Spontaneous symmetry breakings in the singlet scalar Yukawa model within the auxiliary field method

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Abstract

The aim of this work is to investigate the occurrence of two different spontaneous symmetry breakings at two levels of the description of fermion-scalar field model, by means of a set of gap equations and with a background field effective action. For that, we consider the Yukawa model, as a toy model for interactions between non-massive fermions intermediated by a self-interacting real scalar field. This model has at stakes two symmetries at the classical level that, as we know, might be spontaneously or dynamically broken with mass generation for the particles. The auxiliary field method is considered and it produces coupled renormalized gap equations. The effective action is then written with quantum contributions by the external background field method. We brought to light how the renormalization procedure affects the physical gaps, investigate its properties, and discuss the connection between the auxiliary fields not only to define composite states but also to compute the effective action.

Keywords: Quantum Field Theory, Yukawa model, Spontaneous symmetry breaking, Mechanism of masses of particles, Coupling constants, Low energy effective model
1 Introduction

A well known method to deal with polynomial self interactions in field theories is the auxiliary field method (AFM), or the so called Hubbard-Stratonovich (HS) transformations \[1,2\], that provides an interesting way to deal with the non linear dynamics. This method is also suitable to investigate spontaneous or dynamical symmetry breakings (SSB or DSB) that have important roles inside the Standard Model (SM) \[3,4\]. The auxiliary field method may have a particular extra advantage that is to define the composite states that might correspond to quasi-particles of the system, as it is the case for the definition of mesons states in Strong Interactions \[5-7\]. This method might be extended for higher order interactions in different ways \[8-10\]. Besides that, observables in a quantum field theory might be described in terms of vacuum expected values of composite fields that can give rise to different condensates \[11,12\]. In the present work we intend to revisit some of these issues. For that, we will consider the Yukawa model (YuM) with a self interacting scalar singlet wherein we consider massless fermions whose masses are generated by SSB \[13\]. This model can be seen as a prototype to understand the Higgs sector of the SM \[13,14\] and, for example, to address the role of discretization of fermions in lattices \[15,16\]. In this program, the spontaneous symmetry breaking of the $\mathbb{Z}_2$ discrete symmetry takes place \[17\]. Besides the interest in considering toy models, such as the scalar singlet Higgs-Yukawa, to develop analytical techniques and to have a path to analyze phenomenology, extensions of the Standard Model with an additional singlet Higgs have been envisaged \[18\]. As a model, it might eventually reproduce aspects of the fundamental and more intrincated theory. It has also been considered, for example, for the analysis of a scalar field with some modifications \[19\], in an external gravitational field \[20\], analyzed by means of holography \[21\], envisaged for dark matter investigations \[22\] and its phase diagram has been addressed extensively in the large $N_f$ limit \[23,24\]. In spite of the different approaches employed to understand further the YuM and its scalar sector, one can rather reach also upper and lower bounds for masses and coupling constants \[25,26\]. Its renormalization has been very often employed at one loop level by means of the effective action technique introduced by Coleman-Weinberg to describe the origin of spontaneous symmetry breaking \[27\]. One can expect that the YuM, in particular its ground state, eventually can be suitably described by a series of condensates, $<\phi>$, $<\phi^2>$, $<\bar{\psi}\psi>$ and so on. These three condensates, by the way, can be considered the leading lower dimensional ones. The other one would be $<\phi^3>$ that will be considered to be factorized into the first two of them.

The exchange of scalar field in the Yukawa model provides a mechanism for fermion interaction that cannot be currently tested, although power counting arguments can also lead to quark contact interactions at the energies scales of LHC \[28\]. The Higgs particle might participate into this type of mechanism for heavier quarks. The investigation of the role of the quartic vertices and their relationship with symmetry and mass has a relevant role in physics, corresponding to an effective description of the interactions that eventually should find justification in a more fundamental boson mediation processes explored in many other contexts \[29-34\]. The relation of fundamental (renormalizable) theories and the effective models has been explored in the last decades and it helped the construction
of the SM with its interplay with phenomenology and therefore it allowed the discovery of many effects and phenomena. Effective models are expected to be valid in a restricted range of energies, usually low energies with respect to an energy/momentum scale or cut-off $\Lambda$. Usually they are non-renormalizable. The model may be renormalized at a particular level of calculation and for each new quantum correction there might arise the need of systematic changes or corrections [11,35–40]. Although the SSB effect keeps some different characteristics from Dynamical Chiral Symmetry Breaking (DChSB), they have several properties in common that might be implemented not only in the full version of the SM but also in effective models [41]. The occurrence of SSB or dynamical symmetry breaking is usually directly related to the phenomenology of gap equations, widely present in effective models for the strong and weak interactions, with the implicit discussion of the mechanisms for mass generation. Historically this has been initially envisaged in the scalar case [42] and also in the vector case with its connection with the gauge symmetry [43,44]. Later the renormalization of the scalar or gauge theories with and without symmetry breaking have been established [45–50].

In this work, we investigate the Yukawa interaction between massless fermions and a self-interacting scalar by means of the auxiliary field technique. When reducing the original model by the auxiliary composite fields, both mechanisms of spontaneous and dynamical symmetry breakings, providing mass generations, can be explored independently or simultaneously, depending on the values of the masses and coupling constants. As a second level of analysis, we address the renormalization aspect of the resulting auxiliary field effective action, using the logics of [51] for a different model. We show the consequences of the gap renormalization, unveiling the properties of the renormalized fermion-boson system, for a single component scalar field. We find out the link between the auxiliary fields and the vertices in the effective action by the current expansion methodology. The paper is organized as follows: In Sec. 2 we derive the coupled renormalized gap equations: one for the YuM composite scalar field, $\Psi$, and the other for a composite fermion condensate $<\bar{\psi}\psi>$ as a single flavor chiral condensate. We investigate too the proprieties of the coupled gap equations for particular limits of $m_R$, $\lambda_R$ and $g_R$. In Sec. 3 we write the effective action from quantum contributions considering background field methods and study the consequence of renormalization in the masses and coupling constants. We explore also the conditions for the existence of two-boson state and fermion-antifermion state. For that, mixing interaction between the composite fields describing the two-boson and two-fermion states is identified. In Sec. 4 there are final remarks.

2 Composite fields and the coupled gap equations

By starting with the YuM with SSB one reaches a non-renormalizable effective model that includes the Fermi-type of fermion interactions, with the contribution of a boson condensate $\phi_0 = <\bar{\psi}\psi>$. The subscript 0 indicates a condensate valued in the vacuum. In the limit of a resulting effective four-fermion interaction reasonably strong, one can also obtain a DChSB.
2.1 Composite-scalar field

The generating functional of the Yukawa model for massless fermions coupled to massive self interacting scalar field can be written as:

\[ Z = N \int D[\phi, \bar{q}, q] \exp[i \int d^4x (\mathcal{L} + \mathcal{L}_s)], \]

\[ \mathcal{L} = \bar{q}(i\partial - g\phi)q + \frac{1}{2}(\partial_\mu \phi)^2 + m^2 \phi^2 - \frac{\lambda}{4!} \phi^4, \]  

wherein the scalar and fermion field sources (\( J \) and \( \eta, \bar{\eta} \)) are encoded in the term: \( \mathcal{L}_s = (J\phi + \bar{\eta}q + \bar{q}\eta) \). The scalar field sector is subject to the spontaneous symmetry breaking depending on the scalar field mass, therefore at this tree level the usual conditions for the emergence of the so-called scalar field condensate is the following:

\[ \bar{\phi}_0^2 = \frac{12m^2}{\lambda}, \]  

(2.2)

in which one needs \( m^2 > 0 \). These conditions will receive corrections due to the quantization of the scalar and of the fermion fields and it will be discussed again below.

With the renormalization procedure we establish the following relation between the naked and dressed fields and parameters:

\[ \phi = \mathcal{Z}_\phi^{\frac{1}{2}} \phi_R, \quad q = \mathcal{Z}_q^{\frac{1}{2}} q_R, \quad \bar{q} = \mathcal{Z}_{\bar{q}}^{\frac{1}{2}} \bar{q}_R; \]

\[ m = \mathcal{Z}_m \frac{1}{\mathcal{Z}_\phi} m_R, \quad \lambda = \mathcal{Z}_{\lambda} \frac{1}{\mathcal{Z}_\phi^2} \lambda_R, \quad g = \mathcal{Z}_g \frac{1}{\mathcal{Z}_q \mathcal{Z}_\phi} g_R. \]  

(2.3)

With these redefinitions the model is then be written by:

\[ \mathcal{L} = \mathcal{L}_q \mathcal{Z}_{q}^{\frac{1}{2}} \bar{q}_R i\partial q_R - \mathcal{Z}_s \mathcal{Z}_{s}^{\frac{1}{2}} \bar{q}_R q_R + \mathcal{Z}_\phi \frac{1}{2} (\partial_\mu \phi_R)^2 + \mathcal{Z}_m m_R^2 \phi_R^2 - \mathcal{Z}_\lambda \frac{\lambda_R}{4!} \phi_R^4. \]  

(2.4)

wherein we see a linear combination of all terms of the original Lagrangian respecting the discrete \( Z_2 \) symmetry and global charge conservation. In terms of the counter-terms notation \( \mathcal{Z}_i = 1 + \delta \mathcal{Z}_i \) we write the previous equation as

\[ \mathcal{L} = \mathcal{L}_R^{\text{free}} + \mathcal{L}_R^{\text{ct}}, \]

\[ \mathcal{L}_R^{\text{ct}} = \delta \mathcal{Z}_q \mathcal{Z}_{q}^{\frac{1}{2}} \bar{q}_R i\partial q_R - \delta \mathcal{Z}_s \mathcal{Z}_{s}^{\frac{1}{2}} \bar{q}_R q_R + \delta \mathcal{Z}_\phi \frac{1}{2} (\partial_\mu \phi_R)^2 + \delta \mathcal{Z}_m m_R^2 \phi_R^2 - \delta \mathcal{Z}_\lambda \frac{\lambda_R}{4!} \phi_R^4. \]  

(2.5)

From now on, we will only present the renormalization of some of the resulting equations in which a more complete discussion can be done.

