Monolithic Finite Element Methods for the simulation of thixo-viscoplastic flows

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6th ECCOMAS Young Investigators Conference YIC2021
7-9 July 2021, Valencia Spain
Motivation

Why “Thixotropic materials?"

- Processing of thixotropic materials relevant for industrial applications
  → Lubrication, asphalt, self-compacting concrete…

- Physically fascinating due to improved mechanical properties

Goal:

- Modern CFD methods with high accuracy, robustness and efficiency for thixotropic materials
  → Saving time, money and resources

Investigation of solid/liquid and liquid/solid transitions based on micro-structure
Introduction

- Thixotropy means
  - combination of two greek words
  - Thixis: shaking/stirring
  - trepo: turning/changing

- Thixotropy concept
  - Based on viscosity
  - Flow induced by time-dependent decrease of viscosity
  - The phenomena is reversible

- Rejuvenation / Breakdown
  - “Faster” flow: fluid rejuvenates
    Decreases of viscosity with acceleration of the flow

- Aging / Build-up
  - At rest or under slow flow: fluid ages
    Increases of the viscosity in time
Realization in FeatFlow

HPC features:
- Moderately parallel
- GPU computing
- Open source

Non-Newtonian flow module:
• generalized Newtonian model (Power-law, Carreau, Houska, …)
• viscoelastic differential model (Giesekus, FENE, Oldroyd, …)

Multiphase flow module (resolved interfaces):
• l/l – interface capturing (Level Set)
• s/l – interface tracking (FBM)
• s/l/l – combination of l/l and s/l

Numerical features:
• Higher order FEM in space & (semi-) Implicit FD/FEM in time
• Semi-(un)structured meshes with dynamic adaptive grid deformation
• Fictitious Boundary (FBM) methods
• Newton-Multigrid-type solvers

Hardware-oriented Numerics

Engineering aspects:
• Geometrical design
• Modulation strategy
• Optimization

Here: FEM-based tools for the accurate simulation of (thixotropic) flow problems, particularly with complex rheology

For details, please visit: www.featflow.de

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Starting point: Generalized Navier-Stokes equations (+initial and boundary conditions)

\[ \rho \left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \mathbf{u} - \nabla \cdot \mathbf{\sigma} + \nabla p = \rho f, \]

\[ \nabla \cdot \mathbf{u} = 0, \]

- velocity- and pressure field \( \mathbf{u} \) and \( p \)
- stress tensor \( \mathbf{\sigma} \)
- linear material behaviour - Newtonian fluids
  \[ \mathbf{\sigma} = 2\eta_s D(\mathbf{u}) \quad : \quad \eta_s \text{ is constant viscosity} \]
- non-linear material behaviour - structurally viscous / viscoplastic
  \[ \mathbf{\sigma} = 2\eta_s (D_\Pi, p, \Theta, \lambda) D(\mathbf{u}), \quad D_\Pi = \text{tr} \left( \frac{1}{2} D(\mathbf{u})^2 \right) \]
  - Power-law, Carreau, Bingham, Herschel-Bulkley, Houska, …
- structure parameter \( \lambda \)
Rheological Models

- **Archetypical thixotropic viscoplastic (TVP) models**

\[
\sigma = 2 \eta(D_\Pi, \lambda) D(u) + \sqrt{2} \tau(\lambda) \frac{D(u)}{\sqrt{D_\Pi}} \quad \text{if } D_\Pi \neq 0
\]
\[
\sigma_\Pi \leq \tau(\lambda) \quad \text{if } D_\Pi = 0
\]

- **Relations between rheological parameters and structural parameter**

|                  | \(\eta(D_\Pi, \lambda)\)         | \(\tau(\lambda)\) |
|------------------|-----------------------------------|---------------------|
| Worrall and Tulliani\(^1\) | \(\lambda \eta_0\)               | \(\tau_0\) |
| Coussot et al.\(^2\)      | \(\lambda^a \eta_0\)             | \(-\)              |
| Houska\(^3\)            | \((\eta_0 + \eta_1 \lambda) D_\Pi^{(n-1)}\) | \((\tau_0 + \tau_1 \lambda)\) |
| Mujumbar et al.\(^4\)    | \((\eta_0 + \eta_1 \lambda) D_\Pi^{(n-1)}\) | \(\lambda^{a+1} G_0 \Lambda_c^*\) |
| Burgos et al.\(^5\)      | \(\eta_0\)                       | \(\lambda \tau_0\) |
| Dullaert & Mewis\(^6\)   | \(\lambda \eta_0\)               | \(\lambda G_0 \left(\lambda D_\Pi^{\frac{1}{2}}\right) \Lambda_c^*\) |

\(\Lambda_c^*\) is a constant/variable elastic strain.

