Dynamical systems analysis of anisotropic cosmologies in $R^n$-gravity

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Received 9 August 2007, in final form 17 September 2007
Published 6 November 2007
Online at stacks.iop.org/CQG/24/5689

Abstract
In this paper, we study the dynamics of orthogonal spatially homogeneous Bianchi cosmologies in $R^n$-gravity. We construct a compact state space by dividing the state space into different sectors. We perform a detailed analysis of the cosmological behaviour in terms of the parameter $n$, determining all the equilibrium points, their stability and corresponding cosmological evolution. In particular, the appropriately compactified state space allows us to investigate static and bouncing solutions. We find no Einstein static solutions, but there do exist cosmologies with bounce behaviours. We also investigate the isotropization of these models and find that all isotropic points are flat Friedmann-like.

PACS numbers: 98.80.JK, 04.50.+h, 05.45.−a

1. Introduction
Over the past few years, there has been growing interest in higher order theories of gravity (HOTG). This is in part due to the fact that these theories contain extra curvature terms in their equations of motion, resulting in a dynamical behaviour which can be different to general relativity (GR). In particular these additional terms can mimic cosmological evolution which is usually associated with dark energy [1], dark matter [2, 3] or a cosmological constant [4]. The isotropization of anisotropic cosmologies can also be significantly altered by these higher order corrections. In a previous work [5], the existence of an isotropic past attractor within the class of Bianchi type I models was found for a power law Lagrangian of the form $R^n$. This feature was also found for Bianchi type I, II and IX models in quadratic theories of gravity [6, 7]. In these cases, the extra curvature terms can dominate at early times and consequently allow for isotropic initial conditions. This is not possible in GR, where the shear term dominates at early times.
A natural extension of this analysis is to investigate the effect of spatial curvature on the isotropization in HOTG. In GR, it is well known that spatial curvature can source anisotropies for Bianchi models [8, 9]. In this paper, we extend the analysis in [5] to the case of orthogonal spatially homogeneous (OSH) Bianchi models [10], in order to investigate the effect of spatial curvature on the isotropization of $R^n$ models. OSH Bianchi models exhibit local rotational symmetry (LRS), and include the LRS Bianchi types I (BI), III (BIII) and the Kantowski–Sachs (KS) models. For a review of this class of cosmologies see [10–12].

In GR, a cosmological constant or scalar field is required to obtain an Einstein static solution in a closed ($k = +1$) Friedmann–Lemaître–Robertson–Walker (FLRW) model [13, 14]. The existence of Gödel and Einstein static universes has been investigated for gravitational theories derived from functions of linear and quadratic contractions of the Riemann curvature tensor [15]. Recently, the stability of Einstein static models in some $f(R)$-theories of gravity was investigated [16]. It was shown that the modified Einstein static universe is stable under homogeneous perturbations, unlike its GR counterpart [18]. Static solutions are interesting in their own right, but are often an important first step in finding cosmologies that have a ‘bounce’ during their evolution [17].

The existence conditions for a bounce to occur for FLRW universes in $f(R)$-gravity have been determined recently [19]. Bouncing cosmological models have been found for FLRW models in $R^n$-gravity [20, 21]. This should in principle be possible for anisotropic models as well, since the higher order corrections can mimic a cosmological constant, and so prevent the model from collapsing to a singularity. In [22], it was shown that bounce conditions for OSH Bianchi models cannot be satisfied in GR with a scalar field, but can be satisfied for KS models in the Randall–Sundrum type braneworld scenario.

As in [5], we make use of the dynamical systems approach [14, 23, 24] in this analysis. This approach has been applied to study the dynamics of a range of extended theories of gravity [5, 6, 20, 25–31]. However, in these works, the dynamical variables were non-compact, i.e. their values did not have finite bounds. This non-compactness of the state space has certain disadvantages (see [32] for detailed discussion of this issue). The standard expansion-normalized variables, for example, only define a compact state space for simple classes of ever expanding models such as the open and flat FLRW models and the spatially homogeneous Bianchi type I models in GR [24]. As soon as a wider class of models or more complicated underlying gravitational theories are considered, the expansion rate may pass through zero, making the state space non-compact (see e.g. [5, 25–27]). The points at infinity then correspond to a vanishing Hubble factor, and the non-compact expansion-normalized state space can only contain the expanding (or by time-reversal collapsing) models. In order to obtain the full state space, one would have to carefully attempt to match the expanding and collapsing copies at infinity.

While static solutions correspond to equilibrium points at infinity and can be analysed by performing a Poincaré projection [33, 34], bouncing or recollapsing behaviours on the other hand are very difficult to study in this framework. In both cases ambiguities at infinity can easily occur, since in general only the expanding copy of the state space is studied. A point at infinity may for example appear as an attractor in the expanding non-compact analysis, even though it corresponds to a bounce when also including the collapsing part of the state space.

In order to avoid these ambiguities, we will here construct compact variables that include both expanding and collapsing models, allowing us to study static solutions and bounce behaviour in $R^n$-theories of gravity. This approach is a generalization of [13], which has been adapted to more complicated models in [35–38]. We refer to the accompanying work [32] for a detailed comparison between the approach established here and differently constructed non-compact state spaces applied to the class of BI or flat FLRW models in $R^n$-gravity.
We note that we recover the isotropic past attractor found in [5] in this analysis, and we only obtain flat \((k = 0)\) isotropic equilibrium points. Bounce behaviour is found for BI, BIII and KS cosmologies, but no Einstein static solutions could be found in the phase space. Our analysis also reveals that we can have cosmologies that bounce from expansion to contraction and vice versa, depending on the value of the parameter \(n\).

The outline of this paper is as follows: in section 2, we state the field equations and the evolution equations for the OSH Bianchi models. In section 3, we construct a compact state space and then analyse the BIII and KS subspaces separately. Section 4 is devoted to a discussion of the isotropization of these cosmologies.

The following conventions will be used in this paper: the metric signature is \((-+++\)) ; Latin indices run from 0 to 3; units are used in which \(c = 8\pi G = 1\).

2. Preliminaries

The general action for a \(f(R)\)-theory of gravity reads

\[
A = \int dx^4 \sqrt{-g} f(R) + \int L_M dx^4,
\]

where \(L_M\) is the Lagrangian of the matter fields. The fourth-order field equations can be obtained by varying (1):

\[
T_{ab}^M = f' R_{ab} - \frac{1}{2} f g_{ab} + S_{cd}(g^{cd} g_{ab} - g^c_a g^d_b),
\]

where primes denote derivatives with respect to \(R\) and \(S_{ab} = \nabla_a \nabla_b f'(R)\). The field equation (2) can be rewritten in the standard form

\[
G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R = T_{ab}^{\text{eff}},
\]

(when \(f'(R) \neq 0\), where the effective stress–energy–momentum tensor \(T_{ab}^{\text{eff}}\) is given by

\[
T_{ab}^{\text{eff}} = f'^{-1} \left[ T_{ab}^M + \frac{1}{2} g_{ab} (f - f' R) + S_{cd} (g^c_a g^d_b - g^{cd} g_{ab}) \right].
\]

It is easy to show that the contracted Bianchi identities \(\nabla^a G_{ab} = 0\) give rise to the conservation laws for standard matter [25]. The propagation and constraint equations can be obtained straightforwardly for these field equations (see [5, 39]).

We consider here the case \(f(R) = R^n\) for OSH Bianchi spacetimes, where the Raychaudhuri equation becomes

\[
\dot{\Theta} + \frac{1}{3} \Theta^2 + 2\sigma^2 - \frac{1}{2n} R = (n - 1) \frac{\dot{R}}{R} \Theta + \frac{\mu}{n R^{n-1}} = 0,
\]

and the trace-free Gauss–Codazzi equation is given by

\[
\dot{\sigma} = - \left( \Theta + (n - 1) \frac{\dot{R}}{R} \right) \sigma + \frac{1}{2\sqrt{3}} R.
\]

Here \(\Theta\) is the volume expansion which defines a length scale \(a\) along the flow lines via the standard relation \(\Theta = \frac{3a}{\dot{a}}\), and \(\mu\) is the standard matter energy density. The magnitude of the shear tensor is given by \(\sigma^2 = \frac{1}{2} \sigma^{ab} \sigma_{ab}\), and the 3-Ricci scalar by \(3 \cdot 3\) (see [10]).

