GLOBAL SYMPLECTIC COORDINATES ON GRADIENT KÄHLER–RICCI SOLITONS

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Abstract. A classical result of D. McDuff [14] asserts that a simply-connected complete Kähler manifold \((M,g,\omega)\) with non positive sectional curvature admits global symplectic coordinates through a symplectomorphism \(\Psi: M \rightarrow \mathbb{R}^{2n}\) (where \(n\) is the complex dimension of \(M\)), satisfying the following property (proved by E. Ciriza in [4]): the image \(\Psi(T)\) of any complex totally geodesic submanifold \(T \subset M\) through the point \(p\) such that \(\Psi(p) = 0\), is a complex linear subspace of \(\mathbb{C}^n \simeq \mathbb{R}^{2n}\). The aim of this paper is to exhibit, for all positive integers \(n\), examples of \(n\)-dimensional complete Kähler manifolds with non-negative sectional curvature globally symplectomorphic to \(\mathbb{R}^{2n}\) through a symplectomorphism satisfying Ciriza’s property.

1. Introduction

D. McDuff [14] (see also [1]) proved a global version of Darboux theorem for \(n\)-dimensional complete and simply-connected Kähler manifolds with nonpositive sectional curvature. She shows that there exists a diffeomorphism \(\Psi: M \rightarrow \mathbb{R}^{2n} = \mathbb{C}^n\) satisfying and \(\Psi^*(\omega_0) = \omega\), where \(\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j\) is the standard symplectic form on \(\mathbb{R}^{2n}\). The interest for these kind of questions comes, for example, after Gromov’s discovery [9] of the existence of exotic symplectic structures on \(\mathbb{R}^{2n}\). E. Ciriza [4] (see also [3] and [5]) proves that the image \(\Psi(T)\) of any complete complex and totally geodesic submanifold \(T \subset M\) passing through the point \(p\) such that \(\Psi(p) = 0\), is a complex linear subspace of \(\mathbb{C}^n\). A global symplectomorphism satisfying this property has been constructed by the first author and A. J. Di Scala [7] (see also [8]) for Hermitian symmetric spaces of noncompact type and by the authors of the present paper for the Calabi’s inhomogeneous Kähler–Einstein metric on tubular domains (cfr. [12]). It is then natural and interesting to investigate the existence of positively curved complete Kähler manifolds globally symplectomorphic to \(\mathbb{R}^{2n}\) through a symplectomorphic satisfying the above Ciriza’s property. In this paper we construct explicit
global symplectic coordinates for the positively curved complete gradient Kähler–Ricci solitons built by H. D. Cao in [2]. Moreover, we exhibit, for all positive integers \( n \), an example of gradient Kähler–Ricci solitons (the product of \( n \) copies of the Cigar soliton) where Ciriza’s property holds true. Our results are summarized in the following two theorems (see next section for details and terminology).

**Theorem 1.** A gradient Kähler–Ricci soliton \((C^n, \omega_{RS})\) is globally symplectomorph to \((\mathbb{R}^{2n}, \omega_0)\).

**Theorem 2.** Let \((C^n, \omega_{C,n})\) be the product of \( n \) copies of the Cigar soliton. Then there exists a symplectomorphism \( \Psi_{C,n} : (C^n, \omega_{C,n}) \to (\mathbb{R}^{2n}, \omega_0) \), with \( \Psi_{C,n}(0) = 0 \), taking complete complex totally geodesic submanifolds through the origin to complex linear subspaces of \( C^n \cong \mathbb{R}^{2n} \).

The paper consists of two other sections containing respectively the basic material on gradient Kähler–Ricci solitons and the proofs of the main results.

