THE ADDITIVITY OF TRACES IN MONOIDAL DERIVATORS

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Abstract. We develop the theory of monoidal structures on derivators, culminating in a proof that generalized trace maps in a closed symmetric monoidal stable derivator are additive along distinguished triangles. This includes the additivity of classical Euler characteristics and Lefschetz numbers, as well as many generalizations of these invariants.

The proof of additivity closely follows that of May for triangulated categories, but the derivator context makes the underlying ideas more transparent, showcasing the advantages of derivators over triangulated categories, model categories, and (\infty,1)-categories. We expect many other generalizations of classical stable results to be possible in this context as well.

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1. Introduction

The Euler characteristic for finite CW complexes is determined by its value on a point and its additivity on subcomplexes: if $A$ is a subcomplex of $X$, then

$$\chi(A) + \chi(X/A) = \chi(X).$$

For generalizations of the Euler characteristic, such as the Lefschetz number of an endomorphism of a finite CW complex, axiomatizations are more complicated (see [AB04, Bro71]); but a similar additivity on subcomplexes is an essential component.

The classical Euler characteristic can also be generalized to Euler characteristics for dualizable objects in a symmetric monoidal category—and further, to traces of (twisted) endomorphisms of such, which include Lefschetz numbers and topological transfers. In this generality, many fundamental properties are easy to prove, but additivity is not among them. Despite this, for most specific examples, additivity remains important.

In this paper, we will show that if our symmetric monoidal category has the additional structure of a stable derivator (see below), then the Euler characteristic is automatically additive, relative to the appropriate generalization of subcomplexes. Similarly, if an endomorphism of a dualizable object $X$ restricts to a “subcomplex” $A$ of $X$, then its trace on $X$ is the sum of the traces of the induced endomorphisms of $A$ and of $X/A$. Using a suggestive notation, we write this as

$$\text{tr}(\phi_A) + \text{tr}(\phi_{X/A}) = \text{tr}(\phi_X).$$

Essentially this same theorem, in slightly less generality, was proven by [May01], who worked with triangulated categories and stable model categories. In this paper we have chosen to use stable derivators instead. Indeed, the point of this paper is not so much the truth of the additivity theorem, which has been known since [May01], but the fact that derivators are a much more convenient context in which to state it, prove it, and generalize it.

1.1. Why derivators? Intuitively, an additivity theorem for traces should be true in any “stable homotopy theory”. (The case of ordinary Euler characteristics and Lefschetz numbers corresponds to the classical stable homotopy theory of spectra.) There are various different formal axiomatizations of “stable homotopy theories”: in addition to stable derivators, one may consider stable model categories, stable $(\infty, 1)$-categories, or triangulated categories. Derivators are perhaps less well-known than any of the others, but we find that they have many advantages. In particular, as compared with derivators, each of the other notions contains either too little information (triangulated categories) or too much (stable model categories and stable $(\infty, 1)$-categories).

The fact that triangulated categories contain too little information is well-known, and was particularly evident in [May01]. The major contribution of ibid. was to describe “compatibility axioms” between a triangulation and a monoidal structure, which hold in the homotopy category of any stable, monoidal, enriched model category, and which imply the additivity of Euler characteristics. One major difficulty is that a triangulation is merely a “remnant” of the homotopy theory visible in the homotopy category, so its axioms assert that certain objects and morphisms exist but do not characterize them uniquely. For instance, any morphism $A \to X$
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(regarded as a “subcomplex inclusion”) can be extended to some distinguished triangle, whose third object may be regarded as the “quotient” $X/A$, but the quotient map $X \to X/A$ is not uniquely determined.

In examples, there are particular good choices of these objects and morphisms arising from homotopy limit and colimit constructions, but their universal properties are not visible to the triangulated category. Therefore, new axioms which build on previous ones tend to be of the form “there exist data as in all the previous axioms which is compatible in such-and-such precise way and which moreover satisfies...”. This starts to become evident in May’s axiom (TC5b), and it gets much worse when trying to generalize further.

In particular, the axioms given in the first part of [May01] were insufficient to imply the more general additivity result for traces (and it is in fact known that this fails to be true for some morphisms between distinguished triangles [Fer06]). Instead, May proved the stronger result by working directly with a stable monoidal model category. This avoids the problem of too little information, because in a model category we can construct homotopy limits and colimits using ordinary limits and colimits along with fibrations and cofibrations. For instance, if we represent a subcomplex inclusion by a cofibration $A \to X$, then its “quotient” in the sense of stable homotopy theory is just its ordinary category-theoretic quotient $X/A$ in the model category.

However, the problem with model categories is also evident in [May01]: there are too many different objects which represent the “same” object in the underlying homotopy theory. Cofibrant objects are good for mapping out of and are preserved by colimits and tensor products, while fibrant objects are good for mapping into and are preserved by limits and internal-homs. When combining both types of construction, as is necessary for duality and traces, one needs fairly tricky fibrant and cofibrant replacements, passing back and forth across zigzags of weak equivalences. Indeed, for this reason May did not even prove the additivity theorem in full generality for model categories: if not all objects are fibrant, then “a fairly elaborate diagram chase using functorial fibrant approximation” is required which is left “to the interested reader”.

One attempt to avoid these problems is the modern theory of $(\infty, 1)$-categories. Here homotopy limits and colimits have $(\infty, 1)$-categorical universal properties, with no need for fibrant and cofibrant replacements, and all equivalences are invertible. This makes for quite a beautiful and powerful theory. However, $(\infty, 1)$-categories introduce a great deal of their own complexity, as witnessed by the length of the books [Lur09, Lur]. Additionally, one must constantly deal explicitly with questions of homotopy coherence, generally by using various types of fibration (and other model-categorical techniques) at the level of the $(\infty, 1)$-categories themselves. That is, while an individual $(\infty, 1)$-category has “forgotten” all the extra information contained in a model category, the theory of $(\infty, 1)$-categories retains too much information about each individual $(\infty, 1)$-category.

We believe that the theory of derivators avoids all of these problems. Derivators were developed independently (under various names) by Grothendieck [Gro90], Heller [Hel88], and Franke [Fra96]. They have recently been studied further by Maltsiniotis [Mal], Cisinski [Cis03], and others including the first author [Gro12]; see the website [Gro90] for a comprehensive bibliography.
A derivator $\mathcal{D}$ consists of, among other things, a category $\mathcal{D}(A)$ for each small category $A$. We think of $\mathcal{D}(A)$ as the homotopy category of $A$-shaped homotopy coherent diagrams in $\mathcal{D}$. In particular, when $A$ is the terminal category $\mathbb{1}$, the category $\mathcal{D}(\mathbb{1})$ should be regarded as analogous to a triangulated category, or the homotopy category of a model category or $(\infty, 1)$-category. The categories $\mathcal{D}(A)$ are related by restriction and left and right (homotopy) Kan extension functors.

A derivator avoids the problems of triangulated categories, because the additional structure of all the categories $\mathcal{D}(A)$ allows us to characterize homotopy limits and colimits using ordinary universal properties. They are simply the left and right adjoints of the restriction functors $\mathcal{D}(\mathbb{1}) \to \mathcal{D}(A)$. Thus, instead of searching for more and more axioms we can discover and prove new lemmas as we need them.

On the other hand, a derivator avoids the problems of model categories, because each category $\mathcal{D}(A)$ is already a “homotopy category”. Weakly equivalent objects have been made isomorphic, so there are no zigzags to deal with. Also, the limit and colimit functors can be applied to any object of $\mathcal{D}(A)$ (that is, any “coherent diagram” in $\mathcal{D}$), with no need to worry about fibrancy or cofibrancy; homotopy meaningfulness is already built in.

Finally, a derivator avoids the problems of $(\infty, 1)$-categories, because their definition and basic properties are simple and easy to understand, requiring only ordinary category-theoretic machinery. For instance, the present paper is almost entirely self-contained, except that we build on the theory of [Gro12]. Moreover, derivators themselves form a well-behaved 2-category in which various 2-categorical constructions are also homotopically meaningful; thus there is no superfluous information at this level either.

Of course, we may need to start from a model category or $(\infty, 1)$-category in order to obtain a derivator in the first place, but we can package this up in a general black-box theorem. Specifically, the homotopy category of any model category can be enhanced to a derivator; this was proven by Cisinski [Cis03] (an easier proof that applies only to combinatorial model categories can be found in [Gro12]). We expect that the same is true for any complete and cocomplete $(\infty, 1)$-category. Similarly, when computing with concrete examples, we may have to present them by objects of a model category and use fibrant and cofibrant replacements here and there. The point of derivators is that when proving abstract, general facts like the additivity of traces, these technicalities can be packaged up once and never mentioned again.

One does not have to regard derivators as objects of interest in their own right. They can be considered as simply a convenient abstract context in which to express a calculus of homotopy Kan extensions that is valid in any model category or $(\infty, 1)$-category. However, there are some indications that the notion of a derivator is also intrinsically interesting. For instance, the derivator of topological spaces (or simplicial sets, or $\infty$-groupoids) is the free cocompletion of a point. This is known to be true in the world of $(\infty, 1)$-categories, but it is more surprising to see it in the world of derivators, which does not “build in” any information about homotopy or higher groupoids. In essence, derivators tell us that many notions of homotopy theory arise automatically out of the most obvious formal calculus of limits, colimits, and Kan extensions.

1.2. An outline of this paper. Our goal is to prove the additivity of traces in any stable monoidal derivator. More specifically, we will show that the underlying triangulated category of any stable monoidal derivator satisfies May’s axioms, and
that his proof of general additivity for model categories can be performed basically word-for-word in a derivator. By itself, this does not really increase the applicability of May’s result appreciably (since most examples of derivators arise from model categories). However, for all the reasons cited above, we believe that it clarifies the concepts involved and provides a more flexible foundation for further developments and generalizations.

In fact, the majority of this paper is spent developing the abstract theory of stable monoidal derivators. This may be regarded as the beginning of a formal study of the interaction of stability with monoidal structure. We expect to take advantage of this in future work; for instance, in [PS13] we will use it to prove a much more general additivity theorem, using an entirely different approach to that of [May01].

The notion of stable derivator is not a new one; indeed, most interest in derivators has centered around the stable ones. In [Gro12] the first author showed that the homotopy category of a stable derivator is triangulated (another proof by Maltsiniotis is yet to appear in print). However, not all of the familiar facts which relate homotopy limits in a stable model category to structure in its underlying triangulated category have yet been extended to derivators. The first contribution of this paper (in sections 2–6) is to improve this situation. In particular, we will prove that any bicartesian square in a stable derivator gives rise to a distinguished triangle, and that a derivator is stable as soon as its suspension functor is an equivalence.

We also give several convenient technical lemmas for the calculus of (homotopy) Kan extensions.

The second contribution of this paper (sections 7–11) is to define and study the new notion of monoidal derivator. In particular, we study “tensor products of functors” and what it means for a monoidal derivator to be “closed”. Neither of these is immediately obvious, but finding their correct formulation is crucial to the proof of additivity. We also show that a monoidal derivator gives rise to a “bicategory of profunctors”, which proves to be a useful technical tool (and will continue to be so in [GS13, PS13]).

Finally, in sections 12–14 we prove May’s axioms and the additivity theorem. The context of derivators transparently reveals May’s axioms (TC3)–(TC5) as simply various functoriality and naturality properties of the “pushout product” in a stable monoidal derivator. Of course, this was always the intuition behind these axioms, and the way to prove them via model category theory. The derivator framework makes this precise at a homotopical level with honest universal properties. The bicategory of profunctors also proves its worth in the analysis of duality.

In order to make the paper maximally self-contained, in the first few sections 2–6 we have also reviewed the basic definitions of derivators, pointed derivators, and stable derivators. (We recommend [Gro12] as further reading.) Additionally, in Appendix A we summarize the “calculus of mates” for natural transformations, which is used extensively in the theory of derivators. Thus, very little background is required of the reader.

2. Review of derivators

In this section we recall the definitions of prederivators and derivators, and fix some notation and terminology. Let $\text{Cat}$ and $\text{CAT}$ denote the 2-categories of small and large categories, respectively.
Definition 2.1. A prederivator is a 2-functor $\mathcal{D} : \text{Cat}^{\text{op}} \to \text{CAT}$.

For a functor $u : A \to B$, we write $u^* : \mathcal{D}(B) \to \mathcal{D}(A)$ for its image under the 2-functor $\mathcal{D}$, and refer to it as restriction along $u$. If $\mathbb{1}$ denotes the terminal category, we call $\mathcal{D}(\mathbb{1})$ the underlying category of $\mathcal{D}$.

Example 2.2. Any (possibly large) category $C$ gives rise to a represented prederivator $y(C)$ defined by

$$y(C)(A) := C^A.$$ 

Its underlying category is $C$ itself. This example is a good one to think about when generalizing from ordinary categories to (pre)derivators.

Example 2.3. Suppose $C$ is a category equipped with a class $W$ of “weak equivalences”; for instance, $C$ could be a Quillen model category (see e.g. [Hov99]). We write $C[W^{-1}]$ for the category obtained by formally inverting the weak equivalences. If $A$ is a small category, then $C^A$ also has a class of weak equivalences, namely the pointwise ones $W^A$. We then have the derived or homotopy prederivator $\mathcal{H}_o(C)$ defined by

$$\mathcal{H}_o(C)(A) := (C^A)[W^A]^{-1}.$$ 

Its underlying category is $C[W^{-1}]$. This example is the primary application of the theory of derivators.

Motivated by the previous two examples, we sometimes refer to the objects of $\mathcal{D}(A)$ as (coherent, $A$-shaped) diagrams in $\mathcal{D}$. Any such $X \in \mathcal{D}(A)$ has an underlying (incoherent) diagram, which is an ordinary diagram in $\mathcal{D}(\mathbb{1})$, i.e. an object of the functor category $\mathcal{D}(\mathbb{1})^A$. For each $a \in A$, the underlying diagram of $X$ sends $a$ to $a^*X$. (Here $a$ also denotes the functor $a : \mathbb{1} \to A$ whose value on the object of $\mathbb{1}$ is $a$.) We may also write $X_a$ for $a^*X$.

More generally, any diagram $X \in \mathcal{D}(B \times A)$ has an underlying “partially coherent” diagram which is an object of $\mathcal{D}(B)^A$, sending $a \in A$ to the diagram $X_a = a^*X = (1 \times a)^*X$.

We will occasionally refer to a coherent diagram as having the form of, or looking like, its underlying diagram, and proceed to draw that underlying diagram using objects and arrows in the usual way. It is very important to note, though, that a coherent diagram is not determined by its underlying diagram, not even up to isomorphism.

Example 2.4. For any prederivator $\mathcal{D}$ and category $B \in \text{Cat}$, we have a “shifted” prederivator $\mathcal{D}^B$ defined by $\mathcal{D}^B(A) := \mathcal{D}(B \times A)$. This is technically very convenient: it enables us to ignore extra “parameter” categories $B$ by shifting them into the (universally quantified) derivator under consideration. Note that $y(C)^B \cong y(C^B)$ and $\mathcal{H}_o(C)^B \cong \mathcal{H}_o(C^B)$.

Example 2.5. Any prederivator $\mathcal{D}$ has an opposite prederivator, defined by setting $\mathcal{D}^{\text{op}}(A) := \mathcal{D}(A^{\text{op}})^{\text{op}}$. Note that $y(C)^{\text{op}} = y(C^{\text{op}})$ and $\mathcal{H}_o(C^{\text{op}}) = \mathcal{H}_o(C)^{\text{op}}$ and $(\mathcal{D}^B)^{\text{op}} = (\mathcal{D}^{\text{op}})^B^{\text{op}}$.

A morphism of prederivators $F : \mathcal{D}_1 \to \mathcal{D}_2$ is a pseudonatural transformation; this consists of functors $F_A : \mathcal{D}_1(A) \to \mathcal{D}_2(A)$ and isomorphisms $F_B u^* \cong u^*F_A$ for all $u : A \to B$, satisfying obvious axioms. Similarly, a transformation of prederivators is a “modification”, which consists simply of transformations $\alpha_A : F_A \to G_A$ satisfying an axiom. This defines the 2-category $\mathcal{PDER}$.
Now a *derivator* is a prederivator with (1) some stack-like properties and (2) well-behaved left and right Kan extensions along functors in $\mathcal{C}$. 

**Definition 2.6.** A derivator is a prederivator $\mathcal{D}$ with the following properties.

(Der1) $\mathcal{D} : \mathcal{C}^{\text{op}} \rightarrow \mathcal{CAT}$ takes coproducts to products. In particular, $\mathcal{D}(\emptyset)$ is the terminal category.

(Der2) For any $A \in \mathcal{C}^{\text{at}}$, the family of functors $a^* : \mathcal{D}(A) \rightarrow \mathcal{D}(1)$, as $a$ ranges over the objects of $A$, is jointly conservative (isomorphism-reflecting).

(Der3) Each functor $u^* : \mathcal{D}(B) \rightarrow \mathcal{D}(A)$ has both a left adjoint $u_!$ and a right adjoint $u_*$. 

(Der4) Given functors $u : A \rightarrow C$ and $v : B \rightarrow C$ in $\mathcal{C}^{\text{at}}$, let $(u/v)$ denote their comma category, with projections $p : (u/v) \rightarrow A$ and $q : (u/v) \rightarrow B$. Then the canonical mate-transformations (see Appendix A)

$$q p^* \rightarrow v^* u_! \quad \text{and} \quad u^* v_* \rightarrow p_* q^*$$

are isomorphisms. (We will discuss this axiom further in §3.)

We say that a derivator is **strong** if it satisfies:

(Der5) For any $A$, the induced functor $\mathcal{D}(A \times 2) \rightarrow \mathcal{D}(A)^2$ is full and essentially surjective, where $2 = (0 \rightarrow 1)$ is the category with two objects and one nonidentity arrow between them.

Following the established terminology of the theory of $(\infty, 1)$-categories, we refer to the functors $u_!$ and $u_*$ in (Der3) simply as **left and right Kan extensions**, respectively, as opposed to calling them homotopy Kan extensions. This is meant to simplify the terminology and does not result in a risk of ambiguity since actual ‘categorical’ Kan extensions are meaningless in the context of an abstract derivator.

Similarly, when $B$ is the terminal category $\mathbb{1}$, we call them **colimits and limits**. In later sections, we will extend additional concepts from classical category theory to derivators, such as ends and coends and the tensor product of functors, and we will continue referring to them by their classical names.

Note that (Der1) and (Der3) together imply that each category $\mathcal{D}(A)$ has (actual) small coproducts and products. These are the only actual 1-categorical (co)limits which must exist in each derivator.

**Remark 2.7.** Axiom (Der5), which makes a derivator “strong”, is necessary whenever we want to perform limit constructions starting with morphisms in the underlying category $\mathcal{D}(\mathbb{1})$, since it enables us to “lift” such morphisms to objects of $\mathcal{D}(2)$. Combined with (Der2), it implies that if two objects of $\mathcal{D}(A \times 2)$ become isomorphic in $\mathcal{D}(A)^2$, then they were already isomorphic in $\mathcal{D}(A \times 2)$ — although such an isomorphism is not in general uniquely determined by its image in $\mathcal{D}(A)^2$.

Axiom (Der5) is also the most negotiable as to its exact form; for instance, Heller [Hel88] assumed a stronger version in which $2$ is replaced by any finite free category.

**Examples 2.8.** A large category $\mathcal{C}$ is both complete and cocomplete if and only if its represented prederivator $y(\mathcal{C})$ is a derivator. And if $\mathcal{C}$ is a model category, then its homotopy prederivator $\mathcal{H}o(\mathcal{C})$ is a derivator (see [Cis03, Gro12]). The functors $u_!$ and $u_*$ in $\mathcal{H}o(\mathcal{C})$ are the left and right derived functors, respectively, of the corresponding functors in $y(\mathcal{C})$. All derivators of either kind are strong.
Examples 2.9. If \( \mathcal{D} \) is a derivator, so are \( \mathcal{D}^B \) and \( \mathcal{D}^{op} \), and both are strong if \( \mathcal{D} \) is. Note that the Kan extension functors of \( \mathcal{D}^B \) are defined in terms of those of \( \mathcal{D} \) by
\[
 f_! := (1_B \times f)! \quad \text{and} \quad f_* := (1_B \times f)_* 
\]
while those of \( \mathcal{D}^{op} \) are defined by
\[
 f_! := ((f^{op})_*)^{op} \quad \text{and} \quad f_* := ((f^{op})_!)^{op}. 
\]
The verification of (Der4) for \( \mathcal{D}^B \) is slightly nontrivial; see [Gro12, Theorem 1.25].

Let us now turn to morphisms. We define a morphism of derivators to be simply a morphism of prederivators whose domain and codomain are derivators. More generally, we take the 2-category \( \mathcal{D}_{\mathcal{D}} \) to be a full sub-2-category of \( \mathcal{P}_{\mathcal{D}} \).

By an adjunction between derivators (or prederivators) we mean, of course, an internal adjunction in \( \mathcal{D}_{\mathcal{D}} \). This can be characterized in more elementary terms as follows.

**Lemma 2.10.** A morphism \( F: \mathcal{D}_1 \rightarrow \mathcal{D}_2 \) of prederivators is a left adjoint if and only if
\[
(i) \quad \text{Each functor } F_A: \mathcal{D}_1(A) \rightarrow \mathcal{D}_2(A) \text{ is a left adjoint, and}
(ii) \quad \text{For each } u: A \rightarrow B \text{ in } \mathcal{C}_{\mathcal{D}}, \text{ the canonical mate-transformation}
\]
\[
(2.11) \quad u^* G_B \Rightarrow G_A F_A u^* G_B \Rightarrow G_A u^* F_B G_B \rightarrow G_A u^* 
\]
is an isomorphism, where \( G_A \) and \( G_B \) are right adjoints of \( F_A \) and \( F_B \) respectively.

**Proof.** If (2.11) are isomorphisms, they assemble into pseudonaturality isomorphisms for \( G \), and the adjunctions \( F_A \dashv G_A \) assemble into adjunctions in \( \mathcal{P}_{\mathcal{D}} \). Conversely, given \( F \dashv G \) in \( \mathcal{P}_{\mathcal{D}} \), we have adjunctions \( F_A \dashv G_A \), while the fact that the unit and counit are transformations of prederivators ensures that the pseudonaturality isomorphism for \( G \) must be (2.11). (This is a particular case of a general fact; see [Kel74].) \( \square \)

As in ordinary category theory, left adjoints preserve colimits (and, more generally, left Kan extensions). The relevant notion of “preserving left Kan extensions” for derivators is the following.

**Definition 2.12.** A morphism of derivators \( F: \mathcal{D}_1 \rightarrow \mathcal{D}_2 \) is **cocontinuous** if for any \( u: A \rightarrow B \) in \( \mathcal{C}_{\mathcal{D}} \), the canonical mate-transformation
\[
(2.13) \quad u_! F_A \rightarrow u_! F_A u^* u_! \Rightarrow u_! u^* F_B u_! \rightarrow F_B u_! 
\]
is an isomorphism.

**Lemma 2.14.** If \( F: \mathcal{D}_1 \rightarrow \mathcal{D}_2 \) is a morphism of derivators such that each functor \( F_A \) has a right adjoint, then \( F \) has a right adjoint if and only if it is cocontinuous.

**Proof.** In this case, (2.13) is the total mate (see Appendix A) of (2.11). \( \square \)

There is of course a dual notion of \( F \) being **continuous.** If \( F: \mathcal{D}_1 \rightarrow \mathcal{D}_2 \) and \( G: \mathcal{D}_2 \rightarrow \mathcal{D}_3 \) are cocontinuous, then by the functoriality of mates with respect to
pasting of the following squares:

$\mathcal{D}_1(A) \xrightarrow{F_A} \mathcal{D}_2(A) \xrightarrow{G_A} \mathcal{D}_3(A)$

$\uparrow u^* \quad \uparrow u^* \quad \uparrow u^*$

$\mathcal{D}_1(B) \xrightarrow{F_B} \mathcal{D}_2(B) \xrightarrow{G_B} \mathcal{D}_3(B)$

their composite $GF: \mathcal{D}_1 \to \mathcal{D}_3$ is also cocontinuous. Similarly, by (Der2) and the functoriality of mates with respect to the following pasting:

$\mathcal{D}_2(u/b) \xleftarrow{\pi_{(u/b)}} \mathcal{D}_2(1) \xrightarrow{F_1} \mathcal{D}_1(1) \xleftarrow{\pi_{(u/b)}} \mathcal{D}_1(B)$

$\uparrow u^* \quad \uparrow u^* \quad \uparrow u^*$

$\mathcal{D}_2(B) \xleftarrow{F_B} \mathcal{D}_2(1) \xrightarrow{F_{(u/b)}} \mathcal{D}_1(u/b) \xleftarrow{\pi_{(u/b)}} \mathcal{D}_1(A)$

$\mathcal{D}_2(A) \xleftarrow{F_A} \mathcal{D}_1(A)$

for $F$ to be cocontinuous it suffices for (2.13) to be an isomorphism when $B = 1$.

**Example 2.15.** Any functor $F: C \to D$ induces a morphism $y(F): y(C) \to y(D)$, and likewise for adjunctions; in fact we have a 2-functor $y: \text{CAT} \to \mathcal{PD}E$ that is an embedding. If $C$ and $D$ are complete and cocomplete, then $F: C \to D$ is continuous or cocontinuous if and only if $y(F)$ is so, in the classical sense of preserving all small limits or colimits. For if $u$ is $\pi_A: A \to 1$, then (2.13) is easily identified with the canonical map $\text{colim}_A \circ F \to F \circ \text{colim}_A$.

**Example 2.16.** Any Quillen adjunction $C \rightleftarrows D$ between model categories induces an adjunction $\mathcal{H}_0(C) \rightleftarrows \mathcal{H}_0(D)$ between their homotopy derivators; see [Gro12, Example 2.10]. In particular, the derived left adjoint is cocontinuous and the derived right adjoint is continuous. This construction can be made into a pseudofunctor as well, and even a “double pseudofunctor” as in [Shu11].

**Example 2.17.** If $\mathcal{D}$ is a derivator, then each morphism $u^*: \mathcal{D}^B \to \mathcal{D}^A$ has a left adjoint $u_!: \mathcal{D}^A \to \mathcal{D}^B$ and a right adjoint $u_*: \mathcal{D}^A \to \mathcal{D}^B$. Example 3.5 will imply that $u_!$ and $u_*$ are morphisms of prederivators, i.e. pseudonatural transformations (see also [Gro12, Example 2.10]). Thus, $u_!$ and $u^*$ are cocontinuous, while $u_*$ and $u^*$ are continuous.

### 3. Homotopy exact squares

Axiom (Der4) deserves a little more discussion. Suppose given any natural transformation in $\text{Cat}$ which lives in a square

$$
\begin{array}{ccc}
D & \xrightarrow{p} & A \\
\downarrow q & & \downarrow u \\
B & \xleftarrow{o} & C.
\end{array}
$$

(3.1)
Then by 2-functoriality of $\mathcal{D}$, we have an induced transformation
$$
\begin{array}{ccc}
\mathcal{D}(C) & \xrightarrow{u^*} & \mathcal{D}(A) \\
v^* & \downarrow & p^* \\
\mathcal{D}(B) & \xrightarrow{q^*} & \mathcal{D}(D).
\end{array}
$$

As summarized in Appendix A, this transformation has mates
\begin{align*}
(3.2) \quad qp^* & \to qp^*u^*u_! \xrightarrow{\alpha^*} q_!q^*v^*v_! \to v^*u_! \quad \text{and} \\
(3.3) \quad u^*v_! & \to p_*p^*u^*v_! \xrightarrow{\alpha^*} p_!q^*v^*v_! \to p_!q^*.
\end{align*}

Axiom (Der4) asserts that when $D = (u/v)$ and $\alpha$ is the canonical transformation in the relevant square, then these two mates are isomorphisms. (Since each is the total mate of the other, one is an isomorphism if and only if the other is.)

**Definition 3.4.** A square (3.1) is **homotopy exact** if the two mate-transformations (3.2) and (3.3) are isomorphisms in any derivator $\mathcal{D}$.

Thus (Der4) says that the universal squares associated to all comma categories are homotopy exact, by definition. By functoriality of mates under pasting (Appendix A), the pasting of two homotopy exact squares is again homotopy exact. On the other hand, axiom (Der2) implies that (3.2) is an isomorphism just when it becomes so after restricting along all functors $b: 1 \to B$, and similarly for (3.3).

Thus, in the presence of (Der2), axiom (Der4) is equivalent to the conjunction of its own special cases when $A$ or $B$ is $1$. In this case, it says intuitively that Kan extensions are calculated “pointwise” in terms of limits and colimits.

Here are some other useful special cases of homotopy exactness.

**Example 3.5.** For any functors $u: A \to B$ and $v: C \to D$, the following (commutative) square is homotopy exact
$$
\begin{array}{ccc}
A \times C & \xrightarrow{1 \times v} & A \times D \\
u \times 1 & \downarrow & u \times 1 \\
B \times C & \xrightarrow{1 \times v} & B \times D.
\end{array}
$$

This means that left and right Kan extension “in one variable” commute with restriction in an unrelated variable, i.e. $(u \times 1)_!(1 \times v)^* \cong (1 \times v)^*(u \times 1)$. See [Gro12, Prop. 2.5] for a proof.