The Friedmann equation is given by

\[
\frac{1}{3} \Theta^2 - \sigma^2 - (n - 1) \frac{\dot{R}}{R} \Theta - \frac{(n - 1)}{2n} R - \frac{\mu}{n R^{n-1}} + \frac{1}{2} R = 0.
\]

Combining the Friedmann and Raychaudhuri equations yields

\[
R = 2\dot{\Theta} + \frac{1}{3} \Theta^2 + 2\sigma^2 + \frac{3}{2} R.
\]
We will assume standard matter to behave like a perfect fluid with barotropic index $w$, so that the conservation equation gives

$$
\dot{\mu} = -(1 + w)\mu \Theta.
$$

(9)

In the following, we assume $n > 0$ and $n \neq 1$.

3. Dynamics of OSH Bianchi cosmologies

3.1. Construction of the compact state space

The overall goal here is to define compact dimensionless expansion-normalized variables and a time variable $\tau$ such that the system of propagation equations above (5)–(9) can be converted into a system of autonomous first-order differential equations. We choose the expansion-normalized time derivative

$$
\dot{\equiv} \frac{d}{d\tau} \equiv \frac{1}{D} \frac{d}{dt}
$$

(10)

and make the following ansatz for our set of expansion normalized variables$^3$:

$$
\begin{align*}
\Sigma &= \sqrt{3} \sigma D, \\
x &= \frac{3R\Theta}{RD^2} (1 - n), \\
y &= \frac{3R}{2nD^2} (n - 1), \\
z &= \frac{3\mu}{nR^{n-1}D^2}, \\
K &= \frac{3^3 R}{2D^2}, \\
Q &= \frac{\Theta}{D}
\end{align*}
$$

(11)

Here $D$ is a normalization of the form

$$
D = \sqrt{\Theta^2 - \Delta},
$$

(12)

where $\Delta$ is a linear combination of the terms appearing on the right-hand side of the Friedmann equation (7) as discussed below. In order to maintain a monotonically increasing time variable, $\Delta$ must be chosen such that the normalization $D$ is real-valued and strictly positive.

Note that we have chosen to define $x$ with an opposite sign to that in $[5]$ in order to have a simple form of the Friedmann equation (see below), and $\sigma$ can be both positive and negative $[10]$. We emphasize that the coordinates (11) are strictly speaking only defined for $R \neq 0$, which means for $y \neq 0$. Even though the case $R = 0$ may not be of physical interest, the limiting case is interesting in the context of the stability analysis, since we obtain equilibrium points with $y = 0$. This means that the system may evolve towards/away from that singular state if these points are attractors or repellers. In the analysis below we will investigate this by taking the limit $y \to 0$ (by letting $R \to 0$) and find that this puts a constraint on the relation between the coordinates.

We now turn to the issue of compactifying the state space. It is useful to rewrite the Friedmann equation (7) as

$$
\Theta^2 = \hat{\Sigma}^2 - \hat{K} + \hat{x} + \hat{y} + \hat{z} \equiv D^2 + \Delta,
$$

(13)

where the quantities with a hat are just the variables defined in (11) without the normalization $D$. If all the contributions ($\hat{\Sigma}^2$, $-\hat{K}$, $\hat{x}$, $\hat{y}$ and $\hat{z}$) to the central term in equation (13) are non-negative, we can simply normalize with $\Theta^2$ (i.e. $\Delta = 0$), but we have to explicitly make the assumption $\Theta \neq 0$. We can then conclude that the state space is compact, since all the non-negative terms have to add up to 1 and are consequently bounded between 0 and 1.

$^3$ It is important to note that this choice of variables excludes GR, i.e., the case of $n = 1$. See [14, 23] for the dynamical systems analysis of the corresponding cosmologies in GR.
However, while $\Sigma^2$ is always positive, $-\dot{K}, \dot{x}, \dot{y}$ and $\dot{z}$ may be positive or negative for the class of models considered here\(^4\). This means that the variables (11) do not in general define a compact state space.

In the following, we will study the class of LRS BIII models with $^3R < 0$ and the class of KS models with $^3R > 0$ separately, as in [13]. While we may in principle normalize with $\Theta^2$ in the Bianchi III subspace, we have to absorb the curvature term into the normalization $D$ in the KS subspace.

For both classes of models, we can construct a compact state space by splitting up the state space into different sectors according to the sign of $\dot{x}, \dot{y}$ and $\dot{z}$. In both the open and the closed subspaces we will have to define $2^3 = 8$ sectors, corresponding to the possible signs of the three variables $\dot{x}, \dot{y}, \dot{z}$. In the following, we will refer to the spatially open BIII sectors as sector 1o to sector 8o, where the subscript 'o' stands for 'open'. Similarly, the spatially closed KS sectors will be labelled sectors 1c–8c, where the 'c' stands for 'closed'.

After defining the appropriate normalizations for the various sectors, we derive the dynamical equations for the accordingly normalized variables in each sector. For each sector we then analyse the dynamical system in the standard way: we find the equilibrium points and their eigenvalues, which determine their nature for each sector. The overall state space is then obtained by matching the different sectors along their common boundaries.

### 3.2. The LRS BIII subspace

If $^3R \leq 0$, we obtain the class of spatially open LRS BIII cosmologies. This class of models contains the flat LRS BI models as a subclass. In this case $K$ enters the Friedmann equation with a non-negative sign and does not have to be absorbed into the normalization. As can be seen from the Friedmann equation in each sector (see table 1), $K \in [-1, 0]$ and $\Sigma \in [-1, 1]$ holds in each sector.

#### 3.2.1. Sector 1o

The first open sector denoted 1o is defined to be that part of the state space where $\dot{x}, \dot{y}, \dot{z} \geq 0$. In this case all the contributions to the right-hand side of (13) are non-negative, and we can choose $\Delta = 0$. This means we can normalize with $D = |\Theta| = \epsilon \Theta$, where $\epsilon$ is the sign function of $\Theta$ and $\epsilon = \pm 1$ for expanding/collapsing phases of the evolution. Note that it is crucial to include $\epsilon$ in the normalization: if we were to exclude this factor, time would decrease for the collapsing models, and any results about the dynamical behaviour of collapsing equilibrium points would be time-reversed.

It is important to note that we have to exclude $\Theta = 0$ in this sector, so we cannot consider static or bouncing solutions here. However, this assumption is not as strong as it first appears: we can see from the Friedmann equation (13) that the only static solution in this sector appears for $\dot{x} = \dot{y} = \dot{z} = \dot{\Sigma} = \dot{K} = 0$, because all the quantities enter (13) with a positive sign in this sector by construction. This means that we only have to exclude the static flat isotropic vacuum cosmologies\(^5\). Under this restriction, the normalization above is strictly positive and thus defines a monotonically increasing time variable via (10). Equation (13) now becomes

$$1 = \Sigma^2 - K + x + y + z. \quad (14)$$

We can directly see from (14) that the appropriately normalized variables (11) define a compact subsector of the total state space:

$$x, y, z \in [0, 1], \quad K \in [-1, 0] \quad \text{and} \quad \Sigma \in [-1, 1]. \quad (15)$$

Here $Q = \epsilon$ is constant and not a dynamical variable.

\(^4\) Note that the sign of $K$ is preserved within the open and the closed sectors.

\(^5\) The same restriction appears in GR, see [13].
This sector is different from all the other sectors in both the open BIII and the closed KS subspaces for the following reasons. When gluing together the different sectors to obtain the total state space, we will actually use two copies of \( L_o \): one copy with \( \epsilon = 1 \) corresponding to expanding cosmologies and one copy with \( \epsilon = -1 \) corresponding to collapsing cosmologies. The two copies are in fact disconnected: the closed sector \( L_c \) from the KS subspace separates the expanding and collapsing copies of open sector \( L_o \). Again, this reflects the fact that we cannot study static solutions in sector \( L_o \). In all the other sectors we allow \( \Theta = 0 \), and the expanding and collapsing sets are connected via the non-invariant subset \( Q = 0 \).

