LOCAL SMOOTHING RESULTS FOR THE RICCI FLOW IN DIMENSIONS TWO AND THREE

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Abstract. We present local estimates for solutions to the Ricci flow, without the assumption that the solution has bounded curvature. These estimates lead to a generalisation of one of the pseudolocality results of G.Perelman in dimension two.

1. Introduction

In this paper, unless otherwise specified, a solution \((M, g(t))_{t \in [0, T)}\) to Ricci flow refers to a family \((M, g(t))_{t \in [0, T)}\) of smooth (in space and time) Riemannian manifolds which are complete for all \(t \in [0, T)\), solve \(\frac{\partial}{\partial t} g(t) = -2 \text{Ric}(g(t))\) and have no boundary. We do not require (unless otherwise stated) that the solution has bounded curvature.

In the paper [12], G.Perelman proved the following fact: if a ball \(0 B_r(x_0)\) in \((M, g(0))\) at time zero is almost euclidean, and \((M, g(t))_{t \in [0, T)}\) is a solution to the Ricci flow with bounded curvature, then for small times \(t \in [0, \varepsilon(n, r))\), we have estimates on how the curvature behaves on balls \(B_{\varepsilon r}(x_0)\). There are a number of versions of his theorem: see Theorem 10.1 and Theorem 10.3 in [12] for proofs and the definitions of almost euclidean. See [3], [11], [5], [10] and [4] for alternative proofs and related results. In dimension two, we show that a similar result holds under weaker initial assumptions.

Theorem 1.1. Let \(1 > \sigma, \alpha > 0, v_0, r > 0, N > 1\) be given. Let \((M^2, (g(t))_{t \in [0, T)})\) be a smooth complete solution to Ricci flow, \(x_0 \in M\), and assume that

- \(\text{vol}^0 B_r(x_0) \geq v_0 r^2\) and
- \(R(g(0)) \geq -\frac{N}{\sigma} \text{ on } 0 B_r(x_0)\).

Then there exists a \(\hat{v}_0 = \hat{v}_0(v_0, \sigma, N, \alpha) > 0\) and a \(\delta_0 = \delta_0(v_0, \sigma, N, \alpha) > 0\) such that

- \(\text{vol}^t B_{r(1-\sigma)}(x_0) \geq \hat{v}_0 r^2\)
- \(R(g(t)) \geq -\frac{(N+\alpha)}{r^2} \text{ on } t B_{r(1-\sigma)}(x_0)\)
- \(|R(g(t))| \leq \frac{1}{\hat{v}_0^2} \text{ on } t B_{r(1-\sigma)}(x_0)\)

as long as \(t \leq (\delta_0)^2 r^2\) and \(t \in [0, T)\).

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Remark 1.2. Notice that we do not require that a region be almost euclidean here (see Thm. 10.1 and 10.3 of [12] for the definition of almost euclidean). If the ball $^0B^r(x_0)$ is almost cone like (that is, it is as close as we like in the Gromov Hausdorff sense to an euclidean cone and has $R \geq -2$) then the Theorem (with $r = 1$) still applies. This means that the interior of regions which are cone like in this sense will be smoothed out by Ricci flow in two dimensions, regardless of what the solution looks like outside of this region. Both of the Theorems of G.Perelman (Thm. 10.1, Thm. 10.3 of [12]) do not apply to this situation.

Remark 1.3. By scaling, it suffices to prove the Theorem for $r = 1$.

Remark 1.4. It is not possible to improve the constant $\delta_0$ in the estimate $|\text{Riem}(g(t))| \leq \frac{1}{8\sigma r}$ to an arbitrary constant $\delta_0 > 0$ for a short time. This is because, solutions coming out of non-negatively curved cones exist which have curvature behaviour immediately like $\frac{c}{r}$ where $c \geq 0$ depends on the cone angle (see [13]). In G.Perelman’s first Pseudolocality result (Theorem 10.1 of [12]), where he assumes that a ball $B_r(y_0)$ at time zero is almost euclidean, he showed that it is possible to obtain an estimate of the form $|\text{Riem}(g(t))| \leq \frac{\alpha}{r^4}$ on a smaller ball for arbitrary $\alpha$ at least for some short time interval depending on $\alpha$, as long as the initial ball is close enough to the euclidean ball. Here close enough means, that $(\text{vol}(\partial \Omega))^n \geq (1 - \delta) c_n (\text{vol}(\Omega))^{n-1}$ for any $\Omega \subset B_r(y_0)$ where $c_n$ is the euclidean isoperimetric constant, $R \geq -\frac{1}{r^4}$ and $\delta = \delta(n, \alpha) > 0$ is small enough.

The second theorem is valid in three dimensions. In contrast to the above theorem, we need to have information on how the curvature is behaving (in time) in the balls we are considering in order to draw (stronger) conclusions.

**Theorem 1.5.** Let $r, v_0 > 0, N > 1, 1 > \sigma > 0, V > 0$ be given. Let $(M^3, (g(t))_{t \in [0, T)})$ be a smooth complete solution to Ricci flow with $T \leq 1$ and let $x_0 \in M$ be a point such that

- $\text{vol}^0 B^r(x_0) \geq v_0 r^3$ and
- $\mathcal{R}(g(0)) \geq -\frac{V}{r^4}$ on $^0B^r(x_0)$,
- $|\text{Riem}(g(t))| \leq \frac{\alpha}{r^4}$ on $^tB^r(x_0)$, for all $t \in (0, T)$.

Then there exists a $\tilde{v}_0 = \tilde{v}_0(v_0, N, \sigma, V) > 0$ and a $\delta_0 = \delta_0(v_0, N, \sigma, V) > 0$ such that

- $\text{vol}(^tB^r(x_0)) \geq \tilde{v}_0 r^3$
- $\mathcal{R}(g(t)) \geq -\frac{400NV}{r^4}$ on $^tB_{r(1-\sigma)}(x_0)$.

as long as $t \leq r^2 (\delta_0)^2$ and $t \in [0, T)$.

Remark 1.6. By scaling arguments it suffices to prove the theorem for $r = 1$ and $V = \frac{1}{200N}$: see Remark 5.2.

Remark 1.7. As in the two dimensional case (Theorem 1.1 above), the regions which are considered are not necessarily almost euclidean at time zero.

Remark 1.8. Related results were proved recently in a pre-print [6]. There the authors require that the curvature of the solution be uniformly bounded by a constant $c$ on a ball for the times $t \in [0, S)$ being considered. See [6].

Remark 1.9. The above results were first presented in Nov. 2011 at the H.I.M workshop in Bonn ‘Geometric Flows’.
2. A local bound for the curvature on regions whose curvature is bounded from below

We use the following notation in this paper. **Notation**

\[d(x, y, t) = d_t(x, y) = d(g(t))(x, y)\]

is the distance from \(x\) to \(y\) in \(M\) with respect to the metric \(g(t)\). 

\[d_t(x) = d_t(x, x_0)\]

is distance from \(x\) to \(x_0\) with respect to the metric \(g(t)\) for some fixed \(x_0\).

\[tB_r(x) := \text{ball of radius } r > 0, \text{ centre point } x \in M \text{ measured with respect to } (M, g(t))\]

and \(\text{vol}(tB_r(x)) := \text{volume of } tB_r(x) \text{ with respect to the volume form } d\mu_t \text{ induced by } g(t)\).

\[\text{Riem}(g(t))(x) = \text{Riem}(x, t) \text{ is the Riemannian curvature Tensor of the metric } g(t) \text{ at the point } x \in M.\]

\[\text{R}(g(t))(x) := \text{R}(x, t) \text{ is the curvature operator of } (M, g(t)) \text{ at the point } x \in M; \text{R}(x, t)(V, W) := \text{Riem}_{ijkl}(x, t)V^{ij}W^{kl}\]

for \(2\)-forms \(V, W\) (\(\text{Riem}_{ijkl}\) is the curvature tensor of \(g(t)\) in local coordinates, and \(V = V_{ij}dx^i \otimes dx^j, W = W_{ij}dx^i \otimes dx^j\), and \(V^{km} = V_{ij}g^{ki}g^{mj}, W^{km} = W_{ij}g^{ki}g^{mj}\))

\[\text{R}(x, t) \text{ the scalar curvature of } (M, g(t)) \text{ at the point } x \in M\]

Let \((M, g(t))_{t \in [0, T]}, T \leq 1\) be a smooth complete solution to Ricci flow. We wish to prove estimates on a ball of radius \(r\) at time \(t \in [0, T]\), assuming the curvature operator stays bounded from below on \(tB_r(x)\) for all \(t \in [0, T]\) and the volume of \(tB_r(x)\) is bounded from below for all \(t \in [0, T]\). The estimates will depend on \(n, r\) and the bounds from below. A local result of this type was obtained by B.-L Chen in the proof of Theorem 3.6 of [4], under the assumption that the curvature operator is non-negative on all of \((M, g(t))_{t \in [0, T]}\). A global result of this type was obtained in Lemma 2.4 in [15], and Lemma 4.3 in [10].

In the proof of Theorem 3.6 of [4] by B.-L Chen, the author uses a point picking argument of G.Perelman before rescaling to obtain a contradiction to Proposition 11.4 of [12] (in the proof Lemma 2.4 in [15], and Lemma 4.3 in [16] we used a more global point picking type argument of R.Hamilton and then also obtained a contradiction to Proposition 11.4 of [12] after scaling). The point picking argument of G.Perelman is more suited to this local situation, and so we use it in the following.

The proof follows the lines given in the proof of Theorem 3.6 of [4]. A number of modifications are necessary.

