On three quantization methods
for particle on hyperboloid

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Abstract

We compare the respective efficiencies of three quantization methods (group theoretical, coherent state and geometric) by quantizing the dynamics of a free massive particle in two-dimensional de Sitter space. For each case we consider the realization of the principal series representation of $SO_{0}(1,2)$ group and its two-fold covering $SU(1,1)$. We demonstrate that standard technique for finding an irreducible representation within the geometric quantization scheme fails. For consistency we recall our earlier results concerning the other two methods, make some improvements and generalizations.

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I. INTRODUCTION

The aim of this work is to compare three quantization methods, namely group theoretical, coherent state and geometric, by quantizing the dynamics of a free massive particle on two-dimensional one-sheet hyperboloid embedded in three dimensional Minkowski space.

The problem of a particle in 1+1 de Sitter space has already been considered in various contexts \[1, 2, 3, 4, 5\]. In Ref. \[6\] we have carried out the coherent state quantization, but without taking into account the time-reversal invariance of the system. The present work feels the gap and presents generalization of the discussion given in Ref. \[6\].

In Sec. II we specify the canonical structure of our physical system: the physical phase space has the topology of a one-sheet hyperboloid, the basic observables satisfy the \( sl(2, \mathbb{R}) \) algebra, and the symmetry group is \( SO_0(1,2) \) or \( SU(1,1) \) (taking into account the time-reversal invariance of the dynamics). We carry out the group theoretical quantization in Sec. III. We get the representation of \( sl(2, \mathbb{R}) \) algebra by applying the Stone theorem to the principal series representation of \( SU(1,1) \) group. For comparison, we recall the derivation of the representation of \( sl(2, \mathbb{R}) \) by using the Schrödinger representation for the canonical variables and applying the symmetrization prescription for basic observables. We also make comment on the importance of the canonical variables topology for quantization procedure. Sec. IV concerns coherent state quantization. We recall basic notions and steps of the method, choose coherent states suitable to our physical system, carry out mapping of classical observables into quantum operators, and identify obtained representation with the Bargmann principal series representation of \( SU(1,1) \). We also extend the analysis in order to include the time-reversal invariance of the system. The geometric quantization of our classical system is presented in Sec. V in its prequantization step. As is known, this method leads usually to reducible representations. We demonstrate that the standard technique for solving the problem does not lead to irreducibility. Our result may be known (in the context of principal series representations) to the community of experts in geometric quantization. However, we have not found the proof. Our results seem to feel the gap. Each section includes the discussion of a given quantization method. General discussion is presented in Sec. VI where we indicate merits and demerits of the quantization schemes under consideration, and we compare their efficiencies.

We believe that a generalization of our results may allow to quantize the dynamics of the particle in realistic four dimensional curved spacetimes. Also, the analysis presented here may be useful for the choice of a quantization scheme in quantum cosmology. For instance, to address the problem of existence of a big-bounce or big-crunch/big-bang transition, during the evolution of the universe. One may examine that issue by considering propagation of p-brane states across an orbifold (e.g. Misner’s type) singularity (see, Ref. \[7\] and references therein). Non-perturbative methods of quantization, discussed in our paper, may overcome technical problems encountered by perturbative approaches.
II. PHASE SPACE

A. Particle in de Sitter spacetime

The two-dimensional de Sitter spacetime $V$ has the topology $\mathbb{R}^1 \times S^1$. It may be visualized as a one-sheet hyperboloid $H_{r_0}$ embedded in 3-dimensional Minkowski space $\mathbb{M}$, i.e.

$$H_{r_0} := \{(y^0, y^1, y^2) \in \mathbb{M} \mid (y^0)^2 + (y^1)^2 - (y^2)^2 = r_0^2, \quad r_0 > 0\}, \quad (1)$$

where $r_0$ is the parameter of the one-sheet hyperboloid $H_{r_0}$. The induced metric, $g_{\mu\nu}$ ($\mu, \nu = 0, 1$), on $H_{r_0}$ is the de Sitter metric.

An action integral, $A$, describing a free relativistic particle of mass $m_0 > 0$ in gravitational field $g_{\mu\nu}$ is proportional to the length of a particle world-line:

$$A = \int_{\tau_1}^{\tau_2} L(\tau) \, d\tau, \quad L(\tau) := -m_0 \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}, \quad (2)$$

where $\tau$ is an evolution parameter, $x^\mu$ are intrinsic coordinates and $\dot{x}^\mu := dx^\mu/d\tau$. It is assumed that $\dot{x}^0 > 0$, i.e., $x^0$ has interpretation of time monotonically increasing with $\tau$.

The action (2) is invariant under the reparametrization $\tau \rightarrow f(\tau)$ of the world-line (where $f$ is an arbitrary function of $\tau$). This gauge symmetry leads to the constraint

$$G := g^{\mu\nu} p_\mu p_\nu - m_0^2 = 0, \quad (3)$$

where $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$ and the $p_\mu := \partial L/\partial \dot{x}^\mu$ are canonical momenta.

Since a test particle does not modify spacetime, the local symmetries of the system are described by the algebra of all Killing vector fields. It is known that a Killing vector field $Y$ defines a dynamical integral $D$ of a test particle moving along a geodesic by

$$D = p_\mu Y^\mu, \quad \mu = 0, 1, \quad (4)$$

where $Y^\mu$ are components of $Y$.

To be more specific we parametrize the hyperboloid $H_{r_0}$ as follows $[4]$

$$y^0 = -\frac{r_0 \cos \rho / r_0}{\sin \rho / r_0}, \quad y^1 = \frac{r_0 \cos \theta / r_0}{\sin \rho / r_0}, \quad y^2 = \frac{r_0 \sin \theta / r_0}{\sin \rho / r_0}, \quad (5)$$

where $0 < \rho < \pi r_0$ and $0 \leq \theta < 2\pi r_0$. The metric tensor on $H_{r_0}$ reads

$$ds^2 = (d\rho^2 - d\theta^2) \sin^{-2}(\rho / r_0). \quad (6)$$

Thus the constraint (3) has the form

$$G = (p_\rho^2 - p_\theta^2) \sin^2(\rho / r_0) - m_0^2 = 0, \quad (7)$$

where $p_\rho := \partial L/\partial \dot{\rho}$ and $p_\theta := \partial L/\partial \dot{\theta}$ are canonical momenta.
B. Dynamical integrals from Killing vectors

The three Killing vector fields $Y_a \ (a = 0, 1, 2)$ of $V$ correspond to the generators of the proper orthochronous Lorentz group $SO_0(1, 2)$. The infinitesimal transformations, in parametrization (5), read

$$
\begin{align*}
\rho, \theta &\rightarrow (\rho, \theta + a_0 r_0), \\
\rho, \theta &\rightarrow (\rho - a_1 r_0 \sin \rho / r_0 \sin \theta / r_0, \theta + a_1 r_0 \cos \rho / r_0 \cos \theta / r_0), \\
\rho, \theta &\rightarrow (\rho + a_2 r_0 \sin \rho / r_0 \cos \theta / r_0, \theta + a_2 r_0 \cos \rho / r_0 \sin \theta / r_0),
\end{align*}
$$

where $(a_0, a_1, a_2) \in \mathbb{R}^3$ are infinitesimal parameters. The transformation (8) corresponds to the infinitesimal spatial de Sitter translations, whereas (9) and (10) define two infinitesimal “boosts” in the embedding spacetime. One of them can be interpreted as Lorentz boost in de Sitter spacetime; the other describes de Sitter ‘time’ translation.

The three corresponding dynamical integrals (4) read

$$
\begin{align*}
J_0 &= p_\theta r_0, \\
J_1 &= -p_\rho r_0 \sin \rho / r_0 \sin \theta / r_0 + p_\theta r_0 \cos \rho / r_0 \cos \theta / r_0, \\
J_2 &= p_\rho r_0 \sin \rho / r_0 \cos \theta / r_0 + p_\theta r_0 \cos \rho / r_0 \sin \theta / r_0.
\end{align*}
$$

Making use of (11)-(13) one may rewrite the constraint (7) as

$$
J_2^2 + J_1^2 = \kappa^2, \quad \kappa := m_0 r_0,
$$

where $m_0$ is the particle mass and $r_0$ the radius of the spacetime hyperboloid.

