Probability Representation of Quantum Mechanics and the Quantizer–Dequantizer Formalism

Vladimir N Chernega\textsuperscript{1}, Olga V Man’ko\textsuperscript{1,2} and Vladimir I Man’ko\textsuperscript{1,3}

\textsuperscript{1} - Lebedev Physical Institute, Russian Academy of Sciences, Leninskii Prospect 53, Moscow 119991, Russia
\textsuperscript{2} - Bauman Moscow State Technical University, The 2nd Baumanskaya Str. 5, Moscow 105005, Russia
\textsuperscript{3} - Moscow Institute of Physics and Technology (State University), Institutskii per. 9, Dolgoprudnyi, Moscow Region 141700, Russia

E-mail: mankovi@lebedev.ru

Abstract
A review of the approach where the states of quantum systems are identified with fair probability distributions is presented. The quantizer–dequantizer operators used to construct the invertible map of the density operators onto the probability distributions are applied to obtain the kinetic equations for probability distributions identified with the quantum system states. For qubit states, the von Neumann evolution equation for the density operator is explicitly given in the form of kinetic equation for the probability distribution. Simplest tomographic probability distributions describing the states of multimode quantum oscillator are constructed.

Keywords: quantum tomography, probability representation, quantizer–dequantizer operators, qubit states.

1. Introduction
In conventional formulation of quantum mechanics, the system states are identified with the wave functions [1] and state vectors $|\psi\rangle$ [2] belonging to a Hilbert space. In the presence of interaction of the system with an environment, the system states are identified with density operators acting on the vectors in the Hilbert space [3, 4]. Different representations of the density operators like the phase-space representation [5] for systems with continuous variables (e.g., harmonic oscillators), including the Wigner function [6], Husimi–Kano function [7,8], Glauber–Sudarshan quasidistribution function [9–11], were introduced. In the phase-space representation, the states are identified with functions depending on variables $q$ and $p$, and these functions are similar in some aspects to the probability densities describing the states of systems in classical statistical mechanics.

In view of the uncertainty relations [12–14], there is no possibility to have for quantum systems the joint probability distributions of two random variables like position and momentum, because these variables can not be measured simultaneously. Thus, the above functions are not probability distributions. The Wigner function, for example, can take negative values. For systems with discrete variables like spin systems, qudits, and $N$-level atoms, the Wigner functions depending on discrete variables were considered in [15]; see also [16,17]).
The method to introduce the representations of quantum states, using the functions on the phase spaces, is the star-product quantization method. This method is based on the employment of the pair of operators acting in a Hilbert space $\mathcal{H}$ called dequantizer–quantizer operators. These operators depend on the set of parameters. Any operator acting in the Hilbert space $\mathcal{H}$ can be expressed in terms of a linear combination of the quantizer operators. The coefficients of the linear combination (called symbols of the given operator) are determined by means of dequantizer operators. Thus, the pair of dequantizer–quantizer operators depending on the set of parameters provides the possibility to construct the invertible map of the operators acting in a Hilbert space $\mathcal{H}$ on the functions (symbols of operators).

The associative product of the operators acting in a Hilbert space is mapped onto the associative product of symbols of operators. The equations for the operators like the von Neumann equation for the density operators of quantum system states are mapped onto the equations for functions (symbols of operators). In the case, where the symbols of density operators are probability distribution functions, the quantum evolution equation for density operators takes the form of kinetic equation for the probability distributions. The examples of tomographic probability distributions describing the quantum states were discussed in [18,19].

The main aim of our paper is to consider such cases of dequantizer–quantizer pairs, which map the system-state density operators onto fair probability distributions. We call such a construction the probability representation of quantum states.

