Noncommutative gauge theory and symmetry breaking in matrix models

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Abstract

We show how the fields and particles of the standard model can be naturally realized in noncommutative gauge theory. Starting with a Yang-Mills matrix model in more than 4 dimensions, a $SU(n)$ gauge theory on a Moyal-Weyl space arises with all matter and fields in the adjoint of the gauge group. We show how this gauge symmetry can be broken spontaneously down to $SU(3)_{c} \times SU(2)_{L} \times U(1)_{Q}$ (resp. $SU(3)_{c} \times U(1)_{Q}$), which couples appropriately to all fields in the standard model. An additional $U(1)_{B}$ gauge group arises which is anomalous at low energies, while the trace-$U(1)$ sector is understood in terms of emergent gravity. A number of additional fields arise which we assume to be massive, in a pattern that is reminiscent of supersymmetry. The symmetry breaking might arise via spontaneously generated fuzzy spheres, in which case the mechanism is similar to brane constructions in string theory.
1 Introduction

While no one knows how to describe physics at the Planck scale, there are suggestions that it may be described by some generalization of ordinary spaces which goes under the generic name of noncommutative geometry \[1, 2, 3, 4\]. Regardless of the details of such a construction, the noncommutative generalization of the coordinate functions will be some matrices which satisfy commutation relations of the type

\[
[x^\mu, x^\nu] = i\theta^{\mu\nu} \tag{1.1}
\]

where $\theta^{\mu\nu}$ is a quantity of the order of the square of Planck’s length. An action is then naturally defined as some kind of matrix model in terms of these noncommutative coordinates, such as the models introduced in \[5, 6, 7\]. These matrix models are known to describe noncommutative gauge theory \[8, 9\], and contain gravity as an emergent phenomenon \[10\] a la Sakharov \[11, 12\]. Thus they are promising candidates for a quantum theory of fundamental interactions. However, the noncommutative gauge theories obtained in this manner are quite restrictive \[13\]: only $U(n)$ gauge groups (or possibly products thereof) are consistent, fermions can be introduced only in the adjoint or possibly (anti-)fundamental representation, and the trace-$U(1)$ sector is afflicted with the notorious UV/IR mixing \[14, 15, 16, 17\]. Hence these models are often thought to be incompatible with particle physics. There are proposals how to formulate the standard model on noncommutative space-time based on different approaches such as a Seiberg-Witten expansion or open Wilson lines \[18, 19\], which however lead to serious drawbacks in particular for the quantizations \[20, 21\].

The main point of this paper is to demonstrate that the simple matrix models for noncommutative gauge theory may nevertheless lead to low-energy gauge theories which are extensions of the standard model. In particular, we show how all fermions in the standard model with their appropriate charges can be accommodated. The principal idea is to consider a matrix model which describes not only the usual Moyal plane $\mathbb{R}^4_\theta$, but also extra dimension encoded by additional matrices. These matrices corresponding to extra dimensions can be equivalently interpreted as scalar fields on $\mathbb{R}^4_\theta$, and can acquire nontrivial vacuum expectation values leading via the usual Higgs effect to spontaneous symmetry breaking. The extra-dimensional matrices are assumed to have a finite spectrum and no massless modes, similar in a
sense to Connes approach to the standard model in noncommutative geometry \cite{1, 23, 24}. The mechanism is essentially the same as the generation of fuzzy extra dimensions in ordinary gauge theory \cite{25}, cf. \cite{26}. The trace-$U(1)$ components and its UV/IR mixing were understood in \cite{10} to be part of gravity sector and are not part of the low-energy gauge theory. This allows us to resolve the problems with the $U(1)$ sector found in previous formulations of the standard model on the Moyal-Weyl plane \cite{16, 17} based on (products of) $U(N)$ gauge groups.

The models we will describe below have some key features of the standard model, mainly regarding symmetry breaking, but are not yet phenomenologically viable, in the sense that there are still several features which are unrealistic. However, the basic mechanism based on spontaneous symmetry breaking of the underlying noncommutative $SU(N)$ (resp. $U(N)$) gauge theory is rather general, and it is quite conceivable that more sophisticated versions might be realistic. In particular, we will see that a promising line of development is to consider the internal space as fuzzy spheres, similar as in \cite{25}. Then the pattern which emerges is quite similar to string-theoretical constructions of (extensions of the) standard model \cite{27, 28, 29}, based on strings stretching between branes. These modes are recovered here as bimodules of $SU(n_i)$ subgroups of the spontaneously broken $SU(N)$ gauge group. One of the main open problems is the origin of chirality, and we only discuss some possible avenues here. This problem is similar as in the commutative case \cite{30}, and can probably be solved by invoking more sophisticated geometrical structures such as orbifolds \cite{31}.

This paper is structured as follows. After recalling the basic constructions of matrix models and noncommutative gauge theory, we discuss in section 3 the symmetry breaking of $SU(n)$ to products of $SU(n_i)$ via extra dimensions. We consider both a simplified effective treatment involving only the low-energy degrees of freedom, as well as a more sophisticated realization in terms of fuzzy spheres in extra dimensions. Section 4 contains the main results of the paper, namely the embedding of the standard model particles and fields in the basic matrices which are in the adjoint of $SU(N)$, focusing on $N=7$. The electroweak symmetry breaking is discussed in section 5, as well as the structure of the Yukawa couplings. Here we only exhibit some qualitative aspects and discuss possible avenues for further studies. Parts of the present paper have been presented in the proceedings \cite{32}.
2 The Matrix Model

We start with a Yang-Mills matrix model which involves \( D = 4 + n \) matrices \( X^a \) and a set of fermions:

\[
S_{YM} = -\left(2\pi\right)^2 \frac{\Lambda^4_{NC}}{g^2} \text{Tr} \left( [X^a, X^b] [X^{a'}, X^{b'}] \eta_{aa'} \eta_{bb'} + \Psi \Gamma_a [X^a, \Psi] \right) \quad (2.1)
\]

where \( X^a \) are infinite-dimensional hermitian matrices \( Mat(\infty, \mathbb{C}) \), or operators in a Hilbert space \( \mathcal{H} \). \( \Gamma_a \) generates the \( SO(1, D - 1) \) Clifford algebra, the metric \( \eta_{aa'} \) is the flat Minkowski (or Euclidean) metric with the mostly minus choice of signs, and \( \Psi \) is a corresponding (Grassmann-valued) spinor taking values in \( Mat(\infty, \mathbb{C}) \). We introduced a scale parameter \( \Lambda_{NC} \) which will be identified with the scale of noncommutativity below, and \( g \) will be identified as a gauge coupling constant. This model is invariant under the symmetry

\[
X^a \rightarrow UX^a U^{-1}, \quad U \in U(\mathcal{H}). \quad (2.2)
\]

The equations of motion of the bosonic part of model are

\[
[X^a, [X^b, X^{a'}]] \eta_{aa'} = 0 \quad (2.3)
\]

we will discuss the fermions later. There are several solutions for this classical equation of motion, which we will often call vacua in the following. Apart from the trivial one \((X^a = 0)\) and the case in which all of the \( X \)'s commute, a relevant vacuum for our model is the “scalar Moyal-Weyl" vacuum:

\[
[X^a_0, X^b_0] = i\theta^{ab} \quad (2.4)
\]

with \( \theta^{ab} \) constant. The functions of the \( X_0 \)'s in this case generate an algebra isomorphic (under appropriate regularity conditions) to the algebra of functions on a \( D \) dimensional space multiplied with the Grönewold-Moyal product. That is, given two functions \( f \) and \( g \), then consider \( f(x), g(x) \) as ordinary functions on the plane, and \( f(X), g(X) \) operators, then

\[
f(X)g(X) = (f \ast g)(X) \quad (2.5)
\]

* The case \( D = 10 \) is of particular interest. In this case it is possible to impose a Majorana-Weyl condition on \( \Psi \), and the model admits an extended supersymmetry [6]. On a 4-dimensional Moyal-Weyl background as discussed below, the model then reduces to the \( N = 4 \) SYM on \( \mathbb{R}^4 \), which is expected to be well-behaved upon quantization.
with
\[
(f \star g)(x) = e^{-i \theta_{ab} \partial_x a \partial_y b} f(x)g(y) \bigg|_{x=y}
\] (2.6)

We interpret this as the fact that the vacuum (2.4) describes a noncommutative space where the coordinates have a nontrivial constant commutator, the noncommutative space \( \mathbb{R}^D_\theta \). The bosonic part action has a gauge invariance for the unitary elements of the algebra, since we are considering functions of \( X \) we consider these unitary elements as unitary matrix functions \( U(X) \) to which corresponds a function of \( x \) which is as usual a phase. We call this association of functions of the matrices with functions on an ordinary space the Moyal-Weyl limit.

