TWISTED ORBI-FOLD K-THEORY

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Abstract. We use equivariant methods to define and study the orbifold \( K \)-theory of an orbifold \( X \). Adapting techniques from equivariant K-theory, we construct a Chern character and exhibit a multiplicative decomposition for \( K^\star_{orb}(X) \otimes \mathbb{Q} \), in particular showing that it is additively isomorphic to the orbifold cohomology of \( X \). A number of examples are provided. We then use the theory of projective representations to define the notion of twisted orbifold \( K \)-theory in the presence of discrete torsion. An explicit expression for this is obtained in the case of a global quotient.

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1. Motivation

During the last twenty years, one major trend has been the constant flow of physical ideas into mathematics. Although many of them have had a significant impact, there are still many others which deserve more attention from mathematicians. Orbifold string theory is such an example: it has been around in physics since 1985; compared to its
popularity in physics, its influence in mathematics is minimal. This project represents an effort to change this picture.

First, let us give a brief introduction to orbifold string theory aiming to explain its mathematical content. In the end, we hope that it will become clear why we have to study orbifold K-theory. In 1985, Dixon-Harvey-Vafa-Witten \[15\] discovered that on a singular manifold such as an orbifold one can also build a smooth consistent string theory. One of the remarkable insights of orbifold string theory is that its consistency demands the introduction of so called twisted sectors. In another words, ordinary cohomology is the wrong theory for orbifolds; the correct one seems to be orbifold cohomology, which has been constructed by Chen-Ruan \[12\]. Orbifold cohomology (not ordinary cohomology) should fulfill the role of cohomology for smooth manifolds. Moreover, on smooth manifolds, orbifold cohomology is the same as ordinary cohomology.

One of the original motivations of orbifold string theory is to capture the information of its crepant resolution (see \[19\] page 126 for definitions). In fact, it is generally believed that orbifold string theory is equivalent to the ordinary string theory of its crepant resolution in some fashion. More precisely, one can associate a super conformal field theory to each 3-dimensional Calabi-Yau orbifold or smooth manifold. The orbifold super conformal field theory and ordinary super conformal field theory of its crepant resolution appear to be two members of a family of superconformal field theories. To explore its mathematical implications, it is useful to consider its low energy effective theory. Low energy effective theory is parameterized by the moduli space of vacua. At each vacuum, one can associate the orbifold cohomology (for the orbifold case) or ordinary cohomology (for the smooth case). Then there is a natural metric on orbifold cohomology corresponding to Poincaré duality. After quantization, the metric is corrected by the quantum corrections given by orbifold quantum cohomology (see \[13\]).

To compare the theory at an orbifold point with that of its crepant resolution, one has to consider the variation of low energy effective theory as we vary the point of moduli space of superconformal field theory. It is known that the orbifold points are often singularities. Its monodromy could interchange $H^{1,1}$ and $H^{2,2}$. Therefore, there is no hope to preserve the grading of orbifold cohomology. However, it still preserves the parity. This naturally leads to the concept of orbifold $K$-theory \[1\] which we shall define in section 4. In this context, a natural conjecture arising from orbifold string theory is

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1 Throughout this paper we shall be dealing exclusively with (equivariant) complex $K$-theory, i.e. the generalized cohomology theory arising from complex vector bundles
**K-Orbifold String Theory Conjecture:** *There is a natural additive isomorphism between orbifold K-theory and the ordinary K-theory of its crepant resolution.*

Another physical motivation comes from the recent discovery in physics that the D-brane charge of an ordinary string theory is described by the K-theory of its underlying smooth manifold. For orbifold string theory, it is obvious that the orbifold D-brane charge should be described by our orbifold K-theory.

An obvious definition of orbifold K-theory is the K-theory associated to orbifold vector bundles. However, to be a correct definition for our purposes, we have to obtain at least an additive isomorphism to orbifold cohomology. It is well-known that any reduced orbifold can be expressed as the quotient of a smooth manifold by an almost free action of a compact Lie group (see section 2). Therefore, we will use methods from equivariant topology to provide an effective K-theoretic approach to orbifolds. In particular, using an appropriate equivariant Chern character we obtain a decomposition theorem for orbifold K-theory as a ring, and we show that it is additively isomorphic to orbifold cohomology. A nice by-product of our orbifold K-theory is a natural notion of orbifold Euler number for a general orbifold. In the context of orbifold cohomology, it only makes sense for $SL$-orbifolds, where the local group is a finite subgroup of $SL(n,C)$.

In order to compare orbifold K-theory with orbifold cohomology, we make use of equivariant Bredon cohomology with coefficients in the representation ring functor. This equivariant theory is the natural target for Chern characters and with real coefficients is additively isomorphic to orbifold cohomology. It can however be defined with very general coefficients and hence seems to be an important technical device for the study of orbifolds.

Orbifold K-theory has a different and interesting ring structure. It suggests that orbifold K-theory should be interesting in its own right, and not just orbifold cohomology with a $\mathbb{Z}/2$ grading. This is indeed the case for several reasons. For example, orbifold K-theory (not orbifold cohomology) is the natural set-up for index theory. There is also an indication that orbifold K-theory behaves better algebro-topologically than orbifold cohomology. In [13], Chen-Ruan introduced the notion of a “good map” to replace a traditional orbifold map as an appropriate map between orbifolds. We shall refer to these as orbifold morphisms; an equivalent notion appeared previously in [20]. From their definition it is not hard to see that an orbifold morphism is natural with respect to orbifold K-theory and that it induces a ring homomorphism on orbifold K-theory. In contrast, such a naturality property is far more subtle and unknown for orbifold cohomology.

A key physical concept of orbifold string theory is the twisting by discrete torsion. In a cohomological context, it motivated the construction of twisted orbifold cohomology
Another important goal of this paper is to introduce twisting for orbifold K-theory. For twisted orbifold cohomology, the second author introduced the more general notion of an “inner local system” to twist orbifold cohomology. We do not know how to twist orbifold K-theory in such a generality; here, we restrict ourselves to the use of discrete torsion. We introduce twisted orbifold K-theory using an explicit geometric model. In the case when the orbifold is of the form $X = Y/G$, where $G$ is a finite group, then our construction can be understood as a twisted version of equivariant K-theory, where the twisting is done using a fixed element $\alpha \in H^2(G, S^1)$. The basic idea is to use the associated central extension, and to consider equivariant bundles with respect to this group which cover the $G$–action on $Y$. A computation of the associated twisted theory can be explicitly obtained (over the complex numbers) using ingredients from the classical theory of projective representations. In fact, we define a bigraded ring denoted the total twisted K-theory which is parametrized over all discrete torsion elements. This construction seems to be of independent interest.

More generally we can define a twisted orbifold K-theory associated to the universal orbifold cover; in this generality it can be computed in terms of twisted Bredon cohomology. This can be understood as the $E_2$–term of the twisted version of a spectral sequence converging to twisted orbifold K-theory, where in all known instances the higher differentials are trivial in characteristic zero (this is a standard observation in the case of the Atiyah–Hirzebruch spectral sequence). In particular, we obtain an additive isomorphism between our twisted orbifold K-theory and twisted orbifold cohomology for the case of global quotients.

It remains to relate our construction to other forms of twisting for K-theory (such as that described in [11]). For ”good” orbifolds, a twisted version of K-theory has been studied by Marcolli-Mathai [23] using $C^*$ algebras. However, we would like to remark that the connection from geometric K-theory to algebraic K-theory of $C^*$ algebras breaks down for non-reduced orbifolds, where an orbifold bundle does not have any nonzero section in general.

The second author would like to thank R. Dijkgraaf, W. Wang, E. Witten and E. Zaslow for many stimulating discussions.

2. Orbifolds and Group Actions

Our basic idea to study orbifold K-theory is to apply methods from equivariant topology. In this section, we collect some basic properties of orbifolds and describe how they relate to group actions. For the reader’s convenience, we include a definition of an orbifold and
of orbifold vector bundles. One can find more details and alternative definitions in [13]. The notion of an orbifold was first introduced by Satake in [32], where a different name, V-manifold, was used. Satake’s V-manifold corresponds to a reduced orbifold in our case.

**Definition 2.1.** An orbifold structure on a Hausdorff, separable topological space $X$ is given by an open cover $U$ of $X$ satisfying the following conditions:

- (2.1.1a) Each element $U$ in $U$ is uniformized, say by $(V, G, \pi)$. Namely, $V$ is a smooth manifold and $G$ is a finite group acting smoothly on $V$ such that $U = V/G$ with $\pi$ as projection map. Let $\text{ker } G$ be the subgroup of $G$ acting trivially on $V$.
- (2.1.1b) If $U' \subset U$, then there is a collection of injections $(V', G', \pi') \rightarrow (V, G, \pi)$. Namely, the inclusion $i: U' \subset U$ can be lifted to maps $\tilde{i}: V' \rightarrow V$ and an injective homomorphism $i_# : G' \rightarrow G$ such that $i_#$ is an isomorphism from $\text{ker } G'$ to $\text{ker } G$ and $\tilde{i}$ is $i_#$-equivariant.
- (2.1.1c) For any point $p \in U_1 \cap U_2$, $U_1, U_2 \in U$, there is a $U_3 \in U$ such that $p \in U_3 \subset U_1 \cap U_2$.

For any point $p \in X$, suppose that $(V, G, \pi)$ is a uniformizing neighborhood and $\bar{p} \in \pi^{-1}(p)$. Let $G_p$ be the stabilizer of $G$ at $\bar{p}$. Up to conjugation, it is independent of the choice of $\bar{p}$ and called the local group of $p$.

