A Note on Distance-Preserving Graph Sparsification

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Abstract

We consider problems of the following type: given a graph \( G \), how many edges are needed in the worst case for a sparse subgraph \( H \) that approximately preserves distances between a given set of node pairs \( P \)? Examples include pairwise spanners, distance preservers, reachability preservers, etc. There has been a trend in the area of simple constructions based on the hitting set technique, followed by somewhat more complicated constructions that improve over the bounds obtained from hitting sets by roughly a log factor. In this note, we point out that the simpler constructions based on hitting sets don’t actually need an extra log factor in the first place. This simplifies and unifies a few proofs in the area, and it improves the size of the +4 pairwise spanner from \( \tilde{O}(np^{2/7}) \) [Kavitha Th. Comp. Sys. ’17] to \( O(np^{2/7}) \).

* A previous version of this paper claimed a proof of a +4 all-pairs additive spanner on \( O(n^{7/5}) \) edges, in addition to the other results presented here. With apologies, this result has been retracted due to a fatal bug in the argument. As of this writing, state-of-the-art for the +4 spanner is \( O(n^{7/5}) \) edges, by Chechik [7].

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1 Introduction

In graph algorithms, an effective preprocessing technique is to replace a large input graph with a “similar” smaller graph, which can thus be stored or analyzed more efficiently in place of the original. In this paper, we will specifically study sparsification problems where the goal is to find a sparse subgraph that approximately preserves some shortest path distances of the input. Our focus will be on the following objects:

Definition 1 (Sparsifier Variants [2,9,10,17,18]). Given a (possibly directed/weighted) graph $G = (V,E)$ and a set of demand pairs $P \subseteq V \times V$, a subgraph $H$ is a $(+k)$ pairwise spanner of $(G,P)$ if we have

$$\text{dist}_H(s,t) \leq \text{dist}_G(s,t) + k \quad \text{for all } (s,t) \in P.$$

We say that a particular demand pair $(s,t)$ is satisfied by a subgraph $H$ when the above inequality holds for $s,t$. When $k = 0$, i.e. distances between demand pairs are preserved exactly, we say that $H$ is a distance preserver. When $k \to \infty$, i.e. the only requirement is to preserve reachability between demand pairs, $H$ is called a reachability preserver.

We will use the general term sparsifier to be deliberately ambiguous to which of these objects is in play, so that we may speak about all of them at once. For applications of these various sparsifiers to algorithms, data structures, routing schemes, etc., we refer to the recent survey [3]. The most common goal in this area is to bound the extremal tradeoff between the error budget $k$, the number of demand pairs $|P|$, the number of nodes in the input graph $n$, and the number of edges needed in the sparsifier $|E(H)|$ (ideally $|E(H)|$ is as small as possible).

We will discuss a variant on the problem called sparsifiers with slack, in which only a constant fraction of the demand pairs need to be satisfied:

Definition 2 (Sparsifiers with Slack). Given a graph $G$ and demand pairs $P$, a subgraph $H$ is a sparsifier with slack if there is $P' \subseteq P, |P'| = \Omega (|P|)$, such that $H$ is a sparsifier of $(G,P')$.

See [6,11,14] for some prior work on various sparsifiers with slack; this is also closely related to the “for-each” setting studied for spectral sparsifiers and related objects [5,9]. We will say “complete” sparsifier when we want to emphasize that we mean a standard sparsifier, with all demand pairs satisfied, rather than a sparsifier with slack. This paper is driven by the following observation relating the two settings:

Lemma 1 (Main Lemma). Let $a, b, c > 0$ be absolute constants, let $G$ be an $n$-node input graph, and let $p^*$ be a parameter. Additionally suppose:

- there is a complete sparsifier on $O(n^a)$ edges for any set of demand pairs of size $|P| \leq p^*$, and
- there is a sparsifier with slack on $O(n^b|P|^c)$ edges for any set of demand pairs of size $|P| \geq p^*$.

Then there is a complete sparsifier on $O(n^a + n^b|P|^c)$ edges.

