SPHERICAL ORBIT CLOSURES IN
SIMPLE PROJECTIVE SPACES AND
THEIR NORMALIZATIONS

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Abstract. Let $G$ be a simply connected semisimple algebraic group over an algebraically closed field $k$ of characteristic 0 and let $V$ be a rational simple $G$-module. If $G/H \subset P(V)$ is a spherical orbit and if $X = G/H$ is its closure, then we describe the orbits of $X$ and those of its normalization $\tilde{X}$. If moreover the wonderful completion of $G/H$ is strict, then we give necessary and sufficient combinatorial conditions so that the normalization morphism $\tilde{X} \to X$ is a homeomorphism. Such conditions are trivially fulfilled if $G$ is simply laced or if $H$ is a symmetric subgroup.

Introduction.

Let $G$ be a simply connected semisimple algebraic group over an algebraically closed field $k$ of characteristic 0; all $G$-modules considered in the following will be supposed to be rational. An algebraic $G$-variety is said to be spherical if it is normal and if it contains an open $B$-orbit, where $B \subset G$ is a Borel subgroup; a subgroup $H \subset G$ is said to be spherical if the homogeneous space $G/H$ is so: any spherical variety can thus be regarded as an open embedding of a spherical homogeneous space, namely its open $G$-orbit. Important classes of spherical varieties are that of toric varieties and that of symmetric varieties: toric varieties are those spherical varieties whose open orbit is an algebraic torus; symmetric varieties are those spherical varieties whose generic stabilizer $H$ is such that $G^\sigma \subset H \subset N_G(G^\sigma)$, where $\sigma : G \to G$ is an algebraic involution and where $G^\sigma$ is the set of its fixed points. Other important classes of spherical varieties are that of flag varieties and the more general one of wonderful varieties: a wonderful variety (of rank $r$) is a smooth projective $G$-variety having an open $G$-orbit which satisfies following properties:

- the complement of the open $G$-orbit is the union of $r$ smooth prime divisors having a non-empty transversal intersection;
- any orbit closure equals the intersection of the prime divisors containing it.

A spherical subgroup $H$ is said to be wonderful if $G/H$ possesses a wonderful completion (which is unique, if it exists). By [CP] every self-normalizing symmetric subgroup is wonderful; more generally every self-normalizing spherical subgroup is wonderful by [Kn3].

However many natural examples of embeddings of a spherical homogeneous space do not need to be normal. For instance, consider a simple $G$-module $V$ (in which case we will call $P(V)$ a simple projective space) possessing a line $[v]$ fixed by a spherical subgroup. Then consider the orbit $G[v] \subset P(V)$, which is spherical, and take its closure $X = \overline{G[v]} \subset P(V)$, which generally is not normal; denote $\tilde{X}$ its normalization. The aim of this work is the study of the orbits of compactifications which arise in such a way, and as well the study of the orbits of their normalizations.

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In [BL] it has been proved that any spherical subgroup which occurs as the stabilizer of a point in a simple projective space is wonderful. If \( M \) is the wonderful completion of \( G[v] \), then the morphism \( G[v] \to X \) extends to \( M \) and thus we get a morphism \( M \to \tilde{X} \to X \); examining such morphism we get a description of the set of orbits of \( X \) and of \( \tilde{X} \). Moreover this leads to a combinatorial criterion to establish whether or not two orbits in \( M \) map onto the same orbit in \( X \), which in particular implies that different orbits in \( X \) are never \( G \)-equivariantly isomorphic.

Our main theorem (Theorem 5.9) is a combinatorial criterion for \( \tilde{X} \to X \) to be bijective; this is done under the assumption that \( M \) is strict, i.e. that all isotropy groups of \( M \) are self-normalizing: strict wonderful varieties, introduced in [Pe2], are those wonderful varieties which can be embedded in a simple projective space; they form an important class of wonderful varieties which generalize the symmetric ones of [CP]. The condition of bijectivity involves the double links of the Dynkin diagram of \( G \) and it is trivially fulfilled whenever \( G \) is simply laced or \( M \) is symmetric; it is easily read off by the spherical diagram of \( M \), which is a useful tool to represent a wonderful variety starting from the Dynkin diagram of \( G \). Main examples of strict wonderful varieties where bijectivity fails arise from the context of wonderful model varieties introduced in [Lu3]; the general strict case is substantially deduced from the model case.

A model space for a connected (possibly non-simply connected) semisimple algebraic group \( G' \) is a quasi-affine homogeneous space whose coordinate ring contains every simple \( G' \)-module with multiplicity one; model spaces were classified in [Lu3], where it is introduced the wonderful model variety \( M_{G'}^{\text{mod}} \), whose orbits naturally parametrize up to isomorphism the model spaces for \( G' \); every orbit of \( M_{G'}^{\text{mod}} \) is of the shape \( G'/N_{G'}(H) \), where \( G'/H \) is a model space, and conversely this correspondence gives a bijection up to isomorphism.

In order to illustrate the above mentioned criterion of bijectivity in the case of a wonderful model variety, let’s set up some further notation. If \( \lambda \) is a dominant weight (w.r.t. a fixed maximal torus \( T \subset G \) and a fixed Borel subgroup \( B \supset T \)), define the support of \( \lambda \) as the set

\[
\text{Supp}(\lambda) = \{ \alpha \in S : \langle \alpha^\vee, \lambda \rangle \neq 0 \},
\]

where \( S \) is the set of simple roots w.r.t. \( T \subset B \). If \( G_i \subset G \) is a simple factor of type \( B \) or \( C \), number the associated subset of simple roots \( S_i = \{ \alpha_i^1, \ldots, \alpha_i^{r(i)} \} \) starting from the extreme of the Dynkin diagram of \( G_i \) which contains the double link; define moreover \( S_i^{\text{even}}, S_i^{\text{odd}} \subset S_i \) as the subsets whose element index is respectively even and odd. If they are defined, set

\[
e_i(\lambda) = \min\{k \leq r(i) : \alpha_k^i \in \text{Supp}(\lambda) \cap S_i^{\text{even}}\}
\]

\[
o_i(\lambda) = \min\{k \leq r(i) : \alpha_k^i \in \text{Supp}(\lambda) \cap S_i^{\text{odd}}\}
\]

or set \( e_i(\lambda) = +\infty \) (resp. \( o_i(\lambda) = +\infty \)) otherwise. Finally, if \( G_i \) is of type \( F_4 \), number the simple roots in \( S_i = \{ \alpha_i^1, \alpha_i^2, \alpha_i^3, \alpha_i^4 \} \) starting from the extreme of the Dynkin diagram which contains a long root.

**Theorem** (see Thm. 5.9). Suppose that \( G[v] \subset \mathbf{P}(V) \) is the open orbit of a wonderful model variety \( M_{G'}^{\text{mod}} \), where \( G' \) is isogenous with \( G \); denote \( \lambda \) the highest weight of \( V \) and set \( X = \tilde{G[v]} \). Then the normalization \( \tilde{X} \to X \) is bijective if and only if the following conditions are fulfilled, for every connected component \( S_i \subset S \):

i) If \( S_i \) is of type \( B \), then either \( \alpha_i^1 \in \text{Supp}(\lambda) \) or \( \text{Supp}(\lambda) \cap S_i^{\text{even}} = \emptyset \);

ii) If \( S_i \) is of type \( C \), then \( o_i(\lambda) \geq e_i(\lambda) - 1 \);

iii) If \( S_i \) is of type \( F_4 \) and \( \alpha_i^2 \in \text{Supp}(\lambda) \), then \( \alpha_i^3 \in \text{Supp}(\lambda) \) as well.
When the generic stabilizer $H$ is a self-normalizing symmetric subgroup, compactifications in simple projective spaces were studied in [Ma]. Under this assumption, an explicit description of the orbits of $X$ was given and it was proved that these orbits are equal to those of the normalization of $X$. Thus our results generalize those contained in [Ma].

In the case of a compactification of the adjoint group $G_{\text{ad}}$ (regarded as a $G \times G$-symmetric variety) obtained as the closure of the orbit of the line generated by these orbits are equal to those of the normalization of $X$. Thus our results generalize those contained in [BGMR].

The paper is organized as follows. In section 1, we set notations and preliminaries; in section 2 we give some general results about spherical orbit closures in projective spaces; in section 3 we recall some results from [BL] about stabilizers of points in simple projective spaces. In section 4, we describe the orbits of the compactifications $X$ and $\bar{X}$; in section 5, we prove the criterion of bijectivity of the normalization map in the strict case; in section 6, we briefly consider the non-strict case giving some sufficient conditions of bijectivity and non-bijectivity of the normalization map.

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1. Preliminaries.

Fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$; denote $\Phi$ the corresponding root system and $S \subset \Phi$ the corresponding set of simple roots. If $H \subset G$ is any subgroup, denote $\mathcal{X}(H)$ its character group; if $V$ is a $G$-module, denote $V(H)$ the set of $H$-eigenvectors of $V$ and, if $\chi \in \mathcal{X}(H)$, denote $V^{(H)}(\chi)$ the subset of $V^{(H)}$ where $H$ acts by $\chi$. If $\lambda \in \mathcal{X}(B)$ is a dominant weight, we will denote $V_\lambda$ the simple $G$-module with highest weight $\lambda$. If $\Lambda$ is a lattice (i.e. a finitely generated free $\mathbb{Z}$-module), then $\Lambda^\vee = \text{Hom}(\Lambda, \mathbb{Z})$ denotes the dual lattice and $\Lambda_Q = \Lambda \otimes \mathbb{Q}$ denotes the rational vector space generated by $\Lambda$. If $\mathcal{C}$ is a cone contained in some vector space $V$, then $\mathcal{C}^\vee$ denotes the dual cone in the dual vector space $V^\vee$.

Let $X$ be a spherical $G$-variety and fix a base point $x_0 \in X$ in such a way that $Bx_0 \subset X$ is an open subset; denote $H = \text{Stab}(x_0)$. Let’s introduce some data associated to $X$:

1. $\Lambda_X = \{B$-weights of rational $B$-eigenfunctions in $k(X)\} \simeq k(X)^{(B)}/k^*$.
2. $\Delta(X) = \{B$-stable prime divisors in $X$ which are not $G$-stable\}, its elements are called the colors of $X$. If $Y \subset X$ is an orbit, then $\Delta_Y(X)$ denotes the set of colors which contain $Y$.
3. If $\nu : k(X)^* \rightarrow \mathbb{Q}$ is a rational discrete valuation of $k(X)$, then $\nu$ defines an element $\rho_X(\nu) \in (\Lambda_X^*)_\mathbb{Q}$ by

$$\langle \rho_X(\nu), \chi \rangle = \nu(f_\chi),$$

where $f_\chi \in k(X)^{(B)}$ is any $B$-semiinvariant function of weight $\chi$: since $X$ possesses an open $B$-orbit, such definition does not depend on the function, but only on the weight. If $D \in \Delta(X)$, by abuse of notation we will denote $\rho_X(D) = \rho_X(\nu_D)$ the image of the respective valuation $\nu_D$.

A rational discrete valuation $\nu$ is said $G$-invariant if $\nu(gf) = \nu(f)$, for any $f \in k(X)$ and for any $g \in G$; set $\mathcal{V}_X$ the set of $G$-invariant rational
valuations of $k(X)$. The map $\rho_X : \mathcal{V}_X \rightarrow (\Lambda^c_X)^{G_H}$ identifies $\mathcal{V}_X$ with a convex cone which generates $(\Lambda^c_X)^{G_H}$ as a vector space \cite[Cor. 4.1]{BP}; together with such embedding, $\mathcal{V}_X$ is called the $G$-invariant valuation cone of $X$.

Both $\Lambda^c_X$ and $\Delta(X)$, as well as the map $\rho_X$, depend only on the open orbit $G/H \subset X$ and they are the main objects of the Luna-Vust Theory (see \cite{Kn1}), which classifies normal equivariant embeddings of a given spherical homogeneous space. A spherical variety is said to be simple if it possesses only one closed orbit; it is said to be toroidal if no color contains a closed orbit. If a spherical homogeneous space $G/H$ possesses a complete, simple and toroidal embedding, then this is uniquely determined and it is called the canonical embedding of $G/H$; we will denote it $M(G/H)$ and it dominates any simple complete embedding of $G/H$. In general, a canonical embedding of $G/H$ exists if and only if the index of $H$ in its normalizer is finite, in which case $H$ is said to be sober.