This paper is organized as follows. In section 2, we review the dequantizer–quantizer method. In section 3, we consider the quantum evolution based on the scheme of quantizer-dequantizer operators. In section 4, the quantizer–dequantizer operators determining the Wigner function are studied. In section 5, the symplectic tomography of oscillator system is discussed. We describe the probability representation for systems like quantum oscillator, where the states can be identified with fair probability distributions called tomographic probability distributions. The case of systems with discrete variables like spin-1/2 or qubits will be presented in probability representation in section 6. Conclusions and prospectives are presented in section 7.

2. Dequantizer–quantizer operators and the star-product of functions

Given a Hilbert space $\mathcal{H}$ and given two sets of operators called dequantizers $\hat{U}(\vec{x}), (\vec{x} = x_1, x_2, \ldots, x_N)$ and quantizers $\hat{D}(\vec{x})$ depending on a set of parameters, which can be either continuous or discrete ones. The operators and parameters are such that, for any arbitrary operator $\hat{A}$ acting in a Hilbert space $\mathcal{H}$, one has the relations

$$f_A(\vec{x}) = \text{Tr} \hat{A}\hat{U}(\vec{x}),$$

$$\hat{A} = \int f_A(\vec{x})\hat{D}(\vec{x}) d\vec{x}.\quad (1)$$

The function $f_A(\vec{x})$ is called the symbol of operator $\hat{A}$ for the scheme with chosen dequantizer–quantizer pair $\hat{U}(\vec{x})$ and $\hat{D}(\vec{x})$. The product $(\hat{A}\hat{B})$ of two operators $\hat{A}$ and $\hat{B}$ has the symbol $f_{AB}(\vec{x})$, which is expressed in terms of symbols $f_A(\vec{x})$ and $f_B(\vec{x})$ as their star-product, i.e.,

$$f_{AB}(\vec{x}) \equiv (f_A \star f_B)(\vec{x}),\quad (3)$$

which is given in the integral form as

$$(f_A \star f_B)(\vec{x}) = \int f_A(\vec{x}_1)f_B(\vec{x}_2)K(\vec{x}_1, \vec{x}_2, \vec{x}) d\vec{x}_1 d\vec{x}_2.\quad (4)$$

If the variables $\vec{x}$ are discrete ones, then in (4) we have the corresponding sums instead of the integrals.
The kernel $K(\vec{x}_1, \vec{x}_2, \vec{x})$ is expressed in terms of dequantizers and quantizers as follows:

$$K(\vec{x}_1, \vec{x}_2, \vec{x}) = \text{Tr} \left( \hat{D}(\vec{x}_1) \hat{D}(\vec{x}_2) \hat{U}(\vec{x}) \right).$$  \hfill (5)

Simple example of the representation of the operator $\hat{A}$ by the function $f_A(\vec{x}) = A(x, x')$, in the case of one-mode oscillator system where $x_1 = x'$ and $x_2 = x$, uses the dequantizer

$$\hat{U}(x, x') = |x'\rangle\langle x|.$$

The vector $|x\rangle$ is the eigenvector of the position operator $\hat{q}$, $|x\rangle = x|x\rangle$. Since $f_A(x', x) = \text{Tr} \hat{A} |x\rangle\langle x| = \langle x|\hat{A}|x\rangle$, the operator $\hat{A}$ reads

$$\hat{A} = \int \langle x|\hat{A}|x'\rangle(|x\rangle\langle x'|) \, dx \, dx'.$$  \hfill (6)

We use the completeness relation of rectangular matrices (vectors $|x\rangle$), i.e.,

$$\int |x\rangle\langle x| \, dx = \int |x'\rangle\langle x'| \, dx' = 1$$  \hfill (7)

and the property of the product of rectangular matrices $|x\rangle$ and $\langle x'|$

$$|x\rangle f_A(x, x')\langle x'| = f_A(x, x')|x\rangle\langle x'|.$$  \hfill (8)

In this example, the quantizer and dequantizer operators are used to connect the matrix elements of operator $\hat{A}$ with its symbol.