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Rheological Models

- **General format of evolution equation for structural parameter:**

  \[ \frac{\partial \lambda}{\partial t} + u \cdot \nabla \lambda = F_{\text{buildup}} - F_{\text{breakdown}} \]

- **Expressions for different thixotropic models:**

| Model                        | $F_{\text{buildup}}$                                      | $F_{\text{breakdown}}$ |
|------------------------------|----------------------------------------------------------|------------------------|
| Worrall and Tulliani         | $c_1 (1 - \lambda) D_{\text{II}}^{\frac{1}{2}}$          | $c_2 \lambda D_{\text{II}}^{\frac{1}{2}}$ |
| Coussot et al.               | $c_1$                                                    | $c_2 \lambda D_{\text{II}}^{\frac{1}{2}}$ |
| Houska                      | $c_1 (1 - \lambda)$                                      | $c_2 \lambda^m D_{\text{II}}^{\frac{1}{2}}$ |
| Mujumbar et al.             | $c_1 (1 - \lambda)$                                      | $c_2 \lambda D_{\text{II}}^{\frac{1}{2}}$ |
| Burgos et al.               | $c_1 (1 - \lambda)$                                      | $c_2 \lambda D_{\text{II}}^{\frac{1}{2}} \exp(aD_{\text{II}}^{\frac{1}{2}})$ |
| Dullaert & Mewis            | $(c_1 + c_3 D_{\text{II}}^{\frac{1}{2}})(1 - \lambda)t^{-b}$ | $c_2 \lambda D_{\text{II}}^{\frac{1}{2}} t^{-b}$ |
Viscoplastic (VP) flow

\[
\begin{cases}
\sigma = 2 \eta_0 D(u) + \sqrt{2} \tau_0 \frac{D(u)}{\sqrt{D}} & \text{if } D_{\Pi} \neq 0 \\
\sigma_{\Pi} \leq \tau_0 & \text{if } D_{\Pi} = 0
\end{cases}
\]

Thixo-viscoplastic (TVP) flow

\[
\begin{cases}
\sigma = 2 \eta(D_{\Pi}, \lambda) D(u) + \sqrt{2} \tau(\lambda) \frac{D(u)}{\sqrt{D}} & \text{if } D_{\Pi} \neq 0 \\
\sigma_{\Pi} \leq \tau(\lambda) & \text{if } D_{\Pi} = 0
\end{cases}
\]

Affine functions

\[
\begin{cases}
\eta(\lambda) = \eta_0 + \eta_1 \lambda \\
\tau(\lambda) = \tau_0 + \tau_1 \lambda
\end{cases}
\]

Structure evolution equation

\[
\frac{\partial \lambda}{\partial t} + u \cdot \nabla \lambda = a(1 - \lambda) - b\lambda D_{\Pi}^{\frac{1}{2}}
\]

\((a, b \text{ are structure parameters})\)
Viscosity model for TVP flow i.e. extended viscosity defined on all domains s.t.

\[
\begin{align*}
I. \quad \eta_s(D_{II}, \lambda) &= \eta(\lambda) + \frac{\sqrt{2}}{2} \tau(\lambda) \frac{1}{\sqrt{(D_{II} + (k^{-1})^2)}} \\
II. \quad \eta_s(D_{II}, \lambda) &= \eta(\lambda) + \frac{\sqrt{2}}{2} \tau(\lambda) \frac{1}{D_{II}^{\frac{1}{2}}} (1 - e^{-kD_{II}^{\frac{1}{2}}})
\end{align*}
\]

(k : regularization parameter)

