Higher Order Periodic Solutions of Coupled $\phi^4$ Models

Avinash Khare
Institute of Physics, Bhubaneswar, Orissa 751005, India

Avadh Saxena
Theoretical Division and Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, NM 87545, USA

Abstract:

We obtain several higher order exact periodic solutions of (i) a coupled symmetric $\phi^4$ model in an external field, (ii) an asymmetric coupled $\phi^4$ model, (iii) an asymmetric-symmetric coupled $\phi^4$ model, in terms of Lamé polynomials of order two and obtain the corresponding hyperbolic solutions in the appropriate limit. These solutions are unusual in the sense that while they are the solutions of the coupled problems, they are not the solutions of the corresponding uncoupled problems.
1 Introduction

Coupled double well ($\phi^4$) one-dimensional potentials are prevalent in both condensed matter physics and field theory. Few examples of current interest include spin configurations, domain walls and magnetic phase transitions in multiferroic materials [1][2] and $\omega$ phase transition in various elements and alloys [3]. In two recent publications [4][5], we obtained a large number of periodic solutions, in terms of the Lamé polynomials of order one [6][7], for (i) a coupled symmetric $\phi^4$ model in an external field and (ii) an asymmetric coupled $\phi^4$ model, both models with a biquadratic coupling. All those solutions had the feature that in the uncoupled limit, they reduce to the well known solutions of the uncoupled symmetric or asymmetric double well problem as the case may be. The purpose of this paper is to point out that both these coupled models have, in addition, truly novel solutions, in terms of the Lamé polynomials of order two, which only exist due to the presence of the coupling between the two fields. In other words, while the Lamé polynomials of order two are the solutions of the coupled problem, they are not the solutions of the decoupled problem. For completeness, we also consider an asymmetric-symmetric coupled $\phi^4$ model (which corresponds to a first order transition in one field and a second order transition in the other) and show that not only Lamé polynomials of order one but also Lamé polynomials of order two are the solutions of this coupled problem, even though only Lamé polynomials of order one are the solutions of the uncoupled problem. This model is relevant for certain martensitic transformations in elements [8][9]

The paper is organized as follows. In Sec. II we provide the novel periodic as well as the corresponding hyperbolic solutions for the coupled symmetric $\phi^4$ model [4] with an explicit biquadratic coupling in the presence of an external field (with an additional linear-quadratic coupling) [4]. Note that the symmetric $\phi^4$ model, in the decoupled limit, corresponds to a second order transition in both the fields. We show that while the solutions of the uncoupled $\phi^4$ problem are the Lamé polynomials of order one, (i.e. sn, cn, dn), for the coupled problem, not only the Lamé polynomials of order one [4], but even the Lamé polynomials of order two are the solutions of the coupled field equations. In Sec. III we provide the periodic (Lamé polynomials of order two) and the corresponding hyperbolic solutions for the coupled asymmetric $\phi^4$ model, which corresponds to a first order transition in both the fields [5]. In Sec. IV and V we consider the Lamé polynomial solutions of order one and two respectively of a coupled asymmetric-symmetric $\phi^4$ model.
Finally, we conclude in Sec. VI with summary and possible extensions.

2 Coupled symmetric $\phi^4$ model in an external field

In [4] we had considered the following potential, with a biquadratic coupling between the two fields and in an external magnetic field ($H_z$)

$$V = \alpha_1 \phi^2 + \beta_1 \phi^4 + \alpha_2 \psi^2 + \beta_2 \psi^4 + \gamma \phi^2 \psi^2 - H_z [\rho_1 \phi + \rho_2 \phi^3 + \rho_3 \phi \psi^2],$$  

(1)

where $\alpha_i$, $\beta_i$, $\gamma$ and $\rho_i$ are material (or system) dependent parameters. For $\alpha_1 < 0, \alpha_2 < 0$ and $\beta_1 > 0, \beta_2 > 0$, this model corresponds to second order transitions in both fields $\phi$ and $\psi$. The corresponding (static) equations of motion are

$$\frac{d^2 \phi}{dx^2} = 2 \alpha_1 \phi + 4 \beta_1 \phi^3 + 2 \gamma \phi \psi^2 - H_z [\rho_1 + 3 \rho_2 \phi^2 + \rho_3 \psi^2],$$  

(2)

$$\frac{d^2 \psi}{dx^2} = 2 \alpha_2 \psi + 4 \beta_2 \psi^3 + 2 \gamma \phi^2 \psi - 2 H_z \rho_3 \phi \psi.$$  

(3)

These coupled set of equations admit several novel periodic solutions (i.e. Lamé polynomials of order 2), which we now discuss one by one systematically.

2.1 Solution I

It is not difficult to show that

$$\phi = F + A \text{sn}^2[D(x+x_0), m], \quad \psi = G + B \text{sn}^2[D(x+x_0), m],$$  

(4)

is an exact solution to coupled field Eqs. (2) and (3) provided the following eight field equations are satisfied

$$2 \alpha_1 F + 4 \beta_1 F^3 + 2 \gamma F G^2 - H_z \rho_1 - 3 H_z \rho_2 F^2 - H_z \rho_3 G^2 = 2 AD^2,$$  

(5)

$$2 \alpha_1 A + 12 \beta_1 F^2 A + 4 \gamma B F G + 2 \gamma A G^2 - 6 H_z \rho_2 A F - 2 H_z \rho_3 B G = -4(1 + m) AD^2,$$  

(6)

$$12 \beta_1 F A^2 + 2 \gamma F B^2 + 4 \gamma A B G - 3 H_z \rho_2 A^2 - H_z \rho_3 B^2 = 6 A m D^2,$$  

(7)

$$2 \beta_1 A^2 + \gamma B^2 = 0,$$  

(8)
\[2\alpha_2 G + 4\beta_2 G^3 + 2\gamma G F^2 - 2H_z\rho_3 G F = 2BD^2,\]  
\[2\alpha_2 B + 12\beta_2 G^2 B + 4\gamma AFG + 2\gamma B F^2 - 2H_z\rho_3 (BF + AG) = -4(1 + m)BD^2,\]  
\[12\beta_2 GB^2 + 2\gamma GA^2 + 4\gamma ABF - 2H_z\rho_3 AB = 6mBD^2,\]  
\[2\beta_2 B^2 + \gamma A^2 = 0.\]  

Here \(A\) and \(B\) denote the amplitudes of the “pulse lattice”, \(F\) and \(G\) are constants, \(D\) is an inverse characteristic length and \(x_0\) is the (arbitrary) location of the pulse. Five of these equations determine the five unknowns \(A, B, D, F, G\) while the other three equations, give three constraints between the nine parameters \(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma, H_z, \rho_1, \rho_2, \rho_3\). In particular, from Eqs. (9) and (12) it follows that

\[\gamma < 0, \quad |\gamma|^2 = 4\beta_1 \beta_2, \quad \sqrt{\beta_1} A^2 = \sqrt{\beta_2} B^2.\]  

Few comments are in order at this stage.

1. From the Eq. (9) it follows that no solution of form (4) exists in case \(G = 0\). Thus no solutions exist with \(\psi = B \text{sn}^2[D(x + x_0), m]\) irrespective of the value of \(F\). In fact one can also show that no solution exists in case \(B = -G\) or if \(B = -mG\) unless \(m = 1\). In other words, even the solutions of the form \(\psi = G \text{cn}^2[D(x + x_0), m]\) or \(\psi = G \text{dn}^2[D(x + x_0), m]\) do not exist, no matter what \(F\) is, except when \(m = 1\).

