ON THE VALUATION OF ARITHMETIC-AVERAGE ASIAN OPTIONS: INTEGRAL REPRESENTATIONS

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This paper derives integral representations for the Black–Scholes price of arithmetic–average Asian options. Their proof is by Laplace inverting the Laplace transform of [GY] using complex analytic methods. The analysis ultimately rests on the gamma function which in this sense is at the base of Asian options. The results of [GY] are corrected and their validity is extended.

Introduction: The aim of this paper is to derive exact closed–form valuation formulas for European–style arithmetic–average Asian options in a Black–Scholes framework. These options are a particular class of path–dependent options with the arithmetic average of the underlying security as the relevant variable.

This paper has its origin in that development in the analysis of Asian options initiated by [Y]. Yor’s valuation formula gives clear evidence that pricing Asian options is a problem of some intrinsic difficulty indeed for which no, in the strict sense, simple solution should be expected. Instead, one should, as a first step, ask for structurally clear solutions, and only then, as a second step, consider in particular questions of a more computational nature.

The main valuation result of this paper is described in paragraph four and identifies the Black–Scholes price of an Asian option as an integral over, roughly speaking, the product of two well–studied higher transcendental functions both given as integrals and built up using so–called Hermite functions. These last functions generalize the familiar complementary error function and come from boundary–value problems in potential theory for domains whose surface is an infinite parabolic cylinder. It seems to be the first time that such parabolic cylinder functions are explicitly identified to characterize solutions to problems of Finance. Their relevant properties are reviewed in paragraph zero.

Both valuation formulas involve three integrations. In a sense, this measures the difficulty of the problem. For getting some intuition, first think of the error integral as an example of how a single integration yields a higher transcendental function. In the best case, a triple integral separates as a product of three such functions. In the worst case, no separation is possible and all one can say is that one has to integrate a function in three variables three times in succession. With its integrand a function $f(x, y, w)$ that cannot be further separated, Yor’s formula belongs to this last category. In our formula complete separation is not achieved either. However, with its integrand a product $g(x, y)h(y, w)$ it is a single integral of a function given as the product of two single integrals.

The method used in this paper is the Laplace transform Ansatz to option pricing of [GY] and its main result is proved by Laplace inverting the Laplace transform they have derived.

Transform methods like the Laplace transform are classical concepts in analysis with applications ranging from number theory to boundary–value problems for partial differential
equations. In particular the Laplace transform approach used in this paper generally consists of two stages. As a first step, a suitably nice function on the positive real line is transformed into a complex–valued function. This Laplace transform exists on a half–plane sufficiently deep within the right complex half–plane and defines there a holomorphic, i.e., complex analytic, function. The methods in this part of the transform analysis have real analysis origin. As a second step, the Laplace transform has to be inverted to give the desired function on the positive real line. The natural methods in this step are complex analytic. Some relevant concepts are recalled in paragraph zero of this paper.

The computation of the Laplace transform already is quite a problem in general. Indeed, with the functions to be transformed generally not known one expressed aim for trying to calculate their Laplace transform actually is to get an explicit expression at all. However, for an actual calculation of a Laplace transform in such a situation additional insights into the mathematical structure are necessary. Thus, the Laplace transform in [GY] is computed introducing a stochastic clock and using general results on Bessel processes.

It should be said that, unfortunately, there are serious problems of a fundamental nature with this result. More precisely, the conceptual error I found in September 1999 kills this approach to the valuation of Asian options as it stands. Luckily, Peter Carr eventually succeeded in convincing me that the Geman–Yor result can nevertheless be used in a very astute way for its original purpose. All this is discussed in paragraph five of this paper.

Mathematically, the inversion of a non–trivial Laplace transform belongs to the nastier problems in analysis. Difficulties are encountered here as a rule, and more than often one has to be satisfied with only knowing the Laplace transform. It thus can be regarded as as the main mathematical contribution of the paper that it indeed provides such an inversion for the above Laplace transform of the price of Asian options. The details of the proof are described in paragraphs six to twelve of this paper. Its methods are complex analytic. It uses the complex inversion formula for the Laplace transform and is ultimately based on the classical Hankel formulas for the gamma function. These methods are such that the results can be extended from the classical Black–Scholes framework essential for the validity of [GY] to one with no restrictions on the drift coefficients.

It came as a surprise to the author that one has such a structural integral representation for the value of Asian options as derived in this paper. He is not sure if this should be attributed to the qualities intrinsic to Asian options or to the fact that, as described above, one takes a somewhat indirect approach to their valuation.

The series representations and asymptotic expansions of our formula appropriate and adequate for computation are discussed in [Sch].

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0. Analytic preliminaries: This paragraph collects some relevant facts about the Laplace transform from [B], [D], and about Hermite functions from [L, §§10.2ff].

Laplace transform: The class of functions considered in the sequel is that of functions of exponential type, i.e., of continuous, real-valued functions $f$ on the non-negative real line such that there is a real number $a$ for which $\exp(at)f(t)$ is bounded for any $t > 0$. The Laplace transform $L$ is the linear operator $L$ that associates to any function $f$ of exponential type the complex-valued function $L(f)$, holomorphic on a suitable complex half-plane, which for any complex number $z$ with sufficiently big real part is explicitly given by:

$$L(f)(z) = \int_0^\infty e^{-zt}f(t)\,dt.$$  

This operator is an injection. Its inverse, the inverse Laplace transform $L^{-1}$, is expressed as a contour integral by the complex inversion formula of Riemann. Applying to any function $H$ analytic on half-planes $\{\Re(z) \geq z_0\}$ with $z_0$ any sufficiently big positive real number, it asserts:

$$L^{-1}(H)(t) = \frac{1}{2\pi i} \int_{z_0 - i\infty}^{z_0 + i\infty} e^{zt} \cdot H(z)\,dz,$$

for any positive real number $t$ if $H$ satisfies a growth condition at infinity such that the above integral exists. For any function $f$ of exponential type one so has in particular $(L^{-1} \circ L)(f) = f$.

For getting an idea of the proof of the complex inversion formula start with any function $f$ holomorphic on a half-plane $\{z|\Re(z) > a\}$. For obtaining a suitable integral representation of $f$, invoke the Cauchy integral formula in the following way. Fix $z$ with $\Re(z) > a$ and choose a line $C(z_0) = \{z|\Re(z) = z_0\}$ such that $z$ is to the right of it, i.e., $z_0 < \Re(z)$, and such that it is contained in the half-plane where $f$ is holomorphic, i.e. $a < z_0$. Make this line into a path of integration by moving upwards from $z_0 - i\infty$ to $z_0 + i\infty$. Shift perspective to the Riemann sphere by adding the point infinity to the complex plane. In this picture, $C(z_0)$ is a closed path that circles clockwise, i.e., in the mathematically negative direction, around $z$. Formally apply the Cauchy integral formula, expressing $2\pi i \cdot f(z)$ as minus the integral of $(w-z)^{-1}f(w)$ over $C(z_0)$. Since $-(w-z)^{-1}$ is the Laplace transform at $z$ of the map on the positive real line given by $\exp(wh)$ for any $h > 0$, it so follows:

$$f(z) = \frac{1}{2\pi i} \int_{C(z_0)} \int_0^\infty e^{(w-z)h}f(w)\,dh\,dw.$$  

On the Riemann sphere the contour $C(z_0)$ can be seen as the limit of closed contours contained in the complex plane. The above integral thus makes sense as such a limit if $f$ is supposed rapidly decreasing, i.e., the absolute value of $f$ exponentially decreases to zero with the absolute value $|w|$ of $w$ going to infinity. Then apply Fubini’s theorem to interchange the integrals in the above formula. The inverse Laplace transform of $f$ is then identified as the function on the positive real line given by

$$\frac{1}{2\pi i} \int_{C(z_0)} e^{wh}f(w)\,dw,$$

for any $h > 0$, which is the complex inversion formula.
Hermite functions: The Hermite function $H_{\nu}$ of degree any complex number $\nu$ is the complex–valued function on the complex plane given as solution to the differential equation $u'' - 2zu' + 2\nu u = 0$ in the complex variable $z$. It is holomorphic on the complex plane as a function of both its variable $z$ and its degree $\nu$. Using the differential equation, for $\nu$ any non–negative integer, $H_{\nu}$ is the $\nu$–th Hermite polynomial and belongs to a standard class of orthogonal functions. If the real part of the degree $\nu$ is negative the Hermite function $H_{\nu}$ has the following integral representation:

$$H_{\nu}(z) = \frac{1}{\Gamma(-\nu)} \int_0^\infty e^{-u^2 - 2zu} u^{-(\nu+1)} \, du,$$

for any complex number $z$ and with $\Gamma(-\nu)$ the value of the gamma function at $-\nu$. Hermite functions thus also generalize the complementary error function Erfc. Recalling that Erfc is for any complex number $z$ given by any of the following expressions:

$$\text{Erfc}(z) = N(-z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-u^2} \, du = e^{-z^2} \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2 - 2zu} \, du,$$

notice $(2/\sqrt{\pi}) \cdot H_{-1}(z) = \exp(z^2)\text{Erfc}(z)$. More generally, one has for any non–negative integer $n$ the identity:

$$H_{-(n+1)}(z) = \sqrt{\frac{\pi}{2}} \cdot \frac{(-1)^n}{n! \cdot 2^n} \cdot \frac{d^n}{dz^n} \left( e^{z^2} \text{Erfc}(z) \right)$$

for any complex number $z$. Hermite functions of degree any complex number $\nu$ satisfy the recurrence relations:

$$H'_{\nu}(z) = 2\nu \cdot H_{\nu-1}(z),$$
$$H_{\nu+1}(z) - 2z \cdot H_{\nu}(z) + 2\nu \cdot H_{\nu-1}(z) = 0.$$

Using the differential equation of the Hermite functions the second of these is implied by the first which can be seen using the following absolutely convergent series expansion:

$$H_{\nu}(z) = \frac{1}{2 \cdot \Gamma(-\nu)} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{\Gamma\left(\frac{n - \nu}{2}\right)}{\Gamma\left(-\nu\right)} \cdot (2z)^n,$$

valid for any complex number $z$ and uniformly convergent on compact sets. Like the exponential series this series has excellent convergence properties for complex numbers $z$ of smaller absolute value. For arguments $z$ of larger absolute value one uses asymptotic expansions. If the real part of $\nu$ is negative the Hermite function of degree $\nu$ has the following asymptotic expansion in the right half–plane:

$$H_{\nu}(z) = (2z)^\nu \sum_{k=0}^n (-1)^k \frac{(-\nu)_{2k}}{k!} \cdot \frac{1}{(2z)^{2k}} + r_{n+1}(z),$$

for any complex number $z$ with positive real part and any non–negative integer $n$. Herein $(-\nu)_{2k}$ is the Pochhammer symbol recalled to be given by $(-\nu)_0 = 1$ and $(-\nu)_{2k} = \Gamma(-\nu + 2k)/\Gamma(-\nu) = (-\nu)(-\nu + 1) \cdots (-\nu + 2k - 1)$ for $k$ positive. For estimating the error term $r_{n+1}$, fix any positive real number $\delta$ less than or equal to $\pi/2$ and choose any complex number $z$ in the wedge contained in the right half–plane that is enclosed between
the two rays emanating from the origin with angles $\pm (\pi/2 - \delta)$ respectively. Equivalently, the absolute value of the argument of $z$ is at most $\pi/2 - \delta$. Then the absolute value of the error term $r_{n+1}(z)$ satisfies the following estimate:

$$|r_{n+1}(z)| \leq \frac{1}{(\sin(\delta))^{2(n+1)-\Re(\nu)}} \cdot \frac{\Gamma(2(n+1) - \Re(\nu))}{(n+1)! |\Gamma(-\nu)|} \cdot \frac{1}{(2|z|)^{2(n+1)-\Re(\nu)}}.$$ 

