A $d$-SHIFTED DARBOUX THEOREM

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Abstract. We give a local model for $d$-shifted symplectic dg-schemes, or Deligne-Mumford dg-stacks [PTVV]. Locally any such is a product of a “twisted shifted cotangent bundle”, where the twist is given by an element $df$, with $f \in H^{1-d}(\mathcal{O})$, and a quadratic bundle in middle degree. The latter only occurs if $d = 4r + 2$.

Let $X$ be a derived algebraic variety, or more generally a derived stack [TV,L1,Pr], defined over a field $k$ of characteristic zero, and let $L_X$ be its cotangent complex.

Let $P \in \text{Pic}(\mathcal{O}_X)$ be an invertible complex of coherent $D_X$-modules, equipped with a quasi-isomorphism $\lambda_P : (P^*)^{-1} \to P$.

A $P$-shifted symplectic structure on $X$ is a $P$-valued deRham closed 2-form

$$\omega = \omega_2 + \omega_3 + \omega_4 + \cdots \in P \otimes F^2 \Omega_X$$

such that i) $\omega_2 : L^*_X \to L_X \otimes P$ is a quis, and ii) the induced map $\omega_2 \otimes \lambda_P : L^*_X \to L_X \otimes P^* \to L_X \otimes P$ coincides with $\omega_2$.

When $P = \mathcal{O}_X[-d]$, and for appropriate choice of sign $\lambda_{\mathcal{O}_X[-d]}$, this is precisely the $d$-shifted symplectic structure of [PTVV]; see also [Co], [AKSZ] and the extensive physics literature for earlier avatars of this definition.

A $P$-shifted symplectic structure on $X$ defines an element $[\omega_2]$ in the Grothendieck-Witt group $KH_0(X,P,\lambda_P)$, the 0'th Hermitian $K$-groups of $X$ [Sc], refining the class of $L_X$ in $K_0(X)$.

There is a nice class of examples of $P$-shifted symplectic structures, the twisted shifted cotangent bundles. We outline their definition, repeating it in §3.2 in more detail.

Let $Y$ be a derived algebraic variety such that $L_Y$ is perfect, let $P = \mathcal{O}_Y[-d]$, for $d > 0$, $\lambda_P = \pm 1$, and suppose the sheaf $H^i(P^{-1} \otimes L^*_Y) = 0$ for $i > 0$.

Then the shifted cotangent bundle $\bar{X} = L_Y \otimes P = \text{Spec Sym}_{\mathcal{O}_Y}(P^{-1} \otimes L^*_Y) \to Y$ canonically has a $P$-shifted symplectic structure $d\lambda$, where $\lambda : \mathcal{O}_{\bar{X}} \to P \otimes L_X$ is the Liouville 1-form, defined just as in classical symplectic geometry.

Now let $\xi \in H^2(Y, H^2(P \otimes F^1 \Omega_Y))$, so $\bar{\xi} = \xi_1 + \xi_2 + \cdots$ is de Rham closed, with $\xi_1 : \mathcal{O}_Y[1] \to P \otimes L_Y$.

Then $\xi_1$ defines a one parameter family of deformations of $\bar{X}$, whose generic fiber is $X = \text{Spec Sym}_{\mathcal{O}_Y}(P^{-1} \otimes L^*_Y) \overset{\xi_1}{\longrightarrow} \mathcal{O}_Y[-1]) \times_{Y \times \mathbb{A}^1} Y \rightarrow Y$, and the deformation

1Many typos. Some imprecisions. Bad prose.
λ of the Liouville form \( \lambda \) to \( X \) combines with \( \pi^* \xi \) to define \( \omega_\xi := (d + D)\lambda - \pi^* \xi \in H^2(X, \mathcal{P} \otimes F^2 \Omega_X) \), a \( \mathcal{P} \)-shifted symplectic structure.

Given a \( \mathcal{P} \)-shifted symplectic stack \( X \), and \( V \in \text{Perf} \mathcal{X} \) a complex with \( H^i(V) = 0 \) for \( i \geq 0 \) and \( \alpha : V^\dagger \to V \) a quis such that \( \alpha = \lambda_{\mathcal{P}} \alpha^\dagger \), where \( V^\dagger = \mathcal{V}^* \otimes \mathcal{P}^{-1} \), then \( X \times V^* = \text{Spec} \text{Sym}_{\mathcal{O}_X} \mathcal{V} \) is also naturally a \( \mathcal{P} \)-shifted symplectic stack.

If \( d \neq 2 \mod 4 \) this will not produce any new examples locally, but for \( d = 2 \mod 4 \) gives us new examples attached to non-zero classes in the Witt group of quadratic forms.

In this paper we prove that locally every \( \mathcal{P} \)-shifted symplectic structure on \( X \) is a product of a twisted shifted cotangent bundle and a quadratic bundle \( V \), with \( V \) zero unless \( d = 2 \mod 4 \). This generalize both the classical Darboux theorem, which is the case \( d = 0 \), and the theorem of \([\text{BBDJ}],^2 \) which is part of the case \( d = 1 \).

We remark that for \( d = 2 \mod 4 \), any quadratic bundle on a variety \( Y \) with a non-trivial class in the Witt group of the function field \( k(Y) \) defines a \( \mathcal{P} \)-shifted symplectic variety \( X \) on the shifted cotangent bundle of \( Y \). If the dimension of the quadratic bundle is even, it is etale locally, but not Zariski locally, a metabolic bundle, and so \( X \) is still etale locally a twisted shifted cotangent bundle, and if the dimension of the bundle is odd we can write it etale locally as the product of a one dimension quadratic bundle with a metabolic bundle.

The proof has 4 steps. We begin by assuming \( X \) is foliated by almost Lagrangian subvarieties—that is, there is a smooth morphism \( \pi : X \to Y \) such that the vertical maps in

\[
\begin{array}{cccc}
\pi^* \mathcal{L}_Y & \longrightarrow & \mathcal{L}_X & \longrightarrow & \mathcal{L}_{X/Y} \oplus 1 \\
\omega_2 & & \omega_2 & & \omega_2 \\
\mathcal{L}^\dagger_{X/Y} & \longrightarrow & \mathcal{L}^\dagger_X & \longrightarrow & \pi^* \mathcal{L}^\dagger_Y \oplus 1
\end{array}
\]

are quasi-isomorphisms, where \( \mathcal{A}^\dagger = \mathcal{A}^* \otimes \mathcal{P}^{-1} \).

This is not necessarily possible—there is an obstruction to doing so if the class of \( \omega_2 \) in the Witt group of \( X \) is non-zero.

In step 1 of the proof we show that the class of \( \omega_2 \) in the Witt group of \( X \) is the only obstruction to finding such a foliation. The main ingredient for this is a dg-Frobenius theorem, stating when a ‘subbundle’ of the cotangent complex can be integrated to a map of dg-schemes, which we formulate and prove in proposition 1.4.

In step 2, we show that, locally on \( X \), the map \( X \to Y \) degenerates in a dg-smooth family to \( \tilde{X} = \mathcal{T}^+[d] \to \tilde{Y} \), the \( d \)-shifted cotangent bundle of \( Y \).

Unlike smooth families of ordinary schemes, this does not imply the de Rham cohomology is constant in the family (any semi-stable family is dg-smooth—for example a smooth elliptic curve degenerating to a nodal one). However we can always define a specialisation map for closed \( p \)-forms which become zero in the de Rham complex of the generic fibre; we get a closed form on the special fibre.\(^3\)

\(^2\)As we were writing this note, Brav, Brussi and Joyce released a preprint, arXiv/1305.6302, with a different normal form for \( -d \)-symplectic varieties. They give a Hamiltonian description. A twisted cotangent bundle has such a description: if \( Y = \text{Spec} \text{Sym} k[z_1, \ldots, z_n][D z_i = h_i] \), \( \xi = df \), \( f \in H^{1-d}(O_Y) \), and \( y_i = d z_i^\dagger \), then \( H = f + \sum D z_i y_i \).

\(^3\)This is only an issue when \( d = 1 \). When \( d > 1 \), we can choose \( Y \) so \( \pi_0(X) \to \pi_0(Y) \)
Using this, we can transport the class $\omega \in F^2 L \Omega_X$; it becomes a class in $F^1 L \Omega_{X'}$, which is the pullback of a class in $F^1 L \Omega_Y$. This class defines a deformation of $X'$ to $X''$, a twisted shifted cotangent bundle.

In step 3 we show that there is an automorphism of $P \otimes L_Y$ which induces an isomorphism from $X$ to $X''$ sending $\omega$ to the standard form on $X''$. This can be accomplished by the Moser technique; our setup is already sufficiently simple that we just do it by hand.

Finally, in step 4 we observe that if the class of $\omega_2$ is non-zero in the Witt group, then over a (Zariski) local neighbourhood we can write $X$ as a product of a quadratic bundle and a $P$-shifted symplectic variety with $\omega_2$ zero in the Witt group. This is an immediate consequence of Witt cancellation and algebraic surgery.

The proofs are written in the body of the paper, perhaps perversely, without using geometric language.

The contents of §0 and §1 are taken from notes for a course one of us taught in 2005, and make no particular claim to originality. §2 is more background, much of which is unnecessary to the proof, but which we find pedagogically reassuring. Finally, in §3 we prove the theorem.

Conventions

In a model category, we write $\sim \rightarrow, \hookrightarrow, \twoheadrightarrow$ to mean morphisms which are, respectively, weak equivalences, cofibrations or fibrations.

We sometimes write $A/B$, and sometimes $\text{cone}(B \rightarrow A)$, for the mapping cone of a morphism $B \rightarrow A$.

Morphisms are strict unless otherwise indicated.

