THE UNIFIED METHOD FOR THE THREE-WAVE EQUATION ON THE HALF-LINE

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Abstract. We present a Riemann-Hilbert problem formalism for the initial-boundary value problem of the three-wave equation:

\[ p_{ij,t} - \frac{b_i - b_j}{a_i - a_j} p_{ij,x} + \sum_k \left( \frac{b_k - b_j}{a_k - a_j} - \frac{b_i - b_k}{a_i - a_k} \right) p_{ik} p_{kj} = 0, \quad i, j, k = 1, 2, 3, \]

on the half-line.

1. Introduction

The 3-wave resonant interaction model described by the equations

\[ p_{ij,t} - \frac{b_i - b_j}{a_i - a_j} p_{ij,x} + \sum_k \left( \frac{b_k - b_j}{a_k - a_j} - \frac{b_i - b_k}{a_i - a_k} \right) p_{ik} p_{kj} = 0, \quad i, j, k = 1, 2, 3; \quad a_i \neq a_j, b_i \neq b_j, \text{ for } i \neq j, \]

(1.1)
is one of the important nonlinear models with numerous applications to physics [14]. The 3- and \( N \)-wave interaction models describe a special class of wave-wave interactions that are not sensitive on the physical nature of the waves and bear an universal character. This explains why they find numerous applications in physics and attract the attention of the scientific community over the last few decades [23, 24, 25, 20, 19, 22, 21, 9] and the references therein.

The 3-wave equations can be solved through the inverse scattering method due to the fact that equation (1.1) admits a Lax representation [24, 25]. But until the 1990s, the inverse scattering method was pursued almost entirely for pure initial value problems. In 1997, Fokas announced a new unified approach for the analysis of initial-boundary
value problems for linear and nonlinear integrable PDEs [1] [2] [3]. The Fokas method provides a generalization of the inverse scattering formalism from initial value to IBV problems, and over the last fifteen years, this method has been used to analyze boundary value problems for several of the most important integrable equations with $2 \times 2$ Lax pairs, such as KdV, Schrödinger, sine-Gordon, and stationary axisymmetric Einstein equations, see e.g. [4] [6]. Just like the IST on the line, the unified method yields an expression for the solution of an initial-boundary value problem with that of a Riemann-Hilbert problem. In particular, the asymptotic behavior of the solution can be analyzed in an effective way by employing the Riemann-Hilbert problem and the steepest descent method introduced by Deift and Zhou [10]. Recently, Lenells develop a methodology for analyzing initial-boundary value problems for integrable evolution equations with Lax pairs involving $3 \times 3$ matrices [7]. He also used this method to analyze the Degasperis-Procesi equation in [8].

Pelloni and Pinotsis also studied the boundary value problem of the $N$–wave equation by using the unified method [11]. Recently, Gerdjikov and Grahovski considered Cauchy problem of the 3-wave equation with with non-vanishing initial values [12]. In this paper we analyze the initial-boundary value problem of the three-wave equation (1.1) on the half-line. Compared with these two papers, there are two differences in our paper. The first difference is that we get the residue conditions of matrix function $M$ in the Riemann-Hilbert problem (see (2.27) in the next section 2). The second difference is that we the jump matrix $J$ is explicitly constructed ( see the equations (2.14) and (2.22) in the next section 2). Of course, the initial-boundary value problem for the 3–wave equation does not need to analysis the global relation, because the initial data and the boundary data are all known.

The organization of the paper is as follows. In the following section 2, we perform the spectral analysis of the associated Lax pair. And we formulate the main Riemann-Hilbert problem in section 3.
2. Spectral Analysis

Our goal in this section is to define analytic eigenfunctions of the Lax pair (2.1) which are suitable for the formulation of a Riemann-Hilbert problem.

