COMPLEXES OF MARKED GRAPHS IN GAUGE THEORY

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Abstract. We review the gauge graph complexes as defined by Kreimer, Sars and van Suijlekom in “Quantization of gauge fields, graph polynomials and graph homology” and compute their cohomology.

1. Introduction

In [KSvS13] Kreimer, Sars and van Suijlekom showed how gauge theory amplitudes can be generated using only a scalar field theory with cubic interaction. On the analytic side this is achieved by means of a new graph polynomial, dubbed the corolla polynomial, that transforms integrands of scalar graphs into gauge theory integrands. On the combinatorial side all graphs relevant in gauge theory can be generated from the set of all 3-regular graphs by means of operators that label edges and cycles. These labels represent edges with different Feynman rules that incorporate contributions from 4-valent vertices and relations between 3- and 4-valent vertices, and similar for gluon and ghost cycles.

Generating and exchanging these labels can be cast as operations on graphs that square to zero, hence define differentials on the free abelian groups generated by all 3-regular graphs (with a fixed number of legs and loops). Modeling particle types and edge-collapses by different labels on edges and cycles, called markings, one thereby obtains cochain complexes whose cohomology encodes physical constraints on scattering amplitudes in gauge theory. Very roughly speaking, the first marking represents modified Feynman rules, such that the full gauge theory amplitude is given by the sum over all marked, 3-regular graphs (representing all ways of expanding 4-gluon into 3-gluon vertices or all ways of exchanging gluon for ghost loops, respectively). The second marking or, more precisely, the two differentials that change the first into the second marking and generate new marked edges of the second type, reflect physical constraints such as unitarity and gauge covariance, in the sense that observable quantities must lie in the kernel of these maps (similar to the approach in BRST quantization). Thus the relevance of understanding the cohomology of these complexes.

However, not much physical input is needed to understand the latter. The gauge graph complexes of [KSvS13] are special cases of a general construction that associates to a graph and a class of subgraphs allowed to be marked a cochain complex. This complex is generated by all possible markings of the graph and the differentials operate on the markings by generating and interchanging them. The connection to physics comes here from the mere choice of marked substructures (i.e. cycles and edges) and differentials, encoding physical relevant information.

For a thorough discussion of the quantum field theoretical motivation and interpretation of these complexes we refer to the original article [KSvS13] and the review in [Kre18]; background material can be found in the classic work [Cvi04].

In the present article we compute the cohomology of the gauge graph complexes, showing that in both cases it is concentrated in degree zero, its generators resembling a combinatorial Green’s function, the graphical representative of the gauge theory amplitude. In our language this translates into the following statement:

Main Theorem: Fix \( r, l \in \mathbb{N} \) and let \((\mathcal{G}, D)\) denote the gluon or ghost cycle graph complexes (defined in Sections 2.2 and 2.3). Define

\[
X := \sum_{\Gamma} \sum_{m} (\Gamma, m),
\]

as the sum over all admissible 1-markings of edges and cycles in 3-regular graphs \( \Gamma \) with \( r \) legs and \( l \) loops. Then \( DX = 0 \) and it represents the only non-trivial (maximal) cohomology class in \( H^\bullet(\mathcal{G}, D) \).

This result is based on two properties of the gluon and ghost cycle graph complexes. Firstly, there is an universal model for complexes of marked graphs allowing to treat both cases at once. Secondly, its differentials are of the form \( D = d_1 + d_2 \) with \( d_1 \) very simple. This allows to compute the cohomology of
D by a spectral sequence argument without the need of explicitly understanding the cohomology with respect to \( d_2 \).

The exposition is organized as follows. In the next section we introduce complexes of marked graphs in general, then specialize to edge-, cycle- or vertex marked graphs, the latter serving as the universal model for the former two cases. We compute its cohomology in Section 3 in two steps. First we study only the homology with respect to the simple differential \( d_1 \), then we apply the result in a spectral sequence associated to the double complex formed by \( d_1 + d_2 \). Last, but not least, we combine the gauge and ghost cycle graph complexes into a large complex with two differentials whose cohomologies are generated by one single element, the full gauge theory amplitude.

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### 2. Gauge graph complexes

We introduce a bit of notation, then describe complexes of marked graphs of which the gluon and ghost cycle graph complexes in [KSvS13] emerge as special cases. We discuss these two in detail and introduce a third one which serves as an universal model for these kind of complexes.