**Definition 2.2.** We call $X$ a reduced orbifold if $G_p$ acts effectively for all $p \in X$.

It is well-known that if a compact Lie group $G$ acts smoothly and effectively on a manifold $M$ with finite stabilizers (isotropy subgroups), then $M/G$ is a reduced orbifold. More generally, $X = M/G$ is an orbifold for any smooth Lie group action $G$ if the following conditions are satisfied:

- (2i) For any $x \in M$ the isotropy subgroup $G_x$ is finite (this is what we will call an almost free $G$–action).
- (2ii) For any $x \in M$ there is a smooth slice $S_x$ at $x$.
- (2iii) For any two points $x, y \in M$ such that $y \notin Gx$, there are slices $S_x$ and $S_y$ such that $G S_x \cap G S_y = \emptyset$.

If $G$ is compact, an almost free $G$-action automatically satisfies (2ii), (2iii). Examples arising from proper actions of discrete groups will also appear in our work.
Conversely, we do not know if every orbifold can be expressed as a quotient of a smooth manifold by an almost free action of a Lie group. We call such an orbifold a quotient orbifold. It is well-known that every reduced orbifold is a quotient:

**Proposition 2.3.** Let $X$ denote an $n$–dimensional reduced orbifold. There exists a smooth, effective, almost free $G = O(n)$ action on a smooth manifold $M$ such that the quotient space $M/G$ is naturally isomorphic as an orbifold to $X$.

In other words we can think of any reduced orbifold as the quotient of a manifold under a certain $G = O(n)$ action. An explicit construction for $M$ is the so-called frame bundle over $X$, described as follows. We can choose an orbifold Riemannian metric on $X$ and consider the frame bundle $P_X$ of the metric. When $X$ is reduced, $P_X$ is a smooth manifold. Indeed if $(V, G_x)$ is a local chart for $X$ at $x$, then above it we have the manifold $V \times_{G_x} G$, and the quotient by the right $G$ action gives $V/G_x$. From our point of view the above can be interpreted as saying that the category of reduced orbifolds can be embedded in the category of smooth $O(n)$–manifolds with the three properties mentioned above; a section is obtained by taking $O(n)$ orbit spaces.

It is not hard to show that if $M/G$ is a quotient orbifold, then there exists a group extension $1 \to G_0 \to G \to G_{eff} \to 1$ where $G_0$ is a finite group which is the kernel of the action (i.e. $G_0$ acts trivially on $M$) and $G_{eff}$ acts effectively on $M$. Hence we see that we can associate a canonical reduced orbifold, $X_{red} = M/G_{eff}$, to any quotient orbifold $X = M/G$.

We will assume for simplicity that our orbifolds are compact. In the case of quotient orbifolds $M/G$ with $G$ a compact Lie group, this is equivalent to the compactness of $M$ itself (see [10], page 38); a fact which we will make use of.

Next we recall the notion of the singular set of an orbifold and its resolution.

**Definition 2.4.** Given an orbifold $X$, its singular set, $\Sigma X$ consists of the set of points $p \in X$ such that the local group $G_p \neq 1$. The orbifold resolution of the singular set is defined as $\tilde{\Sigma}X = \{(p,(g)_{G_p}) \mid p \in \Sigma X, \ 1 \neq g \in G_p\}$, where $(g)_{G_p}$ denotes the conjugacy class of $g$ in $G_p$.

Note that we have a map $\tilde{\Sigma}X \to X$, $(p,(g)) \mapsto p$. It turns out that $\tilde{\Sigma}X$ is naturally an orbifold, which is not necessarily connected. Later we shall see that invariants of an orbifold $X$ can be expressed in terms of topological invariants of its orbifold resolution.

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$^2$Traditionally orbifolds of the form $M/G$, where $G$ is a finite group acting smoothly on a manifold $M$ have been called global quotients, we will adhere to this.
In order to apply methods from algebraic topology in the study of orbifolds, we recall a well known result about manifolds with smooth actions of compact Lie groups (see [17]):

**Theorem 2.5.** If a compact Lie group $G$ acts on a smooth, compact manifold $M$, then the manifold is triangulable as a finite $G$-CW complex.

Hence any such manifold will have a cellular $G$ action such that the orbit space $X/G$ has only finitely many cells.

We recall the notions of an orbifold vector bundle and of an orbifold morphism.

**Definition 2.6.** An orbifold vector bundle over an orbifold $X$ consists of the following data: A compatible cover $U$ of $X$ such that for any injection $i : (V', G', \pi') \to (V, G, \pi)$, there is a continuous map $g_i : V' \to \text{Aut}(C^k)$ giving an open embedding $V' \times C^k \to V \times C^k$ by $(x, v) \mapsto (i(x), g_i(x)v)$, and for any composition of injections $j \circ i$, we have

$$g_{j \circ i}(x) = g_j(i(x)) \circ g_i(x), \forall x \in V.$$  

Two collections of maps $g^{(1)}$ and $g^{(2)}$ define isomorphic bundles if there are maps $\delta_V : V \to \text{Aut}(C^k)$ such that for any injection $i : (V', G', \pi') \to (V, G, \pi)$, we have

$$g_i^{(2)}(x) = \delta_V(i(x)) \circ g_i^{(1)}(x) \circ (\delta_{V'}(x))^{-1}, \forall x \in V'.$$

Since (2.1.2) behaves naturally under constructions of vector spaces such as tensor product, exterior product, etc. we can define these constructions for orbifold vector bundles. Suppose that $X = M/G$ as defined previously and that $E \to M$ is a $G$-equivariant bundle. Then, $E/G \to X$ is an orbifold vector bundle.

Recall that if $p : X \to Y$ is a continuous map and $E$ is a vector bundle over $Y$, then $p^*E$ is a vector bundle over $X$. This naturality property breaks down for orbifold maps. This suggests that orbifold map is a bad notion. One should replace it by a “good” map, as introduced by Chen-Ruan [13], which we prefer to call an orbifold morphism.

**Definition 2.7.** Let $\tilde{f} : X \to X'$ be a $C^\infty$ map between orbifolds $X$ and $X'$ whose underlying continuous map is denoted by $f$. Suppose there is a compatible cover $U$ of $X$, and a collection of open subsets $U'$ of $X'$ satisfying (2.1.1a – c) and the following condition: There is a 1:1 correspondence between elements of $U$ and $U'$, say $U \leftrightarrow U'$, such that $f(U) \subset U'$, and an inclusion $U_2 \subset U_1$ implies an inclusion $U'_2 \subset U'_1$. Moreover, there is a collection of local $C^\infty$ liftings $\{\tilde{f}_{UU'}\}$ of $f$, where $\tilde{f}_{UU'} : (V, G, \pi) \to (V', G', \pi')$.\footnote{We should also point out that an equivalent notion has been defined in [26], pp. 17 in the language of groupoids; they are referred to as strong maps.}
satisfies the following condition: each injection \( i : (V_2, G_2, \pi_2) \to (V_1, G_1, \pi_1) \) is assigned an injection \( \lambda(i) : (V'_2, G'_2, \pi'_2) \to (V'_1, G'_1, \pi'_1) \) such that \( \tilde{f}_{U_1 U'_1} \circ i = \lambda(i) \circ \tilde{f}_{U_2 U'_2} \), and for any composition of injections \( j \circ i \), the following compatibility condition holds:

\[
\lambda(j \circ i) = \lambda(j) \circ \lambda(i).
\]

We call \( \{\tilde{f}_{U U'}, \lambda\} \) a \textit{compatible system} for \( \tilde{f} \).

Given a compatible system \( \xi \), one can construct a pull-back orbifold vector bundle in the following sense. Suppose that \( E \to Y \) is an orbifold vector bundle. By the definition, \( E \) can be viewed as a collection of local orbifold vector bundles satisfying (2.1.2), (2.1.3). One can pull back the local orbifold bundles. The conditions of a compatible system ensures that one can glue them together to form a global orbifold bundle denoted by \( f^*_x E \).

It is clear that there is a bundle map \( \bar{f} : f^*_x E \to E \) covering \( f \).

**Definition 2.8.** Two compatible systems \( \xi_i \) for \( i = 1, 2 \), of a \( C^{\infty} \) map \( \tilde{f} : X \to X' \) are \textit{isomorphic} if for any orbifold vector bundle \( E \) over \( X' \) there is an isomorphism \( \psi \) between the corresponding pull-back bundles \( f^*_x E^*_i \) with \( \bar{f}_i : f^*_x E^*_i \to E \), \( i = 1, 2 \), such that \( \bar{f}_1 = \bar{f}_2 \circ \psi \).

We often drop \( \xi \) from the definition without any confusion.

**Definition 2.9.** An orbifold morphism \( f \) is an orbifold map together with an isomorphism class of compatible systems.

Later we will see that orbifold morphisms are compatible with respect to products in K-theory, making them especially significant.

### 3. Equivariant Cohomology and Orbifolds

In this paper, we only consider quotient orbifolds \( X = M/G \). We will consider actions of both compact and discrete groups. Throughout the paper, we will use \( G \) to denote a compact Lie group and \( \Gamma \) to denote a discrete group. By the previous discussion, a reduced orbifold is a quotient orbifold through the frame bundle construction. Hence, our results work for a rather general class of orbifolds. It is clear that one can study them using equivariant techniques. We begin by considering the equivariant cohomology of \( M \). Recall that the Borel construction associated to the \( G \)–action on \( M \) is the orbit space \( M \times_G EG = (M \times EG)/G \), where \( G \) is the universal \( G \)–space. The \( G \)–equivariant cohomology of \( M \) is by definition the cohomology of the associated Borel construction. We begin by computing this equivariant cohomology under certain assumptions on the isotropy subgroups.
Theorem 3.1. Suppose that $X = M/G$ be a quotient orbifold. If $R$ is a ring such that $|G_x|$ is invertible in $R$ for each $x \in X$, then there is an algebra isomorphism $H^*_G(M,R) \cong H^*(X,R)$.

