A Unified Dissertation on Bearing Rigidity Theory

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Abstract—Accounting for the current state-of-the-art, this work aims at summarizing the main notions about the bearing rigidity theory, namely the branch of knowledge investigating the structural properties for multi-element systems necessary to preserve the inter-units bearings when exposed to deformations. Our original contribution consists in the definition of a unified framework for the statement of the principal definitions and results on the bearing rigidity theory that are then particularized by evaluating the most studied metric spaces.

I. INTRODUCTION

According to the most general definition, rigidity theory aims at studying the stiffness of a given system, understood as reaction to an induced deformation. This branch of knowledge originally emerges in the mathematics and geometry field, and extends to several research areas, ranging from mechanics to biology, from robotics to chemistry, properly declining itself according to the study context.

A. Historical Background

The concept of rigidity dates back to 1776 when Euler has conjectured that every polyhedron is rigid, meaning that every motion of a given polyhedron results in a new polyhedron which is congruent to the first one [1]. This fact has been proven in 1813 by Cauchy for convex polytopes in three dimensional space [2]. Nonetheless, the most interesting part of the Cauchy’s theorem consists in its corollary affirming that if one makes a physical model of a convex polyhedron by connecting together rigid plates for each of the polyhedron faces with flexible hinges along the polyhedron edges, then this ensemble of components necessarily forms a rigid structure. Such a corollary represents a starting point for the study of the rigid structures that affects various research fields, involving mechanical and building structures, biological and artificial compounds and industrial materials, to cite a few (see [3] and the references therein).

An outstanding result in this direction is constituted by the work of Laman who, in 1970, has provided a definition of a family of sparse graphs describing rigid systems of bars and joints in the plane [4]. According to this definition, a plane skeletal structure can be modeled as a graph so that each vertex corresponds to a joint in the structure and each edge represents a bar connecting two elements. This graphical model is then endowed with a map from the vertex set to the two-dimensional Euclidean space associating each joint to its position on the plane. The pair made up of the underlying graph and the corresponding position map is often referred as a plane Euclidean realization of the graph.

In 1978, to model more complex systems composed of different units interconnected by flexible linkages or hinges, Asimow and Roth have introduced the more general notion of framework. This mathematically corresponds to the graph-based representation of the system jointly with a set of elements belonging to \( \mathbb{R}^d \), \( d \geq 2 \), each of one associates to a vertex of the graph in order to describe the position in \( \mathbb{R}^d \) of the corresponding unit composing the structure [5].

The problem of determining whether a given framework is rigid in \( \mathbb{R}^d \), namely if there is no transformation of the graph vertices such that the final configuration is not congruent (in the Euclidean sense) to the original one, has been studied by many authors for well over a century [6]–[15]. In most of the cases, the rigidity or flexibility of a given system can be established by computing the rank of a suitable defined matrix, generically called rigidity matrix, that accounts for the interconnections among the structure components.

Recently, overcoming the standard bar-and-joints frameworks, the rigidity theory has enlarged its focus towards autonomous multi-agent systems wherein the connections among the system elements are virtual, representing the sensing relations among the devices (see [16] and the references therein). The concept of framework has thus been redefined by considering also manifolds more complex than the \((n\text{-dimensional})\) Euclidean space. In these cases, the rigidity theory is turned out to be an important architectural property of many multi-agent systems where a common inertial reference frame should be unavailable but the agents involved are characterized by sensing, communication and movement capabilities. In particular, the rigidity concepts and results suitably fit for applications connected to the motion control of mobile robots and to the sensors cooperation for localization, exploration, mapping and tracking of a target (see, e.g., [17]).

B. Distance vs. Bearing Rigidity

Within the multi-device systems context, rigidity properties for a given framework deals with agents interactions maintenance, according to the available sensing measurements. From this perspective, the literature differentiates between distance rigidity and parallel/bearing rigidity. When the agents are able to gather only range measurements, distance constraints can be imposed to preserve the formation distance rigidity. On the other hand, the formation...
parallel/bearing rigidity properties are determined through the fulfillment of direction constraints defined upon bearing measurements which are available whether the vehicles are equipped with bearing sensors and/or calibrated cameras.

The principal notions about distance rigidity are illustrated in [18]–[28]. These works explain how distance constraints for a framework can be summarized into a properly defined matrix whose rank determines the rigidity properties of the system analogously to the case of frameworks embedded in \( \mathbb{R}^d \). In such a context, indeed, it turns out to be useful to consider the given multi-agent system as a bar-and-joint structure where the agents are represented as particle points (joints) in \( \mathbb{R}^d, d \geq 2 \), and the interacting agent pairs can be thought as being joined by bars whose lengths enforce the inter-agent distance constraints.

Bearing rigidity in \( \mathbb{R}^2 \) (or parallel rigidity) is instead determined by the definition of normal constraints over the directions of interacting agents, namely the edges of the graph associated to the framework, as explained in [29]–[33]. These constraints entail the preservation of the angles formed between pairs of interconnected agents and the lines joining them, i.e., the inter-agent bearings. Similar inter-agent direction constraints can be stated to access the rigidity properties of frameworks embedded in \( \mathbb{R}^d \) with \( d > 2 \), where the bearing measurement between two agents coincides with their normalized relative direction vector [17], [34]–[39]. In both cases, the agents are modeled as particle points in \( \mathbb{R}^d, d \geq 2 \), and the necessary and sufficient condition to guarantee the rigidity properties of a given framework rests upon the rank and eigenvalues of a matrix which summarizes the involved constraints.

