ON THE INVARIANT MEASURE OF THE RANDOM DIFFERENCE EQUATION $X_n = A_nX_{n-1} + B_n$ IN THE CRITICAL CASE

SARA BROFFERIO, DARIUSZ BURACZEWSKI AND EWA DAMEK

Abstract. We consider the autoregressive model on $\mathbb{R}^d$ defined by the following stochastic recursion

$$X_n = A_nX_{n-1} + B_n,$$

where $\{(B_n, A_n)\}_{n \in \mathbb{N}}$ are i.i.d. random variables valued in $\mathbb{R}^d \times \mathbb{R}_+$. The critical case, when $\mathbb{E}\left[\log A_1\right] = 0$, was studied by Babillot, Bougerol and Elie, who proved that there exists a unique invariant Radon measure $\nu$ for the Markov chain $\{X_n\}$. In the present paper we prove that the weak limit of properly dilated measure $\nu$ exists and defines a homogeneous measure on $\mathbb{R}^d \setminus \{0\}$.

1. Introduction and the main result

We consider the autoregressive process on $\mathbb{R}^d$:

$$X_0^x = x,$$

$$X_n^x = A_nX_{n-1}^x + B_n,$$

where the random pairs $\{(B_n, A_n)\}_{n \in \mathbb{N}}$ valued in $\mathbb{R}^d \times \mathbb{R}_+$ are independent, identically distributed (i.i.d.) according to a given probability measure $\mu$. Markov chain (1.1) occurs in various applications e.g. in biology and economics, see [1, 28] and the comprehensive bibliography there.

It is convenient to define $X_n$ in the group language. Let $G$ be the "ax + b" group, i.e. $G = \mathbb{R}^d \rtimes \mathbb{R}_+$, with multiplication defined by $(b,a) \cdot (b',a') = (b + ab', aa')$. The group $G$ acts on $\mathbb{R}^d$ by $(b,a) \cdot x = ax + b$, for $(b,a) \in G$ and $x \in \mathbb{R}^d$. For each $n$, we sample the random variables $(B_n, A_n) \in G$ independently with respect to the measure $\mu$ and we write $W_n = (B_n, A_n) \cdots (B_1, A_1)$ for the left random walk on $G$. Then $X_n^x = W_n \cdot x$.

The Markov chain $X_n^x$ is usually studied under the assumption $\mathbb{E}\left[\log A_1\right] < 0$. Then, if additionally $\mathbb{E}\left[\log^+ |B|\right] < \infty$, there is a unique stationary probability measure $\nu$ [23], i.e. the measure $\nu$ on $\mathbb{R}^d$ satisfying

$$\mu * G \nu(f) = \nu(f),$$

for any positive measurable function $f$. Here

$$\mu * G \nu(f) = \int_G \int_{\mathbb{R}^d} f(ax + b)\nu(dx)\mu(db \, da).$$

In a number of papers [23, 20, 24, 18, 22, 8], under some additional assumptions, behavior of the tail of $\nu$ was studied. Roughly speaking, it is known that, if there is a positive constant $\alpha$ such that $\mathbb{E}[A_1^\alpha] = 1$, then

$$\nu\{\{t : |t| > z\}\} \sim Cz^{-\alpha} \quad \text{as} \quad z \to +\infty,$$

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In [8] not only the size of the tail is studied but also the asymptotic behavior of \( \nu \) at infinity. It is proved there that the weak limit of \( z^\alpha \delta_{(0,z^{-1})} *_G \nu(K) \), when \( z \to \infty \) exists and it is a Radon measure homogeneous of degree \( \alpha \).

Here we study the critical case, when \( \mathbb{E} [ \log A_1 ] = 0 \). Then \( X_n \) has no invariant probability measure. However it was proved by Babillot, Bougerol and Elie [1] that under the following hypotheses

- \( \mathbb{P}[A_1 = 1] < 1 \) and \( \mathbb{P}[A_1 x + B_1 = x] < 1 \) for all \( x \in \mathbb{R}^d \),
- \( \mathbb{E} \left[ (|\log A_1| + \log^+ |B_1|)^{2+\varepsilon} \right] < \infty \),
- \( \mathbb{E} [\log A_1] = 0 \).

there exists a unique (up to a constant factor) invariant Radon measure \( \nu \) (see also [3, 6]). We are going to say that \( \mu \) satisfies hypothesis \((H)\) if all the assumptions above are satisfied.

Our aim is to study behavior of \( \nu \) at infinity, i.e. to understand how the measure \( \nu(zK) = \delta_{(0,z^{-1})} *_G \nu(K) \) of the compact set \( K \) dilated by \( z > 0 \) behaves as \( z \) goes to infinity. The known results concern mainly the one dimensional setting and they have been proved under quite restrictive hypotheses. The first estimate of \( \nu \) is given in [1], where it is assumed that \( d = 1 \) and that the closed semigroup generated by the support of \( \mu \) is the whole group \( G \). Then for every \( \alpha < \beta \)

\[
\nu((\alpha z, \beta z)) \sim \log(\beta/\alpha) \cdot L(z) \quad \text{as } z \to +\infty,
\]

where \( L \) is a slowly varying function.

More recently the second author has proved, in [5], that \( L \) is constant, provided the measure \( \mu_A \) (the projection of \( \mu \) onto \( \mathbb{R}_+^d \)) is spread-out and \( \mu \) has finite small moments. (as defined in (1.3)) Moreover he has proved the positivity of the constant \( C \) in the special case \( B_1 > \varepsilon \) a.s. for \( \varepsilon > 0 \).

The only results concerning the multidimensional critical case have been obtained in [12, 9] in a very particular setting, when the measure \( \mu \) is related to a differential operator. More precisely, let \( \{\mu_t\} \) be the one parameter semigroup of probability measures, whose infinitesimal generator is a second order elliptic differential operator on \( \mathbb{R}^d \times \mathbb{R}_+^* \). Then there exists a unique Radon measure \( \nu \) that is \( \mu_t \)-invariant, for any \( t \). Moreover, \( \nu \) has a smooth density \( m \) such that

\[
m(zu) \sim C(u)z^{-d} \quad \text{as } z \to +\infty,
\]

for some continuous nonzero function \( C \) on \( \mathbb{R}^d \setminus \{0\} \).

Here we consider the multidimensional situation and general measures \( \mu \). We prove that the family of measures \( \delta_{(0,z^{-1})} *_G \nu \) has a weak limit as \( z \) tends to \( +\infty \), the limit measure is homogeneous and so it has a radial decomposition, analogous to the one obtained in the contracting case [22, 8]. For that we do not need hypotheses concerning the support of \( \mu \) and nonsingularity of the measure \( \mu_A \). Moreover we prove nondegeneracy of the limit.

To state our main result we need some further definitions. Let \( G(\mu_A) \) be the closed subgroup of \( \mathbb{R}_+^d \) generated by the support of \( \mu_A \). Since \( \mathbb{P}[A_1 = 1] < 1 \) we have two possibilities, either \( G(\mu_A) = \mathbb{R}_+^d \), in this case \( \mu_A \) is said aperiodic or \( G(\mu_A) \) is countable and \( \mu_A \) is said periodic, then \( G(\mu_A) = \{e^{np} : n \in \mathbb{Z}\} \) for some \( p > 0 \).

Given a unit vector \( w \) in \( \mathbb{R}^d \) let \( \mathbb{R}w \) be the line generated by \( w \) and let \( \pi_w(x) \) be the orthogonal projection of \( x \) onto \( \mathbb{R}w \).

We will say that hypothesis \((G)\) is satisfied if there exists an affine subspace \( W \subset \mathbb{R}^d \) of dimension \( d - 1 \) and a unit vector \( w \) perpendicular to \( W \) such that

- the half-space \( W + \mathbb{R}^+w \) is \( \mu \)-invariant;
- the projection of the action of the support of \( \mu \) onto \( \mathbb{R}w \), has no fixed points, i.e. \( \mathbb{P}[\pi_w(A_1x + B_1) = \pi_w(x)] < 1 \) for every \( x \in \mathbb{R}^d \);
• the following integral condition holds:

\[ \mathbb{E}\left[ \log^{-1} \left| \pi_w(B_1 + (1 - A_1)w_0) \right| \right] < \infty, \]

where \( w_0 \) is the multiple of \( w \) such that \( -w_0 \in W \).

Notice that hypothesis (G) is fulfilled if e.g. one of coordinates of \( B_1 \), say \( \pi_1(B_1) \) is positive a.e., \( \mathbb{E}\left[ \log^{-1} \left| \pi_1(B_1) \right| \right] < \infty \) and the action of \( \mu \) on the first coordinate has no fixed points.

The main results of the paper are the following

**Theorem 1.2.** Assume that hypothesis (H) is satisfied, the measure \( \mu_A \) is aperiodic and either

(1.3) \( \mathbb{E}[A^\delta + A^{-\delta} + |B|^\delta] < \infty \) for some \( \delta > 0 \), that is \( \mu \) has small moments,

or

(1.4) \( \mathbb{E}[\log|A| + \log^+|B|]^{4+\varepsilon} < \infty \) for some \( \varepsilon > 0 \) and hypothesis (G) holds.

Then there exists a probability measure \( \Sigma \) on \( S^{d-1} = \{|u| = 1\} \) and a positive number \( C_+ \) such that the measures \( \delta_{(0,z^{-1})} \ast_G \nu \) converge weakly on \( \mathbb{R}^d \setminus \{0\} \) to \( C_+ \Sigma \otimes \frac{da}{a} \) as \( z \to +\infty \), that is

\[ \lim_{z \to +\infty} \int_{\mathbb{R}^d} \phi(uz^{-1})\nu(du) = C_+ \int_{\mathbb{R}^d} \int_{S^{d-1}} \phi(aw)\Sigma(dw)\frac{da}{a}, \]

for every function \( \phi \in C_c(\mathbb{R}^d \setminus \{0\}) \).

In particular for every \( \alpha < \beta \)

(1.5) \( \lim_{z \to \infty} \nu\left( u : \alpha z < |u| < \beta z \right) = C_+ \log\frac{\beta}{\alpha}. \)

**Theorem 1.6.** If hypothesis (H) is satisfied, the measure \( \mu_A \) is periodic, i.e. \( G(\mu_A) = \langle e^p \rangle \) and either (1.3) or (1.4) holds, then there exists a positive constant \( C_+ \) such that

\[ \lim_{z \to \infty} \nu\left( u : z < |u| < e^{np}z \right) = nC_+, \]

for every \( n \geq 1 \).

The first estimate of the behavior of the tail of \( \nu \) is given in Section 2 under the very mild hypothesis (H). In Theorem 2.1, we prove that \( \delta_{(0,z^{-1})} \ast_G \nu(K) \) is smaller than \( C_K L(z) \), for all compact sets \( K \) and a slowly varying function \( L \), i.e. the family of measures \( \delta_{(0,z^{-1})} \ast_G \nu/L(z) \) is weakly compact. We also show some invariance properties of the accumulation points and an upper-bound for the measure \( \nu \) that implies \( \nu(du) \) integrability of the function \( (1 + |u|)^{-\gamma} \) for every \( \gamma > 0 \). Moreover, if additionally hypothesis (G) is assumed, then (Proposition 2.8) says that the slowly varying function \( L \) is dominated by the logarithm function, and for every \( \gamma > 0 \) the function \( \log^{-1(1+\gamma)}(2 + |u|) \) is \( \nu(du) \) integrable. All the estimates are crucial in the remaining part of the proof.

Next we reduce the problem of describing the tail of \( \nu \) to study asymptotic behavior of positive solutions of the Poisson equation, as in [7]. More precisely, given a positive \( \phi \in C_c(\mathbb{R}^d \setminus \{0\}) \) we define the function

(1.7) \( f_\phi(x) = \delta_{(0,e^{-x})} \ast_G \nu(\phi) = \int_{\mathbb{R}^d} \phi(e^{-x}u)\nu(du). \)

on \( \mathbb{R} \). Let \( \overline{\mu} \) be the law of \( - \log A_1 \) i.e. for a Borel set \( U \)

(1.8) \( \overline{\mu}(U) = \mu_A(\{x : - \log x \in U\}). \)
Observe that the mean of $\overline{\mu}$ is equal to 0. The convolution on $\mathbb{R}$ of $\overline{\mu}$ and a function $f$ is $\overline{\mu} \ast f(x) = \int f(y + x) \overline{\mu}(dy)$ and the function $f_\phi$ satisfies the Poisson equation

$$(1.9) \quad \overline{\mu} \ast f_\phi(x) = f_\phi(x) + \psi_\phi(x), \quad x \in \mathbb{R},$$

for a specific function $\psi_\phi$. $\psi_\phi$ posses some regularity properties and it is easier to study than $f_\phi$. The main problem can be formulated as follows: given a function $\psi_\phi$ describe the behavior at infinity of positive solutions of the Poisson equation. An answer to this rather classical question has been given by Port and Stone [27], under the hypothesis that $\phi$ describe their asymptotic behavior at infinity in the so-called boundary case. Roughly speaking, let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence of random positive numbers such that only first $N$ of them are nonzero, where $N$ is some random number, finite a.s. Given a random variable $Z$ and a sequence $\{Z_n\}_{n \in \mathbb{N}}$ of i.i.d. copies of $Z$ independent both on $N$ and $\{A_n\}_{n \in \mathbb{N}}$, we define new random variable $Z^* = \sum_{i=1}^{N} A_i Z_i$ and the map $Z \rightarrow Z^*$ is called smoothing transformation. The random variable $Z$ is said to be fixed point if $Z^*$ has the same distribution as $Z$. There exists an extensive literature, where the problems of existence, uniqueness and asymptotic behavior of $Z$ were studied, see e.g. Durrett and Liggett [14], Liu [25], Biggins and Kyprianou [4]. The case of solutions having finite mean has been completely described, however in the boundary case, when $Z$ has infinite mean all the results concerning asymptotic behavior of $Z$ are expressed in terms of the Laplace transform. Applying techniques introduced by Guivarc'h [21] and Liu [25] and Theorem 1.2 one can prove (of course under appropriate assumptions) that $\lim_{x \rightarrow +\infty} x P[Z > x]$ exists and is positive. A complete proof of this fact will be the subject of a future work.

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2. Upper bound

The goal of this section is to prove a preliminary estimate of the measure $\nu$ at infinity. We first prove that, under very mild hypothesis (H) on the measure $\mu$, the tail measure of compact set $\delta_{0,z^{-1}} \ast \nu(K)$ is bounded by a slowly varying function $L$, that is a function on $\mathbb{R}^+_\gamma$ such that $\lim_{z \rightarrow +\infty} L(az)/L(z) = 1$ for all $a > 0$. An important property of such functions is that they grow very slowly, namely slower than $z^\gamma$ for any $\gamma > 0$.

Under the hypothesis (G) we will prove in the next subsection a stronger result: any bounded function that is integrable, at infinity, with respect to the measure $d\nu/\overline{\mu}$ (i.e. the Lebesgue measure of $\mathbb{R}^+_\gamma$) is $\nu$-integrable. In particular this implies that $L(z)$ is bounded by $\log z$. 
2.1. Generic case.

