Fusion Algebras of Fermionic Rational Conformal Field Theories via a Generalized Verlinde Formula

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Abstract

We prove a generalization of the Verlinde formula to fermionic rational conformal field theories. The fusion coefficients of the fermionic theory are equal to sums of fusion coefficients of its bosonic projection. In particular, fusion coefficients of the fermionic theory connecting two conjugate Ramond fields with the identity are either one or two. Therefore, one is forced to weaken the axioms of fusion algebras for fermionic theories. We show that in the special case of fermionic $\mathcal{W}(2,\delta)$-algebras these coefficients are given by the dimensions of the irreducible representations of the horizontal subalgebra on the highest weight. As concrete examples we discuss fusion algebras of rational models of fermionic $\mathcal{W}(2,\delta)$-algebras including minimal models of the $\mathcal{N}=1$ super Virasoro algebra as well as $\mathcal{N}=1$ super $\mathcal{W}$-algebras $\mathcal{SW}(\frac{3}{2},\delta)$. 

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1. Introduction

In the last years two-dimensional conformally invariant quantum field theories have found wide applications in various fields of physics and mathematics such as statistical mechanics, string theory, knot theory, number theory and the classification of 3-manifolds [1 – 5]. As was shown by A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov in 1984 [1], rational conformal field theories (RCFTs) are of particular interest because all \( n \)-point functions can be calculated explicitly (at least in principle). Therefore, the classification of all RCFTs is one of the outstanding problems in mathematical physics. To this end many different approaches have been developed [6 – 26]. One direction is the purely algebraic approach where one considers abstract observable algebras (\( C^* \)-algebras) and endomorphisms thereof [27]. Here, the fusion rules appear naturally if one decomposes the product of two endomorphisms into the irreducible ones [27] which can e.g. be calculated under some additional assumptions using algebraic \( K \)-theory [28]. A more concrete ansatz follows from the assumption that all rational models can be described as minimal models of an extension of the conformal algebra. The investigation of these \( \mathcal{W} \)-symmetries in conformal field theory is still one of the main streams of research in this field of mathematical physics [6 – 25]. In this approach one tries to construct an algebra of local fields and searches for rational models by investigating representation theory [11 – 22]. The fusion rules describing the interactions of a RCFT are directly linked to the modular properties of the characters of the chiral algebra via the Verlinde formula [4, 5].

Recently, new methods tried to deal directly with the fusion algebras [23] (cf. references therein) or the fusion algebras induced by representations of the modular group [24] via the Verlinde formula. In a RCFT a representation of the modular group is given by the natural action of \( SL_2(\mathbb{Z}) \) on the characters of the highest weight representations (HWRs) of the (maximally extended) chiral symmetry algebra \( \mathcal{W} \) underlying the RCFT [3, 29]. An important tool in this approach is the famous Verlinde formula [4, 5] which establishes the connection between the representation matrix \( S \) of the modular transformation \( \tau \to -\frac{1}{\tau} \) and the fusion coefficients themselves. In the case of bosonic extended symmetry algebras, several ansätze using the Verlinde formula led to interesting results [23, 16, 24, 30]. The main goal of this article is to establish a generalized Verlinde formula which describes the fusion in all sectors of fermionic theories. This generalization of the Verlinde formula reproduces the correct sector structure of the fusion algebra [31]. We show that the fusion algebras given by the generalized Verlinde formula can be obtained from the fusion algebras of the corresponding bosonic projections applying ‘simple current’ arguments. Such ‘simple current’ arguments have been first proposed by A.N. Schellekens and S. Yankielowicz [32]. Furthermore, we investigate the representations of the horizontal subalgebras on the highest weights in the Ramond sector. For fermionic \( \mathcal{W}(2, \delta) \)-algebras their dimensions are encoded in the corresponding fusion algebra.

This paper is organized as follows. In section 2 we discuss some fundamental properties of fermionic RCFTs and present the explicit form of the generalized Verlinde formula. Here, the main statements of our analysis are formulated. In the next section we prove the generalized Verlinde formula under certain assertions. In section 4 we apply the formula
to the case of $\mathcal{W}$-algebras with one additional fermionic generator of conformal dimension $\delta \geq \frac{5}{2} \ (\mathcal{W}(2, \delta))$. We proceed with a discussion of the fusion algebras of the (unitary as well as non-unitary) minimal models of the $N = 1$ super Virasoro algebra in section 5. Section 6 contains the most complicated examples, namely $N = 1$ super $\mathcal{W}$-algebras with two generators ($\mathcal{SW}(\frac{5}{2}, \delta)$). Finally, we draw conclusions from our results and point out some open questions. Two concrete examples where the fusion algebras of the bosonic projection of the $N = 1$ super Virasoro algebra using the ordinary Verlinde formula are presented in the appendix.

2. Generalized Verlinde formula for fermionic RCFTs

Let $\mathcal{R}$ be a fermionic rational conformally invariant quantum field theory with (maximally extended) chiral symmetry algebra $\mathcal{W}$ containing the Virasoro algebra. One has to distinguish between the Neveu-Schwarz sector ($NS$) and the Ramond sector ($R$) because the symmetry algebra contains bosonic and fermionic fields. We denote the $\mathcal{W}$-primary fields and the corresponding highest weight representations of the symmetry algebra enlarged by the (involutive) ‘chirality’-operator $\Gamma = (-1)^F$ ($F$ is the fermion number operator) in the two sectors by

$$\phi^i_{NS} \leftrightarrow \mathcal{H}^i_{NS} \quad \text{for} \quad i \in \mathcal{I}_{NS}$$

$$\phi^j_R \leftrightarrow \mathcal{H}^j_R \quad \text{for} \quad j \in \mathcal{I}_R$$

where $\mathcal{H}^1_{NS}$ is the vacuum representation. The corresponding characters are defined by:

$$\chi^k_{NS} := \text{tr}_{\mathcal{H}^k_{NS}}(q^{L_0-c/24})$$

$$\chi^k_R := \text{tr}_{\mathcal{H}^k_R}(q^{L_0-c/24}).$$

The sector structure of such a fermionic theory is reflected by the modular properties of the characters. Let $T$ ($S$) be the representation matrix of the modular transformation $\tau \rightarrow \tau + 1$ ($\tau \rightarrow -\frac{1}{\tau}$) for the representation of the modular group given by its natural action on the characters. Because the span of the characters in the $NS$ sector is invariant only under the subgroup of the modular group generated by $T^2$ and $S$, it is useful to define a third (physically irrelevant) sector $\bar{NS}$ (see e.g. [33, 34]) in order to obtain a representation space of the full modular group:

$$\chi^k_{\bar{NS}} := e^{-2\pi i(h(\phi^k_{NS})-c/24)} T \chi^k_{NS}$$

$$= \text{tr}_{\mathcal{H}^k_{NS}}((-1)^F q^{L_0-c/24}).$$

The modular transformation $TST$ intertwines between the $NS$ and the $R$ sectors. Together with the $R$ sector, which is invariant under $T$ and $ST^2S$, these three sectors have the structure of a $SL_2(\mathbb{Z})$ module. The ‘horizontal’ subalgebra is defined as the subalgebra consisting of the zero modes of all fields in $\mathcal{W}$ and the ‘chirality’-operator $\Gamma$. We stress that in our convention the characters $\chi^R$ begin with $q^{h - \frac{c}{2} \frac{d}{d}}$, where $d$ is the dimension of the $L_0$ eigenspace $\mathcal{V}_0$ to the lowest eigenvalue $h$ in the representation module. A highest
weight vector is an eigenvector of a maximal number of zero-modes of fields in the bosonic part of the horizontal subalgebra which commute in $\mathcal{V}_0$. Therefore, $d$ is in general greater than one because zero-modes of fermionic fields act nontrivially on the highest weight vector generating $\mathcal{V}_0$. In the following we call the characters (2.2) ‘energy’ characters. In particular, we do not include a factor $\sqrt{2}$ in the $R$ characters like in ref. [34]. It may happen that some of the irreducible representation modules of the algebra $\mathcal{W}$ are degenerate and have equal energy characters. In this case the sector structure of the representation of the modular group is respected only if one considers the energy characters and identifies the degenerate ones.

In analogy to the $\tilde{N}S$ sector one can define characters in the $\tilde{R}$ sector by

$$\chi_{\tilde{R}}^k := \text{tr} \mathcal{H}_k^{\tilde{R}} \left( (-1)^F q^{L_0 - c/24} \right). \quad (2.2c)$$

This sector is invariant under the action of the modular group, and the $\tilde{R}$ characters (2.2c) are constant. These constants can be identified with Witten indices of the corresponding highest weight representations because they indicate whether the boson-fermion symmetry of the ground state is broken or not [35, 36].

The transformation properties of the characters lead to the following form for the representation matrices of $S$ and $T$ (we omit the $\tilde{R}$ sector because of its modular invariance) [33, 34]:

$$S = \begin{pmatrix}
S^{NS \to NS} & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & S^{R \to \tilde{N}S}
\end{pmatrix}
\begin{pmatrix}
S^{\tilde{N}S \to R}
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
T^{\tilde{N}S \to NS}
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
T^{R \to R}
\end{pmatrix}
$$

$$T = \begin{pmatrix}
T^{NS \to \tilde{N}S}
\end{pmatrix}
\begin{pmatrix}
\delta_{k,l} e^{2\pi i (h(\phi^{NS}_k) - c/24)}
\end{pmatrix}
\begin{pmatrix}
T^{R \to R}
\end{pmatrix}
$$

with $T^{\tilde{N}S \to NS}_{k,l} = T^{NS \to \tilde{N}S}_{k,l} = \delta_{k,l} e^{2\pi i (h(\phi^{NS}_k) - c/24)}$.

It is well-known that the fusion rules of fermionic fields respect the sector structure of the theory in the following way [31]:

$$[\Phi_i^{NS}] [\Phi_j^{NS}] = \sum_{k \in I_{NS}} (N^{NS,NS}_{NS})_{i,j}^k [\Phi_k^{NS}]$$

$$[\Phi_i^{R}] [\Phi_m^{R}] = \sum_{i \in I_{NS}} (N^{NS,R}_{R,R})_{i,m}^i [\Phi_i^{NS}]$$

$$[\Phi_i^{R}] [\Phi_i^{NS}] = \sum_{m \in I_{NS}} (N^{R,R}_{R,NS})_{i,i}^m [\Phi_m^{R}]$$