Dealing with a more realistic scenario, in [40]–[45] bearings are assumed to be expressed in the local frame of each agent composing the framework. This implies that each device in the group is modeled a rigid body having a certain position and orientation w.r.t. a common inertial frame which is supposed to be unavailable to the group. In particular, in [41], [42] the attention is focus on multi-agent systems acting on a plane, in [40], [43], [44] the study is extended to the 3D space although limiting the agents attitude kinematics to rotations along only one axis, while in [45] fully-actuated formations are considered by assuming to deal with systems of agents having six controllable degrees of freedom (dofs). Analogously to the former cases, the rigidity properties of the aforementioned multi-agent systems can be established through the definition and the spectral analysis of a matrix accounting for the inter-vehicle sensing interplay.

C. A Unified Framework for the Bearing Rigidity Theory

Since in the last decades distance rigidity has been deeply investigated from the theoretical perspective and the related multi-agent systems applications are copious, in this work the attention is focused on bearing rigidity theory.

Motivated by the similarities emerging from the existing works that account for the different framework domains due to agent features, we aim at providing a unified framework for the statement of the principal bearing rigidity notions which are thus defined accounting for generic metric spaces. This constitutes the original contribution of this work that also summarizes the state-of-the-art results by properly specifying the given generic notions for the multi-agent scenarios described in Sec. [3]–[6] In all the cases, we state the main definitions jointly with the conditions to determine whether a given system is rigid and we clarify the theoretical results through graphical examples. Tab. [II] provide a comprehensive overview about the principal features of the bearing rigidity theory for different framework domains.

The rest of the paper is organized as follows. In Sec. [II] we recall some notions on graph theory and we state the notation used in the rest of the work. In Sec. [III] we provide the basics of the bearing rigidity theory that is then particularized in Sec. [IV–VI] for specific domains. Sec. [VII] is devoted to the discussion about degeneration formation cases. Finally, we summarize the main conclusions in Sec. [VIII].

II. Preliinisaries and Notation

A graph \( G = (V,E) \) consisting of a vertex set \( V = \{v_1 \ldots v_n\} \), and an edge set \( E = \{e_1 \ldots e_m\} \subseteq V \times V \), having cardinality \( |V| = n \) and \( |E| = m \), respectively. We distinguish between undirected, directed and oriented graphs. An undirected graph is a graph in which edges have no orientation, thus \( e_k = (v_i, v_j) \in E \) is identical to \( e_k = (v_j, v_i) \in E \). Contrarily, a directed graph is a graph in which edges have orientation so that the edge \( e_k = (v_i, v_j) \in E \) is directed from \( v_i \in V \) (head) to \( v_j \in V \) (tail). An oriented graph is an undirected graph jointly with an orientation that is the assignment of a unique direction to each edge, hence only one directed edge \( e_k = (v_i, v_j) \) or \( e_h = (v_j, v_i) \) can exist between two vertices \( v_i, v_j \in V \).

For any graph \( G = (V,E) \), the corresponding complete graph \( K = (V,E_K) \) is the graph characterized by the same vertex set \( V \), while the edge set is completed so that each pair of distinct vertices is joined by an edge if \( G \) is undirected/oriented \( (|E_K| = n(n-1)/2) \) and by a pair of edges (one in each direction) if \( G \) is directed \( (|E_K| = n(n-1)) \).

For a directed/oriented graph, the incidence matrix \( E \in \mathbb{R}^{n \times m} \) is the \( \{0, \pm 1\} \)-matrix defined as

\[
\begin{pmatrix}
-1 & \text{if} & e_k = (v_i, v_j) \in E \text{ (outgoing edge)} \\
1 & \text{if} & e_k = (v_j, v_i) \in E \text{ (ingoing edge)} \\
0 & \text{otherwise}
\end{pmatrix}
\]

and, in a similar way, the matrix \( E_0 \in \mathbb{R}^{n \times m} \) is given by

\[
\begin{pmatrix}
-1 & \text{if} & e_k = (v_i, v_j) \in E \text{ (outgoing edge)} \\
0 & \text{otherwise}
\end{pmatrix}
\]

We introduce also the matrices \( E = E \otimes I_d \in \mathbb{R}^{dn \times dm} \) and \( E_0 = E_0 \otimes I_d \in \mathbb{R}^{dn \times dm} \), where \( \otimes \) indicates the Kronecker product, \( I_d \) is the the \( d \)-dimensional identity matrix, and \( d \geq 2 \) refers to the dimension of the considered space.

In this perspective, the \( d \)-sphere embedded in \( \mathbb{R}^{d+1} \) is denoted as \( S^d \). Thus, \( S^1 \) represents the 1-dimensional manifold on the unit circle in \( \mathbb{R}^2 \), and \( S^2 \) represents the 2-dimensional manifold on the unit sphere in \( \mathbb{R}^3 \). The vectors of the
canonical basis of $\mathbb{R}^d$ are indicated as $e_i$, $i \in \{1 \ldots d\}$, and they have a one in the entry $i \mod d$ and zeros elsewhere.

Given a vector $x \in \mathbb{R}^d$, its Euclidean norm is denoted as $\|x\| \in \mathbb{R}$. In addition, we define the operator $P : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$, 

$$P(x) = I_d - \frac{x x^\top}{\|x\|^2},$$

that maps any (non-zero) vector to the orthogonal complement of the vector $x$ (orthogonal projection operator).

Hence, $P(x)y$ indicates the projection of $y \in \mathbb{R}^d$ onto the orthogonal complement of $x \in \mathbb{R}^d$. Given two vectors $x, y \in \mathbb{R}^3$, their cross product is referred as $xy = [x]_y y = -[y]_x x$, where the map $[\cdot]_x : \mathbb{R}^3 \rightarrow so(3)$ associates each vector $x \in \mathbb{R}^3$ to the corresponding skew-symmetric matrix belonging to the Special Orthogonal algebra $so(3)$.