**Theorem 2.1.** If hypothesis (H) is fulfilled, then there exists a positive slowly varying function \( L \) on \( \mathbb{R}_+^d \) such that the normalized family of measures on \( \mathbb{R}^d \setminus \{0\} \)

\[
\frac{\delta_{(0,z^{-1})} * G \nu}{L(z)}
\]

is weakly compact for \( z \geq 1 \). That is \( \frac{\delta_{(0,z^{-1})} * G \nu(\phi)}{L(z)} \) is bounded for all \( \phi \) bounded and compactly supported. Thus \((1 + |x|)^{-\gamma} \in L^1(\nu)\) for all \( \gamma > 0 \). Furthermore, all accumulation points \( \eta \) are non null and invariant with respect to the group \( G(\mu_A) \), that is

\[
\delta_{(0,a)} * G \eta = \eta \quad \forall a \in G(\mu_A).
\]

Theorem 2.1 is a partial generalization of Proposition 5.2 in [1]. It is proved there that this family of measures converges to the Lebesgue measure of \( \mathbb{R}_+^d \), if \( d = 1 \) and the closed semi-group generated by the support of \( \mu \) is the whole group \( G \).

First we prove that, since the support of the measure \( \mu \) contains contracting and dilating elements, there exists a compactly supported function \( r \) such that the quotient family \( \frac{\delta_{(0,z^{-1})} * G \nu}{\delta_{(0,z^{-1})} * G \nu(\eta)} \) is weakly compact, that is

**Proposition 2.2.** Under hypothesis (H), there exists a bounded compactly supported function \( r \) such that \( \delta_{(0,z^{-1})} * G \nu(r) \) is strictly positive for all \( z \geq 1 \). Furthermore, for every compact set \( K \) there is a positive constant \( C_K \) such that

\[
\delta_{(0,z^{-1})} * G \nu(K) \leq C_K \delta_{(0,z^{-1})} * G \nu(r) \quad \forall z \geq 1.
\]

**Proof.** For all real numbers \( \alpha \) and \( \beta \), consider the annulus

\[
C(\alpha, \beta) = \{ u \in \mathbb{R}^d \mid \alpha \leq |u| \leq \beta \},
\]

Observe that if either \( \alpha > \beta \) or \( \beta < 0 \) the set \( C(\alpha, \beta) \) is void. It is easy to check that for all \((b, a) \in G\) the following implication holds

\[
u(\alpha + |b|, \beta - |b|, a) \Rightarrow au + b \in C(\alpha, \beta)
\]

Let \( U \) be an open set in \( G \) and \( n \in \mathbb{N} \). Since \( \nu \) is invariant with respect to \( \mu^\ast \), we have

\[
\delta_{(0,z^{-1})} * G \nu(C(\alpha, \beta)) = \int_{\mathbb{R}^d} \int_G 1_{C(\alpha, \beta)}(z^{-1}(au + b)) \mu^\ast \nu(du) 
\]

\[
\geq \int_{\mathbb{R}^d} \int_G 1_{C(\alpha, \beta)}(au + b) 1_U(b, a) \mu^\ast \nu(du) 
\]

\[
\geq \mu^\ast(U) \nu \left( C \left( \max_{(b, a) \in U} \alpha + \frac{|b|}{a}, \min_{(b, a) \in U} \beta - \frac{|b|}{a} \right) \right).
\]

First we prove that there exists a sufficiently large \( R > 0 \) such that \( \delta_{(0,z^{-1})} * G \nu(C(1/R, R)) \) is strictly positive for all \( z \geq 1 \). By hypothesis (H) the support of \( \mu \) contains at least two elements \( g_+ = (b_+, a_+) \) and \( g_- = (b_-, a_-) \) with \( a_+ > 1 > a_- \).

Fix \( z \geq 1 \) and take \( n \in \mathbb{N} \) such that \( a_+^{n-1} \leq z \leq a_-^n \). Clearly, if \( g^n = (b(g^n), a(g^n)) \) is the \( n \)-th power of an element \( g = (b, a) \in G \) then

\[
a(g^n) = a^n \quad \text{and} \quad b(g^n) = \sum_{i=0}^{n-1} a^i b = \frac{a^n - 1}{a - 1} b.
\]
Consider the \( \delta \)-neighborhood of \( g^n \)
\[
U_\delta(g^n) = \{(b, a) \in G | e^{-\delta} < a(a) - n < e^{\delta} \text{ and } |b - b(g^n)| < \delta\}.
\]
Observe that \( \mu^n(U_\delta(g^n)) > 0 \) for all \( \delta > 0 \) and for \((b, a) \in U_\delta(g^n)\)
\[
\frac{z/R + |b|}{a} \leq \frac{a^n_+}{R} + |b(g^n_+)| + \delta \leq e^{\delta} \left( \frac{1}{R} + \frac{|b_+|}{a_+ - 1} \right) =: \alpha_R
\]
\[
\frac{Rz - |b|}{a} \geq \frac{Ra_+^{-1} - |b(g^n_+)| - \delta}{e^{\delta} a^n_+} \geq e^{\delta} \left( \frac{R}{a_+} - \frac{|b_+|}{a_+ - 1} - \delta \right) =: \beta_R.
\]
Since \( \nu \) is a Radon measure with infinite mass, its support cannot be compact. Thus, for a fixed \( \delta \), there exists a sufficiently large \( R \) such that: \( \nu(C(\alpha_R, \beta_R)) > 0 \). Then by (2.4):
\[
\delta_{(0, z^{-1})} * G \nu(C(1/R, R)) \geq \mu^n(U_\delta(g^n)) \nu(C(\alpha_R, \beta_R)) > 0
\]
for all \( z \geq 1 \).

For \( R > 2 \) consider the compact sets \( K^n_\pm = C(2a^n_+/R, a^n_+/R/2) \). Observe that for \( \delta < \log(4/3) \), \((b, a) \in U_\delta(g^n_\pm)\) and \( z > z^n_\pm := 2R(|b(g^n_\pm)| + \delta) \):
\[
\frac{z/R + |b|}{a} \leq z e^{\delta} \frac{1}{R} + z^{-1}(|b(g^n_\pm)| + \delta) \leq z \frac{2a^n_+/R}{2} \leq \frac{2a^n_+/R}{2}
\]
\[
\frac{Rz - |b|}{a} \geq z e^{-\delta} R - z^{-1}(|b(g^n_\pm)| + \delta) \geq z \frac{a^n_+/R}{2} \cdot 2e^{-\delta} \left( 1 - z^{-1}(|b(g^n_\pm)| + \delta) \right) \geq \frac{a^n_+/R}{2}.
\]
Thus by (2.4):
\[
\delta_{(0, z^{-1})} * G \nu(C(1/R, R)) \geq \mu^n(U_\delta(g^n_\pm)) \nu(C(z a^n_+/R, z a^n_+/R/2)) = C_n^{-1} \delta_{(0, z^{-1})} * G \nu(K^n_\pm)
\]
for all \( z > z^n_\pm \). Since \( \delta_{(0, z^{-1})} * G \nu(C(1/R, R)) > 0 \), the above inequality holds in fact for all \( z \geq 1 \), possibly with a bigger constant \( C_K \) and sufficiently large \( R \).

We may assume that \( R > 2 \max\{a_+, 1/a_-\} \), then the family of sets \( K^n_\pm \) covers \( \mathbb{R}^d \setminus \{0\} \). Take any function \( r \in C_c(\mathbb{R}^d \setminus \{0\}) \) such that \( r(u) \geq 1_{C(1/R, R)}(u) \). Then a generic compact set \( K \) in \( \mathbb{R}^d \setminus \{0\} \) is covered by a finite number of compacts \( \{K_i\}_{i \in I} \) of the type \( K^n_\pm \) and
\[
\delta_{(0, z^{-1})} * G \nu(K) \leq \sum_{i \in I} \delta_{(0, z^{-1})} * G \nu(K_i) \leq \left( |I| \max_{i \in I} C_{K_i} \right) \delta_{(0, z^{-1})} * G \nu(r)
\]
for all \( z \geq 1 \). Moreover, in view of (2.5), \( \delta_{(0, z^{-1})} * G \nu(r) \) is strictly positive for all \( z \geq 1 \), which finishes the proof.

Now we prove that \( \mu \)-invariance of \( \nu \) implies that the accumulation points are invariant by the action of \( G(\mu_A) \), namely we have

**Lemma 2.6.** Suppose that there exists a function \( L(z) \) such that
\[
\frac{\delta_{(0,z^{-1})} * G \nu}{L(z)}
\]
is weakly compact when \( z \) goes to \( +\infty \), then the accumulation points \( \eta \) are invariant for the action of \( G(\mu_A) \).

**Proof.** Let \( \eta \) be a limit measure i.e. there is a subsequence \( \{z_n\} \) such that
\[
\lim_{n \to \infty} \frac{\delta_{(0,z_n^{-1})} * G \nu(\phi)}{L(z_n)} = \eta(\phi) \quad \forall \phi \in C_c(\mathbb{R}^d \setminus \{0\}).
\]
Fix a function \( \phi \in C^1_c(\mathbb{R}^d \setminus \{0\}) \) and observe that for all \((b,a) \in G\) there is a compact set \( K = K(b) \) and a constant \( C \) such that
\[
|\phi(z^{-1}(au + b)) - \phi(z^{-1}(au))| < C|z^{-1}b| \mathbf{1}_K(z^{-1}(au))
\]
for all \( z > 1 \) and \( u \in \mathbb{R}^d \).

We claim that the function
\[
h(y) = \delta_{(0,y)} *_G \eta(\phi) = \lim_{n \to \infty} \frac{\delta_{(0,z^{-1}n)} *_G \nu(\phi)}{L(z_n)}
\]
on \( \mathbb{R}^+ \) is \( \mu_A \)-superharmonic. Indeed, take a function \( \psi \in C_c(\mathbb{R}^d \setminus \{0\}) \) such that \( \psi \geq 1_{a^{-1}K} \), then
\[
\lim_{n \to \infty} \frac{|\delta_{(0,z^{-1}n)} * G \delta_{(b,a)} * G \nu(\phi) - \delta_{(0,z^{-1})} * G \delta_{(0,a)} * G \nu(\phi)|}{L(z_{n})} \leq \lim_{n \to \infty} \frac{C|z_{n}^{-1}b|\nu(a^{-1}z_{n}K)}{L(z_{n})} \leq \lim_{n \to \infty} \frac{C|z_{n}^{-1}b|\delta_{(0,z^{-1})} * G \nu(\psi)}{L(z_{n})} = C\eta(\psi) \cdot \lim_{n \to \infty} |z_{n}^{-1}b| = 0,
\]
hence
\[
\int_{G} h(ay) \mu_A(da) = \int_{G} \lim_{n \to \infty} \frac{\delta_{(0,z^{-1}n)} * G \delta_{(b,a)} * G \nu(\phi)}{L(z_{n})} \mu(db da) = \int_{G} \lim_{n \to \infty} \frac{\delta_{(0,z^{-1}n)} * G \delta_{(b,a)} * G \nu(\phi)}{L(z_{n})} \mu(db da) \leq \lim_{n \to \infty} \frac{\delta_{(0,z^{-1}n)} * G \mu * G \nu(\phi)}{L(z_{n})} \text{ by Fatou’s Lemma} = \lim_{n \to \infty} \frac{\delta_{(0,z^{-1}n)} * G \nu(\phi)}{L(z_{n})} = h(y)
\]
Since \( h \) is positive, then by Choquet-Deny theorem \( h(ay) = h(y) \) for every \( a \in G(\mu_A) \), that is
\[\delta_{(0,a)} * G \eta(\phi) = \eta(\phi), \quad \forall \phi \in C_c(\mathbb{R}^d \setminus \{0\}).\]

\[\Box\]

**Proof of Theorem 2.1.** Let \( r \) be the function introduced in Proposition 2.2 and take any \( R \in C_c(\mathbb{R}^d \setminus \{0\}) \) such that \( R > r \). Let \( L(z) = \delta_{(0,z^{-1})} * G \nu(R) \), so that, by Proposition 2.2, \( \delta_{(0,z^{-1})} * G \nu/L(z) \) is weakly compact when \( z \) goes to \( +\infty \). It remains to prove that \( L \) is a slowly varying function. Observe that
\[
L(az) = \int_{\mathbb{R}^d} R(z^{-1}a^{-1}u)\nu(du) = \delta_{(0,z^{-1})} * G \nu(R_a)
\]
where \( R_a(u) = R(a^{-1}u) \).

Since \( R_a \) has compact support thus \( L(az)/L(z) \) is bounded. Let \( z_n \) be a sequence such that both \( L(az_n)/L(z_n) \) and \( \delta_{(0,z^{-1}n)} * G \nu/L(z_n) \) converge i.e. there is a number \( l \) and a measure \( \eta \) such that
\[
l = \lim_{n \to \infty} \frac{L(az_n)}{L(z_n)} = \eta(R_a) = \delta_{(0,a^{-1})} * G \eta(R).
\]
Observe that, since \( \eta(R) = 1 \), then for every \( a \) such that \( \delta_{(0,a^{-1})} * G \eta(R) = \eta(R) \)
\[
\lim_{z \to +\infty} \frac{L(az)}{L(z)} = 1.
\]
By Lemma 2.6, $\eta$ is invariant under the action of closed group $G(\mu_A)$ generated by the support of $\mu_A$ and the proof in the case $\mu_A$ aperiodic is completed.

If $G(\mu_A) = \langle e^p \rangle$, consider the function

$$R(u) = \int_{\mathbb{R}^*_+} 1_{[e^{-p},e^p)}(t) r(u/t) \frac{dt}{t}.$$ 

An easy argument shows that $R$ is in $C_c(\mathbb{R}^d \setminus \{0\})$ and it is bigger of some multiple of $r$. We claim that $\delta_{(0,a^{-1})} * G \eta(R) = \eta(R)$ for all $a \in \mathbb{R}^*_+$ not only for $a \in G(\mu_A)$. In fact, let $e^{Kp} \in G(\mu_A)$ such that $e^{Kp} > ae^p$ then

$$\delta_{(0,a^{-1})} * G \eta(R) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^*_+} 1_{[ae^{-p},ae^p)}(t) r(u/t) \frac{dt}{t} \eta(du)$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^*_+} \left(1_{[ae^{-p},e^{Kp})}(t) - 1_{[ae^{-p},e^{Kp})}(t)\right) r(u/t) \frac{dt}{t} \eta(du)$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^*_+} 1_{[ae^{-Kp},e^{Kp})}(t) r(e^{-(K-1)p} u/t) - 1_{[ae^{-Kp},e^{-p})}(t) r(e^{-(K-1)p} u/t) \frac{dt}{t} \eta(du)$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^*_+} 1_{[ae^{-Kp},e^{Kp})}(t) r(u/t) - 1_{[ae^{-Kp},e^{-p})}(t) r(u/t) \eta(du) \frac{dt}{t}$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^*_+} 1_{[e^{-p},e^p)}(t) r(u/t) \frac{dt}{t} \eta(du) = \eta(R),$$

since $\eta$ is $G(\mu_A)$-invariant. \hfill \Box

The following lemma will be used in the sequel to give bounds for integrals against $\nu$. The statement holds for any Radon measure $\rho$ on $\mathbb{R}^d \setminus \{0\}$ for which we can control the growth of $\delta_{(0,a^{-1})} * G \rho(K)$.

