HILBERT’S 16TH PROBLEM.
II. PFAFFIAN EQUATIONS AND VARIATIONAL METHODS

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Abstract. Starting from a Pfaffian equation in dimension $N$ and focusing on compact solutions for it, we place in perspective the variational method used in [29] to solve Hilbert’s 16th problem. In addition to exploring how this viewpoint can help in detecting and finding approximations for limit cycles of planar systems, we recall some of the initial important facts of the full program developed in [29] to motivate that same proposal could eventually be used in other situations. In particular, we make some initial interesting calculations in dimension $N = 3$ that lead to some similar initial conclusions as with the case $N = 2$.

Contents

1. Introduction 2
2. The case $N = 2$ 6
  2.1. Quadratic functional for periodic paths 7
  2.2. Optimality 9
  2.3. The descent procedure 11
  2.4. Some numerical tests 12
3. Hilbert’s 16th problem 15
4. The case $N = 3$ 16
  4.1. The optimality system 16
  4.2. The Pfaffian functional 20
Acknowledgements 21
References 22

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1. Introduction

Pfaffian equations are not among the typical material covered in recent textbooks in Differential Equations. Because of this reason, and to see its potential connection to Hilbert’s 16th problem, we recall here the basic concepts related to it. For deeper and more complete discussions, we refer to the classical textbooks [20], [27]. Pfaffian equations were introduced by J. F. Pfaff [31], and studied later by Carathéodory, Darboux [14] and some others. Euler made some contributions even before its formal introduction by Pfaff.

A Pfaffian differential equation is an expression of the form

\[ \omega = \sum_{i=1}^{N} u_i(x) \, dx_i = 0, \quad x = (x_1, x_2, \ldots, x_N), \]

for \( N \) functions \( u_i(x) \). In vector notation, we also write

\[ \omega = u(x) \cdot dx, \quad u = (u_1, u_2, \ldots, u_N), \quad dx = (dx_1, dx_2, \ldots, dx_N). \]

A \( C^1 \)-manifold \( M \) of dimension \( k \geq 1 \) is called an integral manifold of (1) if the differential 1-form \( \omega \) identically vanishes on \( M \). The Pfaffian equation (1) is said to be completely integrable if there is a unique integral manifold of the highest possible dimension \( N - 1 \) through each point \( x_0 \in \mathbb{R}^N \).

Typically, (1) is considered for \( N \geq 3 \), since the case \( N = 2 \) corresponds to a standard differential equation in the plane

\[ u_1(x_1, x_2) \, dx_1 + u_2(x_1, x_2) \, dx_2 = 0, \]

written usually as

\[ P(x, y) \, dx + Q(x, y) \, dy = 0. \]

The case \( N = 3 \) is, for this reason, studied explicitly most of the time as it amounts to exploring new situations other than standard Differential Equations. It reads in explicit form

\[ P(x, y, z) \, dx + Q(x, y, z) \, dy + R(x, y, z) \, dz = 0. \]

A 2-dimensional manifold embedded in \( \mathbb{R}^3 \) is an integral solution of this equation if the normal to the tangent plane to it in every point is given by the vector field \( (P, Q, R) \).

Compact manifold solutions of (1) play a special role. If \( N = 2 \), limit cycles of the differential system

\[ x' = -Q(x, y), \quad y' = P(x, y) \]

are among those compact solutions for (3). If we change the notation and put

\[ x' = P(x, y), \quad y' = Q(x, y) \]

put
for the underlying differential system, and

$$Q(x, y) \, dx - P(x, y) \, dy = 0$$

(6) instead of (3), then a periodic path

$$\sigma(t) = (x(t), y(t)) : J \to \mathbb{R}^2, \quad (x(0), y(0)) = (x(1), y(1)),$$

is a parameterization of a limit cycle of system (5) if (6) holds over $$\sigma$$, i.e. if the combination

$$Q(x, y)x' - P(x, y)y'$$

identically vanishes in $$J$$. There can be more complicated periodic structures that can be parametrized in $$J$$ for which (6) holds like, for example, policycles.

In a similar way, for the case $$N = 3$$ we can consider compact manifolds parameterized periodically over the unit square $$J^2 = [0, 1] \times [0, 1]$$ which are solutions for (4). Suppose we consider

$$x(s) = (x_1(s), x_2(s), x_3(s)) : J^2 \to \mathbb{R}^3, \quad s = (s_1, s_2),$$

$$x(s_1, 0) = x(s_1, 1), x(0, s_2) = x(1, s_2),$$

for all $$s = (s_1, s_2) \in Q = J^2$$. Such doubly-periodic map $$x(s)$$ will be a parameterization of a compact solution of (4) if

$$x_{s_1}(s) \wedge x_{s_2}(s) \wedge u(x(s)) \equiv 0, \quad u = (P, Q, R).$$

(7) Equivalently, taking advantage of the identity

$$|y \wedge z|^2 + (y \cdot z)^2 = |y|^2 |z|^2$$

valid for vectors $$y, z$$ in $$\mathbb{R}^3$$, (7) is equivalent to the condition

$$\left| \det \left( \frac{\nabla x(s)}{u(x(s))} \right) \right|^2 \equiv |u(x(s))|^2 |x_{s_1}(s) \wedge x_{s_2}(s)|^2,$$

or even further

$$\left| \det \left( \frac{\nabla x(s)}{u(x(s))} \right) \right|^2 \equiv |u(x(s))|^2 (|x_{s_1}(s)|^2 |x_{s_2}(s)|^2 - (x_{s_1}(s) \cdot x_{s_2}(s))^2).$$

These identities are interesting because they make the term with the determinant occur. However, there is a much simpler way of enforcing (7), namely

$$u(x(s)) \cdot x_{s_1}(s) = u(x(s)) \cdot x_{s_2}(s) = 0.$$

Our intention is to explore the functionals

$$E(x) = \frac{1}{2} \int_Q (u(x(s)) \cdot x_{s_1}(s))^2 + u(x(s)) \cdot x_{s_2}(s))^2 \, ds$$

