Calculation of anisotropic transport coefficients for an ultrarelativistic Boltzmann gas in a magnetic field within a kinetic approach

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Abstract: In the present work, we consider the same particle system as in \cite{58}, calculate the anisotropic transport coefficients with BAMPS via the Kubo formulas given in \cite{13, 15, 16}, and compare the results with those obtained in \cite{58}. An agreement of both results will confirm the general use of the derived Kubo formulas for calculating the anisotropic transport coefficients of quark-gluon plasma in a magnetic field.

I. INTRODUCTION

The experiments of ultrarelativistic heavy-ion collisions at the Relativistic Heavy Ion Collider (RHIC) and the Large Hadron Collider (LHC) are believed to reach high enough energies to create the quark-gluon plasma (QGP) \cite{1}, which is composed of deconfined quarks and gluons at or close to thermal equilibrium. The QGP behaves like a nearly perfect fluid with a small value of the shear viscosity to the entropy density ratio \(\eta/s\). Due to the existence of strong magnetic field in the early stage of relativistic heavy-ion collisions \cite{8–11}, the QGP may behave different from the one when ignoring the magnetic field. Actually, the magnetic field breaks the spatial symmetry and leads to the anisotropization of transport coefficients \cite{12–16}. Their values depend on the strength of the magnetic field.

The transport coefficients are important physical quantities characterizing the features of QGP and reflecting the nature of interactions between quarks and gluons. In the last decade, dissipative hydrodynamic models \cite{17–25} have played a very important role in extracting the shear and bulk viscosity of QGP from the flow measurements \cite{26–33}. Now it is necessary to develop models based on relativistic magneto-hydrodynamics (MHD) \cite{34, 35}, in order to study QGP dynamics in the presence of a magnetic field.

How large the magnetic effect (also the chiral magnetic effect \cite{36–47}) is, depends on the strength of the magnetic field. The magnetic field in the early stage of a noncentral heavy-ion collision stems from the pass of two moving ions and its life time is very short. However, the rapid decrease of this external magnetic field will lead to an electromagnetic induction of QGP, so that the total magnetic field can last longer, if the QGP is a good conductor with a large electric conductivity. Therefore, the value of the electric conductivity of the QGP is essential for the possibility of observing magnetic effects in heavy-ion collisions.

According to the Green-Kubo relation \cite{50, 51}, the transport coefficients are related to the correlation functions of the corresponding tensor or flux. In Refs. \cite{13, 15, 16}, Kubo formulas for anisotropic transport coefficients in the presence of a magnetic field are derived with different methods. The calculation of the correlation functions of fluctuating tensor or flux can be realized in kinetic transport models. In this work, we employ the Boltzmann Approach of MultiParton Scattering (BAMPS) \cite{52}, which solves the Boltzmann equation for systems of on-shell particles. In the early studies (in the absence of a magnetic field), BAMPS has been used to calculate the shear viscosity of a pQCD-based gluon gas \cite{53, 54} and of QGP \cite{55}, the electric conductivity of QGP \cite{56} and recently the shear viscosity of ultrarelativistic Boson systems in the presence of a Bose-Einstein condensation \cite{57}.

For a simple case such like a one-component system with massless Boltzmann particles undergoing isotropic binary elastic collisions, the anisotropic transport coefficients in a magnetic field can be calculated analytically by using Grad’s approximations \cite{58}. In this work we consider the same particle system as in \cite{58}, calculate the anisotropic transport coefficients with BAMPS via the Kubo formulas given in \cite{13, 15, 16}, and compare the results with those obtained in \cite{58}. An agreement of both results will confirm the general use of the derived Kubo formulas for calculating the anisotropic transport coefficients of QGP in a magnetic field.

The paper is organized as follows: In Sec. \textsuperscript{II} we briefly review the equations of magneto-hydrodynamics and give the Kubo formulas for the corresponding transport coefficients. In Sec. \textsuperscript{III} we introduce the parton cascade BAMPS and numerical implementations. Subsequently, in Sec. \textsuperscript{IV} we show our numerical results including the influence of the magnetic field on the time evolution of corresponding correlation functions and the values of shear viscosity and electric conductivity coefficients for an one-component system of ultrarelativistic Boltzmann particles with isotropic binary scatterings. Finally, we give a conclusion in Sec. \textsuperscript{V}.