Another vacuum of interest is
\[
\bar{X}^a = X^a_0 \otimes 1_N
\] (2.7)

In this case the Moyal-Weyl limit is given by matrix valued functions on \( \mathbb{R}^D_\theta \) and the gauge symmetry is given by unitary elements of the algebra of \( n \times n \) matrices of functions of the \( X_0 \). We say that this theory has a noncommutative \( U(n) \) gauge symmetry because in the semiclassical limit it corresponds to a nonabelian gauge theory.

### 2.1 Moyal-Weyl, gauge theory and extra dimensions

Let us now consider the case in which not all dimensions have the same significance. Split the \( D \) matrices as
\[
X^a = (X^\mu, \mathcal{X}^i), \quad \mu = 0, ..., 3, \ i = 1, ..., n
\] (2.8)

into 4 “spacetime” generators \( X^\mu \) which will be interpreted as (quantized) coordinate functions, and \( n \) generators \( \mathcal{X}^i \) which are interpreted as extra dimensions. More specifically, we consider a background (i.e. a solution) of the matrix model where 4 “spacetime” generators \( X^\mu \) generate the Moyal-Weyl quantum plane \( \mathbb{R}^4_\theta \)
\[
\bar{X}^\mu = X^\mu_0 \otimes 1_N, \quad \mathcal{X}^i = 0
\] (2.9)

which satisfies
\[
[\bar{X}^\mu, \bar{X}^\nu] = i \theta^{\mu\nu} \otimes 1_N, \quad [\bar{X}^\mu, \mathcal{X}^i] = 0
\] (2.10)
Here we assume $\theta^{\mu\nu} = \text{const}$ for simplicity. This background preserves the symmetry $SU(N)$ which commutes with $X^\mu$.

Now consider small fluctuations around this solution,

$$X^\mu = \bar{X}^\mu + A^\mu, \quad \Phi^i = \Lambda_{NC}^2 \bar{X}^i$$

so that $X^\mu$ has dimension length and $\Phi^i$ has dimension length$^{-1}$ and can be considered (also dimensionally) a field from the four dimensional point of view. As shown in [10] and recalled in Section (2.2), the trace-$U(1)$ fluctuations gives rise to the gravitational degrees of freedom which lead to an effective (“emergent”) metric and gravity. For the sake of the present paper we will ignore these $U(1)$ degrees of freedom and concentrate on traceless fluctuations, assuming a flat Moyal-Weyl background with Minkowski signature. The remaining $SU(N)$-valued fluctuations

$$A^\mu = -\theta^{\mu\nu} A^\nu(x) \otimes \lambda_\alpha$$

then correspond to $SU(N)$-valued gauge fields, while the fluctuations in the internal degrees of freedom

$$\Phi^i = \Phi^{i,\alpha}(x) \otimes \lambda_\alpha$$

correspond to scalar fields in the adjoint. The matrix model action (2.1) therefore describes $SU(N)$ gauge theory on $\mathbb{R}^4_\theta$ coupled to $n$ scalar fields. From now on we will drop the $\otimes$ sign whenever there is no risk of confusion.

Noncommutative gauge theory is obtained from the matrix model using the following basic observation

$$[\bar{X}^\mu + A^\mu, f] = i\theta^{\mu\nu} (\partial_{\bar{x}^\nu} f + i[A^\nu, f]) \equiv i\theta^{\mu\nu} D_{\nu} f$$

The matrix model action (2.1) can then be written as

$$S_{YM} = \frac{1}{g^2} \int d^4 \bar{x} \, \text{tr} \left( G^{\mu\nu} G^{\mu'\nu'} F_{\mu\nu} F_{\mu'\nu'} ight. 
+ 2 G^{\mu\nu} D_\mu \Phi^i D_\nu \Phi^j \delta_{ij} - [\Phi^i, [\Phi^j, \Phi^k]] \delta_{ij} \delta_{kl} 
+ \bar{\Psi} \Gamma_i \Phi^i \Psi + \bar{\Psi} \Gamma_i \Phi^i \Psi \right).$$

5
This is the action of a $SU(N)$ gauge theory on $\mathbb{R}_\theta^4$, with effective metric given by
\[ G^{\mu\nu} = \rho \theta^{\mu\nu'} \eta_{\mu'\nu'}, \quad \rho = (\det \theta^{\mu\nu})^{-1/2} =: \Lambda_{NC}^4, \tag{2.16} \]
which satisfies $\sqrt{|G|} = 1$. Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$ is the field strength on $\mathbb{R}_\theta^4$,
\[ D_\mu \equiv \partial_\mu + i[A_\mu, \cdot] \tag{2.17} \]
is the covariant derivative for fields in the adjoint, and $\text{tr}()$ denotes the trace over the $SU(N)$ components. The effective Dirac operator is given by
\[ D_\mu = \Gamma_\mu [X^\mu, \Psi] \sim i\gamma^\mu D_\mu \Psi \tag{2.18} \]
where
\[ \gamma^\mu = \sqrt{\rho} \Gamma_\nu \theta^{\mu\nu}, \quad \{\gamma^\mu, \gamma^\nu\} = 2G^{\mu\nu}. \tag{2.19} \]
The fermions have been rescaled appropriately, and a constant shift as well as total derivatives in the action are dropped. Note that $g$ is now identified as the coupling constant for the nonabelian gauge fields on $\mathbb{R}_\theta^4$.

### 2.2 Fluctuations of the Vacuum, Emergent Gravity and Gauge Theory

As explained above, fluctuations of $X^a$ can be parametrized in terms of gauge fields $A_\mu$ and scalar fields $\phi^i$ on $\mathbb{R}_\theta^4$. At first sight, this might suggest that the action (2.15) describes $U(n)$ gauge theory on $\mathbb{R}_\theta^4$. However, this interpretation is not quite correct: it turns out that the trace-$U(1)$ fluctuations of both $A^\mu$ and $\Phi^i$ describe gravitational degrees of freedom which modify the geometry of $\mathbb{R}_\theta^4$, defining an effective (“emergent”) geometry given by
\[ G^{\mu\nu}(x) = e^{-\sigma} \theta^{\mu\nu}(x) \theta^{\mu\nu'}(x) g_{\mu'\nu'}(x), \]
\[ i\theta^{\mu\nu}(x) = i\{x^\mu, x^\nu\} \sim [X^\mu, X^\nu], \]
\[ g_{\mu\nu} = \eta_{\mu\nu} + \partial_\mu \phi^i \partial_\nu \phi_i, \]
\[ e^\sigma = \sqrt{\det \theta^{\mu\nu}(x) \det g_{\mu\nu}(x)}. \tag{2.20} \]
The $SU(n)$-valued components of $A^\mu$ and $\Phi^i$ describe nonabelian gauge fields (resp. scalar fields) which in the semi-classical limit (denoted by $\sim$) couple to

\[ ^\dagger\text{The } U(1) \text{ components are gravitational degrees of freedom and will be ignored here.} \]
the effective metric $G^{\mu\nu}(x)$. Note that the “would-be $U(1)$ gauge fields” $A_\mu \mathbb{1}$ are absorbed in the Poisson structure $\theta^{\mu\nu}(x) = \tilde{\theta}^{\mu\nu}(\tilde{x}) - \tilde{\theta}^{\mu\nu'}\tilde{\theta}_{\nu'} F_{\mu\nu'}$, and similarly the $U(1)$ degrees of freedom of $\phi^i$ are absorbed in $g_{\mu\nu}(x)$. In the following we will ignore the gravitational degrees of freedom and only keep the $SU(n)$-valued components of $A^\mu$ and $\phi^i$, focusing on the flat Moyal-Weyl space $\mathbb{R}^d_\theta$.