0. The cotangent complex

The various (Quillen equivalent) modern formulations of $(\infty, 1)$-categories give a clean conceptual picture of the cotangent complex and what it means. This formidable technology is unnecessary to the theorems of this paper, which are about local computations, and for which the straightforward and classical theory of model categories suffice. We recall briefly the definitions [Qu,BG,GS].

0.1 We fix a field $k$, $\text{char}(k) = 0$. Denote by $Ch(k)$ the category of chain complexes of $k$-modules, $\cdots \rightarrow M^i \xrightarrow{\partial^i} M^{i+1} \rightarrow \cdots$, equipped with the projective model category structure and the usual symmetric monoidal structure. Weak equivalences are quasi-isomorphisms, and fibrations are degree-wise surjections. Let $\text{cdga}$ denote the category of commutative differential graded algebras over $k$, the commutative algebra objects in $Ch(k)$. The adjunction

$$\text{Sym} : Ch(k) \rightleftarrows \text{cdga} : \text{Forget}$$

allows us to transport the model category structure on $Ch(k)$ to one on $\text{cdga}$ and the adjunction

$$\text{Include} : \text{cdga}_{\leq 0} \rightleftarrows \text{cdga} : \tau_{\leq 0}$$

is an isomorphism, and we have $H^i(X, L \Omega_X) \xrightarrow{\sim} H^i(Y, L \Omega_Y) \xrightarrow{\sim} H^i(X', L \Omega_{X'})$. In this case the composite can be described explicitly on the level of chain complexes in terms of flat sections of the Gauss-Manin connection of the degenerating family.
allows us to transport the model category structure on $\text{cdga}$ to $\text{cdga}_{\leq 0}$, the category of non-positively graded cdga’s over $k$. Explicitly, a morphism $f : A \to B$ in $\text{cdga}_{\leq 0}$ is a weak equivalence if it is a quis, that is $H^n(f) : H^n(A) \to H^n(B)$ is an isomorphism for all $n$, $f$ is a fibration if $f^n : A^n \to B^n$ is a surjection for all $n \leq -1$, and $f$ is a cofibration if it is one in $\text{cdga}$. These have been described in [BG], and we recall: a set of generating cofibrations are given by $k \to \text{Sym} k[n], k \to \text{Sym} \text{Disc}_n$, and $\text{Sym} k[n] \to \text{Sym} \text{Disc}_n$, for $n \leq 0$, where $\text{Disc}_n = k[n - 1] \xrightarrow{id} k[n]$. Cofibrations are closed under pushouts, a retract of a cofibration is a cofibration, if $X_1 \to X_2 \to \ldots$ is a sequence of cofibrations, then $X_1 \to \text{colim} X_i$ is a cofibration, and if $f_j : X_j \to Y_j, j \in J$ is a set of cofibrations, then $\otimes f_j : \otimes X_j \to \otimes Y_j$ is a cofibration.

The model category $\text{cdga}$, and hence $\text{cdga}_{\leq 0}$, is left proper; that is if $f : A \to B$ is a quis, and $A \to C$ is a cofibration, then $C \to B \otimes_A C$ is a quis.

If $A \in \text{cdga}$, $A\text{-mod}$ is the dg-category whose objects are the objects $M \in Ch(k)$ endowed with a strictly associative action of $A$. The free $A$-module functor gives an adjunction

$$A \otimes_k (\cdot) : Ch(k) \leftrightarrows A\text{-mod} : \text{Forget},$$

the induced model category structure on $A\text{-mod}$ is called the projective model category, and is enriched over $Ch(k)$. More generally, given a cofibration $A \to B$ in $\text{cdga}$, the adjunction $B \otimes_A (\cdot) : A\text{-mod} \leftrightarrows B\text{-mod} : \text{Forget}$ is Quillen when both $A\text{-mod}$ and $B\text{-mod}$ are equipped with the projective model category structure, and is a Quillen equivalence if $A \to B$ is a quis.

An object $B$ in a model category $C$ enriched in chain complexes is said to be compact if $\text{Hom}_C(B, \cdot)$ commutes with filtered homotopy colimits. If $C$ has a fixed set of compact generators, we say $B$ is finitely presented, or f.p., if it is in the smallest subcategory containing the generators and closed under shifts, finite coproducts, mapping cones and weak equivalences. For such a category an object is compact if and only if it is a retract of a f.p. object. We apply these notions to $\text{cdga}_{\leq 0}$, $\text{cdga}_{\leq 0}, \text{cdga}_{A\setminus\cdot}$, and to $A\text{-mod}$, where we denote the category of compact objects $\text{Perf}_A$.

0.2 If $f : A \to B$ is in $\text{cdga}$, we write $L_{B/A}$ for the cotangent complex. Recall the definition: Take a cofibrant replacement for $f$ in the model category $\text{cdga}_{\leq 0}$ of algebras over $A$, i.e. factor $f = pi, A \xrightarrow{i} cB \xrightarrow{p} B$, with $i$ a cofibration and $p$ a weak equivalence. Then $L_{B/A} = B \otimes_{cB} \Omega^1_{cB/A} \in B\text{-mod}$.

As $\text{cdga}$ is left proper, this coincides with the cofibrant replacement in the model category of morphisms in $\text{cdga}$. In other words, if we first cofibrantly replace $A$, $cA \xrightarrow{\sim} A$, and then factor $cA \to B$ by $cA \hookrightarrow X \xrightarrow{\sim} B$, the natural map $B \otimes_X \Omega^1_{X/cA} \to L_{B/A}$ is a quis.

$$A \longrightarrow R$$

We recall (i) a square

$$\begin{array}{ccc}
A & \longrightarrow & R \\
\downarrow & & \downarrow \\
B & \longrightarrow & S
\end{array}$$

induces a morphism $S \otimes_B L_{B/A} \to L_{S/R}$ in $S\text{-mod}$, that this is a weak equivalence if both $B \to S, A \to R$ are, and
ii) morphisms $A \to B \to C$ in cdga induce a triangle
\[ C \otimes_B L_{B/A} \to L_{C/A} \to L_{C/B} \xrightarrow{+1} \]
in $C$-mod.

0.3 Given $f : A \to B$ in cdga, factor $f \circ p_i : A \xrightarrow{i} cB \xrightarrow{p} B$ as in 0.2, and define
\[ F^p \Omega_{B/A} = \prod_{n \geq p} \bigwedge_{cB}^{\Omega_1} \big([-n]\big) \in A$-mod,\]
with differential $D + d$, where $D$ is induced from the differential $D$ on $\Omega^{1}_{cB/A}$, and $d$ is the derivation induced from $d : cB \to \Omega_1^{cB/A}$.

Proof. The derivation $d$.

Moreover, if $f : B \to A$ in cdga is such that $H^0(B) \to H^0(A)$ has nilpotent kernel, then the induced map $L\Omega_B \to L\Omega_A$ is a quis in $k$-mod.

In particular, if $B \in \text{cdga}_{\leq 0}$, then $H^i(\text{Spec } B, L\Omega_B) = 0$ for $i < 0$.

1. Basics

1.1 We say that a morphism $X \to Y$ is $k$-connected if $H^i \text{cone}(X \to Y) = 0$ for all $i \geq -k$.\(^4\) Observe $X \to Y$ is $k$-connected if and only if $H^{-k}X \to H^{-k}Y$ is an epimorphism and, for $i > -k$, $H^i(X) \to H^i(Y)$ is an isomorphism.

Proposition. Let $B \in \text{cdga}_{\leq 0}$ be f.p., and $B \to A$ a morphism in cdga. Then for $d \geq 1$ the following are equivalent.

i) $H^0B \to H^0A$ is a surjection, and $H^i(L_{A/B}) = 0$ for $i \geq -d + 1$, and

ii) $H^iB \to H^iA$ is an isomorphism for all $i > -d + 1$, and $H^{-d+1}B \to H^{-d+1}A$ is a surjection, i.e. $B \to A$ is $(d - 1)$-connected.

Moreover, if $K = \text{cone}(B \to A) \in B$-mod, then $H^{-d}(K) = H^{-d}(A \otimes_B K) \xrightarrow{\sim} H^{-d}(L_{A/B})$.

Proof. Replace $B \to A$ by a cofibration, $B \to cA \xrightarrow{\sim} A$. The small object argument shows that $cA$ is built out of $B$ by a (possibly transfinite) attachment of cells; that is there is an ordinal $I$, and for each $\alpha \in I$ an element $f_\alpha \in B[z_\beta, \beta < \alpha]$, such that $cA := B[z_\alpha, \alpha \in I | Dz_\alpha = f_\alpha]$.\(^5\) Furthermore, if $H^iB \to H^iA$ is an isomorphism for all $i > -r + 1$, and $H^{-r+1}B \to H^{-r+1}A$ is a surjection, then we can insist $\text{deg } z_\alpha \leq -r$.

We have $L_{A/B} = A[dz_\alpha, \alpha \in I | D(dz_\alpha) = -df_\alpha]$.

Let $K$ be the mapping cone of $B \to cA$ in $B$-mod. Then $K = B\{z_\alpha\}$, where, given elements $\gamma_\alpha \in cA$, we write $B\{\gamma_\alpha\}$ for the $B$-subalgebra of $cA$ they generate.

The derivation $d : cA \to L_{cA/B}$ factors through $K$, and the mapping cone of the induced map $1 \otimes d : cA \otimes_B K \to L_{cA/B}$ in $cA$-mod is $Q := cA(1 \otimes z_\alpha z_\beta - z_\alpha \otimes$

\(^4\)This is probably usually called $-k$-co-connected. Whatever.