Full set of equations

\[
\begin{align*}
\left( \frac{\partial}{\partial t} + u \cdot \nabla \right) u - \nabla \cdot \left( 2 \eta_s(D_{II}, \lambda) D(u) \right) + \nabla p &= 0 \quad \text{in } \Omega \\
\nabla \cdot u &= 0 \quad \text{in } \Omega \\
\frac{\partial \lambda}{\partial t} + u \cdot \nabla \lambda - a(1 - \lambda) + b\lambda D_{II}^{\frac{1}{2}} &= 0 \quad \text{in } \Omega
\end{align*}
\]
Newton’s method

Let $\mathcal{U} = (\lambda, u, p)$, and $\mathcal{R}_\mathcal{U} (\mathcal{U})$ be the continuous or the discrete corresponding system’s residum.

- Update of the nonlinear iteration with the correction $\delta \mathcal{U}$ i.e.

$$\mathcal{U}^N = \mathcal{U} + \delta \mathcal{U}$$

- The linearization of the residual provides

$$\mathcal{R}_\mathcal{U} (\mathcal{U}^N) = \mathcal{R}_\mathcal{U} (\mathcal{U} + \delta \mathcal{U})$$

$$= \mathcal{R}_\mathcal{U} (\mathcal{U}) + \mathcal{J} (\mathcal{U}) \cdot \delta \mathcal{U}$$

- The Newton’s method assuming invertible Jacobian

$$\mathcal{U}^N = \mathcal{U} - \mathcal{J}^{-1} (\mathcal{U}) \mathcal{R}_\mathcal{U} (\mathcal{U})$$
Adaptive Newton’s method

Jacobian calculations

$$\mathbf{J}(\mathbf{U}) = \left( \frac{\partial \mathbf{R}_u(\mathbf{U})}{\partial \mathbf{U}} \right)$$

- Continuous Adaptive Newton based on a priori study of Jacobian’s properties and decompositions

$$\mathbf{J}(\mathbf{U}) = \left( \frac{\partial \mathbf{R}_u(\mathbf{U})}{\partial \mathbf{U}} \right) + \delta \left( \frac{\partial \mathbf{R}_u(\mathbf{U})}{\partial \mathbf{U}} \right)$$

- Discrete Adaptive Newton based on the rate of residuum’s convergence

$$\left( \frac{\partial \mathbf{R}}{\partial \mathbf{U}} \right)_{ij} \approx \left( \frac{\mathbf{R}_i(\mathbf{U} + \epsilon \mathbf{e}_j) - \mathbf{R}_i(\mathbf{U} - \epsilon \mathbf{e}_j)}{2\epsilon} \right)$$
Continuous thixotropic problem

• **Flow variables** \((\lambda, u, p)\)

  ➢ Set \(\mathbb{T} := L^2(\Omega), \mathbb{V} := [H^1_0(\Omega)]^2, \mathbb{Q} := L^2_0(\Omega)\)
  
  ➢ Set \(\tilde{\mathbb{u}} := (\lambda, u)\)

  ➢ Find \((\lambda, u, p) \in (\mathbb{T} \cap H^1(\Omega)) \times \mathbb{V} \times \mathbb{Q}\) s.t.

\[
\langle \mathcal{K}(\lambda, u, p), (\xi, v, q) \rangle = \langle \mathcal{L}, (\xi, v, q) \rangle, \quad \forall (\xi, v, q) \in \mathbb{T} \times \mathbb{V} \times \mathbb{Q}
\]

\[
\mathcal{K} = \begin{bmatrix}
\mathcal{A}\tilde{\mathbb{u}} & \mathcal{B}^T \\
\mathcal{B} & 0
\end{bmatrix}
\]

➢ **Compatibly constraints**

\[
\sup_{v \in \mathbb{V}} \frac{\langle \mathcal{B}v, q \rangle}{|v|_v} \geq \beta |q|_{\mathbb{Q}/\text{Ker}\mathcal{B}^T}, \quad \forall q \in \mathbb{Q}
\]
Numerical challenges

• Discretizations have to handle the following challenges
  ➢ Stable FEM spaces
  ➢ Non-symmetric, non-coercive and ill-posedness
  ➢ Convection and positivity preserving
  ➢ Locally adapted meshes for steep gradients