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We adopt natural units, \( h = c = k_B = 1 \). The metric tensor is chosen to be \( g^{\mu\nu} = \text{diag}(+, -, -, -) \).

## II. ANISOTROPIC TRANSPORT COEFFICIENTS AND KUBO FORMULAS

The dynamics of relativistic fluids in external magnetic field has been studied in Ref.\[13\]. The authors have found that due to the breaking of the spatial symmetry in the presence of a magnetic field, the dissipative functions contain anisotropic transport coefficients, namely, two bulk viscosity, five shear viscosity and three electric conductivity coefficients. At first we briefly summarize the results of Ref.\[13\]. The same results can also be found in Ref.\[59\].

For a charged particle system, the basic equations of MHD consist of the conservation laws of energy, momentum, and electric charge, and the constitutive equations for the energy-momentum tensor and the electric current. The conservation laws can be expressed as

\[
\begin{align*}
\partial_t \rho_j + \nabla \cdot (\rho_j \mathbf{v}) &= 0, \\
\partial_t \rho_T^{\mu\nu} + \nabla \cdot (\rho_T^{\mu\nu} \mathbf{v}) &= 0,
\end{align*}
\]

where \( F^{\mu\nu} \) is the electromagnetic field-strength tensor. The electric field is neglected in \[13, 15\] assuming that the electric field is much smaller than the magnetic field, which is a good approximation for QGP produced in heavy-ion collisions for instance. The constitutive equations in the Landau frame read \[13, 59\]

\[
\begin{align*}
\mathbf{j}^\mu &= q n u^\mu + \mathbf{J}^\mu, \\
T^{\mu\nu} &= \varepsilon u^\mu u^\nu - P_1 \Xi^{\mu\nu} + P_1 b^\mu b^\nu + T^{\mu\nu},
\end{align*}
\]

where \( u^\mu \) is the fluid 4-velocity normalized to \( u^2 = 1 \) and \( b^\mu = B^\mu / B \) with \( B^\mu \equiv \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} / 2 \) and \( B \equiv \sqrt{-g} B_\mu B^\mu, \) \( q \) is the particle charge. \( \varepsilon \) and \( n \) denote the energy and particle number density, respectively. The tensor which projects onto the three-dimensional space orthogonal to the flow velocity \( u^\mu \) is defined as \( \Delta^{\mu\nu} \equiv g^{\mu\nu} - \mu^\mu u^\nu, \) \( \Xi^{\mu\nu} \equiv \Delta^{\mu\nu} + b^\mu b^\nu \) is the tensor, which projects onto the two-dimensional space orthogonal to both the flow velocity \( u^\mu \) and the direction of magnetic field. \( P_1 \) and \( P_\perp \) are defined as \( P_1 \equiv b_\mu b^\nu T^{\mu\nu} \) and \( P_\perp \equiv -\Xi^{\mu\nu} T^{\mu\nu} / 2. \) The dissipative terms in Eqs. \[13\] and \[14\] can be obtained by the derivative expansion to the leading order and have the form in terms of viscosity and electric conductivity coefficients,

\[
\begin{align*}
\mathbf{J}^\mu &= T(\kappa \Xi^{\mu\nu} \nabla_{\nu} \alpha - \kappa_\perp b^\mu b^\nu \nabla_{\nu} \alpha - \kappa_\perp b^\mu \nabla_{\mu} \alpha), \\
T^{\mu\nu} &= \frac{3}{2} \kappa \Xi^{\mu\nu} \phi + \frac{3}{2} \kappa_\perp b^\mu b^\nu \phi + 2 \eta_0 (w^{\mu\nu} - \frac{1}{3} \Delta^{\mu\nu} \theta) + \eta_1 (\Delta^{\mu\nu} - \frac{3}{2} \Xi^{\mu\nu}) (\theta - \frac{3}{2} \phi) \\
&\quad - 2 \eta_2 (b^\mu \Xi^{\sigma\nu} b^{\sigma} + b^\nu \Xi^{\alpha\beta} b^{\alpha}) w_{\alpha\beta} \\
&\quad - 2 \eta_3 (\Xi^{\mu\nu} b^\beta + \Xi^{\alpha\beta} b^\mu) w_{\alpha\beta} \\
&\quad + 2 \eta_4 (b^\mu b_\perp b^\beta + b^\nu b_\perp b^\mu) w_{\alpha\beta},
\end{align*}
\]

