World-line Green functions with momentum and source conservations

Haru-Tada Sato

Institut für Theoretische Physik
Universität Heidelberg
Philosophenweg 16, D-69120 Heidelberg, Germany

Abstract

Based on the generating functional method with an external source function, a useful constraint on the source function is proposed for analyzing the one- and two-loop world-line Green functions. The constraint plays the same role as the momentum conservation law of a certain nontrivial form, and transforms ambiguous Green functions into the uniquely defined Green functions. We also argue reparametrizations of the Green functions defined on differently parameterized world-line diagrams.

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*E-mail: sato@thphys.uni-heidelberg.de
I. Introduction

String theory organizes the scattering amplitudes in a very compact form (in the infinite string tension limit), and this fact makes the investigation of field theory scattering amplitudes very nontrivial and potentially useful [1]-[5]. In this spirit, multi-loop scattering amplitudes have also been studied both from the string theory viewpoint [6]-[9] and the field theory based on the first quantization formalism (world-line formalism) [10]-[14]. The general structure of field theory amplitudes (with $N$ external momenta $p_1, p_2, \ldots, p_N$) at the one-loop order is described as [1, 2]

$$\Gamma_N = \int_0^{\infty} \frac{dT}{T} (\frac{1}{4\pi T})^\frac{D}{2} \left( \prod_{n=1}^{N} \int_0^{T} d\tau_n \right) K(\tau_1, \tau_2, \ldots, \tau_n; T) \exp \left[ \frac{1}{2} \sum_{j,k=1}^{N} p_j \cdot p_k G_B(\tau_j, \tau_k) \right], \quad (1.1)$$

where $K$ is a certain function which depends on the detail of theory in question (it can be determined systematically). The exponent including the function $G_B$ is sometimes called the generating kinematical factor, and is a theory independent object. $G_B$ is the (world-line) Green function between two points on a loop of length $T$. One can similarly write down the generalized formulae for certain multi-loop cases [8, 13] with using multi-loop Green functions [8, 11]. In this sense, determinations of multi-loop Green functions are important factors in the world-line formalism.

In this paper, we focus on an ambiguity problem of the multi-loop Green functions. There is an ambiguity involved in the world-line Green functions, raised from the momentum conservation law. All of such Green functions should be reduced to the uniquely defined ones under the constraint of a vanishing identity, without changing the value of a kinematical factor. The problem is trivial in the one-loop case, and summarized as follows. In the original definition

$$\frac{1}{2} \partial^2 G_B(\tau) = \delta(\tau) - \frac{1}{T} \quad (1.2)$$

with imposing rotational invariance and periodicity, $G_B$ is uniquely determined as the rotational symmetric form

$$G_B(\tau_1, \tau_2) = G_B(\tau_1 - \tau_2) = |\tau_1 - \tau_2| - \frac{(\tau_1 - \tau_2)^2}{T}. \quad (1.3)$$

However, we do not necessarily use this functional form as concerns the kinematical factor itself, actually which does not change if we add a polynomial in $\tau_j$ to the $G_B(\tau_j, \tau_k)$ in (1.1)
because of the conservation law $\sum_k p_k = 0$. This ambiguity does not cause any problem in the one-loop case, where the defining equation of $G_B$ is simple and its rotational invariance should be clear. Nevertheless, in multi-loop cases, it is certainly useful to identify the ambiguity which can be canceled by a certain condition such as conservation law, because the definitions and calculations of multi-loop Green functions are in general complicated — in addition that the rotational invariance is unclear in certain cases.

As the simplest nontrivial example, we discuss the two-loop Green functions [11, 12]. We refer to the Green functions containing the ambiguity as the wide sense Green functions. As suggested above, the value of a kinematical factor should be invariant for any set of wide sense Green functions. Various wide sense Green functions can be obtained depending on how to define and evaluate, and we shall verify that all of them can be identified with each other in the sense of keeping the kinematical factor invariant by way of examples. To this end, we obviously need a constraint such as the total momentum conservation law. However, in the generic multi-loop cases, a useful form of the conservation law is in practice not a simple summation (along the single loop as mentioned in the one-loop case), because of the presence of additional internal lines. We generalize the conservation law into a more suitable form to our purpose. In addition to the momentum conservation law, we also present a continuous analogue of the conservation law; that is a constraint on the integrals of external source functions along the two-loop worldline vacuum diagram. This continuous version is very simple and useful to apply practical computations, and is non-trivial since such a constraint does not exist in the source term of the usual formulation of field theory.

In Section II, for notational conveniences, we briefly review the two-loop kinematical factor and the Green functions in $\phi^3$ theory. In Section III, we present a useful two-loop momentum conservation formula, and demonstrate how to apply the formula to the identification of different wide sense Green functions. In Section IV, employing the generating functional method with external source functions [14], we consider another derivation of the Green functions. In this case, we show that there also exists a similar constraint formula on the source functions, and verify that it plays the same role as the momentum conservation method of Section III. In Section V, we further confirm the validity and usefulness of the source constraint in more specific cases (one-loop QED). Section VI is a short note on the previous work [12], concerning
new reparametrization transformations of the two-loop Green functions. Section VII contains conclusions.

II. Notations

For the purpose of setting our notations, we briefly review the world-line Green functions and the master amplitude formulae corresponding to Eq. (1.1) in the two-loop $\phi^3$ theory [9, 11, 13]. The (two-loop) master formula is a fundamental quantity which contains all necessary Feynman diagrams belonging to a certain class of diagrams. The classes are labeled by two or three integers ($N', N_3$) or ($N_1, N_2, N_3$), and amplitudes are certain combinations of these classes [13]. We call the first labeling the loop type, and the latter the symmetric type. The general form of the master formula is as follow:

$$\Gamma^{2\text{-loop}}_M = \frac{1}{12}(-g)^{N+2} \int dM \ (4\pi)^{-D} \Delta \frac{D}{2} \exp[E_G]. \quad (2.1)$$

For the loop type parameterization ($N = N' + N_3$), the integration measure $dM$ is

$$dM = \frac{dT}{T} dT_3 d\tau_\alpha d\tau_\beta \prod_{n=1}^{N'} d\tau_n \prod_{l=1}^{N_3} d\tau_l^{(3)}, \quad (2.2)$$

and $\Delta$ is the determinant factor

$$\Delta = \left( TT_3 + T G_B(\tau_\alpha, \tau_\beta) \right)^{-1}. \quad (2.3)$$

