The two-leg $t - J$ ladder: A mean-field description

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Abstract

Two-leg $t - J$ ladders are investigated in the framework of a combination of the phase string formulation and bond-operator representation. We develop a mean-field theory in the strong rung interaction regime, i.e. $J_\perp \gg J, t$, which provides a unified description of the undoped insulating phase and the low doping phase — the so-called $C1S0$ phase. Both of them are characterized by the resonating-valence-bond (RVB) order parameter, with gap opened up in all spin excitations. The ground state of the doped phase is intrinsically a superconductor with a d-wave symmetry, which is driven by the RVB correlations. The ground-state energy is in good agreement with numerical results. Phase separation is shown to occur beyond some critical value of $J/t$ for given doping concentration. We also show that the spin gap in the doped phase is determined by quasi-particle-like excitations. The local structure of hole pairs as well as the spectra of various spin and charge modes are analyzed in comparison with other approaches.
I. INTRODUCTION

Since the discovery of high-$T_c$ superconductors, the study of $t-J$ type models has become an important topic of strongly correlated electron systems. The $t-J$ model provides us one of the simplest examples of the nontrivial interplay between charge and antiferromagnetism in a doped Mott insulator. It is a problem generally difficult to tackle by conventional many-body methods due to the nature of strong correlations.

Recently, ladder systems have received intensive studies both theoretically [1–13] and experimentally [14–17]. From theoretical point of view, the ladder $t-J$ systems may be easier to investigate both numerically and analytically than the two-dimensional (2D) case related to the high-$T_c$ cuprates. However, the former may already catch some key physics of the latter and offer some important insights into the competition between charge and spin correlations beyond one-dimensional (1D) geometry.

Like in 1D and 2D, the physics of ladder $t-J$ systems has been more or less well-understood at half-filling [3–6] where only the spin degrees of freedom are present. For example, in the two-leg ladder problem, the ground state may be visualized as a spin liquid state of the condensate of short-ranged RVB spin singlets [6]. In the strong rung interaction limit, a bond-operator representation [18] provides a very useful description [5] of such a spin liquid state.

Also similar to 1D and 2D, the doped case in ladder $t-J$ systems poses a real challenge to analytic approaches: The central issue is how to correctly handle the competing charge and spin correlations once holes are introduced. In the two-leg ladder system, it has been established on numerical basis that on the small doping side ($\delta < 0.5$) there exists a so-called $C1S0$ phase [8,9] where spin excitations are all gapped (denoted by $S0$) while the density fluctuations of holes represent the only gapless mode there (denoted by $C1$). Such a phase is a superconducting phase with a d-wave-like symmetry. With the increase of the ratio $J/t$ (where $t$ and $J$ are two parameters of the $t-J$ model to be defined later), eventually a phase separation is found beyond some critical value of $J/t$. 
Previously, a mean-field theory of lightly doped two-leg ladders proposed by Sigrist et al. [7] showed that the short-ranged RVB state evolves into a superconductor with modified d-wave symmetry. It also gave a continuous evolution of the spin gap with doping. However, this theory fails to capture many features of the ground state and the excitation spectrum uncovered later by numerical work [8–13]. The main reason is that the fermionic RVB mean-field theory employed in this approach is not a very good description even at half-filling. It is the purpose of this paper to develop an analytic framework that can give a unified and systematic description of the aforementioned physical properties in both the undoped and doped phases of the two-leg ladder $t – J$ model. We shall start from the strong rung interaction limit where it is natural to adopt the previously mentioned bond-operator representation [5,18] for the undoped case. The corresponding mean-field theory [5] gives a reasonably good description of the ground state and spin excitations at half-filling as mentioned before. In the doped case, additional bond operators are apparently needed and they can be classified as the rung hole pair ($d$) and quasiparticles ($a_\sigma$ and $\bar{a}_\sigma$) besides the spin singlet ($s$) and spin triplet ($t_\alpha$) operators. They compose of a complete basis convenient for describing the low doping case.

For any dimensionality, it is highly nontrivial to get access to the doped phase from the undoped insulator of the $t – J$ model. This is due to the fact that the Marshall signs [19] hidden in the half-filled spin background will be generally “disordered” by the motion of holes, leading to the phase string effect [20]. Such a phase string effect cannot be repaired through spin flip processes as the latter always respects the Marshall sign at low energy. It implies that the nonrepairable “phase strings” left on the hole paths will be present in the ground state of the doped case. These “phase strings” play a role similar to the Fermi-surface phase-shifts originally proposed by Anderson. Indeed, in 1D case, the phase string effect leads to the Luttinger liquid behavior. The phase string formulation [20] provides a systematical method which reproduces correct exponents of various correlation functions. Nontrivial phase string effect in 2D mean-field theory has been also investigated in Ref. [21]. Incorporating the phase string effect thus becomes a necessary step to construct a sensible
mean-field theory in the study of the two-leg ladder systems.

Starting from the half-filling where the mean-field theory is based on a RVB characterization with an order parameter \( \langle s_j \rangle = \bar{s} \), we are able to generalize the theory to the doped regime after incorporating the phase string effect. We find that the ground state in the doped phase is naturally a superconductor with a d-wave-like symmetry, as the consequence of the RVB correlations in the insulating phase. The mean-field ground-state energy is in good agreement with the numerical one at the doping concentration \( \delta \leq 0.5 \). Moreover, an instability of phase separation occurs in our mean-field state as the ratio \( J/t \) increases beyond some critical value, also consistent with numerical results. The present mean-field theory thus accommodates the most important physical properties of the doped two-leg \( t-J \) model previously identified only numerically. We would like to point out that without explicitly dealing with the nonlocal phase string effect at the starting point, a mean-field treatment would lead to a phase which is always unstable against phase separation, similar to the spiral instability in 2D case [22].

Furthermore, a series of detailed features obtained in various numerical work are also reproduced at this mean-field level. The energy spectra of magnons and quasiparticles have been determined in the \( C1S0 \) phase where they all exhibit finite gaps varying with the doping concentration. The gap of the former continuously evolves from the insulating phase while the one of the latter arises from the formation of Cooper pairs between quasiparticles. The minimum gap of creating a pair of quasiparticle excitations is smaller than that of the magnon. This indicates that the spin gap in the two-leg ladder system is generally associated with quasiparticles and shows a discontinuous evolution with doping, as first pointed out by Troyer et al. [9]. We also examine the local structure of hole pairs and show that the pairing on diagonal sites occurs simultaneously with the condensation of rung hole pairs. This point was also noted previously by Sierra et al. [11]. Since the pairing between quasiparticles also results from the condensation of rung hole pairs, a relationship between the pairing on diagonal sites and the spin gap in the two-leg ladders is then established.

The rest of the paper is organized as follows: In Section 2, we introduce the phase
string formulation and bond-operator representation. The general feature of the resulting Hamiltonian is discussed and the mean-field treatment is presented in Section 3. In Section 4, we present our numerical analysis of the mean-field equations. The final section is devoted to a conclusive discussion. For the sake of compactness, some useful and relevant formulae are listed in the Appendices.

II. MATHEMATICAL FORMULATION

We start with the original $t-J$ Hamiltonian

$$H_{t-J} = P_s \{-t \sum_{\langle i,j \rangle} (c_i^+ c_j + H.c.) + J \sum_{\langle i,j \rangle} (S_i \cdot S_j - \frac{1}{4} n_i n_j)\} P_s$$

(1)

where $P_s$ is the projection operator which imposes the no-double-occupancy constraint such that the electron occupation number $n_i \leq 1$. The conventional way to handle the constraint is to introduce the so-called slave-particle representation of electron operator: $c_{j,\sigma} \to f_j^+ b_{j,\sigma}$ such that the constraint $n_i \leq 1$ is replaced by an equality condition: $f_i^+ f_i + \sum_{\sigma} b_{i\sigma}^+ b_{i\sigma} = 1$.

For our purpose, in the following we will use the slave-fermion representation in which $f_j$ and $b_{j,\sigma}$ satisfy the canonical anti-commutation and commutation relations, respectively. Then the $t-J$ Hamiltonian $H_{t-J} = H_t + H_J$ can be rewritten as follows:

$$H_t = -t \sum_{\langle i,j \rangle} \{f_i^+ f_j (\sigma) b_{j\sigma}^+ b_{i\sigma} + H.c.\},$$

$$H_J = -\frac{J}{2} \sum_{\langle i,j \rangle} b_{i\sigma}^+ b_{j-\sigma}^+ b_{j-\sigma'} b_{i\sigma'}.$$  

(2)

In obtaining the above expressions, a replacement was made: $b_{j,\sigma} \to (-\sigma)^j b_{j,\sigma}$ where $\sigma = 1, -1$ for spin up and down, respectively. In this representation, the matrix element of $H_J$ always remains negative-definite which is equivalent to say that the Marshall sign [19] has been built into the basis [20]. But then the sign $\sigma$ appearing in $H_t$ indicates that holes dislike the Marshall sign hidden in the spin background, and their motion generally creates Marshall-sign mismatches on their paths known as phase strings. Since $H_t$ respects the Marshall sign rule at low energy, the phase strings cannot be repaired through the spin.
flip processes. Such a phase-string-type doping effect has been argued to be the key to understanding the evolution of the ground state at finite doping. In the following, we first give a brief review of the phase string formulation developed in Ref. [20] to deal with this nonlocal singular phase effect.

A. Phase string formulation

The basic idea underlying the phase string formulation is to “gauge away” the original singular source of the phase string effect shown in $H_t$ [Eq. (2)] such that the resulting form of the Hamiltonian becomes treatable in a perturbative scheme. According to Ref. [20], this procedure can be realized through a unitary transformation:

$$U \equiv \exp\left\{-\frac{i}{2} \sum_{j \neq l} n^h_j \theta_j(l) (1 - n^h_l - \sum_{\sigma} \sigma n^b_{l,\sigma})\right\}$$

(3)

where $n^h_j$ and $n^b_{j,\sigma}$ are the number densities of holons and spinons with spin $\sigma$ at site $j$. Under the unitary transformation (3), the electron operators become

$$c_{j,\sigma} \rightarrow \bar{h}_j^+ \bar{b}_{j,\sigma} (-\sigma)^j \sigma^{N_h}$$

(4)

where

$$\bar{h}_j^+ \equiv h_j^+ \exp\left\{\frac{i}{2} \sum_{l \neq j} \theta_j(l) (\sum_{\sigma} \sigma n^b_{l,\sigma} - 1)\right\},$$

$$\bar{b}_{j,\sigma} \equiv b_{j,\sigma} \exp\left\{-\frac{i}{2} \sigma \sum_{l \neq j} \theta_j(l) n^h_l\right\},$$

$$h_j \equiv f_j \exp\left\{-i \sum_{l \neq j} \theta_j(l) n^h_l\right\}.$$  

(5)

$N_h$ is the total number of holes. It is easy to verify that $h_j$ is a hard-core boson, i.e. they satisfy the following commutation relations: $[h_i, h_j] = 0 = [h_i, h_j^+]$, $i \neq j$ and $\{h_i, h_i\} = 0$, $\{h_i, h_i^+\} = 1$. Even though in the original definition, $\theta_j(l) = \text{Im} \ln(z_j - z_l)$, the choice of $\theta_j(l)$ is equivalent to a kind of gauge fixing and we will explicitly write down our choice later.

