EXAMPLES OF SCALAR-FLAT HYPERSURFACES IN $\mathbb{R}^{n+1}$

JORGE H. LIRA AND MARC SORET

ABSTRACT. Given a hypersurface $M$ of null scalar curvature in the unit sphere $S^n$, $n \geq 4$, such that its second fundamental form has rank greater than 2, we construct a singular scalar-flat hypersurface in $\mathbb{R}^{n+1}$ as a normal graph over a truncated cone generated by $M$. Furthermore, this graph is 1-stable if the cone is strictly 1-stable.

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1. INTRODUCTION

A consistent theme of research is the use of refined perturbation techniques in the study of constant mean curvature surfaces and metrics with positive constant scalar curvature. New and complex examples and deep results on structure of moduli space of solutions had been achieved with the aid of those techniques.

A kind of prototype of this type of construction may be found at the seminal paper [3]. There, the authors prove the existence of minimal hypersurfaces with an isolated singularity in $\mathbb{R}^{n+1}$. These examples arise as perturbations of cones over minimal hypersurfaces of $S^n$.

Our contribution here focuses on a similar construction but for scalar-flat singular hypersurfaces in Euclidean space $\mathbb{R}^{n+1}$. We consider a truncated cone $\bar{M}^*$ in $\mathbb{R}^{n+1}$ generated by a hypersurface $M$ of $S^n$ that satisfies $S_2 = 0$ and then we take normal graphs over that cone. A priori estimates plus a fixed point theorem assure the existence of a graph with “small” boundary data which also satisfies the equation $S_2 = 0$.

We recall that $S_2$ is one of the elementary symmetric functions $S_r$, $1 \leq r \leq n$, of the principal curvatures of a hypersurface in $\mathbb{R}^{n+1}$. An interesting feature of $S_2$ is that this curvature is intrinsic and coincides with the scalar curvature of the hypersurface.

Our aim here is to provide a test case that gives an evidence that the well succeeded perturbation methods alluded above may be also applicable to deal with some geometric problems involving fully nonlinear elliptic equations. The results we obtained are in some sense local. Global issues may be addressed only if we are able to overcome serious technical difficulties.

Theorem 1. Let $M$ be a scalar-flat hypersurface in $\mathbb{S}^n$, $n \geq 4$. Suppose that the rank of the second fundamental form of $M$ is greater than or equal to 3. Let $\psi$ be a function in $C^{2,\alpha}(M)$. There exists $\Lambda < 1$ depending on $M$ such that for each $\lambda \in [0, \Lambda)$ there exists a function $u_{\lambda}$ defined in $\bar{M}^*$ whose graph $\bar{M}^*_\lambda$ has null scalar curvature and boundary given by $\Pi_J(u_{\lambda}) = \Pi_J(\lambda \psi)$, for some integer $J$.

Here, $\Pi_J$ is the projection map defined in p. 10.

This paper has the following presentation. In Section 2, we deduce the null scalar curvature equation $\bar{R}(u) = 0$ for the normal graph of a function $u$ defined over $\bar{M}^*$. The linearized equation involves the Jacobi operator $L$ in $\bar{M}^*$ which turns to be elliptic in view...
of the hypothesis concerning the rank of the second fundamental form of $M$. Section 3 is devoted to solve in $\bar{M}^*$ a Dirichlet problem for the Jacobi operator with boundary data $\psi$. Following closely [3], the idea is that an adequate control of the data $f$ near the singular point in $\bar{M}^*$ permits to solve $Lu = f$ in terms of separation of variables techniques. Second order estimates for the resulting Fourier series $u$ may be obtained in suitably weighted Hölder spaces. Applying these estimates to the problem

\begin{equation}
Lu = Q(v), \quad u|_M = \psi,
\end{equation}

where $v$ is a function in a weighted Hölder space and $Q$ collects all nonlinear terms in $\bar{R}(v) = 0$, we reduce the nonlinear problem to that one of finding a fixed point for the map that associates $v$ to the solution of (1). This is achieved by showing that for small boundary data $\psi$, this map is a contraction.

In the last section we relate the stability of the normal graphs with the stability of the hypersurface $M \subset S^n$. There, stability refers to the functional $A_1$ defined by the integral of the mean curvature.

**Theorem 2.** If $\bar{M}^*$ is strictly 1-stable, then the graph $\bar{M}^*_{\lambda}$ of the function $u_{\lambda}$ given in Theorem 1 is strictly 1-stable for $\lambda$ sufficiently small.

We point out that the results presented here may be easily adapted to the other higher order mean curvatures $S_r$, $r \geq 3$. It is interesting to produce examples with singular sets with small codimension as Nathan Smale did for minimal hypersurfaces in [12]. This is the subject of current research by the authors.

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2. **Scalar-flat cones**

2.1. **The scalar curvature equation.** Let $M$ be a compact hypersurface of the unit sphere $S^n$ in the Euclidean space $\mathbb{R}^{n+1}$. The cone over $M$ is the hypersurface $\bar{M}$ in $\mathbb{R}^{n+1}$ parametrized by

\begin{equation}
X(t, \theta) = t \theta, \quad t \in \mathbb{R}^+, \theta \in M.
\end{equation}

Let $N$ be an unit normal vector field to $M$. Parallel transporting $N$ along the rays $t \mapsto t \theta$ gives rise to a normal vector field to $\bar{M}$. One then defines the first and second fundamental forms of $\bar{M}$ respectively by

\begin{equation}
I = \langle dX, dX \rangle, \quad II = -\langle dN, dX \rangle.
\end{equation}

Let $x^1, \ldots, x^{n-1}$ be local coordinates in $M$ with corresponding coordinate vector fields denoted by $\partial_1, \ldots, \partial_{n-1}$. A local frame tangent to $\bar{M}$ may be given by adding the vector field $\partial_t$ to that coordinate local frame. In terms of such a frame, the first quadratic form is represented by the matrix

\begin{equation}
(\bar{g}_{\mu\nu}) = \begin{pmatrix}
t^2 \theta_{ij} & 0 \\
0 & 1
\end{pmatrix}
\end{equation}

and the second fundamental form has components

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(\bar{b}_{\mu\nu}) = \begin{pmatrix}
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where $\theta_{ij} = (\partial_i, \partial_j)$ and $b_{ij} = -\langle \partial_j N, \partial_i \rangle$ are the components of the first and second fundamental forms of the immersion $M \subset S^n$. Thus, the Weingarten map $\tilde{A}$ of $M$ has local components given by $\tilde{a}_{ij}^\mu = \tilde{g}^{\mu\rho}\tilde{b}_{\rho\nu}$. We then compute

$$
(\tilde{a}_{ij}^\mu) = \begin{pmatrix} \frac{1}{a_j^1} a_j^i & 0 \\ 0 & 0 \end{pmatrix},
$$

where $a_j^i = \theta^{ik}b_{jk}$ are the components of the Weingarten map $A$ of $M$ defined by $N_i = -a_j^i\partial_j$.

