Noether symmetries and conserved quantities of constrained Hamilton systems with quasi coordinates

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Abstract

The constraint mechanical systems with quasi coordinates are more universal than generalized coordinates, in this paper, we study the Noether symmetries and conserved quantities of nonconservative singular systems in phase space. Firstly, the internal constraints induced by singularity are equivalent considered as extrinsic nonholonomic constraints, the canonical equations of constrained Hamilton systems with quasi coordinates are obtained by using transform to the Euler-Lagrange equations. Secondly, the infinitesimal transformations of time, quasi coordinates and generalized momentum are introduced, the definition, criterion and Noether theorem are obtained according to the regular action quantity keep generalized quasi invariance under the transformation, meanwhile, the inverse problem of the Noether symmetry is also studied. Finally, an example is given to illustrate the application of the content. The results found that the rational use of quasi coordinates will make the constraints caused by the singularity of the system do not affect the standard form of the regular equations and avoid the emergence of constrained multipliers, the conservation is more concise.

Key words: Quasi coordinates, constrained Hamilton systems, Noether symmetries, conserved quantities, inverse problem.

1. Introduction

Study on about the symmetries and conserved quantities of dynamical systems are a main branch of intersection in modern mathematics and physics mechanics. The systems described by singular Lagrange function are called singular systems and expressed in Hamilton form in phase space, it is also called constrained Hamilton systems. Many important dynamic problems of mathematical physics and engineering technology are in accord with the model of constraint Hamilton systems, such as electromagnetic field, light traversing phenomena, quantum electrodynamics, superstring theory etc. In recent years, the research on constrained
Hamilton systems’s symmetries and the conserved quantities are paid more and more attention, and have made some progress. Dirac$^2$ and Li$^{15}$ firstly studied the Hamilton canonical equations for singular systems with the Noether symmetry and conserved quantity and many physical applications. Mei$^6$ and Zhang$^7$ respectively study the Lie symmetries and conserved quantities of singular systems in the form space and phase space. Luo$^8$ study Mei symmetries and conserved quantities of Hamilton canonical equations of Singular systems, the relation between Mei symmetry and Noether symmetry and Lie symmetry are also illustrated. Li$^9$ also studied the theory of regular symmetry of constrained Hamilton systems under external constraints and compared it with the nonsingular systems. The present study to the singular systems’ theory is all based on the generalized coordinates, however in analytical mechanics, the use of quasi coordinates is more universal, if the quasi coordinates is rational, some problems will become great convenient. In document$^{10}$, the Noether symmetries and Noether conserved quantities of holonomic systems with quasi coordinates are discussed. In document$^{11}$, the Lie symmetries and Hojman type conserved quantities of holonomic systems with quasi coordinates are given. In document$^{12}$, they study the form invariance of holonomic systems with quasi coordinates, that is Mei symmetries and conserved quantities. In document$^{13}$, the Lie symmetries and conserved quantities of nonholonomic nonsingular mechanical systems with quasi coordinates are studied. The research literature about the quasi coordinates have$^{14-15}$.

Through literature discovery, the current research of constraint mechanics systems with quasi coordinates are all in the form space, that is, the equations of motion of the systems are in the form of Lagrange instead of Hamilton. As everyone knows, Hamilton systems have a set of perfect mechanical characteristics (such as integral theory and canonical transformation theory), therefore, the symmetries and conserved quantities of the singular systems with quasi coordinates in phase space are of great significance for the constrained dynamics and engineering science. According to this idea “the equations of motion expressed by quasi coordinates and quasi speed have completely single structure, and it do not depend on constraints or not”. By introducing infinite transformations of time, quasi coordinates and generalized momenta, the Noehter symmetry theory (including positive and inverse problems) with quasi coordinates of singular systems in phase space are firstly obtained.

2. Canonical equations of constrained Hamilton systems with quasi coordinates:

The form of mechanical systems is determined by $n$ generalized coordinates $q_s (s = 1,\ldots,n)$ and the Lagrange function of the systems is $L(t, \dot{q}, \ddot{q}) = T - V$, the nonpotential generalized force is $Q_s(t, q, \dot{q})$, according to Euler-Lagrange principle, the differential equations of motion for nonconservative systems is:

$$ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s, (s = 1,2,\ldots,n) \quad (1) $$

For constrained Hamilton systems, since the rank of L's Hess matrix $\left[ \frac{\partial^2 L}{\partial \dot{q}_s \partial \dot{q}_k} \right]$ is $n - r < n$, after the (1) expansion, it is impossible to completely solve all generalized accelerations $\ddot{q}_s$, and only a part generalized acceleration can be solved, so exists some relations that the number is $r$:

$$ f_{\beta} (t, q, \dot{q}) = 0, (\beta = 1,\ldots,r) \quad (2) $$

Take quasi velocities:

$$ \omega_k = \omega_k (q_s, \dot{q}_s, t) = \alpha_k \dot{q}_s, (s, k = 1,2,\ldots,n) \quad (3) $$
It can be solved by formula (3):
\[ \dot{q}_s = \dot{q}_s(q_k, \omega_k, t) = b_{sk}\omega_k \]  
(4)