With eqs. (3) and (5), the $t-J$ Hamiltonian becomes
\[ H_t = -t \sum_{\langle i,j \rangle} \{ (e^{iA^f_{ij}})h_i^\dagger h_j^\dagger (e^{i\sigma A^h_{ij}})b^\dagger_{i\sigma} b_j \}, \]
\[ H_J = -\frac{J}{2} \sum_{\langle i,j \rangle} (e^{i\sigma A^h_{ij}})b^\dagger_{i\sigma} b^\dagger_{j - \sigma} (e^{i\sigma' A^h_{ij}})b_{j - \sigma} b_{i\sigma'}, \]  

where the gauge phases \( A^f_{ij} \) and \( A^h_{ij} \) are defined by

\[ A^f_{ij} = \frac{1}{2} \sum_{l \neq i,j} [\theta_i(l) - \theta_j(l)] \left( \sum_\sigma \sigma n^h_{i\sigma} - 1 \right), \]
\[ A^h_{ij} = \frac{1}{2} \sum_{l \neq i,j} [\theta_i(l) - \theta_j(l)] n^h_{l}. \]  

In the case of one chain, we can choose \( \theta_i(l) \) such that all these gauge phases vanish \[20\]. Thus, all important phases are absorbed into eq. (4) in a form of phase shifts \[20\]. This is the prefect case that the phase string effect is completely “gauged away” from the Hamiltonian. We shall see later that the situation slightly changes in the ladder case.

We now focus ourselves on the two-leg ladder and define \( \theta_{j,m}(l, n) \) as follows:

\[ \theta_{j,m}(l, n) = 0, \; j > l, \]
\[ \pi, \; j < l, \]
\[ \theta_{j,m}(j, n) = \frac{\pi}{2}, \; m = 1, n = 2, \]
\[ -\frac{\pi}{2}, \; m = 2, n = 1 \]  

where \( m, n \in \{1, 2\} \) are labels of legs. 1 and 2 indicate the upper and lower chains, respectively. This convention fixes the gauge phases as

\[ A^{1h}_{j,j+1} = \frac{\pi}{4}(n^h_{j,2} + n^h_{j+1,2}), \]
\[ A^{2h}_{j,j+1} = -\frac{\pi}{4}(n^h_{j,1} + n^h_{j+1,1}), \]
\[ A^{1f}_{j,j+1} = -\frac{\pi}{2}(1 - S^z_{j,2} - S^z_{j+1,2}), \]
\[ A^{2f}_{j,j+1} = \frac{\pi}{2}(1 - S^z_{j,1} - S^z_{j+1,1}) \]  

where \( n^h_{j,m} \) and \( S^z_{j,m} \) are the number density of holes and the \( z \)-component spin operator at site \( j \) and chain \( m \), respectively. \( A^{m(f)}_{j,j+1} \) represents the gauge phase on chain \( m \). In the derivation of eq. (9), we have used two identities: \( \exp i\pi(1 - 2S^z_{j,m} + \sigma n^h_{j,m}) = 1 \) and
\[ \exp(i\pi(\sigma - \sigma')n_{j,m}^h) = 1, \] which are valid in the physical Hilbert space. With the help of eq. (9), we can rewrite the \( t - J \) Hamiltonian on two-leg ladders as follows:

\[ H_t = -t \sum_{j,\sigma} (h_{j,2}^+ b_{j,2,\sigma} h_{j,1} b_{j,1,\sigma} + H.c.) \]

\[ -t \sum_{j,\sigma} \{ i e^{-i\pi/4} (S_{j,2}^z + S_{j+1,2}^z) e^{i\pi/4} \sigma (n_{j,2}^h + n_{j+1,2}^h) h_{j+1,1,\sigma} b_{j+1,1,\sigma} + H.c. \}, \]

\[ H_J = -\frac{J}{2} \sum_j (\sum_{\sigma} b_{j,1,\sigma} b_{j,2,-\sigma})(\sum_{\sigma'} b_{j,2,-\sigma'} b_{j,1,\sigma'}) \]

\[ -\frac{J}{2} \sum_{j,\sigma,m} (b_{j,m,\sigma} b_{j+1,m,-\sigma} + b_{j+1,m,\sigma} b_{j,m,-\sigma}) \]

\[ -\frac{J}{2} \sum_{j,\sigma} \{ e^{i\pi/4} \sigma (n_{j,2}^h + n_{j+1,2}^h) b_{j,1,\sigma} b_{j,1,-\sigma} b_{j+1,1,-\sigma} b_{j+1,1,\sigma} + e^{-i\pi/4} \sigma (n_{j,1}^h + n_{j+1,1}^h) b_{j,2,\sigma} b_{j,2,-\sigma} b_{j+1,2,-\sigma} b_{j+1,2,\sigma} \}. \]

Note that in eq. (10), the coupling between different legs is explicitly distinguished from the one on the same leg. In the following, we will reformulate it in the bond-operator representation under the implicit assumption that \( J_\perp \gg t, J \).

In contrast to the results of the single chain (see Ref. [20]), there are some phase factors left in the Hamiltonian. If we increase the number of chains to infinity, they will turn out to become topological gauge fields. It is these gauge fields that strongly affect the dynamics of holons and spinons in two dimensions [21]. On ladders, there are more than one path to connect two points while there is only one way to do it on the single chain. Consequently, in general, there is no closed path in the latter case [24]. That is why we are unable to see those "gauge interactions" arising from the phase string explicitly present in the 1D \( t - J \) model. As for the ladder case, there is no way to completely gauge away these phase factors no matter how we choose the \( \theta_{j,m}(l,n) \). Nevertheless, those phase factors in eq. (10) will become trivial in the bond-operator representation introduced below.
B. Bond-operator description

In half-filled case, the two-leg ladder $t - J$ model at $J_\perp \gg J$ can be well described at the mean-field level based on the bond-operator representation originally introduced by Sachdev and Bhatt. Such a description can be easily generalized to the doped case as emphasized in the Introduction. At each rung, the physical Hilbert space is spanned by nine states which can be generated by applying the bond operators to the vacuum state $|\phi_0\rangle$ as follows:

$$d_j^+ |\phi_0\rangle = h_{j,1}^+ h_{j,2}^+ |0\rangle,$$

$$a_{j,\sigma}^+ |\phi_0\rangle = (-\sigma)^{j+1} h_{j,1}^+ b_{j,2,\sigma}^+ |0\rangle, \quad \bar{a}_{j,\sigma}^+ |\phi_0\rangle = (-\sigma)^j b_{j,1,\sigma}^+ h_{j,2}^+ |0\rangle,$$

$$s_j^+ |\phi_0\rangle = \frac{(-1)^j}{\sqrt{2}} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle),$$

$$t_{j,0}^+ |\phi_0\rangle = \frac{(-1)^j}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle), \quad t_{j,\sigma}^+ |\phi_0\rangle = |\sigma, \sigma\rangle.$$  \hspace{0.5cm} (11)

Here $|\sigma, \sigma\rangle \equiv b_{j,1,\sigma}^+ b_{j,2,\sigma}^+ |0\rangle$ and $|\phi_0\rangle$ is annihilated by these bond operators. $s$ represents the spin singlet and $t_\alpha$ with $\alpha = \pm 1, 0$ represent spin triplet excitations. $a_\sigma$ and $\bar{a}_\sigma$ particles carry the same quantum numbers as electrons. $d$ particles are spinless and charge two. We choose $s$ and $t$ operators to satisfy canonical commutation relations as in Ref. while $d$, $a_\sigma$, and $\bar{a}_\sigma$ operators are hard-core bosons because they contain $h$ operators. The prefactor, $(-1)^j$, in eq. (11) arises from $(-\sigma)^j$ in eq. (4). The no-double-occupancy condition is replaced by the following one:

$$s_j^+ s_j + \sum_{\alpha=\pm 1, 0} t_{j,\alpha}^+ t_{j,\alpha} + \sum_\sigma (a_{j,\sigma}^+ a_{j,\sigma} + \bar{a}_{j,\sigma}^+ \bar{a}_{j,\sigma}) + d_j^+ d_j = 1.$$  \hspace{0.5cm} (12)

By using eq. (11), one can express the bilinear operators composed of $h$ and $b_\sigma$ by these bond operators. Their detailed forms are left in the Appendix A. We now obtain a Hamiltonian in which those gauge phases can be evaluated explicitly by substituting eq. (A1) into eq. (10). The resulting Hamiltonian can be divided into three parts:

$$H_{t-J} = H_0 + H_1 + H_2$$
where

\[ H_0 = -N(J_\perp + J_2) + \frac{t}{2} \sum_j \sigma(a_{j+1,\sigma}^+ \bar{a}_{j,\sigma} + H.c.) \]

\[ -\frac{t}{2} \sum_j \{ [s_{j+1}s_j^+ + 2d_{j+1}d_j^+ + t_{j+1,0}t_{j,j}^+ - 2t_{j+1,\sigma}t_{j,j}^+] \}
\]

\[ \cdot \{ a_{j+1,\sigma}^+ a_j,\sigma + \bar{a}_{j+1,\sigma}^+ \bar{a}_j,\sigma + H.c. \} \]

\[ + \left( \frac{J_\perp}{4} + \frac{J_2}{2} \right) \sum_j (a_{j,\sigma}^+ a_j,\sigma + \bar{a}_{j,\sigma}^+ \bar{a}_j,\sigma) + \left( \frac{J_\perp}{4} + J \right) \sum_j d_j^+ d_j \]

\[ - \frac{J_2}{4} \sum_j \{ (a_{j,\sigma}^+ a_j,\sigma + \bar{a}_{j,\sigma}^+ \bar{a}_j,\sigma) d_{j+1}^+ d_{j} + (j \leftrightarrow j + 1) \} \]

\[ - \mu \sum_j (2d_j^+ d_j + a_{j,\sigma}^+ a_j,\sigma + \bar{a}_{j,\sigma}^+ \bar{a}_j,\sigma) \]

\[ + \frac{J_\perp}{4} \sum_j (s_j s_j^+ + t_{j,\alpha}^+ t_{j,\alpha}^+ + s_j s_j^+ t_{j,\alpha}^+ t_{j,\alpha}^+ + H.c.) \]

\[ + \frac{J_2}{4} \sum_j (s_j s_j^+ + t_{j,\alpha}^+ t_{j,\alpha}^+ - \frac{1}{4}n_i n_j + d_j^+ d_j - 1) \] (13)

and

\[ H_1 = \frac{t}{\sqrt{2}} \sum_j \sigma \{ (s_{j+1}d_j + d_{j+1}s_j)(a_{j+1,\sigma}^+ \bar{a}_{j,\sigma}^+ - \bar{a}_{j+1,\sigma}^+ a_{j,\sigma}^+) + H.c. \} \]

\[ - \frac{J}{2} \sum_j \{ (\sum_{\sigma} a_{j+1,\sigma}^+ a_{j-\sigma}^+) (\sum_{\sigma'} a_{j,\sigma'} a_{j+1,\sigma'}) + (a \rightarrow \bar{a}) \} \]

\[ - \frac{J_2}{2} \sum_j d_j^+ d_j d_{j+1}^+ d_{j+1}. \] (14)