2. In the special case of \(H_z = 0\), the field equations (5) to (12) are completely symmetrical in \(\phi\) and \(\psi\). It is easily shown that in this case, solution (4) does not exist.

**Solution at** \(m = 1\): In the special case of \(m = 1\), the solution (4) goes over to the hyperbolic nontopological soliton solution

\[\phi = F + A \tanh^2[D(x + x_0)], \quad \psi = G + B \tanh^2[D(x + x_0)],\]  
provided the field Eqs. (5) to (12) with \(m = 1\) are satisfied. This hyperbolic soliton solution takes particularly simple form in two cases which we mention one by one.

(i) \(F = 0, G = -B\): In this case, the nontopological soliton solution (14) takes the simpler form

\[\phi = A \tanh^2[D(x + x_0)], \quad \psi = B \text{sech}^2[D(x + x_0)].\]
By analyzing Eqs. (5) to (12) it is easily shown that such a solution exists provided $\gamma < 0, \alpha_2 < 0, \rho_3 < 0$.

Further, while Eqs. (8) and (12) still continue to hold good, the other field equations take slightly simpler form

\[
D^2 = |\alpha_2| - |\gamma|A^2, \tag{16}
\]
\[
3\alpha_2 = H_z|\rho_3|A + |\gamma|A^2, \tag{17}
\]
\[
2AD^2 = H_z\rho_1 + H_z|\rho_3|B^2, \tag{18}
\]
\[
H_z\rho_1 + 3H_z\rho_2A^2 = 2\alpha_1A + 4\beta_1A^3, \tag{19}
\]
\[
3\alpha_1A + 10\beta_1A^3 = 6H_z\rho_2A^2 + H_z|\rho_3|B^2. \tag{20}
\]

(ii) $F = -A, G = -B$: In this limit the nontopological soliton solution (14) takes the simpler form

\[
\phi = A \operatorname{sech}^2[D(x + x_0)] , \quad \psi = B \operatorname{sech}^2[D(x + x_0)]. \tag{21}
\]

provided field Eqs. (5) and (9) hold good and further

\[
\rho_1 = 0, \quad \gamma < 0, \quad \alpha_1 = \alpha_2 > 0, < 0, \quad \rho_2 > 0, \quad \rho_3 > 0, \tag{22}
\]
\[
D^2 = \frac{\alpha_1}{2}, \quad A = \frac{3\alpha_1}{2H_z\rho_3}, \quad 3\rho_2A^2 = (2A^2 - B^2)\rho_3. \tag{23}
\]

### 2.2 Solution II

It is not difficult to show that

\[
\phi = F + A\operatorname{sn}^2[D(x + x_0), m] , \quad \psi = B\operatorname{sn}[D(x + x_0), m]\operatorname{cn}[D(x + x_0), m], \tag{24}
\]

is an exact solution to coupled field Eqs. (2) and (3) provided the following seven field equations are satisfied

\[
2\alpha_1F + 4\beta_1F^3 - H_z\rho_1 - 3H_z\rho_2F^2 = 2AD^2, \tag{25}
\]
\[
2\alpha_1A + 12\beta_1F^2A + 2\gamma FB^2 - 6H_z\rho_2AF - H_z\rho_3B^2 = -4(1 + m)AD^2, \tag{26}
\]
\[
12\beta_1F^2A + 2\gamma B^2(A - F) - 3H_z\rho_2A^2 + H_z\rho_3B^2 = 6AmD^2, \tag{27}
\]
\[
2\beta_1A^2 - \gamma B^2 = 0. \tag{28}
\]
\[2\alpha_2 + 2\gamma F^2 - 2H_z\rho_3 F = -(4 + m)D^2,\]  
\[4\beta_2 B^2 + 4\gamma AF - 2H_z\rho_3 A = 6mD^2,\]  
\[2\beta_2 B^2 - \gamma A^2 = 0.\]  

Four of these equations determine the four unknowns \(A, B, D, F\) while the other three equations, give three constraints between the nine parameters \(\alpha_{1,2}, \beta_{1,2}, \gamma, H_z, \rho_1, \rho_2, \rho_3\). In particular, from Eqs. (28) and (31) it follows that

\[\gamma > 0, \quad \gamma^2 = 4\beta_1\beta_2, \quad \sqrt{\beta_1}A^2 = \sqrt{\beta_2}B^2.\]  

It is easy to show that the solution (24) continues to exist at \(F = 0, -A, -A/m\) so long as \(H_z \neq 0\).

**Solution at \(m = 1\):** In the special case of \(m = 1\), the solution (24) goes over to the hyperbolic nontopological soliton solution

\[\phi = F + A \tanh^2[D(x + x_0)], \quad \psi = B \tanh[D(x + x_0)] \text{sech}[D(x + x_0)],\]  

provided the field Eqs. (25) to (31) with \(m = 1\) are satisfied.

**Special case \(H_z = 0\):** In the special case of \(H_z = 0\), the field equations (2) and (3) are symmetrical in \(\phi\) and \(\psi\). In this case, the Eqs. (25) to (31) take rather simple form. In particular, in case \(H_z = 0\), it is easily shown that the solution (24) exists provided

\[\gamma = 2\beta_1 = 2\beta_2, \quad A^2 = B^2,\]  

while the remaining equations take the simpler form

\[3mD^2 = (1 + 2x)\gamma A^2,\]  
\[D^2 = \alpha_1 x + \gamma A^2x^3,\]  
\[- 2(1 + m)D^2 = \alpha_1 + x(1 + 3x)\gamma A^2,\]  
\[- (4 + m)D^2 = 2\alpha_2 + 2x^2\gamma A^2,\]  

where \(x = F/A\). On solving these equations, one finds that the only acceptable solution is given by

\[3mx = -(1 + m) + \sqrt{1 - m + m^2},\]  

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using which one can then easily express $D^2, \alpha_2$ and $\gamma A^2$ in terms of $\alpha_1$.

In particular, at $H_z = 0$ and $m = 1$, the solution (33) exists provided relations (34) are satisfied and further

$$F = -\frac{A}{3}, \quad \alpha_1 < 0, \quad \alpha_2 < 0, \quad D^2 = \frac{|\alpha_1|}{4}, \quad \gamma A^2 = \frac{3|\alpha_1|}{2}, \quad |\alpha_2| = \frac{7}{8} \alpha_1. \tag{40}$$

### 2.3 Solution III

It is not difficult to show that unlike the solution (24), the solution

$$\phi = B\text{sn}[D(x + x_0), m]\text{cn}[D(x + x_0), m], \quad \psi = F + A\text{sn}^2[D(x + x_0), m], \tag{41}$$

exists only if $H_z = 0$. But since at $H_z = 0$, the field Eqs. (2) and (3) are symmetrical in $\phi$ and $\psi$, hence (41) is a solution to field Eqs. (2) and (3) provided Eqs. (34) to (40) (with suitable change of parameters) are satisfied.