where \( \alpha \equiv \beta \mu \) (\( \mu \) is the chemical potential), \( b^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} b_{\alpha} u^\beta, \) \( w^{\mu\nu} \equiv (\nabla u^{\mu} + \nabla u^{\nu}) / 2, \) \( \phi \equiv \Xi^{\mu\nu} u_{\mu\nu}, \) \( \psi \equiv b^\mu b^\nu u_{\mu\nu}, \) \( \theta \equiv \partial^n u_\mu, \) with \( \nabla \equiv \Delta_{\mu\nu} \partial^{\nu}. \) The combining coefficients are identified as five shear viscosity \((\kappa_\perp, \kappa_{\perp\parallel}, \kappa_{\parallel\perp})\), two bulk viscosity \((\zeta_\perp, \zeta_\parallel)\), and three electric conductivity coefficients \((\kappa_\parallel, \kappa_{\perp\parallel}, \kappa_{\parallel\perp})\).

We now summarize Kubo formulas for the anisotropic transport coefficients, which are given in Refs.\[13, 15\] and \[16\]. \( T^{\mu\nu} \) and \( j^\mu \) in the formulas given below are taken at the local rest frame. The authors of Ref.\[13\] used Zubarev’s non-equilibrium statistical operator method to relate the anisotropic transport coefficients to correlation functions in equilibrium. The corresponding Kubo formulas are given by \[13, 59\]

\[
\begin{align*}
\eta_0 &= \frac{\partial}{\partial \omega} \text{Im} G_{R_{12}} |_{\omega = 0} , \\
\eta_1 &= -\frac{4}{3} \eta_0 - 2 \frac{\partial}{\partial \omega} \text{Im} G_{R_{12}} |_{\omega = 0} , \\
\eta_2 &= -\eta_0 + \frac{\partial}{\partial \omega} \text{Im} G_{R_{12,T_{12}}^{\perp}} |_{\omega = 0} , \\
\eta_3 &= \frac{1}{4} \frac{\partial}{\partial \omega} \text{Im} G_{R_{12,T_{12}}}^{\perp} |_{\omega = 0} , \\
\eta_4 &= \frac{\partial}{\partial \omega} \text{Im} G_{R_{12,T_{23}}}^{\perp} |_{\omega = 0} ,
\end{align*}
\]

where the retarded Green’s function in quantum statistical theory has the form \( G^{R}_{AB} \equiv i \theta(x^0) \langle A(x) B(0) \rangle \), and the angular brackets denote the ensemble average in equilibrium. Some other symbols in the above formulas are defined as \( P_\parallel \equiv P_1 - \Theta_{\beta\gamma} - \Theta_{\alpha n} \) and \( P_\perp \equiv P_{\perp} - (\Theta_{\beta\gamma} + \Phi_{\beta})(\Theta_{\alpha n} + \Phi_{\alpha n}) \) with \( \Theta_{\beta\gamma} \equiv \partial P / \partial n, \Phi_{\beta} \equiv -B (\partial M / \partial n, B, \Theta_{\alpha n} \equiv (\partial P / \partial n)_{\epsilon, B}, \) \( \Phi_{\alpha n} \equiv -B (\partial M / \partial n)_{\epsilon, B}. \) The coefficients involving magnetization in the definition of \( P_\parallel \) and \( P_\perp \) vanish for particles without dipole moment or spin, which is the case we consider in this work. We also note that a sign mistake in the formula of \( \eta_3 \) occurred in \[13\] has been corrected.

