Bayesian Analysis of Hybrid EoS based on Astrophysical Observational Data

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Abstract

We perform a Bayesian analysis of probability measures for compact star equations of state using new, disjunct constraints for mass and radius. The analysis uses a simple parametrization for hybrid equations of state to investigate the possibility of a first order deconfinement transition in compact stars. The latter question is relevant for the possible existence of a critical endpoint in the QCD phase diagram under scrutiny in heavy-ion collisions.

Introduction

The most basic features of neutron stars (NS) are their radii and masses which so far have not been well determined simultaneously for a single object. In some cases masses are precisely measured like in the case of binary systems but radii are quite uncertain. In the other hand, for isolated NS some radius and mass measurements exist but lack the necessary precision to inquire into their interiors. In fact, it has been conjectured that there exists a unique relation for all NS between the sequence of mass and radius relations on the one hand and their equation of state (EoS) on the other that determines their internal composition [1]. For this reason, accurate observations of masses and radii are crucial to study cold dense nuclear matter as it exists in NS.

However, the present observable data allow to make only probabilistic estimations of the internal structure of the star which can be performed using Bayesian Analysis (BA) and modeling of relativistic configurations of NS. In comparison to previous work in this direction [2] we choose disjunct mass-radius constraints in this work which reveals that probabilistic estimations of the superdense stellar matter EoS have a much more preliminary character at present than obtained previously. Moreover, we focus in this analysis on investigating the possibility of a quark matter inner core in massive NS with $M \sim 2 M_\odot$ (see [3, 4]), separated from the outer core of hadronic matter by a strong first order phase transition.

$\begin{align*}
d m(r) \over dr &= C_1 \varepsilon(r) r^2, \\
d p(r) \over dr &= -C_2 \left(\varepsilon(r) + p(r) + C_1 p(r) r^3\right) \over r(r - 2C_2 m(r))
\end{align*}

as well as the equation for the baryon mass profile

$$m_B(r) = m_N n(r)$$

and with $m_N$ being the atomic mass unit and the constants are defined as

$$C_1 = \frac{4\pi}{c^2} = 1.11269 \times 10^{-5} \frac{M_\odot}{\text{km}^3 \text{MeV}}, \quad C_2 = \frac{GM}{c^2} = 1.4760 \frac{\text{km}}{M_\odot}.$$  

These equations are integrated from the center of the star towards its surface, with the radius of the star $R$ defined by the condition $p(R) = 0$ while the gravitational mass is $M = m(R)$. In a similar manner, the baryon mass is given by $m_B = m_B(R)$.

In order to solve the TOV equations the EoS is required. It is given by the relation $p = p(\varepsilon)$ which carries information about the microscopic state of dense nuclear matter. Thus, the above equations have to be solved simultaneously using as a boundary condition the energy density at the star center $\varepsilon(r = 0)$.

In this way, for a given value of $\varepsilon(0)$ the solution of the TOV equations are the $p(r)$ and $m(r)$ profiles and with them the relation $M(R)$ in the parametric form $M(\varepsilon(0))$ and $R(\varepsilon(0))$. 

\[ \text{Abstract} \]

\[ \text{Introduction} \]

\[ \text{NS structure} \]
The hybrid EoS

For this study we follow the AHP scheme for defining the hybrid EoS in the form

\[ p(\varepsilon) = p_h(\varepsilon)\Theta(\varepsilon - \varepsilon_Q) + p_q(\varepsilon)\Theta(\varepsilon - \varepsilon_Q) + p_h(\varepsilon)\Theta(\varepsilon - \varepsilon_H)\Theta(\varepsilon_Q - \varepsilon) \]

(6)

where \( p_h \) is the pressure of a pure hadronic EoS and \( p_q \) represents the high density EoS assumed here as quark matter with \( c^2_q \), its squared speed of sound, as parametrized by Haensel et al. [9] which describes pretty well the superconducting NJL model derived in [10, 11, 12]. For the hadronic EoS we take the well known model of APR [13] that is in agreement with experimental data of densities about nuclear saturation. For this hadronic branch all the relevant thermodynamical variables: energy density \( \varepsilon \), pressure \( p \), baryon density \( n \) and chemical potential \( \mu \) are well defined and taken as input for determination of the hybrid (hadronic + quark matter) EoS.