This can be interpreted as the conservation of an additional additive $\mathbb{Z}_2$ charge (the $NS$ sector is neutral and the $R$ sector carries charge 1). Taking into account the sector structure
of the fusion rules for fermionic theories the generalized Verlinde formula can be written as:

$$
(N_{NS,NS})^k_{i,j} = \sum_{n \in \mathcal{I}_{NS}} S_{n,i}^{NS\rightarrow NS} S_{n,j}^{NS\rightarrow NS} (S_{n,1}^{NS\rightarrow NS})^{-1} k,n
$$

$$
(N_{NS})^i_{l,m} = d_l d_m \sum_{n \in \mathcal{I}_{NS}} S_{n,l}^{R\rightarrow \tilde{NS}} S_{n,m}^{R\rightarrow \tilde{NS}} (S_{n,1}^{NS\rightarrow NS})^{-1} i,n
$$

$$
(N_{R,NS})^m_{i,l} = (N_{NS,R})^m_{i,l} = \frac{d_l}{d_m} \sum_{n \in \mathcal{I}_{NS}} S_{n,l}^{R\rightarrow \tilde{NS}} S_{n,i}^{NS\rightarrow NS} (S_{n,1}^{R\rightarrow \tilde{NS}})^{-1} m,n
$$

with $i, j, k \in \mathcal{I}_{NS}$ and $l, m \in \mathcal{I}_{R}$. Note that the formula for $N_{NS,NS}$ is the usual Verlinde formula [4, 5]. The $\{d_l \mid l \in \mathcal{I}_{R}\}$ are defined in the following manner. Consider the bosonic projection $\mathcal{P}W$ of the symmetry algebra $W$ and the corresponding rational model of $\mathcal{P}W$. The fusion algebra of this rational model contains a ‘simple current’ of order two describing its extended symmetry. The orbits under this ‘simple current’ correspond to the fields in the fusion algebra of $W$. Now $d_l$ is defined as the order of the ‘simple current’ divided by the length of the orbit corresponding to the $l$th field in the fusion algebra of $W$. For the fields in the $NS$ sector $d$ is equal to one (cf. section 3). We combine these integers to a diagonal matrix $D^R = \text{diag}(\{d_l \mid l \in \mathcal{I}_{R}\})$ and define the $D$-matrix as

$$
D = \begin{pmatrix}
\mathds{1} & 0 & 0 \\
0 & \mathds{1} & 0 \\
0 & 0 & D^R
\end{pmatrix}
$$

The definition (2.5) directly implies that the fusion constants lead to a well-defined, commutative and associative algebra. Furthermore, fusion with the identity field acts trivially in the fusion algebra. However, it is not apparent that the fusion coefficients defined in (2.5) are positive integers. In section 3 we will show that this is an immediate consequence of the fusion coefficients of the bosonic fusion algebra being positive integers.

In general, the representation modules are degenerate and give rise to a diagonal ‘multiplicity’ matrix $M$ defined as

$$
M = \begin{pmatrix}
M^{NS} & 0 & 0 \\
0 & M^{\tilde{NS}} & 0 \\
0 & 0 & M^R
\end{pmatrix}
$$

with $M^{NS} = M^{\tilde{NS}}$. Here the entries of the three diagonal submatrices are the multiplicities of the respective representations in the theory. In all known cases degeneracies can be removed by considering the eigenvalues of some additional zero modes of bosonic fields $B_0$ which commute with $L_0$. In general, there are several HWRs with identical energy characters but with different eigenvalues of the operators in $B_0$. As described above we
must use the energy characters in order to preserve the sector structure so that we have to deal with the multiplicity matrix $M$.

Instead of being unitary, $S$ obeys the equation

$$S^\dagger HS = H \quad H = MD^{-1},$$  \hspace{1cm} (2.8)

as will be explained in the next section. One should note that if $S$ and $M$ are known $D$ is fixed by this equation. Due to (2.8) the ‘fusion charge conjugation matrix’ $N_{ij}^1$ is in general not equal to the usual charge conjugation matrix $C = S^2$ but satisfies $N_{ij}^1 = (MD)_{i,j}$. As we will discuss below, it is not possible to avoid the $D$-matrix by means of an extension of the fusion algebra in contrast to the degeneracies encoded in $M$ which can be resolved (see e.g. ref. [32]). Thus, one is forced to weaken the axioms of fusion algebras of fermionic RCFTs and has to allow more general fusion charge conjugation matrices.

Finally, we add some remarks concerning the chirality operator $\Gamma$ which we always include into the chiral algebra. In general it is very unphysical to consider representations with diagonal fermionic operators. However, if the chirality operator (anticommuting with all fermionic operators) is included into the symmetry algebra the fermionic operators act non-diagonal and do not preserve the $\Gamma$-eigenspaces. Irreducible representations of the fermionic algebra (including $\Gamma$) which correspond to orbits of length two in the fusion algebra of the bosonic projection are also irreducible with respect to the chiral algebra without $\Gamma$. In contrast, irreducible representations corresponding to fixed points in the bosonic fusion algebra are not irreducible with respect to the chiral algebra without $\Gamma$ but decompose into a direct sum of two irreducible representations. This may explain the fact that the fusion coefficients connecting two such conjugate fields with the identity are equal to two. As we will see in section 4 it is not possible to resolve these nontrivial fusion coefficients by extending the fusion algebra.

In section 4 we verify for fermionic $W(2,\delta)$-algebras that the diagonal entries of $D^R$ give exactly the dimensions of the $L_0$-eigenspaces $V_0$ in the respective representation modules. However, this observation is in general not valid for symmetry algebras with more than one fermionic generator like $SW(\frac{3}{2},\delta)$-algebras (cf. section 6).

### 3. Proof of the Generalized Verlinde Formula

Assume that we have a bosonic RCFT with characters $\chi_i^{bos}$ ($i \in I_{bos}$) and unitary $S$-matrix $S^{bos}$. Furthermore, we suppose for simplicity that $(S^{bos})^2 = \mathbb{I}$, i.e. trivial charge conjugation and that there are no degenerate HWRs so that $N_{i,i}^1 = 1$ ($\forall i \in I_{bos}$) is valid. The bosonic RCFT shall also admit a fermionic extension of its underlying chiral symmetry algebra. We require therefore that the fusion algebra $\mathcal{A}$ obtained from $S^{bos}$ via the (ordinary) Verlinde formula possesses a ‘simple current’ $[J]$ of order two ($[J][J] = \mathbb{I}$) with conformal dimension $h([J]) = \delta \in \mathbb{N} + \frac{1}{2}$ [32]. Assume furthermore that the fermionic symmetry algebra is obtained from its bosonic projection by extension with this ‘simple
The basis elements \([i] \ (i \in \mathcal{I}_{bos})\) of \(\mathcal{A}\) are organized into orbits of the 'simple current' \([J]\) of length one (fixed points) or two. These orbits thus define the multiplets of the extended fermionic RCFT. Accordingly, the characters of the fermionic theory are defined as
\[
\chi_{fer}^i := \chi_{bos}^i + \chi_{bos}^J i.
\]
It is possible to define a conserved charge \(Q\) on the fusion algebra \(\mathcal{A}\) in the following way:
\[
Q([i]) := (h([i]) + h([J]) - h([J][i])) \mod 1.
\]
It obeys the addition rule
\[
Q([i][j]) = Q([i]) + Q([j]) \mod 1.
\]
The fields have either charge 0 or charge \(\frac{1}{2}\) with respect to the 'simple current' \([J]\). Due to charge conservation the fusion algebra \(\mathcal{A}\) is \(\mathbb{Z}_2\) graded, i.e. we have \(\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_{\frac{1}{2}}\). Here \(\mathcal{A}_0\) (\(\mathcal{A}_{\frac{1}{2}}\)) denotes the subspace spanned by fields of charge 0 (\(\frac{1}{2}\)). The fusion rules respect this structure in the following way:
\[
\mathcal{A}_0 \cdot \mathcal{A}_0 \subset \mathcal{A}_0 \quad \quad \quad \mathcal{A}_0 \cdot \mathcal{A}_{\frac{1}{2}} \subset \mathcal{A}_{\frac{1}{2}} \quad \quad \quad \mathcal{A}_{\frac{1}{2}} \cdot \mathcal{A}_{\frac{1}{2}} \subset \mathcal{A}_0.
\]
on the characters \( \chi^{fer}_{\alpha(i)} := \chi^{bos}_{i} + \chi^{bos}_{j} \). This implies that \( S^{fer} \) in not unitary but obeys eq. (2.8). To formulate it differently we have to show that the following diagram commutes:

\[
\begin{array}{ccc}
S^{bos} & \xrightarrow{\text{Verlinde}} & S^{fer} \\
\downarrow & & \downarrow \\
\text{bosonic fusion algebra} & \rightarrow & \text{fermionic fusion algebra}
\end{array}
\]

Equation (3.2) corresponds to the bottom line of the diagram (3.3).

Let us first check the result for \( \mathcal{N}^{k}_{i,j} \) with \( i, j, k \in \mathcal{I}_{NS} \). The \( S \)-matrix \( S^{fer}_{NS \rightarrow NS} \) is given by \((S^{fer}_{NS \rightarrow NS})_{\alpha(i), \alpha(j)} = S^{bos}_{i,j} + S^{bos}_{j,i} \). Inserting this in the first relation of (2.5) we obtain (using the fact that \( S^{bos} \) and \( S^{fer}_{NS \rightarrow NS} \) are symmetric and orthogonal):

\[
\mathcal{N}^{\alpha(k)}_{\alpha(i), \alpha(j)} = \frac{1}{2} \sum_{n \in \mathcal{I}_{NS}'} \frac{(S^{bos}_{n,i} + S^{bos}_{n,j})(S^{bos}_{n,j} + S^{bos}_{n,j})(S^{bos}_{k,n} + S^{bos}_{k,n})}{S^{bos}_{n,0} + S^{bos}_{n,0}}
\]

\[
= \frac{1}{4} \sum_{n \in \mathcal{I}_{NS}'} \frac{(S^{bos}_{n,i} + S^{bos}_{n,j})(S^{bos}_{n,j} + S^{bos}_{n,j})(S^{bos}_{k,n} + S^{bos}_{k,n})}{S^{bos}_{n,0}}
\]

\[
= \frac{1}{4} \left( \mathcal{N}_{i,j}^k + \mathcal{N}_{i,j}^{Jk} + \mathcal{N}_{i,j}^{k} + \mathcal{N}_{i,j}^{Jk} + \mathcal{N}_{i,j}^{k} + \mathcal{N}_{i,j}^{Jk} + \mathcal{N}_{i,j}^{k} + \mathcal{N}_{i,j}^{Jk} \right)
\]

\[
= \mathcal{N}_{i,j}^k + \mathcal{N}_{i,j}^{Jk}.
\]

Note that we used the identities \( S^{bos}_{i,j} = S^{bos}_{i,j} \) and \( S^{bos}_{n,0} = S^{bos}_{n,0} \) (formula (4.2) in ref. [32]). This follows directly from the Verlinde formula and the defining property of the ‘simple current’ \([J]\) (fusion of \([J]\) with a basis element of \( \mathcal{A} \) yields only one (different) basis element).

Consider now the by far more interesting case of \( \mathcal{N}^{\alpha(k)}_{\alpha(i), \alpha(j)} \) with \( i, j \in \mathcal{I}_{R} \) (in the case \( i \in \mathcal{I}_{R}, j \in \mathcal{I}_{NS} \) one proceeds in the same way). For simplicity we treat only the case where \( i, j \) correspond to fixed points in \( \mathcal{A}_4 \). For fixed points the characters of the fermionic theory are twice the corresponding bosonic characters. Due to the chirality operator \( \Gamma \) in the underlying chiral symmetry algebra these characters are indeed the characters of irreducible HWRs of the fermionic theory. In particular, for fixed points the dimension \( d \) of the irreducible representation of the horizontal subalgebra of the symmetry algebra in \( \mathcal{V}_0 \) is two. For HWRs corresponding to orbit length two in \( \mathcal{A}_4 \) the dimension \( d \) equals one or two depending whether the conformal dimensions of the fields in the orbit are different or not. With \((S^{fer}_{R \rightarrow NS})_{\alpha(i), \alpha(j)} = S^{bos}_{i,j} \) \((i \in \mathcal{I}_{NS}, j \in \mathcal{I}_{R}, j \) corresponding to a fixed point\) we obtain with the second relation of (2.5) the following result (inserting \( d_i = d_j = 2 \)):

\[
\mathcal{N}^{\alpha(k)}_{\alpha(i), \alpha(j)} = \frac{1}{2} d_i d_j \sum_{n \in \mathcal{I}_{NS}} \frac{S^{bos}_{n,i} S^{bos}_{n,j} (S^{bos}_{k,n} + S^{bos}_{k,n})}{S^{bos}_{n,0} + S^{bos}_{n,0}} = \sum_{n \in \mathcal{I}_{NS}} \frac{S^{bos}_{n,i} S^{bos}_{n,j} (S^{bos}_{k,n} + S^{bos}_{k,n})}{S^{bos}_{n,0}}
\]

\[
= \mathcal{N}_{i,j}^k + \mathcal{N}_{i,j}^{Jk} = 2 \mathcal{N}_{i,j}^k.
\]
The case with orbits of length two in $A_{\frac{1}{2}}$ can be treated similarly. We see that the diagram (3.3) is indeed commutative thus proving our assertion.