**Example 3.6.** Recall that a functor $v: B \to C$ is called a **fibration** if for any $b \in B$ and $\phi: c \to v(b)$ in $C$, there exists $\psi: b' \to b$ in $B$ such that $v(\psi) = \phi$ and $\psi$ is cartesian. The latter property means that for any $\chi: b'' \to b$ and factorization $v(\chi) = \phi \xi$, there exists a unique $\zeta: b'' \to b'$ such that $\chi = \psi \zeta$ and $v(\zeta) = \xi$. This is equivalent to asking that the evident functor $B \to (C/v)$ has a right adjoint over $C$. We say that a functor $u$ is an **opfibration** if $u^{op}$ is a fibration.

Let us say that $\psi$ is lax cartesian if it satisfies the above universal property only when $\xi = 1_c$, and that $v$ is a **lax fibration** if lax cartesian liftings exist as for a fibration. This is equivalent to asking that for any $c \in C$, the functor $v^{-1}(c) \to (c/v)$ has a right adjoint. It is well-known that a lax fibration is a
fibration just when lax cartesian arrows are closed under composition. Similarly, we have **oplax opfibrations**.

It is shown in [Gro12, Prop. 1.24] that a pullback square

\[
P \xrightarrow{p} A \\
q \downarrow \downarrow \\
B \xrightarrow{v} C
\]

is homotopy exact whenever \(v\) is a fibration or \(u\) is an opfibration. Inspecting the proof reveals that it suffices for \(v\) to be a lax fibration or \(u\) an oplax opfibration.

In particular, if \(u\) is an (oplax) opfibration, then for any \(c \in C\), the following pullback square is homotopy exact

\[
u^{-1}(c) \longrightarrow A \\
\downarrow \downarrow \\
\mathbb{1} \xrightarrow{c} C.
\]

If \(u\) is a *discrete* opfibration, then by definition each \(u^{-1}(c)\) is a discrete category, so that we have an additional isomorphism \((u_! X)_c \cong \coprod_{a(a)=c} X_a\).

**Example 3.7.** If \(u: A \rightarrow B\) is fully faithful, then the square

\[
\begin{array}{ccc}
A & \xrightarrow{u} & B \\
\downarrow \downarrow \\
\mathbb{1} & \xrightarrow{u_*} & \mathbb{1}
\end{array}
\]

is homotopy exact; see [Gro12, Prop. 1.20]. This is equivalent to saying that \(u_l\) and \(u_*\) are fully faithful.

**Example 3.8.** A functor \(f: A \rightarrow B\) is called **homotopy final** if the square

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow \downarrow \\
\mathbb{1} & \xrightarrow{u_*} & \mathbb{1}
\end{array}
\]

is homotopy exact. This means that for any \(X \in \mathcal{D}(B)\), the colimits of \(X\) and of \(f^* X\) agree. Any right adjoint is homotopy final; see [Gro12, Prop. 1.18].

We now give a characterization of homotopy exactness in terms of a couple of simpler notions. Together with results of [Hel88] or [Cis06], this will imply that the notion of homotopy exactness is actually not dependent on the notion of derivator, but can equivalently be defined by reference to any other sufficiently general notion of “homotopy theory”, such as model categories or \((\infty, 1)\)-categories.

**Definition 3.9.** A category \(A\) is **homotopy contractible** if the unique functor \(\pi_A: A \rightarrow \mathbb{1}\) is homotopy final, i.e. the counit

\[
(\pi_A)_!(\pi_A)^* \rightarrow 1_{\mathcal{C}(\mathbb{1})}
\]

is an isomorphism in any derivator.
Intuitively, homotopy contractibility says that colimits of constant $A$-shaped diagrams do nothing. Note that in a represented derivator $y(C)$, the colimit of a constant $A$-shaped diagram is simply a copower by the set of connected components of $A$. Thus, in the represented case, $(\pi_A)_!(\pi_A)^*\to 1_{y(C)(A)}$ is an isomorphism whenever $A$ is connected. However, as we will see, being homotopy contractible is a much stronger condition than being connected.

**Remark 3.11.** If $f: A \to B$ is homotopy final, then $A$ is homotopy contractible if and only if $B$ is. In particular, any category with an initial or terminal object is homotopy contractible.

**Definition 3.12.** A functor $f: A \to B$ is a **homotopy equivalence** if the natural transformation

$$(\pi_A)_!(\pi_A)^* \cong (\pi_B)_!f_!(\pi_B)^* \to (\pi_B)_!(\pi_B)^*$$

is an isomorphism in any derivator.

Intuitively, $f: A \to B$ is a homotopy equivalence if colimits of constant $A$-shaped diagrams and constant $B$-shaped diagrams are the same. Of course, $A$ is homotopy contractible if and only if $\pi_A: A \to 1$ is a homotopy equivalence.

**Remark 3.14.** Any homotopy final functor is a homotopy equivalence, but in general the latter is a weaker notion: it refers only to colimits of constant $B$-shaped diagrams. However, Remark 3.11 does generalize: if $f: A \to B$ is a homotopy equivalence, then $A$ is homotopy contractible if and only if $B$ is.

**Remark 3.15.** The notions of homotopy contractible category and homotopy equivalence functor may appear asymmetric, in that they refer to colimits rather than limits. However, this asymmetry is only apparent, for the total mates of (3.10) and (3.13) are the analogous morphisms for limits, and passing to total mates preserves and reflects invertibility.

The following theorem was proven independently by Heller [Hel88] and Cisinski [Cis06]; see [Cis08] for closely related results. We will not need it in this paper, but the consistency it provides is reassuring. Recall that any category $A$ has a **nerve** $NA$, which is a simplicial set whose $n$-simplices are the strings of $n$ composable arrows in $A$.

**Theorem 3.16.** A functor $f: A \to B$ is a homotopy equivalence if and only if $Nf: NA \to NB$ is a weak homotopy equivalence of simplicial sets.

We now give a characterization of the homotopy exact squares. Let

$$(3.17)$$

$$
\begin{array}{ccc}
D & \xrightarrow{p} & A \\
\downarrow{q} & \downarrow{u} & \\
B & \xrightarrow{o} & C
\end{array}
$$

\alpha
be a square in \( \mathbf{Cat} \). By (Der2) and (Der4), homotopy exactness of (3.17) is equivalent to homotopy exactness of all the pasted squares

\[
\begin{array}{ccc}
(a/D/b) & \rightarrow & \mathbb{1} \\
\downarrow \phi & & \downarrow a \\
(q/b) & \rightarrow & D \\
\downarrow q & & \downarrow p \\
\mathbb{1} & \rightarrow & B \\
\downarrow b & & \downarrow v \\
\end{array}
\]

(3.18)

where the objects of \((a/D/b)\) are triples \((d \in D, a \xrightarrow{\phi} p(d), q(d) \xrightarrow{\psi} b)\). The component of (3.18) at \((d, \phi, \psi)\) is the composite

\[
ua \xrightarrow{u\phi} upd \xrightarrow{\alpha_d} vqd \xrightarrow{v\psi} vb.
\]

Now by (Der4) again, the square

\[
\begin{array}{ccc}
C(ua, vb) & \rightarrow & \mathbb{1} \\
\downarrow \alpha & & \downarrow ua \\
\mathbb{1} & \rightarrow & C \\
\downarrow vb & & \downarrow C
\end{array}
\]

(3.19)

is always homotopy exact, since the discrete category \( C(ua, vb) \) is also the comma category \((ua/vb)\). By the universal property of comma categories, (3.18) factors through (3.19) by a functor \( k_{a,b} : (a/D/b) \to C(ua, vb) \). Thus, by the functoriality of mates, (3.18) is homotopy exact if and only if the induced transformation

\[
\pi_1(k_{a,b}) \ast \pi_1^\ast : \mathbb{1} \xrightarrow{\pi_1} \mathbb{1} \xrightarrow{\pi_1^\ast}
\]

is an isomorphism, where \( \pi : C(ua, vb) \to \mathbb{1} \). But this is just to say that \( k_{a,b} \) is a homotopy equivalence. Thus, we have proven (i) \( \Leftrightarrow \) (iv) of the following theorem.

**Theorem 3.20.** For a square (3.17) in \( \mathbf{Cat} \), the following are equivalent.

(i) The square is homotopy exact, i.e. the mate-transformation \( qp^\ast \to v^\ast u^\ast \) is an isomorphism in any derivator \( \mathcal{D} \).

(ii) As in (i), but only for derivators of the form \( \mathcal{H}o(\mathbf{C}) \) for \( \mathbf{C} \) a model category.

(iii) As in (i), but only for the classical homotopy derivator \( \mathcal{H}o(\mathbf{sSet}) \).

(iv) For each \( a \in A \) and \( b \in B \), the functor \( k_{a,b} \) is a homotopy equivalence.

(v) For each \( a \in A \) and \( b \in B \), the nerve \( Nk_{a,b} \) is a weak homotopy equivalence.

(vi) For each \( a \in A \) and \( b \in B \) and \( \gamma : ua \to vb \), the category

\[
(a/D/b)_\gamma := (k_{a,b})^{-1}(\gamma)
\]

is homotopy contractible.

(vii) For each \( a \in A \) and \( b \in B \) and \( \gamma : ua \to vb \), the nerve \( N(a/D/b)_\gamma \) is a weakly contractible simplicial set.

**Proof.** We have shown (i) \( \Leftrightarrow \) (iv). But by (Der1), the functor \( (k_{a,b})^\ast \) is the product of all the induced functors \( \mathcal{D}(\mathbb{1}) \to \mathcal{D}((a/D/b)_\gamma) \), and likewise for \( (k_{a,b})^! \). Thus (vi) \( \Rightarrow \) (iv).

Similarly, since \( Nk_{a,b} \) is a coproduct of the morphisms \( N\pi : N(a/D/b)_\gamma \to N\mathbb{1} \), we have (v) \( \Leftrightarrow \) (vii). Clearly (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii), while Theorem 3.16 gives (iv) \( \Leftrightarrow \) (v)
and (vi)⇔(vii). Finally, (iii)⇒(v) by the observation that $\mathcal{N}A \simeq (\pi_A)_!(\pi_A)^*(\star)$ in $\mathcal{K}\omega(s\text{Set})$. \hfill $\square$

Once it is known that complete and cocomplete $(\infty, 1)$-categories have underlying derivators, it will be immediate to add a further characterization to Theorem 3.20, since these sit in between derivators and model categories in generality.

In the rest of the paper, we will only need the implication (vi)⇒(i) from Theorem 3.20, whose proof does not depend on Theorem 3.16. The same is true of the “if” direction of the following immediate corollary.

**Corollary 3.21.** A functor $f: A \to B$ is homotopy final if and only if for each $b \in B$, the comma category $(b/f)$ is homotopy contractible.

**Example 3.22.** If $f$ is a lax fibration with homotopy contractible fibers, then the adjunction $f^{-1}(b) \rightleftarrows (b/f)$ implies that $(b/f)$ is also homotopy contractible. Thus $f$ is homotopy final, and in particular a homotopy equivalence.

For instance, if $\Gamma: s\text{Set} \to \text{Cat}$ denotes the “category of simplices” functor, then for any category $A$ we have a lax fibration $\gamma: \Gamma \mathcal{N}A \to A$, which picks out the first object in a string of composable arrows. Its fibers have terminal objects, hence are homotopy contractible; thus $\gamma$ is homotopy final, and in particular a homotopy equivalence. This is a key step in Heller’s proof of Theorem 3.16 (see [Hel88, Lemma 3.1], where lax fibrations are called merely “fibrations”). We will also have a use for it in §8.

In general, the easiest way to verify homotopy contractibility of a small category is often to connect it to 1 with a zigzag of adjunctions. Occasionally, it can be quicker to invoke Theorem 3.20(vii), but we will never need to do so.

4. On Kan extensions and cartesian squares

In this section we collect a number of additional useful lemmas for manipulating Kan extensions, limits, and colimits in derivators. Many were proven in [Gro12].

Recall that a functor $u: A \to B$ is called a **sieve** if it is fully faithful, and for any morphism $b \to u(a)$ in $B$, there exists an $a' \in A$ with $u(a') = b$. By Example 3.7, left or right Kan extension along a sieve or cosieve is fully faithful.

**Lemma 4.1** ([Gro12, Prop. 1.23]). If $u: A \to B$ is a sieve and $\mathcal{D}$ is a derivator, then a diagram $X \in \mathcal{D}(B)$ is in the essential image of $u_!$ if and only if $X_b \in \mathcal{D}(1)$ is a terminal object for all $b \notin u(A)$. Dually, if $u$ is a cosieve, then $X \in \mathcal{D}(B)$ is in the essential image of $u^*$ if and only if $X_b$ is an initial object for all $b \notin u(A)$.

**Remark 4.2.** In particular, Lemma 4.1 implies that if $u: A \to B$ is a sieve and we have $X, Y \in \mathcal{D}(B)$ such that $X_b$ and $Y_b$ are terminal for $b \notin u(A)$, and moreover $u^*X \cong u^*Y$, then we have $X \cong u_!u^*X \cong u_!u^*Y \cong Y$. This fact and its dual are very convenient, because one of the trickiest parts of working with derivators is that coherent diagrams which “look the same” (have the same underlying diagram) may not in fact be isomorphic. In the context of the inclusion of a (co)sieve, Lemma 4.1 tells us that if we know that two coherent diagrams become isomorphic when restricted to their “nontrivial parts”, then the whole diagrams are in fact isomorphic.

We also have a version of the familiar theorem from category theory that limits and colimits in functor categories may be computed pointwise. The specialization of this to Kan extensions along fully faithful functors has an especially nice form.
Lemma 4.3 ([Gro12, Corollary 2.6]). If \( u: A \to B \) is fully faithful, then \( X \in \mathcal{D}^C(B) \) lies in the essential image of \( u \) (with respect to \( \mathcal{D}^C \)) if and only if for each \( c \in C \), the diagram \( X_c \in \mathcal{D}(B) \) lies in the essential image of \( u \) (with respect to \( \mathcal{D} \)).

We now give a criterion to detect when certain sub-diagrams of a Kan extension are colimiting cocones. This generalizes a theorem of [Fra96].

For any category \( A \), let \( A \uparrow \) denote the result of freely adjoining a new terminal object to \( A \). Call the new terminal object \( \infty \) and the inclusion \( i: A \hookrightarrow A \uparrow \). Then the following square is homotopy exact by (Der4)

\[
\begin{array}{c}
A \\
\downarrow \downarrow \\
1 \to A \uparrow \\
\end{array}
\]

(4.4)

Thus, left Kan extensions from \( A \) into \( A \uparrow \) are an alternative way to compute and characterize colimits over \( A \). We may refer to a coherent diagram in the image of \( i! \) as a colimiting cocone.

The proof of the following lemma is an immediate generalization of [Gro12, Prop. 3.10]. Its hypotheses may seem technical, but in practice, this is the lemma we reach for most often whenever it seems "obvious" that a certain cocone is colimiting.

Lemma 4.5. Let \( A \in \mathsf{Cat} \), and let \( u: C \to B \) and \( v: A \uparrow \to B \) be functors. Suppose that there is a full subcategory \( B' \subseteq B \) such that

- \( u(C) \subseteq B' \) and \( v(\infty) \notin B' \);
- \( vi: A \to B \) factors through the inclusion \( B' \subseteq B \); and
- the functor \( A \to B'/v(\infty) \) induced by \( v \) has a left adjoint.

Then for any derivator \( \mathcal{D} \) and any \( X \in \mathcal{D}(C) \), the diagram \( v^* u_! X \) is in the essential image of \( i_! \). In particular, \( (v^* u_! X)_\infty \) is the colimit of \( i^* v^* u_! X \).

Proof. We want to show that the mate-transformation associated to the square

\[
\begin{array}{c}
A \\
\downarrow \downarrow \\
1 \to A \uparrow \\
\end{array}
\]

is an isomorphism when evaluated at \( u_! X \). By [Gro12, Lemma 1.21], it suffices to show this for the pasted square

\[
\begin{array}{c}
A \\
\downarrow \downarrow \\
1 \to A \uparrow \\
\end{array}
\]

But this square is also equal to the pasting composite

\[
\begin{array}{c}
A \\
\downarrow \downarrow \\
1 \to B'/v(\infty) \\
\downarrow \downarrow \\
B \\
\end{array}
\]

Therefore, the mate-transformation is an isomorphism when evaluated at \( u_! X \), completing the proof.

\[
\begin{array}{c}
A \\
\downarrow \downarrow \\
1 \to B \\
\end{array}
\]

(4.4)
Now the mates associated to each of these three squares are isomorphisms: the left-hand square by the fact that right adjoints are homotopy final, the middle one by (Der4), and the right-hand one because $B' \hookrightarrow B$ is fully faithful.

Franke’s version of this was the special case for cocartesian squares. Let $\Box$ denote the category $2 \times 2$

\[
\begin{array}{c}
(0,0) \rightarrow (0,1) \\
\downarrow \quad \downarrow \\
(1,0) \rightarrow (1,1).
\end{array}
\]

Let $\Gamma$ and $\Upsilon$ denote the full subcategories $\Box \setminus \{(1,1)\}$ and $\Box \setminus \{(0,0)\}$, respectively, with inclusions $i_\Gamma : \Gamma \hookrightarrow \Box$ and $i_\Upsilon : \Upsilon \hookrightarrow \Box$. Since $i_\Gamma$ and $i_\Upsilon$ are fully faithful, Example 3.7 implies that the left and right Kan extension functors along $i_\Gamma$ and $i_\Upsilon$ are also fully faithful.

**Definition 4.6.** A coherent diagram $X \in \mathcal{D}(\Box)$ is cartesian if it is in the essential image of $(i_\Upsilon)_*$, and cocartesian if it is in the essential image of $(i_\Gamma)_!$.

Taking $A = \Gamma$ in (4.4) implies that if $X \in \mathcal{D}(\Gamma)$ looks like $(y \leftarrow x \rightarrow z)$ and $w = (\pi_\Gamma)_!(X)$ is its pushout, then there is a cocartesian square

\[
\begin{array}{c}
x \rightarrow z \\
\downarrow \quad \downarrow \\
y \rightarrow w
\end{array}
\]

and conversely, if there is a cocartesian square (4.7), then $w \cong (\pi_\Gamma)_!(X)$.

**Remark 4.8.** If we have two cocartesian squares $X, Y \in \mathcal{D}(\Box)$ such that $(i_\Gamma)_!X \cong (i_\Gamma)_!Y$, then in fact $X \cong Y$, and in particular $X_{1,1} \cong Y_{1,1}$.

Since cartesian and cocartesian squares play an essential role in the theory of pointed and stable derivators to be described in the subsequent sections, it is very useful to be able to identify cocartesian squares in larger diagrams. This is the purpose of Franke’s lemma, which we can now derive.

**Lemma 4.9.** Suppose $u : C \rightarrow B$ and $v : \Box \rightarrow B$ are functors, with $v$ injective on objects, and let $b = v(1,1) \in B$. Suppose furthermore that $b \notin u(C)$, and that the functor $\Gamma \rightarrow (B \setminus b)/b$ induced by $v$ has a left adjoint. Then for any derivator $\mathcal{D}$ and any $X \in \mathcal{D}(C)$, the square $v^*u_!X$ is cocartesian.

**Proof.** Since $\Box \cong (\Gamma)^p$, we can apply Lemma 4.5 with $A = \Gamma$ and $B' = B \setminus b$. (Injectivity of $v$ is needed to ensure that $v(A) \subseteq B'$.)

Franke’s lemma immediately implies the usual “pasting lemma” for cocartesian squares. Let $\Box$ denote the category $2 \times 3$

\[
\begin{array}{c}
(0,0) \rightarrow (0,1) \rightarrow (0,2) \\
\downarrow \quad \downarrow \quad \downarrow \\
(1,0) \rightarrow (1,1) \rightarrow (1,2).
\end{array}
\]

Let $i_{jk}$ denote the functor $\Box \rightarrow \Box$ induced by the identity of 2 on the first factors and the functor $2 \rightarrow 3$ on the second factors which sends 0 to $j$ and 1 to $k$. 

\[
\begin{array}{c}
\end{array}
\]
Lemma 4.10 ([Gro12, Prop. 3.13]). If $X \in \mathcal{D}(\square)$ is such that $i_{\square}^0X$ is cocartesian, then $i_{\square}^2X$ is cocartesian if and only if $i_{\square}^1X$ is cocartesian.

Proof. Let $A$ be the full subcategory $00-01-02-10$ of $\square$, with $j: A \hookrightarrow \square$ the inclusion. Then Lemma 4.9 implies that $i_{01}^0j^*Y$ and $i_{02}^0j^*Y$ and $i_{12}^0j^*Y$ are cocartesian for any $Y \in \mathcal{D}(A)$. Thus, for $X \in \mathcal{D}(\square)$ with $i_{\square}^0X$ cocartesian, it will suffice to show that cocartesianness of $i_{\square}^0X$ and of $i_{\square}^2X$ each imply that the counit $\epsilon: j_!j^*X \rightarrow X$ is an isomorphism. Since $j$ is fully faithful, by (Der2) it suffices to check this at the objects $(1,1)$ and $(1,2)$. However, cocartesianness of $i_{\square}^0X$ implies that $\epsilon_{11}$ is an isomorphism, while cocartesianness of $i_{\square}^0X$ and of $i_{\square}^2X$ each imply that $\epsilon_{12}$ is an isomorphism. □

Here is another useful consequence of the general form of Lemma 4.5; it will again be used in the proof of Theorem 6.10.

Corollary 4.11. Coproducts in a derivator are the same as pushouts over the initial object. More precisely, for any objects $x$ and $y$ there is a cocartesian square

$$
\begin{array}{ccc}
\emptyset & \longrightarrow & x \\
\downarrow & & \downarrow \\
y & \longrightarrow & x \sqcup y
\end{array}
$$

Proof. Taking $B = \square$ and $C = B' = A = \{(1,0), (0,1)\}$ and $A^p = j$ in Lemma 4.5, with $X = (x,y) \in \mathcal{D}(C) \cong \mathcal{D}(\square) \times \mathcal{D}(1)$, yields a square of the desired form; its lower-right corner is $x \sqcup y$ by Lemma 4.5, and its upper-left corner is initial by (Der4). And it is cocartesian since the left Kan extension from $C$ to $B$ factors through $\square$. □

Finally, the following lemma says that squares which are constant in one direction are (co)cartesian. For the rest of this paper this observation will be used without comment.

Lemma 4.12 ([Gro12, Prop. 3.12(2)]). Let $\pi_2: \square \rightarrow 2$ denote a projection (either one). Then any square in the image of $(\pi_2)^*$ is cartesian and cocartesian.

5. Pointed derivators

In this and the next section we recall the definitions of pointed and stable derivators, as well as the associated theory of cofiber and fiber sequences and distinguished triangles. Most of these two sections is from [Gro12], but we also establish a couple new results about stable derivators and their triangulations (whose analogues for model categories are already known). In particular, we show that cocartesian squares induce Mayer-Vietoris triangles and that a pointed derivator is stable if and only if the suspension is an equivalence.

Definition 5.1. A derivator $\mathcal{D}$ is pointed if the category $\mathcal{D}(\square)$ has a zero object (an object which is both initial and terminal).

Since $\pi^*_A: \mathcal{D}(\square) \rightarrow \mathcal{D}(A)$ is both a left and a right adjoint, it preserves zero objects. Hence, in a pointed derivator each category $\mathcal{D}(A)$ also has a zero object.

Examples 5.2. A complete and cocomplete category $\mathcal{C}$ is pointed if and only if $y(\mathcal{C})$ is so. If a model category $\mathcal{C}$ is pointed, then so is $Ho(\mathcal{C})$. Finally, if $\mathcal{D}$ is pointed, so are $\mathcal{D}^B$ and $\mathcal{D}^{op}$. 
Lemma 4.1 is especially important for pointed derivators, in which case its two characterizations become identical since initial and terminal objects are the same. Thus, in this case, when $u$ is a sieve we refer to $u_*$ as an extension by zero functor, and similarly for $u!$ when $u$ is a cosieve.

In a pointed derivator $\mathcal{D}$, the suspension functor $\Sigma: \mathcal{D}(\mathbf{1}) \to \mathcal{D}(\mathbf{1})$ is the composite

$$\mathcal{D}(\mathbf{1}) \xrightarrow{(0,0)_*} \mathcal{D}(\mathbf{1}) \xrightarrow{(1,1)_*} \mathcal{D}(\mathbf{1}).$$

Since $(0,0)$ is a sieve in $\mathbf{1}$, the functor $(0,0)_*$ is an extension by zero; thus for any $x \in \mathcal{D}(\mathbf{1})$ we have a co-cartesian square of the form

$$x \to 0 \downarrow \downarrow 0 \to \Sigma x. \tag{5.3}$$

More generally, any co-cartesian square of the form

$$x \to 0 \downarrow \downarrow 0 \to w \tag{5.4}$$

induces a canonical isomorphism $w \cong \Sigma x$. Note that by Remarks 4.2 and 4.8, any two such co-cartesian squares containing the same object $x$ are isomorphic.

It is very important to note that if we restrict a co-cartesian square (5.4) along the automorphism $\sigma: \square \to \square$ which swaps $(0,1)$ and $(1,0)$, we obtain a different co-cartesian square (with the same underlying diagram), and hence a different isomorphism $w \cong \Sigma x$. The relationship between the two is the following.

Lemma 5.5 ([Gro12, Prop. 4.12]). In any pointed derivator, $\Sigma x$ is a cogroup object, and the composite $\Sigma x \Rightarrow w \Rightarrow \Sigma x$ of the two isomorphisms arising from a co-cartesian square (5.4) and its $\sigma$-transpose gives the “inversion” morphism of $\Sigma x$.

We generally write this cogroup structure additively, and thus denote this morphism by “$-1$”.

Remark 5.6. This may seem strange, but it is not really a new sort of phenomenon. Already in ordinary category theory, a universal property is not merely a property of an object, but of that object equipped with extra data, and changing the data can give the same object the same universal property in more than one way. For instance, a cartesian product $A \times A$ comes with two projections $\pi_1, \pi_2: A \times A \Rightarrow A$ exhibiting it as a product of $A$ and $A$, whereas switching these two projections exhibits the same object as a product of $A$ and $A$ in a different way. In that case, the induced automorphism of $A \times A$ is the symmetry, $(a,b) \mapsto (b,a)$. In the case of suspensions, the “universal property data” consists of a co-cartesian square (5.3), and transposing the square is analogous to switching the projections.

The suspension functor of $\mathcal{D}^{op}$ is called the loop space functor of $\mathcal{D}$ and denoted $\Omega$. By definition, $\Omega x$ comes with a coherent diagram of shape $\square$ in $\mathcal{D}^{op}$. In $\mathcal{D}$, this is a diagram of shape $\square^{op}$, hence looks like

$$x \leftarrow 0 \uparrow \uparrow 0 \leftarrow \Omega x. \tag{5.7}$$
Restricting this along the isomorphism \( \tau: \square \cong \square^{op} \) which fixes \((0,1)\) and \((1,0)\) and exchanges \((0,0)\) with \((1,1)\), we obtain a cartesian square in \(\mathcal{D}\) of the form

\[
\begin{array}{ccc}
\Omega x & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & x.
\end{array}
\]

(5.8)

It may seem terribly pedantic to distinguish between diagrams of shape \(\square\) and \(\square^{op}\), but we find that it helps avoid confusion with minus signs. In particular, if instead of the isomorphism \(\tau\) we used the isomorphism \(\tau \sigma\) (which “rotates” (5.7) to make it look like (5.8)), we would obtain a different cartesian square of shape (5.8) in \(\mathcal{D}\). The difference would, again, be the inversion map \(-1: \Omega x \to \Omega x\).

**Lemma 5.9** ([Gro12, Prop. 3.17]). There is an adjunction \(\Sigma \dashv \Omega\).

**Proof.** Since \(i\) is fully faithful, \((i^*)\) exhibits \(\mathcal{D}(\square)\) as equivalent to the coreflective subcategory of \(\mathcal{D}(\square)\) whose objects are the cocartesian squares (the coreflection being \((i^*)^*\)). If we write \(\mathcal{D}(\square)_{00}\) and \(\mathcal{D}(\square)_{00}\) for the full subcategories of each on the diagrams \(X\) such that \(X_{0,1}\) and \(X_{1,0}\) are zero objects, then both \((i^*)^*\) and \((i^*)\) preserve these subcategories, and so \(\mathcal{D}(\square)_{00}\) is likewise equivalent to the coreflective subcategory of cocartesian squares in \(\mathcal{D}(\square)_{00}\). Moreover, by Lemma 4.1, \((0,0)_*: \mathcal{D}(\square)_{00} \to \mathcal{D}(\square)_{00}\) is an equivalence.

