LIFTING DIVISORS WITH IMPOSED RAMIFICATIONS  
ON A GENERIC CHAIN OF LOOPS  

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ABSTRACT. Let $C$ be a curve over an algebraically closed non-archimedean field with non-trivial valuation. Suppose $C$ has totally split reduction and the skeleton $\Gamma$ is a chain of loops with generic edge lengths. Let $P$ be the rightmost vertex of $\Gamma$ and $P \in C$ be a point that specializes to $P$. We prove that any divisor class on $\Gamma$ with imposed ramification at $P$ that is rational over the value group of the base field lifts to a divisor class on $C$ that satisfies the same ramification at $P$, which extends the result in [CJP13].

1. INTRODUCTION

A metric graph which is a generic chain of loops (Definition 2.2) plays a crucial role in connecting classic and tropical Brill-Noether theory. Many properties of these graphs, such as Brill-Noether generality, can be transferred to certain curves with minimal skeleton isometric to them. Related approaches can be found in [CDPR12, JP14, JP16].

Let $\Gamma$ be a generic chain of loops with or without bridges. Let $K$ be an algebraically closed non-archimedean field with nontrivial value group $G$ and valuation ring $R$. Let $C$ be a smooth projective curve of genus $g$ over $K$ which has totally split reduction (by which we mean $C$ admits a split semistable $\overline{K}$-model as in [BR14 §5] whose special fiber only has rational components) and the skeleton is isometric to $\Gamma$. The tropicalization map from $\text{Pic}^d(C)$ to $\text{Pic}^d(\Gamma)$ given by linear expansion of the retraction from $C^\text{an}$ to $\Gamma$ maps $\text{Pic}^d(C)^\text{an}$ to $\text{Pic}^d(\Gamma)$ [Bak08], where $\text{Pic}^d(\Gamma)$ (resp. $\text{Pic}^d(\Gamma)$) parametrizes divisor classes on $C$ (resp. $\Gamma$) with degree $d$ and rank at least $r$. It is then proved in [CJP15] that $\text{Trop}(\text{Pic}^d(C)) = \text{Pic}^d(\Gamma)$ via the classification of divisors in $\text{Pic}^d(\Gamma)$, given in [CDPR12]. In other words, every $G$-rational divisor class on $\Gamma$ of rank $r$ can be lifted to a divisor class on $C$ of the same rank.

On the other hand, let $\alpha = (\alpha_0, \ldots, \alpha_r)$ be a Schubert index of type $(r, d)$, which is a non-decreasing sequence of non-negative numbers bounded by $d - r$. For an arbitrary chain of loops $\Gamma$, Pflueger [Pfl17] provides a straightforward expression for the locus of divisors on $\Gamma$ with ramification at least $\alpha$ at the rightmost vertex $P$ of $\Gamma$:

$$W^r_\alpha(\Gamma, P) = \{ D \in \text{Pic}^d(\Gamma) : r(D - (\alpha_i + i)P) \geq r - i \text{ for } i = 0, 1, \ldots, r \},$$

which is locally a union of translates of coordinate planes possibly of different dimensions in $\mathbb{R}^g$. It follows that $W^r_\alpha(\Gamma, P)$ contains the tropicalization of the corresponding locus on $C$:

$$W^r_\alpha(C, P) = \{ D \in \text{Pic}^d(C) : h^0(\mathcal{O}_C(D - (\alpha_i + i)P)) \geq r + 1 - i \text{ for } i = 0, 1, \ldots, r \}. $$

When $\Gamma$ is a generic chain of loops, [Pfl17] shows that $W^r_\alpha(\Gamma, P)$ has dimension equal to that of $W^r_\alpha(C, P)$, and both of pure dimensions $\rho(g, r, d) - \sum_{0 \leq i \leq r} \alpha_i$ as expected. We prove an analogue of the lifting result of [CJP15]:

**Theorem 1.1.** Let $\Gamma$ be a generic chain of loops, possibly with bridges, and $C$ a smooth projective curve over $K$ which has totally split reduction and skeleton isometric to $\Gamma$. Let $P$ be a point on $C$.
that tropicalize to the rightmost vertex $P$ of $\Gamma$. Let $\alpha$ be a Schubert index of type $(r, d)$. Then every $G$-rational divisor class on $\Gamma$ with ramification at least $\alpha$ at $P$ lifts to a divisor class on $C$ with ramification at least $\alpha$ at $P$.

We now explain the general strategy used in the proof of Theorem 1.1. Let

$$\alpha^j = (\alpha_0, \ldots, \alpha_j, \alpha_j, \ldots, \alpha_j)$$

be a Schubert index of type $(r, d)$ whose last $r - j + 1$ coordinates are all $\alpha_j$s. Denote $W_j(C) = W_{d-\alpha^j}(C, P)$ and $W_j(\Gamma) = W_{d-\alpha^j}(\Gamma, P)$. Denote also

$$X_j(C) = (\alpha_j + j) P + W_{d-\alpha_j-j}^{-j}(C)$$

and

$$X_j(\Gamma) = (\alpha_j + j) P + W_{d-\alpha_j-j}^{-j}(\Gamma)$$

Note that $W_j(C) = W_{j-1}(C) \cap X_j(C)$ and that

$$W_j(\Gamma) = W_{j-1}(\Gamma) \cap X_j(\Gamma) = W_{j-1}(\Gamma) \cap \text{Trop}(X_j(C)).$$

We proceed by induction. At each step, assuming $\text{Trop}(W_{j-1}(C)) = W_{j-1}(\Gamma)$, it suffices to show that

$$\text{Trop}(W_{j-1}(C) \cap X_j(C)) = \text{Trop}(W_{j-1}(C)) \cap \text{Trop}(X_j(C)). \quad (1)$$

Note that $Y_j-1(C)$ is locally isomorphic to a polytopal domain, which is the preimage of an integral $G$-affine polytope in $\mathbb{R}^n$ of some $n$, at points in the relative interior of maximal faces of $Y_j-1(\Gamma) = \text{Trop}(Y_{j-1}(C))$. Since $W_{j-1}(\Gamma)$ and $X_j(\Gamma) = \text{Trop}(X_j(C))$ intersect properly in $Y_j-1(\Gamma)$, namely intersect in expected dimension, the problem boils down to lifting proper tropical intersections within a polytopal domain:

**Theorem 1.2.** Let $\mathcal{U}_\Delta \subset (\mathbb{G}_m^n)_{\text{an}}$ be the preimage of an integral $G$-affine polytope $\Delta$ in $\mathbb{R}^n$ of dimension $n$ and $\mathcal{X}$ and $\mathcal{X}'$ be two Zariski closed analytic subspaces of $\mathcal{U}_\Delta$ of pure dimension. Suppose $\text{Trop}(\mathcal{X})$ and $\text{Trop}(\mathcal{X}')$ intersect properly in $\Delta$. Then we have

$$\text{Trop}(\mathcal{X}) \cap \text{Trop}(\mathcal{X}') \cap \Delta^0 = \text{Trop}(\mathcal{X} \cap \mathcal{X}') \cap \Delta^0.$$ 

This theorem is proved in Section 4. See also [OP13] for an algebraic counterpart, where the authors proved a lifting theorem for subschemes of an algebraic torus whose tropicalizations intersect properly. See Section 3 for the discussion of local property of $Y_{j-1}(C)$, which works for all Brill-Noether loci $W_{d-\alpha^j}(C)$ on $C$. The proof of (1) is in Section 5, where we use the notation of ramification imposed by a partition instead of a Schubert index (Section 2).

**Conventions.** Throughout this paper $K$ will be an algebraically closed non-archimedean field $K$ with nontrivial value group $G$. For a lattice $N$ we denote $T_N$ the algebraic torus over $K$ whose lattice of characters, denoted by $M$, is dual to $N$. Denote also $N_K = N \otimes K$. All polytopes in $N_K$ are assumed integral $G$-affine.

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## 2. PRELIMINARIES

In this section we recall some notions and techniques which are useful for later arguments.
2.1. Special divisors on a generic chain of loops. Let $\Gamma$ be a metric graph that is a chain of $g$ loops with or without bridges, let $\{v_i\}_{1 \leq i \leq g}$ and $\{w_i\}_{1 \leq i \leq g}$ be vertices of $\Gamma$ as in Figure 1 (with the possibility that $w_i = v_{i+1}$). Let $l_i$ (resp. $n_i$) be the length of the top (resp. bottom) segment of the $i$th loop connecting the vertices $v_i$ and $w_i$. The divisors on $\Gamma$ with imposed ramification is classified in [Pfl17], we recall some related concepts from loc.cit..

![Figure 1. A chain of $g$ loops with bridges.](image)

**Definition 2.1.** The torsion profile of $\Gamma$ is a sequence $m = (m_2, \ldots, m_g)$ of $g - 1$ integers. If $l_i/n_i$ is a rational number, then $m_i$ is the minimum positive integer such that $m_i \cdot l_i$ is an integer multiple of $l_i + n_i$, otherwise $m_i = 0$.

Note that we omit $m_1$ because it is immaterial to the properties of the divisors of interest. The following notion of a generic chain of loops was introduced in [CDPR12] for constructing Brill-Noether general curves:

**Definition 2.2.** We say that $\Gamma$ is generic if none of the ratios $l_i/n_i$ is equal to the ratio of two positive integers whose sum does not exceed $2g - 2$, or equivalently if for each $i$ either $m_i > 2g - 2$ or $m_i = 0$.

Let $\lambda$ be a partition, which is a finite, non-increasing sequence of non-negative integers. As in [Pfl17], we will identify partitions with their Young diagrams in French notation.

![Figure 2. The partition as in [Pfl17] Figure 1](image)

**Definition 2.3.** Let $P$ be a point on $\Gamma$. The Brill-Noether locus corresponding to a partition $\lambda$ and the marked graph $(\Gamma, P)$ is

$$W^\lambda(\Gamma, P) = \{D \in \text{Pic}^0(\Gamma) : r(D + d'P) \geq r' \text{ whenever } (g - d' + r', r' + 1) \in \lambda\}.$$
Let $\alpha = (\alpha_0, ..., \alpha_r)$ be a Schubert index of type $(r, d)$ and $\lambda = (\lambda_0, ..., \lambda_r)$ be the induced partition where $\lambda_i = (g - d + r) + \alpha_{r-i}$. Then the locus $W^\lambda(\Gamma, P)$ is isomorphic to $W^\mu(\Gamma, P)$ under the Abel-Jacobi map with respect to $dP$. In particular if $\lambda$ is a $(r+1) \times (g-d+r)$ diagram then $W^\lambda(\Gamma, P)$ is isomorphic to $W_d^\lambda(\Gamma)$.

We next describe the Brill-Noether locus of a partition when $P = w_g$ is the rightmost vertex of $\Gamma$. As in [Pfl17] we identify $\lambda = (\lambda_0, ..., \lambda_r)$ with the set

$$\{(x, y) \in \mathbb{Z}_{\geq 0}^2 | 1 \leq x \leq \lambda_{y-1}, 1 \leq y \leq r + 1\}.$$  

**Definition 2.4.** Let $\lambda$ be a partition, and let $\underline{m} = (m_2, ..., m_g)$ be a $(g - 1)$-tuple of nonnegative integers. An $\underline{m}$-displacement tableaux on $\lambda$ is a function $t: \lambda \to \{1, 2, ..., g\}$ satisfying the following properties:

1. $t$ is strictly increasing in any given row or column of $\lambda$.
2. For any two distinct boxes $(x, y)$ and $(x', y')$ in $\lambda$, if $t(x, y) = t(x', y')$ then $x - y \equiv x' - y' \pmod{m_{t(x, y)}}$.

We denote by $t \vdash \underline{m}$ if $t$ is a $\underline{m}$-displacement tableaux on $\lambda$.

According to [Pfl17, Theorem 1.3 and Corollary 3.8] if $\Gamma$ is a generic chain of loops and $\underline{m}$ is its torsion profile, then every $\underline{m}$-displacement tableaux on $\lambda$ is injective.

**Definition 2.5.** Let $\underline{m}$ be the torsion profile of $\Gamma$. Let $t$ be a $\underline{m}$-displacement tableau on a partition $\lambda$. Denote by $\Upsilon(t)$ the set of divisor classes on $\Gamma$ of the form

$$\sum_{i=1}^{g} \langle \xi_i \rangle_i - g\omega,$$

where $\{\xi_i\}_i$ are real numbers such that $\xi_{t(x, y)} \equiv x - y \pmod{m_{t(x, y)}}$ and the symbol $\langle z \rangle_i$ denotes the point on the $i$-th loop that is located $z \cdot l_i$ units clockwise from $w_i$.