### 2.4 Solution IV

It is not difficult to show that

$$\phi = F + A\text{sn}^2[D(x + x_0), m], \quad \psi = B\text{sn}[D(x + x_0), m]\text{dn}[D(x + x_0), m], \tag{42}$$

is an exact solution to coupled field Eqs. (2) and (3) provided the following seven field equations are satisfied

$$2\alpha_1 F + 4\beta_1 F^3 - H_z \rho_1 - 3H_z \rho_2 F^2 = 2AD^2, \tag{43}$$

$$2\alpha_1 A + 12\beta_1 F^2 A + 2\gamma F B^2 - 6H_z \rho_2 AF - H_z \rho_3 B^2 = -4(1 + m)AD^2, \tag{44}$$

$$12\beta_1 FA^2 + 2\gamma B^2(A - Fm) - 3H_z \rho_2 A^2 + mH_z \rho_3 B^2 = 6AmD^2, \tag{45}$$

$$2\beta_1 A^2 - m\gamma B^2 = 0, \tag{46}$$

$$2\alpha_2 + 2\gamma F^2 - 2H_z \rho_3 F = -(1 + 4m)D^2, \tag{47}$$

$$4\beta_2 B^2 + 4\gamma AF - 2H_z \rho_3 A = 6mD^2, \tag{48}$$

$$2m\beta_2 B^2 - \gamma A^2 = 0. \tag{49}$$
Four of these equations determine the four unknowns $A, B, D, F$ while the other three equations, give three constraints between the nine parameters $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma, \rho_1, \rho_2, \rho_3$. In particular, from Eqs. (46) and (49) it follows that

\[
\gamma > 0, \quad \gamma^2 = 4\beta_1\beta_2, \quad \sqrt{3_1}A^2 = m\sqrt{3_2}B^2. \tag{50}
\]

It is easy to show that the solution (42) continues to exist in case $F = 0, -A, -A/m$ as long as $H_z \neq 0$.

**Solution at $m = 1$:** In the special case of $m = 1$, the solution (42) goes over to the hyperbolic nontopological soliton solution (33) provided the field Eqs. (43) to (49) with $m = 1$ are satisfied.

**Special case $H_z = 0$:** In the special case of $H_z = 0$, the field equations (2) and (3) are symmetrical in $\phi$ and $\psi$. In this case, Eqs. (43) to (49) take rather simple form. In particular, in case $H_z = 0$, it is easily shown that the solution (42) exists provided Eq. (34) is satisfied while the remaining equations take the simpler form

\[
3m^2D^2 = (1 + 2mx)\gamma A^2, \tag{51}
\]

\[
D^2 = \alpha_1 x + \gamma A^2 x^3, \tag{52}
\]

\[
-2(1 + m)mD^2 = m\alpha_1 + x(1 + 3mx)\gamma A^2, \tag{53}
\]

\[
- (1 + 4m)D^2 = 2\alpha_2 + 2x^2\gamma A^2, \tag{54}
\]

where $x = F/A$. On solving these equations, one finds that the only acceptable solution is the one with $x$ again given by Eq. (39) using which one can then easily express $D^2, \alpha_2$ and $\gamma A^2$ in terms of $\alpha_1$. At $H_z = 0$ and $m = 1$, of course the solution goes over to the solution (33), which exists provided relations (34) to (40) are satisfied.

**2.5 Solution V**

It is not difficult to show that unlike the solution (42), the solution

\[
\phi = B\text{sn}[D(x + x_0), m]\text{dn}[D(x + x_0), m], \quad \psi = F + A\text{sn}^2[D(x + x_0), m], \tag{55}
\]

exists only if $H_z = 0$. But since at $H_z = 0$, the field Eqs. (2) and (3) are symmetrical in $\phi$ and $\psi$, hence (55) is a solution to field Eqs. (2) and (3) provided Eqs. (51) to (54) (with suitable change of parameters) are satisfied.
2.6 Solution VI

It is not difficult to show that
\[ \phi = F + Asn^2[D(x + x_0), m], \quad \psi = Bcn[D(x + x_0), m]dn[D(x + x_0), m], \]  

(56)
is an exact solution to coupled field Eqs. (2) and (3) provided the following seven field equations are satisfied
\[ 2\alpha_1 F + 4\beta_1 F^3 + 2\gamma B^2 F - H_z\rho_1 - 3H_z\rho_2 F^2 - H_z\rho_3 B^2 = 2AD^2, \]  

(57)
\[ 2\alpha_1 A + 12\beta_1 F^2 A + 2\gamma B^2 [A - (1 + m)F] - 6H_z\rho_2 AF + (1 + m)H_z\rho_3 B^2 = -4(1 + m)AD^2, \]  

(58)
\[ 12\beta_1 F A^2 + 2\gamma B^2 [mF - (1 + m)A] - 3H_z\rho_2 A^2 - mH_z\rho_3 B^2 = 6AmD^2, \]  

(59)
\[ 2\beta_1 A^2 + \gamma B^2 = 0, \]  

(60)
\[ 2\beta_2 A^2 + 4\gamma F^2 - 2H_z\rho_3 F = -(1 + m)D^2, \]  

(61)
\[ -4(1 + m)\beta_2 B^2 + 4\gamma AF - 2H_z\rho_3 A = 6mD^2, \]  

(62)
\[ 2m\beta_2 B^2 + \gamma A^2 = 0. \]  

(63)

Four of these equations determine the four unknowns \(A, B, D, F\) while the other three equations, give three constraints between the nine parameters \(\alpha_1, \beta_1, \gamma, H_z, \rho_1, \rho_2, \rho_3\). In particular, from above Eqs. (60) and (63) it follows that
\[ \gamma < 0, \quad |\gamma|^2 = 4m\beta_1 \beta_2, \quad \sqrt{\beta_1 A^2} = \sqrt{m\beta_2 B^2}. \]  

(64)

It is easily shown that the solution (56) continues to exist in case \(F = 0, -A, -A/m\) as long as \(H_z \neq 0\).

**Solution at** \(m = 1\): In the special case of \(m = 1\), the solution (56) goes over to the hyperbolic nontopological soliton solution
\[ \phi = F + A \tanh^2[D(x + x_0)], \quad \psi = B \sech^2[D(x + x_0)], \]  

(65)
which is essentially the solution (14) with \(B = -G\) provided the field Eqs. (57) to (63) with \(m = 1\) are satisfied.

**Special case** \(H_z = 0\): In the special case of \(H_z = 0\), the field equations (2) and (3) are symmetrical in \(\phi\) and \(\psi\). In this case, at least at \(m = 1\), one can show that there is no solution to these equations.
Of course this is expected since we know from the discussion of solution (4) that at $m = 1$ and $H_z = 0$, solution (65) does not exist.

3 Coupled Asymmetric $\phi^4$ Model

Recently we had also considered a coupled asymmetric $\phi^4$ model [5], which in the uncoupled limit corresponds to a first order transition in both the fields, and had obtained periodic solutions in terms of the Lamé polynomials of order one. The purpose of this section is to show that the Lamé polynomials of order two also constitute exact solutions of the same model. It is worth pointing out here that while the Lamé polynomials of order one are also solutions of the asymmetric uncoupled model, the Lamé polynomials of order two are in fact not the solutions of the uncoupled equations, thereby giving us genuinely new solutions of the coupled problem.

