A QUANTITATIVE INTERNAL UNIQUE CONTINUATION FOR
STOCHASTIC PARABOLIC EQUATIONS

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Abstract. This paper is addressed to a quantitative internal unique continuation property for stochastic parabolic equations, i.e., we show that each of their solutions can be determined by the observation on any nonempty open subset of the whole region in which the equations evolve. The proof is based on a global Carleman estimate.

1. Introduction. Let $T > 0$, and $G \subset \mathbb{R}^n (n \in \mathbb{N})$ be a given bounded domain. Let $G_0 \subset \subset G$ be a given nonempty subdomain. Put $Q_0 = (0, T) \times G_0$.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, P)$ be a complete filtered probability space (satisfying the usual conditions), on which a standard one-dimensional Brownian motion $\{B_t\}_{t \geq 0}$ is defined, and write $\mathcal{F} = \{\mathcal{F}_t\}_{t \in [0, T]}$. Assume that $H$ is a Fréchet space. Denote $L^2(0, T; H)$ by the Fréchet space consisting of all $H$-valued $\mathcal{F}$-adapted processes $X(\cdot)$ such that $\mathbb{E} \|X(\cdot)\|_{L^2(0, T; H)}^2 < \infty$; by $L^\infty(0, T; H)$ the Fréchet space of all $H$-valued $\mathcal{F}$-adapted bounded processes and by $L^2(\Omega; C([0, T]; H))$ the Fréchet space of all $H$-valued $\mathcal{F}$-adapted continuous processes $X(\cdot)$ with $\mathbb{E} \|X(\cdot)\|_{C([0, T]; H)}^2 < \infty$ (All of these Frechet spaces are equipped with the canonical quasi-norms).

Assume that $a^{ij} \in W^{1, \infty}(0, T; W^{2, \infty}_{loc}(G)), i, j = 1, 2, \cdots, n$, and for any subset $G_1$ of $G$, there exists a constant $\Lambda = \Lambda(G_1) > 0$ such that

$$\sum_{i, j = 1}^n a^{ij} \xi_i \xi_j \geq \Lambda |\xi|^2, \quad \forall (t, x, \xi) \equiv (t, x, \xi_1, \xi_2, \cdots, \xi_n) \in (0, T) \times G_1 \times \mathbb{R}^n. \quad (1)$$

For simplicity, we use the symbol $y_i = y_i(x)$ to stand for $\frac{\partial y}{\partial x_i}$, the partial derivative of $y$ in the $x_i$-direction, where $x_i$ is the $i$-th entry of a generic point $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$. In a similar way, we use the notations $y_{ii}, y_{ij}, v_i, u_i$, etc. By $a \cdot b$ we denote the scalar product of two vectors $a$ and $b$ in $\mathbb{R}^n$.

Consider the following stochastic parabolic equation:

$$dy - \sum_{i, j = 1}^n (a^{ij} y_i) dt = a_1 \cdot \nabla y dt + a_2 y dt + a_3 y dB(t) \quad \text{in} \quad Q. \quad (2)$$
Here, \( a_1 \in L^\infty_\mathbb{P}(0,T;L^\infty_{\text{loc}}(G;\mathbb{R}^n)) \), \( a_2 \in L^\infty_\mathbb{P}(0,T;L^\infty_{\text{loc}}(G)) \) and \( a_3 \in L^\infty_\mathbb{P}(0,T;W^{1,\infty}_{\text{loc}}(G)) \).

We call \( y \in L^2_\mathbb{P}(\Omega;C([0,T];L^2_{\text{loc}}(G))) \cap L^2_\mathbb{P}(0,T;H^1_{\text{loc}}(G)) \) a solution to the equation (2) if

1. For any nonempty open set \( G' \subset \subset G \),
   \[ y \in L^2_\mathbb{P}(\Omega;C([0,T];L^2(G'))) \cap L^2_\mathbb{P}(0,T;H^1(G')); \]

2. For any \( t \in [0,T] \) and any \( \varphi \in H^1_0(G') \), it holds that
   \[
   \int_{G'} y(t,x)\varphi(x)dx - \int_{G'} y(0,x)\varphi(x)dx = \int_0^t \int_{G'} \left[ -\sum_{i,j=1}^n a_{ij}y_{ij}(s,x)\varphi_i(x) + a_1 \cdot \nabla y(s,x)\varphi(x) + a_2 y(s,x)\varphi(x) \right] dxds + \int_0^t \int_{G'} a_3 y(s,x)\varphi(x)dxdB(s), \quad P\text{-a.s.}
   \]

The aim of this paper is to derive a quantitative internal unique continuation result for the equation (1.2), i.e., to show that the value of each solution to (1.2) can be determined by an observation on any nonempty open subset of the whole region \( G \). The main result in this work is stated as follows:

**Theorem 1.1.** For any subdomain \( G' \subset \subset G \) with \( G_0 \subset \subset G' \) and \( \kappa \in (0,\frac{1}{2}) \), there exists a subdomain \( \tilde{G} \subset \subset G \) with \( G' \subset \subset \tilde{G} \), such that for any solution \( y \) to the equation (2), it holds that

\[
\mathbb{E} \int_{\frac{T}{2}-\kappa T}^{\frac{T}{2}+\kappa T} \int_{G'} (|\nabla y|^2 + y^2) dxdt \leq C \exp \left( \frac{C}{\varepsilon} \right) \mathbb{E} \int_{G_0} (|\nabla y|^2 + y^2) dxdt + C\varepsilon \mathbb{E} \int_{0}^{T} \int_{\tilde{G}} (|\nabla y|^2 + y^2) dxdt
\]

for some positive constant \( C \).