The differential equation for the Hermite function of degree any complex number $\nu$ can be transformed into the differential equation $u'' + (\nu + 1/2 - z^2/4)u = 0$ whose solution $D_\nu$ is thus connected with the Hermite function $H_\nu$ by $D_\nu(z) = 2^{-\nu/2} \exp(-z^2/4) H_\nu(2^{-1/2}z)$, for any complex number $z$. Similarly, Hermite functions are related to the second Kummer confluent hypergeometric function $\Psi$ by $H_\nu(z) = 2 \cdot \Psi(-\nu/2, 1/2, z^2)$, for any complex number $z$ in the right half-plane.

**Part I Statement of results**

1. **Black–Scholes framework for valuating contingent claims:** For our analysis we place ourselves in the Black–Scholes framework and use the risk–neutral approach to valuating contingent claims as described in [DSM, Chapters 17, 22].

In this set–up one has two securities. First there is a riskless security, a bond, that has the continuously compounding positive interest rate $r$. Then there is a risky security whose price process $S$ is modelled as follows. Start with a complete probability space equipped with the standard filtration of a standard Brownian motion on it that has the time set $[0, \infty)$. On this filtered space one has the risk neutral measure $Q$, a probability measure equivalent to the given one, and a standard $Q$–Brownian motion $B$ such that $S$ is the strong solution of the following stochastic differential equation:

$$dS_t = \omega \cdot S_t \, dt + \sigma \cdot S_t \, dB_t, \quad t \in [0, \infty).$$

Herein the positive constant $\sigma$ is the volatility of $S$, whereas the specific form of the constant $\omega$ depends on the security modelled. For instance it is the interest rate rate if $S$ is a non–dividend paying stock.

With the existence of $Q$ one has the arbitrage–pricing principle: let $Y$ be any European–style contingent claim on the above filtered probability space written at time $t_0$ and paying $Y_T$ at time $T$. At any time $t$ between $t_0$ and $T$ its time–$t$ price $C_t$ is then:

$$C_t = e^{-r(T-t)} E_t^Q [Y_T]$$

with the expectation $E_t^Q$ conditional on the information available at time $t$ and taken with respect to $Q$.

2. **The notion of arithmetic–average Asian options:** Fix any time $t_0$ and consider in the Black–Scholes framework reviewed above the process $J$ given for any time $t$ by:

$$J(t) = \int_{t_0}^t S_\tau \, d\tau.$$
The European style arithmetic–average Asian option written at time $t_0$, with maturity the time $T$, and fixed strike price $K$ is then the contingent claim on $[t_0, T]$ paying 

$$
\left( \frac{J(T)}{T-t_0} - K \right)^+ := \max \left\{ 0, \frac{J(T)}{T-t_0} - K \right\}
$$

at time $T$. Recall that points in time are taken to be non–negative real numbers.

3. The valuation problem for arithmetic–average Asian options: Using the arbitrage–pricing principle, the price $C_{t,T}(K)$ at any time $t$ between $t_0$ and $T$ of the arithmetic–average Asian option introduced in paragraph two is given by:

$$
C_{t,T}(K) = e^{-r(T-t)} E_t^Q \left[ \left( \frac{J(T)}{T-t_0} - K \right)^+ \right].
$$

From [GY, §3.2] one has the following normalization of this value process:

**Lemma:** For any time $t$ between $t_0$ and $T$, one has:

$$
C_{t,T}(K) = e^{-r(T-t)} \frac{4S_t}{\sigma^2} \cdot C^{(\nu)}(h, q),
$$

where

$$
C^{(\nu)}(h, q) = E^Q \left[ \max \left\{ 0, \int_0^h e^{2(B_\tau + \nu \tau)} d\tau - q \right\} \right].
$$

To explain the notation, the normalized parameters herein are as follows:

$$
\nu = \frac{2\varpi}{\sigma^2} - 1,
$$

$$
h = \frac{\sigma^2}{4} (T - t),
$$

$$
q = \frac{\sigma^2}{4S_t} \left( K \cdot (T-t_0) - \int_{t_0}^t S_\tau \, d\tau \right).
$$

To interpret these quantities, $C^{(\nu)}(h, q)$ is the normalized price for the Asian option. It depends on the normalized adjusted interest rate $\nu$, which for $\varpi$ positive is bigger than minus one, on the normalized time to maturity $h$, which is non–negative, and on the normalized strike price $q$.

To prove the Lemma break up the integral $J(T)$ at the point $t$ and bring $S_t$ in front of the expectation. Modify that part of the integral unknown given the information at time $t$ as follows. Change variables to obtain an integral from time 0 to time $T-t$. The strong Markov property of Brownian motion identifies the exponent $B(t+\tau) - B(t)$ of the new integrand as a standard Brownian motion that is independent of the information at time $t$. Herein change time by dividing through $\sigma^2/4$. Using the scaling property of Brownian motions, this time–changed process is again a standard Brownian motion if multiplied by $\sigma/2$ which completes the proof.
4. **Statement of results:** In the setting of paragraphs one to three consider any time \( t \) between the times \( t_0 \) and \( T \). Using the decomposition

\[
C_{t,T}(K) = \frac{e^{-r(T-t)}}{T-t_0} \cdot \frac{4S_t}{\sigma^2} \cdot C^{(\nu)}(h, q)
\]

of §3 Lemma, the valuation of the arithmetic–average Asian option is reduced to computing its normalized price \( C^{(\nu)}(h, q) \). This price depends on the sign of the normalized strike price \( q \). This parameter is recalled to be given by:

\[
q = \frac{\sigma^2}{4} \cdot \frac{1}{S_{T-(4/\sigma^2) \cdot h}} \left( K \cdot (T-t_0) - \int_{t_0}^{T-(4/\sigma^2) \cdot h} S_u \, du \right).
\]

as a function of time \( t \) which enters via the normalized time to maturity \( h = (\sigma^2/4) \cdot (T-t) \). Generically \( q \) will be positive. It is non–positive if and only if at time \( t \) it is already known that the option will be in the money at its maturity \( T \).

The main result: The main result of this paper is a closed form solution for \( C^{(\nu)}(h, q) \) if \( q \) is positive. It expresses this function as the sum of integral representations. These are obtained by integrating the product of Hermite functions, discussed in paragraph zero, with functions derived from weighted complementary error functions. Proved later in paragraphs nine to twelve, the precise result is as follows:

**Theorem:** If \( q \) is positive, the normalized price \( C^{(\nu)}(h, q) \) of the Asian option is given by the following five–term sum:

\[
C_{\text{trig}} + \frac{\Gamma(\nu+4)}{2\pi(\nu+1)} \cdot \left( \frac{2q}{e^{2q}} \right)^{\frac{\nu+2}{2}} \cdot \left( C_{\text{hyp},\nu+2} + C_{\text{hyp},-(\nu+2)} - C_{\text{hyp},\nu} - C_{\text{hyp},-\nu} \right).
\]

To explain the functions, recall from paragraph zero the Hermite functions \( H_\mu \) of degree any complex number \( \mu \). The trigonometric term \( C_{\text{trig}} \) is then:

\[
C_{\text{trig}} = c \cdot \int_0^\pi H_{-(\nu+4)} \left( -\frac{\cos(\theta)}{\sqrt{2q}} \right) \cos(\nu \cdot \theta) \, d\theta,
\]

recalling \( \nu = 2\pi/\sigma^2 - 1 \) and abbreviating:

\[
c = c(\nu, q) = \Gamma(\nu+4) \cdot \frac{e^{2h(\nu+1)} - 1}{\nu+1} \cdot \frac{(2q)^{\nu+2}}{2\pi \cdot e^{2q}}.
\]

For \( \nu \) bigger than minus one \( C_{\text{trig}} \) has the additional representation:

\[
C_{\text{trig}} = \frac{e^{2h(\nu+1)} - 1}{2(\nu+1)} \left( 1 + 2q(\nu+1) \right) + \sin(\nu\pi) \cdot c \cdot \int_0^\infty H_{-(\nu+4)} \left( \frac{\cosh(x)}{\sqrt{2q}} \right) e^{-\nu x} \, dx.
\]
The hyperbolic terms $C_{\text{hyp},b}$ with $b$ equal to $\pm \nu$ or $\pm (\nu+2)$ are:

$$C_{\text{hyp},b} = \frac{2}{\sqrt{\pi}} \cdot e^{\frac{\pi^2 h^2 + b^2 h}{2}} \int_0^\infty H_{-(\nu+4)} \left( \frac{\cosh(y)}{\sqrt{2q}} \right) \cdot e^{by} \cdot E_b(h)(y) \, dy.$$ 

Herein $E_b(h)$ is in terms of real–valued functions for any real number $y$ given by:

$$E_b(h)(y) = \int_{y + \frac{b}{\sqrt{2h}}}^\infty e^{-u^2} \sin \left( \pi \left( b - u \frac{\sqrt{2h}}{h} \right) \right) \, du.$$ 

It is identified in paragraph ten as imaginary part of a certain weighted complementary error function at complex arguments. Actually, this last interpretation seems to be best suited for a numerical analysis.

**The case $q \leq 0$:** It is a characteristic feature of Asian options that their prices become rather rigid some time before maturity. The present case $q \leq 0$ where at time $t$ it is known that the option will be in the money at its maturity makes this precise. The rigidity of the price is reflected in the formula of [GY, §3.4]. For any $h > 0$ one has:

$$C^{(\nu)}(h, q) = e^{2(\nu+1)h} - 1 - q.$$ 

The proof reduces to computing the $Q$–expectation of $\int_0^h \exp(2(B_\tau + \nu \tau)) \, d\tau$. For an argument independent of [GY, §3.4], applying Fubini’s theorem this expectation is the integral from zero to $h$ with respect to the variable $\tau$ of the product $\exp(2\nu \tau)$ times the expectation of $\exp(2B_\tau)$. This last expectation is the integral over the real line with respect to the variable $x$ of the product $\exp(2x)$ times the density $(2\pi)^{-1/2} \exp(-x^2)$. Changing variables $y = (2\tau)^{-1/2}(x - 2\tau)$ it is seen to be equal to $\exp(2\tau)$. On substitution of this result one so is reduced to calculate the integral from zero to $h$ with respect to $\tau$ of $\exp(2(\nu+1)\tau)$. Distinguishing the cases where $\nu + 1$ is zero and non–zero, this is seen to have the above value, completing the proof.

