Rao’s Score Tests on Correlation Matrices

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Abstract

Even though the Rao’s score tests are classical tests, such as the likelihood ratio tests, their application has been avoided until now in a multivariate framework, in particular high-dimensional setting. We consider they could play an important role for testing high-dimensional data, but currently the classical Rao’s score tests for an arbitrary but fixed dimension remain being still not very well-known for tests on correlation matrices of multivariate normal distributions. In this paper, we illustrate how to create Rao’s score tests, focussed on testing correlation matrices, showing their asymptotic distribution. Based on Basu et al. (2021), we do not only develop the classical Rao’s score tests, but also their robust version, Rao’s $\beta$-score tests. Despite of tedious calculations, their strength is the final simple expression, which is valid for any arbitrary but fixed dimension. In addition, we provide basic formulas for creating easily other tests, either for other variants of correlation tests or for location or variability parameters. We perform a simulation study with high-dimensional data and the results are compared to those of the likelihood ratio test with a variety of

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distributions, either pure and contaminated. The study shows that the classical Rao’s score test for correlation matrices seems to work properly not only under multivariate normality but also under other multivariate distributions. Under perturbed distributions, the Rao’s $\beta$-score tests outperform any classical test.

**Keywords:** Rao’s Score Test; High-dimensional Data; Multivariate Normal Distribution; Correlation Matrix; Uncorrelatedness of Random Variables; Equicorrelation of Random Variables; Classical Tests; Robust Tests.

## 1 Introduction

It is well known that the classical likelihood ratio based tests are generally not applicable in high-dimensional data due to the singularity of sample correlation matrices, occurred when the dimension $p$ is bigger than the sample size $n$. Consequently, based on likelihood ratio tests, this is an important source of novel methodology developments for high-dimensional data, in particular for correlation matrices of multivariate normal.

The Rao’s Score Tests, introduced by C. R. Rao (1948), became popular in Econometrics with an alternative version and refinements through restrictions, called Lagrange multiplier tests, presented by Aitchison and Silvey (1958) and Silvey (1959). In between both publications, Wald (1943) had proposed another test with the same asymptotic distribution as the likelihood ratio test and focussed on maximum likelihood estimators (MLEs) too, but these estimators were not exactly the same. While the likelihood ratio and Wald tests consider the whole parameter space, under the null and alternative hypotheses, the parameter space of the MLEs for the Rao’s score test is only restricted to the null hypothesis. When $p > n$, this important characteristic makes the Rao’s Score Tests stronger in comparison with the others and this
issue has not been exploited enough in high-dimensional tests setting yet. While the classical likelihood ratio tests for a fixed value of $p$ have not tractable expressions for $p > n$, since they involve a null value of the sample Pearson correlation matrix determinant (lack of semidefinite positiveness of Pearson correlation matrices), the formulas of the classical Rao’s score tests derived in Section 3, unknown prior this article, are perfectly defined even for $p > n$.

In the same way as the classical likelihood ratio tests devoted to correlation matrix are based on the sample Pearson correlation matrix, the new Rao’s Score Tests presented in Section 3 are based on it. It is well known that the sample Pearson correlation matrix is very susceptible to outliers or anomalous observations in the data. The proposed Rao’s $\beta$-score Tests (Basu et al., 2021) are based on a class of ‘density power divergences’ (Basu et al., 1998), indexed by a single parameter $\beta > 0$, which controls the trade-off between robustness and efficiency. Choices of $\beta$ near zero retain efficiency but loose robustness. The classical Rao’s score tests are a particular case of the class of the Rao’s $\beta$-score Tests taking a right hand side limit on zero for $\beta$.

This is a completely novel paper with respect to the derivation of the Rao’s $\beta$-Score Tests based on Correlation Matrices of any fixed dimension $p$ and arbitrary value and fixed “whole” correlation matrix as shown in (22), as required for applying the shortened version of the Rao’s Score test (see Remark 10 of Basu et al. (2021)). Recently, for the particular case of a theoretical identity matrix of the correlation matrix for multivariate normal distribution (test of independence), Leung and Drton (2018) have derived the corresponding classical Rao’s Score Test. Taking $\beta \rightarrow 0^+$ in our paper, as a particular case of the Rao’s $\beta$-score test, our expression’s derivation is much more complex, as required from our point of view. The article of Leung and Drton (2008) applies directly the scores and the Fisher’s information matrix taking
the derivative with respect to the variance-covariance matrix, obtaining the expression of the Rao’s tests for the case in which the “whole” variance-covariance matrix is assumed to be fixed “except” for the variances, which must be estimated under the assumption of independence. As proven at the beginning in our Section 5.12 (the details are in (45)), with the change of parameter variables (not data transformation, as done for a complete fixed variance covariance matrix), the final result works properly; however, in case of not taking appropriate variance estimators (as given in our Section 3.1) or the generalized expression of the Rao’s score test (different from [45]), their technique could fail to obtain the correct expression for a general theoretical correlation matrix. Once we have clarified these issues, as far as we are aware, for a general theoretical correlation matrix there is no any publication which derives and proves neither the classical efficient Rao’s Score’s test nor the robust Rao’s $\beta$-Score’s test.

The rest of the paper is organized as follows. Section 2 covers very technical issues related to Rao’s tests of any parameter associated with the multivariate normal distribution. In Subsection 2.1 notational aspects are described, in Subsections 2.2 and 2.3 some basic results are provided, valid for the specific tests of the current paper as well as for any additional Rao’s score test we could construct, quite easily, associated to multivariate normal distributions, either for correlation or for central or dispersion parameters. In Section 3 in Subsections 3.2, 3.3, 3.4 and 3.5 each of the four correlation tests are introduced, and previously in Subsection 3.1 how to compute their corresponding estimators is explained. A discussion, in Section 4, relates the new proposed tests with existing tests for high dimensional data under specific limiting assumptions. The theorems appearing in Sections 2 and 3 are proven in Section 5. A section devoted to a simulation study will be presented in a future version of this paper.
2 Preliminary results

2.1 Notational aspects

Let $X = (X_1, \cdots, X_p)^T \sim \mathcal{N}_p(\mu, \Sigma)$ be a $p$-variate normal distribution, with $\mu = (\mu_1, \cdots, \mu_p)^T$, $\Sigma = (\sigma_{ij})$, such that $\sigma_{ii} = \sigma_i^2 = \text{Var}[X_i]$, $\sigma_{ij} = \text{Cov}[X_i, Y_j]$. Let us consider the parameter vector as $\theta = (\mu^T, \text{vech}^T(\Sigma))^T$, where vech is the so-called vech operation for a symmetric matrix. The vech of $\Sigma$ (vech for vector half) is the $p(p+1)/2$ dimensional vector obtained by stacking the unique part of each column that lies on or below the diagonal of $\Sigma$ into a single vector (for more details, see Henderson & Searle (1979)). For example, when $p = 2$ then $\text{vech}(\Sigma) = (\sigma_{11}, \sigma_{12}, \sigma_{22})^T = (\sigma_1^2, \sigma_{12}, \sigma_2^2)^T$. In this setting, the density function is expressed as

$$f_\theta(x) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} \exp \left\{ -\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right\},$$

being $x = (x_1, \cdots, x_p)^T$ any point in the support, $\mathbb{R}^p$ and $\theta \in \mathbb{R}^p \times \mathbb{R}_{+}^{p(p+1)/2}$.

Let us consider the parameter vector as

$$\theta = (\theta_1^T, \theta_2^T)^T,$$

$$\theta_1 = \mu,$$

$$\theta_2 = \text{vech}(\Sigma),$$

but now the variance-covariance components are reordered and in addition the correlation matrix is considered according to

$$\Sigma = (\sigma_{ij}) = \Lambda^{1/2} R \Lambda^{1/2} = (\sigma_{i} \rho_{ij} \sigma_{j}),$$
where

\[ \Lambda = \text{diag}\{\sigma_j^2\}_{j=1}^p, \]
\[ R = (\rho_{ij}). \]

The first parameter vector variable change is

\[ \phi = (\phi_1^T, \phi_2^T)^T, \]
\[ \phi_1 = \mu = \theta_1, \]
\[ \phi_2 = ((\Lambda 1_p)^T, \text{vecl}^T(\Sigma))^T = M^T \theta_2, \]

where \text{vecl}(\cdot) denotes the vectorization operator of the lower off-diagonal elements of matrix \( \cdot \) (unlike the \text{vech}(\cdot) operator, this operator excludes the diagonal elements) and

\[ M = (P, Q), \]
\[ P = (P_1, \cdots, P_p), \]
\[ P_{\cdot i} = e_{(i-1)(p+1)-\frac{(i-1)}{2}+1}, \]
\[ Q = (Q_1, \cdots, Q_{(p-1)p/2}), \]
\[ Q_{\cdot s} = e_{(i-1)(p+1)-\frac{(i-1)}{2}+1+j-1}, \]

\( i < j \) in lexicographical order. Notice that \( M \) is a permutation matrix and so it is orthogonal, i.e. \( M^{-1} = M^T \). Let

\[ \phi_2 = (\phi_{2,1}^T, \phi_{2,2}^T)^T = ((\Lambda 1_p)^T, \text{vecl}^T(\Sigma))^T, \]
\[ \theta_2 = \text{vech}(\Sigma), \Sigma = \Sigma(\Lambda) = \Lambda^{1/2} R \Lambda^{1/2} \]
be a partition of $\phi_2$, then from previous expressions it is concluded that

$$\phi_{2,1} = P^T \theta_2,$$

$$\phi_{2,2} = Q^T \theta_2.$$  

Ferrari and Yang (2010) used the previous parameter scheme for estimation though the so-called $Lq$-estimators of $\phi$. The second parameter change is

$$\eta = (\eta_1^T, \eta_2^T)^T,$$

$$\eta_2 = (\eta_{2,1}^T, \eta_{2,2}^T)^T,$$

$$\eta_{2,1} = \phi_{2,1} = \Lambda_1 p,$$

$$\eta_{2,2} = \text{vecl}(R) = \text{vecl}(\Lambda^{-1/2} \Sigma \Lambda^{-1/2}).$$

For example, if $p = 3$ then $\theta = (\mu_1, \mu_2, \mu_3, \sigma_1^2, \sigma_{12}, \sigma_{13}, \sigma_2^2, \sigma_{23}, \sigma_3^2)^T$, $\phi = (\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_{12}, \sigma_{13}, \sigma_{23})^T$ and $\eta = (\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \sigma_3^2, \rho_{12}, \rho_{13}, \rho_{23})^T$.

### 2.2 Basic results for the partition of parameter vector $\theta$

**Proposition 1** The expression of the vectorial score function, $s_\theta(x) = (s_\mu(x), s_{\text{vech}(\Sigma)}(x))^T$, is given as follows

$$s_\mu(x) = \frac{\partial}{\partial \mu} \log f_\theta(x)$$

$$= \Sigma^{-1}(x - \mu),$$

(1)

$$s_{\text{vech}(\Sigma)}(x) = -\frac{1}{2} \text{vech} \left( \frac{\partial}{\partial \Sigma} \left( \log |\Sigma| + (x - \mu)^T \Sigma^{-1}(x - \mu) \right) \right)$$

$$= -\frac{1}{2} G_p^T \text{vec}(\Sigma^{-1}) + \frac{1}{2} G_p^T \left[ (\Sigma^{-1}(x - \mu)) \otimes ((x - \mu)^T \Sigma^{-1}) \right],$$

(2)
with \( x \in \mathbb{R}^p \) being any point of the support and \( G_p \) the so-called “duplication matrix” of order \( p \), i.e., the unique \( p^2 \times \frac{p(p+1)}{2} \) matrix such that \( \text{vec}(\Sigma) = G_p \text{vech}(\Sigma) \).

**Theorem 2** The expression of \( J_\beta(\theta) = E_{\theta}[s_{\theta}(X)s_{\theta}^T(X)f_\theta^\beta] \), is given as follows

\[
J_\beta(\theta) = \begin{pmatrix} J_\beta(\mu) & 0_{p \times \frac{p(p+1)}{2}} \\ 0_{p \times \frac{p(p+1)}{2}} & J_\beta(\text{vech}(\Sigma)) \end{pmatrix},
\]

where

\[
J_\beta(\mu) = \frac{(\beta + 1)^{-\frac{\beta}{2}-1}}{(2\pi)^{\frac{3p}{2}} |\Sigma|^\frac{3p}{2}} \Sigma^{-1},
\]

\[
J_\beta(\text{vech}(\Sigma)) = \frac{(\beta + 1)^{-\frac{\beta}{2}-2}}{4(2\pi)^{\frac{3p}{2}} |\Sigma|^\frac{3p}{2}} \left[ \beta^2 C_\theta + 2G_p^T (\Sigma^{-1} \otimes \Sigma^{-1}) G_p \right],
\]

and

\[
C_\theta = G_p^T \text{vech} (\Sigma^{-1}) \text{vec}^T (\Sigma^{-1}) G_p^T.
\]

