Asymmetric random walks in a discrete spacetime as a model for quantum mechanics

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Abstract

1 Introduction

Despite being an extremely successful theory to predict the outcome of experiments with particles and other microscopic objects, quantum mechanics (QM) is believed by many to be incomplete or, at least, not fully understood. In particular, the “strange” or non-classical phenomena of QM, like self-interference and Born’s rule, are described in terms of abstract mathematical objects. Although some have been ready to interpret the complex-valued wavefunction as a real object, wavefunctions are generally seen as mathematical tools serving to calculate probabilities from their square moduli. Contrasting to real-valued mathematics and one-to-one mapping between real variables and observables of classical theories, the standard description is thus sometimes considered as purely operational.

In this paper a model is presented that, although very simple in terms of mathematical development and formalism, seems to be able to predict probability fields of at least free particles, and particularly self-interference, without using Schrödinger equation or wavefunctions. Unlike the standard QM picture, the proposed model only uses integer-valued quantities and arithmetic operations. In particular, it assumes a discrete spacetime under the form of an euclidean lattice. The proposed approach describes individual particle trajectories as random walks. Transition probabilities are simple functions of a few quantities that are either randomly associated to the particles during their preparation, or stored in the lattice sites they visit during the walk. Non-relativistic QM predictions are retrieved as probability distributions of similarly-prepared ensembles of particles.
The proposed model goes beyond ensemble interpretations \cite{3,4} in the sense that it describes the behavior of individual particles. With respect to De Broglie–Bohm mechanics \cite{1}, or deterministic trajectory representation \cite{2}, the proposed model introduces a non-deterministic behavior but does not appeal to non-localities. Intrinsically, determinism is already contained in stochastic interpretations of QM that are, however, mostly aimed at retrieving the Schrödinger equation from a classical equation of motion plus a stochastic force. The Born probability rule remains unexplained in this context \cite{5,6,7}, or is founded on the definition of probability density of particles as the squared intensity of an associated wave \cite{8}. This latter assumption is not used in the proposed approach, which in contrast predicts nonclassical consequences of Born rule (as double-slit interference) only from the random walk features.

The idea of lattice or discrete-time algorithms that reproduce particle propagation in the continuum limit is also not new. However, the proposed model uses the lattice only as the support for particle motion, not for wavefunctions or other mathematical operators as, e.g., in \cite{9,10,11}. While other random walks or spacetime quantizations \cite{12,13,14} are able to reproduce the emergence of Schrödinger equation from pure combinatorics, again the Born probability rule and thus interference are not explained in such models, while naturally emerges in the proposed one. In order to reproduce interference, antiparticles are not appealed to, as in some abstract lattice gas models for waves \cite{15}, nor negative probabilities.

The paper is organized as follows. Section 2 presents the assumptions concerning spacetime, which naturally lead to Heisenberg’s uncertainty principle. Section 3 describes the model for free motion without quantum forces and retrieves de Broglie relation and Schrödinger equation. Section 4 describes the model with quantum forces and shows numerical results for a self-interference scenario. Section 6 is just a short introduction to possible extensions to treat interacting particles and the transition to relativistic QM.

2 The lattice: uncertainty principle

The proposed model assumes that the spacetime is inherently discrete. Limiting for simplicity the analysis to one dimension $x$, that means that only values $x = \xi X$, $\xi \in \mathbb{Z}$ and $t = \tau T$, $\tau \in \mathbb{N}$ are meaningful. Noninteger values of space and time are simply impossible in this picture. The two fundamental quantities $X$ and $T$ are the size of the lattice that constitutes the space
and the fundamental temporal resolution, respectively.

Under this assumption, a particle trajectory consists of a succession of points \( \{\xi, \tau\} \) in the spacetime. Advance in time is unidirectional and unitary, that is, \( \tau + 1 \) follows necessarily \( \tau \). Advance in space is still unitary but bidirectional, according to two separate mechanisms. If at a time \( \tau \) a particle resides at the location \( \xi \) of the spatial lattice, the first mechanism (free motion) implies that at time \( \tau + 1 \) the particle can only reside at locations \( \xi + 1, \xi, \) or \( \xi - 1 \). A second mechanism (interference) can further increase or decrease \( \xi \) (at \( \tau + 1 \)) of one unit. The local velocities of these two mechanisms, \( \beta \) and \( \gamma \), are thus random variables that can take only the three discrete values +1, 0, and -1, as described in Sect. 3 and Sect. 4 respectively.

Consider for the moment only free motion, without interference, such that \( \xi_{\tau+1} = \xi_\tau + \beta_\tau \) (a subscript will denote in the rest of this paper the time index). The observable velocity of the particle as the result of a observation process lasting \( N \) time steps (\( N \) is arbitrary) would be

\[
\bar{v} := \frac{1}{N} \sum_{\tau=1}^{N} \beta_\tau \frac{X}{T}.
\]

The maximum velocity that a particle can reach is the speed of light \( c \). Light trajectory in the positive direction corresponds to \( \beta_\tau = 1, \forall \tau \). Consequently to \( (1) \), one constraint to the fundamental lattice quantities is necessarily

\[
\frac{X}{T} = c.
\]

Another consequence of \( (1) \) is that to determine the average velocity of a particle, an observer should wait in principle a time \( N \) tending to infinity. Every observation lasting a finite amount \( N \) of time steps will give an approximation of \( \bar{v} \). Consider, e.g., \( N = 1 \). The observed velocity can be +c, 0 or -c. Thus the uncertainty on \( \bar{v} \) is c in absolute value. For \( N = 2 \), the possible results for the sample mean are \( c, c/2, 0, -c/2, \) and \( -c \). Thus the uncertainty on \( \bar{v} \) is \( c/2 \) in absolute value. Extending these considerations, the uncertainty on \( \bar{v} \) after an observation lasting \( N \) time steps is \( c/N \) in lattice units.

Moreover, an observation lasting \( N \) time steps necessarily implies a change in the position of the particle. The span of the particle during the observation ranges from \( NX \) to \( -NX \). Thus the uncertainty on the position of the particle at the end of the observation is obviously \( 2N \) in lattice units.

