Analytical Results for Random Band Matrices with Preferential Basis

Klaus Frahm and Axel Müller-Groeling
Service de Physique de l’État condensé. Commissariat à l’Énergie Atomique Saclay, 91191 Gif-sur-Yvette, France
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Abstract

Using the supersymmetry method we analytically calculate the local density of states, the localization length, the generalized inverse participation ratios, and the distribution function of eigenvector components for the superposition of a random band matrix with a strongly fluctuating diagonal matrix. In this way we extend previously known results for ordinary band matrices to the class of random band matrices with preferential basis. Our analytical results are in good agreement with (but more general than) recent numerical findings by Jacquod and Shepelyansky.
Banded matrices with random elements play an important role in the description of both certain systems exhibiting quantum chaos (in particular the kicked rotator [1]) and electron transport in quasi 1d geometries [4]. In a series of papers [3,4], Fyodorov and Mirlin have performed a detailed analytical investigation of the properties of such matrices, making use of Efetov’s supersymmetry technique [5]. In recent times particular interest in random band matrices (RBM) with preferential basis (PB) (realized, e.g., by a very strongly fluctuating diagonal) has emerged for at least two different reasons: First, any attempt to go beyond the quasi 1d case for the electron transport problem, either by employing a Fokker–Planck approach [2] or the (equivalent [7]) σ model formulation [2,8], necessarily leads away from the isotropic situation and introduces a model with preferential basis. Second, Shepelyansky [9] recently studied the problem of two interacting particles in a 1d random potential by reducing the Hamiltonian to a RBM with PB. He found that the two–particle localization length is strongly enhanced as compared to the one–particle localization length. This result was reinforced, made more precise and further investigated by subsequent work [10,11]. Very recently, Jacquod and Shepelyansky [12] studied numerically certain properties of RBM with PB, especially the local density of states, the inverse participation ratio and the level spacing statistics.

In this letter, we extend the analytical treatment of Fyodorov and Mirlin [3,4] to the case of RBM with PB, derive explicit formulas for the local density of states and the localization length, and present expressions for the generalized inverse participation ratios and the distribution function of eigenvector components in terms of previously known results for RBM. Our results are in good agreement with the asymptotic estimates given in [12] and can also explain certain numerical deviations which occur in [12] in the limited range of accessible system parameters.

We consider the random matrix

\[ H_{ij} = \eta_{ij} \delta_{ij} + \zeta_{ij}, \quad (i, j = 1, \ldots, N), \]  

(1)

where the \( \eta_i \equiv \eta_{ii} \) are real random numbers with the distribution function \( \rho_0(\eta) \). We introduce a scale parameter \( W_b \) by setting \( \langle \eta^2 \rangle \approx W_b^2 \). The matrix \( \zeta \) is either symmetric (\( \zeta_{ij} = \zeta_{ji}, \beta = 1 \)) or Hermitian (\( \zeta_{ij} = \zeta_{ji}^*, \beta = 2 \)) with Gaussian random variables satisfying \( \langle |\zeta_{ij}|^2 \rangle = (1 + \delta_{ij}(2 - \beta)) A_{ij}/2. \) \( B_0 = \sum_r a(r) \) and \( B_2 = \sum_r a(r)r^2/2. \) The class of matrices introduced in (1) contains the case considered earlier by Fyodorov and Mirlin [4] and \( \eta \) introduces a PB. For later use we define (as in [3])

\[ B_0 = \sum_r a(r) \]  

and \( B_2 = \sum_r a(r)r^2/2. \) The class of matrices introduced in (1) contains the case considered in [4] as a particular example:

\[ \rho_0(\eta) = \frac{1}{2W_b} \Theta(W_b - |\eta|), \]

\[ a(r) = \frac{2}{3\sqrt{1 + 2b}} \Theta(b - |r|). \]

(2)

The importance of the \( \eta_i \) is governed by the ratio \( W_b/\sqrt{b} \) of the spacing \( \Delta \eta \approx W_b/b \) of those \( \eta_i \) which are coupled and the typical value \( \zeta_\text{typ} \approx b^{-1/2} \) of the coupling matrix elements. We can distinguish three important regimes: (i) \( W_b/\sqrt{b} \gg 1 \) \( \Rightarrow \) the coupling matrix elements are very weak and can be treated perturbatively, (ii) \( 1/\sqrt{b} \ll W_b/\sqrt{b} \ll 1 \) \( \Rightarrow \) the \( \eta_i \) are still much
larger than the $\zeta_{ij}$, but the coupling to the $\zeta_{ij}$ becomes nontrivial, and (iii) $W_b/\sqrt{b} \lesssim 1/\sqrt{b} \Rightarrow$ the $\eta_i$ are comparable to the $\zeta_{ij}$ and the RBM results in [3,4] are applicable.

