Dynamics of impurity in the environment of Dirac fermions

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Abstract
We study the dynamics of a nonmagnetic impurity interacting with the surface states of a 3D and 2D topological insulator (TI). Employing the linked cluster technique we develop a formalism for obtaining the Green’s function of the mobile impurity interacting with the low-energy Dirac fermions. We show that for the non-recoil case in 2D, the Green’s function in the long-time limit has a power-law decay in time implying the breakdown of the quasiparticle description of the impurity. The spectral function in turn exhibits a weak power-law singularity. In the recoil case, however, the reduced phase-space for scattering processes implies a non-zero quasiparticle weight and the presence of a coherent part in the spectral function. Performing a weak coupling analysis we find that the mobility of the impurity reveals a $T^{-3/2}$ divergence at low temperatures. In addition, we show that the Green’s function of an impurity interacting with the helical edge modes (surface states of 2D TI) exhibit power-law decay in the long-time limit for both the non-recoil and recoil case (with low impurity momentum), indicating the break down of the quasiparticle picture. However, for impurity with high momentum, the quasiparticle picture is restored. The mobility of the heavy impurity interacting with the helical edge modes exhibits unusual behaviour. It has an exponential divergence at low temperatures which can be tuned to a power-law divergence, $T^{-4}$, by the application of the magnetic field.

Keywords: mobile impurity, topological insulator, Green’s function, mobility, power-law

(Some figures may appear in colour only in the online journal)

1. Introduction

In a pioneering work, Anderson (1967) showed that in the thermodynamic limit adding an impurity as a perturbation in a many-particle system consisting of free electron gas results in a vanishing overlap between the initial unperturbed ground state and the final ground-state leading to a phenomenon known as the orthogonality catastrophe (OC) [1]. This idea was extended by Nozieres and De Dominicis into a dynamical theory of the absorption process by studying the long-time behaviour of the core-hole Green’s function [2]. A number of studies have uncovered drastic modifications to the fermionic system due to its interaction with a single impurity. Some of the examples include, the x-ray edge effect [3–7], the Kondo problem [8–11], impurity in a semiconductor quantum dot [12, 13], impurity interacting with a quantum liquid [14–18], heavy particle in a fermionic bath [19–22]. The physics of impurity dynamics in a quantum bath remains a topic of intense focus with new directions being pursued in setups involving impurity and Bose–Einstein condensate [23, 24], dissipative baths [25, 26], stochastically driven impurities [27], and in ultracold atomic systems [28–32].

Lately, a new class of materials, the 3D and 2D topological insulators (TI) having spin-momentum locked surface states has generated tremendous interest [33, 34]. Well known examples of the 3D TI include [33] Bi$_1$–Sb$_{x}$, Bi$_2$Se$_3$, Bi$_2$Te$_3$, etc, whereas those of the 2D TI include HgTe quantum well [35, 36], silicene [37], stanene [38] etc. The TIs exhibit insulating behaviour in the bulk but have surface states which are
metallic and are described by the relativistic Dirac equation [35, 39–45]. They have an odd number of gapless Dirac cones in which the spin and momentum are locked together into a spin-helical state and are protected by the time-reversal symmetry (TRS). The physics of 3D TIs have been studied in quite detail, some of the questions addressed include magnetoelectric response in TI [46–49], integer quantum hall effect [50], competition between localization and anti-localization [51, 52], the effects of phonon and disorder on transport [53], bulk–surface coupling [54], impurity dynamics at the particle–hole symmetric point [55], role of magnetic and nonmagnetic impurities from the point of view of their effect on local charge/spin density of states and also on the surface states of 3D TI with Dirac spectrum has been an extremely active area of research [56–65] etc. At the same time, a number of work on the interacting surface states of 2D TI which are the helical Luttinger liquid have been made. These include studies on the Kondo effect in the helical edge liquid [66], Coulomb drag [67], spin susceptibility [68, 69], transport [70, 71], structure factor [72], the role of inelastic scattering channels on transport [73–75], etc.

The 2D Dirac system is known to exhibit behaviour different from the conventional 2D electron system. For example, although the 2D Dirac Hamiltonian belongs to the same symplectic symmetry class as the conventional 2D Hamiltonian with spin–orbit interaction and parabolic spectrum, the former does not undergo metal–insulator transition even with increasing disorder strength and remains metallic [76–79]. Moreover, due to the linear dispersion, the inelastic quasiparticle scattering rate in a 2D Dirac system is qualitatively different from that of the 2D electron system [80–82]. Thus, the unusual surface states of TI provides an interesting new scenario for the study of the phenomenon of orthogonality catastrophe. We, however, find that the physics of the heavy particle in a bath of Dirac fermions is similar to that in a fermionic bath [20–22]. The behaviour of the impurity Green’s function is strongly influenced by the presence or absence of infrared singularity. In $D = 2$, the interaction between the bath and a particle with the recoilless mass generates an infinite number of low-energy particle–hole pairs resulting in an incoherent behaviour of the heavy particle, i.e., the quasiparticle weight vanishes. The spectral function, in turn, exhibits a power-law divergence at the renormalized energy. In contrast, the recoil of the heavy particle suppresses the phase space available for particle–hole generation resulting in non-zero quasi-particle weight and consequently a $\delta$–function peak in the spectral-function. However, a part of the spectral-weight is transferred to the incoherent part which exhibits a square-root singularity. We find that the Maxwell–Boltzmann distribution of the mobile impurity governs the typical momentum transfer between the impurity and the Dirac fermions resulting in a $T^{-3/2}$ temperature dependence of the mobility of the impurity. These results turn out to be consistent with those obtained for an impurity interacting with a bath of fermions having the parabolic spectrum.

The study of interaction effects between the mobile impurity and 1D helical Luttinger liquid reveals that the quasiparticle weight vanishes except for the scenario when the momentum of the mobile impurity with mass $M$ exceeds $Mv$, where $v$ is the sound velocity. This result is in agreement with an earlier study of a single spin-down fermion interacting with the bath of spin-up fermions [83]. As for mobility, in the absence of a magnetic field, the mobility of the impurity is limited by forwarding scattering-processes only and diverges exponentially. However, turning on the magnetic-field results in a power-law divergence at low-temperatures.

The paper is organized as follows: section 2 includes a general description of our model along with the Green’s function of the Dirac fermions in TI for the 2D case. In section 3 the linked cluster technique has been used to develop a formalism for obtaining the Green’s function of impurity interacting with the Dirac fermions. In addition, the long-time behaviour of the impurity Green’s function for the recoilless and the recoil case and the corresponding spectral-function have been studied. In section 4 the temperature dependence of the mobility of the impurity has been obtained. In section 5 we establish the model for the 1D case and discuss impurity Green’s function for the recoilless and the recoil case. The mobility of impurity interacting with 1D helical liquid is discussed in detail in section 6 followed by a section on the summary of our results.

