Quantum even spheres $\Sigma^q_{2n}$ from Poisson double suspension

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Abstract

We define even dimensional quantum spheres $\Sigma^q_{2n}$ that generalize to higher dimension the standard quantum two-sphere of Podleś and the four-sphere $\Sigma^q_4$ obtained in the quantization of the Hopf bundle. The construction relies on an iterated Poisson double suspension of the standard Podleś two-sphere. The Poisson spheres that we get have the same symplectic foliation consisting of a degenerate point and a symplectic plane and, after quantization, have the same $C^*$-algebraic completion. We investigate their $K$-homology and $K$-theory by introducing Fredholm modules and projectors.

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1 Introduction

In the seminal paper [13] by Podleś on $SU_q(2)$-covariant quantum two-spheres it was introduced a family of deformations $S^2_{q,d}$, depending on $d \in \mathbb{R}$ and $q < 1$. They define three distinct quantum topological spaces $S^2_{q,d}$: the case $d = 0$, the so called standard sphere, $d > 0$, the non standard sphere, and $d = -q^{2n}$, $n = 1, 2, \ldots$, the exceptional case.

The geometry of these quantum spaces can be nicely interpreted by looking at the underlying Poisson geometry and considering the sphere as a patching of
symplectic leaves. There exists, in fact, a 1–parameter family of $SU(2)$–covariant Poisson bivectors on the sphere $S^2$ such that $S^2_{q,d}$ can be seen as a quantization of such structures ([13]). In the case $d = 0$ the symplectic foliation is made of a point and a symplectic $\mathbb{R}^2$; after quantization, the symplectic plane is quantized to $K$, the algebra of compact operators, and the degenerate point survives as a character. The $C^*$-algebra $C(S^2_{q,0})$ is then isomorphic to the minimal unitization of compacts $\tilde{K}$, and satisfies the following exact sequence $0 \to K \to C(S^2_{q,0}) \to \mathbb{C} \to 0$. In the case $d > 0$ symplectic foliation consists of an $S^1$-family of degenerate points, and two symplectic disks. After quantization $C(S^2_{q,d})$ satisfies $0 \to K \oplus K \to C(S^2_{q,d}) \to C(S^1) \to 0$. The exceptional spheres correspond to the symplectic case and after quantization $C(S^2_{q,-q^2})$ is isomorphic to the finite dimensional algebra $M_n(\mathbb{C})$ of matrices.

In higher dimension, there exist the so called euclidean spheres $S^q_n$ introduced in [8] as quantum homogeneous spaces of $SO_q(n + 1)$; let us notice that in odd dimension they coincide with the so called Vaksman-Soibelman spheres $S^{2n+1}_{q}$ ([18]) introduced as quantum quotients $U_q(n)/U_q(n - 1)$, as observed in [10].

This family of spheres represents a generalization of the Podleś $d = 1$ case. In fact they satisfy the following exact sequences (see [10, 11, 18])

$$0 \to K \oplus K \to C(S^2_{q}) \to C(S^{2n-1}_{q}) \to 0$$

$$0 \to C(S^1) \otimes K \to C(S^{2n+1}_{q}) \to C(S^{2n-1}_{q}) \to 0.$$ 

This behaviour reflects exactly the Poisson level where each $S^{2n}$ contains an equator $S^{2n-1}$ as a Poisson submanifold and the two remaining hemispheres are open symplectic leaves of dimension $2n$. In particular all these spheres contain an $S^1$-family of degenerate points.

As there are no symplectic forms on higher dimensional spheres, exceptional spheres are possible only in dimension 2.

In this paper we introduce the family of even spheres $\Sigma^2_{q}$ which generalize the Podleś sphere: they all share with it the same kind of symplectic foliation. The idea of the construction relies on two classical subjects: double suspension and Poisson geometry.

The suspension idea is certainly not new and, in fact, appears in many of the papers devoted to the construction of particular deformations of the four-sphere ([1, 2, 3, 11]). When one considers double suspension, however, more interesting possibilities appear: on one hand one could define a purely classical double suspension already at an algebraic level by adding a pair of central selfadjoint generators and modding out suitable relations. A different kind of double suspension was considered, at the $C^*$–algebra level, in [11]. There the authors consider the non reduced double suspension of a $C^*$–algebra $A$ as the middle term $S^2A$ of a short exact sequence

$$0 \to A \otimes K \to S^2A \to \mathcal{C}(S^1) \to 0$$

for a suitable fixed Busby invariant. Let us remark that such a non reduced double suspension always has a naturally defined $S^1$–family of characters. All euclidean
spheres were reconstructed in this way starting either from a two–point space or from $S^1$.

In this paper we will consider the reduced double suspension or reduced topological product $S^2 X = S^2 \times X / S^2 \vee X$ where, given $p \in S^2$ and $q \in X$, $S^2 \vee X = (S^2 \times q) \cup (p \times X)$.