(8) for the case $$N = 3$$, and

$$E(x, y) = \frac{1}{2} \int_J (P(x(t), y(t))y'(t) - Q(x(t), y(t))x'(t))^2 \, dt,$$

(9)
for $N = 2$, in addition to

\begin{equation}
\tilde{E}(x) = \frac{1}{2} \int_Q \left| \det \left( \nabla x(s) \right) \right|^2 \, ds
\end{equation}

defined over their respective classes of periodic maps. Notice that the versions of the two functionals

\begin{equation*}
E(x) = \frac{1}{2} \int_Q \left( u(x(s)) \cdot x(s) \right) \cdot x(s) \, ds
\end{equation*}

and

\begin{equation*}
\tilde{E}(x) = \frac{1}{2} \int_Q \left| \det \left( \nabla x(s) \right) \right|^2 \, ds
\end{equation*}

for dimension $N = 2$ become the same one replacing the field $(P, Q)$ by its normal $(-Q, P)$ as is usually done in elementary courses of Differential Equations when one seeks the normal family of solutions to a given one.

We will identify, for obvious reasons, functional $E$ in (8) as the Pfaffian functional associated with vector field $u$ or Pfaffian equation (2). Note that our previous observations mean that compact solutions of the corresponding Pfaffian equation are absolute minimizers of these functionals because $E \geq 0$ always in (8) or (9), whereas $E = 0$ on those solutions.

We plan to start examining and becoming familiar with this viewpoint as quite relevant consequences can be deduced. In particular, variational methods applied to the functional in (9) has led to the solution of Hilbert’s 16th problem [29]. More specifically, we will deal with the following issues.

1. As a nice way to gain some familiarity with this perspective and see the main issues associated with it, we will perform a basic analysis of $E$ in (9) and compute, in a quite explicit form, its derivative $E'$. Information about critical paths can be gained by studying optimality conditions coming from $E' = 0$. In the area of variational problems, such optimality conditions are usually referred to as the associated Euler-Lagrange equations or system. Furthermore, we will show how to use the calculation of $E'$ to setup an approximation procedure to detect limit cycles of planar differential systems other than by following integral curves until they wind up around a limit cycle. This will also serve as a way to assess the advantages of using variational techniques in the study of planar differential systems. This task requires to clearly describe the various ingredients of the underlying variational problem, derive optimality conditions and check how periodicity constraints are reflected in those; show the already-mentioned closed formula for the derivative of the functional, and use a standard descent procedure to perform simulations. We will illustrate the performance of such a scheme with some classical and well-known examples.
(2) The previous analysis can be a very good training ground to better appreciate the material in [29] for a complete solution of Hilbert’s 16th problem. The basic, driving idea is to count the number of absolute minimizers of $E$, among which limit cycles can be counted, through those critical paths that are not global minima. This is accomplished by means of Morse inequalities. This rough idea requires a lot of work but it is worthwhile, in our opinion, to understand the main steps of such an endeavor stressing the principal difficulties to be overcome, and the key parts of the strategy, so as to envision how one could use the same ideas for other situations, and its potential extension to more general frameworks (see [19]). We limit ourselves here to stress some simple informal calculations that originally ignited the whole strategy to solve Hilbert’s 16th problem. Our intention is to examine the extension of these initial facts to the three-dimensional scenario.

(3) The case $N = 3$ is, comparatively, much more complex. At this stage, our goal is quite modest though we plan to examine this and the general case $N \geq 3$ in the future. We will write down the Euler-Lagrange equations for the functional in (8) as well as the one for (10), to understand the role played by the divergence of the field $u = (P, Q, R)$. We do not pay attention, however, to additional conditions coming from periodicity. The functional in (10) is interesting too. Its absolute minimizers when

$$\det \begin{pmatrix} \nabla x(s) \\ u(x(s)) \end{pmatrix} \equiv 0$$

for a parameterized embedded surface $x(Q)$, correspond to $u$-invariant, compact surfaces, i.e. the restriction of $u$ to $x(Q)$ is tangent to it. In case vector field $u$ represents an ideal steady incompressible fluid flow, such surface would be an invariant torus of the flow [3], [15]. The same ideas for the vorticity field $\nabla \wedge u$ would lead to vortex tubes [16].

Some results in the theory of Pfaffian equations go back to Euler and Frobenius. In particular, it was well-known since the time of Euler that the condition for a Pfaffian equation to be completely integrable in the case $N = 3$, i.e. (4), is that

$$u \cdot \nabla \wedge u = 0, \quad u = (P, Q, R), \quad \text{curl} u = \nabla \wedge u.$$  

We will come back to this condition later. More material about Pfaffian equations and systems can be found in [10], [18], [21], [24]. [30] is an interesting article where the relationship between Pfaffian equations are variational problems is explored. In [23], one can check about the history of these Pfaffian equations.

Pfaffian equations and systems have been thoroughly studied in connection with symplectic geometry. In this regard, there are more related references [5], [28], [33]. Darboux’s theorem about symplectic forms plays an important role here. See [13] for an interesting recent paper on this.
A variational viewpoint about ODEs has been investigated systematically in [1], [2].

2. The case $N = 2$

We would like to consider the functional

$$E(x, y) = \frac{1}{2} \int_0^1 (P(x(t), y(t))y'(t) - Q(x(t), y(t))x'(t))^2 \, dt,$$

as a means to say something about limit cycles of the planar differential system

$$x'(t) = P(x(t), y(t)), \quad y'(t) = Q(x(t), y(t)).$$

Evidently, functional (11) is non-negative, and vanishes (among other possibilities) at limit cycles of system (12) once they are reparameterized in the unit interval $J$. Limit cycles are thus absolute minimizers of functional $E$ in (11), and hence the derivative of $E$ should vanish on those. We would like to calculate, in as explicit terms as possible, such derivative and check what kind of additional information we can gather by demanding its vanishing at a given feasible periodic path. To do so, it is important to clearly specify the functional-analytical ambient space. In vector notation, we will write

$$x = (x, y), \quad F = (P, Q), \quad F^\perp = (-Q, P),$$

and our underlying space will be

$$H^1_O(J; \mathbb{R}^2) = \{ x \in H^1(J; \mathbb{R}^2) : x(0) = x(1) \}.$$

$H^1(J; \mathbb{R}^2)$ is the standard Sobolev space of paths with image in $\mathbb{R}^2$ and weak derivatives that are square-integrable [9].