Eqs. (5) and (11)-(13) lead to the algebraic equations

$$
J_0 y^a = 0, \quad J_2 y^1 - J_1 y^2 = r_0^2 p_\rho.
$$

For our system, the physical phase space $\Gamma$ is defined (Ref. 4) as the space of all particle geodesics consistent with the constraint (7). Since each triple $(J_0, J_1, J_2)$ satisfying (14) defines uniquely a particle geodesic (by solution of (15)), the one-sheet hyperboloid (14) represents $\Gamma$. The dynamical integrals $J_a \ (a = 0, 1, 2)$ are the basic observables of our system.

C. Gauge system

The free particle in curved spacetime, defined by the action (2), may be treated as a gauge system, with reparametrization invariance as gauge symmetry. It is a characteristic feature of such a system that the Hamiltonian corresponding to the Lagrangian (2) identically vanishes. The general treatment of such gauge system within the constrained Hamiltonian formalism and the reduction scheme to gauge invariant variables has been presented elsewhere (see Ref. 8). Here we only outline the method leading to the phase space with independent canonical variables.
For a spacetime dimension $N$, the Hamiltonian formulation of a theory with gauge invariant Lagrangian leads to an extended phase space $\Gamma_e$ of dimension $2N$, with $M$ first-class constraints ($N = 2$, $M = 1$ in our case). $\Gamma_e$ has a natural parametrization by $(\rho, \theta, p_\rho, p_\theta)$. The constraint surface $\Gamma_c \subset \Gamma_e$, defined by (14), plays a special role in the formalism. This type of system may have up to $2N - 2M$ (2 in our case) gauge invariant functionally independent variables on $\Gamma_c$. Those may be used to parametrize the physical phase space and gauge invariant observables. It is known that the dynamical integrals may be used to represent such variables.

The observables $J_0, J_1$ and $J_2$ are gauge invariant (their Poisson bracket with $G$ vanish) and any two of them are functionally independent on $\Gamma_c$ due to (14). There exists a general method for finding the corresponding canonical variables, although it is quite involved. In what follows we recall the simple method used in Ref. 4. It consists of three steps:

First, we identify the algebra of the observables on $\Gamma_e$. The canonical coordinates above define the Poisson bracket on $\Gamma_e$,

$$\{\cdot, \cdot\} := \frac{\partial}{\partial p_\rho} \frac{\partial}{\partial \rho} \cdot - \frac{\partial}{\partial \rho} \frac{\partial}{\partial p_\rho} \cdot + \frac{\partial}{\partial p_\theta} \frac{\partial}{\partial \theta} \cdot - \frac{\partial}{\partial \theta} \frac{\partial}{\partial p_\theta} \cdot. \quad (16)$$

Direct calculations lead to

$$\{J_0, J_1\} = -J_2, \quad \{J_0, J_2\} = J_1, \quad \{J_1, J_2\} = J_0. \quad (17)$$

Hence, the three basic observables $J_a$ satisfy the $sl(2, \mathbb{R})$ algebra.

Second, we parametrize the physical phase space $\Gamma$, identified with the hyperboloid (14), by two coordinates $J, \beta$ defined through

$$J_0 := J, \quad J_1 := J \cos \beta - \kappa \sin \beta, \quad J_2 := J \sin \beta + \kappa \cos \beta, \quad 0 < \kappa < \infty. \quad (18)$$

Thus, we identify it with

$$X := \{x \equiv (J, \beta) \mid J \in \mathbb{R}, \ 0 \leq \beta < 2\pi\}. \quad (19)$$

Third, we write a Poisson bracket in $X$, with the variables $J$ and $\beta$:

$$\{\cdot, \cdot\} := \frac{\partial}{\partial J} \frac{\partial}{\partial \beta} \cdot - \frac{\partial}{\partial \beta} \frac{\partial}{\partial J} \cdot. \quad (20)$$

Since $\{J, \beta\} = 1$, $J$ and $\beta$ are canonical variables. In what follows, we call $X$, expressed in terms of them, the canonical phase space. This specifies completely the canonical structure of our system.

### D. Symmetries

The local symmetries of the phase space coincide with the local symmetries of the Lagrangian of a free particle in de Sitter’s space.

Now, let us discuss the symmetry group of the phase space $\Gamma$. There are infinitely many Lie groups having $sl(2, \mathbb{R}) \sim su(1, 1) \sim so(1, 2)$ as their Lie algebras. The common examples are:

- $SO_0(1, 2)$, the proper orthochronous Lorentz group;
• $SU(1,1) \sim SL(2,\mathbb{R})$, the two-fold covering of $SO_0(1,2) \sim SU(1,1)/\mathbb{Z}_2$.

• $\widetilde{SL}(2,\mathbb{R})$, the infinite fold covering of $SO_0(1,2) \sim \widetilde{SL}(2,\mathbb{R})/\mathbb{Z}$ (the universal covering group).

Let us give a short description of the way in which the $SU(1,1)$ symmetry appears in this problem, and what are the precise connections with the $SO_0(1,2)$ and $\widetilde{SL}(2,\mathbb{R})$ (possible) symmetries. The action of $SU(1,1)$ on the de Sitter spacetime can be understood through the following “space-time” factorization $[14]$ of a generic element $g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$, $a,b \in \mathbb{C}$, $|a|^2 - |b|^2 = 1$ of the group:

$$g = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \begin{pmatrix} \cosh \frac{\psi}{2} & \sinh \frac{\psi}{2} \\ \sinh \frac{\psi}{2} & \cosh \frac{\psi}{2} \end{pmatrix} \begin{pmatrix} \cosh \frac{\varphi}{2} & i \sinh \frac{\varphi}{2} \\ -i \sinh \frac{\varphi}{2} & \cosh \frac{\varphi}{2} \end{pmatrix}$$

$\equiv s(\theta) \ t(\psi) \ l(\varphi) \equiv j(\theta, \psi) \ l(\varphi)$, with $0 \leq \theta < 4\pi, \ \varphi, \psi \in \mathbb{R}$. (21)

Note that the parameter $\varphi$ stands for the Lorentz rapidity.

The right coset $J = SU(1,1)/(\text{Lorentz subgroup})$ is the subset of elements of the $j$ type. The space-time factorization (21) allows us to make $SU(1,1)$ acts on $J$ by left action, $SU(1,1) \ni g : j \to j' \equiv g \cdot j$,

where $j'$ is defined by $gj = j'j''$. From this, we easily infer the action of $SU(1,1)$ on the matrices of the type $jj^t$:

$$g : jj^t \to j'j'' = gjj^t g^t. \quad (22)$$

On the other hand, each group element of the type $jj^t$ is in one-to-one correspondence with a point of the double covering of $H_{r_0}$ defined by Eq. (1), through

$$jj^t = \begin{pmatrix} e^{i\theta} \cosh \psi & \sinh \psi \\ \sinh \psi & e^{-i\theta} \cosh \psi \end{pmatrix} \equiv \begin{pmatrix} r_0^{-1}y^+ & r_0^{-1}y^0 \\ r_0^{-1}y^0 & r_0^{-1}y^- \end{pmatrix}, \text{ with } y^\pm = y^1 \pm iy^2. \quad (23)$$

This provides global coordinates for this double covering:

$$y^0 = r_0 \sinh \psi, \ y^1 = r_0 \cos \theta \cosh \psi, \ y^2 = r_0 \sin \theta \cosh \psi. \quad (24)$$

Hence, we easily derive from (22) the homomorphism which sends $SO_0(1,2)$ into $SU(1,1)$.