**Theorem 3** The expression of \( \xi_\beta(\theta) \) and \( \xi_\beta(\theta)\xi_\beta^T(\theta) \), where \( \xi_\beta(\theta) = E_{\theta}[s_{\theta}(X)f_\theta^\beta(X)] \), are given as follows

\[
\xi_\beta(\theta) = \begin{pmatrix} \xi_\beta(\mu) \\ \xi_\beta(\text{vech}(\Sigma)) \end{pmatrix} = \begin{pmatrix} 0_p \\ \xi_\beta(\text{vech}(\Sigma)) \end{pmatrix},
\]

\[
\xi_\beta(\theta)\xi_\beta^T(\theta) = \begin{pmatrix} 0_{p \times p} & 0_{p \times \frac{p(p+1)}{2}} \\ 0_{\frac{p(p+1)}{2} \times p} & \xi_\beta(\text{vech}(\Sigma))\xi_\beta^T(\text{vech}(\Sigma)) \end{pmatrix}
\]

where

\[
\xi_\beta(\text{vech}(\Sigma)) = \frac{-\beta (\beta + 1)^{-\frac{\beta}{2}-1}}{2(2\pi)^{\frac{3p}{2}} |\Sigma|^\frac{3p}{2}} G_p^T \text{vech} (\Sigma^{-1}),
\]

\[
\xi_\beta(\text{vech}(\Sigma))\xi_\beta^T(\text{vech}(\Sigma)) = \frac{\beta^2 (\beta + 1)^{-\frac{\beta+2}{2}}}{4(2\pi)^{\beta p} |\Sigma|^{\beta}} C_\theta.
\]
Corollary 4  The expression of $K_\beta(\theta)$ is given as follows

$$K_\beta(\theta) = \begin{pmatrix} K_\beta(\mu) & 0_{p \times \frac{p(p+1)}{2}} \\ 0_{\frac{p(p+1)}{2} \times p} & K_\beta(\text{vech}(\Sigma)) \end{pmatrix},$$

where

$$K_\beta(\mu) = J_{2\beta}(\mu) = \frac{(2\beta + 1)^{-\frac{\beta}{2} - 1}}{(2\pi)^{\beta p} |\Sigma|^\beta} \Sigma^{-1},$$

$$K_\beta(\text{vech}(\Sigma)) = \frac{1}{4(2\pi)^{\beta p} |\Sigma|^\beta} G_p^T \left[ \mathcal{J}_{2\beta}(\Sigma^{-1}) + \xi_{\beta}(\Sigma^{-1}) \xi_{\beta}(\Sigma^{-1})^T \right] G_p,$$

with

$$\mathcal{J}_{2\beta}(\Sigma^{-1}) = \kappa_1(p, \beta) (\Sigma^{-1} \otimes \Sigma^{-1}) ,$$

$$\kappa_1(p, \beta) = 2(2\beta + 1)^{-\frac{\beta}{2} - 2} ,$$

$$\xi_{\beta}(\Sigma^{-1})^T(\Sigma^{-1}) = \kappa_2(p, \beta) \text{vec}(\Sigma^{-1}) \text{vec}^T(\Sigma^{-1})$$

$$\kappa_2(p, \beta) = \beta^2 \left[ 4(2\beta + 1)^{-\frac{\beta}{2} - 2} - (\beta + 1)^{-(p+2)} \right] .$$

Proposition 5  The expresión of $K_\beta(\text{vech}(\Sigma))$ and its inverse en terms of $\Lambda$ and $R_0$ are given by

$$K_\beta(\text{vech}(\Lambda^{\frac{1}{2}} R_0^{\frac{1}{2}} \Lambda^{\frac{1}{2}})) = \frac{1}{4(2\pi)^{\beta p} |\Lambda|^\beta |R_0|^\beta}$$

$$\times G_p^T \left( \Lambda^{-\frac{1}{2}} \otimes \Lambda^{-\frac{1}{2}} \right) \left( \mathcal{J}_{2\beta}(R_0^{-1}) + \overline{\xi}_{\beta}(R_0^{-1}) \overline{\xi}_{\beta}(R_0^{-1})^T \right) \left( \Lambda^{-\frac{1}{2}} \otimes \Lambda^{-\frac{1}{2}} \right) G_p,$$

with

$$\mathcal{J}_{2\beta}(R_0^{-1}) = \kappa_1(p, \beta) (R_0^{-1} \otimes R_0^{-1}),$$

$$\overline{\xi}_{\beta}(R_0^{-1})^T(\Sigma_0^{-1}) = \kappa_2(p, \beta) \text{vec}(R_0^{-1}) \text{vec}^T(R_0^{-1}),$$

$$\kappa_2(p, \beta) = \beta^2 \left[ 4(2\beta + 1)^{-\frac{\beta}{2} - 2} - (\beta + 1)^{-(p+2)} \right] .$$
and
\[ K_\beta^{-1}(\text{vech}(\Lambda^\beta R_0 \Lambda^\beta)) = 4(2\pi)^\beta \left| \Lambda \right| \beta \left| R_0 \right| \beta \]
\[ \times L_p \left( \Lambda^\beta \otimes \Lambda^\beta \right) \left( \mathbf{J}_{2\beta}(R_0^{-1}) + \xi_\beta(R_0^{-1})\xi_\beta^T(R_0^{-1}) \right)^{-1} \left( \Lambda^\beta \otimes \Lambda^\beta \right) L_p^T, \]

with
\[ \left( \mathbf{J}_{2\beta}(R_0^{-1}) + \xi_\beta(R_0^{-1})\xi_\beta^T(R_0^{-1}) \right)^{-1} = \kappa_1^{-1}(p, \beta) \left( (R_0 \otimes R_0) - \frac{\kappa_3(p, \beta) \text{vec}(R_0) \text{vec}^T(R_0)}{1 + p\kappa_3(p, \beta)} \right), \]

where
\[ \kappa_3(p, \beta) = \kappa_1^{-1}(p, \beta)\kappa_2^2(p, \beta). \]

Theorem 6 The expression of the \( \beta \)-score statistic,
\[ U_{\beta,n}(\theta) = \frac{1}{n} \sum_{i=1}^{n} u_\beta(X_i, \theta) = (U_{\beta,n}^T(\mu), U_{\beta,n}^T(\mu))^T, \]
\[ u_\beta(x, \theta) = s_\theta(x)f_\theta(x) - \xi_\beta(\theta), \]
is given by
\[ U_{\beta,n}(\mu) = -\frac{1}{2(2\pi)^{\frac{p\beta}{2}} |\Sigma|^{\frac{\beta}{2}}} \sum_{i=1}^{n} \frac{1}{n} \sum_{i=1}^{n} w_{i,\beta}(\theta)(X_i - \mu), \]

with
\[ w_{i,\beta}(\theta) = \exp \left\{ -\frac{\beta}{2} (X_i - \mu)^T \Sigma^{-1} (X_i - \mu) \right\}, \]

and
\[ U_{\beta,n}(\text{vech}(\Sigma)) = G_p^T V_{\beta,n}(\text{vec}(\Sigma)) \]
Proposition 7 Let $V_{\beta,n}(\tilde{\Lambda}_\beta, R_0)$ denote $V_{\beta,n}(\tilde{\theta}_{2,\beta})$, given in (13), a term of $U_{\beta,n}(\tilde{\theta}_{2,\beta})$ according to (34) particularized to $\tilde{\Sigma}_\beta = \tilde{\Lambda}_\beta^{1/2} R_0 \tilde{\Lambda}_\beta^{1/2}$, then

$$V_{\beta,n}(\tilde{\Lambda}_\beta, R_0) = -\frac{\tilde{\kappa}_0(p,\beta)}{2(2\pi)^{\frac{d_p}{2}} |\tilde{\Sigma}_\beta|^\frac{1}{2}} (\tilde{\Lambda}_\beta^{-1/2} \otimes \tilde{\Lambda}_\beta^{-1/2}) (R_0^{-1} \otimes R_0^{-1}) \left[ \text{vec} \left( \tilde{R}_x,\beta \right) - \text{vec} \left( R_0 \right) \right],$$

(14)

where $\tilde{\kappa}_0(p,\beta)$ was given in (37) and

$$\text{vec} \left( \tilde{R}_x,\beta \right) = \frac{1}{n} \sum_{i=1}^{n} \tilde{w}_{i,\beta} \left( \tilde{\Lambda}_\beta^{-1/2} (X_i - \tilde{\mu}) \right) \otimes \left( \tilde{\Lambda}_\beta^{-1/2} (X_i - \tilde{\mu}) \right).$$

2.3 Basic result for the partition of parameter vector $\eta_2$

Theorem 8 The vectorial and matricial expressions of interest for vector $\eta_2$ in relation to $\theta_2$ is given by

$$s_{\eta_2}(x) = (s_{\eta_{2,1}}^T(x), s_{\eta_{2,2}}^T(x))^T, \quad \xi_\beta(\eta_2) = (\xi_\beta^T(\eta_{2,1}), \xi_\beta^T(\eta_{2,2})), \quad \xi_\beta^T(\eta_{2,1}) = P^T \xi_\beta(\theta_2), \quad \xi_\beta^T(\eta_{2,2}) = \text{diag}^\frac{1}{2} (\text{vecl}(\eta_{2,1}^T)) Q^T \xi_\beta(\theta_2);$$

$$J_\beta(\eta_2) = \begin{pmatrix} P^T J_\beta(\theta_2) P & \text{diag}^\frac{1}{2} (\text{vecl}(\eta_{2,1}^T)) \\ \text{diag}^\frac{1}{2} (\text{vecl}(\eta_{2,1}^T)) & Q^T J_\beta(\theta_2) Q \text{diag}^\frac{1}{2} (\text{vecl}(\eta_{2,1}^T)) \end{pmatrix}$$
\[ K_\beta(\eta_2) = \begin{pmatrix} P^T K_\beta(\theta_2) P & P^T K_\beta(\theta_2) Q \text{diag}^{\frac{1}{2}}(\text{vecl}(\eta_{2,1}, \eta_{2,1}^T)) \\ \text{diag}^{\frac{1}{2}}(\text{vecl}(\eta_{2,1}, \eta_{2,1}^T)) Q^T K_\beta(\theta_2) P & \text{diag}^{\frac{1}{2}}(\text{vecl}(\eta_{2,1}, \eta_{2,1}^T)) Q^T K_\beta(\theta_2) Q \text{diag}^{\frac{1}{2}}(\text{vecl}(\eta_{2,1}, \eta_{2,1}^T)) \end{pmatrix} \]

\[ U_{\beta,n}(\eta_2) = \begin{pmatrix} U_{\beta,n}(\eta_{2,1}) \\ U_{\beta,n}(\eta_{2,2}) \end{pmatrix} = \begin{pmatrix} P^T U_{\beta,n}(\theta_2) \\ \text{diag}^{\frac{1}{2}}(\text{vecl}(\eta_{2,1}, \eta_{2,1}^T)) Q^T U_{\beta,n}(\theta_2) \end{pmatrix}. \]

3 Main results

3.1 Restricted maximum likelihood estimators (MLEs) and minimum density power divergences (MDPDs)

Theorem 9 For known correlation matrix, \( R = R_0 \), the restricted minimum DPD estimators of \( (\mu^T, 1_p \Lambda)^T \), are obtained as solution in \( (\tilde{\mu}_\beta, 1_p \tilde{\Lambda}_\beta)^T \) of

\[ \tilde{\mu}_\beta = \frac{1}{n} \sum_{i=1}^n w_i(\tilde{\mu}_\beta, \tilde{\Lambda}_\beta) X_i, \]

\[ 1_p = \text{diag}\{R_0^{-1} R_{X,\beta}(\tilde{\mu}_\beta, \tilde{\Lambda}_\beta)\} 1_p, \]

where

\[ w_i(\tilde{\mu}_\beta, \tilde{\Lambda}_\beta) = \exp \left\{ -\frac{\beta}{2} (X_i - \tilde{\mu}_\beta)^T \tilde{\Lambda}_\beta^{-1/2} R_0^{-1} \tilde{\Lambda}_\beta^{-1/2} (X_i - \tilde{\mu}_\beta) \right\} \]

and

\[ R_{X,\beta}(\tilde{\mu}_\beta, \tilde{\Lambda}_\beta) = \tilde{\Lambda}_\beta^{-1/2} S_{X,\beta}(\tilde{\mu}_\beta, \tilde{\Lambda}_\beta) \tilde{\Lambda}_\beta^{-1/2}, \]

\[ S_{X,\beta}(\tilde{\mu}_\beta, \tilde{\Lambda}_\beta) = \frac{1}{n} \sum_{i=1}^n w_i(\tilde{\mu}_\beta, \tilde{\Lambda}_\beta)(X_i - \tilde{\mu}_\beta)(X_i - \tilde{\mu}_\beta)^T \]

Remark 10 Since \( w_i(\tilde{\mu}_\beta, \tilde{\Lambda}_\beta) = 1 \), for \( \beta = 0 \), the MLE of \( (\mu^T, 1_p \Lambda)^T \) has an explicit
explicit expression but the minimum DPD estimators need recursive computations.

**Remark 11** The expression of $R_{X,\beta}(\tilde{\mu}_\beta, \tilde{\Lambda}_\beta)$ for $\beta = 0$ does not match (in most cases) the sample Pearson correlation matrix since the variance estimators fail to be the ordinary sample variances. For example for $p = 2$,

$$R_0 = \begin{pmatrix} 1 & \rho_0 \\ \rho_0 & 1 \end{pmatrix}, \quad R_{X}(\tilde{\Lambda}) = \begin{pmatrix} \frac{S_1^2}{\sigma_1^2} & \frac{S_{12}}{\sigma_1 \sigma_2} \\ \frac{S_{12}}{\sigma_1 \sigma_2} & \frac{S_2^2}{\sigma_2^2} \end{pmatrix},$$

and the estimating equations for the variances are

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{1 - \rho_0^2} \text{diag}\left\{\begin{pmatrix} 1 & -\rho_0 \\ -\rho_0 & 1 \end{pmatrix} \begin{pmatrix} \frac{S_1^2}{\sigma_1^2} & \frac{S_{12}}{\sigma_1 \sigma_2} \\ \frac{S_{12}}{\sigma_1 \sigma_2} & \frac{S_2^2}{\sigma_2^2} \end{pmatrix}\right\} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

i.e.

$$\frac{1}{1 - \rho_0^2} \left(\frac{S_j^2}{\sigma_j^2} - \rho_0 \frac{S_{12}}{\sigma_1 \sigma_2}\right) = 1, \quad j = 1, 2,$$

or equivalently

$$\frac{S_j^2}{\sigma_j^2} = 1 - \rho_0^2 - \rho_0 \frac{S_{12}}{\sigma_1 \sigma_2}, \quad j = 1, 2,$$

from which the ratio is deducted to be

$$\frac{\tilde{\sigma}_1}{\tilde{\sigma}_2} = \frac{S_1}{S_2},$$

but

$$\begin{pmatrix} \tilde{\sigma}_1^2 \\ \tilde{\sigma}_2^2 \end{pmatrix} = \text{diag}\{R_0^{-1} R_X\} \begin{pmatrix} S_1^2 \\ S_2^2 \end{pmatrix},$$

i.e.

$$\begin{pmatrix} \tilde{\sigma}_1^2 \\ \tilde{\sigma}_2^2 \end{pmatrix} = \frac{1 - \rho_0 R_{12}}{1 - \rho_0^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} S_1^2 \\ S_2^2 \end{pmatrix},$$

13
or

\[ \tilde{\sigma}_j^2 = S_j^2 \frac{1 - \rho_j R_{12}}{1 - \rho_0^2}, \quad j = 1, 2. \]

In the foregoing subsections it is shown that both match under equicorrelation with 
\( \rho_0 = 0 \) (uncorrelatedness) or estimated \( \rho \) by (20).