The exponential part $E_G$, the generating kinematical factor, takes the following bilinear form in $N$ external momenta ($p_j, p_k^{(3)}; j = 1, \ldots, N'; k = 1, \ldots, N_3$):

$$E_G = \frac{1}{2} \sum_{j,k=1}^{N'} p_j p_k G_{00}^{(1)}(\tau_j, \tau_k) + \frac{1}{2} \sum_{j,k=1}^{N_3} p_j p_k^{(3)} G_{33}^{(1)}(\tau_j, \tau_k^{(3)}) + \frac{N'}{2} \sum_{j=1}^{N'} \sum_{k=1}^{N_3} p_j p_k^{(3)} G_{03}^{(1)}(\tau_j, \tau_k^{(3)}), \quad (2.4)$$

where the bilinear momenta should be understood as Lorentz contracted forms (hereafter as well). The explicit forms of these Green functions are [12]

$$G_{00}^{(1)}(\tau, \tau') = G_{B}(\tau, \tau') - \frac{1}{4} \frac{G_B(\tau, \tau_\alpha) - G_B(\tau, \tau_\beta) - G_B(\tau', \tau_\alpha) + G_B(\tau', \tau_\beta))}{T_3 + G_B(\tau_\alpha, \tau_\beta)} \quad (2.5)$$

$$G_{33}^{(1)}(z_1, z_2) = G_{33}^{(1)}(z_1 - z_2) = |z_1 - z_2| - \frac{(z_1 - z_2)^2}{T_3 + G_B(\tau_\alpha, \tau_\beta)}, \quad (2.6)$$

3
\[G_{03}^{(1)}(\tau, z) = \begin{cases} 
G_{00}^{(1)}(\tau, \tau_\alpha) + \frac{1}{T_3 + G_B(\tau_\alpha, \tau_\beta)} (T_3 z - z^2 + z[G_B(\tau, \tau_\beta) - G_B(\tau, \tau_\alpha)]) & \text{for } \tau_\beta < \tau_\alpha \\
G_{00}^{(1)}(\tau, \tau_\beta) + \frac{1}{T_3 + G_B(\tau_\alpha, \tau_\beta)} (T_3 z - z^2 + z[G_B(\tau, \tau_\alpha) - G_B(\tau, \tau_\beta)]) & \text{for } \tau_\alpha < \tau_\beta
\end{cases}\] (2.7)

The \(\tau\) parameters \(\{\tau_\alpha, \tau_\beta, \tau_n | n = 1, \cdots, N'\}\) run from zero to \(T\), which stands for the length of a loop (fundamental loop), and \(\tau_n^{(3)}, n = 1, \cdots, N_3\) run from zero to \(T_3\), the length of the internal line (the rest part of the vacuum diagram). \(T\) and \(T_3\) are to be integrated from zero to infinity. In [9], we pointed out that one may fix and eliminate one of the parameters \(\{\tau_\alpha, \tau_\beta, \tau_n | n = 1, \cdots, N'\}\) because of the rotational symmetry of the fundamental loop. This means that we can set one of these parameters to be zero which corresponds to the origin of world-line coordinate along the fundamental loop. Obviously, \(G_{00}^{(1)}\) is invariant under this rotation, and does not receive any serious change, however \(G_{03}^{(1)}\) does not even possess any translational symmetry such as seen in \(G_{03}^{(1)}\). Hence the explicit form of \(G_{03}^{(1)}\) depends on which parameter will be set zero. For example, if we choose \(\tau_\beta\) as such origin, \(G_{03}^{(1)}\) should follow the form for \(\tau_\beta < \tau_\alpha\). Similarly, if \(\tau_\alpha\), then take for \(\tau_\alpha < \tau_\beta\). There is also a different Green function [11] from Eq. (2.5). However, both coincide under the same momentum conservation constraint (for \(N_3 = 0\)) as the one-loop type.

Using the transformation obtained in Ref. [12], we can transform the above quantities to the other version (symmetric parameterization). It is done by dividing the fundamental loop into two pieces \(T = T_1 + T_2\) with \(N' = N_1 + N_2\) and \(\{\tau_n\} \to \{\tau_n^{(1)}, \tau_n^{(2)}\}\). In this case, we have [11]

\[dM = dT_1dT_2dT_3 \prod_{i=1, n=1}^{3} \prod_{n=1}^{N_i} d\tau_n^{(i)},\] (2.8)

\[\Delta = (T_1T_2 + T_2T_3 + T_3T_1)^{-1},\] (2.9)

and

\[E_G = \frac{1}{2} \sum_{a=1}^{3} \sum_{j,k} N_a p_j^{(a)} p_k^{(a)} G_{aa}^{\text{sym}}(\tau_j^{(a)}, \tau_k^{(a)}) + \sum_{a=1}^{3} \sum_{j,k} N_a N_{a+1} \sum_{j,k} p_j^{(a)} p_k^{(a+1)} G_{aa+1}^{\text{sym}}(\tau_j^{(a)}, \tau_k^{(a+1)}),\] (2.10)

where we set \(\tau^{(4)} = \tau^{(1)}\) and \(N_4 = N_1\) etc. in accord with the cyclic expression. The Green functions are [11]

\[G_{aa}^{\text{sym}}(\tau, \tau') = G_{aa}^{\text{sym}}(\tau - \tau') = |\tau - \tau'| - \frac{T_{a+1} + T_{a+2}}{T_1T_2 + T_2T_3 + T_3T_1}(\tau - \tau')^2,\] (2.11)
\[ G_{aa+1}^{\text{sym}}(\tau, \tau') = \tau + \tau' - \frac{\tau^2 T_{a+1} + \tau'^2 T_a + (\tau + \tau')^2 T_{a+2}}{T_1 T_2 + T_2 T_3 + T_3 T_1}. \] (2.12)

All the formulae in this section are reproduced from string theory \[8, 9\], and in this sense, we refer to these Green functions (2.5)-(2.7), (2.11), and (2.12) as the standard forms.

### III. Momentum conservation constraint

In this section, we encounter the (wide sense) Green functions of different forms, depending on calculation methods (in the symmetric parameterization). However, the value of \(E_G\) should be shown to be invariant under the constraint of total momentum conservation. In the one-loop case, as mentioned in the introduction, the constraint is expressed by the identity

\[ \sum_{j=1}^{N} \sum_{k=1}^{N} p_j \cdot p_k \tau^m_k = 0. \] (3.1)

The single summation over all momenta \(p_j\) is nothing but the summation over the fundamental loop. However, the same structure can not be seen in (2.4) or (2.10) for the \(N_3 \neq 0\) case. Hence, we shall derive a suitable two-loop generalization of this identity, and explain how it works. To illustrate the idea clearly, we need a couple of examples of different Green functions in the first place.