In the remaining part of this paper we apply the generalized Verlinde formula to various fermionic RCFTs. We start with rational models of fermionic $\mathcal{W}(2, \delta)$ algebras ($\frac{5}{2} \leq \delta \in \mathbb{N} + \frac{1}{2}$) where it is believed that the classification is complete. Then we proceed with minimal models (unitary as well as non-unitary) of the $N = 1$ super Virasoro algebra ($= \mathcal{W}(2, \frac{5}{2})$). In two cases we study the fusion rules of the bosonic projection of the $N = 1$ super Virasoro algebra, described by the usual Verlinde formula, and demonstrate concretely that the diagram (3.3) commutes (see appendix). We continue with rational models of $N = 1$ super-$\mathcal{W}$-algebras $\mathcal{SW}(\frac{3}{2}, \delta)$ ($\delta \geq 2$) which are the most complicated examples because in general both the multiplicity matrix $M$ and the $D$-matrix are nontrivial. In particular, we discuss $\mathcal{SW}(\frac{3}{2}, \delta)$-algebras with vanishing self-coupling constant, where the classification also seems to be complete. Using the $ADE$-classification and earlier results about the degeneracies [20] of the representation modules, one obtains the $D$-matrix for all $\mathcal{SW}(\frac{3}{2}, \delta)$-algebras related to the $ADE$-classification.

4. Fermionic $\mathcal{W}(2, \delta)$ algebras

In this section we discuss $\mathcal{W}(2, \delta)$-algebras with one additional generator $W$ with half-integer conformal dimension $\delta \geq \frac{5}{2}$ [11, 15, 16]. Besides the parabolic cases [37] these algebras exist for Virasoro minimal values of the central charge which can be understood by the $ADE$-classification of modular invariant partition functions [33]. These values of the central charge can be organized in two series according to the type of the partition function which is diagonalized by the $\mathcal{W}$-characters. From table 1 below one can read off these $\mathcal{W}$-characters in terms of Virasoro characters. Actually, the $\mathcal{W}$-characters are the quantities which appear in $Z = Z_{NS} + Z_{\tilde{NS}} + Z_{R}$ with their absolute value squared. The irreducible representations of the horizontal subalgebra of $\mathcal{W}$ in the $L_0$-eigenspace $\mathcal{V}_0$ are either one or two-dimensional in the $R$ sector. In the one-dimensional (trivial) representation $W_0$ is represented by zero whereas in the two-dimensional representation $W_0$ acts non-trivially. The representation of the horizontal subalgebra in $\mathcal{V}_0$ is equivalent to the irreducible two-dimensional representation of the Clifford algebra $Cl(2, 0)$ of a two-dimensional euclidian vectorspace:

$$\{W_0, W_0\} = 2w^2, \quad \{W_0, \Gamma\} = 0, \quad \{\Gamma, \Gamma\} = 2.$$ 

The trivial representation only occurs in those highest weight modules for which the $\mathcal{W}$-character is a sum of Virasoro characters with an equal number of terms as in the $NS$ sector. From the representation theory of these algebras we see that this is exactly the case for the HWRs with $w = 0$ [16]. In the case of $\mathcal{W}(2, \delta)$ algebras $M$ is equal to the identity matrix $I$ because there are no additional ‘quantum numbers’ present (and necessary). So one can read off the $D$-matrix from table 1 and verify that the diagonal entries coincide with the dimensions of the corresponding vector spaces $\mathcal{V}_0$. 

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Let us also discuss the parabolic fermionic $\mathcal{W}(2, \delta)$-algebras. The series is given by the algebras $\mathcal{W}(2, 3k)$ existing for $c = 1 - 24k$ with $k \in \mathbb{N} + \frac{1}{2}$. The following HWRs are permitted [16] ($h_{r,r} = k(r^2 - 1)$, $h_{r,-r} = h_{r,r} + r^2$)

$$NS : \begin{cases} h_{\frac{m}{2}, \frac{m}{2}} & m = 0, \ldots, [k], 2k \\ h_{\frac{m}{2k+2}, -\frac{m}{2k+2}} & m = 1, \ldots, [k + \frac{1}{2}] 
\end{cases}$$

$$R : \begin{cases} h_{\frac{2m+1}{4k+4}, -\frac{2m+1}{4k+4}} & m = 0, \ldots, [k+\frac{1}{2}] - 1, [k+\frac{1}{2}] 
\end{cases}$$

The modular invariant partition function $Z = Z^{NS} + Z^{\tilde{NS}} + Z^R$ is given by the expressions [37] (the $\theta$ function is defined below (4.2)):

$$Z^{NS} = \frac{1}{\eta^2} \left( \frac{1}{2} | \theta_{0,k} - \theta_{0,k+1} |^2 + \frac{1}{2} | \theta_{0,k} + \theta_{0,k+1} |^2 + \sum_{m=1}^{[k]} | \theta_{m,k} |^2 + \sum_{m=1}^{[k+\frac{1}{2}]} | \theta_{m,k+1} |^2 \right)$$

$$Z^R = \frac{1}{\eta^2} \left( \frac{1}{2} | \theta_{k,k} |^2 + \frac{1}{2} | \theta_{k+1,k+1} |^2 + \sum_{m=0}^{[k]-1} | \theta_{m+\frac{1}{2},k} |^2 + \sum_{m=0}^{[k+\frac{1}{2}]-1} | \theta_{m+\frac{1}{2},k+1} |^2 \right)$$

The part $Z^{\tilde{NS}}$ of the partition function is obtained from $Z^{NS}$ by applying the modular transformation $\tau \to \tau + 1$. The $\mathcal{W}$-characters are the quantities which appear in $Z$ with their absolute value squared divided by the $\eta$-function, e.g. $\chi_\text{vac}^\mathcal{W} = \eta^{-\frac{1}{2}}(\theta_{0,k} - \theta_{0,k+1})$. From the partition function one can immediately read off the matrix $D^R = \text{diag}(\{2, 2, 1, \ldots, 1\})$ taking into account $M = \mathbb{I}$. Also in this case the diagonal elements of $D^R$ are equal to the dimensions of the corresponding spaces $V_0$. The fusion algebra can easily be obtained by the well-known transformation rules of the theta functions [37].

| algebra                     | $c(p, q)$ | series         | $Z^{NS}, Z^R$                                                                 |
|-----------------------------|-----------|----------------|-------------------------------------------------------------------------------|
| $\mathcal{W}(2, \frac{p-1}{2})$ | $(E_6, A_{p-1})$ | $Z^{NS} = \sum_{s=1}^{p-1} |\chi_{1,s}+\chi_{7,s}+\chi_{5,s}+\chi_{11,s}|^2$ |
|                             |           | $Z^R = \sum_{s=1}^{p-1} \frac{1}{4} |2(\chi_{4,s}+\chi_{8,s})|^2$ |
| $\mathcal{W}(2, \frac{p-2}{2}2k-1))$ | $(D_{2k+1}, A_{p-1})$ | $Z^{NS} = \sum_{s=1}^{p-1} 2k-1 |\chi_{r,s}+\chi_{r,p-s}|^2$ |
|                             |           | $Z^R = \sum_{s=1}^{p-1} \frac{1}{4} |2\chi_{2k,s}|^2 + \sum_{s=1}^{p-1} 2k-2 |\chi_{r,s}+\chi_{r,p-s}|^2$ |

Table 1: partition functions and series of fermionic $\mathcal{W}(2, \delta)$-algebras [33, 11, 16]
In the rest of this section we discuss the fusion rules emerging from the generalized Verlinde formula for two series of $\mathcal{W}(2, \delta)$-algebras, namely those corresponding to $(D_3, A_{p-1})$ and $(E_6, A_{p-1})$. As an example for the general $(D, A)$-series we discuss the case $(D_5, A_2)$ and verify that it is not possible to extend the fusion algebra such that the fusion charge conjugation becomes equal to the usual charge conjugation.

Take as the first concrete example $\mathcal{W}(2, \frac{p-2}{2})$ at the Virasoro minimal value of the central charge $c = c(p, 4) = 1 - \frac{3}{2} \frac{(p-4)^2}{p}$ for odd $p \geq 3$. These algebras are related to the partition functions of the type $(D_3, A_{p-1})$ (specializing the second case in the table to $k = 1$). In order to give the characters of these algebras explicitly, recall the general form of the Virasoro minimal characters [38]

$$\chi^{\nu}_{r,s}(\tau) = \eta(\tau)^{-1} (\theta_{pr-qs,pq}(\tau) - \theta_{pr+qs,pq}(\tau)) \quad \text{for} \quad 1 \leq r \leq q - 1, \quad 1 \leq s \leq p - 1,$$

where we have introduced Dedekind’s eta function and the Riemann-Jacobi theta functions

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n \in \mathbb{N}} (1 - q^n), \quad \theta_{\lambda,k}(\tau) = \sum_{n \in \mathbb{Z}} q^{\frac{(2kn + \lambda)^2}{4k}} \quad \text{with} \quad q = e^{2\pi i \tau}.$$

The $\mathcal{W}$-algebra characters can be expressed in terms of the Virasoro minimal characters as [15]

$$\begin{align*}
NS : \quad & \chi^{W,NS}_{i} = \chi_{1,i} + \chi_{1,p-i} \quad \text{for} \quad i \in \mathcal{I}_{NS} = \{1, \ldots, \frac{p-1}{2}\} \\
R : \quad & \chi^{W,R}_{i} = 2\chi_{2,i} \quad \text{for} \quad i \in \mathcal{I}_{R} = \{1, \ldots, \frac{p-1}{2}\}.
\end{align*}$$

(4.3)

Using the arguments given above, we infer from (4.3) that the matrix $D^R$ is equal to $2\mathbb{I}$. Rewriting the modular transformation matrix $S$ of the Virasoro minimal models [38] in the $\mathcal{W}$-algebra character basis, one obtains (for $S^{NS,NS}$ see [15])

$$\begin{align*}
S^{NS,NS}_{i,j} = & \frac{2}{\sqrt{p}} (-1)^{i+j+1+\left\lfloor \frac{i}{2} \right\rfloor + \left\lfloor \frac{j}{2} \right\rfloor} \sin\left(\frac{4\pi ij}{p}\right) \quad \text{for} \quad i,j \in \mathcal{I}_{NS} \\
S^{\tilde{NS},R}_{i,l} = & \frac{1}{2} S^{R,\tilde{NS}}_{l,i} = \frac{2}{\sqrt{2p}} (-1)^{l+1+\left\lfloor \frac{l}{2} \right\rfloor} \sin\left(\frac{4\pi il}{p}\right) \quad \text{for} \quad i \in \mathcal{I}_{NS}, \quad l \in \mathcal{I}_{R}.
\end{align*}$$

(4.4)

Inserting this result into (2.5) gives the fusion algebra. This fusion algebra is isomorphic to the fusion algebra of the Virasoro minimal model with $c = c(2, p)$ tensor an element $\omega$ with $\omega^2 = 2$, i.e. an additional $\mathbb{Z}_2$ grading. In particular, the vacuum occurs with multiplicity 2 in the fusion of a $R$ field with itself indicating that all $R$ fields correspond to fixed points in the fusion algebra of the bosonic projection. The structure of the fusion algebra is evident because the two matrices $S^{NS,NS}$ and $S^{\tilde{NS},R}$ are, up to a constant, equal to the $S$-matrix of the Virasoro minimal $c(2, p)$ model. The element $\omega$ of order 2 leads to the correct sector structure due to $\mathbb{Z}_2$ charge conservation. Note that a rescaling of the fields in the $R$ sector with a factor $\frac{1}{\sqrt{2}}$ leads to a fusion algebra where all coefficients are equal to zero or one and the fusion charge conjugation is equal to $S$. However, this rescaling is unnatural as our considerations in section 3 have shown.
Note that the fusion algebras of these $\mathcal{W}(2, \delta)$-algebras were also determined in ref. [39] using path space realizations of the corresponding characters.