Proof. Let $\alpha$ be an absolute constant such that the sparsifier with slack satisfies at least an $\alpha$ fraction of the given demand pairs. While $|P| \geq p^*$, compute a sparsifier with slack on $O(n^b|P|^c)$ edges, remove the satisfied demand pairs from $P$, and then repeat on the remaining demand pairs. Once $|P| \leq p^*$, compute one final complete sparsifier on $O(n^a)$ edges, and then union all computed sparsifiers together.

Let $P_i$ denote the demand pairs remaining in the $i^{th}$ round (the initial set of demand pairs is $P_0$). To bound the total size of the sparsifiers with slack $\{H_i\}$ computed in each round, we have:

$$\left| \bigcup_i E(H_i) \right| \leq \sum_i |E(H_i)| = \sum_i O\left( n^b |P_i|^c \right) \leq \sum_i O\left( n^b (|P_0|(1-\alpha)^i)^c \right) \quad \alpha \text{ fraction satisfied each round}$$

$$= O\left( n^b |P_0|^c \right) \cdot \sum_i (1-\alpha)^{ic} = O\left( n^b |P_0|^c \right) \quad \text{telescoping sum.}$$

So we pay $O(n^b|P|^c)$ for the sparsifiers with slack, and $O(n^a)$ for the final complete sparsifier, completing the proof. \qed
The proof now follows by setting $\ell$ on $\leq$.

**Proof.**

The spanner is actually a slightly improved result here; the previous bound was corresponding constructions in prior work, are given in the body of the paper below. The +4 pairwise preserver on $O$ preserver with slack on $O$ distance preservers:

We now begin proving the pieces of Theorem 2. We start with the following foundational result in $2$ Pairwise Sparsifiers with Slack

Theorem 3

Any $n$-node directed weighted graph $G$ and $|P| = p$ demand pairs have a distance preserver on $O(np^{1/2})$ edges.

Before giving a more involved proof of this theorem, Coppersmith and Elkin point out a simple construction that nearly works, which can easily be converted to the following:

Theorem 4

Any $n$-node directed weighted graph $G$ and $|P| = p$ demand pairs have a distance preserver with slack on $O(np^{1/2})$ edges.

**Proof.** Let $\ell$ be a parameter, and say that a demand pair $(s,t) \in P$ is “short” if it has a shortest path on $\leq \ell$ edges, or “long” otherwise.

- For any short demand pair $(s,t) \in P$, add all edges of a shortest path $\pi(s,t)$ to the distance preserver (cost $O(p\ell)$).

- To handle the long demand pairs $(s,t) \in P$, let $R$ be a random sample of nodes obtained by including each node independently with probability $\ell^{-1}$, and add in- and out- shortest path trees rooted at each $r \in R$ (expected cost $O(n^2/\ell)$). With constant probability or higher we sample a node $r \in R$ on a shortest path $\pi(s,t)$, and thus there is a shortest $s \to t$ path included between the in- and out- shortest path trees rooted at $r$. So each long demand pair is satisfied with constant probability or higher.

The proof now follows by setting $\ell := n/p^{1/2}$, giving total cost $|E(H)| = O \left( \ell p + n^2/\ell \right) = O \left( np^{1/2} \right)$.

In fact, by Lemma[1] Theorem[4] implies Theorem[3] and so this simpler proof suffices for Theorem[6]. For another example along these lines, the following facts are proved in[2]:

2 Pairwise Sparsifiers with Slack

We now begin proving the pieces of Theorem[2]. We start with the following foundational result in distance preservers:

Theorem 3 (Informal). The following theorems all have simple proofs based on the hitting set technique. Let $n$ be the number of nodes in the input graph and $p$ the number of demand pairs.

