Uninsured Idiosyncratic Investment Risk and Aggregate Saving

George-Marios Angeletos

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Room E52-251
50 Memorial Drive
Cambridge, MA 02142

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George-Marios Angeletos
MIT and NBER

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Abstract

This paper augments the neoclassical growth model to study the macroeconomic effects of idiosyncratic investment risk. The general equilibrium is solved in closed form under standard assumptions for preferences and technologies. Relative to complete markets, the steady state is characterized by both a lower interest rate and a lower capital stock when the elasticity of intertemporal substitution is sufficiently high. For plausible calibrations of the model, the reduction in aggregate savings and income is quantitatively significant. Finally, cyclical variation in private investment risks can amplify the transitional dynamics.

JEL codes: D52, E13, E32, G11, O16, O41.

Keywords: incomplete markets, heterogeneity, private equity, entrepreneurial risk, precautionary savings, amplification.

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1 Introduction

Following Bewley (1977), Aiyagari (1994) and Krusell and Smith (1998), an extensive literature has examined the macroeconomic implications of labor-income risk and precautionary saving, but has largely neglected idiosyncratic risk in private investment and capital income.¹ Labor-income risk is certainly important for understanding the consumption and saving behavior of the majority of the population. However, more relevant for aggregate investment and growth seems to be the minority of the rich and the entrepreneurs; and for them idiosyncratic capital-income risk can be quite significant. In the United States, for example, the richest 20 percent own 83 percent of net worth and 95 percent of financial (non-housing) assets. At the same time, the median rich household invests about half of his wealth in undiversified private equity.² What is more, idiosyncratic investment risk need not be limited to private equity if individuals do not diversify their holdings of publicly-traded stocks and other forms of savings.

This paper provides a tractable framework for examining the macroeconomic effects of uninsured idiosyncratic investment risk. This is obtained with a minimal modification of the neoclassical growth model.

Households supply labor in a competitive labor market and invest capital in privately-held firms. Firms in turn operate a neoclassical technology, with constants returns to scale (CRS) in capital and labor. Firm-specific productivity shocks translate to household-specific capital-income risk. Households have homothetic (CRRA/CEIS) preferences and can freely borrow and lend in a riskless bond. If they could also diversify their idiosyncratic investment risk, the model would reduce to the standard Ramsey-Cass-Koopman growth model; but they can not.

A key property of the neoclassical growth model is nevertheless unaffected: capital accumulation exhibits diminishing returns at the aggregate level, but linear returns at the individual level. For a given sequence of prices, the households’ decision problem is homothetic and the optimal decision rules are therefore linear in individual wealth. As a result, the aggregate dynamics do not depend on the wealth distribution, which avoids the “curse of dimensionality” and permits closed-form solution of the general-equilibrium recursion.

The steady state is easily characterized. Incomplete markets introduce a risk premium on private investment, which reduces the demand for capital. This effect would unambiguously lead to a lower capital stock if the interest rate were exogenously fixed. However, an Aiyagari-like precautionary-savings effect pushes the interest rate down, below the discount rate. The lower¹ For a review of Bewley models, see Ríos-Rull (1995) and Ljungqvist and Sargent (2000, ch. 13 and 14).

²Privately-held businesses indeed account for almost half of aggregate capital and output in the United States. Quadrini (1999), Gentry and Hubbard (2000) and Carroll (2001) document the importance of private equity for savings and wealth concentration. Moskowitz and Vissing-Jørgensen (2002) further document the dramatic lack of diversification in the private-equity holdings and the overall portfolio of private investors, and the high cross-section variation in the return to private equity.
interest rate in turn tends to stimulate investment. As a result, the general-equilibrium effect on capital accumulation is ambiguous in general.

A simple necessary and sufficient condition for the risk premium to dominate is next identified in the case of small risks: incomplete markets lead to a lower capital stock if and only if $\theta > \phi$, where $\theta$ is the elasticity of intertemporal substitution and $\phi$ is the ratio of private equity to total (financial plus human) wealth.

In US data, private equity is about one half of financial (non-human) wealth, which suggests that $\phi$ can not be higher than 0.5; in the model, $\phi$ is no more than the income share of capital. Micro estimates, on the other hand, of the elasticity of intertemporal substitution for wealthy investors – the agents that matter for the exercise of this paper – suggest a value for $\theta$ above 0.5, probably close to 1.3 Hence, the possibility that $\theta > \phi$, and therefore that idiosyncratic investment risk leads to low aggregate capital and income, appears empirically relevant.

These results may have important welfare and policy implications. Although I will not pursue these implications here, I want to highlight how they contrast with those of Bewley-type models (e.g., Aiyagari, 1994; Huggett, 1993, 1997; Krusell and Smith, 1998, 2004). These models focus on labor-income risk, which increases precautionary savings but has no effect on investment demand. They thus predict that financial innovations, policy reforms and other counterfactual exercises that lower idiosyncratic risk also lower aggregate savings, productivity, and wages.4 The opposite predictions are made here for idiosyncratic investment risk.

In the benchmark model, the entire capital stock is held in private firms. I next extend the model so that a fraction of aggregate savings is in “public equity”, where idiosyncratic risks are pooled. Because the low risk-free rate stimulates investment in public equity, the negative impact of incomplete markets on aggregate savings is significantly mitigated. Nevertheless, incomplete markets now also reduce aggregate total factor productivity by shifting resources away from the more risky but also more productive private equity. As a result, the impact on aggregate output remains quantitatively important.

The lack of estimates for the magnitude of idiosyncratic investment risk does not permit a precise quantitative analysis. I nevertheless show that large effects on savings and income are consistent with modest idiosyncratic risks and low excess returns in private equity. When I calibrate the model so that private equity accounts for half the capital stock and the associated risk premium is as low as 1% or 2%, the saving rate is about 2 or 3 percentage points lower than under complete markets and aggregate income is about 10% lower.

Turning to transitional dynamics, I identify two potential sources of amplification. First, cyclical variation in the level of uninsured investment risk generates cyclical variation in private

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3See the discussion in Section 4.1 and the references therein.
4See, for example, Krusell and Smith (2004), who examine the welfare and distributional effects of eliminating business-cycle variation in idiosyncratic labor-income risk.
risk premia, thus reinforcing the cyclicality in the demand for private equity. Second, the general-equilibrium interaction of wealth and risk taking introduces a macroeconomic complementarity: individual investment depends on the present value of future income, which in turn depends on current aggregate investment. A short of “Keynesian accelerator” thus emerges in a relatively standard neoclassical economy.

In plausible calibrations, the complementarity alone turns out to have a rather modest effect: it is largely offset by a “neoclassical” price effect, namely the endogenous reaction of interest rates. In contrast, cyclical risks can have strong effects: a flight to quality during recessions generates endogenous variation in the Solow residual, thus amplifying the transitional dynamics.

**Related Literature.** As mentioned already, the paper contributes to the Bewley literature by examining the macroeconomic impact of idiosyncratic capital-income, rather than labor-income, risk. In this respect, the paper complements Angeletos and Calvet (2000, 2003), which was probably the first to introduce idiosyncratic production risk in a Bewley economy, but assumed constant *absolute* risk aversion (CARA), thus killing the effect of wealth on precautionary savings, risk taking, and investment. Here instead I assume standard CRRA/CEIS preferences, along with a competitive labor market and a public-equity sector, thus improving both the qualitative and the quantitative content of the analysis. Also related is Meh and Quadrini (2004), which examines the potential welfare benefits of state-contingent contracts as compared to simple debt contracts in an economy where entrepreneurs can not diversify their risks due to agency problems.

The paper also extends previous work on the impact of rate-of-return risk in AK models (e.g., Levhari and Srinivasan, 1969; Sadmo, 1970; Obstfeld, 1994; Devereux and Smith, 1994; Jones, Manuelli, and Stacchetti, 2000; Krebs, 2003). These models also predict that investment risk reduces savings if and only if the elasticity of intertemporal substitution is above a threshold; but ignoring labor income or other sources of wealth biases this threshold upwards, which in turn is critical for assessing whether a negative or a positive effect on savings is more likely empirically (see Section 4.1 for details). Moreover, these models obtain tractability only by eliminating transitional dynamics.

An extensive literature has examined how wealth may affect the *ability* to invest when entrepreneurs face credit constraints (e.g., Bernanke and Gertler, 1989, 1990; Banerjee and Newman, 1993; Kiyotaki and Moore, 1997; Caggeti and De Nardi, 2003; Buera, 2004). This paper shows

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5 The paper, however, does not help explain wealth inequality, which is in contrast to the Bewley tradition (see especially Krusell and Smith, 1997; Castañeda, Diaz-Giménez, and Ríos-Rull, 2003; and Caggeti and De Nardi, 2003).

6 Note that the complementarity identified here – a small contribution to the literature on macroeconomic complementarities (e.g., Matsuyama, 1991; Benhabib and Farmer, 1994; Cooper, 1999) – relies on wealth effects.

7 Some papers consider aggregate rather than idiosyncratic rate-of-return risk; with an AK technology, however, this makes little difference (actually, no difference if the shocks are uncorrelated over time). Also, some studies allow for an additional channel via which incomplete markets can affect growth, namely the allocation of savings across different investment opportunities.
how wealth may affect the willingness to invest even in the absence of borrowing constraints. The distinction may be important for at least two reasons. First, although credit constraints and uninsurable risks share the prediction that investment is sensitive to wealth, the welfare and policy implications may be quite different. For example, redistributing from the rich to the poor has no impact on aggregate returns in the model of this paper. Second, the impact of investment risk, unlike that of credit constraints, need not vanish as agents get wealthier. This may help explain the difference with Kocherlakota (2000), who finds the quantitative importance of credit constraints in baseline calibrations of the neoclassical growth model to be limited.

The rest of the paper is organized as follows. Section 2 introduces the basic model. Sections 3 and 4 analyze the general equilibrium and the steady state. Section 5 introduces public equity. Section 6 examines the transitional dynamics. Section 7 concludes. All proofs are in the Appendix.

