Practical Privacy For Expectation Maximization

Mijung Park∗, Jimmy Foulds†, Kamalika Chaudhuri‡, Max Welling§

Abstract

Expectation maximization (EM) is an iterative algorithm that computes maximum likelihood and maximum a posteriori estimates for models with unobserved variables. While widely used, the iterative nature of EM presents challenges for privacy-preserving estimation. Multiple iterations are required to obtain accurate parameter estimates, yet each iteration increases the amount of noise that must be added to achieve a reasonable degree of privacy. We propose a practical algorithm that overcomes this challenge and outputs EM parameter estimates that are both accurate and private. Our algorithm focuses on the frequent use case of models whose joint distribution over observed and unobserved variables remains in the exponential family. For these models, the EM parameters are functions of moments of the data. Our algorithm leverages this to preserve privacy by perturbing the moments, for which the amount of additive noise scales naturally with the data. In addition, our algorithm uses a relaxed notion of the differential privacy (DP) gold standard, called concentrated differential privacy (CDP). Rather than focusing on single-query loss, CDP provides high probability bounds for cumulative privacy loss, which is well suited for iterative algorithms. For mixture models, we show that our method requires a significantly smaller privacy budget for the same estimation accuracy compared to both DP and its \((\epsilon, \delta)\)-DP relaxation. Our general approach of moment perturbation equipped with CDP can be readily extended to many iterative machine learning algorithms, which opens up various exciting future directions.

1 Introduction

Many devices such as household appliances, phones, drones, watches, cars, and so on, are increasingly equipped with sensors and connected in networks that collect, store and analyze data at an unprecedented scale. However, there are significant concerns that all this data in the hands of a few corporations and/or governments can lead to abuse. Hence there is a need for new machine learning tools to analyze the data but at the same time guarantee the privacy of every individual. Much progress has been made recently in developing privacy preserving algorithms [1]. In particular, differential privacy is emerging as the dominant notion of algorithmic privacy.

In this paper we will analyze the popular EM algorithm and derive privacy preserving variants for it. Expectation maximization iteratively estimates the parameters of models with unobserved variables. EM alternates inferring the unobserved variables given parameter values (the E-step) and optimizing the parameters given the inferred variables (the M-step)[2]. This iterative algorithm is widely used to solve statistical problems in many areas of science including bioinformatics [3], neuroscience [4], and computer vision [5]. We will apply our new privacy preserving EM algorithm to a mixture of Gaussians density estimation model. Having access to a private density estimator is particularly valuable because it provides a means to anonymize the data in a principled way (i.e. with strict privacy guarantees), by simply sampling a dataset from the model and replacing the original data with this sampled data.

When using privacy-sensitive data, iterative algorithms like EM need to handle a privacy-utility trade-off. The number of iterations required to guarantee accurate estimates causes high cumulative privacy loss. To compensate for the loss, one needs to add a significantly high level of noise to the parameters of interest. For example, recent work on the \(k\)-means algorithm, a variant of EM algorithm for mixture of Gaussians, requires adding noise to the parameters where the noise standard deviation is on the order of input dimension times the number of iterations [6]. To avoid adding so much noise, more recent work proposes generating a synopsis of a dataset first, then applying a standard \(k\)-means clustering algorithm on the synopsis [7]. Their synopsis generation method consists of putting rectangular bounding boxes in the data space and counting how many data points are in each box. However, this method mainly works well for a clustering task and for low dimensional data.

∗QUvA Lab, University of Amsterdam. mijungi.p@gmail.com
†California Institute for Telecommunications and Information Technology, University of California, San Diego. jrfoulds@gmail.com
‡Department of Computer Science, University of California, San Diego. kamalika@cs.ucsd.edu
§QUvA Lab, University of Amsterdam, welling.max@gmail.com
In this paper, we propose to resolve the privacy-utility trade-off by using moment perturbation. This is applicable for models where the complete-data likelihood is in the exponential family (even though the marginal over unobserved variables may not be). In such cases, the EM parameters are functions of moments of latent and observed variables, which we perturb for privacy. Since the amount of noise for perturbing the moments scales with the number of datapoints, our algorithm yields asymptotically efficient private EM parameters even when \( \epsilon \) is non-zero. Besides, thanks to the \( \frac{1}{N} \) factor, under the mixture of Gaussians model, the noise standard deviation in our method is smaller than that in other methods [6]. The difference in the amount of additive noise will be more significant when \( N \) gets larger. Moment perturbation for differentially private estimators isn’t a new concept (See [8]). However, unlike existing methods, we do not require subsampling of the data.

Furthermore, our algorithm uses a relaxed version of differential privacy called \textit{concentrated differential privacy} (CDP) [9]. CDP has two major advantages over DP. First, \((\mu, \tau)\)-CDP offers a bounded expected privacy loss, while in \((\epsilon, \delta)\)-DP the privacy loss is infinite with probability \( \delta \) (in other words, the mechanism fails w.p. \( \delta \)). Second, CDP requires adding much less noise for the same expected privacy guarantee compared to the \((\epsilon, \delta)\)-DP relaxation. As we will analyze in Sec 5 for \( J \) iterations, the noise standard deviation is roughly on the order of \( \sqrt{2J \log(1/\delta_1)/\epsilon} \) in \((\epsilon, \delta)\)-DP relaxation\(^1\) where the failure probability is \( \delta = J\delta' + \delta_1 \). To obtain meaningful privacy guarantees, \( \delta \) is set to be tiny, which yields quite a large noise standard deviation. In CDP, the noise standard deviation is on the order of \( \sqrt{J/2\epsilon} \), which effectively lowers the amount of additive noise.

We start by overviewing privacy and the general EM algorithm in Sec 2. In Sec 3, we introduce a general DP EM framework. We then derive the DP EM algorithm for mixture of Gaussians in Sec 4, which we will use for illustrating the effectiveness of our algorithm in Sec 5. In Sec 6, we construct the concentrated differential privacy formulation for EM.

