Enumeration of Latin squares with conjugate symmetry

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Abstract
A Latin square has six conjugate Latin squares obtained by uniformly permuting its (row, column, symbol) triples. We say that a Latin square has conjugate symmetry if at least two of its six conjugates are equal. We enumerate Latin squares with conjugate symmetry and classify them according to several common notions of equivalence. We also do similar enumerations under additional hypotheses, such as assuming the Latin square is reduced, diagonal, idempotent or unipotent. Our data corrected an error in earlier literature and suggested several patterns that we then found proofs for, including (1) the number of isomorphism classes of semisymmetric idempotent Latin squares of order $n$ equals the number of isomorphism classes of semisymmetric unipotent Latin squares of order $n + 1$, and (2) suppose $A$ and $B$ are totally symmetric Latin squares of order $n \equiv 0 \mod 3$. If $A$ and $B$ are paratopic then $A$ and $B$ are isomorphic.

KEYWORDS
idempotent, Latin square, semisymmetric, symmetric, totally symmetric, unipotent

1 | INTRODUCTION

A Latin square is a matrix of order $n$ in which each row and column is a permutation of some (fixed) symbol set of size $n$. Throughout, we will assume that the symbol set is also used to index the rows and columns. The symbols will be $\{1, 2, \ldots, n\}$ unless specified otherwise. It is sometimes convenient to think of a Latin square of order $n$ as a set of $n^2$ triples of the form
(row, column, symbol). The Latin property means that distinct triples never agree in more than one coordinate. Latin squares are well known to be equivalent to operation tables of finite quasigroups. We will usually state our results in terms of Latin squares but will occasionally mention the corresponding interpretation in terms of quasigroups. See [6,11] for background and terminology regarding Latin squares and quasigroups.

For each Latin square there are six conjugate squares obtained by uniformly permuting the coordinates of each triple. These conjugates can be labelled by a permutation giving the new order of the coordinates, relative to the former order of \((1, 2, 3)\). For example, the \((1, 2, 3)\)-conjugate is the square itself and the \((2, 1, 3)\)-conjugate is its transpose. We say that a square possesses a conjugate symmetry if at least two of the square’s conjugates are equal. The number of equal conjugates must be a divisor of 6. A square is said to be totally symmetric if all six of its conjugates are equal. It is semisymmetric if it is equal to (at least) three of its conjugates, which must necessarily include the \((1, 2, 3)\), \((3, 1, 2)\) and \((2, 3, 1)\)-conjugates. It is symmetric if it equals its \((2, 1, 3)\)-conjugate. Any square which equals exactly two of its conjugates has exactly two conjugates which are symmetric.

In this paper we count all of the Latin squares of small order which have a conjugate symmetry. By the above comments, it is sufficient to count the totally symmetric, semisymmetric and symmetric Latin squares.

Let \(S_n\) denote the symmetric group on \(\{1, 2, ..., n\}\) and \(\varepsilon\) denote the identity element in \(S_n\). The cycle structure of a permutation is a list of its cycle lengths in decreasing order, using exponent notation to denote multiplicity. For example, one permutation with the cycle structure \(3^2 \cdot 2 \cdot 1^3\) is \((1, 2, 3)(4, 5, 6)(7, 8)(9)(10)(11)\) \(\in S_{11}\). For any \(\alpha \in S_n\) we use \(\text{ord}(\alpha)\) to denote the order of \(\alpha\) in \(S_n\), which is the least common multiple of its cycle lengths. We will write the image of \(i\) under \(\alpha\) as \(i^\alpha\).

For \((\alpha, \beta, \gamma) \in S_n \times S_n \times S_n\) we can apply \(\alpha, \beta, \gamma\) to, respectively, the rows, columns and symbols of a Latin square \(L\). In other words, we map each triple \((r, c, s)\) to \((r^\alpha, c^\beta, s^\gamma)\). This operation is called isotopism. The resulting Latin square is isotopic to \(L\) and is written \(L(\alpha, \beta, \gamma)\). If \(\alpha = \beta = \gamma\) the isotopism is an isomorphism, if \(\gamma = \varepsilon\) then the isotopism is principal, and if \(\alpha = \beta\) then it is an rrs-isotopism.

Isotopism gives an action of \(S_n \times S_n \times S_n\) on the set of Latin squares of order \(n\). On the same set there is also an action of \(S_n \rtimes S_3\), called paratopism, which combines an isotopism with taking a conjugate. The stabiliser of a Latin square \(L\) under isomorphism, isotopism and paratopism are, respectively, its automorphism group, autotopism group and its autoparatopism group. The set of squares isomorphic, isotopic, rrs-isotopic and paratopic to \(L\) are, respectively, the isomorphism class, isotopism class, rrs-isotopism class and species of \(L\). Species are sometimes known as main classes. A useful observation that follows directly from the definitions is:

**Lemma 1.1.** The species of a totally symmetric Latin square contains a single isotopism class.
possesses a reflective symmetry in the plane through the vertices \((1, 1, 1), (1, 1, n), (n, n, 1)\) and \((n, n, n)\) of the cube. Semisymmetric Latin squares are those whose permutation cube possesses a threefold rotational symmetry about the axis through vertices \((1, 1, 1)\) and \((n, n, n)\) of the cube. Totally symmetric squares are those whose permutation cubes possess both the above symmetries.

A **reduced Latin square** is one in which the elements in the first row and column occur in order. A reduced Latin square is the operation table of a quasigroup in which 1 is an identity element. A quasigroup with an identity element is called a **loop**. A Latin square is **unipotent** if the symbols on the main diagonal are all the same. It is **diagonal** if the symbols on its main diagonal are pairwise distinct. It is **idempotent** if it is diagonal and the symbols occur in natural order down the main diagonal. More generally, an element \(i\) is idempotent if \(i\) occurs in cell \((i, i)\). At several subsequent points we will use that the number of idempotent elements is an isomorphism invariant. We will also repeatedly use the following simple consequences of the definitions.

**Lemma 1.2.**

(i) A symmetric Latin square is diagonal if and only if it has odd order.
(ii) Idempotent Latin squares are necessarily diagonal.
(iii) A unipotent Latin square of order \(n \geq 2\) cannot be diagonal or idempotent.
(iv) There are no reduced idempotent Latin squares of order \(n \geq 2\).
(v) Every symmetric Latin square can be mapped to a reduced square by applying a unique permutation to the symbols.
(vi) A semisymmetric reduced Latin square is necessarily unipotent.

**Proof.** Part (i) follows from the observation that off-diagonal entries in a symmetric Latin square come in pairs. Parts (ii) and (iii) are immediate from the definitions. If \(n \geq 2\) then any reduced Latin square of order \(n\) contains the triple \((1, 2, 2)\), which is incompatible with having the triple \((2, 2, 2)\) that is required by idempotent Latin squares of order \(n\), hence (iv) holds. Any symmetric Latin square can be mapped to a reduced square by permuting the symbols to get the first row in order (and hence also the first column in order). This operation preserves symmetry, which implies (v).

It remains to justify (vi). Any reduced semisymmetric Latin square of order \(n\) has the triples \((1, i, i)\) for \(1 \leq i \leq n\) because it is reduced. Semisymmetry then requires the triples \((i, i, 1)\) for \(1 \leq i \leq n\) to be present, which means the square is unipotent. \(\square\)

Lemma 1.2(v) shows that the numbers of rrs-isotopism classes, isotopism classes and species, respectively, of symmetric Latin squares equal the numbers of rrs-isotopism classes, isotopism classes and species containing symmetric reduced Latin squares. The analogous property does not hold for isomorphism classes, as we will see in Table 4.

Table 1 presents a summary of the classes that are enumerated in this paper. To read the table, the following notes are important:

- There are three sections of the table, one for each of the conjugate symmetries (symmetric, semisymmetric and totally symmetric).
- The first row in each section covers the conjugate symmetry without any further assumptions. Subsequent rows in the section add extra assumptions, as specified in the first column.
| **Symmetric:** | Isomorphism classes | rrs-Isotopism classes | Isotopism classes | Species | All squares |
|----------------|---------------------|----------------------|------------------|--------|------------|
|                | Table 4             | Table 2              | Table 2 species  | Table 2| $n!$ (Table 2 reduced) |
|                |                     |                      | Lemma 2.2        |        | Lemma 1.2(v) |
| Idempotent$^1$ | Table 2 species     | Table 2 species      | Table 2 species  | Table 2| Table 2 reduced |
|                | Theorem 2.3(vi)     | Theorem 2.3(v)       | Theorem 2.3(iv)  | Theorem 2.3(iii) | Theorem 2.4(iv) |
| Unipotent$^0$  | Table 4†            | Table 5              | Table 5 species  | Table 5| $n!$ (Table 2 reduced$^1$) |
|                | Theorem 2.4(vii)    | Theorem 2.4(viii)    | Theorem 2.4(i), (viii) | Theorem 2.4(vii) | Theorem 2.4(v) |
| Reduced unipotent$^0$ | Table 5            | Table 5              | Table 5 species  | Table 5| Table 2 reduced$^1$ |
|                |                     |                      | Lemma 2.2        |        | Theorem 2.4(iv) |
| Reduced        | Table 4 loops       | Table 2              | Table 2 species  | Table 2| Table 2 reduced |
|                |                     | Lemma 1.2(v)         | Lemmas 1.2(v), 2.2 | Lemma 1.2(v) | |
| Diagonal$^1$   | Table 4             | Table 2              | Table 2 species  | Table 2| $n!$ (Table 2 reduced) |
|                | Lemma 1.2(i)        | Lemma 1.2(i)         | Lemmas 1.2(i), 2.2 | Lemma 1.2(i) | (v) |
| Reduced diagonal$^1$ | Table 4 loops | Table 2 | Table 2 species | Table 2 | Table 2 reduced |
|                | Lemma 1.2(i)        | Lemma 1.2(i), (v)    | Lemmas 1.2(i), (v), 2.2 | Lemma 1.2(i), (v) | |