- Every (possibly directed and weighted) graph has a distance preserver on $O(np^{1/2})$ edges. [4]
- Every (possibly directed) graph has a reachability preserver on $O((np)^{2/3} + n)$ edges. [2]
- Every undirected unweighted graph has a +2 pairwise spanner on $O(np^{1/3})$ edges. [1, 14, 15]
- Every undirected unweighted graph has a +4 pairwise spanner on $O(np^{2/7})$ edges. [13]
- Every undirected unweighted graph has a +6 pairwise spanner on $O(np^{1/4})$ edges. [13, 15]

Full proofs of these theorems, which are mostly just simplified and unified expositions of the corresponding constructions in prior work, are given in the body of the paper below. The +4 pairwise spanner is actually a slightly improved result here; the previous bound was $\tilde{O}(np^{2/7})$ [13], as the log factors from the hitting set technique had not been previously shaved.
Theorem 5 ([2]).

1. Any \(|P| = p\) demand pairs in an \(n\)-node directed graph \(G = (V, E)\) has a reachability preserver on \(O((np)^{2/3} + n)\) edges.

2. When \(P \subseteq S \times V\) for some subset of \(|S| = s\) nodes, there is a distance preserver on \(O((nps)^{1/2} + n)\) edges.

The two parts of this theorem are proved separately in [2], each using somewhat involved arguments. We show that the former actually follows from the latter, thus cutting the work in half.

Proof of Theorem 5.1, given Theorem 5.2. When \(p = O(n^{1/2})\), we trivially have \(P \subseteq S \times V\) for some node subset of size \(s \leq p\). Hence, by Theorem 5.2, there is a complete reachability preserver on \(O((np^2)^{1/2} + n) = O(n)\) edges. When \(p = \Omega(n^{1/2})\), we construct a reachability preserver with slack as follows. Like before, let \(\ell\) be a parameter, and say that a demand pair \((s, t)\) is “short” if its shortest path (or any canonical choice of \(s \rightsquigarrow t\) path will work here) has length \(\leq \ell\), or “long” otherwise.

- To handle the short pairs \((s, t)\), add the \(\leq \ell\) edges of a path to the preserver (cost \(O(p\ell)\)).
- To handle the long pairs \((s, t)\), randomly sample a set of nodes \(R\) by including each node independently with probability \(\ell^{-1}\). Let \(P_R\) denote the demand pairs \((s, t)\) whose shortest path intersects a node \(r \in R\), and note that each long pair \((s, t)\) is in \(P_R\) with at least constant probability. We then split each such pair \((s, t) \in P_R\) into two pairs \((s, r), (r, t)\) and add two reachability preservers via Theorem 5.2, to handle all pairs of the form \((s, r)\) and then all pairs of the form \((r, t)\), for cost

\[
O\left(\sqrt{|R|/p_R}|n + n\right) = O\left(np^{1/2}/\ell^{1/2} + n\right).
\]

The proof now follows by setting \(\ell := n^{2/3}/p^{1/3}\), giving total cost

\[
|E(H)| = O\left(p\ell + np^{1/2}/\ell^{1/2} + n\right) = O\left(n^{2/3}p^{2/3} + n\right).
\]

We next turn to pairwise spanners. The following auxiliary lemma will be useful. Let us say that a \(d\)-initialization of a graph \(G\) is a subgraph \(H\) obtained by arbitrarily choosing \(d\) edges incident to each node in \(G\) and including them in \(H\), or including all edges incident to a node of degree \(\leq d\) (this simplifying technique, which replaces the standard clustering step, was first used in [15]).

Lemma 6 (e.g. [7], [15] and others). If \(H\) is a \(d\)-initialization of an undirected unweighted graph \(G\), and there is a shortest path \(\pi\) in \(G\) that is missing \(x\) edges in \(H\), then there are \(\Omega(xd)\) total nodes adjacent in \(H\) to any node in \(\pi\).

Proof. Note that any node \(y\) is adjacent to at most three nodes in \(\pi\), since otherwise there is a path of length 2 (passing through \(y\)) between the first and last such node, which is shorter than the corresponding subpath in \(\pi\). Additionally, for each edge \((u, v) \in \pi \setminus H\), there must be \(\geq d\) edges in \(H\) incident to \(u, v\) since we did not choose to add \((u, v)\) itself in the initialization. Thus, we have:

\[
|\{x \mid x \text{ adjacent to } \pi\}| \geq \frac{\sum_{(u, v) \in \pi \setminus H} \deg_H(u)}{3} = \Omega(xd).
\]

Using this, we now give some hitting-set-based pairwise spanner constructions. We will first prove:

Theorem 7 ([1], [14]). Every set of \(|P| = p\) demand pairs in an \(n\)-node graph \(G\) has a \(+2\) pairwise spanner on \(O(np^{1/3})\) edges.