From (6)–(8) follows that

$$\hat{D}(x, x') = |x\rangle\langle x'|.$$  \hfill (9)

One has the property $\text{Tr} \hat{D}(x, x') \hat{U}(y, y') = \delta(x - y) \delta(x' - y')$. In the case of two-dimensional Hilbert space (qubit system), the operators $|j\rangle\langle k|$, $j, k = 1, 2$ and the basis vectors $|j\rangle$, satisfying the orthogonality condition $\langle j|k\rangle = \delta_{jk}$, provide the system of dequantizer–quantizer operators $\hat{U}(j, k) = |k\rangle\langle j|$ and $\hat{D}(j, k) = |j\rangle\langle k|$.

### 3. Quantum evolution and the quantizer–dequantizer method

The evolution of the system-state density operator $\hat{\rho}(t)$ obeys the von Neumann evolution equation ($\hbar = 1$)

$$\dot{\hat{\rho}} = i \left[ \hat{\rho}, \hat{H} \right].$$

Using the dequantizer operator depending on $\vec{X}$ parameter, i.e., $\hat{U}(\vec{X})$, we obtain the symbol $f_\rho(\vec{X}) = \text{Tr} (\hat{\rho}(t)\hat{U}(\vec{X}))$ of the density operator $\hat{\rho}(t)$ obeying the equation [20]

$$\frac{\partial f_\rho(\vec{X}, t)}{\partial t} = i \text{Tr} \left\{ \hat{U}(\vec{X})\hat{\rho}(t)\hat{H} - \hat{U}(\vec{X})\hat{H}\hat{\rho}(t) \right\}. $$  \hfill (10)

Since the density operator $\hat{\rho}(t) = \int f_\rho(\vec{X}, t)\hat{D}(\vec{X})d\vec{X}$, where $\hat{D}(\vec{X})$ is a given quantizer corresponding to dequantizer $\hat{U}(\vec{X})$, we obtain the linear evolution equation for the symbol of operator $\hat{\rho}(t)$ of the form

$$\frac{\partial f_\rho(\vec{X}, t)}{\partial t} = i \int f_\rho(\vec{X}', t) \left\{ \text{Tr} \hat{H} \left( \hat{U}(\vec{X})\hat{D}(\vec{X}') - \hat{D}(\vec{X}')\hat{U}(\vec{X}) \right) \right\} \, d\vec{X}' .$$  \hfill (11)

Thus, the quantum evolution is described by the linear integral equation for the symbol of the density operator with the integral kernel determined by the trace of the product of the Hamiltonian and the commutator of dequantizer and quantizer operators. In the case of using the dequantizer–quantizer scheme, where symbol of the state density operator is a probability distribution, equation (11) takes the form of kinetic equation for the probability distribution.
4. Quantizer–dequantizer operators determining the Wigner function

Let us recall the construction of the Wigner function, e.g., of harmonic oscillator using the scheme of quantizer–dequantizer operators. The dequantizer–operator $\hat{U}(\vec{q}, \vec{p})$ has the form

$$\hat{U}(\vec{q}, \vec{p}) = \int \frac{du}{\sqrt{2\pi}} |q - u/2\rangle \langle q + u/2| e^{-ipu}. \quad (12)$$

Here $\hat{q}|x\rangle = x|q\rangle$. The quantizer operator $\hat{D}(\vec{q}, \vec{p}) = (2\pi)^{-1/2} \hat{U}(\vec{q}, \vec{p})$. One can check that

$$\text{Tr} (\hat{U}(\vec{q}, \vec{p}) \hat{D}(\vec{q}', \vec{p}')) = \delta(q - q') \delta(p - p').$$

The Wigner function is given in terms of the density operator and dequantizer (12) as follows:

$$W(q, p) = \int \text{Tr} \hat{\rho} |q - u/2\rangle \langle q + u/2| e^{-ipu} du. \quad (13)$$