| **Semisymmetric:** | Isomorphism classes | rrs-Isotopism classes | Isotopism classes | Species | All squares |
|-------------------|---------------------|----------------------|------------------|--------|------------|
|                   | Table 6             | –                    | Table 6          | Table 6| Table 6 |
| Idempotent        | Table 8             | –                    | Table 8          | Table 8| Table 8 |
| Unipotent         | Table 9             | –                    | Table 9          | Table 9| $n$ (Table 9) |
|                   | Lemma 3.1           | –                    | Lemma 3.1        | Lemma 3.1| Lemma 3.1 |
| Reduced$^u$       | Table 9             | –                    | Table 9          | Table 9| Table 9 |
| Diagonal          | Table 7             | –                    | Table 7          | Table 7| Table 7 |
| Isomorphism classes | rrs-Isotopism classes | Isotopism classes | Species | All squares |
|---------------------|-----------------------|-------------------|---------|------------|
| **Totally symmetric:** | Table 10 | – | Table 10 species | Table 10 |
| **Idempotent**: | Table 11 species | – | Table 11 species | Table 11 |
| Theorem 5.2(ii) | Theorem 5.2(iii) | | | |
| **Unipotent**: | Table 11 species† | – | Table 11 species† | Table 11† |
| Theorem 5.2(vi) | Theorem 5.2(viii) | | | | $n$ (Table 11†) |
| | | | | | Lemma 3.1 |
| **Reduced**: | Table 11 species† | – | Table 11 species† | Table 11† |
| Theorem 5.2(vii) | Theorem 5.2(ix) | | | | Table 11† |
| | | | | | Theorem 2.4(vi) |
| **Diagonal**: | Table 10 Lemma 1.2(i) | – | Table 10 species | Table 10 |
| | | | | | Table 10 |
| | | | | | Lemma 1.2(i) |
| | | | | | Lemma 1.2(i) |
We refer to the conjugate symmetry together with any further restrictions added in the first column, as the “category” of Latin squares being counted.

- Columns (after the first) specify what classes are being counted, whether it is isomorphism classes, rrs-isotopism classes, isotopism classes, species or all squares. Another viewpoint for the last column is that it counts “labelled” Latin squares, meaning that they count as different unless they are equal as matrices.
- Each entry in the table gives a reference which says which table to look at in this paper for an enumeration of the class (as specified at the top of the column) of Latin squares of order \( n \) in the category (as determined by the row). In many cases the reference includes mention of a result or results which justify our claim that the indicated table contains the desired numbers.
- Where a reference is given to Table X, the column of Table X that should be consulted will have the same heading as the column containing the reference, unless the reference specifies another column name.
- In some cases the reference includes a \( ^\dagger \) which means that the order needs to be shifted down by 1. In other words, to get the numbers for order \( n \) Latin squares (in the desired category), you should look at order \( n - 1 \) in the referenced table.
- Superscripts on the category title indicate further information as follows:
  - \( ^1 \) means that category is nonempty only for odd orders.
  - \( ^0 \) means that category is nonempty only for even orders.
  - \( ^u \) means that squares in that category are necessarily unipotent, by Lemma 1.2(vi).

The first two of these are an important caveat when following a reference for a category with that superscript, and the third is helpful when understanding the justification we provide for such a reference.

Here and throughout our paper we adopt a convention for the many tables that present counts of classes of Latin squares with particular properties. A class will be counted if it contains any Latin square with the named properties. So, for example, a column headed “species” in a table of symmetric Latin squares would count any species containing at least one symmetric Latin square, even though only some of the Latin squares in the species are symmetric. In text we shorten “class containing at least one” to “class containing” and we use “class of” for the case when every object in the class has the pertinent property.

Table 1 covers all combinations of interest among these properties: symmetric, semisymmetric, totally symmetric, reduced, diagonal, idempotent and unipotent. Guided by Lemma 1.2, we do not list the following categories in Table 1: idempotent diagonal, idempotent unipotent or unipotent diagonal. Moreover, for semisymmetric and totally symmetric Latin squares we do not combine reduced with any of unipotent, idempotent or diagonal. Nor did we count rrs-isotopism classes except for symmetric Latin squares. The notion of rrs-isotopism is particular to the symmetric case since it is the strongest form of paratopism which respects symmetry. It could be argued that, by the same logic, we should not count isotopism classes or species for any of our conjugate symmetries. However, isotopism classes and species are extremely widely studied notions in the Latin squares literature. Also, in the authors’ experience, isotopism classes or species containing Latin squares with conjugate symmetry are often extremal for properties which have no apparent connection to their symmetry. See [7,21] for a number of examples. This makes catalogues of isotopism classes or species that contain Latin squares with conjugate symmetry useful.

The structure of this paper is as follows. We report the results of our enumerations of symmetric, semisymmetric and totally symmetric Latin squares in Sections 2, 3 and 5,
respectively. In each case, we will also count the Latin squares with conjugate symmetries that have the additional properties of being unipotent, idempotent or diagonal. In Section 4 we explain how our results from Section 3 uncovered an error in earlier literature.

All numbers reported in this paper were computed independently by the two authors using algorithms that differed in some details. The total CPU time taken for all of our computations ran to several months. For each problem we also computed small order catalogues by elementary direct searches, to crosscheck the more sophisticated algorithms which we needed for larger cases. We have made catalogues of many of the Latin squares that we generated in our enumerations available online [22].

2 | SYMMETRIC LATIN SQUARES

In this section we count and classify symmetric Latin squares. Table 2 shows data for the symmetric Latin squares of order up to 13, classified by rrs-isotopism and species. The last column counts all reduced symmetric Latin squares. By Lemma 1.2(v), the number of all symmetric Latin squares of order \( n \) can be obtained by multiplying the number of reduced symmetric Latin squares of order \( n \) by \( n! \), the number of ways to permute the symbols.

We now explain how the numbers in Table 2 were obtained, starting with the numbers in the final column. These were computed using a simple adaptation of the method used in [15] to count the Latin squares of order 11. Rather than building symmetric Latin squares we built the (equinumerous) Latin squares that equal their (1, 3, 2)-conjugate. This was achieved by adding one row at a time, ensuring the new row was an involution (when viewed as a permutation in image format). An alternative viewpoint is that we counted factorisations by adding a factor at a time. In [15] the task was to factorise the complete bipartite graph \( K_{n,n} \) into 1-factors. In the present work we factorised the complete graph \( K_n \) with a loop added to each vertex into factors.

| Order | rrs-Isotopism classes | Species | Reduced      |
|-------|------------------------|---------|-------------|
| 2     | 1                      | 1       | 1           |
| 3     | 1                      | 1       | 1           |
| 4     | 2                      | 2       | 4           |
| 5     | 1                      | 1       | 6           |
| 6     | 6                      | 6       | 456         |
| 7     | 7                      | 7       | 6240        |
| 8     | 423                    | 415     | 10,936,320  |
| 9     | 3460                   | 3460    | 1,225,566,720 |
| 10    | 35,878,510             | 35,878,418 | 130,025,295,912,960 |
| 11    | 6,320,290,037          | 6,320,290,037 | 252,282,619,805,368,320 |
| 12    | 4,612,966,007,179,768  | 4,612,965,997,149,292 | 2,209,617,218,725,251,404,267,520 |
| 13    | 15,859,695,832,489,637,513 | 15,859,695,832,489,637,513 | 98,758,655,816,833,727,741,338,583,040 |

Note: The number of isotopism classes equals the number of species by Lemma 2.2. Also, for odd orders, the number of rrs-isotopism classes equals the number of species, by the same result.
consisting of disjoint edges which may be loops. In the case of odd \( n \) each factor had to have exactly one loop (cf. Lemma 1.2(i)).

For any odd order \( n \) the number of symmetric reduced Latin squares of order \( n \) is equal to the number of 1-factorisations of \( K_{n+1} \) (see, e.g., [23]). These numbers have been computed up to \( n = 13 \). We did not recompute the number for \( n = 13 \), but instead relied on the result quoted in [10]. For smaller odd \( n \) we did recompute the numbers, and used the previously published values as a validation of our code.

Next we counted the rrs-isotopism classes of symmetric Latin squares. These numbers can be inferred by generating all symmetric Latin squares that have a nontrivial rrs-autotopism. Any such Latin square \( L \) necessarily possesses an rrs-autotopism \((\alpha, \alpha, \gamma)\) of prime order. The same autotopism combines with \((2, 1, 3)\)-conjugation to produce an autoparatopism, since \( L \) is symmetric. We used the classifications of autotopisms [20] and autoparatopisms [16]. From those lists we deduce that \( \alpha \) and \( \gamma \) must have one of the cycle structures given in Table 3 and that we are at liberty to fix any choice of \( \alpha \) and \( \gamma \) with the appropriate cycle structure. For each possible \((\alpha, \gamma)\), we generated the number of symmetric Latin squares having \((\alpha, \alpha, \gamma)\) as an autotopism. These numbers are shown in Table 3 under the heading “LS”.

It is interesting that only one of the counts in Table 3 is zero. The classifications in [20,16] give necessary and sufficient conditions for autotopisms and autoparatopisms to be achievable. However, when we insist on simultaneously achieving an autotopism and a separate autoparatopism, the conditions are necessary but no longer sufficient. It is not hard to see that no symmetric Latin square can achieve \((\alpha, \alpha, \gamma)\) as an autotopism when \( \alpha \) and \( \gamma \) both have cycle structure \( \cdot 2^{13} 63 6 \). After filling in three of the symbols that are fixed by \( \gamma \) there is nowhere to place the other two such symbols.

For each entry in Table 3 we used one choice of permutations \( \alpha \) and \( \gamma \) with the indicated cycle structures, and generated all symmetric Latin squares possessing \((\alpha, \alpha, \gamma)\) as an autotopism. Each time we built a square we calculated the order of its rrs-autotopism group. If that group had order 2 and \( n \geq 12 \) then we counted the square but did not store it. All other generated squares were stored. Throwing away the squares with rrs-autotopism group size 2 saved substantial disk space. For example, when \((\alpha, \gamma)\) had cycle structure \((2^{6}, 2^{3} \cdot 1^{6})\) there were 1,553,860,785 squares with group size 2, but only 2,237,762 squares with a larger group. Crucially, the only way two squares from our enumeration could be rrs-isotopic to each other is if they had an rrs-isotopism group of order greater than 2 (since, by assumption, they possess two different nontrivial rrs-autotopisms). Thus with the aid of our stored catalogue we were able to find representatives of all rrs-isotopism classes for which the rrs-autotopism group had order greater than 2. Combining with the count of all the squares we generated, we could then infer the number of rrs-isotopism classes with an autoparatopism group of order greater than 2, and the number of Latin squares in those classes. The number of rrs-isotopism classes for which the autoparatopism group had order exactly 2 could then be inferred from the last column of Table 2.

