OTHER REPRESENTATIONS OF THE RIEMANN ZETA FUNCTION AND AN ADDITIONAL REFORMULATION OF THE RIEMANN HYPOTHESIS

STEFANO BELTRAMINELLI AND DANilo MERLINI

Abstract. New expansions for some functions related to the Zeta function in terms of the Pochhammer's polynomials are given (coefficients \( b_k, d_k, \hat{d}_k \) and \( \hat{\hat{d}}_k \)). In some formal limit our expansion \( b_k \) obtained via the alternating series gives the regularized expansion of Maslanka for the Zeta function. The real and the imaginary part of the function on the critical line is obtained with a good accuracy up to \( \Im(s) = t < 35 \).

Then, we give the expansion (coefficient \( \hat{d}_k \)) for the derivative of \( \frac{\ln((s - 1)\zeta(s))}{s} \). The critical function of the derivative, whose bounded values for \( \Re(s) > \frac{1}{2} \) at large values of \( k \) should ensure the truth of the Riemann Hypothesis (RH), is obtained either by means of the primes or by means of the zeros (trivial and non-trivial) of the Zeta function. In a numerical experiment performed up to high values of \( k \) i.e. up to \( k = 10^{13} \) we obtain a very good agreement between the two functions, with the emergence of eleven oscillations with stable amplitude.

For a special case of values of the two parameters entering in the general Pochhammer's expansion it is argued that the bound on the critical function should be given by the Euler constant \( \gamma \).

1. Introduction

Lately there has been new interest in the study of the expansion of the Zeta function via the Pochhammer's polynomials. This is related to the original idea of Riesz [1] and of Hardy-Littlewood [2] at the beginning of the last century. In a pioneering work [3] Maslanka obtained a regularized expansion for the Zeta function (with coefficients \( A_k \)) and Baez-Duarte for the expansion of the reciprocal of the Zeta function (with coefficients \( c_k \)) for the Riesz case [4, 5]. Other cases of interest have also recently been studied [6, 7, 8, 9, 10]. As pointed out in [5], the discrete version by means of the Pochhammer's polynomials \( P_k(s) \), where \( s = \sigma + it \) is the complex variable and \( k \) is an integer, has advantages especially in the context of numerical experiments in connection with some "kind of verification" in the direction to believe that the RH may be true.

In this work we first derive a new expansion for the Zeta function in terms of the Pochhammer's polynomials via the alternating series (with new coefficients \( b_k \)). In some formal limit, a connection with the expansion of Maslanka is also obtained in Section 2. Our expansion is then studied numerically on the critical line where a good agreement with the real function is obtained up to \( \Im(s) = t < 35 \), with the

Date: 15 April 2007.

1991 Mathematics Subject Classification. 11M26.

Key words and phrases. Riemann Zeta function, Riemann Hypothesis, Criteria of Riesz, Hardy-Littlewood and Baez-Duarte, Pochhammer's polynomials.
emergence of the first few low zeros. After this value of \( t \), a divergence possibly of numerical nature set on.

In Section 3 we then obtain the expansion for the function \( \ln((1 - 2^{1-s})\zeta(s)) \) (with new coefficients \( d_k \)) as well for the derivative of \( \ln((s - 1)\zeta(s)) \) (with new coefficients \( \hat{d}_k \)) in terms of the two parameters \( \alpha \) and \( \beta \), already introduced in our previous works \cite{11, 12, 13}. Then the critical function for the derivative (whose boundedness at large \( k \) would ensure the truth of the RH) is then obtained either with the primes or with the trivial and non-trivial zeros of the Zeta function.

In a numerical experiment for the special case \( \alpha = \frac{1}{2} \) and \( \beta = 4 \) up to high values of \( k \), i.e. \( k = 10^{13} \), the results for the two functions are in very good agreement, both with the emergence of the same twelve oscillations of stable amplitude of about 0.01 (Section 4).

Finally, in the limit of large \( \beta \) and \( \alpha = 1 \), it is argued that an upper bound to the critical function should be given by the Euler constant \( \gamma \) (Section 5).

2. \textbf{Zeta function representation via the alternating series}

In this section we derive a formula for \( (1 - 2^{1-s})\zeta(s) \) similar to the one of Maslanka \cite{3} for \( (s - 1)\zeta(s) \) and of Baez-Duarte \cite{4, 5} for \( [\zeta(s)]^{-1} \).