2.1. Lax pair. We first consider the three-wave equations (1.1), with \((x, t) \in \Omega\), and \(\Omega\) denoting the half-line domain

\[
\Omega = \{0 < x < +\infty, 0 < t < T\}
\]

and \(T > 0\) being a fixed final time. We denote the initial and boundary values by \(p_{ij,0}(x)\) and \(q_{ij,0}(t)\), respectively

\[
p_{ij,0}(x) = p_{ij}(x,0), \quad q_{ij,0}(t) = p_{ij}(0,t)
\]

with \(p_{ij,0}(x)\) and \(q_{ij,0}(t)\) are rapidly decaying. Equation (1.1) admits the following Lax representation [14]

\[
\begin{align*}
\phi_x &= M\phi, \\
\phi_t &= N\phi,
\end{align*}
\]

(2.1)

where \(M = i\lambda A + P\) and \(N = i\lambda B + Q\), with

\[
A = \begin{pmatrix}
    a_1 & 0 & 0 \\
    0 & a_2 & 0 \\
    0 & 0 & a_3
\end{pmatrix} \quad P = \begin{pmatrix}
    0 & p_{12} & p_{13} \\
p_{21} & 0 & p_{23} \\
p_{31} & p_{32} & 0
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
    b_1 & 0 & 0 \\
    0 & b_2 & 0 \\
    0 & 0 & b_3
\end{pmatrix} \quad Q = \begin{pmatrix}
    0 & n_{12}p_{12} & n_{13}p_{13} \\
n_{21}p_{21} & 0 & n_{23}p_{23} \\
n_{31}p_{31} & n_{32}p_{32} & 0
\end{pmatrix}
\]

(2.2)

Obviously, \(\text{trace}(A) = \text{trace}(B) = 0\). We also assume that \(a_1 > a_2 > a_3\) and \(b_1 < b_2 < b_3\). By using transformation

\[
\mu = \phi e^{-i\lambda Ax - i\lambda Bt},
\]

we change Lax pair (2.1) in the form

\[
\begin{align*}
\mu_x - i\lambda[A, \mu] &= P\mu, \\
\mu_t - i\lambda[B, \mu] &= Q\mu,
\end{align*}
\]

(2.3)
which can be further written in differential form as

\[ d(e^{-i\lambda \hat{A}x - i\lambda \hat{B}t}) = W(x, t, \lambda), \] (2.4)

where

\[ W(x, t, \lambda) = e^{-i\lambda \hat{A}x - i\lambda \hat{B}t}(Pdx + Qdt)\mu \] (2.5)

and \( e^{\hat{A}X} = e^{A}xe^{-A} \).

2.2. Spectral functions. We define three eigenfunctions \( \{\mu_j\}_{1}^{3} \) of (2.3) by the Volterra integral equations

\[ \mu_j(x, t, \lambda) = \mathbb{I} + \int_{\gamma_j} e^{(i\lambda \hat{A}x + i\lambda \hat{B}t)}W_j(x', t', \lambda) \, dt' \] \( j = 1, 2, 3 \) (2.6)

where \( W_j \) is given by (2.5) with \( \mu \) replaced with \( \mu_j \), and the contours \( \{\gamma_j\}_{1}^{3} \) are showed in Figure 1. And we have the following inequalities

\[ \begin{align*}
\gamma_1 : & \quad x - x' \geq 0, \quad t - t' \leq 0, \\
\gamma_2 : & \quad x - x' \geq 0, \quad t - t' \geq 0, \\
\gamma_3 : & \quad x - x' \leq 0.
\end{align*} \] (2.7)

So, these inequalities imply that the functions \( \{\mu_j\}_{1}^{3} \) are bounded and analytic for \( \lambda \in \mathbb{C} \) such that \( \lambda \) belongs to

\[ \begin{align*}
\mu_1 : & \quad (D_2, \emptyset, D_1), \\
\mu_2 : & \quad \emptyset, \\
\mu_3 : & \quad (D_1, \emptyset, D_2),
\end{align*} \] (2.8)

where \( \{D_n\}_{1}^{2} \) denote two open, pairwisely disjoint subsets of the complex \( \lambda \) plane showed in Figure 2.
The sets $D_n, n = 1, 2$, which decompose the complex $k-$plane.

And the sets $\{D_n\}_{i=1}^3$ has the following properties:

$$D_1 = \{\lambda \in \mathbb{C} | \text{Re} l_1 < \text{Re} l_2 < \text{Re} l_3, \text{Re} z_1 > \text{Re} z_2 > \text{Re} z_3\},$$

$$D_2 = \{\lambda \in \mathbb{C} | \text{Re} l_1 > \text{Re} l_2 > \text{Re} l_3, \text{Re} z_1 < \text{Re} z_2 < \text{Re} z_3\},$$

where $l_i(\lambda)$ and $z_i(\lambda)$ are the diagonal entries of matrices $i\lambda A$ and $i\lambda B$, respectively.

