Criteria for commutative factorization of a class of algebraic matrices

Andrey V. Shanin¹, Eugeny M. Doubravsky²
Russia, 119899, Moscow, Moscow State University,
¹ Department of Physics, ²Department of Mechanics and Mathematics

May 5, 2014

Abstract

The problem of matrix factorization motivated by diffraction or elasticity is studied. A powerful tool for analyzing its solutions is introduced, namely analytical continuation formulae are derived. Necessary condition for commutative factorization is found for a class of “balanced” matrices. Together with Moiseyev’s method and Hurd’s idea, this gives a description of the class of commutatively solvable matrices. As a result, a simple analytical procedure is described, providing an answer, whether a given matrix is commutatively factorizable or not.

Keywords: matrix factorization, Wiener-Hopf method, Riemann surfaces

1 Introduction

A matrix factorization problem (i.e. a problem of finding the factors $Q^+$ and $Q^-$ providing the decomposition (1) for a known matrix $G(k)$) is usually motivated by an elasticity or a wave diffraction problem. Typically its formulation does not contain a requirement of commutative factorization. However, the possibility to perform a commutative factorization is usually studied carefully, since all known factorization methods are based on ideas connected with commutativity.

There are two main methods for commutative factorization. The first one is based on the idea by Heins [1] who proposed to split the logarithm of matrix $G$ additively. Further progress in this direction is connected with the names of Chebotarev [2] and Khrapkov [3]. In the paper by Khrapkov an explicit form of factorization for a certain class of matrices $2 \times 2$ was found.

However, sometimes Khrapkov’s formula leads to exponential growth of the factors at infinity, and this is not acceptable for physical applications. To suppress this growth in some cases, a special technique was proposed by Daniele [4] and Hurd & Lüneburg [5], who thus enlarged the class of explicitly factorizable matrices. A factorization to some other matrices can be obtained if the analytical continuation of the matrix is studied (see
Matrices with dimension more than $2 \times 2$ have been investigated by Lukyanov [8].

Another technique of commutative matrix factorization is based on diagonalization of $G$ and studying the eigenvalues as a multi-valued function. This approach has been developed by Cercignani [9], Lüneburg [10], Moiseyev [11], Meister & Penzel [12] and Antipov & Silvestrov [13]. It is based mainly on the research of Zverovich [14], who described a method for solving scalar boundary-value problem on Riemann surfaces.

It was noticed by Hurd [15] and Daniele [16] that all known methods to factorize a matrix are connected with the fact that the matrix commutes with a polynomial matrix.

A question of finding a rational factor enabling to perform a commutative factorization for matrices $2 \times 2$ was discussed by Williams [17]. Also this question was studied in details by Ehrhardt and Speck [18].

Beside the exact methods, an interesting attempt to solve a matrix factorization problem approximately was proposed by Abrahams [19].

Each of the works having either Kharapkov’s or Moiseyev’s context was dedicated to a distinct class of matrices, i.e. the starting point of such a work was a phrase like “consider a matrix having the following form …”. Sometimes, however, it is not easy to say whether a given matrix can be reduced to one of the known classes by algebraic manipulations. That is why, a separate and interesting issue is a classification of matrices with respect to factorization, i.e. finding a criteria, for example, of the possibility of commutative factorization. Such a theory is known for matrices $2 \times 2$ [2], however the necessary condition for commutative factorization has been found in an “Ansatz” form, i.e. a matrix should have a specific representation including some entire (polynomial) matrices and some arbitrary functions as coefficients.

A necessary condition for commutative factorization for matrices of arbitrary dimension was studied by Jones in [20], however the author restricted himself to the case of matrices having distinct eigenvectors everywhere, while a typical matrix emerging in diffraction theory has branch points, i.e. it has distinct eigenvectors almost everywhere.

The current paper is inspired mainly by the works of Antipov et.al. and the bright work of Hurd. The idea is to take an algebraic matrix $G(k)$ and study the properties of the factors $Q^+$ and $Q^-$ on their Riemann surfaces following a priori from the decomposition (1), i.e. without constructing the factors explicitly. We found that the decomposition (1) taken together with the regularity conditions imposed on the factors define a unique Riemann surface of $Q^+$ or $Q^-$. Moreover, the values of, for instance, $Q^+$ taken on different sheets are connected by simple algebraic relations. Thus, in Section 2 we obtain analytical continuation formulae. The relation obtained by Hurd is a particular case of such formulae.

The problem for $Q^+$ reminds the functional problem for Abelian integrals, but while the value of an Abelian integral is increased by a constant when the argument is carried along a certain contour, the value of $Q^+$ is multiplied by a non-constant bypass matrix.