Here \( \mu \) is the chemical potential of holes. \( \lambda_j \) is the Lagrangian multiplier to impose the constraint (12). \( N \) is the number of sites for the single chain. The attraction between \( d \) particles comes from the \(-1/4n_i n_j\) term while those among quasiparticles arise from the exchange term. All the remaining terms are collected in \( H_2 \), which consists of terms that involve either spin-flip processes or the creation or annihilation of triplet excitations. We shall see later that all spin excitations are gapped. Thus, we expect that the inclusion of \( H_2 \) is supposed not to change the main features of our results and will not consider it in the following calculations. We leave the detailed form of \( H_2 \) in the Appendix B and define our working Hamiltonian as \( H \equiv H_0 + H_1 \) in the following sections.
III. MEAN-FIELD THEORY

A. The low energy effective Hamiltonian

Before taking the mean-field approximation, we would like to re-organize the general form of $H_{t-j}$ in the phase-string and bond-operator representation [Eqs. (13) and (14)]. Then some features of it can be more easily revealed.

We first express the $a$-operators in terms of the bonding and anti-bonding operators as follows:

$$a_{j,\sigma} = \frac{1}{\sqrt{2}}(a_{-j,\sigma} + i\sigma a_{+j,\sigma}), \quad \bar{a}_{j,\sigma} = \frac{1}{\sqrt{2}}(\sigma a_{-j,\sigma} - ia_{+j,\sigma})$$  \hspace{1cm} (15)

where $a_{\pm,\sigma}$ denote the bonding and anti-bonding operators, respectively. Since $a_{\pm,\sigma}$ operators are still hard-core bosons, one may introduce the following Jordan-Wigner transformation to transform them into fermions without changing the Hamiltonian:

$$a_{\pm,j,\sigma}^+ = e_{j,\sigma}^+ U_j, \quad a_{\pm,j,\sigma}^- = \bar{e}_{j,\sigma}^+ U_j.$$  \hspace{1cm} (16)

Here $U_j = \exp\left\{i\pi \sum_{l<j,\sigma} (a_{+l,\sigma}^+ a_{-l,\sigma} + a_{-l,\sigma}^+ a_{+l,\sigma})\right\}$. $e_{j,\sigma}$ and $\bar{e}_{j,\sigma}$ become fermions and satisfy the canonical anti-commutation relations.

Secondly, because under the unitary transformation (3), $S_{j,1}^+$ becomes

$$S_{j,1}^+ = (-1)^j b_{j,1,\sigma}^+ b_{j,1,\bar{\sigma}} \exp\left\{i\pi \sum_{l<j} (n_{l,1}^h + n_{l,2}^h) - i\frac{\pi}{2} n_{j,2}^h\right\};$$
$$S_{j,2}^+ = (-1)^{j+1} b_{j,2,\sigma}^+ b_{j,2,\bar{\sigma}} \exp\left\{i\pi \sum_{l<j} (n_{l,1}^h + n_{l,2}^h) + i\frac{\pi}{2} n_{j,1}^h\right\}$$  \hspace{1cm} (17)

instead of $(-1)^j b_{j,1,\sigma}^+ b_{j,1,\bar{\sigma}}$, the $t$ operators defined based on $b_{j,\sigma}^+$ in (11) do not form a vector. Therefore, the coefficients before the terms $t_{j+1,0} t_{j,0}^+$ and $t_{j+1,\sigma} t_{j,\sigma}^+$ in eq. (13) have different signs. The manifest rotational symmetry can be easily recovered by introducing the following unitary transformation:

$$t_{j,\sigma}^+ \rightarrow t_{j,\sigma}^+ \exp\left\{-i\pi\sigma \sum_{l<j} (n_{l,1}^h + n_{l,2}^h)\right\}.$$  \hspace{1cm} (18)

for $t_{j,\sigma}$ with $\sigma = \pm 1$. 

11
Then $H_{t-1}$ is simplified after substituting eqs. (13) and (16) into eqs. (13) and (14) and performing the transformation (18): 

$$
H_0 = -N\left(\frac{J_1}{4} + \frac{J}{2}\right) - t \sum_j (e_{j,\sigma}^+ e_{j,\sigma} - \bar{e}_{j,\sigma}^+ \bar{e}_{j,\sigma}) \\
- \frac{t}{2} \sum_j \left\{ [s_{j+1}s_j^+ + 2d_{j+1}d_j^+ + t_{j+1,\sigma}t_{j,\sigma}^+ + 2t_{j+1,\sigma}t_{j,\sigma}^+] \right\} \\
\cdot (e_{j+1,\sigma}^+ e_{j,\sigma} + \bar{e}_{j+1,\sigma}^+ \bar{e}_{j,\sigma}) \right\} + H.c. \right\} \\
+ \left(\frac{J_1}{4} + \frac{J}{2}\right) \sum_j (e_{j,\sigma}^+ e_{j,\sigma} + \bar{e}_{j,\sigma}^+ \bar{e}_{j,\sigma}) \right\} + \left(\frac{J_1}{4} + J\right) \sum_j d_j^+ d_j \\
- \frac{J}{4} \sum_j \left\{ (e_{j,\sigma}^+ e_{j,\sigma} + \bar{e}_{j,\sigma}^+ \bar{e}_{j,\sigma})d_{j+1}^+ d_{j+1} + (j \leftrightarrow j + 1) \right\} \\
- \mu \sum_j (2d_j^+ d_j + e_{j,\sigma}^+ e_{j,\sigma} + \bar{e}_{j,\sigma}^+ \bar{e}_{j,\sigma}) \\
+ \frac{J}{2} \sum_j (s_j s_{j+1}^+ t_{j,\sigma}^+ t_{j+1,\sigma} + s_j s_{j+1}^+ t_{j,\sigma}^+ t_{j+1,\sigma}^+ + H.c.) \\
+ \frac{J_1}{4} \sum_j (-3s_j^+ s_j + t_{j,\sigma}^+ t_{j,\sigma}) \\
- \sum_j \lambda_j (s_j^+ s_j + t_{j,\sigma}^+ t_{j,\sigma} + e_{j,\sigma}^+ e_{j,\sigma} + \bar{e}_{j,\sigma}^+ \bar{e}_{j,\sigma} + d_j^+ d_j - 1) \tag{19}
$$

and

$$
H_1 = -\frac{t}{\sqrt{2}} \sum_j \left\{ (s_{j+1}d_j + d_{j+1}^+ s_j) (e_{j+1,\sigma}^+ e_{j,\sigma} - \bar{e}_{j+1,\sigma}^+ \bar{e}_{j,\sigma}^+) + H.c. \right\} \\
- \frac{J}{4} \sum_{j,\sigma,\sigma'} \left\{ (e_{j+1,\sigma}^+ e_{j+1,\sigma'}^+ + \bar{e}_{j+1,\sigma}^+ \bar{e}_{j+1,\sigma'}^+) (e_{j,\sigma'} e_{j+1,\sigma'} - \bar{e}_{j,\sigma'} \bar{e}_{j+1,\sigma'}) \right\} \\
+ (e_{j+1,\sigma}^+ \bar{e}_{j,\sigma}^+ - \bar{e}_{j+1,\sigma}^+ e_{j,\sigma}^+) (e_{j,\sigma'} e_{j+1,\sigma'} - \bar{e}_{j,\sigma'} \bar{e}_{j+1,\sigma'}) \right\} \\
- \frac{J}{2} \sum_j d_j^+ d_j t_{j+1}^+ d_{j+1}. \tag{20}
$$

In addition to the spin rotational symmetry, translation symmetry, and the electromagnetic U(1) symmetry, we note that the Hamiltonian of the two-leg ladder is also invariant under the exchange of chain indices to which we can assign a parity operator. Under the exchange of chain indices, the bond operators transform as follows: $s \to s, \ t_\alpha \to -t_\alpha, \ d \to d$. For quasiparticles, it is $a_\sigma \leftrightarrow -\sigma \bar{a}_\sigma$, or $e_\sigma \to e_\sigma, \ \bar{e}_\sigma \to -\bar{e}_\sigma$. We see that magnons and quasiparticles in the anti-bonding band are parity odd while $d$ bosons and quasiparticles in the bonding band are parity even [24].
We shall see later that both quasiparticles and magnons have gaps at low doping concentration. Consequently, we can integrate them out (treating $s$ as a $c$-number.) and obtain the low energy theory described by the following hard-core boson (HCB) model:

$$H = -t^* \sum_j (d^+_{j+1}d_j + H.c.) + V \sum_j d^+_j d^+_j d^+_j d^+_{j+1}$$  \hspace{1cm} (21)

where $t^*$ is the effective hopping amplitude and the dominant contribution to $V$ comes from the attraction between $d$ particles in eq.(20). This model can be solved by a bosonization approach [27]. For $V < -2 | t^* |$ the system is phase separated and this occurs only at very large values of $J_\perp$ for physically reasonable values of $J/t$. For example, $J_\perp > 31.8t$ for $J/t = 0.5$ (see Troyer et al. [9]). In the region where the system is stable against the phase separation, the low-lying excitation will be the phase fluctuations of $d$-particles, which corresponds to the collective charge mode.

Although the HCB model, eq. (21), is appropriate to describe the low energy properties of two-leg ladders in the lightly doped region, it can not address questions such as the internal structure of hole pairs, which is related to the nature of the superconducting order parameter, and how the spectra of those gapped modes vary with the hole concentration. Later we will answer these questions by a mean-field treatment of eqs. (19) and (20).