The \( b_{sk} \) is inverse matrix elements of the matrix \( [\alpha_{ks}] \), now introducing quasi coordinates \( \pi_s \):
\[ \omega_s = \pi_s \]  
(5)

The internal constraints (2) can be shown as the form with quasi coordinates:
\[ \omega_{\varepsilon+\beta} = 0, \delta\pi_{\varepsilon+\beta} = 0, (\varepsilon = n - r) \]  
(6)

The relation between quasi coordinates and generalized coordinates are \(^{16}\):
\[ \delta q_s = \frac{\partial q_s}{\partial \omega_k} \delta \pi_k, \delta \pi_k = \frac{\partial \omega_k}{\partial q_s} \delta q_s \]  
(7)

The system’s Lagrange function \( \dot{L}^* \) and nonconservative force \( \dot{Q}^*_s \) are represented by quasi velocities:
\[ \dot{L}^* = L^*(t, q_s, \omega_s) = L(t, q_s, b_{sk}\omega_k) \]
\[ \dot{Q}^*_s = \dot{Q}^*(t, q_s, \omega_s) = Q_s b_{sk}, (s, k = 1, 2, \ldots, n) \]  
(8)

Introducing the generalized momentums:
\[ p_s = \frac{\partial \dot{L}^*}{\partial \omega_k} \Rightarrow \omega_s = d_{sk}p_k + e_s \]  
(9)

According to the inverse theorem of Legendre transform, The Hamilton function \( \dot{H}(t, q, p) \) with the quasi coordinates is:
\[ \dot{H}(t, q, p) = \left( p_s \omega_s - \dot{L}^* \right) \bigg|_{p_s \rightarrow \omega_s} \]  
(10)

The generalized Boltzmann-Hamel equations for constrained Hamilton systems with quasi coordinates are \(^{17}\):
\[ \frac{d}{dt} \frac{\partial \dot{L}^*}{\partial \omega_\sigma} - \frac{\partial \dot{L}^*}{\partial \pi_\sigma} + \sum_{r=1}^{n} \sum_{\varepsilon=1}^{n} \frac{\partial L^*}{\partial \omega_r} \gamma'_{\varepsilon\sigma} \omega_r = Q^*_\sigma (\sigma = 1, 2, \ldots, \varepsilon) \]  
(11)

\( \gamma'_{\varepsilon\sigma} \) is three mark for Boltzmann:
\[ \gamma'_{\varepsilon\sigma} = \sum_{m=1}^{n} \sum_{r=1}^{n} \left( \frac{\partial \alpha_{rt}}{\partial q_m} - \frac{\partial \alpha_{mr}}{\partial q_t} \right) b_{mv} b_{\sigma} \]  
(12)

Form (10) has relations to \( \pi_t, p_t \):
\[ \dot{\pi}_t = \frac{\partial \dot{H}}{\partial p_t} \]
\[ \frac{\partial \dot{H}}{\partial \pi_t} = - \frac{\partial \dot{L}^*}{\partial \pi_t} \]  
(13)
By (11) and (13), we obtain the canonical equations of Hamilton systems in phase space with quasi coordinates are:

\[ \begin{align*}
\dot{\pi}_i &= -\frac{\partial H}{\partial p_i} \\
\dot{p}_i &= -\frac{\partial H}{\partial \pi_i} - \sum_{r=1}^{n-r} \sum_{\nu=1}^{n} \gamma_{i\nu}^{r} \pi_{\nu} + Q_{i}^{r} 
\end{align*} \]

(14)

The form is completely similar to the holonomic nonconservative nonsingular systems, but there are more internal constraint equations (6). It shows that the singularity of the systems does not affect the single structure of the canonical equation with quasi coordinates, but it reduces the number of differential equations.