Using the two results above, and denoting \( \Delta v(N) \) and \( \Delta x(N) \) the uncertainties of velocity and position as a function of observation time \( N \), the
relationship
\[ \Delta v(N) \cdot \Delta x(N) = \frac{c}{N} \cdot 2NX = \frac{2X^2}{T} \] (3)
holds.

The latter equation resembles the Heisenberg uncertainty principle since it fixes an inverse proportionality between the uncertainty with which the velocity of a particle can be known and the uncertainty with which its position can be known. Multiplying by the particle mass \( m \), and comparing (3) to Heisenberg uncertainty principle, one obtains that the two fundamental lattice quantities are related to the Planck constant,

\[ m \frac{X^2}{T} = \frac{h}{2} = \frac{\hbar}{4\pi} \] (4)

The term \( 4\pi \) (the solid angle of a sphere) holds for three dimensional spaces. In our example case of a 1D space, this term reduces to 2. Thus, combining (2) with the accordingly modified (4), the values for the fundamental lattice quantities are obtained as

\[ X = \frac{h}{2mc} \] (5)

and

\[ T = \frac{h}{2mc^2} \] (6)

Note that the Compton wavelength is retrieved as twice the fundamental lattice size \( X \).

The role of mass is not completely clear at this point. Likely, general relativity will serve to integrate it into the picture.

3 Free motion without interference

This section will first describe the equations of motion of a free particle in the proposed model. Then, the stochastic variables associated with the motion will be analyzed and characterized. Finally, the equivalence with the wavefunction picture and Schrödinger equation will be retrieved.

3.1 Lagrangian viewpoint: equations of motion

This section describes the propagation rules of a particle on the lattice. At each time \( \tau \), the particle might jump to one of the nearest neighboring sites of the lattice, or stay at rest. The actual local trajectory is not deterministic, i.e., it is not a prescribed function of previous parts of trajectory. Rather, the
local trajectory has the characteristics of a random walk. This point is very important and it implies that an intrinsic randomness affects the particle motion. Generally, there is a different transition probability for each of the three possible transitions. In free motion (without external forces), these probabilities do not change with time. Label the transition probabilities 

\[ a := \Pr(\beta_\tau = 1), \quad b := \Pr(\beta_\tau = 0), \quad c := \Pr(\beta_\tau = -1), \] 

respectively. Of course,

\[ a + b + c = 1. \quad (7) \]

Moreover, the proposed model assumes that the expected value of \( \beta \) is imprinted to the particle. This imprint is to be attributed to the preparation process and is possibly actualized every time the particle interacts with the environment (external forces). We denote as momentum propensity \( p \) this expected value,

\[ p := E[\beta] = a - c. \quad (8) \]

Another characteristic of the random motion is the expected value of the squared velocity, that is,

\[ e := E[\beta^2] = a + c \quad (9) \]

that can be reinterpreted as an energy propensity. Combining (7)–(9), obtain

\[ a = \frac{e + p}{2}, \quad b = 1 - e, \quad c = \frac{e - p}{2}. \quad (10) \]

The energy \( e \) must be a function of \( p \). A well-known result of special relativity states that energy of a particle is the sum of the rest energy and the kinetic energy. Following this suggestion, the proposed model assumes that

\[ e(p) := \frac{1 + p^2}{2}. \quad (11) \]

Consequently, (10) can be rewritten as

\[ a = \left( \frac{1 + p}{2} \right)^2, \quad b = \frac{1 - p^2}{2}, \quad c = \left( \frac{1 - p}{2} \right)^2. \quad (12) \]

The equations above describe completely the free motion, without external forces and interference, of a particle.

Define now for later use two stochastic variables related to particle motion. The first variable is the cumulated sum of the velocity that the particle experiences along the walk,

\[ X_\tau := \sum_{0}^{\tau} \beta_\tau. \quad (13) \]
The second integrator is the cumulated sum of the energy that the particle experiences,

$$ S_\tau := \sum_{0}^{\tau} |\beta_\tau| = \sum_{0}^{\tau} \beta_\tau^2. \quad (14) $$

### 3.2 Eulerian viewpoint: a priori probabilities

The equations above, in fixing the probability of each jump at each time step \( \tau \), define the trajectory of the particle as a random walk. We can now calculate the probability mass functions of the stochastic variables introduced, starting with that of the position, \( \rho^\xi_\tau \), i.e., the probability of finding the particle at time \( \tau \) at site \( \xi \). Consider the scenario where particles are emitted from a source located at the site \( \xi_0 = 0 \) of the lattice with an intrinsic value of \( p \) and thus of \( e \), determined by the preparation. Time interval between two emissions is very large, so to exclude any interactions between successive particles. Moreover, the single source excludes quantum interference. After one time step, the particle has a probability \( a \) to be at the site \( \xi = 1 \), a probability \( b \) to be at the site \( \xi = 0 \), and a probability \( c \) to be at the site \( \xi = -1 \). After two time steps, the probabilities are:

$$
\rho^2_\xi = a^2, \quad \rho^1_\xi = 2ab, \quad \rho^0_\xi = 2ac + b^2, \quad \rho^{-1}_\xi = 2bc, \quad \rho^{-2}_\xi = c^2.
$$

Note that, since the functions \( a(p) \) and \( c(p) \) are symmetric, the probability function is symmetric with respect to \( \xi = 0 \).

In general, the position probability function is described by the recursive expression

$$
\rho^\xi_\tau = a\rho^{\xi-1}_{\tau-1} + b\rho^\xi_{\tau-1} + c\rho^{\xi+1}_{\tau-1}.
$$

Deriving a closed formula for the probability mass function \( \rho^\xi_\tau \) is tedious but straightforward at this point. With the initial condition \( \rho^0_0 = 1 \) the result is

$$
\rho^\xi_\tau = \frac{2\tau}{\tau + \xi} \frac{(1 + p)^{\tau + \xi}(1 - p)^{\tau - \xi}}{(1 + p)^{2\tau}}
$$

and can be verified by inspection. From this formula, the probability that a particle is at the event horizon, i.e., \( \rho^r_\tau \), is easily retrieved as \( a^r \). Similarly, \( \rho^{-r} = c^r \).

