Infinitesimal bendings of submanifolds

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Abstract

This paper deals with the subject of infinitesimal bendings of Euclidean submanifolds with arbitrary dimension and codimension. The main goal is to establish a Fundamental theorem for these geometric objects. Similar to the theory of isometric immersions in Euclidean space, we prove that a system of three equations for a certain pair of tensors are the integrability conditions for the differential equation of an infinitesimal bending. In addition, we give some rigidity results when the submanifold is intrinsically a Riemannian product of manifolds.

The classical theory of bending of surfaces $M^2$ in $\mathbb{R}^3$ was the object of intense study by geometers in the 19th century, initially with no distinction between isometric variations and only infinitesimally isometric variations. While isometric bendings of a submanifold refer to isometric variations, infinitesimal bendings are associated to variations that preserve lengths “up to the first order”. By the end of that century a clear distinction was done by Darboux. In the 20th century there is a vast number of papers on the subject as can be seen in the survey paper [8]. In particular, it is quoted that Efimov observed that “The theory of infinitesimal bendings is the differential of the theory of bendings”. For a modern account of several aspects of the subject we refer to Spivak [12].

The case of infinitesimally bendable Euclidean hypersurfaces has been initially considered around the beginning of the 20th century, when the work of Sbrana [11] in 1908 stands out. A complete local parametric classification of the infinitesimally bendable hypersurfaces is due to Dajczer and Vlachos [7] who, in particular, showed that this class is much larger than the one of

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the isometrically bendable ones. The classification of complete infinitesimally bendable Euclidean hypersurfaces is due to Jimenez [9].

Dajczer and Rodríguez [5] showed that submanifolds in low codimension are generically infinitesimally rigid, that is, only trivial infinitesimal bendings are possible. An infinitesimal bending is called trivial if it relates to a variation that is the restriction to the submanifold of a smooth one-parameter family of isometries of Euclidean space. For hypersurfaces, their result is already contained in the book of Cesàro [2] published in 1886. For recent results on the subject of rigidity of infinitesimal bendings we refer to Dajczer and Jimenez [3].

Our goal in this paper is twofold. In the first part, we obtain a clean version of the Fundamental theorem of infinitesimal bendings that extends to any codimension the result for hypersurfaces given in [7]. A quite cumbersome coordinate version for general codimension has been stated in [10]. Similarly, as in the theory of isometric immersions, in terms of a pair of tensors associated to the bending we obtain a system of three equations that are shown to be the integrability conditions for the equations that determine an infinitesimal bending.

Although the Fundamental theorem is stated and proved for the ambient Euclidean space with the standard Riemannian metric, the same statement and proof work if the ambient Euclidean space has any possible signature.

The second part of the paper is devoted to some rigidity results concerning the situation when the starting manifold is a Riemannian product of manifolds. We provide conditions that imply that any infinitesimal bending is given by infinitesimal bendings of each factor.

1 Preliminaries

We recall that an isometric immersion \( f: M^n \to \mathbb{R}^m \) of an \( n \)-dimensional Riemannian manifold \( M^n \) into Euclidean space with codimension \( m - n \) is said to be isometrically bendable if there exists a nontrivial smooth variation \( \mathcal{F}: I \times M^n \to \mathbb{R}^m \) of \( f \) for some open interval \( 0 \in I \subset \mathbb{R} \) such that the map \( f_t = \mathcal{F}(t, \cdot): M^n \to \mathbb{R}^m \) is an isometric immersion for any \( t \in I \). The variation is said to be trivial if it is produced by a family of isometries of \( \mathbb{R}^m \). That is, if there exist a smooth family \( C: I \to O(m) \) of orthogonal transformations of \( \mathbb{R}^m \) and a smooth map \( v: I \to \mathbb{R}^m \) such that

\[
\mathcal{F}(t, x) = C(t)f(x) + v(t)
\]
for all \((t, x) \in I \times M^n\).

The variational vector field of the isometric bending \(\mathcal{F}\) is the section 
\(\mathcal{T} \in \Gamma(f^*T\mathbb{R}^m)\) defined as 
\(\mathcal{T} = \mathcal{F}_*\partial/\partial t|_{t=0}\). It is easily seen that it satisfies 
the condition 
\[\langle \tilde{\nabla}_X \mathcal{T}, f_*Y \rangle + \langle f_*X, \tilde{\nabla}_Y \mathcal{T} \rangle = 0 \tag{1} \]
for any tangent vector fields \(X, Y \in \mathfrak{X}(M)\). Here and in the sequel we use 
the same notation for the inner products in \(M^n\) and \(\mathbb{R}^m\) and denote by \(\nabla\) and \(\tilde{\nabla}\) the respective associated Levi-Civita connections.

In order to study the infinitesimal analogue of isometric bendings one knows from classical geometry that the right approach is to look at the variational vector field. The bendings under consideration are the ones that preserve lengths just “up to the first order”, and that means that 
\[\frac{\partial}{\partial t}|_{t=0}\langle f_*X, f_*X \rangle = 0 \]
for all \(X \in \mathfrak{X}(M)\). A section \(\mathcal{T}\) of \(f^*T\mathbb{R}^m\) is called an \textit{infinitesimal bending} of an isometric immersion \(f: M^n \to \mathbb{R}^m\) if condition (1) holds. Associated to an infinitesimal bending \(\mathcal{T}\) of \(f\) we have that the variation \(\mathcal{F}: \mathbb{R} \times M^n \to \mathbb{R}^m\) given by 
\[\mathcal{F}(t, x) = f(x) + t\mathcal{T}(x)\]
has variational vector field \(\mathcal{T}\). That the lengths are preserved up to the first order is given by 
\[\|f_*X\|^2 = \|f_*X\|^2 + t^2\|\tau_*X\|^2\]
for any \(X \in \mathfrak{X}(M)\).

A \textit{trivial infinitesimal bending} is the restriction to the submanifold of a Killing vector field of the ambient space. More precisely, there is a skew-symmetric linear endomorphism \(D\) of \(\mathbb{R}^m\) and \(v \in \mathbb{R}^m\) such that \(\mathcal{T} = Df + v\). Then, we have that 
\[\mathcal{F}(t, x) = e^{tD}f(x) + tv\]
is a trivial isometric bending of \(f\). Throughout the paper we identify pairs of infinitesimal bendings that differ by a trivial one.

\section{The Fundamental theorem}

In this section, we first define a pair of tensors associated to any infinitesimal bending and then show that they satisfy a set of three equations that form...
the Fundamental system of equations of the bending. Then we state and prove the Fundamental theorem of infinitesimal bendings by showing that the equations of the system are the integrability conditions for the existence of an infinitesimal bending.

