COUNTING CHARACTERS OF SMALL DEGREE IN UPPER UNITRIANGULAR GROUPS

MARIA LOUKAKI

Abstract. Let $U_n$ denote the group of upper $n \times n$ unitriangular matrices over a fixed finite field $F$ of order $q$. That is, $U_n$ consists of upper triangular $n \times n$ matrices having every diagonal entry equal to 1. It is known that the degrees of all irreducible complex characters of $U_n$ are powers of $q$. It was conjectured by Lehrer that the number of irreducible characters of $U_n$ of degree $q^e$ is an integer polynomial in $q$ depending only on $e$ and $n$. We show that there exist recursive (for $n$) formulas that this number satisfies when $e$ is one of 1, 2 and 3, and thus show that the conjecture is true in those cases.

1. Introduction

We fix a prime $p$. Let $q$ be a fixed power of $p$ and $F_q = F$ the finite field of order $q$. It is standard to write $U_n(q) = U_n(q)$ for the group of upper triangular $n \times n$ matrices over $F$, whose diagonal elements are all equal to 1. We also write $GL_n(q)$ for the general linear group of all $n \times n$ invertible matrices over $F$ and note that $U_n(q)$ is a $p$-Sylow subgroup of $GL_n(q)$. Furthermore, for every finite group $G$ and every integer $k$ we write $N_k(G) = |\{ \chi \in \text{Irr}(G) \mid \chi(1) = k \}|$, for the number of irreducible characters of $G$ of degree $k$.

In 1974 G. I. Lehrer, see [6], conjectured two results. First, he claimed that the degrees of the irreducible representations of $U_n$ are of type $q^e$ for some $e \in \{0, 1, \cdots, \mu(n)\}$, where

$$\mu(n) = \begin{cases} m(m-1) & \text{if } n = 2m \text{ and} \\ m^2 & \text{if } n = 2m + 1. \end{cases}$$

Next he conjectured that for any fixed $n$, the number of irreducible characters of $U_n$ whose degree is $q^e$, i.e., $N_{q^e}(U_n)$ in our notation, is an integer polynomial in $q$ depending only on $e$.

As far as the first of his conjecture is concerned, it was shown by M. Isaacs [5], that every irreducible character of $U_n$ has degree a power of $q$. In addition, B. Huppert, [2], proved that the degrees of the irreducible characters of $U_n$ is exactly the set $\{q^e \mid 0 \leq e \leq \mu(n)\}$.

As for the second part of his conjecture, it still remains open apart for some specific values of $e$.

The case $e = 0$ is well known and easy to compute, that is, $N_1(U_n(q)) = N_1(U_n) = q^{n-1}$. For greater values of $e$, M. Marjoram [7] provided some first formulas. In particular, he proved that there exist formulas for the number of irreducible characters having one of the next two lowest degrees, that is $N_{q}(U_n)$ and $N_{q^2}(U_n)$. Also in his unpublished thesis [8], M. Marjoram established
formulas for the three highest degrees when \( n = 2m \) is even, that is \( N_{q^{\mu(n)}}(U_{2m}) \), \( N_{q^{\mu(n)-1}}(U_{2m}) \) and \( N_{q^{\mu(n)-2}}(U_{2m}) \), as well as a formula for the number of irreducible characters of highest degree when \( n \) is odd, that is \( N_{q^{\mu(n)}}(U_{2m+1}) \).

In addition, I. M. Isaacs, in his paper [4], using a different method, constructed specific polynomials for the number of irreducible characters of \( U_n(q) \) of degree \( q, q^{\mu(n)} \) and \( q^{\mu(n)-1} \). He also suggested a stronger form a Lehrer’s conjecture (see Conjecture B in [4]), that the functions \( N_{q^{\mu}}(U_n) \) are polynomials in \( q - 1 \) with nonnegative integer coefficients.

In this paper we use an elementary method to construct recursive formulas for the number of irreducible characters of degree \( q, q^2 \) and \( q^3 \) satisfy and thus we verify Lehrer’s conjecture. (Of course the cases \( q \) and \( q^2 \) were already known by Marjoram’s formulas.) In a forthcoming paper we prove analogous recursive formulas for the degrees \( q^{\mu(n)}, q^{\mu(n)-1} \) and \( q^{\mu(n)-2} \).

We follow the notation used in [3]. In addition, for any matrix \( X = (x_{i,j}) \in GL_n(q) \) we write \( R_i(X) \) for its \( i \)-row written as an \( 1 \times n \) matrix. We also write \( C_j(X) \) for its \( j \)-column written as an \( n \times 1 \) matrix. Also if \( A = (a_{i,j}) \in U_n \) then we say that its \( i \)-row is trivial if the only nonzero element in that row is the diagonal element \( a_{i,i} = 1 \). Similarly, we say that the \( j \)-column of \( A \) is trivial if every entry in the \( j \)-column of \( A \) is 0 except \( a_{j,j} = 1 \). We will often consider the additive abelian group of the \( st \)-dimensional vector space \( \mathbf{F}^{st} \) (of order \( q^st \)) as the additive group of all \( s \times t \) matrices over \( \mathbf{F} \). When viewed as such we write it as \( \mathbf{F}^{s \times t} \).