Proof. The proof is a straightforward application of the Leray spectral sequence of the map $\phi : M \times G EG \to M/G = X$, noting that the fibers are spaces of the form $BG_x$, which have trivial reduced cohomology with $R$–coefficients.

Consider now the so-called classifying map $p : M \times G EG \to BG$ induced by projection onto the second factor; here $BG = EG/G$ denotes the classifying space of $G$. Assuming that $R$ is a ring where all the $|G_x|$ are invertible, we obtain an algebra homomorphism $q = (\phi^*)^{-1}p^* : H^*(BG,R) \to H^*(X,R)$. We now use this to define characteristic classes for the orbifold $X$.

Definition 3.2. Let $X = P_X/O(n)$ be a reduced orbifold such that all its local transformation groups are of odd order. The $i$–th Stiefel–Whitney class $w_i(X) \in H^i(X,\mathbb{Z}/2)$ is defined as $w_i(X) = q(w_i)$, where $w_1, \ldots, w_n \in H^*(BO(n),\mathbb{Z}/2)$ are the usual universal Stiefel–Whitney classes.

Similarly we can define Chern classes as follows

Definition 3.3. Let $X = P_X/U(n)$ be a reduced complex orbifold. Let $R$ denote a ring where all the orders of the local transformation groups are invertible. The $i$–th Chern class $c_i(X) \in H^{2i}(X,R)$ is defined as $c_i(X) = q(c_i)$, where $c_1, \ldots, c_n \in H^*(BU(n),R)$ are the usual universal Chern classes and $q$ is the analogue of the map previously defined.

An appropriate ring $R$ can be constructed from the integers by inverting the least common multiple of the orders of all the local transformation groups; the rational numbers $\mathbb{Q}$ are of course also a suitable choice.

More generally what we see is that with integral coefficients, the equivariant cohomology of $M$ will have interesting torsion classes. Unfortunately, integral computations are notoriously difficult, especially when finite group cohomology is involved. The mod $p$ equivariant cohomology of the frame bundle $M$ will contain interesting information about the action; in particular its Krull Dimension will be equal to the maximal $p$–rank of the isotropy subgroups (see [31]). However for our geometric applications it is convenient to use an equivariant cohomology theory which has substantial torsion–free information. That is where K-theory naturally comes in, as instead of cohomology the basic object is a representation ring.
Less well known than ordinary equivariant cohomology is the Bredon cohomology associated to a group action. It is in fact the most adequate equivariant cohomology theory available. We briefly sketch its definition for the case of compact Lie groups (we refer to [3], [24], [16] and [13], appendix).

Let $Or(G)$ denote the homotopy category whose objects are orbits $G/H$, $H \subset G$ and where $Mor_{Or(G)}(G/H, G/K)$ is the set of $G$–homotopy classes of $G$ maps between these orbits. A coefficient system for Bredon cohomology is a functor $F : Or(G)^{op} \to Ab$. Now for any $G$–CW complex $M$, define $C_{G}^{*}(M) : Or(G) \to Ab$ by setting $C_{G}^{*}(M)(G/H) = C_{*}(M^{H}/WH_{0})$. Here $C_{*}(-)$ denotes the cellular chain complex and $WH_{0}$ is the identity component of $NH/H$. We now define $C_{G}^{*}(M; F) = Hom_{Or(G)}(C_{G}^{*}(M), F)$ and $H_{G}^{*}(M; F) = H(C_{G}^{*}(M; F))$. One can see that in fact for each $n \geq 0$, $C_{G}^{n}(M; F)$ is the direct product, taken over all orbits $G/H \times D^{n}$ of $n$–cells in $M$, of the groups $F(G/H)$. We have that $C_{G}^{*}(M; F) = Hom_{Or(G)}(C_{G}^{*}(M), F)$; indeed this is determined on $Or(G, M)$, the full subcategory containing all orbit types in $M$. Note that from its definition there will be a spectral sequence (see [16])

$$E_{2} = Ext_{Or(G)}^{1}(H_{*}(M), F) \Rightarrow H_{G}^{*}(M; F)$$

where $H_{*}(M)(G/H) = H_{*}(M^{H}/WH_{0}, \mathbb{Z})$.

In our applications we will assume that the isotropy is all finite. Our basic example will involve the complex representation ring functor $R(-)$; i.e. $G/H \mapsto R(H)$. In this case the fact that $R(H)$ is a ring implies that Bredon cohomology will have a natural ring structure (constructed using the diagonal).

We will be interested in using the rationalized functor $R_{Q} = R(-) \otimes \mathbb{Q}$. For $G$ finite, it was shown in [32] that $R_{Q}$ is an injective functor; similarly for $\Gamma$ a discrete group and for proper actions with finite isotropy it was shown in [24] that $R_{Q}$ is injective. This result will also hold for $G$–CW complexes with finite isotropy, where $G$ is a compact Lie group. This follows by adapting the methods in [34] and described in [16]. The key technical ingredient is the surjectivity of $R_{Q}(H) \to \lim_{K \in F_{p}(H)} R_{Q}(K)$ where $H$ is any finite subgroup of $G$ and $F_{p}(H)$ denotes the family of all proper subgroups in $H$. Hence we have the following basic isomorphism: $H_{G}^{*}(M, R_{Q}) \cong Hom_{Or(G)}(H_{*}(M), R_{Q})$. As we shall see, in the case of a quotient orbifold $M/G$, its orbifold cohomology is additively isomorphic to $H_{G}^{*}(M, R(-) \otimes \mathbb{R})$.

Suppose that $X = M/G$ is a quotient orbifold. Using equivariant K-theory we will show that the Bredon cohomology $H_{G}^{*}(M, R_{Q})$ is independent of the presentation $M/G$ and canonically associated with the orbifold $X$ itself. A direct proof with more general coefficients would be of some interest. In the case of a reduced orbifold, we can canonically
associate to it the Bredon cohomology of its frame bundle; motivated by this we introduce the following

**Definition 3.4.** Let $X$ be a reduced orbifold; we define its orbifold Bredon cohomology with $R_Q$-coefficients as $H^*_\text{orb}(X, R_Q) = H^*_G(P_X, R_Q)$.

**Remark 3.5.** The approach in [26] and [27] using groupoids and their classifying spaces is also an important and basic point of view in studying orbifolds. From our perspective, if $\mathcal{G}$ is the groupoid associated to the quotient orbifold $X = M/G$, then in fact $B\mathcal{G} \simeq M \times_G EG$, where $B\mathcal{G}$ denotes the classifying space of the groupoid $\mathcal{G}$. Hence the ‘cohomology of the orbifold’ can be identified with the usual equivariant cohomology, and the Leray spectral sequence of the projection onto $X = M/G$ gives rise to the basic calculational device for its computation. In fact the $E_2$ term of this spectral sequence can be identified with the Bredon cohomology of the $G$-space $M$ with coefficients in the graded functors $G/H \mapsto H^*(H, \mathbb{Z})$, the ordinary group cohomology. Note that if $X$ is an orbifold which can be presented as a quotient in two different ways, say $X \cong M/G \cong M'/G'$, then we have homotopy equivalences $B\mathcal{G} \simeq M \times_G EG \simeq M' \times_{G'} EG'$, hence the equivariant cohomology is an intrinsic invariant of a quotient orbifold. Similarly one can define the orbifold fundamental group, denoted $\pi_1^{\text{orb}}(X)$ as $\pi_1(B\mathcal{G})$; for a quotient orbifold $X = M/G$ it will be isomorphic to $\pi_1(M \times_G EG)$. We emphasize that the ‘cohomology of the orbifold $X$’ is very different from the ‘orbifold cohomology of $X$’. Indeed as we shall see the latter is actually additively isomorphic to the orbifold Bredon cohomology defined above.

### 4. Orbifold Bundles and Equivariant K-Theory

We will recall (see [30]) an invariant constructed from orbifold vector bundles, as defined in §2.

**Definition 4.1.** Given a compact orbifold $X$ we define $K_{\text{orb}}(X)$ to be the Grothendieck group of isomorphism classes of orbifold vector bundles on $X$.

Recall that an orbifold morphism $f : X \to Y$ is an orbifold map together with an isomorphism class of compatible system. From the definition, one can verify that orbifold bundles over $Y$ pull back to orbifold bundles over $X$ under an orbifold morphism. In fact one can establish the

**Proposition 4.2.** Suppose that $(f, \xi) : X \to Y$ is an orbifold morphism. Then, it induces a ring homomorphism $f_*^\xi : K_{\text{orb}}(Y) \to K_{\text{orb}}(X)$. 
An important example of an orbifold morphism is the projection map $p: M \to M/G$, where $G$ is a compact Lie group acting almost freely on a manifold $M$. Therefore, if $E$ is an orbifold vector bundle over $M/G$, $p^*E$ is a smooth vector bundle over $M$. It is obvious that $p^*E$ is $G$-equivariant. Conversely, if $F$ is a $G$-equivariant bundle over $M$, $F/G \to M/G$ is an orbifold vector bundle over $X = M/G$. Therefore, we have a canonical identification between $K_{orb}(X)$ and $K_G(M)$.

**Proposition 4.3.** Let $X = M/G$ be a quotient orbifold. Then the projection map $p: M \to X$ induces an isomorphism $p^*: K_{orb}(X) \to K_G(M)$.

In particular, if $X$ is a reduced orbifold, we can identify its orbifold K-theory with the equivariant K-theory of its frame bundle. Based on the above we can now introduce an alternative definition of orbifold K-theory for quotient orbifolds.

**Definition 4.4.** Let $X = M/G$ denote a compact quotient orbifold, then we define its orbifold K-theory as $K^*_{orb}(X) = K_G^*(M)$.

