim(\text{Img}(y_s y_{s-1} \cdots y_a) \cap V'_a) = |\tilde{\mu}'|_i\}.
\]
Note that restricting to the stratum $[U_{\tilde{\mu}'} / G]$, $\pi$ can be factored as
\[
[U_{\tilde{\mu}'} / G] \xrightarrow{f_{\tilde{\mu}'}} Z_{\tilde{\mu}'} \times P_{\tilde{\mu}'} \xrightarrow{\pi_{\tilde{\mu}'} \times u_{\tilde{\mu}'} | n_i'} [E_{\nu', \Omega}/G_{\nu'}],
\]
where $f_{\tilde{\mu}'}$ is a vector bundle of fiber dimension $\sum_{a \leq b} (\mu_i^a - \mu_i^b) \mu_i^b$. Therefore, from the decomposition theorem 2.6, our lemma follows.

**3.2.12. Proof of Proposition 3.2.10** Assume $A \in \mathcal{Q}_{\Omega, \nu', \nu} : = \mathcal{Q}_{\Omega, \omega} \cap D([E_{\nu', \Omega}/G_{\nu'}])$. We show the proposition by induction on the norm $|\nu| := \sum_{i \in I} \nu_i$ and the length of $\omega$.

Case (1) $\nu = 0$. That is, $A = \mathcal{L}_{\Omega, 0} = \mathcal{Q}_{l, pt}$. We are done.

Case (2) $A$ is a direct summand of $\mathcal{F}_{\Omega, \omega, \tilde{\mu}} B$ where $B$ is a direct summand of $\mathcal{L}_{\Omega, \tilde{\mu}} B$. Then by Proposition 3.1.7 (4), $A \cong \mathcal{F}_{\Omega, \omega} B$ where $B$ is a direct summand of $\mathcal{L}_{\Omega, \tilde{\mu}} B$. Then by Proposition 3.1.7 (2) and by the inductive hypothesis, both claims of the proposition are true for $A$.

Case (3) $A$ is a direct summand of $\mathcal{F}_{\Omega, \omega, i} \mathcal{L}_{\Omega, \tilde{\mu}} B$ with $\tilde{\mu} \triangleright \omega$, $m \geq 1$, $i \in I$. Assume $i$ is a sink of $\Omega$ and assume $\text{Supp}(A) \subset U_{\geq n}$ and $\text{Supp}(A) \cap U_{\leq n} \neq \emptyset$. By Lemma 3.2.8 $n \geq m$.

Let $\nu' = \nu - ni$. By Lemma 3.2.11 $i'_n A$ is a semisimple complex in $\mathcal{Q}_{\Omega, \nu', \nu'}$. By Lemma 3.2.10 (1), $i'_n|nu'| A|_{U_0'}$ is a simple perverse sheaf. Hence the intermediate extension $B$ of $i'_n|nu'| A|_{U_0'}$ is a simple perverse sheaf in $\mathcal{Q}_{\Omega, \nu', \nu'}$, which is self dual.
by the inductive hypothesis. Then by Lemma 3.2.10(2)(3), $A$ is self dual and $\mathcal{F}_{Q}B \cong A \oplus C$ with $C \in Q_{Q_{\Omega_{\omega}}, \varphi}$ and $\text{Supp}(C) \subset U_{\geq n+1}$.

Applying the argument in the last paragraph on the simple direct summands of $C$, and so on, yields

$$A \oplus A_1[m_1] \oplus A_2[m_2] \oplus \cdots \oplus A_s[m_s] \cong A'_1[m'_1] \oplus A'_2[m'_2] \oplus \cdots \oplus A'_t[m'_t]$$

where $A_r, A'_r$ are in the form $\mathcal{F}_{Q}B$ with $1 \leq n \leq \nu_r$ and $B$ being a simple perverse sheaf in $Q_{Q_{\Omega_{\omega}}, \varphi}$. By the inductive hypothesis, the second claim of the proposition holds for $A_r, A'_r$, thus holds for $A$.

3.3. Main theorems.

3.3.1. Let $G(Q_{\omega})$ denote the Grothendieck group of the additive category $Q_{\omega}$. That is, $G(Q_{\omega})$ is the free $\mathbb{Z}[q, q^{-1}]$-module defined by the generators each for an isomorphism class of objects from $Q_{\omega}$ and the relations

(i) $[A \oplus B] = [A] + [B]$, for $A, B \in Q_{\omega}$;
(ii) $[A[m]] = q^{-1}[A]$, for $A \in Q_{\omega}$.

