Trading classical communication, quantum communication, and entanglement in quantum Shannon theory

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Abstract—We give optimal trade-offs between classical communication, quantum communication, and entanglement for processing information in the Shannon-theoretic setting. We first prove a “unit-resource” capacity theorem that applies to the scenario where only the above three noiseless resources are available for consumption or generation. The optimal strategy mixes the three fundamental protocols of teleportation, super-dense coding, and entanglement distribution. Furthermore, no protocol other than these three fundamental ones is necessary to generate the unit resource capacity region. We then prove the “direct static” capacity theorem that applies to the scenario where a large number of copies of a noisy bipartite state are available (in addition to consumption or generation of the above three noiseless resources). The result is that a coding strategy involving the classically-assisted mother protocol and the three fundamental protocols is optimal. We finally prove the “direct dynamic” capacity theorem. This theorem applies to the scenario where a large number of uses of a noisy quantum channel are available in addition to the consumption or generation of the three noiseless resources. The optimal strategy combines the classically-enhanced father protocol with the three fundamental unit protocols. Interestingly, one octant of the direct-dynamic capacity region applies to an open question concerning the use of entanglement-assisted coding and teleportation for entanglement-and classically-assisted quantum communication.

I. INTRODUCTION

The publication of Shannon’s classic article in 1948 formally marks the beginning of information theory [1]. Shannon’s article states two fundamental theorems: the source coding theorem and the channel coding theorem. The source coding theorem concerns processing of a static resource—an information source that emits a symbol from an alphabet where each symbol occurs with some probability. The proof of the theorem appeals to the asymptotic setting where many copies of the static resource are available, i.e., the information source emits a large number of symbols. The result of the source coding theorem is a lower bound on the compression rate of the static resource. On the other hand, the channel coding theorem applies to a dynamic resource. An example of a dynamic resource is a bit-flip channel that flips each input bit with a certain probability. The proof of the channel coding theorem again appeals to the asymptotic setting where a sender consumes a large number of independent and identically distributed (IID) uses of the channel to transmit information to a receiver. The result of the channel coding theorem is an upper bound on the transmission rate of the dynamic resource.

Quantum Shannon theory has emerged in recent years as the quantum generalization of Shannon’s information theory. Schumacher established a quantum source coding theorem that is a “quantized” version of Shannon’s source coding theorem [2], [3]. Schumacher’s static resource is a quantum information source that emits a given quantum state with a certain probability. Holevo, Schumacher, and Westmoreland followed by proving a quantum channel coding theorem that determines how much classical information a sender can transmit to a receiver [4]. Lloyd, Shor, and Devetak then proved a different quantum channel coding theorem that determines how much quantum information a sender can transmit to a receiver [6]. Both of these quantum channel coding theorems exploit a dynamic resource—a noisy quantum channel that connects a sender to a receiver.

Entanglement is a static resource shared between a sender and receiver and does not have a classical analog. Entanglement is “static” because a sender and receiver cannot exploit entanglement alone to generate either classical communication or quantum communication or both. However, they can exploit entanglement and classical communication to communicate quantum information—this protocol is the well-known teleportation protocol [9]. The super-dense coding protocol [10] doubles the classical capacity of a noiseless quantum channel by exploiting entanglement in addition to the use of the noiseless quantum channel. These two protocols and others demonstrate that entanglement is a valuable resource in quantum information processing.

Several researchers have shown how to process entanglement in the asymptotic setting where a large number of identical copies of an entangled state or a large number of independent uses of a noisy channel are available to generate entanglement. Bennett et al. [11] proved a coding theorem for entanglement concentration that determines how much entanglement (in terms of maximally entangled states) a sender and receiver can generate from noisy bipartite states. The reverse problem of entanglement dilution [12], [13], [14] shows that
entanglement is not an inter-convertible static resource (i.e., simulating noisy bipartite states from maximally entangled states requires a sublinear amount of classical communication). A dynamic resource can also generate entanglement. Devetak [8] proved a coding theorem that determines how much entanglement a sender and receiver can generate by sending quantum states through a noisy quantum channel.

Quantum Shannon theory began with the aforementioned single-resource coding theorems [2]. [3]. [4]. [5]. [6]. [7]. [8]. [11]. [12]. [13]. [14]. [15] and has advanced to include double-resource coding theorems—their corresponding protocols either generate two different resources or they generate one resource with the help of another [16]. [17]. [18]. [19]. [20]. [21]. [22]. [23]. [24]. [25]. The result of each of these scenarios was an optimal two-dimensional trade-off region for the resources involved in the protocols. Quantum information theorists have organized many of the existing protocols into a family tree [23]. [25]. [26]. Furthermore, the authors in [23]. [25] proposed the resource inequality framework that establishes many classical and quantum coding theorems as interconversions between non-local information resources. The language of resource inequalities provides structural insights into the relationships between coding theorems in quantum Shannon theory and greatly simplifies the development of new coding schemes. An example of one of the resource trade-offs is the so-called “father” capacity region. The father protocol exploits a noisy quantum channel and shared noiseless entanglement to generate noiseless quantum communication. The father capacity region illustrates the optimal trade-offs between entanglement consumption and quantum communication.

In this article, we give the triple trade-offs for both the static and dynamic scenarios. The static resource that we consider here is a shared noisy bipartite state, and the dynamic resource that we consider is a noisy quantum channel. We again appeal to the asymptotic setting where a large number of independent copies or uses of the respective static or dynamic noisy resource are available. For both the static and dynamic scenarios, we assume that the sender and receiver either consume or generate noiseless classical communication, noiseless quantum communication, and noiseless entanglement in addition to the consumption of the noisy resource. The result is a three-dimensional capacity region that gives the optimal trade-offs for the three noiseless resources in both the static and dynamic scenarios. The rate of a noiseless resource is negative if a protocol consumes the corresponding resource, and its rate is positive if a protocol generates the resource. The above interpretation of a negative rate first appeared in Refs. [27]. [28] with subsequent appearance, for example, in Refs. [25]. [29]. [30]. The present article’s full solution for the static and dynamic scenarios contains both negative and positive rates.

An interesting aspect of this article is that we employ basic topological arguments and reductio ad absurdum arguments in the optimality proofs of the triple trade-off capacity theorems. Many times, we apply well-known results in quantum Shannon theory to reduce an optimality proof to that of a previously known protocol (for example, we apply the well-known result that forward classical communication does not increase the quantum capacity of a quantum channel [16]. [31]). For the constructive part of the coding theorems, there is no need to employ asymptotic arguments such as typical subspace techniques [22] or the operator Chernoff bound [33] because the resource inequality framework is sufficient to prove the coding theorems. The simplicity of our arguments perhaps reflects the maturation of the field of quantum Shannon theory.

One benefit of the direct dynamic capacity theorem is that we are able to answer a question concerning the use of entanglement-assisted coding [34]. [35]. [36]. [37]. [38]. [39]. [40]. [41] versus the use of teleportation. We show exactly when entanglement-assisted coding is useful, when teleportation is useful, or when a combination of both protocols is useful. We consider this result an important contribution of this article because it is rare that quantum Shannon theory gives insight into practical error correction schemes.

We structure this article as follows. In the next section, we establish some definitions and notation that proves useful for later sections. In Section III, we prove the optimality of a unit resource capacity region. The unit resource capacity region does not include a static or dynamic resource, but includes the resources of noiseless classical communication, noiseless quantum communication, and noiseless entanglement only. The unit resource capacity theorem shows that a mixed strategy combining teleportation [9], super-dense coding [10], and entanglement distribution [23] is optimal whenever a static or dynamic noisy resource is not available. Section IV states and proves the direct static capacity theorem. This theorem determines the optimal trade-offs between the three noiseless resources when a noisy static resource is available. Section V states and proves the direct dynamic capacity theorem. This theorem determines the optimal triple trade-offs when a noisy dynamic resource is available. We end with a discussion of the results in this article and future open problems.

II. DEFINITIONS AND NOTATION

We first establish some notation before proceeding to the main theorems. We review the notation for the three noiseless unit resources and that for resource inequalities. We finally establish some notation for handling geometric objects such as lines, quadrants, and octants in the three-dimensional space of classical communication, quantum communication, and entanglement.

The three fundamental resources are noiseless classical communication, noiseless quantum communication, and noiseless entanglement. Let [c → c] denote one c-bit of noiseless forward classical communication, let [q → q] denote one qubit of noiseless forward quantum communication, and let [qq] denote one ebit of shared noiseless entanglement [23]. [25]. The ebit is a maximally entangled state \(|\Phi^+\rangle^{AB} = (|00\rangle^{AB} + |11\rangle^{AB})/\sqrt{2}\) shared between two parties A and B given the respective names Alice and Bob. The ebit [qq] is a unit static resource and both the cbit [c → c] and the qubit [q → q] are unit dynamic resources.

We consider two noisy resources in this article: a noisy static resource and a noisy dynamic resource. Let \(\rho^{AB}\) denote the noisy static resource—a noisy bipartite state shared between Alice and Bob. Let \(N^{A \rightarrow B}\) denote a noisy dynamic...
resource—a noisy quantum channel that connects Alice to Bob. Throughout this article, we assume the dynamic resource $N^{A^* \rightarrow B}$ to be a completely positive and trace-preserving map that takes density operators in the Hilbert space of Alice’s system $A^*$ to Bob’s system $B$.

Resource inequalities are a compact, yet rigorous, way to state coding theorems in quantum Shannon theory [23], [25]. In this article, we formulate resource inequalities that consume the above noisy resources and either consume or generate the noiseless resources. An example from Refs. [23], [25] is the following “mother” resource inequality:

$$\langle \rho^{AB} \rangle + |Q| |q \rightarrow q| \geq E[qq].$$

It states that a large number $n$ of copies of the state $\rho^{AB}$ and $n |Q|$ uses of a noiseless qubit channel are sufficient to generate $nE$ ebits of entanglement while tolerating an arbitrarily small error in the fidelity of the produced ebits. The optimal rates $Q$ and $E$ of respective qubit channel consumption and entanglement generation are entropic quantities: $|Q| = \frac{1}{2} I(A; E)$ and $E = \frac{1}{2} I(A; B)$ (see Ref. [42] for definitions of entropy and mutual information). The entropic quantities are with respect to a state $|\psi\rangle^{EAB}$ where $|\psi\rangle^{EAB}$ is a purification of the noisy static resource state $\rho^{AB}$ and $E$ is the purifying reference system (it should be clear when $E$ refers to the purifying system and when it refers to the rate of entanglement generation). We take the convention that the rate $Q$ is negative and $E$ is positive because the protocol consumes quantum communication and generates entanglement (this convention is the same as in Refs. [27], [23], [25], [29], [30]).

We make several geometric arguments throughout this article because the static and dynamic capacity regions lie in a three-dimensional space with points that are rate triples $(R, Q, E)$. $R$ represents the rate of classical communication, $Q$ the rate of quantum communication, and $E$ the rate of entanglement consumption or generation. Let $L$ denote a line, $Q$ a quadrant, and $O$ an octant in this space (it should be clear from context whether $Q$ refers to quantum communication or “quadrant”). E.g., $L^{-00}$ denotes a line going in the direction of negative classical communication:

$$L^{-00} \equiv \{ \alpha (-1, 0, 0) : \alpha \geq 0 \}.$$  

$Q^{0+}$ denotes the quadrant where there is zero classical communication, generation of quantum communication, and consumption of entanglement:

$$Q^{0+} \equiv \{ \alpha (0, 1, 0) + \beta (0, 0, -1) : \alpha, \beta \geq 0 \}.$$  

$O^{++}$ denotes the octant where there is generation of classical communication, consumption of quantum communication, and generation of entanglement:

$$O^{++} \equiv \{ \alpha (1, 0, 0) + \beta (0, -1, 0) + \gamma (0, 0, 1) : \alpha, \beta, \gamma \geq 0 \}.$$  

It proves useful to have a “set addition” operation between two regions $A$ and $B$:

$$A + B \equiv \{ a + b : a \in A, b \in B \}.$$  

The following relations hold

$$Q^{0+} = L^{0+0} + L^{00-},$$  

$$O^{++} = L^{000} + L^{00-} + L^{00+},$$  

by using the above definition. Addition of the same line gives the line itself, e.g.,

$$L^{000} + L^{00-} = L^{00-}.$$  

This set equality holds because of the definition of set addition and the definition of the line. A similar result also holds for addition of the same quadrant or octant. We define the set subtraction of two regions $A$ and $B$ as follows:

$$A - B \equiv \{ a - b : a \in A, b \in B \}.$$  

According to this definition, it follows that $L^{000} \subseteq L^{000} - L^{000} = L^{000}$ where $L^{000}$ represents the full line of classical communication.

### III. THE TRIPLE TRADE-OFF BETWEEN UNIT RESOURCES

We now consider what rates are achievable when there is no noisy resource—the only resources available are noiseless classical communication, noiseless quantum communication, and noiseless entanglement. The result is a three-dimensional “unit resource” capacity region that involves the three fundamental noiseless resources.

Three important protocols relate the three fundamental noiseless resources. These protocols are teleportation (TP) [9], super-dense coding (SD) [10], and entanglement distribution (ED) [23]. We can express these three protocols as resource inequalities.$^1$ The resource inequality for teleportation is

$$2|c \rightarrow c| + [qq] \geq |q \rightarrow q|.$$  

while super-dense coding corresponds to

$$|q \rightarrow q| + [qq] \geq 2|c \rightarrow c|.$$  

Entanglement distribution is a trivial resource inequality:

$$|q \rightarrow q| \geq [qq].$$  

A sender implements ED by transmitting half of a locally prepared Bell state $|\Phi^+\rangle$ through a noiseless qubit channel. The above three noiseless protocols are sufficient to recover all other noiseless protocols. For example, we can combine SD and ED to produce the following resource inequality:

$$2|q \rightarrow q| + [qq] \geq 2|c \rightarrow c| + [qq].$$  

The above resource inequality is equivalent to the following one

$$|q \rightarrow q| \geq |c \rightarrow c|,$$  

after removing the entanglement from both sides and scaling by $1/2$ (we can remove the entanglement here because it acts as a catalytic resource [23], [25]). We refer to (5) as “classical coding over quantum channels” (CC).