#### 2.1. Complexes of marked graphs.

Let \( \Gamma = (\Gamma^0, \Gamma^1) \) be a connected graph. We call edges connected to univalent vertices *external (edges)* or *legs*, all other edges are referred to as *internal* or, by abuse of language, simply as *edges*. Thus, the set \( \Gamma^1 \) of edges of \( \Gamma \) splits into \( \Gamma^1 = \Gamma^1_{\text{ext}} \sqcup \Gamma^1_{\text{int}} \).

A similar decomposition holds for the set of vertices of \( \Gamma \), \( \Gamma^0 = \Gamma^0_{\text{ext}} \sqcup \Gamma^0_{\text{int}} \).

Since our operations will focus solely on the internal structure of graphs, we write \( V = V(\Gamma) := \Gamma^0_{\text{int}} \) and \( E = E(\Gamma) := \Gamma^1_{\text{int}} \) for its internal vertices and edges. In that spirit we denote graphs by \( \Gamma = (V, E) \), as is customary in graph theory, tacitly remembering the external structure encoded by the pair \( (\Gamma^0_{\text{ext}}, \Gamma^1_{\text{ext}}) \).

**Definition 2.1.** Fix a graph \( \Gamma \) and \( S = \{0, \ldots, s\} \subset \mathbb{N} \) a finite set and denote by \( \text{Sub}(\Gamma) \) the set of all (internal) subgraphs of \( \Gamma \). An *S-marking* of \( \Gamma \) is then a map \( m : \text{Sub}(\Gamma) \to S \). We call the pair \((\Gamma, m)\) a *marked graph* and think of the subgraphs in \( m^{-1}(0) \) as being not marked. A marking is *admissible* if no two marked elements share a common vertex.

In the following we will describe (co-)chain complexes of marked graphs. For this we consider only admissible markings with \( s = 2 \), but more general settings are obviously possible. Throughout this work all coefficients will be in \( \mathbb{Z} \).

**Definition 2.2.** Fix a graph \( \Gamma \) and \( P \subset \text{Sub}(\Gamma) \) a set of subgraphs\(^1\) of \( \Gamma \) endowed with a total order. Let \( \mathcal{P}(\Gamma) \) denote the free abelian group generated by all markings \((\Gamma, m)\) where \( m : P \to \{0, 1, 2\} \) is admissible.

The marking induces a partition of \( P \). We write \( P = P_0 \sqcup P_m \) where \( P_0 \) denotes the unmarked objects in \( P \) and \( P_m = P_1 \sqcup P_2 \) the 1- and 2-marked ones.

The group \( \mathcal{P}(\Gamma) \) carries two gradings. With \( \mathcal{P}(\Gamma)^i \) denoting the subgroup of \( \mathcal{P}(\Gamma) \) generated by markings with \( |P_1| = i \) and \( |P_2| = j \) we set

\[
\mathcal{P}(\Gamma)^i := \bigoplus_{i \in \mathbb{N}} \mathcal{P}(\Gamma)^i.
\]

We define two differentials on this group by changing and permuting the markings.

**Definition 2.3.** For \((\Gamma, m) \in \mathcal{P}(\Gamma)\) let \((\Gamma, m_{\{p\to 2\}})\) denote the marking that is identical to \( m \) on \( P \setminus p \) and marks \( p \) by 2. Define two linear maps \( d_i : \mathcal{P}(\Gamma)^i \to \mathcal{P}(\Gamma)^{i+1} \) by

\[
d_1(\Gamma, m) := (-1)^{|P^m|} \sum_{p \in P_1} (-1)^{|\{p' \in P_1 : |p'| > |p|\}|} d_1^p(\Gamma, m),
\]

\[
d_2(\Gamma, m) := \sum_{p \in P} (-1)^{|\{p' \in P_m : |p'| < |p|\}|} d_2^p(\Gamma, m),
\]

where \( d_1^p(\Gamma, m) := (\Gamma, m_{\{p\to 2\}}) \) and \( d_2^p(\Gamma, m) := \begin{cases} 0 & \text{if } p \text{ shares a vertex with some } p' \\ (\Gamma, m_{\{p\to 2\}}) & \text{else.} \end{cases} \)

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1This is quite interesting in its own right as its computation is related to NP-hard problems in graph theory [Kn17].

2A subgraph of \( \Gamma \) is a pair of subsets \( \emptyset \neq V' \subset V \) and \( \emptyset \neq E' \subset E \) such that \((V', E')\) is a graph itself.
Remark 2.4. Definitions 2.2 and 2.3 depend on the order on \( P \), but all choices will produce isomorphic complexes. Therefore, we omit this piece of data in the following. See also Remark 2.8.

