Notes on “Some Properties of $L$-fuzzy Approximation Spaces on Bounded Integral Residuated Lattices”

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Abstract: In this note, we continue the works in the paper [Some properties of $L$-fuzzy approximation spaces on bounded integral residuated lattices”, Information Sciences, 278, 110-126, 2014]. For a complete involutive residuated lattice, we show that the $L$-fuzzy topologies generated by a reflexive and transitive $L$-relation satisfy (TC)$^L$ or (TC)$^R$ axioms and the $L$-relations induced by two $L$-fuzzy topologies, which are generated by a reflexive and transitive $L$-relation, are all the original $L$-relation; and give out some conditions such that the $L$-fuzzy topologies generated by two $L$-relations, which are induced by an $L$-fuzzy topology, are all the original $L$-fuzzy topology.

Keywords: Involutive Residuated Lattice, $L$-relation, $L$-fuzzy Topology, $L$-fuzzy Approximation Space

1. Introduction

A residuated lattice (see [1, 10]) is an algebra $L=(L, \wedge, \vee, \cdot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 2, 0, 0)$ satisfying the following conditions:

(L1) $(L, \wedge, \vee)$ is a lattice,

(L2) $(L, \cdot, 1)$ is a monoid, i.e., is associative and $x \cdot 1 = 1 \cdot x = x$ for any $x \in L$,

(L3) $x \cdot y \leq z$ if and only if $x \leq y \rightarrow z$ if and only if $y \leq x \leftarrow z$ for any $x, y, z \in L$.

A residuated lattice with a constant 0 is called an FL-algebra. If $x \leq 1$ for all $x \in L$, then L is called integral residuated lattice. An FL-algebra $L$, which satisfies the condition $0 \leq x \leq 1$ for all $x \in L$, is called an FL$_{leq}$-algebra or a bounded integral residuated lattice (see [1]).

We adopt the usual convention of representing the monoid operation by juxtaposition, writing $ab$ for $a \cdot b$.

Let $L$ be a bounded integral residuated lattice. Define two negations on $L$, $\neg^t$ and $\neg^r$:

$\neg^t x = x \rightarrow 0, \quad \neg^r x = x \leftarrow 0 \quad \forall x \in L$.

A bounded residuated lattice $L$ is called an involutive residuated lattice (see [3]) if

$\neg^t \neg^r x = \neg^r \neg^t x \quad \forall x \in L$.

In the sequel, unless otherwise stated, $L$ always represents any given complete involutive residuated lattice with maximal element 1 and minimal element 0.

Definition 1.1 (see Liu and Luo [5]). Let $\tau \subseteq L^J$ and $J$ be an index set. If $\tau$ satisfies the following three conditions:

(LFT1) $0_x, 1_x \in \tau$,

(LFT2) $\mu, \nu \in \tau \Rightarrow \mu \land \nu \in \tau$,

(LFT3) $\mu_j \in \tau (j \in J) \Rightarrow \mu_{\mu_j} \in \tau$,

then $\tau$ is called an $L$-fuzzy topology on $X$ and $(L^J, \tau)$ L-fuzzy topological space. Every element in $\tau$ is called an open subset in $L^J$.

When $L=[0,1]$, an $L$-fuzzy topological space $(L^J, \tau)$ is
also called an $F$-topological space.

Let $\tau_L = \{ \mu \mid \mu \in \tau \}$ and $\tau_R = \{ \mu \mid \mu \in \tau \}$. The elements of $\tau_L$ and $\tau_R$ are, respectively, called left closed subsets and right closed subsets in $L^X$ (see Wang et al. [12]).

**Definition 1.2** (Wang and Liu [11], Wang et al. [12]). Let $\tau$ be an $L$-fuzzy topology on $X$ and $\mu$ an $L$-fuzzy subset of $X$. The interior, left closure and right closure of $\mu$ w.r.t $\tau$ are, respectively, defined by

$$\text{int}(\mu) = \{ \eta \mid \eta \leq \mu, \eta \in \tau \},$$

$$\text{cl}_L(\mu) = \{ \xi \mid \mu \leq \xi, \xi \in \tau_L \},$$

$$\text{cl}_R(\mu) = \{ \xi \mid \mu \leq \xi, \xi \in \tau_R \}.$$ 

The interior, left closure and right closure operators, respectively, called the interior, left closure and right closure operators.

