Bulk viscous fluid in extended symmetric teleparallel gravity

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In this paper, we investigate the existence of bulk viscous FLRW cosmological models in a recently proposed extended symmetric teleparallel gravity or \( f (Q,T) \) gravity in which \( Q \) is the non-metricity and \( T \) is the trace of the energy-momentum tensor. We consider a simple coupling between matter and non-metricity, specifically, \( f (Q,T) = \alpha Q^{m+1} + \lambda T \) and \( f (Q,T) = \alpha Q + \lambda T \) where \( \alpha, \lambda \) and \( m \) are free model parameters. The exact cosmological solutions are found by assuming the scale factor in the form of the hybrid expansion law. This type of relation generates a time-varying deceleration parameter with the transition of the Universe from the early decelerating phase to the present accelerating phase. In the presence of viscous fluid, we analyze some cosmological parameters of our cosmological model such as the energy density, bulk viscous pressure, bulk viscous coefficient, equation of state parameter, and energy conditions. Finally, we conclude our \( f (Q,T) \) cosmological models agree with the recent astronomical observations.

I. INTRODUCTION

Modern cosmology starts its beginnings with the emergence of the theory of general relativity (GR) at the beginning of the last century, since that time this branch of physics has experienced great prosperity thanks to the development of means and tools of observation. On the other hand, considering the temporal evolution of the Universe, cosmologists argued that the Universe is in decelerated expansion due to Friedmann’s standard equations based on GR. But at the end of the last century, a cluster of astronomical observations emerged that showed the opposite to be true, meaning the Universe is in a phase of accelerated expansion [1–4]. This cosmic puzzle created a challenge for cosmologists, which made them wonder about the reasons for this cosmic acceleration.

In the literature, there are many alternatives to GR that attempt to explain this problem [5–7]. The most popular alternative currently proposed are modified gravity theories such as \( f (R) \) gravity, where \( f (R) \) is an arbitrary function of the Ricci scalar \( R \). Several models have been studied in framework of \( f (R) \) gravity in various cosmological contexts [8–10]. Next, an extension of this theory proposed by Harko et al. [11] named \( f (R,T) \) gravity, where \( T \) is the trace of energy-momentum tensor. For more details on this theory, see our work [12–15].

Recently, \( f (Q) \) gravity (or symmetric teleparallel gravity) appeared, which is a new modified theory of gravity proposed by Jimenez et al. [16] where the non-metricity scalar \( Q \) drives the gravitational interaction. This theory is based on Weyl geometry, which is a generalization of Riemannian geometry that is the mathematical basis of GR. In Weyl geometry, gravitational effects do not occur because of the change in direction of a vector in parallel transport, but because of the change in length of the vector itself. The implications of this theory have been studied in several papers. The first cosmological solutions in \( f (Q) \) gravity studied in Ref. [17, 18] while in other works energy conditions and cosmography in \( f (Q) \) gravity are discussed in [19, 20]. Quantum cosmology with a power-law model has been investigated in [21]. Cosmological solutions and growth index of matter perturbations have been evaluated for a polynomial functional form of \( f (Q) \) gravity in [18]. Assuming the power-law function, the coupling of matter in \( f (Q) \) gravity analyzed by Harko et al. in [22] along with many other works mentioned in [23–27].

Inspired by \( f (R,T) \) gravity, the cosmologist’s Xu et al. proposed an extension of \( f (Q) \) gravity by adding a coupling between the non-metricity scalar \( Q \) and trace of

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the energy-momentum tensor $T$ [28]. The Lagrangian of the gravitational field is presumed to be a general function of both $Q$ and $T$. In addition, the field equations of this theory are derived from the modified Einstein-Hilbert type variational principle. Thus, the action for this modified theory of gravity is given by (in the units $G = c = 1$) [28]

$$S = \int \left( \frac{1}{16\pi} f(Q, T) + L_m \right) \sqrt{-g} d^4x, \quad (1)$$