Set $\tilde{G}(Q_{\omega}) := G(Q_{\omega}) \otimes \mathbb{Q}(q)$. By definition it has a basis

$$B(\omega) := \{ [A] \mid A \in Q_{\omega} \text{ is a simple perverse sheaf in } \mathcal{D} \}.$$ Moreover, the functor $\mathfrak{g}_{\omega}$ induces a linear homomorphism

$$\varphi_{\omega} : \tilde{G}(Q_{\omega}) \to \tilde{G}(Q_{\omega}).$$

Theorem 3.3.2. The followings endow $\tilde{G}(Q_{\omega})$ with a structure of $U$-module.

$$K_i^{\pm 1}[A] := [\mathfrak{k}_i^{\pm 1} A], \quad E_i^{(n)}[A] := [\mathfrak{e}_i^{(n)} A], \quad F_i^{(n)}[A] := [\mathfrak{f}_i^{(n)} A].$$

This is the consequence of Theorem 2.5.2 and Proposition 3.2.1.

Theorem 3.3.3. The following defines a nondegenerate contravariant form on the $U$-module $G(Q_{\omega})$.

\[ ([A], [B]) := \sum_n \dim \text{Ext}_{Q_{\omega}}^n(A, DB) \cdot q^{-n}. \]

By Proposition 3.2.4 and Proposition 3.2.6(1), the above expression values in $\mathbb{N}[q, q^{-1}]$ for $A, B \in Q_{\omega}$. So, by Proposition 2.6.2 it defines a contravariant form of $U$-module.

It is a basic property of a t-structure that for simple perverse sheaves $A, B \in \mathcal{D}$, $\text{Ext}_{Q_{\omega}}^n(A, B)$ vanishes for $n < 0$, and $\text{Ext}_{Q_{\omega}}^n(A, B) \cong \mathbb{Q}_l$ if $A \cong B$ or vanishes otherwise. By Proposition 3.2.6(1), simple perverse sheaves in $Q_{\omega}$ are self dual. Hence $(b, b') \in \delta_{\omega} + q^{-1}\mathbb{N}[q^{-1}]$ for $b, b' \in B(\omega)$. Therefore, the contravariant form under the basis $B(\tilde{\omega})$ yields a unit matrix modulo $q^{-1}$. This proves the nondegeneracy.

Theorem 3.3.4. There exist a unique family of isomorphisms of $U$-modules $\Lambda(\tilde{\omega}) \cong \tilde{G}(Q_{\omega})$ so that

1. the isomorphisms preserve contravariant form,
2. identifying $\Lambda(\tilde{\omega})$ with $\tilde{G}(Q_{\omega})$, we have $B(\emptyset) = \{1\}$ and
3. $\varphi_{\omega}(u) = u \otimes \eta_{\omega}$ for $u \in \tilde{G}(Q_{\omega})$. 


By Proposition 3.1.5(1) \( E_i \phi_\omega(u) = \phi_\omega(E_i u) \) and \( K_i \phi_\omega(u) = q^{-1} \phi_\omega(K_i u) \) for \( u \in \hat{G}(\mathbf{Q}_\omega) \). Moreover, by Proposition 3.1.5(2), the homomorphism \( \phi_\omega \) preserves contravariant form.

Therefore, by the universality of \( M(\omega) \) (cf. 1.1.4), there exists a unique homomorphism of \( U \)-modules

\[
\tilde{\phi}_\omega : \hat{G}(\mathbf{Q}_\omega) \otimes M(\omega) \to \hat{G}(\mathbf{Q}_\omega)
\]

such that \( \tilde{\phi}_\omega(u \otimes \eta_\omega) = \phi_\omega(u), u \in \hat{G}(\mathbf{Q}_\omega) \). Moreover, \( \tilde{\phi}_\omega \) preserves contravariant form.

Since the contravariant forms on \( \hat{G}(\mathbf{Q}_\omega), \hat{G}(\mathbf{Q}_\tilde{\omega}) \) are nondegenerate, the kernel of \( \phi_\omega \) is \( \hat{G}(\mathbf{Q}_\omega) \otimes R(\omega) \). On the other hand, Proposition 3.2.6(2) implies that the set \( \{ L_{\mu} | \mu \ni \tilde{\omega} \} \) generates \( \hat{G}(\mathbf{Q}_\tilde{\omega}) \) as a linear space. Thus \( \phi_\omega(\hat{G}(\mathbf{Q}_\omega)) \) generates \( \hat{G}(\mathbf{Q}_\tilde{\omega}) \) as a \( U \)-module. Thus \( \phi_\omega \) is an epimorphism. Therefore \( \phi_\omega \) induces an isomorphism of \( U \)-modules \( \hat{G}(\mathbf{Q}_\omega) \otimes \Lambda(\omega) \cong \hat{G}(\mathbf{Q}_\tilde{\omega}) \).