If we denote by $\lambda_1, \ldots, \lambda_{n-1}$ the eigenvalues of $A$, then the eigenvalues of $\tilde{A}$ are

$$
0, \frac{1}{t} \lambda_1, \ldots, \frac{1}{t} \lambda_{n-1}.
$$

The $r$-th mean curvature $\tilde{H}_r$ of $M$ is defined by

$$
\tilde{H}_r = \frac{1}{\binom{n}{r}} \tilde{S}_r, \quad 1 \leq r \leq n,
$$

where $\tilde{S}_r$ are the elementary symmetric functions of the eigenvalues of $\tilde{A}$ relative to $I$ given by

$$
\det (\text{Id} - s \tilde{A}) = 1 - s \tilde{S}_1 + s^2 \tilde{S}_2 + \ldots + (-s)^{n-1} \tilde{S}_{n-1} + (-s)^n \tilde{S}_n.
$$

Denoting by $H_r$ and $S_r$ the corresponding functions on $M$, one easily proves that

$$
\tilde{S}_r = \frac{1}{\binom{n}{r}} S_r, \quad 1 \leq r \leq n - 1
$$

and $\tilde{S}_n = 0$. For a given multi-index $i_1 < \ldots < i_r$ with $1 \leq i_k \leq n$, we denote

$$
D_{i_1 \ldots i_r} = \det (\theta_{ij} \ldots b_{i_1 j} \ldots b_{i_r j} \ldots \theta_{n-1 j}),
$$

that is, $D_{i_1 \ldots i_r}$ is the determinant of the matrix obtained replacing in $(\theta_{ij})$ the columns numbered by $i_1, \ldots, i_r$ by the corresponding columns in $(b_{ij})$.

In terms of these determinants, one calculates

$$
\det(\theta_{ij}) S_r = \sum_{i_1 < \ldots < i_r} D_{i_1 \ldots i_r}.
$$

We suppose that $M$ satisfies $S_2 = 0$. Thus, the cone $\tilde{M}$ is a scalar-flat manifold, that is, it holds that $\tilde{S}_2 = 0$.

2.2. The scalar curvature equation for normal graphs over cones. From now on, we will be mainly concerned with linearizing the equation $\tilde{S}_2 = 0$ near $\tilde{M}$. Given a function $u : \tilde{M} \rightarrow \mathbb{R}$ with sufficiently small $C^2$ norm, its normal graph is defined as the hypersurface

$$
\tilde{M}_u = \{ X(t, \theta) + u(t, \theta) N : t \in \mathbb{R}^+, \theta \in M \}.
$$

We denote by $\tilde{S}_2(u)$ the scalar curvature of $\tilde{M}_u$. We then proceed to linearize the equation $\tilde{S}_2(u) = 0$ and to describe the nonlinear part of this equation.

We begin by determining the quadratic fundamental forms in $\tilde{M}_u$. The tangent space to $\tilde{M}_u$ is spanned by the vector fields $\theta + u_t N$ and

$$
\frac{t}{\sqrt{\rho}} (\tilde{S}_i^j - u \tilde{a}_i^j) \partial_j + u_t N,
$$

where $u_t = \frac{\partial u}{\partial t}$ and $u_i = \frac{\partial u}{\partial \theta_i}$. The induced metric in $\tilde{M}_u$ has components

$$
\tilde{g}_{\mu\nu}(u) = \tilde{g}_{\mu\nu} + \delta\tilde{g}_{\mu\nu},
$$
where
\[
(\delta \bar{g}_{\mu \nu}) = \left( -2u \bar{h}_{ij} + u^2 \bar{r}_{ij} + u_i u_j \right) \frac{u_i u_j}{u^2}
\]
and \(\bar{r}_{ij} = t^2 \theta_{kl} \bar{a}_k^i \bar{a}_l^j\) are the components of the third fundamental form \((dN, dN)\) of \(M\).

More briefly, we may write
\[
(\delta \bar{g})_{\mu \nu} = -2u \bar{b}_{\mu \nu} + u^2 \bar{r}_{\mu \nu} + u_\mu u_\nu.
\]

Let \(\bar{R}_{\mu \nu}\) be the Ricci tensor of \(M\). If we denote \(\bar{R} = \bar{S}_2\) and \(\bar{R}'(u) = \bar{S}_2(u)\) then it follows that
\[
\bar{R}(u) = \bar{R} + \delta \bar{R},
\]
where
\[
\delta \bar{R} = \bar{g}^\rho_{\mu \nu} \delta \bar{R}_{\mu \nu} + \delta \bar{g}^{\mu \nu} \bar{R}_{\mu \nu}.
\]

A classical tensorial identity (see [4], p. 398) states that
\[
\bar{g}^\rho_{\mu \nu} \delta \bar{R}_{\mu \nu} = \bar{\nabla}_\rho W^\rho,
\]
where \(\bar{\nabla}\) denotes the Riemannian covariant derivative in \(M\) with respect to the metric \((\bar{g}_{\mu \nu})\) and
\[
W^\rho = \bar{g}^\rho_{\mu \sigma} \bar{g}^{\mu \nu} \bar{\nabla}_\nu \delta \bar{g}_{\mu \sigma} - \bar{g}^\rho_{\mu \sigma} \bar{g}^{\mu \nu} \bar{\nabla}_\nu \delta \bar{g}_{\mu \sigma}.
\]

In what follows, we use the abbreviated notation \(\bar{\nabla}_\rho = \bar{g}^\rho_{\mu \nu} \bar{\nabla}_\nu\).

Since \(\bar{\nabla} \bar{g} = 0\) we may commute the covariant derivatives and the components \(\bar{g}^{\mu \nu}\) in the formula above (16), obtaining
\[
\bar{g}^{\mu \nu} \delta \bar{R}_{\mu \nu} = \bar{\nabla}_\rho \bar{g}^{\mu \sigma} \bar{g}^{\nu \rho} \bar{\nabla}_\nu \delta \bar{g}_{\mu \sigma} - \bar{\nabla}_\rho \bar{g}^{\mu \sigma} \bar{g}^{\nu \rho} \bar{\nabla}_\nu \delta \bar{g}_{\mu \sigma} = -2\bar{\nabla}_\rho \bar{g}^{\mu \sigma} \bar{g}^{\nu \rho} \bar{b}_{\mu \sigma} + 2\bar{\nabla}_\rho \bar{g}^{\mu \sigma} \bar{g}^{\nu \rho} \bar{b}_{\mu \sigma} u + Q_1 = 2\bar{\nabla}_\rho \bar{\nabla}^{\rho} (\bar{S}_2 - \bar{\bar{a}}_2) u + Q_1 + 2\bar{\nabla}^{\rho} \bar{\nabla}_\rho \bar{T}_2 u + Q_1,
\]
where \(\bar{T}_2^\rho\) are the components of the \((1, 1)\) tensor field
\[
\bar{T}_1 = \bar{S}_1 \text{Id} - \bar{\bar{A}}
\]
and
\[
Q_1 = \bar{\nabla}_\rho \bar{\nabla}^{\rho} (u^2 \bar{r}_\mu^\rho + u^\rho u_\mu) - \bar{\nabla}_\rho \bar{\nabla}_\rho (u^2 \bar{r}_\mu^\rho + u^\rho u_\mu).
\]

However, we have
\[
\bar{\nabla}_\rho \bar{\nabla}^{\rho} (u^\rho u_\mu) = \bar{g}^{\mu \rho} \bar{g}^{\rho \sigma} \bar{\nabla}_\rho (u_\tau u_\mu) = \bar{g}^{\mu \rho} \bar{g}^{\rho \sigma} \bar{\nabla}_\rho (u_\tau u_\mu) - \bar{g}^{\rho \sigma} \bar{g}^{\mu \rho} \bar{\nabla}_\sigma (u_\tau u_\mu)
\]
\[
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\]
\[
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\]

