EXTENDED NILHECKE ALGEBRAS AND SYMMETRIC FUNCTIONS
IN TYPE B

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ABSTRACT. We formulate a type B extended nilHecke algebra, following the type A construction of Naisse and Vaz. We describe an action of this algebra on extended polynomials and describe some results on the structure on the extended symmetric polynomials. Finally, following Appel, Egilmez, Hogancamp, and Lauda, we prove a result analogous to a classical theorem of Solomon connecting the extended symmetric polynomial ring to a ring of usual symmetric polynomials and their differentials.

1. Introduction

The nilHecke algebra is an object of fundamental importance in higher representation theory. Type A nilHecke algebras appear as the quiver Hecke algebras associated to a single vertex, which were shown by Lauda [Lau] to categorify the negative half of the idempotented form of the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$. The nilHecke algebras also form an essential building block of the Khovanov–Lauda–Rouquier categorification of the quantum group associated to an arbitrary Kac–Moody algebra.

An extended form of the nilHecke algebra in type A, $\text{NH}_{n}^{\text{ext}}$ was constructed by Naisse and Vaz in [NV1]. The algebra $\text{NH}_{n}^{\text{ext}}$ has an additional set of exterior algebra generators $\omega_i$, which anticommute and square to 0. This algebra was used to construct the first categorifications of Verma modules. It has a faithful representation on a ring of extended polynomials, which are the tensor product of a polynomial ring and an exterior algebra.

In [AEHL], Appel, Egilmez, Hogancamp, and Lauda study the combinatorics of this algebra and its polynomial representation, and provide additional descriptions of the ring of extended symmetric polynomials. They further show that $\text{NH}_{n}^{\text{ext}}$ is a matrix algebra over these invariants (this was also shown independently in [NV2]). They also prove an extended analogue of a theorem of Solomon ([Sol]), which relates the extended symmetric polynomials to the invariants of $\mathbb{Q}[x] \otimes \wedge[dx]$.

In this note, we formulate a type B version of the extended nilHecke algebra, $\text{bNH}_{n}^{\text{ext}}$, and of its extended polynomial representation. A new feature here is the action of the simple reflections $s_i$ and the Demazure operators $\partial_i$ on the odd polynomial generators $\omega_i$: we set

$$s_i(\omega_i) = \omega_i + (x_i^2 - x_{i+1}^2)\omega_{i+1}$$

and

$$\partial_i(\omega_j) = -\delta_{ij}(x_i + x_{i+1})\omega_{i+1}$$

for each $i$. This shifted action is more natural from the viewpoint of type B symmetric polynomials, which are generated by usual symmetric polynomials in the variables $x_i^2$, and facilitates the connection to Solomon’s theorem. We then establish several basis and dimension results about the type B extended symmetric polynomials following [AEHL].
demonstrate that $^b\text{NH}_n^{\text{ext}}$ is a matrix algebra over these invariants, and adapt the extended form of Solomon’s theorem to the type B setting.

The paper is structured as follows. In Section 2, we describe the extended polynomial representation of the Weyl group of type $B_n$, and use it to define an extended polynomial representation of $^b\text{NH}_n^{\text{ext}}$. In Section 3, we describe the extended symmetric polynomial ring. We describe a basis of extended Schur polynomials and show that $^b\text{NH}_n^{\text{ext}}$ is a matrix ring over it. Finally, in Section 4, we formulate an extended type B analogue of Solomon’s theorem, linking the extended type B symmetric polynomials to the type B invariants of $\mathbb{Q}[x] \otimes \Lambda[dx]$.

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2. THE TYPE B EXTENDED NILHECKE ALGEBRA

We review the definition of the type B Weyl group, and define an action of this group on extended polynomials. Using this action, we define a type B extended nilHecke algebra, derive a presentation with generators and relations, and prove that it has a PBW-type basis.

2.1. An action of $W_{B_n}$ on extended polynomials. The Weyl group of type $B_n$, $W_{B_n}$, is generated by $s_1, \ldots, s_n$ with relations such that $s_1, \ldots, s_{n-1}$ generate a subgroup isomorphic to $S_n$, and additional relations

$$s_n s_{n-1} s_n s_{n-1} = s_{n-1} s_n s_{n-1} s_n$$

$$s_n s_i = s_i s_n, \quad (1 \leq i \leq n - 2)$$

$$s_n^2 = 1.$$

Define the extended polynomial ring

$$P_n^{\text{ext}} = \mathbb{Q}[x_1, \ldots, x_n] \otimes \Lambda[\omega_1, \ldots, \omega_n].$$

This ring is graded with $\deg(x_i) = 2$ and $\deg(\omega_i) = -2i$.