With the background or external field method the scalar field can be shifted by a background part as \( \phi \rightarrow \phi_0 + \tilde{\phi} \) where \( \phi_0 \) is the background part. In a first analysis \( \phi_0 \) is the classical homogeneous
condensate $\phi_0$, although we let the freedom for receiving corrections latter. The field $\phi$ will be integrated out. To make possible quantization of this scalar field the AFM is used by means of the following multiplicative identity in the generating functional $[1]$

$$1 = N' \int D\Psi \exp \{ i \int d^4x \frac{4!}{\lambda} [\Psi + \frac{\lambda}{4!} (\phi^2 + 2\phi_0\phi)]^2 \}$$  \hspace{1cm} (2.6)

wherein $N'$ is a normalization, $\Psi$ are the auxiliary field and it has a wave function renormalization factor $Z_\Psi$. It yields the following form for the generating functional:

$$Z = N \int D\bar{q}DqD\Psi \exp [i \int d^4x \{ \bar{q}i\partial q - g(\phi_0 + \bar{\phi})q +$$

$$-\square \phi_0\phi + 2m^2 \phi_0 \phi - \frac{\lambda}{3!} \phi_0^3 \phi + 4\phi_0 \Psi \phi +$$

$$\frac{1}{2} \phi (\square - 2m^2 - 4\Psi - 4\frac{\lambda}{4!} \phi_0^2) \phi + \frac{4!}{\lambda} \Psi^2 + \mathcal{L}_s \} \exp (i\Gamma_0),$$  \hspace{1cm} (2.7)

in which $\Gamma_0$ is the effective action that collected all the terms exclusively associated with the background field and that is given by:

$$\Gamma_0 = \int d^4x [\frac{1}{2}(\partial_{\mu}\phi_0)^2 + m^2 \phi_0^2 - \frac{\lambda}{4!} \phi_0^4].$$  \hspace{1cm} (2.8)

To make possible a latter current expansion for eq. (2.7), the scalar field can be exactly integrated out by means of the following field translation:

$$\phi(x) \rightarrow \phi_0(x) + \int d^4y G(x,y) j(y)$$

where

$$j = -g\bar{q}q - \square \phi_0 + 2m^2 \phi_0 - \frac{\lambda}{3!} \phi_0^3 + 4\phi_0 \Psi,$$

and

$$G^{-1}(x,y) = (\square - 2m^2 - 4\Psi - 4\frac{\lambda}{4!} \phi_0^2) \delta^4(x-y).$$  \hspace{1cm} (2.9)

It is interesting to note that:

$$-\frac{1}{2} \phi G^{-1} \phi + j\phi = -\frac{1}{2} \phi_0 G^{-1} \phi_0 + \frac{1}{2} j G j.$$  

The functional generator can then be written as:

$$Z = N \int D\bar{q}DqD\phi_0 D\Psi \exp [i \int d^4x \{ \bar{q}i\partial q - g\phi_0\bar{q}q +$$

$$+ \frac{4!}{\lambda} \Psi^2 + \int d^4y \left( -\frac{1}{2} \phi_0 G^{-1} \phi_0 + \frac{1}{2} j G j \right) + \mathcal{L}_s \} \exp (i\Gamma_0).$$  \hspace{1cm} (2.10)

Therefore it is possible to define a mass for the scalar field in both cases, when the fields $\phi_0$ and $\Psi$ develop or not a non zero expected value in the vacuum as discussed below. In general the scalar field mass for $\phi_0$, by assuming the possibility of non trivial classical solutions, can be written, except for renormalization effects shown below in Section (2.1.2), as:

$$M^2_\phi = 2m^2 + 4(\Psi_0 + \frac{\lambda}{4!} \phi_0^2).$$  \hspace{1cm} (2.11)
In this equation we see the contribution of the SSB to the scalar particle mass by means of $\phi_0$ and the contribution from an eventual contribution of the expected value in the vacuum of the auxiliary field $\Psi_0$ in the case its gap equation, discussed below, presents non trivial solution. It corresponds to consider higher order contribution for the usual mean field solution of the SSB for $\phi$. The composite field $\Psi$ will be analyzed below. In this equation there is a contribution from the spontaneous symmetry breaking ($\phi_0$) and another from the auxiliary field, whenever it develops a non zero classical value in the vacuum. The auxiliary scalar field $\Psi$ corresponds to a two boson quantum state and its dynamics is completely undetermined so far. With the identity $\det A = \exp[tr \ln A]$ we write the effective action as it follows:

$$S_{\text{eff}} = -\frac{i}{2} Tr \ln[G^{-1}(x,y)] + \int d^4x \frac{4!}{\lambda} \Psi(x)^2,$$

and the corresponding extremization that lead to a gap equation, hopefully defining a ground state. It will be given by:

$$\frac{\partial S_{\text{eff}}}{\partial \Psi} |_{\Psi = \Psi_0} = -\frac{i}{[\Box - M^2_\phi]} \delta^4(x-y) + \frac{4!}{\lambda} \Psi_0 = 0,$$

and so, in the momentum representation we have that

$$\Psi_0 = \frac{\lambda}{4!} \int \frac{d^4p}{(2\pi)^4} \left[ p^2 + 2m^2 + 4(\Psi_0 + \frac{\lambda}{4!} \phi_0^2) \right].$$

This might yield a non trivial solution $\Psi_0$ that eventually contributes for the effective mass of the scalar field $\tilde{\phi}_0$ of the expression (2.11). This equation has a quadratic divergence, typical from gap equations.

### 2.1.1 Higgs-type SSB

Above it was assumed that the scalar field might develop a classical value in the vacuum, as discussed for the tree level, in eq. (2.2). Therefore it is also relevant to consider again the equation that define the expected value of the scalar field $\phi_0$ with quantum corrections. By extremizing eq. (2.8) with the one loop corrections (2.12) with respect to it. Then a corrected gap equation arises and it is given by:

$$\frac{\partial S_{\text{eff}}}{\partial \phi} |_{\phi_0} \to \left( m^2 - \frac{\lambda}{12} \phi_0^2 + \frac{i\lambda}{6[\Box - M^2_\phi]} \delta^4(x-y) \right) 2\phi_0 = 0.$$

The solutions for this equation can be written as:

$$\phi_0 = 0, \quad \phi_0^2 = \left( \frac{12m^2}{\lambda} + \frac{2i}{6[\Box - M^2_\phi]} \delta^4(x-y) \right),$$

in which the non-trivial solutions can be written in the momentum representation

$$\phi_0^2 = \frac{12m^2}{\lambda} - \int \frac{d^4p}{(2\pi)^4} \frac{2i}{p^2 + 2m^2 + 4(\Psi_0 + \frac{\lambda}{4!} \phi_0^2)}.$$
By considering eq. (2.14) this last eq. can be written as:

$$\phi_0^2 = \bar{\phi}_0^2 - \frac{48}{\lambda} \Psi_0.$$  \hspace{1cm} (2.18)

This equation makes explicit the role of the auxiliary field $\Psi$ for the SSB. Note that, when computing the effective action for $\phi_0$ in Section (3) a spacetime dependence will be taken into account.

### 2.1.2 Renormalized SSB gap equations

Now we renormalize the previous gap equations in eq. (2.14) and eq. (2.16) respectively. By means of eq. (2.5) and applying the following renormalized (HS) transformation

$$1 = N' \int D\Psi_R \exp \{ i \int d^4x \frac{4!}{\lambda_R} [\mathcal{Z}_\Psi^2 \Psi_R + \mathcal{Z}_\lambda^2 \frac{\Lambda_R}{4!} (\tilde{\phi}_R^2 + 2 \phi_0 \tilde{\phi}_R) ] \}.$$ \hspace{1cm} (2.19)

we have the general result

$$\phi_0^2 (R, c.t) = \frac{\mathcal{Z}_m^2}{\mathcal{Z}_\lambda^2} \frac{12 m_R^2}{\lambda_R} - \frac{\mathcal{Z}_\Psi^2}{\mathcal{Z}_\lambda^2} \frac{48 \Psi_0 (R, c.t)}{\lambda_R},$$

$$\Psi_0 (R, c.t) = \frac{i \mathcal{Z}_\lambda^2 \lambda_R}{24 \mathcal{Z}_\Psi^2 [\mathcal{Z}_\phi \Box - M_\phi^2 (R, c.t)]} \delta^4 (x - y),$$

$$M_\phi^2 (R, c.t) = 2 \mathcal{Z}_m m_R^2 + 4 [\mathcal{Z}_\Psi^2 \mathcal{Z}_\lambda^2 \Psi_0 (R, c.t) + \mathcal{Z}_\lambda^2 \frac{\lambda_R}{4!} \phi_0^2 (R, c.t)],$$ \hspace{1cm} (2.20)

wherein the notation (R,c.t) is to remind us that now the objects are function of dressed quantities (R) and counter-terms (c.t) due to the renormalization factors $\mathcal{Z}_i$. The first equation, for $\phi_0 (R, c.t)$, presents a quadratic divergence in the correction that is renormalized by $\mathcal{Z}_m$ and compatible with a finite mass condition for $m_R$. Although that equation looks like the classical $\lambda \phi^4$ gap equation it is a highly non linear equation - and this is seen in the second of these equations - which reduces to the classical level equation by setting the one loop contribution to zero. The effective mass $M_\phi^2$ dependence on $\phi_0$, however, introduces further non linearities not only because of its dependence on $\phi_0$ but also because it depends on the auxiliary field expected value $\Psi_0$. At this point it is important to note that, the pole of the two point function of $\tilde{\phi}_0$ is real and positive, $M_\phi^2 (R, c.t) > 0$.