2 The Model

Time is discrete, indexed by $t \in \{0, 1, ..., \infty\}$. The economy is populated by a continuum of infinitely-lived households, indexed by $i$ and distributed uniformly over $[0, 1]$. All firms in the economy are privately held, and each household owns a single firm, so that firm $i$ is identified as the firm owned by household $i$. Firms employ labor in a competitive labor market but use the capital stock accumulated by their respective household-owner. Households, on the other hand, are each endowed with one unit of labor, which they supply inelastically in the competitive labor market; they can invest capital in the firm they own, but in no other firm; and they can freely trade a riskless bond, but cannot diversify the idiosyncratic risk in their capital income.8

Preferences. I assume an Epstein-Zin specification with constant elasticity of intertemporal substitution (CEIS) and constant relative risk aversion (CRRA). A stochastic consumption stream \( \{c^i_t\}_{t=0}^\infty \) generates a stochastic utility stream \( \{u^i_t\}_{t=0}^\infty \) according to the recursion

$$u^i_t = U(c^i_t) + \beta \cdot U(CE_t[U^{-1}(u^i_{t+1})]),$$

where \( CE_t(u^i_{t+1}) \equiv V^{-1}[E_tV(u^i_{t+1})] \) represents the certainty equivalent of \( u_{t+1} \) conditional on period-$t$ information. The utility functions $U$ and $V$ aggregate consumption across dates and states, respectively, and are given by

$$U(c) = \frac{c^{1-1/\theta}}{1-1/\theta} \quad \text{and} \quad V(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

where \( \theta > 0 \) is the elasticity of intertemporal substitution and \( \gamma > 0 \) is the coefficient of relative risk aversion.

8As in Aiyagari (1994), the market structure is exogenously assumed to be incomplete. Cole and Kocherlakota (2001) have provided a microfoundation for Aiyagari’s set-up based on private information and hidden savings, which might extend to the model of this paper. I leave this issue for future research.
None of the results of the paper relies on the Epstein-Zin preference specification. Standard expected utility is nested by letting $\theta = 1/\gamma$, in which case (1) reduces to

$$u^i_t = \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \frac{(c^i_{t+s})^{1-\gamma}}{1-\gamma}.$$ 

I nevertheless find it useful to allow $\theta \neq 1/\gamma$ in order to clarify that the sign of the steady-state effect of incomplete markets depends most critically on the elasticity of intertemporal substitution, not the coefficient of risk aversion. This also guides the appropriate calibration of the model.

**Budgets.** Let $\omega_t$ denote the wage rate in period $t$ and $R_t$ the gross risk-free rate between periods $t-1$ and $t$. The budget constraint of household $i$ in period $t$ is given by

$$c^i_t + k^i_{t+1} + b^i_{t+1} = \pi^i_t + R_t b^i_t + \omega_t,$$  

(3)

where $c^i_t$ denotes consumption, $k^i_{t+1}$ investment in physical capital, $b^i_{t+1}$ savings in the risk-free bond, and $\pi^i_t$ capital income (or the value of firm $i$, to be specified below).\(^9\) Naturally, consumption and physical capital can not be negative: $c^i_t \geq 0$ and $k^i_{t+1} \geq 0$. Finally, households can freely borrow in the riskless bond up to the “natural” solvency constraint that debt is low enough to be paid out even under the worst realization of idiosyncratic uncertainty.\(^10\)

**Technology and idiosyncratic risk.** The capital income of household $i$ is given by the earnings of firm $i$ net of labor costs:

$$\pi^i_t = y^i_t - \omega_t n^i_t,$$  

(4)

where $n^i_t$ denotes the amount of labor firm $i$ hires in period $t$ and $y^i_t$ the gross output it produces in the same period. Output in turn is given by

$$y^i_t = F(k^i_t, n^i_t, A^i_t),$$

where $F : \mathbb{R}^3_+ \rightarrow \mathbb{R}_+$ is a neoclassical production technology – that is, $F$ exhibits constant returns to scale (CRS) with respect to $K$ and $L$, has positive and strictly diminishing marginal products, and satisfies the familiar Inada conditions – and $A^i_t$ represents an exogenous production shock specific to firm $i$.

The shock $A^i_t$ is realized in the beginning of period $t$, after capital $k^i_t$ is installed but before employment $n^i_t$ is chosen. It is independently and identically distributed across $i$ and $t$, with continuous p.d.f. $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. In order to interpret a higher $A^i_t$ as higher productivity (or higher profitability), I impose $F_A > 0$, $F_{KA} > 0$, and $F_{LA} > 0$. I finally let $F(K, L, 0) = 0$, meaning that the worst idiosyncratic event leads to zero output, and normalize $\bar{A} \equiv \int A\psi(A)\,dA = 1$.

\(^9\)The budget constraint in (3) is expressed in terms of stock variables: $R_t$ equals 1 plus the net risk free rate and $\pi^i_t$ includes the value of the beginning-of-period non-depreciated capital stock installed in firm $i$.

\(^10\)As shown in Aiyagari (1994), given the non-negativity of consumption, this constraint is equivalent to imposing a non-Ponzi game condition.
Equilibrium. Households choose plans \( \{c_i^n, n_i^t, k_i^t, b_i^{t+1}\}_{t=0}^\infty \) contingent on the history of their idiosyncratic shocks so as to maximize their life-time utility. Idiosyncratic uncertainty, however, washes out at the aggregate. I thus define an equilibrium as a deterministic sequence of prices \( \{\omega_t, R_t\}_{t=0}^\infty \), a deterministic macroeconomic path \( \{C_t, K_t, Y_t\}_{t=0}^\infty \), and a collection of contingent plans \( \{c_i^n, n_i^t, y_i^t, k_i^t, b_i^{t+1}\}_{t=0}^\infty \), \( i \in [0, 1] \), such that the following conditions hold:\(^{11}\)

(i) Optimality: \( \{c_i^n, n_i^t, y_i^t, k_i^t, b_i^{t+1}\}_{t=0}^\infty \) maximizes \( u_i^t \) for every \( i \).

(ii) Labor-market clearing: \( \int c_i^n = 1 \) in all \( t \).

(iii) Bond-market clearing: \( \int b_i^t = 0 \) in all \( t \).

(iv) Aggregation: \( C_t = \int c_i^n, \ Y_t = \int y_i^t, \ K_t = \int k_i^t \) in all \( t \).

3 Equilibrium Characterization

3.1 Individual behavior

The idiosyncratic state of agent \( i \) in period \( t \) is summarized by \( (k_i^t, b_i^t, A_i^t) \). For a given price sequence, the value function can therefore be denoted by \( V(k, b, A; t) \). Since, by the assumption \( F(K, L, 0) = 0 \), the worst possible realization of capital income is zero, the natural solvency constraint reduces to \( b_{i+1}^t \geq -h_t \), where

\[
h_t = \sum_{j=1}^{\infty} \frac{\omega_{t+j}}{R_{t+1}...R_{t+j}}
\]

denotes the present value of future labor income (a.k.a. “human wealth”). The household’s problem can thus be represented by the following dynamic program:

\[
V(k, b, A; t) = \max_{c, n, k, b} U(c) + \beta \cdot UY^{-1} \left( \int T U^{-1} V(k', b', A'; t + 1) \psi(A') dA' \right)
\]

s.t. \( c + k' + b' = \pi + Rb + \omega \)

\( \pi = F(k, n, A) - \omega n \)

\( c \geq 0, \ k' \geq 0, \ b' \geq -h_t \)

This reduces to the more familiar Bellman equation, \( V(\cdot; t) = \max \{U(\cdot) + \beta \mathbb{E}_t V(\cdot; t + 1)\} \), in the case of expected utility \((\theta = 1/\gamma)\).

I solve this problem in two steps: first for the optimal labor demand of firm \( i \); and then for the optimal consumption, savings and investment of household \( i \).

Since employment \( n_i^t \) affects only earnings \( \pi_i^t \) in period \( t \), and since it is chosen after the capital stock \( k_i^t \) has been installed and the contemporaneous shock \( A_i^t \) has been observed, the optimal \( n_i^t \) maximizes \( \pi_i^t \) state by state. By CRS then the optimal \( n_i^t \) and the maximal \( \pi_i^t \) are linear in \( k_i^t \): the

\(^{11}\) With some abuse of notation, whenever I write \( \int x_i^t \) for some variable \( x \in \{c, n, y, k, b\} \), I mean the cross-sectional expectation of \( x \) in period \( t \).
individual firm can always adjust its employment in proportion to its capital stock, implying that the individual household faces linear returns in his investment.

**Lemma 1** Given $(\omega_t, A^i_t, k^i_t)$, labor demand and capital income are linear in $k^i_t$, decreasing in $\omega_t$, and increasing in $A^i_t$:

$$ n^i_t = n(A^i_t, \omega_t)k^i_t \quad \text{and} \quad \pi^i_t = r(A^i_t, \omega_t)k^i_t, $$

(6)

where $r(A, \omega) \equiv \max_L [F(1, L, A) - \omega L]$ and $n(A, \omega) \equiv \arg \max_L [F(1, L, A) - \omega L]$.

Let $w^i_t \equiv \pi^i_t + R_t b^i_t + \omega_t$ denote the financial (or “non-human”) wealth of household $i$ in period $t$. The budget constraint reduces to $c^i_t + k^i_{t+1} + b^i_{t+1} = w^i_t$ and, by Lemma 1,

$$ w^i_t = r(A^i_t, \omega_t)k^i_t + R_t b^i_t + \omega_t. $$

(7)

Note then that conditioning on $(k^i_t, b^i_t, A^i_t)$ is useful only for evaluating the optimal $n^i_t$ and the associated $w^i_t$ in (7). It follows that the household’s savings problem reduces to

$$ V(w; t) = \max_{(c, k, b) \in \mathbb{R}_+ \times [-h_t, \infty]} U(c) + \beta \cdot U Y^{-1} (\int_Y U^{-1} V(w'; t + 1) \psi(A') dA') $$

(8)

s.t. $c + k + b = w$, $w' = r(A', \omega_{t+1})k' + R_{t+1}b' + \omega_{t+1}$.

(With slight abuse of notation, $V$ now denotes the value function in terms of financial wealth.)

This problem is formally similar to the classic portfolio problem studied by Samuelson (1969) and Merton (1969): preferences are homothetic (by assumption) and wealth is linear in all assets (by CRS/Lemma 1). That the risky asset is physical investment in a privately-held business rather than a financial security, that the payoff of this asset depends on the wage rate and thereby on the aggregate capital stock, or that the risk is idiosyncratic, are important for the general-equilibrium properties of the economy, but do not affect the mathematical structure of the individual decision problem.

**Lemma 2** Given prices, optimal consumption, investment and bond holdings are linear in wealth:

$$ c^i_t = (1 - s_t)(w^i_t + h_t) $$

(9)

$$ k^i_{t+1} = s_t \phi_t(w^i_t + h_t) $$

(10)

$$ b^i_{t+1} = s_t(1 - \phi_t)(w^i_t + h_t) - h_t $$

(11)

where $w^i_t$ and $h_t$ are given by (7) and (5) and

$$ s_t = \frac{1}{1 + \{ \sum_{r=t}^{\infty} \prod_{j=1}^{r} \beta^a \rho_j^{b-1} \}^{-1}} $$

(12)

$$ \rho_t = \rho(\omega_{t+1}, R_{t+1}) \equiv \max_{\varphi \in [0, 1]} \mathbb{E}_t [\varphi r(A_{t+1}, \omega_{t+1}) + (1 - \varphi) R_{t+1}] $$

(13)

$$ \phi_t = \phi(\omega_{t+1}, R_{t+1}) \equiv \arg \max_{\varphi \in [0, 1]} \mathbb{E}_t [\varphi r(A_{t+1}, \omega_{t+1}) + (1 - \varphi) R_{t+1}] $$

(14)
To interpret the above conditions, note that the sum $w_i^t + h_t$ represents the “effective” wealth of household $i$, $s_t$ the saving rate out of effective wealth, $\phi_t$ the fraction of savings allocated to capital, and $\rho_t$ the risk-adjusted return to savings (a.k.a. the certainty equivalent of the overall portfolio return). Condition (12) follows from the Euler condition and gives the saving rate as a function of current and future risk-adjusted returns. Because of the familiar income and substitution effects, this is an increasing function if $\theta > 1$ and a decreasing one if $\theta < 1$.