3 Spontaneous symmetry breaking in extra dimensions

In this section we will present two mechanisms to break the gauge symmetry form $SU(n)$ down to a smaller group. The first with one constant extra dimensions, and the second with inner fuzzy spheres. The two mechanisms are not really different because the former can be seen as an effective model of the latter. Only the bosonic part will be discussed in this section as well. Both models are somewhat analogous to a GUT-like model, where the breaking is realized through a Higgs in the adjoint.

3.1 Single-Coordinate Effective Breaking

The mechanism how to obtain non-trivial low-energy gauge groups and particle spectrum can be understood in a simple way as follows. Consider a model with $\bar{X}^\mu$ as in (2.10), but this time with a single extra dimension which we call $\mathcal{X}^{\Phi}$ which in a suitable vacuum takes the form

$$\langle \mathcal{X}^{\Phi} \rangle = \begin{pmatrix} \alpha_1 \mathbb{1}_2 \\ \alpha_2 \mathbb{1}_2 \\ \alpha_3 \mathbb{1}_3 \end{pmatrix}.$$  (3.1)

Here $\alpha$’s are constant quantities with the dimensions of a length, all different among themselves. These new coordinates are still solutions of the equations of the motion because $[X^\mu, \langle \mathcal{X}^{\Phi} \rangle] = 0$ i.e. $\theta^{\mu\Phi} = 0$, which in turn implies that $G^{\mu\Phi} = 0$ regardless on the value of $\eta^{\Phi\Phi}$. Therefore extra coordinate is not geometric and does not correspond to propagating degrees of freedom from
the four dimensional point of view. The new coordinate is not invariant for the transformation
\[ \langle \mathcal{X}^\Phi \rangle \rightarrow U \langle \mathcal{X}^\Phi \rangle U^\dagger \neq \langle \mathcal{X}^\Phi \rangle \] (3.2)
for a generic \( U \in SU(7) \). The traceless generators commuting with \( \langle \mathcal{X}^\Phi \rangle \) generate the surviving gauge group \( SU(2) \times SU(2) \times SU(3) \times U(1) \times U(1) \).

In the bosonic action as in Section 2.1 the spacetime \((\mu \nu)\) part of action remains unchanged, while for the \( \mu \phi \) components we obtain, in the Moyal-Weyl background,
\[ [\bar{X}_\mu + A_\mu, \mathcal{X}^\Phi] = i \theta^{\mu\nu} D_\nu \mathcal{X}^\Phi = i \theta^{\mu\nu} (\partial_\nu + i A_\nu) \mathcal{X}^\Phi, \]
\[ - (2\pi)^2 \text{Tr} [X^\mu, \mathcal{X}^\Phi][X^\nu, \mathcal{X}^\Phi] \eta_{\mu\nu} = \int d^4x G^{\mu\nu} (\partial_\mu \mathcal{X}^\Phi \partial_\nu \mathcal{X}^\Phi - [A_\mu, \mathcal{X}^\Phi][A_\nu, \mathcal{X}^\Phi]). \] (3.3)

Note that the mixed terms \( \int \partial^\mu \mathcal{X}^\Phi [A_\mu, \mathcal{X}^\Phi] = -\frac{1}{2} \int \mathcal{X}^\Phi [\partial^\mu A_\mu, \mathcal{X}^\Phi] = 0 \) vanish, assuming the Lorentz gauge \( \partial^\mu A_\mu = 0 \).

Now consider the vacuum (3.1). Since \( X^\mu \) and \( \langle \mathcal{X}^\Phi \rangle \) commute, this means \( \langle \mathcal{X}^\Phi \rangle = \text{const} \) and the first term in the integral above vanish. We can therefore separate the fluctuations of this extra dimension which are a field, the (high energy) Higgs field. In the action the first term is nothing but the derivative of it. The second term instead is
\[ [A_\mu, \langle \mathcal{X}^\Phi \rangle] = \begin{pmatrix}
0 & (\alpha_2 - \alpha_1)A^\mu_{12} & (\alpha_3 - \alpha_1)A^\mu_{13} \\
(\alpha_1 - \alpha_2)A^\mu_{21} & 0 & (\alpha_3 - \alpha_2)A^\mu_{23} \\
(\alpha_1 - \alpha_3)A^\mu_{31} & (\alpha_2 - \alpha_3)A^\mu_{32} & 0
\end{pmatrix} \] (3.4)
where we consider the block form of \( A_\mu \)
\[ A_\mu = \begin{pmatrix}
A^\mu_{11} & A^\mu_{12} & A^\mu_{13} \\
A^\mu_{21} & A^\mu_{22} & A^\mu_{23} \\
A^\mu_{31} & A^\mu_{32} & A^\mu_{33}
\end{pmatrix} \] (3.5)
Therefore (3.3) leads to the mass terms for the off-diagonal gauge fields,
\[ - (2\pi)^2 \text{Tr} [X^\mu, \langle \mathcal{X}^\Phi \rangle][X^\nu, \langle \mathcal{X}^\Phi \rangle] \eta_{\mu\nu} = \int d^4x G^{\mu\nu} \left( \sum (\alpha_i - \alpha_j)^2 A_{\mu,ij} A_{\nu,ji} \right) \] (3.6)
which is nothing but the usual Higgs effect. If we now assume that the differences \( \alpha_i - \alpha_j \) are large, say of the grand unification scale, it is easy to
see that all non diagonal blocks of $A^\mu$ acquire large masses $m^2_{ij} \sim (\alpha_i - \alpha_j)^2$, thus effectively decoupling.

In order to approach the standard model, we will assume the following version of the above mechanism

$$\langle X^\Phi \rangle = \begin{pmatrix} \alpha_1 \mathbb{1}_2 & \alpha_2 \sigma_3 \\ \alpha_2 \sigma_3 & \alpha_3 \mathbb{1}_3 \end{pmatrix}$$ (3.7)

with $\alpha_1 \neq \alpha_2 \neq \alpha_3$. Then the surviving traceless gauge group is given by $SU(3) \times SU(2) \times U(1) \times U(1) \times U(1)$, and the off-diagonal gauge fields $A_{\mu,ij}$ for $i,j$ labeling the 4 blocks acquire a mass as in (3.6). Again, this may simply be a crude picture of some more sophisticated mechanism involving fuzzy spheres (branes) in extra dimensions, as discussed below. This then comes very close to some of the proposals how to recover the standard-model using branes and strings stretching between them, cf. [27, 28]. Thus in a sense we show how such a mechanism can be realized in the matrix model framework.

### 3.2 Fuzzy Sphere Breaking

According to the splitting of the matrices into a noncommutative spacetime $\mathbb{R}^4_\theta$ and “extra” generators $X^i$, it is quite natural to add extra terms to the potential and add a potential term involving quadratic and cubic terms in the fluctuations $\Phi^i$ defined in (2.11):

$$V_{soft}(\Phi^i) = 2 \text{Tr} \left( c_2 \Phi^i \Phi^j \delta_{ij} + ic_3 \varepsilon_{ijk} \Phi^i \Phi^j \Phi^k \right)$$

$$= 2\Lambda^4_{NC} \int d^4x \text{ tr} \left( c_2 \Phi^i \Phi^j \delta_{ij} + ic_3 \varepsilon_{ijk} \Phi^i \Phi^j \Phi^k \right)$$ (3.8)

From the point of view of field theory on $\mathbb{R}^4_\theta$, these amount to soft (resp. relevant) terms which may (partially) break the global $SO(n)$ symmetry, as well as supersymmetry if applicable. In particular they may be generated upon quantization. The full 1-loop effective potential can have more complicated effective potentials, but for the present work we will limit our considerations to these terms.