The vector space where this optimization problem is setup is, then, the Hilbert space $H^1_O(J; \mathbb{R}^2)$. Paths $x(t)$ in this space are absolutely continuous, $x(0) = x(1)$, and have a weak derivative in $J$ which is square integrable. As a subspace of $H^1(J; \mathbb{R}^2)$, we can take as a norm, under which $H^1_O(J; \mathbb{R}^2)$ is complete, the usual norm in $H^1(J; \mathbb{R}^2)$ coming from the standard inner product, i.e.

$$\|x\|^2 = \int_J (|x'(s)|^2 + |x(s)|^2) \, ds.$$

However, this norm makes certain things more complicated than necessary. In particular, as we will shortly see, it is most convenient to use an equivalent norm which might allow to write quite explicitly the derivative $E'(x)$ of our functional $E$ at an arbitrary path $x \in H^1_O(J; \mathbb{R}^2)$. We therefore take the equivalent norm (using the same notation)

$$\|x\|^2 = \int_J |x'(s)|^2 \, ds + |x(0)|^2.$$
This is indeed the norm associated with the inner product
\[ \langle x, y \rangle = \int_J x'(s) \cdot y'(s) \, ds + x(0) \cdot y(0), \]
which makes of \( H^1_O(J; \mathbb{R}^2) \) a Hilbert space with the same topology, as the two norms are equivalent. At any rate, it is important to include a contribution to the norm coming from the path \( x \) itself and not only its derivative, because the quantity
\[ \|x\|^2 = \int_J |x'(s)|^2 \, ds \]
is just a seminorm in \( H^1_O(J; \mathbb{R}^2) \). This is standard.

An important remark needs to be made at this point. Given that we are interested in limit cycles themselves and not so much in the way they may be parameterized, it might be interesting to avoid the ambiguity coming from the number of times that piecewise-smooth, plane curves can be run either counter- or clockwise. In this regard, we recall that the rotation index of a smooth closed plane curve is the number of complete rotations that a tangent vector to the curve makes as it goes around the curve. According to a classical theorem of H. Hopf [26], the rotation index of a piecewise smooth, closed plane curve with no self-intersections is +1 or -1, depending on whether the curve is oriented counterclockwise or clockwise. This observation will be fundamental in [29] to be capable of counting, finitely, critical paths for a suitable perturbation of \( E \). However, at this stage it is not necessary to bear this in mind in the calculations that follow.

Both as a preliminary step, and as a good way to gain some familiarity with our functional and our space, we cope first with quadratic functionals.

2.1. Quadratic functional for periodic paths. We would like to deal first with problems of the kind
\[ \text{Minimize in } x \in H^1_O(J; \mathbb{R}^2) : \quad I(x), \]
\[ I(x) = \int_J \left[ \frac{1}{2} |x'(s)|^2 + f(s) \cdot x'(s) + g(s) \cdot x(s) \right] ds + \frac{1}{2} |x(0)|^2. \]
Bear in mind and recall that:

1. The space \( H^1_O(J; \mathbb{R}^2) \) contains periodic paths with a weak derivative which is square-integrable.
2. \( J \) is the unit interval \( J = [0, 1] \).
3. \( f(t) \) is a continuous periodic path.
4. \( g(t) \) is a path whose square is integrable, in short, it belongs to \( L^2(J; \mathbb{R}^2) \).

The presence of the final term with the contribution of the square of the value at 0 (or any other point) has already been explained earlier.
Proposition 2.1. The unique minimizer \( x(t) \in H^1_0(J; \mathbb{R}^2) \) of the above quadratic optimization problem is given explicitly by the formula

\[
x(t) = - \int_0^1 g(s) \, ds + (t - 1) \int_0^1 [f(s) - (1 - s)g(s)] \, ds \\
+ (t - 1) \int_0^t g(s) \, ds + \int_t^1 [f(s) - (1 - s)g(s)] \, ds.
\]

Proof. There is indeed a unique minimizer for our problem. On the one hand, the functional \( I(x) \) is convex in \( x \) because it involves a quadratic, strictly convex contribution plus a linear part; on the other, it is coercive. To show this, note that for arbitrary positive \( \tau \),

\[
\begin{align*}
| x(s) \cdot g(s) | & \leq \tau^2 | x(s) |^2 + \frac{1}{4\tau^2} | g(s) |^2, \\
| x'(s) \cdot f(s) | & \leq \tau^2 | x'(s) |^2 + \frac{1}{4\tau^2} | f(s) |^2.
\end{align*}
\]

But also, by exploiting the identity

\[
x(s) = x(0) + \int_0^t x'(s) \, ds,
\]

we arrive at

\[
\| x \|_{L^2(J; \mathbb{R}^2)}^2 \leq 2 \| x(0) \|^2 + 2 \| x' \|_{L^2(J; \mathbb{R}^2)}^2.
\]

Taking into account all of these inequalities

\[
I(x) \geq (\frac{1}{2} - 3\tau^2) \| x' \|_{L^2(J; \mathbb{R}^2)}^2 + (\frac{1}{2} - 2\tau^2) | x(0) |^2 \\
- \frac{1}{4\tau^2} (\| f \|_{L^2(J; \mathbb{R}^2)}^2 + \| g \|_{L^2(J; \mathbb{R}^2)}^2).
\]

By choosing \( \tau \) sufficiently small, given that the term with a minus sign is finite, we see that \( I \) is coercive. Coercivity plus strict convexity implies the existence of a unique minimizer, this is a well-known standard fact (see for instance, among many other possibilities, [1]).

