PROJECTIONS OF PLANAR SETS IN WELL-SEPARATED DIRECTIONS

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ABSTRACT. This paper contains two new projection theorems in the plane. First, let $K \subset B(0, 1) \subset \mathbb{R}^2$ be a set with $H_\infty^1(K) \sim 1$, and write $\pi_e(K)$ for the orthogonal projection of $K$ into the line spanned by $e \in S^1$. For $1/2 \leq s < 1$, write

$$E_s := \{e : N(\pi_e(K), \delta) \leq \delta^{-s}\},$$

where $N(A, r)$ is the $r$-covering number of the set $A$. It is well-known – and essentially due to R. Kaufman – that $N(E_s, \delta) \preceq \delta^{-s}$. Using the polynomial method, I prove that

$$N(E_s, r) \preceq \min\left\{\delta^{-s} \left(\frac{\delta}{r}\right)^{1/2}, r^{-1}\right\}, \quad \delta \leq r \leq 1.$$

I also construct a family of examples showing that the exponents in the bound are sharp.

The second theorem concerns projections of 1-Ahlfors-David regular sets. Let $A \geq 1$ and $1/2 \leq s < 1$ be given. I prove that, for $p = p(A, s) \in \mathbb{N}$ large enough, the finite set of unit vectors $S_p := \{e^{2\pi ik/p} : 0 \leq k < p\}$ has the following property. If $K \subset B(0, 1)$ is non-empty and 1-Ahlfors-David regular with regularity constant at most $A$, then

$$\frac{1}{p} \sum_{e \in S_p} N(\pi_e(K), \delta) \geq \delta^{-s}$$

for all small enough $\delta > 0$. Thus, if $K$ is compact, $\dim_p \pi_e(K) \geq s$ for some $e \in S_p$.

1. INTRODUCTION

Let $K \subset B(0, 1) \subset \mathbb{R}^2$ be a compact set with $H_\infty^1(K) \sim 1$. For $0 \leq s \leq 1$, a classical result of R. Kaufman [8], sharpening the projection theorem of Marstrand [10], states that

$$\dim\{e \in S^1 : \dim \pi_e(K) \leq s\} \leq s,$$

(1.1)

where $\pi_e$ denotes orthogonal projection onto $\text{span}(e)$ and $\dim$ is Hausdorff dimension. It seems unlikely that this bound is sharp for $s < 1$. It is conjectured (see [12, (1.8)]) that the correct bound is $2s - 1$ instead of $s$, and [12, Theorem 1.2] shows that $\dim\{e : \dim \pi_e(K) < 1/2\} = 0$. A stronger, and significantly harder to prove, improvement is due to Bourgain [1]: a (non-trivial) application of his "discretised sum-product theorem" shows that the left hand side of (1.1) tends to zero as $s \searrow 1/2$. However, even Bourgain’s method of proof only gives an improvement to (1.1) when $s$ is "very close" to $1/2$. So, for example, nothing better than (1.1) is currently known for $s = 3/4$.

The starting point of this paper was to investigate the case where $s$ is far away from $1/2$. In trying to prove statements about Hausdorff dimension, such as (1.1), a natural intermediary step is to find and solve a "$\delta$-discretised" analogue of the problem. In the

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current situation, the simplest such analogue is probably the following: fix \( \delta > 0 \), and let \( E_s \) be the collection of vectors in \( S^1 \) such that \( \pi_e(K) \) can be covered by \( \leq \delta^{-s} \) intervals of length \( \delta \). In symbols,

\[
E_s := \{ e \in S^1 : N(\pi_e(K), \delta) \leq \delta^{-s} \}.
\]

How many \( \delta \)-intervals does it take to cover \( E_s \)? An argument close to Kaufman’s proof of (1.1) shows that

\[
N(E_s, \delta) \lesssim \delta^{-s},
\]

where \( A \lesssim B \) stands for \( A \leq C \log(1/\delta) B \) for some absolute constant \( C \). A significant difference between (1.1) and this \( \delta \)-discretised bound is, however, that the latter is sharp, and almost trivially so: one needs only take \( K \) to be a horizontal unit line segment, and consider its projections (at scale \( \delta \)) in nearly vertical directions. It is worth emphasising that the bound (1.2) is even sharp for \( s = 1/2 \).

At first sight, the observation above might seem fatal to the \( \delta \)-discretised approach of improving (1.1), but this need not be the case. One just needs to modify the discretised question slightly. The line segment is certainly not the only example for which (1.2) is improving (1.1), but this need not be the case. One just needs to modify the discretised bound is, however, that the latter is sharp, but all the configurations appear to have one feature in common: the associated \( \approx \delta^{-s} \) directions in \( E_s \) are very clustered. In the case of the horizontal line segment, for instance, they all lie packed around the vertical direction.

Now, a reasonable conjecture could be the following: if \( E \) is any collection of \( \sim \delta^{-s} \) vectors, which are "quantitatively not packed together", then \( E \) contains a vector \( e \) with \( N(\pi_e(K), \delta) \geq \delta^{-s-\epsilon} \). Here is one possible precise formulation:

**Conjecture 1.3.** Assume that \( K \subset B(0,1) \subset \mathbb{R}^2 \) is a set with \( \mathcal{H}^1_{\infty}(K) \sim 1 \). Let \( E \subset S^1 \) be any \( \delta \)-separated set of directions with cardinality \( |E| \sim \delta^{-s} \), satisfying the non-concentration hypothesis

\[
|E \cap B(x, t)| \lesssim t^\kappa |E|, \quad x \in S^1, \ t \geq \delta,
\]

for some \( \kappa > 0 \). Then \( N(\pi_e(K), \delta) \geq \delta^{-s-\epsilon} \) for some \( e \in E \), where \( \epsilon > 0 \) is a constant depending only on \( \kappa, s \).

The conjecture is true, and due to Bourgain, if \( s \) is sufficiently close to \( 1/2 \); in this case, one can also drop the a priori assumption \( |E| \sim \delta^{-s} \), because (1.4) alone guarantees that \( E \) contains enough directions, see [1, Theorem 3]. Progress in Conjecture 1.3 for a certain \( s \in (1/2, 1) \) would, most likely, lead to an improvement for the Hausdorff dimension estimate (1.1) for the same \( s \).

The first main result of the present paper is a variant of the conjecture, where the non-concentration hypothesis (1.4) is replaced by the requirement that the vectors in \( E \) be \( r \)-separated for some \( \delta \leq r \leq 1 \):

**Theorem 1.5.** Let \( K \subset B(0,1) \subset \mathbb{R}^2 \) be a compact set with \( \mathcal{H}^1_{\infty}(K) \gtrsim 1 \), and let \( 1/2 \leq s < 1 \) and \( \delta \leq r \leq 1 \). Then

\[
N(E_s, r) \lesssim \min \left\{ \delta^{-s} \left( \frac{\delta}{r} \right)^{1/2}, \frac{1}{r} \right\}.
\]

The exponents in the bound are sharp.

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\(^1\)Any union \( K \) of \( \delta^{-s} \) vertical line segments of length \( \delta^s \) works, as long as \( \mathcal{H}^1_{\infty}(K) \sim 1 \).
Remark 1.6. An equivalent formulation of Theorem 1.5 – more reminiscent of Conjecture 1.3 – is the following: if \(|E| \sim \delta^{-s}\), and the separation between the vectors in \(E\) is at least \(r \geq \delta\), then \(N(\pi_e(K), \delta) \gtrsim \delta^{-s}(r/\delta)^{1/2}\) for some \(e \in E\). Assuming that \(\delta \leq r < \delta^{s+\epsilon}\), a set \(E\) satisfying these hypotheses can be found inside an arc of length \(\sim \delta^s\), and such an \(E\) naturally cannot satisfy the non-concentration hypothesis (1.4) with \(t = \delta^s\). So, in fact, the separation assumption in Theorem 1.5 is neither weaker nor stronger than (1.4), and in particular Theorem 1.5 gives some new information even in the \(s\) is close to \(1/2\) regime, which does not follow from [1, Theorem 3]. The proof of Theorem 1.5 is based on the “polynomial method” developed by Dvir, Guth and Katz, and I do not know how – or if – this technique can be combined with the non-concentration hypothesis (1.4).

The case \(s < 1/2\) is systematically ignored in this paper, because the corresponding results in that range are quite straightforward.