**Theorem 2.1.** Let \(r, v_0 > 0, 1 > \sigma > 0\) and \((M^n, g(t))_{t \in [0, T]}\) be a smooth complete solution to Ricci-flow which satisfies

\[
\text{(a)} \quad \text{vol}(tB_r(x_0)) \geq v_0 r^n, \\
\text{(b)} \quad \text{R}(x, t) \geq -\frac{1}{t} \text{ for all } t \in [0, T], x \in tB_r(x_0).
\]

Then, there exists a \(N = N(n, v_0, \sigma) < \infty\) such that

\[
\text{(c)} \quad |\text{Riem}| \leq \frac{N^2}{t} + \frac{N^2}{r^2} \text{ for all } x \in tB_r(1-\sigma)(x_0), t \in [0, \frac{r^2}{N^2}) \cap [0, T).
\]

**Proof.** By scaling, it suffices to prove the case \(r = 1\). Assume that the statement is false. Then we can find solutions \((M^n_t, g_t(t))_{t \in [0, T_i]}, T_i \leq 1, i \in \mathbb{N}, (i \neq 0 \text{ for}}

\[\frac{N^2}{t} + \frac{N^2}{r^2} \text{ for all } x \in tB_r(1-\sigma)(x_0), t \in [0, \frac{r^2}{N^2}) \cap [0, T).
\]

**Proof.** By scaling, it suffices to prove the case \(r = 1\). Assume that the statement is false. Then we can find solutions \((M^n_t, g_t(t))_{t \in [0, T_i]}, T_i \leq 1, i \in \mathbb{N}, (i \neq 0 \text{ for}}
notational reasons: we shall use the symbol $x_0$ in a moment) and points $x_i \in M_i$ such that

\begin{itemize}
  \item [a] \( \text{vol}(B_1(x_i, t)) \geq v_0 \) for all \( t \in [0, T_i] \),
  \item [b] \( \mathcal{R}(x, t) \geq -1 \) for all \( t \in [0, T_i], x \in tB_1(x_i) \)
\end{itemize}

and points $t_i \in [0, T_i], z_i \in t_i B_{(1-\sigma)}(x_i)$ such that $t_i \leq \frac{1}{N_i^2}$ and

\[ |\text{Riem}(z_i, t_i)| \geq \frac{N_i^2}{t_i} + N_i^2 \]

with $N_i \to \infty$ as $i \to \infty$. Fix $i \in \mathbb{N}$ for the moment and define $M := M_i$, $x_0 := x_i$, $s_0 := t_i$, $g(t) := g_i(t)$, $y_0 := z_i$, $T = T_i$, $d_i(x) := \text{dist}(g_i(t))(x, x_i) = \text{dist}(g(t))(x, x_0)$, $A = \frac{8N_i^2}{\varepsilon}$, $\varepsilon := \frac{1}{N_i}$ and $\alpha = N_i^2$. Then $A\varepsilon \leq \frac{\varepsilon}{2}$ and $s_0 \leq \varepsilon^2$ and $g$ solves RF on $[0, T]$ with $T \leq \varepsilon^2$ and $|\text{Riem}(y_0, s_0)| \geq \frac{\varepsilon^2}{8} + \frac{1}{10^i}$. That is, we are in the setup of Claim 1 of Theorem 10.1 of [12], except he requires $g$ be a RF on $[0, \varepsilon^2]$. Examining the argument of Perelman, we see that we only need that $g$ solves RF on $[0, T]$ with $T \leq \varepsilon^2$, as the subsequent point picking argument only looks at times less than or equal to $s_0$ (see the proof of Claim 1 of Theorem 10.1 of [12]). Also, we do not have $d_{s_0}(y_0) \leq \varepsilon$: we have $d_{s_0}(y_0) = d_{s_0}(x_i, z_i) \leq (1 - \sigma)$. This causes no problem in the point picking argument, and merely leads to the term $2A\varepsilon + 1 - \sigma$ appearing in place of $2A\varepsilon + \varepsilon$ in the estimate (2.1) below. Using Claim 1 of Theorem 10.1 of [12], we obtain new points $\bar{y}_0 \in M, \bar{s}_0$ satisfying

\begin{equation}
\bar{s}_0 \leq s_0
\end{equation}

\begin{equation}
d_{\bar{s}_0}(\bar{y}_0) \leq 2A\varepsilon + (1 - \sigma) \quad (\leq 3)
\end{equation}

\begin{equation}
|\text{Riem}(\bar{y}_0, \bar{s}_0)| \geq \frac{\alpha}{\bar{s}_0} = \frac{N_i^2}{\bar{s}_0} \quad (\geq N_i^2)
\end{equation}

and

\[ |\text{Riem}(x, t)| \leq 4|\text{Riem}(\bar{y}_0, \bar{s}_0)| \]

whenever $|\text{Riem}(x, t)| \geq \frac{\alpha}{\bar{s}_0}, t \leq \bar{s}_0(\leq s_0)$ and $d_i(x) \leq d_{\bar{s}_0}(\bar{y}_0) + 4|\text{Riem}|^{-\frac{1}{2}}(\bar{y}_0, \bar{s}_0)$.

Hence a version of Claim 2 of Theorem 10.1 of [12] is applicable. We follow the first part of the argument of B.Kleiner/J.Lott in the proof of Lemma 32.1 of the Arxiv version of their paper [10]. This gives us

\[ |\text{Riem}(x, t)| \leq 4|\text{Riem}(\bar{y}_0, \bar{s}_0)| \]

whenever

\begin{equation}
\bar{s}_0 - \frac{1}{2}Q^{-1} \leq t \leq \bar{s}_0 \quad (**)
\end{equation}

\begin{equation}
d_i(x) \leq d_{\bar{s}_0}(\bar{y}_0) + AQ^{-\frac{1}{2}} \quad (*),
\end{equation}

where here $Q := \text{Riem}(\bar{y}_0, \bar{s}_0)$. Note (*) just says: $x \in tB_{1}d_{\bar{s}_0}(\bar{y}_0) + AQ^{-\frac{1}{2}}(x_0)$. Now we modify the rest of the argument of B.Kleiner/J.Lott given in the proof of Lemma 32.1 of the Arxiv version of their paper [10] in order to obtain a product region on which the curvature is bounded. Notice that we do NOT have $\alpha \leq M_{\text{non}}$, and therefore modifications are necessary. We claim that

\[ M_{\text{bar}}B_{\bar{s}_0}(\bar{s}_0) + M_{\text{bar}}AQ^{-\frac{1}{2}}(x_0) \subseteq tB_{\bar{s}_0}(\bar{y}_0) + \frac{1}{8}AQ^{-\frac{1}{2}}(x_0), \]

whenever $\bar{s}_0 - M_{\text{bar}}Q^{-1} \leq t \leq \bar{s}_0$. 


where $M_0$ is a fixed large constant. We assume in the following that $Q$ and $A$ are large (a lot larger than $M_0$). Let $x \in s_0 B_{d_{s_0} (\bar{y}_0)} + \frac{1}{M_0} A Q^{-\frac{1}{2}} (x_0)$. As long as (going backward in time) $x \in t B_{d_{s_0} (\bar{y}_0)} + \frac{1}{M_0} A Q^{-\frac{1}{2}} (x_0)$, we have $| \text{Riem}(x, t) | \leq 4Q$. Choose $r = \frac{1}{M_0} A Q^{-\frac{1}{2}}$. Then note that $t B_r (x) \subseteq t B_{d_{s_0} (\bar{y}_0)} + A Q^{-\frac{1}{2}} (x_0)$ by the triangle inequality and hence $| \text{Riem}(\cdot, t) | \leq 4Q$ on $t B_r (x)$ (note $t B_r (x_0) \subseteq t B_{d_{s_0} (\bar{y}_0)} + A Q^{-\frac{1}{2}} (x_0)$ trivially, and hence $| \text{Riem}(\cdot, t) | \leq 4Q$ on $t B_r (x_0)$ also). Using Lemma 8.3 (b) of [12] we see that

$$ \frac{\partial}{\partial t} d_t (x_0, x) (t) \geq -2(n-1) \left( \frac{2}{3} 4Q \frac{1}{M_0} A Q^{-\frac{1}{2}} + M_0^2 A^{-1} Q^{\frac{1}{2}} \right) $$

Hence

$$ d_t (x_0, x) - d_{s_0} (x_0, x) \leq M_0 Q^{-1} 2(n-1) \left( \frac{2}{3} 4Q \frac{1}{M_0} A Q^{-\frac{1}{2}} + M_0^2 A^{-1} Q^{\frac{1}{2}} \right) $$

$$ \leq \frac{8(n-1)}{M_0} A Q^{-\frac{1}{2}} + 2(n-1) M_0^2 A^{-1} Q^{\frac{1}{2}} $$

$$ \leq \frac{16(n-1)}{M_0} A Q^{-\frac{1}{2}} $$

where we have used that $A >> M_0$, and $|t - s_0| \leq M_0 Q^{-1}$. That is

$$ d_t (x) \leq d_{s_0} (x_0, x) + \frac{16(n-1)}{M_0} A Q^{-\frac{1}{2}} $$

$$ \leq d_{s_0} (\bar{y}_0) + \frac{1}{10} A Q^{-\frac{1}{2}} + \frac{16(n-1)}{M_0} A Q^{-\frac{1}{2}} $$

$$ \leq d_{s_0} (\bar{y}_0) + \frac{1}{8} A Q^{-\frac{1}{2}} $$

and hence $x \in t B_{d_{s_0} (\bar{y}_0)} + \frac{1}{M_0} A Q^{-\frac{1}{2}} (x_0)$. Hence, $x \in t B_{d_{s_0} (\bar{y}_0)} + \frac{1}{M_0} A Q^{-\frac{1}{2}} (x_0)$ will not be violated for $s_0 - M_0 Q^{-1} \leq t \leq s_0$, as claimed.