The study of unique continuation property for partial differential equations may date back to the classical results due to Holmgren and Carleman at the beginning of the last century. After that, there were many authors working on this topic. Many of the existing results in this respect were successfully applied to control and inverse problems for partial differential equations (see [3, 14], etc.). Until now, there are numerous references on the unique continuation for deterministic parabolic equations (see [1, 2, 5, 8, 9, 10, 12, 15], etc.). It is interesting and meaningful to extend the study of unique continuation properties from the deterministic case to its stochastic counterpart. However, there are only a very few works addressed to the unique continuation for stochastic partial differential equations. In [4], the authors obtained the boundary unique continuation for stochastic parabolic equations. The unique continuation property for stochastic Schrödinger and stochastic hyperbolic equations were established in [6] and [7], respectively. On the other hand, in [13], the author obtained a unique continuation property for (2) without giving the quantitative estimate (4).

There are two well-known tools to deal with the unique continuation property for deterministic partial differential equations, i.e., Carleman estimate and Holmgren-type uniqueness theorem. In general a stochastic equation does not admit a solution which is analytic in time even if all coefficients of the equation are constants. Thus,
it cannot be expected to obtain a Holmgren-type unique continuation result for stochastic equations, except for some very special cases. We employ Carleman estimate as a main tool to prove the desired unique continuation result in the present work.

2. Proof of the main result. In order to prove our main result, Theorem 1.1, we need the following known result, which can be found in [11, Theorem 3.1].

Lemma 2.1. Let \( n \) be a positive integer,

\[
b^{ij} = b^{ji} \in L^2_0(\Omega; C^1([0, T]; W^{2, \infty}(\mathbb{R}^n))), \quad i, j = 1, 2, \ldots, n,
\]

\( \ell \in C^{1,4}((0, T) \times \mathbb{R}^n) \) and \( \Psi \in C^{1,2}((0, T) \times \mathbb{R}^n) \). Assume that \( u \) is an \( H^3(\mathbb{R}^n) \)-valued continuous semi-martingale. Set \( \theta = e^t \) and \( v = \theta u \). Then for a.e. \( x \in \mathbb{R}^n \) and \( \omega \in \Omega \), it holds that

\[
2 \int_0^T \theta \left[- \sum_{i,j=1}^n (b^{ij} v_i)_j + \mathcal{A} v \right] \left[ du - \sum_{i,j=1}^n (b^{ij} u_i)_j dt \right] + 2 \int_0^T \sum_{i,j=1}^n (b^{ij} v_i v_j) dt \\
+ 2 \int_0^T \sum_{i,j=1}^n \left[ \sum_{i', j'=1}^n \left( 2 b^{ij} b^{i'j'} \ell_{i'i'} - b^{ij} b^{i'j'} \ell_{i'i'} \right) - \frac{b^{ij}}{2} + \Psi b^{ij} \right] v_i v_j dt \\
+ \int_0^T B v^2 dt + 2 \int_0^T \left[ - \sum_{i,j=1}^n (b^{ij} v_i)_j + \mathcal{A} v \right] \left[ - \sum_{i,j=1}^n (b^{ij} v_i)_j + (\mathcal{A} - \ell_t) v \right] dt \\
+ \left( \sum_{i,j=1}^n b^{ij} v_i v_j + \mathcal{A} v^2 \right) \int_0^T \theta^2 \mathcal{A} (du)_2, \tag{6}
\]

where

\[
\begin{aligned}
\mathcal{A} &\triangleq - \sum_{i,j=1}^n (b^{ij} \ell_{i} - b^{ij} \ell_{j} - b^{ij} \ell_{ij}) - \Psi, \\
B &\triangleq 2 \left[ \mathcal{A} \Psi - \sum_{i,j=1}^n (\mathcal{A} b^{ij} \ell_{i}) \right] - \mathcal{A}_t - \sum_{i,j=1}^n (b^{ij} \Psi)_i.
\end{aligned}
\tag{7}
\]

Now, we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. We borrow some idea from [4]. Without loss of generality, we can assume that the boundary \( \partial G \) of \( G \) is smooth. Otherwise, we can choose a domain \( G'' \) contained in \( G \) such that \( G' \subset G'' \subset G \) and the boundary \( \partial G'' \) of \( G'' \) is smooth and then study the unique continuation problem on \( G'' \). It is well known that there exists a function \( \psi \in C^4(G) \) such that

\[
\begin{cases}
\psi > 0 & \text{in } G, \\
\psi = 0 & \text{on } \partial G, \\
|\nabla \psi| > 0 & \text{in } G \setminus G_0.
\end{cases}
\]

