VIRTUAL GEOMETRICITY IS RARE

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Abstract. We present the results of computer experiments suggesting that the probability that a random multiword in a free group is virtually geometric decays to zero exponentially quickly in the length of the multiword. We then prove this fact.

1. Introduction

Let $F$ be a finite rank non-abelian free group. Fix, once and for all, a basis $x = \{x_1, \ldots, x_n\}$. A multiword $w = \{w_1, \ldots, w_k\}$ is a finite subset of $F$. The set $w$ determines a collection of conjugacy classes $[w] = \{[w_1], \ldots, [w_k]\}$ in $F$, possibly with multiplicity.

Definition 1.1. If $F = \pi_1 H$ for an orientable 3-dimensional handlebody $H$, then $[w]$ determines a free homotopy class of map $\sqcup_{w \in w} S^1 \to H$. The multiword $w$ is (orientably) geometric if this homotopy class contains an embedding into $\partial H$.

Similarly, a multiword $w$ is non-orientably geometric if there is such an embedding where we allow $H$ to be a non-orientable handlebody.

Definition 1.2. If $[w]$ is a conjugacy class in $F$ and $F < F$ is a finite index subgroup, we can “lift” $[w]$ to $F$ as follows. Let $[w]_F$ be the set of $F$–conjugacy classes of the form $g^{-1}w^\alpha g$, where $g \in F$ and $\alpha = \alpha(w, g) \geq 1$ is minimal subject to the requirement that $g^{-1}w^\alpha g \subset F$. The lift $[w]_F$ of $[w]$ to $F$ is then defined to be $\sqcup_{w \in w}[w]_F$.

(From a topological point of view, let $\tilde{H} \to H$ be the cover corresponding to $F < F$. A conjugacy class $[w]$ corresponds to the free homotopy class of some map $\phi: S^1 \to \tilde{H}$, and $[w]_F$ corresponds to the collection of free homotopy classes of elevations of $\phi$ to $\tilde{H}$.)

Definition 1.3. A multiword $w$ is virtually geometric if there exists a finite index subgroup $F$ of $F$ such that $[w]_F$ is geometric.

The Baumslag-Solitar words $ba^pba^q$ in $F_2 = \langle a, b \rangle$ with $p \neq 0 \neq q$ and $|p| \neq |q|$ are examples of words that are virtually geometric but not geometric [5, Section 6].

In an early version of [5], Gordon and Wilton ask whether every word in $F$ is virtually geometric. This was answered in the negative by the second author [10], who exhibited explicit non-virtually geometric words.

The first author [3] showed that virtual geometricity reduces to showing geometricity or non-orientable geometricity for each vertex group in the cyclic JSJ-decomposition of $F$ relative to $w$. 

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One might still wonder how common virtual geometricity is. We wrote a computer program that determines if a given multiword is virtually geometric or not, and set it to work testing random multiwords in low rank free groups. Our experiments, presented in Section 2, suggest that the probability that a random multiword is virtually geometric decays to zero exponentially quickly in the length of the multiword. We also make explicit estimates for the rate of exponential decay. Surprisingly, our experiments suggest that the ratio of the number of virtually geometric words to the number of geometric words of a given length is bounded above.

**Question.** Does the ratio of virtually geometric words to geometric words stay bounded as the length goes to infinity? Does it tend to 1?

In Sections 3–7 of the paper we apply the technology developed in [4, 3] to establish the result suggested by the experiments:

**Theorem (Theorem 7.2).** Virtual geometricity is exponentially rare.

The rough idea of the proof is to find a “poison” word $v \in \mathbb{F}$ which obstructs the virtual geometricity of any cyclically reduced $w \in \mathbb{F}$ containing $v$ as a subword. This poison word $v$ is a concatenation of words $v_1$ and $v_2$ so that $v_1$ obstructs the existence of a relative splitting, and $v_2$ obstructs non-orientable geometricity. The characterization from [3] implies that $v = v_1v_2$ obstructs virtual geometricity.

Finally we appeal to the well-known fact (Proposition 4.2) that cyclically reduced words exponentially generically contain every short word – in particular they contain $v$.

In fact there is the slight complication that our word $v$ is only poisonous to Whitehead minimal words, but these are exponentially generic by a result of Kapovich–Schupp–Shpilrain [7].

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2. **Experimental Estimates**

In the figures below we present findings of our computer experiments\(^1\) on the proportions of random words which are geometric, virtually geometric, and not full in ranks 2, 3, and 4. (See Definition 6.1 for a definition of *full* words.)

We see in Figure 1 that while the proportion of not full words provides an exponentially decaying upper bound for the proportion of virtually geometric words, it is not very sharp.