**Lemma 2.7.** Let $\rho$ be a Radon measure and $l(z)$ a nondecreasing function such that

$$\rho(z < |u| \leq ze) \leq C_1 l(z)$$

for a constant $C_1$ and for every $z \in \mathbb{R}^*_+$. Then for all $M > 0$ and all nonnegative functions $f$

$$\int_{|u| \geq M} f(|u|) \rho(du) \leq C_1 \int_{e^{-1}M}^{\infty} f(a) l(a) \frac{da}{a}, \text{ if } f \text{ is nonincreasing}$$

and

$$\int_{0 < |u| \leq M} f(|u|) \rho(du) \leq C_1 \int_0^{eM} f(a) l(a) \frac{da}{a}, \text{ if } f \text{ is nondecreasing}.$$

In particular under hypothesis $(H)$,

$$\int_{\mathbb{R}^d} \frac{1}{1 + |u|^\gamma} \nu(du) < \infty$$

for all $\gamma > 0$. 
Proof. If $f$ is nonincreasing

$$
\int_{|u| \geq M} f(|u|) \rho(du) = \sum_{n=0}^{\infty} \int_{Me^n \leq |u| < Me^{n+1}} f(|u|) \rho(du)
\leq \sum_{n=0}^{\infty} f(Me^n) \rho(Me^{n-1}e \leq |u| < Me^{n-1}e^2)
\leq C_1 \sum_{n=0}^{\infty} f(Me^n) l(Me^{n-1})
\leq C_1 \sum_{n=0}^{\infty} \int_{Me^n}^{Me^{n-1}} f(a) l(a) \frac{da}{a}
\leq C_1 \int_{e^{-1}M}^{\infty} f(a) l(a) \frac{da}{a}.
$$

Exactly in the same way we prove the second part of the Lemma.

We can apply this result to $\nu$, since any slowly varying function $L(z)$ is smaller of a multiple of $z^\gamma$ for all $\gamma > 0$.

2.2. Upper bounds under hypothesis (G). The main result of this subsection is the following

**Proposition 2.8.** Assume that hypotheses (H) and (G) are satisfied, then there exists a constant $C$ such that for every bounded nonincreasing nonnegative function $f$ on $\mathbb{R}$

$$
\int_{\mathbb{R}^d} f(|u|) \nu(du) < C(\|f\|_\infty + \int_{1/e}^{\infty} f(y) \frac{dy}{y})
$$

In particular for every $\varepsilon > 0$

$$
\int_{\mathbb{R}^d} \frac{1}{\log^{1+\varepsilon}(2 + |u|)} \nu(du) < \infty
$$

and for $z > 1/e$

$$
\nu\{|u| < z\} < C(2 + \log z).
$$

Let us recall the following [1] explicit construction of the measure $\nu$. Define a random walk on $\mathbb{R}$

$$
S_0 = 0,
S_n = \log(A_1 \ldots A_n), \quad n \geq 1,
$$
and consider the downward ladder times of $S_n$:

$$
L_0 = 0,
L_n = \inf\{k > L_{n-1}; S_k < S_{L_{n-1}}\}.
$$

Let $L = L_1$. The Markov process $\{X_{L_n}^x\}$ satisfies the recursion

$$
X_{L_n}^x = M_n X_{L_{n-1}}^x + Q_n,
$$

where $(Q_n, M_n)$ is a sequence of $G$-valued i.i.d. random variables and $(Q_n, M_n) \equiv_d (X_L, e^{S_L})$. We denote by $\mu_L$ the law of $(Q_n, M_n)$. It is known that $-\infty < \mathbb{E}S_L < 0$ and $\mathbb{E}[\log^+ |X_L|] < \infty$ (see
Let $U$ be the potential associated with the random walk $Y_1 + \ldots + Y_n$, i.e.

$$U(a, b) = \mathbb{E} \left[ \# n : a < Y_1 + \ldots + Y_n \leq b \right].$$

By the renewal theorem $U(k, k + 1)$ is bounded, thus we have

$$\nu(f) \leq \int_{\mathbb{R}^+} \mathbb{E} \left[ \sum_{n=0}^{\infty} f(e^{Y_1 + \ldots + Y_n}x) \right] \nu_L(dx) \leq \sum_{k=0}^{\infty} \int_{\mathbb{R}^+} U(k, k + 1)f(e^kx)\nu_L(dx) \leq C \sum_{k=0}^{\infty} \int_{\mathbb{R}^+} f(e^kx)\nu_L(dx).$$

Next we divide the integral into two parts. First we assume that $x > 1$:

$$\sum_{k=0}^{\infty} \int_{1}^{\infty} f(e^kx)\nu_L(dx) \leq \sum_{k=0}^{\infty} f(e^k) \leq \int_{-1}^{\infty} f(e^y)dy = \int_{-1/e}^{\infty} f(y)\frac{dy}{y}.$$
Secondly, for $0 < x < 1$ we write

$$
\sum_{k=0}^{\infty} \int_0^1 f(e^k x) \nu_L(dx) \leq \int_0^1 \left( \sum_{k=0}^{\infty} + \sum_{k=|\log x|}^{\infty} \right) f(e^k x) \nu_L(dx)
$$

$$
\leq C\|f\| \int_0^1 \left| \log x \right| \nu_L(dx) + \sum_{k=0}^{\infty} \left| f(e^k) \right|
$$

$$
\leq C\|f\| \int_0^1 \left| \log x \right| \nu_L(dx) + \int_{1/e}^{\infty} f(y) \frac{dy}{y}.
$$

Hence to prove (2.9) we have to justify that the first term above is finite. For that we use the integral condition in hypothesis (G), which in this setting says that $E[\left| \log B_1 \right|] < \infty$. Notice that if $x, y \in \mathbb{R}^+$ and $x + y < 1$ then $\left| \log(x + y) \right| < \left| \log x \right|$. We write

$$
\int_0^1 \left| \log x \right| \nu_L(dx) = \int \int_{ax+b<1} \left| \log(ax + b) \right| \mu_L(db da) \nu_L(dx)
$$

$$
= \int_{\mathbb{R}^+} E \left[ \left| \log X_L^x \right| \cdot 1_{\{X_L^x < 1\}} \right] \nu_L(dx)
$$

$$
\leq \int_{\mathbb{R}^+} E \left[ \left| \log \left( \frac{A_1 A_2 \ldots A_L B_1}{A_1} \right) \right| \right] \nu_L(dx)
$$

$$
\leq E[|S_L| + |\log B_1| + |\log A_1|] < \infty,
$$

that completes the proof of (2.9) in this case.

**Step 2.** To generalize the results to higher dimensions, the key observation is that the measure $\nu$ can be compared with the invariant measures for projections of the process $\{X_n\}$ onto one dimensional subspaces. Their behavior at infinity is already controlled.

Let $w \in \mathbb{R}^d \setminus \{0\}$ be the unit vector as in hypothesis (G)

and let $\pi_w$ be the orthogonal projection on the line $\mathbb{R}w = \{sw\}_{s \in \mathbb{R}}$. Consider the random process on the line

$$
X_n^{w,x} = \pi_w(X_n^x) = A_n X_{n-1}^{w,x} + \pi_w(B_n),
$$

where $x = \pi_w(x)$. Let $\mu^w$ be the law of $(\pi_w(B_1), A_1)$, then the measure $\mu^w$ satisfies hypothesis (H). Therefore there exists a unique Radon measure $\nu_w$ on $\mathbb{R}w$, which is the invariant measure of the process $\{X_n^{w,x}\}$.

We claim that $\nu^w$ is the projection of $\nu$ onto $\mathbb{R}w$ that is

$$
\nu^w = \pi_w(\nu).
$$

As in (2.13), we may write

$$
\nu^w(g) = \int_{\mathbb{R}} E \left[ \sum_{n=0}^{L} g(X_n^{w,x}) \right] \nu_L^w(dx),
$$

for any positive function $g$ on $\mathbb{R}$. $\nu_L^w$ is the unique invariant measure for $\{\pi_w(X_n^L)\} = \{X_n^{w,x}\}$. Notice that $\nu_L^w$ is projection of $\nu_L$ onto $\mathbb{R}w$:

$$
\nu_L^w = \pi_w(\nu_L).
$$
Step 3. General case. Let \( w_0 \) be a multiple (possibly null) of \( w \), such that \( 0 \in W + w_0 \). The measure \( \mu_0 = \delta_{(w_0,1)} \ast \mu \ast \mu \) satisfies hypothesis (H), hence there exists a unique \( \mu \)-invariant Radon measure \( \nu_0 \), and one can easily prove that \( \nu_0 = \delta_{w_0} \ast \mu \). \( \mu_0 \) satisfies hypothesis (G) and the \( \mu \)-invariant half-space is \( W + w_0 + \mathbb{R}^+ w = \{ u : \pi_w(u) \in \mathbb{R}^+ w \} \). This implies that \( \pi_w(B_1) \in \mathbb{R}^+ w \) almost surely. Moreover,

\[
\int \log^{-} |\pi_w(\cdot)| \mu(\cdot) = \int G \log^{-} |\pi_w(b + (1-a)w_0)| \mu(\cdot) < \infty.
\]

Therefore, \( \mu_0 w \) satisfies the hypothesis of step 1 and \( \nu_0 w \) satisfies (2.9). Hence for any nonnegative and nonincreasing function \( f \) on \( \mathbb{R} \):

\[
\int f(|u|) \nu(du) \leq \int f(|\pi_w(u)|) \nu(du) \leq \int f(|\pi_w(u + w_0)| - |w_0|) \nu(du)
\]

\[
= \int f(|x| - |w_0|) \nu_0 w (dx) \leq C' \left( \| f \| \infty + \int_{1/e}^{\infty} f(y - |w_0|) \frac{dy}{y} \right)
\]

\[
= C' \left( \| f \| \infty + \int_{1/e}^{1/e + |w_0|} f(y - |w_0|) \frac{dy}{y} + \int_{1/e + |w_0|}^{\infty} f(y - |w_0|) \frac{y - |w_0|}{y} \frac{dy}{y - |w_0|} \right)
\]

\[
\leq C \left( \| f \| \infty + \int_{1/e}^{\infty} f(y) \frac{dy}{y} \right)
\]

To prove (2.10) we set \( f = 1_{[-\infty,z]} \). Then

\[
\nu\{ |u| \leq z \} \leq C(2 + \log z).
\]

3. RECURRENT POTENTIAL KERNEL AND SOLUTIONS OF THE POISSON EQUATION FOR GENERAL PROBABILITY MEASURES

As it has been observed in the introduction, to understand the asymptotic behavior of the measure \( \nu \) one has to consider of the function

\[
f_\psi(x) = \int_{\mathbb{R}^d} \phi(ue^{-x}) \nu(du)
\]

that is a solution of the Poisson equation

(3.1) \[ \Xi \ast f = f + \psi \]
for a peculiar choice of the function $\psi$, that is

$$\psi_\phi = \mu \ast f_\phi - f_\phi.$$  

Studying solutions of such equation for a centered probability measure $\mu$ on $\mathbb{R}$ is a classical problem. Port and Stones in their papers [26] and [27] give an explicit formula describing all bounded from below solutions of (3.1) in term of the recurrent potential kernel $A$ of the function $\psi$. However, to obtain this result, they suppose either that the measure is spread-out or, if not, that functions $\psi$ satisfy conditions to restrictive from our point of view. Therefore, the previous results on the decay of the measure $\nu$ were obtained in [7] under the hypothesis that $\mu$ is spread out. The goal of this section is to generalize Port and Stone techniques to arbitrary measures, that do not satisfy any smoothness conditions, and to an appropriate class of functions $\psi$ depending on the measure $\mu$.

Let $\mu$ be a centered probability measure on $\mathbb{R}$ with second moment $\sigma^2 = \int_\mathbb{R} x^2 \mu(dx)$ (we do not assume in this section that $\mu$ is related to $\mu$). If we exclude the degenerate case when $\mu = \delta_0$, the closed group $G(\mu)$ generated by the support of $\mu$ can be either a discrete group of the type $p \mathbb{Z}$ or $\mathbb{R}$. In the latter case, the measure $\mu$ is said aperiodic and we set $p = 0$.

The Fourier transform of $\mu$

$$\hat{\mu}(\theta) = \int_\mathbb{R} e^{ix\theta} \mu(dx)$$

is a continuous bounded function (of period $2\pi/p$, if $\mu$ is periodic), whose Taylor expansion near zero is

$$\hat{\mu}(\theta) = 1 + O(\theta^2)$$

and such that

$$|1 - \hat{\mu}(\theta)| > 0 \quad \forall \theta \in (0, 2\pi/p)$$

Consider the set $\mathcal{F}(\mu)$ of functions $\psi$ that can be written as

$$\psi(x) = \frac{1}{2\pi} \int_\mathbb{R} e^{-ix\theta} \hat{\psi}(\theta)d\theta$$

for some bounded, integrable, complex valued function $\hat{\psi}$ verifying the following hypothesis

- its Taylor expansion near 0 is
  $$\hat{\psi}(\theta) = J(\psi) + i\theta K(\psi) + O(\theta^2)$$
  for two constants $J(\psi)$ and $K(\psi)$,
- the function $\theta \mapsto \frac{\hat{\psi}(\theta)}{1 - \hat{\mu}(\theta)} \cdot 1_{[-a,a]}(\theta)$ is integrable for some $a \in (0, 2\pi/p)$.

Notice that the first condition is satisfied when $\psi$ is a continuous integrable function, such that $x^2 \psi$ is integrable and whose Fourier transform is integrable. In this case:

$$J(\psi) = \int_\mathbb{R} \psi(x)dx \quad \text{and} \quad K(\psi) = \int_\mathbb{R} x\psi(x)dx.$$  

The second condition is satisfied when the measure is aperiodic and the Fourier transform of $\psi$ has compact support or in the case the measure $\mu$ is spread-out (since is this case $\limsup_{|\theta| \to \infty} |\hat{\mu}(\theta)| < 1$). Thus, the set $\mathcal{F}(\mu)$ contains the set of functions on which Port and Stone define the recurrent potential and, in many cases, it is bigger. We will see that even in the periodic case $\mathcal{F}(\mu)$ contains interesting functions and if we suppose that $\mu$ is $p$-nonlattice (i.e. $\liminf_{|\theta| \to \infty} |\theta|^p |1 - \hat{\mu}(\theta)| > 0$, see [10, 5]), then $\mathcal{F}(\mu)$ contains the Schwartz space.
For $0 < \lambda < 1$ let

$$G^\lambda \ast \psi = \sum_{n=0}^{\infty} \lambda^n \mu^n \ast \psi,$$

where $\mu^n$ denotes the $n$-th convolution power of $\mu$. One can easily see that the foregoing series is convergent when $\psi$ is a bounded measurable function. Next we define

$$A^\lambda \psi = c_\lambda J(\psi) - G^\lambda \ast \psi,$$

where $c_\lambda = G^\lambda \ast g(0)$ for some fixed positive function $g$ in $F(\mu)$ such that $J(g) = 1$.

We are going to generalize the classical results of Port and Stone to functions $\psi \in F(\mu)$ and to show that then the limit value of $A^\lambda \psi$ exists and provides solutions of the Poisson equation (3.1). We state here the main results that are proved in Appendix A.