Now assume $d_{s_0} (x, \bar{y}_0) \leq \frac{1}{10} A Q^{-\frac{1}{2}}$ (i.e. $x \in s_0 B_{\frac{1}{M_0} A Q^{-\frac{1}{2}} (\bar{y}_0)}$) and $s_0 - M_0 Q^{-1} \leq t \leq s_0$. The triangle inequality implies that

$$ d_{s_0} (x, x_0) \leq d_{s_0} (x, \bar{y}_0) + d_{s_0} (x_0, \bar{y}_0) $$

$$ \leq \frac{1}{10} A Q^{-\frac{1}{2}} + d_{s_0} (\bar{y}_0) $$

and hence

$$ x \in s_0 B_{\frac{1}{M_0} A Q^{-\frac{1}{2}} (\bar{y}_0)} \subseteq s_0 B_{\frac{1}{M_0} A Q^{-\frac{1}{2}} + d_{s_0} (\bar{y}_0)} (x_0) \subseteq t B_{\frac{1}{M_0} A Q^{-\frac{1}{2}} + d_{s_0} (\bar{y}_0)} (x_0) $$

( as we just showed ) and hence $| \text{Riem}(x, t) | \leq 4Q$ in view of [25]. That is $| \text{Riem}(\cdot, t) | \leq 4Q$ on $s_0 B_{\frac{1}{M_0} A Q^{-\frac{1}{2}} (\bar{y}_0)}$ for all $s_0 - M_0 Q^{-1} \leq t \leq s_0$. Furthermore, for such $x$ and $t$ we have $x \in t B_{\frac{1}{M_0} A Q^{-\frac{1}{2}} + d_{s_0} (\bar{y}_0)} (x_0)$, as we just showed, and using (2.1), we see that

$$ \frac{1}{2} A Q^{-\frac{1}{2}} + d_{s_0} (\bar{y}_0) \leq \frac{1}{2} A Q^{-\frac{1}{2}} + 2A \varepsilon + (1 - \sigma) $$

$$ \leq \frac{\varepsilon}{4} + \frac{\sigma}{\delta} + (1 - \sigma) $$

$$ \leq (1 - \frac{1}{2}) $$
which gives us that \( x \in \mathcal{B}_{1-\frac{1}{10}}(x_0) \), and there we have that \( \mathcal{R} \geq -1 \). Note we have used here that \( Q \geq N_i^2 \) (follows from the inequality \( 2.2 \)) and the definition of \( A \) and \( \varepsilon \). Using the Bishop-Gromov volume comparison principle, we also see that

\[
\text{vol}(tB_s(x)) \geq \tilde{v}(\sigma, v_0)s^n
\]

for such \( x \) and \( t \) and all \( s \leq \frac{\varepsilon}{10} \) in view of the fact that \( \text{vol}(tB_1(x_0)) \geq v_0 \). Taking \( x = x_0 \in \mathcal{B}_{\mathcal{R}^{-\frac{1}{10}}}A\mathcal{R}^{-\frac{1}{10}}(x_0) \) we get

\[
\text{vol}(tB_s(x_0)) \geq \tilde{v}(\sigma, v_0)s^n
\]

for all \( s \leq \frac{\varepsilon}{10} \) and \( \tilde{s}_0 - M_0Q^{-1} \leq t \leq \tilde{s}_0 \).

Defining \( \tilde{z}_i := \bar{y}_0, \tilde{t}_i := \bar{s}_0 \) and substituting \( \alpha = N_i \) and so on back into the above, we get

\[
\text{vol}(tB_{s}(\tilde{z}_i)) \geq \tilde{v}(\sigma, v_0)s^n
\]

for all \( s \leq \frac{\varepsilon}{10} \), \( \tilde{t}_i - M_0Q^{-1} \leq t \leq \tilde{t}_i \).

Rescaling the solutions by \( Q_i \) and shifting time by \( t_i \) we get solutions to Ricci flow with

\[
\text{vol}(tB_{s}(\bar{z}_i)) \geq \tilde{v}(\sigma, v_0)s^n
\]

for all \( s \leq \frac{\varepsilon}{10} \), \( \tilde{t}_i - M_0Q^{-1} \leq t \leq \tilde{t}_i \).

\[
\text{vol}(tB_{s}(\bar{z}_i)) \geq \tilde{v}(\sigma, v_0)s^n
\]

for all \( s \leq \frac{\varepsilon}{10} \), \( \tilde{t}_i - M_0Q^{-1} \leq t \leq \tilde{t}_i \).

\[
\text{vol}(tB_{s}(\bar{z}_i)) \geq \tilde{v}(\sigma, v_0)s^n
\]

for all \( s \leq \frac{\varepsilon}{10} \), \( \tilde{t}_i - M_0Q^{-1} \leq t \leq \tilde{t}_i \).

Taking the pointed limit of a subsequence as \( i \to \infty \) (see Theorem 1.2 of \([9]\) ) of \((M_i, g_i(t), \bar{z}_i)_{t \in (-M_0, 0]}\), we see that the limiting solution, denoted by \((\Omega, \rho_0, \tilde{h}(t))_{t \in (-M_0, 0]}\), has non-negative curvature operator, is complete, has bounded curvature \( |\text{Riem}(x, t)| \leq 4 \) at all times and points in the limiting manifold, has \( |\text{Riem}(\rho_0, 0)| = 1 \) and \( \lim_{r \to \infty} \frac{\text{vol}(tB_{s}(\rho_0))}{s^n} \geq \tilde{v} > 0 \) (note: \( \tilde{v} = \hat{v}(\sigma, v_0, n) > 0 \) does NOT depend on \( M_0 \)). We repeat the procedure for larger and larger \( M_0, M_0 \to \infty \) to obtain, after taking a pointed limit of a subsequence, a solution \((\Omega, \tilde{\rho}_0, \tilde{h}(t))_{t \in (-M_0, 0]}\), which has non-negative curvature.

\[
\text{vol}(tB_{s}(\bar{z}_i)) \geq \tilde{v}(\sigma, v_0)s^n
\]

for all \( s \leq \frac{\varepsilon}{10} \), \( \tilde{t}_i - M_0Q^{-1} \leq t \leq \tilde{t}_i \).
Then, there exists a different constant. Also, the bound from below on \( R \) is now \( R \geq -V \). Otherwise the proof remains unchanged. □

3. A CUT-OFF FUNCTION AND IT’S PROPERTIES

In the next section we use a cut off function with certain nice properties. We define this cut-off function here and examine some of it’s properties.

**Lemma 3.1.** There exists a smooth cut off function \( \varphi : \mathbb{R} \to \mathbb{R}^+ \) with the following properties.

\[
\begin{align*}
(i) & \quad 0 \leq \varphi \leq 1, \\
(ii) & \quad \varphi(r) = 1 \text{ for all } r \leq 1, \quad \varphi(r) = 0 \text{ for all } r \geq 2, \\
(iii) & \quad \varphi \text{ is decreasing: } \varphi' \leq 0, \\
(iv) & \quad \varphi'' \geq -200\varphi, \\
(v) & \quad |\varphi|^2 \leq 200\varphi^2 \text{ for some constant } 0 < C < \infty.
\end{align*}
\]

**Proof.** To construct a cut-off function with the properties (i)-(iv) stated above is standard. In fact we obtain \( \varphi'' \geq -10\varphi \) and \( (\varphi')^2 \leq 10\varphi \) in place of (iv). Define \( \psi = \varphi^4 \). Then \( \psi \) satisfies properties (i)-(iii) trivially, and \( \psi' = (\varphi^4)' = 4\varphi^3\varphi' \). Then \( (\psi')^2 \leq 16\varphi^6(\varphi')^2 \leq 160\varphi^7 = 160(\varphi^4)^2 = 160\varphi^4 \leq 160\varphi \varphi^2 = 160\varphi^3 \) in view of the fact that \( \varphi \leq 1 \). Also \( \psi'' = (\varphi^4)'' = (4\varphi^3\varphi')' = 16\varphi^2|\varphi'|^2 + 4\varphi^3\varphi'' \geq -40(\varphi^3)\varphi = -40\psi \). Hence (iv) and (v) are also satisfied. □

**Lemma 3.2.** Let \( A, B > 0 \). We may choose a cut-off function satisfying

\[
\begin{align*}
(i) & \quad 0 \leq \varphi \leq 1 \\
(ii) & \quad \varphi(r) = 1 \text{ for all } r \leq A, \quad \varphi(r) = 0 \text{ for all } r \geq A + B \\
(iii) & \quad \varphi \text{ is decreasing: } \varphi' \leq 0, \\
(iv) & \quad \varphi' \geq -k_0(A, B)\varphi, \quad (\varphi')^2 \leq k_0(A, B)\varphi, \\
(v) & \quad |\varphi|^2 \leq k_0(A, B)\varphi^2 \text{ for some constant } 0 < k_0(A, B) < \infty.
\end{align*}
\]

**Proof.** By shifting and scaling: Define \( \tilde{\varphi}(r) := \varphi(\frac{r+B-A}{B}) \) where \( \varphi \) is the function appearing in the above Lemma. Then \( \tilde{\varphi} \) has all of the desired properties □
Construction of a cut-off function on a Riemannian manifold which is evolving by Ricci flow.

Now we construct a cut-off function similar to that constructed by G. Perelman (see proof of Theorem 10.1 in [12]) and similar to that used by B.-L. Chen in [4]. Assume that we have a solution to Ricci flow \((M, g(t))_{t \in [0,T]}\). We do not assume that the curvature is bounded uniformly on some region for all \(t \in [0,T]\) as in the argument of B.-L.-Chen in the proof of proposition 2.1 in [4]. Instead we assume a uniform estimate of the form

\[
(c) \quad \text{Riem} \leq \frac{\varphi}{\sigma} \quad \text{on} \quad \mathcal{B}_{\frac{1}{100}}(x_0) \quad \text{for} \quad t \in [0, S)
\]

for some \(S \leq \frac{1}{100}, \quad S \leq T\). Note: The radius of the ball \(\frac{1}{100}\) is chosen for convenience. If we replace \(\frac{1}{100}\) by \(\sigma > 0\), then all constants occurring in this section also depend on \(\sigma\). This estimate combined with Lemma 8.3 of [12] guarantees that the cut-off function we construct will satisfy estimates which are sufficient for the arguments in the following section.