It is quite natural to look at the interaction between the double suspension and Poisson geometry. One word has to be said about the fact that while suspension is an essentially topological construction, Poisson bivectors are of differential nature, so that, in principle, on the double suspension of a given manifold there’s no reason to have a manifold structure, let aside a Poisson bracket. Still whenever the manifold structure is there one can ask whether such a Poisson structure arises. More precisely the double suspension of a manifold $M$ can be seen as a topological quotient $S^2 M$ of $S^2 \times M$. If we are given a Poisson bivector on the 2–sphere and a Poisson bivector on $M$ we can ask whether the quotient map $S^2 \times M \to S^2 M$ coinduces a Poisson bracket on the quotient. If this is the case we then look for its quantization.

From this point of view the classical double suspension quantizes a double suspension with respect to the trivial Poisson structure on $S^2$ while Hong–Szymański construction corresponds to the standard symplectic structure on $\mathbb{R}^2$ attached along an $S^1$ which will survive as a family of 0–leaves on the suspension.

In this paper the double suspension is built by assuming the Podleś $d = 0$ Poisson structure on the two-sphere and considering $S^{2n}$ as the double suspension of $S^{2(n-1)}$. We will show that a Poisson double suspension of spheres exists for each $n$ and that it quantizes both at an algebraic level and at the $C^*$–algebra level. The spheres $\Sigma^{2n}_q$ that we obtain have all the same symplectic foliation of the two-sphere, and their quantizations are topologically equivalent, $i.e.$ they are the minimal unitization of compacts and satisfy

$$0 \to K \to C(\Sigma^{2n}_q) \to \mathbb{C} \to 0$$

We then have a quite extreme case of quantum degeneracy: these quantum spaces, whatever is the classical dimension, are all topologically equivalent to a zero dimensional compact quantum space. This is an extreme manifestation of a well known fact, that quantum spaces associated to quantum groups have lower dimension than the classical one. Moreover this reminds of canonical quantization in which the Weyl quantization of $C_0(\mathbb{R}^{2n})$ is $K$, for each $n$. The opposite behavior is represented by the so called $\theta$-deformation, whose behavior is almost classical (see [5, 4]).

In the case of the four-sphere $\Sigma^4_q$, the algebra that we get is that obtained in [4], in the context of a quantum group analogue of the Hopf principal bundle $S^1 \to S^4$. It is unclear whether all even spheres $\Sigma^{2n}_q$ can be obtained as coinvariant subalgebras in quantum groups.

In Section 2 we introduce the Poisson double suspension that iteratively defines the Poisson even spheres and we study their symplectic foliation. In Section 3 we introduce the quantization $\text{Pol}(\Sigma^{2n}_q)$ at the level of polynomial functions, we classify the irreducible representations in bounded operators and then show that the
universal $C^*$-algebra $C(\Sigma^2_n)$ is the minimal unitization of compacts. We introduce Fredholm modules for each of these spheres. In Section 4, we give the non trivial generator of $K_0(\Sigma^2_n)$ and compute its coupling with the character of the previously introduced Fredholm modules.

2 The standard Poisson structure.

Let us define a point on $S^2 \times \ldots \times S^2$ by giving to it coordinates $((\alpha_1, \tau_1), \ldots (\alpha_n, \tau_n))$, where $|\alpha_i|^2 = \tau_i(1 - \tau_i)$. Let $M$ be the matrix whose entries are $M_{ii} = 0$, $M_{ij} = 1$ and $M_{ji} = 1/2$ if $i < j$. Let

$$a_i = \alpha_i \prod_k \tau_k^{M_{ik}} \quad t = \prod_i \tau_i.$$  \hfill (1)

Since

\[ \sum_i |a_i|^2 = \sum_i \tau_i (1 - \tau_i) \prod_k \tau_k^{2M_{ik}} = \tau_1 (1 - \tau_1) \tau_2^2 \ldots \tau_n^2 + \tau_2 (1 - \tau_2) \tau_3^2 \ldots \tau_n^2 + \ldots + \tau_n (1 - \tau_n) \tau_1 \ldots \tau_{n-1} \]

\[= t (1 - t), \]

then relation (1) defines a projection into $S^{2n}$. Let us call $\Phi : S^2 \times \ldots \times S^2 \rightarrow S^{2n}$ such projection. One can verify that this map is equivalent to the iterated reduced double suspension of a two-sphere, with preferred point the North Pole $\alpha = \tau = 0$. In fact the map from the cartesian product $S^2 \times X$ to the reduced topological product $S^2 \times X$ is the unique continuous map which is a homeomorphism everywhere but on the counter image of a point over which its fiber is $S^2 \vee X$. Starting from the two-sphere and iterating this procedure one defines a map from $S^2 \times \ldots S^2$ to $S^{2n}$ that is a homeomorphism everywhere but on the North Pole, where its fiber is the topological join of $n$ copies of $\underbrace{S^2 \times \ldots S^2}_{n-1}$. This map is the projection $\Phi$.

Let us equip $S^2$ with the standard Poisson structure, i.e. the limit structure of the Podleś standard two sphere ([13, 15]), and $S^2 \times \ldots S^2$ with the product Poisson structure. The brackets among polynomial functions are defined by giving

$$\{\alpha_i, \tau_j\} = -2 \delta_{ij} \alpha_i \tau_i, \quad \{\alpha_i, \alpha_j^*\} = 2 \delta_{ij} (\tau_i^2 - \alpha_i \alpha_j^*).$$ \hfill (2)

We prove the following result.