The Kubo formulas of the viscosity coefficients were also given in Refs. \[13, 16\], where a variational approach and a derivative method were used, respectively. Despite the sign and/or factor differences after unifying the convention for the transport coefficients, the Kubo formulas for the five shear viscosity coefficients are definitely the same among \[13, 15\] and \[16\]. Since the bulk viscosity vanish by considering a massless Boltzmann gas, there is no need to give their Kubo formulas.

The Kubo formulas for the shear viscosity coefficients can be expressed in real space-time with a integration...
where we choose the z-direction as the direction of the magnetic field, $B^z = (0, 0, B_0)$. Since we consider a homogeneous particle system, the space dependence of the particle flow and the energy-momentum tensor appeared in the above Kubo formulas can be integrated out directly.

Without the magnetic field ($B_0 = 0$), except for $\eta_0$, which should be equal to the standard isotropic shear viscosity $\eta$, all other shear viscosity coefficients should vanish. It is obvious for $\eta_0$, $\eta_1$, $\eta_2$, and $\eta_3$, since $\langle T^{13}(r, t)T^{13}(0, 0) \rangle$ is equal to $\langle T^{12}(r, t)T^{12}(0, 0) \rangle$ and both $\langle T^{11}(r, t) - T^{22}(r, t) \rangle T^{12}(0, 0) \rangle$ and $\langle T^{13}(r, t)T^{23}(0, 0) \rangle$ vanish. For the considered system we have $\langle \mathbf{P}_\parallel(r, t)\mathbf{P}_\perp(0, 0) \rangle = - \langle T^{33}(r, t)T^{33}(0, 0) \rangle / 2$ according to the definitions of $\mathbf{P}_\parallel$ and $\mathbf{P}_\perp$. From $54$, $60$ we realize that $\langle T^{13}(r, t)T^{33}(0, 0) \rangle = 4 \langle T^{12}(r, t)T^{12}(0, 0) \rangle / 3 = 4 \langle T^{12}(r, t)T^{22}(0, 0) \rangle / 3$. Therefore, $\eta_1$ vanishes.

Differences in numerical results of two kind of electric conductivity coefficients will be shown later in Sec. [LV]

Without the magnetic field ($B_0 = 0$), the longitudinal and transverse electric conductivity become equal, while the Hall electric conductivity (or resistivity) is meaningless. From the above Kubo formulas, it is obvious that $\kappa_\parallel = \kappa_\perp$ and $1/\rho_\perp$ is infinite. We will show in Sec. [LV] that $\sigma_\parallel$ is also infinite.

III. THE PARTON CASCADE BAMPS AND NUMERICAL IMPLEMENTATIONS

The time correlation functions in Eqs. (12)-(16) and Eqs. (22)-(29) are evaluated numerically for the considered particle system in a static box with periodic boundary conditions. Initially, particles are distributed homogeneously in coordinate space and thermally in momentum space. The space-time evolution of particles is calculated by employing the parton cascade BAMPS [52].
Coupled to an external electromagnetic field, the Boltzmann equation has the form

$$p^\mu \partial_\mu f(x,p) + qF^{\mu\nu}p_\nu \partial_\nu f(x,p) = C[f(x,p)], \quad (29)$$

where $f(x,p)$ is the one-particle phase-space distribution function, $C[f(x,p)]$ denotes the collision term. Since we restrict ourselves to a single-component gas of particles carrying no dipole moment or spin, the magnetic field $F^{\mu\nu}$ involves only a Lorentz force, which changes the momenta of charged particles. The microscopic interaction processes among particles are simulated via Monte Carlo techniques based on the stochastic interpretation of transition rates. In order to improve the numerical accuracy, the test particle method is introduced. The particle number is artificially increased by a factor of $N_{\text{test}}$, while the interaction cross section is reduced by the same factor simultaneously. Thus, the physical evolution of the particle system is not influenced by this implementation. The correlation probability for binary elastic scattering in a spatial cell of a volume of $\Delta V$ and within a time step $\Delta t$ is

$$P_{22} = v_{\text{rel}} \frac{\sigma_{22}}{N_{\text{test}}} \frac{\Delta t}{\Delta V}. \quad (30)$$