Proposition 2.5. Both maps \( d_1 \) and \( d_2 \) square to zero. Moreover, \( d_1 d_2 + d_2 d_1 = 0 \), so that \( D := d_1 + d_2 \) is a differential on \( \mathcal{P}(\Gamma) \).

Proof. A simple calculation keeping track of the signs. It can be found in [KSvS13]. \( \square \)

In summary, for any type \( P \) of marked subgraphs of a graph \( \Gamma \) we get a cochain complex \( (\mathcal{P}(\Gamma), D) \). In the following we specialize this construction to three cases where \( P \) consists of edges, cycles or vertices. Nevertheless, a priori one needs to take care in regards to graph automorphisms and how they change a chosen ordering. In the usual definition of differentials on graph complexes (see for instance [AV03]) one introduces a sign in the sense that if \( \varphi \in \text{Aut}(\Gamma) \), then

\[ (\Gamma, o) = \text{sgn}(\varphi) \cdot (\Gamma, o\varphi). \]

This allows to relate different orderings, but it also forces all graphs with multi-edges to be equal to zero as then any odd permutation on a multi-edge gives \( (\Gamma, o) = - (\Gamma, o) \). To keep these graphs in the game, we take the ordering as an additional, separately chosen piece of information, tacitly equipping every isomorphism class of graphs with a choice. Ultimately, this works because our differentials do not relate different graphs but operate on the marking only.

The group \( \mathcal{E}(\Gamma) \) carries two gradings. With \( \mathcal{E}(\Gamma)^i \) denoting the subgroup of \( \mathcal{E}(\Gamma) \) generated by marked graphs with \( i \) edges of type 1 and \( j \) edges of type 2 we set

\[ \mathcal{E}(\Gamma)^j := \bigoplus_{i \in \mathbb{N}} \mathcal{E}(\Gamma)^i, \]

Definition 2.6 and Proposition 2.5 produce three differentials \( s, \sigma, S : \mathcal{E}(\Gamma)^j \rightarrow \mathcal{E}(\Gamma)^{j+1} \), given by

\[ s(\Gamma, m) := \sum_{e \in E} (-1)^{1 + \text{val}(e)} \cdot |\{ e' \in E_{\text{mark}} | e' < e \}| \cdot e(\Gamma, m), \]

\[ \sigma(\Gamma, m) := (-1)^{1 + \text{val}(e)} \cdot \sum_{e \in E_1} (-1)^{1 + \text{val}(e')} \cdot |\{ e' \in E_{\text{mark}} | e' > e \}| \cdot e(\Gamma, m), \]

\[ S := s + \sigma, \]

where \( e(\Gamma, m) := (\Gamma, m_{e \rightarrow 2}) \) and \( e(\Gamma, m) := \begin{cases} 0 & \text{if } e \text{ is adjacent to another marked edge} \\ (\Gamma, m_{e \rightarrow 2}) & \text{else}. \end{cases} \)

Example 2.9. Let \( (\Gamma, m) = 1 2 3 4 \) with \( E \) ordered as pictured, the 1- and 2-markings denoted by ‘●’ and ‘●’, respectively. Then

\[ s(\Gamma, m) = - - - - - - - - - - , \quad \sigma(\Gamma, m) = - - - - - - - - - - , \quad \sigma s(\Gamma, m) = - - - - - - - - - - = - s \sigma(\Gamma, m). \]
Let $E := \bigoplus_{\Gamma \in \text{Gr}_r,l} E(\Gamma)$. This group naturally inherits a grading and a differential $S = s + \sigma$ from each of its summands, hence defines a cochain complex.

**Definition 2.10.** The complex $(E, S)$ is called the **gluon graph complex**.

2.3. The **ghost cycle graph complex**. We repeat the above construction for the case of cycle-markings.

**Definition 2.11.** Let $\Gamma$ be a connected graph. A **cycle** $c$ in $\Gamma$ is a closed path without repeated vertices, i.e. a subset $c \subseteq E$ such that

- every $v \in V$ is incident to none or exactly two elements in $c$.
- $c \subseteq \Gamma$ is a connected subgraph.

We denote by $C = C(\Gamma)$ the set of all cycles in $\Gamma$.

**Definition 2.12.** For fixed $r, l \in \mathbb{N}$ and $\Gamma \in \text{Gr}_r,l$ let $C(\Gamma)$ denote the free abelian group generated by all admissible markings $m: C \to \{0, 1, 2\}$ of the (ordered set of) cycles of $\Gamma$. An analogous to the case of edge-markings, every cycle-marking induces a partition of the cycle set, $C = C_0 \sqcup C_1$ and $C_1 = C_1 \sqcup C_2$.