We can now derive the propagation equations for the dynamical systems variables in this sector by using the definitions (10) and (11) and substituting them into the original propagation equations (5)–(9). We obtain five equations, one for each of the dynamical variables defined in (11). These variables are constrained by the Friedmann equation (13), which we use to eliminate \( x \), resulting in a 4-dimensional state space. Note that we have to verify that the constraint is propagated using all five (unconstrained) propagation equations, which we have done for each sector. The effective system is given by

\[
\begin{align*}
K' &= 2\epsilon K \left[ 1 + \Sigma^2 - \frac{n}{n-1} y + \epsilon \Sigma + K \right], \\
\Sigma' &= -\epsilon \left( \epsilon \Sigma \left( \frac{2n-1}{n-1} y + z - 2K - K \right) - K \right), \\
y' &= \frac{\epsilon y}{n-1} \left[ z + (2n - 3)K - (2n - 1)y + (2n - 1) \Sigma^2 + 4n - 5 \right], \\
z' &= -\epsilon z \left[ z - \Sigma^2 + \frac{3n-1}{n-1} y - 3K + 3w - 2 \right].
\end{align*}
\]

Only in this sector does the sign of the expansion-rate appear directly in the dynamical equations, and we can see directly that the stability of the collapsing equilibrium points is given by simple time-reversal of the stability of the expanding points and vice versa.

The subset \( K = 0 \) (Bianchi I) is a two-dimensional invariant sub-manifold, so it is justified to discuss the Bianchi I subspace on its own. This is done in detail in [32]. The vacuum subset \( z = 0 \) and the submanifold \( y = 0 \) are also invariant subspaces. On the other hand, the isotropic subset \( \Sigma = 0 \) is not invariant unless \( K = 0 \). This agrees with GR, where it was found that the spatial curvature can source anisotropies for Bianchi models [8, 9].

We can find the equilibrium points and the corresponding eigenvalues of the dynamical system (16), and classify the equilibrium points according to the sign of their eigenvalues as attractors, repellers and saddle points (see [34]). Because of the large number of sectors that need to be studied, we do not show the results for each sector. Instead we combine the results from the various sectors in table 2.

3.2.2. Sectors 2o–8o. Sectors 2o–8o are defined according to the possible signs of \( \hat{x}, \hat{y}, \hat{z} \) as summarized in table 1. In each sector \( \Delta \) is defined as the sum of the strictly negative contributions to (13), so that \( -\Delta \) is strictly positive, making \( D \) strictly positive even for \( \Theta = 0 \). This means that \( D \) is a well-defined (nonzero) normalization, and (10) defines a well-defined monotonously increasing time variable for each sector, even for static or bouncing solutions. With this choice of normalization, only positive contributions remain in the Friedmann equation, and the appropriately normalized variables define a compact sub-sector of the total state space, as can be seen from the respective versions of the Friedmann

6 If we used the unconstrained 5-dimensional system, we would not constrain the allowed ranges of \( n \) and \( w \) for the different equilibrium points correctly. We would also get a fifth zero-valued eigenvalue for all equilibrium points.
which can be written in terms of the variables (11) in each sector.

### 3.2.3. Equilibrium points of the full LRS BIII state space.

The equilibrium points of the LRS BI subset is indeed an invariant submanifold. We summarize them in Table 2. Note that not all the points occur in all of the sectors, and some

\[ 1 = Q^2 - \frac{\Delta}{D^2}, \quad \text{(17)} \]

which can be written in terms of the variables (11) in each sector.

It is straightforward to derive the dynamical equations for each sector, and again we analyse them as outlined in the previous subsection. We confirm in each sector that the flat LRS BI subset is indeed an invariant submanifold.

#### 3.2.3. Equilibrium points of the full LRS BIII state space.

The equilibrium points of the entire BIII state space are obtained by combining the equilibrium points in each sector. We summarize them in Table 2. Note that not all the points occur in all of the sectors, and some

| Sector | $\hat{x}$ | $\hat{y}$ | $\hat{z}$ | Normalization | Friedmann equation | Range of $(x, y, z)$ |
|--------|----------|----------|----------|---------------|-------------------|------------------|
| $l_1$  | $\geq 0$ | $\geq 0$ | $\geq 0$ | $\Delta = 0$ | $1 = x + y + z + \Sigma^2 - K$ | $[0, 1] \times [0, 1] \times [0, 1]$ |
| $l_2$  | $< 0$   | $> 0$    | $> 0$    | $\Delta = \hat{y}$ | $1 = y + z + \Sigma^2 - K$ | $[-1, 0] \times [0, 1] \times [0, 1]$ |
| $l_3$  | $> 0$   | $< 0$    | $> 0$    | $\Delta = \hat{y}$ | $1 = x + y + \Sigma^2 - K$ | $[0, 1] \times [-1, 0] \times [0, 1]$ |
| $l_4$  | $> 0$   | $> 0$    | $< 0$    | $\Delta = \hat{z}$ | $1 = x + y + \Sigma^2 - K$ | $[0, 1] \times [0, 1] \times [-1, 0]$ |
| $l_5$  | $< 0$   | $< 0$    | $> 0$    | $\Delta = \hat{x} + \hat{z}$ | $1 = z + \Sigma^2 - K$ | $[-1, 0] \times [-1, 0] \times [0, 1]$ |
| $l_6$  | $< 0$   | $< 0$    | $< 0$    | $\Delta = \hat{x} + \hat{z}$ | $1 = y + \Sigma^2 - K$ | $[-1, 0] \times [0, 1] \times [-1, 0]$ |
| $l_7$  | $> 0$   | $< 0$    | $< 0$    | $\Delta = \hat{x} + \hat{z}$ | $01$ | $[0, 1] \times [-1, 0] \times [-1, 0]$ |
| $l_8$  | $< 0$   | $< 0$    | $< 0$    | $\Delta = \hat{x} + \hat{y} + \hat{z}$ | $1 = \Sigma^2 - K$ | $[-1, 0] \times [-1, 0] \times [-1, 0]$ |

### Table 1. Choice of normalization in the different LRS Bianchi III sectors, where the subscripts in the sector labels stand for open, differentiating the labels for the open sectors from those defined in the closed KS subspace below. We abbreviate $\hat{x} = (1 - n)R\hat{\theta}/R$, $\hat{y} = (1 - n)R/n$ and $\hat{z} = \mu/(nR^{n-1})$.
points only occur in a given sector for certain ranges of \( n \) or a specific equation of state \( w \). For this reason, we cannot express all the equilibrium points in terms of the same variables. When possible we state the coordinates in terms of the dimensionless variables defined for sector 1\(_o\), i.e. if the given point occurs in this sector. This is true for all the points except the line \( L_2 \), whose coordinates are described in terms of the variables defined in sector 2\(_o\) (see below for more details on the relation between \( L_1 \) and \( L_2 \)).

We emphasize that if the same point occurs in different sectors, it will have different coordinates in each of these sectors. In particular, \( Q \) can be a function of \( n \) or \( w \) in sectors 2\(_o\)–8\(_o\) even if \( Q = \epsilon \) is a constant in sector 1\(_o\). This simply reflects the fact that we have to exclude the static solutions in sector 1\(_o\) but not in the other sectors. This issue will be of importance when looking for static solutions in section 3.4.3. In order to ensure that equilibrium points obtained in different sectors correspond to the same solution, we have to look at the exact solution at these points. This is outlined in section 3.4.

Note that each of the isolated equilibrium points has an expanding (\( \epsilon = 1 \)) and a collapsing (\( \epsilon = -1 \)) version as indicated in the labelling of the points via the subscript \( \epsilon \) in table 2. Similarly, the lines each have an expanding and a contracting branch (see below). We will however drop the subscript in the following unless we explicitly address an expanding or contracting solution.

We find the three equilibrium points \( A \), \( B \) and \( C \) corresponding to spatially flat Friedmann cosmologies. The expanding versions of these points correspond to the equally labelled points in the BI analysis [5] (see [32] for detailed comparison). These points were also found in the Friedmann analysis [25]. \( A \) and \( B \) are vacuum Friedmann points, while \( C \) represents a non-vacuum Friedmann point whose scale factor evolution resembles the well-known Friedmann-GR perfect fluid solution with \( a \propto t^{1/3} \).

We now address the two lines of equilibrium points denoted by \( L_1 \) and \( L_2 \). Both these lines correspond to the spatially flat anisotropic BI cosmologies. The ratio of shear \( \Sigma \) and curvature component \( \kappa \) changes as we move along both lines. We note that in [5] a single line of equilibrium points denoted by \( L_1^* \) was found. In section 3.6, we will discuss in more detail how \( L_1 \) and \( L_2 \) are related to \( L_1^* \).