In Section 3 we study the question of the possibility of commutative matrix factorization. To describe the matrices, for which the necessary condition for commutative factorization is fulfilled, we introduce the class of branch-commutative matrices $G(k)$, i.e. the matrices whose values on different sheets of their Riemann surface commute. Branch-commutativeness is a much weaker condition than commutativeness introduced by Cheb-
otarev [2], since we demand the commutation only of matrices having the same affix \( k \). We demonstrate that branch-commutative matrices can be factorized by Moiseyev’s technique [11], so the branch-commutativeness is necessary and sufficient condition for commutative factorization.

In Section 4 we study the Hurd’s idea in our terms. Bypass matrices are introduced. They are the matrices connecting the values of unknown function \( Q^+ \) on different sheets. If all bypass matrices commute, we call matrix \( G \) bypass-commutative. The class of bypass-commutative matrices is wider than the class of branch-commutative, for example all algebraic matrices, to which Hurd’s method can be applied are bypass-commutative. We show that for any bypass-commutative matrix \( G \) a rational factor can be found, transforming \( G \) into a branch commutative matrix.

2 Analytical continuation of the factors \( Q^\pm \)

2.1 Problem under consideration

The initial problem of matrix factorization is as follows:

**Problem 1** For a matrix \( G(k) \) defined in a narrow strip along the real axis \((-\epsilon < \text{Im}[k] < \epsilon)\) find matrices \( Q^+(k), Q^-(k) \) analytical (maybe except some isolated poles), continuous, having algebraic growth in the upper (\( \text{Im}[k] > -\epsilon \)) and lower (\( \text{Im}[k] < \epsilon \)) half-planes, respectively, and satisfying the equation

\[
G(k) = Q^+(k)Q^-(k). \quad (1)
\]

Algebraic growth hereafter means that there exists a number \( l \), such that all elements of corresponding matrices grow at the corresponding half-plane no faster than \( |k|^l \). We cannot expect that the elements will grow exactly as some powers of \( k \), since the solutions can have logarithmic behaviour.

We assume that \( G(k) \) is an algebraic function, as it happens in many applications. Thus, the function itself and the relation (1) can be continued from the strip \(-\epsilon < \text{Im}[k] < \epsilon \).

Besides, we assume everywhere that the determinant of \( G \) is not equal to zero identically.

2.2 Notations for bypasses

Let \( \mathcal{R}_G \) be the Riemann surface of matrix \( G(k) \). Below we shall call \( k \) an affix of a point \((k, G(k)) \in \mathcal{R}_G\).

Let branch points of \( G(k) \) have affixes \( \tau^+_m \) and \( \tau^-_m \), where \( \text{Im}[\tau^+_m] > 0 \) and \( \text{Im}[\tau^-_m] < 0 \). Make \( G \) single-valued on \( \mathbb{C} \) by performing cuts going from branch points to infinity. The cuts can be chosen as \( \gamma^+_m = (\tau^+_m, +i\infty) \) and \( \gamma^-_m = (\tau^-_m, -i\infty) \). It is important that the cuts should not cross the real axis and each other. As a result, the surface \( \mathcal{R}_G \) becomes split.
into several sheets. There is a special sheet of $\mathcal{R}_G$, on which equation (1) is assumed to be valid. Name this sheet a **physical sheet**.

Here and below, the structure of Riemann surface is displayed by graphical diagrams. Horizontal lines correspond to the sheets, nodes correspond to branch points, and vertical lines link sheets, which are connected at a certain branch point.

In some artificial cases an affix may correspond to several branch points having different orders. Define for each affix its order $n_m^\pm$, which is the least common multiple of all orders of branch points with corresponding affix. For example, in Fig. 1 a fragment of a Riemann surface is shown. Parameter $n$ for the affix $k_0$ is equal to 6.

![Figure 1: Order of an affix](image)

Introduce a notation for the sheets of $\mathcal{R}_G$. Note that later the same notation will be used for the sheets of the Riemann surfaces of $Q^\pm$. Each point of the surface will be denoted by $(k)\{w\}$, where $k$ is an affix, and $\{w\}$ is a word describing the path, along which the argument $k$ should be carried from physical sheet to a selected sheet. The structure of this word is explained below.

Denote bypasses about points $\tau_i^+$ in positive direction by letters $a_i$ and bypasses about points $\tau_i^-$ in positive direction by $b_i$ (Fig. 2).

![Figure 2: Notation for bypasses](image)

A series of consecutive bypasses will be denoted by a word of letters $a_i$ and $b_i$. The word must be read from left to right, i.e. the first performed bypass corresponds to the
Define the composition of words \( w \) and \( v \) as the bypass performed along the way composed of \( w \) and \( v \). The bypass \( w \) is performed first. Denote this composition by \( wv \). Let \( \mathcal{W} \) be the set of all words, and let \( \mathcal{W}_a, \mathcal{W}_b \) be the sets of words composed only of the letters \( a_i \), and only of letters \( b_i \), respectively.

Let \( G(k)\{e\} \) be the value of function \( G(k) \) on the “physical sheet”. Denote by \( G(k)\{\chi \} \) the value of \( G(k) \) on the sheet that can be reached by performing the bypass \( \chi \) starting from the point \( (k, G(k)\{e\}) \).

