METRIC PROPERTIES OF INCOMPARABILITY GRAPHS WITH AN EMPHASIS ON PATHS

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Abstract. We describe some metric properties of incomparability graphs. We consider the problem of the existence of infinite paths, either induced or isometric, in the incomparability graph of a poset. Among other things, we show that if the incomparability graph of a poset is connected and has infinite diameter, then it contains an infinite induced path. Furthermore, if the diameter of the set of vertices of degree at least 3 is infinite, then the graph contains as an induced subgraph either a comb or a kite.

1. Introduction and presentation of the results

In this paper, we highlight the special properties of incomparability graphs by considering the behavior of paths. We consider the problem of the existence of infinite paths, either induced or isometric, in the incomparability graph of a poset. We apply one of our results in the theory of hereditary classes of certain permutation classes that are well quasi ordered by embeddability.

The graphs we consider are undirected, simple and have no loops. That is, a graph is a pair $G := (V, E)$, where $E$ is a subset of $[V]^2$, the set of 2-element subsets of $V$. Elements of $V$ are the vertices of $G$ and elements of $E$ its edges. The graph $G$ be given, we denote by $V(G)$ its vertex set and by $E(G)$ its edge set. The complement of a graph $G = (V, E)$ is the graph $G^c$ whose vertex set is $V$ and edge set $E^c := [V]^2 \setminus E$.

Throughout, $P := (V, \leq)$ denotes an ordered set (poset), that is a set $V$ equipped with a binary relation $\leq$ on $V$ which is reflexive, antisymmetric and transitive. We say that two elements $x, y \in V$ are comparable if $x \leq y$ or $y \leq x$, otherwise we say they are incomparable.

The comparability graph, respectively the incomparability graph, of a poset $P := (V, \leq)$ is the undirected graph, denoted by $\text{Comp}(P)$, respectively $\text{Inc}(P)$, with vertex set $V$ and edges the pairs $\{u, v\}$ of comparable distinct vertices (that is, either $u < v$ or $v < u$) respectively incomparable vertices.

A result of Gallai, 1967 [8], quite famous and nontrivial, characterizes comparability graphs among graphs in terms of obstructions: a graph $G$ is the comparability graph of a poset if and only if it does not contain an induced subgraph a graph belonging to a minimal list of finite graphs. Since the complement of a comparability graph is an incomparability graph,
Gallai’s result yields a similar characterization of incomparability graphs. In this paper, we consider incomparability graphs as metric spaces by means of the distance of the shortest path. The metric properties of a graph, notably of an incomparability graph, and metric properties of its complement seem to be far apart. In general, metric properties of graphs are based on paths and cycles. It should be noted that incomparability graphs have no induced cycles of length at least five ([8]; for a short proof see after Lemma 10) while comparability graphs have no induced odd cycles but can have arbitrarily large induced even cycles.

In the sequel, we will study the specificity of the metric properties of incomparability graphs by emphasising the properties of paths.

We start with few definitions. Let \( G := (V, E) \) be a graph. If \( A \) is a subset of \( V \), the graph \( G_A := (A, E \cap [A]^2) \) is the graph induced by \( G \) on \( A \). A path is a graph \( P \) such that there exists a one-to-one map \( f \) from the set \( V(P) \) of its vertices into an interval \( I \) of the chain \( \mathbb{N} \) of nonnegative integers in such a way that \( \{u, v\} \) belongs to \( E(P) \), the set of edges of \( P \), if and only if \( |f(u) - f(v)| = 1 \) for every \( u, v \in V(P) \). If \( I \) is finite, say \( I = \{1, \ldots, n\} \), then we denote that path by \( P_n \); its length is \( n - 1 \) (so, if \( n = 2 \), \( P_2 \) is made of a single edge, whereas if \( n = 1 \), \( P_1 \) is a single vertex). We denote by \( P_\infty \) the one way infinite path i.e. \( I = \mathbb{N} \). If \( x, y \) are two vertices of a graph \( G := (V, E) \), we denote by \( d_G(x, y) \) the length of a shortest path joining \( x \) and \( y \) if any, and \( d_G(x, y) := \infty \) otherwise. This defines a distance on \( V \), the graphic distance. A graph is connected if any two vertices belong to some path. The diameter of \( G \), denoted by \( \delta_G \), is the supremum of the set \( \{d_G(x, y) : x, y \in V\} \). If \( A \) is a subset of \( V \), the graph \( G' \) induced by \( G \) on \( A \) is an isometric subgraph of \( G \) if \( d_G'(x, y) = d_G(x, y) \) for all \( x, y \in A \). The supremum of the length of induced finite paths of \( G \), denoted by \( D_G \), is sometimes called the (induced) detour of \( G \) [1].

The main results of the paper are presented in the next four subsections. Section 1.5 is devoted to an application of one of our main results (Theorem 3). The remaining sections contain intermediate results and proofs of our main results.

1.1. Induced paths of arbitrarily large length in incomparability graphs and in arbitrary graphs. We now consider the question of the existence of infinite induced paths in incomparability graphs with infinite detour. In order to state our main result of this subsection we need to introduce the notions of direct sum and complete sum of graphs. Let \( G_n := (V_n, E_n) \) for \( n \in \mathbb{N} \) be a family of graphs having pairwise disjoint vertex sets. The direct sum of \( (G_n)_{n \in \mathbb{N}} \), denoted \( \oplus_n G_n \), is the graph whose vertex set is \( \bigcup_{n \in \mathbb{N}} V_n \) and edge set \( \bigcup_{n \in \mathbb{N}} E_n \). The complete sum of \( (G_n)_{n \in \mathbb{N}} \), denoted \( \sum_n G_n \), is the graph whose vertex set is \( \bigcup_{n \in \mathbb{N}} V_n \) and edge set \( \bigcup_{i \neq j} \{v, v' : v \in V_i \land v' \in V_j\} \cup \bigcup_{n \in \mathbb{N}} E_n \).

A necessary condition for the existence of an infinite induced path in a graph is to have infinite detour. On the other hand, the graphs consisting of the direct sum of finite paths of arbitrarily large length and the complete sum of finite paths of arbitrarily large length are (incomparability) graphs with infinite detour and yet do not have an infinite induced path. We should mention that in the case of incomparability graphs, having infinite detour is equivalent to having a direct sum or a complete sum of finite paths of arbitrarily large length. This is Theorem 2 from [21].
Theorem 1 ([21]). Let $G$ be the incomparability graph of a poset. Then $G$ contains induced paths of arbitrarily large length if and only if $G$ contains $\sum_{n \geq 1} P_n$ or $\oplus_{n \geq 1} P_n$ as an induced subgraph.

For general graphs, the statement of Theorem 1 is false. Indeed, in [21] we exhibited uncountably many graphs of cardinality $\aleph_0$, containing finite induced paths of unbounded length and neither a direct sum nor a complete sum of finite paths of unbounded length. In particular, these graphs do not have an infinite induced path.

In the case of incomparability graphs of posets coverable by two chains, having infinite detour is equivalent to the existence of an infinite induced path. Our first result is this.

Theorem 2. Let $P$ be a poset coverable by two chains (that is totally ordered sets). If $\text{Inc}(P)$, the incomparability graph of $P$, is connected then the following properties are equivalent:

(i) $\text{Inc}(P)$ contains the direct sum of induced paths of arbitrarily large length;
(ii) the detour of $\text{Inc}(P)$ is infinite;
(iii) the diameter of $\text{Inc}(P)$ is infinite;
(iv) $\text{Inc}(P)$ contains an infinite induced path.

A proof of Theorem 2 will be provided in Section 5.

The implication $(i) \Rightarrow (iv)$ of Theorem 2 becomes false if the condition "coverable by two chains" is dropped (see Figure 1 for an example). Indeed,

**Example 1.** There exists a poset with no infinite antichain whose incomparability graph is connected and embeds the direct sum of finite induced paths of arbitrarily large length and yet does not have an infinite induced path (See Figure 1).

Example 1 and a proof that it verifies the required properties will be given in Section 6.

![Figure 1](image-url)
1.2. Infinite induced paths, combs and kites. We now consider the question of the existence of infinite induced paths in incomparability graphs with infinite diameter. In order to state our main result of this subsection we need to introduce two types of graphs: comb and kite.

Let us recall that a graph $G := (V, E)$ is a caterpillar if the graph obtained by removing from $V$ the vertices of degree one is a path (finite or not, reduced to one vertex or empty). A comb is a caterpillar such that every vertex is adjacent to at most one vertex of degree one. Incidentally, a path on three vertices is not a comb. It should be mentioned that caterpillars are incomparability graphs of interval orders coverable by two chains (see Lemma 14 of [27]).

We now give the definition of a kite. This is a graph obtained from an infinite path $P_\infty := (x_i)_{i \in \mathbb{N}}$ by adding a new set of vertices $Y$ (finite or infinite). We distinguish three types of kites (see Figure 2) depending on how the vertices of $Y$ are adjacent to the vertices of $P_\infty$.

- A kite of type (1): every vertex of $Y$ is adjacent to exactly two vertices of $P_\infty$ and these two vertices are consecutive in $P_\infty$. Furthermore, two distinct vertices of $Y$ share at most one common neighbour in $P_\infty$.

- A kite of type (2): every vertex of $Y$ is adjacent to exactly three vertices of $P_\infty$ and these three vertices must be consecutive in $P_\infty$. Furthermore, for all $x, x' \in Y$, if $x$ is adjacent to $x_i, x_{i+1}, x_{i+2}$ and $x'$ is adjacent to $x_{i'}, x_{i'+1}, x_{i'+2}$ then $i + 2 \leq i'$ or $i' + 2 \leq i$.

- A kite of type (3): every vertex of $Y$ is adjacent to exactly two vertices of $P_\infty$ and these two vertices must be at distance two in $P_\infty$. Furthermore, for all $x, x' \in X$, if $x$ is adjacent to $x_i$ and $x_{i+2}$ and $x'$ is adjacent to $x_{i'}$ and $x_{i'+2}$ then $i + 2 \leq i'$ or $i' + 2 \leq i$.

![Figure 2. A comb and the three types of kites.](image-url)

**Theorem 3.** If $G$ is a connected incomparability graph with infinite diameter. Then

1. Every vertex of $G$ has an induced path of infinite diameter starting at it.
2. If the set of vertices of degree at least 3 in $G$ has infinite diameter, then $G$ contains an induced comb or an induced kite having an infinite diameter and infinitely many vertices of degree at least 3.

**Theorem 3** will be proved in Section 9 (an important ingredient of its proof is Theorem 7 below).
1.3. **Infinite isometric paths in incomparability graphs.** A basic result about the existence of an infinite isometric path in a graph is König’s lemma [12]. Recall that a graph is *locally finite* if every vertex has a finite degree.

**Theorem 4 ([12]).** Every connected, locally finite, infinite graph contains an isometric infinite path.

Moreover,

**Theorem 5.** If a connected graph $G$ has an infinite isometric path, then every vertex has an isometric path starting at it.

Theorem 5 was proved by Watkins in the case of locally finite graphs (see [28], Lemma 3.2). The general case is contained in Theorem 3.5 and Lemma 3.7 of [19].

A necessary condition for a graph to have an infinite isometric path is to have infinite diameter. Note that a graph has an infinite diameter if and only if it has finite isometric paths of arbitrarily large length. The existence of such paths does not necessarily imply the existence of an infinite isometric path even if the graph is connected.

![Figure 3. The Hasse diagram of a poset of width two whose incomparability graph is connected, has infinite diameter but no infinite isometric path.](image)

**Example 2.** There exists a poset coverable by two chains whose incomparability graph is connected, having infinite diameter and no isometric infinite path (see Figure 3).

We provide Example 2 and a proof that it verifies the required properties in Section 10.

We obtain a positive result in the case of incomparability graphs of interval orders with no infinite antichains. A poset $P$ is an interval order if $P$ is isomorphic to a subset $J$ of the set $Int(C)$ of non-empty intervals of a chain $C$, ordered as follows: if $I, J \in Int(C)$, then

$$I < J \text{ if } x < y \text{ for every } x \in I \text{ and every } y \in J.$$
Interval orders were considered in Fishburn \[1\,\,2\] and Wiener \[29\] in relation to the theory of measurement.

**Theorem 6.** If \( P \) is an interval order with no infinite antichains so that \( \text{Inc}(P) \) is connected and has infinite diameter, then \( \text{Inc}(P) \) has an infinite isometric path.

The proof of Theorem 6 will be provided in Section \[\text{11}\].

The conclusion of Theorem 6 becomes false if the condition ”no infinite antichains” is removed. Indeed,

**Example 3.** There exists an interval order whose incomparability graph is connected, has an infinite diameter and no infinite isometric path.

Example 3 and a proof that it verifies the required properties will be provided in Section \[\text{11}\].