In the regime (i), one may apply simple perturbation theory to calculate the eigenfunctions. This yields the behavior $|\psi(jb)| \sim |\psi(0)| (\zeta_{\text{typ}}/\Delta \eta)^j$. The corresponding localization length is then estimated as $\xi \sim 2b/\ln(W_b^2/b)$, a behavior that was also anticipated (and numerically confirmed) in [3]. In the present work, we are mostly concerned with regime (ii) and the crossover to (iii).

We are interested in calculating the (position dependent) generalized inverse participation ratios (we use the notation of [4]):

$$P_q(E, n) = \frac{1}{\rho(E)} \sum_k |\psi_k(n)|^{2q} \delta(E - E_k)$$

$$= \lim_{\varepsilon \to 0} \frac{i^{l+m}}{2\pi \rho} (2\varepsilon)^{q-1} (l-1)!(m-1)! \left\langle \left( G_{nn}^+ \right)^l (G_{nn}^-)^m \right\rangle_{\zeta} \quad (q = l + m).$$

(3)

To perform the average over the product of Green’s functions we use the supersymmetry method. For all details and the standard notation we have to refer the reader to [3,4].

We define a supersymmetric functional

$$F(J) = \left\langle \text{sdet}^{-1/2}(E - H + i\varepsilon \Lambda + \hat{J}) \right\rangle_{\zeta} = \left\langle \text{sdet}^{-1/2}(1 + g\hat{J}) \right\rangle_{\zeta}$$

$$= \left\langle (1 + x_+ G_{nn}^+)^{-1} (1 + x_- G_{nn}^-) \right\rangle_{\zeta},$$

(4)

where $g = \text{diag}(G^+, G^-)$, and $\hat{J} = \text{diag}(x_+ P_B, x_- P_B) \otimes e_n e_n^\dagger = J \otimes e_n e_n^\dagger$. Here, $x_+$ and $x_-$ are simple numbers, $P_B$ is a projector on bosonic variables, $e_n$ is a vector with $(e_n)_i = \delta_{ni}$, and $\langle \ldots \rangle_{\zeta}$ denotes averaging over $\zeta$. Our formulas are valid for both orthogonal and unitary symmetry, provided the four–dimensional graded space in the unitary case is simply doubled.

Following standard procedures [3,4], we express (4) as an integral over supervectors, average over $\zeta_{ij}$, perform a Hubbard–Stratonovitch transformation and finally arrive at

$$F(J) = \int D[\sigma_i] \exp\{-\mathcal{L}_1(\sigma)\},$$

$$\mathcal{L}_1(\sigma) = \frac{1}{8} \sum_{ij} \text{str}(\sigma_i (A^{-1})_{ij} \sigma_j) + \frac{1}{2} \sum_j \text{str} \ln(E - \eta_j + i\varepsilon \Lambda + \sigma_j/2 + J\delta_{jn}),$$

(5)

where the $\sigma_i$ are $8 \times 8$ supermatrices. Note that the functional still depends on the particular realization of the $\eta_j$. No average over these variables has been performed yet. The saddle–point condition for (5) reads

$$\sigma_j = -\sum_l A_{jl} \frac{1}{E - \eta_l + i\varepsilon \Lambda + \sigma_l/2}.$$ 

(6)

With the ansatz $\sigma = \Gamma_0 + i\Gamma_1 Q$ for the solution of (5), where $Q$ satisfies $Q^2 = 1$ and is a proper element of the coset space introduced in [3], we have for $z = \Gamma_0 + i\Gamma_1$:

$$z = -B_0 \int d\eta \rho_0(\eta) \frac{1}{E - \eta + z/2} \quad (\text{Im}(z) > 0).$$