Here, $f(Q, T)$ is the general function of the nonmetricity scalar $Q$ and trace of energy-momentum tensor $T$, $g$ is the determinant of the metric tensor $g_{\mu\nu}$ i.e. $g = \det \left( g_{\mu\nu} \right)$, and $L_m$ represents the matter Lagrangian density. If $L_m$ depends only on the metric components and not on its derivatives, one has, for the energy-momentum tensor $T_{\mu\nu}$ the following

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta \left( \sqrt{-g} L_m \right)}{\delta g^{\mu\nu}}, \quad (2)$$

so that $T = g^{\mu\nu} T_{\mu\nu}$, and

$$\theta_{\mu\nu} = g^{\alpha\beta} \frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}}. \quad (3)$$

The variation of energy-momentum tensor with respect to the metric tensor $g_{\mu\nu}$ read as,

$$\frac{\delta g^{\mu\nu} T_{\mu\nu}}{\delta g^{\alpha\beta}} = T_{\mu\nu} + \theta_{\mu\nu}. \quad (4)$$

The non-metricity scalar $Q$ can be given as,

$$Q \equiv -g^{\mu\nu} (L^\beta_{\mu\alpha} L^\alpha_{\nu\beta} - L^\beta_{\alpha\nu} L^\alpha_{\mu\beta}), \quad (5)$$

where the disformation tensor $L_{\alpha\gamma}$ is given by,

$$L_{\alpha\gamma} = -\frac{1}{2} g^{\beta\eta} (\nabla_\gamma g_{\alpha\eta} + \nabla_\alpha g_{\eta\gamma} - \nabla_\eta g_{\alpha\gamma}). \quad (6)$$

Further, the non-metricity tensor $Q_{\gamma\mu}$ is defined as,

$$Q_{\gamma\mu} = \nabla_\gamma g_{\mu}, \quad (7)$$

and the trace of the non-metricity tensor is obtained as,

$$Q_\beta = g^{\mu\nu} Q_{\mu\nu} \quad (8)$$

The superpotential of the model is defined as,

$$P_\mu^\beta = \frac{1}{2} L_\mu^\beta + \frac{1}{4} (Q^\beta - Q_\beta) g_{\mu\nu} - \frac{1}{4} \delta_\beta^{\beta\nu} Q_\nu, \quad (9)$$

and using this definition above, the non-metricity scalar in terms of superpotential is given as,

$$Q = -Q_{\beta\mu} P^\beta_{\mu\nu}. \quad (10)$$

Now, varying the gravitational action (1) with respect to metric tensor $g_{\mu\nu}$, the corresponding field equations of $f(Q, T)$ gravity is obtained as,

$$-\frac{2}{\sqrt{-g}} \nabla_\beta (f_Q \sqrt{-g} P_\mu^\beta) - \frac{1}{2} f_Q g_{\mu\nu} + f_T (T_{\mu\nu} + \Theta_{\mu\nu}) - f_Q (P_{\mu\beta\alpha} Q_{\nu}^\beta - 2Q^{\beta\mu} P_{\beta\mu\nu}) = 8\pi T_{\mu\nu}. \quad (11)$$

In the present paper we use the notations as $f(Q, T) \equiv f$, $f_Q = \frac{df(Q, T)}{dQ}$, $f_T = \frac{df(Q, T)}{dT}$, and $\nabla_\beta$ denotes the covariant derivative. From Eq. (11) it appears that the field equations of $f(Q, T)$ gravity depends on the tensor $\theta_{\mu\nu}$. Hence, depending on the nature of the source of matter, several theoretical models corresponding to different matter sources in $f(Q, T)$ gravity are possible. Originally, Xu et al. [28] obtained three models using following functional forms of $f(Q, T)$ as

- $f(Q, T) = \alpha Q + \lambda T$,
- $f(Q, T) = \alpha Q^{m+1} + \lambda T$,
- $f(Q, T) = -\alpha Q - \lambda T^2.$