The second main result, Theorem 6.2 below, is directly motivated by Bourgain’s proof of [1, Theorem 3] (which is essentially Conjecture 1.3 for \(s\) close enough to \(1/2\), and without the assumption \(|E_s| \sim \delta^{-s}\)). Here is a prestissimo explanation of some parts of [1]. Assuming that the result fails, one can, for arbitrarily small \(e, \delta > 0\), find a set \(K\) as in Conjecture 1.3, and three vectors \(e_1, e_2, e_3 \in S^1\) with separation \(\sim 1\), such that \(N(\pi_{e_i}(K), \delta) \leq \delta^{-1/2-\epsilon}\) for \(i \in \{1, 2, 3\}\). Then, this counter-assumption is used to extract strong structural information about \(K\): in particular, \(K\) is quantitatively not \(1\)-Ahlfors-David regular. In the second part of the proof of [1, Theorem 3], the structural information is applied to show that \(K\) must, after all, have plenty of reasonably big projections.

A major (but not the only) obstacle in applying Bourgain’s method to Conjecture 1.3 is that the same structural conclusions cease to hold, if one replaces the assumption

\[
\max N(\pi_e(K), \delta) \leq \delta^{-1/2-\epsilon}
\]

by

\[
\max N(\pi_e(K), \delta) \leq \delta^{-s}
\]

for some \(s < 1\), possibly very close to 1. Indeed, the 1-dimensional four corners Cantor set \(K\) is \(1\)-Ahlfors-David regular with very modest constants, yet it has three well-separated projections \(\pi_e(K)\) (vertical, horizontal and \(45^\circ\)) such that \(N(\pi_{e_i}(K), \delta) \lesssim \delta^{-s}\) with \(s = \log 3/\log 4 < 1\).

So, three directions are not enough, but how about a million? More precisely: fix \(s < 1\), and assume that \(N(\pi_e(K), \delta) \leq \delta^{-s}\) for, say, \(p(s) \in \mathbb{N}\) well-separated vectors \(e \in S^1\). Is it, then, true that \(K\) cannot be \(1\)-Ahlfors-David regular with bounded constants? A positive answer to this question is the content of the second main theorem:

**Theorem 1.7.** Given \(1/2 \leq s < 1\) and \(A > 0\), there are numbers \(p = p(s, A) \in \mathbb{N}\) and \(\delta(A, s) > 0\) with the following property. Let

\[
S_p := \{e^{2\pi ik/p} : 0 \leq k < p\} \subset S^1,
\]

and let \(\emptyset \neq K \subset B(0, 1)\) be a \(1\)-Ahlfors-David regular set with regularity constant at most \(A\). Then

\[
\frac{1}{p} \sum_{p \in S_p} N(\pi_e(K), \delta) \geq \delta^{-s}, \quad 0 < \delta \leq \delta(A, s).
\]

**Corollary 1.8.** With the notation of Theorem 6.2, and assuming that \(K\) is compact,

\[
\dim_p K \geq s
\]
for some \( e \in S_p \), where \( \dim_p \) stands for packing dimension.

Remark 1.9. The precise form of the vectors in \( S_p \) is not important for the argument: it is only needed that the difference

\[
\left| \frac{1}{|S_p|} \sum_{e \in S_p} f(e) - \int_{S^1} f(\xi) \, d\sigma(\xi) \right|
\]

can be made arbitrarily small for functions \( f \) on \( S^1 \) with a reasonable modulus of continuity, depending on \( A \) and \( s \). The precise form of \( S_p \) is quite convenient, however, and so I chose not to pursue maximal generality in this matter.

Another point is that there is no analogue of Corollary 1.8 for Hausdorff dimension. Indeed, given any countable collection of vectors \( E \subset S^1 \), it is straightforward to construct a \( 1 \)-Ahlfors-David regular set such that \( \dim_\pi(e)(K) = 0 \) for all \( e \in E \) (and indeed for all \( e \in G \), where \( G \supset E \) is a suitable \( G_\delta \)-set). For the details, see [13, Theorem 1.5].

It is a somewhat less trivial question, whether Ahlfors-David regularity is, in fact, necessary for Theorem 6.2 and Corollary 1.8. For instance: given \( s < 1 \), is it possible to find a finite set \( E_s \subset S^1 \) such that \( \overline{\dim}_B \pi_s(K) \geq s \) for some \( e \in E_s \), whenever \( K \subset B(0, 1) \) is a compact set with \( \mathcal{H}^s_{\infty}(K) \sim 1 ? \) Most likely, the answer is negative. Given any \( \epsilon > 0 \) and any finite set \( D \subset \mathbb{R} \), an example of B. Green – which appears as [9, Remark 2] – can be modified to produce a finite set \( A \subset \mathbb{R} \) with the property that \( |A + tA| \leq |A|^{1+\epsilon} \) for all \( t \in D \). Then, it seems probable that a self-similar construction with \( |A|^2 \) homotheties (mapping 0 to the points in \( A \times A \), with contraction ratios \( 1/|A|^2 \)) produces a set \( K \subset \mathbb{R}^2 \) with \( 0 < \mathcal{H}^1(K) < 1 \), or at least \( \dim K = 1 \), such that \( \overline{\dim}_B \pi_1(K) \leq 1/2 + \epsilon \) for \( t \in D \).

The rest of the paper is organised as follows. Section 3 discusses the sharpness of the bound in Theorem 1.5. Section 4 reviews some basic concepts used in the proof of Theorem 1.5, and gives a quick – and well-known – argument for the discrete Kaufman bound (1.2). The proof of Theorem 1.5 is given in Section 5, and Section 6 contains the proof of Theorem 6.2.

Some notational remarks: \( B(x, r) \) stands for a closed ball of radius \( r > 0 \) and centre \( x \in \mathbb{R}^2 \). The side-length of a cube \( Q \subset \mathbb{R}^d \) is denoted by \( \ell(Q) \). The inequality \( A \lesssim B \) means that \( A \leq C B \) for an absolute constant \( C > 0 \); the two-sided inequality \( A \lesssim B \lesssim A \) is abbreviated to \( A \sim B \). As mentioned above, \( A \lesssim B \) means that \( A \lesssim \log(1/\delta) B \). The Hausdorff measure of dimension \( s \) is denoted by \( \mathcal{H}^s \), and Hausdorff content by \( \mathcal{H}_{\infty}^s \). Thus

\[
\mathcal{H}_{\infty}^s(A) := \inf \left\{ \sum_{i=1}^{\infty} \text{diam}(E_i)^s : A \subset \bigcup_{i=1}^{\infty} E_i \right\}.
\]

For information about Hausdorff dimension or measures, packing dimension, or any other geometric measure theoretic concept in the text, see [11].

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3. SOME WORST-CASE EXAMPLES

Fix $\delta > 0$, $1/2 \leq s < 1$ and $\delta \leq r \leq 1$, and choose $\tau \in [0, 1]$ so that $r = \delta^\tau$ (this clears up notation later on). The worst case of Theorem 1.5 appears in a constellation of vertical line segments of length $\delta^{s+(1-\tau)/2}$ in a "squashed" grid formation, depicted in Figure 1. The precise definition is the following. Let $G$ be the grid $G = \left\{ \frac{1}{\sqrt{n}}, \frac{2}{\sqrt{n}}, \ldots, 1 \right\} \times \left\{ \frac{1}{\sqrt{n}}, \frac{2}{\sqrt{n}}, \ldots, 1 \right\}$ with $n = |G| = \delta^{s-(1-\tau)/2}$. Then, let $A$ be the linear mapping $A(x, y) := (ax, y)$, where $a = \delta^{-s+(1+\tau)/2}$. Finally, let $K_0 := A(G)$, and let $K$ be a union of vertical line segments of length $\delta^{s+(1-\tau)/2}$ centred at the points in $K_0$. One can check that $K$ satisfies $H_\infty^1(K) \sim 1$ for any $\delta > 0$ and $0 \leq \tau, s \leq 1$. The following computations show that $N(E_s, \delta^\tau) \gtrsim \delta^{-s-(1-\tau)/2}$.