### 2.3. Matrix valued FUNCTIONS $M_n$’s.

For each $n = 1, 2$, define a solution $M_n(x, t, \lambda)$ of (2.3) by the following system of integral equations:

$$(M_n)_{ij}(x, t, \lambda) = \delta_{ij} + \int_{\gamma_{ij}^n} \left( e^{(-i\lambda A x - i\lambda B t)} W_n(x', t', \lambda) \right)_{ij}, \quad \lambda \in D_n, \quad i, j = 1, 2, 3.$$  \tag{2.9}$$

where $W_n$ an $M_n$ are given by (2.5), and the contours $\gamma_{ij}^n$, $n = 1, 2$, $i, j = 1, 2, 3$ are defined by

$$\gamma_{ij}^n = \begin{cases} 
\gamma_1 & \text{if } \text{Re} l_i(\lambda) < \text{Re} l_j(\lambda) \text{ and } \text{Re} z_i(\lambda) \geq \text{Re} z_j(\lambda), \\
\gamma_2 & \text{if } \text{Re} l_i(\lambda) < \text{Re} l_j(\lambda) \text{ and } \text{Re} z_i(\lambda) < \text{Re} z_j(\lambda), \quad \text{for } \lambda \in D_n. \\
\gamma_3 & \text{if } \text{Re} l_i(\lambda) \geq \text{Re} l_j(\lambda) 
\end{cases}$$  \tag{2.10}$$

The following proposition ascertains that the $M_n$’s defined in this way have the properties required for the formulation of a Riemann-Hilbert problem.

**Proposition 2.1.** For each $n = 1, 2$, the function $M_n(x, t, \lambda)$ is well-defined by equation (2.9) for $\lambda \in \bar{D}_n$ and $(x, t) \in \Omega$. For any fixed point $(x, t)$, $M_n$ is bounded and analytic as a function of $\lambda \in D_n$ away...
from a possible discrete set of singularities \(\{\lambda_j\}\) at which the Fredholm determinant vanishes. Moreover, \(M_n\) admits a bounded and continuous extension to \(\bar{D}_n\) and

\[
M_n(x, t, \lambda) = \mathbb{I} + O\left(\frac{1}{\lambda}\right), \quad \lambda \to \infty, \quad \lambda \in D_n.
\] (2.11)

Proof. The boundedness and analyticity properties are established in appendix B in [7]. And substituting the expansion

\[
M = M_0 + \frac{M^{(1)}}{\lambda} + \frac{M^{(2)}}{\lambda^2} + \cdots, \quad \lambda \to \infty.
\]

into the Lax pair (2.3) and comparing the terms of the same order of \(\lambda\) yield the equation (2.11).

2.4. The jump matrices. We define spectral functions \(S_n(\lambda), \ n = 1, 2,\) and

\[
S_n(\lambda) = M_n(0, 0, \lambda), \quad \lambda \in D_n, \quad n = 1, 2.
\] (2.12)

Let \(M\) denote the sectionally analytic function on the complex \(\lambda\)–plane which equals \(M_n\) for \(\lambda \in D_n\). Then \(M_n\) satisfies the jump conditions

\[
M_1 = M_2 J, \quad \lambda \in \mathbb{R},
\] (2.13)

where the jump matrices \(J(x, t, \lambda)\) are defined by

\[
J = e^{(i\lambda Ax + i\lambda Bt)} (S_2^{-1}S_1).
\] (2.14)

According to the definition of the \(\gamma^n\), we find that

\[
\gamma^1 = \begin{pmatrix} \gamma_3 & \gamma_1 & \gamma_1 \\ \gamma_3 & \gamma_3 & \gamma_1 \\ \gamma_3 & \gamma_3 & \gamma_3 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} \gamma_3 & \gamma_3 & \gamma_1 \\ \gamma_3 & \gamma_3 & \gamma_3 \\ \gamma_1 & \gamma_1 & \gamma_3 \end{pmatrix}
\] (2.15)

2.5. The adjugated eigenfunctions. We will also need the analyticity and boundedness properties of the minors of the matrices \(\{\mu_j(x, t, \lambda)\}_{1}^{3}\). We recall that the adjugate matrix \(X^A\) of a \(3 \times 3\) matrix \(X\) is defined by

\[
X^A = \begin{pmatrix} m_{11}(X) & -m_{12}(X) & m_{13}(X) \\ -m_{21}(X) & m_{22}(X) & -m_{23}(X) \\ m_{31}(X) & -m_{32}(X) & m_{33}(X) \end{pmatrix}
\]
where $m_{ij}(X)$ denote the $(ij)$th minor of $X$.