Let us now discuss the algebras $\mathcal{W}(2, \frac{p-1}{2})$ existing for the Virasoro minimal values $c(p, 12)$ with $p \geq 5$ odd and $p, 3$ coprime. The partition function is of the type $(E_6, A_{p-1})$. From table 1 we find that the $\mathcal{W}$ characters are given by [11, 16]:

\[
\begin{align*}
NS &: \quad \chi^{W,NS}_i = \chi_{1,i} + \chi_{7,i} + \chi_{5,i} + \chi_{11,i} \\
\tilde{NS} &: \quad \chi^{W,\tilde{NS}}_i = \text{sgn}(p - 4i)(\chi_{1,i} + \chi_{7,i} - \chi_{5,i} - \chi_{11,i}) \\
R &: \quad \chi^{W,R}_i = 2(\chi_{4,i} + \chi_{8,i})
\end{align*}
\quad \text{for } i \in \mathcal{I}_{NS} = \{1, \ldots, \frac{p-1}{2}\}
\]

Using the $S$-matrix of the minimal models of the Virasoro algebra we obtain for the $S$-matrix in the $\mathcal{W}$-character basis:

\[
\begin{align*}
S^{NS,NS}_{i,j} &= \frac{2}{\sqrt{p}}(-1)^{i+j+1+\lfloor \frac{p}{12} \rfloor}\sin\left(\frac{12\pi ij}{p}\right) \quad \text{for } i, j \in \mathcal{I}_{NS} \\
S^{\tilde{NS},R}_{i,l} &= \frac{1}{2}S^{R,\tilde{NS}}_{l,i} = \frac{2}{\sqrt{2p}}\text{sgn}(p - 4i)(-1)^{l+(p \mod 3)}\sin\left(\frac{12\pi il}{p}\right) \quad \text{for } i \in \mathcal{I}_{NS}, \ l \in \mathcal{I}_{R}
\end{align*}
\quad \text{(4.5)}
\]

The $S$-matrix obeys (2.8) with $H^R = (D^R)^{-1} = \frac{1}{2}I$. As in the first example the resulting fusion algebra is isomorphic to the fusion algebra of the Virasoro $(2, p)$ model tensor an element of order two generating the right sector structure. This can be inferred from the form (4.6) of the $S$-matrix being equal to that of the $(2, p)$ model modulo factors.

We remark that for $c(p, 12)$ also the bosonic $\mathcal{W}$-algebras $\mathcal{W}(2, p - 3)$ exist and diagonalize the modular invariant partition functions given by

\[
Z_{bos} = \sum_{s=1}^{\frac{p-1}{2}} (|\chi_{1,s} + \chi_{7,s}|^2 + |\chi_{5,s} + \chi_{11,s}|^2 + |\chi_{4,s} + \chi_{8,s}|^2)
\quad \text{(4.7)}
\]

The first realization for low spins is $\mathcal{W}(2, 8) \subset \mathcal{W}(2, \frac{7}{2})$ at $c = \frac{21}{22}$ [11]. The subalgebra $\mathcal{W}(2, p - 3)$ is the bosonic projection of the fermionic algebra $\mathcal{W}(2, \frac{p-4}{2})$ for $c = c(12, p)$ with $p \geq 5$ odd and $p, 3$ coprime. One can show that the field with conformal dimension $\frac{p-4}{2}$ is a ‘simple current’ in the fusion algebra of the bosonic algebra. It is not very hard to find the fusion algebra of the fermionic algebra from that of the bosonic subalgebra (which can be calculated with the ordinary Verlinde formula) using a ‘simple current’ argument. In the appendix we discuss this in detail for the $N = 1$ super Virasoro algebra. For a detailed discussion of ‘simple currents’ in the context of modular invariants of RCFTs we refer the reader to ref. [32].

In the two series discussed above it was possible to obtain a fusion algebra with fusion coefficients $\mathcal{N}_{ii}^1 = 1$ by rescaling the $R$ fields with a factor $\frac{1}{\sqrt{2}}$. In the general case of the $(A, D)$
series this rescaling would result in irrational fusion coefficients. Already the simplest non-trivial example demonstrates this: We show that the coefficients \( \mathcal{N}_i^1 = 2 \) (for some \( i \in \mathcal{I}_R \)) are essential and cannot be removed by an extension of the fusion algebra.

Our example is \( \mathcal{W}(2, \frac{3}{2}) \) at \( c = c(3, 8) = -\frac{21}{4} \) with the modular invariant \( (D_5, A_2) \). The model is the non-unitary super Virasoro minimal model \( c(2, 8) \) and consists of two HWRs in each sector:

\[
NS: \{0, -\frac{1}{4}\} \equiv \{\mathbb{I}, \sigma\} \quad \text{and} \quad R: \{-\frac{3}{32}, -\frac{7}{32}\} \equiv \{\phi, \psi\}.
\]

The \( \mathcal{W} \)-characters in terms of Virasoro characters are

\[
\chi_{0}^{W,NS} = \chi_{1,1} + \chi_{1,2} \quad \chi_{-\frac{1}{4}}^{W,NS} = \chi_{3,1} + \chi_{3,2} \\
\chi_{-\frac{3}{32}}^{W,R} = 2\chi_{4,1} \quad \chi_{-\frac{7}{32}}^{W,R} = \chi_{2,1} + \chi_{2,2}.
\]

The matrix \( D^R \) is equal to \( \text{diag}(\{2, 1\}) \). The fusion rules are given by (using the formulae of section 2):

\[
[\sigma][\sigma] = [\mathbb{I}] + 2[\sigma] \quad [\sigma][\phi] = [\phi] + 2[\psi] \quad [\sigma][\psi] = [\phi] + [\psi] \\
[\phi][\phi] = 2[\mathbb{I}] + 2[\sigma] \quad [\phi][\psi] = 2[\sigma] \quad [\psi][\psi] = [\mathbb{I}] + [\sigma]
\]

We have indeed \( \mathcal{N}_i^1 = D_i^R, i \in \mathcal{I}_R \) but a simple rescaling of the \( R \) characters does not remove the coefficient \( \mathcal{N}_i^1 = 2 \). Let us now show that an extension of this fusion algebra by splitting the field \( \phi \) is not possible. This is in contrast to the case of degeneracies. To this end one makes the most general ansatz \( (a, b, a^\pm, b^\pm, d, e, f^\pm, g^\pm, h^\pm \in \mathbb{N}_0) \):

\[
[\phi] = [\phi^+] + [\phi^-] \quad [\phi^+][\phi^+] = a^+ [\mathbb{I}] + b^+ [\sigma] \quad [\phi^-][\phi^-] = a^- [\mathbb{I}] + b^- [\sigma] \\
[\phi^+][\phi^+] = f^+ [\phi^+] + f^- [\phi^-] + h^+ [\psi] \quad [\phi^-][\sigma] = g^+ [\phi^+] + g^- [\phi^-] + h^- [\psi]
\]

Using the fusion rules for \([\phi][\psi], [\phi][\phi] \) and \([\sigma][\phi] \) we get from this ansatz the following equations:

\[
a + b = 2 \quad a^+ + a^- + 2d = 2 \quad b^+ + b^- + 2e = 2 \\
f^+ + g^+ = 1 \quad f^- + g^- = 1 \quad h^+ + h^- = 2
\]

From the associativity of \([\psi][\phi^+][\phi^+] \) and \([\psi][\phi^-][\phi^-] \) we have

\[
a f^+ = b^+ \quad a f^- = b^+ \quad a h^+ = a^+ + b^+ \quad b g^+ = b^- \quad b g^- = b^- \quad b h^- = a^- + b^-
\]

and from \([\psi][\phi^+][\phi^-] \) one obtains

\[
ag^+ = e \quad ag^- = e \quad ah^- = d + e \quad bf^+ = e \quad bf^- = e \quad bh^+ = d + e
\]

Because of the triviality of the solutions \( a = 0 \) or \( b = 0 \) (in the sense that one recovers the original fusion algebra), we conclude from \( a + b = 2 \) that \( a = b = 1 \). Inserting this into \( ag^+ + bf^+ = 2e \), we get the contradiction \( 1 = f^+ + g^+ = 2e \) (q.e.d.).
Due to these facts one is forced to weaken the axioms of fusion algebras for fermionic theories, i.e. one has to admit more general fusion charge conjugation matrices.