2. Model for the 2D case

We consider the motion of a heavy particle with mass $M$ having a parabolic dispersion (for example in a 2D-semiconductor) and constrained to move in two-dimensions. The semiconductor is placed on top of the surface of a 3D topological insulator separated by a thin insulating layer (see figure 1). We make following three assumptions: the bulk is insulating and does not influence the physics, absence of tunneling between the TI and the semiconductor and that the heavy mass $M$ interacts with the Dirac fermions via a contact potential. The low energy effective Hamiltonian of the 2D Dirac fermions has the following form [84, 85]

$$H = \frac{p^2}{2M} + \hbar v_F \hat{\sigma} \cdot \hat{k} + \sum_q V(q)\rho_0(q)m(-q),$$

where $p$ is the momentum of the particle and the second term, $\hbar v_F \hat{\sigma} \cdot \hat{k}$, represents the transformed low-energy effective Hamiltonian of the Dirac fermions. Henceforth we will work in the $h = 1$ and $v_F = 1$ units unless specified otherwise. The third term is the interaction term, where the potential $V(q) = U/A$ is momentum independent and the density operators $\rho_0(q)$ and $m(q)$ correspond to the Dirac fermions and the impurity particle, respectively. The second quantized form of the Hamiltonian in equation (1) acquires the following form

$$H = \sum_p \epsilon_p \hat{a}_p^\dagger \hat{a}_p + \sum_{k,\alpha,\beta} \epsilon_k^{\alpha} c_{k\alpha}^\dagger (\hat{\sigma} \cdot \hat{k}) c_{k\beta} + V,$$

where $\epsilon_p = p^2/2M$ is the energy of the particle and the interaction potential in the second quantized notation is

$$V = \frac{U}{A} \sum_{\sigma, k_1, k_2, \alpha} \hat{a}_{k_2}^\dagger \hat{a}_{k_1} \epsilon_{k_2}^{\alpha} \hat{a}_{k_1+q, \sigma} \hat{a}_{k_1, \alpha}.$$
in the upper band only, the expression for the Green’s function which opens up a gap in the TI. Considering the Dirac fermions completely filled valence band) behaves like an impurity in the environment of Dirac fermions.

The corresponding zero temperature Matsubara Green’s function for the Dirac fermions on the surface of a 3D TI has the following form,

$$G(k, i\omega) = \sum_{\eta=\pm 1} \left[ \frac{i + \eta(\vec{\sigma} \cdot \vec{k})/\xi_k}{i\omega - \eta \xi_k + \mu_F} \right],$$

where $\vec{k} = k_x \hat{e}_x + k_y \hat{e}_y + \Delta \hat{e}_z$ and $\xi_k = \sqrt{k^2 + \Delta^2}$ is the dispersion relation of Dirac fermions and $\Delta$ is the mass term which opens up a gap in the TI. Considering the Dirac fermions in the upper band only, the expression for the Green’s function in the momentum–time representation is given by

$$\hat{G}(k, t) = \left[ \frac{i + (\vec{\sigma} \cdot \vec{k})/\xi_k}{2i} \right] [\theta(t)(1 - n_k) - \theta(-t)n_k] e^{-i\xi_k t},$$

where $\xi_k = \xi_k - \mu_F$.

3. Impurity Green function

In the following, we will utilize the linked cluster method to obtain the expression for the Green’s function of an impurity particle interacting with the surface states of a 3D TI. The approach is similar to the one used for the polaron problem, here instead, we will incorporate the interaction between the impurity particle and the Dirac fermions. The expression for the impurity Green’s function to all orders in interaction has the following form:

$$G(p, t) = \sum_{n=0}^{\infty} M_n(p, t),$$

where

$$M_n(p, t) = (-i)^{n+1}/n! \int_0^t dt_1 \cdots \int_0^t dt_n C_n,$$

and

$$C_n = \langle \hat{a}_p(t) V(t_1) \cdots V(t_n) \hat{a}_p^\dagger(0) \rangle.$$
and from the impurity creation and annihilation operators we obtain:

\[ Z = \langle T \{ a_p(t)a_{p}^{\dagger}(t_0) \} \rangle = e^{-i\text{\varepsilon}_p t_0} \sum_{q} \Theta(\text{\varepsilon}_k - \text{\varepsilon}_p - q) \exp[i\text{\varepsilon}_p(t_0 - t_1)]. \]

Thus, \( S_2 \) is given by

\[ S_2 = \frac{G_0^{-1}(p,t)}{2} \sum_{k_1,k_2} \int dt_1 dt_2 \text{Tr} \left[ G_{k_1}(t_1-t_2)G_{k_2}(t_2-t_1) \right]. \]

Performing the integration on time, we obtain

\[ S_2 = \frac{U^2}{A^2} \sum_{k_1,k_2} \left[ 1 + \hat{k}_1 \cdot \hat{k}_2 \right] (1 - n_{k_1})n_{k_2} \left[ \frac{i\text{\varepsilon}_p}{\Delta} - \frac{1 - e^{-i\Delta t}}{\Delta^2} \right], \tag{12} \]

where, \( \Delta(k_1,k_2) = \text{\varepsilon}_p + k_1 - k_2 - \text{\varepsilon}_p + \xi_{k_1} - \xi_{k_2} \). Note that the chiral form in (12) is a feature of the particle–hole pairs in the Dirac sea. The contributions to \( S_2 \) are enhanced for \( k_1 \) and \( k_2 \) pointing in the same direction, while they are suppressed for large angles.

Putting together \( S_1 \) and \( S_2 \) we obtain the following expression for the impurity Green’s function

\[ iG(p,t) = \Theta(t) \exp[-i\text{\varepsilon}_p t + \mathcal{X}(t)], \tag{13} \]

where the renormalized energy \( \tilde{\text{\varepsilon}}_p \) is given by

\[ \tilde{\text{\varepsilon}}_p = \text{\varepsilon}_p + \frac{U}{A} \sum_{k} n_k - \frac{U^2}{A^2} \sum_{k_1,k_2} \left[ 1 + \hat{k}_1 \cdot \hat{k}_2 \right] (1 - n_{k_1})n_{k_2} \frac{1 - e^{-i\Delta t}}{\Delta(k_1,k_2)}, \]

while the function \( \mathcal{X}(t) \) which will be our object of interest encodes the non-trivial \( t \) dependence and is given by

\[ \mathcal{X}(t) = -\frac{U^2}{A^2} \sum_{k_1,k_2} \left[ 1 + \hat{k}_1 \cdot \hat{k}_2 \right] (1 - n_{k_1})n_{k_2} \frac{1 - e^{-i\Delta t}}{\Delta^2}. \tag{14} \]
The largest contribution is obtained from the $q_1 < q < q_2$ region wherein the polarization function is given by

$$\text{Im } \Pi_\lambda(q, \omega) = -\frac{1}{2\pi \sqrt{q^2 - \omega^2}} \sum_{\nu = \pm} \eta F(2\mu_F + \eta \omega). \quad (18)$$