For the sake of convenience, we denote $\text{int}(\mu)$, $\text{cl}_L(\mu)$ and $\text{cl}_R(\mu)$ by $\mu^\tau$, $\mu^{\leq}$ and $\mu^{\geq}$, respectively.

Zhang et al. [14, 15] studied some properties of rough sets and rough approximation operators. Ouyang et al. [6, 7] investigated some generalized models of fuzzy rough sets, Liu and Lin [4] considered the intuitionistic fuzzy rough set model, Wu et al. [13] discussed the axiomatic characterizations of fuzzy rough approximation operators, Radzikowska and Kerre [9] studied $L$-fuzzy rough sets and lower (upper) $L$-fuzzy approximation based on commutative residuated lattices. Recently, Wang et al. [12] discussed the notion of left (right) lower and left (right) upper $L$-fuzzy approximation based on complete bounded integral residuated lattices.

**Definition 1.3** (Wang et al. [12]). Let $R$ be an $L$-relation on $X$. A pair $(X, R)$ is called an $L$-fuzzy approximation space. Define the following four mappings $R \downarrow_L$, $R \uparrow_L$, $R \downarrow_R$, $R \uparrow_R$ : $L^X \rightarrow L^X$, called a left lower, left upper, right lower, and right upper $L$-fuzzy rough approximation operators, respectively, as follows: for every $\mu \in L^X$ and $x \in X$,

$$R \downarrow_L(\mu)(x) = \land_{x \in X} (R(x, y) \rightarrow \mu(y)), $$

$$R \uparrow_L(\mu)(x) = \lor_{x \in X} \mu(y)R(y, x),$$

$$R \downarrow_R(\mu)(x) = \land_{x \in X} (R(y, x) \leftarrow \mu(y)), $$

$$R \uparrow_R(\mu)(x) = \lor_{x \in X} \mu(y)R(x, y).$$

$R \downarrow_L(\mu)$, $R \uparrow_L(\mu)$, $R \downarrow_R(\mu)$ and $R \uparrow_R(\mu)$ are called left lower, left upper, right lower, and right upper $L$-fuzzy rough approximations of $\mu$, respectively.

A pair $(\lambda, \xi) \in L^X \times L^X$ such that $\lambda = R \downarrow_L(\mu)$ ($\lambda = R \downarrow_R(\mu)$) and $\xi = R \uparrow_L(\mu)$ ($\xi = R \uparrow_R(\mu)$) for some $\mu \in L^X$, is called a left (right) $L$-fuzzy rough set in $(X, R)$.

When $L = [0, 1]$, $L$-fuzzy rough approximation operators, $L$-fuzzy approximation space and left (right) $L$-fuzzy rough sets are, respectively, called fuzzy rough approximation operators, fuzzy approximation space and left (right) fuzzy rough sets.

### 2. The $L$-fuzzy Topologies Generated by a Reflexive and Transitive $L$-relation

In this section, we supplement some properties of the $L$-fuzzy topologies generated by a reflexive and transitive $L$-relation.

If $R$ is a reflexive and transitive $L$-relation on $X$, then it follows from Theorem 6.1 in [12] that

$$\tau_1 = \{ \xi \mid R \downarrow_L(\xi) = \xi, \xi \in L^X \},$$

$$\tau_2 = \{ \xi \mid R \downarrow_R(\xi) = \xi, \xi \in L^X \},$$

are all $L$-fuzzy topologies on $X$ and $R \downarrow_L$ and $R \downarrow_R$ are just the interior operators w.r.t $\tau_1$ and $\tau_2$, respectively.

Here, $\tau_1$ and $\tau_2$ are called the $L$-fuzzy topologies generated by the $L$-relation $R$ or by left lower $L$-fuzzy rough approximation operator $R \downarrow_L$ and right lower $L$-fuzzy rough approximation operator $R \downarrow_R$, respectively.