**Theorem 12** Under fixed equicorrelation,

\[ R(\rho_0) = (1 - \rho_0) I_p + \rho_0 1_p 1_p^T, \]

the restricted minimum DPD estimators of \((\mu^T, 1_p^T \Lambda)^T\), are obtained as solution in (15) and

\[ \tilde{R}_{jj, \beta} = (1 - \rho_0) = \frac{\rho_0}{1 + (p - 1)\rho_0} \tilde{R}_{j, \beta}, \quad (18) \]

where \( \tilde{R}_{jj, \beta} = S_j^2 / \tilde{\sigma}_j^2 \) are the diagonal elements of (17).

**Theorem 13** Under non-fixed equicorrelation,

\[ R(\rho_{12}) = (1 - \rho_{12}) I_p + \rho_{12} 1_p 1_p^T, \quad (19) \]

being \( \rho_{12} \) an unknown parameter, apart from the estimating equations of Theorem 9, we have an additional one

\[ \tilde{\rho}_{12, \beta} = \frac{2}{p(p - 1)} \sum_{i<j} R_{ij, \beta}, \quad (20) \]

where \( R_{ij, \beta} \) are the elements of (17) when

\[ \tilde{\sigma}_{j, \beta}^2 = S_j^2, \quad j = 1, \ldots, p. \]

**Remark 14** The minimum DPD estimators of \( \rho_{12} \) need recursive computations for
\(\beta > 0\), along with the ones of \(\mu_j\) and \(\sigma_j^2, j = 1, \ldots, p\), according to

\[
\begin{align*}
\bar{\mu}_{j,\beta} &= \frac{1}{n} \sum_{i=1}^{n} \bar{w}_{i,\beta} X_{ij}, \\
S^2_{j,\beta} &= \frac{1}{n} \sum_{i=1}^{n} \bar{w}_{i,\beta} (X_{ij} - \bar{\mu}_{j,\beta})^2, \\
\bar{w}_{i,\beta} &= \exp \left\{ -\frac{\beta}{2(1 - \rho_{12,\beta})} \left[ \sum_{j=1}^{p} \tilde{X}_{ij,\beta}^2 - \frac{\tilde{\rho}_{12,\beta}}{1 + (p - 1)\tilde{\rho}_{12,\beta}} \left( \sum_{j=1}^{p} \tilde{X}_{ij,\beta} \right)^2 \right] \right\}, \\
\tilde{X}_{ij,\beta} &= \frac{X_{ij} - \bar{\mu}_{j,\beta}}{S_{j,\beta}}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, p,
\end{align*}
\]

where

\[
\tilde{\kappa}_0(p, \beta) = \frac{1}{n} \sum_{i=1}^{n} \bar{w}_{i,\beta} - \beta (\beta + 1)^{-\left(\frac{p}{2}+1\right)}. \tag{21}
\]

For \(\beta = 0\), since \(\bar{w}_{i,\beta=0} = 1, i = 1, \ldots, n\), and \(\tilde{\kappa}_0(p, \beta = 0) = 1\), the MLEs of \(\rho_{12}\), \(\mu_j\) and \(\sigma_j^2, j = 1, \ldots, p\), have explicit expressions.

### 3.2 Testing specified values for correlation matrix

For testing \(R = R_0\) or more formally

\[H_0: \eta_{2,2} = \text{vecl}(R_0) \quad \text{vs.} \quad H_1: \eta_{2,2} \neq \text{vecl}(R_0), \tag{22}\]

let us consider a transformation of the original sample, \(\tilde{X}_{i,\beta}, i = 1, \ldots, n\), where

\[
\tilde{X}_{i,\beta} = \tilde{\Lambda}_{\beta}^{-1/2} (X_i - \tilde{\mu}_{\beta}),
\]

being \(\tilde{\mu}_{\beta}\) and \(\tilde{\Lambda}_{\beta}\) the minimum DPD estimators of \(\mu\) and \(\Lambda\) under \(R = R_0\) (see Section 3.1).
Theorem 15  The Rao’s $\beta$-score test-statistic for (22) is

$$\tilde{R}_{\beta,n} = n\tilde{\kappa}_0^2(p, \beta)\kappa_1^{-1}(p, \beta)\text{trace} \left( \left( R_0^{-1}\tilde{R}_{X,\beta} - I_p \right)^2 \right),$$  \hspace{1cm} (23)

with $\tilde{R}_{X,\beta} = R_{X,\beta}(\tilde{\mu}_\beta, \tilde{\Lambda}_\beta)$, given previously in (17), i.e.

$$\tilde{R}_{X,\beta} = \frac{1}{n}\sum_{i=1}^{n} \tilde{w}_{i,\beta} \tilde{X}_{i,\beta} \tilde{X}_{i,\beta}^T,$$

$\tilde{w}_{i,\beta} = w_{i,\beta}(\tilde{\theta})$, given previously in (12), i.e.

$$\tilde{w}_{i,\beta} = \exp \left\{ -\frac{\beta}{2} \tilde{X}_{i,\beta}^T R_0^{-1}\tilde{X}_{i,\beta} \right\},$$

$\tilde{\kappa}_0(p, \beta)$ was given in (21) and $\kappa_1(p, \beta)$ in (9). The asymptotic distribution of (23) is $\chi^2$ with $\frac{p(p-1)}{2}$ degrees of freedom.

Remark 16  Notice that from symmetry

$$\text{trace} \left( \left( R_0^{-1}\tilde{R}_{X,\beta} - I_p \right)^2 \right) = 2\text{vech}^T \left( R_0^{-1}\tilde{R}_{X,\beta} - I_p \right) \text{vech} \left( R_0^{-1}\tilde{R}_{X,\beta} - I_p \right)$$

$$- \text{diag} \{ R_0^{-1}\tilde{R}_{X,\beta} - I_p \},$$

and from the estimating equations from the variances, it holds \text{diag} \{ $R_0^{-1}\tilde{R}_{X,\beta} - I_p$ \} = 0_{p \times p}, hence

$$\text{trace} \left( \left( R_0^{-1}\tilde{R}_{X,\beta} - I_p \right)^2 \right) = 2\text{vecl}^T \left( R_0^{-1}\tilde{R}_{X,\beta} \right) \text{vecl} \left( R_0^{-1}\tilde{R}_{X,\beta} \right).$$  \hspace{1cm} (24)

Remark 17  For $\beta = 0$, i.e. for all the classical Rao’s score tests, it hold

$$\tilde{\kappa}_0^2(p, \beta = 0)\kappa_1^{-1}(p, \beta = 0) = \frac{1}{2}.$$
Remark 18 Particularization of the Rao’s $\beta$-score test-statistic for testing specified values of correlation in the bidimensional case ($p = 2$):

$$\tilde{R}_{\beta,n} = 2n \tilde{\kappa}_0^2(p = 2, \beta) \left( \frac{R_{12,\beta} - \rho_0}{1 - \rho_0 R_{12,\beta}} \right)^2,$$

where the weights to calculate the sample correlation is given by

$$\tilde{w}_{i,\beta} = \exp \left\{ -\frac{\beta}{2(1 - \rho_0)} \left( \tilde{X}_i^T \tilde{X}_i - \frac{\rho_0}{1 + (p - 1)\rho_0} (\tilde{X}_i^T 1_2)^2 \right) \right\}.$$

The likelihood ratio test statistic can be found in Chapter 4 of Anderson (2003), but to our knowledge, the explicit expression of the classical score test statistic ($\beta = 0$),

$$\tilde{R}_n = n \left( \frac{R_{12} - \rho_0}{1 - \rho_0 R_{12}} \right)^2,$$

had not been published yet. It suits results such as (5.4.34)-(5.4.40) of Lehman (1999, page 316) for any distribution, but it is not an equivalent expression. In fact, the most well-known test-statistic for the correlation coefficient of the bivariate normal distribution is the one for the Fisher’s transform of the sample correlation coefficient (see page (5.4.41)-(5.4.42) in Lehman (1999)).

3.3 Testing fixed equicorrelation

The equicorrelation structure establishes homogeneity for all off-diagonal elements of the correlation matrix, i.e.

$$H_0: \rho_{ij} = \rho_0, \forall i \neq j \quad \text{vs.} \quad H_1: \exists i \neq j \text{ s.t. } \rho_{ij} \neq \rho_0,$$
with $\rho_0$ being the so called intraclass correlation satisfying

$$-\frac{1}{p-1} < \rho_0 < 1,$$

to guarantee positive definiteness. In matrix form, the equicorrelation structure can be expressed as

$$H_0: \eta_{2,2} = \text{vecl}(R(\rho_0)) \; \text{vs.} \; H_1: \eta_{2,2} \neq \text{vecl}(R(\rho_0)),$$

where

$$R(\rho_0) = (1 - \rho_0)I_p + \rho_01_p1_p^T.$$

**Corollary 19** The Rao's $\beta$-score test-statistic for testing equicorrelation, as particular case of (23), is given by

$$\tilde{R}_{\beta,n} = \frac{2n\kappa_0^2(p, \beta)}{(1 - \rho_0)^2} \sum_{i<j} \left( R_{ij,\beta} \frac{S_{i,\beta} S_{j,\beta}}{\sigma_{i,\beta} \sigma_{j,\beta}} - \frac{S_{j,\beta}^2}{\sigma_{j,\beta}^2} + (1 - \rho_0) \right)^2,$$  \hspace{1cm} (26)

where

$$\frac{S_{j,\beta}^2}{\sigma_{j,\beta}^2} = \tilde{R}_{jj,\beta},$$

$$R_{ij,\beta} \frac{S_{i,\beta} S_{j,\beta}}{\sigma_{i,\beta} \sigma_{j,\beta}} = \frac{S_{ij,\beta}}{\sigma_{i,\beta} \sigma_{j,\beta}} = \tilde{R}_{ij,\beta},$$

are diagonal and extradiagonal elements of (17),

$$\tilde{w}_{i,\beta} = \exp \left\{ -\frac{\beta}{2(1-\rho_0)} \left[ \sum_{j=1}^{p} \tilde{X}_{ij,\beta}^2 - \frac{\rho_0}{1+(p-1)\rho_0} \left( \sum_{j=1}^{p} \tilde{X}_{ij,\beta} \right)^2 \right] \right\},$$

$$\tilde{X}_{ij,\beta} = \frac{X_{ij} - \mu_{ij,\beta}}{\sigma_{j,\beta}}, \; i = 1, \ldots, n, \; j = 1, \ldots, p.$$
Its asymptotic distribution is $\chi^2$ with $\frac{p(p-1)}{2}$ degrees of freedom.

### 3.4 Testing complete uncorrelatedness or independence

**Corollary 20**  The Rao’s $\beta$-score test-statistic for testing uncorrelatedness or independence ($R(\rho_0 = 0)$), as particular case of (23), is given by

$$\tilde{R}_{\beta,n} = 2n\tilde{\kappa}_0^2(p, \beta)\kappa_1^{-1}(p, \beta) \sum_{i<j} R_{ij,\beta}^2,$$

(27)

where $R_{ij,\beta}$ are extradiagonal elements of (17) when $\tilde{\sigma}_{j,\beta}^2 = S_{j,\beta}^2$, $j = 1, \ldots, p$. Its asymptotic distribution is $\chi^2$ with $\frac{p(p-1)}{2}$ degrees of freedom.

For the specific case of $p = 2$, $\tilde{R}_{\beta,n} = 2n\tilde{\kappa}_0^2(p = 2, \beta)\kappa_1^{-1}(p = 2, \beta) R_{12,\beta}^2$, with $2\kappa_1^{-1}(p, \beta) = (2\beta + 1)^3$ (see Example 4 in Basu et al. (2021)).

**Remark 21**  Notice that (27) has an explicit expression for $\beta = 0$, which is based on the classical Pearson sample correlations (without constraints), $\tilde{R}_n = \tilde{R}_{\beta=0,n} = n \sum_{i<j} R_{ij,\beta}$, while for $\beta > 0$, the correlations are calculated as

$$R_{X,\beta} = \text{diag}^{-1}\{S_{j,\beta}\}_{j=1}^p S_{X,\beta} \text{diag}^{-1}\{S_{j,\beta}\}_{j=1}^p,$$

$$S_{X,\beta} = \frac{1}{n} \sum_{i=1}^n \tilde{w}_{i,\beta}(X_i - \tilde{\mu}_\beta)(X_i - \tilde{\mu}_\beta)^T,$$

$$S_{i,\beta}^2 = \text{diag}(S_{X,\beta}),$$

$$\tilde{\mu}_\beta = \frac{1}{n} \sum_{i=1}^n \tilde{w}_{i,\beta} X_i,$$

$$\tilde{\kappa}_0(p, \beta) = \frac{1}{\kappa_0(p, \beta)},$$

$$\tilde{w}_{i,\beta} = \exp \left\{ -\frac{\beta}{2}(X_i - \tilde{\mu}_\beta)^T \text{diag}^{-1}\{S_{j,\beta}\}_{j=1}^p (X_i - \tilde{\mu}_\beta) \right\},$$

\(\tilde{\kappa}_0(p, \beta)\) was given in (21).
Remark 22 In the traditional multivariate analysis, when $p$ is small relative to $n$, Bartlett (1954) established the likelihood ratio test for the complete independence as

$$- \left( n - 1 - \frac{2p + 5}{6} \right) \log |R_{X,\beta=0}| \xrightarrow{p \to \infty} \chi^2_{\frac{p(p-1)}{2}},$$

which is not longer valid for $p > n$, since the lack of positive definiteness makes the determinant to be null. Such a problem of the likelihood ratio test does not exist for the Rao’s score test.