Our next task was to count species. To do that, we will need the following result. A permutation is said to be semiregular if it has no fixed points and each of its cycles has the same length.

**Lemma 2.1.** If \((\alpha, \beta, \varepsilon)\) is a principal autotopism of some Latin square and \( \alpha \neq \varepsilon \), then \( \alpha \) and \( \beta \) have the same cycle structure, and both are semiregular.

**Proof.** Let \( c \) be the minimum length among the cycles that occur in the cycle decomposition of \( \alpha \) or \( \beta \). Then \( \alpha^c \) or \( \beta^c \) will have fixed points and \((\alpha^c, \beta^c, \varepsilon)\) is an autotopism. Applying [13, Thm. 1], we find that \( \alpha^c = \beta^c = \varepsilon \), from which the result follows. \( \square \)
Define $\Omega_n$ to be the set of symmetric Latin squares of order $n$ that possess an autotopism of the form $(\theta, \theta^{-1}, \varepsilon)$ where $\theta \in S_n$ is semiregular of prime order.

**Lemma 2.2.** Suppose $A$ and $B$ are paratopic symmetric Latin squares of order $n$. Then $A$ and $B$ are isotopic. Also, if $A$ and $B$ are not rrs-isotopic then $n$ is even and $A, B \in \Omega_n$.

**Proof.** The fact that $A$ and $B$ are isotopic follows immediately from [23, Lem. 15]. So assume that $A(\alpha, \beta, \gamma) = B$. Since both $A$ and $B$ are symmetric, we also have $A(\beta, \alpha, \gamma) = B$. Hence

$$B = A(\alpha, \beta, \gamma) = B(\beta^{-1}, \alpha^{-1}, \gamma^{-1})(\alpha, \beta, \gamma) = B(\beta^{-1}\alpha, \alpha^{-1}\beta, \varepsilon).$$

(1)

Now either $\alpha = \beta$, in which case $A$ and $B$ are rrs-isotopic, or $B$ has a nontrivial autotopism of the form $(\theta, \theta^{-1}, \varepsilon)$ by (1). By replacing $\theta$ by an appropriate power of $\theta$, we...
may assume that it has prime order. Also, it follows from Lemma 2.1 that \( \vartheta \) is semi-
regular. Hence \( B \in \Omega_n \). The proof that \( A \in \Omega_n \) is similar.

Finally, suppose that \( n \) is odd. In that case, by Lemma 1.2(i), we can convert \( A, B \) to
idempotent symmetric Latin squares \( A', B' \) by permuting symbols. By [23, Lem. 6], \( A' \) is
isomorphic to \( B' \). It follows that \( A \) is rrs-isotopic to \( B \). \( \square \)

Lemma 2.2 allowed us to count the species that contain symmetric Latin squares of order \( n \)
as follows. We may assume that \( n \) is even, since otherwise the number of species equals the
number of rrs-classes, which we have already counted. We generated \( \Omega_n \) by considering the
primes \( p \) that divide \( n \). For each such prime we generated the symmetric Latin squares that
have an autotopism \((\vartheta, \vartheta^{-1}, \varepsilon)\) where \( \vartheta \) is one fixed semiregular permutation of order \( p \) (in the
case \( p = 2 \) this is a task we have already done when counting rrs-isotopism classes, given that
\( \vartheta^{-1} = \vartheta \)). The hardest case we had to handle was \( n = 12 \). In that case, we only need to consider
\( p = 2 \) and \( p = 3 \). By Lemma 2.2, among the symmetric Latin squares of order \( n \) the number of
species overall is the number of species in \( \Omega_n \) plus the number of rrs-isotopism classes outside
\( \Omega_n \). We got the latter number indirectly, by counting the number of rrs-isotopism classes in \( \Omega_n \)
and subtracting them from the total, which we had already calculated. In this way, we counted
the species just by examining \( \Omega_n \). Some care was required in the hardest case, when \( n = 12 \),
because we had not kept the symmetric Latin squares with an autotopism with cycle structure
\((2^6, 2^6, \varepsilon)\) unless they had an additional rrs-autotopism. These discarded squares had the po-
tential to appear in our catalogue of the squares with a principal autotopism \((\vartheta, \vartheta^{-1}, \varepsilon)\) of order
3. Indeed, in that catalogue there were 40 rrs-isotopism classes (34 species) of symmetric Latin
squares that also possessed a principal autotopism of order 2. All but one of those classes had
been discarded due to having no additional rrs-autotopism.

To be extra careful, we also computed the number of species a second (slower) way. In this
approach we generated every symmetric Latin square with one of the autotopisms in Table 3, but
this time screened them for isotopism. We counted the isotopism classes with autotopism group of
order exactly 2, but discarded their representatives. We stored representatives of all classes where
the order of the autotopism group exceeded 2 for further comparison. In this and other respects, the
computation was similar to how we counted rrs-isotopism classes. The result gave us independent
confirmation of the number of isotopism classes (which equals the number of species).

It is clear from Table 2 that \( n = 8 \) is the smallest order for which there are isotopic sym-
metric Latin squares that are not rrs-isotopic. An example of this behaviour is the following pair
of Latin squares:

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 \\
3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 \\
4 & 3 & 2 & 1 & 8 & 7 & 6 & 5 \\
5 & 6 & 7 & 8 & 2 & 1 & 3 & 4 \\
6 & 5 & 8 & 7 & 1 & 3 & 4 & 2 \\
7 & 8 & 5 & 6 & 3 & 4 & 2 & 1 \\
8 & 7 & 6 & 5 & 4 & 2 & 1 & 3 \\
\end{array}
\quad
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 \\
3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 \\
4 & 3 & 2 & 1 & 8 & 7 & 6 & 5 \\
5 & 6 & 7 & 8 & 2 & 1 & 3 & 4 \\
6 & 5 & 8 & 7 & 1 & 3 & 4 & 2 \\
7 & 8 & 5 & 6 & 3 & 4 & 2 & 1 \\
8 & 7 & 6 & 5 & 4 & 2 & 1 & 3 \\
\end{array}
\]

Applying the isotopism \(((1324)(78), (12)(5867), (1423)(56))\) to the left-hand square, we see
that the two squares are isotopic. However, they are not rrs-isotopic, since the right-hand
square has a symbol that appears six times on the main diagonal and the left-hand square has
no such symbol.
Next, we turn to the issue of counting isomorphism classes of symmetric Latin squares (in other words, counting commutative quasigroups up to isomorphism). For this task we made use of the following result, which allows us to count isomorphism classes using the catalogue of rrs-isotopism classes that we had computed above. For permutations $\alpha, \beta \in S_n$, we write $\alpha^\beta$ as shorthand for $\beta^{-1}\alpha\beta$.

**Theorem 2.1.** Let $L$ be a symmetric Latin square, and $\Gamma$ its rrs-autotopism group. Then the number of isomorphism classes of symmetric Latin squares rrs-isotopic to $L$ is

$$\frac{1}{|\Pi_\Gamma|} \sum_{(\alpha, \alpha^\beta) \in \Gamma} \psi(\alpha, \beta),$$

where

$$\psi(\alpha, \beta) = \begin{cases} \prod_{i=1}^k (n_i)!c_{i}^{n_i} & \text{if } \alpha, \beta \text{ have the same cycle structure } c_1^{n_1}...c_k^{n_k}, \\ 0 & \text{otherwise}. \end{cases}$$

**Proof.** The rrs-isotopism class of $L$ is $\mathcal{L} = \{ L(\sigma, \tau) : \sigma, \tau \in S_n \}$. Define $\Lambda = \{ (\rho, \sigma, \rho) : \rho \in S_n \}$. The number $N$ of isomorphism classes in $\mathcal{L}$ is the number of orbits of the action of $\Lambda$ on $\mathcal{L}$. By the Frobenius–Burnside lemma, we have

$$N = \frac{1}{n!} \sum_{\rho \in S_n} |\{ \mathcal{M} \in \mathcal{L} : \mathcal{M}(\rho, \rho, \rho) = \mathcal{M} \}|$$

$$= \frac{1}{n!} |\{ \sigma, \tau, \rho \in S_n : L(\sigma, \sigma, \tau)(\rho, \rho, \rho) = L(\sigma, \sigma, \tau) \}|$$

$$= \frac{1}{n!} |\{ \sigma, \tau, \rho \in S_n : (\rho^{\sigma^{-1}}, \rho^{\sigma^{-1}}, \rho^{\sigma^{-1}}) \in \Gamma \}|$$

$$= \frac{1}{n!} |\{ \sigma, \rho \in S_n : \alpha^\sigma = \beta^\rho \}|$$

$$= \frac{1}{|\Pi_\Gamma|} \sum_{(\alpha, \alpha^\beta) \in \Gamma} |\{ \sigma \in S_n : \alpha^\sigma = \beta^\rho \}| = \frac{1}{|\Pi_\Gamma|} \sum_{(\alpha, \alpha^\beta) \in \Gamma} \psi(\alpha, \beta). \quad \Box$$

Theorem 2.1 allowed us to complete the middle column of Table 4. In the last column of the same table we count isomorphism classes of symmetric reduced Latin squares (which are equinumerous with the isomorphism classes of commutative loops). To do that, we applied the following result to the representatives of rrs-isotopism classes of symmetric Latin squares, which we found in the production of Table 2.

