Trace anomaly and black holes evaporation

F. Canfora and G. Vilasi*

*Istituto Nazionale di Fisica Nucleare, GC di Salerno, Italy.
Dipartimento di Fisica "E.R.Caianiello", Università di Salerno.

November 19, 2018

Abstract

A model is proposed to describe a transition from a Schwarzschild black hole of mass $M_0$ to a Schwarzschild black hole of mass $M_1 \leq M_0$. The basic equations are derived from the non-vacuum Einstein field equations taking a source representing a null scalar field with a nonvanishing trace anomaly. It is shown that the nonvanishing trace anomaly of the scalar field prevents a complete evaporation.

PACS numbers: 04.70.Dy, 04.20Gz

In the Hawking seminal paper in 1974 [3] it was shown that a Schwarzschild black hole, the most attractive object of theoretical physics, does emit thermal radiation. This discovery provided deeper physical basis to the Bekenstein entropy [2] and to the black-hole thermodynamics.

However, this new understanding of the black hole physics raised some highly nontrivial questions about the foundations of Quantum Field Theory (QFT). In particular, the black-hole evaporation led to an apparent breakdown of predictability [4] because of the thermal character of the emitted flux. Eventually, if the evaporation is complete, a loss of information and of unitarity occur since, in this case, a pure state evolves in a density matrix. Among all these proposals to overcome these difficulties, probably the three most popular are:

(i) QFT has to be generalized to allow nonunitary processes [4] since it seems that a complete evaporation cannot be ruled out. However, up to now, this idea has been found to be incompatible with locality or conservation of energy [9], [8];

(ii) a Planckian remnant, which could store the lost information, is left [11]. Moreover, as showed by Bekenstein [12], the actual information loss due to the Hawking radiation is lower than that of a pure blackbody. Thus, a part of lost information could be detained by the remnant itself and the

*Corresponding author: Gaetano Vilasi, Università di Salerno, Dipartimento di Fisica E.R.Caianiello, Via S.Allende, 84081 Baronissi (Salerno) Italy; e-mail: vilasi@sa.infn.it, Phone: +39-089-965317, Fax: +39-089-965275.
remaining part could be encoded in the Hawking radiation through the Bekenstein mechanism;

(iii) a quantum theory of gravity, such as superstring theory \cite{23} or loop quantum gravity \cite{22}, will disclose the information hidden in the detailed structure of the emitted quanta \cite{4}.

Here, a model is proposed which shows that a complete evaporation does not occur if one takes into account the trace anomaly of the null quantum fields that carry away energy from the black hole to infinity. Furthermore, the approach sheds light on the relation between the conservation of energy and the black hole evaporation.

The model It is believed that a full understanding of the black hole evaporation requires a full quantum theory of gravity such as the superstring theory or the loop quantum gravity. In recent years, even more "conventional" approaches to the quantum gravity, such as the effective action and the renormalization group method \cite{16}, gave new hints for approaching this problem. However, until the final stages, this is unnecessary because a black hole evaporation cannot be considered a strong gravitational field phenomenon since there is a slow rate of change of the black hole mass with time for almost all the process \cite{6}. Then, it must be possible to describe how the metric changes, owing to the evaporation, by the standard Einstein equations with a suitable matter source. Furthermore, a very interesting application of loop quantum gravity, shows that the discrete eigenvalues of the geometric operators converge very fast, actually in a few Plank lengths, to their semiclassical approximations \cite{19} \cite{20}.

For this reason we can assume that, even if near the horizon there could be quantum gravity effects, far enough (with respect to the Planck length) from the horizon a semiclassical description should give the right results. Therefore, the semiclassical framework is well suited to study the blackhole evaporation, since, to study the evaporation, one has to look at the energy radiated at infinity. Moreover, it is desirable for the self-consistency of QFT to solve the problems raised by the blackhole evaporation by only using general relativity and QFT itself.