**Theorem 2.1.** If $R$ is a reflexive and transitive $L$-relation on $X$, then

$$\tau_1 = \{ \xi \mid R \downarrow_L(\xi) = \xi, \xi \in L^X \},$$

$$\tau_2 = \{ \xi \mid R \downarrow_R(\xi) = \xi, \xi \in L^X \},$$

$R \uparrow_L$ and $R \uparrow_R$ are, respectively, the right closure operator w.r.t $\tau_1$ and the left closure operator w.r.t $\tau_2$.

Proof. When $L$ is an involutive residuated lattice, $L^X$ is called an $F$-topological space.

If $R$ is a reflexive and transitive $L$-relation on $X$, then it follows from Theorems 4.1(5) and Remark 5.2 in [12] that

$$R \downarrow_L(\nabla^\tau R \uparrow_R(\xi)) = R \downarrow_L(\nabla^\tau R \uparrow_R(\xi)),$$

i.e., $\nabla^\tau R \uparrow_R(\xi) \in \tau_1$ for any $\xi \in L^X$. If $\xi \in \tau_1$ and $\mu \in L^X$, then it follows from Theorems 3.1(3) and 4.1(5) in [12] that

$$\xi = R \downarrow_L(\xi) = R \downarrow_L(\nabla^\tau R \uparrow_R(\eta))$$

$$= R \downarrow_L(\nabla^\tau R \uparrow_R(\eta)) \forall \eta \in L^X,$$

$$\xi = R \uparrow_R(\xi) \in \tau_2$$

$$\nabla^\tau R \uparrow_R(\eta) \in \tau_2$$

$$= R \uparrow_R(\eta) \in \tau_2.$$
where $\eta = -^\delta \xi$. So, $\tau_1 = \{ -^\delta R \uparrow_{L_h} (\eta) | \eta \in L^X \}$ and $R \uparrow_{L_h}$ is the right closure operator w.r.t. $\tau_2$.

Similarly, we can show that $\tau_2 = \{ -^\delta R \uparrow_{L_h} (\eta) | \eta \in L^X \}$ and $R \uparrow_{L_h}$ is the left closure operator w.r.t. $\tau_2$.

The theorem is proved.

Recently, Qin et al. [2, 8] studied the topological properties of fuzzy rough sets. The following left and right (TC) axioms are generalizations of (TC) axiom in [8].

(TC)$_L$ axiom: for any $x, y \in X$ and $\mu \in \tau$ there exists $\mu' \in \tau$ such that $\mu' (x) = 0$ and $\mu' (y) \rightarrow \mu' (x) \leq \mu (y) \rightarrow \mu (x)$.

(TC)$_R$ axiom: for any $x, y \in X$ and $\nu \in \tau$ there exists $\nu' \in \tau$ such that $\nu' (y) = 0$ and $\nu' (x) \leftarrow \nu' (y) \leq \nu (x) \leftarrow \nu (y)$.

Theorem 2.2. If $R$ is a reflexive and transitive L-relation on $X$, then the L-fuzzy topologies $\tau_1$ and $\tau_2$, generated by $R$, satisfy (TC)$_R$ and (TC)$_L$ axioms, respectively.

Proof. For any $x, y \in X$ and $\mu \in \tau_1$, let $\mu' = -^\delta \left( R^\delta \uparrow_{L} (1_{(x,y)}) \right)$, then

$$\mu' (y) = -^\delta \left( R^\delta \uparrow_{L} (1_{(y,x)}) \right)(y) = -^\delta R^\delta (y, y) = -^\delta 1 = 0,$$

$$\mu' (x) \leftarrow \mu' (y) = -^\delta \left( R^\delta \uparrow_{L} (1_{(y,x)}) \right)(x) \leftarrow 0 = -^\delta -^\delta R^\delta (x, y) = R^\delta (x, y) = \land_{x,y} \langle \xi (x) \leftarrow \xi (y) \rangle \leq \mu (x) \leftarrow \mu (y),$$

i.e., $\tau_1$ satisfies (TC)$_R$ axiom; for any $\nu \in \tau_2$, let $\nu' = -^\delta \left( R^\delta \uparrow_{L} (1_{(x,y)}) \right)$,