### 2.2 Current expansion and fermion-effective action

Consider the non linear term in eq. (2.10), which depends on the fermion current $j$, and that can be written as:

$$\frac{1}{2} \int d^4x d^4y jGj = - \frac{1}{2} \int d^4x [g \bar{q}(x)q(x) - \Box \phi_0 + 2m^2 \phi_0 - \frac{\lambda}{3!} \phi_0^3 + 4 \phi_0 \Psi(x)] \times$$

$$\int d^4y \int \frac{d^4p}{(2\pi)^4} \frac{[g \bar{q}(y)q(y) - \Box \phi_0 + 2m^2 \phi_0 - \frac{\lambda}{3!} \phi_0^3 + 4 \phi_0 \Psi(y)]}{[p^2 + M_\phi^2]} \exp[ip(x - y)],$$ \hspace{1cm} (2.21)
We now assume that the kinetic part of the scalar field is suppressed by the total (large) mass term \( p^2 \ll M_\phi^2 \), in a way similar to the relation between the electroweak theory and the effective Fermi theory due to the large mass of the interaction mediators. In this case, the following local limit can be taken:

\[
\frac{1}{2} \int d^4x d^4y jGj \approx -\frac{1}{2} \int d^4x [a(\bar{q}q)^2 + 2b(\bar{q}q) + c\Psi + d\Psi^2 + f] 
\]

(2.22)

wherein

\[
a = \frac{g^2}{M_\phi^2}, \\
b = \frac{g}{M_\phi^2} [2m^2\phi_0 - \frac{\lambda}{3!}\phi_0^3 + 4\phi_0\Psi], \\
c = \frac{8\phi_0}{M_\phi^2} [2m^2\phi_0 - \frac{\lambda}{3!}\phi_0^3], \\
d = \frac{16\phi_0^2}{M_\phi^2}, \\
f = \frac{2m^2\phi_0 - \frac{\lambda}{3!}\phi_0^3}{M_\phi^2}.
\]

(2.23)

The following expression is obtained for the effective action of the model:

\[
Z = N \int D\tilde{q}DqD\tilde{\phi}_0D\Psi \exp[i \int d^4x \{\bar{q}i\partial^4q - g\phi_0\bar{q}q - b\bar{q}q + C_F(\bar{q}q)^2 + \\
-\frac{1}{2} \tilde{\phi}_0 G^{-1}\tilde{\phi}_0 - \frac{c}{2} \Psi + \frac{4!}{\lambda} - \frac{d}{2} \Psi^2 - \frac{f}{2} \} \exp(i\Gamma_0),
\]

(2.24)

in which we could define the renormalized Fermi constant \( C_F^R = -\frac{1}{2} \frac{g^2}{M_\phi^2} \) with \( C_F = C_g^2 \). Also note that, as \( \phi_0 \) is a constant, or very slowly varying, background, we can set \( \Box \phi_0 = 0 \).

2.3 Coupled one loop gap equations

Fermion degrees of freedom were kept so far intact and now the AFM is considered again to reduce the fermion self interactions into bilinears. Before doing that however, firstly let us introduce a background fermion current by shift of the bilinear, that is needed for the one loop calculation, \( (\bar{q}q)_0 \) as \( \bar{q}q \rightarrow (\bar{q}q)_0 + (\bar{q}q) \) in which \( (\bar{q}q) \) is the quantum field. The auxiliary field for the fluctuations are introduced again by means of the following unit integral in the generating functional:

\[
1 = N'' \int DS \exp[-i \int d^4x \frac{1}{4C_F}(S + 2C_F(\bar{q}q))^2].
\]

(2.25)

We are left with the following functional generator:

\[
Z = N \int D\tilde{q}DqD\tilde{\phi}_0DSD\Psi \exp(i\Gamma_0) \exp[i \int d^4x \{\bar{q}(i\partial - g\phi_0 - b - S + 2C_F(\bar{q}q)_0)q + \\
\]

\[1 \text{In the regime where } \Psi_0 \text{ suppresses the other masses we have that } C_F^R = -\frac{1}{2} \frac{g^2}{M_\phi^2}. \]
\[-\frac{1}{4C_F} S^2 - \frac{1}{2} \bar{\phi}_0 \left( \Box - 2m^2 - 4\Psi - 4\frac{\lambda}{4!} \phi_0^2 \right) \phi_0 - \frac{c}{2} \Psi + \left[ \frac{4!}{\lambda} - \frac{d}{2} \right] \Psi^2 + \frac{1}{4C_F} S^2 - \frac{f}{2} + \Gamma_{q_0} \right] (2.26)\]

where $\Gamma_{q_0}$ contains the other terms that depend exclusively on the background fermion field $(\bar{q}q)_0$. It is possible to define a total dressed mass for the fermion field in the case that the fields $\phi_0$ and $S$ develop classical solutions in the corresponding gap equations. This effective mass can be written as

\[M_q = g\phi_0 + b + S, \quad (2.27)\]

wherein each of the terms has a precise physical meaning: the first represents a SSB of the Higgs-type, the second $(b)$ contains different contributions from the Higgs-type order parameter, $\phi_0$, and a correction to the Higgs mechanism due to two-scalars correlations in the vacuum, $\Psi$, and, at the last, a dynamical symmetry breaking by means of $S$.

By proceeding with the integration in the fluctuation fields we have:

\[Z = N \int DSD\Psi \det[S_F^{-1}] \det[S_B^{-1}]^{-\frac{1}{2}} \exp \left[ i \int d^4x \left\{ \frac{1}{4C_F} S^2 - \frac{c}{2} \Psi + \left[ \frac{4!}{\lambda} - \frac{d}{2} \right] \Psi^2 - \frac{f}{2} \right\} \right] \exp(i\Gamma_0) \quad (2.28)\]

where

\[S_F(x,y) = \left( i\bar{\partial} - M_q \right)^{-1} \delta^4(x-y), \]

\[S_B(x,y) = \left( \Box - M_\phi^2 \right)^{-1} \delta^4(x-y). \quad (2.29)\]

To calculate the full set of (gap) equations that define the ground state of the system, let us write the effective potential $V_{eff}$ from eq. (2.28). By means of the identity $\det A = \exp(\text{tr} \ln A)$ we write

\[-V_{eff} = -i\text{tr} \int d^4y \ln[S_F(x,y)] \delta^4(x-y) + \frac{1}{4C_F} S^2(x) + \frac{1}{2} \int d^4y \ln[S_B(x,y)] \delta^4(x-y) + \]

\[-\frac{c}{2} \Psi(x) + \left[ \frac{4!}{\lambda} - \frac{d}{2} \right] \Psi^2(x) - \frac{f}{2} \quad (2.30)\]

By extremizing this effective potential with respect to the auxiliary fields $\Psi$ and $S$ the following gap equations are obtained:

\[\frac{\partial V_{eff}}{\partial \Psi} \bigg|_{\Psi = \Psi_0} = i\text{tr} \frac{2}{\left[ i\bar{\partial} - M_q \right]} \left[ b + g\phi_0 \right] M_\phi^2 \delta^4(x-y) - \frac{2}{g} S_0^2 \]

\[-i \left[ \Box - M_\phi^2 \right] \delta^4(x-y) + \frac{2\Psi_0}{M_\phi^2} - \frac{1}{2} c + 2 \left[ \frac{4!}{\lambda} - \frac{d}{2} \right] \delta^4(x-y) + \frac{4\Psi_0}{M_\phi^2} - \frac{1}{2} d \right] \Psi_0 + \frac{2 f}{M_\phi^2} = 0, \]

\[\frac{\partial V_{eff}}{\partial S} \bigg|_{S = S_0} = i\text{tr} \frac{1}{\left[ i\bar{\partial} - M_q \right]} \delta^4(x-y) + \frac{1}{2C_F} S_0 = 0, \quad (2.31)\]
where the first of these equations provides corrections to the eq. \((2.13)\) and the second equation provides the usual DChSB. A non trivial solution for the first equation \((\Psi_0)\) leads to a contribution for the scalar boson mass and a redefinition of the scalar field condensate \((\text{SSB})\) \((2.18)\). It can be noted that this pair of equations, together with the quantum gap eq. for \(\phi_0\), provides further account of YuM interactions than usual one loop equation derived for a single auxiliary field.

### 2.3.1 Renormalized coupled one loop gap equations

Now let us renormalize the coupled gap equations in eq. \((2.31)\) and eq. \((2.31)\), by means of eq. \((2.5)\) by applying the following renormalized (HS) transformation:

\[
1 = N' \int D\Psi_R \exp\{i \int d^4x \frac{4!}{\lambda_R} \left[ \mathscr{Z}_\Psi^\dagger \Psi_R + \mathscr{Z}_\lambda^\dagger \lambda_R + (\tilde{\phi}_R + 2 \phi_0 \tilde{\phi}_R) \right] \}
\]

\[
1 = N'' \int D\phi_R D\phi_0 D\phi_R D\Psi_R \exp(i\Gamma_0) \exp\{i \int d^4x \}
\]

- \(\tilde{q}_R (\mathscr{Z}_q \partial - \mathscr{Z}_\phi \Phi_0 - \frac{b(R,c.t)}{2}) - \mathscr{Z}_S^\dagger \mathscr{Z}_S \mathbf{R}_\phi + 2 \mathscr{Z}_C \mathbf{C}_F (\tilde{q}_R \phi_0) q_R\)

- \(-\frac{1}{4C_F^2} \mathbf{S}_S S_R - \frac{1}{2} \phi_0 (\mathscr{Z}_S \partial - 2 \mathbf{R}_m m_R^2 - 4 \mathbf{Z}_\Psi \mathbf{Z}_\lambda \Psi_R - 4 \mathbf{Z}_\lambda \frac{\lambda_R}{4!} \phi_0^2) \phi_0\)

- \(-\frac{c(R,c.t)}{2} \Psi_R + \left[ \frac{4!}{\lambda_R} \mathbf{Z}_\Psi - \frac{d(R,c.t)}{2} \right] \Psi_R + \frac{1}{4C_F^2} \mathbf{Z}_C S_R^2 - \frac{f(R,c.t)}{2} \}

(\text{2.32}\)

We are left with the following functional generator

\[
\mathcal{Z} = N \int D\tilde{q}_R D\phi_R D\tilde{\phi}_0 D\mathbf{S}_R D\mathbf{C}_F \mathbf{Z}_R \exp\{i\Gamma_0\} \exp\{i \int d^4x \}
\]

and from it we find the following set of renormalized gap equations:

\[
(\Psi_0 : \quad \text{itr} \quad \mathbf{Z}_q \partial - \mathbf{M}_q (R,c.t) \quad \mathbf{M}^2_\phi (R,c.t) \quad \delta^4 (x-y) - \frac{2 \mathbf{Z}_\Psi \mathbf{Z}_\lambda \mathbf{C}_F}{8} \mathbf{S}_0^2 +
\]

\[
- \frac{\mathbf{Z}_\Psi \mathbf{Z}_\lambda}{\mathbf{M}_\phi (R,c.t)} \delta^4 (x-y) + \frac{2 \mathbf{Z}_\Psi \mathbf{Z}_\lambda \Psi_0}{\mathbf{M}_\phi (R,c.t)} + \frac{1}{2} c(R,c.t) +
\]

\[
+ 2 \left[ \frac{4!}{\lambda_R} \mathbf{Z}_\Psi + \left( \frac{4 \mathbf{Z}_\Psi \mathbf{Z}_\lambda \Psi_0}{\mathbf{M}_\phi (R,c.t)} - \frac{1}{2} d(R,c.t) \right) \Psi_0 + \frac{2 \mathbf{Z}_\Psi \mathbf{Z}_\lambda f(R,c.t)}{\mathbf{M}_\phi (R,c.t)} \right] = 0,
\]

\[
(S_0 : \quad \text{itr} \quad \mathbf{Z}_q \partial - \mathbf{M}_q (R,c.t) \quad \delta^4 (x-y) + \frac{1}{2C_F^2} \mathbf{Z}_C S_0 = 0,
\]

in which

\[
\mathbf{M}_\phi^2 (R,c.t) = 2 \mathbf{Z}_m m_R^2 + 4 (\mathbf{Z}_\Psi \mathbf{Z}_\lambda \Psi_R + \mathbf{Z}_\lambda \frac{\lambda_R}{4!} \phi_0^2)
\]