If $X$ is a simple spherical variety with complete closed orbit $Y$, then the Picard group $\text{Pic}(X)$ is freely generated by the classes $[D]$, with $D \in \Delta(X) \setminus \Delta_Y(X)$; moreover, a divisor is generated by global sections (resp. ample) if and only if it is equivalent to a linear combination of such colors with non-negative (resp. positive) coefficients \cite[Prop. 2.6 and Thm. 2.6]{Bri}.

Wonderful varieties are always spherical (see \cite{Lu1}) and a spherical variety is wonderful if and only if it is complete, toroidal, simple and smooth. A spherical subgroup which appears as the generic stabilizer of a wonderful variety is said wonderful.

If $H$ is a spherical subgroup, then the normalizer $N_G(H)$ acts on the right on $G/H$ by $n \cdot gH = gn^{-1}H$. Consider the induced action of $N_G(H)$ on $\Delta(G/H)$: the kernel of such action is called the spherical closure of $H$; if $H$ coincides with its spherical closure, then it is called spherically closed. Spherically closed subgroups are always wonderful \cite[Cor. 7.6]{Kn3}; a wonderful variety is said to be spherically closed if its generic stabilizer is so.

Suppose now $M$ is a wonderful variety with open $B$-orbit $Bx_0$ and set $H = \text{Stab}(x_0)$; suppose moreover that the center of $G$ acts trivially on $M$. Denote $\Delta$ the set of colors and $Y \subset M$ the closed orbit; denote $z \in Y^{B^-}$ the $B^-$-fixed point (where $B^-$ denotes the opposite Borel subgroup of $B$ with respect to $T$). Since $G$ is semisimple and simply connected, $\text{Pic}(Y)$ is identified with a sublattice of $X(B)$, while $\text{Pic}(G/H)$ is identified with $X(H)$: if $L \in \text{Pic}(Y)$, then $L$ will be identified with the character of $B^-$ acting on the fiber of $L$ over $z$, while if $L \in \text{Pic}(G/H)$, then $L$ will be identified with the character of $H$ acting on the fiber over $eH$.

Let's introduce some more data attached to a wonderful variety $M$, together with some results which can be found with more details and references in \cite{Lu2} and in \cite{BL}.

### (4) $\Sigma = \{T$-weights of the $T$-module $T_zM/T_zY\}$: its elements are called the spherical roots of $M$. Spherical roots form a basis of the lattice $\Lambda_{G/H}$; they also coincide with the minimal set of generators of the free semigroup $\Lambda_{G/H} \cap -\mathcal{V}_{G/H}^c$. The cardinality of $\Sigma$ coincides with the rank of $M$, i.e. with the number of $G$-stable prime divisors of $M$, which are naturally in correspondence with spherical roots. If $\sigma \in \Sigma$, denote $M^\sigma$ the corresponding $G$-stable prime divisor: it is a wonderful $G$-subvariety whose set of spherical roots is $\Sigma \setminus \{\sigma\}$. Spherical roots are either positive roots or a sum of two positive roots; the support of a spherical root $\sigma$ is the set of simple roots where $\sigma$ is supported.

### (5) The Cartan pairing of $M$ is the natural pairing $c : \Delta \times \Sigma \rightarrow \mathbb{Z}$ between colors and spherical roots defined by the equality $[M^\sigma] = \sum_{D \in \Delta} c(D, \sigma)[D]$.
in the Picard group $\text{Pic}(M) = \mathbb{Z}\Delta$. If $D \in \Delta$ and $\sigma \in \Sigma$, then $c(D, \sigma) = \langle \rho_G/\rho_H(D), \sigma \rangle$.

(6) $\Delta(\alpha) = \{D \in \Delta : P_\alpha D \neq D\}$ is the set of colors moved by $\alpha$, where $\alpha \in S$ and where $P_\alpha \supset B$ is the minimal parabolic subgroup associated to $\alpha$. For every $\alpha \in S$ it holds $0 \leq \text{card} \, \Delta(\alpha) \leq 2$.

(7) $S^p = \{\alpha \in S : \Delta(\alpha) = \emptyset\}$; it coincides with the set of simple roots associated to the stabilizer of the open $B$-orbit, which is a parabolic subgroup. As well, $S^p$ coincides with the set of simple roots associated with the stabilizer of the $B^-$-fixed point $z$ in the closed orbit $Y$.

(8) $S^a = \{\alpha \in S : \text{card} \, \Delta(\alpha) = 2\} = S \cap \Sigma$ is the set of simple spherical roots; correspondingly, set $A = \bigcup_{\alpha \in S^a} \Delta(\alpha)$ the set of colors of type $a$. If $\alpha \in S^a$, set $\Delta(\alpha) = \{D^\alpha_+, D^\alpha_-\}$; then for every spherical root $\sigma$ it holds

$$c(D^\alpha_+, \sigma) + c(D^\alpha_-, \sigma) = \langle \alpha^\vee, \sigma \rangle.$$ 

(9) $S^{2a} = \{\alpha \in S : 2\alpha \in \Sigma\}$; correspondingly, set $\Delta^{2a} = \bigcup_{\alpha \in S^{2a}} \Delta(\alpha)$ the set of colors of type $2a$. If $\alpha \in S^{2a}$, set $\Delta(\alpha) = \{D^\alpha\}$; then for every spherical root $\sigma$ it holds

$$c(D^\alpha, \sigma) = \langle \alpha^\vee, \sigma \rangle/2.$$ 

(10) $S^b = S \setminus (S^p \cup S^a \cup S^{2a})$; correspondingly, set $\Delta^b = \bigcup_{\alpha \in S^b} \Delta(\alpha)$ the set of colors of type $b$. If $\alpha \in S^b$, set $\Delta(\alpha) = \{D^\alpha\}$; then for every spherical root $\sigma$ it holds

$$c(D^\alpha, \sigma) = \langle \alpha^\vee, \sigma \rangle.$$ 

(11) $\mathcal{S} = (\Sigma, S^p, A)$ is the spherical system of $M$, where $A$ has to be thought of as an abstract set together with the pairing $c : A \times \Sigma \to \mathbb{Z}$. This is the combinatorial datum which expresses a wonderful variety: each wonderful variety is uniquely determined by its spherical system (see [Lu]). There is also an abstract combinatorial definition of spherical system (see [BL]), introduced in order to obtain the classification of wonderful varieties (Luna’s conjecture): the geometrical realizability of spherical systems has been checked in many cases (see [Bra], [BC], [BPe1], [BPe2], [Lu], [Pe]) and recently a general proof which avoids a case-by-case approach has been proposed in [Cu]. A very useful tool to represent spherical systems are spherical diagrams, obtained adding information to the Dynkin diagram of $\Phi$ (see [BL]).

(12) Denote $\omega : \text{Pic}(M) \to \mathcal{X}(B)$ and $\psi : \text{Pic}(M) \to \mathcal{X}(H)$ the restrictions to the closed and to the open orbit; then we get a commutative diagram

$$
\begin{array}{ccc}
\text{Pic}(M) & \xrightarrow{\psi} & \mathcal{X}(H) \\
\downarrow & & \downarrow \\
\mathcal{X}(B) & \xrightarrow{\psi} & \mathcal{X}(B \cap H)
\end{array}
$$

which identifies $\text{Pic}(M)$ with the fiber product [Bri Prop. 2.2.1]

$$\mathcal{X}(B) \times_{\mathcal{X}(B \cap H)} \mathcal{X}(H) = \{(\lambda, \chi) \in \mathcal{X}(B) \times \mathcal{X}(H) : \chi|_{B \cap H} = \chi|_{B \cap H}\}.$$ 

On the combinatorial level, the map $\omega$ is described on colors as follows ([Fo Thm. 2.2]):

$$\omega(D) = \begin{cases} 
\sum_{D \in \Delta(\alpha)} \omega_\alpha & \text{if } D \in A \cup \Delta^b \\
2\omega_\alpha & \text{if } D = D^\alpha \in \Delta^{2a}
\end{cases}$$

where $\omega_\alpha$ is the fundamental dominant weight associated to $\alpha \in S$. 


(13) If $\Sigma' \subset \Sigma$ is a subset of spherical roots, then the localization at $\Sigma'$ of $M$ is the $G$-stable subvariety

$$M_{\Sigma'} = \bigcap_{\sigma \in \Sigma \setminus \Sigma'} M^\sigma;$$

it is a wonderful variety whose spherical system is $\mathcal{S}' = (\Sigma', S^p, A')$, where $A' = \bigcup_{\alpha \in \Sigma' \cap \Sigma} \Delta(\alpha)$. Denote $\Delta'$ the set of colors of $M_{\Sigma'}$; if $\alpha \in S \cap \Sigma'$ and $\beta \in \mathcal{S} \setminus (\Sigma' \cup S^p)$ set $\Delta'(\alpha) = \{D^+_{\alpha}, D^-_{\alpha}\}$ and $\Delta'(\beta) = \{D^+_{\beta}\}$. Let $q : \text{Pic}(M) \to \text{Pic}(M_{\Sigma'})$ be the pullback map; then $\omega$ factors through $\text{Pic}(M_{\Sigma'})$ and by its combinatorial description it follows that:

- if $\alpha \in S \cap \Sigma'$ then $q(D^+_{\alpha})$ (resp. $q(D^-_{\alpha})$) is supported on $D^+_{\alpha}$ (resp. on $D^-_{\alpha}$) with multiplicity one, while it is not supported on $'D^+_{\alpha}$ (resp. on $'D^-_{\alpha}$);
- if $\alpha \in S \cap (\Sigma' \setminus \Sigma')$ then $q(D_{\alpha})$ are supported on $D'_{\alpha}$ with multiplicity one;
- if $\alpha \in S \cap (\Sigma \setminus \Sigma')$, then $q(D_{\alpha}) = 2D'_{\alpha};$
- if $\alpha \in S^p$, then $q(D_{\alpha})$ is supported on $D'_{\alpha}$ with multiplicity one and on at most one more color.

(14) $M$ is said to be strict if the stabilizer of any point is self-normalizing; equivalently, we will say also that $H$ is strict. A wonderful variety is strict if and only if it can be embedded in a simple projective space (see [Pe2]).

(15) Consider the following sets of spherical roots

$$\Sigma^D_\ell = \left\{ \sigma \in \Sigma \setminus S : \text{there exists a rank one wonderful variety whose spherical system is } (2\sigma, S^p, \emptyset) \right\};$$

$$\Sigma^S_\ell = \left\{ \sigma \in S \cap \Sigma : c(D^+_{\alpha}, \sigma) = c(D^-_{\alpha}, \sigma) \forall \sigma \in \Sigma \right\};$$

set $\Sigma_\ell = \Sigma^D_\ell \cup \Sigma^S_\ell$ the set of loose spherical roots. Loose spherical roots of the first kind are easily identified, they are those of the following types (where $S = \{\alpha_1, \ldots, \alpha_n\}$ and simple roots are labelled as in Bourbaki):

- spherical roots $\sigma = \alpha_{i+1} + \ldots + \alpha_{i+r}$ with support of type $B_r$ and with $\alpha_{i+r} \in S^p$;
- spherical roots $\sigma = 2\alpha_{i+1} + \alpha_{i+2}$ with support of type $G_2$.

For every $\sigma \in \Sigma_\ell$, it is defined a $G$-equivariant automorphism $\gamma(\sigma) \in \text{Aut}_G(M)$ of order 2 which fixes pointwise the $G$-stable divisor $M^\sigma$ associated to $\sigma$. If $\sigma \in \Sigma^D_\ell$, then $\gamma(\sigma)$ acts trivially on $\Delta$, while if $\sigma \in \Sigma^S_\ell$, then $\gamma(\sigma)$ exchanges $D^+_{\alpha}$ and $D^-_{\alpha}$ and acts trivially on $\Delta \setminus \Delta(\sigma)$. Such automorphisms commute and generate $\text{Aut}_G(M)$ (see [Ld]).

By the natural identification $\text{Aut}_G(M) = N_G(H)/H$, it follows that

- $H$ is self-normalizing if and only if $\Sigma_\ell = \emptyset$;
- $H$ is spherically closed if and only if $\Sigma^D_\ell = \emptyset$;
- $H$ is strict if and only if $S \cap \Sigma = \emptyset$ and $\Sigma_\ell = \emptyset$.