It follows from (2.3) that the adjugated eigenfunction $\mu^A$ satisfies the Lax pair

\[
\begin{cases}
\mu^A_x + [i\lambda A, \mu^A] = -P^T \mu^A,
\mu^A_t + [i\lambda B, \mu^A] = -Q^T \mu^A.
\end{cases}
\tag{2.16}
\]

where $V^T$ denote the transform of a matrix $V$. Thus, the eigenfunctions $\{\mu^A_j\}_{j=1}^3$ are solutions of the integral equations

\[
\mu^A_j(x, t, \lambda) = I - \int_{\gamma_j} e^{-i\lambda \hat{A}(x-x')-i\lambda \hat{B}(t-t')} (P^T dx + Q^T) \mu^A_j, \quad j = 1, 2, 3.
\tag{2.17}
\]

Then we can get the following analyticity and boundedness properties:

\[
\begin{align*}
\mu^A_1 : & \quad (D_1, \emptyset, D_2), \\
\mu^A_2 : & \quad \emptyset, \\
\mu^A_3 : & \quad (D_2, \emptyset, D_1).
\end{align*}
\tag{2.18}
\]

2.6. The computation of jump matrices. Let us define the $3 \times 3$–matrix value spectral functions $s(\lambda)$ and $S(\lambda)$ by

\[
\begin{align*}
\mu_3(x, t, \lambda) & = \mu_2(x, t, \lambda) e^{(i\lambda \hat{A}x + i\lambda \hat{B}t)} s(\lambda), \\
\mu_1(x, t, \lambda) & = \mu_2(x, t, \lambda) e^{(i\lambda \hat{A}x + i\lambda \hat{B}t)} S(\lambda).
\end{align*}
\tag{2.19a,b}
\]

Thus,

\[
\begin{align*}
s(\lambda) & = \mu_3(0, 0, \lambda), \\
S(\lambda) & = \mu_1(0, 0, \lambda).
\end{align*}
\tag{2.20}
\]

And we deduce from the properties of $\mu_j$ and $\mu^A_j$ that $s(\lambda)$ and $S(\lambda)$ have the following boundedness properties:

\[
\begin{align*}
s(\lambda) : & \quad (D_1, \emptyset, D_2), \\
S(\lambda) : & \quad (D_2, \emptyset, D_1), \\
s^A(\lambda) : & \quad (D_2, \emptyset, D_1), \\
S^A(\lambda) : & \quad (D_1, \emptyset, D_2).
\end{align*}
\]

Moreover,

\[
M_n(x, t, \lambda) = \mu_2(x, t, \lambda) e^{(i\lambda \hat{A}x + i\lambda \hat{B}t)} S_n(\lambda), \quad \lambda \in D_n.
\tag{2.21}
\]
Proposition 2.2. The $S_n$ can be expressed in terms of the entries of $s(\lambda)$ and $S(\lambda)$ as follows:

\[ S_1 = \begin{pmatrix} s_{11} & \frac{m_{33}(s)M_{21}(S) - m_{23}(s)M_{31}(S)}{(s^T s^A)_{11}} & \frac{S_{11}}{(s^T s^A)_{33}} \\ s_{21} & \frac{m_{33}(s)M_{11}(S) - m_{13}(s)M_{31}(S)}{(s^T s^A)_{11}} & \frac{S_{21}}{(s^T s^A)_{33}} \\ s_{31} & \frac{m_{33}(s)M_{11}(S) - m_{13}(s)M_{21}(S)}{(s^T s^A)_{11}} & \frac{S_{31}}{(s^T s^A)_{33}} \end{pmatrix}, \]