$^1$These three protocols are finite and exact protocols. Representing the finite protocols using resource inequalities is also possible even though the resource inequality framework applied originally to the asymptotic setting [23], [25].
In the unit resource capacity theorem, we exploit the following geometric objects that lie in the \((R, Q, E)\) space:

1. Teleportation is the point \((-2, 1, -1)\). The “line of teleportation” \(L_{TP}\) is the following set of points:
\[
L_{TP} \equiv \{ \alpha (-2, 1, -1) : \alpha \geq 0 \}.
\]

2. Super-dense coding is the point \((2, -1, -1)\). The “line of super-dense coding” \(L_{SD}\) is the following set of points:
\[
L_{SD} \equiv \{ \beta (2, -1, -1) : \beta \geq 0 \}.
\]

3. Entanglement distribution is the point \((0, -1, 1)\). The “line of entanglement distribution” \(L_{ED}\) is the following set of points:
\[
L_{ED} \equiv \{ \gamma (0, -1, 1) : \gamma \geq 0 \}.
\]

**Definition 1:** Let \(\tilde{C}_U\) denote the unit resource achievable region. It consists of all linear combinations of the above protocols:
\[
\tilde{C}_U \equiv L_{TP} + L_{SD} + L_{ED}.
\]

The following matrix equation gives all achievable triples \((R, Q, E)\) in \(\tilde{C}_U\):
\[
\begin{bmatrix}
R \\
Q \\
E
\end{bmatrix} = \begin{bmatrix}
-2 & 2 & 0 \\
1 & -1 & -1 \\
-1 & -1 & 1
\end{bmatrix} \begin{bmatrix}
\alpha \\
\beta \\
\gamma
\end{bmatrix},
\]
where \(\alpha, \beta, \gamma \geq 0\). We can rewrite the above equation with its matrix inverse:
\[
\begin{bmatrix}
\alpha \\
\beta \\
\gamma
\end{bmatrix} = \begin{bmatrix}
-1/2 & -1/2 & -1/2 \\
0 & -1/2 & -1/2 \\
-1/2 & -1 & 0
\end{bmatrix} \begin{bmatrix}
R \\
Q \\
E
\end{bmatrix},
\]
in order to express the coefficients \(\alpha, \beta, \gamma\) as a function of the rate triples \((R, Q, E)\). The restriction of non-negativity of \(\alpha, \beta, \gamma\) and \(\gamma\) follows the restriction on the achievable rate triples \((R, Q, E)\):
\[
R + Q + E \leq 0, \quad Q + E \leq 0, \quad \frac{1}{2} R + Q \leq 0.
\]

The above result implies that the achievable region \(\tilde{C}_U\) in \(\tilde{C}_U\) is equivalent to all rate triples satisfying \((10)\). Figure 1 displays the full unit resource achievable region.

**Definition 2:** The unit resource capacity region \(C_U\) is the closure of the set of all points \((R, Q, E)\) in the \(R, Q, E\) space, satisfying the following resource inequality:
\[
0 \geq R[c \rightarrow c] + Q[q \rightarrow q] + E[qq].
\]

The above notation may seem confusing at first glance until we recall that a resource with a negative rate implicitly belongs on the left-hand side of the resource inequality.

Theorem 1 below gives the optimal three-dimensional capacity region for the three unit resources.

**Theorem 1:** The unit resource capacity region \(C_U\) is equivalent to the unit resource achievable region \(\tilde{C}_U\):
\[
C_U = \tilde{C}_U.
\]
communication, quantum communication, and entanglement. (They cannot generate a noiseless resource from nothing!)

(\(++,+-\)). This octant of \(C_U\) is empty because entanglement alone cannot generate either classical communication or quantum communication or both.

(\(+-,+-\)). The task for this octant is to generate a noiseless classical channel of \(R\) bits and \(E\) ebits of entanglement using \(Q\) qubits of quantum communication. We thus consider all points of the form \((R, Q, E)\) where \(R \geq 0, Q \leq 0,\) and \(E \geq 0\). It suffices to prove the following inequality:

\[
R + E \leq |Q|,
\]

because combining \([15]\) with \(R \geq 0\) and \(E \geq 0\) implies \([10][12]\). The achievability of \((R, -|Q|, E)\) implies the achievability of the point \((R + 2E, -|Q| - E, 0)\), because we can consume all of the entanglement with super-dense coding \([3]\). This new point implies that there is a protocol that consumes \(|Q| + E\) noiseless qubit channels to send \(R + 2E\) classical bits. The following bound then applies

\[
R + 2E \leq |Q| + E,
\]

because the Holevo bound \([32]\) states that we can send only one classical bit per qubit. The bound in \([15]\) then follows.

(\(+-,-,-\)). The task for this octant is to simulate a classical channel of size \(R\) bits using \(|Q|\) qubits of quantum communication and \(|E|\) ebits of entanglement. We consider all points of the form \((R, Q, E)\) where \(R \geq 0, Q \leq 0,\) and \(E \leq 0\). It suffices to prove the following inequalities:

\[
R \leq 2|Q|,
\]

\[
R \leq |Q| + |E|,
\]

because entanglement alone cannot generate quantum communication. The bound in \([16]\) then follows from the above bound. The achievability of \((R, -|Q|, -|E|)\) implies the achievability of \((0, -|Q| + R/2, -|E| - R/2)\), because we can consume all of the classical communication with teleportation \([2]\). The following bound applies (quantum communication cannot be positive)

\[-|Q| + R/2 \leq 0,
\]

because combining them with \(R \leq 0\) implies \([10][12]\). The achievability of the point \((-|R|, Q, -|E|)\) implies the achievability of the point \((-|R|, 0, Q - |E|)\), because we can consume all of the quantum communication for entanglement distribution \([4]\). The following bound applies (entanglement cannot be positive)

\[
Q - |E| \leq 0,
\]

because classical communication alone cannot generate entanglement. The bound in \([19]\) follows from the above bound. The achievability of \((-|R|, -|Q|, E)\) implies the achievability of \((-|R| + 2Q, 0, -|Q| - E)\), because we can consume all of the quantum communication for super-dense coding \([3]\). The following bound applies (classical communication cannot be positive)

\[
-|R| + 2Q \leq 0,
\]

because entanglement alone cannot create classical communication. The bound in \([19]\) follows from the above bound.

(\(-,-,+-\)). The task for this octant is to create \(E\) ebits of entanglement using \(|Q|\) qubits of quantum communication and \(|R|\) bits of classical communication. We consider all points of the form \((R, Q, E)\) where \(R \leq 0, Q \leq 0,\) and \(E \geq 0\). It suffices to prove the following inequality:

\[
E \leq |Q|,
\]

because combining it with \(Q \leq 0\) and \(R \leq 0\) implies \([10][12]\). The achievability of \((-|R|, -|Q|, E)\) implies the achievability of \((-|R| - 2E, -|Q| + E, 0)\), because we can consume all of the entanglement with teleportation \([2]\). The following bound applies (quantum communication cannot be positive)

\[-|Q| + E \leq 0,
\]

because classical communication alone cannot generate quantum communication. The bound in \([20]\) follows from the above bound.

(-, - , - ). \(\bar{C}_U\) completely contains this octant.

IV. DIRECT STATIC TRADE-OFF

In this section, we consider what rates are achievable when a sender and receiver consume a noisy static resource. They additionally consume or generate noiseless classical communication, noiseless quantum communication, and noiseless entanglement. The result is a three-dimensional “direct static” capacity region that gives the full trade-off between the three fundamental noiseless resources when a noisy static resource is available.

Definition 3: The direct static capacity region \(C_{DS}(\rho^{AB})\) of a noisy bipartite state \(\rho^{AB}\) is a three-dimensional region in the \((R, Q, E)\) space. It is the closure of the set of all points \((R, Q, E)\) satisfying the following resource inequality:

\[
\langle \rho^{AB} \rangle \geq R[c \rightarrow c] + Q[q \rightarrow q] + E[qq].
\]

The rates \(R, Q,\) and \(E\) can either be negative or positive with the same interpretation as in the previous section. The statement of the resource inequality is precise (see Ref. \([25]\) for further details). For example, suppose that \(R \leq 0, Q \leq 0,\)
and $E \geq 0$. Then we interpret the resource inequality as the following one:

$$\langle \rho^{AB} \rangle + R[c \rightarrow c] + Q[q \rightarrow q] \geq E[qq].$$

It states that a large number $n$ of copies of the bipartite state $\rho^{AB}$, $nR$ bits of forward noiseless classical communication, $nQ$ qubits of noiseless forward quantum communication are sufficient to generate $nE$ ebits of entanglement with a fidelity $1 - \epsilon$ for any $\epsilon > 0$ and sufficiently large $n$.

We first recall the following “classically-assisted mother” resource inequality from Ref. [23] (Ref. [23] referred to it as the “grandmother” resource inequality). It gives the optimal rates for the above resource inequality and proves to be useful in determining the achievable region for the static case.

**Theorem 2:** The following “classically-assisted mother” resource inequality holds

$$\langle \rho^{AB} \rangle + \frac{1}{2} I(A'; EE')[X]_{\sigma}[q \rightarrow q] + I(X; BE)_{\sigma}[c \rightarrow c] \geq \frac{1}{2} I(A'; B|X)_{\sigma}[qq]$$

(22)

for a static resource $\rho^{AB}$ and for any remote instrument $T^{A \rightarrow A'X}$. In the above resource inequality, the state $\sigma^{X'A'EE'}$ is defined by

$$\sigma^{X'A'EE'} = \tilde{T}(\psi^{ABE}),$$

(23)

where $|\psi^{ABE}\rangle$ is some purification of $\rho^{AB}$ and $\tilde{T}^{A \rightarrow A'X}$ is an isometric extension of $T$.

We refer the reader to Refs. [23], [43] for definitions of the above entropic quantities and the definition of a “quantum instrument.” Ward’s thesis also provides a great introduction to the direct static capacity region contains the direct static achievable region:

$$\tilde{C}_{DS}(\rho^{AB}) \subseteq C_{DS}(\rho^{AB}).$$

It follows directly from combining the classically-assisted mother resource inequality (22) with TP (2), SD (3), and ED (4) and considering the definition of the direct static achievable region in (25) and the definition of the direct static capacity region in (21).

**B. Proof of the Direct Coding Theorem**

The converse theorem is the statement that the direct static capacity region contains the direct static achievable region:

$$C_{DS}(\rho^{AB}) \subseteq \tilde{C}_{DS}(\rho^{AB}).$$

In order to prove it, we consider one octant of the $(R, Q, E)$ space at a time and use the notation from Section A. We omit writing $\rho^{AB}$ in what follows and instead write $\rho$ to denote the noisy bipartite state $\rho^{AB}$.

The converse proof exploits the optimality and the relation among the previously known capacity regions: the classically-assisted mother [25], the mother [23], [25], noisy super-density coding [19], [25], noisy teleportation [23], [25], and entanglement distillation [25]. We illustrate the relation between these protocols in Figure 2.

The mother protocol (MP) consumes a noisy static resource and noiseless quantum communication to generate noiseless entanglement. The mother’s achievable region $C_{MP}(\rho)$ and capacity region $C_{MP}(\rho)$ lie in the $Q^{0-}$ quadrant of the $(R, Q, E)$ space. The mother’s respective regions are special cases of the direct static achievable region $C_{DS}(\rho)$ and the direct static capacity region $C_{DS}(\rho)$ with the following correspondences:

$$C_{MP}(\rho) = \tilde{C}_{DS}^{0-0}(\rho) \equiv \tilde{C}_{DS}(\rho) \cap Q^{0-},$$

(26)

$$C_{MP}(\rho) = C_{0-0}(\rho) \equiv C_{DS}(\rho) \cap Q^{0-}.$$  

(27)

The mother capacity theorem states that the mother’s achievable region $C_{MP}(\rho)$ is optimal: $C_{MP}(\rho) = C_{MP}(\rho)$. It follows that

$$\tilde{C}_{DS}^{0-0}(\rho) = C_{DS}^{0-0}(\rho).$$

(28)
The mother’s achievable region $\bar{C}_{MP}(\rho)$ is a special case of the classically-assisted mother achievable region $\bar{C}_{CAM}(\rho)$ where there is no consumption of classical communication:

$$\bar{C}_{MP}(\rho) = \bar{C}_{CAM}(\rho) \cap Q^{0-+}.$$  \hfill (29)

The noisy super-dense coding protocol (NSD) consumes a noisy static resource and noiseless quantum communication to generate noiseless classical communication. NSD’s achievable region $\bar{C}_{NSD}(\rho)$ and capacity region $C_{NSD}(\rho)$ lie in the $Q^{+0}$ quadrant of the $(R,Q,E)$ space. NSD’s respective regions are special cases of the direct static achievable region $\bar{C}_{DS}(\rho)$ and the direct static capacity region $C_{DS}(\rho)$. We have the following correspondences:

$$\bar{C}_{NSD}(\rho) = \bar{C}_{DS}(\rho) \cap Q^{+0},$$  \hfill (30)

$$C_{NSD}(\rho) = C_{DS}(\rho) \cap Q^{+0}.  \hfill (31)$$

The NSD capacity theorem states that NSD’s achievable region is optimal: $\bar{C}_{NSD}(\rho) = C_{NSD}(\rho)$. It follows that

$$\bar{C}_{DS}(\rho) \cap Q^{+0} = C_{DS}(\rho) \cap Q^{+0}. \hfill (32)$$

NSD’s achievable region $\bar{C}_{NSD}(\rho)$ is obtainable from the mother’s achievable region $\bar{C}_{M}(\rho)$ by combining it with super-dense coding and keeping the points with zero entanglement:

$$\bar{C}_{NSD}(\rho) = (\bar{C}_{MP}(\rho) + L_{SD}) \cap Q^{+0},$$  \hfill (33)

where the second equality comes from (29).

The noisy teleportation protocol (NTP) consumes a noisy static resource and noiseless classical communication to generate noiseless quantum communication. NTP’s achievable region $\bar{C}_{NTP}(\rho)$ and capacity region $C_{NTP}(\rho)$ lie in the $Q^{-0+}$ quadrant of the $(R,Q,E)$ space. NTP’s respective regions are special cases of the direct static achievable region $\bar{C}_{DS}(\rho)$ and the direct static capacity region $C_{DS}(\rho)$. We have the following correspondences:

$$\bar{C}_{NTP}(\rho) = \bar{C}_{DS}(\rho) \cap Q^{-0+},$$  \hfill (34)

$$C_{NTP}(\rho) = C_{DS}(\rho) \cap Q^{-0+}. \hfill (35)$$

The NTP capacity theorem states that NTP’s achievable region is optimal: $\bar{C}_{NTP}(\rho) = C_{NTP}(\rho)$. It follows that

$$\bar{C}_{DS}(\rho) \cap Q^{-0+} = C_{DS}(\rho) \cap Q^{-0+}. \hfill (36)$$

NTP’s achievable region $\bar{C}_{NTP}(\rho)$ is obtainable from the classically-assisted mother’s achievable region $\bar{C}_{CAM}(\rho)$ by combining it with teleportation and keeping the points with zero entanglement:

$$\bar{C}_{NTP}(\rho) = (\bar{C}_{CAM}(\rho) + L_{TP}) \cap Q^{-0+}. \hfill (37)$$

The entanglement distillation protocol (ED) consumes a noisy static resource and noiseless classical communication to generate noiseless entanglement. ED’s achievable region $\bar{C}_{ED}(\rho)$ and capacity region $C_{ED}(\rho)$ lie in the $Q^{-0+}$ quadrant of the $(R,Q,E)$ space. ED’s respective regions are special cases of the direct static achievable region $\bar{C}_{DS}(\rho)$ and the direct static capacity region $C_{DS}(\rho)$. We have the following correspondences:

$$\bar{C}_{ED}(\rho) = \bar{C}_{DS}(\rho) \cap Q^{-0+},$$  \hfill (38)

$$C_{ED}(\rho) = C_{DS}(\rho) \cap Q^{-0+}. \hfill (39)$$

The ED capacity theorem states that ED’s achievable region is optimal: $\bar{C}_{ED}(\rho) = C_{ED}(\rho)$. It follows that

$$\bar{C}_{DS}(\rho) \cap Q^{-0+} = C_{DS}(\rho) \cap Q^{-0+}. \hfill (40)$$

ED’s achievable region $\bar{C}_{ED}(\rho)$ is obtainable from the classically-assisted mother’s achievable region $\bar{C}_{CAM}(\rho)$ by combining it with teleportation and keeping the points with zero quantum communication:

$$\bar{C}_{ED}(\rho) = (\bar{C}_{CAM}(\rho) + L_{TP}) \cap Q^{-0+}. \hfill (41)$$

1) Entanglement-Generating Octants: We first consider the four octants with corresponding protocols that generate entanglement, i.e., those of the form $(\pm, \pm, \pm)$. The proof of one octant is trivial and the proof of another corresponds to showing that the classically-assisted mother protocol is optimal. The proofs of the remaining two octants are similar to each other and show the optimality of a particular octant by
consuming all of the entanglement in them and resorting to the previously established optimality of a simpler two-dimensional quadrant.