(7)
Here we have replaced each element of the sum in (3) by its average over \( \eta \). This procedure is only justified in the limit of “overlapping resonances”, i.e. if the widths \( \Gamma_1 \) of the superimposed Lorentzians are larger than their spacings determined by those \( \eta_i \) contributing to the sum. With the result \( \Gamma_1 \sim \rho(E) \sim 1/W_b \) (see (4) below) this immediately leads back to the condition \( W_b \ll \sqrt{b} \), which is fulfilled in our regimes (ii) and (iii) defined above. Also, in precisely this limit we can expect the saddle–point solving (6) to be homogeneous in space as in our ansatz.

Equation (4) determines \( \Gamma_0 \) and \( \Gamma_1 \) and hence the local density of states \( \langle \rho(j, E) \rangle_\zeta \). Using (3) it is not difficult to show that \( \langle G_{jj}^+ \rangle_\zeta = (E - \eta_j + z/2)^{-1} \) and therefore

\[
\langle \rho(j, E) \rangle_\zeta = \frac{1}{\pi} \text{Im} \langle G_{jj}^+ \rangle_\zeta = \frac{\Gamma_1/2\pi}{(E - \eta_j + \Gamma_0/2)^2 + \Gamma_1^2/4}.
\]

This is exactly the Breit–Wigner form found numerically in [12]. Averaging this expression over all sites (assuming that \( N^{-1} \sum_j (\ldots) = \langle \ldots \rangle_\eta \)) and using the saddle–point equation (4) we get

\[
\rho(E) \equiv \langle \langle \rho(j, E) \rangle_\zeta \rangle_\eta = \frac{\Gamma_1}{\pi B_0}
\]

for the average density of states. From (4) and (3), we find the limiting cases \( \rho(E) \sim \rho_0(E) \) for \( W_b \gg 1 \) and \( \rho(E) \sim \sqrt{2B_0 - E^2/(\pi B_0)} \) for \( W_b \ll 1 \).

Inserting the saddle–point solution in (3) we proceed in analogy to [3,4] and arrive at

\[
F(J) = \int D[Q_i] \exp\{-\mathcal{L}_2(Q)\},
\]

\[
\mathcal{L}_2(Q) = -\frac{\xi}{16} \sum_j \text{str}[(Q_{j+1} - Q_j)^2] + \frac{1}{2} \sum_j \text{str} \ln[1 + M_j(Q_j)(i\varepsilon \Lambda + J\delta_{jn})],
\]

\[
M_j(Q) = (E - \eta_j + \Gamma_0/2 + i\Gamma_1/2Q)^{-1}.
\]

Here, \( \xi = 2\Gamma_1^2 B_2/\sqrt{B_0^2} = 2\pi^2 \rho(E)^2 B_2 \) denotes the localization length [14,5] for the wave functions [15], \( |\psi(n)| \sim |\psi(0)| \exp(-n/\xi) \). Due to the one–dimensional structure of the \( Q \)–functional (10) \( F(J) \) can be written as

\[
F(J) = \int dQ Y(Q, n) Y(Q, N - n) \text{sdet}^{-1/2}(1 + M_n(Q)J),
\]

where the function \( Y(Q, n) \) arises from integrating out all \( Q \)–matrices up to site \( n \). The properties of these functions have been discussed in detail in [3,4]. From (3) it is clear that

\[
\langle (G_{nn}^+)^l (G_{nn}^-)^m \rangle = \left. \frac{(-1)^{l+m}}{l!m!} \partial_{x^+}^l \partial_{x^-}^m F(J) \right|_{x^\pm = 0}.
\]

Performing the source term derivatives in (11) and using that

\[
M_j(Q) = \frac{E - \eta_j + \Gamma_0/2 - i(\Gamma_1/2) Q}{(E - \eta_j + \Gamma_0)^2 + \Gamma_1^2/4}
\]

we finally get in analogy to [4] (for \( \varepsilon \to 0 \) and \( \beta = 2 \)).
\[
\langle (G_{nn}^+)^l (G_{nn}^-)^m \rangle = \binom{l+m}{m} (-i \pi \rho C(\eta_n))^{l+m} \int dQ Y(Q, n) Y(Q, N-n) Q_{11, BB}^l Q_{22, BB}^m \tag{14}
\]

with
\[
C(\eta_n) = \frac{1}{\pi \rho (E - \eta_n + \Gamma_0/2)^2 + \Gamma_1^2/4} \tag{15}
\]