For these reasons, some authors claimed that the issues raised by Hawking radiation could be resolved by including in the models the back-reaction effects in the framework of the semiclassical program \cite{6}, \cite{7}, \cite{10}, \cite{11}, \cite{12}, \cite{14}, \cite{17}, \cite{21}. The standard semiclassical program for studying the back-reaction in the black-hole evaporation consists, first, in evaluating an effective vacuum stress-energy tensor of a null quantum field on the black hole space-time of interest \cite{7} and, then, using this tensor as a source for the Einstein equations \cite{10}.

This kind of approach has the advantage that results are easy to interpret in terms of the unperturbed quantity, thus making transparent the influence of the source on the back-ground geometry \cite{10}.

However, the calculation of the vacuum stress-energy tensor of quantum fields on a curved space-time is not easy. The approximated form of the stress-energy tensor is computed in a four-dimensional geometry only in static situ-
ations\textsuperscript{7}, while, in non static models, it is usually computed in effective two-dimensional geometries\textsuperscript{13, 24}. Therefore, in such effective models, it is not possible to take into account the nontrivial effects of the four-dimensional trace anomaly\textsuperscript{1}.

Thus, an approach is developed in which the form of the stress-energy tensor is kept general but the trace anomaly is fully taken into account. It will be shown, via the exact Einstein equations, that this has important physical consequences.

The Einstein tensor A transition from a Schwarzschild blackhole of initial mass $M$ to a Schwarzschild blackhole of final mass $\overline{M}$ will be described outside the collapsed matter by the means of a well suited source that carries away energy from the blackhole to infinity. The metric, valid only for $r > R^\ast > 0$, $R^\ast$ denoting the initial radius of the horizon, will be written as in [6]:

$$ds^2 = \left(1 - \frac{m(r, v)}{r}\right) e^{2j(r, v)} dv^2 - 2e^{j(r, v)} dr dv - r^2 d\Omega.$$  \hspace{1cm} (1)

For $m = const$ and $j = const$, metric (1) reduces to the Schwarzschild metric in Eddington-Finkelstein coordinates.

Thus, in order to describe a transition from a Schwarzschild blackhole of mass $M$ to a Schwarzschild black hole of mass $\overline{M}$, we will search for a solution of the Einstein equations satisfying the following boundary conditions:

$$m(r, 0) = M > 0; \quad m(r, T) = \overline{M} \geq 0; \quad j(r, 0) = k_1; \quad j(r, T) = k_2,$$

where $k_1$ and $k_2$ are constants and at $v = T$ the blackhole reaches its final state.

The above metric has the following scalar curvature

$$R = \left\{ - \exp(-j)2r^2\partial_r^2 j + 3r\partial_r j\partial_r m + m\partial_r j \right. \right.$$ 
\hspace{2cm} $$+ 2\partial_r m - 4r\partial_r j - 2r^2 \left(1 - \frac{m}{r}\right) \partial_r^2 j \right. \right.$$ 
\hspace{2cm} $$\left. + -2r^2 \left(1 - \frac{m}{r}\right)(\partial_r j)^2 + r\partial_r^2 m \right\} \frac{1}{r^2}$$

\textsuperscript{1}Some attempts to take into account the four-dimensional trace anomaly in the framework of dilatonic gravity can be found in [18].
and the nonvanishing independent components of the Einstein tensor are

\[
G_{11} = \frac{\exp(2j)}{r^3} (m \partial_r m - r \exp(-j) \partial_v m - r \partial_r m),
\]

\[
G_{12} = \frac{\exp(j) \partial_r m}{r^2},
\]

\[
G_{22} = -2 \frac{\partial_r j}{r},
\]

\[
G_{33} = \frac{1}{2} \left\{ - (2r + m) \partial_r j - \exp(-j) 2r^2 \partial_r^2 j + 3r \partial_r j \partial_r m - 2r^2 \left( 1 - \frac{m}{r} \right) \partial_r^2 j - 2r^2 \left( 1 - \frac{m}{r} \right) (\partial_r j)^2 + r \partial_r^2 m \right\},
\]

\[
G_{44} = \sin^2 \theta G_{33}.
\]

