A Cyclic Analogue of Stanley’s Shuffle Theorem

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Abstract. We introduce the cyclic major index of a cycle permutation and give a bivariate analogue of enumerative formula for the cyclic shuffles with a given cyclic descent number due to Adin, Gessel, Reiner and Roichman, which can be viewed as a cyclic analogue of Stanley’s Shuffle Theorem. This gives an answer to a question of Adin, Gessel, Reiner and Roichman, which has been posed by Domagalski, Liang, Minnich, Sagan, Schmidt and Sietsema again.

Keywords: descent, major index, permutation, shuffle, cyclic permutation, cyclic descent.

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1 Introduction

The main theme of this note is to establish a cyclic analogue of Stanley’s Shuffle Theorem. Recall that Stanley’s Shuffle Theorem establishes an explicit expression for the generating function of the number of shufflings of two disjoint permutation σ and π with a given cyclic descent number and a given major index. Here we adopt some common notation and terminology on permutations as used in [13, Chapter 1]. We say that π = π₁π₂ ... πₙ is a permutation of length n if it is a sequence of n distinct letters (not necessarily from 1 to n). For example, π = 9 2 8 10 12 3 7 is a permutation of length 7. Let Sₙ denote the set of all permutations of length n.

Let π ∈ Sₙ, we say that 1 ≤ i ≤ n − 1 is a descent of π if πᵢ > πᵢ₊₁. The set of descents of π is called the descent set of π, denoted Des(π), viz.,

Des(π) := {1 ≤ i ≤ n − 1 : πᵢ > πᵢ₊₁}.

The number of its descents is called the descent number, denoted des(π), namely,

des(π) := #Des(π),

where the hash symbol #S stands for the cardinality of a set S. The major index of π, denoted maj(π), is defined to be the sum of its descents. To wit,

maj(π) := ∑_{k ∈ Des(π)} k.

Let σ ∈ Sₙ and π ∈ Sₘ be disjoint permutations, that is, permutations with no letters in common. We say that α ∈ Sₙ₊ₘ is a shuffle of σ and π if both σ and π are subsequences of α. The set of shuffles of σ
and \( \pi \) is denoted \( S(\sigma, \pi) \). For example,

\[
S(63, 14) = \{6314, 6134, 6143, 1463, 1634, 1643\}.
\]

Clearly, the number of permutations in \( S(\sigma, \pi) \) is \( \binom{m+n}{n} \) for two disjoint permutations \( \sigma \in S_n \) and \( \pi \in S_m \).

Stanley’s Shuffle Theorem states that

**Theorem 1.1.** Let \( \sigma \in S_m \) and \( \pi \in S_n \) be disjoint permutations, where \( \text{des}(\sigma) = r \) and \( \text{des}(\pi) = s \). Then

\[
\sum_{\alpha \in S(n, \pi) \atop \text{des}(\alpha) \geq k} q^{\text{maj}(\alpha)} = \left[ \begin{array}{c} m - r + s \\ k - r \end{array} \right] \left[ \begin{array}{c} n - s + r \\ k - s \end{array} \right] q^{\text{maj}(\sigma)+\text{maj}(\pi)+(k-s)(k-r)}.
\]

(1.1)

Here

\[
\left[ \begin{array}{c} n \\ m \end{array} \right] = \frac{(1-q^n)(1-q^{n-1}) \cdots (1-q^{n-m+1})}{(1-q^m)(1-q^{m-1}) \cdots (1-q)}
\]

is the Gaussian polynomial (also called the \( q \)-binomial coefficient), see Andrews [2, Chapter 1].

Stanley [12] obtained the above expression in light of the \( q \)-Pfaff-Saalschütz identity in his setting of \( P \)-partitions. The bijective proofs of Stanley’s shuffle theorem have been given by Goulden [6], Stadler [11], Ji and Zhang [10], respectively.

Recently, Adin, Gessel, Reiner and Roichman [1] introduced a cyclic version of quasisymmetric functions with a corresponding cyclic shuffle operation. A cyclic permutation \( \pi \) of length \( n \) can be viewed as an equivalence class of linear permutations \( \pi = \pi_1\pi_2 \cdots \pi_n \) of length \( n \) under the cyclic equivalence relation \( \pi_1\pi_2 \cdots \pi_n \sim \pi_1 \cdots \pi_{n-1}\pi_n \) for all \( 2 \leq i \leq n \). For example,

\[
[4 2 3 1] = \{4 2 3 1, 2314, 3142, 1423\}
\]

(1.2)

is a cyclic permutation of length 4, where

\[
[4 2 3 1] = [2314] = [3142] = [1423].
\]