$(+,+,+)$. This octant is empty because a noisy static resource alone cannot generate a dynamic resource.

$(-,-,+)$. The converse of this octant is the converse of the classically-assisted mother protocol. See the Appendix for the proof. It is interesting that the classically-assisted mother protocol is optimal for this octant, i.e., there is no need to extend the protocol with the unit resource inequalities.

$(-,+,+)$. Define

$$C_{DS}^{++} (\rho) \equiv C_{DS} (\rho) \cap O^{++},$$

and recall the definition of $C_{DS}^{-0} (\rho)$ in (31). We exploit the line of super-dense coding $L_{SD}$ as defined in (3). Define the following maps:

$$f : S \rightarrow (S + L_{SD}) \cap Q^{+0},$$

$$\hat{f} : S \rightarrow (S - L_{SD}) \cap O^{++}.$$  

The map $f$ translates the set $S$ in the super-dense coding direction and keeps the points that lie on the $Q^{+0}$ quadrant. The map $\hat{f}$, in a sense, undoes the effect of $f$ by moving the set $S$ back to the $(+,+,+)$ octant $O^{++}$.

The inclusion $C_{DS}^{++} (\rho) \subseteq \hat{f}(f(C_{DS}^{++} (\rho)))$ holds because

$$C_{DS}^{++} (\rho)
= C_{DS}^{++} (\rho) \cap O^{++}
\subseteq (((C_{DS}^{++} (\rho) + L_{SD}) \cap Q^{+0}) - L_{SD}) \cap O^{++}
= (f(C_{DS}^{++} (\rho)) - L_{SD}) \cap O^{++}
= \hat{f}(f(C_{DS}^{++} (\rho))).$$

The first set equivalence is obvious from the definition of $C_{DS}^{++} (\rho)$. The first inclusion follows from the following logic. Pick any point $a \equiv (R, Q, E) \in C_{DS}^{++} (\rho) \cap O^{++}$ and a particular point $b \equiv (2E, -E, -E) \in E_{SD}$. It follows that $a + b = (R + 2E, Q - E, 0) \in (C_{DS}^{++} (\rho) + L_{SD}) \cap Q^{+0}$. We then pick a point $\neg b = (-2E, E, E) \in E_{SD}$. It follows that $a + b - b = ((C_{DS}^{++} (\rho) + L_{SD}) \cap Q^{+0} - L_{SD}) \cap O^{++}$ and that $a + b - b = (R, Q, E) = a$. The first inclusion thus holds because every point in $C_{DS}^{++} (\rho) \cap O^{++}$ is in $(((C_{DS}^{++} (\rho) + L_{SD}) \cap Q^{+0}) - L_{SD}) \cap O^{++}$. The second set equivalence follows from the definition of $f$ and the third set equivalence follows from the definition of $\hat{f}$.

It is operationally clear that the following inclusion holds

$$f(C_{DS}^{++} (\rho)) \subseteq C_{DS}^{-0} (\rho),$$

because the mapping $f$ converts any achievable point $a \in C_{DS}^{++} (\rho)$ to an achievable point in $C_{DS}^{-0} (\rho)$ by consuming all of the entanglement at point $a$ with super-dense coding.

The converse proof of the noisy super-dense (NSD) coding protocol [25] is useful for us:

$$C_{DS}^{-0} (\rho) = C_{NSD} (\rho) \subseteq \tilde{C}_{NSD} (\rho).$$

Recall the relation in (33) between the NSD achievable region $C_{NSD} (\rho)$ and the classically-assisted mother achievable region $\tilde{C}_{CAM} (\rho)$. The following set inclusion holds

$$C_{NSD} (\rho) \subseteq \tilde{C}_{CAM} (\rho) + L_{SD} \cap Q^{+0},$$

by dropping the intersection with $Q^{0+}$ in (33).

The inclusion $\hat{f}(\tilde{C}_{NSD} (\rho)) \subseteq C_{DS}^{++} (\rho)$ holds because

$$\hat{f}(\tilde{C}_{NSD} (\rho))
\subseteq (((\tilde{C}_{CAM} (\rho) + L_{SD}) \cap Q^{+0}) - L_{SD}) \cap O^{++}
\subseteq ((\tilde{C}_{CAM} (\rho) + L_{SD}) - L_{SD}) \cap O^{++}
= ((\tilde{C}_{CAM} (\rho) + L_{SD}) \cap O^{++}) \cup ((\tilde{C}_{CAM} (\rho) - L_{SD}) \cap O^{++})
\subseteq C_{DS}^{++} (\rho).$$

The first inclusion follows from (45). The second inclusion follows by dropping the intersection with $Q^{+0}$. The second set equivalence follows because $(\tilde{C}_{CAM} (\rho) + L_{SD}) - L_{SD} = (\tilde{C}_{CAM} (\rho) + L_{SD}) \cup (\tilde{C}_{CAM} (\rho) - L_{SD})$, and the last inclusion follows because $(\tilde{C}_{CAM} (\rho) - L_{SD}) \cap O^{++} = (0, 0, 0)$.

Putting (42), (43), (44), and (46) together, the inclusion $C_{DS}^{++} (\rho) \subseteq C_{DS}^{-0} (\rho)$ holds because

$$C_{DS}^{++} (\rho) \subseteq \hat{f}(f(C_{DS}^{++} (\rho)))
\subseteq \hat{f}(C_{DS}^{-0} (\rho)) \subseteq \hat{f}(\tilde{C}_{NSD} (\rho)) \subseteq \tilde{C}_{DS}^{++} (\rho).$$

The above inclusion is the statement of the converse theorem for this octant.

$(-,-,+)$. The technique for handling this octant is similar to the technique for handling the previous octant. We give the full proof for completeness. Define

$$C_{DS}^{++} (\rho) \equiv C_{DS} (\rho) \cap O^{++},$$

and recall the definition of $C_{DS}^{-0} (\rho)$ in (35). Recall the line of teleportation $L_{TP}$ as defined in (4).

Define the following maps

$$f : S \rightarrow (S + L_{TP}) \cap Q^{+0},$$

$$\hat{f} : S \rightarrow (S - L_{TP}) \cap O^{++}.$$  

The map $f$ translates the set $S$ in the teleportation direction and keeps the points in the $Q^{+0}$ quadrant. The map $\hat{f}$, in a sense, undoes the effect of $f$ by moving the set $S$ back to the $O^{++}$ octant.

The inclusion $C_{DS}^{++} (\rho) \subseteq \hat{f}(f(C_{DS}^{++} (\rho)))$ holds because

$$C_{DS}^{++} (\rho)
= C_{DS}^{++} (\rho) \cap O^{++}
\subseteq (((C_{DS}^{++} (\rho) + L_{TP}) \cap Q^{+0}) - L_{TP}) \cap O^{++}
= (f(C_{DS}^{++} (\rho)) - L_{TP}) \cap O^{++}
= \hat{f}(f(C_{DS}^{++} (\rho))).$$

The first set equivalence is obvious from the definition of $C_{DS}^{++} (\rho)$. The first inclusion follows from the following logic. Pick any point $a \equiv (R, Q, E) \in C_{DS}^{++} (\rho) \cap O^{++}$ and a particular point $b \equiv (2E, -E, -E) \in L_{TP}$. It follows that the point $a + b \in (C_{DS}^{++} (\rho) + L_{TP}) \cap Q^{+0}$. We then pick a point $\neg b = (-2E, E, E) \in L_{TP}$. It follows that $a + b - b = ((C_{DS}^{++} (\rho) + L_{TP}) \cap Q^{+0} - L_{TP}) \cap O^{++}$ and that $a + b - b = (R, Q, E) = a$. The first inclusion thus holds because every point in $C_{DS}^{++} (\rho) \cap O^{++}$ is in $(((C_{DS}^{++} (\rho) + L_{TP}) \cap Q^{+0}) - L_{TP}) \cap O^{++}$. The second
set equivalence follows from the definition of $f$ and the third set equivalence follows from the definition of $\tilde{f}$.

It is operationally clear that the following inclusion holds

$$f(C_{DS}^{++} (\rho)) \subseteq C_{DS}^{-0} (\rho),$$  

(48)

because the mapping $f$ converts any achievable point $\rho \in C_{DS}^{-0} (\rho)$ to an achievable point in $C_{DS}^{++} (\rho)$ by consuming all of the entanglement in $\rho$ with teleportation.

The converse proof of the noisy teleportation (NTP) protocol is useful for us:

$$C_{DS}^{-0} (\rho) = C_{\text{NTP}} (\rho) \subseteq \tilde{C}_{\text{NTP}} (\rho).$$  

(49)

The inclusion $\hat{f}(\tilde{C}_{\text{NTP}} (\rho)) \subseteq \tilde{C}_{DS}^{++} (\rho)$ holds because

$$\hat{f}(\tilde{C}_{\text{NTP}} (\rho)) = (((\tilde{C}_{\text{CAM}} (\rho) + L_{\text{TP}}) \cap Q^{-0}) - L_{\text{TP}}) \cap O^{;++}$$

$$\subseteq (((\tilde{C}_{\text{CAM}} (\rho) + L_{\text{TP}}) - L_{\text{TP}}) \cap O^{;++}$$

$$= ((\tilde{C}_{\text{CAM}} (\rho) + L_{\text{TP}}) \cap O^{;++}) \cup ((\tilde{C}_{\text{CAM}} (\rho) - L_{\text{TP}}) \cap O^{;++})$$

$$\subseteq \tilde{C}_{DS}^{++} (\rho).$$  

(50)

The first set equivalence follows by definition. The first set equivalence follows from the mother capacity theorem in (28), the third set equivalence from (51), and the last from linearity of the map $f$. The above inclusion implies the following one:

$$f(C_{DS}^{0-} (\rho \otimes \Phi |E\rangle)) \subseteq \tilde{C}_{DS}^{0-} (\rho) + \tilde{C}_{DS}^{++} (\Phi |E\rangle).$$

The lemma follows because

$$f(C_{DS}^{0-} (\rho \otimes \Phi |E\rangle)) = f(\tilde{C}_{DS}^{0-} (\rho \otimes \Phi |E\rangle)) = \tilde{C}_{DS}^{0-} (\rho \otimes \Phi |E\rangle) = \tilde{C}_{DS}^{0-} (\rho \otimes \Phi |E\rangle),$$

where we apply the mother capacity theorem in (28) and the NSD capacity theorem in (32). 

Observe that

$$\tilde{C}_{DS}^{++} (\Phi |E\rangle) = \tilde{C}_{U}^{++} E.$$

(52)

Hence for all $E \leq 0$,

$$C_{DS}^{++} (\rho) = C_{DS}^{0-} (\rho \otimes \Phi |E\rangle) \subseteq \tilde{C}_{DS}^{0-} (\rho) + \tilde{C}_{U}^{++} E,$$

(53)

where we apply Lemma 1 and (52). Thus, the inclusion

$$C_{DS}^{++} (\rho) \subseteq C_{DS}^{+--} (\rho)$$

holds because

$$C_{DS}^{+--} (\rho) = \bigcup_{E \leq 0} C_{DS}^{++} (\rho)$$

$$\subseteq \bigcup_{E \leq 0} (\tilde{C}_{DS}^{0-} (\rho) + \tilde{C}_{U}^{++} E)$$

$$= (\tilde{C}_{DS}^{0-} (\rho) + \tilde{C}_{U}) \cap O^{;++}$$

$$\subseteq (\tilde{C}_{\text{CAM}} (\rho) + \tilde{C}_{U}) \cap O^{;++}$$

$$= \tilde{C}_{DS}^{+--} (\rho).$$

The following relation holds

$$f(C_{DS}^{0-} (\rho)) = \tilde{C}_{DS}^{-0} (\rho),$$

(51)

because applying dense coding to the mother resource inequality given noisy dense coding [25]. The inclusion $C_{DS}^{0-} (\rho \otimes \Phi |E\rangle) \subseteq f^{-1}(\tilde{C}_{DS}^{0-} (\rho) + \tilde{C}_{DS}^{++} (\Phi |E\rangle))$ holds because

$$C_{DS}^{0-} (\rho \otimes \Phi |E\rangle) = C_{DS}^{0-} (\rho) + (0, 0, E)$$

$$\subseteq C_{DS}^{0-} (\rho) + C_{DS}^{++} (\Phi |E\rangle)$$

$$= C_{DS}^{0-} (\rho) + C_{DS}^{++} (\Phi |E\rangle)$$

$$= f^{-1}(\tilde{C}_{DS}^{0-} (\rho)) + f^{-1}(\tilde{C}_{DS}^{++} (\Phi |E\rangle))$$

$$= f^{-1}(\tilde{C}_{DS}^{0-} (\rho)) + \tilde{C}_{DS}^{++} (\Phi |E\rangle).$$

The first set equivalence follows because the capacity region of the noisy resource state $\rho$ combined with a rate $E$ maximally entangled state is equivalent to a translation of the capacity region of the noisy resource state $\rho$. The first inclusion follows because the capacity region of a rate $E$ maximally entangled state contains the rate triple $(0, 0, E)$. The second set equivalence follows from the mother capacity theorem in (28), the third set equivalence from (51), and the last from linearity of the map $f$. The above inclusion implies the following one:

$$f(C_{DS}^{0-} (\rho \otimes \Phi |E\rangle)) \subseteq \tilde{C}_{DS}^{0-} (\rho) + \tilde{C}_{DS}^{++} (\Phi |E\rangle).$$

The lemma follows because

$$f(C_{DS}^{0-} (\rho \otimes \Phi |E\rangle)) = f(\tilde{C}_{DS}^{0-} (\rho \otimes \Phi |E\rangle)) = \tilde{C}_{DS}^{0-} (\rho \otimes \Phi |E\rangle) = \tilde{C}_{DS}^{0-} (\rho \otimes \Phi |E\rangle),$$

where we apply the mother capacity theorem in (28) and the NSD capacity theorem in (32).