Then

$$\nu' (x) = -^\delta \left( R^\delta \uparrow_{L} (1_{(y,x)}) \right)(x) = -^\delta R^\delta (x, x) = -^\delta 1 = 0,$$

$$\nu' (y) \rightarrow \nu' (x) = -^\delta \left( R^\delta \uparrow_{L} (1_{(y,x)}) \right)(y) \rightarrow 0 = -^\delta -^\delta R^\delta (y, x) = R^\delta (y, x) = \land_{x,y} \langle \xi (y) \rightarrow \xi (x) \rangle \leq \nu (y) \rightarrow \nu (x),$$

i.e., $\tau_2$ satisfies (TC)$_L$ axiom.

The theorem is proved.

3. The L-relations Induced by an L-fuzzy Topology

In this section, we supplement some properties of the L-relations induced by an L-fuzzy topology.

Let $\tau$ be an L-fuzzy topology on $X$. For any $x, y \in X$, let

$$R^{\delta}_{x,y} (x, y) = \land_{\mu \in \tau} \langle \mu (x) \rightarrow \mu (y) \rangle,$$

Clearly, $R^{\delta}_{x,y}$ and $R^{\delta}_{y,x}$ are reflexive L-relations on $X$. Moreover, it follows from Theorem 2.1(5) in [12] that

$$R^{\delta}_{x,y} (x, y) R^{\delta}_{y,z} (y, z) = \land_{\mu \in \tau} \langle \mu (x) \rightarrow \mu (y) \rangle \land \langle \mu (y) \rightarrow \mu (z) \rangle \forall x, y, z \in X,$$

Thus, $R^{\delta}_{x,y}$ and $R^{\delta}_{y,x}$ are all transitive L-relations on $X$. Let

$$R_{x,y} = R^{\delta}_{x,y} \land R^{\delta}_{y,x} (x, y) = \land_{\mu \in \tau} \langle \mu (x) \rightarrow \mu (y) \rangle \land \langle \mu (y) \rightarrow \mu (x) \rangle \forall x, y \in X.$$

It is easy to see that $R_1 = R^{\delta}_{1} \land R^{\delta}_{1}$ is also a reflexive and transitive L-relations on $X$.

Theorem 3.1. If $R$ is a reflexive and transitive L-relation on $X$, then

$$R = R^{\delta}_{x,y} = R^{\delta}_{y,x}.$$

Proof. For any $x, y \in X$, by virtue of Definitions 1.2 and 1.3 and Theorem 2.1, we see that

$$R(x, y) = R \uparrow_{L} (1_{(x,y)}) = \{ (x, y) | x \in X \} = \land_{\xi \in \tau} \langle \xi (x) \leq \xi (y) \rangle \land \langle \xi (y) \leq \xi (x) \rangle \forall x, y \in X,$$

$$R(x, y) = R \uparrow_{L} (1_{(y,x)}) = \{ (y, x) | y \in X \} = \land_{\xi \in \tau} \langle \xi (y) \leq \xi (x) \rangle \land \langle \xi (x) \leq \xi (y) \rangle \forall x, y \in X,$$

$$R(x, y) = R \uparrow_{L} (1_{(x,y)}) = \{ (x, y) | x \in X \} = \land_{\xi \in \tau} \langle \xi (x) \leq \xi (y) \rangle \land \langle \xi (y) \leq \xi (x) \rangle \forall x, y \in X,$$

$$R(x, y) = R \uparrow_{L} (1_{(y,x)}) = \{ (y, x) | y \in X \} = \land_{\xi \in \tau} \langle \xi (y) \leq \xi (x) \rangle \land \langle \xi (x) \leq \xi (y) \rangle \forall x, y \in X,$$

$$R(x, y) = R \uparrow_{L} (1_{(x,y)}) = \{ (x, y) | x \in X \} = \land_{\xi \in \tau} \langle \xi (x) \leq \xi (y) \rangle \land \langle \xi (y) \leq \xi (x) \rangle \forall x, y \in X,$$

$$R(x, y) = R \uparrow_{L} (1_{(y,x)}) = \{ (y, x) | y \in X \} = \land_{\xi \in \tau} \langle \xi (y) \leq \xi (x) \rangle \land \langle \xi (x) \leq \xi (y) \rangle \forall x, y \in X.$$
Thus, \( R \geq R^* \) and \( R \geq R^\# \).