Here the starting series is convergent for \( \Re(s) = \sigma > 0 \) and the formula is obtained still in terms of the so called Pochhammer's polynomials of degree \( k \), in the complex variable \( s = \sigma + it \).

\begin{equation}
(1 - 2^{1-s}) \zeta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \quad \forall \Re(s) = \sigma > 0
\end{equation}

(2.2)

we have using the trick as in \cite{4} that:

\( (1 - 2^{1-s}) \zeta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} (1 - (1 - \frac{1}{n^s})) \frac{x^s}{x^s} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \sum_{k=0}^{\infty} (-1)^k (1 - \frac{1}{n^s})^k \frac{x^s}{x^s} \)

Since

\( (-1)^k \left( \frac{x^s}{k} \right) = \left( -1 \right)^k \left( \frac{x^s}{k} \right) = \prod_{r=1}^{k} \left( 1 - \frac{x^s}{r} \right) = P_k \left( \frac{x^s}{k} \right) \)

we obtain:

\begin{equation}
(1 - 2^{1-s}) \zeta(s) = \sum_{k=0}^{\infty} P_k \left( \frac{x^s}{k} \right) + 1 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} (1 - \frac{1}{n^s})^k
\end{equation}

(2.3)
Since from (2.2)
\[
(1 - 2^{1-(\alpha+\beta j)}) \zeta(\alpha + \beta j) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{\alpha+\beta j}}
\]
substitution in (2.3) gives:
\[
(2.4) \quad (1 - 2^{1-s}) \zeta(s) = \sum_{k=0}^{\infty} P_k\left(\frac{s-\alpha}{\beta} + 1\right) \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left(1 - 2^{1-(\alpha+\beta j)}\right) \zeta(\alpha+\beta j)
\]

With the definition
\[
(2.5) \quad b_k := \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left(1 - 2^{1-(\alpha+\beta j)}\right) \zeta(\alpha+\beta j)
\]
(2.4) becomes:
\[
(2.6) \quad (1 - 2^{1-s}) \zeta(s) = \sum_{k=0}^{\infty} b_k P_k\left(\frac{s-\alpha}{\beta} + 1\right)
\]
where \(P_0(\frac{s-\alpha}{\beta} + 1) = 1\) and \(b_0 = (1 - 2^{1-s})\zeta(\alpha)\).

The series above, is expected to represent \((1 - 2^{1-s})\zeta(s)\) for \(s\) in some compact subset of the plane as for the Maslanka case [3]. In that case, the central point has been investigated and elucidated by Baez-Duarte [14]. Here many choices of \(\alpha\) and \(\beta\) are possible. For \(\alpha = \beta = 2\) we have the Riesz case [1] and it is the analogon to the regularized version of Maslanka but the representation of the Zeta function is not the same. For \(\alpha = 1 + \delta\) \((\delta \not\in \mathbb{N})\) and \(\beta = 2\) we obtain the Hardy-Littlewood case [2] which was also discussed numerically in a different way using other polynomials [15].

In fact, from Lemma 2.3 of Baez-Duarte [5] which states that at large \(k\):
\[
(2.7) \quad |P_k(s)| \leq Ck^{-\Re(s)}
\]
where \(C\) is a constant depending on \(|s|\), we obtain here that:
\[
\left| P_k\left(\frac{s-\alpha}{\beta} + 1\right) \right| \leq Ck^{-\left(\Re(s)-\alpha\right)/\beta + 1})
\]

We thus suspect and expect that the above series represents \((1 - 2^{1-s})\zeta(s)\) for all \(\Re(s) > \frac{1}{2} + \delta, \delta > 0\) if we assume \(|b_k| \leq Dk^{-\gamma}\) with \(\gamma \geq \frac{\alpha-1/2-\delta}{\beta}\) at large values of \(k\) and for some constant \(D\). In fact with this assumption we have that:
\[
|(1 - 2^{1-s}) \zeta(s)| \leq \sum_{k=0}^{\infty} b_k P_k\left(\frac{s-\alpha}{\beta} + 1\right) \leq \text{const.} \sum_{k=0}^{\infty} k^{-\frac{\alpha-1/2-\delta}{\beta}} k^{-\left(\Re(s)-\alpha\right)/\beta + 1})
\]
\[
\leq \text{const.} \sum_{k=0}^{\infty} k^{-\left(1+\Re(s)-1/2-\delta\right)} < \infty
\]
if \(\Re(s) > \frac{1}{2} + \delta\).

For \(\alpha = \beta = 2\) (Riesz) we should have \(|b_k| \leq Dk^{-1/2+\epsilon}\). For the case \(\alpha = 1\) and \(\beta = 2\) (Hardy-Littlewood) we should have \(|b_k| \leq Dk^{-1/2+\epsilon}\). Another case of interest is the one where \(\alpha = \frac{3}{2}\) and \(\beta = 1\). In this case one should have \(|b_k| \leq Dk^{-1+\epsilon}\).