5. \( N = 1 \) super Virasoro minimal models

For the \( N = 1 \) super Virasoro minimal models the central charge, the conformal dimensions of the HWRs and the characters are given by (for the unitary case see ref. [40])

\[
c = c(p, q) = \frac{3}{2} (1 - 2 \frac{(p - q)^2}{pq}) \quad p, q \in \mathbb{N}, \quad (p \mid q) = 1, \quad p + q \in 2\mathbb{N} \quad \text{or}
\]

\[
h(r, s) = \frac{(pr - qs)^2 - (p - q)^2}{8pq} + \frac{1 - (-1)^{r+s}}{32} \quad 1 \leq r \leq q - 1, \quad 1 \leq s \leq p - 1
\]

\[
\chi_{r,s}^{SV_{ir,NS}}(\tau) = \frac{e^{-2\pi i \frac{r}{p} \eta(\frac{r}{q})}}{\eta(\tau)^2} (\theta_{pr-qs,pq}(\frac{\tau}{2}) - \theta_{pr+qs,pq}(\frac{\tau}{2}))
\]

\[
\hat{\chi}_{r,s}^{SV_{ir,R}}(\tau) = \frac{\eta(2\tau)}{\eta(\tau)^2} (2 - \delta_{r,\frac{p}{2}} \delta_{s,\frac{q}{2}}) (\theta_{pr-qs,pq}(\frac{\tau}{2}) - \theta_{pr+qs,pq}(\frac{\tau}{2}))
\]

(5.1)

where \( r + s \) even (odd) corresponds to the \( NS \) (\( R \)) sector. Note that we do not include a ‘global’ factor \( \sqrt{2} \) in the \( R \) characters as in ref. [34] but use definition (2.2). Using the reflection symmetry in the superconformal grid the set of linear independent characters is labelled by:

\[
NS: \quad \mathcal{I}_{NS} = \{ (r, s) \mid r + s \text{ even}, \quad 1 \leq r \leq q - 1, \quad 1 \leq s \leq \lfloor \frac{p-1}{2} \rfloor \text{ or } 1 \leq r \leq \frac{q}{2}, \quad s = \frac{q}{2} \}
\]

\[
R: \quad \mathcal{I}_{R} = \{ (r, s) \mid r + s \text{ odd}, \quad 1 \leq r \leq q - 1, \quad 1 \leq s \leq \lfloor \frac{p-1}{2} \rfloor \text{ or } 1 \leq r \leq \frac{q}{2}, \quad s = \frac{q}{2} \}. \quad (5.2)
\]

Using the well-known transformation properties of the theta functions under modular transformations we obtain with a straightforward calculation the following expressions for the \( S \)-matrix:

\[
S_{r_1, s_1; r_2, s_2}^{NS, NS} = \frac{2}{\sqrt{pq}} (\cos(\frac{2\pi \lambda_1 \lambda_2}{4pq}) - \cos(\frac{2\pi \bar{\lambda}_1 \bar{\lambda}_2}{4pq}))
\]

\[
S_{r_1, s_1; r_2, r_2}^{R, NS} = \frac{2}{\sqrt{2pq}} (\cos(\frac{2\pi \lambda_1 \lambda_2}{4pq}) - (-1)^{r_2}s_2 \cos(\frac{2\pi \lambda_1 \bar{\lambda}_2}{4pq}))
\]

\[
S_{r_1, s_1; r_2, s_2}^{\bar{N}S, R} = \frac{2}{\sqrt{2pq}} (1 + \delta_{r_2,\frac{p}{2}} \delta_{s_2,\frac{q}{2}}) (\cos(\frac{2\pi \lambda_1 \lambda_2}{4pq}) - \cos(\frac{2\pi \lambda_1 \bar{\lambda}_2}{4pq}))
\]

(5.3)

with \( \lambda_i = pr_i - qs_i \), \( \bar{\lambda}_i = pr_i + qs_i \). One can check that with the standard definition of \( T \) these formulæ define a proper representation of the modular group. It is well known that
the unitary minimal models are given by \((p, q) = (m, m + 2)\) for \(m \geq 2\). For this special choice of \((p, q)\) formula (5.3) reduces to the corresponding one already given in [41].

We emphasize that the multiplicity matrix \(M\) is equal to the identity matrix in the case of the \(N = 1\) super Virasoro minimal models because there are no additional independent ‘quantum numbers’.

Let us first consider the case \(p\) and \(q\) odd. Here the reflection symmetry in the superconformal grid has no fixed point and the \(D\)-matrix is given by \(D^R = 2I\) since the \(S\)-matrix obeys (2.8) with \(H^R = \frac{1}{2}I\). In order to calculate the fusion algebra corresponding to the super Virasoro minimal models one has to insert (5.3) into (2.5). It is not necessary to calculate the fusion coefficients directly if one remembers that the matrices \(S^{NS,NS}\) and \(S^\tilde{NS},R\) are (modulo constants) equal to the \(S\)-matrix of the Virasoro minimal models with central charge \(c = c(p, q)\), so that the well-known selection rules for the Virasoro minimal models can be applied to the fusion coefficients of the super Virasoro minimal models. Since the selection rules of the Virasoro minimal models respect the sector structure given by odd or even \(r + s\) in the same way as the super minimal models, the corresponding fusion algebras are isomorphic. Obviously, the only difference between the fusion algebras is the fact that the fusion coefficients connecting two \(R\) fields with a \(NS\) field are elements of \(2\mathbb{N}\) for the \(N = 1\) supersymmetric model. Note that the fusion algebra has a \(\mathbb{Z}_2\)-structure like the first two examples in section 4. As in the case of the \(\mathcal{W}(2, \frac{k-2}{2})\) algebras, \(N^R_{ii} = 2 \ (\forall i \in I_R)\) indicates that the fields in the \(R\) sector correspond to fixed points in the fusion algebra of the bosonic projection. Furthermore, the dimensions of the representations of the horizontal subalgebra in the \(L_0\) eigenspace \(\mathcal{V}_0\) are equal to the corresponding diagonal entries of \(D^R\).

In the case \(p\) and \(q\) even the structure is different. Because \(M\) is trivial the matrix \(D\) can be immediately read off from (5.3) and (2.8): \(D^R_{i,j} = (2 - \delta_{i,i_0}) \delta_{i,j}\), where \(i_0\) is the label of the HWR \(h = \frac{c}{24}\) in the \(R\) sector. Note that this is exactly the only fixed point under the reflection symmetry in the superconformal grid: \(i_0 \equiv (r, s) = (\frac{q}{2}, \frac{p}{2})\). This implies that in this representation \(G^2_0\) is represented by zero so that the irreducible representation of the horizontal subalgebra in \(\mathcal{V}_0\) is one-dimensional (in contrast to the generic case \(G^2_0 \neq 0\) where the irreducible representations are two-dimensional). Consequently, the corresponding \(\tilde{R}\) character is nontrivial and encodes the broken boson-fermion symmetry of the ground state. Hence the Witten index [35, 36] of this HWR is nontrivial. This shows that the diagonal entries of \(D^R\) are equal to the dimensions of the spaces \(\mathcal{V}_0\). Calculating the fusion algebra with (2.5) yields an associative and commutative algebra with nontrivial fusion charge conjugation \(N^R_{i,j} = D^R_{i,j}\). The example \(c(2, 8)\) was already treated in section 4.

Finally, we present as a second example the fusion algebra of the unitary model \(m = 4\) with \(c(4, 6) = 1\) [31] corresponding to the \(N = 2\) supersymmetric point of the Ashkin-Teller model. There are 4 fields in the Neveu-Schwarz (Ramond) sector: \(h \in \{0, \frac{1}{16}, 1, \frac{1}{6}\}\) \((h \in \{\frac{3}{8}, \frac{1}{24}, \frac{9}{16}, \frac{1}{10}\})\). The fusion algebra reads:
The fusion algebra (5) the supersymmetric fusion algebra in a consistent way. In the bosonic projection of the $N$-Φ normalization of the fields in the $R$ sector is fixed if one requires that the fusion algebra of the bosonic projection of the $N = 1$ supersymmetric algebra under consideration induces the supersymmetric fusion algebra in a consistent way.

The fusion rules for this model have also been calculated in [31] using the Coulomb-gas approach. However, this approach shows only if a fusion coefficient vanishes or not. Furthermore, null vector methods do not work because the $G_0$-diagonal spin field is not well-defined [31].

Note that in (5) a coefficient 2 appears in front of the vacuum representation in the fusion of all $R$ fields besides $Φ^{R\Phi}_{\frac{3}{2}}$ with itself. Usually, one demands that the fusion of a field with its conjugate contains the vacuum only once. However, one can check that the fusion coefficients in (5) are sufficient and necessary for the associativity of the fusion algebra. In contrast to the (odd, odd) case discussed above, here it is impossible to rescale the fields in the $R$ sector in such a way that the resulting fusion algebra is integer-valued and has a trivial fusion charge conjugation. As was shown in section 3, the choice of the normalization of the fields in the $R$ sector is fixed if one requires that the fusion algebra of the bosonic projection of the $N = 1$ supersymmetric algebra under consideration induces the supersymmetric fusion algebra in a consistent way.

The fusion algebra (5) contains a ‘simple current’ of conformal dimension 1 and a subalgebra generated from the fields $[Φ^N_{0\Phi}]$, $[Φ^N_{1\Phi}]$, $[Φ^N_{0\Phi}]$, $[Φ^R_{\frac{1}{2}}]$. This is reminiscent of the additional $N = 2$ supersymmetry of the $c = 1$ model. Taking into account the fact that the $N = 2$ super Virasoro algebra has a minimal model at $c = 1$ it is obvious to consider the extension of the symmetry algebra by this ‘simple current’.

6. $N = 1$ $SW(\frac{3}{2}, δ)$-algebras

In this section we investigate $N = 1$ super $W$-algebras $SW(\frac{3}{2}, δ)$ with two generators for $δ \geq 2$. These are the most complicated examples of fermionic RCFTs as far as the fusion algebra is concerned because both the multiplicity matrix $M$ as well as the $D$-matrix are different from the identity matrix in the general case. After some general comments on the representation theory in the Ramond sector we discuss the $SW(\frac{3}{2}, δ)$-algebras fitting into the $ADE$-classification [34]. Then we proceed with the parabolic $N = 1$ $SW$-algebras with vanishing and non-vanishing self-coupling constant. We give the matrices $M$ and $D$ for all
these series. As an example for ADE-cases we present the fusion algebra for the rational model of $\mathcal{SW}(\frac{3}{2}, 2)$ at $c = -\frac{16}{5}$. We conclude with some further remarks concerning fusion algebras of fermionic RCFTs possessing a $\mathcal{SW}(\frac{3}{2}, \delta)$ symmetry algebra.

We found by explicit computer calculations that for the $\mathcal{SW}(\frac{3}{2}, \delta)$-algebras with $2 \leq \delta \leq \frac{9}{2}$ the identity $[\psi_0, G_0 \phi_0] = 0$ holds on the corresponding highest weight vectors where $\psi$ ($\phi$) is the bosonic (fermionic) component of the additional super field. Therefore, one considers only HWRs in which $G_0 \phi_0$ is represented by a scalar on the highest weight leading to an additional quantum number which can take at most two values (since $G_0 \phi_0$ satisfies a quadratic equation for fixed $L_0$ and $\psi_0$ eigenvalues of the highest weight) (cf. section 2). However, this quantum number is redundant in the supersymmetric theory because $V_0$ contains for all possible eigenvalues of $G_0 \phi_0$ an eigenvector. Nevertheless, in the bosonic projection it distinguishes between different highest weight representations. Because $G_0 \phi_0$ is represented by a scalar on the highest weight the dimension of $V_0$ is at most two, it is one-dimensional exactly if $G_0^2 = \psi_0^2 = 0$ holds on the highest weight. Indeed, for $\delta \in \mathbb{N}$ the identity $G_0^2 = 0$ implies $\phi_0^2 = 0$ due to $[G_0, \psi_0] = \phi_0$. However, computer results show that even for $9 \geq \delta \in \mathbb{N} + \frac{1}{2}$ this implication is true. In the case $\delta \in \mathbb{N} + \frac{1}{2}$, $G_0^2 \neq 0$ the representation of the horizontal subalgebra in $V_0$ is equal to the two-dimensional representation of the Clifford algebra $Cl(2, 0)$ (cf. section 4). For $\delta \in \mathbb{N}$ the structure of the representation of the horizontal subalgebra is more complicated.