3. Noether symmetries and conserved quantities of the systems:

The regular action of the constrained Hamilton systems with quasi coordinates in phase space is:

\[ S = \int_{t_1}^{t_2} [p_i \dot{q}_i - H(t, p, q)] dt \]

(15)

The infinitesimal transformation of a continuous Lie group involving a single parameter is:

\[ \begin{align*}
t^{\ast} &= t + \Delta t = t + \varepsilon \xi_0^\ast(t, q, p) \\
\pi_{i}^{\ast}(t^{\ast}) &= \pi_i(t) + \Delta \pi_i = \pi_i(t) + \varepsilon \xi_i(t, q, p) \\
p_{i}^{\ast}(t^{\ast}) &= p_i(t) + \Delta p_i = p_i(t) + \varepsilon \eta_i(t, q, p)
\end{align*} \]

(16)

Under the infinitesimal transformations (16), the total variation of the (15) is:

\[ \Delta S = \delta S + \dot{S} \Delta t = \int_{t_1}^{t_2} \left( p_i \delta \omega_i + \omega_i \delta p_i - \frac{\partial H}{\partial \pi_i} \delta \pi_i - \frac{\partial H}{\partial p_i} \delta p_i \right) dt + (p_i \omega_i - H) \Delta t \]

\[ = \int_{t_1}^{t_2} \left\{ \frac{d}{dt} [p_i \delta \omega_i + (p_i \omega_i - H) \Delta t] + (-\dot{p}_i - \frac{\partial H}{\partial \pi_i} \delta \pi_i + (\omega_i - \frac{\partial H}{\partial p_i} \delta p_i) \right\} dt \]

(17)

In formula (17), \( \frac{\partial H}{\partial \pi_i} \delta \pi_i = \frac{\partial H}{\partial q_i} \delta q_i \), and it also because:

\[ \delta \pi_i = \Delta \pi_i - \omega_i \Delta t = \varepsilon (\xi_i - \omega_i \xi_0) = \varepsilon \xi_i \]

\[ \delta p_i = \Delta p_i - \dot{p}_i \Delta t = \varepsilon (\eta_i - \dot{p}_i \xi_0) = \varepsilon \eta_i \]

(18)

So formula (17) may be changed as:

\[ \Delta S = \int_{t_1}^{t_2} \left\{ \frac{d}{dt} [p_i \xi_i + (p_i \omega_i - H) \xi_0] + (-\dot{p}_i - \frac{\partial H}{\partial \pi_i} \xi_i + (\omega_i - \frac{\partial H}{\partial p_i} \eta_i) \right\} dt \]
4. The inverse problem of Noether symmetry:

The inverse problem of Noether symmetry for constrained Hamilton systems is known the first integral
(conserved quantity of motion) of holonomic systems in phase space, the generating element of infinitesimal transformations and gauge function are obtained by using the invariance condition of the regular action quantity and the Noether theorem. The conserved quantities of singular systems generally do not necessarily have corresponding Noether symmetries, therefore, the inverse problem of Noether symmetry for constrained Hamilton systems may be solvable, and may not be solvable.

Suppose the systems have first integral:

\[ I = I(t, q, p) = \text{const} \]  

(24)

The second equation in the canonical equations (14) multiply \( \xi_\sigma = \xi_\sigma - \omega_\sigma \xi_0 \), it can obtained:

\[ (-\dot{p}_\sigma - \frac{\partial H}{\partial \pi_\sigma} + \gamma_\nu_\sigma p_\nu \frac{\partial H}{\partial p_\nu} + Q_\sigma')(\xi_\sigma - \frac{\partial H}{\partial p_\sigma} \xi_0) = 0 \]  

(25)

Because of the rule of total derivatives:

\[ \frac{dI}{dt} = \frac{\partial I}{\partial t} + \frac{\partial I}{\partial \pi_s} \omega_s + \frac{\partial I}{\partial p_\sigma} \dot{p}_\sigma = 0 \]  

(26)

Make formula (25) and formula (26) add up, and make the coefficient of the item \( \dot{p}_\sigma \) is equal to zero:

\[ \xi_\sigma = \frac{\partial H}{\partial p_\sigma} \xi_0 + \frac{\partial I}{\partial p_\sigma} \]  

(27)

By formula (23) and (27), the all generators of infinitesimal transform \( \xi_0, \xi_s \) can be obtained, the generator \( \eta_s \) can be given by the following formula:

\[ \eta_s = \frac{\partial p_s}{\partial t} \xi_0 + \frac{\partial p_s}{\partial \pi_k} \xi_k + \frac{\partial p_s}{\partial \omega_s} (\xi_s - \omega_s \xi_0) \]  

(28)

The form(28) is easy to prove, because:

\[ \Delta p_s = \varepsilon \eta_s = \frac{\partial p_s}{\partial t} \Delta t + \frac{\partial p_s}{\partial \pi_k} \Delta \pi_k + \frac{\partial p_s}{\partial \omega_s} \Delta \omega_s \]  

(29)

Combined with formula (16), it can be proved.

5. Example

The Lagrange function and the nonpotential generalized force of the holonomic mechanical systems with two degrees of freedom are established:

\[ L = \frac{1}{2} q_1^2 + q_1 \dot{q}_2 \]

\[ Q_1 = -\frac{q_1}{q_1}, Q_2 = q_1 \dot{q}_2 \]  

(30)

Try to investigate the Noether symmetry and inverse problem of the systems with quasi coordinates.

