Abstract

The model of binary aggregation with constant kernel is subjected to stochastic resetting: aggregates of any size explode into monomers at independent stochastic times. These resetting times are Poisson distributed, and the rate of the process is called the resetting rate. The master equation yields a Bernoulli-type equation in the generating function of the concentration of aggregates of any size, which can be solved exactly. This resetting prescription leads to a non-equilibrium steady state for the densities of aggregates, which is a function of the size of the aggregate, rescaled by a function of the resetting rate. The steady-state density of aggregates of a given size is maximised if the resetting rate is set to the quotient of the aggregation rate by the size of the aggregate (minus one).

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1 Introduction

Resetting a stochastic process to its initial configuration effectively cuts off long excursions in the space of configurations. In particular, the first-passage time of a single diffusive random walker is made finite by resetting the random walker to its initial position at Poisson-distributed stochastic times [1]. Moreover the expectation value of the first passage time at a fixed target can be optimised as a function of the resetting rate [2]. Optimisation properties of diffusive search times and relaxation dynamics are illustrated in [3–5]. Moreover, stochastic resetting induces non-equilibrium steady states: the steady state of the diffusive random walker with resetting to the origin has been shown to be an exponentially decaying function of the distance to the origin [1]. These rich features of stochastic resetting have found numerous applications to active matter [6,7], predator-prey dynamics [8,9], population dynamics [10–12], as well as stochastic processes [13–18] (see [19] for a recent review, and references therein).

Extensions to many-body interacting systems include reaction-diffusion systems. In particular, the coagulation-diffusion model under resetting has been studied in [20]. On the other hand, in models of aggregation, diffusion or mixing is supposed to be fast enough so that concentrations are globally well defined at all times. Aggregation provides illustrations of features of non-equilibrium phenomena, such as steady states and scaling. In the simplest model of aggregation, clusters of all sizes merge pairwise at a uniform rate. This model was solved for the first time by Smoluchowski in [21] (see [22,23] and Chapter 5 of [24] for reviews).

Coupling aggregation to fragmentation gives rise to a very broad family of models, whose kinetics may be studied to model physical phenomena at various scales, such as phase separation in alloys, nucleation of droplets or the formation of galaxies. The convergence properties and structure of equilibrium of the corresponding equations have been studied mathematically (see [25] for a review). Cluster sizes can vary by one unit at a time in the Becker–Doering theory [26–28], generalisations including all possible processes were presented in [29].

On the other hand, large systems of particles evolving stochastically can be mapped to a random walk in the space of population sizes. This mapping has been used to study models of relaxation with entropy barriers [30,31], such as the backgammon model [32–34]. The recent applications of stochastic resetting in statistical physics suggest to couple aggregation to fragmentation processes that correspond to resetting the constituents to their original situation as monomers. This fragmentation prescription is the opposite as the one taken in Becker–Doering theory. It is natural to expect steady states to emerge (together with an exact description). Optimisation properties w.r.t. the resetting rate would generalise to clustering the results obtained for diffusion in [2]. In this work we therefore subject the Smoluchowski model to resetting according to a process in which any cluster can explode into monomers at Poisson-distributed times. The resetting rate introduces an additional time scale into the model, and low resetting rates should favour large aggregate sizes. Moreover, it is natural to ask whether the scaling properties of the Smoluchowski model (in which large aggregate sizes are scaled by a function of time) are reflected in the steady state of the system under resetting (with sizes rescaled by some function of the resetting rate).
We will make the same assumptions as in the Smoluchowski model: the kinetics of the reactions does not depend on the shape of the aggregates, and the transport phenomena are fast enough for the concentration of aggregates of any size to be a well-defined function of time. With these assumptions, the concentrations evolve according to a set of coupled master equations. These master equations induce a non-linear equation in the generating function of concentrations.

In Section 2 we set the notations and work out the master equation induced by the resetting prescription. In Section 3 the total density of clusters is expressed as a function of time, which allows to solve the master equation as a Bernoulli equation. In Section 4 the stationary state is studied: in particular, the concentration of aggregates of all masses are expressed, and maximised in the resetting rate. The limit of low resetting rate is shown to yield the Smoluchowski model. In Section 5 initial conditions consisting of aggregates of uniform size are studied. To obtain an idea of the typical size of aggregates, the second moment of the density is expressed as a function of time.