• Solvers have to deal with
  ➢ Different source of nonlinearities
  ➢ Strong coupling of equations
  ➢ Robustness and efficiency
Approximated problem

• Conforming approximations

\[ T_h \subset T, \quad V_h \subset V, \quad Q_h \subset Q \]
\[ A\tilde{u}_h = A\tilde{u}, \quad B_h = B \]

• Discrete inf-sup condition

\[ \sup_{\boldsymbol{v}_h \in V_h} \frac{\langle B_h \boldsymbol{v}_h, q_h \rangle}{\|\boldsymbol{v}_h\|_{\tilde{V}}} \geq \beta_h \|q_h\|_{Q/Ker B_h^T}, \quad \forall q_h \in Q_h \]
The family of conforming FEM $Q_r/Q_r/P_{r-1}^{\text{disc}}, r \geq 2$ for $(\lambda, u, p)$ with stabilization

$$J_u(u_h, v_h) = \gamma_u \sum_{e \in \mathcal{E}_h} h^2 \int_e [\nabla u_h] : [\nabla v_h] \, d\Omega$$

$$J_\lambda(\lambda_h, \xi_h) = \gamma_\lambda \sum_{e \in \mathcal{E}_h} h \int_e [\nabla \lambda_h] : [\nabla \xi_h] \, d\Omega$$

- Inf-sup conditions is satisfied
- Discontinuous pressure
  - Good for the solver
  - Element-wise mass conservation
- Discrete problem is well-posed
- Highly consistent and symmetric stabilization
- Robust solver w.r.t. the monolithic approach
- Efficient solver w.r.t. multigird solver
Monolithic-multigrid linear solver

- Standard geometric multigrid solver for linearized system

- Full $Q_r$ and $P_{r-1}^{\text{disc}}$ restriction and prolongation

- Local Multilevel Pressure Schur Complement via Vanka-like smoother

\[
\begin{pmatrix}
\lambda_{l+1} \\
u_{l+1} \\
p_{l+1}
\end{pmatrix}
= \begin{pmatrix}
\lambda_{l} \\
u_{l} \\
p_{l}
\end{pmatrix} + \omega^l \sum_{T \in \mathcal{T}_h} \begin{pmatrix}
(K_h + J) |_{T}
\end{pmatrix}^{-1}
\begin{pmatrix}
\mathcal{R}_{\lambda^l} \\
\mathcal{R}_{u^l} \\
\mathcal{R}_{p^l}
\end{pmatrix} |_{T}
\]

Coupled Monolithic Multigrid Solver!
Starting point: consider flow in a cavity with unit height
- Steady, incompressible flow
- Constant speed at upper lid
- No-slip Dirichlet boundary conditions

Newtonian, Viscoplastic, and Thixo-viscoplastic (TVP)
Newtonian Lid-driven cavity flow

- **Point-wise convergence for Newtonian flow**

Re=1000  
Re=5000  
Re=10000

\begin{align*}
\text{Level 5} & \\
\text{Level 6} & \\
\text{Level 7} & \\
\text{Level 8} & \\
\text{Level 9} & \\
\end{align*}

\begin{align*}
\text{Level 5} & \\
\text{Level 6} & \\
\text{Level 7} & \\
\text{Level 8} & \\
\text{Level 9} & \\
\end{align*}

\begin{align*}
\text{Level 5} & \\
\text{Level 6} & \\
\text{Level 7} & \\
\text{Level 8} & \\
\text{Level 9} & \\
\end{align*}
Newtonian Lid-driven cavity flow

- Global and point-wise quantities and solver behaviour

| Level | cells | $\text{Energy} \times 10^2$ | N/M | $\text{Energy} \times 10^2$ | N/M | $\text{Energy} \times 10^2$ | N/M |
|-------|-------|-----------------|-----|-----------------|-----|-----------------|-----|
| 5     | 1024  | 4.541506        | 5/1 | 6.082524        | 6/1 | 7.940472        | 7/1 |
| 6     | 4096  | 4.458877        | 5/1 | 4.955858        | 6/1 | 5.369527        | 6/1 |
| 7     | 16384 | 4.452357        | 3/1 | 4.768669        | 4/1 | 4.868399        | 5/1 |
| 8     | 65536 | 4.451904        | 3/1 | 4.744815        | 3/2 | 4.783917        | 4/2 |
| 9     | 262144| 4.451846        | 3/1 | 4.742921        | 3/1 | 4.773500        | 3/2 |
| 10    | 1048576| 4.451834      | 2/1 | 4.742815        | 3/1 | 4.772692        | 3/1 |