Remark 23 In Kallenberg et al. (1997) it is mentioned $\tilde{R}_n = nR_{12}^2$ (27), with $p = 2$ and $\beta = 0$) to be the classical Rao test statistic for testing independence for bivariate normal random variables and its asymptotic standard normality is very well-known from different sources such as the example given in page 293 of Lehman (1999).

3.5 Testing non-fixed equicorrelation

The non-fixed equicorrelation structure establishes homogeneity for all off-diagonal elements of the correlation matrix, i.e.

$$H_0: \rho_{ij} = \rho_{12}, \forall i \neq j \quad \text{vs.} \quad H_1: \exists i \neq j \text{ s.t. } \rho_{ij} \neq \rho_{12},$$

for unknown value of $\rho_{12}$. In addition, it is assumed

$$\frac{-1}{p - 1} < \rho_{12} < 1,$$

for imposing positive definiteness. In matrix form, the equicorrelation can be expressed as

$$R(\rho_{12}) = (1 - \rho_{12})I_p + \rho_{12}1_p1^T_p.$$
Corollary 24 The Rao’s β-score test-statistic for testing equicorrelation, according to (23), is given by

\[
\tilde{R}_{\beta,n} = \frac{2n\tilde{\kappa}_0^2(p,\beta)\kappa_1^{-1}(p,\beta)}{(1-\tilde{\rho}_{12,\beta})^2} \sum_{i<j}(R_{ij,\beta} - \tilde{\rho}_{12,\beta})^2,
\]

where \(R_{ij,\beta}\) are extradiagonal elements of (17) when \(\tilde{\sigma}_{j,\beta}^2 = S_{j,\beta}^2\), \(j = 1, ..., p\), \(\tilde{\rho}_{12,\beta}\) was defined in Theorem 13 and it is assumed to belong to \((-\frac{1}{p-1},1)\). The asymptotic distribution is \(\chi^2\) with \(\frac{p(p-1)}{2}\) degrees of freedom.

The expression is the same as the one of Corollary 19 with \(\rho_0\) replaced by \(\tilde{\rho}_{12,\beta}\), according to Theorem 13.

Remark 25 In this case it holds \(\tilde{\sigma}_{j,\beta}^2 = S_{j,\beta}^2\), \(j = 1, ..., p\) and hence for \(\beta = 0\) the classical sample Pearson correlations are used.

4 Discussion

Apart from the interesting developed results, an important contribution of this paper is to let publicly know that Rao’s score tests could be satisfactorily exploited in a high-dimensional setting. The test statistic proposed in Nagao (1973) for testing, \(H_0: \theta_2 = \text{vech}(\Sigma_0)\) vs. \(H_1: \theta_2 \neq \text{vech}(\Sigma_0)\), similar to (22) but not the same, seems to be a Rao’s score tests but it is not being recognized as a Rao’s score test. In Fujikoshi et al. (2010, Section 8.1.3) provided a detailed explanation of the test given in Nagao (1973) and Ledoit and Wolf (2002) analyzed the robustness of such a test against high dimensionality, i.e. the asymptotic behavior of a (Rao’s score) test-statistic when \(\lim_{n,p \to \infty} \frac{p}{n} = c \in (0, +\infty)\), with \(c\) being the so-called “concentration”. The hypothesis testing is similar to (22), but not the same as the variances are fixed under the null hypothesis. Schott (2005) proposed (27) for testing complete independence.
in high-dimensional data, i.e. under the aforementioned limiting assumption for \( n \) and \( p \). These publications, jointly this one, motivate us to consider such limiting assumption in a future paper.

5 Proofs

5.1 Proof of Proposition 1

The detailed proof of the expressions of the score functions was given in McCulloch (1982), and in particular the expression of the second partition one needs careful derivations,

\[
\begin{align*}
  s_\Sigma(X) &= -\frac{1}{2} \frac{\partial}{\partial \Sigma} \left( \log |\Sigma| + (X - \mu)^T \Sigma^{-1} (X - \mu) \right) \\
  &= -\frac{1}{2} \Sigma^{-1} + \frac{\partial}{\partial \Sigma} \left[ (X - \mu)^T \Sigma^{-1} (X - \mu) \right] \\
  &= -\frac{1}{2} \Sigma^{-1} - \Sigma^{-1} (X - \mu)(X - \mu)^T \Sigma^{-1}
\end{align*}
\]

and hence,

\[
\begin{align*}
  s_{\text{vech}(\Sigma)}(X) &= -\frac{1}{2} \text{vech} \left( \Sigma^{-1} - \Sigma^{-1} (X - \mu)(X - \mu)^T \Sigma^{-1} \right) \\
  &= -\frac{1}{2} G_p^T \text{vec}(\Sigma^{-1}) + \frac{1}{2} G_p^T \text{vec} \left( \Sigma^{-1} (X - \mu)(X - \mu)^T \Sigma^{-1} \right) \\
  &= -\frac{1}{2} G_p^T \text{vec}(\Sigma^{-1}) + \frac{1}{2} G_p^T \left( \Sigma^{-1} \otimes \Sigma^{-1} \right) \text{vec} \left( (X - \mu)(X - \mu)^T \right) \\
  &= -\frac{1}{2} G_p^T \text{vec}(\Sigma^{-1}) + \frac{1}{2} G_p^T \left( \Sigma^{-1} \otimes \Sigma^{-1} \right) ((X - \mu) \otimes (X - \mu)) \\
  &= -\frac{1}{2} G_p^T \text{vec}(\Sigma^{-1}) + \frac{1}{2} G_p^T \left[ (\Sigma^{-1}(X - \mu)) \otimes ((X - \mu)^T \Sigma^{-1}) \right].
\end{align*}
\]
5.2 Proof of Theorem 2

Let us consider

\[ f_{\theta}^{\beta+1}(x) = \frac{1}{(2\pi)^{\frac{d+1}{2}} |\Sigma|^{\frac{\beta+1}{2}}} \exp \left\{ -\frac{1}{2} (x - \mu)^T \left( \frac{1}{\beta+1} \Sigma \right)^{-1} (x - \mu) \right\} \]

\[ = \frac{(2\pi)^{\frac{\beta}{2}} |\Sigma|^{\frac{1}{2}}}{(2\pi)^{\frac{\beta+1}{2}} |\Sigma|^{\frac{\beta+1}{2}}} \frac{1}{(2\pi)^{\frac{\beta+1}{2}} |\Sigma|^{\frac{\beta+1}{2}}} \exp \left\{ -\frac{1}{2} (x - \mu)^T \left( \frac{1}{\beta+1} \Sigma \right)^{-1} (x - \mu) \right\} \]

\[ = \frac{(2\pi)^{\frac{\beta}{2}} |\Sigma|^{\frac{1}{2}}}{(2\pi)^{\frac{\beta+1}{2}} |\Sigma|^{\frac{\beta+1}{2}}} \frac{1}{(2\pi)^{\frac{\beta+1}{2}} |\Sigma|^{\frac{\beta+1}{2}}} \exp \left\{ -\frac{1}{2} (x - \mu)^T \left( \frac{1}{\beta+1} \Sigma \right)^{-1} (x - \mu) \right\} \]

\[ = \frac{(\beta + 1)^{-\frac{\beta}{2}}}{(2\pi)^{\frac{\beta+1}{2}} |\Sigma|^{\frac{\beta+1}{2}}} f_{\theta}^{\mu^*}(x), \]

where the parameter vector is given by \( \theta^* = (\mu^T, vech^T(\Sigma^*))^T, \mu^* = \mu \) and \( \Sigma^* = \frac{1}{\beta+1} \Sigma \). Notice that if we call

\[ \vartheta = (\mu^T, (\beta + 1)vech^T(\Sigma))^T \quad (29) \]

then

\[ f_{\vartheta}^{\beta+1}(x) = \frac{(\beta + 1)^{-\frac{\beta}{2}}}{(2\pi)^{\frac{\beta}{2}} |(\beta + 1)\Sigma|^{\frac{\beta}{2}}} f_{\theta}(x) \]

\[ = \frac{(\beta + 1)^{-\frac{\beta}{2}}}{(2\pi)^{\frac{\beta}{2}} |\Sigma|^{\frac{\beta}{2}}} f_{\theta}(x), \]
\[
J_\beta(\vartheta) = E_\vartheta[\vartheta(X)\vartheta^T(X)f_\theta^\beta]
\]
\[
= \int_{\mathbb{R}^p} \vartheta(x)\vartheta^T(x)f_\theta^{\beta+1}(x)dx
\]
\[
= \frac{(\beta + 1)^{-\frac{p}{2}(\beta+1)}}{(2\pi)^{\frac{dp}{2}}|\Sigma|^\frac{p}{2}} \int_{\mathbb{R}^p} \vartheta(x)\vartheta^T(x)f_\theta(x)dx
\]
\[
= \frac{(\beta + 1)^{-\frac{p}{2}(\beta+1)}}{(2\pi)^{\frac{dp}{2}}|\Sigma|^\frac{p}{2}}E_\theta[\vartheta(X)\vartheta^T(X)],
\]
where

\[
\vartheta(x) = \frac{\partial}{\partial \vartheta} \log f_\vartheta(x) = \left( \frac{\partial}{\partial \mu} \log f_\vartheta(x) \right) \left( \frac{1}{\beta+1} \frac{\partial}{\partial \text{vech}(\Sigma)} \log f_\vartheta(x) \right)
\]
\[
= \left( \frac{1}{\beta+1} \frac{\partial}{\partial \mu} \log f_\vartheta(x) \right) \left( -\frac{p}{2} \frac{\partial}{\partial \text{vech}(\Sigma)} \log |\Sigma| + \frac{\partial}{\partial \text{vech}(\Sigma)} \log f_\vartheta(x) \right),
\]
\[
\frac{\partial}{\partial \mu} \log f_\vartheta(x) = \frac{\partial}{\partial \mu} \left( -\frac{p}{2} \log(2\pi) - \frac{1}{2} \log |(\beta + 1)\Sigma| - \frac{1}{2(\beta + 1)}(x - \mu)^T\Sigma^{-1}(x - \mu) \right)
\]
\[
= -\frac{\partial}{\partial \mu} \left( \frac{1}{2(\beta + 1)}(x - \mu)^T\Sigma^{-1}(x - \mu) \right)
\]
\[
= -\frac{1}{\beta + 1} \frac{\partial}{\partial \mu} \left( \frac{1}{2}(x - \mu)^T\Sigma^{-1}(x - \mu) \right)
\]
\[
= \frac{1}{\beta + 1} \frac{\partial}{\partial \mu} \log f_\vartheta(x),
\]
\[
\begin{align*}
\frac{\partial}{\partial (\beta + 1) \text{vech}(\Sigma)} \log f_\theta(x) &= \frac{\partial}{\partial (\beta + 1) \text{vech}(\Sigma)} \left( -\frac{p}{2} \log(2\pi) - \frac{1}{2} \log |(\beta + 1)\Sigma| - \frac{1}{2} (x - \mu)^T (\beta + 1)^{-1} \Sigma^{-1} (x - \mu) \right) \\
&= \frac{\partial}{\partial (\beta + 1) \text{vech}(\Sigma)} \left( -\frac{1}{2} \log |(\beta + 1)\Sigma| - \frac{1}{2} (x - \mu)^T (\beta + 1)^{-1} \Sigma^{-1} (x - \mu) \right) \\
&= \frac{\partial}{\partial \text{vech}(\Sigma)} \left( \frac{\partial}{\partial (\beta + 1) \text{vech}(\Sigma)} \left( -\frac{1}{2} \log |\Sigma| - \frac{1}{2 (\beta + 1)} (x - \mu)^T \Sigma^{-1} (x - \mu) \right) \right) \\
&= \frac{1}{(\beta + 1)^2} \frac{\partial}{\partial \text{vech}(\Sigma)} \left( -\frac{\beta + 1}{2} \log |\Sigma| - \frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right) \\
&= \frac{1}{(\beta + 1)^2} \left[ -\frac{\beta}{2} \frac{\partial}{\partial \text{vech}(\Sigma)} \log |\Sigma| + \frac{\partial}{\partial \text{vech}(\Sigma)} \left( -\frac{1}{2} \log |\Sigma| - \frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right) \right] \\
&= \frac{1}{(\beta + 1)^2} \left( -\frac{\beta}{2} \frac{\partial}{\partial \text{vech}(\Sigma)} \log |\Sigma| + \frac{\partial}{\partial \text{vech}(\Sigma)} \log f_\theta(x) \right),
\end{align*}
\]