$v_{\text{rel}} = s/(2E_1E_2)$ is the relative velocity of the incoming particles with energy $E_1$ and $E_2$, $s$ is the invariant mass, and $\sigma_{22}$ is the total cross section of elastic binary scatterings. We consider the magnetic field $B$ to be constant and homogeneous, pointing in $z$ direction. Thus, the Lorentz force, $\vec{F}_L = q\vec{v} \times \vec{B}$, will change the directions (while not the magnitude) of particles’ transverse momenta for every computational time step $\Delta t$. Between the collisions the particles will move in a circle in the transverse plane, while they propagate via free streaming in the $z$ direction.

According to the physical definition, the electric current $j^\mu$ and energy-momentum tensor $T^{\mu\nu}$ are calculated as

$$j^\mu(t) = \frac{q}{V N_{\text{test}}} \sum_{i=1}^N \frac{p_i^\mu}{E_i}, \quad (31)$$

$$T^{\mu\nu}(t) = \frac{1}{V N_{\text{test}}} \sum_{i=1}^N \frac{p_i^\mu p_i^\nu}{E_i}, \quad (32)$$

where the sum is running over all the $N$ test particles in the box at time $t$ and $V$ is the volume of the box. The correlation is calculated at discrete and equally distributed time steps $t_i = t_0, t_1, ..., t_K$ by time average in the limit $t_K \rightarrow \infty$,

$$C(t_i) = \frac{1}{s_{\text{max}}} \sum_{s=0}^{s_{\text{max}}} A(t_s) B(t_s + t_i), \quad s_{\text{max}} = K - l. \quad (33)$$

$A(t)$ and $B(t)$ represent the component of the particle flow or energy-momentum tensor. The ensemble average is realized by $N_{\text{run}}$ individual initialization. In the presence of a magnetic field, some of the correlation functions oscillate instead of an exponential decrease (as we will show in the next section). Therefore, we have to evolve the systems to a much longer time until the correlations become small.

IV. NUMERICAL RESULTS

We consider a system of massless Boltzmann particles with a positive charge $e$. Initially particles are sampled in momentum space according to the Boltzmann distribution $f(x,p) = e^{-E/T}$ with a temperature of $T = 400$ MeV. Since the magnetic fields in noncentral Au-Au collisions at RHIC can reach $eB \sim m_0^2 = (0.443 \text{ GeV})^2$, we choose in this work $eB = 0, 0.01, 0.03, 0.05, 0.1 \text{ GeV}^2$. We neglect the Landau quantization of the particles’ cyclotron motion and assume that particles carry no dipole moment or spin, so that the gas has vanishing magnetization and polarization. We consider binary elastic collisions only. The total cross section is set to be a constant value ($\sigma_{22} = 1 \text{ mb}$), and the particles scatter isotropically.

In the following we show the results of shear viscosity and electric conductivity coefficients in the presence of a magnetic field, respectively.

A. Shear viscosity coefficients

The time evolution of correlation functions $\langle T^{12}(t)T^{12}(0) \rangle$ and $\langle T^{13}(t)T^{13}(0) \rangle$, which determine the shear viscosity coefficients $\eta_1$ and $\eta_2$, are shown in Fig. 1. We can see that the correlation functions behave quite different from those without magnetic field. The correlation functions decrease no longer exponentially. The presence of the magnetic field induces the oscillations of the correlation functions, because the Lorentz force changes the sign of $\langle T^{12}(t)T^{12}(0) \rangle$. Compared with $\langle T^{12}(t)T^{12}(0) \rangle$, the oscillation frequencies of $\langle T^{13}(t)T^{13}(0) \rangle$ are smaller, since the momenta in $z$-direction are not influenced by the Lorentz force.