E. Orbits of the Lie algebra $su(1,1)$

Let us chose for $su(1,1)$, the Lie algebra of $SU(1,1)$, a basis $(Y_s, Y_t, Y_l)$ corresponding to the above space-time-Lorentz parametrization:

$$Y_s = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad Y_t = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y_l = \frac{1}{2} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \quad (25)$$
These generators obey the commutation rules
\[ [Y_t, Y_l] = -Y_s, \quad [Y_t, Y_s] = Y_l, \quad [Y_s, Y_l] = Y_t. \] (26)

A generic element \( Y \) of \( su(1,1) \) reads:
\[ Y = \xi_s Y_s + \xi_t Y_t + \xi_l Y_l = \left( \begin{array}{cc} \zeta & iu \\ -i\bar{\zeta} & -iu \end{array} \right). \] (27)

Since \( su(1,1) \) is a simple Lie algebra, the bilinear form \( \langle Y_1, Y_2 \rangle \equiv \text{Tr} Y_1 Y_2 \) is nondegenerate and allows us to identify \( su(1,1) \) and its vector space dual. Therefore, the classification of the co-adjoint orbits as possible phase spaces for motions on the de Sitter space amounts to classify orbits of the adjoint representation:
\[ su(1,1) \ni Y \xrightarrow{g} Y' = g Y g^{-1}. \] (28)

The trivial one apart, three types of orbits are found. The first family corresponds to the transport of the particular element \( 2\kappa Y_t \). The subgroup stabilizing this element under the adjoint action is the non-compact \( SO(1,1) \) corresponding to the time-translation subgroup for the de Sitter space-time, since the phase space for a test particle in de Sitter space-time can be viewed as the group coset \( SU(1,1)/(\text{"Time-translation subgroup"}) \). By using Lorentz boosts and dS space translations only for transporting the latter, we obtain orbit generic elements as
\[ Y(J, \beta) = s(\theta)l(\varphi)2\kappa Y_t l(-\varphi) s(-\theta) \]
\[ = \kappa \left( \begin{array}{cc} \sinh \varphi & \cosh \varphi e^{i\theta} \\ \cosh \varphi e^{-i\theta} & -i \sinh \varphi \end{array} \right) \]
\[ \equiv \left( \begin{array}{cc} iJ & p_0 e^{i\beta} \\ p_0 e^{-i\beta} & -iJ \end{array} \right), \]
with \( J = \kappa \sinh \varphi, \quad p_0 = \sqrt{\kappa^2 + J^2}, \quad \theta = \beta + \arctan \frac{\kappa}{J}. \) (29)

We recognize here the phase space parameters introduced in [15]. Note that the invariant \( \kappa^2 = \det Y \) and (the Minkowskian-like) "energy" \( p_0 \) is also equal to \( p_0 = \sqrt{J_1^2 + J_2^2} \). This matrix realization of the phase-space for massive particle is very convenient for describing the action of the symmetry group, since we simply have \( Y(J', \beta') = g Y(J, \beta) g^{-1} \).

Apart from continuous transformations, the phase space may be also invariant under discrete transformations. The most important seems to be the time-reversal invariance, since our system is not dissipative one. It has been shown in Ref. [5] that taking into account of this invariance leads to the conclusion that the symmetry group of the phase space \( \Gamma \) of our system must be \( SU(1,1) \).

III. GROUP THEORETICAL QUANTIZATION

Now we intend to find (essentially) self-adjoint representations of the \( su(1,1) \) algebra, integrable to unitary irreducible representations (UIR’s) of the \( SU(1,1) \) group.

The set of unitary irreducible representations of the group \( SU(1,1) \simeq SL(2, \mathbb{R}) \) is well known since the seminal work of Bargmann [11]. It is made up of three series: the principal
They are labelled by the parameter \( \tau \) and \( \rho \). In what follows we describe these representations in the so-called “compact” realization. \[ \text{[12]} \]

In other words, the principal series is induced by a character of the subgroup \( A \). Actually, \( \sigma \) reduces here to \( \pm l \) ("even" or "odd") and \( \tau \) is a character of \( \mathbb{R} \). The principal series of representations constitute the family of representations

\[ \pi(\sigma, \tau) = \text{Ind}_B^{SU(1,1)}(\sigma \times \tau). \]  

In what follows we describe these representations in the so-called “compact” realization. \[ \text{[12]} \]

They are labelled by the parameter \( \chi = (l, \lambda) \).

- Here \( l = -\frac{1}{2} - i\rho \), \( \rho \in \mathbb{R} \), and \( \lambda = 0 \) or \( \frac{1}{2} \). Parameter \( \kappa \) of this paper is related to \( \rho \) by \( \kappa = -\rho \) for \( \rho < 0 \).

- They act in the Hilbert space \( L^2([0, 2\pi]) = \left\{ f(e^{i\beta}) \mid \|f\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\beta})|^2 \, d\beta < \infty \right\} \) of the exponential Fourier series.

- The representation operator \( T^{ps}_\chi \) is given by:

\[ T^{ps}_\chi(g)f(e^{i\beta}) = \left( be^{i\beta} + a \right)^{l-\lambda} \left( be^{-i\beta} + \alpha \right)^{l+\lambda} f \left( \frac{ae^{i\beta} + \bar{b}}{be^{i\beta} + \bar{a}} \right). \]  

\[ \text{[32]} \]

- The representations \( T^{ps}_{(l,\lambda)} \) and \( T^{ps}_{(-l-1,\lambda)} \) are unitarily equivalent.

Applying the Stone theorem \[ \text{[15]} \] to \( \text{[32]} \) leads to the following operators corresponding to the generators of \( SU(1,1) \) group

\[ \hat{J}_0 = \lambda - i \frac{d}{d\beta}, \quad \lambda = 0, \ 1/2, \quad \text{with} \quad T^{ps}_\chi(s(\theta)) \underset{\theta \to 0}{\sim} \mathbb{I} + \theta i \hat{J}_0, \]  

\[ \text{[33]} \]

\[ \hat{J}_1 = \cos \beta \hat{J}_0 - (\kappa - i/2) \sin \beta, \quad \text{with} \quad T^{ps}_\chi(l(\varphi)) \underset{\varphi \to 0}{\sim} \mathbb{I} - \varphi i \hat{J}_1, \]  

\[ \text{[34]} \]

\[ \hat{J}_2 = \sin \beta \hat{J}_0 + (\kappa - i/2) \cos \beta, \quad \text{with} \quad T^{ps}_\chi(t(\psi)) \underset{\psi \to 0}{\sim} \mathbb{I} - \psi i \hat{J}_2. \]  

The operators \( \hat{J}_a \quad (a = 0, 1, 2) \) are essentially self-adjoint on a suitable common dense subspace \( \Omega_\lambda \subset L^2[0,2\pi] \). These operators cannot be defined on the entire Hilbert space because they are unbounded.
The quantum Casimir operator corresponding to (33)-(35) reads
\[ \hat{C} := \hat{J}_2^2 + \hat{J}_1^2 - \hat{J}_0^2 = (\kappa^2 + 1/4) \mathbb{I}. \] (36)

Eq. (36) shows that there exists \( \Omega_\lambda \) such that the representation (33)-(35) may be lifted to the UIR of the \( SU(1, 1) \) group. Since \( 0 < \kappa < \infty \) the representation belongs to the principal series.

For comparison, let us recall the method of mapping of the basic observables into quantum operators commonly used by physicists and applied in Ref. 4:

First, one maps the canonical observables \( J \) and \( \beta \) as follows
\[ J \rightarrow \hat{J} \Psi(\beta) := (\lambda - i \frac{d}{d\beta}) \Psi(\beta), \] (37)
\[ \beta \rightarrow \hat{\beta} \Psi(\beta) := \beta \Psi(\beta), \] (38)

where \( \Psi \in L^2[0, 2\pi] \).

Next, one applies the symmetrization prescription to all products in (18), i.e.
\[ AB \rightarrow \frac{1}{2}(\hat{A}\hat{B} + \hat{B}\hat{A}). \] (39)

As the result one gets the operators \( \hat{J}_a \) \((a = 0, 1, 2)\) in the form (33)-(35). It was shown in Ref. 4 that the heuristic mapping (37)-(38) leads to the homomorphism
\[ [\hat{J}_a, \hat{J}_b] = -i \{\hat{J}_a, \hat{J}_b\}, \quad a, b = 0, 1, 2 \] (40)

and leads to the essentially self-adjoint representation of \( sl(2, \mathbb{R}) \) algebra on \( \Omega_\lambda \subset L^2[0, 2\pi] \) defined to be
\[ \Omega_\lambda := \{ \Psi \in L^2[0, 2\pi] \mid \Psi \in C^\infty[0, 2\pi], \Psi^{(n)}(0) = e^{i\lambda \pi} \Psi^{(n)}(2\pi), \quad n = 0, 1, 2... \}. \] (41)

We emphasize here that the problem of quantization of the algebra of basic observables (17) does not depend on the problem of quantization of the canonical algebra
\[ \{J, \beta\} = 1, \] (42)
in the sense that one can find an (essentially) self-adjoint representation of (17) without the requirement that one should first find an (essentially) self-adjoint representation of (12).