As explained in Ref. \[13\], Eq.(2.1) is obtained from the path integral

\[
\Gamma_{2\text{-loop}}^M = \frac{(-g)^{N+2}}{2 \cdot 3!} \int d^D x_1 d^D x_2 \prod_{a=1}^{3} \int_0^\infty dT_a e^{-m^2 T_a} \times \int_{y_a(0)=x_2}^{y_a(T_a)=x_1} \mathcal{D}y_a(\tau) \exp \left[ - \int_0^{T_a} \frac{1}{4g_a^2} d\tau(\tau) \right] \prod_{n=1}^{N_a} \int_0^{T_a} d\tau_n(\tau) e^{i\tau_n(\tau)} y_n(\tau) (3.2)
\]

by using the mode expansion

\[ y_a(\tau) = x_1 + \frac{\tau}{T_a} (x_2 - x_1) + \sum_{m=1}^{N} y_m \sin \left( \frac{m \pi \tau}{T_a} \right). \] (3.3)

A straightforward computation in this case shows that the \(E_G\) part is composed of the following Green functions instead of \(G_{ab}^{\text{sym}}\)

\[
G_{aa}^M(\tau, \tau') = |\tau - \tau'| - (\tau + \tau') + \frac{2 \tau \tau'}{T_a} \left( 1 - \frac{T_1 T_2 T_3}{T_a} \right), \tag{3.4}
\]

\[
G_{aa+1}^M(\tau, \tau') = -2 \Delta T_1 T_2 T_3 \frac{\tau \tau'}{T_a T_{a+1}}. \tag{3.5}
\]
Note that the $x_1$ integration generates the total momentum conservation factor

$$(2\pi)^D \delta \left( \sum_n \sum_a \delta_n^{(a)} \right). \quad (3.6)$$

A second example is from Ref. [14]. The world-line Green function should also be derived as a two-point function in the sense of ordinary field theory:

$$G_{\mu\nu}(\tau_1, \tau_2) = \langle x_\mu(\tau_1) x_\nu(\tau_2) \rangle = \frac{\delta}{\delta J_\mu^{(a)}(\tau_1)} \frac{\delta}{\delta J_\nu^{(a)}(\tau_2)} \ln Z[J] \bigg|_{J=0}, \quad (3.7)$$

where the generating functional is given by

$$Z[J] \equiv \int d^D y_1 d^D y_2 \left( \prod_{a=1}^3 \int_{x_a(T_0) = y_1}^{x_a(T_T) = y_2} D x_a \right) \exp \left[ -\frac{1}{4} \sum_a \int_0^{T_a} \dot{x}_a^2(\tau) \, d\tau + \sum_a \int_0^{T_a} J_\mu^{(a)}(\tau) x_\mu^{(a)}(\tau) \, d\tau \right]. \quad (3.8)$$

For later convenience, we here write the intermediate expression (putting $w = \frac{y_1+y_2}{2}$, $z = y_2-y_1$)

$$Z[J] = \left( \prod_{a=1}^3 (4\pi T_a)^{-D/2} \right) \exp \left[ -\frac{1}{2} \sum_{a=1}^3 \int_0^{T_a} J_\mu^{(a)}(\tau) \tilde{G}_{\mu\nu}^{(a)}(\tau, \tau') J_\nu^{(a)}(\tau') \, d\tau \, d\tau' \right] \times \int dz \, dw \prod_{a=1}^3 \exp[w \int_0^{T_a} J_\mu^{(a)}(\tau) \, d\tau - \frac{1}{4} z^\mu A^{(a)}_{\mu\nu} z^\nu + z^\nu \int_0^{T_a} J_\mu^{(a)} R_{\mu\nu}^{(a)}] \quad (3.9)$$

as well as the final expression

$$Z[J] = i\delta^D \left( \sum_{a=1}^3 \int_0^{T_a} J_\mu^{(a)}(\tau) \, d\tau \right) (4\pi)^D \left( \prod_{a=1}^3 (4\pi T_a)^{-D/2} \right) \text{Det}^{-\frac{1}{2}} \left( \sum_a A^{(a)} \right) \times \exp \left[ -\frac{1}{2} \sum_a \int_0^{T_a} J_\mu^{(a)}(\tau) \tilde{G}_{\mu\nu}^{(a)}(\tau, \tau') J_\nu^{(a)}(\tau') \, d\tau \, d\tau' \right] \times \exp \left[ \left( \sum_a A^{(a)} \right)^{-1} \left( \sum_a \int_0^{T_a} R_{\mu\nu}^{(a)} J_\mu^{(a)}(\tau) \, d\tau \right) \left( \sum_c \int_0^{T_c} R_{\nu\rho}^{(c)} J_\rho^{(c)}(\tau) \, d\tau \right) \right], \quad (3.10)$$

where we take

$$A^{(a)}_{\mu\nu} = \delta_{\mu\nu} T_a^{-1}, \quad R^{(a)}_{\mu\nu} = \left( \frac{T_a}{T_a} - \frac{1}{2} \right) \delta_{\mu\nu}, \quad (3.11)$$

and

$$\tilde{G}_{\mu\nu}^{(a)}(\tau_1, \tau_2) = \delta_{\mu\nu} \left( |\tau_1 - \tau_2| - (\tau_1 + \tau_2) + 2 \frac{\tau_1 \tau_2}{T_a} \right). \quad (3.12)$$

A main difference from the first example is the existence of non-constant source terms, where further $\tau$ integrations are formally impossible. This is the reason of having a different form of Green function. Further decoupling the metric factor ($g_{\mu\nu} = -\delta_{\mu\nu}$)

$$G_{\mu\nu}(\tau^{(a)}, \tau^{(b)}) = -g_{\mu\nu} G_{ab}^{J}(\tau^{(a)}, \tau^{(b)}), \quad (3.13)$$
we actually derive the second different form
\[
G_{ab}'(\tau_1^{(a)}, \tau_2^{(a)}) = \delta_{ab} \left( |\tau_1^{(a)} - \tau_2^{(b)}| - (\tau_1^{(a)} + \tau_2^{(b)}) + 2 \frac{\tau_1^{(a)} \tau_2^{(b)}}{T_a} \right) - \frac{1}{2} (2\tau_1^{(a)} - T_a)(2\tau_2^{(b)} - T_b) \frac{T_1 T_2 T_3}{T_a T_b} \Delta .
\]

(3.14)