It follows that $\Upsilon(t)$ is a real torus of dimension $d_t = g - |t(\lambda)|$. Moreover, under the identification $\text{Pic}^0(\Gamma) = \prod_{1 \leq i \leq g} \mathbb{R}/(m_i + l_i)\mathbb{Z}$ induced by the Abel-Jacobi map [M105, §6], the torus $\Upsilon(t)$ is the image of a translate of a coordinate $d_t$-plane in $\mathbb{R}^g$. The following is the description of the Brill-Noether locus of $\lambda$ ([Pfl17, Theorem 1.4]).

**Proposition 2.6.** We have

$$W^\lambda(\Gamma, w_g) = \bigcup_{t \vdash \underline{m}} \Upsilon(t).$$

In particular, if $\Gamma$ is generic then $W^\lambda(\Gamma, w_g)$ is of pure dimension $g - |\lambda|$.  

2.2. **Curves with special skeletons and their tropicalizations.** Let $C$ be a smooth projective curve of genus $g$ over $K$ which has totally split reduction and the skeleton is isometric to $\Gamma$. Let $\tau: C^\text{an} \to \Gamma$ be the retraction map. The Jacobian variety of $C$ is totally degenerate in the sense of [Gub07, §6]. In other words, $\text{Pic}^0(C)^\text{an}$ is isomorphic to $(T_N)^\text{an}/L$ where $N$ is a lattice of rank $g$ and $L$ is a discrete subgroup of $T_N(\mathbb{K})$ which maps isomorphically onto a complete lattice of $N_{\mathbb{R}}$ under the tropicalization map. Moreover, the induced tropicalization map on $\text{Pic}^0(C)^\text{an}$ is compatible with the retraction to its skeleton, which is canonically identified with $\text{Pic}^0(\Gamma)$ ([BR14, §6]):

$$\begin{align*}
C^\text{an} \xrightarrow{\alpha_p} \text{Pic}^0(C)^\text{an} & \xrightarrow{\tau} (T_N)^\text{an} \\
\Gamma \xrightarrow{\alpha_p} \text{Pic}^0(\Gamma) & \xrightarrow{Trop} N_{\mathbb{R}}.
\end{align*}$$

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where $\alpha_P$ and $\alpha_P'$ are the Abel-Jacobi maps associated to $P \in C$ and $P \in \Gamma$ with $\tau(P) = P$.

**Definition 2.7.** Let $P$ be a point in $C$. As in Definition 2.3, the **Brill-Noether locus** corresponding to a partition $\lambda$ and the marked curve $(C, \mathcal{P})$ is

$$W^\lambda(C, \mathcal{P}) = \{D \in \text{Pic}^0(C) : h^0(\mathcal{O}_C(D + d'P)) \geq r' + 1 \text{ whenever } (g - d' + r', r' + 1) \in \lambda\}.$$

Let $\alpha = (\alpha_0, ..., \alpha_r)$ be a Schubert index of type $(r, d)$ and $\lambda = (\lambda_0, ..., \lambda_r)$ be the induced partition as above. Then as in the graph case the locus $W^\lambda(C, \mathcal{P})$ is isomorphic to $W^\lambda_d(C, \mathcal{P})$ under the Abel-Jacobi map with respect to $d\mathcal{P}$. In particular if $\lambda$ is a $(r + 1) \times (g - d + r)$ diagram then $W^\lambda(C, \mathcal{P})$ is isomorphic to $W^\lambda_d(C)$.

The following theorem is a (partial) summary of [Pfl17, Theorem 1.13 and Theorem 5.1] and [CJPT15, Theorem 1.1].

**Theorem 2.8.** Let $C$ be as above and $\Gamma$ is a generic chain of loops, let $\mathcal{P}$ be a point of $C$ that tropicalize to $P = w_\mathcal{P}$. Then $\text{Trop}(W^\lambda(C, \mathcal{P})) \subset W^\lambda(\mathcal{P}, \Gamma)$ and $W^\lambda(C, \mathcal{P})$ is of pure dimension $g - |\lambda|$. Moreover, if $\lambda$ is an $(r + 1) \times (g - d + r)$ diagram then $\text{Trop}(W^\lambda(C, \mathcal{P})) = W^\lambda(\mathcal{P}, \Gamma)$.

As the tropicalization preserves dimension [Gub07, §6], in the theorem above, if $\lambda$ is induced by $\alpha$ then the dimension of $W^\lambda(\mathcal{P}, \Gamma)$ is equal to the (expected) dimension of $W^\lambda(C, \mathcal{P})$ (or $W^\lambda_d(C, \mathcal{P})$).

### 2.3. Intersection multiplicities in a polytopal domain

Let $N$ be a lattice of rank $n$. Let $\Delta \subset N_\mathbb{R}$ be a polytope. We denote $U_\Delta$ the preimage of $\Delta$ in $(T_N)_{\text{an}}$ under the tropicalization map, then $U_\Delta$ is an affinoid domain in $(T_N)_{\text{an}}$ by [Gub07] and called a **polytopal domain**. Denote $K\langle U_\Delta \rangle$ the corresponding affinoid algebra, whose basic properties can be found in [Rab12, Proposition 6.9].

**Definition 2.9.** Let $f_1, ..., f_k \in K\langle U_\Delta \rangle$. Let $Y$ be a Zariski-closed analytic subspace of $U_\Delta$ of dimension $n - k$. Denote $Y_i = V(f_i)$ and $Z = Y \cap (\cap_{1 \leq i \leq k} Y_i)$. The **intersection multiplicity** of $Y$ and $Y_i$ at an isolated point $\xi$ of $Z$ is

$$i(\xi, Y \cdot Y_1 \cdot Y_k; U_\Delta) = \dim_K(\mathcal{O}_{Z, \xi}).$$

If $Z$ is finite we define the **intersection number** of $Y$ and $Y_1, ..., Y_k$ as:

$$i(Y \cdot Y_1 \cdot Y_k; U_\Delta) = \sum_{\xi \in Z} \dim_K(\mathcal{O}_{Z, \xi}).$$

This definition agrees with [Rab12, Definition 11.4] and is also compatible with the intersection multiplicities of algebraic varieties. We refer to [Rab12, §11] about intersection multiplicities of tropical hypersurfaces in $\Delta$ when $\Delta$ is of maximal dimension, which is compatible with the stable intersection of tropical cycles in $N_\mathbb{R}$.