Let $\mathcal{T} \in \Gamma(f^*T\mathbb{R}^m)$ be an infinitesimal bending of a given isometric immersion $f: M^n \to \mathbb{R}^m$. We first argue that $\mathcal{T}$ and the second fundamental form $\alpha: TM \times TM \to N_fM$ of $f$ determine an associate pair of tensors $(\beta, \mathcal{E})$ where $\beta: TM \times TM \to N_fM$ is symmetric and $\mathcal{E}: TM \times N_fM \to N_fM$ satisfies the compatibility condition

$$\langle \mathcal{E}(X, \eta), \xi \rangle + \langle \mathcal{E}(X, \xi), \eta \rangle = 0 \quad (2)$$

for any $X \in \mathfrak{X}(M)$ and $\eta, \xi \in \Gamma(N_fM)$.

Let $L \in \Gamma(\text{Hom}(TM, f^*T\mathbb{R}^m))$ be the tensor defined by

$$LX = \tilde{\nabla}_X \mathcal{T} = \mathcal{T}_X$$

for any $X \in \mathfrak{X}(M)$. Notice that (1) in terms of $L$ has the form

$$\langle LX, f_*Y \rangle + \langle f_*X, LY \rangle = 0 \quad (3)$$

for any $X, Y \in \mathfrak{X}(M)$. Let $B: TM \times TM \to f^*T\mathbb{R}^m$ be the tensor given by

$$B(X, Y) = (\tilde{\nabla}_X L)Y = \tilde{\nabla}_X LY - L\nabla_X Y$$

for any $X, Y \in \mathfrak{X}(M)$. Since

$$B(X, Y) = \tilde{\nabla}_X \tilde{\nabla}_Y \mathcal{T} - \tilde{\nabla}_{\nabla_X Y} \mathcal{T}$$

then $B$ is symmetric due to the flatness of the ambient space. Then also the tensor $\beta: TM \times TM \to N_fM$ defined by

$$\beta(X, Y) = (B(X, Y))_{N_fM}$$

is symmetric. In particular, we define $B_\xi \in \Gamma(\text{End}(TM))$ symmetric by

$$\langle B_\xi X, Y \rangle = \langle \beta(X, Y), \xi \rangle$$

for any $X, Y \in \mathfrak{X}(M)$ and $\xi \in \Gamma(N_fM)$. 

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Let \( Y \in \Gamma(\text{Hom}(N_fM, f^*TM)) \) be given by
\[
\langle Y\eta, f^*X \rangle + \langle \eta, LX \rangle = 0.
\]
Then, the tensor \( \mathcal{E}: TM \times N_fM \to N_fM \) is defined by
\[
\mathcal{E}(X, \eta) = \alpha(X, Y\eta) + (LA_\eta X)_{N_fM}
\]
where \( A_\eta \in \Gamma(\text{End}(TM)) \) is given by
\[
\langle A_\eta X, Y \rangle = \langle \alpha(X, Y), \eta \rangle.
\]
We have
\[
\langle \mathcal{E}(X, \eta), \xi \rangle = \langle \alpha(X, Y\eta) + LA_\eta X, \xi \rangle = \langle A_\xi X, Y\eta \rangle - \langle Y\xi, A_\eta X \rangle
\]
and hence (2) is satisfied.

**Lemma 1.** We have that
\[
(B(X, Y))_{TM} = y\alpha(X, Y)
\]
for any \( X, Y \in \mathfrak{X}(M) \).

**Proof:** We need to show that
\[
C(X, Y, Z) = \langle (B - y\alpha)(X, Y), f_*Z \rangle
\]
vanishes for any \( X, Y, Z \in \mathfrak{X}(M) \). The derivative of (3) gives
\[
0 = \langle \nabla_Z LX, f_*Y \rangle + \langle LX, \nabla_Z f_*Y \rangle + \langle \nabla_Z L Y, f_*X \rangle + \langle LY, \nabla_Z f_*X \rangle
\]
\[
= \langle B(Z, X), f_*Y \rangle + \langle L\nabla_Z X, f_*Y \rangle + \langle LX, f_*\nabla_Z Y + \alpha(Z, Y) \rangle
\]
\[
+ \langle B(Z, Y), f_*X \rangle + \langle L\nabla_Z Y, f_*X \rangle + \langle LY, f_*\nabla_Z X + \alpha(Z, X) \rangle
\]
\[
= \langle B(Z, X), f_*Y \rangle + \langle LX, \alpha(Z, Y) \rangle + \langle B(Z, Y), f_*X \rangle + \langle LY, \alpha(Z, X) \rangle
\]
\[
= \langle (B - y\alpha)(Z, X), f_*Y \rangle + \langle (B - y\alpha)(Z, Y), f_*X \rangle.
\]
From the symmetry of \( B \) and the above, we obtain
\[
C(X, Y, Z) = C(Y, X, Z) \quad \text{and} \quad C(Z, X, Y) = -C(Z, Y, X)
\]
for any \( X, Y, Z \in \mathfrak{X}(M) \). Then
\[
C(X, Y, Z) = -C(X, Z, Y) = -C(Z, X, Y) = C(Z, Y, X) = C(Y, Z, X) = -C(Y, X, Z) = -C(X, Y, Z) = 0,
\]
as we wished. \( \blacksquare \)

The last manipulation of the above proof is known as the Braid Lemma; for instance see [1] p. 224.

**Proposition 2.** The pair of tensors \((\beta, \varepsilon)\) associated to an infinitesimal bending \(T\) satisfy the following system of three equations:
\[
A_{\beta(Y,Z)}X + B_{\alpha(Y,Z)}X = A_{\beta(X,Z)}Y + B_{\alpha(X,Z)}Y \tag{6}
\]
\[
(\nabla^X_\beta(Y, Z) - (\nabla^Y_\beta(X, Z) = \varepsilon(Y, \alpha(X, Z)) - \varepsilon(X, \alpha(Y, Z)) \tag{7}
\]
and
\[
(\nabla^X_\varepsilon(Y, \eta) - (\nabla^X_\varepsilon)(X, \eta)
= \beta(X, A_\eta Y) - \beta(A_\eta X, Y) + \alpha(X, B_\eta Y) - \alpha(B_\eta X, Y) \tag{8}
\]
for all \( X, Y, Z \in \mathfrak{X}(M) \) and \( \eta \in \Gamma(N_fM) \). Moreover (7) is equivalent to
\[
(\nabla_X B_\eta)Y - (\nabla_Y B_\eta)X - B_{\nabla^X_\eta Y} + B_{\nabla^Y_\eta X} = A_{\varepsilon(X, \eta)Y} - A_{\varepsilon(Y, \eta)X} \tag{9}
\]
for all \( X, Y, Z \in \mathfrak{X}(M) \) and \( \eta \in \Gamma(N_fM) \).