Let \( \pi_l \) be the largest element in \( \pi \), the linear permutation \( \hat{\pi} = \pi_l\pi_{l+1} \cdots \pi_n\pi_1 \cdots \pi_{l-1} \) corresponding to the cyclic permutation \( \pi \) is called the representative of the cyclic permutation \( \pi \). For the example above, \( [4 2 3 1] \) is the representative of the cyclic permutation \( [4 2 3 1] \). Here and in the sequel, we use the representative to represent each cyclic permutation \( [\pi] \). For example, we use \( [4 2 3 1] \) to represent the equivalence class in (1.2). In this way, all cyclic permutations of \( \{1, 2, 3, 4\} \) are listed as follows:

\[
\{4 1 2 3, 4 3 1 2, 4 1 3 2, 4 2 1 3, 4 2 3 1, 4 3 2 1\}.
\]

Let \( cS_n \) denote the set of all cyclic permutations of length \( n \) and let \( [\sigma] \in cS_n \) and \( [\pi] \in cS_m \) be disjoint cyclic permutations, that is, cyclic permutations with no letters in common. We say that \( [\alpha] \in cS_{n+m} \) is a cyclic shuffle of two cyclic permutations \( [\sigma] \) and \( [\pi] \) if both \( [\sigma] \) and \( [\pi] \) are circular subsequences of \( [\alpha] \). The set of cyclic shuffles of \( [\sigma] \) and \( [\pi] \) is denoted \( cS([\sigma], [\pi]) \). For example,

\[
cS([6 3], [4 1]) = \{[6 3 1 4], [6 3 4 1], [6 1 4 3], [6 4 1 3], [6 1 3 4], [6 4 3 1]\}.
\]

(1.3)

The elements of \( [\pi] \) in \( [\alpha] \) are in boldface to distinguish them from the elements of \( [\sigma] \). Figure 1 lays out the circular representations of cyclic shuffles of \( [6 3] \) and \( [4 1] \).
Figure 1: The circular representations of cyclic shuffles of [63] and [41].

Evidently,
\[
\#cS([\sigma], [\pi]) = (m + n - 1) \binom{m + n - 2}{m - 1},
\]
for two disjoint cyclic permutations \([\sigma] \in cS_n\) and \([\pi] \in cS_m\), see [5, Eq. (7)].

In order to study Solomon’s descent algebra, Cellini [3, 4] introduced the cyclic descent set. Let \(\pi = \pi_1 \pi_2 \ldots \pi_n\) be a linear permutation, the cyclic descent set of \(\pi\) is defined to be
\[
cDes(\pi) = \{1 \leq i \leq n: \pi_i > \pi_{i+1}\}
\]
with the convention \(\pi_{n+1} = \pi_1\). The number of its cyclic descents is called the cyclic descent number, denoted \(cdes(\pi)\), viz.,
\[
cdes(\pi) := \#cDes(\pi).
\]
Let \([\pi]\) be a cyclic permutation of length \(n\), note that all linear permutations corresponding to \([\pi]\) have the same number of cyclic descents, so we may define the cyclic descent number of \([\pi]\) as
\[
cdes([\pi]) = cdes(\pi),
\]
where \(\pi\) is any linear permutation corresponding to \([\pi]\).

Based on their setting of cyclic quasi-symmetric functions, Adin, Gessel, Reiner and Roichman [1] established the following enumerative formula for the cyclic shuffles with a given cyclic descent number.

**Theorem 1.2** (Adin-Gessel-Reiner-Roichman). Let \([\sigma] \in cS_m\) and \([\pi] \in cS_n\) be disjoint cyclic permutations, where \(cdes([\sigma]) = r\) and \(cdes([\pi]) = s\). Let \(cS([\sigma], [\pi], k)\) denote the set of cyclic shuffles of \([\sigma]\) and \([\pi]\) with \(k\) cyclic descent number. Then
\[
\#cS([\sigma], [\pi], k) = \frac{k(m-r)(n-s) + (m+n-k)rs}{(m-r+s)(n-s+r)} \binom{m-r+s}{k-r} \binom{n-s+r}{k-s}.
\]

Summing (1.6) over all \(k\) gives (1.4) upon using the Chu-Vandermond identity [2, Eq. (3.3.10) with \(q = 1\)]. At the end of their paper, Adin, Gessel, Reiner and Roichman [1] asked a question about looking
for a notion of cyclic major index, which provides a bivariate analogue of Theorem 1.2. This question has been posed by Domagalski, Liang, Minnich, Sagan, Schmidt and Sietsema in [5, Question 4.1] again.