Several authors have explored applications of this theory in many contexts: Arora et al. [29] have recently explored $f(Q, T)$ gravity models with observational constraints. Yang et al. [30] formulated the geodesic deviation and Raychaudhuri equations in $f(Q, T)$ gravity placed on the observation that the curvature-matter coupling significantly modifies the nature of current forces and the equation of motion in the Newtonian limit. Pati et al. [31] considered some rip cosmological models in the form of $f(Q, T) = aQ^m + bT$ gravity. Also in another work, Pati et al. [32] studied the dynamical aspects of some accelerating models in $f(Q, T)$ gravity using the hybrid scale factor.
Most of the authors have analyzed cosmological models using the perfect fluid as the content of the Universe to account for various problems in the scientific domain such as present cosmic acceleration and dark energy. In line with recent observations, the cosmic acceleration is due to a strange form of energy wearing negative pressure. Inspired by these observations, in this article, we will establish a cosmological model without taking dark energy into account by choosing a further realistic fluid, like a viscous fluid. Various researchers point out the idea that cosmic viscosity acts as an alternative to dark energy, which could play an important role in establishing the accelerated expansion phase of the Universe by devouring negative effective pressure [33]. Diverse research has exhibited that in the early Universe, matter behaved as a viscous fluid during the neutrino decoupling in the radiation era [34, 35]. For further justifications for choosing a viscous fluid instead of a perfect fluid, see [36]. In addition, many viscous fluid cosmological models have been considered in the literature. Srivastava and Singh [37] evaluated a new holographic dark energy model within the framework of an FLRW model with bulk viscous matter content regarding \( p_v = p - 3\zeta H \), where \( \zeta \) is the constant bulk viscosity coefficient and \( H \) is the Hubble parameter. Also, Brevik et al. [38] analyzed viscous FLRW cosmology in modified gravity. Singh and Kumar [39] created bulk viscosity in \( f(R, T) \) gravity with the viscous term as \( \zeta = \zeta_0 + \zeta_1 H \), where \( \zeta_0 \) and \( \zeta_1 \) are constants. To get the accelerating expansion of the Universe, Ren et al. [40] supposed the form of the viscosity coefficient depends on the velocity and acceleration as \( \zeta = \zeta_0 + \zeta_1 H + \zeta_2 \left( \frac{H}{N} + H \right) \) where \( \zeta_0, \zeta_1 \) and \( \zeta_2 \) are constants. This manuscript is organized as follows: In Sec. II we derive the exact solutions of \( f(Q, T) \) gravity for the flat FLRW space-time. In Secs. III and IV we analyze the physical behavior of some cosmological parameters such as the energy density, bulk viscous pressure, bulk viscous coefficient, equation of state (EoS) parameter, and energy conditions for both models i.e. \( f(Q, T) = aQ^{n+1} + \lambda T \) and \( f(Q, T) = aQ + \lambda T \), respectively. The last Sec. V is devoted to discuss the results and conclusion.

II. FIELD EQUATIONS AND SOLUTIONS

Taking into account the spatial isotropy and homogeneity of the Universe, we consider the following flat FLRW metric for our analysis [28]

\[
ds^2 = -N^2(t) \left[ dt^2 + a^2(t) \left( dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right) \right],
\]