The algebras existing for super Virasoro minimal values of $c$ can be organized into four series according to the partition function which is diagonalized by the $\mathcal{SW}$-characters. From table 2 we can directly read off these characters in terms of super Virasoro characters (the quantities appearing with their absolute value in $Z$ are the $\mathcal{SW}$-characters). Obviously, one can obtain $D$ from the form of the partition functions given in table 2 if the multiplicity matrix $M$ is known. This matrix follows from earlier studies of the degeneracies of the representation modules [20, 43]. The representations whose characters are a sum of Virasoro characters with maximal number of summands are non-degenerate. The representations whose characters are a sum of Virasoro characters with half the number of summands are doubly degenerate. Note that one has to consider the NS and $R$ sectors separately.
| $SW(\frac{3}{2}, \delta)$ | $c(p, q)$ | series | $Z^{NS}, Z^{R}$ |
|----------------|----------------|---------|----------------|
| $k-\frac{2}{2}$ | $c(12, 2k)$ | $(D_{k+1}, E_6)$ | $Z^{NS} = \sum_{r=1 \atop odd}^{k-1} |\chi_{r,1}+\chi_{r,7}+\chi_{r,5}+\chi_{r,11}|^2 + 2|\chi_{k,1}+\chi_{k,5}|^2$  
| & | | $Z^{R} = \sum_{r=1 \atop odd}^{k-1} |\tilde{\chi}_{r,4}+\tilde{\chi}_{r,8}|^2 + 2|\tilde{\chi}_{k,4}|^2$ |
| $\frac{2k-3}{2}$ | $c(12, 2k)$ | $(A_{2k-1}, E_6)$ | $Z^{NS} = \sum_{r=2 \atop even}^{k-1} (\frac{1}{2}|\chi_{r,1}+\chi_{r,7}|^2+|\tilde{\chi}_{r,5}+\tilde{\chi}_{r,11}|^2) +$  
| & | | $\frac{1}{2}\sum_{r=1 \atop odd}^{k-1} |\tilde{\chi}_{r,4}+\tilde{\chi}_{r,8}|^2 + |\tilde{\chi}_{k,4}|^2$ |
| $\frac{2k-5}{2}$ | $c(30, 2k)$ | $(A_{2k-1}, E_8)$ | $Z^{NS} = \sum_{r=2 \atop even}^{k-1} (\frac{1}{2}|\chi_{r,1}+\chi_{r,7}|^2+|\tilde{\chi}_{r,5}+\tilde{\chi}_{r,11}|^2) +$  
| & | | $\frac{1}{2}\sum_{r=1 \atop odd}^{k-1} |\tilde{\chi}_{r,4}+\tilde{\chi}_{r,8}|^2 + |\tilde{\chi}_{k,4}|^2$ |
| $\frac{(q-2)(k-1)}{4}$ | $c(2k, q)$ | $(A_{q-1}, D_{k+1})$ | $Z^{NS} = \sum_{r=1 \atop odd}^{\frac{q-1}{2}} (\sum_{s=1 \atop odd}^{k-1} |\chi_{r,s}+\chi_{q-r,s}|^2 + 2|\chi_{r,k}|^2)$  
| & | | $Z^{R} = \sum_{r=2 \atop even}^{\frac{q-1}{2}} (\frac{1}{2}|\chi_{r,s}+\tilde{\chi}_{q-r,s}|^2 + \sum_{s=1 \atop odd}^{k-1} |\chi_{r,s}|^2 +$  
| & | | $\sum_{r=2 \atop even}^{\frac{q-1}{2}} |\tilde{\chi}_{r,k}|^2 + 2|\tilde{\chi}_{k,\frac{q}{2}}|^2$ |

Table 2: partition functions and series of $SW(\frac{3}{2}, \delta)$-algebras [33, 19, 20]

We continue with the discussion of parabolic $SW$-algebras. There are two series, one with non-vanishing and one with vanishing self-coupling constant [37]. The series with $C_{\phi\phi} \neq 0$ consists of the algebras $SW(\frac{3}{2}, 8k)$ at $c = \frac{3}{4}(1 - 16k)$ with $4k \in \mathbb{N}$. They possess the following HWRs [20, 43] ($h_{r,r} = k(r^2 - 1)$, $h_{r,-r} = h_{r,r} + \frac{1}{2}r^2$):

$NS: \quad h_{\frac{m}{4k}, \frac{m}{4k}} \quad m = 0, \ldots, 4k, 8k$  
$h_{\frac{m}{4k+2}, -\frac{m}{4k+2}} \quad m = 1, \ldots, 4k + 1$
For doubly degenerate HWRs with two-dimensional $V$ subalgebra (this is only different for the representation with conformal dimension $(k$ distinguish the two cases

It turned out in the course of our calculations that in the Ramond sector one has to other representations occur only once. We conclude that $M - D$ two

Firstly, we state that the two representation modules to the $h$-value $h_{0,0} + \frac{1}{16} = \frac{c}{24}$ corresponding to the first two summands in $Z^R$ are different and that the dimension of the representation of the horizontal subalgebra in $\mathcal{V}_0$ is equal to one in both cases. In all other representations this dimension is equal to two. Furthermore, the representations to $h_{1,1} + \frac{1}{16}$ third summand and $h_{1,-1} + \frac{1}{16}$ fourth summand are doubly degenerate, whereas all other representations occur only once. We conclude that $M^R = diag(\{1, 1, 2, 2, 1, \ldots, 1\})$ so that $D^R = diag(\{1, 1, 2, 2, 2, \ldots, 2\})$ (again the diagonal entries of $D^R$ equal the dimensions of the corresponding spaces $\mathcal{V}_0$).

The series with $C^\phi_\phi = 0$ is given by the algebras $\mathcal{SW}(\frac{3}{2}, 3k)$ existing for $c = \frac{3}{2}(1 - 16k)$ with $2k \in \mathbb{N}$. They possess the following HWRs [20, 43]:

$$Z_{NS}^R(\tau) = \left| \frac{\eta(2\tau)}{\eta(\tau)} \right|^2 \left( | \frac{1}{2}(\theta_{4k,4k}(\tau) - \theta_{4k,4k+2}(\tau)) |^2 + | \frac{1}{2}(\theta_{4k,4k}(\tau) + \theta_{4k,4k+2}(\tau)) |^2 + \sum_{m=1}^{4k+1} | \theta_{4k+2+m,4k+2}(\tau) |^2 + \sum_{m=1}^{4k-1} | \theta_{4k+m,4k}(\tau) |^2 \right)$$

$$Z^R(\tau) = \left| \frac{\eta(2\tau)}{\eta(\tau)} \right|^2 \left( | \theta_{0,4k+2}(\tau) |^2 + | \theta_{0,4k}(\tau) |^2 + | \theta_{4k,4k}(\tau) |^2 + | \theta_{4k+2,4k+2}(\tau) |^2 + \sum_{m=1}^{4k-1} \frac{1}{2} | 2\theta_{4k+m,4k}(\tau) |^2 + \sum_{m=1}^{4k+1} \frac{1}{2} | 2\theta_{4k+2+m,4k+2}(\tau) |^2 \right)$$

(6.1)

It turned out in the course of our calculations that in the Ramond sector one has to distinguish the two cases $k \in \mathbb{N}$ and $k \in \mathbb{N} + \frac{1}{2}$. In the case $k \in \mathbb{N}$ there are two doubly degenerate HWRs with two-dimensional $d = 2$ representation of the horizontal subalgebra $(h_{\frac{1}{2}, \frac{1}{2}} + \frac{1}{16}, h_{\frac{1}{2}, -\frac{1}{2}} + \frac{1}{16})$. In the single representation with $h_{0,0} + \frac{1}{16} = \frac{c}{24}$ this dimension $d$ is equal to one. All other representations are non-degenerate and have $d = 2$. For $k \in \mathbb{N} + \frac{1}{2}$ there is only one doubly degenerate representation $(h_{\frac{1}{2}, \frac{1}{2}} + \frac{1}{16})$ which has $d = 2$. Furthermore, the representation $h_{0,0} + \frac{1}{16} = \frac{c}{24}$ has $d = 1$ and all other representations have $d = 2$ and are not degenerate, clarifying some unexplained subtleties in ref. [37]. From the corresponding modular invariant partition function $Z$ one obtains $D^R$. For $k \in \mathbb{N}$ the diagonal entries of $D^R$ are given by the dimensions $d$ of $\mathcal{V}_0$. For $k \in \mathbb{N} + \frac{1}{2}$ this is only different for the representation with conformal dimension $(h_{\frac{1}{2}, \frac{1}{2}} + \frac{1}{16})$ where
d = 2 but the $D^R$ entry is equal to one. Here we see that in general the entries of the diagonal matrix $D^R$ are different from the corresponding dimensions of $V_0$.

We present now an example for a fusion algebra of a rational model of a $SW(\frac{3}{2}, \delta)$-algebra which shows the most general features of fusion algebras of fermionic RCFTs. Our example is $SW(\frac{3}{2}, 2)$ at $c = -\frac{5}{2}$ corresponding to the partition function $(A_3, D_6)$. The model consists of three HWRs in each sector: $NS : h \in \{0, -\frac{1}{10}, -\frac{1}{5}\}$, $R : h \in \{\frac{3}{2}, \frac{5}{2}, -\frac{7}{20}\}$. The last $h$-value is the fixed point of the superconformal grid and hence equal to $\frac{c}{24}$. The representations with $h = \frac{1}{5}, \frac{3}{2}, \frac{3}{20}, -\frac{7}{20}$ are doubly degenerate. The dimensions of the representations of the horizontal subalgebra in $V_0$ are equal to two with the exception of the representation $h = -\frac{1}{20} = \frac{c}{24}$, where it is equal to one. These dimensions coincide again with the corresponding diagonal entries of $D^R$ obtained from $M$ and $S$. Using the explicit form of the $SW$-characters in terms of super Virasoro characters, we obtain the fusion algebra via formulæ (4.3) and (2.5):

$$\begin{align*}
[\phi^{NS}_1][\phi^{NS}_1] &= [\phi^{NS}_{-1}] + [\phi^{NS}_0] + [\phi^{NS}_1] \\
[\phi^{NS}_0][\phi^{NS}_1] &= 2[\phi^{NS}_{-1}] + 2[\phi^{NS}_0] + [\phi^{NS}_1] \\
[\phi^{NS}_1][\phi^{NS}_{-1}] &= 2[\phi^{NS}_0] + 2[\phi^{NS}_1] + [\phi^{NS}_{-1}] \\
[\phi^{NS}_1][\phi^{NS}_0] &= 2[\phi^{NS}_{-1}] + 2[\phi^{NS}_0] + [\phi^{NS}_1] \\
[\phi^{NS}_0][\phi^{NS}_{-1}] &= 2[\phi^{NS}_0] + 2[\phi^{NS}_{-1}] + [\phi^{NS}_1] \\
[\phi^{NS}_{-1}][\phi^{NS}_{-1}] &= 2[\phi^{NS}_0] + 2[\phi^{NS}_1] + [\phi^{NS}_{-1}]
\end{align*}$$

We recognize that the equality $N^i_{ii} = (MD)_{i,i}$ ($i \in I_R$) is indeed satisfied in this example. Using the fact that the fusion algebra of the super Virasoro minimal model $c(10, 4) = -\frac{5}{2}$ possesses a $\mathbb{Z}_2$ ‘simple current’ of conformal dimension 2 one can recover (6.1) from the fusion algebra of the super Virasoro minimal model.