First of all, in this section, it is convenient to identify orthogonal projections with the family of mappings $\pi_t(x, y) := x + ty$, so the task is to find $\gtrsim \delta^{-s+(1-\tau)/2}$ numbers $t$, which are $\delta^\tau$-separated and such that $N(\pi_t(K), \delta) \leq \delta^{-s}$ for these $t$. Observe that $\pi_t(A(x, y)) = \pi_t(ax, y) = a \left( x + \frac{t}{a}y \right) = a\pi_{t/a}(x, y), \quad t > 0,$ so $\pi_t(K_0) = a\pi_{t/a}(G)$. It is an easy exercise to check that for any $\kappa \in [1/2, 1)$, there are $\sim n^{2\kappa-1}$ points $t \in (0, 1)$, which are $n^{1-2\kappa}$-separated and such that $|\pi_t(G)| \leq |G|^\kappa = n^\kappa$. Apply this fact with $\kappa := \frac{2s}{2s + 1 - \tau},$
and let \( \{t_1, t_2, \ldots, t_N\} \) be the set of points in \((0, 1)\) such that \(|\pi_{t_i}(G)| \leq n^\kappa\). Thus
\[
N \sim n^{2\kappa - 1} = (\delta^{-s-(1-r)/2})^{4s/(2s+1-r) - 1} = \delta^{-s-(1-r)/2},
\]
and \(|t_i - t_j| \geq n^{1-2\kappa}\) for \(i \neq j\). Next, let \(E := \{at_1, \ldots, at_N\}\), so that
\[
|\pi_{at_i}(K_0)| = |a\pi_{t_i}(G)| = |\pi_{t_i}(G)| \leq n^\kappa = (\delta^{-s-(1-r)/2})^{2s/(2s+1-r)} = \delta^{-s}
\]
for all \(at_i \in E\). Furthermore, the distance between distinct points \(at_i \in E\) is at least
\[
an^{1-2\kappa} = \delta^{-s+(1+r)/2} \cdot (\delta^{-s-(1-r)/2})^{1-4s/(2s+1-r)} = \delta^r.
\]
So, it suffices to show that
\[
N(\pi_{at_i}(K), \delta) = N(\pi_{at_i}(K_0), \delta), \quad at_i \in E.
\] (3.1)
To this end, note that \(at_i < a = \delta^{-s+(1+r)/2}\), and recall that \(K\) was a union of vertical line segments of the form \(\{x_0\} \times I\), where \(\ell(I) = \delta^{s+(1-r)/2}\), centred at points in \(K_0\). Now, (3.1) follows immediately from
\[
\ell(\pi_{at_i}(\{x_0\} \times I)) < \delta,
\]
and the construction is complete.

4. Basic Concepts and Kaufman’s Bound

As in the previous section, fix the parameters \(\delta, r, s, \) and choose \(r \in [0, 1]\) so that \(r = \delta^r\).

The task is to estimate \(N(E_s, r) = N(E_s, \delta^r)\) from above, which is equivalent to bounding the cardinality of a maximal \(\delta^r\)-separated subset of \(E_s\) from above. With this in mind, and from this point on, assume that \(E_s\) is a \(\delta^r\)-separated subset of \(\{e \in S^1 : N(\pi_e(K), \delta) \leq \delta^s\}\).

It is also convenient to discretise the set \(K\) at the scale \(\delta\). The following definition is due to Katz and Tao [7]:

**Definition 4.1** ((\(\delta, 1\))-sets). A finite set \(P \subset \mathbb{R}^2\) is called a \((\delta, 1)\)-set, if \(P\) is \(\delta\)-separated, and
\[
|P \cap B(x, r)| \lesssim \frac{r}{\delta^3}, \quad x \in \mathbb{R}^2, \quad r \geq \delta.
\]
Here \(| \cdot |\) means cardinality.

**Lemma 4.2.** Let \(\delta > 0\), and let \(K \subset \mathbb{R}^2\) be a set with \(H^1_{\infty}(K) =: \kappa > 0\). Then, there exists a \((\delta, 1)\)-set \(P \subset K\) with \(|P| \gtrsim \kappa \cdot \delta^{-1}\).

**Proof.** Choose a \(\delta\)-net inside \(K\) and discard surplus points. For more details, see [2, Proposition A.1].

**Definition 4.3** (Incidences). Let \(T\) be a family of infinite tubes of width \(\delta\), and let \(P \subset \mathbb{R}^2\) be a finite set of points. The set of incidences \(I(P, T)\) between \(P\) and \(T\) is the following family of pairs:
\[
I(P, T) := \{(p, T) : p \in P, \ T \in T \text{ and } p \in T\}.
\]

The definition will be applied to subsets of set \(P\) from Lemma 4.2, and subsets of the following family \(T\) of tubes:
**Definition 4.4** (Tubes $T$). Let $P \subset K$. For each $e \in E_s$, cover $\pi_e(P)$ by $\leq \delta^{-s}$ intervals $I$ of length $\delta$ and bounded overlap (this is possible since $N(\pi_e(P), \delta) \leq N(\pi_e(K), \delta) \leq \delta^{-s}$), and let $T_e$ be the family of $\delta$-tubes of the form $\pi_e^{-1}(I)$. Finally, let

$$T := \bigcup_{e \in E_s} T_e.$$

The basic strategy in the proofs will be to bound $|I(P, T)|$ from above and below. The lower bound is trivial:

**Lemma 4.5.** Let $P$ be an arbitrary finite set in $\mathbb{R}^2$, and construct $T$ as in Definition 4.4. Then

$$|I(P, T)| \geq |P||E_s|.$$

**Proof.** Each point $p \in P$ is contained in at least one tube from each family $T_e, e \in E_s$. □

Kaufman’s $\delta^{-s}$-bound (1.2) will follow from comparing the previous bound with the one provided by the next proposition.

**Proposition 4.6.** Assume that $P \subset K$ is a $(\delta, 1)$-set, and $T$ is the collection of tubes from Definition 4.4, associated with $P$. Then

$$|I(P, T')| \lesssim |P||T'|^{1/2} + |T'| + \sqrt{\delta^{-s}}|P||T'|$$

for any subset $T' \subset T$.

**Proof.** Using the definition of $I(P, T')$ and Cauchy-Schwarz,

$$|I(P, T')| = \sum_{T \in T'} |\{(p, q) : p, q \in P \cap T\}| \leq |T'|^{1/2} \left(\sum_{T \in T'} |\{(p, q) : p, q \in P \cap T\}|\right)^{1/2}. \quad (4.7)$$

It remains to estimate the sum on the right hand side:

$$\sum_{T \in T'} |\{(p, q) : p, q \in P \cap T\}| = \sum_{p, q \in P} |\{T \in T' : p, q \in P \cap T\}|$$

$$= \sum_{p \in P} |\{T \in T' : p \in P \cap T\}| + \sum_{p \neq q} |\{T \in T' : p, q \in P \cap T\}|.$$

The first sum equals $|I(P, T')|$ again, which gives rise to the $|T'|$-term in (4.7). To estimate the second sum, one uses the finite overlap of the tubes in any fixed family $T_e$ to estimate

$$\sum_{p \neq q} |\{T \in T' : p, q \in P \cap T\}| \lesssim \sum_{p \neq q} |\{e \in E_s : p, q \in T\}|.$$

At this point, one applies the standards geometric fact that the set of vectors $e \in S^1$ such that $p, q$ can share a common $\delta$-tube in $T_e$ is contained in two arcs of length $\lesssim \delta/|p - q|$. Since the vectors in $E_s$ are $\delta^{-s}$-separated, this leads to

$$|\{e \in E_s : p, q \in T\}| \lesssim \max \left\{ \frac{\delta^{1-s}}{|p - q|}, 1 \right\}. \quad (4.8)$$

Observe that the "1" is really needed here, because if $|p - q|$ is far greater than $\delta^{1-s}$, the arcs mentioned above have length far smaller than $\delta^s$, but it is still perfectly possible for
one $\delta^r$-separated vector to land in any such arc. The bound (4.8) leads to
\[
\sum_{p \neq q} |\{e \in E_s : p, q \in T \text{ for some } T \in T_e\}| \lesssim \sum_{p \neq q} \max \left\{ \frac{\delta^{1 - r}}{|p - q|}, 1 \right\}
\]
\[
= \sum_{p \in P} \left( \sum_{q : |p - q| \leq \delta^{1 - r}} \frac{\delta^{1 - r}}{|p - q|} + \sum_{q : |p - q| > \delta^{1 - r}} 1 \right)
\]
\[
\lesssim \sum_{p \in P} \left( \delta^{-r} \log \left( \frac{1}{\delta} \right) + |P| \right) \lesssim \delta^{-r} |P| + |P|^2.
\]

The inequality between the last two lines was obtained by splitting $P$ around $p$ in annuli of radius $\sim 2^{-j}$, $0 \leq j \leq \log(1/\delta)$, and using the $(\delta, 1)$-set hypothesis. Moving terms completes the proof. \hfill \Box

To prove Kaufman’s $\delta^{-s}$-bound (1.2) (or the $r = \delta$ case of Theorem 1.5), one uses Lemma 4.2 to find a $(\delta, 1)$-set $P \subset K$ with $|P| \sim \delta^{-1}$. Then, the lower and upper bounds of Lemma 4.5 and Proposition 4.6 (with $T' = T$) combined yield
\[
\delta^{-1} |E_s| \lesssim |P||E_s| \leq |I(P, T)| \lesssim \delta^{-1} |T|^{1/2} + |T|.
\]
Since $|T_e| \leq \delta^{-s}$, this gives
\[
|E_s| \lesssim (\delta^{-s}|E_s|)^{1/2} + |E_s|\delta^{1-s}.
\]
Given that $s < 1$, the term $|E_s|\delta^{1-s}$ cannot dominate the left hand side, and the proof is finished by taking squares and moving terms.