2 Model and quantities of interest

Consider the aggregation process of identical monomers with constant kernel. Each of the processes in which a cluster of size $i$ (denoted by $A_i$) and a cluster of size $j$ join to form a cluster of size $i+j$, is described by a reaction

$$A_i + A_j \rightarrow A_{i+j}$$

(1)

of rate $K > 0$, independent of the size (and shape) of the clusters. Let us introduce reversibility into the process under the form of resetting. In an infinitesimal interval $d\tau$ of time, any aggregate of size $k$ has a probability $\rho d\tau$ of exploding into $k$ monomers in the reaction

$$A_k \rightarrow kA_1.$$  

(2)

Let us rescale time so that the rate of aggregation equals 2. The rescaled time is denoted by $t$. The resetting rate is denoted by $r$ in the rescaled time:

$$t := K\tau/2, \quad r = \rho \times \frac{\tau}{t} = \frac{2\rho}{K}.$$  

(3)

The main quantities of interest are the concentrations of aggregates of all sizes:

$$\{c_k(t) := \text{concentration of aggregates of size } k \text{ at time } t, \ k \geq 1, \ t > 0\}.$$  

(4)

The aggregates are assumed to be well mixed in a solvent, so that the above densities are well defined at all times, and the monomers resulting from the resetting processes of Eq. (2) are immediately available for aggregation.

Consider the concentration of aggregates of size $k$, for some $k \geq 1$. It satisfies the following master equation:

$$\frac{dc_k}{dt} = \sum_{i+j=k} c_i c_j - c_k \sum_i c_i - rc_k + r\delta_{k1} \sum_i i c_i.$$  

(5)
The first two terms on the r.h.s. correspond to aggregation of pairs of clusters (of sizes $i$ and $j$) into one cluster of size $k$, the second one to the aggregation of a cluster of size $k$ and another cluster of any size $i$. These two terms are those present in the irreversible model [21]. The third term corresponds to the resetting of an aggregate of size $k$ to $k$ monomers at rate $r$, and the last term expresses the contribution of the resulting monomers to the concentration $c_1$. As the dissociation of aggregates into monomers is the only dissociation channel we consider, this term can be added to the master equation for all values of $k$, with a factor of $\delta_{k1}$. For $k = 1$, the contribution of the resetting processing from $c_1$ to the time derivative reads $(-rc_1 + r \times 1 \times c_1) = 0$, which is consistent because the resetting of an aggregate of size 1 leaves it unchanged.

Let us denote by $C$ the generating function of the densities of aggregates, and by $N$ the total density of aggregates:

$$C(t, z) := \sum_{k \geq 1} c_k(t) z^k,$$

$$N(t) := \sum_{k \geq 1} c_k(t) = C(t, 1).$$

The generating function of the densities not only allows to read off the coefficients $(c_k(t))_{k \geq 1}$ from a series expansion, but it also gives access to the moments of the distribution of densities, by taking derivatives w.r.t. the variable $z$ at the value $z = 1$. These moments give physical insights into the distribution of mass (for instance the ratio of the moment of order 1 to the moment of order zero, $N(t)^{-1} \sum_k k c_k(t)$ is the average mass of aggregates at time $t$).

For our purposes it is enough to restrict the variable $z$ to $[0, 1]$. Both aggregation and resetting (Eqs 1, 2) conserve mass. The total mass density is therefore a constant, as in the model without resetting. Let us denote it by $M$:

$$M := \sum_{k \geq 1} k c_k(t).$$

The master equations for aggregates of fixed size (Eq. 5) induce the following master equation for the generating function:

$$\frac{\partial C(t, z)}{\partial t} = C(t, z)^2 - (2N(t) + r)C(t, z) + rMz.$$

Setting the total mass density $M$ to unity is equivalent to picking a unit of volume, just as setting the rate of aggregation to 2 is equivalent to rescaling time:

$$M := 1.$$  

**Monomer-only initial conditions.** All the equations so far are independent of the initial conditions. For definiteness we can consider the monomer-only initial conditions, where all the aggregates have unit size, with a unit total mass density:

$$c_k(0) = \delta_{k1}.$$  

We will use these boundary conditions in Section 3.2, but eventually we will consider more general initial configurations of densities.
Figure 1: Total density of aggregates as a function of time. The plot corresponds to monomer-only initial conditions ($N(0) = 1$). The large-time limit is a growing function of the resetting rate.