Let \(\varphi : [0, \infty) \rightarrow \mathbb{R}_+^*\) be one of the cut-off functions defined above with \(A \leq 1\).

Let \(r_0(t) = \sqrt{t}\) and \(K(t) = \frac{\sigma}{\varphi}\). Then \(|\text{Ric}(x, t)| \leq (n - 1)K\) whenever \(d(x, x_0, t) \leq r_0(t)\) for all \(t \leq S\) in view of \((c)\). Hence, using Lemma 8.3 of [12], we have

\[
\frac{\partial}{\partial t} d_t(x) - \Delta d_t(x) \geq -(n - 1)\left(\frac{2}{3}K r_0 + r_0^{-1}\right) = -(n - 1)\left(\frac{2}{3}c + 1\right) \sqrt{t},
\]

in view of condition \((c)\), where this inequality is valid for points \((x, t)\) where \(d_t(x) = d(x, x_0, t)\) is differentiable and \(t \leq S\), and \(d(x, x_0, t) \geq r_0(t) = \sqrt{t}\). Note that for \(t \leq \frac{A^2}{100}\), the last condition is satisfied for all \(x\) outside of \(\mathcal{B}_{\frac{1}{100}}(x_0)\). The right hand side is integrable.

That is,

\[
\frac{\partial}{\partial t} (d_t(x) + 4(n - 1)(c_0 + 1)\sqrt{t}) - \Delta (d_t(x) + 4(n - 1)(c_0 + 1)\sqrt{t}) \geq \frac{2(n - 1)(c_0 + 1)}{\sqrt{t}} > 0.
\]

for such points. Let us denote the constant appearing here as \(m_0 = m_0(c_0, n) = (n - 1)(c_0 + 1)\). Let \(k(x,t) = \varphi(d_t(x) + 4m_0\sqrt{t})\). Using the above information, we obtain the following evolution inequality for \(k\):

\[
(\frac{\partial}{\partial t} - \Delta)k = \varphi' \cdot (\frac{\partial}{\partial t} - \Delta)(d_t + 4m_0\sqrt{t})(x,t) - (\varphi'')^* \leq \varphi' \frac{m_0}{2\sqrt{t}} + k_0 \varphi \leq k_0 \varphi
\]

where \(k_0 = k_0(A,B)\) comes from the above Lemma, Lemma 3.2. Note that \(\varphi(d_t(x) + 4m_0\sqrt{t}) = 1\) for all \(x \in \mathcal{B}_{\frac{1}{100}}(x_0)\) and \(t \leq \frac{A^2}{100m_0^2}\) and hence \(\varphi' = 0\) for all points \(x\) inside \(\mathcal{B}_{\frac{1}{100}}(x_0)\) as long as \(t \leq \frac{A^2}{100m_0^2}\), and hence the above evolution inequality (3.1) is valid for all \(x \in M\) and all \(t \leq \frac{A^2}{100m_0^2}\) (we assume that \(m_0 >> 1\)) as long \((x,t)\) is a point where \(d_t(x) = d(x_0, x, t)\) is differentiable. Hence
$h(x, t) := e^{-2kt}k(x, t)$ satisfies

$$\frac{\partial}{\partial t} - \Delta h(x, t) \leq 0$$

for all $x$ and all $t \leq \frac{A^2}{100m_0^2}$, as long as $d(x_0, \cdot, \cdot)$ is differentiable there. We collect the definitions and observations made above in the following.

**Proposition 3.3.** Let $(M, g(t))_{t \in [0, T)}$ be a smooth complete solution to Ricci flow and $\varphi$ be one of the functions appearing in Lemma 3.2 with $A \leq 100$. We assume that

$$(c) \ |\text{Riem}| \leq \frac{m}{\sigma} \text{ on } tB_{\frac{1}{4}}(x_0) \text{ for } t \in [0, S)$$

for some $S \leq \frac{1}{100}$, $S \leq T$. $h : M \to \mathbb{R}$ is the function $h(x, t) := e^{-2kt}((\varphi(d_t(x)) + 4m_0 \sqrt{t}))$ where $d_t(x) := d_t(x_0, x)$, and $x_0$ is a fixed point in $M$ and $m_0 = m_0(c_0, n) = (n - 1)(c_0 + 1)$, $k_0 = k_0(A, \beta)$. For $t \leq \frac{A^2}{100m_0^2}$, we have

$$\frac{\partial}{\partial t} - \Delta h(x, t) \leq 0,$$

as long as $d(x_0, \cdot, \cdot)$ is differentiable at $(x, t)$. $h \equiv 0$ for all $d_t(x) \geq (A + \beta)$ and $h \equiv e^{-2kt}$ for all $d_t(x) \leq A - 4m_0 \sqrt{t}$ and $h(x, t) \leq e^{-2kt} \leq 1$.

4. A LOCAL RESULT IN TWO DIMENSIONS

In this section we restrict ourselves to the two dimensional case. We consider a ball of radius $r$ in a two dimensional manifold which has curvature operator and volume bounded from below by known constants. We show that a ball of smaller radius will smooth out quickly at least for a short time. The rate of this smoothing depends on the bounds from below and $r$.

**Theorem 4.1.** Let $(M^2, g(t))_{t \in [0, T)}$ be a smooth complete solution to Ricci flow and let $x_0 \in M$, $N, v_0, r > 0$ and $1 > \sigma, \alpha > 0$. Assume that

- $\text{vol}^0 B_{r}(x_0) \geq v_0 r^2$ and
- $R(g(0)) \geq -\frac{N}{\alpha} \text{ on } B_{r}(x_0)$,

and $1 > \sigma > 0$. Then there exists a $\tilde{v}_0 = \tilde{v}_0(v_0, \sigma, N, \alpha) > 0$ and a $\delta_0 = \delta_0(v_0, \sigma, N, \alpha) > 0$ such that

- $\text{vol}(tB_{r}(x_0)) \geq \tilde{v}_0 r^2$
- $R(g(t)) \geq -\frac{(N+\alpha)}{\sigma t} \text{ on } tB_{(1-\sigma)r}(x_0)$,
- $|R(g(t))| \leq \frac{1}{\delta_0 t} \text{ on } tB_{(1-\sigma)r}(x_0)$

as long as $t \leq r^2(\delta_0)^2$ and $t \in [0, T)$.

**Remark 4.2.** In Theorem 3.1 of [3] B.-L. Chen proved the following similar result: if we assume the above conditions with $r = 1$ but replace the lower bound on the scalar curvature by the condition that $|R(g_0)| \leq 1$ on $0B_2(x_0)$, then $|R(g(t))| \leq 2$ for all $t \in [0, T(n, v_0))$ on a smaller ball. This a version of G.Perelman’s second Pseudolocality result, Theorem 10.3 of [12] in dimension two. Note that in this case, the curvature bound and volume bound from below guarantee that balls of
radius \( r \leq R = R(n, \varepsilon, v_0) \) which are sufficiently small satisfy the almost euclidean condition \( \text{vol}(\mathcal{B}_r(x_0)) \geq (1 - \varepsilon)r^2 \).

**Proof.** By scaling, it suffices to prove the theorem for \( r = 1 \). W.l.o.g. \( \sigma > \frac{1}{3} \). By the Bishop-Gromov volume comparison principle we have \( \text{vol}(\mathcal{B}_s(x_0)) \geq c(N, v_0)s^n \) for all \( s \leq 1 \). In particular \( \text{vol}(\mathcal{B}_{\frac{1}{1000}}(x_0)) \geq v_0(N, v_0) > 0 \). For some maximal time interval \( [0, S_{\text{max}}) \), \( 0 < S_{\text{max}} \leq T \) we have (due to smoothness) that

\[
\begin{align*}
(a) & \quad R \geq -(N + 2\alpha) \quad \text{on } tB_{1-\sigma}(x_0) \\
(b) & \quad \text{vol}(tB_{1-\sigma}(x_0)) \geq \frac{v_0(N, v_0)}{2^\alpha} > 0 \text{ trivially.}
\end{align*}
\]

Note that \( b) \implies \bar{b}) : \text{vol}(tB_1(x_0)) \geq \frac{v_0(N, v_0)}{2} > 0 \text{ trivially.}

Our aim is to obtain an estimate from below for the time \( S_{\text{max}} \) which only depends on \( N, v_0, \sigma, \alpha \) \( (n = 2 \text{ is fixed here}). \) According to Theorem \[2.1\] above, we have that

\[
(c) \quad |R| \leq \frac{c_0(v_0 N, \sigma, \alpha)}{2} \text{ on } tB_{1-\sigma}(x_0)
\]

for \( t \in [0, S_{\text{max}}) \cap [0, S(N, v_0, \sigma, \alpha)) =: [0, S(N, v_0, \sigma, \alpha)) \)

for some constant \( c_0 = c_0(N, v_0, \sigma, \alpha) = c_0(N, v_0, \sigma, \alpha) \). In the rest of the proof we often shorten time intervals \( [0, S(N, v_0, \sigma, \alpha)) \) to \( [0, S(N, v_0, \sigma, \alpha)) \) where \( 0 < S(N, v_0, \sigma, \alpha) < S(N, v_0, \sigma, \alpha) \). We will denote \( S(N, v_0, \sigma, \alpha) \) also by \( S(N, v_0, \sigma, \alpha) \).