Acknowledgments Most of this work was done while I was visiting Georgia Institute of Technology. I would like to thank the Math department for its hospitality. I’m also grateful to the referee whose suggestions improved dramatically the original paper. The short proof of Theorem 1 is his contribution. Furthermore, as he pointed out the recursive formulas we prove below for \( N_{q^{3}}(U_n) \) (see (6.3), (7.2) and (7.3)) imply that \( N_{q^{3}}(U_n) \) is a polynomial function in \( q - 1 \) with nonnegative integer coefficients. Hence Isaacs conjecture holds for \( e = 3 \). (Of course it holds for \( e = 1 \) and \( e = 2 \).)

2. Orbits of Unitriangular Actions on \( \mathbf{F}^{s \times t} \)

The aim of this section is to compute the orbits of a specific action of \( H = U_s \times U_t \) on \( \mathbf{F}^{s \times t} \).

**Definition 1.** Let \( T \) be an \( s \times t \) matrix over \( \mathbf{F} \). We call \( T \) **quasimonomial** if it has at most one non-zero entry in every column and row.

We write \( E_{i,j} \) for the matrix that has 1 in it’s \((i, j)\)-entry and 0 everywhere else. Clearly \( E_{i,j} \) is quasimonomial. Furthermore, every nonzero quasimonomial matrix \( T \) can be written as

\[
T = f_1E_{i_1,j_1} + f_2E_{i_2,j_2} + \cdots + f_kE_{i_k,j_k}
\]

with \( j_1 < j_2 < \cdots < j_k \), all \( i_1, \ldots, i_k \) distinct, and \( f_1, \ldots, f_k \) non-zero elements in \( \mathbf{F} \). We call 2.1 the **standard form** of the non-zero \( T \) and we say that \( k \) is the **length** of \( T \).

**Theorem 1.** Assume that the group \( H = U_s \times U_t \) acts on \( \mathbf{F}^{s \times t} \) in the following way

\[
X^{(A,B)} = A^{-1}XB,
\]
for all $X \in \mathbb{F}^{s \times t}$, $A \in U_s$ and $B \in U_t$. Then the set of distinct quasimonomial matrices in $\mathbb{F}^{s \times t}$ forms a complete set of orbit representatives of the action of $H$ on $\mathbb{F}^{s \times t}$.

Proof. Let $X \in \mathbb{F}^{s \times t}$. We show that by performing admissible transformations we can get a quasimonomial matrix. By an admissible transformation we mean adding to a row (respectively a column) a multiple of a subsequent row (resp. a previous column). By induction we can suppose that the $(s-1) \times t$ submatrix of $X$ formed by all rows except the first one is quasimonomial. Let $x_{i_1,j_1}, \ldots, x_{i_l,j_l}$, $2 \leq i_1 < \ldots < i_l$, be the non-zero elements in this submatrix. Then we can suppose that $x_{1,j_1} = \ldots = x_{1,j_l} = 0$. If the rest of elements in the first row are now zero we are done.

Otherwise let $x_{1,j}$ be the first non-zero element in the first row. Then, except for $x_{1,j}$, the $j$th column is zero and, by performing admissible column transformations, we can have that $x_{1,j}$ is the unique non-zero element in the first row, and thus obtain a quasimonomial matrix.

To prove uniqueness, we argue again by induction on $s$ and $t$. If $X, Y$ are quasimonomial matrices in the same orbit, we can suppose that the last $s-1$ rows and the first $t-1$ columns of $X$ and $Y$ are the same. We only need to show that $x_{1,t} = y_{1,t}$. If some element in the first row or in the last column different from the $(1,t)$-entry is non-zero, then $x_{1,t} = y_{1,t} = 0$ and $X = Y$. Otherwise comparing the $(1,t)$-entry in $XB = AY$ we get $x_{1,t} = y_{1,t}$ and $X = Y$.

When a first version of this paper appeared, Vera-López, Arregi and Ormaetxea told me (I thank them for that) about a more general result concerning conjugacy classes in unitriangular groups (see [9], [10], [11]), whose special case is Theorem 1.

3. Irreducible characters in $U_n$

In this section we will show how Section 2 is connected to Lehrer’s conjecture. We follow Marjoram’s approach on the problem, and Proposition 1 below follows from his paper [7].

For a fixed but arbitrary integer $n$ we consider the upper unitriangular group $U_n$ over $\mathbb{F}_q$, and its two subgroups $M_{n,t}$ and $H_{n,t}$ defined in the following way. If $1 \leq t \leq n$ and $s = n - t$ then

$$M_{n,t} = \left\{ \begin{pmatrix} I_t & C \\ 0 & I_s \end{pmatrix} : C \in \mathbb{F}^{t \times s} \right\} = \left\{ X \in U_n \text{ with } x_{i,j} = 0 \text{ if either } i < j \leq t \text{ or } t < i < j \right\}$$

and

$$H_{n,t} = \left\{ \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} : A \in U_s \text{ and } B \in U_t \right\} = \left\{ X \in U_n \text{ with } x_{i,j} = 0 \text{ if } i \leq t \text{ and } j > t \right\}.$$

It is easy to see that for all $t = 1, \ldots, n$, the group $M_{n,t}$ is an abelian normal subgroup of $U_n$, isomorphic to $\mathbb{F}^{t \times s}$. In addition, $H_{n,t}$ complements $M_{n,t}$ in $U_n$, and is isomorphic to $U_s \times U_t$. We identify $H_{n,t}$ with $U_s \times U_t$ and we write its elements as $(A,B)$ with $A \in U_s$ and $B \in U_t$. We also identify the $M_{n,t}$ with $\mathbb{F}^{t \times s}$, and thus we write the elements of $M_{n,t}$ as $C \in \mathbb{F}^{t \times s}$. Note that with these identifications, the conjugation action of $H_{n,t}$ on $M_{n,t}$ in $U = M_{n,t} \rtimes H_{n,t}$ is given as

$$C^{(A,B)} = B^{-1}CA,$$
for all $(A, B) \in H_{n,t}$ and $C \in M_{n,t}$. The product that appears at the right hand side of the equation above is the standard product of matrices.