\(^5\)If $A \in \text{cdga}_{\leq 0}$ and $\alpha \in A^n$ is an $n$-cycle, we write $A[x \mid Dx = \alpha]$ for the pushout $A \otimes_{\text{Sym } k[n]} \text{Disc}_n$, and we say $\text{deg } x = n - 1$.\)
$z_β - (-1)^{deg z_α deg z_β} z_β ⊗ z_α$, $α, β ∈ I$ \{1\}. Hence if $deg z_α ≤ -r$ for all $α ∈ I$, then $H^i Q = 0$ for $i > -2r - 1$.

Now, $H^i K = 0$ for $i > -r$ if and only if $H^i (cA ⊗ B K) = 0$ for $i > -r$, as is evident from the spectral sequence $Tor_\mathcal{H}^{HB}(HA, HK) ⇒ H(cA ⊗ B K)$.

Fix $r$ such that $H^i K = 0$ when $i ≥ -r + 1$ and $r ≥ 1$; this is possible as $H^0 B → H^0 A$ is a surjection. Then $H^{i-1} Q = H^i Q = 0$ for $i ≥ -2r$, $H^i (cA ⊗ B K) = 0$ for $i ≥ -r + 1$ and the triangle $cA ⊗ B K → L_{cA/B} → Q \xrightarrow{+1}$ gives $H^i (L_{cA/B}) = 0$ for $i ≥ -r + 1$ and an isomorphism $H^{−r}(K) = H^{−r}(cA ⊗ B K) \xrightarrow{H^{−r}(cA ⊗ B)} H^{−r}(L_{cA/B})$.

Hence (i) implies (ii), on taking $r = d$, but also (ii) implies (i).

**Corollary.** Let $f : B → A$ in $\mathcal{cdga}_{≤0}$ have $L_{A/B} = 0$.

(i) If $H^0 B → H^0 A$ is an isomorphism, then $f$ is a quis.

(ii) If $H^0 B → H^0 A$ is a surjection, then $f$ is a quis.

1.2 We say that $M ∈ A$-mod has Tor amplitude in $[a, b]$ if for all $N ∈ A$-mod such that $H^i (N) = 0$ for $j ≠ i$, $H^i (M ⊗_A N) ≠ 0$ implies $a ≤ i ≤ b$, and that it has Tor dimension $d$, $TorDim M = d$, if it has Tor amplitude in $[−d, 0]$.

Observe $A[d]$ has Tor dimension $d$ if $d ≥ 0$, and that if $M' → M → M'' \xrightarrow{+1}$ is a triangle, then $TorDim M ≤ TorDim M' + TorDim M''$. Hence if $M ∈ \text{Perf} A$, then $M$ has finite Tor dimension.

**Lemma.** If $M ∈ A$-mod has $H^i M = 0$ for $i ≥ -r$ and if $TorDim M ≤ r$, then $M = 0$.

**Proof.** Suppose $M ≠ 0$, and let $j$ be the maximum integer such that $H^j (M) ≠ 0$, so $j < -r$. Then the spectral sequence $Tor_\mathcal{H}^j (H(M), H^0 (A)) ⇒ H(M ⊗^L_A H^0 (A))$ gives $H^j (M ⊗_A H^0 (A)) ≠ 0$, so $-r ≤ j$, a contradiction.

1.3 A morphism $A → B ∈ \mathcal{cdga}_{≤0}$ is a Zariski open embedding if it is quis in $\mathcal{cdga}_A \setminus_1$ to a morphism of the form $A → A[t, ξ | Dξ = tf − 1]$ where $f ∈ A^0$. A cover of a f.p. algebra $A$ is a finite set $f_i : A → B_i$ of Zariski open embeddings such that if $B = B_i$, the augmented bar complex $A → B → B ⊗_A B → · · ·$ is exact; equivalently the maps $H^0 (f_i) : H^0 (A) → H^0 (B_i)$ form a cover in the usual sense. We say a property of an $M ∈ A$-mod holds Zariski locally on $A$ if there is a cover of $A$ for which the the property holds for the pullback $B_i ⊗^L_A M$ on $B_i$ for all $i$.

**Lemma.** Let $A ∈ \mathcal{cdga}_{≤0}$, $H^0 (A)$ finitely presented, $M ∈ \text{Perf} A$. The following are equivalent.

(i) $M$ has Tor amplitude in $[a, b]$.

(ii) $M \xrightarrow{−1} (M_a → · · · → M_b)$, where each $M_i[−i]$ is a summand of a free module $A^{n_i}$, for some $n_i ≥ 0$.

(iii) There is a Zariski cover $A → A'$ such that $M ⊗_A A' \xrightarrow{−1} (M'_a → · · · → M'_b)$, where each $M'_i[−i]$ is a free module $A'^{n_i}$, for some $n_i ≥ 0$.

1.4 Let $R → A ∈ \mathcal{cdga}_{≤0}$, and suppose given (i) a triangle $S → L_{A/R} → L_{A/R} / S \xrightarrow{+1}$ in $A$-mod, with $L_{A/R}$ and $L_{A/R} / S$ cofibrant, and (ii) a map $d : L_{A/R} → L_{A/R} / S$ in $R$-mod with $d^2 = 0$ such that $(strictly)$ commutes.

$L_{A/R} / S → \wedge^2 (L_{A/R} / S)$ in $R$-mod with $d^2 = 0$ such that $(strictly)$ commutes.
Call such data *foliation data*; observe that if \( B \to A \) is a morphism in \( \text{cdga}_{R/\cdot} \), we get foliation data from \( L_{B/R} \otimes_B A \to L_{A/R} \to L_{A/B} \) after cofibrantly replacing \( B \to A \).

Define \( L\Omega_{A/S} = \prod_{i \geq 0} \wedge^i (L_{A/R}/S)[-i] \in R\text{-mod} \) with differential \( D + d \), where \( D \) is the internal differential, and observe we have a morphism \( L\Omega_{A/R} \to L\Omega_{A/S} \) in \( R\text{-mod} \).

**Proposition.** Let \( A \in \text{cdga}_{<0} \) have \( H^0(A) \) finitely presented as a commutative algebra, and suppose given either

1. \( S \in \text{Perf} A \), and morphism \( S \to L_A \), such that, writing \( L_A/S = \text{cone}(S \to L_A) \), there is an \( s > 0 \) with \( H^i(L_A/S) = 0 \) for \( i > s \), and \( \text{TorDim} S \leq 2s - 1 \), or
2. foliation data \( S \to L_A \to L_A/S \xrightarrow{+1} \) with \( S \in \text{Perf} A \), as above.

Suppose also that \( S \to L_A \) is 0-connected.\(^6\)

Then there exists a finitely presented \( B \in \text{cdga}_{<0} \), morphism \( B \to A \), with \( H^0(B) \to H^0(A) \) a surjection, and quis \( L_B \otimes_B A \to S \) factoring \( L_B \otimes_B A \to L_A \). In case (ii) we also have a weak equivalence of triangles \( (L_B \otimes_B A \to L_A \to L_{A/B}) \to \)

\[
\begin{align*}
L_{A/B} & \xrightarrow{d} \wedge^2 L_{A/B} \\
(S \to L_A & \to L_A/S) \quad \text{such that} \\
L_A/S & \xrightarrow{d} \wedge^2 (L_A/S)
\end{align*}
\]

(strictly commutes.

**Proof.** We inductively build algebras \( B_r \to A \), and morphisms \( L_{B_r} \otimes_{B_r} A \to S \to L_A \) factoring \( L_{B_r} \otimes_{B_r} A \to L_A \) such that, writing \( S/L_{B_r} \) for \( \text{cone}(L_{B_r} \otimes_{B_r} A \to S) \), i) \( B_r \) is finitely presented, with \( \text{TorDim} L_{B_r} \leq r \), ii) for \( r \geq 1 \), \( H^0(B_r) \to H^0(A) \) is a surjection, iii) \( H^i(S/L_{B_r}) = 0 \) for \( i > -r \), iv) \( S/L_{B_r} \) is perfect, and \( \text{TorDim} S/L_{B_r} \leq \)

\[
\begin{align*}
L_{A/B_r} & \xrightarrow{d} \wedge^2 L_{A/B_r} \\
L_A/S & \xrightarrow{d} \wedge^2 (L_A/S)
\end{align*}
\]

(strictly commutes.

Then for \( r > \text{TorDim} S + 1 \) lemma 1.2 gives \( S/L_{B_r} = 0 \), proving the proposition.

To begin, as \( H^0(A) \) is finitely presented we can choose \( x_1, \ldots, x_n \in A^0 \) which generate \( H^0(A) \), and \( \xi_1, \ldots, \xi_r \in A^{-1} \) such that the \( D\xi_i \) generate the ideal \( D(A^{-1}) \) in \( A^0 \). Define \( B_0 = k[x_1, \ldots, x_n] \), and \( L_{B_0} \otimes_{B_0} A \to S \) by sending \( dx_i \) to some choice of a lift of \( dx_i \in L_A^0 \) to \( S^0 \); this is possible as \( H^0(S) \) surjects onto \( H^0(L_A) \).

Now suppose we have defined \( B_r \). If \( H^{-r}(S/L_{B_r}) = 0 \), set \( B_{r+1} = B_r \); otherwise as \( S \) and \( L_{B_r} \) are perfect and \( H^0(A) \) is finitely presented, \( H^{-r}(S/L_{B_r}) \) is a finitely generated \( H^0(A) \)-module; let \( w_1, \ldots, w_l \) be elements which generate it.

As in proposition 1, let \( K = \text{cone}(B_r \to A) \in B_r\text{-mod} \), \( d : K \to L_{A/B_r} \). Consider the image of \( w_i \) in \( H^{-r}(L_{A/B_r}) \). We claim this is in the image of \( d : H^{-r}K \to H^{-r}(L_{A/B_r}) \).