**Claim (i)** The scalar curvature is bounded from below by \( -(N + \alpha) \) on \( tB_{1-\sigma}(x_0) \) for \( t \leq S \) where \( S = S(v_0, \sigma, \alpha, N) > 0 \), as long as \( t \leq S_{\text{max}} \). That is, \( (a) \) is not violated for \( t \leq S \) as long as \( (b) \) still holds.

**Proof of Claim (i):** We modify the argument of B.-L. Chen given in the proof of Theorem 3.6 in [4]. Let \( f := hR \) where we have chosen \( \varphi \) in the definition of \( h \) of Proposition \[3.3\] to be a smooth function with \( \varphi(r) = 0 \) for \( r \geq (1 - \frac{\alpha}{2}) \), and \( \varphi(r) = 1 \) for \( r \leq (1 - \frac{\alpha}{2}) \). Note that \( m_0 = m_0(c_0, n) = m_0(c_0(v_0, N, \sigma, \alpha), 2) = m_0(v_0, \sigma, \alpha, N) \) in this case, where \( m_0 \) is the constant appearing in Proposition \[3.3\]. In the following we assume without loss of generality, that \( t \leq \frac{\alpha^2}{100m_0^2} \), so that Proposition \[3.3\] is valid.

Using the evolution inequality for \( h \) and the evolution equation for \( R \) we see that at any point \( (x, t) \) where \( R(x, t) < 0 \) and \( d(x_0, \cdot, \cdot) : M \times [0, T) \to \mathbb{R} \) is differentiable we have

\[
\begin{align*}
\frac{\partial}{\partial t}(hR + \sqrt{t}) - \Delta(hR + \sqrt{t}) &= R(\frac{\partial}{\partial t} - \Delta)(h) + h(\frac{\partial}{\partial t} - \Delta)(R) - 2g(\nabla h, \nabla R) + \frac{1}{2\sqrt{t}} \\
&\geq 2|h| \text{Ric} |^2 - 2g(\nabla h, \nabla R) + \frac{1}{2\sqrt{t}}.
\end{align*}
\]

(4.1)

If \( (x, t) \) is a first time and point where \( (h(x, t)R(x, t) + \sqrt{t}) = -(N + \frac{\alpha}{2}) \) then the gradient term at \( (x, t) \) can be estimated as follows.

\[
-2g(\nabla R, \nabla h) = -2h^{-\frac{\alpha}{2}} g(\nabla (hR), \nabla h) + 2R \frac{|\nabla h|^2}{h}
\]

\[
= 2R \frac{|\nabla h|^2}{h}
\]
\[ \geq -\left(\frac{R}{4} \right)^2 h - \left(\frac{4}{h^3}\right) |\nabla h|^4 \]
\[ \geq -\left(\frac{R}{4} \right)^2 h - 2C(\sigma) \]

where in the last line we have used that \(|\nabla d| = 1\) and \(|\varphi'|^2 \leq C(A,B)\varphi^3\) with \(B = \frac{1}{4}, A = (1 - \frac{1}{4})\). Now using this inequality in (4.1) we get
\[
\frac{\partial}{\partial t} (hR + \sqrt{t}) - \Delta (hR + \sqrt{t}) \geq 2h|\mathsf{Ric}|^2 - \left(\frac{R}{n}\right)^2 h - 2C + \frac{1}{2\sqrt{t}}
\]
at the point \((x,t)\) in question, since \((h(x,t)R(x,t) + \sqrt{t}) = -(N + \frac{4}{3})\) guarantees that \(R(x,t) < 0\), as long as \(d(x_0, \cdot, \cdot)\) is differentiable at \((x,t)\) and \(t \leq S(v_0, N, \sigma, \alpha)\).

Hence, in view of the maximum principle, we see that \(hR + \sqrt{t} \geq 0\) for all \(t \leq S(N, \sigma, v_0, \alpha)\) as long as \(t \leq S_{\max}\) (for the case that \((x,t)\) is not a point where \(d\) is differentiable, then the argument is still valid, as we explain in Claim (iii) at the end of the proof). In particular, this shows that \(R \geq -(N + \frac{4}{3})\) for \(x \in \{B_{1-\sigma}(x_0)\}\) as long as \(t \leq S(N, \sigma, v_0, \alpha)\) (possibly a smaller \(S\)) and \(t \leq S_{\max}\), in view of the definition \(h(x,t) := e^{-2\alpha t}(\varphi(d(x) + 8m_0\sqrt{t}))\), which is close as we like to one on \(tB_{1-\sigma}(x_0)\) for \(t \leq S(N, \sigma, v_0, \alpha)\) small enough.

This finishes the proof of the Claim (i).

Claim (ii)

The volume condition (b) is not violated for a well defined time interval, as long as (a) holds. Let \(x, y \in tB_{\frac{1}{100}}(x_0)\) then \(d(x,y,t) \leq d(x, x_0, t) + d(y, x_0, t) \leq \frac{100}{100}\) and hence any shortest geodesic between \(x\) and \(y\) must lie in \(tB_{\frac{1}{100}}(x_0)\) (proof: If it didn’t, smooth out the union of the two radial curves (measured with respect to \(g(t)\)) going from \(x\) to \(x_0\) and then back to \(y\) of length \(\frac{200}{100}\). This would result in a curve of length less then \(\frac{100}{100}\). Any curve which starts in \(tB_{\frac{1}{100}}(x_0)\), reaches \(\partial tB_{\frac{1}{100}}(x_0)\) and finishes in \(tB_{\frac{1}{100}}(x_0)\) must have length larger than or equal to \(\frac{1}{100}\). Hence, if a minimising Geodesic between \(x\) and \(y\) leaves \(tB_{\frac{1}{100}}(x_0)\) we obtain a contradiction). Hence using the estimate of Hamilton (see Theorem 17.2 of [7] and the Editors’ comment thereon in [2] or, alternatively, see Appendix B in [16] ) and the fact that (a) and (c) hold on \(tB_{\frac{1}{100}}(x_0)\), we get
\[
-(N + 2\alpha)d(y, x, t) \geq \frac{\partial}{\partial t} d(y, x, t) \geq -\frac{c_1(c_0)}{\sqrt{t}}
\]
for all \(s \leq t \leq \min(S_{\max}, S(N, v_0, \sigma, \alpha))\) \(x, y \in tB_{\frac{1}{100}}(x_0)\),

where \(r \geq \frac{\partial}{\partial t} d \geq m\) is meant in the sense of forward difference quotients (see Theorem 17.2 of [7] ). Note \(c_1(c_0) = c_1(v_0, \sigma, N, \alpha)\). Integrating in time we get
\[
e^{-\left(N+\alpha\right)(t-s)}d(x_0, x, s) \geq d(x, y, t) \geq d(x, y, s) - 2c_1\sqrt{t}
\]
for all \(s \leq t \leq \min(S_{\max}, S(N, v_0, \sigma, \alpha))\) \(x, y \in tB_{\frac{1}{100}}(x_0)\).

Arguing as in Corollary 6.2 of [16], we see that \(\text{vol}(tB_{\frac{1}{100}}(x_0)) \geq \frac{1}{4}v_0\) for all \(t \leq S(N, v_0, \sigma, \alpha)\) (we have possibly decreased \(S\) once again here) as long as \(t \leq S_{\max}\). That is the volume condition is not violated for the time interval \([0, S(N, \sigma, v_0, \alpha)]\) (as long as \(t \leq S_{\max}\)).
In particular we see that the second condition (b) will not be violated for some well defined time interval \([0, S(N, \sigma, v_0, \alpha)]\) (as long as \(t \leq S_{\text{max}}\)). This finishes the proof of Claim (ii) and of the theorem if we accept Claim (iii) below.

Claim (iii) If \(d_s(\cdot)\) is not differentiable at \(x \in M\) then we use the trick of E. Calabi (\([\Pi]\)) as follows. Let \(y_0\) be a point on a shortest geodesic between \(x_0\) and \(x\) which is very close to \(x_0\). By smoothness, we can find a small open neighbourhood \(P\) of \((x, t)\) in \(M \times (0, T)\) such that \(d(y_0, \cdot, s)\) is differentiable at \(y\) for each \((y, s) \in P\). We define \(\overset{\sim}{d}_s(y) = d(x_0, y_0, s) + d(y_0, y, s)\). Then \(\overset{\sim}{d}_s(\cdot)\) is differentiable at \(y\) for all \((y, s) \in P\). Furthermore, \(d_s(\cdot) \geq \overset{\sim}{d}_s(\cdot)\) for all \((y, s) \in P\) due to the triangle inequality. Since \(\varphi\) is non-increasing, we therefore have \(\overset{\sim}{d}'_s(y) = \frac{\partial}{\partial s} d(x_0, y_0, s) + \frac{\partial}{\partial s} d(y_0, y, s)\) very close to \(d(x_0, y_0, s)\). Remembering that \(\overset{\sim}{d}'_s(y) \leq \varphi(d + m_0\sqrt{s})\) is as small as we like. Hence, there is no such \((y, s) \in P\) so that \(\overset{\sim}{d}'_s(y) = \varphi(d + m_0\sqrt{s})\) if \((y, s) \in P\) and per definition \(\overset{\sim}{d}'_s(y) = k\) in \(P\) and per definition \(k(x, t) = k(x, t)\) if \((x, t)\) is the point given at the beginning of the claim. Also, if we pick \(y_0\) very close to \(x_0\) we still have

\[
\frac{\partial}{\partial t} (\overset{\sim}{d}_s(x) + m_0\sqrt{t}) - \Delta (\overset{\sim}{d}_s(x) + m_0\sqrt{t}) \geq \frac{m_0}{2\sqrt{t}} > 0.
\]