Now observe that \( \hat{F}_{i,n}(\Omega, pt) \in \mathcal{N}_{i,n,\Omega, \omega} \) for \( n \geq 1, i \in I \). Thus \( L_{\mu} \) vanishes for nonempty \( \mu \ni \emptyset \). Thus \( L_{\mu} \sim D(\text{pt}) \) and \( \hat{G}(\Omega_0) \) is a trivial \( U \)-module. So, there is a unique isomorphism of \( U \)-modules \( \Lambda(\emptyset) \cong \hat{G}(\Omega_0) \) sending \( 1 \) to the unique element \( [\Omega_{l, \emptyset}] \) of \( B(\emptyset) \).

Then an induction on the length of \( \omega \) establishes the desired isomorphisms \( \Lambda(\omega) \cong \hat{G}(\Omega_\omega) \). The uniqueness is clear from the proof.

**Theorem 3.3.5.** Verdier duality induces a family of \( \mathbb{Q} \)-linear involutions \( \Psi : \hat{G}(\mathbf{Q}_\omega) \to \hat{G}(\mathbf{Q}_\omega) \). They satisfy

1. \( \Psi(xu) = \bar{x} \Psi(u) \), for \( x \in U \), \( u \in \hat{G}(\mathbf{Q}_\omega) \);
2. \( \Psi(u \otimes \eta_\omega) = \psi(u) \otimes \eta_\omega \), for \( u \in \hat{G}(\mathbf{Q}_\omega) \).

By Proposition 3.2.6(1), the involutions \( \Psi \) are well defined. (1) follows from Proposition 2.6.1 (2) follows from Proposition 3.1.5(2).

**Theorem 3.3.6.** The basis \( B(\omega) \) of \( \hat{G}(\mathbf{Q}_\omega) \) satisfies

1. \( (b, b') \in \delta_{b,b'} + q^{-1} [q^{-1}] \) for \( b, b' \in B(\omega) \);
2. \( \Psi(b) = b \), for \( b \in B(\omega) \);
3. the subset \( [q, q^{-1}] [B(\omega)] \subset \hat{G}(\mathbf{Q}_\omega) \) is stable under \( K_i^{\pm 1}, E_i^{(n)}, F_i^{(n)} \);
4. \( b \otimes \eta_\omega \in B(\omega) \) for \( b \in B(\omega) \).

(1) has been proved in Theorem 3.3.3 (2) follows from Proposition 3.2.6(1). (3) is clear from definition. (4) follows from Proposition 3.1.5(4).

**Remark 3.3.7.** (1) Note that the involutions \( \Psi \) from Theorem 3.3.5 are uniquely determined by the properties (1)/2) therein along with the normalization \( \Psi(B_0) = B_0 \). Therefore, they coincide with those involutions expressed in terms of quasi-\( R \)-matrix in [Lu93], which share the same properties and normalization.

(2) The basis \( B(\omega) \) is identical to the canonical basis introduced by Lusztig [Lu93 14.4.12, 27.3]. First, both bases satisfy \( 3.3.6(1) \) and both are contained in \( G(\mathbf{Q}_\omega) \), so they may differ at most by signs. Moreover, both bases satisfy the recursive formula \( 3.3.6(4) \). Comparing the crystal graph on Lusztig’s basis and the positivity property \( 3.3.6(3) \) of \( B(\omega) \), one is able to see the ambiguity of sign does not happen.
(3) The positivity result stated in Theorem 3.3.6(3) is new. It was proved for single highest weight integrable modules in [Lu93 22.1] when \((I, H)\) is simply-laced. It was also proved for quantum \(sl_n\) independently in [Su08] by using the categorification [Su07].

In fact, the canonical bases of the highest weight integrable \(U\)-modules (they coincide with Kashiwara’s global crystal bases [Ka91]) were originally specialized from a common basis, also referred to as canonical basis or global crystal basis, of the negative part \(U^-\) of \(U\) (the subalgebra generated by \(F_i, i \in I\)). Lusztig’s geometric construction of the basis of \(U^-\) yields further a positivity property [Lu91, Theorem 11.5]. From that Theorem 3.3.6(3) is naturally expected for single highest weight integrable modules.