As a starting point in the derivation of the high density EoS we introduce the pressure as function of energy density in the quark matter phase

\[ p_q(\varepsilon) = c_q^2 \varepsilon - B, \]

(7)

with \( B \) playing the role of a bag constant. To determine the remaining thermodynamical quantities \( n_q \) and \( p_q \) of the quark matter phase we use the following relations

\[ n_q(\varepsilon) = n_Q \exp \left( \int_{\varepsilon_Q}^{\varepsilon} \frac{d\varepsilon'}{\varepsilon' + p(\varepsilon')} \right) \]

(8)

\[ \mu = \frac{\varepsilon + p}{n} \]

(9)

where \( \varepsilon_Q \) is the quark matter energy density right after the phase transition following the jump \( \Delta \varepsilon \) as density increases. Therefore one arrives at the following formula

\[ n_q(\varepsilon) = n_Q \frac{p_c + \varepsilon_Q}{p_c + \varepsilon_H} \left( \frac{1 + c_q^2 \varepsilon - B}{1 + c_q^2 \varepsilon_Q - B} \right)^{\frac{1}{2 + \frac{n}{n_H}}}, \]

(10)

obtained by enforcing conditions of equal pressure and chemical potential at the transition (Gibbs conditions):

\[ \mu_c = \frac{p_c + \varepsilon_Q}{n_Q} = \frac{p_c + \varepsilon_H}{n_H}, \]

(11)

where upper case subscripts stand for the values of the thermodynamic functions at the phase transition: \( \varepsilon_Q = \varepsilon_q(\mu_c) \), \( \varepsilon_H = \varepsilon_h(\mu_c) \), \( n_Q = n_q(\mu_c) \), \( n_H = n_h(\mu_c) \), etc.

The free parameters of the model are the transition density \( \varepsilon_H \), the energy density jump \( \Delta \varepsilon \equiv \gamma \varepsilon_H \) and \( c_q^2 \), the quark matter speed of sound squared.

The resulting EoS in the plane pressure versus density is depicted in Fig. 1 for a given set of input parameters.

Figure 1: Hybrid EoS scheme for two different sets of the three parameters \( (\varepsilon_h, c_q^2) \).

Bayesian Analysis Formulation

We define the vector of free parameters \( \pi = (\varepsilon_H, c_q^2, \gamma) \), which define the EoS with phase transition from nuclear to quark matter. The nuclear equation of state can be taken to be APR [13].

These parameters were sampled as

\[ \pi_i = \pi \left( \varepsilon_H(k), \gamma(l), c_q^2(m) \right), \]

(12)

where \( i = 0 \ldots N - 1 \) with \( N = N_1 \times N_2 \times N_3 \) such that \( i = N_1 \times N_2 \times k + N_2 \times l + m \) and \( k = 0 \ldots N_1 - 1, l = 0 \ldots N_2 - 1, m = 0 \ldots N_3 - 1 \), here \( N_1, N_2 \) and \( N_3 \) number of parameters \( \varepsilon_k, \gamma_l \) and \( c_q^2(m) \), respectively.

Using the EoS one can calculate the neutron star structure by solving the TOV equations. Then it is possible to use different neutron star observations to check the probability for this EoS. We use here three observations as constraints: a mass constraint [1], a radius constraint [14] and a constraint for the gravitational binding energy (relation between gravitational mass and baryon mass) [15, 16].

The goal is to find the set of the most probable \( \pi_i \), based on the given constraints using Bayesian Analysis (BA). For initializing the BA we propose that \( a priori \) each vector of parameters \( \pi_i \) has a probability equal to one, \( P(\pi_i) = 1 \), for all \( i \).
Mass Constraint

We assume that error of the mass measurement for the high-mass pulsar PSR J0348+0432 [1] is normal distributed with \( N(\mu_A, \sigma_A^2) \), where the mean value is \( \mu_A = 2.01 \, M_\odot \) and the variance is \( \sigma_A = 0.04 \, M_\odot \). Using this assumption we can calculate the conditional probability of the event \( E_A \) that the mass of a neutron star corresponds to measurement:

\[
P(E_A | \pi_i) = \Phi(M_i, \mu_A, \sigma_A), \tag{13}
\]

where \( M_i \) - maximal mass constructed by \( \pi_i \) and \( \Phi(x, \mu, \sigma) \) is the cumulative distribution function for the normal distribution

\[
\Phi(x, \mu, \sigma) = \frac{1}{2} \left[ 1 + \text{erf} \left( \frac{x - \mu}{\sqrt{2\sigma^2}} \right) \right]. \tag{14}
\]