The source  The commonly accepted picture describing the evaporation is that the Schwarzschild black-hole emits black-body radiation at the Hawking temperature. Since the emission of massive particles by the blackhole is highly suppressed, the natural choice is to take as source a null quantum fluid, i.e. a null fluid violating all the energy conditions:

\[
T_{\mu\nu} = g_{\mu\nu}V (v, r) + \rho (v, r) U_\mu U_\nu.
\]

Thus, \(T_{\mu\nu}\) has to be interpreted as a suitable average over quantum fields that share to Hawking radiation and take away energy from the collapsing matter to infinity, while \(V\) corresponds to the trace anomaly of quantum fields. Since no constraints will be imposed on \(\rho\) and \(V\), the tensor \(T_{\mu\nu}\) can describe a classical, a semiclassical or a purely quantum source, so that neither the weak nor the null energy conditions need to be satisfied. When \(\rho \geq 0\), \(T_{\mu\nu}\) can be thought as the energy-momentum tensor of a null scalar field with trace anomaly and, in this case, the weak energy condition can still be violated but the null cannot.

At a first glance it seems that we would have to specify the explicit form of \(\rho\) and \(V\), but the Einstein field equations will give a closed system of equations for \(m(v, r)\) and \(j(v, r)\) which does not depend on the detailed form of \(T_{\mu\nu}\). Thus, the gravitational part "decouples" from the matter source. Thanks to this fact, it is possible to keep general the form of \(T_{\mu\nu}\).

The Einstein equations  Taking into account that \(g_{\mu\nu}U^\mu U^\nu = 0\), the Einstein equations \(G_{\mu\nu} = \kappa T_{\mu\nu}\) give:

\[
\kappa^{-1} G_{11} = V \left( 1 - \frac{m}{r} \right) \exp(2j) + \rho (U_v)^2,
\]

\[
\kappa^{-1} G_{12} = - \exp(j) V + \rho U_r U_v, \quad \kappa^{-1} G_{22} = \rho (U_r)^2, \quad \kappa^{-1} G_{33} = g_{33} V, \quad - R = 4\kappa V.
\]

The last two equations give for the scalar curvature

\[
R = \frac{4}{r^2} \left[ (m - r) \partial_r j + \partial_r m \right].
\]
Moreover, from
\[
G^2_{12} - G_{11}G_{22} = \kappa^2 \left(T^2_{12} - T_{11}T_{22}\right) = \kappa^2 (g_{12})^2 V^2
\]
it follows that
\[
(m - r)^2 \left(\partial_{r,j}\right)^2 = -2r \exp \left(-j\right) \partial_{r,j} \partial_r m.
\]
Thus, by assuming\(^2\) \(\partial_{r,j} \neq 0\), the equations for \(m\) and \(j\) reduce to
\[
\exp \left(j\right) \frac{\partial_{r,j}}{2} = \partial_v \left(\frac{r}{m - r}\right),
\]
\[
3m \partial_{r,j} = -\exp \left(-j\right) 2r^2 \partial_{r,j}^2 + (3r \partial_{r,j} - 2) \partial_r m
- r \partial_r^2 m - 2r^2 \left(1 - \frac{m}{r}\right) \left(\partial_{r,j}^2 + \left(\partial_{r,j}\right)^2\right),
\]
where it is worth to note that only Eq. (3) depends on the assumption \(\partial_{r,j} \neq 0\). Thus, the problem reduces to solve Eqs. (3) and (4) with the boundary conditions
\[
m, j \in C^2 ([0, T] \times [R^*, \infty]),
\]
\[
|m|, |j| \leq K_1 < +\infty, \quad |\partial_{r,j}|, |\partial_r m| \leq \frac{K_2 r^{-2}}{r \to +\infty}
\]
\[
m(0, r) = M > 0; \quad j(0, r) = k_1; \quad m(T, r) = \overline{M} \geq 0; \quad j(T, r) = k_2,
\]
where the upper bounds (6) are needed to have an asymptotically flat metric. It is important to stress that Eqs (3) and (4), not depending on the explicit form of \(\rho\) and \(V\), are decoupled from the matter source but, however, are a direct consequence of the non vanishing trace anomaly.

