Redefining the magic square on numerical characters

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Abstract. As a number system, the magic square is different from the others. Characteristic depends not only on size but also depends on numerical character in computation. This paper has redefined the term of magic square formally, by exposing the inductive general characteristics of cases to numerical ordering of numbers.

1. Introduction

The magic square is a numerical system that has long filled the human life [1], which has a special character [2]. The magic square although square, but computational nature is different from matrix and also other number systems [3]. There is a term of different understanding with formal formulation in mathematics about magic square [4, 5]. In addition, different types of magic square also give effect to the formalization of this system [6]. Therefore, redefinition needs to be done formally by first exploring some of the basic characteristics of numbering cases.

Redefining an object is not as easy as building a definition from scratch [7, 8]. Sometimes certain terms in language deny the truth of the statement to be built [9]. Therefore, some characteristics generated so far are only disclosed as axiom of definition support, and as assumptions. Characteristics excavated from the cases are inductively generalized. Thus, this paper aims to rediscover the formal definition of magic square by considering the odd size of magic square and some of numerical characters.

2. Problem Definition

Sometime the dictionary does not precisely define something: The term of magic square has been termed as

\[ T \text{ A square containing a number of integers arranged so that the sum numbers is the same in each row, column, and main diagonal and often in some or all of the other diagonals.} \]

\[ T \text{ in Merriam-Websters online dictionary}^{1}, \text{ thereby it provides improper understanding to who makes the study about the magic square.} \]

The magic square is a two-dimensional array \( a_{ij} \) with size \( n \times n \) or \( i, j = 1, \ldots, n \), \( n \) can be odd or even. For \( n \) is odd, the magic square contains a number with the index \( k \in I, k = 1, \ldots, n \times n \)

\(^1\) https://www.merriam-webster.com/dictionary/magic%20square
being entered in a way into the two-dimensional array with which \( n \mod 2 = 1 \) \[2\]. So each row, column, and two main diagonals have the same number. In other words, for first row we have the sum of numbers in first, second, \ldots \) rows

\[
\Sigma r_1 = \sum_{i=1,j=1}^{1,n} a_{ij}, \quad \Sigma r_2 = \sum_{i=2,j=1}^{2,n} a_{ij}, \quad \sigma r_n = \sum_{i=n,j=1}^{n,n} a_{ij}.
\]  

(1)

For first column we obtain the first, second, \ldots \) columns

\[
\Sigma c_1 = \sum_{i=1,j=1}^{n,1} a_{ij}, \quad \Sigma c_2 = \sum_{i=1,j=2}^{n,2} a_{ij}, \quad \Sigma c_n = \sum_{i=1,j=n}^{n,n} a_{ij}.
\]  

(2)

\[
\begin{array}{ccc}
  8 & 1 & 6 \\
  3 & 5 & 7 \\
  4 & 9 & 2 \\
\end{array}
\]

**Figure 1.** Positive integers in \( 3 \times 3 \) magic-square.

For two main diagonals we have the sum of numbers are

\[
\Sigma d_l = \sum_{j=i,1}^{j=n,i,1} a_{ij}, \quad \Sigma d_r = \sum_{j=n,i,1}^{j=1,n} a_{ij},
\]  

(3)

where

\[
\Sigma r_1 = \Sigma r_2 = \ldots = \Sigma r_n = \Sigma c_1 = \Sigma c_2 = \ldots = \Sigma c_n = \Sigma d_l = \Sigma d_r.
\]  

(4)

Suppose we take the simplest example of a magic square measuring \( 2 \times 3 \) or \( n = 3 \) (odd). Entries involve integers 1, 2, 3, 4, 5, 6, 7, and 9 and the result is like Fig. 1, but for the same size can also be submitted the numbers \(-1,-2,-3,-4,-5,-6,-7,-8 \) and \(-9 \) with which a magic square (see Fig. 2).

\[
\begin{array}{ccc}
  -8 & -1 & -6 \\
  -3 & -5 & -7 \\
  -4 & -9 & -2 \\
\end{array}
\]

**Figure 2.** Negative integers in \( 3 \times 3 \) magic-square.

or by involving the numbers 2,4,6,8,10,12,14,16, and 18, and we get a magic square (see Fig. 3).

\[
\begin{array}{ccc}
  16 & 2 & 12 \\
  6 & 10 & 14 \\
  8 & 18 & 4 \\
\end{array}
\]

**Figure 3.** Integers with a differentiator 2 in \( 3 \times 3 \) magic-square.

or with the real numbers 0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8 and 0.9, and a magic square of them is Fig. 4.

\[
\begin{array}{ccc}
  0.8 & 0.1 & 0.6 \\
  0.3 & 0.5 & 0.7 \\
  0.4 & 0.9 & 0.2 \\
\end{array}
\]

**Figure 4.** Decimal numbers in \( 3 \times 3 \) magic-square.
or etc. Thus, numbers have a uniform pattern or a difference of up and down the same number for all entrain of a magic square,

$$e_k = k \times b$$

(5)

for $$k = 1, \ldots, n \times n$$, where $$b$$ is the difference between two adjacent numbers in a sequence.