To derive optimality conditions that such unique critical path should comply with, we perform a perturbation of the kind \( x + rX \) for an arbitrary path \( X \) in our ambient space, and compute the derivative

\[
\langle I'(x), X \rangle \equiv \frac{d}{dr} I(x + rX) \bigg|_{r=0}.
\]

It is elementary to calculate this derivative (differentiation under the integral sign is easy to justify) to find

\[
\langle I'(x), X \rangle = \int_J [x'(s) \cdot X'(s) + f(s) \cdot X'(s) + g(s) \cdot X(s)] \, ds + x(0) \cdot X(0).
\]
If we perform an integration by parts in the first two terms, we have
\[
\langle I'(x), X \rangle = \int_0^1 \left[ -(x'(s) + f(s)') + g(s) \right] \cdot X(s) \, ds \\
+ (x'(1) + f(1)) \cdot X(1) - (x'(0) + f(0) - x(0)) \cdot X(0).
\]
In our space \(X(1) = X(0) = a\) is a vector that can be chosen freely. Hence, equating this directional derivative to zero we arrive at
\[
0 = \int_0^1 \left[ -(x'(s) + f(s)') + g(s) \right] \cdot X(s) \, ds + (x'(1) + f(1)) - (x'(0) - f(0) + x(0)) \cdot a.
\]
Since vector \(a\) and path \(X(s)\) for \(s \in [0, 1]\) can be chosen independently and freely of each other, we conclude that
\[
(14) - (x'(t) + f(t))' + g(t) = 0 \quad \text{in} \quad J, \quad x'(1) + f(1) - x'(0) - f(0) + x(0) = 0.
\]
Integrating the differential system in \(J\), and bearing in mind the other condition, we deduce that
\[
x(0) + \int_0^1 g(s) \, ds = 0.
\]
Once we have this information, it is elementary to check that the path given in the statement is the only one that complies with the differential system (14). It certainly belongs to \(H^1_0(J; \mathbb{R}^2)\). \(\square\)

2.2. Optimality. Proposition 2.1 is the basis for the computation of the derivative \(E'(x)\) of our functional in (13) at an arbitrary path \(x\).

**Proposition 2.2.** Let \(x \in H^1_0(J; \mathbb{R}^2)\). Then \(X = -E'(x)\) is given by
\[
X(t) = - \int_0^1 g(s) \, ds + (t - 1) \int_0^1 [f(s) - (1 - s)g(s)] \, ds \\
+ (t - 1) \int_0^t g(s) \, ds + \int_t^1 [f(s) - (1 - s)g(s)] \, ds,
\]
where
\[
f(t) = [F^\perp(x(t)) \cdot x'(t)]^\perp F^\perp(x(t)),
\]
\[
g(t) = [F^\perp(x(t)) \cdot x'(t)]^\perp x'(t) \nabla F^\perp(x(t))^T.
\]

**Proof.** Take \(x \in H^1_0(J; \mathbb{R}^2)\), and let \(X \in H^1_0(J; \mathbb{R}^2)\) be an arbitrary perturbation of it. We would like to compute the Gateaux (directional) derivative
\[
\langle E'(x), X \rangle = \lim_{r \to 0} \frac{1}{r} (E(x + rX) - E(x)) = g'(0)
\]
if
\[
g(r) = E(x + rX).
\]
It is straightforward to write
\[
g(r) = \frac{1}{2} \int_J [F^\perp(x(t) + rX(t)) \cdot (x'(t) + rX'(t))]^2 \, dt,
\]
and then the corresponding directional derivative is given by

$$\langle E'(x), X \rangle = g'(0)$$

$$= \int \left[ F^\perp(x(t)) \cdot x'(t) \right]\left[ \nabla F^\perp(x(t)) \cdot x'(t) + F^\perp(x(t)) \cdot x'(t) \right] dt.$$  

That it is legitimate to differentiate under the integral sign is straightforward to justify, under appropriate smoothness hypotheses. Since in every abstract Hilbert space \(H\), we always have that the unique minimizer of the quadratic optimization problem

$${\text{Minimize in }} X \in H : \frac{1}{2} \| X \|^2 + \langle Y, X \rangle$$

is precisely \(X = -Y\), we conclude in our specific situation for \(Y = E'(x)\), that \(X = -E'(x)\) is the unique minimizer of the problem

$${\text{Minimize in }} X \in H^1_0(J; \mathbb{R}^2) : \frac{1}{2} \| X \|^2 + \langle E'(x), X \rangle$$

where

$$\| X \|^2 = \int_j |X'(t)|^2 dt + |X(0)|^2$$

and \(\langle E'(x), X \rangle\) is given explicitly above. But Proposition 2.1 yields in a very precise way such minimizer under the identification

\[ f(t) = \left[ F^\perp(x(t)) \cdot x'(t) \right] F^\perp(x(t)), \]

\[ g(t) = \left[ F^\perp(x(t)) \cdot x'(t) \right] x'(t) \nabla F^\perp(x(t))^T. \]

As indicated earlier, optimality information comes from the condition \(E'(x) = 0\). Since we now have a quite explicit form of \(E'(x)\), we are entitled to say that a feasible path \(x\) is critical for \(E\) if

\[ 0 = -\int_0^1 g(s) ds + (t - 1) \int_0^1 [f(s) - (1 - s)g(s)] ds \]

\[ + (t - 1) \int_0^t g(s) ds + \int_0^1 [f(s) - (1 - s)g(s)] ds, \]

for fields \(g\) and \(f\) given above in terms of \(F\). If we differentiate twice with respect to \(t\), it is an elementary exercise to find that for a critical path we must have, in addition to some further information involving values at \(t = 0\) and \(t = 1\), that

\[ (15) \quad g(t) - f'(t) = 0 \text{ in } J, \]

i.e.

\[ [(F^\perp(x(t)) \cdot x'(t))F^\perp(x(t))]' - [F^\perp(x(t)) \cdot x'(t)]x'(t)\nabla F^\perp(x(t))^T = 0. \]
This is a vector equation that critical paths \( x \) ought to verify. If we write this system component-wise by setting
\[
x = (x,y), \quad \mathbf{F}^\perp = (-Q,P),
\]
some elementary algebra leads to
\[
(P(x,y)y' - Q(x,y)x')^2 = k^2, \quad k^2(P_x(x,y) + Q_y(x,y)) = 0,
\]
for a positive constant \( k^2 \). These computations are carefully performed in [29]. We can infer two fundamental pieces of information from here:

1. Absolute minimizers of \( E \) correspond to \( k = 0 \);
2. Critical paths, other than absolute minimizers, can only occur when \( k > 0 \) but then, their image is contained in the curve of equation
\[
P_x(x,y) + Q_y(x,y) = 0.
\]

A different matter is to show the existence of such critical paths, as optimality only yields information that such paths should comply with. The existence of such critical paths and how many of them there could be is part of the program for Hilbert’s 16th problem [29].