It is obvious that the L's Hess matrix rank is \( 1 < 2 \), so it is constrained Hamilton systems, then the
Euler-Lagrange equation and the internal constraint equation of the systems are:

\[
\begin{align*}
\dot{q}_1 - \dot{q}_2 + \frac{q_1}{q_2} &= 0 \\
\dot{q}_1 &= q_1 \dot{q}_2 \\
\end{align*}
\] (31)

Take the quasi velocities are \(\omega_1 = \dot{q}_1, \omega_2 = \dot{q}_1 - q_1 \dot{q}_2\), it can be solved by reverse: \(\dot{q}_1 = \omega_1, \dot{q}_2 = (\omega_1 - \omega_2) \frac{1}{q_1}\), then the internal constraint becomes:

\[
\omega_2 = 0
\] (32)

After calculating the systems’s three mark of Boltzmann, it can obtain \(\gamma_1^1 = \gamma_1^2 = 0\), then the canonical equations of the systems are:

\[
\begin{cases}
\dot{\pi}_1 &= p_1 - 1 \\
\dot{\pi}_2 &= 0 \\
\dot{p}_1 &= \frac{\pi_1}{\pi_1} - \frac{\pi_1}{\pi_1} \\
\dot{p}_2 &= 0
\end{cases}
\] (33)

According to the generalized quasi Noether equation (22), the system’s infinitesimal generator \(\xi_0, \xi_1\) satisfies the following equations:

\[
\begin{align*}
p_1 \dot{\xi}_1 &= \frac{(\omega_2 - \omega_1)}{q_1} \xi_1 - H \dot{\xi}_0 - \frac{q_1}{\omega_1} (\xi_1 - \omega_1 \xi_0) + \dot{G} = 0 \\
\xi_2 - \omega_2 \xi_0 &= 0 \\
\omega_2 &= 0
\end{align*}
\] (34)

The corresponding generating elements of infinitesimal transformations and gauge function are preferable to:

\[
\begin{align*}
\xi_0 &= 0, \xi_1 = 1, \xi_2 = 0, G = -p_1 \\
\xi_0 &= 1, \xi_1 = \xi_2 = 0, G = -\int q_1
\end{align*}
\] (35)

It can obtained by formula (28):

\[
\eta_1 = \eta_2 = 0
\] (36)

Take formula (35) and (36) into the system’s Noether conserved quantities formula (23), it can obtained:

\[
\begin{align*}
I_1 &= \text{const} \\
I_2 &= -p_1 \omega_1 - \frac{1}{2} \omega_1^2 - \omega_1 - \int q_1
\end{align*}
\] (37)
It can be seen that the first set of Noether symmetry generators leads to an ordinary conserved quantity, and the second set of generators leads to the conservation of the energy of nonconservative systems.

Secondly, we study the inverse problem of Noether symmetry. Suppose the systems have first integral:

\[ I = -p_i \omega_i - \frac{1}{2} \omega_i^2 - \omega_i - \int q_i \]  

(38)

Given by equations (27), (6) and (28):

\[ \xi_0 = \omega_1 \xi_0 - \omega_1 \]

\[ \xi_2 - \omega_2 \xi_0 = 0 \]

\[ \eta_1 = (\dot{\xi}_1 - \omega_1 \xi_0) \]

\[ \eta_2 = 0 \]  

(39)

To (39), a set of solutions is preferred:

\[ \xi_0 = 1, \xi_1 = \xi_2 = 0, \eta_1 = \eta_2 = 0 \]  

(40)

Then, according to the conserved quantities of the Noether symmetry (23), the gauge function is:

\[ G = -\int q_i \]  

(41)

An example of Noether symmetry for constrained Hamilton systems with quasi coordinates in phase space is given.

6. Conclusion

In summary, the Noether symmetry theory of nonconservative constrained Hamilton systems in phase space with quasi coordinates is studied in this paper. By constructing quasi velocities, the canonical equations of dimension reduction are obtained by transforming the Euler-Lagrange equations, the existence of bound multipliers are avoided, the equations preserves the single structure form of the canonical equations in the complete with nonsingular mechanical systems. Based on the generalized quasi invariance of regular function under infinitesimal transformations of regular coordinates, the generating elements of constrained Hamilton systems satisfying the generalized quasi Noether equation are obtained by variational computation. The corresponding conserved quantity of Noether type is also found, at the same time, the inverse problem of Noether symmetry of constrained Hamilton systems with quasi coordinates in phase space is studied. With quasi coordinates, the form of the constrained Hamilton systems is more concise and convenient and the conservation quantity has multiple expression forms. However, this article does not consider the existence of external constraints, when external nonholonomic constraints and singularities exist at the same time, how to construct quasi coordinates is a big problem, this is a question that can be further studied.

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