Granting this for the moment, let \( w'_1, \ldots, w'_l \in H^{-r}K \) be elements for which \( dw'_i \) equals the image of \( w_i \) in \( H^{-r}(L_{A/B_r}) \). These define elements in \( H^{-r+1}(B_r) \) under the transgression in the exact sequence

\[
\begin{align*}
&\to H^{-r}A \to H^{-r}(K) \to H^{-r+1}B_r \to H^{-r+1}A \to
\end{align*}
\]

\(^6\)A lazy assumption, but we do not need the general case in this paper.
and we let $\gamma_1, \ldots, \gamma_s \in (B_r)^{-r+1}$ be lifts of these elements from $H^{-r+1}(B_r)$.

Write $\gamma_1, \ldots, \gamma_t$ in $A^{-r+1}$ for the images of the $\gamma_i$ under the morphism $B_r \to A$. As these elements are zero in $H^{-r+1}A$, there exist $z_i \in A^{-r}$ such that $Dz_i = \gamma_i$.

We may further assume that $dz_i = dw_i$ in $H^{-r}(L_{A/B_r})$. To see this, choose a cofibrant replacement $B_r \to \mathcal{A}$ for $B_r \to A$. As in proposition 1.1, we can assume $K \subseteq A$, and so $w_i = 0$ in $A^{-r}$ has $dw_i' = \gamma_i' \in B^{-r+1} \subseteq A^{-r+1}$. Now choose $z_i = w_i'$.

Set $B_{r+1} = B_r[z_i' | Dz_i' = \gamma_i']$, with deg $z_i' = -r$, and map $B_{r+1} \to A$ by sending $z_i' \mapsto z_i$.

As $d\gamma_i' \in H^{-r+1}(L_{B_r})$ are transgressed from $w_i \in H^{-r}(\mathbb{S}/L_{B_r})$, we can find $\tilde{w}_i \in S^{-r}$ lifting $w_i$ with $D\tilde{w}_i = -\gamma_i'$ in $S^{-r+1}$. Extend the map $L_{B_r} \to S$ to $L_{B_{r+1}} \to S$ by sending $z_i \mapsto \tilde{w}_i$.

Then $B_{r+1}$ satisfies the inductive conditions (i)-(iv).

It remains to prove the claim. For case (i) of the proposition, note that if $H^i(L_{A/S}) = 0$ for $i > -s$, then for $r \leq s$ and $i > -r \geq -s$, the exact sequence $\rightarrow H^i(S/L_{B_r}) \to H^i(L_{A/B_r}) \to H^i(L_{A/S})$ gives $H^i(L_{A/B_r}) = 0$. Hence proposition 1 gives $H^{-r}(K) = H^{-r}(A \otimes_{B_r} K) = H^{-r}(L_{A/B_r})$, and the claim. If $s < r \leq 2s$, we reduce to case (ii) by constructing foliation data Zariski locally.

Replace $B_r \to A$ by $B_r \to \mathcal{A}$, so $L_{A/B_r}$ is cofibrant and $(L_{A/B_r})^r = 0$ for $i > -s$. We have the Tor amplitude of $S/L_{B_r}$ is contained in $[-2s + 1, -r]$.

Suppose

$$S/L_{B_r} \to A[\eta_1, \ldots, \eta_s] = A \otimes_k M[1],$$

where $D\eta_i \in A[\eta_1, \ldots, \eta_{i-1}], M[1] = \otimes^k \eta_i, -2s + 1 \leq \deg \eta_i \leq -r$, and $S/L_{B_r} \to L_{A/B_r}$ is induced by $\xi : M[1] \to L_{A/B_r}$.

Then writing $C = \text{cone}(S/L_{B_r} \to L_{A/B_r}) = L_{A/B_r} \otimes (A \otimes M)$ (with differentials), $\otimes^2 C = \otimes^2 L_{A/B_r} \otimes (L_{A/B_r} \otimes M) \otimes \otimes^2 M$ (with differentials), we can define $d : C \to \otimes^3 C$ by $(\omega, a \otimes \eta) \mapsto (da, da \otimes \eta, 0)$. This is a morphism of complexes, as $d(a \otimes \eta_i) = da \otimes \eta_i + a \otimes d\eta_i = da \otimes \eta_i$, as $d\eta_i \in \otimes^2 L_{A/B_r}^{\text{deg} \eta_i + 1} = 0$, as $-\deg \eta_i - 1 < 2s$. Moreover, $d^2 : C \to \otimes^3 C$ is zero, and $d$ defines foliation data on $S/L_{B_r} \to L_{A/B_r} \to C$.

To finish the proof, for both cases of the proposition we have, by construction, foliation data $S/L_{B_r} \to L_{A/B_r} \to L_{A/S}$, and so we have a morphism $L_{\Omega A/B_r} \to L_{\Omega A/S}$. Let $Q[1]$ be its mapping cone. We have $Q \to \prod_{i \geq 1} Q[i \neq 1]$, where each $Q_i \in \otimes^i L_{A/B_r} \to \otimes^i (L_{A/S}/S) \in A\text{-mod}$ has a filtration with subquotients $\otimes^a (L_{A/S}) \otimes \otimes^b (S/L_{B_r}), a + b = i$, and $b > 0$.

As $H^k(L_{A/S}) = 0$ for $k \geq 0$, and $H^k(S/L_{B_r}) = 0$ for $k > -r$, $H^k(\otimes^b (L_{A/S}) \otimes \otimes^b (S/L_{B_r})) = 0$ when $k > -br - a$. Hence $H^{1-r}(Q_i) = 0$ when $i > 1$, and any $w \in H^{-r}(S/L_{B_r}) = Q[1]$ defines a class in $H^{1-r}Q$, and hence a class in $\text{Ext}(H^{1-r}(F^1 L_{\Omega A/B_r}), H^{1-r}(L_{\Omega A/B_r}))$.

This is the case even when assumption (**) does not hold, as it always holds Zariski locally, and so there is a Zariski cover $\phi : A \to A'$ such that, for $u \in \text{Ext}(H^{1-r}(S/L_{B_r}), H^{-r}(L_{A/B_r})), d\phi(u) = 0$. But $d\phi(u) = \phi du$, and as the kernel of $\phi : H^{-r}(\otimes^2 L_{A/B_r}) \to H^{-r}(\otimes^2 L_{A'/B_r}) = H^{-r}(\otimes^2 L_{A'/B_r}) \otimes_{H^0 A} H^0 A'$ is zero, $du = 0$.

For $r \geq 1$, $\text{Ext}(H^{1-r}(F^1 L_{\Omega A/B_r}), H^{1-r}(L_{\Omega A/B_r}))$ is zero, so the triangle $F^1 L_{\Omega A/B_r} \to L_{\Omega A/B_r} \to L_{\Omega A/B_r}/F^1 L_{\Omega A/B_r}$ gives that the image of $w$ is of the form $dw'$, for some $w' \in A^{-r}, Dw' = 0$, as required.
Finally, we must check the inductive condition (v). Again, choose cofibrant replacements $B_i \to cA$ for $B_i \to A$, and $c(S/L_{B_i}) \to S/L_{B_i}$ such that $(cS/L_{B_i})^i = 0$ for $i > r$. Then $Q_i = 0$ for $i > r$, so $dw = 0$ for $w \in (L_A/S)^r$, ensuring (v) for $B_{r+1}$.

**Corollary.** An algebra $A \in \cdga_{\leq 0}$ is f.p. if and only if $H^0(A)$ is finitely presented as a commutative algebra and $L_A$ is perfect.

**Proof.** The ‘only if’ implication is clear. For the converse, apply the proposition with $S \to L_A$ equal to $L_A \xrightarrow{Id} L_A$. This gives a f.p. algebra $B$ with $H^0(B) \xrightarrow{\sim} H^0(A)$ an isomorphism and $L_A/B = 0$. Corollary 1.1 gives $B \to A$ is a quis.

Note that the proof shows that if $L_A$ has Tor dimension $d > 0$ then $A$ is quis to a cdga built by attaching $r$-cells for $r \geq -d$, $A \xrightarrow{\sim} k[z_1, \ldots, z_n \mid Dz_i = f_i]$ for some elements $f_i \in k[z_1, \ldots, z_n]$ with deg $z_i \leq -d$ for all $i$.

1.5 **Corollary.** Let $A \in \cdga_{\leq 0}$ have $H^0(A)$ f.p., and $L_A[-d] \xrightarrow{\sim} L_A$ for some $d \geq 0$. Then TorDim $L_A \leq d$, and $A$ is f.p.

**Proof.** We have $L_A^* \xrightarrow{\sim} L_A$, and so $L_A$ is perfect. A perfect complex $M$ has Tor amplitude in $[a, b]$ if and only if $M^*$ has Tor amplitude in $[-b, -a]$, so $L_A$ has Tor dimension $d$.

1.6 **Lemma.** Suppose $B \to A$ in $\cdga_{\leq 0}$ has $H^0(B) \to H^0(A)$ an isomorphism. Let $M, N \in \Perf_B$. If $\phi : M \otimes_B A \to N \otimes_B A$ is a quis, then there exists a quis $\tilde{\phi} : \tilde{M} \to \tilde{N}$ in $B$-mod.

**Proof.** Let $M, N$ have Tor amplitude in $[a, b]$, and $H^k(M) = H^k(N) = 0$ for $k > j$, and suppose $j$ is minimal. We induct on $j - a + b$, if this is negative $M = N = 0$ and the lemma is true. Otherwise, as $H^0(B) \to H^0(A)$ is an isomorphism, $H^j(M) \to H^j(N)$ is also an isomorphism, and we can choose $\gamma_1, \ldots, \gamma_i \in M$ which generate $H^j(M)$ as an $H^0(B)$-module. Put $\Gamma = \oplus B\gamma_i$, $\tilde{M} = \text{cone}(\Gamma \to M)$, $\tilde{N} = \text{cone}(\Gamma \xrightarrow{\tilde{\phi}} N)$. Then $\tilde{M}, \tilde{N}$ still have Tor amplitude in $[a, b]$, but $H^k(M) = H^k(\tilde{N}) = 0$ for $k > j$.