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\(^1\)IPython [12] was used in the development of the computer scripts and in running the experiments. The program *heegaard* [2] was used to test geometricity. Scripts for testing virtual geometricity and reproducing our experiments can be found at https://bitbucket.org/christopher_cashen/virtuallygeometric.
Fit curves are computed for each data series by taking the subseries that comes after the first word length for which the proportion of words falls below 50%, taking logarithms, computing a best fit line by weighted least squares approximation, and then exponentiating.

Figure 1. Geometricity, Virtual Geometricity, and Full Words

Figure 2 plots logarithm(proportion ± standard error) and omits the full words data.

3. Whitehead Graphs

Definition 3.1. A multiword $w = \{w_1, \ldots, w_k\}$ such that each $w_i$ is not a proper power, and such that $w_i$ is not conjugate to $w_j$ or $\overline{w_j}$ for all $i \neq j$, is called unramified.

A multiword is called cyclically reduced if all of its elements are cyclically reduced with respect to the fixed basis $x$ of $F$.

Let $T$ be the Cayley graph of $F$ with respect to $x$, which is a $2|\langle x \rangle|$-valent tree. Let $\overline{T} = T \cup \partial T$ denote the compactification of $T$ by its Gromov boundary $\partial T$. 
**Figure 2.** Geometricity and Virtual Geometricity

**Definition 3.2.** If $w$ is a cyclically reduced multiword, define $L_w$ to be the collection of distinct bi-infinite geodesics $[fw^\infty, fw^-\infty] \subset T$ where $w \in w$ and $f \in F$.

**Definition 3.3.** Let $w$ be a cyclically reduced multiword. Let $\mathcal{X}$ be a connected subset of $T$. The Whitehead graph of $w$ over $\mathcal{X}$, denoted $\mathfrak{W}(\mathcal{X})$, is a graph whose vertices are in bijection with connected components of $T \setminus \mathcal{X}$. Distinct vertices are joined by an edge for each $L \in L_w$ with endpoints in the corresponding complementary components of $\mathcal{X}$.

**Remark.** The Whitehead graph depends on $w$ via $L_w$. We suppress $w$ from the notation, as it will always be clear from context.

The following easy lemma clarifies the definition:

**Lemma 3.4.** Let $C_1$ and $C_2$ be components of $T \setminus \mathcal{X}$. Then $C_1$ and $C_2$ are connected by an edge in $\mathfrak{W}(\mathcal{X})$ if and only if the label of the shortest path joining them is a subword of $w^\infty$ or $w^-\infty$ for some $w \in w$.

**Remark.** If $w$ is unramified and cyclically reduced, and if $\mathcal{X} = \ast$ is a single vertex, then the vertices of $\mathfrak{W}(\ast)$ are in bijection with $\pm$, and there is one edge from vertex...
VIRTUAL GEOMETRICITY IS RARE

6 to vertex \( y \) for every occurrence of the subword \( xy \) in \( w \), with words of \( w \) treated as cyclic words. This is the classical definition of the Whitehead graph.

Let \(|w|\) denote the word length of \( w \) with respect to the basis \( \mathcal{X} \) of \( \mathbb{F} \). Let \(|[w]|\) denote the minimal word length of an element of the conjugacy class of \( w \).

**Definition 3.5.** An unramified, cyclically reduced multiword \( w = \{w_1, \ldots, w_k\} \) is *Whitehead minimal* if for every automorphism \( \alpha \in \text{Aut}(\mathbb{F}) \) we have:

\[
\sum_{i=1}^{k} |w_i| \leq \sum_{i=1}^{k} |[\alpha(w_i)]|.
\]

4. **Generic Sets**

Our definitions in this section follow Kapovich, Schupp, and Shpilrain [7]. A sequence \((c_n) \subset \mathbb{R}\) with \( \lim_{n \to \infty} c_n = c \in \mathbb{R} \) converges *exponentially fast* if there exist \( a > 0 \) and \( b \in \mathbb{R} \) such that \( |c - c_n| \leq \exp(-an + b) \) for all sufficiently large \( n \).

Let \(|w|\) denote the length of \( w \) in the word metric on \( \mathbb{F} \) corresponding to \( \mathcal{X} \).

**Definition 4.1.** Let \( A \subset B \subset \mathbb{F} \). The set \( A \) is *generic* in \( B \) if:

\[
\lim_{n \to \infty} \frac{\# \{w \in A \mid |w| \leq n\}}{\# \{w \in B \mid |w| \leq n\}} = 1
\]

\( A \) is *exponentially generic* in \( B \) if the convergence is exponentially fast.

A subset is *rare*, or *negligible*, if the complement is generic. It is *exponentially rare*, or *exponentially negligible*, if the complement is exponentially generic.

A property \( P \) is said to be (exponentially) generic/rare in \( B \) if the set of words having \( P \) is (exponentially) generic/rare.