### 1.4. Convexity and isometry of metric balls in incomparability graphs.

In this subsection we compare the notions of order convexity and metric convexity with respect to the distance on the incomparability graph of a poset. Before stating our result we need few definitions.

An initial segment of a poset \( P := (V, \leq) \) is any subset \( I \) of \( V \) such that \( x \in V, y \in I \) and \( x \leq y \) imply \( x \in I \). If \( X \) is a subset of \( V \), the set \( \downarrow X := \{ y \in P : y \leq x \text{ for some } x \in X \} \) is the least initial segment containing \( X \), we say that it is generated by \( X \). If \( X \) is a one element set, say \( X = \{ x \} \), we denote by \( \downarrow x \), instead of \( \downarrow X \), this initial segment and say that it is principal. Final segments are defined similarly.

Let \( P := (V, \leq) \) be a poset. A subset \( X \) of \( V \) is order convex or convex if for all \( x, y \in X \), \([x,y] := \{ z : x \leq z \leq y \} \subseteq X \). For instance, initial and final segments of \( P \) are convex. Note that any intersection of convex sets is also convex. In particular, the intersection of all convex sets containing \( X \), denoted \( \text{Conv}_P(X) \), is convex. This is the smallest convex set containing \( X \). Note that

\[
\text{Conv}_P(X) = \{ z \in P : x \leq z \leq y \text{ for some } x, y \in X \} = \downarrow X \cap \uparrow X.
\]

Let \( G := (V,E) \) be a graph. We equip it with the graphic distance \( d_G \). A ball is any subset \( B_G(x,r) := \{ y \in V : d_G(x,y) \leq r \} \) where \( x \in V, r \in \mathbb{N} \). A subset of \( V \) is convex w.r.t. the distance \( d_G \) if this is an intersection of balls. The least convex subset of \( G \) containing \( X \) is

\[
\text{Conv}_G(X) := \bigcap_{x \in B_G(x,r)} B_G(x,r).
\]

Let \( X \subseteq V \) and \( r \in \mathbb{N} \). Define

\[
B_G(X,r) := \{ v \in V : d_G(v,x) \leq r \text{ for some } x \in X \}.
\]

With all needed definitions in hand we are now ready to state the following theorem.

**Theorem 7.** Let \( P := (V,\leq) \) be a poset, \( G \) be its incomparability graph, \( X \subseteq V \) and \( r \in \mathbb{N} \).

(a) If \( X \) is an initial segment, respectively a final segment, respectively an order convex subset of \( P \) then \( B_G(X,r) \) is an initial segment, respectively a final segment, respectively an order convex subset of \( P \). In particular, for all \( x \in V \) and \( r \in \mathbb{N} \), \( B_G(x,r) \) is order convex;

(b) If \( X \) is order convex then the graph induced by \( G \) on \( B_G(X,r) \) is an isometric subgraph of \( G \). In particular, if \( X \) is included into a connected component of \( G \) then the graph induced by \( G \) on \( B_G(X,r) \) is connected.
It follows from Theorem 7 that every ball in an incomparability graph $G$ of a poset is order convex and that the graph induced on it is an isometric subgraph of $G$.

The proof of Theorem 7 is provided in Section 8.

1.5. **An application of Theorem 3 in the theory of well quasi order.** The purpose of this subsection is to provide an application of Theorem 3 in the theory of well quasi order. Let us first recall some notions from the Theory of Relations [7]. A graph $G$ is embeddable in a graph $G'$ if $G$ is isomorphic to an induced subgraph of $G'$. The embeddability relation is a quasi order on the class of graphs. A class $C$ of graphs, finite or not, is hereditary if it contains every graph which embeds in some member of $C$. The age of a graph $G$ is the collection of finite graphs, considered up to isomorphy, that embed in $G$ (or alternatively, that are isomorphic to some induced subgraph of $G$). We recall that an age of finite graphs, and more generally a class of finite graphs, is well quasi ordered (w.q.o. for short) if it contains no infinite antichain, that is an infinite set of graphs pairwise incomparable with respect to embeddability. There are several results about w.q.o. hereditary classes of graphs, see for examples [13, 14], [15] and [18].

We recall that a graph $G := (V, E)$ is a permutation graph if there is a linear order $\leq$ on $V$ and a permutation $\sigma$ of $V$ such that the edges of $G$ are the pairs $\{x, y\} \in [V]^2$ which are reversed by $\sigma$. The study of permutations graphs became an important topic due to the Stanley-Wilf Conjecture, formulated independently by Richard P. Stanley and Herbert Wilf in the late 1980s, and solved positively by Marcus and Tardós [16] 2004. It was proved by Lozin and Mayhill 2011 [15] that a hereditary class of finite bipartite permutation graphs is w.q.o. by embeddability if and only if there is a bound on the length of the double ended forks (see Figure 4) it may contain (for an alternative proof see [20]). In [20], we extend results of Lozin and Mayhill [15] and present an almost exhaustive list of properties of w.q.o. ages of bipartite permutation graphs. One of our results is a positive answer, in the case of an age of bipartite permutation graphs, to a long standing unsolved question by the first author, of whether the following equivalence is true in general: an age is not w.q.o. if and only if it contains $2^{\aleph_0}$ subages (see subsection I-4 Introduction à la comparaison des âges, page 67, [22]). This result, Theorem 8 below, is a consequence of (2) of Theorem 3.

**Figure 4.** Double-ended forks: an antichain of finite graphs with respect to embeddability.

**Theorem 8** ([20]). Let $C$ be an age that consists of finite bipartite permutation graphs. Then $C$ is not w.q.o. if and only if it contains the age of a direct sum $\oplus_{i \in I} \text{DF}_i$ of double ended forks of arbitrarily large length for some infinite subset $I$ of $\mathbb{N}$. In particular, if $C$ is not w.q.o., it contains $2^{\aleph_0}$ subages which are not w.q.o.

A proof is given in [20]. For completeness we provide the proof here.

**Proof.** The set of double-ended forks forms an infinite antichain, hence if $C$ contains the direct sum $\oplus_{i \in I} \text{DF}_i$ of double ended forks of arbitrarily large length for some infinite subset $I$ of $\mathbb{N}$, it is not w.q.o. Conversely, suppose $C$ is not w.q.o. Then it embeds double-ended forks of unbounded length. This important result is due Lozin and Mayhill (see Theorem 7
in [15]). Let \( G \) be a graph with \( \text{Age}(G) = \mathcal{C} \). We consider two cases:

1. Some connected component of \( G \), say \( G_i \), embeds double forks of unbounded length. In this case, the detour of \( G_i \), that is the supremum of the lengths of induced paths in \( G_i \), is unbounded. Since \( G_i \) is the incomparability graph of a poset of width at most two, its diameter is unbounded (See Corollary 18). In fact, since the vertices of degree 3 in the forks are end vertices of induced paths, the diameter of the set of vertices of degree 3 in \( G_i \) is unbounded. Thus from (2) of Theorem 3, \( G_i \) embeds an induced caterpillar or an induced kite with infinitely many vertices of degree at least 3. Since \( G \) is bipartite, it can only embed a kite of type (3). As it is easy to see, this caterpillar or that kite embeds a direct sum \( \bigoplus_{i \in I} DF_i \) of double-ended forks of arbitrarily large length, as required.

2. If the first case does not hold, there are infinitely many connected components \( G_i \), each embedding some double-ended fork \( DF_i \), and the length of these double-ended forks is unbounded. This completes the proof of Theorem 8. \( \square \)

The paper is organised as follows. In Section 2 we present some prerequisites on graphs and posets. In Section 3 we state a fundamental lemma on paths in incomparability graphs and some consequences. In Section 4 we present few metric properties of posets of width 2. In Section 5 we present the proof of Theorem 2. In Section 6 we present Example 1. In Section 7 we present various metric properties of incomparability graphs. In Section 8 we present a proof of Theorem 7 and some consequences. In Section 9 we give a proof of Theorem 3 (an important ingredient of the proof is Theorem 7). In Section 10 we present Example 2. Finally a proof of Theorem 6 and Example 3 are provided in Section 11.

2. Graphs and Posets

2.1. Posets. Throughout, \( P := (V, \leq) \) denotes an ordered set (poset). The dual of \( P \) denoted \( P^* \) is the order defined on \( V \) as follows: if \( x, y \in V \), then \( x \leq y \) in \( P^* \) if and only if \( y \leq x \) in \( P \). Let \( P := (V, \leq) \) be a poset. We recall that two elements \( x, y \in V \) are comparable if \( x \leq y \) or \( y \leq x \), otherwise, we say they are incomparable, denoted \( x \parallel y \). A set of pairwise comparable elements is called a chain. On the other hand, a set of pairwise incomparable elements is called an antichain. The width of a poset is the maximum cardinality of its antichains (if the maximum does not exist, the width is set to be infinite). Dilworth’s celebrated theorem on finite posets [2] states that the maximum cardinality of an antichain in a finite poset equals the minimum number of chains needed to cover the poset. This result remains true even if the poset is infinite but has finite width. If the poset \( P \) has width 2 and the incomparability graph of \( P \) is connected, the partition of \( P \) into two chains is unique (picking any vertex \( x \), observe that the set of vertices at odd distance from \( x \) and the set of vertices at even distance from \( x \) form a partition into two chains). According to Szpilrajn [25], every order on a set has a linear extension. Let \( P := (V, \leq) \) be a poset. A realizer of \( P \) is a family \( \mathcal{L} \) of linear extensions of the order of \( P \) whose intersection is the order of \( P \). Observe that the set of all linear extensions of \( P \) is a realizer of \( P \). The dimension of \( P \), denoted \( \text{dim}(P) \), is the least cardinal \( d \) for which there exists a realizer of cardinality \( d \). It follows from the Compactness Theorem of First Order Logic that an order is intersection of at most \( n \) linear orders \( (n \in \mathbb{N}) \) if and only if every finite restriction of the order has this property. Hence the class of posets with dimension at most \( n \) is determined by a set of finite obstructions, each obstruction is a poset \( Q \) of dimension \( n + 1 \) such that the deletion of any element of \( Q \) leaves a poset of dimension \( n \); such a poset is said critical. For \( n \geq 2 \) there are infinitely many
critical posets of dimension \( n + 1 \). For \( n = 2 \) they have been described by Kelly [10]; beyond, the task is considered as hopeless.

2.1.1. Comparability and incomparability graphs, permutation graph. A graph \( G := (V, E) \) is a comparability graph if the edge set is the set of comparabilities of some order on \( V \). From the Compactness Theorem of First Order Logic, it follows that a graph is a comparability graph if and only if every finite induced subgraph is a comparability graph. Hence, the class of comparability graphs is determined by a set of finite obstructions. The complete list of minimal obstructions was determined by Gallai [8]. A graph \( G := (V, E) \) is a permutation graph if and only if each finite induced graph is a permutation graph (sometimes these graphs are called permutation graphs, while there is no possible permutation involved). For more about permutation graphs, see [11].

2.1.2. Lexicographical sum. Let \( I \) be a poset such that \(|I| \geq 2\) and let \( \{P_i := (V_i, \leq_i)\}_{i \in I} \) be a family of pairwise disjoint nonempty posets that are all disjoint from \( I \). The lexicographical sum \( \sum_{i \in I} P_i \) is the poset defined on \( \bigcup_{i \in I} V_i \) by \( x \leq y \) if and only if

(a) There exists \( i \in I \) such that \( x, y \in V_i \) and \( x \leq_i y \) in \( P_i \); or
(b) There are distinct elements \( i, j \in I \) such that \( i < j \) in \( I \), \( x \in V_i \) and \( y \in V_j \).

The posets \( P_i \) are called the components of the lexicographical sum and the poset \( I \) is the index set. If \( I \) is a totally ordered set, then \( \sum_{i \in I} P_i \) is called a linear sum. On the other hand, if \( I \) is an antichain, then \( \sum_{i \in I} P_i \) is called a direct sum. Henceforth we will use the symbol \( \oplus \) to indicate direct sum.

The decomposition of the incomparability graph of a poset into connected components is expressed in the following lemma which belongs to the folklore of the theory of ordered sets.

**Lemma 9.** If \( P := (V, \leq) \) is a poset, the order on \( P \) induces a total order on the set \( \text{Connect}(P) \) of connected components of \( \text{Inc}(P) \), the incomparability graph of \( P \), and \( P \) is the lexicographical sum of these components indexed by the chain \( \text{Connect}(P) \). In particular, if \( \leq \) is a total order extending the order \( \leq \) of \( P \), each connected component \( A \) of \( \text{Inc}(P) \) is an interval of the chain \((V, \leq)\).

The next two sections introduce the necessary ingredients to the proof of Theorem 2.

3. A fundamental lemma

We state an improvement of I.2.2 Lemme, p.5 of [23].
Lemma 10. Let $x, y$ be two vertices of a poset $P$ with $x < y$. If $x_0, \ldots, x_n$ is an induced path in the incomparability graph of $P$ from $x$ to $y$ then $x_i < x_j$ for all $j - i \geq 2$.