The set \( \mathcal{W} \) can be considered as a group of words, a subject of combinatorial group theory. Its generators are the letters \( a_i \), \( b_i \), and the relations have the form

\[
\begin{align*}
a_i^{n_i} &= e, & b_i^{-n_i} &= e. 
\end{align*}
\]

As we shall see below, the same relations are valid for the words describing the Riemann surfaces of \( Q^\pm \).

Relations (2) enable one to determine an inverse element for each \( w \in \mathcal{W} \) without introducing new letters for bypasses in negative direction (or without using the symbols \( a_i^{-1} \) and \( b_i^{-1} \)). Using (2), below we assume that for each word \( w \) there exists a word \( w^{-1} \), such that \( ww^{-1} = w^{-1}w = e \).

Let us demonstrate an example of Riemann surfaces for \( G \) and \( Q^+ \). Take matrix \( G \) from Daniele’s paper [4]:

\[
G(k) = \begin{pmatrix} 1 & k_1 - s(k) \\ k_2 - s(k) & k_1 + s(k) \end{pmatrix},
\]

where \( s(k) = \sqrt{k_0^2 - k^2} \); \( k_0, k_1 \) and \( k_2 \) are some complex constants.

In this case the Riemann surface of \( G(k) \) has two sheets and two quadratic branch points, namely \( k = \pm k_0 \). Let be \( \text{Re} [k_0] > 0 \). Let letter \( a \) denote a bypass about \( k_0 \), and letter \( b \) denote a bypass about \( -k_0 \).

The scheme for the Riemann surface of \( G \) is shown in Fig. 3 a. The upper sheet is physical (i.e. it contains the “physical” real axis).

The scheme of \( Q^+ \) corresponding to this problem is shown in Fig. 3 b. The number of sheets is infinite, but all branch points are of second order, and the positive physical half-plane contains no branch points. This structure can be revealed, e.g. from [4].

### 2.3 Truncation operators

Let be \( w = \alpha_1 \alpha_2 \ldots \alpha_n \) where \( \alpha_i \) substitutes an arbitrary single letter. Denote by \( p \) the maximal number, such that the word \( \alpha_1 \alpha_2 \ldots \alpha_p \in \mathcal{W}_a \). Analogically let \( m \) be the maximal number, such that \( \alpha_1 \alpha_2 \ldots \alpha_m \in \mathcal{W}_b \). Obviously, one of this integers is zero, since the first letter of the word is either \( a_j \) or \( b_j \).

Define the truncation operators \( ^+ \) and \( ^- \) by

\[
\begin{align*}
w^+ &= \alpha_{p+1} \alpha_{p+2} \ldots \alpha_n, \\
w^- &= \alpha_{m+1} \alpha_{p+2} \ldots \alpha_n.
\end{align*}
\]
2.4 Formulae of analytical continuation

Consider equation (1). Both right and left sides of this equation are analytic functions in some neighbourhood of the real axis of the physical sheet. Continue $Q^+$ and $Q^-$ analytically to this domain and, further, onto some Riemann surfaces. Continue also the relation (1) onto the Riemann surfaces of $G$, $Q^+$ and $Q^-$. Obviously, the continuation of the relation (1) can be written in the form:

$$Q^+(k)\{w\} Q^-(k)\{w\} = G(k)\{w\}. \quad (4)$$

At this formula (4) has sense only for geometrically fixed bypasses.

Here we are going to find the formulae of analytical continuation for $Q^\pm$, i.e. algebraic relations connecting $Q^\pm(k)\{w\}$ with $Q^\pm(k)\{e\}$.

General formulae of analytical continuation can be written in a recursive form as follows:

**Theorem 1** Let $Q^+(k)$ and $Q^-(k)$ form a solution of Problem [1]. Then the following relations are valid

$$Q^+(w) = G\{w^+\} G^{-1}\{w^+\} Q^+(w^+), \quad (5)$$

$$Q^-(w) = Q^-\{w^+\} G^{-1}\{w^-\} G\{w^+\}. \quad (6)$$

(A dependence on $k$ is implied for all functions in (5), (6)).

The proof is rather straightforward and based on the relations following from the regularity conditions

$$Q^+(w) = Q^+(w^+), \quad Q^-(w) = Q^-\{w^-\} \quad (7)$$
literally denoting that $Q^+$ is analytical at the points $\tau_j^+$, while $Q^-$ is analytical at the points $\tau_j^-$. Let us prove (5) (relation (6) is similar). First, according to (7),

$$Q^+\{w\} = Q^+\{w^+\}. \tag{8}$$

Then, according to (4),

$$Q^+\{w\} = Q^+\{w^+\} = G\{w^+\}(Q^-\{w^+\})^{-1}. \tag{9}$$

According to the second relation of (7),

$$Q^+\{w\} = Q^+\{w^+\} = G\{w^+\}(Q^-\{w^+\})^{-1} = G\{w^+\}(Q^-\{w^+\})^{-1}. \tag{10}$$

Finally, according to (4)

$$(Q^-\{w^+\})^{-1} = (G\{w^+\})^{-1}Q^+\{w^+\} \tag{11}$$

and we get (5).

