Two results on ill-posed problems *†

A.G. Ramm
Mathematics Department, Kansas State University,
Manhattan, KS 66506-2602, USA
ramm@math.ksu.edu

Abstract
Let $A = A^*$ be a linear operator in a Hilbert space $H$. Assume that equation $Au = f$ (1) is solvable, not necessarily uniquely, and $y$ is its minimal-norm solution. Assume that problem (1) is ill-posed. Let $f_\delta$, $||f - f_\delta|| \leq \delta$, be noisy data, which are given, while $f$ is not known. Variational regularization of problem (1) leads to an equation $A^*Au + \alpha u = A^*f_\delta$. Operation count for solving this equation is much higher, than for solving the equation $(A + ia)u = f_\delta$ (2). The first result is the theorem which says that if $a = a(\delta), \lim_{\delta \to 0} a(\delta) = 0$ and $\lim_{\delta \to 0} \frac{\delta}{a(\delta)} = 0, then the unique solution $u_\delta$ to equation (2), with $a = a(\delta)$, has the property $\lim_{\delta \to 0} ||u_\delta - y|| = 0$. The second result is an iterative method for stable calculation of the values of unbounded operator on elements given with an error.

1. Introduction
The results of this note are formulated as Theorems 1 and 2 and proved in Sections 1 and 2 respectively. For the notions, related to ill-posed problems, one may consult [1] and [2] and the literature cited there.

Let $A = A^*$ be a linear operator in a Hilbert space $H$. Assume that equation

$$Au = f,$$ (1)

is solvable, not necessarily uniquely, and $y$ is its minimal-norm solution, $y \perp N := \{u : Au = 0\}$. Assume that problem (1) is ill-posed. In this case small perturbations of $f$ may cause large perturbations of the solution to (1) or may throw $f$ out of the range of $A$. Let $f_\delta$, $||f - f_\delta|| \leq \delta$, be noisy data, which are given, while $f$ is not known. Variational regularization of problem (1) leads to an equation $A^*Au + \alpha u = A^*f_\delta$, where $A^* = A$ since we assume $A$ to be selfadjoint. Operation count for solving this equation is much higher, than for solving the equation

$$(A + ia)u = f_\delta,$$ (2)

*Math subject classification: 47A05, 45A50, 35R30
†key words: linear operators, ill-posed problems, regularization, discrepancy principle
The result of this paper is the following theorem.

**Theorem 1.** Let \( A = A^* \) be a linear bounded, or densely defined, unbounded, self-adjoint operator in a Hilbert space. Assume that \( a = a(\delta) > 0, \lim_{\delta \to 0} a(\delta) = 0 \) and \( \lim_{\delta \to 0} \frac{\delta}{a(\delta)} = 0 \), then the unique solution \( u_\delta \) to equation (2) with \( a = a(\delta) \) has the property

\[
\lim_{\delta \to 0} ||u_\delta - y|| = 0. \tag{3}
\]

Why should one be interested in the above theorem? The answer is: because the solution to equation (2) requires less operations than the solution of the equation \((A^*A + \alpha I)u = A^*f_\delta\) basic for the variational regularization method for stable solution of equation (1). Here \( I \) is the identity operator. Also, a discretized version of (2) leads to matrices whose condition number is of the order of square root of the condition number of the matrix corresponding to the operator \( A^*A + \alpha(\delta)I \), where \( I \) is the identity operator.

**Proof of Theorem 1.** One has

\[
||u_{a,\delta} - y|| \leq ||(A + ia)^{-1}(f_\delta - f)|| + ||(A + ia)^{-1}Ay - y|| \leq \frac{\delta}{a} + a||(A + ia)^{-1}y||. \tag{4}
\]

Moreover,

\[
\lim_{a \to 0} a^2||A + ia\rangle^{-1}Ay - y||^2 = \lim_{a \to 0} a^2 \int_{-\infty}^{\infty} \frac{d(E_s y, y)}{s^2 + a^2} = 0, \tag{5}
\]

where we have used the spectral theorem, \( E_s \) is the resolution of the identity corresponding to the selfadjoint operator \( A \), and we have taken into account that

\[
\lim_{a \to 0} a^2 \int_{-\infty}^{\infty} \frac{d(E_s y, y)}{s^2 + a^2} = 0
\]

because \( y \perp N \). From formulas (4) and (5) one concludes that if \( a = a(\delta) > 0, \lim_{\delta \to 0} a(\delta) = 0 \) and \( \lim_{\delta \to 0} \frac{\delta}{a(\delta)} = 0 \), then the unique solution \( u_\delta \) to equation (2) with \( a = a(\delta) \) satisfies equation (3).

Theorem 1 is proved.

2. Calculation of values of unbounded operators

Assume that \( A \) is a densely defined closed linear operator in \( H \). We do not assume in this Section that \( A \) is selfadjoint. If \( f \in D(A) \), then we want to compute \( Af \) given noisy data \( f_\delta, ||f_\delta - f|| \leq \delta \). Note that \( f_\delta \) may not belong to \( D(A) \). The problem of stable calculation of \( Af \) given the data \( \{f_\delta, \delta, A\} \) is ill-posed. It was studied in the literature (see, e.g., [1]) by a variational regularization method. Our aim is to reduce this problem to a standard equation with a selfadjoint bounded operator \( 0 \leq B \leq I \), and solve this equation stably by an iterative method.