### B. Mean-field equations

To proceed with the mean-field approximation, we first note that the undoped two-leg ladder is characterized as a spin liquid with non-vanishing RVB order parameter $\langle s_j \rangle$. Following Gopalan et al. [5], we take the ansatz for the spin part as: $\langle s_j \rangle = \bar{s}$, $\langle t_{j,\alpha} \rangle = 0$, $Q_\alpha = \langle t_{j+1,\alpha} t^+_{j,\alpha} \rangle$, and $P_\alpha = \langle t^+_j t_{j,\alpha} \rangle$ where $\alpha = \pm 1, 0$. Because of the rotational symmetry, $Q_+ = Q_- = Q_0$ and $P_+ = P_- = P_0$. We set $P = P_+ + P_- + P_0$ and $Q = Q_+ + Q_- + Q_0$.

For the charge part at finite doping, we define the following mean-field parameters:

$\chi_\sigma = \langle e^+_{j+1,\sigma} e_{j,\sigma} \rangle$ and $\bar{\chi}_\sigma = \langle \bar{e}^+_{j+1,\sigma} \bar{e}_{j,\sigma} \rangle$. Again due to the rotational symmetry, $\chi_+ = \chi_-$.
and $\chi_+ = \chi_-$. We define $\chi = \chi_+ + \chi_- \text{ and } \bar{\chi} = \chi_+ + \bar{\chi}_-$. Moreover, we take $\lambda_j = \lambda$ in accordance with the translational invariance along the chain direction.

Note that there is a linear $d$-operator appearing in the first term of $H_1$ [Eq. (20)], describing the process that a rung hole pair dissolves into two quasiparticles or a pair of quasiparticles is recombined into a rung hole pair. Such a term (with $s_j \rightarrow \bar{s}$) looks similar to a Hubbard-Stratonovich transformed four-fermion interaction in the BCS theory with $d$ playing the role of the order parameter. Thus, we will treat $d_j$ as a c-number by assuming $d_j = \bar{d}$ and neglect its phase fluctuations. A solution with non-vanishing $\bar{d}$ will then immediately lead to forming Cooper pairs for $e_\uparrow$ and $e_\downarrow$ according to the first term in $H_1$. To make things more transparent, let us define the following pairing field for quasiparticles:

$$\Delta \equiv \sum_\sigma \langle e_{j+1, \sigma}^+ e_{j, -\sigma}^+ \rangle, \quad \bar{\Delta} \equiv \sum_\sigma \langle \bar{e}_{j+1, \sigma}^+ \bar{e}_{j, -\sigma}^+ \rangle.$$  

Then

$$\Delta + \bar{\Delta} = -\sum_\sigma \langle a_{j+1, \sigma}^+ a_{j, -\sigma}^+ - \bar{a}_{j+1, \sigma}^+ \bar{a}_{j, -\sigma}^+ \rangle;$$

$$\Delta - \bar{\Delta} = -\sum_\sigma \langle a_{j+1, \sigma}^+ \bar{a}_{j, -\sigma}^+ - \bar{a}_{j+1, \sigma}^+ a_{j, -\sigma}^+ \rangle.$$  

Here $\Delta + \bar{\Delta}$ and $\Delta - \bar{\Delta}$ represent the hole pairing along the chain direction and diagonal sites, respectively and are shown schematically in Fig. 1. Now it is easy to see that the pairing on the rung and diagonal sites must occur simultaneously. This is what the first term of $H_1$ tells us and it is consistent with the picture emerged from numerical studies [10,11].

Naively, it seems that $\Delta$ and $\bar{\Delta}$ are not tensors under the spin rotation and thus our mean-field ansatz may break the rotational symmetry. This is, in fact, disguised by the phase string effect as we have discussed before. Our mean-field ansatz indeed respects rotational symmetry which is left to be discussed in the Appendix C.

Based on the above mean-field ansatz, we finally obtain the following mean-field Hamiltonian:

$$H_{MF} = N\{\lambda - \frac{J_\perp}{4} - \frac{J}{2} - (\frac{3}{4}J_\perp + \lambda)\bar{s}^2 + (\chi + \bar{\chi})Qt + \frac{J}{8}(\Delta + \bar{\Delta})^2\}.$$
\[-\frac{J}{16}(\chi - \bar{\chi})^2 - \left(\frac{J}{2}d^2 + 2\mu + \lambda - \frac{J_\perp}{4} - Jd^2\right)\]
\[-(t(d^2 + \bar{s}^2 + \bar{Q}) - \frac{J}{16}(\chi - \bar{\chi})) \sum_j (e_{j+1,\sigma}^+ e_{j,\sigma} + H.c.)\]
\[-(t(d^2 + \bar{s}^2 + \bar{Q}) + \frac{J}{16}(\chi - \bar{\chi})) \sum_j (\bar{e}_{j+1,\sigma}^+ \bar{e}_{j,\sigma} + H.c.)\]
\[+(-t - \mu - \lambda + \frac{J_\perp}{4} + \frac{J}{2} - \frac{J}{2}d^2) \sum_j e_{j,\sigma}^+ e_{j,\sigma}\]
\[+(t - \mu - \lambda + \frac{J_\perp}{4} + \frac{J}{2} - \frac{J}{2}d^2) \sum_j \bar{e}_{j,\sigma}^+ \bar{e}_{j,\sigma}\]
\[-\left(\frac{J}{8}(\Delta + \bar{\Delta}) + \sqrt{2}t\bar{s}\bar{d}\right) \sum_j (e_{j+1,\sigma}^+ e_{j,-\sigma} + H.c.)\]
\[-\left(\frac{J}{8}(\Delta + \bar{\Delta}) - \sqrt{2}t\bar{s}\bar{d}\right) \sum_j (\bar{e}_{j+1,\sigma}^+ \bar{e}_{j,-\sigma} + H.c.)\]
\[+\frac{1}{2} \sum_j \{(Js^2 - t(\chi + \bar{\chi}))t_{j,\alpha}^+ t_{j+1,\alpha} + Js^2 t_{j,\alpha}^+ t_{j+1,-\alpha} + H.c.\} \]
\[+(\frac{J}{4} - \lambda) \sum_j t_{j,\alpha}^+ t_{j,\alpha}.\]

Eq. (23) can be diagonalized by Bogolioubov transformations. We leave the procedure in
the Appendix D and write down the diagonalized Hamiltonian in the following:

\[H_{MF} = \Omega_0 + \sum_k \{E_k(\alpha_k^+ \alpha_k + \beta_k^+ \beta_k) + \bar{E}_k(\bar{\alpha}_k^+ \bar{\alpha}_k + \bar{\beta}_k^+ \bar{\beta}_k)\} + \sum_k \omega_k \gamma_k^+ \gamma_k\]

where

\[\Omega_0 = \sum_k \left(\frac{3}{2}\omega_k - E_k - \bar{E}_k\right) + N \left\{\frac{\lambda}{2} - 2\mu - \frac{J_\perp}{8} + \frac{J}{2} + (\chi + \bar{\chi})Qt - \left(\frac{3}{4}J_\perp + \lambda\right)s^2\right\} \]
\[+\frac{J}{8}(\Delta + \bar{\Delta})^2 - \frac{J}{16}(\chi - \bar{\chi})^2 - \left(\frac{J}{2}d^2 + 2\mu + \lambda - \frac{J_\perp}{4}\right)d^2\]

is the mean-field ground state energy. (In fact, \(\Omega_0\) is the zero temperature grand potential.)

The parameters \(\bar{d}, \bar{s}, \mu, \lambda, \chi, \bar{\chi}, \Delta, \bar{\Delta}, P,\) and \(Q\) are determined by solving eq. (22), the
following self-consistent equations:

\[Q = \sum_\alpha \langle t_{j+1,\alpha}^+ t_{j,\alpha}\rangle, \quad P = \sum_\alpha \langle t_{j,\alpha}^+ t_{j,\alpha}\rangle,\]
\[\chi = \sum_\sigma \langle e_{j+1,\sigma} e_{j,\sigma}\rangle, \quad \bar{\chi} = \sum_\sigma \langle \bar{e}_{j+1,\sigma} \bar{e}_{j,\sigma}\rangle,\]

(26)
and the saddle-point equations:

\[
\frac{\partial \Omega_0}{\partial \lambda} = 0, \quad \frac{\partial \Omega_0}{\partial \bar{s}} = 0, \\
\frac{\partial \Omega_0}{\partial \bar{d}} = 0, \quad \frac{\partial \Omega_0}{\partial \mu} = -2N\delta,
\]

where \(\delta\) is the hole concentration. We obtain the following mean-field equations from eqs. (22), (26), and (27):

\[
1 - \bar{s}^2 + \bar{d}^2 = P + 2\delta, \quad (28)
\]

\[
1 + \bar{d}^2 - \delta = \frac{1}{2N} \sum_k \left( \frac{\epsilon_k}{E_k} + \bar{\epsilon}_k / \bar{E}_k \right), \quad (29)
\]

\[
(2t/J)(\chi + \bar{\chi}) + 2\mu/J + \frac{\lambda}{J} - 1 + \delta)\bar{d} = -\sqrt{2t/J}\bar{s}(\Delta - \bar{\Delta}), \quad (30)
\]

\[
\frac{3J_1}{4J} - Q + \frac{t}{J}(\chi + \bar{\chi}) + \sqrt{2t/J}\bar{d}(\Delta - \bar{\Delta}) = -3\frac{1}{N} \sum_k \cos k \frac{\Pi_k}{\omega_k}, \quad (31)
\]

\[
\Delta = \frac{1}{N} \sum_k \sin k \frac{\Gamma_k}{E_k}, \quad \bar{\Delta} = \frac{1}{N} \sum_k \sin k \frac{\bar{\Gamma}_k}{\bar{E}_k}, \quad (32)
\]

\[
\chi = -\frac{1}{N} \sum_k \cos k \frac{\epsilon_k}{E_k}, \quad \bar{\chi} = -\frac{1}{N} \sum_k \cos k \frac{\bar{\epsilon}_k}{\bar{E}_k}, \quad (33)
\]

\[
Q = \frac{3}{2N} \sum_k \cos k \frac{\Lambda_k}{\omega_k}, \quad P = \frac{3}{2N} \sum_k \frac{\Lambda_k}{\omega_k} - \frac{3}{2}. \quad (34)
\]

Notice that eq. (30) says that \(\Delta - \bar{\Delta} = 0\) as long as \(\bar{d} = 0\).