**Theorem 2.2.** Let $L$ be a symmetric Latin square, and $\Gamma$ its rrs-autotopism group. Then the number of isomorphism classes containing reduced symmetric Latin squares rrs-isotopic to $L$ is

$$\frac{1}{|\Pi_\Gamma|} \sum_{(\alpha, \alpha^\beta) \in \Gamma} \lambda(\alpha, \beta),$$

where $\lambda(\alpha, \beta)$ is the number of fixed points of $\alpha$, if $\alpha$ and $\beta$ have the same cycle structure, and $\lambda(\alpha, \beta) = 0$ otherwise.
Proof. First note that any isomorphism between reduced Latin squares must fix \((1, 1, 1)\). Moreover, any isomorphism that fixes \((1, 1, 1)\) preserves the property of being reduced. Let \(H = \{(\rho, \rho, \rho) : \rho \in S_3 \text{ and } 1^\rho = 1\}\) and consider \(H\) acting on the set of symmetric reduced Latin squares rrs-isotopic to \(L\). Each orbit of this action is the set of reduced Latin squares within one of the isomorphism classes that we wish to count. By the Frobenius–Burnside lemma, \(|H|\times \text{the number of orbits} = \text{the number of distinct triples } M_{\gamma}(\alpha, \beta) \text{ such that } M \text{ is a reduced square rrs-isotopic to } L \text{ and } \gamma(\alpha, \beta) \text{ is an automorphism of } M\). These triples have the form \((L(\sigma, \sigma, \tau), \alpha^\sigma, \beta^\tau)\) where \((\alpha, \alpha, \beta) \in \Gamma, \alpha^\sigma = \beta^\tau\) and \(\sigma, \tau \in S_n\).

Now consider fixed \((\alpha, \alpha, \beta) \in \Gamma\) and consider the ways of choosing \(\sigma, \tau \in S_n\). Since we need \(\alpha^\sigma = \beta^\tau\), it must be that \(\alpha\) and \(\beta\) have the same cycle structure, so we assume this is the case. Suppose \(L = (\ell_{ij})\) and define \(\xi_i \in S_n\) by \(j^{\xi_i} = \ell_{ij}\) for each \(j\). Let \(\delta \in S_n\) fix 1. Then for \(\sigma = \xi_i(1 \ell_{ij})\delta\) and \(\tau = (1 \ell_{ij})\delta\), the rrs-isotopism \((\sigma, \sigma, \tau)\) reduces \(L\). Moreover, all rrs-isotopisms that map \(L\) to a reduced square can be uniquely parameterised by \((i, \delta)\) in this way. Since \((\alpha^\sigma, \alpha^\tau, \beta^\tau)\) is an autotopism of \(L(\sigma, \sigma, \tau)\), it remains to determine when it is an automorphism; that is, when \(\alpha^\sigma = \beta^\tau\), equivalently when \(\alpha^{\xi_i} = \beta^{\xi_i}\). For any \(j\), the triple \((i, j^\alpha, \ell_{ij}) = (i, j^\alpha, j^{\xi_i})\) appears in \(L\). Also, \(L\) contains \((i^\alpha, j^\alpha, \ell_{ij}^{\beta^\tau}) = (i^\alpha, j^\alpha, j^{\xi_i})\) since \((\alpha, \alpha, \beta) \in \Gamma\). However, the condition \(\alpha^{\xi_i} = \beta^{\xi_i}\) implies that \(j^{\alpha^{\xi_i}} = j^{\xi_i}\). Since two triples in a Latin square cannot have exactly two entries in common, it follows that \(i = i^\alpha\). That is, for \((\alpha^\sigma, \alpha^\tau, \beta^\tau)\) to be an automorphism of \(L(\sigma, \sigma, \tau)\) it is necessary that \(i\) is fixed by \(\alpha\). Conversely, suppose that \(i\) is fixed by \(\alpha\). Since \((\alpha, \alpha, \beta) \in \Gamma\) we see that \(L\) contains the triple \((i^\alpha, j^\alpha, \ell_{ij}^{\beta^\tau}) = (i, j^\alpha, j^{\xi_i})\) for all \(j\). From the definition of \(\xi_i\) it then follows that \(j^{\alpha^{\xi_i}} = j^{\xi_i}\) for all \(j\) and hence \(\alpha^{\xi_i} = \beta^{\xi_i}\), as required.

By the above argument, for fixed \((\alpha, \alpha, \beta) \in \Gamma\), there are \((n - 1)!\ \lambda(\alpha, \beta)\) choices of \(\sigma, \tau\) such that \((\alpha^\sigma, \alpha^\tau, \beta^\tau)\) is an automorphism of \(L(\sigma, \sigma, \tau)\), since we have \(\lambda(\alpha, \beta)\) choices for \(i\) and \((n - 1)!\) choices for \(\delta\).
However, different choices of $\alpha, \beta, \sigma, \tau$ may give the same triple $(L(\sigma, \tau, \beta), \alpha^\tau, \beta^\sigma)$. If $L(\sigma_1, \sigma_2, \tau_1) = L(\sigma_2, \sigma_3, \tau_2)$ for $\sigma_1, \tau_1, \sigma_2, \tau_2 \in S_n$, then there is $(\mu, \mu, \nu) \in \Gamma$ such that $\sigma_2 = \mu \sigma_1$ and $\tau_2 = \nu \tau_1$. If we also have $\alpha_2^\sigma = \alpha_1^\sigma$ and $\beta_2^\sigma = \beta_1^\sigma$ then $\alpha_2 = \alpha_1^{-1}$ and $\beta_2 = \beta_1^{-1}$. Furthermore, it follows from $(\alpha, \alpha, \beta), (\mu, \mu, \nu) \in \Gamma$ that $(\alpha_2, \alpha_3, \beta_2) \in \Gamma$. In summary, each triple $(M, \gamma, \gamma)$ such that $M$ is a reduced square rrs-isotopic to $L$ and $(\gamma, \gamma, \gamma)$ is an automorphism of $M$ occurs exactly $|\Gamma|$ times in our counting, once for each choice of $(\mu, \mu, \nu)$. This completes the proof.

We next turn to the enumeration of symmetric Latin squares with additional properties. Lemma 1.2(i) shows there is no further work to do to count diagonal symmetric Latin squares. It also shows that symmetric Latin squares can be idempotent only for odd orders and can be unipotent only for even orders. Nevertheless there are connections between the two classes, as we see shortly.

There is a well-known process called prolongation (see, e.g., [6]) which can be applied to any diagonal Latin square $L$ of order $n$ as follows. We remove the triples

$$T = \{(i, i, L[i, i]) : 1 \leq i \leq n\}$$

from $L$ and install in their place the triples

$$T^* = \{(i, i, n + 1) : 1 \leq i \leq n + 1\} \cup \{(i, n + 1, L[i, i]), (n + 1, i, L[i, i]) : 1 \leq i \leq n\}$$

(3)

to create a new Latin square of order $n + 1$ that we will denote $L^\star$. It is immediate that $L^\star$ is unipotent. Also, it is not hard to check that $L^\star$ is symmetric if and only if $L$ is symmetric. Analogous claims also hold for semisymmetry and total symmetry.

The reverse process to prolongation is called either antiprolongation or contraction [6]. In this process, if the triples in (3) are present, then we replace them by the triples in (2), thereby reducing the order of the Latin square by 1.

For $n \leq 13$, the first quantity listed in our next result appears in Table 2, allowing us to infer all the other quantities.

**Theorem 2.3.** Let $n$ be odd. The following objects are equinumerous:

(i) species containing symmetric Latin squares of order $n$,

(ii) species containing diagonal symmetric Latin squares of order $n$,

(iii) species containing idempotent symmetric Latin squares of order $n$,

(iv) isotopism classes containing idempotent symmetric Latin squares of order $n$,

(v) rrs-isotopism classes containing idempotent symmetric Latin squares of order $n$,

(vi) isomorphism classes of idempotent symmetric Latin squares of order $n$,

(vii) isomorphism classes containing unipotent symmetric reduced Latin squares of order $n + 1$.

**Proof.** By Lemma 1.2(i), any symmetric Latin square of odd order is diagonal, which means that it can be made idempotent by permuting the symbols. The equality between (i), (ii) and (iii) follows. Also, Lemma 2.2 shows that (iii), (iv) and (v) are equal, and [23, Lem. 6] adds (vi) to that list.
It thus suffices to show equality between (vi) and (vii). Since we will do this using prolongation, it will be more convenient for us to replace (vii) with a related (and equinumerous) set. Let \( U_{n+1} \) be the set of symmetric unipotent Latin squares of order \( n + 1 \) that have their last row and column in natural order.

Now consider two idempotent symmetric Latin squares \( L_1 \) and \( L_2 \) of order \( n \), which we prolong to \( L_1^* \in U_{n+1} \) and \( L_2^* \in U_{n+1} \). Suppose there is an isomorphism \( I = (\alpha, \alpha, \alpha) \) which maps \( L_1 \) to \( L_2 \). We extend the permutation \( \alpha \in S_n \) to a permutation \( \alpha^* \in S_{n+1} \) by \((n + 1)^* = n + 1\) and define \( I^* = (\alpha^*, \alpha^*, \alpha^*) \). Observe that \( I \) must fix \( T \) setwise and \( I^* \) must fix \( T^* \) setwise, where \( T \) and \( T^* \) are defined by (2) and (3). Since these sets contain the only triples changed by the prolongations it follows easily that \( I^* \) is an isomorphism from \( L_1^* \) to \( L_2^* \). We conclude that the number of isomorphism classes of symmetric idempotent Latin squares of order \( n \) does not exceed the number of isomorphism classes within \( U_{n+1} \).