Using Ricci identity
\[
u_\nu u_\nu = \bar{R}_{\mu \rho \sigma} u^\sigma
\]
where $\bar{R}_{\tau \rho \sigma \nu}$ is the Riemann curvature tensor in $\bar{M}$, we rewrite the terms with third order derivatives as follows

\[
\begin{align*}
\bar{g}^{\mu \nu} \bar{g}^{\rho \tau} (u_{\tau} u_{\rho} u_{\mu} + u_{\tau} u_{\mu} u_{\rho}) - \bar{g}^{\rho \tau} \bar{g}^{\mu \nu} (u_{\mu} u_{\tau} u_{\rho} + u_{\mu} u_{\rho} u_{\tau}) \\
= \bar{g}^{\mu \nu} \bar{g}^{\rho \tau} (u_{\tau} u_{\rho} u_{\mu} - u_{\tau} u_{\mu} u_{\rho}) + \bar{g}^{\mu \nu} \bar{g}^{\rho \tau} u_{\tau} u_{\mu} - \bar{g}^{\rho \tau} \bar{g}^{\mu \nu} u_{\mu} u_{\rho} \\
= \bar{g}^{\mu \nu} \bar{g}^{\rho \tau} (u_{\tau} u_{\rho} u_{\mu} - u_{\tau} u_{\mu} u_{\rho}) + \bar{g}^{\mu \nu} \bar{g}^{\rho \tau} u_{\tau} u_{\mu} - \bar{g}^{\rho \tau} \bar{g}^{\mu \nu} u_{\mu} u_{\rho} \\
= \bar{g}^{\mu \nu} \bar{g}^{\rho \tau} \bar{R}_{\tau \rho \sigma \nu} u_{\sigma} u_{\nu} + \bar{g}^{\mu \nu} \bar{g}^{\rho \tau} u_{\mu} u_{\nu} u_{\tau} + \bar{g}^{\mu \nu} \bar{R}_{\rho \sigma \nu \mu} u_{\sigma} u_{\nu} - \bar{g}^{\rho \tau} \bar{g}^{\mu \nu} u_{\mu} u_{\nu} u_{\rho}.
\end{align*}
\]

The antisymmetry of the curvature tensor in the last two indices implies that $\bar{g}^{\rho \tau} \bar{R}_{\tau \rho \sigma \nu} u_{\sigma} u_{\nu} = 0$. Therefore, one has

\[
\begin{align*}
\bar{g}^{\mu \nu} \bar{g}^{\rho \tau} (u_{\tau} u_{\rho} u_{\mu} + u_{\tau} u_{\mu} u_{\rho}) - \bar{g}^{\rho \tau} \bar{g}^{\mu \nu} (u_{\mu} u_{\tau} u_{\rho} + u_{\mu} u_{\rho} u_{\tau}) \\
= \bar{g}^{\mu \nu} \bar{g}^{\rho \tau} (u_{\tau} u_{\rho} u_{\mu} - u_{\tau} u_{\mu} u_{\rho}) + \bar{g}^{\mu \nu} \bar{g}^{\rho \tau} u_{\tau} u_{\mu} - \bar{g}^{\rho \tau} \bar{g}^{\mu \nu} u_{\mu} u_{\rho} \\
= \bar{g}^{\mu \nu} \bar{g}^{\rho \tau} \bar{R}_{\rho \sigma \nu \mu} u_{\sigma} u_{\nu} + \bar{g}^{\mu \nu} \bar{R}_{\rho \sigma \nu \mu} u_{\sigma} u_{\nu} - \bar{g}^{\rho \tau} \bar{g}^{\mu \nu} u_{\mu} u_{\nu} u_{\rho}.
\end{align*}
\]

Thus, one concludes that

\[
\begin{align*}
\nabla_\rho \nabla_\mu (u_\rho u_\mu) - \nabla_\mu \nabla_\rho (u_\rho u_\mu) &= \bar{R}_{\rho \sigma \nu \mu} u_\sigma u_\nu + \bar{g}^{\mu \nu} \bar{R}_{\rho \sigma \nu \mu} u_\sigma u_\nu - \bar{g}^{\rho \tau} \bar{g}^{\mu \nu} u_{\mu} u_{\tau} \\
&= \bar{R}_{\rho \sigma \nu \mu} u_\sigma u_\nu + \bar{g}^{\mu \nu} \bar{R}_{\rho \sigma \nu \mu} u_{\mu} u_{\nu} - \bar{g}^{\rho \tau} \bar{g}^{\mu \nu} u_{\mu} u_{\nu}.
\end{align*}
\]

These calculations imply that

\[
\begin{align*}
Q_1 &= \nabla_\rho \nabla_\mu (u_\rho u_\mu) - \nabla_\mu \nabla_\rho (u_\rho u_\mu) \\
&= \bar{g}^{\mu \nu} \bar{R}_{\rho \sigma \nu \mu} u_\sigma u_\nu + \bar{g}^{\mu \nu} \bar{R}_{\rho \sigma \nu \mu} u_{\mu} u_{\nu} - \bar{g}^{\rho \tau} \bar{g}^{\mu \nu} u_{\mu} u_{\tau} \\
&= \bar{g}^{\mu \nu} \bar{R}_{\rho \sigma \nu \mu} u_\sigma u_\nu + \bar{g}^{\mu \nu} \bar{R}_{\rho \sigma \nu \mu} u_{\mu} u_{\nu} - \bar{g}^{\rho \tau} \bar{g}^{\mu \nu} u_{\mu} u_{\tau}.
\end{align*}
\]

where $\bar{r} = \bar{r}_\mu \mu$.

It is a well-known fact that the tensor $\bar{T}_1$ is divergence-free. Indeed, one computes using Codazzi’s equation

\[
(\delta_\mu \bar{S}_1 - \bar{a}_\mu)^\rho = \delta_\mu \bar{a}_\rho - \bar{a}_\mu = \bar{a}_\rho - \bar{a}_\mu = 0.
\]

Using this, one gets

\[
\begin{align*}
\bar{g}^{\mu \nu} \delta \bar{R}_{\mu \nu} &= 2 \nabla_\nu (\nabla_\mu (\bar{T}_\mu^{\nu} u + \bar{T}_\mu^{\nu} (\nabla_\mu u)) + \bar{Q}_1 = 2 \nabla_\nu \bar{T}_\mu^{\nu} \nabla_\mu u + \bar{Q}_1 \\
&= 2 \nabla_\nu (\bar{T}_\mu^{\nu} \nabla_\mu u) + \bar{Q}_1 = 2 \text{div} \bar{T}_1 \nabla u + \bar{Q}_1.
\end{align*}
\]

On the other hand, we infer from Gauss equation that

\[
\bar{R}_{\mu \nu} = \bar{g}^{\rho \sigma} \bar{R}_{\mu \nu \rho \sigma} = \bar{g}^{\rho \sigma} (\bar{b}_{\mu \nu} \bar{b}_{\rho \sigma} - \bar{b}_{\mu \rho} \bar{b}_{\nu \sigma}) = \bar{b}_{\mu \nu} \bar{S}_1 - \bar{r}_{\mu \nu}
\]

and since

\[
\delta \bar{g}^{\mu \nu} \bar{R}_{\mu \nu} = \delta \bar{g}^{\mu \nu} g_{\mu \nu} \bar{R}_\nu = -\delta g_{\mu \rho} \bar{g}^{\mu \nu} \bar{R}_\nu = -\delta g_{\mu \rho} \bar{R}_{\nu \rho},
\]

one obtains

\[
\begin{align*}
\delta \bar{g}^{\mu \nu} \bar{R}_{\mu \nu} &= 2 u \bar{S}_1 \bar{b}_{\mu \nu} \bar{b}_{\rho \sigma} - 2 u \bar{b}_{\mu \rho} \bar{b}_{\nu \sigma} + \bar{Q}_2 = 2 u \bar{S}_1 \text{tr} \bar{A}^2 - 2 u \text{tr} \bar{A}^2 + \bar{Q}_2 \\
&= 2 \text{tr} (\bar{S}_1 \text{Id} - \bar{A} \bar{A}) + \bar{Q}_2 = 2 \text{tr} \bar{T}_1 \bar{A}^2 + \bar{Q}_2.
\end{align*}
\]
where
\[ Q_2 = -\bar{R}^{\mu \rho} (u^2 \bar{r}_{\mu \rho} + u_{\rho} u_{\mu}). \]