Since \( G' \subset G \), then by the choice of \( \psi \) we can fix an integer \( N \) large enough such that

\[
G' \subset \left\{ x \in G, \psi(x) > \frac{4}{N} \| \psi \|_{L^\infty(G)} \right\}. \tag{8}
\]
For any $\kappa \in (0, \frac{1}{2})$, take $\delta = \frac{1}{\sqrt{2}} \left( \frac{1}{2} - \kappa \right) T > 0$. It follows that $\sqrt{2}\delta = (\frac{1}{2} - \kappa)T$, $T - \sqrt{2}\delta = (\frac{1}{2} + \kappa)T$. We pick a constant $c$ satisfying
\[ c\delta^2 < \|\psi\|_{L^\infty(G)} < 2c\delta^2. \] (9)

Let $t_0 \in [\sqrt{2}\delta, T - \sqrt{2}\delta]$, and set
\[ \phi(t, x) = \psi(x) - c(t - t_0)^2, \quad \alpha(t, x) = \exp(\mu \phi). \]

Put
\[ \beta_k = \exp \left( \mu \left[ \frac{k}{N} \|\psi\|_{L^\infty(G)} - \frac{c}{N} \delta^2 \right] \right), \quad k = 1, 2, 3, 4. \]
Define
\[ Q_k = \{(t, x) \mid t \in [0, T], x \in \mathcal{G}, \alpha(t, x) > \beta_k \}. \]

It is clear that $Q_{k+1} \subset Q_k$, for $k = 1, 2, 3$. Further, there is a $\mathcal{O} \subset \subset G$ such that $Q_1 \subset \mathcal{O} \times (0, T)$. For any $(t, x)$ satisfying
\[ |t - t_0| < \frac{\delta}{\sqrt{N}}, \quad x \in G', \] (10)

it is clear from (8) that $\psi(x) > \frac{4}{N} \|\psi\|_{L^\infty(G)}$ and $-c(t - t_0)^2 > -\frac{2}{N} \delta^2$. Then
\[ \psi(x) - c(t - t_0)^2 > \frac{4}{N} \|\psi\|_{L^\infty(G)} - \frac{c}{N} \delta^2. \]

Therefore, $(t, x) \in Q_4$ whenever $(t, x)$ satisfies (10).

If $(t, x) \in Q_1$, then by the definition of $Q_1$, we have
\[ \psi(x) - c(t - t_0)^2 > \frac{1}{N} \|\psi\|_{L^\infty(G)} - \frac{c}{N} \delta^2. \]

Therefore, it follows from (9) that
\[ c(t - t_0)^2 < \psi(x) - \frac{1}{N} \|\psi\|_{L^\infty(G)} + \frac{c}{N} \delta^2 < \left( 2 - \frac{1}{N} \right) c\delta^2. \]

Thus
\[ |t - t_0| < \sqrt{2} \delta. \]

As a result, we conclude that
\[ \left( t_0 - \frac{\delta}{\sqrt{N}}, t_0 + \frac{\delta}{\sqrt{N}} \right) \times G' \subset Q_4 \subset Q_1 \subset (t_0 - \sqrt{2}\delta, t_0 + \sqrt{2}\delta) \times \mathcal{G}. \] (11)

Let $\eta \in C_0^\infty(Q_2)$ satisfy
\[ \eta \in [0, 1] \text{ and } \eta = 1 \text{ in } Q_3. \]

Write $z = \eta y$. It is easy to show that
\[ \begin{cases} dz - \sum_{i,j=1}^n \left( a^{ij}_1 z_i \right) dt = a_1 \cdot \nabla z dt + a_2 z dt + f dt + a_3 z dB(t) & \text{in } Q_1 \\ z = 0 & \text{on } \partial Q_1. \end{cases} \] (12)

Here, $f$ is given by
\[ f = \eta y - 2 \sum_{i,j=1}^n a^{ij} \eta_i y_j - y \sum_{i,j=1}^n (a^{ij}_1) y_j - y a_1 \cdot \nabla \eta. \]

Clearly, $f$ is supported in $Q_2 \setminus \mathcal{G}_3$. 

Assume that $\lambda$ and $\mu$ are two parameters with $\lambda > 1$ and $\mu > 1$. Let $\ell = \lambda \alpha$. Hereafter, we use the symbol $O(1)$ to stand for a bounded quantity and $O(\mu^k)$ a function with the same order to $\mu^k$, where $k$ is a positive integer.

Replace $b^j$ by $a^j$ and $u$ by $z$ in (6) and then integrate (6) over $G \times (0, T)$. Noticing that $z$ is supported in $Q_1$, after taking mathematical expectation in both sides, we obtain the following inequality:

\[
2\mathbb{E} \int_{Q_1} \theta \left[ - \sum_{i,j=1}^{n} (a^{ij}v_i + \mathcal{A}v) \right] dz - \sum_{i,j=1}^{n} (a^{ij}z_i) dt \ dx + \mathbb{E} \int_{Q_1} \left[ \sum_{i,j=1}^{n} \left( 2a^{ij}a^{i'j'} \ell_i v_i v_{j'} - a^{ij}a^{i'j'} \ell_i v_i v_{j'} \right) \right] dt dx + 2\mathbb{E} \int_{Q_1} \left[ \sum_{i,j=1}^{n} (a^{ij}v_i v_j) \right] dt dx 
\]