Kavitha and Varma [14] implicitly proved a pairwise spanner with slack of this quality, while the complete version was subsequently proved in [1] with a more involved argument. The former proof is:

Theorem 8 ([14]). Every set of \(|P| = p\) demand pairs in an \(n\)-node graph \(G\) has a \(+2\) pairwise spanner with slack on \(O(np^{1/3})\) edges.
Proof. Let $\ell, d$ be parameters, and like before, say that a demand pair $(s, t) \in P$ is “short” if its shortest path is currently missing $\leq \ell$ edges in the spanner, or “long” otherwise. Start the spanner as a $d$-initialization of $G$ (cost $O(nd)$). Then:

- For the short pairs $(s, t)$, add the $\leq \ell$ missing edges of a shortest path to the spanner (cost $O(p\ell)$).

- To handle the long pairs $(s, t)$, randomly sample a set of nodes $R$ by including each node independently with probability $(ld)^{-1}$. Add to the spanner a shortest path tree rooted at each $r \in R$ (cost $O(n^2/(ld))$). By Lemma 4 there are $\Omega(ld)$ nodes adjacent to the shortest $s \sim t$ path, so with constant probability or higher, we sample a node $r \in R$ adjacent to a node $u$ on this shortest path. In this event, we compute:

$$
\text{dist}_H(s, t) \leq \text{dist}_H(s, r) + \text{dist}_H(r, t) \quad \text{triangle inequality}
= \text{dist}_G(s, r) + \text{dist}_G(r, t) \quad \text{shortest path inequality at $r$}
\leq \text{dist}_G(s, u) + \text{dist}_G(u, t) + 2 \quad \text{triangle inequality}
= \text{dist}_G(s, t) + 2 \quad u \text{ on shortest $s \sim t$ path.}
$$

To complete the proof we then set $\ell := n/p^{2/3}$ and $d := p^{1/3}$, giving

$$
|E(H)| = O(nd + p\ell + n^2/(ld)) = O(np^{1/3}).
$$

\[\square\]

A similar story holds for the $+\ell$ pairwise spanner. Kavitha [13] proved:

**Theorem 9 ([13]).** Every set of $|P| = p$ demand pairs in an $n$-node graph $G$ has a $+\ell$ pairwise spanner on $O(np^{1/4})$ edges.

Kavitha also mentions a simpler proof that results in a pairwise spanner with slack on $O(np^{1/4})$ edges. By our Lemma 11 in fact, this simpler proof implies Theorem 9. The spanner with slack is constructed by reduction to the following key lemma in the area, which has been repeatedly rediscovered:

**Theorem 10 ([3][10][12][19]).** For every $n$-node undirected unweighted graph $G = (V, E)$ and set of demand pairs with the structure $P = S \times S$ for some $S \subseteq V, |S| = s$, there is a $+2$ pairwise spanner of $(G, P)$ on $O(ns^{1/2})$ edges.

We will not recap the proof of Theorem 10 here. Given this theorem, the spanner with slack is proved as follows:

**Theorem 11 ([13]).** Every set of $|P| = p$ demand pairs in an $n$-node graph $G$ has a $+\ell$ pairwise spanner with slack on $O(np^{1/4})$ edges.

**Proof.** Let $\ell, d$ be parameters, and start the spanner as a $d$-initialization of $G$ (cost $O(nd)$). A demand pair $(s, t) \in P$ is “short” if the shortest $s \sim t$ path is missing $\leq \ell$ edges in the spanner, or “long” otherwise.

- To handle the short pairs $(s, t)$, add the $\leq \ell$ missing edges in its shortest path to the spanner (cost $O(p\ell)$).