Conditions (13) and (14), on the other hand, mean that the allocation of savings between private equity and bonds maximizes the risk-adjusted return to savings. Note that

$$\phi_t \approx (\ln \bar{\tau}_{t+1} - \ln R_{t+1})/(\gamma \sigma^2_{t+1}) \quad \text{and} \quad \rho_t \approx R_{t+1} \exp \left\{(\ln \bar{\tau}_{t+1} - \ln R_{t+1})^2/(2\gamma \sigma^2_{t+1})\right\},$$

(15)

where $\bar{\tau}_{t+1} \equiv E_t [r(A_{t+1}, \omega_{t+1})]$ in the mean and $\sigma_{t+1} \equiv \text{Var}_t [\ln r(A_{t+1}, \omega_{t+1})]$ the volatility of private returns (see Appendix). Assuming that $\sigma_{t+1}$ and $\omega_{t+1}$ are independent, $\phi_t$ and $\rho_t$ are decreasing in both $\omega_{t+1}$ and $\sigma_{t+1}$. The first effect is simply because a higher wage reduces firm earnings and capital returns for every realization of the productivity shock; the second effect is simply due to risk aversion.

3.2 General equilibrium

By Lemma 1 and the fact that there is a continuum of agents and the shocks are i.i.d. across them, aggregate employment and capital income are given by $N_t = \bar{n}(\omega_t)K_t$ and $\Pi_t = \bar{r}(\omega_t)K_t$, where $\bar{n}(\omega) \equiv \int n(A, \omega)\psi(A)dA$ and $\bar{r}(\omega) \equiv \int r(A, \omega)\psi(A)dA$. It follows that the labor market clears in period $t$ if and only if $\omega_t = \omega(K_t)$, where $\omega(K) \equiv \bar{n}^{-1}(1/K)$. Similarly, aggregate gross output – including non-depreciated capital – is given by $Y_t = \Pi_t + \omega_t = f(K_t)$, where $f(K) \equiv \bar{r}(\omega(K))K + \omega(K)$. By Lemma 2, in turn, consumption, bond holdings, and private investment are linear in individual wealth and therefore the corresponding aggregates are not affected by wealth inequality. Using these properties and aggregating across agents, we conclude to the following closed-form recursive characterization of the general equilibrium.

**Proposition 1 (General Equilibrium)** In equilibrium, the aggregate dynamics satisfy

$$C_t + K_{t+1} = Y_t = f(K_t)$$

(16)

$$C_t = (1 - s_t) [f(K_t) + H_t]$$

(17)

$$(1 - s_t)^{-1} = 1 + \beta^t \rho^t_{t+1} (1 - s_{t+1})^{-1}$$

(18)

$$K_{t+1} = \phi_t s_t [f(K_t) + H_t]$$

(19)

$$\bar{n}(\omega_t)K_t = 1$$

(20)

$$H_t = \frac{\omega_{t+1} + H_{t+1}}{R_t}$$

(21)

where $\phi_t = \phi(\omega_{t+1}, R_{t+1})$ and $\rho_t = \rho(\omega_{t+1}, R_{t+1})$. 

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The interpretation of these conditions is straightforward. (16) is the resource constraint. (17) gives aggregate consumption and (18) the associated Euler condition. (19) gives the aggregate capital stock and (20) the clearing condition for the labor market. Finally, (21) is the present value of aggregate labor income in recursive form.

To see more clearly that the system is recursive, use (16), (20) and (21) to eliminate $C_t$, $\omega_t$, and $R_{t+1}$. The equilibrium dynamics then reduce to a three-dimension, first-order, difference-equation system in $(K_t, H_t, s_t)$. This is a dramatic gain in tractability as compared to most other incomplete-markets models, in which the entire wealth distribution – an infinitely-dimensional object – is a relevant state variable. The simple structure of the equilibrium recursion is further exploited in Section 6.2, when I analyze the transitional dynamics.

4 Steady State

4.1 Characterization

A steady state is a fixed point of the dynamic system (16)-(20). Since the general equilibrium was characterized in closed form for any kind of idiosyncratic risk, so does the steady state as well. For expositional simplicity, however, it is most useful to consider the case that the productivity shock is augmented to capital and lognormally distributed. I thus henceforth assume the following.

Assumption A1. $F(K, L, A) = F(AK, L, 1)$ and $\ln A \sim \mathcal{N}(-\sigma^2/2, \sigma^2)$.

The standard deviation $\sigma$ then parsimoniously parameterizes the amount of uninsured idiosyncratic risk in private production and investment.13

Proposition 2 (Steady State) *In steady state, the capital stock $K$ and the interest rate $R$ solve*

$$\beta^\theta \rho^{\theta-1} \left[ \phi f'(K) + (1 - \phi)R \right] = 1$$

(22)

$$\frac{f(K) - f'(K)K}{(R - 1)K} = \frac{1 - \phi}{\phi}$$

(23)

*where $\phi = \phi(\omega(K), R)$ and $\rho = \rho(\omega(K), R)$.*

Condition (22) follows from combining the resource constraint with the Euler condition and has a simple interpretation. The first term in the left-hand side of (22), $s = \beta^\theta \rho^{\theta-1}$, is the steady-state value of the saving rate; this is increasing (respectively, decreasing) in the risk-adjusted

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12 Although aggregates are well defined at the steady state, individual wealth is a martingale and there is no stationary wealth distribution. This is not uncommon in incomplete-market models, but here it can easily be fixed with the following modification: in every period, let a mass $\lambda \in (0, 1)$ of randomly selected households die and be replaced with an equal mass of new-born households; and let the assets of the dead households be distributed uniformly among the new-born households.

13 The first part of Assumption A1 implies that $f(K) = F(K, 1, 1)$ and $\bar{r}(\omega(K)) = F_K(K, 1, 1) = f'(K)$, for every $\sigma$ and $K$. An increase in $\sigma$ is hence equivalent to a mean-preserving spread in individual returns.
return $\rho$ if and only if $\theta > 1$ ($\theta < 1$) and reduces to $s = \beta$ when $\theta = 1$. The second term, $\phi f'(K) + (1 - \phi)R$, represents the aggregate return to savings; this is a weighted average of the marginal product of capital and the risk-free rate. The product of these two terms gives the growth rate of aggregate effective wealth. In the steady state, aggregate wealth must be constant, which gives (22). Condition (23), on the other hand, follows from clearing the bond market and requires that the ratio of the present value of labor income to the capital stock is consistent with the individuals’ optimal allocation of savings between private equity and the riskless bond.

When markets are complete, the optimality condition for $\phi$ reduces to the familiar arbitrage condition $f'(K) = R$. Condition (22) then reduces to $R = 1/\beta$ and finally (23) pins down $\phi$. When instead markets are incomplete, (22) pins a unique $K$ for any given $R$. Condition (23) then can be solved for $R$.

Clearly, it must be that $R < 1/\beta$, or otherwise aggregate consumption would explode to infinity and a steady state would not exist. This is similar to Aiyagari (1994) and reflects the precautionary motive for savings. But whereas in Aiyagari (1994) there is no investment risk and therefore $f'(K) = R$, here it must be that $f'(K) > R$, or otherwise agents would hold no capital in equilibrium and a steady state would again fail to exist. In other words, there is a tension between the precautionary motive for savings and the risk premium on investment. This leaves open the possibility that $f'(K) > 1/\beta$. Unlike Aiyagari (1994), the impact of incomplete markets on the capital stock is ambiguous in general.

The following result establishes that the steady-state effect on savings and income is negative if the elasticity of intertemporal substitution is sufficiently high.

**Proposition 3** For any $\sigma > 0$, there exists $\theta < 1$ such that the steady-state levels of capital, output, and consumption are lower under incomplete markets if and only if $\theta > \underline{\theta}$. For $\sigma > 0$ small enough,

$$\underline{\theta} \approx \phi \approx \frac{\alpha - \chi}{1 - \chi} \leq \alpha,$$

where $\alpha$ is the income share of capital and $\chi$ the rate of saving out of income.\(^\text{14}\)

The critical value $\underline{\theta}$ for the elasticity of intertemporal substitution (EIS) equals the contribution of private equity to effective wealth. The main intuition behind this result is the following. As first noted by Sadmo (1970), an increase in capital risk, by reducing the effective return to savings, has opposing income and substitution effects; and $\theta$ is what determines the strength of the substitution effect. What is new here is how the income effect depends on $\phi$: the income effect is weaker the smaller the contribution of risky capital to total wealth. In AK models where risky capital accounts for all wealth (e.g., Levhari and Srinivasan, 1969; Sadmo, 1970; Obstfeld, 1994; Devereux and Smith, \(^\text{14}\)That is, $\alpha \equiv f'(K) K / \bar{f}(K)$ and $\chi = I / \bar{f}(K)$, where $\bar{f}(K) = f(K) - (1 - \delta) K$ is GDP and $I = \dot{f}(K) - C = \delta K$ is gross saving (both evaluated at steady state). Note in particular that $\chi \neq s$.\)
1994), we have $\theta = \phi = 1$. In the neoclassical economy of this paper, instead, $\theta = \phi \leq \alpha < 1$. In other words, it is the existence of labor income – or other sources of income beyond the risky asset – that explains why the critical threshold for the EIS is lower in the model of this paper than in AK models.\textsuperscript{15}

The above highlights a potentially important margin that is absent in the model: occupational choice and the distribution of human wealth between workers and “capitalists”. Occupational choice can not be accommodated without destroying the tractability of the model. The role of human wealth, on the other hand, can be partially captured by introducing “hand-to-mouth” workers. These are agents that do not hold any assets, consume their entire labor income every period, and serve a single purpose in the model: they absorb a fraction $\xi \in (0, 1)$ of aggregate human wealth. Condition (24) then becomes

$$\theta \approx \phi \approx \left( \frac{(1 - \xi)}{(1 - \chi)} \left( \frac{\alpha - \chi}{1 - \chi} \right)^{-1} + \xi \right)^{-1}. $$

Therefore, $\theta$ increases with $\xi$, meaning that over-accumulation becomes more likely, and $\theta \to 1$ as $\xi \to 1$. Nevertheless, $\theta$ remains quite low for plausible parameters. For example, with $\chi = 20\%$ and $\alpha = 36\%$ (approximately US data), the threshold is $\theta = 0.32$ when $\xi$ is calibrated so that hand-to-mouth agents account for 50\% of aggregate consumption (the fraction estimated by Campbell and Mankiw, 1989) as compared to $\theta = 0.18$ when $\xi = 0$.\textsuperscript{16}

Some information about the threshold $\theta$ is contained also in the data. Carroll (2001) and Moskowitz and Vissing-Jørgensen (2002) document that private equity accounts for about one half of total financial wealth for either the rich or the entire US economy. Adjusting this for human wealth implies that 0.5 is an upper bound for $\phi$ and therefore for $\theta$ as well.\textsuperscript{17}

The empirical estimates of the EIS, on the other hand, vary a lot. Hall (1988) and Campbell and Mankiw (1989) obtain very low estimates using US macro data, but such estimates suffer

\textsuperscript{15}These intuitions and the formula for $\theta$ in (24) are exact only in the limit as $\sigma \to 0$. Nevertheless, (24) remains a good approximation at least for the range of the numerical exercises conducted in Section 4.2. Note also that the result relies on the other source of income being riskless; I would conjecture that this is not crucial as long as the idiosyncratic risk in other income is not highly correlated with the idiosyncratic risk in private equity.