\[ S_2 = \begin{pmatrix} s_{11} & \frac{m_{33}(s)M_{33}(S) - m_{31}(s)M_{31}(S)}{(s^T s^A)_{33}} & \frac{S_{11}}{(s^T s^A)_{33}} \\ s_{13} & \frac{m_{31}(s)M_{33}(S) - m_{33}(s)M_{31}(S)}{(s^T s^A)_{33}} & \frac{S_{13}}{(s^T s^A)_{33}} \\ s_{31} & \frac{m_{11}(s)M_{33}(S) - m_{33}(s)M_{13}(S)}{(s^T s^A)_{33}} & \frac{S_{31}}{(s^T s^A)_{33}} \\ s_{33} & \frac{m_{11}(s)M_{23}(S) - m_{23}(s)M_{13}(S)}{(s^T s^A)_{33}} & \frac{S_{33}}{(s^T s^A)_{33}} \end{pmatrix}, \]

where $m_{ij}$ and $M_{ij}$ denote that the $(i, j)$-th minor of $s$ and $S$, respectively.

Proof. Let $\gamma_3^{X_0}$ denote the contour $(X_0, 0) \to (x, t)$ in the $(x, t)$-plane, here $X_0 > 0$ is a constant. We introduce $\mu_3(x, t, k; X_0)$ as the solution of (2.6) with $j = 3$ and with the contour $\gamma_3$ replaced by $\gamma_3^{X_0}$. Similarly, we define $M_n(x, t, \lambda; X_0)$ as the solution of (2.9) with $\gamma_3$ replaced by $\gamma_3^{X_0}$. We will first derive expression for $S_n(\lambda; X_0) = M_n(0, 0, \lambda; X_0)$ in terms of $S(\lambda)$ and $s(\lambda; X_0) = \mu_3(0, 0, \lambda)$ in $S_2$. Then (2.22) will follow by taking the limit $X_0 \to \infty$.

First, We have the following relations:

\[ \begin{cases} M_n(x, t, \lambda; X_0) = \mu_1(x, t, \lambda)e^{(i\lambda \hat{A}x + i\lambda \hat{B}t)}R_n(\lambda; X_0), \\ M_n(x, t, \lambda; X_0) = \mu_2(x, t, \lambda)e^{(i\lambda \hat{A}x + i\lambda \hat{B}t)}S_n(\lambda; X_0), \\ M_n(x, t, \lambda; X_0) = \mu_3(x, t, \lambda)e^{(i\lambda \hat{A}x + i\lambda \hat{B}t)}T_n(\lambda; X_0). \end{cases} \]  

(2.23)

Then we get $R_n(\lambda; X_0)$ and $T_n(\lambda; X_0)$ are defined as follows:

\[ R_n(\lambda; X_0) = e^{-i\lambda \hat{B}T}M_n(0, T, \lambda; X_0), \]

(2.24a)

\[ T_n(\lambda; X_0) = e^{-i\lambda \hat{A}X_0}M_n(X_0, 0, \lambda; X_0). \]

(2.24b)

The relations (2.23) imply that

\[ s(\lambda; X_0) = S_n(\lambda; X_0)T_n^{-1}(\lambda; X_0), \quad S(\lambda) = S_n(\lambda; X_0)R_n^{-1}(\lambda; X_0). \]

(2.25)
These equations constitute a matrix factorization problem which, given \( \{s(\lambda), S(\lambda)\} \) can be solved for the \( \{R_n, S_n, T_n\} \). Indeed, the integral equations (2.9) together with the definitions of \( \{R_n, S_n, T_n\} \) imply that

\[
\begin{align*}
(R_n(\lambda; X_0))_{ij} &= 0 \quad \text{if} \quad \gamma^n_{ij} = \gamma_1, \\
(S_n(\lambda; X_0))_{ij} &= 0 \quad \text{if} \quad \gamma^n_{ij} = \gamma_2, \\
(T_n(\lambda; X_0))_{ij} &= 0 \quad \text{if} \quad \gamma^n_{ij} = \gamma_3.
\end{align*}
\] (2.26)

It follows that (2.25) are 18 scalar equations for 18 unknowns. By computing the explicit solution of this algebraic system, we find that \( \{S_n(\lambda; X_0)\}_1^2 \) are given by the equation obtained from (2.22) by replacing \( \{S_n(\lambda), s(\lambda)\} \) with \( \{S_n(\lambda; X_0), s(\lambda; X_0)\} \). Taking \( X_0 \to \infty \) in this equation, we arrive at (2.22). □