Comparing (14) with the corresponding expressions in [4] we can immediately conclude that
\[
P_q(E, n) = C(\eta_n)^q P_{FM}^q(E, n). \tag{16}
\]

The superscript “FM” refers to the result by Mirlin and Fyodorov [4] for ordinary RBM, where their rescaled length \( x \) has to be identified with \( x = 2N/(\beta \xi) \) [\( \xi \) as in eq. (10)].

Similarly, following the steps in [4] to derive the local distribution function \( \mathcal{P}(y, n) \) for \( y = N|\psi(n)|^2 \) defined by \( P_q(E, n) = N^{1-q} \int_0^\infty dy \mathcal{P}(y, n) y^q \), we find
\[
\mathcal{P}(y, n) = \frac{1}{C(\eta_n)} \mathcal{P}_{FM}^f \left( \frac{y}{C(\eta_n)}, n \right). \tag{17}
\]

We see that the local distribution is rescaled by the energy dependent factor \( C(\eta_n) \). In the resonant case, \( E \approx \eta_n \), the typical amplitude \( y \) exhibits a strong enhancement (in the limit \( W_b \gg 1 \), i.e. \( \Gamma_1 \sim W_b^{-1} \)), whereas for \( (E - \eta_n)^2 \gg B_0 \) a strong suppression is observed.

Up to now we have considered a fixed realization of the diagonal elements \( \eta_j \). The additional average over the sites (or equivalently over \( \eta_n \)) results in
\[
\mathcal{P}(y) = \int d\eta \rho_0(\eta) \frac{1}{C(\eta)} \mathcal{P}_{FM} \left( \frac{y}{C(\eta)}, \right). \tag{18}
\]

This completes our formal derivations for the case of RBM with PB. In the following we derive some more explicit results for the special case of Shepelyansky (see (2)).

With \( B_0 = 2/3 \) we have \( \rho(E) = 3 \Gamma_1(E)/2\pi \), where \( \Gamma_1(E) \) is determined by (7) and depends on \( E \) and \( \rho_0(\eta) \) (i.e. on \( W_b \) for the case (2)). In the main part of Fig.1 we show \( \rho(E) \) for different values of \( W_b \), demonstrating the crossover from a semicircle to the box shape (2). For \( E = 0 \) we must have \( \Gamma_0 = 0 \) and (7) simplifies to
\[
\Gamma_1(0) = \frac{2}{3W_b} \arctan \left( \frac{2W_b}{\Gamma_1(0)} \right). \tag{19}
\]

With \( B_2 = b^2/9 \) we have \( \xi = \Gamma_1(0)^2 b^2/2 \) so that we can express \( \xi \) as a function of \( W_b \). We recall that our treatment is valid for all \( W_b \) with \( 0 \leq W_b \ll \sqrt{b} \), i.e. for RBM, RBM with PB and the whole crossover. In the inset of Fig.1 we have plotted \( \xi \) versus \( W_b \). Interestingly, the crossover region between the two limiting cases \( \xi/b^2 = const. \) (RBM) and \( \xi/b^2 \sim 1/W_b^2 \) (RBM with PB as specified in (2)) is a comparatively small interval around \( W_b = 1 \).

For the asymptotic case \( W_b \gg 1 \) considered in the estimates in [9,12] we get by expanding (19)
\[
\xi \approx \frac{\pi^2}{18} \frac{b^2}{W_b^2} (1 - \frac{2}{3W_b^2}). \tag{20}
\]
This is in very good agreement with the estimate \( b^2/2W_b^2 \) in [12]. We also remark that a naive application of the RBM results in [3] (where one simply modifies the function \( a(|r|) \) to account for the large diagonals) does reproduce the parameter dependence in (20) but yields a (wrong) prefactor \( 4/3 \) instead of \( \pi^2/18 \).