Define an action of $W_{B_n}$ on the ring $P_n^{\text{ext}}$ of extended polynomials by setting

$$s_i(x_j) = x_{s(i)(j)}, \quad s_n(x_i) = x_i, \quad (1 \leq i \leq n - 1),$$

$$s_n(x_n) = -x_n,$$

$$s_i(\omega_j) = \omega_j + \delta_{ij}(x_i^2 - x_{i+1}^2)\omega_{i+1}, \quad (1 \leq i \leq n - 1),$$

$$s_n(\omega_n) = \omega_n$$

and letting the $s_i$ act as automorphisms.

Lemma 1. The above defines an action of $W_{B_n}$ on $P_n^{\text{ext}}$.

Proof. First, we check that the type A relations are satisfied. Note that, for any $i < n$, we have

$$s_i^2(\omega_i) = \omega_i + (x_i^2 - x_{i+1}^2)\omega_{i+1} + (x_{i+1}^2 - x_i^2)\omega_i,$$

$$-\omega_i.$$

Next, note that, for any $i < n - 1$,

$$s_i s_{i+1} s_i(\omega_i) = \omega_i + (x_i^2 - x_{i+1}^2)\omega_{i+1} + (x_{i+1}^2 - x_i^2)\omega_i + (x_{i+1}^2 - x_{i+2}^2)(x_{i+2}^2 - x_i^2)\omega_{i+2} = s_{i+1} s_i(\omega_i).$$
Clearly \( s_{i+1}s_is_{i+1}(\omega_i) = s_{i+1}s_i(\omega_i) \), so this braid relation is satisfied. Finally, we have

\[
s_is_{i+1}s_i(\omega_{i+1}) = s_is_{i+1}(\omega_{i+1}) = \omega_{i+1} + (x_i^2 - x_{i+2}^2)\omega_{i+2}.
\]

On the other hand,

\[
s_{i+1}s_is_{i+1}(\omega_{i+1}) = s_{i+1}(\omega_{i+1}) + (x_i^2 - x_{i+2}^2)\omega_{i+2} = \omega_{i+1} + (x_i^2 - x_{i+2}^2)\omega_{i+2}.
\]

Hence, all type A relations are satisfied.

Now it remains to check that the action of \( W_{B_n} \) on \( \omega_n \) satisfies the type B relations. We compute:

\[
s_{n-1}s_ns_{n-1}s_n(\omega_n) = s_ns_{n-1}s_{n-1}(\omega_n) = \omega_n.
\]

Next, note that

\[
s_{n-1}s_ns_{n-1}s_n(\omega_{n-1}) = \omega_{n-1} + 2x_{n-1}^2\omega_n.
\]

Furthermore, since both \( \omega_{n-1} \) and \( \omega_{n-1} + 2x_{n-1}^2\omega_n \) are symmetric with respect to \( s_n \), it follows that the latter is equal to \( s_ns_{n-1}s_{n-1}(\omega_{n-1}) \). It’s easy to see that the action of \( s_n \) commutes with that of \( s_i \) for any \( 1 \leq i \leq n \).

This action induces an action of type B Demazure operators. Define the Demazure operator \( \partial_i \) by

\[
\partial_i = \frac{1 - s_i}{x_i - s_i(x_i)}
\]

In particular, note that

\[
\partial_n = \frac{1 - s_n}{2x_n}.
\]

It’s easy to check the following actions: for \( 1 \leq i \leq n - 1 \) and \( 1 \leq j \leq n \) we have

\[
\partial_i(x_j) = \begin{cases} 
1 & \text{if } i = j \\
-1 & \text{if } j = i + 1 \\
0 & \text{else},
\end{cases}
\]

\[
\partial_i(\omega_j) = -\delta_{ij}(x_i + x_{i+1})\omega_{i+1}.
\]

Finally, we have

\[
\partial_n(x_j) = \delta_{jn} \quad \text{and} \quad \partial_n(\omega_j) = 0
\]

for all \( 1 \leq j \leq n \). Extend this action to an arbitrary polynomial by the Leibniz rule

\[
\partial_i(fg) = \partial_i(f)g + s_i(f)\partial_i(g).
\]

**Lemma 2.** The above defines an action of the type B Demazure operators on \( \mathbb{P}^{ext}_n \).

**Proof.** We again first check the type A relations. Note that for any \( i \),

\[
\partial_i^2(\omega_i) = \partial_i(-x_i + x_{i+1})\omega_{i+1} = 0.
\]

Now, for any \( i < n - 1 \), we have

\[
\partial_i\partial_{i+1}\partial_i(\omega_i) = \partial_i\partial_{i+1}(-(x_i + x_{i+1})\omega_{i+1})
\]

\[
= \partial_i((x_i + x_{i+1})(x_2 + x_3)\omega_3)
\]

\[
= (x_3 - x_3)\omega_3
\]

\[
= 0.
\]

Clearly \( \partial_2\partial_1\partial_2(\omega_1) = 0 \). For the other braid relation, note that

\[
\partial_2\partial_1\partial_2(\omega_2) = \partial_2(-\omega_3) = 0.
\]
whereas again $\partial_1 \partial_2 \partial_1 (\omega_2) = 0$ immediately. Hence, the type A relations are satisfied.