Of interest also, is the limiting case of large values of \(\beta\), where barely \(b_k\) should behave as \(|b_k| \leq D\).
For a strong argument (a Theorem) in favour of the validity of the Maslanka representation of \((s - 1)\zeta(s)\) in some regions of the complex plane (compact subsets), the reader should consult the works of Baez-Duarte [14] already mentioned and it is expected that using the same methods, the proof of (2.6) may be obtained for all \(\Re(s) > \frac{1}{2}\). Here, for our series we limit ourselves to a numerical analysis just illustrating the kind of accuracy of some representations.

**Remark.** Let us consider the Riesz case \(\alpha = \beta = 2\). We can write:

\[
(1 - e^{(1-s)\ln 2}) \zeta(s) = \sum_{k=0}^{\infty} P_k(\frac{s}{2}) \sum_{j=0}^{k} (-1)^j \binom{k}{j} (1 - e^{-(1+2j)\ln 2}) \zeta(2 + 2j)
\]

and using the Taylor’s expansion of \(e^x\), we obtain:

\[
(s - 1) \zeta(s) = \sum_{k=0}^{\infty} A_k P_k(\frac{s}{2})
\]

where

\[
A_k = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (2j + 1) \zeta(2j + 2)
\]

i.e. the representation obtained originally by a different method by Maslanka in a pioneering work [3]. We remark that (2.8) and (2.9) should not be considered as an approximation of our formulas (2.5) and (2.6) and vice versa. (2.5), (2.6) and (2.8), (2.9) are simply two different representations of functions related to the Riemann Zeta function, the first one given by \((s - 1)\zeta(s)\), the second one by \((1 - 2^{1-s})\zeta(s)\).

As an example, for \(s = \sigma\) with \(\sigma\) in \([0, 1]\), both representations give a good description of the real function \(\zeta(\sigma)\) as may easily be computationally checked.

We now proceed to obtain a representation of \(\zeta(s)\) possibly correct on the critical line \(s = \frac{1}{2} + it\), with the help of (2.5) and (2.6), in which we are free to set \(\alpha = \frac{1}{2}\) and \(\beta = i\). Then:

\[
(1 - 2^\frac{1}{2} - it) \zeta(\frac{1}{2} + it) = \sum_{k=0}^{\infty} b_k P_k(t + 1)
\]

where now

\[
b_k = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (1 - 2^\frac{1}{2} - ij) \zeta(\frac{1}{2} + ij)
\]

We now check the series in (2.10) restricting \(k\) up to 20 for \(t \leq 18\) and up to 50 for \(t > 18\). We compare the result with the exact functions \(\Re((1 - 2^t)\zeta(s))\) and \(\Im((1 - 2^t)\zeta(s))\), for \(s = \frac{1}{2} + it\) with \(t\) up to 40. The plots are given below. We obtain a good approximation with the emergence of the first five non-trivial zeros located at \(t_1 = 14.13472\ldots, t_2 = 21.02204\ldots, t_3 = 25.01085\ldots, t_4 = 21.02204\ldots, t_5 = 32.93505\ldots\). The numerical results are satisfactory until \(t \cong 35\).

This concludes the first part of our work. Below, in the second part we develop two new representations of the functions \(\ln((1 - 2^{1-s})\zeta(s))\) and \(\frac{d}{ds} \ln((s - 1)\zeta(s))\) which may possibly constitute a satisfactory approximation to the exact functions.
Figure 1. The plot of the real part of \( \sum_{k=0}^{20(50)} b_k P_k(t+1) \) [red] vs. \( \Re((1 - 2^\tau)\zeta(s)) \) [black]

Figure 2. The plot of the imaginary part of \( \sum_{k=0}^{20(50)} b_k P_k(t+1) \) [red] vs. \( \Im((1 - 2^\tau)\zeta(s)) \) [black]

3. A representation for the logarithm of the Zeta Function and an additional criterion for the truth of the RH

We will start as before but instead of writing \( \zeta(s) \) as a sum, i.e. \( \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \), we will use the Euler product formula to derive a new representation for \( \ln((1 - 2^{1-s})\zeta(s)) \), which of course should be carefully investigated by means of some numerical experiments. Thus:

\[
(3.1) \quad \ln[(1 - 2^{1-s})\zeta(s)] = \ln[(1 - 2^{1-s}) \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}] \quad \forall \Re(s) > 1
\]
For any prime $p$, we have:

$$\ln(1 - p^{-s}) = -\sum_{n=1}^{\infty} \frac{p^{-ns}}{n}$$

so that introducing the parameters $\alpha$ and $\beta$ as before we have that:

$$\sum_{n=1}^{\infty} \frac{p^{-\alpha n}}{n} (1 - (1 - p^{-\beta n}))^{\frac{s-\alpha}{\beta}} = \sum_{n=1}^{\infty} \frac{p^{-\alpha n}}{n} \sum_{k=0}^{\infty} (-1)^k (1 - p^{-\beta n})^{k} \left(\frac{s-\alpha}{k}\right)$$