By induction there exists a quis $\tilde{\phi} : \tilde{M} \to \tilde{N}$, and we set $\tilde{\phi}$ to be the cone of $\tilde{\phi}[-1] \oplus \text{Id} : \text{cone}(\tilde{M}[-1] + \Gamma) \to \text{cone}(\tilde{N}[-1] + \Gamma)$.

1.7 **Lemma.** Let $A \in \cdga_{\leq 0}$ be f.p., and suppose $P, Q \in \Perf_A$ satisfy $P \xrightarrow{\phi} Q \to A$. Then Zariski locally on $A$ there is a quis $P \to A[d]$, for some integer $d$ (which may depend on the open set, if $H^0(A)$ is not connected).

2. **More background, didactically**

2.1 Given $M \in B$-mod, $\xi \in H^0(\text{Hom}_B(M, B[1]))$, write $\text{Sym}_B^\xi M$ for the pushout $\text{Sym}_B \tilde{M} \otimes_{\text{Sym}_B B} B$, where $\tilde{M} = \text{cone}(M[-1] \xrightarrow{\xi} B)$.

2.2 Fix $B \in \cdga_{\leq 0}$, $M \in \Perf_B$ with $H^i(M) = 0$ for $i > 0$, and $\xi \in H^0(\text{Hom}_B(M, B[1]))$ as above. The algebra $\text{Sym}_B^\xi M$ is filtered, with $B$ in filtration degree $0$ and $M$ in filtration degree $1$; the associated graded algebra is $\text{Sym}_B M$.

Write $A = \text{Sym}_B^\xi M, \tilde{A} = \text{Sym}_B M$. The grading on $A$ is defined by a $G_m$-action; differentiating we get a map $\text{Lie} G_m = k \to L_A^*$. Denote the image of $1$ by $\tilde{E}$, this is the ‘Euler vector field’, $\tilde{E} \in H^0(\text{Spec} \tilde{A}, L_A^*) = H^0(\text{Hom}(L_A, \tilde{A}))$. If we choose a cofibrant representative for $M$, so $M = B \langle y_1, \ldots, y_n \mid Dy_i = \sum_{j<i} \mu_{ij} y_j \rangle$, $\mu_{ij} \in B$, then $\tilde{E} = \sum y_i(dy_i)^*, D_A \tilde{E} = 0$. 


2.3 The grading on $\bar{A}$ induces one on $L_{\bar{A}}$ and on $L\Omega_{\bar{A}}$; the $i$'th graded piece is the $i$'th eigenspace of $\mathcal{L}_E$: $\bar{A}$, $\wedge^i L_{\bar{A}}$ and $L\Omega_{\bar{A}}$ are the sums of their graded pieces.

**Lemma.** i) If $\lambda \neq 0$, the $\lambda$'th graded piece of $H^i(\text{Spec } \bar{A}, F^p L \Omega_{\bar{A}})$ is the $\lambda$'th graded piece of $\ker(d: H^{i-p}(\wedge^{p+1} L_{\bar{A}}) \to H^{i-p}(\wedge^p L_{\bar{A}}))$.

ii) The $\lambda$'th graded piece of $H^i(\text{Spec } \bar{A}, F^p L \Omega_{\bar{A}})$ is $H^i(\text{Spec } B, F^p L \Omega_{B})$.

**Proof.** Choose a cofibrant replacement for $M$ as above. Define $D_{Kos}$ to be contraction with $E$, so $[D, D_{Kos}] = 0$, and $[d, D_{Kos}] = [d + D, D_{Kos}] = \mathcal{L}_E$.

Let $\omega = \omega_d + \omega >_p \in H^i(\text{Spec } \bar{A}, F^d \Omega \Lambda_{\bar{A}})$, $\mathcal{L}_E \omega = \lambda \omega$. Then $[D_{Kos}, d + D] \omega >_p = \mathcal{L}_E \omega >_p = \lambda \omega >_p$, and $d \omega >_p = -(d + D) \omega >_p$, so $\omega >_p = D_{Kos}(-d \omega >_p / \lambda) + (d + D)(D_{Kos} \omega >_p / \lambda)$, i.e. $\omega = \omega_d + D_{Kos}(-d \omega >_p / \lambda) = d D_{Kos} \omega >_p / \lambda) \in H^i(\text{Spec } \bar{A}, F^p \Omega \Lambda_{\bar{A}})$. This shows the inclusion $\ker(d: \wedge^{p+1} L_{\bar{A}}[-p] \to \wedge^p L_{\bar{A}}[-p-1]) \to F^p L_{\bar{A}}$ is a quis on $\lambda$'th graded pieces, $\lambda \neq 0$.

By definition the $0$'th graded piece of $F^p L \Omega_{\bar{A}}$ is $F^p L \Omega_{B}$, proving (ii).

2.4 Regard the class $\xi: M \to B[1]$ as a vector field on $\bar{A}$ via the map $M^* \otimes_B \bar{A} \to L_{\bar{A}}^*$.

The vector field $\bar{E} \in H^0(\mathcal{L}_{\bar{A}}^*)$ does not deform to a vector field on $A$, and neither does the cohomology class of $\xi \in H^1(\mathcal{L}_{\bar{A}}^*)$. Instead, there is an $E \in (L_{\bar{A}})^0$ and $\xi \in (L_{\bar{A}})^1$ such that $D_A E = \xi$, and the specialisation of $E$, $\xi$ to $\bar{A}$ are $E$, $\xi$, respectively. On choosing a cofibrant representative for $M$ as in 2.2, $E = \sum y_i (d y_i)^*$, $\xi = \sum \xi_i (d y_i)^*$.

Choose cofibrant replacements for $B$ and $M$, and an element $\xi \in \text{Hom}(M[-1], B)^0$ whose class is $\xi$. This defines cofibrant representatives for $A$ (resp. $\Lambda A$) having the same underlying algebra as that for $\bar{A}$ (resp. as $\Lambda \bar{A}$), but $D_A = D_{\bar{A}} + \mathcal{L}_\xi$, where $\mathcal{L}_\xi = [d, \xi]: \wedge^i L_{\bar{A}} \to \wedge^i \Lambda A$, $i \geq 0$.

Write $\bar{D} = D_A$. The operators $d, \bar{D}, \xi, \mathcal{L}_\xi, D_{Kos}, \mathcal{L}_E$ act on the underlying chain complex of $L \Omega A$, and these operators are subject to the relations $[d, D_{Kos}] = 0$, $[D_{Kos}, \mathcal{L}_\xi] = [\xi, \mathcal{L}_E, \mathcal{L}_\xi] = [\mathcal{L}_E, \mathcal{L}_\xi] = [-\mathcal{L}_\xi, [D_{Kos}, \xi] = [D, \mathcal{L}_E] = [D, d] = 0$.

2.5 The filtration on $A = \text{Sym}_B^* M$ induces an increasing filtration on $\wedge \Lambda_{\bar{A}}$, denoted $^* \text{Filt}^\delta \wedge \Lambda_{\bar{A}}$. There is a related filtration that is also useful. The triangle

\[ L_B \otimes_B A \to L_A \to L_{B/A} \] 

induces an increasing filtration $^* \text{Filt}^\delta \wedge \Lambda_{\bar{A}}$ of $\wedge \Lambda_{\bar{A}}$: $^* \text{Filt}^\delta \wedge \Lambda_{\bar{A}}$ is the $A$-submodule of $\wedge \Lambda_{\bar{A}}$ spanned by terms $d \gamma_1 \ldots d \gamma_r$ with at most $\delta$ of the $\gamma_i \in M$.

Define a decreasing exhaustive filtration on $L \Omega A$ by

\[ \text{Filt}^\delta L \Omega A = \prod_{r \geq 0} \text{Filt}^{r-\delta} \wedge \Lambda_{\bar{A}}, \]

and on $F^p L \Omega A$ by $F^p \text{Filt}^\delta L \Omega A = \prod_{r \geq p} \text{Filt}^{r-\delta} \wedge \Lambda_{\bar{A}} = \text{Filt}^\delta L \Omega A \cap F^p L \Omega A$.

Define $gr^\delta F^p L \Omega A = \text{Filt}^\delta F^p L \Omega A / \text{Filt}^{\delta+1} F^p L \Omega A$.

2.6 If $N \in \text{Perf } B$, and $\xi: N \to B$ a morphism in $B$-mod, define the completion of $B$ at the ideal generated by $N$,

\[ \hat{B}_N = \lim_{\leftarrow} B \otimes_{\text{Sym}_B^* N \otimes_B} B \]

where $B$ is a $\text{Sym}_B N \otimes_B B$-algebra in two ways: via the augmentation morphism sending $N \otimes_B B$ to zero, and via the morphism $\xi : \text{Sym}_B N \otimes_B B \to B$.

Observe that $L_{\hat{B}_N} = L_B \otimes_B \hat{B}_N$. 


2.7 We use this notation when \( N = M[-1], \xi : M[-1] \to B \) as above, and denote the completion \( \hat{B}_{M[-1]} \) by \( \hat{B} \). Also write \( \hat{A} = \hat{B} \otimes_B \hat{A} \).