However, Theorem 3.3.6(3) for tensor product modules is far from clear from Lusztig’s original algebraic construction. So our geometric realization of canonical bases is yet another illustration of the advantage of geometric approach to the representation theory.

(4) Nakajima [Na01] associated to each module \(\Lambda(\vec{\omega})\) a Lagrangian subvariety \(\tilde{\mathcal{Z}}\) of the Nakajima’s quiver variety \(\bigcup \nu \mathfrak{M}_\nu\) (see also Malkin’s tensor product variety [Ma03]). In fact, there is a one-to-one correspondence

\[
\{ \text{“characteristic varieties” of the simple perverse sheaves in } \Omega_{\vec{\omega}} \} \leftrightarrow \{ \text{irreducible components of } \tilde{\mathcal{Z}} \}.
\]

So the category \(\Omega_{\vec{\omega}}\) can be reformulated in a more intrinsic way as the full subcategory of \(\mathfrak{D}\) formed by the semisimple complexes whose characteristic varieties are contained in \(\tilde{\mathcal{Z}}\).

3.4. Examples.

3.4.1. Quantum \(sl_2\). The underlying finite graph consists of a single vertex: \((I, H) = (\{i\}, \emptyset)\) and \((\hat{I}, \hat{H}) = (\{i, i\}, \{i \rightarrow \hat{i}, \hat{i} \rightarrow i\})\). We choose \(\Omega = \{i \rightarrow \hat{i}\}\).

Assume \(\nu = ri + d\hat{i}\). Observe that \([\tilde{\mathcal{E}}_{\nu, \Omega, i}/G_{\nu}]\) is precisely the Grassmannian

\[
Gr(r, d) = \{ V \subset \mathbb{F}^d | \dim V = r \}.
\]

Moreover, the thick subcategory \(\tilde{\mathcal{N}}_{\nu, \Omega, i}\) of \(\mathfrak{D}(\tilde{\mathcal{E}}_{\nu, \Omega, i}/G_{\nu})\) is generated by zero. So

\[
\mathfrak{D}_{ri+d\hat{i}} = \mathfrak{D}(Gr(r, d)).
\]

Assume \(\vec{\omega} = (d_1i, d_2\hat{i}, \ldots, d_t\hat{i})\) and \(d = \sum_n d_n\). Let \(P_{\vec{\omega}} \subset GL(\mathbb{F}^d)\) be the parabolic subgroup preserving the partial flag

\[
0 \subset \mathbb{F}^{d_1} \subset \mathbb{F}^{d_1+d_2} \subset \cdots \subset \mathbb{F}^d.
\]

Note that the simple perverse sheaves in \(\Omega_{\vec{\omega}}\) are all \(P_{\vec{\omega}}\)-equivariant. Counting the number of \(P_{\vec{\omega}}\)-orbits and comparing it with the dimension of \(G(\Omega_{\vec{\omega}})\) show that the simple perverse sheaves in \(\Omega_{\vec{\omega}}\) are exactly those \(P_{\vec{\omega}}\)-equivariant simple perverse sheaves in \(\mathfrak{D}(Gr(r, d))\), \(0 \leq r \leq d\).

Therefore, we recover the categorification described in [Zh07].
3.4.2. Quantum $sl_3$. The underlying finite graph consists of two vertices and a single edge joining them. Let $\Omega, \Omega'$ be orientations as below.

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{ccc}
& j & \\
& i & \\
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{ccc}
& j & \\
& i & \\
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{ccc}
& j & \\
& i & \\
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{ccc}
& j & \\
& i & \\
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{ccc}
& j & \\
& i & \\
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{ccc}
& j & \\
& i & \\
\end{array}
\end{array}
\end{array}
\end{array}
\]

\[(I, H) \quad (I', H) \quad \Omega \quad \Omega'
\]

The whole category $\mathcal{D}$ has already been too complicated to be studied in general within an example. Below are some typical cases for reference.