where here \(m_0\) is the constant appearing in Proposition 3.3. Hence we may argue with \(h(y, s) := e^{-2k_0s}(\overset{\sim}{d}_s(y, s))\) everywhere above. If for example \((x, t)\) is a first time and point where \((h(x, t)R(x, t) + \sqrt{t}) = -(N + 4\pi)\), then \((x, t)\) is a first time and point for which the function \((h(x, t)R(x, t) + \sqrt{t}) = -(N + 4\pi)\) on the set \(P\) and hence we may argue as above with \((h(x, t)R(x, t) + \sqrt{t})\) replaced by \((h(x, t)R(x, t) + \sqrt{t})\) ( note that without loss of generality \(R < 0\) on \(P\), since \(R(x, t) < 0\) and hence \(h(y, s)R(y, s) + \sqrt{t} \geq h(y, s)R(y, s) + \sqrt{t} \geq -(N + \frac{4\pi}{\alpha})\) on \(P \cap \{ (y, s) | s \leq t \})\). We must also consider the case that \(\overset{\sim}{d}'(1, x)\) is not differentiable in time at the time \(t\) we are considering. In this case, all the estimates are still valid if we understand the inequalities \(\frac{\partial}{\partial t} d_i(x) \geq m\) or \(\frac{\partial}{\partial t} d_i(x) \leq m\) in the sense of forward difference quotients: see \([8]\). At times \(s < t\) very close to \(t\) ( \((x, t)\) as above ) we have ( due to smoothness )

\[
\Delta (h(x, s)R(x, s) + \sqrt{s}) \geq -e, \quad |\nabla (h(x, s)R(x, s))| \leq e
\]

where \(e\) is as small as we like. remembering that \(h(x, t) > 0\), we see that the term \(-\frac{2}{h(x, s)^2}g(\nabla (R\overset{\sim}{h}), \nabla \overset{\sim}{h})(x, s)\) which is zero at \((x, t)\) is also as small as we like for \(s < t\) very close to \(t\). Hence, examining the proof of Claim (i) again, we see that

\[
\frac{\partial}{\partial t} (h(x, s)R(x, s) + \sqrt{s}) \geq 0
\]

in the sense of forward difference quotients for \(s < t\) close to \(t\).

In particular, using Lemma 3.1 in \([8]\), we see that \(h(x, t)R(x, t) + \sqrt{t} > -(N + \frac{4\pi}{\alpha})\), which is a contradiction. Hence, there is no such \((x, t)\).

\[\square\]

5. A LOCAL RESULT IN THREE DIMENSIONS

In this section we restrict ourselves to the three dimensional case. We first consider a ball of radius 1 in a three dimensional manifold which has curvature operator and
volume bounded from below by known constants at time zero. For later times we assume a bound on the curvature of the form \( |\text{Riem}(g(t))| \leq \frac{N}{t} \) on the time \( t \) ball of radius 1, where \( N \) depends on the curvature bound from below. We show that the curvature cannot become too negative too quickly on smaller balls.

**Theorem 5.1.** Let \( v_0 > 0 \) and \( N \geq 1 \) be given. Let \((M^3, (g(t))_{t \in [0, T]}\) be a smooth complete solution to Ricci flow with \( T \leq 1 \), and let \( x_0 \in M \) be a point such that

- \( \text{vol}(0)B_1(x_0) \geq v_0 \) and
- \( R(g(0)) \geq -\frac{1}{10N} \) on \( 0B_1(x_0) \),
- \( |\text{Riem}(g(t))|(x) \leq \frac{N}{t^2} \) for all \( t \in (0, T) \) for all \( x \in tB_1(x_0) \)

Then, for all \( 1 > \sigma > 0 \) there exists \( \delta = \delta(v_0, N, \sigma) > 0 \) and \( \tilde{v}_0 = \tilde{v}_0(N, v_0, \sigma) > 0 \) such that

- \( \text{vol}(tB_1(x_0)) \geq \tilde{v}_0 \)
- \( R(g(t)) \geq -1 \) on \( tB_{(1-\sigma)}(x_0) \).

as long as \( t \leq (\delta_0)^2 \) and \( t \in [0, T) \).

**Proof.** By the Bishop-Gromov volume comparison principle we have \( \text{vol}(0)B_1(x_0) \geq c(N, v_0)s^n \) for all \( s \leq 1 \). In particular \( \text{vol}(0)B_{\frac{1}{1000}}(x_0) \geq \tilde{v}_0(N, v_0) > 0 \).

For some maximal time interval \([0, S_{\text{max}}], S_{\text{max}} \leq T\) we have (due to smoothness) that

(a) \( R \geq -1 \) on \( tB_{(1-\sigma)}(x_0) \)
(b) \( \text{vol}(tB_{\frac{1}{1000}}(x_0)) \geq \frac{\tilde{v}_0}{2} \).

Our convenience we denote the constant \( \frac{1}{10N} \) by \( \varepsilon_0 := \frac{1}{10N} \).

**Claim (i)** The scalar curvature is bounded from below by \(-2\varepsilon_0 \) on \( tB_{(1-\frac{\sigma}{N})}(x_0) \) for \( t \leq S \) where \( S = S(v_0, \sigma, N) > 0 \), as long as \( t \leq S_{\text{max}} \).

proof of Claim (i): This is proved using the same argument as that given in the proof of Claim (i) in the proof of Theorem 4.1 above.

This finishes the proof of the **Claim (i)**.

For convenience we introduce \( \alpha < \beta < \gamma \) to be the eigenvalues of \( R \) as in Hamilton. Then \( R = \alpha + \beta + \gamma \) and \( |\text{Ric}|^2 = \frac{1}{2}(\alpha^2 + \beta^2 + \gamma^2 + \alpha\beta + \alpha\gamma + \beta\gamma) \).

**Claim (ii)** \( \alpha + 2R \geq -2\varepsilon_0 \) on \( tB_{(1-\frac{\sigma}{N})}(x_0) \) as long as \( t \leq S_{\text{max}} \) and \( t \leq S(N, \sigma, v_0) \).

**Remark:** A local result of this type was first shown by B.L-Chen under the extra assumption that the Ricci curvature remains bounded on \( tB_1(x_0) \): see Proposition 2.2 in [4] (for a related result see Lemma 4.1 of [14]).

proof of Claim (ii): Let \( L(V, V) = \text{Riem}(V, V) + 2R \text{Id}(V, V), P(V, V) = hL(V, V) + \varepsilon_0(1 + \frac{h}{\varepsilon_0}) \) (\( = h(\text{Riem}(V, V) + 2R \text{Id}(V, V)) + \varepsilon \)) for 2-Forms, where we have chosen \( \varepsilon \) to satisfy \( \varphi(r) = 1 \) for all \( r \leq (1 - \frac{\varepsilon}{\varepsilon_0}) \) and \( \varphi(r) = 0 \) for all \( r \geq (1 - \frac{\varepsilon}{\varepsilon_0}) \) in the definition of \( h \) in Proposition 3.3. We shall only be concerned with points where \( h \neq 0 \), and so
we may use freely the results of Claim (i) for \( t \leq S(N, v_0, \sigma) \). We do so, sometimes without further comment. \( k = k(N, \sigma, v_0) \) is a large constant which we shall choose later in the proof.

Also we have introduced the notation \( \varepsilon = \varepsilon(t) = \varepsilon_0(1 + \frac{k}{\varepsilon_0}) \). For the time intervals we are considering, we have \( \varepsilon_0 \leq \varepsilon(t) \leq 2\varepsilon_0 \), as we shall assume that \( t \leq \frac{\varepsilon}{20} \).

In all of the following arguments (also for the proofs of claims (iii),(iv) and (v)) we shall be calculating the evolution of the curvature in the setting of \( S \). That is, we are using the trick of K.Uhlenbeck. In particular, the metric \( G_{ab}(x) := g_{ij}(x, t)u_a^i(x, t)u_b^j(x, t) \) is the pullback of the metric \( g(x, t) \), and it is time independent: \( \frac{\partial}{\partial t} G_{ab}(x) = 0 \). Id here is the operator on two forms given by \((\text{Id}(V, W)) := G^{ab}G^{cd}V_{ac}W_{bd} \). In particular \( \frac{\partial}{\partial t} (\text{Id}(V, W)) = 0 \) for a time independent vector field. The connection, \( \nabla \), is the pullback connection of \( g(t)\nabla \). We still have \( \Delta f(x) = \Delta g(t)f(x) \) for smooth functions \( f : M \to \mathbb{R} \) (the left hand side is the laplacian with respect to the pullback connection and the right hand side is the laplacian with respect to \( g(t) \)). We also have \( \nabla \text{Id} = 0 \). See [8] for details. Once again we consider only \( t \leq \frac{A^2}{100m_0} = \bar{m}_0(\sigma, N, v_0) \) so that Proposition 3.3 is applicable. Then \( P(V, V) = h(\alpha + 2R) + \varepsilon \) for a 2-form \( V \) with length one which minimises \( P \) at any point in space and time. We first estimate the reaction term coming from the evolution equation for \( L \). At the end of the proof we explain how to deal with the reaction diffusion equation for \( P \) (in particular the gradient terms). In Lemma 4.1 of [15] it is shown (with \( \varepsilon := 1 \) there) that the reaction equation for \( L = \alpha + 2R \) is given by

\[
\frac{\partial}{\partial t}(\alpha + 2R) = \alpha^2 + \beta\gamma + 2(\alpha^2 + \beta^2 + \gamma^2 + \alpha\beta + \alpha\gamma + \beta\gamma)
\]