**Lemma.** i) There is a well defined morphism of chain complexes \( e^\xi : F^p\Omega_{\hat{A}} \to \Omega_{\hat{A}} \).

\[
w = w_p + w_{p+1} + \cdots \mapsto e^\xi \omega = \sum_{a \geq 0} \left( \sum_{k \geq \max(0, p-a)} \frac{\xi^k \omega_{k+a}}{k!} \right).
\]

ii) This descends to a morphism \( e^\xi : \text{Filt}^\delta F^p\Omega_{\hat{A}} \to \text{Filt}^\delta F^\delta\Omega_{\hat{A}} \).

iii) If the image of \( \omega \in H^i(\text{Spec } A, \text{Filt}^\delta \Omega_{\hat{A}}) \) in \( H^i(\text{Spec } \hat{A}, \text{Filt}^\delta F^\delta\Omega_{\hat{A}}) \) is zero, then \( e^\xi \omega \) is the image of a class in \( H^i(\text{Spec } \hat{A}, \text{Filt}^\delta F^\delta\Omega_{\hat{A}}) \), with \( [e^\xi \omega]_\delta = \frac{e^{p+\delta} - \xi^{p+\delta}}{(p-\delta)!} \omega_p \).

**Proof.** \( \xi^k \omega_{k+a} \) is manifestly zero in \( \hat{A} \otimes_{\text{Sym}(M[-1]+e)} \hat{A} \) if \( k \geq 1 \), so the sum \( \sum_k \xi^k \omega_{k+a} \) is a well defined element of the completion \( \text{Filt}^\delta \Omega_{\hat{A}} = \text{Filt}^\delta (\hat{A} \otimes_{\hat{A}} \hat{A}) \). Moreover \( (d + D_A - L_\xi) e^\xi = e^\xi (d + D_A) \), as \( L_\xi = [d, \xi] \) commutes with \( \xi \).

If \( \omega_{k+a} \in \text{Filt}^{k-\delta} \Lambda^\delta \Omega_{\hat{A}} \), then \( \xi^k \omega_{k+a} \in \text{Filt}^{k-\delta} \Lambda^\delta \Omega_{\hat{A}} \), which is zero if \( a < \delta \), giving (ii). For (iii), observe the triangle

\[
F^p\Omega_{\hat{A}} \to \Omega_{\hat{A}} \to \Omega_{\hat{A}}/F^p\Omega_{\hat{A}} \xrightarrow{+1}
\]
gives that \( \omega \) is the transgression of a class \( \nu_0 + \nu_1 + \cdots + \nu_{p-1} \in H^{i-1}(\text{Spec } A, \Omega_{\hat{A}}/F^p\Omega_{\hat{A}}) \), so \( \omega = \nu_0 + \cdots + \nu_{p-1} \in (\Lambda^\delta \Omega_{\hat{A}})^{-p} \) and \( D\nu_{p-1} = D\nu_{p-2} = 0 \).

Hence \( e^\xi \omega = \omega + \xi \omega + \cdots + \frac{e^{p+\delta} - \xi^{p+\delta}}{(p-\delta)!} \omega \), a finite sum.

**Remark.** If \( \omega \in F^p\Omega_{\hat{A}} \) is transgressed from a class in \( \text{Filt}^{\delta+1} (\Omega_{\hat{A}}/F^p\Omega_{\hat{A}}) \), then \( \omega \in \text{Filt}^{\delta+1} F^p\Omega_{\hat{A}} \). In particular, its class in \( gr^\delta F^p\Omega_{\hat{A}} \) is zero.

**Remark.** The action of vector fields \( \lambda^A \) on \( \Lambda^\delta \Omega_{\hat{A}} \) is well defined (independant of the choice of a cofibrant replacement for \( A \)). The independence of \( e^\xi \) on choices is a different phenomena, as \( e^\xi + D_{\partial^p} \omega \) need not equal \( e^\xi \omega \) in the de Rham complex. To define \( e^\xi \), we choose a cofibrant replacement for \( B \), and a representative \( \xi \in \text{Hom}_B(M[-1], B)^0 \) for the class of \( \xi \). This defines a cofibrant representative for \( A = \text{Sym}^\delta_B, M \), an \( E \in (\lambda^A)^0 \) with \( DE = \xi \), and the morphism \( e^\xi \). Different cofibrant choices for representatives of \( B, M \) and \( \xi \) produce canonically quis results. So our notation is harmlessly imprecise.

**Remark.** We only use part (iii) of the lemma, but the following may be clarifying. The morphism \( e^\xi : \Omega_{\hat{A}} \to \Omega_{\hat{A}} \) is a quis, \( \hat{A} = \hat{B} \otimes_B \hat{A} \). If \( H^i(M) = 0 \) for \( i \geq 1 \), then \( \Omega_{\hat{A}} \to \Omega_{\hat{A}}, \Omega_{\hat{B}} \to \Omega_{\hat{A}} \) are quis; regardless \( H^i(\text{Spec } \hat{A}, \Omega_{\hat{A}}) \) equals \( H^i(\text{Spec } \hat{B}^{\partial}, \Omega_{\hat{A}}) \) and this coincides with Hartshorne-Deligne’s algebraic de Rham cohomology of \( H^0(\hat{A}) \).

\[\text{This is a special case of the Campbell-Baker-Hausdorff formula, or a straightforward computation in the quotient of the completed free algebra in two variables } x, y \text{ by the ideal } [y, [x, y]].\]
3. A Darboux theorem

Throughout this section we fix an invertible element \( \mathcal{P} \) of \( Ch(k) \), so \( \mathcal{P} = k[\delta] \), a sign \( \lambda_\mathcal{P} \in \{ \pm 1 \} \), and we assume \( d \geq 1 \). Given \( B \in \text{cdga}_{\leq 0} \), and \( M \in B\text{-mod} \), write \( \mathcal{P}^{\pm 1} \) for \( \mathcal{P}^{\pm 1} \otimes_k B \), and define \( M^1 = \text{Hom}_B(M, \mathcal{P}^{-1}) \in B\text{-mod} \). Note that if \( M \) is cofibrant and perfect then so is \( M^1 \), and \( M \xrightarrow{\sim} M^{1\dagger} \).

For \( A \in \text{cdga} \), \( \text{Perf} A \) becomes an exact category with weak equivalences and duality [Sc,7.4], where we define \( \eta_M : M \to M^{1\dagger} \) to be the natural duality quis [Sc,6.1] composed with \( \lambda_\mathcal{P} \).

Given integers \( a \leq b \) with \( a + b = -d \), consider the full subcategory of \( \text{Perf} A \) consisting of complexes with Tor amplitude in \( [a,b] \); this is the subcategory of complexes quis to one of the form \( M^a \to \cdots \to M^b \) with each \( M_i[\ell] \) a summand of a free \( A \)-module \( A^n \). This subcategory inherits the structure of an exact category with weak equivalences and duality.

We remark that there is a unique choice of \( \lambda_\mathcal{P} \) for which \( \mathcal{P}\text{-shifted symplectic} \) structures exist, and we may as well fix it.

Moreover, we may also fix a cofibrant finitely presented \( R \in \text{cdga}_{\leq 0} \), and assume all cdga's lie over \( R \), \( A \in \text{cdga}_{R\text{-}} \). Then all of the theorems below are true, where we interpret \( L_A \) to be \( L_{A/R} \) etc.

3.1 A \( \mathcal{P} \text{-symmetric complex} \) on \( A \) is a pair \( M \in \text{Perf} A \), and \( \varphi : M^1 \xrightarrow{\sim} M \) a quis such that \( \varphi^1 = \eta_M \circ \varphi : M^1 \to M \xrightarrow{\sim} M^{1\dagger} \).

If \( M \) is a \( \mathcal{P} \text{-symmetric complex} \) on \( A \), and \( N \hookrightarrow M \) is a cofibration in \( \text{Perf} A \) equipped with a factorisation \( (M/N)^1 \to N \to M \) of \( (M/N)^1 \to M \), we say that \( N \to M \) is co-isotropic. For co-isotropic \( N \to M \), the quotient \( N/(M/N)^1 \) is a \( \mathcal{P} \)-symmetric complex on \( A \). When this quotient is quis to zero, we say that \( N \to M \), or the triangle \( N \to M \to N^1 \), is Lagrangian.

If \( M \) is a \( \mathcal{P} \text{-symmetric complex} \) on \( A \), then the class of \( M \) is metabolic if there is a Lagrangian \( N \hookrightarrow M \); and it is zero in the Witt group if and only if it is metabolic. Moreover, if this is so there is a Lagrangian \( S \to M \to S^1 \) with \( H^i(S^1) = 0 \) for \( \tilde{i} \geq \left\lfloor \frac{d-1}{2} \right\rfloor \).

**Proof.** This is a consequence of ‘algebraic surgery’; see [Sc,§6] for example, for a careful proof. Here is a sketch of the first statement: If the class of \( M \) is zero in the Witt group, we have a triangle \( L \to \text{cone}(P^1[-1] \to P) \oplus M \to L^1 \xrightarrow{+1} \) for some \( P \in \text{Perf} A \) and morphism \( P^1 \to P \), hence we have a triangle \( \tilde{L} \to M \to \tilde{L}^1 \xrightarrow{+1} \) where \( \tilde{L} = \text{cone}(L \to P^1)[-1] \).

3.2 For any \( M \in \text{Perf} B \), the inclusion \( M \hookrightarrow \text{Sym}_B M = \hat{A} \) induces \( B \to \text{Sym}_B M \otimes_B M^* \) by adjunction, and hence \( \hat{A} 

\to \hat{A} \otimes_B M^* \). When \( M = L^1_B \), so \( M^* = (P^{-1})^* \otimes L_B \), the composite map \( \hat{A} \to \mathcal{P} \otimes L_B \) is the Liouville form, we have \( D\hat{\lambda} = 0 \), and \( d\hat{\lambda} = (d + D)\hat{\lambda} = \omega_{\text{std}} \) is the “standard” shifted symplectic structure on \( \hat{A} \), which is manifestly deRham closed and non-degenerate.