2.7. The residue conditions. Since \( \mu_2 \) is an entire function, it follows from (2.21) that \( M \) can only have singularities at the points where the \( S_n's \) have singularities. We infer from the explicit formulas (2.22) that the possible singularities of \( M \) are as follows:

- \([M]_1\) could have poles in \( D_2 \) at the zeros of \((S^T s^A)_{11}(\lambda)\);
- \([M]_2\) could have poles in \( D_1 \) at the zeros of \((s^T S^A)_{11}(\lambda)\);
- \([M]_2\) could have poles in \( D_2 \) at the zeros of \((s^T S^A)_{33}(\lambda)\);
- \([M]_3\) could have poles in \( D_1 \) at the zeros of \((S^T s^A)_{33}(\lambda)\).

We denote the above possible zeros by \( \{\lambda_j\}_1^N \) and assume they satisfy the following assumption.

**Assumption 2.3.** We assume that

- \((S^T s^A)_{11}(\lambda)\) has \( n_0 \) possible simple zeros in \( D_2 \) denoted by \( \{\lambda_j\}_{1}^{n_0} \);
- \((s^T S^A)_{11}(\lambda)\) has \( n_1 - n_0 \) possible simple zeros in \( D_1 \) denoted by \( \{\lambda_j\}_{n_0 + 1}^{n_1} \);
- \((s^T S^A)_{33}(\lambda)\) has \( n_2 - n_1 \) possible simple zeros in \( D_2 \) denoted by \( \{\lambda_j\}_{n_1 + 1}^{n_2} \);
- \((S^T s^A)_{33}(\lambda)\) has \( n_3 - n_2 \) possible simple zeros in \( D_1 \) denoted by \( \{\lambda_j\}_{n_2 + 1}^{N} \).
and that none of these zeros coincide. Moreover, we assume that none of these functions have zeros on the boundaries of the $D_n$’s.

We determine the residue conditions at these zeros in the following:

**Proposition 2.4.** Let $\{ M_n \}_1^2$ be the eigenfunctions defined by (2.9) and assume that the set $\{ \lambda_j \}_1^N$ of singularities are as the above assumption. Then the following residue conditions hold:

\[
\text{Res}_{\lambda_j} [M]_1 = \frac{1}{(S^T s^2_{11}(\lambda_j))} \frac{(S_{11}s_{23}-S_{21}s_{13})(\lambda_j)}{m_{31}(\lambda_j)} e^{\theta_{21}(\lambda_j)} [M(\lambda_j)]_2, \quad (2.27a)
\]

\[
1 \leq j \leq n_0, \lambda_j \in D_2
\]

\[
\text{Res}_{\lambda_j} [M]_2 = -\frac{1}{(s^T S^3_{13}(\lambda_j))} \frac{M_{23}(S^T s^2_{11}(\lambda_j))}{(s_{13}(\lambda_j)S_{31}(\lambda_j)-S_{33}(\lambda_j)s_{11}(\lambda_j))} e^{\theta_{32}(\lambda_j)} [M(\lambda_j)]_1, \quad (2.27b)
\]

\[
n_0 < j \leq n_1, \lambda_j \in D_1
\]

\[
\text{Res}_{\lambda_j} [M]_2 = -\frac{1}{(s^T S^3_{13}(\lambda_j))} \frac{M_{23}(S^T s^2_{11}(\lambda_j))}{(s_{13}(\lambda_j)S_{31}(\lambda_j)-S_{33}(\lambda_j)s_{11}(\lambda_j))} e^{\theta_{32}(\lambda_j)} [M(\lambda_j)]_3, \quad (2.27c)
\]

\[
n_1 < j \leq n_2, \lambda_j \in D_2
\]

\[
\text{Res}_{\lambda_j} [M]_3 = \frac{1}{(S^T s^2_{13}(\lambda_j))} \frac{(S_{13}s_{21}-S_{23}s_{11})(\lambda_j)}{m_{33}(\lambda_j)} e^{\theta_{23}(\lambda_j)} [M(\lambda_j)]_2, \quad (2.27d)
\]

\[
n_2 < j \leq N, \lambda_j \in D_1
\]

where $\dot{f} = \frac{df}{dt}$, and $\theta_{ij}$ is defined by

\[
\theta_{ij}(x, t, \lambda) = (l_i - l_j)x + (z_i - z_j)t, \quad i, j = 1, 2, 3. \quad (2.28)
\]