Ref. values $\approx$: 4.45 4.74 4.77

| Level | $\psi_{\text{max}}$ | $\psi_{\text{min}} \times 10^3$ | $\psi_{\text{max}}$ | $\psi_{\text{min}} \times 10^3$ | $\psi_{\text{max}}$ | $\psi_{\text{min}} \times 10^3$ |
|-------|---------------------|------------------------------|---------------------|------------------------------|---------------------|------------------------------|
| 6     | 0.1190073           | $-1.72813$                  | 0.1249471           | $-3.145666$                  | 0.1586626           | $-5.7535749$                |
| 7     | 0.1189360           | $-1.72649$                  | 0.1225439           | $-3.077555$                  | 0.1236127           | $-3.2070181$                |
| 8     | 0.1189361           | $-1.72851$                  | 0.1222499           | $-3.072411$                  | 0.1225210           | $-3.1831353$                |
| 9     | 0.1189362           | $-1.72963$                  | 0.1222269           | $-3.073524$                  | 0.1224097           | $-3.1910101$                |
| 10    | 0.1189366           | $-1.72965$                  | 0.1222259           | $-3.073589$                  | 0.1223892           | $-3.1797390$                |

- Mesh convergence of the solutions irrespective of Re number
- Efficient non-linear solver
- Mesh independent linear solver

Accurate, robust and efficient Monolithic Multigrid Solver

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• Boundary limit for rigid-zone w.r.t regularization $k$

$$\quad \begin{align*}
(a) \quad \tau_0 &= 2.0 \\
(b) \quad \tau_0 &= 5.0
\end{align*}$$

• Accurate track of interface requires
  ✓ larger $k$ solutions
  ✓ finer mesh refinement

• Existence of pair $(k, L)$ beyond which no further improvement in solutions is expected
Viscoplastic flow in Lid-driven cavity

- progressive growth of unyielded zones for non-thixotropic (Bingham Plastic) flow

✓ Unyielded zones’s shape and extent is in agreement with Ref. Results
Solver behaviour w.r.t. Regularization and mesh refinement

| $k \backslash L$ | $\tau_0 = 1$ | | | $\tau_0 = 2$ | | | $\tau_0 = 5$ | | |
|----------------|------------|------------|------------|------------|------------|------------|
|                | 5          | 6          | 7          | 5          | 6          | 7          | 5          | 6          | 7          |
| $1 \times 10^1$| 3/1        | 3/1        | 3/1        | 3/1        | 3/1        | 3/1        | 4/1        | 4/1        | 4/1        |
| $5 \times 10^1$| 2/1        | 2/1        | 2/1        | 2/1        | 2/1        | 2/1        | 3/1        | 3/1        | 3/1        |
| $1 \times 10^2$| 3/1        | 3/1        | 3/1        | 3/1        | 3/1        | 3/1        | 4/1        | 4/1        | 4/1        |
| $5 \times 10^2$| 3/1        | 2/1        | 2/1        | 3/1        | 2/1        | 3/1        | 3/1        | 3/1        | 3/1        |
| $1 \times 10^3$| 2/2        | 3/2        | 3/1        | 3/1        | 3/1        | 4/1        | 4/1        | 5/2        | 5/2        |
| $5 \times 10^3$| 2/1        | 2/1        | 4/1        | 3/1        | 3/2        | 6/2        | 4/1        | 8/2        | 6/1        |
| $1 \times 10^4$| 2/1        | 2/2        | 5/1        | 3/1        | 3/1        | 6/1        | 4/1        | 5/4        | 6/3        |
|                | $\tau_0 = 10$ | | | $\tau_0 = 20$ | | | $\tau_0 = 50$ | | |
| $1 \times 10^1$| 5/1        | 5/1        | 5/1        | 6/1        | 6/1        | 6/1        | 5/1        | 7/1        | 7/1        |
| $5 \times 10^1$| 4/1        | 3/1        | 3/1        | 4/1        | 4/1        | 3/2        | 5/4        | 4/2        | 4/2        |
| $1 \times 10^2$| 5/2        | 4/1        | 4/1        | 5/2        | 5/2        | 5/1        | 6/5        | 5/4        | 5/1        |
| $5 \times 10^2$| 5/3        | 3/2        | 3/1        | 4/4        | 3/4        | 4/3        | 5/4        | 4/2        | 4/3        |
| $1 \times 10^3$| 5/2        | 7/4        | 9/1        | 5/5        | 7/2        | 8/1        | 5/5        | 9/2        | 9/2        |
| $5 \times 10^3$| 5/1        | 7/3        | 8/2        | 6/3        | 6/4        | 6/4        | 6/4        | 7/2        | 8/2        |
| $1 \times 10^4$| 6/1        | 7/2        | 8/3        | 6/3        | 5/5        | 7/3        | 6/3        | 7/3        | 8/2        |