**Proof:** We first show that
\[
(\tilde{\nabla}_X Y)\eta = -f_* B_\eta X - LA_\eta X + \varepsilon(X, \eta) \tag{10}
\]
for any \( X \in \mathfrak{X}(M) \) and \( \eta \in \Gamma(N_fM) \), where \((\tilde{\nabla}_X Y)\eta = \tilde{\nabla}_X Y\eta - Y\nabla^X_\eta \). We have from (4) and the derivative of (3) that
\[
0 = (\tilde{\nabla}_X Y\eta, f_* Y) + (Y\eta, f_* \nabla_X Y) + (\tilde{\nabla}_X L Y, \eta) + (L Y, \tilde{\nabla}_X \eta)
= (\tilde{\nabla}_X Y\eta, f_* Y) + (B_\eta X, Y) + (LA_\eta X, f_* Y).
\]
Since \((Y\eta, \xi) = 0\), we obtain
\[
0 = (\tilde{\nabla}_X Y\eta, \xi) + (Y\eta, \tilde{\nabla}_X \xi) = ((\tilde{\nabla}_X Y)\eta, \xi) - (\alpha(X, Y\eta), \xi)
= ((\tilde{\nabla}_X Y)\eta, \xi) + (LA_\eta X - \varepsilon(X, \eta), \xi)
\]
for any $X \in \mathfrak{X}(M)$ and $\eta, \xi \in \Gamma(N_f M)$, and (10) follows.

Using

$$(\tilde{\nabla}_X B)(Y, Z) = \tilde{\nabla}_X (\tilde{\nabla}_Y L)Z - (\tilde{\nabla}_{\nabla_X Y} L)Z - (\tilde{\nabla}_Y L)\nabla_X Z$$

(11)

it is easy to see that

$$(\tilde{\nabla}_X B)(Y, Z) - (\tilde{\nabla}_Y B)(X, Z) = -LR(X, Y)Z$$

(12)

for all $X, Y, Z \in \mathfrak{X}(M)$. It follows using (5) that

$$\langle (\tilde{\nabla}_X B)(Y, Z), f^* W \rangle = \langle (\tilde{\nabla}_Y B)(X, Z), f^* W \rangle$$

for any $X, Y, Z, W \in \mathfrak{X}(M)$. Then (12) and the Codazzi equation give

$$\langle (\tilde{\nabla}_X B)(Y, Z), f^* W \rangle = \langle LR(Y, X)Z + A_\beta(Y, Z) - A_\beta(X, Y), W \rangle$$

Now using the Gauss equation, we obtain

$$\langle (\tilde{\nabla}_X B)(Y, Z), f^* W \rangle = \langle LR(Y, X)Z + A_\beta(Y, Z) - A_\beta(X, Y), W \rangle$$

On the other hand, it follows from (10) that

$$\langle (\tilde{\nabla}_X B)(Y, Z), f^* W \rangle = \langle LR(Y, X)Z + A_\beta(Y, Z) - A_\beta(X, Y), W \rangle$$

From the last two equations, we obtain

$$\langle (\tilde{\nabla}_X B)(Y, Z), f^* W \rangle = \langle LR(Y, X)Z + A_\beta(Y, Z) - A_\beta(X, Y), W \rangle$$

and this is (6).

Using (11) we obtain

$$(\tilde{\nabla}_X B)(Y, Z))_{N_f M} = \alpha(X, y \alpha(Y, Z)) + (\nabla_X \beta)(Y, Z).$$

Then, we have from (12) and the Gauss equation that

$$(\nabla_X \beta)(Y, Z) - (\nabla_Y \beta)(X, Z)$$

$$= (LR(Y, X)Z)_{N_f M} - \alpha(X, y \alpha(Y, Z) + \alpha(Y, y \alpha(X, Z)$$

$$= (LA_{\alpha(X, Z)}Y - LA_{\alpha(Y, Z)}X)_{N_f M} - \alpha(X, y \alpha(Y, Z) + \alpha(Y, y \alpha(X, Z),$$
and this is (7). Since $\mathcal{E}$ satisfies the compatibility condition (2), then
\[
\langle \mathcal{E}(X, \alpha(Y, Z)), \eta \rangle = -\langle A \mathcal{E}(X, \eta) Y, Z \rangle.
\]
and this gives (9).

We have
\[
(\nabla_X^\perp \mathcal{E})(Y, \eta) = \nabla_X^\perp \mathcal{E}(Y, \eta) - \mathcal{E} (\nabla_X Y, \eta) - \mathcal{E}(Y, \nabla_X^\perp \eta)
\]
\[
= (\nabla_X^\perp \alpha)(Y, \eta) + (L(\nabla_X A)(Y, \eta))_{N_f M} + \alpha(Y, \nabla_X Y \eta)
\]
\[- \alpha(Y, \eta \nabla_X^\perp \eta) - (L\nabla_X A_{\eta} Y)_{N_f M} + \nabla_X^\perp(LA_{\eta} Y)_{N_f M}.
\]
Then (10) gives
\[
(\nabla_X^\perp \mathcal{E})(Y, \eta) = (\nabla_X^\perp \alpha)(Y, \eta) + (L(\nabla_X A)(Y, \eta))_{N_f M} - \alpha(Y, B_{\eta} X)
\]
\[- \alpha(Y, (L A_{\eta} X)_{TM}) - (L\nabla_X A_{\eta} Y)_{N_f M} + \nabla_X^\perp(LA_{\eta} Y)_{N_f M}.
\]

Using the Codazzi equation, we obtain
\[
(\nabla_X^\perp \mathcal{E})(Y, \eta) - (\nabla^\perp_Y \mathcal{E})(X, \eta) = \alpha(X, B_{\eta} Y) - \alpha(Y, B_{\eta} X) + \alpha(X, (L A_{\eta} Y)_{TM})
\]
\[- \alpha(Y, (L A_{\eta} X)_{TM}) - (L\nabla_X A_{\eta} Y)_{N_f M} + \nabla_X^\perp(LA_{\eta} Y)_{N_f M}
\]
\[+ (L\nabla_Y A_{\eta} X)_{N_f M} - \nabla_Y^\perp(LA_{\eta} X)_{N_f M}.
\]

Since
\[
\beta(X, A_{\eta} Y) = \alpha(X, (L A_{\eta} Y)_{TM}) - (L\nabla_X A_{\eta} Y)_{N_f M} + \nabla_X^\perp(LA_{\eta} Y)_{N_f M},
\]
then (8) follows.