**Theorem 2.10.** ([Rab12, Theorem 11.7]) Let $\Delta$ be a polytope in $N_\mathbb{R}$ and $f_1, ..., f_n \in K\langle U_\Delta \rangle$. Let $Y_i = V(f_i)$ for all $i$ and $w \in \cap_{1 \leq i \leq n} \text{Trop}(Y_i)$ be an isolated point contained in the interior of $\Delta$. Let $Z = \cap_{1 \leq i \leq n} Y_i$. Then:

$$\sum_{\xi \in Z, \text{Trop}(\xi) = w} i(\xi, Y_1 \cdot Y_n; U_\Delta) = i(w, \text{Trop}(Y_1) \cdot \text{Trop}(Y_n); \Delta)$$

On the other hand, for two Zariski closed subspace $\mathcal{X}$ and $\mathcal{X}'$ of $U_\Delta$ and an isolated point $\xi$ of $\mathcal{X} \cap \mathcal{X}'$, the **intersection multiplicity** of $\mathcal{X}$ and $\mathcal{X}'$ at $\xi$ is defined to be:

$$i(\xi, \mathcal{X} \cdot \mathcal{X}'; U_\Delta) = \sum_{i \geq 0} (-1)^i \dim_K \text{Tor}_i^{U_\Delta, \xi} (\mathcal{O}_{\mathcal{X}, \xi}, \mathcal{O}_{\mathcal{X}', \xi})$$

...
If $X \cap X'$ is finite, the intersection number of $X$ and $X'$ is

$$i(X \cdot X'; U_\Delta) = \sum_{\xi \in X \cap X'} i(\xi, X \cdot X'; U_\Delta).$$

3. LOCAL PROPERTIES OF $W^r_d(C)^{an}$

Let $C$ and $\Gamma$ and $T_N$ be as in §2.2 and suppose $\Gamma$ is generic. We prove in this section that $W^r_d(C)$ is locally analytically isomorphic to a polytopal domain at a “tropically general” point of $W^r_d(\Gamma)$. Before we start, we specify the following notation:

**Notation 3.1.** Let $P$ be the rightmost vertex of $\Gamma$ and fix $P \in C$ that tropicalizes to $P$. Let $e_1, ..., e_g \in N$ be the standard basis of $N$ and $e'_1, ..., e'_g \in M$ the dual basis. For each partition $\lambda$ we write $W^\lambda(C)$ (resp. $W^\lambda(\Gamma)$) instead of $W^\lambda(C, P)$ (resp. $W^\lambda(\Gamma, P)$). For a polytope $\Delta \subset N_\mathbb{R}$, if $\Delta$ maps to $\text{Pic}^0(\Gamma)$ isomorphically with image $\Delta$ in $\text{Pic}^0(C)^{an}$ under the tropicalization map. Let $N_\Delta = N \cap L_\Delta$. For a given projection $\pi: N \rightarrow N_\Delta$ we denote $\tilde{U}_\Delta$ the preimage of $\pi(\Delta)$ in $(T_{N_\Delta})^{an}$. For a pure polyhedral complex $\gamma$ in $N_\mathbb{R}$ denote $\text{relint}(\gamma)$ the union of relative interior of all maximal faces of $\gamma$.

Now let $\lambda$ be the $(r + 1) \times (g - d + r)$ diagram. As discussed in Section 2 we have $W^\lambda(C)$ isomorphic to $W^r_d(C)$ and $\text{Trop}(W^\lambda(C)) = W^\lambda(\Gamma)$, and $W^\lambda(\Gamma)$ is a union of translates of the images of the coordinate $\rho$-planes in $N_\mathbb{R}$, where $\rho = \rho(g, r, d)$.

Let $\delta$ be a maximal face of $W^\lambda(\Gamma)$. Take a polytope $\Lambda$ such that $\Lambda \subset \text{relint}(\delta)$. Let $\Delta = \Lambda \times I \subset N_\mathbb{R}$ where $I = [-\epsilon, \epsilon]^{g-\rho}$ such that $\Delta$ maps to $\text{Pic}^0(\Gamma)$ isomorphically. We then have that $U_\Delta$ is isomorphic to $\tilde{U}_\Delta$. Hence we may consider $W^\lambda(C)^{an}$ as a Zariski-closed analytic subspace of the polytopal domain $U_\Delta$. We may assume that $L_{\Lambda}$ is generated by $e_1, ..., e_\rho$. The canonical projection from $N$ to $N_{\Lambda}$ gives rise to a projection $\pi_{\Lambda}: (T_{N})^{an} \rightarrow (T_{N_{\Lambda}})^{an}$ which is compatible with the tropicalization map. Denote $W_{\Lambda} = W^\lambda(C)^{an} \cap U_\Lambda$. The argument in [BPR12, Theorem 4.31] shows that $\pi_{\Lambda}: W_{\Lambda} \rightarrow U_\Lambda$ is finite, and maps every irreducible component of $W_{\Lambda}$ surjectively onto $\tilde{U}_{\Lambda}$.  


Proposition 3.2. The map $\pi_\Lambda: W_\Lambda \to \tilde{U}_\Lambda$ is an isomorphism.

Proof. We first show that $\pi_\Lambda$ is of degree one in the sense of [BPR12, §3.27]. Take $\rho$ general translates of theta divisors $\Theta^1 = \text{Trop}(\Theta^1_1), ..., \Theta^\rho = \text{Trop}(\Theta^\rho_1)$ on $\text{Pic}^0(\Gamma)$ such that $\Lambda \cap (\cap_i \Theta^1_i)$ is nonempty and consists of finitely many points, where $\Theta^i_1$ are theta divisors on $\text{Pic}^0(\Gamma)$. According to [CJP15, §2] we may also assume that $\cap_i \Theta^1_i$ intersects $W^\lambda(\Gamma)$ transversally at $m$ points, where $m = g! \prod_{i=0}^r \frac{i!}{(g-d+r+i)!} = i(W^\lambda(\Gamma) \cdot \Theta^1_1 \cdots \Theta^\rho; \text{Pic}^0(\Gamma))$.