In particular, if $S \cap \Sigma = \emptyset$, then $H$ is self-normalizing if and only if it is spherically closed if and only if it is strict.

(16) $\Sigma(G)$ denotes the set of the spherical roots of $G$: its elements are the spherical roots of all possible rank one $G$-wonderful varieties. Following the classification of the latter (see [AK]), such set is classified for any $G$ (see [Lu2] and [BL]).
2. Projective varieties with an open $B$-orbit.

Let $V$ be a $G$-module and let $X \subset \mathbf{P}(V)$ be a projective variety with an open $B$-orbit; denote $G/H \hookrightarrow X$ the open orbit. Since $X$ contains finitely many $B$-orbits [Kn2 Cor. 2.6], every $G$-orbit in $X$ is spherical. Let $p : \tilde{X} \to X$ be the normalization; then $\tilde{X}$ is a complete spherical variety with the same open orbit of $X$ whose orbits are naturally in bijection with those of $X$:

**Proposition 2.1** ([IT] Prop. 1). The normalization morphism $p : \tilde{X} \to X$ induces a bijection between $G$-orbits.

If $Z \subset X$ is an orbit, set $Z' = p^{-1}(Z) \subset \tilde{X}$ the corresponding orbit. Denote $Z_B \subset Z$ and $Z'_B \subset Z'$ the open $B$-orbits; fix base points $z_0 \in Z_B$ and $z'_0 \in Z'_B$ so that we have isomorphisms

$$Z' \simeq G/K', \quad Z \simeq G/K$$

with $K' \subset K$. Let’s recall a result which will be useful in the following:

**Theorem 2.2** ([BP] Prop 5.1 and Cor. 5.2). Let $H$ be a spherical subgroup of $G$.

i) The algebraic group $N_G(H)/H$ is diagonalizable; moreover, if $H^0$ is the identity component of $H$, then $N_G(H) = N_G(H^0)$.

ii) If $B$ is any Borel subgroup such that $BH$ is open in $G$, then $N_G(H)$ equals the right stabilizer of $BH$.

Combing back to our situation, then we obtain:

**Corollary 2.3.** $K' \subset K$ is a normal subgroup with finite index; in particular $K/K'$ is a finite diagonalizable group.

**Proof.** Since $p$ is a finite morphism, it preserves dimensions of orbits: so we have $\dim(K') = \dim(K)$. Then $K' \subset K$ implies $(K')^0 = K^0$ and we obtain

$$N_G(K') = N_G((K')^0) = N_G(K^0) = N_G(K).$$

This shows

$$(K')^0 = K^0 \subset K' \subset K \subset N_G(K) = N_G(K')$$

and the claim follows. $\square$

**Lemma 2.4.** Let $K' \subset K$ be two spherical subgroups of $G$ with $K'$ normal in $K$; fix a Borel subgroup $B$ such that $BK'$ is open in $G$ and consider the projection $\pi : G/K' \to G/K$. Then $\pi^{-1}(BK/K) = BK'/K'$ and $\pi^* : \Lambda_{G/K} \to \Lambda_{G/K'}$ identifies $\Lambda_{G/K}$ with a sublattice of $\Lambda_{G/K'}$ such that

$$\Lambda_{G/K'}/\Lambda_{G/K} \simeq \chi(K/K').$$

**Proof.** First claim follows by the equality $BK' = BK$, which stems immediately from Theorem 2.2(ii).

If $B' \subset B$, denote $\chi(B)^{B'}$ the kernel of the restriction $\chi(B) \to \chi(B')$, which is surjective by the following argument: if $U \subset B$ is the unipotent radical, then $\chi(B) = \chi(B/U)$ and $\chi(B') = \chi(B'/B' \cap U)$ and the restriction $\chi(B/U) \to \chi(B'/B' \cap U)$ is surjective since $B'/B' \cap U \subset B/U$ is a diagonalizable subgroup of the torus $B/U$.

By definition, we have isomorphisms $\Lambda_{G/K} \simeq \chi(B)^{B \cap K}$ and $\Lambda_{G/K'} \simeq \chi(B)^{B \cap K'}$; thus the restriction gives a surjective homomorphism

$$\Lambda_{G/K'} \to \chi(B \cap K)^{B \cap K'} = \chi(B \cap K/B \cap K')$$
whose kernel is $\Lambda_{G/K}$. On the other hand $BK/K \simeq B/(B \cap K)$ and $BK'/K' \simeq B/(B \cap K')$, hence the equality $BK' = BK$ implies

$$B \cap K / B \cap K' \simeq K/K'.$$

Therefore we get

$$\Lambda_{G/K} / \Lambda_{G/K} \simeq \chi \left( B \cap K / B \cap K' \right) \simeq \chi (K/K').$$

$\Box$

Going back to our situation, since $K/K'$ is a finite diagonalizable group, it is isomorphic to its character group and we get the following corollary.

**Corollary 2.5.** Let $G/K \simeq Z \subset X$ be an orbit and let $G/K' \simeq Z' = p^{-1}(Z)$, with $K' \subset K$; then

$$\Lambda_{Z'} / \Lambda_{Z} \simeq K/K'.$$

Fix a closed orbit $Y \subset X$. Since parabolic subgroups are self-normalizing, $Y$ and $p^{-1}(Y)$ are isomorphic; from now on, we will denote both of them with the same letter $Y$. Let $y = [v^-] \in Y^{-}$ be the unique fixed point by $B^-$ (where $v^- \in V$ is a lowest weight vector) and let $\eta \in (V^*)^{(B)}$ be a highest weight vector such that $\langle \eta, v^- \rangle = 1$. If $P$ is the stabilizer of $[\eta]$, then $P$ and $\text{Stab}(y)$ are opposite parabolic subgroups; denote $L = P \cap \text{Stab}(y)$ the associated Levi subgroup.

Consider the affine $P$-stable open subset $X_0 = X \cap P(V)_Y$ defined by the non-vanishing of $\eta$ and recall that there exists an affine closed $L$-stable subvariety $S_X \subset X_0$ containing $y$ and possessing an open ($B \cap L$)-orbit such that the multiplication morphism

$$P^u \times X_0 \longrightarrow X_0$$

$$(g, s) \mapsto gs,$$

(where $P^u$ denotes the unipotent radical of $P$) is a $P$-equivariant isomorphism [BLV Prop. 1.2]. Since $k[S_X] = k[S_X]^L = k$, we get that $S_X$ possesses a unique closed $L$-orbit, namely the fixed point $y$.

If $D \in \Delta(G/H)$, denote $\overline{D}$ and $\overline{D}$ its closure respectively in $X$ and in $\tilde{X}$.

**Lemma 2.6.** Let $D \in \Delta(G/H)$, let $Z \subset X$ be an orbit and set $Z' = p^{-1}(Z)$. Then $\overline{D} \supset Z$ if and only if $\overline{D} \supset Z'$.

**Proof.** By Lemma 2.4 $p^{-1}(Z_B) = Z'_B$ is the open $B$-orbit of $Z'$. Suppose that $\overline{D} \supset Z$ and fix $z_0 \in Z_B$; if $z_0 \in p^{-1}(z_0) \cap \overline{D}$, then we obtain $Z_B = Bz_0 \subset \overline{D}$, which implies $Z' \subset \overline{D}$. Suppose conversely that $\overline{D} \supset Z'$: then $\overline{D} = p(\overline{D}) \supset p(Z') = Z$. $\Box$

**Lemma 2.7.** Let $D \in \Delta(G/H)$; then $\overline{D} \supset Y$ if and only if $\eta |_D \neq 0$.

**Proof.** Notice that $P\overline{D}$ is closed and that it contains $Y$ if and only if $\overline{D}$ contains $Y$. If $\eta |_D \neq 0$, then $\overline{D} \cap X_0 \neq \varnothing$, thus $P\overline{D} \cap S_X$ is non-empty, $L$-stable and closed in $S_X$; hence $y \in P\overline{D} \cap S_X$, which implies that $Y_B = Bg \subset P\overline{D}$. If conversely $\eta |_D = 0$, then $\overline{D} \subset P(\ker(\eta))$ and we get $\overline{D} \not\supset Y$. $\Box$

**Remark 2.8.** Combining previous lemmas it follows that the set of colors $\Delta_Y(\tilde{X}) \subset \Delta(G/H)$ whose closure in $\tilde{X}$ contains the closed orbit $Y$ is

$$\Delta_Y(X) = \{ D \in \Delta(G/H) : \eta |_D \neq 0 \}.$$

If $X$ possesses a unique closed orbit $Y$, then previous lemmas allow us to compute the colored cone of the normal embedding $G/H \hookrightarrow \tilde{X}$: this is the couple $(\mathcal{C}_Y(\tilde{X}), \Delta_Y(\tilde{X}))$, where $\mathcal{C}_Y(\tilde{X}) \subset (\Lambda_{G/H}^0)_{\mathbb{Q}}$ is the cone generated by the elements
\[\rho_X(D),\] where \(D \subset \tilde{X}\) is any \(B\)-stable (possibly \(G\)-stable) prime divisor which contains \(Y\) (see [Kn1]). In fact, since \(\tilde{X}\) is simple and complete, then \(\mathcal{C}_Y(\tilde{X})\) contains the \(G\)-invariant valuation cone \(\mathcal{V}_{G/H}\) ([Kn1] Thm. 5.2): therefore \(\mathcal{C}_Y(X)\) is the cone generated by \(\mathcal{V}_{G/H}\) together with \(\rho_X(\Delta_Y(X))\).

Let \(Z \subset X\) be an orbit. Set \(Z_0 = \overline{Z} \cap \mathcal{P}(V)_0, \tilde{X}_0 = p^{-1}(X_0)\) and \(Z'_0 = p^{-1}(Z_0)\); then \(Z_0 = X_0 \cap \overline{Z}\) and \(Z'_0 = \tilde{X}_0 \cap \overline{Z'}\). Considering the rings of functions, we get a commutative diagram

\[
\begin{array}{ccc}
 k[X_0] & \longrightarrow & k[\tilde{X}_0] \\
 \downarrow & & \downarrow \\
 k[Z_0] & \longrightarrow & k[Z'_0]
\end{array}
\]

**Theorem 2.9** ([Kn1] Thm. 2.3). Every \(B\)-semi-invariant function \(f \in k[Z_0]^{(B)}\) (resp. \(f \in k[Z'_0]^{(B)}\)) can be extended to a \(B\)-semi-invariant function \(f' \in k[X_0]^{(B)}\) (resp. \(f' \in k[X'_0]^{(B)}\)).

If \(\Lambda\) is a finitely generated free \(\mathbb{Z}\)-module and \(\Gamma \subset \Lambda\) is a submonoid, then the saturation of \(\Gamma\) in \(\Lambda\) is the submonoid \(\overline{\Gamma} = \Gamma_{\mathbb{Q}^+} \cap \Lambda\), where \(\Gamma_{\mathbb{Q}^+} \subset \Lambda_{\mathbb{Q}}\) is the cone generated by \(\Gamma\). If \(\Omega \subset \Lambda\) is a submonoid containing \(\Gamma\), then we will say that \(\Gamma\) is saturated in \(\Omega\) if \(\Gamma = \Gamma_{\mathbb{Q}^+} \cap \Omega\).

**Proposition 2.10.** Fix an orbit \(Z \subset X\) and set \(Z' = p^{-1}(Z)\).

i) \(\Lambda_{Z'}\) is the saturation of \(\Lambda_Z\) in \(\Lambda_{G/H}\).

ii) If \(Z \simeq Z'\), then \(\overline{Z} \subset \tilde{X}\) is the normalization of \(\overline{Z} \subset X\).

**Proof.** Set \(\Omega(\tilde{X}) = k[\tilde{X}_0]^{(B)}/k^*\) and \(\Omega(\overline{Z'}) = k[Z'_0]^{(B)}/k^*\); in terms of colored cones, such monoids are described as follows ([Kn1] Thm. 3.5):

\[
\Omega(\tilde{X}) = \Lambda_{G/H} \cap \mathcal{C}_Y^*(\tilde{X}), \quad \Omega(\overline{Z'}) = \Lambda_{Z'} \cap \mathcal{C}_Y^*(\overline{Z'}). \]

Since every \(B\)-semi-invariant function is uniquely determined by its weight up to a scalar factor, by previous theorem restriction gives an isomorphism of multiplicative monoids

\[
\left\{ f \in k[\tilde{X}_0]^{(B)} \mid f|_{Z'_0} \neq 0 \right\} \sim k[Z'_0]^{(B)} : \]

since the first one is saturated in \(k[\tilde{X}_0]^{(B)}\), we may then identify \(\Omega(\overline{Z'})\) with a saturated submonoid of \(\Omega(\tilde{X})\). Hence we may as well consider \(\Lambda_{Z'}\) with a sublattice of \(\Lambda_{G/H}\): in fact every \(B\)-semi-invariant rational function on \(\tilde{X}\) (resp. on \(Z'\)) can be written as a quotient of two \(B\)-semi-invariant regular functions on \(\tilde{X}_0\) (resp. on \(Z'_0\)).