The function \( (16) \) has a continuum limit for large \( \tau \)'s that can be derived in two equivalent ways. On the one hand, the equation of motion

$$
\xi_{\tau+1} = \xi_\tau + \beta_\tau.
$$

6
can be reviewed in the continuum limit as a stochastic differential equation reading

\[ d\xi = E[\beta] d\tau + \text{Var}[\beta] dB, \]

where \( dB \) is a Brownian motion with zero mean and unit variance. Now, from (8)–(12), the identity \( \text{Var}[\beta] = e - p^2 = b \) follows. Consequently, the continuum limit of \( \rho \) is a Gaussian function with mean \( p\tau \) and variance \( b\tau \), that is,

\[ \rho(\xi, \tau) := \lim_{\tau \to \infty} \rho_0^\tau \approx \frac{1}{\sqrt{2\pi b\tau}} \exp \left( -\frac{(\xi - p\tau)^2}{2b\tau} \right). \]  

On the other hand, consider (16) as a binomial distribution \( f(k; n, q) \) with \( k = \xi + \tau, \ n = 2\tau, \ q = (1 + p)/2. \) For \( n \) large enough an approximation of \( \rho \) is a normal distribution with mean \( \mu = nq = \tau + \tau p \) and variance \( \sigma^2 = nq(1 - q) = b\tau. \) Recalling that \( k - \mu = \xi - p\tau, \) obtain (19).

Yet a third possible method would start from expressing the recursive equation (15) as

\[ \rho_\xi^\tau - \rho_{\xi - 1}^\tau = -p \cdot \frac{\rho_{\xi + 1}^\tau - \rho_{\xi - 1}^\tau}{2} + e \cdot \frac{\rho_{\xi + 1}^\tau - 2\rho_{\xi - 1}^\tau + \rho_{\xi - 1}^\tau}{2}. \]  

The latter difference equation has a continuum limit described by the differential equation

\[ \frac{\partial \rho}{\partial \tau} = -p \frac{\partial \rho}{\partial \xi} + e \frac{\partial^2 \rho}{\partial \xi^2} \]

which is a convective–diffusion equation with \( e \) playing the role of the diffusivity and \( p \) of the convection velocity. Notice, however, that in (19) the correct result for the diffusivity is \( b = 1 - e \) and not \( e \) as it would be predicted by (21).

### 3.3 Eulerian viewpoint: site variables

The probability mass function \( \rho \) can be also regarded as \( E[O], \) i.e., the expected value of a stochastic variable (occupancy) defined as

\[ O_\xi^\tau = \begin{cases} 
1, & \text{if the particle occupies the site } \xi \\
0, & \text{otherwise}
\end{cases} \]

Analogously, one may define other stochastic site variables, such as the momentum \( P \) and and the energy \( E \) seen by a certain site at a certain time, defined as

\[ P_\xi^\tau = \begin{cases} 
1, & Pr = a\rho_{\xi - 1}^\tau \\
0, & Pr = b\rho_{\xi + 1}^\tau \\
-1, & Pr = c\rho_{\xi - 1}^\tau
\end{cases} \]

\[ E_\xi^\tau = \begin{cases} 
1, & Pe = d\rho_{\xi - 1}^\tau \\
0, & Pe = e\rho_{\xi + 1}^\tau \\
-1, & Pe = f\rho_{\xi - 1}^\tau
\end{cases} \]
and, respectively

\[ E^\xi_{\tau} = \begin{cases} 1, & \Pr = a\rho^\xi_{\tau-1} + c\rho^\xi_{\tau+1} \\ 0, & \Pr = b\rho^\xi_{\tau-1} \end{cases} \]  

(24)

Denote their respective expected values as the *average momentum* \( q := E[P] \) and the *average energy* \( \epsilon := E[E] \) (not to be confused with energy propensity \( e \)) and calculate them using (16) and (12) as

\[ q^\xi_{\tau} = \frac{\xi}{\tau}, \]  

(25)

and

\[ \epsilon^\xi_{\tau} = \frac{\xi^2 + \tau^2 - \tau}{\tau(2\tau - 1)}. \]  

(26)

Also the Lagrangian stochastic variables \( \mathcal{X} \) and \( S \) can be transformed into Eulerian variables, i.e., as functions of the site, by imposing that \( \sum_{\beta} \beta_{\tau} = \xi \). Denote the Eulerian variables with a time subscript *and* a position superscript, instead of the only time subscript as for the Lagrangian variables. Clearly, \( \mathcal{X}^\xi_{\tau} \) can only take the value \( \xi \), while \( S^\xi_{\tau} \) remains a fully stochastic variable.

### 3.4 Frequency and matter waves

In some interpretations of quantum phenomena, a particle is associated with a matter wave, whose frequency is proportional to its energy via the Planck constant. In the proposed model, the frequency is retrieved as the reciprocal of the *average return time* to any position of the lattice. To see that, define the probability mass function \( P(n) \) as the probability that a particle returns at an arbitrary position for the first time after a time \( 2n \). For example, \( P(1) = 4ac = b^2/2 \), \( P(2) = 2a^2c^2 + 2ab^2c = 5/8b^4 \), etc. The general expression for \( P(n) \) is

\[ P(n) = 2b^{2n} \left( \frac{1}{4} + \sum_{k=2}^{n} \frac{1 + 4(n-k)}{4^k} \right) = 2b^{2n} \left( \frac{n}{3} - \frac{4}{9} + \frac{13}{9} \left( \frac{1}{4} \right)^n \right) \]  

(27)

Now, define the average return time as

\[ \tau_r := E[n] = \frac{\sum_{n=1}^{\infty} nP(n)}{\sum_{n=1}^{\infty} P(n)} \]  