A dual argument using \(\mathcal{D}(\square)_{00}\) shows that \(\mathcal{D}(\square)_{00}\) is also equivalent to the reflective subcategory of cartesian squares in \(\mathcal{D}(\square)_{00}\). Thus, we have a composite adjunction \(\mathcal{D}(\square)_{00} \rightleftharpoons \mathcal{D}(\square)_{00} \rightleftharpoons \mathcal{D}(\square),\) which is easily verified to be \(\Sigma \dashv \Omega\). \(\square\)

The **cofiber functor** \(\text{cof}: \mathcal{D}(2) \to \mathcal{D}(2)\) in a pointed derivator is the composite

\[
\mathcal{D}(2) \xrightarrow{(0,-)_*} \mathcal{D}(\square) \xrightarrow{(i^*)^*} \mathcal{D}(\square^*). \]

Here \((0,-): 2 \to \square\) indicates the inclusion as the objects with first coordinate 0, and similarly for \((-,1): 2 \to \square\). Since \((0,-)\) is a sieve, \((0,-)_*\) is an extension by zero; thus by stopping after the first two functors we have a cocartesian square

\[
\begin{array}{ccc}
x & \xrightarrow{f} & y \\
\downarrow & & \downarrow \\
0 & \xrightarrow{\text{cof}(f)} & z.
\end{array}
\]

(5.10)

By Remarks 4.2 and 4.8, any two cocartesian squares (5.10) with the same underlying object \((x \xrightarrow{f} y)\) of \(\mathcal{D}(2)\) are isomorphic.

**Remark 5.11.** In a strong pointed derivator, every morphism in \(\mathcal{D}(\square)\) underlies some object of \(\mathcal{D}(2)\). Thus, we can construct “the” cofiber of any morphism in \(\mathcal{D}(\square)\) by first lifting it to an object of \(\mathcal{D}(2)\). By Remark 2.7, the result is independent of the chosen lift, up to non-unique isomorphism.

Dually, the **fiber functor** \(\text{fib}: \mathcal{D}(2) \to \mathcal{D}(2)\) is the cofiber functor of \(\mathcal{D}^{op}\), which can be identified with the composite

\[
\mathcal{D}(2) \xrightarrow{(-1)_*} \mathcal{D}(\square) \xrightarrow{(i_*)^*} \mathcal{D}(\square^*) \xrightarrow{(0,-)_*} \mathcal{D}(2) \xrightarrow{(1_*)^*} \mathcal{D}(\square) \xrightarrow{(i^*)^*} \mathcal{D}(2) \xrightarrow{(-1)_*} \mathcal{D}(\square^*) \xrightarrow{(0,-)_*} \mathcal{D}(2).
\]
so that we have a cartesian square

\[
\begin{array}{ccc}
   w & \xrightarrow{\text{fib}(f)} & x \\
   \downarrow & & \downarrow f \\
   0 & \rightarrow & y.
\end{array}
\]

**Lemma 5.12** ([Gro12, Prop. 3.20]). There is an adjunction \(\text{cof} \dashv \text{fib}\).

**Proof.** Repeat the proof of Lemma 5.9 using the subcategory \(\mathcal{D}(\square)_0\) of squares \(X\) such that \(X_{1,0}\) is a zero object. \(\Box\)

In a pointed derivator \(\mathcal{D}\), we define a **cofiber sequence** to be a coherent diagram of shape \(\square = 2 \times 3\) whose \((0, 2)\)- and \((1, 0)\)-entries are zero objects

\[
\begin{array}{ccc}
   x & \xrightarrow{f} & y & \rightarrow & 0 \\
   \downarrow & & \downarrow g & & \downarrow \\
   0 & \rightarrow & z & \rightarrow & h & \rightarrow & w
\end{array}
\]

and in which both squares are cocartesian. Recall that \(i_{jk}\) denotes the functor \(\square \rightarrow \square\) induced by the identity of 2 on the first factors and the functor 2 \(\rightarrow\) 3 on the second factors which sends 0 to \(j\) and 1 to \(k\). Then a cofiber sequence is an \(X \in \mathcal{D}(\square)\) such that \(X_{(0,2)}\) and \(X_{(1,0)}\) are zero objects and \(i^*_{01}X\) and \(i^*_{12}X\) are cocartesian. By Lemma 4.10, \(i^*_{02}X\) is also cocartesian, and therefore induces an isomorphism \(w \cong \Sigma x\). (As always, \(\sigma^*i^*_{02}X\) is also cocartesian, but would induce the opposite isomorphism \(w \cong \Sigma x\).) Of course, by restricting to the two cocartesian squares, we also obtain canonical isomorphisms \(g \cong \text{cof}(f)\) and \(h \cong \text{cof}(g)\).

The identification of \(w\) with \(\Sigma x\) can also be made functorial.

**Lemma 5.13.** The functor \(\text{cof}^3 : \mathcal{D}(2) \rightarrow \mathcal{D}(2)\) is naturally isomorphic to the suspension functor \(\Sigma\) of \(\mathcal{D}^2\).

**Proof.** Let \(A\) be the full subcategory of \(3 \times 3\) which omits \((2, 0)\). Using a combination of extension by zero functors and left Kan extensions, we have a functor \(\mathcal{D}(2) \rightarrow \mathcal{D}(A)\) which sends \(x \xrightarrow{f} y\) to a diagram of the following form

\[
(5.14)
\]

\[
\begin{array}{ccc}
   x & \xrightarrow{f} & y & \rightarrow & 0_2 \\
   \downarrow & & \downarrow g & & \downarrow \\
   0_1 & \rightarrow & z & \rightarrow & h & \rightarrow & w \\
   \downarrow & & \downarrow h & & \downarrow k & & \downarrow \\
   0_3 & \rightarrow & w & \rightarrow & v.
\end{array}
\]

(Ignore the subscripts for now; all objects denoted \(0_k\) are zero objects.) Lemma 4.9 implies that all squares and rectangles in this diagram are cocartesian. Thus we have a canonical identification of \(g \in \mathcal{D}(2)\) with \(\text{cof}(f)\), and similarly of \(h\) and \(k\) with \(\text{cof}^2(f)\) and \(\text{cof}^3(f)\).
Now let $C = 2^3$ be the shape of a cube, and let $q: C \to A$ be the functor such that $q^*$ of (5.14) has the following form

Here the subscripts match those in (5.14) to indicate the definition of $q$ precisely. This cube may be regarded as a coherent square in $\mathcal{D}$ (with the 2-direction going left-to-right). Moreover, since its left and right faces are cocartesian in $\mathcal{D}$, by Lemma 4.3 it is cocartesian in $\mathcal{D}^2$. Thus, it naturally identifies $k \cong \text{cof}^3(f)$ with $\Sigma(f)$. □

Remark 5.15. The identification of $k$ with $\Sigma(f)$ in the proof of Lemma 5.13 mandates that we identify $w$ and $v$ with $\Sigma x$ and $\Sigma y$ using the cocartesian squares respectively. Of course, if we were to instead use the transpose of one of these squares, then $k$ would instead be identified with $-\Sigma f$. This is exactly what happens in the proof in [Gro12, Theorem 4.16] that distinguished triangles can be "rotated" (axiom (T2) of a triangulated category).

We define a fiber sequence in $\mathcal{D}$ to be a cofiber sequence in $\mathcal{D}^\text{op}$. Thus, it is a diagram of shape $\square$ in $\mathcal{D}^\text{op}$, which looks like

By restricting along the "rotation" isomorphism $\rho: \square \cong \square^\text{op}$, we can draw this as a diagram of shape $\square$ in $\mathcal{D}$

in which both squares are cartesian. As before, it follows that the outer rectangle is also cartesian, and hence we can identify $w$ with the loop space object $\Omega z$. Note, though, that the isomorphism $\square \cong \square^\text{op}$ induced on the outer rectangle by $\rho$ is $\tau \sigma$, not $\tau$ — whereas $\tau$ itself cannot be extended to an isomorphism $\square \cong \square^\text{op}$.
6. Stable derivators

In contrast to pointedness, which at least makes sense in the representable case (even if suspensions and loops are generally not very interesting there), stability is entirely a homotopical notion. The following definition is the most convenient one, but as we will see it is possible to simplify it.

**Definition 6.1.** A pointed derivator $\mathcal{D}$ is stable if its classes of cartesian and cocartesian squares coincide.

We sometimes refer to these squares as bicartesian. In particular, in a stable derivator, the bicartesian square (5.3) induces an isomorphism $x \cong \Omega \Sigma x$, and similarly (5.8) induces an isomorphism $\Sigma \Omega x \cong x$. Thus, the adjunction $\Sigma \dashv \Omega$ is an equivalence. Similarly, $\text{cof} \dashv \text{fib}$ is also an equivalence. We will see below (Theorem 6.10) that each of these properties actually characterizes stable derivators. It also follows that in a stable derivator each category $\mathcal{D}(A)$ is additive (though this does not characterize the stable derivators); see [Gro12, Prop. 4.7].

**Example 6.2.** A stable model category is generally defined to be a pointed model category whose suspension functor is an equivalence of homotopy categories. However, it is also known that this implies that homotopy pushout squares and homotopy pullback squares coincide; see [Hov99, Remark 7.1.12]. (Our Theorem 6.10 is a generalization of this fact to derivators.) Thus, any stable model category gives rise to a stable derivator. Stable model categories abound in homological algebra (unbounded chain complexes) and stable homotopy theory (all types of spectra).

In a stable derivator, if we have a cofiber sequence

\begin{equation}
\begin{array}{ccc}
x & \overset{f}{\rightarrow} & y \\
\downarrow & & \downarrow \\
0 & \overset{g}{\rightarrow} & z \\
\downarrow & & \downarrow \\
0 & \overset{h'}{\rightarrow} & w,
\end{array}
\end{equation}

then we say that the induced string of composable arrows in $\mathcal{D}(1)$

\begin{equation}
x \overset{f}{\rightarrow} y \overset{g}{\rightarrow} z \overset{h'}{\rightarrow} \Sigma x
\end{equation}

is a distinguished triangle. Here $h'$ is the composite $z \overset{h}{\rightarrow} w \overset{\sim}{\rightarrow} \Sigma x$, the isomorphism being induced by the outer rectangle of (6.3). Note that a distinguished triangle is an incoherent diagram, i.e. an object of $\mathcal{D}(1)^3$ rather than $\mathcal{D}(1)$. As usual, we also extend the term distinguished triangle to any such incoherent diagram which is isomorphic to one obtained in this way.

**Theorem 6.4** ([Gro12, Theorem 4.16]). If $\mathcal{D}$ is a strong, stable derivator, then the suspension functor and distinguished triangles defined above make $\mathcal{D}(1)$ into a triangulated category in the sense of Verdier.

**Remark 6.5.** The assumption that $\mathcal{D}$ is strong is necessary for this theorem because the triangulation axioms for $\mathcal{D}(1)$ refer only to morphisms of $\mathcal{D}(1)$ (having no other option), whereas to prove the axioms we need to lift those morphisms to objects of $\mathcal{D}(2)$. For instance, for the axiom which says that any morphism $f: x \rightarrow y$ occurs in a distinguished triangle, we need $f$ to be an object of $\mathcal{D}(2)$ so as to be able to extend it to a cofiber sequence.
It is crucial that in passing from cofiber sequences to distinguished triangles, we use the isomorphism \( w \cong \Sigma x \) obtained from the outer rectangle of (6.3) and not its \( \sigma \)-transpose. In particular, this implies that although fiber sequences and cofiber sequences essentially coincide in a stable derivator (modulo \( \rho^* \)), they do not induce the same notion of “distinguished triangle”. This is expressed by the following lemma, whose analogue for homotopy categories of stable model categories is well-known (see e.g. [Hov99, Theorem 7.1.11]).

**Lemma 6.6.** If \( \mathcal{D} \) is a stable derivator, then the distinguished triangles in \( \mathcal{D}^{\text{op}}(1) \) are the negatives of those in \( \mathcal{D}(1) \).

Recall that the negative of a triangulation is obtained by negating an odd number of the morphisms in each given distinguished triangle.

**Proof.** Suppose \( X \in \mathcal{D}(\square) \) is a cofiber sequence in \( \mathcal{D} \) that looks like (6.3). Then since cocartesian squares in \( \mathcal{D} \) are also cartesian, \( \rho^* X \in \mathcal{D}(\square^{\text{op}}) = \mathcal{D}^{\text{op}}(\square)^{\text{op}} \) is a fiber sequence in \( \mathcal{D} \), i.e. a cofiber sequence in \( \mathcal{D}^{\text{op}} \). It therefore induces a distinguished triangle in \( \mathcal{D}^{\text{op}} \), which interpreted in \( \mathcal{D} \) looks like

\[
(6.7) \quad w \leftarrow^h z \leftarrow^g y \leftarrow^{f'} \Omega w.
\]

Since \( \Sigma \) is inverse to \( \Omega \), we can turn this around and write it as

\[
\Omega w \xrightarrow{f'} y \xrightarrow{g} z \xrightarrow{h} \Sigma \Omega w
\]

and then use (either) isomorphism \( x \cong \Omega w \) to write it as

\[
x \xrightarrow{f'} y \xrightarrow{g} z \xrightarrow{h} \Sigma x.
\]

However, because in (6.7) \( x \) is identified with \( \Omega w \) (the suspension of \( w \) in \( \mathcal{D}^{\text{op}} \)) not via the outer cartesian rectangle of (6.3), but its \( \sigma \)-transpose (since \( \rho \) restricts to \( \tau \sigma \) on the outer rectangle of \( \square \)), we have \( f' = -f \) and not \( f \). \( \square \)

The following result is also known to be true for homotopy categories of stable model categories (see [May01, Lemma 5.7]).

**Proposition 6.8.** In a stable derivator \( \mathcal{D} \), if we have a cocartesian square

\[
\begin{array}{ccc}
x & \xrightarrow{f} & y \\
\downarrow g & & \downarrow j \\
z & \xrightarrow{k} & w
\end{array}
\]

then there is a cocartesian square

\[
\begin{array}{ccc}
x & \xrightarrow{(-f,g)} & y \oplus z \\
\downarrow & & \downarrow \left[ j,k \right] \\
0 & \xrightarrow{} & w
\end{array}
\]

and hence a distinguished triangle

\[
x \xrightarrow{(-f,g)} y \oplus z \xrightarrow{\left[ j,k \right]} w \xrightarrow{} \Sigma x.
\]
The proof of Proposition 6.8 is long and not particularly enlightening, so we postpone it to Appendix B.

Remark 6.9. Inspecting the proof in [Gro12, Theorem 4.16] that strong stable derivators are triangulated, we see that with Proposition 6.8 and Lemma 4.9 we can conclude that such a triangulation is always strong in the sense of [May01, Definition 3.8]. (This use of “strong” is unrelated to the notion of a derivator being strong.)

Finally, we show that stable derivators have a simpler characterization.

Theorem 6.10. For a pointed derivator $\mathcal{D}$, the following are equivalent.

(i) $\mathcal{D}$ is stable, i.e. a square

\[
\begin{array}{cc}
x & \rightarrow & y \\
\downarrow & & \downarrow \\
z & \rightarrow & w
\end{array}
\]

is cartesian if and only if it is cocartesian.

(ii) As in (i), but only when $z$ is a zero object.

(iii) The adjunction $\text{cof} \dashv \text{fib}$ is an equivalence.

(iv) As in (i), but only when $y$ and $z$ are zero objects.

(v) The adjunction $\Sigma \dashv \Omega$ is an equivalence.

Proof. Clearly (i)$\Rightarrow$(ii)$\Rightarrow$(iv). By the construction of the adjunctions $\Sigma \dashv \Omega$ and $\text{cof} \dashv \text{fib}$ in Lemmas 5.9 and 5.12, we easily deduce (ii)$\Leftrightarrow$(iii) and (iv)$\Leftrightarrow$(v).

On the other hand, assuming (v), the suspension functor of $\mathcal{D}$ is an equivalence, and therefore (using Lemma 4.3) the suspension functor of $\mathcal{D}^\square$ is an equivalence. Since $\text{cof}^3 = \Sigma$, this implies that $\text{cof}$ is also an equivalence (e.g. by using the “two-out-of-six property” for equivalences); hence (iii) holds. (This argument can be found in [Hel97], among other places.)

It remains to show (iii)$\Rightarrow$(i), and here we can mostly mimic the proof of [Lur, 1.1.3.4]. Let $X \in \mathcal{D}(\square)$ be of the form $z \leftarrow x \rightarrow y$; we want to show that the cocartesian square $(\nu_\square)X$ is also cartesian. (The dual argument will be identical.)

Now $X$ can be left extended by zero to a diagram of the following form

\[
\begin{array}{cc}
0 & \\
\downarrow & \\
z & \leftarrow \rightarrow & y
\end{array}
\]

Let $B$ denote the shape of (6.11), and let $A$ be the category $(\cdot \leftarrow \cdot \rightarrow \cdot \leftarrow \cdot \rightarrow \cdot)$. Then there is a functor $r: A^\square \times \Gamma \rightarrow B$ such that if $Y$ is (6.11), then $r^*Y$ has the form shown in Figure 1. It is straightforward to conclude that each vertical level of this diagram is in the image of $(i_A)_!$. Therefore, by Lemma 4.3, the whole diagram is in the image of $(i_A \times 1_\Gamma)_!$. It follows that $(\pi_A)_!(i_A \times 1_\Gamma)^* r^*Y \cong X$, where $\pi_A$ denotes the projection $A \times \Gamma \rightarrow \Gamma$. That is, if we view $(i_A \times 1_\Gamma)^* r^*Y$ as an $A$-shaped diagram in $\mathcal{D}$, then its colimit is $X$.

Now $(\nu_\square)_!: \mathcal{D} \rightarrow \mathcal{D}^\square$ is a cocontinuous morphism of derivators, and hence preserves this colimit. Therefore, $(1_A \times (\nu_\square)_!)((i_A \times 1_\Gamma)^* r^*Y) \in \mathcal{D}^\square(A)$ has colimit $(\nu_\square)_! X$. Therefore, (1) $\Rightarrow$ (i) and (ii) $\Rightarrow$ (iv). By the construction of the adjunctions $\Sigma \dashv \Omega$ and $\text{cof} \dashv \text{fib}$ in Lemmas 5.9 and 5.12, we easily deduce (ii)$\Leftrightarrow$(iii) and (iv)$\Leftrightarrow$(v).

On the other hand, assuming (v), the suspension functor of $\mathcal{D}$ is an equivalence, and therefore (using Lemma 4.3) the suspension functor of $\mathcal{D}^\square$ is an equivalence. Since $\text{cof}^3 = \Sigma$, this implies that $\text{cof}$ is also an equivalence (e.g. by using the “two-out-of-six property” for equivalences); hence (iii) holds. (This argument can be found in [Hel97], among other places.)

It remains to show (iii)$\Rightarrow$(i), and here we can mostly mimic the proof of [Lur, 1.1.3.4]. Let $X \in \mathcal{D}(\square)$ be of the form $z \leftarrow x \rightarrow y$; we want to show that the cocartesian square $(\nu_\square)X$ is also cartesian. (The dual argument will be identical.)

Now $X$ can be left extended by zero to a diagram of the following form

\[
\begin{array}{cc}
0 & \\
\downarrow & \\
z & \leftarrow \rightarrow & y
\end{array}
\]

Let $B$ denote the shape of (6.11), and let $A$ be the category $(\cdot \leftarrow \cdot \rightarrow \cdot \leftarrow \cdot \rightarrow \cdot)$. Then there is a functor $r: A^\square \times \Gamma \rightarrow B$ such that if $Y$ is (6.11), then $r^*Y$ has the form shown in Figure 1. It is straightforward to conclude that each vertical level of this diagram is in the image of $(i_A)_!$. Therefore, by Lemma 4.3, the whole diagram is in the image of $(i_A \times 1_\Gamma)_!$. It follows that $(\pi_A)_!(i_A \times 1_\Gamma)^* r^*Y \cong X$, where $\pi_A$ denotes the projection $A \times \Gamma \rightarrow \Gamma$. That is, if we view $(i_A \times 1_\Gamma)^* r^*Y$ as an $A$-shaped diagram in $\mathcal{D}$, then its colimit is $X$.

Now $(\nu_\square)_!: \mathcal{D} \rightarrow \mathcal{D}^\square$ is a cocontinuous morphism of derivators, and hence preserves this colimit. Therefore, (1) $\Rightarrow$ (i) and (ii) $\Rightarrow$ (iv). By the construction of the adjunctions $\Sigma \dashv \Omega$ and $\text{cof} \dashv \text{fib}$ in Lemmas 5.9 and 5.12, we easily deduce (ii)$\Leftrightarrow$(iii) and (iv)$\Leftrightarrow$(v).
Figure 1. Building a span as a colimit of simpler ones

However, this diagram has the following form in $D$

and when regarded as an $A$-diagram in $D^\square$, all its objects are cartesian as well as cocartesian squares in $D$, since they are constant in at least one direction (see Lemma 4.12). Therefore, it will suffice to show that cartesian squares are closed under $A$-shaped colimits in $D^\square$.

Towards this end, we first note that $A$-shaped colimits can be constructed from pushouts. Namely, if $D$ denotes the following category

with $j: A \hookrightarrow D$ the inclusion of the solid arrows, then in a diagram of the form $j!Z$ both the square and the rectangle are cocartesian (by Lemma 4.9) while the lower-right corner is the colimit over $A$ (by Lemma 4.5 with $B' = C = A$). Thus, if cartesian squares are closed under pushouts, they are also closed under $A$-colimits.

Now since $\text{cof}: D^{\square \times 2} \to D^{\square \times 2}$ is an equivalence of derivators, it preserves cartesian squares; i.e. cartesian squares are closed under cofibers. In particular, they are closed under suspension (and under loop spaces).
On the other hand, if $X$ and $Y$ are cartesian squares in $\mathcal{D}$, then the following squares in $\mathcal{D}^{\Box}$ (i.e. objects of $\mathcal{D}^{\Box}(\Box)$) are cocartesian

$$\Omega X \rightarrow 0 \quad \text{and} \quad 0 \rightarrow Y$$

$$\downarrow \quad \downarrow$$

$$0 \rightarrow X \quad \downarrow \quad \downarrow$$

and hence so is their coproduct

$$\Omega X \rightarrow Y$$

$$\downarrow$$

$$0 \rightarrow X \sqcup Y.$$  

Thus, $X \sqcup Y$ is the cofiber of a map from $\Omega X$ to $Y$, both of which are cartesian; hence it is also cartesian. Thus, cartesian squares are closed under coproducts.

Finally, for an arbitrary cocartesian square

$$X \rightarrow Y$$

$$\downarrow \quad \downarrow$$

$$Z \rightarrow W$$

in $\mathcal{D}^{\Box}$, arguing as in the proof of Proposition 6.8 yields a cocartesian square

$$X \rightarrow Y \sqcup Z$$

$$\downarrow \quad \downarrow$$

$$0 \rightarrow W.$$  

Thus, if $X$, $Y$, $Z$, and hence also $Y \sqcup Z$ are cartesian, then $W$ is the cofiber of a map between cartesian squares and hence is also cartesian. Thus, cartesian squares are closed under pushouts in $\mathcal{D}^{\Box}$, as desired.

Remark 6.12. Theorem 6.10 can be regarded as a converse to Theorem 6.4: if the structure defined on a pointed derivator $\mathcal{D}$ in §5 makes $\mathcal{D}(1)$ triangulated, then in particular the suspension functor must be an equivalence; hence $\mathcal{D}$ is stable.

7. Monoidal derivators

We now move on to study monoidal structures on derivators, and the circle of related ideas. For the following definition, note that the 2-category of prederivators $\mathcal{PDEr}$ is cartesian monoidal, with products computed pointwise in $\mathcal{CAT}$.

Definition 7.1. A monoidal prederivator is a pseudomonoid object in $\mathcal{PDEr}$.

Thus, a monoidal prederivator $\mathcal{D}$ has a product $\otimes: \mathcal{D} \times \mathcal{D} \rightarrow \mathcal{D}$, a unit $S: 1 \rightarrow \mathcal{D}$, and coherence isomorphisms expressing associativity and unitality. Since the monoidal structure of $\mathcal{PDEr}$ is pointwise, this is equivalent to asking that $\mathcal{D}$ lift to a 2-functor valued in the 2-category $\mathcal{MONCAT}$ of monoidal categories and strong monoidal functors. Similarly, we have braided and symmetric monoidal prederivators, and notions of strong, lax, and colax monoidal morphisms of monoidal prederivators.
Examples 7.2. If $C$ is a monoidal category, then its represented prederivator $y(C)$ is monoidal, with the pointwise product on each category $C^A$. If $D$ is a monoidal prederivator, so are $D^{op}$ and $D^B$ (since $(\cdot)^{op}$ and $(\cdot)^B$ are product-preserving endo-2-functors of $PDER$).

Example 7.3. If $C$ is a monoidal model category, it does not automatically follow that $C^A$ is a monoidal model category, even when it admits a model structure. However, the pointwise tensor product on $C^A$ does preserve weak equivalences between pointwise cofibrant diagrams, and every diagram admits a pointwise weak equivalence from a pointwise cofibrant one (apply a cofibrant replacement functor from $C$ pointwise). This is sufficient to ensure that the pointwise tensor product has a left derived functor (see e.g. [DHKS04]). The associativity and unit constraints and axioms can then be constructed exactly as for $C$ itself (see [Hov99, 4.3.2] or [Shu06, Prop. 15.4]). Thus, $\mathcal{H}(C)$ is a monoidal prederivator.

A monoidal derivator is not just a monoidal prederivator which is a derivator; we require an extra compatibility condition. Before we can make this condition precise we first have to talk about external variants of monoidal structures (or, more generally, morphisms of two variables). As we have defined it, a monoidal structure on a prederivator consists only of pointwise or internal monoidal product functors, which in the represented case $y(C)$ are

$$\otimes_A: C^A \times C^A \to C^A$$

defined by $(X \otimes_A Y)_a = X_a \otimes Y_a$.

However, in the represented case there are also external monoidal product functors

$$\otimes: C^A \times C^B \to C^{A \times B}$$

defined by $(X \otimes Y)_{(a,b)} = X_a \otimes Y_b$.

In fact, such external products exist in any monoidal prederivator. More generally, given any morphism of prederivators $\oplus: \mathcal{D}_1 \times \mathcal{D}_2 \to \mathcal{D}_3$, with components denoted $\oplus_A: \mathcal{D}_1(A) \times \mathcal{D}_2(A) \to \mathcal{D}_3(A)$, we can define a functor

$$\oplus: \mathcal{D}_1(A) \times \mathcal{D}_2(B) \to \mathcal{D}_3(A \times B)$$

(7.4)

$$\oplus: \mathcal{D}_1(A) \times \mathcal{D}_2(B) \to \mathcal{D}_3(A \times B)$$

($7.4$)

to be the composite

$$\mathcal{D}_1(A) \times \mathcal{D}_2(B) \xrightarrow{\pi_1^A \times \pi_2^A} \mathcal{D}_1(A \times B) \times \mathcal{D}_2(A \times B) \xrightarrow{\oplus_{A \times B}} \mathcal{D}_3(A \times B).$$

($7.5$)

In general, we write $\pi_A$ to denote a projection functor in which the category $A$ is projected away. This includes the map $\pi_A: A \to 1$ to the terminal category, but also projections such as $A \times B \to B$ and $B \times A \to B$.

The functors (7.4) form a pseudonatural transformation from the 2-functor

$$\mathcal{D}_1 \times \mathcal{D}_2: \text{Cat}^{op} \times \text{Cat}^{op} \to \text{CAT}$$

$$(A, B) \mapsto \mathcal{D}_1(A) \times \mathcal{D}_2(B)$$

to the 2-functor

$$\mathcal{D}_3 \circ \times: \text{Cat}^{op} \times \text{Cat}^{op} \to \text{CAT}$$

$$(A, B) \mapsto \mathcal{D}_3(A \times B).$$

In particular, we have natural isomorphisms

$$\oplus \times \pi^*(X \otimes Y) \cong u^*X \oplus v^*Y.$$
Moreover, the internal product $\oplus_A$ can be recovered from the external one $\oplus$ as the following composite

\[
\mathcal{D}_1(A) \times \mathcal{D}_2(A) \xrightarrow{\oplus} \mathcal{D}_3(A \times A) \xrightarrow{\Delta_A} \mathcal{D}_3(A)
\]

where $\Delta_A : A \to A \times A$ is the diagonal functor of $A$. More precisely, we have the following theorem.

**Theorem 7.7.** For prederivators $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$, there is an equivalence

\[
\mathcal{PD} \mathcal{E} \mathcal{R}(\mathcal{D}_1 \times \mathcal{D}_2, \mathcal{D}_3) \simeq \mathcal{P} \mathcal{sNat}(\mathcal{D}_1 \boxtimes \mathcal{D}_2, \mathcal{D}_3 \circ \times).
\]

**Proof.** Note that $\mathcal{D}_1 \times \mathcal{D}_2 \cong (\mathcal{D}_1 \boxtimes \mathcal{D}_2) \circ \Delta$, where $\Delta : \text{Cat}^{op} \to \text{Cat}^{op} \times \text{Cat}^{op}$ is the diagonal. Thus the desired equivalence is between pseudonatural transformations

\[
\begin{array}{ccc}
\text{Cat}^{op} \times \text{Cat}^{op} & \xrightarrow{\mathcal{P}_1 \boxtimes \mathcal{P}_2} & \text{CAT} \\
\times & \Downarrow \varphi & \\
\text{Cat}^{op} & \xrightarrow{\mathcal{P}_3} & \text{CAT}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\text{Cat}^{op} \times \text{Cat}^{op} & \xrightarrow{\mathcal{P}_1 \boxtimes \mathcal{P}_2} & \text{CAT} \\
\Delta & \Downarrow \Delta & \\
\text{Cat}^{op} & \xrightarrow{\mathcal{P}_3} & \text{CAT}
\end{array}
\]

But $\Delta$ is right 2-adjoint to $\times$ (passing from $\text{Cat}$ to $\text{Cat}^{op}$ reverses the handedness of adjunctions), so this is just a categorified version of the mate correspondence; see for instance [Lau06, Prop. 3.5]. (Another way to say this is that because the monoidal structure of $\text{Cat}$ is cartesian, the induced Day convolution monoidal structure on $\mathcal{PD} \mathcal{E} \mathcal{R} = [\text{Cat}^{op}, \text{CAT}]$ coincides with its pointwise monoidal structure.)