(12)

where \( a(t) \) is the scale factor of the Universe, \( N(t) \) is the lapse function considered to be 1 in the standard case and \( t, r, \theta, \phi \) are the comoving coordinates. The rates of expansion and dilation are determined as \( H \equiv \frac{\dot{a}}{a} \) and \( T \equiv \frac{\dot{N}}{N} \) respectively. Thus, the corresponding non-metricity scalar is given by \( Q = 6H^2 \), choosing \( N = 1 \) we have \( Q = 6H^2 \). In the study of cosmological model, for the description of bulk viscosity following two main formalism are mentioned:

a) The first is the non-casual theory in which the deviation of only first-order is considered and one can find that the heat flow and viscosity propagate with infinite speed,

b) The second one is the casual theory which propagates with finite speed. To analyze the late acceleration of the universe, the casual theory of bulk viscosity has been used. Cataldo et al. have investigated the late time acceleration using the casual theory [41]. Basically, they used an ansatz for the Hubble parameter (inspired by the Eckart theory) and they have shown the transition of the universe from the big rip singularity to the phantom behaviour. Hence, here we consider the energy momentum tensor in the form of bulk viscous fluid as,

\[
T_{\mu\nu} = (\rho + \overline{p}) u_{\mu}u_{\nu} + \overline{p}g_{\mu\nu}.
\]

(13)

where \( u_{\mu} = (1, 0, 0, 0) \) is the four velocity vector in co-moving coordinate system satisfying \( u^{\mu}u_{\nu}\Gamma_{\mu\nu}^{\gamma} = -1 \). Thus, for tensor \( \theta_{\mu\nu} \), the expression is obtained as \( \theta_{\mu\nu} = \eta_{g_{\mu\nu} - 2T_{\mu\nu}}, \) and \( \eta \) is the bulk viscous pressure given by

\[
\eta = p - 3\zeta H,
\]

(14)

which satisfies the linear equation of state \( p = \gamma\rho \), \( 0 \leq \gamma \leq 1 \), \( \zeta \) is the bulk viscous coefficient, \( H \) is the Hubble parameter, \( p \) is normal pressure and \( \rho \) is the energy density of the Universe. The trace of energy momentum tensor is given as

\[
T = 3\eta - \rho.
\]

(15)

Using the metric (12), the Einstein field equations are given as [28],

\[
8\pi\rho = \frac{f}{2} - 6FH^2 - \frac{2G}{1 + G} \left( \dot{F}H + FH \right),
\]

(16)

\[
8\pi\eta = -\frac{f}{2} + 6FH^2 + 2 \left( \dot{F}H + FH \right).
\]

(17)
where (·) dot represents a derivative with respect to cosmic time \( t \). In this case, \( F \equiv f_Q \) and \( 8\pi G \equiv f_T \) represent differentiation with respect to \( Q \) and \( T \) respectively. The evolution of equation for the Hubble function \( H \) can be obtained by combining Eqs. (16) and (17) as,

\[
\dot{H} + \frac{\dot{F}}{F} H = \frac{4\pi}{F} \left( 1 + \tilde{G} \right) \left( \rho + \overline{\rho} \right).
\]  

(18)

From the above equations, there is a system of nonlinear equations with four unknowns: \( H, \rho, \overline{\rho}, f \). Therefore, to find the exact solutions and reduce the number of unknowns, additional constraints must be added. Several researchers in the literature use additional constraints of the scale factor \( a(t) \) in the form of the power-law and exponential law to construct a cosmological model that describes the evolution of the Universe. However, one of the negative aspects of these models is that they do not take into account the transition of the Universe from early decelerating to the current accelerating stage, meaning that the deceleration parameter i.e.

\[
q = -1 + \frac{d}{dt} \left( \frac{1}{H} \right) = -1 + n\eta \left( \beta t + \eta \right)^{-2}.
\]  

(20)

In cosmology, it is a good idea to write cosmological parameters in terms of the redshift. Using the relation between the scale factor and redshift of the Universe as \( a(t) = a_0 (1 + z)^{-1} \) where \( a_0 = 1 \) represents the present value of the scale factor, the time-redshift relation is given by,

\[
t(z) = \frac{\eta}{\beta} W \left( \frac{\beta (z + 1)^{-\frac{\eta}{\beta}}}{\eta} \right). \]  

(21)

where \( W \equiv \text{Lambert} \) denotes the Lambert function (also known as "product logarithm").