Finally, note that it is possible to obtain the fusion algebras of the $SW(\frac{3}{2}, \delta)$-algebra rational models related to the $(A_{q-1}, D_{\frac{q+2}{2}})$-series in the $ADE$-classification from the super Virasoro fusion algebras using the simple current of order two and conformal dimension $\delta$. However, this is not possible for $SW(\frac{3}{2}, \delta)$-algebras fitting into one of the other three series because the field with conformal dimension $\delta$ is no ‘simple current’ in the fusion algebra of the corresponding super Virasoro minimal model any more. Nevertheless, it is possible to obtain the fusion algebra with the generalized Verlinde formula from the $S$-matrix of the $SW(\frac{3}{2}, \delta)$-minimal model. Furthermore, one can show that – in perfect analogy to the second example in section 4 – the fusion algebras of the models related to $(D_{k+1}, E_6)$ factorize into a $\mathbb{Z}_2 \otimes A$ fusion algebra ($A$ denotes the fusion algebra of the $NS$ sector). In the first example – $SW(\frac{3}{2}, \frac{5}{2})$ at $c = c(12, 14) = \frac{10}{7}$ – we verified by explicit calculation that it is possible to resolve the degeneracies in the $NS$ sector by a suitable extension of the fusion algebra.

Unfortunately, we are not able to calculate the fusion algebras of the exceptional rational models of $SW(\frac{3}{2}, \delta)$ algebras since no explicit formulæ for the $S$-matrices are known.
7. Conclusions

We proved a generalized Verlinde formula for fermionic RCFTs by showing that the fusion algebras coincide with those obtained from the corresponding bosonic projection by the ordinary Verlinde formula and ‘simple current’ arguments. Using this generalized Verlinde formula we were able to calculate the fusion algebras of several fermionic RCFTs. The S-matrix is in general neither unitary nor symmetric but it obeys the equation $S^\dagger H S = H$ with $H = M D^{-1}$. $M$ is a diagonal matrix encoding the multiplicities of the HWRs of the theory whereas $D$ is a diagonal matrix which is defined through the orbit lengths under the action of the ‘simple current’ in the fusion algebra of the bosonic projection (cf. section 2). In a concrete example we showed that it is not possible to avoid the $D$-matrix by extending the fusion algebra. There is strong evidence that this holds in general. Furthermore, we considered the representation theory of the horizontal subalgebra on the highest weights in the Ramond sector for fermionic $W$-algebras. For fermionic $\mathcal{W}(2, \delta)$-algebras the dimensions of the irreducible representations are encoded in the fusion algebra of the corresponding rational model. In particular, we considered fermionic $\mathcal{W}(2, \delta)$-algebras, minimal models of the $N = 1$ super Virasoro algebra and finally $N = 1$ $S\mathcal{W}$-algebras with two generators. In some cases we verified explicitly that our results agree with the fusion algebras calculated by ‘simple current’ arguments from bosonic projections of the corresponding fermionic theories. Furthermore, we pointed out that in the case of the $N = 1$ super Virasoro minimal models with central charge $c = c(p, q)$, $p, q$ even, null-state methods cannot be applied. These examples show that one has to weaken the axioms of fusion algebras for fermionic RCFTs allowing more general fusion charge conjugation matrices $N^\dagger_{ij} = (MD)_{i,j}$. We have shown that in general fusion coefficients greater than one appear, so the consideration of 3-point functions in the Coulomb-gas picture which only tells whether a certain field appears in the fusion of two other fields or not, cannot yield the full information about the fusion algebra.

Using ‘simple current’ arguments it is possible to define in a very natural way the fusion of twisted fields in the cases of bosonic $\mathcal{W}(2, \delta)$-algebras admitting an outer $\mathbb{Z}_2$-automorphism. It is not clear to us how to write down a generalized Verlinde formula in these cases.

It will be interesting to use the generalized Verlinde formula in the future to set up a classification program for the fusion algebras of fermionic theories in complete analogy to the bosonic case considered in [23, 24]. Especially it would be very interesting to show whether fermionic RCFTs exist which cannot be obtained by a $\mathbb{Z}_2$ ‘simple current’ extension of its bosonic projection.

Acknowledgements

We are very grateful to W. Nahm for encouraging us to study fusion algebras of bosonic projections and many valuable discussions.

We thank R. Blumenhagen, M. Flohr, A. Honecker, J. Kellendonk, S. Mallwitz, N. Mohammadi, A. Recknagel, M. Rösgen, N.-P. Skoruppa, M. Terhoeven and R. Varnhagen for useful discussions.

W.E. thanks the Max-Planck-Institut für Mathematik for financial support.

R.H. is supported by a research studentship of the NRW-Graduiertenförderung.
Appendix

In this appendix the formulae of section 3 are verified by considering the bosonic projection of the $N = 1$ super Virasoro algebra. We calculate the fusion algebra — using the ordinary Verlinde formula — of the bosonic projection for two special minimal values of $c$ where this projection yields a $\mathcal{W}(2, 4)$. One recognizes that these fusion algebras contain a ‘simple current’ of dimension $\frac{3}{2}$. The definition of a new basis, given by the sum of the fields lying in one orbit, divided by the length of the orbit, yields the fusion algebra of the $N = 1$ super Virasoro minimal model. The fusion algebras obtained this way coincide exactly with the fusion algebras calculated directly using the generalized Verlinde formula.

It is well-known that the projection of the $N = 1$ super Virasoro algebra onto the bosonic sector yields a $\mathcal{W}(2, 4, 6)$ [44–46]. As one can easily show by direct computation, the primary field of conformal dimension six turns out to be a null field for some special values of $c$, among them $-11$ and $-\frac{11}{14}$, so that the $\mathcal{W}(2, 4, 6)$ reduces to a $\mathcal{W}(2, 4)$ (the primary field of dimension four has nonzero norm for these values of $c$) [46]. It has been found earlier that at these two values of $c$ rational models of $\mathcal{W}(2, 4)$ exist and the possible $h$-values are known [16].

Firstly let us determine the fusion algebra of the $N = 1$ super Virasoro minimal model at $c(2, 12) = -11$ using (4.3) and (2.5). There are three HWRs in each sector: $NS : h \in \{0, -\frac{1}{3}, -\frac{1}{2}\}$, $R : h \in \{-\frac{1}{8}, -\frac{3}{8}, -\frac{11}{24}\}$. Note that the last $h$-value is the fixed point of the superconformal grid and is equal to $\tilde{c} \frac{c}{24}$. The fusion algebra reads:

\[
\begin{align*}
[\phi^0_{\frac{NS}{2}}[\phi^0_{\frac{NS}{2}} &= [\phi^0_{0}^NS] + [\phi^0_{\frac{1}{3}}^NS] + [\phi^0_{\frac{1}{2}}^NS] \quad [\phi_{\frac{NS}{4}}^NS][\phi_{\frac{NS}{4}}^NS] &= [\phi_{\frac{1}{4}}^NS] + 2[\phi_{\frac{1}{2}}^NS] \\
[\phi^0_{\frac{NS}{2}}[\phi^0_{\frac{NS}{2}} &= [\phi^0_{0}^NS] + 2[\phi^0_{\frac{1}{3}}^NS] + 2[\phi^0_{\frac{1}{2}}^NS] \quad [\phi_{\frac{NS}{4}}^NS][\phi_{\frac{NS}{4}}^NS] &= [\phi_{\frac{1}{4}}^NS] + 2[\phi_{\frac{1}{2}}^NS] \\
[\phi^R_{\frac{1}{8}}[\phi^R_{\frac{1}{8}} &= 2[\phi^R_{0}^NS] + 2[\phi^R_{\frac{1}{3}}^NS] \quad [\phi_{\frac{NS}{4}}^R][\phi_{\frac{NS}{4}}^R] &= 2[\phi_{\frac{1}{4}}^NS] + 2[\phi_{\frac{1}{2}}^NS] \\
[\phi^R_{\frac{3}{8}}[\phi^R_{\frac{3}{8}} &= 2[\phi^R_{0}^NS] \quad [\phi_{\frac{NS}{4}}^R][\phi_{\frac{NS}{4}}^R] &= 2[\phi_{\frac{1}{4}}^NS] + 2[\phi_{\frac{1}{2}}^NS] + 4[\phi_{\frac{1}{2}}^NS] \\
[\phi^R_{\frac{5}{8}}[\phi^R_{\frac{5}{8}} &= 2[\phi^R_{0}^NS] + 2[\phi^R_{\frac{1}{3}}^NS] \quad [\phi_{\frac{NS}{4}}^R][\phi_{\frac{NS}{4}}^R] &= 2[\phi_{\frac{1}{4}}^NS] + 2[\phi_{\frac{1}{2}}^NS] + 4[\phi_{\frac{1}{2}}^NS] \\
[\phi^R_{\frac{7}{8}}[\phi^R_{\frac{7}{8}} &= 2[\phi^R_{0}^NS] \quad [\phi_{\frac{NS}{4}}^R][\phi_{\frac{NS}{4}}^R] &= 2[\phi_{\frac{1}{4}}^NS] + [\phi_{\frac{1}{2}}^NS] + [\phi_{\frac{1}{2}}^NS] 
\end{align*}
\]

(A.1)

For $\mathcal{W}(2, 4)$ at $c = -11$ the effective central charge $\tilde{c}$ is equal to 1, so that this model belongs to the parabolic $\mathcal{W}(2, \delta)$-algebras which have been studied by M. Flohr [37]. Using the explicit form of the $S$-matrix one is able to calculate the fusion algebra of this model [47,37]. There exist 10 HWRs of $\mathcal{W}(2, 4)$ at $c = -11$:

\(h \in \{0, \frac{3}{7}, -\frac{1}{7}, -\frac{1}{5}, -\frac{1}{7}, 0, -\frac{1}{8}, -\frac{3}{8}, -\frac{11}{24}, \frac{13}{24}\}\). The fusion algebra is given by:
\[
\begin{align*}
[\Phi_\frac{1}{2}][\Phi_\frac{1}{2}] &= [\Phi_0] \\
[\Phi_\frac{1}{2}][\Phi_\frac{3}{2}] &= [\Phi_\frac{1}{2}] \\
[\Phi_\frac{1}{2}][\Phi_0] &= [\Phi_\frac{1}{2}] \\
[\Phi_\frac{3}{2}][\Phi_\frac{1}{2}] &= [\Phi_\frac{3}{2}] \\
[\Phi_\frac{3}{2}][\Phi_0] &= [\Phi_\frac{1}{2}] \\
\end{align*}
\]