Now, let us derive the two-loop version of the constraint (3.1). It can be derived by combining the trivial identities
\[
\left( \sum_{a=1}^{N_a} \sum_{j=1}^{N_b} p_j^{(a)} \sum_{k=1}^{N_b} p_k^{(b)} (\tau_k^{(b)})^m \right) = 0 ; \quad b = 1, 2, 3
\]
with multiplying weight coefficients \(C_m^{(b)}\). The result is arranged in the form suitable to \(E_G\):
\[
0 = \sum_{a=1}^{N_a} \sum_{j,k} p_j^{(a)} p_k^{(b)} C_m^{(a)} (\tau_j^{(a)})^m + \sum_{a=1}^{N_a} \sum_{j} p_j^{(a)} p_{k+1}^{(a+1)} \left( C_m^{(a)} (\tau_j^{(a)})^m + C_m^{(a+1)} (\tau_k^{(a+1)})^m \right) ,
\]
where \(m\) is an arbitrary integer and the \(m\)th coefficient \(C_m^{(a)}\) may depend only on \(T_a\). One can add an arbitrary number of copies of this identity to \(E_G\) with different choices of \(C_m^{(a)}\). Consider the \(E_G\) where \(G_{ab}^{(M)}\) is substituted for \(G^{(sym)}\) in (2.10), and add the identity (3.16) to the \(E_G\). Then a new set of Green functions can be read from the modified \(E_G\) as
\[
G_{a0}'(\tau_1^{(a)}, \tau_2^{(a)}) = G_{a0}^{(M)}(\tau_1^{(a)}, \tau_2^{(a)}) + 2C_m^{(a)} (\tau_1^{(a)})^m + \cdots ,
\]
\[
G_{aa+1}'(\tau_1^{(a)}, \tau_2^{(a+1)}) = G_{aa+1}^{(M)}(\tau_1^{(a)}, \tau_2^{(a+2)}) + C_m^{(a)} (\tau_1^{(a)})^m + C_m^{(a+1)} (\tau_2^{(a+1)})^m + \cdots ,
\]
where \(\cdots\) means the additions of further different copies mentioned above. These relations suggest that a variety of Green function’s representations can be derived starting from \(G_{ab}^{(M)}\). In practice, the three representations listed here \((G^{(sym)}, G^{(M)} G^{(J)})\) are connected in the following choices of the \(C_m^{(a)}\) coefficients. We obtain \(G_{a0}' = G_{a0}^{(J)}\), if we choose
\[
C_0^{(a)} = -\frac{1}{2} \Delta T_1 T_2 T_3 , \quad C_1^{(a)} = \Delta \frac{T_1 T_2 T_3}{T_a} , \quad \text{others} = 0 ,
\]
and we obtain \(G_{a0}' = G_{a0}^{(sym)}\), if we choose
\[
C_1^{(a)} = 1 , \quad C_2^{(a)} = -\frac{1}{T_a} \left( 1 - \Delta \frac{T_1 T_2 T_3}{T_a} \right) , \quad \text{others} = 0 .
\]
In this way, every possible form is related to the standard form by the transformation rules (3.17) and (3.18), or in other words, by the two-loop momentum constraint formula (3.16).
IV. Source constraint

In this section, we discuss what identity in the generating functional method should play the same role as the momentum conservation constraint (3.10).

Let us first recall the computation process from (3.9) to (3.10). The \( w \) integration in (3.9) gives rise to the similar \( \delta \)-function divergence as before (cf. Eq. (3.6)) in the sense of Minkowski formulation, and we then have

\[
Z[J] = i\delta \left( \sum_{a=1}^{3} \int_{0}^{T_a} J^a_\mu(\tau) d\tau \right) \prod_{a=1}^{3} (4\pi T_a)^{-D/2} \times \exp \left[ -\frac{1}{2} \sum_{a=1}^{3} \int_{0}^{T_a} J^a_\mu(\tau) \tilde{G}^a_\mu(\tau, \tau') J^a_\nu(\tau) d\tau d\tau' \right] I[J] \tag{4.1}
\]

with

\[
I[J] = \int dz \prod_{a=1}^{3} \exp \left[ -\frac{1}{4} z^\mu A^a_{\mu\nu} z^\nu + z^\nu \int_{0}^{T_a} J^a_\mu(\tau) R^a_{\mu\nu}(\tau) d\tau \right]. \tag{4.2}
\]

Here, the \( R^a_{\mu\nu} \) given in (3.11) is a symmetric tensor, however it is not a general property. Rather, the following reflection anti-symmetry is general and important:

\[
R^a_{\mu\nu}(\tau) = -R^a_{\nu\mu}(T^a - \tau). \tag{4.3}
\]

Suppose that \( J^a_\mu \) behaves as an even or odd function w.r.t. the center point \( T^a/2 \) for the interval \( 0 \leq \tau^{(a)} \leq T_a \); i.e.,

\[
J^a_\mu(\tau^{(a)}) = \pm J^a_\mu(T_a - \tau^{(a)}). \tag{4.4}
\]

Using these properties, we have

\[
z^\nu \int_{0}^{T_a} J^a_\mu(\tau) R^a_{\mu\nu}(\tau) d\tau = \mp z^\nu \int_{0}^{T_a} R^a_{\nu\mu}(\tau) J^a_\mu(\tau) d\tau. \tag{4.5}
\]

By this formula, we perform the Gaussian integral in (4.2):

\[
I[J] = (4\pi)^{D/2} \det L_{\mu} \left( \sum_{a} A^a \right) \exp \left[ \mp \left( \sum_{a} \int_{0}^{T_a} R^a_{\mu\rho} J^a_\rho(\tau) \left( \sum_{a} A^a \right)^{-1} \sum_{a} \int_{0}^{T_a} R^a_{\sigma\nu} J^a_\nu(\tau) d\tau \right) \right]. \tag{4.6}
\]