(28)
Using the results
\[ \sum_{k=1}^{\infty} kr^k = \frac{r}{(1-r)^2}, \quad \sum_{k=1}^{\infty} k^2 r^k = \frac{r(1+r)}{(1-r)^3}, \] (29)
one can find that
\[ \sum_{n=1}^{\infty} P(n) = \frac{2}{3} \frac{b^2}{(1-b^2)^2} - \frac{8}{9} \left( \frac{1}{(1-b^2)^2} - 1 \right) + \frac{26}{9} \left( \frac{1}{(1-b^2/4)^2} - 1 \right), \] (30)
\[ \sum_{n=1}^{\infty} n P(n) = \frac{2}{3} \frac{b^2(1+b^2)}{(1-b^2)^3} - \frac{8}{9} \frac{b^2}{(1-b^2)^2} + \frac{26}{9} \frac{b^2/4}{(1-b^2/4)^2}, \] (31)
and consequently \( \tau_r \) as a function of \( b \). Let me now introduce the energy with the substitution \( b = 1 - e \). After some tedious but straightforward manipulations of (30)–(31), find the frequency \( f = 1/\tau_r \) as
\[ f = \frac{e(2-e)(-1-e)(3-e)(e^4 - 4e^3 + 5e^2 - 2e + 1)}{(e^2 - 2e - 1)(5e^4 - 20e^3 + 29e^2 - 18e + 6)} \] (32)
that is the relationship sought. It is easy to verify (Fig. 1) that for \( e = 0 \), \( f = 0 \), while for \( e = 1 \), also \( f = 1 \). Moreover, for small values of \( e \), the relationship (32) is approximated by
\[ f = e, \] (33)
which is precisely the de Broglie relation in lattice units.

### 3.5 Wavefunction: probability

To retrieve the predictions of Schrödinger’s equation, a delicate passage in the theory is introduced. The proposed model assumes that whenever the particle interacts with the environment, the propensity \( p \) is set to the actual average momentum \( q \) of the particle, see Sect. 5.1. Now, in the free motion scenario, consider for the moment that \( p \) is a continuous variable determined randomly during the preparation at the particle source. Consequently, the probability of releasing a particle with a momentum propensity \( p \) is uniform over the interval between -1 and +1, spanning 2, and thus the probability density of the momentum propensity is \( f(p) = 1/2 \).

When the source releases a large number of particles in succession, each one with a randomly determined value of \( p \), the probability of finding a particle at the location \( \xi \), \( \tau \) is clearly given by
\[ P^\xi_\tau = \int_{-1}^{1} f(p) \rho^\xi_\tau dp = \frac{1}{2} \int_{-1}^{1} \rho^\xi_\tau dp. \] (34)
Introducing (16) into (34), and after some manipulations, obtain

\[ P_\tau^\xi = \left( \frac{2\tau}{\tau + \xi} \right) 2^{2\tau + 1} B(\tau + \xi + 1, \tau - \xi + 1) \]  

where \( B(\cdot) \) is here the Beta function,

\[ B(v, w) := \int_0^1 z^{v-1} \cdot (1 - z)^{w-1} dz. \]  

From the properties of such function, it follows that

\[ P_\tau^\xi = \left( \frac{2\tau}{\tau + \xi} \right) \frac{(\tau + \xi)!(\tau - \xi)!}{(2\tau + 1)!}, \]  

that is,

\[ P_\tau^\xi = \frac{1}{2\tau + 1}, \forall \xi \in [-\tau, \tau]. \]  

Moreover, it is easily verified that

\[ \sum_{\xi=-\tau}^{\xi=\tau} P_\tau^\xi = 1 \]  

as obviously required.
Now, compare this result with the predictions of QM, i.e., the particular solution of the Schrödinger equation. The wavefunction for a free particle is

$$\Psi(x, t) = \frac{1}{\sqrt{2\pi}} \int e^{i(kx-\omega t)} \varphi(k, 0) dk,$$  \hspace{1cm} (40)

where the wavenumber $k$ is related to the momentum of the particle and

$$\varphi(k, 0) = \frac{1}{\sqrt{2\pi}} \int \Psi(x, 0) e^{-ikx} dx.$$ \hspace{1cm} (41)

For a single perfectly localized source at $x = 0$ (41) reads

$$\varphi(k, 0) = \frac{1}{\sqrt{2\pi}}$$ \hspace{1cm} (42)

and consequently (40) is rewritten as

$$\Psi(x, t) = \frac{1}{\sqrt{2\pi}} \int \exp\left[i(kx - \frac{\hbar k^2}{2m} t)\right] dk \hspace{1cm} (43)$$

that, integrated, yields

$$\Psi(x, t) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{m}{i\hbar t}} \exp\left[i\frac{mx^2}{2\hbar t}\right].$$ \hspace{1cm} (44)

The probability density is easily calculated as

$$|\Psi(x, t)|^2 = \frac{m}{2\pi\hbar t} = \frac{m}{\hbar t}$$ \hspace{1cm} (45)

thus it is inversely proportional to time and it does not depend on $x$. Normalizing time to lattice units and using (4) allows reducing (45) to

$$\text{prob. density} = \frac{1}{2\tau X^2}.$$ \hspace{1cm} (46)

This result compares with (38), with $2\tau$ replacing $2\tau + 1$. The two functions of $\tau$ are very similar and, indeed, practically coincident for $\tau$ sufficiently large. In other terms, the square modulus of the wavefunction predicted by the Schrödinger equation is the continuum limit of the probability in the proposed model.
3.6 Wavefunction: phase

Define the action of the particle \( \sigma^\xi := E[S^\xi] \). The following recursive equation is easily derived for \( \sigma \),

\[
\sigma^\xi = \frac{1}{\rho^\tau} \left( a \rho^\xi_{\tau-1}(\sigma^\xi_{\tau-1} + 1) + b \rho^\xi_{\tau-1} \sigma^\xi_{\tau-1} + c \rho^\xi_{\tau-1}(\sigma^\xi_{\tau-1} + 1) \right),
\]

with the initial condition \( \sigma^0_0 = 0 \).