The bosonic part of the action (2.1) now becomes the following gauge
theory action on $\mathbb{R}^4_\text{g}$,

$$
S_Y M = \int d^4 x \frac{1}{g^2} \text{tr} F_{\mu\nu} F_{\mu'\nu'} G^{\mu\nu} G^{\mu'\nu'} + 2\Lambda^4_{NC} \int d^4 x \text{ tr} \left( \frac{1}{g^2} G^{\mu\nu} D_\mu \Phi^i D_\nu \Phi^i \right) - \frac{\Lambda^4_{NC}}{2g^2} [\Phi^i, \Phi^j][\Phi^{i'}, \Phi^{j'}] \delta_{i'j'} \delta_{ij} + c_2 \Phi^i \Phi^j \delta_{ij} + ic_3 \varepsilon_{ijk} \Phi^i \Phi^j \Phi^k 
$$

(3.9)

with $G^{\mu\nu'}$ is as in (2.16). We omit here surface terms (such as $\int d^4 x F_{\mu\nu} \theta_{\mu\nu}$), as well as all trace-$U(1)$ degrees of freedom which are part of the gravitational sector.

**Scalar fields and spontaneous symmetry breaking.** Assuming that the extra coordinates commute with the space time ones, $[\mathcal{X}, X^\mu] = [\Phi^i, X^\mu] = 0$, or that in other words, in the Moyal-Weyl vacuum they commute with the $x$’s, the above action leads to the following equation of motion for the scalar fields

$$
\frac{2\Lambda^4_{NC}}{g^2} [\Phi^j, [\Phi^i, \Phi^{j'}]] \delta_{jj'} + 2c_2 \Phi^i + \frac{3}{2} ic_3 \varepsilon_{ijk} [\Phi^j, \Phi^k] = 0
$$

(3.10)

This equation has as solution

$$
\langle \Phi^i \rangle = a J^i_N
$$

(3.11)

where $J^i_N$ are generators of the $N \times N$ representation of $SU(2)$

$$
[J^i_N, J^j_N] = i \varepsilon_{ijk} J^k_N, \quad J^i_N J^j_N = \frac{N^2 - 1}{4}. \quad (3.12)
$$

This solution is interpreted as fuzzy sphere \[35\] with radius

$$
\alpha = \frac{a}{2} \sqrt{N_i^2 - 1}
$$

(3.13)

as in \[25\]. The equations of the motion then reduce to

$$
\frac{2\Lambda^4_{NC}}{g^2} a^2 + 2c_2 - 3c_3 a = 0,
$$

(3.14)

which generically has 2 solutions. It is important to note that one of them really is a global minimum of the potential for $\Phi^i$ (3.9):

$$
V[\Phi^i] = \text{ tr} \left( - \frac{\Lambda^4_{NC}}{2g^2} [\Phi^i, \Phi^j][\Phi^{i'}, \Phi^{j'}] \delta_{i'j'} \delta_{ij} + c_2 \Phi^i \Phi^j \delta_{ij} + ic_3 \varepsilon_{ijk} \Phi^i \Phi^j \Phi^k \right) = -N \left( \frac{\alpha^2 \Lambda^4_{NC}}{2g^2} + c_2 \alpha^2 - c_3 \alpha^3 \right).
$$

(3.15)
This potential has either one or two degenerate minima as a function of $\alpha$, and (3.11) is the physical vacuum of the model. The $SU(N)$ gauge symmetry is then broken completely by the presence of a fuzzy sphere in the internal space.

A physically more relevant vacuum could be the one corresponding to a “stack” of fuzzy spheres as proposed in [25], in particular

$$\langle \Phi_i \rangle = \begin{pmatrix} a_1 J_{N_1}^i \otimes 1_2 & 0 & 0 \\ 0 & a_2 J_{N_2}^i \otimes \sigma_3 & 0 \\ 0 & 0 & a_3 J_{N_3}^i \otimes 1_3 \end{pmatrix}, \quad (3.16)$$

which gives an explicit realization of (3.7). Each of the blocks corresponds to a fuzzy sphere $S^2_{N_i}$ with radius $\alpha_i$. More precisely, the last block has a 3-fold multiplicity, which can be interpreted as stack of 3 coinciding fuzzy spheres with radius $\alpha_3$. If these fuzzy spheres are large, then the fluctuations around this vacuum effectively “see” only the radius of the fuzzy spheres. Thus at low energies we are very close to the case of the previous subsection. The symmetry in the vacuum (3.16) is broken down to $SU(3) \times SU(2) \times U(1) \times U(1)$, which is very close to what we want. Note that by setting e.g. $\alpha_1 = \alpha_2$ and $N_1 = N_2$ the symmetry is enhanced, and more sophisticated symmetry breaking processes with several steps (resp. scales) are conceivable.

An important bonus compared with standard Higgs scenarios is that the above Higgs fields have a natural geometrical interpretation in terms of compact fuzzy spaces. The double commutator has an interpretation in terms of a higher-dimensional curvature, and the additional potential (3.8) is cubic. This should lead to milder renormalization properties and less fine-tuning compared with the standard $\phi^4$ case, as observed in [25].

**Massive vector bosons and Higgs effect.** The masses of the four-dimensional nonabelian gauge bosons $A_\mu$ in the presence of such a fuzzy sphere vacuum were studied in [25]. The result is essentially the same as in section 3.1, i.e. the off-diagonal components $A_{\mu,ij}$ for $i \neq j$ acquire a mass due to the term $\text{tr} G^{\mu\nu} D_\mu \Phi^i D_\nu \Phi^i$, provided $(\alpha_i, N_i) \neq (\alpha_j, N_j)$. For the diagonal blocks, only the $l = 0$ mode of the decomposition of $\text{Mat}(N_i) = \bigoplus_{l=0}^{2N_i-1} Y_{l,m}$ into fuzzy spherical harmonics remains massless, while all higher Kaluza-Klein modes

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8More sophisticated versions are of course conceivable [33].
acquire a mass \( m^2 \sim \alpha^2 \Lambda_{NC}^4 l(l + 1) \). This is nothing but a geometrical version of the usual Higgs effect. Therefore the low-energy sector of such a fuzzy sphere vacuum is essentially captured by the effective single-variable description in section 3.1. However, the fuzzy sphere scenario provides a natural origin of a Higgs potential with nontrivial minimum, which is not seen in the single-variable description.

4 Particle assignments, charges and symmetries

In this section we show how the fermions in the standard model can be naturally accommodated in the framework of matrix models. This is nontrivial because the fermions in the matrix model are necessarily in the adjoint of some basic \( SU(N) \) gauge group. In a later section we will also show how the electroweak symmetry can be broken through a somewhat modified Higgs sector, and the Yukawa couplings are obtained.

We start with the matrix model of Section 3 in \( D = 4 + n \) dimensions. Thus the fermions are realized as \( D \)-dimensional spinors \( \Psi \) in the adjoint of \( SU(N) \), and there are \( n \) scalar fields \( \Phi^i \) in the adjoint of \( SU(N) \) as well as the 4-dimensional gauge fields \( A_\mu \).

The basic idea is to assume that the fundamental \( SU(N) \) gauge group is spontaneously broken in several steps down to the low-energy gauge group \( SU(3) \times U(1) \) of the standard model. It is natural to assume that the various (intermediate and low-energy) gauge groups are realized as block-diagonal subgroups of \( SU(N) \). We will show in later section how this can be realized by spontaneous compactification on fuzzy internal spaces. In this section, we simply assume that in a first step (at very high energy) the block-matrix decomposition in Sect 3 has occurred and that therefore the symmetry is broken to \( SU(3) \times SU(2) \times U(1) \times U(1) \times U(1) \) as in (3.7). We assign the fermions accordingly by the matrix

\[
\Psi = \begin{pmatrix}
L_{4 \times 4} & Q \\
Q' & 0_{3 \times 3}
\end{pmatrix}
\]  

(4.1)