**Remark 2.3.** Typically, optimality conditions in the form of Euler-Lagrange equations are derived directly without the need of the formula in Proposition 2.2. Equation (15) can be deduced directly from (14) by setting \( x \equiv 0 \). Path \( x \) in (14) plays the role of the derivative \( E' \).

### 2.3. The descent procedure.

The explicit formula for \( E'(x) \) provided in Proposition 2.2 can serve to some other interesting purpose. The idea is to use it in the context of a typical descent algorithm to find (approximate) minimizers of \( E \) among periodic paths. One needs to be well aware, though, that functional \( E \) in (11) is far from being convex, and so only local minima will be approximated. In addition, such local minima may depend on the initial guess utilized in computations. Said differently, the landscape of \( E \) in terms of valleys, hills, etc, might be quite intricate. Depending on particular circumstances in examples, it may very hard to detect some limit cycles, as reliable information is necessary to start the numerical scheme itself in a good initial path.

Such descent procedures are standard in approximation theory (check [7] for a full treatment in a finite-dimensional framework). For the convenience of the reader we simply transcribe here the basic scheme. The standard strategy is easy to understand: if we start out at some path \( x_0 \in H^1_O(J;\mathbb{R}^2) \), called initialization, we would like to move to another one \( x_1 \in H^1_O(J;\mathbb{R}^2) \) in such a way that \( E(x_1) < E(x_0) \), and proceed successively in this fashion until no more decreasing of \( E \) is possible at the level of accuracy that is being used. The passage from \( x_0 \) to \( x_1 \) is designed in two steps:
(1) standing at $x_0$, choose an appropriate direction $v \in H^1_{\partial}(J; \mathbb{R}^2)$ in such a way that the value of $E$ will decrease as we move along it: we will take $v = -E'(x_0)$ as given in Proposition 2.2, though there are more efficient ways to select $v$;
(2) once $v$ has being chosen, decide on how far to go from $x_0$ in that direction $v$: this is the decision about the step size, and, again, there are various alternatives to do so.

2.4. Some numerical tests. For the sake of illustration, we will approximate, through our variational procedure, some limit cycles of very well-known planar systems. We do not claim that our numerical experiments have been setup in the most efficient way. It might be interesting to pursue further this viewpoint in order to look for limit cycles for quadratic systems, for instance, in a more systematic way to see if more than four limit cycles could possibly be detected. Four is the maximum found so far, see for instance [11] and [32]. Note that our approach is quite different from the typical one of following integral curves until they wind up around a limit cycle.

Our initial example is the classical van der Pol oscillator

$$x''(t) + x(t) + (x(t)^2 - 1)x'(t) = 0.$$  

The equivalent first-order differential system is

(17) $$x' = y, \quad y' = -x - (x^2 - 1)y,$$

and the associated functional

$$E(x, y) = \frac{1}{2} \int_0^2 (yy' + xx' + x'y(x^2 - 1))^2 dt.$$  

The use of our descent method for this functional starting from the circle of radius 2.5 and centered at the origin, produces the approximation shown in Figure 1.

The second example comes from a simplified model of glycolysis (check [34] for a more in-depth analysis)

(18) $$x'(t) = -x(t) + ay(t) + x(t)^2y(t), \quad y'(t) = b - ay(t) - x(t)^2y(t), \quad a, b > 0.$$  

For a certain range of the parameters $a, b$, the system has exactly one limit cycle. Figure 2 depicts our approximation for some specific values of the parameters in that range, applied to the functional

$$E(x, y) = \frac{1}{2} \int_0^2 (x'(b - ay - x^2y) + y'(x - ay - x^2y))^2 dt.$$
The next example is a quadratic case taken from [32]. The specific system is

\begin{align}
    x' &= y + y^2, \\
    y' &= -x + 0.2y - xy + 1.2y^2.
\end{align}

\[\text{Figure 1. Initial guess (circle), and limit cycle for the van der Pol oscillator (17).}\]

\[\text{Figure 2. Unique limit cycle for the model of glycolysis (18).}\]
This quadratic system has a unique limit cycle which is approximated through a descent method for the functional

\[ E(x, y) = \frac{1}{2} \int_0^1 (y'(y + y^2) + x'(x - 0.2y + xy - 1.2y^2))^2 \, dt. \]

The limit cycle is the one in Figure 3.

Finally, a slight modification of the previous example

\[ (20) \quad x' = y + y^2, \quad y' = -0.5x + 0.2y - xy + y^2, \]

produces two limit cycles instead of one, see [32]. Check those in Figure 4.

**Figure 3.** Unique limit cycle for the quadratic system (19).

**Figure 4.** Two limit cycles for the quadratic system (20).
3. Hilbert’s 16th problem

Full details for the resolution of Hilbert’s 16th problem through variational methods can be found in \[29\]. We review here, in an informal way, some of the seminal facts that led to the full solution of this problem to see (in the next section) to what extent these same ideas could be extended to more general cases.

The functional in (11)

\[ E(x, y) = \frac{1}{2} \int_0^1 (P(x(t), y(t))y'(t) - Q(x(t), y(t))x'(t))^2 \, dt, \]

is the starting point of everything, as one can try to count absolute minimizers of that functional, among which limit cycles of (12) can be counted, through critical paths other than global minimizers. This can be accomplished by means of Morse inequalities.