\[
\geq 2\mathbb{E} \int_{Q_1} \left[ \sum_{i,j=1}^{n} (a^{ij}v_i + \mathcal{A}v) \right] dz - \sum_{i,j=1}^{n} (a^{ij}v_i + \mathcal{A}v) \quad dt dx 
+ \mathbb{E} \int_{Q_1} \left[ \sum_{i,j=1}^{n} (a^{ij}v_i v_j) \right] dt dx - 2\mathbb{E} \int_{Q_1} \left[ \sum_{i,j=1}^{n} (a^{ij}v_i + \mathcal{A}v) \right] \quad dt dx 
- \mathbb{E} \int_{Q_1} \left[ \sum_{i,j=1}^{n} (a^{ij}(dz_i + \ell_i dz)(dz_j + \ell_j dz)) \right] dt dx, 
\]

where

\[
\begin{align*}
\Psi &= 2 \sum_{i,j=1}^{n} a^{ij} \ell_{ij}, \\
\mathcal{A} &= \sum_{i,j=1}^{n} \left( a^{ij} \ell_{ij} + a^{ij} \ell_i - a^{ij} \ell_i \ell_j \right) - \Psi, \\
\mathcal{B} &= 2\mathcal{A} \Psi - 2 \sum_{i,j=1}^{n} \left( \mathcal{A} a^{ij} \ell_i \right) - \mathcal{A} - \sum_{i,j=1}^{n} \left( a^{ij} \Psi \right), \\
c^{ij} &= \sum_{i',j'=1}^{n} \left[ 2a^{ij'(i'j'} \ell_{i'} - a^{ij}a^{i'j'} \ell_i \ell_{j'} \right] - \frac{1}{2} a^{ij'} + \Psi a^{ij}.
\end{align*}
\]

Put

\[ F = a_1 \cdot \nabla z + a_2 z + f. \]

According to the property of $\eta$, both the second and the third terms in the left hand side of (13) vanish. It is easy to verify that

\[
2\mathbb{E} \int_{Q_1} \theta \left[ - \sum_{i,j=1}^{n} (a^{ij}v_i) \right] dz - \sum_{i,j=1}^{n} (a^{ij}z_i) dt \ dx 
\]

\[
= 2\mathbb{E} \int_{Q_1} \theta \left[ - \sum_{i,j=1}^{n} (a^{ij}v_i) \right] F dx dt 
+ 2\mathbb{E} \int_{Q_1} \theta \left[ - \sum_{i,j=1}^{n} (a^{ij}v_i) \right] a_3 dB(t) dx 
\]
Note that

\[ \ell^2 F^2 dx dt \]

\[ \leq E \int_{Q_1} \left| - \sum_{i,j=1}^n (a_{ij} v_i)_j + Av \right|^2 dx dt + E \int_{Q_1} \theta^2 F^2 dx dt. \]

Therefore,

\[ E \int_{Q_1} \theta^2 F^2 dt dx \]

\[ \geq 2E \int_{Q_1} \sum_{i,j=1}^n c_{ij} v_i v_j dt dx + E \int_{Q_1} Bu^2 dx dt - \int_{Q_1} \theta^2 A(dz)^2 dx \]

\[ - E \int_{Q_1} \ell^2 v^2 dt dx - E \int_{Q_1} \theta^2 \sum_{i,j=1}^n a_{ij}[(dz_i + \ell_i dz)(dz_j + \ell_j dz)] dx. \]

Furthermore, by

\[ E \int_{Q_1} \theta^2 f^2 dx dt \]

\[ = E \int_{Q_1} \theta^2 \left| \eta y - 2 \sum_{i,j=1}^n a_{ij} \eta_i y_j - y \sum_{i,j=1}^n (a_{ij} \eta_i)_j - ya_1 \cdot \nabla \eta \right|^2 dx dt \]

\[ \leq CE \int_{Q_2 \setminus Q_3} \theta^2 (|\nabla y|^2 + y^2) dx dt \]

holding for any \( \eta \in C_0^\infty(Q_2) \) satisfying \( \eta = 1 \) in \( Q_3 \), we get

\[ E \int_{Q_1} \theta^2 F^2 dx dt \]

\[ \leq 3 \|a_1\|_{L^2(0,T;L^\infty(\Omega))} E \int_{Q_2} \theta^2 |\nabla z|^2 dx dt + 3 \|a_2\|_{L^2(0,T;L^\infty(\Omega;\mathbb{R}^n))} E \int_{Q_1} \theta^2 z^2 dx dt \]

\[ + CE \int_{Q_2 \setminus Q_3} \theta^2 (|\nabla y|^2 + y^2) dx dt. \]