**Theorem 3.2.** Assume that $\psi \in F(\mu)$. Then the potential

$$A\psi(x) = \lim_{\lambda \rightarrow 1} A^\lambda \psi(x)$$

is a well defined continuous solution of the Poisson equation (3.1). Furthermore if $J(\psi) \geq 0$ then $A\psi$ is bounded from below and

$$\lim_{x \rightarrow \pm \infty} \frac{A\psi(x)}{x} = \pm \sigma^{-2} J(\psi).$$

If additionally $J(\psi) = 0$, then $A\psi$ is bounded and has a limit at infinity

$$\lim_{x \rightarrow \pm \infty} A\psi(x) = \mp \sigma^{-2} K(\psi).$$

**Corollary 3.5.** If $J(\psi) = 0$, then every continuous solution of the Poisson equation bounded from below is of the form

$$f = A\psi + h$$

where $h$ is constant if $\mu$ is aperiodic, and it is periodic of period $p$ if the support of $\mu$ is contained in $p\mathbb{Z}$. Thus every continuous solution of the Poisson equation is bounded and the limit of $f(x)$ exists when $x$ goes to $+\infty$ and $x \in G(\mu)$.

Conversely if there exists a bounded solution of the Poisson equation, then $A\psi$ is bounded and $J(\psi) = 0$. In particular the first part of corollary is valid.

Using results of Baldi [2] it is possible to give an explicit decomposition of the solutions of the Poisson equation, also in the case $J(\psi) \neq 0$. The following result, although not needed in the sequel, is stated for completeness.

**Corollary 3.6.** For every continuous solution $f$ of the Poisson equation (3.1) bounded from below there are two constants $C_1$ and $C_2$ such that

$$f(x) = A\psi(x) + C_1 J(\psi)x + C_2$$

for all $x \in G(\mu)$.

The next lemma describes a class of functions in $F(\mu)$ that we will be used later on and that have the same type of decay at infinity as $\mu$. In particular we see that if $\mu$ has exponential moment then $F(\mu)$ contains functions with exponential decay.

**Lemma 3.7.** Let $Y$ a random variable with the law $\mu$, then the function

$$r(x) = \mathbb{E}[|Y - x|]$$

is nonnegative and

$$\hat{r}(\theta) = C \cdot \frac{\hat{\mu}(\theta) - 1}{\theta^2}$$

for $\theta \neq 0$. Moreover
(3.8) if \( \mathbb{E}[e^{\phi Y} + e^{-\phi Y}] < \infty \), then \( r(x) \leq C e^{-\delta_1|x|} \) for \( \delta_1 < \delta \);
(3.9) if \( \mathbb{E}|Y|^{4+\varepsilon} < \infty \) for some \( \varepsilon > 0 \) then \( r(x) \leq \frac{C}{1+|x|^{4+\varepsilon}} \).

Hence if (3.9) holds, \( r \) is in \( \mathcal{F}(\pi) \) and for every function \( \zeta \in L^1(\mathbb{R}) \) such that \( x^2 \zeta \) is integrable the convolution \( r \ast \zeta \) is in \( \mathcal{F}(\pi) \).

4. PROOFS OF THEOREMS 1.2 AND 1.6 - EXISTENCE OF THE LIMIT

First we are going to prove the following result that holds for generic \( \pi \) not necessarily aperiodic.

**Proposition 4.1.** Suppose that hypothesis \((\mathcal{H})\) is satisfied and either (1.3) or (1.4) holds. Then the family of measures \( \delta_{(0,e^{-t})} \ast \eta \nu \) is relatively compact in the weak topology on \( \mathbb{R}^d \setminus \{0\} \) and, when \( x \) goes to infinity, every limit measure \( \eta \) is invariant by the action of \( G(\mu_A) \) that is

\[
\delta_{(0,a)} \ast \eta = \eta \quad \forall a \in G(\mu_A).
\]

Furthermore for any function of the type

\[
(4.2) \quad \phi(u) = \int_{\mathbb{R}} r(t) \zeta(e^t u) dt,
\]

where

\[
(4.3) \quad r(t) = \mathbb{E} [ | - \log A_1 - t | - | t | ]
\]

and \( \zeta \) is a nonnegative Lipschitz function on \( \mathbb{R}^d \setminus \{0\} \) such that \( \zeta(u) \leq e^{-\gamma \log |u|} \) for some \( \gamma > 0 \), the limit

\[
\lim_{x \to +\infty} \int_{\mathbb{R}^d} \phi(u e^{-x}) \nu(du) =: T(\phi)
\]

exists, it is finite and equal to \( \eta(\phi) \) for any limit measure \( \eta \).

We have already observed that the function \( f_\phi(x) = \int_{\mathbb{R}^d} \phi(u e^{-x}) \nu(du) \) is a solution of the Poisson equation associated to the function \( \psi_\phi \). Our aim is to apply the results of section 3. For we need to show that \( \psi_\phi \) is sufficiently integrable. The upper bound of the tail of \( \nu \) given in section 2 will guarantee integrability for positive \( x \). To control the function for \( x \) negative we need to perturb slightly the measures \( \mu \) and \( \nu \) in order to have more integrability near \( 0 \). This is included in the following lemma

**Lemma 4.4.** For all \( x_0 \in \mathbb{R}^d \) the translated measure \( \nu_0 = \delta_{x_0} \ast \nu \) is the unique invariant measure for \( \mu_0 = \delta_{(x_0,1)} \ast \mu \ast \delta_{(-x_0,1)} \) and it has the same behavior as \( \nu \) at infinity, that is:

\[
\lim_{x \to +\infty} \left( \int_{\mathbb{R}^d} \phi(u e^{-x}) \nu(du) - \int_{\mathbb{R}^d} \phi(u e^{-x}) \nu_0(du) \right) = 0
\]

for every function \( \phi \in C^1_c(\mathbb{R}^d \setminus \{0\}) \).

Furthermore there is \( x_0 \in \mathbb{R}^d \) such that:

- if \( \mu \) satisfies (1.3) then the same holds for \( \mu_0 \) and the measure \( \nu_0 \) satisfies

\[
(4.5) \quad \int_{\mathbb{R}^d} \frac{1}{|u|^\gamma} \nu_0(du) < \infty \text{ for all } \gamma \in (0,1)
\]

- if \( \mu \) satisfies (1.4) then

\[
(4.6) \quad \int_G (|a| + \log^+ |b|)^{4+\varepsilon} \mu_0(da db) < \infty, \text{ supp} \nu_0 \subset W_0 + \mathbb{R}^+w \text{ and dist}(0; W_0 + \mathbb{R}^+w) > 2 \]

where \( W_0 \) is an affine subspace of dimension \( d - 1 \) orthogonal to the unit vector \( w \).
Proof. If $\phi \in C_c^1(\mathbb{R}^d \setminus \{0\})$, by the Lipschitz property there exists a compact set $K = K(x_0)$ such that if $|x_0e^{-x}| < 1$, then
\[
|\phi(ue^{-x}) - \phi((u + x_0)e^{-x})| \leq C e^{-x}|x_0|1_K(ue^{-x})
\]
for every $u \in \mathbb{R}^d$. Hence by Theorem 2.1,
\[
\lim_{x \to +\infty} \int_{\mathbb{R}^d} \phi(ue^{-x})\nu(du) - \int_{\mathbb{R}^d} \phi(ue^{-x})\nu_0(du) \leq C \lim_{x \to +\infty} e^{-x}|x_0| \int_{\mathbb{R}^d} 1_K(ue^{-x})\nu(du)
\]
\[
\leq C|x_0| \lim_{x \to +\infty} e^{-x}L(e^x) = 0
\]

It is easy to check that the integrability at infinity of $\mu$ and $\mu_0$ are the same since
\[
\int_G f(a, b + (1 - a)x_0)\mu(db da) = \int_G f(a, b)\mu_0(db da).
\]

Notice also that the projections of $\mu$ and $\mu_0$ onto the $A$-part coincide i.e. $\mu_A = \pi_A(\mu_0)$, in particular the measure $\overline{\nu}$ defined by (1.8) is the same for both $\mu$ and $\mu_0$.

In the case (1.4) the support of $\nu$ is contained in some half-space $W + \mathbb{R}^+w$. Therefore the support of $\nu_0$ is contained in $W + x_0 + \mathbb{R}^+w$. Let $W_0 = W + x_0$, we may choose $x_0$ in such a way that $\text{dist}(0, W_0 + \mathbb{R}^+w) > 2$.

In the case (1.3), since by Theorem 2.1 $\delta_{(0,e^{-x})} * \nu_0(K)$ is smaller than a slowly varying function as $\nu$, then $|u|^{-\gamma} \nu_0(K)$ is $\nu_0$-integrable on $|u| \geq 1$.

For integrability near zero observe that
\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{|u + x||\log|u + x||^2} 1_{|u + x| < 1} |x| < 1 \nu(du) dx \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{|x||\log|x||^2} 1_{|x| < 1} |u| < 2 \nu(du) dx < \infty.
\]

Then there exists $x_0$ such that
\[
\int_{\mathbb{R}^d} \frac{1}{|u||\log|u||^2} 1_{|u| < 1} \nu_0(du) = \int_{\mathbb{R}^d} \frac{1}{|u + x_0||\log|u + x_0||^2} 1_{|u + x_0| < 1} \nu(du) < \infty.
\]

Since for $|u| < 1$ the function $|u|^{-\gamma}$ is bounded by a multiple of $|u|^{-1}(\log |u|)^{-2}$, (4.5) follows. \hfill \Box

If we suppose that (4.5) (or (4.6)) hold for $\nu$, we can guarantee that the function $\psi_\phi$ decays quickly at infinity, as it is proved in the two following lemmas, in the first one for the generic case and the second under the hypothesis (G).

Lemma 4.8. Assume that (1.3) is satisfied and that the function $|u|^{-\gamma}$ is $\nu(du)$ integrable for all $\gamma \in (0, 1)$. Let $\phi$ be a continuous function on $\mathbb{R}^d$ such that
\[
|\phi(u)| \leq \frac{C}{(1 + |u|)^\beta}
\]
for some $\beta, C > 0$. Then $\phi_\psi$ and $\overline{\nu} * \phi$ are well defined and continuous.

Furthermore if $\phi$ is Lipschitz, then
\[
\int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \phi(e^{-x}(au + b)) - \phi(e^{-x}au) \right| \nu(du)\mu(db da) dx < \infty
\]
and
\[
|\psi_\phi(x)| \leq Ce^{-\zeta|x|},
\]
for $\zeta < \min\{\delta/4, \beta, 1\}$.
Proof. If $\zeta < \min\{\beta, 1\}$, then
\[
|f_\phi(x)| \leq \int_{\mathbb{R}^d} |\phi(e^{-x} u)| \nu(du) \leq \int_{\mathbb{R}^d} \frac{C}{e^{-\zeta x}|u|^\zeta} \nu(du) \leq C e^{\zeta x}.
\]
If we suppose also $\zeta \leq \delta$, we have that
\[
|\mathcal{P} * f_\phi(x)| \leq \int_{\mathbb{R}} |f_\phi(x + y)| \mathcal{P}(dy) \leq C e^{\zeta x} \int_{\mathbb{R}^+} a^{-\zeta} \mu_A(da) \leq C e^{\zeta x}.
\]
Thus $\psi_\phi = \mathcal{P} * f_\phi - f_\phi$ is well defined, continuous and
\[
|\psi_\phi(x)| \leq C e^{\zeta x},
\]
that gives the required estimates for negative $x$. In order to prove (4.9) we divide the integral into two parts. For negative $x$ we use the estimates given above:
\[
\int_{-\infty}^{0} \int_{\mathbb{R}^d} |\phi(e^{-x}(au + b)) - \phi(e^{-x} au)| \nu(du) \mu(db \, da) \, dx
\]
\[
\leq \int_{-\infty}^{0} \int_{\mathbb{R}^d} |\phi(e^{-x}(au + b))| \nu(du) \mu(db \, da) \, dx + \int_{-\infty}^{0} \int_{\mathbb{R}^d} |\phi(e^{-x} au)| \nu(du) \mu(db \, da) \, dx
\]
\[
\leq \int_{-\infty}^{0} \int_{\mathbb{R}^d} |\phi(e^{-x} u)| \nu(du) \, dx + \int_{-\infty}^{0} \int_{\mathbb{R}^d} |\phi(e^{-x} au)| \nu(du) \mu(db \, da) \, dx
\]
\[
\leq \int_{-\infty}^{0} |f_\phi|(x) \, dx + \int_{-\infty}^{0} \mathcal{P} * f_\phi(x) \, dx < \infty.
\]
To estimate the integral of $|\phi(e^{-x} au) - \phi(e^{-x}(au + b))|$ for $x$ positive, we use the Lipschitz property of $\phi$ to obtain the following inequality for $0 \leq \theta \leq 1$
\[
|\phi(s) - \phi(r)| \leq 2|\phi(s) - \phi(r)|^\theta \max\left\{|\phi(s)|^{1-\theta}, |\phi(r)|^{1-\theta}\right\}
\]
\[
\leq C|s - r|^{\theta} \max_{\xi \in \{|s|, |r|\}} \frac{1}{(1 + \xi)^{\beta(1-\theta)}}.
\]
Again we divide the integral into two parts. First we consider the integral over the set where $|au + b| \geq \frac{1}{2}|au|$. We choose $\theta < \min\{\delta/2, 1\}$, $\gamma < \min\{\theta/2, \beta(1 - \theta)\}$. Then
\[
\int \int_{|au + b| \geq \frac{1}{2}|au|} |\phi(e^{-x} au) - \phi(e^{-x}(au + b))| \mu(db \, da) \nu(du)
\]
\[
\leq \int \int_{\mathbb{R}^d} \frac{C|e^{-x} b|^\theta}{(1 + |e^{-x} au|)^{\beta(1-\theta)}} \nu(du) \mu(db \, da) \leq \int \int_{\mathbb{R}^d} \frac{C|e^{-x} b|^\theta}{|e^{-x} au|^\gamma} \nu(du) \mu(db \, da)
\]
\[
\leq C e^{-(\theta - \gamma)x} \int_{\mathbb{R}} |b|^\theta |a|^{-\gamma} \mu(db \, da) \int_{\mathbb{R}} |u|^{-\gamma} \nu(du)
\]
\[
\leq C e^{-(\theta - \gamma)x} \int_{\mathbb{R}} (|b|^{2\theta} + |a|^{-2\gamma}) \mu(db \, da) \leq C e^{-\gamma x}.
\]
If \( |au + b| < \frac{1}{2}|a|u \) then \( |u| \leq \frac{2|b|}{a} \). Therefore choosing \( \theta \) as above and \( \gamma < \frac{\delta}{2} - \theta \), in view of Theorem 2.1, for the remaining part we have

\[
\int \int_{|au + b| \leq \frac{1}{2}|au|} |\phi(e^{-x}au) - \phi(e^{-x}(au + b))| \mu(db \, da) \nu(du) 
\leq \int \int_{|u| \leq \frac{2|b|}{a}} |e^{-x}b|^{\gamma} \mu(db \, da) \leq C \int_{G} |e^{-x}b|^{\gamma} \left(1 + \frac{2|b|}{a}\right) \mu(db \, da) 
\leq Ce^{-\theta x} \int_{G} |b|^{\gamma} \left(1 + \frac{2|b|}{a}\right) \mu(db \, da) 
\leq Ce^{-\theta x} \int_{G} (|b|^{\gamma} + |b|^{2(\beta+\gamma)} + a^{-2\gamma}) \mu(db \, da) 
\leq Ce^{-\theta x}.
\]