We close this section by a remark. When \(\delta = 0\), i.e., the undoped case, our mean-field equations are not exactly reduced to those in Ref. [5]. The difference arises from the fact that there are three components for \(t\) operators and each contributes \(\sum_k \frac{1}{2} \omega_k\) to \(\Omega_0\). Thus, the zero-point energy is \(\sum_k \frac{3}{2} \omega_k\) instead of \(\sum_k \frac{1}{2} \omega_k\). A similar effect also appears in eq. (25) in which the coefficient of \(N\lambda\) changes from 3/2 to 1/2. This point was neglected in Ref. [5]. We will see later that this difference increases the spin gap in comparison with that obtained in Ref. [5].

IV. RESULTS
A. Undoped case

As a reference point, we first examine the results of our mean-field equations for the undoped case. By defining $\tilde{\Lambda}_k = \Lambda_k |_{\chi=0}$, the relevant integrals in the mean-field equations can be expressed as elliptic integrals [23]:

\[
\frac{1}{N} \sum_k \tilde{\Lambda}_k \omega_k = \frac{1}{\pi} \left\{ \frac{1}{\sqrt{1 + \nu}} K(\sqrt{\frac{2\nu}{1 + \nu}}) + \sqrt{1 + \nu} E(\sqrt{\frac{2\nu}{1 + \nu}}) \right\},
\]

\[
\frac{1}{N} \sum_k \cos k \frac{\tilde{\Lambda}_k - 2\Pi_k}{\omega_k} = -\frac{2}{\pi \nu} \left\{ \frac{1}{\sqrt{1 + \nu}} K(\sqrt{\frac{2\nu}{1 + \nu}}) \right. \\
\left. - \sqrt{1 + \nu} E(\sqrt{\frac{2\nu}{1 + \nu}}) \right\}
\]

where $K(\xi)$ and $E(\xi)$ are respectively the complete elliptic integrals of the first and second kind with modulus $\xi$. The dimensionless parameter $\nu$ is defined in the Appendix D.

Then, in the undoped limit, eqs. (28), (29), (30), (31), (32), (33), and (34) are reduced to the following forms:

\[
\frac{5}{2} - \bar{s}^2 = \frac{3}{2\pi} \left\{ \frac{1}{\sqrt{1 + \nu}} K(\sqrt{\frac{2\nu}{1 + \nu}}) \right. \\
\left. + \sqrt{1 + \nu} E(\sqrt{\frac{2\nu}{1 + \nu}}) \right\},
\]

\[
\frac{3}{4} + \frac{\lambda}{J_\perp} = -\frac{3\eta}{\pi \nu} \left\{ \frac{1}{\sqrt{1 + \nu}} K(\sqrt{\frac{2\nu}{1 + \nu}}) \right. \\
\left. - \sqrt{1 + \nu} E(\sqrt{\frac{2\nu}{1 + \nu}}) \right\}
\]

where $\eta = J/J_\perp$. The spin-triplet excitation spectrum is given by

\[
\omega_k = J_\perp \left( \frac{1}{4} - \frac{\lambda}{J_\perp} \right) \sqrt{1 + \nu \cos k}.
\]

In eq. (38), we have to assume $0 \leq \nu \leq 1$; otherwise the mean-field equations would break down. This is verified in the following calculations. The band minimum is at $k = \pi$ and the spin gap is determined by

\[
\Delta_s = J_\perp \left( \frac{1}{4} - \frac{\lambda}{J_\perp} \right) \sqrt{1 - \nu}.
\]
We can analytically study the asymptotic behavior of the spin gap for small values of \( \eta \) (and hence \( \nu \)) to see the difference between our equations and those in Ref. [5]. For small values of \( \xi \) the elliptic integrals \( K(\xi) \) and \( E(\xi) \) can be expanded in a power series as

\[
K(\xi) = \frac{\pi}{2} \left( 1 + \frac{1}{4} \xi^2 + \frac{9}{64} \xi^4 + \cdots \right),
\]
\[
E(\xi) = \frac{\pi}{2} \left( 1 - \frac{1}{4} \xi^2 - \frac{3}{64} \xi^4 + \cdots \right),
\]

and we obtain

\[
\nu = 2\eta \left( 1 + \frac{23}{8} \eta^2 + O(\eta^4) \right),
\]
\[
\frac{1}{4} - \lambda J_\perp = 1 + \frac{3}{4} \eta^2 + O(\eta^4),
\]

and the spin gap is given by

\[
\Delta_t = J_\perp \left( 1 - \eta + \frac{1}{4} \eta^2 + O(\eta^3) \right).
\]  

(40)

We find that our result, eq. (40), is larger than that in Ref. [5], and is much closer to the strung rung interaction result, which is \( J_\perp \left( 1 - \eta + \frac{1}{2} \eta^2 + O(\eta^3) \right) \) [4]. At the end of Sec. III we have pointed that there is a numerical factor missing in the zero-point energy in Ref. [5] which is responsible for this discrepancy.

To obtain \( \Delta_t \) at any value of \( J/J_\perp \), we numerically solved eqs. (36) and (37). The results are shown in Fig. 2. The spin gap at the instropic point, \( \eta = 1 \), is about 0.501\( J \), which is very close to the numerical result — 0.504\( J \) [6]. Of course, the mean-field approximation is only justified in the strong rung interaction regime and we do not expect the theory to be extended into the region with \( \eta > 1 \) where the coupling between spins on the same leg becomes dominant over the rung coupling. In fact, our calculation shows that the spin gap continuously increases beyond \( \eta > 1 \), but according to Ref. [3], it should smoothly diminish to zero as \( \eta \) approaches 0.

### B. Phases in doped case
1. The C1S0 phase

The phase diagram obtained from numerical studies [8,9,12] shows that at most values of $J/t$ the two-leg ladders fall into the universality classes of Luther-Emery and Luttinger liquids for small and large doping concentration, respectively. The former and the latter are respectively denoted by C1S0 and C1S1 phases. ($CmSn$ means that there are $m$ gapless charge modes and $n$ gapless spin modes [23].)

The present theory using the bond-operator description presumably works at small doping for the Luther-Emery type phase. A mean-field solution with non-vanishing $\bar{d}$ is found at $\delta < 0.5$ which is stable against the phase separation when $J/t$ is not too large. We interpret this mean-field state as the C1S0 phase. This is because in this situation there are gaps in the spectra of both quasiparticles and magnons. This can be understood as the following:

To let quasiparticles be gapless, $\Gamma_k$ and $\bar{\Gamma}_k$ must be zero [29]. From eq. (32), it results in vanishing $\Delta$ and $\bar{\Delta}$. This implies that $\bar{d} = 0$ due to eq. (30). Therefore, a non-zero $\bar{d}$ will induce a gap for quasiparticles. The gap in the spectrum of magnons is a continuation of the one in the undoped case, which ensures the validity of the extension of the mean-field ansatz from the undoped case to the finite doping. The only gapless excitation in this region is the density fluctuations of hole pairs as discussed before.

**Ground state energy:** To examine the validity of our mean-field ansatz, we first compute the ground state energy and compare it with numerical results. The mean-field ground state energy per site, $E_0$, is given as the following:

$$E_0 = \frac{1}{2N}\{\Omega_0 + \mu \sum_j (2\bar{d}^2 + \langle \bar{a}_{j\sigma}^+ a_{j\sigma} + \bar{a}_{j\sigma}^+ \bar{a}_{j\sigma} \rangle)\}$$

$$= \frac{1}{2N}\Omega_0 + \mu \delta.$$

In the above derivation, we have used eq. (29). Also note that the number of sites is $2N$. We calculated $E_0$ with $J/t = 0.5$ and its doping dependence at various $J_J/J$’s is plotted in Fig. 3. By comparing with the numerical results in Ref. [11], we find that both the tendency of the energy versus the doping concentration and its absolute magnitude agree well with
the data obtained by the recurrent variational ansatz (RVA) as well as the density matrix
renormalization group (DMRG) method. For comparison, in Fig. 3 the DMRG results \[11\]
at $\delta = 1/8$ and $1/2$ are also shown. (Note that we set $J = 1$ in Fig. 3 while $J = 0.5$ in Ref.
\[11\] but $J/t = 0.5$ is the same.) We see that the agreement is especially good in the region
with large values of $J_\perp/J$ and small doping concentration as expected for the bond-operator
representation. (The comparisons with the RVA results \[11\] are even better over the whole
$\delta \leq 0.5$ region.) Such a good agreement over a wide range of parameters indicates that
our mean-field treatment based on the phase string and bond-operator formalism indeed
captures the basic physics of doped two-leg ladders in the strong rung interaction regime.
In the following, we focus on some detailed properties by solving the mean-field equations
at $J/t = 0.5$ and $J_\perp/J = 10$.

**Local structure of hole pairs:** Next, we would like to discuss the local structure of
hole pairs in the C1S0 phase. As has been discussed in Ref. \[11\], holes will form pairs along
the diagonal sites as well as along the rung and chain directions. This diagonal pairing is
energetically favored by the $t$ term and the most probable configuration of two dynamical
holes in a two-leg ladder \[10\]. In our formalism, the amplitudes for pairing along the rung,
diagonal, and chain directions can be respectively represented by $\bar{d}$, $\Delta - \bar{\Delta}$ and $\Delta + \bar{\Delta}$. We
calculated $(|\Delta - \bar{\Delta}|)/\bar{d}$ and $(|\Delta + \bar{\Delta}|)/\bar{d}$ and plot them in Fig. 4. We found that both
decrease with increasing $\delta$. In the low doping region, the amplitude of hole pairs on diagonal
sites is almost comparable to the one of rung hole pairs and always larger than that on the
chain direction. The amplitude of hole pairs along the leg being smaller than other hole
configurations reflects the fact that the rung bonds are stronger than the leg bonds in the
underlying two-leg spin ladder. The above results were also pointed out by Sierra et al. \[11\].
They proposed a dimer hard-core boson (DHCB) model to describe the low energy properties
of two-leg ladders in the strong rung interaction regime, which contains both the charge and
spin degrees of freedom in contrast to the HCB model. The bond-operator formulation also
retains these high energy modes. In particular, the coexistence of the diagonal pairing and
rung hole pairs is further manifested in our approach.
Pairing symmetry: We also calculated the expectation values of pairing fields along rung and chain directions. The corresponding operators are defined as follows:

\[
\Delta_x(j) = \frac{1}{\sqrt{2}} \sum_{\sigma} \sigma c_{j+1,\sigma} c_{j,\sigma},
\]

\[
\Delta_y(j) = \frac{1}{\sqrt{2}} \sum_{\sigma} \sigma c_{j,\sigma} c_{j+1,\sigma}.
\]

These pairing fields can be represented by bond operators and we list their explicit forms in Appendix E. In terms of the above mean-field parameters, their vacuum expectation values are given as follows:

\[
i\langle \Delta_x \rangle = -\frac{1}{4\sqrt{2}}(\Delta + \bar{\Delta})[s^2 - 2d^2 - \frac{3J_\perp}{4J} - \frac{\lambda}{J}]
+ Q - \frac{t}{J}(\chi + \bar{\chi}) - \sqrt{2}\frac{td}{Js}(\Delta - \bar{\Delta}),
\]

\[
ii\langle \Delta_y \rangle = \frac{1}{2}d. \] (43)

The results are shown in Fig. 5. It is clear that there is a critical hole concentration \( \delta_c \). (In our case, \( \delta_c = 0.37 \) for \( J/t = 0.5 \) and \( J_\perp/J = 10 \).) In the low doping regime \( \delta < \delta_c \), the pairing symmetry shows d-wave-like behavior while for \( \delta > \delta_c \), it becomes s-wave-like symmetry. The difference can be attributed to different internal structures of hole pairs. [11]

When \( \delta < \delta_c \), holes doped into a spin liquid state with RVB correlations form pairs with \( dx^2-\gamma^2 \)-like structure. However, in the overdoped region, one moves into the low density limit characterized by electrons doped into a background with an internal s-wave-like symmetry.