In what follows, we analyze the terms in the right hand side of (14) one by one. Recalling the definition of \( \alpha \) and \( \psi \), we see that

\[ \ell_i = \lambda \mu \psi_i, \quad \ell_{ij} = \lambda \mu \psi_i \psi_j + \lambda \mu \psi_{ij}, \]

\[ \ell_{ijj'} = \lambda \mu \psi_i \psi_j \psi_{j'} + \lambda \mu \alpha (\psi_{ij} \psi_{j'} + \psi_{ij} \psi_{j'}) + \lambda \mu \psi_{ijj'}, \]

\[ \ell_{ijj'} = \lambda \mu \psi_i \psi_j \psi_{j'} + \lambda \mu \alpha (\psi_{ij} \psi_{j'} + \psi_{ij} \psi_{j'}) + \lambda \mu \alpha \psi_{ijj'}, \]

\[ + \psi_{ij} \psi_{ijj'} + \psi_{ij} \psi_{ij} \psi_{j'} + \psi_{ij} \psi_{ij} \psi_{j'} + \psi_{ij} \psi_{ij} \psi_{j'} + \psi_{ij} \psi_{ij} \psi_{j'} + \psi_{ij} \psi_{ij} \psi_{j'}, \]

\[ + \psi_{ij} \psi_{ijj'} + \psi_{ij} \psi_{ijj'} + \psi_{ij} \psi_{ijj'} + \psi_{ij} \psi_{ijj'} + \lambda \mu \alpha \psi_{ijijj'}. \]

Noticing that \( \Psi = 2 \sum_{i,j=1}^n a_{ij} \ell_{ij} \), we have that
We turn to estimate the terms in the right hand side of (21) one by one. Firstly, clearly, thus, after some direct computations, it follows that

\[
\sum_{i,j=1}^{n} c^{ij} v_i v_j = \sum_{i,j=1}^{n} \sum_{i',j'=1}^{n} \left\{ 2a^{ij}(a^{i'j} \xi_{i'}) - (a^{ij} a^{i'j}) \xi_{i'} \right\} v_i v_j
\]

\[
= \sum_{i,j=1}^{n} \sum_{i',j'=1}^{n} \left( 2a^{ij} a^{i'j} \xi_{i'} + a^{ij} a^{i'j} \xi_{i'} \xi_{i'} - 2a^{ij} a^{i'j} \xi_{i'} - \frac{\ell^{ij}}{2} \right) v_i v_j
\]

\[
= 2\lambda \mu^2 \alpha \left( \sum_{i,j=1}^{n} a^{ij} v_i v_j \right)^2 + \lambda \mu^2 \alpha \sum_{i,j=1}^{n} a^{ij} v_i v_j \sum_{i,j=1}^{n} a^{ij} v_i v_j + \lambda \alpha O(\mu)|\nabla v|^2 + O(1)|\nabla v|^2.
\]

It is easy to show that

\[
{\cal A} = \sum_{i,j=1}^{n} a^{ij} \xi_{i} + \sum_{i,j=1}^{n} a^{ij} \xi_{i} - \sum_{i,j=1}^{n} a^{ij} \xi_{i} - \Psi
\]

\[
= - \sum_{i,j=1}^{n} a^{ij} \xi_{i} + \sum_{i,j=1}^{n} a^{ij} \xi_{i} - \sum_{i,j=1}^{n} a^{ij} \xi_{i}
\]

\[
= - \lambda^2 \mu^2 \alpha^2 \sum_{i,j=1}^{n} a^{ij} v_i v_j + \lambda \alpha O(\mu).
\]

Thus,

\[
\theta^2 A(dx)^2 = \left[ - \lambda^2 \mu^2 \alpha^2 a_3^2 \sum_{i,j=1}^{n} a^{ij} v_i v_j - \lambda \mu^2 \alpha a_3^2 \sum_{i,j=1}^{n} a^{ij} v_i v_j + \lambda \alpha O(1) \right] v^2 dt.
\]

Our next goal is to estimate the term \( B \) in (14). By (16) and (18), we find that

\[
2 \lambda^3 A^2 = -4 \lambda^3 \mu^2 \alpha^3 \left( \sum_{i,j=1}^{n} a^{ij} v_i v_j \right)^2 - 4 \lambda^3 \mu^2 \alpha^3 \sum_{i,j=1}^{n} a^{ij} v_i v_j \sum_{i,j=1}^{n} a^{ij} v_i v_j
\]

\[
-4 \lambda^2 \mu^2 \alpha^2 \left( \sum_{i,j=1}^{n} a^{ij} v_i v_j \right)^2 + \lambda^2 \alpha^2 O(\mu^2).
\]

Clearly,

\[
2 \sum_{i,j=1}^{n} (Aa^{ij} \xi_{i})_j = 2 \sum_{i,j=1}^{n} a^{ij} \xi_{i} A_j + 2 \sum_{i,j=1}^{n} Aa^{ij} \xi_{i} + 2 \sum_{i,j=1}^{n} Aa^{ij} \xi_{i}.
\]