### 2. Gradient Kähler–Ricci solitons

We recall here what we need about the gradient Kähler–Ricci solitons described by H-D. Cao in [2] (to whom we refer for references and further details). Let \( g_{RS} \) be the Kähler metric on \( C^n \) generated by the radial Kähler potential \( \Phi(z, \bar{z}) = u(t) \), where for all \( t \in (-\infty, +\infty) \), \( u \) is a smooth function of \( t = \log(||z||^2) \) and as \( t \to -\infty \) it has an expansion:

\[
u(t) = a_0 + a_1 e^t + a_2 e^{2t} + \ldots, \quad a_1 = 1.
\]  

(1)

Denote by \( \omega_{RS} = \frac{i}{2} \partial \bar{\partial} \Phi \) the Kähler form associated to \( g_{RS} \). If \( u \) satisfies the equation:

\[(u')^{n-1} u'' e^u = e^{nt},\]

then the conditions:

\[u'(t) > 0, \quad u''(t) > 0, \quad \forall t \in (-\infty, +\infty),\]

(2)

\[
\lim_{t \to +\infty} \frac{u'(t)}{t} = n, \quad \lim_{t \to +\infty} u''(t) = n.
\]

(3)

are fulfilled and \((C^n, \omega_{RS})\) is a gradient Kähler–Ricci soliton. The metric \( g_{RS} \) is complete and positively curved and for \( n = 1 \) one recovers the Cigar metric on \( C \) whose associated Kähler form reads:

\[
\omega_C = \frac{dz \wedge d\bar{z}}{1 + ||z||^2},
\]

which was introduced by Hamilton in [10] as first example of Kähler–Ricci soliton on non-compact manifolds. Observe that a Kähler potential for \( \omega_C \) is given by (see also [15]):

\[
\Phi_C = \int_0^{||z||} \log(1 + s^2) \frac{ds}{s}.
\]
Furthermore, in this case the Riemannian curvature reads:

$$R = \frac{1}{(1 + |z|^2)^3}. \quad (4)$$

It is interesting observing that the Kähler metric $\omega_{C,n}$ on $\mathbb{C}^n = \frac{i}{2} \partial \bar{\partial} \Phi_{C,n}$ defined as product of $n$ copies of Cigar metric $\omega_C$, satisfies $\Phi_{C,n} = \Phi_C \oplus \cdots \oplus \Phi_C$ and it is still a complete and positively curved (i.e. with non-negative sectional curvature) gradient Kähler–Ricci soliton, namely it satisfies (1), (2) and (3) above. In particular its Riemannian tensor satisfies $R_{ijk\ell} = 0$ whenever one of the indexes is different from the others and by (4) it is easy to see that the nonvanishing components are given by:

$$R_{jjjj} = \frac{1}{(1 + |z|^2)^3}. \quad (5)$$

3. PROOF OF THE MAIN RESULTS

In [13] the first author of the present paper, jointly with F. Zuddas, proved the following result on the existence of a symplectomorphism between a rotation invariant Kähler manifold of complex dimension $n$ and $(\mathbb{R}^{2n}, \omega_0)$. For the reader’s convenience, we summarize here that result and the proof in the case when the manifold is $\mathbb{C}^n$. This will be the main ingredient in the proof of our main results.

**Lemma 3.** Let $\omega_\Phi = \frac{i}{2} \partial \bar{\partial} \Phi$ be a rotation invariant Kähler form on $\mathbb{C}^n$ i.e. the Kähler potential only depends on $|z_j|^2$, $j = 1, \ldots, n$.\footnote{Notice that the rotation invariant condition on the potential $\Phi$ is more general than the radial one which requires $\Phi$ depending only on $|z_1|^2 + \cdots + |z_n|^2$.} If

$$\frac{\partial \Phi}{\partial |z_k|^2} \geq 0, \quad k = 1, \ldots, n. \quad (6)$$

then the map:

$$\Psi : (M, \omega_\Phi) \to (\mathbb{C}^n, \omega_0), \quad z = (z_1, \ldots, z_n) \mapsto (\psi_1(z)z_1, \ldots, \psi_n(z)z_n),$$

where

$$\psi_j = \sqrt{\frac{\partial \Phi}{\partial |z_j|^2}}, \quad j = 1, \ldots, n,$

is a symplectic immersion. If in addition:

$$\lim_{z \to +\infty} \sum_{j=1}^n \frac{\partial \Phi}{\partial |z_j|^2} |z_j|^2 = +\infty, \quad (7)$$

then $\Psi$ is a global symplectomorphism.
Proof. Assume condition (6) holds true. Let us prove first that $F^*\omega_0 = \omega$. We have:

$$
\begin{align*}
\Psi^*\omega_0 &= \frac{i}{2} \sum_{j=1}^{n} d\Psi_j \wedge d\bar{\Psi}_j \\
&= \sum_{j=1}^{n} \left( \frac{\partial \Psi_j}{\partial z_j} dz_j + \frac{\partial \Psi_j}{\partial \bar{z}_j} d\bar{z}_j \right) \wedge \left( \frac{\partial \bar{\Psi}_j}{\partial z_j} dz_j + \frac{\partial \bar{\Psi}_j}{\partial \bar{z}_j} d\bar{z}_j \right) \\
&= \sum_{j,k=1}^{n} \left( \left| \frac{\partial \Psi_j}{\partial z_j} \right|^2 - \left| \frac{\partial \Psi_j}{\partial \bar{z}_j} \right|^2 \right) dz_j \wedge d\bar{z}_j
\end{align*}
$$

Since

$$
\frac{\partial \Psi_j}{\partial z_j} = \frac{\partial \psi_j}{\partial z_j} z_j + \psi_j, \quad \frac{\partial \Psi_j}{\partial \bar{z}_j} = \frac{\partial \psi_j}{\partial \bar{z}_j} \bar{z}_j,
$$

and

$$
\frac{\partial \psi_j}{\partial z_j} = \frac{1}{2} \psi_j^{-1} \left( \frac{\partial^2 \Phi}{\partial |z_j|^4} \right) z_j,
$$

it follows:

$$
\begin{align*}
\Psi^*\omega_0 &= \sum_{j=1}^{n} \left( \left| \frac{\partial \psi_j}{\partial z_j} z_j + \psi_j \right|^2 - \left| \frac{\partial \psi_j}{\partial \bar{z}_j} \bar{z}_j \right|^2 \right) dz_j \wedge d\bar{z}_j \\
&= \sum_{j=1}^{n} \left( \left| \frac{\partial \psi_j}{\partial z_j} z_j + \frac{\partial \psi_j}{\partial \bar{z}_j} \bar{z}_j \right| \bar{z}_j + \psi_j^2 \right) dz_j \wedge d\bar{z}_j \\
&= \sum_{j=1}^{n} \left( \left( \frac{\partial^2 \Phi}{\partial |z_j|^4} \right) |z_j|^2 + \left( \frac{\partial \Phi}{\partial |z_j|^2} \right) \right) dz_j \wedge d\bar{z}_j \\
&= \sum_{j=1}^{n} \frac{\partial^2 \Phi}{\partial z_j \partial \bar{z}_j} dz_j \wedge d\bar{z}_j.
\end{align*}
$$

Observe now that since $\omega$ and $\omega_0$ are non-degenerate, it follows by the inverse function theorem that $\Psi$ is a local diffeomorphism. If in addition condition (7) holds true, then $\Psi$ is a proper map and hence a global diffeomorphism.

We are now in the position of proving Theorem 1.

**Proof of Theorem 1.** Let $\Phi(z, \bar{z}) = u(t)$, where $u(t)$ is given by (1). Then for all $j = 1, \ldots, n$

$$
\frac{\partial \Phi}{\partial |z_j|^2} = \frac{\partial \Phi}{\partial ||z||^2} = \frac{u'(\log(||z||^2))}{||z||^2},
$$

which is greater than zero for all $||z||^2 \neq 0$ by (2), and evaluated at $||z||^2 = 0$ gives the value 1 by (5). Notice now that by the first of the limit conditions
given in \([3]\) it follows that condition \([7]\) in Lemma \([3]\) holds true. Therefore by Lemma \([3]\) the map:

\[
F : (\mathbb{C}^n, g_{RS}) \rightarrow (\mathbb{R}^{2n}, g_0), \quad z = (z_1, \ldots, z_n) \mapsto \frac{u'(\log(||z||^2))}{||z||^2}(z_1, \ldots, z_n),
\]

is the desired global symplectomorphism. \(\square\)

In order to prove Theorem \([2]\) we need the following lemma which classifies all totally geodesic submanifolds of \((\mathbb{C}^n, \omega_{C,n})\) through the origin.