\[
\begin{align*}
[\Phi_{-\frac{1}{2}}][\Phi_{-\frac{1}{2}}] &= [\Phi_0] + [\Phi_{-\frac{1}{2}}] + [\Phi_{-\frac{1}{2}}] \\
[\Phi_{-\frac{1}{2}}][\Phi_{-\frac{1}{2}}] &= [\Phi_{-\frac{1}{2}}] + [\Phi_{-\frac{1}{2}}] + [\Phi_{0}] \\
[\Phi_{-\frac{1}{2}}][\Phi_0] &= [\Phi_{-\frac{1}{2}}] + [\Phi_{-\frac{1}{2}}] + [\Phi_0] \\
[\Phi_{-\frac{3}{2}}][\Phi_{-\frac{1}{2}}] &= [\Phi_{-\frac{3}{2}}] + [\Phi_{-\frac{1}{2}}] + [\Phi_{0}] \\
[\Phi_{-\frac{3}{2}}][\Phi_{-\frac{3}{2}}] &= [\Phi_{-\frac{3}{2}}] + [\Phi_{-\frac{3}{2}}] + [\Phi_{-\frac{1}{2}}] + [\Phi_{-\frac{1}{2}}] + [\Phi_{0}] \\
[\Phi_{-\frac{5}{2}}][\Phi_{-\frac{3}{2}}] &= [\Phi_{-\frac{5}{2}}] + [\Phi_{-\frac{3}{2}}] + [\Phi_{-\frac{3}{2}}] + [\Phi_{-\frac{3}{2}}] + [\Phi_{-\frac{1}{2}}] + [\Phi_{-\frac{1}{2}}] + [\Phi_{-\frac{1}{2}}] + [\Phi_{0}] \\
[\Phi_{-\frac{7}{2}}][\Phi_{-\frac{5}{2}}] &= [\Phi_{-\frac{7}{2}}] + [\Phi_{-\frac{5}{2}}] + [\Phi_{-\frac{5}{2}}] + [\Phi_{-\frac{5}{2}}] + [\Phi_{-\frac{3}{2}}] + [\Phi_{-\frac{3}{2}}] + [\Phi_{-\frac{3}{2}}] + [\Phi_{-\frac{3}{2}}] + [\Phi_{-\frac{1}{2}}] + [\Phi_{-\frac{1}{2}}] + [\Phi_{-\frac{1}{2}}] + [\Phi_{-\frac{1}{2}}] + [\Phi_{0}] \\
\end{align*}
\]

\[
\begin{align*}
[\Phi_{-\frac{1}{2}}][\Phi_{-\frac{1}{2}}] &= [\Phi_0] + [\Phi_{-\frac{1}{2}}] + [\Phi_{-\frac{1}{2}}] + [\Phi_{-\frac{1}{2}}] + [\Phi_{0}] \\
[\Phi_{-\frac{3}{2}}][\Phi_{-\frac{1}{2}}] &= [\Phi_{-\frac{3}{2}}] + [\Phi_{-\frac{1}{2}}] + [\Phi_{0}] \\
[\Phi_{-\frac{3}{2}}][\Phi_{-\frac{3}{2}}] &= [\Phi_{-\frac{3}{2}}] + [\Phi_{-\frac{3}{2}}] + [\Phi_{-\frac{1}{2}}] + [\Phi_{-\frac{1}{2}}] + [\Phi_{0}] \\
[\Phi_{-\frac{5}{2}}][\Phi_{-\frac{3}{2}}] &= [\Phi_{-\frac{5}{2}}] + [\Phi_{-\frac{3}{2}}] + [\Phi_{-\frac{3}{2}}] + [\Phi_{-\frac{3}{2}}] + [\Phi_{-\frac{1}{2}}] + [\Phi_{-\frac{1}{2}}] + [\Phi_{-\frac{1}{2}}] + [\Phi_{0}] \\
[\Phi_{-\frac{7}{2}}][\Phi_{-\frac{5}{2}}] &= [\Phi_{-\frac{7}{2}}] + [\Phi_{-\frac{5}{2}}] + [\Phi_{-\frac{5}{2}}] + [\Phi_{-\frac{5}{2}}] + [\Phi_{-\frac{3}{2}}] + [\Phi_{-\frac{3}{2}}] + [\Phi_{-\frac{3}{2}}] + [\Phi_{-\frac{3}{2}}] + [\Phi_{-\frac{1}{2}}] + [\Phi_{-\frac{1}{2}}] + [\Phi_{-\frac{1}{2}}] + [\Phi_{-\frac{1}{2}}] + [\Phi_{0}] \\
\end{align*}
\]

\[
\begin{align*}
[\Phi_{-\frac{1}{2}}][\Phi_{-\frac{1}{2}}] &= [\Phi_0] + [\Phi_{-\frac{1}{2}}] + [\Phi_{-\frac{1}{2}}] + [\Phi_{-\frac{1}{2}}] + [\Phi_{0}] \\
[\Phi_{-\frac{3}{2}}][\Phi_{-\frac{1}{2}}] &= [\Phi_{-\frac{3}{2}}] + [\Phi_{-\frac{1}{2}}] + [\Phi_{0}] \\
[\Phi_{-\frac{3}{2}}][\Phi_{-\frac{3}{2}}] &= [\Phi_{-\frac{3}{2}}] + [\Phi_{-\frac{3}{2}}] + [\Phi_{-\frac{1}{2}}] + [\Phi_{-\frac{1}{2}}] + [\Phi_{0}] \\
[\Phi_{-\frac{5}{2}}][\Phi_{-\frac{3}{2}}] &= [\Phi_{-\frac{5}{2}}] + [\Phi_{-\frac{3}{2}}] + [\Phi_{-\frac{3}{2}}] + [\Phi_{-\frac{3}{2}}] + [\Phi_{-\frac{1}{2}}] + [\Phi_{-\frac{1}{2}}] + [\Phi_{-\frac{1}{2}}] + [\Phi_{0}] \\
[\Phi_{-\frac{7}{2}}][\Phi_{-\frac{5}{2}}] &= [\Phi_{-\frac{7}{2}}] + [\Phi_{-\frac{5}{2}}] + [\Phi_{-\frac{5}{2}}] + [\Phi_{-\frac{5}{2}}] + [\Phi_{-\frac{3}{2}}] + [\Phi_{-\frac{3}{2}}] + [\Phi_{-\frac{3}{2}}] + [\Phi_{-\frac{3}{2}}] + [\Phi_{-\frac{1}{2}}] + [\Phi_{-\frac{1}{2}}] + [\Phi_{-\frac{1}{2}}] + [\Phi_{-\frac{1}{2}}] + [\Phi_{0}] \\
\end{align*}
\]
One recognizes that the field $[\phi_\frac{3}{2}]$ is a ‘simple current’ reflecting the supersymmetric structure of this model. We conclude that the symmetry algebra of this model can be extended by this ‘simple current’ which can be viewed as the inverse procedure of the projection onto the bosonic part of the fermionic algebra. Doing so we sum up the fields lying in one orbit under $[\phi_\frac{3}{2}]$ and arrive at the following natural definitions:

\[
\begin{align*}
[\Phi^0_0] &:= \frac{1}{2}([\phi_0] + [\phi_\frac{3}{2}]), \\
[\Phi^0_{-\frac{3}{2}}] &:= \frac{1}{2}([\phi_\frac{-3}{2}] + [\phi_0]), \\
[\Phi^{NS}_0] &:= \frac{1}{2}([\phi_\frac{-3}{2}] + [\phi_\frac{3}{2}]), \\
[\Phi^{NS}_{\frac{3}{2}}] &:= \frac{1}{2}([\phi_\frac{-3}{2}] + [\phi_0]).
\end{align*}
\]

With this definition we recover exactly the fusion algebra (A.1) of the $N = 1$ super Virasoro minimal model $c(2, 12)$ from the fusion algebra (A.2) of the $\mathcal{W}(2, 4)$ rational model. This verifies the consistency of the formulae presented in section 2.

As a second example we consider the fusion algebra of the $N = 1$ super Virasoro minimal model $c(3, 7) = -\frac{1}{14}$. This model has three HWRs per sector: $\text{NS}: h \in \{0, \frac{7}{4}, -\frac{11}{14}\}, \text{R}: h \in \{\frac{11}{16}, -\frac{3}{12}, \frac{13}{12}\}$. There is no fixed point in the superconformal grid. With the formulae of sections 2 and 4 we obtain the following fusion algebra:

\[
\begin{align*}
[\Phi^{NS}_0][\Phi^{NS}_{\frac{3}{2}}] &= [\Phi^0_0] + [\Phi^{NS}_0], \\
[\Phi^{NS}_{\frac{3}{2}}][\Phi^{NS}_{\frac{3}{2}}] &= [\Phi^0_0] + 2[\Phi^{NS}_0], \\
[\Phi^0_0][\Phi^0_{-\frac{3}{2}}] &= 2[\Phi^{NS}_0], \\
[\Phi^0_{-\frac{3}{2}}][\Phi^0_{-\frac{3}{2}}] &= 2[\Phi^{NS}_0], \\
[\Phi^{NS}_0][\Phi^{NS}_{-\frac{3}{2}}] &= 2[\Phi^{NS}_0] + 2[\Phi^{NS}_{\frac{3}{2}}].
\end{align*}
\]

We already pointed out that this fusion algebra has a $\mathbb{Z}_2$-structure. As mentioned above the bosonic projection of the $N = 1$ super Virasoro algebra yields a $\mathcal{W}(2, 4)$ for $c = -\frac{1}{14}$. This value of the central charge is also contained in the minimal series of the Virasoro algebra and can be parametrized by $c = c_{\mathcal{W}}(7, 12) = -\frac{1}{14}$. The calculations in [16] showed that $\mathcal{W}(2, 4)$ has a rational model at this value of $c$ and that the $\mathcal{W}$-characters diagonalize the modular invariant partition function $(E_6, A_6)$ given by:

\[
Z = \sum_{s=1}^{3} \left( |\chi_{1,s} + \chi_{7,s}|^2 + |\chi_{4,s} + \chi_{8,s}|^2 + |\chi_{5,s} + \chi_{11,s}|^2 \right).
\]

Hence the $\mathcal{W}(2, 4)$ characters read:

\[
\chi^{\mathcal{W},1}_s = \chi_{1,s} + \chi_{7,s} \quad \chi^{\mathcal{W},2}_s = \chi_{4,s} + \chi_{8,s} \quad \chi^{\mathcal{W},3}_s = \chi_{5,s} + \chi_{11,s} \quad s = 1, 2, 3.
\]

Thus, this $\mathcal{W}(2, 4)$ minimal model has the following 9 HWRs:

$h \in \{0, \frac{3}{2}, \frac{7}{4}, \frac{11}{14}, -\frac{1}{14}, \frac{3}{7}, \frac{11}{16}, -\frac{3}{12}, \frac{13}{12} \}$. Using the $S$-matrix for Virasoro minimal models and the ordinary Verlinde formula one arrives – after performing the change of basis (A.5) – at the following fusion algebra for the $\mathcal{W}(2, 4)$-minimal model $c = -\frac{11}{14}$:
\[
\begin{align*}
[\Phi_2^\pm][\Phi_2^\pm] &= [\Phi_0] \\
[\Phi_2^\pm][\Phi_2^\mp] &= [\Phi_\mp] \\
[\Phi_2^\pm][\Phi_2^\pm] &= [\Phi_1] \\
[\Phi_2^\pm][\Phi_2^\mp] &= [\Phi_{-1}] \\
[\Phi_2^\pm][\Phi_{1/2}] &= [\Phi_{1/2}] \\
[\Phi_2^\pm][\Phi_{5/2}] &= [\Phi_{5/2}] \\
\end{align*}
\]

Once again we recognize that the field \([\Phi_2^\pm]\) is a ‘simple current’ which reflects the additional supersymmetry of this model. Using the above conclusions and summing up the fields belonging to the same orbit under the action of \([\Phi_2^\pm]\), we make the following definition:

\[
\begin{align*}
[\Phi_0^{NS}] := \frac{1}{2} \left( [\Phi_0] + [\Phi_\mp] \right) \\
[\Phi_2^{NS}] := \frac{1}{2} \left( [\Phi_2^\pm] + [\Phi_{1/2}] \right) \\
[\Phi_{-1}^{NS}] := \frac{1}{2} \left( [\Phi_{-1}] + [\Phi_\mp] \right) \\
\end{align*}
\]

Using this definition we recover the fusion algebra (A.4) from the fusion algebra (A.6) of the \(\mathcal{W}(2, 4)\)-minimal model.
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