At this point, it is worth pausing to emphasize our notation. If we have a morphism of prederivators $\oplus : \mathcal{D}_1 \times \mathcal{D}_2 \to \mathcal{D}_3$, then:

- The internal product of $X \in \mathcal{D}_1(A)$ and $Y \in \mathcal{D}_2(A)$ is denoted with a subscript as $X \oplus_A Y \in \mathcal{D}_3(A)$, and
- The external product of $X \in \mathcal{D}_1(A)$ and $Y \in \mathcal{D}_2(B)$ is denoted without a subscript as $X \oplus Y \in \mathcal{D}_3(A \times B)$.

More generally, we can have operations that are partly internal and partly external, such as

\[
\mathcal{D}_1(A \times B) \times \mathcal{D}_2(B \times C) \to \mathcal{D}_3(A \times B \times C).
\]

(This particular operation takes $X$ and $Y$ to $(1_A \times \Delta_B \times 1_C)^*(X \oplus Y)$.) We always include subscripts for indexing categories that are treated internally and leave them off for those treated externally; thus (7.8) would be denoted $(X, Y) \mapsto X \oplus_B Y$.

This notational convention is sufficient to determine the type of any such expression unless one of the indexing categories appears more than once somewhere (and in that case context usually disambiguates).

**Remark 7.9.** There are two alternative ways of viewing (7.8). On the one hand, by the functoriality of shifting, $\oplus$ induces a morphism of prederivators

\[
\mathcal{D}_1^B \times \mathcal{D}_2^B \to \mathcal{D}_3^B
\]

of which (7.8) is an external component. On the other hand, we can lift the definition (7.5) to a morphism of prederivators

\[
\mathcal{D}_1^A \times \mathcal{D}_2^C \xrightarrow{\pi^A \times \pi^A} \mathcal{D}_1^A \times \mathcal{D}_2^A \times \mathcal{D}_3^A \times \mathcal{C}
\]

and (7.8) is the internal component of this morphism at $B$. 
Remark 7.12. In the special case of a monoidal prederivator, the associativity for \( \otimes : \mathcal{D} \times \mathcal{D} \to \mathcal{D} \), together with the compatibility between the restriction functors and the monoidal product, induce associativity isomorphisms for the external product
\[
(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)
\]
for \( X \in \mathcal{D}(A), Y \in \mathcal{D}(B), \) and \( Z \in \mathcal{D}(C) \). Similarly, we have unit isomorphisms
\[
X \otimes S_2 \cong X \quad \text{and} \quad S_2 \otimes X \cong X
\]
(where in all cases we decline to notate restriction along the associativity and unit isomorphisms of \( \mathcal{C}at \)). These isomorphisms satisfy appropriate versions of the axioms for a monoidal category. Conversely, from a coherent family of isomorphisms (7.13) and (7.14) we can reconstruct coherent associativity and unitality isomorphisms for the internal components making \( \mathcal{D} \) a monoidal prederivator. (See [Shu08] for a general theorem along these lines.) The more general operations such as (7.8) are similarly associative and unital.

One important advantage of the external product is that it allows us to formalize cocontinuity of monoidal structures, and more generally of a two-variable morphism.

Definition 7.15. A two-variable morphism of derivators \( \otimes : \mathcal{D}_1 \times \mathcal{D}_2 \to \mathcal{D}_3 \) is cocontinuous in the first variable if the canonical mate-transformation
\[
(u \times 1)_!(X \otimes Y) \to (u \times 1)_!(u^*u_!X \otimes Y) \cong (u \times 1)_!(u_!X \otimes Y) \to u_!(X \otimes Y)
\]
is an isomorphism for all \( X \in \mathcal{D}_1(A), Y \in \mathcal{D}_2(C), \) and \( u : A \to B \).

By (Der2), the functoriality of mates, and the pseudonaturality of the external product (7.6), it suffices for (7.16) to be an isomorphism when \( B = 1 \) and \( C = 1 \). There is of course a dual notion, and a combined notion of \( \otimes \) being cocontinuous in each variable (separately).

Definition 7.17. A monoidal derivator is a monoidal prederivator that is a derivator and whose product \( \otimes : \mathcal{D} \times \mathcal{D} \to \mathcal{D} \) is cocontinuous in each variable. A monoidal derivator is braided or symmetric if and only if it is so as a monoidal prederivator.

Example 7.18. If \( \otimes : \mathbf{C}_1 \times \mathbf{C}_2 \to \mathbf{C}_3 \) is an ordinary two-variable functor between complete and cocomplete categories, then for \( B = C = \mathbf{1} \), a diagram \( X \in (\mathbf{C}_1)^A \), and an object \( Y \in \mathbf{C}_2 = (\mathbf{C}_2)^\mathbf{1} \), the transformation (7.16) is easily identified with the canonical map
\[
\text{colim}^A(X \otimes Y) \to \text{colim}^A(X) \otimes Y.
\]
Thus, the induced two-variable morphism \( y(\mathbf{C}_1) \times y(\mathbf{C}_2) \to y(\mathbf{C}_3) \) is cocontinuous in each variable, as defined above, if and only if the original functor \( \otimes \) was so, in the ordinary sense. In particular, if \( \mathbf{C} \) is a complete and cocomplete monoidal category, then \( y(\mathbf{C}) \) is a monoidal derivator if and only if the tensor product of \( \mathbf{C} \) is cocontinuous in each variable in the usual sense.

Example 7.19. If \( \otimes : \mathbf{C}_1 \times \mathbf{C}_2 \to \mathbf{C}_3 \) is a two-variable Quillen left adjoint, then for any cofibrant object \( Y \in \mathbf{C}_2 \), the induced functor \( (\cdot \otimes Y) \) is left Quillen, hence induces a cocontinuous morphism of derivators. Thus, the induced two-variable morphism of derivators is cocontinuous in each variable. In particular, if \( \mathbf{C} \) is a monoidal model category, then \( \mathcal{H}o(\mathbf{C}) \) is a monoidal derivator.
Examples 7.20. If $\mathcal{D}$ is a monoidal derivator, then so is $\mathcal{D}^B$. On the other hand, if $\mathcal{D}$ is a monoidal derivator, then $\mathcal{D}^{op}$ need not be. This fails already in the represented case: colimits in $\mathcal{D}^{op}$ are limits in $\mathcal{D}$, and $\otimes$ need not (and rarely does) preserve limits in either variable.

Warning 7.21. The definition of a monoidal derivator is based on the notion of cocontinuous morphisms of two variables, which we have phrased using the external monoidal product. One might guess that this condition could also be rephrased using the internal products, but the obvious guess, at least, does not work. Specifically, given a morphism $\otimes: \mathcal{D}_1 \times \mathcal{D}_2 \to \mathcal{D}_3$, objects $X \in \mathcal{D}_1(A)$, $Y \in \mathcal{D}_2(B)$, and a functor $u: A \to B$, one can form the following canonical mate transformation

$$u_!(X \otimes_A u^*Y) \to u_!(u^*u_!X \otimes_A u^*Y) \cong u_!u^*(u_!X \otimes_B Y) \to u_!X \otimes_B Y. \tag{7.22}$$

However, this morphism will not be an isomorphism in the typical examples. For example, if $u: 1 \to 2$ classifies the object 0, then the domain of (7.22) is $X \otimes Y_0 \to X \otimes Y_0$ while its target is $X \otimes Y_0 \to X \otimes Y_1$.

8. Ends and coends

The correspondence between internal and external monoidal structures only uses the fact that prederivators are contravariant functors with cartesian monoidal domain $\textbf{Cat}$ (see [Shu08] for a general theory). However, if $\mathbf{C}$ is a cocomplete monoidal category, there is a third type of operation induced on the values of its represented prederivator: the (canceling) tensor product of functors

$$\otimes_{[A]} : \mathbf{C}^{A_{\text{op}}} \times \mathbf{C}^A \to \mathbf{C} \text{ defined by } X \otimes_{[A]} Y := \int_{a \in A} X_a \otimes Y_a. \tag{8.1}$$

This requires not only the monoidal structure but the notion of coend in $\mathbf{C}$ (denoted by an integral sign with a variable at the top).

In classical category theory, there are several ways to define coends. For a generalization to derivators, we choose one which is particularly easy to work with. However, we will show below that the resulting definition has many other equivalent reformulations, reinforcing its correctness.

Let $A$ be a small category; its twisted arrow category $\text{tw}(A)$ is defined to be the category of elements of the hom-functor $A(-,-): A^{op} \times A \to \textbf{Set}$. Thus, its objects are morphisms $a \xrightarrow{f} b$ in $A$, while its morphisms from $a_1 \xrightarrow{f_1} b_1$ to $a_2 \xrightarrow{f_2} b_2$ are pairs of morphisms $b_1 \xrightarrow{b} b_2$ and $a_2 \xrightarrow{a} a_1$ such that $f_2 = h f_1 g$ (that is, “two-sided factorizations” of $f_2$ through $f_1$). It comes with a projection

$$(s,t) : \text{tw}(A) \to A^{op} \times A,$$

where for $a \xrightarrow{f} b$ we have $s(f) = a$ and $t(f) = b$. We will also be interested in its opposite category $\text{tw}(A)^{op}$, which of course comes with a projection

$$(t^{op},s^{op}) : \text{tw}(A)^{op} \to A^{op} \times A.$$

Definition 8.2. If $\mathcal{D}$ is a derivator, then the coend of $X \in \mathcal{D}(A^{op} \times A)$ is defined to be

$$\int_A X := (\pi_{\text{tw}(A)^{op}})(t^{op},s^{op})^* X,$$

which is an object of the underlying category $\mathcal{D}(1)$. Dually, the end of $X$, denoted $\int_A X$, is its coend in $\mathcal{D}^{op}$. 


Put differently, the coend functor $\int^A$ is the composite
\[
\mathcal{D}(A^{\text{op}} \times A) \xrightarrow{(t^{\text{op}}, s^{\text{op}})} \mathcal{D}(\text{tw}(A)^{\text{op}}) \xrightarrow{(\pi_{\text{tw}(A)^{\text{op}}})} \mathcal{D}(1).
\]
This definition also makes sense “with parameters”, i.e. if $X \in \mathcal{D}(A^{\text{op}} \times A \times B)$, then we have $\int^A X \in \mathcal{D}(B)$. The naturality in parameters is then phrased concisely by defining the coend as the following morphism of derivators
\[
\mathcal{D}^{A^{\text{op}} \times A} \xrightarrow{(t^{\text{op}}, s^{\text{op}})} \mathcal{D}(\text{tw}(A)^{\text{op}}) \xrightarrow{(\pi_{\text{tw}(A)^{\text{op}}})} \mathcal{D}.
\]
When $\mathcal{D} = y(C)$ is a represented derivator, it is easy to verify that this definition reproduces the usual notion of coend. But there are other, perhaps more familiar, definitions of coends in the classical case. For instance, if $C$ is cocomplete, then $\int^a X(a, a)$ is equivalently a coequalizer
\[
\coprod_{\alpha: a_1 \to a_2} X(a_2, a_1) \rightrightarrows \coprod_a X(a, a).
\]
Our coends in derivators can also be rephrased in a similar way, but (as usual in a homotopical context) coequalizers must be generalized to “geometric realizations of simplicial bar constructions”. In a derivator, “geometric realization” just means the colimit of a diagram of shape $\Delta^{\text{op}}$; we now construct such a diagram whose colimit is the coend of $X \in \mathcal{D}(A^{\text{op}} \times A)$.

Let $(\Delta/A)^{\text{op}}$ denote the opposite of the category of simplices of $A$. Its objects are functors $[n] \to A$, where $[n]$ denotes the $(n + 1)$-element totally ordered set regarded as a category, and its morphisms from $[n] \to A$ to $[m] \to A$ are functors $[m] \to [n]$ over $A$. (The opposite of the category $A^b$ from [ML98, §IX.5] is contained in $(\Delta/A)^{\text{op}}$ as a final, but not homotopy final, subcategory. This reflects the fact that mere coequalizers suffice to define coends in non-homotopical category theory.)

Now there is a functor $c: (\Delta/A)^{\text{op}} \to \text{tw}(A)^{\text{op}}$ which simply composes up a string of arrows; we claim it is homotopy final. By Corollary 3.21, it suffices to show that for any $a: a_1 \to a_2$ in $A$, the category $(\alpha/c)$ is homotopy contractible. However, $(\alpha/c)$ can be identified with the category of simplices of the nerve of the category $(a_1/A/a_2)_{\alpha}$, constructed as in Theorem 3.20 for the square
\[
\begin{array}{ccc}
A & \xrightarrow{1_A} & A \\
1_A \downarrow & & \downarrow 1_A \\
A & \xrightarrow{1_A} & A
\end{array}
\]
Since this square is homotopy exact, $(a_1/A/a_2)_{\alpha}$ is homotopy contractible. Thus, by Example 3.22, so is its category of simplices $(\alpha/c)$. Therefore, the coend $\int^A X$ can be identified with the colimit of the restriction of $X$ along
\[
(\Delta/A)^{\text{op}} \xrightarrow{c} (\text{tw}(A)^{\text{op}} \to A^{\text{op}} \times A.
\]
Moreover, since $\pi_{(\Delta/A)^{\text{op}}}$ factors through $\Delta^{\text{op}}$ by a functor $p: (\Delta/A)^{\text{op}} \to \Delta^{\text{op}}$, this colimit is also isomorphic to the colimit of $p e^{(t^{\text{op}}, s^{\text{op}})^*} X$; this is our “simplicial bar construction”. To see that it deserves the name, note that $p$ is a discrete
opfibration, so that each pullback square

\[
P_n \rightarrow (\Delta/A)^{op} \\
\downarrow \quad \downarrow \\
\mathbb{1} \rightarrow [n] \Delta^{op}
\]
is homotopy exact, where \(P_n\) is the discrete category on the set of composable strings of \(n\) arrows in \(A\). Thus, \(p_{!}c^\ast(e^{op}, s^{op})^\ast X\) looks like

\[
\cdots \rightarrow \prod_{a_1 \rightarrow a_2 \rightarrow a_3} X(a_3, a_1) \equiv \prod_{a_1 \rightarrow a_2} X(a_2, a_1) \equiv \prod_a X(a, a)
\]
as we expect of a bar construction.

A third definition of the coend, familiar in enriched category theory, is as the weighted colimit of \(X\) weighted by the hom-functor \(\text{hom}_A : A^{op} \times A \rightarrow \text{Set}\). There is no direct notion of “weighted colimit” in a derivator, but in this case we can mimic one using left Kan extensions into collages (see also [GS13]). Specifically, consider the category \(\text{coll} (\text{hom}_A)\) which contains \(A^{op} \times A\) as a full subcategory, together with one additional object \(\infty\), and with the morphisms from \((a_1, a_2)\) to \(\infty\) being the homset \(A(a_1, a_2)\) and no nonidentity morphisms with domain \(\infty\). Classically, we can obtain the \(\text{hom}_A\)-weighted colimit of \(X\) by left Kan extending from \(A^{op} \times A\) to \(\text{coll} (\text{hom}_A)\), then evaluating at \(\infty\). However, it is easy to see that there is a comma square

\[
\begin{array}{c}
tw(A)^{op} \\
\Pi_{\text{tw}(A)^{op}} \rightarrow A^{op} \times A \\
\downarrow \pi_{\text{tw}(A)^{op}} \downarrow \ \\
\Pi_{\text{tw}(A)^{op}} \rightarrow \text{coll} (\text{hom}_A)
\end{array}
\]

so that by (Der4), the same is true in any derivator.

Thus reassured that there is only one sensible notion of coend in a derivator, from now on we will mostly consider only our original definition, using twisted arrow categories. (We will use (8.3) once again, in the proof of Lemma 9.7.)

Moving on to the basic properties of coends, we have the usual Fubini-type theorem (which again has obvious variants with parameters).

**Lemma 8.4.** Let \(\mathcal{D}\) be a derivator,

\[
s : (A^{op} \times A) \times (B^{op} \times B) \rightarrow (A \times B)^{op} \times (A \times B)
\]

be the canonical isomorphism, and \(X \in \mathcal{D}((A \times B)^{op} \times (A \times B))\). Then there are natural isomorphisms

\[
\int^A \int^B s^\ast X \cong \int^A \int^B X \cong \int^B \int^A s^\ast X.
\]

**Proof.** This follows immediately from the observation that there is a canonical isomorphism between \(\text{tw}(A \times B)\) and \(\text{tw}(A) \times \text{tw}(B)\), which is compatible with the
source and target maps in the sense that the following diagram commutes
\[
\begin{array}{c}
tw(A)^{\text{op}} \times tw(B)^{\text{op}} \\
\downarrow \\
A^{\text{op}} \times A \times B^{\text{op}} \times B \\
\downarrow \\
(A \times B)^{\text{op}} \times (A \times B).
\end{array}
\]
\[\square\]

We will be particularly interested in taking coends of diagrams of the form \(X \otimes Y\), where \(\otimes : \mathcal{D}_1 \times \mathcal{D}_2 \to \mathcal{D}_3\) is a two-variable morphism, and \(X \in \mathcal{D}_1(B^{\text{op}})\) and \(Y \in \mathcal{D}_2(B)\). In the represented case, this yields (8.1). We denote such a **tensor product of functors** by

\[X \otimes_{[B]} Y := \int^B (X \otimes Y).\]

The brackets around \([B]\) indicate that it is “canceled”, no longer appearing at all as an indexing category in the result. This gives us a third kind of “product” in addition to the internal and external ones; functorially it is the composite

\[(8.5)\] \[\mathcal{D}_1(B^{\text{op}}) \times \mathcal{D}_2(B) \xrightarrow{\otimes} \mathcal{D}_3(B^{\text{op}} \times B) \xrightarrow{f^B} \mathcal{D}_3(1).\]

The canceling tensor product over one category \(B\) can also be combined with internal or external products over other categories. For instance, we have a functor

\[(8.6)\] \[\mathcal{D}_1(A \times B^{\text{op}}) \times \mathcal{D}_2(A \times B) \to \mathcal{D}_3(A)\]

which is internal in \(A\) and canceling in \(B\); thus we write it \(X \otimes_{A,[B]} Y\). As with (7.8), we can view (8.6) either as the tensor product (8.5) over \(B\) constructed from the induced morphism

\[\mathcal{D}_1^A \times \mathcal{D}_2^A \to \mathcal{D}_3^A\]

or as the internal component at \(A\) of the lifted version of (8.5)

\[(8.7)\] \[\mathcal{D}_1^B \times \mathcal{D}_2^B \xrightarrow{\otimes} \mathcal{D}_3^B \times B \xrightarrow{f^B \times 1} \mathcal{D}_3.\]

Similarly, we have a functor

\[\mathcal{D}_1(A \times B^{\text{op}}) \times \mathcal{D}_2(B \times C) \to \mathcal{D}_3(A \times C)\]

denoted \((X, Y) \mapsto X \otimes_{A,[B]} Y\), in which \(A\) and \(C\) are treated externally. This can be viewed either as an external component of (8.7), or as the canceling tensor product constructed from (7.11). We leave it to the reader to write down the most general form with all three of internal, external, and canceled indexing categories, and reinterpret it in all the possible ways.

Finally, we mention an “adjointness” property which will be useful later.

**Lemma 8.8.** *Let* \(f : A \to B\).

(i) *For* \(X \in \mathcal{D}(B^{\text{op}} \times A)\), *we have a natural isomorphism*

\[\int^A (f^{\text{op}} \times 1)^* X \cong \int^B (1 \times f)_! X.\]
(ii) For $X \in \mathcal{D}(A^{\text{op}} \times B)$, we have a natural isomorphism
\[
\int^A (1 \times f)^* X \cong \int^B (f^{\text{op}} \times 1)_! X.
\]

Proof. We prove (i); (ii) is analogous. Since $(L, s^{\text{op}}) : \mathcal{D}(B) \to \mathcal{D}(B \times B)$ is a fibration, the pullback square below is homotopy exact.

Thus, by unraveling definitions, it will suffice to show that the induced functor $\mathcal{D}(A)^{\text{op}} \to \mathcal{D}(A)^{\text{op}}$ is homotopy final. By Theorem 3.20(vi) and the definition of $\mathcal{D}(A)^{\text{op}}$ as a pullback, this is equivalent to asking that for any $a, b \in B$, and $\phi : f\alpha \to b$, the following category $K_{a,b,\phi}$ is homotopy contractible:

- Its objects consist of a pair of objects $a_1, a_2 \in A$, morphisms $\alpha : a \to a_1$ and $\xi : a_1 \to a_2$ in $A$, and $\psi : f\alpha \to b$ in $B$, such that $\psi \circ f\xi \circ f\alpha = \phi$.
- A morphism from $(a_1, a_2, \alpha, \xi, \psi)$ to $(a_1', a_2', \alpha', \xi', \psi')$ consists of a pair of morphisms $\zeta_1 : a_1 \to a_1'$ and $\zeta_2 : a_2 \to a_2'$ in $A$ such that $\zeta_1 \circ \alpha = \alpha'$, $\psi \circ f\zeta_2 = \psi'$, and $\xi = \zeta_2 \circ \xi'$.

Let $L_{a,b,\phi}$ be the full subcategory of $K_{a,b,\phi}$ whose objects have $a_1 = a$ and $\alpha = 1_a$. Then $L_{a,b,\phi}$ is coreflective; the coreflection of $(a_1, a_2, \alpha, \xi, \psi)$ is $(a, a_2, 1_a, \xi \circ \alpha, \psi)$. Thus, since the coreflection is a right adjoint and hence homotopy final, $K_{a,b,\phi}$ is homotopy contractible if and only if $L_{a,b,\phi}$.

However, $L_{a,b,\phi}$ has a terminal object, namely $(a, a, 1_a, 1_a, \phi)$. Thus, since the inclusion $1 \to L_{a,b,\phi}$ of this terminal object is a right adjoint and $1$ is homotopy contractible, so is $L_{a,b,\phi}$. \hfill \Box

Corollary 8.9. Let $f : A \to B$, and let $\otimes : \mathcal{D}_1 \times \mathcal{D}_2 \to \mathcal{D}_3$ be cocontinuous in both variables.

(i) If $X \in \mathcal{D}_1(B^{\text{op}})$ and $Y \in \mathcal{D}_2(A)$, we have a natural isomorphism
\[
(f^{\text{op}})^* X \otimes_{[A]} Y \cong X \otimes_{[B]} f_! Y.
\]

(ii) If $X \in \mathcal{D}_1(A^{\text{op}})$ and $Y \in \mathcal{D}_2(B)$, we have a natural isomorphism
\[
X \otimes_{[A]} f^* Y \cong (f^{\text{op}})_! X \otimes_{[B]} Y.
\]

Proof. Pseudonaturality of the external product $\otimes$ implies
\[
(f^{\text{op}})^* X \otimes Y \cong (f^{\text{op}} \times 1)^*(X \otimes Y),
\]
while its cocontinuity in the second variable implies
\[
X \otimes f_! Y \cong (1 \times f)_!(X \otimes Y).
\]
Thus (i) follows from Lemma 8.8(i). Similarly, (ii) follows from Lemma 8.8(ii). \hfill \Box
9. The bicategory of profunctors

We remarked in §7 that associativity and unitality of a monoidal structure on a derivator can equivalently be expressed in terms of the internal or the external products. In this section we explain the corresponding notion of “associativity” and “unitality” for the canceling tensor product of functors. It can be concisely expressed as follows.

Theorem 9.1. If \( \mathcal{D} \) is a monoidal derivator, then there is a bicategory \( \mathcal{P}rof(\mathcal{D}) \) described as follows:

- Its objects are small categories.
- Its hom-category from \( A \) to \( B \) is \( \mathcal{D}(A \times B^{op}) \).
- Its composition functors are the external-canceling tensor products
  \[ \otimes_B : \mathcal{D}(A \times B^{op}) \times \mathcal{D}(B \times C^{op}) \to \mathcal{D}(A \times C^{op}) \].
- The identity 1-cell of a small category \( B \) is
  \[ \mathbb{1}_B = (t, s)_{tw(B)} : (\pi_{tw(B)})^* S_1 \to \mathcal{D}(B \times B^{op}) \].

Note that in the definition of \( \mathbb{1}_B \) we use \( tw(B) \), rather than its opposite as we did in §8.

Remark 9.2. On the one hand, the definition of \( \mathcal{P}rof(\mathcal{D}) \) can be regarded as a simple generalization of the usual bicategory of profunctors enriched in a monoidal category (with the objects restricted to unenriched categories rather than enriched ones). Indeed, the tensor product of functors is exactly how classical profunctors are composed. As for the identities, since \( (t, s) \) is a discrete opfibration, for any \( b_1, b_2 \in B \) the following pullback square is homotopy exact

\[
\begin{array}{ccc}
B(b_1, b_2) & \to & tw(B) \\
\downarrow & & \downarrow (t, s) \\
\mathbb{1} & \to & B \times B^{op}
\end{array}
\]

where \( B(b_1, b_2) \) is the homset of \( B \) regarded as a discrete category. Therefore, \( (b_2, b_1)^* \mathbb{1}_B \in \mathcal{D}(1) \) is a coproduct of \( B(b_1, b_2) \) copies of the unit object \( S_1 \in \mathcal{D}(1) \). So in the represented case \( \mathcal{D} = y(C) \), this definition of \( \mathbb{1}_B \) agrees with the usual identity profunctor of (the free \( C \)-enriched category on) the ordinary category \( B \).

Remark 9.3. On the other hand, the construction of \( \mathcal{P}rof(\mathcal{D}) \) can also be regarded as analogous to the construction of a bicategory from an indexed monoidal category (a.k.a. monoidal fibration) in [Shu08, PS12a]. Indeed, a monoidal derivator is an indexed monoidal category, where the indexing is over the cartesian monoidal category \( \text{Cat} \). However, instead of the Beck-Chevalley condition for pullback squares, as assumed in \textit{ibid.}, a monoidal derivator satisfies the Beck-Chevalley condition for \textit{comma} squares, by (Der4). This is a manifestation of the fact that the objects of \( \text{Cat} \) are “directed”. As a consequence, we must replace the diagonal maps \( \Delta_A : A \to A \times A \) used in \textit{ibid.} with either \( (t, s) : tw(B) \to B \times B^{op} \), as used in the definition of the identity 1-cells, or \( (t^{op}, s^{op}) : tw(B)^{op} \to B^{op} \times B \), as used in the definition of composition. Similarly, \( \pi_A : A \to \mathbb{1} \) is replaced by \( \pi_{tw(A)} \) or \( \pi_{tw(A)^{op}} \).
We will prove Theorem 9.1 in a sequence of lemmas. The details of the proof are not at all important for the rest of the paper, so the uninterested reader is free to skip ahead to §10 beginning on page 40.

**Lemma 9.4.** For any diagrams

\[ X \in \mathcal{D}(A \times B^{op}), \quad Y \in \mathcal{D}(B \times C^{op}), \quad \text{and} \quad Z \in \mathcal{D}(C \times D^{op}), \]

we have a natural isomorphism

\[ X \otimes_{[B]} (Y \otimes_{[C]} Z) \cong (X \otimes_{[B]} Y) \otimes_{[C]} Z \]

which satisfies the pentagon identity.

**Proof.** Since the external product \( \otimes \) preserves colimits in each variable, it also preserves coends; thus we have a natural isomorphism

\[ (X \otimes_{[B]} Y) \otimes Z = \left( \int^B X \otimes Y \right) \otimes Z \cong \int^B ((X \otimes Y) \otimes Z). \]

Therefore, we have a natural isomorphism

\[ (X \otimes_{[B]} Y) \otimes_{[C]} Z = \int^C (X \otimes_{[B]} Y) \otimes Z \cong \int^C \int^B ((X \otimes Y) \otimes Z). \]

Similarly, we have

\[ X \otimes_{[B]} (Y \otimes_{[C]} Z) \cong \int^B \int^C (X \otimes (Y \otimes Z)). \]

Thus, composing (9.5) and (9.6) with the associativity isomorphism of \( \otimes \) and the Fubini isomorphism \( \int^B \int^C \cong \int^C \int^B \), we obtain a natural isomorphism

\[ (X \otimes_{[B]} Y) \otimes_{[C]} Z \cong X \otimes_{[B]} (Y \otimes_{[C]} Z). \]

The pentagon identity for this associativity isomorphism follows from the pentagon identity for \( \otimes \), together with the pasting coherence properties of the mates that define the isomorphisms (9.5) and (9.6) and the Fubini isomorphisms. \( \square \)

It will be convenient later on to have the following more general version of the unitality isomorphism.