Let $C(\Gamma)$ be graded by the number of 1- and 2-markings. Let $C(\Gamma)_i$ denote the subgroup generated by marked graphs with $i$ cycles of type 1 and $j$ cycles of type 2 and let

$$C(\Gamma)^j := \bigoplus_{i \in \mathbb{N}} C(\Gamma)_i.$$

There are three differentials $t, \tau, T: C(\Gamma)^j \to C(\Gamma)^{j+1}$, given by

$$t(\Gamma, m) := \sum_{c \in C} (-1)^{|c|} \tau c(\Gamma, m),$$

$$\tau(\Gamma, m) := (\sum_{c \in C_1} (-1)^{|c|} \tau c(\Gamma, m),$$

$$T := t + \tau,$$

where $\tau c(\Gamma, m) := (\Gamma, m_c \mapsto 2)$ and $t_c(\Gamma, m) := \begin{cases} 0 & \text{if } c \text{ is adjacent to another marked cycle} \\ (\Gamma, m_{c \mapsto 2}) & \text{else} \end{cases}$.

**Example 2.13.** Let $(\Gamma, m) = \circ \circ \circ \circ \circ \circ \circ$ with $C$ ordered as pictured, the 1- and 2-markings drawn as dotted and dashed cycles, respectively. Then

$$t(\Gamma, m) = \circ \circ \circ \circ \circ \circ \circ, \quad \tau(\Gamma, m) = \circ \circ \circ \circ \circ \circ \circ,$$

$$\tau t(\Gamma, m) = \circ \circ \circ \circ \circ \circ \circ = -t\tau(\Gamma, m).$$

Let $C := \bigoplus_{\Gamma \in \text{Gr}_r,l} C(\Gamma)$, graded by number of 1- and 2-marked cycles and furnished with the differential $T = t + \tau$.

**Definition 2.14.** The complex $(C, T)$ is called the **ghost cycle graph complex**.

2.4. **Marking vertices.** Note that all of the previously defined differentials do not alter the topology of graphs, they only change their markings. Furthermore, $\sigma$ and $\tau$ act only on 1-marked edges or cycles, respectively, of a graph $\Gamma$, hence are completely independent of its topology. On the other hand, $s$ and $t$ generate new 2-markings, so they depend on the incidence structure and the marking of $\Gamma$ in a non-trivial way. Nevertheless, it is possible to construct universal models for all cases of (admissible) markings. In abstract terms, these models form a subcategory of the category of complexes of marked graphs which is equivalent to the larger category.

**Definition 2.15.** Given a (not necessarily connected or 3-regular) graph $\Gamma$ (without external legs and self-loops) let $V(\Gamma)$ denote the free abelian group generated by all markings $(\Gamma, m)$ where $m: V \to \{0, 1, 2\}$ marks the (ordered set of) vertices of $\Gamma$ such that no two marked vertices are connected by an edge.

We write $V = V_0 \sqcup V_m$ with $V_m = V_1 \sqcup V_2$ for the partition of $V$ induced by a marking.
Let \( V(\Gamma)^j \) denote the subgroup generated by marked graphs with \( i \) vertices of type 1 and \( j \) vertices of type 2 and let

\[
V(\Gamma)^j := \bigoplus_{i \in \mathbb{N}} V(\Gamma)^j_i.
\]

Mimicking the previous constructions (only the notion of admissible markings has changed) we obtain three differentials \( u, \mu, U \) : \( V(\Gamma)^j \to V(\Gamma)^{j+1} \), given by

\[
u(\Gamma, m) := \sum_{v \in V} (-1)^{|\{v' \in V \mid v' < v\}|} u_v(\Gamma, m),
\]

\[
\mu(\Gamma, m) := (-1)^{|V_1|} \sum_{v \in V_1} (-1)^{|\{v' \in V_1 \mid v' > v\}|} \mu_v(\Gamma, m),
\]

\[
U := u + \mu,
\]

where \( \mu_v(\Gamma, m) := (\Gamma, m_{|_{v \to 2}}) \) and \( u_v(\Gamma, m) := \begin{cases} 0 & \text{if } v \text{ is adjacent to another marked vertex} \\ (\Gamma, m_{|_{v \to 2}}) & \text{else.} \end{cases} \)