3 Solution of the master equation

3.1 Total density of clusters

The evolution equation for the total density of clusters is obtained by substituting 1 to $z$ in Eq. (8):

$$\frac{dN}{dt} = -N^2 - rN + r,$$

where we used the value of the mass concentration defined in Eq. (9). The r.h.s. is quadratic in $N$, the two roots have opposite signs, let us denote by $N_-$ and $N_+$. The long-time limit $N(\infty)$ of the total density is equal to the positive root:

$$N(\infty) = N_+ := \frac{r}{2} \left( -1 + \sqrt{1 + \frac{4}{r}} \right), \quad N_- := \frac{r}{2} \left( -1 - \sqrt{1 + \frac{4}{r}} \right).$$

The value of $N(\infty)$ depends on the resetting rate. At low resetting rate it is close to zero (equivalent to $\sqrt{r}$), and at large resetting rate it is close to 1 (it grows towards 1 when the resetting rate goes to infinity, in which limit the resetting process destroys the aggregation process).

With these notations we can rewrite Eq. (11) as

$$-1 = \frac{1}{(N(t) - N_+)(N(t) - N_-)} \frac{dN}{dt} = \frac{1}{N_+ - N_-} \left( -\frac{1}{N - N_-} + \frac{1}{N - N_+} \right) \frac{dN}{dt},$$

(13)
Figure 2: Stationary density of aggregates $c_k$ (of fixed size $k$) as a function of the resetting rate $r$. (a) The maximum of each plot is at $r_k^* = 2/(k - 1)$. (b) A log-log plot shows that the steady-state density at large resetting rate decays like a power of $r$. 
Figure 3: Optimal density of aggregates $c_k(r_k^*)$ as a function of the size $k$ of the aggregate. The large-$k$ approximation is given by $\sqrt{2/(e\pi)}k^{-2}$.

The aggregation process tends to decrease the total density of aggregates, while the resetting process increases it by turning large clusters into monomers. If the total density of aggregates starts with $N(0) > N_+$ (as in the monomer-only initial conditions), aggregation drives the density of aggregates towards lower values at the beginning of the process (in the case of monomer-only boundary conditions, aggregation is the only phenomenon that can take place at the beginning of the process). If the total density of clusters starts from below $N_+$, resetting drives the total density of clusters towards lower values at the beginning of the process (indeed, in the case of initial conditions with only polymers of large size, aggregation decreases the total density at a slow rate, compared to the rate of increase contributed by the decay of large polymers into monomers).

Moreover, the time-derivative of the total density can only be zero if the density equals $N_+$ (because $N_-$ is negative). If the process starts at $N(0) > N_+$, the density cannot go below $N_+$ because it would need to reach a minimum before approaching the asymptotic value $N_+$. If the process starts at $N(0) < N_+$, the density cannot go above $N_+$ because it would need to reach a maximum. The quantity $N(t) - N_+$ therefore has constant sign. Moreover the quantities $N(t) - N_-$ and $N(0) - N_-$ are both positive because $N_-$ is negative. Integrating Eq. (14) between time 0 and time $t$ therefore yields:

$$- (N_+ - N_-) = \frac{d}{dt} \left( - \log |N(t) - N_-| + \log |N(t) - N_+| \right).$$

(14)

$$- \sqrt{r(r + 4)}t = \log \left( \frac{(N(t) - N_+)(N(0) - N_-)}{(N(t) - N_-)(N(0) - N_+)} \right),$$

(15)
from which obtain the exponential convergence of the total density of cluster to the steady state value:

\[ N(t) = \frac{N_+(N(0) - N_-) - N_-(N(0) - N_+)}{N(0) - N_- - (N(0) - N_+)e^{-\sqrt{r(r+4)}t}}. \]  

(16)

The total density obtained in Eq. (16) holds for any initial distribution of cluster sizes with a unit mass concentration. The dependence on the initial condition is entirely contained in the initial density of clusters \( N(0) \). This result is plotted on Fig. 1 for monomer-only initial conditions.

### 3.2 Generating function for monomer-only initial conditions

Going back to Eq. 8 and introducing the new function

\[ D(t,z) = C(t,z) - N(t), \]  

(17)

we obtain a non-linear differential equation in time:

\[ \frac{\partial D}{\partial t} = D^2 - rD + r(z - 1). \]  

(18)

The last term does not depend on the variable \( t \) and can therefore be treated as a constant. Let us denote by \( X(z) \) a root (we will specify which one later) of the quadratic equation on the r.h.s.:

\[ X(z)^2 - rX(z) + r(z - 1) = 0. \]  

(19)

We can convert Eq. (18) into a Bernoulli equation by changing unknown from \( D \) to \( F \) as follows:

\[ D(t,z) =: F(t,z) + X(z), \]  