The potential that we considered in [5] is given by $(\beta_1 > 0, \beta_2 > 0)$

$$V = \alpha_1 \phi^2 + \delta_1 \phi^3 + \beta_1 \phi^4 + \alpha_2 \psi^2 + \delta_2 \psi^3 + \beta_2 \psi^4 + \gamma \phi^2 \psi^2 + \eta \phi \psi^2, \quad (66)$$

where $\alpha_i, \delta_i, \beta_i, \gamma$ and $\eta$ are material (or system) dependent parameters. Note that we have changed the notation slightly from that followed in [5], in order to be in conformity with the notation in the previous section. Hence the (static) equations of motion are

$$\frac{d^2 \phi}{dx^2} = 2\alpha_1 \phi + 3\delta_1 \phi^2 + 4\beta_1 \phi^3 + 2\gamma \phi \psi^2 + \eta \phi \psi^2, \quad (67)$$

$$\frac{d^2 \psi}{dx^2} = 2\alpha_2 \psi + 3\delta_2 \psi^2 + 4\beta_2 \psi^3 + 2\gamma \phi^2 \psi + 2\eta \phi \psi. \quad (68)$$

Observe that as long as $\eta \neq 0$, the two field equations are asymmetric in $\phi$ and $\psi$. We shall consider solutions of these coupled field equations in case $\alpha_i \neq 0, \delta_i \neq 0, \beta_i > 0$, as only then the model corresponds to a first order transition in both the fields.

There is only one solution in this case. It is not difficult to show that

$$\phi = F + A \text{sn}^2[D(x + x_0), m], \quad \psi = G + B \text{sn}^2[D(x + x_0), m], \quad (69)$$
is an exact solution to coupled field Eqs. (67) and (68) provided the following eight field equations are satisfied

\[ 2\alpha_1 F + 3\delta_1 F^2 + 4\beta_1 F^3 + 2\gamma FG^2 + \eta G^2 = 2 AD^2, \]  
\[ 2\alpha_1 A + 6\delta_1 AF + 12\beta_1 F^2 A + 4\gamma BFG + 2\gamma AG^2 + 2\eta BG = -4(1 + m)AD^2, \]  
\[ 3\delta_1 A^2 + 12\beta_1 FA^2 + 2\gamma FB^2 + 4\gamma ABG + \eta B^2 = 6AmD^2, \]  
\[ 2\beta_1 A^2 + \gamma B^2 = 0, \]  
\[ 2\alpha_2 G + 3\delta_2 G^2 + 4\beta_2 G^3 + 2\gamma GF^2 + 2\eta FG = 2 BD^2, \]  
\[ 2\alpha_2 B + 6\delta_2 BG + 12\beta_2 G^2 B + 4\gamma AFG + 2\gamma BF^2 + 2\eta(AG + BF) = -4(1 + m)BD^2, \]  
\[ 3\delta_2 B^2 + 12\beta_2 GB^2 + 2\gamma GA^2 + 4\gamma ABF + 2\eta AB = 6mBD^2, \]  
\[ 2\beta_2 B^2 + \gamma A^2 = 0. \]

Five of these equations determine the five unknowns \( A, B, D, F, G \), while the other three equations, give three constraints between the eight parameters \( \alpha_1, \delta_1, \beta_1, \gamma, \eta \). In particular, from the above equations it follows that

\[ \gamma < 0, \quad |\gamma|^2 = 4\beta_1 \beta_2, \quad \sqrt{\beta_1} A^2 = \sqrt{\beta_2} B^2. \]  

From Eq. (74) it follows that no solution of the form (69) exists in case \( G = 0 \). Thus no solutions exist with \( \psi = B\text{sn}^2[D(x + x_0), m] \), irrespective of the value of \( F \). In fact one can also show that no solution exists even in case \( B = -G \) or if \( B = -mG \) unless \( m = 1 \). In other words, solutions of the form \( \psi = G\text{cn}^2[D(x + x_0), m] \) or \( \psi = G\text{dn}^2[D(x + x_0), m] \) do not exist, no matter what \( F \) is except when \( m = 1 \).

**Solution at \( m = 1 \):** In the special case of \( m = 1 \), the solution (69) goes over to the hyperbolic nontopological soliton solution

\[ \phi = F + A\tanh^2[D(x + x_0)], \quad \psi = G + B\tanh^2[D(x + x_0)], \]

provided the field Eqs. (70) to (77) with \( m = 1 \) are satisfied. This hyperbolic soliton solution takes a particularly simple form in two cases which we mention one by one.
(i) $F = 0, G = -B$: In this limit the nontopological soliton solution (79) takes the simpler form

$$\phi = A \tanh^2[D(x + x_0)], \quad \psi = B \sech^2[D(x + x_0)].$$ (80)

By analyzing Eqs. (70) to (77) it is easily shown that such a solution exists provided $\gamma < 0, \alpha_2 < 0$. Further, while Eqs. (73) and (77) still continue to hold good, the other field equations take a slightly simpler form

$$2AD^2 = \eta B^2,$$ (81)

$$6AD^2 = 3\delta_1 A^2 + 4|\gamma|AB^2 + \eta B^2,$$ (82)

$$-4AD^2 = \alpha_1 A - |\gamma|AB^2 - \eta B^2,$$ (83)

$$2D^2 = \alpha_2 - |\gamma| A^2 + \eta A,$$ (84)

$$6D^2 = 3\delta_2 B - 4|\gamma| A^2 + 2\eta A.$$ (85)

From here one can easily solve for $A, B, D$ and further obtain four constraints between the eight parameters.

(ii) $F = -A, G = -B$: In this limit the nontopological soliton solution (14) takes the simpler form

$$\phi = A \sech^2[D(x + x_0)], \quad \psi = B \sech^2[D(x + x_0)],$$ (86)

provided field Eqs. (73) and (77) hold good and further

$$\alpha_1 = \alpha_2 > 0, \quad 2D^2 = \alpha_1,$$ (87)

$$-6AD^2 = 3\delta_1 A^2 + \eta B^2, \quad -6BD^2 = 3\delta_2 B^2 + 2\eta AB.$$ (88)

It turns out that as long as $\delta_2 \neq 0$, no other Lamé polynomials of order two form a solution of field Eqs. (67) and (68).

4 Asymmetric-Symmetric $\phi^4$ Model: Lamé Polynomial Solutions of Order one

In the last section we considered solutions in case both $\delta_1$ and $\delta_2$ are nonzero, i.e. solutions of the asymmetric $\phi^4$ problem such that in both $\psi$ and $\phi$ fields one has a first order phase transition. In this
section, we consider the case when $\delta_1 \neq 0$ while $\delta_2 = 0$. This corresponds to having a first order transition in $\phi$ and a second order transition in $\psi$. There are interesting physical situations such as a face-centered cubic to a hexagonal close packed (FCC-HCP) reconstructive structural transition \[8\] and the martensitic transition in cobalt \[9\] where this model is relevant. Therefore, in this section we consider such a coupled model and obtain various solutions of this coupled model in terms of Lamé polynomials of order one, and their hyperbolic limit. In the next section, we shall show that the Lamé polynomials of order two are also the exact solutions of this coupled model, even though they are not the solutions in the decoupled limit.