Given a matrix $A \in \mathbb{R}^{p \times q}$, its null space and image space are denoted as $\ker(A)$ and $\text{Im}(A)$, respectively. The dimension of $\text{Im}(A)$ (or equivalently the number of linearly independent columns of $A$) is indicated as $\text{rank}(A)$, whereas $\ker(A)$ stands for the nullity of the matrix, namely $\ker(A) = \text{dim(}\ker(A)\text{)}$. The well-known rank-nullity theorem asserts that $\text{rank}(A) + \text{null}(A) = q$.

Finally, we use the notation $\text{diag}(A_k) \in \mathbb{R}^{p \times q}$ to indicate the block diagonal matrix associated to the set $\{A_k \in \mathbb{R}^{p \times q}\}^{r}_{k=1}$. In the following we will deal with block diagonal matrices related to the edges of a given graph $G$: each block $A_k$ thus depends on the vertices $v_i, v_j$ so that $e_k = (v_i, v_j)$, i.e., $A_k = f(i, j)$ and $r = m$.

### III. Main definitions

In this section we introduce the main concepts related to the bearing rigidity theory. These will be particularized for the specific metric spaces in the rest of the paper.

#### A. Framework Formation Model

Consider a generic formation of $n \geq 3$ agents, wherein each agent is associated to an element of the metric space $\mathcal{D}$ describing its state in terms of controllable variables. In addition, each agent is provided with bearing sensing capabilities. For this reason, we introduce the bearing measurements domain $\mathcal{M}$. According to the framework formation model $(G, \chi)$ in Def. III.1 any edge $e_k = e_{ij} = (v_i, v_j) \in \mathcal{E}$ represents the bearing measurement $b_k = b_{ij} \in \mathcal{M}$ recovered from the $i$-th agent which is able to sense the $j$-th agent, $i, j \in \{1 \ldots n\}, i \neq j$. The set of the available measurements is related to the framework configuration according to the following definition where an arbitrary edges labeling is introduced.

**Definition III.2** (Non-Degenerate Formation). A $n$-agent formation $(n \geq 3)$ modeled as a framework $(G, \chi)$ in $\mathcal{D}$ is non-degenerate if the agents are not all collinear, namely if the matrix of the coordinates describing their positions is of rank greater than 1.

For a given formation, the bearing rigidity properties are related to the agents sensing capabilities. For this reason, we introduce the bearing measurements domain $\mathcal{M}$. According to the framework formation model $(G, \chi)$ in Def. III.1 any edge $e_k = e_{ij} = (v_i, v_j) \in \mathcal{E}$ represents the bearing measurement $b_k = b_{ij} \in \mathcal{M}$ recovered from the $i$-th agent which is able to sense the $j$-th agent, $i, j \in \{1 \ldots n\}, i \neq j$. The set of the available measurements is related to the framework configuration according to the following definition where an arbitrary edges labeling is introduced.

**Definition III.3** (Bearing Rigidity Function). Given a $n$-agent formation $(n \geq 3)$ modeled as a framework $(G, \chi)$ in $\mathcal{D}$, the bearing rigidity function is the map $b_G : \mathcal{D}^n \rightarrow \mathcal{M}^m$

$$b_G(x) = [b_1^\top \ldots b_m^\top]^\top,$$

where the vector $b_G(x) \in \mathcal{M}^m$ stacks all the available bearing measurements.

Hereafter, the framework model is adopted to refer a $n$-agents formation and the two concepts are assumed to be equivalent. Moreover, we always suppose that $n \geq 3$.

#### B. Static Rigidity Properties

Def. III.3 allows to introduce the first two notions related to the bearing rigidity theory, namely the equivalence and the congruence of different frameworks.

**Definition III.4** (Bearing Equivalence). Two frameworks $(G, \chi)$ and $(G', \chi')$ are bearing equivalent (BE) if $b_G(x) = b_{G'}(x')$.

**Definition III.5** (Bearing Congruence). Two frameworks $(G, \chi)$ and $(G', \chi')$ are bearing congruent (BC) if $b_K(x) = b_{K'}(x')$, where $K$ is the complete graph associated to $G$.

Two frameworks defining by the same graph $G = (\mathcal{V}, \mathcal{E})$ (and different configurations) are BE if they are characterized by the same set of bearing measurements for the interacting agents, i.e., $b_k = b_{K'}$ for all $e_k \in \mathcal{E}$. On the contrary, they are BC when the bearing measurements are the same for each pair of agents in the formation, namely $b_{ij} = b_{ij}'$ for all $(v_i, v_j) \in \mathcal{V} \times \mathcal{V}$. Accounting for the preimage under the bearing rigidity function, the set $\mathcal{E}$.
For a given (dynamic) framework \( (\mathcal{G}, \chi(t)) \), the bearing rigidity matrix is the matrix that satisfies the relation
\[
\mathbf{b}_{\mathcal{G}}(\chi(t)) = \mathbf{B}_{\mathcal{G}}(\chi(t)) \delta(t).
\]
(7)

The dimensions of the bearing rigidity matrix depend on the spaces \( \mathcal{M} \) and \( \mathcal{I} \). Nevertheless, one can observe that the null space of \( \mathbf{B}_{\mathcal{G}}(\chi(t)) \) always identifies all the (first-order) deformations of the configuration \( \chi(t) \) that keep the bearing measurements unchanged. From a physical perspective, such variations of \( (\mathcal{G}, \chi(t)) \) can be considered as sets of command inputs to provide to the agents to instantaneously drive the formation from the initial state \( \chi = \chi(t) \) to a final state \( \chi \) belonging to \( \mathcal{Q}(\chi) \).