The Wigner function determined by (12) reads

$$W(q, p) = \int \rho(q + u/2, q - u/2) e^{-ipu} du. \quad (14)$$

For the ground state of harmonic oscillator $\hat{\rho}_0 = |0\rangle \langle 0|$, the Wigner function is

$$W_0(q, p) = 2e^{-q^2 - p^2}. \quad (15)$$

The density matrix in the position representation is expressed in terms of the Wigner function as follows:

$$\rho(x, x') = \frac{1}{2\pi} \int W\left(\frac{x + x'}{2}, p\right) e^{ip(x - x')} dp. \quad (16)$$

The Wigner function can take negative values. For example, the first excited state of harmonic oscillator $W_1(0, 0)$ is given by the integral

$$\int \psi_1(u/2)\psi_1^*(-u/2) du = -\int |\psi_1(u/2)|^2 du < 0.$$

5. Symplectic tomography of the multimode oscillator

Now we consider the probability representation of quantum-system states with continuous variables like $N$-dimensional harmonic oscillator. We introduce two operators, first one is dequantizer

$$\hat{U}(\vec{X}, \vec{\mu}, \vec{\nu}) = \prod_j \delta(X_j \hat{1} - \mu_j \hat{q}_j - \nu_j \hat{p}_j) ; \quad j = 1, 2, \ldots N \quad (17)$$

and second one is quantizer

$$\hat{D}(\vec{X}, \vec{\mu}, \vec{\nu}) = (2\pi)^{-N} \prod_{j=1}^N \exp \left[ i \left( X_j \hat{1} - \mu_j \hat{q}_j - \nu_j \hat{p}_j \right) \right]. \quad (18)$$

Here, $-\infty < X_j, \mu_j, \nu_j < \infty$ and $\hat{q}_j$ and $\hat{p}_j$ are the position and momentum operators. These operators provide the possibility to map any operator $\hat{A}$ onto its symbol $w_A(\vec{X}|\vec{\mu}, \vec{\nu})$, using the formalism of quantizer–dequantizer operators, i.e.,

$$w_A(\vec{X}|\vec{\mu}, \vec{\nu}) = \text{Tr} \left( \hat{A} \prod_{j=1}^N \delta(X_j \hat{1} - \mu_j \hat{q}_j - \nu_j \hat{p}_j) \right). \quad (19)$$
qubit (spin-1/2) states. We consider density matrices, which play the role of dequantizers. Using the method of dequantizer operators, one can construct the probability representation of 6. Spin 1/2 evolution equations for the oscillator in [22].

The integrals of motion of the oscillator can be related to symplectic tomograms. The integral of motion for the parametric oscillator, quadratic in the position and momentum operators, was studied using the nonlinear Riccati equation and applied to analyze the solutions of quantum evolution equations for the oscillator in [22].

6. Spin 1/2 states in the probability representation

Using the method of dequantizer operators, one can construct the probability representation of qubit (spin-1/2) states. We consider density matrices, which play the role of dequantizers

\[
\rho(1) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \rho(2) = \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \quad \rho(3) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\]

In view of Born’s rule for given density matrix \(\rho^j = \rho\), \(\text{Tr} \rho = 1\), \(\rho \geq 0\) which is given as