$$= \sum_{k=0}^{\infty} P_k(\frac{s-\alpha}{\beta} + 1) \sum_{n=1}^{\infty} \frac{1}{n} \sum_{j=0}^{k} (-1)^j \binom{k}{j} p^{-\beta(n+j)}$$

$$= \sum_{k=0}^{\infty} P_k(\frac{s-\alpha}{\beta} + 1) k \sum_{j=0}^{k} (-1)^j \binom{k}{j} \ln(1 - p^{-(\alpha+j)})$$

the same treatment for the function $\ln(1 - 2^{1-s})$, gives:

$$\ln(1 - 2^{1-s}) = \sum_{k=0}^{\infty} P_k(\frac{s-\alpha}{\beta} + 1) k \sum_{j=0}^{k} (-1)^j \binom{k}{j} \ln(1 - 2^{1-(\alpha+j)})$$

where $P_k$ are still the Pochhammer’s polynomials.

Finally, the representation of $\ln((1 - 2^{1-s})\zeta(s))$, we propose is given by:

$$\ln\left[(1 - 2^{1-s})\zeta(s)\right] = \sum_{k=0}^{\infty} d_k P_k\left(\frac{s-\alpha}{\beta} + 1\right)$$

where now:

$$d_k := \sum_{j=0}^{k} (-1)^j \binom{k}{j} \ln\left[(1 - 2^{1-(\alpha+j)})\zeta(\alpha + \beta j)\right]$$

Remark. Another formal derivation of the above equations is the following:

$$\ln\left[(1 - 2^{1-s})\zeta(s)\right] = \ln\left[\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}\right]$$

Supposing now that the right hand side may be given as an unknown series

$$\sum_{r=1}^{\infty} \frac{a_r}{r^s}$$

we then have:

$$\sum_{r=1}^{\infty} \frac{a_r}{r^s} (1 - (1 - \frac{1}{r^\beta}))^{\frac{s-\alpha}{\beta}} = \sum_{k=0}^{\infty} P_k(\frac{s-\alpha}{\beta} + 1) \sum_{r=1}^{\infty} \frac{a_r}{r^s} (1 - \frac{1}{r^\beta})^k$$

$$= \sum_{k=0}^{\infty} P_k(\frac{s-\alpha}{\beta} + 1) \sum_{j=0}^{k} (-1)^j \binom{k}{j} \sum_{r=1}^{\infty} \frac{a_r}{r^{s+j\beta}}$$

$$= \sum_{k=0}^{\infty} P_k(\frac{s-\alpha}{\beta} + 1) \sum_{j=0}^{k} (-1)^j \binom{k}{j} \ln\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{s+j\beta}}\right)$$

which coincide with (3.2) and (3.3), obtained with the Euler product formula for $\Re(s) > 1$. (3.2) with (3.3), is the new formula possibly representing the logarithm of the Zeta function in terms of the two parameters Pochhammer’s polynomials.

To the best of our knowledge the above representation is new and it is our aim to carry out some numerical experiments in the sequel in order to support its validity also in some compact subset of the critical strip.
We now investigate the representation of the derivative of the function $\ln((s-1)\zeta(s))$:

\begin{equation}
\frac{d}{ds} \ln((s-1)\zeta(s)) = \frac{1}{s-1} + \frac{\zeta'(s)}{\zeta(s)}
\end{equation}

Then with $\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p^s}}$ we obtain:

\begin{align*}
\frac{\zeta'(s)}{\zeta(s)} &= -\sum_p \frac{d}{ds} \ln(1 - p^{-s}) = -\sum_p \frac{1}{1-p^{-s}} \frac{d}{ds} \left(1 - e^{-s \ln p}\right) \\
&= -\sum_p \frac{p^{-s}}{1-p^{-s}} \ln p = -\sum_p \ln p \sum_{q=1}^\infty \frac{1}{p^{sq}}
\end{align*}