2 Background

In this section, we provide background information on the definitions of algorithmic privacy that we use, as well as the general formulation of the EM algorithm.

\textbf{Differential privacy and concentrated differential privacy.} Differential privacy (DP) is a formal definition of the privacy properties of data analysis algorithms [1]. A randomized algorithm \( \mathcal{M}(X) \) is said to be \((\epsilon, \delta)\)-differentially private if
\[
Pr(\mathcal{M}(X) \in S) \leq \exp(\epsilon)Pr(\mathcal{M}(X') \in S) + \delta
\]
for all measurable subsets \( S \) of the range of \( \mathcal{M} \) and for all datasets \( \mathbf{X}, \mathbf{X}' \) differing by a single entry. If \( \delta = 0 \), the algorithm is said to be \( \epsilon \)-differentially private. Intuitively, the definition states that the output probabilities must not change very much when a single individual’s data is modified, thereby limiting the amount of information that the algorithm reveals about any one individual.

Concentrated differential privacy (CDP) is a recently proposed relaxation of differential privacy which aims to make privacy-preserving iterative algorithms more practical than for DP while still providing strong privacy guarantees. The CDP framework treats the privacy loss of an outcome,
\[
L^{(\alpha)}(\mathcal{M}(\mathbf{X}||\mathcal{M}(\mathbf{X'})) = \log \frac{Pr(\mathcal{M}(\mathbf{X}) = o)}{Pr(\mathcal{M}(\mathbf{X'}) = o)}
\]
as a random variable. An algorithm is \((\mu, \tau)\)-CDP if this privacy loss has mean \( \mu \), and after subtracting \( \mu \) the resulting random variable \( l \) is subgaussian with standard deviation \( \tau \), i.e. \( \forall \lambda \in \mathbb{R} : E[e^{\lambda l}] \leq \exp(\lambda^2\tau^2/2) \). While \( \epsilon \)-DP guarantees bounded privacy loss, and \((\epsilon, \delta)\)-DP ensures bounded privacy loss with probability \( 1 - \delta \), \((\mu, \tau)\)-CDP requires the privacy loss to be near \( \mu \) w.h.p.

\textbf{The general EM algorithm.} Given \( N \ i.i.d. \) observations \( X := \{x_i\}_{i=1}^N \), with each observation \( x_i \in \mathbb{R}^d \), and hidden variables \( Z := \{z_i\}_{i=1}^N \), computing the maximum likelihood estimator of a vector of model parameters \( \theta = [\theta_1, \cdots, \theta_L] \) is analytically intractable, due to the integral or summation inside the logarithm,
\[
\mathcal{L}(\theta) = \log p(X|\theta) = \log \int dZ \ p(X, Z|\theta).
\]

\(^1\)By applying the Advanced composition theorem for \( J \)-composition of \((\epsilon', \delta')\)-DP mechanisms.
Instead, one can lower-bound \( \mathcal{L}(\theta) \) using the posterior distribution over latent variables \( q(Z) \):

\[
\mathcal{L}(\theta) = \log \int dZ \, q(Z) \frac{p(X,Z|\theta)}{q(Z)} \geq \int dZ \, q(Z) \log \frac{p(X,Z|\theta)}{q(Z)} \overset{\text{def}}{=} \mathcal{F}(q, \theta),
\]

where the lower bound is often called free energy \( \mathcal{F}(q, \theta) = \langle \log p(X,Z|\theta) \rangle_{q(Z)} + H(q) \), where \( H(q) \) is the entropy of \( q(Z) \). EM alternates between: (1) the E-step: optimizing \( \mathcal{F} \) wrt distribution over unobserved variables holding parameters fixed

\[
q^{(j)}(Z) = \arg \max_{q(Z)} \mathcal{F}(q(Z), \theta^{(j-1)})
\]

and (2) the M-step: maximizing \( \mathcal{F} \) wrt parameters holding the latent distribution fixed

\[
\theta^{(j)} = \arg \max_{\theta} \mathcal{F}(q^{(j)}(Z), \theta) = \arg \max_{\theta} \langle \log p(X,Z|\theta) \rangle_{q^{(j)}(Z)}
\]

where the second equality holds since \( H(q) \) does not directly depend on \( \theta \). To understand what EM does, one can rewrite the free energy in terms of the log-likelihood and the KL divergence terms, \( \mathcal{F}(q, \theta) = \mathcal{L}(\theta) - D_{KL}[q(Z)||p(Z|X, \theta)] \). During the E-step, we set \( q^{(j)}(Z) = p(Z|X, \theta^{j-1}) \), which makes the second term zero and the free energy equals the likelihood. Then, in the M-step, we get the maximum likelihood estimate (MLE). For the maximum a posteriori (MAP) estimate, we add the log prior for the parameters \( \log p(\theta) \) to the right hand side of eq (6) and optimize for \( \theta \).

3 The general differentially private EM algorithm

EM is widely used for models whose joint distribution over observed and unobserved variables has exponential family form: \( p(X,Z) = h(X,Z) \exp(\theta^\top T(X,Z)) / A(\theta) \) while the marginal \( p(X) \) does not. In this case, the free energy can be rewritten as

\[
\mathcal{F}(q, \theta) = \theta^\top (T(X,Z))_{q(Z)} - N \log A(\theta) + \text{constant wrt } \theta,
\]

where \( \theta^\top (T(X,Z))_{q(Z)} = \sum_{i=1}^N \mathbb{E}_{q(z_i)} \sum_{l=1}^L \theta_l T_l(x_i, z_i) \). So, in the E-step, all we need to compute is the expected sufficient statistics under \( q \), i.e., \( \langle T(X,Z) \rangle_{q(Z)} \). Then, in M-step, we compute partial derivatives wrt each parameter,

\[
\frac{\partial}{\partial \theta_l} \mathcal{F}(q, \theta) = \frac{1}{N} \sum_{i=1}^N \mathbb{E}_{q(z_i)} T_l(x_i, z_i) - \frac{\partial}{\partial \theta_l} \log A(\theta) = 0.
\]