\[
\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}, \quad \rho_{11} + \rho_{22} = 1,
\]
The numbers
\[ p_1 = \text{Tr} \rho \rho(1), \quad p_2 = \text{Tr} \rho \rho(2), \quad p_3 = \text{Tr} \rho \rho(3) \] (27)
are probabilities of spin-1/2 projections \( m = +1/2 \) on the axes \( x, y, z \), respectively. This means that the density matrix \( \rho \) can be presented in the form [23–25]
\[ \rho = \begin{pmatrix} p_1 & p_2 & p_3 \\ p_2 & p_1 - 1/2 + i(p_2 - 1/2) \\ p_3 & 1 - p_3 \end{pmatrix} \] (28)
This form shows that the qubit state with density matrix (28) can be determined by six-vector \( \vec{P} \) with components \( P_1 = k_1 p_1, \ P_2 = k_2 p_2, \ P_3 = k_3 p_3, \ P_4 = k_1(1 - p_1), \ P_5 = k_2(1 - p_2), \ P_6 = k_3(1 - p_3) \), where \( 0 \leq k_1, k_2, k_3 \leq 1 \) satisfy the condition \( k_1 + k_2 + k_3 = 1 \). For example, we can take \( k_1 = k_2 = k_3 = 1/3 \). The numbers \( k_1, k_2, k_3 \) can be interpreted as probabilities. The probabilities \( p_1, p_2, \) and \( p_3 \) satisfy the condition
\[ (p_1 - 1/2)^2 + (p_2 - 1/2)^2 + (p_3 - 1/2)^2 \leq 1/4. \] (29)
For the pure state, \( \rho^2 = \rho \) and the probabilities \( p_1, p_2, p_3 \) satisfy the condition (29) in the form of equality. The spin-1/2 density matrix satisfies the von Neumann equation with Hamiltonian matrix
\[ H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}. \] (30)
Using the explicit form of density matrix (28) and the von Neumann evolution equation, we obtain the kinetic equation for the probability six-vector \( \vec{P}(t) \) of the form
\[ \frac{d\vec{P}(t)}{dt} = \vec{M} \vec{P}(t) + \vec{f}(t). \] (31)
Here, for \( k_1 = k_2 = k_3 = 1/3 \),
\[ \vec{M} = \begin{pmatrix} 0 & H_{22} - H_{11} & -2 \text{Im} H_{12} & 0 & 0 & 0 \\ H_{11} - H_{22} & 0 & -2 \text{Re} H_{12} & 0 & 0 & 0 \\ 2 \text{Im} H_{12} & 2 \text{Re} H_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & H_{22} - H_{11} & 0 & 2 \text{Re} H_{12} \\ 0 & 0 & 0 & -2 \text{Im} H_{12} & -2 \text{Re} H_{12} & 0 \end{pmatrix} \] (32)
and
\[ \vec{f}(t) = \frac{1}{3} \begin{pmatrix} \text{Im} H_{12} \\ \text{Re} H_{12} \\ -\text{Im} H_{12} - \text{Re} H_{12} \\ \text{Im} H_{12} \\ -\text{Re} H_{12} \\ \text{Re} H_{12} \end{pmatrix}. \] (33)
Thus, the quantum evolution equation for spin-1/2 system is presented in the form of kinetic equation for the probability six-vector \( \vec{P}(t) \). At any time moment, the solution to (31) preserves the property
\[ \mathcal{P}_1(t) + \mathcal{P}_4(t) = k_1, \ \mathcal{P}_2(t) + \mathcal{P}_5(t) = k_2, \ \mathcal{P}_3(t) + \mathcal{P}_6(t) = k_3. \]
Thus, we show that the qubit quantum state can be described by the probability distribution obeying to the linear kinetic classical-like equation.
7. Conclusions
To conclude, we point out the main results of our work.

We reviewed the probability representation of quantum states based on recently introduced [26] tool of employing the pair of quantizer–dequantizer operators. The dequantizer operator is applied to map a quantum observable operator (“quantum” object) onto a function (“classical” object). We considered not exact but more or less intuitively clear semantic understanding that physical objects described by functions are usually classical objects and described by the operators acting in a Hilbert spaces are usually quantum ones. The quantizer operator is applied to map a function (“classical” object) onto quantum observable operator (“quantum” object). So, the aim of dequantizer operator is to “dequantize” and the aim of quantizer operator is to “quantize.” We point out that the standard quantization procedure used to associate the description of classical systems with the description of quantum systems has the rigorous meaning, which is not equivalent to the application of quantizer–dequantizer operators.