**Estimate of the final mass** Now, it will be proven that the trace anomaly prevents a complete evaporation. Indeed, by using the boundary conditions \(j(0, r) = 0\) and \(m(0, r) = M\), from Eq. (3) it follows that the function \(m\) can be expressed in closed form in terms of \(j\) in the following way:
\[
m(v, r) = r - \frac{r}{r - \overline{M}} - \frac{r}{\partial_v} \int_0^v \exp \left[j(\tau, r)\right] d\tau,
\]
so that the final mass \(\overline{M}\) may be expressed as
\[
\overline{M} = r - \frac{r}{r - \overline{M}} - \frac{r}{\partial_v} \int_0^v \exp \left[j(\tau, r)\right] d\tau.
\]
It is worth to note that \(m(v, r)\), and then \(\overline{M}\), cannot be negative. In fact, by using Eqs (6) and (9) we have:
\[
m \sim \frac{r}{r \to +\infty} - \frac{r}{1 + \frac{\overline{M}}{r} - v K_2 \exp \left[K_1\right]} \geq 0
\]
\(^2\)It can be easily seen that \(\partial_{r,j} = 0\) gives only static solutions, or solutions with vanishing trace anomaly.
being \( M > 0 \). Furthermore, by using Eqs. \( \text{[2]} \) it is trivial to show that:

\[
\rho \geq 0 \Rightarrow \partial_r j \leq 0 \Rightarrow \overline{M} \geq M
\]

\[
\rho < 0 \Rightarrow \partial_r j > 0 \Rightarrow \overline{M} < M.
\]

Then, when \( \rho \geq 0 \), so that \( T_{\mu \nu} \) describes a null scalar field violating the weak energy condition but not the null one, the evaporation is absent. Moreover, in this case, the boundary conditions \( \text{[9]} \) do not play any role.

When \( \rho < 0 \), the evaporation takes place but leaves a remnant. In fact, Eq.\( \text{[9]} \) with \( \overline{M} = 0 \) reads:

\[
\frac{1}{2} \partial_r \int_0^T \exp[j(\tau, r)] d\tau = \frac{M}{r - M}
\]

which gives

\[
\frac{1}{2} \int_0^T \exp[j(\tau, r)] d\tau = M \ln \left| \frac{r}{M} - 1 \right| + N,
\]

where \( N \) is an integration constant.

The above relation is not compatible with the boundary conditions because:

\[
\infty > \frac{1}{2} T e^{K_i} \geq \frac{1}{2} \int_0^T e^{j(\tau, r)} d\tau \rightarrow r \rightarrow +\infty \infty,
\]

so that, even with the "worse" energy-momentum tensor (i.e. a \( T_{\mu \nu} \) violating all the energy conditions), \( \overline{M} \) cannot vanish, that is, the evaporation stops due to the trace anomaly of the scalar fields that carry away energy from the blackhole to infinity. Furthermore, by requiring a finite total Bondi flux \( \text{[1]} \), it is easy to show that the same results also hold in the limit \( T \rightarrow \infty \).

In conclusion, the information loss paradox could be overcome by taking into account the informations detained by the remnant and by the Hawking radiation through the Bekenstein mechanism \( \text{[12]} \). This is very appealing since the key ingredient to stop the evaporation, namely the trace anomaly, is a typical feature of QFT, so QFT itself could care for the loss of unitarity and this would be a good self-consistency test for QFT.