It remains to check the extra type B relations on generators: first note that

$$\partial_{n-1} \partial_n \partial_{n-1} \partial_n (\omega_n) = 0 = \partial_n \partial_{n-1} \partial_n \partial_{n-1} (\omega_n)$$

Also, we have

$$\partial_{n-1} \partial_n \partial_{n-1} \partial_n (\omega_{n-1}) = 0,$$

whereas

$$\partial_n \partial_{n-1} \partial_n \partial_{n-1} (\omega_{n-1}) = -\partial_n \partial_{n-1} \partial_n (\omega_n) = 0.$$

It is clear that the action of $\partial_n$, $1 \leq i \leq n$. □

2.2. **Extended nilHecke algebra.** Define the extended nilHecke algebra of type $B$, $\mathfrak{bNH}^{\text{ext}}$, to be the $\mathbb{Q}$-superalgebra generated by multiplication in $P_n^{\text{ext}}$, together with the action of the Demazure operators on extended polynomials. The generators $\omega_i$ are odd, and all other generators are even. There is an additional $\mathbb{Z}$-grading, with $\deg(x_i) = 2$, $\deg(\partial_i) = -2$, and $\deg(\omega_i) = -2i$. We give a presentation of this algebra in terms of generators and relations and prove a PBW-type theorem.

Recall the type B nilHecke algebra $\mathfrak{bNH}_n$ is the $\mathbb{Q}$-algebra generated by $\partial_1, \ldots, \partial_n$ and $x_1, \ldots, x_n$, with relations $x_i x_j = x_j x_i$ and, for $i, j < n$,

$$\partial_i^2 = 0, \quad \partial_i \partial_j = \partial_j \partial_i \text{ if } |i - j| > 1, \quad 1 \leq i \leq n,$$

$$\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}, \quad 1 \leq i < n - 1,$$

$$\partial_{n-1} \partial_n \partial_{n-1} \partial_n = \partial_n \partial_{n-1} \partial_n \partial_n,$$

$$\partial_i x_j = x_j \partial_i \text{ if } |i - j| > 1, \quad \partial_i x_i - x_{i+1} \partial_i = 1, \quad \partial_i x_{i+1} - x_i \partial_i = -1, \quad 1 \leq i < n,$$

$$\partial_n x_n + x_n \partial_n = 1, \quad x_n \partial_n + \partial_n x_n = -1.$$

This is a $\mathbb{Z}$-graded algebra with $\deg(x_i) = 2$ and $\deg(\partial_i) = -2$.

**Proposition 1.** There is an isomorphism of $\mathbb{Q}$-superalgebras

$$\mathfrak{bNH}^{\text{ext}} \cong \mathfrak{bNH}_n \times \bigwedge [\omega_1, \ldots, \omega_n]$$

with relations $x_i \omega_j = \omega_j x_i$ for all $i$ and $j$, as well as

$$\partial_i \omega_j = \omega_j \partial_i, \quad i \neq j,$$

$$\partial_i (\omega_i - x_{i+1} \omega_{i+1}) = (\omega_i - x_{i+1} \omega_{i+1}) \partial_i, \quad i < n,$$

$$\partial_n \omega_i = \omega_i \partial_n, \quad 1 \leq i \leq n.$$

**Proof.** We only need to prove the additional relations. The first two and the last relations are clear. The remaining relation follows from

$$\partial_i (\omega_i) = -(x_i + x_{i+1}) \omega_{i+1} = \partial_i (x_{i+1} \omega_{i+1}).$$

□

Using these relations and the action on $P_n^{\text{ext}}$, we obtain the following PBW basis for $\mathfrak{bNH}^{\text{ext}}$. For $w \in W_{B_n}$ with reduced expression $w = s_{i_1} \ldots s_{i_k}$, set $\partial_w = \partial_{i_1} \ldots \partial_{i_k}$.

**Theorem 1.** The superalgebra $\mathfrak{bNH}^{\text{ext}}$ has a $\mathbb{Q}$-basis given by

$$\{ x_1^{k_1} \ldots x_n^{k_n} \omega_1^{\epsilon_1} \ldots \omega_n^{\epsilon_n} \partial_w | k_i \in \mathbb{N}, \epsilon_i \in \{0, 1\}, w \in W_{B_n} \}.$$


Proof. Using the relations, we can put any given monomial in this form, so this set spans \( b \text{NilHecke}_{\text{ext}} \). But \( b \text{NilHecke}_{\text{ext}} \) acts faithfully on \( P_{\text{ext}}^n \), which implies that this set is linearly independent and thus a basis. \( \square \)

3. Extended symmetric polynomials

In this Section, we turn to investigate some of the combinatorial aspects of the extended type B nilHecke algebra and its action on extended polynomials. Following [AEHL], we define the extended symmetric polynomial ring:

\[
b \Lambda_{\text{ext}}^n := \bigcap_{i=1}^n \ker(\partial_i).
\]

We will describe several bases of this algebra and demonstrate that \( b \text{NilHecke}_{\text{ext}}^n \) is a matrix algebra over it.