Spin excitations: There are two kinds of excitations which carry non-trivial spin quantum numbers. One is the magnon, which is represented by \( t \) operators in our formulation and is the spin triplet excitation around \( \mathbf{q} = (\pi, \pi) \). The other type of spin excitations are quasiparticles, which carry spin-\( 1/2 \). The band minima of quasiparticles in bonding and anti-bonding bands are at \( \mathbf{q} = (0, 0) \) and \( \mathbf{q} = (0, \pi) \), respectively. The behaviors of their gaps varying with hole concentration are shown in Fig. 6. We found that the gap of magnons increases while quasiparticle gaps decrease with increasing \( \delta \). In addition, it is easy to see that the low-lying spin modes with odd and even parity are magnons and quasiparticles in
the bonding band, respectively. Real spin-1 excitations in two-leg ladders are composed of magnons or pairs of quasiparticles. The gap of the latter is still smaller than that of the former. Thus, the spin gap in the C1S0 phase is determined by quasiparticles instead of magnons. This was also shown by numerical studies [9].

2. Phase separation

The above-discussed C1S0 superconducting phase may become unstable against phase separation when the value of $J/t$ becomes large in the $t-J$ ladders [9]. The reason is that in the large $J$ limit the gain in exchange energy by maximizing the number of AF bonds outweighs the cost in kinetic energy.

The stability of the C1S0 solution against the phase separation can be examined by studying the compressibility $\kappa$:

$$\kappa^{-1} = \delta^2 \frac{\partial \mu}{\partial \delta}.$$ 

At $J_{\perp}/J = 10$, we found that $\kappa$ diverges at $J/t = 1.2$ and becomes negative when $J/t > 1.2$. This implies that our mean-field solutions with uniform hole density become unstable against phase separation when $J/t \geq 1.2$ [see Fig. 7(a)]. For $J_{\perp}/J = 5$, the situation is similar as shown in Fig. 7(b). The only difference is that the boundary between the C1S0 phase and the phase separation region is moved to $J/t = 1.6$. At $J_{\perp}/J = 2$, the critical value of $J/t$ where the phase separation occurs not only increases but also becomes strongly doping-dependent as shown in Fig. 7(c). The latter trend is quite similar to that found in numerical studies [9][11] for the isotropic case, i.e. $J_{\perp}/J = 1$. We note, however, that further reducing $J_{\perp}/J$ towards the isotropic limit in our mean-field theory does not improve more of the comparison with the numerical results since the ground-state energy starts to visibly deviate from numerical data at $J_{\perp}/J < 2$ even for small $\delta$ as shown in Fig. 3.
V. DISCUSSIONS

In this paper, we proposed a mean-field description on doped two-leg ladders in the strong rung interaction limit based on the phase string formulation. The transformation to bond operators is a natural choice in this formalism. With the help of bond operators, we can easily separate the high energy and low energy processes in the $t-J$ Hamiltonian. Thus, a mean-field treatment becomes straightforward. Naively, there are two competing pairing channels in eq. (20) - the pairing between $e_\sigma$ and $\bar{e}_{-\sigma}$ and the one between $e_\sigma$ and $e_{-\sigma}$ (or $\bar{e}_\sigma$ and $\bar{e}_{-\sigma}$). However, the condensation of the hard-core boson $d$ and a non-vanishing RVB order parameter $\bar{s}$ demand that the latter dominates. Furthermore, this type of pairing implies the formation of spin singlet bonds not only along the chain direction but also on diagonal sites. The latter turns out to be an important low energy structure in various types of $t-J$ models [10]. This singlet bond becomes a strong nearest-neighbor one after one of the holes hops next to the other. As a consequence, the formation of this kind of singlets can maximize the hopping overlap with other hole configurations and lower its energy. This feature is a necessary result in our formulae as shown in eq. (30) or the linear $d$ term in $H_1$ [Eq. (20)]. We have to emphasize that this mechanism for pairing comes from the $t$ term and is quite different from the "broken-bond" effect though the latter does enhance hole pairing somewhat. Also, the pairing must cause a gap opened up in the quasiparticle spectrum. Thus, all spin excitations are gapped in this region.

Here we would like to make some comments on the effects of phase string in two-leg ladders. If we directly apply the bond-operator representation to the original $t-J$ Hamiltonian without explicitly taking into account the nonlocal effect of phase string, then we would obtain a Hamiltonian with a different form in the $t$-term. For example, the $t$-term with a linear $d$ in $H_1$ would involve quasiparticle pairing between different bands which is a high energy process now. Subsequently, the pairing between quasiparticles would only come from the four-fermion attraction in the $J$-term of the Hamiltonian. If we still use the similar mean-field ansatz to treat this Hamiltonian, the results would be incorrect because
the compressibility is always negative. This implies that the solution is thermodynamically unstable. Such an instability is actually similar to the spiral instability \[22\] in 2D case when one tries to generalize the Schwinger-boson mean-field theory to the doped case without considering the phase string effect \[21\]. Therefore, it is necessary to take into account the phase string effect in order to acquire a correct mean-field theory on the doped antiferromagnets regardless of dimensionality.

Müller and Rice have suggested a possible $C2S2$ phase existing between the Nogaoka phase with very small vaules of $J/t$ and the $C1S0$ phase with intermediate values of $J/t$ \[12\]. They provided some numerical evidence to support this conjecture. Physically, the appearance of this phase can be understood as the following: When the holons move fast, i.e. $t \gg J, J_\perp$, the gain in kinetic energy may outweights the cost by breaking the rung hole pairs and rung singlets. Thus, the phase coherence between bonding and anti-bonding bands is lost and the two-leg ladder is effectively decoupled into two chains at low energy. If this picture is correct, then the bond operators (especially the $d$-operator) are no longer a good description of the low energy degrees of freedom in this region. On the other hand, as we have seen in eq. \[21\], there are always attractions between quasiparticles on the nearest-neighbor sites arising from breaking the singlet bonds. If they are not completely compensated by some repulsive forces at least at the intermediate scale, the underlying magnetic structure may still be a gapped spin liquid with a small spin gap and the observed $C2S2$ phase perhaps is a finite size effect. But the present mean-field theory, which works in the limit $J_\perp \gg J, t$, cannot be directly applied to this regime to address those issues.

For large doping concentration, the anti-bonding band of electrons is empty and the system falls into the $C1S1$ phase. To describe this phase, the bond-operator representation is not convenient. We have to go back to eq. \[11\]. Nevertheless, the problem that there are non-trivial phase factors in the Hamiltonian rears its head again. These phase factors are the interactions arising from the phase string effect and entail careful treatment. Otherwise, important physics may be lost. The pursuit along this direction is beyond the scope of the present paper.
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APPENDIX A: THE BOND-OPERATOR REPRESENTATIONS OF BILINEAR OPERATORS IN THE HAMILTONIAN

In this Appendix, we list the bond-operator representations of those bilinear operators appearing in the Hamiltonian in the following:

\[
\begin{align*}
\sum_{\sigma} b_{j,1,\sigma} b_{j,2,-\sigma} &= (-1)^j \sqrt{2} s_j, \\
 b_{j,1,\sigma} b_{j,1,\sigma} &= \frac{1}{2} \left\{ s_j^+ s_j + \sum_{\alpha=\pm 1, 0} t_{j,\alpha}^+ t_{j,\alpha}^+ + \sigma \sum_{\sigma'=\pm1} \sigma' t_{j,\sigma'}^+ t_{j,\sigma'}^+ + \sigma (s_{j,0}^+ t_{j,0} + t_{j,0}^+ s_j) \right\} \\
&\quad + a_{j,\sigma}^+ a_{j,\sigma}, \\
b_{j,2,\sigma} b_{j,2,\sigma} &= \frac{1}{2} \left\{ s_j^+ s_j + \sum_{\alpha=\pm 1, 0} t_{j,\alpha}^+ t_{j,\alpha}^+ + \sigma \sum_{\sigma'=\pm1} \sigma' t_{j,\sigma'}^+ t_{j,\sigma'}^+ - \sigma (s_{j,0}^+ t_{j,0} + t_{j,0}^+ s_j) \right\} \\
&\quad + a_{j,\sigma}^+ a_{j,\sigma}, \\
b_{j,1,\sigma} b_{j,1,-\sigma} &= \frac{(-1)^j}{\sqrt{2}} \left\{ t_{j,\sigma}^+ s_j - \sigma t_{j,0}^+ + (s_{j,0}^+ + \sigma t_{j,0}^+ ) t_{j,-\sigma} \right\} \\
&\quad + (-1)^j a_{j,\sigma}^+ a_{j,-\sigma}.
\end{align*}
\]
\[
\begin{align*}
    b_{j_2,\sigma}^+ b_{j,-\sigma}^- &= \frac{(-1)^j}{\sqrt{2}} \{ t_{j,\sigma}^+ (s_j + \sigma t_{j,0}) + (s_j^+ - \sigma t_{j,0}^-) t_{j,-\sigma}^- \} \\
    &+ (-1)^{j+1} a_{j,\sigma}^+ a_{j,-\sigma}^-.
\end{align*}
\] (A1)

The above equations should be considered only in the physical Hilbert space.