\textsuperscript{16}To see this, let $\lambda$ denote the fraction of aggregate consumption that goes to hand-to-mouth agents (the Cambell-Mankiw fraction), $C'$ the total consumption of the other “regular” agents, and $Y$ the GDP in steady state. By definition, $C' = (1 - \lambda)Y$; by the resource constraint, $C' = Y - \xi(1 - \alpha)Y - I$, where $I = \chi Y$. Combining and solving for $\xi$ gives $\xi = (\lambda - \chi)/(1 - \alpha)$. With $\alpha = 36\%$ and $\chi = 20\%$, $\theta < 0.5$ as long as $\lambda < 68\%$.

\textsuperscript{17}In the model analyzed so far, private equity accounts for the entire financial wealth, but this can be relaxed either by introducing public equity (which I do in Section 5) or more simply by allowing the riskless asset to be in positive net supply. In particular, let this supply be $B \geq 0$ and let $\lambda \equiv K/(B + K)$ be the ratio of private equity to financial wealth evaluated at steady state (in US data, $\lambda \approx 0.5$). Then, it is easy to show that (24) becomes

$$\theta \approx \phi \approx \left( \left( \frac{\alpha - \chi}{1 - \chi} \right)^{-1} + \left( \frac{\lambda}{1 - \lambda} \right)^{-1} \right)^{-1} \leq \lambda. $$

Moreover, $\theta \leq \lambda$ is robust to $\xi > 0$ (hand-to-mouth workers).
from aggregation bias and ignore heterogeneity. Using British data, Attanasio and Weber (1993) show that correcting for aggregation bias raises the estimated EIS from around 0.3−0.35 to around 0.6−0.8. Moreover, the estimated EIS tends to increase with wealth and/or asset holdings (Blundell, Browning and Meghir, 1994; Attanasio and Browning, 1995; Vissing-Jørgensen, 2002; Attanasio, Banks and Tanner, 2002; and Guvenen 2005b). The exact estimates vary a lot across studies and different specifications, but most are above 0.5 and frequently around 1, especially if one concentrates on the top layers of wealth or asset holdings. For example, using data from the US Consumer Expenditure Survey (CEX) and an Epstein-Zin specification as in this paper, Vissing-Jørgensen and Attanasio (2003) report baseline estimates around 1−1.4 for stockholders.

Although my model cannot accommodate heterogeneity in the EIS, what seems most relevant for the exercises conducted here is the EIS of rich investors, for they hold almost the entire capital stock in the economy. This suggests a value for \( \theta \) close to 1. Consistent with this approach is Guvenen (2005a). He considers an RBC model where wealthy stockholders have a higher EIS than poor non-stockholders and shows that, whereas the poor are important for the business-cycle properties of aggregate consumption (and hence for the EIS estimates obtained from Hall-type regressions), they are virtually irrelevant for the dynamics of capital: investment behaves as if only the rich high-EIS agents were present in the economy.

To recap, an EIS close to 1 and certainly higher than the threshold \( \theta \) seems most appropriate for the purposes of this paper.\(^{18}\) That incomplete markets lead to lower savings and lower income thus appears to be an empirically relevant scenario. I make a first attempt at quantifying these effects in the next section.\(^{19}\)

### 4.2 Numerical results

Thanks to the tractability of the model, the numerical solution of the steady state is trivial: substituting \( \phi = \phi(\omega(K), R) \) and \( \rho = \rho(\omega(K), R) \) into (22)-(23) gives a simple system of two equations in two unknowns, the steady-state levels of \( K \) and \( R \).

With a Cobb-Douglas technology and a lognormal productivity, the economy is fully parameterized by \( (\beta, \gamma, \theta, \alpha, \delta, \sigma) \). For my benchmark calibration, I let the time period be one year, the discount rate \( 1 - \beta^{-1} = 5\% \), the coefficient of relative risk aversion \( \gamma = 2 \), the elasticity of

\(^{18}\)Note also that micro estimates of the EIS are usually based on high-frequency data; if habit persistence or consumption commitments (e.g., Campbell and Cochrane, 1999) imply that the effective EIS is lower in shorter horizons, then the aforementioned micro evidence can underestimate the parameter that is relevant in this paper. Related is also Kevin Murphy’s remark (cited in Lucas, 1990) that, given the observed cross-country variation in growth rates, an EIS lower than 0.5 would imply an implausibly high cross-country variation in real returns.

\(^{19}\)Levine and Zamme (2002) have shown that the effect of market incompleteness in Bewley economies vanishes as the discount rate converges to zero. An analogue of this limit result appears to hold here: as \( \beta \to 1 \), (22) and (23) can hold only if \( R \to 1 \) and \( \phi \to 0 \), which in turn implies that the risk premium vanishes. Nevertheless, the simulations presented below deliver large quantitative effects with low discount rates and modest i.i.d. shocks.
intertemporal substitution $\theta = 1$, the income share of capital $\alpha = 40\%$, and the depreciation rate $\delta = 5\%$. These values are standard in the literature. What remains is $\sigma$, the standard deviation of the individual return in private equity.\footnote{To see that $\sigma$ is the standard deviation of investment returns, note that, under A1, $\ln r(A, \omega) = \ln A + \ln \tau(\omega)$ and therefore $\text{Var}[\ln r(A_{t+1}, \omega_{t+1})]^{1/2} = \sigma$. Alternatively, one can interpret $A_{t+1}$ as an idiosyncratic shock in the depreciation of capital, in which case $\delta$ is the mean and $\sigma$ the standard deviation of the rate of depreciation.}

There are many indications that idiosyncratic investment risks are quantitatively large. The estimated value of private equity in the United States is about as high as the value of public equity today – it was about twice as large in the 70’s and 80’s. More than 75% of aggregate private equity is owned by households for whom private equity constitutes at least half of their total net worth. The median rich (top 5% or so) household holds almost half of his total wealth, or almost 60% of his non-housing wealth, in private equity; and more than 70% of this is invested in a single company in which the household has an active management interest. The probability that a privately-held firm survives over the first 5 years of its life is less than 40 percent; and the variation in private investment returns is very large even conditional on survival.\footnote{For further details on these facts, see Carroll (2001) and Moskowitz and Vissing-Jørgensen (2002).}

Unfortunately, however, there are no precise estimates of the level of idiosyncratic investment risk. For example, Moskowitz and Vissing-Jørgensen (2002) and Bitler, Moskowitz, and Vissing-Jørgensen (2005) explore the cross-section of private-equity investors in the Survey of Consumer Finances, but are unable to provide a reliable measure for the standard deviation of individual returns due to lack of enough time-series variation in the data. For their numerical exercises, they instead proxy $\sigma$ with the standard deviation of the annual return to an individual publicly-traded stock. In want of a better option, I follow a similar strategy.

Campbell et al. (2001) report that the standard deviation of the annual return to a publicly-traded stock is around 50%. One possibility is that privately-held firms, being on average younger and smaller than publicly-held firms, face even higher risks. Perhaps a more likely possibility however is that publicly-held firms are willing to engage in more risky projects than privately-held firms. Hence, I would take 50% as an upper bound for $\sigma$. On the other hand, I would expect that $\sigma$ is no lower than the aggregate stock-market volatility, which pools all idiosyncratic risk and is about 17%. I thus choose $\sigma = 20\%$ as my benchmark, but also consider $\sigma = 40\%$ and $\sigma = 10\%$ for comparison.

The results are presented in Table 1. The first raw corresponds to the benchmark calibration; the rest check robustness to different values for $\sigma, \gamma, \theta, \beta, \alpha,$ and $\delta$.\footnote{For the quantitative results, income is measured by $GDP_{t} \equiv \tilde{f}(K_{t}) \equiv f(K_{t}) - (1 - \delta) K_{t} = K_{t}^{\alpha} L^{1-\alpha}$; the risk-free rate and the mean excess return on private equity by $R_{t}$ and $\bar{r}_{t} - R_{t}$, respectively; and the saving rate by $I_{t}/GDP_{t}$, where $I_{t} = K_{t+1} - (1 - \delta) K_{t}$.}
Table 1: Steady-state effects in the benchmark model. $\Delta$(Saving Rate) is the change in the aggregate saving rate between complete and incomplete markets; $\Delta$(GDP) is the corresponding percentage change in the aggregate level of income; interest rate is the rate of return on the riskless bond under incomplete markets; private premium is the mean excess return on private equity.

Under the benchmark calibration, the steady-state saving rate falls by 2.4 percentage points, from 20% under complete markets to 17.6% under incomplete markets. Consequently, aggregate capital falls by 19% and aggregate income by 8%. The risk-free rate, on the other hand, falls from 5% to 4.6%; the associated risk premium on private equity is 1.7%.

Not surprisingly, the effects are much stronger if $\sigma = 40\%$ and more modest if $\sigma = 10\%$. Raising risk aversion $\gamma$ from 2 to 4 (which is within the range of empirically plausible values) raises the reduction in saving rates from 2.4 to 4.1 percentage points and the reduction in income from 8% to 14%. Perhaps more surprisingly, letting the elasticity of intertemporal substitution be as low as 0.5 (or as high as 2) has little effect on the magnitude of capital and output losses. Also modest are the effects of the discount and depreciation rates, whereas quite strong is the impact of a higher $\alpha$, suggesting that human-capital accumulation could amplify the effects.