(20)

\[ \frac{\partial F}{\partial t} = F^2 + (2X - r)F. \]  

(21)

Changing function again through the definition

\[ G(t,z) := \frac{1}{F(z,t)} \]  

(22)

and dividing Eq. (21) by \( F^2 \) yields

\[ \frac{\partial G}{\partial t}(t,z) = (r - 2X(z))G(t,z) - 1. \]  

(23)

Solving this first-order ODE in \( t \) involves a \( z \)-dependent integration constant, denoted by \( Y(z) \), such that

\[ G(t,z) = Y(z)e^{(r - 2X(z))t} + \frac{1}{r - 2X(z)}. \]  

(24)

The generating function is therefore expressed (using Eqs (22,20,17)) as

\[ C(t,z) = N(t) + X(z) + \frac{r - 2X(z)}{(r - 2X(z))Y(z)e^{(r - 2X(z))t} + 1}. \]  

(25)
For the generating function to have a finite limit at large time, we must pick the negative root of Eq. (19):

\[ X(z) := \frac{1}{2} \left( r - \sqrt{r^2 + 4r(1 - z)} \right). \] (26)

We notice that \( X(z) \) does not depend on the choice of initial conditions. The integration constant introduced in Eq. (24) can be traded for:

\[ K(z) := (r - 2X(z))Y(z). \] (27)

Imposing the monomer-only initial condition of Eq. (10) yields

\[ \forall z \in [0, 1], \quad \mathcal{C}(0, z) = z, \]

hence

\[ z = 1 + X(z) + \frac{r - 2X(z)}{K(z) + 1}. \] (28)

Hence, using Eq. (26), we obtain

\[ K(z) = -1 + \frac{2\sqrt{r^2 + 4r(1 - z)}}{2(z - 1) - r + \sqrt{r^2 + 4r(1 - z)}}. \] (29)

We notice that \(|K(z)| \) goes to infinity when \( z \) goes to 1. Moreover, at all times \( C(t, 1) = N(t) \), hence Eq. (25) reduces to

\[ X(1) + \frac{r - 2X(1)}{K(1)e^{(r - 2X(1))t} + 1} = 0, \] (30)

which is consistent since \( X(1) = 0 \) and \(|K(1)| = \infty\).

Rearranging Eqs (27, 25) we obtain the generating function of concentrations for monomer-only initial conditions:

\[ \mathcal{C}(t, z) = N(t) + \frac{1}{2} \left( r - \sqrt{r^2 + 4r(1 - z)} \right) + \frac{\sqrt{r^2 + 4r(1 - z)}e^{-\sqrt{r^2 + 4r(1 - z)t}}}{-1 + \frac{2\sqrt{r^2 + 4r(1 - z)}}{2(z - 1) - r + \sqrt{r^2 + 4r(1 - z)}} + e^{\sqrt{r^2 + 4r(1 - z)t}}}. \] (31)

4 Stationary state

4.1 Stationary density profile as a function of the resetting rate

The large-time limit of the generating function reads

\[ \mathcal{C}(\infty, z) = N(\infty) + X(z) = \frac{r}{2} \sqrt{1 + \frac{4}{r}} - \frac{1}{2} \sqrt{r^2 + 4r(1 - z)}. \] (32)

As a check, we can solve directly the equation satisfied by the steady state \( \mathcal{C}^{\text{stat}} \), which is obtained by putting time-derivatives to zero in the master equation (Eq. [8]):

\[ \mathcal{C}^{\text{stat}}(z)^2 - (2N(\infty) + r)\mathcal{C}^{\text{stat}}(z) + rz = 0. \] (33)
Using the fact that $C_{\text{stat}}(0) = 0$ selects the solution

$$C_{\text{stat}}(z) = \frac{\sqrt{r(r+4)}}{2} \left( 1 - \sqrt{1 - \frac{4z}{r+4}} \right), \quad (34)$$

which is indeed equal to the large-time limit of the generating function $C(\infty, z)$, obtained in Eq. [32]. Moreover, this state is independent of the initial conditions, because $N(\infty)$ and $X(z)$ are. At large time the system forgets its initial conditions, because they only enter the expression of the generating function through the quantity we denoted by $Y(z)$ in Eq. (25), whose contribution is exponentially suppressed at large time.