Although it is not straightforward to derive a closed-form expression for each parameter update due to the dependence on other parameters in \( A(\theta) \), it is easy to see that each parameter update depends on each expected sufficient statistics, i.e., moments, denoted by \( M_l = \frac{1}{N} \sum_{i=1}^N \mathbb{E}_{q(z_i)} T_l(x_i, z_i) \). So, to output privatized parameters, all we need is to perturb the moments to compensate any single data point’s change. The sensitivity of expected sufficient statistics is given by

\[
\Delta M_l = \max_{|D-D'|=1} |M_l(D) - M_l(D')| = \max_{x_j, \tilde{x}_j} \frac{1}{N} |\mathbb{E}_{q(z_j)} T_l(x_j, z_j) - \mathbb{E}_{q(z_j)} T_l(\tilde{x}_j, \tilde{z}_j)|,
\]

\[
\leq \max_{x_j, \tilde{x}_j} \frac{1}{N} |\mathbb{E}_{q(z_j)} T_l(x_j, z_j)| + \frac{1}{N} |\mathbb{E}_{q(z_j)} T_l(\tilde{x}_j, \tilde{z}_j)|,
\]

where the last line is due to the triangle inequality. The expectation over \( z \) can be rewritten as an inner product, and using Hölder’s inequality we obtain \( |\mathbb{E}_{q(z_j)} T_l(x_j, z_j)| = |\langle q(z_j), T_l(x_j, z_j) \rangle| \leq |q(z_j)|_1 |T_l(x_j, z_j)|_\infty \). To further bound this quantity, we assume that the unobserved variables are discrete\(^2\) so that \( q(z) \) is bounded between 0 and 1. We also assume that datasets are pre-processed such that the \( L_2 \) norm of any \( x_i \) is less than 1. In such a case, any \( x_i \) and \( z_i \) stay within a unit ball. Under these two assumptions, now the sensitivity is given by

\[
\Delta M_l = \max_{(x_j, z_j) \in B_1(x, z)} \frac{2}{N} |T_l(x_j, z_j)|.
\]

Using this sensitivity, we add noise to each moment and the perturbed moments are mapped by a model-specific deterministic function \( g \) to the vector of privatized parameters, given as \( \hat{\theta}^* = g(M_l) \), where \( M_l \) are perturbed moments. Using this general framework, we derive the differentially private EM algorithm for mixture of Gaussians in the following.

\(^2\)When the unobserved variables are continuous, the boundedness of weights depends on whether or not the variational distribution \( q(Z) \) has limited sensitivity. We leave this as our future work.
4 DPEM for mixture of Gaussians

4.1 Mixture of Gaussians

We consider the mixture of Gaussians (MoG) model as an example to derive the differentially private EM algorithm. For $K$ Gaussians and $N$ data points $X := \{x_i\}_{i=1}^N$, the log-likelihood under MoG is given by

\[
\log p(X; \pi, \mu, \Sigma) = \sum_{i=1}^N \log \sum_{k=1}^K \pi_k N(x_i | \mu_k, \Sigma_k),
\]

where $\sum_{k=1}^K \pi_k = 1$. We denote the parameters by $\theta := \{\pi, \mu, \Sigma\} = \{\pi_k, \mu_k, \Sigma_k\}_{k=1}^K$.

Introducing a binary vector of length $K$ for each data point, $z_i \in \mathbb{R}^K$, to represent the membership to which Gaussian each datapoint belongs, e.g., $z_{ij} \in \{0, 1\}$ and $\sum_{k=1}^K z_{ik} = 1$, the distribution over each $z_i$ is given by $p(z_i) := \prod_{k=1}^K \pi_k$, and the distribution over all unobserved variables $Z = \{z_i\}_{i=1}^N$ is given by $p(Z) := \prod_{i=1}^N p(z_i)$. The joint distribution over observed and unobserved variables which is in exponential family is given by $p(X, Z; \pi, \mu, \Sigma) = \sum_{\alpha=1}^N \sum_{k=1}^K \sum_{i=1}^N \gamma_{i,k} \log \tau_k + \log N(x_i | \mu_k, \Sigma_k)$. In E-step, we compute the responsibilities by $\gamma_{i,k} := p(z_{i,k} = 1 | x_i, \theta_{MPE})$.

\[
\gamma_{i,k} = p(z_{i,k} = 1 | x_i, \theta_{MPE}) = \pi_k N(x_i | \mu_k, \Sigma_k) / \sum_{k=1}^K \pi_k N(x_i | \mu_k, \Sigma_k),
\]

and in M-step, we update the parameters $\theta$ by eq (8)

\[
\pi_k^{MLE} = \frac{N_k}{N}, \quad \mu_k^{MLE} = \frac{1}{N} \sum_{i=1}^N \gamma_{i,k} x_i, \quad \Sigma_k^{MLE} = \frac{1}{N} \sum_{i=1}^N \gamma_{i,k} (x_i - \mu_k^{MLE}) (x_i - \mu_k^{MLE})^\top,
\]

where $N_k = \sum_{i=1}^N \gamma_{i,k}$. One could assume each Gaussian is isotropic, in which case the covariance is parameterized by $\Sigma_k^{MLE} := \sigma_k^2 I = \frac{1}{N} \sum_{i=1}^N \gamma_{i,k} (x_i - \mu_k^{MLE}) (x_i - \mu_k^{MLE})^\top$.