Case (1). Let $\omega = i + j$. The canonical basis of $\Lambda(\omega)$ consists of eight elements

\[
\begin{align*}
\eta_{\omega}, & \quad F_i \eta_{\omega}, \quad F_j \eta_{\omega}, \\
F_i^{(2)} F_j \eta_{\omega}, & \quad F_j^{(2)} F_i \eta_{\omega}, \quad F_i F_j^{(2)} \eta_{\omega}, \\
F_j F_i^{(2)} \eta_{\omega}, & \quad F_i F_j \eta_{\omega}.
\end{align*}
\]

So, totally seven $\mathcal{D}_\nu$ are involved here:

(i) $\nu = \omega$. Clearly $\mathcal{D}_\nu = \mathcal{D}([E_{\nu,\Omega}/G_\nu]) = \mathcal{D}(pt)$. The unique simple perverse sheaf gives rise to the basis element $\eta_{\omega}$.

(ii) $\nu = i + \omega$. $[E_{\nu,\Omega,i}/G_\nu] = \mathbb{F}^x/GL(\mathbb{F}) = pt$. Thus $\mathcal{D}_\nu = \mathcal{D}(pt)$. The unique simple perverse sheaf gives rise to $F_i \eta_{\omega}$.

(iii) $\nu = j + \omega$. Similarly using $\Omega'$ one establishes $\mathcal{D}_\nu = \mathcal{D}(pt)$. The unique simple perverse sheaf gives rise to $F_j \eta_{\omega}$.

(iv) $\nu = 2i + j + \omega$. $[\hat{E}_{\nu,\Omega,i}/G_\nu] = \mathbb{F}^x/GL(\mathbb{F})$. $\tilde{N}_{\nu,\Omega,i}$ is generated by the simple perverse sheaf supported on $[\mathbb{F}^0/GL(\mathbb{F})]$. Thus $\mathcal{D}_\nu \cong \mathcal{D}([\mathbb{F}^x/GL(\mathbb{F})]) = \mathcal{D}(pt)$. The unique simple perverse sheaf gives rise to $F_i^{(2)} F_j \eta_{\omega}$.

(v) $\nu = i + 2j + \omega$. Similarly $\mathcal{D}_\nu \cong \mathcal{D}(pt)$ and the unique simple perverse sheaf gives rise to $F_j^{(2)} F_i \eta_{\omega}$.

(vi) $\nu = 2i + 2j + \omega$. The only simple perverse sheaf on $[E_{\nu,\Omega}/G_\nu]$ not contained in $\mathcal{N}_{\nu,\Omega}$ is the intermediate extension of $\mathcal{Q}_{[U/G_a]}[-1]$, where $U$ is the $G_a$-orbit $\{x \in E_{\nu,\Omega} \mid \text{dim } \text{Img } x_1 = \text{dim } \text{Img } x_2 = \text{dim } \text{Img } x_3 = 1, \text{ Ker } x_1 \cap \text{ Ker } x_3 = 0\}$. So, $\mathcal{D}_\nu$ has a unique simple perverse sheaf, which gives rise to $F_i F_j^{(2)} F_i \eta_{\omega}$.

Moreover, a direct computation shows $(F_i F_j^{(2)} F_i \eta_{\omega}, F_i F_j^{(2)} F_j \eta_{\omega}) = 1$. That being said, the unique simple perverse sheaf in $\mathcal{D}_\nu$ has no self extensions. So, $\mathcal{D}_\nu \cong \mathcal{D}(pt)$.

(vii) $\nu = i + j + \omega$. We partition $E_{\nu,\Omega}$ into eight strata $U_{d_1,d_2,d_3} = \{x \in E_{\nu,\Omega} \mid \text{dim } \text{Img } x_a = d_a\}$. The simple perverse sheaves in $\mathcal{D}_\nu$ are provided by the intermediate extensions of those on the strata $[U_{1,0,0}/G_\nu] = [\text{pt}/GL(\mathbb{F})], [U_{1,0,1}/G_\nu] = \text{pt}, [U_{0,1,1}/G_\nu] = \text{pt}, [U_{1,1,1}/G_\nu] = \mathbb{F}^x$. The intermediate extensions of $\mathcal{Q}_{[U_{1,0,1}/G_a]}, \mathcal{Q}_{[U_{1,1,1}/G_a]}[1]$ give rise respectively to $F_j F_i \eta_{\omega}, F_j F_i \eta_{\omega}$.

Case (2). The canonical basis of $\Lambda(i) \otimes \Lambda(j)$ is as follows.

\[
\begin{align*}
(\eta \otimes \eta), & \quad F_i(\eta \otimes \eta), \quad F_j(\eta \otimes \eta), \\
F_i^{(2)} F_j(\eta \otimes \eta), & \quad F_j^{(2)} F_i(\eta \otimes \eta), \quad F_j F_i^{(2)} F_i(\eta \otimes \eta), \\
F_j F_i(\eta \otimes \eta), & \quad F_j F_i^{(2)} F_i(\eta \otimes \eta).
\end{align*}
\]

The first eight elements are provided by the same simple perverse sheaves as $\Lambda(\omega)$. The last one is given by the intermediate extension of $\mathcal{Q}_{[U_{1,0,0}/G_a]}[-1]$ in $\mathcal{D}_{i+j+\omega}$. 