3.1. Low rank examples. Here we describe the structure of the extended symmetric polynomial ring for several low rank cases. These bases are computed directly by analyzing the action of each Demazure operator on a general extended polynomial, e.g. \( \omega_1 + A \omega_2 \), etc., and provide a framework for the general structure.

\( n = 2 \): The algebra \( b \Lambda_{\text{ext}}^2 \) is a free module of rank 4 over \( B \Lambda_{\text{ext}}^2 \) with basis

\[
\{1, \omega_1 + A \omega_2, \omega_2, \omega_1 \omega_2\}
\]

where \( A \) is a solution to the system

\[
\partial_1(A) = x_1 + x_2, \quad \partial_2(A) = 0.
\]

For example, we could have \( A = x_1^2 \) or \( A = -(x_2^2 + x_3^2) \).

\( n = 3 \): The algebra \( b \Lambda_{\text{ext}}^3 \) is a free module of rank 8 over \( B \Lambda_{\text{ext}}^2 \) with basis

\[
\frac{1}{\omega_1 \omega_2 + C_1 \omega_1 \omega_3 + C_2 \omega_2 \omega_3}, \quad \frac{1}{\omega_1 \omega_3 + D \omega_2 \omega_3}, \quad \omega_1 + A \omega_2 + A_2 \omega_3, \quad \omega_2 + B \omega_3, \quad \omega_3, \quad \omega_1 \omega_2 \omega_3,
\]

where \( A_i, B, C_i, D \) satisfy

\[
\partial_3(A_i) = 0, \quad \partial_2(x) = 0, \quad \partial_2(A_2) = s_2(A_1)(x_2 + x_3) \quad \text{quad} \partial_1(A_1) = x_1 + x_2, \quad \partial_1(A_2) = 0;
\]

\[
\partial_1(B) = \partial_3(B) = 0, \quad \partial_2(B) = x_2 + x_3;
\]

\[
\partial_3(C_1) = \partial_3(C_2) = 0, \quad \partial_2(C_1) = x_2 + x_3, \quad \partial_2(C_2) = 0, \quad \partial_1(C_1) = 0, \quad \partial_1(C_2) = s_1(C_1)(x_1 + x_2);
\]

\[
\partial_3(D) = \partial_2(D) = 0, \quad \partial_1(D) = x_1 + x_2.
\]

Hence, for example, the second basis element could take the form \( \omega_1 + x_1^2 \omega_2 + x_2^2 \omega_3 \), or \( \omega_1 - (x_2^2 + x_3^2) \omega_2 + x_3^4 \omega_3 \).
3.2. **Extended Schur polynomials.** The above bases can be realized as collections of extended Schur polynomials. Let $w_0 \in W_{B_n}$ be the longest element; this takes the form

$$s_1 s_2 \ldots s_n s_{n-1} \ldots s_1 (s_2 s_3 \ldots s_n s_{n-1} \ldots s_2) \ldots (s_{n-1} s_n s_{n-1}) s_n.$$ 

For any $w \in W_B$ with reduced expression $w = s_{i_1} \ldots s_{i_k}$, recall that

$$\partial_w = \partial_{i_1} \ldots \partial_{i_k}.$$ 

For $\alpha = (\alpha_1, \ldots, \alpha_n)$ a partition (possibly with trailing zeros), denote by

$$x^{\delta+\alpha} = x_1^{2n-1+\alpha_1} x_2^{2n-3+\alpha_2} \ldots x_n^{\alpha_n}.$$ 

For a bounded strict partition $\beta$ of length $k \leq n$ with parts no larger than $n$, define

$$\omega_\beta = \omega_\beta_1 \omega_\beta_2 \ldots \omega_\beta_k.$$ 

Finally, for any such partitions $\alpha$ and $\beta$, define the extended Schur polynomial $S_{\alpha,\beta}$ by the following formula:

$$S_{\alpha,\beta} = \partial_{w_0}(x^{\delta+\alpha} \omega_\beta).$$ 

We have the following low rank examples.

**n = 2:**

1. Let $\alpha = (2n + 1, 2m + 1)$ for any positive integers $n$ and $m$. Then $x^{\alpha+\delta}$ is type B symmetric, so $S_{\alpha,\beta} = 0$ for any $\beta$.

