On the number-phase problem

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(Dated: November 1, 2018)

Abstract

The known approaches of number-phase problem (for a quantum oscillator) are mutually contradictory. All of them are subsequent in respect with the Robertson-Schrödinger uncertainty relation (RSUR). In opposition here it is proposed a new approach aimed to be ab origine as regard RSRUR. From the new perspective the Dirac’s operators for vibrational number and phase appear as correct mathematical tools while the alluded problem receives a natural solution.

PACS numbers: 03.65.-w, 03.65.Ca, 03.65.Fd, 03.65.Ta

Keywords: quantum oscillator, number and phase, uncertainty relations

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I. INTRODUCTION

A recent outstanding work [1] reviews the publications referring to the problem of theoretical description for the vibrational number \( N \) and phase \( \Phi \) of a quantum oscillator (QO). So one discloses the fact that the respective problem known various approaches differing among them both quantitatively and qualitatively. Moreover it is pointed out that until now in scientific literature an agreement regarding the mentioned problem does not exist. As we shall show below all the alluded approaches are subsequent in respect with the Robertson-Schrödinger uncertainty relation (RSUR). In such a context we think that another approach, ab origine regarding RSRUR, as the one which we present in the next sections, can be of nontrivial interest.

II. BRIEFLY ON KNOWN FACTS

The story of \( N - \Phi \) problem [1] started with the Dirac’s idea to transcribe the ladder (annihilation and creation) operators \( \hat{a} \) and \( \hat{a}^+ \) in the forms

\[
\hat{a} = e^{i\Phi} \sqrt{\hat{N}} \quad \hat{a}^+ = \sqrt{\hat{N}} e^{-i\Phi}
\]  

(1)

For an oscillator \( \hat{N} \) and \( \hat{\Phi} \) were identified with the operators of vibrational number respectively of phase. Due to the fact that \([\hat{a}, \hat{a}^+] = \hat{a}\hat{a}^+ - \hat{a}^+\hat{a} = 1\) from (1) it follows

\[
[\hat{N}, \hat{\Phi}] = i
\]  

(2)

Then this relation was regarded in connection with RSUR

\[
\Delta A \cdot \Delta B \geq \frac{1}{2} |<[\hat{A}, \hat{B}]>| \]  

(3)

In (3) the standard deviations \( \Delta A \) and \( \Delta B \) respectively the operators \( A \) and \( B \) refer to the two arbitrary observables \( A \) and \( B \). By a direct application of (3) to the \( N - \Phi \) case it was introduced the relation

\[
\Delta N \cdot \Delta \Phi \geq \frac{1}{2}
\]  

(4)

But, lately it was found that relation (4) is false - at least in some well-specified situations. Such a situation appears in the case of QO eigenstate corresponding to the energy eigenvalue \( E_n = \hbar \omega (n + \frac{1}{2}) \). The respective state is described by the wave function \( \Psi_n \) for which one
obtains $\hat{N}\Psi_n = \hat{a}^\dagger \hat{a} \Psi_n = n \Psi_n$, $\Delta N = 0$ respectively $\Delta \Phi \leq 2\pi$ (the noted restriction for the value of $\Delta \Phi$ results from the fact that the range of definition for $\Phi$ is the interval $[0, 2\pi]$). With the mentioned falsity of $\text{(4)}$ the $N - \Phi$ problem reached a deadlock. For avoiding the mentioned deadlock in literature various alternative approaches were promoted (see $\text{[1]}$ and references). But it is easy to remark that all the alluded approaches are subsequent (and dependent) in respect with the RSUR ($\text{(3)}$) in the following sense. The respective approaches consider ($\text{(3)}$) as an absolutely valid formula and try to adjust accordingly the description of the pair $N - \Phi$ for QO. So the operators $\hat{N}$ and $\hat{\Phi}$ defined in $\text{(1)}$ were replaced by some alternative operators $\hat{N}_a$ and $\hat{\Phi}_a$ whose standard deviations $\Delta N_a$ and $\Delta \Phi_a$ satisfy relations resembling more or less with ($\text{[3]}$). But it is very doubtful that the variables $N_a$ and $\Phi_a$ have a natural (or even useful) physical significance. Probably that this is one of the reasons why until now, in scientific community, it does not exist an agreement regarding the $N - \Phi$ problem.