Case (3). The canonical basis of $\Lambda(j) \otimes \Lambda(i)$ is as follows.

$$(\eta_j \otimes \eta_i), \quad F_i(\eta_j \otimes \eta_i), \quad F_j(\eta_j \otimes \eta_i),$$

$$(\eta_j \otimes \eta_i), \quad F_i(\eta_j \otimes \eta_i), \quad F_j(\eta_j \otimes \eta_i),$$

$$(\eta_j \otimes \eta_i), \quad F_i(\eta_j \otimes \eta_i), \quad F_j(\eta_j \otimes \eta_i).$$

The first eight elements are provided by the same simple perverse sheaves as $\Lambda(\omega)$. The last one is given by the intermediate extension of $\mathbb{Q}[u_0,1,1/G_x]$ in $\mathcal{D}_{i+j+\omega}$.

3.5. Abelian categorification.

3.5.1. Let $P_{\omega,\nu}$ be the set of simple perverse sheaves (up to isomorphism) in $\Omega_\omega \cap \mathcal{D}_\nu$. $P_{\omega,\nu}$ is finite since there are only finitely many $\mu > \omega$ with $|\mu| = \nu$. Let $L_{\omega,\nu}$ denote the direct sum $\bigoplus_{A \in P_{\omega,\nu}} A \in \Omega_\omega \cap \mathcal{D}_\nu$.

Define a graded $\mathbb{Q}_l$-algebra

$$A^*_\omega := \bigoplus_\nu A^*_{\omega,\nu}$$

where

$$A^*_{\omega,\nu} := \text{Ext}^*_\mathcal{D}(L_{\omega,\nu}, L_{\omega,\nu}).$$

Note that for every complex $A \in \mathcal{D}$,

$$\text{Ext}^*_\mathcal{D}(L_{\omega,\nu}, A) \quad \text{(resp. \text{Ext}^*_\mathcal{D}(A, L_{\omega,\nu}))}$$

defines a graded left (resp. right) $A^*_\omega$-module.

Let $A^*_\omega$-mof denote the category of finite-dimensional graded left $A^*_\omega$-modules and let $A^*_\omega$-pof denote the full subcategory formed by the projectives.

3.5.2. Basic properties of $A^*_\omega$ and $A^*_\omega$-mof are as follows. The second one follows readily from the basic properties of a t-structure. The others are clear from the implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4)(5) $\Rightarrow$ (6)(7).

1. $\dim A^*_{\omega,\nu} < \infty$, which is immediate from Proposition 3.2.5.
2. $A^*_\omega$ is $\mathbb{Q}$-graded and $A^*_\omega = \bigoplus_\nu \bigoplus_{A \in P_{\omega,\nu}} \text{Hom}_{\mathcal{D}}(A, A)$ each summand of which is isomorphic to $\mathbb{Q}_l$.
3. The elements $\text{Id}_A \in A^*_{\omega,\nu}$, $A \in P_{\omega,\nu}$ are precisely the indecomposable idempotents of $A^*_{\omega}$.
4. The $\mathbb{Q}_l$-summands of $A^*_{\omega}/A^*_{\omega}^{>0}$ enumerate the simple objects of $A^*_\omega$-mof up to grading shifts.
5. The modules $A^*_{\omega}\text{Id}_A = \text{Ext}^*_\mathcal{D}(L_{\omega,\nu}, A)$, $A \in P_{\omega,\nu}$ enumerate the indecomposable projectives of $A^*_\omega$-mof up to grading shifts.
6. The obvious map $\text{Hom}_{\mathcal{D}}(A, B) \rightarrow \text{Hom}_{A^*_\omega}(\text{Ext}^*_\mathcal{D}(L_{\omega,\nu}, A), \text{Ext}^*_\mathcal{D}(L_{\omega,\nu}, B))$ is a bijection for $A, B \in \Omega_\omega \cap \mathcal{D}_\nu$.
7. The obvious map $\text{Ext}^*_\mathcal{D}(A, L_{\omega,\nu}) \otimes_{A^*_\omega} \text{Ext}^*_\mathcal{D}(L_{\omega,\nu}, B) \rightarrow \text{Ext}^*_\mathcal{D}(A, B)$ is a bijection for $A, B \in \Omega_\omega \cap \mathcal{D}_\nu$.