The leading term obtained upon the momentum integration yields linear in $\omega$ term given by

$$\int \frac{q}{2\pi} \text{Im } \Pi(q, \omega) \approx -\frac{\omega}{2\pi^2} \int_{q_1}^{q_2} q \frac{\partial F(x)}{\partial x} \bigg|_{x=2\mu_F} = -\frac{1}{\pi^2} \int_{q_1}^{q_2} q \frac{\omega(4\mu_F^2 - q^2)}{\sqrt{4\mu_F^2(q^2 - \omega^2) - q^2(4\Delta^2 + q^2 - \omega^2)}}$$

$$\approx -\frac{k_F^2 \omega}{\pi}. \quad (19)$$

In addition, we obtain a second linear in $\omega$ term, which however, is smaller by a factor of $\Delta^2/\mu_F^2$. The linear in $\omega$ term is also obtained for the case of an impurity interacting with a bath of fermions having a parabolic spectrum [19, 21, 22, 86]. Keeping the dominant term in $\chi(t)$ and in the long-time limit we obtain [86]

$$x(t) \approx -\frac{k_F^2 U^2}{\pi^2} \log(1 + i\omega \omega_c), \quad (20)$$

where $\omega_c$ is the bandwidth and is taken to be of the order of Fermi-energy. Thus the behaviour of the Green’s function (13) in the long-time limit is determined by the $t$ and the log$t$ term both of which are in the exponential. The latter term leads to a power-law decay of the Green’s function $\propto 1/t^\nu$, where $\nu = k_F^2 U^2/\pi^2$ and is responsible for the orthogonality catastrophe.

Besides the Green’s function, the spectral function of the heavy particle acquires drastic modifications as compared to the free case. The spectral function is given by

$$A(\epsilon) = -2 \text{Im } \left[ \int_{-\infty}^{\infty} d\epsilon' e^{i\epsilon' G(t)} \right] = \frac{e^{-\frac{\epsilon}{\eta}}}{\epsilon} \int_{-\infty}^{1+i\infty} dz \frac{e^{\frac{\omega}{\mu}}}{\sqrt{\omega}}.$$

First consider the case $\epsilon > 0$, since $\epsilon^2/\omega^2$ is analytic everywhere for $\text{Re}(z) > 1$, the contour of integration can be pushed to $\text{Re}(z) > 1$ and $|z| \to \infty$. The integrand vanishes everywhere for the modified contour, therefore $A(\epsilon) = 0$ for $\epsilon < 0$. On the other hand, for $\epsilon > 0$, the integrand is analytic everywhere except for the negative real axis where it has a branch cut. Therefore, the contour can be deformed on to the negative real axis and we obtain

$$A(\epsilon) = -\frac{2}{\mu_F} \text{Im } \left[ \int_{0}^{\infty} dt e^{-t} e^{-\epsilon t} e^{i\nu t} \right] = \Theta(\epsilon) \frac{2\pi}{\mu_F} \frac{1}{\Gamma(\nu)} \epsilon^{-\nu-1}, \quad (21)$$

Thus the spectral function is no longer a delta-function peaked at the renormalized energy $\epsilon_F$, instead due to the large number of particle–hole excitations it has a power-law singularity given by $A(\epsilon) \propto \Theta(\epsilon - \epsilon_F)/\epsilon^{1-\nu}$. Thus the localized impurity acts as an incoherent excitation due to its interaction with the Dirac electrons and decays with time.

3.2. Recoil case: suppression of orthogonality catastrophe

The above-discussed scenario is significantly modified when considering an impurity with finite mass. In a typical scattering event involving an impurity atom and a particle–hole pair with momentum $q$ and energy $\omega$ (where $q \nu_F \geq \omega$) the impurity momentum changes by $q \sim \sqrt{2M\Delta}$. Thus for $\sqrt{2M\Delta} \ll 2k_F$, the phase-space available for low-energy scattering is severely restricted. This, in turn, is reflected in the deviation of $\rho(\omega)$ from the linear behaviour and results in a modified Green’s function.

Following earlier discussion, $\rho(\omega)$ for the recoil case is given by,

$$\rho(\omega) = \int d^3q \text{Im } \Pi(q, \omega - \epsilon_{\tilde{p}}, \tilde{q} + \epsilon_{\tilde{p}}). \quad (22)$$

In the limit of small frequency and vanishingly small momentum of the impurity, the limits of integration (see figure 4) are from $\omega$ to $\sqrt{2M\omega}$, where $\sqrt{2M\Delta} \ll 2k_F$ and we have assumed $\Delta \ll \mu_F$. Using the expression for the polarization operator given in equation (58) and replacing $\omega \rightarrow \omega - \tilde{q}^2/2m_F$, $\rho(\omega)$ acquires the following form,

$$\rho(\omega) = \left(\frac{\omega}{\omega} - \frac{\omega^2}{\Delta^2} \right) \left(\frac{\omega}{\omega} - \frac{\omega^2}{\Delta^2} \right) \left(\frac{4\mu_F^2 - q^2}{4\mu_F^2 - \omega^2} \right),$$

where the leading order result is given by $\rho(\omega) = -\omega g^3/2$, with the proportionality constant being $g = 4\sqrt{2M\Delta} \mu_F$. Thus compared to the infinite mass scenario, recoil of the impurity causes suppression of the particle–hole excitation and as will be shown below the impurity quasiparticle weight remains non-zero.

The quasiparticle weight $Z_0$ is obtained from evaluating the time independent part of $\chi(t)$ (15), i.e.,

$$U^2 \int \frac{d\omega}{\pi} \frac{\rho(\omega)}{\omega^2}. \quad \text{ }}$$
yielding $Z_0 \approx \exp\left[-2gU^2/\sqrt{\omega_c/\pi}\right]$. As in the infinite mass case the linear in time term in the exponential, $\exp(-i\epsilon_p t)$, gets trivially renormalized to $\epsilon_p$. However, unlike the log term which is responsible for the strong suppression of the Green’s function of the infinite mass, here the time-dependent integral of $\chi(t)$ results in a $t^{-1/2}$ term, specifically

$$-U^2 \int \frac{d\omega}{\pi} \frac{\rho(\omega)}{\omega^2} e^{-i\omega t} \approx gU^2 \frac{e^{-i\epsilon_p t}}{\sqrt{\pi} t} \approx gU^2 \frac{e^{-i\epsilon_p t}}{\sqrt{\pi}} t^{-1/2}.$$  

Therefore the long-time behaviour of the Green’s function acquires the form

$$G(p,t) = -i\Theta(t)Z_0 \exp\left(-i\epsilon_p t + gU^2 \frac{e^{-i\epsilon_p t}}{\sqrt{\pi}} t^{-1/2}\right).$$ (23)

We note that for $t \to \infty$ the second term in the exponential vanishes and we are left with a Green function describing a well defined quasiparticle excitation with $Z_0 < 1$.