Expanding in powers of $z$ yields the expression of the steady-state density $c_k(\infty)$ of the clusters of size $k$. Indeed, using $\Gamma(1/2) = \sqrt{\pi}$, we may substitute $4z/(r+4)$ to $s$ in the expansion

$$\sqrt{1 - s} = 1 - \sum_{k \geq 1} \frac{\Gamma(k - \frac{1}{2})}{\sqrt{\pi} \Gamma(k+1)} s^k, \quad (35)$$

to obtain

$$C_{\text{stat}}(z) = \sum_{k \geq 1} c_k(\infty) z^k, \quad c_k(\infty) = \frac{\sqrt{r(r+4)}}{2\sqrt{\pi} \Gamma(k+1)} \frac{\Gamma(k - \frac{1}{2})}{2} \left( 1 + \frac{r}{4} \right)^{-k}. \quad (36)$$

In the original time denoted by $\tau$, consider a rescaling of the rate of aggregation $K$ of aggregation by a factor of $\alpha$ (at fixed resetting rate $\rho$, in the notations of Eq. (3)). Because of the redinition of time, we are led to the same equations of motion as before, with primed symbols for time and resetting rate $t'$ and $r'$ substituted to $t$ and $r$ respectively:

$$K' := \alpha K, \quad t' = K' \tau = \alpha t, \quad r' = \rho \times \frac{\tau}{t'} = \frac{2\rho}{K'} = \frac{1}{\alpha K}. \quad (37)$$

The effect on the equation of motion is therefore identical to a rescaling of the rate $r$ by $\alpha^{-1}$. For this value of $r'$, we can read off the steady-state concentrations from Eq. (40). The large-time behaviour of the model is therefore captured by the single parameter $r$, instead of the pair $(K, \rho)$. We can therefore reason on the large sizes using the resetting rate only, at fixed aggregation rate. Intuitively, low resetting rates favour large values of $k$, so we can look for a characteristic size $\sigma(r)$, where $\sigma$ is a decreasing function of $r$, such that the stationary concentration of large aggregates assumes the scaling form

$$c_k(\infty) \sim \nu(r) g \left( \frac{k}{\sigma(r)} \right), \quad (38)$$

where $g$ is a scaling function to be determined, and the prefactor $\nu$ ensures the conservation of mass, $\sum_k k c_k(\infty) = 1$. 

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Inspection of the exact result of Eq. (36) proves that we only need an equivalent at large $k$ of the quotient of the two values of the Gamma function to read off the scaling form. Using the equivalent

$$\Gamma(k - 1/2) \sim \Gamma(k + 1) k^{-3/2}$$

yields the following equivalent for the stationary concentration of aggregates of large size:

$$c_k(\infty) \sim \frac{\sqrt{r(r + 4)}}{2\sqrt{\pi}} k^{-\frac{3}{2}} \left(\frac{4}{r + 4}\right)^k.$$ (40)

The concentration $c_k(\infty)$ therefore assumes a scaling form, with the gamma distribution of parameters $-1/2$ and 1:

$$c_k(\infty) \sim \frac{\sqrt{r(r + 4)}}{2\sqrt{\pi}} \left(\log \left(1 + \frac{r}{4}\right)\right)^{\frac{3}{2}} g \left(\frac{k}{\sigma(r)}\right),$$ (41)

with $g(x) := x^{-\frac{3}{2}} e^{-x}$, and $\sigma(r) := \left(\log \left(1 + \frac{r}{4}\right)\right)^{-1}$.

### 4.2 Size-dependent optimal resetting rate

For any value $k$ of the cluster size, the steady-state density depends on the resetting rate through the function

$$\varphi_k(r) := \sqrt{r(r + 4)} \left(\frac{4}{r + 4}\right)^k.$$ (42)

For any size $k > 1$, the steady-state density therefore goes to zero in the limit of large resetting (and it goes to zero as $2\sqrt{r}$ in the limit of small $r$). There is therefore a value of the resetting rate that maximises the steady-state density at cluster size $k$ (except for $k = 1$, as the density of monomers is maximised in the limit of infinite resetting rate, which destroys aggregation altogether). Calculating the derivative of $\varphi_k$ yields the unique optimal value $r_k^*$ of the resetting rate:

$$\frac{1}{2r_k^*} = \left(\frac{k - \frac{1}{2}}{r_k^* + 4}\right) \frac{1}{r_k^* + 4},$$ (43)

$$r_k^* = \frac{2}{k - 1}, \quad \text{for } k > 1.$$ (44)