We say that an isometric immersion $f : M^n \to \mathbb{R}^m$ has full first normal space $N^f_1(x)$ at $x \in M^n$ if
\[
N^f_1(x) = \text{span}\{\alpha(X, Y) : X, Y \in T_x M\}
\]
satisfies $N^f_1(x) = N_f M(x)$.

The following result shows that for a submanifold with full first normal spaces the tensor $\beta$ determines $\mathcal{E}$.

**Proposition 3.** Let $f : M^n \to \mathbb{R}^m$ be an isometric immersion with full first normal space. If $(\beta, \mathcal{E})$ is the associated pair of tensors to an infinitesimal bending $\mathcal{J}$ of $f$ then $\mathcal{E}$ is the unique tensor that satisfies (2) and (7).
Proof: If \( \mathcal{E}_0 : T M \times N_f M \to N_f M \) is a tensor that satisfies (2) and (7), it follows from (7) that
\[
(\mathcal{E} - \mathcal{E}_0)(X, \alpha(Y, Z)) = (\mathcal{E} - \mathcal{E}_0)(Y, \alpha(X, Z))
\]
for any \( X, Y, Z \in \mathfrak{X}(M) \). Since both \( \mathcal{E} \) and \( \mathcal{E}_0 \) satisfy (2), we have
\[
\langle (\mathcal{E} - \mathcal{E}_0)(X_1, \alpha(X_2, X_3)), \alpha(X_4, X_5) \rangle = -\langle (\mathcal{E} - \mathcal{E}_0)(X_1, \alpha(X_4, X_5)), \alpha(X_2, X_3) \rangle
\]
where \( X_i \in \mathfrak{X}(M), 1 \leq i \leq 5 \). Denoting
\[
\langle (\mathcal{E} - \mathcal{E}_0)(X_1, \alpha(X_2, X_3)), \alpha(X_4, X_5) \rangle = (X_1, X_2, X_3, X_4, X_5)
\]
it follows from the relations above and the symmetry of \( \alpha \) that
\[
(X_1, X_2, X_3, X_4, X_5) = -(X_1, X_4, X_5, X_2, X_3) = -(X_1, X_4, X_5, X_3, X_2)
\]
\[
= (X_5, X_2, X_3, X_4, X_1) = (X_3, X_2, X_5, X_4, X_1) = -(X_3, X_4, X_1, X_2, X_5)
\]
\[
= -(X_4, X_3, X_1, X_2, X_5) = (X_4, X_2, X_5, X_3, X_1) = (X_2, X_4, X_5, X_3, X_1)
\]
\[
= -(X_2, X_3, X_1, X_4, X_5) = -(X_2, X_1, X_3, X_4, X_5) = -(X_1, X_2, X_3, X_4, X_5)
\]
\[
= 0.
\]
Thus \( \mathcal{E} - \mathcal{E}_0 = 0 \). \( \blacksquare \)

Remark 4. An alternative way to obtain the equations in Proposition 2 is to follow the “classical” procedure, which goes as follows. Since the metrics \( g_t \) induced by the infinitesimal variation \( f_t = f + t \mathcal{T} \) satisfy \( \partial/\partial t \big|_{t=0} g_t = 0 \), hence the Levi-Civita connections and curvature tensors of \( g_t \) satisfy
\[
\partial/\partial t \big|_{t=0} \nabla^t_X Y = 0
\]
and
\[
\partial/\partial t \big|_{t=0} g_t(R^t(X, Y)Z, W) = 0
\]
for any \( X, Y, Z, W \in \mathfrak{X}(M) \). Then use this to compute the derivatives with respect to \( t \) at \( t = 0 \) of the Gauss, Codazzi and Ricci equations for \( f_t \). In fact, as can be seen in [3] this works quite nicely to obtain (6). On the contrary, the computation for the other two equations becomes really cumbersome outside the hypersurface case. For hypersurfaces this was done in [4], [7] and the result in coordinates for general codimension has been stated in [10].
Let \( f : M^n \to \mathbb{R}^m \) be an isometric immersion and \( \mathcal{J} \) be a trivial infinitesimal bending of \( f \), that is,
\[
\mathcal{J} = Df + w
\]
where \( D \in \text{End}(\mathbb{R}^m) \) is skew-symmetric and \( w \in \mathbb{R}^m \). Then
\[
L = D|_{f^*TM} \quad \text{and} \quad B(X,Y) = D\alpha(X,Y).
\]
Let \( D^N \in \Gamma(\text{End}(N_fM)) \) skew-symmetric be given by
\[
D^N \eta = (D\eta)_{N_fM}
\]
for any \( \eta \in \Gamma(N_fM) \). Then we have
\[
\beta(X,Y) = D^N\alpha(X,Y) \quad \text{and} \quad \mathcal{E}(X,\eta) = -(\nabla^\perp_X D^N)\eta,
\]
where the second equation follows computing \((\tilde{\nabla}_X D)\eta = 0\).

**Proposition 5.** An infinitesimal bending \( \mathcal{J} \) is trivial if and only if there is \( C \in \Gamma(\text{End}(N_fM)) \) skew-symmetric such that
\[
\beta(X,Y) = C\alpha(X,Y) \quad \text{and} \quad \mathcal{E}(X,\eta) = -(\nabla^\perp_X C)\eta.
\]  

**Proof:** Define \( D \in \Gamma(\text{End}(f^*\mathbb{R}^m)) \) by
\[
D(x)X = L(x)X \quad \text{and} \quad D(x)\eta = Y(x)\eta + C(x)\eta
\]
for any \( X \in T_xM \) and \( \eta \in N_{f(x)}M \). Using the assumption on \( \beta \) we obtain
\[
\tilde{\nabla}_X DY = (\tilde{\nabla}_X L)Y + L\nabla_X Y = Y\alpha(X,Y) + C\alpha(X,Y) + L\nabla_X Y = D\tilde{\nabla}_X Y
\]
for any \( X,Y \in \mathfrak{X}(M) \). The assumptions on \( \mathcal{E} \) and (10) give
\[
\tilde{\nabla}_X D\eta = \tilde{\nabla}_X Y\eta + \tilde{\nabla}_X C\eta
\]
\[
= (\tilde{\nabla}_X Y)\eta + Y\nabla^\perp_X \eta + (\nabla^\perp_X C)\eta + C\nabla^\perp_X \eta - f_*A_{C\eta}X
\]
\[
= -f_*B_\eta X - LA_\eta X + Y\nabla^\perp_X \eta + C\nabla^\perp_X \eta - f_*A_{C\eta}X
\]
for any \( X \in \mathfrak{X}(M) \) and \( \eta \in \Gamma(N_fM) \). But \( B_\eta = -A_{C\eta} \) from \( \beta = C\alpha \), then
\[
\tilde{\nabla}_X D\eta = -LA_\eta X + Y\nabla^\perp_X \eta + C\nabla^\perp_X \eta = D\tilde{\nabla}_X \eta.
\]
Therefore, we have shown that \( D(x) = \mathcal{D} \) is constant along \( M^n \). Thus the map \( \mathcal{T} - \mathcal{D} f \) is constant, and this concludes the proof.