2) Entanglement-Consuming Octants: We now consider all the octants with corresponding protocols that consume entanglement, i.e., those of the form $(\pm, \pm, -)$. The proofs for two of the octants are trivial and the proofs for the two non-trivial octants each contain an additivity lemma that shows how to relate the optimality to the optimality of a quadrant. ($+, +, -$). This octant is empty because a noisy static resource assisted by noiseless entanglement cannot generate a dynamic resource (the two static resources cannot generate classical communication or quantum communication or both).

($-, -, -$). $\tilde{C}_{DS}$ completely contains this octant.

($+, -, -$). Define $\Phi |E\rangle$ to be a state of size $|E|$ ebits. We need the following lemma.

Lemma 1: The following inclusion holds

$$C_{DS}^{++} (\rho \otimes \Phi |E\rangle) \subseteq C_{DS}^{0-} (\rho) + \tilde{C}_{DS}^{++} (\Phi |E\rangle).$$

Proof: Super-dense coding induces a linear bijection $f : C_{DS}^{0-} (\rho) \rightarrow C_{DS}^{++} (\rho)$ between the mother achievable region $C_{DS}^{0-} (\rho)$ and the noisy dense coding achievable region $C_{DS}^{++} (\rho)$. The bijection $f$ behaves as follows for every point $(0, Q, E) \in C_{DS}^{0-} (\rho)$:

$$f : (0, Q, E) \rightarrow (2E, Q - E, 0).$$

The first set equivalence follows by dropping the intersection with $O^{;++}$ because

$$\tilde{C}_{DS}^{++} (\rho) \subseteq \tilde{C}_{DS}^{0-} (\rho) + \tilde{C}_{DS}^{++} (\Phi |E\rangle),$$

(51) because applying dense coding to the mother resource inequality given noisy dense coding [25].
$C_{DS}^{+,-} (\rho) \subseteq \widetilde{C}_{DS}^{+,-} (\rho)$ is the statement of the converse theorem for this octant. 

($-, +, -$). The proof is similar to the proof of the previous octant $(+, -, -)$. We need the following additivity lemma.

**Lemma 2:** The following inclusion holds

$$
C_{DS}^{-0} (\rho \otimes \Phi^{[1]:E}] \subseteq \widetilde{C}_{DS}^{-0} (\rho) + \widetilde{C}_{DS}^{-0} (\Phi^{[E]})
$$

**Proof:** Teleportation induces a linear bijection $f : \widetilde{C}_{DS}^{-0} (\rho) \rightarrow \widetilde{C}_{DS}^{-0} (\rho)$ between the entanglement distillation achievable region $C_{DS}^{-0} (\rho)$ and the noisy teleportation achievable region $C_{DS}^{-0} (\rho)$ [25]. The bijection $f$ behaves as follows for every point $(R, 0, E) \in C_{DS}^{-0} (\rho)$,

$$
f : (R, 0, E) \rightarrow (R - 2E, E, 0).
$$

The following relation holds

$$
f(\widetilde{C}_{DS}^{-0} (\rho)) = \widetilde{C}_{DS}^{-0} (\rho),
$$

(54)

because applying teleportation to entanglement distillation gives noisy teleportation [25]. The inclusion $C_{DS}^{-0} (\rho \otimes \Phi^{[E]}) \subseteq f^{-1}(\widetilde{C}_{DS}^{-0} (\rho) + \widetilde{C}_{DS}^{-0} (\Phi^{[E]}))$ holds because

$$
C_{DS}^{-0} (\rho \otimes \Phi^{[E]}) = C_{DS}^{-0} (\rho) + (0, 0, E)
\leq C_{DS}^{+0} (\rho) + C_{DS}^{-0} (\Phi^{[E]})
= \widetilde{C}_{DS}^{+0} (\rho) + \widetilde{C}_{DS}^{-0} (\Phi^{[E]})
= f^{-1}(\widetilde{C}_{DS}^{-0} (\rho)) + f^{-1}(\widetilde{C}_{DS}^{-0} (\Phi^{[E]}))
= f^{-1}(\widetilde{C}_{DS}^{+0} (\rho) + \widetilde{C}_{DS}^{-0} (\Phi^{[E]})).
$$

The first set equivalence follows because the capacity region of the noisy resource state $\rho$ combined with a rate $R$ maximally entangled state is equivalent to a translation of the capacity region of the noisy resource state $\rho$. The first inclusion follows because the capacity region of a rate $R$ maximally entangled state contains the rate triple $(0, 0, E)$. The second set equivalence follows from (40), the third set equivalence from (54), and the fourth set equivalence from linearity of the map $f$. The above inclusion implies the following one:

$$
f(C_{DS}^{+0} (\rho \otimes \Phi^{[E]})) \subseteq \widetilde{C}_{DS}^{-0} (\rho) + \widetilde{C}_{DS}^{-0} (\Phi^{[E]}).
$$

The lemma follows because

$$
f(C_{DS}^{+0} (\rho \otimes \Phi^{[E]})) = f(\widetilde{C}_{DS}^{+0} (\rho \otimes \Phi^{[E]}))
= \widetilde{C}_{DS}^{+0} (\rho \otimes \Phi^{[E]})
= C_{DS}^{+0} (\rho \otimes \Phi^{[E]}),
$$

where we apply the relations in (54) and (40). 

Observe that

$$
\widetilde{C}_{DS}^{+0} (\Phi^{[E]}) = \widetilde{C}_{U}^{+E}.
$$

(55)

Hence for all $E \leq 0$,

$$
C_{DS}^{+E} (\rho) = C_{DS}^{+0} (\rho \otimes \Phi^{[E]}) \subseteq \widetilde{C}_{DS}^{-0} (\rho) + \widetilde{C}_{U}^{+E},
$$

(56)

where we apply Lemma 2 and (55). Thus,

$$
C_{DS}^{+E} (\rho) = \bigcup_{E \leq 0} \widetilde{C}_{DS}^{+E} (\rho)
\subseteq \bigcup_{E \leq 0} (\widetilde{C}_{DS}^{-0} (\rho) + \widetilde{C}_{U}^{+E})
= (\widetilde{C}_{DS}^{-0} (\rho) + \widetilde{C}_{U}) \cap O^{-\infty+}
\subseteq (\widetilde{C}_{CAM} (\rho) + \widetilde{C}_{U}) \cap O^{-\infty+}
= \widetilde{C}_{DS}^{+E} (\rho).
$$

The first set equivalence holds by definition. The first inclusion follows from (56). The second set equivalence follows because $\bigcup_{E \leq 0} \widetilde{C}_{DS}^{+E} (\rho) = \widetilde{C}_{U} \cap O^{-\infty+}$. The second inclusion holds because $\widetilde{C}_{DS}^{+E} (\rho)$ is equivalent to noisy teleportation and the classically-assisted mother combined with the unit resource region generates noisy teleportation. The above inclusion $C_{DS}^{+E} (\rho) \subseteq \widetilde{C}_{DS}^{+E} (\rho)$ is the statement of the converse theorem for this octant.

**V. DIRECT DYNAMIC TRADE-OFF**

In this section, we consider what rates are achievable when a sender and receiver consume a noisy dynamic resource. They additionally consume or generate noiseless classical communication, noiseless quantum communication, and noiseless entanglement. The result is a three-dimensional “direct dynamic” capacity region that gives the full trade-off between the three fundamental noiseless resources when a noisy dynamic resource is available. The interpretation of negative and positive rates is as before.

**Definition 6:** The direct dynamic capacity region $C_{DD} (\mathcal{N})$ of a noisy channel $\mathcal{N}^{A' \rightarrow B}$ is a three-dimensional region in the $(R, Q, E)$ space defined by the closure of the set of all points $(R, Q, E)$ satisfying the following resource inequality:

$$
\langle \mathcal{N} \rangle \geq R[\sigma \rightarrow c] + Q[\sigma \rightarrow q] + E[qq].
$$

(57)

We first recall a few theorems concerning the classically-enhanced father protocol [33], because this protocol proves useful in determining the achievable region for the dynamic case. Briefly, the classically-enhanced father protocol is an optimal protocol for the simultaneous transmission of classical and quantum information with an entanglement-assisted quantum channel.

**Theorem 5:** The following classically-enhanced father resource inequality holds

$$
\langle \mathcal{N} \rangle + \frac{1}{2} I(A; EE'|X)_{\sigma}[qq] \geq
\frac{1}{2} I(A; B|X)_{\sigma}[q \rightarrow q] + I(X; B)_{\sigma}[c \rightarrow c]
$$

(58)

for a noisy dynamic resource $\mathcal{N}^{A' \rightarrow B}$. In the above resource inequality, the state $\sigma^{XABEE'}$ is defined as follows

$$
\sigma^{XABEE'} = \sum_{x} p(x) |x\rangle \langle x|^{X} \otimes U_{\mathcal{N}} (\psi_{x}^{AA'E'})
$$

(59)

where the states $\psi_{x}^{AA'E'}$ are pure and $U_{\mathcal{N}}^{A' \rightarrow BE}$ is an isometric extension of $\mathcal{N}$.
Again, we refer the reader to Refs. [25], [42], [43] for more detail on the above terminology.

**Definition 7:** The “one-shot” classically-enhanced father achievable region \( \tilde{C}_{CEF}^{(1)}(N) \) is as follows:

\[
\tilde{C}_{CEF}^{(1)}(N) = \left\{ \mathbf{I}(X, B)_{\sigma}, \frac{1}{2} \mathbf{I}(A; B|X)_{\sigma}, -\frac{1}{2} \mathbf{I}(A; EE'|X)_{\sigma} \right\},
\]

where \( \sigma \) is defined in (59). The classically-enhanced father achievable region \( \tilde{C}_{CEF}(N) \) is as follows:

\[
\tilde{C}_{CEF}(N) = \bigcup_{k=1}^{\infty} \frac{1}{k} \tilde{C}_{CEF}^{(1)}(N^{\otimes k}).
\]

**Theorem 6:** The classically-enhanced father capacity region \( C_{CEF}(N) \) is equivalent to the classically-enhanced father achievable region:

\[
C_{CEF}(N) = \tilde{C}_{CEF}(N).
\]

The following definition follows from Theorem 6 and the definition of \( C_{DD}(N) \) in (57).

**Definition 8:** The classically-enhanced father capacity region \( C_{CEF}(N) \) is the closure of all achievable rate triples \( (R, Q, E) \) that lie in the \((+, +, -)\) octant of \( C_{DD}(N) \).

We now state this section’s main theorem, the direct dynamic capacity theorem. We prove the theorem by first proving the corresponding direct coding theorem and then with the proof of its converse theorem.

**Theorem 7:** The direct dynamic capacity region \( C_{DD}(N) \) is equivalent to the direct dynamic achievable region \( \tilde{C}_{DD}(N) \):

\[
C_{DD}(N) = \tilde{C}_{DD}(N). \tag{60}
\]

The direct dynamic achievable region \( \tilde{C}_{DD}(N) \) is the set addition of the classically-enhanced father achievable region and the unit resource achievable region:

\[
\tilde{C}_{DD}(N) = \tilde{C}_{CEF}(N) + \tilde{C}_{U}. \tag{61}
\]

**A. Proof of the Direct Coding Theorem**

The direct coding theorem is the statement that the direct dynamic capacity region \( C_{DD}(N) \) contains the direct dynamic achievable region \( \tilde{C}_{DD}(N) \):

\[
C_{DD}(N) \subseteq \tilde{C}_{DD}(N).
\]

It follows immediately from combining the classically-enhanced father resource inequality in [58] with the unit resource inequalities and considering the definition of \( \tilde{C}_{DD}(N) \) in Theorem 7 and \( C_{DD}(N) \) in (57).

**B. Proof of the Converse Theorem**

The statement of the converse theorem is that the direct dynamic achievable region \( \tilde{C}_{DD}(N) \) contains the direct dynamic capacity region:

\[
\tilde{C}_{DD}(N) \subseteq C_{DD}(N).
\]

We consider one octant of the \((R, Q, E)\) space at a time in order to prove the converse theorem.

The converse theorem exploits the optimality of several other protocols: the father protocol [23], [25], classically-enhanced quantum communication and entanglement generation [21], classically-assisted quantum communication and entanglement generation [51], and entanglement-assisted classical communication [17], [18], [24]. We review each of these protocols and their relation to the classically-enhanced father achievable region \( \tilde{C}_{CEF}(N) \) in what follows. We illustrate the relation between these protocols in Figure S.
achievable region \( \tilde{C}_{CE}(\mathcal{N}) \) and capacity region \( C_{CE}(\mathcal{N}) \) lie in the \( Q^{0+} \) quadrant of the \( (R, Q, E) \) space. CE’s respective regions are special cases of the direct dynamic achievable region \( \tilde{C}_{DD}(\mathcal{N}) \) and the direct dynamic capacity region \( C_{DD}(\mathcal{N}) \) with the following correspondences:

\[
\tilde{C}_{CE}(\mathcal{N}) = \tilde{C}_{DD}^{0+}(\mathcal{N}) \equiv \tilde{C}_{DD}(\mathcal{N}) \cap Q^{0+}, \quad (70)
\]

\[
C_{CE}(\mathcal{N}) = C_{DD}^{0+}(\mathcal{N}) \equiv C_{DD}(\mathcal{N}) \cap Q^{0+}. \quad (71)
\]

The CE capacity theorem states that the achievable region \( \tilde{C}_{CE}(\mathcal{N}) \) is equivalent to the capacity region \( C_{CE}(\mathcal{N}) \) [21]:

\[
\tilde{C}_{CE}(\mathcal{N}) = (\tilde{C}_{CEF}(\mathcal{N}) + L_{ED}) \cap Q^{0+}. \quad (73)
\]

The entanglement-assisted classical communication protocol (EAC) consumes a noisy dynamic resource and noiseless entanglement to generate noiseless classical communication. EAC’s achievable region \( \tilde{C}_{EAC}(\mathcal{N}) \) and capacity region \( C_{EAC}(\mathcal{N}) \) lie in the \( Q^{0+} \) quadrant of the \( (R, Q, E) \) space. EAC’s respective regions are special cases of the direct dynamic achievable region \( \tilde{C}_{DD}(\mathcal{N}) \) and the direct dynamic capacity region \( C_{DD}(\mathcal{N}) \) with the following correspondences:

\[
\tilde{C}_{EAC}(\mathcal{N}) = \tilde{C}_{DD}^{0+}(\mathcal{N}) \equiv \tilde{C}_{DD}(\mathcal{N}) \cap Q^{0+}, \quad (74)
\]

\[
C_{EAC}(\mathcal{N}) = C_{DD}^{0+}(\mathcal{N}) \equiv C_{DD}(\mathcal{N}) \cap Q^{0+}. \quad (75)
\]

The EAC capacity theorem states that the achievable region \( \tilde{C}_{EAC}(\mathcal{N}) \) is equivalent to the capacity region \( C_{EAC}(\mathcal{N}) \) [17], [18], [24], [25]:

\[
\tilde{C}_{EAC}(\mathcal{N}) = C_{EAC}(\mathcal{N}). \quad (76)
\]