In the proof of Theorem 1.5, one has to make more efficient use of Proposition 4.6: the key point is that it gives a reasonably good bound for $|I(P, T)|$, when $|P| \approx \delta^{-r}$ – which is crucially better than the best possible bound obtainable with mere $\delta$-separation. So, the strategy will be to use an algebraic variety – a zero-set of a polynomial in two variables – to partition $P$ into chunks of approximately this size, and then control the incidences in each chunk separately. As is common with such a cell-decomposition argument, one also has to handle separately the case where most of $P$ is concentrated in the $\delta$-neighbourhood of the said variety.

5. PROOF OF THE FIRST MAIN THEOREM

A central tool is the polynomial cell decomposition theorem of Guth and Katz, see [4, Theorem 4.1], which is quoted below:

**Theorem 5.1 (Guth-Katz).** Let $P \subset \mathbb{R}^2$ be a finite set of points, and let $D \geq 1$ be an integer. Then, there exists a polynomial surface $Z$ of degree $\deg(Z) \leq D$ with the following property: the complement $\mathbb{R}^2 \setminus Z$ is the union of $\leq D^2$ open cells $O_i$, and each cell contains $\leq |P|/D^2$ points of $P$.

To begin the proof of Theorem 1.5 in earnest, apply the partitioning theorem with the $(\delta, 1)$-set $P \subset K$ of cardinality $|P| \sim \delta^{-1}$, obtained from Lemma 4.2, and with some large integer $D \geq 1$ to be optimised later. Let $Z$ be the ensuing polynomial surface of degree $\leq D$, and let $O_i$, $1 \leq i \leq N \leq D^2$ be the components of the complement $\mathbb{R}^2 \setminus Z$. Finally, let
\[
O_i := \tilde{O}_i \setminus Z(\delta),
\]
where \( Z(\delta) := \{ x : \text{dist}(x, Z) \leq \delta \} \) is the closed \( \delta \)-neighbourhood of \( Z \). The reason for defining the cells \( O_i \), so is the following simple consequence of Bézout’s theorem (first observed in [3]):

**Lemma 5.2.** Let \( T \) be an infinite tube of width \( \delta \). Then \( T \) can intersect at most \( D + 1 \) cells \( O_i \).

**Proof.** Let \( L_T \) be the central line of \( T \). For every \( i \) such that \( T \cap O_i \neq \emptyset \), one has \( L_T \cap \hat{O}_i \neq \emptyset \), and this is only possible for \( \leq D + 1 \) values of \( i \); namely, if there were \( D + 2 \) values or more, then \( L_T \) would contain at least \( D + 1 \) points on the polynomial surface \( Z \), and by Bézout’s theorem, this would force \( L_T \) to be contained on \( Z \). Consequently, \( T \) would be contained in the \( \delta/2 \)-neighbourhood of \( Z \) and could not, in fact, touch any of the cells \( O_i \). \( \square \)

The proof of Theorem 1.5 now divides into two main cases, according to whether or not most of the points in \( P \) are contained in the union of the cells \( O_i \). The argument in the first, “cellular” case closely resembles a (by now) standard proof of the Szemerédi-Trotter incidence theorem, while the “non-cellular” situation arguably requires more case-specific reasoning. As a final remark, the proof of Theorem 1.5 would be shorter and require no polynomials, if the set \( P \) had a product form, say \( P = A \times A \), to begin with. Then one could perform the cell-decomposition by hand using two perpendicular families of straight lines, and the “non-cellular” case could not even occur.

\[ \text{Figure 2. The cellular and non-cellular cases} \]

5.1. **The cellular case.** In this subsection, assume that \(|\hat{P}| \geq |P|/2 \sim \delta^{-1}\), where

\[ \hat{P} := P \cap \bigcup_{i=1}^{N} O_i. \]  (5.3)

First, discard all the cells, and the points of \( \hat{P} \) within, such that \(|\hat{P} \cap O_i| < \delta^{-\tau}\). Since the number of cells is bounded by \( D^2 \), this results in the removal of at most \( D^2 \delta^{-\tau} \) points of \( \hat{P} \), and this is smaller than \(|\hat{P}|/2\) as long as

\[ D \leq c\delta^{(\tau-1)/2} \]  (5.4)

for some small absolute constant \( c \). Assume (5.3) in the sequel, and note that the remaining points of \( \hat{P} \) still satisfy (5.3): hence, keep the notation \( \hat{P} \) without loss of generality, and observe that \(|O_i \cap \hat{P}| \geq \delta^{-\tau}\) for the non-empty cells \( O_i \).
Let $\mathcal{T}$ be the collection of tubes introduced in Definition (4.4), with $P$ replaced by $\hat{P}$ (the definition of $E_s$ need not be changed to reflect the projections of $\hat{P}$). Then

$$|I(\hat{P}, \mathcal{T})| \geq |\hat{P}| |E_s| \sim \delta^{-1} |E_s|$$

(5.5)

by Lemma 4.5, and it remains to find an upper bound in the spirit of the end of the previous section.

First, write

$$|I(\hat{P}, \mathcal{T})| = \sum_{i=1}^{N} |I(\hat{P} \cap O_i, \mathcal{T})| = \sum_{i=1}^{N} |I(\hat{P} \cap O_i, \mathcal{T}^i)|,$$

where $\mathcal{T}^i$ is the collection of tubes $T \in \mathcal{T}$ with $T \cap O_i \neq \emptyset$, and the sum only runs over the non-empty cells $O_i$. Observe that $\hat{P} \cap O_i$ and $\mathcal{T}^i \subset \mathcal{T}$ satisfy the assumptions of Proposition 4.6, so

$$|I(\hat{P} \cap O_i, \mathcal{T}^i)| \preceq |\hat{P} \cap O_i| |\mathcal{T}^i|^{1/2} + |\mathcal{T}^i| \sqrt{|\hat{P} \cap O_i||\mathcal{T}^i|}$$

$$\preceq |\hat{P} \cap O_i| |\mathcal{T}^i|^{1/2} + |\mathcal{T}^i|,$$

where the latter inequality used $|\hat{P} \cap O_i| \geq \delta^{-\tau}$. Plugging the estimate into the previous displayed formula, recalling that $|\hat{P} \cap O_i| \leq |\hat{P}| / D^2$ and $N \leq D^2$, and using Cauchy-Schwarz yields

$$|I(\hat{P}, \mathcal{T})| \preceq \sum_{i=1}^{N} |\hat{P} \cap O_i| |\mathcal{T}^i|^{1/2} + \sum_{i=1}^{N} |\mathcal{T}^i|$$

$$\leq |\hat{P}| D \left( \sum_{i=1}^{N} |\mathcal{T}^i| \right)^{1/2} + \sum_{i=1}^{N} |\mathcal{T}^i|.$$

Finally, by Lemma 5.2,

$$\sum_{i=1}^{N} |\mathcal{T}^i| = \sum_{T \in \mathcal{T}} \sum_{T \cap O_i \neq \emptyset} \chi_{(T \cap O_i \neq \emptyset)} \leq (D + 1) \sum_{T \in \mathcal{T}} = (D + 1) |\mathcal{T}|,$$

so that

$$\delta^{-1} |E_s| \lesssim |I(\hat{P}, \mathcal{T})| \lesssim \frac{|\hat{P}|}{D^{1/2}} |\mathcal{T}|^{1/2} + D |\mathcal{T}| \lesssim D^{-1} \left( |E_s| \delta^{-s} \right)^{1/2} + D |E_s| \delta^{-s},$$

using (5.5) in the left-hand side inequality. The second term on the right hand side cannot dominate the left hand side, if $D$ is significantly smaller than $\delta^{s-1}$: tracking the constants behind the $\lesssim$-notation, and combining with the restriction coming from (5.4), the correct thing to assume is

$$D \leq c \min \{ \delta^{s-1} / \log(1/\delta), \delta^{(\tau-1)/2} \}.$$

For such a choice of $D$,

$$|E_s| \gtrsim \frac{\delta^{-s}}{D}.$$

(5.6)

This finishes the proof of the cellular case, because obviously the bound for $|E_s|$ can be taken of the form $|E_s| \lesssim \delta^{-s + \epsilon}$ for some $\epsilon$ depending only on $s, \tau$. 