Since \(\tilde{X}\) is normal, \(\Omega(\tilde{X}) \subset \Lambda_{G/H}\) is saturated; therefore, being saturated in \(\Omega(\tilde{X})\), we see that \(\Omega(\overline{Z'}) \subset \Lambda_{G/H}\) as well is saturated. Since the colored cone of a spherical variety is strictly convex ([Kn1] Thm. 4.1), \(\mathcal{C}_Y^*(\overline{Z'}) \subset (\Lambda_{Z'})_{\mathbb{Q}}\) has maximal dimension: thus the equality

\[
\Lambda_{Z'} \cap \mathcal{C}_Y^*(\overline{Z'}) = \Omega(\overline{Z'}) = \Lambda_{G/H} \cap \mathcal{C}_Y^*(\overline{Z'})
\]

implies that \(\Lambda_{Z'} \subset \Lambda_{G/H}\) is a saturated sublattice. Hence by \([\Lambda_{Z'} : \Lambda_Z] = [K : K'] < \infty\) we get i).

Finally, ii) stems from the fact that a \(G\)-stable subvariety of a spherical variety is normal ([HP] Prop. 3.5) together with the fact that the restriction \(p : \overline{Z'} \rightarrow \overline{Z}\) is finite and birational. \(\square\)
3. Faithful divisors.

Let $V$ be a simple $G$-module and suppose $G/H \cong Gx_0 \subset \mathbf{P}(V)$ is a spherical orbit. A necessary and sufficient condition so that a spherical subgroup $H$ arises in such a way has been given in [BL]:

**Proposition 3.1** ([BL] Cor. 2.4.2). A spherical subgroup occurs as the stabilizer of a point in a simple projective space if and only if it is spherical closed.

Let $M$ be the wonderful completion of $G/H$: then the embedding $G/H \hookrightarrow \mathbf{P}(V)$ extends to a morphism $\phi : M \to \mathbf{P}(V)$ and it determines a $B$-stable effective divisor $\delta \in \text{Pic}(M)$ which is generated by global sections. Conversely, it is possible to start with an arbitrary spherically closed wonderful variety $M$ together with an effective $B$-stable divisor $\delta$ generated by global sections and to consider then the associated morphism $\phi_{\delta} : M \to \mathbf{P}(V)$, where $V = \langle Gs \rangle^*$ is the dual of the simple module generated by the canonical section $s \in \Gamma(M, \mathcal{O}(\delta))$. In [BL], there have been given necessary and sufficient conditions on such a divisor $\delta$ so that $\phi_{\delta}$ restricts to an embedding of the open orbit $G/H$; the aim of this subsection is to recall some facts which lead to such conditions.

Let $M$ be a wonderful variety with base point $x_0$ and set $H = \text{Stab}(x_0)$. Set $\mathcal{S} = (\Sigma, S^p, A)$ its spherical system and $\Delta = \Delta(G/H)$ its set of colors. Recall that a subset $\Delta^* \subset \Delta$ is said to be distinguished if there exists $\delta \in \mathbb{N}_{>0} \Delta^*$ such that $c(\delta, \sigma) \geq 0$, for every $\sigma \in \Sigma$.

If $H' \supset H$ is a sober subgroup and if $\phi : G/H \to G/H'$ is the projection, then the subset of colors

$$
\Delta_{\phi} = \{ D \in \Delta : \overline{\phi(D)} = G/H' \}
$$

is distinguished; conversely, if $\Delta^* \subset \Delta$ is a distinguished subset, then there exists a unique wonderful subgroup $H' \supset H$ with $H'/H$ connected such that $\Delta^* = \Delta_{\phi}$.

This is the content of the following theorem:

**Theorem 3.2** ([Kn1] Thm. 5.4, [La2] Prop. 3.3.2, [Bra2] Thm. 3.1.1).

There is an inclusion-preserving bijection as follows

$$
\{ \Delta^* \subset \Delta \text{ distinguished} \} \leftrightarrow \left\{ \begin{array}{l} H' \subset G \text{ wonderful} : \\
H \subset H' \text{ and } H'/H \text{ connected} \end{array} \right\}
$$

Moreover, if $H' \supset H$ is a wonderful subgroup with $H'/H$ connected and if $\Delta^* \subset \Delta$ is the corresponding distinguished subset, then

i) the projection $G/H \to G/H'$ identifies $\Delta(G/H')$ with $\Delta \setminus \Delta^*$;

ii) the spherical system of the wonderful completion of $G/H'$ is

$$
\mathcal{S}/\Delta^* = (\Sigma/\Delta^*, S^p/\Delta^*, A/\Delta^*),
$$

defined as follows:

- $\Sigma/\Delta^*$ is the set of indecomposable elements of the free semigroup

$$
\mathbb{N}\Sigma/\Delta^* = \{ \sigma \in \mathbb{N}\Sigma : c(D, \sigma) = 0, \forall D \in \Delta^* \};
$$

- $S^p/\Delta^* = S^p \cup \{ \alpha \in S : \Delta(\alpha) \subset \Delta^* \};$

- $A/\Delta^* = \bigcup_{\alpha \in S \cap \Sigma/\Delta^*} A(\alpha)$, and the pairing is obtained by restriction.

In the notations of previous theorem, the wonderful completion of $G/H'$ is denoted $M/\Delta^*$ and it is called the quotient wonderful variety of $M$ by $\Delta^*$, while $\mathcal{S}/\Delta^*$ is called the quotient spherical system of $\mathcal{S}$ by $\Delta^*$.

**Remark 3.3.** By [Kn1] Lemma 5.3 and Thm. 5.4 together with [La2] Lemma 3.3.1, there is an inclusion-preserving bijection between distinguished subsets $\Delta^* \subset \Delta$ and sober subgroups $H' \supset H$ such that $H'/H$ is connected. In [La2] Cor. 5.6.2 it was proved that, in case $G$ is of type $A$, then such a subgroup $H'$ is necessarily
wonderful; although this was claimed in general in [Lu3], a general proof (which stems from the classification of spherical systems) appeared only recently in [Bra2, Thm. 3.3.1].

Suppose that $H' \supset H$ is a sober subgroup such that $H'/H$ is connected and denote $\Delta^* \subset \Delta$ the distinguished subset of colors which map dominantly to $G/H'$. Denote $G/H' \hookrightarrow M'$ the canonical embedding and extend the projection $\phi : G/H \to G/H'$ to a morphism $\phi : M \to M'$; consider $\Lambda_{G/H'}$ as a sublattice of $\Lambda_{G/H}$. Set $N(\Delta^*) = \Lambda_{G/H'} \cap (\Lambda_{G/H})'q$: it is a linear subspace which contains $\rho_{G/H}(\Delta^*)$ and which intersects the valuation cone $\mathcal{V}_{G/H}$ in a face; as a cone, $N(\Delta^*)$ is generated by this face together with $\rho_{G/H}(\Delta^*)$ [Kn1, Lemma 3.3.1]. While the valuation cone $\mathcal{V}_{G/H'}$ is the image of $\mathcal{V}_{G/H}$ under the quotient map $(\Lambda_{G/H})'q \to (\Lambda_{G/H'})'q$ by $N(\Delta^*)$, the lattice $\Lambda_{G/H'}$ is identified with a sublattice of $\Lambda_{G/H}$ as follows

\[
\Lambda_{G/H'} = \Lambda_{G/H} \cap N(\Delta^*)^\perp
\]

[Kn1] Lemma 5.3 and Thm. 5.4; as a consequence, $\Lambda_{G/H'}$ is saturated in $\Lambda_{G/H}$.

If $\Delta'$ is the set of colors of $M'$ and if $M'_0 = M' \setminus \Delta' \cup \Delta$, then $\phi^{-1}(M'_0) = M \setminus \bigcup_{\Delta' \setminus \Delta} D_i$; since the fibers of $\phi$ are complete and connected, it follows that $k[M'_0] = k[\phi^{-1}(M'_0)]$. Considering the $B$-semilinear invariant functions, we get then the identification of semigroups

\[
k[M'_0]^{(B)}/k^* \cong -N\Sigma/\Delta^*.
\]

If $M'$ is smooth, then such semigroup is free; conversely, given any distinguished subset $\Delta^* \subset \Delta$, the semigroup $N\Sigma/\Delta^*$ is free [Bra2, Thm. 3.1.1], which means that $M'$ is necessarily smooth.

Recall the restrictions to the closed and to the open orbit $\omega : \text{Pic}(M) \to \mathcal{X}(B)$ and $\psi : \text{Pic}(M) \to \mathcal{X}(H)$ and let $\delta = \sum_{\Delta} n(\delta, D)D$ with $n(\delta, D) \geq 0$ for every $D \in \Delta$. If $s \in \Gamma(M, \mathcal{O}(\delta))$ is the canonical section, then the submodule $\langle Gs \rangle \subset \Gamma(M, \mathcal{O}(\delta))$ generated by $s$ is identified with the simple module $V_{\omega(\delta)}$ (which contains a unique $H$-invariant line where $H$ acts by $\psi(\delta)$) and we get a morphism

\[
\phi_{\delta} : M \to \mathbf{P}(V_{\omega(\delta)}^*).
\]

Define the support of $\delta$ as

\[
\text{Supp}_\Delta(\delta) = \{ D \in \Delta : n(\delta, D) > 0 \}.
\]

As a consequence of Theorem \ref{thm:main}, we get the following corollary.

**Corollary 3.4.** Let $M$ be a wonderful variety and let $\delta \in N\Delta$ be a divisor generated by global sections and consider the associated morphism $\phi_{\delta} : M \to \mathbf{P}(V_{\omega(\delta)}^*)$. Then the correspondence of Theorem \ref{thm:main} gives an inclusion-preserving bijection as follows

\[
\left\{ \begin{array}{l}
\Delta^* \subset \Delta \text{ distinguished :} \\
\Delta^* \cap \text{Supp}_\Delta(\delta) = \emptyset
\end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l}
H' \subset G \text{ wonderful :} \\
H \subset H' \subset \text{Stab}(\phi_{\delta}(x_0))
\end{array} \right. \text{and } H'/H \text{ connected}
\]

**Proof.** Let $H' \supset H$ be a wonderful subgroup with $H'/H$ connected and set $\Delta^* \subset \Delta$ the corresponding distinguished subset. If $G/H' \hookrightarrow M'$ is the wonderful embedding, then the projection $G/H \to G/H'$ extends to a morphism $M \to M'$ and pullback identifies $\text{Pic}(M')$ with the submodule $\mathbf{Z}[\Delta \setminus \Delta^*] \subset \mathbf{Z}[\Delta] = \text{Pic}(M)$. Thus the map $M \to \mathbf{P}(V_{\omega(\delta)}^*)$ factors through a map $M' \to \mathbf{P}(V_{\omega(\delta)}^*)$ if and only if $\text{Supp}_\Delta(\delta) \subset \Delta \setminus \Delta^*$. \hfill $\Box$

**Definition 3.5 ([BL]).** Let $M$ be a spherically closed wonderful variety and denote $\Delta$ its set of colors. A divisor generated by global sections $\delta = \sum n(\delta, D)D \in N\Delta$ is called faithful if it satisfies the following conditions:
(FD1): Every non-empty distinguished subset of $\Delta$ intersects $\text{Supp}_\Delta(\delta)$;

(FD2): If $\alpha \in \Sigma_\ell$ is a loose spherical root, then $n(\delta, D_\alpha^+) \neq n(\delta, D_\alpha^-)$.

**Proposition 3.6** ([HI] Prop. 2.4.3). Let $M$ be a spherically closed wonderful variety and let $\delta \in N\Delta$. Then the morphism $\phi_\delta : M \to \mathbb{P}(V^*_{\omega(\delta)})$ restricts to an embedding $G/H \hookrightarrow \mathbb{P}(V^*_{\omega(\delta)})$ if and only if $\delta$ is faithful.