That proves (4.9) and finally

\[
|\psi_{\phi}(x)| = \left| \int_{G} \int_{\mathbb{R}^{d}} \phi(e^{-x}au) \nu(du) - \phi(e^{-x}(au + b)) \nu(du) \mu(db \, da) \right| 
\leq \int_{G} \int_{\mathbb{R}^{d}} \left| \phi(e^{-x}au) \nu(du) - \phi(e^{-x}(au + b)) \nu(du) \right| \mu(db \, da) < Ce^{-\zeta|x|}. 
\]

for \( \zeta < \min\{\delta/4, \beta, 1\} \)

\[\square\]

**Lemma 4.10.** If (4.6) is satisfied and \( \phi \) is a continuous function on \( \mathbb{R}^{d} \) such that for \( \beta > 2 \)

\[
|\phi(u)| \leq \frac{C}{(1 + \log^{+}|u|)^{\beta}}.
\]

then the functions \( f_{\phi} \) and \( \overline{\eta} * f_{\phi} \) are well defined. Furthermore if \( \phi \) is Lipschitz and \( \beta > 4 \), then

\[
\int_{\mathbb{R}} \int_{G} \int_{\mathbb{R}^{d}} \phi(e^{-x}(au + b)) - \phi(e^{-x}au) \nu(du) \mu(db \, da) \nu(dx) < \infty.
\]

and

\[
|\psi_{\phi}(x)| \leq \frac{C}{1 + |x|^{\chi}},
\]

for \( \chi = \min\{\beta - 1, 3 + \varepsilon\} \).

\[\text{Proof.}\] Assume first \( x < -1 \). In view of Proposition 2.8 we have

\[
|f_{\phi}(x)| = \int_{|u| > 2} |\phi(e^{-x}u)| \nu(du) \leq \int_{|u| > 2} \frac{C}{\log^{\beta}(e^{-x}|u|)} \nu(du)
\leq C \sum_{n=0}^{\infty} \int_{e^{n} \leq |u| < e^{n+1}} \frac{1}{(n-x)^{\beta}} \nu(du)
\leq C \sum_{n>|x|} \frac{1}{n^{\beta}} \int_{e^{n+1} \leq |u| < e^{n+1+1}} \nu(du)
\leq C \sum_{m=1}^{\infty} \frac{1}{m^{\beta} |x|^{\beta}} \int_{e^{n+1} \leq |u| < e^{n+1+1}} \nu(du)
\leq C \sum_{m=1}^{\infty} \frac{1}{m^{\beta} |x|^{\beta}} \int_{|u| < e^{n+1+1} |x|} \nu(du) \leq \frac{C}{|x|^{\beta-1}} \sum_{m=1}^{\infty} \frac{1}{m^{\beta-1}}
\leq \frac{C}{|x|^{\beta-1}}.
\]
To proceed with positive \( x \) notice that, by Proposition 2.8, for every \( y \in \mathbb{R}^+ \) and \( \beta' > 2 \), arguing as above, we obtain:

\[
\int_{\mathbb{R}^d} \frac{1}{1 + (\log^+ (y|u|))^\beta'} \nu(du) \leq \int_{|y| < 1} \nu(du) + \sum_{n=0}^{\infty} \int_{e^n \leq |y| < e^{n+1}} \frac{1}{1 + n^\beta'} \nu(du)
\]

\[
\leq C + C|\log y| + C \sum_{n=1}^{\infty} \frac{1}{1+n^{\beta'}} \leq C(1 + |\log y|)
\]

Hence \( |f_\phi(x)| \leq C(1 + x) \) if \( x > 0 \).

Finally \( f_\phi \) is continuous, hence for \( x \in (-1, 0) \) is bounded. Thus

\[
|f_\phi(x)| \leq C \left( (1 + |x|)1_{x > 0} + \frac{1}{1 + |x|^{\beta-1}}1_{x \leq 0} \right)
\]

Consider now the convolution of \( f_\phi \) with \( \overline{\mu} \). First if \( x > 0 \), then

\[
\left| \overline{\mu} * f_\phi(x) \right| \leq C \int_{\mathbb{R}} (1 + |x + y|) \overline{\mu}(dy) \leq C(1 + |x|).
\]

Next if \( x < -1 \), then since \( \mathbb{E}|\log A|^{4+\varepsilon} < \infty \), we have

\[
\left| \overline{\mu} * f_\phi(x) \right| \leq \int_{\mathbb{R}} \frac{C}{1 + |x + y|^{\beta-1}} \overline{\mu}(dy)
\]

\[
\leq \int_{|y| < |x|} \frac{C}{1 + |x + y|^{\beta-1}} \overline{\mu}(dy) + \frac{C}{|x|^{4+\varepsilon}} \int_{|y| \geq |x|} \overline{\mu}(dy)
\]

\[
\leq \frac{C}{1 + |x|^{\chi_0^{-1}}}
\]

for \( \chi_0 = \min\{\beta - 1, 4 + \varepsilon\} \). The function \( \overline{\mu} * f_\phi \) is also continuous, hence finally we obtain

\[
\left| \overline{\mu} * f_\phi(x) \right| \leq C \left( (1 + |x|)1_{x > 0} + \frac{1}{1 + |x|^{\chi_0}}1_{x \leq 0} \right).
\]

Proceeding as in the previous lemma we prove

\[
\int_{-\infty}^{0} \int_{\mathbb{R}^d} \left| \phi(e^{-x}(au + b)) - \phi(e^{-x}au) \right| \nu(du) \mu(db \, du) \, dx
\]

\[
\leq \int_{-\infty}^{0} |f_{1\phi}(x)| \, dx + \int_{-\infty}^{0} |\overline{\mu} * f_{1\phi}(x)| \, dx < \infty
\]

For \( x > 0 \) we divide the integral of \( \left| \phi(e^{-x}au) - \phi(e^{-x}(b + au)) \right| \) into several parts and we use the following inequality, being a consequence of the Lipschitz property of \( \phi \):

\[
|\phi(s) - \phi(r)| \leq C|s - r|^{\theta} \max_{\xi \in \{s, |r|\}} \frac{1}{1 + (\log^+ \xi)^{\beta'}}
\]

where \( \theta < 1 - 2/\beta \) and \( \beta' = \beta(1 - \theta) > 2 \).
Case 1. First we assume $|b| \leq e^{\frac{\beta}{2}}$. Then by (4.12)
\[
\int_{|b| \leq e^{\frac{\beta}{2}}} \int_{\mathbb{R}^d} |\phi(e^{-x} au) - \phi(e^{-x} (b + au))| \nu(du) \mu(db da)
\leq C \int_{|b| \leq e^{\frac{\beta}{2}}} \int_{\mathbb{R}^d} e^{-\theta x} |b|^{\beta} \left( \frac{1}{1 + (\log^+ (e^{-x} a|u|))^{\beta} + \log^+ (e^{-x} a|u| + b))} \right) \nu(du) \mu(db da)
\leq C e^{-\theta x/2} \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1 + (\log^+ (e^{-x} a|u|))^{\beta} \nu(du) \mu_A(da) + \int_{\mathbb{R}^d} 1 + (\log^+ (e^{-x} a|u|))^{\beta} \nu(du) \right)
\leq C e^{-\theta x/2} \left[ 1 + x + \int_{\mathbb{R}^d} |\log a| \mu_A(da) \right] < C e^{-\theta x/4}.
\]

Case 2. We assume $a|u| < 2|au + b|$ and $|b| > e^{\frac{\beta}{2}}$. Notice first
\[
\int_{|b| > e^{\frac{\beta}{2}}} \mu(db da) \leq C \int_{\mathbb{R}^d} \left( 1 + (|\log a| + |b| + \log^+ |b|) \right) \mu(db da) \leq \frac{C}{1 + x^{4+\varepsilon}}.
\]
and
\[
\int_{|b| > e^{\frac{\beta}{2}}} (|\log a| + |b|) \mu(db da)
\leq \frac{C}{1 + x^{4+\varepsilon}} \int_{G} \left( 1 + (|\log a| + |b|)^{3+\varepsilon} (\log^+ |b| + |\log a|) \right) \mu(db da) \leq \frac{C}{1 + x^{3+\varepsilon}}.
\]

Then, proceeding as previously, we have
\[
\int \int_{a|u| < 2|au + b|} \phi(e^{-x} au) - \phi(e^{-x} (b + au)) |\nu(du) \mu(db da)
\leq 2 \int \int_{a|u| < 2|au + b|} \max \left\{ |\phi(e^{-x} au)|, |\phi(e^{-x} (b + au))| \right\} \nu(du) \mu(db da)
\leq C \int \int_{|b| > e^{\frac{\beta}{2}}} \int_{\mathbb{R}^d} 1 + (\log^+ (e^{-x} a|u|))^{\beta} \nu(du) \mu(db da)
\leq C \int \int_{|b| > e^{\frac{\beta}{2}}} (x + |\log a| + 1) \mu(da db) \leq \frac{C}{1 + x^{3+\varepsilon}}.
\]

Case 3. The last case is $a|u| \geq 2|au + b|$ and $|b| > e^{\frac{\beta}{2}}$. Then $|u| < \frac{2|b|}{e}$ and we obtain
\[
\int \int_{a|u| \geq 2|au + b|} \phi(e^{-x} au) - \phi(e^{-x} (b + au)) |\nu(du) \mu(db da)
\leq C \int \int_{|b| > e^{\frac{\beta}{2}}} \int \int_{a|u| < \frac{2|b|}{e}} \nu(du) \mu(db da) \leq C \int \int_{|b| > e^{\frac{\beta}{2}}} \left( 1 + \log |b| + |\log a| \right) \mu(db da) \leq \frac{C}{1 + x^{3+\varepsilon}}.
\]

We conclude (4.11) and the required estimates for $\psi_\phi$. \hfill \Box

Proof of Proposition 4.1. Step 1. First we suppose that $\mu$ satisfies either (1.3) and (4.5) or (4.6).

We are going to show that for functions of type (4.2) the limit
\[
\lim_{x \to \infty} \int_{\mathbb{R}^d} \phi(ue^{-x}) \nu(du) = T(\phi) := -2\sigma^{-2} K(\psi_\phi)
\]
is finite. To do this we will prove that $\psi_\phi$ is an element of $\mathcal{F}(\mathcal{P})$ and $J(\psi_\phi) = 0$. Thus, by Corollary 3.5, $f_\phi(x) = \int_{\mathbb{R}^d} \phi(ue^{-x}) \nu(du)$, that is the solution of the corresponding Poisson equation,
is bounded and it has a limit when $x$ converge to $+\infty$. We will prove that the limit exists even if values of $x$ are not restricted to $G(\overline{\mathcal{P}})$.

First observe that if $E \left[ |A_1|^{\delta} + |A_1|^{-\delta} \right] < \infty$, then by lemma 3.7, for $\beta < \min\{\delta, \gamma\}$, we have

$$
|\phi(u)| \leq C \int_{\mathbb{R}} e^{-\beta|t|} e^{-\gamma|t+\log|u||} dt \leq C \int_{\mathbb{R}} e^{-\beta(|t-\log|u||)} e^{-\gamma|t|} dt \\
\leq C \int_{\mathbb{R}} e^{-\beta(|t|+|\log|u||)} e^{-\gamma|t|} dt = Ce^{-\beta|\log|u||}.
$$

In the same way if $E \left[ |\log A|^{4+\varepsilon} \right] < \infty$, then

$$
|\phi(u)| \leq C \int_{\mathbb{R}} \frac{1}{1+|t-\log|u||^{3+\varepsilon}} e^{-\gamma|t|} dt \\
\leq C \int_{\mathbb{R}} \frac{1+|t-\log|u||^{3+\varepsilon} + |t|^{3+\varepsilon} e^{-\gamma|t|}}{1+|t-\log|u||^{3+\varepsilon}} dt \\
\leq \frac{C}{1+|\log|u||^{3+\varepsilon}} \int_{\mathbb{R}} (1+|t|^{3+\varepsilon}) e^{-\gamma|t|} dt \leq \frac{C}{1+|\log|u||^{3+\varepsilon}}.
$$

Thus by Lemmas 4.8 and 4.10, $f_\phi, f_\zeta, \mathcal{P} * f_\phi$ and $\mathcal{P} * f_\zeta$ are well defined. Furthermore since $\zeta$ is Lipschitz $\psi_\zeta$ is bounded, and $x^2 \psi_\zeta(x)$ is integrable on $\mathbb{R}$. We cannot guarantee that $\phi$ is Lipschitz, but we can observe that

$$
f_\phi(x) = \int_{\mathbb{R}^4} \int_{\mathbb{R}} r(t) \zeta(e^{-x+t}u) dt \nu(du) \\
= \int_{\mathbb{R}^4} \int_{\mathbb{R}} r(t+x) \zeta(e^t u) dt \nu(du) \\
= \int_{\mathbb{R}} r(t+x) f_\zeta(-t) dt \\
= r * f_\zeta(x)
$$

and

$$
\mathcal{P} * f_\phi(x) = r * (\mathcal{P} * f_\zeta)(x).
$$

Hence

$$
\psi_\phi = f_\phi - \mathcal{P} * f_\phi = r * (f_\zeta - \mathcal{P} * f_\zeta) = r * \psi_\zeta.
$$

Therefore, by Lemma 3.7, $\psi_\phi \in \mathcal{F}(\overline{\mathcal{P}})$.

Furthermore if $\zeta$ is radial then $J(\psi_\phi) = 0$. In fact, let $\zeta_r$ be the radial part of $\zeta$, i.e. $\zeta_r(|u|) = \zeta(u)$, then

$$
\int_{\mathbb{R}} \psi_\zeta(x) dx = \int_{\mathbb{R}} \int_{\mathbb{R}^4} \left[ \zeta(au^{-x}) - \zeta(e^{-x}(au+b)) \right] \nu(du) \mu(db da) dx \\
= \int_{\mathbb{R}} \int_{\mathbb{R}^4} \left[ \zeta_r(e^{-x+\log(|au|)}) - \zeta_r(e^{-x+\log(|au+b|)}) \right] dx \nu(du) \mu(db da) \\
= \int_{\mathbb{R}} \int_{\mathbb{R}^4} \left( \int_{\mathbb{R}} \zeta_r(e^{-x}) dx - \int_{\mathbb{R}} \zeta_r(e^{-x}) dx \right) \nu(du) \mu(db da) = 0.
$$

Observe that we can apply the Fubini theorem since $\zeta$ is Lipschitz and, by Lemmas 4.8 and 4.10, the absolute value of the integrand in the second line above is integrable. Hence

$$
J(\psi_\phi) = \int_{\mathbb{R}} \psi_\phi(x) dx = \int_{\mathbb{R}} r * \psi_\zeta(x) dx = \int_{\mathbb{R}} r(x) dx \cdot \int_{\mathbb{R}} \psi_\zeta(x) dx = 0.
$$
If $\zeta$ is radial, then by Corollary 3.5, we have
\[(4.13)\quad f_\phi = A\psi_\phi + h_\phi\]
where $h_\phi$ is a constant if $\mu_A$ is aperiodic and a continuous periodic function if $\mu_A$ is periodic. In any case $f_\phi$ is bounded.

