THE NUMBER OF UP-RUNS OF LENGTH ONE IN DEUTSCH PATHS

HELMUT PRODINGER

ABSTRACT. A variation of Dyck paths allows for down-steps of arbitrary length, not just one. This is motivated by ideas due to Emeric Deutsch. We use the adding-a-new-slice technique and the kernel method to compute the number of maximal runs of up-step runs of length 1.

1. ENUMERATION OF DEUTSCH PATHS BY THE ADDING-A-NEW-SLICE TECHNIQUE

Deutsch paths are like Dyck paths, but extra down-steps of the form \((1, -j)\), any \(j \geq 2\), are also allowed. They were analyzed recently in [6]. Here, we want to enumerate them in a different manner which is quite versatile when certain parameters of Deutsch paths should be analyzed.

We decompose a Deutsch paths into maximal runs of up- resp. down-steps. First, we restrict our attention to instance when the path ends with down-steps. If a path is closed, this happens anyway (except for the empty path), but for open-ended paths, the last step of the path might be an up-step.

The technique we are using can be found in [1].

We consider a generating function \(F_k(z, u)\), where \(k\) is the number of “mountains” (runs of up-steps, followed by runs of down-steps), \(z\) marks the length of the path and \(u\) is used to remember the last level reached (coefficient of \(u^i\)).

The extra down-steps require some preparations. If one want to go down by \(h \geq 1\) levels, and do this in \(n \geq 1\) steps, it can be done in

\[
[z^h]\left(\frac{z}{1-z}\right)^n = [z^{h-n}](1-z)^{-n} = \binom{h-1}{h-n}
\]

ways. If one wants to keep \(n\) variable, using a generating function, we get

\[
\sum_{n=1}^{h} \binom{h-1}{h-n}z^n = z(1+z)^{h-1}.
\]

And now we have to compute this:

\[
\sum_{k>i} z^{k-i} \sum_{0 \leq j < k} u^j z(1+z)^{k-1-j} = \frac{z^2 (1+z)^{i+1}}{(1-z-z^2)(1+z-u)} - \frac{z^2 u^{i+1}}{(1-zu)(1+z-u)}.
\]

Written for my own benefit. Potential coauthors who want to polish/extend this material send private email.
Consequently we have

$$F_{k+1}(z, u) = \frac{z^2(1+z)}{(1-z-z^2)(1+z-u)} F_k(z, 1+z) - \frac{z^2 u}{(1-zu)(1+z-u)} F_k(z, u).$$

With

$$\Phi(z, u) := \sum_{k \geq 0} F_k(z, u)$$

(arbitrary number of mountains) this leads to

$$\Phi(z, u) - 1 = \frac{z^2(1+z)}{(1-z-z^2)(1+z-u)} \Phi(z, 1+z) - \frac{z^2 u}{(1-zu)(1+z-u)} \Phi(z, u)$$

or

$$\frac{z(u-u_1)(u-u_2)}{(1-zu)(1+z-u)} \Phi(z, u) = 1 + \frac{z^2(1+z)}{(1-z-z^2)(1+z-u)} \Phi(z, 1+z),$$

with

$$u_{1,2} = \frac{1+z \pm \sqrt{1-2z-3z^2}}{2z}.$$  

As it was shown already in [6], we are in the Motzkin world. Now we use arguments from the kernel method [5]. The factor $(u-u_2)$ must also be a factor of the right-hand side, otherwise there would not be power series expansion around $z = 0$. This leads to

$$\frac{1 + \frac{z^2(1+z)}{(1-z-z^2)(1+z-u)} \Phi(z, 1+z)}{u-u_2} = -\frac{1}{1+z-u}.$$

Further simplification leads to

$$\Phi(z, u) = -\frac{(1-zu)}{z(u-u_1)} = \frac{(1-zu)}{zu_1(1-u/u_1)}.$$  

Setting $u = 0$ means that the path ends on the $x$-axis:

$$\Phi(z, 0) = \frac{1}{zu_1} = 1 + z - \sqrt{1-2z-3z^2} = 1 + z^2 + 3z^3 + 6z^4 + 15z^5 + 36z^6 + \cdots.$$  

As was used already in [6], the substitution $z = \frac{v}{1+v+v^2}$ makes everything prettier:

$$\Phi(z, 0) = \frac{1 + v + v^2}{1+v}.$$  

The function

$$\Phi(z, u) \frac{1}{1-zu}$$

describes Deutsch paths that can also end with up-steps. And if one replaces now $u := 1$, we get so-called open Deutsch paths, that can end at any level:

$$\Phi(z, 1) \frac{1}{1-z} = \frac{1}{z(u_1-1)} = 1 + v + v^2,$$

which also enumerates Motzkin paths. This was explained via a bijection in [6].
2. Counting runs of single up-steps

The paper [4] counted the number of maximal runs of up-steps of length one. This was greatly extended in [3]; the method of choice in the first incarnation of this paper was indeed the adding-a-new-slice technique combined with the kernel method. We will do this now for Deutsch paths. We keep this simple and only consider the basic case, leaving more general considerations for later.