Recall that we identify two infinitesimal bendings \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \) of \( f \) whenever \( \mathcal{T}_0 = \mathcal{T}_2 - \mathcal{T}_1 \) is a trivial infinitesimal bending. In this case, if \((\beta_1, \mathcal{E}_1)\) and \((\beta_2, \mathcal{E}_2)\) are the associated pairs to \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \), respectively, then the associated pair to \( \mathcal{T}_0 \) is \((\beta_2 - \beta_1, \mathcal{E}_2 - \mathcal{E}_1)\) that satisfies (13) for \( \mathcal{D}^N \in \Gamma(\text{End}(N_f M)) \) skew-symmetric. Thus, in this situation we identify \((\beta_1, \mathcal{E}_1)\) and \((\beta_2, \mathcal{E}_2)\).

The following is the Fundamental theorem of infinitesimal bendings.

**Theorem 6.** Let \( f : M^n \to \mathbb{R}^m \) be an isometric immersion of a simply connected Riemannian manifold. Let \( \beta : TM \times TM \to N_f M \) be a symmetric tensor and let the tensor \( \mathcal{E} : TM \times N_f M \to N_f M \) satisfy the compatibility condition (2). If the pair \( 0 \neq (\beta, \mathcal{E}) \) satisfies (3), (7) and (8), then there is a unique infinitesimal bending \( \mathcal{T} \) of \( f \) having \((\beta, \mathcal{E})\) as associated pair.

**Proof:** Given a pair \((\beta, \mathcal{E})\) as in the statement, we first argue that there is \( \mathcal{D} \in \Gamma(\text{End}(f^* \mathbb{R}^m)) \) satisfying

\[
(\tilde{\nabla}_X \mathcal{D})(Y + \eta) = -f_* B_\eta X + \beta(X, Y) + \mathcal{E}(X, \eta)
\] (14)

for any \( X, Y \in \mathfrak{X}(M) \) and \( \eta \in \Gamma(N_f M) \). To prove this, henceforth we check that the integrability condition

\[
(\tilde{\nabla}_X \tilde{\nabla}_Y \mathcal{D} - \tilde{\nabla}_Y \tilde{\nabla}_X \mathcal{D} - \tilde{\nabla}_{[X,Y]} \mathcal{D})(Z + \eta) = 0
\]

holds for any \( X, Y, Z \in \mathfrak{X}(M) \) and \( \eta \in \Gamma(N_f M) \). For simplicity, in the following we write \( X \) instead of \( f_* X \). We have

\[
(\tilde{\nabla}_X \tilde{\nabla}_Y \mathcal{D} - \tilde{\nabla}_Y \tilde{\nabla}_X \mathcal{D} - \tilde{\nabla}_{[X,Y]} \mathcal{D})(Z + \eta)
\]

\[
= \tilde{\nabla}_X (\tilde{\nabla}_Y \mathcal{D})(Z + \eta) - (\tilde{\nabla}_Y \mathcal{D}) \tilde{\nabla}_X (Z + \eta) - \tilde{\nabla}_Y (\tilde{\nabla}_X \mathcal{D})(Z + \eta)
\]

\[
+ (\tilde{\nabla}_X \mathcal{D}) \tilde{\nabla}_Y (Z + \eta) - (\tilde{\nabla}_{[X,Y]} \mathcal{D})(Z + \eta)
\]

\[
= \tilde{\nabla}_X [-B_\eta Y + \beta(Y, Z) + \mathcal{E}(Y, \eta)] + B_\alpha(X, Z) + \nabla_\alpha \eta Y - \beta(Y, \nabla_\alpha Z - A_\eta X)
\]

\[
- \mathcal{E}(Y, \alpha(X, Z) + \nabla_\alpha \eta) + \tilde{\nabla}_Y [B_\eta X - \beta(X, Z) - \mathcal{E}(X, \eta)]
\]

\[
- B_\alpha(Y, Z) + \nabla_\alpha \eta X + \beta(X, \nabla_Y Z - A_\eta Y) + \mathcal{E}(X, \alpha(Y, Z) + \nabla_\alpha \eta)
\]

\[
+ B_\eta [X, Y] - \beta([X, Y], Z) - \mathcal{E}([X, Y], \eta).
\]
Hence

\[
\tilde{\nabla}_X \tilde{\nabla}_Y \mathcal{D} - \tilde{\nabla}_Y \tilde{\nabla}_X \mathcal{D} - \tilde{\nabla}_{[X,Y]} \mathcal{D})(Z + \eta)
\]

\[
= -A_{\beta(Y,Z)}X + B_{\alpha(X,Z)}Y + A_{\beta(X,Z)}Y - B_{\alpha(Y,Z)}X
\]

\[
+ (\nabla_X \beta)(Y,Z) - (\nabla_Y \beta)(X,Z) + \mathcal{E}(X,\alpha(Y,Z)) - \mathcal{E}(Y,\alpha(X,Z))
\]

\[
- (\nabla_X B_\eta)Y + (\nabla_Y B_\eta)X + B_{\nabla_X \eta}Y - B_{\nabla_Y \eta}X - A_{\varepsilon(Y,\eta)}X + A_{\varepsilon(X,\eta)}Y
\]

\[
+ (\nabla_X \varepsilon)(Y,\eta) - (\nabla_Y \varepsilon)(X,\eta) - \alpha(X,B_\eta Y) + \alpha(Y,B_\eta X)
\]

\[
+ \beta(Y,A_\eta X) - \beta(X,A_\eta Y)
\]

\[
= 0,
\]

where for the last equality we made use (6), (7), (8) and (9).

Fix a solution \(\mathcal{D}^* \in \Gamma(\text{End}(f^*T\mathbb{R}^m))\) of (14) and a point \(x_0 \in M^n\). Set \(\mathcal{D}_0 = \mathcal{D}^*(x_0)\) and let \(\phi: f^*T\mathbb{R}^m \times f^*T\mathbb{R}^m \to \mathbb{R}\) be the tensor defined by

\[
\phi(\rho, \sigma) = \langle (\mathcal{D}^* - \mathcal{D}_0)\rho, \sigma \rangle + \langle (\mathcal{D}^* - \mathcal{D}_0)\sigma, \rho \rangle.
\]

Using (2) and (14) we have \((\tilde{\nabla}_X \phi)(\rho, \sigma) = 0\). Hence \(\phi = 0\), and thus the maps \(\mathcal{D}(x) = \mathcal{D}^*(x) - \mathcal{D}_0\) are skew-symmetric endomorphisms of \(\mathbb{R}^m\).