For the maximum a posteriori estimate, we impose the Dirichlet prior on $\pi \sim \text{Dir}(\alpha)$ and Normal-inverse-Wishart prior on $p(\mu_k, \Sigma_k) = \text{NIW}(0, \kappa_0, \nu_0, S_0)$, where the MAP estimates are

\[
\pi_k^{MAP} = \frac{N_k \pi_k^{MLE} + \alpha_k - 1}{N + \sum \alpha_k - K}, \quad \mu_k^{MAP} = \frac{N_k \mu_k^{MLE} + \kappa_0}{N_k + \kappa_0}, \quad \Sigma_k^{MAP} = \frac{S_0 + N_k \Sigma_k^{MLE} + \kappa_0 N_k}{\nu_0 + N_k + d + 2}.
\]

In this paper we follow conventional ways to set hyperparameters, e.g. setting $\alpha = [2, 2, \cdots, 2], \kappa_0 = 1, \nu_0 = d + 2, S_0 = \text{diag}(0.1, \cdots, 0.1)$, rather than optimizing them [12].

Before moving to the next section, we would like to motivate why it is important to construct a privacy preserving algorithm for MoG. In Fig. 1 we show that if one runs the conventional EM algorithm for the given dataset, an individual’s information can be easily revealed by just looking at the EM parameters, while the noised-up parameters obtained by the method, which will be described next, protect private information effectively.

4.2 DPEM for MoG

Under MoG, we plug in the responsibilities given in eq (11) to the parameter update expressions given in eq (12).

We then perturb each of these by taking into account one datapoint’s worst-case difference between two neighboring datasets. We use $\epsilon'$ to denote a privacy budget allocated for each type of parameter perturbation.

$\epsilon'$-DP mixing coefficients. For two datasets $D, \tilde{D} \in \mathbb{N}^{|X|}$ with a single data point difference, the maximum difference in $\pi$ occurs when the data point $x_j$ is assigned to the $k$-th Gaussian with $\gamma_{j,k} = 1$ and the altered data point $\tilde{x}_j$ is assigned to another, e.g., the $k'$-th Gaussian, with $\gamma_{j,k'} = 1$. Hence, we get the following sensitivity: $\Delta \pi^{MLE} = \max_{x_j, \tilde{x}_j} \sum_{k=1}^K \frac{1}{K} |\gamma_{j,k} - \tilde{\gamma}_{j,k}| \leq 2/K$, since $0 \leq \gamma_{j,k} \leq 1$ and $\sum_{k=1}^K \gamma_{j,k} = 1$. We add noise to compensate the maximum difference

\[
\tilde{\pi}_k^{MLE} = \pi_k^{MLE} + (Y_1, \cdots, Y_K), \quad \text{where } Y_i \sim i.i.d. \text{ Lap}(2/\epsilon').
\]
For $\pi_k^{MAP}$, we do not need any additional sensitivity analysis, since the MAP estimate is a deterministic mapping of the MLE.

$\epsilon'$-DP mean parameters. Using the noised-up $\tilde{N}_k$ obtained from the noised-up mixing coefficients, i.e., $\tilde{N}_k = N\hat{\pi}_k$, the maximum difference in mean parameters due to one datapoint’s difference is given by

$$\Delta \mu^{MLE}_k = \max_{x_j, x_j'} \left| \frac{1}{\tilde{N}_k} (A_k + \gamma_{j,k} x_j) - \frac{1}{\tilde{N}_k} (A_k + \gamma_{j,k} x_j') \right| \leq 2\sqrt{d}/\tilde{N}_k,$$

where $A_k := \sum_{i=1}^N \gamma_{i,k} x_i$, and the $L_1$ term is bounded due to eq (10) and $\sum_{i=1}^d |x_{i,l}| = |(\bar{x}_l, 1)| \leq (\sum_{i=1}^d |x_{i,l}|^2)^{1/2} (\sum_{i=1}^d 1)^{1/2} \leq \sqrt{d}$. We add Laplace noise to the MLE via

$$\mu^{MLE}_k = \mu^{MLE}_k + (Y_1, \cdots, Y_d), \text{ where } Y_i \sim^i.i.d. \text{ Lap}\left(\frac{2\sigma}{\pi\sqrt{N_k}}\right).$$

$(\epsilon', \delta')$-DP mean parameters. In order to use the Gaussian mechanism, we can straightforwardly derive the $L_2$-sensitivity from the $L_1$ sensitivity of $\mu^{MLE}$: $\Delta_2 \mu^{MLE} = 2/\tilde{N}_k$.

$$\mu^{MLE}_k = \mu^{MLE}_k + (Y_1, \cdots, Y_d), \text{ where } Y_i \sim^i.i.d. \mathcal{N}(0, \tau^2).$$

where $\tau \geq c\Delta_2 \mu^{MLE}/\epsilon'$ and $c^2 > 2 \log(1.25/\delta')$.

$\epsilon'$-DP variance parameters. To make the sensitivity analysis easier, we plug in the noised-up mean parameters obtained from the previous section to $\mu^{MLE}_k$, and the noised-up version of $\tilde{N}_k$ to $\sigma^2_k I = \frac{1}{\tilde{N}_k} \sum_{i=1}^N \gamma_{i,k} x_i^\top x_i - \frac{1}{\tilde{N}_k} \mu^{MLE}_k \mu^{MLE}_k^\top$. Now, we perturb the first term, using Gamma noise as below in order to obtain non-negative $\epsilon'$-DP variances:

$$\tilde{\sigma}^2_k = \sigma^2_k + Y, \text{ where } Y \sim^i.i.d. \text{ Gam}(1, \beta).$$

To set $\beta$, we look at the probability ratio given two neighboring datasets $\mathcal{D}$ and $\tilde{\mathcal{D}}$:

$$p(\tilde{\sigma}^2_k - \sigma^2_k(D)) / p(\tilde{\sigma}^2_k - \sigma^2_k(\tilde{D})) = \exp \left( \beta (\sigma^2_k(D) - \sigma^2_k(\tilde{D})) \right),$$

due to the Gamma noise

$$\leq \max_{x_j, x_j'} \exp \left( \frac{\beta}{\nabla_{x_j, x_j'} } \left( (B_k + \gamma_{j,k} x_j^\top x_j) - (B_k + \gamma_{j,k} \tilde{x}_j^\top \tilde{x}_j) \right) \right) \leq \exp \left( \frac{\beta}{\nabla_{x_j, x_j'} } \right),$$

The MAP estimate only differs from the MLE in the denominator: $\mu^{MAP}_k = \frac{1}{\tilde{N}_k + \kappa_0} \sum_{i} \gamma_{i,k} x_i$. We simply replace $\tilde{N}_k$ with $\tilde{N}_k + \kappa_0$ in eq (15) in the MAP estimation case.
where $B_k := \sum_{i=1,i \neq j}^N \gamma_{i,k} x_i x_i^\top$, and the last line is due to eq (10). Hence, $\beta = \epsilon' dN_k$.