Note that for any word $w$ there exists some constant $c$, such that $w^{(+-)-c} = e$, therefore formula (5) being repeated several times connects $Q^+\{w\}$ with $Q^+\{e\}$. Analogously, $Q^-\{w\}$ is connected with $Q^-\{e\}$. The coefficients are always products of known matrices.

Analytical continuation in the form (5) has been obtained by Hurd\cite{7} for a particular case of a single bypass. Hurd’s ideas are discussed later in details.

Using analytical continuation we can investigate the structure of Riemann surface of unknown function $Q^+$. For example, the following proposition can be easily proved:

**Proposition 1** Let $G(k)$ be an algebraic matrix, and let the functions $Q^+(k)$ and $Q^-(k)$ form a solution of Problem 1. Then the functions $Q^+(k)$ and $Q^-(k)$ can be analytically continued onto some Riemann surfaces; both functions have branch points only at affixes $\tau_i^\pm$. The order of each branch point is a divisor of corresponding $n_i^\pm$.

A formal proof can be conducted by induction with respect to the length of the word $w$, which is the argument of $Q^+(k)\{w\}$.

Generally, solution of Problem 1 is not unique: for example the behaviour of different solutions at infinity can be different. However, it is easy to prove that all solutions are similar up to a meromorphic matrix factor.

### 3 Necessary condition for commutative matrix factorization

#### 3.1 Necessary condition in the “check-up” form

**Definition 1** Let $G(k)$ be an algebraic matrix, let its branch points have affixes $\tau_i^\pm$ and lie aside from the real axis. Let the sets $W$, $W_a$, and $W_b$ be defined as described above. Riemann surface $R_G$ will be called balanced if for any $w \in W$ there exist words $w_a \in W_a$ and $w_b \in W_b$ such that

$$G(k)\{w_a\} = G(k)\{w_b\} = G(k)\{w\}. \tag{12}$$
An example of a balanced Riemann surface is the Riemann surface of scalar function \( \sqrt{1 + \sqrt{2 + k^2}} \) with an arbitrary choice of the physical sheet. Besides, a surface of an arbitrary matrix function, which is a rational combination of \( k \) and several square roots \( \sqrt{\tau_j^2 - k^2} \), is balanced.

An example of a Riemann surface that is not balanced is the surface of the function \( \sqrt{i + k + \sqrt{-i + k}} \).

**Definition 2** Algebraical matrix \( G(k) \) is called branch-commutative, if for any \( k \) the values of \( G \) on different sheets of its Riemann surface commute, i.e.

\[
[G(k)\{w_1\}, G(k)\{w_2\}] \equiv G(k)\{w_1\}G(k)\{w_2\} - G(k)\{w_2\}G(k)\{w_1\} = 0. \tag{13}
\]

for any different words \( w_1 \) and \( w_2 \).

To illustrate the definition of branch-commutativeness consider a simple example. Take a matrix

\[
G(k) = \begin{pmatrix}
  k & 2k & s(k) \\
  2k & k & -s(k) \\
  -s(k) & s(k) & k
\end{pmatrix}, \quad s(k) = \sqrt{k_0^2 - k^2}. \tag{14}
\]

Let be \( \text{Im}[k_0] > 0 \). There are two letters, \( a \) and \( b \), corresponding to bypasses about \( k_0 \) and \( -k_0 \). The value of \( G \) on the sheet \( \{a\} \) is equal to

\[
G(k)\{a\} = \begin{pmatrix}
  k & 2k & -s(k) \\
  2k & k & s(k) \\
  s(k) & -s(k) & k
\end{pmatrix}.
\]

Since \( \mathcal{R}_G \) has two sheets, to check branch-commutativeness one should check only the identity

\[
[G(k)\{e\}, G(k)\{a\}] = 0, \tag{15}
\]

where \( G(k)\{e\} \) is defined by (14). Simple calculations show that (15) is fulfilled, so (14) is a branch-commutative matrix.

The necessary condition of commutative factorization is given by the following theorem:

**Theorem 2** If a matrix \( G \) having balanced Riemann surface admits commutative factorization

\[
Q^+(k)Q^-(k) = Q^-(k)Q^+(k) = G(k), \tag{16}
\]

then it is a branch-commutative matrix.