Marjoram has given a nice characterization for the abelian group $\text{Irr}(M_{n,t})$, and the way $H_{n,t}$ acts on that group. We collect his results (Lemma 2 and 3 in [7]) in the next proposition.

**Proposition 1.** Let $M_{n,t}$ and $H_{n,t}$ be defined as above, for fixed but arbitrary $n$ and $t$. Then $\text{Irr}(M_{n,t})$ is isomorphic to the abelian additive group $F^{s \times t}$ of all the $s \times t$ matrices over $F = F_q$. The isomorphism is given by $D \in F^{s \times t} \to \lambda_D \in \text{Irr}(M_{n,t})$, where the map $\lambda_D : M_{n,t} \to C$ is defined as

$$\lambda_D(C) = \omega^{T(\text{tr}(DC))}, \text{ for all } C \in M_{n,t},$$

where $\omega$ is a primitive $p$-root of unity, $T : F_q \to F_p$ is the usual trace map from the extension field of $q$ elements $F_q$ to the ground field $F_p$ of $p$ elements and $\text{tr}(DC)$ denotes the trace of the square $s \times s$ matrix $DC$. Furthermore identifying $H_{n,t}$ with $U_s \times U_t$, we get that the action of $H_{n,t}$ on $\text{Irr}(M_{n,t})$ is given as

$$\lambda_D^{(A,B)}(C) = \lambda_D(C^{(A^{-1},B^{-1})}) = \lambda_D(BCA^{-1}) = \omega^{T(\text{tr}(DBCA^{-1}))} = \omega^{T(\text{tr}(A^{-1}DBC))} = \lambda_{A^{-1}DB}(C),$$

for all $D \in F^{s \times t}, C \in F^{t \times s}$ and all $(A, B) \in U_s \times U_t \cong H_{n,t}$. Thus $U_s \times U_t \cong H_{n,t}$ acts on $\text{Irr}(M_{n,t}) \cong F^{s \times t}$ as

$$D^{(A,B)} = A^{-1}DB.$$

What the above proposition says is that, identifying $\text{Irr}(M_{n,t})$ with $F^{s \times t}$ and $H_{n,t}$ with $U_s \times U_t$, then Theorem provides a complete set of orbit representatives of the action of $H_{n,t}$ on $\text{Irr}(M_{n,t})$. In particular,

$$\Omega_{n,t} = \{ T \in F^{s \times t} \mid T \text{ quasimonomial}\}.$$  

is such a set of representatives.

Now, let $G$ be any finite group $N$ an abelian normal subgroup of $G$ and $H$ a complement of $N$ in $G$, then it is easy to characterize the irreducible characters of $G$. In particular, if $\lambda \in \text{Irr}(N)$, and $G_\lambda$ is the stabilizer of $\lambda$ in $G$, then Gallagher’s theorem and Clifford Theory implies that $\lambda$ extends to $G_\lambda$ and a canonical extension $\lambda^e$ is given as $\lambda^e(hn) = \lambda(n)$, for all $h \in H_\lambda = G_\lambda \cap H$ and $n \in N$. Every character $\Psi \in \text{Irr}(H_\lambda)$ defines a unique irreducible character $\Psi \cdot \lambda^e$ of $G_\lambda$ lying above $\lambda$ and inducing irreducibly on $G$. Distinct irreducible characters $\Psi \in \text{Irr}(H_\lambda)$ define distinct irreducible characters $(\Psi \cdot \lambda^e)^G$ of $G$. In addition, every $\chi \in \text{Irr}(G)$ lies above some $\lambda \in \text{Irr}(N)$ and thus $\chi = (\Psi \cdot \lambda^e)^G$, for some $\Psi \in \text{Irr}(H_\lambda)$. Note that $\chi(1) = \Psi(1)(|H|/|H_\lambda|)$.

The group $G$ acts on $\text{Irr}(N)$ and divides its members into conjugacy classes. (Observe that the $G$-classes of $\text{Irr}(N)$ are also the $H$-conjugacy classes of $\text{Irr}(N)$.) Let $\Omega \subseteq \text{Irr}(N)$ consisting of one representative from every $G$-conjugacy class of irreducible characters of $N$. Then

$$\text{Irr}(G) = \bigcup_{\lambda \in \Omega} \{(\Psi \lambda^e)^G \mid \Psi \in \text{Irr}(H_\lambda)\}.$$  