**Lemma 9.7.** For any diagrams

\[ X \in \mathcal{D}(A \times B^{op}) \quad \text{and} \quad Y \in \mathcal{D}(C) \]

we have a natural isomorphism

\[ X \otimes_{[B]} \left( (t,s)_!(\pi_{tw(B)})^* Y \right) \cong X \otimes Y. \]

**Proof.** Since \((t,s): tw(B) \to B \times B^{op}\) is a Grothendieck opfibration, the following pullback square is homotopy exact

\[
\begin{array}{ccc}
tw(B)^{op} \times_B tw(B) & \xrightarrow{\epsilon^{op} \times 1} & B^{op} \times tw(B) \\
1 \times s & \downarrow & \downarrow 1 \times (t,s) \\
tw(B)^{op} \times B^{op} & \xrightarrow{(t^{op}, s^{op}) \times 1} & B^{op} \times B \times B^{op}.
\end{array}
\]
Thus, in the following diagram, the square commutes up to isomorphism for any derivator $\mathcal{E}$:

\[
\begin{align*}
\mathcal{E}(\text{tw}(B)^{\text{op}} \times_{B} \text{tw}(B)) & \xleftarrow{(\pi_{\text{tw}(B)})^{\ast}} \mathcal{E}(B^{\text{op}}) \\
\xrightarrow{\sigma} & \mathcal{E}(B^{\text{op}}) \\
\xrightarrow{(t,s)} & \mathcal{E}(B^{\text{op}}) \\
\xrightarrow{(\pi_{\text{tw}(B)^{\text{op}}})^{\ast}} & \mathcal{E}(B^{\text{op}})
\end{align*}
\]

Letting $\mathcal{E} = \mathcal{G}^{A \times C}$ and using the fact that $\otimes$ preserves colimits in each variable, for $X$ and $Y$ as in the statement of the lemma we have

\[
X \otimes_{[B]} ((t,s)(\pi_{\text{tw}(B)})^{\ast} Y) = (\pi_{\text{tw}(B)^{\text{op}}})^{\ast} (t^{\text{op}}, s^{\text{op}})^{\ast} (X \otimes (t, s)(\pi_{\text{tw}(B)})^{\ast} Y) \\
\cong (\pi_{\text{tw}(B)^{\text{op}}})^{\ast} (t^{\text{op}}, s^{\text{op}})^{\ast} (t, s)(\pi_{\text{tw}(B)})^{\ast} (X \otimes Y) \\
\cong (\pi_{\text{tw}(B)^{\text{op}}})^{\ast} s^{\ast} (t^{\text{op}})^{\ast} (\pi_{\text{tw}(B)})^{\ast} (X \otimes Y)
\]

Thus, it suffices to show that the upper-left composite in (9.8) is isomorphic to the identity functor. For this, it will suffice to show that the following square is homotopy exact:

\[
\begin{align*}
tw(B)^{\text{op}} \times_{B} \text{tw}(B) & \xrightarrow{t^{\text{op}}} B^{\text{op}} \\
\xrightarrow{s} & B^{\text{op}} \\
\xrightarrow{\beta} & B^{\text{op}}
\end{align*}
\]

The objects of $tw(B)^{\text{op}} \times_{B} tw(B)$ are composable pairs $(b_1 \xrightarrow{\beta} b_2 \xrightarrow{\gamma} b_3)$ of morphisms in $B$, and its morphisms from $(\gamma, \beta)$ to $(\gamma', \beta')$ are commutative diagrams:

We have $t^{\text{op}}(\gamma, \beta) = b_3$ and $s(\gamma, \beta) = b_1$, and the natural transformation in (9.10) simply composes $\beta$ and $\gamma$ (this goes in the direction shown because its target is $B^{\text{op}}$ rather than $B$). Now by Theorem 3.20(vi), it suffices to show that for any $b_0, b_4 \in B$ and $\varphi: b_0 \to b_4$, the category

\[
\left( b_4 / (tw(B)^{\text{op}} \times_{B} tw(B)) / b_0 \right)_\varphi
\]

is homotopy contractible. Denote this category by $E_4$: its objects are composable quadruples $(b_0 \xrightarrow{\gamma} b_1 \xrightarrow{\beta} b_2 \xrightarrow{\gamma'} b_3 \xrightarrow{\beta'} b_4)$ in $B$ whose composite is $\varphi$, where the objects $b_0$ and $b_4$ are fixed but the other three can vary. Its morphisms are
Now the full subcategory $E_3 \subset E_4$ of objects where $\delta$ is an identity is coreflective in $E_4$. Then the further subcategory $E_2 \subset E_3$ where both $\gamma$ and $\delta$ are identities is reflective in $E_3$. Finally, the third subcategory $E_1 \subset E_2$ where $\beta$, $\gamma$, and $\delta$ are all identities is coreflective in $E_2$, and $E_1$ contains only the single object $(b_0 \rightsquigarrow b_4 = b_4 = b_4)$. Thus, $E_4$ is connected by a chain of adjunctions to $1$, hence is homotopy contractible.

Taking $Y = S_1$ in Lemma 9.7, we have an isomorphism $\rho: X \otimes_{[B]} I_B \cong X$. Dually, of course, we have $\lambda: \coprod A \otimes_{[A]} X \cong X$.

**Lemma 9.11.** These associativity and unitality isomorphisms satisfy the unit axiom

$$
\begin{array}{c}
(X \otimes_{[B]} I_B) \otimes_{[B]} Y \xrightarrow{\cong} X \otimes_{[B]} (I_B \otimes_{[B]} Y) \\
X \otimes_{[B]} Y \xrightarrow{\cong} \int^B (X \otimes S_1) \otimes Y \xrightarrow{\cong} \int^B X \otimes (S_1 \otimes Y)
\end{array}
$$

**Proof.** Consider coherent diagrams $X \in \mathcal{D}(A \times B^{\text{op}})$ and $Y \in \mathcal{D}(B \times C^{\text{op}})$, and our $I_B \in \mathcal{D}(B \times B^{\text{op}})$. By passing to the derivator $\mathcal{D}^{A \times C^{\text{op}}}$, we can suppress the additional parameters $A$ and $C^{\text{op}}$. We will construct a commutative diagram

in which the unlabeled vertical isomorphisms are given by Lemma 9.7 and the horizontal ones are the respective associativity constraints. Of course, the lower triangle commutes by the unit axiom for the external monoidal structure $\otimes$, so it suffices to produce the upper square.

First of all, since $\otimes$ is pseudonatural and cocontinuous in both variables, we may pull all restriction and left Kan extension functors out of all tensor products, as in (9.9). By the functoriality of mates, this does not affect the commutativity of any diagrams. Thus, we may consider $(X \otimes_{[B]} I_B) \otimes_{[B]} Y$ to be obtained from
We have the 2-cell (8.3) rendering – with and is literally equal to. The right vertical composite are obviously identical, so it remains to show are homotopy exact, as is the rectangle are homotopy exact since all of them are pull-

and endowed with the same 2-cell as in that proof.

Thus, it will suffice to show that \( \mu_{23} \mu_2^{-1} \), which as before is equal to \( \mu_{5678} \mu_7^{-1} \), which as before is equal to \( \mu_{5678} \mu_7^{-1} \mu_6^{-1} \).

Similarly, we may consider \( X \otimes [B] (1_B \otimes [B] Y) \) to be obtained from \( X \otimes (\mathbb{S}_n \otimes Y) \) by transporting along the diagonal in the following diagram:

As before, the rectangles \( \square_5 \square_7 \) are homotopy exact, as is the rectangle \( \square_{17} \) with an analogous 2-cell, while \( \square_9 \) is literally equal to \( \square_1 \). The right vertical composite in (9.12) is then by definition \( \mu_{5678} \mu_7^{-1} \), which as before is equal to \( \mu_{5678} \mu_7^{-1} \mu_6^{-1} \).

However, from the proof of the Fubini Theorem 8.4, the associativity isomorphism from Lemma 9.4 is composed of the associativity of \( \otimes \) together with \( \mu_6 \mu_1^{-1} \). Thus, it will suffice to show that \( \mu_{5678} \mu_7^{-1} \mu_6^{-1} = \mu_{0123} \mu_0^{-2} \mu_2^{-1} \). But the commutative rectangles \( \square_0 \square_2 \) and \( \square_5 \square_7 \) are obviously identical, so it remains to show \( \mu_{5678} = \mu_{0123} \).
Now the 2-cells that live in regions 0123 and 5678 are not equal, and indeed it doesn’t even make sense to ask whether they are, since their codomains are different. The equality $\mu_{5678} = \mu_{0123}$ only makes sense after postwhiskering with $(\pi_{tw(B)\circ P})$. However, since regions 4 and 9 are homotopy exact, to show the equality of these two whiskerings, it suffices to show the equality of the pasted 2-cells in the regions 01234 and 56789, whose domains and codomains are equal. It is straightforward to check that both of these 2-cells just “compose up” a composable string of three arrows. Hence they are the same, and (9.12) commutes. □

10. Two-variable adjunctions and closed monoidal derivators

In classical category theory, we say that a functor $C_1 \times C_2 \to C_3$ is a two-variable left adjoint if each functor $(X \otimes -)$ and $(- \otimes Y)$ is a left adjoint. This is equivalent to the existence of functors $\triangleright: C_2^{op} \times C_3 \to C_1$ and $\triangleleft: C_3 \times C_1^{op} \to C_2$ and natural isomorphisms

$$C_3(X \otimes Y, Z) \cong C_1(X, Y \triangleright Z) \cong C_2(Y, Z \triangleleft X).$$

Our notational convention is chosen so that these isomorphisms preserve the cyclic ordering of $X, Y, Z$. In this case we say that we have a two-variable adjunction $(\otimes, \triangleright, \triangleleft): C_1 \times C_2 \rightleftarrows C_3$.

In particular, a monoidal category is closed (sometimes called biclosed in the non-symmetric case) if and only if its tensor product is a two-variable left adjoint. Thus, in order to define closed monoidal derivators, we must generalize two-variable adjunctions to derivators. For the same reason as Warning 7.21, we must formulate this notion using the external rather than the internal products.

**Definition 10.1.** A morphism $\otimes: D_1 \times D_2 \to D_3$ of derivators is a two-variable left adjoint if each external component

$$(10.2) \quad \otimes: D_1(A) \times D_2(B) \to D_3(A \times B)$$

is a two-variable left adjoint, with adjoints

$$\triangleright_{[B]}: D_2(B)^{op} \times D_3(A \times B) \to D_1(A)$$
$$\triangleleft_{[A]}: D_3(A \times B) \times D_1(A)^{op} \to D_2(B)$$

and such that

- For any $Y \in D_2(B)$ and $Z \in D_3(A \times B)$ and functor $u: A' \to A$, the canonical mate-transformation

$$(10.3) \quad u^* (Y \triangleright_{[B]} Z) \to (Y \triangleright_{[B]} (u \times 1)^* Z)$$

is an isomorphism, and

- For any $X \in D_1(A)$ and $Z \in D_3(A \times B)$ and functor $v: B' \to B$, the canonical mate-transformation

$$(10.4) \quad v^* (Z \triangleleft_{[A]} X) \to ((1 \times v)^* Z \triangleleft_{[A]} X)$$

is an isomorphism.

In this case we say that $(\otimes, \triangleright, \triangleleft)$ is a two-variable adjunction $D_1 \times D_2 \rightleftarrows D_3$.

**Definition 10.5.** A monoidal derivator is closed if its tensor product $\otimes$ is a two-variable left adjoint.
Lemma 10.8. If each functor \( \otimes \): \( \mathcal{D}_1 \times \mathcal{D}_2 \to \mathcal{D}_3 \) is a two-variable left adjoint, then the transformations (10.3) and (10.4) make the functors \( \triangleright_{[B]} \) (for fixed \( B \)) and \( \ll_{[A]} \) (for fixed \( A \)) into lax natural transformations.

Proof. By functoriality of the mate-construction with respect to pasting. \( \square \)

Thus, in this case \( \otimes \) is a two-variable left adjoint if and only if these lax transformations are pseudonatural. We also have a two-variable version of Lemma 2.14.

Lemma 10.8. If each functor \( \otimes \): \( \mathcal{D}_1(A) \times \mathcal{D}_2(B) \to \mathcal{D}_3(A \times B) \) is a two-variable left adjoint, then \( \otimes \): \( \mathcal{D}_1 \times \mathcal{D}_2 \to \mathcal{D}_3 \) is a two-variable left adjoint if and only if it is cocontinuous in each variable.

Proof. In this case, (10.3) is the total mate of (7.16), and similarly for the second variable. \( \square \)

Example 10.9. If \( \otimes \): \( \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}_3 \) is an ordinary two-variable left adjoint between complete and cocomplete categories, then the induced morphism \( y(\mathcal{C}_1) \times y(\mathcal{C}_2) \to y(\mathcal{C}_3) \) is a two-variable left adjoint of derivators. We remarked in Example 7.18 that it is cocontinuous in each variable, so by Lemma 10.8 it suffices to exhibit adjoints to its external components. Using the usual notation for ends, we define the adjoint \( \triangleright_{[B]} \) by

\[
(\triangleright_{[B]} Z)_{\alpha} := \int_{b \in B} (Y_b \triangleright Z_{(\alpha, b)})
\]

and dually for \( \ll_{[A]} \). In particular, if \( \mathcal{C} \) is a complete and cocomplete closed monoidal category, then \( y(\mathcal{C}) \) is a closed monoidal derivator.

Example 10.11. If \( \otimes \): \( \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}_3 \) is a two-variable Quillen left adjoint between combinatorial model categories, then \( \otimes \): \( (\mathcal{C}_1)^A \times (\mathcal{C}_2)^B \to (\mathcal{C}_3)^{A \times B} \) is a two-variable Quillen left adjoint relative to the injective model structures, and hence induces a two-variable adjunction on homotopy categories. Together with Example 7.19, this implies that the induced morphism of derivators \( \mathcal{H}o(\mathcal{C}_1) \times \mathcal{H}o(\mathcal{C}_2) \to \mathcal{H}o(\mathcal{C}_3) \) is a two-variable left adjoint. In particular, if \( \mathcal{C} \) is a combinatorial monoidal model category, then \( \mathcal{H}o(\mathcal{C}) \) is a closed monoidal derivator. In fact, the hypothesis of combinatoriality can be weakened; see Theorem 11.10.

The following lemma is also useful for constructing adjunctions of two variables.

Lemma 10.12. Let \( \otimes \): \( \mathcal{D}_1 \times \mathcal{D}_2 \to \mathcal{D}_3 \) be a two-variable left adjoint and let \( L_1: \mathcal{E}_1 \to \mathcal{D}_1 \), \( L_2: \mathcal{E}_2 \to \mathcal{D}_2 \), and \( L_3: \mathcal{D}_3 \to \mathcal{E}_3 \) be left adjoints. Then

\[
L_3 \circ \otimes \circ (L_1 \times L_2): \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3
\]

is also a two-variable left adjoint.

Proof. The functoriality of mates implies that the given composite is cocontinuous in each variable. The result then follows easily from the corresponding result in ordinary category theory, Lemma 2.14, and Lemma 10.8. \( \square \)
Using Lemmas 10.8 and 10.12, we now observe that if \( \otimes : \mathcal{D}_1 \times \mathcal{D}_2 \to \mathcal{D}_3 \) is a two-variable left adjoint, then so are each of the “liftings” of its internal, external, and canceling components to morphisms of derivators.

**Example 10.13.** If \( \otimes : \mathcal{D}_1 \times \mathcal{D}_2 \to \mathcal{D}_3 \) is a two-variable left adjoint, then so is the induced morphism \( \oplus_B : \mathcal{D}_1^B \times \mathcal{D}_2^B \to \mathcal{D}_3^B \) from (7.10). The external component of \( \oplus_B \) at \( A \) and \( C \) is the composite

\[
\mathcal{D}_1(B \times A) \times \mathcal{D}_2(B \times C) \xrightarrow{\otimes} \mathcal{D}_3(B \times A \times B \times C) \xrightarrow{(s\Delta_B)^*} \mathcal{D}_3(B \times A \times C).
\]

By the version of Lemma 10.12 for ordinary categories, this composite has a right adjoint in each variable. The functoriality of mates, together with cocontinuity of \( \oplus \) and \((s\Delta_B)^*\), implies that \( \oplus_B \) is also cocontinuous in each variable. Thus, by Lemma 10.8, \( \oplus_B \) is a two-variable left adjoint.

In particular, it follows that the ordinary functor \( \oplus_B : \mathcal{D}_1(B) \times \mathcal{D}_2(B) \to \mathcal{D}_3(B) \) is a two-variable left adjoint for all \( B \). One might thus hope to be able to reformulate the notion of two-variable adjunction of derivators in terms of the internal monoidal structures only, but the obvious version of the “naturality” condition for these adjunctions fails, for the same reason as Warning 7.21.

It does follow, however, that if \( \mathcal{D} \) is a closed monoidal derivator, then so is \( \mathcal{D}^B \).

**Example 10.14.** If \( \otimes : \mathcal{D}_1 \times \mathcal{D}_2 \to \mathcal{D}_3 \) is a two-variable left adjoint, then so is the induced morphism \( \mathcal{D}_1^A \times \mathcal{D}_2^B \to \mathcal{D}_3^{A \times B} \) from (7.11). This follows immediately from its definition using Example 10.13 and Lemma 10.12.

**Example 10.15.** If \( \otimes : \mathcal{D}_1 \times \mathcal{D}_2 \to \mathcal{D}_3 \) is a two-variable left adjoint, then so is the induced morphism \( \mathcal{D}_1^{B^{op}} \times \mathcal{D}_2^{B^{op}} \to \mathcal{D}_3^{B^{op}} \) from (8.7). This follows immediately from its definition as the composite

\[
\mathcal{D}_1^{B^{op}} \times \mathcal{D}_2^{B^{op}} \xrightarrow{\otimes} \mathcal{D}_3^{B^{op} \times B^{op}} \xrightarrow{(s^{op}, s^{op})^*} \mathcal{D}_3^{tw(B)^{op}} \xrightarrow{[\pi_{tw(B)^{op}}]} \mathcal{D}_3
\]

along with Example 10.14 and Lemma 10.12.

Recall that a bicategory is said to be **closed** if its composition functors are two-variable left adjoints.

**Corollary 10.16.** If \( \mathcal{D} \) is a closed monoidal derivator, then \( \text{Prof}(\mathcal{D}) \) is closed. \( \square \)

Thus, we have natural isomorphisms

\[
\mathcal{D}(A \times C^{op})(X \otimes_{[B]} Y, Z) \cong \mathcal{D}(A \times B^{op})(X, Y \triangleright_{[C]} Z) \cong \mathcal{D}(B \times C^{op})(Y, Z \triangleleft_{[A]} X).
\]

11. **Cycling two-variable adjunctions**

In an ordinary two-variable adjunction

\[
(\otimes, \triangleright, \triangleleft) : C_1 \times C_2 \rightleftarrows C_3,
\]

we can “cycle” the categories involved to obtain two other two-variable adjunctions

\[
(\triangleleft^{op}, \otimes^{op}, \triangleright^{op}) : C_3^{op} \times C_1 \rightleftarrows C_2^{op},
\]

\[
(\triangleright^{op}, \triangleleft^{op}, \otimes^{op}) : C_2 \times C_3^{op} \rightleftarrows C_1^{op}.
\]

This is a two-variable version of the fact that an adjunction \((F, G) : C_1 \rightleftarrows C_2\) can equally be regarded as an adjunction \((G^{op}, F^{op}) : C_2^{op} \rightleftarrows C_1^{op}\). (The placement of the opposites can also be made more symmetrical; see [CGR12].)
By contrast, in our definition of a two-variable adjunction for derivators, the morphisms $\rhd$ and $\lhd$ appear to play very different roles from $\otimes$ (e.g. their components have different forms). We now show that this asymmetry is only apparent: our two-variable adjunctions can be “cycled” just like the ordinary ones.

**Theorem 11.1.** From any two-variable adjunction of derivators

\[
(\otimes, \rhd, \lhd): \mathcal{D}_1 \times \mathcal{D}_2 \leftrightarrow \mathcal{D}_3
\]

we can construct a “cycled” two-variable adjunction

\[
(\lhd^{op}, \otimes^{op}, \rhd): \mathcal{D}_3^{op} \times \mathcal{D}_1 \rightarrow \mathcal{D}_2^{op}. \tag{11.2}
\]

**Proof.** Invoking the definition of opposite derivators, we see that the external components have different forms. We now show that this asymmetry is only apparent:

We have used our standard notation with canceling subscripts for these right adjoints, which immediately tells us how to define them. Namely, we let $\otimes^{op}_{[B^{op}]}$ be the opposite of an instance of the canceling product for $\otimes$ with $A^{op}$ as an external parameter, i.e. the opposite of the composite

\[
\mathcal{D}_1(B) \times \mathcal{D}_2(A^{op} \times B^{op}) \xrightarrow{\otimes^{op}_{[B^{op}]}} \mathcal{D}_3(B \times A^{op} \times B^{op}) \overset{B^{op}}{\longrightarrow} \mathcal{D}_1(A^{op})
\]

so that for $X \in \mathcal{D}_1(B)$ and $Y \in \mathcal{D}_2(A^{op} \times B^{op})$ we have

\[
X \otimes^{op}_{[B^{op}]} Y = (\pi_{tw(B^{op})})_!(t^{op}, s^{op})^*(X \otimes Y).
\]

By **Example 10.15**, the functor (11.3) is a two-variable left adjoint. Thus, its right adjoints give us definitions for $\lhd^{op}$ and $\rhd_{[A^{op}]}$. Explicitly, for $X \in \mathcal{D}_1(B)$, $Y \in \mathcal{D}_2(A^{op} \times B^{op})$, and $Z \in \mathcal{D}_3(A^{op})$ we have

\[
\begin{align*}
X \otimes^{op}_{[B^{op}]} Y &= (t^{op}, s^{op})_!(\pi_{tw(B^{op})})^*Z \lhd^{op}[B^{op}] X \quad \text{and} \quad Y \rhd_{[A^{op}]} Z = Y \rhd_{[A^{op} \times B^{op}]} ((t^{op}, s^{op})_!(\pi_{tw(B^{op})})^*Z).
\end{align*}
\]

According to **Definition 10.1**, it remains to show that $\lhd^{op}$ is pseudonatural in both variables, that $\otimes^{op}_{[B^{op}]}$ is pseudonatural in $A$, and that $\rhd_{[A^{op}]}$ is pseudonatural in $B$. Pseudonaturality of $\lhd^{op}$ in $A$ is equivalent to pseudonaturality of $\lhd$ in $A^{op}$, which follows from cocontinuity of $\otimes^{op}_{[B^{op}]}$ in $A^{op}$, as in **Lemma 10.8**. Pseudonaturality of $\lhd^{op}$ in $B$ means an isomorphism

\[
(Z \lhd^{op} u^* X) \simeq (1 \times u^{op})^*(Z \lhd^{op} X)
\]

for $u: B \to B'$, $X \in \mathcal{D}_1(B')$, and $Z \in \mathcal{D}_3(A^{op})$. We take this to be the total mate of the isomorphism

\[
(u^* X) \otimes^{op}_{[B^{op}]} Y \simeq X \otimes^{op}_{[(B')^{op}]} (1 \times u^{op})_1 Y
\]

which exists for $Y \in \mathcal{D}_2(A^{op} \times B^{op})$, by **Corollary 8.9** applied to the shifted two-variable adjunction $\mathcal{D}_1 \times \mathcal{D}_2(A^{op}) \to \mathcal{D}_3(A^{op})$. \hfill $\square$
Pseudonaturality of $\otimes_{[B^\op]}^{\op}$ in $A$, of course, is equivalent to pseudonaturality of $\ltimes_{[A^\op]}$ in $A^\op$. Finally, pseudonaturality of $\ltimes_{[A^\op]}$ in $B$ means an isomorphism

$$u^*(Y \ltimes_{[A^\op]} Z) \cong ((1 \times u^\op)^*Y) \ltimes_{[A^\op]} Z$$

for $u : B \to B', Y \in \mathcal{D}_2(A^\op \times (B')^\op)$, and $Z \in \mathcal{D}_3(A^\op)$. The total mate of the isomorphism

$$(u_!X) \otimes_{[[B^\op]\op]} Y \cong X \otimes_{[B^\op]} (1 \times u^\op)^*Y$$

which exists for $X \in \mathcal{D}_1(B)$, again by Corollary 8.9, is such an isomorphism. We leave it to the reader to verify that the induced isomorphism (11.5) is, in fact, the canonical transformation (10.3) which we require to be an isomorphism. □

Analogously to (10.6), the cycled two-variable adjunction has isomorphisms

$$\mathcal{D}_2(A^\op \times B^\op)^{\op}(Z \ltimes_{[B^\op]} X, Y) \cong \mathcal{D}_1(A)(X, Y \ltimes_{[B]} Z) \cong \mathcal{D}_3(B^\op)^{\op}(Z, X \otimes_{[A^\op]} Y).$$

If we iterate Theorem 11.1, we obtain its dual version, which gives a two-variable adjunction

$$(\ltimes^{\op}, \ltimes, \otimes^{\op}) : \mathcal{D}_2 \times \mathcal{D}_3^\op \xrightarrow\cong \mathcal{D}_1^\op.$$ (11.6)

From the proof of Theorem 11.1, we see that the functor $\otimes^{\op}$ occurring in (11.6), which has components

$$\otimes^{\op}_{[A^\op]} : \mathcal{D}_1(A^\op \times B^\op)^{\op} \times \mathcal{D}_2(A)^{\op} \to \mathcal{D}_3(B^\op)^{\op},$$

is defined in terms of the original functor $\otimes$ by

$$X \otimes^{\op}_{[A^\op]} Y = X \otimes_{[A^\op \times B^\op]} (t, s)!(\pi_{tw(B^\op)})^*Y \cong X \otimes_{[A^\op]} Y \quad \text{(by Lemma 9.7).}$$

By uniqueness of adjunctions, it follows that the functors $\ltimes^{\op}$ and $\ltimes$ in (11.6) must be the right adjoints of this, as constructed in Example 10.15. Namely, their components

$$\ltimes^{\op} : \mathcal{D}_2(A) \times \mathcal{D}_2(B^\op)^{\op} \to \mathcal{D}_1(A^\op \times B^\op)^{\op}$$

$$\ltimes_{[B]} : \mathcal{D}_2(B^\op) \times \mathcal{D}_1(A^\op \times B^\op)^{\op} \to \mathcal{D}_2(A)$$

must be isomorphic to

$$(11.7) \quad Y \ltimes^{\op} Z = Y \ltimes_{[A]} (t^{\op}, s^{\op})_*(\pi_{tw(A^\op)})^*Z$$

$$Z \ltimes_{[B^\op]} X = (t^{\op}, s^{\op})_*(\pi_{tw(A^\op)})^*Z \ltimes_{[A^\op \times B^\op]} X.$$ Note that these definitions are exactly symmetrical to those obtained in Theorem 11.1. Finally, applying Theorem 11.1 one more time, we obtain a two-variable adjunction of the original form

$$(\otimes, \ltimes, \ltimes) : \mathcal{D}_1 \times \mathcal{D}_2 \xrightarrow\cong \mathcal{D}_3$$

whose functor $\otimes$ is defined, for $X \in \mathcal{D}_1(A)$ and $Y \in \mathcal{D}_2(B)$, by

$$X \otimes Y = (t, s)!((\pi_{tw(B)})^*X \otimes_{[B]} Y.$$ Applying Lemma 9.7 again, we see that this is isomorphic to the original functor $\otimes$, and hence so must its adjoints be isomorphic to the original adjoints. Thus, up to isomorphism, Theorem 11.1 describes a cyclic action on two-variable adjunctions. Abstractly, we could say that derivators and two-variable adjunctions form a “pseudo cyclic double multicategory” [CGR12].
Also, by the construction in Theorem 11.1, the functor $\triangleright$ appearing in (11.8) should be given by a canceling version of (11.7), which is to say an end in $\mathcal{D}_1$:

$$Y \triangleright_B Z = \int_B (Y \rhd Z) = (\pi_{tw(B)^{op}})_*(t^{op}, s^{op})^*(Y \triangleright Z).$$

Since this is isomorphic to the original adjoint $\triangleright_B$ of $\otimes$, we obtain a version of the formula (10.10) that holds in any derivator.

Remark 11.9. It should be possible to extract from the proof of Theorem 11.1 an equivalent characterization of two-variable adjunctions of derivators in terms of three morphisms

$$\otimes: \mathcal{D}_1 \times \mathcal{D}_2 \to \mathcal{D}_3$$
$$\triangleright: \mathcal{D}_2^{op} \times \mathcal{D}_3 \to \mathcal{D}_1$$
$$\triangleleft: \mathcal{D}_3 \times \mathcal{D}_1^{op} \to \mathcal{D}_2$$

together with “extraordinary natural transformations” [EK66] satisfying some identities. This could then be used as a definition of a two-variable adjunction for prederivators.

One useful application of Theorem 11.1 is to extend Example 10.11 to the non-combinatorial case.

Theorem 11.10. If $\mathbf{C}$ is a cofibrantly generated monoidal model category, then $\mathcal{H}_0(\mathbf{C})$ is a closed monoidal derivator.