Combining (12)–(16) with (47), yields

\[
\sigma^\xi = \frac{\xi^2 + \tau^2 - \tau}{2\tau - 1} = \sigma^0_\tau + \frac{\xi^2}{2\tau - 1}.
\]

Equation (48) can be easily verified by inspection of a few points, provided that \( \{\xi, \tau\} > 0 \). For example, one easily obtains \( \sigma^1_0 = 1 \). The action of every particle reaching the point \( \{1, 1\} \) is obviously 1. Generalizing this result, clearly \( \sigma^\xi = \tau \). For a point like \( \{0, 2\} \) the prediction is less trivial. The possible values of action can be 2, if the particle follows a back-and-forth path, or 0, if it stays at rest for two time steps. Using (47), one obtains for this case \( \sigma^0_2 = 2/3 \), which is a weighted mean between the two possible values of action. Note from (26) that

\[
\sigma^\xi = \tau \xi^\xi,
\]

as expected from the usual definition of action as the integral of energy. Analogously, \( \xi = E[\mathcal{X}^\xi] = \tau q^\xi \). Finally, note that the action does not depend on \( p \). Therefore, \( \int_{-1}^1 f(p) \sigma^\xi dp = \sigma^\xi \).

Now, compare this result with the phase of the wavefunction (44). The latter, usually interpreted as the action of the particle is

\[
S(x, t) = \frac{m x^2}{2\hbar t},
\]

which, in lattice units, becomes

\[
S(\xi, \tau) = \pi \frac{\xi^2}{2\tau}.
\]

The latter equation corresponds to the second term in the right-hand side of (48), that is, \( \sigma^\xi - \sigma^0_\tau \), multiplied by \( \pi \) to obtain a phase angle. The correspondence is almost perfect, except for the term \( 2\tau - 1 \) that in the proposed model replaces the term \( 2\tau \) predicted by QM. For large values of \( \tau \), however, the two results are practically coincident. In other terms, the phase of the wavefunction predicted by the complex Schrödinger equation is the continuum limit of the action in the proposed model.
3.7 Schrödinger equation: de Broglie-Bohm formulation

The results in the previous sections have been derived for a probability density of the momentum propensity \( f(p) = 1/2 \). Consider now a generic function \( f(\cdot) \). Apply (34) to the continuum limit (large \( \tau \)'s) of the probability function \( \rho_\xi^\tau \). It can be shown that (19) approximates a Dirac delta function, whence

\[
P(\xi, \tau) := \lim_{\tau \to \infty} \int_{-1}^{1} f(p) \rho_\xi^\tau dp \approx \frac{f(\xi/\tau)}{\tau},
\]

or, equivalently, \( P(\xi, \tau) \approx \frac{f(q)}{\tau} \).

The continuous function \( P \) obeys the following partial differential equation

\[
\frac{\partial P}{\partial \tau} = -\frac{\xi}{\tau} \frac{\partial P}{\partial \xi} - \frac{1}{\tau} P = -\frac{\partial}{\partial \xi} \left( \frac{\xi}{\tau} P \right).
\]

(53)

Introducing now the continuum-limit approximation of \( \sigma \)

\[
\sigma(\xi, \tau) := \lim_{\tau \to \infty} \sigma_\xi^\tau \approx \frac{\xi^2 + \tau^2}{2\tau},
\]

(54)

in (52) one recognizes the continuity equation

\[
\frac{\partial P}{\partial \tau} = -\frac{\partial}{\partial \xi} \left( P \frac{\partial \sigma}{\partial \xi} \right)
\]

(55)

On the other hand, the relationship

\[
\frac{\partial \sigma}{\partial \tau} = -\frac{1}{2} \left( \frac{\partial \sigma}{\partial \xi} \right)^2
\]

(56)

also holds.

Sect. 3.5 has shown the equivalence of \( P \) to \( |\Psi|^2 x^2 \), while Sect. 3.6 that of \( \sigma \) with \( S/\pi = \angle \Psi/\pi \) for large \( \tau \)'s. With these two substitutions, and reintroducing physical units instead of lattice units, equations (55)–(56) become the continuity equation

\[
\frac{\partial |\Psi|^2}{\partial t} = -\frac{\partial}{\partial x} \left( |\Psi|^2 \frac{\partial S/h}{\partial x} \right),
\]

(57)

and the Hamilton–Jacobi equation of the de Broglie–Bohm formulation of QM.

\[
\frac{\partial S}{\partial t} = -\frac{1}{2m} \left( \frac{\partial S/h}{\partial x} \right)^2
\]

(58)

which in turn are equivalent to Schrödinger’s equation for a free particle.

It is also interesting to note that \( \partial \sigma/\partial \xi \) approximates \( q \) for large \( \tau \)'s, while \( \partial \sigma/\partial \tau \approx -\epsilon \).
Table 1: Pseudocode used for the simulations of Fig. 2

```
for particle \( i = 1 \) to \( N_P \)
    \( p(i) = \) random value between -1 and +1
    \( \xi(0) = 0 \)
    \( s(0) = 0 \)
    for time \( \tau = 1 \) to \( N_T \)
        \( \beta = \) random value +1, 0 or -1 with prob. given by (10)
        \( \xi(\tau) = \xi(\tau - 1) + \beta \)
        \( s(\tau) = s(\tau - 1) + |\beta| \)
    end for
    \( \nu(\xi(N_T)) = \nu(\xi(N_T)) + 1 \)
end for
\( \nu(\xi(N_T)) = \nu(\xi(N_T))/N_P \)
```

3.8 A posteriori probabilities: numerical results

In the last sections, the predictions of the proposed model were shown in closed form, using mathematical equations in terms of \emph{a priori} probabilities and probability fluxes. The probability of a number of observable were calculated and found to be in accord to the predictions of QM. Now, I will present numerical simulations of the random walk of \emph{single} particles and I will calculate the \emph{a posteriori} probabilities as frequencies over a large number of emissions. Thus, this section is aimed at reproducing numerically a true experiment.