- To handle the long demand pairs $(s, t)$, there are two steps. First, add the first and last $\ell$ missing edges of the shortest $s \sim t$ path to the spanner (cost $O(p\ell)$). Then, randomly sample a set $R$ by including each node with probability $(ld)^{-1}$. Using Theorem 10 add a $+2$ pairwise spanner on demand pairs $R \times R$; this costs

$$
O\left(n\sqrt{|R|}\right) = O\left(n^{3/2}/\sqrt{ld}\right)
$$

edges. By Lemma 6 the added prefix and suffix of the shortest $s \sim t$ path each have $\Omega(ld)$ adjacent nodes. Thus, with constant probability or higher, we sample $r_1, r_2 \in R$ such that $r_1$ is
adjacent to \( u_1 \) in the added prefix and \( r_2 \) is adjacent to \( u_2 \) in the added suffix. In this event we can compute:

\[
\text{dist}_H(s, t) \leq \text{dist}_H(s, r_1) + \text{dist}_H(r_1, r_2) + \text{dist}_H(r_2, t) \quad \text{triangle inequality}
\]

\[
\leq \text{dist}_H(s, r_1) + \text{dist}_G(r_1, r_2) + \text{dist}_H(r_2, t) \quad R \times R + 2 \text{ pairwise spanner}
\]

\[
\leq (\text{dist}_H(s, u_1) + 1) + (\text{dist}_G(r_1, r_2) + 2) + (\text{dist}_H(u_2, t) + 1) \quad \text{triangle inequality}
\]

\[
= \text{dist}_G(s, u_1) + \text{dist}_G(r_1, r_2) + \text{dist}_G(u_2, t) + 4 \quad \text{added prefix/suffix}
\]

\[
= \text{dist}_G(s, t) + 6 \quad \text{triangle inequality}
\]

\( u_1, u_2 \) on \( s \rightsquigarrow t \) shortest path.

To complete the proof we set \( \ell := n/p^{3/4} \) and \( d := p^{1/4} \), giving

\[
|E(H)| = O \left( nd + p\ell + n^{3/2}/\sqrt{nd} \right) = O \left( np^{1/4} \right). \quad \square
\]

Finally, we discuss the +4 pairwise spanner. Kavitha \[13\] proved a +4 pairwise spanner on \( \tilde{O}(np^{2/7}) \) edges, which can easily be turned into a +4 pairwise spanner with slack on \( O(np^{2/7}) \) edges. By Lemma \[1\] in fact this implies a complete +4 pairwise spanner on \( O(np^{2/7}) \) edges, thus shaving the log factors from the original result in \[13\]. Kavitha’s proof is as follows:

**Theorem 12** (\[13\]). Every set of \(|P| = p\) demand pairs in an \( n \)-node graph has a +4 pairwise spanner with slack on \( O(np^{2/7}) \) edges.

**Proof.** Let \( \ell, d \) be parameters, and let the spanner be a \( d \)-initialization of \( G \) (cost \( O(nd) \)). This time there are three cases: a demand pair \((s, t)\) is “short” if its shortest path is missing \( \leq \ell \) edges, it is “medium” if its shortest path is missing \( > \ell \) and \( \leq n/d^2 \) edges, or it is “long” otherwise.

- To handle the short pairs \((s, t)\), add the \( \leq \ell \) missing edges of the shortest path to the spanner (cost \( O(p\ell) \)).