Moreover, \( \omega_{\text{std}} : L^1_A \to L_A \) is a \( \mathcal{P} \text{-symmetric complex} \), zero in the Witt group of
Now let \( \tilde{\xi} \in H^2(\text{Spec } B, \mathcal{P} \otimes F^1L\Omega_B) \) be a one form, and write \( [\tilde{\xi}]_1 = \tilde{\xi}_1 \in H^1(\mathcal{P} \otimes L_B^\vee) \). Regard \( \tilde{\xi}_1 \) as an element of \( \mathcal{P}[1] \otimes L_A \) via \( L_B \otimes_B \tilde{A} \to L_A \). Then there exists an element \( \xi' \in H^1(\mathcal{P} \otimes L_A^\vee) \) which is the image of a class \( \xi \in \mathcal{P}[1] \otimes L_B^\vee \). Let \( y \in L_B \otimes_B \tilde{A} \), and \( \xi \in \mathcal{P}[1] \otimes L_A^\vee \) such that \( \xi' \otimes \xi_1 = [\xi]_1 \), where \( \xi' = \xi \otimes 1 \in L_A^\vee \otimes \mathcal{P}[1] = L^*_A[1] \).

Now let \( \xi : L_B^* \to B[1] \) defines a deformation \( A = \text{Sym}_B^\xi L_B^1 \); writing \( \lambda, \omega^{\text{std}} \) for the deformations of \( \lambda, \tilde{\omega}^{\text{std}} \) to \( A \), we get \( D_A \lambda = \mathcal{L}_\xi \lambda = \tilde{\xi}_1 \in (\mathcal{P} \otimes L_A)^1 \), and \( d\lambda = \omega^{\text{std}} \), so \( (d + D_A)\lambda = \tilde{\xi}_1 + \omega^{\text{std}} \).

Let \( \omega^{\text{std}}_{\xi} = -\xi + (d + D_A)\lambda \in L\Omega_A \). This defines a class in \( H^2(\text{Spec } A, \mathcal{P} \otimes F^2L\Omega_A) \) with \( [\omega^{\text{std}}_{\xi}]_2 = -\tilde{\xi}_1 + \omega^{\text{std}}_A \). As \( \text{gr}[\omega^{\text{std}}_{\xi}]_2 \) is the identity map in \( L_B \otimes L_B^* \otimes_B A \), \( [\omega^{\text{std}}_{\xi}]_2 \) is non-degenerate, and \( \omega^{\text{std}}_{\xi} \) defines a twisted symmetric structure on \( A = \text{Sym}_B^\xi L_B \), which we call the “standard” twisted structure attached to \( \xi \).

### 3.3 Proposition

Let \( A \in \text{cdga}_{\leq 0} \) have \( H^0(A) \) f.p., and let \( \omega_2 : L_A^* \to L_A \) be a \( \mathcal{P} \)-symmetric complex.

If the class of \( \omega_2 \) in the Witt group of \( \mathcal{P} \)-symmetric forms is zero, then there exists \( B \to A \) in cdga\(_{\leq 0}\) such that \( L_B^\vee \otimes_B A \xrightarrow{\sim} L_{A/B} \), \( B \) is f.p., and \( H^i(L_{A/B}) = 0 \) for \( i \geq -\left[\frac{d-1}{2}\right] \).

**Proof.** By proposition 3.1 we have a Lagrangian \( S \to L_A \to S^\vee \xrightarrow{+1} \) with \( H^i(S^\vee) = 0 \) for \( i \geq -\left[\frac{d+1}{2}\right] \). The Tor amplitude of \( S^\vee \) is in \( [r, -\left[\frac{d+1}{2}\right]] \) for some \( r \), hence the Tor amplitude of \( S \) is in \( [-\left[\frac{d+1}{2}\right], 0] \), and so \( \text{Tor}^d S \leq \left[\frac{4}{d}\right] \), and \( r = -d \). The result is now immediate from the ‘Frobenius theorem’ proposition 1.4.

**Lemma.** Let \( B \to A \) be in cdga\(_{\leq 0}\) such that \( \omega_2 : L_B^\vee \otimes_B A \xrightarrow{\sim} L_{A/B} \), \( B \) is f.p., and \( H^i(L_{A/B}) = 0 \) for \( i \geq -\left[\frac{d+1}{2}\right] \). Then there exists \( \xi \in H^0\text{Hom}_B(L_{B^1}, B[1]) \) and a quis \( \alpha : \text{Sym}_B^\xi L_B^1 \xrightarrow{\sim} A \).

Moreover, if \( [d, \omega_2] = 0 \), then we can choose this quis so that \( \alpha^*\tilde{\omega}_2 : L_B^\vee \otimes_B \text{Sym}_B^\xi L_B^1 \to L_B^\vee \otimes_B \text{Sym}_B^\xi L_B^1 \) is the pullback of a morphism \( L_B^1 \to L_B^1 \).

**Proof.** Arguing as in the previous proposition, the Tor amplitude of \( L_{A/B} \) is in \( [-d, -\left[\frac{4d+1}{2}\right]] \).

Hence by proposition 1.4, we may assume \( B \to A \) is cofibrant, with \( A = B[y_1, \ldots, y_n] \) for some \( h_i \in B[y_1, \ldots, y_{i-1}] \) with \( \left[\frac{d+1}{2}\right] \leq -\deg y_i \leq d \) for all \( i \). Hence \( -\deg y_i - \deg y_j \geq 2 \left[\frac{d+1}{2}\right] > d - 1 \), and only terms at most linear in \( y_i \) appear in \( D_{y_i} \). Write \( D_{y_i} = h_i = \lambda_i + \sum_{j<i} \mu_{ij} y_j \) for some \( \lambda_i, \mu_{ij} \in B \).

Set \( M = B[y_1, \ldots, y_n] \) \( D_{y_i} = h_i - \lambda_i \in B \)-mod, and \( \xi : M \to B[1] \), \( \xi(y_i) = \lambda_i \), so that \( \text{Sym}_B^\xi M \to A \) is a quis. As \( 1 \otimes_B d : A \otimes_B M \to L_{A/B} \) is a quis, we have there exists a quis \( L_B^1 \xrightarrow{\sim} M \) by lemma 1.6, proving the result.
Now write \( A = \text{Sym}_{\mathbb{C}} L_B^1 \), and \( \bar{\omega}_2 \) for the pullback of \( \omega_2 \) to \( A \). The filtration by degree on \( A = \text{Sym}_{\mathbb{C}} L_B^1 \) induces one on \( \text{Hom}_A(L_A, L_A) \), the \( i \)'th piece of which are endomorphisms which raise filtration degree by \( i \); a morphism is in filtration degree 0 precisely when it is the pullback of an endomorphism in \( \text{Hom}_B(L_B^1, L_B^1) \).

Write \( \bar{\omega}_2(dy_i) = \sum a_{ij} dy_j, \) with \( \deg a_{ij} = -\deg y_j + \deg y_i \leq 0 \). As \( \{ \frac{d-1}{2} \} \leq -\deg y_i \leq d \), \( a_{ij} \in B \) unless \( d \) is even, \( \deg y_i = -d \), \( \deg a_{ij} = -d/2 \) is even, and \( \bar{\omega}_2(dy_i) \) is zero unless \( 1/2 \deg y_i = \deg y_j = -d/2 \).

By assumption \( d\bar{\omega}_2(dy_i) = 0 \) in \( \wedge^2 L_{A/B} \), so \( \sum_{j,k} \gamma_{i,jk} dy_j dy_k = 0 \), and hence \( \bar{\omega}_2(dy_i) = d(\sum_{j<k} \gamma_{i,jk} y_j y_k) \).

Now define \( L_B^1 \to A \) by \( y_i \to y_i - \sum_{j<k} \gamma_{i,jk} y_j y_k \), and let \( \alpha : A \to A \) be the induced morphism of cdgas. Note that \( DB(y_i) = 0 \) if \( \deg y_j = -d/2 \), so \( \alpha \) is even, and \( \alpha^\ast \bar{\omega}_2 : L_B^1 \otimes_B A \to L_B^1 \otimes_B A \) is in filtration degree zero.

**Remark.** Note that \( \alpha^\ast \xi \neq \xi \), so the isomorphism chosen in the second part of the lemma changes the choice of \( \xi \) from the first part.

### 3.4 Lemma

Let \( B \in \text{cdga}_{\leq 0} \) with \( H^0(B) \) flat, \( \xi \in H^1(L_B^1) \), \( A = \text{Sym}_{\mathbb{C}} L_B^1 \), and suppose given \( \omega \in H^2(Spec A; \mathcal{P} \otimes F^2 \Omega_A) \) such that

i) \( [\xi] \in H^0(\mathcal{P} \otimes \wedge^2 L_A) \) defines a quis \( L_B \otimes_B A \to L_B^1 \otimes_B A \), and

ii) the class of \( \omega \) in \( H^2(Spec A; \mathcal{P} \otimes \Omega_A) \) is zero.

Then there exists an \( f : B \to \mathcal{P}[1] \) such that

\[
\epsilon^\xi \omega = \pi^\ast (df) \in H^2(Spec \bar{A}, \mathcal{P} \otimes F^1 \Omega_{\bar{A}}),
\]

where \( \pi : B \hookrightarrow \bar{A} = \text{Sym}_{\mathbb{C}} L_B^1 \) is the inclusion.

**Remark.** We have that the image of \( H^i(Spec A; \mathcal{P} \otimes F^2 \Omega_A) \to H^i(Spec A; \mathcal{P} \otimes \Omega_A) \) is zero for \( i \leq d \), so the condition (ii) in the lemma is always satisfied if \( d > 1 \).