Table 1 shows the pseudocode used for such simulations. The two for-cycles are for the successively released \( N_P \) particles, and for time up to \( N_T \). Each particle experiences the choice of two randomly-selected values: (i) the momentum propensity and (ii) at each time step, its local velocity \( \beta \) as a function of \( p \). Note that for this scenario the term \( \gamma \) is identically null. The final code line represents the counting of the particle that arrive at a certain location at time \( N_T \). From this number of arrivals, an \emph{a posteriori} frequency \( \nu(\xi) \) is calculated as the ratio to the total number of particles emitted.

Figure 2 shows the frequency \( \nu(\xi) \) after a time \( N_T = 300 \) for different values of \( N_P \). As the the number of particles emitted in the ensemble increases, a frequency distribution builds up. For large \( N_P \), the frequency clearly tends to the \emph{a priori} probability \( P(\xi, \tau = N_T) \), that is, a constant value given by (38).
Figure 2: Probability of arrival of a particle emitted at $\xi = 0$ as a function of $\xi$ after $N_T = 300$. From top-left to bottom-right, $N_P = 500, 5000, 10000$, and 50000, respectively.

4 Interference

4.1 Double-slit preparation

After having reproduced the predictions of the Schrödinger equation for a free particle, let me proceed now to a second puzzling aspect of QM: particle self-interference. Double-slit experiment usually serves to visualize this phenomenon. However, the core of self-interference is isolated and better illustrated by a double-source preparation. Instead of having a single source, a two-slit barrier, and a screen behind the barrier, I will represent the same process with two independent and mutually alternative sources of particles, separated by a certain distance $2\delta X$, and a screen. The sources are equivalent to very narrow, i.e., punctiform slits. Being in a one-dimensional space, the location of the “screen” is clearly fictitious. Pictorially, the geometry of the system can be still imagined in two dimensions. One dimension is $\xi$, ...
Figure 3: Pictorial equivalence between a double-slit experiment in two dimensions and a double-source test case in one dimension.

along which the particle move with a momentum propensity $p$. The second dimension is perpendicular to $\xi$ and is traversed by the particle with momentum propensity 1 (certainty of advancing in the positive direction). The “screen” is thus located at a distance $\tau$ from the sources. Figure 3 illustrates this equivalence.

The solution of the Schrödinger equation for this case is based on the linear superimposition of the two waveforms relative to the two sources,

$$\Psi(x, t) = \frac{\Psi_1(x, t) + \Psi_2(x, t)}{\sqrt{2}},$$

(59)

where both $\Psi_1(x, t)$ and $\Psi_2(x, t)$ are obtained from (44) by replacing the term $x^2$ (that was valid for a source at $x = 0$) with a term $(x - \delta X)^2$ and $(x + \delta X)^2$, respectively. Thus,

$$\Psi(x, t) = \frac{1}{2\sqrt{\pi}} \sqrt{\frac{m}{i\hbar t}} \left\{ \exp \left[ \frac{i m(x - \delta X)^2}{2\hbar t} \right] + \exp \left[ \frac{i m(x + \delta X)^2}{2\hbar t} \right] \right\}.$$  

(60)

The probability density is given by

$$|\Psi(x, t)|^2 = \frac{2m}{\hbar t} \cos^2 \left( \frac{S_1 - S_2}{2} \right),$$

(61)

where $S_1$ and $S_2$ are the two independent action values, that is, the phases of the two exponentials in (60). Using the equivalence $P \approx |\Psi|^2 X^2$ derived
in Sect. 3, the probability in lattice units is found as

\[ P(\xi, \tau) = \frac{1}{\tau} \cos^2 \left( \frac{\pi (\xi - \delta)^2 - (\xi + \delta)^2}{4\tau} \right) = \frac{1 + \cos \left( \frac{2\pi \delta \xi}{\tau} \right)}{2\tau}, \] (62)

and thus an interference term arises due to the presence of two possible sources. The interference is related to the phase difference between the two waveforms.

The representation of the same process in the proposed model using only the process \( \beta \) (see Sect. 3) would give just the superimposition of two probability densities of the type (58), if the same initial conditions are taken as in the single-source case. No interference term would arise in this case. The missing element is the second process \( \gamma \).

### 4.2 Lagrangian viewpoint: quantum force mechanism

The equation of motion in the general case of free motion (without external forces) reads

\[ \xi_\tau = \xi_{\tau-1} + \beta_\tau + \gamma_\tau \] (63)

where the probability mass function of \( \beta \) has been given in Sect. 3.1. Contrarily to \( a, b \) and \( c \), the transition probabilities \( \Pr(\gamma_\tau = 0, \pm 1) \) depend on the site location. The \( \gamma \)-process (quantum force) is thus equivalent to a further jump of the particle in either directions of the lattice, which is activated when the conditions below are met.

To describe quantum forces, consider the ensemble of particles emitted from either of the two sources. The proposed model assumes that each particle leaves two traces at each \( \xi \)-site of the lattice visited, these traces consisting in the values of the variables \( S \) and \( X \) defined in Sect. 3.1, i.e., the cumulated energy and the distance travelled by the particle when it passes through the site \( \xi \) at time \( \tau \). The traces are stored in the lattice site, as denoted by the following relation

\[ S_\tau \mapsto \Lambda^{\xi_\tau}, \quad X_\tau \mapsto M^{\xi_\tau}. \] (64)

A particle will continue its walk by leaving subsequent values of its traces in the lattice sites it visits. When another particle of the ensemble emitted from the two sources visits the site \( \xi \), it will have traces that are generally different from the stored values. A quantum force then arises as a function of the difference of the traces

\[ \lambda_\tau = |S_\tau - \Lambda^{\xi_\tau}| \] (65)
\[ \mu_\tau = |X_\tau - M\xi_\tau|. \]  
(66)

Define the function
\[ \nu_\tau := \frac{2}{\mu_\tau} \left( -\lambda_\tau - 2 \left\lfloor \frac{1}{2} - \frac{\lambda_\tau}{2} \right\rfloor \right), \]  
(67)

where \(\lfloor \cdot \rfloor\) denotes the “floor” function. The constant 2 in (67) reflects the number of different values of \(\mu_\tau\) experienced, i.e., the cardinality of the support of \(\mu\) that coincides with the number of sources. The proposed model assumes the following transition probabilities as a function of \(\nu_\tau\):
\[ \gamma_\tau = \begin{cases} 
1, & \text{Pr} = \frac{|\nu_\tau| + \nu_\tau}{2} \\
0, & \text{Pr} = 1 - |\nu_\tau| \\
-1, & \text{Pr} = \frac{|\nu_\tau| - \nu_\tau}{2}
\end{cases} \]  
(68)

By comparison with (8)–(11), note that \(\nu_\tau\) plays the role of (variable) momentum propensity for the \(\gamma\)-process, while \(|\nu_\tau|\) plays the role of energy.