As before, an insightful perspective into the nature of excitations is revealed from the behaviour of the spectral function $A(\epsilon)$. The small contribution to the Green’s function due to the $t^{-1/2}$ term allows for a perturbative treatment of the spectral function. Therefore, the spectral function can be split into a coherent and incoherent part. The coherent part is given by $A^{\text{coh}}(\epsilon) \approx Z_0\delta(\epsilon - \epsilon_p)$. On the other hand, the incoherent part has a square-root singularity with the following expression,

$$A^{\text{incoh}}(\epsilon) \approx 2gU^2Z_0\Theta(\epsilon - \epsilon_p)/\sqrt{\epsilon - \epsilon_p},$$ (24)

and is obtained by performing a partial series expansion of the above Green’s function and taking the imaginary part of the Fourier transform of $\delta G \propto -i\Theta(t)e^{-i\epsilon_p t}t^{-1/2}$, where we have made use of the following result: $\int_0^\infty e^{\omega t} dt/\sqrt{t} = \sqrt{\pi} e^{\pi\omega^2/4}/\sqrt{[\omega]}$.

The non-zero quasiparticle weight and the delta-function in the spectral function attest to the well-behaved quasiparticle like excitation. At the same time, the weaker square-root singularity in the incoherent part is indicative of the remnants of the orthogonality physics that is significantly subdued due to the relatively fewer number of particle–hole excitations generated in the recoil process.

4. Mobility of impurity

In this section we will obtain the low temperature behaviour of the DC mobility which is given by $\mu = e\tau/M$, where $\tau$ is the transport time [22]. We estimate $\tau$ by first calculating the inverse quasiparticle lifetime for a mobile impurity with momentum $p$ using the Fermi’s golden rule [88]

$$\frac{1}{\tau_p} = -\int \frac{d\omega}{2\pi^2} \frac{d^2q}{(2\pi)^2} U^2(q) \frac{1}{e^{\beta \omega} - 1} \text{Im} \left[ \Pi(q,\omega) \delta(\omega + \epsilon_p - (\epsilon_p + q)) \right],$$ (25)

where the above expression is a modified version of the standard formula for the life-time of fermions which has an additional $[1 - n_F(\epsilon_p + q)]$ factor. The term represents the probability that the scattered state is unoccupied, which in our case is simply set to unity as the corresponding impurity state remains unoccupied. An identical expression is obtained from the on-shell imaginary part of the self-energy of the mobile impurity.

The statistical average of $1/\tau_p$ is performed with respect to the Boltzmann weight factor. We denote the average as $\langle 1/\tau_p \rangle$ given by

$$\langle 1/\tau_p \rangle = \frac{\beta}{2\pi M} \int dp \frac{1}{\tau_p} e^{-\beta \epsilon_p}.$$ (26)

For our purpose the above expression is useful as the time-scale obtained from it yields the same order of magnitude and the temperature dependence as the transport time.

The energy scale in the integral of equation (26) is set by the temperature. Therefore, the contribution to the integrals are dominated by the regions $p, q \sim \sqrt{2MT}$ and $\omega \sim T$. In the low temperature regime ($T \ll k_B/\hbar$) the typical momentum transferred $q$ satisfies $q \ll k_F$, moreover, $\omega/\sqrt{q^2v^2} \ll 1$ which implies the polarization operator can be expanded in the ratio $\omega/\sqrt{q^2v^2}$ yielding $\text{Im} \Pi(q,\omega) \approx -(4/\pi)\mu(\epsilon_p + q)/\sqrt{q^2v^2}$. Performing the angular integration removes the $\delta$-function and yields

$$\langle 1/\tau_p \rangle = \frac{4\mu U^2}{\pi^3 v_F} \left(\frac{MT}{\alpha}\right)^{1/2} \int_0^\infty \frac{d\bar{q}}{\bar{q}} \frac{\bar{q}}{e^{\beta T^2} - 1} \sqrt{\bar{q}^2 - (\omega^2/4)^2},$$ (27)

where we have used dimensionless variables $\bar{p} = p/\sqrt{MT}$, $\bar{q} = q/\sqrt{MT}$ and $\bar{\omega} = \omega/T$. The lower cut-off on the frequency integration is imposed by the $\delta$-function which forbids the frequency range $\omega < -\epsilon_p$. We note that the dimensionless integral is of order $O(1)$, while a change of variables allows us to extract the $T^{3/2}$ temperature dependence of the inverse scattering time. The above result implies that the mobility of impurity interacting with the Dirac fermions on the surface of TI in the low temperature region diverges with decreasing temperature as $\mu \propto T^{-3/2}$. This temperature dependence of the mobility is qualitatively similar to the one discussed by Rosch and Kopp [22] for a heavy mass-particle in a fermionic bath.

5. Interaction of impurity with the helical edge state

So far we have considered the interaction of an isolated impurity with that of the surface states of a 3D TI. Similar to a 3D TI, a 2D TI has an insulating bulk and metallic edge states. The pair of gapless-edge states have specific chirality (also called helical edge-states) and are time-reversed partners of each other. These are the 1D helical modes in which backscattering due to the nonmagnetic impurities is forbidden. A gap in the spectrum can be introduced by breaking time-reversal symmetry which is typically achieved by an external magnetic field. In this section, we will first develop the formalism to describe the interaction of an isolated mobile impurity with that of an interacting helical liquid followed by the study of Green’s function in the non-recoil and recoil case.
The non-interacting Hamiltonian of a helical liquid in the presence of a magnetic field has the following form
\[ \hat{H}_\text{HLL} = \int dx \psi^\dagger(x)(-i\hbar\partial_x \sigma_z + B \sigma_x - \epsilon_p)\psi(x), \tag{28} \]
where \( B \) is the Zeeman field applied along the \( x \)-direction (taken to be perpendicular to the spin-quantization axis) and the dispersion is given by \( \epsilon_{\pm} = \pm \sqrt{v_l^2 p_x^2 + B^2} \).\( \gamma_p = \tan^{-1}(p/B) \), while \( \hat{a} \) and \( \hat{\bar{a}} \) correspond to upper and lower bands respectively. We consider the scenario wherein the lower band is completely filled (henceforth it will be ignored) whereas the upper band is filled till the Fermi momentum \( \pm k_F \). Thus the field operator \( \psi(x) \) has the following form
\[ \psi(x) = \left[ \hat{a}(k_F)\psi_p(x)e^{i2k_Fx} + \hat{\bar{a}}(-k_F)\psi_L(x)e^{-i2k_Fx} \right], \]
where \( \psi_p(x) \) and \( \psi_L(x) \) are the slow degrees of freedom about the points \( k_F \) and \( -k_F \), respectively, and the fermion spin texture is given by
\[ \hat{a}(p) = \frac{1}{2} \left\{ a_+ + \frac{p}{|p|} a_-, a_+ - \frac{p}{|p|} a_- \right\}, \tag{29} \]
where \( a_\pm = \sqrt{1 \pm \frac{p}{|p|} \sqrt{B^2 + p^2}} \).