\[
\begin{align*}
\varsigma_\theta(X) &= \begin{pmatrix} \varsigma_\mu(X) \\ \varsigma_{(\beta + 1) \text{vech}(\Sigma)}(X) \end{pmatrix}, & \varsigma_\theta(X) &= \frac{\partial}{\partial \theta} \log f_\theta(X), \\
\varsigma_\mu(X) &= \frac{\partial}{\partial \mu} \log f_\theta(x), & \varsigma_{(\beta + 1) \text{vech}(\Sigma)}(X) &= \frac{\partial}{\partial (\beta + 1) \text{vech}(\Sigma)} \log f_\theta(X), \\
\varsigma_\theta(X) &= \begin{pmatrix} s_\mu(X) \\ s_{\text{vech}(\Sigma)}(X) \end{pmatrix}, & \varsigma_\theta(X) &= \frac{\partial}{\partial \theta} \log f_\theta(X), \\
s_\mu(X) &= \frac{\partial}{\partial \mu} \log f_\theta(x), & s_{\text{vech}(\Sigma)}(X) &= \frac{\partial}{\partial \text{vech}(\Sigma)} \log f_\theta(X), \\
\varsigma_\theta(X)^T \varsigma_\theta(X) &= \begin{pmatrix} \varsigma_\mu(X)^T \varsigma_\mu(X) & \varsigma_\mu(X)^T \varsigma_{(\beta + 1) \text{vech}(\Sigma)}(X) \\ \varsigma_{(\beta + 1) \text{vech}(\Sigma)}(X)^T \varsigma_\mu(X) & \varsigma_{(\beta + 1) \text{vech}(\Sigma)}(X)^T \varsigma_{(\beta + 1) \text{vech}(\Sigma)}(X) \end{pmatrix}, \\
\varsigma_\mu(X)^T \varsigma_\mu(X) &= \frac{1}{(\beta + 1)^2} s_\mu(X)^T s_\mu(X),
\end{align*}
\]
\[ \Psi(\beta+1) \text{vech}(\Sigma)(X) \Psi^T_{\mu}(X) \]
\[
= \frac{1}{(\beta + 1)^2} \left( \frac{1}{\beta + 1} \left( -\frac{\beta}{2} \partial \log |\Sigma| + s_{\text{vech}(\Sigma)}(X) \right) \right) \Psi^T_{\mu}(X)
\]
\[
= \frac{1}{(\beta + 1)^2} \left( \frac{\beta}{2(\beta + 1)} \partial \log |\Sigma| \Psi^T_{\mu}(X) + \frac{1}{\beta + 1} s_{\text{vech}(\Sigma)}(X) s^T_{\mu}(X) \right)
\]

\[ \Psi(\beta+1) \text{vech}(\Sigma)(X) \Psi^T_{(\beta+1)\text{vech}(\Sigma)}(X) \]
\[
= \frac{1}{(\beta + 1)^2} \left( \frac{1}{(\beta + 1)} \left( -\frac{\beta}{2} \partial \log |\Sigma| + s_{\text{vech}(\Sigma)}(X) \right) \right) \left( -\frac{\beta}{2} \partial \log |\Sigma| + s_{\text{vech}(\Sigma)}(X) \right)^T
\]
\[
= \frac{1}{(\beta + 1)^2} \left( \frac{\beta^2}{4(\beta + 1)^2} \partial \log |\Sigma| \partial \log |\Sigma| - \frac{\beta}{2(\beta + 1)^2} \partial \log |\Sigma| s^T_{\text{vech}(\Sigma)}(X) - \frac{\beta}{(\beta + 1)^2} s_{\text{vech}(\Sigma)}(X) \frac{\partial}{\partial \text{vech}^T(\Sigma)} \log |\Sigma| + \frac{1}{(\beta + 1)^2} s_{\text{vech}(\Sigma)}(X) s^T_{\text{vech}(\Sigma)}(X) \right),
\]

its expectation is

\[ E_{\theta}[\Psi_\phi(X) \Psi^T_\phi(X)] = \frac{1}{(\beta + 1)^2} \left( E_{\theta}[s_{\mu}(X) s^T_{\mu}(X)] \frac{1}{3 + 1} E_{\theta}[s_{\text{vech}(\Sigma)}(X) s^T_{\mu}(X)] \frac{1}{3 + 1} E_{\theta}[s_{\text{vech}(\Sigma)}(X) s^T_{\text{vech}(\Sigma)}(X)] \right) \]

where

\[ C_\theta = \frac{\partial}{\partial \text{vech}(\Sigma)} \log |\Sigma| \frac{\partial}{\partial \text{vech}^T(\Sigma)} \log |\Sigma| = G_p \text{vec} (\Sigma^{-1}) \text{vec}^T (\Sigma^{-1}) G_p \]

with \( G_p \) being the so-called “duplication matrix” of order \( p \), i.e., the unique \( p^2 \times \frac{p(p+1)}{2} \) matrix such that \( \text{vec}(\Sigma) = G_p \text{vec}(\Sigma) \) and the last derivatives are deduced from McCulloch (1982). Notice that some terms are cancelled since they appear multiplied...
by expectations, which were null, \( E_\theta[s_\theta(X)] = 0_{p+p(p+1)/2} \). Hence

\[
J_\beta^*(\vartheta) = \frac{(\beta + 1)^{-\frac{p}{2}(\beta+1)}}{(2\pi)^{p/2} |\Sigma|^{\frac{p}{2}}} E_\theta[s_\theta(X)s_\theta^T(X)] \\
= \frac{(\beta + 1)^{-\frac{p}{2}(\beta+1)-2}}{(2\pi)^{p/2} |\Sigma|^{\frac{p}{2}}} \left( \begin{array}{ll}
E_\theta[s_\mu(X)s_\mu^T(X)] & \frac{1}{\beta+1} E_\theta[s_\mu(X)s_{\text{vech}(\Sigma)}^T(X)] \\
\frac{1}{\beta+1} E_\theta[s_{\text{vech}(\Sigma)}(X)s_\mu^T(X)] & \frac{1}{(\beta+1)^2} \left( \frac{\nu^2}{4} \mathbf{C}_\theta + E_\theta[s_{\text{vech}(\Sigma)}(X)s_{\text{vech}(\Sigma)}^T(X)] \right) 
\end{array} \right)
\]

(30)

\[
s_\mu(X) = \frac{\partial}{\partial \mu} \log f_\theta(X) = \frac{\partial}{\partial \mu} \left( -\frac{p}{2} \log(2\pi) - \frac{1}{2} \log |\Sigma| - \frac{1}{2} (X - \mu)^T \Sigma^{-1} (X - \mu) \right)
= -\frac{\partial}{\partial \mu} \frac{1}{2} (X - \mu)^T \Sigma^{-1} (X - \mu)
= \Sigma^{-1} (X - \mu),
\]

and

\[
E_\theta[s_\mu(X)s_\mu^T(X)] = E_\theta \left[ \Sigma^{-1} (X - \mu)(X - \mu)^T \Sigma^{-1} \right]
= \Sigma^{-1} E_\theta \left[ (X - \mu)(X - \mu)^T \right] \Sigma^{-1}
= \Sigma^{-1} \Sigma \Sigma^{-1}
= \Sigma^{-1}.
\]

We omit the rest of the terms, since they were calculated by McCulloch (1982) in detailed way. Accordingly, the Fisher information matrix is

\[
\begin{pmatrix}
E_\theta[s_\mu(X)s_\mu^T(X)] & E_\theta[s_{\text{vech}(\Sigma)}(X)s_\mu^T(X)] \\
(E_\theta[s_\mu(X)s_{\text{vech}(\Sigma)}^T(X)])^T & E_\theta[s_{\text{vech}(\Sigma)}(X)s_{\text{vech}(\Sigma)}^T(X)]
\end{pmatrix},
\]

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where

\[ E_{\theta}[s_{\mu}(X)s_{\mu}^T(X)] = \Sigma^{-1}, \]
\[ E_{\theta}[s_{\mu}(X)s_{vech(\Sigma)}^T(X)] = 0_{p \times \frac{p(p+1)}{2}}, \]
\[ E_{\theta}[s_{vech(\Sigma)}(X)s_{vech(\Sigma)}^T(X)] = \frac{1}{2}G_p^T(\Sigma^{-1} \otimes \Sigma^{-1})G_p. \]

Hence

\[ J^*_\beta(\vartheta) = E_{\theta}[s_{\vartheta}(X)s_{\vartheta}^T(X)f_\vartheta^\beta(X)] \]
\[ J^*_\beta(\vartheta) = \begin{pmatrix} J^*_\beta(\mu) & 0_{p \times \frac{p(p+1)}{2}} \\ 0_{\frac{p(p+1)}{2} \times p} & J^*_\beta((\beta + 1)vech(\Sigma)) \end{pmatrix}, \quad (31) \]

where

\[ J^*_\beta(\mu) = \frac{(\beta + 1) - \frac{p}{2}(\beta + 1)^{-2}}{(2\pi)^{p/2} |\Sigma|^{p/2}} \Sigma^{-1}, \quad (32) \]
\[ J^*_\beta((\beta + 1)vech(\Sigma)) = \frac{(\beta + 1)^{-\frac{p}{2}(\beta + 1)^{-4}}}{4(2\pi)^{3p/2} |\Sigma|^{p/2}} \left[ \beta^2 C_{\vartheta} + 2G_p^T(\Sigma^{-1} \otimes \Sigma^{-1})G_p \right]. \quad (33) \]

According to the original problem, from (31)-(32)-(33) associated to (29), taking into account

\[ J^*_\beta(\mu) = \frac{(\beta + 1) - \frac{p}{2}}{(2\pi)^{p/2} |(\beta + 1)\Sigma|^{p/2}} ((\beta + 1)\Sigma)^{-1}, \]
\[ J^*_\beta((\beta + 1)vech(\Sigma)) = \frac{(\beta + 1)^{-\frac{p}{2} - 2}}{4(2\pi)^{3p/2} |(\beta + 1)\Sigma|^{p/2}} \left[ \beta^2 C_{\vartheta} + 2G_p^T((\beta + 1)\Sigma^{-1} \otimes (\beta + 1)\Sigma^{-1})G_p \right], \]

we deduce the final expression of \( J_\beta(\theta) \) for \( \theta = (\mu^T, \text{vech}^T(\Sigma))^T \).

\[ \frac{(\beta + 1)^{-\frac{p}{2} - 2}}{4(2\pi)^{3p/2} |\Sigma|^{p/2}} \left[ \beta^2 C_{\vartheta} + 2G_p^T(\Sigma^{-1} \otimes \Sigma^{-1})G_p \right] \]
5.3 Proof of Theorem 3

Following the parametrization of the Proof 2 (see 5.2), we obtain

\[ \xi^*(\mu) = \int_{\mathbb{R}^p} s_\mu(x) f_\theta^{\beta+1}(x) dx \]
\[ = \frac{\beta + 1}{(2\pi)^{\frac{\beta p}{2}}} |\Sigma|^{\frac{\beta}{2}} \int_{\mathbb{R}^p} s_\mu(x) f_\theta(x) dx \]
\[ = 0_p, \]

\[ \xi^*((\beta + 1)\text{vech}(\Sigma)) = \int_{\mathbb{R}^p} s_{(\beta+1)\text{vech}(\Sigma)}(x) f_\theta^{\beta+1}(x) dx \]
\[ = \frac{\beta + 1}{(2\pi)^{\frac{\beta p}{2}}} |\Sigma|^{\frac{\beta}{2}} \int_{\mathbb{R}^p} \left( -\frac{\beta}{2} \frac{\partial}{\partial \text{vech}(\Sigma)} \log |\Sigma| + s_{\text{vech}(\Sigma)}(x) \right) f_\theta(x) dx \]
\[ = -\beta \frac{(\beta + 1)^{-\frac{\beta}{2}(\beta+1)-2}}{2 (2\pi)^{\frac{\beta p}{2}} |\Sigma|^{\frac{\beta}{2}}} \partial \Sigma \]
\[ = -\frac{\beta}{2} \frac{(\beta + 1)^{-\frac{\beta}{2}(\beta+1)-2}}{2 (2\pi)^{\frac{\beta p}{2}} |\Sigma|^{\frac{\beta}{2}}} G_p^T \text{vec} (\Sigma^{-1}) \]
\[ = -\frac{\beta}{2} \frac{(\beta + 1)^{-\frac{\beta}{2}(\beta+1)}}{2 (2\pi)^{\frac{\beta p}{2}} |(\beta + 1)|^{\frac{\beta}{2}}} G_p^T \text{vec} (((\beta + 1)\Sigma^{-1})^{-1}) \]

and their cross products

\[ \xi^*(\mu)\xi^{*T}(\mu) = 0_{p \times p}, \]

\[ \xi^*((\beta + 1)\text{vech}(\Sigma))\xi^{*T}((\beta + 1)\text{vech}(\Sigma)) \]
\[ = \frac{\beta^2}{4} (\beta + 1)^{-\frac{\beta p}{2}} G_p^T \text{vec} (((\beta + 1)\Sigma^{-1})^{-1}) \text{vec}^T (((\beta + 1)\Sigma^{-1})^{-1}) G_p \]
\[ = \frac{\beta^2}{4} (\beta + 1)^{-(\beta+2)} C_\theta. \]
Translating these terms to the original parametrization \( \theta = (\mu^T, \text{vech}^T(\Sigma))^T \), we obtain the final expression of \( \xi_\beta(\theta) \) and \( \xi_\beta(\theta)\xi_\beta^T(\theta) \).