In case \( \beta, \gamma \geq 0 \), or \( \beta, \gamma \leq 0 \) (which implies \( \beta\gamma \geq 0 \)) we get

\[
\frac{\partial}{\partial t}(\alpha + 2R) \geq 3\alpha^2 + 3\beta^2 + 3\gamma^2 + 2\alpha\beta + 2\alpha\gamma + 2\beta\gamma + 2\alpha^2 + 2\beta^2 + 2\gamma^2 \geq \frac{1}{1000}(\alpha + 2R)^2
\]

in view of Young’s inequality. In case \( \beta \leq 0, \gamma \geq 0 \) (which implies \( \alpha\beta \geq 0 \)) we get by applying Young’s inequality a number of times

\[
\frac{\partial}{\partial t}(\alpha + 2R) = \alpha^2 + \beta\gamma + 2(\alpha^2 + \beta^2 + \gamma^2 + \alpha\beta + \alpha\gamma + \beta\gamma) \\
\geq 3\alpha^2 + 2\beta\gamma + 2\alpha\gamma + 2\beta^2 + 2\gamma^2 \\
\geq 2\alpha^2 + 2\beta\gamma + 2\beta^2 + 2\gamma^2 \\
\geq \frac{1}{3}(\alpha^2 + \beta^2 + \gamma^2) \\
\geq \frac{1}{1000}(\alpha + 2R)^2
\]

At a first time and point \((y, s)\) where \( h(\alpha + 2R) = -\varepsilon \), we must clearly have that \( \alpha < 0 \) (otherwise \( -\varepsilon = h(\alpha + 2R) \geq 0 \) which is a contradiction). Let \( V \) be a
local smoothing for the ricci flow in dimensions two and three

2-form with length one such that $P(V, V) = 0$. We have

$$
\frac{\partial}{\partial t} P(V, V) \geq (\Delta P)(y, s)(V, V) - 2(g^{ij}\nabla_j h)(y, s)(\nabla_i hL)(y, s)(V, V)
+ \frac{1}{1000} h(y, s)(L(y, s)(V, V))^2 + k
$$

in view of the above reaction equation for $L$.

We estimate the gradient term in the above as follows

$$
-2g^{ij}\nabla_j h(y, s)(\nabla_i hL)(y, s)(V, V) = \frac{2}{h(y, s)} (g^{ij}\nabla_j h)(y, s)(\nabla_i hL)(y, s)(V, V)
+ 2L(y, s)(V, V) \frac{\nabla h}{h}^2 (y, s)
= 2L(y, s)(V, V) \frac{\nabla h}{h}^2 (y, s)
\geq -\frac{L^2}{2000} (y, s) h(y, s) - \frac{4000}{h^3(y, s)} \nabla h^4 (y, s)
\geq -\frac{L^2}{2000} (y, s) h(y, s) - 2\tilde{C}
$$

(5.1)

where in the last line we have once again used that $|\nabla h|^4 \leq \tilde{C}(\sigma) h^3$. Hence we obtain

$$
\frac{\partial}{\partial t} P(V, V) \geq (\Delta P)(y, s)(V, V) - 2\tilde{C} + k
$$

(5.2)

at $(y, s)$, which leads to a contradiction if $k$ is chosen appropriately (here $n = 3$).

Hence $P$ remains non-negative in the time interval considered. In particular, using the definition of $h$, we see that $h(\alpha + 2R) + \varepsilon_0(1 + \frac{k}{\varepsilon_0}) \geq 0$ implies

$$
\alpha + 2R + 2\varepsilon_0 \geq 0
$$

on $\tilde{t}B_1(x_0)$ for $t \leq S(N, v_0, \sigma)$ (possibly a smaller $S$ now) as required.

This finishes the proof of Claim (ii).

Claim (iii) The volume condition (b) is not violated for a well defined time interval $t \leq S(N, v_0, \sigma)$, as long as $t \leq S_{\text{max}}$.

The proof may be taken from Claim (ii) in the proof of Theorem 4.1 above, with two changes: we use $R \geq -1$ on in place of $R \geq -1$ and we use the assumption that $|\text{Riem}(g(t))| \leq \frac{N}{T}$ on $\tilde{t}B_1(x_0)$.

This finishes the proof of Claim (iii).

Claim (iv) The curvature condition (a) will also not be violated for a well defined time interval $[0, S(N, v_0, \sigma))$ as long as $t \leq S_{\text{max}}$. The proof of this claim is initially similar to that of Claim (i) and Claim (ii). In order to estimate the gradient term we require some different arguments.

Define $\varepsilon(t) := \varepsilon_0(\frac{1}{4} + \frac{kt}{\varepsilon_0})$. Let $Y := hR + (\frac{1}{100} + \varepsilon t) \text{Id}$ (note here: $\varepsilon t = \varepsilon(t)t$), where $h$ is a cut-off function coming from Proposition 3.3 with $\varphi(r) = 1$ for all $r \leq 1 - \frac{\sigma}{2}$ and $\varphi(r) = 0$ for all $r \geq 1 - \frac{\sigma}{2}$. This is a local version of the tensor appearing in Lemma 5.2 of [15]. $k = k(N, \sigma, v_0)$ is a large positive constant which shall be chosen later in the proof. We wish to show that $Y$ remains larger than zero for $t \leq S(N, \sigma, v_0)$ in $\tilde{t}B_1(x_0)$ as longs as $t \leq S_{\text{max}}$. Assuming we have
a first time and point \((y, s)\), \(y \in \mathcal{B}_{(1-\frac{\varepsilon}{100})}(x_0)\) and a two form \(V\) of length one where \(Y(y, s)(V, V) = 0\), then we must have \(h(y, s) > 0\), otherwise \(Y = h\alpha + (\varepsilon tR + \frac{|Ric|^2}{100}) = (\varepsilon tR + \frac{|Ric|^2}{100}) > -4\varepsilon_0 s + \frac{\varepsilon}{100} > 0\) for \(s \leq S(N, v_0, \sigma)\) small in view of Claim (i), which is a contradiction. Henceforth, we shall only be concerned with points where \(h > 0\) and so we may freely use the results of both Claims (i) and (ii) for \(t \leq S(N, v_0, \sigma)\) in view of the definition of \(\varphi\) we have chosen here. We do so, sometimes without further comment. We assume that \(\varepsilon\) is given by

\[
\begin{align*}
\frac{\partial}{\partial t} \alpha h & \geq h\alpha^2 + h\beta \gamma \\
& \geq h\alpha^2 + h\alpha \gamma \\
& = h\alpha^2 + (h\alpha + \frac{\varepsilon}{100} + \varepsilon tR)\gamma - \left(\frac{\varepsilon}{100} + \varepsilon tR\right)\gamma.
\end{align*}
\]

If \(\gamma \leq 0\) then \(0 \geq \gamma, \beta, \alpha \geq -2\varepsilon_0\) in view of Claim (i) and hence we have

\[
\begin{align*}
\frac{\partial}{\partial t} \alpha h & \geq h\alpha^2 + h\beta \gamma \\
& \geq h\alpha^2 \\
& \geq h\alpha^2 + h\alpha \gamma - 10 \\
& = h\alpha^2 + (h\alpha + \frac{\varepsilon}{100} + \varepsilon tR)\gamma - \left(\frac{\varepsilon}{100} + \varepsilon tR\right)\gamma - 10.
\end{align*}
\]

The reaction equation for \((\frac{\varepsilon}{100} + \varepsilon tR)\) Id is

\[
\frac{\partial}{\partial t} \left(\frac{\varepsilon}{100} + \varepsilon tR\right) = \frac{k}{400t^\frac{3}{2}} + \varepsilon R + 2\varepsilon t|\text{Ric}|^2 + \frac{ktR}{4t^\frac{3}{2}} \\
\geq \frac{800t^\frac{3}{2}}{t^\frac{3}{2}} + \varepsilon R + \varepsilon t(\alpha^2 + \beta^2 + \gamma^2 + \alpha \beta + \alpha \gamma + \beta \gamma),
\]

for \(t \leq S(N, v_0, \sigma)\), in view of Claim (i). Combining these three inequalities we get

\[
\begin{align*}
\frac{\partial}{\partial t} (\alpha h + (\frac{\varepsilon}{100} + \varepsilon tR)) & \geq h\alpha^2 + (h\alpha + \frac{\varepsilon}{100} + \varepsilon tR)\gamma + \varepsilon[R - 0] + \frac{k}{800t^\frac{3}{2}} \\
& \geq h\alpha^2 + (h\alpha + \frac{\varepsilon}{100} + \varepsilon tR)\gamma \\
& \geq h\alpha^2 + (h\alpha + \frac{\varepsilon}{100} + \varepsilon tR)\gamma + \varepsilon[R - 0] + \frac{k}{800t^\frac{3}{2}}.
\end{align*}
\]

(5.3)
Assuming we have a first time and point \((y,s)\) where \(Y(y,s) = 0\), \(y \in \mathcal{B}_1(x_0)\) then we must have \(h(y,s) > 0\) as we explained at the beginning of the proof of this Claim. At \((y,s)\) we have \(\alpha(y,s) = -\frac{1}{h} (s \varepsilon(s) R(y,s) + \frac{\varepsilon(s)}{100}) < \frac{1}{h} (4s \varepsilon(s) \varepsilon_0 - \frac{\varepsilon(s)}{50}) < 0\) (in view of Claim (i)) and hence the reaction equation of \(Y = h \alpha + \varepsilon t R + \frac{\varepsilon}{100}\) may be estimated by

\[
\frac{\partial}{\partial t} (ah + \left(\frac{\varepsilon}{100} + \varepsilon t R\right)) \geq ha^2 + (ha + \frac{\varepsilon}{100} + \varepsilon t R) \gamma + \varepsilon [R - \gamma \left(\frac{1}{100}\right) + \frac{k}{800t^2}]
\]

in view of the estimate \((5.3)\), where we have used that \(Y(y,s) = 0\). \([-\frac{1}{100} \gamma + R = \frac{(\alpha + \beta)}{100} + \frac{99}{100} R \geq \frac{2}{100} \alpha + \frac{4}{100} R + \frac{95}{100} R \geq \frac{2}{100} (\alpha + 2R) - \frac{95}{100} \varepsilon_0 \geq -10 \varepsilon_0\) in view of Claim (i) and (ii). Hence

\[
\frac{\partial}{\partial t} Y \geq \frac{h(y,s)}{2} \alpha^2 + \frac{k}{801t^2}.
\]