We turn to estimate the terms in the right hand side of (21) one by one. Firstly,

\[
2 \sum_{i,j=1}^{n} a^{ij} \xi_{i} A_j
\]

\[
= -2 \sum_{i,j=1}^{n} a^{ij} \xi_{i} \sum_{i',j'=1} \left( a^{i'j'} \xi_{i'} \xi_{j'} + 2a^{i'j'} \xi_{i'} \xi_{j'} - a^{i'j'} \xi_{i'} \xi_{j'} - a^{i'j'} \xi_{i'} \xi_{j'} + a^{i'j'} \xi_{i'} \xi_{j'} + a^{i'j'} \xi_{i'} \xi_{j'} \right)
\]
Secondly,
\[
2 \sum_{i,j=1}^{n} \mathcal{A}_{ij} \ell_{ij} = 2 \sum_{i,j=1}^{n} a_{ij} \ell_{ij} \sum_{i', j'=1}^{n} \left( -a_{i'j'} \ell_{i'j'} + a_{ij} \ell_{ij} - a_{i'j'} \ell_{i'j'} \right)
\]
\[= \lambda^2 \alpha^2 O(\mu^3). \tag{23}\]

Thirdly, it is easy to verify that
\[
2 \sum_{i,j=1}^{n} \mathcal{A}_{ij} \ell_{ij} = 2 \sum_{i,j=1}^{n} a_{ij} \ell_{ij} \sum_{i', j'=1}^{n} \left( -a_{i'j'} \ell_{i'j'} + a_{ij} \ell_{ij} - a_{i'j'} \ell_{i'j'} \right)
\]
\[= -2\lambda^3 \mu^4 \alpha^3 \left( \sum_{i,j=1}^{n} a_{ij} \psi_i \psi_j \right)^2 - 2\lambda^2 \mu^2 \alpha^2 \left( \sum_{i,j=1}^{n} a_{ij} \psi_i \psi_j \right)^2 \tag{24}\]
\[= -2\lambda^2 \mu^2 \alpha^2 \left( \sum_{i,j=1}^{n} a_{ij} \psi_i \psi_j \right)^2 + \lambda^2 \alpha^2 O(\mu^3).
\]

Therefore, by (22), (23) and (24), we end up with that
\[
2 \sum_{i,j=1}^{n} \left( \mathcal{A}_{ij} \ell_{ij} \right)_j = -6\lambda^3 \mu^3 \alpha^3 \left( \sum_{i,j=1}^{n} a_{ij} \psi_i \psi_j \right)^2 - 4\lambda^2 \mu^4 \alpha^2 \left( \sum_{i,j=1}^{n} a_{ij} \psi_i \psi_j \right)^2
\]
\[+ \lambda^3 \alpha^3 O(\mu^3) + \lambda^2 \alpha^2 O(\mu^3). \tag{25}\]

It is easy to see that
\[
\mathcal{A}_{ij} = -2\lambda^2 \mu^2 \alpha^2 \phi_t \sum_{i,j=1}^{n} a_{ij} \psi_i \psi_j - 2\lambda \mu^2 \alpha^2 \sum_{i,j=1}^{n} a_{ij} \psi_i \psi_j - 2\lambda^2 \mu^2 \alpha^2 \sum_{i,j=1}^{n} \left( a_{ij} \psi_i \psi_j - a_{ij} \psi_i \psi_j \right)
\]
\[= \lambda^2 \alpha^2 O(\mu^3) + \lambda \alpha O(\mu^3). \tag{26}\]

Furthermore, we have that
\[
\sum_{i,j=1}^{n} a_{ij} \psi_j = \sum_{i,j=1}^{n} a_{ij} \sum_{i', j'=1}^{n} \left( a_{i'j'} \ell_{i'j'} + a_{i'j'} \ell_{i'j'} \right) = \lambda \alpha O(\mu^3) \tag{27}\]

and
\[
\sum_{i,j=1}^{n} a_{ij} \psi_j = \sum_{i,j=1}^{n} a_{ij} \sum_{i', j'=1}^{n} \left( a_{i'j'} \ell_{i'j'} + a_{i'j'} \ell_{i'j'} + a_{i'j'} \ell_{i'j'} + a_{i'j'} \ell_{i'j'} \right)
\]
\[= 2\lambda \mu^4 \alpha \sum_{i,j=1}^{n} \left( a_{ij} \psi_i \psi_j \right)^2 + \lambda \alpha O(\mu^3). \tag{28}\]

From (27) and (28), we obtain that
\[
\sum_{i,j=1}^{n} (a_{ij} \psi_j)_i = \sum_{i,j=1}^{n} a_{ij} \psi_j + \sum_{i,j=1}^{n} a_{ij} \psi_j = 2\lambda \mu^4 \alpha \left( \sum_{i,j=1}^{n} a_{ij} \psi_i \psi_j \right)^2 + \lambda \alpha O(\mu^3). \tag{29}\]
According to (20), (25), (26) and (29), it holds that
\[ B = 2\lambda^3\mu^4\alpha^3 \left( \sum_{i,j=1}^{n} a_{ij} \psi_i \psi_j \right)^2 + 2\lambda^2\mu^4\alpha^2 \left( \sum_{i,j=1}^{n} a_{ij} \psi_i \psi_j \right)^2 - 2\lambda^4\alpha \left( \sum_{i,j=1}^{n} a_{ij} \psi_i \psi_j \right)^2 + \lambda^3\alpha^3O(\mu^3). \]  
(30)

It is easy to check the following equality
\[ -\ell_t^2 = -\lambda^2\mu^2\alpha^2\phi_t^2. \]  
(31)