Proof. Induction on $n$. If $n \leq 2$ the property holds trivially. Suppose $n \geq 3$. Taking out $x_0$, induction applies to $x_1, \ldots, x_n$. Similarly, taking out $x_n$, induction applies to $x_0, \ldots, x_{n-1}$. Since the path from $x_0$ to $x_n$ is induced, $x_0$ is comparable to every $x_j$ with $j \geq 2$ and $x_n$ is comparable to every $x_j$ with $j < n - 1$. In particular, since $n \geq 3$, $x_0$ is comparable to $x_{n-1}$. Necessarily, $x_0 < x_{n-1}$. Otherwise, $x_{n-1} < x_0$ and then by transitivity $x_{n-1} < x_n$ which is impossible since $\{x_{n-1}, x_n\}$ is an edge of the incomparability graph. Thus, we may apply induction to the path from $x_0, \ldots, x_{n-1}$ and get $x_0 < x_j$ for every $j > 2$. Similarly, we get $x_1 < x_n$ and via the induction applied to the path from $x_1$ to $x_n$, $x_j < x_n$ for $j < n - 1$. The stated result follows.

An immediate corollary is this.

Corollary 11. Let $P$ be a poset such that $Inc(P)$ is connected and let $a < b$. If $(a, b)$ is a covering relation in $P$, then $2 \leq d_{Inc(P)}(a, b) \leq 3$.

Another consequence of Lemma 10 is that incomparability graphs have no induced cycles of length at least five [8]. Indeed, let $P$ be a poset and let $x_0, \ldots, x_l, x_0$ be an induced cycle of $Inc(P)$. Suppose for a contradiction that $l \geq 4$. We will apply Lemma 10 successively to the induced paths $x_0, \ldots, x_{l-1}$ and $x_1, \ldots, x_l$ and will derive a contradiction. We may assume without loss of generality that $x_0 < x_{l-1}$. It follows from Lemma 10 applied to $x = x_0$ and $y = x_{l-1}$ that $x_0 < x_{l-2}$ (recall that $l \geq 4$) and $x_1 < x_{l-1}$. We now consider the induced path $x_1, \ldots, x_l$. Then $x_1$ and $x_l$ are comparable. It follows from $x_1 < x_{l-1}$ and Lemma 10 applied to $x = x_1$ and $y = x_l$ that $x_1 < x_l$. Hence, $x_{l-2} < x_l$. By transitivity we get $x_0 < x_l$ which is impossible.

Here is yet another consequence of Lemma 10.

Proposition 12. Let $P := (V, \leq)$ be a poset. A sequence $a_0, \ldots, a_n$ of vertices of $V$ forms an induced path in $Inc(P)$ originating at $a_0$ if and only if for all $i \in \mathbb{N}$, $a_i, a_{i+1}, a_{i+2}, a_{i+3}$ is an induced path of $Inc(P)$ with extremities $a_i, a_{i+3}$.

Proof. $\Rightarrow$ Obvious.

$\Leftarrow$ Suppose that for all $i \in \mathbb{N}$, $a_i, a_{i+1}, a_{i+2}, a_{i+3}$ is an induced path with extremities $a_i, a_{i+3}$. We prove by induction that for all $n \in \mathbb{N}$, $a_0, \ldots, a_n$ is an induced path in $G$. Suppose $a_0, \ldots, a_n$ is an induced path in $G$ and assume without loss of generality that $a_0 < a_n$. Then $a_i < a_n$ for all $i \leq n-2$ (follows from Lemma 10). From $a_{n-2}, a_{n-1}, a_n, a_{n+1}$ is an induced path with extremities $a_{n-2}, a_{n+1}$ and $a_{n-2} < a_n$ we deduce that $a_{n-2} < a_{n+1}$ and $a_{n-1} < a_{n+1}$. Therefore, $a_i < a_{n+1}$ for all $i \leq n-1$ proving that $a_0, \ldots, a_n, a_{n+1}$ is an induced path in $G$.

We should mention that the value 3 is the previous proposition is best possible. Indeed, if $P$ the direct sum of two copies of the chain of natural numbers, then $Inc(P)$ is a complete bipartite graph and every path on 3 vertices is an induced path. Yet an infinite sequence of vertices that alternates between the copies of $\mathbb{N}$ does not constitute an infinite induced path of $Inc(P)$.

4. Posets of width 2 and their distances

4.1. Posets of width 2 and bipartite permutation graphs. In this subsection we recall some properties about posets of width at most 2 and permutation graphs. We start with a
characterization of bipartite permutation graphs, next we give some properties of the graphic
distance and the detour in comparability graphs of posets of width at most 2. We recall the
existence of a universal poset of width at most 2 [23]. We describe the incomparability graph
of a variant of this poset more appropriate for our purpose.

Figure 5. Critical posets of dimension 3 and height 2.

We note that a poset $P$ of width at most 2 has dimension at most 2, hence its compara-
rability graph is an incomparability graph. As previously mentioned, a finite graph $G$ is a
comparability and incomparability graph if and only if it is a permutation graph. Incompara-
brability graphs of finite posets of width 2 coincide with bipartite permutation graphs. For
arbitrary posets, the characterization is as follows.

**Lemma 13.** Let $G$ be a graph. The following are equivalent.

(i) $G$ is bipartite and is the comparability graph of a poset of dimension at most two;
(ii) $G$ is bipartite and embeds no even cycles of length at least six and none of the compa-
rability graphs of the posets depicted in Figure 5.
(iii) $G$ is the incomparability graph of a poset of width at most 2.
(iv) $G$ is a bipartite incomparability graph.

**Proof.** $(i) \iff (ii)$. If $G$ is finite, this is Theorem 1 of [26]. Hence, the equivalence between $(i)$
and $(ii)$ holds for the restrictions of $G$ to every finite set $F$ of vertices. This gives immediately
the implication $(i) \implies (ii)$. For the converse implication, we get that every finite induced
subgraph of $G$ is bipartite and the comparability graph of a poset of dimension at most two.
The Compactness Theorem of First Order Logic implies that these properties extend to $G$.
$(iii) \implies (i)$. Suppose $G$ is the incomparability graph of a poset of width at most 2. Then $G$
has no 3-element cycles. Also, $G$ has no induced odd cycles of length at least five (see [8],
Section 3.8, Table 5). This shows that $G$ is bipartite. Since $P$ is coverable by two chains it
has order dimension two (the dimension of a poset is at most its width [2]) and therefore its
incomparability graph is also a comparability graph [3]. Thus $G$ is a comparability graph of
a poset of dimension at most two.

$(i) \iff (iv)$. Follows from the fact that a graph $G$ is the incomparability graph of a poset of
dimension at most 2 if and only if this is the comparability graph of a poset of dimension at
most 2 [3]. $(iii) \iff (iv)$. Implication $(iv) \implies (iii)$ is trivial. For the converse, suppose that
$G$ is a bipartite incomparability graph of a poset $P$, apply Dilworth’s theorem [2] or pick any
vertex $x$, and observe that the set of vertices at odd distance from $x$ and the set of vertices
at even distance from $x$ form a partition of $P$ into two chains, hence $G$ is bipartite.

We should mention the following result (this is essentially Lemma 14 from [27]) which
states that a bipartite permutation graph without cycles must embed a caterpillar. A key
observation is that if a vertex has at least three neighboring vertices in $Inc(P)$, then at least
one has degree one. Otherwise, $Inc(P)$ would have a spider (see Figure 5) as an induced
subgraph, which is impossible.

**Lemma 14.** Let $P$ be a poset of coverable by two chains. Then the following properties are
equivalent.
(i) The incomparability graph of \( P \) has no cycles of length three or four.

(ii) The incomparability graph of \( P \) has no cycle.

(iii) The connected components of the incomparability graph of \( P \) are caterpillars.

4.2. Detour of bipartite permutation graphs. We are going to evaluate the detour of connected components of the incomparability graph of a poset of width at most 2.

Let \( P := (V, \leq) \) be a poset of width 2. Suppose that \( \text{Inc}(P) \) is connected. In this case, the partition of \( P \) into two chains is unique. An alternating sequence in \( P \) is any finite monotonic sequence \((x_0, \ldots, x_i, \ldots, x_n)\) of elements of \( V \) (i.e., increasing or decreasing) such that no two consecutive elements \( x_i \) and \( x_{i+1} \) belong to the same chain of the partition. The integer \( n \) is the oscillation of the sequence; \( x \) and \( y \) are its extremities.

We recall that the oscillation of an alternating sequence with extremities \( x, y \) is either 0 or at most \( d_{\text{Inc}(P)} \) (see I.2.4. Lemme p.6 of [23]). This allows to define the following map. Let \( d_P \) be the map from \( V \times V \) into \( \mathbb{N} \) such that:

1. \( d_P(x, x) = 0 \) for every \( x \in V \);
2. \( d_P(x, y) = 1 \) if \( x \) and \( y \) are incomparable;
3. \( d_P(x, y) = 2 \) if \( x \) and \( y \) are comparable and there is no alternating sequence from \( x \) to \( y \);
4. \( d_P(x, y) = n + 2 \) if \( n \neq 0 \) and \( n \) is the maximum of the oscillation of alternating sequences with extremities \( x \) and \( y \).

We recall a result of [23] II.2.5 Lemme, p. 6.

**Lemma 15.** The map \( d_P \) is a distance on any poset \( P \) of width 2 such that the incomparability graph is connected. Moreover, for every \( x, y \in P \) the following inequalities hold:

\[
0 \leq d_{\text{Inc}(P)}(x, y) - d_P(x, y) \leq 2\left[d_{\text{Inc}(P)}(x, y)/3\right].
\]

We give a slight improvement of [23] I.2.3. Corollaire, p. 5.

**Lemma 16.** Let \( P \) be poset of width 2 such that \( \text{Inc}(P) \) is connected. Let \( n \in \mathbb{N} \), \( r \in \{0, 1\} \) and \( x, y \in P \) such that \( \text{Inc}(P) \) contains an induced path of length \( 3n + r \) and extremities \( x \) and \( y \). If \( r \neq 1 \) and \( n \geq 1 \) (resp. \( r = 1 \) and \( n \geq 2 \)) then there is an alternating sequence with extremities \( x, y \) and oscillation \( n \) (resp. \( n - 1 \)).

**Proof.** Since \( n \geq 1 \), \( x \) and \( y \) are comparable and we may suppose \( x < y \). Let \( x_0, \ldots, x_{3n+r} \) be a path with \( x_0 = x \), \( x_{3n+r} = y \). According to Lemma 10 the sequence \( x_0, \ldots, x_{3i}, \ldots, x_{3n} \) is alternating. If \( r \neq 1 \), we may replace \( x_{3n} \) by \( x_{3n+r} \) in the above sequence and get an alternating sequence with extremities \( x, y \) and oscillation \( n \). If \( r = 1 \), we delete \( x_{3n} \) and replace \( x_{3(n-1)} \) by \( x_{3n+r} \) in the above sequence. We get an alternating sequence of oscillation \( n - 1 \). \( \square \)

From Lemma 15 the oscillation between two vertices \( x \) and \( y \) of \( P \) is bounded above. With this lemma, the length of induced paths between \( x \) and \( y \) is bounded too, that is the detour \( D_{\text{Inc}(P)}(x, y) \) is an integer. In fact we have:

**Proposition 17.** Let \( P \) be poset of width 2 such that \( \text{Inc}(P) \) is connected and let \( x, y \in P \). Then:

1. \( d_{\text{Inc}(P)}(x, y) = d_P(x, y) = D_{\text{Inc}(P)}(x, y) \) if either \( x = y \), in which case this common value is 0, or \( x \) and \( y \) are incomparable, in which case this common value is 1.
\( (2) \ d_{Inc(P)} \geq d_P(x,y) \geq \lceil D_{Inc(P)}(x,y)/3 \rceil + \epsilon \) where \( \epsilon = 1 \) if \( D_{Inc(P)}(x,y) \equiv 1 \mod 3 \) and \( \epsilon = 2 \) otherwise.