**Remark:** The author is very grateful to a number of people, including a referee of this paper, for having drawn his attention to Yor’s formula [Y, (6.e), p.528]. His integral representation is a triple integral with a trigonometric–hyperbolic type integrand:

$$C^{(\nu)}(h, q) = e^{\frac{\pi^2 h^2 - \nu^2 h}{2}} \sqrt{\frac{\pi}{2h}} \int_0^\infty x^{\nu} \int_0^\infty e^{-\frac{(1+x^2)y}{2}} \cdot \left( \frac{1}{y} - q \right) \psi_{xy}(h) \, dy \, dx,$$

where for any positive real number $a$, the function $\psi_a$ is given for any $h > 0$ by:

$$\psi_a(h) = \int_0^\infty e^{-\frac{y^2}{2h}} e^{-a \cosh(w)} \sinh(w) \cdot \sin \left( \frac{\pi}{h} w \right) \, dw.$$ 

In contrast, our formula separates into a trigonometric and a hyperbolic part. It is given as a sum of single integrals whose integrands have a structural interpretation as products of two functions. It moreover identifies the higher transcendental functions occuring as factors in these products, and shows how they are given by or built up from Hermite functions. On
a technical level, these differences can be regarded as consequences of the different mathematical approaches for proving the valuation formula. Which of the above representations is better suited for which purpose in which situation remains to be analyzed.

Part II  Proof of the valuation formula

5. Computing the price of the Asian option using Laplace transforms: The basic idea for the Laplace transform approach to valuating Asian options is not to consider a single above Asian option. Instead, take the exercise time $T$ of the option as variable and consider today at time $t$ all above Asian options whose exercise times $T$ range from $t$ to points in time very far in the future. Then, one wants to average the values of these options. This, however, is not possible in a naive way. One has to suitably weight these values and only then compute their average. This is effected by the Laplace transform.

In the generic case with $q$ positive a candidate for the Laplace transform of the factor $C^{(\nu)}(\, q)$ of the price for the Asian option was computed in [GY, pp.361ff]. Unfortunately, I found a conceptional error in their computations. Luckily however, Peter Carr eventually succeeded in making me see that this error is not as fatal as I originally believed. Indeed, in his interpretation the results of [GY] are sufficient for valuating Asian options as follows.

**Proposition:** Suppose $\nu = 2\omega/\sigma^2 - 1 > -1$. If the positive real number $h$ is such that $q(h) = kh + q^*$ is positive, the normalized price of the Asian option at $h$ is given by:

$$C^{(\nu)}(h, q(h)) = L^{-1}\left(F_{GY}(q(h), z)\right)(h)$$

as the Laplace inverse of $F_{GY}(q(h), z)$ at $h$.

Leaving the function $q$ unexplained for a moment, for any positive real number $a$ define:

$$F_{GY}(a, z) = \frac{D_\nu(a, z)}{z \cdot (z - 2(\nu + 1))},$$

for any complex number $z$ with positive real part bigger than $2(\nu+1)$, where on choosing the principal branch of the logarithm:

$$D_\nu(a, z) = \frac{e^{-\frac{1}{2a}}}{a} \int_0^\infty e^{-\frac{x^2}{2a}} \cdot x^{\nu+3} \cdot I_{2\nu+1}(\frac{x}{a}) \, dx.$$

Here $I_\mu$ is, for any complex number $\mu$, the modified Bessel function of order $\mu$ more fully discussed in paragraph six below or in [L, Chapter 5], for instance.

What regards the function $q$, the first general observation is that $T$ does not enter directly in the analysis but indirectly via the normalized time to maturity $h = (\sigma^2/4) \cdot (T-t)$ of paragraph three. Letting $T$ vary from $t$ to infinity, $h$ thus varies from zero to infinity. Take
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$h$ as new variable. The crucial difficulty now is that $q$ of paragraph three also depends on $h$ as a function of $T$. Indeed, abbreviating $k = K/S_t$, on chasing definitions one gets:

$$q(h) = k \cdot h + q^* \quad \text{where} \quad q^* = \frac{\sigma^2}{4S_t} \left( K \cdot (t-t_0) - \int_{t_0}^{t} S \, d\tau \right).$$

Herein $q^*$ is independent of $T$ and thus of $h$. The upshot is that $q$ is affine linear in $h$ and non–constant in particular. This is a nasty situation since the computations of the normalized value of the Asian option in [GY, pp.361ff] erroneously take $q$ as a constant.

Indeed, they compute for any positive real number $a$ and any complex number $z$ with sufficiently big positive real part the Laplace transform:

$$F_{GY}(a, z) := \int_{0}^{\infty} e^{-zx} \cdot E\left[ A^{(\nu)} - a \right] \, dx,$$

setting:

$$A^{(\nu)}(x) = \int_{0}^{x} e^{2(\nu \cdot u + B_u)} \, du,$$

for any non–negative real number $x$, to adopt Yor’s notation.

The proof of the Proposition then proceeds in two steps. Reduce to the above Geman–Yor Laplace transforms. Then sketch the essential steps of their computation.

The basic idea of Peter Carr for reducing the Proposition to the Laplace transforms of [GY] was to fix any positive $h$ and introduce an auxiliary function that coincides with the normalized value of the Asian option at $h$. More precisely, recalling §3 Lemma one has to consider the function $\psi$ given for any positive real number $x$ by:

$$\psi(x) = C^{(\nu)}(x, kx + q^*) = E\left[ A^{(\nu)} - kx - q^* \right].$$

Deform this function into the function $\phi$ given for any positive real number $x$ by:

$$\phi(x) = E\left[ A^{(\nu)} - kh - q^* \right].$$

Then $\psi$ and $\phi$ are identical at $h$ by construction. Granted it exists, the Laplace transform of $\phi$ is equal to $F_{GY}(q(h), z)$ recalling $q(h) = kh + q^*$. Its Laplace inverse is the function $\phi$ by definition, and so equals $\psi$ when evaluated at $h$. One so has tautologically:

$$C^{(\nu)}(h, q(h)) = \psi(h) = \phi(h) = L^{-1}(F_{GY}(q(h), z))(h),$$

and is reduced to have the above Geman–Yor Laplace transforms $F_{GY}(a, z)$.

The basic idea for computing these Laplace transforms in [GY, pp.361ff] is to introduce a suitable stochastic clock for the process $J$ of paragraph two. This is made possible by the Bessel factorization result, attributed to Williams in [Y, §1.5] and to Lamperti in [RY, XI 1.28, p.432], that the exponential of any Brownian motion with drift is a time–changed Bessel process in the following way:

$$e^{B_t + \nu t} = R^{(\nu)} \left( \int_{0}^{t} e^{2(B_\tau + \nu \tau)} \, d\tau \right) = R^{(\nu)}(A^{(\nu)}_t).$$
Here $R^{(\nu)}$ is the Bessel process on $[0, \infty)$ with index $\nu$, starting at 0 with the value 1. At the level of a definition, the square of this process is a continuous diffusion process $\rho$ with values in the non-negative real numbers satisfying:

$$d\rho_t = 2(\nu + 1) \, dt + 2\sqrt{\rho_t} \, dB_t, \quad \rho_0 = 1.$$ 

To be able to compare times, let $\tau_{\nu,u}$ for any non-negative real number $u$ denote the least upper bound of all non-negative real numbers $s$ such that $\int_0^s \exp (2(B_t + \nu t)) \, d\tau = u$. Notice that this equals $\int_0^u (R^{(\nu)}(s))^{-2} \, ds$.

The computation of $F_{GY}(a, z)$ is then by deftly using the double role of $A^{(\nu)}$ as both the stochastic clock in its Bessel factorization and the actual underlying of the option. Indeed, fix any positive real number $x$, and consider the process $A^{(\nu)}$ at $x$ on the set of all events where the passage time $\tau_{\nu,a}$ takes values less than or equal to $x$. Break the integral defining $A^{(\nu)}(x)$ at $\tau_{\nu,a}$. The first summand then is $A^{(\nu)}$ at time $\tau_{\nu,a}$ and so is equal to $x$. In the second summand one wants to restart the Brownian motion in the exponent of the integrand at $\tau_{\nu,a}$. Thus shift the variable of integration accordingly. The second integral then is the product of $\exp (2 \cdot (B(\tau_{\nu,a}) + \nu \cdot \tau_{\nu,a}))$ times $A^{(\nu)}$ at $x-\tau_{\nu,a}$, by abuse of language after having applied Strong Markov. This last process is such that it is independent of the information at time $\tau_{\nu,a}$. Recalling the role of $\tau_{\nu,a}$, the first factor is the square of the Bessel process $R^{(\nu)}$ at time $a$. Now taking the expectation conditional on the information at $\tau_{\nu,a}$, one thus gets:

$$E^Q \left[ (A^{(\nu)}(x) - a)^+ \mid F_{\tau_{\nu,a}} \right] = (R^{(\nu)}(a))^2 \cdot E^Q \left[ A^{(\nu)} \left( \left[ x - \tau_{\nu,a} \right]^+ \right) \right].$$

The $Q$-expectation of $A^{(\nu)}(w) = \int_0^w \exp (2(B_u + \nu u)) \, du$ is $\exp (2(\nu+1)w)/((2(\nu+1)))$, as shown in §4 or using the more general results of [Y92, pp.69ff]. It thus follows:

$$E^Q \left[ (A^{(\nu)}_x - a)^+ \right] = E^Q \left[ (R^{(\nu)}(a))^2 \cdot \frac{e^{2(\nu+1)[x-\tau_{\nu,a}]^+}}{2(\nu+1)} - 1 \right].$$

To avoid a direct calculation of this expectation adopt the common strategy to first calculate its Laplace transform at arguments $z$ whose real parts are positive and sufficiently big. The hope is to thus arrive at a simpler situation from which one is able to identify the original function itself. There is a technical point in that one wants to interchange the expectation with the Laplace transform integral. If $z$ is real it is at this stage possibly best to follow Yor’s proposal for justifying this. Indeed with the integrand of the double integral in question positive and measurable, apply Tonelli’s theorem now but justify only in a later step that any of the resulting integrals is finite. The case of a general argument $z$ is reduced to this case considering the absolute value of the integrand, and the result is:

$$F_{GY}(a, z) = \frac{E^Q \left[ e^{-z \cdot \tau_{\nu,a}} \cdot (R^{(\nu)}(a))^2 \right]}{z \cdot (z - 2(\nu + 1))}.$$ 

One has enough information about Bessel processes for computing the $Q$–expectation of the right hand side. Using [Y80, Théorème 4.7, p.80], the expectation of $\exp (-z \cdot \tau_{\nu,a})$
conditional upon $R^{(\nu)}(a)$ being the positive real number $w$ is given by the following quotient of modified Bessel functions:

$$E^Q \left[ e^{-z_{\nu,a}} \middle| R^{(\nu)}(a) = w \right] = \frac{I_{\sqrt{2z+\nu^2}} \left( \frac{w}{a} \right)}{I_{\nu}}.$$  

This result ultimately depends on the following standard fact about Bessel processes [Y80, (4.3), p.78]. If $\nu$ is bigger than minus one, the density $p_{\nu,a}(1,w)$ of the Bessel semigroup with index $\nu$ and starting point 1 at time $a$ is given by:

$$p_{\nu,a}(1,w) = \frac{w^{\nu+1}}{a} \cdot e^{-\frac{1+w^2}{2a}} \cdot I_{\nu} \left( \frac{w}{a} \right).$$

It is for this result that the hypothesis $\nu$ bigger than minus one of the Proposition is crucially required. The numerator of the above Laplace transform now is obtained by integrating the product of these last two expressions and $w$ square with respect to $w$ over the positive real line. With the resulting integral finite for any complex number $z$ with real part bigger than $2(\nu+1)$, this completes the proof of the Proposition.