Definition III.9 (Infinitesimal Variation). For a given (dynamic) framework \( (\mathcal{G}, \chi(t)) \), an infinitesimal variation is an instantaneous variation \( \delta(t) \in \mathcal{I} \) that allows to preserve the relative directions among the interacting agents.

Lemma III.2. For a given (dynamic) framework \( (\mathcal{G}, \chi(t)) \), an infinitesimal variation is an instantaneous variation \( \delta(t) \in \mathcal{I} \) such that \( \delta(t) \in \ker (\mathbf{B}_{\mathcal{G}}(\chi(t))) \).

On the other hand, when the bearing measurements remain unchanged for each pair of agents in the formation (\( \chi \in \mathcal{C}(\chi) \)) the shape uniqueness is guaranteed.

Definition III.10 (Trivial Variation). For a given (dynamic) framework \( (\mathcal{G}, \chi(t)) \), a trivial variation is an instantaneous variation \( \delta(t) \in \mathcal{I} \) such that \( \delta(t) \in \ker (\mathbf{B}_{\mathcal{K}}(\chi(t))) \), where \( \mathbf{B}_{\mathcal{K}}(\chi(t)) \) is the bearing rigidity matrix computed for the complete graph \( \mathcal{K} \) associated to \( \mathcal{G} \).

Accounting for Lemmas [III.2 and III.3], one can realize that the bearing rigidity theory for dynamic formations rests upon the comparison of \( \ker (\mathbf{B}_{\mathcal{G}}(\chi(t))) \) and \( \ker (\mathbf{B}_{\mathcal{K}}(\chi(t))) \).

Theorem III.4. Given a (dynamic) framework \( (\mathcal{G}, \chi(t)) \) and denoting as \( \mathcal{K} \) the complete graph associated to \( \mathcal{G} \), it holds
that

$$\ker (B_K(\chi(t))) \subseteq \ker (B_G(\chi(t))).$$  \hspace{1cm} (8)

Proof. Since each edge of the graph $G$ belongs to the graph $K$, the equations set $\ker (B_G(\chi(t))) \delta(t) = 0$ constitutes a subset of the equations set $\ker (B_K(\chi(t))) \delta(t) = 0$. Then $\delta(t) \in \ker (B_K(\chi(t)))$ implies $\delta(t) \in \ker (B_G(\chi(t)))$. \hfill $\square$

Condition (8) is fundamental for the next definition that constitutes the core of the rigidity theory.

**Definition III.11 (Infinitesimal Bearing Rigidity in $D$).** A (dynamic) framework $(G, \chi(t))$ is infinitesimally bearing rigid (IBR) in $D$ if

$$\ker (B_G(\chi(t))) = \ker (B_K(\chi(t))).$$  \hspace{1cm} (9)

Otherwise, it is infinitesimally bearing flexible (IBF).

A framework $(G, \chi(t))$ is IBR if all its infinitesimal variations are also trivial. Contrarily, a framework is IBF if there exists at least an infinitesimal variation that warps the configuration $\chi = \chi(t)$ in $\chi' \in Q(\chi) / C(\chi)$.

**Remark 1.** It is notable how the trivial variation $\delta(t) \in \ker (B_K(\chi(t)))$ assumes a specific physical meaning when the non-degenerate formation case is specified according to the domain of interest, as detailed in the following. This leads to a characterization of the dimension of $\ker (B_K(\chi(t)))$ that is exploited to derive a (necessary and sufficient) condition to check whether a given framework is IBR.

In the rest of the paper we investigate the rigidity properties of formation characterized by a specific metric space $D$. In detail, we limit our analysis to the dynamic frameworks case, however the time dependency is dropped out to simplify the notation.

**IV. BEARING RIGIDITY THEORY IN $\mathbb{R}^d$**

In this section we focus on dynamic (non-degenerate) formations of agents controllable in $\mathbb{R}^d$, $d \geq 2$, and endowed with bearing sensing capabilities. This is, for example, the case of a team of mobile sensors interacting in a certain (two-dimensional or three-dimensional) area of interest. We aim at describing the rigidity properties of these multi-agent structures recasting the results provided in [17], [29]–[39] within the generic context described in Sec. [III].

**A. Formation Description**

Each element of a formation composed of $n \geq 3$ agents acting in the metric space $D = \mathbb{R}^d$ can be modeled as a particle point whose state is defined by its (controllable) position $p_i \in \mathbb{R}^d$, with $i \in \{1 \ldots n\}$, in the global inertial frame $\tilde{F}_W$ that is assumed to be known by all the agents.\footnote{Note that two agents can not have the same position, hence the agents non-overlapping condition $p_i \neq p_j$ for $i \neq j$ stands. Moreover, because of non-degenerate formation assumption, for each $k$-th component of the position vectors $(k \in \{1 \ldots d\})$ it also holds $p^k_i \neq p^k_j$ for any $c \in \mathbb{R}$.

Adopting the framework formation model introduced in Sec. [III-A] a formation in $\mathbb{R}^d$ can thus be described by the pair $(G, \chi)$, where the configuration $\chi \in \mathbb{R}^{dn}$ is associated to the position vector $p = [p_1 \ldots p_n]^T \in \mathbb{R}^{dn}$ and the graph $G = (V, \mathcal{E})$ is undirected.