**Proof:** The formula of analytical continuation (5) has been derived for right factorization. One can obtain a similar formula for left factorization \( G(k) = Q^-(k)Q^+(k) \):

\[
Q^+\{w\} = Q^+\{w^+\}G^{-1}\{w^+\}G\{w^+\}. \tag{17}
\]

Perform the rest of the proof step by step. Here we mark the statements and make some comments if the statements are not obvious:
1. For any word \( w \quad Q^+\{ w \} Q^-\{ w \} = Q^-\{ w \} Q^+\{ w \} = G\{ w \} \). It is an analytical continuation of (16).

2. For any word \( w \quad \left[ G\{ w \}, (Q^+\{ w \})^{-1} \right] = 0 \). This follows from \( G\{ w \}(Q^+\{ w \})^{-1} = Q^-\{ w \} = (Q^+\{ w \})^{-1}G\{ w \} \).

3. For any word \( w \quad \left[ G\{ w \}, Q^+\{ w \} \right] = 0 \). This can be obtained from the previous point by multiplication by \( G^{-1} \) at left and right.

4. For any \( v \in W_a \quad \left[ G\{ v \}, Q^+\{ e \} \right] = 0 \). This follows from the previous point and (7).

5. For any word \( w \quad \left[ G\{ w \}, Q^+\{ e \} \right] = 0 \). Note that for a matrix with balanced Riemann surface for any word \( w \) there exists a word \( v \in W_a \), such that \( G\{ v \} = G\{ w \} \), and \( Q^+\{ v \} = Q^+\{ e \} \).

6. For any \( v_1 \in W_b, \ v_2 \in W_a \quad \left[ G\{ v_1 v_2 \}, G^{-1}\{ v_2 \} \right] = 0 \). This statement can be obtained by applying left and right analytical continuation formulae to the word \( v_1 v_2 \) and by using the previous point.

7. For any \( v_1 \in W_b, \ v_2 \in W_a \quad \left[ G\{ v_1 v_2 \}, G\{ v_2 \} \right] = 0 \).

8. The statement of the theorem, by noting that for any \( w_1 \) and \( w_2 \) one can find the words \( v_b \in W_b \) and \( v_a \in W_a \), such that \( G\{ v_b v_a \} = G\{ w_1 \}, \ G\{ v_a \} = G\{ w_2 \} \).

Theorem 2 is an important result of the paper. Note that since the number of sheets of \( G \) is finite, the necessary condition can be established by checking a finite number of matrix identities.

### 3.2 Diagonalization and properties of eigenvectors

Let an algebraic (not necessarily branch-commutative) matrix \( G(k) \) have distinct eigenvalues almost everywhere (i.e. on the whole complex plane excluding several points). Represent this matrix in the form

\[
G(k) = M(k) \text{ diag } [\lambda_1, \ldots, \lambda_N] M(k)^{-1}
\]  

(18)

Here matrix \( M(k) \) consists of vector-columns, which are right eigenvectors of \( G \); \( \lambda_1 \ldots \lambda_N \) are corresponding eigenvalues; \( N \) is dimension of \( G \). Normalize the columns of \( M \) by making all elements of the first row of \( M \) equal to 1.

Obviously, for obtaining representation (18) one should first solve the characteristic equation for \( G \), and then find a solution of an inhomogeneous linear system for each eigenvector.

Denote Riemann surface of matrix \( M(k) \) by \( \mathcal{R}_M \). Now we have associated with a matrix \( G \) two Riemann surfaces: \( \mathcal{R}_G \) and \( \mathcal{R}_M \). Typically, say for Khrapkov matrices, \( \mathcal{R}_M \) has a structure very different from \( \mathcal{R}_G \).

A lot of authors studied matrix factorization problems by formulating a functional problem on a Riemann surface. It is important to mention that most of them had in mind
the surface \( \mathcal{R}_M \), not \( \mathcal{R}_G \). Typically, the surface \( \mathcal{R}_M \) is studied in Moiseyev’s context, and \( \mathcal{R}_G \) in Hurd’s one.

Obviously, Riemann surface for the eigenvalues \( \lambda_j(k) \) should contain branch points of both structures, i.e. of \( G \) and of \( M \).

Let \( G(k) \) be a branch-commutative matrix. In this case the set of normalized eigenvectors must be the same on all sheets of \( \mathcal{R}_M \). Therefore, matrix \( M(k) \) possesses an important property: any bypass about branch points leads to a permutation of the columns, i.e. an analytical continuation of each column along a closed contour \( c \) on \( \mathbb{C} \) is some other column of \( M \).

As an example, consider matrix (3), which is branch-commutative. As it was mentioned, it has only two branch points, namely \( \pm k_0 \). The scheme of Riemann surface for this matrix is shown is Fig. 3 a. It is easy to find that matrix \( M \) for this \( G \) is as follows:

\[
M(k) = \begin{pmatrix}
\frac{1}{\sqrt{k_0^2 - k^2 - k_1^2}} & \frac{1}{\sqrt{k_0^2 - k^2 - k_2^2}} \\
\frac{1}{\sqrt{k_0^2 - k^2 - k_1^2}} & \frac{1}{\sqrt{k_0^2 - k^2 - k_2^2}}
\end{pmatrix}.
\] (19)

Matrix \( M \) has four branch points, namely \( \pm \sqrt{k_0^2 - k^2 - k_1^2} \) and \( \pm \sqrt{k_0^2 - k^2 - k_2^2} \). Generally (i.e. if \( k_1 \neq 0 \) and \( k_2 \neq 0 \)) the branch points of \( \mathcal{R}_M \) are different from the branch points of \( \mathcal{R}_G \). The scheme of \( \mathcal{R}_M \) is shown in Fig. 4.