### III. AN AB ORIGINE APPROACH

In contrast with the known approaches of $N - \Phi$ problem alluded above we think that the same problem can be approached on a new way which is ab origine (i.e. non-subsequent) in respect with the RSUR ($\text{(3)}$). Such a new approach can be done by investigating the true origin of the relation ($\text{[3]}$) as well as the conditions of validity for the respective relation. For putting in practice our thinking we act as follows. Firstly let us remind some elements/notations from quantum mechanics. We consider a system (particularly an oscillator) of quantum nature. The state of the system and its observables $A_j (j = 1, 2, \ldots, r)$ are described by the wave function $\Psi$ respectively by the operators $\hat{A}_j$. The scalar product of two functions $\Psi_\alpha$ and $\Psi_\beta$ will be denoted by $(\Psi_\alpha, \Psi_\beta)$. In the state described by $\Psi$ the expected (mean) value of the quantity $A_j$ is given by $\langle A_j \rangle = (\Psi, \hat{A}_j \Psi)$ and the operator $\delta \hat{A}_j = \hat{A}_j - \langle A_j \rangle$ can be defined. Then for two observables $A_1 = A$ and $A_2 = B$ one can write the following Schwartz relation

$$\langle (\delta \hat{A} \Psi, \delta \hat{A} \Psi) (\delta \hat{B} \Psi, \delta \hat{B} \Psi) \rangle \geq |(\delta \hat{A} \Psi, \delta \hat{B} \Psi)|^2$$  \hspace{1cm} (5)

But $(\delta \hat{A} \Psi, \delta \hat{A} \Psi) = (\Delta A)^2$ where $\Delta A$ denotes the standard deviation of $A$. So from $\text{(5)}$ one obtains

$$\Delta A \cdot \Delta B \geq |(\delta \hat{A} \Psi, \delta \hat{B} \Psi)|$$  \hspace{1cm} (6)

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Note that the relation (6) is generally valid for any wave function $\Psi$ and any observables $A$ and $B$. The respective relation imply the less general formula which is RSUR (3) only in particular circumstances. The alluded circumstances can be specified as follows. If in respect with the wave function $\Psi$ the operators $\hat{A} = \hat{A}_1$ and $\hat{B} = \hat{A}_2$ satisfy the conditions
\[
(\hat{A}_j \Psi, \hat{A}_k \Psi) = (\Psi, \hat{A}_j \hat{A}_k \Psi) \quad (j = 1, 2; \ k = 1, 2)
\] (7)
one can write
\[
(\delta \hat{A} \Psi, \delta \hat{B} \Psi) = \frac{1}{2}(\Psi, \{\delta \hat{A}, \delta \hat{B}\} \Psi) - \frac{i}{2}(\Psi, i[\hat{A}, \hat{B}] \Psi)
\] (8)
where both $(\Psi, \{\delta \hat{A}, \delta \hat{B}\} \Psi) = (\Psi, (\delta \hat{A} \delta \hat{B} + \delta \hat{B} \delta \hat{A}) \Psi)$ and $(\Psi, i[\hat{A}, \hat{B}] \Psi)$ are real quantities. This means that in the circumstances strictly delimited by the conditions (7), the relation (6) imply directly the RSUR (3). In all other circumstances RSUR (3) is false but the relation (6) remains always valid. The above-presented considerations, regarded in connection with the here investigated $N - \Phi$ problem, justify the following observations (Ob):

Ob.1: For a state described by an eigenfunction $\Psi_n$ mentioned in Sec.2, by using the formula
\[
\hat{N} \Psi_n = n \Psi_n
\] together with the relation (2), it results
\[
(\hat{N} \Psi_n, \hat{\Phi} \Psi_n) = (\Psi_n, \hat{N} \hat{\Phi} \Psi_n) + i
\] (9)
Such a result clearly shows that in the considered situation $N$ and $\Phi$ do not satisfy the conditions (7). This means that, in the respective situation, for $N$ and $\Phi$ the RSUR (6) and (consequently) the formula (4) are not valid. However in the same situation the relation (6) remain true. But then $\delta \hat{N} \Psi_n = 0$, $\Delta N = 0$ and $\Delta \Phi \leq 2\pi$ (more exactly $\Delta \Phi = \pi/\sqrt{3}$ - see below (A.4) in Appendix). So in the alluded state (6) degenerates into trivial equality $0 = 0$.

Ob.2: The cases when the state of the oscillator is described by a wave function of the form
\[
\Psi = \sum_n C_n \Psi_n \quad (\text{with} \ \sum_n |C_n|^2 = 1)
\] must be discussed distinctly. Depending on the values of coefficients $C_n$ in respect with such a function $\hat{N}$ and $\hat{\Phi}$ defined by (1) can satisfy the conditions (7). Then relation (1) is valid with
\[
\Delta N = \left[\sum_n |C_n|^2 n^2 - \left(\sum_n |C_n|^2 n\right)^2\right]^{1/2}, \quad \Delta \Phi \leq 2\pi
\] (10)

Ob.3: By means of the operators $\hat{N}$ and $\hat{\Phi}$ from (1) can be composed a simple procedure able to reveal the important characteristics of a QO (see below the Appendix).
Conclusions