This result looks quite surprising. In fact, many works on the back reaction \( \text{[6]} \), \( \text{[14]} \) suggest that the blackhole evaporates adiabatically in a time \( O(M^3) \) as first predicted by Hawking \( \text{[3]} \). However, the four dimensional trace anomaly brings in an important new feature which give rise to the question: when is the trace anomaly not anymore negligible? This question, as we will now explain, cannot be answered by only looking at Eqs. \( \text{[3]} \) and \( \text{[4]} \).

Trace anomaly, cosmological constant and scale invariance Let us observe that Eqs \( \text{[3]} \) and \( \text{[4]} \) are formally scale-invariant. Namely, if \( m_0 \) and \( j_0 \) are a solution of the system, then \( \lambda m_0 \) and \( \lambda j_0 \) are a solution too (provided we also make the transformation \( r \rightarrow \lambda r, t \rightarrow \lambda t \)). Thus, Eqs \( \text{[3]} \) and \( \text{[4]} \) by themselves do not provide us with a lower bound on \( M \).

However, the scale invariance is broken by the introduction of a nonvanishing cosmological constant since scale transformations on \( \Lambda \) cannot be performed,
being $\Lambda$ a physical measurable parameter whose value is fixed by observations. Roughly speaking, this means that $\Lambda$ introduces a characteristic energy scale. The same is true for the trace anomaly, because it also breaks the scale invariance.

To get a lower bound on $M$ and a new insight into the mechanism stopping the evaporations, we can proceed as follows.

**Thermodynamical features**  As originally found by Hawking [3], the expectation value of the operator number, as measured by an asymptotically static observer in a Schwarzschild blackhole of mass $M$, is

$$n_i = \frac{1}{\exp(\hbar \omega_i \sigma M) - 1}$$

(11)

$$\sigma = \frac{\kappa}{\hbar^2 c^3}$$

where, in this formulas, the constants $\hbar$ and $c$ have been restored. Since only a magnitude order estimate will be performed, in the following the greybody factors will be neglected.

Relation (11) basically follows from the existence of a horizon and from the fact that the notion of a particle is not an invariant concept on curved spacetime. Thus, in some sense, the Hawking effect is only "kinematical" [15], since the Einstein equations are not needed to get the relation (11). This obviously means that, in deducing relation (11), we are neglecting the backreaction effects of the emitted particles on the metric.

To take into account the backreaction, we have to impose the conservation of energy, i.e. that this thermal particles are created by the mass lost by the blackhole. In the following, it will be assumed that the expectation value of the operator number $n_i = n_i (M)$ varies adiabatically as a function of $M$, that is, if $M \to M_t$ then $n_i \to n_i t = n_i (M t)$. This assumption is well verified in all situations in which the blackhole thermodynamics is applicable [14]. Therefore, if $M \to M + \delta M$, then $n_i \to n_i + \delta n_i$ with

$$\delta n_i = \frac{\partial n_i}{\partial M} \delta M = -\frac{\hbar \omega_i \exp(\hbar \omega_i \sigma M)}{(\exp(\hbar \omega_i \sigma M) - 1)^2} \sigma \delta M.$$ 

To assure that the new particles are created by the mass lost by the black-hole, we have to impose the conservation of energy:

$$c^2 \delta M + \sum_i \hbar \omega_i \delta n_i \leq 0,$$

(12)

where, since the blackhole is evaporating, $\delta M < 0$, and in Eq. (12) the inequality has been used to take into account the fact that the mass lost by the blackhole gives rise to other kind of particles (such as fermions, bosons, etc.) too. Then, the second term in Eq. (12) encodes, in the thermodynamical limit, the backreaction effects of the emitted particles on the gravitational field. It is clear from Eq. (12) that, as long as $M$ is big enough, the second term in
the inequality is completely negligible and the usual blackhole thermodynamics should apply.