The correlation function $\langle P_{\parallel}(t)P_{\perp}(0) \rangle$, which determines the shear viscosity coefficient $\eta_3$, are shown in Fig. 2. For the system we have considered in this work, we have $P_{\parallel} = T^{33} - T^{00}/3$ and $P_{\perp} = (T^{11} + T^{22})/2 - T^{00}/3 = T^{00}/6 - T^{33}/2$. From Fig. 2 we see that the magnetic field has no influence on this correlation function despite the small deviation due to numerical fluctuations. This is because the correlation function only involve particles’ energy and momentum in the $z$-direction, both of them are not affected by the magnetic field.

It is obvious that the correlation functions $\langle (T^{11}(t) - T^{22}(t))T^{12}(0) \rangle$ and $\langle (T^{13}(t)T^{23}(0) \rangle$ will vanish, if there is no magnetic field. With the magnetic field the two correlation functions, corresponding to $\eta_3$ and $\eta_4$ respectively, oscillate due to the same reason for $\langle T^{12}(t)T^{13}(0) \rangle$ and $\langle T^{13}(t)T^{13}(0) \rangle$, as seen in Fig. 3.
lytically in Ref.[58] by using the method of 14-moment collisions in a magnetic field have been derived analytically in Ref.[58] by using the method of 14-moment collisions in a magnetic field. We see that except for $\eta = 4\lambda_{mfp}P/3$ obtained without the magnetic field. We see that except for $\eta_1$, which is almost constant at large magnetic field, all the other shear viscosity coefficients are decreasing.

The five anisotropic shear viscosity coefficients of a massless Boltzmann gas undergoing binary isotropic elastic collisions in a magnetic field have been derived analytically in Ref.[58] by using the method of 14-moment collisions in a magnetic field. We see that except for $\eta_1$, which is almost constant at large magnetic field, all the other shear viscosity coefficients are decreasing.

The two correlation functions behave similarly. The stronger the magnitude of the magnetic field, the earlier the maximum value is reached and the larger is the value.

Figure 2 shows the five anisotropic shear viscosity coefficients for various values of $\xi_B = \lambda_{mfp}/R_T$, where the mean free path $\lambda_{mfp} = 1/(n\sigma_{22})$ is fixed and $R_T = T/(eB)$ varies by varying $eB$. $R_T$ denotes the Larmor radius of a particle with the transverse momentum being equal to the temperature. The five shear viscosity coefficients are normalized by the standard isotropic shear viscosity $\eta = 4\lambda_{mfp}P/3$ obtained without the magnetic field. We see that except for $\eta_1$, which is almost constant at large magnetic field, all the other shear viscosity coefficients are decreasing.

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Grad’s approximation. We list the results below:

$$
\eta_0 = \frac{12\lambda_{mfp}P}{9 + 4\xi_B^2}, \quad \eta_1 = \frac{64\xi_B^2\lambda_{mfp}P}{9 + 4\xi_B^2},
$$

$$
\eta_2 = \frac{36\xi_B^2\lambda_{mfp}P}{9 + 4\xi_B^2}, \quad \eta_3 = \frac{4\xi_B^2\lambda_{mfp}P}{9 + 4\xi_B^2},
$$

$$
\eta_4 = \frac{4\xi_B^2\lambda_{mfp}P}{9 + 4\xi_B^2}.
$$

The analytical results are depicted by different curves in Fig. 2. We see excellent agreements with our numerical results. From the formulas Eq. (34) it is clear that $\eta_1/\eta_2$ goes to a constant of $4/3$ for large $\xi_B$ (large $eB$), while all the other shear viscosity coefficients go to zero.