Although the sum of entry numbers for each row, column, and main diagonal are the same. The arrangement of numbers on the first magic square reveals that for diagonal the numbers 1, 7 and 4 have a sum is 12, whereas the diagonal of the numbers 3, 9, and 6 have a sum is 18, on the other hand, the diagonal with the number 1, 3 and 2 have a sum 6 while for the diagonal of numbers 9, 7 and 8 have a sum is 24. Therefore, this sample example cancels the explication of the magic square T above.

Also, suppose $$n = 5$$, the size of the magic square is $$5 \times 5$$, for example by involving integers of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, and 25 the result is like Fig. 5.

![Figure 5. Positive integers in 5 × 5 magic-square.](image)

The sum of numbers on each row, column, and main diagonal is 65. However, the sum of numbers on the diagonals parallel to one of the main diagonals is also 65, whereas the sum of numbers on diagonals parallel to the other diagonal is not equal to 65.

Therefore, each magic square has a common and special characteristics. The common characteristics generally hold as axioms in definition, whereas the special characteristics as the conditions in assumptions or statements for theories.

### 3. An Approach

To express the definition of magic squares formally, we will classify the magic square characteristics into two parts [2]. That is generally applicable characteristics, or characteristics that apply specifically. This classification is based on the properties and order of numbers that fill the square-shaped array based on numerical characteristics [10].

To reveal the general characteristics of magic squares, we do a study of some cases of odd-sized magic squares, ranging from $$n = 3$$ to $$n = 15$$. This general characteristics is inductively formed and becomes the axioms that support the definition about magic square. While the special characteristics are derived will form the theory specifically about the properties of magic square [11, 12]. Excavation of special characteristics is disclosed on the basis of cases that inductively also form specific formulations.

### 4. An Investigation of Axioms

To extract the general axioms of the magic squares we re-present a sequence of the magic squares of $$n = 3, 5, 7, 9, 11, 13, 15$$ or odd only, like Figs. 1., 5., 6., 7., 8., 9., and Fig. 10. From the entire magic square described above that all satisfy Eqs. (1) to (9) so that a general definition of magic
squares can be formally expressed as follows.

\[
\begin{array}{ccccccccc}
30 & 39 & 48 & 1 & 10 & 19 & 28 \\
38 & 47 & 7 & 9 & 18 & 27 & 29 \\
46 & 6 & 8 & 17 & 26 & 35 & 37 \\
5 & 14 & 16 & 25 & 34 & 36 & 45 \\
13 & 15 & 24 & 33 & 42 & 44 & 4 \\
21 & 23 & 32 & 41 & 43 & 3 & 12 \\
22 & 31 & 40 & 49 & 2 & 11 & 20
\end{array}
\]

**Figure 6.** Positive integers in 7 \( \times \) 7 magic-square.

\[
\begin{array}{cccccccccccc}
47 & 58 & 69 & 80 & 1 & 12 & 23 & 34 & 45 \\
57 & 68 & 79 & 9 & 11 & 22 & 33 & 44 & 46 \\
67 & 78 & 8 & 10 & 21 & 32 & 43 & 54 & 56 \\
77 & 7 & 18 & 20 & 31 & 42 & 53 & 55 & 66 \\
6 & 17 & 19 & 30 & 41 & 52 & 63 & 65 & 76 \\
16 & 27 & 29 & 40 & 51 & 62 & 73 & 75 & 5 \\
26 & 28 & 39 & 50 & 61 & 72 & 74 & 4 & 15 \\
36 & 38 & 49 & 60 & 71 & 82 & 73 & 3 & 14 & 25 \\
37 & 48 & 59 & 70 & 81 & 2 & 13 & 24 & 35
\end{array}
\]