To show equality we make use of antiprolongation. Suppose that we have an isomorphism \( I^* = (\alpha^*, \alpha^*, \alpha^*) \) mapping \( L_1^* \in U_{n+1} \) to \( L_2^* \in U_{n+1} \). By definition, \( L_1^*[i, i] = L_2^*[i, i] = n + 1 \) for \( 1 \leq i \leq n + 1 \). Since \( I^* \) is an isomorphism it must map the main diagonal of \( L_1^* \) to the main diagonal of \( L_2^* \), which requires that \((n + 1)^* = n + 1\). Thus we can define \( \alpha \in S_n \) as the restriction of \( \alpha^* \) to \( \{1, ..., n\} \), and define \( I = (\alpha, \alpha, \alpha) \). It is now routine to check that \( I \) is an isomorphism between the antiprolongation \( L_1 \) of \( L_1^* \) and the antiprolongation \( L_2 \) of \( L_2^* \). Moreover, both \( L_1 \) and \( L_2 \) are symmetric idempotent Latin squares by construction. \( \square \)

Theorem 2.3 and Table 2 combine to tell us the numbers of isomorphism classes of unipotent symmetric Latin squares as listed in Table 5. For the rrs-isotopism classes we need the following result, together with [10]:

**Theorem 2.4.** Let \( n \) be even. The following hold:

(i) The number of species containing symmetric unipotent Latin squares of order \( n \) equals the number of isotopism classes containing symmetric unipotent Latin squares of order \( n \).

(ii) The number of rrs-isotopism classes containing symmetric unipotent Latin squares of order \( n \) equals the number of isomorphism classes of 1-factorisations of \( K_n \).

(iii) If \( n \equiv 2 \mod 4 \) then the numbers in parts (i) and (ii) above are equal.

(iv) The number of symmetric unipotent reduced Latin squares of order \( n \) equals the number of symmetric idempotent Latin squares of order \( n - 1 \) which in turn equals the number of symmetric reduced Latin squares of order \( n - 1 \).

(v) The number of symmetric unipotent Latin squares of order \( n \) equals \( n! \) times the number of symmetric reduced Latin squares of order \( n - 1 \).

(vi) The number of totally symmetric unipotent reduced Latin squares of order \( n \) equals the number of totally symmetric idempotent Latin squares of order \( n - 1 \).

(vii) The number of isomorphism classes of symmetric unipotent Latin squares of order \( n \) equals the number of isomorphism classes of symmetric Latin squares of order \( n - 1 \).

(viii) All species, isotopism classes and rrs-isotopism classes that contain a symmetric unipotent Latin square of order \( n \), contain a reduced such square.
Proof. Lemma 2.2 implies (i). Also, (ii) is a consequence of a standard encoding of 1-factorisations of complete graphs as symmetric unipotent Latin squares (e.g., as spelled out in [8]).

We next show (iii). The case \( n = 2 \) is trivial, so assume that \( n > 2 \). By Lemma 2.2, it is enough to argue that if \( \Omega_n \) contains a unipotent symmetric Latin square then \( n \equiv 0 \mod 4 \). Suppose that \( L \) is a unipotent symmetric Latin square of order \( n \) with an autotopism \((\vartheta, \vartheta^{-1}, \varepsilon)\). Let \( u \) be the symbol on the main diagonal of \( L \). For \( 1 \leq i \leq n \) we know that \( L \) contains the triple \((i, i, u)\) and \((i^\vartheta, i^{\vartheta^{-1}}, u)\), from which it follows that \( \vartheta = \vartheta^{-1} \). Hence \( \text{ord}(\vartheta) = 2 \), and Lemma 2.1 then tells us that \( \vartheta \) has cycle structure \( 2^{n/2} \).

The combined action of transposition and the autotopism \((\vartheta, \vartheta, \varepsilon)\) forms \( n \) orbits of size 2 on triples, and all other orbits have size 4. The symbol \( u \) occupies \( n/2 \) of the orbits of size 2. Since \( n - n/2 < n - 1 \), there must be some symbol \( s \) that does not appear in any of the orbits of size 2. As \( s \) must occur \( n \) times in \( L \), it follows that \( n \) must be a multiple of 4, completing the proof of (iii).

We next justify (iv). Prolongation is a bijection between symmetric idempotent Latin squares of order \( n - 1 \) and symmetric unipotent Latin squares of order \( n \) that have their last row and column in natural order. It is clear that the latter set is equinumerous with the symmetric unipotent reduced Latin squares of order \( n \).

For any symmetric idempotent Latin square of order \( n - 1 \) there is a unique symbol permutation that maps it to a symmetric reduced Latin square, namely, the permutation which puts the first row in order. Conversely, for any symmetric reduced Latin square of order \( n - 1 \) there is a unique symbol permutation that maps it to a symmetric idempotent Latin square, namely, the permutation which puts the main diagonal in order (note that \( n - 1 \) is odd, and we have Lemma 1.2(i)). It follows that among the symmetric Latin squares of order \( n - 1 \) the number of idempotent squares equals the number of reduced squares. These observations combine to prove (iv).

Symbol permutations provide an \( n! \) to 1 map from symmetric unipotent Latin squares of order \( n \) to symmetric unipotent reduced Latin squares of order \( n \). Hence (v) follows from (iv).

The proof of (vi) is identical to the proof of the first claim in (iv).

For (vii) it helps observe that isomorphisms can be used to change the symbol on the main diagonal of a unipotent Latin square into the symbol \( n \). After that, the argument is identical to the proof that Theorem 2.3(vi) and Theorem 2.3(vii) are equal.

| Order | Isomorphism classes | rrs-Isotopism classes | Species |
|-------|--------------------|-----------------------|---------|
| 2     | 1                  | 1                     | 1       |
| 4     | 1                  | 1                     | 1       |
| 6     | 1                  | 1                     | 1       |
| 8     | 7                  | 6                     | 6       |
| 10    | 3460               | 396                   | 396     |
| 12    | 6,320,290,037      | 526,915,620           | 526,915,616 |
| 14    | 15,859,695,832,489,637,513 | 1,132,835,421,602,062,347 | 1,132,835,421,602,062,347 |

Note: The number of species equals the number of isotopism classes by Lemma 2.2.

**Table 5** Counts of symmetric unipotent reduced Latin squares

| Order | Isomorphism classes | rrs-Isotopism classes | Species |
|-------|--------------------|-----------------------|---------|
| 2     | 1                  | 1                     | 1       |
| 4     | 1                  | 1                     | 1       |
| 6     | 1                  | 1                     | 1       |
| 8     | 7                  | 6                     | 6       |
| 10    | 3460               | 396                   | 396     |
| 12    | 6,320,290,037      | 526,915,620           | 526,915,616 |
| 14    | 15,859,695,832,489,637,513 | 1,132,835,421,602,062,347 | 1,132,835,421,602,062,347 |
To prove (viii) we apply Lemma 1.2(v) and note that symbol permutations preserve unipotence.

We remark that Theorem 2.4(viii) shows that the counting problems for species, isotopism classes and rrs-isotopism classes that contain a symmetric unipotent Latin square are unchanged by adding an extra condition that the squares should be reduced. However, this is not true for isomorphism classes, as can be seen by comparing Table 4 to Table 5, in light of Theorem 2.4(vii).

It only remains to explain how we deduced the numbers of species in Table 5. As seen in the proof of Theorem 2.4(iii), it suffices to find the difference between the number of species and number of rrs-isotopism classes of unipotent symmetric Latin squares with a principal autotopism \((\theta, \theta, \epsilon)\) where \(\theta\) has cycle type \(2^{n/2}\). We had already generated rrs-isotopism class representatives of such squares in the process of compiling Table 3. We know from Theorem 2.4(iii) that there will be equal numbers of species and rrs-isotopism classes when \(n \equiv 2 \mod 4\), but we found this was also true when \(n \in \{4, 8\}\). For order 12 there was a small difference, with 4851 rrs-isotopism classes but only 4847 species of unipotent Latin squares in \(\Omega_{12}\). Thus there are 4 fewer species than rrs-isotopism classes overall, among the unipotent symmetric Latin squares of order 12. We finish the section by giving an example demonstrating this phenomenon. Let \(A_{12}\) be the unipotent symmetric Latin square below and let \(B_{12}\) be the square that is obtained from \(A_{12}\) by replacing each of the four highlighted subsquares by the other possible subsquare on the same symbols.

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 & 10 & 9 & 12 & 11 \\
3 & 4 & 1 & 2 & 7 & 9 & 5 & 11 & 8 & 12 & 6 & 10 \\
4 & 3 & 2 & 1 & 9 & 7 & 11 & 5 & 12 & 8 & 10 & 6 \\
5 & 6 & 7 & 9 & 1 & 2 & 10 & 12 & 3 & 11 & 4 & 8 \\
6 & 5 & 9 & 7 & 2 & 1 & 12 & 10 & 11 & 3 & 8 & 4 \\
7 & 8 & 5 & 11 & 10 & 12 & 1 & 2 & 6 & 4 & 3 & 9 \\
8 & 7 & 11 & 5 & 12 & 10 & 2 & 1 & 4 & 6 & 9 & 3 \\
9 & 10 & 8 & 12 & 3 & 11 & 6 & 4 & 1 & 2 & 5 & 7 \\
10 & 9 & 12 & 8 & 11 & 3 & 4 & 6 & 2 & 1 & 7 & 5 \\
11 & 12 & 6 & 10 & 4 & 8 & 3 & 9 & 5 & 7 & 1 & 2 \\
12 & 11 & 10 & 6 & 8 & 4 & 9 & 3 & 7 & 5 & 2 & 1
\end{bmatrix}
\]

Then \(A_{12}\) is isotopic to \(B_{12}\) using the isotopism \((\alpha, \gamma, \gamma)\) where \(\alpha = (3, 10, 7, 6, 11, 4, 9, 8, 5, 12)\) and \(\gamma = (1, 2)(3, 9, 7, 5, 11)(4, 10, 8, 6, 12)\), but \(A_{12}\) is not rrs-isotopic to \(B_{12}\).

3 | SEMISYMMETRIC LATIN SQUARES

In this section we report on the results of our enumeration of semisymmetric Latin squares of small order up to various notions of equivalence, and with or without certain extra properties. We state the results first, and then at the end of the section we offer some discussion of how they were obtained. We start with Table 6, which gives the counts of semisymmetric Latin squares (without any additional restrictions) of orders up to 11.
The isomorphism classes of semisymmetric squares for orders up to 6 were listed by Sade [18], who in the same paper gave the species of semisymmetric squares of order 7. Later, the same author [19] listed the isomorphism classes of semisymmetric squares of order 7. In all these cases, Table 6 agrees with Sade’s results. As well as enumerating classes of general semisymmetric squares Sade also noted how many of these squares were diagonal, idempotent or unipotent. Motivated by his work, we also count these classes.