Hence if $N_k(G) = |\{ \chi \in \text{Irr}(G) \mid \chi(1) = k \}|$, for any finite group $G$, and any $k = 1, 2, \ldots$, then

$$N_k(G) = \sum_{\lambda \in \Omega} N_{|H_\lambda|}(H_\lambda) = \sum_{\lambda \in \Omega} N_{|\Omega_{k\lambda}|}(H_\lambda)$$

where $O_\lambda$ is the $H$-orbit of $\lambda$ in $\text{Irr}(N)$.
Applying the above argument to the groups $U_n = M_{n,t} \times H_{n,t}$ for any arbitrary but fixed integer $n$ and any $t = 1, \ldots, n - 1$, we conclude that

$$N_k(U_n) = \sum_{T \in \Omega_{n,t}} N_{k/|O_T|}((H_{n,t}, T)) = \sum_{T \in \Omega_{n,t}} N_{k/|O_T|}(H_{n,t}, T),$$

where $\Omega_{n,t}$ is the set of quasimonomial matrices in $F^{s \times t}$, $O_T$ is the $H_{n,t}$-orbit of $T \in F^{s \times t} \cong \text{Irr}(M_{n,t})$ and $H_{n,t,T}$ is the stabilizer of $T$ in $H_{n,t} \cong U_s \times U_t$.

**Case 1:** $t = 1$. So $s = n - 1$ and the groups $H_{n,1}$ and $M_{n,1}$ become $U_{n-1} \times U_1 \cong U_{n-1}$ and $F^{1 \times n-1}$ respectively. Furthermore, $\text{Irr}(M_{n,1}) \cong F^{n-1 \times 1}$ and $\Omega_{n,1} = \{ T \in F^{n-1 \times 1} \mid T \text{ quasimonomial} \}$ consists of the matrices $T_i = fE_{i,1}$, for all $i = 1, \ldots, n - 1$, and $f \neq 0 \in F$, along with the zero matrix. So we get $q - 1$ matrices of type $fE_{i,1}$. For any $n$ and any $i = 1, \ldots, n$ we define $P_{n,i}$ as

$$P_{n,i} = \{ A \in U_n \mid C_i(A) \text{ is trivial} \}.$$ 

Then it is easy to check that $H_{n,1,T} = P_{n-1,i}$ while $|O_{T_i}| = q^{i-1}$. Thus in view of equation (3.3) we get

$$N_k(U_n) = \sum_{T \in \Omega_{n,1}} N_{k/|O_T|}(H_{n,1}, T) = (q - 1) \sum_{i=1}^{n-1} N_{k/|O_{T_i}|}(P_{n-1,i}) + N_k(U_{n-1}),$$

where the last summand is the contribution of the zero matrix whose orbit size is 1 and the stabilizer group is $H_{n,1} \cong U_{n-1}$ itself. For $k = q^e$, $e = 0, 1, \ldots, \mu(n)$ the above equation, along with the fact that $P_{n-1,1} = U_{n-1}$, implies

$$N_k(U_n) = qN_k(U_{n-1}) + (q - 1) \sum_{i=2}^{n-1} N_{k/|O_{T_i}|}(P_{n-1,i}).$$

Observe that for $k = 1$ equation (3.5) provides the well known formula $N_1(U_n) = qN_1(U_{n-1})$, for all $n \geq 2$.

**Case 2:** $t = 2$ and thus $s = n - 2$. (Assume $n \geq 4$ for the rest of the section.) Now the groups $H_{n,2}$ and $M_{n,2}$ become $U_{n-2} \times U_2 \cong U_{n-2}$ and $F^{2 \times n-2}$ respectively. Furthermore, $\text{Irr}(M_{n,2}) \cong F^{n-2 \times 2}$ and $\Omega_{n,2} = \{ T \in F^{n-2 \times 2} \mid T \text{ quasimonomial} \}$ consists of matrices whose length is either 1 or 2 along with the zero matrix. In particular, the non-zero matrices in $\Omega_{n,2}$ are of the following two types:

Those of length 1, i.e. $T_{i,j} = fE_{i,j}$, $j = 1, 2$ and $i = 1, \ldots, n - 2$, while $f \neq 0 \in F$. For any fixed $i$ and $j$ we get $q - 1$ such. If $j = 1$ then $T_{i,1} = fE_{i,1}$, for $i = 1, \ldots, n - 2$. In this case it is left to the reader to check that $|O_{T_{i,1}}| = q^i$ while $H_{n,2,T_{i,1}} = \{(A, B) \mid A \in U_{n-2}, B \in U_2 \text{ with } C_i(A) \text{ and } R_1(B) \text{ trivial} \}$. Thus $H_{n,2,T_{i,1}} \cong P_{n-2,i}$.

If $j = 2$ then $T_{i,2} = fE_{i,2}$, for some $i = 1, \ldots, n - 2$. In this case $|O_{T_{i,2}}| = q^{i-1}$ while $H_{n,2,T_{i,2}} = \{(A, B) \mid A \in U_{n-2}, B \in U_2 \text{ with } C_i(A) \text{ and } R_2(B) \text{ being trivial} \} \cong P_{n-2,i} \times F$.

The second type are those of length 2, i.e., $T_{i_1,i_2} = f_1E_{i_1,1} + f_2E_{i_2,2}$ for some $i_1 \neq i_2$ and $f_1, f_2$ non-zero elements in $F$. We get exactly $(q - 1)^2$ such distinct quasimonomial characters.