We emphasize that for \( L_1 \) the two expanding and contracting branches are disconnected and appear as two copies \( L_{1,\epsilon} \) of the line labelled by \( \epsilon \) in table 2. Each of these two branches ranges from purely shear dominated (\( \Sigma = 1 \)) to isotropic (\( \Sigma = 0 \)), to purely shear dominated with opposite orientation (\( \Sigma = -1 \)). For \( L_2 \) on the other hand, the expanding and contracting branches are connected: each \( L_{2,\epsilon} \) ranges from expanding (\( Q_\epsilon > 0 \)) and static (\( Q_\epsilon = 0 \)) to collapsing (\( Q_\epsilon < 0 \)). The two disconnected copies \( L_{2,\epsilon} \) correspond to positive and negative values of the shear, respectively. Note that there is no isotropic subset of \( L_2 \) in analogy to the fact that there is no static subset of \( L_1 \).

A closer look shows that \( L_1 \) and \( L_2 \) are actually the same object in different sectors: \( L_1 \) has \( \dot{\kappa} > 0 \) hence occurs in sectors 1\(_o\), 3\(_o\), 4\(_o\) and 7\(_o\), while \( L_2 \) is the analogue with \( \dot{\kappa} < 0 \) occurring in sectors 2\(_o\), 5\(_o\), 6\(_o\) and 8\(_o\). This statement is confirmed by looking at the exact solutions corresponding to the points on both lines; we find that both these lines have the same parametric solution of scale factor and shear (see the section below). For this reason, we could in fact give the two lines the same label. However, it is useful to treat them separately, since we obtain different bifurcations in the sectors with \( \dot{\kappa} > 0 \) and \( \dot{\kappa} < 0 \) respectively. Furthermore, the subset of the line denoted by \( L_2 \) allows for static solutions unlike the subset labelled \( L_1 \). This is due to the fact that a negative curvature contribution \( \dot{\kappa} \) can effectively act as a cosmological constant by counter-balancing other contributions in the Friedmann equation. This is explored in section 3.4.3.
The curvature can be obtained from (17), which in this sector becomes
\[ \Delta = -\hat{K}. \]  
(18)

The curvature can be obtained from (17), which in this sector becomes
\[ 1 = Q^2 + K. \]  
(19)

From (18) and (19) it is clear that the appropriately normalized variables (11) define a compact subsector of the total state space with
\[ x, y, z \in [0, 1], \quad K \in [0, 1] \quad \text{and} \quad Q, \Sigma \in [-1, 1]. \]  
(20)

Note that the variable \( K \) will not be used explicitly in any of the closed sectors.

As in the BIII case, we derive the propagation equations for the dynamical systems variables in this sector and reduce the dimensionality of the state space to four by eliminating

\begin{table}
\centering
\caption{Choice of normalization in different KS sectors, where the subscripts in the sector labels stand for closed. See text and caption of table 1 for details on the notation used here.}
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
Sector & \( \hat{x} \) & \( \hat{y} \) & \( \hat{z} \) & Normalization & Friedmann equation & Range of \((x, y, z)\) \\
\hline
1, & \( \geq 0 \) & \( \geq 0 \) & \( \geq 0 \) & \( \Delta = -\hat{K} \) & \( 1 = x + y + z + \Sigma^2 \) & \([0, 1] \times [0, 1] \times [0, 1] \) \\
2, & \( < 0 \) & \( > 0 \) & \( > 0 \) & \( \Delta = \hat{x} - \hat{K} \) & \( 1 = y + z + \Sigma^2 \) & \([-1, 0] \times [0, 1] \times [0, 1] \) \\
3, & \( > 0 \) & \( < 0 \) & \( > 0 \) & \( \Delta = \hat{y} - \hat{K} \) & \( 1 = x + z + \Sigma^2 \) & \([0, 1] \times [-1, 0] \times [0, 1] \) \\
4, & \( > 0 \) & \( > 0 \) & \( < 0 \) & \( \Delta = \hat{z} - \hat{K} \) & \( 1 = x + y + \Sigma^2 \) & \([0, 1] \times [0, 1] \times [-1, 0] \) \\
5, & \( < 0 \) & \( < 0 \) & \( > 0 \) & \( \Delta = \hat{x} + \hat{y} - \hat{K} \) & \( 1 = z + \Sigma^2 \) & \([-1, 0] \times [-1, 0] \times [0, 1] \) \\
6, & \( < 0 \) & \( > 0 \) & \( < 0 \) & \( \Delta = \hat{x} + \hat{z} - \hat{K} \) & \( 1 = y + \Sigma^2 \) & \([-1, 0] \times [0, 1] \times [-1, 0] \) \\
7, & \( > 0 \) & \( < 0 \) & \( > 0 \) & \( \Delta = \hat{y} + \hat{z} - \hat{K} \) & \( 1 = x + \Sigma^2 \) & \([0, 1] \times [-1, 0] \times [-1, 0] \) \\
8, & \( < 0 \) & \( < 0 \) & \( < 0 \) & \( \Delta = \hat{x} + \hat{y} + \hat{z} - \hat{K} \) & \( 1 = \Sigma^2 \) & \([-1, 0] \times [-1, 0] \times [-1, 0] \) \\
\hline
\end{tabular}
\end{table}
We recover the following features from the BIII subspace: the flat subset $K = 0$ (here corresponding to $Q^2 = 1$) is invariant, as can be seen from the $Q'$-equation together with the Friedmann equation (18). Other invariant subspaces are the hyper-surfaces $y = 0$ and $z = 0$. The isotropic subset $\Sigma = 0$ is not invariant unless $K = 0$.

Sectors 2_c–8_c are defined according to the possible signs of $\hat{x}$, $\hat{y}$, $\hat{z}$ as summarized in table 3: in each sector $\Delta$ is defined as the sum of the strictly negative contributions to (13). The dynamical equations analogous to (21) can be derived straightforwardly for each sector. We then solve these equations in each sector for their respective equilibrium points and the corresponding eigenvalues, and classify the equilibrium points according to their dynamical properties. The results are combined with the results from the open sectors and summarized in tables 6–10.

3.4. Exact solutions corresponding to the equilibrium points

We now derive the solutions corresponding to the various equilibrium points. Special attention has to be paid to the points with $y = 0$, since these correspond to the limit $R \to 0$, which may make the coordinate $x$ singular. We will study this issue in detail below. Note that it is legitimate to take the limit $R \to 0$ in the original field equations as long as $n > 1$, which results in the constraint $T_{ab} \to 0$. Consequently it is only possible to study the limit $R \to 0$ for $\mu \to 0$ and $n > 1$ when solving for the solutions corresponding to the equilibrium points with $y = 0$.

It is important to emphasize that the dynamical system by itself is well defined for $y = 0$; only when going back to the original equations to solve for the exact solutions corresponding to the equilibrium points with $y = 0$ do we notice that there may not be an exact solution corresponding to these coordinates.

We now proceed to find the exact solutions corresponding to the non-static ($\Theta, Q \neq 0$) equilibrium points. As usual, we can solve the energy conservation equation (9) for the non-vacuum solutions to obtain

$$\mu = \mu_0 a^{-3(1+w)},$$

(22)

where $\mu_0$ is determined by the $z$-coordinate of the given equilibrium point. We require $\mu_0 \geq 0$, which constrains the allowed range of $n$ or $w$ for a given equilibrium point (see below).

In order to determine the scale factor evolution at each equilibrium point, we rewrite the Raychaudhuri equation (5) as

$$\dot{\Theta} = -(1 + q_i) \frac{\Theta^2}{3},$$

(23)
where we express the deceleration parameter \( q_i \) at each point in terms of the dimensionless variables \((11)\):

\[
q_i = 2 \frac{\Sigma_i^2}{Q_i^2} + \frac{x_i}{Q_i^2} - \frac{y_i}{(n-1)Q_i^2} + \frac{z_i}{Q_i^2}. \tag{24}
\]

Note that this equation is invariant in different sectors: for a given equilibrium point, each coordinate divided by \( Q^2 \) is the same in all sectors. This ensures that the corresponding solution is invariant, no matter with which coordinates we describe the equilibrium point.

Similarly, we rewrite the trace-free Gauss–Codazzi equation \((6)\) as

\[
\dot{\sigma} = -\frac{1}{\sqrt{3}Q_i^2} \left[ \left( \frac{Q_i}{3Q_i} \right) \Sigma_i - \frac{K_i}{3} \right] \Theta^2, \tag{25}
\]

and the curvature constraint \((8)\) as

\[
R = \frac{2}{3} \Theta^2 \left[ 1 - q_i + \frac{\Sigma_i^2}{Q_i^2} - \frac{K_i}{Q_i^2} \right] \tag{26}
\]

for a given equilibrium point with coordinates \((Q_i, K_i, \Sigma_i, x_i, y_i, z_i)\) and deceleration parameter \( q_i \).