Let \( v = Af \). This relation is equivalent to

\[
Bv = Ff, \tag{6}
\]
where \( B := (I + Q)^{-1}, F := BA, Q = AA^* \) is a densely defined, non-negative, selfadjoint operator, the range of \( I + Q \) is the whole space \( H \), and \( B \) is a selfadjoint operator, \( 0 \leq B \leq I \), where the inequalities are understood in the sense of quadratic forms, e.g., \( B \geq 0 \) means \( \langle Bg,g \rangle \geq 0 \) for all \( g \in H \), and \( F := (I + Q)^{-1}A \).

**Lemma 1.** (see [3]) The operator \((I + Q)^{-1}A\), originally defined on \( D(A) \), is closable. Its closure is a bounded, defined on all of \( H \) linear operator with the norm \( \leq \frac{1}{2} \). One has \((I + Q)^{-1}A = A(I + T)^{-1}, \) where \( T = A^*A \) is a non-negative, densely defined selfadjoint operator, and \( \|A(I + T)^{-1}\| \leq \frac{1}{2} \).

If \( f_\delta \) is given in place of \( f \), then we stably solve equation \((6)\) for \( v \) using the following iterative process:

\[
v_{n+1} = (I - B)v_n + Ff_\delta, \quad v_0 \perp N^*,
\]

where \( N^* := \{ u : A^*u = 0 \} \). Let \( y \) be the unique minimal-norm solution to equation \((6)\), \( By = Ff \). Note that \( y = Hy + Ff \), where \( H := I - B \).

**Theorem 2.** If \( n = n(\delta) \) is an integer, \( \lim_{\delta \to 0} n(\delta) = \infty \) and \( \lim_{\delta \to 0} [\delta n(\delta)] = 0 \), then

\[
\lim_{\delta \to 0} \|v_\delta - y\| = 0,
\]

where \( v_\delta := v_{n(\delta)} \), and \( v_n \) is defined in \((7)\).

**Proof of Theorem 2.** From \((7)\) one gets \( v_{n+1} = \sum_{j=0}^{n} H^j F f_\delta + H^{n+1} u_0 \), where \( H := I - B \). One has \( y = Hy + Ff \). Let \( w_n := v_n - y \). Then \( w_n = \sum_{j=0}^{n-1} H^j f_\delta + H^n w_0 \), where \( g_\delta := f_\delta - f \), and \( w_0 \) is an arbitrary element such that \( w_0 \perp N^* \). Since \( 0 \leq H \leq I, ||F|| \leq \frac{1}{2} \), and \( ||g_\delta|| \leq \delta \), one gets

\[
||w_n|| \leq \frac{n\delta}{2} + \left[ \int_{0}^{1} (1 - s)^{2n} d(E_s, w_0, w_0) \right]^{1/2},
\]

where \( E_s \) is the resolution of the identity corresponding to the selfadjoint operator \( B \).

If \( w_0 \perp N^* \), then

\[
\lim_{h \to 1} \int_{h}^{1} (1 - s)^{2n} d(E_s, w_0, w_0) = ||Pw_0||^2,
\]

where \( P \) is the orthoprojector onto the subspace \( \{ u : Bu = u \} = \{ u : Qu = 0 \} = N^* \), and \( Pw_0 = 0 \) because \( w_0 \perp N^* \) by the assumption. The conclusion of Theorem 2 can now be derived. Given an arbitrary small \( \epsilon > 0 \), find \( h \) sufficiently close to \( 1 \) such that \( \int_{h}^{1} (1 - s)^{2n} d(E_s, w_0, w_0) < \epsilon \). Fix this \( h \) and find \( n = n(\delta) \), sufficiently large, so that \( \delta n(\delta) < \epsilon \) and, at the same time, \( (1 - h)^{2n}(< \epsilon \). This is possible if \( \delta \) is sufficiently small, because \( \lim_{\delta \to 0} n(\delta) = \infty \) and \( \lim_{\delta \to 0} [\delta n(\delta)] = 0 \). Then \( \int_{0}^{1} (1 - s)^{2n} d(E_s, w_0, w_0) \leq \int_{0}^{h} (1 - s)^{2n} d(E_s, w_0, w_0) + \int_{h}^{1} (1 - s)^{2n} d(E_s, w_0, w_0) < \epsilon \), and inequality \((9)\) shows that \((8)\) holds. Theorem 2 is proved. \( \square \)

**Remark 1.** It is not possible to estimate the rate of convergence in \((8)\) without making additional assumptions on \( y \) or on \( f \). In [2] one can find examples illustrating similar statements concerning various methods for solving ill-posed problems.
References

[1] V. Morozov, Methods of solving incorrectly posed problems, Springer Verlag, New York, 1984.

[2] A. G. Ramm, Inverse Problems, Springer, New York, 2005.

[3] A. G. Ramm, On unbounded operators and applications, (submitted)