**Proposition 1** The map $\Phi$ is a Poisson map. The coinduced brackets on $S^{2n}$ read:

\[\{a_k, a_\ell\} = a_k a_\ell \quad (k < \ell), \quad \{a_k, a_\ell^*\} = -3 a_k a_\ell^* \quad (k \neq \ell), \]

\[\{a_i, t\} = -2 a_i t, \quad \{a_k, a_\ell\} = 2 t^2 + 2 \sum_{\ell < k} a_\ell a_\ell^* - 2 a_k a_k^* . \] \hfill (3)
Proof. The result is obtained by explicitly computing the brackets on $\mathbb{S}^2 \times \ldots \times \mathbb{S}^2$.

The only relation that deserves some attention is the last one. By direct computation we obtain

$$\{a_k, a_k^*\} = 2\tau_k^2 \prod_i \tau_i^{2M_{ki}} - 2a_k a_k^*.$$ Let us show by induction on $k$ that

$$\tau_k^2 \prod_i \tau_i^{2M_{ki}} = t^2 + \sum_{\ell<k} a_{\ell} a_{\ell}^*.$$ It is clearly true for $k = 1$. Let it be true for $k$. We then have

$$\tau_{k+1}^2 \prod_i \tau_i^{2M_{ki}} = \tau_1 \ldots \tau_k \tau_{k+1}^2 \ldots \tau_n^2 = \tau_1 \ldots (\tau_k + \alpha_k a_k^*) \tau_{k+1}^2 \ldots \tau_n^2$$

$$= t^2 + \sum_{\ell<k} a_{\ell} a_{\ell}^* + a_k a_k^* = t^2 + \sum_{\ell<k+1} a_{\ell} a_{\ell}^*.$$}

The Poisson manifold $(\mathbb{S}^{2n}, \{,\})$ is usually called coinduced from the bracket on $\mathbb{S}^2 \times \ldots \times \mathbb{S}^2$ (see [17]). Its symplectic foliation is described in the following proposition.

**Proposition 2** There are two distinct symplectic leaves in $(\mathbb{S}^{2n}, \{,\})$:

i) a zero dimensional leaf given by the north pole $P_N = (a_i = 0, t = 0)$;

ii) $\mathbb{R}^{2n} = \mathbb{S}^{2n} \setminus P_N$.

The Poisson brackets on $\mathbb{R}^{2n}$ read:

$$\{z_k, z_\ell\} = z_k z_\ell \quad (k \leq \ell),$$

$$\{z_k, z_k^*\} = 2(1 + \sum_{\ell \leq k} z_\ell z_\ell^*),$$

$$\{z_k, z_\ell^*\} = z_k z_\ell^* \quad (k \neq \ell).$$

Proof. It is clear that the north pole $P_N$ defined by $a_i = t = 0$ is a degenerate point. We are going to show that $\mathbb{R}^{2n} = \mathbb{S}^{2n} \setminus P_N$ is symplectic. Relations (4) are obtained by direct computation of the brackets among the complex coordinates $z_i = a_i / t$.

Let us define the $2n \times 2n$ antisymmetric matrix $S^{(n)}$ as $S^{(n)}_{ij} = \{w_i, w_j\}$, where $w_{2k-1} = z_k$ and $w_{2k} = z_k^*$, for $k = 1, \ldots, n$. It is clear that $S^{(n)}_{ij} = S^{(n-1)}_{ij}$ for $i, j = 1, \ldots, 2(n-1)$. To compute the determinant of this matrix let us introduce a set of $2n$ fermionic variables $\eta_i$, i.e. $\eta_i \eta_j + \eta_j \eta_i = 0$. The pfaffian of $S^{(n)}$ can be expressed as

$$Pf(S^{(n)}) = \int d\eta_{2n} \ldots d\eta_1 e^{\frac{i}{2} \sum_{ij} S^{(n)}_{ij} \eta_i \eta_j}$$
Let us define Proposition 3
representation $\sigma$. Let us define $\epsilon$ operators, the first is one dimensional $<q<$
Let $0 $
\begin{align*}
Pf(S^{(n)}) &= Pf(S^{(n-1)}) \int du_{2n}du_{2n-1}(1 + S^{(n)}_{2n-1,2n}u_{2n-1}u_{2n}) \\
&= Pf(S^{(n-1)})S^{(n)}_{2n-1,2n} = Pf(S^{(n-1)})\{z_\tau, z_\tau\} \\
&= 2Pf(S^{(n-1)})(1 + \sum_{\ell \leq n} |z_\ell|^2).
\end{align*}
Since $Pf(S^{(1)}) = 2(1 + |z|^2)$ we conclude that $Pf(S^{(n)}) \neq 0$ and that $\mathbb{R}^{2n}$ is symplectic. ■

The Poisson structure on the other chart $\mathbb{R}^{2n} = S^{2n} \setminus \{t = 1\}$ is symplectic everywhere but the origin. In [13] Zakrzewski introduced a family of $SU(n)$-covariant Poisson structures on $\mathbb{R}^{2n}$ with this foliation. It is an interesting problem to understand the relations between them.