Since the degree of $\pi_\Lambda$ is preserved under flat base change, we may shrink $\Lambda$ so that $\Lambda \cap (\cap_i \Theta^1_i)$ consists of exactly one point. Also, take $\epsilon$ small enough such that $\Theta^1_1 \cap \Delta$ is of the form $\tilde{\Theta}^1_1 \times I_\epsilon$ where $\tilde{\Theta}^1_1$ is a codimension one polyhedral complex in $\Lambda$. Note that by [Wil09, §6] we know that $K(\tilde{U}_\Lambda)$ is a UFD, hence we can take $f_i \in K(\tilde{U}_\Lambda)$ to be the function that defines $(\Theta^1_1)_{an}$ in $U_\Delta$.

According to [Rab12, §8] there is a Laurent polynomial $f'_i$ which is a sum of monomials in $f_i$ such that $\text{Trop}(V(f'_i)) \cap \Delta = \text{Trop}(V(f_i)) = \tilde{\Theta}^1_1 \times I_\epsilon$. Moreover for all $w \in \Delta$ the monomials in $f'_i$ which obtains minimal $w$-weight is the same as those in $f'_i$. Let $A = \{ u_1, ..., u_k \} \subset M$ be the set of vertices of the Newton complex of $f'_i$ corresponding to the maximal faces of $\Delta$, whose polyhedral complex structure is induced by $\tilde{\Theta}^1_1 \times I_\epsilon$. We must have that $A$ is contained in a $\rho$ dimensional plane in $M_\mathbb{R}$ that is parallel to the one generated by $e'_1, ..., e'_\rho$. We may assume that $A$ is contained in the sublattice generated by $e'_1, ..., e'_\rho$. Consequently, if $g_i = \sum x^{u_i}$ and $h_i = f_i - g_i$, then for every $a \in K$ with $\text{val}(a) \geq 0$ we have $\text{Trop}(V(g_i + ah_i)) = \text{Trop}(V(f_i))$ (with the same multiplicities, which are all ones by [CJP15, Theorem 3.1]). Moreover, $g_i$ is contained in $K(\tilde{U}_\Lambda)$.

We next denote by $B^1_K$ the unit ball in $(G_m)^an$ with coordinate ring $K(t)$, and prove the following lemma:

Lemma 3.3. Let $W$ be a Cohen-Macaulay Zariski-closed analytic subspace of $U_\Delta$ of pure codimension $k$. Let $l_1, ..., l_k \in K(\tilde{U}_\Lambda) \times K(t)$ be global sections on $U_\Delta \times B^1_K$. Let $Y_i$ be the subspace of $U_\Delta \times B^1_K$ defined by $l_i$. For $t \in B^1_K$ let $Y_i(t) = \pi^{-1}(t) \cap Y_i$ where $\pi$ is the projection from $U_\Delta \times B^1_K$ to $B^1_K$. Suppose $\text{Trop}(W \cap (\cap_i \text{Trop}(Y_i(t))))$ is finite and contained in $\Delta^\circ$ for all $i$ and $t \in B^1_K$. Then the intersection number $\Lambda(W \cdot Y_1(t) \cdots Y_k(t); U_\Delta)$ is constant on $B^1_K$.

Proof. We proceed by showing that the analytic space $\tilde{W} = (W \times B^1_K) \cap (\cap_i Y_i)$ is finite and flat over $B^1_K$, hence every fiber has the same length.

It is obvious that $\tilde{W}$ has finite fiber. To show it is proper over $B^1_K$, we use the ideas in [OR11, §4.9]. Since all analytic space appeared are affinoid, hence compact Hausdorff, we have that $\pi$ is compact on $\tilde{W}$ and separated. On the other hand, let $\pi': U_\Delta \times B^1_K \to U_\Delta$ be the other projection. By [OR11, Lemma 4.14] we have

$$\tilde{W} \subset (\text{Trop} \circ \pi')^{-1}(\Delta^\circ) \subset \text{Int}(U_\Delta \times B^1_K/B^1_K).$$

According to the sequence of morphisms $\tilde{W} \to U_\Delta \times B^1_K \to B^1_K$ we have

$$\text{Int}(\tilde{W}/B^1_K) = \text{Int}(\tilde{W}/U_\Delta \times B^1_K) \cap \text{Int}(U_\Delta \times B^1_K/B^1_K) = \text{Int}(\tilde{W}/U_\Delta \times B^1_K) = \tilde{W}.$$

Hence $\pi$ is boundaryless on $\tilde{W}$. Consequently $\pi$ is proper, and hence finite on $\tilde{W}$.

Now the flatness of $\pi$ follows from [Liu02, Exercise 1.2.12] and induction. Note that the finiteness of fibers of $\pi$ and the Cohen-Macaulay-ness of $W$ ensures that each $l_i$ is not a zero divisor on $W \cap Y_1(t) \cap \cdots \cap Y_{i-1}(t)$ for all $t$. \qed
We return to the proof of Proposition 3.2. In Lemma 3.3 let $W = W_\Lambda$ and $l_i = g_i + th_i$. It follows that (set $t = 0$)

$$i(W_\Lambda \cdot \prod_{i=1}^{\rho} (\Theta_C)^{an}; \mathcal{U}_\Delta) = i(W_\Lambda \cdot \prod_{i=1}^{\rho} V(f_i); \mathcal{U}_\Delta) = i(W_\Lambda \cdot \prod_{i=1}^{\rho} V(g_i); \mathcal{U}_\Delta) = m_\Lambda \cdot i(\prod_{i=1}^{\rho} V(g_i); \tilde{\mathcal{U}}_\Lambda)$$

where the last equation is the projection formula in [Gub98, Proposition 2.10] and $m_\Lambda$ is the degree of $\pi_\Lambda$. By Theorem 2.10 we have

$$i(\prod_{i=1}^{\rho} V(g_i); \tilde{\mathcal{U}}_\Lambda) = i(\prod_{i=1}^{\rho} \text{Trop}(V(g_i)); \Lambda) = 1,$$

therefore $i(W_\Lambda \cdot \prod_{i=1}^{\rho} (\Theta_C)^{an}; \mathcal{U}_\Delta) = m_\Lambda$.