6. Preliminaries on integral representations of gamma and Bessel functions:

Recall the occurrence of modified Bessel functions in §5 Proposition. They are given for any complex number $\mu$ by the following series:

$$I_{\mu}(z) = \sum_{m=0}^{\infty} \frac{1}{m! \Gamma(\mu + m + 1)} \cdot \left( \frac{z}{2} \right)^{\mu+2m},$$

for any complex number $z$ in $\mathbb{C}\setminus\mathbb{R}_{<0}$. In this section preliminary results on integral representations of these functions are reviewed.

Hankel contours and the Hankel representation of the gamma function: Recall that the gamma function at any complex number $s$ with $\text{Re}(s) > 0$ is given by:

$$\Gamma(s) = \int_0^{\infty} e^{-x} x^{s-1} \, dx.$$  

In this section, following [D, pp.225f], the Hankel form of the gamma function is described, an integral representation of the reciprocal of $\Gamma(s)$ valid for any complex number $s$.

First notice that the Laplace transform at any $z$ with $\text{Re}(z) > 0$ of the function $x^{s-1}$ on the positive real line is $z^{-s} \cdot \Gamma(s)$. The following classical result of Laplace now gives the inverse of this Laplace transform. For any complex number $s$ with $\text{Re}(s) > 0$ one has:

$$\frac{1}{2\pi i} \int_{\xi_0 - i\infty}^{\xi_0 + i\infty} e^{z\xi} \cdot \xi^{-s} \, d\xi = \mathbf{L}^{-1} \left( z^{-s} \right) (x) = \frac{x^{s-1}}{\Gamma(s)},$$

for any $x > 0$ and any fixed $\xi_0 > 0$. Herein deform the path of integration as follows. Fix any positive numbers $P$ bigger than $R$ and with $R$ bigger or equal $a$. Starting at $\xi_0 - iP$ move parallel to the imaginary axis until $\xi_0 + iP$. From that point move parallel to the real axis to the point $iP$ on the imaginary axis. Now follow counterclockwise the circle with radius $P$ around the origin. When the parallel to the axis through $ia$ is hit follow this parallel until the circle with radius $R$ around the origin is hit. Continue clockwise on this circle until $R$ on the positive real axis. Thereafter move to $\xi_0 - iP$ in such a fashion that the path traced out in the complex plane is symmetric with respect to the real axis.
The integral of \( \exp(x \xi) \xi^{-s} \) over the closed path of integration just constructed is zero using the Cauchy Theorem. The integrand is rapidly decreasing in the radius \( |\xi| \) of \( \xi \). Letting \( P \) go to infinity the above inverse Laplace transform so equals the integral over the Hankel contour \( C_{a,R} \): the boundary of the pan in the complex plane of radius \( R \) around zero with handle of diameter \( 2a \) stretching to minus infinity along the negative real line, on which one comes in from \(-\infty\) on the branch in \( \{ z | \Re(z), \Im(z) < 0 \} \), passes counterclockwise around zero, and leaves on the branch in \( \{ z | \Re(z) < 0, \Im(z) > 0 \} \). The Hankel contour \( C_R \) is defined as the limit of \( a \) going to zero of the Hankel contours \( C_{a,R} \).

Returning to the gamma function, the main point now is that on any Hankel contour \( C \) the absolute value of \( \exp(x \xi) \) is exponentially decreasing for any positive \( x \) with the absolute value of \( \xi \) going to infinity. Hence one obtains the following Hankel formula:

\[
\frac{x^{s-1}}{\Gamma(s)} = \frac{1}{2\pi i} \int_C e^{x \cdot \xi} \cdot \xi^{-s} \, d\xi,
\]

valid now for any complex number \( s \), and any positive real number \( x \).

**A Hankel type integral representation of Bessel functions:** In this section, following [WW, 17.231, p.362], the following Hankel–type integral representations on the right half–plane of modified Bessel functions are recalled.

**Lemma:** For any modified Bessel function \( I_\mu \) one has:

\[
I_\mu(z) = \frac{1}{2\pi i} \int_{\log C} e^{-\mu \cdot \xi + z \cdot \cosh(\xi)} \, d\xi,
\]

for any complex number \( z \) with positive real part and any Hankel contour \( C \).

The key step in the proof is the following integral representation:

\[
I_\mu(z) = \frac{(z/2)^\mu}{2\pi i} \int_C \xi^{-(\mu+1)} \exp \left( \xi + \frac{z^2}{4\xi} \right) \, d\xi,
\]

valid for any complex number \( z \) in \( \mathbb{C} \setminus \mathbb{R}_{<0} \). Indeed, substitute for the reciprocal gamma values in the series of \( I_\mu(z) \) the respective Hankel formulas with \( x = 1 \). Interchange the order of summation and integration. The resulting factor \( \sum_m (z^2/4\xi)^m / m! \) is the exponential function at \( z^2/4\xi \), as was to be shown.
Change variables $\xi = z\eta/2$ in the above integral representation of $I_\mu$ to get:

$$I_\mu(z) = \frac{1}{2\pi i} \int_C \eta^{-(\mu+1)} e^{\frac{z}{2}(\eta + \frac{1}{\eta})} d\eta.$$  

Herein change variables $\eta = \exp (\xi)$ to complete the proof of the Lemma.

**The contour $\log C_R$:** In the sequel, it is the Hankel contours $C_R$ that are used. For a description of the logarithmicalized contour $\log C_R$, recall the principal branch of the logarithm on the complex plane with the non–positive real axis deleted:

$$\log (z) = \log |z| + i \cdot \arg(z) \quad \text{where} \quad -\pi < \arg(z) < \pi.$$  

Returning to the Hankel contour $C_R$, the argument of the complex numbers in the upper branch of the panhandle in $C_R$ is $+\pi$, of those in the lower branch it is $-\pi$. The elements on the circle part of $C_R$ have the form $R \cdot \exp (i\theta)$ with $-\pi < \theta < \pi$. The contour $\log C_R$ thus has the following shape: Coming in from plus infinity, move on the parallel through the point $-i\pi$ to the real axis to the point $\log R - i\pi$. From this point move up to the point $\log R + i\pi$ on a parallel to the imaginary axis. Finally exit from $\log R + i\pi$ to plus infinity on the parallel through $i\pi$ to the real axis. Notice $\log C_R \subseteq \{z | \Re(z) > 0\}$ if and only if the radius of the circle in $C_R$ is bigger than 1.

**Odds and ends:** To conclude, explicit consequences of the above development are indicated. First of all using the explicit coordinates discussed above in the logarithmicalized Hankel contour with radius $R$ equal to one, the integral representation for $I_\mu$ of the Lemma specializes to the following *Schläfli integral representation*:

$$I_\mu(z) = \frac{1}{\pi} \int_0^\pi e^{z \cos \theta} \cos(\mu \theta) \, d\theta - \frac{\sin(\mu \pi)}{\pi} \int_0^\infty e^{-z \cosh(x) - \mu x} \, dx,$$

for any complex number $z$ with positive real part [Wa, 6 · 22]. Direct calculations using the series of the respective modified Bessel functions or their Hankel type integral representations of the Lemma, prove the *recursion rule*:

$$z \cdot I_\mu(z) = 2(\mu + 1) \cdot I_{\mu+1}(z) + z \cdot I_{\mu+2}(z),$$

valid for any complex numbers $z$ with $|\arg(z)| < \pi$ and $\mu$ [Wa, 3 · 71].
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The picture is finally completed with Weber’s integral:

\[
\int_0^\infty e^{-ax^2} x^{\mu+1} I_{\mu}(x) \, dx = \frac{1}{(2a)^{\mu+1}} \cdot e^{\frac{1}{4a}},
\]

valid for any positive number \(a\) and any complex number \(\mu\) with real part bigger than minus one. For its proof following [Wa, 13 · 3] develop the Bessel function factor of the integrand into its series. Interchange the order of integration and summation. Changing variables \(y = ax^2\), any \(n\)–th term of the resulting series is \((2a)^{-(n+1)}\) times the quotient of \((4a)^{-n}\) over \(n\) factorial times the quotient of \(\int_0^\infty \exp(-y) y^{\mu+n} \, dy\) over \(\Gamma(\mu+n+1)\). With the real part of \(\mu\) bigger than minus one, the numerator integral of the third quotient is the gamma function at \(\mu+n+1\) and thus cancels with the denominator. Thus the series is \((2a)^{-(n+1)}\) times the series of the exponential function at the reciprocal of \(4a\). Applying Lebesgue Dominated Convergence then completes the calculation of Weber’s integral.