The deceleration parameter is a good tool to describe the evolution of the Universe from the initial deceleration to the current acceleration. If \( q > 0 \) the Universe decelerates, and if \( q < 0 \) then it accelerates. Recent observations in astronomy such as SNIa (type Ia Supernovae) [1, 2] and CMB (Cosmic Microwave Background) anisotropy [3, 4] have shown that the Universe is already in an acceleration phase and the current value is in the range \(-1 \leq q \leq 0\). By using Eqs. (20) with \( \eta = 1.35 \) and \( \beta = 2.05 \), the variation of the deceleration parameter \( (q) \) in terms of the redshift \( z \) for the hybrid scale factor is exhibited in Fig. 1 for the three representative values of \( n \) i.e. 2.50, 2.60, 2.70. It is clear that for all \( n \) the deceleration parameter is positive at early phase of the Universe and negative for the late Universe. Hence, it indicates that the Universe shows a transition from deceleration to acceleration. Also, the transition from the early deceleration phase to the current accelerated phase is done with a certain redshift, called a transition redshift \((z_{tr})\). From the figure, the value of the transition redshift for \( n = 2.5, 2.6, 2.7 \) are respectively \( z_{tr} = 0.582, 0.624, 0.812 \). The transition redshift value \( z_{tr} = 0.582 \) at \( n = 2.50 \) is therefore consistent with the results of the observation [46–48]. Hence here we fix \( n = 2.50 \) throughout analysis.

III. COSMOLOGICAL MODEL WITH

\[ f(Q,T) = \alpha Q^m + \lambda T \]

In the first choice, we consider the cosmic evolution for a function of the form \( f(Q,T) = \alpha Q^{m+1} + \lambda T \), where \( \alpha, \lambda \) and \( m \) are free model parameters. In this case, we will consider that \( m \neq 0 \). Hence, the expressions of \( f_Q \) and \( f_T \) in the field equation (11) are derived as \( F = f_Q = \alpha (m + 1) Q^m \) and \( 8\pi G = f_T = \lambda \). By using Eqs (20) with this choice, and solving the field equations (16) and (17), the values of \( \rho \) and \( \overline{\rho} \) are obtained as
\[
\rho = \frac{\alpha 2^{m_1} 3^m (2m + 1)}{\lambda + 4\pi} \left[ \lambda (m + 1) \left( -\frac{\eta}{nt^2} \right) - 3(\lambda + 8\pi) \left( \frac{bt + \eta}{nt} \right)^2 \right] \left( \frac{bt + \eta}{nt} \right)^{2m},
\]

\[
\bar{p} = \frac{\alpha 2^{m_1} 3^m (2m + 1)}{\lambda + 4\pi} \left[ (3\lambda + 16\pi)(m + 1) \left( -\frac{\eta}{nt^2} \right) + 3 \left( \frac{bt + \eta}{nt} \right)^2 (\lambda + 8\pi) \right] \left( \frac{bt + \eta}{nt} \right)^{2m}.
\]

In thermodynamics, the behavior of the bulk viscosity coefficient \(\xi\) is positive, ensuring that viscosity pushes the bulk viscous pressure toward negative values. From Fig. 4, it is clear that the bulk viscosity coefficient \(\xi\) is an increasing function of redshift \(z\), remains positive for all values of \(z\), and approaches a constant amount close to zero, which ressembles well with the physical behavior of \(\xi\) [14].

\[
\xi = -\frac{\alpha 6^{m_1} (2m + 1)}{\lambda + 4\pi} \left[ \left( -\frac{\eta}{nt^2} \right) (m + 1)(16\pi - (\gamma - 3)\lambda) + 3(\gamma + 1)(\lambda + 8\pi) \left( \frac{bt + \eta}{nt} \right)^2 \right] \left( \frac{bt + \eta}{nt} \right)^{2m-1},
\]

\[
p = \gamma \rho = \frac{\alpha \gamma 2^{m_1} 3^m (2m + 1)}{\lambda + 4\pi} \left[ \lambda (m + 1) \left( -\frac{\eta}{nt^2} \right) - 3(\lambda + 8\pi) \left( \frac{bt + \eta}{nt} \right)^2 \right] \left( \frac{bt + \eta}{nt} \right)^{2m}.
\]