In this paper, we introduce the cyclic major index of a cycle permutation $[\pi]$. Let $[\pi]$ be a cycle permutation of length $n$ and its representative $\hat{\pi} = \hat{\pi}_1 \hat{\pi}_2 \cdots \hat{\pi}_n$, where $\hat{\pi}_1$ is the largest element in $[\pi]$. The cyclic major index of the cyclic permutation $[\pi]$ is defined to be

$$\text{maj}([\pi]) = \text{maj}(\hat{\pi}).$$

(1.7)

For example, the representative of the cycle permutation $[4 1 3 2]$ is $\hat{\pi} = 4 1 3 2$, and so its cyclic major index is defined to be the major index of $\hat{\pi} = 4 1 3 2$. It gives that $\text{maj}([4 1 3 2]) = 1 + 3 = 4$.

In order to state the cyclic analogue of Stanley’s Shuffle Theorem, we will need to introduce the cyclic descent-bottom set of a cyclic permutation and recall the splitting map $S_i$ defined by Domagalski, Liang, Minnich, Sagan, Schmidt and Sietsema in [5], which maps a cyclic permutation to a linear permutation. Let $[\pi]$ be a cyclic permutation of length $n$, the cyclic descent-bottom set of $[\pi]$ is defined as:

$$cB_d([\pi]) = \{ \pi_i + 1 : \pi_i > \pi_{i+1}, \, for \, 1 \leq i \leq n \}$$

(1.8)

with the convention $\pi_{n+1} = \pi_1$. It should be mentioned that the descent-bottom set of a linear permutation has been studied by Haglund and Visontai [7] and Hall and Remmel [8,9] respectively.

It is manifest from (1.5) and (1.8) that

$$\#cB_d([\pi]) = c\text{des}([\pi]).$$

For example,

$$cB_d([6 4 1 3]) = \{1, 4\}.$$

Let $[\pi]$ be a cyclic permutation of length $n$. For $i \in [\pi]$, Domagalski, Liang, Minnich, Sagan, Schmidt and Sietsema [5] defined the map $S_i([\pi])$ to be the unique permutation corresponding to $[\pi]$ which starts with $i$. For example,

$$S_5([5 1 3 4]) = 5 1 3 4, \quad S_1([5 1 3 4]) = 1 3 4 5, \quad S_3([5 1 3 4]) = 3 4 5 1,$$

and

$$S_4([5 1 3 4]) = 4 5 1 3.$$

We obtain the following generating function of the number of cyclic shufflings of two disjoint cyclic permutations with a given cyclic descent number and a given cyclic major index.

**Theorem 1.3 (Cyclic Stanley’s Shuffle Theorem).** Let $[\sigma] \in c\mathcal{S}_m$ and $[\pi] \in c\mathcal{S}_n$ be disjoint cyclic permutations, where $c\text{des}([\sigma]) = r$ and $c\text{des}([\pi]) = s$. Moreover, the largest element of $[\sigma]$ and $[\pi]$ is in $[\sigma]$. Then

$$\sum_{[\alpha] \in c\mathcal{S}_m([\sigma]), c\text{des}([\alpha]) = k} q^{\text{maj}([\alpha])}$$

$$= \binom{m - r + s}{k - r} \binom{n - s + r - 1}{k - s - 1} q^{\text{maj}([\sigma]) + (k-s)(k-r)} \sum_{i \in cB_d([\pi])} q^{\text{maj}(S_i([\pi]))}$$

$$+ \binom{m - r + s - 1}{k - r} \binom{n - s + r}{k - s} q^{\text{maj}([\sigma]) + (k-s+1)(k-r)} \sum_{i \in cB_d([\pi])} q^{\text{maj}(S_i([\pi]))}. \quad (1.9)$$
Proof of Theorem 1.3

This section is denoted to the proof of Theorem 1.3 with the aid of Stanley’s Shuffle Theorem 1.1.

Proof of Theorem 1.3. Assume that \([\sigma] \in cS_m\) and \([\pi] \in cS_n\) are two disjoint cyclic permutations, where \(c\text{des}(\{\sigma\}) = r\) and \(c\text{des}(\{\pi\}) = s\). Moreover, the largest element of \([\sigma]\) and \([\pi]\) is in \([\sigma]\). Let \(\hat{\sigma} = \hat{\sigma}_1 \hat{\sigma}_2 \cdots \hat{\sigma}_m\) be the representative of the cyclic permutation \([\sigma]\), that is, \(\hat{\sigma}_1\) is the largest element of \([\sigma]\). Under the precondition in this theorem, we see that \(\hat{\sigma}_1\) is greater than all elements in \([\pi]\). Define

\[
\hat{\sigma}' = \hat{\sigma}_2 \cdots \hat{\sigma}_m.
\]