A transition from one sheet of \( \mathcal{R}_M \) to another leads to a permutation of the columns of \( M \).

3.3 “Ansatz” form of necessary condition

**Theorem 3** Let \( G \) be a branch-commutative matrix \( N \times N \), whose eigenvalues are distinct almost everywhere. Then it can be represented in the form

\[
G = \sum_{m=0}^{N-1} g_m(k) A^m(k),
\] (20)

where \( A(k) \) is a rational matrix, \( g_m(k) \) are algebraic functions. Vice versa, any matrix admitting a decomposition of the form (20) is branch-commutative.

**Proof:** The second part of the theorem is obvious, so we are concentrating our efforts on the first one. Consider matrix \( M(k) \). Let \( \pi_c \) be a permutation of columns of \( M \) occuring
when the argument is carried along a contour \( c \) on \( \mathbb{C} \) starting and terminating at \( k \). Let \( \Pi_c \) be a matrix containing only numbers 0 and 1, describing permutation \( \pi_c \) in matrix language, i.e.

\[
(\Pi_c)_n^m = \delta_{m,\pi_c(n)}, \tag{21}
\]

and the permutation of columns of \( M \) looks like \( M \rightarrow M\Pi_c \).

Construct \( N \) functions \( f_m(k), m = 1 \ldots N \) as follows. Take \( N \) constants \( \beta_1 \ldots \beta_N \) such that the combinations

\[
f_m(k) = \sum_{n=1}^{N} \beta_n (M)_m^n, \tag{22}
\]

almost everywhere obey the relation \( f_{m_1}(k) \neq f_{m_2}(k) \) as \( m_1 \neq m_2 \). (Here \( (M)_m^n \) are the elements of \( M \).) Obviously, \( f_m \rightarrow f_{\pi_c(m)} \) when the argument is carried along \( c \).

Construct a combination

\[
A(k) = M(k) \text{ diag } [f_1(k), \ldots, f_N(k)] M^{-1}(k). \tag{23}
\]

Note that the diagonal matrix obeys the relation

\[
\text{diag } [f_{\pi_c(1)}, \ldots, f_{\pi_c(N)}] = \Pi_c^{-1} \text{ diag } [f_1, \ldots, f_N] \Pi_c. \tag{24}
\]

Substituting (21) and (24) into (23), conclude that \( A \) remains unchanged after any bypass \( c \). Since \( A \) is an algebraic matrix by construction, it should be a rational matrix.

Finally, let us show that \( G \) can be expressed in the form (20) with matrix \( A \) constructed above. The matrix composed of the elements \( (F)_m^n = f_{n-1}^m \) (here \( m - 1 \) is a power, \( m = 1 \ldots N \)) has a non-zero determinant almost everywhere. In the opposite case it would happen that \( N \) distinct numbers are roots of a polynomial of order smaller than \( N \). Therefore any set of \( N \) numbers, for example the eigenvalues of \( G \), can be represented as

\[
\lambda_n(k) = \sum_{m=1}^{N} g_m(k) f_{n-1}^m(k) \tag{25}
\]

for almost all \( k \). By construction, \( g_n \) are algebraic functions.

The theorem is proved.

The form (20) is close to that of [20], however on one hand we impose no restrictions on the behaviour of the matrices \( Q^\pm \), and on the other hand, we do not specify the form of equation, which matrix \( A \) should obey.

Theorem 3 states that there are two alternative ways to check, whether a diffraction matrix \( G \) can be factorized commutatively: 1) by checking whether a matrix can be represented in a certain form, and 2) by checking conditions (13) between different sheets. The second variant seems more easy.

**Note:** Commutative factorization of the matrices having form (20) was considered in [11], where an explicit formula for the factors was constructed.

Thus, branch-commutativeness (or, alternatively, the form (20)) is a necessary and sufficient condition form commutative factorization of matrices with balanced Riemann surfaces.
4 Bypass matrices and Hurd’s method

4.1 Bypass matrices

Let $G$ be an algebraic matrix with a balanced Riemann surface, and let $W$ be the set of words associated with this matrix. Let the right factorization problem (1) be studied.