The potential we consider is given by $(\beta_1 > 0, \beta_2 > 0)$

$$V = \alpha_1 \phi^2 - \delta_1 \phi^3 + \beta_1 \phi^4 + \eta \phi \psi^2 + \gamma \phi^2 \psi^2 + \alpha_2 \psi^2 + \beta_2 \psi^4,$$  \hspace{1cm} (89)

where $\alpha_{1,2}, \beta_{1,2}, \delta_1, \eta, \gamma$ are system dependent parameters. The static field equations that follow from here are

$$\phi_{xx} = 2\alpha_1 \phi - 3\delta_1 \phi^2 + 4\beta_1 \phi^3 + \eta \phi \psi^2 + 2\gamma \phi^2 \psi^2, \hspace{1cm} (90)$$

$$\psi_{xx} = 2\alpha_2 \psi + 4\beta_2 \psi^3 + 2\eta \phi \psi + 2\gamma \phi^2 \psi. \hspace{1cm} (91)$$

Observe that as long as $\eta \neq 0$, the two field equations are asymmetric and hence kink-pulse and pulse-kink solitons would be distinct.

For the uncoupled model ($\eta = \gamma = 0$), it is easy to show that the potential in $\phi$ corresponds to a first order transition while in $\psi$ corresponds to a second order transition. In particular, while $\phi = 0$ is the only minimum in case $\delta_1^2 < (32/9)\alpha_1 \beta_1$, for $(32/9)\alpha_1 \beta_1 < \delta_1^2 < 4\alpha_1 \beta_1$, $\phi = 0$ is the absolute minimum while $\phi = \phi_c$ is the local minimum, whereas for $\delta_1^2 > 4\alpha_1 \beta_1$, the opposite is true. At $\delta_1^2 = 4\alpha_1 \beta_1$ we have degenerate minima at $\phi = 0$ and $\phi = \phi_c$. It is well known that while at $T = T_c^I$ (i.e. $\delta_1^2 = 4\alpha_1 \beta_1$), one has a kink solution, for both $T > T_c^I$ and $T < T_c^I$ one has a pulse solution around the local minimum. On the other hand, while $\psi = 0$ is the only extremum, i.e. minimum in case $\alpha_2 > 0$ (i.e. $T > T_c^{II}$), for $\alpha_2 < 0$, $\psi = 0$ is the maximum, while there are degenerate absolute minima at $\psi = \pm \psi_c$ (i.e. $T < T_c^{II}$). In this case, one has a kink solution for $T < T_c^{II}$.

Let us now write down the periodic soliton solutions of this coupled asymmetric-symmetric model in terms of Lamé polynomials of order one. We shall see that at $\delta_1^2 = 4\alpha_1 \beta_1$ (corresponding to the uncoupled
We look for the most general periodic solutions in terms of the Jacobi elliptic functions $\text{sn}(x,m)$, $\text{cn}(x,m)$ and $\text{dn}(x,m)$ \cite{6}. Since this model is almost similar to the asymmetric coupled $\phi^4$ model \cite{5}, except that while $\delta_2$ is nonzero in that case, in the present case $\delta_2 = 0$, hence most of the results about the Lamé polynomial solutions are very similar in the two cases. We shall therefore only focus on those results which are different in the two cases.

**Solution I**

It is not difficult to show that

$$
\phi = F + A\text{sn}[D(x + x_0), m], \quad \psi = G + B\text{sn}[D(x + x_0), m],
$$

(92)

is an exact solution to coupled Eqs. \eqref{90} and \eqref{91} provided the following eight coupled equations are satisfied

\begin{align}
2\alpha_1 F - 3\delta_1 F^2 + 4\beta_1 F^3 + \eta G^2 + 2\gamma FG^2 &= 0, \\
2\alpha_1 A - 6\delta_1 AF + 12\beta_1 F^2 A + 2\eta BG + 2\gamma AG^2 + 4\gamma FBG &= -(1 + m)AD^2, \\
-3\delta_1 A^2 + 12\beta_1 FA^2 + \eta B^2 + 2\gamma FB^2 + 4\gamma ABG &= 0, \\
2\beta_1 A^2 + \gamma B^2 &= mD^2, \\
2\alpha_2 G + 4\beta_2 G^3 + 2\eta FG + 2\gamma GF^2 &= 0, \\
2\alpha_2 B + 12\beta_2 G^2 B + 2\eta AG + 2\eta FB + 4\gamma AFG + 2\gamma BF^2 &= -(1 + m)BD^2, \\
12\beta_2 GB^2 + 2\eta AB + 2\gamma GA^2 + 4\gamma ABF &= 0, \\
2\beta_2 B^2 + \gamma A^2 &= mD^2.
\end{align}

(93) \quad (94) \quad (95) \quad (96) \quad (97) \quad (98) \quad (99) \quad (100)

Five of these equations determine the five unknowns $A, B, D, F, G$ while the other three equations, give three constraints between the seven parameters $\alpha_{1,2}, \beta_{1,2}, \delta_1, \eta, \gamma$. In particular, $A$ and $B$ are given by

$$
A^2 = \frac{mD^2(2\beta_2 - \gamma)}{(4\beta_1\beta_2 - \gamma^2)}, \quad B^2 = \frac{mD^2(2\beta_1 - \gamma)}{(4\beta_1\beta_2 - \gamma^2)}.
$$

(101)
It is easily shown that, while no solution exists in case $F = 0$, irrespective of whether $G$ is zero or nonzero, a solution exists in case $G = 0, F \neq 0$. In fact the analysis becomes somewhat simpler in that case and we shall restrict our discussion to that case.

**G=0, F\neq0:**

In this case $A, B$ are again given by Eq. (101) while $D$ and $F$ are given by

$$D^2 = \frac{\alpha_1}{(1 + m)}, \quad F = \sqrt{\frac{\alpha_1}{4\beta_1}},$$

and further the following three constraints have to be satisfied

$$\delta_1^2 = 4\alpha_1\beta_1, \quad \eta^2 = 2\gamma(\alpha_1 + 2\alpha_2), \quad \delta_1\eta + 2\alpha_1\gamma = 0.$$ \hspace{1cm} (103)

Thus this solution exists at $T = T_c^I$ for $\phi$ and $T < T_c^{II}$ for $\psi$.

**m=1**

In this limiting case we have a kink-kink solution given by

$$\phi = F + A \tanh[D(x + x_0)], \quad \psi = B \tanh[D(x + x_0)],$$

with $A, B, F$ and $D$ given by Eqs. (101) and (102) with $m = 1$ while the three constraints are again given by Eq. (103).

**Solution II**

A different type of solution (pulse lattice) is given by

$$\phi = F + A \text{cn}[D(x + x_0), m], \quad \psi = G + B \text{cn}[D(x + x_0), m],$$

which is an exact solution provided eight coupled equations similar to those given by Eqs. (93) to (100) are satisfied. In particular, in this case $A$ and $B$ are given by

$$A^2 = \frac{mD^2(2\beta_2 + |\gamma|)}{(\gamma^2 - 4\beta_1\beta_2)}, \quad B^2 = \frac{mD^2(2\beta_1 + |\gamma|)}{(\gamma^2 - 4\beta_1\beta_2)}.$$ \hspace{1cm} (106)

Notice that unlike the previous solution, this solution exists only when $\gamma < 0$ and $\gamma^2 > 4\beta_1\beta_2$.