For \(i \in [\pi]\), recall that \(S_i([\pi])\) is the unique permutation corresponding to \([\pi]\) which starts with \(i\). We claim that there is a bijection \(\psi\) between the set \(cS([\sigma],[\pi])\) and the set \(\bigcup_{i \in [\pi]} S(\hat{\sigma}', S_i([\pi]))\), where \(cS([\sigma],[\pi])\) denotes the set of cyclic shuffles of \([\sigma]\) and \([\pi]\) and \(S(\hat{\sigma}', S_i([\pi]))\) denotes the set of linear shuffles of \(\hat{\sigma}'\) and \(S_i([\pi])\). Moreover, for \(\alpha \in cS([\sigma],[\pi])\), we have \(\psi(\alpha) = \hat{\alpha}'\) such that

\[
c\text{des}([\alpha]) = c\text{des}(\hat{\alpha}') + 1 \quad (2.1)
\]

and

\[
c\text{maj}([\alpha]) = c\text{maj}(\hat{\alpha}') + \text{des}(\hat{\alpha}') + 1. \quad (2.2)
\]

Let \(\alpha \in cS([\sigma],[\pi])\) and let \(\hat{\alpha} = \hat{\alpha}_1 \hat{\alpha}_2 \cdots \hat{\alpha}_{n+m}\) be the representative of \([\alpha]\), which is a linear permutation corresponding to \([\alpha]\) such that \(\hat{\alpha}_1\) is the largest element in \([\alpha]\). Then \(\hat{\alpha}_1 = \hat{\sigma}_1\) and

\[
c\text{des}([\alpha]) = c\text{des}(\hat{\alpha}). \quad (2.3)
\]

Define

\[
\hat{\alpha}' = \hat{\alpha}_2 \hat{\alpha}_3 \cdots \hat{\alpha}_{n+m},
\]

which clearly belongs to \(\bigcup_{i \in [\pi]} S(\hat{\sigma}', S_i([\pi]))\). From the construction of \(\hat{\alpha}'\) and (2.3), we see that \([\alpha]\) and \(\hat{\alpha}'\) satisfy (2.1) and (2.2). Moreover, this process is reversible. This proved the claim. Hence it follows from (2.1) and (2.2) that

\[
\sum_{[\alpha] \in cS([\sigma],[\pi])} q^{c\text{maj}([\alpha])} \sum_{c\text{des}([\alpha]) = k} q^{c\text{maj}(\hat{\alpha}') + k}
\]

\[
= \sum_{i \in [\pi]} \sum_{\hat{\alpha}' \in S(\hat{\sigma}', S_i([\pi])} q^{c\text{maj}(\hat{\alpha}') + k}
\]

\[\]

5
\[ \sum_{i \notin \text{cB}(\pi)} \sum_{\hat{\alpha}' \in S_{\hat{\sigma}'}(\hat{\sigma}', S_i(\pi)) \text{des}(\hat{\alpha}') = k-1} q^{\text{maj}(\hat{\alpha}') + k} + \sum_{i \in \text{cB}(\pi)} \sum_{\hat{\alpha}' \in S_{\hat{\sigma}'}(\hat{\sigma}', S_i(\pi)) \text{des}(\hat{\alpha}') = k-1} q^{\text{maj}(\hat{\alpha}') + k}. \] (2.4)

Observe that \( \text{des}(\hat{\sigma}') = \text{cdes}(\sigma) - 1 = r - 1 \) and \( \text{des}(S_i(\pi)) = \text{cdes}(\pi) - 1 = s - 1 \) if \( i \in \text{cB}(\pi) \), otherwise, \( \text{des}(S_i(\pi)) = \text{cdes}(\pi) = s \). Moreover,
\[ \text{maj}(\hat{\sigma}') = \text{maj}(\sigma) - r. \] (2.5)

Hence, by Theorem 1.1, we obtain
\[ \sum_{i \notin \text{cB}(\pi)} \sum_{\hat{\alpha}' \in S_{\hat{\sigma}'}(\hat{\sigma}', S_i(\pi)) \text{des}(\hat{\alpha}') = k-1} q^{\text{maj}(\hat{\alpha}') + k} \]
\[ = \sum_{i \notin \text{cB}(\pi)} \frac{m - r + s}{k - r} \left[ \frac{n - s + r - 1}{k - s - 1} \right] q^{\text{maj}(\hat{\sigma}') + \text{maj}(S_i(\pi)) + (k-s-1)(k-r)+k} \] (2.6)
and
\[ \sum_{i \in \text{cB}(\pi)} \sum_{\hat{\alpha}' \in S_{\hat{\sigma}'}(\hat{\sigma}', S_i(\pi)) \text{des}(\hat{\alpha}') = k-1} q^{\text{maj}(\hat{\alpha}') + k} \]
\[ = \sum_{i \in \text{cB}(\pi)} \frac{m - r + s - 1}{k - r} \left[ \frac{n - s + r}{k - s} \right] q^{\text{maj}(\hat{\sigma}') + \text{maj}(S_i(\pi)) + (k-s)(k-r)+k}. \] (2.7)

Substituting (2.6) and (2.7) into (2.4) and using (2.5), we obtain (1.9). This completes the proof. \( \square \)

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