**Figure 7.** Positive integers in 9 \( \times \) 9 magic-square.

\[
\begin{array}{cccccccccccc}
68 & 81 & 94 & 107 & 120 & 1 & 14 & 27 & 40 & 53 & 66 \\
80 & 93 & 106 & 119 & 11 & 13 & 26 & 39 & 52 & 65 & 67 \\
92 & 105 & 118 & 10 & 12 & 25 & 38 & 51 & 64 & 77 & 79 \\
104 & 117 & 9 & 22 & 24 & 37 & 50 & 63 & 76 & 78 & 91 \\
116 & 8 & 21 & 23 & 36 & 49 & 62 & 75 & 88 & 90 & 103 \\
7 & 20 & 33 & 35 & 48 & 61 & 74 & 87 & 89 & 102 & 115 \\
19 & 32 & 34 & 47 & 60 & 73 & 86 & 99 & 101 & 114 & 6 \\
31 & 44 & 46 & 59 & 72 & 85 & 98 & 100 & 113 & 5 & 18 \\
43 & 45 & 58 & 71 & 84 & 97 & 110 & 112 & 4 & 17 & 30 \\
55 & 57 & 70 & 83 & 96 & 109 & 111 & 3 & 16 & 29 & 42 \\
56 & 69 & 82 & 95 & 108 & 121 & 2 & 15 & 28 & 41 & 54
\end{array}
\]

**Figure 8.** Positive integers in 11 \( \times \) 11 magic-square.

**Definition 1.** Let \( e_k \) is a sequence of numbers on pattern Eq. (4) \( e_k = k \ast b \), \( b \) is difference between two adjacent numbers and \( k = 1, \ldots, m \), and let \( a_{ij} \) is an array where \( i, j = 1, \ldots, n \) such that \( m = n \times n, n \mod 2 = 1 \). An odd magic square is \( a_{ij} \), we notated by \( [a_{ij}] \), with a unary-operation \( f \)

\[
\sum a_{ij} = f(e_k), \tag{6}
\]

and

\[
\sum r_i = \sum_{i,j=1}^{i,n} a_{ij}, i = 1, \ldots, n \tag{7}
\]

and

\[
\sum c_i = \sum_{i=1,j}^{n,j} a_{ij}, j = 1, \ldots, n \tag{8}
\]
and

\[ \Sigma d_l = \sum_{i=1, j=i}^{n, n} a_{ij} \]  

(9)

and

\[ \Sigma d_r = \sum_{i=1, j=n}^{n, 1} a_{ij} \]  

(10)

so that \( \Sigma r_1 = \ldots = \Sigma r_n = \Sigma c_1 = \ldots = \Sigma c_n = \Sigma d_l = \Sigma d_r \).

![Figure 9. Positive integers in 13 x 13 magic-square.](image)

![Figure 10. Positive integers in 15 x 15 magic-square.](image)

In particular it is found that there is one main diagonal having the same number of sequences: Fig. 1 revealing the numbers 4, 5, 6 filling the main diagonal for the 3 x 3 magic square; Fig. 5 illustrates the order of numbers 11, 12, 14, 14, and 15 in the main diagonal of the 5 x 5 magic square; Fig. 6 reveals that the order of numbers in the main diagonal of 7 x 7 magic square are 22, 23, 24, 25, 26, 27, 28; and etc. This characteristic applies specifically, but cannot be taken as a general axiom. For example for 5 x 5 magic square (Fig. 5) because if two left columns are
removed and alternate with two in the right column and vice versa, a different arrangement will be obtained, whereas Eq. (1) to Eq. (4) remain valid, see Fig. 11.

\[ \begin{array}{cccc}
8 & 15 & 1 & 17 \\
14 & 16 & 7 & 23 \\
20 & 22 & 13 & 4 \\
21 & 3 & 19 & 10 \\
2 & 9 & 25 & 11 \\
\end{array} \]

\textbf{Figure 11.} Positive integers in other $5 \times 5$ magic-square.

Therefore, there is a general presumption of characteristic as follows

\textbf{Assumption 1.} If $[a_{ij}]$ is an odd magic square where $i, j = 1, \ldots, n$ and $n \mod 2 = 1$, then there is one main diagonal $d_r$ with a same difference $b$.

Under the Assumption 1, the other main diagonal $d_l$ does not have the same difference $b$.

Another diagonal $d_{(l)}$ and $d_{(r)}$, or not main diagonal $d_r$ and $d_l$, parallel to $d_l$ and $d_r$ have different characteristics, respectively. All diagonals $d_{(r)}$ not only have different results, but their characteristics depend on $n$. Thus there are characteristics that are not generally applicable, i.e.