**Proof.** We will prove (2.27a), (2.27b), the other conditions follow by similar arguments. Equation (2.21) implies the relation

\[
M_1 = \mu_2 e^{(i\lambda \dot{A}x + i\lambda Bt)} S_1, \quad (2.29a)
\]

\[
M_2 = \mu_2 e^{(i\lambda \dot{A}x + i\lambda Bt)} S_2. \quad (2.29b)
\]
In view of the expressions for $S_1$ and $S_2$ given in (2.22), the three columns of (2.29a) read:

$$[M_1]_1 = [\mu_2]_1 s_{11}(\lambda) + [\mu_2]_2 e^{\theta_{21}} s_{21}(\lambda) + [\mu_2]_3 e^{\theta_{31}} s_{31}(\lambda),$$  \hspace{1cm} (2.30a)

$$[M_1]_2 = [\mu_2]_1 e^{\theta_{12}} \frac{m_{33} M_{33} - m_{23} M_{33}}{(s^{T} s^A)_{11}}(\lambda) + [\mu_2]_2 \frac{m_{11} M_{13} - m_{13} M_{13}}{(s^{T} s^A)_{11}}(\lambda),$$

$$+ [\mu_2]_3 e^{\theta_{32}} \frac{m_{23} M_{33} - m_{13} M_{33}}{(s^{T} s^A)_{11}}(\lambda),$$  \hspace{1cm} (2.30b)

$$[M_1]_3 = [\mu_2]_1 e^{\theta_{13}} \frac{S_{13}}{(s^{T} s^A)_{33}}(\lambda) + [\mu_2]_2 e^{\theta_{23}} \frac{S_{23}}{(s^{T} s^A)_{33}}(\lambda) + [\mu_2]_3 \frac{S_{33}}{(s^{T} s^A)_{33}}(\lambda).$$  \hspace{1cm} (2.30c)

while the three columns of (2.29b) read:

$$[M_2]_1 = [\mu_2]_1 \frac{S_{11}}{(s^{T} s^A)_{11}}(\lambda) + [\mu_2]_2 e^{\theta_{21}} \frac{S_{21}}{(s^{T} s^A)_{11}}(\lambda) + [\mu_2]_3 e^{\theta_{31}} \frac{S_{31}}{(s^{T} s^A)_{11}}(\lambda),$$  \hspace{1cm} (2.31a)

$$[M_2]_2 = [\mu_2]_1 e^{\theta_{12}} \frac{m_{21} M_{21} - m_{31} M_{23}}{(s^{T} s^A)_{33}}(\lambda) + [\mu_2]_2 \frac{m_{11} M_{13} - m_{13} M_{13}}{(s^{T} s^A)_{33}}(\lambda),$$

$$+ [\mu_2]_3 e^{\theta_{32}} \frac{m_{13} M_{13} - m_{23} M_{23}}{(s^{T} s^A)_{33}}(\lambda),$$  \hspace{1cm} (2.31b)

$$[M_2]_3 = [\mu_2]_1 s_{13} e^{\theta_{13}} + [\mu_2]_2 s_{23} e^{\theta_{23}} + [\mu_2]_3 s_{33}.$$  \hspace{1cm} (2.31c)

We first suppose that $\lambda_j \in D_2$ is a simple zero of $(s^{T} s^A)_{11}(\lambda)$. Solving (2.31b) and (2.31c) for $[\mu_2]_1$ and $[\mu_2]_2$ and substituting the result in to (2.31a), we find

$$[M_1]_1 = \frac{S_{11} s_{23} - S_{21} s_{13}}{(s^{T} s^A)_{11} m_{31}} e^{\theta_{21}} [M_2]_2 + \frac{M_{33}}{m_{31}} e^{\theta_{31}} [M_2]_3 + \frac{1}{m_{31}} e^{\theta_{31}} [\mu_2]_3.$$  \hspace{1cm} (2.27a)

Taking the residue of this equation at $\lambda_j$, we find the condition (2.27a) in the case when $\lambda_j \in D_2$. Similarly, we can get the equation (2.27b).