Define \(L \in \Gamma(\text{Hom}(TM, f^*T\mathbb{R}^m))\) by \(L(x) = \mathcal{D}(x)\big|_{T_x M}\). Using (14) we obtain

\[
(\tilde{\nabla}_X L)Y = \tilde{\nabla}_X \mathcal{D}Y - \mathcal{D}\nabla_X Y = \beta(X,Y) + \mathcal{D}\alpha(X,Y).
\]

Then

\[
(\tilde{\nabla}_X L)Y = (\tilde{\nabla}_Y L)X.
\]

Hence, there is \(\mathcal{T} \in \Gamma(f^*T\mathbb{R}^m)\) such that

\[
\tilde{\nabla}_X \mathcal{T} = LX
\]

for any \(X \in \mathfrak{X}(M)\). Since \(\mathcal{D}\) is skew-symmetric then \(L\) satisfies

\[
\langle LX, Y \rangle + \langle LY, X \rangle = 0,
\]

proving that \(\mathcal{T}\) is an infinitesimal bending of \(f\). Moreover, its associate pair of tensors \((\tilde{\beta}, \tilde{\mathcal{E}})\) is

\[
\tilde{\beta}(X,Y) = \beta(X,Y) + \mathcal{D}^N\alpha(X,Y) \quad \text{and} \quad \tilde{\mathcal{E}}(X,\eta) = \mathcal{E}(X,\eta) - (\nabla_X^\perp \mathcal{D}^N)\eta,
\]
In fact, in this case $Y\eta = (D\eta)_{TM}$. Using (14), we have
\[
\tilde{E}(X, \eta) = \alpha(X, (D\eta)_{TM}) + (LA_\eta X)_{N_fM}
\]
\[
= (\tilde{\nabla}_X (D\eta)_{TM})_{N_fM} + (LA_\eta X)_{N_fM}
\]
\[
= (\tilde{\nabla}_X D\eta)_{N_fM} - \nabla^N_X (D^N\eta)_{N_fM} + (LA_\eta X)_{N_fM}
\]
\[
= \mathcal{E}(X, \eta) + (D\tilde{\nabla}_X \eta)_{N_fM} - \nabla^N_X (D^N\eta)_{N_fM} + (LA_\eta X)_{N_fM}
\]
\[
= \mathcal{E}(X, \eta) - (LA_\eta X)_{N_fM} - (\nabla^N_X (D^N\eta)_{N_fM} + (LA_\eta X)_{N_fM}
\]
\[
= \mathcal{E}(X, \eta) - (\nabla^N_X (D^N\eta)_{N_fM}.
\]

Another solution $D^*_1$ of (14) gives rise to an infinitesimal bending $T_1$ of $f$. It follows from Proposition 5 that $T - T_1$ is a trivial infinitesimal bending and this concludes the proof.

The following result is Theorem 13 in [7] or Theorem 14.11 in [4].

**Corollary 7.** Let $f : M^n \to \mathbb{R}^{n+1}$ be an isometric immersion of a simply connected Riemannian manifold. Let $0 \neq B \in \Gamma(End(TM))$ be a symmetric Codazzi tensor that satisfies
\[
BX \land AY - BY \land AX = 0
\]
for all $X, Y \in \mathfrak{X}(M)$. Then there exists a unique infinitesimal bending $\mathcal{T}$ of $f$ having $B$ as associated tensor.

**Proof:** In this case the tensor $\mathcal{E}$ vanishes. Let $\beta : TM \times TM \to N_fM$ be the symmetric tensor given by $\beta(X, Y) = \langle BX, Y \rangle N$. Then (8) trivially holds for $\beta$ and $\mathcal{E} = 0$. Moreover, by the assumptions on $B$ we have that $(\beta, 0)$ satisfies (6) and (7). Thus, by Theorem 6 there is an infinitesimal bending $\mathcal{T}$ of $f$ having $(\beta, 0)$ as associated pair, and this concludes the proof.

### 3 Bending of a product of manifolds

In this section, we consider infinitesimal bendings of submanifold that are intrinsically a Riemannian product of manifolds.

Let $M^n = M^{n_1}_1 \times \cdots \times M^{n_r}_r$ be a Riemannian product of manifolds of dimensions $n_i \geq 2$, $1 \leq i \leq r$. Given isometric immersions $f_i : M^{n_i}_i \to \mathbb{R}^{m_i}$,
1 \leq i \leq r$, then the extrinsic product \( f \colon M^n \to \mathbb{R}^m \) of the submanifolds is the isometric immersion given by

\[
f(x) = (f_1(x), \ldots, f_r(x)),
\]

where \( x = (x_1, \ldots, x_r) \) and \( \mathbb{R}^m = \bigoplus_{i=1}^r \mathbb{R}^{m_i} \). The normal space of \( f \) at \( x = (x_1, \ldots, x_r) \in M^m \) is given by

\[
N_f M(x) = \bigoplus_{i=1}^r N_{f_i} M_i(x_i).
\]

where \( N_{f_i} M_i(x_i) \) is the normal space of \( f_i \) at \( x_i \in M_i^{m_i}, 1 \leq i \leq r \).

Let \( \iota_i^\bar{x} \colon M_i^{m_i} \to M^n \) denote the inclusion map for \( \bar{x} = (\bar{x}_1, \ldots, \bar{x}_r) \), that is,

\[
\iota_i^\bar{x}(x_i) = (\bar{x}_1, \ldots, x_i, \ldots, \bar{x}_r),
\]

and let \( \tilde{\iota}_i^\bar{y} \) be the inclusion of \( \mathbb{R}^{m_i} \) into \( \mathbb{R}^m \) defined in a similar manner. Then the second fundamental form \( \alpha \) of \( f \) at \( x \) satisfies

\[
\alpha(\iota_i^\bar{x} X, \iota_j^\bar{x} Y) = \begin{cases} 
\tilde{\iota}_i^\bar{x}(x) \alpha_i(X, Y) & \text{if } i = j \\
0 & \text{if } i \neq j,
\end{cases}
\]  

for any \( X \in \mathfrak{X}(M_i) \) and \( Y \in \mathfrak{X}(M_j) \), where \( \alpha_i \) is the second fundamental form of \( f_i \).