Since we are assuming that \( \bar{S}_2 = 0 \) one easily verifies that
\[ \text{tr} \bar{A}^2 = -3 \bar{S}_3. \]

We then conclude that the equation \( \bar{R}(u) = 0 \) may be written as
\[ Lu + Q(u) = 0, \]
where
\[ Lu = \text{div} \bar{T}_1 \bar{u} - 3 \bar{S}_3 u \]
is the Jacobi operator for the scalar curvature and \( Q = Q_1 + Q_2. \)

The quadratic term \( Q \) has the form
\[ Q(u, \bar{\nabla} u, \bar{\nabla}^2 u) = |\Delta_{\bar{M}} u|^2 - |\bar{\nabla}^2 u|^2 + \text{tr} P_0(u) \cdot \bar{\nabla}^2 u + P_1 (u, \bar{\nabla} u) \]
where \( \Delta_{\bar{M}} \) is the Laplace-Beltrami operator in \( \bar{M} \) and
\[ (P_0)^\mu_{\mu} = 2(\bar{r}_{\mu \rho} - \delta_{\mu} \bar{r}) u \]
and
\[ P_1 = (2\bar{r}_{\rho \mu} - 2\bar{g}_{\rho \mu} \bar{r}) u^\rho u^\mu + 4(\bar{r}_{\mu \rho} - \bar{r}_{\mu \rho}) u u^\rho \]
\[ + (\bar{g}^{\mu \nu} \bar{r}_{\mu \nu} - \bar{g}^{\mu \nu} \bar{r}_{\mu \nu} - \bar{R}^{\mu \nu}) u^2. \]

### 3. The Dirichlet Problem for the Jacobi Operator

As we proved above, a normal graph \( \bar{M}_u \) is scalar-flat if \( u \) satisfies the fully nonlinear equation (22). Our goal in this section is to solve the corresponding linearized equation for small boundary data by using Fourier analysis in some suitably weighted spaces.

Following the notation previously fixed, we denote
\[ \bar{L}_1 u = \text{div} \bar{T}_1 \bar{u}. \]
The corresponding tensor and operator in \( M \) are respectively
\[ T_1 = S_1 \text{Id} - A \]
and
\[ L_1 u = \text{div} T_1 u, \]
where the divergence and gradient are taken this time on \( M \). In [1], it is proved that the operators \( L \) and \( \bar{L}_1 \) decomposes as follows
\[ \bar{L}_1 u = \frac{1}{t} S_1 u_{tt} + \frac{n - 2}{t^2} S_1 u_t + \frac{1}{t^3} L_1 u(t, \cdot) \]
and
\[ L u = \frac{1}{t} S_1 u_{tt} + \frac{n - 2}{t^2} S_1 u_t + \frac{1}{t^3} L_1 u(t, \cdot) - 3 \frac{1}{t^3} S_3 u. \]

From now on, we assume that \( S_3 \) never vanishes along \( M \) or equivalently that \( \text{rk} A \geq 3 \). In [6], it is proved that this assumption assures the ellipticity of the second-order differential operator \( L \). This is a crucial ingredient in our analysis. We point out that there are examples of hypersurfaces fitting our assumptions in \( \mathbb{S}^n \) like certain products of spheres.
As an example, if we fix the lowest dimension \( n = 4 \), we may consider the product of spheres \( M = \mathbb{S}^2(a_1) \times \mathbb{S}^1(a_2) \) immersed in \( \mathbb{S}^4 \), where \( a_1 = \sqrt{1/3} \) and \( a_2 = \sqrt{2/3} \). With these choices one has \( S_2 = 0 \) and
\[
S_1 = 2\sqrt{2} - \sqrt{1/2} \quad \text{and} \quad S_3 = -\sqrt{2}.
\]
For a detailed explanation on these products of spheres, we refer the reader to [1].

We begin our analysis of the equation (22) by solving first the non-homogeneous linear Dirichlet problem for the Jacobi operator
\[
Lu = f \quad \text{in} \quad \bar{M}^*, \quad u = \psi \quad \text{in} \quad M
\]
where \( \bar{M}^* \) is the truncated cone obtained restricting the variable \( t \) to \((0, 1]\). Using (28), we reduce the linear equation \( Lu = f \) to
\[
t^2S_1u_{tt} + (n-2)tS_1u_t + L_1u(t, \cdot) - 3S_3u = t^3f(t, \cdot).
\]
The hypothesis on \( S_3 \) implies that \( S_1 \) also never vanishes. We then may choose an orientation for \( M \) in such a way that \( S_1 > 0 \). Hence, the operator in \( M \) defined by
\[
-S_1^{-1}(L_1 - 3S_3)
\]
has \( L^2(M, S_1d\theta) \) discrete spectra given by a set of diverging eigenvalues
\[
\mu_1 \leq \mu_2 \leq \ldots \rightarrow +\infty
\]
with corresponding eigenfunctions \( \{\phi_1, \phi_2, \ldots\} \). These facts permit to separate variables in (30) and reduce the problem to the determination of a Fourier series for \( u \). We will see that a formal solution of (30) in Fourier series gives rise to convergent solutions if we consider functions \( f = f(t, \theta) \) such that
\[
|f|_2^2 := \int_M f(t, \theta)^2S_1(\theta)^{-1}d\theta < \infty, \quad t \in (0, 1].
\]
Let \( m > 2 \) and \( \epsilon > 0 \) be real constants to be chosen later. It is required too that the function \( t \mapsto |f|_t \) satisfies
\[
\sup_{(0, 1)} t^{2-m-\epsilon}|f|_t < \infty.
\]
This implies that \( f(0, \cdot) = 0 \) and
\[
||f|| := \left( \int_0^1 t^{4-2m}|f|_t^2dt \right)^{1/2} < \infty.
\]
Under the assumptions above on \( f \), it is possible to decompose it in its Fourier series
\[
\frac{f}{S_1} = \sum_{j=1}^\infty f_j(t)\phi_j(\theta)
\]
with \( f_j(t) = \int f \phi_j d\theta \). Let \( u \) be a formal solution
\[
u(t, \theta) = \sum_{j} a_j(t)\phi_j(\theta)
\]
of equation (30). Thus, the coefficients \( a_j \) are determined by the sequence of ODE’s
\[
t^2a_j'' + (n-2)taj' - \mu_ja_j = t^3f_j, \quad j = 1, 2, \ldots
\]
The homogeneous equations associated to \((38)\) have solutions of the form \(t^{\gamma_j}\) where \(\gamma_j\) is root of the characteristic equation \(\gamma^2 + (n - 3)\gamma - \mu_j = 0\). Its roots are the indicial roots

\[
\gamma_j = -\frac{n - 3}{2} \pm \sqrt{\frac{(n - 3)^2}{4} + \mu_j}.
\]

We observe that \(\gamma_j\) may be complex since \(\mu_j\) may be negative. In these cases, one has \(\Re \gamma_j = (3 - n)/2\). Since the eigenvalues \(\mu_j\) diverge to \(+\infty\), there exists an index \(J\) such that \(\Re (\gamma_{J+1}) = \gamma_{J+1} > 0\). This index may be chosen so that for a given \(m > 2\) it holds that

\[
\frac{3 - n}{2} \leq \Re(\gamma_J) < m \leq \Re(\gamma_{J+1}) \leq \Re(\gamma_{J+2}) \leq \ldots
\]

From now on, we consider these choices for \(m\) and \(J\).