**Example 2.16.** Let \( (\Gamma, m) = \begin{array}{c} 1 \\ 2 \end{array} \begin{array}{c} \otimes 1 \\ \otimes 2 \end{array} \) with \( V \) ordered as pictured, the 1- and 2-markings drawn as “\(^\circ\)” and “\(^\bullet\)”, respectively. Then

\[
u(\Gamma, m) = \begin{array}{c} \otimes 1 \\ \otimes 2 \end{array} + \begin{array}{c} \otimes 1 \\ \otimes 2 \end{array} - \begin{array}{c} \otimes 1 \\ \otimes 2 \end{array}, \quad \mu(\Gamma, m) = - \begin{array}{c} \otimes 1 \\ \otimes 2 \end{array},
\]

\[
u(\Gamma, m) = \begin{array}{c} \otimes 1 \\ \otimes 2 \end{array} + \begin{array}{c} \otimes 1 \\ \otimes 2 \end{array} - \begin{array}{c} \otimes 1 \\ \otimes 2 \end{array} = -u(\mu(\Gamma, m)).
\]

Adapting the proof of Proposition 2.5 we conclude that \( (V(\Gamma), U) \) is a cochain complex. The universality of this complex is established by

**Theorem 2.17.** Given a graph \( \Gamma \) and \( P \subset \text{Sub}(\Gamma) \) let \( \mathcal{P}(\Gamma, D) \) be the associated complex constructed in Section 2.7. Define a graph \( \Gamma' = (V', E') \) by

\[
V' := P, \quad E' := \{ (p, p') \mid p \text{ and } p' \text{ share a common vertex} \} \subset V' \times V'.
\]

Then \( \mathcal{P}(\Gamma, D) \cong (V(\Gamma'), U) \) as cochain complexes.

**Proof.** Let the order on \( V' \) be induced by the one on \( P \). Define a linear map \( \Psi : \mathcal{P}(\Gamma)^j_1 \to \mathcal{V}(\Gamma')^j_1 \) by \( \Psi(\Gamma, m) := (\Gamma', m') \) with \( m'(v) := m(p) \). Then

\[
\Psi d_1(\Gamma, m) = (-1)^{|P_1|} \sum_{p \in P_1} (-1)^{|\{v' \in P_1 \mid v' > p\}|} \Psi(\Gamma, m_{|_{v \to 2}})
\]

\[
= (-1)^{|P_1|} \sum_{v \in P_1} (-1)^{|\{v' \in P_1 \mid v' > v\}|} (\Gamma', m'_{|_{v \to 2}})
\]

\[
= (-1)^{|V_1|} \sum_{v \in V_1} (-1)^{|\{v' \in V_1 \mid v' > v\}|} (\Gamma', m'_{|_{v \to 2}}) = \mu(\Gamma, m)
\]

and similar for \( \Psi d_2 = u \Psi \). Furthermore, \( \Psi \) is bijective by construction, hence the result follows. \( \square \)

Consider the inclusion of the full subcategory of the category of complexes of marked graphs given by all complexes of the form \( V(\Gamma) \). This functor is fully-faithful by construction and the previous theorem shows that it is also essentially surjective, hence part of an equivalence.

**Example 2.18.** For the marked graphs in the examples 2.10 and 2.13 the associated graphs \( (\Gamma', m') \) are given by \( \begin{array}{c} 1 \\ 2 \end{array} \begin{array}{c} \otimes 1 \\ \otimes 2 \end{array} \) and \( 1 \cdot \otimes 2 \), respectively. The complex \( (V(\Gamma'), U) \) contains thus all information of \( (\mathcal{P}(\Gamma), D) \) while being as simple as possible.

### 3. Gauge graph cohomology

In this section we prove our main result, Theorem 3.6. By Theorem 2.17 it suffices to consider the complex \( (V, U) \). Moreover, since taking homology commutes with taking direct sums of complexes, we focus on the smaller complexes \( (V(\Gamma), U) \).

The proof is based on two steps. First we calculate the cohomology the complex \((V(\Gamma), \mu)\), then we use the result in a spectral sequence to find the cohomology of the total complex \((V(\Gamma), u + \mu)\).
3.1. The homology of the complex \((\mathcal{V}, \mu)\). Fix a graph \(\Gamma\) and let \((\Gamma, m) \in \mathcal{V}(\Gamma)^+\), so \(m\) specifies \(i\) and \(j\) 2-marked vertices. Since the map \(\mu\) changes 1-markings into 2-markings, the image \(\mu(\Gamma, m)\) is an element of \(\mathcal{V}(\Gamma)^{+1}\), given by the sum over \(i\) copies of \(\Gamma\) where a single 1-marked vertex has been replaced with a 2-marked vertex.