The EoS parameter \(\omega\) is given by

\[
\omega = \frac{(3\lambda + 16\pi)(m + 1) \left( -\frac{\eta}{nt^2} \right) + 3 \left( \frac{bt + \eta}{nt} \right)^2 (\lambda + 8\pi)}{\lambda (m + 1) \left( -\frac{\eta}{nt^2} \right) - 3(\lambda + 8\pi) \left( \frac{bt + \eta}{nt} \right)^2}.
\]
−1 the model is like a phantom model. Also, the several cosmological analysis have constrained the numerical value of the EoS parameter such as: Supernovae Cosmology Project, \( \omega = -1.035^{+0.055}_{-0.059} \) [49]; WMAP+CMB, \( \omega = -1.073^{+0.090}_{-0.089} \) [50]; Planck 2018, \( \omega = -1.03 \pm 0.03 \) [51].

Here, the behavior of EoS parameter \( \omega \) in terms of redshift \( z \) is shown in Fig. 5. The EoS parameter indicates that the bulk viscous fluid behaves like the quintessence dark energy model at \( z = 0 \) and is finally approached to \( \Lambda \)CDM region at \( z \to -1 \) which describe the late-time acceleration of the expanding universe without invoking any dark energy component. Further, the present value of EoS parameter corresponding to the parameters of the model \( \alpha = -4.5, \, \lambda = -3\pi, \, \gamma = 0.5 \) and \( m = 0.2 \) is \( \omega_0 = -0.7698 \). Thus, the cosmic fluid with bulk viscosity is the most viable candidate and thus the EoS parameter of the model is in good agreement with the astronomical observations [4].

As we have the cosmological parameters studied above play an important role in understanding the evolution of the Universe. But to predict the cosmic acceleration in modern cosmology, a set of energy conditions can be obtained from the equation of Raychaudhuri. In GR, the role of these energy conditions is to demonstrate the theorems for the existence of space-time singularity and black holes [52]. Various authors have worked on energy conditions in different backgrounds. In this work, we will consider the renowned energy conditions to test the validity of the model in the context of cosmic acceleration. There are several forms of energy conditions such as weak energy conditions (WEC), null energy conditions (NEC), dominant energy conditions (DEC), and strong energy conditions (SEC) are given for the content of the Universe in form of a viscous fluid in the \( f (Q, T) \) gravity as follows [53]

- **WEC**: if \( \rho \geq 0, \rho + p \geq 0 \);
- **NEC**: if \( \rho + p \geq 0 \);
- **DEC**: if \( \rho \geq 0, |p| \leq \rho \);
- **SEC**: if \( \rho + 3p \geq 0 \).

The importance of the above energy conditions is that when the NEC is violated, all more energy conditions are violated. This violation of NEC constitutes the depletion of energy density as the Universe expands. Further, the violation of SEC constitutes the acceleration of the Universe. To account for the late-time cosmic acceleration with \( \omega \approx -1 \), it must be \( \rho (1 + 3\omega) < 0 \). Using Eqs. (23) and (24) we can get the behavior of energy conditions in terms of \( z \) and \( n \) of the model as shown in Fig 6. We observe that WEC, NEC and DEC are satisfied while the SEC is violated in the present and future. Hence, the violation of SEC leads to the acceleration of the Universe.
Fig. 6. NEC, DEC and SEC versus redshift \( z \) for the model 
\[ f(Q, T) = aQ^{m+1} + \lambda T. \]

![Graph showing energy conditions](image)