Now for all $w_j \in W^\lambda(\Gamma) \cap (\cap_i \Theta_C^i)$ where $1 \leq j \leq m$ we pick a polytope $\Lambda_j$ as above and get a degree $m_{\Lambda_j}$ of the corresponding projection map, which yields

$$\sum_{j=1}^{m} m_{\Lambda_j} = i(W^\lambda(C) \cdot \Theta_C^1 \cdots \Theta_C^\rho; \text{Pic}^0(C)) = m.$$

Note that the first equality follows from the fact that the $K$-dimension of the local ring of $W^\lambda(C) \cap (\cap_i \Theta_C^i)$ at a point is equal to that of $W^\lambda(C)^{an} \cap (\cap_i (\Theta_C^i)^{an})$. Hence we must have $m_{\Lambda_j} = 1$ for all $j$. Therefore $\pi_\Lambda$ is of degree one.

It follows that $W_\Lambda$ is irreducible and generically reduced. However, $W_\Lambda$ is Cohen-Macaulay since $W^\lambda(C)$ is, so it is everywhere reduced, hence integral. Now $\pi_\Lambda$ induces a finite morphism of degree one between integral domains whose source $K(\tilde{\mathcal{U}}_\Lambda)$ is normal, it must be an isomorphism.

\[\square\]

Remark 3.4. An algebraic analogue of Proposition 3.2 is that a reduced (or Cohen-Macaulay) closed subscheme $Z$ of $T_N$ is local analytically isomorphic to a torus at a point that tropicalize to the relative interior of a maximal face of $\text{Trop}(Z)$ of multiplicity one, see for example [He16, Lemma 6.2].

4. LIFTING TROPICAL INTERSECTIONS IN A POLYHEDRAL DOMAIN.

Let $N$ be an arbitrary lattice of rank $n$ as in Theorem 1.2 and $T_N$ the induced torus. Let $\Delta$ be a polytope of maximal dimension in $N_\mathbb{R}$. In this section we use Osserman and Rabinoff’s continuity theorem [ORT, §5] of analytic intersection numbers to prove Theorem 1.2. Let $\mathcal{U}_0 \subset T_N^{an}$ be the preimage of the origin in $N_\mathbb{R}$. Then $\mathcal{U}_0$ acts on $\mathcal{U}_\Delta$. Denote the action by $\mu: \mathcal{U}_0 \times \mathcal{U}_\Delta \to \mathcal{U}_\Delta$ and let $\pi: \mathcal{U}_0 \times \mathcal{U}_\Delta \to \mathcal{U}_0$ be the projection. This gives an isomorphism:

$$(\pi, \mu): \mathcal{U}_0 \times \mathcal{U}_\Delta \to \mathcal{U}_0 \times \mathcal{U}_\Delta.$$

The following lemma is a consequence of [ORT, Proposition 5.8]:

Lemma 4.1. Let $\pi$ be as above. Let $\mathcal{Y}, \mathcal{Y}' \subset \mathcal{U}_0 \times \mathcal{U}_\Delta$ be Zariski-closed subspaces, flat over $\mathcal{U}_0$, such that $\mathcal{Y} \cap \mathcal{Y}'$ is finite over $\mathcal{U}_0$. Then the map

$$s \mapsto i(\mathcal{Y}_s \cdot \mathcal{Y}'_s; \mathcal{U}_\Delta): |\mathcal{U}_0| \to \mathbb{Z}$$

is constant on $\mathcal{U}_0$, where $\mathcal{Y}_s$ and $\mathcal{Y}'_s$ are fibers of $\pi$.

We refer to [Duc11] or [ORT, §5] about the notion of flatness for analytic spaces, which is preserved under composition and change of base. Any analytic space is flat over $K$. We now prove Theorem 1.2.

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Proof of Theorem 1.2: Let $\mathcal{X}, \mathcal{X}' \subset \mathcal{U}_\Delta$ be Zariski closed analytic subspaces of $\mathcal{U}_\Delta$ of pure dimension. We first assume that $\dim(\mathcal{X}) + \dim(\mathcal{X}') = n$, hence $\mathcal{X} \cap \mathcal{X}'$ is finite. As the statement is local, we may also assume that $\Trop(\mathcal{X}) \cap \Trop(\mathcal{X}')$ contains only one point $w$ that lies in $\Delta^\circ$. It suffices to show that $\mathcal{X} \cap \mathcal{X}'$ is nonempty.

In Lemma 4.1 let $\mathcal{Y} = (\pi, \mu)(\mathcal{U}_0 \times \mathcal{X})$ and $\mathcal{Y}' = \mathcal{U}_0 \times \mathcal{X}'$ where $\pi$ and $\mu$ are as above. Then both $\mathcal{Y}$ and $\mathcal{Y}'$ are flat over $\mathcal{U}_0$. On the other hand, the argument in Lemma 3.1 shows that $\mathcal{Y} \cap \mathcal{Y}'$ is finite over $\mathcal{U}_0$, as $\Trop(\mathcal{Y}_s) \cap \Trop(\mathcal{Y}'_s) = \Trop(\mathcal{X}') \cap \Trop(\mathcal{X}')$ is finite for every $s \in |\mathcal{U}_0|$. Therefore $\iota(\mathcal{Y}_s \cdot \mathcal{Y}'_s; \mathcal{U}_\Delta)$ is constant on $|\mathcal{U}_0|$ by Lemma 4.1. Take $\xi \in |\mathcal{X}|$ and $\xi' \in |\mathcal{X}'|$ such that $\Trop(\xi) = \Trop(\xi') = w$. Take also $t \in |\mathcal{U}_0|$ such that $t(\xi) = \xi'$. Then $\mathcal{Y}_t \cap \mathcal{Y}'_t = \mathcal{Y} \cap \mathcal{X}'$ contains $\xi'$, hence $\iota(\mathcal{Y}_t \cdot \mathcal{Y}'_t; \mathcal{U}_\Delta) > 0$. Thus $\iota(\mathcal{Y}_s \cdot \mathcal{Y}'_s; \mathcal{U}_\Delta) > 0$ for all $s \in |\mathcal{U}_0|$. Taking $s$ to be the identity in $\mathcal{U}_0$ implies that $\mathcal{X} \cap \mathcal{X}' = \mathcal{Y}_s \cap \mathcal{Y}_s'$ is nonempty. Thus $w \in \Trop(\mathcal{X}) \cap \Trop(\mathcal{X}')$.