**Proof.** Fix $v_0 \in \left(V^*_{\omega(\delta)}\right)_{\phi(\delta)}^{(H)}$ a representative of the line $\phi_\delta(x_0)$ and suppose that $H = \text{Stab}[v_0]$; then previous corollary implies (FD1). Suppose by absurd that (FD2) fails and let $\alpha \in \Sigma_\ell \subset S \cap \Sigma$ be a loose spherical root such that $n(\delta, D_\alpha^+) = n(\delta, D_\alpha^-)$. If $\gamma(\alpha) \in \text{Aut}_G(M) = N_G(H)/H$ is the corresponding automorphism, then $\gamma(\alpha)$ exchanges $D_\alpha^+$ and $D_\alpha^-$ and fixes every other color $D \in \Delta \setminus \Delta(\alpha)$: therefore $\gamma(\alpha)$ fixes $\delta$. The action of $\text{Aut}_G(M)$ on $\text{Pic}(M) = \mathbb{Z}\Delta \simeq \chi(\Delta) \times_{\chi(H \cap H)} \chi(H)$ is defined extending by linearity the right action of $N_G(H)/H$ on $\Delta$, i.e. by the action of $N_G(H)$ on $\chi(H)$. Therefore, if $g \in N_G(H)$ is a representative of $\gamma(\alpha)$, then $\psi(\delta)_g = \psi(\delta)$, i.e. $g$ moves the line $[v_0]$ in a line where $H$ acts by the same character: since $H$ is spherical, such a line is unique, thus $g \in H = \text{Stab}[v_0]$ which is absurd.

Suppose conversely that $\delta$ is a faithful divisor. By (FD1) it follows that $\dim H = \dim \text{Stab}[v_0]$, therefore by Theorem 2.2 we get $H \subset \text{Stab}[v_0] \subset N_G(H)$. Suppose by absurd that there exists $g \in \text{Stab}[v_0] \setminus H$. Then $\psi(\delta)_g = \psi(\delta)$ and the equivariant automorphism corresponding to the coset $gH$ fixes $\delta$: therefore by (FD2) we get that every color $D \in \text{Supp}_\Delta(\delta)$ is fixed by $g$. On the other hand, since $H$ is spherically closed, every element in $N_G(H) \setminus H$ acts non-trivially on $\Delta$. Take $\alpha \in S$ such that $g$ moves $D \in \Delta(\alpha)$, then we get $\alpha \in \Sigma_\ell \subset S \cap \Sigma$ and $\Delta(\alpha) = \{D, D \cdot g\}$; therefore $n(\delta, D) = n(\delta, D \cdot g) = 0$, which contradicts (FD2). 

**Corollary 3.7.** In the same hypotheses of Corollary 3.3, suppose moreover that every distinguished subset of $\Delta$ intersects $\text{Supp}_\Delta(\delta)$ and set

$$
\Sigma(\delta) = \left\{ \alpha \in \Sigma_\ell : \alpha \notin S \text{ or } n(\delta, D_\alpha^+) = n(\delta, D_\alpha^-) \right\}.
$$

Then the spherical system of $\text{Stab}(\phi_\delta(x_0))$ is $\mathcal{S}' = (\Sigma', S', \mathbf{A}')$, where

$$
\Sigma' = (\Sigma \setminus \Sigma(\delta)) \cup 2\Sigma(\delta) \quad \text{and} \quad \mathbf{A}' = \bigcup_{\alpha \in \Sigma \cap S} \mathbf{A}(\alpha).
$$

**Proof.** For every loose spherical root $\sigma \in \Sigma_\ell$, the quotient $M/\gamma(\sigma)$ is easily proved to be a wonderful variety, whose spherical system is $\mathcal{S}' = (\Sigma', S', \mathbf{A}')$, where $\Sigma' = (\Sigma \setminus \Sigma(\sigma)) \cup \{2\sigma\}$ and where $\mathbf{A}' = \bigcup_{\alpha \in \Sigma \cap S} \mathbf{A}(\alpha)$. If $g \in N_G(H)$ is a representative of the coset corresponding to $\gamma(\sigma)$, then $M/\gamma(\sigma) = M(G/H_\sigma)$, where $H_\sigma$ is the subgroup generated by $H$ together with $g$. By the first part of the proof of previous theorem it follows that $H_\sigma$ fixes $\phi_\delta(x_0)$, thus we get a commutative diagram

$$
\begin{array}{ccc}
M & \xrightarrow{\gamma(\sigma)} & \mathbb{P}(V^*_{\omega(\delta)}) \\
| \downarrow & & | \\
M/\gamma(\sigma) & \longrightarrow & \mathbb{P}(V^*_{\omega(\delta)})
\end{array}
$$

Consider now the quotient variety $M/\Gamma_\delta$, where $\Gamma_\delta \subset \text{Aut}_G(X)$ is the subgroup generated by the elements $\gamma(\sigma)$, with $\sigma \in \Sigma(\delta)$: then, by previous discussion and by Proposition 3.3, it follows that $M/\Gamma_\delta$ is a spherically closed wonderful variety endowed with a faithful divisor whose associated characters are the same of $\delta$. 

**Remark 3.8.** In the hypotheses of previous corollary, the assumption that every distinguished subset of colors intersects $\text{Supp}_\Delta(\delta)$ (which is equivalent to assume
that $H$ and $\text{Stab}(\phi_\delta(x_0))$ have the same dimension) involves no loss of generality: we can always reduce to that case considering, instead of $M$, the quotient wonderful variety $M/\Delta(\delta)$, where $\Delta(\delta) \subset \Delta$ is the maximal distinguished subset which does not intersect $\text{Supp}_\Delta(\delta)$.

4. Orbits in $X_\delta$ and in $\tilde{X}_\delta$.

Let $M$ be a spherically closed wonderful variety with base point $x_0$ and set $H = \text{Stab}(x_0)$; set $\mathcal{F} = (\Sigma, S^p, A)$ its spherical system and $\Delta = \Delta(G/H)$ its set of colors.

If $\delta \in \text{Pic}(M)$ is a faithful divisor, set $V = V^*_\omega(\delta)$ and consider the morphism $\phi_\delta : M \to \mathbb{P}(V)$. Set $X_\delta = \phi_\delta(M)$ and set $p : \tilde{X}_\delta \to X_\delta$ the normalization; then we get a commutative diagram

$$
\begin{array}{ccc}
M & \xrightarrow{\phi_\delta} & \tilde{X}_\delta \\
\downarrow & & \downarrow p \\
X_\delta & \subset & \mathbb{P}(V)
\end{array}
$$

If $Y \subset X_\delta$ is the closed orbit, then by Lemma 2.6 and Lemma 2.7 we get that $\Delta_Y(X_\delta) = \Delta_Y(\tilde{X}_\delta)$ is canonically identified with $\Delta \setminus \text{Supp}_\Delta(\delta)$.

If $W \subset M$ is an orbit, in the following $\delta_W \in \text{Pic}(W)$ will denote the pullback of $\delta \in \text{Pic}(M)$. Notice that if $\alpha \in S$ then

$$
\text{Supp}_\Delta(\delta) \cap \Delta(\alpha) \neq \emptyset \iff \alpha \notin S^p(Y) \iff \text{Supp}_{\Delta(W)}(\delta_W) \cap \Delta(W)(\alpha) \neq \emptyset \quad \forall W \subset M
$$

where $S^p(Y)$ denotes the set of simple roots associated to the closed orbit $Y$.

If $M$ is strict, then the variety $X_\delta$ depends only on the support of $\delta$ [BGMR, Lemma 1]: if $\delta, \delta'$ are faithful divisors on $M$, then

$$
X_\delta \simeq X_{\delta'} \iff \text{Supp}_\Delta(\delta) = \text{Supp}_\Delta(\delta')
$$

As will be shown by Lemma 6.3, this is not true if $M$ is not strict.

**Proposition 4.1.** Let $G/K \simeq Z \subset X_\delta$ be an orbit and let $G/K' \simeq Z' = p^{-1}(Z)$; let $G/K_W \simeq W \subset M$ be any orbit which maps on $Z$ and choose the stabilizers so that $K_W \subset K' \subset K$. Then $K'$ is the maximal subgroup such that

$$
K_W \subset K' \subset K
$$

In particular, $Z \simeq Z'$ if and only if $K/K_W$ is connected.

**Proof.** Set $K^* = K_W K^0$ the maximal subgroup of $K$ containing $K_W$ such that $K^*/K_W$ is connected. Since $K_W \subset K'$ and since $K^0 = (K')^0$, by Theorem 2.7 we get that $K^* \subset K'$ is a normal subgroup; thus by Lemma 2.4 it follows that $K^* = K'$ if and only if $\Lambda_{G/K^*} = \Lambda_{Z'}$.

Consider the inclusions $\Lambda_Z \subset \Lambda_{Z'} \subset \Lambda_W \subset \Lambda_{G/H}$: since $\Lambda_W$ is saturated in $\Lambda_{G/H}$, Proposition 2.10 shows that $\Lambda_{Z'}$ is the saturation of $\Lambda_Z$ in $\Lambda_W$. On the other hand, by Remark 3.3 it follows that $\Lambda_{G/K^*}$ is saturated in $\Lambda_W$: since $[\Lambda_{G/K^*} : \Lambda_Z] = [K : K^*] < \infty$, we get the equality $\Lambda_{G/K^*} = \Lambda_{Z'}$.

Combining previous proposition together with [Bra2, Thm. 3.3.1] and Corollary 3.7 we get the following corollary.

**Corollary 4.2.** Let $G/K \simeq Z \subset X_\delta$ be an orbit and let $p^{-1}(Z) \simeq G/K'$ with $K' \subset K$. Then $K'$ is a wonderful subgroup. If moreover $M$ is strict, then $K$ is the spherical closure of $K'$. 

If \( Z \subset X_\delta \) is an orbit and if \( Z' \subset \bar{X}_\delta \) is the corresponding orbit, denote \( \Sigma_Z, \Sigma_{Z'} \subset \mathcal{N}\Sigma \) the sets of spherical roots of the respective wonderful completions. By Corollary 3.7 there exists a bijection between \( \Sigma_Z \) and \( \Sigma_{Z'} \), which associates to \( \gamma \in \Sigma_Z \) the unique \( \gamma' \in \Sigma_{Z'} \) which is proportional to \( \gamma \); more precisely, if \( \gamma \neq \gamma' \), then \( \gamma = 2\gamma' \).

**Definition 4.3.** If \( \sigma \in \Sigma(G) \), then we say that:

- \( \sigma \) is of type \( B_i^1 \) if \( \sigma = a_{i+1} + \ldots + a_{i+r} \) has support of type \( B_i \);
- \( \sigma \) is of type \( B_i^2 \) if \( \sigma = 2a_{i+1} + \ldots + 2a_{i+r} \) has support of type \( B_i \);
- \( \sigma \) is of type \( G_i \) if \( \sigma = 2a_{i+1} + a_{i+2} \) has support of type \( G_i \);
- \( \sigma \) is of type \( G_2^3 \) if \( \sigma = 4a_{i+1} + 2a_{i+2} \) has support of type \( G_2 \).

Consider a spherical root \( \sigma \in \Sigma(G) \) such that \( 2\sigma \in \Sigma(G) \); following the explicit description of \( \Sigma(G) \), such a root either is a simple root, or it is of type \( B_i^1 \) or it is of type \( G_i \). If \( Z \subset X_\delta \) is any orbit and if \( Z' \subset \bar{X}_\delta \) is the corresponding orbit, define \( \Sigma(\delta_Z) \subset \Sigma_{Z'} \) to be the subset of spherical roots which have to be doubled to get the spherical roots of \( Z \).

**Lemma 4.4.** An orbit \( Z \subset X_\delta \) is not isomorphic to its corresponding orbit \( Z' \subset \bar{X}_\delta \) if and only if \( Z \) possesses a spherical root \( \gamma \) of the shape \( \gamma = 2\sigma_1 + \ldots + 2\sigma_k \), where \( \sigma_1, \ldots, \sigma_k \in \Sigma \) are pairwise distinct elements (and where \( \gamma' = \sigma_1 + \ldots + \sigma_k \in \Sigma_{Z'} \)).