On the other hand, \( R \downarrow_1 \) and \( R \uparrow_1 \) are, respectively, the interior and right closure operators w.r.t. \( \tau_1 \) and \( R \downarrow_2 \) and \( R \uparrow_2 \) are, respectively, the interior and left closure operators w.r.t. \( \tau_2 \). Thus, by virtue Theorem 3.1(3) and Remark 5.2 in [12], we can see that
\[
R \uparrow_2 \{ \neg \rho (R \downarrow_2 (\mu)) \} = \neg \rho \{ R \downarrow_1 (\mu) \}_1,
\]
\[
= \neg \rho \{ \neg \rho (R \downarrow_2 (\mu))) \} = \neg \rho \{ R \downarrow_1 (\mu) \}_2 = \neg \rho \{ R \downarrow_2 (\mu) \}_2
\]
\[
= \neg \rho \{ R \downarrow_1 (\mu) \} \forall R \downarrow_2 (\mu) \in \tau_1,
\]
\[
R \uparrow_2 \{ \neg \rho (R \downarrow_2 (\mu)) \} = \neg \rho \{ R \downarrow_1 (\mu) \}_2 = \neg \rho \{ R \downarrow_2 (\mu) \}_2
\]
\[
= \neg \rho \{ R \downarrow_1 (\mu) \} \forall R \downarrow_2 (\mu) \in \tau_2.
\]

So, it follows from the proof of Theorem 7.2 in [12] that \( R \leq R^* \) and \( R \leq R^\# \).

Therefore, \( R = R^* = R^\# \).

The theorem is proved.

This result shows that the reflexive and transitive \( R^* \) and \( R^\# \) induced by, respectively, the \( L \)-fuzzy topologies \( \tau_1 \) and \( \tau_2 \) are all the original reflexive and transitive \( L \)-relation.

For any \( \mu \in L^X \) and \( R \in L^{X \times X} \),
\[
\mu = \nu_{ax} (\mu(x) \wedge 1_{\{i\}}).
\]

Thus, by Definition 1.3 and Theorem 4.1(3) in [12], we see that
\[
R \uparrow_1 (\mu) = \nu_{ax} R \uparrow_1 (\mu(x) \wedge 1_{\{1\}})
\]
\[
= \nu_{ax} \mu(x) \wedge \uparrow_1 (1_{\{1\}}),
\]
\[
= \nu_{ax} \mu(x) \wedge \uparrow_2 (1_{\{1\}}),
\]
\[
= \nu_{ax} \mu(x) \wedge \uparrow_2 (1_{\{1\}}) \cdot \mu(x).
\]

Theorem 3.2. Let \( \tau \) be an \( L \)-fuzzy topology on \( X \) and \( J \) index set. Then the following properties hold.

(1) If \( \tau \) satisfies (TC)\(_L\) axiom and the left closure operator w.r.t. \( \tau \) satisfies the following two conditions:

CL1) \( (\forall_{j \in J}) \mu^* \subseteq \nu_{ax} (\mu) \forall \mu \in L^X \),

CL2) \( (a \wedge 1_{\{1\}}) \subseteq \nu(1_{\{1\}}) \forall a \in L, x \in X \),

then \( R^L \uparrow_1 \) and \( R^L \downarrow_1 \) are, respectively, just the left closure operator and the interior operator w.r.t. \( \tau \) and
\[
\tau = \{ \xi | R^L \downarrow_2 (\xi) = \xi, \xi \in L^\mu \}.
\]

(2) If \( \tau \) satisfies (TC)\(_L\) and the right closure operator w.r.t. \( \tau \) satisfies the following two conditions:

CR1) \( (\forall_{j \in J}) \mu^* \subseteq \nu_{ax} (\mu) \forall \mu \in L^X \),

CR2) \( (a \wedge 1_{\{1\}}) \subseteq \nu(1_{\{1\}}) \forall a \in L, x \in X \),

then \( R^L \uparrow_1 \) and \( R^L \downarrow_2 \) are, respectively, just the right closure operator and the interior operator w.r.t. \( \tau \) and
\[
\tau = \{ \xi | R^L \downarrow_1 (\xi) = \xi, \xi \in L^\mu \}.
\]

Proof. We only prove (1).