We express \( \psi_p(x) \) and \( \psi_L(x) \) in terms of the slowly varying bosonic fields \( \phi(x) \) and \( \theta(x) \) as follows
\[ \psi_p(x) = \frac{1}{\sqrt{2\pi a_0}} e^{i\theta(x)} \quad \psi_L(x) = \frac{1}{\sqrt{2\pi a_0}} e^{i\phi(x)}, \tag{30} \]
where \( a_0 \) is the short distance cutoff and the bosonic fields satisfy the commutation relation: \([\phi(x), \theta(y)] = -i\pi \text{sgn}(x-y)/2 \). Plugging (30) into (28) the Hamiltonian acquires the standard quadratic form in terms of the bosonic fields \[ \hat{H}_\text{HLL} = v \int \frac{dx}{2\pi} \left[ (\partial_x \phi)^2 + (\partial_x \theta)^2 \right]. \tag{31} \]

The Hamiltonian (31) is modified by including the interaction terms \( 1/2 \int dxdy U(x-x')\psi_p(x)\psi_p(x') \), where the density operator is given by
\[ \rho(t) = \psi_p^\dagger(x)\psi_p(x) + \psi_L^\dagger(x)\psi_L(x) + \frac{B}{\sqrt{2\pi + k_F^2}} \]
\[ \times \left[ \psi_p^\dagger(x)\psi_L(x)e^{-i2k_Fx} + \psi_L^\dagger(x)\psi_p(x)e^{i2k_Fx} \right]. \tag{32} \]

It is worth noting that the \( 2k_F \) component of the density in a helical liquid is allowed due to the presence of the magnetic-field. The interaction corrections arising from the forward-scattering terms: \( \psi_{\text{R}/L}^\dagger(x)\psi_{\text{R}/L}(x')\psi_{\text{R}/L}(y) \psi_{\text{R}/L}(y') \) yield \( \int \frac{dx}{2\pi} (\partial_x \phi)^2 \) term to the Hamiltonian, where \( U_\text{R}(k) \) is the \( k \)th mode of the \( U_\text{R} \) potential. On the other hand, from the back-scattering terms
\[ \psi_{\text{R}/L}^\dagger(x)\psi_{\text{R}/L}(x)\psi_{\text{R}/L}(y)\psi_{\text{R}/L}(y')e^{i2k_F(x-y)}, \]
one obtains correction to the Hamiltonian which is proportional to the square of the field-strength and given by \[ -\frac{B^2}{2\pi^2 + k_F^2} \frac{U_\text{R}(2k_F)}{2}\int dx (\partial_x \phi)^2. \]
The interaction modified Hamiltonian thus acquires the following form
\[ \hat{H}_\text{HLL} = v \int \frac{dx}{2\pi} \left[ (\partial_x \phi)^2 + K[\pi \Pi(x)]^2 \right], \tag{33} \]
where \( K = \sqrt{e^2/(eF + r)} \) and \( r = [U_\text{R}(0) - B^2/(2\pi^2 + k_F^2)U_\text{R}(2k_F)]/\pi \).

In terms of the bosonic annihilation operator
\[ b_p = \frac{1}{\sqrt{2}|p|} \left[ -\frac{|p|\phi_p}{\sqrt{\pi}} + i\frac{K}{\sqrt{\pi}} \Pi \right], \tag{34} \]
the potential term due to the interaction of the mobile impurity with the bosonic excitation, \( V = U \int dx a_\dagger(x)a(x)/2\), is given by
\[ V = U \sum_{kq} \text{sgn}(q) \sqrt{\frac{|q|}{\pi}} \sum_{\alpha} a_\alpha^\dagger(k+q) \epsilon_{\alpha}(b_q + b_{-q}). \tag{35} \]

With this expression for the Hamiltonian, we will employ the linked cluster expansion technique to describe the modifications to the impurity Green’s function \[ \langle \Phi \rangle. \] As before, the interaction modified impurity Green’s function has the form \( G(t,t') = G_0(t,t')\text{e}^{-\sum_{t'}\Sigma^I_{t-t'}}, \) where \( G_0(t,t') = -i\theta(t-t')e^{-\Sigma^I_{t-t'}} \). It suffices to focus till the second cumulant. The first cumulant, \( \Sigma^I_1 = -i \int dt_1 \int dr_1 \langle [a_\alpha(t)v(t)v(t_1)a_\beta(0)] \rangle / G_0 \) vanishes as it involves averaging over a single boson operator. The non-vanishing contribution arises from the second cumulant: \( \Sigma^I_2 = G_0^{-1}M_2 - S_2^2 / 2, \) where\[ M_2 = \frac{(-i)^3}{2} \int dr_1 \int dr_2 \langle [a_\alpha(t)v(t)v(t_1)a_\beta(0)] \rangle. \]
As in the 2D case only the connected diagrams need be considered. In the unperturbed Green’s function the second cumulant has the following form,
\[ S_2(t) = (-i)^3 \sum_q V^2(q) \int_0^{t} dt_1 \int_0^{t} dt_2 G_0(k, t-t_1) G_0(k, t-t_2) D_0(q, t-t_1-t_2) G_0(k, t-t_1-t_2), \tag{36} \]
where \( D_0(q, t) = \text{e}^{-i\theta(t-t_2)e^{-\Sigma^I_{t-t_2}}} \) is the zero temperature time ordered bosonic
Green’s function. Performing the integration over \( t_2 \) and \( t_1 \) we obtain

\[
S_2(k, t) = -\int d\omega \rho(\omega, k) \left[ -\frac{\nu}{\omega} + 1 - e^{-i\omega t} \right],
\]

(37)

where

\[
\rho(\omega, k) = \frac{U^2}{2\pi} \int \frac{dq}{2\pi} \left| q \right| \delta(\omega - \epsilon_k + q + v|q|).
\]

(38)

We note that similar to the 2D case, the first term (linear in time term) in (37) renormalizes the impurity, whereas it is again the second term which determines the long-time asymptotics of the impurity Green’s function.