By (17), (19), (30) and (31), there exists a constant \( \mu_0 > 1 \), such that for \( \mu \geq \mu_0 \), one can find a \( \lambda_0(\mu) > 1 \), so that for \( \lambda \geq \lambda_0(\mu) \),
\[
\begin{cases}
\mathbb{E} \int_{Q_1} \sum_{i,j=1}^{n} c_{ij} v_i v_j dx dt \geq C\lambda_0^2 \mathbb{E} \int_{Q_1 \setminus Q_0} \alpha |\nabla v|^2 dx dt, \\
\mathbb{E} \int_{Q_1} \theta^2 A(dx)^2 dx \geq -C\lambda^2\mu^2 \|a_3\|_{L^\infty(0, T; W^{1, \infty}(\Omega)))}^2 \mathbb{E} \int_{Q_1 \setminus Q_0} \alpha^2 v^2 dx dt, \\
\mathbb{E} \int_{Q_1} \mathbb{B} v^2 dx dt \geq C\lambda^3\mu^4 E \int_{Q_1 \setminus Q_0} \alpha^3 v^2 dx dt, \\
\mathbb{E} \int_{Q_1} \ell_t^2 v^2 dx dt \geq -C\lambda^2\mu^2 \mathbb{E} \int_{Q_1} \alpha^2 v^2 \phi_t^2 dx dt.
\end{cases}
\]
(32)

It follows from Itô’s formula that
\[(dz_i + \ell_t dz_i)(dz_j + \ell_t dz_j) = (a_3 z_i)(a_3 z_j) dt + \ell_t dz_i dz_j + \ell_t dz_j dz_i + \ell_t \ell_j a_3^2 z_i z_j dt.\]

Thus
\[
\mathbb{E} \int_{Q} \theta^2 \sum_{i,j=1}^{n} a_{ij}(dz_i + \ell_t dz_i)(dz_j + \ell_t dz_j) dx \\
\leq C \|a_3\|_{L^\infty(0, T; W^{1, \infty}(\Omega)))}^2 \left[ \mathbb{E} \int_{Q} \theta^2 (z^2 + |\nabla z|^2) dx dt + \lambda^2 \mu^2 \mathbb{E} \int_{Q} \theta^2 \alpha^2 |z|^2 dx dt \right].
\]
(33)

Noticing that \( v = \theta z \) and \( z = \theta^{-1} v \), one can show that
\[
\frac{1}{C} \theta^2 \left( |\nabla z|^2 + \lambda^2 \mu^2 \alpha^2 z^2 \right) \leq |\nabla v|^2 + \frac{\lambda^2 \mu^2 \alpha^2}{C} |v|^2 \leq C \theta^2 \left( |\nabla z|^2 + \lambda^2 \mu^2 \alpha^2 z^2 \right).
\]  
(34)

From (14), (15), (32), (33) and (34), there exists a constant \( \mu_1 > \mu_0 \), such that for \( \mu \geq \mu_1 \), one can find a \( \lambda_1(\mu_1) > \lambda_0(\mu_0) \), so that for \( \lambda \geq \lambda_1(\mu_1) \),
\[
\mathbb{E} \int_{Q_1 \setminus Q_0} \theta^2 (|\nabla y|^2 + y^2) dx dt \\
\geq C\lambda_1^3 \mu_1^4 \mathbb{E} \int_{Q_1 \setminus Q_0} \alpha^3 v^2 dx dt + C\lambda_1^2 \mu_1^2 \mathbb{E} \int_{Q_1 \setminus Q_0} \alpha |\nabla v|^2 dx dt.
\]
(35)

As a direct consequence of (35), we obtain that
\[
\lambda_1^3 \mu_1^4 \mathbb{E} \int_{Q_1} \alpha^3 \theta^2 z^2 dx dt + \lambda_1^2 \mu_1^2 \mathbb{E} \int_{Q_1} \alpha \theta^2 |\nabla z|^2 dx dt \\
\leq C\lambda_1^3 \mu_1^4 \mathbb{E} \int_{Q_0} \alpha^3 \theta^2 y^2 dx dt + C\lambda_1^2 \mu_1^2 \mathbb{E} \int_{Q_0} \alpha \theta^2 |\nabla y|^2 dx dt \\
+ C \mathbb{E} \int_{Q_2 \setminus Q_3} \theta^2 (|\nabla y|^2 + y^2) dx dt.
\]
Here $C$ is a constant depending on $\Lambda$.

Noticing that $z = y$ in $Q_3$ and $Q_3 \subset Q_1$, we see that the above inequality implies that

$$
\lambda^3 \mu^4 \mathbb{E} \int_{Q_3} \alpha^2 \theta^2 y^2 \, dx \, dt + \lambda \mu^2 \mathbb{E} \int_{Q_3} \alpha \theta^2 |\nabla y|^2 \, dx \, dt
\leq C \lambda^3 \mu^4 \mathbb{E} \int_{Q_0} \alpha^2 \theta^2 y^2 \, dx \, dt + \lambda \mu^2 \mathbb{E} \int_{Q_0} \alpha \theta^2 |\nabla y|^2 \, dx \, dt
+ C \mathbb{E} \int_{Q_2 \setminus Q_3} \theta^2 (|\nabla y|^2 + y^2) \, dx \, dt.
$$