We can model this situation in a rather simple way. Let \(\Lambda_n\) denote the graph on \(n\) disconnected vertices, ordered by \(v_1 < \ldots < v_n\), and define a chain complex \(\mathcal{L}(\Lambda_n, \mu)\) by \(\mathcal{L}(\Lambda_n) := \{ (\Lambda_n, m) \mid m : V(\Lambda_n) \to \{1, 2\}; |m^{-1}(1)| = i \}\). Note that here the markings are required to mark every vertex of \(\Lambda_n\). This complex captures the action of \(\mu\) on a single configuration of marked vertices, i.e. on markings \(m, m'\) with \(V_m = V_{m'}\) as subsets of \(V\). There are however also other configurations of marked vertices which have to be taken into account.

**Definition 3.1.** An independent set of size \(n\) in a graph \(\Gamma\) is a subset of \(V\) of size \(n\) such that no two of its elements are adjacent. We write \(I_n = I_n(\Gamma) \subset 2^V\) for the set of all independent sets with size \(n\) in \(\Gamma\).

Rephrasing Definition 2.15 we see that every marking \(m\) on \(\Gamma\) corresponds bijectively to an independent set \(V_m \subset V\) with a choice of labelling or 2-partition \(V_m = V_1 \sqcup V_2\) of its elements. Taking as many copies of \(\mathcal{L}(\Lambda_n)\) as there are independent sets of size \(n\) in \(\Gamma\) allows to model the action of \(\mu\) on \(\mathcal{V}(\Gamma)\).

**Lemma 3.2.** Given \(\Gamma \in \text{Gra}_{\mathbb{N}}\) define a chain complex \(\mathcal{L}(\mu, \mu)\) by \(\mathcal{L}_i := \bigoplus_{j \in \mathbb{N}} \bigoplus_{m \in I_{i+j}(\Gamma)} \mathcal{L}(\Lambda_{i+j})\). Then there is an isomorphism of chain complexes \((\mathcal{V}(\Gamma), \mu) \cong (\mathcal{L}, \mu)\).

**Proof.** Recall that the vertices of \(\Gamma\) are ordered. Thus, for each marking \(m\) on \(\Gamma\) with \(|V_m| = i + j\) there is an induced order on \(V_m(\Gamma)\) and an unique order preserving bijection \(\varphi_m\) between \(V_m(\Gamma)\) and the vertices of \(\Lambda_{i+j}\) (which are all marked). Sending \((\Gamma, m) \in \mathcal{V}(\Gamma)^+\) to \((\Lambda_{i+j}, m')\) where \(m' = m \circ \varphi_m^{-1}\) defines a chain map from \(\mathcal{V}(\Gamma)\) to \(\mathcal{L}\) which is clearly bijective.

**Lemma 3.3.** \(H_*(\mathcal{L}(\Lambda_n), \mu) = 0\) for all \(n > 0\).

**Proof.** To prove the lemma, we define a chain isomorphism \(\Phi\) between \((\mathcal{L}(\Lambda_n), \mu)\) and the augmented (and degree shifted) simplicial chain complex \(\tilde{C}(\Delta)[+1]\) of the standard \((n-1)\)-simplex \(\Delta = [v_1, \ldots, v_n]\) which is contractible, hence has vanishing reduced cohomology.

For \(\tilde{C}_i := C_{i-1}(\Delta), \tilde{C}_0 = \mathbb{Z}\) and \(\partial_0 = \varepsilon : \sum \lambda_i v_i \mapsto \sum \lambda_i\) define

\[
\Phi_i(\Lambda_n, m) := \begin{cases} 
\{ [v_k \mid v_k \text{ is 1-marked}] \} & i > 0 \\
1 & i = 0.
\end{cases}
\]

Let \(m_v\) mark a single vertex by \(1\), then we compute

\[
\varepsilon \Phi(\Lambda_n, m_v) = \varepsilon [0] = 1 = \Phi \mu(\Lambda_n, m \equiv 2).
\]

For \(l > 0\) and \(v_{k_1} < \ldots < v_{k_l}\) the 1-marked vertices of \(\Lambda_n\)

\[
\partial \Phi(\Lambda_n, m) = \sum_{i=1}^{l} (-1)^{i+1} [v_{k_1}, \ldots, \hat{v}_{k_i}, \ldots, v_{k_l}]
\]

and

\[
\Phi \mu(\Lambda_n, m) = (-1)^n \sum_{i=1}^{l} (-1)^{l-i} \Phi(\Lambda_n, m_{[i_{<}, \ldots, i_{<}]})
\]

\[
= (-1)^n \sum_{i=1}^{l} (-1)^{l-i} [v_{k_1}, \ldots, \hat{v}_{k_i}, \ldots, v_{k_l}].
\]

The expressions match up to a sign \((-1)^{n-i+1}\) which may be absorbed into the definition of \(\partial\) without changing the homology of this complex. The map \(\Phi\) is clearly bijective, thus inducing an isomorphism on homology.