2. We are particularly interested in the Schur polynomials where $\alpha = (0, \ldots)$. Let $\alpha = (0, 0)$ and $\beta = \emptyset$. Then $x^{\delta+\alpha} = x_1^3 x_2$, so we have

$$S_{\alpha,\beta} = \partial_1 \partial_2 \partial_1 \partial_2 (x_1^3 x_2)$$

$$= \partial_1 \partial_2 \partial_1 (x_1^3)$$

$$= \partial_1 \partial_2 (x_1^3 + x_1 x_2 + x_2^2)$$

$$= \partial_1 (x_1)$$

$$= 1.$$ 

3. Let $\beta = (1)$. Then we have

$$S_{\alpha,\beta} = \partial_2 \partial_1 \partial_2 \partial_1 (x_1^3 x_2 \omega_1)$$

$$= \partial_1 \partial_2 \partial_1 (x_1^3 \omega_1)$$

$$= \partial_1 \partial_2 (x_1^3 \omega_1 + x_1 x_2 \omega_1 + x_2^2 \omega_1 - x_2^3 (x_1 + x_2) \omega_1)$$

$$= \partial_1 (x_1 \omega_1 - x_1 x_2 \omega_2)$$

$$= \omega_1 + x_1^2 \omega_2.$$ 

Note in particular that this is one of the basis elements described in the previous section.

4. Let $\beta = (2)$. Then we have

$$\partial_1 \partial_2 \partial_1 \partial_2 (x_1^3 x_2 \omega_2) = \partial_1 \partial_2 \partial_1 (x_1^3 \omega_2)$$

$$= \partial_1 \partial_2 (x_1^3 \omega_2 + x_1 x_2 \omega_2 + x_2^2 \omega_2)$$

$$= \partial_1 (x_1 \omega_2)$$

$$= \omega_2.$$
This is another of the basis elements described above.

(5) Finally, let $\beta = (1, 2)$. Then we have
\[
\partial_1 \partial_2 \partial_1 \partial_2 (x_1^2 x_2 \omega_1 \omega_2) = \partial_1 \partial_2 (x_1^2 \omega_1 \omega_2 + x_1 x_2 \omega_1 \omega_2 + x_2^2 \omega_1 \omega_2)
= \partial_1 (x_1 \omega_1 \omega_2) = \omega_1 \omega_2.
\]

This completes the basis of $b\Lambda_2^{ext}$ described in the previous section.

Note also that
\[
S_0,0,0 \cdot S_0,0,0 = S_0,0,1, \\
S_0,0,0 \cdot S_0,0,1 = 0, \\
S_0,0,1 \cdot S_0,0,1 = 0.
\]

$p = 3$: Here we abbreviate the lengthy computations.

1. $S_{(0,0,0),\emptyset} = 1$.
2. $S_{(0,0,0),(1)} = \omega_1 + x_1^2 \omega_2 + x_1^2 x_2 \omega_3$.

In particular, we obtain a basis of the extended symmetric polynomials in this case as well.

3.3. Extended symmetric polynomials as a matrix ring. The Schur polynomials with $\alpha = (0)$ and $\beta$ of length 1 have a regular structure in terms of homogenous symmetric polynomials.

**Lemma 3.** We have
\[
S_{0,i} = \sum_{\ell \geq i} b_{h_{\ell-i}}(1, \ell - 1) \omega_\ell,
\]
where $b_{h_{\ell-i}}(1, \ell - 1)$ is the complete homogeneous symmetric polynomial in the variables $(x_1^2, x_2^2, \ldots, x_{\ell-1}^2)$.

**Proof.** This follows by similar arguments as [NV2 Lemma 2.13], using the type B longest word in place of the type A longest word and taking into account the new $\delta$. □

This immediately implies a multiplication formula following the methods in [NV2 Proposition 2.14]:

**Lemma 4.** For $\beta, \beta'$ strict partitions as above, we have
\[
S_{0,\beta} S_{0,\beta'} = S_{0,\beta \beta'},
\]
where $\beta \beta'$ is the unique strict partition that can be formed from the set $\beta \sqcup \beta'$ and $\epsilon_{\beta,\beta'}$ is the length of the minimal permutation taking $\beta \sqcup \beta'$ to $\beta \beta'$.

We also have a basis for the extended symmetric polynomials.

**Lemma 5.** The abelian group $(b\Lambda_n^{ext})_{2k}$ consisting of all elements with $k$ total nonzero $\omega$'s has a $b\Lambda_n$ basis given by $\{S_0,\nu\}$, where $\nu$ ranges over strict partitions with $k$ parts.
Proof. This is a formal argument which carries over without modification from [NV2, Proposition 2.15].