Substituting this optimal value to the resetting rate in Eq. (36), the maximum value of the density of aggregates of size $k$ therefore reads

$$c_k^*(\infty) = \frac{\Gamma \left(\frac{k - 1}{2}\right)}{\sqrt{\pi}\Gamma(k + 1)} \frac{2}{k - 1} \left(\frac{1}{1 + \frac{1}{2(k - 1)}}\right)^{-k}.$$ (45)

The optimal value of resetting goes to zero at large sizes, which is intuitive as rare resetting events favour the formation of large aggregates. The steady-state density (normalised by the maximum $c_k^*(\infty)$) is plotted for a few values of $k$ on Fig. (2(a)). At fixed resetting rate, the steady-state density goes to zero in the limit of large size, because large clusters are penalised by the resetting
process. Moreover, the density goes to zero at large resetting rate and fixed size as \( r^{-(k-1)} \), which is illustrated by the log-log plot on Fig. (2(b)).

Moreover, Eq. (39) yields the large-\( k \) equivalent of the optimal value

\[
c_k^* (\infty) \sim \sqrt{\frac{2}{\pi e}} \frac{1}{k^2}.
\]

The optimal values of the densities are plotted as a function of the size of the aggregate of Fig. (3), where the large-size equivalent is also plotted. To estimate how fast the large-size regime is reached, we can work out the leading correction to the asymptotic behaviour. We need the next term in Eq. (39). Let us start from the following large-\( m \) asymptotic expansion [35]:

\[
\frac{\Gamma (m + \frac{1}{2})}{\Gamma (m)} = \sqrt{m} \left( 1 - \frac{1}{8m} + O \left( m^{-2} \right) \right),
\]

from which we can work out a correction of order \( k^{-1} \) to Eq. (46). Indeed, we need the large-\( k \) behaviour of

\[
\frac{\Gamma \left( k - \frac{1}{2} \right)}{\Gamma (k + 1)} = \frac{\Gamma \left( k - \frac{1}{2} \right)}{k(k-1)\Gamma (k-1)}
\]

\[
= \frac{1}{k^2} \frac{1}{1 - \frac{1}{k}} \sqrt{k-1} \left( 1 - \frac{1}{8k} + O \left( k^{-2} \right) \right)
\]

\[
= k^{-3/2} \left( 1 + \frac{1}{k} + O(k^{-2}) \right) \left( 1 - \frac{1}{2k} + O(k^{-2}) \right) \left( 1 - \frac{1}{8k} + O \left( k^{-2} \right) \right)
\]

\[
= k^{-3/2} \left( 1 + \frac{3}{8k} + O(k^{-2}) \right),
\]

where in the second line we used Eq. (47) with \( m = k - 1 \).

Another correction of order \( k^{-1} \) comes from

\[
\left( 1 + \frac{1}{2(k-1)} \right)^{-k} = \exp \left( -k \log \left( 1 + \frac{1}{2(k-1)} \right) \right)
\]

\[
= \exp \left( -k \log \left( 1 + \frac{1}{2k} + \frac{1}{2k^2} + O(k^{-3}) \right) \right)
\]

\[
= \exp \left( -k \left( \frac{1}{2k} + \frac{1}{2k^2} - \frac{1}{8k^2} + O(k^{-3}) \right) \right)
\]

\[
= \frac{1}{\sqrt{e}} \left( 1 - \frac{3}{8k} + O(k^{-2}) \right).
\]

Going back to Eq. (45), we collect the leading corrections as

\[
c_k^* (\infty) = \sqrt{\frac{2}{\pi e}} \frac{1}{k^2} \left( 1 + \frac{1}{2k} + O(k^{-2}) \right).
\]

The asymptotic value of Eq. (46) is therefore approached from above in the limit of large size.
4.3 Low resetting rate and consistency with the Smoluchowski model

The large-time limit of our model presents a non-zero total density, whereas the total density in the Smoluchowski model goes to zero at large time. Indeed, denoting the ordinary system without resetting with the symbol $(r = 0)$ in an exponent, we have in the case of monomer-only initial conditions

$$N^{(r=0)}(t) = \frac{1}{1 + t}, \tag{51}$$

which is the solution of Eq. (11) for $r = 0$. Of course the master equation of the Smoluchowski model of aggregation is recovered if we set the resetting rate to zero in our master equation (Eq. (8)), but working out the low resetting-rate limit of our results should yield the densities predicted by the Smoluchowski model.