Noticing $Q_4 \subset Q_3$ and by (11), it is easy to show that

$$
\lambda^3 \mu^4 \mathbb{E} \int_{Q_3} \alpha^2 \theta^2 y^2 \, dx \, dt + \lambda \mu^2 \mathbb{E} \int_{Q_3} \alpha \theta^2 |\nabla y|^2 \, dx \, dt
\geq C \exp(2\lambda \beta_4) \mathbb{E} \int_{Q_0} \theta^2 (\lambda^3 y^2 + \lambda |\nabla y|^2) \, dx \, dt.
$$

Further, from the definition of $Q_2$ and $Q_3$, we find that

$$
\mathbb{E} \int_{Q_2 \setminus Q_3} \theta^2 (|\nabla y|^2 + y^2) \, dx \, dt \leq \exp(2\lambda \beta_3) \mathbb{E} \int_{Q_0} (|\nabla y|^2 + y^2) \, dx \, dt.
$$

Therefore, it holds that

$$
\exp(2\lambda \beta_4) \mathbb{E} \int_{Q_0} \theta^2 (\lambda^3 y^2 + \lambda |\nabla y|^2) \, dx \, dt
\leq C \lambda^3 \mu^4 \mathbb{E} \int_{Q_0} \alpha^2 \theta^2 y^2 \, dx \, dt + C \lambda \mu^2 \mathbb{E} \int_{Q_0} \alpha \theta^2 |\nabla y|^2 \, dx \, dt
+ C \exp(2\lambda \beta_3) \mathbb{E} \int_{Q_1} (y^2 + |\nabla y|^2) \, dx \, dt.
$$

Fixing $\mu = \mu_1$ in $\beta_3, \beta_4$ and $\theta$, we conclude that

$$
\exp(2\lambda \beta_4) \mathbb{E} \int_{Q_0} \theta^2 (\lambda^3 y^2 + \lambda |\nabla y|^2) \, dx \, dt
\leq C \exp(C \lambda) \mathbb{E} \int_{Q_0} (\lambda^3 y^2 + \lambda |\nabla y|^2) \, dx \, dt + C \exp(2\lambda \beta_3) \mathbb{E} \int_{Q_1} (y^2 + |\nabla y|^2) \, dx \, dt.
$$

Thus, for any $\lambda \geq \lambda_1(\mu_1)$,

$$
\mathbb{E} \int_{t_0 - \frac{\delta}{\sqrt{N}}}^{t_0 + \frac{\delta}{\sqrt{N}}} \mathcal{G}(y^2 + |\nabla y|^2) \, dx \, dt
\leq C \exp(C \lambda) \mathbb{E} \int_{Q_0} (y^2 + |\nabla y|^2) \, dx \, dt
+ C \exp \left( -2\lambda (\beta_4 - \beta_3) \right) \mathbb{E} \int_{Q_1} (y^2 + |\nabla y|^2) \, dx \, dt.
$$
Put \( \varepsilon_0 = \exp \left( -2\lambda_1 (\beta_4 - \beta_3) \right) \). Then one finds that for any \( 0 < \varepsilon < \varepsilon_0 \), it holds that
\[
E \int_{t_0 - \frac{\sqrt{N}}{\varepsilon}}^{t_0 + \frac{\sqrt{N}}{\varepsilon}} \int_{G'} (y^2 + |\nabla y|^2) dx dt
\leq C \exp \left( \frac{C}{\varepsilon} \right) E \int_{Q_0} (y^2 + |\nabla y|^2) dx dt + C \varepsilon E \int_{Q_1} (y^2 + |\nabla y|^2) dx dt.
\] (36)

It is obvious that (36) holds for any \( \varepsilon > 0 \).

Noticing that \( t_0 \in [\sqrt{2}\delta, T - \sqrt{2}\delta] \), take \( t_0 = \sqrt{2}\delta + \frac{k\delta}{\sqrt{N}} \), \( k = 0, 1, 2, \cdots, m \) such that
\[
\sqrt{2}\delta + \frac{m\delta}{\sqrt{N}} \leq T - \sqrt{2}\delta \leq \sqrt{2}\delta + \frac{(m+1)\delta}{\sqrt{N}}.
\]

As a direct result, we have that
\[
E \int_{T + \kappa T}^{T - \kappa T} \int_{G'} (|\nabla y|^2 + y^2) dx dt
\leq E \int_{T - \sqrt{2}\delta}^{T - \sqrt{2}\delta} \int_{G'} (|\nabla y|^2 + y^2) dx dt
\leq \sum_{k=1}^{m} E \int_{\sqrt{2}\delta + \frac{k\delta}{\sqrt{N}}}^{\sqrt{2}\delta + \frac{(k+1)\delta}{\sqrt{N}}} \int_{G'} (|\nabla y|^2 + y^2) dx dt
\leq C \exp \left( \frac{C}{\varepsilon} \right) E \int_{Q_0} (|\nabla y|^2 + y^2) dx dt + C \varepsilon E \int_{Q_1} (|\nabla y|^2 + y^2) dx dt.
\] (37)

Taking \( \tilde{G} \times (0, T) \supset Q_1 \) in (37), we complete the proof of Theorem 1.1.

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