**Proof.** Assertion (1) is obvious. For (2), we may suppose \( x < y \). The first inequality is embodied in Lemma \([13]\). As observed above, \( D_{Inc(P)}(x,y) \) is bounded. We may write \( D_{Inc(P)}(x,y) = 3n + r \) with \( r \) be the remainder of \( D_{Inc(P)}(x,y) \mod 3 \). Let \( \alpha := \lceil D_{Inc(P)}(x,y)/3 \rceil + \epsilon \). We have \( \alpha = n + 1 \) if \( r = 1 \) and \( \alpha = n + 2 \) otherwise. If \( n = 0 \) then since \( x < y \), \( r \neq 1 \), hence \( \alpha = 2 \), since \( d_P(x,y) = 2 \), the inequality holds. We may suppose \( n \geq 1 \). If \( r \neq 1 \) then \( \alpha = n + 2 \), while by definition of \( d_P \) and Lemma \([16]\) \( d_P(x,y) \geq n + 2 \). Hence, the second inequality holds. If \( r = 1 \) then \( \alpha = n + 1 \). If \( n = 1 \) \( d_P(x,y) \geq 2 \) and the second inequality holds. Suppose \( n \geq 2 \). Then, by definition of \( d_P(x,y) \) and by Lemma \([16]\) \( d_P(x,y) \geq n + 1 \). Thus second inequality holds.

**Corollary 18.** If a bipartite permutation graph has diameter at most \( k \) it contains no induced path of length \( 3k \).

5. A proof of Theorem \([2]\)

**Proof.** The implication \((i) \Rightarrow (ii)\) is obvious. The implication \((ii) \Rightarrow (iii)\) follows from Proposition \([17]\) given in Subsection \([4.2]\). The implication \((iii) \Rightarrow (iv)\) follows from Theorem \([3]\). The implication \((iv) \Rightarrow (i)\) is obvious.

6. Example \([1]\)

**Proof.** Let \( X := \{y, x_0, x_1, x_2, \ldots\} \) and for every integer \( i \geq 0 \) let \( Z_i := \{z_{0,i}, z_{1,i}, \ldots, z_{i+3,i}\} \) be disjoint sets. We set \( V := \bigcup_{i \geq 0} Z_i \cup X \) and \( P := (V, \leq) \) where \( \leq \) is the binary relation on \( V \) defined as follows: \( X \setminus \{y\} \) is totally ordered by \( \leq \) and \( x_0 < x_1 < x_2 < \cdots < x_i < \cdots \). For all \( 0 \leq i < j \), every element of \( Z_i \) is below every element of \( Z_j \). For all \( i \geq 0 \), \( y \) is smaller than all elements in \( Z_i \) and is incomparable to \( x_i \). For all \( i \geq 0 \), \( x_i \) is smaller than all element of \( Z_i \setminus \{z_{0,i}\} \) and \( x_i \) is incomparable to all elements in \( \bigcup_{j<i} Z_j \cup \{z_{0,j}\} \). For all integers \( i \geq 0 \) and for all \( j \geq i + 1 \), \( x_{i+1} \) is smaller than all element in \( Z_j \). Finally, the restriction of \( Inc(P) \) to \( Z_i \) is the induced path \( z_{0,i}, z_{1,i}, \ldots, z_{i+3,i} \) so that \( z_{0,i} < z_{1,i} < z_{4,i} < \cdots \) and \( z_{1,i} < z_{3,i} < z_{5,i} < \cdots \) (see Figure \([1]\)). It is not difficult to see that \( \leq \) is an order relation and that the corresponding poset \( P \) can be covered by three chains.

**Claim 1:** The diameter of \( Inc(P) \) is 3.

Let \( a, b \) be two distinct vertices of \( Inc(P) \). If \( a, b \in X \), then either \( a = y \) or \( b = y \) in which case \( d_{Inc(P)}(a,b) = 1 \), or \( a \neq y \) and \( b \neq y \) in which case \( d_{Inc(P)}(a,b) = 2 \) (indeed, say \( a = x_i \) and \( b = x_j \) with \( i < j \), then \( a, z_{0,i}, b \) is an induced path in \( Inc(P) \)). Suppose now \( a \in X \) and \( b \notin X \), say \( b \in Z_i \) for some \( i \geq 0 \). If \( a = y \), then \( d_{Inc(P)}(a,b) = 2 \) (indeed, \( a, x_{i+1}, b \) is an induced path in \( Inc(P) \)). Else if \( a = x_j \) for some \( j \geq 0 \), then \( d_{Inc(P)}(a,b) = 2 \) if \( i < j \) and \( d_{Inc(P)}(a,b) = 3 \) otherwise (indeed, \( a, z_{0,j}, x_{i+1}, b \) is the shortest path joining \( a \) to \( b \)). Next we suppose that \( \{a,b\} \cap X = \emptyset \). If \( a, b \in Z_i \) for some \( i \geq 0 \), then \( d_{Inc(P)}(a,b) = 2 \) (indeed, \( a, x_{i+1}, b \) is an induced path in \( Inc(P) \)). Else if \( a \in Z_i \) and \( b \in Z_j \) for some \( i \neq j \), then \( d_{Inc(P)}(a,b) = 2 \) (indeed, \( a, x_{i+j}, b \) is an induced path in \( Inc(P) \)).

**Claim 2:** An induced infinite path in \( Inc(P) \) contains necessarily finitely many elements of \( X \).

Suppose an induced infinite path \( C \) contains infinitely many vertices from \( X \). Since \( Inc(P) \) induces an independent set on \( X \setminus \{y\} \) and \( C \) is connected we infer that \( C \) must meet
ininitely many $Z_i$’s. Hence, there exists some $x_i \in C$ which has degree at least 3 in $C$ and this is not possible.

**Claim 3:** Deleting all vertices of $X$ from $Inc(P)$ leaves a disconnected graph.

Clearly, for all $i \geq 0$, $Z_i$ is a connected component of $Inc(P) \setminus X$.

Now suppose for a contradiction that $Inc(P)$ embeds an infinite induced path $C$. It follows from Claim 2 that we can assume $V(C) \cap X = \emptyset$. Hence, $C$ is an induced infinite path of $Inc(P) \setminus X$. We derive a contradiction since all connected components of $Inc(P) \setminus X$ are finite (indeed, the connected components of $Inc(P) \setminus X$ are finite paths i.e. the subgraphs of $Inc(P) \setminus X$ induced on the $Z_i$’s).

**Claim 4:** The vertex $y$ has an infinite induced detour.

Indeed, $Inc(P)$ induces a path on $\{y, x_i, \} \cup Z_i$ of length $i + 5$ for all $i \geq 0$. \qed

7. ORDER AND METRIC CONVEXITIES OF INCOMPARABILITY GRAPHS

In this section we compare the notions of order convexity and metric convexity with respect to the distance on the incomparability graph of a poset.

We recall few definitions already provided in the introduction. Let $P := (V, \leq)$ be a poset. We recall $Conv_P(X)$ is the smallest convex set containing $X$ and that

$$Conv_P(X) = \{ z \in P : x \leq z \leq y \text{ for some } x, y \in X \} = \downarrow X \cap \uparrow X.$$

Let $G := (V, E)$ be a graph. We equip it with the graphic distance $d_G$. A ball is any subset $B_G(x, r) := \{ y \in V : d_G(x, y) \leq r \}$ where $x \in V, r \in \mathbb{N}$. A subset of $V$ is convex with respect to the distance $d_G$ if this is an intersection of balls. The least convex subset of $G$ containing $X$ is

$$Conv_G(X) := \bigcap_{x \in B_G(x, r)} B_G(x, r).$$

Let $X \subseteq V$ and $r \in \mathbb{N}$. Define

$$B_G(X, r) := \{ v \in V : d_G(v, x) \leq r \text{ for some } x \in X \}.$$  

The proof of the following lemma is elementary and is left to the reader.

**Lemma 19.** Let $G$ be a graph, $X \subseteq V(G)$ and $r \in \mathbb{N}$. Then

(1) $B_G(X, r) = B_G(B_G(X, 1), r - 1) = B_G(B_G(X, r - 1), 1)$ for all $r \geq 1$.

(2) $B_G(X \cup Y, r) = B_G(X, r) \cup B_G(Y, r)$.

**Lemma 20.** Let $P := (V, \leq)$ be a poset and $G$ be its incomparability graph, $X \subseteq V$ and $r \in \mathbb{N}$. Then

(3) $B_G(\downarrow X, r) = (\downarrow X) \cup B_G(X, r) = \downarrow B_G(X, r)$.

(4) $B_G(\uparrow X, r) = (\uparrow X) \cup B_G(X, r) = \uparrow B_G(X, r)$.

(5) $B_G(\uparrow X \cap \downarrow X, r) = B_G(\uparrow X, r) \cap B_G(\downarrow X, r)$.

(6) $B_G(Conv_P(X), r) = Conv_P(X) \cup B_G(X, r) = Conv_P(B_G(X, r)).$
Proof. We mention at first that all above equalities are clearly true for \( r = 0 \). We claim that it is enough to prove (3). Indeed, (1) is obtained from (3) applied to \( P^* \). We now show how to obtain (5) using (3) and (4). The proof is by induction on \( r \).

Basis step: \( r = 1 \).

Clearly, \( B_G(\uparrow X \cap \downarrow X, 1) \subseteq B_G(\uparrow X, 1) \cap B_G(\downarrow X, 1) \). Let \( x \in B_G(\uparrow X, 1) \cap B_G(\downarrow X, 1) \). There are \( y_1 \in \downarrow X \) and \( y_2 \in \uparrow X \) such that \( x \) is equal to \( y_1 \) or incomparable to \( y_1 \) and similarly \( x \) is equal to \( y_2 \) or incomparable to \( y_2 \). Since \( y_1 \in \downarrow X \) and \( y_2 \in \uparrow X \) there are \( x_1, x_2 \in X \) such that \( y_1 \leq x_1 \) and \( x_2 \leq y_2 \). If \( x \) is incomparable or equal to \( x_1 \) or to \( x_2 \), then \( x \in B_G(X, 1) \subseteq B_G(\uparrow X \cap \downarrow X, 1) \) as required. If not, \( x_2 \leq x \leq x_1 \) (since \( x \) is equal to \( y_1 \) or incomparable to \( y_1 \) and \( x \) is equal to \( y_2 \) or incomparable to \( y_2 \)), hence \( x \in \downarrow X \cap \uparrow X \subseteq B_G(\downarrow X \cap \uparrow X, 1) \), as required.

Inductive step: Suppose \( r > 1 \). We have

\[
B_G(\uparrow X \cap \downarrow X, r) = B_G(B_G(\uparrow X \cap \downarrow X, r - 1), 1)
\]

\[
= B_G(B_G(\uparrow X, r - 1) \cap B_G(\downarrow X, r - 1), 1) \text{ (by the induction hypothesis)}
\]

\[
= B_G(\uparrow B_G(X, r - 1), 1) \cap B_G(\downarrow B_G(X, r - 1), 1) \text{ (by equations (3) and (4))}
\]

\[
= B_G(\uparrow B_G(X, r - 1), 1) \cap B_G(\downarrow B_G(X, r - 1), 1) \text{ (follows from the basis step \( r = 1 \))}
\]

\[
= \uparrow B_G(B_G(X, r - 1), 1) \cap \downarrow B_G(B_G(X, r - 1), 1) \text{ (follows from (3) and (4))}
\]

\[
= \uparrow B_G(X, r) \cap \downarrow (B_G(X, r))
\]

\[
= B_G(\uparrow X, r) \cap (\downarrow B_G(X, r)) \text{ (follows from (3)).}
\]

We now show how to obtain (6) using (3), (4) and (5).

From (3) and (4) we obtain

\[
B_G(\downarrow X, r) \cap B_G(\uparrow X, r) = ((\downarrow X) \cup B_G(X, r)) \cap ((\uparrow X) \cup B_G(X, r)) = (B_G(X, r)) \cap (B_G(X, r)).
\]

This is equivalent to

\[
B_G(\downarrow X, r) \cap B_G(\uparrow X, r) = \downarrow (B_G(X, r)) \cap \uparrow (B_G(X, r)).
\]

Using (5) we have

\[
B_G(\downarrow X \cap \uparrow X, r) = (\downarrow X \cap \uparrow X) \cup B_G(X, r)) = \downarrow (B_G(X, r)) \cap \uparrow (B_G(X, r)).
\]

The required equalities follow by definition of the operator \( \text{Conv} \).

We now prove (3).

Basis step: \( r = 1 \).

Since \( X \subseteq \downarrow X \) we have \( B_G(X, 1) \subseteq B_G(\downarrow X, 1) \). Hence, we have \( B_G(\downarrow X, 1) \subseteq (\downarrow X) \cup B_G(X, 1) \). From \( X \subseteq B_G(X, 1) \) we deduce that \( \downarrow X \subseteq B_G(X, 1) \). Hence, \( (\downarrow X) \cup B_G(X, 1) \subseteq B_G(X, 1) \).

Next, we prove that \( B_G(\downarrow X, 1) \subseteq (\downarrow X) \cup B_G(X, 1) \). Let \( x \in B_G(\downarrow X, 1) \). There exists \( y \in X \) at distance at most 1 from \( x \) that is either \( y = x \) or \( y \parallel x \). If \( y = x \) then \( x \in \downarrow X \). Otherwise, since \( y \in X \) there is \( y_1 \in X \) such that \( y \leq y_1 \). If \( y_1 \) is incomparable or equal to \( x \) then \( x \in B_G(X, 1) \). Otherwise \( y_1 \) is comparable to \( x \). Necessarily, \( x \leq y_1 \) since \( x \parallel y \). Hence \( x \in X \).