There is one summand on the right side of \((3.1)\) which has non-trivial homology, the piece corresponding to the empty graph \(\Lambda_0\) representing the case where \(\Gamma\) has no marked vertices at all. This element is \(\mu\)-closed, but not exact.

**Lemma 3.4.** \(H_k(\mathcal{L}(\Lambda_n), \mu) = 0\) for all \(k > 0\) and isomorphic to \(\mathbb{Z}\) in degree 0.

\[\text{In this subsection we use homological conventions for convenience.}\]
Proof. \(\mathcal{L}(\Lambda_0)\) consists of a single element, the empty graph, concentrated in degree 0, and \(\mu\) maps it to 0. 

Putting everything together we arrive at

**Proposition 3.5.** The homology of the complex \((\mathcal{V}(\Gamma), \mu)\) is given by

\[
H_k(\mathcal{V}(\Gamma), \mu) \cong \begin{cases} 
\mathbb{Z} & k = 0 \\
0 & \text{else.}
\end{cases}
\]

3.2. **The cohomology of the complex** \((\mathcal{V}, u + \mu)\). Having understood the homology of \(\mathcal{V}\) with respect to \(\mu\), we now consider the full differential \(U = u + \mu\). Again, it suffices to study each summand \(\mathcal{V}(\Gamma)\) individually. The bigrading is given by the number of 1- and 2-marked vertices so that \((0, 1)\) and \((-1, 1)\). In the following it will be convenient to change this bigrading into the total number of marked vertices and those of type 1. From now on we work with cohomological grading, i.e. we introduce a sign for the second part of the bigrading.

Hence, given a graph \(\Gamma\) we define

\[
T := T(\Gamma), \quad T^{i,j} := \{(\Gamma, m) \mid m : V \to \{0, 1, 2\}, |V_m| = i, |V_1| = -j\} = \mathcal{V}(\Gamma)^{i,j}.
\]

The differentials \(u\) and \(\mu\) are then of bidegree \((1, 0)\) and \((0, 1)\), respectively. The associated total complex is \((T, u + \mu)\) where \(T^n = \bigoplus_{i+j=n} T^{i,j}\), so \(n\) is the number of 2-marked vertices.

**Theorem 3.6.** For any graph \(\Gamma\) the total complex \((T, U)\) satisfies \(H^n(T, U) = 0\) for all \(n > 0\) and isomorphic to \(\mathbb{Z}\) in degree 0.

**Proof.** It is possible to give a constructive proof making the isomorphism explicit, but we opt for a spectral sequence argument.

Let \(\mathcal{T}\) be filtered by \(\mathcal{T} = F^0\mathcal{T} \supseteq \ldots \supseteq F^p\mathcal{T} \supseteq \ldots \supseteq F^{n+1}\mathcal{T} = 0\) with

\[
F^p\mathcal{T}^n := \bigoplus_{i+j=n, i \geq p} T^{i,j}.
\]

The associated spectral sequence starts with

\[
E^0_{p,q} = F^p(T^{p+q})/F^{p+1}(T^{p+q}) = T^{p,q}, \quad d^0_{p,q} : E^0_{p,q} \to E^0_{p,q+1} = \mu : T^{p,q} \to T^{p,q+1}.
\]

On its first page we have \(E^1_{p,q} = H^2(T^{p,\bullet}, \mu)\) and \(d^1_{p,q}\) induced by \(u\). But according to Proposition 3.5, the only non-zero entry is \(E^1_{0,0} = H^2(T^0, \mu)\). All the maps \(d^1_{p,q}\) are then zero, so that the sequence collapses at its first page and we have

\[
E^\infty_{p,q} \cong G^pH^{p+q}(T, u + \mu) = \begin{cases} 
H^0(T^0, \mu) & p, q = 0 \\
0 & \text{else.}
\end{cases}
\]

Thus, \(H^n(T, U) \cong \mathbb{Z}\) and all other cohomology groups of \((T, U)\) are trivial. 