5.4 Proof of Corollary 4

From the (3)-(4)-(5) and (7)-(8) we can obtain

\[
K_\beta(\text{vech}(\Sigma)) = J_{2\beta}(\text{vech}(\Sigma)) - \xi_\beta(\text{vech}(\Sigma))\xi_\beta^T(\text{vech}(\Sigma))
\]

\[
= \frac{(2\beta + 1)^{-\frac{p}{2} - 2}}{4(2\pi)^{\beta p} |\Sigma|^{\beta/2}} \left[ 4\beta^2 C_\theta + 2G_p^T (\Sigma^{-1} \otimes \Sigma^{-1}) G_p \right] - \frac{\beta^2 (\beta + 1)^{-(p+2)}}{4(2\pi)^{\beta p} |\Sigma|^{\beta/2}} C_\theta
\]

\[
= \frac{1}{4(2\pi)^{\beta p} |\Sigma|^{\beta/2}} \left\{ 2(2\beta + 1)^{-\frac{p}{2} - 2} G_p^T (\Sigma^{-1} \otimes \Sigma^{-1}) G_p + \beta^2 \left[ 4(2\beta + 1)^{-\frac{p}{2} - 2} - (\beta + 1)^{-(p+2)} \right] C_\theta \right\}
\]

\[
= \frac{1}{4(2\pi)^{\beta p} |\Sigma|^{\beta/2}} G_p^T \left[ J_{2\beta}(\Sigma^{-1}) + \xi_\beta(\Sigma^{-1})\xi_\beta^T(\Sigma^{-1}) \right] G_p.
\]

5.5 Proof of Proposition 5

For calculating (10) we take into account the following version of the Woodbury’s formula

\[
(G + uv^T)^{-1} = G^{-1} - (1 + v^T G^{-1} u)^{-1} G^{-1} uv^T G^{-1},
\]

i.e.

\[
\left( J_{2\beta}(R_0^{-1}) + \xi_\beta(R_0^{-1})\xi_\beta^T(R_0^{-1}) \right)^{-1} = J_{2\beta}^{-1}(R_0^{-1}) - \frac{J_{2\beta}^{-1}(R_0^{-1})\xi_\beta(R_0^{-1})\xi_\beta^T(R_0^{-1})J_{2\beta}^{-1}(R_0^{-1})}{1 + \xi_\beta^T(R_0^{-1})J_{2\beta}^{-1}(R_0^{-1})\xi_\beta(R_0^{-1})}
\]

\[
= \kappa_1^{-1}(p, \beta) \left( (R_0 \otimes R_0) - \frac{\kappa_3(p, \beta) \text{vec}(R_0)\text{vec}^T(R_0)}{1 + p\kappa_3(p, \beta)} \right).
\]
where $\kappa_3(p, \beta)$ is (11). In the last equality

\[
\overline{J}_{2\beta}^{-1}(R_0^{-1})\overline{\xi}_\beta(R_0^{-1}) = \kappa_1^{-1}(p, \beta)\kappa_2(p, \beta) (R_0 \otimes R_0) \text{vec}(R_0^{-1})
\]

\[
= \kappa_1^{-1}(p, \beta)\kappa_2(p, \beta) \text{vec}(R_0R_0^{-1}R_0)
\]

\[
= \kappa_1^{-1}(p, \beta)\kappa_2(p, \beta) \text{vec}(R_0),
\]

and

\[
\overline{\xi}_\beta^T(R_0^{-1})J_{2\beta}^{-1}(R_0^{-1})\overline{\xi}_\beta(R_0^{-1}) = \kappa_1^{-1}(p, \beta)\kappa_2^2(p, \beta) \text{vec}^T(R_0)\text{vec}(R_0^{-1})
\]

\[
= \kappa_1^{-1}(p, \beta)\kappa_2^2(p, \beta) \text{vec}^T(R_0^{1/2}I_p R_0^{1/2}) \text{vec}(R_0^{-1/2}I_p R_0^{-1/2})
\]

\[
= \kappa_1^{-1}(p, \beta)\kappa_2^2(p, \beta) \text{vec}^T(I_p)\text{vec}(I_p)
\]

\[
= \kappa_1^{-1}(p, \beta)\kappa_2^2(p, \beta)\text{trace}(I_p)
\]

\[
= \kappa_3(p, \beta)p.
\]

### 5.6 Proof of Theorem 6

The $\beta$-score function is

\[
U_{\beta,n}(\mu) = \frac{1}{n} \sum_{i=1}^{n} f^\beta_{\theta}(X_i) s_\mu(X_i) - \xi_\beta(\mu)
\]

\[
= -\frac{1}{2n} \sum_{i=1}^{n} f^\beta_{\theta}(X_i)(X_i - \mu)
\]

\[
= -\frac{1}{2(2\pi)^{p/2}|\Sigma|^{p/2}} \sum_{i=1}^{n} w_{i,\beta}(\theta)(X_i - \mu),
\]

with

\[
f^\beta_{\theta}(X_i) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{p/2}} w_{i,\beta}(\theta),
\]
$w_{i,\beta}(\theta)$ is (12) and

$$U_{\beta,n}(\text{vech}(\Sigma)) = \frac{1}{n} \sum_{i=1}^{n} f_{\beta}(X_i) \text{vech}(\Sigma) (X_i) - \xi_{\beta}(\Sigma)$$

$$= - \frac{1}{2(2\pi)^{\frac{p}{2}} |\Sigma|^\frac{p}{2}} G_p^T \text{vec}(\Sigma^{-1}) \frac{1}{n} \sum_{i=1}^{n} w_{i,\beta}(\theta)$$

$$+ \frac{1}{2(2\pi)^{\frac{p}{2}} |\Sigma|^\frac{p}{2}} G_p^T (\Sigma^{-1} \otimes \Sigma^{-1}) \frac{1}{n} \sum_{i=1}^{n} w_{i,\beta}(\theta) ((X_i - \mu) \otimes (X_i - \mu))$$

$$+ \frac{\beta (\beta + 1)^{-\frac{p}{2}+1} G_p^T \text{vec}(\Sigma^{-1})}{(2\pi)^{\frac{p}{2}} |\Sigma|^\frac{p}{2}}$$

$$= - \frac{1}{2(2\pi)^{\frac{p}{2}} |\Sigma|^\frac{p}{2}} G_p^T \text{vec}(\Sigma^{-1}) \left( \frac{1}{n} \sum_{i=1}^{n} w_{i,\beta}(\theta) - \beta (\beta + 1)^{-\frac{p}{2}+1} \right)$$

$$+ \frac{1}{2(2\pi)^{\frac{p}{2}} |\Sigma|^\frac{p}{2}} G_p^T \frac{1}{n} \sum_{i=1}^{n} w_{i,\beta}(\theta) (\Sigma^{-1}(X_i - \mu)) \otimes (\Sigma^{-1}(X_i - \mu))$$

$$= G_p^T V_{\beta,n}(\text{vec}(\Sigma)) \quad (34)$$

with $V_{\beta,n}(\text{vec}(\Sigma))$ given in (13).

### 5.7 Proof of Proposition 7

According with [34]

$$- 2(2\pi)^{\frac{p}{2}} |\tilde{\Sigma}_{\beta}|^\frac{p}{2} V_{\beta,n}(\tilde{\Lambda}_{\beta}, R_0)$$

$$= (\tilde{\Lambda}_{\beta}^{-1/2} \otimes \tilde{\Lambda}_{\beta}^{-1/2}) (R_0^{-1} \otimes R_0^{-1}) \left[ \frac{1}{n} \sum_{i=1}^{n} \tilde{w}_{i,\beta}(\tilde{\Lambda}_{\beta}^{-1/2} (X_i - \tilde{\mu}_{\beta})) \otimes (\tilde{\Lambda}_{\beta}^{-1/2} (X_i - \tilde{\mu}_{\beta})) \right]$$

$$- \tilde{\kappa}_0(p, \beta)(\tilde{\Lambda}_{\beta}^{-1/2} \otimes \tilde{\Lambda}_{\beta}^{-1/2}) \text{vec}(R_0^{-1})$$

$$= \tilde{\kappa}_0(p, \beta)(\tilde{\Lambda}_{\beta}^{-1/2} \otimes \tilde{\Lambda}_{\beta}^{-1/2}) \left[ (R_0^{-1} \otimes R_0^{-1}) \text{vec}(\tilde{R}_{X,\beta}) - \text{vec}(R_0^{-1}) \right]$$

$$= \tilde{\kappa}_0(p, \beta)(\tilde{\Lambda}_{\beta}^{-1/2} \otimes \tilde{\Lambda}_{\beta}^{-1/2}) \left[ \text{vec}(R_0^{-1} \tilde{R}_{X,\beta} R_0^{-1}) - \text{vec}(R_0^{-1} R_0 R_0^{-1}) \right]$$

$$= \tilde{\kappa}_0(p, \beta)(\tilde{\Lambda}_{\beta}^{-1/2} \otimes \tilde{\Lambda}_{\beta}^{-1/2}) (R_0^{-1} \otimes R_0^{-1}) \left[ \text{vec}(\tilde{R}_{X,\beta}) - \text{vec}(R_0) \right] .$$
5.8 Proof of Theorem

The expressions of $s_\theta(x)$, $J_\beta(\theta)$, $\xi_\beta(\theta)$, $K_\beta(\theta)$, $U_{\beta,n}^T(\theta)$ given in the previous section can be adjusted taking into account

$$\frac{\partial}{\partial \eta^T} \theta = I_p \oplus \frac{\partial}{\partial \eta^T} \theta_2$$

$$\frac{\partial}{\partial \eta^T} \theta_2 = M \frac{\partial}{\partial \eta^T} \phi_2 = M \left( I_p \oplus \frac{\partial}{\partial \eta^T} \phi_{2,2} \right)$$

$$= M \left( I_p \oplus \text{diag}^\frac{1}{2} \left( \text{vecl}^T(\eta_{2,1}\eta_{2,1}^T) \right) \right)$$

$$\text{diag}^\frac{1}{2} \left( \text{vecl}^T(\eta_{2,1}\eta_{2,1}^T) \right) = \text{diag}\{\sigma_1\sigma_2, \sigma_1\sigma_3, ..., \sigma_{p-1}\sigma_p\}.$$ 

In fact,

$$s_\eta(x) = \frac{\partial}{\partial \eta} \theta^T s_\theta(x) = \left( I_p \oplus \frac{\partial}{\partial \eta^T} \theta_2 \right) s_\theta(x) = \begin{pmatrix} s_{\theta_1}(x) \\ \frac{\partial}{\partial \eta^T} \theta_2 s_{\theta_2}(x) \end{pmatrix},$$

$$s_{\eta_1}(x) = s_{\theta_1}(x), \quad s_{\eta_2}(x) = \frac{\partial}{\partial \eta^T} \theta_2 s_{\theta_2}(x) = \begin{pmatrix} P^T s_{\theta_2}(x) \\ \text{diag}^\frac{1}{2} \left( \text{vecl}(\eta_{2,1}\eta_{2,1}^T) \right) Q^T s_{\theta_2}(x) \end{pmatrix},$$

$$s_{\eta_2,1}(x) = P^T s_{\theta_2}(x), \quad s_{\eta_2,2}(x) = \text{diag}^\frac{1}{2} \left( \text{vecl}(\eta_{2,1}\eta_{2,1}^T) \right) Q^T s_{\theta_2}(x);$$

$$\xi_\beta(\eta) = \frac{\partial}{\partial \eta} \theta^T = \left( I_p \oplus \frac{\partial}{\partial \eta^T} \theta_2 \right) \xi_\beta(\theta) = \begin{pmatrix} \xi_\beta(\theta_1) \\ \frac{\partial}{\partial \eta^T} \theta_2 \xi_\beta(\theta_2) \end{pmatrix},$$

$$\xi_\beta(\eta_1) = \xi_\beta(\theta_1), \quad \xi_\beta(\eta_2) = \frac{\partial}{\partial \eta^T} \theta_2 \xi_\beta(\theta_2) = \begin{pmatrix} \text{diag}^\frac{1}{2} \left( \text{vecl}(\eta_{2,1}\eta_{2,1}^T) \right) Q^T \xi_\beta(\theta_2) \end{pmatrix},$$

$$\xi_\beta(\eta_{2,1}) = P^T \xi_\beta(\theta_2), \quad \xi_\beta(\eta_{2,2}) = \text{diag}^\frac{1}{2} \left( \text{vecl}(\eta_{2,1}\eta_{2,1}^T) \right) Q^T \xi_\beta(\theta_2);$$
\(J_\beta(\eta) = \frac{\partial}{\partial \eta} \theta^T J_\beta(\theta) \frac{\partial}{\partial \eta^T} \theta = \left( I_p \oplus \frac{\partial}{\partial \eta_2} \theta^T_2 \right) J_\beta(\theta) \left( I_p \oplus \frac{\partial}{\partial \eta_2} \theta_2 \right)
\)
\[= \left( \begin{array}{cc}
J_\beta(\theta_1) & 0_{p(p-1)} \\
0_{p(p-1)} & \frac{\partial}{\partial \eta_2} \theta_2 \nu_2 J_\beta(\theta_2) \frac{\partial}{\partial \eta_2} \theta_2 
\end{array} \right)\]

\(J_\beta(\eta_1) = J_\beta(\theta_1),\)

\(J_\beta(\eta_2) = \frac{\partial}{\partial \eta_2} \theta_2 J_\beta(\theta_2) \frac{\partial}{\partial \eta_2} \theta_2
\)
\[= \left( \begin{array}{cc}
P^T J_\beta(\theta_2) P & P^T J_\beta(\theta_2) Q \text{diag}^{\frac{1}{2}}(\text{vecl}(\eta_{2,1}^T \eta_{2,1}^T)) \\
\text{diag}^{\frac{1}{2}}(\text{vecl}(\eta_{2,1}^T \eta_{2,1}^T)) Q^T J_\beta(\theta_2) P & \text{diag}^{\frac{1}{2}}(\text{vecl}(\eta_{2,1}^T \eta_{2,1}^T)) Q^T J_\beta(\theta_2) Q \text{diag}^{\frac{1}{2}}(\text{vecl}(\eta_{2,1}^T \eta_{2,1}^T))
\end{array} \right)\]

\(K_\beta(\eta) = \frac{\partial}{\partial \eta} \theta^T K_\beta(\theta) \frac{\partial}{\partial \eta^T} \theta = \left( I_p \oplus \frac{\partial}{\partial \eta_2} \theta^T_2 \right) K_\beta(\theta) \left( I_p \oplus \frac{\partial}{\partial \eta_2} \theta_2 \right)
\)
\[= \left( \begin{array}{cc}
K_\beta(\theta_1) & 0_{p(p-1)} \\
0_{p(p-1)} & \frac{\partial}{\partial \eta_2} \theta_2 \nu_2 K_\beta(\theta_2) \frac{\partial}{\partial \eta_2} \theta_2 
\end{array} \right)\]

\(K_\beta(\eta_1) = K_\beta(\theta_1),\)

\(K_\beta(\eta_2) = \frac{\partial}{\partial \eta_2} \theta_2 K_\beta(\theta_2) \frac{\partial}{\partial \eta_2} \theta_2
\)
\[= \left( \begin{array}{cc}
P^T K_\beta(\theta_2) P & P^T K_\beta(\theta_2) Q \text{diag}^{\frac{1}{2}}(\text{vecl}(\eta_{2,1}^T \eta_{2,1}^T)) \\
\text{diag}^{\frac{1}{2}}(\text{vecl}(\eta_{2,1}^T \eta_{2,1}^T)) Q^T K_\beta(\theta_2) P & \text{diag}^{\frac{1}{2}}(\text{vecl}(\eta_{2,1}^T \eta_{2,1}^T)) Q^T K_\beta(\theta_2) Q \text{diag}^{\frac{1}{2}}(\text{vecl}(\eta_{2,1}^T \eta_{2,1}^T))
\end{array} \right)\]