One can easily check that if $i_1 > i_2$, then $|O_{T_{i_1,i_2}}| = q^{i_1 + i_2 - 1}$, while the stabilizer of $T_{i_1,i_2}$ in $H_{n,2}$ equals

$$H_{n,2,T_{i_1,i_2}} = \{(A, 1) \mid A \in U_{n-2} \text{ with } C_{i_1}(A) \text{ and } C_{i_2}(A) \text{ being trivial} \} \cong P_{n-2,i_1} \cap P_{n-2,i_2}.$$
On the other hand if $i_1 < i_2$, then $|O_{T_{i_1,i_2}}| = q^{i_1+i_2-2}$, while the stabilizer $H_{n,2,T_{i_1,i_2}}$ of $T_{i_1,i_2}$ in $H_{n,2} = U_{n-2} \times U_2$ consists of all matrices $(A, B) \in U_{n-2} \times U_2$ that satisfy $a_{i_1,i_2} = -f_1/f_2 \cdot b_{i_1,2}$ while $C_{i_1}(A)$ is a trivial column and $a_{x,i_2} = 0$ for all $i_1 \neq x = 1, \ldots, i_2 - 1$. For $1 \leq i_1 < i_2 \leq n$ we define

$$Q_{n,i_1,i_2} = \{ A \in U_n \mid a_{y,i_1} = 0 = a_{x,i_2}, \text{ for all } i_1 \neq x = 1, \ldots, i_2 - 1 \text{ and } y = 1, \ldots, i_1 - 1 \}.$$ 

Then it is easy to see that $H_{n,2,T_{i_1,i_2}} \cong Q_{n,2,i_1,i_2}$. Finally the zero matrix has orbit length 1 and its stabilizer in $H_{n,2}$ is $H_{n,2} \cong U_{n-2} \times \mathbf{F}$. Collecting all the above and applying equation (3.3) along with equation (3.5) and the fact that $N_k(M \times \mathbf{F}) = |\mathbf{F}| \cdot N_k(M) = qN_k(M)$ for any group $M$, we get

$$N_k(U_n) = qN_k(U_{n-1}) + N_{\frac{k}{q}}(U_{n-1}) - N_{\frac{k}{q}}(U_{n-2}) +$$

$$\sum_{1 \leq i_2 < i_1 \leq n-2} (q-1)^2 N_{\frac{k}{q^{i_1+i_2-2}}}(P_{n-2,i_1} \cap P_{n-2,i_2}) +$$

$$(q-1)^2 \sum_{1 \leq i_1 < i_2 \leq n-2} N_{\frac{k}{q^{i_1+i_2-2}}}(Q_{n-2,i_1,i_2}),$$

for all $n \geq 4$ and all $k$. Some of the summands above are easy to compute. First observe that $P_{n-2,i} \cap P_{n-2,n-2} \cong P_{n-3,i}$ for all $i = 1, \ldots, n-3$, and all $n \geq 5$. Thus (3.5) implies

$$(q-1)^2 \sum_{i=1}^{n-3} N_{\frac{k}{q^{n-4+i}}}(P_{n-2,i} \cap P_{n-2,n-2}) = (q-1)[N_{\frac{k}{q}}(U_{n-2}) - N_{\frac{k}{q}}(U_{n-3})].$$

In addition, $P_{n-2,i} \cap P_{n-2,1} = P_{n-2,i}$, for all $i = 2, \ldots, n-3$ and all $n \geq 5$. Hence

$$(q-1)^2 \sum_{i=2}^{n-3} N_{\frac{k}{q}}(P_{n-2,i} \cap P_{n-2,1}) = (q-1)[N_{\frac{k}{q}}(U_{n-1}) - qN_{\frac{k}{q}}(U_{n-2}) - (q-1)N_{\frac{k}{q^{n-2}}}(U_{n-3})].$$

Furthermore, for all $i = 1, \ldots, n-3$ and all $n \geq 5$, the group $Q_{n-2,i,n-2}$ is isomorphic to $P_{n-3,i} \times \mathbf{F}$. This along with (3.5) implies

$$(q-1)^2 \sum_{i=1}^{n-3} N_{\frac{k}{q^{n-3+i}}}(Q_{n-2,i,n-2}) = q(q-1)[N_{\frac{k}{q^{n-3}}}(U_{n-2}) - N_{\frac{k}{q^{n-3}}}(U_{n-3})].$$

We finally observe that $Q_{n-2,1,2} = U_{n-2}$. Replacing all the above in equation (3.7) we get

$$N_k(U_n) = qN_k(U_{n-1}) + qN_{\frac{k}{q}}(U_{n-1}) - qN_{\frac{k}{q}}(U_{n-2}) +$$

$$\sum_{2 \leq i_2 < i_1 \leq n-3} (q-1)^2 N_{\frac{k}{q^{i_1+i_2-2}}}(P_{n-2,i_1} \cap P_{n-2,i_2}) +$$

$$(q-1)^2 \sum_{1 \leq i_1 < i_2 \leq n-3} \text{ and } (i_1,i_2) \neq (1,2) N_{\frac{k}{q^{i_1+i_2-2}}}(Q_{n-2,i_1,i_2}),$$

for all $n \geq 5$ and all $k$. Note that in the equation above, the sum $i_1 + i_2$ is greater or equal to 5 when $i_2 < i_1$, while $i_1 + i_2 \geq 4$ when $i_1 < i_2$. 