From Eq. (12) we obtain

$$c^2 \geq \sigma \sum_i \frac{(\hbar \omega_i)^2 \exp(\hbar \omega_i \sigma M)}{(\exp(\hbar \omega_i \sigma M) - 1)^2}. \quad (13)$$

The above relation, in which neither infrared nor ultraviolet divergences appear, implies that the conservation of energy can be satisfied only for $M \geq \overline{M}$, where $\overline{M}$ is implicitly defined by the following equation:

$$c^2 = \sigma \sum_i \frac{(\hbar \omega_i)^2 \exp(\hbar \omega_i \sigma \overline{M})}{(\exp(\hbar \omega_i \sigma \overline{M}) - 1)^2}. \quad (14)$$

Of course, $\overline{M}$ is of the order of the Planck mass.

Thus, the conservation of energy, and then the backreaction, prevent the blackhole mass $M$ from being lower then $\overline{M}$, so the remnant is stable, since a smaller mass is not allowed by the energy conservation. This conclusion clarifies the relation between energy conservation and complete evaporation. In fact, if one tries to generalize QFT to allow nonunitary processes, such as a complete evaporation, then the energy conservation is lost [9], [8]. Moreover, this, besides to be very interesting in itself, fits very well with the numerical results obtained by R. Casadio [21] in a Hamiltonian framework.

Acknowledgments

The authors wish to thank D. Grumiller, G. Esposito and S. Sonego for remarks and bibliographic suggestions.

References

[1] H. Bondi, M.G. J. Van der Burg, A. W. K. Metzner, Proc. Roy. Soc. A269 (1962) 21.
[2] J. D. Bekenstein, Phys. Rev. D 7 (1973) 2333;
[3] S. W. Hawking, Nature (London) 243 (1974) 30;
[4] S. W. Hawking, Phys. Rev. D 14 (1976) 2460;
[5] D. N. Page, Phys. Rev. Lett. 44 (1980) 301;
[6] J. Bardeen, Phys. Rev. Lett. 46 (1981) 382;
[7] D. N. Page, Phys. Rev. D 25 (1982) 1499;
[8] T. Banks, M. E. Peskin, L. Susskind, Nucl. Phys. B 244 (1984) 125;
[9] J. Ellis, J. Hagelin, D. V. Nanopoulos, M. Srednicki, Nucl. Phys. B 241 (1984) 381;

[10] J. W. York, Phys. Rev. D 31 (1985) 775;

[11] Y. Aharonov, A. Casher, S. Nussinov, Phys. Lett. B 191 (1987) 51;

[12] J. D. Bekenstein, Phys. Rev. Lett. 70 (1993) 3680;

[13] R. Parentani, T. Piran, Phys. Rev. Lett. 73 (1994) 2805;

[14] S. Massar, Phys. Rev. D 52 (1995) 5861 and references therein;

[15] M. Visser, Phys. Rev. Lett. 80 (1998) 3436 and references therein;

[16] A. Bonanno, M. Reuter, Phys. Rev. D 62 (2000) 043008 and references therein;

[17] S. Massar, R. Parentani, Nucl. Phys. B 575 (2000) 333;

[18] S. Nojiri, S. Odintsov, Int. J. Mod. Phys. A 16 (2001) 1015 and references therein;

[19] M. Bojowald, Phys. Rev. Lett. 86 (2001) 5227;

[20] M. Bojowald, Phys. Rev. D 64 (2001) 084018;

[21] R. Casadio, Phys. Lett. B 511 (2001) 285;

[22] T. Thiemann, gr-qc/0110034 and references therein;

[23] J. R. David, G. Mandal, S. R. Wadia, Phys. Rept. 369 (2002) 549;

[24] D. Grumiller, W. Kummer, D. V. Vassilevich, Phys. Rept. 369 (2002) 327 and references therein;