### 3.4.1. Power-law solutions.

We first study the non-stationary \((q \neq -1)\) cosmologies, for which \((23)\) has the solution

\[
\Theta = \frac{3}{(1 + q_i)t}. \tag{27}
\]

We have set the big bang time \( t_0 = 0 \). Given \( \Theta \), we can solve for all the other dynamical quantities for a given equilibrium point to obtain the scale factor evolution

\[
a = a_0 |t|^\alpha, \quad \text{where} \quad \alpha = (1 + q_i)^{-1}, \tag{28}
\]

the shear

\[
\sigma = \frac{\beta}{t^2} + \text{const}, \quad \text{where} \quad \beta = \frac{\sqrt{3}}{(1 + q_i)^2} \left[ \frac{\Sigma_i}{Q_i} \left( \frac{3}{Q_i} - \frac{x_i}{Q_i^2} \right) - \frac{K_i}{Q_i^2} \right], \tag{29}
\]

and the curvature scalar

\[
R = \frac{\gamma}{t^2}, \quad \text{where} \quad \gamma = \frac{6}{(1 + q_i)^2} \left[ 1 - q_i + \frac{\Sigma_i^2}{Q_i^2} + \frac{K_i}{Q_i^2} \right]. \tag{30}
\]

Again, we point out that even though a given equilibrium point formally has different coordinates in the different sectors, the exact solutions corresponding to the point are invariant, since the coordinates only enter the solutions \((28)-(30)\) with a factor \(1/Q_i^2\). The solutions for each point are summarized in table 4, where constants of integration were obtained by substituting the solutions into the original equations.

When substituting the points with \( y = 0 \) into the original field equations, we find that these are only satisfied for special values of \( n \). This is reflected in table 4. Point B only has a solution for \( n = 5/4 \) and \( w = 2/3 \). The solutions for points D and E only satisfy the original equations for \( n = 1 \), which has been excluded from the start. These points therefore do not have any physical power-law solutions. The points on the lines \( L_{1,2} \) only have corresponding solutions for special coordinate values, making only two points on each line physical (see below).
Table 4. Solutions for scale factor, shear, curvature and energy density corresponding to the equilibrium points.

| Point | Scale factor ($a$) | Shear ($\sigma$) | Ricci scalar ($R$) | $\mu$ |
|-------|--------------------|------------------|------------------|------|
| $A$   | $n \neq 2$         | $a_0 [t]^{\frac{1+2-(2n-1)}{2n-1}}$ | 0                | $\frac{4n(1-\alpha(2n-1)(4n-15)}{(2n-2)^2}$ | 0    |
|       | $n = 2$            | $a_0 e^{\frac{1}{2} \phi_0}$ | $\frac{4}{\phi_0^2}$ | $\mu_0$ | 0    |
| $B$   | $w = \frac{2}{3}$  | $a_0 [t]^{\frac{1}{2}}$ | 0                | 0    | 0    |
|       | $w = -1$           | $a_0 [t]^{\frac{1}{2\phi_0}}$ | 0                | $\mu_0^{(2-2n)}$ | $\mu_1^{(2n)}$ |
| $C$   | $w \neq -1$        | $a_0 [t]^{\frac{1}{2\phi_0}}$ | 0                | $\mu_0^{(2-2n)}$ | $\mu_1^{(2n)}$ |
|       | $w = -1$           | $a_0 [t]^{\frac{1}{2\phi_0}}$ | $\frac{1}{2\phi_0}$ | $\mu_0^{(2-2n)}$ | $\mu_1^{(2n)}$ |
| $C'$  | $n \neq 2$         | $a_0 [t]^{\frac{1}{2\phi_0}}$ | 0                | $\mu_0^{(2-2n)}$ | $\mu_1^{(2n)}$ |
|       | $n = 2$            | $a_0 [t]^{\frac{1}{2\phi_0}}$ | $\frac{1}{2\phi_0}$ | $\mu_0^{(2-2n)}$ | $\mu_1^{(2n)}$ |
| $G$   | $w \neq -1$        | $a_0 [t]^{\frac{1}{2\phi_0}}$ | 0                | $\mu_0^{(2-2n)}$ | $\mu_1^{(2n)}$ |
|       | $w = -1$           | $a_0 [t]^{\frac{1}{2\phi_0}}$ | $\frac{1}{2\phi_0}$ | $\mu_0^{(2-2n)}$ | $\mu_1^{(2n)}$ |
| Line  | $\Sigma_0^2 = \frac{5-4n}{2(3n^2)}$, $n \in (1, \frac{5}{2})$ | $a_0 [t]^{\frac{1}{2\phi_0}}$ | $\sqrt{\frac{N_0}{(2n+1)^2}}$ | 0    | 0    |
|       | $\Sigma_1^2 = \frac{2n-1}{3(n^2)}$, $n \in (\frac{1}{2}, 1)$ | $a_0 [t]^{\frac{1}{2\phi_0}}$ | $\sqrt{\frac{N_1}{(2n+1)^2}}$ | 0    | 0    |

Excluding these non-physical points, we find that the only non-vacuum solutions are given by $C$ and $G$. Substituting the solution (22) into the definition of $\varepsilon$, we find that the constant $\mu_0$ must satisfy

$$\mu_0 = z_1 Y_0^{n-1} \left( \frac{\alpha}{Q_l} \right)^{2n} (3n)^{2n} \left( \frac{2}{n-1} \right)^{n-1}.$$

In order for these solutions to be physical, we require that $\mu > 0$ and therefore $\mu_0 > 0$. For $C$ we find that this condition is satisfied for

$$1 < n < \frac{1}{4(4 + 3w)(13 + 9w + \sqrt{9w^2 + 66w + 73})}, \quad w > -1,$$

while for $G$ it is only valid for

$$1 < n < N_+, \quad -1 < w \leq 0,$$

$$N_- < n < N_+, \quad 0 < w < \frac{1}{15}(-15 + 4\sqrt{15}),$$

where $N_+ = \frac{1}{4(4 + 3w)}(9 + 5u \pm \sqrt{1 - 30w - 15u^2})$.

For points $A$ and $F$ the solutions only depend on $n$, while the solutions at $C$ and $G$ depend on both $n$ and $w$. We can see from these solutions that points $A$ and $C$ are the isotropic analogues of points $F$ and $G$, respectively.

The lines $L_1$ and $L_2$ have the same solutions for shear and energy density. As noted above, they are the same line but for different ranges of $\lambda$ and hence $\alpha$. $L_1$ contains the isotropic subset of solutions ($\Sigma_0 = 0$) while $L_2$ contains the static subset ($\alpha = Q_0 = 0$).

In table 5, we summarize the behavior of the deceleration parameter $q$. By studying the deceleration parameter, we can determine whether the power law solutions above correspond to accelerated ($-1 < q < 0$) or decelerated ($q > 0$) expansion or contraction. The expansion (or contraction) of point $A$ is decelerating for $n \in (0, 1/2)$ or $n \in (1, \frac{1}{2}(1 + \sqrt{5}))$ and accelerating for $n \in (\frac{1}{2}(1 + \sqrt{5}), 2)$. Point $B$ and lines $L_{1,2}$ only admit decelerating behaviours. Point
Table 5. Deceleration parameter for the equilibrium points. In the last three columns we state explicitly for which values of \( n \) the deceleration parameter \( q \) (stated in the second column) is less, equal to or larger than 0, i.e. whether the have accelerated, de—Sitter—like or decelerated behaviours. The parameters are \( P = \frac{1}{2}(1 + \sqrt{3}) \) and \( S_n = \frac{1}{2}(1 + w) \).

| Point | \( q \) | \( w \) | \( q = -1 \) | \( -1 < q < 0 \) | \( q > 0 \) |
|-------|--------|--------|--------------|---------------|-------------|
| \( A \) | \( \frac{1 + 2n - 2e^{-2\sqrt{2}a}}{1 - 2n + 2e^{-2\sqrt{2}a}} \) | all | 2 | \( (P_+, 2) \) | \( (1, P_+) \) |
| \( B \) | 1 | all | – | – | \( (0, \infty) \) |
| \( C \) | \( \frac{3(1 + w) - 2e^{-2\sqrt{2}a}}{e^{-2\sqrt{2}a}} \) | all | – | – | \( (S_n, \infty) \) |
| \( F \) | \( \frac{1 + 2n - 2e^{-2\sqrt{2}a}}{3 - 5n + 2e^{-2\sqrt{2}a}} \) | all | – | \( (P_+, 2) \) | \( (0, P_+) \) |
| \( G \) | \( \frac{3(1 + w) - 2e^{-2\sqrt{2}a}}{e^{-2\sqrt{2}a}} \) | all | – | \( (0, \infty) \) | – |

\( \mathcal{F} \) has a decelerated behaviour for \( n \in \left(0, \frac{1}{2}(1 + \sqrt{3})\right) \) and an accelerated behaviour for \( n \in \left(\frac{1}{2}(1 + \sqrt{3}), 2\right) \). The equilibrium points \( C \) and \( G \) for \( w \in \left[0, 1\right] \), have decelerated behaviours when \( n \in \left(0, \frac{3}{2}(1 + w)\right) \) and accelerated behaviours when \( n \in \left(\frac{3}{2}(1 + w), \infty\right) \).