3 Quantization of the standard Poisson structure.

The algebra $Pol(S^2_{q,0})$ of the Podleś standard sphere is generated by $\{\alpha, \alpha^*, \tau\}$, where $\tau$ is real, with the following relations:

$$
\alpha \tau = q^2 \tau \alpha, \quad q^2 \alpha^* \alpha = \tau (1 - \tau), \quad \alpha \alpha^* = q^2 \alpha^* \alpha + (1 - q^2) \tau^2.
$$

Let $0 < q < 1$. There are two irreducible representations of $Pol(S^2_{q,0})$ with bounded operators, the first is one dimensional $\epsilon(\alpha) = \epsilon(\tau) = 0$, the second $\sigma : Pol(S^2_{q,0}) \to B(\ell^2(\mathbb{N}))$ is defined by

$$
\sigma(\alpha)|n\rangle = q^{n-1}(1 - q^{-2})^{1/2}|n-1\rangle, \\
\sigma(\tau)|n\rangle = q^{2n}|n\rangle.
$$

Let us define $\sigma^{\otimes n} : Pol(S^2_{q,0})^{\otimes n} \to B(\ell^2(\mathbb{N})^{\otimes n})$ as the $n^{th}$-tensor product of the representation $\sigma$, we denote by $\{\alpha_i, \alpha_i^*, \tau_i\}$ the generators of the $i^{th}$ $Pol(S^2_{q,0})$.

**Proposition 3** Let us define $Pol(S^2_{q})$ the algebra generated by $\{a_i, a_i^*, t\}$ with relations

$$
a_i t = q^2 t a_i, \quad a_i a_j = q^{-1} a_j a_i, \quad a_i a_j^* = q^2 a_j^* a_i \quad (i < j), \\
a_i a_i^* = q^2 a_i^* a_i + q^2 (1 - q^2) \sum_{\ell < i} a_i^* a_\ell + (1 - q^2) t^2, \quad \sum_{i=1}^n q^2 a_i^* a_i = t - t^2.
$$
The mapping $\sigma_n : \text{Pol}(\Sigma_q^{2n}) \to B(\ell^2(N)^{\otimes n})$ given by

$$\sigma_n(a_i) = \sigma^{\otimes n}(\alpha_i \prod_k \tau_k^{M_{ik}}) , \quad \sigma_n(t) = \sigma^{\otimes n}(\prod_i \tau_i) ,$$

(6)
is a representation of $\text{Pol}(\Sigma_q^{2n})$.

**Proof.** From the relation $\sigma^{\otimes n}(\alpha_k \tau_k^{M_{ik}}) = q^{M_{ik}} \sigma^{\otimes n}(\tau_k^{M_{ik}} \alpha_k)$ where $M_{ij}$ is the matrix with $M_{ii} = 0$, $M_{ij} = 1$ and $M_{ji} = 1/2$ for $i < j$, it is straightforward to verify the first line of relations. In order to prove the relations in the second line we need the following equality (we will omit the application of $\sigma^{\otimes n}$):

$$\tau_i^2 \prod_k \tau_k^{2M_{ik}} = t^2 + q^2 \sum_{\ell < i} a^*_\ell a_\ell .$$

For $i = 1$ it is true, we will verify that it is true for $i + 1$ assuming it true for $i$:

$$\tau_{i+1}^2 \prod_k \tau_k^{2M_{i+1k}} = \tau_1 \cdots \tau_i \tau_{i+1}^2 \cdots \tau_n^2 = \tau_1 \cdots \tau_{i-1}^2 (q^2 \alpha_i^* \alpha_i + \tau_i^2) \cdots \tau_n^2 = q^2 \alpha_i^* \alpha_i + \tau_i^2 \prod_k \tau_k^{2M_{ik}} = q^2 \alpha_i^* \alpha_i + t^2 + q^2 \sum_{\ell < i} a^*_\ell a_\ell = t^2 + q^2 \sum_{\ell < i+1} a^*_\ell a_\ell .$$

To verify the modulus relation we simply needs the same computation of the classical case presented at the beginning of Section 2. ■

**Remark 4** The semiclassical limit, defined by $\{f, g\} = \lim_{q \to 1} \frac{1}{1-q} [f, g]$, of the relations of Proposition 3 coincides with the Poisson structure defined by the map $\Phi$ in Proposition 1. ■

**Proposition 5** If $\varphi : \text{Pol}(\Sigma_q^{2n}) \to B(\mathcal{H})$ is an irreducible representation on some Hilbert space, then $\varphi = \epsilon$ or $\varphi = \sigma_n$.