### 4.3 Eulerian viewpoint: a priori probabilities

With the mechanism illustrated in Sect. 4.2 derive now the probability function of the particles for the two-source scenario (two lattice sources separated by \(\delta\) units). Consider for simplicity the probability density function \(\rho_\xi(\xi, \tau)\), i.e., the continuum-limit approximation of \(\rho_\xi^\tau\), as done in Sect. 3.2. What changes with respect to the derivation of (19) is that now \(E[\beta + \gamma] \neq p\) and \(Var[\beta + \gamma] \neq b\).

To calculate the expected value of \(X\), start with considering the variables \(\lambda_\tau\) and \(\mu_\tau\). These variables have the following properties: \(\Pr(\mu_\tau = 0) = 1/2\), \(\Pr(\mu_\tau = 2\delta) = 1/2\) (a property of three-dimensional walks extended for the sake of illustration to 1D walks here). On the other hand, \(E(\lambda_\tau | \mu_\tau = 0) = 0\), while

\[
E[\lambda_\tau | \mu_\tau = 2\delta] = \sigma_{\xi-\delta}^\tau - \sigma_{\xi+\delta}^\tau = \frac{(\xi_\tau + \delta)^2}{\tau - 1} - \frac{(\xi_\tau - \delta)^2}{\tau - 1} \approx \frac{2\delta \xi_\tau}{\tau} = 2\delta q_\tau.
\]  
(69)

Consequently, \(E[\mu_\tau] = \delta\), \(E[\lambda_\tau] = \delta q_\tau\). Moreover, from the definition (67), obtain
\[
E[\nu_\tau] = \frac{1}{2\delta} \left( -2\delta q_\tau - 2n_J \right) = -q_\tau - \frac{n_J}{\delta}, \quad n_J = 0, \pm 1, \ldots, \pm \lfloor \delta \rfloor
\]  
(70)
where $n_J = \left\lfloor \frac{1-2\delta q}{2} \right\rfloor$. Finally, $E[\gamma \tau] = E[\nu \tau]$.

Now evaluate $E[X]$ after having recognized that, after a sufficiently long time, $E[X] \to q\tau$ or, equivalently, that $q\tau \to q$. Since $E[\beta \tau] = p$ and $E[\gamma \tau] = \nu \tau$, the equation of motion becomes, for large $\tau$'s,

$$q = p - q + \frac{n_J}{\delta},$$

leading to various solutions of the type

$$q^{(n_J)} = \frac{p + n_J}{2}, \quad n_J = 0, \pm 1, \ldots, \pm \lfloor \delta \rfloor.$$

Therefore, under the combined effect of $\beta$ and $\gamma$ processes, the probability density of the particle (in the continuum-limit approximation) $\rho$ tends to split into $1 + 2\lfloor \delta \rfloor$ pulses $\rho^{(n_J)}$ centered at $q^{(n_J)}$. If, for example, $\delta = 1$, there will be three such pulses, centered at $q^{(0)} = p/2$ and $q^{(\pm 1)} = (\pm 1 + p)/2$. After sufficiently long time, these pulses behave like independent Gaussian-like pulses, i.e.,

$$\rho(\xi, \tau) = \sum_{n_J = -\lfloor \delta \rfloor}^{\lfloor \delta \rfloor} \rho^{(n_J)} \approx \sum_{n_J = -\lfloor \delta \rfloor}^{\lfloor \delta \rfloor} \frac{\alpha^{(n_J)}}{\sqrt{2\pi V^{(n_J)\tau}}} \exp \left( -\frac{(\xi - q^{(n_J)\tau})^2}{2V^{(n_J)\tau}} \right),$$

with the coefficients $\alpha^{(n_J)} = \int \rho^{(n_J)} d\xi$ such that

$$\sum_{n_J = -\lfloor \delta \rfloor}^{\lfloor \delta \rfloor} \alpha^{(n_J)} = 1.$$

### 4.4 A posteriori probabilities: numerical results

Instead of attempting deriving an analytical calculation of $\alpha^{(n_J)}$, $V^{(n_J)}$, and thus $\rho(\xi, \tau)$, this section presents numerical result obtained with the pseudocode of Table 2.

First, results for $N_P$ particles having the same momentum propensity $p$ are shown. Figure 4 shows the frequency of arrival of the particles at the site $\xi$ after a time $\tau = N_T$. Clearly recognizable are the two Gaussian-like pulses $\rho^{(0)}$ and $\rho^{(1)}$, while $\rho^{(-1)}$ is virtually null (positive $p$).

On the other hand, Fig. 5 shows the frequency of arrival when the momentum propensity is randomly attributed at either source, for increasing values of $N_P$. As the the number of particles emitted in the ensemble increases, a sinusoidal frequency distribution builds up. Figure 6 confirms
Table 2: Pseudocode used for the simulations of Fig. 5.

for particle \( i = 1 \) to \( N_P \)
\[ p(i) = \text{random value between -1 and +1} \]
\[ \xi(0) = \text{random value between } -\delta \text{ and } +\delta \]
\[ s(0) = 0 \]

for time \( \tau = 1 \) to \( N_T \)
\[ \beta = \text{random value +1, 0 or -1 with prob. given by (10)} \]
\[ \xi(\tau) = \xi(\tau - 1) + \beta \]
\[ s(\tau) = s(\tau - 1) + |\beta| \]
\[ \lambda(\tau) = |s(\tau) - \Lambda(\xi(\tau))| \]
\[ \mu(\tau) = |\xi(\tau) - \xi(0) - M(\xi(\tau))| \]
\[ \Lambda(\xi(\tau)) = s(\tau) \]
\[ M(\xi(\tau)) = x(\tau) - \xi(0) \]
\( \nu(\xi(\tau)) \) given by [67]
\[ \gamma = \text{random value +1, 0 or -1 with prob. given by (68)} \]
\[ \xi(\tau) = \xi(\tau) + \gamma \]
end for
\[ \nu(\xi(N_T)) = \nu(\xi(N_T)) + 1 \]
end for
\[ \nu(\xi(N_T)) = \nu(\xi(N_T))/N_P \]