5.2. The non-cellular case. In this subsection, assume that $|\bar{P}| \geq |P|/2 \sim \delta^{-1}$, where $\bar{P} := P \cap Z(\delta)$.

The strategy is to use the existence of many small projections to force $Z$ to contain many lines, which is impossible if $D$ is small enough.

Since every point in $p \in \bar{P}$ lies in the $\delta$-neighbourhood of $Z$, there exists a point $z_p \in Z$ with $|p - z_p| \leq \delta$. Let $C_p$ be the component of $Z$ containing $z_p$. Given a number $r > 0$ to be specified momentarily, call $p$ an $r$-bad point, if there exist two vectors $e_1, e_2 \in E_s$ with $|e_1 - e_2| \gtrsim |E_s|\delta$ such that the maximal (component) interval of $\pi_{e_i}(B(p, 2\delta) \cap C_p)$ containing $\pi_{e_i}(z_p)$ has length $\leq r$ for $i = 1, 2$ (including the case where the component interval is just the single point $\pi_{e_i}(z_p)$). The claim is that there cannot be many $r$-bad points in $\bar{P}$. Figure 3 is relevant to the following argument.

![Figure 3](image_url)

**Figure 3.** The picture near an $r$-bad point $p \in \bar{P}$.

Fix an $r$-bad point $p \in \bar{P}$, so that the corresponding component intervals of both $\Pi_1 := \pi_{e_1}(B(p, 2\delta) \cap C_p)$ and $\Pi_2 := \pi_{e_2}(B(p, 2\delta) \cap C_p)$ have length $\leq r$. Then, one can find two open intervals $I_1$ and $I_2$, containing $\pi_{e_1}(z_p)$ and $\pi_{e_2}(z_p)$, respectively, of length $\leq 2r$, and such that

$$\partial I_1 \cap \Pi_1 = \emptyset = \partial I_2 \cap \Pi_2. \quad (5.7)$$

By elementary geometry, the box $Q := \pi_{e_1}^{-1}(I_1) \cap \pi_{e_2}^{-1}(I_2)$ has diameter

$$\text{diam}(Q) \lesssim \frac{r}{|E_s|\delta + r}.$$ 

Hence, $Q$ is an open box containing $z_p$, and contained in $B(p, 2\delta)$ if $r \leq c|E_s|\delta^{1+\tau}$ for a sufficiently small constant $c > 0$ (recall that $z_p \in B(p, \delta)$). It follows from these observations that, for such $r > 0$, in fact $C_p \subset Q \subset B(p, 2\delta)$: otherwise $C_p$ should intersect the boundary of $Q$, and since this happens inside $B(p, 2\delta)$, one has either $\partial I_1 \cap \Pi_1 \neq \emptyset$ or $\partial I_2 \cap \Pi_2 \neq \emptyset$ contrary to $(5.7)$.

To summarise, if

$$r = c|E_s|\delta^{1+\tau} \quad (5.8)$$

for a suitable small constant $c > 0$, then for every $r$-bad point $p \in \bar{P}$, there exists a component of $Z$ inside $B(p, 2\delta)$. By Harnack’s curve theorem, see [5], the number of
components of $Z$ is bounded by $\lesssim D^2$, so as long as (5.8) holds, and $D^2 \leq |\tilde{P}|/2$, one can discard the $r$-bad points from $P$ and assume, without loss of generality, that $\tilde{P}$ contains no $r$-bad points.

Assuming that $|E_s| \geq 2$, as one may – pick two vectors $e_1, e_2 \in E_s$ with $|e_1 - e_2| \gtrsim |E_s|\delta^r$. Since no point in $\tilde{P}$ is $r$-bad, the following holds for either $i = 1$ or $i = 2$: there is a subset $P' \subset \tilde{P}$ of cardinality $|P'| \geq |\tilde{P}|/2$ such that $\mathcal{H}^1(\pi_{e_i}(B(p, 2\delta) \cap Z)) \geq r$ for all $p \in P'$.

Assume that this holds for $i = 1$. Since the points $P'$ are $5\delta$-separated (if not, discard additional points and observe that $\gtrsim \delta^{-1}$ points in $P' \cap T$ remain).

Pick a line $l$ passing through – and parallel to – $5T_0$ uniformly at random, and for a given point $p \in P' \cap T_0$, consider the random variable

$$X_p := \chi\{l \cap Z \cap B(p, 2\delta) \neq \emptyset\}.$$ 

Since $\mathcal{H}^1(B(p, 2\delta) \cap Z) \geq r = c|E_s|\delta^{1+\tau}$, and $\pi_{e_1}(B(p, 2\delta)) \subset 5I_0$, one has $E[X_p] \gtrsim |E_s|\delta^r$, and

$$E\left[\sum_{p \in P' \cap T_0} X_p\right] \gtrsim \delta^{s+\tau-1}|E_s|.$$ 

Since the points $p \in P' \cap T_0$ are $5\delta$-separated, the sum $\sum X_p(l)$ gives a lower bound for distinct intersections of $l$ with $Z$. On the other hand, by Bézout’s theorem, almost every line in any fixed direction hits $Z$ in at most $D$ distinct spots, so

$$|E_s| \lesssim D\delta^{1-s-\tau}.$$ 

5.3. **Conclusion of the proof.** With $r = \delta^\tau$, the claim was that

$$N(E_s, \delta^\tau) \lesssim \min\{\delta^{-s}\min(1-\tau)/2, \delta^{-\tau}\}. \quad (5.9)$$

Obviously,

$$N(E_s, \delta^\tau) \leq \delta^{-\tau},$$

and this coincides with the minimum in (5.9), if $\tau \leq 2s - 1$. So, one may assume that $\tau > 2s - 1$. Now, the previous two subsections have shown that

$$|E_s| \lesssim \max \left\{ \frac{\delta^{-s}}{D}, D\delta^{1-s-\tau} \right\},$$

where $D$ is any integer satisfying

$$D \leq c \min\{\delta^{-s}/\log(1/\delta), \delta^{(\tau-1)/2}\}. \quad (5.10)$$

Since $\tau > 2s - 1$, one has $\delta^{(\tau-1)/2} \leq \delta^{-s}/\log(1/\delta)$, so one is allowed to choose $D = c\delta^{(\tau-1)/2}$, and this results in

$$|E_s| \lesssim \delta^{-s+1-\tau)/2}.$$ 

---

2Obviously this is a weaker requirement than the existence of a long component interval in $\pi_{e_i}(B(p, 2\delta) \cap \tilde{C}_p)$: the stronger claim was simply introduced, because it was easier to prove.
The proof of (5.9), and Theorem 1.5, is complete.

6. PROJECTIONS OF 1-AD REGULAR MEASURES

This section contains the proof of Theorem 6.2.

**Definition 6.1.** A Radon probability measure \( \mu \) on \([0, 1]^d\) is said to be \((1, A)\)-Ahlfors-David regular – or \((1, A)\)-AD regular in short – if

\[
\frac{r}{A} \leq \mu(B(x, r)) \leq Ar
\]

for all \( x \in \text{spt} \mu \) and \( 0 < r \leq \text{diam}(\text{spt} \mu) \). A \( \mathcal{H}^1 \)-measurable set \( K \subset [0, 1] \) is called \((1, A)\)-AD regular, if \( 0 < \mathcal{H}^1(K) < \infty \), and the normalised restriction

\[
\mu := \frac{1}{\mathcal{H}^1(K)} \mathcal{H}^1|_K
\]

of \( \mathcal{H}^1 \) to \( K \) is \((1, A)\)-AD regular.

Here is the statement of Theorem 6.2 once again:

**Theorem 6.2.** Given \( 0 < s < 1 \) and \( A > 0 \), there are numbers \( p = p(s, A) \in \mathbb{N} \) and \( \delta(A, s) > 0 \) with the following property. Let

\[
S_p := \{ e^{2\pi ik/p} : 0 \leq k < p \} \subset S^1.
\]

Then, for any \((1, A)\)-AD regular set \( K \subset B(0, 1) \),

\[
\frac{1}{p} \sum_{e \in S_p} N(e \pi_e(K), \delta) \geq \delta^{-s}, \quad 0 < \delta \leq \delta(A, s).
\]  

(6.3)

The proof of Theorem 6.2 will use the notion of entropy, and in fact (6.3) will be deduced from an intermediary conclusion of the form "the measure \( \mathcal{H}^1|_K \) has at least one projection with large entropy."