If we write down the associated Euler-Lagrange system as in Section 2 and manipulate it in an elementary way, it is not difficult to find that critical pairs \((x, y)\) that need to be considered for an upper bound of the number of limit cycles, ought to verify (16)

\[ (P(x, y)y' - Q(x, y)x')^2 = k^2, \quad P_x(x, y) + Q_y(x, y) = 0, \]

for a non-vanishing constant \(k\). The case \(k = 0\) corresponds to absolute minimizers. In particular, their image must be contained in the algebraic curve

\[ \text{Div} \equiv P_x + Q_y = 0, \]

and, in addition, the integrand of \(E\) in (21) must be constant and equal to \(k^2\). Even further, differentiating

\[ P_x(x, y) + Q_y(x, y) = 0 \]

with respect to \(t\), we also deduce that

\[ -Q(x, y)x' + P(x, y)y' = \pm k, \]

\[ (P_{xx}(x, y) + Q_{yx}(x, y))x' + (P_{xy}(x, y) + Q_{yy}(x, y))y' = 0. \]

These are two (associated with \(\pm\) signs) implicit ODE systems with singular points corresponding to places where the matrix multiplying the vector derivative \((x', y')\) vanishes, i.e. to the solutions of the non-linear system

\[ (P_{xx}(x, y) + Q_{yx}(x, y))P(x, y) + (P_{xy}(x, y) + Q_{yy}(x, y))Q(x, y) = 0, \]

\[ P_x(x, y) + Q_y(x, y) = 0, \]

which is the system for contact points of (12).

In this way, it is shown in \[29\] that

\[ H(n) \leq n(M + N) \]
where $H(n)$ is the maximum number of limit cycles that a differential system of degree $n$ might have, $M$ is the number of components of the curve $\text{Div} = 0$ while $N$ is the number of contact points of the system. The sum $M + N$ is the maximum number of possible asymptotic behaviors of branches of critical paths of a suitable perturbation $E_\varepsilon$ of $E$, to which Morse inequalities can be applied, while $n$ is the maximum number of branches tending to a given asymptotic behavior of the $M + N$ possible ones. By means of the classical theorems of Harnack [17], about the maximum number of components of an algebraic planar curve, and Bezout [22], about the maximum number of roots of a polynomial systems of equations, to write $M$ and $N$, respectively, in terms of $n$, one finds the bound
\[
H(n) \leq \frac{5}{2}n^3 - \frac{13}{2}n^2 + 6n \quad \text{if } n \text{ is even, and}
\]
\[
H(n) \leq \frac{5}{2}n^3 - \frac{13}{2}n^2 + 5n \quad \text{if } n \text{ is odd.}
\]
which yields the solution of Hilbert’s 16th problem (see [29]). We would like to argue how some of these initial facts are identical in dimension $N = 3$ appropriately interpreted.

4. The case $N = 3$

Let $\mathbf{u}(\mathbf{x}) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a smooth vector field, and $\mathbf{x}(s) : Q \equiv [0,1]^2 \rightarrow \mathbb{R}^3$, a doubly-periodic, smooth mapping. We would like to focus first on the functional
\[
\tilde{E}(\mathbf{x}) = \frac{1}{2} \int_Q \left| \det \left( \begin{pmatrix} \nabla \mathbf{x}(s) \\ \mathbf{u}(\mathbf{x}(s)) \end{pmatrix} \right) \right|^2 \, ds.
\]
Here
\[
\mathbf{A}(\mathbf{x}, \mathbf{u}) \equiv \left( \begin{pmatrix} \nabla \mathbf{x}(s) \\ \mathbf{u}(\mathbf{x}(s)) \end{pmatrix} \right)
\]
is the $3 \times 3$-matrix whose first two rows are $\mathbf{x}_{s_i}$, $i = 1, 2$, $\mathbf{s} = (s_1, s_2)$, and the third one is the given field $\mathbf{u}$ itself.

Most definitely $\tilde{E}$ is non-negative, and smooth. Moreover, it vanishes when the three rows of $\mathbf{A}(\mathbf{x}, \mathbf{u})$ are dependent for every $\mathbf{s}$. If we assume that $\mathbf{x}$ is the parameterization of a non-singular embedded surface in $\mathbb{R}^3$, so that the two tangent vectors $\mathbf{x}_{s_1}$ and $\mathbf{x}_{s_2}$ are non-vanishing and independent, then $\tilde{E}(\mathbf{x})$ vanishes exactly on such parameterized surfaces that are invariant under $\mathbf{u}$ because $\mathbf{u}$ is always tangent to $\mathbf{x}(Q)$.

4.1. The optimality system. Under sufficient smooth assumptions that may enable calculations without difficulty, it is a matter of careful algebra to find the explicit form of the Euler-Lagrange system corresponding to the functional $\tilde{E}(\mathbf{x})$. First, we provide the Euler-Lagrange system for a general functional in which competing maps depend on two variables. This is again a standard and well-known result [12].
We consider a functional of the form
\[ I(x) = \int_{\Omega} F(x(s), \nabla x(s)) \, ds \]
where
\[ s = (s_1, s_2) \in \Omega \subset \mathbb{R}^2, \quad x(s) : \Omega \to \mathbb{R}^3, \]
\[ \nabla x(s) = (x_{s_1}, x_{s_2}) \in \mathbb{R}^{2 \times 3}, \quad F(y, z) : \mathbb{R}^3 \times \mathbb{R}^{2 \times 3} \to \mathbb{R}. \]
This integrand \( F \) is assumed to be smooth and enjoy every property necessary for the computations that follow to be valid. Moreover the set of competing maps \( A \) is a certain subset of an appropriate Hilbert space with the property that if \( x \in A \) and \( X \) is such that \( X|_{\partial \Omega} = 0 \) then \( x + rX \in A \) for every value of \( r \in \mathbb{R} \). We focus on the optimization problem
\[ \text{Minimize in } x \in A : \quad I(x). \]

**Proposition 4.1.** If \( x \in A \) is critical for \( I \), then
\[ -[F_{z_1}(x, x_{s_1}, x_{s_2})]_{s_1} - [F_{z_2}(x, x_{s_1}, x_{s_2})]_{s_2} + F_y(x, x_{s_1}, x_{s_2}) = 0 \quad \text{in } \Omega. \]