Note that by Bott periodicity, this invariant is $\mathbb{Z}/2$-graded. It is of course possible to extend the original definition of orbifold K-theory in the usual way; indeed if $X$ is an orbifold, then $X \times \mathbb{S}^n$ is also an orbifold and moreover the inclusion $i: X \to X \times \mathbb{S}^n$ is an orbifold morphism. Let $i^*_n: K_{orb}(X \times \mathbb{S}^n) \to K_{orb}(X)$; then we can define $K^*_{orb}(X) = \ker i^*_n$. However the canonical identification outlined above shows that for a quotient orbifold this extension must agree with the usual extension for equivariant complex K-theory to a $\mathbb{Z}/2$-graded theory. Given this, we have chosen to define orbifold K-theory by using equivariant K-theory, as it will enable us to make some meaningful computations.

Another point to make is that the homomorphism $G \to G_{eff}$ will induce a ring map $K^*_{orb}(X_{red}) \to K^*_{orb}(X)$.

We also introduce the (K-theoretic) orbifold Euler characteristic.

**Definition 4.5.** The orbifold Euler characteristic of $X$ is

$$\chi_{orb}(X) = \dim \mathbb{Q} K^0_{orb}(X) \otimes \mathbb{Q} - \dim \mathbb{Q} K^1_{orb}(X) \otimes \mathbb{Q}$$

It remains to show that these invariants are tractable or even well-defined.

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4This has also been proposed by J.Morava [28] and it also appears implicitly in [36]

5This definition extends the string-theoretic orbifold Euler characteristic which has been defined for global quotients.
Proposition 4.6. If $X = M/G$ is a compact quotient orbifold for a compact Lie group $G$, then $K^*_\text{orb}(X)$ is a finitely generated abelian group, and the orbifold Euler characteristic is well-defined.

Proof. We know that $M$ is a finite, almost free $G$–CW complex. It follows from [33] that there is a spectral sequence converging to $K^*_\text{orb}(X) = K^*_G(M)$, with $E_1^{p,q}$–term equal to zero if $q$ is odd and equal to $\bigoplus_{\sigma \in X^{(p)}} R(G_\sigma)$ where $X^{(p)}$ denotes the collection of $p$–dimensional cells in $X$ and $R(G_\sigma)$ denotes the complex representation ring of the stabilizer of $\sigma$ in $M$. In fact the $E_2$ term is simply the homology of a chain complex assembled from these terms. By our hypotheses, each $G_\sigma$ is finite, and there are finitely many such cells, hence each term is finitely generated as an abelian group and there are only finitely many of them. We conclude that $E_1$ satisfies the required finiteness conditions, and so must its subquotient $E_\infty$, whence the same holds for $K^*_\text{orb}(X) = K^*_G(M)$. \[\square\]

Corollary 4.7. \[\chi_{\text{orb}}(X) = \sum_{\sigma \in X} (-1)^{\dim \sigma} \rank R(G_\sigma)\]

where the $\sigma$ range over all cells in $X$, with corresponding isotropy subgroup $G_\sigma$.

The spectral sequence used above is in fact the equivariant analogue of the Atiyah–Hirzebruch spectral sequence. We have described the $E_1$–term as a chain complex assembled from the complex representation rings of the isotropy subgroups. The $E_2$–term coincides with the equivariant Bredon cohomology of $M$ with coefficients in the representation ring functor, denoted $H^*_G(M, R(-))$. We shall see later that in fact this spectral sequence collapses rationally at the $E_2$–term (see [24], page 28). As a consequence of this we have that $H^*_\text{orb}(X)$, $K^*_\text{orb}(X) \otimes \mathbb{R}$ and $H^*_G(M, R(-) \otimes \mathbb{R})$ are all additively isomorphic and in fact the last two invariants have the same ring structure (provided we take the $\mathbb{Z}/2$–graded version of Bredon cohomology).

Computations for equivariant K-theory can be quite complicated. Our approach will be to study the case of global quotients arising from actions of finite and more generally discrete groups. The key computational tool will be an equivariant Chern character, which we will define for almost free actions of compact Lie groups. This will be used to establish the additive rational equivalences outlined above. However it should be noted that $K^*_\text{orb}(X)$ can contain important torsion classes and its rationalization is a rather crude approximation.

Let us review the special case of a global quotient, where the K-theoretic invariant above is more familiar.
Example 4.8. Let $G$ denote a finite group acting on a manifold $Y$ so that $X = Y/G$ is an orbifold (a global quotient). In this case we know that there is an isomorphism $K_{orb}(X) \cong K_G(Y)$. When tensored with the rationals this equivariant K-theory decomposes as a direct sum and we obtain the well–known formula

$$K^*_{orb}(X) \otimes \mathbb{Q} \cong \bigoplus_{\{g \mid g \in G\}} K^*(Y^{(g)}/Z_G(g)) \otimes \mathbb{Q}$$

where $(g)$ is the conjugacy class of $g \in G$ and $Z_G(g)$ denotes the centralizer of $g$ in $G$. Note that this decomposition appears in [3] but it can be traced back (independently) to [34], [35] and [23].

One of the key elements in the theory of orbifolds is the orbifold resolution of the singular set. For the case of a global quotient $X = Y/G$, it can be shown (see [12]) that we have a homeomorphism

$$\tilde{\Sigma}X \cong \bigsqcup_{\{g \mid g \neq 1\}} Y^{(g)}/Z_G(g)$$

whence in fact we see that $K^*_{orb}(X) \cong \mathbb{Q} K^*(\tilde{\Sigma}X \sqcup X)$. The conjugacy classes are used to index the so-called ‘twisted sectors’ arising in this decomposition. We will use this as a model for our result in the following section.

5. A Decomposition for Orbifold K-theory

We will now prove a decomposition for orbifold K-theory using the methods developed by Lück and Oliver in [24]. The basic technical result we will use is the construction of an equivariant Chern character. Cohomology will be assumed $\mathbb{Z}/2$–graded in the usual way.

Theorem 5.1. Let $X = M/G$ be a compact, quotient orbifold where $G$ is a compact Lie group. Then there is an equivariant Chern character which defines a rational isomorphism of rings

$$K^*_{orb}(X) \cong \bigotimes_{\{(C) \mid C \subset G \text{ cyclic}\}} [H^*(M^C/Z_G(C)) \otimes \mathbb{Q}(\zeta_C)]^{W_G(C)}$$

where $\zeta_C$ ranges over conjugacy classes of cyclic subgroups and $W_G(C) = N_G(C)/Z_G(C)$, a necessarily finite group.

Proof. As has been remarked, we can assume that $M$ is a finite, almost free $G$–CW complex. Now, as in [24] and [3], the main idea of the proof is to construct a natural Chern character for any $G$–space as above, and then prove that it induces an isomorphism
for orbits of the form $G/H$, where $H \subset G$ is finite. Using induction on the number of orbit types and a Mayer–Vietoris sequence will complete the proof.

To begin we recall the existence (see [24], Prop. 3.4) of a ring homomorphism

$$\psi : K_{N_G(C)}^*(M^C) \to K_{Z_G(C)}^*(M^C) \otimes R(C);$$

in this much more elementary setting it can be defined by its value on vector bundles, namely

$$\psi([E]) = \sum_{V \in \text{Ir}(C)} [\text{Hom}_C(V, E)] \otimes [V]$$

for any $N_G(C)$–vector bundle $E \to M^C$. We make use of the natural maps

$$K_{Z_G(C)}^*(M^C) \otimes R(C) \to K_{Z_G(C)}^*(EG \times M^C) \otimes R(C) \to K^*(EG \times Z_G(C) M^C) \otimes R(C)$$

as well as the Chern map

$$K^*(EG \times Z_G(C) M^C) \otimes R(C) \to H^*(EG \times Z_G(C) M^C; \mathbb{Q}) \otimes R(C) \cong H^*(M^C/Z_G(C); \mathbb{Q}) \otimes R(C)$$

Note that the isomorphism above is due to the crucial fact that all the fibers of the projection map $EG \times Z_G(C) M^C \to M^C/Z_G(C)$ are rationally acyclic, as they are classifying spaces of finite groups. Finally we make use of the ring map $R(C) \otimes \mathbb{Q} \to \mathbb{Q}(\zeta_{|C|})$, with kernel the ideal of elements whose characters vanish on all generators of $C$. Putting all of this together, and using the restriction map, we obtain a natural ring homomorphism

$$K_G^*(M) \otimes \mathbb{Q} \to H^*(M^C/Z_G(C), \mathbb{Q}(\zeta_{|C|}))^{N_G(C)/Z_G(C)}.$$    

Here we have taken invariants on the right hand side, as the image naturally lands there. Verification of the isomorphism on $G/H$ is an elementary consequence of the isomorphism $K^*_G(G/H) \cong R(H)$ and details are left to the reader.

\begin{claim}
Corollary 5.2. Let $X = M/G$ be a compact quotient orbifold; then there is an additive decomposition

$$K^*_G(M) \otimes \mathbb{Q} \cong \bigoplus_{\{u \in G\}} K^*(M^{(u)}/Z_G(u)) \otimes \mathbb{Q}$$

\end{claim}
Note that the (finite) indexing set will consist of the $G$–conjugacy classes of elements in the isotropy subgroups–all of finite order. These should be considered analogues of the ‘twisted sectors’ which arise in the case of global quotients.

Our immediate goal is to relate this decomposition to specific information about the orbifold $X$.

**Theorem 5.3.** Let $X = M/G$ denote a compact quotient orbifold; then there is a homeomorphism

$$\bigsqcup_{\{a \mid a \in G\}} M^{(a)}/Z_G(a) \cong \tilde{\Sigma}X \sqcup X$$

and in particular $K^*_\text{orb}(X) \cong_\mathbb{Q} K^*(\tilde{\Sigma}X \sqcup X)$.