10
\[ M_q(R, c.t) = \mathcal{D}_{g g R} \phi_0 + b(R, c.t) + \mathcal{D}_{S}^{\frac{1}{2}} \mathcal{D}_{C_r}^{\frac{1}{2}} S_R, \]  

(2.35)

and

\[
\begin{align*}
    b(R, c.t) &= \frac{\mathcal{D}_{g g R}}{M_\phi^2(R, c.t)} \left[ 2Z_m m_R^2 \phi_0 - \mathcal{D}_{\lambda}^R \frac{\lambda_R}{3!} \phi_0^3 + 4 \mathcal{D}_{g}^{\frac{1}{2}} \mathcal{D}_{\lambda}^{\frac{1}{2}} \phi_0 \Psi_R \right], \\
    c(R, c.t) &= \frac{8 \mathcal{D}_{g}^{\frac{1}{2}} \mathcal{D}_{\lambda}^{\frac{1}{2}} \phi_0}{M_\phi^2(R, c.t)} \left[ 2Z_m m_R^2 \phi_0 - \mathcal{D}_{\lambda}^R \frac{\lambda_R}{3!} \phi_0^3 \right], \\
    d(R, c.t) &= \frac{16 \mathcal{D}_{\phi} \mathcal{D}_{\lambda} \phi_0^2}{M_\phi^2(R, c.t)}, \\
    f(R, c.t) &= \frac{\left[ 2Z_m m_R^2 \phi_0 - \mathcal{D}_{\lambda}^R \frac{\lambda_R}{3!} \phi_0^3 \right]^2}{M_\phi^2(R, c.t)}. 
\end{align*}
\]

(2.36)

There are two mass generation mechanisms in play, dynamic mass generation due to the dynamical chiral symmetry breaking and the mass generated by the spontaneous symmetry breaking due to the $Z_2$ discrete symmetry. As we see in eq. (2.24) we can associate two physical conditions for the fermion sector and three physical conditions for the scalar sector (propagators and vertices). So, with five counter-terms we adjust the theory. Actually we have three coupled gap equations, $\phi_0$ and $\Psi_0$ for the boson and $S_0$ for the fermions. All of them can contribute for both boson and fermion masses according to eqs. (2.35). On the other hand, when the dynamic mass generation happens, we have one gap associated with the fermion mass and other associated with the scalar mass, and the gap coupled equations are divergent. Therefore, in addition to the regularization process [41] (cut-off, for example), apparently we need two more counter-terms to extract a physical gap.

It is not clear whether these coupled gap equations might have an unique solution or not. To make sure these are solutions of minimum action a second derivative might be taken. Therefore, we write the Hessian matrix from the renormalized effective action in eq. (2.33)

\[
[H] = \begin{pmatrix}
\frac{\partial^2 S_{\text{eff}}}{\partial \Psi \partial \Psi} & \frac{\partial^2 S_{\text{eff}}}{\partial \Psi \partial S} \\
\frac{\partial^2 S_{\text{eff}}}{\partial S \partial \Psi} & \frac{\partial^2 S_{\text{eff}}}{\partial S \partial S}
\end{pmatrix}
\]

(2.37)

in which we have the following terms

\[
\begin{align*}
    \frac{\partial^2 S_{\text{eff}}}{\partial \Psi \partial \Psi} &= u_1 + u_2 + u_3 + u_4 + u_5 \\
    u_1 &= i \mathrm{tr} \frac{8 \mathcal{D}_{g} \mathcal{D}_{\lambda}}{[\mathcal{D}_q \ddagger - M_q(R, c.t)]^2} \left[ b(R, c.t) + \mathcal{D}_{g g R} \phi_0 \right]^2 \delta^4(x-y) - i \mathrm{tr} \frac{8 \mathcal{D}_{g} \mathcal{D}_{\lambda}}{[\mathcal{D}_q \ddagger - M_q(R, c.t)]^2} b(R, c.t) \delta^4(x-y) + \\
    -i \mathrm{tr} \frac{8 \mathcal{D}_{g} \mathcal{D}_{\lambda}}{[\mathcal{D}_q \ddagger - M_q(R, c.t)]} \frac{b(R, c.t) + \mathcal{D}_{g g R} \phi_0}{M_\phi(R, c.t)} \delta^4(x-y)
\end{align*}
\]
\[ u_2 = i tr \frac{8 \partial^2_{\psi} \partial^2_{\lambda}}{\left[ \partial_q i \partial - M_q(R, c, t) \right]^2} \delta^4(x - y) \]

\[ u_3 = (\frac{2 \partial^2_{\psi} \partial^2_{\lambda}}{M^2_\phi(R, c, t)} - \frac{8 \partial^2_{\psi} \partial^2_{\lambda} \psi_0}{M^4_\phi}) c(R, c, t) - \left( \frac{2 \partial^2_{\psi} \partial^2_{\lambda} \psi_0}{M^2_\phi(R, c, t)} - \frac{1}{2} \right) \frac{4 \partial^2_{\psi} \partial^2_{\lambda} c(R, c, t)}{M^2_\phi} \]

\[ u_4 = 2 \left( \frac{4 \partial^2_{\psi} \partial^2_{\lambda}}{M^2_\phi(R, c, t)} - \frac{16 \partial^2_{\psi} \partial^2_{\lambda} \psi_0}{M^4_\phi} \right) d(R, c, t) \psi_0 - \left( \frac{4 \partial^2_{\psi} \partial^2_{\lambda} \psi_0}{M^2_\phi(R, c, t)} - \frac{1}{2} \right) \frac{32 \partial^2_{\psi} \partial^2_{\lambda} d(R, c, t)}{M^2_\phi} \psi_0 + \]

\[ + 2 \left[ \frac{4!}{\lambda_R} \partial^2_{\psi} + \left( \frac{4 \partial^2_{\psi} \partial^2_{\lambda} \psi_0}{M^2_\phi(R, c, t)} - \frac{1}{2} \right) d(R, c, t) \right] \]

\[ u_5 = - \frac{16 \partial^2_{\psi} \partial^2_{\lambda} f(R, c, t)}{M^2_\phi(R, c, t)} , \]

\[ (2.38) \]

\[ \frac{\partial^2 S_{eff}^R}{\partial \Psi \partial S} = i tr \left[ \frac{\partial^2_{\psi} \partial^2_{\lambda} \partial^2_{\psi} \partial^2_{\lambda} \partial^2_{\phi} \partial^2_{\phi} [ b(R, c, t) + \partial^2_{\phi} g \phi_0 ]}{\left[ \partial_q i \partial - M_q(R, c, t) \right]^2} \delta^4(x - y) - \frac{2 \partial^2_{\psi} \partial^2_{\lambda} \partial^2_{\phi} \partial^2_{\phi} S_0}{\partial^2_{\phi} \partial^2_{\phi} g_R} \right] , \]

\[ (2.39) \]

\[ \frac{\partial^2 S_{eff}^R}{\partial S \partial \Psi} = i tr \left[ \frac{\partial^2_{\psi} \partial^2_{\lambda} \partial^2_{\psi} \partial^2_{\lambda} \partial^2_{\phi} \partial^2_{\phi} [ b(R, c, t) + \partial^2_{\phi} g \phi_0 ]}{\left[ \partial_q i \partial - M_q(R, c, t) \right]^2} \delta^4(x - y) \right] , \]

\[ (2.40) \]

\[ \frac{\partial^2 S_{eff}^R}{\partial S \partial \phi} = i tr \left[ \frac{\partial^2_{\phi} \partial^2_{\phi} \partial^2_{\phi} \partial^2_{\phi} S_0}{\left[ \partial_q i \partial - M_q(R, c, t) \right]^2} \delta^4(x - y) + \frac{1}{2 C_F^R} \partial^2_{\phi} \partial^2_{\phi} \right] . \]

\[ (2.41) \]

So the condition of local minimum can be achieved for the extremal points \((\psi_0, S_0)\) if the characterist equation \(det([H] - \nu I)\) has positive eigenvalues \((\nu_{\pm})\). In the large mass limit for the intermediate boson \(M_\phi\) the off-diagonal terms of the Hessian do not contribute and we have following conditions

\[ i tr \left[ \frac{8 \partial^2_{\psi} \partial^2_{\lambda}}{\left[ \partial_q i \partial - M_q(R, c, t) \right]^2} \delta^4(x - y) + 2 \left( \frac{4!}{\lambda_R} \partial^2_{\psi} - \frac{1}{2} \right) d(R, c, t) \right] > 0 \]

\[ i tr \left[ \frac{\partial^2_{\phi} \partial^2_{\phi} \partial^2_{\phi} \partial^2_{\phi} S_0}{\left[ \partial_q i \partial - M_q(R, c, t) \right]^2} \delta^4(x - y) + \frac{1}{2 C_F^R} \partial^2_{\phi} \partial^2_{\phi} \right] > 0 . \]

\[ (2.42) \]

### 2.3.2 Coupled gap equations for particular limits of \(m, \lambda\) and \(g\)

As we discussed previously, to solve the previous coupled gap equations in eq. \((2.20)\) and eq. \((2.34)\) may not be an easy task, due to the arbitrariness of couplings and masses along with the concept of renormalization. Below some particular prescriptions for solving the gap equations are presented by considering the usual logics of low energy effective models since the resulting model in terms of auxiliary fields is non renormalizable. Formally, we have 3 equations and 3 gaps \((\psi_0, \Phi_0, S_0)\), that can be written momentum representation as follows \((\mathcal{L}_i \to 1)\):

\[ \Phi_0^2 = \frac{12 m^2}{\lambda} - \int \frac{d^4p}{(2\pi)^4} \frac{2i}{p^2 + M^2_\phi} . \]
\[
\frac{2\Psi_0}{M_\phi^2} - \frac{1}{2}c = -8i\frac{[b + g\phi_0]}{M_\phi^2} \int \frac{d^4 p}{(2\pi)^4} \frac{M_q}{p^2 + M_q^2} - 2 \frac{g^2 \phi_0^2}{g^2} + 2i \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 + M_\phi^2} + 2[4! \frac{1}{\lambda} + (\frac{4\Psi_0}{M_\phi^2} - \frac{1}{2})d]\frac{1}{2} + \frac{2f}{M_\phi^2}
\]

\[
S_0 = -8C_F i \int \frac{d^4 p}{(2\pi)^4} \frac{M_q}{[p^2 - M_q^2]}.
\]

Thus, we will explore solutions of the gap coupled equations in the following limits:

a) \(\lambda \phi_0^2 \sim m^2, \phi_0 \neq 0\) (solution for SSB without quantum contributions)

b) \(m \rightarrow \infty\), (scalar particle, mediator of the interaction, with large mass)

c) \(g\phi_0 \sim m\) ("ad doc" constrain in the Yukawa coupling).