Proposition 2. There is a $\mathbb{Z}$-algebra isomorphism

$$\mathbb{b}NH^{ext} \sim \to \text{End}_{\mathbb{b}\Lambda^{ext}}(P^{ext}) \cong \text{Mat}_{q^2[2n]!!}(\mathbb{b}\Lambda^{ext}).$$

Proof. That this map is injective follows from the fact that $\mathbb{b}NH^{ext}$ acts on $P^{ext}$ via linearly independent operators.

By Lemma 5, the graded rank of $P^{ext}$ as a $\mathbb{b}\Lambda^{ext}$-module is equal to the rank of type B spin polynomials as a module over the type B spin symmetric polynomials, which is described in [JW, Proposition 5.3]. The surjectivity of the map follows. □

4. Extended Solomon’s theorem for type B

Fix an integer $n \geq 1$, and let $x = \{x_1, x_2, \ldots, x_n\}$ and $dx = \{dx_1, dx_2, \ldots dx_n\}$ be sets of formal even and odd variables, respectively. We use the following shorthand for the superpolynomials in $x$ and $dx$:

$$Q[x, dx] := \mathbb{Q}[x_1, \ldots, x_n] \otimes \bigwedge[dx_1, \ldots dx_n].$$

This ring is bigraded with $\deg(x_i) = (1,0)$ and $\deg(dx_i) = (0,1)$.

Solomon’s theorem gives the following description of the $W$-invariants of this ring for any Weyl group $W$.

Theorem 2. [Sol] For any family $f = \{f_1, \ldots, f_n\}$ of algebraically independent generators of $Q[x]^W$,

$$Q[x, dx]^W = Q[f, df]$$

where for $f \in Q[x]$,

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$

For $1 \leq i \leq n-1$, let $\alpha_i^{-1} = \frac{1}{x_i - x_{i+1}}$ and define $\alpha^{-1} = \{\alpha_1^{-1}, \ldots, \alpha_{n-1}^{-1}\}$. In [AEHL], this theorem is enlarged to the extended polynomial ring $Q[x, \omega]$ by showing that there is an $A^{NH^{ext}}$-equivariant isomorphism $Q[x, \omega] \to Q[x, dx, \alpha^{-1}]$ which induces a canonical identification of $S_n$-invariants:

$$Q[x, \omega]^{S_n} \sim \to Q[f, df].$$

We aim to prove an analogous result for the type $B$ invariants.

It is first necessary to define an action of the type $B$ divided difference operators $\partial_i$ on $Q[x, dx]$. Define the denominator $\alpha_n^{-1} = \frac{1}{x_{n}}$ and set $\alpha^{-1} = \{\alpha_1, \ldots, \alpha_n\}$ and consider the algebra $Q[x, dx, \alpha^{-1}]$. This algebra is bigraded, with $\deg(\alpha_i) = (-1,0)$.

There is an action of the type $B_n$ divided difference operators on this algebra given by the usual action on polynomials and

$$\partial_i(dx_i) = \frac{dx_i - s_i(dx_i)}{x_i - s_i(x_i)},$$
for \(1 \leq i \leq n\). Solomon’s theorem implies that for any set \(f = \{f_1, \ldots, f_n\}\) of algebraically independent generators of \(W_{B_n}\), the subalgebra \(Q[x, df]\) is closed under the action of the divided difference operators.

We will construct a map \(Q[x, dx] \to Q[x, \omega]\) which will furnish an action of \(\Lambda_3^{\text{ext}}\) on \(Q[x, df]\), and show that it is isomorphic an \(\Lambda_3^{\text{ext}}\)-module to \(Q[x, \omega]\).

As in [AEHL], call a tuple \(p \subset Q[x]\) admissible if \(p_j \in Q[x]\) \(W_{B_n}\), \(s_3(p_j) = p_j\), \(\deg(p_j) = 2(n - j)\), and \(\partial_{c[j]}(p_j) \in \mathbb{Q}^\times\) for any \(1 \leq j \leq n\), where

\[c[j] := s_{j+1}s_js_{j+2}s_{j+1}\cdots s_ns_{n-1}\]

and \(c[n] = 1\). We choose this last condition so that for an admissible tuple \(p\), the matrix

\[P = (\partial_{c[j]}(p_i))_{1 \leq i, j \leq n}\]

is upper triangular and invertible, and so that \(P\) contains the admissible tuple in its last column.