$\epsilon'$-DP covariance parameters. For full covariances, we follow the symmetric noise (SN) algorithm \[13\], which provides strong privacy guarantees and significantly higher utility than other methods (e.g., \[14\] \[15\] \[16\]) as illustrated in \[13\] when perturbing positive definite matrices. We first draw Gaussian random variables $z_i \sim \mathcal{N}(0, \lambda I_d)$, for $i = \{1, \cdots, d + 1\}$, and construct a matrix $Z_k := [z_1, \cdots, z_{d+1}] \in \mathbb{R}^{d \times (d+1)}$, which we add to perturb each of the covariance matrices as

$$
\tilde{\Sigma}_k^{MLE} := \Sigma_k^{MLE} + Z_k Z_k^\top.
$$

The perturbed covariance $\tilde{\Sigma}_k^{MLE}$ are made $\epsilon'$-differentially private by setting $\lambda$ to meet this privacy budget. We follow the proof of \[13\] to set $\lambda$ using the fact that the matrix $Z_k Z_k^\top$ is a sample from a Wishart distribution with covariance $\lambda I_d$ and $d + 1$ degrees of freedom. See Appendix for full derivation, where we set $\lambda = 1/(2\epsilon' N_k)$. We provide the derivation for $\epsilon'$-DP MAP estimates of covariances in the Appendix.

In summary, we perturb the mixing coefficients by the Laplace mechanism ($\epsilon'$-DP); mean parameters either by using the Laplace mechanism ($\epsilon'$-DP) or using the Gaussian mechanism ($\epsilon', \delta'$-DP); and the variance or covariance parameters by Gamma or Wishart noise ($\epsilon'$-DP).

5 Concentrated differential privacy for EM

To construct a CDP formulation for EM, we use two key theorems from \[9\]. Their Theorem 3.5 states that any $\epsilon$-DP algorithm is $(\epsilon (\exp(\epsilon') - 1)/2, \epsilon)$-CDP, which means that each pure-$\epsilon$ perturbation introduced in Sec \[4\] produces $(\epsilon (\exp(\epsilon') - 1)/2, \epsilon')$-CDP parameters. For simplicity, we rework the $\epsilon' (\exp(\epsilon') - 1)/2 \approx \epsilon^2/2$. Also, their Theorem 3.4 states that after $(2K + 1)$-composition we obtain $((2K + 1)\epsilon^2/2, \sqrt{(2K + 1)\epsilon^2})$-CDP parameters. All we need is to set $\epsilon'$ such that the expected cumulative privacy loss meets the privacy budget $\epsilon$. For example, when we perturb all the parameters with $\epsilon'$-DP mechanisms, which equal $(\epsilon^2/2, \epsilon')$-CDP, we set $\epsilon' = \sqrt{2\epsilon / (2K + 1)}$. When we use the Gaussian mechanism for mean perturbation, on the other hand, this introduces $(\epsilon', \delta')$-DP parameters. In this case, we use Theorem 3.2 in \[9\], which states that the Gaussian mechanism with noise magnitude $\sigma$ is $(\tau^2/2, \tau)$-CDP, where $\tau = \Delta_2 \mu_k / \sigma$. We further express $\tau$ as a function of $\epsilon'$: $\tau^2 = (\Delta_2 \mu_k)^2 / \sigma^2 \leq \epsilon'^2 \sigma^2 / \epsilon$, since $\sigma^2 \geq \epsilon^2 (\Delta_2 \mu_k)^2 / \epsilon$. Combined with $\epsilon'$-DP covariance and mixing coefficient perturbations, we set $\epsilon' = \sqrt{\frac{2\epsilon}{\tau (2K + 1)}}$. Notice that using the Gaussian mechanism for mean perturbation, a larger value of $\epsilon'$ is assigned to each parameter perturbation compared to using the Laplace mechanism for mean perturbation. Algorithm 4 summarizes our algorithm.

Algorithm 1 Concentrated differential privacy for EM under MoG

Require: Dataset $\mathcal{D}$
Ensure: $(\epsilon, \sqrt{2\epsilon})$-CDP parameters $\tilde{\theta}$

Iterate until convergence ($J$ iterations)

1. Compute parameters with plugging in the responsibilities given in eq (11).
2. Noise up $\pi$ by eq (13), $\mu$ by eq (15), and $\sigma$ by eq (16) (or $\Sigma$ by eq (18)).

6 Illustration

Synthetic data We first used synthetic data to illustrate how much improvement we gain by formulating the private EM algorithm under the CDP composition as opposed to the advanced $(\epsilon, \delta)$-DP composition. We

\footnote{Using Taylor’s expansion $\epsilon' (\exp(\epsilon') - 1)/2 = \epsilon' (1 + \epsilon' + \sum_{j=2}^{\infty} \epsilon'^j / j! - 1)/2$, which we can lower bound by ignoring the infinite sum, $\epsilon'^2/2$.}

\footnote{We perturb $\pi$ once, $\mu_k$, $K$ times, and $\sigma_k^2$ (or $\Sigma_k$) $K$ times per iteration. Hence, $2K + 1$ parameter perturbations per iteration, and after $J$ iterations, $(2K + 1) J$ compositions in total.}
generated data from three isotropic Gaussians, where each Gaussian lies in $2D$ space. We generated 100 training datasets with varying $N$. For each training set, we also generated a test set in the amount of 10% of training data points, which we used to compute the test likelihood given the trained parameters. Since EM is well-known for being sensitive to parameter initialization, for each training set, we ran EM and (C)DP-EM with different initial values of parameters. We then calculated the interquantile range (25 − 75%) of the test likelihoods per training set, and computed the average of the interquantile range across 100 training sets. We report the test likelihood per data point by dividing the average by the number of test points in Fig. 2A. We set $\epsilon = 0.9$ in this experiment. For $(\epsilon, \delta)$-DP, we set $\delta = 0.01$.