3.3. **The cohomology of the gauge graph complexes.** In terms of the gluon and ghost cycle graph complexes Theorem 3.6 translates into

\[
H^k(\mathcal{E}, S) = \bigoplus_{\Gamma \in \text{Gra}_{r,t}} H^k(\mathcal{E}(\Gamma), S) \cong \bigoplus_{\Gamma \in \text{Gra}_{r,t}} H^k(\mathcal{V}(\Gamma), U) \cong \bigoplus_{\Gamma \in \text{Gra}_{r,t}} \mathbb{Z} \quad k = 0
\]

and similar for \((\mathcal{C}, T)\).

It is possible to write down explicit generators for these cohomology groups.

**Proposition 3.7.**

1. Let \(\chi^+ : \mathcal{E} \to \mathcal{E}\) be defined by \(\chi^+(\Gamma, m) := \sum_{e \in \Gamma} \chi^+_e(\Gamma, m)\) where

\[
\chi^+_e(\Gamma, m) := \begin{cases} 
0 & \text{if } e \text{ is adjacent to another marked edge} \\
(\Gamma, m_{e,+1}) & \text{else.}
\end{cases}
\]

Denote by \(m_0 : \mathcal{E} \to \{0\}\) the trivial marking. Then \(Se\chi^+(\Gamma, m_0) = 0\).

2. Let \(\delta^+ : \mathcal{C} \to \mathcal{C}\) be defined by

\[
\delta^+_e(\Gamma, m) := \sum_{c \in \mathcal{C}} \delta^+_c(\Gamma, m), \quad \delta^+_e(\Gamma, m) := \begin{cases} 
0 & \text{if } c \text{ is adjacent to a marked edge} \\
(\Gamma, m_{e,+1}) & \text{else.}
\end{cases}
\]

Then \(T\mathcal{e}\delta^+(\Gamma, m_0) = 0\).
Proof. Propostions 4.29 and 4.35 in [KSvS13], a straightforward calculation.

From a physical point of view this establishes the expected result; the cohomology of \((E, S)\) or \((C, T)\) is generated (up to scaling) by the sum over all graphs marked accordingly which resembles the pure gluon or gluon/ghost amplitudes.

In fact, we can combine the gluon and ghost cycle graph complexes into a single complex by considering graphs with admissible markings \(m : E \sqcup C \to \{0, 1, 2\}\). Define the free abelian groups

\[
G_n := \bigoplus_{i+j=n} \mathcal{G}^{i,j}, \quad \mathcal{G}^{i,j} := \langle (\Gamma, m) \mid \Gamma \in \text{Gra}_{r,l}, m : E \sqcup C \to \{0, 1, 2\} : |E_m| = i, |C_m| = j \rangle,
\]
equipped with two differentials \(S\) and \(T\). Then \(H(\mathcal{G}^{i,j}, S) \cong H(E^*, S)\) and \(H(\mathcal{G}^{i,j}, T) \cong H(C^*, T)\). Moreover, in this symmetrized picture the very same element generates both cohomology groups. Let \(m_0 : E \sqcup C \to \{0\}\) denote the trivial marking. If

\[
X_{r,l} := \sum_{\Gamma \in \text{Gra}_{r,l}} (\Gamma, m_0)
\]
evaluates the \(l\)-th order \(r\)-point amplitude for a scalar cubical field, then its general gauge theoretic counterpart is

\[
\tilde{X}_{r,l} := \sum_{\Gamma \in \text{Gra}_{r,l}} e^{\delta+} e^{\chi+} (\Gamma, m_0),
\]
the sum over all type 1 edge- and cycle-marked graphs. The equations \(S\tilde{X}_{r,l} = 0\) and \(T\tilde{X}_{r,l} = 0\), i.e. the fact that \(\tilde{X}_{r,l}\) is a generator of both \(H(\mathcal{G}^{i,j}, S)\) and \(H(\mathcal{G}^{i,j}, T)\), reveal thus \(\tilde{X}_{r,l}\) as the correct physical gauge theory amplitude.

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\[^{4}\text{Note that “real” ghost cycles are directed. This can be included without changing any of the results by declaring a marked cycle to represent the sum of two ghost cycles with opposite orientation.}\]

\[^{5}\text{Including fermions into the picture is a mere technicality. The compatibility of symmetry factors however is not so simple. See [KSvS13], Lemma 4.10 and 4.24, as well as [Kis19].}\]