Behind these findings there is a familiar decision-theoretic effect: an increase in risk discourages private investment for any level of wealth (provided $\theta > \phi$). But there is also an indirect general-equilibrium effect: as all agents cut back in their investments, aggregate income falls in equilibrium, which further discourages risk taking and investment. This interaction between wealth and risk taking introduces a complementarity – a “multiplier” – that increases the overall impact of market incompleteness. I will revisit the role of this complementarity in transitional dynamics.
5 Two Sectors: Private and Public Equity

The analysis so far has assumed that all investment is subject to idiosyncratic risk. This is not necessarily a bad benchmark for less developed economies, in which production is dominated by privately-held firms. Nevertheless, it is important to understand the robustness of the results to the availability of a safe asset that is in positive net supply.\(^{23}\) I thus introduce a second sector of production, to be called “public equity”, in which ownership of capital is freely traded across agents and therefore idiosyncratic risks are fully diversified.

5.1 General equilibrium

Let \(X_t\) and \(L_t\) denote the total capital and labor allocated to the public-equity sector in period \(t\). Total output for this sector is given by \(G(X_t, L_t)\), where \(G\) is a neoclassical production function. Since public equity is risk-free, simple arbitrage implies that its return must equal the return to the riskless bond. Moreover, by profit maximization, \(\omega_t = G_L(X_t, L_t)\) and \(R_t = G_X(X_t, L_t)\). The rest of the equilibrium characterization is like in the benchmark model. Lemma 1 remains unaffected, whereas Lemma 2 extends with a minor modification, namely replacing bond holdings with the sum of bond and public-equity holdings. We can thus show the following.

**Proposition 4 (General Equilibrium)** In an equilibrium in which both sectors are active, the aggregate dynamics satisfy

\[
C_t + K_{t+1} + X_{t+1} = Y_t = F(K_t, N_t, 1) + G(X_t, L_t)
\]

\(25\)

\[
C_t = (1 - s_t) (Y_t + H_t)
\]

\(26\)

\[
(1 - s_t)^{-1} = 1 + \beta \theta \frac{\rho_t}{\rho_t} (1 - s_{t+1})^{-1}
\]

\(27\)

\[
R_t = G_X (X_t, L_t) \\
\omega_t = G_L (X_t, L_t)
\]

\(28\)

\[
K_{t+1} = \phi_t s_t (Y_t + H_t) \\
N_t = \bar{n}(\omega_t) K_t
\]

\(29\)

\[
N_t + L_t = 1
\]

\(30\)

\[
H_t = (\omega_{t+1} + H_{t+1}) / R_{t+1}
\]

\(31\)

where \(\rho_t = \rho(\omega_{t+1}, R_{t+1})\) and \(\phi_t = \phi(\omega_{t+1}, R_{t+1})\).

The above conditions have a simple interpretation. \((25)\) is the resource constraint of the economy. \((26)\) and \((27)\) give the equilibrium consumption and the Euler condition. \((28)\) gives the familiar conditions characterizing the equilibrium capital and employment in public equity, whereas \((29)\) gives the analogues for private equity. Finally, \((30)\) is the clearing condition for the labor market and \((31)\) the present value of aggregate labor income in recursive form.

\(^{23}\)In the benchmark model, the net supply of the riskless bond was zero (except for the variant in footnote 17). Nevertheless, agents had an implicit safe asset, the present value of their labor income.
5.2 Steady state

A steady state in which both sectors are active is a fixed point of the dynamic system (25)-(31). To simplify the analysis, it is useful to assume that the capital intensity of the technology used by public-equity firms is identical to the one in privately-held firms, in which case the productivity difference between two sectors can be parameterized by a single scalar.

**Assumption A2.** \( G(X, L) = F(X, L, 1/\mu) \) for some \( \mu > 1 \).

The restriction \( \mu > 1 \) simply means that private equity has a higher mean return than public equity; given that private equity is riskier than public equity, this is a necessary condition for private equity to be held in equilibrium.

Letting \( R(\omega) \equiv \max_l [G(1, l) - \omega l] \) and \( l(\omega) \equiv \arg \max_l [G(1, l) - \omega l] \), we can restate (28) as \( R_t = R(\omega_t) \) and \( L_t = l(\omega_t) X_t \). Moreover, the combination of A1 and A2 gives \( \bar{r}(\omega) = \mu R(\omega) \), so that \( \mu \) pins down the risk premium on private equity. It follows that \( \phi(\omega, R) = \varphi \) and \( \rho(\omega, R) = \varrho R \), where \( \varphi \) and \( \varrho \) are determined by the exogenous parameters \( \mu, \sigma \) and \( \gamma \) alone:

\[
\varphi \equiv \arg \max_{\phi \in [0, 1]} \mathbb{E}_t (\phi \mu A_{t+1} + 1 - \phi) \quad \text{and} \quad \varrho \equiv \max_{\phi \in [0, 1]} \mathbb{E}_t (\phi \mu A_{t+1} + 1 - \phi). \tag{32}
\]

These properties greatly simplify the characterization of the steady state.\(^{24}\)

**Proposition 5 (Steady State)** In a steady state in which both sectors are active:

(i) The interest rate is

\[
R = \beta^{-1} \varrho^{1/\theta - 1}(\varphi \mu + 1 - \varphi)^{-1/\theta} < 1/\beta \tag{33}
\]

where \( \varphi \) and \( \varrho \) are given by (32); the wage rate is then given by \( R(\omega) = R \) and the capital stocks by

\[
K = \frac{1/l(\omega) + \omega/(R - 1)}{\mu + 1/\phi - 1} \quad \text{and} \quad X = 1/l(\omega) - \mu K. \tag{34}
\]

(ii) There exists \( \theta < 1 \) such that \( \theta > \theta \) suffices for a local increase in \( \sigma \) to raise the interest rate and reduce private equity, total factor productivity, aggregate output, and aggregate consumption.

As compared to the benchmark model, public equity introduces three novel effects.

First, incomplete markets reduce the demand for private equity but increase the demand for public equity. Indeed, since the risk-free rate is necessarily lower than the discount rate, the capital-labor ratio in publicly-traded firms is unambiguously higher than what under complete markets. As a result, the impact of incomplete markets on aggregate savings may be ambiguous even for relatively high elasticities of intertemporal substitution.

\(^{24}\) If \( \mu \) is too high relative to \( \sigma \), the equilibrium may involve zero resources allocated to the public-equity sector. In what follows, I focus on the case where public equity remains active in equilibrium; the model otherwise reduces the one-sector benchmark examined before.
Second, an increase in idiosyncratic risk triggers a reallocation of resources (both capital and labor) from the more risky but more productive sector (private equity) to the less risky but less productive one (public equity), thus causing a reduction in aggregate total factor productivity. As a result, aggregate output can fall with \( \sigma \) even if aggregate capital does not.

Third, even though the risk-free rate is always below the discount rate, an increase in \( \sigma \) may locally increase the risk-free rate when both private and public equity are held. This is unlike either the Bewley class of models or the one-sector model of the previous section. The reason for this new effect is that the technology in the public equity sector imposes a negative relation between the wage rate and the interest rate, namely the relation implied by the equation of the input price ratio with the marginal rate of technical substitution. When an increase in \( \sigma \) causes a reallocation of resources from private to public equity, thus reducing aggregate productivity and wages, the reduction in wages is necessarily associated with an increase in interest rates. In the Bewley class of models, the same negative relation between wages and interest rates is present, since all capital is public, but it works the other way round – higher labor-income risk leads to a lower interest rate and thereby to a higher capital-labor ratio and a higher wage rate. In the one-sector model of the previous sections, on the other hand, the negative relation between wages and interest rates was broken, because the interest rate is not equated to the marginal product of capital.

5.3 Numerical results

The new parameter that needs to be calibrated is \( \mu \). Other things equal, \( \mu \) determines the cross-sectoral allocation of resources. As mentioned earlier, private and public equity each claim roughly one half of aggregate wealth in the United States. For any given set of values for \((\sigma, \beta, \gamma, \theta, \alpha, \delta)\), I thus calibrate \( \mu \) so that the implied steady-state shares of private and public equity in the aggregate capital stock are 50% each. The results are reported in Table 2.25

In the benchmark calibration (first row of Table 2), the reduction in the saving rate is 1.1 percentage point with public equity as compared to 2.4 percentage points in the benchmark model. Similarly, the reduction in the capital stock is now 12% as compared to 19% before. On the other hand, the reduction in the steady-state level of income is 7% as compared to 9% before. Hence, the impact of incomplete markets on aggregate savings is significantly mitigated by the introduction of public equity, but the effect on aggregate income remains strong – the reason is that incomplete markets now distort also aggregate total factor productivity.

This finding seems robust across different parameter specifications (rest of Table 2). For example, the reduction in steady-state income is 12% when \( \gamma = 4 \) and 19% when \( \sigma = 40\% \). The premium on private equity, on the other hand, is considerably lower. In the benchmark calibration,

\[25\text{Calibrating } \mu \text{ to the shares of employment or output, or varying the target shares to 40\% or 60\%, yields similar quantitative results.}\]
for example, the private premium is as low as 0.9% (compared to 1.7% without public equity). Even when \( \gamma = 4 \) or \( \sigma = 40\% \), the premium is no more than 3%. This is simply because the excess return required for holding a risky asset (here private equity) is lower the lower the fraction of savings invested in this asset. Large effects on savings and income thus appear consistent with modest investment risks and low private premia.

| \( \sigma \) | \( \gamma \) | \( \theta \) | \( \beta^{t-1} \) | \( \alpha \) | \( \delta \) | \( \Delta(\text{Saving Rate}) \) | \( \Delta(\text{GDP}) \) | Interest Rate | Private Premium |
|-------|-------|-------|-------|-------|-------|----------------|----------------|---------------|----------------|
| 20%   | 2     | 1     | 5%    | 40%   | 5%    | -1.2%          | -7%            | 4.89%         | 0.91%          |
| 40%   |       |       |       |       |       | -3.4%          | -19%           | 4.69%         | 2.99%          |
| 10%   |       |       |       |       |       | -0.3%          | -2%            | 4.97%         | 0.24%          |
| 20%   | 4     |       |       |       |       | -2.1%          | -12%           | 4.80%         | 1.72%          |
| 40%   |       |       |       |       |       | -5.4%          | -29%           | 4.53%         | 5.22%          |
| 10%   |       | 1     | 2     | 0.5   |       | -0.6%          | -4%            | 4.94%         | 0.48%          |
| 20%   |       |       |       |       |       | -0.6%          | -4%            | 4.94%         | 0.47%          |
|       | 1     | 5%    | 2     | 0.5   | 10%   | -1.2%          | -7%            | 4.92%         | 0.92%          |
|       |       |       |       |       |       | -1.1%          | -6%            | 4.83%         | 0.91%          |
|       | 1%    | 5%    | 2     | 0.5   |       | -0.8%          | -4%            | 4.94%         | 0.69%          |
|       | 5%    | 60%   |       |       |       | -1.4%          | -5%            | 0.98%         | 0.37%          |

Table 2: Steady-state effects in the model with public equity.