Again using (4.3), we can eliminate the complex signature symbol

\[
I[J] = (4\pi)^{D/2} \det L_{\mu} \left( \sum_{a} A^a \right) \exp \left[ \left( \sum_{a} \int_{0}^{T_a} R^a_{\mu\rho} J^a_\rho(\tau) \left( \sum_{a} A^a \right)^{-1} \sum_{a} \int_{0}^{T_a} R^a_{\sigma\nu} J^a_\nu(\tau) d\tau \right) \right]. \tag{4.7}
\]
This result (4.7) holds for any linear combination of even and odd $J^a_\mu$ functions. It should be noted that the odd source case implies
\[
\int_0^{T_a} J^a_\mu(\tau^{(a)})d\tau^{(a)} = 0, \quad (a = 1, 2, 3).
\] (4.8)
It means a strong sense 'momentum' conservation which is subjected to only one of three internal lines, and trivially satisfies
\[
\sum_{a=1}^{3} \int_0^{T_a} J^a_\mu(\tau^{(a)})d\tau^{(a)} = 0.
\] (4.9)
Mimicking this property, we in general impose this identity as the total 'momentum' conservation (sum of three lines), as advocated by the $\delta$-function in (4.1). Now, let us compare the roles of discrete and continuous constraints (3.16) and (4.9) in an example. We notice the following term in $G^J_{ab}$ (q.v. (3.14))
\[
- \frac{1}{2} (2\tau^{(a)}_1 - T_a)(2\tau^{(b)}_2 - T_b)\frac{T_1T_2T_3}{T_aT_b} \Delta
\] (4.10) and its corresponding term in the generating functional (4.7):
\[
\ln I[J] = \delta_{\mu\nu} T_1 T_2 T_3 \Delta \sum_a \sum_b \int_0^{T_a} \frac{\tau^{(a)}}{T_a} - \frac{1}{2} J^a_\mu d\tau^{(a)} \int_0^{T_b} (\frac{\tau^{(b)}}{T_b} - \frac{1}{2}) J^b_\nu d\tau^{(b)} + \cdots.
\] (4.11)
If we subtract the $C^{(a)}_m$ terms from (4.10) with the choice (3.19), we obtain $G^M$ as understood from (3.17) and (3.18). On the other hand, using the source constraint (4.9), we can remove from (4.11) the linear terms in $\tau^{(a)}$ and $\tau^{(b)}$ as well as the constant term:
\[
\ln I[J] \approx \delta_{\mu\nu} T_1 T_2 T_3 \Delta \sum_a \sum_b \int_0^{T_a} \int_0^{T_b} \frac{\tau^{(a)} \tau^{(b)}}{T_aT_b} J^a_\mu(\tau^{(a)})J^b_\nu(\tau^{(a)})d\tau^{(a)}d\tau^{(b)} + \cdots.
\] (4.12)
This manipulation leads to the Green function $G^M_{ab}$ as expected; i.e.the removal of the $\tau^{(a)}$ and $\tau^{(b)}$ linear terms corresponds to the subtraction of $C^{(a)}_1$ given in (3.19), and the constant term removal to $C^{(a)}_0$. In this way, the source constraint (4.9) plays the same role as the actual momentum conservation constraint (3.16) on the kinematical factor $E_G$, thus on the wide sense world-line Green functions. It is worth noting that the constraint (4.9) is simpler than (3.16).

V. Examples in QED

The idea of the source constraint gives a family of equivalent world-line Green functions as seen in the previous section. This property is useful for identifying different (wide sense) Green
functions obtained by various computations. In this section, we verify its usefulness in more specific cases. The examples discussed here is the one-loop photon scatterings in the scalar and spinor QED cases. First we discuss the scalar case, and then the spinor case.

The $N$-point function for a complex boson loop is known to be given by the closed path integral of one-dimensional bosonic field $x^\mu(\tau)$:

$$
\Gamma_N(p_1, \cdots, p_N) \equiv \int_0^\infty \frac{dT}{T} \int Dx(\prod_{j=1}^N d\tau_j d\theta_j d\bar{\theta}_j) \exp\left[ \int_0^T \left( -\frac{1}{4} \dot{x}^2 + J \cdot x \right) d\tau \right] \tag{5.1}
$$

with the following specific source function

$$
J^\mu(\tau) = \sum_{j=1}^N \delta(\tau - \tau_j)(\bar{\theta}_j \theta_j \epsilon^\mu_j \frac{\partial}{\partial \tau_j} + i p^\mu_j) , \tag{5.2}
$$

where $\epsilon^\mu_j$ are photon polarization vectors, and $\theta_j$ and $\bar{\theta}_j$ are the Grassmann variables. This source is neither an even function nor an odd one in $\tau$, and we assume the one-loop version of the constraint (4.9) to be

$$
\int_0^T J^\mu(\tau) d\tau = 0 . \tag{5.3}
$$

This leads to the constraint similar to the momentum conservation law

$$
\sum_{j=1}^N J^\mu_j = 0; \quad J^\mu_j = \bar{\theta}_j \theta_j \epsilon^\mu_j \frac{\partial}{\partial \tau_j} + i p^\mu_j . \tag{5.4}
$$

The second term in $J_j$ exactly corresponds to the momentum conservation law, while the first term does not vanish in the sum at all. In this sense, the present constraint (5.4) assumes a nontrivial conservation law. Let us see how our idea works in the following. We perform the path integral (5.1) as the mode integrations with expanding

$$
x^\mu(\tau) = x^\mu_0 + \sum_{n>0} x^\mu_n \sin\left( \frac{n\pi \tau}{T} \right) , \quad (-\infty \leq x_n \leq \infty) . \tag{5.5}
$$

Note that $x_0$ integration diverges as the $\delta$-function corresponding to the constraint (5.3). (This is similar to the $\delta$-function in Eq.(4.1)). Remember that this kind of divergence is usually removed by hand (so-called zero mode divergence). The resulting expression is then

$$
\Gamma_N(p_1, \cdots, p_N) = \int \frac{dT}{T} \left( \frac{1}{4\pi T} \right)^N (\prod_{j=1}^N d\tau_j d\theta_j d\bar{\theta}_j) \exp\left[ \frac{1}{2} g_{\mu\nu} \sum_{j,l=1}^N J^\mu_j J^\nu_l \tilde{G}_B(\tau_j, \tau_l) \right] , \tag{5.6}
$$

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where we formally put $g_{\mu\nu} = -\delta_{\mu\nu}$ and

$$
\tilde{G}_B(\tau_i, \tau_j) = \sum_{m=1}^{\infty} \frac{4T}{m^2\pi^2} \sin\left(\frac{\pi m \tau_i}{T}\right) \sin\left(\frac{\pi m \tau_j}{T}\right) \quad (5.7)
$$

$$
= |\tau_i - \tau_j| - (\tau_i + \tau_j) + 2\frac{\tau_i \tau_j}{T} \quad . \quad (5.8)
$$

Here we have used the following formula at the 2nd line of the above:

$$
\sum_{m=1}^{\infty} \frac{\cos(mx)}{m^2} = \frac{1}{4}(|x| - \pi)^2 - \frac{\pi^2}{12} \quad . \quad (5.9)
$$

Under the constraint (5.4), we realize that $\tilde{G}_B$ in the exponent (the generating kinematical factor) in Eq.(5.6) behaves as the one-loop Green function (1.3) exactly, and we thus rederive the same result

$$
\Gamma_N(p_1, \cdots, p_N) = \int \frac{dT}{T} \left( \frac{1}{4\pi T} \right)^2 \prod_{j=1}^{N} d\tau_j d\theta_j d\bar{\theta}_j \exp\left[ -\frac{1}{2} \sum_{j,l=1}^{N} J_j \cdot J_l G_B(\tau_j, \tau_l) \right] \quad . \quad (5.10)
$$

In this example, it is clear that the source constraint helps us obtain a correct kinematical factor even if a different (wide sense) Green function appears in an intermediate step.