- To handle the long pairs \((s, t)\), randomly sample a set of nodes \( R_1 \) by including each node independently with probability \( d/n \), and add the edges of a BFS tree rooted at each \( r \in R_1 \) to the spanner (cost \( O(nd) \)). By Lemma \[6\] there are \( \Omega(n/d) \) nodes adjacent to the shortest \( s \rightsquigarrow t \) path, so with constant probability or higher we sample a node \( r \in R_1 \) adjacent to a node \( u \) on this path. In this event, we compute:

\[
\text{dist}_H(s, t) \leq \text{dist}_H(s, r) + \text{dist}_H(r, t) \quad \text{triangle inequality}
\]

\[
= \text{dist}_G(s, r) + \text{dist}_G(r, t) \quad \text{shortest path tree}
\]

\[
\leq \text{dist}_G(s, u) + \text{dist}_G(u, t) + 2 \quad \text{triangle inequality}
\]

\[
= \text{dist}_G(s, t) + 2 \quad u \text{ on a shortest } s \rightsquigarrow t \text{ path.}
\]

- There are two steps to handle the medium pairs \((s, t)\). First, add the first and last \( \ell \) missing edges in the shortest path to a spanner (cost \( O(p\ell) \)). Then, randomly sample a set of nodes \( R_2 \) by including each node independently with probability \( (\ell d)^{-1} \). For each pair of nodes \( r, r' \in R_2 \), check to see if there exist nodes \( u, u' \) adjacent to \( r, r' \) (respectively) in the current spanner \( H \) such that the shortest \( u \rightsquigarrow u' \) path is missing \( \leq n/d^2 \) edges. If so, then choose nodes \( u, u' \) with this property minimizing \( \text{dist}_G(u, u') \), and add all missing edges in the shortest \( u \rightsquigarrow u' \) path to the spanner. If no such nodes \( u, u' \) exist, then do nothing for this pair \( r, r' \). This step costs

\[
O \left( |R_2|^2 \cdot \frac{n}{d^2} \right) = O \left( \frac{n^2}{\ell^2 d^2} \cdot \frac{n}{d^2} \right) = O \left( \frac{n^3}{\ell^2 d^4} \right)
\]

edges. For a medium demand pair \((s, t)\), by Lemma \[6\] there are \( \Omega(\ell d) \) nodes adjacent to the added prefix and suffix, so with constant probability or higher we sample nodes \( r, r' \in R_2 \) adjacent to nodes \( x, x' \) on the added prefix, suffix (respectively). In this event, note that there are \( \leq n/d^2 \) missing edges on the shortest \( x \rightsquigarrow x' \) path, since \( x, x' \) are on the \( s \rightsquigarrow t \) shortest path and \((s, t)\)
is a medium pair. Thus, when \( r, r' \in R_2 \) are considered in the construction, we will indeed add a new shortest path to the spanner (as opposed to the case where we do nothing). Letting \( u, u' \) be the endpoints of this added shortest path, we compute:

\[
\begin{align*}
\text{dist}_H(s, t) &= \text{dist}_H(s, x) + \text{dist}_H(x, x') + \text{dist}_H(x', t) & x, x' \text{ on shortest } s \rightsquigarrow t \text{ path} \\
&= \text{dist}_G(s, x) + \text{dist}_H(x, x') + \text{dist}_G(x', t) & x, x' \text{ on added prefix, suffix} \\
&\leq \text{dist}_G(s, x) + (\text{dist}_H(u, u') + 4) + \text{dist}_G(x', t) & \text{triangle inequality} \\
&= \text{dist}_G(s, x) + \text{dist}_G(u, u') + \text{dist}_G(x', t) + 4 & \text{shortest } u \rightsquigarrow u' \text{ path added} \\
&\leq \text{dist}_G(s, x) + \text{dist}_G(x, x') + \text{dist}_G(x', t) + 4 & \text{dist}_G(u, u') \text{ minimal} \\
&= \text{dist}_G(s, t) + 4 & x, x' \text{ on shortest } s \rightsquigarrow t \text{ path.}
\end{align*}
\]

We then complete the proof by setting \( \ell := n/p^{5/7} \) and \( d := p^{2/7} \), giving

\[
|E(H)| = O \left( nd + p\ell + n^3/(\ell^2 d^4) \right) = O \left( np^{2/7} \right).
\]

This completes the proof(s) of Theorem 2.

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1 A technical detail here is that this step requires that shortest paths are chosen consistently, i.e. the canonical shortest \( x \rightsquigarrow x' \) path is a subpath of the canonical shortest \( s \rightsquigarrow t \) path.
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