The stability parameter \( \vartheta \), there are three types of particles are available in the universe called sub-
luminal, luminal and super-luminal. Out of which the sub-luminal particles moving very slow in comparison of the speed of light (example are electrons and neutrons while the luminal particles move with exactly the same speed as that of the speed of light (example are photon and graviton) whereas super-luminal particles are moving faster than the speed of light (tachyons). There are two possibilities for the existence of super-luminal particles: either they do not exist or if they do, then they do not interact with an ordinary matter. If the speed of sound is less than the local light speed, then only we can say about the non-violation of causality. The positive square sound speed is necessary for the classical stability of the universe. The speed of sound \( \vartheta_z^2 \) is defined by 
\[ \vartheta_z^2 = \frac{\partial p}{\partial \rho} \] and observed as

\[ \vartheta_z^2 = \frac{(3\lambda + 16\pi)n(\eta + \eta m + \beta t) - 3(\lambda + 8\pi)(\eta + \beta t)^2}{\lambda(n(\eta + \eta m + \beta t) + 3(\eta + \beta t)^2)} + 24\pi(\eta + \beta t)^2. \] (28)

The Fig. 7, shows that the stability parameter \( \vartheta_z^2 < 0 \) throughout the evolution of the universe. Hence, the model remains unstable with the expansion of the Universe.

IV. COSMOLOGICAL MODEL WITH \( f(Q, T) = aQ + \lambda T \)

For the second model, we presume a functional form of \( f(Q, T) \) as \( f(Q, T) = aQ + \lambda T \), where \( a \) and \( \lambda \) are free model parameters \((m = 0)\). In this case, we get \( F = f_Q = a \) and \( 8\pi G = f_T = \lambda \). Here, using Eqs. (18) with this case, and solving the field equations (16) and (17), the values of \( \rho \) and \( \overline{\rho} \) are obtained as

\[ \rho = \frac{\alpha}{2(\lambda + 4\pi)(\lambda + 8\pi)} \left[ \lambda \left( -\frac{\eta}{nt^2} \right) - 3(\lambda + 8\pi) \left( \frac{\beta t + \eta}{nt} \right)^2 \right], \] (29)

\[ \overline{\rho} = \frac{\alpha}{2(\lambda + 4\pi)(\lambda + 8\pi)} \left[ (3\lambda + 16\pi) \left( -\frac{\eta}{nt^2} \right) + 3(\lambda + 8\pi) \left( \frac{\beta t + \eta}{nt} \right)^2 \right]. \] (30)

Fig. 8 represents the evolution of the energy density \( \rho \) of the Universe in terms of the redshift \( z \) for the fixed \( \alpha = -4.5, \lambda = -3\pi \) and \( \gamma = 0.5 \). We can observe that the energy density is an increasing function of redshift and remains positive. Initially, \( \rho \) starts with a large positive value and approaches zero at \( z \to -1 \). The bulk viscous pressure \( \overline{\rho} \) behavior in terms of redshift \( z \) is shown in Fig. 9. We can see that the bulk viscous pressure of this model is initially positive and then becomes negative in the present and the future, and it is an increasing function of redshift for all fixed \( n, a, \lambda \).

For this choice of model \( f \), the values of bulk viscosity coefficient \( \xi \) and normal pressure \( p \) are obtained as
\[ \xi = - \frac{\alpha}{6(\lambda + 4\pi)(\lambda + 8\pi)} \left[ (16\pi - \lambda(\gamma - 3)) \left( -\frac{\eta}{nt^2} \right) + 3(\gamma + 1)(\lambda + 8\pi) \left( \frac{\beta t + \eta}{nt} \right)^2 \right] \left( \frac{nt}{\beta t + \eta} \right), \]  
\[ (31) \]

\[ p = \gamma \rho = \frac{\alpha \gamma}{2(\lambda + 4\pi)(\lambda + 8\pi)} \left[ \lambda \left( -\frac{\eta}{nt^2} \right) - 3(\lambda + 8\pi) \left( \frac{\beta t + \eta}{nt} \right)^2 \right]. \]  
\[ (32) \]

Through Fig. 10, it is clear that the bulk viscosity coefficient \( \xi \) is an increasing function of redshift \( z \), remains positive for all values of \( z \) i.e. throughout the evolution of the Universe.