7. First steps of the Laplace inversion: From paragraph five the Geman–Yor functions \(F_{GY}\) are for any positive real number \(a\) recalled to be given by:

\[
F_{GY}(a, z) = \frac{D_{\nu}(a, z)}{z \cdot (z-2(\nu+1))},
\]

where

\[
D_{\nu}(a, z) = \frac{e^{-\frac{1}{2a}}}{a} \int_0^\infty e^{-\frac{x^2}{2a}} \cdot z^{\nu+3} \cdot I_{\sqrt{2z+\nu^2}} \left( \frac{x}{a} \right) \, dx,
\]

for any complex number \(z\) with real part bigger than \(2(\nu+1)\). In the sequel the problem of inverting this Laplace transform is reduced to the following:

**Lemma:** For \(\nu\) bigger than minus one, the inverse Laplace transform at any positive real number \(h\) of any function \(F_{GY}(a, z)\) is given by:

\[
c_1 \cdot \int_0^\infty \frac{1}{2\pi i} \int_{\log C_R} e^{-x^2+x\sqrt{\frac{\nu}{2a}} \cosh(w)} x^{\nu+3} \left\{ \mathbf{L}^{-1} \left( \frac{e^{-w\sqrt{z}}}{z-(\nu+2)^2} \right) - \mathbf{L}^{-1} \left( \frac{e^{-w\sqrt{z}}}{z-2\nu^2} \right) \right\} \left( \frac{h}{2} \right) \, dw \, dx,
\]

where

\[
c_1 = e^{-h \frac{\nu^2}{2}} \cdot \frac{(2a)^{\nu+2}}{(\nu+1) \cdot e^{\frac{1}{2a}}},
\]

where \(C_R\) is any Hankel contour with \(R \geq R_0\), and with integrands absolutely integrable.

The proof of the Lemma is based on the complex inversion formula for the Laplace transform reviewed in paragraph zero.

To prove the Lemma, let \(z_0'\) be any positive real number such that the line \(\{z | \text{Re}(z) = z_0'\}\) is contained in the half–plane where \(F_{GY}(a, z)\) is a holomorphic function. Suppose for a moment proved that this function is rapidly decreasing, and apply the complex inversion
formula to it. Writing out $D_\nu(a,z)$, the inverse Laplace transform of $F_{GY}(a,z)$ is the function on the positive real line given for any positive real number $h$ by:

$$e^{-\frac{2\pi}{a}} \cdot \frac{1}{2\pi i} \int_{z_0^+ - i\infty}^{z_0^+ + i\infty} e^{hz} \int_0^{\infty} e^{-\frac{x^2}{4}} \cdot x^{\nu+3} \cdot \frac{I_{\sqrt{2z+\nu^2}}(x)}{z(z-2(\nu+1))} \, dx \, dz.$$ 

For an equivalent expression, change variables $\eta = 2z+\nu^2$, put $z_0 = 2z_0'+\nu^2$, and substitute the Hankel–type integral representation of §6 Lemma for the modified Bessel function. With $c_1' = 4(\nu + 1) \cdot (2a)^{-(\nu+1)/2} \cdot c_1$, the following integral is then to be computed:

$$c_1' \cdot \left(\frac{1}{2\pi i}\right)^2 \int_{z_0^+ - i\infty}^{z_0^+ + i\infty} \int_0^{\infty} e^{-\frac{2\eta}{2a} x} \cdot x^{\nu+3} \cdot \frac{\cosh(w)}{\sqrt{\eta}} \cdot \frac{h \cdot z}{(z-\nu^2)(z-(\nu+2)^2)} \, dw \, dx \, dz.$$ 

We claim that for $R$ sufficiently big the absolute value of the integrand of this integral is exponentially decreasing to zero with the absolute values of $z$, $x$, or $w$ going to infinity.

Granting this result, the above triple integral then gives the desired Laplace inverse, and, using Fubini’s theorem, the order of its integrals can be interchanged. Take the integral for Laplace inversion, i.e., the integral over the line $\{z|\Re(z) = z_0\}$, as inner integral. Change variables $x = (2a)^{1/2}t$. The Lemma follows on decomposing the denominator of the integrand in partial fractions.

One is thus reduced to prove the last claim about the asymptotic behaviour. For the calculations recall $|\exp(\xi)| = \exp(\Re(\xi))$, for any complex number $\xi$. For the asymptotic behaviour in the absolute value of $w$, reduce to elements $w = x \pm i\pi$ in $\log C_R$. For these $\cosh(w) = -\cosh(x)$. Hence the absolute value of the hyperbolic cosine factor of the numerator equals $\exp(-(x/a) \cosh(\Re w))$ whence the required asymptotic behaviour in the absolute value of $w$.

For the asymptotic behaviour in $z$, notice $|\exp(-wz^{1/2})| = \exp(-\Re(wz^{1/2}))$ and recall $z^{1/2} = \exp((1/2) \cdot (\log |z| + i \arg(z)))$. The argument of $z$ converges to $\pi/2$ with $|z|$ going to infinity. Now $\Re(wz^{1/2})$ equals $|z|^{1/2}$ times $\Re(w) \cos(\arg(z)/2) - \Im(w) \sin(\arg(z)/2)$. Herein the cosine is positive and bigger than $\cos(\pi/4)$. Thus $\Re(wz^{1/2})$ is positive if the real part of any $w$ is big enough. Hence choose $R$ big enough to have the desired asymptotic behaviour in the absolute value of $z$.

The asymptotic behaviour in $x$ is determined by $\exp(-x^2/2a)$, thus completing the proof.

8. **Computation of certain Laplace transforms:** Lacking a suitable reference, this section computes the inverse Laplace transforms identified in §7 Lemma.

For any $\alpha$ and $\beta$ in $\mathbb{C}$, consider the functions on the positive real line given by:

$$f_{\alpha,\beta}(t) = \frac{t^{-1/2}}{\sqrt{\pi}} e^{-\frac{\alpha^2}{4t}} - \beta \cdot e^{\alpha \cdot \beta - \beta^2 \cdot t} \text{Erfc}\left(\frac{\beta \sqrt{t}}{\alpha^{1/2}} + \frac{\alpha}{2\sqrt{t}}\right),$$

$$g_{\alpha,\beta}(t) = \frac{\beta^2 \cdot t}{2} \left( e^{\alpha \cdot \beta} \text{Erfc}\left(\frac{\alpha}{2\sqrt{t}} + \beta \sqrt{t}\right) + e^{-\alpha \cdot \beta} \text{Erfc}\left(\frac{\alpha}{2\sqrt{t}} - \beta \sqrt{t}\right) \right),$$
for any positive real number $t$. Then one has the following two results:

**Lemma:** If the real parts of $\alpha$ and $\alpha^2$ are positive, one has:

$$L(\phi_{\alpha,\beta})(z) = \frac{e^{-\alpha \sqrt{z}}}{\sqrt{z} + \beta},$$

for any complex number $z$ in $\mathbb{C} \setminus \mathbb{R}_{<0}$ with real part bigger than $|\text{Re}(\beta)|$.

**Corollary:** If the real parts of $\alpha$ and $\alpha^2$ are positive, one has:

$$L(\chi_{\alpha,\beta})(z) = \frac{e^{-\alpha \sqrt{z}}}{z - \beta^2},$$

for any complex number $z$ in $\mathbb{C} \setminus \mathbb{R}_{<0}$ with real part bigger than $\text{Re}(\beta^2)$.

The Corollary follows from the Lemma upon decomposing the denominator in partial fractions and using the linearity of the Laplace transform.

We use the following two results proved mutatis mutandis in [D, Beispiel 8, p.50f]:

$$L(\psi_{\alpha})(z) = e^{-\alpha \sqrt{z}} \quad \text{where} \quad \psi_{\alpha}(t) = \frac{\alpha}{2\sqrt{\pi}} \cdot t^{-\frac{3}{2}} \cdot e^{-\frac{\alpha^2}{4t}},$$

$$L(\chi_{\alpha})(z) = \frac{e^{-\alpha \sqrt{z}}}{\sqrt{z}} \quad \text{where} \quad \chi_{\alpha}(t) = \frac{1}{\sqrt{\pi}} \cdot t^{-\frac{1}{2}} \cdot e^{-\frac{\alpha^2}{4t}},$$

for any complex number $z$ in $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ and any positive real number $t$.

Subtracting $\chi_{\alpha}$ from $\phi_{\alpha,\beta}$, the proof of the Lemma reduces to show the identity:

$$L\left( -\beta \cdot e^{\alpha \cdot \beta^2 \cdot t} \text{Erfc}\left( \frac{\beta \sqrt{t} + \alpha}{2\sqrt{t}} \right) \right)(z) = -\beta \frac{e^{-\alpha \sqrt{z}}}{\sqrt{z}(\sqrt{z} + \beta)}.$$

Multiplying any nice function $f$ with $\exp(a \cdot \cdot)$ induces a shift by $-a$ in its Laplace transform: $L(\exp(at) f(t))(z) = L(f)(z - a)$. Using this with $a = \beta^2$ one is further reduced to calculating the Laplace transform of the above complementary error function factor only.

Since $\text{Re}(\alpha)$ is positive, the real part of $\beta u^{1/2} + (\alpha/2) u^{-3/2}$ goes to plus infinity with $u$ converging from the right to zero. The Fundamental Theorem of Calculus thus gives:

$$\text{Erfc}\left( \frac{\beta \sqrt{t} + \alpha}{2\sqrt{t}} \right) = -\frac{2}{\sqrt{\pi}} \int_0^t e^{-\left(\frac{\beta \sqrt{u} + \alpha}{2\sqrt{u}}\right)^2} \left( \frac{\beta}{2} u^{-1/2} - \frac{\alpha}{4} u^{-3/2} \right) du.$$
Using the transform–of–an–integral property $L \left( \int_0^w f(u) \, du \right) = w^{-1} L(f)(w)$ the Laplace transform of this complementary error function at $w = z - \beta^2$ is given by:

$$- \frac{2\beta}{2(z - \beta^2)} \left( \frac{1}{\sqrt{\pi}} \cdot t^{-\frac{1}{2}} \cdot e^{-\left( \beta \sqrt{u} + \frac{\alpha}{2\sqrt{u}} \right)^2} \right) (z - \beta^2)$$

$$+ \frac{2}{2(z - \beta^2)} \left( \frac{\alpha}{2\sqrt{\pi}} \cdot t^{-\frac{3}{2}} \cdot e^{-\left( \beta \sqrt{u} + \frac{\alpha}{2\sqrt{u}} \right)^2} \right) (z - \beta^2).$$

Using the Laplace transform of $\chi_\alpha$, the first Laplace transform of this sum equals

$$e^{-\alpha \beta} e^{-\alpha \sqrt{z}}.$$  

Using the Laplace transform of $\psi_\alpha$, the second Laplace transform of this sum equals

$$e^{-\alpha \beta} e^{-\alpha \sqrt{z}}.$$  

The above sum hence equals $\exp (-\alpha \beta) \exp (-\alpha \sqrt{z})/(z^{1/2} (z^{1/2} + \beta))$. The identity to be proved follows upon substituting this last expression. This completes the proof.

9. Two intermediate results: In this section the Laplace inversion of §7 is resumed concentrating on the two summands of the expression derived in §7 Lemma.