In order to find a particular solution of the non-homogeneous equation \((38)\), we consider functions of the form \(a_j(t) = t^{\gamma_j} v_j(t)\). Plugging this expression of \(a_j\) in \((38)\) we obtain

\[
t^{\gamma_j + 2} v'' + (2\gamma_j + n - 2)t^{\gamma_j + 1} v' = t^3 f_j
\]

and after multiplying this equation by \(t^{\gamma_j + n - 4}\) one has

\[
(t^{n-2+2\gamma_j} v_j)' = t^{n-1+\gamma_j} f_j.
\]

Integrating twice we get

\[
v_j = \alpha_j + \int_{\beta_j}^t s^{2-n-2\gamma_j} \int_0^s \tau^{n-1+\gamma_j} f_j \, d\tau \, ds, \quad j = 1, 2, \ldots
\]

where \(\alpha_j\) and \(\beta_j\) are constants of integration to be specified in the sequel. We conclude that the formal solution \(u = \sum_j a_j \phi_j\) to equation \((30)\) has coefficients of the form

\[
a_j(t) = \Re \left( \alpha_j t^{\gamma_j} + \int_{\beta_j}^t s^{2-n-2\gamma_j} \int_0^s \tau^{n-1+\gamma_j} f_j(\tau) \, d\tau \, ds \right).
\]

We claim that the integrals in the definition of these coefficients converge in \((0, 1]\) if we choose \(\alpha_j = \beta_j = 0\) for \(j \leq J\) and \(\beta_j = 1\) for \(j \geq J + 1\). In fact, one has

\[
f_j(t) = \int_{M} \frac{f}{\sqrt{S_1}} \phi_j \sqrt{S_1} \, d\theta \leq \sqrt{\int_{M} \frac{f^2}{S_1} \, d\theta} \sqrt{\int_{M} \phi_j^2 S_1 \, d\theta} = |f|_{L_1}.
\]

Thus, using the hypothesis \((35)\) and Cauchy-Schwarz inequality we estimate, for a constant \(c\) that does not depend on \(f\),

\[
\int_0^s \tau^{n-1+\gamma_j} f_j(\tau) \, d\tau \leq \sqrt{\int_0^s \tau^{2(n-1+\gamma_j)} \tau^{2m-4} \, d\tau} \sqrt{\int_0^s \tau^{4-2m} |f|_2^2 \, d\tau} = c ||f|| ||s^{n-3+m+\gamma_j+\frac{1}{2}}||,
\]

where we used the fact that \(m > \Re \gamma_j\) for \(j \leq J\) in order to assure convergence of the integral at \(s = 0\). This estimate implies that

\[
t^{\gamma_j} \int_{\beta_j}^t s^{2-n-2\gamma_j} \int_0^s \tau^{n-1+\gamma_j} f_j(\tau) \, d\tau \, ds \leq c ||f|| |t^{\gamma_j} \int_{\beta_j}^t s^{m-\gamma_j-\frac{1}{2}} \, ds|
\]

For \(j \leq J\), the right hand side converges at \(t = 0\) if one sets \(\beta_j = 0\). For \(j \geq J + 1\), it converges if we consider \(\beta_j = 1\). This proves the claim.
EXAMPLES OF SCALAR-FLAT HYPERSURFACES IN $\mathbb{R}^{n+1}$

The values of $\alpha_j$ for $j \geq J + 1$ are determined by

\begin{equation}
\alpha_j := \int_M \lim_{t \to 1} u(t, \cdot) \phi_j, \quad j \geq J + 1.
\end{equation}

Let $\Pi_J$ be the projection of $L^2(M, S_1 d\theta)$ in the linear subspace spanned by the eigenfunction $\phi_j$, $j \geq J + 1$. Thus,

\begin{equation}
\Pi_J(u) = \Pi_J(\psi)
\end{equation}

if and only if

\begin{equation}
\Pi_J(\psi) = \sum_{j=J+1}^{\infty} \alpha_j \phi_j.
\end{equation}

Thus, since $\psi \in L^2(M, S_1 d\theta)$, one has

\begin{equation}
\sum_{j=J+1}^{\infty} \alpha_j^2 < \infty.
\end{equation}

In this case, we then had verified that the problem (29) has as solution the convergent Fourier series $u$ defined by the coefficients $a_j$ above.

In particular we have found a solution to the equation $Lu = 0$ with boundary Dirichlet data $\psi$ referred to in what follows as the $L$-harmonic extension of $\psi$. In other terms we denote by $H_J(\psi)$ the Fourier series solution of

$$Lu = 0 \text{ in } \bar{M}^*, \quad \Pi_J(u) = \Pi_J(\psi) \text{ in } M.$$ 

Notice that our previous calculations imply that

\begin{equation}
H_J(\psi) = \sum_{j=J+1}^{\infty} \alpha_j t^{s_j} \phi_j.
\end{equation}

and $H_J$ is a right inverse to $\Pi_J$.

In order to obtain integral estimates for $u$, we notice that since

\begin{equation}
|uS_1|^2 = \int_M u^2(t, \theta) S_1(\theta) d\theta = \sum_{j=1}^{\infty} a_j^2(t)
\end{equation}

it follows that

\begin{equation}
|u|^2 \leq c \sum_{j=1}^{\infty} a_j^2(t)
\end{equation}

where $c = 1/(\inf_M S_1^2(\theta))$. On the other hand, using (44) and (45), one obtains from Cauchy-Schwarz inequality

\begin{equation}
t^{-m} |u|_t \leq t^{-m} \sqrt{\sum_{j=1}^{\infty} a_j^2(t)} \leq c\|f\| + \sqrt{\sum_{j=J+1}^{\infty} \alpha_j^2},
\end{equation}

where $c > 0$ is a positive constant which depends on $M, m$ and $J$. In a similar way, using (44) and (50) one proves that

\begin{equation}
t^{-m} |u - H_J(\psi)|_t \leq c\|f\|.
\end{equation}

We summarize the facts above in the following proposition.
Proposition 1. Let $m > 2$ be a constant and let $J$ be an integer such that $0 < \Re(\gamma_j) < m < \Re(\gamma_{j+1})$ for $\gamma_j$ given by (39). Given a function $f$ defined in $\bar{M}^*$ satisfying
\[
\sup_{(0,1)} t^{2-m-\epsilon} |f|_t < \infty
\]
and a function $\psi \in L^2(M, S_1d\theta)$, the series
\[
u = \sum_{j=1}^{\infty} a_j \phi_j
\]
with $a_j$ defined by (44) is the unique solution of
\[
Lu = f \text{ in } \bar{M}^* \quad \text{and} \quad \Pi_j(u) = \Pi_j(\psi) \text{ in } M
\]
satisfying
\[
\sup_{(0,1)} t^{-m} |u|_t < \infty.
\]
Moreover, we have the following estimates for $u$
\[
t^{-m} |u|_t \leq c \left( ||f|| + ||\Pi_j(\psi)|| \right),
\]
\[
t^{-m} |u - H_j(\psi)|_t \leq c ||f||,
\]
where the constant $c$ does not depend on $f$.

Proof of the uniqueness. In view of the previous discussion, it remains to prove the uniqueness of the solution. If we consider two solutions $u_1$ and $u_2$ of the equation $Lu = f$, then their difference $v = u_1 - u_2$ is decomposed as $v = \sum j b_j \phi_j$ where the functions $b_j$ are solutions of the homogeneous ODE associated to (38). Notice that (55) implies that $u_1$ and $u_2$ vanish at the origin. Thus, $b_j(t) \rightarrow 0$ as $t \rightarrow 0$ for all $j$. Moreover, if $j \geq J + 1$ then $\gamma_j$ is real and positive. So, $\mu_j$ is necessarily positive. Therefore the maximum principle guarantees that $b_j = 0$ for all $j \geq J + 1$. For $j \leq J$ we have that $b_j$ is of the form $b_j = ct^{\gamma_j} + ct^{\bar{\gamma}_j}$ where $\gamma_j, \bar{\gamma}_j$ are the roots of the characteristic equation. Thus $|t^{-m}b_j| \rightarrow \infty$ unless that $b_j = 0$ for $j \leq J$. So, we have proved the proposition.