The approach is quite similar to the previous section; however, we use a third variable, $t$, to count the up-runs of length one.

We have to compute this:

$$\sum_{k \geq i+1} z^k \sum_{0 \leq j < k} u^j z(1+z)^{k-1-j} + tz \sum_{0 \leq j \leq i} u^j z(1+z)^{i-j},$$

which leads to

$$F_{k+1}(z,u) = \alpha F_k(z,u) + \beta F_k(z, 1+z)$$

with

$$\alpha = \frac{z^2 u (zu + tzu - t)}{(zu - 1)(u - 1 - z)}$$

and

$$\beta = -\frac{z^2(1+z)(-z - z^2 + tz + tz^2 - t)}{(u - 1-z)(z + z^2 - 1)}.$$

This leads to

$$\Phi(z,u) - 1 = \alpha \Phi(z,u) + \beta \Phi(z, 1+z)$$

or

$$\frac{z(1+z^2-tz^2)(u-u_1)(u-u_2)}{(1-zu)(1+z-u)} \Phi(z,u) = 1 + \beta \Phi(z, 1+z),$$

The two roots are now

$$u_{1,2} = \frac{-tz^2 + z + 1 + z^2 \pm \sqrt{t^2z^4 + 2t^3z^2 - 2tz^2 + 2t^2z^4 - z^2 - 2z - 2z^3 + 1 - 3z^4}}{2z(1+z^2-tz^2)}.$$

Simplification, after dividing out the factor $u-u_z$ from the equation, leads to

$$\Phi(z,u) = \frac{(1-zu)}{z(1+z^2-tz^2)(u_1-u)}.$$

This time, we confine ourselves to the instance $u = 0$, i.e., Deutsch paths returning to the $x$-axis. We get

$$\Phi(z,0) = \frac{1}{z(1+z^2-tz^2)u_1}$$

$$= \frac{-tz^2 + z + 1 + z^2 - \sqrt{t^2z^4 + 2t^3z^2 - 2tz^2 + 2t^2z^4 - z^2 - 2z - 2z^3 + 1 - 3z^4}}{2z(1+z)(z^2-tz^2+1)}$$

$$= 1 + tz^2 + z^3 + (t^2 + 2)z^4 + (3 + 3t)z^5 + (7 + 7t + t^3)z^6 + (17 + 13t + 6t^2)z^7 + \cdots.$$
Once one has this generating function, one can state many results as a corollary. We will only provide one such result, namely, compute the average of the parameter labelled by the variable \( t \). So, we differentiate \( \Phi(z, 0) \) w.r.t. \( t \) and then set \( t := 1 \). This leads to

\[
\frac{v^2}{(1-v)(1+v)^2(1+v+v^2)}.
\]

One could even read off the coefficients from this, but this would we a sum, so we refrain from doing this. However, we are interested in asymptotics. We will use singularity analysis, as is now customary, see [2].

The relevant singularity is at \( z = \frac{1}{3} \), and we find, as \( z \to \frac{1}{3} \),

\[
v \sim 1 - \sqrt[3]{1-3z}.
\]

Furthermore,

\[
\frac{v^2}{(1-v)(1+v)^2(1+v+v^2)} \sim \frac{\sqrt[3]{3}}{36 \sqrt{1-3z}}.
\]

Therefore

\[
[z^n] \frac{v^2}{(1-v)(1+v)^2(1+v+v^2)} \sim [z^n] \frac{\sqrt[3]{3}}{36 \sqrt{1-3z}} \sim \frac{\sqrt[3]{3}}{36} 3^n \frac{1}{\sqrt{\pi n}}.
\]

This needs to be divided by the total number of such paths, viz.

\[
[z^n] \frac{1+v+v^2}{1+v} \sim [z^n] \left( \frac{3}{2} - \frac{3 \sqrt[3]{3}}{4 \sqrt{1-3z}} \right) \sim \frac{3 \sqrt[3]{3}}{8} 3^n \frac{1}{\sqrt{\pi n^{3/2}}}.
\]

The quotient is

\[
\sim \frac{2n}{27} = 0.074 n.
\]

So, a Deutsch path of length \( n \) has about \( 0.074 n \) up-runs of length 1. Many such results could be derived with some patience and a computer.

**References**

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