Figure 4: A posteriori pdf for the case \( \delta = 1, \ p = 0.4, \ N_P = 10^5, \ N_T = 500. \)
Figure 5: A posteriori pdf for the case $\delta = 1$, random $p$, $N_T = 500$. From top-left to bottom-right, $N_P = 100$, 1000, 10000, and 100000, respectively.

Figure 6: A posteriori pdf for the case $\delta = 1$, random $p$, $N_P = 10^5$, $N_T = 500$ (black), and $P(\xi)$ calculated by QM (red).
that, for large ensembles of particles, the prediction of the proposed model tends to coincide with the QM prediction given by \[62\].

To explain such a tendential coincidence, consider Fig. 7 that shows the computed values of \(\alpha(0)\) and \(\alpha(\pm1)\) as a function of \(p\) for the case \(\delta = 1\). Clearly, the obtained functions are well approximated as

\[
\alpha(0) \approx \frac{1}{2} (1 + \cos(\pi p)),
\]

\[
\alpha(\pm1) \approx \frac{1}{2} (1 - \cos(\pi p)), \quad \text{for } p \in [0, \pm1].
\]

Now compute

\[P(\xi, \tau) = \frac{1}{2} \int \rho dp = \frac{1}{2} \int \left( \rho^{(0)} + \rho^{(1)} + \rho^{(-1)} \right) dp\]

using the fact that Gaussian pulses behave like Dirac delta functions of \(\xi\) and thus

\[
\int \rho^{(0)} dp = \frac{2}{\tau} \alpha^{(0)}(2q), \quad q \in [-1/2, 1/2],
\]

\[
\int \rho^{(1)} dp = \frac{2}{\tau} \alpha^{(1)}(2q - 1), \quad q \in [1/2, 1].
\]

\[
\int \rho^{(-1)} dp = \frac{2}{\tau} \alpha^{(-1)}(2q + 1), \quad q \in [-1, -1/2].
\]

Recalling that \(q = \xi/\tau\), finally obtain

\[P(\xi, \tau) = \frac{1}{2\tau} \left\{ \begin{array}{ll}
1 - \cos(\pi(2q + 1)), & \text{if } q \in [-1, -1/2] \\
1 + \cos(\pi 2q), & \text{if } q \in [-1/2, 1/2] \\
1 - \cos(\pi(2q - 1)), & \text{if } q \in [1/2, 1]
\end{array} \right\} = \frac{1 + \cos 2\pi q}{2\tau},
\]

which is the same as \[62\].

Similar consideration can be applied to any integer value of \(\delta\), equally leading to the retrieval of \[62\]. The extension to non-integer values of \(\delta\) requires treating trajectories that might temporarily go beyond the speed of light barrier, see Sect. 5.2. The extension to any number of sources (slits) is also possible. However, it presents several tedious technicalities and for this reason it is not shown here.
Figure 7: A posteriori functions $\alpha$’s as a function of $p$ for the case $\delta = 1$ (shown values are mirrored for negative $p$’s).

5 Discussion and Future Work

The proposed approach has been proven capable of describing in a simple and realistic way trajectories of individual particles in an ensemble of similarly-prepared particles. Simple and realistic means that the ontology of the proposed model includes real particles, a real discrete spacetime, both capable of storing a few pieces of information (momentum propensity, traces, etc.), and arithmetic operations. The predictions of the model have been shown to tend to the predictions of QM in the continuum limit for free particles and, most remarkably, also in the case of quantum interference. In the opinion of the author, the latter evidence makes the model a successful candidate to provide both qualitative and quantitative explanation for several quantum phenomena.

Besides interference, other QM aspects have still to be added to the model. Two possible extensions are briefly discussed below.

5.1 Interacting Particles

The results of this paper concern free particles only. However, the proposed model seems naturally capable to integrate also external forces into the picture. Each interaction of the particle with its surrounding is indeed expected to modify its intrinsic properties, namely, its momentum propensity.
The rule
\[ \frac{X_\tau}{\tau} \mapsto p \]  
(82)
is assumed and will be tested as a further work.

5.2 Link to Special Relativity

Special relativity is not discussed in the paper, however, there is at least
an effect that can be briefly addressed. In the proposed model, the aver-
age momentum \( q_\tau \) of the particle could temporarily become larger than 1
in absolute value (corresponding to the speed of light in natural units), by
virtue of the \( \gamma \)-mechanism that can move the particle further from its event
horizon. When this happens, the particle is “reflected” at the boundary
\( q = \pm 1 \) in the sense that its momentum propensity \( p \) switches to the value
\( \text{sign}(p) - p \). This additional rule is particularly relevant for cases with frac-
tional \( \delta \) (such as \( \delta = 1/2, 3/2, \ldots \)). This is due to the fact that, for such
cases, the function \( \nu \) is nonzero for \( q = \pm 1 \) or it has a discontinuity there.
The reflection of the momentum propensity is almost equivalently modeled
with letting the particle diffuse in the regions \( |q| > 1 \) and then mirroring
the arrival frequency obtained at the end of the simulation. An example of
such result is shown in Fig. 8 for \( \delta = 5/2 \). The figure clearly shows that
the mirroring technique is not strictly equivalent to the QM result expected
for values of \( \xi \) close to the event horizon. This phenomenon thus requires
further investigation.

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Figure 8: A posteriori pdf for the case $\delta = 5/2$, random $p$, $N_P = 10^5$, $N_T = 500$ (black), and $P(\xi)$ calculated by QM (red).

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