The discussion concerning the quantization of the algebra (17) has a long history (see Ref. 16 and references therein). One knows that the commonly used Schrödinger representation of (12) defined by
\[ J \rightarrow \hat{J} \Psi(\beta) := -i \frac{d}{d\beta} \Psi(\beta) \quad \beta \rightarrow \hat{\beta} \Psi(\beta) := \beta \Psi(\beta), \quad \Psi \in \mathcal{H} \] (43)

(where \( \mathcal{H} \) is a Hilbert space) can be made (essentially) self-adjoint, if \( (J, \beta) \in \mathbb{R} \times \mathbb{R} \), i.e. in case both operators \( \hat{J} \) and \( \hat{\beta} \) are unbounded, and if the Weyl relations are satisfied \[ 16 \]. In such a case one can find a common dense invariant subspace \( \Omega \subset L^2(\mathbb{R}) \) such that both \( \hat{J} \) and \( \hat{\beta} \) are essentially self-adjoint on \( \Omega \). Such result is a consequence of the Stone-von Neumann theorem.
It has been shown (see App. B of Ref. 17) that, if $\beta \in [a, b] \subset \mathbb{R}$ the self-adjoint representation of (42) obtained by making use of (43) does not exist. This result can be probably extended to the case of any representation of (42), but such that the quantum operator $\hat{J}$ (or $\hat{\beta}$) is unbounded whereas $\hat{\beta}$ (or $\hat{J}$) is bounded. It was proved in Ref. 18 that in case both operators $\hat{J}$ and $\hat{\beta}$ are bounded, the self-adjoint representation of (42) cannot exist.

The problem discussed above concerns canonical variables with a trivial topology. As it is known the entire programme connected with higher dimensional theories rests heavily on canonical variables with non-trivial topology. The problem of quantizing a particle on one-sheet hyperboloid, considered in this paper, is a simple example of the situation when one of the variables has a non-trivial topology, i.e. $(J, \beta) \in \mathbb{R}^1 \times S^1$. In such a case (42) should be replaced by

$$\{J, U\} = U, \quad U := e^{i\beta}. \quad (44)$$

It is shown in Ref. 5 that (44) may be used to impose the self-adjointness onto the algebra of basic observables already at the canonical level. It is clear that the choice of canonical variables in the form compatible with their topologies has basic significance (see [19] for more discussion).

IV. COHERENT STATE QUANTIZATION

We start from the canonical phase space $X$ equipped with some measure $\mu$, e.g. its canonical phase space measure. Let $\mathcal{H}$ be a separable Hilbert space. Suppose there exists a continuous mapping

$$X \ni x \longrightarrow |x\rangle \in \mathcal{H} \quad (45)$$

(in Dirac notations), defining a family of states $\{|x\rangle\}_{x \in X}$ obeying the following two conditions:

- **Normalisation**
  $$\langle x | x \rangle = 1, \quad (46)$$

- **Resolution of the unity in $\mathcal{H}$**
  $$\int_X |x\rangle\langle x| \nu(dx) = \mathbb{I}_\mathcal{H}, \quad (47)$$

  where $\nu(dx)$ is another measure on $X$, usually absolutely continuous with respect to $\mu(dx)$: there exists a positive measurable $h(x)$ such that $\nu(dx) = h(x)\mu(dx)$.

The resolution of the unity on $\mathcal{H}$ can alternatively be understood in terms of the scalar product $\langle x | x' \rangle$ of two states of the family. Indeed, Eq. (41) implies that to any vector $|\phi\rangle$ in $\mathcal{H}$ one can isometrically associate the function $\phi(x) \equiv \sqrt{h(x)} \langle x | \phi \rangle$ in $L^2(X, \mu)$, and this function obeys $\phi(x) = \int_X \sqrt{h(x)h(x')} \langle x | x' \rangle \phi(x') \mu(dx')$. Hence, $\mathcal{H}$ is isometric to a reproducing Hilbert space $\mathcal{K}$, closed subspace of $L^2(X, \mu)$, with kernel $K(x, x') = \sqrt{h(x)h(x')} \langle x | x' \rangle$.

The quantization of a classical observable, that is to say of a function $f(x)$ on $X$, having specific properties in relationship with the topological structure allocated to $X$, simply
consists in associating to \( f(x) \) the operator

\[
A(f) \equiv \int_X f(x) \langle x \mid x \rangle \nu(dx).
\] (48)

In this context, \( f(x) \) is said to be upper (or contravariant [22]) symbol of the operator \( A(f) \), whereas the mean value \( \langle x \mid f(x) \mid x \rangle \) is said lower (or covariant) symbol of \( A(f) \). Of course, such a particular quantization scheme is intrinsically limited to all those classical observables for which the expansion (48) is mathematically justified within the theory of operators in Hilbert spaces (e.g. weak convergence).

A method of construction [20, 21] of \( \langle x \mid \) has a wave packet flavor, in the sense that it is obtained from some superposition of elements of an orthonormal basis \( \{ |n\rangle \}_{n \in \mathbb{N}} \) of \( \mathcal{H} \). Suppose that the basis \( \{ |n\rangle \}_{n \in \mathbb{N}} \) is in one-to-one correspondence with an orthonormal set \( \{ \phi_n(x) \}_{n \in \mathbb{N}} \) (as elements of \( L^2(X, \mu) \)). Furthermore, and this is a decisive step in the wave packet construction, we assume that

\[
N(x) \equiv \sum_n |\phi_n(x)|^2 < \infty \text{ almost everywhere.} \tag{49}
\]

Then, the states

\[
|x\rangle \equiv \frac{1}{\sqrt{N(x)}} \sum_n \phi_n(x) |n\rangle, \tag{50}
\]

satisfy both our requirements (46) and (47). Indeed, the normalization follows automatically from the orthonormality of the set \( \{ |n\rangle \} \) and from the presence of the normalization factor (49). The resolution of the unity in \( \mathcal{H} \) also follows from the orthonormality of the same set \( \{ \phi_n(x) \} \), so long as the measure \( \nu(dx) \) is related to \( \mu(dx) \) by

\[
\nu(dx) = N(x) \mu(dx). \tag{51}
\]

A. Choice of coherent states

Let us recall, modify and extend the quantization scheme presented in Ref. [6]. First, we define the measure \( \mu \) on \( X \) to be the canonical one

\[
\mu(dx) := d\beta dJ/2\pi. \tag{52}
\]

Next, we introduce a subsidiary abstract separable Hilbert space \( \mathcal{H} \) with an orthonormal basis \( \{ |m\rangle \}_{m \in \mathbb{Z}} \), i.e.

\[
< m_1 | m_2 > = \delta_{m_1, m_2}, \quad \sum_m |m\rangle < m| = \mathbb{I}, \tag{53}
\]

and an orthonormal set of vectors \( \{ \phi^\epsilon_m \}_{m \in \mathbb{Z}} \) which spans the Hilbert space \( L^2(X, \mu) \) appropriate to our physical system. For regularization purposes we introduce here an arbitrary small real parameter \( \epsilon > 0 \).

Being inspired by the choice of the coherent states for the motion of a particle on a circle [23, 24, 25, 26, 27], we define \( \phi^\epsilon_m \)’s to be suitably weighted Fourier exponentials

\[
\phi^\epsilon_m(\beta, J) := \left( \frac{\epsilon}{\pi} \right)^{1/4} \exp \left( -\frac{\epsilon}{2} (J - m)^2 \right) \exp(im\beta), \quad m \in \mathbb{Z}. \tag{54}
\]
The coherent states $|x, \epsilon > \in \mathcal{H}$ are defined as follows

$$X \ni x \rightarrow |x, \epsilon > \equiv |\beta, J, \epsilon > := \frac{1}{\sqrt{N_{\epsilon}(\beta, J)}} \sum_{m \in \mathbb{Z}} \phi_{m}^{\epsilon}(\beta, J)|m >, \quad (55)$$

where the normalization factor $N_{\epsilon}$ reads

$$N_{\epsilon}(x) := \sum_{m \in \mathbb{Z}} |\phi_{m}^{\epsilon}|^2 = (\frac{\epsilon}{\pi})^{1/2} \sum_{m \in \mathbb{Z}} \exp(-\epsilon(J - m)^2) < \infty, \quad (56)$$

and it is proportional to an elliptic theta function.