**Proof.** We begin by considering the situation locally. Suppose that we have a chart in $M$ of the form $V \times_H G$, mapping onto $V/H$ in $X$, where we assume $H \subset G$ is a finite group. Then

$$(V \times_H G)^{(a)} = \{H(x, u) \mid H(x, ua) = H(x, u)\} = \{H(x, u) \mid uau^{-1} = h \in H, x \in V^h\}$$

Let us now define an $H$ action on $\bigsqcup_{t \in H} (V^{(t)}, t)$ by $k(x, t) = (kx, ktk^{-1})$. We define a map

$$\phi : (V \times_H G)^{(a)} \to \bigsqcup_{t \in H} (V^{(t)}, t)/H$$

by $\phi(H(x, u)) = [x, uau^{-1}]$. We check that this is well–defined; indeed if $H(x, u) = H(y, v)$ then there exists a $k \in H$ with $(y, v) = k(x, u)$, so $y = kx, v = ku$. This means that $vav^{-1} = kua^{-1}k^{-1}$ and so $[y, vav^{-1}] = [kx, kua^{-1}k^{-1}] = [x, uau^{-1}]$ as $k \in H$. Now suppose that $z \in Z_G(a)$; then $\phi(H(x, u)z) = \phi(H(x, uz)) = [x, uzaz^{-1}u^{-1}] = [x, uau^{-1}] = \phi(H(x, u))$; hence we have a well-defined map on the orbit space

$$\bar{\phi} : (V \times_H G)^{(a)}/Z_G(a) \to \bigsqcup_{t \in H} (V^{(t)}, t)/H.$$ 

This map turns out to be injective, indeed if $(x, uau^{-1}) = k(y, vav^{-1})$ for some $k \in H$, then $x = ky$ and $a = u^{-1}kva^{-1}k^{-1}u$, hence $u^{-1}kv \in Z_G(a)$ and $H(x, u)(u^{-1}kv) = H(x, kv) = H(ky, kv) = H(y, v)$. The image of $\bar{\phi}$ consists of the $H$–equivalence classes of pairs $(x, t)$ where $x \in V^{(t)}$ and $t$ is conjugate to $a$ in $G$.

Putting this together and noting that $(V \times_H G)^{(a)} = \emptyset$ unless $a$ is conjugate to an element in $H$, we observe that we obtain a homeomorphism
To complete the proof of the theorem it suffices to observe that by the compatibility of charts, the local homeomorphisms on fixed-point sets can be assembled to yield the desired global homeomorphism on $M$.

**Remark 5.4.** The result above in fact shows that for a compact quotient orbifold $X = M/G$, the orbifold resolution $\tilde{\Sigma}X$ is isomorphic as an orbifold to $\bigcup\{a \in G \} M^\langle a \rangle / Z_G(a)$.

**Corollary 5.5.** Up to regrading there is an additive isomorphism between the orbifold cohomology of $X$ and the rationalized orbifold $K$-theory of $X$.

*Proof.* Indeed we recall that in [12], the orbifold cohomology $H^\ast_{orb}(X, \mathbb{Q})$ is defined to be additively isomorphic to $H^\ast(\tilde{\Sigma}X \sqcup X, \mathbb{Q})$. Hence via the Chern character map we get the isomorphism above. □

**Corollary 5.6.** $\chi_{orb}(X) = \chi(X \sqcup \tilde{\Sigma}X)$

**Example 5.7.** We will now consider the case of a weighted projective space $\mathbb{C}P(p,q)$ where $p$ and $q$ are assumed to be distinct prime numbers. Let $S^1$ act on the unit sphere in $\mathbb{C}^2$ via $(v, w) \mapsto (z^p v, z^q w)$. The space $\mathbb{C}P(p,q)$ is the quotient under this action, and it has two singular points, $x = [1,0]$ and $y = [0,1]$. In this case the Lie group used to present the orbifold is $SO(2) = S^1$ and the corresponding isotropy subgroups are precisely $\mathbb{Z}/q$ and $\mathbb{Z}/p$. Their fixed point sets are disjoint circles in $S^3$, hence the formula for the orbifold $K$-theory yields

$$K^\ast_{orb}(\mathbb{C}P(p,q)) \cong \mathbb{Q}(\zeta_p \times \mathbb{Q}(\zeta_q) \times \Lambda(b_2))$$

where $\zeta_p$, $\zeta_q$ are the corresponding primitive roots of unity (compare with Corollary 2.7.6 in [2]). More explicitly we have an isomorphism

$$K^\ast_{orb}(X) \otimes \mathbb{Q} \cong \mathbb{Q}[x]/(x^{p-1} + x^{p-2} + \cdots + x + 1)(x^{q-1} + x^{q-2} + \cdots + x + 1)(x^2),$$

from which we see that the orbifold Euler characteristic is given by $\chi_{orb}(\mathbb{C}P(p,q)) = p + q$.

**Remark 5.8.** The decomposition described above is based on entirely analogous results for proper actions of discrete groups (see [24]). In particular this includes the case of arithmetic orbifolds, also discussed in [1] and [20]. Let $G(\mathbb{R})$ denote a semisimple $\mathbb{Q}$-group,
and $K$ a maximal compact subgroup. Let $\Gamma \subset G(\mathbb{Q})$ denote an arithmetic subgroup. Then $\Gamma$ acts on $X = G(\mathbb{R})/K$, a space diffeomorphic to Euclidean space. Moreover if $H$ is any finite subgroup of $\Gamma$, then $X^H$ is a totally geodesic submanifold, hence also diffeomorphic to Euclidean space. We can make use of the Borel-Serre completion $\overline{X}$ (see [8]). This is a contractible space with a proper $\Gamma$–action such that the $X^H$ are also contractible (we are indebted to A.Borel and G. Prasad for outlining a proof of this [7]) but having a compact orbit space $\Gamma \backslash \overline{X}$. In this case we obtain the multiplicative formula

$$K^*_\Gamma(X) \otimes \mathbb{Q} \cong K^*_\Gamma(\overline{X}) \otimes \mathbb{Q} \cong \prod_{\{(C) \mid C \subset \Gamma \text{ cyclic}\}} H^*(BZ_{\Gamma}(C), \mathbb{Q}(\zeta_{|C|}))^{N_{\Gamma}(C)}.$$ 

This allows us to express the orbifold Euler characteristic of $\Gamma \backslash X$ in terms of group cohomology:

$$\chi_{\text{orb}}(\Gamma \backslash X) = \sum_{\{(\gamma) \mid \gamma \in \Gamma \text{ of finite order}\}} \chi(BZ_{\Gamma}(\gamma)).$$

**Example 5.9.** Another example of some interest is that of compact, two–dimensional hyperbolic orbifolds. They are described as quotients of the form $\Gamma \backslash PSL_2(\mathbb{R})/SO(2)$, where $\Gamma$ is a Fuchsian subgroup. The groups $\Gamma$ can be expressed as extensions of the form

$$1 \to \Gamma' \to \Gamma \to G \to 1$$

where $\Gamma'$ is the fundamental group of a closed orientable Riemann surface, and $G$ is a finite group (i.e. they are virtual surface groups). Geometrically we have an action of $G$ on a surface $\Sigma$ with fundamental group $\Gamma'$; this action has isolated singular points, with cyclic isotropy. The group $\Gamma$ is $\pi_1(EG \times_G \Sigma)$, the fundamental group of the associated Borel construction. Assume that $G$ acts on $\Sigma$ with $n$ orbits, having respective isotropy groups $\mathbb{Z}/v_1, \ldots, \mathbb{Z}/v_n$ and with quotient a surface $W$ of genus equal to $g$. The formula then yields (compare with the description in [25], pg. 563)

$$K^*_{\text{orb}}(W) \otimes \mathbb{Q} \cong \tilde{R}(\mathbb{Z}/v_1) \otimes \mathbb{Q} \times \cdots \times \tilde{R}(\mathbb{Z}/v_n) \otimes \mathbb{Q} \times K^*(W) \otimes \mathbb{Q}$$

In this expression $\tilde{R}$ denotes the reduced representation ring, which arises because the trivial cyclic subgroup only appears once. From this we see that

$$\dim_{\mathbb{Q}} K^0_{\text{orb}}(W) \otimes \mathbb{Q} = \sum_{i=1}^{n} (v_i - 1) + 2, \quad \dim_{\mathbb{Q}} K^1_{\text{orb}}(W) \otimes \mathbb{Q} = 2g,$$

and so we have that $\chi_{\text{orb}}(W) = \sum_{i=1}^{n} (v_i - 1) + \chi(W)$. 

Remark 5.10. It has been brought to our attention that the decomposition formula is
analogous to a decomposition of equivariant algebraic K-theory which appears in work of
Vezzosi–Vistoli [39] and B.Toen (see [37], page 29) in the context of algebraic Deligne–Mumford stacks. A detailed comparison would seem worthwhile.

Remark 5.11. It should also be observed that the decomposition above could equally
well have been stated in terms of the computation of Bredon cohomology mentioned
previously, namely $H^*_G(M, R_Q) \cong Hom_{Or(G)}(H_*(M), R_Q)$ and the collapse at $E_2$ of the rationalized Atiyah–Hirzebruch spectral sequence: $K^*_\text{orb}(X) \otimes \mathbb{Q} \cong H^*_G(M, R_Q)$. It had
been shown previously that a Chern character with expected naturality properties inducing
such an isomorphism cannot exist; in particular [16] contains an example where such
an isomorphism is impossible. However the example is for a circle action with stationary
points, our result shows that almost free actions of compact Lie groups do indeed give
rise to appropriate equivariant Chern characters. A different equivariant Chern character
for abelian Lie group actions was defined in [5], using a $\mathbb{Z}/2$–indexed de Rham coho-
mology (called delocalized equivariant cohomology). Presumably it must agree with our
decomposition in the case of almost free actions. E.Getzler has pointed out an alternative
approach involving cyclic cohomology (see [6]).