Proof. If $\mathbf{C}$ is not combinatorial, the injective model structure on $\mathbf{C}^A$ may not exist, but the projective model structure always does, in which the fibrations and weak equivalences are objectwise. Moreover, under the isomorphism

$$(\mathbf{C}^{B^{op}})^{op} \cong (\mathbf{C}^{op})^B$$

the dual of the projective model structure on the left-hand side becomes identified with an injective model structure on the right-hand side. Since projective cofibrations are in particular objectwise cofibrations, it follows that the induced two-variable adjunction

$$\triangleright^{op}: \mathbf{C}^A \times (\mathbf{C}^{B^{op}})^{op} \to (\mathbf{C}^{A^{op} \times B^{op}})^{op}$$

is a two-variable Quillen left adjoint. As in the combinatorial case, therefore, we obtain a two-variable adjunction of derivators

$$(\triangleright^{op}, \triangleleft, \otimes^{op}): \mathcal{H}_0(\mathbf{C}) \times \mathcal{H}_0(\mathbf{C})^{op} \to \mathcal{H}_0(\mathbf{C})^{op}$$

which we can therefore cycle forwards into a two-variable adjunction

$$(\otimes', \triangleright, \triangleleft): \mathcal{H}_0(\mathbf{C}) \times \mathcal{H}_0(\mathbf{C}) \to \mathcal{H}_0(\mathbf{C}).$$

Thus, to make $\mathcal{H}_0(\mathbf{C})$ into a closed monoidal derivator, it suffices to identify this functor $\otimes'$ with the monoidal structure $\otimes$ on $\mathcal{H}_0(\mathbf{C})$ constructed in Example 7.3. By definition, $\otimes'$ is given by (11.4) with $\otimes'_{[B]}$ replacing $\triangleleft_{[B^{op}]}$:

$$Z \otimes' X = ((t, s)_!(\pi_{tw(B)^{op}})^*Z) \otimes'_{[B]} X.$$
Here $\text{tw}(B^{\text{op}})^{\text{op}}$ and $(t^{\text{op}}, s^{\text{op}})_*$ have become $\text{tw}(B^{\text{op}})$ and $(t, s)_!$, respectively, because of passage to the opposite derivator. Moreover, $\otimes'_B$ denotes, not the canceling tensor product $\otimes_B$ constructed in §8, but the left derived functor of the point-set-level canceling tensor product 

$$C^{A \times B \times B^{\text{op}}} \times C^B \xrightarrow{\otimes} C^{A \times B}$$ 

obtained by applying it to projectively cofibrant diagrams. However, because this point-set-level functor is the composite 

$$C^{A \times B \times B^{\text{op}}} \times C^B \xrightarrow{\otimes_B} C^{A \times B \times B^{\text{op}}} \xrightarrow{(t^{\text{op}}, s^{\text{op}})_*} C^{A \times B \times \text{tw}(B^{\text{op}})} \xrightarrow{\pi_{\text{tw}(B^{\text{op}})}_!} C^{A \times B},$$ 

its left derived functor is naturally isomorphic to the composite of the left derived functors of each of these ingredients. But these left derived functors are precisely the corresponding functors in $\mathcal{H}_o(C)$ which went into the construction of $\otimes_B$, so we have $\otimes'_B \cong \otimes_B$. Finally, applying Lemma 9.7, we obtain $\otimes' \cong \otimes$. \hfill \Box

The proof actually shows that a two-variable Quillen adjunction between cofibrantly generated model categories induces a two-variable adjunction of derivators. The hypothesis of cofibrant generation is not really necessary either, but in the general case a more involved argument is needed. See, for instance [Shu06, Corollary 20.10].

12. Pushout products in stable monoidal derivators

With the basic theory of stable monoidal derivators under our belts, we can move on to prove May’s axioms and the additivity theorem. In this section we consider May’s axioms (TC1)–(TC4); in the next section we consider (TC5).

Let $\mathcal{D}$ be a closed monoidal, stable derivator; thus $\mathcal{D}(1)$ is both monoidal and triangulated. May [May01] assumed symmetry as well, but for the axioms we consider in this section, that does not play an essential role. In fact, for the most part, we could be talking about any two-variable adjunction of derivators. The hypothesis of cofibrant generation is not really necessary either, but in the general case a more involved argument is needed. See, for instance [Shu06, Corollary 20.10].

Theorem 12.1. For $x, y \in \mathcal{D}(1)$, there are natural isomorphisms 

$$\Sigma x \otimes y \xrightarrow{\sim} \Sigma(x \otimes y) \xrightarrow{\sim} x \otimes \Sigma y$$ 

which commute with the associativity and unit isomorphisms in an obvious way. If $\mathcal{D}$ is symmetric, they also commute with the symmetry, in the sense that the following square commutes 

$$\begin{CD} 
\Sigma(x \otimes y) @>>> \Sigma(x \otimes y) \\
@VVV @VVV 
\Sigma(y \otimes x) @>>> y \otimes \Sigma x. 
\end{CD}$$

Moreover, the induced composite 

$$\Sigma \Sigma(x \otimes y) \xrightarrow{\sim} \Sigma(\Sigma x \otimes y) \xrightarrow{\sim} \Sigma x \otimes \Sigma y \xrightarrow{\sim} \Sigma(x \otimes \Sigma y) \xrightarrow{\sim} \Sigma \Sigma(x \otimes y)$$ 

is multiplication by $-1$.

Proof. By definition, $\Sigma y$ comes with a cocartesian square of the form 

$$\begin{array}{ccc} 
y & \xrightarrow{\sim} & 0 \\
\downarrow & & \downarrow 
0 & \xrightarrow{\sim} & \Sigma y. 
\end{array}$$
Applying the external product \( \mathcal{D}(1) \times \mathcal{D}(\Box) \to \mathcal{D}(\Box) \) to \( x \) and to this square, we obtain a square of the form

\[
\begin{array}{ccc}
x \otimes y & \to & 0 \\
\downarrow & & \downarrow \\
0 & \to & x \otimes \Sigma y.
\end{array}
\]

We have used the fact that the tensor product preserves zero objects since it is cocontinuous in each variable. Cocontinuity also implies that this square is again cocartesian, hence induces the second isomorphism in (12.2); the first is similar. Commutativity with the associativity, unit, and symmetry isomorphisms follows since this entire construction was functorial and these isomorphisms are natural.

For the second part, note that an isomorphism \( \Sigma \Sigma z \cong w \) can be determined by giving a coherent diagram of the form

\[
\begin{array}{ccc}
z & \to & 0 \\
\downarrow & & \downarrow \\
0 & \to & \bullet & \to & 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \to & w
\end{array}
\]

in which both squares are cocartesian. Now from \( x \) and \( y \) we can obtain cocartesian squares

\[
\begin{array}{ccc}
x & \to & 0_2 \\
\downarrow & & \downarrow \\
0_1 & \to & \Sigma x \\
\end{array} \quad \text{and} \quad \begin{array}{ccc}
y & \to & 0_4 \\
\downarrow & & \downarrow \\
0_3 & \to & \Sigma y
\end{array}
\]

which we denote \( X, Y \in \mathcal{D}(\Box) \). Their external product \( X \otimes Y \in \mathcal{D}(\Box \times \Box) \) is a hypercube, which we have shown as the solid arrows in Figure 2. Of course, since \( \otimes \) preserves zero objects in each variable, there are a lot of zero objects in this diagram. However, we have notated them all as tensor products rather than merely as “0”, since for purposes of identifying minus signs it is important to distinguish between squares and their transposes.

Now by restriction from Figure 2, we obtain the two coherent diagrams shown in Figure 3 (in which we have abbreviated \( a \otimes b \) by \( a.b \) for conciseness). These two diagrams induce, respectively, the composite of the first two arrows in (12.4) and the composite of the last two arrows therein. Thus, it will suffice to show that these two diagrams become isomorphic after transposing one of the squares in one of them, by an isomorphism which restricts to the identity on \( x.y \) and \( \Sigma x.\Sigma y \).

Let \( A \) denote the category \( \Box \times \Box \) extended with two new objects 5 and 6 as shown with the dotted arrows in Figure 2, with inclusion \( j: \Box \times \Box \to A \). Let \( Z = j_!(X \otimes Y) \); then by the dual of Lemma 4.9, we conclude that the squares

\[
\begin{array}{ccc}
0_5 & \to & 0_1.y \\
\downarrow & & \downarrow \\
x.0_3 & \to & 0_1.0_4 \\
\end{array} \quad \text{and} \quad \begin{array}{ccc}
0_6 & \to & 0_2.y \\
\downarrow & & \downarrow \\
x.0_4 & \to & 0_2.0_4
\end{array}
\]

are cartesian, so that the objects labeled \( 0_5 \) and \( 0_6 \) in \( Z \) are zero objects.
Figure 2. The hypercube for (TC1)

Figure 3. Two ways that $\Sigma x \otimes \Sigma y$ is $\Sigma \Sigma (x \otimes y)$

Now by restriction from $Z$ along a suitable functor, we obtain a coherent diagram looking like the solid arrows in Figure 4, with the middle object $\Sigma(x.y)$ also omitted (but with the composite arrows from each of $0_5$ and $0_6$ to each of $0_2.0_4$ and $0_1.0_4$ included, so that the shape is a full subcategory of the shape of the whole of Figure 4). Finally, we perform a left Kan extension to add the object denoted $\Sigma(x.y)$ and the dotted arrows connecting to it. Since all the solid horizontal arrows are isomorphisms, and the left, middle, and right faces are composed of cocartesian squares, the horizontal dotted arrows are also isomorphisms. However, the left and right faces of Figure 4 are the left- and right-hand diagrams in Figure 3, with one square transposed on the right side. This completes the proof.

May’s version of (TC1) involves the symmetry more irreducibly.

Corollary 12.5 (TC1). Suppose $\mathcal{D}$ is symmetric. Then there is a natural isomorphism $\alpha: \Sigma x \xrightarrow{\sim} x \otimes \Sigma S$ such that the composite

\[(12.6) \quad \Sigma \Sigma S \xrightarrow{\alpha} \Sigma S \otimes \Sigma S \xrightarrow{\sigma} \Sigma S \otimes \Sigma S \xrightarrow{\alpha^{-1}} \Sigma \Sigma S\]
is multiplication by \(-1\), where \(\mathfrak{s}\) denotes the symmetry isomorphism.

**Proof.** Take \(y = \mathfrak{s}\) and let \(\alpha\) be the second isomorphism in (12.2) (composed with a unit isomorphism). Naturality of the unit isomorphism and (12.3) then implies that the composite

\[
\Sigma \mathfrak{s} \otimes x \xrightarrow{\mathfrak{s}} x \otimes \Sigma \mathfrak{s} \xrightarrow{\alpha^{-1}} \Sigma x
\]

is an instance of the first isomorphism in (12.2) (again composed with a unit isomorphism). In fact, we can recover the general case of (12.2) from \(\alpha\) and (12.7), using associativity and unitality, e.g.

\[
\Sigma(x \otimes y) \xrightarrow{\alpha} (x \otimes y) \otimes \Sigma \mathfrak{s} \xrightarrow{\sim} x \otimes (y \otimes \Sigma \mathfrak{s}) \xrightarrow{\alpha^{-1}} x \otimes \Sigma y.
\]

(May used this as a **definition** of the isomorphisms (12.2) in terms of his \(\alpha\).) Finally, using (12.3) we can identify (12.6) with (12.4). \(\square\)

In fact, in Theorem 12.1 we did not need stability of \(\mathcal{D}\), only pointedness. The next axiom could also be stated in a pointed version by using fiber and cofiber sequences, but for simplicity we stick to the notation of the stable case.

**Theorem 12.8 (TC2).** For any distinguished triangle

\[
x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} \Sigma x
\]

in \(\mathcal{D}(1)\) and any \(w \in \mathcal{D}(1)\), each of the following triangles is distinguished

\[
x \otimes w \xrightarrow{f \otimes 1} y \otimes w \xrightarrow{g \otimes 1} z \otimes w \xrightarrow{h \otimes 1} \Sigma(x \otimes w),
\]

\[
w \otimes x \xrightarrow{1 \otimes f} w \otimes y \xrightarrow{1 \otimes g} w \otimes z \xrightarrow{1 \otimes h} \Sigma(w \otimes x),
\]

\[
w \triangleright x \xrightarrow{1 \triangleright f} w \triangleright y \xrightarrow{1 \triangleright g} w \triangleright z \xrightarrow{1 \triangleright h} \Sigma(w \triangleright x),
\]

\[
\Sigma^{-1}(x \triangleright w) \xrightarrow{-h \triangleright 1} z \triangleright w \xrightarrow{g \triangleright 1} y \triangleright w \xrightarrow{f \triangleright 1} x \triangleright w.
\]
along with the analogous triangles involving $\triangleleft$ (which, in the symmetric case, are isomorphic to the last two above).

Proof. By assumption, we have a cofiber sequence

$$
x \rightarrow y \rightarrow 0 \rightarrow \\
0 \rightarrow z \rightarrow \Sigma x.
$$

Taking the external product with $w$ on both sides, and arguing as in the proof of Corollary 12.5, we obtain the first two triangles above.

Now we invoke the fact that by Theorem 11.1,

$$
\triangleright^{\text{op}} : \mathcal{D} \times \mathcal{D}^{\text{op}} \rightarrow \mathcal{D}^{\text{op}}
$$

is itself a two-variable left adjoint, and hence cocontinuous in each variable. In particular, the morphism $(w \triangleright -) : \mathcal{D}^{\text{op}} \rightarrow \mathcal{D}^{\text{op}}$ preserves cofiber sequences, which (since $\mathcal{D}^{\text{op}}$ is stable) is equivalent to preserving fiber sequences. But fiber sequences in $\mathcal{D}^{\text{op}}$ are the same as cofiber sequences in $\mathcal{D}$, so preservation of these yields the third triangle above.

Finally, $( - \triangleright w) : \mathcal{D} \rightarrow \mathcal{D}^{\text{op}}$ also preserves cofiber sequences, which is to say that it takes cofiber sequences in $\mathcal{D}$ to fiber sequences in $\mathcal{D}$. Since by Lemma 6.6 the fiber sequences induce the negative of the triangulation on $\mathcal{D}$, we get the fourth triangle above together with the minus sign as indicated.

The next axiom relates to the pushout product of two morphisms, which we can construct as a canceling tensor product. Namely, suppose given diagrams $X \in \mathcal{D}(2^{\text{op}})$ and $X' \in \mathcal{D}(2)$ of the form $(y \leftarrow^f x)$ and $(x' \rightarrow^f y')$. Their external product $X \otimes X' \in \mathcal{D}(2^{\text{op}} \times 2)$ looks like

$$
y \otimes x' \leftarrow x \otimes x' \downarrow \\
y \otimes y' \leftarrow x \otimes y'
$$

while its restriction to $\text{tw}(2)^{\text{op}}$ looks like

$$
y \otimes x' \leftarrow x \otimes x' \downarrow \\
x \otimes y'.
(12.9)
$$

Thus, the canceling tensor product $X \otimes_{\mathcal{D}[1]} X'$ is the pushout of this cospan, which is the usual definition of a pushout product.

In a triangulated category, of course, such a construction is unavailable. Thus May’s version of the following axiom asserts only that there is some object $v$, without specifying it further.

Theorem 12.10 (TC3). Suppose we have $X = (y \leftarrow^f x)$ and $X' = (x' \rightarrow^{f'} y')$, giving rise to distinguished triangles

$$
x \underbrace{\rightarrow^f y} \underbrace{\rightarrow^g z} \rightarrow^h \Sigma x, \\
x' \underbrace{\rightarrow^{f'} y'} \underbrace{\rightarrow^{g'} z'} \rightarrow^{h'} \Sigma x'.
$$
Then if we write $v = X \otimes_{C^l} X'$, there exist morphisms $p_1, p_2, p_3, j_1, j_2,$ and $j_3,$ along with cofiber sequences giving rise to distinguished triangles

$$y \otimes x' \xrightarrow{p_1} v \xrightarrow{j_1} x \otimes z' \xrightarrow{f \otimes h} \Sigma(y \otimes x'),$$

$$\Sigma^{-1}(z \otimes z') \xrightarrow{p_2} v \xrightarrow{j_2} y \otimes y' \xrightarrow{g \otimes g'} z \otimes z',$$

$$x \otimes y' \xrightarrow{p_3} v \xrightarrow{j_3} z \otimes x' \xrightarrow{h \otimes f'} \Sigma(x \otimes y'),$$

and a coherent diagram of the form of Figure 5.

May’s version of (TC3) follows from this when $\mathcal{D}$ is strong, since then we can lift any pair of morphisms in $\mathcal{D}(1)$ to objects of $\mathcal{D}(2^\op)$ and $\mathcal{D}(2)$.

Proof. The basic idea of the following proof is to break the symmetry between $X$ and $X'$. We will construct from $X$ an object $U \in \mathcal{D}(C \times 2^\op)$, where $C$ is the poset that is the shape of Figure 5, such that the tensor product $U \otimes_{C^l} X' \in \mathcal{D}(C)$ is the desired Figure 5 itself. Yoneda-type arguments suggest that the object $U$ should be determined by its tensor products with the two “representable” 2-diagrams

$$\mathbb{S} \to \mathbb{S} \quad \text{and} \quad 0 \to \mathbb{S}.$$  

By substituting these two diagrams for $x' \xrightarrow{f'} y'$ in Figure 5, we can conclude that that $U$ should look like Figure 6 (as in earlier proofs, each object denoted $0_k$ is a zero object).

We can obtain such an object $U \in \mathcal{D}(C \times 2^\op)$ by restricting a known object along some functor out of $C \times 2^\op$. Recall that by definition of the triangulation,
we have a diagram

\[
\begin{array}{c}
\Sigma^{-1} y \to \Sigma^{-1} z \to 0_2 \\
\downarrow \Sigma^{-1} h \downarrow \downarrow f \downarrow y \downarrow 0_4 \\
0_1 \to x \to z h \to \Sigma x \Sigma f \\
\downarrow \downarrow g \downarrow \downarrow \downarrow \\
0_3 \to 0_5 \to \Sigma y
\end{array}
\]

Figure 7. The origin of \( U \)

in which both squares are bicartesian. Using left and right extensions by zero and Kan extension, we can extend this to the diagram of the form shown in Figure 7. Again, all the objects labeled \( 0_k \) are zero objects. According to Remark 5.15, the
lack of minus signs means that we have identified the objects labeled $\Sigma^{-1}y$, $\Sigma^{-1}z$, $\Sigma x$, and $\Sigma y$ in Figure 7 using the respective bicartesian squares.

\[
\begin{array}{cccc}
\Sigma^{-1}y & 0_2 & \Sigma^{-1}z & 0_2 \\
\downarrow & & \downarrow & \\
0_1 & y_1 & \downarrow & z_1 \\
\end{array}
\quad
\begin{array}{cccc}
x & 0_4 & y & 0_4 \\
\downarrow & & \downarrow & \\
\downarrow & \Sigma x & \downarrow & \Sigma y \\
\end{array}
\]

(12.11)

and not their transposes.

Now, if $A$ is the shape of Figure 7, then there is a functor $q: C \times 2^{op} \to A$ such that restricting Figure 7 along $q$ produces the diagram in Figure 6. The subscripts indicate which zero objects in Figure 6 map to which zero objects in Figure 7, and this fact uniquely determines $q$ since its domain and codomain are posets. We leave it to the reader to verify that $q$ does actually exist, i.e. that every arrow in Figure 6 corresponds to some arrow in Figure 7. (The objects $0_5$ and $\Sigma y$ are not used yet; they will be important later.)

Thus, we have our $U \in \mathcal{D}(C \times 2^{op})$ that looks like Figure 6; we must now verify that $U \otimes [2] X'$ looks like Figure 5. By naturality of coends in parameters, for any $c \in C$, the object $(U \otimes [2] X')_c$ can be identified with $U_c \otimes [2] X'$, where of course $U_c \in \mathcal{D}(2^{op})$ is the arrow occurring at the appropriate spot in $U$. Thus, for instance, the object at the top of $U \otimes [2] X'$ is the tensor product of $(x \leftarrow 0) \in \mathcal{D}(2^{op})$ with $X' = (x' \to y') \in \mathcal{D}(2)$. By the argument above, this is the pushout of

\[
x \otimes x' \leftarrow 0 \otimes x' \cong 0 \\
\downarrow \\
0 \otimes y' \cong 0
\]

which is canonically isomorphic to $x \otimes x'$, as it should be. The same argument identifies the objects of $U \otimes [2] X'$ that should be $y \otimes x'$, $z \otimes x'$, and $\Sigma (x \otimes x') \cong (\Sigma x) \otimes x'$.

Next, consider the object at the bottom left-hand corner of $U \otimes [2] X'$: the tensor product of $(z \leftarrow 0)$ with $(x' \to y')$. This is the pushout of

\[
z \otimes x' \leftarrow z \otimes x' \\
\downarrow^{1z \otimes f'} \\
z \otimes y',
\]

which is canonically isomorphic to $z \otimes y'$ by Lemma 4.12, as it should be. The same argument identifies the objects that should be $y \otimes y'$, $x \otimes y'$, and $\Sigma^{-1}(z \otimes y')$.

Now consider the object at the bottom right-hand corner of $U \otimes [2] X'$: the tensor product of $(0 \leftarrow y)$ with $(x' \to y')$. This is the pushout of

\[
0 \cong 0 \otimes x' \leftarrow y \otimes x' \\
\downarrow^{1y \otimes f'} \\
y \otimes y',
\]

which is to say the cofiber of $1_y \otimes f'$. But $(y \otimes -)$ is cocontinuous, and $z'$ is the cofiber of $f'$, so this is canonically isomorphic to $y \otimes z'$, as it should be. The same argument identifies the objects that should be $x \otimes z'$, $\Sigma^{-1}(z \otimes z')$, and $\Sigma^{-1}(y \otimes z')$. 

This completes our identification of the objects in $U \otimes \mathbb{P}_2 X'$ with their corresponding objects in Figure 5 — except for $v$, of course, which we simply define to be the appropriate object in $U \otimes \mathbb{P}_2 X'$. This is the tensor product of $(y \leftarrow x)$ with $(x' \rightarrow y')$, which is the pushout of (12.9) as we intended.

Now we need to identify the arrows occurring in $U \otimes \mathbb{P}_2 X'$ — apart from $p_1, p_2, p_3, j_1, j_2, \text{ and } j_3$, of course, which we make correct by definition. Naturality of coends again tells us that for any morphism $\gamma: c \rightarrow c'$ in $C$, the arrow $(U \otimes \mathbb{P}_2 X') \gamma \in \mathcal{D}(2)$ can be identified with $U \gamma \otimes \mathbb{P}_2 X'$, which is a left Kan extension along the projection $(2 \times \ast) \rightarrow 2$. For instance, the arrow towards the top left of $U \otimes \mathbb{P}_2 X'$ which should be $f \otimes 1 x'$ is obtained from the solid-arrow $(2 \times \ast)$-diagram below by extending to the dotted arrows:

\[
\begin{array}{ccc}
x \otimes x' & \xleftarrow{f \otimes 1 x'} & 0_1 \otimes x' \cong 0 \\
\downarrow & \downarrow & \downarrow \\
y \otimes x' & \xleftarrow{g \otimes 1 x'} & 0_1 \otimes x' \cong 0 \\
\downarrow & \downarrow & \downarrow \\
x \otimes x' & \xleftarrow{h \otimes 1 x'} & 0_1 \otimes y' \cong 0 \\
\downarrow & \downarrow & \downarrow \\
y \otimes x' & \xleftarrow{f \otimes 1 y'} & 0_1 \otimes y' \cong 0 \\
\downarrow & \downarrow & \downarrow \\
x \otimes x' & \xleftarrow{f \otimes 1 z'} & 0_1 \otimes z' \cong 0.
\end{array}
\]

Up to isomorphism, the solid arrow diagram is obtained by restriction from a 3-diagram

\[
0 \rightarrow x \otimes x' \xrightarrow{f \otimes 1 x'} y \otimes x'
\]

along a particular functor $2 \times \ast \rightarrow 3$. However, this functor extends to a functor $2 \times \square \rightarrow 3$, restriction along which produces some diagram looking like the entire cube above, with the morphism $f \otimes 1 x'$ as labeled. By Lemma 4.12 the front and back face of this cube are cocartesian, and hence by Lemma 4.3 it must be the left Kan extension which computes the desired morphism in Figure 5. Thus, that morphism must be $f \otimes 1 x'$, as desired.

Similar arguments easily identify the morphisms that should be $g \otimes 1 x'$, $h \otimes 1 x'$, $\Sigma^{-1}(h \otimes 1 y')$, $f \otimes 1 y'$, $g \otimes 1 y'$, $\Sigma^{-1}(g \otimes 1 z')$, $\Sigma^{-1}(h \otimes 1 z')$, and $f \otimes 1 z'$ — all those whose primed factor is an identity. In some cases, we must restrict from a $\square$-diagram instead of a 3-diagram, such as for $g \otimes 1 x'$ where we have

\[
\begin{array}{c}
0 \cong 0_1 \otimes x' \rightarrow x \otimes x' \\
\downarrow \\
0 \cong 0_3 \otimes x' \rightarrow y \otimes x'.
\end{array}
\]

But since any zero object is uniquely isomorphic to any other zero object, this makes no essential difference.
Now consider the morphism that should be $1_x \otimes f'$. The above argument implies that it is the left Kan extension to $2$ of the solid arrow diagram

\begin{equation}
\begin{array}{c}
x \otimes x' & \xymatrix{ & 0_1 \otimes x' \cong 0} & \xymatrix{ & x \otimes x'} \\
\downarrow & & \downarrow \\
x \otimes x' & \xymatrix{ & 0_1 \otimes y' \cong 0} & \xymatrix{ & 1_x \otimes f'} \\
\downarrow & & \downarrow \\
x \otimes y' & \xymatrix{ & 1_x \otimes f'} & \xymatrix{ & x \otimes y'}. \\
\end{array}
\end{equation}

But here again, the solid arrow part of (12.12) is obtained by restriction from a diagram of the form

$$0 \longrightarrow x \otimes x' \xymatrix{ & 1_x \otimes f'} x \otimes y'.$$

And we can obtain an entire cube of the form (12.12), with the morphism $1_x \otimes f'$ as labeled, by restricting along a further functor $2 \times \Box \to 3$. We conclude as before that this cube is cocartesian, hence determines the desired morphism to be $1_x \otimes f'$. The same argument applies for $1_y \otimes f'$ and $1_z \otimes f'$.

The morphisms $\Sigma^{-1}(1_y \otimes g')$, $1_x \otimes g'$, and $1_y \otimes g'$ are likewise similar to each other. For $1_y \otimes g'$, we use the following cube

\begin{equation}
\begin{array}{c}
y \otimes x' & \xymatrix{ & y \otimes x'} \\
\downarrow & & \downarrow \\
y \otimes y' & \xymatrix{ & 0_1 \otimes x' \cong 0} & \xymatrix{ & y \otimes x'} \\
\downarrow & & \downarrow \\
y \otimes y' & \xymatrix{ & y \otimes y'} & \xymatrix{ & 1_y \otimes f'} \\
\downarrow & & \downarrow \\
y \otimes z' & \xymitt{y \otimes z'} & \xymitt{1_y \otimes g'} \\
\end{array}
\end{equation}

which can be obtained by restriction from the cocartesian square

\begin{equation}
\begin{array}{c}
y \otimes x' \xymitt{1_y \otimes f'} \longrightarrow 0_1 \otimes x' \cong 0 \\
\downarrow \\
y \otimes y' \xymitt{1_y \otimes g'} \longrightarrow y \otimes z'. \\
\end{array}
\end{equation}

The other two are analogous.
Now there are three morphisms left: $\Sigma^{-1}(1_y \otimes h')$, $\Sigma^{-1}(1_z \otimes h')$, and $-1_x \otimes h'$.

For $\Sigma^{-1}(1_y \otimes h')$, the relevant cube is

As in all the previous cases, the back and the front faces are cocartesian. However, in this case the top face is also cocartesian by definition of $U$ as a restriction of Figure 7. But then the bottom square is cocartesian by Lemma 4.10. Using the definition of the triangulation, we can use the back and the bottom faces in order to identify the morphism labeled by "?" as $\Sigma^{-1}(1_y \otimes h')$. The remaining two cases can be established in a similar way.

We have not yet explained the one minus sign in Figure 5. It appears on the morphism $-1_x \otimes h'$, but it should really not be regarded as fundamentally attached to that morphism; rather, it is merely a marker that the square with vertices $v$, $x \otimes z'$, $z \otimes x'$, and $\Sigma(x \otimes x')$ anticommutes rather than commutes.

The point is that in identifying the morphisms labeled $h \otimes 1_{x'}$ and $1_x \otimes h'$, we have identified their common codomain with $\Sigma(x \otimes x')$ in two different ways, which differ by the transposition automorphism $\sigma$ of $\Box$. To see this, let us restrict $U$ to the $(\downarrow \times 2^3)$-shaped diagram corresponding to these two morphisms. After taking the external product with $X'$ and pulling back to $\text{tw}(2)^{op}$, we obtain the solid arrow diagram in Figure 8, with the morphisms of interest being obtained by left extension to the dotted arrows.