Introducing as above the Pochhammer’s polynomials we obtain further:

\begin{align*}
\frac{\zeta'(s)}{\zeta(s)} &= -\sum_p \ln p \sum_{q=1}^\infty \frac{1}{p^{sq}} \left(1 - \left(1 - \frac{1}{p^{sq}}\right)\right) \frac{\frac{\sigma}{p}}{1+eta} \\
&= -\sum_p \ln p \sum_{k=0}^\infty P_k \left(\frac{s-\alpha}{\beta} + 1\right) \sum_{j=0}^k (-1)^j \frac{k!}{j!(k-j)!} \sum_{q=1}^\infty \frac{1}{p^{q(\alpha+\beta j)}} \\
&= \sum_{k=0}^\infty P_k \left(\frac{s-\alpha}{\beta} + 1\right) \sum_{j=0}^k (-1)^j \frac{k!}{j!(k-j)!} \frac{\ln p}{p^{q(\alpha+\beta j)}} \\
&= \sum_{k=0}^\infty P_k \left(\frac{s-\alpha}{\beta} + 1\right) \sum_{j=0}^k (-1)^j \frac{k!}{j!(k-j)!} \frac{\ln \left(1 - \frac{1}{p^{\alpha+\beta j}}\right)}{p^{\alpha+\beta j}} \\
&= \sum_{k=0}^\infty P_k \left(\frac{s-\alpha}{\beta} + 1\right) \sum_{j=0}^k (-1)^j \frac{k!}{j!(k-j)!} \ln (\alpha + \beta j)
\end{align*}

For $\frac{1}{s-1}$, using $\frac{1}{s-1} = \int_0^\infty e^{-\lambda(s-1)}d\lambda$ we have similarly:

\begin{align*}
\frac{1}{s-1} &= \int_0^\infty e^\lambda \frac{1}{e^{\alpha}} d\lambda = \int_0^\infty e^\lambda \frac{k}{e^{\alpha \lambda}} \left(1 - \left(1 - \frac{1}{e^{\alpha \lambda}}\right)\right)^{\frac{\sigma}{p}} d\lambda \\
&= \int_0^\infty e^\lambda \sum_{k=0}^\infty P_k \left(\frac{s-\alpha}{\beta} + 1\right) \sum_{j=0}^k (-1)^j \frac{k!}{j!(k-j)!} \frac{1}{e^{\alpha \lambda} + \beta j} d\lambda \\
&= \sum_{k=0}^\infty P_k \left(\frac{s-\alpha}{\beta} + 1\right) \sum_{j=0}^k (-1)^j \frac{k!}{j!(k-j)!} \frac{1}{e^{\alpha \lambda} + \beta j} \\
&= \sum_{k=0}^\infty P_k \left(\frac{s-\alpha}{\beta} + 1\right) \sum_{j=0}^k (-1)^j \frac{k!}{j!(k-j)!} \frac{1}{e^{\alpha \lambda} + \beta j - 1} \\
&= \sum_{k=0}^\infty P_k \left(\frac{s-\alpha}{\beta} + 1\right) \sum_{j=0}^k (-1)^j \frac{k!}{j!(k-j)!} \ln (\alpha + \beta j - 1)
\end{align*}

Thus, along these lines we obtain:

\begin{equation}
\frac{d}{ds} \ln((s-1)\zeta(s)) = \sum_{k=0}^\infty d_k P_k \left(\frac{s-\alpha}{\beta} + 1\right)
\end{equation}
where:

\[ (3.6) \quad \hat{d}_k = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{\partial}{\partial \alpha} \ln[(\alpha + \beta j - 1) \zeta(\alpha + \beta j)] \]

From the formula (7) in [16], where \( \rho \) represents a non-trivial zero of the Zeta function, i.e.:

\[
\frac{1}{s-1} + \frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} - \sum_{\rho} \frac{1}{s-\rho} + \sum_{\rho} \frac{1}{s+2n} + \sum_{n=1}^{\infty} \frac{1}{s+2n} + \frac{\zeta'(0)}{\zeta(0)}
\]

\[
= \frac{\zeta'(0)}{\zeta(0)} - 1 + \frac{1}{\rho} - \sum_{n=1}^{\infty} \frac{1}{s+2n} + \sum_{n=1}^{\infty} \frac{1}{s+2n} \]

Setting \( C = \frac{\zeta'(0)}{\zeta(0)} - 1 \), this equation applied to \( s = \alpha + \beta j \) in (3.6) gives:

\[
\hat{d}_k = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left( C + \int_{0}^{\infty} \left( \sum_{\rho} e^{-\lambda(\alpha+j\beta-j\rho)} + e^{-\lambda\rho} + \sum_{n=1}^{\infty} e^{-\lambda(\alpha+j\beta+2n)} - e^{-\lambda 2n} \right) d\lambda \right)
\]

\[
= \int_{0}^{\infty} \sum_{\rho} \left( e^{-\lambda(\alpha-j\rho)}(1 - \frac{1}{e^{\lambda\rho}})^{k} + e^{-\lambda}(1 - \frac{1}{e^{\lambda\rho}})^{k} \delta_{k,0} \right) d\lambda + \int_{0}^{\infty} \left( \sum_{n=1}^{\infty} e^{-\lambda(\alpha+2n)}(1 - \frac{1}{e^{\lambda\rho}})^{k} - e^{-\lambda 2n}(1 - \frac{1}{e^{\lambda\rho}})^{k} \delta_{k,0} \right) d\lambda
\]