EAC’s achievable region \( \tilde{C}_{EAC}(\mathcal{N}) \) is obtainable from the classically-enhanced father’s achievable region \( \tilde{C}_{CEF}(\mathcal{N}) \) by combining it with the super-dense coding protocol and keeping the points with zero quantum communication [43]:

\[
\tilde{C}_{EAC}(\mathcal{N}) = (\tilde{C}_{CEF}(\mathcal{N}) + L_{SD}) \cap Q^{0+}. \quad (77)
\]

Forward classical communication does not increase the entanglement generation capacity or the quantum communication capacity [16], [31]. Thus, there is a simple relation between the achievable region \( \tilde{C}_{DD}^{0+}(\mathcal{N}) \) and the achievable entanglement generation capacity region \( \tilde{C}_{DD}^{00+}(\mathcal{N}) \):

\[
\tilde{C}_{DD}^{00+}(\mathcal{N}) = \tilde{C}_{DD}^{0+}(\mathcal{N}) + L^{-00},
\]

where \( L^{-00} \equiv \{ \lambda (-1, 0, 0) : \lambda \geq 0 \} \). The converse proof of the entanglement generation capacity region states that the achievable region \( \tilde{C}_{DD}^{00+}(\mathcal{N}) \) is optimal [8]:

\[
\tilde{C}_{DD}^{00+}(\mathcal{N}) = C_{DD}^{00+}(\mathcal{N}).
\]

Optimality of the region \( \tilde{C}_{DD}^{00+}(\mathcal{N}) \) then follows:

\[
C_{DD}^{00+}(\mathcal{N}) = C_{DD}^{00+}(\mathcal{N}) + L^{-00}
\]

\[
= \tilde{C}_{DD}^{00+}(\mathcal{N}) + L^{-00} = \tilde{C}_{DD}^{-00}(\mathcal{N}). \quad (78)
\]

The entanglement generation achievable region results from combining the father achievable region in (65) with entanglement distribution [23]:

\[
\tilde{C}_{DD}^{00+}(\mathcal{N}) = (\tilde{C}_{CEF}(\mathcal{N}) \cap Q^{0+}) + L_{ED}) \cap L^{00+}. \quad (79)
\]

Similar results for the classically-assisted quantum communication capacity region \( C_{DD}^{-00}(\mathcal{N}) \) hold:

\[
C_{DD}^{-00}(\mathcal{N}) = C_{DD}^{00+}(\mathcal{N}) + L^{-00}
\]

\[
= \tilde{C}_{DD}^{00+}(\mathcal{N}) + L^{-00} = \tilde{C}_{DD}^{-00}(\mathcal{N}). \quad (80)
\]

The quantum communication achievable region results from combining the father achievable region in (65) with entanglement distribution [23]:

\[
\tilde{C}_{DD}^{00+}(\mathcal{N}) = (\tilde{C}_{CEF}(\mathcal{N}) \cap Q^{0+}) + L_{ED}) \cap L^{00+}. \quad (81)
\]
1) Octants That Generate Quantum Communication: We first consider all of the octants with corresponding protocols that generate quantum communication, i.e., octants of the form \((\pm, \pm, \pm)\). The proof of one of the octants is the converse proof of the classically-enhanced father protocol. The proofs of two of the remaining three octants are similar to the entanglement-generating octants from the static case. The similarity holds because these octants generate the noiseless version of the noisy dynamic resource. The proof of the last remaining octant is different from any we have seen so far. We discuss later how its proof gives insight into the question of using entanglement-assisted coding versus teleportation.

\((+, +, -)\). This octant is the converse theorem of the classically enhanced father protocol (Ref. 43 contains the proof). It is interesting that the classically-enhanced father protocol is optimal for this octant, i.e., there is no need to extend the protocol with unit resource protocols.

\((+, +, +)\). Define 
\[
C_{DD}^{++} (N) = C_{DD} (N) \cap O^{+++}
\]
and recall the definition of \(C_{DD}^{0+} (N)\) in (71). We exploit the line of entanglement distribution \(L_{ED}\) as defined in (8). Define the following maps 
\[
f : S \to (S + L_{ED}) \cap Q^{0+},
\]
\[
\hat{f} : S \to (S - L_{ED}) \cap O^{+++}.
\]
The map \(f\) translates the set \(S\) in the entanglement distribution direction and keeps the points that lie on the \(Q^{0+}\) quadrant. The map \(\hat{f}\), in a sense, undoes the effect of \(f\) by moving the set \(S\) back to the \(O^{+++}\) octant.

The inclusion \(C_{DD}^{++} (N) \subseteq \hat{f}(f(C_{DD}^{++} (N)))\) holds because
\[
\begin{align*}
C_{DD}^{++} (N) &= C_{DD}^{++} (N) \cap O^{+++} \\
&\subseteq ((C_{DD}^{++} (N) + L_{ED}) \cap Q^{0+}) - L_{ED}) \cap O^{+++} \\
&= (f(C_{DD}^{++} (N)) - L_{ED}) \cap O^{+++} \\
&= \hat{f}(f(C_{DD}^{++} (N)))).
\end{align*}
\] (82)
The first set equivalence is obvious from the definition of \(C_{DD}^{++} (N)\). The first inclusion follows from the following logic. Pick any point \(a \equiv (R, Q, E) \in C_{DD}^{++} (N)\) and a particular point \(b \equiv (0, -Q, Q) \in L_{ED}\). It follows that \(a + b = (R, 0, E + Q) \in (C_{DD}^{++} (N) + L_{ED}) \cap Q^{0+}\). Pick a point \(-b = (0, Q, -Q) \in -L_{ED}\). It follows that \(a + b + (-b) = ((C_{DD}^{++} (N) + L_{ED}) \cap Q^{0+}) - L_{ED}) \cap O^{+++}\) and that \(a + b - b = (R, Q, E) = a\). The first inclusion thus holds because every point in \(C_{DD}^{++} (N) \cap O^{+++}\) is in \(((C_{DD}^{++} (N) + L_{ED}) \cap Q^{0+}) - L_{ED}) \cap O^{+++}\). The second set equivalence follows from the definition of \(f\) and the third set equivalence follows from the definition of \(\hat{f}\).

It is operationally clear that the following inclusion holds
\[
f(C_{DD}^{++} (N)) \subseteq C_{DD}^{0+} (N),
\] (83)
because the mapping \(f\) converts any achievable point \(a \in C_{DD}^{++} (N)\) to an achievable point in \(C_{DD}^{0+} (N)\) by consuming all of the quantum communication at point \(a\) with entanglement distribution.

The converse proof of the classically-enhanced entanglement generation protocol (CE) [21] states that the following inclusion holds
\[
C_{DD}^{0+} (N) = C_{CE} (N) \subseteq C_{CE} (N).
\] (84)
The inclusion \(f(C_{CE} (N)) \subseteq C_{DD}^{0+} (N)\) holds because
\[
\begin{align*}
\hat{f}(C_{CE} (N)) &= (((C_{CE} (N) + L_{ED}) \cap Q^{0+}) - L_{ED}) \cap O^{+++} \\
&\subseteq ((C_{CE} (N) + L_{ED}) - L_{ED}) \cap O^{+++} \\
&= ((C_{CE} (N) + L_{ED}) \cap O^{+++}) \cup ((C_{CE} (N) - L_{ED}) \cap O^{++}) \\
&\subseteq C_{DD}^{0+} (N).
\end{align*}
\] (85)
The first set equivalence follows from (73) the definition of \(\hat{f}\). The first inclusion follows by dropping the intersection with \(Q^{0+}\). The second set equivalence follows because \((C_{CE} (N) + L_{ED}) - L_{ED} = (C_{CE} (N) + L_{ED}) \cup (C_{CE} (N) - L_{ED})\), and the second inclusion holds because \((C_{CE} (N) - L_{ED}) \cap O^{+++} = (0, 0, 0)\) and (61).

Putting (82), (83), (84), and (85) together, we have the following inclusion:
\[
C_{DD}^{++} (N) \subseteq f(f(C_{DD}^{++} (N))) \\
\subseteq \hat{f}(C_{DD}^{++} (N)) \subseteq \hat{f}(C_{CE} (N)) \subseteq C_{DD}^{++} (N).
\]

The above inclusion \(C_{DD}^{++} (N) \subseteq C_{DD}^{++} (N)\) is the statement of the converse theorem for this octant.

\((-, +, +)\). Define 
\[
C_{DD}^{0-} (N) = C_{DD} (N) \cap O^{-++},
\]
\[
C_{DD}^{0+} (N) = C_{DD} (N) \cap Q^{-0+}.
\]
Recall the definition of the line of entanglement distribution \(L_{ED}\) in (8). Define the following maps 
\[
f : S \to (S + L_{ED}) \cap Q^{-0+},
\]
\[
\hat{f} : S \to (S - L_{ED}) \cap O^{-++}.
\]
The map \(f\) translates the set \(S\) in the entanglement distribution direction and keeps the points that lie on the \(Q^{-0+}\) quadrant. The map \(\hat{f}\), in a sense, undoes the effect of \(f\) by moving the set \(S\) back to the \(O^{-++}\) octant.

The inclusion \(C_{DD}^{0-} (N) \subseteq \hat{f}(f(C_{DD}^{0-} (N)))\) holds because
\[
\begin{align*}
C_{DD}^{0-} (N) &= C_{DD}^{0-} (N) \cap O^{-++} \\
&\subseteq ((C_{DD}^{0-} (N) + L_{ED}) \cap Q^{-0+}) - L_{ED}) \cap O^{-++} \\
&= (f(C_{DD}^{0-} (N)) - L_{ED}) \cap O^{-++} \\
&= \hat{f}(f(C_{DD}^{0-} (N)))).
\end{align*}
\] (86)
The first set equivalence is obvious from the definition of \(C_{DD}^{0-}\). The first inclusion follows from the following logic. Pick any point \(a \equiv (R, Q, E) \in C_{DD}^{0-} (N) \cap O^{-++}\) and a particular point \(b \equiv (0, -Q, Q) \in L_{ED}\). It follows that the
point \(a + b = (R, 0, E + Q) \in (C_{DD}^+ (N) + L_{ED}) \cap Q^{-0+}\). We then pick a point \(-b = (0, Q, -Q) \in -L_{ED}\). It follows that \(a + b - b \in (((C_{DD}^+ (N) + L_{ED}) \cap Q^{-0+}) - L_{ED}) \cap O^{+0+}\) and that \(a + b - b = (R, Q, E) = a\). Thus, the first inclusion follows because every point in \(C_{DD}^+ \cap O^{+0+}\) belongs to \(((C_{DD}^+ (N) + L_{ED}) \cap Q^{-0+}) - L_{ED}) \cap O^{+0+}\). The second set equivalence follows from the definition of \(f\), and the third set equivalence follows from the definition of \(\tilde{f}\).

It is operationally clear that the following inclusion holds

\[
f(C_{DD}^+ (N)) \subseteq C_{DD}^{-0+} (N),
\]

because the mapping \(f\) converts any achievable point in \(C_{DD}^+ (N)\) to an achievable point in \(C_{DD}^{-0+} (N)\) by combining it with entanglement distribution.

Forward classical communication does not increase the entanglement generation capacity \([16, 31]\). Thus, the following result from \([78]\) applies

\[
C_{DD}^{-0+} (N) = \tilde{C}_{DD}^{-0+} (N).
\]

It then follows that

\[
\begin{align*}
\tilde{C}_{DD}^{0+} (N) &= C_{DD}^{0+} (N) + L_{ED}^{-0} \\
&= ((C_{CEF} (N) \cap Q^{0+}) + L_{ED}^{-0}) + L_{ED}^{-0} \\
&\subseteq \tilde{C}_{CEF} (N) + L_{ED} + L_{ED}^{-0} \\
&= \tilde{C}_{CEF} (N) + L_{ED} + L_{TP}.
\end{align*}
\]

The first set equivalence follows from \([78]\). The second set equivalence follows from \([79]\). The first inclusion follows by dropping the intersections with \(Q^{0+}\) and \(L_{ED}^{-0}\), and the last inclusion follows because entanglement distribution and teleportation can generate any point along \(L_{ED}^{-0}\).

The inclusion \(\tilde{f}(\tilde{C}_{DD}^{0+} (N)) \subseteq \tilde{C}_{DD}^{-0+} (N)\) holds because

\[
\begin{align*}
\tilde{f}(\tilde{C}_{DD}^{0+} (N)) &\subseteq ((C_{CEF} (N) + L_{TP} + L_{ED} - L_{ED}) \cap O^{++}) \\
&= ((C_{CEF} (N) + L_{TP} + L_{ED}) \cap O^{++}) \\
&\subseteq \tilde{C}_{DD}^{-0+} (N).
\end{align*}
\]

The first inclusion follows from \([89]\) and the definition of \(\tilde{f}\). The first set equivalence follows because \((C_{CEF} (N) + L_{TP} + L_{ED} - L_{ED} = (C_{CEF} (N) + L_{TP} + L_{ED}) \cup (C_{CEF} (N) + L_{TP} - L_{ED})\), and the last inclusion follows because \((C_{CEF} (N) + L_{TP} - L_{ED}) \cap O^{++} = (0, 0, 0)\) and \([61]\).

Putting \([86, 87, 88, 90]\) together, the following inclusion holds

\[
C_{DD}^{-0+} (N) \subseteq \tilde{f}(f(C_{DD}^{-0+} (N))) \subseteq \tilde{f}(C_{DD}^{0+} (N)) \subseteq \tilde{C}_{DD}^{-0+} (N).
\]

The above inclusion \(C_{DD}^{-0+} (N) \subseteq \tilde{C}_{DD}^{-0+} (N)\) is the statement of the converse theorem for this octant.

\((-+, +, -)\). This octant is perhaps the most interesting one for the direct dynamic trade-off. Its proof uses ideas similar to those of previous octants, but it requires four different cases.

Interestingly, this octant gives insight into optimal strategies for entanglement-assisted coding \([34, 35, 36, 37, 38, 39, 40, 41]\). We discuss these insights in Section 7C.

The four different cases for this octant are as follows: 1) \(Q \geq |R| / 2\) and \(|E| \geq |R| / 2\), 2) \(Q \geq |R| / 2\) and \(|E| \leq |R| / 2\), 3) \(Q \leq |R| / 2\) and \(|E| \geq |R| / 2\), 4) \(Q \leq |R| / 2\) and \(|E| \leq |R| / 2\).

We first prove the case where \(Q \geq |R| / 2\) and \(|E| \geq |R| / 2\).

Define

\[
C_{DD}^{-0+} (N) \equiv C_{DD} (N) \cap O^{0+},
\]

\[
C_{DD}^{0+} (N) \equiv C_{DD} (N) \cap Q^{0+}.
\]

We make it implicit for this case that \(C_{DD}^{0+} (N)\) denotes the region of the octant \(O^{0+}\) where \(Q \geq |R| / 2\) and \(|E| \geq |R| / 2\). Recall the definition of the line of teleportation \(L_{TP}\) in \([6]\). Define the following maps

\[
f : S \rightarrow (S - L_{TP}) \cap Q^{0+},
\]

\[
\tilde{f} : S \rightarrow (S + L_{TP}) \cap O^{0+}.
\]

The map \(f\) translates the set \(S\) in the negative teleportation direction and keeps the points that lie on \(Q^{0+}\). The map \(\tilde{f}\), in a sense, undoes the effect of \(f\) by moving the set \(S\) back to the \(O^{0+}\) octant.