Let us first address the total density of aggregates. Of course, the asymptotic value $N_+$ goes to zero when the resetting rate $r$ goes to zero, but the time-dependence displayed in Eq. (51) should be recovered at low resetting in a transitory regime. We can define this regime by fixing time, and looking for a range of resetting rates that ensures that the exponential terms in the expression of the total density are close to 1. At fixed time $t$, the density expressed in Eq. (16) is in a transitory regime if $r$ is low enough for the quantity $\sqrt{r(4 + r)}t$ to be close to zero. At low $r$, the quantity is equivalent to $2\sqrt{rt}$, and the low-resetting regime is defined by

$$r \ll \frac{1}{4t^2}. \tag{52}$$

In this limit, $N_+ \simeq \sqrt{r}$ and $N_- \simeq -\sqrt{r}$. Equivalently, we can reason at fixed resetting rate, and declare that times much smaller than $(2\sqrt{r})^{-1}$ constitute a transitory regime, in which the system is in the situation of the ordinary model of aggregation. Both reasonings lead to the approximation

$$N(t) \simeq \frac{N_+ - N_-}{N_+ - N_- + (1 - N_-)2\sqrt{rt}}, \tag{53}$$

Indeed, at fixed $t$ and low $r$, the term $N_-(1 - N_-)\sqrt{rt}$ (present in the numerator of Eq. (16)) is equivalent to $2rt$, and it therefore subdominant, as all the other terms in the fraction are of order $\sqrt{r}$. Keeping only the dominant terms yields a low-resetting time-independent limit

$$N(t) \simeq \frac{2\sqrt{r}}{2\sqrt{r} + 2\sqrt{rt}} = \frac{1}{1 + t}, \tag{54}$$

which as expected coincides with $N^{(r=0)}(t)$.

The same approach can be taken to study the generating function at low resetting rate. Let us fix some time $t$, and some $z$ in $[0, 1]$. The resetting rate $r$ is deemed small for these particular values if the exponential relaxation of the generating function has not taken place yet. This yields the condition

$$2\sqrt{(1 - z)rt} \ll 1, \quad \text{i.e.} \quad r \ll \frac{1}{4(1 - z)t^2}. \tag{55}$$
In this limit, the combination \( r - \sqrt{r^2 + 4r(1 - z)} \) is equivalent to \( 2\sqrt{r(1 - z)} \). Both the numerator and the denominator in the last term in Eq. (31) are of order \( \sqrt{r} \) (where the exponential factor contributes a factor of 1 in the numerator and the term \( 2\sqrt{r(1 - z)}t \) by Taylor expansion in the denominator), so their quotient has a finite limit:

\[
C(t, z) \simeq N(r=0)(t) + \frac{2\sqrt{r(1 - z)}}{\frac{4\sqrt{r(1 - z)}}{2(z - 1)} - 2\sqrt{r(1 - z)}t} = \frac{1}{1 + t} + \frac{1}{(z - 1)^{-1} - t} = \frac{1}{1 + t} \left( \frac{z}{1 - (z - 1)t} \right),
\]

which is indeed the generating function of the Smoluchowski model, which satisfies the master equation at \( r = 0 \).

5 Typical size of aggregates for polymer-only boundary conditions

Consider slightly more general initial conditions in which the total mass density \( M = 1 \) results from polymers of fixed size \( A > 1 \):

\[
c_k(0) = \frac{1}{A} \delta_{kA}.
\]

We keep the same unit of time and volume, so the master equation is unchanged. The total cluster density is still given by Eq. (16), with \( N(0) = A^{-1} \). The only modification in the solution comes from the initial condition on the generating function (still denoted by \( C \)):

\[
C(0, z) = \frac{z^A}{A},
\]

which enters Eq. (28). The generating function therefore reads

\[
C(t, z) = N(t) + \frac{1}{2} \left( r - \sqrt{r^2 + 4r(1 - z)} \right) + \frac{\sqrt{r^2 + 4r(1 - z)}e^{-\sqrt{r^2 + 4r(1 - z)}t}}{-1 + \frac{2\sqrt{r^2 + 4r(1 - z)}}{z^A - 1 - r + \sqrt{r^2 + 4r(1 - z)}} + e^{-\sqrt{r^2 + 4r(1 - z)}t}}.
\]