**Real dataset I: Gowalla dataset.** We first used the Gowalla dataset to compare the clustering performance of our method (in “$k$-means mode”) to a differentially private $k$-means clustering algorithm, DPLloyd [6]. As summarized in [7], the DPLloyd adds Laplace noise to the number of data points assigned to each cluster as well as to the sum of each coordinate of the data points assigned to each cluster. Due to the conventional composition theorem for DP, their noise distribution follows Lap($J/\epsilon$) for $J$ iterations. We also tested two variants of our algorithm, in which we perturb the mean by the Laplace mechanism (CDPlap), and by the Gaussian mechanism (CDPgau). In both algorithms, we set $\epsilon'$ such that each algorithm satisfies $(\epsilon, \sqrt{2\epsilon})$-CDP. In CDPgau, we set $\delta = 0.001$. As shown in Fig. 3 even with a very small value of $\epsilon$, our methods achieve significantly smaller NICV than DPLloyd.

**Real dataset II: Lifescience dataset.** We tested our method on Life science data from the UCI repository [17]. The dataset contains 26,733 records where each of them consists of 10 principal components for a chemistry or biology experiment ($d = 10$). Following other approaches (e.g., [18]), we set $k = 3$. We divided the dataset into 10 different pairs of training (90%) and test sets (10%), and show the average log test likelihood per data point across the 10 independent trials. In this experiment, we tried our CDP-EM algorithms with full covariances, as well as the EM algorithm with $(\epsilon, \delta)$ relaxation.
7 Discussion

We developed a practical algorithm that outputs accurate and privatized EM parameters based on moment perturbation under the CDP formulation, which effectively decreases the amount of additive noise for the same expected privacy guarantee compared to the \((\epsilon, \delta)\)-DP relaxation. The private EM algorithm for the mixture of Gaussians model we discussed in this paper is clearly only one example of a much broader class of models to which our private EM framework applies. Our positive empirical results with EM strongly suggest that these ideas are likely to be beneficial for privatizing many other iterative machine learning algorithms. In future work we plan to apply this general framework to other inference methods which fits in our broader vision that practical privacy preserving machine learning algorithms will have an increasingly relevant role to play in our field.

Acknowledgements

The work of M. Welling and M. Park was supported in part by Qualcomm, and the work of K. Chaudhuri was supported in part by NSF under IIS 1253942. We also thank Eric Nalisnick and Babak Shahbaba for helpful discussions.
Appendix

A Sensitivity of mean parameters

A.1 L-1 sensitivity of MLE

We plug in the noised-up version of \( \tilde{N}_k \) from \( \tilde{N}_k = \tilde{\pi}_k N \) in the definition of mean

\[
\Delta \mu_{mle}^k = \max_{|D - \tilde{D}|_1 = 1} |\mu_k(D) - \mu_k(\tilde{D})|_1 = \max_{|D - \tilde{D}|_1 = 1} |\frac{1}{N_k} \sum_{i=1}^{N} \gamma_{i,k} x_i - \frac{1}{N_k} \sum_{i=1}^{N} \tilde{\gamma}_{i,k} \tilde{x}_i|_1,
\]

where \( A_k := \sum_{i=1, i \neq j}^{N} \gamma_{i,k} x_i \). The L1 term is further bounded by

\[
|\frac{1}{N_k} (A_k + \gamma_{j,k} x_j) - \frac{1}{N_k} (A_k + \tilde{\gamma}_{j,k} \tilde{x}_j)|_1 \leq |\frac{1}{N_k} \gamma_{j,k} x_j - \frac{1}{N_k} \tilde{\gamma}_{j,k} \tilde{x}_j|_1 \leq \frac{1}{N_k} \gamma_{i,k} |x_j|_1 + \frac{1}{N_k} \tilde{\gamma}_{j,k} |\tilde{x}_j|_1,
\]

where the last line is due to \( \sum_{i=1}^{d} |x_{i,l}| := |\langle x_i, 1 \rangle| \leq \left( \sum_{i=1}^{d} |x_{i,l}|^2 \right)^{1/2} \left( \sum_{i=1}^{d} 1 \right)^{1/2} \leq \sqrt{d} \). Hence, the sensitivity of MLE is given by

\[
\Delta \mu_{mle}^k = \frac{2 \sqrt{d}}{N_k},
\]

(21)

A.2 L-2 sensitivity of MLE

The only difference from L-1 sensitivity is rather than considering \( |x_i|_1 \), we consider \( |x_i|_2 \). Due to the assumption that \( |x_i|_2 \leq 1 \), the L-2 sensitivity of MLE is given by

\[
\Delta_{2\mu}^k_{mle} = \frac{2}{N_k},
\]

(22)

A.3 L-1 sensitivity of MAP estimate

The MAP estimate only differs the MLE in the denominator: \( \mu_{map}^k = \frac{1}{N_k + \kappa_0} \sum_{i} \gamma_{i,k} x_i \). So we replace \( \tilde{N}_k \) with \( \tilde{N}_k + \kappa_0 \) in section A.1 and obtain the following sensitivity

\[
\Delta \mu_{map}^k = \frac{2 \sqrt{d}}{N_k + \kappa_0},
\]

(23)

which simplifies to eq (21), when \( \kappa_0 = 0 \).