Let consider an arbitrary orientation for $G$ obtaining an oriented graph. The bearing measurement associated to the (directed) edge $e_k = (v_i, v_j) \in \mathcal{E}$ results to be

$$b_k = b_{ij} = -\frac{p_j - p_i}{\|p_j - p_i\|} = d_{ij}p_{ij} = \hat{p}_{ij} \in \mathbb{R}^{d-1},$$  \hspace{1cm} (10)

where $p_{ij} = p_j - p_i \in \mathbb{R}^d$, and $d_{ij} = \|p_{ij}\|^{-1} \in \mathbb{R}$. Note that $b_{ij} = -b_{ji}$, namely any orientation for $G$ entails the same amount of bearing information, and that $\mathcal{M} = \mathbb{S}^{d-1}$.

Exploiting (10), it is possible to characterize the bearing rigidity function introduced in Def. III.3 for frameworks embedded in $\mathbb{R}^d$. Specifically, given $(G, \chi)$, we obtain

$$b_G(\chi) = \text{diag}(d_{ij}I_d) \hat{E}^T p \in \mathbb{S}^{(d-1)m},$$  \hspace{1cm} (11)

where $\hat{E} \in \mathbb{R}^{dn \times dm}$, introduced in Sec. [II] is obtained from the incidence matrix of the (oriented) graph $G$. Note that the non-degenerate formation assumption in this case translates into a condition about the non-collinearity of the bearing measurements, thus ensuring that the bearing rigidity function is such that $b_G(\chi) \neq \text{diag}(b_1 \ldots b_m)(I_n \otimes v)$, where the vector $v$ identifies a direction in $\mathcal{M}$ and $b_i \in \mathbb{R}$ for $i \in \{1 \ldots m\}$.

**B. Rigidity Properties**

To characterize the rigidity properties of a formation $(G, \chi)$ in $\mathbb{R}^d$, it is necessary to derive a suitable expression of its corresponding rigidity matrix (Def. III.8).

To this end, note that each agent in $(G, \chi)$ is characterized by $d$ translational degrees of freedom (tdofs) as its position can vary over time in a controllable manner. In this perspective, the instantaneous variation vector introduced in Sec. III-C can be selected as

$$\delta = \delta_p = [p_1^T \ldots p_n^T]^T \in \mathbb{R}^{dn}.$$  \hspace{1cm} (12)

Thus the variation domain $\mathcal{I}$ coincides with $\mathbb{R}^{dn}$ and the selection (12) corresponds to assuming a first-order model for the agents dynamics.

Furthermore, using (10), we observe that the dynamics of the bearing measurements depends on the position variation of the interacting agents. Indeed, it holds that

$$b_{ij} = d_{ij}P(\hat{p}_{ij})(p_j - p_i), \forall (v_i, v_j) \in \mathcal{E}.$$  \hspace{1cm} (13)

Combining (7), (12) and (13), the bearing rigidity matrix for a given framework $(G, \chi)$ can be written as

$$B_G(\chi) = \text{diag}(d_{ij}P(\hat{p}_{ij})) \hat{E}^T \in \mathbb{R}^{dn \times dm}.$$  \hspace{1cm} (14)

One can observe that the matrix (14) coincides with the null space of the bearing rigidity matrix (12) allows to identify the infinitesimal variations of $(G, \chi)$. However, because of Lemma III.3 and
Def. [III.11] to check the infinitesimal rigidity of the framework is necessary to account also for its trivial variations, namely to study the null space of the bearing rigidity matrix computed by considering the complete graph $\mathcal{K}$ associated to $\mathcal{G}$. Given a (non-degenerate) $n$-agents formation $(\mathcal{G}, \chi)$, in the following we prove that its trivial variation set coincides with the $d+1$-dimensional set

$$S_t = \text{span} \{1_n \otimes I_d, p\},$$

(15)
describing the (instantaneous) translation and uniform scaling of the entire configuration $\chi$.

**Lemma IV.1** (Lemma 4 in [36]). For a framework $(\mathcal{K}, \chi)$ in $\mathbb{R}^d$, it holds that $S_t \subseteq \ker (B_{\mathcal{K}} (\chi))$.

**Proof.** We prove that the all the vectors spanning $S_t$ belong to $\ker (B_{\mathcal{K}} (\chi))$. From (14), for any graph $\mathcal{G}$, one has that $(1_n \otimes I_d) \in \ker (E) \subseteq \ker (B_{\mathcal{G}} (\chi))$. Furthermore, since for each pair of non-coincident vertices $v_i, v_j \in \mathcal{V}$, $P (p_{ij}) p_{ij} = 0$, then also $p \in \ker (B_{\mathcal{G}} (\chi))$ for any graph $\mathcal{G}$. Choosing $\mathcal{G} = \mathcal{K}$ proves the thesis. □

Lemma IV.1 holds also for degenerate frameworks and implies $\text{rank} (B_{\mathcal{K}} (\chi)) \leq dn - d - 1$ for both degenerate and non-degenerate cases. This preliminary result is exploited in the proof of the next statement where the attention is restricted to the cases of (non-degenerate) formations with $d = 2, 3$, which are of interest for real world scenarios.