It is an easy computation to see that the intersection of finitely many (exponentially) generic sets is (exponentially) generic. It follows that if \( A \) is (exponentially) generic in \( B \), then for every fixed \( k \) the set \( A^k \) is (exponentially) generic in \( B^k \) with the \( L^\infty \) metric. In particular, any property that is (exponentially) generic for \( B \) is (exponentially) generic for \( k \)-element multiwords in \( B \), thought of as elements in \( B^k \).

Let \( C \) be the set of cyclically reduced words in \( \mathbb{F} \) with respect to the basis \( \mathcal{X} \).

**Proposition 4.2.** Let \( w \) be a reduced word in \( \mathbb{F} \). The subset of words that contain \( w \) as a subword is exponentially generic in \( \mathbb{F} \) and in \( C \).

**Proof.** This fact is well known. For \( \mathbb{F} \), see [6, Section 2] or [8, Corollary 4.4.9]. For \( C \), see [9, Lemma 2.5]. Note that the statement of the latter result includes the assumption that \(|w| > 4\), but in fact the estimate and proof given there are valid for any subword \( w \) once the random word length is greater than 16. \( \square \)

**Definition 4.3.** A word \( w \) is *poison to property* \( P \) if no word containing \( w \) as a subword enjoys \( P \).

A word \( w \) is *(exponentially) generically poison to* \( P \) if there exists a (exponentially) generic set \( A \) such that if \( v \) is a word of \( A \) containing \( w \) as a subword, then \( v \) does not enjoy \( P \).

**Corollary 4.4.** If there exists a word that is (exponentially) generically poison to \( P \) then \( P \) is (exponentially) rare in \( \mathbb{F} \) and in \( C \).
Proposition 4.5. For every \( k \geq 1 \), the set of cyclically reduced, unramified, \( k \)-element multiwords is exponentially generic in \( C^k \).

Proof. Arzhantseva and Ol’shanski˘ı [1] show that the set of cyclically reduced words which are not a proper power is exponentially generic in \( C \).

As noted in [7], it is an easy computation to see that every conjugacy class in \( F \) is exponentially negligible, so the set of multiwords in which one element is conjugate to one of the others, or to the inverse of one of the others, is exponentially negligible. \( \Box \)

Theorem 4.6 ([7, Theorem A(1)]). The set of cyclically reduced, Whitehead minimal elements is exponentially generic in \( C \).

Corollary 4.7. For every \( k \geq 1 \), the set of unramified, cyclically reduced, Whitehead minimal \( k \)-element multiwords is exponentially generic in \( C^k \).

5. Relative Splittings

Definition 5.1. A splitting of \( F \) relative to \( w \) is a splitting of \( F \) as a graph of groups such that each \( w \in \mathbb{w} \) is elliptic.

The following lemma is essentially due to Whitehead [15]. See also [11, 13, 14].

Lemma 5.2. If \( \mathbb{W}(\ast) \) is connected and has no cut vertices then \( F \) does not split freely relative to \( w \).

The next lemma is a consequence of [4, Lemma 4.9]:

Lemma 5.3. If \( F \) splits over \( \langle v \rangle \) relative to \( w \) then \( \mathbb{W}(v^\infty) \) has more than one connected component.

6. Full Words

If \( w \) and \( v \) are words in \( x^\pm \), we say \( w \) cyclically contains \( v \) if the free reduction of \( v \) appears as a subword of the cyclic reduction of a power of \( w \). We say a multiword \( w \) cyclically contains \( v \) if one of the words of \( w \) cyclically contains \( v \).

Definition 6.1. A multiword \( w \) is full if for every word \( v \) in \( (x^\pm)^3 \) either \( v \) or \( v^{-1} \) is cyclically contained in \( w \).

Lemma 6.2. Full multiwords are exponentially generic.

Proof. Fix a reduced word \( w \) containing every word of \( (x^\pm)^3 \) as a subword. Then \( w \) is a poison subword for being non-full. Apply Corollary 4.4. \( \Box \)

The rest of this subsection is devoted to establishing that \( F \) cannot split freely or cyclically relative to a full multiword.

Lemma 6.3. \( F \) does not split freely relative to a full multiword.

Proof. Let \( w \) be a full multiword. Its Whitehead graph contains the complete graph, so it is connected without cut vertices. By Lemma 5.2, \( F \) does not split freely relative to \( w \). \( \Box \)

Proposition 6.4. If \( \gamma \subset T \) is a line and \( w \) is full, then \( \mathbb{W}(\gamma) \) is connected.
Proof. For \( K \subseteq \gamma \) compact, let \( C^\pm \) be the components of \( \mathcal{T} \setminus K \) containing the rest of \( \gamma \). Let \( V_K \) be the vertex set of \( \mathfrak{W}(K) \). Define \( W(K) \) to be the full subgraph of \( \mathfrak{W}(K) \) on vertices \( V_K \setminus \{ C^\pm \} \).