Inductive step: Let \( r > 1 \). We suppose true the equalities

\[
B_G(\downarrow X, r - 1) = (\downarrow X) \cup B_G(X, r - 1) \subseteq B_G(X, r - 1).
\]

We apply the operator \( T \rightarrow B_G(T, 1) \) to each term of the previous equalities and obtain

\[
B_G(B_G(\downarrow X, r - 1), 1) = B_G((\downarrow X) \cup B_G(X, r - 1), 1) = B_G(\downarrow B_G(X, r - 1), 1).
\]
We have
\[ B_G(B_G(\downarrow X, r - 1), 1) = B_G(\downarrow X, r) \] (see (1) of Lemma 19).

Also,
\[
\begin{align*}
B_G((\downarrow X) \cup B_G(X, r - 1), 1) &= B_G(\downarrow X, 1) \cup B_G(B_G(X, r - 1), 1) \quad \text{(see (2) of Lemma 19)} \\
&= B_G(\downarrow X, 1) \cup B_G(X, r) \quad \text{(follows from (3) with } r = 1). \\
&= (\downarrow X) \cup B_G(X, r).
\end{align*}
\]

Finally we have
\[
\begin{align*}
B_G((B_G(X, r - 1)), 1) &= \downarrow B_G((B_G(X, r - 1)), 1) \\
&= \downarrow (B_G(X, r)).
\end{align*}
\]

8. A proof of Theorem 7 and some consequences

We now proceed to the proof of Theorem 7.

Proof. (a) Apply successively equations (3), (1) and (5) of Lemma 20.

(b) Suppose \( r = 1 \). Let \( G' := G_{1B_G(X, 1)} \) and \( x, y \in B_G(X, 1) \). Let \( n := d_G(x, y) \). Clearly, \( n \leq d_G(x, y) \). To prove that the equality holds, we may suppose that \( 2 \leq n < \infty \). We argue by induction on \( n \). Let \( u_0, \ldots, u_n \) be a path in \( G \) connecting \( x \) and \( y \). If \( n \geq 4 \), we have \( x_0 < x_2 < x_n \) by Lemma 10. Since \( B_G(X, 1) \) is convex, it contains \( x_2 \), hence, by induction, \( d_G(x, x_2) = d_G(x, x_2) = 2 \) and \( d_G(x_2, y) = d_G(x_2, y) = n - 2 \), hence \( d_G(x, y) = d_G(x, y) \). Thus, to conclude, it suffices to solve the cases \( n = 2 \) and \( n = 3 \). Let \( x', y' \in X \) with \( x' \) incomparable or equal to \( x \) and \( y' \) incomparable or equal to \( y \). If \( u_{n-1} \) is incomparable or equal to \( y' \), then \( x_{n-1} \in B_G(X, 1) \). From the induction, \( d_G'(x, x_{n-1}) = d_G(x, x_{n-1}) \) hence \( d_G'(x, y) = d_G(x, y) \) as required. Hence, we may suppose \( u_{n-1} \) comparable to \( y' \), and similarly \( u_1 \) comparable to \( x' \).

Also, if \( x' \) is incomparable or equal to \( x_2 \) then \( x, x', x_2 \) is a path in \( B_G(X) \); if \( n = 2 \) we have \( d_G'(x, y) = 2 \) as required, if \( n = 3 \), then \( x, x', x_2, y \) is a path in \( B_G(X) \) and \( d_G'(x, y) = 3 \) as required. Thus we may suppose \( x' \) comparable to \( x_2 \) and, similarly, \( y' \) comparable to \( x_{n-2} \).

Since \( x' \) is incomparable or equal to \( x_0 \) and, by Lemma 10, \( x_0 < x_2 \), we have \( x' < x_2 \). Similarly, we have \( x_{n-2} < y' \). Since \( x_1 \) is comparable to \( x' \) and incomparable to \( x_2 \) we deduce \( x' \leq x_1 \) from \( x' < x_2 \). Similarly, we deduce \( x_{n-1} \leq y' \). For \( n = 2 \) we have \( x' \leq x_1, x_2 \) and for \( n = 3, x' \leq x_1, x_2 \leq y' \). By order convexity of \( X \), \( x_1 \in X \) (and also \( x_2 \in X \) if \( n = 3 \), hence the path \( x = x_0, x_1, x_2 = y \) if \( n = 2 \) and the path \( x = x_0, x_1, x_2, x_3 = y \) if \( n = 3 \) in \( B_G(X) \) and thus \( d_G'(x, y) = n \).

Suppose \( r > 1 \). Then from (a) above, \( B_G(X, 1) \) order convex. Via the induction hypothesis, \( G_{1B_G(X, 1), r-1} \) is an isometric subgraph of \( G \). Since \( B_G(X, r) = B_G(B_G(X, 1), r-1) \), \( G_{1B_G(X, r)} \) is an isometric subgraph of \( G \). \( \square \)

As the proof of the Lemma 20 suggests, balls are not necessarily geodesically convex (for an example, look at the ball \( B(x, 1) \) in a four element cycle). A consequence of Theorem 7 is that the order convexity of balls is equivalent to the following inequality:

**Corollary 21.** Let \( P \) be a poset and let \( G \) be its incomparability graph. Then
\[
d_G(u, v) \leq d_G(x, y) \quad \text{for all } x \leq u \leq v \leq y \text{ in } P.
\]
Proof. The inequality above amounts to $d_G(u, v) \leq d_G(x, v) \leq d_G(x, y)$. We prove the first inequality; the second inequality follows by the same argument applied to the dual of $P$. We may suppose that $x < u < v$, otherwise nothing to prove. Let $n := d_G(v, x)$. By (a) of Theorem\ref{thm:p}, $B_G(v, n)$ is order convex. Since $x, v \in B_G(v, n)$ and $x \leq u \leq v$, then $u \in B(v, n)$ amounting to $d_G(u, v) \leq n = d_G(x, v)$. Conversely, assuming that inequality (1) holds, observe that every ball $B_G(x, r)$ is order-convex. We may suppose $r \geq 1$, otherwise the conclusion is obvious. Let $u, v \in B_G(x, r)$ and $w \in P$ with $u < w < v$. If $x \parallel w$, then $d_G(x, w) = 1 \leq r$ hence $w \in B_G(x, r)$. If not, then either $x < w$ or $w < x$. In the first case, from $x < w < v$, inequality (2) yields $d_G(x, w) \leq d_G(x, v) \leq r$ hence $w \in B_G(x, r)$, whereas in the second case, from $u < w < x$, inequality (1) yields $d_G(w, x) \leq d_G(u, x) \leq r$ hence $w \in B_G(x, r)$. □

**Corollary 22.** $\delta_G(X) = \delta_G(Conv_P(X)) = \delta_G(Conv_G(X))$ for every subset $X$ of a poset $P$.

Proof. Since by (a) of Theorem\ref{thm:p}, each ball $B_G(x, r)$ is order convex, $Conv_P(X) \subseteq Conv_G(X)$. Hence $\delta_G(X) \leq \delta_G(Conv_P(X)) \leq \delta_G(Conv_G(X))$.

The equality $\delta_G(X) = \delta_G(Conv_G(X))$ is a general convexity property of metric spaces. Let $r := \delta_G(X)$. Let $x, y \in Conv_G(X)$. We prove that $d_G(x, y) \leq r$. First $X \subseteq B_G(x, r)$. Indeed, let $z \in X$; since $\delta_G(X) = r$, $X \subseteq B_G(x, r)$. Since $Conv_G(X)$ is the intersections of balls containing $X$, we have $Conv_G(X) \subseteq B_G(z, r)$, hence $z \in B_G(x, r)$. Next, from $X \subseteq B_G(x, r)$ we deduce $Conv(X) \subseteq B_G(x, r)$ hence $y \in B_G(x, r)$ that is $d_G(x, y) \leq r$.

**Lemma 23.** Let $P := (V, \leq)$ be a poset and $G$ be its incomparability graph. Let $x, y, z \in V$ be such that $x < z < y$. Then

$$\max\{d_G(x, z), d_G(z, y)\} \leq d_G(x, y) \leq d_G(x, z) + d_G(z, y) \leq d_G(x, y) + 2.$$

Proof. The first inequality follows from Corollary\ref{cor:gamma}. The second inequality is the triangular inequality. We now prove the third inequality. Let $p := d_G(x, z)$, $q := d_G(z, y)$, $r := d_G(x, y)$.

**Claim:** Let $x_0 := x, \ldots, x_r := y$ be a path from $x$ to $y$. Then there exist $i \notin \{0, r\}$ such that $z$ is incomparable to $x_i$.

**Proof the claim.** By induction on $r$. Note that since $x < y$ we have $r \geq 2$. If $r = 2$, then necessarily $z$ is incomparable to $x_1$. Suppose $r > 2$. Then $z \not\leq x_1$. If $z$ is incomparable to $x_1$, then we are done. Otherwise $x_1 < z$ and we may apply the induction hypothesis to $x_1, y$ and the path $x_1, \ldots, x_r = y$. This completes the proof of the claim.

Let $i$ be such in the Claim. Then $x_0 := x, \ldots, x_i, z$ is a path from $x$ to $z$ of length $i + 1$ and $z, x_i, x_{i+1}, \ldots, x_r$ is a path from $z$ to $y$ of length $r - i + 1$. Then $p + q \leq i + 1 + r - i + 1 = r + 2$. The proof of the lemma is now complete.

**Lemma 24.** Let $x_0, \ldots, x_n$ be an isometric path in a graph $G$ with $n \geq 2$. There exists a vertex $x_{n+1}$ such that $x_0, \ldots, x_n, x_{n+1}$ is an isometric path in $G$ if and only if $B_G(x_0, 1) \subseteq B_G(x_n, 0)$.\[17\]

**Proof.** ⇒ is obvious.

⇐ Suppose $B_G(x_0, 1) \not\subseteq B_G(x_n, 0)$ and let $x_{n+1} \in B_G(x_n, 1) \setminus B_G(x_0, n)$.

**Claim 1:** $d_G(x_0, x_{n+1}) = n + 1$.

Indeed, since $x_{n+1} \in B_G(x_n, 1) \setminus B_G(x_0, n)$ we have $d_G(x_0, x_{n+1}) > n$. From the triangular inequality $d_G(x_0, x_{n+1}) \leq d_G(x_0, x_n) + d_G(x_n, x_{n+1}) = n + 1$.

**Claim 2:** $d_G(x_j, x_{n+1}) = n + 1 - j$ for all $0 \leq j \leq n$.

Indeed, from the triangular inequality $d_G(x_j, x_{n+1}) \leq d_G(x_j, x_n) + d_G(x_n, x_{n+1}) = n - j + 1$. Similarly, $d_G(x_0, x_{n+1}) \leq d_G(x_0, x_j) + d_G(x_j, x_{n+1})$ and therefore $d_G(x_j, x_{n+1}) \geq d_G(x_0, x_{n+1}) - d_G(x_0, x_j) = n + 1 - j$. The equality follows. □
We could restate the previous lemma as follows. There is an isometric path of length \( n + 1 \) starting at some vertex \( x_0 \) if there is some \( x_n \in B_G(x_0, n) \) such that \( B_G(x_0, 1) \not\subset B_G(x_0, n) \).

An other consequence of the convexity of balls in an incomparability graph is the following:

**Lemma 25.** Let \( G \) be the incomparability graph of a poset \( P \). If a ball contains infinitely many vertices of a one way infinite induced path then it contains all vertices except may be finitely many vertices of that path.

**Proof.** Let \( P_\infty \) be an infinite induced path of \( G \) and \( (x_n)_{n\in\mathbb{N}} \) be an enumeration of its vertices, so that \( (x_n, x_{n+1}) \in E(G) \) for \( n \in \mathbb{N} \). Without loss of generality we may suppose that \( x_0 < x_2 \) (otherwise, replace the order of \( P \) by its dual). By Lemma 20 we have \( x_i < x_j \) for every \( i+2 \leq j \). Let \( B_G(x, r) \) be a ball of \( G \) containing infinitely many vertices of \( P_\infty \). Let \( x_i \in P_\infty \cap B(x, r) \). We claim that \( x_j \in P_\infty \cap B(x, r) \) for all \( j \geq i + 2 \). Indeed, due to our hypothesis, we may pick \( x_r \in P_\infty \cap B(x, r) \) with \( r \geq j + 2 \). We have \( x_i < x_j < x_r \). Due to the convexity of \( B(x, r) \) we have \( x_j \in B(x, r) \). This proves our claim. \( \square \)

Said differently:

**Lemma 26.** If a one way infinite induced path \( P_\infty \) has an infinite diameter in the incomparability graph \( G \) of a poset then every ball of \( G \) with finite radius contains only finitely many vertices of \( P_\infty \).