6.1. Preliminaries on entropy and projections. The presentation of this subsection follows closely that of M. Hochman’s paper [6], although I only need a fraction of the machinery developed there. In the interest of being mostly self-contained, I will repeat some of the arguments in [6].

**Definition 6.4** (Measures and their blow-ups in \( \mathbb{R}^d \)). Let \( \mathcal{P}(\Omega) \) stand for the space of Borel probability measures on \( \Omega \). In what follows, \( \Omega \) will be \( \mathbb{R}^d \), or a cube in \( \mathbb{R}^d \), and \( d \in \{1, 2\} \). If \( Q = r[0, 1]^d + a \) is a cube in \( \mathbb{R}^d \), let \( T_Q(x) := (x - a)/r \) be the unique homothety taking \( Q \) to \([0, 1]^d\). Given a measure \( \mu \in \mathcal{P}(\mathbb{R}^d) \) and a cube \( Q \) as above, with \( \mu(Q) > 0 \), define the measures

\[
\mu_Q := \frac{1}{\mu(Q)} \mu|_Q \quad \text{and} \quad \mu^Q := T_{D^d} (\mu_Q),
\]

where \( \mu|_Q \) is the restriction of \( \mu \) to \( Q \), and \( T_{D^d} \) is the push-forward under \( T_Q \). So, \( \mu^Q \) is a "blow-up" of \( \mu_Q \) into \([0, 1]^d\).
Definition 6.5 (Entropy). Let $\mu \in \mathcal{P}(\Omega)$, and let $\mathcal{F}$ be a countable $\mu$-measurable partition of $\Omega$. Set

$$H(\mu, \mathcal{F}) := -\sum_{F \in \mathcal{F}} \mu(F) \log \mu(F),$$

where the convention $0 \cdot \log 0 := 0$ is used. If $\mathcal{E}$ and $\mathcal{F}$ are two $\mu$-measurable partitions, one also defines the conditional entropy

$$H(\mu, \mathcal{E}|\mathcal{F}) := \sum_{F \in \mathcal{F}} \mu(F) \cdot H(\mu|F, \mathcal{E}),$$

where $\mu|F := \mu|/\mu(F)$, if $\mu(F) > 0$.

The notion of conditional entropy is particularly useful, when $\mathcal{E}$ refines $\mathcal{F}$, which means that every set in $\mathcal{E}$ is contained in a (unique) set in $\mathcal{F}$:

Proposition 6.6 (Conditional entropy formula). Assume that $\mathcal{E}, \mathcal{F}$ are partitions as in Definition 6.5, and $\mathcal{F}$ refines $\mathcal{E}$. Then

$$H(\mu, \mathcal{E}|\mathcal{F}) = H(\mu, \mathcal{E}) - H(\mu, \mathcal{F}).$$

In particular, $H(\mu, \mathcal{E}) \geq H(\mu, \mathcal{F})$.

Proof. For $F \in \mathcal{F}$, let $\mathcal{E}(F) := \{E \in \mathcal{E} : E \subset F\}$. A direct computation gives

$$H(\mu, \mathcal{E}|\mathcal{F}) = -\sum_{F \in \mathcal{F}} \mu(F) \cdot \sum_{E \in \mathcal{E}} \mu(E) \log \mu(E)$$

$$= -\sum_{F \in \mathcal{F}} \sum_{E \in \mathcal{E}(F)} \mu(E) \log \frac{\mu(E)}{\mu(F)}$$

$$= -\left( \sum_{E \in \mathcal{E}} \mu(E) \log \mu(E) - \sum_{F \in \mathcal{F}} \log \mu(F) \sum_{E \in \mathcal{E}(F)} \mu(E) \right)$$

$$= H(\mu, \mathcal{E}) + \sum_{F \in \mathcal{F}} \mu(F) \log \mu(F) = H(\mu, \mathcal{E}) - H(\mu, \mathcal{F}),$$

as claimed. \qed

The partitions $\mathcal{E}, \mathcal{F}$ used below will be the dyadic partitions of $\mathbb{R}^d$: $\mathcal{E}, \mathcal{F} = \mathcal{D}_n$. The lemma below contains two more useful and well-known – or easily verified – properties of entropy. The items are selected from [6, Lemma 3.1] and [6, Lemma 3.2].

Lemma 6.7. Let $\mathcal{E}, \mathcal{F}$ be countable $\mu$-measurable partitions of $\Omega$.

(i) The functions $\mu \mapsto H(\mu, \mathcal{E})$ and $\mu \mapsto H(\mu, \mathcal{E}|\mathcal{F})$ are concave.

(ii) If $\text{spt} \mu \subset B(0, R)$, and $f, g : B(0, R) \to \mathbb{R}$ are functions so that $|f(x) - g(x)| \leq R2^{-n}$ for $x \in B(0, R)$, then

$$|H(f \mu, \mathcal{D}_n) - H(f \mu, \mathcal{D}_n)| \leq C,$$

where $C > 0$ only depends on $R$.

Finally, for $n \in \mathbb{N}$, write $H_n$ for the normalised scale $2^{-n}$-entropy

$$H_n(\mu) := \frac{1}{\log 2^n} \cdot H(\mu, \mathcal{D}_n) = \sum_{Q \in \mathcal{D}_n} \mu(Q) \cdot \left( \frac{\log \mu(Q)}{\log 2^{-n}} \right).$$
Now, all the definitions and tools are in place to state and prove the key auxiliary result from Hochman’s paper, namely \([6, \text{Lemma 3.5}]\), in slightly modified form:

Lemma 6.8. Let \(\mu \in \mathcal{P}([0,1])\), \(e \in S^1\), and \(m, n \in \mathbb{N}\) with \(m < n\). Then

\[
H_n(\pi e^\sharp \mu) \geq \frac{m}{n} \sum_{Q \in D_{km}} \mu(Q) \cdot H_m(\pi e^\sharp \mu^Q) - Cm
\]

where \(C > 0\) is an absolute constant.

Proof. Write \(n = k_0 m + r\), where \(0 \leq r < m\), and \(k_0 = \lfloor n/m \rfloor\). Then

\[
H(\pi e^\sharp \mu, D_n) \geq H(\pi e^\sharp \mu, D_{km}) = \sum_{k=0}^{k_0} H(\pi e^\sharp \mu, D_{(k+1)m} | D_{km}) + H(\pi e^\sharp \mu, D_0) \geq \sum_{k=0}^{k_0} H(\pi e^\sharp \mu, D_{(k+1)m} | D_{km})
\]

by repeated application of Proposition 6.6. Next, observe that

\[
\pi e^\sharp \mu = \pi e^\sharp \left( \sum_{Q \in D_{km}} \mu|_Q \right) = \sum_{Q \in D_{km}} \pi e^\sharp \mu|_Q = \sum_{Q \in D_{km}} \mu(Q) \cdot \pi e^\sharp \mu|_Q,
\]

so, by Jensen’s inequality and the concavity of (conditional) entropy,

\[
H(\pi e^\sharp \mu, D_{(k+1)m} | D_{km}) \geq \sum_{Q \in D_{km}} \mu(Q) \cdot H(\pi e^\sharp \mu|_Q, D_{(k+1)m} | D_{km}).
\]

Here

\[
H(\pi e^\sharp \mu|_Q, D_{(k+1)m} | D_{km}) = H(\pi e^\sharp \mu^Q, D_m | D_0) = H(\pi e^\sharp \mu^Q, D_m) - H(\pi e^\sharp \mu^Q, D_0),
\]

by Proposition 6.6 once again, where \(H(\pi e^\sharp \mu^Q, D_0) \leq 3\), because \(\pi e^\sharp \mu^Q\) is supported in an interval of length \(\sqrt{2}\). This leads to

\[
H_n(\pi e^\sharp \mu) \geq \frac{1}{\log 2^n} \sum_{k=0}^{k_0-1} \sum_{Q \in D_{km}} \mu(Q) \cdot (H(\pi e^\sharp \mu|_Q, D_m) - 3) = \frac{m}{n} \sum_{k=0}^{k_0-1} \sum_{Q \in D_{km}} \mu(Q) \cdot H_m(\pi e^\sharp \mu^Q) - \frac{3k_0}{\log 2^n},
\]

where \(3k_0/\log 2^n \leq 10/m\) as claimed. \(\square\)