Following [3] we now define some weighted Hölder spaces in terms of that it is possible to obtain second order estimates for the solution of the linear problem.

More precisely, we introduce as in [3] and [9], the norms
\[
|u|_{k,\alpha,t} = \sum_{l=0}^{k} t^l |\nabla^l v|_{0,A_t} + t^{k+\alpha} |\nabla^k u|_{\alpha, A_t},
\]
for $t \in (0, 1/2)$, $k$ a positive integer and $\alpha \in (0, 1)$. Here, $A_t$ is the truncated cone corresponding to $t < |x| < 2t$ and $| \cdot |_{0,\alpha, A_t}$ denotes the usual Hölder norm in $A_t$.

Proposition 2. Under the hypothesis of the Proposition[7] the function $u$ satisfies
\[
t^{-m} |u|_{2,\alpha,t} \leq c \left( ||f||_\alpha + ||\Pi_j(\psi)||_\alpha \right),
\]
\[
t^{-m} |u - H_j(\psi)|_{2,\alpha,t} \leq c ||f||_\alpha,
\]
for $t \in (0, 1/2)$, $\psi \in C^2,\alpha(M)$ and
\[
||f||_\alpha \equiv \sup_{0 < t < 1/2} t^{2-m-\epsilon} |f|_{0,\alpha,t}
\]
where $\epsilon$ is a fixed positive number. The constants do not depend on $f$.

**Sketch of the proof.** A similar estimate for the Laplacian could be found in [9] and [9]. We may obtain the estimates for elliptic linear operators with constant coefficients and only second order terms. The general case could be handled by freezing coefficients in $L$. For usual Hölder norms, this method is nicely exposed in Chapters 4 and 6 of [5].

4. **Solving the nonlinear problem**

Using the weighted Hölder spaces we just defined above, we then introduce the subspace $B$ of $C^{2,\alpha}(\tilde{M}^*)$ consisting of the functions $v$ for which

$$
\|v\| = \sup_{0 < t < 1/2} t^{-m}|v|_{2,\alpha,t}
$$

is finite.

We define a map $U$ in the unit ball in $B$ in the following way: given a function $v \in B$ with $\|v\| < 1$, $U(v)$ is the solution of the linear problem

$$
LU = Q(v) \text{ in } M^*, \quad \Pi_J(U) = \Pi_J(\psi) \text{ in } M
$$

as defined in Proposition 1. Our task now is to exhibit a convex subset $K$ of the unit ball in $B$ so that $U|_K$ is a contraction map.

With this purpose, we begin by estimating $Q(v)$ for $v$ with $\|v\| < 1$. We have, using that $t < 1$,

$$
|Q(v)|_{0,\alpha,t} \leq 2|\nabla^2 v|_{0,\alpha,t} + |P_0|_{0,\alpha,t}|\nabla^2 v|_{0,\alpha,t} + |P_1|_{0,\alpha,t}
$$

$$
\leq 2(t^{-2}|v|_{2,\alpha,t})^2 + |P_0|_{0,\alpha,t}t^{-2}|v|_{2,\alpha,t} + |P_1|_{0,\alpha,t}
$$

$$
\leq 2t^{-4}|v|_{2,\alpha,t}^2 + C_0^2 t^{-2}|v|_{2,\alpha,t} + C_1(|v|_{0,\alpha,t} + |\nabla v|_{0,\alpha,t})^2
$$

$$
\leq 2t^{-4}|v|_{2,\alpha,t}^2 + C_0^2 t^{-2}|v|_{2,\alpha,t} + C_1(1 + t^{-1})^2|v|_{2,\alpha,t}^2
$$

$$
\leq \mu|v|_{2,\alpha,t}^2 \leq \mu^2 m^2\|v\|^2,
$$

where $C_0, C_1$ and $\mu$ are positive constants depending only on $M$.

We choose $\epsilon$ such that $m + 2 \geq \epsilon$. Since $t < 1$ we have $t^{2m} \leq t^{m-2+\epsilon}$. Thus we obtain

$$
|Q(v)|_{0,\alpha,t} \leq \mu^{m-2+\epsilon}\|v\|^2
$$

and similarly one easily verifies that

$$
|Q(v) - Q(w)|_{0,\alpha,t} \leq \mu(||v|| + ||w||)(||v - w||) t^{m-2+\epsilon}.
$$

It follows from estimates stated in Proposition 2 that $U(v)$ satisfies

$$
\|U(v) - H_J\psi\| = \sup_{0 < t < 1/2} t^{-m}|U(v) - H_J\psi|_{2,\alpha,t} \leq \epsilon \|f\|_\alpha
$$

$$
\leq c \sup_{0 < t < 1/2} t^{2-m-\epsilon}|Q(v)|_{0,\alpha,t} \leq c \mu \|v\|^2.
$$

Moreover since $L(U(v) - U(w)) = Q(v) - Q(w)$ and $\Pi_J(U(v)) = \Pi_J(U(w))$ then using the first estimate in Proposition 2 we obtain

$$
\|U(v) - U(w)\| = \sup_{0 < t < 1/2} t^{-m}|U(v) - U(w)|_{2,\alpha,t} \leq \epsilon \|Q(v) - Q(w)\|_\alpha
$$

$$
\leq c \sup_{0 < t < 1/2} t^{2-m-\epsilon}|Q(v) - Q(w)|_{0,\alpha,t}
$$

$$
\leq c \mu(||v|| + ||w||)(||v - w||).
$$
In view of the last inequality, it is necessary to distinguish two cases. We suppose first that 
\( c\mu < \frac{\lambda}{2} \) for some constant \( \lambda < 1 \). Then, given \( u, v \) with \( \|u\| \leq 1 \) and \( \|v\| \leq 1 \) we have 
\[
\|U(u) - U(v)\| \leq \lambda \|u - v\|.
\]
Moreover,
\[
\|U(v)\| \leq c\mu \|v\|^2 + \|H_J\psi\| \leq 1
\]
if we assume that
\[
\|H_J\psi\| \leq 1 - c\mu \|v\|^2.
\]
Since \( \|v\| \leq 1 \) the last inequality holds if we suppose
\[(63) \quad \|H_J\psi\| \leq 1 - c\mu,\]
which is true for sufficiently small \( \psi \). Hence, assuming this we conclude that \( U|_K : K \to K \) is a contraction map where \( K \) is the intersection of the unit open ball in \( B \) with the affine subspace \( \mathcal{P} = \{ v \in B : \Pi_J v = \Pi_J \psi \} \). Notice that the smallness of \( \psi \) also guarantees that \( K \) is not empty.