If $\nu > -1$, for any real numbers $a$, $h > 0$ and $b$, thus define more generally:

$$I_b^a(h) = \frac{1}{2\pi i} \int_{\log C_R} \int_0^\infty e^{-x^2 + x \sqrt{2} \cosh(w)} x^{\nu+3} L^{-1} \left( \frac{e^{-w \sqrt{z}}}{z - b^2} \right) \left( \frac{h}{2} \right) \, dx \, dw$$

with $\log C_R$ the logarithmicalized Hankel contour discussed in §6. For these integrals one has the following two results, the first of a more structural and the second of a more technical nature:

**Lemma:** If $\nu > -1$, for any real numbers $R \geq 1$, and $a$, $h > 0$, and $b$ one has:

$$I_b^a(h) = c_2 \cdot \Gamma(\nu + 4) \cdot \frac{1}{2\pi i} \int_{\log C_R} \left( F_b(h) + F_{-b}(h) \right) (w) \cdot H_{-(\nu + 4)} \left( -\frac{\cosh(w)}{\sqrt{2a}} \right) \, dw,$$

where on both sides the respective integrands are absolutely integrable.

**Corollary:** If $\nu > -1$, for any real numbers $\rho = \log R \geq 0$, and $a$, $h > 0$, and $b$ one has:

$$I_b^a(h) = \frac{c_2}{\pi} \int_0^\infty e^{-x^2} x^{\nu+3} \left\{ \int_0^\pi \Re \left( G_{x,b}^a(h) \right) (\rho + i\theta) \, d\theta + \int_0^\infty \Im \left( G_{x,b}^a(h) \right) (y + i\pi) \, dy \right\} \, dx,$$

with $\Re(\xi)$ and $\Im(\xi)$ the real respectively the imaginary part of any complex number $\xi$. 
To explain the notation in these two results, $c_2$ denotes the following constant:

$$c_2 = c_2(h, b) = \frac{1}{2} \cdot e^{b^2 h}.$$  

The Hermite functions $H_\mu$ of degree $\mu$ are discussed in paragraph zero, and the two other functions that occur are given by:

$$F_b(h)(w) = e^{wb} \cdot \text{Erfc} \left( \frac{w}{\sqrt{2h}} + \frac{b}{2 \sqrt{2h}} \right),$$

$$G_{x,b}(h)(w) = e^{\frac{\sqrt{2} \cosh(w)}{\sqrt{a}}} \left( F_b(h)(w) + F_{-b}(h)(w) \right),$$

for any complex number $w$.

**Proof of the Lemma:** For the proof of the Lemma choose a Hankel contour $C_R$ with $R$ so big that for any element $w$ in $\log C_R$ also the real parts of $w$ and $w^2$ are both positive. At any $w$ on $\log C_R$ substitute in $I^0(h)$ the inverse Laplace transforms calculated in §8 Corollary. Interchange the order of integration using the absolute integrability of the integrand. This gives the expression of the Lemma for $I^0(h)$. The integrand herein is a holomorphic function on $\mathbb{C} \setminus \mathbb{R}_{<0}$. Using the Cauchy Theorem, the value of the integral is independent of the Hankel contour $C_R$ chosen as long as $\log R$ is non-negative. This completes the proof.

**Proof of the Corollary:** Given the Lemma, the proof of the Corollary is an exercise in path integration. Put $G_x = G^a_{x,b}(h)$. Changing the order of integration in the Lemma,

$$c_2 \int_0^\infty e^{-x^2} x^{\nu+3} \frac{1}{2\pi i} \int_{\log C_R} G_x(w) \, dw \, dx,$$

is to be calculated. Concentrate on its inner integral and consider the following subpath

$$P = P_c + P_\infty \quad \text{with} \quad P_c = P_{c,+} - P_{c,-}$$

of the path $\log C_R$. Here the path $P_{c,-}$ starts from $\log R$ and moves parallel to the imaginary axis to the point $\log R - i\pi$. The path $P_{c,+}$ starts from $\log R$ and moves parallel to the imaginary axis to the point $\log R + i\pi$. The path $P_\infty$ moves from $\log R + i\pi$ parallel to the real axis to $+\infty$. For any $x > 0$, the inner integral thus breaks up as follows:

$$\frac{1}{2\pi i} \int_{P_\infty} \left( G_x(w) - G_x(\overline{w}) \right) \, dw + \frac{1}{2\pi i} \int_{P_{c,+}} G_x(w) \, dw - \frac{1}{2\pi i} \int_{P_{c,-}} G_x(w) \, dw,$$

with $\overline{w}$ the complex conjugate of $w$. Using the series expansion of the exponential and the complementary error functions, $G_x$ is compatible with complex conjugation, i.e., $G_x$ evaluated at the complex conjugate of any complex number $w$ is the complex conjugate of $G_x$ at $w$:

$$G_x(\overline{w}) = \overline{G_x(w)}.$$
Write the elements of $P_\infty$ as $w = y + i\pi$ with $y \geq \log R$, notice $dw = dy$ and change variables. Using the compatibility of $G_x$ with complex conjugation, one obtains the improper integral from $\log R$ to infinity of the imaginary parts of $G_x(y + i\pi)$.

On the circle part $P_c$ of $P$ one has to be a bit more careful about the volume forms. The elements of $P_{c,+}$ are parametrized by $w = \log R + i\theta$ with $0 \leq \theta \leq \pi$, whereas those of $P_{c,-}$ are parametrized by $\log R - i\theta$ with $0 \leq \theta \leq \pi$. Changing variables accordingly and integrating from zero to $\pi$, the induced volume form in the $P_{c,+}$–integral is $dw = i\,d\theta$, whereas that on the $P_{c,-}$–part is $dw = -i\,d\theta$. Using the compatibility of $G_x$ with complex conjugation, this completes the proof of the Corollary.

10. The final explicit calculations: The aim of this section is to explicitly compute the integrals of §9 Corollary thus essentially proving the valuation formula §4 Theorem.

Recall the integral $I_a^b(h)$ of paragraph nine, the functions Erfc, $H_\mu$ introduced in paragraph zero, and the function $E_b(h)$ introduced in paragraph four. In $c_2(h, b)$ of paragraph nine divide by $\pi$, add a gamma factor, and drop the exponential in $h$ and $b$ to obtain:

$$c_3 = c_3(\nu) = \frac{1}{2\pi} \cdot \Gamma(\nu + 4).$$

In view of §7 Lemma, the following result is the key for proving the valuation formula:

**Lemma:** If $\nu > -1$, for any real numbers $a$, $h > 0$ and $b$, the function $I_b^a(h)$ is given by:

$$2 \cdot c_3 \cdot e^{\frac{b^2 \pi^2}{2}} \cdot \int_0^\pi H_{-(\nu+4)} \left( -\frac{\cos(\theta)}{\sqrt{2}a} \right) \cos(\nu\theta) \, d\theta$$

$$+ c_3 \cdot \frac{2}{\sqrt{\pi}} \cdot e^{\frac{\pi^2 + h^2}{2\pi} + \frac{h^2}{2}} \cdot \int_0^\infty H_{-(\nu+4)} \left( \frac{\cosh(y)}{\sqrt{2}a} \right) \left( e^{yb} E_b(h)(y) + e^{-yb} E_{-b}(h)(y) \right) \, dy.$$

The Lemma is proved by computing the two integrals of $G_x := G_{x,b}^a$ of §9 Corollary upon choosing $R = 1$ there. Start with the following integral:

$$\int_0^\infty \text{Im} \left( G_x(y + i\pi) \right) \, dy.$$

Abbreviate $\beta_{\pm} = y/\sqrt{2h} \pm (b/2)\sqrt{2h}$ and notice $\cosh(y + i\pi) = -\cosh(y)$. Write the complementary error functions occuring in $G_x(y + i\pi)$ as improper integrals starting from zero as in paragraph zero. Multiply out the expressions of the exponents to obtain:

$$e^{\pm(y+i\pi)b} \text{Erfc} \left( \beta_{\pm} + i\frac{\pi}{\sqrt{2h}} \right) = \frac{2}{\sqrt{\pi}} \cdot e^{\frac{\pi^2}{2h}} \cdot e^{\pm yb} \int_0^\infty e^{-(u+\beta_{\pm})^2} \cdot e^{i\pi \left( \pm b - (u+\beta_{\pm}) \sqrt{2h} \right)} \, du.$$

Since $\beta_{\pm}$ is a real number and $u$ can be taken as real numbers, the imaginary parts in $G_x$ are determined by the imaginary parts of the exponentials in the integral. These are given by the functions $E_{\pm,b}(h)$ as indicated, completing the calculation.
As a next step,
\[ \int_0^\pi \text{Re} \left( G_x(i\theta) \right) d\theta \]
is calculated. In contrast to the above argument, in the complementary error functions occurring in \( G_x(i\theta) \) now the following paths of integration are used: Abbreviating \( \beta_{\pm} = \pm (b/2) \sqrt{2h} \), first move from \( \beta_{\pm} + i\theta \) to \( \beta_{\pm} \), then continue from \( \beta_{\pm} \) to plus infinity along the real line. Since \( \cosh(i\theta) = \cos(\theta) \), the above integral then equals:
\[ \int_0^\pi e^{x \sqrt{2q} \cos(\theta)} \left\{ \left( \text{Erfc} \left( \beta_+ \right) + \text{Erfc} \left( \beta_- \right) \right) \cdot \cos(\theta b) + \frac{2}{\sqrt{\pi}} (\phi_+ + \phi_-)(\theta) \right\} d\theta, \]
upon abbreviating for any angle \( \theta \):
\[ \phi_{\pm}(\theta) = -\text{Re} \left( e^{\pm i\theta b} \int_{\beta_{\pm}}^{\beta_{\pm} + i\theta} \frac{e^{-u^2}}{\sqrt{2h}} \, du \right). \]

In the first of these two last integrals notice \( \text{Erfc} \left( \beta_+ \right) + \text{Erfc} \left( \beta_- \right) = 2 \) since \( \beta_- \) is minus \( \beta_+ \). To calculate \( \phi_{\pm} \) change variables \( u = \beta_{\pm} + iw/\sqrt{2h} \) in the integral. Write the factor \( i/\sqrt{2h} \) that is picked up as \( \exp \left( i(\pi/2) / \sqrt{2h} \right) \). Multiplying out the expression obtained in the exponent, it follows
\[ e^{\pm i\theta b} \int_{\beta_{\pm}}^{\beta_{\pm} + i\theta} e^{-u^2} \, du = e^{-\frac{b^2 h}{2}} \int_0^\theta e^{\frac{2q}{\sqrt{2h}}} e^{i \left( \frac{\pi}{2} \pm b(\theta - u) \right)} \, du. \]
The real part of this expression is determined by the real part of the exponential functions in the integral. Abbreviating \( x_u = b(\theta - u) \), the values of the cosine at \( \pi/2 \pm x_u \) thus appear as factors. A shift by \( \pi/2 \) turns a cosine into a sine as follows: \( \cos(\pi/2 \pm x_u) = \mp \sin(x_u) \). Thus \( \phi_- \) is minus \( \phi_+ \), completing the proof.