So that, the massive particle associated with the \(\phi\) is in some sense localized. We have as consequences the results below

\[
g^2 \sim \lambda,\]

\[
C_F \sim -\frac{g^2}{m^2} \sim -\frac{1}{\phi_0^2}.
\]

Therefore the dynamical mass mechanism are dictated by the value of \(\phi_0\) and the SSB. As a consequence, in the weak regime for the couplings \((g, \lambda)\), \(C_F \rightarrow 0\) and \(\phi_0^2 \rightarrow \infty\). In another hand, in the regime where the couplings \((g, \lambda)\) are not too weak, or in another words, stronger, \(C_F < 0\) and \(\phi_0^2 \rightarrow 0\). In the limit of large mass discussed above, we can seek a solution for the gap coupled equations, saying that \(M_\phi\) and \(M_q\) have the following behavior

\[
M_\phi^2 = x m^2 = 4m^2 + 4\Psi_0
\]

\[
M_q = y m = m + \frac{2}{x m}\Psi_0 + S_0,
\]

wherein \(x\) and \(y\) are the unknown variables that we will find later. This can be seen just an algebraic trick to re-write eq. (2.43) in a more suitable form to seek solutions. Being that, at this point, we are dealing with a non-renormalizable effective model for the low energy Yukawa Model, it is justifiable to reduce the gap equations to the bare ones, by neglecting the renormalization constants whose calculations remain outside the scope of the work, and within the usual logics of low energy effective models.

Hence, in the case of large mass approximation defined earlier, the coupled gap equations for \(\Psi_0\) and \(S_0\) are simplified

\[
\frac{[b + g\phi_0]}{M_\phi^2} = \frac{1}{m} \frac{(2x - 4)}{x^2} \rightarrow 0,
\]
\[-\frac{1}{2}c + \frac{2f}{M_\phi^2} = 0,\]

\[
\frac{4!}{\lambda} + \left(\frac{\Psi_0}{M_\phi^2} - \frac{1}{2}\right)d + \frac{c}{M_\phi^2} = \frac{24 \left(-3x^2 + 16x - 64\right)}{x^3},
\]

\[(2.46)\]

where the second of these equations corresponds to a constraint among the model parameters and resulting masses: \(M_\phi^2 = 4m^2 - \frac{4}{\lambda}\phi_0^2\). This equation must be compatible with the solution of Eq. (2.45). The other two equations can be written as:

\[
\Psi_0 = \frac{\lambda}{24\phi_0^2} i \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - M_\phi^2} + \frac{\lambda}{24g^2} \left[-3x^2 + 16x - 64\right] S_0^2,
\]

\[
S_0 = \frac{1}{8} C_F i \int \frac{d^4p}{(2\pi)^4} \frac{2C_f}{p^2 - M_q^2}.
\]

\[(2.47)\]

Thus, after simplifications, the following system of equations is obtained:

\[
xm^2 = 4m^2 + \frac{2m^2}{\phi_0^2} \left[-3x^2 + 16x - 64\right] i \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - xm^2} + \frac{8}{x^2\phi_0^2} \left[-3x^2 + 16x - 64\right] \left[\int \frac{d^4p}{(2\pi)^4} \frac{ym}{p^2 - y^2m^2}\right] \left[\int \frac{d^4p}{(2\pi)^4} \frac{ym}{p^2 - y^2m^2}\right],
\]

\[
ym = \frac{m}{2x} (x - 2) + \frac{4}{x\phi_0^2} i \int \frac{d^4p}{(2\pi)^4} \frac{ym}{p^2 - y^2m^2}.
\]

\[(2.48)\]

Observe that the masses \(M_\phi^2\) and \(M_q\) do not obey a simple scaling behavior. As we can see, we have a system of two equations and two unknown variables \((x, y)\) whose the real and positive solution for the unknown variables depends on the values of \(m\) and \(\phi_0\). Therefore with the fractions \(x\) and \(y\) we determine too \(\Psi_0\) and \(S_0\), from the system of equations in eq. (2.45). The integrals in eq. (2.48) can be solved by implementing, for example, the cut-off methodology.

3 Effective action from external field methods

Above it was presented how renormalization affects and refines the study of the gap equations for \(\phi_0\), \(\Psi_0\) and \(S_0\). Now renormalization and the gap solutions will be shown to affect the resulting effective action with its two contributions \(\Gamma_{eff} = \Gamma_0 + \tilde{\Gamma}\), wherein \(\Gamma_0\) is the background and \(\tilde{\Gamma}\) the fluctuations.

By assuming the auxiliary field expected value in the vacuum are non zero there emerges modifications in particles and fields interactions in such non trivial background. With the expansion in terms of the fluctuations, \(\Psi_R \to \Psi_0 + \Psi_R\), \(S_R \to S_0 + S_R\), we obtain the action from eq. (2.33) in terms of two determinants:

\[
\exp[i\Theta] = \det\{1 + \tilde{S}_\Sigma [\mathcal{Z}_\Sigma ] \tilde{C}_f \tilde{S}_R + 2 \mathcal{Z}_C \mathcal{C}_f (\bar{q}q)0\} \times
\]
where
\[
\tilde{S}_F^{-1} = \mathcal{S}_q i \partial - \tilde{M}_q(R, c.t.), \quad \tilde{M}_q(R, c.t.) = \mathcal{S}_g R \phi_0 + \frac{b(R, c.t.)}{2} + \mathcal{S}_R \frac{1}{4!} \mathcal{S}_R S_0,
\]
\[
\tilde{S}_B^{-1} = \mathcal{S}_\phi \Box - \tilde{M}_\phi^2(R, c.t.), \quad \tilde{M}_\phi^2(R, c.t.) = 2 \mathcal{S}_m m_R^2 + 4 \mathcal{S}_R \mathcal{S}_C \mathcal{S}_R \Psi_0
\]

So we have the following contribution \(i \Theta = i \Theta_F + i \Theta_B\) with the terms:
\[
i \Theta_F \simeq \int d^4 x tr \left\{ 1 + \tilde{S}_F \left[ -2 \mathcal{S}_S \mathcal{S}_C \mathcal{S}_R + 2 \mathcal{S}_C r c_F R (\tilde{\bar{q}} q_0) \right] \right\} \delta^4(x-y),
\]
\[
i \Theta_B \simeq -\frac{1}{2} \int d^4 x \ln \left\{ 1 - 2 \tilde{S}_B \left[ \mathcal{S}_Q^2 \mathcal{S}_C \mathcal{S}_R + \mathcal{S}_S \frac{\lambda_R}{4!} \phi_0^2 \right] \right\} \delta^4(x-y).
\]

Because of the SSB and DChSB it is reasonable to perform large fermion and scalar field effective masses expansions. For the zero order derivative expansion, we get the subsequent leading terms for each of the determinants:
\[
i \Theta_F \simeq \int d^4 x tr \tilde{S}_F \left[ -2 \mathcal{S}_S \mathcal{S}_C \mathcal{S}_R + 2 \mathcal{S}_C r c_F R (\tilde{\bar{q}} q_0) \right] \delta^4(x-y) +
\]
\[-\frac{1}{2} \int d^4 x \ln \left\{ 1 - 2 \tilde{S}_B \left[ \mathcal{S}_Q^2 \mathcal{S}_C \mathcal{S}_R + \mathcal{S}_S \frac{\lambda_R}{4!} \phi_0^2 \right] \right\} \delta^4(x-y),\]

and
\[
i \Theta_B \simeq \int d^4 x \tilde{S}_B \left[ \mathcal{S}_Q^2 \mathcal{S}_C \mathcal{S}_R + \mathcal{S}_S \frac{\lambda_R}{4!} \phi_0^2 \right] \delta^4(x-y) +
\]
\[+ \int d^4 x \tilde{S}_B \left[ \mathcal{S}_Q^2 \mathcal{S}_C \mathcal{S}_R + \mathcal{S}_S \frac{\lambda_R}{4!} \phi_0^2 \right] \delta^4(x-y).\]

The above leading terms can be rearranged in the effective action such that one writes:
\[
\Gamma_{\text{eff}} = \int d^4 x \left\{ \left[ -\mathcal{S}_Q \phi_0 \Box \phi_0 + \left[ \mathcal{S}_m m_R^2 + \delta m(R, c.t.) \right] \phi_0^2 \right] + \left[ \mathcal{S}_S \frac{\lambda_R}{4!} \phi_0^2 \right] \right\} +
\]
\[+ \left[ \mathcal{S}_R \tilde{\bar{q}} q_0 i \partial q_0 + \left[ -\mathcal{S}_g \phi_0 - \frac{b(R, c.t.)}{2} + \delta M(R, c.t.) \right] \tilde{\bar{q}} q_0 +
\]
\[+ \left[ \mathcal{S}_C r c_F + \delta C_F (R, c.t.) \right] \left( \bar{q} q_0 \right)^2 - \delta G \tilde{S}_R \bar{\Psi} R \right\},
\]

wherein we can see the quantum contributions to the masses and coupling constants in the low energy or long-wavelength local limit and a mixing-type interaction for the two auxiliary fields for 2-fermion and 2-boson states. The resulting interactions becomes punctual:
\[
- i \delta M(R, c.t.) = -2 \mathcal{S}_C r c_F tr \tilde{S}_F \delta^4(x-y) = 8 \mathcal{S}_C r c_F R \left[ \frac{\tilde{M}_q(R, c.t.)}{(2\pi)^4} \frac{\tilde{M}_q(R, c.t.)}{[\mathcal{S}_Q^2 p^2 - \tilde{M}_q^2(R, c.t.)]} \right],
\]
\[-i \delta C_F(R,c.t) = -2[\mathcal{Z}_C R F]^2 \text{tr}(\bar{S}_F S_F) \delta^4(x-y) = -8[\mathcal{Z}_C R F]^2 \int \frac{d^4 p}{(2\pi)^4} \frac{\mathcal{Z}_q^2 p^2 + \tilde{M}_q^2(R,c.t)}{[\mathcal{Z}_q^2 p^2 - \tilde{M}_q^2(R,c.t)]^2}, \quad (3.8)\]