Remark 5.12. If $X = M/G$ is a quotient orbifold, then the K-theory of $M \times_G EG$ and the
orbifold K-theory are related by the Atiyah–Segal Completion Theorem in [4]. Considering
the equivariant K-theory $K^*_G(M)$ as a module over $R(G)$, it states that $K^*(M \times_G EG) \cong
K^*_G(M)\hat{}$, where the completion is taken at the augmentation ideal $I \subset R(G)$.

6. Projective Representations, Twisted Group Algebras and Extensions

We will now extend many of the constructions and concepts used previously to an
appropriately twisted setting. This twisting occurs naturally in the framework of mathe-
matical physics and leads to an interesting notion of ‘totally twisted’ equivariant notions.
In this section we will always assume that we are dealing with finite groups, unless stated
otherwise. Most of the background results which we list appear in [21], Chapter III.

Definition 6.1. let $V$ denote a finite dimensional complex vector space. A mapping
$\rho : G \to GL(V)$ is called a projective representation of $G$ if there exists a function
$\alpha : G \times G \to \mathbb{C}^*$ such that $\rho(x)\rho(y) = \alpha(x, y)\rho(xy)$ for all $x, y \in G$ and $\rho(1) = Id_V$.

6I. Moerdijk has informed us that in unpublished work (1996), he and J.Svensson obtained essentially
the same Chern character construction as that appearing in this paper.
Note that $\alpha$ defines a $C^*$-valued cocycle on $G$, i.e. $\alpha \in Z^2(G, C^*)$. Also there is a one-to-one correspondence between projective representations of $G$ as above and homomorphisms from $G$ to $PGL(V)$. We will be interested in the notion of linear equivalence of projective representations.

**Definition 6.2.** Two projective representations $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ are said to be linearly equivalent if there exists a vector space isomorphism $f : V_1 \to V_2$ such that $\rho_2(g) = f \rho_1(g) f^{-1}$ for all $g \in G$.

If $\alpha$ is the cocycle attached to $\rho$, we say that $\rho$ is an $\alpha$-representation on the space $V$.

We list a couple of basic results

**Lemma 6.3.** Let $\rho_i$, $i = 1, 2$ be an $\alpha_i$-representation on the space $V_i$. If $\rho_1$ is linearly equivalent $\rho_2$, then $\alpha_1$ is equal to $\alpha_2$.

It is easy to see that given a fixed cocycle $\alpha$, we can take the direct sum of any two $\alpha$-representations. Hence we can introduce

**Definition 6.4.** We define $M_\alpha(G)$ as the monoid of linear isomorphism classes of $\alpha$-representations of $G$. Its associated Grothendieck group will be denoted $R_\alpha(G)$.

In order to use these constructions we need to introduce the notion of a twisted group algebra. If $\alpha : G \times G \to C^*$ is a cocycle, we denote by $C^\alpha G$ the vector space over $C$ with basis $\{\bar{g} \mid g \in G\}$ with product $\overline{\tau \cdot \gamma} = \alpha(x, y) \overline{\tau \gamma}$ extended distributively. One can check that $C^\alpha G$ is a $C$-algebra with $1$ as the identity element. This algebra is called the $\alpha$-twisted group algebra of $G$ over $C$. Note that if $\alpha(x, y) = 1$ for all $x, y \in G$, then $C^\alpha G = CG$.

**Definition 6.5.** If $\alpha$ and $\beta$ are cocycles, then $C^\alpha G$ and $C^\beta G$ are equivalent if there exists a $C$ algebra isomorphism $\psi : C^\alpha G \to C^\beta G$ and a mapping $t : G \to C^*$ such that $\psi(\bar{g}) = t(g) \bar{g}$ for all $g \in G$, where $\{\overline{\tau}\}$ and $\{\bar{g}\}$ are bases for the two twisted algebras.

We have a basic result which classifies twisted group algebras.

**Theorem 6.6.** We have an isomorphism between twisted group algebras, $C^\alpha G \simeq C^\beta G$, if and only if $\alpha$ is cohomologous to $\beta$; hence if $\alpha$ is a coboundary, $C^\alpha G \simeq CG$. Indeed, $\alpha \mapsto C^\alpha G$ induces a bijective correspondence between $H^2(G, C^*)$ and the set of equivalence classes of twisted group algebras of $G$ over $C$.

Next we recall how these twisted algebras play a role in determining $R_\alpha(G)$. 
Theorem 6.7. There is a bijective correspondence between $\alpha$–representations of $G$ and $\mathbb{C}^\alpha G$–modules. This correspondence preserves sums and bijectively maps linearly equivalent (respectively irreducible, completely reducible) representations into isomorphic (respectively irreducible, completely reducible) modules.

Definition 6.8. Let $\alpha \in \mathbb{Z}^2(G, \mathbb{C}^*)$. An element $g \in G$ is said to be $\alpha$–regular if $\alpha(g, x) = \alpha(x, g)$ for all $x \in Z_G(g)$.

Note that the identity element is $\alpha$–regular for all $\alpha$. Also one can see that $g$ is $\alpha$–regular if and only if $\overline{g} \cdot x = x \cdot \overline{g}$ for all $x \in Z_G(g)$.

If an element $g \in G$ is $\alpha$–regular, then any conjugate of $g$ is also $\alpha$–regular, hence we can speak of $\alpha$–regular conjugacy classes in $G$. For technical purposes we also want to introduce the notion of a ‘standard’ cocycle; it will be a cocycle $\alpha$ with values in $\mathbb{C}^*$ such that (1) $\alpha(x, x^{-1}) = 1$ for all $x \in G$ and (2) $\alpha(x, g)\alpha(xg, x^{-1}) = 1$ for all $\alpha$–regular $g \in G$ and all $x \in G$. Expressed otherwise, this simply means that $\alpha$ is standard if and only if for all $x \in G$ and for all $\alpha$–regular elements $g \in G$, we have $x^{-1} = x^{-1}$ and $xyx^{-1} = x^{-1}$.

It can be shown that in fact any cohomology class $c \in H^2(G, \mathbb{C}^*)$ can be represented by a standard cocycle, hence we will make this assumption from now on.

The next result is basic:

Theorem 6.9. If $r_{\alpha}$ is equal to the number of non–isomorphic irreducible $\mathbb{C}^\alpha G$–modules, then this number is equal to the number of distinct $\alpha$–regular conjugacy classes of $G$. In particular $R_{\alpha}(G)$ is a free abelian group of rank equal to $r_{\alpha}$.

In what follows we will be using cohomology classes in $H^2(G, \mathbb{S}^1)$, where the $G$–action on the coefficients is assumed to be trivial. Note that $H^2(G, \mathbb{S}^1) \cong H^2(G, \mathbb{C}^*) \cong H^2(G, \mathbb{Q}/\mathbb{Z}) \cong H^3(G, \mathbb{Z})$. We will always use standard cocycles to represent any given cohomology class.

An element $\alpha \in H^2(G, \mathbb{S}^1)$ corresponds to an equivalence class of group extensions

$$1 \to \mathbb{S}^1 \to \widetilde{G}_{\alpha} \to G \to 1$$

The group $\widetilde{G}_{\alpha}$ can be given the structure of a compact Lie group, where $\mathbb{S}^1 \to \widetilde{G}_{\alpha}$ is the inclusion of a closed subgroup. The elements in the extension group can be represented by pairs \{(g, a) | g \in G, a \in \mathbb{S}^1\} with the product $(g_1, a_1)(g_2, a_2) = (g_1g_2, \alpha(g_1, g_2)a_1a_2)$.

Consider the case when $z \in Z_G(g)$; then we can compute the following commutator of lifts:

$$(z, 1)(g, 1)((g, 1)(z, 1))^{-1} = (zg, \alpha(z, g))(z^{-1}g^{-1}, \alpha(g, z)^{-1})$$
This computation is independent of the choice of lifts. It can be seen that this defines a character \( L^\alpha_g \) for the centralizer \( Z_G(g) \), via the correspondence \( z \mapsto \alpha(z, g)\alpha(g, z)^{-1} \). Note that this character is trivial if and only if \( g \) which restrict to scalar multiplication on the central \( S \) character \( L \) fine an associated \( \alpha \) such that \( S \) to an action of \( \tilde{G}_\alpha \) 

Definition 7.1. An \( \alpha \)-twisted \( G \)-vector bundle on \( X \) is a complex vector bundle \( E \to X \) such that \( S^1 \) acts on the fibers through complex multiplication, so that the action extends to an action of \( \tilde{G}_\alpha \) on \( E \) which covers the given \( G \)-action on \( X \).
In fact \( E \to X \) is a \( \tilde{G}_\alpha \)-vector bundle, where the action on the base is via the projection onto \( G \) and the given \( G \)-action. Note that if we divide out by the action of \( S^1 \), we obtain a projective bundle over \( X \). These twisted bundles can be added, hence forming a monoid.

**Definition 7.2.** The \( \alpha \)-twisted \( G \)-equivariant K-theory of \( X \), denoted by \( {}^\alpha K_G(X) \), is defined as the Grothendieck group of isomorphism classes of \( \alpha \)-twisted \( G \)-bundles over \( X \).

As with \( \alpha \)-representations, we can describe this twisted group as the subgroup of \( K_{\tilde{G}_\alpha}(X) \) generated by isomorphism classes of bundles that restrict to multiplication by scalars on the central \( S^1 \). As the \( S^1 \)-action on \( X \) is trivial, we have a natural isomorphism \( K_{S^1}(X) \cong K(X) \otimes R(S^1) \). Composing the restriction with the map \( K(X) \otimes R(S^1) \to R(S^1) \) we obtain a homomorphism \( K_{\tilde{G}_\alpha}(X) \to R(S^1) \); we can define \( {}^\alpha K_G(X) \) as the inverse image of the subgroup generated by the representations defined by scalar multiplication.