This situation easily generalizes to intersections of three or more subschemes as in [OP13, §5.2]. Namely, we have the following lemma:

Lemma 4.2. Let $\mathcal{X}_1, \ldots, \mathcal{X}_m$ be Zariski-closed subspaces of $\mathcal{U}_\Delta$ of (pure) codimension $d_1, \ldots, d_m$ respectively, where $d_1 + \cdots + d_m = n$. Suppose $\Trop(\mathcal{X}_1) \cap \cdots \cap \Trop(\mathcal{X}_m)$ is finite and contained in $\Delta^\circ$. Then

$$\Trop(\mathcal{X}_1) \cap \cdots \cap \Trop(\mathcal{X}_m) = \Trop(\mathcal{X}_1 \cap \cdots \cap \mathcal{X}_m).$$

Now suppose $\Trop(\mathcal{X}) \cap \Trop(\mathcal{X}')$ has dimension $l > 0$. For any $G$-rational point $v \in \Trop(\mathcal{X}) \cap \Trop(\mathcal{X}') \cap \Delta^\circ$ we can find a Zariski-closed subspace $\mathcal{Z}$ of $\mathcal{U}_\Delta$ of codimension $l$ such that $\Trop(\mathcal{Z})$ contains $v$ and intersect properly with $\Trop(\mathcal{X}) \cap \Trop(\mathcal{X}')$ near $v$. Hence Lemma 4.2 implies that

$$v \in \Trop(\mathcal{Z} \cap \mathcal{X} \cap \mathcal{X}') \subset \Trop(\mathcal{X} \cap \mathcal{X}').$$

As $G$-rational points are dense in $\Trop(\mathcal{X}) \cap \Trop(\mathcal{X}')$ this implies that $\Trop(\mathcal{X}) \cap \Trop(\mathcal{X}') \cap \Delta^\circ = \Trop(\mathcal{X} \cap \mathcal{X}') \cap \Delta^\circ$. □

As mentioned in the proof above, we can also state Theorem 1.2 for the intersection of more than two analytic subspaces of $\mathcal{U}_\Delta$:

Corollary 4.3. Let $\mathcal{X}_1, \ldots, \mathcal{X}_m$ be Zariski-closed subspaces of $\mathcal{U}_\Delta$ of pure dimensions whose tropicalizations intersect properly. Then

$$\Trop(\mathcal{X}_1) \cap \cdots \cap \Trop(\mathcal{X}_m) \cap \Delta^\circ = \Trop(\mathcal{X}_1 \cap \cdots \cap \mathcal{X}_m) \cap \Delta^\circ.$$

Remark 4.4. As the statement in Corollary 4.3 is local, it is still true if we replace $\Delta$ by the support of a polyhedral complex with integral $G$-affine faces. In particular, we have

$$\Trop(\mathcal{X}_1) \cap \cdots \cap \Trop(\mathcal{X}_m) = \Trop(\mathcal{X}_1 \cap \cdots \cap \mathcal{X}_m)$$

if $\mathcal{X}_1, \ldots, \mathcal{X}_m$ are Zariski-closed analytic subspaces of $T^G_N$ with proper tropical intersections.

It is necessary to only consider the interior of $\Delta$ in Theorem 1.2 or Corollary 4.3. See the example below.
Example 4.5.

Let $\mathcal{X}$ and $\mathcal{X}'$ be two curves in $(K^*)^2$ defined by $x + y + 1 = 0$ and $x + y = 0$ respectively. Let $\Delta = [0, 1]^2$. Then $\text{Trop}(\mathcal{X}) \cap \text{Trop}(\mathcal{X}') \cap \Delta$ is the origin, hence $\text{Trop}(\mathcal{X})$ and $\text{Trop}(\mathcal{X}')$ intersect properly in $\Delta$. But $\mathcal{X} \cap \mathcal{X}' \cap U_\Delta$ is empty.

5. LIFTING DIVISORS WITH IMPOSED RAMIFICATION

In this section we prove Theorem 1.1. We will use the notations in Notation 3.1. Let $\alpha = (\alpha_0, ..., \alpha_r)$ be a Schubert index of type $(d, r)$. Let $\lambda = (\lambda_0, ..., \lambda_r)$ be the induced partition. Hence $\lambda_i = g - d + r + \alpha_{r-i} - 1$.

After translating every divisor class on $C$ (resp. $\Gamma$) of degree $d$ to its image in $\text{Pic}^0(C)$ (resp. $\text{Pic}^0(\Gamma)$) under the Abel-Jacobi map induced by $dP$ (resp. $dP$) we may assume that the ramification is imposed by $\lambda$ (instead of $\alpha$). Hence it remains to prove the following:

Theorem 5.1. We have $\text{Trop}(W^\lambda(C)) = W^\lambda(\Gamma)$.

Let $\lambda_j$ be the partition corresponding to the $(r+1-j) \times (g-d+r+\alpha_j)$ diagram. Then $W^\lambda_j(C)$ is isomorphic to $W^{r-j}_{d-\alpha_j-j}(C)$, and $W^\lambda(C) = \bigcap_{0 \leq i \leq r} W^\lambda_i(C)$, while $W^\lambda(\Gamma)$ is isomorphic to $W^{r-j}_{d-\alpha_j-j}(\Gamma)$, and $W^\lambda(\Gamma) = \bigcap_{0 \leq i \leq r} W^\lambda_i(\Gamma)$. Let $\lambda^j$ be the union of $\lambda_1, ..., \lambda_j$ and $W_j(C) = \bigcap_{0 \leq i \leq j} W^\lambda_i(C)$ and $W_j(\Gamma) = \bigcap_{0 \leq i \leq j} W^\lambda_i(\Gamma)$ for $0 \leq j \leq r$.

Let also $\mu_j$ be the partition corresponding to the $(r-j) \times (g-d+r+\alpha_j)$ diagram (this is the intersection of $\lambda^j$ and $\lambda_{j+1}$). As above we have $W^\mu_j(C)$ isomorphic to $W^{r-j-1}_{d-\alpha_j-j-1}(C)$, and $W^\mu_j(\Gamma)$ isomorphic to $W^{r-j-1}_{d-\alpha_j-j-1}(\Gamma)$. Moreover, we have $W_j(C) \subset W^\mu_j(C)$ and $W^{\lambda_{j+1}}(C) \subset W^\mu_j(C)$ and $W_j(\Gamma) \subset W^\mu_j(\Gamma)$ and $W^{\lambda_{j+1}}(\Gamma) \subset W^\mu_j(\Gamma)$. 