Unlike the previous case, it turns out that in this case a solution exists both when $G = 0, F \neq 0$ and $F = 0, G \neq 0$ and we discuss both the cases one by one.
In this case $A,B$ are again given by Eq. (106) while $D$ and $F$ are given by

$$D^2 = \frac{\alpha_1}{(1 - 2m)}, \quad F = \sqrt{\frac{\alpha_1}{4\beta_1}}.$$ (107)

Further, there are three constraints given by

$$\eta\delta_1 = 2|\gamma|\alpha_1, \quad \delta_1^2 = 4\alpha_1\beta_1, \quad \eta^2 + 2|\gamma|\alpha_1 = -4|\gamma|\alpha_2.$$ (108)

Notice that this solution exists only if $\alpha_1 > 0, \alpha_2 < 0$ and $m < 1/2$. Thus this solution exists at $T = T_{c_I}^I$ for $\phi$ and $T < T_{c_{II}}^I$ for $\psi$.

In this limiting case we have a pulse-pulse solution given by

$$\phi = A\text{sech}[D(x + x_0)], \quad \psi = G + B\text{sech}[D(x + x_0)],$$ (111)

provided the relations (106), (109) and (110) are satisfied with $m = 1$.

Yet another pulse lattice solution is given by

$$\phi = F + A\text{dn}[D(x + x_0), m], \quad \psi = G + B\text{dn}[D(x + x_0), m],$$ (112)

Notice that this solution exists only if $\alpha_1 > 0, \alpha_2 < 0$ and $m > 1/2$. Thus this solution exists at $T = T_{c_I}^I$ for $\phi$ and $T < T_{c_{II}}^I$ for $\psi$.

In this case, solution (105) exists only if $\eta = 0$ and $\alpha_2 < 0, \gamma < 0$. While $A,B$ are again given by Eq. (106), $D$ and $G$ are given by

$$D^2 = \frac{4|\alpha_2|}{(2m - 1)}, \quad G = \sqrt{\frac{|\alpha_2|}{2\beta_2}}.$$ (109)

Further, there are three constraints given by

$$2\alpha_1\beta_2 = |\alpha_2|(4\beta_2 + |\gamma|), \quad 3\delta_1 A + 4|\gamma|BG = 0, \quad |\gamma|A^2 = 6\beta_2 B^2.$$ (110)

Notice that this solution exists only if $\alpha_1 > 0, \alpha_2 < 0$ and $m > 1/2$. Thus this solution exists at $T = T_{c_I}^I$ for $\phi$ and $T < T_{c_{II}}^I$ for $\psi$.

Yet another pulse lattice solution is given by

$$\phi = F + A\text{dn}[D(x + x_0), m], \quad \psi = G + B\text{dn}[D(x + x_0), m],$$ (112)
which is an exact solution provided eight coupled equations similar to Eqs. (93) to (100) are satisfied. In particular, \( A \) and \( B \) are given by

\[
A^2 = \frac{D^2(2\beta_2 + |\gamma|)}{(\gamma^2 - 4\beta_1\beta_2)}, \quad B^2 = \frac{D^2(2\beta_1 + |\gamma|)}{(\gamma^2 - 4\beta_1\beta_2)}.
\]  (113)

Thus this solution too exists only if \( \gamma < 0 \) and \( \gamma^2 > 4\beta_1\beta_2 \).

Unlike the solution II, this solution exists only if \( F = 0, G \neq 0 \). In this case, solution (112) exists only if \( \eta = 0 \) and \( \alpha_2 < 0, \gamma < 0 \). While \( A, B \) are again given by Eq. (113), \( D \) and \( G \) are given by

\[
D^2 = \frac{4|\alpha_2|}{(2 - m)}, \quad G = \sqrt{\frac{|\alpha_2|}{2\beta_2}}.
\]  (114)

Further, there are three constraints given by Eq. (110).

At \( m = 1 \), this solution too goes over to the pulse-pulse solution as given by Eq. (111).

**Solution IV**

We shall now discuss mixed solutions to the coupled Eqs. (90) and (91). It turns out that all these solutions exist only if \( G \) is necessarily zero while \( F \) is necessarily nonzero.

It is easily shown that

\[
\phi = F + A\sin[D(x + x_0), m], \quad \psi = G + B\csc[D(x + x_0), m],
\]  (115)

is an exact solution provided

\[
G = 0, \quad \eta + 2\gamma F = 0, \quad \eta < 0, \quad 2\beta_1 > \gamma > 2\beta_2 > 0.
\]  (116)

In particular, \( A \) and \( B \) are given by

\[
A^2 = \frac{mD^2(\gamma - 2\beta_2)}{(\gamma^2 - 4\beta_1\beta_2)}, \quad B^2 = \frac{mD^2(2\beta_1 - \gamma)}{(\gamma^2 - 4\beta_1\beta_2)},
\]  (117)

while \( D \) is given by

\[
(1 + m)D^2 = \alpha_1 - 2\gamma B^2.
\]  (118)

Further, the three constraints are

\[
\delta_1^2 = 4\alpha_1\beta_1, \quad 2\alpha_1\gamma + \eta\delta_1 = 0, \quad (2m - 1)D^2 = 2\alpha_2 + 2\gamma(A^2 - F^2).
\]  (119)
In this limiting case we have a kink-pulse solution given by

$$
\phi = F + A \tanh[D(x + x_0)], \quad \psi = B \sech[D(x + x_0)],
$$

(120)

with $A$, $B$, $F$ and $D$ given by Eqs. (116) to (118) with $m = 1$ while the three constraints are again given by Eq. (119).

**Solution V**

It is easy to show that another such solution is

$$
\phi = F + A \text{sn}[D(x + x_0), m], \quad \psi = G + B \text{dn}[D(x + x_0), m].
$$

(121)

This is an exact solution provided Eq. (116) is satisfied. Further, $A$ and $B$ are given by

$$
A^2 = \frac{mD^2(\gamma - 2\beta_2)}{(\gamma^2 - 4\beta_1\beta_2)}, \quad B^2 = \frac{D^2(2\beta_1 - \gamma)}{(\gamma^2 - 4\beta_1\beta_2)},
$$

(122)

while $D$ is again given by Eq. (118). In addition, two of the three constraints are again given by Eq. (119) while the third one is now given by

$$
(2 - m) m D^2 = 2m \alpha_2 + 2\gamma (A^2 - m F^2).
$$

(123)

In the limiting case of $m = 1$, we again have the same kink-pulse solution as given by Eq. (120).

**Solution VI**

We now discuss two periodic solutions which at $m = 1$ reduce to pulse-kink solutions. In particular, it is easily shown that

$$
\phi = F + A \text{cn}[D(x + x_0), m], \quad \psi = G + B \text{sn}[D(x + x_0), m],
$$

(124)

is an exact solution provided Eq. (116) is satisfied. Further, $A$ and $B$ are given by

$$
A^2 = \frac{mD^2(\gamma - 2\beta_2)}{(4\beta_1\beta_2 - \gamma^2)}, \quad B^2 = \frac{mD^2(2\beta_1 - \gamma)}{(4\beta_1\beta_2 - \gamma^2)},
$$

(125)

while $D$ is given by

$$
(2m - 1) D^2 = 2\gamma B^2 - \alpha_1.
$$

(126)
In addition, two of the three constraints are again given by Eq. (119) while the third constraint is

\[-(1 + m)D^2 = 2\alpha_2 + 2\gamma(A^2 - F^2).
\]

m=1

In this limiting case we have a pulse-kink solution given by

\[\phi = F + A\text{sech}[D(x + x_0)], \quad \psi = B\text{tanh}[D(x + x_0)],\]

with \(A, B, F\) and \(D\) given by Eqs. (125) to (127) with \(m = 1\).