6.2. An entropy version of Marstrand’s theorem.

Proposition 6.9. Assume that \(\mu \in \mathcal{P}([0,1])\) satisfies the linear growth condition \(\mu(B(x,r)) \leq Ar\) for \(x \in \mathbb{R}^2\), \(r > 0\) and some \(A \geq 1\). Then

\[
\int_{S^1} H_n(\pi e^\sharp \mu) \, d\sigma(e) \geq s - ACm \cdot 2^{(s-1)m}, \quad 0 < s < 1,
\]

where \(\sigma\) is the unit-normalised length measure on \(S^1\), and \(C > 0\) is an absolute constant.
Proof. Fix \( m \in \mathbb{N} \). It follows from the linear growth condition for \( \mu \) that

\[
\int_{S^1} 2^m \sum_{Q \in D_m} |\pi_{e^c \mu}(Q)|^2 \, d\sigma(e) \lesssim Am. \tag{6.10}
\]

This is standard, so I only sketch the details: observe that for any \( \nu \in \mathcal{P}([0,1]^2) \)

\[
\int_{S^1} \|\pi_{e^c \nu}\|^2 \, d\sigma(e) = \int_{S^1} \int_{\mathbb{R}} |\hat{\nu}(te)|^2 \, dt \, d\sigma(e)
\]

\[
\sim \int_{\mathbb{R}^2} |\hat{\nu}(\xi)|^2 |\xi|^{-1} \, d\xi \sim \int \frac{d\nu \, d\nu}{|x-y|} =: I_1(\nu).
\]

Apply this with \( \nu := \mu \ast \psi_m \), where \( \psi_m(x) := 2^{2m} \psi(2^m x) \) and \( \psi \) is a radial bump function with \( \chi_{B(0,3)} \leq \psi \leq \chi_{B(0,10)} \). Using the linear growth condition for \( \mu \), it is easy to verify that \( I_1(\mu \ast \psi_m) \lesssim Am \), for \( A, m \geq 1 \). Further, since \( \psi \) is radial, the projection \( \pi_{e^c}(\mu \ast \psi_m) \) has the form \( \pi_{e^c} \ast \phi_m \), where \( \phi_m \) is a bump in \( \mathbb{R} \) at scale \( 2^{-m} \), independent of \( e \). Finally, the left hand side of (6.10) is controlled by an absolute constant times \( \|\pi_{e^c(\mu \ast \phi_m)}\|_2^2 \). The inequality now follows by combining all the observations.

Let

\[
C_e := 2^m \sum_{Q \in D_m} |\pi_{e^c \mu}(Q)|^2.
\]

Then, for \( s < 1 \) fixed,

\[
\pi_{e^c \mu} \left( \bigcup \left\{ Q \in D_m : \pi_{e^c \mu}(Q) \geq 2^{-ms} \right\} \right) \leq C_e 2^{(s-1)m},
\]

and so

\[
\int_{S^1} \pi_{e^c \mu} \left( \bigcup \left\{ Q \in D_m : \frac{\pi_{e^c \mu}(Q)}{\log 2^{-m}} \leq s \right\} \right) \, d\sigma(e) \lesssim Am \cdot 2^{(s-1)m}. \tag{6.11}
\]

Inspired by (6.11), let

\[
D_m^{e^c} := \left\{ Q \in D_m : \frac{\log \pi_{e^c \mu}(Q)}{\log 2^{-m}} \leq s \right\},
\]

and denote by \( \beta_e \) the \( \pi_{e^c \mu} \)-measure of the \( e \)-bad intervals. Then,

\[
\int_{S^1} H_m(\pi_{e^c \mu}) \, d\sigma(e) \geq \int_{S^1} \sum_{Q \in D_m \setminus D_m^{e^c}} \pi_{e^c \mu}(Q) \left( \frac{\log \pi_{e^c \mu}(Q)}{\log 2^{-m}} \right)
\]

\[
\geq \int_{S^1} s(1 - \beta_e) \, d\sigma(e) \geq s - ACm \cdot 2^{(s-1)m},
\]

as claimed. \( \square \)

Corollary 6.12. Let \( \mu \) be as in Proposition 6.9, and let \( S_{2m} := \{ e^{2\pi ik/2^m} : 0 \leq k < 2^m \} \subset S^1 \).

Then

\[
\frac{1}{|S_{2m}|} \sum_{e \in S_{2m}} H_m(\pi_{e^c \mu}) \geq s - AC(m \cdot 2^{(s-1)m} + 1/m).
\]

Proof. Partition \( S^1 \) into arcs \( S^1_e \) of equal length \( \sigma(S^1_e) := 1/|S_{2m}| \) such that \( |e' - e| \leq 2^{m+1} \)

for \( e' \in S^1_e \). Lemma 6.7(ii) says that

\[
|H_m(\pi_{e^c \mu}) - H_m(\pi_{e^c \mu})| \lesssim \frac{1}{m}.
\]
Then,
\[ \frac{1}{|S_{2m}|} \sum_{e \in S_{2m}} H_m(\pi_{e\xi}\mu) = \frac{1}{|S_{2m}|} \sum_{e \in S_{2m}} \int_{S_e^c} H_m(\pi_{e\xi}\mu) d\sigma(\xi) \]
\[ \geq \sum_{e \in S_{2m}} \int_{S_e^c} (H_m(\pi_{e\xi}\mu) - C/m) d\sigma(\xi) \]
\[ \geq s - AC(m \cdot 2^{(s-1)m} + 1/m), \]
as claimed. \( \square \)

6.3. Conclusion of the proof. Fix \( 0 \leq s < s' < 1 \), and let \( K \subset [0,1]^2 \) be \((1,\nu)\)-AD regular. Write \( \mu := H|_K \), fix \( m \in \mathbb{N} \), and let \( S_{2m} := \{e^{2\pi i k/2^m} : 0 \leq k < 2^m\} \subset S^1 \) as in Corollary 6.12. The key observation is that if \( Q \subset [0,1]^2 \) is a cube with \( \mu(Q) > 0 \), then the blow-up \( \mu^Q \in P([0,1]^2) \) is \((1,\nu(Q)/\mu(Q))\)-AD regular. Thus, from Lemma 6.8 and Corollary 6.12, one infers that, for \( n \geq m \),
\[ \frac{1}{|S_{2m}|} \sum_{e \in S_{2m}} H_n(\pi_{e\xi}\mu) \geq \frac{m}{n} \sum_{k=0}^{n/m} \mu(Q) \left[ \frac{1}{|S_{2m}|} \sum_{e \in S_{2m}} H_m(\pi_{e\xi}\mu^Q) \right] - \frac{C}{m} \]
\[ \geq \frac{m}{n} \sum_{k=0}^{n/m} \mu(Q) \left( s' - \frac{AC\ell(Q)}{\mu(Q)} (m \cdot 2^{(s'-1)m} + 1/m) \right) - \frac{C}{m} \]
\[ = s' \cdot \frac{m}{n} \cdot \frac{n/m}{\mu(Q)} \sum_{k=0}^{n/m} \ell(Q) (m \cdot 2^{(s'-1)m} + 1/m) - \frac{C}{m}. \]
To proceed further, observe that, for any fixed generation of squares \( Q \) with \( \ell(Q) = r \), there are at most \( AC/r \) squares \( Q \) such that \( \mu(Q) > 0 \). Indeed, by the \((1,\nu)\)-AD regularity of \( \mu \), each such square \( Q \) with \( \mu(Q) > 0 \) is adjacent to a square \( Q' \) with \( \ell(Q') = r \) and \( \mu(Q') \geq r/(100A) \). Since each such “good” square \( Q' \) is adjacent to at most eight other squares \( Q \) with \( \mu(Q) > 0 \), the claim follows. This leads to the estimate
\[ \frac{1}{|S_{2m}|} \sum_{e \in S_{2m}} H_n(\pi_{e\xi}\mu) \geq s' \cdot \frac{m}{n} \cdot \frac{n/m}{\mu(Q)} - \frac{ACm}{n} \cdot \frac{n/m}{\mu(Q)} \cdot (m \cdot 2^{(s'-1)m} + 1/m) - \frac{C}{m}, \]
valid for any \( 0 \leq s' < 1 \) and any \((1,\nu)\)-AD regular measure \( \mu \in P([0,1]^2) \). Specialising to \( s' := (1+s)/2 \), say, and choosing \( m \geq m(A,s) \), where \( m(A,s) \in \mathbb{N} \) depends only on \( A \) and \( s \), one obtains
\[ \frac{1}{|S_{2m}|} \sum_{e \in S_{2m}} H_n(\pi_{e\xi}\mu) \geq s, \quad n \in \mathbb{N}. \]
Via the following lemma, this immediately leads to the desired statement about the covering numbers \( \overline{N}(\pi_{e}(K),\delta) \) with \( \delta = 2^{-n}, n \in \mathbb{N} \). The proof of Theorem 6.2 is complete.