If \( \tau \) satisfies (TC)\(_L\) axiom and the left closure operator w.r.t. \( \tau \) satisfies the conditions (CL1) and (CL2), then it follows from Definition 1.3 and the proof of Theorem 3.1 that
\[
R^L \uparrow_1 (1_{\{1\}}) (y) = (R^L \downarrow_1 (x, y)) \wedge \mu(x) \wedge \mu(y) = (1_{\{1\}}) (y) \forall x, y \in X
\]
\[
= (1_{\{1\}}) (y) \forall x, y \in X,
\]
\[
i.e., \ R^L \uparrow_1 (1_{\{1\}}) \text{ for any } x \in X . \text{ Thus, for any } \mu \in L^X, \text{ we have that}
\]
\[
\mu^* = \nu_{ax} (\mu(x) \wedge 1_{\{1\}}) \subseteq \nu_{ax} \mu(x) \wedge \mu(1_{\{1\}}) \subseteq \nu_{ax} \mu(x) \wedge 1_{\{1\}} \subseteq R^L \uparrow_1 (1_{\{1\}})
\]
\[
= \nu_{ax} \mu(x) \wedge \uparrow_1 (1_{\{1\}}) \subseteq \nu_{ax} \mu(x) \wedge \uparrow_2 (1_{\{1\}}) \subseteq \nu_{ax} \mu(x) \wedge \uparrow_2 (1_{\{1\}}) \cdot \mu(x)
\]
\[
= \nu_{ax} \mu(x) \wedge \uparrow_2 (1_{\{1\}}) \cdot \mu(x).
\]

Therefore,
\[
\tau = \{ \xi | R^L \downarrow_2 (\xi) = \xi, \xi \in L^\mu \}.
\]

The theorem is proved.

This result shows that the \( L \)-topologies generated by two reflexive and transitive \( L \)-relations \( R^L \) and \( R^L \), which are induced by an \( L \)-topology \( \tau \), are all the original \( L \)-topology \( \tau \) when \( \tau \) satisfies some conditions.

Moreover, if \( \tau \) satisfies (CL1) or (CR1), then it follows from Remark 2.1 and Theorem 3.1(2) in [12] that
\[
(\wedge_{j \in J} \mu)^* = \neg \rho (\wedge_{j \in J} \mu)_2 \subseteq \nu_{ax} (\mu) \forall \mu \in L^X
\]
\[
= \neg \rho (\wedge_{j \in J} \mu)_2 \subseteq \nu_{ax} \mu(x) \wedge \mu(1_{\{1\}}) \subseteq \nu_{ax} \mu(x) \wedge 1_{\{1\}} \subseteq R^L \uparrow_1 (1_{\{1\}})
\]
\[
= \nu_{ax} \mu(x) \wedge \uparrow_2 (1_{\{1\}}) \cdot \mu(x).
\]

Therefore,
\[
\tau = \{ \xi | R^L \downarrow_2 (\xi) = \xi, \xi \in L^\mu \}.
\]
i.e., the interior operator int of $\tau$ distributes over arbitrary intersection of $L$-fuzzy sets. Thus, the intersection of arbitrarily many open subsets is still an open subset.

### 4. Conclusions and Future Work

In this note, we continue the works in [12]. For a complete involutive residuated lattice, we have supplemented some properties of the $L$-fuzzy topologies generated by a reflexive and transitive $L$-relation; showed that the $L$-fuzzy topologies generated by a reflexive and transitive $L$-relation satisfy $(TC)_L$ or $(TC)_R$ axioms; and given out some conditions such that the $L$-fuzzy topologies generated by two $L$-relations, which are induced by an $L$-fuzzy topology, are all the original $L$-fuzzy topology.

In a forthcoming paper, we will discuss the relationships between the $L$-fuzzy topological spaces and the $L$-fuzzy rough approximation spaces on a complete involutive residuated lattice.

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