Let \( \mathcal{T}_i \) be an infinitesimal bending of \( f_i \) in \( \mathbb{R}^{m_i} \) for each \( 1 \leq i \leq r \). Then, \( \mathcal{T}(x) = \sum_{i=1}^r \tilde{\iota}_i^\bar{x}(x) \mathcal{T}_i(x_i) \) is an infinitesimal bending of \( f \) in \( \mathbb{R}^m \). Let \( L \) be associated to \( \mathcal{T} \) and let \( L_i \) be associated to \( \mathcal{T}_i \). Then

\[
L \iota_i^\bar{x} X = \tilde{\iota}_i^\bar{x}(x) L_i X
\]

for any \( X \in \mathfrak{X}(M_i) \). If \( B_i \) is associated to \( \mathcal{T}_i \), it follows that

\[
B(\iota_i^\bar{x} X, \iota_j^\bar{x} Y) = (\tilde{\nabla} \iota_i^\bar{x} X L) \iota_j^\bar{x} Y = \begin{cases} 
\tilde{\iota}_i^\bar{x}(x) B_i(X, Y) & \text{if } i = j \\
0 & \text{if } i \neq j,
\end{cases}
\]

for any \( X \in \mathfrak{X}(M_i) \) and \( Y \in \mathfrak{X}(M_j) \). In particular,

\[
\beta(\iota_i^\bar{x} X, \iota_j^\bar{x} Y) = \begin{cases} 
\tilde{\iota}_i^\bar{x}(x) \beta_i(X, Y) & \text{if } i = j \\
0 & \text{if } i \neq j,
\end{cases}
\]  

(17)
and

$$\mathcal{E}(i^*_i X, i^*_j Y) = \begin{cases} i^f(x) E_i(X, \eta) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{18}$$

for any $X \in \mathcal{X}(M_i)$, $Y \in \mathcal{X}(M_j)$, $\eta \in \Gamma(N_f M_i)$ and $(\beta, E_i)$ is the pair associated to $T_i$, where (18) follows from (15), (16) and the definition of $Y$.

If $(\beta, E_i)$ is the associated pair to an infinitesimal bending $\mathcal{T}$ of an extrinsic product $f = (f_1, \ldots, f_r)$ we say that $\beta$ is adapted to the product structure if

$$\beta(i^*_i X, i^*_j Y) = 0$$

for any $X \in \mathcal{X}(M_i)$ and $Y \in \mathcal{X}(M_j)$ with $i \neq j$.

**Proposition 8.** Let $f: M^n \to \mathbb{R}^m$ be an extrinsic product of isometric immersions $f_i: M_i^{n_i} \to \mathbb{R}^{n_i}$, $n_i \geq 2$, $1 \leq i \leq r$, such that the first normal spaces $N_i^f$ are full at any point of $M^n$. If the tensor $\beta$ associated to an infinitesimal bending $\mathcal{T}$ of $f$ is adapted, then there exist locally infinitesimal bendings $\mathcal{T}_i$ of $f_i$, $1 \leq i \leq r$, such that $\mathcal{T}(x) = \sum_{i=1}^r i^f(x) \mathcal{T}_i$.

**Proof:** From (6) we obtain

$$\langle \beta(i^*_i X, i^*_i Y), \alpha(i^*_i Z, i^*_i W) \rangle + \langle \alpha(i^*_i X, i^*_i Y), \beta(i^*_i Z, i^*_i W) \rangle = 0 \tag{19}$$

for any $X, Y \in \mathcal{X}(M_i)$ and $Z, W \in \mathcal{X}(M_j)$ with $i \neq j$.

Let $\alpha_i(X_k, Y_k)$, $1 \leq k \leq \dim N_{f_i} M_i$ with $X_k, Y_k \in \mathcal{X}(M_i)$ be a basis of $N_{f_i} M_i$, and set

$$C_{ij} \alpha_i(X_k, Y_k) = \beta_{N_{f_i} M_i}(i^*_i X_k, i^*_i Y_k), j \neq i,$$

where $\beta_{N_{f_i} M_j}$ is the component of $\beta$ in $N_{f_j} M_j$. We claim that the linear extension to a map $C_{ij}: N_{f_i} M_i \to N_{f_j} M_j$, $i \neq j$, is well defined. In fact, if we have $\alpha_i(X, Y) = \sum_k \alpha_i(X_k, Y_k)$ for $X, Y \in \mathcal{X}(M_i)$, we obtain from (19) that

$$\langle \beta(X, Y) - \sum_k \beta(i^*_i X_k, i^*_i Y_k), \alpha(t_{i,j} Z, t_{i,j} W) \rangle = 0$$

for any $Z, W \in \mathcal{X}(M_j)$, $i \neq j$, and the claim follows.

We have from (19) that the map $C \in \Gamma(\text{End}(N_f M))$ defined by

$$C_{ij}(x) \eta_i = \sum_{j \neq i} i^f(x) C_{ij} \eta_i$$
where \( \eta_i \in \Gamma(N_{fi}M_i) \), \( 1 \leq i \leq r \), is skew-symmetric.

We see that \( \beta(t_{is}^x X, t_{is}^x Y) \) decomposes orthogonally as
\[
\beta(t_{is}^x X, t_{is}^x Y) = \beta_{N_{fi}M_i}(t_{is}^x X, t_{is}^x Y) + C\alpha(t_{is}^x X, t_{is}^x Y)
\]
for any \( X, Y \in \mathfrak{X}(M_i) \).

Let \( L_i: TM_i \to f_i^*T\mathbb{R}^{m_i} \) be given by
\[
L_i X = (L_{is}^x X)_{\mathbb{R}^{m_i}}.
\]
Since \( f \) is an extrinsic product of immersions, we have for any \( X \in \mathfrak{X}(M_i) \) and \( Y \in \mathfrak{X}(M_j) \) with \( i \neq j \) that
\[
\tilde{\nabla}_{i^*} Y L_{is}^x X = \tilde{\nabla}_{i^*} Y (L_{is}^x X)_{\mathbb{R}^{m_i}} = (\tilde{\nabla}_{i^*} Y L_{is}^x X)_{\mathbb{R}^{m_i}} = (B(t_{js}^x Y, t_{is}^x X))_{\mathbb{R}^{m_i}}
\]
\[
= (Y\alpha(t_{js}^x Y, t_{is}^x X) + \beta(t_{js}^x Y, t_{is}^x X))_{\mathbb{R}^{m_i}} = 0,
\]
where the last steps follow using (5) and the assumption on \( \beta \). Thus the tensors \( L_i \) are well defined on \( M_i \), \( 1 \leq i \leq r \). Moreover, since \( B \) is symmetric, these tensors verify
\[
(\tilde{\nabla}_X L_i) Y = (\tilde{\nabla}^i Y L_i) X
\]
for any \( X, Y \in \mathfrak{X}(M_i) \), where \( \tilde{\nabla}^i \) is the connection in \( \mathbb{R}^{m_i} \). Thus there exist, locally, vector fields \( \mathcal{J}_i \in \Gamma(f_i^*T\mathbb{R}^{m_i}) \) with \( \tilde{\nabla}^i_x \mathcal{J}_i = L_i X \) for any \( X \in \mathfrak{X}(M_i) \), \( 1 \leq i \leq r \). In particular, since \( L_i \) verifies (3), \( \mathcal{J}_i \) is an infinitesimal bending of \( f_i \) and, if \( \beta_i \) is associated to \( \mathcal{J}_i \), we have
\[
\tilde{\tau}_{is}^f \beta_i(X, Y) = \beta_{N_{fi}M_i}(t_{is}^x X, t_{is}^x Y)
\]
for any \( X, Y \in \mathfrak{X}(M_i) \), \( 1 \leq i \leq r \).