**Proof.** By Corollary 3.7, \( Z \) and \( Z' \) are not isomorphic if and only if \( \Sigma(\delta_Z) \neq \emptyset \); suppose \( \gamma' \in \Sigma(\delta_Z) \). By Proposition 3.1 the wonderful completion of \( Z' \) is the quotient of a wonderful subvariety \( M' \subset M \); therefore we can write \( \gamma' = a_1\sigma_1 + \ldots + a_k\sigma_k \), where \( a_1, \ldots, a_k \) are spherical roots of \( M' \).

Since \( 2\gamma' \in \Sigma(G) \), by the discussion preceding the lemma \( \gamma' \) is either a simple root, or it is of type \( B_i^1 \) or it is of type \( G_i \). If \( \gamma' \) is a simple root or if it is of type \( B_i^1 \) then it follows immediately that every \( a_i \) is equal to one. Suppose instead that \( \gamma' \) is of type \( G_i \); in order to show the thesis it is enough to consider the case wherein \( M' \) is a wonderful variety whose spherical roots are all supported on a subset \( S' = \{\alpha_1, \alpha_2\} \subset S \) of type \( G_2 \). An easy computation shows that, if \( \Sigma' = \Sigma' \) and if \( \Delta' \) is any distinguished subset of colors of \( M' \), then the quotient \( M'/\Delta' \) never possesses \( 2\alpha_1 + \alpha_2 \) as a spherical root. Therefore, if \( \gamma' = 2\alpha_1 + \alpha_2 \), it must be either \( \Sigma' = \{2\alpha_1 + \alpha_2\} \) or \( \Sigma' = \{\alpha_1, \alpha_1 + \alpha_2\} \) and the claim follows. \( \square \)

As exemplified in the following sections (Example 5.5 and Example 6.2), Proposition 3.1 together with Corollary 3.7 allow to compute explicitly the set of orbits of \( X_\delta \) and that of \( \bar{X}_\delta \) in terms of their spherical systems. This is further simplified by the following proposition, which shows that, given an orbit \( Z \subset X_\delta \), there exists a minimal orbit \( W_Z \subset M \) mapping on \( Z \). If \( \gamma = \sum_{\sigma \in \Sigma} n_\sigma \sigma \in \Sigma_Z \), define

\[
\text{Supp}_\Sigma(\gamma) = \{ \sigma \in \Sigma : n_\sigma \neq 0 \}
\]

its support over \( \Sigma \); define

\[
\Sigma(Z) = \bigcup_{\gamma \in \text{Supp}_\Sigma(\gamma)} \gamma \subset \Sigma_Z.
\]

**Proposition 4.5.** Let \( Z \subset X_\delta \) be an orbit and let \( W_Z \subset M \) the orbit whose closure has \( \Sigma(Z) \) as set of spherical roots. Then \( W_Z \) maps on \( Z \) and on every other orbit \( W \) maps on \( Z \) contains \( W_Z \) in its closure.

**Proof.** Let \( W \subset M \) be an orbit mapping on \( Z \) and let \( \Sigma_W \subset \Sigma \) be the associated set of spherical roots. Since \( \phi_\delta(W) = Z \), we get \( \Sigma_Z \subset \mathcal{N}\Sigma_W \); this shows \( \Sigma(Z) \subset \Sigma_W \), i.e. \( W_Z \subset \Sigma_W \). In order to prove that \( \phi_\delta(W_Z) = Z \) it is enough to notice that \( \Lambda_{\phi_\delta(W_Z)} = \Lambda_{W_Z} \cap \Lambda_Z = \Lambda_Z \). \( \square \)
**Lemma 5.2.** Let $\Sigma$ be a spherical root of type $B^s_2$ and let $\delta$ be a faithful divisor on it; let $Z \subset X_\delta$ be an orbit. Then $Z \neq Z'$ if and only if there exists a spherical root $\gamma \in \Sigma_Z$ of type $B^s_2$ and a spherical root $\sigma \in \text{Supp}_Z(\gamma)$ of type $B^s_2$.

**Proof.** By Lemma 4.4, we may assume that $Z'$ possesses a spherical root $\gamma$ of type $B^s_2$ or of type $G^s_2$. Since $S \cap \Sigma = \emptyset$, it is uniquely determined a spherical root $\sigma \in \text{Supp}_Z(\gamma)$ which is of type $B^s_2$ (with $2 \leq s \leq r$) in the first case and of type $G^s_2$ in the second case. Since $M$ is strict, the latter cannot happen; thus we are in the first case.

Suppose that $s > 2$ and $2\gamma \in \Sigma_Z$; let $\beta \in S$ be the short root in the support of $\sigma$. Since $M$ is strict, $\beta$ moves a color $D_\beta \in \Delta$, while $s > 2$ implies $c(D_\beta, \tau) \geq 0$ for every $\tau \in \Sigma$; therefore $\{D_\beta\}$ is distinguished and by the faithfulness of $\delta$ we get $D_\beta \in \text{Supp}_\Delta(\delta)$, which implies $\beta \not\in S^p(Y)$. But this is a contradiction since $2\gamma \in \Sigma_Z$ implies $\beta \in S^p(Z) \subset S^p(Y)$.

If $\sigma \in \Sigma$ is a spherical root of type $B^s_2$, write $\sigma = \alpha^s_+ + \alpha^s_-$, where $\alpha^s_+, \alpha^s_- \in S$ are respectively the long simple root and the short simple root in the support of $\sigma$. Since $M$ is strict, both $\alpha^s_+$ and $\alpha^s_-$ move exactly one color; set $\Delta(\alpha^s_+) = \{D^s(\sigma)\}$ and $\Delta(\alpha^s_-) = \{D^s(\sigma)\}$.

**Lemma 5.2.** Let $M$ be a strict wonderful variety and let $\delta$ be a faithful divisor on it; let $\sigma \in \Sigma$ be a spherical root of type $B^s_2$.

**Remark 4.6.** Unlike the symmetric case (see [Ma]), in the general spherical case there does not need to exist a maximal orbit in $M$ mapping on a fixed orbit $Z \subset X_\delta$: for instance this is shown by Example 5.5 and by Example 6.2.
i) If $D^p(\sigma) \in \text{Supp}_\Delta(\delta)$, then no orbit $Z \subset X_\delta$ possesses a spherical root $\gamma \in \Sigma_Z$ of type $B_2^\delta$ with $\sigma \in \text{Supp}_\Sigma(\gamma)$.
ii) If $\text{Supp}_\Delta(\delta) \cap \{D^2(\sigma), D^p(\sigma)\} = \{D^2(\sigma)\}$, then there exists an orbit $Z \subset X_\delta$ such that $2\sigma \in \Sigma_Z$; in particular $Z \neq Z'$ and the normalization $p: \tilde{X}_\delta \to X_\delta$ is not bijective.

Proof. i). If $Z \subset X_\delta$ possesses a spherical root $\gamma$ of type $B_2^\delta$ supported on $\sigma$, then $\alpha_\gamma^p \in S^p(Z) \subset S^p(Y)$. But this is a contradiction since $D^p(\sigma) \in \text{Supp}_\Delta(\delta)$ implies $\alpha_\gamma^p \not\in S^p(Y)$.

ii). Consider the rank one orbit $W \subset M$ whose unique spherical root is $\sigma$. If $\Delta(W)(\alpha_\gamma^p) = \{D^p(\sigma)\}$ and $\Delta(W)(\alpha_\gamma^p) = \{D^2(\sigma)\}$, then
$$\text{Supp}_{\Delta(W)}(\delta_W) \cap \{D^2(\sigma), D^p(\sigma)\} = \{D^p(\sigma)\}.$$ 
Set $Z = \phi_\delta(W)$ and $Z' = p^{-1}(Z)$; set $\Delta(\delta_W) \subset \Delta(W)$ the maximal distinguished subset not intersecting the support of $\delta_W$. Since $c(D^p(\sigma), \sigma) = 0$ and since $D^p(\sigma)$ is the unique color $D \in \Delta(W)$ such that $c(D, \sigma) > 0$, we get
$$D^p(\sigma) \in \Delta(\delta_W) = \{D \in \Delta(W) : c(D, \sigma) = 0\} \setminus \text{Supp}_{\Delta(W)}(\delta_W)$$
which shows $\Sigma_{Z'} = \{\sigma\}$. On the other hand $\Delta(Z')(\alpha_\gamma^p) = \emptyset$, thus $Z'$ is not spherically closed and $\Sigma_Z = \{2\sigma\}$. 

Corollary 5.3. i) If $M$ is a strict wonderful symmetric variety and if $\delta$ is a faithful divisor on it, then the normalization $p: \tilde{X}_\delta \to X_\delta$ is bijective.
ii) Suppose that the Dynkin diagram of $G$ is simply laced. If $M$ is any strict wonderful variety for $G$ and if $\delta \in \text{Pic}(M)$ is any faithful divisor, then the normalization $p: \tilde{X}_\delta \to X_\delta$ is bijective.
iii) If $D^p(\sigma) \in \text{Supp}_\Delta(\delta)$ for every $\sigma \in \Sigma$ of type $B_2^1$, then the normalization morphism $p: \tilde{X}_\delta \to X_\delta$ is bijective.

Proof. By the classification of symmetric varieties, we deduce that a strict wonderful symmetric variety never possesses a spherical root of type $B_2^1$. Then all the claims above follow straightforward by previous lemmas.

Another proof of Corollary 5.3 i) was given in [Ma]. Following examples show some cases wherein the conditions of Lemma 5.1 are fulfilled:

Example 5.4. Consider the wonderful model variety $M$ of Spin(7), whose spherical system is expressed by the spherical diagram

Then the divisor $\delta = D_{\alpha_2}$ is faithful. Consider the codimension one orbit $W \subset M$ having spherical root $\alpha_2 + \alpha_3$; following Proposition 4.11 and Corollary 3.7 we get the following sequence of spherical diagrams

where the first one represents the orbit $W \subset M$, the second one represents the orbit $\tilde{\phi}_\delta(W) \subset \tilde{X}_\delta$ and the third one represents the orbit $\phi_\delta(W) \subset X_\delta$.

Example 5.5. Consider the wonderful model variety $M$ of SO(11), whose spherical system is expressed by the spherical diagram
Then the divisor $\delta = D_{\alpha_2}$ is faithful. See Table 1 for a full list of the orbits in $X_\delta$ and in $\tilde{X}_\delta$ (for simplicity, in the table orbits in $M$ are described by giving a subset of its spherical root index set).

As illustrated by previous examples, main examples of strict wonderful varieties possessing a faithful divisor $\delta$ such that the normalization $p : \tilde{X}_\delta \to X_\delta$ is not bijective arise from the context of wonderful model varieties (see \cite{Lu3}); as will be shown in the following, the case of a general strict wonderful variety substantially follows from this special case.

Consider a strict wonderful variety $M$ and let $\delta$ be a faithful divisor on it. Let $\sigma \in \Sigma$ be a spherical root of type $B^I_j$ and set $\Gamma(\sigma)$ the connected component of the Dynkin diagram of $G$ where $\sigma$ is supported. If $\Gamma(\sigma)$ is of type $B$ or $C$, number the simple roots in $\Gamma(\sigma)$ which are not in $S^p$ starting from the extreme of the diagram which contains the double link.

If $\{D^p_\sigma, D^p_\delta\}$ contains a distinguished subset, then by Lemma 5.2 we get that there is no orbit $Z \subset X_\delta$ possessing a spherical root $\gamma$ of type $B^I_j$ with $\sigma \in \text{Supp}_2(\gamma)$ if and only if $D^p_\gamma \in \text{Supp}_\Delta(\delta)$. For instance, this is the case if one of the following conditions is fulfilled:

- $\Gamma(\sigma)$ is of type $B$ or $C$ and $\sigma$ is the unique spherical root supported on $\alpha_2$;
- $\Gamma(\sigma)$ is of type $C$ and $2\alpha_2 \in \Sigma$.