**Example 1.** Let \(n = 3\) and choose \(p = ((-1)^{3-i}h_{3-i}(x_3^2))_{i=1}^3\). Certainly the \(p_i\) have the correct degrees and are symmetric with respect to \(S_2\) and \(s_3\). Also,

\[
\partial_{c[1]}(p_1) = \partial_2\partial_1\partial_3\partial_2(x_3^4)
= \partial_2\partial_1(-(x_2^2 + x_3^2))
= \partial_2(x_1 + x_2)
= 1 \in \mathbb{Q}^\times;
\]

\[
\partial_{c[2]}(p_2) = \partial_3\partial_2(-x_3^2)
= -\partial_3(-x_2 + x_3)
= 1 \in \mathbb{Q}^\times;
\]

and \(\partial_{c[3]}(p_3) = 1\). Hence \(p\) is admissible. The corresponding matrix is

\[
P = \begin{pmatrix}
1 & -(x_2^2 + x_3^2) & x_3^4 \\
0 & 1 & -x_3^2 \\
0 & 0 & 1
\end{pmatrix}
\]

Note in particular that

\[P\omega^T = \begin{pmatrix}
\omega_1 - (x_2^2 + x_3^2)\omega_2 + x_3^4\omega_3 \\
\omega_2 - x_2^2\omega_3 \\
\omega_3
\end{pmatrix}\]

which are degree 1 basis elements for \(\Lambda_3^{\text{ext}}\) as a \(\Lambda_3\)-module.

Define the maps

\[\gamma_k(A)_{ij} = \delta_{j,k+1}A_{ik}\]

and

\[p_k(A)_{ij} = \delta_{ik}A_{k+1,j}.
\]

In other words, \(\gamma_k\) returns the \(k\)th column of \(A\) in the \(k + 1\)st column, and \(p_k\) returns the \(k + 1\) row in the \(k\)th row. We have the following characterization of admissible tuples.
Lemma 6. (1) If \( p = \{p_1, \ldots, p_n\} \) is an admissible tuple, then \( P \) satisfies
\[
\partial_{k+1} \partial_k (P) = \gamma_k (P)
\]
for any \( k = 1, \ldots, n-1 \), and \( \partial_n (P) = 0 \).
(2) For any invertible \( Q = \{q_{ij}\} \in M_n (\mathbb{Q}[x]) \) such that
\[
\partial_{k+1} \partial_k (Q) = \gamma_k (Q)
\]
for \( 1 \leq k \leq n-1 \), \( \partial_n (Q) = 0 \), and \( \deg (q_{ij}) = 2(j-i) \), the tuple \( q = \{q_{1n}, \ldots, q_{nn}\} \) is admissible and \( Q_{ij} = \partial_{c[j]} q_{in} \).

Proof. The first part follows from the fact that \( p_i \) is symmetric with respect to \( S_{n-1} \) and \( s_n \), so that
\[
\partial_{k+1} \partial_k \partial_{c[j+1]} p_i = \delta_{j,k+1} \partial_{c[j]} p_i.
\]
Indeed, this is obviously zero unless \( j = k+1 \), and, if \( k < n-1 \)
\[
\partial_{k+1} \partial_k \partial_{c[k+1]} = \partial_{k+1} \partial_k \partial_{c[k+2]} \partial_{k+1} \cdots \partial_n \partial_{n-1} = \partial_{c[k]};
\]
if \( k = n-1 \), then \( \partial_n \partial_{n-1} \partial_{c[n]} = \partial_n \partial_{n-1} = \partial_{c[n-1]} \). Clearly, \( \partial_n (P) = 0 \).

The second part follows from similar calculations using the same proof as in type A, cf [AEHIL Lemma 4.3] \( \Box \)

Example 2. Let \( n = 3 \) and \( p \) be as before. Then
\[
\partial_3 \partial_2 (P) = \begin{pmatrix} 0 & 0 & -(x_2^2 + x_3^2) \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.
\]
This is clearly \( \gamma_2 (P) \).

Also,
\[
\partial_2 \partial_1 (P) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \gamma_1 (P).
\]

We have an additional characterization of matrices of admissible tuples which requires extra notation. Let \( \Theta = \{\theta_i\} \) and \( \Xi = \{\xi_i\} \) be two sets of algebraically independent elements in \( \mathbb{Q}[x, dx] \) such that \( \deg (\theta_i) = \deg (\xi_i) = 2(n-i) \), and define an invertible matrix \( P \in M_n (\mathbb{Q}[x]) \) by the relation
\[
\Xi = P \Theta.
\]
We must therefore have \( \deg (p_{ij}) = 2(j-i) \).

Lemma 7. Any two of these equations imply the third:
(1) \( \partial_{k+1} \partial_k (P) = \gamma_k (P) \) for \( 1 \leq k \leq n-1 \), and \( \partial_n (P) = 0 \).
(2) \( \partial_k (\Xi) = 0 \) for \( 1 \leq k \leq n \).
(3) \( \partial_k (\Theta) = -\rho_k (\Theta) \) for \( 1 \leq k \leq n \).
Proof. We follow [AEHL] Lemma 4.4. Let \( k \leq n-1 \) and suppose \( \partial_{k+1}\partial_k(P) = \gamma_k(P) \). Note that we have

\[
s_{k+1}s_k(\gamma_{k+1}\gamma_k(P)) = s_{k+1}s_k(\partial_{k+1}\partial_k(P)) = \partial_{k+1}\partial_k(P) = \gamma_k(P),
\]