- Efficient non-linear solver
- Mesh independent linear solver
- Solutions are obtained with continuation strategy w.r.t. $k$
- Integration of continuation strategy w.r.t. $k$ in the solver
• Impact of thixotropic yield stress on morphology of unyielded zones in TVP flow

✓ Main rheological characteristics of materials with yield stress is preserved
• **Solver behaviour w.r.t. Regularization and mesh refinement**

| $k \div L$ | 5 | 6 | 7   | 5 | 6 | 7   | 5 | 6 | 7   |
|-----------|---|---|-----|---|---|-----|---|---|-----|
| $\tau_1 = 0.5$ |   |   |     |   |   |     |   |   |     |
| $1 \times 10^1$ | 5/2 | 5/3 | 6/2 | 5/2 | 5/2 | 9/1 | 5/2 | 5/2 | 9/1 |
| $5 \times 10^1$ | 4/2 | 4/2 | 4/2 | 3/2 | 3/3 | 7/1 | 3/2 | 3/3 | 8/1 |
| $1 \times 10^2$ | 4/1 | 4/2 | 5/1 | 4/1 | 4/2 | 7/1 | 4/2 | 4/2 | 8/1 |
| $5 \times 10^2$ | 4/1 | 4/1 | 5/1 | 3/1 | 4/1 | 6/1 | 4/1 | 6/1 | 7/1 |
| $1 \times 10^3$ | 4/1 | 4/1 | 4/1 | 4/2 | 4/2 | 8/1 | 4/4 | 6/1 | 7/1 |
| $5 \times 10^3$ | 4/1 | 4/1 | 3/2 | 7/1 | 9/1 | 5/1 | 6/1 | 9/1 | 8/1 |
| $1 \times 10^4$ | 4/1 | 4/2 | 4/2 | 5/1 | 7/1 | 4/1 | 7/1 | 10/1 | 8/2 |
| $\tau_1 = 5.0$ |   |   |     |   |   |     |   |   |     |
| $1 \times 10^1$ | 6/2 | 6/2 | 10/1 | 11/1 | 8/2 | 11/1 | 10/1 | 9/2 | 11/1 |
| $5 \times 10^1$ | 4/2 | 3/2 | 11/1 | 11/1 | 4/2 | 7/1 | 12/1 | 5/3 | 9/1 |
| $1 \times 10^2$ | 4/2 | 5/2 | 11/1 | 10/1 | 5/3 | 8/1 | 12/1 | 6/3 | 10/1 |
| $5 \times 10^2$ | 5/2 | 4/2 | 10/1 | 9/1 | 5/3 | 5/1 | 8/1 | 5/5 | 11/1 |
| $1 \times 10^3$ | 5/2 | 9/1 | 10/1 | 10/1 | 9/1 | 7/1 | 8/2 | 9/1 | 9/2 |
| $5 \times 10^3$ | 5/1 | 5/1 | 5/1 | 8/1 | 8/2 | 6/1 | 8/1 | 7/1 | 11/1 |
| $1 \times 10^4$ | 5/1 | 5/2 | 5/1 | 8/3 | 7/1 | 5/1 | 8/2 | 7/1 | 9/1 |
| $\tau_1 = 10.0$ |   |   |     |   |   |     |   |   |     |
| $1 \times 10^1$ |   |   |     |   |   |     |   |   |     |
| $5 \times 10^1$ |   |   |     |   |   |     |   |   |     |
| $1 \times 10^2$ |   |   |     |   |   |     |   |   |     |
| $5 \times 10^2$ |   |   |     |   |   |     |   |   |     |
| $1 \times 10^3$ |   |   |     |   |   |     |   |   |     |
| $5 \times 10^3$ |   |   |     |   |   |     |   |   |     |
| $1 \times 10^4$ |   |   |     |   |   |     |   |   |     |
| $\tau_1 = 20.0$ |   |   |     |   |   |     |   |   |     |