Solution VII

Another periodic solution which at \(m = 1\) reduces to the same pulse-kink solution (128) is given by

\[\phi = F + A\text{dn}[D(x + x_0), m], \quad \psi = G + B\text{sn}[D(x + x_0), m],\]

provided Eq. (116) is satisfied. Further, \(A\) and \(B\) are given by

\[A^2 = \frac{D^2(\gamma - 2\beta_2)}{(4\beta_1\beta_2 - \gamma^2)}, \quad B^2 = \frac{mD^2(2\beta_1 - \gamma)}{(4\beta_1\beta_2 - \gamma^2)},\]

while \(D\) is given by

\[m(2 - m)D^2 = 2\gamma B^2 - 2\alpha_1 m.
\]

In addition, two of the three constraints are again given by Eq. (119) while the third constraint is given by Eq. (127).

Solution VIII

We now discuss two mixed solutions which at \(m = 1\) reduce to pulse-pulse solution. In particular, it is easy to check that

\[\phi = F + A\text{cn}[D(x + x_0), m], \quad \psi = G + B\text{dn}[D(x + x_0), m],\]

is an exact solution to Eqs. (90) and (91) provided

\[G = 0, \quad \eta + 2\gamma F = 0, \quad \gamma < 0, \quad \gamma^2 > 4\beta_1\beta_2.
\]

Further, \(A\) and \(B\) are given by

\[A^2 = \frac{mD^2(|\gamma| + 2\beta_2)}{(\gamma^2 - 4\beta_1\beta_2)}, \quad B^2 = \frac{D^2(|\gamma| + 2\beta_1)}{(\gamma^2 - 4\beta_1\beta_2)}.
\]
while $D$ is given by
\[(1 - 2m)D^2 = 2|\gamma|(1 - m)B^2 + \alpha_1.\] (135)

In addition, two of the three constraints are again given by Eq. (119) while the third constraint is given by
\[(2 - m)D^2 = 2\alpha_2 + 2|\gamma|[(1 - m)A^2 - mE^2].\] (136)

At $m = 1$, the solution (132) goes over to the pulse-pulse solution
\[\phi = F + \text{Asech}[D(x + x_0)], \quad \psi = B\text{sech}[D(x + x_0)],\] (137)
satisfying conditions (133) to (135) with $m = 1$.

**Solution IX**

Another mixed solution is
\[\phi = F + \text{Adn}[D(x + x_0), m], \quad \psi = G + B\text{cn}[D(x + x_0), m],\] (138)
provided Eq. (131) is satisfied. Further, $A$ and $B$ are given by
\[A^2 = \frac{D^2(|\gamma| + 2\beta_2)}{(\gamma^2 - 4\beta_1\beta_2)}, \quad B^2 = \frac{mD^2(|\gamma| + 2\beta_1)}{(\gamma^2 - 4\beta_1\beta_2)},\] (139)
while $D$ is given by
\[m(2 - m)D^2 = 2|\gamma|(1 - m)B^2 - m\alpha_1.\] (140)

In addition, two of the three constraints are again given by Eq. (119) while the third constraint is given by
\[(2m - 1)D^2 = 2\alpha_2 - 2|\gamma|[(1 - m)A^2 - F^2].\] (141)

At $m = 1$, the solution (138) also goes over to the pulse-pulse solution (137).

In addition to these, there are four other solutions which have been discussed in [5] (i.e. in case $\delta_2 \neq 0$) which continue to be valid even when $\delta_2 = 0$. As an illustration, we discuss only one of these solutions at $m = 1$ which in the uncoupled case corresponds to $\delta_1^2 > 4\alpha_1\beta_1$, i.e. $T < T_c^I$.

**Solution X**
It is easily shown that

\[ \phi = \frac{A \text{sech}[D(x + x_0)]}{1 + H \text{sech}[D(x + x_0)]}, \quad \psi = \frac{B \text{sech}[D(x + x_0)]}{1 + H \text{sech}[D(x + x_0)]}, \quad (142) \]

is an exact solution to the field Eqs. (90) and (91) provided \( A \) and \( H \) are given by

\[ A^2 = \frac{(H^2 - 1)D^2(2\beta_2 - \gamma)}{(4\beta_1\beta_2 - \gamma^2)}, \quad B^2 = \frac{(H^2 - 1)D^2(2\beta_1 - \gamma)}{(4\beta_1\beta_2 - \gamma^2)}, \quad (143) \]

and

\[ D^2 = 2\alpha_1 = 2\alpha_2, \quad \eta A + 3\alpha_1 H = 0, \quad \eta(\gamma + 2\beta_1) + 3\beta_1\gamma = 0. \quad (144) \]

5 \hspace{1em} \textbf{Asymmetric-Symmetric} \hspace{1em} \phi^4 \hspace{1em} \textbf{Model: Lamé Polynomial Solutions of Order Two}

In the last section we discussed the Lamé polynomial solutions of order one for the field Eqs. (90) and (91) which are also the solutions of the uncoupled problem. We now show that the field Eqs. (90) and (91) also admit Lamé polynomial solutions of order two, which are not the solutions of the uncoupled problem.

5.1 \hspace{1em} \textbf{Solution I}

It is not difficult to show that

\[ \phi = F + A \text{sn}^2[D(x + x_0), m], \quad \psi = G + B \text{sn}^2[D(x + x_0), m], \quad (145) \]

is an exact solution to coupled field Eqs. (90) and (91) provided the field Eqs. (70) to (77) with \( \delta_2 = 0 \) are satisfied. Five of these equations determine the five unknowns \( A, B, D, F, G \), while the other three equations, give three constraints between the seven parameters \( \alpha_1, \alpha_2, \delta_1, \beta_1, \beta_2, \gamma, \eta \). In particular, Eq. (78) must be satisfied. Most of the conclusions drawn in the last section for this solution also apply in this case.

**Solution at \( m = 1 \):** In the special case of \( m = 1 \), the solution (145) goes over to the hyperbolic nontopological soliton solution (79). This hyperbolic soliton solution takes a particularly simple form in two cases which we mention one by one.
(i) $F = 0, G = -B$: In this limit the nontopological soliton solution (14) takes the simpler form

$$\phi = A \tanh^2[D(x + x_0)], \quad \psi = B \sech^2[D(x + x_0)].$$  

By analyzing Eqs. (70) to (77) it is easily shown that such a solution exists provided $\gamma < 0, \alpha_2 < 0$. Further, while Eqs. (73) and (77) still continue to hold good, the other field equations take a slightly simpler form as given by Eqs. (81) to (84). However, instead of Eq. (85), we now have

$$6D^2 = -4|\gamma|A^2 + 2\eta A.$$  

(ii) $F = -A, G = -B$: In this limit the nontopological soliton solution (14) takes the simpler form

$$\phi = A \sech^2[D(x + x_0)], \quad \psi = B \sech^2[D(x + x_0)],$$

provided field Eqs. (73) and (77) hold good and further

$$\alpha_1 = \alpha_2 > 0, \quad 2D^2 = \alpha_1,$$

$$-6AD^2 = 3\delta_1 A^2 + \eta B^2, \quad -3D^2 = 2\eta A.$$ 

**Solution II**

Unlike the previous section, it turns out that in view of $\delta_2 = 0$, now three more Lamé polynomial solutions of order two are allowed which we present one by one.