9. **Induced infinite paths in incomparability graphs:** A proof of Theorem 3

The proofs of (1) and (2) of Theorem 3 are similar. We construct a strictly increasing sequence \( (y_n)_{n\in\mathbb{N}} \) of vertices such that \( 3 \leq d_G(y_n, y_{n+1}) < +\infty \) for all \( n \in \mathbb{N} \) and we associate to each \( n \in \mathbb{N} \) a finite path \( P_n := z_{(n,0)}, z_{(n,1)}, \ldots, z_{(n,r_n)} \) of \( G \) of length \( n \) := \( d_G(y_n, y_{n+1}) \) joining \( y_n \) and \( y_{n+1} \). We show first that the graph \( G' := G(|\bigcup_{n\in\mathbb{N}} V(P_n)|) \) is connected and has an infinite diameter. Next, we prove that it is locally finite. Hence from König’s Lemma 4, it contains an isometric path. This path yields an induced path of \( G \). The detour via König’s Lemma is because the union of the two consecutive paths \( P_n \) and \( P_{n+1} \) do not form necessarily a path. In the first proof, our paths have length 3. In the second proof, their end vertices have degree at least 3.

**Lemma 27.** Let \( P := (V, \leq) \) be a poset so that its incomparability graph \( G \) is connected and has infinite diameter. Let \( x \in V \) be arbitrary. Then at least one of the sets \( d^+_G(x) := \{d_G(x, y) : x < y \in V\} \) or \( d^-_G(x) := \{d_G(x, y) : y \in V \text{ and } y < x\} \) is unbounded in \( \mathbb{N} \). Furthermore, if \( d^+_G(x) := \{d_G(x, y) : x < y \in V\} \) is unbounded in \( \mathbb{N} \) and \( z \geq x \), then \( d^+_G(z) := \{d_G(z, y) : z < y \in V\} \) is unbounded in \( \mathbb{N} \) (in particular, \( z \) cannot be maximal in \( P \)).

**Proof.** Suppose for a contradiction that the sets \( d^-_G(x) := \{d_G(x, y) : x < y \in V\} \) and \( d^+_G(x) := \{d_G(x, y) : y \in V \text{ and } y < x\} \) are bounded. Let \( r := \max d^+_G(x) \) and \( r' := \max d^-_G(x) \). Then \( V := B_G(x, \text{max}\{2, r, r'\}) \) and therefore the diameter of \( G \) is bounded contradicting our assumption. Now let \( z > x \) and suppose for a contradiction that \( d^+_G(z) := \{d_G(z, y) : z < y \in V\} \) is bounded and let \( r := \max d^+_G(z) \). Let \( x < y \). If \( y \leq z \) then \( d(x, y) \leq d(x, z) \) by Lemma 24; if \( z \parallel y \) then \( d(x, y) \leq d(x, z) + 1 \); if \( z \leq y \) then we have \( d(x, y) \leq d(x, z) + d(z, y) \leq d(x, z) + r \), hence, the set \( d^+_G(x) \) is bounded, contradicting our assumption. \( \square \)

**Proof of (1) of Theorem 3** We construct a sequence \( (x_n)_{n\in\mathbb{N}} \) of vertices (see Figure 6). We pick \( x_0 \in V \). According to Lemma 27 one of the set \( d^+_G(x_0) := \{d_G(x_0, y) : x_0 < y \in V\} \)
and \( d_G(x_0) := \{d_G(x_0, y) : y \in V \text{ and } y < x_0 \} \) is unbounded. We may assume without loss of generality that the set \( d_G(x_0) \) is unbounded. Choose an element \( x_3 > x_0 \) at distance three from \( x_0 \) in \( G \) and let \( x_0, x_1, x_2, x_3 \) be a path joining \( x_0 \) to \( x_1 \). Note that necessarily we have \( x_0 < x_2 \) and \( x_1 < x_3 \). Now suppose constructed a sequence \( x_0, x_1, \ldots, x_{3n} \) such that \( x_0 < x_3 \cdots < x_{3n} \) and such that \( x_{3i}, x_{3i+1}, x_{3i+2}, x_{3i+3} \) is a path of extremities \( x_{3i} \) and \( x_{3(i+1)} \) for \( i < n \). According to Lemma 27, the set \( d_G^0(x_{3n}) \) is unbounded. Hence, it contains a vertex \( x_{3(n+1)} \) at distance three from \( x_{3n} \). Let \( x_{3n}, x_{3n+1}, x_{3n+2}, y_{3n+3} \) be a path of extremities \( x_{3n} \) and \( x_{3(n+1)} \). By Lemma 10 we have necessarily:

\[
(8) \quad x_{3n} < x_{3n+2} \text{ and } x_{3n+1} < x_{3n+3}.
\]

Let \( P' \) be the poset induced on the set \( V' := \{x_n : n \in \mathbb{N} \} \) and \( G' \) be the incomparability graph of \( P' \). According to our construction \( G' \) contains a spanning path (not necessarily induced), hence it is connected.

**Claim 1.** \( d_G(x_0, x_{3n}) \geq n + 2 \) for every \( n \geq 1 \).

Since \( d_G(x_0, x_{3n}) \geq d_G(x_0, x_{3n}) \), it follows that the diameter of \( G' \) is infinite.

**Proof of Claim 1.** We prove the inequality of the claim by induction on \( n \geq 1 \). By definition, the inequality holds for \( n = 1 \). Suppose the inequality holds for \( n \). It follows from Lemma 23 that \( n + 5 \leq d_G(x_0, x_{3n}) + d_G(x_{3n}, x_{3(n+1)}) \leq d_G(x_0, x_{3(n+1)}) + 2 \) and therefore the inequality holds for \( n + 1 \).

**Claim 2.** The incomparability graph of \( P' \) is locally finite, that is for all \( x \in P' \), \( \text{inc}_P(x) := \{ y \in V' : x \parallel y \} \) is finite.

In fact, \( \text{inc}_P(x) \) has at most six elements.

**Proof of Claim 2.** We have

\[(a) \quad \text{inc}_P(x_{3n}) \subseteq \{x_{3n-1}, x_{3n+1}\} \text{ for } n \geq 1,\]

\[(b) \quad \text{inc}_P(x_{3n+1}) \subseteq \{x_{3(n-2)+2}, x_{3(n-1)+1}, x_{3(n-1)+2}, x_{3n}, x_{3n+2}, x_{3(n+1)+1}\} \text{ for } n \geq 2,\]

\[(c) \quad \text{inc}_P(x_{3n+2}) \subseteq \{x_{3n-1}, x_{3(n+1)}, x_{3(n+1)+1}, x_{3(n+1)+2}, x_{3(n+2)+1}\} \text{ for } n \geq 1,\]

\[(d) \quad \text{inc}_P(x_0) = \{x_1\}, \quad \text{inc}_P(x_1) \subseteq \{x_0, x_2, x_4\}, \quad \text{inc}_P(x_2) \subseteq \{x_1, x_3, x_5, x_7\} \text{ and } \text{inc}_P(x_4) \subseteq \{x_1, x_3, x_5, x_7\}.
\]
Proof. (a) Let \( n \in \mathbb{N} \). By inequalities (8) stated above, we have \( x_{3n-2} < x_{3n} < x_{3n+2} \). Let \( n' \in \mathbb{N} \) be such that \( n < n' \). By construction, \( x_{3n} < x_{3n'} \). By inequalities (8) again we have \( x_{3n'} < x_{3n'+2} \), hence \( x_{3n} < x_{3n'+2} \). Since \( x_{3n} < x_{3n'} \) and \( x_{3n'+1} \) is incomparable to \( x_{3n'} \) we infer that \( x_{3n'+1} \notin x_{3n} \). We have \( d_G(x_{3n}, x_{3n'}) \geq 3 \); indeed, if \( n' = n + 1 \), \( d_G(x_{3n}, x_{3n'}) = 3 \) by construction, otherwise apply the first inequality of Lemma 22 with \( x = x_{3n}, \ z = x_{3(n+1)} \) and \( y = x_{3n'} \). Since \( d_G(x_{3n}, x_{3n'}) \geq 3 \) and \( x_3 \) is incomparable to \( x_{3n+1} \), the vertices \( x_3 \) and \( x_{3n+1} \) cannot be incomparable; it follows that \( x_{3n} < x_{3n'+1} \).

Since a poset and its dual have the same incomparability graph, we deduce that if \( n' < n \), then \( x_{3n'}, x_{3n'+1}, x_{3n'+2} < x_{3n} \). Hence, \( inc_{P'}(x_{3n}) \subseteq \{x_{3n-1}, x_{3n+1}\} \) for \( n \geq 1 \).

(b) Since \( x_{3n-3} < x_{3n} \) and \( x_{3n} \) and \( x_{3n+1} \) are incomparable we infer that \( x_{3n+1} \notin x_{3n-3} \). It follows that \( x_{3n-3} < x_{3n+1} \) because otherwise \( x_{3n-3}, x_{3n+1}, x_{3n} \) would be a path of length two contradicting our assumption that \( d_G(x_{3n-3}, x_{3n}) = 3 \). From Lemma 10 we deduce that if \( k < 3n - 4 \), then \( x_k < x_{3n-3} \) and hence \( x_k < x_{3n+1} \). Hence, if \( k < 3n-1 \) and \( x_k \) is incomparable to \( x_{3n+1} \), then \( k \in \{3n - 4, 3n - 2\} \). Since \( x_{3n+1} < x_{3n+3} \), it follows from Lemma 10 that if \( k > 3n + 4 \), then \( x_k > x_{3n+4} \) and hence \( x_k \notin inc_{P'}(x_{3n+1}) \). Hence, \( x_{3n} \) and \( x_{3n+1} \) are possible elements incomparable to \( x_{3n+1} \), hence the required inclusion.

(c) Since \( x_{3n} < x_{3n+2} \) it follows from Lemma 10 that if \( x_k \), for \( k < 3n \), is incomparable to \( x_{3n+2} \) then \( k \in \{3n - 1, 3n + 3\} \). Now observe that \( x_{3n+2} < x_{3n+6} \) because otherwise \( x_{3n+3}, x_{3n+2}, x_{3n+6} \) is a path of length two contradicting \( d_G(x_{3n+3}, x_{3n+6}) = 3 \). By duality we infer that if \( k > 3n + 4 \), then \( x_k \) incomparable to \( x_{3n+2} \) implies \( k \in \{3n + 5, 3n + 7\} \). The required inclusion readily follows.

(d) We have \( x_0 < x_3 \) and \( x_0 < x_2 \). Since \( d_G(x_0, x_3) = 3 \) and \( x_3 \) incomparable to \( x_4 \) we must have \( x_0 < x_4 \). From \( inc_{P'}(x_3) \subseteq \{x_2, x_4\} \) we deduce that \( x_1 \) is the only element incomparable to \( x_0 \). From \( x_1 < x_3 \) we deduce that \( inc_{P'}(x_1) \subseteq \{x_0\} \cup inc_{P'}(x_3) \) and therefore \( inc_{P}(x_1) \subseteq \{x_0, x_2, x_4\} \). From \( x_2 < x_6 \) and \( inc_{P}(x_6) \subseteq \{x_5, x_7\} \) we derive \( inc_{P'}(x_2) \subseteq \{x_1, x_3, x_5, x_7\} \). Similarly, we have \( inc_{P'}(x_4) \subseteq \{x_1, x_3, x_5, x_7\} \).

From Claim 1 and Claim 2, \( Inc(P') \) is connected, locally finite and has an infinite diameter. From König’s Lemma, \( G' \) contains an infinite isometric path, hence \( G \) contains an infinite induced path. This completes the proof of (1).

Proof of (2) of Theorem 8: We break the proof into two parts.

Claim 3. If \( G \) is a connected incomparability graph of infinite diameter and if the set of vertices of degree at least 3 in \( G \) has infinite diameter, then \( G \) contains an infinite induced path such that the set of vertices of this path with degree at least 3 in \( G \) has an infinite diameter.