✓ Efficient non-linear solver  
✓ Mesh independent linear solver  
✓ Solutions are obtained with continuation strategy w.r.t. $k$  
⇒ Integration of continuation strategy w.r.t. $k$ in the solver
• Material micro-structural level w.r.t. breakdown parameter

• Interplay of yield stress and thixotropy
  ✓ Structuring level is predicting shape and extent of rigid zones
  ✓ Induction of more breakdown layers
  ✓ Shear localization
  ✓ Shear band
Thixo-viscoplastic flow in Couette

Continuous axial-Flow Couette device:

- The material is sheared in the annulus between the interior and exterior cylinder shells of radii $r_{in}$ and $r_{out}$ respectively.
  - Concentric cylinders
  - Rotating inner cylinder with $\omega_{in} = 1 \text{rads}^{-1}$
  - Stationary outer cylinder
  - Vertical flow super-imposed in radial direction

Investigations of thixo-viscoplastic phenomena
- Shear localization
- Shear banding
- Consistent transition points between velocity and structure
Thixo-viscoplastic flow in Couette

- Velocity profile at cut-line positions $c; c \in [0, 2\pi]$ in a Couette device w.r.t breakdown parameter

✓ Localization
✓ Shear banding
Thixo-viscoplastic flow in Couette

- Shear rate at cut-line positions $c; c \in [0, 2\pi]$ in a Couette device w.r.t breakdown parameter

 ✓ Smooth and sharp transition are possible
 ✓ Transition point matches with the velocity
Thixo-viscoplastic flow in Couette

- **Structure parameter at cut-line positions** \( c; c \in [0, 2\pi] \) **in a couette w.r.t.** breakdown parameter

✓ Transition point matches with the velocity
✓ Structuring level is predicting shape and extent of rigid zones
• 2D-FEM simulation results for thixo-viscoplastic flow- validation of 1D tool
• Specifying the “unidirectional profiles as boundary Data” in 2D for contraction
Thixo-viscoplastic flow in curved contractions

- 2D-FEM simulation results for thixotropic flow - validation of 1D tool
- Specifying the “1D-profiles as boundary Data” in 2D simulations for contraction domain
As predicted \((u, \lambda, p)\) solutions

Structuring level is predicting shape and extent of rigid zones
Thixo-viscoplastic flow in curved contractions

- progressive growth of unyielded zones (shaded)

(a) $\tau_1 = 0.125$

(b) $\tau_1 = 0.25$

(c) $\tau_1 = 0.50$

(d) $\tau_1 = 2.0$

Unyielded zones in upstream and downstream are not merging
• Material micro-structural level w.r.t. breakdown

(i) Upstream channel & Entrance zone

(ii) Downstream channel

• Inherent thixotropy speed-up the breakdown
  ✓ Appearance of more breakdown layers
  ✓ Applications: restart pressure in pipelines should not be over-estimated

“Further investigation” regarding material structuring in thixo-elasovisoplastic
An accurate, robust, and efficient numerical solver for TVP flows is developed using

- Higher order finite element method
- Monolithic Newton-multigrid
  - Adaptive discrete Newton’s method with global convergent property
  - Geometric multigrid with local MPSC

To analyze the quasi-Newtonian model for TVP materials for different flow simulations

- Newtonian, VP, and TVP flow in Lid-driven cavity
- TVP flow in Couette devices
- TVP flow in curved contractions
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