Just as in non-twisted equivariant K-theory, this definition extends to a \( \mathbb{Z}/2 \)-graded theory. In fact we can define \( {}^\alpha K^0_G(X) = {}^\alpha K_G(X) \) and \( {}^\alpha K^1_G(X) = \ker [{}^\alpha K_G(S^1 \times X) \to {}^\alpha K_G(X)] \). We can also extend the description given above to express \( {}^\alpha K^*_G(X) \) as a subgroup of \( K^*_G(X) \).

We begin by considering the case \( \alpha = 0 \); this corresponds to the split extension \( G \times S^1 \).

Any ordinary \( G \)-vector bundle can be made into a \( G \times S^1 \)-bundle via scalar multiplication on the fibers; conversely a \( G \times S^1 \)-bundle restricts to an ordinary \( G \)-bundle. Hence we readily see that \( {}^\alpha K^*_G(X) = K^*_G(X) \).

Now we consider the case when \( X \) is a trivial \( G \)-space.

**Lemma 7.3.** Let \( X \) denote a CW-complex with a trivial \( G \)-action; then there is a natural isomorphism \( K(X) \otimes R_\alpha(G) \to {}^\alpha K_G(X) \).

**Proof.** This result is the analogue of the untwisted version (see [33], page 133). The natural map \( R(\tilde{G}_\alpha) \to K_{\tilde{G}_\alpha}(X) \) can be combined with the map \( K(X) \to K_{\tilde{G}_\alpha}(X) \) (which gives any vector bundle the trivial \( G \)-action) to yield a natural isomorphism \( K(X) \otimes R(\tilde{G}_\alpha) \to K_{\tilde{G}_\alpha}(X) \) which covers the restriction to the \( S^1 \)-action; the result follows from looking at inverse images of the subgroup generated by the scalar representation.

The inverse of the map above is given by

\[
[E] \mapsto \bigoplus_{\{[M] \in \text{Irr}(\tilde{G}_\alpha)\}} [\text{Hom}_{\tilde{G}_\alpha}(M, E)] \otimes [M].
\]

Note that only the \( M \) which restrict to scalar multiplication on \( S^1 \) are relevant—these are precisely the irreducible \( \alpha \)-representations.
Let $X$ be a $G$–space and $Y$ a $G'$–space, and let $h: G \to G'$ denote a group homomorphism. If $f : X \to Y$ is a continuous map equivariant with respect to this homomorphism we obtain a map $\alpha f^* : \alpha K_{G'}(Y) \to ^{h^*(\alpha)}K_G(X)$, where $h^* : H^2(G', \mathbb{S}^1) \to H^2(G, \mathbb{S}^1)$ is the map induced by $h$ in cohomology. Let $H \subset G$ be a subgroup; the inclusion defines a restriction map $H^2(G, \mathbb{S}^1) \to H^2(H, \mathbb{S}^1)$. In fact if $\tilde{G}_\alpha$ is the group extension defined by $\alpha \in H^2(G, \mathbb{S}^1)$; then $\text{res}_H^G(\alpha)$ defines the ‘restricted’ group extension over $H$, denoted $\tilde{H}_{\text{res}(\alpha)}$; we have a restriction map $\alpha K_G(X) \to ^{\text{res}(\alpha)}K_H(X)$.

Now consider the case of an orbit $G/H$; then we have $\alpha K_G(G/H) = R_{\text{res}_H^G(\alpha)}(H)$. Indeed we can identify $K_{\tilde{G}_\alpha}(G/H) = K_{\tilde{G}_\alpha}(\tilde{G}_\alpha/\tilde{H}_\alpha) \cong R(\tilde{H}_\alpha)$, and by restricting to the representations that induce scalar multiplication on $\mathbb{S}^1$ we obtain the result.

We are now ready to state a basic decomposition theorem for our twisted version of equivariant K-theory.

**Theorem 7.4.** Let $G$ denote a finite group and $X$ a finite $G$–CW complex. For any $\alpha \in H^2(G, \mathbb{S}^1)$ we have a decomposition

$$\alpha K^*_G(X) \otimes \mathbb{C} \cong \bigoplus_{\langle g \rangle \in G} (K^*(X^{(g)}) \otimes L^\alpha_g)^{Z_G(g)}.$$  

**Proof.** Fix the class $\alpha \in H^2(G, \mathbb{S}^1)$; for any subgroup $H \subset G$, we can associate $H \mapsto R_{\text{res}(\alpha)}(H)$. Note the special case when $H = \langle g \rangle$, a cyclic subgroup. As $H^2(\langle g \rangle, \mathbb{S}^1) = 0$, $R_{\text{res}(\alpha)}(\langle g \rangle)$ is isomorphic to $R(\langle g \rangle)$.

Now consider $E \to X$, an $\alpha$–twisted bundle over $X$; it restricts to an $\text{res}(\alpha)$–twisted bundle over $X^{(g)}$. Recall that we have an isomorphism $\text{res}(\alpha) K^{\ast}_{\langle g \rangle}(X^{(g)}) \cong K^*(X^{(g)}) \otimes R_{\text{res}(\alpha)}(\langle g \rangle)$. Let $u : R_{\text{res}(\alpha)}(\langle g \rangle) \to L^\alpha_g$ denote the $\mathbb{C}Z_G(g)$–map $\chi \mapsto \chi(g)$ described previously, where the centralizer acts on the projective representations as described above. Then the composition

$$\alpha K^*_G(X) \otimes \mathbb{C} \to ^{\text{res}(\alpha)}K^*_\langle g \rangle (X^{(g)}) \otimes \mathbb{C} \to K^*(X^{(g)}) \otimes R_{\text{res}(\alpha)}(\langle g \rangle) \otimes \mathbb{C} \to K^*(X^{(g)}) \otimes L^\alpha_g$$

has image lying in the invariants under the $Z_G(g)$–action. Hence we can put these together to yield a map

$$\alpha K^*_G(X) \otimes \mathbb{C} \to \bigoplus_{\langle g \rangle} (K^*(X^{(g)}) \otimes L^\alpha_g)^{Z_G(g)}.$$
One checks that this induces an isomorphism on orbits \( G/H \); the desired isomorphism follows from using induction on the number of \( G \)-cells in \( X \) and a Mayer-Vietoris argument (as in \([3]\)).

From the definition of twisted orbifold cohomology appearing in \([30]\), we conclude

**Corollary 7.5.** After regrading there is an additive isomorphism between the \( \alpha \)-twisted orbifold cohomology of a global quotient \( M/G \), and the \( \alpha \)-twisted equivariant K-theory \( \alpha K^*_G(M) \otimes \mathbb{C} \).

Note that in the case when \( X \) is a point, we are saying that \( R^*_\alpha(G) \otimes \mathbb{C} \) has rank equal to the number of conjugacy classes of elements in \( G \) such that the associated character \( L^\alpha_g \) is trivial. This of course agrees with the number of \( \alpha \)-regular conjugacy classes, as indeed \( \alpha K^*_G(*) = R^*_\alpha(G) \).

The reader may have noticed that our twisted equivariant K-theory does not have a product structure. Moreover it depends on a choice of a particular cohomology class in \( H^2(G, \mathbb{S}^1) \). Our next goal is to relate the different twisted versions by using a product structure inherited from the additive structure of group extensions.

Suppose we are given \( \alpha, \beta \) in \( H^2(G, \mathbb{S}^1) \), represented by central extensions \( 1 \to \mathbb{S}^1 \to \tilde{G}_1 \to G \to 1 \) and \( 1 \to \mathbb{S}^1 \to \tilde{G}_2 \to G \to 1 \). These give rise to a central extension of the form

\[
1 \to \mathbb{S}^1 \times \mathbb{S}^1 \to \tilde{G}_1 \times \tilde{G}_2 \to G \times G \to 1.
\]

Now we make use of the diagonal embedding \( \Delta : G \to G \times G \) and the product map \( \mu : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1 \) to obtain a central extension

\[
1 \to \mu(\mathbb{S}^1 \times \mathbb{S}^1) \to \tilde{G} \to \Delta(G) \to 1.
\]

This operation corresponds to the sum of cohomology classes, i.e. the extension above represents \( \alpha + \beta \). Note that \( \ker \mu = \{(z, z^{-1})\} \subset \mathbb{S}^1 \times \mathbb{S}^1 \).

Now consider an \( \alpha \)-twisted bundle \( E \to X \) and a \( \beta \)-twisted bundle \( F \to X \). Consider the tensor product bundle \( E \otimes F \to X \). Clearly it will have a \( \tilde{G}_1 \times \tilde{G}_2 \) action on it, which we can restrict to the inverse image of \( \Delta(G) \). Now note that \( \ker \mu \) acts trivially on \( E \otimes F \), hence we obtain a \( \tilde{G} \) action on \( E \otimes F \), covering the \( G \)-action on \( X \). This is an \( \alpha + \beta \)-twisted bundle over \( X \). Hence we have defined a product

\[
\alpha K_G(X) \otimes^\beta K_G(X) \to^{\alpha + \beta} K_G(X)
\]

which prompts us to introduce the following definition.
Definition 7.6. The total twisted equivariant $K$-theory of a $G$-CW complex $X$ is defined as

$$TK^*_G(X) = \bigoplus_{\alpha \in H^2(G, S^1)} \alpha K^*_G(X)$$

Using the product above, we deduce that $TK^*_G(X)$ is a bigraded algebra, as well as a module over $K^*_G(X)$. Note that the indexing set is finite, and that in particular given any homogeneous element a sufficiently high power of it will land in $0^{K^*_G(X)}$.