By construction, the states (55) are normalized and lead to the resolution of the identity in $\mathcal{H}$

$$\frac{1}{2\pi} \int_{0}^{2\pi} \text{d}\beta \int_{-\infty}^{\infty} \text{d}J N_{\epsilon}(\beta, J)|\beta, J, \epsilon > < \beta, J, \epsilon| = \mathbb{I}. \quad (57)$$

B. Mapping of classical observables

A class of quantum operators is naturally given in “CS diagonal representation” by the mapping

$$C^{\infty}(X, \mathbb{R}) \ni f \rightarrow \hat{f}^{\epsilon} := A_{\epsilon}(f) := \int_{X} \mu(dx) N_{\epsilon}(x)f(x)|x, \epsilon > < x, \epsilon|, \quad (58)$$

where

$$\hat{f}^{\epsilon} : L^2(X, \mu) \rightarrow L^2(X, \mu),$$

for any classical observable $f$, i.e., a function of $(\beta, J)$ with reasonable properties.

For an arbitrary function of $J$ alone, we get the diagonal operator:

$$A_{\epsilon}(f) = \int_{X} \mu(dx) N_{\epsilon}(J)f(J)|\beta, J, \epsilon > < \beta, J, \epsilon|$$

$$= \sum_{m \in \mathbb{Z}} \langle f \rangle_{\epsilon,m}|m > < m|, \quad (59)$$

where $\langle f \rangle_{\epsilon,m}$ designates the mean value of $f(J)$ with respect to the Gauss normal distribution $\sqrt{\frac{2}{\pi}} e^{-\epsilon(J - m)^2}$ centred at $m$ and with width $\sqrt{\frac{2}{\epsilon}}$.

For a function of $\beta$ alone, we have

$$A_{\epsilon}(f) = \int_{X} \mu(dx) N_{\epsilon}(J)f(\beta)|\beta, J, \epsilon > < \beta, \epsilon|$$

$$= \sum_{m, m'} e^{-\frac{\epsilon}{4}(m - m')^2} c_{m-m'}(f)|m > < m'|, \quad (60)$$

where $c_{m}(f) = \frac{1}{2\pi} \int_{0}^{2\pi} e^{-im\beta} f(\beta) d\beta$ is the $m$th Fourier coefficient of $f$. 

12
C. Homomorphism

Let us introduce three suitably shifted versions of the fundamental classical observables (18), namely:

\[ J_0^\lambda = J_0 + \lambda, \quad J_1^\lambda = J_1 + \lambda \cos \beta, \quad J_2^\lambda = J_2 + \lambda \sin \beta. \] (61)

Note that they have the same Poisson brackets as \( J_0, J_1 \) and \( J_2 \) (see (17)):

\[ \{ J_0^\lambda, J_1^\lambda \} = -J_2^\lambda, \quad \{ J_0^\lambda, J_2^\lambda \} = J_1^\lambda, \quad \{ J_1^\lambda, J_2^\lambda \} = J_0^\lambda. \] (62)

A straightforward application of (59) and (60) (compare to Ref. 6) yields the quantum counterparts of (61):

\[ \hat{J}^\epsilon_0^\lambda = A_\epsilon(J_0^\lambda) = \sum_{m \in \mathbb{Z}} (m + \lambda) \langle m > m \rangle. \] (63)

\[ \hat{J}^\epsilon_1^\lambda = A_\epsilon(J_1^\lambda) = \frac{1}{2} e^{-\epsilon/4} \sum_{m \in \mathbb{Z}} \left( (m + 1 + \lambda + i\kappa) \langle m + 1 > m \rangle + c.c. \right), \] (64)

\[ \hat{J}^\epsilon_2^\lambda = A_\epsilon(J_2^\lambda) = \frac{1}{2i} e^{-\epsilon/4} \sum_{m \in \mathbb{Z}} \left( (m + 1 + \lambda + i\kappa) \langle m + 1 > m \rangle - c.c. \right), \] (65)

where \( c.c. \) stands for the complex conjugate of the preceding term.

The commutation relations corresponding to (17) are found to be

\[ [\hat{J}^\epsilon_0^\lambda, \hat{J}^\epsilon_1^\lambda] = i\hat{J}^\epsilon_2^\lambda = -iA_\epsilon(\{ J_0^\lambda, J_1^\lambda \}), \] (66)

\[ [\hat{J}^\epsilon_0^\lambda, \hat{J}^\epsilon_2^\lambda] = -i\hat{J}^\epsilon_1^\lambda = -iA_\epsilon(\{ J_0^\lambda, J_2^\lambda \}), \] (67)

\[ [\hat{J}^\epsilon_1^\lambda, \hat{J}^\epsilon_2^\lambda] = -ie^{-\epsilon/2} \hat{J}^\epsilon_0^\lambda = e^{-\epsilon/2} (-iA_\epsilon(\{ J_1^\lambda, J_2^\lambda \})). \] (68)

Now, we consider the asymptotic case \( \epsilon \to 0 \). All operators and equations (63)- (68) in this limit are represented as infinite matrices in the basis \( \{|m\rangle\} \) of the abstract Hilbert space \( \mathcal{H} \). The equations (66)-(67) prove that in the asymptotic case the mapping (58) is a homomorphism sending the \( su(1,1) \) Lie Poisson algebra into the Lie algebra generated by \( \{ \hat{J}_0^\lambda, \hat{J}_1^\lambda, \hat{J}_2^\lambda \} \), where \( \hat{J}_a^\lambda := \lim_{\epsilon \to 0} \hat{J}_a^\epsilon \lambda \) \( (a = 0, 1, 2) \).

It has been proved in Ref. 3 for \( \lambda = 0 \), that the operators \( \hat{J}_a^\lambda \) are essentially self-adjoint. The corresponding proof for \( \lambda = 1/2 \) can be done by analogy.

D. “Angular” representation

In agreement with our general framework we denote by \( \mathcal{K}_\epsilon \) the closure of the linear span of the orthonormal set \( \{ \phi_m \}_{m \in \mathbb{Z}} \) given in (54). The space \( \mathcal{K}_\epsilon \) is a reproducing Hilbert subspace of \( L^2(X, \mu_\epsilon) \), isomorphic to \( \mathcal{H} \) through the one-to-one map

\[ \mathcal{H} \ni |\phi\rangle \to \phi^\epsilon(\beta, J) \equiv \sqrt{N_\epsilon(J)} \langle \beta, J, \epsilon | \phi \rangle \in \mathcal{K}_\epsilon. \] (69)
The reproducing kernel for elements of $K$ is given by

$$K_\epsilon(\beta, J; \beta', J') = \sum_{m \in \mathbb{Z}} \phi^\epsilon_m(\beta, J) \overline{\phi^\epsilon_m(\beta', J')}.$$  \hfill (70)

If we specify the map (69) to the basis elements of $\mathcal{H}$, we obtain

$$|m\rangle \rightarrow \sqrt{N_\epsilon(J)}\langle \beta, J, \epsilon|m\rangle = \phi^\epsilon_m(\beta, J) = \left(\frac{\epsilon}{\pi}\right)^{1/4} \exp \left(-\frac{\epsilon}{2}(J - m)^2\right) \exp(im\beta).$$  \hfill (71)

It is clear from this correspondence that the basis $|m\rangle$'s could as well be considered as Fourier exponentials $e^{im\beta}$ forming the orthonormal basis of the Hilbert space $L^2(S^1) \simeq \mathcal{H}$. They are the spatial modes in this “angular position” representation and they are also obtained as the limit, after suitable rescaling, of the $\phi^\epsilon_m(\beta, J)$'s at $\epsilon = 0$.