\[-i \delta m(R,c.t) = \mathcal{Z}_R \frac{\lambda R}{4!} \text{tr}(\bar{S}_B S_B) \delta^4(x-y) = \frac{1}{24} \mathcal{Z}_R \lambda R \int \frac{d^4 p}{(2\pi)^4} \frac{1}{[\mathcal{Z}_q^2 p^2 - \tilde{M}_q^2(R,c.t)]^2}, \quad (3.9)\]

\[-i \delta \lambda (R,c.t) = \left[\mathcal{Z}_R \lambda \right]^2 \text{tr}(\bar{S}_B S_B) \delta^4(x-y) = \left[\mathcal{Z}_R \lambda \right]^2 \int \frac{d^4 p}{(2\pi)^4} \frac{1}{[\mathcal{Z}_q^2 p^2 - \tilde{M}_q^2(R,c.t)]^2}. \quad (3.10)\]

\[\delta G(R,c.t) = (\mathcal{Z}_R \mathcal{Z}_C \mathcal{Z}_q \mathcal{Z}_q)^2 \text{tr}(\bar{S}_B S_F) \delta^4(x-y) = 4(\mathcal{Z}_R \mathcal{Z}_C \mathcal{Z}_q \mathcal{Z}_q)^2 \int \frac{d^4 p}{(2\pi)^4} \frac{1}{[\mathcal{Z}_q^2 p^2 - \tilde{M}_q^2(R,c.t)]^2} \frac{\tilde{M}_q(R,c.t)}{[\mathcal{Z}_q^2 p^2 - \tilde{M}_q^2(R,c.t)]^2}. \quad (3.11)\]

For the (long-wavelength) local limit for the second order terms of the determinants expansions, the fermion bilinears $(\bar{q}q)$ and fields $\bar{S}_R, \bar{\Psi}_R, \phi_0$ were considered to be at the same spacetime point, being the leading terms of the derivative expansion [12]. We are left with an effective model for background fermions interacting with two auxiliary fields, one corresponding to a two $\phi$-boson quantum state, $\Psi$, and another one corresponding to a fermion-antifermion state, $S$. In this equations, $\delta M$ and $\delta m$ are quadratically UV-divergent mass corrections with the same shape of the gap equations (2.47). They renormalize differently however being that the parameters $b$ and $m_F^2$ must be used to the elimination of the UV divergence. The coupling constants, on the other hand, $\delta C_F$ and $\delta \lambda$, are respectively quadratic-UV and log-UV divergent being eliminated by the $\mathcal{Z}_q, \phi$ coefficients and by the renormalization prescription of the $S_0$ gap equation. The scalar field $\phi$ in the original model is responsible for the emergence of fermion effective self interactions of current-current type, and higher orders. We can also extract from eq. (3.4) the free Lagrangian terms for the auxiliary fields and latter verify if they can be bound states quasiparticles.

For the field $S_R$, by considering the leading terms of the derivative expansion, it can be written as:

\[-\frac{1}{4C_F^2} \mathcal{Z}_S \bar{S}_R^2 - \frac{1}{2} \mathcal{Z}_S \mathcal{Z}_C r[\bar{S}_F \bar{S}_R \bar{S}_F \bar{S}_R] \delta^4(x-y) = \frac{1}{2} \alpha(R,c.t) \partial_\mu \bar{S}_R \partial^\mu \bar{S}_R + \frac{1}{2} \beta(R,c.t) \bar{S}_R^2, \quad (3.12)\]

wherein the field normalization and its effective mass are respectively given by:

\[\alpha(R,c.t) = \mathcal{Z}_S \mathcal{Z}_C \mathcal{Z}_q \int \frac{d^4 p}{(2\pi)^4} \frac{1}{[\mathcal{Z}_q^2 p^2 - \tilde{M}_q^2(R,c.t)]^2} \]

\[\beta(R,c.t) = -\mathcal{Z}_S \mathcal{Z}_C \int \frac{d^4 p}{(2\pi)^4} \frac{\tilde{M}_q^2(R,c.t)}{[\mathcal{Z}_q^2 p^2 - \tilde{M}_q^2(R,c.t)]^2} - \frac{1}{4C_F^2} \mathcal{Z}_S. \quad (3.13)\]

Again the quadratic divergence is renormalized by a subtraction incorporated by a mass counterterm. The Yukawa-type effective interaction of this particle with an external background $(\bar{q}q)_0$ is given by:

\[2 \mathcal{Z}_S^4 \mathcal{Z}_C^2 \mathcal{Z}_F^R \text{tr}[\bar{S}_F S_F] \delta^4(x-y) \bar{S}(\bar{q}q)_0 = 2 \mathcal{Z}_S^4 \mathcal{Z}_C^2 \mathcal{Z}_F^R \mathcal{Z}_q^2 \int \frac{d^4 p}{(2\pi)^4} \frac{\mathcal{Z}_q^2 p^2 + \tilde{M}_q^2(R,c.t)}{[\mathcal{Z}_q^2 p^2 - \tilde{M}_q^2(R,c.t)]^2} \bar{S}(\bar{q}q)_0. \quad (3.14)\]
These composite boson-fermion system is analogous to the emergence of mesons in the quark dynamics obtained from low energy effective models for QCD [52], although there is, in the present model, only one single scalar fermion-antifermion state emerging from the scalar field exchange.

In the same way, from eq. (3.5) we note the emergence of dynamics of a composite field \( \tilde{\Psi} \), eventually corresponding to a two-boson state. The leading effective Lagrangian terms obtained from the derivative expansion can be written as:

\[
\frac{4!}{\lambda_R} \mathcal{Z}_\Psi - \frac{d(R,c.t)}{2} \mathcal{Z}_R^2 + \mathcal{Z}_\lambda \mathcal{Z}_B \tilde{\Psi}_R \tilde{\Psi} \mathcal{Z}_B \delta^4(x-y) = \frac{\epsilon(R,c.t)}{2} \mathcal{Z}_R^2 \square \mathcal{Z}_R + \frac{\sigma(R,c.t)}{2} \mathcal{Z}_R^2 \mathcal{Z}_R \tag{3.15}
\]

wherein the field normalization and mass can be expressed as:

\[
\epsilon(R,c.t) = 2 \mathcal{Z}_\Psi \mathcal{Z}_\lambda \mathcal{Z}_\phi \int \frac{d^4p}{(2\pi)^4} \frac{1}{[\mathcal{Z}_\phi p^2 - \tilde{M}_\phi^2(R,c.t)]^2[\mathcal{Z}_\phi p^2 + \tilde{M}_\phi^2(R,c.t)]^2} + \frac{4!}{\lambda_R} \mathcal{Z}_\Psi - \frac{d(R,c.t)}{2} \mathcal{Z}_R^2 \tag{3.16}
\]

where an effective coupling constant was naturally resolved.

The effective interaction of this composite field with an external background \( \phi_0^2 \) in the long wavelength local limit can be written as:

\[
\left(2 \mathcal{Z}_\Psi \mathcal{Z}_\lambda \mathcal{Z}_B \frac{\lambda_R}{4!} \tilde{\Psi}_R \tilde{\Psi} \mathcal{Z}_B \delta^4(x-y) \right) \mathcal{Z}_R \phi_0^2 = \left(12 \mathcal{Z}_\Psi \mathcal{Z}_\lambda \mathcal{Z}_B \frac{\lambda_R}{4!} \tilde{\Psi}_R \tilde{\Psi} \mathcal{Z}_B \delta^4(x-y) \right) \mathcal{Z}_R \phi_0^2 \tag{3.17}
\]

where an effective coupling constant was naturally resolved.

From the above equations, it is useful to resolve effective interactions between the remaining fields. Consider the following quantities:

\[
\mathcal{Y}_F(R,c.t) = 2 \mathcal{Z}_S \mathcal{Z}_G \mathcal{Z}_B \frac{\lambda_R}{4!} \tilde{\Psi}_R \tilde{\Psi} \mathcal{Z}_B \delta^4(x-y) \int \frac{d^4p}{(2\pi)^4} \frac{\mathcal{Z}_G^2 p^2 + \tilde{M}_G^2(R,c.t)}{[\mathcal{Z}_\phi p^2 - \tilde{M}_\phi^2(R,c.t)]^2},
\]

and

\[
\mathcal{Y}_B(R,c.t) = \frac{1}{12} \mathcal{Z}_\Psi \mathcal{Z}_\lambda \mathcal{Z}_B \frac{\lambda_R}{4!} \tilde{\Psi}_R \tilde{\Psi} \mathcal{Z}_B \delta^4(x-y) \int \frac{d^4p}{(2\pi)^4} \frac{1}{[\mathcal{Z}_\phi p^2 - \tilde{M}_\phi^2(R,c.t)]^2}.
\]

An effective model for the YuM can be written as

\[
\tilde{Z} = \int D\tilde{S}_R D\tilde{\Psi}_R \exp \{i \int d^4x(-i)[\tilde{S}_R[-\alpha(R,c.t)\square + \beta(R,c.t)]\tilde{S}_R + \mathcal{Y}_F(R,c.t)C^R_F(\bar{q}q)\tilde{S}_R + \mathcal{Y}_B(R,c.t)\lambda_R \phi_0^2 \tilde{\Psi}_R + \mathcal{Y}_B(R,c.t)\lambda_R \phi_0^2 \tilde{\Psi}_R] - \delta G \tilde{S}_R \tilde{\Psi}_R\}, \tag{3.18}
\]

in which we see the kinetic term for the free composite fields and their interactions with the external background condensates similarly to other models, for earlier reviews see [5–7]. If we continue the expansion we would have further interactions between these composite fields and derivative couplings as well.
The mixing interaction $S_R$ and $\Psi_R$ suggests one can rotate the system of states $S_R$ and $\Psi_R$ to diagonalize the corresponding masses. Let us define a mixing angle $\theta_{22}$ that allow to rotate the auxiliary fields of 2-fermion states to 2-boson states:

\[
\begin{align*}
\tilde{S}_R &= \cos(\theta_{22})\tilde{S}_{mass,R} + \sin(\theta_{22})\tilde{\Psi}_{mass,R}, \\
\tilde{\Psi}_R &= \cos(\theta_{22})\tilde{\Psi}_{mass,R} - \sin(\theta_{22})\tilde{S}_{mass,R},
\end{align*}
\]  