A.4 L-2 sensitivity of MAP estimate

Similar to the L-2 sensitivity for MLE, the L-2 sensitivity of MAP estimate is given by

\[
\Delta_{2\mu}^k_{map} = \frac{2}{N_k + \kappa_0},
\]

(24)

which simplifies to eq (22), when \( \kappa_0 = 0 \).
B Sensitivity of covariance parameters

B.1 Sensitivity of MLE

As explained in the main text, we add symmetric noise drawn from a Wishart distribution to the covariance matrix. We need to quantify how much noise to add to obtain $\epsilon_3$-differentially private covariances. Consider two second moment matrices $\Sigma^\text{mle}_k$ given a dataset $\mathcal{D}$ and $\Sigma^\text{mle}_k$ given a neighbouring dataset $\mathcal{D}$, and an output $\Sigma^\text{mle}_k$ from the SN algorithm. The density of $\Sigma^\text{mle}_k$ is $W(\Sigma^\text{mle}_k|\lambda I_d, d+1)$ under the input $\Sigma^\text{mle}_k$ and $W(\Sigma^\text{mle}_k - \Sigma^\text{mle}_k|\lambda I_d, d+1)$ under the input $\Sigma^\text{mle}_k$. Hence, the probability ratio is given by

$$
\frac{W(\Sigma^\text{mle}_k - \Sigma^\text{mle}_k|\lambda I_d)}{W(\Sigma^\text{mle}_k - \Sigma^\text{mle}_k|\lambda I_d)} = \exp\left(-\frac{1}{2\lambda} \text{tr}(\Sigma^\text{mle}_k - \Sigma^\text{mle}_k)\right) \exp\left(-\frac{1}{2\lambda} \text{tr}(\Sigma^\text{mle}_k - \Sigma^\text{mle}_k)\right) = \exp\left(\frac{1}{2\lambda} \text{tr}(\Sigma^\text{mle}_k - \Sigma^\text{mle}_k)\right),
$$

where we plug in the noised-up mean parameters to $\mu_k$ and $\tilde{N}_k$, which makes the difference in trace is simply

$$
\frac{1}{2\lambda} \text{tr}(\Sigma^\text{mle}_k - \Sigma^\text{mle}_k) = \frac{1}{2\lambda} \left( \frac{1}{N_k} \sum_{i=1}^{N} \gamma_{i,k} x_i^\top x_i - \frac{1}{N_k} \sum_{i=1}^{N} \tilde{\gamma}_{i,k} \tilde{x}_i^\top \tilde{x}_i \right).
$$

One-datapoint difference affects only two Gaussians’ covariances. Hence, the difference in trace is given by

$$
\text{tr}(\Sigma^\text{mle}_k - \Sigma^\text{mle}_k) = \max_{x_j,\tilde{x}_j} \frac{1}{N_k} \left( B_k + \gamma_{j,k} x_j^\top x_j - \frac{1}{N_k} \tilde{B}_k + \tilde{\gamma}_{j,k} \tilde{x}_j^\top \tilde{x}_j \right),
$$

where $B_k := \sum_{i=1, i\neq j}^{N} \gamma_i x_i^\top x_i$. We look at the two terms in eq. 25

$$
\frac{1}{N_k} \left( B_k + \gamma_{j,k} x_j^\top x_j \right) - \frac{1}{N_k} \left( B_k + \tilde{\gamma}_{j,k} \tilde{x}_j^\top \tilde{x}_j \right) = \frac{1}{N_k} \gamma_{j,k} x_j^\top x_j - \frac{1}{N_k} \tilde{\gamma}_{j,k} \tilde{x}_j^\top \tilde{x}_j, \leq \frac{1}{N_k} \gamma_{j,k},
$$

since $0 \leq x_j^\top x_j \leq 1$. Therefore, the probability ratio is bounded by

$$
\frac{1}{2\lambda} \text{tr}(\Sigma^\text{mle}_k - \Sigma^\text{mle}_k) \leq \frac{1}{2\lambda N_k}
$$

which sets $\lambda = \frac{1}{2e'N_k}$.

B.2 Sensitivity of MAP estimate

We again use the symmetric noise (SN) algorithm \[\text{SN}\] to perturb the MAP estimate of covariance matrix, where the difference in trace $\text{tr}(\Sigma^\text{map}_k - \Sigma^\text{map}_k)$ is given by (with plugging in the noised-up version of $\tilde{N}_k$ to $N_k$ in the definition of covariance)

$$
\frac{\text{sum}(\text{diag}(S_0))}{\tilde{N}_k + c} + \frac{1}{N_k + c} \sum_{i=1}^{N} \gamma_i x_i^\top x_i - \mu_k^\text{mle}^\top \mu_k^\text{mle} + \frac{\kappa_0 \tilde{N}_k}{(N_k + c)(\tilde{N}_k + \kappa_0)} \mu_k^\text{mle}^\top \mu_k^\text{mle},
$$

$$
- \frac{\text{sum}(\text{diag}(S_0))}{\tilde{N}_k + c} - \frac{1}{N_k + c} \sum_{i=1}^{N} \tilde{\gamma}_i \tilde{x}_i^\top \tilde{x}_i - \tilde{\mu}_k^\text{mle}^\top \tilde{\mu}_k^\text{mle} + \frac{\kappa_0 \tilde{N}_k}{(N_k + c)(\tilde{N}_k + \kappa_0)} \tilde{\mu}_k^\text{mle}^\top \tilde{\mu}_k^\text{mle},
$$

$$
= \frac{1}{N_k + c} \sum_{i=1}^{N} \gamma_i x_i^\top x_i - \frac{\tilde{N}_k}{N_k + c} \mu_k^\text{mle}^\top \mu_k^\text{mle} + \frac{\kappa_0 \tilde{N}_k}{(N_k + c)(\tilde{N}_k + \kappa_0)} \mu_k^\text{mle}^\top \mu_k^\text{mle},
$$