\footnote{Which in turn are obtained as fluctuations of the covariant coordinates in the matrix model.}
Here the $4 \times 4$ block $\mathcal{L}$ will contain the leptons which are color-blind, and $Q$ (resp. $Q'$) will contain the quarks (which we assume to be in $(\bar{3})$ here for convenience). We drop all fermions in the adjoint of an unbroken gauge group, i.e. we assume that they are very massive. This is plausible if this block arises from Kaluza-Klein modes on some fuzzy sphere as discussed above; in principle such fermions would correspond to gauginos. We denote the off-diagonal blocks according to (3.7) as

$$L = \begin{pmatrix} 0_{2 \times 2} & L_L \\ L'_L & e_R \\ e'_R & 0 \end{pmatrix} ,$$

$$L_L = (\tilde{l}_L \ l_L), \quad l_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} , \quad \tilde{l}_L = \begin{pmatrix} \tilde{e}_L \\ \tilde{\nu}_L \end{pmatrix} \quad (4.2)$$

Here $l_L$ will be the standard (left-handed) leptons, and $e_R$ the right-handed electron. Fields with a prime may either be related to the unprimed ones through some conjugation, or they be independent new fields, or they may vanish for some reason; this will be discussed below. In particular $\tilde{l}$ will correspond to additional leptons, which may be present at some energy, or which may be null; at present the model allows them and we will keep the term. The quarks split accordingly as

$$Q = \begin{pmatrix} Q_L \\ Q_R \end{pmatrix} ,$$

$$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} , \quad Q_R = \begin{pmatrix} d_R \\ u_R \end{pmatrix} \quad (4.3)$$

which will again correspond to the standard quarks. This gives the following general fermionic matrix,

$$\Psi = \begin{pmatrix} 0_{2 \times 2} & L_L & Q_L \\ L'_L & 0 & e_R \\ e'_R & 0 & Q'_R \end{pmatrix} \quad (4.4)$$

The correct hypercharge, electric charge and baryon number are then repro-
duced by the following traceless generators

\[
Y = \begin{pmatrix}
0_{2 \times 2} & -\sigma_3 \\
-\frac{1}{3} \mathbb{1}_{3 \times 3} & 0
\end{pmatrix} + \frac{1}{7} \mathbb{1} \tag{4.5}
\]

\[
Q = T_3 + \frac{Y}{2} = \frac{1}{2} \begin{pmatrix}
\sigma_3 & -\sigma_3 \\
-\frac{1}{3} \mathbb{1}_{3 \times 3} & \sigma_3
\end{pmatrix} + \frac{1}{14} \mathbb{1} \tag{4.6}
\]

\[
B = \begin{pmatrix}
0_{2 \times 2} & -1 \\
-\frac{1}{3} \mathbb{1}_{3 \times 3} & 0
\end{pmatrix} + \frac{1}{7} \mathbb{1} \tag{4.7}
\]

which act in the adjoint. For easy reference we display the charges \((Q, Y, B)\) of these block-matrices for the above generators:

\[
(Q, Y, B)\Psi = \begin{pmatrix}
0_{2 \times 2} & \begin{pmatrix}
(1, 1, 0) & (0, -1, 0) \\
(0, 1, 0) & (-1, -1, 0)
\end{pmatrix} & \begin{pmatrix}
(\frac{2}{3}, \frac{1}{3}, \frac{4}{3}) & (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \\
(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}) & (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})
\end{pmatrix} \\
* & 0 & \begin{pmatrix}
(-1, -2, 0) & (-\frac{1}{2}, -\frac{2}{3}, \frac{1}{3}) \\
(-\frac{1}{2}, -\frac{2}{3}, \frac{1}{3}) & (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})
\end{pmatrix}
\end{pmatrix} \tag{4.8}
\]

as it should be, omitting the obvious lower-diagonal entries. In particular, the charges of the exotic leptons \(\tilde{l}\) are those of Higgsinos.

The main result is that all particles of the standard model with the correct quantum numbers fit naturally into this framework, based on matrices in the adjoint of a fundamental \(SU(N)\) gauge symmetry. This is very important because the only representations which can be realized in noncommutative gauge theory\footnote{At a fundamental level i.e. without resorting to an effective Seiberg-Witten expansion.} are fundamental, anti-fundamental and adjoint representations of \(U(N)\) gauge groups. Matrix models provides a natural framework to study the quantization of NC gauge theory in a non-perturbative way. In order to be free of UV/IR mixing, it appears that only the IKKT matrix model or close relatives are consistent at the quantum level, which contain only matrices in the adjoint. This is a strong and predictive restriction, which restricts the freedom in model building, with the added bonus of an intrinsic gravity sector\footnote{At a fundamental level i.e. without resorting to an effective Seiberg-Witten expansion.}. Perhaps the main result of this paper is that realistic models for particle physics appear feasible within this framework.
There is some freedom in relation to the primed fermions appearing in the lower block of the matrix (4.1), and we need to understand the relation among the upper and lower diagonal blocks. It is likely that a supersymmetric version of this model can be built, and in that framework some of zeros of the matrix can be filled by supersymmetric partners, and the additional leptons $\tilde{\ell}$ could be identified with Higgsinos.

Let us discuss some possibilities how fermions may arise in the off-diagonal blocks. We first need to understand the relation between the upper-diagonal and the lower-diagonal components. We note that the upper and lower triangles in the matrix are exchanged under hermitean conjugation, which is part of charge conjugation. Therefore, as we will see in detail in the following, the role of primed and unprimed elements are exchanged in $\bar{\Psi}$. There are three obvious choices for the primed fermions.

1. If $\Psi = \Psi^C$ is a Majorana-Weyl fermion in the fundamental matrix model, then the lower-diagonal components are related with the upper-diagonal ones directly by charge conjugation. This choice may be natural in the presence of 10 dimension, in which case it is possible to have fermions which are both Majorana and Weyl.

2. One can set the primed fermions equal to zero, so that $\Psi$ is an upper triangular matrix. The lower part of the matrix will appear in $\bar{\psi}$, which will be lower triangular. This choice has several advantage, as will see in the following, but it seems “ad hoc”, without an explanation at the present. This may be related to the presence of a magnetic flux [30].

3. If both upper and lower-diagonal components are non-vanishing and not related via conjugation, the model is non-chiral, corresponding to a mirror model. Then has to explain why each single sector has an independent cancellation of anomalies, which would be canceled by the mirrors anyway; this of course apart form consideration on the presence of dark matter which are still distant from the present state of the art.

In the latter two case the lower-triangular case can be seen as an instance of the presence of fermion doubling, which is known phenomenon in noncommutative geometry [34, 24]. At this stage it is really a matter of taste if one prefers to eliminate the lower triangle of the matrix setting it to zero, or to
keep it as a mirror world. With the former choice one is setting to zero a sector which is in principle present, but which can give unwanted couplings.

The correct chirality assignment is put in by hand here. There is also a slot which could naturally accommodate additional leptons $\tilde{l}_L$. Notice also that the scheme is naturally suited for supersymmetry, since all particles and fields arise from matrices in the adjoint of $SU(N)$.

The full $U(N)$ model is certainly free of anomalies. After symmetry breaking, the $U(1)_B$ gauge symmetry may turn out to be anomalous, as it often does in string theory [29], and we assume that the corresponding gauge boson becomes massive through some version of the Green-Schwarz mechanism. Furthermore the additional leptons $\tilde{l}$ lead to an anomaly unless their lower-diagonal partners $\tilde{l}$ are also present; this strongly suggest that $\tilde{l}$ should be set to zero.

These ambiguities indicate that an additional mechanism is required to single out the correct physical result. In particular, it is very interesting that the above scheme is very similar to constructions in string theory [27, 28, 29], where the standard model is realized in terms of 4 stacks of branes with exactly the above gauge groups, and particles realized as strings stretched between these branes. The latter correspond precisely to the off-diagonal blocks, and there seems to be a correspondence between the possibilities indicated above and the different versions of this construction in string theory. This suggests that additional structures such as intersecting branes should be considered in the matrix model framework. This is probably possible, and e.g. branes with fluxes were recently realized in [30]. In particular, fuzzy orbifolds [31] appear to realize the above structures in a chiral model. We will not investigate these in the present paper.

Here we do not claim to have the final answer, rather we want to point out possible directions which should be pursued elsewhere. We take this similarity with string theory as additional encouragement. However, we want to stress that our approach offers advantages over string theory, simply because the matrix model is a very specific and predictive framework. For example, the branes realized as fuzzy spheres are naturally obtained as stable minima of the potential (3.15). Furthermore, this result shows clearly that there is no obstacle to describe (an extension of) the standard model within the framework of noncommutative gauge theory. The mechanism is applicable to models which are expected to be well-defined at the quantum level, in
particular the IKKT model [9].