Lemma 6.13. Let \( \nu \in P(\mathbb{R}^d) \), and assume that \( H_n(\nu) \geq s \). Then
\[ |\{Q \in \mathcal{D}_n : \nu(Q) > 0\}| > 2^{nt} \]
for any \( t < s - 1/(n \log 2) \). In particular, \( \overline{N}(\text{spt} \nu, 2^{-n}) \geq 2^{nt} \) for such \( t \).
Remark 6.14. Note that the converse of the lemma is false: a large covering number certainly does not guarantee large entropy.

Proof of Lemma 6.13. Assume that $|\{Q \in D_n : \nu(Q) > 0\}| \leq 2^{nt}$ for some $t$, and let $D_n^{\lambda \text{-bad}}$, $\lambda \geq 0$, be the cubes $Q \in D_n$ such that $\nu(Q) \leq 2^{-\lambda n}$. Then

$$
\sum_{Q \in D_n^{\lambda \text{-bad}}} \nu(Q) \leq 2^{(t-\lambda)n}, \quad \lambda \geq t,
$$

so that

$$
s \leq H_n(\nu) = \int_0^\infty \nu \left( \bigcup \left\{ Q : \frac{\log \nu(Q)}{\log 2^{-n}} \geq \lambda \right\} \right) d\lambda
\leq t + \int_t^\infty \nu \left( \bigcup \left\{ Q : \nu(Q) \leq 2^{-\lambda n} \right\} \right) d\lambda
\leq t + \int_t^\infty 2^{(t-\lambda)n} d\lambda = t + \frac{1}{n \log 2}.
$$

This proves the lemma. $\square$

Finally, the proof of Corollary 1.8 about the packing dimension of projections:

Proof of Corollary 1.8. Recall that

$$
\dim_p(B) = \inf \left\{ \sup \text{\em upper} \dim B_i : B \subset \bigcup_i F_i, \, F_i \text{ closed} \right\},
$$

where

$$
\text{\em upper} \dim B_i = \limsup_{\delta \to 0} \frac{\log N(F_i, \delta)}{-\log \delta}
$$

is the upper box-dimension. Now, assume that $K$ is $(1, A)$-AD regular, and $\mathcal{H}^1(K) = 1$ without loss of generality. Let $p = p(100A, s) \in \mathbb{N}$ be so large that for all $(1, 100A)$-regular sets $B \subset [0, 1]^2$,

$$
\max_{e \in S_p} \text{\em upper} \dim B \pi_e(B) \geq s. \quad (6.15)
$$

Now, assume that, nevertheless,

$$
\max_{e \in S_p} \dim \pi_e(K) =: t' < s,
$$

and pick $t' < t < s$. Then, each projection $\pi_e(K)$, $e \in S_p$ can be covered by closed sets $\{F_i^e\}_{i \in \mathbb{N}}$ such that $\text{\em upper} \dim F_i^e \leq t$ for $i \in \mathbb{N}$ and $e \in S_p$. Assume, without loss of generality, that $F_i^e \subset \pi_e(K)$. Enumerate the vectors in $S_p$, say $S_p = \{e_1, \ldots, e_p\}$. Since $\pi_{e_1}(K)$ is closed, Baire’s theorem states that one of the sets $F_i^{e_1}$ must have interior in the relative topology of $\pi_{e_1}(K)$: thus

$$
U^{e_1} \cap \pi_{e_1}(K) \subset F_i^{e_1}
$$

for some open set $U^{e_1} \subset \mathbb{R}$ and some $i_1 \in \mathbb{N}$. Now, one can find a dyadic cube $Q_1 \subset [0, 1]^2$, such that

$$
\mathcal{H}^1(K \cap Q_1) \geq \frac{\ell(Q_1)}{100A}
$$

and

$$
\pi_{e_1}(K \cap 2Q_1) \subset U^{e_1} \cap \pi_{e_1}(K) \subset F_i^{e_1},
$$
where $2Q_1$ is the cube with the same center but twice the side-length of $Q_1$. In particular, 
\[ \dim \pi_{e_2}(K \cap Q_1) \leq t. \] 
On the other hand, writing $\mu := \mathcal{H}^1|_K$, the measure $\mu^{Q_1}$ is $(1, 100A)$-AD regular, so $K \cap Q_1$ is a $(1, 100A)$-AD regular set.

Next, repeat the drill with $e_2$. Since $\pi_{e_2}(K \cap 2Q_1)$ is a compact set, one $F_i^{e_2}$ must have interior in the relative topology of $\pi_{e_2}(K \cap Q_1)$. Thus
\[ U^{e_2} \cap \pi_{e_2}(K \cap 2Q_1) \subset F_i^{e_2} \]
for some open $U^{e_2} \subset \mathbb{R}$ and some $i_2 \in \mathbb{N}$. Use this to find a dyadic cube $Q_2$ such that $2Q_2 \subset 2Q_1$,
\[ \mathcal{H}^1(K \cap Q_2) \geq \frac{\ell(Q_2)}{100A} \]
and
\[ \pi_{e_2}(K \cap 2Q_2) \subset U^{e_2} \cap \pi_{e_2}(K \cap 2Q_1) \subset F_i^{e_2}. \]
Now $\dim \pi_{e_1}(K \cap Q_2) \leq t$ for $i \in \{1, 2\}$, and $K \cap Q_2$ is again $(1, 100A)$-AD regular.

After $p$ iterations of this procedure, one ends up with a dyadic cube $Q_p \subset [0, 1]^2$ such that $K \cap Q_p$ is $(1, 100A)$-AD regular, and
\[ \dim \pi_{e_i}(K \cap Q_p) \leq t < s, \quad 1 \leq i \leq p, \]
which contradicts (6.15) and completes the proof of the corollary. (In fact, the proof did not use the fact that the $K \cap Q_i$ is $(1, 100A)$-AD regular for $i < p$. It was only needed that $\mathcal{H}^i(K \cap Q_i) > 0$ for these $i$, so that $Q_{i+1} \subset 2Q_i$ could always be found with a substantial value of $\mathcal{H}^i(K \cap Q_{i+1})$.)

\section*{References}

[1] J. Bourgain: The discretised sum-product and projection theorems, J. Anal. Math 112 (2010), pp. 193–236
[2] K. Fassler and T. Orponen: On restricted families of projections in $\mathbb{R}^2$, Proc. London Math. Soc. 109 (2014), p. 353-381, available at arXiv:1302.6550
[3] L. Guth: A restriction estimate using polynomial partitioning, to appear in J. Amer. Math. Soc. (2015)
[4] L. Guth and N. Katz: On the Erdős distinct distance problem in the plane, Ann. of Math. 181, Issue 1 (2015), p. 155–190
[5] C. G. A. Harnack: Über Vieltheiligkeit der ebenen algebraischen Curve, Math. Ann. 10 (1876), p. 189–199
[6] M. Hochman: Self-similar sets with overlaps and inverse theorems for entropy, Ann. of Math. 180, No. 2 (2014), p. 773–822
[7] N. Katz and T. Tao: Some connections between Falcolner’s distance set conjecture, and sets of Furstenberg type, New York J. Math. 7 (2001), pp. 149–187
[8] R. Kaufman: On Hausdorff dimension, Mathematika 15 (1968), pp. 153–155
[9] I. Laba and S. Konyagin: Distance sets of well-distributed planar sets for polygonal norms, Israel J. Math. 152 (2006), p. 157–179
[10] J.M. Marstrand: Some fundamental geometrical properties of plane sets of fractional dimensions, Proc. London Math. Soc. 3 (4) (1954), pp. 257-302
[11] P. Mattila: Geometry of sets and measures in Euclidean spaces: fractals and rectifiability, Cambridge University Press, 1995
[12] D. Oberlin: Restricted Radon transforms and projections of planar sets, published electronically in Canadian Math. Bull. (2014), available at arXiv:0805.1678
[13] T. Orponen: On the packing dimension and category of exceptional sets of orthogonal projections, published online in Ann. Mat. Pura Appl. (2015), available at arXiv:1204.2121

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