6 Transient Dynamics and Amplification

Since aggregate uncertainty is not allowed, I can not examine literally the business-cycle properties of the model. We can nevertheless get some guidance by studying the transitional dynamics.

For this purpose, I modify the model as follows. Let \( Z_t \) denote the aggregate labor productivity in period \( t \); with a Cobb-Douglas production function, variation in \( Z_t \) is equivalent to variation in total factor productivity for both sectors. I assume that \( Z_t \) follows the deterministic analogue of an \( AR(1) \) process:

\[
\ln Z_{t+1} = \rho \ln Z_t, \quad (35)
\]

where \( \rho \in [0,1) \) measures the persistence of productivity. I also allow for the level of idiosyncratic risk to vary with the state of the economy:

\[
\sigma_t \equiv \text{Var}_t \left( \ln A_{t+1}^i \right)^{1/2} = \sigma \left[ 1 - \eta \ln Z_t \right], \quad (36)
\]
where \( \sigma \geq 0 \) and \( \eta \geq 0 \) parameterize, respectively, the steady-state level and the cyclical elasticity of idiosyncratic risk. I can then mimic a recession with a reduction in \( Z_0 \) starting from steady state.

Thanks to the simple closed-form recursive structure of the general equilibrium, it is easy to compute the response of the economy to such a shock – or, more generally, the transitional dynamics from any given initial conditions. Letting \( \theta = 1 \) (benchmark calibration) further simplifies the equilibrium recursion by ensuring that \( s_t = \beta \) for all \( t \).

**Lemma 3** Suppose \( \theta = 1 \) and \((Z_t, \sigma_t)\) satisfy (35)-(36). There is a unique mapping \( \Omega : \mathbb{R}^4 \to \mathbb{R}^8 \) such that, for all \( t \geq 0 \),

\[
((Z_{t+1}, K_{t+1}, X_{t+1}, H_t), (C_t, Y_t, \omega_t, R_t)) = \Omega (Z_t, K_t, X_t, H_{t-1}).
\]

For any given \((Z_0, K_0, X_0, H_{-1})\), the whole path is computed simply by iterating \( \Omega \). Since \((Z_0, K_0, X_0)\) are historically given, one only needs to find the equilibrium value for \( H_{-1} \). Starting with an arbitrary guess for \( H_{-1} \), iterating \( \Omega \) to compute the implied \( \{\omega_t, R_t\}_{t=0}^T \) for \( T \) large enough, and letting \( H'_{-1} = \sum_{t=1}^{T} \omega_t / (R_1 \ldots R_{t-1}) \), gives a mapping from \( H_{-1} \) to \( H'_{-1} \). Iterating this mapping till \( H'_{-1} \approx H_{-1} \) gives the equilibrium.27

### 6.1 Some partial insights

Before simulating the transitional dynamics, let me give some intuition about the two potential sources of amplification – the cyclical variation in the level of uninsurable idiosyncratic risk and the equilibrium interaction of wealth and risk taking.

Consider first the role of (exogenous) cyclical variation in \( \sigma_t \). As the economy enters a recession (that is, after a negative shock in \( Z_0 \)), the level \( \sigma_t \) of uninsurable investment risk increases, implying a reduction in the willingness to invest in private equity. That is, the demand for investment is low during a recession, not only because interest rates are high – the standard reason in the complete-markets neoclassical model – but also because private risk premia are high. Moreover, as resources are diverted away from private equity towards public equity, an endogenous reduction in aggregate productivity takes place – the Solow residual itself is amplified.

Consider next the interaction of wealth and risk taking. To gain some insight, ignore for a moment the presence of public equity and the equilibrium variation in \( R, s, \) or \( \phi \). Conditions (18) and (19) then reduce to the following:

\[
K_{t+1} = s \phi [f (K_t) + H_t],
\]
\[
H_t = \sum_{j=1}^{\infty} R^{-j} \omega(K_{t+j}).
\]

26 If, instead, \( \theta \neq 1 \), the “state vector” \((Z_{t+1}, K_{t+1}, X_{t+1}, H_t)\) must be expanded to include \( s_t \).

27 Although this is not a contraction mapping, I obtained a fixed point for all the simulations reported below.
On the one hand, (37) implies that, other things equal, $K_{t+1}$ increases with either $K_t$ or $H_t$ and therefore the path of capital $\{K_{t+1}, K_{t+2}, \ldots\}$ increases with the path of human wealth $\{H_t, H_{t+1}, \ldots\}$. This effect reflects a decision-theoretic property: the dependence of individual risk taking on wealth. On the other hand, (38) implies that $\{H_{t+1}, H_{t+2}, \ldots\}$ increases with $\{K_{t+1}, K_{t+2}, \ldots\}$. This feedback reflects a general-equilibrium effect: the dependence of individual labor income and wealth on aggregate capital. The combination of these two effects gives rise to a dynamic macroeconomic complementarity: the anticipation of low income tomorrow leads every agent to invest less today, which in turn implies lower aggregate income tomorrow.

Few remarks are worth making about this complementarity. First, it introduces a short of “Keynesian accelerator” in an RBC economy: investment demand depends on anticipated income. Second, it derives from a pecuniary externality: it would be absent if wages and interest rates (and therefore $H_t$) were exogenously fixed. Third, it relies on investment being subject to uninsured idiosyncratic risk and risk taking being sensitive to future income – a combination that, to the best of my knowledge, is novel to the literature.  

However, this is only part of the story – the transmission channel discussed above contributes to amplifying the transitional dynamics, but there may be other counteracting effects in general equilibrium. To obtain a more complete picture and make a first pass at the quantitative potential of the amplification effect, I next simulate the response of the economy to an unanticipated shock in aggregate productivity $Z_0$.

### 6.2 A numerical example

There are various indications that idiosyncratic investment risks are highly cyclical – proxies such as bankruptcy rates, firm-exit rates, and firm-specific volatility in publicly traded firms vary a lot over the business cycle – but there is no hard evidence on which to base the calibration of their cyclical sensitivity ($\eta$). I am thus forced to make a plausible yet random guess: I calibrate $\eta$ so that a 2% reduction in output below steady state is associated with a 5% increase in $\sigma_t$. For the rest of the parameters, I use the benchmark calibration ($\sigma = 20\%$, $\gamma = 2$, $\theta = 1$, $1 - \beta^{-1} = 5\%$, $\alpha = 40\%$, $\delta = 5\%$) along with $\rho = 95\%$ and $\mu$ such that, again, 50% of capital is private equity.

Figure 1 illustrates the response of the economy to a negative 1% productivity shock starting from steady state (that is, $\ln Z_0 = -1\%(1 - \alpha)$, $K_0 = K_\infty$, and $X_0 = X_\infty$). The solid lines correspond to incomplete markets; the dashes ones to complete markets. The amplification effect is quite strong: the impact of the exogenous shock on aggregate output, consumption, and investment

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28Idiosyncratic investment risk is absent from Bewley models (e.g., Aiyagari, 1994), while the effect of anticipated income on risk taking is absent from $AK$ models (e.g., Obstfeld, 1994, Krebs, 2003), credit-constraint models with risk-neutral entrepreneurs (e.g., Bernanke and Gertler, 1989; Kiyotaki and Moore, 1997), and the CARA-normal economy of Angeletos and Calvet (2003, 2004). The paper thus makes a small but independent contribution to the literature on macroeconomic complementarities (e.g., Matsuyama, 1991; Benhabib and Farmer, 1994; Cooper, 1999).
Figure 1: The response of the economy to a negative 1% shock in aggregate productivity. Solid lines for incomplete markets with cyclical idiosyncratic risk; dashed lines for complete markets. All variables normalized by their respective steady-state levels.
Figure 2: The response of the economy with acyclical idiosyncratic risk.

is, respectively, 63%, 92%, and 53% higher than the impact of the shock under complete markets. Importantly, note how the amplification shows up in aggregate productivity: the reallocation of resources away from private equity contributes 42% of the overall reduction in the Solow residual.29

These results do not distinguish whether the main source of amplification is the cyclical variation in risk or the macroeconomic complementarity discussed earlier. To isolate the role of the latter, Figure 2 repeats the same exercise as Figure 1 setting \( \eta = 0 \). Clearly, the amplification effect is now much smaller. The reason the complementarity is weak is the presence of an offsetting general-equilibrium effect that the earlier intuitive discussion overlooked: the reduction in real interest rates during the recession counteracts the reduction in expected future wages and thereby mitigates the reduction in the demand for private equity. In other words, the “Keynesian accelerator” is here offset by a “neoclassical” price effect.

To recap, the complementarity is of theoretical interest on its own – for it is likely to extend to a larger class of models where agents face idiosyncratic investment risk – but it fails to generate strong amplification in the closed-economy model of this paper.30 In contrast, cyclical variation in idiosyncratic risk appears to have a more significant quantitative potential.

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29 For each variable \( X \), the amplification effect is measured by taking the maximal value of the ratio \( \hat{X}_t^{inco} / \hat{X}_t^{com} \) over the first 4 periods, where \( \hat{X}_t^{inco} = (X_t - X_\infty) / X_\infty \) denotes the period-\( t \) percentage change relative to steady state under incomplete markets, and \( \hat{X}_t^{com} \) the corresponding change under complete markets; the numbers reported in the text are \( \hat{X}_t^{inco} / \hat{X}_t^{com} - 1 \).

30 If the economy is open to an international market for the riskless bond, the risk-free rate is exogenously fixed and the offsetting effect discussed above is absent. Moreover, the dependence of risk premia on wealth can introduce co-movement between domestic savings and investment. Finally, a multi-country extension can generate a stationary wealth distribution for the world economy, with cross-country differences in wealth being explained by cross-country differences in within-country financial incompleteness.
7 Concluding Remarks

The merit but also the limitation of the analysis was the focus on idiosyncratic investment risk per se. I abstracted from aggregate uncertainty, occupational choice, and risk in human capital, which may have important implications for the theme of this paper. I also did not allow for any interesting mean-reverting force in individual wealth dynamics, such as the one introduced by labor-income risk (e.g., Aiyagari, 1994) or diminishing returns at the individual level (e.g., Banerjee and Newman, 1993; Caggeti and De Nardi, 2003; Covas, 2005). Extending the analysis in these dimensions is essential for a more complete understanding, and a more precise quantitative assessment, of the macroeconomic effects of idiosyncratic investment risk.

The goal of this paper was only to provide a positive benchmark, but I wish to conclude with a note on efficiency and potential policy implications.