3.4.2. Stationary solutions. If \( q = -1 \), we obtain stationary solutions (\( \Theta = 0 \)), which have an exponentially increasing scale factor. As reflected in table 5, the vacuum points \( A \) and \( \mathcal{F} \) correspond to de Sitter solutions for the bifurcation value \( n = 2 \) for all equations of state, while the matter points \( C \) and \( G \) are de Sitter-like for all \( n > 0 \) but \( w = -1 \) only, and \( E \) appears to be de Sitter-like for \( w = 1 \) for all values of \( n > 0 \). Since \( E \) has \( y = 0 \), we will have to study this case in more detail below.

For a constant expansion rate
\[
\Theta = \Theta_0, \tag{33}
\]
the scale factor has the following solution:
\[
a = a_0 e^{\frac{1}{3}\Theta_0 t}. \tag{34}
\]
The energy conservation equation becomes
\[
\dot{\mu} = 0 \Rightarrow \mu = \mu_0. \tag{35}
\]
The trace-free Gauss–Codazzi equation (6) can be rewritten as
\[
\dot{\sigma} = \beta_0, \quad \text{where} \quad \beta_0 = \frac{\Theta_0^2}{3\sqrt{3}Q_i} \left[ K_i - \left(\frac{3Q_i - x_i}{Q_i}\right) \Sigma_i \right], \tag{36}
\]
which on integration yields
\[
\sigma = \beta_0 t + \sigma_0, \tag{37}
\]
where \( \sigma_0 \) is an integration constant. The evolution of the Ricci scalar can be obtained by substituting the solutions above into (8), to find
\[
R = \frac{2}{3} \left(2 + \frac{K_i}{Q_i}\right) \Theta_0^2 + 2(\beta_0 t + \sigma_0)^2. \tag{38}
\]
As before, we substitute the solutions at each equilibrium point into the definition of the coordinates, which constrains the constants of integration for each point. In particular, \( \beta_0 = 0 \) holds for all stationary equilibrium points, which means that we only have constant or vanishing shear.

As in the power-law case, we see that all the equilibrium points except for \( C \) and \( G \) correspond to vacuum solutions \( \mu = 0 \). For point \( C \) the energy density is given by

\[
\mu = \mu_1^C = 4^{n-1}3^{-n}(2-n)\Theta_0^{2n},
\]

and for \( G \) the energy density is given by

\[
\mu = \mu_1^G = 4^{n-1}(2-n)\Theta_0^{2n}.
\]

Both of these solutions only hold for \( 1 < n \leq 2 \) with \( w = -1 \).

Again, we substitute the generic solutions into the original field equations for each point, and find that the original equations are satisfied for all points with \( y \neq 0 \). It is however not possible to find a stationary solution at point \( E \) (which has \( y = 0 \)), even after carefully considering the limit \( y \to 0 \).

### 3.4.3. Static solutions

The static equilibrium points are characterized by \( \Theta = \dot{\Theta} = 0 \). These points satisfy \( Q = x = 0 \), where the second identity comes from the fact that if \( Q = 0 \), then we require that \( x = 0 \) from the definition of the variables, as discussed below\(^7\).

We will now explore which of the equilibrium points obtained above correspond to static solutions. As indicated above, even though \( Q = \pm \epsilon \) holds in the first sector as stated in Table 2, \( Q \) can be a function of \( n \) and/or \( w \) in the other sectors. In order to find the static equilibrium points, we have to look at the coordinates that each equilibrium point takes in each sector, and find the values of \( n \) and/or \( w \) for which \( Q = 0 \) in the given sector.

An obvious static solution appears to be the subset \( Q = 0 \) on line \( L_{2,\pm} \) for all values of \( n \) and \( w \). We can however not find a solution corresponding to this limit, since \( Q = 0 \) implies \( \sigma = 0 \), which contradicts the value of the shear coordinate of this equilibrium point. We can study the eigenvalues associated with the line \( L_2 \) in the limit \( Q \to 0^\pm \) and find that the static subset is an unstable saddle point for all values of \( n \) for both \( L_{2,+} \) and \( L_{2,-} \).

The point \( A \) appears to admit a static solution for the bifurcation value \( n = 1/2 \). This bifurcation only occurs in sectors 2, 3, 6 and 7 of the open and the closed sectors. However, it is not possible to find a solution satisfying the coordinates of the static equilibrium point that satisfies the original field equations. For this reason, this static equilibrium point is unphysical. We explore the stability of the static solution in the limit \( n \to 1/2 \) from the appropriate sides: for example, point \( A \) only lies in the open sector 2 for \( n \in [0, 1/2] \) or \( n \in [2, \infty] \), making only the limit \( n \to 1/2^- \) well defined. We find that this bifurcation represents a saddle point in the state space since two of the eigenvalues approach \( \infty \) from the left and \( -\infty \) from the right, making the point unstable.

Even though the \( Q \)-coordinate of point \( B \) is a function of \( w \) in sectors 2, 4, 5 and 7, \( Q \) cannot be zero for any values of \( w \). This means point \( B \) does not admit any static solutions.

Point \( C \) can only be static in the limit \( n \to 0 \) in sector 6 for \( w = 0 \), \( 1/3 \), 1 and in sector 3 and 5 for \( w = -1 \). Again, we cannot find a solution for this special case, but this case is physically uninteresting either way.

\[^7\] Note that unlike in the bouncing or recollapsing case below, we do not consider \( Q = y = 0, x \neq 0 \) here, since this corresponds to the limit \( R \to 0 \). While we may want to study a bounce where the Ricci scalar approaches zero and then grows again, we are not interested in static solutions that have vanishing Ricci curvature at all times.
Table 6. Nature of the expanding \((\epsilon = +1)\) spatially flat BI equilibrium points. The collapsing analogs are simply time reversed. The parameters are \(P_+ = \frac{1}{2} \left(1 + \sqrt{3} \right)\) and \(V_+ = \frac{1}{14} \left(11 + \sqrt{37} \right)\).

| Point | \(w\) | Range of \(n\) |
|-------|-------|----------------|
| \(A_-\) | \(-1\) | Saddle Attractor Repeller Repeller Saddle Saddle Saddle Attractor |
| 0 | Saddle | Attractor | Repeller | Repeller | Saddle | Saddle | Attractor |
| \(B_0\) | \(-1\) | Saddle | Saddle | Saddle | Saddle | Saddle | Saddle | Saddle | Saddle |
| 0 | Saddle | Saddle | Saddle | Saddle | Saddle | Saddle | Saddle | Saddle | Saddle |
| \(C_0\) | \(-1\) | Repeller | Repeller | Saddle | Saddle | Saddle | Repeller | Repeller |
| 0 | Saddle | Saddle | Saddle | Saddle | Saddle | Saddle | Saddle | Saddle | Saddle |
| \(D_0\) | \(-1\) | Saddle | Saddle | Attractor | Attractor | Attractor | Attractor | Saddle |
| 0 | Saddle | Saddle | Saddle | Saddle | Saddle | Saddle | Saddle | Saddle | Saddle |
| \(E_0\) | \(-1\) | Saddle | Saddle | Repeller | Repeller | Repeller | Saddle |
| 0 | Saddle | Saddle | Saddle | Saddle | Saddle | Saddle | Saddle | Saddle | Saddle |

Table 7. Nature of the line of expanding spatially flat anisotropic equilibrium points \(L_{1+}\). Here \(\Sigma_\pm(n) = \sqrt{\frac{1}{1 - 4n^2}}\) is a bifurcation value depending on \(n\).

| \(w\) | \(\Sigma\) |
|-------|--------|
| \(-1, 0, 1/3\) | Repeller |
| \((0, 1)\) | Repeller |
| \((1, 5/4)\) | Repeller |
| \((n > 5/4)\) | Repeller |
| All \(n\) | Saddle |

The \(Q\)-coordinate of point \(E\) is zero in sectors 6–8 for \(w = 2/3\), but again there is no solution corresponding to this limit.