**Lemma 4.** Let \(S\) be a totally geodesic complex submanifold (of complex dimension \(k\)) of \((\mathbb{C}^n, \omega_{C,n})\). Then, up to unitary transformation of \(\mathbb{C}^n\), \(S = (\mathbb{C}^k, \omega_{C,k})\).

**Proof.** Let us first prove the statement for \(n = 2\). For \(k = 0, 2\) there is nothing to prove, thus fix \(k = 1\). Let

\[
f : (S, \tilde{\omega}) \rightarrow (\mathbb{C}^2, \omega_{C,2}), \quad f(z) = (f_1(z), f_2(z)).
\]

be a totally geodesic embedding of a 1-dimensional complex manifold \((S, \tilde{\omega})\) into \((\mathbb{C}^2, \omega_{C,2})\). By \(\tilde{\omega} = f^* (\omega_{C,2})\) we get:

\[
\tilde{\omega} = \frac{i}{2} \left( \frac{1}{1 + |f_1(z)|^2} \left| \frac{\partial f_1}{\partial z} \right|^2 + \frac{1}{1 + |f_2(z)|^2} \left| \frac{\partial f_2}{\partial z} \right|^2 \right) dz \wedge d\bar{z}. \quad (8)
\]

Let \(\tilde{R}, R_C\) be the curvature tensor of \((S, \tilde{\omega})\) and \((\mathbb{C}^2, \omega_C)\) respectively. Since \((S, \tilde{\omega})\) is totally geodesic in \((\mathbb{C}^2, \omega_C)\) we have

\[
\tilde{R}(X, JX, X, JX) = R_C(X, JX, X, JX)
\]

for all the vector fields \(X\) on \(S\) (see e.g. \([11]\) p. 176). Taking \(X = \partial / \partial z\), we have:

\[
\tilde{R} \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial z'}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z'} \right) = -\frac{\partial^2 \tilde{g}}{\partial z \partial \bar{z}} + \tilde{g}^{-1}(z) \left| \frac{\partial \tilde{g}(z)}{\partial z} \right|^2,
\]

where \(\tilde{g}\) is the Kähler metric associated to \(\tilde{\omega}\), i.e.

\[
\tilde{g} = \left| \frac{\partial f_1}{\partial z} \right|^2 \frac{1}{1 + |f_1(z)|^2} + \left| \frac{\partial f_2}{\partial z} \right|^2 \frac{1}{1 + |f_2(z)|^2}.
\]

Further, since the vector field \(\frac{\partial}{\partial z}\) corresponds through \(df\) to \(\frac{\partial f_1}{\partial z} \frac{\partial}{\partial z} + \frac{\partial f_2}{\partial z} \frac{\partial}{\partial z'}\), by \([5]\) we get:

\[
R_C \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial z'}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z'} \right) = \left| \frac{\partial f_1}{\partial z} \right|^4 \frac{1}{(1 + |f_1(z)|^2)^3} + \left| \frac{\partial f_2}{\partial z} \right|^4 \frac{1}{(1 + |f_2(z)|^2)^3}.
\]

Since

\[
\frac{\partial \tilde{g}}{\partial z} = \sum_{j=1}^{2} \left( \frac{2}{1 + |f_j(z)|^2} \frac{\partial f_j}{\partial z} \frac{\partial^2 f_j}{\partial z^2} - \left| \frac{\partial f_j}{\partial z} \right|^2 \frac{\tilde{j}_j}{(1 + |f_j(z)|^2)^2} \frac{\partial f_j}{\partial z} \right),
\]

...
\[\frac{\partial^2 \tilde{g}}{\partial z \partial \bar{z}} = \sum_{j=1}^{2} \left[ \left| \frac{\partial f_j}{\partial z} \right|^4 \frac{2 |f_j|^2}{(1 + |f_j(z)|^2)^3} + \left| \frac{\partial^2 f_j}{\partial z^2} \right|^2 \frac{1}{1 + |f_j(z)|^2} + \frac{1}{(1 + |f_j|^2)^2} \left( \tilde{f}_j \left( \frac{\partial f_j}{\partial z} \right)^2 \frac{\partial^2 f_j}{\partial z^2} + \left| \frac{\partial f_j}{\partial z} \right|^4 + f_j \left( \frac{\partial f_j}{\partial z} \right)^2 \frac{\partial^2 f_j}{\partial z^2} \right) \right] \]