Suppose that $\{D^p_\sigma, D^p_\delta\}$ does not contain any distinguished subset. If $\Gamma(\sigma) \neq F_4$, then there exists $\tau \in \Sigma$ supported on $\alpha_2$ different both from $\sigma$ and from $2\alpha_2$; by a case-by-case check, it turns out that either $\tau$ has support of type $A_2$ or $\Gamma(\sigma)$ is of type $C$ and $\tau$ has support of type $A_1 \times A_1$. Thus the spherical diagram of $M$ in $\Gamma(\sigma)$ has one of the following shapes:

(B1) \[ \text{Diagram B1} \]
Suppose that we are not in case C2 and that $\Gamma(\sigma)$ is not of type $A_2$: then we are substantially reduced to the case of a wonderful model variety. Let $m(\sigma) \geq 3$ be the first integer such that the simple root $\alpha_{m(\sigma)}$ occurs in the support of one and only one spherical root with support of type $A_2$. For $1 \leq k \leq m(\sigma)$, set $\Delta(\alpha_k) = \{D_k\}$; set $\Delta(\sigma) = \{D_1, \ldots, D_{m(\sigma)}\}$ and define $\Delta(\sigma)^{\text{even}}, \Delta(\sigma)^{\text{odd}} \subset \Delta(\sigma)$ as the subsets whose element index is respectively even and odd.

**Lemma 5.6.** Let $M$ be a strict wonderful variety possessing a spherical root $\sigma$ of type $B_2^r$ such that the spherical diagram of $M$ in $\Gamma(\sigma)$ is of type $B_1$; let $\delta$ be a faithful bijection between $\Delta(1)$ and $\Delta^\prime M$ such that the spherical diagram of whose element index is respectively even and odd.

**Lemma 5.6.** Let $M$ be a strict wonderful variety possessing a spherical root $\sigma$ of type $B_2^r$ such that the spherical diagram of $M$ in $\Gamma(\sigma)$ is of type $B_1$; let $\delta$ be a faithful bijection between $\Delta(1)$ and $\Delta^\prime M$ such that the spherical diagram of $M$. Then there does not exist any orbit $Z \subset X_2$ possessing a spherical root $\gamma$ of type $B_2^r$ with $\sigma \in \text{Supp}_2(\gamma)$ if and only if $D_1 \in \text{Supp}_\Delta(\delta)$ or the following conditions are both satisfied:

i) $\text{Supp}_\Delta(\delta) \cap \Delta(\sigma)^{\text{even}} = \emptyset$;

ii) If $M$ possesses a spherical root supported on $\alpha_{m(\sigma)+1}$, then $m(\sigma)$ is odd.

**Proof.** By Lemma 5.2 we may assume that $\text{Supp}_\Delta(\delta) \cap \{D_1, D_2\} = \emptyset$. Notice that $\Delta(\sigma) \setminus \{D_m(\sigma)\}$ is distinguished and that conversely any distinguished subset which intersects $\Delta(\sigma)$ contains $\Delta(\sigma) \setminus \{D_m(\sigma)\}$. Number the $m(\sigma)$ spherical roots supported on $\{\alpha_1, \ldots, \alpha_{m(\sigma)}\}$ from the right to the left: set $\sigma_i = 2\alpha_1$ and, if $2 \leq i \leq m(\sigma)$, set $\sigma_i = \sigma_{i-1} + \alpha_i$.

If $W \subset M$ is an orbit, denote $\Sigma' \subset \Sigma$ its set of spherical roots and $\Delta'$ its set of colors; for $1 \leq i \leq m(\sigma)$ set $\Delta'(\alpha_i) = \{D'_1\}$ and set $\Delta'(\sigma) = \{D'_1, \ldots, D'_{m(\sigma)}\}$. Denote $q : \text{Pic}(M) \rightarrow \text{Pic}(W)$ the pullback map and observe that $q$ induces a bijection between $\Delta(\sigma)$ and $\Delta'(\sigma)$. More precisely, $q(D_i) = D'_i$ for every $1 < i \leq m(\sigma)$, while

$$q(D_1) = \begin{cases} D'_1 & \text{if } 2\alpha_1 \in \Sigma' \\ 2D'_1 & \text{if } 2\alpha_1 \notin \Sigma' \end{cases}$$

therefore, if $i \leq m(\sigma)$, $\delta$ is supported on $D_i$ if and only if $\delta_W = q(\delta)$ is supported on $D'_i$.

(\Rightarrow) Consider the codimension one orbit $W$ whose spherical root set is $\Sigma' = \Sigma \setminus \{\alpha_1\}$; set $Z = \phi_W(W)$ and $Z' = p^{-1}(Z)$. Denote $\Delta^* \subset \Delta'$ the maximal distinguished subset of colors which does not intersect the support of $\delta_W$; since $D'_1 \notin \text{Supp}_\Delta(\delta_W)$ and since it is non-negative against any spherical root, we get $D'_1 \notin \Delta^*$. Suppose that i) or ii) fails. Notice that, in order to show that $Z \neq Z'$, it is enough to show that $D'_2 \notin \Delta^*$. On one hand, by Proposition 4.1 together with Corollary 3.3 this implies $\sigma \in \Lambda_Z$: in fact $e(D', \sigma) = 0$ for every $D' \in \Delta' \setminus \{D'_2, D'_3\}$ and $D'_2 \notin \Delta^*$ implies $D'_3 \notin \Delta^*$. On the other hand, since $D'_1 \notin \Delta^*$, we get $\Delta(Z)(\alpha_1) = \Delta(Z)(\alpha_1) = \emptyset$, which implies that $\sigma \notin \Lambda_Z$. Therefore, if $D'_2 \notin \Delta^*$, then $\sigma \in \Lambda_Z \setminus \Lambda_Z$ and $2\sigma \in \Sigma_Z$.

Suppose first that i) fails and that $D'_2 \notin \Delta^*$. Then it must be either $\Delta'(\sigma)^{\text{even}} \subset \Delta^*$ or $\Delta'(\sigma) \setminus \{D'_{m(\sigma)}\} \subset \Delta^*$: this follows by considering the conditions defining...
Proof. Let $M$ with one further spherical root 2

Theorem 5.7. Let $M$ be a strict wonderful variety possessing a spherical root $\sigma$ of type $B_1$ such that the spherical diagram of $M$ in $\Gamma(\sigma)$ is of type $B_2$; let $\delta$ be a faithful divisor on $M$. Then there does not exist any orbit $Z \subset X_3$ possessing a spherical root $\gamma$ of type $B_2^s$ with $\sigma \in \Supp_2(\gamma)$ if and only if $D_1 \in \Supp_\Delta(\delta)$. Then $\Sigma'(\sigma) \in \Delta'$. Observe that $\Delta' \cap \Delta'(\sigma)^{odd} = \emptyset$: in fact otherwise it should be $\Delta'(\sigma)^{even} \subset \Delta'$. Therefore $\Delta' \cap \Delta'(\sigma)^{even}$ and we get $\sigma \notin \Sigma(\phi_{\delta}(W))$: in fact, since $D_2 \notin \Delta'$, a spherical root $\gamma \in \Sigma(\phi_{\delta}(W))$, with support of type $B_1$, is necessarily a multiple of $\sigma$, and this cannot happen since $c(D_2, \sigma) = 1$. To conclude, it is enough to notice that, if $\sigma \notin \phi_{\delta}(M')$, then $\Sigma(Z) \subset \Sigma(\phi_{\delta}(W))$. \qed

Corollary 5.7. Let $M$ be a strict wonderful variety possessing a spherical root $\sigma$ of type $B_1$ such that the spherical diagram of $M$ in $\Gamma(\sigma)$ is of type $B_2$; let $\delta$ be a faithful divisor on $M$. Then there does not exist any orbit $Z \subset X_3$ possessing a spherical root $\gamma$ of type $B_2^s$ with $\sigma \in \Supp_2(\gamma)$ if and only if $D_1 \in \Supp_\Delta(\delta)$. Then $\Sigma'(\sigma) \in \Delta'$. Observe that $\Delta' \cap \Delta'(\sigma)^{odd} = \emptyset$: in fact otherwise it should be $\Delta'(\sigma)^{even} \subset \Delta'$. Therefore $\Delta' \cap \Delta'(\sigma)^{even}$ and we get $\sigma \notin \Sigma(\phi_{\delta}(W))$: in fact, since $D_2 \notin \Delta'$, a spherical root $\gamma \in \Sigma(\phi_{\delta}(W))$, with support of type $B_1$, is necessarily a multiple of $\sigma$, and this cannot happen since $c(D_2, \sigma) = 1$. To conclude, it is enough to notice that, if $\sigma \notin \phi_{\delta}(M')$, then $\Sigma(Z) \subset \Sigma(\phi_{\delta}(W))$. \qed

Corollary 5.7. Let $M$ be a strict wonderful variety possessing a spherical root $\sigma$ of type $B_1$ such that the spherical diagram of $M$ in $\Gamma(\sigma)$ is of type $B_2$; let $\delta$ be a faithful divisor on $M$. Then there does not exist any orbit $Z \subset X_3$ possessing a spherical root $\gamma$ of type $B_2^s$ with $\sigma \in \Supp_2(\gamma)$ if and only if $D_1 \in \Supp_\Delta(\delta)$. \qed

Proof. Let $M'$ be the wonderful variety whose spherical system is the same one of $M$ with one further spherical root $2\alpha_1$: then $M$ is identified with a $G$-stable prime divisor of $M'$ and the spherical diagram of $M'$ in $\Gamma(\sigma)$ is of the type considered in previous lemma. Denote $\Sigma'$ and $\Delta'$ the set of spherical roots and the set of colors of $M'$; observe that the pullback map $q : \Pic(M') \rightarrow \Pic(M)$ induces an isomorphism between the sublattices generated by $\Delta \setminus \{D_{\alpha_1}\}$ and $\Delta' \setminus \{D_{\alpha_1}'\}$. If $D_1 \in \Supp_\Delta(\delta)$ then the claim follows by Lemma 5.2; thus we may assume $D_1 \notin \Supp_\Delta(\delta)$ and we may identify $\delta$ with a divisor $\delta'$ on $M'$ which is still faithful.

If $Z \subset \phi_{\delta'}(M')$ is an orbit possessing a spherical root $\gamma$ of type $B_2^s$ with $\sigma \in \Supp_2(\gamma)$, then $2\alpha_1 \notin \Sigma'(Z)$ and by Proposition 4.5 we get $Z \subset X_3 = \phi_{\delta}(M)$; therefore such an orbit exists in $X_3$ if and only if it exists in $\phi_{\delta'}(M')$ and we can
apply previous lemma. In order to get the claim it is enough to observe that if condition ii) of Lemma 5.2 holds, then (in the notations of that lemma) $\Delta(\sigma)^{\text{even}} = q(\Delta(\sigma)^{\text{even}}) \subset \Delta$ is distinguished: thus $\text{Supp}_\Delta(\delta) \cap \Delta(\sigma)^{\text{even}} \neq \varnothing$ and consequently i) fails.

If they are defined, set
\[ e_\sigma(\delta) = \min\{k \leq m(\sigma) : D_k \in \text{Supp}_\Delta(\delta) \cap \Delta(\sigma)^{\text{even}}\}, \]
\[ o_\sigma(\delta) = \min\{k \leq m(\sigma) : D_k \in \text{Supp}_\Delta(\delta) \cap \Delta(\sigma)^{\text{odd}}\}. \]

**Lemma 5.8.** Let $M$ be a strict wonderful variety possessing a spherical root $\sigma$ of type $B_3^1$ such that the spherical diagram of $M$ in $\Gamma(\sigma)$ is of type $C_1$; let $\delta$ be a faithful divisor on $M$. Then there does not exist any orbit $Z \subset X_\delta$ possessing $2\sigma$ as a spherical root if and only if the following conditions are both satisfied:

i) $\text{Supp}_\Delta(\delta) \cap \Delta(\sigma)^{\text{even}} \neq \varnothing$;

ii) If $\text{Supp}_\Delta(\delta) \cap \Delta(\sigma)^{\text{odd}} \neq \varnothing$, then $o_\sigma(\delta) \geq e_\sigma(\delta) - 1$.

**Proof.** Notice that if $m(\sigma)$ is even then $\Delta(\sigma)^{\text{odd}}$ is distinguished, while if $m(\sigma)$ is odd then $\Delta(\sigma)^{\text{even}}$ is distinguished: thus at least one between $e_\sigma(\delta)$ and $o_\sigma(\delta)$ is well defined. By Lemma 5.2 we may suppose $\min\{e_\sigma(\delta), o_\sigma(\delta)\} > 2$. Number the $m(\sigma) - 1$ spherical roots supported on $\{\alpha_1, \ldots, \alpha_m(\sigma)\}$ from the right to left: if $i < m(\sigma)$, set $\sigma_i = \alpha_i + \alpha_{i+1}$.