**APPENDIX B: THE FORM OF H\_2**

We list \( H_2 \) in the following:

\[
H_2 = \frac{t}{2} \sum_j \sigma (s_{j+1} t_{j,0}^+ + t_{j+1,0}^+ s_j^+)(a_{j+1,\sigma}^+ a_{j,\sigma} - a_{j+1,\sigma}^+ \tilde{a}_{j,\sigma})
\]

\[
+ i \frac{t}{\sqrt{2}} \sum_j \sigma (s_{j+1} t_{j,-\sigma}^- - t_{j+1,\sigma}^- s_j^+)(a_{j+1,\sigma}^+ a_{j,-\sigma}^- + \tilde{a}_{j+1,\sigma}^+ \tilde{a}_{j,-\sigma}^-)
\]

\[
- i \frac{t}{\sqrt{2}} \sum_j (t_{j+1,0}^+ t_{j,-\sigma}^- + t_{j+1,\sigma}^- t_{j,0}^-)(a_{j+1,\sigma}^+ a_{j,-\sigma}^- - \tilde{a}_{j+1,\sigma}^+ \tilde{a}_{j,-\sigma}^-)
\]

\[
- it \sum_j (t_{j+1,\sigma} d_j - d_{j+1} t_{j,\sigma})(a_{j+1,\sigma}^+ \tilde{a}_{j,\sigma}^- - \tilde{a}_{j+1,\sigma}^+ a_{j,-\sigma}^-)
\]

\[
- \frac{J}{8} \sum_j \sigma \{(s_j t_{j,0}^+ + t_{j,0}^+ s_j^-)(a_{j+1,\sigma}^+ a_{j+1,\sigma} - \tilde{a}_{j+1,\sigma}^+ \tilde{a}_{j+1,\sigma}) + (j \leftrightarrow j + 1) \}
\]

\[
+ \frac{J}{8} \sum_j \sigma \{( \sum_{\sigma' = \pm 1} \sigma' t_{j,\sigma}^+ t_{j,\sigma'}^-)(a_{j+1,\sigma}^+ a_{j+1,\sigma} + \tilde{a}_{j+1,\sigma}^+ \tilde{a}_{j+1,\sigma}) + (j \leftrightarrow j + 1) \}
\]

\[
- \frac{J}{2} \sum_{j,\alpha = \pm 1} (t_{j,0}^+ t_{j+1,0}^+ t_{j,-\sigma}^- - t_{j,0}^+ t_{j+1,0}^- t_{j,-\sigma}^+)(a_{j+1,\sigma}^+ a_{j+1,\sigma} - \tilde{a}_{j+1,\sigma}^+ \tilde{a}_{j+1,\sigma})
\]

\[
- \frac{J}{2 \sqrt{2}} \sum_j \sigma \{a_{j+1,\sigma}^+ a_{j+1,-\sigma} t_{j,-\sigma}^- (s_j - \sigma t_{j,0}) + \tilde{a}_{j+1,\sigma}^+ \tilde{a}_{j+1,-\sigma} t_{j,-\sigma}^+ (s_j + \sigma t_{j,0}) - (j \leftrightarrow j + 1) \}
\]

\[
+ \frac{J}{4} \sum_{\sigma = \pm 1} ( \sum_{\sigma' = \pm 1} t_{j,\sigma}^+ t_{j,\sigma'}^-)( \sum_{\sigma' = \pm 1} t_{j+1,\sigma}^+ t_{j+1,\sigma'}^-) + H.c.
\] (B1)

**APPENDIX C: \( \Delta \) AND \( \tilde{\Delta} \) ARE SPIN SINGLETS**

We would like to discuss the transformation properties of \( \Delta \) and \( \tilde{\Delta} \) under the spin rotation and verify that they are spin singlets. Due to the same reason as that for \( t \) operators, \( a_\sigma \)
and $\bar{a}_\sigma$ defined in (11) are not spinors. Therefore, we are unable to directly conclude that $\Delta$ and $\bar{\Delta}$ are not tensors under the spin rotation. To make the rotational symmetry manifest, we perform the following unitary transformation:

\begin{align}
  a_{j,\sigma}^+ &\rightarrow a_{j,\sigma}^+ \exp \{-i \sigma \frac{\pi}{4} - i \sigma \frac{\pi}{2} \sum_{l<j} (n_{l,1}^h + n_{l,2}^h)\}, \\
  \bar{a}_{j,\sigma}^+ &\rightarrow \bar{a}_{j,\sigma}^+ \exp \{i \sigma \frac{\pi}{4} - i \sigma \frac{\pi}{2} \sum_{l<j} (n_{l,1}^h + n_{l,2}^h)\}. 
\end{align}

(C1)

Under the above transformation (C1), $\Delta + \bar{\Delta}$ and $\Delta - \bar{\Delta}$ become

\begin{align}
  \Delta + \bar{\Delta} &\rightarrow i \sum_\sigma \sigma \langle a_{j+1,\sigma}^+ a_{j,-\sigma}^- - \bar{a}_{j+1,\sigma}^+ \bar{a}_{j,-\sigma}^- \rangle, \\
  \Delta - \bar{\Delta} &\rightarrow \sum_\sigma \sigma \langle a_{j+1,\sigma}^+ \bar{a}_{j,-\sigma}^- + \bar{a}_{j+1,\sigma}^+ a_{j,-\sigma}^- \rangle. 
\end{align}

(C2)

Now it is clear that $\Delta$ and $\bar{\Delta}$ are indeed spin singlets. We also examine other terms in $H$ with eq. (C1) and confirm that our mean-field ansatz respects the rotational symmetry.

**APPENDIX D: DIAGONALIZATION OF EQ.(23)**

Here we present the procedure to diagonalize eq.(23). After performing Fourier transformations on all operators by $\hat{O}_j = \frac{1}{\sqrt{N}} \sum_k \hat{O}_k e^{i k x_j}$, the mean-field Hamiltonian in eq.(23) becomes

\begin{align}
  H_{MF} &= N \{ \lambda - \frac{J_4}{2} \frac{3}{4} J_\perp + \lambda \bar{s}^2 + (\chi + \bar{\chi}) Q t + \frac{J}{8} (\Delta + \bar{\Delta})^2 \} \\
  &\quad - \frac{J}{16} (\chi - \bar{\chi})^2 - \frac{J}{2} (d^2 + 2 \mu + \lambda - \frac{J_4}{4} - J) d^2 \} \\
  &\quad + \sum_k \{ \epsilon_k e_{k\alpha}^e e_{k\sigma}^e + \bar{e}_k \bar{e}_{k\sigma}^e e_{k\alpha}^e \} \\
  &\quad + \sum_k \{ i \Gamma_k e_{k\alpha}^e e_{k\alpha} ^+ + i \bar{\Gamma}_k \bar{e}_k e_{k\alpha} ^+ e_{k\alpha}^e + H.c. \} \\
  &\quad + \sum_k \{ \Lambda_k t_{k\alpha} t_{k\alpha} + \Pi_k (t_{k\alpha}^+ t_{-k\alpha}^- + t_{k\alpha} t_{-k\alpha}) \}
\end{align}

(D1)

where

\begin{align}
  \epsilon_k &= -t(2 \bar{d}^2 + \bar{s}^2 + Q) - \frac{J}{8} (\chi - \bar{\chi}) \cos k \\
  &\quad - t - \mu - \lambda \frac{J_4}{4} + \frac{J}{2} (1 - \bar{d}^2),
\end{align}
\[ \epsilon_k = -(t(2d^2 + s^2 + Q) + \frac{J}{8}(\chi - \bar{\chi})) \cos k \]
\[ + t - \mu - \lambda + \frac{J_\perp}{4} + \frac{J}{2}(1 - d^2), \]
\[ \Gamma_k = (\frac{J}{4}(\Delta + \bar{\Delta}) + 2\sqrt{2}tsd) \sin k, \]
\[ \bar{\Gamma}_k = (\frac{J}{4}(\Delta + \bar{\Delta}) - 2\sqrt{2}tsd) \sin k, \]
\[ \Lambda_k = (Js^2 - t(\chi + \bar{\chi})) \cos k + \frac{J_\perp}{4} - \lambda, \]
\[ \Pi_k = \frac{J}{2} \bar{s}^2 \cos k. \]

Here the lattice spacing has been taken to be unity.

Eq. (D1) can be diagonalized by the following Bogoliubov transformations:

\[ \alpha_k = u_k e_{k\uparrow} - v_k e_{-k\downarrow}^+, \quad \beta_k = u_k e_{-k\downarrow} + v_k e_{k\uparrow}^+, \quad (D2) \]
\[ \bar{\alpha}_k = \bar{u}_k e_{k\uparrow} - \bar{v}_k e_{-k\downarrow}^+, \quad \bar{\beta}_k = \bar{u}_k e_{-k\downarrow} + \bar{v}_k e_{k\uparrow}^+, \quad (D3) \]
\[ \gamma_{ka} = \cosh \theta_k t_{ka} + \sinh \theta_k t_{-k-a}^+. \quad (D4) \]

The coefficients \( u_k, v_k, \bar{u}_k, \bar{v}_k, \cosh \theta_k, \) and \( \sinh \theta_k \) are given by

\[ u_k = \cos \phi_k e^{i\frac{\pi}{4}}, \quad v_k = \sin \phi_k e^{-i\frac{\pi}{4}}, \]
\[ \bar{u}_k = \cos \bar{\phi}_k e^{i\frac{\pi}{4}}, \quad \bar{v}_k = \sin \bar{\phi}_k e^{-i\frac{\pi}{4}}, \]
\[ \cos^2 \phi_k = \frac{1}{2}(1 + \frac{\epsilon_k}{E_k}), \quad \sin^2 \phi_k = \frac{1}{2}(1 - \frac{\epsilon_k}{E_k}), \]
\[ \cos^2 \bar{\phi}_k = \frac{1}{2}(1 + \frac{\bar{\epsilon}_k}{\bar{E}_k}), \quad \sin^2 \bar{\phi}_k = \frac{1}{2}(1 - \frac{\bar{\epsilon}_k}{\bar{E}_k}), \]
\[ E_k = \sqrt{\epsilon_k^2 + \Gamma_k^2}, \quad \bar{E}_k = \sqrt{\bar{\epsilon}_k^2 + \bar{\Gamma}_k^2}, \quad (D5) \]
\[ \cosh^2 \theta_k = \frac{1}{2}(\frac{\Lambda_k}{\omega_k} + 1), \quad \sinh^2 \theta_k = \frac{1}{2}(\frac{\Lambda_k}{\omega_k} - 1), \]
\[ \omega_k = \sqrt{\Lambda_k^2 - 4\Pi_k^2}. \quad (D6) \]

In eq. (D6), both \( \Lambda_k \) and \( \Lambda_k^2 - 4\Pi_k^2 \) have to be positive. This constraint will be enforced in our numerical analysis. If we define \( \bar{\nu} = (\chi + \bar{\chi})t/(\frac{J_\perp}{4} - \lambda) \) and \( \nu = 2Js^2/(\frac{J_\perp}{4} - \lambda) \), then the band minimum of \( t \) particles occurs at \( k = \pi \) when \( \bar{\nu} \leq \nu/2 \) and at \( k = 0 \) when \( \bar{\nu} > \nu/2 \).