**Proof.** Choose a cofibrant replacement for \( B \), and \( \xi \in (L_B^1)^1 \) with class \( \xi \), and consider \( \epsilon^\xi : \Omega_A \to \Omega_{\bar{A}} \) as in lemma 2.7.

We have \( \omega_2 \in \text{Filt}^1 \wedge^2 L_A \) by hypothesis, and by hypothesis \( \omega \) is transgressed from a class in \( L_B \otimes F^2 \Omega_{\bar{A}} \). So, by lemma 2.7(iii) we may assume that \( \omega = \omega_2 \) and \( \epsilon^\xi \omega = \omega_2 + \omega_2 \in H^2(Spec A, \mathcal{P} \otimes \text{Filt}^1 F^1 \Omega_A) \).

If we write \( (\xi \omega_2 + \omega_2)(\lambda) = (d+D)D_{Kos}(\xi \omega_2 + \omega_2)(\lambda) = (d+D)D_{Kos}(\omega_2(\lambda), \alpha \in H^2(Spec A, \mathcal{P} \otimes \Omega_{\bar{A}}) \).

Hence \( \xi \omega_2 + \omega_2 = (\xi \omega_2 + \omega_2)(0) = \pi^\ast \gamma \), for \( \gamma \in H^2(Spec B, \mathcal{P} \otimes F^1 \Omega_B) \).

Moreover, the class of \( \gamma \) in \( H^2(Spec A, \mathcal{P} \otimes \Omega_A) \) is zero. This is because the class of \( \epsilon^\xi \omega \) is zero in \( H^2(Spec A, \mathcal{P} \otimes \Omega_{\bar{A}}) \) by hypothesis (ii), and \( \pi^\ast : H^i(Spec B, \mathcal{P} \otimes \Omega_B) \to H^i(Spec A, \mathcal{P} \otimes \Omega_A) \) is an isomorphism for all \( i \).

Hence \( \gamma = df = \pi^\ast \gamma \), for some element \( f \in H^1(Spec B, \mathcal{P}) \).

### 3.5 Let \( B, \xi, A, \omega \) be as in the lemma and its proof, and suppose also that \( \bar{\omega}_2 : L_B^1 \otimes_B A \to L_B^1 \otimes_B A \) is the pullback of a map \( \sigma : L_B^1 \to L_B^1 \). Define \( A' = \text{Sym}_{\mathbb{C}} L_B^1 \), and \( \sigma : A' = \text{Sym}_{\mathbb{C}} L_B^1 \to A = \text{Sym}_{\mathbb{C}} L_B^1 \) to be the induced map in \( \text{cdga}_{B \setminus \cdot} \). Observe that the map \( L_B^1 \otimes_B A \to L_A \otimes A' A \to L_B^1 \otimes_B A \)
is induced by \([\omega]_2\) and so is a quis, i.e. \(L_{A'/A} = 0\). Moreover, as \(H^0(B) \to H^0(A')\), \(H^0(B) \to H^0(A)\) are surjections, so is \(H^0(A') \to H^0(A)\), hence \(A' \to A\) is a quis, by corollary 1.1(ii).

Put \(\tilde{\eta} = (\xi \omega_2 + \omega_2)^{(0)} = \xi \omega_2 + \omega_2^{(0)}\). Then \(\eta = \xi \sigma\), and \(\sigma^* \omega_{std} = (\omega_{std}) \in \wedge^2 L_A\) satisfies \(\sigma^* \omega_{std} = \omega \in \mathrm{Filt}^0 \wedge^2 L_A\).

**Lemma.** \(\sigma^* \omega_{std} = \omega) in \(H^2(\mathrm{Spec} A, \mathcal{P} \otimes F^2 \Omega_A)\)

**Proof.** Put \(\delta = \sigma^* \omega_{std} - \omega\). Consider the filtration by degree, "Filt. If \(\delta\) is in \(\wedge^2 L_A \) and \(\lambda > 0\), then \(\langle \mathrm{gr}^\lambda \rangle \rangle = \langle (d + D) D_{Kos} (\langle \mathrm{gr}^\lambda \rangle) / \lambda\). But \(D_{Kos} (\langle \mathrm{gr}^\lambda \rangle) = 0\), as \(\delta\) is in \(\mathrm{Filt}^0\) and so pulled back from \(B\). It follows that by construction \(\delta\) is zero.

3.6 To summarise, it seems we have proved the following.

**Theorem.** Let \(X \to R\) be a connected Deligne-Mumford dg-stack over some base \(\mathcal{D}\) stack, with \(\mathcal{P}\) an invertible complex of \(D\)-modules on \(X\), \(H^d(\mathcal{P}) \neq 0\) for some \(d > 0\), and \(\omega \in H^0(X, H^2(\mathcal{P} \otimes F^2 \Omega_X/X))\) a \(\mathcal{P}\)-shifted symplectic form.

\[
\begin{array}{ccc}
\mathrm{Spec} A & \xrightarrow{i} & \mathcal{X} \\
\downarrow & & \downarrow \\
\mathrm{Spec} R & \xrightarrow{f} & \mathcal{R}
\end{array}
\]

Suppose that \(f\) is etale map for which

i) the class of \(i^* [\omega]_2\) is zero in the Witt group of \(A\),

ii) the class of \(i^* \omega\) is zero in \(H^2(\mathrm{Spec} A, \mathcal{P} \otimes L_{\Omega A/R})\), and

iii) the underlying \(D\)-module of \(i^* \mathcal{P}\) is trivial. \(^8\)

Then there is a f.p. cdga \(B, f : B \to \mathcal{P}[1]\), and quis \(\sigma : A' = \mathrm{Sym}^\eta_B L_B^{1} \to A\) such that \(\sigma^* \omega_{df} = i^* \omega\), where \(\eta : L_B^{1} \to B[1]\) is induced from \(df\).

**Proof.** Given \(A\) as above, observe that by lemma 1.7 the underlying \(A\)-module of \(i^* \mathcal{P}\) is \(O[-d]\), necessarily with \(d \geq 0\). As the class of \([\omega]_2\) in the Witt group is zero, by construction 3.3 and lemma 3.3(i) we have a finitely presented cdga \(B \to A\), and \(\xi : L_B^1 \to B[1]\) such that \(\mathrm{Sym}^\xi_B L_B^1 \to A\) is a quis. As the class of \(i^* \omega\) is zero in the deRham complex of \(A\), we can assume that \(i^* \omega\) is transgressed from \(L_A/F^2 L_A\), so by lemma 3.3(ii) we can further assume \(\omega_2\) is pulled back from a morphism \(L_B^1 \to L_B^1\). Lemma 3.4 now gives an \(f : B \to \mathcal{P}[1]\) with \(e^f \omega = \pi^* df\), and the discussion and lemma in \(§3.5\) gives the result.

**Remark.** If \(d \neq 2 \mod 4\), every geometric point \(x \in X\) has a neighbourhood for which (i) holds. We can do slightly better. Let \(X\) be a Deligne-Mumford dg-stack, \(\omega\) a \(\mathcal{P}\)-shifted symplectic form on \(X\), Spec \(A' \to \mathcal{X}\) an etale map, and \(x \in X\) a closed point.

Algebraic surgery and Witt cancellation gives the localisation \((L_{A'})_{(x)}\) is quis to a direct sum of perfect complexes \(P_{(x)} \oplus M_{(x)}\), where \(M_{(x)}\) is metabolically perfect and \(P_{(x)}\) has Tor amplitude in \([-d/2, -d/2]\); moreover we can insist that \(P_{(x)}\) is zero if \(d \neq 2 \mod 4\). As \(X\) is locally finitely presented, and \(M_{(x)}\), \(P_{(x)}\) are perfect, there is some Zariski neighbourhood \(A' \to A\) of \(x\), and \(P, M \in \mathrm{Perf} A\), such that

\(^8\)This assumption is somewhat harsh. We will repost this note with a more general statement later.
$L_A = P \oplus M$, the localisations of $P, M$ are $P_{(x)}, M_{(x)}$, and $M$ is metabolic and $P$ has Tor amplitude $[-d/2, -d/2]$.\(^9\)

Put $C = \text{Sym}^*_A P$, where $\gamma : P \to P$ is the identity map. Then $M \otimes_A C \to L_C$ is a quis, so $\text{Spec} C$ carries a $P$-shifted symmetric structure with $L_C$ zero in the Witt group. Hence assuming (ii) and (iii) of the theorem, we get that $C$ is a twisted shifted cotangent bundle. But $A \to \text{Sym}_C(P)$ is a quis. We have proved

**Corollary.** Let $x \in X$, $P = \mathcal{O}[-d]$. Then there is a neighbourhood $\text{Spec } A \to X$ and quis $\sigma : \text{Sym}^*_B(L^*_B + P) \to A$, where $B \in \text{cdga}_{R_2}$ is f.p., $P \in \text{Perf}_B$ is a $P$-shifted symmetric complex of Tor amplitude $[-d/2, -d/2]$, zero unless $d = 2 \mod 4$, and $f : B \to \mathcal{P}[1]$, such that $i^* \omega$ is the pullback by $\sigma$ of the sum of the standard symplectic form $\omega_{std}$ with the form induced from $P$.

### 3.7 It seems that the theorem extends to Artin dg-stacks with little extra effort: if $X \to R$ is an Artin dg-stack, $\omega$ a $P$-shifted symplectic structure, and $d > 0$, then locally $X$ is a twisted shifted cotangent bundle $\text{Sym}^*_B(L^*_B + P)$, where $Y$ is an Artin dg-stack, $H^i(L^*_Y) = 0$ for $i \geq 0$, and $P \in \text{Perf}_Y$ is a $P$-shifted symmetric complex of Tor amplitude $[-d/2, -d/2]$, zero unless $d = 2 \mod 4$, as above. We will repost with details shortly.

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