In particular the same holds for $f_{\Phi_\gamma}$, where
\[
\Phi_\gamma(u) = \int_\mathbb{R} r(t)e^{-\gamma|t+h_\phi u|}dt.
\]

For a generic non-radial function $\phi$ of the type (4.2), there exists $\gamma > 0$ such that $\phi \leq \Phi_\gamma$. Hence $f_\phi \leq f_{\Phi_\gamma}$ and $f_\phi$ is a bounded solution of the Poisson equation associated to $\psi_\phi$. Therefore, by Corollary 3.5, $J(\psi_\phi) = 0$ and
\[
f_\phi = A\psi_\phi + h_\phi
\]
for a periodic function $h_\phi$.

Since the measure $\nu$ has no mass in zero $\lim_{x \to -\infty} f_\phi(x) = 0$ and by Theorem 3.2
\[
\lim_{x \to -\infty} A\psi_\phi(x) = \sigma^{-2}K(\psi_\phi).
\]
Thus when $x$ goes to $-\infty$ the limit (not necessarily restricted to $G(\mathcal{P})$) of $h_\phi$ exists which is possible only if $h_\phi$ is constant and is equal to $-\sigma^{-2}K(\psi_\phi)$. Finally
\[
\lim_{x \to -\infty} f_\phi(x) = \lim_{x \to -\infty} A\psi_\phi(x) - \sigma^{-2}K(\psi_\phi) = -2\sigma^{-2}K(\psi_\phi).
\]

**Step 2.** Fix a $\gamma > 0$. Since $\Phi_\gamma > 0$ for every function $\phi \in C_c(\mathbb{R}^d \setminus \{0\})$ there exists a constant $C_\phi$ such that $|\phi| \leq C_\phi \Phi_\gamma$. Thus the family of measures on $\mathbb{R}^d \setminus \{0\}$
\[
\delta_{(0,e^{-r})} * G \nu(\phi) = \int_{\mathbb{R}^d} \phi(e^{-r}u)\nu(du)
\]
is bounded hence it is relatively compact in the weak topology.

Let $\eta$ be an accumulation point for a subsequence $\{x_n\}$ that is
\[(4.14)\quad \lim_{n \to \infty} \delta_{(0,e^{-x_n})} * G \nu(\phi) = \eta(\phi) \quad \forall \phi \in C_c(\mathbb{R}^d \setminus \{0\}).
\]
By Lemma 2.6 the measure $\eta$ is $G(\mu_A)$ invariant. We claim that for any continuous non negative function such that $\phi \leq C_\phi \Phi_\gamma$, not necessarily compactly supported,
\[
\eta(\phi) = \lim_{n \to \infty} \delta_{(0,e^{-x_n})} * G \nu(\phi).
\]
Indeed, fix a large constant $M$, take $\varepsilon > 0$ and define
\[
\phi_M(u) = \phi(u) \cdot \int_{\mathbb{R}} 1_{[1/(1+\varepsilon)M,M(1+\varepsilon)]}(t)|u|h(t)dt,
\]
where $h \in C_C((1+\varepsilon)^{-1}, 1+\varepsilon)$ and $\int_{\mathbb{R}} h(t)dt = 1$. Then $\phi_M$ is continuous, its support is contained in the annulus $\{u: 1/(M(1+\varepsilon)^2) \leq |u| < M(1+\varepsilon)^2\}$ and moreover $\phi_M(u) = \phi(u)$ for $1/M < |u| < M$.

Notice that by the monotone convergence theorem
\[
\eta(\phi) = \lim_{M \to \infty} \int_{\mathbb{R}^d} \phi_M(u)\eta(du)
\]
and by (4.14), for a fixed $M$
\[
\lim_{n \to \infty} \delta_{(0,e^{-x_n})} * G \nu(\phi_M) = \eta(\phi_M).
\]
Therefore,
\[
|\eta(\phi) - \lim_{n \to \infty} \delta_{(0,e^{-\epsilon n})} \ast \mu(\phi)| \leq \lim_{M \to \infty} \left[ |\eta(\phi) - \eta(\phi_M)| + |\eta(\phi_M) - \lim_{n \to \infty} \delta_{(0,e^{-\epsilon n})} \ast \mu(\phi_M)| \right] \\
+ \left| \lim_{n \to \infty} \delta_{(0,e^{-\epsilon n})} \ast \mu(\phi - \phi_M) \right| \\
\leq \lim_{M \to \infty} \lim_{n \to \infty} \delta_{(0,e^{-\epsilon n})} \ast \mu(\phi - \phi_M)
\]

and we have to prove that the last limit is 0. For that observe that for every compact set \( K \), there exists a constant \( C_K \) such that \( 1_K \leq C_K \Phi, \) hence
\[
\delta_{(0,e^{-\epsilon})} \ast \mu(K) \leq C_K',
\]
for every \( x \in \mathbb{R} \). Now, applying Lemma 2.7 with \( l(z) \equiv 1 \), since \( \Phi(1) \leq \frac{C}{1 + \log |u|^{\epsilon K}} \), we obtain
\[
\lim_{n \to \infty} \delta_{(0,e^{-\epsilon n})} \ast \mu(\phi - \phi_M) \leq \sup_{x \in \mathbb{R}} \int_{\mathbb{R}^d} \Phi(1) (1 - 1_{[1/M,M]}(e^{-x} u)) \nu(du)
\leq \sup_{x \in \mathbb{R}} \left( \int_{M/e}^{e/M} \frac{1}{1 + \log (e^{-x} a)^{1/3} a} \right) + \left( \int_{0}^{e/M} \frac{1}{1 + \log (e^{-x} a)^{1/3} a} \right)
\leq C \left( \int_{1/M}^{e/M} \frac{1}{1 + \log a^{1/3}} \right) + \left( \int_{0}^{e/M} \frac{1}{1 + \log a^{1/3}} \right)
\]

letting \( M \) to go to infinity, we conclude.

**Step 3.** Now we return to the general case of a measure \( \mu \) satisfying (1.3) (or (1.4)). Then by lemma 4.4 there exists \( \nu_0 = \delta_{x_0} \ast \mu \) for which (1.3) and (4.5) (or (4.6)) hold. We have proved in Lemma 4.4 that \( \delta_{(0,e^{-\epsilon})} \ast \mu \) and \( \delta_{(0,e^{-\epsilon})} \ast \nu_0 \) have the same behavior on compactly supported functions when \( x \) go to +\( \infty \). Thus we still need to prove that they have the same limit for functions \( \phi \) of the type (4.2) even if they do not have compact support. Observe that both sets of measures are bounded on compact set, and in particular:
\[
\sup_{x \in \mathbb{R}} \delta_{(0,e^{-\epsilon})} \ast \mu(1 \leq |u| \leq e) = K < \infty \quad \sup_{x \in \mathbb{R}} \delta_{(0,e^{-\epsilon})} \ast \nu_0(1 \leq |u| \leq e) = K_0 < \infty.
\]

By the Lipschitz property
\[
|\zeta(e^{t-x} u) - \zeta(e^{t-x} (u + x_0))| \leq C e^{\theta(t-x)} (\zeta(e^{t-x} u)^{1-\theta} + \zeta(e^{t-x} (u + x_0))^{1-\theta})
\]
for all \( \theta \in [0,1] \). Hence by Lemma 2.7
\[
|\delta_{(0,e^{-\epsilon})} \ast \nu(\phi) - \delta_{(0,e^{-\epsilon})} \ast \nu_0(\phi)| \leq \int_{\mathbb{R}} \int_{\mathbb{R}^d} r(t) \left| \zeta(e^{t-x} u) - \zeta(e^{t-x} (u + x_0)) \right| dt \nu(du)
\leq C \left[ \int_{\mathbb{R}} r(t) dt + \int_{-\infty}^{x} r(t) e^{\theta(t-x)} \left( \int_{\mathbb{R}^d} \zeta(e^{t-x} u)^{1-\theta} \nu(du) + \int_{\mathbb{R}^d} \zeta(e^{t-x} u)^{1-\theta} \nu_0(du) \right) dt \right]
\leq C \left[ \int_{\mathbb{R}} r(t) dt + \int_{-\infty}^{x} r(t) e^{\theta(t-x)} \left( \int_{0}^{\infty} e^{\gamma(1-\theta)} \log a \frac{da}{a} dt \right) \right]
\leq C \left[ \int_{\mathbb{R}} r(t) dt + \int_{-\infty}^{x} r(t) e^{\theta(t-x)} dt \right] \leq C \left[ \int_{\mathbb{R}} r(t) dt + \int_{-\infty}^{x} r(t) dt \right]
\]
Since, by the dominated convergence theorem, the last term goes to zero when \( x \) go +\( \infty \), we conclude.
Proof of Theorem 1.2 - existence of the limit. We assume that $\mu_A$ is aperiodic. Then in view of Proposition 4.1 the family of measures $\delta_{(0,e^{-t})} *_G \nu$ is relatively compact in the weak topology and if $\eta$ is an accumulation point, then it is $\mathbb{R}_+^*$ invariant i.e. for every $a \in \mathbb{R}_+^*$

$$\int_{\mathbb{R}^d} \phi(au) \eta(du) = \int_{\mathbb{R}^d} \phi(u) \eta(du).$$

Therefore there exists a probability measure $\Sigma_\eta$ on $S^{d-1}$ and a constant $C_\eta$ such that

$$\eta = C_\eta \Sigma_\eta \otimes \frac{da}{a}$$

(see [17], Proposition 1.15). It remains to prove that $C_\eta$ and $\Sigma_\eta$ do not depend on $\eta$. We have proved in Proposition 4.1 that for any function $\phi$ of type (4.2), the limit exists (that is it does not depend on the subsequence along which one tends to $\eta$)

$$\lim_{x \to +\infty} \int_{\mathbb{R}^d} \phi(ue^{-x}) \nu(du) = \eta(\phi) = T(\phi).$$

Consider the radial function $\Phi_\gamma(u) = \int_{\mathbb{R}} r(t)e^{-|t+\log |u||} dt$, since $\eta(\Phi_\gamma) = C_\eta \int_{\mathbb{R}_+^*} \Phi_\gamma(a) \frac{da}{a}$. Then:

$$C_\eta = \frac{T(\Phi_\gamma)}{\int_{\mathbb{R}_+^*} \Phi_\gamma(a) \frac{da}{a}}$$

does not depend on $\eta$. Set $C_+ = C_\eta$.

For any Lipschitz function $\zeta_0$ of $S^{d-1}$ consider the function $\zeta(u) = e^{-|\log |u||}\zeta_0(u/|u|)$ and

$$\phi(u) = \int_{\mathbb{R}} r(t) \zeta(e^t u) dt = \Phi_\gamma(u) = \int_{\mathbb{R}} r(t)e^{-|t+\log |u||}\zeta_0(e^-t u/|e^-t u|) dt = \Phi_\gamma(u)\zeta_0(u/|u|).$$

Then

$$\eta(\phi) = C_+ \Sigma_\eta(\zeta_0) \cdot \int_{\mathbb{R}_+^*} \Phi_\gamma(a) \frac{da}{a} = T(\phi)$$

thus $\Sigma_\eta(\zeta_0)$ does not depend on $\eta$.

\[\square\]

Proof of Theorem 1.6 - existence of the limit. We proceed as in the previous proof. Assume that $\mu_A$ is aperiodic and $G(\mu_A) = \langle e^p \rangle$. Let $D = \{x : 1 \leq x < e^p\}$ be the fundamental domain for the action of $G(\mu_A)$ on $\mathbb{R}^d \setminus \{0\}$. Then every $x \in \mathbb{R}^d \setminus \{0\}$ can be uniquely written as $x = aw$, where $a \in G(\mu_A)$ and $w \in D$. Denote by $l$ the counting measure on $G(\mu_A)$.

First we will prove that the exists a radial function $\Phi$ of type (4.2) such that for every $a \in \mathbb{R}_+^*$, $\delta_{(0,a)} *_G l(\Phi) = l(\Phi)$, where now $l$ is considered as a measure on $\mathbb{R}^d \setminus \{0\}$, i.e. for any unit vector $x \in \mathbb{R}^d$, $l(\Phi) = \sum_{k \in \mathbb{Z}} \Phi(e^{kp}x)$. For radial functions $l(\Phi)$ does not depend on the choice of $x$. Indeed, let $R \in C(\mathbb{R}^d \setminus \{0\})$ be the function defined in the proof of Theorem 2.1. Clearly, we may assume that $R$ is radial. Then, since $\delta_{(0,a)} *_G l(R) = l(R)$ for all $a \in G(\mu_A)$ and repeating the argument given at the end of the proof of Theorem 2.1, we prove that $\delta_{(0,a)} *_G l(R) = l(R)$ for all $a \in \mathbb{R}_+^*$. Take $\Phi(u) = \int_{\mathbb{R}} r(t) R(e^tu) dt$, then for every $a \in \mathbb{R}_+^*$ we have

$$\delta_{(0,a)} *_G l(\Phi) = \int_{\mathbb{R}} r(t) \delta_{(0,e^t)} *_G \delta_{(0,a)} *_G l(R) dt = \int_{\mathbb{R}} r(t) \delta_{(0,e^t)} *_G l(R) dt = l(\Phi).$$

Let $\eta$ be an accumulation point of the family of measures $\delta_{(0,e^{-t})} *_G \nu$. Then, in view of Proposition 4.1, $\eta$ is $G(\mu_A)$ invariant. Therefore there exists a probability measure $\Sigma_\eta$ on $D$ and a constant $C_\eta$.
such that \( \eta = C_\eta \Sigma_\eta \otimes l \). By Proposition 4.1, the value \( T(\Phi) = \eta(\Phi) \) does not depend on \( \eta \). Notice that

\[
T(\Phi) = C_\eta \int_D \int_{G(\mu_A)} \Phi(aw) l(da) \Sigma_\eta(dw) = C_\eta \int_D \delta_{(0,|w|)} * G l(\Phi) \Sigma_\eta(dw)
\]

Therefore \( C_\eta = T(\Phi) / \langle \Phi \rangle \) does not depend on \( \eta \). Define \( C_+ = C_\eta \). Finally, we have

\[
\eta(1_{C(z,e^{np}z)}) = C_+ \int_D \int_{G(\mu_A)} 1_{C(z,e^{np}z)}(aw) l(da) \Sigma_\eta(dw) = C_+ \int_D \Sigma_\eta(dw) = nC_+
\]

and this value also does not depend on \( \eta \).