All triples have an orbit of length 3 under $(2, 3, 1)$-conjugation except for the constant triples $(i, i, i)$ that arise from idempotent elements. Hence the following result is an easy consequence of counting triples.

**Theorem 3.1.** In a semisymmetric Latin square of order $n$ the number of idempotent elements is congruent to $n^2 \mod 3$. Hence, for there to exist an idempotent semisymmetric Latin square of order $n$ it is necessary that $n \not\equiv 2 \mod 3$. For there to exist a unipotent semisymmetric Latin square of order $n$ it is necessary that $n \not\equiv 0 \mod 3$.

Theorem 3.1 was noted in [5], which established that idempotent semisymmetric Latin squares exist for all orders $n \not\equiv 2 \mod 3$ except $n = 6$. Also unipotent semisymmetric Latin squares exist for all orders $n \not\equiv 0 \mod 3$ except $n = 7$. These results are consistent with what we found in our work.

The following three tables omit orders which are immediately eliminated by Theorem 3.1. For other orders up to order 12 we give the numbers of semisymmetric squares that are, respectively, diagonal in Table 7 and idempotent in Table 8. Similarly, Table 9 gives counts up to order 13 of semisymmetric reduced Latin squares.

**Lemma 3.1.** The number of unipotent semisymmetric Latin squares of order $n$ is $n$ times the number of reduced semisymmetric Latin squares of order $n$. The numbers of isomorphism classes, isotopism classes and species containing unipotent semisymmetric Latin squares of order $n$ are equal, respectively, to the numbers of isomorphism classes, isotopism classes and species containing reduced semisymmetric Latin squares of order $n$. The above statements also hold with “semisymmetric” replaced throughout by “totally symmetric”.

### Table 6: Semisymmetric Latin squares

| Order | Isomorphism classes | Isotopism classes | Species | All squares |
|-------|---------------------|-------------------|---------|------------|
| 2     | 1                   | 1                 | 1       | 2          |
| 3     | 2                   | 1                 | 1       | 3          |
| 4     | 3                   | 2                 | 2       | 18         |
| 5     | 4                   | 2                 | 2       | 120        |
| 6     | 9                   | 7                 | 7       | 2880       |
| 7     | 41                  | 33                | 28      | 140,256    |
| 8     | 595                 | 557               | 366     | 20,782,080 |
| 9     | 26,620              | 26,511            | 13,899  | 9,569,532,672 |
| 10    | 3,908,953           | 3,908,091         | 1,968,997 | 14,175,610,675,200 |
| 11    | 1,867,918,845       | 1,867,909,542     | 934,327,507 | 74,559,788,174,868,480 |
### Table 7  Diagonal semisymmetric Latin squares

| Order | Isomorphism classes | Isotopism classes | Species | All Squares |
|-------|---------------------|-------------------|---------|-------------|
| 2     | 0                   | 0                 | 0       | 0           |
| 3     | 2                   | 1                 | 1       | 3           |
| 4     | 1                   | 1                 | 1       | 2           |
| 5     | 1                   | 1                 | 1       | 30          |
| 6     | 0                   | 0                 | 0       | 0           |
| 7     | 7                   | 5                 | 5       | 3000        |
| 8     | 2                   | 2                 | 2       | 20,160      |
| 9     | 112                 | 91                | 76      | 19,571,328  |
| 10    | 2369                | 2341              | 1285    | 8,136,806,400 |
| 11    | 347,299             | 347,299           | 175,105 | 13,826,847,640,320 |
| 12    | 237,570,420         | 237,569,195       | 118,815,560 | 113,788,019,281,305,600 |

### Table 8  Idempotent semisymmetric Latin squares

| Order | Isomorphism classes | Isotopism classes | Species | All squares |
|-------|---------------------|-------------------|---------|-------------|
| 3     | 1                   | 1                 | 1       | 1           |
| 4     | 1                   | 1                 | 1       | 2           |
| 6     | 0                   | 0                 | 0       | 0           |
| 7     | 4                   | 3                 | 3       | 480         |
| 9     | 20                  | 19                | 17      | 2,274,048   |
| 10    | 241                 | 238               | 141     | 757,555,200 |
| 12    | 9,801,188           | 9,801,140         | 4,905,666 | 4,693,077,997,977,600 |

### Table 9  Reduced semisymmetric Latin squares

| Order | Isomorphism classes | Isotopism classes | Species | All squares |
|-------|---------------------|-------------------|---------|-------------|
| 2     | 1                   | 1                 | 1       | 1           |
| 4     | 1                   | 1                 | 1       | 1           |
| 5     | 1                   | 1                 | 1       | 2           |
| 7     | 0                   | 0                 | 0       | 0           |
| 8     | 4                   | 4                 | 3       | 480         |
| 10    | 20                  | 20                | 18      | 2,274,048   |
| 11    | 241                 | 241               | 143     | 757,555,200 |
| 13    | 9,801,188           | 9,801,188         | 4,905,693 | 4,693,077,997,977,600 |
Proof. Note that the presence of the triple \((i, x, x)\) implies the presence of the triple \((x, x, i)\), for each \(x\), and conversely. Hence, semisymmetric quasigroups are unipotent if and only if they are loops. Table 9 thus also counts unipotent semisymmetric Latin squares (semisymmetric loops), with one caveat about the last column in the table. There are the same number of semisymmetric loops with identity element \(i\) as there are with identity element 1 (an isomorphism interchanging \(i\) with 1 maps one set to the other), regardless of what value \(i\) has. Hence the total number of unipotent semisymmetric Latin squares of order \(n\) is \(n\) times the number of reduced semisymmetric Latin squares of the same order.

The above argument applies without change to totally symmetric Latin squares. □

On examining Table 7 we were drawn to wonder whether isotopic diagonal semisymmetric Latin squares of order \(n \equiv n \mod 3\) are necessarily isomorphic. It turns out that the answer is negative, as can be seen from the following counterexample of order 35. Consider the direct products \(A_1 \times B\) and \(A_2 \times B\) where

\[
A_1 = \begin{bmatrix}
1 & 3 & 2 & 5 & 4 & 7 & 6 \\
3 & 4 & 1 & 2 & 6 & 5 & 7 \\
2 & 1 & 5 & 7 & 3 & 6 & 4 \\
5 & 2 & 7 & 6 & 1 & 4 & 3 \\
4 & 6 & 3 & 1 & 7 & 2 & 5 \\
7 & 5 & 6 & 4 & 2 & 3 & 1 \\
6 & 7 & 4 & 3 & 5 & 1 & 2 \\
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
1 & 7 & 6 & 2 & 3 & 4 & 5 \\
4 & 2 & 5 & 3 & 6 & 7 & 1 \\
5 & 4 & 3 & 7 & 2 & 1 & 6 \\
6 & 1 & 2 & 4 & 7 & 5 & 3 \\
7 & 3 & 1 & 6 & 5 & 2 & 4 \\
3 & 5 & 7 & 1 & 4 & 6 & 2 \\
2 & 6 & 4 & 5 & 1 & 3 & 7 \\
\end{bmatrix}
\]

and \(B\) has order 5 and is defined by \(B_{ij} \equiv -i - j \mod 5\). By inspection, \(A_1, A_2\) and \(B\) are diagonal semisymmetric Latin squares. Also, applying the isotopism \(((265)(374), (256)(347), \varepsilon)\) to \(A_1\), we find that it is isotopic to \(A_2\) (in fact it can be shown that both squares are isotopic to the Cayley table of \(\mathbb{Z}_7\)). However, \(A_1\) and \(A_2\) are not isomorphic, since they have different numbers of idempotent elements (\(A_1\) has one idempotent element, while \(A_2\) has seven). It is simple to check that \(A_1 \times B\) and \(A_2 \times B\) inherit the properties of being isotopic diagonal semisymmetric Latin squares. They are not isomorphic, since they have different numbers of idempotent elements.

The following special case of \([1, \text{Thm. 1}]\) explains why the “isomorphism classes” column matches the “isotopism classes” column in Table 9.

**Theorem 3.2.** Any two isotopic semisymmetric loops are isomorphic.

Another pattern that became evident when we compiled Tables 8 and 9 is the following.

**Theorem 3.3.** The number of isomorphism classes of semisymmetric idempotent Latin squares of order \(n\) equals the number of isomorphism classes of semisymmetric unipotent Latin squares of order \(n + 1\).

Proof. Consider two idempotent semisymmetric Latin squares \(L_1\) and \(L_2\) of order \(n\), which we prolong to unipotent semisymmetric Latin squares \(L_1^*\) and \(L_2^*\). Any isomorphism which maps \(L_1\) to \(L_2\) can be extended to an isomorphism which maps \(L_1^*\) to \(L_2^*\), for exactly the same reasons as we saw in the proof of Theorem 2.3.
To work in the other direction, we will make use of antiprolongation. First, we argue that every isomorphism class \( C \) of semisymmetric unipotent squares of order \( n + 1 \) has a representative containing the triples

\[
\{(i, i, n + 1), (i, n + 1, i), (n + 1, i, i) : 1 \leq i \leq n + 1\}.
\] (4)

Suppose that \( L \in C \) and that \( L \) has the symbol \( u \) in every position on its main diagonal. Let \( \tau \) be the transposition \((u, n + 1) \in S_{n+1}\). Then by applying the isomorphism \((\tau, \tau, \tau)\) to \( L \) we get a (necessarily semisymmetric) square \( L' \) containing the triples (4).