In this representation, the operator $\hat{J}_0^{\epsilon,\lambda}$ is nothing but the $\lambda$-shifted angular momentum operator: $\hat{J}_0^{\epsilon,\lambda} = \lambda - i\frac{\partial}{\partial \beta}$. On the other hand, $A_\epsilon(e^{i\beta})$ is multiplication operator by $e^{i\beta}$ up to the (arbitrarily close to 1) factor $e^{-\frac{\epsilon}{4}}$. The “canonical” commutation rule

$$[A_\epsilon(J), A_\epsilon(e^{i\beta})] = A_\epsilon(e^{i\beta})$$

is canonical in the sense that it is in exact correspondence with the classical Poisson bracket

$$\{J, e^{i\beta}\} = ie^{i\beta}$$

It is actually the only non trivial one having this exact correspondence.

Although the function “angle” is strictly speaking not a classical observable since it is not continuous at $\theta = 0 \ mod \ 2\pi$, we take here the freedom to consider its quantum counterpart defined by

$$A_\epsilon(\beta) = \pi \mathbb{I}_\mathcal{H} + \sum_{m \neq m'} \frac{e^{-\frac{\epsilon}{2}(m - m')^2}}{m - m'} |m\rangle\langle m'|.$$  \hfill (72)

Difficulties about correctly defining angle operator in quantum mechanics and related questions of angular localization and uncertainties are famous (see Ref. 43). As a matter of fact, there could be serious interpretative difficulties with commutation rules of the type

$$[A_\epsilon(J), A_\epsilon(f(\beta))] = \sum_{m, m'} (m - m') e^{-\frac{\epsilon}{2}(m - m')^2} c_{m-m'}(f) |m\rangle\langle m'|,$$

when $f$ is not as regular as a Fourier exponential. In particular, we obtain for the angle operator:

$$[A_\epsilon(J), A_\epsilon(\beta)] = i \sum_{m, m'} e^{-\frac{\epsilon}{2}(m - m')^2} |m\rangle\langle m'|,$$  \hfill (73)

to be compared with the classical $\{J, \beta\} = 1$ !

Actually, these difficulties are just apparent and are due to the discontinuity of the $2\pi$ periodic function $B(\beta)$ equal to $\beta$ on $[0, 2\pi]$. They can be circumvented if we examine, for instance the behavior of lower symbols $\langle \beta', J'|A_\beta|\beta', J'\rangle$ and $\langle \beta', J'|[A_J, A_\beta]|\beta', J'\rangle$ for $0 < \beta' < 2\pi$ at the limit $\epsilon \rightarrow 0$. They behave like $\beta'$ and $-i$, respectively.
E. Group representation

To identify the $SU(1,1)$ representation which integrates within the Bargmann classification, we consider the Casimir operator. The classical Casimir operator $C$ for the algebra \( (61) \) (as well as for the algebra \( (17) \)) has the form
\[
C = (J^2_\lambda)^2 + (J^1_\lambda)^2 - (J^0_\lambda)^2 = \kappa^2.
\] (74)

Making use of \( (58) \) we map $C$ into the corresponding quantum operator $\hat{C}$ as follows
\[
\hat{C} := \lim_{\epsilon \to 0} \hat{C}^\epsilon := \lim_{\epsilon \to 0} \left( \hat{J}^{\epsilon,2}_2 \hat{J}^{\epsilon,2}_2 + \hat{J}^{\epsilon,1}_1 \hat{J}^{\epsilon,1}_1 - \hat{J}^{\epsilon,0}_0 \hat{J}^{\epsilon,0}_0 \right) = \lim_{\epsilon \to 0} \sum_{m \in \mathbb{Z}} \left( e^{-\epsilon/2(m+\lambda)^2} + \frac{1}{4} - (m+\lambda)^2 \right) |m><m| = (\kappa^2 + \frac{1}{4}) \sum_{m \in \mathbb{Z}} |m><m| = (\kappa^2 + \frac{1}{4}) \mathbb{I} =: q \mathbb{I}.
\] (75)

Thus, we have obtained that our choice of coherent states \( (55) \) and the mapping \( (58) \) leads, as $\epsilon \to 0$, to the representation of $su(1,1)$ algebra with the Casimir operator to be an identity in $\mathcal{H}$ multiplied by a real constant $1/4 < q < \infty$.

In the asymptotic case ($\epsilon \to 0$) all operators are defined in the Hilbert space $\mathcal{H}$. The specific realization of $\mathcal{H}$ may be obtained by finding the space of eigenfunctions of the set of all commuting observables of the system. We choose $\hat{C}$ and $\hat{J}^0_\lambda$. It is easy to verify that the set of common eigenfunctions of $\hat{C}$ and $\hat{J}^0_\lambda$ is independent of $\lambda$ and reads as
\[
\phi_m(\beta) := \frac{1}{2\pi} e^{im\beta}, \quad m \in \mathbb{Z}.
\] (76)

The set of eigenfunctions (76) is obviously complete in $\mathcal{H}$. Note that we could as well choose the pair $\hat{C}$, $\hat{J}_0 = -i \frac{d}{d\beta} = \hat{J}^0_\lambda - \lambda$ with the basis
\[
\phi_m(\beta) := \frac{1}{2\pi} e^{im\beta}, \quad m \in \mathbb{Z},
\] (77)

which corresponds to different boundary conditions according to whether $\lambda = 0$ or $\lambda = 1/2$. The Bargmann classification \( [11] \) is characterized by the range of $q$ and $m$. Since $1/4 < q < \infty$, the class with $m$ integer corresponds to the first branch, $\lambda = 0$, of the principal series of $SO_0(1,2)$ group already presented in Section III. The second branch, $\lambda = 1$, is realized by the class with $m$ half-integer.

Comparing the results of quantization by group theoretical method \( [5] \) with the results of this section we can see that the choice of the coherent states in the form induced by \( (55) \) ensures that our coherent state results are time-reversal invariant.

F. Vector coherent states

In the previous sections, the occurrence of the parameter $\lambda$ looks rather artificial. Let us indicate here another way of defining the coherent states which takes into account the
time-reversal invariance in a more natural fashion. It is based on the vector coherent state construction (see Ref. [21]):

Apart from a Hilbert space $\mathcal{H}$ with an orthonormal basis $\{|m>\}_{m \in \mathbb{Z}}$, we introduce the Hilbert space $\mathbb{C}^2$ with the orthonormal basis $\chi^k \ (k = 1, 2)$ defined as

$$\chi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$  (78)

The set of vectors $\chi^k \otimes |m> \ (k = 1, 2; \ m \in \mathbb{Z})$ forms an orthonormal basis of $\mathbb{C}^2 \otimes \mathcal{H}$.

Next, we define a Hilbert-Schmidt $F^e_m$ type operator on $\mathbb{C}^2$

$$X \ni x \longrightarrow F^e_m(x) \in \mathcal{B}(\mathbb{C}^2), \quad m \in \mathbb{Z},$$  (79)

as follows

$$F^e_m(x) := \overline{\phi^e_m(x)} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\beta/2} \end{pmatrix},$$  (80)

where $\phi^e_m$ are defined by [21]. $\mathcal{B}(\mathbb{C}^2)$ is the space of Hilbert-Schmidt operators on $\mathbb{C}^2$ with the scalar product

$$<Y|Z> := Tr[Y^*Z], \quad Y, Z \in \mathcal{B}(\mathbb{C}^2),$$  (81)

where $Tr[Z] := (\chi^1)^*Z\chi^1 + (\chi^2)^*Z\chi^2$.

Then, we define $\tilde{N}_e$ corresponding to (56) as

$$\tilde{N}_e(x) := \sum_{m \in \mathbb{Z}} Tr[|F^e_m(x)|^2] = 2N_e(x) < \infty,$$  (82)

where $|F^e_m(x)| := [F^e_m(x)F^e_m(x)^*]^{1/2}$ denotes the positive part of the operator $F^e_m(x)$.

One may verify that

$$\int_X \mu(dx) F^e_m(x) F^e_n(x)^* = \delta_{mn}I_{\mathbb{C}^2}, \quad m, n \in \mathbb{Z},$$  (83)

where $I_{\mathbb{C}^2}$ denotes the identity operator in $\mathbb{C}^2$.