When determining the morphism labeled $h \otimes 1_{x'}$, we identified the middle object with $\Sigma(x \otimes x') \cong (\Sigma x) \otimes x'$ by way of the top face of the right-hand cube in Figure 8 (together with the very middle face, which is trivial). This square is shown again.
on the left below:

\[
\begin{align*}
\displaystyle (y \leftarrow 0_1) & \longrightarrow (y \leftarrow x) \longrightarrow (0_4 \leftarrow 0_1) \\
\downarrow & \quad \downarrow \quad \downarrow \\
\displaystyle (0_4 \leftarrow 0_1) & \longrightarrow (0_4 \leftarrow x) \longrightarrow (\Sigma y \leftarrow 0_5) \\
\downarrow & \quad \downarrow \quad \downarrow \\
\displaystyle (0_2 \leftarrow \Sigma^{-1}z) & \longrightarrow (y \leftarrow x) \longrightarrow (0_5 \leftarrow 0_3) \\
\downarrow & \quad \downarrow \quad \downarrow \\
\displaystyle (0_2 \leftarrow 0_2) & \longrightarrow (y \leftarrow y) \longrightarrow (0_5 \leftarrow z) \\
\downarrow & \quad \downarrow \quad \downarrow \\
\displaystyle (x \leftarrow x) & \longrightarrow (y \leftarrow x) \longrightarrow (0_4 \leftarrow 0_4) \\
\downarrow & \quad \downarrow \quad \downarrow \\
\displaystyle (0_4 \leftarrow 0_3) & \longrightarrow (z \leftarrow 0_3) \longrightarrow (\Sigma x \leftarrow \Sigma x) \\
\end{align*}
\]

Figure 9. Cofiber sequences for (TC3)

It is oriented as shown in order to match (12.11). On the other hand, when determining the morphism labeled \( f \otimes h' \), we identified the middle object with \( \Sigma(x \otimes x') \approx x \otimes (\Sigma x') \) using the cofiber sequence shown on the right in (12.13), which consists of the right and bottom faces of the right-hand cube in Figure 8. Note that the two zero objects 0_3 and 0_4 have been transposed between the two squares in (12.13). Thus, these two identifications of the middle object with \( \Sigma(x \otimes x') \) differ by a minus sign.

It remains to construct the distinguished triangles. Consider the following diagrams of shape \((3 \times 2^{op})\), obtained from \( U \) by restriction.

\[
\begin{align*}
\displaystyle x \otimes x' & \longrightarrow 0_3 \otimes x' \\
\downarrow & \quad \downarrow \\
\displaystyle (\Sigma x) \otimes x' & \longrightarrow 0_4 \otimes x' \\
\downarrow & \quad \downarrow \\
\displaystyle x \otimes (\Sigma x') & \longrightarrow x \otimes z' \\
\end{align*}
\]

Applying the functor \((- \otimes_{[2]} X')\) to these diagrams yields the three 3-shaped diagrams \((\bullet \overset{P}{\longrightarrow} \bullet \overset{\iota}{\longrightarrow} \bullet)\) in Figure 5. Now since \( U \) is a restriction of Figure 7, so is each of these diagrams. Moreover, each one also underlies a cofiber sequence in \( \mathcal{D}^{2^{op}} \) which may also be obtained from Figure 7 by restriction; these cofiber sequences are shown in Figure 9. If we tensor these cofiber sequences with \( X' \), we obtain three cofiber sequences in \( \mathcal{D} \) that will give rise to our desired distinguished triangles. It remains to identify the third morphism in each such sequence.

The morphism that should be \( f \otimes h' \) is obtained by tensoring

\[
(12.14) \quad (0_4 \leftarrow x) \rightarrow (\Sigma y \leftarrow 0_5)
\]
with $X' = (x' \to y')$. But (12.14) factors as

$$(0_4 \leftarrow x) \to (\Sigma x \leftarrow 0_3) \to (\Sigma y \leftarrow 0_5),$$

and when tensored with $X'$ these two morphisms yield $1_x \otimes h'$ and $\Sigma f \otimes 1_{x'}$, by the sort of arguments given above. Thus, their composite is

$$(\Sigma f \otimes 1_{x'}) \circ (1_x \otimes h') = (f \otimes 1_{\Sigma x'}) \circ (1_x \otimes h') = f \otimes h'.$$

Similarly, $(y \leftarrow y) \to (0_5 \leftarrow z)$ factors through $(0_5 \leftarrow y)$, yielding $1_y \otimes g'$ and $g \otimes 1_z'$, while $(z \leftarrow 0_3) \to (\Sigma x \leftarrow \Sigma x)$ factors through $(\Sigma x \leftarrow 0_3)$, yielding $h \otimes 1_{x'}$ and $1_{\Sigma x} \otimes f'$. There is a minus sign in the second distinguished triangle because the outer rectangle of the second sequence in Figure 9 involves the square

$$
\begin{array}{ccc}
\Sigma^{-1} z & \longrightarrow & 0_3 \\
\downarrow & & \downarrow \\
0_2 & \longrightarrow & z
\end{array}
$$

which is transposed from the square used in (12.11) to identify $\Sigma^{-1} z$. \qed

Henceforth, we will use the notations $v, j_1, p_1$, etc. to refer to the specific objects and morphisms constructed in the proof of Theorem 12.10, which come equipped with certain coherent diagrams and cocartesian squares. We will also abuse notation by denoting the lifts $(y \overset{f}{\leftarrow} x) \in \mathcal{D}(2^{\text{op}})$ and $(x' \overset{f'}{\rightarrow} y') \in \mathcal{D}(2)$ also by $f$ and $f'$, so that we can write $v = f \otimes [2] f'$.

We stress again that $v, j_1, p_1$, etc. are determined up to unique canonical isomorphism as soon as we choose these objects of $\mathcal{D}(2^{\text{op}})$ and $\mathcal{D}(2)$. This is in contrast to the situation in [May01, §4], where the axioms can only assert that some objects and morphisms exist, without any uniqueness. As mentioned in the introduction, this is the main advantage of derivators over triangulated categories.

The following lemma, combined with Proposition 6.8, implies that we actually have the “stronger” form of (TC3) from [May01, Definition 4.11].

**Lemma 12.15.** In the coherent diagram Figure 5 constructed in Theorem 12.10, the six squares that have $v$ as a vertex are cocartesian.

**Proof.** Since the tensor product $\otimes_2$ is cocontinuous in both variables, it suffices to show that the corresponding squares in Figure 6 are all cocartesian. But all of these either occur in a cofiber sequence or are constant in one direction. \qed

We may also consider rotating one or the other of the input triangles in (TC3). The following lemma implies that it doesn’t matter which one we rotate. Recall the functor $\text{cof}: \mathcal{D}(2) \to \mathcal{D}(2)$ from §5.

**Lemma 12.16.** Given $(y \overset{f}{\leftarrow} x) \in \mathcal{D}(2^{\text{op}})$ and $(x' \overset{f'}{\rightarrow} y') \in \mathcal{D}(2)$, we have a canonical isomorphism

$$
\text{cof}(f) \otimes [2] f' \cong f \otimes [2] \text{cof}(f').
$$

This isomorphism identifies $p_1, p_2,$ and $p_3$ on the left with $p_2, p_3,$ and $p_1$ on the right, respectively, and similarly for the $j$’s.
Proof. We extend \( f \) and \( f' \) to cocartesian squares

\[
\begin{array}{ccc}
  z & \xleftarrow{\text{cof}(f)} & y \\
  & f & \\
  & & \\
 0 & \xleftarrow{f} & x
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
  x' & \xrightarrow{f'} & y' \\
  & \text{cof}(f') & \\
  & & \\
 0 & \xrightarrow{f'} & z'
\end{array}
\]

Denote these objects by \( Q \in \mathcal{D}(\square^{op}) \) and \( Q' \in \mathcal{D}(\square) \), respectively. Now since \( Q' \) is cocartesian, we have \( Q' \cong (i^r)_!(i^r)^*Q' \). Thus, by Corollary 8.9 we have

\[
Q \otimes [\square] Q' \cong Q \otimes [\square] (i^r)_!(i^r)^*Q' \\
\cong (i^{op})^*Q \otimes _{[\square]} (i^r)^*Q'.
\]

On the other hand, we also have \((i^{op})^*Q \cong (i^{op})_!(\text{cof}(f))\), where \( i: 2 \to \Gamma \) is the inclusion of \((0,0) \to (0,1)\). Thus, by Corollary 8.9 again, we have

\[
(i^{op})^*Q \otimes _{[\square]} (i^r)^*Q' \cong (i^{op})_!(\text{cof}(f)) \otimes _{[\square]} (i^r)^*Q' \\
\cong \text{cof}(f) \otimes _{[2]} (i^r)^*Q' \\
\cong \text{cof}(f) \otimes _{[2]} f'.
\]

A dual argument shows \( Q \otimes [\square] Q' \cong f \otimes _{[2]} \text{cof}(f') \), yielding (12.17).

To identify the \( p_i's \) and \( j'_i's \), we use the functoriality of this identification in \( f \). According to the proof of Theorem 12.10, \( p_1 \) and \( j_1 \) for \( f \otimes _{[2]} \text{cof}(f') \) are obtained by tensoring \( \text{cof}(f') \) with the diagram on the left below

\[
\begin{array}{ccc}
  0 & \xleftarrow{f} & x \\
  & \text{cof}(f) & \\
  & & \\
  y & \xleftarrow{f} & x \\
  & & \\
  y & \xleftarrow{0} & 0
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
  0 & \xleftarrow{\Sigma x} & 0 \\
  & \text{cof}(f) & \\
  & & \\
  y & \xleftarrow{y} & y \\
  & & \\
  y & \xleftarrow{y} & y
\end{array}
\]

But by the above argument, this is isomorphic to what we obtain by tensoring the diagram on the right with \( f' \). Again, the proof of Theorem 12.10 tells us that this yields \( p_1 \) and \( j_1 \) for \( \text{cof}(f) \otimes _{[2]} f' \), as intended. The other two cases are established in a similar way. \( \square \)

Following [May01], we denote the common value of \( \text{cof}(f) \otimes _{[2]} f' \) and \( f \otimes _{[2]} \text{cof}(f') \) by \( w \). By Lemma 12.16, along with Lemma 5.13, \( w \) can equivalently be defined as

\[
w = \text{cof}(f) \otimes _{[2]} f' \\
\cong \text{cof}^2(\text{fib}(f)) \otimes _{[2]} \text{cof}(\text{fib}(f')) \\
\cong \text{cof}^3(\text{fib}(f)) \otimes _{[2]} \text{fib}(f') \\
\cong \Sigma(\text{fib}(f)) \otimes _{[2]} \text{fib}(f') \\
\cong \Sigma(\text{fib}(f) \otimes _{[2]} \text{fib}(f')).
\]

The last is how it is defined in [May01]: by rotating both triangles backwards, obtaining \((-\Sigma^{-1}h, f, g)\) and \((-\Sigma^{-1}h', f', g')\), then suspending the result once. The
object $w$ comes with distinguished triangles

$$
x ⊗ z' \xrightarrow{k_1} w \xrightarrow{q_1} z ⊗ y' \xrightarrow{h ⊗ g'} \Sigma(x ⊗ z'),
$$

$$
y ⊗ y' \xrightarrow{k_2} w \xrightarrow{q_2} \Sigma(x ⊗ x') \xrightarrow{- \Sigma(f ⊗ f') \Sigma(y ⊗ y')},
$$

$$
z ⊗ x' \xrightarrow{k_3} w \xrightarrow{q_3} y ⊗ z' \xrightarrow{g ⊗ h'} \Sigma(z ⊗ x')
$$
equipped with coherent diagrams similar to those in (TC3).

In [May01] the existence of $w$ and its associated data is called (TC3'). The next axiom is about the interaction of (TC3) and (TC3').

**Theorem 12.18** (TC4). With all the data chosen as above, there is a cocartesian square

$$
\begin{array}{ccc}
\nu & \xrightarrow{j_2} & y ⊗ y' \\
\downarrow{(j_1, j_3)} & & \downarrow{k_2} \\
(x ⊗ z') ⊕ (z ⊗ x') & \xrightarrow{[k_1, k_3]} & w.
\end{array}
$$

May’s (TC4), in the strong form of his Definition 4.11, follows from this by Proposition 6.8.

**Proof.** Consider the following square in $\mathcal{D}(2^{op})$

$$
\begin{array}{ccc}
(y ← x) & \xrightarrow{(y ← y)} & (y ← y) \\
\downarrow & & \downarrow \\
(z ← x) & \xrightarrow{(z ← y)} & (z ← y).
\end{array}
$$

(12.19)

This is easily obtained by restriction from the canonical square

$$
\begin{array}{ccc}
x & \xrightarrow{y} & 0 \\
\downarrow & & \downarrow \\
0 & \xrightarrow{z} & z.
\end{array}
$$

(12.20)

Moreover, (12.19) is cocartesian, since its two underlying squares in $\mathcal{D}$ are so. Now take the tensor product of (12.19) with $(x' → y') \in \mathcal{D}(2)$, obtaining a cocartesian square in $\mathcal{D}$

$$
\begin{array}{ccc}
\nu & \xrightarrow{j_2} & y ⊗ y' \\
\downarrow & & \downarrow{k_2} \\
? & \xrightarrow{w} & w.
\end{array}
$$

(12.21)

The objects $\nu$ and $y ⊗ y'$ and the morphism $j_2$ in this diagram are identified purely by their definition in the proof of Theorem 12.10 (TC3). Similarly, the lower-right corner is $\text{cof}(f) ⊗ [2] f'$, which is by definition the object $w$. And since $k_2$ is by definition the morphism $p_3$ for $\text{cof}(f)$ and $f'$, the definition of $p_3$ in the proof of Theorem 12.10 identifies the morphism $k_2$ in (12.21). It remains to identify the missing object and the two morphisms connected to it.
Note that the morphism \((z \leftarrow x)\) at the lower-left corner of (12.19) is the zero morphism. Moreover, we can also obtain by restriction from (12.20) a \(\triangledown\)-diagram in \(\mathcal{D}(2^{\text{op}})\)

\[
(0 \leftarrow x) \\
\downarrow \\
(z \leftarrow 0) \longrightarrow (z \leftarrow x).
\]

Since its two underlying \(\triangledown\)-diagrams in \(\mathcal{D}\) are coproduct diagrams, this diagram is a coproduct in \(\mathcal{D}(2^{\text{op}})\). Thus, since \(\otimes[2]\) is cocontinuous in each variable, the object in (12.21) labeled \(?\) is the coproduct \((0 \leftarrow x) \otimes[2] (x' \rightarrow y') \oplus (z \leftarrow z') \oplus (z \otimes x').\)

We can now identify the two missing morphisms by composing them with the projections and coprojections of this biproduct, which in turn must be obtained from those of the biproduct \((z \leftarrow x) \cong (0 \leftarrow x) \oplus (z \leftarrow 0)\). We leave it to the reader to verify that this yields precisely the definitions of \(j_1\) and \(j_3\) from the proof of Theorem 12.10, and those of \(k_1\) and \(k_3\) coming from the construction of \(w\). \(\square\)

13. The braid duality axiom

May’s final axiom (which comes in two parts) is about the interaction of the pushout products with duality. Following [May01], in a closed symmetric monoidal category with unit object \(S\), we refer to the internal-hom \(x \triangledown S\) as the canonical dual of \(x\) and write it as \(Dx\). There is a canonical evaluation map \(\epsilon: Dx \otimes x \rightarrow S\), defined by adjunction from the identity of \(Dx\). See [DP80, LMSM86].

More generally, the two-variable morphism \(\triangledown: \mathcal{D}^{\text{op}} \times \mathcal{D} \rightarrow \mathcal{D}\) induces a one-variable morphism

\[
D(-) := (- \triangledown S): \mathcal{D}^{\text{op}} \rightarrow \mathcal{D}.
\]

If \(X \in \mathcal{D}(A^{\text{op}})\), we will sometimes refer to \(DX \in \mathcal{D}(A)\) as its pointwise (canonical) dual, since by naturality we have \((DX)_a \cong D(X_a)\). We have a generalized evaluation map \(\epsilon: DX \otimes [A] X \rightarrow S\) defined analogously.

Since \(\triangledown^{\text{op}}\) is cocontinuous in both variables, \(D(-)\) takes colimits in \(\mathcal{D}\) to limits in \(\mathcal{D}\). In particular, it takes cofiber sequences to fiber sequences, and when \(\mathcal{D}\) is stable, it preserves bicartesianness of squares. Expressed at the level of \(\mathcal{D}(1)\), this says that the functor \(D(-): \mathcal{D}(1)^{\text{op}} \rightarrow \mathcal{D}(1)\) preserves distinguished triangles, in the sense that if \((f, g, h)\) is distinguished, then so is \((- Dh, Dg, Df)\). (This is also a special case of Theorem 12.8 (TC2).) By rotating the latter, we obtain another distinguished triangle \((Dg, Df, D\Sigma^{-1}h)\).

**Theorem 13.1 (TC5a).** If \(w\) is the object chosen in (TC3') for the triangles \((Dg, Df, D\Sigma^{-1}h)\) and \((f, g, h)\), then there is a map \(\overline{\epsilon}: w \rightarrow S\) such that the following diagram commutes:

\[
\begin{array}{ccc}
(Dz \otimes z) \oplus (Dx \otimes x) & \xrightarrow{[\overline{\epsilon}_1, \overline{\epsilon}_3]} & w \\
\downarrow \cong & \quad & \downarrow \overline{\epsilon}_2 \quad & \quad \downarrow \overline{\epsilon}_1 \\
S & \xleftarrow{\epsilon} & Dy \otimes y
\end{array}
\]
Proof. As in the proof of Lemma 12.16, we can obtain \( \overline{w} \) as the canceling tensor product over \( \square \) of the two canonical diagrams

\[
\begin{array}{ccc}
Dx & \xleftarrow{df} & Dy \\
\uparrow & & \uparrow \\
0 & \xleftarrow{dz} & Dz
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
x & \xrightarrow{f} & y \\
\downarrow & & \downarrow \\
0 & \xrightarrow{gz} & z
\end{array}
\]

The second of these is the standard cofiber sequence of \((x \xrightarrow{f} y) \in \mathcal{D}(2)\); let us denote it by \( X \). The first is obtained from \( X \) by the morphism \( D(\cdot) \). But now, the counit of the adjunction

\[
\mathcal{D}(1)(M \otimes_\square X, N) \cong \mathcal{D}(\square^{op})(M, X \rhd N)
\]

at \( S \) is a morphism

\[
\overline{w} = DX \otimes_\square X \rightarrow S,
\]

which we may denote by \( \overline{\tau} \).

It remains to show that \( \tau \circ \overline{k}_i = \epsilon \) for \( i = 1, 2, 3 \). Let \( Y \) denote the square

\[
\begin{array}{ccc}
y & \xrightarrow{f} & y \\
\downarrow & & \downarrow \\
0 & \xrightarrow{gz} & 0
\end{array}
\]

Then by the proof of Lemma 12.16, \( \overline{k}_2 \) is induced by tensoring \( X \) with the map of squares \( DY \to DX \) which is induced by the evident map of squares \( X \to Y \). Thus, by extraordinary naturality of the counits of adjunctions, we have a commutative square

\[
\begin{array}{ccc}
DY \otimes_\square X & \xrightarrow{\gamma} & DY \otimes_\square Y \\
\downarrow{\overline{k}_2} & & \downarrow{\tau} \\
DX \otimes_\square X & \xrightarrow{\tau} & S.
\end{array}
\]

It is easy to see that the top map here is an isomorphism, both objects being isomorphic to \( Dy \otimes y \), and that \( \epsilon \) is simply the counit of \( y \). Thus, we have \( \tau \circ \overline{k}_2 = \epsilon \), as desired. The cases \( i = 1, 3 \) are nearly identical. \( \square \)

The second half of (TC5) is the only axiom which makes essential use of the symmetry of \( \mathcal{D} \). If \((f, g, h)\) and \((f', g', h')\) are distinguished triangles, then (after choosing objects of \( \mathcal{D}(2) \) representing \( f \) and \( f' \)) we have pushout products \( f \otimes_{\square} f' \) and \( f' \otimes_{\square} f \), which following [May01] we denote by \( v \) and \( \overline{v} \). Similarly, we have \( w \) and \( \overline{w} \) as in (TC3'), and we denote the maps \( p_i, j_i, k_i, q_i \) related to \( f' \otimes_{\square} f \) likewise with bars.

It is easy to see that the symmetry \( s \) of \( \mathcal{D} \) induces canonical isomorphisms \( \gamma : v \xrightarrow{s} \overline{v} \) and \( \gamma : w \xrightarrow{s} \overline{w} \). However, the asymmetry of the proof of Theorem 12.10 means that some argument is required to deduce that these isomorphisms and \( s \) are compatible with the maps \( p_i, j_i, k_i, q_i \).
Lemma 13.2. In the situation above, we have
\[ \gamma \circ p_1 = \overline{p}_1 \circ s, \quad \gamma \circ p_2 = \overline{p}_2 \circ s, \quad \gamma \circ p_3 = \overline{p}_3 \circ s. \]
\[ s \circ j_1 = \overline{j}_1 \circ \gamma, \quad s \circ j_2 = \overline{j}_2 \circ \gamma, \quad s \circ j_3 = \overline{j}_3 \circ \gamma. \]

Proof. Consider the first equation, involving \( p_1 \) and \( p_3 \). By two-variable functoriability of \( \otimes \mathbb{P}[2] \), the following square commutes
\[
(y \leftarrow 0) \otimes \mathbb{P}[2] (x' \rightarrow x') \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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Proof. Suppose given a cofiber sequence \( X \in \mathcal{D}(\mathbb{1}) \)
\[
\begin{array}{c}
x \xrightarrow{f} y \\
\downarrow \quad \downarrow \quad \downarrow \\
0 \xrightarrow{g} z \xrightarrow{h} \Sigma x.
\end{array}
\]
Then we have the evaluation map of the pointwise dual, \( \epsilon : DX \otimes \mathbb{1} X \to \mathbb{S} \), which
we can tensor with \( u \in \mathcal{D}(\mathbb{1}) \) to obtain a morphism
\[
(u \otimes DX) \otimes \mathbb{1} X \xrightarrow{\sim} u \otimes (DX \otimes \mathbb{1} X) \xrightarrow{\sim} u \otimes \mathbb{S} \xrightarrow{\sim} u
\]
in \( \mathcal{D}(\mathbb{1}) \). The adjunct of this is a morphism
\[
\mu_{X,u} : u \otimes DX \to X \triangleright u
\]
in \( \mathcal{D}(\mathbb{1}) \). By naturality, we have \((\mu_{X,u})(1,2) = \mu_{x,u} \) and so on; thus the underlying
morphism of \( \mu_{X,u} \) in \( \mathcal{D}(\mathbb{1}) \) gives a morphism of distinguished triangles
\[
\Sigma^{-1}(u \otimes Dx) \xrightarrow{-Dh} u \otimes Dz \xrightarrow{Dg} u \otimes Dy \xrightarrow{Df} u \otimes Dx
\]
\[
\Sigma^{-1}\mu_{x,u} \quad \mu_{x,u} \quad \mu_{x,u} \quad \mu_{x,u}
\]
\[
\Sigma^{-1}(x \triangleright u) \xrightarrow{-h_{\triangleright 1}} z \triangleright u \xrightarrow{g_{\triangleright 1}} y \triangleright u \xrightarrow{f_{\triangleright 1}} x \triangleright u.
\]
However, both the domain and codomain of \( \mu_{x,u} \) are cofiber sequences, and a
morphism of bicartesian squares is an isomorphism as soon as it restricts to an
isomorphism on \( \triangleright \) or \( \triangleleft \), since \( (iv) \) and \( (i)_* \) are fully faithful. The result follows. \( \square \)

The last step in this proof can be regarded as a stable-derivator version of the
“five-lemma” for triangulated categories, which says that a morphism of distin-
guished triangles which is an isomorphism at two places is also an isomorphism at
the third. The derivator version refers instead to morphisms of (coherent) cofiber
sequences, and is true whether or not the derivator is strong.

Corollary 13.4. The pushout of any span of dualizable objects in a stable monoidal
derivator is dualizable.

Proof. By Lemma 13.3 together with Proposition 6.8, and the fact that finite co-
products of dualizable objects in an additive monoidal category are dualizable. \( \square \)

We will also need the more general notion of bicategorical duality, in the special
case of \( \mathcal{P}rof(\mathcal{D}) \). Namely, we say that \( X \in \mathcal{P}rof(\mathcal{D})(A,B) = \mathcal{D}(A \times B^\text{op}) \) is right
dualizable if the canonical map
\[
(13.5) \quad \mu_{X,U} : U \otimes_{[B]} (X \triangleright_{[B]} 1_B) \to X \triangleright_{[B]} U
\]
is an isomorphism, for all \( U \in \mathcal{P}rof(\mathcal{D})(C,B) \). As before, it suffices to require this
when \( C = A \) and \( U = X \); see e.g. [MS06, Ch. 16] for the general theory of duality
in bicategories.

We write \( D_rX = X \triangleright_{[B]} 1_B \in \mathcal{P}rof(\mathcal{D})(B,A) \) and call it the canonical right
dual. Similarly, we have the canonical left dual \( D_lX = 1_A \triangleleft_{[A]} X \). If \( X \) is right
dualizable, then \( D_rX \) is left dualizable and \( X \cong D_lD_rX \). We remark on some easy
ways to obtain right dualizable objects.
Lemma 13.6. For any \( f: A \to B \), the diagram \((f \times 1)^*1_B \in \mathcal{P}rof(\mathcal{D})(A,B)\) is right dualizable.

Proof. For any \( U \in \mathcal{P}rof(\mathcal{D})(C,B) \), the total mate of Corollary 8.9(ii) gives an isomorphism
\[
( f \times 1 )^*1_B \triangleright [B] \cong (1 \times f^{op})^*[B] \cong (1 \times f^{op})^*U.
\]
Thus, when \( X = (f \times 1)^*1_B \), (13.5) becomes
\[
U \otimes [B] (1 \times f^{op})^*1_B \longrightarrow (1 \times f^{op})^*U
\]
which is just pseudonaturality of \( \otimes_{[B]} \) in parameters, together with the unit isomorphism of \( \mathcal{P}rof(\mathcal{D}) \).

In particular, the proof of Lemma 13.6 shows that the canonical right dual of \((f \times 1)^*1_B\) is \((1 \times f^{op})^*1_B\). These are analogues of the classical representable profunctors \( B(-,f) \) and \( B(f,-) \).

Lemma 13.7. If \( X \in \mathcal{P}rof(\mathcal{D})(A,B) \) and \( Y \in \mathcal{P}rof(\mathcal{D})(B,C) \) are right dualizable, so is \( X \otimes [B] Y \).

Proof. \( \mu_{X \otimes [B] Y, U} \) factors as \( \mu_{X,Y,U} \) followed by \( 1_X \triangleright [B] \mu_{Y,U} \).

We can also identify the canonical right dual of \( X \otimes [B] Y \) as \( D_r Y \otimes [B] D_r X \).

Lemma 13.8. A coherent diagram \( X \in \mathcal{P}rof(\mathcal{D})(A,B) \) is right dualizable if and only if \( X_a \in \mathcal{P}rof(\mathcal{D})(1,B) \) is right dualizable for every \( a \in A \).

Proof. By (Der2), the map (13.5) is an isomorphism just when it is so when restricted to each \( a \in A \). But all the functors involved are pseudonatural in the extra variables, so the restriction of \( \mu_{X,U} \) to \( a \in A \) is \( \mu_{X,a,U} \). Thus, \( \mu_{X,U} \) is an isomorphism if and only if each \( \mu_{X,a,U} \) is so.

In particular, Lemma 13.8 tells us that \( X \in \mathcal{D}(A) \) is right dualizable, when regarded as a morphism from \( A \) to \( 1 \) in \( \mathcal{P}rof(\mathcal{D}) \), if and only if each object in its underlying diagram is dualizable in \( \mathcal{D}(1) \). For instance, an object \((x \overset{f}{\to} y) \in \mathcal{D}(2)\), regarded as a morphism from \( 2 \) to \( 1 \) in \( \mathcal{P}rof(\mathcal{D}) \), is right dualizable just when \( x \) and \( y \) are dualizable, and in this case its right dual is just its “pointwise dual” \((D_x \overset{Df}{\leftarrow} D_y) \in \mathcal{D}(2^{op})\).

On the other hand, in general, right dualizability of \( X \in \mathcal{P}rof(\mathcal{D})(1,A) \) depends not just on the objects of \( X \) but on \( A \). But in the stable case, we have the following.

Lemma 13.9. Let \( X \in \mathcal{P}rof(\mathcal{D})(1,2) = \mathcal{D}(2^{op}) \). If the underlying objects of \( X \) are dualizable, then \( X \) is right dualizable.

Proof. Suppose \( X = (y \overset{\xi}{\leftarrow} x) \). As usual, we may extend \( X \) to a bicartesian square of the form
\[
\begin{array}{ccc}
u & \xrightarrow{k} & x \\
\downarrow & & \downarrow f \\
0 & \xrightarrow{j} & y 
\end{array}
\]
By restriction, we obtain a cube
\[
\begin{array}{c c c c c c c}
u & \rightarrow & 0 \\
\downarrow & & \downarrow \\
x & \rightleftarrows & x \\
0 & \leftleftarrows & 0 \\
\downarrow & & \downarrow \\
y & \leftarrow & f & \rightarrow & x.
\end{array}
\]
(13.10)

Its left and right face are bicartesian, so by Lemma 4.3 it is bicartesian in \(\mathcal{C}^{op}\). Therefore, by the bicategorical version of Lemma 13.3 (whose proof is identical), it suffices to show that \((u \leftarrow 0)\) and \((x \leftarrow x)\) are right dualizable. But we have
\[
(u \leftarrow 0) \cong u \otimes (S \leftarrow 0)\quad \text{and} \quad (x \leftarrow x) \cong x \otimes (S \leftarrow S),
\]
so since \(x\) and \(u\) are dualizable (the latter by Lemma 13.3), by Lemma 13.7 it suffices to show that \((S \leftarrow 0)\) and \((S \leftarrow S)\) are right dualizable. But since \(1_2 \in \mathcal{P}rof(\mathcal{D})(2, 2) = \mathcal{D}(2 \times 2^{op})\) is given by
\[
\begin{array}{c c c c}
S & \leftarrow & 0 \\
\downarrow & & \downarrow \\
S & \leftarrow & S
\end{array}
\]
these are just the “representable profunctors” from Lemma 13.6 for the two functors \(1 \to 2\).