An cornerstone in the Ramsey paradigm is that the optimal tax on capital income is zero under complete markets (Chamley, 1986; Judd, 1985; Lucas, 1990; Atkeson, Chari and Kehoe, 1999) and positive in Bewley economies (Aiyagari, 1995). The Mirrlees paradigm, on the other hand, has focused on insurance against labor-income risk (e.g., Golosov, Kocherlakota and Tsyvinski, 2000). A natural question, therefore, is how investment risk may affect the properties of optimal taxation in either paradigm. A related issue is constrained inefficiency in the GEI31 tradition (e.g., Geanakoplos and Polemarchakis, 1986). In particular, the pecuniary externality involved in investment choices may introduce a wedge between the equilibrium and the constrained-efficient level of investment.

In a two-period variant of the model consider here, where the lack of insurance can be generated by assuming private information of the productivity shock and where the GEI and Mirrlees notions of constrained efficiency coincide, it can be shown that incomplete markets lead to lower investment if and only if the EIS is sufficiently high, much alike in the steady-state results of this paper. Yet, the equilibrium level of investment is always lower than the constrained efficient level, due to the pecuniary externality. These results need not extend to more periods or to different informational assumptions. They nevertheless illustrate the possibility that an investment subsidy can be optimal even in cases where there is over-saving relative to the first best.

31 GEI is the acronym for general equilibrium with (exogenously) incomplete markets.
Appendix: Proofs

Proof of Lemma 1. By the linear homogeneity of \( F(K, L, A) \) in \((K, L)\),
\[
\frac{\pi^i_t}{k^i_t} = F(1, \frac{n^i_t}{k^i_t}, A^i_t) - \omega \frac{n^i_t}{k^i_t}.
\] (39)
Since \( k^i_t \) and \( A^i_t \) are known when \( n^i_t \) is chosen, the optimal \( n^i_t/k^i_t \) maximizes (39) for any \( A^i_t \), which gives (6). By definition of \( n(\cdot) \) and \( r(\cdot) \), \( F_t(1, n(A, \omega), A) \equiv \omega \) and \( r(A, \omega) \equiv F(1, n(A, \omega), A) - \omega n(A, \omega). \) Hence, \( F(K, L, 0) = 0 \) implies \( n(0, \cdot) = r(0, \cdot) = 0 \), whereas \( n(A, \cdot) > 0 \) and \( r(A, \cdot) > 0 \) for \( A > 0 \). Applying the implicit function theorem and using \( F_L > 0, F_{LL} < 0, F_A > 0, \) and \( F_{LA} > 0 \), we infer \( n_\omega < 0 < n_A \) and \( r_\omega < 0 < r_A \). Finally, the Inada conditions imply, for any \( A > 0 \), \( \lim_{\omega \to 0} n(A, \omega) = \lim_{\omega \to 0} r(A, \omega) = \infty \) and \( \lim_{\omega \to \infty} n(A, \omega) = \lim_{\omega \to \infty} r(A, \omega) = 0 \).

Proof of Lemma 2. For notational simplicity, I drop the superscript \( i \) and use \( r_{t+1} \) as a short-cut for \( r(A_{t+1}, \omega_{t+1}) \). I propose the following solution:
\[
V(w; t) = U(a_t(w + h_t)), \quad c(w; t) = (1 - s_t)(w + h_t), \quad k(w; t) = \phi_t s_t(w + h_t),
\] (40)
where \( a_t, s_t, \) and \( \phi_t \) are coefficients (time-varying but non-stochastic) to be determined. From the budget constraint and (40), we then infer \( b(w; t) = (1 - \phi_t)s_t(w + h_t) - h_t \). From (2) and (40), the certainty equivalent of the value of wealth is
\[
\mathbb{E}_t [U^{-1}V_{t+1}(w_{t+1})] = \mathcal{Y}^{-1} (\mathbb{E}_t [\mathcal{Y}U^{-1}V_{t+1}(w_{t+1})]) = a_{t+1} \left[ \mathbb{E}_t (w_{t+1} + h_{t+1})^{1-\gamma} \right]^{1/(1-\gamma)}.
\]
Hence, the first-order conditions with respect to \( k_{t+1} \) and \( b_{t+1} \) give:
\[
(c_t)^{-1/\theta} = \beta a_{t+1}^{-1/\theta} \left[ \mathbb{E}_t (w_{t+1} + h_{t+1})^{1-\gamma} \right]^{(\gamma-1/\theta)/(1-\gamma)} \cdot \mathbb{E}_t \left[ (w_{t+1} + h_{t+1})^{-\gamma} r_{t+1} \right],
\] (41)
\[
(c_t)^{-1/\theta} = \beta a_{t+1}^{-1/\theta} \left[ \mathbb{E}_t (w_{t+1})^{1-\gamma} \right]^{(\gamma-1/\theta)/(1-\gamma)} \cdot \mathbb{E}_t \left[ (w_{t+1} + h_{t+1})^{-\gamma} R_{t+1} \right].
\] (42)
Combining the two conditions, and using
\[
w_{t+1} = r_{t+1}k_{t+1} + R_{t+1}b_{t+1} + \omega_{t+1} = [\phi_t r_{t+1} + (1 - \phi_t)R_{t+1}] s_t(w_t + h_t) - h_{t+1},
\] (43)
we get \( \mathbb{E}_t \{[R_{t+1} + \phi_t(r_{t+1} - R_{t+1})]^{-\gamma} (r_{t+1} - R_{t+1}) \} = 0 \), or equivalently \( \phi_t = \phi(\omega_{t+1}, R_{t+1}). \)
Next, the envelope condition, \( V'(w_t; t) = U'(c_t) \), or equivalently \( a_t^{-1/\theta}(w_t + h_t)^{-1/\theta} = (c_t)^{-1/\theta} \), along with \( c_t = (1 - s_t)(w_t + h_t) \) from (40), implies
\[
a_t^{-1/\theta} = (1 - s_t)^{-1/\theta}.
\] (44)
Multiplying (41) and (42) with \( \phi_t \) and \( (1 - \phi_t) \), respectively, summing up, substituting \( w_{t+1} \) in the resulting relation from (43), and rearranging, gives the saving rate in recursive form:
\[
(1 - s_t)^{-1} = 1 + \beta^\theta \rho_t^\theta (1 - s_{t+1})^{-1},
\] (45)
where $\rho_t = \rho(\omega_{t+1}, R_{t+1})$. For any $\{\omega_t, R_t\}_{t=0}^{\infty}$ that is part of an equilibrium, $\sum_{s=t}^{\infty} \prod_{r=t}^{s} [\beta^\rho \rho_t^{\theta-1}]$ is finite. Forward iteration of (45) thus yields (12), with $s_t \in (0, 1)$. Using (40), we then verify that $c_t > 0$, $k_{t+1} > 0$, and $b_{t+1} > -h_t$. Finally, we verify that (40) solves the Bellman equation. Substituting (40) into (8) gives

$$U(a_t(w_t + h_t)) = U((1 - s_t)(w_t + h_t)) + \beta U(a_{t+1}[\mathbb{E}_t(w_{t+1} + h_{t+1})]^{1-\gamma})^{1/(1-\gamma)}.$$  

Dividing both sides by $U(w_t + h_t)$ and using (43) and (45), the above reduces to $a_t^{1-1/\theta} = (1 - s_t)^{-1/\theta}[(1 - s_t) + a_{t+1}^{1-1/\theta}(1 - s_{t+1})^{1/\theta} s_t]$, which is satisfied by (44). \[ \blacksquare \]

**Proof of Condition (15).** To simplify notation, let $r_{t+1}^i = r(A_{t+1}^i, \omega_{t+1})$, $\tilde{r}_{t+1} = \mathbb{E}_t r_{t+1}^i$, and $\sigma_{t+1}^2 = \text{Var}_t[\ln r_{t+1}^i]$. A second-order Taylor approximation for $\ln \rho_t$ around $\sigma_t = 0$ gives

$$\ln \rho_t \approx \phi_t \mathbb{E}_t[\ln r_{t+1}^i] + (1 - \phi_t) \ln R_{t+1} + \frac{1}{2} \phi_t (1 - \phi_t) \sigma_{t+1}^2 + \frac{1}{2} \gamma^2 \phi_t^2 \sigma_{t+1}^2. \quad (46)$$

Since $\phi_t$ maximizes $\rho_t$, the above also implies

$$\phi_t \approx \frac{\mathbb{E}_t \ln r_{t+1}^i - \ln R_{t+1} + \sigma_{t+1}^2/2}{\gamma \sigma_{t+1}^2}. \quad (47)$$

(These two equations are the analogues of (2.24) and (2.25) in Chapter 2 of Campbell and Viceira (2002); see there for a detailed derivation.) Combining the two conditions above and using $\mathbb{E}_t \ln r_{t+1}^i \approx \ln \mathbb{E}_t r_{t+1}^i - \text{Var}_t[\ln r_{t+1}^i]/2 = \ln \tilde{r}_{t+1} - \sigma_{t+1}^2/2$ gives (15). \[ \blacksquare \]

**Proof of Proposition 1.** Note that $\phi_t$ and $s_t$ are identical across agents. Aggregating the conditions in Lemma 2 over all $i$ and using the facts that $A_{t}^i$ and $k_{t}^i$ are independent and that $\Pi_t + \omega_t = \bar{r}(\omega_t)K_t + \omega_t = f(K_t) = Y_t$, we infer

$$W_t = \Pi_t + RB_t + \omega_t = f(K_t) + RB_t \quad (48)$$

$$C_t = (1 - s_t)(W_t + H_t) \quad (49)$$

$$K_{t+1} = s_t \phi_t (W_t + H_t) \quad (50)$$

$$B_{t+1} = s_t (1 - \phi_t)(W_t + H_t) - H_t \quad (51)$$

where $B_t = \int b_t di$. The bond market clears if and only if $B_t = 0$ and therefore $W_t = f(K_t) = Y_t$. Along with (49) and (50), this immediately gives (17) and (18). Next, adding up (49)-(51) gives the resource constraint (16), whereas (19) follows directly from (5). Finally, the labor market clears if and only if $1 = \int n_t^i = \bar{n}(\omega_t)K_t$, which gives (20). \[ \blacksquare \]

**Proof of Proposition 2.** Evaluating (48)-(51) [equivalently, (16)-(20)] in the steady state and combining, we get $K + H = s(W + H) = s[\bar{r}(\omega)K + RH] = s[\phi \bar{r}(\omega) + (1 - \phi)R](K + H)$, or equivalently

$$1 = s[\phi \bar{r}(\omega) + (1 - \phi)R], \quad 25$$
which is simply the stationarity condition for aggregate wealth. Substituting \( s = \beta^\theta \rho^{\theta-1} \) into the above gives condition (22) in the Proposition. Next, by (21), (50), (51), and the property that, under A1, \( \bar{r}(\omega) = F_K(K, l, 1) = f'(K) \) and \( \omega = f(K) - f'(K)K \), we have

\[
\frac{H}{K} = \frac{1 - \phi}{\phi} \quad \text{and} \quad H = \frac{\omega}{R - 1} = \frac{f(K) - f'(K)K}{R - 1}.
\]

Combining gives condition (23) in the Proposition.