Now, we suppose that \( c\mu \geq 1/2 \). In this case, we assume that \( \|v\| \leq a \) for some constant \( a \) to determine. One gets
\[
\|U(v)\| \leq c\mu \|v\|^2 + \|H_J\psi\| \leq c\mu a^2 + \|H_J\psi\|.
\]
Thus in order that \( \|U(v)\| \leq a \) it is sufficient that
\[
c\mu a^2 - a + \|H_J\psi\| \leq 0.
\]
Then \( a \) must be chosen as \( a \leq \frac{1 + \sqrt{1 - 4c\mu \|H_J\psi\|}}{2c\mu} \). We must assume that
\[
\|H_J\psi\| \leq \frac{1}{4c\mu}
\]
in order to assure that the square root above is well-defined. Since
\[
\frac{1}{2c\mu} < 1 + \frac{\sqrt{1 - 4c\mu \|H_J\psi\|}}{2c\mu},
\]
we may choose \( a = 1/(2c\mu) \). So, we must suppose simultaneously that \( \|v\| \leq 1 \) and that \( \|v\| \leq a \). However, the hypothesis \( c\mu \geq 1/2 \) implies that \( a = 1/(2c\mu) \leq 1 \). So, we prove that \( U(K_1) \subset K_1 \) and \( U|_{K_1} \) is a contraction mapping, where \( K_1 \) is the intersection of the ball of radius \( a \) in \( B \) with the affine plane \( \mathcal{P} \).

In both cases, we had just verified that \( U \) defines a contraction map in properly chosen convex sets of the Banach space \( B \). So, by Leray’s fixed point theorem (see, e.g., [5], Chapter 11), we assure the existence of a solution for the equation (22).

**Theorem 3.** Let \( M \) be a scalar-flat hypersurface in \( \mathbb{S}^n, \ n \geq 4 \). Suppose that the rank of the second fundamental form of \( M \) is greater than or equal to 3. Let \( \psi \) be a function in \( C^{2,\alpha}(M) \). There exists \( \Lambda < 1 \) depending on \( M \) such that for each \( \lambda \in [0, \Lambda) \) there exists a function \( u_\lambda \) defined in \( M^* \) such that the graph \( M^*_\lambda \) of \( u_\lambda \) has null scalar curvature and boundary given by \( \Pi_J(u_\lambda) = \Pi_J(\lambda \psi) \), for some integer \( J \).
5. Stability of scalar-flat cones

It is well-known that scalar-flat hypersurfaces in $\mathbb{R}^{n+1}$ are locally characterized as extrema of the action

$$\mathcal{A}_1 = \int_{\tilde{M}} \tilde{S}_1 \, d\tilde{M}. \tag{64}$$

In this context, the Jacobi operator $L$ is naturally linked to stability of the hypersurface. For details, we refer the reader to [10], [11] and [2].

In this section, we are concerned with the stability of the scalar-flat cones and graphs we had defined above. For that, we consider a function $u \in C^2_0(\tilde{M}^*)$. The first and second variation formulae for $\mathcal{A}_1$ are:

$$\mathcal{A}_1'(0) = 0, \quad \mathcal{A}_1''(0) = -\int_{\tilde{M}^*} u Lu \, d\tilde{M}. \tag{65}$$

We recall that the Jacobi operator in the last formula is

$$Lu = \bar{L}_1 u - 3\bar{S}_3 u = S_1 t^{1-n} (t^{n-2} u_t)_t + \frac{1}{t^3} (L_1 u(t, \cdot) - 3S_3 u).$$

We decompose $u$ in its Fourier coefficients with respect to the eigenfunctions $\{\phi_j\}$ of $-\frac{1}{S_j} (L_1 - 3S_3)$ obtaining $u = \sum_j b_j \phi_j$ with $b_j(0) = b_j(1) = 0$ and

$$Lu = \sum_j S_1 (t^{1-n}(t^{n-2} b'_j)_t - t^{-3}\mu_j b_j) \phi_j. \tag{66}$$

Since the metric of $\tilde{M}^*$ in spherical coordinates $(t, \theta)$ is written in the form $dt^2 + t^2 \theta^2 d\theta$, one has $d\tilde{M} = t^{n-1}dt \, d\theta$, where $d\theta$ is the volume form in $\tilde{M}$. Since $b_j(1) = 0$, for all $j$, it results that

$$\int_{\tilde{M}^*} u Lu \, d\tilde{M} = \sum_{j,k} \int_0^1 (t^{n-2} b'_j)_t - t^{-4}\mu_j b_j) \int_{\tilde{M}^*} \phi_j \phi_k S_1(\theta) \, d\theta$$

$$= \int_0^1 \sum_j (t^{n-2} b'_j)_t b_j - t^{-4}\mu_j b_j^2) \, dt$$

$$= -\int_0^1 \sum_j (t^{n-2} b'_j)^2 + t^{-4}\mu_j b_j^2) \, dt. \tag{67}$$

The first term in the last integral is given by

$$\int_{\tilde{M}^*} u^2 \tilde{S}_1 \, d\tilde{M} = \int_{\tilde{M}^*} u^2 t^{-1} \tilde{S}_1 \, d\tilde{M} = \int_0^1 t^{n-2} \sum_j (b'_j)^2 \, dt. \tag{68}$$

Denote $\mu^- = \max\{-\mu_1, 0\}$, where $\mu_1$ is the smallest eigenvalue of the operator $-\frac{1}{S_1} (L_1 - 3S_3)$. Thus, one obtains

$$\int_{\tilde{M}^*} u Lu \, d\tilde{M} \leq -\int_{\tilde{M}^*} u^2 \tilde{S}_1 \, d\tilde{M} + \mu^- \int_0^1 t^{n-4} \sum_j b_j^2 \, dt. \tag{69}$$

However, one has

$$\int_{\tilde{M}^*} u^2 t^{-2} \tilde{S}_1 \, d\tilde{M} = \int_{\tilde{M}^*} u^2 t^{-3} \tilde{S}_1 \, d\tilde{M} = \int_0^1 t^{n-4} \sum_j b_j^2 \, dt \tag{70}$$
and the expression on the right hand side of (66) may be calculated as follows
\[
\int_0^1 t^{n-4} \sum_j b_j^2 \, dt = \frac{1}{n-3} \int_0^1 (t^{n-3} \sum_j b_j^2) \, dt - \frac{2}{n-3} \int_0^1 t^{n-3} \sum_j b_j b'_j \, dt \\
\leq \frac{2}{n-3} \left( \int_0^1 t^{n-2} \sum_j (b'_j)^2 \, dt \right)^{1/2} \left( \int_0^1 t^{n-4} \sum_j b_j^2 \, dt \right)^{1/2}.
\]
Therefore, it follows that
\[
\int_{\tilde{M}^*} u^2 t^{-2} \tilde{S}_1 \, d\tilde{M} = \int_0^1 t^{n-4} \sum_j b_j^2 \, dt \leq \frac{4}{(n-3)^2} \int_0^1 (n-3)^2 \sum_j (b'_j)^2 \, dt
\]
(67)
\[
= \frac{4}{(n-3)^2} \int_{\tilde{M}^*} u^2 \tilde{S}_1 \, d\tilde{M}.
\]
Finally, we conclude that
\[
\int_{\tilde{M}^*} u \, L u \, d\tilde{M} \leq \left( \frac{4\mu_1^-}{(n-3)^2} - 1 \right) \int_{\tilde{M}^*} u^2 \tilde{S}_1 \, d\tilde{M}.
\]
Suppose \( n \geq 4 \) and define
\[
\mu_{\tilde{M}} := (1 - 4\mu_1^-/(n-3)^2).
\]
We suppose that \( \mu_{\tilde{M}} \geq 0 \). Hence, it follows from (67) that
\[
- \int_{\tilde{M}^*} u \, L u \geq \mu_{\tilde{M}} \int_{\tilde{M}^*} u^2 \tilde{S}_1 \, d\tilde{M} \geq \mu_{\tilde{M}} \frac{(n-3)^2}{4} \int_{\tilde{M}^*} u^2 t^{-2} \tilde{S}_1 \, d\tilde{M}.
\]
Now, we define the truncated cone \( \tilde{M}_{\sigma, \tau} \) as the set of points \( t \theta \) in \( \tilde{M}^* \) with \( 0 < \sigma < t < \tau \leq 1 \). Let \( \lambda_{\sigma, 1} \) be the smallest eigenvalue of the Dirichlet eigenvalue problem
\[
Lu + t^{-2} \tilde{S}_1 \lambda u = 0 \quad \text{on} \quad \tilde{M}_{\sigma, 1}, \quad u = 0 \quad \text{on} \quad \partial\tilde{M}_{\sigma, 1}.
\]
Hence, we may characterize \( \lambda_{\sigma, 1} \) as the Rayleigh quotient
\[
\lambda_{\sigma, 1} = - \inf_{u \in C_c^0(\tilde{M}_{\sigma, 1}), \, u \neq 0} \frac{\int_{\tilde{M}^*} u \, L u \, d\tilde{M}}{\int_{\tilde{M}^*} u^2 \tilde{S}_1 \, d\tilde{M}}.
\]
We define
\[
I := \inf_{u \in C_c^0(\tilde{M}^*), \, u \neq 0} \left( - \int_{\tilde{M}^*} u \, L u \, d\tilde{M} \right)
\]
and
\[
I_+ := \inf_{u \in C_c^0(\tilde{M}^*), \, u \neq 0} \frac{- \int_{\tilde{M}^*} u \, L u \, d\tilde{M}}{\int_{\tilde{M}^*} u^2 \tilde{S}_1 \, d\tilde{M}}.
\]
Therefore, if \( \mu_{\tilde{M}} \geq 0 \) (respectively, \( \mu_{\tilde{M}} > 0 \)), then \( I \geq 0 \) and \( \inf_{\sigma} \lambda_{\sigma, 1} \geq 0 \) (respectively, \( I_+ \geq 0 \) and \( \inf_{\sigma} \lambda_{\sigma, 1} > 0 \)). In the first case, we say that \( \tilde{M}^* \) is 1-stable. In the second case, \( \tilde{M}^* \) is said to be strictly 1-stable.