3.5.3. Let $\mathcal{O}$ be an endofunctor of the triangulated category $\mathcal{D}$. Assume $\mathcal{O}$ has a left adjoint $\mathcal{O}'$ and assume both $\mathcal{O}, \mathcal{O}'$ restrict to endofunctors of $\Omega_\omega$. We associate to $\mathcal{O}$ a graded $A^*_\omega$-bimodule

$$\mathcal{O}^* := \oplus_{\nu,\nu'} \text{Ext}^*_\mathcal{D}(L_{\omega,\nu'}, \mathcal{O}L_{\omega,\nu}) = \oplus_{\nu,\nu'} \text{Ext}^*_\mathcal{D}(\mathcal{O}'L_{\omega,\nu'}, L_{\omega,\nu}).$$

By property 3.5.2(5), $\mathcal{O}^*$ is (left and right) projective, therefore defines an exact endofunctor of $A^*_\omega$-mof by tensoring from the left.
It follows from property 3.5.2 (7) that
\[ \mathcal{G}^\bullet \otimes_{\mathcal{A}_n^\bullet} \text{Ext}^\bullet_{\mathcal{D}}(L_{\mathcal{A}_n^\bullet}, A) = \oplus_{\nu} \text{Ext}^\bullet_{\mathcal{D}}(L_{\mathcal{A}_n^\bullet, \nu}, \mathcal{G} A), \quad \text{for } A \in \mathcal{D}_\mathcal{A} \cap \mathcal{D}_\nu, \]
and that for two such endofunctors \( \mathcal{G}_1, \mathcal{G}_2 \) of \( \mathcal{D} \),
\[ \mathcal{G}_1^\bullet \otimes_{\mathcal{A}_n} \mathcal{G}_2^\bullet = (\mathcal{G}_1 \mathcal{G}_2)^\bullet. \]

3.5.4. Let \( \mathcal{K}_i^{\pm 1}, \mathcal{E}_i^{(n)} \), \( \mathcal{F}_i^{(n)} \) be the graded projective \( \mathcal{A}_n^\bullet \)-bimodules associated to the functors \( \mathcal{K}_i^{\pm 1}, \mathcal{E}_i^{(n)} \), \( \mathcal{F}_i^{(n)} \) and regard them as exact endofunctors of \( \mathcal{G}(\mathcal{A}_n^\bullet, \text{mo}) \). This makes sense according to Proposition 2.5.1 and Proposition 3.2.4.

Then Theorem 2.5.2 implies

**Theorem 3.5.5.** There are isomorphisms of graded projective \( \mathcal{A}_n^\bullet \)-bimodules.

1. \( \mathcal{K}_i^\bullet \otimes \mathcal{K}_j^{\pm 1} = \mathcal{K}_i^{\pm 1} \otimes \mathcal{K}_j^\bullet = \mathcal{A}_n^\bullet, \mathcal{K}_i^\bullet \otimes \mathcal{K}_j^\bullet = \mathcal{K}_j^\bullet \otimes \mathcal{K}_i^\bullet; \)
2. \( \mathcal{K}_i^\bullet \otimes \mathcal{E}_j^{(n)} = \mathcal{E}_j^{(n)} \otimes \mathcal{K}_i^{\pm n \alpha_{ij}}; \)
3. \( \mathcal{K}_i^\bullet \otimes \mathcal{F}_j^{(n)} = \mathcal{F}_j^{(n)} \otimes \mathcal{K}_i^{\pm m \alpha_{ij}}; \)
4. \( \mathcal{E}_i^{(n-1)} \otimes \mathcal{E}_j^{(n)} \cong \bigoplus_{0 \leq m < n} \mathcal{E}_i^{(n)} \otimes \mathcal{E}_j^{(n) + m + n - 1 - 2m}; \)
5. \( \mathcal{F}_i^{(n-1)} \otimes \mathcal{F}_j^{(n)} \cong \bigoplus_{0 \leq m < n} \mathcal{F}_i^{(n)} \otimes \mathcal{F}_j^{(n) + m + n - 1 - 2m}; \)
6. \( \mathcal{E}_i^\bullet \otimes \mathcal{F}_j^\bullet \cong \bigoplus_{0 \leq m < n} \mathcal{E}_i^\bullet \otimes \mathcal{F}_j^\bullet \mathcal{A}_n^\bullet; \)
7. \( \mathcal{F}_i^\bullet \otimes \mathcal{F}_j^\bullet \cong \bigoplus_{0 \leq m < n} \mathcal{F}_i^\bullet \otimes \mathcal{F}_j^\bullet \mathcal{A}_n^\bullet, \quad i \neq j; \)
8. \( \bigoplus_{0 \leq m \leq 1 - m} \mathcal{E}_i^{(m)} \otimes \mathcal{E}_j^{(1 - m - a_{ij})} \cong \bigoplus_{0 \leq m \leq 1 - m} \mathcal{E}_i^{(m)} \otimes \mathcal{E}_j^{(1 - m - a_{ij})}; \)
9. \( \bigoplus_{0 \leq m \leq 1 - m} \mathcal{F}_i^{(m)} \otimes \mathcal{F}_j^{(1 - m - a_{ij})} \cong \bigoplus_{0 \leq m \leq 1 - m} \mathcal{F}_i^{(m)} \otimes \mathcal{F}_j^{(1 - m - a_{ij})}. \)