**Theorem IV.2.** For a non-degenerate framework $(\mathcal{K}, \chi)$, with $d = 2, 3$ and $n \geq 3$, it holds that

$$\begin{align*}
\ker (B_{\mathcal{K}} (\chi)) &= S_t, \quad \text{or equivalently,} \\
\text{rank} (B_{\mathcal{K}} (\chi)) &= dn - d - 1.
\end{align*}$$

(16) (17)

**Proof.** The bearing rigidity matrix associated to $(\mathcal{K}, \chi)$ has the form

$$B_{\mathcal{K}} (\chi) = \begin{bmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & -B_{ij} & 0 & B_{ij} & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}$$

(18)

where $B_{ij} = d_{ij} P (\bar{p}_{ij}) \in \mathbb{R}^{d \times d}$ is the block corresponding to the edge $e_k = (v_i, v_j) \in \mathcal{E}_\mathcal{K}$. For $d = 2$, this explicitly is

$$B_{ij} = d_{ij} \begin{bmatrix}
p_{ij}^y & r_{ij}^x \\
-p_{ij}^x & r_{ij}^y
\end{bmatrix} \in \mathbb{R}^{2 \times 2},$$

(19)

with $r_{ij} = [p_{ij}^x, -p_{ij}^y]^T \in \mathbb{R}^2$. For each edge $e_k = (v_i, v_j) \in \mathcal{E}_\mathcal{K}$, we consider only one opportune scaled row of $B_{\mathcal{K}} (\chi)$ in (18), obtaining the matrix

$$B(n) = \begin{bmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & -r_{ij}^x & 0 & r_{ij}^x & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix} \in \mathbb{R}^{(n-1)n \times 2n}.$$  

(20)

This has the same rank of $B_{\mathcal{K}} (\chi)$ but lower dimensions, so hereafter, we consider $B(n)$ instead of $B_{\mathcal{K}} (\chi)$ and we prove thesis by induction on the number $n$ of agents in the formation.

**Base case:** $n = 3$

We aim at proving that $\text{rank} (B(3)) = 3$. To do so, we observe that

$$B(3) = \begin{bmatrix}
-r_{12}^T & r_{12}^T & 0 \\
-r_{13}^T & 0 & r_{13}^T \\
0 & -r_{23}^T & r_{23}^T
\end{bmatrix} \in \mathbb{R}^{3 \times 6}$$

(21)
is full-rank if there not exist $c \in \mathbb{R}$ such that $r_{13} = cr_{23}$, i.e., whether the agents are not collinear. Because of non-degenerate formation hypothesis the thesis is thus proved.

**Inductive step:** $n = \bar{n}$

Note that, given a set of $n$ agents, for each subset containing $\bar{n} - 1$ elements, it is possible to partition $B(\bar{n})$ so that

$$B(\bar{n}) = \begin{bmatrix}
B(\bar{n} - 1) & 0 & \vdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & 0 \\
0 & \ddots & 0 & -r_{1\bar{n}} \\
0 & \cdots & 0 & -r_{(n-1)\bar{n}}
\end{bmatrix}$$

(22)

where the first block has $\bar{n} - 1 \times (\bar{n} - 2)$ rows related to the edges incident to the first $\bar{n} - 1$ agents, while the second block has $\bar{n}$ rows related to the edges connecting the $\bar{n}$-th agent with the first $\bar{n} - 1$ agents.

For inductive hypothesis the thesis holds for $n = \bar{n} - 1 \geq 3$, i.e., $\text{rank} (B(\bar{n} - 1)) = 2n - 5$. Exploiting this fact we aim at showing that for $n = \bar{n}$ we get $\text{rank} (B(\bar{n})) = 2\bar{n} - 3$.

For the inductive hypothesis, the first block of $B(\bar{n})$ in (22) contains $2\bar{n} - 5$ linearly independent rows. Moreover, there are at least two agents, for instance the $i$-th and $j$-th agent, that are not aligned with the $\bar{n}$-th agent, hence it does not exist $c \in \mathbb{R}$ such that $r_{i1} = cr_{j1}$ and the rows related to the edges $(v_i, v_{\bar{n}})$, and $(v_j, v_{\bar{n}})$ are linearly independent w.r.t. the rows of the first block. $B(\bar{n})$ has thus at least $2\bar{n} - 3$ linearly independent rows, and, since $\text{rank} (B(\bar{n})) \leq 2\bar{n} - 3$ for Lemma IV.1, then it must be $\text{rank} (B(\bar{n})) = 2\bar{n} - 3$.

For $d = 3$ (19) is substituted by

$$B_{ij} = d_{ij} \begin{bmatrix}
(p_{ij}^x)^2 + (p_{ij}^y)^2 & -p_{ij}^x p_{ij}^y & -p_{ij}^x p_{ij}^y \\
-p_{ij}^x p_{ij}^y & (p_{ij}^x)^2 + (p_{ij}^y)^2 & -p_{ij}^x p_{ij}^y \\
-p_{ij}^x p_{ij}^y & -p_{ij}^x p_{ij}^y & (p_{ij}^x)^2 + (p_{ij}^y)^2
\end{bmatrix}$$

(23)

where $p_{ij}^x, p_{ij}^y, p_{ij}^\gamma \in \mathbb{R}$ are the (scalar) components of vector $p_{ij} \in \mathbb{R}^d$ along the $x$-axis, $y$-axis, and $z$-axis of the global inertial frame, respectively. The proof for this case thus follows the same inductive reasoning performed for $d = 2$.

□

**Theorem IV.3** (Condition for IGR, Thm. 4 in [36]). A non-degenerate framework $(\mathcal{G}, \chi)$ in $\mathbb{R}^d$ is IGR if and only if $\text{rank} (B_{\mathcal{G}} (\chi)) = dn - d - 1$.

**Proof.** From Lemma III.4 and Lemma IV.1, it holds that $S_t = \ker (B_{\mathcal{K}} (\chi)) \subseteq \ker (B_{\mathcal{G}} (\chi))$. According to Def. III.11 the framework is IGR if and only if $\ker (B_{\mathcal{G}} (\chi)) = S_t$, which, for the rank-nullity theorem, is equivalent to $\text{rank} (B_{\mathcal{G}} (\chi)) = dn - d - 1$. □
Remark 2. It can be proved that $\mathbb{S}^1$ is isomorphic to the interval $[0, 2\pi]$ and also to the two-dimensional Special Orthogonal group $SO(2) = \{R \in \mathbb{R}^{2 \times 2} \mid RR^\top = I_2, \det(R) = +1\}$, namely to the set of length-preserving linear transformations in $\mathbb{R}^2$ whose matrix representations have unitary determinant.