We consider only \( k > 0 \). Now we make the variable change \( e^{-\lambda \beta} = x \) and finally we obtain:

\[
\hat{d}_k = \frac{1}{\beta} \int_{0}^{1} (1-x)^{k+1-1} \sum_{\rho} x^{\frac{\alpha-j\rho}{\beta}-1} dx + \frac{1}{\beta} \int_{0}^{1} (1-x)^{k+1-1} \sum_{n=1}^{\infty} x^{\frac{-2n}{\beta}-1} dx
\]

\[
= \frac{1}{\beta} \sum_{\rho} B\left(\frac{\alpha-j\rho}{\beta}, k+1\right) + \frac{1}{\beta} \sum_{n=1}^{\infty} B\left(\frac{-2n}{\beta}, k+1\right)
\]

where \( B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)} \) is the Beta function. Thus for large \( k \) we can write:

\[ (3.7) \quad \hat{d}_k = \frac{1}{\beta} \sum_{\rho} \Gamma\left(\frac{\alpha-j\rho}{\beta}\right) k^{\frac{-\alpha-j\rho}{\beta}} + \frac{1}{\beta} \sum_{n=1}^{\infty} \Gamma\left(\frac{\alpha+2n}{\beta}\right) k^{\frac{-2n-\alpha}{\beta}} \]

For the critical function [12] corresponding to \( \Re(s) = \sigma \) we have an analogous expression to the Baez-Duarte formula for the \( c_k \) appearing in the expansion of \( \zeta(s)^{-1} \) [4, 5]:

\[ (3.8) \quad k^{\frac{-\sigma}{\beta}} \hat{d}_k = \frac{1}{\beta} \sum_{\rho} \Gamma\left(\frac{\alpha-j\rho}{\beta}\right) k^{\frac{-\alpha-j\rho}{\beta}} + \frac{1}{\beta} \sum_{n=1}^{\infty} \Gamma\left(\frac{\alpha+2n}{\beta}\right) k^{\frac{-2n-\alpha}{\beta}} =: \psi_1(k) \]

On the other hand we can express \( \hat{d}_k \) and then the critical function with a second formula:

\[ (3.9) \quad \hat{d}_k = \frac{1}{\beta} \Gamma\left(\frac{\alpha-1}{\beta}\right) k^{\frac{-\alpha-1}{\beta}} - \sum_{p \text{ prime}} \ln p \sum_{q=1}^{\infty} \frac{1}{p^{\alpha q}} \left(1 - \frac{1}{p^{\beta q}}\right) \]

\[ (3.10) \quad k^{\frac{-\sigma}{\beta}} \hat{d}_k = \frac{1}{\beta} \Gamma\left(\frac{\alpha-1}{\beta}\right) k^{\frac{1-\sigma}{\beta}} - k^{\frac{-\sigma}{\beta}} \sum_{p \text{ prime}} \ln p \sum_{q=1}^{\infty} \frac{1}{p^{\alpha q}} \left(1 - \frac{1}{p^{\beta q}}\right) \]
In fact (see above) the Pochhammer expansion for \( \frac{1}{s-1} \) is:

\[
\frac{1}{s-1} = \sum_{k=0}^{\infty} s_k P_k \left( \frac{s-\alpha}{\beta} + 1 \right)
\]

where

\[
s_k = \int_0^\infty e^{-\lambda (s-1)} \left( 1 - e^{-\lambda \beta} \right)^k d\lambda
\]

which for large \( k \) behaves as \( \frac{1}{\beta} \Gamma \left( \frac{s-1}{\beta} \right) k^{-\frac{s-1}{\beta}} \). Indeed with the substitution \( e^{-\lambda \beta} = x \) we obtain:

\[
s_k = \frac{1}{\beta} \int_0^1 x^\frac{s-1}{\beta} - 1 \left( 1 - x \right)^k dx = \frac{1}{\beta} \int_0^1 x^\frac{s-1}{\beta} - 1 \left( 1 - x \right)^{k+1-1} dx = \frac{1}{\beta} B \left( \frac{s-1}{\beta}, k+1 \right)
\]

It is interesting to note that one can express the critical function in terms of the zeros of the Zeta function (3.8) or in terms of the primes (3.10). We will investigate numerically these two functions for the case \( \alpha = \frac{9}{2}, \beta = 4, \sigma = \frac{1}{2} \).

4. Numerical experiments

As a test of the goodness of (3.2) we draw in Figure 3 the plots of the function \( \ln \left( (1 - 2^{1-\sigma}) \zeta(\sigma) \right) \) and of its polynomial representation in the interval \( \sigma \in [-1, 1] \). Figure 3 shows a good match between them also in the “critical real interval” \([0, 1]\).