### 5.1. Non-recoil case

For the non-recoil case which also corresponds to \( M = \infty \), the impurity energy terms drop out from the \( \delta \)-function, therefore the \( \rho \) term acquires the simple form

\[
\rho(\omega) = \frac{U^2}{2\pi} \int \frac{dq}{2\pi} \left| q \right| \delta(\omega - v|q|) = \frac{U^2}{2\pi^2 v^2} \omega,
\]

(39)

where \( \omega > 0 \). The long-time asymptotics in particular the decay of impurity Green’s function is determined by the following term of \( S_2 \)

\[
-\frac{U^2}{2\pi^2 v^2} \int \frac{d\omega(1 - e^{-i\omega t})}{\omega} \approx -\frac{U^2}{2\pi^2 v^2} \log(\omega c).
\]

The Green’s function thus has a power-law decay given by

\[
G(t) \propto t^{-\frac{\nu}{2\pi v^2}},
\]

(40)

resulting in a non-Lorentzian spectral function similar to that obtained in equation (21). The above calculation confirms the well-known fact that in a 1D system the introduction of heavy impurity leads to orthogonality catastrophe.

### 5.2. Recoil case

Consider first the scenario for small impurity momentum, in particular \( k \ll Mv \). Unlike the 2D case, where the impurity exhibits quasiparticle behaviour even at very low momenta, in 1D the decay-behaviour of the Green’s function remains unchanged and is given by equation (40) implying a non-quasiparticle behaviour. Consider next the scenario \( k < Mv \), but \((Mv - k)/k \sim 1 \). The long-time behaviour of the impurity is determined by \( \rho \) near the small frequencies and the corresponding \( \omega \) expansion of \( \rho \) yields the following form

\[
\rho(\omega) = \frac{U^2 M}{2\pi} \int \frac{dq}{2\pi} \left| q \right| \sum_{i=1}^{2} \frac{\delta(\omega - q Mv (-1)^i)}{Mv + (-1)^i k} \approx \frac{U^2 M^2}{2\pi^2} \omega \left[ \frac{M^2 v^2 + k^2}{(M^2 v^2 - k^2)^2} \right].
\]

(41)

The Green’s function, therefore, exhibits power-law decay given by

\[
G(k, t) \propto t^{-\frac{\nu}{2\pi^2} \frac{M^2 v^2 + k^2}{(M^2 v^2 - k^2)^2}},
\]

(42)

where the exponent is now \( k \)-dependent and the \( k/Mv \ll 1 \) limit (40) is recovered from the above equation. Insipite of the decay behaviour, for \( k \gg \sqrt{2M/\tau_0} \) (where \( \tau_0 = 2\pi^2 / \omega_i \)) a quasiparticle type behaviour is expected till time \( t \sim \tau_0 \).

Finally consider the scenario wherein the initial impurity momentum is large, i.e., \( k > Mv \). In this case, the main contribution from the \( \delta \)-function integration yields a frequency independent term, \( \rho(\omega) = U^2 M^2 / 2\pi^2 \), arising from the \( q \approx 2(k - Mv) \) region. Thus from equation (37), it is easy to deduce that the decay term of the Green’s function results in a conventional Fermi-liquid type term \([83]\), i.e., \( \sim e^{-\nu t}/\omega \) where the life-time is given by \( 1/\tau \approx U^2 M / 4\pi \) (thus for \( v \gg \tau \) the excitation is well defined). The oscillatory term on the other hand acquires contribution from a rather unusual term given by \( (U^2 M^2 / 2\pi^2) \log(\omega c) \), which can be neglected in comparison to \( k^2 / 2M \) for \( t < \tau \) as long as \( v \gg U^2 \log(\omega c)/U^2 M \). This criterion on \( v \) also implies that the subleading contribution from the second \( q \)-region \( (\approx M \omega c / (Mv + k)) \) where the \( \delta \)-function is non-zero) can be neglected.

### 6. Mobility of impurity in 1D

The temperature dependence of the mobility of impurity constrained to move in 1D and interacting with the helical edge modes exhibit contrasting behaviour in the presence and the absence of a magnetic field. We will again focus our attention on the low temperature regime \( T \ll k_B^2 / M \). Consider first the scenario without the magnetic field, as discussed earlier the back-scattering processes will be absent and only the forward scattering processes governed by the interaction term \( (35) \) are allowed. Focussing on the weak-coupling limit we utilize the Boltzmann equation approach to analyze the temperature dependence of the mobility. In the presence of an external electric field \( E \), the steady state Boltzmann equation for the momentum distribution function \( f_{\rho \alpha} \) is given by

\[
eE \frac{\partial f_{\rho \alpha}}{\partial p} = \sum_i \left\{ f_{\rho \alpha} \Gamma(k; p) - f_{\rho \alpha} \Gamma(p, k) \right\},
\]

(43)

where \( e \) is taken to be the charge of the heavy particle. The effects of the scattering processes are encoded on the RHS which is also the collision integral. As in the 2D case, the equilibrium distribution function of the impurity is given by the Maxwell–Boltzmann distribution function \( f^0 = N e^{-p^2/2M} \), where the normalization constant is \( N = \sqrt{2\pi/\beta M} \). Indeed, in the equilibrium scenario the LHS vanishes, therefore, the following detailed balance equation \( f^0 \Gamma(k; p) = f^0 \Gamma(p, k) \) is necessarily satisfied. The scattering rate \( \Gamma \) obtained using the Fermi–Golden rule has the form

\[
\Gamma(k; p) = \frac{U^2}{vL} \left[ \frac{\omega_q(n_q + 1)\delta(p^2/2M - k^2/2M + \omega_q)}{\omega_q n_q} + \frac{\omega_q n_q \delta(p^2/2M - k^2/2M - \omega_q)}{\omega_q n_q} \right],
\]

(44)

where \( q = p - k \) and \( n_q \) is the equilibrium Bosonic distribution function. Consider the first term of (43) where the summation
in $k$ implies that the typical impurity momentum is $k \sim \sqrt{MT}$ while the energy conservation criterion forces phonons with momentum $q \sim Mv \approx Mv_F$ to take part in the scattering process; however, this is an exponentially rare process since $T \ll Mv_F^2$. Thus the contribution to friction due to this term and following similar arguments due to the second term is exponentially suppressed. Taking advantage of the detailed balance equation and the $\delta$-function we find that the mobility diverges exponentially as $\mu \sim e^{-2\beta M^2}/U^2M^2$, and hence the scattering rate is suppressed as $1/\tau \sim U^2M e^{-2\beta M^2}$.