The similar argument applies to the fermion loop case as well. For simplicity, we discuss only the spin part (world-line fermion $\psi^\mu(\tau)$), since the bosonic part is essentially the same as the above case. The world-line fermion part of the $N$-point amplitude is given by

$$
\tilde{\Gamma}_N \equiv \oint D\psi \left( \prod_{j=1}^{N} \int_0^T d\tau_j d\theta_j d\bar{\theta}_j \right) \exp\left[ \int_0^T \left( -\frac{1}{2} \psi^\mu \partial_\tau \psi_\mu + \eta^\mu \psi_\mu \right) d\tau \right] \quad (5.11)
$$

with the source function

$$
\eta^\mu(\tau) = \sum_{j=1}^{N} \sqrt{2}(\theta_j e^\mu_j + i\bar{\theta}_j p^\mu_j) \delta(\tau - \tau_j) \quad . \quad (5.12)
$$

Assuming the source constraint

$$
\int_0^T \eta^\mu(\tau) d\tau = 0 \quad (5.13)
$$

or equivalently

$$
\sum_{j=1}^{N} K_j = 0 \ , \quad K^\mu_j = \sqrt{2}(\theta_j e^\mu_j + i\bar{\theta}_j p^\mu_j) \quad , \quad (5.14)
$$

and performing the path integral with the mode expansion

$$
\psi^\mu(\tau) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} b^\mu_r \cos\left(\frac{2\pi r \tau}{T}\right) \quad , \quad (5.15)
$$

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we obtain (the detail is in Appendix A)

\[ \tilde{\Gamma}_N = \prod_{j=1}^{N} \int d\tau_j d\theta_j d\bar{\theta}_j \exp\left[ \frac{1}{4} \sum_{j,l=1}^{N} K_j \cdot K_l \tilde{G}_F(\tau_j, \tau_l) \right] \]

(5.16)

with

\[ \tilde{G}_F(\tau_i, \tau_j) = \text{sign}(\tau_j - \tau_i) + \frac{2}{T} (\tau_i - \tau_j) . \]

(5.17)

Under the constraint (5.14), \( \tilde{G}_F \) in the exponent plays the same role as the standard fermion Green function

\[ G_F(\tau_i, \tau_j) = \text{sign}(\tau_j - \tau_i) , \]

(5.18)

and we reproduce the correct answer \[ \tilde{\Gamma}_N = \prod_{j=1}^{N} \int d\tau_j d\theta_j d\bar{\theta}_j \exp\left[ \frac{1}{4} \sum_{j,l=1}^{N} K_j \cdot K_l G_F(\tau_j, \tau_l) \right] . \]

(5.19)

VI. Reparametrization type transformation

This section is independent of the previous sections. In this section, we consider reparametrizations and transformations between the standard two-loop Green functions. As mentioned in Sect.II, it is known that \( G^{\text{sym}} \) and \( G^{(1)} \) are connected by a certain transformation \[ . \]

Here, we point out another transformation between them, using periodicities of the Green functions, and also discuss reparametrizations of \( G^{(1)} \) through exchanging two of internal lines.

The symmetric Green functions \( G^{\text{sym}} \) and also \( G^{(1)}_{33} \) satisfy the following properties of periodicity

\[ G^{\text{sym}}_{ab}(T_a - \tau^{(a)}, T_b - \tau^{(b)}) = G^{\text{sym}}_{ab}(\tau^{(a)}, \tau^{(b)}), \quad (a, b = 1, 2, 3) \]

(6.1)

\[ G^{\text{sym}}_{aa}(\tau - P_a) = G^{\text{sym}}_{aa}(\tau), \quad G^{(1)}_{33}(\tau - P_{11}) = G^{(1)}_{33}(\tau), \]

(6.2)

where

\[ P_a = T_a + \frac{T_{a+1}T_{a+2}}{T_{a+1} + T_{a+2}}, \quad P_{11} = T_3 + G_B(\tau_\alpha, \tau_\beta) = \frac{1}{\Delta T} . \]

(6.3)

Putting \( P_3 = P_{11} \) with identifying \( T_1 = T(1 - u) \) and \( T_2 = Tu \), we easily find

\[ u = \frac{|\tau_\alpha - \tau_\beta|}{T} , \]

(6.4)
and the necessary relations for the transformation between $G^{\text{sym}}$ and $G^{(1)}$:

$$
T_1 = T - |\tau_\alpha - \tau_\beta|, \quad T_2 = |\tau_\alpha - \tau_\beta|.
$$

(6.5)

For later convenience, we assign more concrete notations to $\tau_n$ in $G^{(1)}$ on the loop type parameterization:

$$
\tau_n = \begin{cases} 
  x_n & (\tau^* < \tau_n) \\
  y_n & (\tau_n < \tau^*) \\
  z_n & \text{on the internal line } T_3 
\end{cases}
$$

(6.6)

where $\tau^*$ is given below (see Eq.(6.8) and Fig.1).

![Figure 1: The directions of $\tau$ parameters.](image)

With these relations, the transformation rule between $G^{\text{sym}}$ and $G^{(1)}$ is allowed to be expressed as

$$
\begin{align*}
\tau^{(1)}_n &= x_n - \tau^*, \\
\tau^{(2)}_n &= \tau^* - y_n, \\
\tau^{(3)}_n &= z_n,
\end{align*}
$$

(6.7)

where

$$
\tau^* = \tau_\alpha \theta(\tau_\alpha - \tau_\beta) + \tau_\beta \theta(\tau_\beta - \tau_\alpha),
$$

(6.8)

$$
\tau^* - T_2 \leq y_n \leq \tau^* \leq x_n \leq T_1 + \tau^*.
$$

(6.9)

We can obtain another transformation by combining the property (6.1) with (6.7); i.e., replacing
\( \tau^{(a)} \rightarrow T_a - \tau^{(a)} \) on the right-hand side in (6.7)

\[
\begin{align*}
\tau_n^{(1)} &= T_1 - x_n + \tau^*, \\
\tau_n^{(2)} &= T_2 + y_n - \tau^*, \\
\tau_n^{(3)} &= T_3 - z_n.
\end{align*}
\]

(6.10)

From these (two-loop) transformations, we can generate an infinite number of transformations for the one-loop case, since \( \tau^* \) is reduced to an arbitrary number when the both edges of the \( z \) line approach each other along the fundamental loop (of course, with vanishing \( T_3 \)); for example,

\[
\begin{align*}
\text{for } \tau^* = 0 : & \quad \begin{cases} 
\tau^{(1)} = x \\
\tau^{(2)} = -y
\end{cases}, & \quad \text{or } \begin{cases} 
\tau^{(1)} = T_1 - x \\
\tau^{(2)} = T_2 + y
\end{cases},
\end{align*}
\]

(6.11)

\[
\begin{align*}
\text{for } \tau^* = T_2 : & \quad \begin{cases} 
\tau^{(1)} = x - T_2 \\
\tau^{(2)} = T_2 - y
\end{cases}, & \quad \text{or } \begin{cases} 
\tau^{(1)} = T - x \\
\tau^{(2)} = y
\end{cases}.
\end{align*}
\]

(6.12)

Although these one-loop transformations are certainly trivial by themselves, an interesting deduction is that one can generate a set of transformations of \( h \)-loop Green functions from \((h + 1)\)-loop transformations by setting one of \( h \) copies of \( \tau^* \) to be an arbitrary value.