![Figure 3. The function \( \ln \left( (1 - 2^{1-\sigma}) \zeta(\sigma) \right) \) [black] and its polynomial representation [red]](image_url)

In the next figures we present the results of the numerical experiment performed on our representation (3.5) for the case \( \alpha = \frac{9}{2} \) and \( \beta = 4 \). We calculated the critical functions \( \psi_1 \) and \( \psi_2 \) for \( \Re(z) = \sigma = \frac{1}{2} \). In our calculations we considered only the first 10 non-trivial zeros of the Zeta function, the first 20 trivial ones and the first 5'000 primes. Furthermore using the usual substitution \( x = \log k, \psi_1 \) and
The critical function \( \psi_1 \) calculated with the zeros of the Zeta function.

The critical function \( \psi_2 \) calculated with the primes.

It is interesting to study the single contribution of a prime to the critical function \( \psi_2 \). In Figure 6 we computed the contributions of the 10th prime \((p = 29)\), of the
50th prime \((p = 229)\) and of the 100th prime \((p = 541)\), all the calculations were performed until \(q = 100\). The computations indicate that not only the contributions decrease with increasing \(p\) but also that great primes have an influence only on big values of \(k\).

![Figure 6](image)

**Figure 6.** The contribution to the critical function \(\psi_2\) of the primes \(p = 29\) [black], \(p = 229\) [red] and \(p = 541\) [green].

### 5. Infinite \(\beta\) Limit

In a numerical context we are also interested in the case of large \(\beta\) values. We start with the equation (7) in [16], given by:

\[
(5.1) \quad f(s) := \left( \frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1} \right) \frac{1}{s} = \sum_{\rho} \frac{1}{\rho(s-\rho)} - \sum_{n=1}^{\infty} \frac{1}{2n(s+2n)} + \left( \frac{\zeta'(0)}{\zeta(0)} - 1 \right) \frac{1}{s}
\]

and we set \(C = \frac{\zeta'(0)}{\zeta(0)} - 1 = \ln 2\pi - 1\). Then, using the formula \(\frac{1}{A} = \int_0^\infty e^{-\lambda A} d\lambda\) as above (\(\Re(A) > 0\)), we obtain:

\[
(5.2) \quad f(s) = \sum_{k=0}^{\infty} \hat{d}_k \mathcal{P}_k \left( \frac{s-\alpha}{\beta} + 1 \right)
\]

where

\[
(5.3) \quad \hat{d}_k = \left( \frac{1}{\beta} \right) \left( \sum_{\rho} \frac{\Gamma\left(\frac{\alpha-\rho}{\beta}\right)}{\rho} k^{-\frac{\alpha-\rho}{\beta}} - \sum_{n=1}^{\infty} \frac{\Gamma\left(\frac{n+2\alpha}{\beta}\right)}{2n} k^{-\frac{n+2\alpha-\sigma}{\beta}} + C \Gamma\left(\frac{\alpha}{\beta}\right) k^{-\frac{\sigma}{\beta}} \right)
\]

We now analyze \(\psi(k)\), the absolute value of the critical function, at large \(\beta\) values where \(\frac{1}{\beta} \Gamma\left(\frac{\alpha-\rho}{\beta}\right) \sim \frac{1}{\alpha-\rho}\) is valid.

\[
(5.4) \quad \psi(k) := \left| \frac{\hat{d}_k}{k^{\frac{n+2\alpha}{\beta}}} \right| = \left| \sum_{\rho} \frac{1}{\rho(\alpha-\rho)} k^{-\frac{n+2\alpha-\sigma}{\beta}} - \sum_{n=1}^{\infty} \frac{1}{2n(\alpha+2n)} k^{-\frac{n+2\alpha-\sigma}{\beta}} + \frac{C}{\alpha} k^{-\frac{\sigma}{\beta}} \right|
\]

Here the second and third term in the bracket converge for all \(\sigma > 0\) (in particular for \(\frac{1}{2} \leq \sigma \leq 1\)). If we choose \(\alpha = 1\), (5.4) would become in the \(\beta\) limit (supposing...
that this limit may be performed and has a meaning):

\[
\lim_{\beta \to \infty} \psi(k) = \left| \sum_{\rho} \frac{1}{\rho(1 - \rho)} - \sum_{n=1}^{\infty} \frac{1}{2n(1 + 2n)} + C \right| = \lim_{x \to 1} \left| \frac{d}{dx} \ln \zeta(x) + \frac{1}{x - 1} \right| = \gamma
\]

where \(\gamma \approx 0.577216\) is the Euler constant (see also [17]).