after a long but straightforward computation, we get that \(\tilde{R} \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right) - RC \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right)\) assumes the form:

\[
-|A(f_1, f_2)|^2 \left( \frac{\partial f_1}{\partial z} \right)^2 (1 + |f_2|^2) + \left| \frac{\partial f_2}{\partial z} \right|^2 (1 + |f_1|^2) \right) (1 + |f_1|^2)^2 (1 + |f_2|^2)^2
\]

where

\[A(f_1, f_2) = \left( \frac{\partial^2 f_2}{\partial z^2} \frac{\partial f_1}{\partial z} - \frac{\partial^2 f_1}{\partial z^2} \frac{\partial f_2}{\partial z} \right) (1 + |f_1|^2)(1 + |f_2|^2) + \left( \frac{\partial f_1}{\partial z} \right)^2 \frac{\partial f_2}{\partial z} \tilde{f}_1 (1 + |f_2|^2) - \left( \frac{\partial f_2}{\partial z} \right)^2 \frac{\partial f_1}{\partial z} \tilde{f}_2 (1 + |f_1|^2).\]

Thus, \(\tilde{R} \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right) - RC \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right) = 0\) iff \(A(f_1, f_2) = 0\), i.e. iff

\[
\frac{\partial f_1}{\partial z} (1 + |f_2|^2) \left( \frac{\partial^2 f_2}{\partial z^2} (1 + |f_1|^2) + \frac{\partial f_1}{\partial z} \frac{\partial f_2}{\partial z} \tilde{f}_1 \right) = \frac{\partial f_2}{\partial z} (1 + |f_1|^2) \left( \frac{\partial^2 f_1}{\partial z^2} (1 + |f_2|^2) + \frac{\partial f_2}{\partial z} \frac{\partial f_1}{\partial z} \tilde{f}_2 \right),
\]

which is verified whenever one between \(f_1(z)\) and \(f_2(z)\) is constant (and thus zero since we assume \(f(0, 0) = 0\), or when \(f_1(z) = f_2(z)\)). In order to prove that these are the only solutions, write (9) as

\[
\frac{\partial f_1}{\partial z} (1 + |f_2|^2) \frac{\partial}{\partial z} \left( \frac{\partial f_2}{\partial z} (1 + |f_1|^2) \right) = \frac{\partial f_2}{\partial z} (1 + |f_1|^2) \frac{\partial}{\partial z} \left( \frac{\partial f_1}{\partial z} (1 + |f_2|^2) \right).
\]

Assuming \(f_1, f_2\) not constant, it leads to the equation:

\[
\left( \frac{\partial f_1}{\partial z} (1 + |f_2|^2) \right)' \frac{\partial f_2}{\partial z} (1 + |f_1|^2)^2 = 0,
\]

which implies that for some complex constant \(\lambda \neq 0\),

\[
\frac{\partial f_1}{\partial z} (1 + |f_2|^2) = \lambda \frac{\partial f_2}{\partial z} (1 + |f_1|^2),
\]

that is:

\[
\frac{\partial \log f_1}{\partial z} \tilde{f}_1 = \lambda \frac{\partial \log f_2}{\partial z} \tilde{f}_2.
\]

Comparing the antiholomorphic parts we get \(\tilde{f}_1 = \alpha \tilde{f}_2\), for some complex constant \(\alpha\). Substituting in (10) we get:

\[
\alpha (1 + |f_2|^2) = \lambda (1 + |\alpha|^2 |f_2|^2).
\]
Since \( f(0,0) = 0 \), from this last equality follows \( \alpha = \lambda \) and thus immediately \( |\alpha|^2 = 1 \). We have been proven that a totally geodesic submanifold of \((\mathbb{C}^2, \omega_{C,2})\) is, up to unitary transformation of \(\mathbb{C}^2\), \((\mathbb{C}, \omega_C)\) realized either via the map \(z \mapsto (f_1,0)\) (or equivalently \(z \mapsto (0, f_1)\)) or via \(z \mapsto (f_1(z), \alpha f_1(z))\), with \(|\alpha|^2 = 1\).