The inclusion \(C_{DD}^{-0+} (N) \subseteq \tilde{f}(f(C_{DD}^{-0+} (N)))\) holds because

\[
C_{DD}^{-0+} (N) = C_{DD} (N) \cap O^{0+}
\]

\[
\subseteq ((C_{DD} (N) - L_{TP}) \cap Q^{0+}) + L_{TP}) \cap O^{0+}.
\]

The first set equivalence is obvious from the definition of \(C_{DD}^{-0+} (N)\). The first inclusion follows from the following logic. Pick any point \(a = (R, Q, E) \in C_{DD}^{-0+} (N)\) and a particular point \(b = (|R|, -|R| / 2, |R| / 2) \in -L_{TP}\). It follows that \(a + b = (0, Q - |R| / 2, E + |R| / 2) \in (C_{DD}^{-0+} (N) - L_{TP}) \cap Q^{0+}\). Pick a point \(-b = (|R|, |R| / 2, -|R| / 2) \in L_{TP}\). It follows that \(a + b - b \in (((C_{DD}^{-0+} (N) - L_{TP}) \cap Q^{0+}) + L_{TP}) \cap O^{0+}\) and that \(a + b - b = (R, Q, E) = a\). The first inclusion thus holds because every point in \(C_{DD}^{-0+} (N) \cap O^{0+}\) is in \(((C_{DD}^{-0+} (N) - L_{TP}) \cap Q^{0+}) + L_{TP}) \cap O^{0+}\). The second set equivalence follows from the definition of \(f\) and the third set equivalence follows from the definition of \(\tilde{f}\).

Now suppose that a “negative teleportation protocol” exists. We define such a negative teleportation protocol as the following resource inequality:

\[
[q \rightarrow q] \geq 2 |e \rightarrow c| + [qq] - [c \rightarrow c].
\]

This protocol may be an imaginary one, but it is useful mathematically because it serves as a way to map achievable points in \(C_{DD}^{0+} (N)\) to those in \(C_{DD}^{-0+} (N)\) with this negative teleportation protocol, it is operationally clear that the

(2)Devetak and Shor imagined that the resource inequality \([qq] \geq |e \rightarrow c|\) existed to relate the classically-enhanced entanglement generation capacity region to the entanglement-assisted classical capacity region \([21]\).
following inclusion holds

\[
(f(C_{\text{DD}}^{-+}^+)(N)) \subseteq C_{\text{DD}}^{-+}^+(N),
\]

(92)
because the mapping \( f \) converts any achievable point \( a \in C_{\text{DD}}^{-+}^+(N) \) to an achievable point in \( C_{\text{DD}}^{-+}^+(N) \) by consuming qubits at a rate \(|R|/2\) at point \( a \) with the negative teleportation protocol.

The father capacity theorem in \[64\] states that the achievable region \( \overline{C}_{\text{DD}}^{-+}^+(N) \) contains the capacity region \( C_{\text{DD}}^{-+}^+(N) \):

\[
C_{\text{DD}}^{-+}^+(N) \subseteq \overline{C}_{\text{DD}}^{-+}^+(N).
\]

(93)
The inclusion \( \hat{f}(\overline{C}_{\text{DD}}^{-+}^+(N)) \subseteq \overline{C}_{\text{DD}}^{-+}^+(N) \) holds because

\[
\hat{f}(\overline{C}_{\text{DD}}^{-+}^+(N)) = (\overline{C}_{\text{DD}}^{-+}^+(N) + L_{\text{TP}}) \cap O^{-+} \\
\subseteq (\overline{CEF} + \overline{CU}) \cap O^{-+} \\
= \overline{C}_{\text{DD}}^{-+}^+(N).
\]

(94)
The first set equivalence follows from the definition of \( \hat{f} \). The first inclusion follows because \( C_{\text{DD}}^{-+}^+(N) \subseteq C_{\text{CEF}}(N) \) and \( L_{\text{TP}} \subseteq \overline{CU} \), and the second set equivalence follows from \[61\].

Putting \[91\], \[92\], \[93\], and \[94\] together, we have the following inclusion

\[
C_{\text{DD}}^{-+}^+(N) \subseteq \hat{f}(f(C_{\text{DD}}^{-+}^+(N))) \\
\subseteq \hat{f}(C_{\text{DD}}^{-+}^+(N)) \subseteq \hat{f}(\overline{C}_{\text{DD}}^{-+}^+(N)) \subseteq \overline{C}_{\text{DD}}^{-+}^+(N).
\]

The above inclusion \( C_{\text{DD}}^{-+}^+(N) \subseteq \overline{C}_{\text{DD}}^{-+}^+(N) \) is the statement of the converse theorem of octant \( O^{-+} \) for the case where \( Q \geq |R|/2 \) and \( |E| \geq |R|/2 \).

We now consider the case where \( Q \geq |R|/2 \) and \( |E| \leq |R|/2 \). Define

\[
C_{\text{DD}}^{-0+}^+(N) \equiv C_{\text{DD}}(N) \cap Q^{-0+}.
\]

We make it implicit for this case that \( C_{\text{DD}}^{-+}^+(N) \) denotes the region of the octant \( O^{-+} \) where \( Q \geq |R|/2 \) and \( |E| \leq |R|/2 \). Recall the definition of the line of teleportation \( L_{\text{TP}} \) in \[6\]. Define the following maps

\[
f : S \to (S - L_{\text{TP}}) \cap Q^{-0+}, \\
\hat{f} : S \to (S + L_{\text{TP}}) \cap O^{-+}.
\]

The map \( f \) translates the set \( S \) in the negative teleportation direction and keeps the points that lie on \( Q^{-0+} \). The map \( \hat{f} \), in a sense, undoes the effect of \( f \) by moving the set \( S \) back to the \( O^{-+} \) octant.

The inclusion \( C_{\text{DD}}^{-+}^+(N) \subseteq \hat{f}(f(C_{\text{DD}}^{-+}^+(N))) \) holds because

\[
C_{\text{DD}}^{-+}^+(N) \\
= f(C_{\text{DD}}^{-+}^+(N)) \cap O^{-+} \\
\subseteq ((C_{\text{DD}}^{-+}^+(N) - L_{\text{TP}}) \cap Q^{-0+}) + L_{\text{TP}} \cap O^{-+} \\
= (f(C_{\text{DD}}^{-+}^+(N)) + L_{\text{TP}}) \cap O^{-+} \\
= \hat{f}(f(C_{\text{DD}}^{-+}^+(N))).
\]

(95)
The first set equivalence is obvious from the definition of \( C_{\text{DD}}^{-+}^+(N) \). The first inclusion follows from the following logic. Pick any point \( a \equiv (R, Q, E) \in C_{\text{DD}}^{-+}^+(N) \) and a particular point \( b \equiv (|E|, -|E|, |E|) \in -L_{\text{TP}} \). It follows that \( a + b = (-|R| + 2|E|, Q - |E|, 0) \in (C_{\text{DD}}^{-+}^+(N) - L_{\text{TP}}) \cap Q^{-0+} \). Pick a point \( -b = (-2|R|, |E|, -|E|) \in L_{\text{TP}} \). It follows that \( a + b - b \in (((C_{\text{DD}}^{-+}^+(N) - L_{\text{TP}}) \cap Q^{-0+}) + L_{\text{TP}}) \cap O^{-+} \) and that \( a + b - b = (R, Q, E) = a \). The first inclusion thus holds because every point in \( C_{\text{DD}}^{-+}^+(N) \cap O^{-+} \) is in \( (((C_{\text{DD}}^{-+}^+(N) - L_{\text{TP}}) \cap Q^{-0+}) + L_{\text{TP}}) \cap O^{-+} \). The second set equivalence follows from the definition of \( f \), and the third set equivalence follows from the definition of \( \hat{f} \).

Suppose again that a “negative teleportation protocol” exists. We can then exploit it to map points in \( C_{\text{DD}}^{-+}^+(N) \) to points in \( \overline{C}_{\text{DD}}^{-+}^+(N) \). With this negative teleportation protocol, it is operationally clear that the following inclusion holds

\[
f(C_{\text{DD}}^{-+}^+(N)) \subseteq C_{\text{DD}}^{-0+}^+(N).
\]

(96)
It holds because the mapping \( f \) converts any achievable point \( a \in C_{\text{DD}}^{-+}^+(N) \) to an achievable point in \( C_{\text{DD}}^{-0+}^+(N) \) by consuming qubits at a rate \(|E| \) from point \( a \) with the negative teleportation protocol.

The classically-assisted quantum communication capacity theorem in \[80\] states that the achievable region \( C_{\text{DD}}^{-0+}^+(N) \) contains the capacity region \( C_{\text{DD}}^{-0+}^+(N) \):

\[
C_{\text{DD}}^{-0+}^+(N) \subseteq \overline{C}_{\text{DD}}^{-0+}^+(N).
\]

(97)
The inclusion \( \hat{f}(C_{\text{DD}}^{-0+}^+(N)) \subseteq \overline{C}_{\text{DD}}^{-0+}^+(N) \) holds because

\[
\hat{f}(C_{\text{DD}}^{-0+}^+(N)) = (C_{\text{DD}}^{-0+}^+(N) + L_{\text{TP}}) \cap O^{-+} \\
\subseteq (\overline{CEF} + \overline{CU}) \cap O^{-+} \\
= \overline{C}_{\text{DD}}^{-0+}^+(N).
\]

(98)
The first set equivalence follows from the definition of \( \hat{f} \). The first inclusion follows because \( C_{\text{DD}}^{-0+}^+(N) \subseteq C_{\text{CEF}}(N) \) and \( L_{\text{TP}} \subseteq \overline{CU} \), and the second set equivalence follows from \[61\].

Putting \[95\], \[96\], \[97\], and \[98\] together, we have the following inclusion:

\[
C_{\text{DD}}^{-+}^+(N) \subseteq \hat{f}(f(C_{\text{DD}}^{-+}^+(N))) \\
\subseteq \hat{f}(C_{\text{DD}}^{-+}^+(N)) \subseteq \hat{f}(\overline{C}_{\text{DD}}^{-+}^+(N)) \subseteq \overline{C}_{\text{DD}}^{-+}^+(N).
\]

The above inclusion \( C_{\text{DD}}^{-+}^+(N) \subseteq \overline{C}_{\text{DD}}^{-+}^+(N) \) is the statement of the converse theorem of octant \( O^{-+} \) for the case where \( Q \geq |R|/2 \) and \( |E| \leq |R|/2 \).

The next case is where \( Q \leq |R|/2 \) and \( |E| \geq |R|/2 \). Define

\[
C_{\text{DD}}^{-0-}^+(N) \equiv C_{\text{DD}}(N) \cap Q^{-0-}.
\]

We make it implicit for this case that \( C_{\text{DD}}^{-+}^+(N) \) denotes the region of the octant \( O^{-+} \) where \( Q \leq |R|/2 \) and \( |E| \geq |R|/2 \). Recall the definition of the line of teleportation \( L_{\text{TP}} \) in \[6\]. Define the following maps

\[
f : S \to (S - L_{\text{TP}}) \cap Q^{-0-}, \\
\hat{f} : S \to (S + L_{\text{TP}}) \cap O^{-+}.
\]

The map \( f \) translates the set \( S \) in the negative teleportation direction and keeps the points that lie on \( Q^{-0-} \). The map \( \hat{f} \), in a sense, undoes the effect of \( f \) by moving the set \( S \) back to the \( O^{-+} \) octant.
The inclusion $C_{DD}^{+\to-}(N) \leq \tilde{f}(f(C_{DD}^{+\to-}(N)))$ holds because
\[
C_{DD}^{+\to-}(N) = C_{DD}^{+\to-}(N) \cap O^{+-} \\
\subseteq (\{(C_{DD}^{+\to-}(N) - L_{TP}) - Q^{-0-}\} + L_{TP}) \cap O^{+-} \\
= (f(C_{DD}^{+\to-}(N)) + L_{TP}) \cap O^{+-} \\
= \tilde{f}(f(C_{DD}^{+\to-}(N))).
\] (99)

The first set equivalence is obvious from the definition of $C_{DD}^{+\to-}(N)$. The first inclusion follows from the following logic. Pick any point $a \equiv (R, Q, E) \in C_{DD}^{+\to-}(N)$ and a particular point $b \equiv (2Q, -Q, Q) \in -L_{TP}$. It follows that $a + b = (-R + 2Q, 0, E + Q) \in (C_{DD}^{+\to-}(N) - L_{TP}) - Q^{-0-}$. Pick a point $-b = (-2|Q|, |Q|, -|Q|) \in L_{TP}$. It follows that $a + b - b = (R, Q, E) = a$. The sets equivalence thus holds because every point in $C_{DD}^{+\to-}(N) \cap O^{+-}$ is in $\{(C_{DD}^{+\to-}(N) - L_{TP}) - Q^{-0-}\} + L_{TP}) \cap O^{+-}$.

Now consider the four octants with corresponding protocols that consume quantum communication, i.e., octants of the form $(\pm, -, \pm)$. The proof of one of the octants is trivial. The proofs of the remaining three octants are similar to the proofs of the entanglement-consuming octants from the static case.