It is not difficult to show that

$$\phi = F + A \text{sn}^2[D(x + x_0), m], \quad \psi = B \text{sn}[D(x + x_0), m] \text{cn}[D(x + x_0), m],$$

is an exact solution to coupled field Eqs. (90) and (91) provided the following seven field equations are satisfied

$$2\alpha_1 F + 3\delta_1 F^2 + 4\beta_1 F^3 = 2AD^2,$$

$$2\alpha_1 A + 6\delta_1 AF + 12\beta_1 F^2 A + 2\gamma FB^2 + \eta B^2 = -4(1 + m)AD^2,$$

$$3\delta_1 A^2 + 12\beta_1 FA^2 + 2\gamma B^2(A - F) - \eta B^2 = 6AmD^2,$$

$$2\beta_1 A^2 - \gamma B^2 = 0.$$ 

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\[ 2\alpha_2 + 2\gamma F^2 + 2\eta F = -(4 + m)D^2, \quad (156) \]
\[ 4\beta_2 B^2 + 4\gamma AF + 2\eta A = 6mD^2, \quad (157) \]
\[ 2\beta_2 B^2 - \gamma A^2 = 0. \quad (158) \]

Four of these equations determine the four unknowns \( A, B, D, F \) while the other three equations, give three constraints between the seven parameters \( \alpha_{1,2}, \beta_{1,2}, \gamma, \delta_{1,2}, \eta \). In particular, from Eqs. (156) and (158) it follows that

\[
\gamma > 0, \quad \gamma^2 = 4\beta_1\beta_2, \quad \sqrt{\beta_1}A^2 = \sqrt{\beta_2}B^2. \quad (159)
\]

From Eq. (152) it is clear that no solution exists in case \( F = 0 \). In fact, one can also show that no solution exists even if \( A = -F \) or \( A = -mF \) unless \( m = 1 \).

**Solution at \( m = 1 \):** In the special case of \( m = 1 \), the solution (151) goes over to the hyperbolic nontopological soliton solution

\[
\phi = F + A \tanh^2[D(x + x_0)], \quad \psi = B \tanh[D(x + x_0)] \text{sech}[D(x + x_0)], \quad (160)
\]

provided the field Eqs. (152) to (158) with \( m = 1 \) are satisfied.

It is not difficult to show that unlike the solution (151), the solution

\[
\phi = B \text{sn}[D(x + x_0), m] \text{cn}[D(x + x_0), m], \quad \psi = F + A \text{sn}^2[D(x + x_0), m], \quad (161)
\]

does not exist as long as \( \delta_1 \) and \( \eta \) are nonzero.

### 5.2 Solution III

Another allowed solution is

\[
\phi = F + A \text{sn}^2[D(x + x_0), m], \quad \psi = B \text{sn}[D(x + x_0), m] \text{dn}[D(x + x_0), m], \quad (162)
\]

provided seven field equations similar to Eqs. (152) to (158) are satisfied. In particular, one can show that solution (162) exists provided

\[
\gamma > 0, \quad \gamma^2 = 4\beta_1\beta_2, \quad \sqrt{\beta_1}A^2 = m\sqrt{\beta_2}B^2. \quad (163)
\]

**Solution at \( m = 1 \):** In the special case of \( m = 1 \), the solution (162) goes over to the hyperbolic nontopological soliton solution (160).
5.3 Solution IV

Finally, another allowed solution is

\[ \phi = F + A \text{sn}^2[D(x + x_0), m], \quad \psi = B \text{cn}[D(x + x_0), m] \text{dn}[D(x + x_0), m], \]  

(164)

provided coupled equations similar to (152) to (158) are satisfied.

In particular, one can show that such a solution exists only if

\[ \gamma < 0, \quad |\gamma|^2 = 4m\beta_1\beta_2, \quad \sqrt{\beta_1}A^2 = \sqrt{m\beta_2}B^2. \]  

(165)

Solution at \( m = 1 \): In the special case of \( m = 1 \), the solution (165) goes over to the hyperbolic nontopological soliton solution

\[ \phi = F + A \tanh^2[D(x + x_0)], \quad \psi = B \text{sech}^2[D(x + x_0)], \]  

(166)

which is essentially the solution (14).

6 Conclusion

In this paper we have shown that the Lamé polynomials of order two are the periodic solutions of the coupled \( \phi^4 \) problems when either the potentials in both the fields are symmetric, or when both are asymmetric or when the potential is symmetric in one and asymmetric in the other field. The latter model is also relevant for reconstructive phase transitions in many materials [8, 9]. These are novel solutions in the sense that while they are the solutions of the coupled problems, they are not the solutions of the corresponding uncoupled problems. In particular, in all the three cases we have shown that while the Lamé polynomials of order one are the solutions of both the coupled and the uncoupled problems, the Lamé polynomials of order two are the solutions of the coupled problems, but not of the uncoupled ones.

It is worth emphasizing here that there are three Lamé polynomials of order one and as a result one has nine different solutions for the coupled \( \phi^4 \) problems which we have presented in [4, 5] and in Sec. IV of this paper. Since there are five Lamé polynomials of order two one would have naively expected sixteen solutions of order two for the coupled \( \phi^4 \) problems [note that two of the Lamé polynomials are of the form]
\[ F + \text{Asn}^2(x, m) \]. However, it turned out that while there are six allowed solutions in the symmetric \( \phi^4 \) case in an external field, only one solution is allowed in the asymmetric case and four solutions are allowed in the asymmetric-symmetric case.

It may be noted here that previously, Hioe and Salter [10] had shown similar features for coupled nonlinear Schrödinger (NLS) equations. They also pointed out that precisely when such solutions exist, the coupled NLS equations are integrable and they pass the Painlevé test [11]. Thus one might get the impression that the existence of higher order Lamé polynomials as solutions of the coupled problem (but not that of the uncoupled problem) could be related to the integrability of the coupled as well as the uncoupled systems. However, our work has clearly shown that this is not so. In particular, it is well known that the \( \phi^4 \) field theory (both symmetric or asymmetric or mixed) is a nonintegrable field theory.

As a further support to our argument, we consider elsewhere [12] a coupled \( \phi^6 \) model studied by us recently [13], and show that provided we add extra interaction terms which are quadratic-quartic in the two fields, then Lamé polynomials of order two are also the solutions of the coupled problem (though they are not the solutions of the uncoupled problem).

Based on these examples, we conjecture that for most of the coupled models, novel solutions (i.e. those which are solutions of the coupled but not the uncoupled problem) will exist as long as there is a coupling term between the fields which is of the same order as the highest power of the uncoupled fields. Further, in those cases where Lamé polynomials, of say order one, are solutions of the uncoupled problem, we conjecture that if there are \( n \)-coupled fields with coupling terms being of the same order as the highest power of the uncoupled fields, then Lamé polynomials of order \( n \) will also be the solutions of the coupled problem. Note that four coupled \( \phi^4 \) fields are required to model different magnetic phases of the hexagonal multiferroic materials [14]. It will be interesting to examine our conjecture in a few coupled field theory models. We hope to address these issues in future.

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