(3.19)  

(3.20)

where $\tilde{S}_{R,mass}$ and $\tilde{\Psi}_{R,mass}$ are mass eigenstates. With this rotation the following effective action is obtained by omitting the indices mass:

\[
S_{eff} = (-i)[\tilde{S}_R[-\alpha(R,c.t)\cos^2(\theta_{22})\Box + \cos^2(\theta_{22})\beta(R,c.t) + \delta \phi \sin(2\theta_{22})]\tilde{S}_R
+ Y_\beta(R,c.t)C^F_\beta(\bar{q}q)[\cos(\theta_{22})\tilde{S}_{mass,R} + \sin(\theta_{22})\tilde{\Psi}_{mass,R}]
+ Y_\phi(R,c.t)\lambda R\phi_0^2[\cos(\theta_{22})\tilde{\Psi}_{mass,R} - \sin(\theta_{22})\tilde{S}_{mass,R}]
+ \tilde{\Psi}_R[\epsilon(R,c.t)\cos^2(\theta_{22})\Box + \cos^2(\theta_{22})\sigma(R,c.t) - \delta \phi \sin(2\theta_{22})]\tilde{\Psi}_R
+ \sin(2\theta_{22})\tilde{\Psi}_R[\epsilon(R,c.t)\Box + \sigma(R,c.t)\tilde{S}_R]
+ \sin(2\theta_{22})\tilde{S}_R[-\alpha(R,c.t)\Box + \beta(R,c.t)]\tilde{\Psi}_R
- \delta \phi \cos(2\theta_{22})\tilde{S}_{mass,R}\tilde{\Psi}_{mass,R}],
\]

(3.21)

With two conditions of final mass eigenstates and normalization

\[
\tan(2\theta_{22}) = \frac{\delta \phi}{\epsilon(R,c.t) - \alpha(R,c.t)},
\]

(3.22)

At this point it becomes interesting to define the bound state conditions for both the composite scalar field $\tilde{\Psi}_R$ and for the composite fermion-antifermion state $\tilde{S}_R$. They can be simply identified to the condition of a pole for real positive masses

\[
m_{\tilde{S}}^2(R,c.t) = -\frac{\beta(R,c.t)}{\alpha(R,c.t)} + 2\delta \phi \tan(\theta_{22}), \quad m_{\tilde{\Psi}}^2(R,c.t) = \frac{\sigma(R,c.t)}{\epsilon(R,c.t)} + 2\delta \phi \tan(\theta_{22}).
\]

(3.23)

They can be written, by considering eq. (3.13) and eq. (5.10) as:

\[
\mathcal{L}_q^2 m_{\tilde{S}}^2(R,c.t) = \mathcal{L}_q^2 \tilde{M}_{\tilde{S}}^2(R,c.t) + \frac{1}{4}\mathcal{L}_F^2 C_F^R \int \frac{d^4p}{(2\pi)^4} \frac{1}{[\mathcal{L}_q^2 p^2 - \tilde{M}_{\tilde{S}}^2(R,c.t)]^2} + 2\delta \phi \tan(\theta_{22}),
\]

\[
\mathcal{L}_q^2 m_{\tilde{\Psi}}^2(R,c.t) = \tilde{M}_{\tilde{\Psi}}^2(R,c.t) + \left[ 4! \lambda R \phi_0^2 \left( \sigma(R,c.t) \frac{4\phi_0^2(R,c.t)}{\tilde{M}_{\tilde{\Psi}}^2(R,c.t)} \right) \times \right.
\]

\[
\times \left. \frac{1}{\mathcal{L}_F^2 C_F^R \int \frac{d^4p}{(2\pi)^4} [\mathcal{L}_F^2 p^2 - \tilde{M}_{\tilde{\Psi}}^2(R,c.t)]^2 [\mathcal{L}_F^2 p^2 + \tilde{M}_{\tilde{\Psi}}^2(R,c.t)]} \right] + 2\delta \phi \tan(\theta_{22}).
\]

(3.24)
3.1 Mass and existence of the composite particles

From the pole condition for the composite fields in eq. (3.24), the bound state conditions for the composite particle from the scalar field $\Psi$ can be write in the following form

$$\mathcal{Z}_\phi m_\Psi^2(R, c.t) = \tilde{M}_\Psi^2(R, c.t) + \frac{4!}{\mathcal{Z}_\lambda \lambda R} \left[ 1 - \left( \frac{\mathcal{Z}_\phi}{\mathcal{Z}_\lambda} \frac{12m_R^2 - \mathcal{Z}_\phi^2}{12M_\phi^2(R, c.t)} \right) \right] \times \int \frac{d^4p}{(2\pi)^4} \frac{1}{\left[ \mathcal{Z}_\phi^2 p^2 - M_\phi^2(R, c.t) \right]^2} + 2\delta G \tan(\theta_{22}),$$

(3.25)

in which we consider the constant background $\phi_0$ with the quantum contributions, seen in eq. (2.20). As we know $\lambda_R > 0$, so we have a composite scalar field particle due to the existence of real pole for the two point function when

$$\tilde{M}_\Psi^2(R, c.t) \geq \frac{4!}{\mathcal{Z}_\lambda \lambda R} \left[ 1 - \left( \frac{\mathcal{Z}_\phi}{\mathcal{Z}_\lambda} \frac{12m_R^2 - \mathcal{Z}_\phi^2}{12M_\phi^2(R, c.t)} \right) \right] \times \int \frac{d^4p}{(2\pi)^4} \frac{1}{\left[ \mathcal{Z}_\phi^2 p^2 - M_\phi^2(R, c.t) \right]^2} + 2\delta G \tan(\theta_{22}),$$

(3.26)

In another hand $C^g_F = -\frac{1}{2M_\phi^2(R,c.t)}$ and $\mathcal{Z}^g_R \phi_0(R, c.t) = \mathcal{Z}^g_{m} m_R$, so the bound state conditions for composite particle from the fermion-antifermion field $\tilde{S}_R$ can be write in the following form

$$\mathcal{Z}^g_{m}^2 \tilde{S}_R^2(R, c.t) = \mathcal{Z}^g_{C_F} \tilde{S}_R^2(R, c.t) - \frac{1}{2} \frac{M_\phi^2(R, c.t)}{\mathcal{Z}_{m}^2 m_R^2} \phi_0^2(R, c.t) \int \frac{d^4p}{(2\pi)^4} \frac{1}{\left[ \mathcal{Z}^g_{m}^2 p^2 - M_\phi^2(R, c.t) \right]^2} + 2\delta G \tan(\theta_{22}).$$

(3.27)

So in the weak regime for the couplings $(g_R, \lambda_R)$, $C_F \to 0$ and $\phi_0^2 \to \infty$ and so we do not have a composite fermion-antifermion due to the existence of a imaginary pole for the two point Green function. Contrarily, in the strong regime for the couplings $(g_R, \lambda_R)$ we have that

$$\mathcal{Z}_m^2 \tilde{S}_R^2(R, c.t) = \mathcal{Z}_C^g \tilde{S}_R^2 + \frac{1}{2} \mathcal{Z}^g_R (2 \mathcal{Z}_m^2 m_R^2 \phi_0 - \mathcal{Z}_\lambda^\frac{1}{2} \phi_0^3 + 4 \mathcal{Z}_\phi^\frac{1}{2} \mathcal{Z}_\lambda^\frac{1}{2} \phi_0 \Psi_0) + \mathcal{Z}_S^\frac{1}{2} \mathcal{Z}_C^\frac{1}{2} S_0 \phi_0^2 - \frac{1}{2} \frac{M_\phi^2(R, c.t)}{\mathcal{Z}_m^2 m_R^2} \phi_0^2(R, c.t) \int \frac{d^4p}{(2\pi)^4} \frac{1}{\left[ \mathcal{Z}_m^2 p^2 - M_\phi^2 \right]^2} + 2\delta G \tan(\theta_{22}).$$

(3.28)

After simplifications we arrived in the inequality for the existence of composite fermion-antifermion state

$$\frac{1}{2} \frac{M_\phi^2(R, c.t)}{\mathcal{Z}_m^2 m_R^2} \int \frac{d^4p}{(2\pi)^4} \frac{1}{\left[ \mathcal{Z}_m^2 p^2 - M_\phi^2 \right]^2} \phi_0^2(R, c.t) \leq \mathcal{Z}_\phi^\frac{1}{2} \mathcal{Z}_C^\frac{1}{2} \mathcal{Z}_m^2 m_R + \frac{1}{2} \frac{M_\phi^2(R, c.t)}{\mathcal{Z}_m^2 m_R^2} \left( \mathcal{Z}_m^2 m_R^2 + 4 \mathcal{Z}_\phi^\frac{1}{2} \mathcal{Z}_\lambda^\frac{1}{2} \Psi_0 + 4 \mathcal{Z}_\phi^\frac{1}{2} \mathcal{Z}_\lambda^\frac{1}{2} \phi_0 \Psi_0 \right) + \mathcal{Z}_S^\frac{1}{2} \mathcal{Z}_C^\frac{1}{2} S_0 \phi_0^2 + 2\delta G \tan(\theta_{22}),$$

(3.29)
or we can write the previous inequality in terms of $\lambda_R$, i.e.

$$
\mathcal{L}_\lambda \lambda_R \geq \frac{1}{\int d^4 p \frac{1}{(2\pi)^4 |\mathcal{L}_p^2 p^2 - M_\phi^2(R,c.t)|^2}} \frac{1}{2 \mathcal{L}_\phi m_R^2} \times \mathcal{L}_g^2 M_\phi^2(R,c.t) \left[ \frac{1}{\mathcal{L}_\phi} 12 m_R^2 - \mathcal{L}_\lambda^2 \mathcal{L}_\phi^2 48 \Psi_0(R,c.t) \right] \times \mathcal{L}_m \left[ \mathcal{L}_m^2 m_R^2 + \frac{1}{2} \frac{\mathcal{L}_m m_g}{M_\phi^2} (\mathcal{L}_m^2 m_R^2 + 4 \mathcal{L}_\phi^2 \mathcal{L}_\xi^2 \Psi_0) + \mathcal{L}_S^2 \mathcal{L}_C^2 S_0 \right]^2 + \mathcal{L}_g^2 2 \delta G \tan(\theta_{22})
$$

We can conclude too that the existence or not of composite fermion-antifermion state are dictated by the value of $\phi_0^2$ (SSB). As we restrict to cases in which $m_R$ is very large, $M_\phi^2 = x \mathcal{L}_m m_R^2$ and $M_q = y \mathcal{L}_m m_R$, so with the solution in eq. (2.45) for $(x, y, S_0, \Psi_0)$ we can solve the above inequalities in eq. (3.26) and eq. (3.30) by the cut-off methodology ($\mathcal{L}_i \rightarrow 1$).