With the help of eqs. (D2), (D3), and (D4), we obtain eq. (24).
APPENDIX E: THE BOND-OPERATOR REPRESENTATIONS OF PAIRING FIELDS

Here we give the bond-operator representations of pairing fields in the phase string formulae. First, we need the corresponding representation of electron operators. They are as the following:

\[
c_{j,1,\sigma} = \exp \left( -i \frac{\pi}{4} (1 + \sigma) \right) \exp \left\{ i \frac{\pi}{2} \sum_{l>j,m} \left( 2S_{l,m}^z - 1 - \sigma n_{l,m}^h \right) \right\}
\]

\[
\left\{ \frac{\sigma}{\sqrt{2}} a_{j,-\sigma}^+(s_j + \sigma t_{j,0}) - ia_{j,\sigma}^+ t_{j,\sigma} + d_{j,\sigma}^+ \bar{a}_{j,\sigma} \right\} \sigma^{N_h},
\]

\[
c_{j,2,\sigma} = \exp \left( i \frac{\pi}{4} (1 + \sigma) \right) \exp \left\{ i \frac{\pi}{2} \sum_{l>j,m} \left( 2S_{l,m}^z - 1 - \sigma n_{l,m}^h \right) \right\}
\]

\[
\left\{ -\frac{\sigma}{\sqrt{2}} \bar{a}_{j,-\sigma}^+(s_j - \sigma t_{j,0}) + i\bar{a}_{j,\sigma}^+ t_{j,\sigma} + d_{j,\sigma}^+ \bar{a}_{j,\sigma} \right\} \sigma^{N_h}.
\]

(E1)

By plugging the above formulae into eq. (E2), we obtain

\[
\Delta_y(j) = -\frac{i}{2} d_j^+ s_j \exp \left( 2\pi i \sum_{l>j,m} S_{l,m}^z \right),
\]

\[
\Delta_x(j) = \exp \left( 2\pi i \sum_{l>j+1,m} S_{l,m}^z \right)
\]

\[
\left\{ \frac{i}{\sqrt{2}} \left( d_{j+1}^+ d_j^+ a_{j+1,\sigma} a_{j,-\sigma} - \bar{a}_{j+1,\sigma}^+ a_{j,-\sigma} t_{j+1,\sigma} t_{j,-\sigma} \right)
\]

\[
-\frac{1}{\sqrt{2}} \left( d_j^+ t_{j+1,\sigma} a_{j+1,\sigma} a_{j,-\sigma} + d_{j+1}^+ t_{j,-\sigma} a_{j+1,\sigma} \bar{a}_{j,-\sigma} \right)
\]

\[
+\frac{i}{2} \left[ d_{j+1}^+ a_{j+1,\sigma} \bar{a}_{j,\sigma}^+ (s_j + \sigma t_{j,0}) - d_j^+ \bar{a}_{j+1,\sigma} a_{j,\sigma} (s_{j+1} + \sigma t_{j+1,0}) \right]
\]

\[
+\frac{\sigma}{2} \bar{a}_{j+1,\sigma}^+ \bar{a}_{j,\sigma}^+ \left[ t_{j,\sigma} (s_{j+1} + \sigma t_{j+1,0}) - t_{j+1,\sigma} (s_j + \sigma t_{j,0}) \right]
\]

\[
+\frac{i}{2 \sqrt{2}} \bar{a}_{j+1,\sigma}^+ \bar{a}_{j,\sigma}^+ (s_{j+1} + \sigma t_{j+1,0}) (s_j - \sigma t_{j,0}) \right\}.
\]

(E2)

We have to emphasize that it is necessary to take into account \( \sigma^{N_h} \) in eq. (E1). Otherwise, the representations of pairing fields would be incorrect.
REFERENCES

[1] For a review, see E. Dagotto and T.M. Rice, Science 271, 618 (1996).

[2] E. Dagotto, J. Riera, and D. Scalapino, Phys. Rev. B 45, 5744 (1992); T.M. Rice, S. Gopalan, and M. Sigrist, Europhys. Lett. 23, 445 (1993).

[3] T. Barnes, E. Dagotto, J. Riera and E.S. Swanson, Phys. Rev. B 47, 3196 (1993).

[4] M. Reigrotzki, H. Tsunetsugu and T.M. Rice, J. Phys: Condens. Matter 6, 9235 (1994).

[5] S. Gopalan, T. M. Rice, and M. Sigrist, Phys. Rev. B 49, 8901 (1994).

[6] S.R. White, R.M. Noack, and D.J. Scalapino, Phys. Rev. Lett 73, 886 (1994).

[7] M. Sigrist, T.M. Rice, and F.C. Zhang, Phys. Rev. B 49, 12058 (1994).

[8] D. Poilblanc, D.J. Scalapino, and W. Hanke, Phys. Rev. B 52, 6796 (1995).

[9] M. Troyer, H. Tsunetsugu, and T.M. Rice, Phys. Rev. B 53, 251 (1995); C.A. Hayward and D.Poilblanc, ibid., 53, 11721 (1996).

[10] S.R. White and D.J. Scalapino, Phys. Rev. B 55, 6504 (1997).

[11] G. Sierra, M.A. Martín-Delgado, J. Dukelsky, S.R. White, and D.J. Scalapino, Phys. Rev. B 57, 11666 (1998).

[12] T. F. A. Müller and T. M. Rice, Phys. Rev. B 58, 3425 (1998).

[13] B. Ammon, M. Troyer, T.M. Rice, and N. Shibata, cond-mat/9812144

[14] M. Azuma, Z. Hiroi, M. Takano, K. Ishida, and Y. Kitaoka, Phys. Rev. Lett. 73, 3463 (1994).

[15] M. Uehara, T. Nagata, J. Akimitsu, H. Takahashi, N. Móri, and K. Kinoshita, J. Phys. Soc. Jpn. 65, 2764 (1996); T. Osafune, N. Motoyama, H. Eisaki, and S. Uchida, Phys. Rev. Lett. 78, 1980 (1997); N. Motoyama, T. Osafune, T. Kakeshita, H. Eisaki, S. Uchida, Phys. Rev. B 55, R3386 (1997).
[16] H. Mayaffre, P. Auban-Senzier, M. Nardone, D. Jérome, D. Poilblanc, C. Bourbonnais, U. Ammerahl, G. Dhalenne, and A. Revcolevschi, Science 279, 345 (1998).

[17] T. Imai, K.R. Thurber, K.M. Shen, A.W. Hunt, and F.C. Chou, Phys. Rev. Lett. 81, 220 (1998); S. Katano, T. Nagata, J. Akimitsu, M. Nishi, and K. Kakurai, ibid., 82, 636 (1999).

[18] S. Sachdev and R.N. Bhatt, Phys. Rev. B 41, 9323 (1990).

[19] W. Marshall, Proc. R. Soc. London Ser. A 232, 48 (1955).

[20] Z. Y. Weng, D. N. Sheng, Y.-C. Chen, and C. S. Ting, Phys. Rev. B 55, 3892 (1997).

[21] Z. Y. Weng, D. N. Sheng, and C. S. Ting, Phys. Rev. Lett. 80, 5401 (1998); Phys. Rev. B 59, April 1 (1999).

[22] B.I. Shraiman and E.D. Siggia, Phys. Rev. Lett. 62, 1564 (1989).

[23] L. Balents and M.P.A. Fisher, Phys. Rev. B 53, 12133 (1996).

[24] In the case with periodic boundary conditions or the ring, we can form closed paths by winding around it. However, these closed paths only give constant phases for fixed hole concentration. They do not affect the dynamics of holons and spinons.

[25] We will not perform the unitary transformation on $H_2$. For those terms with single $t_{j\sigma}$ in $H_2$, this transformation will introduce a nonlocal phase factor, which only depends on the hole density. However, these terms are still high energy processes and will not change the qualitative behaviors obtained from $H_0 + H_1$.

[26] If we do not consider the phase string and apply the bond-operator representation to the $t - J$ model directly, then $d$ would be parity odd. As a consequence, exchange symmetry forbids the condensation of $d$ particles. However, $d$ can condense after taking into account the phase string. This will result in very different physics.

[27] See, for example, F.D.M. Haldane, J. Phys. C 14, 2585 (1981).
[28] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, 5th edition, (Academic Press, 1994).

[29] Of course, there can be gapless excitations at some points of momentum space if we tune those parameters delicately and this does not need a vanishing $\Gamma_k$. However, we do not find such a situation.
Figure Captions:

Fig. 1 The local structure of hole pairs: (a) diagonal site. (b) chain direction. The solid line and open circles represent the spin singlet bonds and holes, respectively.

Fig. 2 The spin gap, $\Delta_t$, as a function of $\eta = J/J_\perp$ where $E = \Delta_t/J_\perp$.

Fig. 3 The ground state energy per site with $J/t = 0.5$. Different curves correspond to $J_\perp/J = 1, 2, 4, 6, 8, \text{ and } 10$. The data denoted by the cross are obtained with DMRG and taken from Ref. [11].

Fig. 4 The weight of different types of hole pairs with $J/t = 0.5$ and $J_\perp/J = 10$. The solid line and open circles represent the weights of hole pairs on diagonal sites and chain direction, respectively.

Fig. 5 The vacuum expectation values of pairing fields as functions of the doping concentration at $J/t = 0.5$ and $J_\perp/J = 10$. The dashed line and open circles represent $\Delta_x$ and $\Delta_y$, respectively.

Fig. 6 The gaps of spin excitations at $J/t = 0.5$ and $J_\perp/J = 10$. The solid line, open circles, and stars correspond to quasiparticles in bonding band, anti-bonding band, and magnons, respectively.

Fig. 7 The boundary between the C1S0 phase and the phase separation region with (a) $J_\perp/J = 10$, (b) $J_\perp/J = 5$, and (c) $J_\perp/J = 2$. 
Fig. 1
Fig. 3

![Graph with various lines and symbols representing data points.

- Lines labeled 1, 2, 4, 6, 8, 10.

- Symbols representing DMRG.

- Axes: 
  - X-axis: $\delta$
  - Y-axis: $E_0/J$

- Legend:
  - 1
  - 2
  - 4
  - 6
  - 8
  - 10
  - DMRG

- Graph shows trends and comparisons based on the data presented.
Fig. 6
Fig. 7(a)
Fig. 7(b)
Fig. 7(c)