**Proof of Proposition 3.** By (23), \( \phi \in (0, 1) \). For that to be true, it must be that \( f'(K) > R \), or otherwise the bond would dominate private equity. By risk aversion, then, \( R < \rho < \phi f'(K) + (1 - \phi)R \), which together with (22) gives also \( \rho < 1/\beta \). Combining, we have

\[
R < \rho < \phi f'(K) + (1 - \phi)R < f'(K) \quad \text{and} \quad R < \rho < 1/\beta.
\]

Next, taking logarithms of (22) and rearranging gives

\[
\theta \log[\beta f'(K)] = -\log[\phi + (1 - \phi)R/f'(K)] - (\theta - 1) \log[\rho/f'(K)].
\]

It follows that \( \beta f'(K) > 1 \) if and only if \( \theta > \underline{\theta} \) where

\[
\underline{\theta} = 1 - \frac{\log[\phi + (1 - \phi)R/f'(K)]}{\log[\rho/f'(K)]}.
\]

Note that \( \underline{\theta} \) above is expressed in terms of endogenous variables, but (52) ensures that \( \underline{\theta} < 1 \). Next, presuming continuity of the steady state in \( \sigma \), consider the limit as \( \sigma \to 0 \). Letting stars indicate the steady-state values under complete markets, we have that \( R, \rho, f'(K) \to R^* = \beta^{-1}, K \to K^* = f^{-1}(1/\beta) \), and \( \phi \to \phi^* \), where

\[
\frac{1 - \phi^*}{\phi^*} = \frac{f(K^*) - f'(K^*)K^*}{(R^* - 1)K^*} = \frac{f(K^*)/K^* - R^*}{R^* - 1} \in (0, 1).
\]

Thus, using L'Hopital's rule in (53), we have that \( \underline{\theta} \to 1 - (1 - \phi^*) = \phi^* \) as \( \sigma \to 0 \). Finally, note that GDP is given by \( \hat{f}(K) = f(K) - (1 - \delta)K \) and let \( \chi = \delta K/\hat{f}(K) \) and \( \alpha = \hat{f}'(K)K/\hat{f}(K) \) denote, respectively, the saving rate out of income and the income share of capital, both evaluated at the complete-markets steady state. Then (54) gives

\[
\phi^* = \frac{R^* - 1}{f(K^*)/K^* - 1} = \frac{\alpha - \chi}{1 - \chi} \leq \alpha,
\]

(with equality if and only if \( \alpha = 1 \) or \( \chi = 0 \)), which completes the proof.

**Proof of Proposition 4.** The budget constraint of household \( i \) in period \( t \) reduces to

\[
c_t^i + k_{t+1}^i + (x_{t+1}^i + b_{t+1}^i) \leq w_t^i = r(A_t, \omega_t)k_t^i + R_t(x_t^i + b_t^i) + \omega_t.
\]

26
Hence, Lemma 2 continues to apply provided we replace $b$ with $x+b$; that is,

\[ c^i_t = (1 - s_t)(w^i_t + h_t) \]
\[ k^i_{t+1} = s_t \phi_t(w^i_t + h_t) \]
\[ x^i_{t+1} = b^i_{t+1} = s_t(1 - \phi_t)(w^i_t + h_t) - h_t \]

where $\phi_t$, $\rho_t$, and $s_t$ are defined again as in Lemma 2. Conditions (25), (26), (29), and (31) then follow from aggregating across agents and using the bond market clearing condition, as in Proposition 1. Finally, (28) follows from profit maximization in the public-equity sector and (30) from labor market clearing. ■

**Proof of Proposition 5.** We first prove that $\rho(\omega, R) = \rho R$ and $\phi(\omega, R) = \phi$, where $\rho$ and $\phi$ are given by (32). Under A1, $n(A, \omega) = A \bar{\eta}(\omega)$, $r(A, \omega) = A \bar{r}(\omega)$, and $\bar{r}(\omega) = F_K(1, \bar{n}(\omega), 1)$. It follows that

\[ \phi(\omega, R) = \arg \max \{ \int \phi AF_K(1, \bar{n}(\omega), 1) + (1 - \phi)R \psi(A) dA \}^{1/(1 - \gamma)} \]
\[ \rho(\omega, R) = \max \{ \int \phi AF_K(1, \bar{n}(\omega), 1) + (1 - \phi)R \psi(A) dA \}^{1/(1 - \gamma)} \]

Under Assumption A2, on the other hand, $R = G_K(1, l(\omega)) = F_K(1, \bar{n}(\omega), 1) / \mu$. Combining gives the result. We now prove the proposition.

(a) As in the one-sector case, stationarity of aggregate savings requires $s[\varphi \bar{r}(\omega) + (1 - \varphi)R] = 1$, where $s = \beta^\alpha \rho^{\beta - 1}$. Using $R = R(\omega)$ and $\bar{r}(\omega) = \mu R(\omega)$, we have $\rho = \rho R$ and $[\varphi \bar{r}(\omega) + (1 - \varphi)R] = (\varphi + 1 - \varphi)R$, and therefore the stationarity condition reduces to (33). This together with $R = R(\omega)$ gives a unique $R$ and a unique $\omega$. Next, in steady state, $K = \varphi s[W + H]$ and $X + H = (1 - \varphi)[W + H]$, and therefore $(X + H) / K = (1 - \varphi) / \varphi$. On the other hand, the clearing condition for the labor market gives $\bar{n}(\omega)K + l(\omega)X = 1$. Using $\bar{n}(\omega) = \mu l(\omega)$, and solving the above two conditions for $K$ and $X$, we get

\[ K = \frac{\varphi [1 + l(\omega)H]}{(\varphi + 1 - \varphi)l(\omega)} \]
\[ X = \frac{1 - \varphi - \varphi \mu l(\omega)H}{(\varphi + 1 - \varphi)l(\omega)} \]

or equivalently (34). This completes the characterization of the steady state. Uniqueness is obvious. As for existence, note that any $\mu > 1$ implies $\varphi > 0$ and therefore $K > 0$ necessarily. On the other hand, $X > 0$ if and only if $\varphi$ is sufficiently small, which is the case as long as $\sigma$ is sufficiently large.

(b) Since $R'(\omega) < 0$, $\omega$ decreases with $\sigma$ if and only if $R$ increases with $\sigma$. From condition (33),

\[ \frac{d \ln R}{d \sigma} = \frac{1}{\theta} \left[ (\theta - 1) \frac{d \ln \rho}{d \sigma} + \frac{\mu - 1}{\varphi \mu + 1 - \varphi} \frac{d \varphi}{d \sigma} \right] \]

It follows that $dR/d\sigma > 0$, and therefore $d\omega/d\sigma < 0$, if and only if $\theta > \theta(\mu, \sigma, \gamma)$, where

\[ \theta(\mu, \sigma, \gamma) \equiv 1 - \frac{\mu - 1}{\varphi \mu + 1 - \varphi} \frac{d \varphi}{d \sigma} \frac{d \ln \rho}{d \sigma} \]
Clearly, \( d\varphi/d\sigma < 0 \), \( d\ln \varphi/d\sigma < 0 \), and therefore \( \theta < 1 \). Next, since \( l'(\omega) < 0 \) and \( R'(\omega) < 0 \), from (34) we infer that \( K \) is increasing in \( \omega \) and decreasing in \( \varphi \), and \( X \) is increasing in \( \varphi \) but (possibly) non-monotonic in \( \omega \). Since

\[
K + X = 1/l(\omega) - (\mu - 1) \left( \frac{1/l(\omega) + \omega/(R - 1)}{\mu + 1/\varphi - 1} - 1/l(\omega) - (\mu - 1)K, \right.
\]

total capital \( K + X \) is increasing in \( \varphi \) but non-monotonic in \( \omega \). Finally, aggregate output is \( Y = F(1, \bar{n}(\omega), \bar{A})K + G(1, l(\omega))X \). Using \( \bar{n}(\omega) = \mu l(\omega) \) and \( F(K/\mu, N, A) = F(K, N, A/\mu) = G(K, N) \), we have \( F(1, \bar{n}(\omega), 1) = F(1, \mu l(\omega), 1) = G(1, l(\omega)) \mu \) and therefore \( Y = G(1, l(\omega))\mu K + X \), which together with (34) gives \( Y = G(1/l(\omega), 1) \). Output thus increases with \( \omega \), reflecting the fact that \( \omega \) increases if and only if resources are shifted from less productive public equity to more productive private equity. By implication, \( Y/(K + X) \) and \( C = Y - (K + X) \) also increase with \( \omega \) and decrease with \( \varphi \). Hence, \( \theta' > \theta \) suffices for a higher \( \sigma \) to raise \( R \) and reduce \( \omega, K, Y, Y/(N + L), \) and \( Y/(K + X) \). \( \blacksquare \)

**Proof of Lemma 3.** Let \( \omega_t \) denote the wage rate per effective unit of labor and take a given \((Z_t, K_t, X_t, H_{t-1}) \). The labor-market clearing condition, \( \bar{n}(\omega_t)K_t + l(\omega_t)X_t = Z_t \), gives a unique \( \omega_t \). Next, let \( R_t = R(\omega_t), H_t = R(\omega_t)H_{t-1} - \omega_t Z_t, \) and \( Y_t = f(\omega_t)K_t + g(\omega_t)X_t \), where \( f(\omega_t) \equiv F(1, \bar{n}(\omega_t), 1) \) and \( g(\omega_t) \equiv G(1, l(\omega_t)) \). Next, denote with \( \psi_t(A) \) the p.d.f. for the lognormal distribution \( \ln A \sim \mathcal{N}(-\sigma_t^2/2, \sigma_t^2) \), where \( \sigma_t = \sigma [1 - \eta \ln Z_t] \), and let \( \phi_t = \arg \max \phi \{ f(\varphi A \mu + 1 - \mu)^{1-\gamma} \psi_t(A) dA \}^{1/(1-\gamma)} \approx \mu/ \left( \gamma \sigma_t^2 \right) \). Finally, let \( C_t = (1 - \beta) [Y_t + H_t], K_{t+1} = \phi_t \beta [Y_t + H_t], X_{t+1} = (1 - \phi_t) \beta [Y_t + H_t] - H_t, \) and \( Z_{t+1} = Z_t^\rho \), which completes the construction of \((Z_{t+1}, K_{t+1}, X_{t+1}, H_t) \) and \((C_t, Y_t, \omega_t, R_t) \). \( \blacksquare \)

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