Assume now \(S\) to be a \(k\)-dimensional complete totally geodesic complex submanifold of \((\mathbb{C}^n, \omega_{C,n})\) and let \(\pi_j, j = 1, \ldots, n\), be the projection into the \(j\)th \(\mathbb{C}\)-factor in \(\mathbb{C}^n\), \(\pi_{jk} j, k = 1, \ldots, n\), the projection into the space \(\mathbb{C}^2\) corresponding to the \(j\)th and \(k\)th \(\mathbb{C}\)-factors. Since \(\pi_j(S), j = 1, \ldots, n\), is totally geodesic into \((\mathbb{C}, \omega_C)\), it is either a point or the whole \(\mathbb{C}\). Thus, up to unitary transformation of the ambient space, we can assume \(S\) to be of the form:

\[
(z_1, \ldots, z_k) \mapsto (0, \ldots, 0, h_{11}(z_1), \ldots, h_{1r}(z_1), \ldots, h_{k1}(z_k), \ldots, h_{ks}(z_k)).
\]

(11)

Since also the projections \(\pi_{jk}(S)\) have to be totally geodesic into \((\mathbb{C}^2, \omega_{C,2})\), by what we have proven for \(n = 2\), we can reduce (11) into the form:

\[
(z_1, \ldots, z_k) \mapsto (0, \ldots, 0, h_1(z_1), \ldots, \alpha_s h_1(z_1), \ldots, h_k(z_k), \ldots, \alpha_s h_k(z_k)),
\]

where \(|\alpha_t|^2 = 1\) for all \(t\) appearing above. Thus, either \(S = (\mathbb{C}^k, \omega_{C,k})\) or \(S\) is a \(k\)-dimensional diagonal, which with a suitable unitary transformation can be written again as \((\mathbb{C}^k, \omega_{C,k})\), and we are done.

**Proof of Theorem** The existence of a global symplectomorphism \(\Psi_{C,n} : (\mathbb{C}^n, \omega_{C,n}) \to (\mathbb{R}^{2n}, \omega_0)\) is guaranteed again by Lemma \(3\). In fact for all \(j = 1, \ldots, n\)

\[
\frac{\partial}{\partial |z_j|^2} \Phi_{C,n} = 2 \frac{\partial}{\partial |z_j|^2} \sum_{j=1}^n \int_0^{|z_j|} \frac{\log(1 + s^2)}{s} ds = \frac{1}{|z_j|} \frac{d}{d|z_j|} \int_0^{|z_j|} \frac{\log(1 + s^2)}{s} ds = \frac{\log(1 + |z_j|^2)}{|z_j|^2} > 0.
\]

Moreover, condition (7) in Lemma \(3\) is fulfilled by:

\[
\lim_{z \to +\infty} |z_j|^2 \sum_{j=1}^n \frac{\partial \Phi_{C,n}}{\partial |z_j|^2} = \lim_{z \to +\infty} \sum_{j=1}^n \log(1 + |z_j|^2) = +\infty.
\]

Thus by Lemma \(3\) the map:

\[
\Psi_{C,n} : (\mathbb{C}^n, \omega_{C,n}) \to (\mathbb{R}^{2n}, \omega_0), \ z = (z_1, \ldots, z_n) \mapsto (\psi_1(z_1)z_1, \ldots, \psi_n(z_n)z_n),
\]

with

\[
\psi_j = \sqrt{\frac{\log(1 + |z_j|^2)}{|z_j|^2}},
\]

is a global symplectomorphism.
In order to prove the second part of the theorem, let \( S \) be a \( k \) dimensional totally geodesic complex submanifold of \((\mathbb{C}^n, \omega_{\mathbb{C},n})\) through the origin, which by Lemma 4 is given by \((\mathbb{C}^k, \omega_{\mathbb{C},k})\). The image \( \Psi_{\mathbb{C},n}(S) \) is of the form:

\[
\left( \sqrt{\frac{\log(1 + |z_1|^2)}{|z_1|^2}} z_1, \ldots, \sqrt{\frac{\log(1 + |z_k|^2)}{|z_k|^2}} z_k, 0, \ldots, 0 \right) \cong \mathbb{C}^k,
\]

concluding the proof. \( \square \)

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