5 Electroweak breaking

Now we show how electroweak symmetry breaking might be realized in this framework. To explain the idea we will first present a simplified version where the Higgs is realized in terms of a single extra coordinate (resp. scalar) field. In section 5.2 we then discuss a more elaborate version involving extra coordinates (resp. Higgs fields), which form an “electroweak” fuzzy sphere. This is again not intended as a realistic model, but it shows that suitable Higgs potential can naturally arise within the present framework.

5.1 Electroweak Higgs and Yukawa coupling.

Higgs field connects the left with the right sectors of leptons, and is otherwise colour blind, it is therefore natural to consider, along the lines of Sect. 3.1, another extra coordinate which will have to necessarily be off-diagonal. The following matrix has the correct characteristics:

\[
\mathcal{X}^{\phi} = \Lambda_{\text{NC}}^{-2} \begin{pmatrix}
0_{2 \times 2} & \phi & 0_{2 \times 3} \\
\phi^\dagger & 0_{2 \times 2} & 0_{2 \times 2} \\
0_{3 \times 2} & 0_{3 \times 2} & 0_{3 \times 3}
\end{pmatrix}
\]

(5.9)

where we use again the notations of section 3.1 and consider the extra variable \( \mathcal{X} \), its vacuum expectation value and the fluctuations which are a physical field. The Higgs \( \phi \) is a \( 2 \times 2 \) matrix which is actually composed of two doublets (which form the Higgs content of the minimal supersymmetric standard model), i.e. two scalar doublets with opposite hypercharges:

\[
\phi = (\tilde{\phi}, \varphi)
\]

(5.10)

The vacuum expectation value of \( \phi \) is an off-diagonal matrix:

\[
\langle \phi \rangle = \begin{pmatrix}
0 & \nu \\
\bar{\nu} & 0
\end{pmatrix}
\]

(5.11)

All other components (possibly even some of the components of \( \phi \)) are assumed to be very massive, e.g. due to the commutator with the high-energy breaking discussed before.
Now consider the fermionic part of the action (2.1), which can be written on $\mathbb{R}^4$ in the form (2.15). The part involving $X^\mu$ gives the usual Dirac action as in (2.15), and the part involving $X^\phi$ yields the Yukawa couplings
\[ S_Y = \text{Tr} \bar{\Psi} \gamma_\phi [X^\phi, \Psi] \] (5.12)
giving mass to the fermions. Here $\gamma_\phi$ is an extra-dimensional gamma matrix corresponding to $X^\phi$. We write
\[ \bar{\Psi} = \Psi^\dagger \gamma_0 = \begin{pmatrix} 0_{2\times2} & \overline{l}_L & \overline{e}_R^\dagger \\ \overline{l}_L & 0 & \tau_R^\dagger \\ \overline{e}_R & \tau_R & 0_{3\times3} \end{pmatrix} \] (5.13)
Then the full Yukawa term without any omissions or further assumptions is
\[ S_Y = \text{Tr} \left( -\overline{L} \gamma_\phi \begin{pmatrix} 0 & e_R^\dagger \\ e_R & 0 \end{pmatrix} \phi^\dagger - \overline{Q}_L \gamma_\phi Q_R^\dagger \phi \right) + \overline{L} \gamma_\phi \begin{pmatrix} 0 & e_R^\dagger \\ e_R & 0 \end{pmatrix} \phi^\dagger (\phi^\dagger L - L^\phi) - \overline{Q}_R \gamma_\phi Q_L^\dagger \phi + \overline{Q}_L \gamma_\phi \phi Q_R + \overline{Q}_R \gamma_\phi \phi^\dagger Q_L \] (5.14)
We now impose $\gamma_\phi = \gamma_5$, which is natural since in this way the five-dimensional Clifford algebra is closed, and
\[ \gamma_5 \overline{L} = + \overline{L}, \quad \gamma_5 \overline{Q}_L = + \overline{Q}_L, \quad \gamma_5 Q_R = - Q_R \quad \gamma_5 e_R = - e_R \] (5.15)
with this assumption and we see that the couplings come to be the correct ones. Only opposite chiralities couple in such a Yukawa term.

The construction is quite solid and works in all three case for the primed and unprimed fermions. In the case for which the primed fermions vanish only a few terms will survive. In the case of Majorana fermions the couplings are the ones needed to give Dirac masses to Majorana fermions. In the case of mirror fermions there is no coupling among mirror and ordinary fermions, so that the mirror world effectively decouples.

The extra fermion doublet $\tilde{l}$ does not couple with the remaining leptons with the option of setting the primed fermions to zero. If the primed sector is the conjugate sector of Majorana fermions there is a coupling of $e_R$ and $\tilde{e}_L$ which may cause problems. Note that in this model the masses and the Yukawa couplings of leptons and quarks come to be the same and there is
no way to differentiate them. A breaking with fuzzy spheres discussed below
give more structure to the extra dimensions and may create differences.

Assuming

$$\langle \phi \rangle = \begin{pmatrix} 0 & \nu \\ \bar{\nu} & 0 \end{pmatrix}$$  \hfill (5.16)

with $\nu$ and $\bar{\nu}$ real, we get that the quark contribution to the action is

$$Q_5 \gamma_5 \begin{pmatrix} 0 & \phi \\ \phi^\dagger & 0 \end{pmatrix} Q - Q'_5 Q' \begin{pmatrix} 0 & \phi^\dagger \\ \phi & 0 \end{pmatrix} =$$

$$-\nu(\bar{d}_Rd_L - \bar{d}_Ld_R - \bar{d}'_Rd'_L + \bar{d}'_Ld'_R) - \bar{\nu}(\bar{u}_Ru_L - \bar{u}_Lu_R - \bar{u}'_Ru'_L + \bar{u}'_Lu'_R)$$  \hfill (5.17)

the lepton part of the action is instead

$$\bar{\mathcal{L}}_\gamma_5[\mathcal{L}, \phi] = -\bar{\nu}(-\bar{e}_Le_R + \bar{e}'_Le'_R + \bar{\epsilon}_Le_R - \bar{\epsilon}_Le'_R) - \nu(-\bar{e}_Re'_L + \bar{e}'_Re_L - \bar{\epsilon}_Le'_R + \bar{\epsilon}_Le_R)$$  \hfill (5.18)

Note that with choice (5.16) the leptons of the $\bar{l}$ doublet do not have mass
terms, but have spurious coupling to the ordinary leptons. It is possible
to set them to zero in this scheme, but then we get that the mass of the
electron is the same as the one of the up quark. Note also that if we relax
the reality requirement on the $\nu$'s then the coefficients of $\bar{e}_Le_R$ and $\bar{e}'_Le'_L$ are
complex conjugate of each other, and the same will hold for quarks. There is
no problem in setting the primed fermions to zero, the hermitean conjugated
appear naturally because of $\bar{\Psi}$, and if we set $\bar{l} = 0$ then there also is no
problem for Majorana spinors. Mirror fermions have again no problems,
except that it is still not clear the mechanism to give then large mass. It is
still too early for a complete analysis of the various choices for the couplings
since we are not yet at the stage to be building a completely realistic model.
For example there are no generations, nor different couplings for the different
gauge groups, and this points to the necessity of the refinement of the model.

5.2 Electroweak Breaking by fuzzy sphere

Consider the fuzzy sphere breaking at high energies (at the GUT scale, say)
described in section 3.2 and in particular the stack of fuzzy sphere breaking
described in (3.16). The residual gauge symmetry in this case is $SU(3) \times
SU(2) \times U(1)_Q \times U(1)_Y \times U(1)_B$. We will later discuss the splitting into
$SU(4) \times SU(3) \times U(1)$ corresponding to the 4 × 4 lepton block the 3 × 3 plus color block.