**Proof.** Let \( x \) be a critical map for \( I \) over \( A \). According to our hypotheses, the following should be true
\[ \langle I'(x), X \rangle \equiv \frac{d}{dr} \bigg|_{r=0} I(x + rX) = 0 \]
for every \( X \). Under suitable smooth assumptions on \( F \), that we tacitly assume, it is easy to obtain
\[ \langle I'(x), X \rangle = \int_{\Omega} [F_y(x, x_{s_1}, x_{s_2}) \cdot X + F_{z_1}(x, x_{s_1}, x_{s_2}) \cdot x_{s_1} + F_{z_2}(x, x_{s_1}, x_{s_2}) \cdot x_{s_2}] \, ds, \]
for every \( X \) with vanishing boundary conditions around \( \partial \Omega \). An integration by parts in the two last terms yields (boundary terms drop out precisely because \( X \) vanishes on \( \partial \Omega) \)
\[ \langle I'(x), X \rangle = \int_{\Omega} [-[F_{z_1}(x, x_{s_1}, x_{s_2})]_{s_1} - [F_{z_2}(x, x_{s_1}, x_{s_2})]_{s_2} + F_y(x, x_{s_1}, x_{s_2})] \cdot X \, ds. \]
The arbitrariness of \( X \), under the sole condition that it vanishes on \( \partial \Omega \), in the condition
\[ \langle I'(x), X \rangle = 0 \]
implies the conclusion in the statement. \( \square \)
The way in which system (22) should be satisfied would require more comments. Yet our aim here is just to write down the form of the corresponding Euler-Lagrange system regardless of whether it should be understood in a weak or strong sense, in order to apply it to our two functionals $E$ and $\tilde{E}$ for the case $N = 3$. Indeed, the application of the previous result to our functional $\tilde{E}$ yields the following calculation.

**Proposition 4.2.** If we put

$$\text{Det}(s) \equiv \det \left( \begin{pmatrix} \nabla x(s) \\ u(x(s)) \end{pmatrix} \right)$$

then the Euler-Lagrange system associated with functional $\tilde{E}(x)$ is

$$-\text{Det}(s)_{s_1} (x_{s_2} (s) \wedge u(x(s)))$$

$$+ \text{Det}(s)_{s_2} (x_{s_1} (s) \wedge u(x(s)))$$

$$- \text{Det}(s) \text{div} u(x(s))(x_{s_1}(s) \wedge x_{s_2}(s)) = 0.$$

**Proof.** Because we can put

$$\text{Det}(s) \equiv \det \left( \begin{pmatrix} \nabla x(s) \\ u(x(s)) \end{pmatrix} \right) = x_{s_1} (s) \cdot x_{s_2} (s) \wedge u(x(s))$$

$$= -x_{s_2} (s) \cdot x_{s_1} (s) \wedge u(x(s))$$

$$= u(x(s)) \cdot x_{s_1} (s) \wedge x_{s_2} (s),$$

it is easy to write down the Euler-Lagrange system for our functional $E(x)$, namely,

$$-\text{Det}(s)_{s_1} (x_{s_2} (s) \wedge u(x(s))) + \text{Det}(s)_{s_2} (x_{s_1} (s) \wedge u(x(s))) + \text{Det}(s) (x_{s_1} (s) \wedge x_{s_2} (s)) \nabla u(x(s)) = 0.$$

This system comes directly from the application of Proposition 4.1 to the integrand

$$F(y, z) = \frac{1}{2} \left| \det \left( \begin{pmatrix} z \\ u(y) \end{pmatrix} \right) \right|^2.$$

Let just expand the first two terms in (23) with a bit of care. Because they are a product of three factors, we find for the first term

$$\text{Det}(s)_{s_1} (x_{s_2} (s) \wedge u(x(s)))$$

that it is equal to

$$\text{Det}(s)_{s_1} (x_{s_2} (s) \wedge u(x(s))) + \text{Det}(s) (x_{s_2, s_1} (s) \wedge u(x(s)))$$

$$+ \text{Det}(s) (x_{s_2} (s) \wedge [u(x(s))]_{s_1}).$$

Expanding $[u(x(s))]_{s_1}$ leads to

$$\text{Det}(s)_{s_1} (x_{s_2} (s) \wedge u(x(s))) + \text{Det}(s) (x_{s_2, s_1} (s) \wedge u(x(s)))$$

$$+ \text{Det}(s) (x_{s_2} (s) \wedge \nabla u(x(s)) x_{s_1} (s))$$
Putting the three terms in (23) together, the optimality system becomes
\[
- \text{Det}(s)(x_{s_1}(s) \wedge u(x(s))) + \text{Det}(s)(x_{s_2}(s) \wedge u(x(s)))
+ \text{Det}(s)(x_{s_1}(s) \wedge \nabla u(x(s))x_{s_1}(s))
+ \text{Det}(s)(x_{s_1}(s) \wedge \nabla u(x(s))x_{s_2}(s))
+ \text{Det}(s)(x_{s_1}(s) \wedge x_{s_2}(s))\nabla u(x(s)) = 0.
\]

The two terms involving second derivatives of \( x \) drop out, and then we are left with
\[
- \text{Det}(s)(x_{s_1}(s) \wedge u(x(s))) + \text{Det}(s)(x_{s_2}(s) \wedge u(x(s)))
+ \text{Det}(s)[-(x_{s_2}(s) \wedge \nabla u(x(s))x_{s_1}(s))]
+ (x_{s_1}(s) \wedge \nabla u(x(s))x_{s_2}(s))
+ (x_{s_1}(s) \wedge x_{s_2}(s))\nabla u(x(s)) = 0.
\]

A careful computation shows that the sum
\[
-(x_{s_2}(s) \wedge \nabla u(x(s))x_{s_1}(s)) + (x_{s_1}(s) \wedge \nabla u(x(s))x_{s_2}(s))
+ (x_{s_1}(s) \wedge x_{s_2}(s))\nabla u(x(s))
\]
is precisely
\[
- \text{div} u(x(s))(x_{s_1}(s) \wedge x_{s_2}(s)),
\]
and thus system (23) reads
\[
- \text{Det}(s)(x_{s_1}(s) \wedge u(x(s)))
+ \text{Det}(s)(x_{s_2}(s) \wedge u(x(s)))
- \text{Det}(s)\text{div} u(x(s))(x_{s_1}(s) \wedge x_{s_2}(s)) = 0.
\]

It is then straightforward to show the following.