As a second application of (6.7), let us consider some reparametrizations of \( G_{ab}^{(1)}; a, b = 0, 3 \). We show that the transformation of the Green functions \( G_{ab}^{(1)}(\tau_1, \tau_2) \) living on the \( zx \)-loop (loop made of internal lines where the \( z \) and \( x \) variables are defined) into \( G_{00}^{(1)}(\tau'_1, \tau'_2) \) on the \( xy \)-loop can be found through the cyclic permutation symmetry of \( G_{\text{sym}} \) (exchanging \( z \)-line and \( y \)-line).

Namely, transforming \( G^{(1)} \rightarrow G_{\text{sym}} \rightarrow G^{(1)} \) successively, we can read how to transform like

\[
\begin{align*}
G_{00}^{(1)}(x_1, x_2) & \rightarrow G_{00}^{(1)}(x'_1, x'_2) \quad (6.13) \\
G_{03}^{(1)}(x, z) & \rightarrow G_{00}^{(1)}(x', y') \quad (6.14) \\
G_{33}^{(1)}(z_1, z_2) & \rightarrow G_{00}^{(1)}(y'_1, y'_2). \quad (6.15)
\end{align*}
\]

Suppose that each of \( G_{ab}^{(1)} \) on the \( xz \)-loop is related to \( G_{ij}^{\text{sym}}(\tau^{(i)}, \tau^{(j)}) \) by the rule (6.7), and that each of \( G_{00}^{(1)} \) on the \( xy \)-loop is related to \( G_{ij}^{\text{sym}}(\tau^{(i)}, \tau^{(j)}) \) by the same rule as (6.7): \( \tau^{(1)} = x' - \tau^* \), \( \tau^{(2)} = \tau^* - y', \tau^{(3)} = z' \). Putting \( \tau^{(2)} = \tau^{(3)} = \tau^{(2)} \) (corresponding to the exchange of \( z \)- and \( y \)-lines), and eliminating \( \tau^{(a)} \) and \( \tau^{(a)} \) from these transformation rules, we find the
following transformation rule attributed from the exchange between $z$- and $y$-lines:

\[
\begin{align*}
\begin{cases}
x' = x \\
y' = \tau^* - z \\
z' = \tau^* - y
\end{cases}
\quad \text{and} \quad T_3 \leftrightarrow T_2 .
\end{align*}
\]

(6.16)

Remember that $\Delta^{-1}$ is invariant in any exchange of $T_a$. The simplest check of this rule is the following case:

\[
\begin{align*}
G_{33}^{(1)}(z_1, z_2) &= |y_1' - y_2'| - \frac{(y_1' - y_2')^2}{(T_3 + G_B(\tau_{\alpha}, \tau_{\beta}))T_1(T_1 + T_2)}_{T_2 \leftrightarrow T_3} \\
&= |y_1' - y_2'| - \frac{(y_1' - y_2')^2}{(T_3 + G_B(\tau_{\alpha}, \tau_{\beta}))T_1(T_3 + T(1 - u))} \\
&= G_{00}^{(1)}(y_1', y_2') .
\end{align*}
\]

(6.17)

Similarly, we derive another transformation rule from the $yz$-loop to the $xy$-loop (exchange of $z$-line and $x$-line):

\[
\begin{align*}
\begin{cases}
x' = \tau^* + z \\
y' = y \\
z' = x - \tau^*
\end{cases}
\quad \text{and} \quad T_3 \leftrightarrow T_1 .
\end{align*}
\]

(6.18)

One can organize these two sets of transformations in a unified way: Let us express the untransforming (identical) variables in (6.16) and (6.18) as

\[
\tau = x\theta(\tau - \tau^*) + y\theta(\tau^* - \tau) ,
\]

(6.19)

and assign $\tilde{\tau}$ to be the parameter transforming to the $z'$ variable

\[
\tilde{\tau} = y\theta(\tau - \tau^*) + x\theta(\tau^* - \tau) ,
\]

(6.20)

with considering the transformation

\[
\begin{align*}
G_{00}^{(1)}(\tau_1, \tau_2) &\quad \rightarrow \quad G_{00}^{(1)}(\tau'_1, \tau'_2) \\
G_{03}^{(1)}(\tau, z) &\quad \rightarrow \quad G_{00}^{(1)}(\tau', \tilde{\tau}') \\
G_{33}^{(1)}(z_1, z_2) &\quad \rightarrow \quad G_{00}^{(1)}(z'_1, z'_2) .
\end{align*}
\]

(6.21)
The above two sets of rules (6.16) and (6.18) are now expressed in the compact form

\[
\begin{align*}
\tau' &= \tau \\
\tilde{\tau}' &= \tau^* - z \text{sign}(\tau - \tau^*) \quad \text{and} \quad T_3 \leftrightarrow T^*, \quad (6.24)
\end{align*}
\]

where

\[
T^* = T_2 \theta(\tau - \tau^*) + T_1 \theta(\tau^* - \tau). \quad (6.25)
\]

In order to verify these relations, one should note that \(TP_{11} (= \Delta^{-1})\) is invariant under this transformation rule, and also that \(T\) transforms as

\[
T = T^* + \tilde{T}^* \rightarrow T_3 + \tilde{T}^* \quad (6.26)
\]

where

\[
\tilde{T}^* = T - T^* = T_1 \theta(\tau - \tau^*) + T_2 \theta(\tau^* - \tau). \quad (6.27)
\]

It is rather convenient to rewrite the Green function (2.7) as

\[
G^{(1)}_{03}(\tau, z) = z + |\tau - \tau^*| - \frac{1}{TP_{11}} \left[ z^2 T + 2z(\tau - \tau^*)[T^* + (\tau - \tau^*)^2(T_3 + T^*)] \right] \quad (6.28)
\]

than considering the original form

\[
G^{(1)}_{03}(\tau, z) = G^{(1)}_{00}(\tau, \tau^*) + \frac{1}{P_{11}} \left\{ T_3 z - z^2 - \text{sign}(\tau_\alpha - \tau_\beta)[G_B(\tau, \tau_\alpha) - G_B(\tau, \tau_\beta)] \right\}. \quad (6.29)
\]