If such a limit is permitted our conjecture is that for \(\Re(s) \geq \sigma + \delta, \delta > 0\), as \(\beta \to \infty\):

\[
|f(s)| \sim B \frac{t}{\delta} \gamma
\]

where \(B\) is some constant and \(t = \Im(s)\).

Since from the definition \(P_k(\frac{x - \alpha}{\beta} + 1) = \frac{x - \alpha}{\beta} \frac{1}{k} P_{k-1}(\frac{x - \alpha}{\beta})\) we obtain:

\[
|f(s)| \sim \sum_{k=2}^{\infty} \left| \frac{1}{k} \psi(k) \right| \sim \sum_{k=2}^{\infty} \frac{B}{\delta} \frac{\alpha - s}{\beta} \sum_{k=2}^{\infty} \frac{1}{k^{1+\frac{\alpha}{\beta}}} \psi(k)
\]

and finally:

\[
|f(s)| \sim B \left( \frac{\alpha - s}{\beta} \right) \frac{k - \alpha}{\beta} \sum_{k=2}^{\infty} \frac{1}{k^{1+\frac{\alpha}{\beta}}} \psi(k)
\]

A similar (of course not rigorous) limit is formally obtained for \(\psi_2(k)\) using the primes along the lines for (3.4) to (3.10), which, as \(\beta \to \infty\) is given by:

\[
\lim_{\beta \to \infty} k^{1-\alpha} \frac{1}{\alpha} \frac{1}{\alpha - 1} - \frac{1}{\alpha} \sum_{p \text{ prime}} \ln p \sum_{q=1}^{\infty} \frac{1}{p^{\alpha q}}
\]

and thus [17]:

\[
\lim_{\alpha \to 1^{+}} \frac{1}{\alpha} \left( \frac{1}{\alpha - 1} - \sum_{p \text{ prime}} \ln p \sum_{q=1}^{\infty} \frac{1}{p^{\alpha q}} \right) = \lim_{\alpha \to 1^{+}} \frac{1}{\alpha} \frac{d}{d\alpha} \log((\alpha - 1) \zeta(\alpha)) = \gamma
\]

We carried out some numerical experiments restricted to large \(\beta\) values (until \(\beta = 10^6\)), using the first 3600 known zeros [18]. The computations in Figure 7 indicate that for a fixed \(k\), within the limit of accuracy of our computations, the difference between (5.4) and \(\gamma\) approximately stabilizes to less than 0.001 independently from the choice of \(k\). The difference is largely due only to the term involving the non-trivial zeros. That is if we need a higher precision we have to consider more non-trivial zeros in (5.4).
6. Conclusions

In this work we have found some new representations of functions related to the Riemann Zeta function in terms of the Pochhammer’s polynomials, i.e. for the Zeta function via the alternating series, for \((1-2^{-s})\zeta(s)\), for \(\ln((1-2^{-s})\zeta(s))\) and for the derivative of \(\ln((s-1)\zeta(s))\).

1. A numerical experiment for the first function give satisfactory results both for the real part as well for the imaginary part even on the critical line \(\Re(s) = 1/2\) (we have used the values \(\alpha = 1/2\), \(\beta = i\) and \(t\) up to \(\Im(s) = t < 35\)).

2. In a formal limit of our representations (2.6) for the special case \(\alpha = \beta = 2\) we obtain the Maslanka’s representation of \((s-1)\zeta(s)\).

3. For the expansion of the derivative of the function \(\ln((s-1)\zeta(s))\) in terms of the Pochhammer’s polynomials \(P_k(s)\) we have found two expressions (\(\psi_1\) and \(\psi_2\)) for the so called critical function: \(\psi_1\) in terms of the primes and \(\psi_2\) in terms of the trivial as well as the non-trivial zeros. We have then carried out a numerical experiment which gives a very satisfactory agreements between the two, which up to very high values of \(k\) remain bounded. The existence of absolut upper bounds for the critical functions at \(k\)-infinity may be considered as being equivalent to the truth of the RH.

4. Concerning the critical function in the large \(\beta\) limit, using \(\alpha = 1\), we may conjecture that \(\frac{1}{s}\) time the derivative of \(\ln((s-1)\zeta(s))\), using the inequality for the Pochhammer’s polynomials has, for \(\Re(s) > 1/2 + \delta\), a bound of the form \(\frac{B_t}{\delta}\gamma\) where \(\gamma\) is the Euler constant and \(t = \Im(s)\).

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S. Beltraminelli, CERFIM, Research Center for Mathematics and Physics, P.O. Box 1132, 6600 Locarno, Switzerland
E-mail address: stefano.beltraminelli@ti.ch

D. Merlini, CERFIM, Research Center for Mathematics and Physics, P.O. Box 1132, 6600 Locarno, Switzerland
E-mail address: merlini@cerfim.ch