11. **First part of the proof of the valuation formula:** The proof of the valuation formula of §4 Theorem is in two steps. As a first step, §4 Theorem is in this paragraph established for \( \nu \) bigger than minus one. As a second step, this equality is extended in the next paragraph to any complex number \( \nu \) using analytic continuation.

Thus let \( \nu \) be bigger than one. Recalling §7 Lemma, the valuation formula of §4 Theorem then is obtained by subtracting \( I_{\nu}^q(h) \) from \( I_{\nu+2}^q(h) \) and thereafter multiplying this difference with the constant \( c_1 \). Substitute the expressions computed in §10 Lemma. Using \( \exp \left( h(\nu + 2)^2/2 \right) = \exp \left( h\nu^2/2 \right) \exp \left( 2h(\nu + 1) \right) \) with the trigonometric summands one is reduced to show that
\[ C^*_{\text{trig}} = \int_0^\pi H_{-(\nu+4)} \left( -\frac{\cos(\theta)}{\sqrt{2q}} \right) \cos(\nu \theta) \, d\theta \]
upon multiplication with \( c \) gives the first two terms in the sum of §4 Theorem. As a first step, write \( C^*_{\text{trig}} \) as a double integral using the defining integral representation of the Hermite function factor of its integrand. Applying Fubini’s theorem and interchanging the order of integration the integrand of its inner integral is \( \exp \left( 2x(2q)^{-1/2} \cos(\theta) \right) \cdot \cos(\nu \theta) \).
Substitute for this last integral using Schl{"a}fli’s integral representation of paragraph six. Then reverse the order of integration to get:

\[ C_{\text{trig}}^{\nu} = \frac{\pi}{\Gamma(\nu + 4)} \cdot I + \sin(\nu \pi) \cdot \int_{0}^{\infty} H_{-(\nu + 4)} \left( \frac{\cosh(x)}{\sqrt{2q}} \right) e^{-\nu x} dx, \]

where

\[ I = \int_{0}^{\infty} e^{-x^2} x^{\nu + 3} I_\nu \left( \frac{2x}{\sqrt{2q}} \right) dx. \]

The proof thus reduces to show:

\[ I = \frac{1}{2} \cdot e^{\frac{1}{2}q} \cdot (2q)^{-\nu-2} \left( 1 + 2q \cdot (\nu + 1) \right). \]

Indeed, abbreviating \( a = q/2 \), change variables \( w = a^{-1/2} x \) in \( I \) to obtain:

\[ I = a^{\nu+3} \left( 2(\nu + 1) \int_{0}^{\infty} e^{-aw^2} w^{\nu+2} I_{\nu+1}(w) dw + \int_{0}^{\infty} e^{-aw^2} w^{\nu+3} I_{\nu+2}(w) dw \right). \]

The evaluation of \( I \) is thus reduced to evaluating two Weber’s integrals as reviewed in paragraph six. Substituting their respective values, the above identity follows. This completes the first step of the proof of §4 Theorem

12. Second part of the proof of the valuation formula: This second part of the proof of §4 Theorem extends its validity from \( \nu \) bigger than minus one, as established in the previous paragraph, to \( \nu \) any complex number.

This reduces to show the following two results. First, the normalized price is an entire function in \( \nu \). Second, the right hand side of §4 Theorem is a meromorphic function on the complex plane. Indeed, these two functions agree on real numbers \( \nu \) bigger than minus one. Using the identity theorem they so agree on the complex plane as meromorphic functions. With one of them entire, the other one is entire, too.

In Yor’s notation \( A^{(\nu)}(h) = \int_{0}^{h} \exp \left( \left( 2(B_u + \nu u) \right) \right) du \) recall \( C^{(\nu)}(h, q) = E[f(A^{(\nu)}(h))] \), where \( f \) is given by \( f(x) = (x - q)^{+} \), for any real number \( x \). The proof of this normalized price being entire in \( \nu \) is then further reduced to show

\[ E\left[ f(A_h) \cdot e^{\nu B_h} \right] \]

an entire function in \( \nu \) where \( A \) is the process \( A^{(0)} \). Indeed, this follows using the Girsanov identity \( E[f(A^{(\nu)}(h))] = \exp (-\nu^2 h/2) \cdot E[f(A(h)) \cdot \exp(\nu B(u))] \) of [Y, (1.c), p.510].
For proving the above function entire develop the factor \(\exp(\nu B_h)\) of \(f(A_h) \cdot \exp(\nu B_h)\) into its exponential series. Suppose computing the expectation of the resulting series term by term is justified for any complex number \(\nu\). For any complex number \(\nu\), this then gives a power series in \(\nu\) and thus explicitly shows \(E[f(A_h) \cdot \exp(\nu B_h)]\) holomorphic at \(\nu\).

Interchanging the order of integration and summation is justified using Lebesgue Dominated Convergence if the following is true. The expectations of the absolute value of any single term of the series for \(f(A_h) \cdot \exp(\nu B_h)\) exist and the series so obtained converges. Using the Cauchy–Schwarz inequality this is implied by the series:

\[
E\left[f^2(A_h)\right]^{1/2} \cdot \sum_{n=0}^{\infty} \frac{|\nu|^n}{n!} \cdot E\left[B_h^{2n}\right]^{1/2}
\]

being convergent for any complex number \(\nu\). Implicit herein is that \(f(A_h)\) is square integrable. This is implied by \(A_h\) being square integrable. Any \(n\)–th moment in particular of \(A_h\) has been computed in [Y, (4.d”), p.519] as:

\[
n! \cdot \left(\frac{(-1)^n}{(n!)^2} + 2 \sum_{k=0}^{n} \frac{(-1)^{n-k}}{(n-k)! \cdot (n+k)!} \cdot e^{-h \cdot k^2}\right).
\]

In particular the second moment of \(A_h\) is thus finite, as was to be shown. What regards the even order moments of the Brownian motion \(B\) at time \(h\), they are computed as:

\[
E\left[B_h^{2n}\right] = \frac{(2h)^n}{n!} \cdot \Gamma(n + 1/2),
\]

for any non–negative integer \(n\). The above series thus converges using the ratio test. This completes the proof of the normalized price being an entire function in the parameter \(\nu\).

The proof of the five–term sum of \(\S 4\) Theorem being a meromorphic function in \(\nu\) is based on the Hermite functions being entire also in their degree. Indeed, from [L, (10.2.8), p.285] one has for any complex numbers \(\mu\) and \(z\) the representation:

\[
H_{\mu}(z) = \frac{2^\mu \cdot \Gamma(1/2)}{\Gamma((1-\mu)/2)} \cdot \Phi\left(-\frac{\mu}{2}, \frac{1}{2}, z^2\right) + \frac{2^\mu \cdot \Gamma(-1/2)}{\Gamma(-\mu/2)} \cdot \Phi\left(-\frac{1-\mu}{2}, \frac{3}{2}, z^2\right).
\]

Herein the reciprocal of the gamma function is entire by construction, and the confluent hypergeometric function \(\Phi\) is entire in its first and third variable.

There is a localization principle for proving entire a function on the complex plane. Indeed, one has to show analyticity at any fixed complex number. For this, one can restrict the function to any relatively compact or compact neighborhood of this fixed complex number.

Apply this localization principle to the factor of the trigonometric term:

\[
A(\nu) = \int_{0}^{\pi} H_{-(\nu+4)}\left(-\frac{\cos(\theta)}{\sqrt{2q}}\right) \cos(\nu \theta) \, d\theta,
\]

and restrict \(\nu\) to belong to any sufficiently small compact neighborhood \(U\) of any point fixed in the complex plane. The above integrand is analytic as a function in \(\nu\) and smooth as a function in \(\theta\). Thus the absolute values of its derivatives with respect to \(\nu\) of any order are bounded on the product of \(U\) and the closed interval between zero and \(\pi\). In this local
situation, using a standard consequence of Lebesgue Dominated Convergence, differentiation of $A$ with respect to $\nu$ is thus by partial differentiation with respect to the parameter $\nu$ under the integral sign. This proves $A$ entire as a function in $\nu$. The trigonometric term $C_{\text{trig}}$ is obtained by multiplying $A$ with $c$. Dividing off from $c$ the gamma factor $\Gamma(\nu+4)$, it has a removable singularity in $\nu = -1$. Thus $C_{\text{trig}}$ is meromorphic in $\nu$ with at most simple poles in the integers less than or equal to minus four.

What regards the hyperbolic terms, the claim is that as a function of $\nu$ they can be extended as analytic functions from $\nu$ bigger than minus one to to the whole complex plane. For this question it is sufficient to consider the function $B$ given by:

$$B(\nu) = \int_0^\infty H_{-(\nu+4)} \left( \frac{\cosh(y)}{\sqrt{2q}} \right) \cdot e^{b(\nu)y} \cdot E_{b(\nu)}(h)(y) \, dy,$$

where $b(\nu)$ equals $\pm \nu$ or $\pm (\nu+4)$ and with:

$$E_\xi(h)(y) = \int_{\frac{y}{\sqrt{2h}}}^\infty \frac{e^{-u^2}}{\sqrt{2\pi}} e^{\xi u} \sin \left( \pi \left( \xi - u \frac{\sqrt{2h}}{h} \right) \right) du.$$

for any real number $y$. Extension of this function is essentially by reduction to the case where the real part of the degree $-(\nu+4)$ of the Hermite function factor in the integrand of $B(\nu)$ is negative, or equivalently, the real part of $\nu$ is bigger than minus four.

To fix ideas first consider the case where $\nu$ is such that the degree of the Hermite function factor in the integrand of $B(\nu)$ is a non–negative integer. This Hermite function is then the corresponding Hermite polynomial. In the integrand of $B(\nu)$ the absolute values of the Hermite function factor and the exponential function factor so have linear exponential order in the variable $y$. The decay to zero of the absolute value of the function $E_{b(\nu)}$, however, is of square exponential order in $y$. It thus dominates the asymptotic behaviour with $y$ to infinity. The absolute value of the integrand of $B(\nu)$ is so majorized by an integrable function, and $B$ can be extended to the above values of $\nu$.