\(U_{\beta,n}(\eta_1) = U_{\beta,n}(\theta_1) = U_{\beta,n}(\mu) = -\frac{1}{2(2\pi)^{\frac{d_p}{2}}|\Sigma|^\frac{d_p}{2}} \Sigma^{-1} \frac{1}{n} \sum_{i=1}^{n} w_{i,\beta}(\theta)(X_i - \mu),\)

\(U_{\beta,n}(\eta_2) = \frac{\partial}{\partial \eta_2} \theta^T U_{\beta,n}(\theta_2)\)
\[= \left( \begin{array}{cc}
P^T U_{\beta,n}(\theta_2) & Q^T U_{\beta,n}(\theta_2) \\
\text{diag}^{\frac{1}{2}}(\text{vecl}(\eta_{2,1}^T \eta_{2,1}^T)) Q^T U_{\beta,n}(\theta_2)
\end{array} \right)\]

\(= \left( \begin{array}{cc}
P^T U_{\beta,n}(\text{vecl}(\Sigma)) & Q^T U_{\beta,n}(\text{vecl}(\Sigma)) \\
\text{diag}^{\frac{1}{2}}(\text{vecl}(\eta_{2,1}^T \eta_{2,1}^T)) Q^T U_{\beta,n}(\text{vecl}(\Sigma))
\end{array} \right),\)
\[ U_{\beta,n}(\eta_{2,1}) = -\frac{1}{2(2\pi)^{\frac{d_p}{2}}|\Sigma|^\frac{n}{2}}(G_pP)^T \left[ \left( \beta(\beta + 1)^{-\left(\frac{\beta}{2}+1\right)} - \frac{1}{n} \sum_{i=1}^{n} w_{i,\beta}(\theta) \right) \text{vec} \left( \Sigma^{-1} \right) \right. \]
\[ \left. + \frac{1}{n} \sum_{i=1}^{n} w_{i,\beta}(\theta) \left( \Sigma^{-1}(X_i - \mu) \right) \otimes \left( (X_i - \mu)^T \Sigma^{-1} \right) \right] \]
\[ = P^T U_{\beta,n}(\theta_2), \]

\[ U_{\beta,n}(\eta_{2,2}) = -\frac{1}{2(2\pi)^{\frac{d_p}{2}}|\Sigma|^\frac{n}{2}} \text{diag}^\frac{1}{2} \left( \text{vecl}(\eta_{2,1}^T, \eta_{2,1}^T) \right) (G_pQ)^T \]
\[ \times \left[ \left( \beta(\beta + 1)^{-\left(\frac{\beta}{2}+1\right)} - \frac{1}{n} \sum_{i=1}^{n} w_{i,\beta}(\theta) \right) \text{vec} \left( \Sigma^{-1} \right) \right. \]
\[ \left. + \frac{1}{n} \sum_{i=1}^{n} w_{i,\beta}(\theta) \left( \Sigma^{-1}(X_i - \mu) \right) \otimes \left( (X_i - \mu)^T \Sigma^{-1} \right) \right] \]
\[ = \text{diag}^\frac{1}{2} \left( \text{vecl}(\eta_{2,1}^T, \eta_{2,1}^T) \right) Q^T U_{\beta,n}(\theta_2) \]

5.9 Proof of Theorem 9

For known correlation matrix, \( R = R_0 \), the variance covariance matrix has less parameters to be estimated,

\[ \Sigma(\Lambda) = \Lambda^{1/2} R_0 \Lambda^{1/2}, \]
\[ \Lambda = \text{diag} \{ \sigma_j^2 \}_{j=1}^p. \]

The estimating equations must be cautiously calculated, in comparison with the non-restricted estimators.
The MLEs \((\beta = 0)\) are obtained as solution of

\[
\sum_{i=1}^{n} s_\mu(X_i) = 0_p, \quad (35)
\]

\[
\sum_{i=1}^{n} s_\Lambda(X_i) = 0_{p \times p}, \quad (36)
\]

where

\[
s_\Lambda(X) = -\frac{1}{2} \frac{\partial}{\partial \Lambda} \left( \log |\Lambda|^{1/2} R_0 \Lambda^{1/2} \right) + (X - \mu)^T \Lambda^{-1/2} R_0^{-1} \Lambda^{-1/2} (X - \mu).
\]

The last term’s derivation is

\[
s_\Lambda(X) = -\frac{1}{2} \left( \frac{\partial}{\partial \Lambda} \log |\Lambda| + 2 R_0^{-1} \Lambda^{-1/2} (X - \mu)(X - \mu)^T \frac{\partial}{\partial \Lambda} \Lambda^{-1/2} \right)
\]

\[
= -\frac{1}{2} \left( \Lambda^{-1} - R_0^{-1} \Lambda^{-1/2} (X - \mu)(X - \mu)^T \Lambda^{-3/2} \right)
\]

\[
= -\frac{1}{2} \left( I_p - R_0^{-1} \Lambda^{-1/2} (X - \mu)(X - \mu)^T \Lambda^{-1/2} \right) \Lambda^{-1},
\]

hence from (36) it holds that the MLE of \(\Lambda\) under \(R = R_0, \tilde{\Lambda}\), is the solution of

\[
-\frac{n}{2} \left( I_p - R_0^{-1} R_X(\Lambda) \right) \Lambda^{-1} = 0_{p \times p},
\]

where

\[
R_X(\Lambda) = \Lambda^{-1/2} S_X \Lambda^{-1/2},
\]

\[
S_X = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n)(X_i - \bar{X}_n)^T,
\]

from which are estimated the diagonal elements of \(\Lambda\), as solution in \(\Lambda\), of

\[
1_p = \text{diag}\{R_0^{-1} R_X(\Lambda)\} 1_p
\]

(37)
or equivalently
\[ 1_p = \text{diag}\{R_0^{-1}\Xi(\Lambda)R_X\Xi^T(\Lambda)\}1_p, \quad (38) \]

where
\[ R_X = \text{diag}^{-\frac{1}{2}}\{S_X\}S_X\text{diag}^{-\frac{1}{2}}\{S_X\}, \]
\[ \Xi(\Lambda) = \Lambda^{-\frac{1}{2}}\text{diag}^{\frac{1}{2}}\{S_X\} = \text{diag}\{\frac{S}{\sigma_j}\}_{j=1}^p. \]

The minimum DPD estimators are obtained as a solution of the system the non-linear system of equations
\[
\begin{align*}
\frac{1}{n} \sum_{i=1}^{n} f_{\beta}(X_i) s_\mu(X_i) - \xi_\beta(\mu) &= 0_p, \\
\frac{1}{n} \sum_{i=1}^{n} f_{\beta}(X_i) s_\Lambda(X_i) - \xi_\beta(\Lambda) &= 0_{p \times p},
\end{align*}
\]

where
\[
\begin{align*}
s_\mu(X) &= \Lambda^{-1/2} R_0^{-1} \Lambda^{-1/2} (X - \mu), \\
\xi_\beta(\mu) &= E\left[ f_{\beta}(X) s_\mu(X) \right] = 0_p, \\
s_\Lambda(X) &= -\frac{1}{2} \left( I_p - R_0^{-1} \Lambda^{-1/2} (X - \mu)(X - \mu)^T \Lambda^{-1/2} \right) \Lambda^{-1}, \\
\xi_\beta(\Lambda) &= E\left[ f_{\beta}(X) s_\Lambda(X) \right], \\
\theta &= (\mu^T, 1^T \Lambda)^T.
\end{align*}
\]
Since

\[
E \left[ f_\beta^\theta(X) h(X) \right] = \int_{\mathbb{R}^p} f_\beta^\theta(x) h(x) dx
\]

\[
= \frac{(\beta + 1)^{-\frac{\theta}{2}}}{(2\pi)^{\frac{d_p}{2}} |\Sigma(\Lambda)|^{\frac{d_p}{2}}} \int_{\mathbb{R}^p} f_{\mathcal{N}_p(\mu_\beta, \frac{1}{\beta+1} \Sigma(\Lambda))}(x) h(x) dx,
\]

it holds

\[
E \left[ f_\beta^\theta(X)(X - \mu) \right] = \frac{(\beta + 1)^{-\frac{\theta}{2}}}{(2\pi)^{\frac{d_p}{2}} |\Sigma(\Lambda)|^{\frac{d_p}{2}}} \left( E[\mathcal{N}_p(\mu_\beta, \frac{1}{\beta+1} \Sigma(\Lambda))] - \mu \right) = 0_p,
\]

\[
\xi_\beta(\mu) = E \left[ f_\beta^\theta(X)s_\mu(X) \right] = \Lambda^{-1/2}R_0^{-1}\Lambda^{-1/2}E \left[ f_\beta^\theta(X)(X - \mu) \right] = 0_p,
\]

and

\[
E \left[ f_\beta^\theta(X)(X - \mu)(X - \mu)^T \right] = \frac{(\beta + 1)^{-\frac{\theta}{2}}}{(2\pi)^{\frac{d_p}{2}} |\Sigma(\Lambda)|^{\frac{d_p}{2}}} \text{Var}[\mathcal{N}_p(\mu_\beta, \frac{1}{\beta+1} \Sigma(\Lambda))]
\]

\[
= \frac{(\beta + 1)^{-\frac{\theta}{2}}}{(2\pi)^{\frac{d_p}{2}} |\Sigma(\Lambda)|^{\frac{d_p}{2}}} \Sigma(\Lambda),
\]

\[
E \left[ f_\beta^\theta(X) \right] = \frac{(\beta + 1)^{-\frac{\theta}{2}}}{(2\pi)^{\frac{d_p}{2}} |\Sigma(\Lambda)|^{\frac{d_p}{2}}}.
\]

\[
\xi_\beta(\Lambda) = E \left[ f_\beta^\theta(X)s_\Lambda(X) \right]
\]

\[
= -\frac{1}{2} \left( I_p E \left[ f_\beta^\theta(X) \right] - R_0^{-1}\Lambda^{-1/2}E \left[ f_\beta^\theta(X)(X - \mu)(X - \mu)^T \right] \Lambda^{-1/2} \right) \Lambda^{-1}
\]

\[
= -\frac{1}{2} \left( I_p \frac{(\beta + 1)^{-\frac{\theta}{2}}}{(2\pi)^{\frac{d_p}{2}} |\Sigma(\Lambda)|^{\frac{d_p}{2}}} - \frac{(\beta + 1)^{-\frac{\theta}{2}}}{(2\pi)^{\frac{d_p}{2}} |\Sigma(\Lambda)|^{\frac{d_p}{2}}} R_0^{-1}\Lambda^{-1/2}\Sigma(\Lambda) \Lambda^{-1/2} \right) \Lambda^{-1}
\]

\[
= -\frac{1}{2} \left( \frac{(\beta + 1)^{-\frac{\theta}{2}}}{(2\pi)^{\frac{d_p}{2}} |\Sigma(\Lambda)|^{\frac{d_p}{2}}} \right) \left( I_p - \frac{1}{\beta+1}R_0^{-1}R_0 \right) \Lambda^{-1}
\]

\[
= -\frac{1}{2} \frac{\beta(\beta + 1)^{-\frac{\theta}{2}}}{(2\pi)^{\frac{d_p}{2}} |\Sigma(\Lambda)|^{\frac{d_p}{2}}} \Lambda^{-1}.
\]
Hence,

\[-\frac{1}{2} \Sigma^{-1}(\Lambda) \frac{1}{n} \sum_{i=1}^{n} f_{\theta}^\beta(X_i)(X_i - \mu) = 0_p,\]

\[-\frac{1}{2n} \sum_{i=1}^{n} f_{\theta}^\beta(X_i) \left[ I_p - R_0^{-1} \Lambda^{-1/2} (X_i - \mu)(X_i - \mu)^T \Lambda^{-1/2} \right] \Lambda^{-1} + \frac{\beta \beta_1 \rho_1}{2} \left( \frac{\beta}{\frac{\beta}{2}} \right) \Lambda^{-1} = 0_{p \times p},\]

equivalent to

\[\sum_{i=1}^{n} f_{\theta}^\beta(X_i)(X_i - \mu) = 0_p,\]

\[R_0^{-1} \Lambda^{-1/2} \left( \frac{1}{n} \sum_{i=1}^{n} f_{\theta}^\beta(X_i)(X_i - \mu)(X_i - \mu)^T \right) \Lambda^{-1/2} = \left( \frac{1}{n} \sum_{i=1}^{n} f_{\theta}^\beta(X_i) - \beta \frac{(\beta + 1)^{\rho_1}}{(2\pi)^{\rho_1} |\Sigma(\Lambda)|^{\beta}} \right) I_p,\]

i.e.

\[\frac{\sum_{i=1}^{n} f_{\theta}^\beta(X_i) X_i}{\sum_{i=1}^{n} f_{\theta}^\beta(X_i)} = \mu,\]

\[R_0^{-1} \Lambda^{-1/2} \frac{1}{n} \sum_{i=1}^{n} f_{\theta}^\beta(X_i)(X_i - \mu)(X_i - \mu)^T \Lambda^{-1/2} = I_p,\]

From the previous expression it is obtained the system of equations (15)-(16).