The EoS parameter \( \omega \) is given by

\[ \omega = \frac{(3\lambda + 16\pi) \left( -\frac{\eta}{nt^2} \right) + 3(\lambda + 8\pi) \left( \frac{\beta t + \eta}{nt} \right)^2}{\lambda \left( -\frac{\eta}{nt^2} \right) - 3(\lambda + 8\pi) \left( \frac{\beta t + \eta}{nt} \right)^2}. \]  
\[ (33) \]

For this case, the variation of EoS parameter \( \omega \) in terms of redshift \( z \) is shown in Fig. 11. It can be seen that the bulk viscous fluid behaves like the quintessence dark energy model at \( z = 0 \) and is finally approached to \( \Lambda \text{CDM} \) region at \( z \to -1 \) like the first case. In addition, the present value of EoS parameter corresponding to the parameters of the model \( \alpha = -4.5, \lambda = -3\pi \) and \( \gamma = 0.5 \) is \( \omega_0 = -0.8122 \). Thus, the EoS parameter for this case is in good agreement with the astronomical observations. From Fig. 12, we observe that WEC, NEC and DEC are satisfied while the SEC is violated in the present and future. The stability parameter

The speed of sound \( \theta_s^2 \) is observed as
accelerating, which means that the current deceleration parameter should be in the range of $-1 \leq q \leq 0$. Using the assumed redshift-time relation and the hybrid scale factor, the behavior of the deceleration parameter was plotted in terms of redshift in Fig. 1. Knowing that the parameters of the model $\eta$, $\beta$ and $n$ have their values carefully chosen to comply with the observational constraints i.e. $\eta = 1.35$, $\beta = 2.05$ and three values for $n = 2.5, 2.6, 2.7$. Based on these data, we found that the deceleration parameter for our model is positive in the early phase of the Universe and negative for the present and late Universe. It indicates that the Universe shows a transition from deceleration to acceleration at the transition redshift value $z_{tr} = 0.582$ for $n = 2.50$ with current values of the deceleration parameter for our model within the range of the observational data.

From Figs. 2 and 8, we observed that for both models, the energy density is an increasing function of redshift and remains positive. It starts with a large positive value and approaches to zero at $z \to -1$. The behavior of bulk viscous pressure for $m \neq 0$ and $m = 0$ is represented in Figs. 3 and 9 respectively, and we see that it is negative in the present and the future which is consistent with recent observations. Further, the behavior of the bulk viscosity coefficient for both the models is a positive and an increasing function of redshift $z$. Thus, this behavior is consistent with thermodynamics. The variation of EoS parameter versus redshift for both models is shown in Figs. 5 and 11 respectively. In both models, the EoS parameter behaves like the quintessence model in the present and the cosmological constant model in the future. Finally, from the energy conditions we can conclude that WEC, NEC and DEC are satisfied while the SEC is violated in the present and future. Also, both models are unstable throughout the evolution of the universe. The physical behavior of all the cosmological parameters discussed in this work of both models is remarkably consistent with the observational data.

V. DISCUSSION AND CONCLUSION

In this paper, we investigated the flat FLRW cosmological models in existence of bulk viscosity in the framework of extended symmetric teleparallel gravity. We considered two $f(Q,T)$ models, specifically, $f(Q,T) = \alpha Q^{n+1} + \lambda T$ and $f(Q,T) = \alpha Q + \lambda T$ where $\alpha$, $\lambda$ and $n$ are free models parameters. To find the exact solutions of the field equations we used the hybrid expansion law. According to the relevant recent observational data, the expansion phase of the Universe is

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