**Proposition 4.3.** Let \( c \geq 0 \) be a critical value of \( \tilde{E} \), and \( x \) a parameterization of a smooth, non-singular critical map for \( \tilde{E} \) such that \( \tilde{E}(x) = c \). Then
\[
\frac{1}{2} \text{Det}(s)^2 \equiv c,
\]
and if \( c > 0 \),
\[
x(Q) \subset \{ z \in \mathbb{R}^3 : \text{div} u(z) = 0 \}.
\]

**Proof.** If we multiply the optimality system of the preceding proposition by \( x_{s_1} \) and \( x_{s_2} \), we find immediately that
\[
\text{Det}(s)(s_1) \equiv 0, \quad \text{Det}(s)(s_2) \equiv 0.
\]

Hence the function
\[
\frac{1}{2} \text{Det}(s)^2
\]
which is exactly the integrand of \( \tilde{E} \), must be identically the critical value \( c \). If we go back to the optimality system, multiply through by \( \text{Det}(s) \) and use the information we already have, we discover that

\[
c \text{div } u(x(s)) = 0
\]

since the normal vector \( x_s \) never vanishes if \( x(Q) \) is a smooth manifold. This is the second conclusion of the statement. \( \square \)

The version of this proposition for the case \( N = 2 \) corresponds to (16), and has been an important ingredient of our discussion on Hilbert’s 16th problem in Section 3. It explains the role played by the zero set of the divergence of the field \( u \), as well as the fact that for critical maps the integrand of the functional must remain constant.

4.2. The Pfaffian functional. We turn to the functional

\[
E(x) = \frac{1}{2} \int_Q \left( u(x(s)) \cdot x_{s_1}(s)^2 + u(x(s)) \cdot x_{s_2}(s)^2 \right) ds
\]

defined for the same class of embedded compact surfaces as before. This time the integrand is given by

\[
F(y, z) = \frac{1}{2} (u(y) \cdot z_1)^2 + \frac{1}{2} (u(y) \cdot z_2)^2, \quad z = \left( \begin{array}{c} z_1 \\ z_2 \end{array} \right).
\]

Based on our previous calculations, it is easy to find the optimality system for \( E \), and to derive some elementary consequences.

**Proposition 4.4.** Let the mapping \( x \) be a smooth, critical map for the Pfaffian functional such that \( x(Q) \) is a non-singular, embedded, compact manifold.

1. If the condition

\[
u \cdot (\nabla \wedge u) = 0
\]

holds, then \( x \) is a solution of the corresponding Pfaffian system

\[
u \cdot dx = 0.
\]

2. In any case, in subsets of \( Q \) where the integrand of \( E \) does not vanish, we must have

\[
(u(x) \cdot x_{s_1})_{s_1} + (u(x) \cdot x_{s_2})_{s_2} = 0.
\]

**Proof.** For each \( i = 1, 2 \), the optimality system incorporates two terms of the form

\[-[u(x) \cdot x_{s_i} u(x)]_{s_i} + u(x) \cdot x_{s_i} x_{s_i} \nabla u(x)\]

Taking into account the identity

\[
v \nabla u - \nabla u v = v \wedge (\nabla \wedge u)
\]

valid for three-dimensional vectors, this sum can be rewritten as

\[-[u(x) \cdot x_{s_1}]_{s_1} u(x) + u(x) \cdot x_{s_1} x_{s_1} \wedge (\nabla \wedge u(x)).\]
Note that $\nabla \wedge u$ is the curl of the vector field $u$. Altogether, we find that the Euler-Lagrange system can be cast in the form

$$-[u(x) \cdot x_{s_1}]_{s_1} + [u(x) \cdot x_{s_2}]_{s_2} u(x)$$

$$+[(u(x) \cdot x_{s_1})x_{s_1} + (u(x) \cdot x_{s_2})x_{s_2}] \wedge (\nabla \wedge u(x)) = 0.$$ 

The vector field

$$\Pi u \equiv [(u(x) \cdot x_{s_1})x_{s_1} + (u(x) \cdot x_{s_2})x_{s_2}]$$

belongs to the tangent space spanned by the two basic tangent vectors $x_{s_1}$ and $x_{s_2}$. It would be the orthogonal projection onto it, if $\{x_{s_1}, x_{s_2}\}$ is an orthonormal basis.

(1) Under the condition

$$u \cdot (\nabla \wedge u) = 0,$$ 

the optimality system amounts to

$$[(u(x) \cdot x_{s_1})_{s_1} + (u(x) \cdot x_{s_2})_{s_2}] = \Pi u = 0.$$ 

Since the two basic tangent vectors $x_{s_i}$, $i = 1, 2$, are assumed independent for every $s \in Q$, if $x(Q)$ is a non-singular manifold, we can conclude that critical mappings $x$ can only be absolute minimizers of the Pfaffian functional.

(2) In any case, if we multiply the Euler-Lagrange system by $\Pi u$, we immediately are led to

$$[(u(x) \cdot x_{s_1})_{s_1} + (u(x) \cdot x_{s_2})_{s_2}] (u(x(s)) \cdot x_{s_1}(s)^2 + u(x(s)) \cdot x_{s_2}(s)^2) = 0.$$ 

The second factor is exactly the integrand for $E$. We can deduce that either this integrand identically vanishes and $x$ is an absolute minimizer, a flow tube for $u$; or, else, in places where the integrand does not vanish, the function

$$(u(x) \cdot x_{s_1})_{s_1} + (u(x) \cdot x_{s_2})_{s_2}$$

does.

□

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