Finally, we calculate the quantity \( \xi_{IPR} \), defined by

\[
\xi_{IPR}^{-1} = P_2 = N^{-1} \sum_n P_2(E,n)
\]

in the limit \( W_b \gg 1 \). We get for arbitrary \( \xi \) and \( N \)

\[
\xi_{IPR} = \frac{\pi^2}{12W_b} \left( \frac{3}{2N} + \frac{1}{\xi} \right)^{-1},
\]

with the metallic limit (\( N \ll \xi \))

\[
\xi_{IPR} = \frac{\pi^2}{18W_b^2} \approx 0.548 \frac{N}{W_b^2}
\]

and the localized limit (\( N \gg \xi \))

\[
\xi_{IPR} = \frac{\pi^2}{12W_b^4} \xi = \frac{\pi^4}{6^3W_b^4} \approx 0.451 \frac{b^2}{W_b^4}.
\]

These analytical results compare quite favourably with the estimates \( N/2W_b^2 \) (resp. \( b^2/4W_b^4 \)) in [12]. Also, in view of the exact formula (27), which interpolates between the metallic and the localized regime, the numerical deviations from \( \xi_{IPR} \sim W_b^{-2} \) (resp. \( W_b^{-4} \)) found in [12] for non–asymptotic system parameters come as no surprise.

Concerning the distribution (18), let us briefly discuss the simplest case (with \( \beta = 2 \) and \( \xi \gg N \)), where \( P_{FM}(y) = \exp(-y) \) as for a GUE random matrix. At \( E = 0 \) and \( W_b \gg 1 \) \[\Gamma_1 = \pi/(3W_b)\], we find:

\[
P(y) = \frac{1}{W_b} \int_0^{W_b} d\eta 3(\eta^2 + \Gamma_1^2/4) \exp[-3(\eta^2 + \Gamma_1^2/4)y].
\]

The probability is shifted to very small amplitudes with \( y \ll W_b^{-2} \), i.e. \( P(y) \sim W_b^2 \), and also to higher amplitudes \( y \gg W_b^{-2} \), \( P(y) \sim \exp[-\pi^2 y/(36W_b^2)] \).

In conclusion we have applied the supersymmetry method to the class of RBM with PB. The key to technical progress was the derivation of a \( \sigma \) model for a fixed realization of the strongly fluctuating diagonal matrix elements. The average over these diagonal variables was then performed in a second step, whenever appropriate. Our results account for the complete crossover between ordinary RBM (as treated earlier in [3,4]) and RBM with strong PB. For the specific model chosen in recent numerical calculations [12] the agreement with our findings is very satisfactory. In particular, the existence of two different scales characterizing the wavefunction as evidenced by the difference between \( \xi_{IPR} \) and \( \xi \) in (21) has been demonstrated analytically. We believe that further investigations of RBM with PB along the lines given in this letter are interesting and feasible.

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[15] We have to consider that \( \langle |G_{1n}|^2 \rangle \sim \exp[-n/(2\xi)] \) when applying the established \( \sigma \) model results of \([14,5]\) to determine the localization length. Note that \( |G_{1,n}|_{\text{typical}} \sim |\psi(n)|^2 \sim \exp(-2n/\xi) \) due to the log–normal distribution of \( |G_{1,n}|^2 \).

[16] In Ref. \([3]\), the site averaged distribution function \( \mathcal{P}^{FM}(y) = N^{-1} \sum_n \mathcal{P}^{FM}(y,n) \) was calculated. The “local” distribution \( \mathcal{P}^{FM}(y,n) \) depends only weakly on the site \( n \), as long as we deal with the metallic regime \( (\xi \gg N) \) or with the localized regime \( (\xi \ll N) \) far from the boundary \( (n, N-n \gg \xi) \). In Eq. (17) the main \( n \)-dependence arises from the rescaling factor \( C(\eta_n) \).
FIG. 1. The average density of states for the particular band matrix considered in Refs. [9,12] (see Eq. (2)) and the values $W_b = 0, 1, 2, 10$. The inset shows the localization length $\xi$ normalized by $b^2$ at $E = 0$ as a function of $W_b$ (in a doubly logarithmic representation).