**Lemma 3**: The following inclusion holds
\[
C_{DD}^{+0+}(N \otimes id^{\otimes |Q|}) \subseteq \tilde{C}_{DD}^{+0+}(N) + C_{DD}^{+0+}(id^{\otimes |Q|}).
\]

**Proof**: Entanglement distribution induces a linear bijection $f : C_{DD}^{+0+}(N) \rightarrow \tilde{C}_{DD}^{+0+}(N)$ between the classically-enhanced quantum communication achievable region $\tilde{C}_{DD}^{+0+}$ and the classically-enhanced entanglement generation achievable region $C_{DD}^{+0+}$. The linear bijection $f$ behaves as follows for every point $(R, Q, 0) \in C_{DD}^{+0+}$:
\[
f : (R, Q, 0) \rightarrow (R, 0, Q).
\]

The following relation holds
\[
f(C_{DD}^{+0+}(N)) = \tilde{C}_{DD}^{+0+}(N),
\] (103)

because applying entanglement distribution to the classically-enhanced quantum communication resource inequality gives classically-enhanced entanglement generation $\tilde{C}_{DD}^{+0+}$. The inclusion $C_{DD}^{+0+}(N \otimes id^{\otimes |Q|}) \subseteq f^{-1}(C_{DD}^{+0+}(N) + C_{DD}^{+0+}(id^{\otimes |Q|}))$ holds because
\[
C_{DD}^{+0+}(N \otimes id^{\otimes |Q|}) \\
= C_{DD}^{+0+}(N) + (0, Q, 0) \\
\subseteq C_{DD}^{+0+}(N) + C_{DD}^{+0+}(id^{\otimes |Q|}) \\
= C_{DD}^{+0+}(N) + C_{DD}^{+0+}(id^{\otimes |Q|}) \\
\subseteq f^{-1}(C_{DD}^{+0+}(N)) + f^{-1}(C_{DD}^{+0+}(id^{\otimes |Q|})) \\
= f^{-1}(C_{DD}^{+0+}(N) + C_{DD}^{+0+}(id^{\otimes |Q|})).
\]

The first set equivalence follows because the capacity region of the noisy channel $N$ combined with a rate $Q$ noiseless qubit channel is equivalent to a translation of the capacity region of the noisy channel $N$. The first inclusion follows because the capacity region of a rate $Q$ noiseless qubit channel contains the rate triple $(0, Q, 0)$. The second set equivalence follows from the classically-enhanced quantum communication capacity theorem in [65]. The third set equivalence follows from (103), and the fourth set equivalence from linearity of the map $f$. The above inclusion implies the following one:
\[
f(C_{DD}^{+0+}(N \otimes id^{\otimes |Q|})) \subseteq \tilde{C}_{DD}^{+0+}(N) + C_{DD}^{+0+}(id^{\otimes |Q|}).
\]
The lemma follows because

\[
f(C_{DD}^{0+}(N \otimes \text{id}^{|Q|})) = f(\tilde{C}_{DD}^{+0}(N \otimes \text{id}^{|Q|}))
= \tilde{C}_{DD}^{+0}(N \otimes \text{id}^{|Q|})
= C_{DD}^{+0}(N \otimes \text{id}^{|Q|}),
\]

where we apply the relations in \((68), (103),\) and \((72).\)

Observe that

\[
\tilde{C}_{DD}^{+0}(\text{id}^{|Q|}) = C_{U}^{+0}. \tag{104}
\]

Hence, for all \(Q \leq 0,\)

\[
C_{DD}^{+0}(N) = C_{DD}^{0+}(N \otimes \text{id}^{|Q|}) \subseteq \tilde{C}_{DD}^{+0}(N) + C_{U}^{+0}, \tag{105}
\]

where we apply Lemma \(3\) and \((104).\) The inclusion

\[
C_{DD}^{0+}(N) \subseteq C_{DD}^{+0}\]

follows because

\[
C_{DD}^{+0}(N) = \bigcup_{Q \leq 0} C_{DD}^{0+}(N)
\subseteq \bigcup_{Q \leq 0} (\tilde{C}_{DD}^{+0}(N) + C_{U}^{+0})
= (\tilde{C}_{DD}^{+0}(N) + C_{U}^{+0}) \cap O^{++}
\subseteq (\tilde{C}_{CEF}(N) + C_{U}^{+0}) \cap O^{++}
= \tilde{C}_{DD}^{+0}(N).
\]

The first and equivalence hold by definition, the first inclusion follows from \((105),\) the second set equivalence follows because \(\bigcup_{Q \leq 0} C_{U}^{+0} = C_{U} \cap O^{++},\) and the second inclusion follows because combining the classically-enhanced father region with entanglement distribution gives the region for classically-enhanced entanglement generation. The above inclusion \(C_{DD}^{0+}(N) \subseteq \tilde{C}_{DD}^{+0}\) is the statement of the converse theorem for this octant.

\((+,-,-,-).\) The proof for this octant is similar to that of the previous one. It exploits the bijection between the father achievable region \(C_{DD}^{0+}\) and the entanglement-assisted classical communication achievable region \(C_{DD}^{+0}.\) We need the following lemma.

**Lemma 4:** The following inclusion holds

\[
C_{DD}^{+0}(N \otimes \text{id}^{|Q|}) \subseteq \tilde{C}_{DD}^{+0}(N) + \tilde{C}_{DD}^{+0}(\text{id}^{|Q|}).
\]

**Proof:** Super-dense coding induces a linear bijection \(f : C_{DD}^{0+}(N) \to C_{DD}^{+0}(N)\) between the father achievable region \(C_{DD}^{0+}(N)\) and the entanglement-assisted classical communication achievable region \(\tilde{C}_{DD}^{+0}(N)\) \([23].\) The map behaves as follows for any point \((0, Q, E) \in C_{DD}^{0+}(N):\)

\[
f : (0, Q, E) \to (2Q, 0, E - Q).
\]

The following relation holds

\[
f(\tilde{C}_{DD}^{0+}(N)) = \tilde{C}_{DD}^{+0}(N), \tag{106}
\]

because applying super-dense coding to the father resource inequality produces entanglement-assisted classical communication \([\tilde{C}, \tilde{C}].\) The inclusion \(C_{DD}^{+0}(N \otimes \text{id}^{|Q|}) \subseteq f^{-1}(\tilde{C}_{DD}^{+0}(N) + \tilde{C}_{DD}^{+0}(\text{id}^{|Q|}))\) holds because

\[
C_{DD}^{+0}(N \otimes \text{id}^{|Q|})
= C_{DD}^{0+}(N) + (0, Q, 0)
\subseteq C_{DD}^{+0}(N) + C_{DD}^{+0}(\text{id}^{|Q|})
= \tilde{C}_{DD}^{+0}(N) + \tilde{C}_{DD}^{+0}(\text{id}^{|Q|})
= f^{-1}(\tilde{C}_{DD}^{+0}(N) + f^{-1}(\tilde{C}_{DD}^{+0}(\text{id}^{|Q|}))
= f^{-1}(\tilde{C}_{DD}^{+0}(N) + \tilde{C}_{DD}^{+0}(\text{id}^{|Q|})).
\]

The first set equivalence follows because the capacity region of the noisy channel \(N\) combined with a rate \(Q\) noiseless qubit channel is equivalent to a translation of the capacity region of the noisy channel \(N\). The first inclusion follows because the capacity region of a rate \(Q\) noiseless qubit channel contains the rate triple \((0, Q, 0).\) The second set equivalence follows from the father capacity theorem in \((64),\) the third set equivalence from \((106),\) and the fourth set equivalence from linearity of the map \(f.\) The above inclusion implies the following inclusion:

\[
f(C_{DD}^{0+}(N \otimes \text{id}^{|Q|})) \subseteq \tilde{C}_{DD}^{+0}(N) + \tilde{C}_{DD}^{+0}(\text{id}^{|Q|}).
\]

The lemma follows because

\[
f(C_{DD}^{0+}(N \otimes \text{id}^{|Q|})) = f(\tilde{C}_{DD}^{0+}(N \otimes \text{id}^{|Q|}))
= \tilde{C}_{DD}^{+0}(N \otimes \text{id}^{|Q|})
= C_{DD}^{+0}(N \otimes \text{id}^{|Q|}),
\]

where we apply the relations in \((64), (106),\) and \((76).\)

Observe that

\[
\tilde{C}_{DD}^{+0}(\text{id}^{|Q|}) = C_{U}^{+0} - \tilde{C}_{DD}^{+0}(N). \tag{107}
\]

Hence, for all \(Q \leq 0,\)

\[
C_{DD}^{0+}(N) = C_{DD}^{+0}(N \otimes \text{id}^{|Q|}) \subseteq \tilde{C}_{DD}^{+0}(N) + C_{U}^{+0} - C_{DD}^{+0}(N), \tag{108}
\]

where we apply Lemma \(4\) and \((107).\) The inclusion

\[
C_{DD}^{0+}(N) \subseteq C_{DD}^{+0}(N)
\]

holds because

\[
C_{DD}^{+0}(N) = \bigcup_{Q \leq 0} C_{DD}^{0+}(N)
\subseteq \bigcup_{Q \leq 0} (\tilde{C}_{DD}^{+0}(N) + C_{U}^{+0})
= (\tilde{C}_{DD}^{+0}(N) + C_{U}^{+0}) \cap O^{++}
\subseteq (\tilde{C}_{CEF}(N) + C_{U}^{+0}) \cap O^{++}
= \tilde{C}_{DD}^{+0}(N).
\]

The first set equivalence holds by definition. The first inclusion follows from \((108).\) The second set equivalence follows because \(\bigcup_{Q \leq 0} C_{U}^{+0} = C_{U} \cap O^{++},\) and the second inclusion follows because combining the classically-enhanced father region with super-dense coding gives the region for entanglement-assisted classical communication. The above inclusion \(C_{DD}^{+0}(N) \subseteq C_{DD}^{+0}(N)\) is the statement of the converse theorem for this octant.

\((-,-,+,+).\) The proof technique for this octant is similar to that of the previous one. We exploit the bijection between the quantum communication achievable region \(C_{DD}^{0+}\) and the
entanglement generation achievable region \( \tilde{C}_{DD}^{00+} \). We need the following lemma.

**Lemma 5:** The following inclusion holds
\[
C_{DD}^{-0+}(\mathcal{N} \otimes \text{id}^{\otimes|Q|}) \subseteq \tilde{C}_{DD}^{-0+}(\mathcal{N}) + \tilde{C}_{DD}^{-0+}(\text{id}^{\otimes|Q|}).
\]

**Proof:** Entanglement distribution induces a bijective mapping \( f : C_{DD}^{-0+}(\mathcal{N}) \to C_{DD}^{-0+}(\mathcal{N}) \) between the classically-assisted quantum communication achievable region and the classically-assisted entanglement generation achievable region. It behaves as follows for every point \((R, Q, 0) \in C_{DD}^{-0+}(\mathcal{N})\):
\[
f : (R, Q, 0) \to (R, 0, Q).
\]

The following relation holds
\[
f(\tilde{C}_{DD}^{-0+}(\mathcal{N})) = \tilde{C}_{DD}^{-0+}(\mathcal{N}), \tag{109}
\]

because applying entanglement distribution to the classically-assisted quantum communication protocol produces classically-assisted entanglement generation. The inclusion \(C_{DD}^{-0+}(\mathcal{N} \otimes \text{id}^{\otimes|Q|}) \subseteq f^{-1}(\tilde{C}_{DD}^{-0+}(\mathcal{N}) + \tilde{C}_{DD}^{-0+}(\text{id}^{\otimes|Q|}))\) holds because
\[
C_{DD}^{-0+}(\mathcal{N} \otimes \text{id}^{\otimes|Q|}) = C_{DD}^{-0+}(\mathcal{N}) + C_{DD}^{-0+}(\text{id}^{\otimes|Q|}) \subseteq \tilde{C}_{DD}^{-0+}(\mathcal{N}) + \tilde{C}_{DD}^{-0+}(\text{id}^{\otimes|Q|}) = f^{-1}(\tilde{C}_{DD}^{-0+}(\mathcal{N} \otimes \text{id}^{\otimes|Q|})).
\]

The first set equivalence follows because the capacity region of the noisy channel \(\mathcal{N}\) combined with a rate \(Q\) noiseless qubit channel is equivalent to a translation of the capacity region of the noisy channel \(\mathcal{N}\). The first inclusion follows because the capacity region of a rate \(Q\) noiseless qubit channel contains the rate triple \((0, Q, 0)\). The second set equivalence follows from the classically-assisted quantum communication theorem in \(\text{(78)}\), the third set equivalence from \(\text{(109)}\), and the fourth set equivalence from linearity of the map \(f\). The above inclusion implies the following one:
\[
f(C_{DD}^{-0+}(\mathcal{N} \otimes \text{id}^{\otimes|Q|})) \subseteq \tilde{C}_{DD}^{-0+}(\mathcal{N}) + \tilde{C}_{DD}^{-0+}(\text{id}^{\otimes|Q|}).
\]

The lemma follows because
\[
f(C_{DD}^{-0+}(\mathcal{N} \otimes \text{id}^{\otimes|Q|})) = f(\tilde{C}_{DD}^{-0+}(\mathcal{N} \otimes \text{id}^{\otimes|Q|})) = \tilde{C}_{DD}^{-0+}(\mathcal{N} \otimes \text{id}^{\otimes|Q|}) = C_{DD}^{-0+}(\mathcal{N} \otimes \text{id}^{\otimes|Q|}),
\]

where we apply the relations in \(\text{(78)}\), \(\text{(109)}\), and \(\text{(80)}\). \hfill \blacksquare

Observe that
\[
\tilde{C}_{DD}^{-0+}(\text{id}^{\otimes|Q|}) = C_{U}^{-Q+}, \tag{110}
\]

Hence, for all \(Q \leq 0\),
\[
C_{DD}^{-0+}(\mathcal{N}) = C_{DD}^{-0+}(\mathcal{N} \otimes \text{id}^{\otimes|Q|}) \subseteq \tilde{C}_{DD}^{-0+}(\mathcal{N}) + \tilde{C}_{U}^{-Q+}, \tag{111}
\]

where we apply Lemma 3 and \(\text{(110)}\). The inclusion \(C_{DD}^{-0+}(\mathcal{N}) \subseteq C_{DD}^{-0+}(\mathcal{N})\) holds because
\[
C_{DD}^{-0+}(\mathcal{N}) = \bigcup_{Q \leq 0} C_{DD}^{-0+}(\mathcal{N}) \subseteq \bigcup_{Q \leq 0} (\tilde{C}_{DD}^{-0+}(\mathcal{N}) + \tilde{C}_{U}^{-Q+}) = (\tilde{C}_{DD}^{-0+}(\mathcal{N}) + \tilde{C}_{U}^{-Q+}) \cap O^{+++} \subseteq (C_{CEF}(\mathcal{N}) + \tilde{C}_{U}^{-Q+}) \cap O^{+++} = \tilde{C}_{DD}^{-0+}(\mathcal{N}).
\]

The first set equivalence holds by definition. The first inclusion follows from \(\text{(111)}\). The second set equivalence follows because \(\bigcup_{Q \leq 0} \tilde{C}_{U}^{-Q+} = \tilde{C}_{U}^{-Q+} \cap O^{+++}\). The second inclusion follows because combining the classically-enhanced father region with entanglement distribution and teleportation gives the region for classically-assisted entanglement generation.

The above inclusion \(C_{DD}^{-0+}(\mathcal{N}) \subseteq \tilde{C}_{DD}^{-0+}(\mathcal{N})\) is the statement of the converse theorem for this octant.

### C. Discussion

The proof of the octant \(O^{+++}\) is perhaps the most interesting of all those for the direct dynamic trade-off. Its proof answers directly the following question concerning the use of entanglement-assisted codes \[34, 35, 36, 37, 38, 39, 40, 41\]: Why even use entanglement-assisted coding if teleportation is a way to consume entanglement for the purpose of transmitting quantum information? The proof of this octant gives a practical answer to the above question by showing exactly when entanglement-assisted coding is useful and when it is not.

Consider the first case where \(Q \geq |R| / 2\) and \(|E| \geq |R| / 2\). The first step of \(\text{(94)}\) indicates that the optimal strategy for this case is a combination of entanglement-assisted coding and teleportation so that entanglement-assisted coding is useful here.