**Proof.** In order to prove the existence of an eigenvector of $\varphi(t)$ we will adapt the proof of Theorem 4.5 in [9]. By using relations we see that $\varphi(t - t^2) > 0$ so that $\text{Sp}(\varphi(t)) \subset [0, 1]$. If $\text{Sp}(\varphi(t)) = \{0\}$ then $\varphi(t) = 0$ and $\varphi = \epsilon$; if $\text{Sp}(\varphi(t)) = \{1\}$ then $\varphi(t) = 1$ and this contradicts relations; if $\text{Sp}(\varphi(t)) = \{0, 1\}$ then $\lambda = 0$ would be an eigenvalue and $\text{Ker}(\varphi(t))$ would be an invariant subspace. So in order to have $\varphi$ irreducible and $\varphi \neq \epsilon$ we must have $\text{Sp}(\varphi(t)) \setminus \{0, 1\} \neq \emptyset$. 7
Let $\lambda \in Sp(\varphi(t)) \setminus \{0, 1\}$ and let $\{\xi_s\}$ be a set of approximate unit eigenvectors, i.e. unit vectors such that $\lim_{s \to \infty} ||\varphi(t)\xi_s - \lambda\xi_s|| = 0$. By writing $t - t^2 = \lambda(1 - \lambda) + (t - \lambda)(1 - \lambda - t)$ we get

$$||\varphi(t - t^2)\xi_s|| \geq ||\lambda(1 - \lambda)|| - ||\varphi(t - \lambda)\varphi(1 - \lambda - t)\xi_s|| \geq ||\lambda(1 - \lambda)|| - ||\varphi(t - \lambda)\xi_s|| ||\varphi(1 - \lambda - t)|| \geq C'\lambda(1 - \lambda),$$

for $s$ bigger than some $s_o$ and for some $C' > 0$. Moreover we have

$$||\sum_{k=1}^n \varphi(a^*_k\xi_s)|| \leq \sum_{k=1}^n ||\varphi(a^*_k)|| ||\varphi(a_k)\xi_s|| \leq C''n||\varphi(a_{k(s)}\xi_s||,$$

where $C''$ and $k(s)$ are such that $||\varphi(a^*_k)|| \leq C''$ and $||\varphi(a_k)\xi_s|| \leq ||\varphi(a_{k(s)}\xi_s||$ for all $k$. We conclude that for each $s > s_o$ there exists $1 \leq k(s) \leq n$ such that $||\varphi(a_{k(s)}\xi_s)|| \geq C''\lambda(1 - \lambda)$. We can define $\nu_s = \varphi(a_{k(s)}\xi_s)/||\varphi(a_{k(s)}\xi_s||$ and verify that they are approximating unit eigenvectors for $q^{-2}\lambda$; in fact

$$||(|\varphi(t) - q^{-2}\lambda)\nu_s|| = \frac{q^{-2}}{||\varphi(a_{k(s)}\xi_s)||} ||\varphi(a_{k(s)}(t - \lambda))\xi_s|| \leq \frac{q^{-2}\lambda(1 - \lambda)}{C''} ||\varphi(a_{k(s)})|| ||\varphi(t - \lambda)\xi_s||.$$

We then showed that if $\lambda \in Sp(\varphi(t)) \setminus \{0, 1\}$ then $q^{-2}\lambda \in Sp(\varphi(t))$. In order to keep $Sp(\varphi(t))$ bounded it is necessary that for each $\lambda$ there exists $k$ such that $q^{-2k}\lambda = 1$, i.e. $Sp(\varphi(t)) \setminus \{0\} = \{q^{2k}, k \in \mathbb{N}\}$. Since each $q^{2k}$ is isolated, we conclude that it is an eigenvalue.

Let $\psi$ the eigenvector corresponding to $\lambda = 1$: since it is the biggest eigenvalue we get that $\varphi(a)\psi = 0$. By direct computation one recovers:

$$\varphi(a_i) \prod_j \varphi(a^*_j)^{m_j} \psi = q^{3\sum_{j<i}m_j}q^4\sum_{j>i}m_jq^{2(m_i-1)}(1 - q^{2m_i}) \varphi(\prod_{j<i}a^*_j a^*_i a^{*(m_i-1)} \prod_{j>i} a^*_j m_j) \psi. \quad (7)$$

Let us define, with $m = (m_1, \ldots, m_n)$,

$$\psi^m = C^m \varphi(\prod_i a^*_i m_i) \psi, \quad C^m = q^{-(\sum_i m_i^2 + \sum_i m_i(m_i+1)/2)} \prod_i (q^2; q^2)^{-1/2}_{m_i},$$

where $(\alpha; q)_s = \prod_{j=1}^s (1 - q^{j-1}\alpha)$ and $(\alpha; q)_0 = 1$. Formula (7) implies

$$||\varphi(a^*_i)^m \psi||^2 = q^{2\sum_{j<i}m_j}q^4\sum_{j>i}m_j(1 - q^{2(m_i+1)})||\psi^m||^2,$$

from which we conclude that $\psi^m \neq 0$ for each $m$. The space generated by $\{\psi^m\}$ is invariant and it coincides with $\mathcal{H}$. Finally it can be verified that the mapping $T: \ell^2(\mathbb{N})^n \to \mathcal{H}$ defined by $|m_1, \ldots, m_n) \mapsto \psi^m$ intertwines the representations
σₙ and ϕ. Since one can verify that the ψₘ’s are orthonormal, we conclude that T is unitary.