Furthermore, by equating Eqs. (12) and (13) to $\eta_0$ and $\eta_2$ in Eq. (34) we find that

$$
\frac{V}{T} \int_0^\infty dt \langle T^{12}(r,t)T^{12}(0,0) \rangle = \frac{12\lambda_{mfp}P}{9 + 4\xi_B^2}, \quad (35)
$$

$$
\frac{V}{T} \int_0^\infty dt \langle T^{13}(r,t)T^{13}(0,0) \rangle = \frac{12\lambda_{mfp}P}{9 + 4\xi_B^2}. \quad (36)
$$

Therefore, the correlation function $\langle T^{12}(t)T^{12}(0)\rangle$ at $\xi_B$ (or $B$) is same as $\langle T^{13}(t)T^{13}(0)\rangle$ at $2\xi_B$ (or $2B$). This behavior can be observed in Fig. 2. A similar scaling behavior between $\langle (T^{11}(t) - T^{22}(t))T^{12}(0)\rangle$ and $\langle (T^{13}(t)T^{23}(0)\rangle$ seen in Fig. 3 can also be explained by equating Eqs. (15) and (16) to $\eta_3$ and $\eta_4$ in Eq. (34).

### B. Electric conductivity coefficients

Firstly, we calculate the longitudinal electric conductivity within BAMPS by applying the Kubo formulas Eq. (22a) and Eq. (22b), respectively. $\sigmaparallel$ is induced purely by an electric field, while $\kappaparallel$ is related to the diffusion (or heat transfer). The correlation functions that determine $\kappaparallel$ and $\sigmaparallel$ are $\langle G^3(t)G^3(0) \rangle$ and $\langle j^3(t)j^3(0) \rangle$. We note
The results are normalized by their initial values for comparisons.

FIG. 5. Time evolution of \( \langle (T^{11}(t) - T^{22}(t))T^{12}(0) \rangle /2 \) and \( \langle (T^{13}(t))T^{23}(0) \rangle \) with various magnetic field strengths. The results are normalized by \( \langle (T^{12}(0))^2 \rangle \). The results without the magnetic field are zero.

FIG. 4. The magnetic field dependence of the shear viscosity coefficients scaled by the standard isotropic shear viscosity \( \eta \). The analytical results from Eq. (54) are shown by the curves for comparisons.

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FIG. 5. Time evolution of \( \langle G^3(t)G^3(0) \rangle \) and \( \langle j^3(t)j^3(0) \rangle \). The results are normalized by their initial values.

that these two correlation functions (also \( \kappa_{||} \) and \( \sigma_{||} \)) are not influenced by the magnetic field, since the Lorentz force does not affect the dynamics in the direction of the magnetic field. In Fig. 5 we show the time evolution of the two correlation functions and see that \( \langle G^3(t)G^3(0) \rangle \) decreases to zero, while \( \langle j^3(t)j^3(0) \rangle \) approaches to a non-zero value. The latter indicates an infinite large electric conductivity \( \sigma_{||} \), which is true for a one-component system of charged particles, since there is no energy loss of charged particles in each collisions. We mention that the electric conductivity of a multi-component system such like the quark-gluon plasma has been calculated in 56 within BAMPS without a magnetic field.

The electric conductivity \( \kappa_{||} \), which is related to the diffusion (or heat transfer), is finite. In the top panel of Fig. 5 \( \kappa_{||} \) scaled by the temperature \( T \) are shown by the solid symbols. Charges are multiplied out in the results with \( e^2 = 4\pi/137 \). The electric conductivity calculated here is related with the diffusion coefficient by the Wiedemann-Franz law. In Ref. 67 the diffusion coefficients for a one-component system without a magnetic field are calculated by using the 14, 23, 32, and 41-moment Grad’s method. To make comparisons, the results obtained in 67 are multiplied by \( e^2/T \) according to the Wiedemann-Franz law and shown in the top panel of Fig. 5 by the dashed and dotted line corresponding to the 14 and 41-moment approximation. We see that our numerical results are about 17% smaller than those in the 14-moment approximation, while they agree nicely with those in the 41-moment approximation.