\( \square \)

5. Positivity of the limiting constant

In this section we are going to discuss non degeneracy of the limit measure (1.5) and finish the proofs of theorems 1.2 and 1.6 proving that the constant \( C_+ \) is positive. A partial result was obtained in [7] in the one dimensional case and \( B \geq \varepsilon \) a.s in [7]. Now we are going to prove

**Theorem 5.1.** If hypothesis (H) is satisfied, then for all \( \alpha, \beta > 0 \)

\[
\limsup_{x \to \infty} \nu\{ u \in \mathbb{R}^d : z\alpha < |u| \leq z\beta \} > 0.
\]

First we will prove a version of the duality lemma generalized to time-reversible functions. Although the technic of proof is classical (see for instance [16]), we present here complete argument for reader’s convenience. Let \( W_i = (Y_i, Z_i) \) be a sequence of i.i.d. random variables on \( \mathbb{R} \times \mathbb{R} \) and let

\[
S_n = \sum_{i=1}^n Y_i \quad \text{if} \quad n \geq 1 \quad \text{and} \quad S_0 = 0
\]

(later we will take \( W_i = (\log A_i, B_i) \)). We define a sequence of stopping times:

\[
T_0 = 0; \quad T_i = \inf\{n > T_{i-1} : S_n > S_{T_{i-1}}\}
\]

and we put

\[
L = \inf\{n : S_n < 0\}.
\]

If the events are void then the stopping times are equal to \( \infty \).

**Lemma 5.3** (Duality Lemma). Consider a sequence of non negative functions

\[
\alpha_n : (\mathbb{R} \times \mathbb{R})^n \to \mathbb{R},
\]

for \( n \geq 1 \), \( \alpha_0 \) equal to some constant and \( \alpha_\infty = 0 \), that are time reversible, that is

\[
\alpha_n(w_1, \ldots, w_n) = \alpha_n(w_n, \ldots, w_1) \quad \forall (w_1, \ldots, w_n) \in (\mathbb{R} \times \mathbb{R})^n.
\]

Then

\[
E \left[ \sum_{i=0}^{L-1} \alpha_i(W_1, \ldots W_i) \right] = E \left[ \sum_{i=0}^{\infty} \alpha_{T_i}(W_1, \ldots W_{T_i}) \right].
\]
Proof. We have
\[
E \left[ \sum_{i=0}^{L-1} \alpha_i(W_1, \ldots, W_i) \right] = \sum_{i=0}^{\infty} E \left[ \mathbf{1}_{[i<L]} \alpha_i(W_1, \ldots, W_i) \right] = \alpha_0 + \sum_{i=1}^{\infty} E \left[ \mathbf{1}_{[S_j>0 \, \forall j=1, \ldots, i]} \alpha_i(W_1, \ldots, W_i) \right].
\]

For fixed $i$ consider the reversed time sequence
\[
\bar{W}_k = W_{i-k+1}
\]
and observe that the vector $(\bar{W}_1, \ldots, \bar{W}_i) = (W_i, \ldots, W_1)$ has the same law as $(W_1, \ldots, W_i)$. Thus
\[
E \left[ \mathbf{1}_{[S_j>0 \, \forall j=1, \ldots, i]} \alpha_i(W_1, \ldots, W_i) \right] = E \left[ \mathbf{1}_{[\sum_{k=1}^i Y_k>0 \, \forall j=1, \ldots, i]} \alpha_i(W_1, \ldots, W_i) \right] = E \left[ \mathbf{1}_{[\sum_{k=1}^i Y_{i-k+1}>0 \, \forall j=1, \ldots, i]} \alpha_i(W_1, \ldots, W_i) \right],
\]
\[
E \left[ \mathbf{1}_{[S_i>S_j \, \forall j=0, \ldots, i-1]} \alpha_i(W_1, \ldots, W_i) \right] = E \left[ \mathbf{1}_{[\exists k \geq 1; i=k]} \alpha_i(W_1, \ldots, W_i) \right].
\]

Then
\[
E \left[ \sum_{i=0}^{L-1} \alpha_i(W_1, \ldots, W_i) \right] = \alpha_0 + \sum_{i=1}^{\infty} E \left[ \mathbf{1}_{[\exists k \geq 1; i=k]} \alpha_i(W_1, \ldots, W_i) \right] = E \left[ \sum_{k=0}^{\infty} \alpha_{T_k}(W_1, \ldots, W_{T_k}) \right].
\]

\[\square\]

Proof of Theorem 5.1. \textbf{Step 1.} First we claim that there exist two positive constants $C$ and $M$ such that for every positive nonincreasing $f$ on $\mathbb{R}_+$
\[
(5.4) \quad \int_{\mathbb{R}^d} f(|u|) \nu(du) \geq C \int_{M}^{\infty} f(a) \frac{da}{a}.
\]

Take $Y_i = \log A_i$ and $S_n = \sum_{i=1}^{n} Y_i$. As it was proved in [1]
\[
\nu(f) = \int_{\mathbb{R}^d} E \left[ \sum_{n=0}^{L-1} f(X_n^x) \right] \nu_L(dx),
\]

were $\nu_L$ is the invariant probability measure of the process $X_{L,n}$. Take a ball $B$ of $\mathbb{R}^d$ of radius $R$ such that $\nu_L(B) = C_R > 0$. We have
\[
\int_{\mathbb{R}^d} f(|u|) \nu(du) \geq \int_{B} E \left[ \sum_{n=0}^{L-1} f \left( |A_1 A_2 \ldots A_n x + A_2 A_3 \ldots A_n B_1 + \cdots + B_n| \right) \right] \nu_L(dx)
\]
\[
\geq C_R E \left[ \sum_{n=0}^{L-1} f \left( A_1 A_2 \ldots A_n (R + |B_1| + \cdots + |B_n|) \right) \right]
\]
\[
= C_R E \left[ \sum_{n=0}^{L-1} f \left( e^{S_n + \log(R + \sum_{i=1}^{n} |B_i|)} \right) \right]
\]
\[
= C_R E \left[ \sum_{n=0}^{\infty} f \left( e^{S_{T_n} + \log(R + \sum_{i=1}^{T_n} |B_i|)} \right) \right].
\]
In the last line we applied the duality Lemma since the function:
\[ \alpha_n((Y_1, B_1), \ldots, (Y_n, B_n)) = f\left( e^{\sum_{i=1}^n Y_i + \log(R + \sum_{i=1}^n |B_i|)} \right) \]
is time reversible. Consider the sequences of i.i.d. variables
\[ W_j = \max\{\log(1 + R + |B_i|) : i = T_{j-1} + 1, \ldots, T_j\} \]
and
\[ V_j = S_{T_j} - S_{T_{j-1}} + \log(T_j - T_{j-1}) + W_j. \]

Observe that for \( n \geq 1 \)
\[ S_{T_n} + \log(R + \sum_{i=1}^n |B_i|) \leq S_{T_n} + \log \left( \sum_{j=1}^n (R + \sum_{i=T_{j-1}+1}^{T_j} |B_i|) \right) \]
\[ \leq \sum_{j=1}^n \left( (S_{T_j} - S_{T_{j-1}}) + \log \left( 1 + R + \sum_{T_{j-1}+1 \leq i \leq T_j} |B_i| \right) \right) \]
\[ \leq \sum_{j=1}^n V_j. \]

We claim that the variables \( V_j \) are integrable. In fact since \( Y_i = \log A_i \) has a moment of order \( 2 + \varepsilon \),
then classical results guarantee that \( S_{T_j} - S_{T_{j-1}} \) is integrable and \( T_j - T_{j-1} \) has a moment of order \( 1/(2 + \varepsilon) \). So we need only to prove that the variable \( W_j \) has the first moment (see [11], page 1279).

By the Borel-Cantelli Lemma it sufficient to show that
\[ \limsup_{n \to \infty} \frac{1}{n} W_n < M \quad \text{a.s.} \]
for some constant \( M \). We have
\[ \frac{1}{n} W_n = \frac{\sum_{j=1}^n (T_j - T_{j-1})^{1/(2+\varepsilon)}}{\sum_{j=1}^n (T_j - T_{j-1})^{1/(2+\varepsilon)}} W_n. \]

By the strong law of large numbers the first term converges. For the second term we have
\[ \left( \frac{W_n}{\sum_{j=1}^n (T_j - T_{j-1})^{1/(2+\varepsilon)}} \right)^{2+\varepsilon} \leq \frac{W_n^{2+\varepsilon}}{T_n^{2+\varepsilon}} \leq \frac{\sum_{k=1}^n \log(1 + R + |B_k|)^{2+\varepsilon}}{T_n}, \]
which converges since \( (\log^+ |B_i|)^{2+\varepsilon} \) is integrable.

Let \( \mathcal{U}(y, x) = \sum_{n=1}^\infty \mathbb{E}\left[ 1_{(y, x)} \left( \sum_{i=1}^n V_i \right) \right] \). Since \( 0 < \mathbb{E}V_1 < \infty \), by renewal theorem
\[ \lim_{x \to \infty} \frac{\mathcal{U}(0, x)}{x} = \frac{1}{\mathbb{E}V_1} > 0. \]
hence for any \( m > 1 \) there exist large \( N \) such that \( \inf_{k \geq N} \frac{U(m^k, m^{k+1})}{m^k} = C_1 > 0 \). Therefore,
\[ \int_{\mathbb{R}^d} f(|x|) \nu(dx) \geq C_R \mathbb{E} \left[ \sum_{n=0}^\infty f(e^{\sum_{i=1}^n V_i}) \right] \geq C_R \sum_{k > N} U(m^k, m^{k+1}) f(e^{m^{k+1}}) \]
\[ \geq C_R C_1 m^k f(e^{m^{k+1}}) \geq C_R C_1 \frac{m^k}{m^2} \sum_{k > N} \int_{m^{k+1}}^{m^{k+2}} f(e^x) dx \geq C \int_{m^{N+1}}^{\infty} f(e^x) dx, \]
that proves (5.4).

**Step 2.** Suppose that \( \limsup_{z \to \infty} \nu\left\{ u : z \alpha < |u| \leq z \beta \right\} = 0 \), that is for any fixed small \( \varepsilon > 0 \) there exists \( N \) such that
\[ \nu\left\{ u : \frac{\beta_k}{\alpha_k} \alpha < |u| \leq \frac{\beta_k}{\alpha_k} \beta \right\} < \varepsilon \]
for every $k > N$. Consider now the functions $f_n = \mathbf{1}_{\left[0, \frac{\beta^{n+1}}{\alpha} \right]}$ on $\mathbb{R}_+$. Observe that for $n > N$

$$
\int_{\mathbb{R}^d} f_n(|u|)\nu(du) = \int_{\mathbb{R}^d} f_N(|u|)\nu(du) + \sum_{k=N+1}^{n} \int_{\mathbb{R}^d} 1_{\left[\frac{\beta^k}{\alpha}, \frac{\beta^{k+1}}{\alpha} \right]}(du)\nu(du)
$$

$$
\leq \int_{\mathbb{R}^d} f_N(|u|)\nu(du) + \varepsilon(n-N).
$$

Thus $\limsup_{n \to \infty} \frac{1}{n} \int_{\mathbb{R}^d} f_n(|u|)\nu(du) < \varepsilon$, that is $\limsup_{n \to \infty} \frac{1}{n} \int_{\mathbb{R}^d} f_n(|u|)\nu(du) = 0$ since $\varepsilon$ is arbitrary small.

On the other hand

$$
\limsup_{n \to \infty} \frac{1}{n} \int_{\mathbb{R}^d} f_n(a) \, da = \limsup_{n \to \infty} \frac{1}{n} \left( \log\left(\frac{\beta^{n+1}}{\alpha^n}\right) - \log M \right) = \log(\beta/\alpha) > 0
$$

which leads to a contradiction with (5.4).

\[\square\]

**Appendix A. Proof of Theorem 3.2 and related corollaries**

To prove Theorem 3.2 we will present a few consecutive lemmas.

**Lemma A.1.** If $\psi \in \mathcal{F}(\mathbb{P})$ the limit

$$
A\psi(x) = \lim_{\lambda \to 1} A^\lambda \psi(x)
$$

exists and is a continuous function. There exists a constant $C$ such that for all $x \in \mathbb{R}$ and $\lambda \in (1/2, 1]$

$$
|A^\lambda \psi(x)| \leq C(1 + x^2)
$$

Furthermore for any fixed $x$

$$
\lim_{y \to \pm\infty} (A\psi(x-y) - A\psi(-y)) = \mp xJ(\psi)\sigma^{-2},
$$

and if $J(\psi) = 0$ then

$$
\lim_{x \to \pm\infty} A\psi(x) = \mp K(\psi)\sigma^{-2}.
$$

**Proof.** Observe that the Fourier transform of the measure $G^\lambda$ is

$$
\hat{G^\lambda}(\theta) = \sum_{n=0}^{\infty} \lambda^n \hat{\mu}(\theta)^n = \frac{1}{1 - \lambda \hat{\mu}(\theta)}
$$

Since $G^\lambda$ is a finite measure and $\hat{\psi} \in L^1(\mathbb{R})$, using the Fubini Theorem, one has

$$
G^\lambda * \psi(x) = \int_{\mathbb{R}} \psi(x+y)G^\lambda(dy) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i\theta(x+y)}\hat{\psi}(\theta)d\theta G^\lambda(dy)
$$

$$
= \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\theta x} \hat{\psi}(-\theta) \int_{\mathbb{R}} e^{i\theta y} G^\lambda(dy) d\theta = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\theta x} \hat{\psi}(-\theta) \hat{G^\lambda}(\theta) d\theta
$$

$$
= \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\theta x} \hat{\psi}(-\theta) \frac{1}{1 - \lambda \hat{\mu}(\theta)} d\theta
$$

Notice that :

$$
\frac{\hat{\psi}(-\theta)}{1 - \lambda \hat{\mu}(\theta)} = \frac{J(\psi) - iK(\psi)\theta}{1 - \lambda \hat{\mu}(\theta)} 1_{[-a,a]}(\theta) + \psi^\lambda_0(\theta)
$$

where

$$
\psi^\lambda_0(\theta) = \frac{O(\theta^2)}{1 - \lambda \hat{\mu}(\theta)} 1_{[-a,a]}(\theta) + \frac{\hat{\psi}(-\theta)}{1 - \lambda \hat{\mu}(\theta)} 1_{[-a,a]^c}(\theta).
$$
Moreover,

\begin{align}
|1 - \lambda \hat{\mu}(\theta)| &\geq \lambda |1 - \hat{\mu}(\theta)| \quad \text{and} \quad |1 - \lambda \hat{\mu}(\theta)| \geq \lambda c|\theta|^2
\end{align}

for every \( \theta \in [-a, a] \). Therefore, for \( 1/2 \leq \lambda \leq 1 \)

\begin{align}
|\psi^0(\theta)| < C \left( |1_{[-a, a]}(\theta) + \frac{\hat{\psi}(\theta)}{1 - \hat{\mu}(\theta)} |_{[-a, a]}(\theta) \right) \in L^1(d\theta)
\end{align}

Take \( \phi \in \mathcal{F}(\mu) \) such that \( J(\phi) = J(\psi) \) then

\[
G^\lambda \ast \phi(-y) - G^\lambda \ast \psi(x - y) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-iy\theta} \hat{\phi}(-\theta) - e^{i(x-y)\theta} \hat{\psi}(\theta)}{1 - \lambda \hat{\mu}(\theta)} d\theta
\]

\[
= \frac{1}{2\pi} \int_{|\theta| < a} \left( e^{-iy\theta} \left( J(\phi) - i\theta K(\phi) \right) - e^{i\theta \phi} \right) \frac{d\theta}{1 - \lambda \hat{\mu}(\theta)}
\]

\[
= \frac{1}{2\pi} \int_{|\theta| < a} \left( e^{-iy\theta} \left( (J(\psi) - i\theta K(\psi)) \right) \right) \left( 1 + i\theta x + (e^{i\theta \phi} - 1 - i\theta x) \right) d\theta.
\]