For the general case of the reduction, fix any $\nu$ of real part less than or equal to minus one that is not an integer. Apply the above localization principle and let $\nu$ belong to a sufficiently small compact neighborhood $U$ in the half–plane $\{ \text{Re}(z) \leq -1 \}$. Shrinking $U$ if necessary assume that for any $\nu$ in $U$ the degree of the Hermite function in the integrand of $B(\nu)$ is not an integer. Using the recursion rule for Hermite functions of paragraph zero express the Hermite function factor of $B(\nu)$ in terms of weighted Hermite functions of negative degrees. Further shrinking $U$ if necessary, assume that the so obtained relation represents $B$ on $U$. Herein the weighting factor for the respective Hermite functions are given as powers of $z = (2q)^{-1/2} \cosh(y)$ times polynomials in $\nu$. The absolute values of the polynomials in $\nu$ can be majorized uniformly on $U$. The problem thus reduces to majorize by an integrable function on the positive real line in the variable $y$ finitely many functions on $U$ times the positive real line sending $\nu$ and $y$ to:

$$H_{-(\nu+4+k)} \left( \frac{\cosh(y)}{\sqrt{2q}} \right) \cdot \cosh^\ell(y) \cdot e^{b(\nu)y} \cdot E_{b(\nu)}(h)(y),$$

where $k, \ell$ range over finitely many non–negative integers and $k$ is such that $\nu+4+k$ is positive. The leading terms of the asymptotic expansion of paragraph zero for any Hermite
function $H_\mu(z)$ with degree $\mu$ any complex number with negative real part has order $z^\mu$. Asymptotically with $y$ to infinity, the Hermite function with the smallest positive number $\nu+4+k$ thus dominates the other Hermite function factors in the above functions. With $\nu$ ranging over a compact set, there is a minimal such degree on $U$. Similarly, there are such majorizing choices $\ell^*$ for the factors $\cosh^\ell(y)$, and $\nu^*$ for the absolute values of the factors $\exp(b(\nu)y)$, and $\nu^{**}$ for the absolute values of the factors $E_{b(\nu)}(h)(y)$. A four–factor–majorizing function on the positive real line thus results whose asymptotic behaviour with $y$ to infinity is governed by the square–exponential decay to zero of the corresponding factor $E_{b(\nu^{**})}(h)$ and which is integrable.

If the real part of $\nu$ is bigger than minus one, the above argument holds in a simplified form. The upshot so is that any complex number not an integer has a sufficiently small compact neighborhood such that the absolute value of the integrand of $B(\nu)$ on $U$ times the positive real line can be majorized by an integrable function on the positive real line. Herein, compact neighborhoods can be replaced by relatively compact neighborhoods mutatis mutandis. Using Lebesgue Dominated Convergence, $B$ can thus be extended as a continuous function to the whole complex plane with the integers less than or equal to minus four deleted.

The idea for showing $B$ analytic as a function of $\nu$ on the complex plane with the integers less than or equal to minus four deleted is as follows. Show that differentiation of $B$ is by differentiation under the integral sign and use that its integrand is entire as function of $\nu$. For this again first localize to $\nu$ in any sufficiently small compact neighborhood containing no integers less than or equal to minus four. The aim is then to majorize the absolute value of the derivative with respect to $\nu$ of the integrand of $B$ by an integrable function independent of $\nu$ as above. The above argument for getting such a majorizing function is based on a comparison of decay rates. The integrand of $B$ has one factor which on the positive real line decays to zero of square exponential order whereas the other factors explode of at most linear exponential order. This situation is preserved on differentiation with respect to the parameter $\nu$. In particular, differentiating with respect to the degree the asymptotic expansion for Hermite functions on the right half–plane gives an asymptotic expansion for this function’s partial derivative with respect to the degree.

At this stage it remains to extend $B$ analytically to the integers less than or equal to minus four. However, $B$ remains bounded in any punctured compact neighbourhood of such an integer. Thus $B$ can be extended to an entire function, completing the proof of §4 Theorem.

Part III

13. Remarks about hedging: This paragraph’s aim is to compute the Asian option’s Delta and discuss how the seller’s hedging portfolio is determined by it.

Delta, and similarly the other local hedging parameters, are computed by partially differentiating the price function $C_{t,T}(K)$ of paragraph three:

$$C_{t,T}(K) = e^{-r(T-t)} \cdot S_t \cdot \frac{4}{\sigma^2(T-t_0)} \cdot C^{(\nu)}(h,q).$$
To simplify notation, in the sequel as many arguments of a function as possible are suppressed, thus writing \( C_t = C_{t,T}(K) \) and \( C^{(\nu)} = C^{(\nu)}(h,q) \) in particular.

Hedging of the Asian option as a particular case of the general theory of hedging European–style contingent claims in a complete Black–Scholes economy has been discussed in [K, p.23f]. The seller’s hedging portfolio \( \Pi \) is determined using the martingale representation of the conditional expectation of the discounted pay–out of the Asian option at its time of maturity, i.e., using the following stochastic differential equation:

\[
e^{-r(t-t_0)}C_{t,T}(K) = C_{t_0,T}(K) + \sigma \int_{t_0}^{t} e^{-r(s-t_0)} \Pi_s \, dB_s.
\]

Apply to the martingale of the left hand side the Itô formula. Comparing diffusion coefficients, the hedging portfolio \( \Pi_t \) at any time \( t \) between \( t_0 \) and \( T \) thus is given by:

\[
\Pi_t = S_t \cdot \Delta_t \quad \text{where} \quad \Delta_t = \frac{\partial C_t}{\partial S_t}.
\]

The option’s Delta \( \Delta_t \) is computed in the case where \( q \) is positive and §4 Theorem applies. The partial derivative of \( q \) with respect to \( S_t \) times \( S_t \) being minus \( q \), one has:

\[
\frac{\partial C_t}{\partial S_t} = e^{-r(T-t)} \frac{4}{\sigma^2(T-t_0)} \left( C^{(\nu)}(h,q) - q \cdot \frac{\partial C^{(\nu)}}{\partial q}(h,q) \right).
\]

One is thus reduced to computing the partial derivative with respect to \( q \) of \( C^{(\nu)} \) at \( (h,q) \). Think of this as the option’s normalized Delta. Further reduce as follows. Write the five–term sum of §4 Theorem for \( C^{(\nu)} \) in the form:

\[
C^{(\nu)} = d \cdot D_{\text{trig}} + d \cdot D_{\text{hyp}},
\]

setting:

\[
d = d(\nu,q) = \frac{\Gamma(\nu+4)}{2\pi(\nu+1)} \cdot \frac{(2q)^{\nu+2}}{2} \cdot e^{-\frac{1}{2q}},
\]

and where the modified trigonometric term \( D_{\text{trig}} \) and the modified hyperbolic term:

\[
e^{\nu^2h} \cdot D_{\text{hyp}} = C_{\text{hyp},\nu+2} + C_{\text{hyp},-(\nu+2)} - C_{\text{hyp},\nu} - C_{\text{hyp},-\nu}
\]

are functions in the variables \( \nu \), \( h \) and in \( q = q(h) \). The partial derivative of \( C^{(\nu)} \) with respect to \( q \) then is:

\[
\left( D_{\text{trig}} + D_{\text{hyp}} \right) \cdot \frac{\partial d}{\partial q} + d \cdot \left( \frac{\partial D_{\text{trig}}}{\partial q} + \frac{\partial D_{\text{hyp}}}{\partial q} \right).
\]

Thus, one is further reduced to compute the partial derivatives with respect to \( q \) of the above functions \( d \), \( D_{\text{trig}} \), and \( C_{\text{hyp},b} \) with \( b \) equal to \( \pm \nu \) or \( \pm(\nu+2) \). The first of these is:

\[
\frac{\partial d}{\partial q} = \frac{\Gamma(\nu+4)}{\pi(\nu+1)} \cdot \left( 2 + (\nu + 2)2q \right) \cdot (2q)^{\nu-2} \cdot e^{-\frac{1}{2q}}.
\]
To compute the partial derivatives of the other two functions recall from paragraph zero
the recursion formula for the derivative of Hermite functions. Granting for a moment
that one can justify the differentiation under the integral sign, the partial derivative with
respect to \( q \) of \( D_{\text{trig}} \) at \((\nu, h, q)\) is:

\[
2 \cdot \left(1 - e^{2h(\nu+1)}\right) \cdot \frac{2(\nu+4)}{(2q)^{3/2}} \int_0^\pi H_{-(\nu+5)} \left(\frac{-\cos(\theta)}{\sqrt{2q}}\right) \cos(\nu \theta) \cdot \cos(\theta) \, d\theta.
\]

With the functions \( E_b(h) \) of paragraph four independent of \( q \), the partial derivative with
respect to the variable \( q \) of any \( C_{\text{hyp},b} \) at \((\nu, h, q)\) similarly is:

\[
\frac{2}{\sqrt{\pi}}e^{\frac{\pi^2}{2h}} \cdot \frac{2(\nu+4)}{(2q)^{3/2}} \int_0^\infty H_{-(\nu+5)} \left(\frac{\cosh(y)}{\sqrt{2q}}\right) \cdot \cosh(y) \cdot e^{yb} E_b(h)(y) \, dy.
\]

To justify the differentiations under the integral sign first notice that differentiation is a
local concept and thus the parameters \( \nu, h \) can be restricted to vary in a fixed compact
set not containing points with \( q \) equal to zero. The integrands of \( D_{\text{trig}} \) and \( D_{\text{hyp}} \) then are
integrable and differentiable functions not only in the variable \( \theta \) respectively
\( y \) but also in the variables \( \nu, h \), and in \( q \). Consider the maxima in \( h \) of the absolute value of the
partial derivatives with respect to \( q \) of \( D_{\text{trig}} \) and \( D_{\text{hyp}} \), for any fixed triple of the other
variables. In particular with \( q \) bounded and bounded away from zero the so obtained
functions in the variables \( \theta \) respectively \( y \), \( \nu \) then are majorized by integrable functions.
A standard application of Lebesgue Dominated Convergence thus completes the argument
for the differentiations under the integral sign, and thus completes the calculations.

References

[B] R. Beals: \textit{Advanced mathematical analysis}, GTM 12, Springer 1973
[D] G. Doetsch: \textit{Handbuch der Laplace Transformation} I, Birkhäuser Verlag 1971
[DSM] D. Duffie: \textit{Security markets}, Academic Press 1988
[GY] H. Geman, M. Yor: Bessel processes, Asian options, and perpetuities, \textit{Math. Finance}
3(1993), 349-375
[K] I. Karatzas: \textit{Lectures on the mathematics of finance}, CRM Monographs 8, American
Mathematical Society, Providence 1997
[L] N.N. Lebedev: \textit{Special functions and their applications}, Dover Publications 1972
[RY] D. Revuz, M. Yor: \textit{Continuous martingales and Brownian motion}, 2nd ed., Springer
1994
[Sch] M. Schröder: On the valuation of arithmetic–average Asian options: explicit formulas,
Universität Mannheim, März 1999
[WW] E.T. Whittacker, G.N. Watson: \textit{A course in modern analysis}, Cambridge UP, repr.
1965
[Y80] M. Yor: Loi d’indice du lacet Brownien, et distribution de Hartman–Watson, \textit{Z. Wahrscheinlichkeitsstatheorie} 53(1980), 71–95
[Y] M. Yor: On some exponential functionals of Brownian motion, \textit{Adv. Appl. Prob.}
24(1992), 509–531
[Y92] M. Yor: \textit{Some aspects of Brownian motion} I, Birkhäuser 1992
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