Radius Constraint

In the BA by Steiner et al. [2] the luminosity radius extracted for burst sources has been used to constrain a combined mass-radius relationship. This method is problematic, in particular because of the unknown stellar atmosphere composition, uncertainties in the distance to the source, the bias of the parabolic M-R constraint with the shape of stellar sequences in the M-R diagram for typical EoS and last not least due to unknown details of the burst mechanism. A very promising technique to measure radii of neutron stars is based on the analysis of pulsar timing residuals since it does not rely on knowledge of the luminosity of thermal radiation. Such a radius measurement gives \( \mu_B = 15.5 \, \text{km} \) and \( \sigma_B = 1.5 \, \text{km} \) for PSR J0437-4715 [14]. A similar range of radius values has recently been obtained for RXJ 1856 by Hambaryan et al. Now it is possible to calculate the conditional probability of the event \( E_B \) that the radius of neutron star corresponds to the given measurement

\[
P(E_B | \pi_i) = \Phi(R_i, \mu_B, \sigma_B). \tag{15}
\]

\( M_G - M_B \) Relation Constraint

This constraint corresponds to a region in the \( M_G - M_B \) plane. We need to estimate the probability that a point \( M_i = (M_{Gi}, M_{Bi}) \) is close to the point \( \mu = (\mu_G, \mu_B) \). The mean values \( \mu_G = 1.249 \), \( \mu_B = 1.36 \) and standard deviations \( \sigma_{M_G} = 0.001 \), \( \sigma_{M_B} = 0.002 \) are given in [16]. The needed probability can be calculated by the formula

\[
P(E_K | \pi_i) = \Phi(\xi_G) - \Phi(-\xi_G) \cdot [\Phi(\xi_B) - \Phi(-\xi_B)]. \tag{16}
\]

where \( \Phi(x) = \Phi(x, 0, 1) \), \( \xi_G = \sigma_{M_G}/d_{M_G} \) and \( \xi_B = \sigma_{M_B}/d_{M_B} \) are the absolute values of the components of the vector \( d = \mu - M_i \). Here \( \mu = (\mu_G, \mu_B)^T \) is given in [16] and \( M_i = (M_{Gi}, M_{Bi})^T \) is the solution of the TOV equations using the \( i^{\text{th}} \) vector of EoS parameters \( \pi_i \). Note that formula (16) does not correspond to a multivariate normal distribution.

Calculation of a posteriori Probabilities

Note, that these measurements are independent of each other. This means that we can calculate the complete conditional probability of an event \( E \) given \( \pi_i \) corresponds to the product of the conditional probabilities of all measurements, in our case resulting from the three constraints \( E_A, E_B, E_K \),

\[
P(E | \pi_i) = P(E_A | \pi_i) \cdot P(E_B | \pi_i) \cdot P(E_K | \pi_i). \tag{17}
\]

Now, we can calculate the probability of \( \pi_i \) using Bayes’ theorem:

\[
P(\pi_i | E) = \frac{P(E | \pi_i) \cdot P(\pi_i)}{\sum_{j=0}^{N-1} P(E | \pi_j) \cdot P(\pi_j)}. \tag{18}
\]

Results and Discussion

We apply the scheme of BA for the probabilistic estimation of the EoS given by the vector parameter \( \pi \). Varying the parameters in the intervals \( 400 < \varepsilon_B [\text{MeV/fm}^3] < 1000, 0 < \gamma < 1 \) and \( 0.3 < c_s^2 < 1 \) we explore the calculations for \( N = 10^3 \) and the results are presented in Fig. 2. The mass-radius relation is shown in the upper panel and the pressure as a function of the energy density is in the lower one. The results are shown for different sets of neutron star configurations corresponding to different sets of EoS parameters. The thickness of the lines is chosen to be proportional to the probability value for the parameter vector \( \pi \). We show that the chosen constraints are not sufficient to distinguish two cases: the one with the existence of a third family of twin stars (two stars with same masses but different radii due to different internal composition) from the case where only neutron star family is possible. This result can also be obtained from the lower panel of Fig. 2 where the EoS are plotted with different line thickness corresponding to the probability value for the parameter vector \( \pi \). It is apparent that the two types of EoS, with and without the phase transition to quark matter, have approximately the same probability. Nevertheless, when the phase transition to quark matter is possible then our constraint requires that the phase transition should occur for energy densities exceeding 900 MeV/fm\(^3\) with a jump in energy density up to \( 10^5 \) MeV/fm\(^3\).

We conclude that the current state of knowledge of observables for masses and radii of compact stars
Figure 2: Mass-radius relations (upper panel) for different sets of NS configurations corresponding to pressure vs. energy density relations (lower panel) which are obtained by varying three EoS parameters, see text. The thickness of the lines is proportional to the probability value for the parameter vector \( \pi \).

does not yet allow to extract with certainty a statement about the possible existence of a quark matter inner core with a strong first order phase transition to the outer core.

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