### 3.2 Effective action for the original model

However the auxiliary fields might not be quasiparticles of the system, in which case a current expansion can be performed based in the contributions of the two-fermion or two-boson states. By eliminating the auxiliary fields at the level discussed above by considering the only quadratic terms it can be written that:

$$
\bar{Z} = \det[-\alpha(R,c.t)\Box + \beta(R,c.t)]^{-\frac{1}{2}} \det[\epsilon(R,c.t)\Box + \lambda(R,c.t)]^{-\frac{1}{2}} \times \exp\{i \int d^4 x d^4 y [-\frac{1}{4} j_F V j_F - \frac{1}{4} j_B W j_B]\}. \quad (3.31)
$$

Now we define following quantities:

$$
\bar{S}_R \rightarrow S_R = \frac{1}{2} \int d^4 y V(x,y) j_F
$$

$$
j_F = Y_F(R,c.t) C_F^R (\bar{q} q)_0
$$

$$
V^{-1}(x,y) = [-\alpha(R,c.t)\Box + \beta(R,c.t)] \delta^4(x-y), \quad (3.32)
$$

$$
\bar{\Psi}_R \rightarrow \Psi_R = -\frac{1}{2} \int d^4 y W(x,y) j_B
$$

$$
j_B = Y_B(R,c.t) \lambda_R \phi_0^2
$$

$$
W^{-1} = [\epsilon(R,c.t)\Box + \sigma(R,c.t)] \delta^4(x-y), \quad (3.33)
$$

So that the vertices in eq. (3.31) can be written respectively as:

$$
-\frac{1}{4} j_F V j_F = \delta C_F^V(R,c.t)(\bar{q} q)_0^2,
$$
where \( \delta C_F^\gamma(R, c.t) = \left[ Y_F(R, c.t) C_F^\gamma \right]^2 \int \frac{d^4p}{(2\pi)^4} \frac{\exp[ip(x-y)]}{\left[ \alpha(R, c.t) p^2 + \beta(R, c.t) \right]} \),

\[
-\frac{1}{4} j_B W j_B = \delta \lambda^W(R, c.t) \phi_0^4,
\]

where \( \delta \lambda^W(R, c.t) = \left[ \Gamma_B(R, c.t) \lambda^R \right]^2 \int \frac{d^4p}{(2\pi)^4} \frac{\exp[ip(x-y)]}{\left[ -\nu(R, c.t) p^2 + \sigma(R, c.t) \right]}.
\]

Therefore the effective action for the YuM can be written as

\[
\Gamma_{eff} = \int d^4x \left\{ -\frac{1}{2} \langle \mathcal{L}_\phi \rangle \phi_0 \phi_0 + \frac{\delta m(R, c.t) \phi_0^2}{4!} + \frac{\delta \lambda(R, c.t) \phi_0^4}{4!} + \frac{\delta \lambda^W(R, c.t) \phi_0^4}{4!} \right\}
\]

\[
+ \mathcal{L}_q \delta q \delta q_0 + \left[ \mathcal{L}_g \delta g \phi_0 - \frac{b(R, c.t)}{2} + \delta M(R, c.t) \phi_0 q_0 + \delta C_F(R, c.t) \phi_0 q_0 \right]
\]

\[
- \frac{1}{2} \text{tr} \ln \left[ \alpha(R, c.t) + \beta(R, c.t) \right] - \frac{1}{2} \text{tr} \ln \left[ \nu(R, c.t) + \sigma(R, c.t) \right].
\]

In this calculation the role of the auxiliary fields is encoded in the non-linear behavior and dependencies of the resulting corrections for the masses and vertices of the original and effective parameters defined along the work on the ground state values of these auxiliary fields: \( \delta m, b, \delta M, \delta \lambda, \delta \lambda^W, C_F^R, \delta C_F, \) and \( \delta C_F^\gamma \), besides further contributions from the integration over the composite auxiliary fields.

We finished this section by adding a brief comment on the previous analysis. As we can see we achieved a way of analyzing how the counter-terms and the spontaneous symmetry breaking contribute to the chiral symmetry breaking and dynamical mass generation mechanism. This is seen not only in the influence of these ingredients in the gap equations above but also in the investigation of its effects in the construction of an effective action by external field methods as just noticed. The analysis presented here is based in the assumption that there are values (regions) of the coupling constants \( (g_R, \lambda_R) \) that permit solutions to the coupled gap equations, eq. (2.20) and eq. (2.34). In the limit of \( \mathcal{L}_i \to 1 \) we recover the usual gap equation with the ultraviolet divergences corresponding to the emergence of a (non renormalizable) low energy effective model. For the resulting effective action we can see four types of ultraviolet divergences in the fermion sector due to the fourth order expansion in large fermion effective mass, and three types of ultraviolet divergences in the boson sector. So that we have seven types of infinity and seven counter-terms \( \delta \mathcal{L}_q, \delta \mathcal{L}_g, \delta \mathcal{L}_\phi, \delta \mathcal{L}_m, \delta \mathcal{L}_\lambda, \delta \mathcal{L}_\psi, \delta \mathcal{L}_S \). In the effective action, eq. (3.35), we have five renormalization conditions, three in the scalar sector associated with the on shell behavior not only of the propagator (residue equal to 1 and pole in physical mass) but also the vertex, and two in the fermion sector again associated with behavior of the propagator. Finally with the two conditions from the gap equations, we determine all the counter-terms. The renormalization of the resulting effective model is a result of the original model (YuM), which is renormalizable, but as we have seen, the gap equations already correspond to the non-renormalizable model (Fermi), requiring more counter-terms to adjust the theory.
4 Outcomes and final comments

By considering the Yukawa model, throughout this work, different ways of dealing with field interactions in terms of equivalent linear quadratic or quartic structures were investigated, wherein the renormalization procedure plays an important role not only in the gap equations but also in the effective action due to quantum contributions, eliminating the infinities and adjusting the coupling parameters, masses and gaps to physical results. A modification in the Hubbard-Stratonovich auxiliary field (HSAF) identity [10], seen in eq. (2.6), that account for SSB order parameter contribution was also implemented. In the limit of $\phi_0 \to 0$ we have an usual Hubbard-Stratonovich identity with no spontaneous symmetry breaking of $Z_2$ symmetry. The resulting functional generator was then written in a better form by implementing the expansion in terms of currents. This change in the HSAF identity affects all the following analysis. We have a generator functional in eq. (2.26) with a residual fermion current-current effective interaction and from it we compute the effective potential with the coupled gap equations in eq. (2.34). If we restrict, as a physical approximation, to a large mass approximation for $m$, so that the massive mediator particle associated with the $\phi$ should be in some sense confined, the corresponding coupled gap equations are simplified and can be re-written in a form similar to the one known in the literature ($Z_i \to 1$). As a consequence we can explore solutions for the coupled gap equations in a particular regime of coupling constants ($g, \lambda$), seen in (2.45) and eq. (2.48) respectively. In the literature not only the large mass approximation are investigated, where two coupled gap equations were analyzed [51,53], but also the strong Yukawa coupling was explored [54].

As a second outcome the effective action of the model was calculated by the background external field method, seen in eq. (3.6) with eq. (3.18), in which we are left with an effective model for background fields interacting with two auxiliary fields, one may correspond to a two-boson quantum states and another one to a fermion-antifermion state. Bound state conditions for these two composite states were established in eq. (3.25) and eq. (3.27), wherein we see in eq. (3.29) that the existence or not of composite fermion-antifermion state are dictated by the value of SSB order parameter $\phi_0^2$. Finally from the YuM effective action, for example by means of eq (3.6) or eq. (3.35), we can see the appearance of seven types of (UV) infinities and seven counter-terms ($\delta \mathcal{L}_q, \delta \mathcal{L}_g, \delta \mathcal{L}_\phi, \delta \mathcal{L}_m, \delta \mathcal{L}_\lambda, \delta \mathcal{L}_\Psi, \delta \mathcal{L}_S$) adjusted by the physical conditions. The physical masses of the two boson composite state $\tilde{\Psi}$ and fermion-antifermion composite state $\tilde{S}$, whenever they are formed, were given in Eqs. (3.25, 3.26) and Eqs. (3.27, 3.30) respectively, corresponding to the poles of their propagators. They can only be obtained by considering the (ground state) solutions of the gap equations for all the fields involved being that the gap equations turn out to be coupled. This highly non linear set of equations cannot be easily diagonalized. The resulting off-diagonal terms, associated with the boson-boson and fermion-antifermion scalar fields ($S, \Psi$) interactions/mixings, were discussed in two levels. Firstly, the condition for minima of the effective potential that yield the gap equations can be done with the Hessian characteristic equation. Secondly, a mixing interaction in the effective action of the type $\delta G S R \Psi R$ required a rotation of eigenstates by defining a mixing angle $\theta_{22}$ to find the mass eigenstates of the 2-boson and 2-fermion composite states. Note that the corresponding mixing mechanism is
very different from the quark mixing in the Standard Model [38]. This mixing emerges from the above one loop calculation, one internal fermion line and one internal boson line, similarly to vacuum polarization worked out in [55] for neutral mesons mixing mechanism that is usually addressed in low energies effective models for QCD by means of other mechanisms [5–8]. Despite we present the phenomenon in the work, a further investigation is needed. Although the initial toy model considered here is pertubatively renormalizable, the resulting current-current fermion interaction is not. This is easily seen in the fact that the coupling constants $g$ and $\lambda$ are dimensionless and the Fermi constant $C_F$ has the dimension of squared length. So the investigation of how the renormalization affects the gap equations and the resulting effective action analysis is quite important to understand. As far as the authors know, the renormalization technique for the gaps in various models is not properly explored in the literature. A next step of investigation is the relationship between the auxiliary field formalism and the gap equations in a thermal environment to articulate eventual symmetries restorations. This matter will be investigated and requires elaborations.

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