4. Linear characters of $P_{n,i}$ and $Q_{n,i,j}$

The aim of this section is to compute the number of linear characters of $P_{n,i}$ and $Q_{n,i,j}$. These groups are examples of pattern groups, a term introduced by M. Isaacs. We give here the basic definitions and properties we need, for more details the reader could see [4].

Let $\mathcal{P}$ be a subset of the set of pairs $\{(i,j) \mid 1 \leq i < j \leq n\}$. $\mathcal{P}$ is called a closed pattern if it has the property that $(i,k) \in \mathcal{P}$ whenever $(i,j),(j,k) \in \mathcal{P}$, for some $j \in \{i+1, \ldots, k-1\}$. The set of unitriangular matrices $X \in U_n$ whose nonzero entries are restricted to lie at positions in the pattern $\mathcal{P}$ is a subgroup of $U_n$ called a pattern group. If $G$ is a pattern group corresponding to the closed pattern $\mathcal{P}$ with $|\mathcal{P}| = k$, then $G$ is generated by the matrices $I_n + aE_{i,j}$, $(i,j) \in \mathcal{P}$, $a \in \mathbb{F}^*$ and $|G| = |\mathbb{F}|^k$.

Direct computations show that $[I_n + aE_{i,j}, I_n + bE_{i,k}] = I_n + abE_{i,k}$ if $j = l$ and $I_n$ otherwise. A pair $(i,j) \in \mathcal{P}$ is called minimal if it is not possible to find numbers $j_1 < j_2 < \ldots < j_l, l \geq 1$, such that $(i, j_1), (j_1, j_2), \ldots, (j_l, k) \in \mathcal{P}$. Then $G'$ is the pattern group associated to $\mathcal{P}_0 = \{(i,k) \in \mathcal{P} \mid (i,k) \text{ is not minimal}\}$. Thus $|G : G'| = q^t$, where $t$ is the number of minimal pairs in $\mathcal{P}$ (see Theorem 2.1 in [4]).

For the group $P_{n,i}$, $n \geq 3, i \leq n-1$ observe that there are $n-1$ minimal pairs: $(k, k+1), k \neq i-1, 1 \leq k \leq n-1$ and $(i-1, i+1)$. Therefore

$$N_1(P_{n,i}) = q^{n-1}. \tag{4.1}$$

For the group $Q_{n,i,i+1}$ with $2 \leq i \leq n-2$ there are $n-1$ minimal pairs: $(k, k+1), k \neq i-1, 1 \leq k \leq n-1$ and $(i-1, i+2)$. For the group $Q_{n,i,j}$ with $1 < i < j-1 \leq n-1$, there are $n$ minimal pairs: $(k, k+1), k \neq i-1, j-1, 1 \leq k \leq n-1$ and $(i-1, i+1), (i, j), (j-1, j+1)$. Therefore

$$N_1(Q_{n,i,j}) = \begin{cases} q^{n-1} \text{ if } i = j - 1 \\ q^n \text{ if } i < j - 1. \end{cases} \tag{4.2}$$

5. Computing $N_q(P_{n,2})$

With the aid of $N_1(P_{n,i})$ and $N_1(Q_{n,i,j})$ we give the recursive formulas for $N_k(U_n)$ when $k = q$ and $k = q^2$ and compute $N_q(P_{n,2})$. For $k = q$ and $n \geq 5$, equation (3.5), implies

$$N_q(U_n) = qN_q(U_{n-1}) + (q-1)N_1(P_{n-1,2}) = qN_q(U_{n-1}) + q^{n-2}(q-1). \tag{5.1}$$

It is straightforward to see that $N_q(U_3) = q - 1$ while $N_q(U_4) = q(q-1)(q+1)$. So the formula $N_q(U_n) = q^{n-3}(q-1)((n-3)q+1)$ for $N_q(U_n)$ obtained by both Marjoram [7] and Isaacs [4] satisfies (5.1).

For $k = q^2$ and $n = 5$ equation (3.9) implies $N_{q^2}(U_5) = q(q-1)(2q^2 + q - 1)$. In addition, for all $n \geq 6$ we have

$$N_{q^2}(U_n) = qN_{q^2}(U_{n-1}) + qN_q(U_{n-1}) - qN_q(U_{n-2}) + (q-1)^2N_1(Q_{n-2,1,3}) = qN_{q^2}(U_{n-1}) + q^{n-4}(q-1)[q^3 + (n-5)q^2 - (n-6)q - 1]. \tag{5.2}$$
It is straightforward to check that the above recursive formula is satisfied by
\begin{equation}
N_q^2(U_n) = q^{n-4}(q-1)((n-5)q^3 + \frac{(n-5)(n-4)}{2} + 2)q^2 + [1 - \frac{(n-6)(n-5)}{2}]q - n + 4.
\end{equation}

On the other hand, equation (3.5) for $k = q^2$ and $n \geq 5$, implies
\begin{equation}
N_q^2(U_n) = qN_q^2(U_{n-1}) + (q - 1)[q^{n-2} + N_q(P_{n-1,2})].
\end{equation}
Combining the above with (5.2) we get
\begin{equation}
N_q(P_{4,2}) = \frac{1}{q-1}(N_q^2(U_5) - qN_q^2(U_4)) = q^3 = q(q^2 - 1),
\end{equation}
while for $n \geq 6$
\begin{equation}
N_q(P_{n-1,2}) = q^{n-4}(q-1)[q^2 + (n-5)q + 1].
\end{equation}

6. The group $Q_{n,1,3}$.

The aim in this section is to compute $N_q(Q_{n,1,3})$, for all $n \geq 4$. We point out that we are not able to compute the number of irreducible characters of degree $q$ for every group $Q_{n,i,j}$ where $i$ and $j$ are arbitrary. But we can do it for the group $Q_{n,1,3}$, and this is enough for the computation of $N_q^3(U_n)$.