The proof of Lemma 13.9 also enables us to identify the bicategorical right dual of \((y \leftarrow f \leftarrow x)\). Namely, applying \(D_r\) to (13.10) regarded as an object of \(\mathcal{P}rof(\mathcal{D})(\square, 2)\), we obtain an object of \(\mathcal{P}rof(\mathcal{D})(2, \square)\), which we claim must look like
\[
\begin{array}{c c c c c c c}
Du & \rightarrow & Du & \leftarrow & Dk \\
\downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & Dx & \leftarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
Dz & \leftarrow & Dy & \leftarrow & Dg \\
Df & \rightarrow & Df & \rightarrow & Df
\end{array}
\]
(13.11)

where \((y \rightarrow z) = \text{cof}(x \rightarrow y)\). By Lemma 13.6, we have \(D_r(S \leftarrow S) \cong (0 \to S)\) and \(D_r(0 \leftarrow S) \cong (S \rightarrow S)\). By Lemma 13.7, taking tensor products of these with \(x\) and \(u\) respectively corresponds in the dual to taking tensor products with \(Dx\) and \(Du\). This (together with its functoriality) identifies the top face of (13.11).

Since (13.11) is also bicartesian, its right and left faces must also be bicartesian. This identifies the arrow \(Df\), since it must be the fiber of \(Dk\). Since the back face of (13.11) is trivially bicartesian, its front face must be as well. This identifies the arrow \(Dg\), since it must be the fiber of \(Df\). (Of course, we have \(z \cong \Sigma u\).) Thus, the right dual of \(f\) is canonically isomorphic to the pointwise dual of its cofiber:
\[
D_r(f) \cong D(\text{cof}(f)).
\]

We can now construct what in [May01] are called duals of diagrams.
Lemma 13.12. For $x \xrightarrow{f} y$ and $x' \xrightarrow{f'} y'$, where $x$ and $y$ are dualizable, we have
\begin{equation}
D(f \otimes_{[2]} f') \cong Df' \otimes_{[2]} D(\text{cof}(f)).
\end{equation}

Moreover, under this isomorphism we have the following left-to-right correspondences between the morphisms from (TC3)
\begin{align*}
p_1 &\leftrightarrow j_2, \quad p_2 \leftrightarrow j_3, \quad p_3 \leftrightarrow j_1, \\
j_1 &\leftrightarrow p_2, \quad j_2 \leftrightarrow p_3, \quad j_3 \leftrightarrow p_1.
\end{align*}

Proof. By Lemma 13.9, the chosen object $(y \xrightarrow{f} x) \in \mathcal{D}(2^{\text{op}}) = \text{Prof}(\mathcal{D})(1, 2)$ is right dualizable. Thus, using Lemma 13.8 for the final step, we have
\begin{align*}
D(f \otimes_{[2]} f') &\cong (f \otimes_{[2]} f') \triangleright_{[2]} I_{[2]} \\
&\cong f \triangleright_{[2]} (f' \triangleright_{[2]} I_{[2]}) \\
&\cong f \triangleright_{[2]} D_r f' \\
&\cong D_{r} f' \otimes_{[2]} D_{r} f \\
&\cong Df' \otimes_{[2]} D(\text{cof}(f)).
\end{align*}

Of course, this isomorphism is natural in $f$ and $f'$. Moreover, it is easy to verify that when $f$ or $f'$ has one of the “degenerate” forms appearing in the outer parts of Figure 6, this isomorphism is essentially an identity. Thus, we can easily trace through the construction of the $p$'s and $j$'s to find the desired identifications. \hfill \Box

In particular, if we let $f'$ be $D(\text{cof}(f))$, then $f \otimes_{[2]} f'$ in (13.13) occurs in a (TC3) diagram for distinguished triangles $(f, g, h)$, say, and $(Dg, Df, D\Sigma^{-1} h)$, while $Df' \otimes_{[2]} D(\text{cof}(f))$ occurs in a (TC3') diagram for these same triangles in the reverse order (using the identification of $f$ with its double pointwise dual). Following [May01], we call the (TC3') diagram for $(Dg, Df, D\Sigma^{-1} h)$ and $(f, g, h)$ the dual of the (TC3) diagram for $(f, g, h)$ and $(Dg, Df, D\Sigma^{-1} h)$.

Theorem 13.14 (TC5b). If $x$, $y$, and $z$ are dualizable, then the (TC3') diagram specified in (TC5a) for the triangles $(Dg, Df, D\Sigma^{-1} h)$ and $(f, g, h)$ is isomorphic to the dual of a (TC3) diagram for the same triangles in reverse order, and satisfies the additivity axiom (TC4) with respect to an involution of the latter diagram.

Proof. Since all of our diagrams are canonically specified by universal properties, we can say that the dual of the (TC3) diagram for $(f, g, h)$ and $(Dg, Df, D\Sigma^{-1} h)$ is the (TC3') diagram for $(Df, Dg, D\Sigma^{-1} h)$ and $(f, g, h)$, while the involution of the former diagram is the (TC3) diagram for $(Dg, Df, D\Sigma^{-1} h)$ and $(f, g, h)$. Thus, since Theorem 12.18 (TC4) is a statement about the (TC3) and (TC3') diagrams for any pair of triangles, it holds for these. \hfill \Box

14. THE ADDITIVITY OF TRACES IN STABLE MONOIDAL DERIVATORS

Finally, we are ready to consider traces. For the reader’s convenience, we recall the basic definitions. First of all, if $x$ is dualizable in a symmetric monoidal category, then the canonical map
$$\mu_{x,x} : x \otimes Dx \rightarrow x \triangleright x$$
is an isomorphism. We also have a morphism $S \rightarrow x \triangleright x$ which is the adjunct of the unit isomorphism $S \otimes x \cong x$. The composite of this with $(\mu_{x,x})^{-1}$ defines a map
$$\eta : S \rightarrow x \otimes Dx$$
which is called the coevaluation of $x$. (Dualizability can equivalently be defined as the existence of such a coevaluation, satisfying the “triangle” or “zig-zag” identities relating it to the evaluation map $\epsilon: Dx \otimes x \to S$.) See [DP80, LMSM86] for further details.

The lemma below is identical to [May01, Lemma 4.14], and follows from Theorems 13.1 (TC5a) and 13.14 (TC5b).

**Lemma 14.1.** For $v$ as defined in (TC3) for the distinguished triangles $(f, g, h)$ and $(Dg, Df, D\Sigma^{-1}h)$, there is a map $\bar{\eta}: S \to v$ such that the following diagram commutes

\[
\begin{array}{c}
(z \otimes Dz) \oplus (x \otimes Dx) \\
\downarrow (j_3, j_1) \\
\downarrow v \\
\downarrow \eta \\
\downarrow \eta \\
\downarrow \bar{\eta} \\
\downarrow \eta \\
y \otimes Dy.
\end{array}
\]

Now, if $x$ is dualizable in any symmetric monoidal category, and $f: p \otimes x \to x \otimes q$ is a morphism, then the trace of $f$ is defined to be the composite

\[
p \cong p \otimes S \xrightarrow{id \otimes \eta} p \otimes x \otimes Dx \xrightarrow{s} Dx \otimes p \otimes x \xrightarrow{id \otimes f} Dx \otimes x \otimes q \xrightarrow{\epsilon \otimes id} S \otimes q \cong q.
\]

An important special case is when $p$ and $q$ are the unit object $S$, in which case the trace of $f: x \to x$ is an endomorphism of $S$. In the even more special case when $f$ is the identity morphism $1_x$, its trace is called the **Euler characteristic** of $x$.

The desired additivity theorem for traces says that given a distinguished triangle

\[
\begin{array}{c}
x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} \Sigma x
\end{array}
\]

with $x$ and $y$ (hence also $z$) dualizable, and a diagram of the form

\[
\begin{array}{c}
p \otimes x \xrightarrow{1_p \otimes f} p \otimes y \xrightarrow{1_p \otimes g} p \otimes z \xrightarrow{1_p \otimes h} p \otimes \Sigma x \\
x \otimes q \xrightarrow{f \otimes 1_q} y \otimes q \xrightarrow{g \otimes 1_q} z \otimes q \xrightarrow{h \otimes 1_q} \Sigma x \otimes q
\end{array}
\]

then we have

\[
\text{tr}(\phi_x) + \text{tr}(\phi_z) = \text{tr}(\phi_y).
\]

However, the problem is that if by “diagram” we mean an incoherent diagram, i.e. a diagram in a triangulated category, then this might not be true (see for instance [Fer06]). We need $(\phi_x, \phi_y, \phi_z)$ to be coherent in an appropriate sense.

We will first prove the theorem for an explicitly coherent choice of $\phi$'s. Then we will deduce a corollary whose statement, at least, makes sense in the language of triangulated categories.

**Theorem 14.2.** Let $\mathcal{D}$ be a closed symmetric monoidal stable derivator. Suppose we have a bicartesian square

\[
\begin{array}{c}
x \xrightarrow{f} y \\
\downarrow \quad \downarrow g \\
0 \xrightarrow{} z
\end{array}
\]
denoted $X \in \mathcal{D}(\square)$, objects $p, q \in \mathcal{D}(\square)$, and a morphism $\phi : p \otimes X \to X \otimes q$ in $\mathcal{D}(\square)$. If $x$ and $y$ (hence also $z$) are dualizable in $\mathcal{D}(1)$, then

$$\text{tr}(\phi_x) + \text{tr}(\phi_z) = \text{tr}(\phi_y)$$

as morphisms $p \to q$ in $\mathcal{D}(1)$.

Proof. We will construct the (incoherent) commutative diagram in $\mathcal{D}(1)$ shown in Figure 10. This will prove the theorem, since the composite around the left is $\text{tr}(\phi_x) + \text{tr}(\phi_z)$ and the composite around the right is $\text{tr}(\phi_y)$. As in §13 (TC5), we are using the notations:

- $v$ occurs in the (TC3) diagram for $(f, g, h)$ and $(DG, DF, D\Sigma^{-1}h)$.
- $w$ occurs in the (TC3') diagram for $(f, g, h)$ and $(DG, DF, D\Sigma^{-1}h)$.
- $\overline{w}$ occurs in the (TC3') diagram for $(DG, DF, D\Sigma^{-1}h)$ and $(f, g, h)$.

Thus, the lower triangles in Figure 10 commute by Theorem 13.1 (TC5a) tensored on the right with $q$, while the upper triangles commute by Lemma 14.1 tensored on the left with $p$. Similarly, the quadrilateral involving $p \otimes v$ and $p \otimes w$ commutes by Theorem 12.18 (TC4) tensored on the left with $p$.

Now since the tensor product is cocontinuous in each variable, it preserves all the constructions that go into (TC3) and (TC3'); thus $p \otimes w$ is (canonically isomorphic to) the object occurring in the (TC3') diagram for $(1_p \otimes f, 1_p \otimes g, 1_p \otimes h)$ and $(DG, DF, D\Sigma^{-1}h)$. (Technically, of course, the former means the bicartesian square $p \otimes X$.) Similarly, $\overline{w} \otimes q$ is the object occurring in the (TC3') diagram for $(DG, DF, D\Sigma^{-1}h)$ and $(f \otimes 1_q, g \otimes 1_q, h \otimes 1_q)$.

We define $\overline{w}$ to be the object defined in the (TC3') diagram for $(DG, DF, D\Sigma^{-1}h)$ and $(1_p \otimes f, 1_p \otimes g, 1_p \otimes h)$. Then Lemma 13.2 yields the dotted map $p \otimes w$ making the two adjacent trapezoids commute. Finally, since $\phi$ is a map of
bicartesian squares, the functoriality of the \((\text{TC3}')\) construction yields the second dotted map \(\tilde{\omega} \rightarrow \omega \otimes q\) making the remaining squares commute. \(\square\)

**Corollary 14.3.** Let \(\mathcal{D}\) be a closed symmetric monoidal stable derivator which is additionally strong. Suppose we have a distinguished triangle in \(\mathcal{D}(1)\)

\[
x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} \Sigma x
\]

in which \(x\) and \(y\) (hence also \(z\)) are dualizable, and morphisms \(\phi_x\) and \(\phi_y\) making the following diagram of solid arrows commute in \(\mathcal{D}(1)\)

\[
\begin{array}{c}
p \otimes x \xrightarrow{1_p \otimes f} p \otimes y \\
\downarrow \phi_x & & \downarrow \phi_y \\
x \otimes q \xrightarrow{f \otimes 1_q} y \otimes q
\end{array}
\begin{array}{c}
p \otimes z \xrightarrow{1_p \otimes h} p \otimes \Sigma x \\
\downarrow \phi_z & & \downarrow \Sigma \phi_x \\
z \otimes q \xrightarrow{g \otimes 1_q} \Sigma x \otimes q.
\end{array}
\]

Then there exists a morphism \(\phi_z\) making both squares commute, such that

\[
\text{tr}(\phi_x) + \text{tr}(\phi_z) = \text{tr}(\phi_y).
\]

**Proof.** By definition, a distinguished triangle in \(\mathcal{D}(1)\) is isomorphic to one arising from a cofiber sequence in \(\mathcal{D}\). Thus, we may assume given a bicartesian square \(X\) as in Theorem 14.2. Let \(X_0\) denote the restriction of \(X\) to an object \((x \xrightarrow{f} y) \in \mathcal{D}(2)\).

Since \(\mathcal{D}\) is strong, the “underlying diagram” functor \(\mathcal{D}(2) \rightarrow \mathcal{D}(1)^2\) is full. Thus, since \((\phi_x, \phi_y)\) defines a morphism \(1_p \otimes f \rightarrow f \otimes 1_q\) in \(\mathcal{D}(1)^2\), it must be in the image of some morphism \(\phi_0: p \otimes X_0 \rightarrow X_0 \otimes q\) in \(\mathcal{D}(2)\). Applying extension by zero and a left Kan extension to \(\phi_0\), we obtain \(\phi: p \otimes X \rightarrow X \otimes q\). Thus, we can apply Theorem 14.2. \(\square\)

Of course, when \(\phi_x\) and \(\phi_y\) are identities, we may choose \(\phi_0\) to be the identity of \(X_0\), so that \(\phi_z\) is also an identity. Thus we recover the additivity of Euler characteristics as a special case.

As remarked in the introduction, Corollary 14.3 is more general than [May01, Theorem 1.9] in three ways, of which the first two are fairly minor.

(i) Our traces are “twisted” on both sides, rather only in the target. This is just a choice; the model-categorical methods of [May01] can be extended to the general case. (The triangulated-category methods used therein for Euler characteristics, on the other hand, would require significantly more complicated axioms to generalize to traces.)

(ii) We use derivators rather than model categories. By itself, this is not important, since all known interesting derivators arise from model categories.

(iii) But, even for a derivator that does arise from a model category, our proof does not need any information about the model category. In particular, there is no requirement that all objects be fibrant, which is the only case treated precisely in [May01]. This may be useful, since there are interesting stable monoidal model categories in which not all objects are fibrant, such as sheaves of chain complexes or the parametrized spectra of [MS06].

Of course, as we have said, our main goal in using derivators is to provide a more flexible framework for future generalizations. In [PS13] we will use the theory developed in this paper to prove a much more general additivity theorem for traces, including bicategorical traces [Pon10, PS12b] for bicategorical duality, such as the
Costenoble–Waner duality of [MS06]. And in [GS13] we will study enrichment of a derivator over a monoidal derivator.

**APPENDIX A. THE CALCULUS OF MATES**

We briefly recall a very useful tool called the *calculus of mates* for natural transformations. Suppose given a square of functors containing a natural transformation

\[
\begin{array}{ccc}
A & \xrightarrow{f^*} & B \\
\downarrow h^* & & \downarrow k^* \\
C & \xleftarrow{g^*} & D.
\end{array}
\]

If the functors \(f^*\) and \(g^*\) have left adjoints \(f_1\) and \(g_1\) respectively, then \(\alpha\) has a **mate** transformation \(\alpha_! : g_! k^* \to h^* f_1\), defined to be the composite

\[
g_! k^* \xrightarrow{g_! k^* \eta} g_! k^* f^* f_1 \xrightarrow{g_! \alpha f_1} g_! g^* h^* f_1 \xrightarrow{\varepsilon h^* f_1} h^* f_1
\]

where \(\eta\) and \(\varepsilon\) denote the unit and counit of the adjunctions \(f_1 \dashv f^*\) and \(g_1 \dashv g^*\) respectively. Similarly, if instead \(k^*\) and \(h^*\) have right adjoints \(k_*\) and \(h_*\) respectively, then \(\alpha\) has another mate \(\alpha_* : f^* h_* \to k_* g^*\)

\[
f^* h_* \xrightarrow{\eta f^* h_*} k_* k^* f^* h_* \xrightarrow{k_* \alpha h_*} k_* g^* h^* h_* \xrightarrow{k_* g^* \varepsilon} k_* g^*.
\]

These operations are inverses, in that if we reorient \(\alpha_!\) to look like \(\alpha\)

\[
\begin{array}{ccc}
B & \xrightarrow{k^*} & D \\
\downarrow f_1 & & \downarrow g_1 \\
C & \xleftarrow{h_*} & A,
\end{array}
\]

then apply the second mate-construction to it (which we can do since \(f_1\) and \(g_1\) have right adjoints, namely \(f^*\) and \(g^*\)), then we have \((\alpha_!)_* = \alpha\), and dually. (This follows from the triangle identities for the adjunctions \(f_1 \dashv f^*\) and \(g_1 \dashv g^*\).)

On the other hand, if all four functors \(f^*, g^*, h^*, k^*\) have left adjoints \(f_1, g_1, h_1, k_1\) respectively, then we can apply the **first** mate-construction to \(\alpha_!\) to obtain \((\alpha_!)_1 : h_1 g_! h^* \to f_1 k_* f^*.\) In this case, we can also regard \(\alpha\) as a transformation

\[
\begin{array}{ccc}
A & \xrightarrow{g^* h^*} & A \\
\downarrow k^* f^* & & \downarrow k^* f^* \\
D & \xleftarrow{g^* h^*} & D.
\end{array}
\]

Then since \(g^* h^*\) and \(k^* f^*\) have left adjoints \(h_1 g_! h^*\) and \(f_1 k_* f^*\) respectively, we can construct a mate \(\alpha_!: h_1 g_! h^* \to f_1 k_* f^*\), and we have \(\alpha_! = (\alpha_!)_1\).

The mate-construction is functorial with respect to horizontal and vertical pastings of squares. This can be expressed formally as an isomorphism of double categories; see [KS74]. However, it is not a functor in the ordinary sense, and in particular the mate of an isomorphism need not be an isomorphism.

It is true, however, that if \(h^*\) and \(k^*\) are identities (or even equivalences), then \(\alpha\) is an isomorphism if and only if \(\alpha_!\) is so, and dually. When \(h^*\) and \(k^*\) are identities, mates are sometimes called **conjugates**; we prefer to call them **total mates**. In particular, if \(f^*\) and \(g^*\) have left adjoints and \(h^*\) and \(k^*\) have right adjoints, then
\( \alpha_1 : g^* k^* \to h^* f_1 \) is an isomorphism if and only if \( \alpha_* : f^* h_* \to k_* g^* \) is so. This is an example of total mates by the above fact about “iterated first mate-constructions”.

**Appendix B. Proof of Proposition 6.8**

We prove Proposition 6.8. Thus, suppose given a cocartesian square

\[
\begin{array}{ccc}
x & \xrightarrow{f} & y \\
g & \downarrow & j \\
z & \xleftarrow{k} & w.
\end{array}
\]

We will construct a cofiber sequence

\[
\begin{array}{ccc}
y \oplus z & \xrightarrow{[j,k]} & w & \xrightarrow{0} \\
\downarrow & & \downarrow & & \downarrow \\
0 & \xrightarrow{\Sigma x} & \Sigma x & \xrightarrow{\Sigma(f,g)} & \Sigma y \oplus \Sigma z
\end{array}
\]

and hence a distinguished triangle

\[
\begin{array}{ccc}
y \oplus z & \xrightarrow{[j,k]} & w \\
\Sigma x & \xrightarrow{\Sigma(f,g)} & \Sigma y \oplus \Sigma z
\end{array}
\]

Rotating backwards will produce the desired result.

Let \( B \) denote the full subcategory of \( 3 \times 3 \times 2 \) consisting of the objects shown in Figure 11. We intend to build, first of all, a coherent diagram of shape \( B \) in \( \mathcal{D} \) that looks like Figure 12. We begin with the upper left-hand corner of (B.1), regarded as a diagram on the 000-100-010 subcategory of \( B \):

\[
\begin{array}{ccc}
x & \xrightarrow{f} & y \\
g & \downarrow & j \\
z & \xleftarrow{k} & w.
\end{array}
\]
Using right extension by zero along a sieve (Lemma 4.1), we obtain a diagram of the form

\[
\begin{array}{c}
x \xrightarrow{f} y \xrightarrow{0} 0 \\
g \downarrow \\
z \xrightarrow{k} w \xleftarrow{j}
\end{array}
\]

whose shape is the subcategory of \( B \) consisting of the objects 000, 100, 200, 010, 020, and 111. Now we perform a left Kan extension to obtain a diagram of shape \( B \); we claim it has the form of Figure 12. We can identify the object \( w \) and the morphisms \( j \) and \( k \) because the 000-100-010-110 square is cocartesian by Lemma 4.9, hence isomorphic to \( (B.1) \). Similarly, applying Lemma 4.9 to the squares 100-200-111-211 and 010-020-111-121, we identify the objects \( \Sigma y \) and \( \Sigma z \) respectively. We identify the object \( \Sigma x \) in a similar way using the square 000-200-020-220, but rather than Lemma 4.9 we have to use Lemma 4.5 with \( B' = C \) being the shape of \( (B.3) \). Finally, to identify the object \( \Sigma y \oplus \Sigma z \), we observe that our diagram of shape \( B \) could have been obtained by first left Kan extending from \( (B.3) \) to a diagram of shape

\[
\begin{array}{c}
x \xrightarrow{f} y \xrightarrow{0} 0 \\
g \downarrow \\
z \xrightarrow{k} w \xleftarrow{j}
\end{array}
\]

and then to all of \( B \). The objects appearing in \( (B.4) \) are identified as before. But now, in performing the second extension from \( (B.4) \) to all of \( B \), we can apply Lemma 4.5 with \( B \) being \( B \) while \( C = B' \) is the shape of \( (B.4) \), \( u \) the inclusion,
\( A = \gamma, \) and \( v \) picking out the square 111-211-121-221. Thus, the square 111-211-121-221 in Figure 12 is cocartesian, so the missing object is the pushout of \( \Sigma y \) and \( \Sigma z \) over 0. By Corollary 4.11, therefore, it is \( \Sigma y \oplus \Sigma z \).

Now if we instead perform the left Kan extension from (B.3) to Figure 12 by way of

\[
\begin{array}{ccc}
x & \xrightarrow{f} & y \\
\downarrow{g} & & \downarrow{0} \\
z & \xrightarrow{k} & w \\
\downarrow & & \downarrow \\
0 & \xrightarrow{j} & \Sigma x \\
\end{array}
\]

then we can apply Lemma 4.9 to the square 110-220-111-221, so that the square

\[
\begin{array}{ccc}
w & \xrightarrow{} & \Sigma x \\
\downarrow & & \downarrow \\
0 & \xrightarrow{} & \Sigma y \oplus \Sigma z \\
\end{array}
\]

appearing in Figure 12 is cocartesian. This looks like the second square in a cofiber sequence that would give rise to the triangle (B.2). In order to construct the first square, we instead extend from (B.3) to Figure 12 by way of a diagram

\[
(B.5)
\]

Let \( A \in \mathcal{C} at \) be the shape of (B.5), and let \( A' \) and \( B' \) be the subcategories of \( A \) and \( B \), respectively, which omit 000. We claim that the commutative square

\[
(B.6)
\]

is homotopy exact. To prove this, we verify the hypotheses of Theorem 3.20(vi). If the object \( a \in A \) is not 000, then the category \( (a/B/b)_\gamma \) from Theorem 3.20(vi) has an initial object, hence is homotopy contractible. Thus assume \( a = 000 \). Now if \( b \in B' \) is 010, 020, 100, 200, 110 or 111, then \( (a/B/b)_\gamma \) has a terminal object. If \( b = 220 \), then \( (a/B/b)_\gamma \) is the subcategory 020-010-110-100-200 of \( B \); this contains 010-110-100 as a coreflective subcategory, which in turn has a terminal object. An analogous argument applies in the other three cases \( b = 121, 221, \) and 211.

This completes the proof that (B.6) is homotopy exact. Therefore, the \( B' \)-diagram obtained from Figure 12 by restriction (i.e. forgetting the object \( x \)) is the
left Kan extension of the corresponding restriction of (B.5):

\[
\begin{array}{c}
 y \\
 \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
 0 \\
 \end{array}
\]

(B.7)

However, starting from (B.7), we can also perform a left Kan extension to a diagram of the following form, where we have added two new objects to the shape $B$ which might naturally be called $(1, 1, -1)$ and $(2, 2, -1)$.

\[
\begin{array}{c}
 y \oplus z \\
 \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
 0 \\
 \end{array}
\]

(B.8)

By Lemma 4.5 applied to coproducts (i.e. with $A = 1 \sqcup 1$), the objects at the top can be identified with $y \oplus z$ and $0$, as shown. To identify the rest of the objects correctly, let $C'$ denote the shape of (B.8). Then the square

\[
\begin{array}{c}
 A' \\
 \downarrow \quad \downarrow \\
 B' \\
 \end{array} \quad \begin{array}{c}
 A' \\
 \downarrow \quad \downarrow \\
 C' \\
 \end{array}
\]

is homotopy exact, because for any $b \in B'$, we have

\[
\begin{array}{c}
 (A'/b) \\
 \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
 1 \\
 \end{array} = \begin{array}{c}
 (A'/b) \\
 \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
 1 \\
 \end{array}
\]

and the right-hand square is homotopy exact by (Der4). Thus, the restriction of (B.8) to a $B'$-diagram agrees with the similar restriction of Figure 12, and in particular, all the objects in (B.8) are labeled correctly.
Now we observe that the left Kan extension from (B.7) to (B.8) could have been performed by way of a diagram of the following form:

(B.9)

An application of Lemma 4.9 to the extension from (B.9) to (B.8) then implies that the square in (B.8) is cocartesian. Thus, inside (B.8) we find our desired cofiber sequence (in transposed form)

It remains only to identify the map $\Sigma x \to \Sigma y \oplus \Sigma z$ with $\Sigma(f, -g)$. For this, it will suffice to show that its composites with the two projections $\Sigma y \oplus \Sigma z \to \Sigma y$ and $\Sigma y \oplus \Sigma z \to \Sigma z$ are $\Sigma f$ and $-\Sigma g$, respectively. For this we regard $B$ as a subcategory of $3 \times 3 \times 3$, and let $E$ be the subcategory of $3 \times 3 \times 3$ obtained by adding $112$, $212$, $122$, and $222$ to $B$, as shown in Figure 13. We extend by zero from $B$ to $B \cup \{112, 212\}$, then left extend to all of $E$; this produces a diagram as shown in Figure 14. The square 111-121-112-122 is cocartesian by Lemma 4.9, enabling us to identify the object at 122 with $\Sigma z$. Lemma 4.9 also tells us that the square 211-221-222 is cocartesian; hence, by the pasting law for cocartesian squares, so is the lower square 112-212-122-222, enabling us to also identify the object at 222 with $\Sigma z$. Moreover, since the 121-221-222 composite $\Sigma z \to \Sigma y \oplus \Sigma z \to \Sigma z$ is the identity and the 211-221-222 composite $\Sigma y \to \Sigma y \oplus \Sigma z \to \Sigma z$ is zero, it follows that the induced map $\Sigma y \oplus \Sigma z \to \Sigma z$ is in fact the projection out of $\Sigma y \oplus \Sigma z$ regarded as a product.
Now consider the functor $\square \times 2 \to E$ determined by the following labeling (where the factor $\square$ is the horizontal squares):

\[
\begin{array}{c}
000 \rightarrow 100 \rightarrow 200 \\
\downarrow \quad \downarrow \quad \downarrow \\
010 \rightarrow 110 \rightarrow 220 \\
\downarrow \quad \downarrow \quad \downarrow \\
020 \rightarrow 121 \rightarrow 221 \\
\downarrow \quad \downarrow \quad \downarrow \\
122 \rightarrow 222 \\
\end{array}
\]
Restricting Figure 14 along this functor yields a diagram of the following form:

Here the top square is cocartesian since it is the square 000-200-020-220 from Figure 12, and the bottom square is cocartesian since it is isomorphic to the square 010-111-020-121 from Figure 12. Therefore, the induced map Σx → Σz is identifiable with Σg.

Finally, a symmetrical argument implies we can identify the map Σx → Σy with Σf. However, the two squares used in these arguments to identify Σy and Σz are mutually transposed relative to how they were used to identify Σy ⊕ Σz above. Thus, if we make a consistent choice of orientation for these squares, one of the morphisms Σf and Σg will end up negated.

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