and it follows that \( \gamma_k(P)\Theta = s_{k+1}s_k(P)s_{k+1}s_k(\Theta) = s_{k+1}s_k(P)\rho_{k+1}\rho_k(\Theta) \). Now, using the definition of \( P \), we have that if \( \partial_k(\Xi) = 0, \partial_{k+1}\partial_k(\Xi) = 0 \), so by Lemma 7, we must have \( \partial_k(\Xi) = 0, \partial_{k+1}\partial_k(\Xi) = 0 \),

\[
0 = \partial_{k+1}\partial_k(P)\Theta + s_{k+1}(\partial_k(P))\partial_{k+1}(\Theta)
+ \partial_{k+1}s_k(P)\partial_k(\Theta) + s_{k+1}s_k(P)\partial_{k+1}\partial_k(\Theta)
= \gamma_k(P)\Theta + s_{k+1}(\partial_k(P))\partial_{k+1}(\Theta)
+ \partial_{k+1}s_k(P)\partial_k(\Theta) + s_{k+1}s_k(P)\partial_{k+1}\partial_k(\Theta).
\]

Thus,

\[
s_{k+1}s_k(P)(\rho_{k+1}\rho_k(\Theta) + \partial_{k+1}\partial_k(\Theta)) = -s_{k+1}\partial_k(P)\partial_{k+1}(\Theta) - \partial_k(s_k(P))\partial_{k+1}(\Theta).
\]

Acting on both sides by \( s_k s_{k+1} \) and using the identity \( \partial_k s_k = -s_{k+1}\partial_k \) gives

\[
P(\rho_k(\Theta) + \partial_{k+1}\partial_k(\Theta)) = -s_k\partial_k(P)\partial_{k+1}(\Theta) + s_k\partial_k(P)\partial_{k+1}(\Theta) = 0.
\]

Hence \( \partial_k(\Xi) = 0 \) if and only if \( \partial_{k+1}\partial_k(\Theta) = -\rho_k(\Theta) \), since \( P \) is invertible (and the proof applies for \( k = n \) because \( s_n(P) = P \)).

The other equivalence follows using the same elementwise techniques as in type A. \( \square \)

Finally, we may construct our isomorphism. Let \( f = \{f_1, \ldots, f_n\} \) be a set of algebraically independent generators of \( \mathbb{Q}[x]^{W_{B_n}} \), with \( \deg(f_i) = 2(n-i) \). Let \( p = \{p_1, \ldots, p_n\} \subset \mathbb{Q}[x] \) be an admissible tuple and \( P \) its associated matrix.

**Theorem 3.** For any choice of \( f \) and \( p \), there is a unique \( \mathbb{Q}[x] \)-linear homomorphism

\[
b J^f_p : \mathbb{Q}[x, \omega] \to \mathbb{Q}[x, dx, \alpha^{-1}]
\]

defined by the relation \( df = P^b J^f_p(\omega) \). Further, \( b J^f_p \) is injective and \( B \text{NH}_{n} \)-equivariant.

Proof. We follow [AEHL] Proposition 4.5. Since \( p \) is admissible, the matrix \( P \) is invertible, and thus \( b J^f_p \) is uniquely determined by this condition and linearity in \( \mathbb{Q}[x] \). Injectivity of \( b J^f_p \) follows from the invertibility of \( P \) and the algebraic independence of the sets \( f \) and \( df \).

It remains to show that \( b J^f_p \) respects the action of the divided difference operators. Since \( p \) is admissible, we have by Lemma 6 that \( \partial_{k+1}\partial_k(P) = \gamma_k(P) \). Note also that \( \partial_k(df) = 0 \), so by Lemma 7 we must have \( \partial_k(b J^f_p(\omega)) = -\rho_k(b J^f_p(\omega)) \). Then

\[
\partial_k(b J^f_p(\omega_j)) = \delta_{ij} b J^f_p(\omega_{j+1})
\]

It follows that

\[
s_k(b J^f_p(\omega_j)) = b J^f_p(\omega_j) + \delta_{jk}(x_k - x_{k+1})b J^f_p(\omega_{j+1})
\]

for all \( j, k < n \). Note that this matches the \( W_{B_n} \) action on \( \omega \). Finally, if \( k = n \), we have

\[
\partial_n(b J^f_p(\omega_j)) = b J^f_p(\omega_j)
\]

since \( \rho_k(b J^f_p(\omega_j)) = 0 \). Hence \( b J^f_p \) is \( b \text{NH}_{ext} \)-equivariant.
Hence $b J_p^f$ descends to a canonical identification of $W_{B_n}$-invariants

$$\mathbb{Q}[x, \omega]^{W_{B_n}} \cong \mathbb{Q}[x, dx]^{W_{B_n}}.$$ 

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