Assume that $n \geq 4$. Note that, according to its definition, $Q_{n,1,3}$ consists of all $n \times n$ unitriangular matrices whose $(2,3)$-entry is zero. We write $Q_{n,1,3}$ as a semidirect product using the following groups. Let $M$ be the subgroup of $Q_{n,1,3}$ consisting of matrices all of whose non-diagonal elements are zero except for the first row. Assume further that $H$ is the subgroup of $Q_{n,1,3}$ consisting of matrices whose non-diagonal entries in the first row are zero. Then it is clear that $M$ is an abelian normal subgroup of $Q_{n,1,3}$ isomorphic to $F^{1 \times n-1} \cong F^{n-1}$. Observe that $H$ is isomorphic to $P_{n-1,2}$. Furthermore, $Q_{n,1,3} = M \rtimes H$ and the conjugation action of $H$ on $M$ is given as
\[
\begin{pmatrix}
1 & 0 \\
0 & X^{-1}
\end{pmatrix}
\begin{pmatrix}
1 & C \\
0 & I_{n-1}
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & X
\end{pmatrix} = \begin{pmatrix}
1 & CX \\
0 & I_{n-1}
\end{pmatrix},
\]
where $X \in P_{n-1,2}$ and $C \in H^{1 \times n-1}$.

Now we apply Proposition 1 to the group $M = F^{1 \times n-1}$ (with $M$ in the place of $M_{n,t}$ for $t = 1$). So the group of irreducible characters $\text{Irr}(M)$ of $M$ is isomorphic to the abelian additive group $F^{n-1 \times 1}$. Thus we regard the irreducible characters of $M$ as column vectors over $F$, and for every $\chi \in \text{Irr}(M)$ we write $\chi = (\chi_1, \ldots, \chi_{n-1})^t$ with $\chi_i \in F$. Under the isomorphism between $\text{Irr}(M)$ and $F^{n-1 \times 1}$ the action of an element $\begin{pmatrix}
1 & 0 \\
0 & X
\end{pmatrix}$ becomes multiplication on the left by $X^{-1}$. It is straightforward to see that the $H$-invariant irreducible characters of $M$ are those with $\chi_3 = \chi_4 = \ldots = \chi_{n-1} = 0$, and thus they look like $(\chi_1, \chi_2, 0, \ldots, 0)^t$ with $\chi_1, \chi_2 \in F$. Hence we get $q^2$ such irreducible characters.

Furthermore, if $\chi = (\chi_1, \ldots, \chi_{n-1})^t \neq (0, \ldots, 0)^t$ is any character in $\text{Irr}(M)$, and $k$ is the biggest index with $\chi_k \neq 0$, then if $k = 1, 2$ the character $\chi$ is $H$-invariant, while if $k \geq 3$ the $H$-orbit of $\chi$ contains all the characters of type $(f_1, \ldots, f_{k-1}, \chi_k, 0, \ldots, 0)^t$, where $f_i \in F$ are arbitrary. Hence we get orbits of length $q^{k-1}$. Therefore for any $\chi \in \text{Irr}(M)$ either $\chi$ is $H$-invariant or its stabilizer $H_\chi$
in $H$ has index at least $q^2$. That is, there are no irreducible characters in $\text{Irr}(M)$ whose stabilizer in $H$ has index $q$ in $H$.

Now we follow the argument after equation (3.2), for the group $Q_{n,1,3} = M \times H$, to get $N_q(Q_{n,1,3}) = q^2 \cdot N_q(P_{n-1,2})$, for all $n \geq 4$. If $n = 4$ then $P_{3,2} \cong \mathbb{F}^2$ and thus

$$N_q(P_{3,2}) = N_q(Q_{4,1,3}) = 0.$$  

If $n = 5$ then in view of (5.5) we get

$$N_q(Q_{5,1,3}) = q^3(q^2 - 1)$$

In addition, for all $n \geq 6$, we use (5.6) to get

$$N_q(Q_{n,1,3}) = q^{n-2}(q - 1)[q^2 + (n - 5)q + 1].$$

7. Computing $N_{q^3}(U_n)$.

For $n = 5$, equation (3.9) implies that $N_{q^3}(U_5) = q(q - 1)(2q - 1)$. Furthermore, when $k = q^3$ and $n = 6$ equation (3.9) along with (6.1) and (4.2) implies

$$N_{q^3}(U_6) = q^2(q - 1)(4q^2 + q - 3).$$

For the case $n = 7$ we similarly get

$$N_{q^3}(U_7) = q^2(q - 1)[3q^4 + 6q^3 - 2q^2 - 5q + 1].$$

In general, for all $n \geq 8$ equation (3.9) implies

$$N_{q^3}(U_n) = qN_{q^3}(U_{n-1}) + qN_{q^3}(U_{n-1}) - qN_{q^2}(U_{n-2}) + (q - 1)^2 [N_q(Q_{n-2,1,3}) + N_1(Q_{n-2,2,3}) + N_1(Q_{n-2,1,4})].$$