The first integral can be decomposed as

\[
\frac{1}{2\pi} \int_{|\theta| < a} \frac{e^{-iy\theta} \left( (J(\psi) - i\theta K(\psi)) \right) \left( 1 + i\theta x + (e^{i\theta \phi} - 1 - i\theta x) \right) d\theta}{1 - \lambda \hat{\mu}(\theta)}
\]

\[
= \frac{1}{2\pi} \int_{|\theta| < a} \left( e^{-iy\theta} \left( (J(\phi) - K(\phi) + xJ(\psi)) \right) \frac{d\theta}{1 - \lambda \hat{\mu}(\theta)}
\]

Let

\[
C^\lambda_{\mu}(y) = \frac{i}{2\pi} \int_{|\theta| < a} \frac{e^{-iy\theta} \theta}{1 - \lambda \hat{\mu}(\theta)} d\theta.
\]

By Theorem 3.17 in [26] as \( \lambda \) goes to 1 the limit

\[
\lim_{\lambda \to 1} C^\lambda_{\mu}(y) = C^1_{\mu}(y)
\]

exists, it is finite and \( \lim_{y \to \pm \infty} C^1_{\mu}(y) = \pm \sigma^{-2} \). Summing up we may write

\begin{align}
G^\lambda \ast \phi(-y) - G^\lambda \ast \psi(x - y) = - (K(\phi) - K(\psi) + xJ(\psi)) C^\lambda_{\mu}(y) + \int_{\mathbb{R}} e^{-iy\theta} h^\lambda_{\psi, \phi, x}(\theta) d\theta,
\end{align}

where the functions

\[
h^\lambda_{\psi, \phi, x}(\theta) = \frac{1}{2\pi} \left( |1_{|\theta| < a}| \left( (J(\psi) - i\theta K(\psi)) \right) \left( 1 + i\theta x + (e^{i\theta \phi} - 1 - i\theta x) \right) \right)
\]

are bounded by a function in \( L^1(d\theta) \) uniformly for all \( \lambda \in [1/2, 1] \) and \( x \) in compact sets. More precisely by (A.2) and (A.3), there is an integrable function \( H = H^\lambda_{\psi, \phi} \) such that for all \( x \in \mathbb{R} \) and \( \lambda \in [1/2, 1] \):

\begin{align}
|h^\lambda_{\psi, \phi, x}(\theta)| \leq C(1 + x^2) H(\theta).
\end{align}
By the Lebesgue’s dominate convergence theorem, the following limit exists
\[(A.6) \quad \lim_{\lambda \to 1} G^\lambda * \psi(-y) - G^\lambda * \psi(x - y) = -(K(\phi) - K(\psi) + xJ(\psi)) C^\lambda_{\psi}(y) + \int_{\mathbb{R}} e^{-iy\theta} h^1_{\psi,\phi,x}(\theta) d\theta \]

For \( \phi = J(\psi)g \) and \( y = 0 \), we have \( A^\lambda \psi(x) = G^\lambda * \psi(0) - G^\lambda * \psi(x) \) thus
\[A^\lambda \psi(x) = -(J(\phi) K(g) - K(\psi) + xJ(\psi)) C^\lambda_{\psi}(0) + \int_{\mathbb{R}} h^1_{\psi,\phi,x}(\theta) d\theta \]

and
\[A\psi(x) = \lim_{\lambda \to 1} A^\lambda \psi(x) = -(J(\phi) K(g) - K(\psi) + xJ(\psi)) C^0_{\psi}(0) + \int_{\mathbb{R}} h^1_{\psi,\phi,x}(\theta) d\theta.\]

Hence we have proved the existence of the recurrent potential kernel. The continuity of \( A\psi \) follows from uniform integrability. Furthermore by (A.5) and since \( C^0_{\psi}(0) \) is finite, we also have
\[|A^\lambda \psi(x)| \leq C (1 + x^2)\]

Take now \( \phi = \psi \) then by (A.4) and (A.6)
\[A\psi(x - y) - A\psi(-y) = \lim_{\lambda \to 1} G^\lambda * \psi(-y) - G^\lambda * \psi(x - y) \]
\[= -xJ(\psi) C^0_{\psi}(y) + \hat{h}^1_{\psi,\phi,x}(-y).\]

Since \( h^1_{\psi,\phi,x} \in L^1(d\theta) \), then \( \hat{h}^1_{\psi,\phi,x} \) is in \( C_0(\mathbb{R}) \) for any fixed \( x \). Thus
\[\lim_{y \to \pm \infty} A\psi(-y) - A\psi(x - y) = -xJ(\psi) \lim_{y \to \pm \infty} C^0_{\psi}(y) = \mp xJ(\psi) \sigma^{-2}.\]

If \( J(\psi) = 0 \) then \( A^\lambda \psi = -G^\lambda * \psi \) and we can take \( \phi = 0 \) and \( x = 0 \). Thus by (A.6)
\[A\psi(-y) = K(\psi) C^0_{\psi}(y) + \hat{h}^1_{\psi,\phi,x}(-y)\]

and passing with \( y \) to \( \pm \infty \) we obtain the expected limit. \( \square \)

**Lemma A.7.** Assume \( J(\psi) \geq 0 \). For all \( 1 > \varepsilon > 0 \) there exists a constant \( M \) such that for any \( x \in \mathbb{R} \)
\[-M + 1 - \varepsilon)|J(\psi)\sigma^{-2}|x| \leq \sup_{|x| \leq K+1} |A\psi(x)| \leq M + (1 + \varepsilon)J(\psi) \sigma^{-2}|x|.\]

**Proof.** If \( J(\psi) = 0 \) the previous lemma guarantees that \( A\psi \) is continuous and has limit at infinity. Hence \( \hat{A}\psi \) is bounded.

Suppose \( J(\psi) > 0 \) and fix \( \varepsilon > 0 \). By the previous Lemma there is \( K > 0 \) such that for all \( y > K \)
\[(1 - \varepsilon)J(\psi) \sigma^{-2} < A\psi(y + 1) - A\psi(y) < (1 + \varepsilon)J(\psi) \sigma^{-2}\]
and for \( y < -K \)
\[(1 - \varepsilon)J(\psi) \sigma^{-2} < A\psi(y - 1) - A\psi(y) < (1 + \varepsilon)J(\psi) \sigma^{-2}.\]

Since \( A\psi \) is continuous,
\[\sup_{|x| \leq K+1} |A\psi(x)| = M' < \infty.\]

Set \( M = M' + (1 + \varepsilon)J(\psi) \sigma^{-2}(K + 1) \) then the bound holds for \(|x| \leq K\).

For \( x > K \) let \([x - K]\) be the integer part of \( x - K \) then, since
\[A\psi(x) = \left( \sum_{i=0}^{[x-K]-1} \left( A\psi(x - [x-K] + i - 1) - A\psi(x - [x-K] + i) \right) \right)\]
and \( x - [x-K] + i > K \) then
\[(1 - \varepsilon)J(\psi) \sigma^{-2}[x-K] \leq A\psi(x) - A\psi(x - [x-K]) < (1 + \varepsilon)J(\psi) \sigma^{-2}[x-K].\]
Thus
\[ A\psi(x) = A\psi(x - [x - K]) + A\psi(x) - A\psi(x - [x - K]) < M' + (1 + \varepsilon)J(\psi)\sigma^{-2}[x - K] \]
\[ < M + (1 + \varepsilon)J(\psi)\sigma^{-2}x \]
and
\[ A\psi(x) = A\psi(x - [x - K]) + A\psi(x) - A\psi(x - [x - K]) > -M' + (1 - \varepsilon)J(\psi)\sigma^{-2}[x - K] \]
\[ > -M + (1 - \varepsilon)J(\psi)\sigma^{-2}x. \]
In the same way we prove the bound for \( x < -K \).

\[ \square \]

**Proof of Theorem 3.2.** In view of Lemma A.1 the potential \( A\psi \) is well defined. To prove that \( A\psi \) is a solution of the Poisson equation observe that
\[ \mu^* A \lambda \psi = c \lambda^2 J(\psi) - \sum_{n=0}^{\infty} \lambda^n \mu^{*n+1} \psi = A^\lambda \psi + G^\lambda * (\psi - \mu^* \psi). \]
Notice that
\[ G^\lambda * (\psi - \mu^* \psi)(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\theta} \hat{\psi}(-\theta) \frac{1 - \hat{\mu}(\theta)}{1 - \lambda \hat{\mu}(\theta)} d\theta, \]
and, by (A.2), the integrand is dominated by \( 2|\hat{\psi}| \in L^1(d\theta) \) for all \( 1/2 < \lambda \leq 1 \). Therefore, by Lebesgue’s dominated convergence theorem
\[ \lim_{\lambda \rightarrow 1} \mu^* A^\lambda \psi = A\psi + \psi. \]
By Lemma A.1 there exists \( C \) such for all \( \lambda, x \in \mathbb{R} \) and any fixed \( x \):
\[ |A^\lambda \psi(x + y)| \leq C(1 + (x + y)^2) \in L^1(\mu(dy)), \]
then by dominate convergence
\[ \mu^* A\psi(x) = \int_{\mathbb{R}} \lim_{\lambda \rightarrow 1} A^\lambda \psi(x + y) \mu(dy) = \lim_{\lambda \rightarrow 1} \int_{\mathbb{R}} A^\lambda \psi(x + y) \mu(dy) = A\psi(x) + \psi(x). \]
The limit behavior is a direct consequence of Lemmas A.1 and A.7.

\[ \square \]

**Proof of Corollary 3.5.** First suppose \( J(\psi) = 0 \). Assume that \( f \) is a continuous solution of the Poisson equation. Since
\[ \mu^* f = f + \psi, \]
\[ \mu^* A\psi = A\psi + \psi, \]
the function \( h = f - A\psi \) is \( \mu \)-harmonic. It is bounded from below because both \(-A\psi \) and \( f \) are bounded from below. Therefore by the Choquet-Deny theorem [13] one has \( h(x + y) = h(x) \) for all \( y \) in the closed subgroup generated by the support of \( \mu \).

Conversely, suppose that there exists a bounded solution \( f_0 \) of the Poisson equation. Then \( A\psi - f_0 \) is \( \mu \)-harmonic and bounded from below, and so the Choquet-Deny theorem implies that \( A\psi \) is bounded. Thus
\[ \lim_{x \rightarrow \infty} \frac{A\psi(x)}{x} = 0 \]
and by (3.3), we deduce \( J(\psi) = 0 \).

\[ \square \]
Proof of Corollary 3.6. We apply a results of [2]. First observe that the measure $\overline{\mu}$ is strictly aperiodic (in the sense of [2]) on $G(\overline{\mu})$. Consider the function

$$h(x) = \left(\frac{\sin x}{x}\right)^4.$$  

Clearly,

$$\hat{h}(x) = C_1 1_{[-1,1]} \ast 1_{[-1,1]} \ast 1_{[-1,1]} \ast 1_{[-1,1]}(x);$$

so it compactly supported. Thus the function $h$ belongs to the class of functions $\mathcal{F}$ defined in [2] (see also [27]).

Let $m$ be the Haar measure of the $G(\overline{\mu})$, that is $m$ is either the Lebesgue measure on $\mathbb{R}$ or the counting measure. Then

$$\int_{G(\overline{\mu})} |A\psi(x)h(x)|m(dx) \leq C \int_{G(\overline{\mu})} (1 + |x|)h(x)m(dx) < +\infty.$$

In view of Corollary 2 of [2], we conclude. \hfill \Box

Proof of Lemma 3.7. To prove the first part of the Lemma it is enough to establish the following formula

(A.8)

$$r(x) = \left\{ \begin{array}{ll}
-2E[(Y + x)1_{Y+x\leq 0}] & \text{for } x \geq 0 \\
2E[(Y + x)1_{Y+x>0}] & \text{for } x < 0
\end{array} \right.$$  

Indeed, since $EY = 0$, for $x \geq 0$ we write

$$r(x) = E[(Y + x)1_{Y+x\geq 0} - (Y + x)1_{Y+x<0} - x]$$

$$= E[(Y + x) - 2(Y + x)1_{Y+x<0} - x] = -2E[(Y + x)1_{Y+x<0}].$$

For $x < 0$ we proceed exactly in the same way. By (A.8), the function $r$ is nonnegative.

The Fourier transform can be computed in distribution sense. Let $a(x) = |x|$, then $r = (\overline{\mu} - \delta_0) \ast a$ and $\hat{a}(\theta) = \frac{\hat{\mu}(\theta)}{|\theta|}$, hence $\hat{r}(\theta) = C \cdot \frac{\hat{\mu}(\theta)}{|\theta|^3}$.

To prove (3.8), by (A.8), for $x \geq 0$ we write

$$|r(x)| = 2E[|Y + x|1_{Y+x<0}] = 2 \int_{x+y<0} |x + y|\overline{\mu}(dy) \leq 2 \int_\mathbb{R} |x + y|e^{-\delta_1|y|}\overline{\mu}(dy) \leq Ce^{-\delta_1x},$$

for some constants $\delta_1 < \delta_0 < \delta$.

Assume now $E|Y|^{4+\varepsilon} < \infty$. Then for $t > 0$ we have

$$|r(x)| = 2 \int_{y<-x} |y + x|\overline{\mu}(dy) = 2 \cdot \sum_{m=1}^{\infty} \int_{-(m+1)x \leq y < -mx} |y + x|\overline{\mu}(dy)$$

$$\leq 2 \cdot \sum_{m=1}^{\infty} mx \int_{|y|>mx} \overline{\mu}(dy) \leq 2 \cdot \sum_{m=1}^{\infty} mx \int_{\mathbb{R}} \frac{|y|^\varepsilon}{m^{1+\varepsilon}x^{4+\varepsilon}}\overline{\mu}(dy) \leq \frac{C}{x^{3+\varepsilon}}.$$

It is clear that if $E|Y|^{4+\varepsilon} < \infty$ then $r \in \mathcal{F}(\overline{\mu})$. If $\psi = r \ast \zeta$ with $\zeta$ and $x^2\zeta$ in $L^1(\mathbb{R})$ then it is easily checked that both $\psi$ and $x^2\psi$ are integrable. Since $\hat{r}\hat{\zeta} = C \frac{\hat{\mu}^{-1}\zeta}{|\theta|^4}$ and $\zeta$ vanish at infinity then $\psi \in \mathcal{F}(\overline{\mu})$. \hfill \Box
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[22] S. Brofferio, Université Paris-Sud, Laboratoire de Mathématiques, 91405 Orsay Cedex, France. E-mail address: sara.brofferio@math.u-psud.fr
D. Buraczewski and E. Damek, Institute of Mathematics, University of Wroclaw, Pl. Grunwaldzki 2/4, 50-384 Wroclaw, Poland,

E-mail address: dcura@math.uni.wroc.pl, edamek@math.uni.wroc.pl