Suppose that we have an isomorphism \( I^* = (\alpha^*, \alpha^*, \alpha^*) \) mapping \( L_1^* \in C \) to \( L_2^* \in C \). By the above argument, we may assume that both squares contain the triples in (4). Since \( I^* \) is an isomorphism it must map the main diagonal of \( L_1^* \) to the main diagonal of \( L_2^* \), which requires that \((n + 1)^{\alpha^*} = n + 1\). Thus we can define \( \alpha \in S_n \) as the restriction of \( \alpha^* \) to \([1, ..., n]\), and define \( I = (\alpha, \alpha, \alpha) \). It is now routine to check that \( I \) is an isomorphism between the antiprolongation \( L_1 \) of \( L_1^* \) and the antiprolongation \( L_2 \) of \( L_2^* \). Moreover, both \( L_1 \) and \( L_2 \) will be semisymmetric idempotent Latin squares. The theorem follows. □

It is important to note that Theorem 3.3 does not generalise to isotopism classes or species. Consider the following pair of semisymmetric idempotent Latin squares of order 9.

\[
\begin{array}{cccccccc}
1 & 3 & 2 & 7 & 8 & 9 & 4 & 5 \\
3 & 2 & 1 & 8 & 9 & 7 & 6 & 4 \\
2 & 1 & 3 & 9 & 7 & 8 & 5 & 6 \\
7 & 8 & 9 & 4 & 6 & 5 & 1 & 2 \\
8 & 9 & 7 & 6 & 5 & 4 & 3 & 1 \\
9 & 7 & 8 & 5 & 4 & 6 & 2 & 3 \\
4 & 6 & 5 & 1 & 3 & 2 & 7 & 9 \\
5 & 4 & 6 & 2 & 1 & 3 & 9 & 8 \\
6 & 5 & 4 & 3 & 2 & 1 & 8 & 7 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
1 & 3 & 2 & 7 & 8 & 9 & 5 & 6 \\
3 & 2 & 1 & 8 & 9 & 7 & 4 & 5 \\
2 & 1 & 3 & 9 & 7 & 8 & 6 & 4 \\
7 & 8 & 9 & 4 & 6 & 5 & 1 & 2 \\
8 & 9 & 7 & 6 & 5 & 4 & 3 & 1 \\
9 & 7 & 8 & 5 & 4 & 6 & 2 & 3 \\
4 & 6 & 5 & 1 & 3 & 2 & 7 & 9 \\
5 & 4 & 6 & 2 & 1 & 3 & 9 & 8 \\
6 & 5 & 4 & 3 & 2 & 1 & 8 & 7 \\
\end{array}
\]

Applying the isotopism \(((789), (465), (456)(798))\) to the left-hand square, we learn that the two squares are isotopic (in fact, it can be shown that both squares are isotopic to the Cayley table of \( Z_3 \times Z_3 \)). However, the prolongations of these squares to semisymmetric unipotent squares of order 10 do not even belong to the same species, as can easily be established by counting their subsquares or transversals.

To finish the section we briefly describe how we obtained the data that we have given in Tables 6–9. For each table the first task was to find a list of representatives of the isomorphism classes. By filtering such a list it is a simple matter to count isotopism classes and species. Also the number of “all squares” with the relevant properties can be calculated using the orbit-stabiliser theorem, by finding the order of the automorphism group for each isomorphism class representative. Hence the main challenge was finding the isomorphism class representatives. To do this, we first filled the main diagonal in all possible ways (up to isomorphism) that were consistent with Theorem 3.1. Of course, for Tables 8 and 9 the main diagonal was completely determined, whilst there were more choices in the other cases. It was also helpful that for each of the classes that we generated, membership of the class was determined by the main diagonal, together with semisymmetry. Hence the generation for each class was the same after the initial step. Indeed, some of the classes that we generated are obviously subclasses of others, and proceeding as we did allowed us to reuse results from the subclasses without repeating the work.
After filling the main diagonal we proceeded row by row in backtrack fashion. Any time that a triple was added we also added every triple which followed from it by semisymmetry. Our two independent computations did isomorphism screening at different points, but both screened after filling the main diagonal and after filling the whole square (and at some points in between).

The hardest case was the generation of unrestricted semisymmetric Latin squares of order 11, which took roughly 4 days of computation. There were 4586 initial options for the main diagonal. Of those, 117 turned out to have no completion to a semisymmetric Latin square. The most productive diagonal could be completed in 35,046,912 nonisomorphic ways.

4  |  CORRECTION TO SADE

In this section we demonstrate an error which our enumerations have uncovered in the pioneering works of Sade on semisymmetric Latin squares. In his terminology, a left autotopism is an isotopism from a Latin square to its \((3, 1, 2)\)-conjugate. (Note that Sade described the \((3, 1, 2)\)-conjugate of a Latin square as its “transpose”, but we shall avoid this confusing name since in ordinary matrix terminology the transpose is the \((2, 1, 3)\)-conjugate. Also, it should be noted that Sade expressed all his results in terms of quasigroups, but we describe them in terms of Latin squares). We say that a square has a semisymmetric form if some member of its species is semisymmetric. Sade observed rightly that a square may possess a left autotopism without the square having a semisymmetric form. His error occurred when he tried to determine the smallest order for which this happens.

**Theorem 4.1.** A necessary and sufficient condition that a Latin square \(L\) has a semisymmetric form is that \(L\) possesses a left autotopism \((\alpha, \beta, \gamma)\) such that the permutation \(\gamma\beta\alpha\) has order not divisible by 3.

Sade proved the above theorem in [17], where he also gave an example that he claimed was of the lowest possible order for a square possessing a left autotopism but having no semisymmetric form. This example, which has order 10, was reproduced in [6, p. 63]. However, as we shall now point out, Sade’s claim is false since there are 11 species of Latin squares of order 9 which have the desired properties. One example, \(L\), of such a Latin square is given in (5). It is isotopic to its \((3, 1, 2)\)-conjugate by applying the permutation \(\tau = (456)(789)\) to its symbols. It also has an automorphism \((\tau, \tau, \tau)\). These two symmetries generate an autoparatopism group of order 9. Clearly then, the three left autotopisms which \(L\) possesses are all of order 3 so, by Theorem 4.1, \(L\) has no semisymmetric form.

\[
\begin{array}{cccccccc}
2 & 1 & 3 & 5 & 6 & 4 & 9 & 7 \\
1 & 3 & 2 & 6 & 4 & 5 & 8 & 9 \\
3 & 2 & 1 & 7 & 8 & 9 & 5 & 6 \\
5 & 4 & 8 & 2 & 7 & 1 & 6 & 3 \\
6 & 5 & 9 & 1 & 2 & 8 & 7 & 4 \\
4 & 6 & 7 & 9 & 1 & 2 & 3 & 8 \\
7 & 8 & 6 & 4 & 9 & 3 & 1 & 5 \\
8 & 9 & 4 & 3 & 5 & 7 & 2 & 1 \\
9 & 7 & 5 & 8 & 3 & 6 & 4 & 2 \\
\end{array}
\] (5)
Sade’s error surfaced when we compared our count of 13,899 species containing semisymmetric squares of order 9 to the number of species of squares with left autotopisms. This latter number, which was found as part of the enumeration in [13], turns out to be 13,910 and the 11 extra species must all be counterexamples to Sade’s claim. By comparing the corresponding numbers for smaller orders, we can be sure that order 9 is the smallest order for which a square without a semisymmetric form can have a left autotopism. Note that Kolesova, Lam and Thiel [12] established that there are 366 species and 557 isotopism classes of squares of order 8 which possess a left autotopism. These numbers match the corresponding numbers of semisymmetric squares as given in Table 6. Sade’s enumeration in [18] is sufficient to handle the smaller orders. Our discovery of Sade’s error was communicated to Keedwell in time for it to be noted in [11].

5 | **TOTALLY SYMMETRIC SQUARES**

In this section we report on the results of our enumeration of totally symmetric Latin squares of small order. Again, we save discussion of algorithmic details until after we have given the results. The counts of totally symmetric squares for orders up to 13 are shown in Table 10. Note that these results confirm and extend the results of Bailey, Preece and Zemroch [2]. They listed the isomorphism classes and species of totally symmetric squares for orders up to 7, and calculated the total number of such squares of these orders. Later, in [3,4] Bailey extended these results up to order 10, although she did not count the species for the new orders.

A similar result to the next lemma was proved in [23] in the odd order case.

**Lemma 5.1.** Let $P$ be the group of principal autotopisms of a symmetric Latin square $L$ of order $n$. Then $P$ is abelian. Also, if $(\alpha, \beta, \varepsilon) \in P$ then $\alpha = \beta^{-1}$ and $\text{ord}(\alpha)$ divides $n$.

| Order | Isomorphism classes | Species | All squares |
|-------|---------------------|---------|-------------|
| 2     | 1                   | 1       | 2           |
| 3     | 2                   | 1       | 3           |
| 4     | 2                   | 2       | 16          |
| 5     | 1                   | 1       | 30          |
| 6     | 3                   | 2       | 480         |
| 7     | 3                   | 3       | 1290        |
| 8     | 13                  | 13      | 163,200     |
| 9     | 12                  | 8       | 471,240     |
| 10    | 139                 | 139     | 386,400,000 |
| 11    | 65                  | 65      | 2,269,270,080 |
| 12    | 25,894              | 25,888  | 12,238,171,545,600 |
| 13    | 24,316              | 24,316  | 149,648,961,369,600 |
| 14    | 92,798,256          | 92,798,256 | 8,089,070,513,113,497,600 |
| 15    | 122,859,802         | 122,859,796 | 160,650,421,233,958,656,000 |
Proof. Suppose \((x, \gamma, \varepsilon) \in P\). For \(1 \leq i \leq n\) define \(k_i\) to be the symbol such that \(L\) contains the triple \((i, i^2, k_i)\). Since \((x, y, z)\) is an autotopism of \(L\) we have that \((i^x, i^y, k_i) \in L\), and since \(L\) is symmetric we know that \((i^x, i, k_i) \in L\). Any two triples of \(L\) that agree in two coordinates must be equal, so \(i^x = i\). As \(i\) was arbitrary, we see that \(x = \gamma\) and \(\gamma = x^{-1}\).

Next, suppose that \((\alpha_i, \alpha^{-1}_i, \varepsilon)\) and \((\beta_i, \beta^{-1}_i, \varepsilon)\) are two arbitrary elements of \(P\). By composition we know that \((\alpha_i \alpha_2, \alpha^{-1}_i \alpha^{-1}_2, \varepsilon) \in P\). Hence, by the characterisation just shown, \(\alpha_i \alpha_2 = (\alpha^{-1}_i \alpha^{-1}_2)^{-1} = \alpha_2 \alpha_1\). It follows that \(P\) is abelian as claimed.