According to (5.3), for all $n \geq 8$ we get

$$N_{q^2}(U_{n-1}) - N_{q^2}(U_{n-2}) = q^{n-6}(q - 1)\{ (n - 6)q^4 + \frac{(n - 6)(n - 5)}{2} - (n - 7) + 2\}q^3 - [(n - 7)(n - 6) + 1]q^2 + [4 - n + \frac{(n - 8)(n - 7)}{2}]q + n - 6\}$$

Furthermore, using (4.2) and (6.3) along with (7.3) in equation (7.2) and we get

$$N_{q^3}(U_n) = qN_{q^3}(U_{n-1}) + q^{n-5}(q - 1)\{ q^5 + (2n - 14)q^4 + [25 - 3n + \frac{(n - 6)(n - 5)}{2}]q^3 + [n - 11 - (n - 7)(n - 6)]q^2 + [5 - n + \frac{(n - 8)(n - 7)}{2}]q + n - 6\},$$

for all $n \geq 8$. As we have already computed the formula for $N_{q^3}(U_7)$, we can easily check that the following equation satisfies the recursive formula (7.4) for all $n \geq 8$.

$$N_{q^3}(U_n) = q^{n-5}(q - 1)\{ A_nq^5 + B_nq^4 + C_nq^3 + D_nq^2 + E_nq + F_n\},$$

where
\[ A_n = n - 7 \]
\[ B_n = 3 + (n - 7)(n - 6) \]
\[ C_n = 40(n - 7) - \frac{17}{4}(n + 8)(n - 7) + \frac{1}{12}n(n + 1)(2n + 1) - 64 \]
\[ D_n = (n - 7)(7n + 3) - \frac{1}{6}n(n + 1)(2n + 1) + 138 \]
\[ E_n = (n - 7)(-\frac{17}{4}n - 1) + \frac{1}{12}n(n + 1)(2n + 1) - 75 \]
\[ F_n = 1 + \frac{(n - 7)(n - 4)}{2} \]

It is clear that the polynomials \( A_n, B_n \) and \( F_n \) in \( n \) are integer valued for every \( n \). To show that the same holds for the polynomials \( C_n, D_n \) and \( E_n \) we make use of the following lemma.

**Lemma 1.** Let \( P(n) \) be a polynomial in \( n \) of degree \( m \) with rational coefficients. If \( P(n) \) is an integer for \( m + 1 \) consecutive integers, then the polynomial is integer valued.

**Proof.** We will use induction on the degree \( m \) of \( P(n) \). It is clear that for \( m = 1 \) holds.

Assume it holds for all polynomials of degree less that \( m \), we will show that it also holds for those of degree \( m \). The polynomial \( Q(n) := P(n + 1) - P(n) \) has degree smaller than \( m \). In addition, if \( P \) has integer values for \( m + 1 \) consecutive integers \( k, k + 1, \ldots, k + m \), then \( Q(n) \) is integer valued for the \( m \) consecutive integers \( k, k + 1, \ldots, k + m - 1 \). Therefore the inductive hypothesis implies that \( Q(n) \) is integer valued for every \( n \). This along with the fact that \( P(n) \) is integer valued for \( n = k + m \), implies that \( P(n) \) is an integer for every \( n \). \( \square \)

Now, it is straight forward to check that \( C_7, C_8, C_9 \) and \( C_{10} \) are integers. Hence the above lemma implies that \( C_n \) is an integer valued polynomial. Similarly we show that \( D_n \) and \( E_n \) are integer valued. Hence \( N_{q^3}(U_n) \) is a polynomial in \( q \) with integer coefficients.

**References**

[1] G. Higman, Enumerating \( p \)-groups. I. Inequalities, Proc. London Math. Soc. (3) 10 (1960), 24–30.
[2] B. Huppert, Character Theory of Finite Groups, Walter de Gruyter, Berlin, 1998.
[3] I.M. Isaacs, Character Theory of Finite Groups, Dover, New York, 1994.
[4] I.M. Isaacs, Counting characters of upper triangular groups, J. of Algebra, 315 (2007), 698-719.
[5] I.M. Isaacs, Characters of groups associated with finite algebras, J. of Algebra, 177 (1995), 708–730.
[6] G.I. Lehrer, Discrete series and unipotent subgroup, Compositio Math. 28 (1974), 9–19.
[7] M. Marjoram, Irreducible characters of small degree of the unitriangular group, Irish Math.Soc. Bull. 42 (1999), 21–31.
[8] M Marjoram, Irreducible characters of Sylow \( p \)-subgroups of classical groups, PH.D. Thesis, (1997), National University of Ireland, Dublin.
[9] A. Vera-López, J.M. Arregi, Conjugacy classes in Sylow \( p \)-subgroups of \( GL(n, q) \), J. of Algebra, 152 (1992), 1–19.
[10] A. Vera-López, J.M. Arregi, Some algorithms for the calculation of conjugacy classes in the Sylow \( p \)-subgroups of \( GL(n, q) \), J. of Algebra, 177 (1995), 889–925.
[11] A. Vera-López, J.M. Arregi, Polynomial properties in unitriangular matrices, J. of Algebra, 244 (2001), 343–351.
Department of Mathematics, University of Crete, 71409 Iraklio Crete, Greece

Email address: loukaki@gmail.com