5.10 Proof of Theorem 12

From the particular case of the Woodbury’s formula

\[(I_p + UV)^{-1} = I_p - U(I_r + VU)^{-1} V,\]
it holds

\[
\frac{1}{1 - \rho_0} R_0(\rho_0) = I_p + \frac{\rho_0}{1 - \rho_0} 1_p 1_p^T
\]

\[
(1 - \rho_0) R_0^{-1}(\rho_0) = \begin{bmatrix}
U = \frac{\rho_0}{1 - \rho_0} 1_p, V = 1_p^T
\end{bmatrix}
\]

\[
= I_p - \frac{\rho_0}{1 - \rho_0} 1_p (1 + p \frac{\rho_0}{1 - \rho_0})^{-1} 1_p^T
\]

\[
= I_p - \frac{\rho_0}{1 - \rho_0} 1_p \left(\frac{1 + (p - 1) \rho_0}{1 - \rho_0}\right)^{-1} 1_p^T
\]

\[
= I_p - \frac{\rho_0}{1 + (p - 1) \rho_0} 1_p 1_p^T,
\]

and hence

\[
R_0^{-1}(\rho_0) = \frac{1}{1 - \rho_0} \left( I_p - \frac{\rho_0}{1 + (p - 1) \rho_0} 1_p 1_p^T \right),
\]

\[
R_0^{-1}(\rho_0) \tilde{R}_{X,\beta} = \frac{1}{1 - \rho_0} \left( \tilde{R}_{X,\beta} - \frac{\rho_0}{1 + (p - 1) \rho_0} 1_p 1_p^T \tilde{R}_{X,\beta} \right)
\]

\[
= \frac{1}{1 - \rho_0} \left( \tilde{R}_{X,\beta} - \frac{\rho_0}{1 + (p - 1) \rho_0} 1_p \otimes \left( \tilde{R}_{1,\beta}, \ldots, \tilde{R}_{p,\beta} \right) \right)
\]

\[
\text{diag}\{R_0^{-1}(\rho_0) \tilde{R}_{X,\beta}\} = \frac{1}{1 - \rho_0} \text{diag}\{\tilde{R}_{jj,\beta} - \frac{\rho_0}{1 + (p - 1) \rho_0} \tilde{R}_{j,\beta}\}_{j=1}^p.
\]

From (40), taking into account (16), can be constructed the estimating equations

\[
\tilde{R}_{jj,\beta} - \frac{\rho_0}{1 + (p - 1) \rho_0} \tilde{R}_{j,\beta} = 1 - \rho_0,
\]

or equivalently (18).
5.11 Proof of Theorem [13]

From the expression of (14), it is deducted that the sum of the estimating equations for \((\Lambda_1, \rho_{21})\) is

\[ k \mathbf{1}^T_p \mathbf{R}_0^{-1} \otimes \mathbf{R}_0^{-1} \left[ \text{vec} \left( \mathbf{\tilde{R}}_{X,\beta} \right) - \text{vec} \left( \mathbf{R}_0 \right) \right] = 0, \]

where \(k\) is a scalar, and taking into account that \(\mathbf{R}_0 = \mathbf{R}(\rho_{12})\), defined by (19), it holds from (39)

\[
\mathbf{1}_p^T \mathbf{R}_0^{-1} = \frac{1}{1 - \rho_{12}} \left( \mathbf{I}_p - \frac{\rho_{12}}{1 + (p-1)\rho_{12}} \mathbf{1}_p \mathbf{1}_p^T \right) \\
= \mathbf{1}_p^T
\]

and since the sum of every row of \(\mathbf{G}_p\) is 1, we have \(\mathbf{1}_{p(p+1)/2}^T \mathbf{G}_p^T = \mathbf{1}_{p^2}^T\), and

\[
\mathbf{1}_{p(p+1)/2}^T \mathbf{G}_p^T (\mathbf{R}_0^{-1} \otimes \mathbf{R}_0^{-1}) = \mathbf{1}_{p^2}^T (\mathbf{R}_0^{-1} \otimes \mathbf{R}_0^{-1}) \\
= (\mathbf{1}_p^T \otimes \mathbf{1}_p^T) (\mathbf{R}_0^{-1} \otimes \mathbf{R}_0^{-1}) \\
= (\mathbf{1}_p^T R_0^{-1} \otimes \mathbf{1}_p^T R_0^{-1}) \\
= (\mathbf{1}_p^T \otimes \mathbf{1}_p^T) \\
= \mathbf{1}_{p^2}^T.
\]

This means that the sum of the estimating equations for \((\Lambda_1, \rho_{21})\) is equivalent to

\[
\sum_{j=1}^{p} \mathbf{\tilde{R}}_{jj,\beta} + 2 \sum_{i<j} \mathbf{\tilde{R}}_{ij,\beta} = p + 2 \frac{p(p-1)}{2} \rho_{21,\beta}. \tag{41}
\]
On the other hand, by following (18), the estimating equations for $\Lambda_1$ are

$$\frac{\tilde{\rho}_{21,\beta}}{1 + (p - 1)\tilde{\rho}_{21,\beta}} \tilde{R}_{j,\beta} = \tilde{R}_{jj,\beta} - (1 - \tilde{\rho}_{21,\beta}), \quad j = 1, \ldots, p.$$  (42)

Now, we will check in the previous $p + 1$ equations that for $j = 1, \ldots, p$ it holds

$$\tilde{R}_{jj,\beta} = 1, \quad j = 1, \ldots, p, \quad (43)$$

$$\tilde{R}_{j,\beta} = 1 + (p - 1)\tilde{\rho}_{21,\beta}. \quad (44)$$

Summing up (43)-(44) separately for $j = 1, \ldots, p$ we get (42), while summing up the total of $2p$ terms, we get (41). From equations (43)-(44), it is concluded that

$$\tilde{R}_{jj,\beta} = 1, \quad j = 1, \ldots, p,$$

$$\sum_{i<j} \tilde{R}_{ij,\beta} = \frac{p(p - 1)}{2} \tilde{\rho}_{21,\beta},$$

or equivalently

$$\tilde{\sigma}^2_{j,\beta} = S^2_{j,\beta}, \quad j = 1, \ldots, p,$$

and (28), since $\tilde{R}_{ij,\beta} = R_{ij,\beta}, \quad j = 1, \ldots, p$ if only if $\tilde{R}_{jj,\beta} = 1, \quad j = 1, \ldots, p$.  

5.12 Proof of Theorem 15

The expression given in (23) is derived from the first remark of Section 4 of Basu et al. (2021), devoted to Rao’s score tests for composite composite hypothesis fixing the
value of a subvector of the parameter vector,

\[
\tilde{R}_{\beta,n} = R_{\beta,n}(\tilde{\eta}_{\beta}) = nU_{\beta,n}(\tilde{\eta}_{2,2,\beta})K_{\beta}^{-1}(\tilde{\eta}_{2,2,\beta})U_{\beta,n,1}(\tilde{\eta}_{2,2,\beta})
\]

\[
= nU_{\beta,n}(\tilde{\eta}_{2,2,\beta})\text{diag}^{-\frac{1}{2}}(\text{vecl}(\tilde{\eta}_{2,1,\beta}^{T}\tilde{\eta}_{2,1,\beta}))Q^{T}K_{\beta}^{-1}(\tilde{\theta}_{2,\beta})Q
\]

\[
\times \text{diag}^{-\frac{1}{2}}(\text{vecl}(\tilde{\eta}_{2,1,\beta}^{T}\tilde{\eta}_{2,1,\beta}))U_{\beta,n,1}(\tilde{\eta}_{2,1,\beta})
\]

\[
= nU_{\beta,n}(\tilde{\theta}_{2,\beta})K_{\beta}^{-1}(\tilde{\theta}_{2,\beta})U_{\beta,n,1}(\tilde{\theta}_{2,\beta}),
\]

(45)

where

\[
K_{\beta}^{-1}(\tilde{\theta}_{2,\beta}) = 4(2\pi)^{\beta}p|\tilde{\Sigma}_{\beta}|^{\beta}L_{p}\left[\mathcal{J}_{2\beta}(\tilde{\Sigma}_{\beta}^{-1}) + \tilde{\xi}_{\beta}(\tilde{\Sigma}_{\beta}^{-1})\tilde{\xi}_{\beta}^{T}(\tilde{\Sigma}_{\beta}^{-1})\right]^{-1}L_{p}^{T},
\]

\[
L_{p} = (G_{p}G_{p}^{T})^{-1}G_{p}
\]

is the elimination matrix which verifies to be a full rank Moore-Penrose pseudoinverse of matrix \(G_{p}^{T}(L_{p}G_{p}^{T} = I_{p})\) and \(\tilde{\theta}_{2,\beta} = \text{vech}(\tilde{\Sigma}_{\beta})\), with \(\tilde{\Sigma}_{\beta} = \tilde{\Lambda}_{\beta}^{\frac{1}{2}}R_{0}\tilde{\Lambda}_{\beta}^{\frac{1}{2}}\). For the calculation of this inverse see Magnus and Nedecker (1980) and Browne (1974) and according to Proposition 5

\[
\left[\mathcal{J}_{2\beta}(\tilde{\Sigma}_{\beta}^{-1}) + \tilde{\xi}_{\beta}(\tilde{\Sigma}_{\beta}^{-1})\tilde{\xi}_{\beta}^{T}(\tilde{\Sigma}_{\beta}^{-1})\right]^{-1} = \kappa_{1}^{-1}(p,\beta)(\tilde{\Lambda}_{\beta}^{\frac{1}{2}} \otimes \tilde{\Lambda}_{\beta}^{\frac{1}{2}})\left(R_{0} \otimes R_{0}\right)(\tilde{\Lambda}_{\beta}^{\frac{1}{2}} \otimes \tilde{\Lambda}_{\beta}^{\frac{1}{2}})
\]

\[
- \kappa_{1}^{-1}(p,\beta)(\tilde{\Lambda}_{\beta}^{\frac{1}{2}} \otimes \tilde{\Lambda}_{\beta}^{\frac{1}{2}})\frac{\kappa_{3}(p,\beta)\text{vec}(R_{0})\text{vec}^{T}(R_{0})}{1 + \kappa_{3}(p,\beta)}(\tilde{\Lambda}_{\beta}^{\frac{1}{2}} \otimes \tilde{\Lambda}_{\beta}^{\frac{1}{2}}).
\]
From the expression of (14), it holds

$$\tilde{R}_{\beta,n} = n \kappa_0(p, \beta) \tilde{\kappa}_2(p, \beta) \vec{T}(\tilde{R}_{X,\beta} - R_0) (R_0^{-1} \otimes R_0^{-1}) \vec{T}(\tilde{R}_{X,\beta} - R_0)$$

$$- n \kappa_0(p, \beta) \tilde{\kappa}_2(p, \beta) \vec{T}(\tilde{R}_{X,\beta} - R_0) \kappa_1(p, \beta) \vec{T}(R_0^{-1}) \vec{T}(R_0^{-1}) \vec{T}(\tilde{R}_{X,\beta} - R_0)$$

$$= n \kappa_0(p, \beta) \tilde{\kappa}_2(p, \beta) \vec{T}(R_0^{-1/2} \tilde{R}_{X,\beta} R_0^{-1/2} - I_p) \vec{T}(R_0^{-1/2} \tilde{R}_{X,\beta} R_0^{-1/2} - I_p)$$

$$= n \kappa_0(p, \beta) \kappa_1(p, \beta) \vec{T}(R_0^{-1/2} \tilde{R}_{X,\beta} R_0^{-1/2} - I_p) \vec{T}(R_0^{-1/2} \tilde{R}_{X,\beta} R_0^{-1/2} - I_p)$$

$$= n \kappa_0(p, \beta) \kappa_1(p, \beta) \vec{T}(R_0^{-1/2} \tilde{R}_{X,\beta} R_0^{-1/2} - I_p) \vec{T}(R_0^{-1/2} \tilde{R}_{X,\beta} R_0^{-1/2} - I_p)$$


The expression in the second row vanishes since

$$\vec{T}(R_0^{-1}) \vec{T}(\tilde{R}_{X,\beta} - R_0) = \vec{T}(R_0^{-1}) \vec{T}(\tilde{R}_{X,\beta}) - \vec{T}(R_0^{-1}) \vec{T}(R_0)$$

$$= \text{trace}(R_0^{-1} \tilde{R}_{X,\beta}) - \text{trace}(I_p)$$

$$= p - p$$

$$= 0,$$

and \(\text{trace}(R_0^{-1} \tilde{R}_{X,\beta}) = p\) from the corresponding estimating equation.

5.13 Proof of Corollary 19

$$\frac{2}{(1 - \rho_0)^2} \sum_{i<j}^{2} \left( \tilde{R}_{ij,\beta} - \frac{\rho_0}{1 + (p-1)\rho_0} \tilde{R}_{ij,\beta} \right)^2 = \frac{2}{(1 - \rho_0)^2} \sum_{i<j}^{2} \left( (\tilde{R}_{ij,\beta} - \rho_0) + (1 - \tilde{R}_{jj,\beta}) \right)^2$$

$$= \frac{2}{(1 - \rho_0)^2} \sum_{i<j}^{2} \left( \left( \tilde{R}_{ij,\beta} - \frac{s_{ij,\beta}}{\sigma_{ij,\beta}} s_{ij,\beta} - \rho_0 \right) + \left( 1 - \frac{s_{ij,\beta}^2}{\sigma_{ij,\beta}} \right) \right)^2.$$
5.14 Proof of Corollary 20

The particularization for the uncorrelatedness or independence test with respect to Theorem 15, where \( R_0 = I_p \), gives

\[
\tilde{R}_{\beta,n} = \frac{n}{n} \tilde{\kappa}_0^2(p, \beta) \text{trace} \left( \left( \tilde{R}_{X,\beta} - I_p \right)^2 \right) \\
= n \frac{\tilde{\kappa}_0^2(p, \beta)}{\kappa_1(p, \beta)} \text{vec}^T \left( \tilde{R}_{X,\beta} - I_p \right) \text{vec} \left( \tilde{R}_{X,\beta} - I_p \right) \\
= 2n \frac{\tilde{\kappa}_0^2(p, \beta)}{\kappa_1(p, \beta)} \text{vec}^T \left( \tilde{R}_{X,\beta} \right) \text{vec} \left( \tilde{R}_{X,\beta} \right) \\
= 2n \frac{\tilde{\kappa}_0^2(p, \beta)}{\kappa_1(p, \beta)} \sum_{i<j} \tilde{R}_{ij,\beta}^2. \tag{46}
\]

In addition, from Theorem 12 it is concluded that \( S_{j,\beta}^2 = \tilde{\sigma}_{j,\beta}^2, \ j = 1, ..., p \), and hence \( \tilde{R}_{ij,\beta}^2 = R_{ij,\beta}^2 \) for all pairs such that \( i < j \).
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