The fact that \(\text{ord}(\alpha)\) divides \(n\) follows from Lemma 2.1.

Our next result shows why in Table 10 the number of isomorphism classes always matches the number of species for orders that are not divisible by 3.

**Theorem 5.1.** Suppose \(A\) and \(B\) are totally symmetric Latin squares of order \(n \neq 0 \mod 3\). If \(A\) and \(B\) are paratopic then \(A\) and \(B\) are isomorphic.

Proof. Suppose \(A\) is paratopic to \(B\). Then \(A\) is isotopic to \(B\), by Lemma 1.1. Moreover, by replacing \(B\) by an isomorph of \(B\) if necessary, we may assume that \(A = B(\alpha, \beta, \varepsilon)\). From total symmetry it then follows that
\[
A = B(\alpha, \beta, \varepsilon) = B(\beta, \alpha, \varepsilon) = B(\alpha, \varepsilon, \beta) = B(\varepsilon, \alpha, \beta) = B(\beta, \varepsilon, \alpha) = B(\varepsilon, \beta, \alpha). \quad (6)
\]

These relationships will be used repeatedly in what follows. For starters, we have
\[
A = B(\alpha, \varepsilon, \beta) = A(\varepsilon, \alpha^{-1}, \beta^{-1})(\alpha, \varepsilon, \beta) = A(\alpha, \alpha^{-1}, \varepsilon)
\]
and a similar argument shows that \((\beta, \beta^{-1}, \varepsilon)\) is an autotopism of \(A\). Thus, by Lemma 5.1, we see that \(x\) and \(\beta\) commute and that \(\text{ord}(\alpha)\) and \(\text{ord}(\beta)\) both divide \(n\). In particular, \(\text{ord}(\alpha) \neq 0 \mod 3\) and \(\text{ord}(\beta) \neq 0 \mod 3\). Now by (6),
\[
A = B(\alpha, \beta, \varepsilon)(\beta^{-1}, \alpha^{-1})(\alpha, \beta, \varepsilon)(\alpha^{-1}, \beta^{-1})(\beta, \alpha, \varepsilon) = B(\alpha \beta, \alpha \beta, (\alpha \beta)^{-1})
\]
and
\[
B = B(\alpha, \beta, \varepsilon)(\alpha^{-1}, \beta^{-1})(\beta, \alpha, \varepsilon)(\varepsilon, \beta^{-1}) = B(\beta, \beta, \beta^{-2}).
\]

Similarly \(B = B(\alpha, \alpha^{-2})\). So for any \(k \in \mathbb{Z}\),
\[
A = B(\alpha, \alpha^{-k})(\beta, \beta^{-k})(\varepsilon, \alpha \beta, (\alpha \beta)^{-1}) = B((\alpha \beta)^{k+1}, (\alpha \beta)^{k+1}, (\alpha \beta)^{-2k-1}). \quad (7)
\]

Now \(\text{ord}(\alpha \beta)\) divides the least common multiple of \(\text{ord}(\alpha)\) and \(\text{ord}(\beta)\). In particular, \(\text{ord}(\alpha \beta) \neq 0 \mod 3\), so there exists \(k \in \mathbb{Z}\) such that \(k + 1 \equiv -2k - 1 \mod \text{ord}(\alpha \beta)\). Thus (7) shows that \(A\) is isomorphic to \(B\), as required.

Table 10 shows that the requirement for \(n \neq 0 \mod 3\) cannot be abandoned in Theorem 5.1. In fact we have:
Lemma 5.2. For any order \( n \equiv 0 \mod 3 \) there exist isotopic totally symmetric Latin squares that are not isomorphic.

Proof. Define Latin squares \( A, B \) on symbols \( \mathbb{Z}_n \) by \( A[i, j] \equiv -i - j \mod n \) and \( B[i, j] \equiv 1 - i - j \mod n \). It is easy to check that both \( A \) and \( B \) are totally symmetric. Clearly \( A \) is isotopic to \( B \) via the symbol permutation \( x \mapsto x + 1 \mod n \). However \( A \) and \( B \) are not isomorphic. To see this, note that \( A \) has three idempotent elements because \( A[\mathbb{C}_n/3, \mathbb{C}_n/3] \equiv -2\mathbb{C}_n/3 \equiv \mathbb{C}_n/3 \mod n \) for \( c \in \{0, 1, 2\} \). However, \( B \) has no idempotent elements since \( B[i, i] = 1 - 2i \not\equiv i \mod n \) for all \( i \in \mathbb{Z}_n \). \( \square \)

Our final result discusses idempotent totally symmetric Latin squares, and the different guises in which they appear.

Theorem 5.2. Let \( n \) be given. The following objects are equinumerous:

(i) isomorphism classes of Steiner triple systems on \( n \) points,
(ii) isomorphism classes of idempotent totally symmetric Latin squares of order \( n \),
(iii) isotopism classes containing idempotent totally symmetric Latin squares of order \( n \),
(iv) species containing idempotent totally symmetric Latin squares of order \( n \),
(v) isomorphism classes of totally symmetric loops of order \( n + 1 \),
(vi) isomorphism classes of totally symmetric unipotent Latin squares of order \( n + 1 \),
(vii) isomorphism classes containing totally symmetric reduced Latin squares of order \( n + 1 \),
(viii) isotopism classes containing totally symmetric unipotent Latin squares of order \( n + 1 \),
(ix) isotopism classes containing totally symmetric reduced Latin squares of order \( n + 1 \),
(x) species containing totally symmetric unipotent Latin squares of order \( n + 1 \),
(xi) species containing totally symmetric reduced Latin squares of order \( n + 1 \).

Proof. The correspondence between Steiner triple systems on \( n \) points and idempotent totally symmetric Latin squares of order \( n \) is well known (see, e.g., [6, Thm. 2.2.3]). Hence (i) \( = \) (ii). To see that (ii) \( = \) (vi) we use the same argument used to prove Theorem 3.3. Also, [23, Lem. 6] shows that (ii) \( = \) (iii) and Theorem 3.2 shows that (vi) \( = \) (viii) and (vii) \( = \) (ix). Lemma 1.1 shows that (iii) \( = \) (iv), (viii) \( = \) (x) and (ix) \( = \) (xi). Lemma 3.1 tells us that (vi) \( = \) (vii), (viii) \( = \) (ix) and (x) \( = \) (xi). Finally, (v) \( = \) (vii) follows from the definition of a loop, and the fact that any loop has an isomorph in which 1 is the identity element. \( \square \)

Using Theorem 5.2 and the known results on enumeration of Steiner triple systems [9], we immediately have the numbers of isomorphism classes shown in Table 11 (hence we performed no computations for this table). Orders below 19 which are not shown in the table are known not to have any Steiner triple systems.

We finish by briefly describing the computations which produced Table 10. These were similar to the computations for semisymmetric Latin squares in Section 3. We first installed the entries on the main diagonal (and any entries they implied), and screened for isomorphism. All other entries come in sets of six: \( L[i, j] = L[j, i] = k, L[i, k] = L[k, i] = j \) and \( L[j, k] = L[k, j] = i \) for distinct \( i, j, k \). These were filled in one 6-tuple at a time in backtracking fashion, while respecting the Latin property. Just as we did for semisymmetric Latin
squares, our two independent computations screened for isomorphism at different points by canonically labelling and sorting down to inequivalent subcases.

The various types of equivalence of Latin squares can be tested by converting the squares to graphs, as described in [13, Theorem 7], and processed using nauty [14]. With squares having conjugate symmetry, we can often take advantage of the symmetry to construct a smaller graph, which allows faster processing. We illustrate with the example of testing isomorphism of a totally symmetric square $L$ of order $n$. Define a directed graph $G(L)$ with vertices $V_1 \cup V_2$ where $V_1 = \{1, \ldots, n\}$ and

$$V_2 = \{X_{u,v,w} : (u, v, w) \text{ is a triple of } L \text{ and } |\{u, v, w\}| = 3\}.$$

There is a directed edge $(u, v)$ for each triple $(u, u, v)$ of $L$ with $u \neq v$. Also, each $X_{u,v,w} \in V_2$ is adjacent (via undirected edges) to $u$, $v$, and $w$ in $V_1$. The vertices of $V_1$ are coloured differently from the vertices in $V_2$. It is clear that $L$ can be uniquely reconstructed from $G(L)$ (there is a triple $(u, u, u)$ for each $u \in V_1$ with no directed edge leaving $u$). Moreover, totally symmetric Latin squares $L_1$ and $L_2$ are isomorphic if and only if $G(L_1)$ and $G(L_2)$ are isomorphic as coloured graphs. Relabelling $L$ according to the order induced on $V_1$ by a canonical labelling of $G(L)$ produces a canonical representative of the isomorphism class of $L$.

Since the graphs produced by these constructions tend to be highly regular, the efficiency of nauty can be enhanced by using invariants to separate inequivalent vertices. Two invariants that proved useful were the cycle structure of the rows (columns, symbols) relative to other rows (resp., columns, symbols), and the distribution of intercalates ($2 \times 2$ Latin subsquares).

### 6 | CONCLUDING REMARKS

In this paper we have enumerated Latin squares that are equal to 2, 3 or 6 of their conjugates. A possibility for future research would be to enumerate Latin squares that are isotopic to 2, 3 or 6 of their conjugates. Another natural direction to extend our work would be to increase the dimension by considering similar questions for Latin cubes and Latin hypercubes more generally.

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**Table 11**: Number of totally symmetric idempotent Latin squares

| Order | Species | All squares |
|-------|---------|-------------|
| 3     | 1       | 1           |
| 7     | 1       | 30          |
| 9     | 1       | 840         |
| 13    | 2       | 1,197,504,000|
| 15    | 80      | 60,281,712,691,200|
| 19    | 11,084,874,829 | 1,348,410,350,618,155,344,199,680,000|

*Note: This table also counts other objects, by Theorem 5.2.*
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