**Definition 3** A bypass matrix $P_w(k)$ for a word $w$ is defined by the relation:

$$P_w(k) = Q^+(k)\{w\} (Q^+(k)\{e\})^{-1}.$$  \hspace{1cm} (26)

According to the formulae of analytical continuation (5),

$$P_w(k) = (G\{w^+\}G^{-1}\{w^+\})(G\{w^{+-}\}G^{-1}\{w^{+-}\}) \ldots$$ \hspace{1cm} (27)

The product in the r.-h.s. is finite, since after some truncations the matrices become equal to $G\{e\}$.

Let the number of sheets of $\mathcal{R}_G$ be equal to $n$. Among all bypass matrices we can select a finite set of $n - 1$ basic bypass matrices

$$\hat{P}_j(k) = G(k)\{w_j\}G^{-1}(k)\{e\}$$ \hspace{1cm} (28)

where $w_1 \ldots w_{n-1}$ are any words belonging to $W_b$, such that all $G(k)\{w_j\}$ belong to different sheets of $\mathcal{R}_G$, and none of these sheets is the physical one. Any bypass matrix can be written as a product of several basic bypass matrices $\hat{P}_j$ taken in positive or negative powers.

4.2 Hurd’s idea and its formalization

The idea of Hurd [7] can be expressed as follows: sometimes the bypass matrices $P_w(k)$ can have a structure simpler than that of the matrix $G(k)$. Here we express this simplicity in the following form.

**Definition 4** Let $G(k)$ be an algebraic matrix with a balanced Riemann surface. Let $P_w$ be a set of corresponding bypass matrices. Let all basic bypass matrices commute with each other:

$$[\hat{P}_j(k), \hat{P}_m(k)] = 0, \quad j, m = 1 \ldots n - 1,$$ \hspace{1cm} (29)

The matrix $G$ will be called bypass-commutative.

The approach introduced by Hurd is closely connected with bypass-commutativity. His idea was to study the matrix boundary value problem on cuts made from branch points to infinity. The matrix coefficient for each cut is one of the bypass matrices. All known techniques available for such problems at the current moment require commutation of these bypass matrices and their analytical continuations. That is why all known matrices, to which Hurd’s method was applied successfully are bypass-commutative.

The class of bypass-commutative matrices is quite wide. For example, all matrices $G$ with hyperelliptic Riemann surface are bypass-commutative, since there is a single basic
bypass matrix $G\{a\}G^{-1}\{e\}$. An example of bypass-commutative matrix with a more sophisticated Riemann surface is

$$G(k) = \begin{pmatrix} s_1(k) & s_2(k) \\ -s_2(k) & ks_1(k) \end{pmatrix},$$

(30)

where $s_1 = \sqrt{k_1^2 - k^2}$, $s_2 = \sqrt{k_2^2 - k_1^2 - k^2}$, $k_1$ and $k_2$ are constants. The Riemann surface for such a matrix has four sheets; affixes of branch points are $\pm k_1$ and $\pm \sqrt{k_1^2 - k_2^2}$. Thus, there are three basic bypass matrices, and their commutativity can be checked explicitly.

**Theorem 4** Let matrix $G(k)$ be bypass-commutative. Then there exists a rational matrix $S(k)$ such that the matrix $G(k)S(k)$ is branch-commutative.

**Proof.** Obviously, if basic bypass matrices commute then all bypass matrices commute.

Let $R_G$ has $n$ sheets, and the dimension of $G$ be $N \times N$.

The matrix $S(k)$ can be constructed as follows:

$$S(k) = \sum_{j=1}^{n} f(k)\{w_j\}G^{-1}(k)\{w_j\},$$

(31)

where $w_j$ is a set of words listing all sheets of $R_G$. For example, the words from (28) can be taken as $w_j$ for $j = 1 \ldots n - 1$, and $w_n = e$.

Function $f(k)$ is an arbitrary function, such that it is single-valued on $R_G$, and the r.h.s. of (31) has non-zero determinant almost everywhere. As a possible choice, one can construct $f(k)$ by the formula

$$f(k) = \beta_{0,0} + \sum_{i=1}^{N} \sum_{j=1}^{N} \beta_{i,j}(G^{-1}(k))^i_j$$

(32)

with almost arbitrary constants $\beta_{i,j}$.

Since the sum in (31) is taken over all sheets of an algebraic function $fG$, the result is single-valued on $\mathbb{C}$, and therefore it is a rational matrix function.

By construction, the combination $G(k)S(k)$ is a linear combination of some bypass matrices or their products. The analytical continuations of $G(k)S(k)$ can also be represented as products of bypass matrices. Since the bypass matrices commute with each other, $G(k)S(k)$ is a branch-commutative matrix.

An example (maybe quite simple) of such a consideration can be constructed using matrix (30). Take function $f(k) \equiv 1$. Matrix $S$ is constructed by summation of $G^{-1}$ over four sheets of its Riemann surface:

$$S(k) = \frac{4(k_2^2 - k^2)}{k_2^4 + k_1^4k^2 + (k^2 - k_1^2)(1 - 2k_2^2k + 2k^4)} \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$$

(33)

Simple calculations show that $GS$ is a branch-commutative matrix.
Note that Moiseyev’s approach can be applied to GS, i.e. an explicit commutative factorization can be constructed.