Even though point \(F\) has \(Q\) as a function of \(n\) in open sectors 2 and 6, \(Q(n)\) is nonzero for the allowed ranges of \(n\).

Point \(G\) becomes static in the limit \(n \to 0\) in sectors 4, 6 and 8, which again is not physically relevant.

3.5. The full state space

The full state space is obtained by matching the various sectors along their common boundaries. Since the full state space is 4-dimensional it is not easily visualized, so we refer to [32] for an illustration of the 2-dimensional Bianchi I vacuum subspace and the 2-dimensional flat FLRW subspace with matter. We emphasize that we have to formally exclude the subset with \(Q = 0\) and \(x \neq 0\) from the state space unless \(y \to 0\). This is an artefact of the definition of the variable \(x\), and reflected by the fact that there are no orbits crossing this subset—the only trajectories crossing the plane \(Q = 0\) pass through the points with \(x = 0\) or \(y = 0\).

3.6. Qualitative analysis

We summarize the dynamical behaviour of the equilibrium points and lines of equilibrium points in tables 6, 9 and 10 and tables 7 and 8 respectively. For the stability analysis, we only consider the four cases: cosmological constant \(w = -1\), dust \(w = 0\), radiation \(w = 1/3\) and stiff matter \(w = 1\). We only state the results for the equilibrium points corresponding...
Table 8. Nature of the line of spatially flat anisotropic equilibrium points $L_{2,\pm}$. Here $Q_b(n) = \sqrt{\frac{8}{\pi n\pi}}$ is a bifurcation value depending on $n$. We discuss the bifurcation $Q = 0$ in the section on static solutions below. Note that the dynamical behaviour of $L_{2,\pm}$ and $L_{2,-}$ is identical.

| $n$          | Range of $Q$                                      | Attractor | Saddle | Repeller | Repeller |
|--------------|--------------------------------------------------|-----------|--------|----------|----------|
| $n \in [0, 1/2]$ | $Q \in [-1, -Q_b(n))$ | Attractor | Saddle | Repeller | Repeller |
| $n \in (1/2, 1)$ | $Q \in (-Q_b(n), 0)$ | Attractor | Saddle | Repeller | Repeller |
| $n > 1$      | $Q \in (0, Q_b(n))$ | Attractor | Saddle | Repeller | Repeller |
|              | $Q \in (Q_b(n), 1]$ | Attractor | Saddle | Repeller | Repeller |

Table 9. Nature of the spatially open Bianchi III equilibrium points, where $P_+ = \frac{1}{2}(1 + \sqrt{3})$.

| Point | $w$          | Range of $n$ | $(0, 1)$ | $(1, 5/4)$ | $(5/4, P_+)$ | $(P_+, 3/2)$ | $(3/2, 3)$ | $(3, \infty)$ |
|-------|--------------|--------------|----------|------------|--------------|--------------|------------|-------------|
| $\mathcal{D}_+$ | All          | Saddle       | Saddle   | Saddle     | Saddle       | Saddle       | Saddle     | Saddle      |
| $\mathcal{E}_+$ | $-1$         | Attractor    | Saddle   | Saddle     | Saddle       | Saddle       | Saddle     | Saddle      |
|          | $0, 1/3, 1$  | Saddle       | Saddle   | Saddle     | Saddle       | Saddle       | Saddle     | Saddle      |
| $\mathcal{F}_+$ | $-1$         | Saddle       | Saddle   | Saddle     | Repeller     | Repeller     | Repeller   | Repeller    |
|          | $0$          | Saddle       | Saddle   | Saddle     | Repeller     | Repeller     | Repeller   | Repeller    |
|          | $1/3$        | Attractor    | Attractor| Saddle     | Saddle       | Saddle       | Saddle     | Saddle      |
|          | $1$          | Attractor    | Attractor| Repeller   | Repeller     | Repeller     | Repeller   | Repeller    |
| $\mathcal{G}_+$ | $-1$         | Saddle       | Saddle   | Saddle     | Saddle       | Saddle       | Saddle     | Saddle      |
|          | $0$          | Saddle       | Saddle   | Saddle     | Saddle       | Saddle       | Saddle     | Saddle      |
|          | $1/3$        | Saddle       | Saddle   | Saddle     | Saddle       | Saddle       | Saddle     | Saddle      |
|          | $1$          | Saddle       | Saddle   | Saddle     | Saddle       | Saddle       | Saddle     | Saddle      |

Table 10. Nature of the spatially closed Kantowski-Sachs equilibrium points, where $P_+ = \frac{1}{2}(1 + \sqrt{3})$ and $X \approx 1.13$.

| Point | $w$          | Range of $n$ | $(0, 1)$ | $(1, X)$ | $(X, P_+)$ | $(P_+, 3/2)$ | $(3/2, 3)$ | $(3, \infty)$ |
|-------|--------------|--------------|----------|----------|------------|--------------|------------|-------------|
| $\mathcal{F}_+$ | $-1$         | Saddle       | Saddle   | Saddle   | Saddle     | Saddle       | Saddle     | Saddle      |
|          | $0$          | Saddle       | Saddle   | Saddle   | Saddle     | Saddle       | Saddle     | Saddle      |
|          | $1/3$        | Saddle       | Saddle   | Saddle   | Saddle     | Saddle       | Saddle     | Saddle      |
|          | $1$          | Saddle       | Saddle   | Saddle   | Saddle     | Saddle       | Saddle     | Saddle      |
| $\mathcal{G}_+$ | $-1$         | Saddle       | Saddle   | Saddle   | Saddle     | Saddle       | Saddle     | Saddle      |
|          | $0$          | Saddle       | Saddle   | Saddle   | Saddle     | Saddle       | Saddle     | Saddle      |
|          | $1/3$        | Saddle       | Saddle   | Saddle   | Saddle     | Saddle       | Saddle     | Saddle      |
|          | $1$          | Saddle       | Saddle   | Saddle   | Saddle     | Saddle       | Saddle     | Saddle      |

Table 6 consists of all the BI subspace equilibrium points (excluding the lines); their behaviour is similar to the flat Friedmann points which were found previously [5, 25]. We note that some of the solutions corresponding to the Friedmann and BI equilibrium points have also been found in [40, 41].
The lines of equilibrium points have to be treated more carefully. We summarize their dynamical behaviour in tables 7 and 8. As noted above, the two lines include the same parametric solutions, but \( L_1 \) corresponds to \( x \geq 0 \), while \( L_2 \) has \( x \leq 0 \).

Since these lines correspond to flat solutions they should have been found in [5]. In fact, the authors of [5] found a line of equilibrium points denoted by \( L_1^* \), extending over \( \Sigma_{ss} \in [0, \infty) \), where \( \Sigma_{ss} \) measures the shear contribution to the Friedmann equation. The range \( 0 \leq \Sigma_{ss} \leq 1 \) corresponds to our \( L_{1,+} \), while \( \Sigma_{ss} > 1 \) corresponds to \( L_{2,+} \) as can be seen from the solutions of the scale factor in [5]. In [5] the isotropic solution was at \( \Sigma_{ss} = 0 \) and the static one occurred for \( \Sigma_{ss} \to \infty \). Note that [5] did not address the collapsing solutions \( L_{1,-} \) or opposite orientation of the shear \( L_{2,-} \) because the phase space for BI is symmetric about the plane \( \Sigma = 0 \). The stability for \( \Sigma < 0 \) can be obtained by time reversal from the corresponding points in the \( \Sigma > 0 \) subspace.

As stated above, \( L_{1,2} \) are only physical for certain special coordinates. These are

\[
\Sigma_{ss} = \pm \sqrt{\frac{5 - 4n}{2n - 1}}, \quad n \in (1, 5/4), \quad \text{and} \quad Q_{ss} = \pm \sqrt{\frac{2n - 1}{5 - 4n}}, \quad n \in (1/2, 1),
\]

for \( L_1 \) and \( L_2 \) respectively. These equilibrium points are always saddles in nature.