Then let us consider that $\lambda_j \in D_1$ is a simple zero of $(s^{T} s^A)_{11}(\lambda)$. Solving (2.30a) and (2.30c) for $[\mu_2]_1$ and $[\mu_2]_3$ and substituting the result in to (2.30b), we find

$$[M_1]_2 = -\frac{M_{21} (s^{T} s^A)_{33}}{(s^{T} s^A)_{11} (s^{T} s^A)_{33}} e^{\theta_{12}} [M_1]_1 - \frac{(s^{T} s^A)_{33}}{S_{13} s_{31} - S_{33} s_{11}} e^{\theta_{32}} [M_1]_3.$$  \hspace{1cm} (2.27b)
Taking the residue of this equation at $\lambda_j$, we find the condition \[(2.27b)\] in the case when $\lambda_j \in D_1$. Similarly, we can get the equation \[(2.27c)\].

\[\square\]

3. The Riemann-Hilbert problem

The sectionally analytic function $M(x, t, \lambda)$ defined in section 2 satisfies a Riemann-Hilbert problem which can be formulated in terms of the initial and boundary values of $p_{ij}(x, t)$. By solving this Riemann-Hilbert problem, the solution of \[(1.1)\] can be recovered for all values of $x, t$.

**Theorem 3.1.** Suppose that $p_{ij}(x, t)$ are a solution of \[(1.1)\] in the half-line domain $\Omega$ with sufficient smoothness and decays as $x \to \infty$. Then $p_{ij}(x, t)$ can be reconstructed from the initial value \[\{p_{ij,0}(x)\}_{i,j=1}^3\] and boundary values \[\{q_{ij,0}(t)\}_{i,j=1}^3\] defined as follows,

\[p_{ij,0}(x) = p_{ij}(x, 0), \quad q_{ij,0}(t) = p_{ij}(0, t). \quad (3.1)\]

Use the initial and boundary data to define the jump matrices $J(x, t, \lambda)$ as well as the spectral $s(\lambda)$ and $S(\lambda)$ by equation \[(2.19)\]. Assume that the possible zeros $\{\lambda_j\}_{1}^{N}$ of the functions $(S^T S^A)_{33}(\lambda), (s^T S^A)_{11}(\lambda), (s^T S^A)_{33}(\lambda)$ are as in assumption 2.3.

Then the solution \[\{p_{ij}(x, t)\}_{i,j=1}^3\] is given by

\[p_{ij}(x, t) = -i(a_i - a_j) \lim_{\lambda \to \infty} \mathbb{I}M(x, t, \lambda))_{ij}, \quad (3.2)\]

where $M(x, t, \lambda)$ satisfies the following $3 \times 3$ matrix Riemann-Hilbert problem:

- $M$ is sectionally meromorphic on the complex $\lambda$–plane with jumps across the contour $\mathbb{R}$, see Figure 2.
- Across the contour $\mathbb{R}$, $M$ satisfies the jump condition
  
  \[M_1(x, t, \lambda) = M_2(x, t, \lambda)J(x, t, \lambda), \quad \lambda \in \mathbb{R}. \quad (3.3)\]

  where the jump $J$ is defined by the equation \[(2.14)\].
- $M(x, t, \lambda) = \mathbb{I} + O(1/\lambda), \quad \lambda \to \infty.$
The residue condition of $M$ is showed in Proposition 2.4.

Proof. It only remains to prove (3.2) and this equation follows from the large $\lambda$ asymptotics of the eigenfunctions.

We write the large $\lambda$ asymptotics of $M$ as follows:

$$M(x, t, \lambda) = M_0(x, t) + \frac{M_1(x, t)}{\lambda} + \cdots \quad \lambda \to +\infty. \tag{3.4}$$

And insert this equation into the equation (2.3) and compare the coefficients of the same order $\lambda$, for the $O(\lambda)$, we have $M_0$ is a diagonal matrix; for the $O(1)$, we get $M_0 = I$ by comparing the diagonal elements, and we can have the following equation by comparing the other elements

$$-i[A, M_1] = P, \tag{3.5}$$

this equation reads the required result of $p_{ij}(x, t)$

$$p_{ij}(x, t) = -i(a_i - a_j)M_{1,ij}(x, t). \tag{3.6}$$

\[ \square \]

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