Note also that if for some matrix $G$ there exists a rational matrix $S$, such that $GS$ or $SG$ is branch-commutative, then obviously all bypass matrices of $G$ commute. That is why, bypass-commutativity is a necessary and sufficient condition for a balanced algebraic matrix of a possibility to be converted into a branch-commutative matrix.

5 A short summary

The main results of this paper are as follows:

1. Formulae of analytical continuation are derived.

2. Necessary condition of commutative factorization is found. Namely, a balanced algebraic matrix should be branch-commutative. This property can be easily checked.

3. Connection with the “Ansatz” form of the necessary condition is established.

4. Hurd’s method is formalized. It can be applied if a matrix is bypass-commutative. It is shown that in this case one can reduce the problem to the commutative factorization case by multiplication by a rational matrix.

The necessary condition of commutative matrix factorization for balanced matrix is checked as follows. First, one should check, whether the values $G(k)$ taken on different sheets commute. If they commute, then the matrix can be factorized by Moiseyev’s method. Second, one should construct the basic bypass matrices $P_j$ and check whether they commute with each other. If they commute, then there exists a rational matrix $S$, multiplication by which transforms matrix $G$ into a commutatively factorizable case.

References

[1] Heins, A.E. The radiation and transmission properties of a pair of semi-infinite parallel plates I and II. Quart. Appl. Math. 6, 157–166, 215–220 (1948).

[2] Chebotarev, G.N. On closed form solution of a Riemann boundary value problem for $n$ pairs of functions. Uchen. Zap. Kazan. Univ. 116, 4, 31–58 (1956).

[3] Khrapkov, A.A. Certain cases of the elastic equilibrium of an infinite wedge with a non-symmetric notch at the vertex, subjected to concentrated forces. J. Appl. Math. Mech. (PMM), 35, 625–637 (1971).

[4] Daniele, V.G. On the solution of two coupled Wiener-Hopf equations. SIAM J. Appl. Math. 44, 667–680 (1984).

[5] Hurd, R.A. and Lüneburg, E. Diffraction by an anisotropic impedance half-plane Can. J.Phys. 63, 1135-1140 (1985).
[6] Rawlins, A.D. The solution of mixed boundary-value problem in the theory of diffraction by a semi-infinite plane. Proc. Roy. Soc. Lond. A, 346, 469–484 (1975).

[7] Hurd, R.A. The Wiener-Hopf-Hilbert method for diffraction problems. Can. J. Phys. 54, 775–780 (1976).

[8] Lukyanov, V.D. Exact solution of the problem of diffraction of an obliquely incident wave at a grating. Dokl. Akad. Nauk SSSR 255, 78–80 (1980).

[9] Cercignani, C. Analytic solution of the temperature jump problem for the BGK model. Transport Theory and Statistical Physics. 6, 29–56 (1977).

[10] Lüneburg, E. Diffraction by an infinite set of soft / hard parallel half-planes: the Riemann approach. Can. J. Phys. 60, 1125–1138 (1982).

[11] Moiseyev, N.G. Factorization of matrix functions of special form. Soviet Math. Dokl. 39, 264–267 (1989).

[12] Meister, E., and Penzel, F. On the reduction of the factorization of matrix functions of Daniele-Khrapkov type to a scalar boundary value problem on a Riemann surface.

[13] Antipov, Y.A. and Silvestrov, V.V. Factorization on a Riemann surface in scattering theory. Quart. J. Mech. Appl. Math. 55, 607–654 (2002).

[14] Zverovich E.I., Boundary value problems in the theory of analytic functions in Holder classes on Riemann surfaces. Russian Math. Surveys. 26, 117–192 (1971).

[15] Hurd, R.A. The explicit factorization of $2 \times 2$ Wiener-Hopf matrices. Technischen Hochschule Darmstadt, Preprint-Nr. 1040, March 1987.

[16] Daniele, V.G. On the solution of vector Wiener-Hopf equations occurring in scattering problems. Radio Science, 19, 1173–1178 (1984).

[17] Williams, W.E. Recognition of some readily Wiener-Hopf factorizable matrices. IMA Journal of Applied Mathematics, 32, 367–378 (1984).

[18] Ehrhardt, T. and Speck, F.-O. Transformation techniques towards the factorization of non-rational $2 \times 2$ matrix functions. Linear Algebra and its Applications, 353, 53–90 (2002).

[19] Abrahams, I.D. On the non-commutative factorization of Wiener-Hopf kernels of Khrapkov type. Proc. R. Soc. A 454, 1719–1743 (1998).

[20] Jones, D.S. Commutative Wiener-Hopf factorization of a matrix. Proc. R. Soc. Lond. A, 393, 185–192 (1984).