As in section 3.2, we assume that there are additional quadratic and cubic terms as in (3.8) in the effective potential

$$V_H(X^{(i)}_\phi) = \text{Tr} \left( -\frac{1}{g^2} [X^{(i)}_\phi, X^{(j)}_\phi]^2 + c_2 X^{(i)}_\phi X^{(j)}_\phi \delta^{ij} + ic_3 \varepsilon^{ijk} X^{(i)}_\phi X^{(j)}_\phi X^{(k)}_\phi \right)$$

at the electroweak scale. A possible minimum of this potential is given by the following fuzzy sphere $S^2_{EW-1}$:

$$
\langle X^{(i)}_\phi \rangle = \alpha_H \begin{bmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{bmatrix} = \mathbb{1} \otimes \sigma_3,
$$

$$
\langle X^{(2)}_\phi \rangle = \alpha_H \begin{bmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix} = \sigma_1 \otimes \sigma_2,
$$

$$
\langle X^{(3)}_\phi \rangle = \alpha_H \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix} = \sigma_1 \otimes \sigma_1,
$$

This breaks the symmetry $SU(3) \times SU(2) \times U(1)_Q \times U(1)_Y \times U(1)_B$ of (3.16) down to $SU(3) \times U(1)_Q \times U(1)_B$, and we have indeed achieved the desired electroweak symmetry breaking. Observe that $\langle X^{(2)}_\phi \rangle$ and $\langle X^{(3)}_\phi \rangle$ are very similar to the two Higgs doublets $H, \tilde{H}$ in the MSSM, with an additional 3rd Higgs $\langle X^{(1)}_\phi \rangle$ in the diagonal blocks. Since the off-diagonal blocks are assumed to have definite chirality as in the standard model, this diagonal Higgs does not contribute to the Yukawa couplings. However, it does contribute to the mass of the $W^\pm$ and $Z$ bosons. This will be discussed below.

Alternatively, if we start from a vacuum

$$
\langle \Phi_1 \rangle = \begin{pmatrix} a_1 J^i_{N_1} \otimes \mathbb{1}_4 & 0 \\ 0 & a_3 J^i_{N_3} \otimes \mathbb{1}_3 \end{pmatrix},
$$

with $SU(4) \times SU(3) \times U(1)$ symmetry, then the single fuzzy sphere (5.20) is

**Here we indicate only the relevant 4 × 4 block in square brackets and drop the color blocks.**
not sufficient, since it commutes with the generators $\mathcal{X}_i^\phi = Q\mathcal{X}_i^\phi$,

\[
\begin{align*}
\langle \mathcal{X}_1^\phi \rangle &= \alpha_H' \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \sigma_3 \otimes \mathbb{1}, \\
\langle \mathcal{X}_2^\phi \rangle &= \alpha_H' \begin{bmatrix} 0 & -i \sigma_1 \\ i \sigma_1 & 0 \end{bmatrix} = \sigma_2 \otimes \sigma_1, \\
\langle \mathcal{X}_3^\phi \rangle &= \alpha_H' \begin{bmatrix} 0 & i \sigma_2 \\ -i \sigma_2 & 0 \end{bmatrix} = -\sigma_2 \otimes \sigma_2; 
\end{align*}
\tag{5.22}
\]

The above six matrices close a $SO(4)$ Lie Algebra

\[
\begin{align*}
[\langle \mathcal{X}_i^\phi \rangle, \langle \mathcal{X}_j^\phi \rangle] &= -2i\varepsilon_{ijk}\alpha_H\langle \mathcal{X}_k^\phi \rangle, \\
[\langle \mathcal{X}_i^\phi \rangle, \langle \mathcal{X}_j^\phi \rangle] &= 2i\varepsilon_{ijk}\alpha'_H\langle \mathcal{X}_k^\phi \rangle, \\
[\langle \mathcal{X}_i^\phi \rangle, \langle \mathcal{X}_j^\phi \rangle] &= -2i\varepsilon_{ijk}\frac{\alpha_H^2}{\alpha_H}\langle \mathcal{X}_k^\phi \rangle, \\
\frac{1}{\alpha_H^2}\langle \mathcal{X}_i^\phi \rangle\langle \mathcal{X}_i^\phi \rangle &= \frac{1}{\alpha'_H^2}\langle \mathcal{X}_i^\phi \rangle\langle \mathcal{X}_i^\phi \rangle = \mathbb{1}_{4 \times 4} 
\end{align*}
\tag{5.23}
\]

The two commuting $SO(3)$ algebras are then

\[
\langle \mathcal{X}_i^\pm \rangle = \frac{1}{2\alpha_H}\langle \mathcal{X}_i^\phi \rangle \pm \frac{1}{2\alpha'_H}\langle \mathcal{X}_i^\phi \rangle 
\tag{5.24}
\]

and they represent two fuzzy spheres which commute with $Q$ (and of course the identity).

Thus in that case we can achieve the desired symmetry breaking down to $SU(3) \times U(1)_Q \times U(1)_B$ using these two fuzzy spheres. There may be important differences between these scenarios depending on the energy scales of these spheres, and more detailed work is required before claiming any direct phenomenological relevance. In any case, our main point is that it seems feasible to obtain a (near-) realistic extension of the standard model by these or similar mechanisms. The essential ingredients, notably the stacks of various fuzzy spheres, are similar to string-theoretical constructions of (extensions of the) standard-model using branes in extra dimensions. Similar ideas are also used in \cite{25, 31}.

The coordinates of the fuzzy spheres couple with the fermions in the action (2.15) via the term $\bar{\Psi}\Gamma_i[\Phi^i, \bar{\Psi}]$, with the $\Phi^i$'s proportional to the $\mathcal{X}$'s.
as in (2.11) and the $\gamma$'s of the internal dimensions represented as diagonal matrices in the $7 \times 7$ gauge matrix space. The Yukawa couplings for the fuzzy sphere (5.20) are then (omitting the proportional factor $\alpha_H$):

$$
\begin{align*}
\text{tr } \overline{\Psi}[x_1^\phi, \Psi] &= -d_L d_L + d_R d_R + 2\bar{e} e_R + \bar{u} u_L - \bar{u} R u_R + 2\bar{\nu} \nu - 2\bar{\nu} \nu' \\
&\quad + d_L d_L' - d_R d_R' - \bar{e} e_R' + \bar{u} u_R' - \bar{u} R u_R' - 2\bar{\nu} \nu' + 2\bar{\nu} \nu'
\text{tr } \overline{\Psi}[x_2^\phi, \Psi] &= -i\bar{d} R d_L + i\bar{d} L d_R + i\bar{e} e_L + i\bar{u} R u_L - i\bar{u} u_R \\
&\quad - i\bar{e} L e_R - i\bar{d} R d_L + i\bar{d} L d_R + i\bar{e} R e_L - i\bar{e} R e_R - i\bar{e} R e_L + i\bar{e} R e_L' \\
&\quad -i\bar{e} L e_R' + i\bar{e} L e_R' + i\bar{u} R u_L - i\bar{u} L u_R' \\
\text{tr } \overline{\Psi}[x_3^\phi, \Psi] &= \bar{d} R d_L + \bar{d} L d_R + \bar{e} e_R + \bar{u} R u_L + \bar{u} L u_R \\
&\quad - \bar{e} L e_R - \bar{d} R d_L - \bar{d} L d_R + \bar{e} R e_L - \bar{e} R e_R + \bar{e} R e_L - \bar{e} R e_L' \\
&\quad - \bar{e} L e_R' - \bar{e} L e_R' - \bar{u} R u_L' - i\bar{u} L u_R'
\end{align*}
$$

while if one considers the pair of spheres (5.23) (setting $\alpha_H = \alpha_H' = 1$ for simplicity) one obtains:

$$
\begin{align*}
\text{tr } \overline{\Psi}[x_1^{+\phi}, \Psi] &= \bar{v} L e_L + \bar{e} L e_L + \bar{e} R e_R + \bar{u} L u_L - \bar{u} R u_R + 2\bar{\nu} \nu \\
&\quad - \bar{e} L e_L' - \bar{e} L e_L' - \bar{e} R e_R' - \bar{u} R u_R' - 2\bar{\nu} \nu' \\
\text{tr } \overline{\Psi}[x_2^{+\phi}, \Psi] &= i\bar{u} R u_R - i\bar{u} L u_R - i\bar{e} L e_R - i\bar{e} R e_L + i\bar{e} R e_L' - i\bar{e} L e_R' + i\bar{u} R u_R' - i\bar{u} L u_R' \\
\text{tr } \overline{\Psi}[x_3^{+\phi}, \Psi] &= \bar{u} R u_R + \bar{u} L u_R - \bar{e} R e_R + \bar{e} R e_R' - \bar{e} L e_R + \bar{e} L e_R' - \bar{u} R u_R' - \bar{u} L u_R' \\
\text{tr } \overline{\Psi}[x_1^{-\phi}, \Psi] &= -\bar{d} L d_L + \bar{d} R d_R - \bar{e} L e_L - \bar{e} L e_L + \bar{e} R e_R - 2\bar{\nu} \nu + \\
&\quad \bar{d} L d_L' - \bar{d} R d_R' + \bar{e} L e_L' + \bar{e} L e_L' - \bar{e} R e_R' - \bar{e} R e_R' + 2\bar{\nu} \nu' \\
\text{tr } \overline{\Psi}[x_2^{-\phi}, \Psi] &= -i\bar{d} R d_L + i\bar{d} L d_R - i\bar{e} R e_R + i\bar{e} L e_R - i\bar{d} R d_L' + i\bar{d} L d_R' - i\bar{e} R e_L + i\bar{e} L e_R' \\
\text{tr } \overline{\Psi}[x_3^{-\phi}, \Psi] &= \bar{d} R d_L + \bar{d} L d_R + \bar{e} R e_R + \bar{e} L e_L - \bar{d} R d_L' - \bar{d} L d_R' - \bar{e} R e_L' - \bar{e} L e_R' \hspace{1em} (5.26)
\end{align*}
$$

The couplings which appear are all "reasonable", meaning that they are either Majorana or Dirac masses, or coupling among the primed particles or the spurious leptons. Setting all of these to zero we obtain:

$$
\begin{align*}
\text{tr } \overline{\Psi}[x_1^{+\phi}, \Psi] &= \bar{v} L e_L + \bar{e} R e_R + \bar{u} L u_L - \bar{u} R u_R + 2\bar{\nu} \nu \\
\text{tr } \overline{\Psi}[x_2^{+\phi}, \Psi] &= i\bar{u} R u_R - i\bar{u} L u_R \\
\text{tr } \overline{\Psi}[x_3^{+\phi}, \Psi] &= \bar{u} R u_R + \bar{u} L u_R \\
\text{tr } \overline{\Psi}[x_1^{-\phi}, \Psi] &= -\bar{d} L d_L + \bar{d} R d_R - \bar{e} L e_L + \bar{e} R e_R \\
\text{tr } \overline{\Psi}[x_2^{-\phi}, \Psi] &= -i\bar{d} R d_L + i\bar{d} L d_R - i\bar{e} R e_R + i\bar{e} L e_R \\
\text{tr } \overline{\Psi}[x_3^{-\phi}, \Psi] &= \bar{d} R d_L + \bar{d} L d_R + \bar{e} R e_R + \bar{e} L e_L \hspace{1em} (5.27)
\end{align*}
$$
These are the couplings of the standard model in the absence of right handed neutrinos. Some of these terms may vanish depending on the specific chirality assignment, as discussed before. As it is the model does not allow for different masses, apart from some freedom afforded by the tuning of $\alpha_H$ and $\alpha_\prime_H$.

Note that the remaining eight generators of $Mat(4,\mathbb{C})$ do not commute with these two fuzzy spheres, thus the gauge symmetry is indeed broken to $Q$ and the generators of colour and baryon number. According to what we explained in the previous sections this implies that they are massive.

It is worthwhile to elaborate in some detail the explicit form of the low-energy electroweak Higgs. Consider first the vacuum without fluctuations. Using

$$
\langle \mathbf{x}_i^\phi \rangle \langle \mathbf{x}_j^\phi \rangle \delta_{ij} = \alpha_H^2,
\varepsilon^{ijk} \langle \mathbf{x}_i^\phi \rangle \langle \mathbf{x}_j^\phi \rangle \langle \mathbf{x}_k^\phi \rangle = -2i\alpha_\prime_H^3
$$

so that the effective potential for $\alpha_H$ becomes

$$
V_H(\langle \mathbf{x}_i^\phi \rangle) = \text{Tr} \left( \frac{4}{g^2} \alpha_H^4 + c_2 \alpha_H^2 + 2c_3 \alpha_\prime_H^3 \right).
$$

(5.29)

This has a non-trivial minimum in $\alpha_H \neq 0$ provided $c_3 \neq 0$ or $c_2 < 0$, and the sign of $\alpha_H$ depends on the sign of $c_3$. Note that these terms have a geometrical interpretation in terms of field strength on $S^2_N$, leading to some protection from quantum corrections [25].

The VEV’s of $\mathbf{x}_2^\phi$ and $\mathbf{x}_3^\phi$ contains the expected degrees of freedom of the two Higgs doublets $\phi$ as in (5.10), parametrizing one complex scalar. This is as in the MSSM, however the two doublet are related to each other. They become independent in the presence of the second fuzzy sphere (5.22). The VEV of $\mathbf{x}_1^\phi$ contains scalar degrees of freedom which are in the adjoint of the electroweak $SU(2)$, with the same VEV. This is different from the standard model and should have observable signatures. We therefore obtain an interesting geometrical interpretation of these scalar Higgs fields.

Now consider fluctuations around this vacuum. Again, these fluctuations contain fluctuations of the two Higgs doublets $\phi$ as in (5.10), and also fluctuations of $\mathbf{x}_1^\phi$ which is in the adjoint of the electroweak $SU(2)$. More generally, fluctuations on the fuzzy sphere can be interpreted as scalar (resp. gauge fields) on $S^2_N$. 

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Vector boson masses. As explained in section 3.2, the vector bosons corresponding to the $4 \times 4$ leptonic block will acquire particular mass terms in the presence of these electroweak fuzzy spheres. For example, $X_1^\phi$ will contribute a mass proportional to $\alpha_H^2$ to the $W$ bosons.

Furthermore, note that $X_1^\phi$ breaks $SU(4)$ into $SU(2) \times SU(2)$, which seems quite appealing; this suggests that $\alpha_H'$ should have higher scale than $\alpha_H$, on the other hand then $X_2^\phi$ lead to EW symmetry breaking which is strange. This suggests some interplay between the two spheres.

6 Conclusions and Outlook

In this paper we have shown how the matrix model which gives rise to non-commutative spaces and emergent gravity can also accommodate a gauge theory with the features of the standard model. We have seen that a simple solution with extra (non-propagating) dimensions contains all the necessary fundamental fermionic degrees of freedom, with a few extra particles which can be set to zero without prejudice to the model. Also the basic gauge symmetries can be accommodated and, with the use of extra dimensions, the pattern of symmetry breakings can be substantially reproduced. The breaking happen in two stages, first some sort of grand-unification breaking, and then the electroweak breaking. Both stages can be accomplished either with the presence of a simple (effective) extra matrix dimension, or with the use of fuzzy spheres. The former mechanism can be considered an effective version of the latter.

There are several gaps in the constructions, and several lines of developments which hopefully can fill the gaps. The list of shortcomings includes the fact that there are some extra $U(1)$ symmetries, the lack of generations, and the fact that couplings do not differentiate between fermions and bosons. Clearly the solutions discussed here are not phenomenologically viable, but we find rather inspiring the fact that we have a semi-realistic matrix model from which gravity and gauge theories of a realistic kind emerge naturally. Among the lines of development there is the possibility to have a supersymmetric version of the model. This type of matrix model is in fact well suited for supersymmetric extensions, the most notable example being the IKKT model [6]. Moreover, the introduction of additional geometrical structures such as intersecting branes and orbifolds [31] is likely to resolve at least
some of these problems, in particular the issue of chirality. Another line of development is a better understanding of the connections among the extra dimensions in the guise of fuzzy spheres and the results obtained in string and brane theory. This supports the hope that the framework of matrix models might be suitable to approach the goal of a consistent quantum theory of fundamental interactions including gravity.

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