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In order to prove the Theorem above, we first show the following lemma:

**Lemma 5.2.** $W_j(\Gamma)$ and $W^{\lambda_{j+1}}(\Gamma)$ intersect properly in $W^{\mu_j}(\Gamma)$, and there is an open dense subset $U_j$ of $W_j(\Gamma) \cap W^{\lambda_{j+1}}(\Gamma) = W_{j+1}(\Gamma)$ which is contained in $\text{relint}(W^{\mu_j}(\Gamma))$.

**Proof.** The properness follows directly from dimension counting (Proposition 2.6), as $\lambda^i + 1$ is the union of $\lambda^i$ and $\lambda^i_{j+1}$ while $\mu_j$ is the intersection of $\lambda^i$ and $\lambda^i_{j+1}$.

For the second conclusion it suffices to show that every real torus in $W_{j+1}(\Gamma)$ is contained in exactly one torus in $W^{\mu_j}(\Gamma)$. Take two tori $T(t) \subset W_{j+1}(\Gamma)$ and $T(t') \subset W^{\mu_j}(\Gamma)$, where $t$ and $t'$ are $m_t$-displacement tableux on $\lambda^i + 1$ and $\mu_j$ respectively, such that $T(t) \subset T(t')$. We claim that $t' = t|_{\mu_j}$.

It is easy to see that $t'|(\mu_j) \subset t(\lambda^i + 1)$. On the other hand, let $S_k = \{(x, y) | x - y = k\}$ for all $k \in \mathbb{Z}$. If $t(x, y) = t'(x', y')$, then $x - y \equiv x' - y' \pmod{m_{t(x,y)}}$, hence $x - y = x' - y'$ by the generality of $\Gamma$. It follows that $t(\mu_j \cap S_k) \subset t'(\lambda^i + 1 \cap S_k)$ for all $k$. In particular, let $k_j = g - d + \alpha_j + j$, we have

$$t(\mu_j \cap S_{k_j}) = t'(\lambda^i + 1 \cap S_{k_j})$$

since $\mu_j \cap S_{k_j} = \lambda^i + 1 \cap S_{k_j}$. Therefore $t|_{\mu_j \cap S_{k_j}} = t'|_{\mu_j \cap S_{k_j}}$ as both $t$ and $t'$ are strictly increasing along rows and columns.

It follows that $t'(r - j + k_j, r - j + 1) \not\subset t(\mu_j \cap S_{k_j-1})$, since this number is bigger than all numbers in $t'(\lambda^i + 1 \cap S_{k_j}) = t(\mu_j \cap S_{k_j})$, thus greater that numbers in $t(\mu_j \cap S_{k_j-1})$. It then follows that $t(\mu_j \cap S_{k_j-1}) = t'(\mu_j \cap S_{k_j-1})$, therefore $t|_{\mu_j \cap S_{k_j-1}} = t'|_{\mu_j \cap S_{k_j-1}}$. Now one can check by induction that $t|_{\mu_j \cap S_k} = t'|_{\mu_j \cap S_k}$ for all $k \leq k_j$. Same argument shows that $t|_{\mu_j \cap S_k} = t'|_{\mu_j \cap S_k}$ for all $k \geq k_j$. \hfill $\Box$

**Proof of Theorem 5.3** We prove by induction that $Trop(W_{k}(C)) = W_{k}(\Gamma)$ for all $0 \leq k \leq r$. The $k = 0$ case is in Theorem 2.8. Now assume $Trop(W_{j}(C)) = W_{j}(\Gamma)$, we need to show that $Trop(W_{j}(C) \cap W^{\lambda_{j+1}}(C)) = Trop(W_{j+1}(C)) = W_{j+1}(\Gamma)$. \hfill (2)

Let $U_j$ be as in Lemma 5.2 and fix $w \in U_j$. As we only care about the local geometry near $w$, we may assume all Brill-Noether loci corresponding to $(C, P)$ (resp. $(\Gamma, P)$) are contained in a polytopal domain (resp. polytope) in $T_{N}$ (resp. $N_{R}$). We may also assume that $w$ is the origin. Take a polytope $\Lambda \subset \text{relint}(W^{\mu_j}(\Gamma))$ such that $w \in \text{relint}(\Lambda)$. According to Proposition 3.2 we
have the following commutative diagram:

$$
\begin{array}{c}
W^{\mu_j}(C)^{an} \cap \mathcal{U}_\Lambda \xrightarrow{T\text{rop}} \Lambda \\
\downarrow \pi_\Lambda \\
\tilde{U}_\Lambda \xrightarrow{T\text{rop}} \pi(\Lambda)
\end{array}
$$

where both vertical arrows are isomorphisms induced by the natural projection from $N$ to $N_\Lambda$ as in loc.cit..

Denote

$$W^{\lambda_j+1}_\Lambda = W^{\lambda_j+1}(C)^{an} \cap \mathcal{U}_\Lambda$$

and

$$W^{\lambda_j}_\Lambda, j = W_j(C)^{an} \cap \mathcal{U}_\Lambda.$$

According to Lemma 5.2 $T\text{rop}(\pi_\Lambda(W^{\lambda_j+1}_\Lambda))$ and $T\text{rop}(\pi_\Lambda(W^{\lambda_j}_\Lambda))$ intersect properly in $\pi(\Lambda)$, which is a polytope of maximal dimensional in $(N_\Lambda)_R$ that contains $\pi(w)$ as an interior point.

Hence Theorem 1.2 implies that $\pi(w) \in T\text{rop}(\pi_\Lambda(W^{\lambda_j+1}_\Lambda) \cap \pi_\Lambda(W^{\lambda_j}_\Lambda))$, and that

$$w \in T\text{rop}(W^{\lambda_j+1}_\Lambda \cap W^{\lambda_j}_\Lambda) \subset T\text{rop}(W_j(C) \cap W^{\lambda_j+1}(C)).$$

As $U_j$ is dense in $W^{\lambda_j+1}(\Gamma)$ and $U_j \subset T\text{rop}(W_j(C) \cap W^{\lambda_j+1}(C))$, we have $W^{\lambda_j+1}(\Gamma) \subset T\text{rop}(W_j(C) \cap W^{\lambda_j+1}(C))$. This proves (2), as the other direction of containment is trivial. □

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