In scientific literature the $N - \Phi$ problem persists as a controversial question. In the main the controversies originate from the fact that the operators $\hat{N}$ and $\hat{\Phi}$ defined by (1) are incompatible with the RSUR (3)/(4). The known approaches of the problem are subsequent in the respect with RSUR (3). In opposition we propose a new approach which is ab origine in relation with RSUR (3). Within the framework of the proposed approach we point out the delimitative conditions in which the RSUR (3) is valid. Also we noted that in fact RSUR (3) originates from a more general formula namely from the Schwartz relation (6)/(5) which is ab origine and always valid. Then as it is shown in Ob.1 in the cases of energy eigenstates of a QO the operators $\hat{N}$ and $\hat{\Phi}$ do not satisfy the conditions of type (7) required by RSUR (3). However in the respective cases the pair $N - \Phi$ satisfies the Schwartz’s relation of type (6) which reduces to the trivial equality $0 = 0$. As we have shown in Ob.1 the mentioned facts give a natural elucidation of the known incompatibility (and resulting troubles) existing between the Dirac’s operators $\hat{N}$ and $\hat{\Phi}$ from (1) and the RSUR (3)/(4). In direct connection with the respective elucidation in Ob.2 we indicate a situation when the Dirac’s operators $N$ and $\Phi$ satisfy RSUR (3)/(4). Additionally in Ob.3 we noted that the Dirac’s operators $\hat{N}$ and $\hat{\Phi}$ prove themselves to be useful tools for composing a simple mathematical procedure able to reveal the important characteristics of a QO The resulting conclusion of the above discussions is that that the ab origine approach presented here gives a complete and natural solution to the controversial $N - \Phi$ problem. Accordingly the Dirac’s operators $\hat{N}$ and $\hat{\Phi}$ [defined in (1)] remains as correct theoritical concepts with clear significance and utility for physics. From the perspective of the mentioned approach, in the case of a QO, the appeals to other “alternative operators for number and phase” seem to be pure mathematical exercises without major significance for physics.

APPENDIX: THE OSCILLATOR EQUATION IN $\Phi$-REPRESENTATION

In obtaining the characteristics of a QO the usual procedures operate with the wave function $\Psi$ taken in x-representation - i.e. $\Psi = \Psi(x)$ ($x =$ coordinate). Mathematically the respective procedures imply relative lablaborious work (for a direct solution of QO Schrödinger equation or, equivalently, for the iterative handling of the ladder operators à
and \( \hat{a}^+ \) in the same equation). A more simple procedure can be done if the wave function \( \Psi \) taken in \( \Phi \)-representation - i.e. \( \Psi = \Psi(\Phi) \) [with \( \Phi \) from (1)]. In such a representation from (2) it results directly that \( \hat{N} \) has the form \( \hat{N} = (i\partial/\partial\Phi) \). Then for the QO Hamiltonian \( \hat{H} = (\hat{p}^2/2m) + (m\omega^2\hat{x}^2/2) = (\hbar\omega/2)(\hat{a}\hat{a}^+ + \hat{a}^+\hat{a}) \) one obtains

\[
\hat{H} = \hbar\omega \left( \hat{N} + \frac{1}{2} \right) = \hbar\omega \left( i \frac{\partial}{\partial\Phi} + \frac{1}{2} \right)
\]

(A.1)

Consequently the Schrödinger equation for QO becomes

\[
\hbar\omega \left( i \frac{\partial}{\partial\Phi} + \frac{1}{2} \right) \Psi = E \Psi
\]

(A.2)

From this equation it is very easy to infer the results

\[
\Psi(\Phi) = \Psi_n(\Phi) = Ce^{-in\Phi}, \quad E = E_n = \hbar\omega \left( n + \frac{1}{2} \right)
\]

(A.3)

The integration constants \( C \) and \( n \) can be precised as follows. The condition \( \Psi_n(0) = \lim_{\Phi \to 2\pi} \Psi_n(\Phi) \) requires for \( n \) to be an integer number. From normalisation condition \( (\Psi_n, \Psi_n) = 1 \) it results \( C = 1/\sqrt{2\pi} \). Then requiring that, similarly with the case of classical oscillator, to have \( E > 0 \) one finds that \( n \geq 0 \). With the wave function \( \Psi_n(\Phi) \) determined as above can be evaluated the characteristics of the QO. So for the states described by \( \psi_n(\Phi) \) given by (A.3) (which regard just the situations debated in literature [1]) one obtains

\[
\Delta N = 0, \quad \Delta \Phi = \pi/\sqrt{3}, \quad (\delta \hat{N}\psi, \delta \hat{\Phi}\psi) = 0
\]

(A.4)

[1] Y. I. Vorontsov, Usp.Fiz. Nauk 172, 907 (2002).