We obtain a purely algebraic construction from the above when $X$ is a point. Namely we obtain the total twisted representation ring of $G$, defined as

$$TR(G) = \bigoplus_{\alpha \in H^2(G, S^1)} R_\alpha(G),$$

endowed with the graded algebra structure defined above. Note that if a cohomology class is represented by a cocycle $\mu$, then its negative is represented by $\mu^{-1}$. Hence we see that $\rho \mapsto \rho^*$ defines an isomorphism between $R_\alpha(G)$ and $R_{-\alpha}(G)$. Indeed, using vector bundles instead we can easily extend this to show that $\alpha K^*_G(X)$ is isomorphic to $-\alpha K^*_G(X)$. We now provide some examples to illustrate the properties of this construction.

Remark 7.7. It is apparent that the constructions introduced in this section can be extended to the case of a proper action on $X$ of a discrete group $\Gamma$. The group extensions and vector bundles used for the finite group case have natural analogues, and so we can define $\alpha K^*_G(X)$ for $\alpha \in H^2(\Gamma, S^1)$. We will make use of this in the next section.

Example 7.8. Consider the group $G = \mathbb{Z}/2 \times \mathbb{Z}/2$; then $H^2(G, S^1) = \mathbb{Z}/2$ (as can be seen from the Kunneth formula). If $a, b$ are generators for $G$, we have a projective representation $\mu : G \to PGL_2(\mathbb{C})$ given by

$$a \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Note that this gives rise to an extension $\tilde{G} \to GL_2(\mathbb{C})$. Restricted to $\mathbb{Z}/2 \subset S^1$, we get an extension of the form $1 \to \mathbb{Z}/2 \to \tilde{D} \to \mathbb{Z}/2 \times \mathbb{Z}/2 \to 1$; however this is precisely the embedding of the dihedral group in $GL_2(\mathbb{C})$. Hence the extension $\tilde{G}$ must also be non-split, and so represents the non-trivial element $\alpha$ in $H^2(G, S^1)$. One can easily verify that there is only one conjugacy class of $\alpha$ regular elements in $G$, the trivial element. The representation $\mu$ is clearly irreducible, hence up to isomorphism is the unique irreducible
α-twisted representation of $G$. In particular, $R_α(G) \cong \mathbb{Z}(\mu)$. Computing the different products, we obtain

$$TR(G) = \mathbb{Z}[x_0, y_0, \mu_1]/(x_0^2 = 1, y_0^2 = 1, x_0 \mu = \mu, y_0 \mu = \mu, \mu^2 = 1 + x_0 + y_0 + x_0 y_0)$$

The ring is $\mathbb{Z}/2$-graded, as we have indicated with the indices. It has total rank equal to five. More generally, if $M$ is any compact manifold with a $G = \mathbb{Z}/2 \times \mathbb{Z}/2$-action, then one can easily verify that $\chi(TK^*_G(M) \otimes \mathbb{C}) = 6 \chi(M/G) - \chi(M)$.

**Example 7.9.** Let $G = \Sigma_n$, the symmetric group on $n$ symbols. Assume that $n \geq 4$; it is well-known that in this range $H^2(G, \mathbb{S}^1) = \mathbb{Z}/2$; denote the non-trivial class by $\alpha$. Using the decomposition formula, one can calculate (see [38]) $K^*_\Sigma_n(M^n)$, where the group acts on the $n$-fold product of a manifold $M$ by permutation of coordinates (the quotient orbifold is the symmetric product). From this one can recover a corrected version of a formula which appears in [13] for twisted symmetric products. This correction was first observed and corrected by W.Wang in [40]. This can be expressed as follows:

$$\sum q^n \chi(\alpha K^*_\Sigma_n(M^n) \otimes \mathbb{C}) =$$

$$\prod_{n>0} (1 - q^{2n-1})^{-\chi(M)} + \prod_{n>0} (1 + q^{2n-1})^{\chi(M)}[1 + \frac{1}{2} \prod_{n>0} (1 + q^{2n})^{\chi(M)} - \frac{1}{2} \prod_{n>0} (1 - q^{2n})^{\chi(M)}]$$

8. **Twisted Orbifold K-theory and Twisted Bredon Cohomology**

In this section, we shift back to the orbifold point of view. In the case of twisted orbifold cohomology, one can construct twisting using any inner local system [29]. We do not know how to twist orbifold K-theory in such generality. Here, we have the more limited goal of twisting orbifold K-theory using discrete torsion.

Recall that a discrete torsion $\alpha$ of an orbifold $X$ is defined as a class $\alpha \in H^2(\pi_{1\text{orb}}(X), S^1)$. Here, the orbifold fundamental group $\pi_{1\text{orb}}(X)$ is the group of deck translations of the orbifold universal cover of $Y \to X$.

For example, if $X = Z/G$ is a global quotient, the universal cover $Y$ of $Z$ is the orbifold universal cover of $X$. In fact, if $Z \times_G EG$ is the Borel construction for $Z$, then we have a fibration sequence $Z \to Z \times_G EG \to BG$ which gives rise to the group extension $1 \to \pi_1(Z) \to \pi_{1\text{orb}}(X) \to G \to 1$; here we are identifying $\pi_{1\text{orb}}(X)$ with $\pi_1(Z \times_G EG)$. Note that a class $\alpha \in H^2(G, S^1)$ induces a class $f^*(\alpha)$ in $H^2(\pi_{1\text{orb}}(X), S^1)$.

Now suppose that $X = M/G$ is a quotient manifold for a compact Lie group $G$ and $p : Y \to X$ is the orbifold universal cover. Note that $p$ is an orbifold morphism. The same
argument used in pulling back orbifold bundles implies that we can pull back the orbifold principal bundle $M \to X$ to obtain an orbifold principal $G$-bundle $\tilde{M} \to Y$. Furthermore, $\tilde{M}$ is smooth and has a free left $\pi_1^{orb}(X)$-action, as well as a right $G$-action. These can be combined to yield a left $\pi = \pi_1^{orb}(X) \times G$-action. Note that we have
\[
K^*_\pi(\tilde{M}) \cong K^*_G(\tilde{M}/\pi_1^{orb}(X)) = K^*_\text{orb}(X).
\]

Consider a group $\pi$ of the form $\Gamma \times G$, where $\Gamma$ is a discrete group and $G$ is a compact Lie group. Now let $Z$ denote a proper $\pi$-complex such that the orbit space $Z/\pi$ is a compact orbifold. We now fix a cohomology class $\alpha \in H^2(\pi_1^{orb}(X), S^1)$, corresponding to a central extension $\Gamma_{\alpha}$. From this we obtain an extension $\tilde{\pi}_{\alpha} = \tilde{\Gamma}_{\alpha} \times G$. We can define the $\alpha$–twisted $\pi$–equivariant K-theory of $Z$ as $^\alpha K^*_\pi(Z) = ^\alpha K^*_\text{orb}(X)$.

We can also define the total twisted orbifold K-theory of $X$ as
\[
TK^*_\text{orb}(X) = \bigoplus_{\alpha \in H^2(\pi_1^{orb}(X), S^1)} ^\alpha K^*_\text{orb}(X).
\]
This will also have a bigraded ring structure, and it will exhibit an isomorphism between the $\alpha$ component and the $-\alpha$ component.

If $Y$, the orbifold universal cover of $X$, is actually a manifold, then $X$ is said to be a good orbifold (see [25]). The $G$ action on $\tilde{M}$ is free, and in this case the $\alpha$–twisted orbifold cohomology will simply be $^\alpha K^*_\pi(X) = ^\alpha K^*_\text{orb}(Y)$. For the case of a global quotient $X = Z/G$ and a class $\alpha \in H^2(G, S^1)$ it is not hard to verify that in fact $f^*(\alpha) K^*_\text{orb}(X) \cong ^\alpha K^*_G(Z)$, where $f : \pi_1^{orb}(X) \to G$ is defined as before.

In the general case we note that $\pi = \pi_1^{orb}(X) \times G$ acts on $\tilde{M}$ with finite isotropy, hence we can make use of ‘twisted Bredon cohomology’ and a twisted version of the usual Atiyah–Hirzebruch spectral sequence. Fix $\alpha \in H^2(\pi_1^{orb}(X), S^1)$, where $X$ is a compact

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7Alternatively we could have used an equivariant version of orbifold bundles and introduced the twisting geometrically. This works for general orbifolds but we will not elaborate on this here.
orbiﬁed. There is a spectral sequence of the form
\[ E_2 = H^*_\pi(\tilde{M}; R_\alpha(-)) =^\alpha K^*_{\text{orb}}(X) \]

The \( E_1 \) term will be a chain complex built out of the twisted representation rings of the stabilizers, all of which are ﬁnite. In many cases, this twisted Atiyah–Hirzebruch spectral sequence will also collapse at \( E_2 \) after tensoring with the complex numbers. We believe that in fact this must always be the case. In particular we conjecture that if \( X \) denotes a compact good orbifold, with orbifold universal cover the manifold \( Y \), with \( \Gamma = \pi^\text{orb}_1(X) \), and \( \alpha \in H^2(\Gamma, S^1) \), then we have an additive decomposition
\[ ^\alpha K^*_\Gamma(X) \otimes \mathbb{C} \cong \bigoplus_{(\gamma)} H^*(\text{Hom}_{Z_\Gamma(\gamma)}(C_*(Y^{(\gamma)}), L^*_\alpha)) \cong H^*_{\text{orb}}(X, L_\alpha) \]

where \( (\gamma) \) ranges over conjugacy classes of elements of ﬁnite order in \( \Gamma \), \( C_*(\cdot) \) denotes the singular chains and \( L^*_\alpha \) is the character for \( Z_\Gamma(\gamma) \) associated to the twisting.

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