Consider the second case where \(Q \geq |R| / 2\) and \(|E| \leq |R| / 2\). The first step of \(\text{(95)}\) indicates the optimal strategy to be a combination of classically-assisted quantum communication and teleportation. A strategy that performs just as well is quantum communication and teleportation because classical assistance does not improve the rate of quantum communication \[16, 31\]. An implementation of this strategy consists of two steps:

1. Quantum channel coding to build up entanglement
2. Teleportation with the classical communication and built-up entanglement

A simple modification of the first step of the above strategy gives another optimal strategy: use entanglement-assisted quantum channel coding (where the code has a positive catalytic rate \(\text{(58)}\) to build up a positive net amount of entanglement. This latter optimal technique may be more useful than the former in practical schemes because one of the main benefits of entanglement-assisted coding is the ability to produce an entanglement-assisted quantum code from two arbitrary classical codes. One can then use two high-performance classical codes, either a low-density parity-check
code [44], [45] or a turbo code [46], to construct a high-performance entanglement-assisted quantum code.

Next, consider the case where $Q \leq |R|/2$ and $|E| \geq |R|/2$. In this case, there is enough entanglement and classical communication available so that teleportation is optimal. The first step of (102) confirms this intuition because it shows that the optimal strategy is a combination of teleportation and the other two unit protocols.

The final case is when $Q \leq |R|/2$ and $|E| \leq |R|/2$. When $Q \leq |E|$, it is optimal to perform teleportation only. When $Q \geq |E|$, the optimal strategy is the same as the two steps in the second case above.

This last scenario, where $|E| \leq Q \leq |R|/2$, is perhaps the most likely to occur in practice. We can consider classical communication as virtually free nowadays because it is cheap and ubiquitous. A sender and receiver may share a certain amount of entanglement before quantum communication begins, where this amount of entanglement does not exceed the amount of quantum information that they would like to communicate. Our intuition before was that entanglement-assisted coding is useful in this scenario even when classical communication is available for free. It is pleasing that the proof of the octant $O^{+++}$ for this case confirms this intuition.

It is rare that quantum Shannon theory gives insight into practical quantum error correction schemes. Devetak’s proof of the quantum channel coding theorem shows that codes with a CSS-like structure are good enough for achieving capacity [8]. Another case occurs with the classically-enhanced father protocol [43], regarding the structure of optimal classically-enhanced entanglement-assisted quantum codes, and yet another occurs in multiple-access quantum coding [47], regarding the structure of optimal multiple-access quantum codes.

This octant proves to be another case where quantum Shannon theory gives some interesting guidelines for the optimal strategy of a quantum error correction scheme.

VI. Conclusion

We have provided a unifying treatment of many of the important results in quantum Shannon theory. Our first result is a solution of the unit resource capacity region—the optimal strategy mixes super-dense coding, teleportation, and entanglement distribution. Our next result is the full triple trade-off for the static scenario where a sender and receiver share a noisy bipartite state. The optimal strategy combines the classically-assisted mother protocol with the three unit protocols. Our last result is a solution of the dynamic capacity theorem—the scenario where a sender and receiver have access to a large number of independent uses of a noisy quantum channel. The optimal strategy combines the classically-enhanced father protocol with the three unit protocols.

The discussion in the previous section demonstrates another case where quantum Shannon theory has practical implications for quantum error correction schemes. We are able to give the exact scenarios where one benefits from entanglement-assisted coding or teleportation or both. The result is an answer to the question regarding the use of entanglement-assisted coding versus teleportation.

Our work was originally inspired by the work in [48] in which the authors solved a triple trade-off problem called generalized remote state preparation (GRSP). The relation between our capacity regions and theirs is yet unknown due to incompatible definitions of a resource [25]. The GRSP uses “pseudo-resources” that resemble our definition of a resource but fail to satisfy the quasi-i.i.d. requirement. We can possibly remedy this by generalizing our definition of a resource.

In this article, we have discussed only the triple trade-off scenario for when a protocol consumes a noisy resource to generate noiseless resources. An interesting open research question is then the triple trade-off scenario for when a protocol generates or simulates a noisy resource rather than consumes it. A special case of this type of triple trade-off is the quantum reverse Shannon theorem, because the protocol corresponding to it consumes classical communication and entanglement to simulate a noisy channel [49], [26], [50]. The discussion in the last section of Ref. [17] speaks of the usefulness of the quantum reverse Shannon theorem and its role in simplifying quantum Shannon theory. One could imagine several other protocols that would arise as special cases of the triple trade-off for simulating a noisy resource, but the usefulness of such triple trade-offs is unclear to us at this point.

An interesting open research question concerns the triple trade-offs for the static and dynamic cases where the noiseless resources are instead public classical communication, private classical communication, and a secret key. We expect the proof strategies to be similar to those in this article, but the capacity regions should be different from those found here. As a starting point, one might first consider that Devetak’s solution of the private classical capacity theorem is actually a publicly-enhanced private classical code (though he neglected this fact in his statement of the theorem) [8]. Another useful protocol is the secret-key-assisted private classical communication protocol (or private father protocol) [51]. These are initial steps for finding the full triple trade-off of the dynamic case. The static case should employ previously found protocols, such as that for secret key distillation. As a last suggestion, one might also consider using these techniques for determining the optimal sextuple trade-offs in multiple-access coding [47], [52] and broadcast channel coding [53], [54].

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Appendix

Converse of the Classically-Assisted Mother Protocol: We consider the most general classically-assisted mother protocol for proving the converse theorem illustrated in Figure 4. The converse proof follows logic similar to that of the converse proof of the classically-enhanced father protocol [43]. Suppose
Fig. 4. (Color Online) The figure above illustrates the most general protocol for classically- and quantum-communication-assisted entanglement distillation. Alice, Bob, and the reference system share a state $|\psi\rangle^{A'B'n}$. Alice performs a quantum instrument $\mathcal{T}^{A''\rightarrow A'A_1'M}$ on her system $A'$. Alice transmits $M$ and $A_1$ to Bob. Bob performs a decoding operation $\mathcal{D}^{B''M'A''\rightarrow B_1}$ that outputs the system $B_1$. The result of the protocol is a state close to the maximally entangled state $\Phi^{A_1'B_1}$. The following bounds apply to the elements of its rate triple $(R, Q, E - \delta)$:

$$ R + \delta \geq \frac{I(M; B^n E^n)_{\omega}}{n}, $$

$$ Q + \delta \geq \frac{I(A; E^n E'|M)_{\omega}}{2n}, $$

$$ E - \delta \leq \frac{I(A; B^n|M)_{\omega}}{2n}, $$

for any $\epsilon, \delta > 0$ and all sufficiently large $n$. In the ideal case, the protocol produces the maximally entangled state $\Phi^{A_1'B_1}$. So for our case, the following inequality

$$ \| (\omega')^{A_1'B_1} - \Phi^{A_1'B_1} \| _1 \leq \epsilon, \quad (115) $$

holds because the protocol is $\epsilon$-good for entanglement generation. We first prove a useful bound:

$$ n(E - \delta) \geq I(A_1B_1)_{\Phi^{A_1'B_1}} $$

$$ \leq I(A_1B_1)_{\omega'} + 4nE\epsilon + H(\epsilon) $$

$$ \leq I(A_1A'B''M)_{\omega} + 4nE\epsilon + H(\epsilon) $$

$$ \leq I(A_1A'B''M)_{\omega} + H(A'|M)_{\omega} + 4nE\epsilon + H(\epsilon) $$

$$ \leq I(A_1A'B''M)_{\omega} + n(Q + \delta) + 4nE\epsilon + H(\epsilon) $$

The first equality follows from evaluating the coherent information of the maximally entangled state $\Phi^{A_1'B_1}$. The first inequality follows from the Alicki-Fannes’ inequality [55]. The second inequality follows from quantum data processing [56]. The third inequality follows because conditioning reduces entropy $(H(A'|B''M)_{\omega} \leq H(A'|M)_{\omega})$, and the last inequality follows because $H(A'|M)_{\omega} \leq H(A'|\omega)_{\omega} \leq n(Q + \delta)$ (the entropy of system $A'$ has to be less than the log of the dimension of the system). Define $\delta' \equiv 4E\epsilon + \delta + H(\epsilon)/n$. It then follows that

$$ n(E - \delta) \leq I(A_1A'B''M)_{\omega} + nQ + n\delta' $$

$$ \leq I(A_1A'B''M)_{\omega} + nQ - Q + n\delta' $$

$$ \leq I(A_1A'B''M)_{\omega} + nQ + n\delta' $$

The second line follows because $H(A_1|M)_{\omega} \leq H(A_1)_{\omega} \leq n(E - \delta)$ and the last lines follows from strong subadditivity and $H(A'|M)_{\omega} \leq nQ$. The following identity holds

$$ I(A_1A'B''M)_{\omega} = H(B^n|M)_{\omega} - H(A_1A'B''|M)_{\omega} $$

$$ = H(A_1A'E^n E'|M)_{\omega} - H(E^n E'|M)_{\omega}. $$

Combining the above identity with the last line above, we have the following inequality:

$$ I(A_1A'; E^n E'|M)_{\omega} \leq nQ + n\delta' $$

$$ \leq 2n $$

We next prove the lower bound in (112) on the classical communication consumption rate:

$$ n(R + \delta) \geq H(M) $$

$$ \geq H(M) - H(M|B^n E^n) $$

$$ = I(M; B^n E^n). $$

The first inequality follows because the entropy of $M$ is less than that of the uniform distribution. The second inequality holds because the conditional entropy of a classical variable cannot be negative, and the last equality holds by definition. We now prove the upper bound in (114) on the entanglement generation rate $E$. First, let us suppose that Alice and Bob have some extra classical communication available in the form of $I(A; B^n|M)_{\omega}$ cbits. Then they can use
Suppose for now that the entanglement generation rate $Q_E$ exceeds $Q$ because the original entanglement is $\epsilon$-good. The new protocol uses at least $I(M; B^n|E^n)_\omega + I(A; B^n|M)_\omega$ bits and is a protocol for noisy teleportation. The upper bound from the noisy teleportation converse theorem then applies to the total quantum communication rate $Q_E - Q$ of this new protocol because it is a protocol for $2\epsilon$-good noisy teleportation:

$$Q_E - Q - \delta' - \delta \leq \frac{I(A; B^n|M)_\omega}{n}.$$ 

Recall the identity:

$$I(A; B^n|M)_\omega = \frac{1}{2} I(A; B^n|M)_\omega - \frac{1}{2} I(A; A'; E^n|E'|M)_\omega.$$

Suppose for now that the entanglement generation rate $E$ exceeds $I(A; B^n|M)_\omega/2n$:

$$E - \delta > \frac{I(A; B^n|M)_\omega}{2n}.$$ 

This assumption then leads to the possibility that the sum rate $Q_E - Q$ exceeds the bound from noisy teleportation, contradicting the optimality of that protocol. Thus, the bound in (114) holds for the entanglement generation rate $E$.

REFERENCES

[1] C. E. Shannon, “A mathematical theory of communication,” Bell System Technical Journal, vol. 27, pp. 379–423, 623–656, 1948.
[2] R. Jozsa and B. Schumacher, “A new proof of the quantum noisless coding theorem,” J. Mod. Opt., vol. 41, p. 2343, 1994.
[3] B. Schumacher, “Quantum coding,” Physical Review A, vol. 51, pp. 2738–2747, 1995.
[4] A. S. Holevo, “The capacity of the quantum channel with general signal states,” IEEE Transactions on Information Theory, vol. 44, pp. 269–273, 1998.
[5] B. Schumacher and M. D. Westmoreland, “Sending classical information via noisy quantum channels,” Physical Review A, vol. 56, pp. 131–138, 1997.
[6] S. Lloyd, “The capacity of a noisy quantum channel,” Physical Review A, vol. 55, pp. 1613–1622, 1997.
[7] P. W. Shor, “The quantum channel capacity and coherent information,” MSRI workshop on quantum computation, 2002. [Online]. Available: http://www.msri.org/publications/lns/lns2002/quantumcrypto/shor/1/
[8] I. Devetak, “The private classical capacity and quantum capacity of a quantum channel,” IEEE Transactions on Information Theory, vol. 51, no. 1, pp. 44–55, 2005.
[9] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, “Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels,” Physical Review Letters, vol. 70, pp. 1895–1899, 1993.
[10] C. H. Bennett and S. J. Wiesner, “Communication via one- and two-particle operators on Einstein-Podolsky-Rosen states,” Physical Review A, vol. 53, pp. 2046–2052, 1996.
[11] H.-K. Lo and S. Popescu, “The classical communication cost of entanglement manipulation: Is entanglement an inter-convertible resource?” Physical Review Letters, vol. 83, pp. 1459–1462, 1999.
[12] P. Hayden and A. Winter, “On the communication cost of entanglement transformations,” Physical Review A, vol. 67, p. 012306, 2003.
[13] A. W. Harrow and H.-K. Lo, “A tight lower bound on the classical communication cost of entanglement dilution,” IEEE Transactions on Information Theory, vol. 50, no. 2, pp. 319–327, 2004.
[14] I. Devetak and A. Winter, “Relating quantum privacy and quantum coherence: an operational approach,” Physical Review Letters, vol. 93, p. 080501, 2004.
[45] M. C. Davey and D. J. C. MacKay, "Low density parity check codes over GF(q)," IEEE Communications Letters, vol. 2, pp. 165–167, 1998.

[46] C. Berrou, A. Glavieux, and P. Thitimajshima, "Near shannon limit error-correcting coding and decoding: Turbo-codes," in Proceedings of the International Conference on Communications, Geneva, Switzerland, May 1993, pp. 1064–1070.

[47] J. Yard, P. Hayden, and I. Devetak, "Capacity theorems for quantum multiple-access channels: Classical-quantum and quantum-quantum capacity regions," IEEE Transactions on Information Theory, vol. 54, no. 7, pp. 3091–3113, 2008.

[48] A. Abeyesinghe and P. Hayden, "Generalized remote state preparation: Trading cbits, qubits, and ebits in quantum communication," Physical Review A, vol. 68, no. 6, p. 062319, December 2003.

[49] C. Bennett, I. Devetak, A. Harrow, P. Shor, and A. Winter, "The quantum reverse shannon theorem," 2005, in Preparation.

[50] I. Devetak, "Triangle of dualities between quantum communication protocols," Physical Review Letters, vol. 97, no. 14, p. 140503, 2006.

[51] M.-H. Hsieh, Z. Luo, and T. Brun, "Secret-key-assisted private classical communication capacity over quantum channels," Physical Review A, vol. 78, no. 4, p. 042306, 2008.

[52] M.-H. Hsieh, I. Devetak, and A. Winter, "Entanglement-assisted capacity of quantum multiple-access channels," IEEE Transactions on Information Theory, vol. 54, no. 7, pp. 3078–3090, 2008.

[53] J. Yard, P. Hayden, and I. Devetak, "Quantum broadcast channels," arXiv:quant-ph/0603098, March 2006.

[54] F. Dupuis and P. Hayden, "A father protocol for quantum broadcast channels," arXiv:quant-ph/0612155, December 2006.

[55] R. Alicki and M. Fannes, "Continuity of quantum conditional information," Journal of Physics A: Mathematical and General, vol. 37, no. 5, pp. L55–L57, 2004. [Online]. Available: http://stacks.iop.org/0305-4470/37/L55

[56] B. Schumacher and M. A. Nielsen, "Quantum data processing and error correction," Physical Review A, vol. 54, pp. 2629–2635, 1996.