We now turn to calculate the transverse and Hall electric conductivity. The time evolution of the correlation functions corresponding to these conductivity coefficients [see Eqs. (24), (25), (27), and (28)] are shown in Fig. 6 and Fig. 7. The transverse conductivity coefficient \( \kappa_{\perp} \) and \( 1/\rho_{\perp} \) scaled by \( T \) are depicted in the middle panel of Fig. 5 by the solid and open symbols, respectively. The values of \( 1/\rho_{\perp} \) are divided by a factor of 100. At \( B = 0 \),
1/ρ⊥ is infinite [see Eq. (27)], while κ⊥ is finite. Both are equal to the longitudinal electric conductivity σ∥ and κ∥, respectively. From the middle panel of Fig. 8 we also see that both κ⊥ and 1/ρ⊥ become smaller for stronger magnetic field strength and 1/ρ⊥ is roughly 100 times larger than κ⊥.

In the bottom panel of Fig. 8 we show the Hall electric conductivity κ× and 1/ρ⊥ scaled by T. The latter is divided by a factor of 5 for comparisons. At B = 0, 1/ρ⊥ is infinite [see Eq. (28)], while κ× is zero. 1/ρ⊥ agrees with the classical result en/B when comparing with the dotted curve in the bottom panel of Fig. 8. (Remember that 1/ρ⊥ has been divided by a factor of 5.) With increasing B, κ× increases first and then decreases as en/(5B). Thus, 1/ρ⊥ is almost 5 times larger than κ×. We realize that the electric conductivity coefficients induced by an electric field are always larger than those related with the diffusion (or heat transfer).

The anisotropic diffusion coefficients of a one-component system in a magnetic field have been calculated in Ref. [58] by using the 14-moment Grad’s method. We multiply these results by e²/T to obtain the electric conductivity coefficients according to the Wiedemann-Franz law:

\[ \kappa_{\parallel} = \frac{3e^2\lambda_{mf}n}{16T}, \quad \kappa_{\perp} = \frac{48e^2\lambda_{mf}n}{(256 + 225\xi_B^2)T}, \]

\[ \kappa_{\times} = \frac{45e^2\xi_B\lambda_{mf}n}{(256 + 225\xi_B^2)T}. \]

(37)

κ∥ is exactly the same as that obtained from [67] and has been shown in the top panel of Fig. 8. κ⊥ and κ× from Eq. (37) are depicted by the dashed curves (scaled by T) in the middle and bottom panel of Fig. 8. We see agreements with the numerical results.

V. CONCLUSIONS

In this work, we have calculated the anisotropic transport coefficients of relativistic fluids in the presence of a
magnetic field according to the Kubo formulas given in Refs. [13, 10]. The time correlations of the components of the energy-momentum tensor and electric current, which are fluctuating in time at thermal equilibrium, are calculated numerically within the kinetic transport approach BAMPS. For comparisons with results from the early studies we have considered a massless one-component Boltzmann gas with isotropic binary collisions, although calculations within BAMPS can be performed for multi-component systems with more complicated scattering processes such like pQCD (in)elastic scatterings of gluons and quarks [55].

We have found that the magnetic field dependence of the five shear viscosity coefficients that we achieved agrees perfectly with the analytical results obtained by using the 14-moment Grad’s approximations [58]. For strong magnetic field $\eta$ approaches $4/3$-fold of $\eta$ (the standard shear viscosity without the magnetic field) and all the other shear viscosity coefficients decrease to zero.

We have also compared two kind of electric conductivity coefficients with each other. One electric conductivity coefficients are associated with the diffusion (or heat transfer), another coefficients are induced by an electric field and have no cross effect with the diffusion constant (or heat conductivity). We found that the three electric conductivity coefficients associated with the diffusion are always smaller than those induced by an electric field. In addition, the magnetic field dependence of the three electric conductivity coefficients associated with the diffusion agrees well with the results from the 14-moment Grad’s approximations [58]. A better agreement for the longitudinal electric conductivity is seen when comparing the result from the 41-moment approximation [67].

The agreements between the numerical and analytical results on the anisotropic transport coefficients for a one-component system of Boltzmann particles with isotropic scatterings confirm the general use of the derived Kubo formulas for multi-component particle systems with more complicated scattering processes. Calculations of the anisotropic transport coefficients for a multi-component system in a strong magnetic field such like the QGP produced in heavy-ion collisions are in progress. New results will be shown in a future publication.

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