We reviewed a particular choice of quantizer–dequantizer operators [24,27] which, in addition to obtaining the invertible map of operators onto functions and vice-versa, provides the possibility to map density operators of quantum states onto functions, which are probability distributions of classical-like random variables. It turned out that quantum formalism like the description of quantum states by wave functions or density operators together with linear equations like Schrödinger and von Neumann equations can be reformulated and given in the form of classical-like formalism of probability distributions describing the quantum states and linear kinetic equations for the classical-like probability distributions mathematically analogous to kinetic equations used in classical statistical mechanics.

We presented the examples of quantum oscillator description and the spin-1/2 (qubit system) description in the introduced recently probability representation of quantum mechanics. The developed formalism is completely equivalent to the conventional formalism of quantum mechanics, where density operators are used in the phase-space representation (Wigner function, Husimi–Kano, and Glauber–Sudarshan quasidistributions). But the probability representation of quantum mechanics nevertheless provides the possibility to use the well-developed methods of standard probability theory to get some new relations in quantum theory like entropic-information inequalities for matrix elements of density matrices, which are not obvious in conventional formulation of quantum mechanics. These aspects of the probability representation of quantum mechanics will be studied in future publications.

References
[1] Schrödinger E 1926 Naturwissenschaften 14 664
[2] Dirac P A M 1958 The Principles of Quantum Mechanics (Clarendon Press: Oxford)
[3] Landau L 1927 Z. Phys. 45 430
[4] von Neumann J 1927 Gött. Nach. 1 245
[5] Schleich W P 2001 Quantum Optics in Phase Space (Weinheim: Wiley-VCH)
[6] Wigner E 1932 Phys. Rev. 40 749
[7] Husimi K 1940 Proc. Phys. Math. Soc. Jpn. 22 264
[8] Kano Y 1965 J. Math. Phys. 6 1913
[9] Glauber R J 1963 Phys. Rev. Lett. 10 84
[10] Sudarshan E C G 1963 Phys. Rev. Lett. 10 277
[11] Klauder J R and Sudarshan E C G 1968 Fundamentals of Quantum Optics (Benjamin: New York, USA)
[12] Heisenberg W 1927 Z. Phys 43 172
[13] Schrödinger E 1930 Ber. Kgl. Akad. Wiss. Berlin 24 296
[14] Robertson H P 1930 Phys. Rev. A 35 667
[15] Stratonovich S L 1957 Sov. Phys. JETP 4 891
[16] Muñoz C, Klimov A B and Sánchez-Soto L 2017 Quantum Inf. Process. 16 158
[17] Adam P, Andreev V A, Man’ko M A, Meckler M and Man’ko V I 2019 Star-product formalism for probability and mean-value representations of qudits Preprint arXiv:1912.06893
[18] Mancini S, Man’ko V I and Tombesi P 1995 Quantum Semiclass. Opt. 7 615
[19] Man’ko V I 1997 Quantum mechanics and classical probability theory, in: Gruber B and Ramek M (Eds.) Symmetries in Science (New York: Plenum Press) p. 225
[20] Korennoy Ya A and Man’ko V I 2017 J. Phys. A: Math. Theor. 50 155302
[21] Man’ko V I and Mendes R V 1999 Phys. Lett. A 263 53
[22] Schuch D. 2018 Quantum Theory from a Nonlinear Perspective (Springer International Publishing AG)
[23] Terra-Cuña M O, Man’ko V I and Scully M 2001 Found. Phys. 14 103
[24] Man’ko O V, Man’ko V I, Marmo G and Vitale P 2007 Phys. Lett. A 360 522
[25] Chernega V N, Man’ko O V, Man’ko V I 2017 J. Russ. Laser Res. 38 324
[26] Man’ko O V, Man’ko V I and Marmo G 2002 J. Phys. A: Math. Gen. 35 699
[27] Chernega V N, Belolipetskiy S N, Man’ko O V and Man’ko, V I 2019 J. Phys. Conf. Ser. 1348 012101