Define the infinitesimal bending \( \tilde{\mathcal{T}} \) of \( f \) by \( \tilde{\mathcal{T}} = \sum_{i=1}^r \tilde{\tau}_{is}^f \mathcal{J}_i \). Hence, from (17), (20) and (21) we have that \( \mathcal{T} - \tilde{\mathcal{T}} \) has the associated tensor \( \beta - \tilde{\beta} = C\alpha \). A straightforward computation using the skew-symmetry of \( C \) and the Ricci equation yields that \( C\alpha \) and \( -(\nabla^1 C) \) verify equations (6), (7) and (8).

Then Proposition 3 gives \( \mathcal{E} - \tilde{\mathcal{E}} = -(\nabla^1 C) \), and the proof now follows from Proposition 5. \( \blacksquare \)

**Remark 9.** If for \( f: M^n \to \mathbb{R}^m \) it fails that the first normal spaces \( N_i^f \) are full then any smooth normal vector field in \( N_i^f \) is an infinitesimal bending.
The \( s \)-nullity \( \nu_s(x) \), \( 1 \leq s \leq m - n \), at \( x \in M^n \) of an isometric immersion \( f: M^n \to \mathbb{R}^m \) is defined as

\[
\nu_s(x) = \max \{ \dim N(\alpha_{U^s})(x) : U^s \subset N_fM(x) \}
\]

where \( \alpha_{U^s} = \pi_{U^s} \circ \alpha \), \( \pi_{U^s}: N_fM \to U^s \) is the orthogonal projection and

\[
N(\alpha_{U^s})(x) = \{ Y \in T_xM : \alpha_{U^s}(Y, X) = 0 \text{ for all } X \in T_xM \}.
\]

Theorem 10. Let \( f: M^n \to \mathbb{R}^{n+p} \), \( p < n \), be an extrinsic product of isometric immersions \( f_i: M^n_i \to \mathbb{R}^{n_i+p_i} \), \( n_i \geq 2 \) and \( 1 \leq i \leq r \). Assume that the \( s \)-nullities of \( f \) satisfy \( \nu_s < n - s \), \( 1 \leq s \leq p \), at any point of \( M^n \). Then any infinitesimal bending \( \mathcal{T} \) of \( f \) is locally of the form \( \mathcal{T}(x) = \sum_{i=1}^{r} t_i'(x)\mathcal{T}_i \), where \( \mathcal{T}_i, 1 \leq i \leq r \), is an infinitesimal bending of \( f_i \).

Proof: We argue that the associated tensor \( \beta \) to any infinitesimal bending \( \mathcal{T} \) of \( f \) has to be adapted. Since \( f \) is an extrinsic product of immersions, then we have \( \sum_{j \neq i} t_j'^i TM_j \subset N(\alpha_{N_{M_i}^j}) \). Thus the assumption on the \( s \)-nullities yields \( \sum_{j \neq i} n_j < n - p_i \), that is,

\[
p_i < n_i, \quad 1 \leq i \leq r. \tag{22}
\]

We assume \( X \in \mathfrak{X}(M_i), W \in \mathfrak{X}(M_j) \) and \( Y, Z \in \mathfrak{X}(M_k) \) with \( i \neq j \). If also \( k \neq i, j \), we obtain from (6) that

\[
\langle \beta(i_{i*}^x X, t_j'^i W), \alpha(i_{k*}^x Y, t_k'^x Z) \rangle = 0. \tag{23}
\]

On the other hand, for \( k = i \) it follows from (6) that

\[
\langle \beta(i_{i*}^x X, t_j'^i W), \alpha(i_{k*}^x Y, t_k'^x Z) \rangle - \langle \beta(i_{i*}^x Y, t_j'^x W), \alpha(i_{i*}^x X, t_k'^x Z) \rangle = 0. \tag{24}
\]

Let \( \beta_W^i: TM_i \to N_{f_i}M_i \) be given by

\[
\beta_W^i X = \beta(i_{i*}^x X, t_j'^i W)_{N_{f_i}M_i}.
\]

Suppose that \( \dim \text{Im} \beta_W^i = s > 0 \). Then (22) gives \( \dim \ker \beta_W^i = n_i - s > 0 \). It follows from (24) that

\[
\langle \beta(i_{i*}^x X, t_j'^i W), \alpha(i_{i*}^x T, t_k'^x Z) \rangle = 0.
\]
for any $T$ in $\ker \beta_W$. This implies that $\nu_s \geq n - s$, which contradicts our assumption and proves that $\beta(v_i^x X, v_j^x W)_{N_i M_i} = 0$ for any $X \in \mathfrak{X}(M_i)$ and $W \in \mathfrak{X}(M_j)$. This together with (23) imply that

$$\beta(v_i^x X, v_j^x W) = 0 \text{ if } i \neq j.$$ 

Thus $\beta$ is adapted, and the proof follows from Proposition 8.

The following is the main result of this section.

**Theorem 11.** Let $f : M^n \to \mathbb{R}^{n+p}$, $2p < n$, be an isometric immersion of a Riemannian product $M^n = M_1^{n_1} \times \cdots \times M_r^{n_r}$ with $n_j \geq 2, 1 \leq j \leq r$. Assume that the $s$-nullities of $f$ satisfy $\nu_s < n - 2s, 1 \leq s \leq p$, at any point of $M^n$. Then $f$ is an extrinsic product of isometric immersions $f = (f_1, \ldots, f_r)$ and any infinitesimal bending $\mathcal{T}$ of $f$ is locally of the form $\mathcal{T}(x) = \sum_{i=1}^r \tilde{\mathcal{T}}_i^{(x)} \mathcal{T}_i$, where $\mathcal{T}_i$ is an infinitesimal bending of $f_i : M_i^{n_i} \to \mathbb{R}^{m_i}$, $1 \leq i \leq r$.

**Proof:** From Theorem 5 in [6] or Theorem 8.14 in [4] we obtain that $f$ is an extrinsic product of isometric immersions, and the proof follows from Theorem 10.

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