Proof of Claim 3. Let \( x \) be any vertex in \( G \), \( I := inc_{P}(x) \cup \downarrow x \) and \( F := inc_{P}(x) \cup \uparrow x \). According to Theorem 4, \( I \) and \( F \) are order convex and \( G_{I} \) and \( G_{F} \) are isometric subgraphs of \( G \). Since, trivially, \( V(G) = I \cup F \), every vertex of degree at least 3 belongs to \( I \) or to \( F \). Since the diameter in \( G \) of the set of vertices of degree at least 3 is infinite and \( G_{I} \) and \( G_{F} \) are isometric subgraphs we infer that the diameter in \( G_{I} \) or in \( G_{F} \) of the set of vertices of degree at least 3 is infinite. Choose \( y \) of degree at least 3. We may assume without loss of generality that the diameter in \( G_{I} \) of the set of vertices of degree at least 3 is infinite. We start by showing that \( P \) contains an infinite chain of elements whose degree is at least 3 in \( G \). Suppose constructed a sequence \( y_0 := y < y_1 < \cdots < y_{n-1} \) of vertices of degree at least 3
such that \( d_G(y_i, y_{i+1}) > 3 \) for all \( i \leq n-2 \). Let \( y_n > x_0 \) be a vertex of degree at least 3 such that \( d_G(y_{n-1}, y_n) > \sum_{j=0}^{n-2} d_G(y_j, y_{j+1}) \). This choice of \( y_n \) is possible since the diameter in \( G'_{|F} \) of the set of vertices of degree at least 3 is infinite. Then \( y_{n-1} \) and \( y_n \) are comparable in \( P \). It follows from Corollary 21 that \( y_{n-1} < y_n \). Hence, the sequence \( (y_i)_{i \in \mathbb{N}} \) forms a chain in \( P \).

For all \( n \in \mathbb{N} \), let \( P_n := z(n,0), z(n,1), \ldots, z(n,r_n) \) be a path in \( G \) of length \( r_n := d_G(y_n, y_{n+1}) \) joining \( y_n \) and \( y_{n+1} \). The graph \( G' := G'_{|V(P_n)} \) is connected and has infinite diameter.

**Subclaim 1.** \( G' \) is locally finite.

**Proof of Subclaim 1.** It suffices to prove that for \( n+2 \leq m \), every vertex of \( P_n \) is comparable to every vertex of \( P_m \). Let \( z_{n,i} \in P_n \) and \( z_{m,j} \in P_m \).

- Suppose first \( i = r_n - 1 \).
  
  (a) \( z(n,r_n-1) \leq y_m, z(m,1) \). Indeed, \( z(n,r_n-1) \) and \( y_m \) are comparable, otherwise \( y_{n+1}, z(n,r_n-1), y_m \) form a path with extremities \( y_{n+1} \) and \( y_m \) hence \( d_G(y_{n+1}, y_m) \leq 2 \). This is impossible since \( d_G(y_{n+1}, y_m) \geq d_G(y_{n+1}, y_{n+2}) \geq 4 \). Furthermore, \( z(n,r_n-1) < y_m \), otherwise, since \( y_{n+1} < y_m \), we obtain \( y_{n+1} < z(n,r_n-1) \) by transitivity, while these vertices are incomparable. Similarly, \( z(n,r_n-1) \) and \( z(m,1) \) are comparable otherwise \( y_{n+1}, z(n,r_n-1), y_m \) form a path with extremities \( y_n \) and \( y_m \) hence \( d_G(y_{n+1}, y_m) \leq 3 \), while this distance is at least 4. Necessarily, \( z(n,r_n-1) < z(m,1) \), otherwise since \( z(n,r_n-1) < y_m \), we have \( z_{m,1} < y_m \) which is impossible.

(b) By symmetry, \( y_{n+1}, z(n,r_n-1) \leq z(m,1) \).

(c) \( z(n,r_n-1) \leq z(m,j) \). We just proved it for \( j = 0, 1 \). If \( j > 1 \), this follows from \( y_m < z_{m,i} \) by transitivity.

- Next, suppose \( i = r_n \). In this case \( z_{(n,i)} = y_{n+1} \). If \( j \geq 2 \), we have \( z_{(n,i)} = y_{n+1} < y_m = z_{(m,0)} < z_{(m,j)} \). If \( j = 1 \), this is just item (c) above.

- Finally, suppose that \( i < r_n - 1 \). In this case, \( z_{(n,i)} < y_{n+1} < z_{(m,j)} \).

Since \( G' \) is connected, locally finite and has an infinite diameter, König’s Lemma ensures that it contains an infinite isometric path \( P_\infty \). We claim that \( P_\infty \) contains an infinite number of vertices of degree at least 3 in \( G \). Clearly, \( V(P_\infty) \) meets infinitely many \( P_i \)’s. For each \( i \in \mathbb{N} \) let \( j_i \in V(P_i) \) be the largest such that \( z_{(i,j_i)} \in V(P_\infty) \). Then the degree of \( z_{(i,j_i)} \) is at least 3 in \( G \). Indeed, if \( z_{(i,j_i)} \in \{ y_i, y_{i+1} \} \), then we are done. Otherwise \( z_{(i,j_i)} \) is not an end vertex of \( P_i \). Then \( z_{(i,j_i)} \) must have a neighbour in \( P_\infty \) which is not in \( P_i \) and therefore must have degree three. So far we have proved that \( G' \) contains an infinite isometric path \( P_\infty \) containing infinitely many vertices of degree at least 3. Hence, \( G \) contains an infinite induced path \( P_\infty \) containing infinitely many vertices of degree at least 3. This proves our claim.

**Claim 4.** If \( G \) is a connected incomparability graph containing an infinite induced path such that the set of vertices of this path with degree at least 3 in \( G \) has an infinite diameter then \( G \) contains either a caterpillar or a kite.

**Proof of Claim 4.** Let \( (x_n)_{n \in \mathbb{N}} \) be a sequence of vertices of \( G \) with \( (x_n, x_{n+1}) \in E(G) \) for \( n \in \mathbb{N} \) forming an infinite induced path \( P_\infty \). Suppose that this path contains infinitely many vertices with degree at least 3 in \( G \) forming a set of infinite diameter in \( G \).

**Subclaim 2.** There is an infinite sequence \( (y_n)_{n} \) of vertices in \( V \setminus P_\infty \) forming an independent set and a family of disjoint intervals \( I_n := [l(n), r(n)] \) of \( \mathbb{N} \) such that \( \{l(n), r(n)\} \subseteq B_G(y_n, 1) \cap P_\infty \subseteq I_n \) for all \( n \in \mathbb{N} \).
Proof of Subclaim 2: Pick \( x_{i_0} \in P_\infty \) with degree at least 3 in \( G \) and set \( y_0 \) arbitrary in \( B_G(x_{i_0}, 1) \setminus P_\infty \). According to Lemma 26, \( P \) be the least, resp., the largest integer \( k \) such that \( x_k \in B_G(y_0, 1) \). Let \( n > 0 \). Suppose \((y_m)_m, I_m := [l(m), r(m)]\) be defined for \( m < n \). By Lemma 26, \( P \cap (\bigcup_{m < n} B_G(y_m, 2)) \) is finite, hence there is a vertex \( x_{i_n} \in P_\infty \) with degree at least 3 such that every vertex in the infinite subpath of \( P_\infty \) starting at \( x_{i_n} \) is at distance at least 3 of any \( y_m \). Pick \( y_n \in B(x_{i_n}, 1) \setminus P_\infty \) and set \( I_n = [l(n), r(n)] \) where \( l(n) \), resp., \( r(n) \) be the least, resp., the largest integer \( k \) such that \( x_k \in B_G(y_n, 1) \) \( \square \).

In order to complete the proof of Claim 4 we show that the graph \( G' \) induced on \( P_\infty \cup \{y_n : n \in \mathbb{N}\} \) contains a caterpillar or a kite. For that, we classify the vertices \( y_n \). We say that \( y_n \) has type \((0)\) if \( l(n) = r(n) \) (that is \( y_n \) has just one neighbour on \( P_\infty \). If the set \( Y_0 \) of vertices of type \((0)\) is infinite then trivially \( G'_{|P_\infty \cup Y_0} \) is a caterpillar (see Figure 2). We say that \( y_n \) has type \((1)\) if \( r(n) = l(n) + 1 \). Again, trivially, if the set \( Y_1 \) of vertices of type \((1)\) is infinite then \( G'_{|P_\infty \cup Y_1} \) is a kite of type \((1)\). We say that \( y_n \) has type \((2)\) if \( r(n) = l(n) + 2 \). It has type \((2,1)\) if \( (y(n), x_{(l(n)+1)}) \in E(G) \) while it has type \((2,2)\) if \( (y(n), x_{(l(n)+1)}) \notin E(G) \). If for \( i = 1, 2 \) the set \( Y_{2,i} \) of vertices of type \((2,i)\) is infinite then \( G'_{|P_\infty \cup Y_{2,i}} \) is a kite of type \((i+1)\) (see Figure 2). We say that \( y_n \) has type \((3)\) if \( r(n) \geq l(n) + 3 \). It has type \((3,1)\) if \( (y(n), x_{(l(n)+1)}) \in E(G) \) while it has type \((3,2)\) if \( (y(n), x_{(l(n)+1)}) \notin E(G) \). If the set \( Y_{3,3} \) of vertices of type \((3,3)\) is infinite delete from \( P_\infty \) the set \( Y := \bigcup_{i \in Y_{3,3}} \{x : m \in [l(n+2, \ldots, r(n)-1) \}. Then \( G'_{|P_\infty \cup Y_{3,3}, Y} \) is a kite of type \((2)\) if \( i = 1 \) or a caterpillar if \( i = 2 \) (see Figure 2). \( \square \).

10. Example 2

We define the poset satisfying the conditions stated in Example 2. For a poset \( P = (V, \leq) \) we set for every \( x \in V \) we set \( inc_P(x) := \{y \in V : x \parallel y\} \).

Let \( P := (X, \leq) \) be the poset defined on \( X := \mathbb{N} \times \mathbb{N} \times \{0, 1\} \) as follows. We let \( (m, n, i) \leq (m', n', i') \) if

\[
i = i' \text{ and } [n < n' \text{ or } (n = n' \text{ and } m \leq m')] \]

or

\[
i \neq i' \text{ and } [n + 1 < n' \text{ or } (n + 1 = n' \text{ and } m \leq m')] \]

We set \( A_m := \{(m, n, 1) : m \in \mathbb{N}\} \) for all \( n \geq 0 \) and \( B_m := \{(m, n, 0) : m \in \mathbb{N}\} \) and note that \( \cup_{n \in \mathbb{N}} A_n \) and \( \cup_{n \in \mathbb{N}} B_n \) are two total orders of order type \( \omega^2 \). In particular \( P \) is coverable by two chains and hence has width two.

Claim 1: \( \leq \) is an order relation.

Reflexivity and antisymmetry are obvious. We now prove that \( \leq \) is transitive. Let \( (m, n, i), (m', n', i'), (m'', n'', i'') \) be such that \( (m, n, i) \leq (m', n', i') \leq (m'', n'', i'') \). Note that since \( \{i, i', i''\} \subseteq \{0, 1\} \) at least two elements of \( \{i, i', i''\} \) are equal. If \( i = i' = i'' \) then clearly \( (m, n, i) \leq (m'', n'', i'') \). Next we suppose that there are exactly two elements of \( \{i, i', i''\} \) that are equal. There are three cases to consider.

\* \( i = i' \)

Since \( (m, n, i) \leq (m', n', i') \) we have

\[\text{(9)} \text{ } n < n' \text{ or } (n = n' \text{ and } m \leq m').\]

Since \( i' \neq i'' \) and \((m', n', i') \leq (m'', n'', i'')\) we have

\[\text{(10)} \text{ } n' + 1 < n'' \text{ or } (n' + 1 = n'' \text{ and } m' \leq m'').\]
If \( n + 1 < n'' \), then since \( i \neq i'' \) it follows that \((m,n,i) \leq (m'',n'',i'')\). Suppose \( n'' \leq n + 1 \). If \( n' + 1 < n'' \), then \( n' < n \). It follows from \((9)\) that \( n = n' \) and hence \( n + 1 < n'' \) proving that \((m,n,i) \leq (m',n',i'')\). Else, \( n'' \leq n' + 1 \). It follows from \((10)\) that \( n' + 1 = n'' \) and \( m' \leq m'' \). If \( n < n' \), then \( n + 1 < n'' \) and once again we have \((m,n,i) \leq (m'',n'',i'')\). Otherwise it follows from \((9)\) that \( n = n' \) and \( m \leq m' \). Hence, \( n + 1 = n'' \) and \( m \leq m'' \) proving that \((m,n,i) \leq (m'',n'',i'')\).

• \( i = i'' \).