\textbf{Assumption 2.} If $[a_{ij}]$ is an odd magic square where $i, j = 1, \ldots, n$, then all diagonals $d_{(r)}$ have the sum of $n$ numbers $\Sigma d_{(r)}$ equal to $\Sigma d_r$, $n = 5, 7, 11, 13, \ldots$, or

\[ \Sigma d_{(r)} = \Sigma d_r; \ell = 1, 2, \ldots, (n-1); n = 5, 7, 11, 13, \ldots \]  

(11)

Under the Assumption 2, for $n = 3, 9, 15, \ldots$ in general the $\Sigma d_{(r)\ell}$, $\ell = 1, 2, \ldots, (n-1)$ do not equal to $\Sigma d_r$. For example, the sum result of

\[ \begin{array}{ccc}
6 & 1 & 8 \\
3 & \neq & 7 \\
9 & 4 & 2 \\
\end{array} \]

$\Sigma d_{(r)1} = 18 \neq \Sigma d_{(r)2} = 12 \neq \Sigma d_{r} = 15$, but $(\Sigma d_{(r)1} + \Sigma d_{(r)2})/2 = \Sigma d_r$. While for $n = 9$ and $n = 15$, we have different results of sum about numbers in diagonals. For $n = 9$ from $d_r$ to right-up (left-down) we have

\[ \begin{array}{c}
\Sigma d_{(r)1} = 57 + 78 + 18 + 30 + 51 + 72 + 3 + 24 + 45 = 378 \\
\Sigma d_{(r)2} = 67 + 7 + 19 + 40 + 61 + 73 + 13 + 34 + 46 = 360 \\
\Sigma d_{(r)3} = 77 + 17 + 29 + 50 + 71 + 2 + 23 + 44 + 56 = 369 \\
\end{array} \]

so $\Sigma d_{(r)4} = 378$, $\Sigma d_{(r)5} = 360$, $\Sigma d_{(r)6} = 369$, $\Sigma d_{(r)7} = 378$, and $\Sigma d_{(r)8} = 360$. We have one characteristic for $\ell = 3$ and $\ell = 6$ that is $\Sigma d_{(r)3} = \Sigma d_r$. For $n = 15$ from $d_r$ to right-up (left-down) we have

\[ \begin{array}{c}
\Sigma d_{(r)1} = 138 + 171 + 204 + 12 + 45 + 63 + 96 + 129 + 162 + 195 + 213 + 21 + 54 + 87 + 120 \\
= 1710 \\
\Sigma d_{(r)2} = 154 + 187 + 220 + 28 + 46 + 79 + 112 + 145 + 178 + 196 + 4 + 37 + 70 + 103 + 121 \\
= 1680 \\
\Sigma d_{(r)3} = 170 + 203 + 11 + 44 + 62 + 95 + 128 + 161 + 194 + 212 + 20 + 53 + 86 + 119 + 137 \\
= 1695 \\
\end{array} \]

in pattern similar to them are $\Sigma d_{(r)4} = \Sigma d_{(r)7} = \Sigma d_{(r)10} = \Sigma d_{(r)13} = 1710$, $\Sigma d_{(r)5} = \Sigma d_{(r)8} = \Sigma d_{(r)11} = \Sigma d_{(r)14} = 1680$, whereas $\Sigma d_{(r)6} = \Sigma d_{(r)9} = \Sigma d_{(r)12} = 1695$. Thus, for $\ell = 3, 6, 9, 12$ we have $\Sigma d_{(r)\ell} = \Sigma d_r$. Or
Assumption 3. If \([a_{ij}]\) is an odd magic square where \(i, j = 1, \ldots, n\), then all diagonals \(d_{(r)}\), \(\ell = 3, 6, 9, 12, \ldots\), have the sum of \(n\) numbers \(\Sigma d_{(r)}\) equal to \(\Sigma d_r\), \(n = 9, 15, \ldots\), or

\[
\Sigma d_{(r)} = \Sigma d_r; \ell = 3, 6, 9, \ldots, (n - 1); n = 9, 15, \ldots
\]  

(12)

Based on cases in Figs. 1, 5, 6, 7, 8, 9, all diagonals parallel to \(d_l\) have different sums of numbers in diagonals and does not equal to \(\Sigma d_l\) (sum of numbers in main diagonal), i.e.

Assumption 4. If \([a_{ij}]\) is an odd magic square where \(i, j = 1, \ldots, n\), then all diagonals \(d_{(l)}\) have the sum of \(n\) numbers \(\Sigma d_{(l)}\) does not equal to \(\Sigma d_l\), \(n = 3, 5, \ldots\), or

\[
\Sigma d_{(l)} = \Sigma d_l, n = 3, 5, 7, 11, 13, \ldots
\]  

(13)

In some cases, we have formulated some of the assumptions underlying the redefinition of the magic squares.

5. Conclusion

Some characteristics of the magic square have been inductively expressed, and redefined the intent of the term magic square formally, albeit based only on odd magic squares. In some cases some assumptions are derived inductively, but theoretically may be generally applicable, so there needs to be proof. Therefore, the future work will be to produce the characteristic of the even magic square, then to math the assumptions into a generally proven statement.

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