Finally, the vector coherent states $|x, \epsilon; \chi^k> \in \mathbb{C}^2 \otimes \mathcal{H}$ are defined as (see Ref. [21] for details)

$$|x, \epsilon; \chi^k> := \tilde{N}_e(x)^{-1/2} \sum_{m \in \mathbb{Z}} F^e_m(x) \chi^k \otimes |m>, \quad k = 1, 2.$$  (84)

One may check that due to (82) and (83) the states (84) are coherent, i.e. they satisfy the normalization condition and lead to the resolution of the identity in $\mathbb{C}^2 \otimes \mathcal{H}$ :

$$\sum_{k=1}^2 \langle x, \epsilon; \chi^k | x, \epsilon; \chi^k \rangle = 1, \quad \sum_{k=1}^2 \int_X \mu(dx) \tilde{N}_e(x) \langle x, \epsilon; \chi^k | x, \epsilon; \chi^k \rangle = I_{\mathbb{C}^2} \otimes I_{\mathcal{H}}$$  (85)

Extension of the method to the finite or infinite covering of $SU(1, 1)$ is straightforward. Now, the quantization of non-shifted classical observables $J_0, J_1, J_2$ through the computation of the operator-valued integrals

$$\left\{ \tilde{J}_0, \tilde{J}_1, \tilde{J}_2 \right\} := \sum_{k=1}^2 \int_X \mu(dx) \tilde{N}_e(x) \left\{ J_0, J_1, J_2 \right\} |x, \epsilon; \chi^k><x, \epsilon; \chi^k|$$  (86)
leads to three operators acting on $\mathbb{C}^2 \otimes \mathcal{H}$. Direct calculations yields:

\[ \tilde{J}_0 = \sum_{k=1}^{2} \sum_{m \in \mathbb{Z}} m \chi_k \otimes |m \rangle \rangle_m \otimes \overline{\chi^k}, \]  
\[ \tilde{J}_1 = \frac{1}{2} e^{-\epsilon/4} \sum_{k=1}^{2} \sum_{m \in \mathbb{Z}} (m + 1/2 + i\kappa) \chi_k \otimes |m+1 \rangle \rangle_m \otimes \overline{\chi^k} + c.c., \]  
\[ \tilde{J}_2 = \frac{1}{2i} e^{-\epsilon/4} \sum_{k=1}^{2} \sum_{m \in \mathbb{Z}} (m + 1/2 + i\kappa) \chi_k \otimes |m+1 \rangle \rangle_m \otimes \overline{\chi^k} - c.c., \]

where $c.c.$ stands for the Hermitian conjugate of the preceding term. Hence, both possibilities, $\lambda = 0$ and $\lambda = 1/2$, are considered within the same Hilbertian framework.

V. PREQUANTIZATION

A. Homomorphism and irreducibility

For the choice of parametrization of the phase space $X$ in the form (19), there exists well defined symplectic potential $\theta$ and 2-form $\omega$ given by

\[ \theta := Jd\beta, \quad \omega := d\theta = dJ \wedge d\beta. \]  

The volume element of the symplectic manifold $(X, \omega)$ is normalized as

\[ \mu(dJd\beta) := \omega/2\pi. \]

The Hilbert space associated with $(X, \omega)$ is defined to be $L^2(X, \mu)$ with the standard scalar product. The Hamiltonian vector fields $X_a$ ($a = 0, 1, 2$) corresponding to the observables (18) are solutions to the equations

\[ X_a \omega + dJ_a = 0, \quad a = 0, 1, 2. \]

They are found to be

\[ X_1 = \cos \beta X_0 + (J_0 \sin \beta + \kappa \cos \beta) \frac{\partial}{\partial J}, \quad X_2 = \sin \beta X_0 - (J_0 \cos \beta - \kappa \sin \beta) \frac{\partial}{\partial J}, \]

and

\[ X_0 = \frac{\partial}{\partial \beta}, \quad J_0 := J + \lambda, \quad \lambda = 0, 1/2. \]

We choose $\lambda = 0$ or $\lambda = 1/2$ to fit the previous sections. The pre-quantum observables $\tilde{J}_a$ corresponding to (18) are defined by

\[ \tilde{J}_a := -iX_a - X_a \theta + J_a, \quad a = 0, 1, 2 \]

and finally read

\[ \tilde{J}_0 = -i \frac{\partial}{\partial \beta} + \lambda, \]
\[\tilde{J}_1 = \cos \beta \tilde{J}_0 - \kappa \sin \beta - i(J_0 \sin \beta + \kappa \cos \beta) \frac{\partial}{\partial J}, \]  
(97) 

\[\tilde{J}_2 = \sin \beta \tilde{J}_0 + \kappa \cos \beta + i(J_0 \cos \beta - \kappa \sin \beta) \frac{\partial}{\partial J}. \]  
(98) 

It is known \[29\] that (95) leads to the homomorphism. Thus we have 

\[[\tilde{J}_a, \tilde{J}_b] = -i\{\tilde{J}_a, \tilde{J}_b\}, \quad a, b = 0, 1, 2. \]  
(99) 

The quantum Casimir operator \(\tilde{C}\) corresponding to prequantum operators (96)-(98) reads 

\[\tilde{C} := \tilde{J}_2^2 + \tilde{J}_1^2 - \tilde{J}_0^2 = (\kappa^2 + i\kappa) - (\kappa^2 + J_0^2) \frac{\partial^2}{\partial J^2} - 2\kappa \frac{\partial^2}{\partial J \partial \beta} + 2(i\kappa J - J_0) \frac{\partial}{\partial J}. \]  
(100) 

Now we intend to find a suitable dense subspace \(\tilde{H}\) of the Hilbert space \(L^2(X, \mu)\) (if the algebra consists of unbounded operators that is our case) such that the lift of the (essentially) self-adjoint representation of the algebra to the unitary representation of the corresponding Lie group is irreducible. Here, we try to find a dense subspace \(\tilde{H} \subset L^2(X, \mu)\) which consists of the functions \(G_{\lambda \kappa m}\) satisfying the following two equations 

\[\tilde{C} G_{\lambda \kappa m}(\beta, J) = (\kappa^2 + 1/4) G_{\lambda \kappa m}(\beta, J), \]  
(101) 

\[\tilde{J}_0 G_{\lambda \kappa m}(\beta, J) = m G_{\lambda \kappa m}(\beta, J), \quad m \in \mathbb{Z}. \]  
(102) 

We impose (101) and (102) because we intend to get the connection with the principal series representation of \(sl(2, \mathbb{R})\) algebra in the Bargmann form \[11\]. In the case where \(\{G_{\lambda \kappa m}\}_{m \in \mathbb{Z}}\) consists of a total set of orthonormal functions, it can be used as a basis of \(\tilde{H}\).

### B. Square-integrability

Now, we examine the square-integrability of the functions \(G_{\lambda \kappa m}\). First, let us look for the non-trivial common solutions to (101) and (102), in the form:

\[G_{\lambda \kappa m}(\beta, J) = e^{i(m-\lambda)\beta} f_{\lambda \kappa m}(J). \]  
(103) 

Replacing \(\partial/\partial \beta\) in (101) by \(\partial/\partial \beta\) of (102), using (103) and making simple algebraic rearrangement of the terms in the resulting form of (101), we finally obtain the equation 

\[(J - \sigma)(J - \sigma) \frac{\partial^2 f}{\partial J^2} + (\delta + \rho J) \frac{\partial f}{\partial J} + \tau f = 0, \]  
(104) 

where 

\[\sigma := -\lambda - i\kappa, \quad \sigma = -\lambda + i\kappa, \quad \delta := 2\lambda + 2i\kappa(m - \lambda), \quad \rho := 2(1 - i\kappa), \quad \tau := 1/4 - i\kappa, \]  

and where \(f(J) := f_{\lambda \kappa m}(J)\) for fixed values of \(\lambda, \kappa\) and \(m\).
It is clear that (104) is the Gauss equation with the three regular singularities, namely at 
\( J = \sigma, \bar{\sigma} \) and \( \infty \). Its solution may be obtained from the solution to the standard Gauss equation

\[
z(1 - z) \frac{d^2 h(z)}{dz^2} + (c - (a + b + 1)z) \frac{dh(z)}{dz} - ab h(z) = 0,
\]

(105) having three regular singularities at \( z = 0, 1 \) and \( \infty \). For this purpose we first rewrite (105) in the form of (104), and then compare the coefficients of the corresponding derivatives to get the expressions for \( a, b \) and \( c \):

Making use of the transformation

\[
\mathbb{C} \ni z \longrightarrow J(z) := \sigma + 2i\kappa z,
\]

(106)

we map the singular points \( (0, 1, \infty) \) of (105) onto the singular points \( (\sigma, \bar{\sigma}, \infty) \) of (104).