The second moment \( M_2(t) \) of the family of densities \( c_k(t) \) gives an order of magnitude of the square of the typical mass of the aggregates at time \( t \). Using the exact expression of the generating function \( C(t, z) \), we can obtain this second moment from a Taylor expansion around \( z = 1 \) (using \( \sum_{k \geq 1} kc_k(t) = 1 \) from the monomer-only boundary condition):

\[
M_2(t) = \sum_{k \geq 1} k^2 c_k(t) = 1 + \frac{\partial^2 C}{\partial z^2}(t, 1).
\]
With the notations
\[ \xi(z) := \sqrt{r^2 + 4r(1 - z)}, \]
\[ \tau(z) := \frac{2}{A}(z^A - 1) - r + \xi(z), \]  \tag{61} \]
the generating function reads for initial conditions consisting of polymers of size \( A \):
\[ C(t, z) = N(t) + \frac{1}{2}(r - \xi(z)) + \frac{\xi(z)\tau(z)}{(-\tau(z) + \xi(z))} e^{\xi(z)t + \tau(z)}. \]  \tag{62} \]

We have the following expansions around \( z = 1 \):
\[ \xi(1 - h) = r \left( 1 + \frac{2}{r}h - \frac{2}{r^2}h^2 + o(h^2) \right) \]
\[ \tau(1 - h) = (A - 1)h^2 - \frac{2}{r}h^2 + o(h^2) = h^2 \left( -\frac{2}{r} + A - 1 \right) + o(h^2). \]  \tag{63} \]
The numerator in the last term of Eq. (62) is therefore \( O(h^2) \), and equivalent to \( \xi(1)\tau(1 - h) \) at small \( h \). The denominator is equivalent to \( \xi(1) \exp(\xi(1)t) \), with \( \xi(1) = r \). The term of order \( h^2 \) in the expansion of Eq. (62) yields:
\[ \frac{1}{2} \frac{\partial^2 C}{\partial z^2}(t, 1) = \frac{1}{r} + \left( -\frac{2}{r} + A - 1 \right) e^{-rt}. \]  \tag{64} \]
The second moment therefore reads
\[ M_2(t) = \sum_{k \geq 1} k^2 c_k(t) = 1 + \frac{2}{r} + 2 \left( -\frac{2}{r} + A - 1 \right) e^{-rt}. \]  \tag{65} \]
The typical size of the aggregates converges exponentially to the steady-state value, at the resetting rate. Moreover, if the resetting rate is set to the special value
\[ r_A := \frac{2}{A - 1}, \]  \tag{66} \]
the second moment is kept constant (at \( A \)).

6 Discussion

In this paper we have obtained the generating function of the aggregation model with constant kernel subjected to resetting at a constant rate (in the sense that aggregates of any size explode into monomers at independent Poisson-distributed times). We have solved the master equation instead of relying on renewal equations. This approach is natural in many-body systems whose constituents are reset independently, and has already been used for local resetting in [17][18].
The steady state of the model is independent of the initial conditions and contains aggregates of all sizes, whose average density is a decreasing function of the size of the aggregate. Moreover, this density assumes a scaling form, in which the size of the aggregate is rescaled according to the resetting rate. In the absence of resetting, the aggregation process is irreversible and the concentration of any aggregate of fixed size goes to zero, but the typical size of the aggregates grows systematically with time, and scaling occurs because a change in time scale preserves the mass distribution (provided the mass is rescaled by a time-dependent factor). When the model is subjected to resetting, the large-time limit of the distribution of concentration is invariant if the size is rescaled by a rate-dependent factor. Indeed low rates of resetting probe the large-time and large-size behaviour of the aggregation process.

The density of aggregates of fixed size in the non-equilibrium steady state is maximised if the resetting rate equals the inverse of the size (multiplied by the rate of the aggregation process). The optimal resetting rate is higher for aggregates of low size, and of the same order of magnitude as the rate of the aggregation process (which is set to 2 in our calculations by picking the unit of time). The assumption of good mixing is therefore valid for in model under resetting at optimal values, as long as it is valid for the model without resetting. In the large-size limit, the optimal resetting rate goes to zero, which is intuitive as a lower resetting rate is more favourable to large aggregates.

Moreover, the generating function has been used to compute the second moment of the densities as a function of time. For initial conditions consisting of polymers of fixed size (larger than 1) and generic values of the resetting rate, the second moment goes exponentially to the steady state, and the rate of convergence is equal to the resetting rate. However, the second moment can be constant, for a unique value of the resetting rate.

The constant kernel provides a workbench for modelling aggregation, as its simplicity allows to display remarkable properties of the phenomenon, such as scaling. We have seen that it serves the same purpose when subjected to resetting. Models of aggregation with size-dependent kernels, such as the sum and product kernels, have been proposed and solved [36][38]. It would be interesting to subject them to resetting. Moreover, the resetting prescription itself could be generalised to become size-dependent.

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