Thus, we have proved that \( \mu_{\tilde{M}} \geq 0 \) (respectively, \( \mu_{\tilde{M}} > 0 \)) implies that \( \tilde{M}^* \) is 1-stable (respectively, strictly 1-stable).

Conversely, if \( \mu_{\tilde{M}} < 0 \), then \( \tilde{M}^* \) is not 1-stable. In fact, in this case, we have \( \mu_1 < -(n-3)^2/4 \). Thus, the root \( \gamma_1 = \sqrt{(n-3)^2}/2 - \mu_1 \) of \( \gamma_1^2 + (n-3)\gamma - \mu_1 = 0 \) is not real. Moreover, the function \( u_1 = \Re(t^{\gamma_1} \phi_1) \) is a Jacobi field, i.e., a solution for \( \tilde{L}_1 u - 3\tilde{S}_1 u = 0 \). Notice that \( u_1(t, \theta) = 0 \) for all \( \theta \) whenever \( t^{\gamma_1} \) is a pure imaginary number. This happens if
and only if \( \ln t \geq \gamma_1 = k\pi /2 \), where \( k \) is a negative integer. Thus, we choose \( \sigma, \tau \) so that \( u_1(\sigma, \cdot) = u_1(\tau, \cdot) = 0 \) and define the test function for the Rayleigh quotient
\[
w(t, \theta) = u(t, \theta) \quad \text{if} \quad \sigma < t < \tau \quad \text{and} \quad w = 0 \quad \text{otherwise}.
\]

It is clear that \( w \) is a piecewise differentiable function which satisfies
\[
\int_{M^*} \left( \langle \bar{T}_1 \nabla w, \nabla w \rangle + 3\bar{S}_3 w \right) \, d\bar{M} = 0.
\]

So, \( \lambda_{\sigma, \tau, 1} < 0 \) since the compact support of \( w \) is strictly contained in the truncated cone \( \bar{M}_{\tau, 1}^* \). We conclude that \( \inf_w \lambda_{\sigma, 1} < 0 \).

In a similar way, we may prove that if \( \mu_{\bar{M}} = 0 \), then \( \bar{M}^* \) is not strictly 1-stable.

These results can now be used to prove

**Theorem 4.** If \( \bar{M}^* \) is strictly 1-stable, then the graph \( \bar{M}_\lambda^* \) of the function \( u_\lambda \) given in Theorem 1 is strictly 1-stable for \( \lambda \) sufficiently small.

**Proof.** Let \( \bar{S}_r(\lambda), 1 \leq r \leq n \), denote the elementary symmetric functions of the eigenvalues of the Weingarten map \( A(\lambda) \) of \( M_\lambda^* \). We also denote \( \bar{T}_1(\lambda) = \bar{S}_1(\lambda) I - \bar{A}(\lambda) \).

As \( \bar{S}_3(\lambda) \) depends on the Hessian of \( u_\lambda \), it follows from the \( C^{2, \alpha} \) estimates on \( u_\lambda \) given in Proposition 2 that
\[
\sup_{\lambda} \frac{1}{\lambda} \sup_{M_\lambda^*} \frac{1}{t^3} \left( \bar{S}_{3}(\lambda) - S_{3} \right) < \infty.
\]

Consequently, for small \( \lambda \), it holds that
\[
\int_{M_\lambda^*} \left( \langle \bar{T}_1(\lambda) \nabla u, \nabla u \rangle - \bar{S}_3(\lambda) u^2 \right) \, d\bar{M} \geq \mu_{\bar{M}} / 2 > 0,
\]
for all \( u \in C^1_0(\bar{M}_\lambda^*) \) with
\[
\int_{M_\lambda^*} \frac{u^2}{t^2} \bar{S}_1(\lambda) \, d\bar{M} = 1.
\]

This finishes the proof of the theorem.

**References**

[1] Barbosa, J. L. M., do Carmo, M. P.: *On stability of cones in \( \mathbb{R}^{n+1} \) with zero scalar curvature*. Ann. Global Anal. Geom. **28**, 107-122, (2005).

[2] Barbosa, J. L. M., Colares, G.: *Stability of hypersurfaces with constant \( r \)-mean curvature*. Ann. Global Anal. Geom. **15**, 3, 277-297, (1997).

[3] Caffarelli, L., Hardt, R., Simon, L.: *Minimal surfaces with isolated singularities*. Manuscripta Math. **48**, 1-18, (1984).

[4] Dubrovin, B., Fomenko, A., Novikov, S.: *Modern geometry*. I. Springer-Verlag, New York, 1992.

[5] Gilbarg, D., Trudinger, N.: *Elliptic partial differential equations of second order*. Springer-Verlag, Heidelberg, 2nd edition, (1998).

[6] Hounie, J., Leite, M. L.: *The maximum principle for hypersurfaces with vanishing curvature functions*. J. Differential Geom. **41**, 2, 247-258, (1995).

[7] Leite, M. L.: *The tangency principle for hypersurfaces with vanishing curvature functions*. XI Escola de Geometria Diferencial, UFF (2000).

[8] Pacard, F.: *Connected sum construction in geometry and nonlinear analysis* at http://perso.math.univ-mlv.fr/users/pacard.frank/Lecture-Part-I.pdf

[9] Pacard, F., Riviere, T.: *Linear and nonlinear aspects of vortices. The Ginzburg-Landau model*. Birkhauser, Boston, (2000).

[10] Reilly, R.: *Variational properties of functions of the mean curvatures for hypersurfaces in space forms*. J. Differential Geometry **8**, 465-477, (1973).

[11] Rosenberg, H.: *Hypersurfaces of constant curvature in space forms*. Bull. Sci. Math. **117**, 2, 211-239, (1993).
[12] Smale, N.: An equivariant construction of minimal surfaces with nontrivial singular sets. Indiana Univ. Math. J. 40, 2, 595-616, (1991).

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