$$
- \frac{1}{N_k + c} \sum_{i=1}^{N} \tilde{\gamma}_i \tilde{x}_i^\top \tilde{x}_i + \frac{\tilde{N}_k}{N_k + c} \tilde{\mu}_k^\text{mle}^\top \tilde{\mu}_k^\text{mle} - \frac{\kappa_0 \tilde{N}_k}{(N_k + c)(\tilde{N}_k + \kappa_0)} \tilde{\mu}_k^\text{mle}^\top \tilde{\mu}_k^\text{mle},
$$

$$
\leq \frac{1}{N_k + c} \gamma_{j,k},
$$

where $c := \nu_0 + d + 2$. We have the following upper bound:

$$
\text{tr}(\Sigma^\text{map}_k - \Sigma^\text{map}_k) \leq \frac{1}{N_k + c}
$$
With the maximum of $\text{Tr}(\Sigma_k^{map} - \Sigma_k^{map'})$ is given in eq (29), to ensure $\epsilon'$ DP, we set $\lambda$ to

$$\lambda = \frac{1}{2\epsilon'(N_k + c)}.$$  \hfill (30)

### C Advanced composition for $(\epsilon, \delta)$-DP relaxation

For $(\epsilon, \delta)$-DP, we assign the privacy budget using the advanced composition theorem (3.20 in [1]). Recall that we perform $2K + 1$ parameter perturbations in each EM iteration. After $J$ iterations, the privacy budget assigned to each parameter perturbation $\epsilon'$ is given by (by directly applying the Corollary 3.21 in [1])

$$\epsilon' = \frac{\epsilon}{2\sqrt{2(2K+1)J\log(1/\delta_1)}}.$$  \hfill (31)

If we perturb parameters with $\epsilon'$-DP mechanisms, the resulting failure probability $\delta$ equals $\delta_1$. However, if we use the $\epsilon', \delta'$-DP mechanism like the Gaussian mechanism for mean estimates, the failure probability $\delta = (J\delta' + \delta_1)$.  

References

[1] Cynthia Dwork and Aaron Roth. The algorithmic foundations of differential privacy. *Found. Trends Theor. Comput. Sci.*, 9:211–407, August 2014.

[2] D. B. Rubin, A. P. Dempster, N. M. Laird. Maximum likelihood from incomplete data via the EM algorithm. *Journal of the Royal Statistical Society. Series B (Methodological)*, 39(1):1–38, 1977.

[3] Timothy L. Bailey, Charles Elkan, et al. Fitting a mixture model by expectation maximization to discover motifs in bipolymers. 1994.

[4] Yongyue Zhang, Michael Brady, and Stephen Smith. Segmentation of brain MR images through a hidden Markov random field model and the expectation-maximization algorithm. *Medical Imaging, IEEE Transactions on*, 20(1):45–57, 2001.

[5] Chad Carson, Serge Belongie, Hayit Greenspan, and Jitendra Malik. Blobworld: Image segmentation using expectation-maximization and its application to image querying. *Pattern Analysis and Machine Intelligence, IEEE Transactions on*, 24(8):1026–1038, 2002.

[6] Avrim Blum, Cynthia Dwork, Frank McSherry, and Kobbi Nissim. Practical privacy: The SuLQ framework. In *Proceedings of the Twenty-fourth ACM SIGMOD-SIGACT-SIGART Symposium on Principles of Database Systems*, PODS ’05, pages 128–138, New York, NY, USA, 2005. ACM.

[7] Dong Su, Jianneng Cao, Ninghui Li, Elisa Bertino, and Hongxia Jin. Differentially private k-means clustering. In *Proceedings of the Sixth ACM Conference on Data and Application Security and Privacy*, CODASPY ’16, pages 26–37, New York, NY, USA, 2016. ACM.

[8] Adam D. Smith. Efficient, differentially private point estimators. *CoRR*, abs/0809.4794, 2008.

[9] C. Dwork and G. N. Rothblum. Concentrated Differential Privacy. *ArXiv e-prints*, March 2016.

[10] Radford M Neal and Geoffrey E Hinton. A view of the EM algorithm that justifies incremental, sparse, and other variants. In *Learning in graphical models*, pages 355–368. Kluwer Academic Publishers, 1998.

[11] R. P. Feynman. *Statistical Mechanics: A Set of Lectures*. Perseus, 1972.

[12] C. M. Bishop et al. *Pattern recognition and machine learning*. Springer New York; 2006.

[13] Rutgers Hafiz Imtiaz, Anand D. Sarwate. Symmetric matrix perturbation for differentially-private principal component analysis. In *ICCASP*, 2016.

[14] Cynthia Dwork, Kunal Talwar, Abhradeep Thakurta, and Li Zhang. Analyze gauss: optimal bounds for privacy-preserving principal component analysis. In *Symposium on Theory of Computing, STOC 2014, New York, NY, USA, May 31 - June 03, 2014*, pages 11–20, 2014.

[15] Moritz Hardt and Eric Price. The noisy power method: A meta algorithm with applications. In Z. Ghahramani, M. Welling, C. Cortes, N. D. Lawrence, and K. Q. Weinberger, editors, *Advances in Neural Information Processing Systems 27*, pages 2861–2869. Curran Associates, Inc., 2014.

[16] Kamalika Chaudhuri, Anand Sarwate, and Kaushik Sinha. Near-optimal differentially private principal components. In F. Pereira, C. J. C. Burges, L. Bottou, and K. Q. Weinberger, editors, *Advances in Neural Information Processing Systems 25*, pages 989–997. Curran Associates, Inc., 2012.

[17] M. Lichman. UCI machine learning repository, 2013.

[18] Prashanth Mohan, Abhradeep Thakurta, Elaine Shi, Dawn Song, and David E. Culler. Gupt: privacy preserving data analysis made easy. In K. Selçuk Candan, Yi Chen, Richard T. Snodgrass, Luis Gravano, and Ariel Fuxman, editors, *SIGMOD Conference*, pages 349–360. ACM, 2012.