If $W \subset M$ is an orbit, denote $\Sigma' \subset \Sigma$ its set of spherical roots and $\Delta'$ its set of colors; for $1 \leq i \leq m(\sigma)$ set $\Delta'(\alpha_i) = \{D'_i\}$ and set $\Delta'(\sigma) = \{D'_1, \ldots, D'_{m(\sigma)}\}$. Denote $q : \text{Pic}(M) \to \text{Pic}(\overline{W})$ the pullback map and observe that $q$ induces a bijection between $\Delta(\sigma)$ and $\Delta'(\sigma)$. Since $q(D_i) = D'_i$ for every $i \leq m(\sigma)$, $\delta$ is supported on $D_i$ if and only if $\delta_W = q(\delta)$ is supported on $D'_i$.

$(\Rightarrow)$ Suppose that $o_\sigma(\delta)$ is defined and, in case $e_\sigma(\delta)$ is defined too, suppose that $o_\sigma(\delta) < e_\sigma(\delta) - 1$: this implies $o_\sigma(\delta) < m(\sigma)$, since otherwise $\Delta(\sigma)^{\text{even}}$ would be distinguished and it would be $e_\sigma(\delta) < o_\sigma(\delta)$.

Consider the orbit $W \subset M$ whose spherical roots are $\sigma_1, \ldots, \sigma_j$; set $Z = \phi_\delta(W)$ and $Z' = p^{-1}(Z)$. Then the maximal distinguished subset of $\Delta'$ which does not intersect the support of $\delta_W$ is
\[ \Delta^* = \Delta' \setminus \{\Delta'(\sigma_j)^{\text{odd}} \cup \text{Supp}_\Delta(\delta_W)\}, \]

which by hypothesis contains $\Delta'(\sigma_j)^{\text{odd}}_{j+1}$ (where the notations are the obvious ones); thus $\Delta^* \cap \{D'_1, D'_2, D'_3\} = \{D'_2\}$. Since $c(D'_1, \sigma) = 0$ for every $D'_i \in \Delta' \setminus \{D'_1, D'_3\}$, by Proposition 4.4 together with Corollary 4.3 we get $\sigma \in \Sigma_{Z'}$. On the other hand, $D'_2 \in \Delta^*$ implies $\Delta(Z)(\alpha_2) = \varnothing$: since $Z$ is spherically closed, we get then $\sigma \notin \Sigma_Z$ and $2\sigma \in \Sigma_Z$.

$(\Leftarrow)$ Suppose that $e_\sigma(\delta)$ is defined and, in case $o_\sigma(\delta)$ is defined too, suppose that $o_\sigma(\delta) \geq e_\sigma(\delta) - 1$. Fix an orbit $W \subset M$, set $Z = \phi_\delta(W)$ and $Z' = p^{-1}(Z)$. We may assume that $\sigma \in \Sigma'$, since otherwise there is nothing to prove. Set $\Delta^* \subset \Delta'$ the maximal distinguished subset which does not intersect the support of $\delta_W$ and notice that $2\sigma \in \Sigma_Z$ if and only if $\Delta^* \cap \{D'_1, D'_2, D'_3\} = \{D'_2\}$. Such condition does not hold if $\sigma \notin \Sigma'$ or if $\sigma_2 \notin \Sigma'$, since then it would be $\Delta'_1 \in \Delta^*$; thus we may assume that $\Sigma' \supset \{\sigma_1, \sigma_2, \sigma_3\}$.

Set $k < m(\sigma)$ the maximum such that $\sigma_i \in \Sigma'$ for every $i \leq k$. By considering the conditions defining a distinguished set only for $\sigma_1, \ldots, \sigma_k$ it follows that, if $D'_2 \in \Delta^*$, then either $\Delta'(\sigma)^{\text{odd}}_{k+1} \subset \Delta^*$ or $\Delta'(\sigma)^{\text{odd}}_{k+1} \subset \Delta^*$. If we are in the first case, then we are done; suppose we are in the second case. Then it must be $e_\sigma(\delta) > k + 1$ and, by the hypothesis, we get $o_\sigma(\delta) > k$. Since it is distinguished and it does not intersect the support of $\delta_W$, we get then $\Delta'(\sigma)^{\text{odd}}_{k+1} \subset \Delta^*$: therefore the condition $\Delta^* \cap \{D'_1, D'_2, D'_3\} = \{D'_2\}$ is not satisfied whenever conditions i) and ii) hold and the claim follows. □
Combining together Lemma 5.6, Corollary 5.7, and Lemma 5.8, we get the following theorem (the cases wherein the spherical diagram of \( M \) in \( \Gamma(\sigma) \) is of type C2, F1, F2 or F3 are easily treated directly).

**Theorem 5.9.** Let \( M \) be a strict wonderful variety and let \( \delta \) be a faithful divisor on it. Then the normalization \( p : \tilde{X}_\delta \rightarrow X_\delta \) is bijective if and only if the following conditions are fulfilled, for every spherical root \( \sigma \in \Sigma \) of type \( B_2^1 \):

i) If the spherical diagram of \( M \) in \( \Gamma(\sigma) \) is of type \( B_1 \), then \( D^+(\sigma) \in \text{Supp}_\delta(\delta) \) or the following conditions are both satisfied:

- \( \text{Supp}_\delta(\delta) \cap \Delta(\sigma)_{\text{even}} = \emptyset \);
- If \( M \) possesses a spherical root supported on \( \alpha_{m(\sigma)+1} \), then \( m(\sigma) \) is odd.

ii) If the spherical diagram of \( M \) in \( \Gamma(\sigma) \) is of type \( B_2 \), then \( D^+(\sigma) \in \text{Supp}_\delta(\delta) \).

iii) If the spherical diagram of \( M \) in \( \Gamma(\sigma) \) is of type \( C_1 \), then the following conditions are both satisfied

- \( \text{Supp}_\delta(\delta) \cap \Delta(\sigma)_{\text{even}} \neq \emptyset \);
- If \( \text{Supp}_\delta(\delta) \cap \Delta(\sigma)_{\text{odd}} \neq \emptyset \), then \( \alpha_{s}(\delta) \geq e_s(\delta) - 1 \).

iv) Otherwise, if \( D^+(\sigma) \in \text{Supp}_\delta(\delta) \), then \( D^+(\sigma) \in \text{Supp}_\delta(\delta) \) as well.

6. Bijectivity in the non-strict case.

Keeping the notations of previous sections, suppose that \( M \) is not strict and let \( \delta = \sum_{\Delta} n(\delta, D)D \) be a faithful divisor on \( M \). Suppose that \( Z \subset X_\delta \) is an orbit such that \( \Sigma(\delta_2) \) contains a non-simple spherical root \( \gamma \). Following examples show that, unlike from the strict case (Lemma 5.1), \( \gamma \) may be as well of type \( G_2^1 \) and, in case \( \gamma \) is of type \( B_2^1 \), then it does not necessarily come from a spherical root of type \( B_2^1 \).

**Example 6.1.** Consider the wonderful variety \( M \) whose spherical system is expressed by the spherical diagram

\[
\begin{array}{c}
\circ \\
\circ \\
\circ
\end{array}
\]

Then the divisor \( \delta = D^+_{\alpha_1} \) is faithful. Consider the codimension one orbit \( W \subset M \) whose spherical roots are \( \alpha_2 \) and \( \alpha_2 + \alpha_3 \); following Proposition 4.1 and Corollary 3.7, we get the sequence of spherical diagrams

\[
\begin{array}{c}
\circ \\
\circ \\
\circ
\end{array} \rightarrow \begin{array}{c}
\circ \\
\circ \\
\circ
\end{array} \rightarrow \begin{array}{c}
\circ \\
\circ \\
\circ
\end{array} \rightarrow \begin{array}{c}
\circ \\
\circ \\
\circ
\end{array} \rightarrow \begin{array}{c}
\circ \\
\circ \\
\circ
\end{array}
\]

where the first one represents the orbit \( W \subset M \), the second one the orbit \( \tilde{\phi}_\delta(W) \subset \tilde{X}_\delta \) and the third one the orbit \( \phi_\delta(W) \subset X_\delta \).

**Example 6.2.** Consider the wonderful variety \( M \) whose spherical system is expressed by the spherical diagram

\[
\begin{array}{c}
\circ \\
\circ \\
\circ
\end{array}
\]

Then the divisor \( \delta = D^+_{\alpha_1} \) is faithful. See Table 2 for a full list of the orbits in \( \tilde{X}_\delta \) and in \( X_\delta \) (for simplicity, in the table orbits in \( M \) are described by giving a subset of its spherical root index set).

**Lemma 6.3.** Suppose that \( M \) is a spherically closed wonderful variety and let \( \delta = \sum_{\Delta} n(\delta, D)D \) be a faithful divisor on it; let \( \alpha \in S \cap \Sigma \).

i) If \( Z \subset X_\delta \) is an orbit such that \( 2\alpha \in \Sigma_Z \), then \( n(\delta, D^\alpha_\delta) = n(\delta, D^-_\delta) \).
Suppose that \( W \subset M \) is an orbit with set of spherical roots \( \Sigma' \subset \Sigma \) and set of colors \( \Delta' \). If \( \alpha \in S \cap \Sigma' \), set \( \Delta'(\alpha) = \{ D_{\alpha}^+, D_{\alpha}^- \} \); then by the description of the pullback map \( q : \text{Pic}(M) \to \text{Pic}(\mathcal{W}) \) it follows that
\[
\begin{align*}
  n(\delta_W, D_{\alpha}^+) &= n(\delta, D_{\alpha}^+), \\
  n(\delta_W, D_{\alpha}^-) &= n(\delta, D_{\alpha}^-),
\end{align*}
\]

where \( \delta_W = q(\delta) \).

i). Let \( Z \subset X_{\delta} \) be an orbit possessing \( 2\alpha \) as a spherical root; let \( Z' = p^{-1}(Z) \) and let \( W \subset M \) be an orbit which maps on \( Z \). Then by Proposition 2.10 we get that \( \alpha \in \Sigma_{Z'} \), while by Corollary 6.4 together with Theorem 3.2 we get \( n(\delta_W, D_{\alpha}^+) = n(\delta_W, D_{\alpha}^-) \); by the remark at the beginning of the proof this implies the thesis.

ii). Consider the rank one orbit \( W \) whose unique spherical root is \( \alpha \), set \( Z = \phi_{\delta}(W) \) and \( Z' = p^{-1}(Z) \). Then \( \alpha \in \Sigma_{Z'} \) is a loose spherical root and by the remark at the beginning of the proof we get \( n(\delta_{Z'}, D_{\alpha}^+) = n(\delta_{Z'}, D_{\alpha}^-) \), where \( \delta_{Z'} \) is the pullback of a hyperplane section and where \( \Delta(Z') \) is identified with a subset of \( \Delta(W) \). Then by Corollary 6.4 we get that \( 2\alpha \in \Sigma_{Z'} \).

Suppose that \( \alpha \in S \cap \Sigma \). As shown by Example 6.2 if \( n(\delta, D_{\alpha}^+) = n(\delta, D_{\alpha}^-) = 0 \), then it may not exist any orbit \( Z \subset X_{\delta} \) possessing \( 2\alpha \) as a spherical root; conversely, if there exists such an orbit, it may be as well \( n(\delta, D_{\alpha}^+) = n(\delta, D_{\alpha}^-) = 0 \).

As a corollary of previous lemma, we get the following sufficient conditions.

**Corollary 6.4.** Suppose that \( M \) is a spherically closed wonderful variety and let \( \delta = \sum_{\Delta} n(\delta, D)D \) be a faithful divisor on it.
i) If there exists $\alpha \in S \cap \Sigma$ such that $n(\delta, D_{\alpha}^+) = n(\delta, D_{\alpha}^-)$ is non-zero, then the normalization $p : \tilde{X}_{\delta} \to X_{\delta}$ is non-bijective.

ii) If the Dynkin diagram of $G$ is simply laced and if $n(\delta, D_{\alpha}^+) \neq n(\delta, D_{\alpha}^-)$ for every $\alpha \in S \cap \Sigma$, then the normalization $p : \tilde{X}_{\delta} \to X_{\delta}$ is bijective.

Reasoning as in Lemma 5.2 and in Corollary 5.3, other sufficient conditions of bijectivity can be obtained imposing further conditions on the support of $\delta$ on the multiple links of the Dynkin diagram of $G$ and on the simple spherical roots of $M$.

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