Turning on the magnetic field opens up the back-scattering channel thus resulting in an additional suppression of the mobility. The interaction term now has an extra term given by

$$ \frac{U}{2\pi \alpha_0} \sqrt{B^2 + k_f^2} \int dx a^\dagger(x)a(x) \cos(2\phi - 2k_F x). \quad (45) $$

Consider the possibility of $2k_F$ momentum transfer to the impurity particle with momentum $k \sim \sqrt{MT}$; in this case the energy transferred will be $\sim k_F^2/M$. Since the temperature regime we are considering is much smaller than this energy scale, the scattering process is again exponentially suppressed with the scattering-rate estimated to be $\sim (UB/e\hbar k_F)^2M e^{-2\beta k_F^2/M}$.

It turns out that even though the second order process arising from (45) is perturbatively weaker in comparison to the first order back-scattering process, yet it yields dominant contribution to the scattering rate at low temperatures. The interaction term for a second order process can be written as [90]

$$ I(p) = \sum_q \left[ -f_p \Gamma_2(p; p + q) + f_{p+q} \Gamma_2(p + q; p) \right], \quad (48) $$

where $\Gamma_2$ is the scattering rate. Defining the non-equilibrium distribution function as $f_p = f_p^0$ and using a similar detailed balance equation as discussed earlier one obtains, $I(p) = \sum_q f_p^0(h_{p+q} - h_p) \Gamma_2(p + q; p)$. Using Fermi’s golden rule the full expression for the collision integral can be written as

$$ I(p) = \frac{V_2^2}{32\pi^3} \int dq \, dq_2 |k_1k_2(h_{p+q} - h_p) (K^2 + K^{-2} + 2\frac{k_1k_2}{|k_1k_2|}) \times f_{p+q}^0 n_1(n_2 + 1)\delta \left( \frac{p^2}{2M} - \frac{(p + q)^2}{2M} + \omega_{k_2} - \omega_{k_1} \right), \quad (49) $$

where $k_2 = -(q + \hat{q})/2$ and $k_1 = (q - \hat{q})/2$.

The evaluation of the $\delta$-function can be divided into the following two cases, $\omega_{q+\hat{q}} - \omega_{q-\hat{q}} = \pm vq$ and $\omega_{q+\hat{q}} - \omega_{q-\hat{q}} = \pm vq$. These are the unshaded and the shaded regions of figure 5, respectively. The former scenario is irrelevant since the $\delta$-function imposes constraint similar to the one discussed before, i.e., the requirement that the phonons have energy $\sim Mv_F^2$. The later case on the other hand is achieved for the range $q^2 > \hat{q}^2$ where the momentum transfer $q$ changes the direction of phonons, i.e., $k_1$ and $-k_2$ are in the opposite direction, however, the energy of phonon hardly changes (see figure 6). This is reflected from the $\delta$-function constraint which fixes the energy transfer to $|q| \approx \epsilon_{p+q} - \epsilon_p$, where $\hat{q}/q \sim \sqrt{T/Mv_F^2} \ll 1$. Therefore (49) reduces to

$$ I(p) = \frac{V_2^2(K + K^{-1})^2}{128\pi^3 v} \int dq_2 f_{p+q}^0(h_{p+q} - h_p)n_2(n_2 + 1). \quad (50) $$

We will next consider the limit of weak electric field $E$. Following Feynman et al., [91] $h_p$ is expanded to linear order in $E$.
as $h_p = 1 + pE\mathcal{H}$, where $\mathcal{H}$ is a weakly varying even function of $p$. The integral is evaluated to yield

$$I(p) = \frac{2\pi T^2}{15\nu^6} \frac{\sqrt{2}}{2!} (K + K^{-1})^2 E\mathcal{H} \frac{\partial \rho_p}{\partial p}. \quad (51)$$

Under the steady-state condition, the LHS of (43) is simply given by $eE\partial \rho_p \approx eE\partial \rho_p(1 + pE\mathcal{H})$. Thus comparing it with (51), we obtain

$$\mathcal{H} \approx \frac{15}{2\pi T^2 \nu^6} \frac{\sqrt{2}}{2!} (K + K^{-1})^2.$$

With the non-equilibrium distribution determined, the mobility, $\mu$, of impurity in the presence of electric-field $E$ can be easily calculated and is given by

$$\mu = \frac{\int dp p^2 f_0(p) E\mathcal{H}/M}{E \int dp f_0(p)} = \mathcal{H} T \propto \frac{1}{T^4 B^4}. \quad (52)$$

Thus the second-order back-scattering rate is proportional to $\tau_{\text{back}}^{-1} \propto T^4 B^4$. It is worth noting that an alternate approach of Neto and Fisher [90] yielded a similar power-law divergent temperature term, however, in here the power-law divergent behaviour is achieved only in the presence of a magnetic field. At very low temperatures and non-zero magnetic field the second-order back-scattering mechanism will be dominant compared to other mechanisms resulting in a power-law divergence of the mobility.

7. Summary and conclusion

To summarize, we have explored the physics of a single non-magnetic impurity interacting with the bath of 2D and 1D Dirac fermions by focussing on the impurity Green’s function and its mobility.

In the 2D scenario, impurity Green’s function exhibits different behaviour in the non-recoil and the recoil case. A crucial ingredient for the analysis is the density of particle–hole excitations evaluated by the momentum integration of the imaginary part of the polarization function of the Dirac fermions. In spite of the spin-momentum locked helical Dirac fermions favouring the creation of low-energy particle–hole pairs with small momentum, the interaction of the impurity with the non-recoil case results in the generation of a large number of particle–hole excitations whose density varies linearly with $\omega$. Thus, similar to the case of a fermionic bath with the quadratic spectrum, this results in a power-law decay of the Green’s function $\propto 1/t'$, where $\nu = k_0^2 U^2/\pi^2$. This implies that the impurity cannot be described in terms of the quasiparticle picture, in particular, the spectral-function is modified from $\delta$-function and manifests a sharp cut-off for energies less than the renormalized impurity energy, while a power-law suppression is exhibited for energies greater than it, given by $\mathcal{H}(\omega) \propto \Theta(\omega - \epsilon_p)/(\omega - \epsilon_p)^{1-v}$. In contrast, the energy–momentum constraint in the recoil case implies reduced phase-space for the particle–hole excitations resulting in a $\omega^{3/2}$ dependence of the density of states. The resulting Green’s function has a pure oscillatory part, implying non-zero quasiparticle weight, in addition, an oscillatory part multiplied by a decaying $t'^{-1/2}$ term. The former is responsible for a delta-function peak in the spectral function, whereas the latter yields an incoherent part that exhibits square-root singularity. The results discussed are expected to be robust as long as the doping is away from the degeneracy point. At the degeneracy point electron–electron interactions need to be incorporated in the analysis [92–97].