3.5.6. Let \( \mathcal{G}(\mathcal{A}_n^\bullet, \text{mo}) \) denote the Grothendieck group of the abelian category \( \mathcal{A}_n^\bullet \text{mo} \). That is, \( \mathcal{G}(\mathcal{A}_n^\bullet, \text{mo}) \) is the free \( \mathbb{Z}[q, q^{-1}] \)-module defined by the generators each for an isomorphism class of objects from \( \mathcal{A}_n^\bullet \text{mo} \) and the relations

(i) \( [M^\bullet] = [M'^\bullet] + [M'^{-1}] \), for an exact sequence \( M'^\bullet \rightarrow M^\bullet \rightarrow M'^{-1}; \)
(ii) \( [M'^{-1}] = q^{-1}[M^\bullet], \) for \( M^\bullet \in \mathcal{A}_n^\bullet \text{mo}. \)

**Corollary 3.5.7.** The followings endow \( \mathcal{G}(\mathcal{A}_n^\bullet, \text{mo}) \otimes \mathbb{Q}(q) \) with a structure of \( U \)-module.

\[ K_i^{\pm 1}[M^\bullet] := [\mathcal{K}_i^{\pm 1} \otimes M^\bullet], \]
\[ E_i^{(n)}[M^\bullet] := [\mathcal{E}_i^{(n)} \otimes M^\bullet], \]
\[ F_i^{(n)}[M^\bullet] := [\mathcal{F}_i^{(n)} \otimes M^\bullet]. \]
3.5.8. By property 3.5.2(5)(6), the functor
\[ Q \vec{\omega} \to \mathfrak{A}_\vec{\omega}^\bullet\text{-pmof}, \quad A \mapsto \oplus_\nu \text{Ext}^\bullet_{D}(L_{\vec{\omega},\nu}, A), \]
defines an equivalence of categories, which in turn induces a linear isomorphism
\[ G(Q_{\vec{\omega}}) \otimes \mathbb{Q}(q) \cong G(\mathfrak{A}_\vec{\omega}^\bullet\text{-mof}) \otimes \mathbb{Q}(q). \]

It is clear from 3.5.3 that this is an isomorphism of \( U \)-modules.

Moreover, the above functor sends the simple perverse sheaves to the indecomposable projectives of \( \mathfrak{A}_\vec{\omega}^\bullet\text{-mof} \)
\[ \bigcup_\nu \{ \mathfrak{A}_\vec{\omega}^\bullet \text{Id}_A = \text{Ext}^\bullet_{D}(L_{\vec{\omega},\nu}, A) \mid A \in P_{\vec{\omega},\nu} \}. \]

Summarizing, we obtain

**Theorem 3.5.9.** There is an isomorphism of \( U \)-modules
\[ G(\mathfrak{A}_\vec{\omega}^\bullet\text{-mof}) \otimes \mathbb{Q}(q) \cong \Lambda(\vec{\omega}) \]
so that the set of indecomposable projectives of \( \mathfrak{A}_\vec{\omega}^\bullet\text{-mof} \)
\[ \{ \mathfrak{A}_\vec{\omega}^\bullet e \mid e \text{ is an indecomposable idempotent of } \mathfrak{A}_\vec{\omega}^\bullet \} \]
gives rise to the canonical basis of \( \Lambda(\vec{\omega}) \).

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