Applying the transformation (6.24) to Eqs.(6.21), (6.28), and (6.23), we obtain

\[
G^{(1)}_{00}(\tau_1, \tau_2) = G_B(\tau_1, \tau_2) - \frac{\Delta}{T} T^*2(\tau_1 - \tau_2)^2, \quad (6.30)
\]

\[
G^{(1)}_{00}(\tau, \tilde{\tau}) = G_B(\tau, \tilde{\tau}) - \frac{\Delta}{T} [T(\tau^* - \tilde{\tau}) - T^*(\tau - \tilde{\tau})]^2, \quad (6.31)
\]

\[
G^{(1)}_{00}(\tilde{\tau}_1, \tilde{\tau}_2) = G_B(\tilde{\tau}_1, \tilde{\tau}_2) - \frac{\Delta}{T} \tilde{T}^*2(\tilde{\tau}_1 - \tilde{\tau}_2)^2. \quad (6.32)
\]

These representations are independent of the choice of either \(\tau^* = \tau_\alpha\) or \(\tau_\beta\), and reproduce Eq.(23) of [12] for the particular choice \(\tau^* = \tau_\beta\) (correcting an error in the literature).
VII. Conclusions

In this paper, we have investigated two types of the constraints on the two-loop kinematical factor and the world-line Green functions. One is nothing but the momentum conservation law on external legs, and the other is the vanishing constraint on the source term integrals along the whole of world-line. Although there is no direct connection between two of them, the latter can be regarded as a continuous version of the former. Because of the ambiguity raised by the constraints, an infinite number of wide sense Green functions are in fact possible to take part in the kinematical factor exponent. However, we verify that all these Green functions can be identified with the standard (restricted) Green functions, all of which are reduced from a world-sheet Green function [8], and some of which are related to actual solutions of defining differential equations with possessing the rotational invariance along the fundamental loop [11].

Conversely, the constraints loosen some imposed restrictions on the standard Green functions, and eventually make various evaluations and approaches possible. These constraints will be useful for analyzing higher loop’s world-line Green functions. Especially it is clear that the source constraint is much easier to apply than the momentum conservation constraint in the multi-loop cases. In two-loop Yang-Mills theory, there arises a different Green function in the calculation in a constant background field [14], and the source constraint is actually useful to identify the Green function with the standard one in the vanishing limit of constant background field (as demonstrated in Sections III and IV). Obviously, the similar thing is expected to occur in the multi-loop cases. Since expressions of multi-loop Green functions are complicated, these constraints will be useful for simplifying the expressions or for transforming into convenient forms together with the transformation property (suggested below (6.12)). It might be interesting to speculate a usefulness of our techniques in the thermal world-line cases [15].

In the final part of the paper, we have considered the transformations among the Green functions of standard forms, associated with the reparametrizations of the two-loop world-line diagram. On the one hand, the form of world-sheet Green function is independent of the orderings of two vertices, which join the internal line and the fundamental loop. On the other hand, the crossing type Green functions (2.7) and (2.12), which belong to the type of a correlation between the fundamental loop and the internal line, are neither translational
invariant nor ordering independent. It might be that this gap will be filled in some way around
by taking account of the discussed transformations into the loop type Green function $G^{(1)}_{00}$. The
crossing type Green functions are necessary in non-Abelian gauge theory, and a complexity in
the combinatorics problem will be caused by this type (similarly to the $\phi^3$ theory case [14]).
We hope for a useful parameterization or a transformation to overcome these problems.

Appendix A. Fermion mode integration

We show the derivation of (5.16) in this appendix. The fermion field (5.15) is an expansion
which satisfies $\psi(0) = -\psi(T) \neq 0$ and $\int_0^T \psi^\mu(\tau) d\tau = 0$. First, we rewrite

$$H \equiv \int_0^T \left( -\frac{1}{2} \psi^\mu \partial_{\tau} \psi_\mu + \eta^\mu \psi_\mu \right) d\tau = I - J \ ,$$

where

$$I \equiv \int_0^T d\tau_1 d\tau_2 d\tau' (\psi(\tau_1) - \frac{1}{2} \eta(\tau) G_I(\tau - \tau_1)) (\delta(\tau_1 - \tau_2) - \frac{1}{2} \partial_{\tau_2} (\psi(\tau_2) - \frac{1}{2} \eta(\tau) G_I(\tau - \tau_2) ) \ ,$$

(A.1)

$$J \equiv -\left( \frac{1}{2} \right)^3 \int_0^T d\tau_1 d\tau_2 d\tau' \eta(\tau) \eta(\tau') G_I(\tau - \tau_1) \delta(\tau_1 - \tau_2) \frac{\partial}{\partial \tau_2} G_I(\tau' - \tau_2) \ .$$

(A.2)

with introducing the function

$$G_I(\tau) = \frac{2}{\pi} \sum_{m \geq 1} \frac{1}{m} \sin\left( \frac{2 \pi m \tau}{T} \right) ,$$

(A.4)

which satisfies

$$\frac{1}{2} \partial_{\tau} G_I(\tau) = \delta(\tau) - \frac{1}{T} ,$$

(A.5)

and

$$G_I(\tau_1 - \tau_2) = \frac{\partial}{\partial \tau_1} G_B(\tau_1, \tau_2) .$$

(A.6)

Putting (5.12) and (A.4) into (A.3), and performing the integrals, we obtain

$$J = -\left( \frac{1}{2} \right)^3 \sum_{j,l} K_j K_l \left( \frac{\partial}{\partial \tau_j} \right) \sum_{m \geq 1} \frac{2T}{\pi^2 m^2} \cos\left( \frac{2 \pi m (\tau_i - \tau_j)}{T} \right) .$$

(A.7)

Using the summation formula (5.9), we have

$$J = -\left( \frac{1}{2} \right)^2 \sum_{j,l} K_j K_l \left( \frac{2}{T} (\tau_j - \tau_i) - \text{sign}(\tau_j - \tau_i) \right) .$$

(A.8)
Shifting $\psi \rightarrow \psi + \frac{1}{2} \eta G_I$ in the path integral (5.11), the quantity $I$ is reduced to the free integral

$$I \rightarrow -\frac{1}{2} \int_0^T \psi \cdot \dot{\psi} ,$$

(A.9)

and this yields nothing but the path integral normalization

$$\oint D\psi e^{-\frac{1}{4} \int_0^T \psi \dot{\psi} d\tau} = 1 .$$

(A.10)

This can be checked by integrating the modes (5.15). Therefore we derive (5.16) owing to (A.8) and (A.10)

$$\tilde{\Gamma}_N = \oint D\psi(\prod_{j=1}^N \int d\tau_j d\theta_j d\bar{\theta}_j) e^H = \prod_{j=1}^N \int d\tau_j d\theta_j d\bar{\theta}_j e^J .$$

(A.11)

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