The universal C*-algebra generated by Pol(Σ²ⁿ) is then the norm closure of σₙ(Pol(Σ²ⁿ)). Since σₙ(aᵢ) and σₙ(t) are trace-class operators, then σₙ(Pol(Σ²ⁿ) \ ℂ) ⊂ K. By using Proposition 15.16 of [7], which states that a norm-closed *-subalgebra A of K, such that the representation A → K is irreducible, coincides with K, we prove the following result.

**Corollary 6** The C*-algebra generated by Pol(Σ²ⁿ) is isomorphic to \( \tilde{K} \), the minimal unitization of compacts.

**Remark 7** The C*-algebra of quantum even spheres is independent of the classical dimension. This is not as strange as it may appear; the C*-algebra level usually reflects the topology of the space of leaves on the underlying Poisson bracket which is the same in all cases.

**Remark 8** As already hinted in the introduction, starting from any C*-algebra A with at least one character εₓ and from a quantum two-sphere B with a character ε₀ one could define a topological double suspension:

\[
S²_q A := \{ f ∈ A ⊗ B \mid (εₓ ⊗ id)(f) = (id ⊗ ε₀)(f) ∈ ℂ \}.
\]

It is then a trivial remark that such construction applied to standard Podleś sphere is stable. What is less trivial is the fact that such algebras quantize a whole family of polynomial quantum even spheres.

Thanks to Corollary (6) we can conclude that \( K₀(C(Σ²ⁿ)) = ℤ² \) for each n, see [12]; each polynomial sphere Pol(Σ²ⁿ) will provide different representatives of the same class in K-homology. Let us describe them explicitly, along the same lines of [12].

The first one \( [ε] \) is the pullback by ε : C(Σ²ⁿ) → ℂ of the generator of \( K₀(ℂ) \). By analogy with the classical case we say that its character ε computes the rank of the vector bundle.

Let us describe the second and more interesting generator. Let the Hilbert space be \( H = ℓ²(N)⊗² \oplus ℓ²(N)⊗² \) and \( π = \begin{pmatrix} σ_n & 0 \\ 0 & ε \end{pmatrix} \), \( F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). We have that \((H, π, F)\) is a 1–summable Fredholm module whose character is the zero cyclic cocycle \( tr_{σ_n} = tr(σ_n - ε) \). Let us denote with \( [tr_{σ_n}] \) its class; we say that \( tr_{σ_n} \) computes the charge. In particular, from (6) we get

\[
tr_{σ_n}(1) = 0, \quad tr_{σ_n}(t) = \frac{1}{(1 - q²)^n}.
\]
4 Algebraic projectors and the Chern–Connes pairing

Let us now come to the construction of non trivial quantum vector bundles on spheres. Since \( C(\Sigma_q^{2n}) = k \) we have that \( K_0(C(\Sigma_q^{2n})) = \mathbb{Z}^2 \). In this section we will introduce two algebraic generators, i.e. two projectors with entries in \( \text{Pol}(\Sigma_q^{2n}) \) whose classes generate \( K_0 \). The first one is the trivial one \([1]\); in order to get the non trivial generator let us go back to the classical case.

The non trivial generator of \( K \)-theory for the classical even sphere can be explicitly written in the following way. Let \( G_{2n}^{2n} \in M_{2k}(\mathbb{S}^{2n}) \) be defined iteratively by

\[
G_{2(k+1)}^{2n} = \left( \begin{array}{cc}
G_{2k}^{2n} & a_{k+1}^* \\
a_{k+1} & 1 - G_{2k}^{2n}
\end{array} \right), \quad G_{0}^{2n} = 1 - t .
\]

It is easy to verify that \( G_{2n} \equiv G_{2n}^{2n} \) is an idempotent for \( n \geq 0 \) defining the vector bundle \( E_{2n} \) of rank \( 2^n - 1 \) and charge \(-1\). This construction has the following geometrical interpretation. Let \( i : \mathbb{S}^{2n} \rightarrow \mathbb{S}^{2(n+1)} \) defined by \( i(t) = (t, a_1, \ldots, a_n) \) be an embedding of \( \mathbb{S}^{2n} \) in \( \mathbb{S}^{2(n+1)} \) and let \( i^* (E_{2(n+1)}) \) the pullback vector bundle on \( \mathbb{S}^{2n} \). It is clear that \( i^*(E_{2(n+1)}) = E_{2n} \oplus E_{2n}' \), where \( E_{2n}' \) is the conjugated vector bundle on \( \mathbb{S}^{2n} \) of charge 1.

In the quantum case not every step of this procedure can be obviously extended. In fact there is no embedding of \( \Sigma_q^{2n} \) in \( \Sigma_q^{2(n+1)} \); i.e. there are no algebra projections from \( \text{Pol}(\Sigma_q^{2(n+1)}) \) to \( \text{Pol}(\Sigma_q^{2n}) \). It is possible to adapt the procedure in order to produce a rank \( 2^n - 1 \) idempotent for \( \Sigma_q^{2n} \); but it is convenient to lift the construction to \( \mathbb{R}^{2n+1} \), a deformation of the odd plane where the even sphere lives. Let us introduce \( \text{Pol}(\mathbb{R}^{2n+1}_q) \) as the algebra generated by \( \{ x_i, x_i^*, y = y^* \}_{i=1}^n \) with the relations

\[
\begin{align*}
&x_i y = q^2 y x_i, \quad x_i x_j = q^{-1} x_j x_i, \quad x_i x_j^* = q^3 x_j^* x_i, \quad i < j, \\
x_i x_i^* = q^2 x_i^* x_i + q^2 (1 - q^2) \sum_{k < i} x_k^* x_k + (1 - q^2) y^2 .
\end{align*}
\]