The low temperature ($T \ll k_0^2/M$) dependence of the mobility of the impurity has been estimated by performing a statistical average on the inverse quasiparticle lifetime with respect to the Boltzmann weight factor. The mobile impurity interacts with a particle–hole excitation having typical energy $\omega \sim T$ and momentum $q \sim \sqrt{2MT}$. In this regime, the polarization function acquires a particularly simple form and the temperature dependence of the mobility is revealed to be $T^{-3/2}$.

For the case of a mobile impurity interacting with the 1D helical modes, similar to the Green’s function behaviour in 2D, the Green’s function in the non-recoil case exhibits power-law suppression at long times with $G \sim t^{-\nu}$, where $\nu = -U^2/2\pi^2 v^2$. However, a crucial difference from the 2D case is that this behaviour persists even for the recoil scenario, albeit for a finite range of momentum. In particular, for $k < Mv$, the long-time decay exponent of the Green’s function acquires a momentum dependence in the exponent given by

$$\nu = -\frac{U^2}{2\pi^2} \frac{v^2 + k^2/M^2}{(v^2 - k^2/M^2)^2},$$

whereas for $k > Mv$ the Green’s function has a conventional Fermi-liquid type of decay with the decay time given by $t^{-1} = U^2 M/4\pi$. Thus the impurity spectral function will exhibit change in behaviour ranging from non-Lorentzian features for $k < Mv$ to a Lorentzian peak above this momentum.
The temperature dependence of the mobile impurity interacting with the 1D helical modes exhibits contrasting behaviour with or without the magnetic field. In the absence of a magnetic field only the forward scattering process is allowed, however, the energy and momentum constraint forces exponential suppression of the scattering. Turning on the magnetic field opens up the previously forbidden back-scattering channel. Nevertheless, at the lowest order in interaction, the scattering process remains exponentially suppressed. Interestingly, the second-order back-scattering process favours a process in which the energy transferred between the mobile impurity and the phonons is negligible compared to the temperature and therefore the scattering processes are relevant. At low temperatures the scattering rate due to this mechanism will dominate and result in a unique \((TB)^{-4}\) divergence of the mobility. The magnetic field can in principle be utilized to tune from the exponential to power-law divergent mobility regimes.

In analogy to the original x-ray edge problem, the spectral function of the impurity can be probed by experimental techniques like the angle resolved photoemission spectroscopy [98, 99]. Besides the solid-state based Dirac systems, techniques have been developed to create honeycomb optical lattice out of ultracold atoms from the periodic potential of interfering laser beams [100]. The tunability of the lattice interfering laser beams [100]. The tunability of the lattice

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Appendix

The noninteracting generalized polarization function for the Dirac fermion in the TI is given by

\[
\Pi(q, \omega_n) = -\frac{1}{2\pi} \int K \left[ \sigma_0 G_K \sigma_0 G_{K+Q} \right], \quad (53)
\]

where \(\text{Tr}\) denotes the trace. \(K = (\tilde{k}, \Omega)\) and \(Q = (\tilde{q}, \omega)\). The corresponding zero temperature single particle Matsubara Green’s function used in the above equation has the following form

\[
\tilde{G}(k, i\Omega) = \frac{1}{2} \sum_{\alpha = \pm 1} \left[ 1 - \alpha (\sigma \cdot q)/\xi_k \right] \frac{1}{1 + \alpha \epsilon_k^q + \mu_F}, \quad (54)
\]

where \(\alpha = \pm 1\) represents valence and conduction bands respectively, \(\tilde{k} = k_x e_1 + k_y e_2 + \Delta \tilde{e}_3\), and \(\xi_k = \sqrt{k^2 + \Delta^2}\). The Pauli matrix \(\sigma\) acts on the spin degrees of freedom. Following the standard frequency summation and the analytical continuation \(i\omega \rightarrow \omega + i0^+\), we obtain the following form of the polarization function,

\[
\Pi(q, \omega) = -\frac{1}{2\pi} \int q^2 \left[ 1 + \frac{\tilde{k}^2}{\Xi_{k+q}} \right] \times \left( n_F(\xi_k) - n_F(\xi_{k+q})\right) (\omega + \xi_k - \xi_{k+q}) \quad (55)
\]

The nonzero contribution to the imaginary part of the polarization function from the upper to upper band \((u \rightarrow u)\) transitions is as follows,

\[
\text{Im} \quad \Pi(q, \omega) = -\frac{1}{2\pi} \int q^2 \left[ 1 + \frac{\tilde{k}^2}{\Xi_{k+q}} \right] \times \left( n_F(\xi_k) - n_F(\xi_{k+q})\right) (\omega + \xi_k - \xi_{k+q}) \quad (56)
\]

After delta-function integration we obtain

\[
\text{Im} \quad \Pi(q, \omega) = -\frac{1}{2\pi} \int q^2 \left[ 1 + \frac{\tilde{k}^2}{\Xi_{k+q}} \right] \times \left\{ F(2\mu + \omega) - F(2\mu + \omega - \Delta) + \omega \right\} \quad (57)
\]

where

\[
\zeta = \sqrt{q^2 + 4q^2\Delta^2/(q^2 - \omega^2)} \quad \text{and} \quad F(x) = \frac{1}{4} \left[ (\zeta^2 - 2q^2) \log \left( x^2 - \zeta^2 + x \sqrt{x^2 - \zeta^2} \right) + x \sqrt{x^2 - \zeta^2} \right]. \quad (58)
\]

The allowed regions for the transitions are

\(1A : \omega < \mu - \sqrt{(q - k_F)^2 + \Delta^2}\)

\(2A : \pm \mu \mp \sqrt{(q - k_F)^2 + \Delta^2} < \omega < -\mu + \sqrt{(q + k_F)^2 + \Delta^2}\).

Next the similar contribution from lower to upper band \((l \rightarrow u)\) transitions are,

\[
\text{Im} \quad \Pi(q, \omega) = -\frac{1}{2\pi} \int q^2 \left[ 1 + \frac{\tilde{k}^2}{\Xi_{k+q}} \right] \times \left( 2\xi_k - \omega^2 \right)^2 \sqrt{1 - (\frac{2\xi_k - \omega^2}{\omega})^2} \quad (59)
\]
Similarly the allowed regions in the \((q, \omega)\) plane are

\[1B: \mu + \sqrt{(q-k)^2 + \Delta^2} < \omega < \mu + \sqrt{(q+k)^2 + \Delta^2} \]

\[2B: \omega > \mu + \sqrt{(q+k)^2 + \Delta^2} \]

\[3B: \omega > (2k); \quad & \quad \sqrt{q^2 + 4\Delta^2} < \omega < \mu + \sqrt{(q-k)^2 + \Delta^2}. \]

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