The nature of the BI11 equilibrium points is stated in table 9. For the sake of completeness, we have included the stability of points \( D \) and \( E \), but will not discuss them any further since they are not physical. The point \( F \) lies in the BI11 subspace for \( n \in (0, \frac{1}{2}(1 + \sqrt{3})) \). \( F_+ \) is a saddle for \( w = -1 \) and for \( n \in (1, 5/4) \) when \( w = 0 \), but an attractor otherwise. \( G_+ \) lies in the BI11 subspace for \( n \in (0, 3/2) \) when \( w = 0 \), for all \( n \) when \( w = 1/3 \) and \( n \in (0, 1) \) and \( n \in (3, \infty) \) when \( w = 1 \). \( G_+ \) is saddle except for \( n \in (1, 5/4) \) when \( w = 0 \) where it becomes a repeller.

The nature of the KS equilibrium points is stated in table 10. Point \( F \) lies in the KS subspace for \( n > \frac{1}{2}(1 + \sqrt{3}) \) and \( F_+ \) is always a saddle. Similarly, \( G_+ \) lies in the KS subspace for all \( n \) when \( w = -1 \), for \( n > 3/2 \) when \( w = 0 \) and \( n \in (1, 3) \) when \( w = 1 \). \( G_+ \) is saddle except for \( n \in (1, 1.13) \) when \( w = 1 \), where it is a repeller.

We can identify the following global attractors and repellers: \( A_+ \) is a global attractor for \( n \in (P_+, 2) \) when \( w = 0 \), 1/3 and 1, and for \( n \in (2, \infty) \) (all \( w \)). When \( w = -1 \), \( C_+ \) is a global attractor for \( n \in (1, 2) \) and \( E_+ \) for \( n \in (0, 1/2) \). Point \( F_+ \) is a global attractor for \( n \in (0, 1/2) \) and \( n \in (5/4, P_+) \) when \( w = 0 \), 1/3 and 1, and for \( n \in (1, 5/4) \) when \( w = 1/3 \) and 1. \( G_+ \) is only a global attractor for \( n \in (1, 5/4) \) when \( w = 0 \). By time reversal the corresponding contracting solutions are global repellers. There are no global repellers in the expanding subspace since the lines \( L_{1,2} \) contain repellers and hence there are no global attractors in the collapsing subspace.

### 3.7. Bouncing or recollapsing trajectories

As motivated above, any trajectory corresponding to a bouncing or recollapsing solution must pass through \( x = Q = 0 \) or \( y = Q = 0 \).

The existence of bouncing orbits for Bianchi I models has been studied in [32]. In the vacuum case, it was found that there exist bouncing/recollapsing trajectories, but only for \( y < 0 \). If \( n > 1 \), \( R \) has to be negative and there can only be re-collapse \( (\Theta < 0) \). For \( n \in [0, 1/2] \) re-collapse may occur if \( R > 0 \), and for \( n \in [0, 1/2] \) there may be a bounce \( (\Theta > 0) \) for positive \( R \). In all cases, the bouncing trajectories have to pass through the single point \( x = Q = 0 \) (denoted by \( M \) in [32]) in the 2-dimensional BI vacuum subspace. Note that it is not possible to achieve a bounce through \( y = Q = 0 \) here, since a line of equilibrium points passes through that point in this subspace.
When matter is added, we obtain another degree of freedom, and unlike in GR, the matter term may enhance bouncing or recollapsing behaviour due to the $R^{n-1}$ term coupled to the energy density. The corresponding trajectories now have to pass through the 1-dimensional lines with $x = Q = 0$ or $y = Q = 0$ instead of the single point $\mathcal{M}$.

In the presence of spatial curvature, it is yet easier to achieve bouncing or recollapsing behaviour. If $3R < 0$, the results from the flat Bianchi I case are qualitatively recovered. For $3R > 0$ however, there are differences to the Bianchi I case. In particular, positive spatial curvature allows $\Theta = 0$ even for positive $y$.

4. Isotropization of OSH Bianchi models

It is possible to study isotropization by looking at the stability of the Friedmann points in the state space (see [24] and references therein). When such an isotropic point is an attractor, then we have asymptotic isotropization in the future. If the point is a repeller we have an isotropic future attractor in these ranges. $A_r$ is an isomorphic past attractor for $n \in (1, 5/4)$ when $w = -1$, 0 or 1/3, and for $n \in (1, 14(11 + \sqrt{37})/37)$ when $w = 1$. As pointed out in [5], this is an interesting feature, since the existence of an isotropic past attractor implies that we do not require special initial conditions for inflation to take place. The contracting analogue $A_-$ is an isotropic future attractor in these ranges. $A_r$ is a future attractor for $n > 2$ when $w = -1$ and for $n > 1/2(1 + \sqrt{3})$ when $w = 0, 1/3$ or 1. By time reversal, $A_-$ is a past attractor for these parameter values.

The equilibrium point $C_+ = 1$ is an isotropic past attractor for $n \in (1, 14(11 + \sqrt{37})/37)$ when $w = 1$ and an isotropic future attractor for $n \in (1, 2)$ when $w = -1$. When $w = 0$ or $w = 1/3$ this point is a saddle for all values of $n$. This means that in this case we have a transient matter/radiation dominated phase in which the model is highly isotropic and hence potentially compatible with observations.

We note that all isotropic equilibrium points found in this analysis are flat Friedmann-like, unlike in [25], where the isotropic points $A$ and $C$ with nonzero spatial curvature were found. The reason for this discrepancy is that the plane $\Sigma = 0$ is no longer invariant when allowing for nonzero spatial curvature ($k \neq 0$); as in GR spatial curvature causes anisotropies to grow in models with $R^n$-gravity. For this reason, the points $A$ and $C$ no longer remain equilibrium points in the full OSH Bianchi state space.

There are two equilibrium points of interest with nonzero shear: the vacuum point $\mathcal{F}$ and the non-vacuum point $\mathcal{G}$. These points are isotropic for certain bifurcation values of $n$ and $w$: $\mathcal{F}$ is isotropic for $n = 1/2(1 + \sqrt{3})$ for all $w$, and $\mathcal{G}$ is isotropic for $n = 1/2(1 + w)$ if $w > -1$. The KS point $G_r$ is a past attractor for $n \in (1, 1.13)$ when $w = 1$ and a saddle for $n > 3/2$ when $w = 0$. This means that we can have initial conditions which are anisotropic, or we can have intermediate anisotropic conditions which are conducive for structure formation, provided that the anisotropies are sufficiently small. When $w = 1/3$, the point $G_r$ lies in the BIII state space and is a saddle for all values of $n$, and when $w = 0$ the same applies for $n \in (0, 1)$ or $n \in (5/4, 3/2)$.
5. Remarks and conclusions

Our main aim in this paper was to investigate the effects of spatial curvature on the isotropization of OSH Bianchi models in $R^n$-gravity, and to possibly identify static solutions and bounce behaviours. To achieve this goal, we constructed a compact state space which allows one to obtain a complete picture of the cosmological behaviour for expanding, contracting and static as well as bouncing or recollapsing models. This is not possible with the non-compact variables used in [5], since the equilibrium points with static solutions do not have finite coordinates in this framework. The Poincaré projection also does not allow one to patch together the expanding and contracting copies of the state space, so bounce behaviour cannot be investigated. This is discussed in detail in [32] for the BI subspace, where the results obtained here are compared to the results obtained in [5].

We do not find any exact Einstein static solutions in this analysis. However, we do find orbits that exhibit cyclic behaviour, which was expected from previous work examining the conditions for bouncing solutions in $f(R)$ gravity [19]. We also recover all the isotropic equilibrium points that were found in [5]. The expanding vacuum point $A_e$ is a past attractor for $n \in (1,5/4)$ as in the BI case. We emphasize that we only find flat ($k = 0$) isotropic equilibrium points ($A$, $B$ and $C$). Therefore for these types of theories, isotropization also implies cosmological behaviours which evolve towards spatially flat spacetimes. Late time behaviour with nonzero spatial curvature will have a growth in anisotropies, as in GR.

In conclusion, we have shown that spatial curvature does indeed affect the isotropization of cosmological models in $R^n$-gravity. While no exact static solutions could be found, we did find that bounces can occur in these cosmologies.

Acknowledgments

This research was supported by the National Research Foundation (South Africa) and the Italian *Ministero Degli Affari Esteri-DG per la Promozione e Cooperazione Culturale* under the joint Italy/South Africa Science and Technology agreement. NG is funded by the Claude Leon Foundation. We thank Salvatore Capozziello for useful discussions. We thank the referees for their useful comments.

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