Hence, a rotation on a plane can be equivalently described by a rotation matrix in $SO(2)$ or by an angle in $[0, 2\pi)$.

Remark 3. When we consider a formation on a plane, i.e., for $d = 2$, the orientation of each agent is (completely) specified by an angle $\alpha_i \in [0, 2\pi)$, $i \in \{1, \ldots, n\}$, that is univocally associated to a rotation matrix $R_i \in SO(2)$, i.e., $R_i = R(\alpha_i)$. When we account for the 3D case ($d = 3$), instead, the single angle $\alpha_i \in [0, 2\pi)$ specifies the $i$-th agent orientation only along a single direction in $\mathbb{F}_W$. In this case, the matrix $R_i = R(\alpha_i) \in SO(3)$, belonging to the three-dimensional Special Orthogonal group, denotes the rotation of angle $\alpha_i \in [0, 2\pi)$ around the arbitrary unit vector $d \in \mathbb{R}^3$ that identifies the unique controllable direction in $\mathbb{F}_W$.

According to the discussion in Sec. III-A, the described formation can be modeled as a framework $(\mathcal{G}, \chi)$ embedded in $D = \mathbb{R}^d \times \mathbb{S}^1$. In this case, it results that $\chi = \{(p_1, \alpha_1), \ldots, (p_n, \alpha_n)\} \subseteq (\mathbb{R}^d \times [0, 2\pi])^n$ and we can distinguish the position vector $p = [p_1^\top \ldots p_n^\top]^\top \in \mathbb{R}^{dn}$ and the orientation vector $\alpha = [\alpha_1^\top \ldots \alpha_n^\top]^\top \in [0, 2\pi]^n$. In addition, because of the mutual visibility constraints deriving from the use of cameras as bearing sensors, the sensing capabilities are not necessarily reciprocal between pair of agents. Moreover, we assume that agents do not have access to the global coordinate system, so the gathered measurements are inherently expressed in the local frames. These facts entail that the graph $\mathcal{G}$ is assumed to be directed.

Given these premises, the directed edge $e_k = (v_i, v_j) \in \mathcal{E}$ refers to the bearing measurement of the $j$-th agent obtained by the $i$-agent. Although measured in the $i$-agent local frame, this can be expressed in terms of the relative position and orientation of the agents in the inertial frame, namely

$$b_k = b_{ij} = R^\top_i \bar{p}_{ij} \in \mathbb{S}^{d-1},$$

where $\bar{p}_{ij} \in \mathbb{R}^d$ is the normalized relative position vector introduced in Sec. Sec. III-A and $R_i = R(\alpha_i) \in SO(d)$ is the rotation matrix that describe the orientation of $\mathcal{F}_i$ w.r.t. $\mathcal{F}_W$. Note that $\mathcal{M} = \mathbb{S}^{d-1}$ as in the previous case.

From (23), according to Def. III.3 the bearing rigidity function can be compactly expressed as

$$b_\mathcal{G}(\chi) = \text{diag}(d_{ij}R_i^\top) \bar{E}^\top p \in \mathbb{S}^{(d-1)m},$$

where $\bar{E} \in \mathbb{R}^{dn \times dm}$ is computed accounting for the incidence matrix of the directed graph $\mathcal{G}$.

B. Rigidity Properties

To evaluate the infinitesimal rigidity properties of a formation modeled by a framework in $\mathbb{R}^d \times \mathbb{S}^1$, one can observe that each agent belonging to this system is characterized by $d$ dofs and only one rotational dof (r dof) that are assumed to be independently controllable. Hence, the instantaneous variation vector $\delta$ belonging to $\mathcal{I} = \mathbb{R}^{d+1}$ results from the contribution of two components related to variation of the position and of the orientation, namely $\delta = [\delta_p^\top \delta_o^\top]^\top$ where

$$\delta_p = [p_1^\top \ldots p_n^\top]^\top \in \mathbb{R}^{dn},$$

$$\delta_o = [\alpha_1^\top \ldots \alpha_n^\top]^\top \in \mathbb{R}^n.$$
Remark 4. Note that, for $d = 2$, the variation of the angle $\alpha_i$ corresponds to the variation of the $i$-th agent orientation on the plane. For $d = 3$, instead, it identifies a variation of the $i$-th agent orientation only along the direction determined by $d \in \mathbb{R}^3$, according to Rmk. 3.

To determine the rigidity matrix, we focus on the time derivative of the generic bearing measurement $b_{ij} \in S^{d-1}$ in (28). For $d = 2$, this results to be

$$b_{ij} = d_{ij} R_i^\top P (\hat{p}_{ij}) (\hat{p}_i - \hat{p}_j) - R_i^\top \hat{p}_{ij} \alpha_i$$

(28)

where $\hat{p}_{ij} = R \left( \frac{\pi}{2} \right) p_{ij} \in \mathbb{R}^2$ with $R \left( \frac{\pi}{2} \right) \in SO(2)$. For $d = 3$, the expression of $b_{ij}$ is more complex involving the skew-symmetric of the unit rotation vector $d \in \mathbb{R}^3$, i.e.,

$$b_{ij} = d_{ij} R_i^\top P (\hat{p}_{ij}) (\hat{p}_i - \hat{p}_j) - R_i^\top \{ d \} \times \hat{p}_{ij} \alpha_i$$