It is clear that \( \text{Pol}(\Sigma_q^{2n}) = \text{Pol}(\mathbb{R}^{2n+1}_q)/I_n \), where \( I_n \) is the ideal generated by \( q^2 \sum_i x_i x_i - y + y^2 \) and \( a_i = p(x_i), t = p(y) \) if \( p \) is the projection map. Let \( \phi_n : \text{Pol}(\mathbb{R}^{2n+1}_q) \rightarrow \text{Pol}(\mathbb{R}^{2n+1}_q) \) be the algebra automorphism defined by \( \phi_n(x_i) = q^2 x_i \) and \( \phi_n(y) = q^2 y \). The existence of this automorphism, that doesn’t pass to the quotient, is actually the reason for this lifting.

For each \( 0 \leq k \leq n \) let us define \( e_{2k}^{2n} \in M_{2k}(\mathbb{R}^{2n+1}_q) \) using the following recursive formula

\[
e_{2(k+1)}^{2n} = \left( \begin{array}{cc}
e_{2k}^{2n} & C_{2k}^{2n} x_{k+1}^* \\
C_{2k}^{2n} x_{k+1} & 1 - \phi_n(e_{2k}^{2n})\end{array} \right), \quad e_{0}^{2n} = 1 - y , \tag{10}
\]

where \( C_{2k}^{2n} \) is the diagonal complex matrix given by

\[
C_{2k}^{2n} = \left( \begin{array}{cc}
C_{2(k-1)}^{2n} & 0 \\
0 & q C_{2(k-1)}^{2n}\end{array} \right) \in M_{2k}(\mathbb{C}) , \quad C_{0}^{2n} = q . \tag{11}
\]

10
We need the following Lemma.

**Lemma 9** For each \( \ell \) such that \( k < \ell \leq n \) we have that

\[
M_{2k,\ell}^{2n} \equiv e_{2k}^{2n}C_{2k}^{2n}x_{\ell}^* - C_{2k}^{2n}x_{\ell}^*\phi(e_{2k}^{2n}) = 0 .
\] (12)

**Proof.** We prove the result by induction on \( k \). For \( k = 0 \) it is equivalent to \( yx_{\ell}^* = q^2x_{\ell}^*y \). Let us suppose \((12)\) true for \( k = 1 \) and let us show it for \( k + 1 \). By direct computation we get \([M_{2k+1,\ell}^{2n}])_{11} = M_{2k,\ell}^{2n}, [M_{2k+1,\ell}^{2n}]_{22} = -q^{-1}\phi_n(M_{2k,\ell}^{2n})\) and \([M_{2k+1,\ell}^{2n}]_{12} = q(C_{2k}^{2n})(x_{k+1}^*x_{\ell}^* - qx_{\ell}^*x_{k+1})\). Using the inductive hypothesis and relations in \( \text{Pol}(\mathbb{R}_q^{2n+1}) \) we conclude that \( M_{2(k+1),\ell}^{2n} = 0 \). \( \blacksquare \)

We now prove the main result of this section.

**Proposition 10** For each \( k \leq n \) we have

\[
(e_{2k}^{2n})^2 - e_{2k}^{2n} = [q^2 \sum_{i=1}^{k} x_i^*x_i - y(1 - y)]q^{-2}(C_{2k}^{2n})^2.
\] (13)

**Proof.** We show the result by induction on \( k \). For \( k = 0 \) it is easy to see that it is true. Let us suppose it true for \( k \). By direct computation, using the inductive hypothesis and equation (12) we get

\[
[(e_{2k+1}^{2n})^2 - e_{2k+1}^{2n}]_{11} = [q^2 \sum_{i=1}^{k+1} x_i^*x_i - y(1 - y)]q^{-2}(C_{2k}^{2n})^2
\]

\[
[(e_{2k+1}^{2n})^2 - e_{2k+1}^{2n}]_{22} = (C_{2k}^{2n})^2x_{k+1}^*x_{k+1} + \phi_n(e_{2k}^{2n})^2 - \phi_n(e_{2k}^{2n})
\]

\[
= (C_{2k}^{2n})^2(x_{k+1}^*x_{k+1} + \sum_{i=1}^{k} q^4 x_i^*x_i - y(1 - q^2y))
\]

\[
= (C_{2k}^{2n})^2[q^2 \sum_{i=1}^{k+1} x_i^*x_i - y(1 - y)]
\]

\[
[(e_{2k+1}^{2n})^2 - e_{2k+1}^{2n}]_{12} = e_{2k}^{2n}C_{2k}^{2n}x_{k+1}^* - C_{2k}^{2n}x_{k+1}^*\phi(e_{2k}^{2n}) = 0 .
\]