Since \((m,n,i) \leq (m',n',i')\) and \( i \neq i' \) we have
\[
(11) \quad n + 1 < n' \text{ or } (n + 1 = n' \text{ and } m \leq m').
\]

Since \((m',n',i') \leq (m'',n'',i'')\) and \( i' \neq i'' \) we have
\[
(12) \quad n' + 1 < n'' \text{ or } (n' + 1 = n'' \text{ and } m' \leq m'').
\]

We prove that \( n < n'' \). We suppose \( n'' \leq n \) and we argue to a contradiction. We Claim that none of \( n + 1 < n' \) and \( n' + 1 < n'' \) can hold. Indeed, suppose \( n + 1 < n' \). Then \( n'' < n' \) and hence \( n' + 1 < n'' \) cannot be true. It follows from \((9)\) that \( n' + 1 = n'' \). But then \( n'' = n' + 1 > n' > n'' \) which is impossible. Now suppose \( n' + 1 < n'' \). Then \( n' + 1 < n < n + 1 < n' \) and this is impossible. It follows from \((11)\) and \((12)\) that \( n + 1 = n' \) and \( m \leq m' \) and \( n' + 1 = n'' \) and \( m' \leq m'' \). Hence, we have proved our claim that none of \( n + 1 < n' \) and \( n' + 1 < n'' \) can hold. It follows from \((11)\) and \((12)\) that \( n + 1 = n' \) and \( n' + 1 = n'' \), and in particular \( n + 2 = n'' \). This contradicts \( n'' \leq n \). Hence, \( n < n'' \) and therefore \((m,n,i) \leq (m'',n'',i'')\) since \( i = i'' \).

• \( i' = i'' \).

Since \((m,n,i) \leq (m',n',i')\) and \( i \neq i' \) we have
\[
(13) \quad n + 1 < n' \text{ or } (n + 1 = n' \text{ and } m \leq m').
\]

Since \((m',n',i') \leq (m'',n'',i'')\) and \( i' = i'' \) we have
\[
(14) \quad n' < n'' \text{ or } (n' = n'' \text{ and } m' \leq m'').
\]

If \( n + 1 < n'' \), then \((m,n,i) \leq (m'',n'',i'')\) since \( i \neq i'' \). We Claim that none of \( n + 1 < n' \) and \( n' < n'' \) can hold. Suppose \( n + 1 < n' \). Then \( n'' < n \) and it follows from \((14)\) that \( n' = n'' \). But then \( n'' \leq n + 1 < n' = n'' \) which is impossible. Suppose \( n' < n'' \). Then \( n' < n + 1 \) and it follows from \((13)\) that \( n + 1 = n' \). But then \( n + 1 = n' < n'' \) which is impossible. Hence, none of \( n + 1 < n' \) and \( n' < n'' \) can hold. It follows from \((13)\) and \((14)\) that \( n + 1 = n' \) and \( m \leq m' \) and \( n' = n'' \) and \( m' \leq m'' \). Therefore, \((n + 1 = n'' \text{ and } m \leq m'')\) proving that \((m,n,i) \leq (m'',n'',i'')\) as required.

**Claim 2:** Let \( j \in \mathbb{N} \). Then for all \( x \in A_j, \ |B_{Inc(P)}(x,1) \cap B_{j+1}| \) is finite.

Let \( x := (m,j,1) \in A_j \). Then \( B_{Inc(P)}(x,1) \cap B_{j+1} = \{(k,j + 1,0) : 0 \leq k \leq m - 1\} \).

**Claim 3:** Let \( j \in \mathbb{N} \). Then for all \( x \in B_j, \ |B_{Inc(P)}(x,1) \cap A_{j+1}| \) is finite.

Let \( x := (m,j,0) \in B_j \). Then \( B_{Inc(P)}(x,1) \cap A_{j+1} = \{(k,j + 1,1) : 0 \leq k \leq m - 1\} \).

**Claim 4:** Let \( j \in \mathbb{N} \). Then for all \( x \in A_j \) and for all \( y \in B_{Inc(P)}(x,1) \cap B_{j+1}, \ |B_{Inc(P)}(y,1) \cap A_{j+2}| < |B_{Inc(P)}(x,1) \cap B_{j+1}| \).

Let \( x := (m,j,1) \in A_j \). It follows from Claim 2 that \(|N(x) \cap B_{j+1}| = m \). Let \( y \in B_{Inc(P)}(x,1) \cap B_{j+1}, \) say \( y = (m',j + 1,0) \) and note that \( m' < m \). Then it follows from Claim 3 that \( |B_{Inc(P)}(x,1) \cap B_{j+2}| = m' \). Since \( m' < m \) we are done.

**Claim 5:** Let \( j \in \mathbb{N} \). Then for all \( x \in B_j \) and for all \( y \in B_{Inc(P)}(x,1) \cap A_{j+1}, \ |B_{Inc(P)}(y,1) \cap B_{2j+1}| < |B_{Inc(P)}(x,1) \cap A_{j+1}| \).
Symmetry and Claim 4.

**Claim 6:** If there exists an infinite isometric path \((x_i)_{i \in \mathbb{N}}\) in \(\text{Inc}(P)\) starting at \(x_0 = (0,0,1)\), then \(x_{2n} \in A_{2n-1}\) and \(x_{2n+1} \in B_{2n}\).

From \(B_0 = \text{inc}_P(x_0) := \{y \in X : y \text{ incomparable to } x \text{ in } P\}\) follows that \(x_1 \in B_0\). Suppose for a contradiction that \(x_2 \notin A_1\). Then \(x_2 \in A_0\). In this case \(x_3 \in B_1\) (this is because \(x_0 < x_3\)). But then \(x_4 \in A_2\) because otherwise \(x_4 \in A_1\) and hence the distance from \(x_0\) to \(x_4\) would be two which is not possible. By the same token \(x_5 \in B_3\) and more generally \(x_{2n+1} \in B_{2n-1}\) and \(x_{2n-2} \in A_n\). This is impossible. Indeed, suppose \(x_2 = (i,0,0)\) then \(x_3 = (j,1,1)\) with \(j < i\) and then \(x_4 = (k,2,0)\) with \(k < j\). Continuing this way we have a decreasing sequence of nonnegative integers.

**Claim 7:** Let \(y \in B_{\text{inc}(P)}(x_0, 1) \cap B_0\). Then the lengths of isometric paths starting at \(x_0\) and going through \(y\) is bounded.

Follows from Claims 4, 5 and 6.

We conclude that there is no isometric path in \(\text{Inc}(P)\) starting at \((0,0,1)\). It follows from Theorem 5 that \(\text{Inc}(P)\) has no isometric infinite path.

### 11. Interval orders: A proof of Theorem 6 and Example 3

We recall that an order \(P\) is an interval order if \(P\) is isomorphic to a subset \(J\) of the set \(\text{Int}(C)\) of non-empty intervals of a chain \(C\), ordered as follows: if \(I, J \in \text{Int}(C)\), then

\[(15)\quad I < J \text{ if } x < y \text{ for every } x \in I \text{ and every } y \in J.\]

The following proposition encompasses some known equivalent properties of interval orders. Its proof is easy and is left to the reader.

**Proposition 28.** Let \(P := (V, \preceq)\) be a poset. The following propositions are equivalent.

(i) \(P\) is an interval order.

(ii) \(P\) does not embed \(2 \oplus 2\).

(iii) The set \(\{(\downarrow x) \setminus \{x\} : x \in V\}\) is totally ordered by set inclusion.

(iv) The set \(\{(\uparrow x) \setminus \{x\} : x \in V\}\) is totally ordered by set inclusion.

**Lemma 29.** Let \(P = (V, \preceq)\) be an interval order and \(x \in V\). Then the neighbours of \(x\) (in \(\text{Inc}(P)\)) that lay on an induced path of length at least two in \(\text{Inc}(P)\) and starting at \(x\) and whose vertices are in \(\text{inc}_P(x) \cup \uparrow x\) form an antichain in \(P\).

**Proof.** Let \(x = x_0, x_1, \ldots, x_n\) and \(x = x'_0, x'_1, \ldots, x'_{n'}\) be two induced paths in \(\text{Inc}(P)\) with \(n, n' \geq 2\) and whose vertices are in \(\text{inc}_P(x) \cup \uparrow x\). Note that necessarily \(x < x_2\) and \(x < x'_2\).

Suppose for a contradiction that \(x_1\) and \(x'_1\) are comparable. Suppose \(x_1 < x'_1\). Since \(x < x_2\) and \(x_1\) is incomparable to \(x_2\) and to \(x_2\) and \(x\) is incomparable to \(x'_1\) and \(P\) is an interval order we infer that \(x'_1\) is comparable to \(x_2\) and hence \(x < x'_1\) or \(x_1 < x_2\), which is impossible.

The case \(x'_1 < x_1\) can be dealt with similarly by considering the comparabilities \(x'_1 < x_1\) and \(x < x'_2\).

**Proof.** (Of Theorem 6) Let \(x_0 \in P\) and set \(I_0 := \text{inc}_P(x_0) \cup \downarrow x_0\) and \(F_0 := \text{inc}_P(x_0) \cup \uparrow x_0\). Clearly, \(V(G) = I_0 \cup F_0\). Furthermore, since the diameter of \(G\) is infinite and \(G_{I_0}\) and \(G_{F_0}\) are connected graphs we infer that the diameter in \(G_{I_0}\) or in \(G_{F_0}\) is infinite. We may assume without loss of generality that the diameter of \(G_0 := G_{F_0}\) is infinite. Hence, the lengths of isometric paths in \(G_0\) starting at \(x_0\) are unbounded.

**Claim 1:** There exists \(x_1 \in \text{inc}_P(x_0)\) such that the lengths of isometric paths in \(G_0\) starting
at \( x_0 \) and going through \( x_1 \) are unbounded.
Since the antichains of \( P \) are finite, there are only finitely many neighbours of \( x_0 \) in \( G_0 \) laying on isometric paths starting at \( x_0 \) and of length at least two. Hence there must be a neighbour \( x_1 \) of \( x \) in \( G_0 \) such that the lengths of isometric paths in \( G_{iF_0} \) starting at \( x_0 \) and going through \( x_1 \) are unbounded.

Now suppose constructed an isometric path \( x_0, \ldots, x_n \) such that \( x_i < x_j \) for all \( j - i \geq 2 \) and that the lengths of isometric paths starting at \( x_0 \) and going through \( x_0, \ldots, x_n \) are unbounded.
From Lemma 22 we deduce that there are only finitely many neighbours of \( x_n \) that lay on such isometric paths. Applying Claim 1 to \( x_n \), we deduce that there exists \( x_{n+1} > x_{n-1} \) such that \( x_0, \ldots, x_n, x_{n+1} \) is an isometric path of length \( n + 1 \).

We now proceed to the proof of Example 3.

Proof. We totally order the set \( \mathbb{N} \times \mathbb{N} \) as follows: \( (n, m) \leq (n', m') \) if \( m < m' \) or \( m = m' \) and \( n \leq n' \). Consider the set \( Q \) of intervals \( X_{n,m} := [(n, m), (n, m + 1)] \) ordered as in \( \mathbb{N} \) above and set \( G := Inc(Q) \). Then \( X_{n,m} \leq X_{n',m'} \) if and only if \( m + 1 < m' \) or \( m + 1 = m' \) and \( n \leq n' \).
Equivalently, \( \{X_{n,m}, X_{n',m'}\} \) is an edge of \( G \) if and only if \( m = m' \) or \( m' = m + 1 \) and \( n' < n \) or \( (m = m' + 1 \) and \( n < n' \).

Claim 1: \( G \) is connected and has infinite diameter.
Let \( X_{n,m} \) and \( X_{n',m'} \) be two elements of \( Q \) so that \( n \leq n' \). We may suppose without loss of generality that \( X_{n,m} \cap X_{n',m'} = \emptyset \). We may suppose without loss of generality that \( m < m' \).
Consider the sequence of intervals \( X_{n,m}, X_{n',m}, X_{n+1,m+1}, X_{n,m+2}, X_{n',m+2}, \ldots, X_{n',m'} \). This is easily seen to be a path in \( G \) proving that \( G \) is connected.

Claim 2: \( G \) has no isometric infinite path starting at \( X_{0,0} \).
Let \( X_{0,0} := Y_0, \ldots Y_r \ldots \) be an isometric path. Then \( Y_1 = X_{n,0} \) for some \( n_1 \in \mathbb{N} \). Now \( Y_2 \) must intersect \( Y_1 \) but not \( Y_0 \). Hence, \( Y_2 = X_{n_2,1} \) for some \( n_2 < n_1 \). Now \( Y_3 \) must intersect \( Y_2 \) but not \( Y_1 \). Suppose \( Y_3 = X_{n',1} \). Then \( n_1 < n' \). But then \( X_{n'+1,0} \) intersects \( Y_3 \) and \( Y_0 \) and therefore the distance in \( G \) between \( Y_0 \) and \( Y_3 \) is two contradicting our assumption that \( X_{0,0} = Y_0, \ldots Y_n \ldots \) is isometric. Hence, we must have \( Y_3 = X_{n,2} \) for some \( n_3 < n_2 \). An induction argument shows that \( Y_r = X_{n, r-1} \) with \( n_r < n_{r-1} < \cdots < n_1 \). Since there are no infinite strictly decreasing sequences of positive integers the isometric path \( X_{0,0} := Y_0, \ldots Y_r \ldots \) must be finite. This completes the proof of Claim 2.

It follows from Theorem \( \Box \) that \( G \) has no isometric path.

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