Notice that for \( K \subseteq K' \) we have \( W(K) \subseteq W(K') \), and moreover \( \mathfrak{W}(\gamma) = \lim_{\rightarrow} W(K) \). The following claim thus suffices to establish the proposition.

Claim. \( W(K) \) is connected, for any nonempty compact subsegment \( K \subseteq \gamma \).

Proof of Claim. We may suppose that \( \gamma \) is parametrized to have unit speed, so that \( \gamma \) sends integers to vertices of \( \mathcal{T} \). For \( z \in \mathbb{Z} \), let \( s_z \in \mathbb{Z}^\pm \) be the label of the edge \( \gamma|_{[z,z+1]} \).

The segment \( K \) is equal to \( \gamma|_{[p,q]} \) for some integers \( p, q \). Each vertex \( C \) of \( W(K) \) is a component of \( \mathcal{T} \setminus K \), connected to some vertex \( \gamma(n_C) \in K \) by an edge of \( \mathcal{T} \) labeled by an element \( s_C \) of \( \mathbb{Z}^\pm \). The pair \( (n_C, s_C) \) completely determines \( C \), so we can also refer to the vertices of \( W(K) \) by these pairs. Namely, \( (n, s) \in \mathbb{Z} \times \mathbb{Z}^\pm \) is a vertex of \( W(K) \) if and only if:

\[
p \leq n \leq q \text{ and } s \notin \{ s_{n-1}, s_n \}.
\]

Two vertices \( (n, s) \) and \( (n, t) \) of \( W(K) \) are connected by an edge if and only if one of \( st \) or \( ts \) is a subword of some \( w^{\infty} \) for \( w \in \mathfrak{w} \). Since \( \mathfrak{w} \) is full, \( (n, s) \) and \( (n, t) \) are indeed connected by an edge. Two vertices \( (n, s) \) and \( (n+1, t) \) are connected if and only if \( s_{n+t} \) or its inverse occurs in some \( w^{\infty} \) for \( w \in \mathfrak{w} \). Again, since \( \mathfrak{w} \) is full, all of these edges occur. It follows that \( W(K) \) is connected.

\( \diamond \) \hspace{1cm} \Box

Corollary 6.5. \( \mathbb{F} \) does not split freely or cyclically relative to a full multiword.

Proof. Let \( \mathfrak{w} \) be a full multiword. Lemma 6.3 tells us there is no free splitting.

Suppose that \( \mathbb{F} \) splits over \( \langle v \rangle \) relative to \( \mathfrak{w} \). Then by Lemma 5.3 the Whitehead graph \( \mathfrak{W}(\langle v^{\infty}, v^{\infty} \rangle) \) has more than one connected component. But this contradicts Proposition 6.4.

\( \Box \)

7. Virtual Geometricity Is Rare

Lemma 7.1 ([2, Lemma p. 18]). Let \( \mathfrak{w} \) be unramified, cyclically reduced, Whitehead minimal and either geometric or nonorientably geometric. Suppose that \( x_1 \) and \( \mathfrak{x}_1 \) do not form a separating pair of vertices in \( \mathfrak{W}(\mathfrak{x}) \). Up to absolute value, at most three different powers of \( x_1 \) appear in \( \mathfrak{w} \).

Proof idea. Using a theorem of Zieschang [2, Theorem p. 11] one can show that if there were four we would be able to find four non-intersecting parallelism classes of properly embedded arcs in a punctured torus or punctured Klein bottle. An Euler characteristic argument shows that there are at most three such classes.

Remark. Berge’s [2] results are stated for orientable handlebodies, but the proofs of [2, Lemma p. 18] and Zieschang’s theorem [2, Theorem p. 11] do not require orientability.

Theorem 7.2. Virtual geometricity is exponentially rare.

Proof. Let \( v' \) be a word with first letter \( x_1 \) that contains every word in \( (\mathbb{Z}^\pm)^3 \). Let \( v = x_1 x_2^2 x_2^3 x_2 x_2^2 x_2 v' \).

Let \( A \) be the set of unramified, cyclically reduced, Whitehead minimal multiwords. Let \( \mathfrak{w} \) be a multiword in \( A \) containing \( v \). Then \( \mathfrak{w} \) is full, so, by Corollary 6.5,
the JSJ decomposition of $F$ relative to $w$ is trivial. [3] then says $w$ is virtually geometric if and only if it is either orientably or non-orientably geometric. However, $\Pi(*)$ contains the complete subgraph on its vertices, so it has no separating pairs of vertices. By Lemma 7.1, $w$ can not be geometric or non-orientably geometric, since it contains at least four distinct powers of $x_1$. Therefore, $w$ is not virtually geometric.

$A$ is exponentially generic by Corollary 4.7, so $v$ is exponentially generically poison to virtual geometricity. The theorem follows from Corollary 4.4. □

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