Recalling the iterative definition of \( C_{2(k+1)}^{2n} \) we finally get the result

\[
(e_{2(k+1)}^{2n})^2 - e_{2(k+1)}^{2n} = [q^2 \sum_{i=1}^{k+1} x_i^*x_i - y(1 - y)]q^{-2}(C_{2k+1}^{2n})^2 .
\]

\( \blacksquare \)

It is then clear that for each \( n > 0 \), \( G_{2n} = p(e_{2n}^{2n}) \in M_{2n}(\text{Pol}(\Sigma_q^{2n})) \) is a projector; let us denote with \([G_{2n}]\) its class in K-theory (both algebraic and topological). By using the recursive definition of \( e_{2n}^{2n} \) it is easy to compute the matrix trace of \( G_{2n} \). In fact the equation

\[
\text{Tr}(e_{2(n+1)}^{2n}) = 2^k + \text{Tr}(e_{2k}^{2n} - \phi_n(e_{2k}^{2n})) , \quad \text{Tr}(e_{0}^{2n}) = 1 - y ,
\]

11
is solved by $\text{Tr}(e^{2n}_{2k}) = 2^{k-1} - (1 - q^2)^k y$ (for $k \geq 1$), so that we have
\[ \text{Tr}(G_{2n}) = 2^{n-1} - (1 - q^2)^n t. \] (14)

By recalling the definition of the Fredholm modules $[\epsilon]$ and $[\text{tr}_{\sigma_n}]$, we compute their Chern–Connes pairing with $G_{2n}$:
\[ \langle [\epsilon], [G_{2n}] \rangle = 2^{n-1}, \quad \langle [\text{tr}_{\sigma_n}], [G_{2n}] \rangle = -1. \]

**Corollary 11** The projector $G_{2n}$ defines a non trivial class both in $K_0(\text{Pol}(\Sigma_q^{2n}))$ and $K_0(C(\Sigma_q^{2n}))$.

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**References**

[1] Bonechi F., Ciccoli N. and Tarlini M.: Non commutative instantons on the 4–sphere from quantum groups. Commun. Math. Phys. 226, 419–432 (2002).

[2] Bonechi F., Ciccoli N. and Tarlini M.: Quantum 4–sphere: the infinitesimal approach. Banach Center Pubbl. at press.

[3] Brzezinski T. and Gonera C.: Non commutative 4–spheres based on all Podleś 2–spheres and beyond. Lett. Math. Phys. 54, 315–321 (2000).

[4] Connes A. and Dubois-Violette M.: Noncommutative finite-dimensional mani-

folds. I. Spherical manifolds and related examples. [math.QA/0107076]

[5] Connes A. and Landi G.: Noncommutative manifolds, the istanton algebra and isospectral deformations. Commun. Math. Phys. 221, 141–159 (2001).

[6] Dabrowski L., Landi G. and Masuda T.: Instantons on the quantum sphere $\mathbb{S}^4_q$. Commun. Math. Phys. 221, 161–168 (2001).

[7] Doran, R.S. and Fell J.M.G.: *Representations of *–Algebras, Locally Compact Groups, and Banach *–Algebraic Bundles: Vol I. Academic Press New York, (1988).

[8] Faddeev L., Reshtekhin N. and Takhtajan L.: Quantization Of Lie Groups And Lie Algebras. Leningrad Math. J. 1 193 (1990). [Alg. Anal. 1 178 (1990)].
[9] Hajac P.M., Matthes R., Szymanski W.: Quantum Real Projective Space, Disc and Sphere. Algebras and Repr. Th. at press. math.QA/0009185

[10] Hawkins E. and Landi G.: Fredholm modules for quantum Euclidean spheres. math.KT 0201039

[11] Hong J.H. and Szymański W.: Quantum spheres and projective spaces as graph algebras. Commun. Math. Phys. at press.

[12] Masuda T., Nakagami Y. and Watanabe J.: Noncommutative differential geometry on the quantum two sphere of Podleś I: an algebraic viewpoint. K–theory 5, 151-175 (1991).

[13] Podleś, P.: Quantum Spheres. Lett. Math. Phys. 14, 193–202 (1987).

[14] Ramond, P.: Field Theory: a modern primer. Addison-Wesley, (1990).

[15] Sheng A.J.L. (with an appendix by Lu J.H. and Weinstein A.): Quantization of the Poisson SU(2) and its Poisson homogeneous space – the 2-sphere, Commun. Math. Phys. 135, 217-232 (1991).

[16] Sitarz A.: More non commutative 4–spheres. Lett. Math. Phys. 55, 127–131 (2001).

[17] Vaisman I.: Lectures on the geometry of Poisson manifolds. Progress in Math., 118, Birkhäuser Verlag, (1994).

[18] Vaksman, L.L. and Soibelman, Ya.: Algebra of functions on the quantum group $SU(N+1)$ and odd dimensional quantum spheres. Leningrad Math. J. 2, 1023–1042 (1991).

[19] Zakrzewski S.: Poisson structures on $\mathbb{R}^{2n}$ having only two symplectic leaves: the origin and the rest. In Poisson Geometry, J. Grabowski and P. Urbański eds., Banach Center Pubbl. 51, Warszaw (2000).