The equation (105) in terms of the variable \( J \), defined by (106), reads

\[
(J - \sigma)(J - \bar{\sigma}) \frac{\partial^2 f}{\partial J^2} + (-2i\kappa c + (1 + a + b)(J - \sigma)) \frac{\partial f}{\partial J} + ab f = 0,
\]

(107)

where

\[
f(J) := h(J - \bar{\sigma})^{1/2i\kappa}.
\]

The comparison of (104) and (107) leads to

\[
a = 1/2 - 2\kappa i, \quad b = 1/2, \quad c = 1 - m - \kappa i.
\]

(108)

Finally, the solution of (104) may be obtained by the insertion of \( z = (J - \sigma)/2i\kappa \) and \((a, b, c)\) defined by (108) into the solution of (105).

As it is known \([34, 35]\), in case none of the numbers \( a, b, c - a, c - b \) is an integer (our case) the two linearly independent solutions of (105) can be obtained from any non-trivial solution by analytic continuation along a path which encircles one of the point \( z = 0, 1, \infty \). This way one can get the solution of (105) in the neighborhood of \( z = 0 \) in the form

\[
h_1(z) = A_1 F(a, b, c; z) + B_1 z^{1-c} F(a + 1 - c, b + 1 - c, 2 - c; z),
\]

(109)

with the convergence range \(|z| < 1 \). In the neighborhood of \( z = 1 \) the solution reads

\[
h_2(z) = A_2 F(a, b, 1 + a + b - c; 1 - z) + B_2 (1 - z)^{c-a-b} F(c - b, c - a, 1 + c - a - b; 1 - z),
\]

(110)

with \(|z - 1| < 1 \). Finally, in the neighborhood of \( z = \infty \) the solution is the following

\[
h_3(z) = A_3 z^{-a} F(a, 1 + a - c, 1 + a - b; z^{-1}) + B_3 z^{-b} F(b, 1 + b - c, 1 + b - a; z^{-1}),
\]

(111)

with \(|z| > 1 \).

The coefficients \( A_k, B_k \) \((k = 1, 2, 3)\) in (109)-(111) are any complex numbers, whereas \( F(a, b, c; z) \) is the well known hypergeometric function defined as

\[
F(a, b, c; z) = 1 + \frac{ab}{1! c} z + \frac{a(a + 1)b(b + 1)}{2! c(c + 1)} z^2 + \ldots
\]

(112)

\[
+ \frac{a(a + 1) \ldots (a + n - 1)b(b + 1) \ldots (b + n - 1)}{n! c(c + 1) \ldots (c + n - 1)} z^n + \ldots
\]

19
At this stage we are ready to discuss the square-integrability of $G_{\lambda\kappa m}$. For that purpose we consider the following integral

$$I := \frac{1}{2\pi} \int_0^{2\pi} d\beta \int_{-\infty}^{\infty} dJ |G_{\lambda\kappa m}(\beta, J)|^2 = \int_{-\infty}^{\infty} dJ |f_{\lambda\kappa m}(J)|^2. \quad (113)$$

We do not intend to calculate $I$ but only make estimate of it. In the neighborhood of $z = \infty$ the solution (111, due to (112), reads

$$h_3(z) = A_3 z^{-a} + B_3 z^{-b} + O(z^{-a-1}) + O(z^{-b-1}). \quad (114)$$

For large value of $|J|$ the function $f_{\lambda\kappa m}$ (with $a, b, c$ defined by (108) and $z = (J - \sigma)/2i\kappa$), due to (114), has the property

$$\left| f_{\lambda\kappa m} \left( \frac{J - \sigma}{2\pi i} \right) \right|^2 = \text{const} \cdot |J|^{-1} + O(|J|^{-2}). \quad (115)$$

Since the singular points $\sigma$ and $\bar{\sigma}$ do not belong to the real axis of $J$, the integrand of (113) is a well defined analytic function for finite $J$. Therefore, due to (115), the integral (113) diverges logarithmically. It means that the non-trivial solutions $G_{\lambda\kappa m}$ to (101) and (102) are not square-integrable with respect to the standard measure (91).

VI. MERITS AND DEMERITS OF QUANTIZATION SCHEMES

The group theoretical method is powerful, and currently used by physicists. It operates when the search of the representation of the algebra of observables of a physical system is the main concern, and when an irreducible (possibly projective) unitary representation of the corresponding Lie group is available in the mathematical literature. In such a case, the Stone theorem leads to an (essentially) self-adjoint representation of the algebra. Our Sec. III demonstrates the efficiency of such a method. However, it is not always so simple. Although physicists usually identify the Lie algebra of observables, the representation of the corresponding Lie group may remain unknown. Difficulties occur when the classical observables are higher than second order polynomials in the canonical variables [41]. In this case, a problem appears with the ordering of the canonical operators mapping the classical observables into the corresponding quantum operators (see, e.g. Ref. 42 and references therein). Then, a general rule is to choose the ordering which provides self-adjoint operators in a certain dense subspace of a Hilbert space (quantum observables are usually unbounded operators). However, the proof of self-adjointness may happen to be quite involved. It may require examination of solutions to the deficiency indices equations, which for some orderings and non trivial functional forms of observables are very complicated non-linear equations. Clearly, this may provide more than one representation of the algebra of observables and it is sometimes difficult to prove their unitary (non)equivalence. In principle an ambiguity may be lifted by comparison with experimental data, but in many cases one considers only toy models to 'get insight into the physical problem'. In our case, for the particle dynamics on hyperboloid, this problem was solvable because we found two canonical variables, such that the observables are linear functions in one of them. Making use of the Schrödinger representation for the canonical variables and the symmetrization prescription for operators ordering we have obtained the deficiency indices equations easy to solve [4].
The **coherent state quantization method** presents no problem with the ordering of operators, since the mapping (48) includes only classical expression for observables. The main difficulty is the definition of the coherent states, in particular the construction of the set of orthonormal vectors which spans a closed subspace $\mathcal{H}$ of the Hilbert space $L^2(X, \mu)$. Here, we have solved this problem, following inspiration by the choice of coherent states for dynamics of a particle on a circle. Application of coherent state quantization for a particle on a sphere may be found in Refs. [36, 37]. For general structure of the coherent state quantization method we recommend Refs. [20, 21, 22, 38, 39, 40].

The **geometric quantization method** (and to some extent the coadjoint orbit method in the Lie group theoretical context [31, 32, 33]) has the ambition of being a rigorous canonical quantization method. Its main problem is that at prequantization step it leads to a **reducible** representation of the symmetry group. One cannot remove this reducibility in any nontrivial physical situations [28, 29, 30]. In this paper we have demonstrated that the standard method leads to reducibility in the considered case. By standard method, we mean the determination of quantum operators by prequantization rule and the choice of a measure identified to the symplectic two-form. It can be argued that another measure, or another mapping of basic observables, would do better. But different choices for the quantum observables, or for the measure defining the scalar product, could easily spoil the self-adjointness of quantum operators and the homomorphism of the prequantization mapping. The geometric quantization procedure suffers from another demerit: it applies comparatively easily only when canonical variables exist which are well defined globally. We emphasize that the coherent state method does not require global, or canonical variables on the phase space. However, it does not guarantee that the mapping of classical observables into corresponding operators is always a homomorphism (in case considered in this paper the mapping is the homomorphism). The geometric quantization method always leads, by its construction, to the homomorphism.

For other discussions concerning comparison of the three methods considered in our paper we recommend Ref. [42] (general case) and Ref. [43] (case of $sl(2, \mathbb{R})$ algebra).

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