Now we examine the reaction diffusion equation. Using the estimate on the reaction equation above, we see that at time \((y,s)\) in direction \(V\) where \(Y(y,s)((V,V) = 0\)

\[
\frac{\partial}{\partial t} Y(V,V) \geq (\Delta Y)(y,s)(V,V) - 2(g^{ij} \nabla_j h)(y,s)(\nabla_i R)(y,s)(V,V)
\]

\[
+ h(y,s) \frac{1}{2} \alpha^2(y,s) + \frac{k}{801t^2}.
\]

We estimate the second term (the gradient term) of the right hand side of this inequality:

\[
-2(g^{ij} \nabla_j h)(y,s)(\nabla_i R)(y,s)(V,V)
\]

\[
= -2 \frac{\varepsilon}{h(y,s)^2} g^{ij} \nabla_j h \nabla_i [Rh(y,s)](V,V) + 2 \alpha(y,s) \frac{\nabla h^2}{h}
\]

\[
= -2 \frac{\varepsilon}{h(y,s)} g^{ij} \nabla_j h \nabla_i [Rh + \frac{\varepsilon}{100} Id] + \varepsilon t R Id](y,s)(V,V)
\]

\[
+ 2 \alpha(y,s) \frac{\nabla h^2}{h} + \frac{2 \varepsilon}{h(y,s)} g^{ij} (\nabla_i R \nabla_j h)(y,s)(V,V)
\]

\[
= 2 \alpha(y,s) \frac{\nabla h^2}{h} + \frac{2 \varepsilon}{h(y,s)} g^{ij} (\nabla_i R \nabla_j h)(y,s)(V,V)
\]

\[
\geq -\frac{1}{4} \alpha^2(y,s) h(y,s) - 2 \frac{\nabla h^4}{h^3}(y,s) - \frac{2 \varepsilon}{h(y,s)} |\nabla \text{Riem} (y,s)| \nabla h(y,s)
\]

\[
\geq -\frac{1}{4} \alpha^2(y,s) h(y,s) - C(\sigma) - \frac{2 \varepsilon}{h(y,s)} |\nabla \text{Riem} (y,s)| \nabla h(y,s),
\]

since \(|\nabla \varphi|^2 \leq \varphi^4 C(\sigma)\). Using the estimates of Shi, see [7] Theorem 3.1, and the fact that \(|\text{Riem} |(x,t) \leq \frac{N}{t^2}\), we see that \(|\nabla \text{Riem} |^2 \leq \frac{\varepsilon \varepsilon_0^3}{t^2}\) where \(\varepsilon = \hat{c}(N,\sigma)\). We explain this in more detail. \(x \in \mathcal{B}_1(x_0)\) implies \(\mathcal{B}_{\text{Riem}} \leq \mathcal{B}_1(x_0)\) for the \(x\) we are considering. We work in the ball \(\mathcal{B}_{\text{Riem}}(x)\), and we have \(|\text{Riem} (\cdot, t)| \leq \frac{N}{t^2}\) there. Scale so that \(t = 1\). Note that \(|\text{Riem}| \leq 2N\) for \(t \in \left(\frac{3}{2}, 1\right]\) after scaling, so distances change in a controlled manner near time 1. This allows one to find a parabolic region of the form \(g^{(1)}B_r(x) \times [-r^2, 1]\) for some \(r = r(\sigma, N) > 0\) close to one on which \(|\text{Riem}| \leq 2N\). Now we may use the estimates of Shi, see [7] Theorem
3.1, at \( t = 1 \), and then scale back to \( t \) (this completes the explanation of the estimate \( |\nabla \mathrm{Riem}|^2 \leq \frac{\hat{\epsilon} N^3}{\nu} \)). Using \( |\nabla \mathrm{Riem}|^2 \leq \frac{\hat{\epsilon} N^3}{\nu} \), we get
\[
\frac{2\epsilon s}{h(y, s)} |\nabla \mathrm{Riem}||y, s||\nabla h||y, s| \leq \frac{4\hat{\epsilon} N^3 \epsilon_0}{h(y, s) s^2} |\nabla h||(y, s).
\]

Notice that we must have \( h(y, s) \geq \frac{\epsilon_0}{500} \). If not, then
\[
h\alpha + (\epsilon s R + \frac{\epsilon}{100}) \text{Id} \geq -h|\nabla \mathrm{Riem}| - \frac{\epsilon_0}{400} + \frac{\epsilon_0}{200}
\geq -h|\nabla \mathrm{Riem}| + \frac{\epsilon_0}{400}
\geq \frac{N \epsilon_0}{N500} + \frac{\epsilon_0}{400}
\geq 0,
\]

(at \((y, s)\) if \( s \leq S(N, v_0, \sigma) \), which is a contradiction (here we have used again that \( |\nabla \mathrm{Riem}(g(t))| \leq \frac{\epsilon_0}{N} \) and \( \epsilon(s) s R \geq -\frac{\epsilon_0}{400} \) for \( s \leq S(N, \sigma, v_0) \) small enough in view of Claim (i)).

Hence
\[
\frac{2\epsilon s}{h(y, s)} |\nabla \mathrm{Riem}||(y, s)||\nabla h||y, s| \leq \frac{4\hat{\epsilon} N^3 \epsilon_0}{h(y, s) s^2} |\nabla h||(y, s)
\geq \frac{4\hat{\epsilon} N^3 \epsilon_0 \alpha}{h(y, s) s^2}
= \frac{4\hat{\epsilon} N^3 \epsilon_0 \alpha}{h(y, s) s^2}
= \frac{4\hat{\epsilon} N^3 \epsilon_0 \alpha}{4(500 N)^2 \hat{\epsilon} \epsilon_0}
\leq \frac{\epsilon_0 c (500 N)^2}{s^4},
\]

where we have once again used that \( |\nabla h|^4 \leq Ch^3 \).

Substituting this inequality into (5.5), we obtain
\[
(5.6) - 2g^{ij} \nabla_j h \nabla_i \mathcal{R}(y, s)(V, V) \geq \frac{1}{4} \alpha^2 (y, s) h(y, s) - C - \frac{\epsilon_0 c (500 N)^2}{s^4}.
\]

Substituting the inequality (5.6) into (5.4) we see that
\[
\frac{\partial}{\partial t} Y(y, s)(V, V) > (\Delta Y)(y, s)(V, V)
\]

if \( k = k(N, \sigma, v_0) \) is chosen large enough. This contradicts \((y, s)\) being a first time point where \( Y \) is zero. This implies that \( Y \) remains larger than zero for a well defined time interval \([0, S(N, v_0, \sigma)]\) as long as \( t \leq S_{\text{max}} \).

Using the definition of \( \varphi \) (which is used in \( h \)) we see that
\[
\alpha \geq -\frac{2\epsilon}{100} - 2\epsilon R \geq -4\epsilon_0 N > -1
\]
on \( B_{(1-\epsilon)}(x_0) \) for \( t \leq S(N, v_0, \sigma) \) as long as \( t \leq S_{\text{max}} \). Here we have used \( |\mathcal{R}(\cdot, t)|t \leq N \).
That is, condition (a) will also not be violated for a well defined time interval. That is $S_{\text{max}} \geq S(n, N, v_0) > 0$. This finishes the proof of Claim (iv) and the proof of the theorem if one accepts Claim (v) (note that (\tilde{h}) also holds).

**Claim (v)** If $d_t(\cdot)$ is not differentiable at $x \in M$ then we use the trick of E.Calabi ([1]) as explained in the proof of Claim (iii) of the proof of Theorem 4.1.

Hence we may argue with $\tilde{h}(y, s) := e^{-2k_0s} \tilde{h}(y, s)$ everywhere above. If for example $(x, t)$ is a first time and point where $(h(\mathcal{R} + 2R \text{ Id}) + \varepsilon \text{ Id})(V, V) = 0$, then $(x, t)$ is a first time and point where $(\tilde{h}(\mathcal{R} + 2R \text{ Id}) + \varepsilon \text{ Id})(V, V) = 0$, at least locally (see the proof of Claim (iii) of the proof of Theorem 4.1 for further details). Arguing as in the proof of Claim (iii) of the proof of Theorem 4.1, we see that such an $(x, t)$ cannot exist.

A similar argument holds for the tensor in Claim (iv) of this proof.

This finishes the proof of Claim (v) of this proof and the proof of the theorem. □

**Remark 5.2.** Theorem 5.1 establishes Theorem 1.5 for the case $V = \frac{1}{400N}$, $N \geq 1$. The case of general $V > 0$ may be obtained as follows. Scale so that $\mathcal{R} \geq \frac{1}{400N}$ on $B_r(x_0)$ where $r = \sqrt{\frac{1}{400N}}$ (notice that we require $V > 0$ to do this). We now have $\text{vol}(B_r(x_0)) \geq \hat{v}_0 = \hat{v}_0(V, N, v_0) > 0$. Now repeat the proof of Theorem 5.1 with the following changes: replace $v_0$ by $\hat{v}_0$, replace all balls $B_s(x_0)$ that appear in the proof by $B_r(x_0)$, choose a cut-off function $\varphi$ from Proposition 3.3 with $\varphi(s) = 1$ for all $s \leq r(1 - \sigma_6)$ and $\varphi(s) = 0$ for all $s \geq r(1 - \sigma_7)$ in Claim (ii) respectively with $\varphi(s) = 1$ for all $s \leq r(1 - \sigma_2)$ and $\varphi(s) = 0$ for all $s \geq r(1 - \sigma_3)$ in Claim (iv). The proof then works without any further changes, except that the constants that occur now also depend on $V$ (this dependence also appears in the statement of the Theorem).

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