(29)

As a consequence of (28)-(29), according to Def. III.8 the bearing rigidity matrix can be written as

$$B_\mathcal{G} (\chi) = [D_1 E^\top \hspace{0.2cm} D_2 E_\alpha] \in \mathbb{R}^{dn \times (d+1)n}$$

(30)

where

$$D_1 = \text{diag}(d_{ij} R_i^\top P (\hat{p}_{ij})) \quad \text{for } d \in \{2, 3\}$$

(31)

$$D_2 = \begin{cases} \text{diag}(R_i^\top \hat{p}_{ij}) & \text{if } d = 2 \\ \text{diag}(R_i^\top \{ d \} \times \hat{p}_{ij}) & \text{if } d = 3 \end{cases}$$

(32)

and $E \in \mathbb{R}^{dn \times dm}$, $E_\alpha \in \mathbb{R}^{n \times m}$ are derived from $\mathcal{G}$.

Note that the two matrix blocks in (30) correspond to the gradients of the bearing rigidity function along vectors $p$ and $\alpha$, i.e., to $\nabla_p b_{ij} (\chi) \in \mathbb{R}^{dm \times d}$ and $\nabla_\alpha b_{ij} (\chi) \in \mathbb{R}^{dm \times n}$ respectively. In addition, evaluating (30), we can state the next theorem about the instantaneous infinitesimal variations of a framework in $\mathbb{R}^{d \times S^1}$.

**Theorem VI.1** (Thm. III.7 in [41]). Given a non-degenerate framework $(\mathcal{G}, \chi)$ in $\mathbb{R}^d \times S^1$, with $d \in \{2, 3\}$ and $n \geq 3$, any infinitesimal variation $\delta = [\delta_p^\top \hspace{0.2cm} \delta_\alpha]^\top \in \mathbb{R}^{(d+1)n}$ satisfies the condition

$$G_\mathcal{G} (\chi \chi_p) \delta_p = -F_\mathcal{G} (\chi) \delta_\alpha$$

(33)

where

$$G_\mathcal{G} (\chi) = \text{diag}(d_{ij} P (\hat{p}_{ij})) E^\top \quad \text{for } d \in \{2, 3\}$$

(34)

$$F_\mathcal{G} (\chi) = \begin{cases} \text{diag}(\hat{p}_{ij} E_\alpha) & \text{if } d = 2 \\ \text{diag}(\hat{p}_{ij} \{ d \} \times E_\alpha) & \text{if } d = 3 \end{cases}$$

(35)

being $E \in \mathbb{R}^{dn \times dm}$ and $E_\alpha \in \mathbb{R}^{n \times m}$ related to the graph $\mathcal{G}$.

**Proof.** Due to Def. III.9 any infinitesimal variations $\delta$ satisfies the condition $B_\mathcal{G} (\chi) \delta = 0$. Pre-multiplying this relation for $\text{diag}(f (i, j))$ with $f (i, j) = R_i$, for each edge $e_k = (v_i, v_j) \in \mathcal{E}$ the relation (33) holds with $d \in \{2, 3\}$.

We observe that matrix $G_\mathcal{G} (\chi)$ has a similar structure to the bearing rigidity matrix (14) defined in Sec. IV] for frameworks embedded in $\mathbb{R}^d$, eventually with different dimension. Indeed, $G_\mathcal{G} (\chi)$ is determined upon a directed graph, while matrix (14) is derived imposing an orientation on an undirected graph. Despite this difference, the results on the null space of the bearing rigidity matrix derived in Sec. IV-B can be applied also to $G_\mathcal{G} (\chi)$. Furthermore, according to Def. III.10 and Thm. IV.1, a trivial variation for a framework $(\mathcal{G}, \chi)$ in $\mathbb{R}^d \times SO(2)$ satisfies condition (33) once substituting $\mathcal{G}$ with the corresponding complete graph $\mathcal{K}$.

In this perspective, imposing $\delta_\alpha = 0$ (i.e., assuming that the agents do not change their orientation), the solution $\delta_p$ of (33) in correspondence of $\mathcal{K}$ coincides with the translation and uniform scaling of the entire configuration. On the other hand, when $\delta_\alpha \neq 0$, we can prove that the unique trivial variation for a framework $(\mathcal{G}, \chi)$ in $\mathbb{R}^d \times S^1$ corresponds to a coordinated rotation, namely the equal rotation of all the agents jointly with the equal rotation of the whole formation around its center. Moreover, the coordinated rotation subspace $\mathcal{R}_\mathcal{G}$ is formally determined as

$$\mathcal{R}_\mathcal{G} = \begin{cases} \text{span} \left\{ \left[ (I_d \otimes R (\pi/2)) p \right] \right\} & \text{if } d = 2 \\ \text{span} \left\{ \left[ (I_d \otimes \{ d \} \times p \right] \right\} & \text{if } d = 3 \end{cases}$$

(36)

where the vector $1_n \in \mathbb{R}^n$ has all the entries equal to one. Note that dim $(\mathcal{R}_\mathcal{G}) = 1$ both for $d = 2$ and $d = 3$. As a consequence, the set including all the instantaneous variation vectors related to translations, scalings, and coordinated rotations of a framework, namely

$$\mathcal{S}_\mathcal{G} = \text{span} \left\{ \left[ 1_n \otimes I_d \right], \left[ p \right], \mathcal{R}_\mathcal{G} \right\}$$

(37)

has dimension dim $(\mathcal{S}_\mathcal{G}) = d + 2$. Furthermore, in the following we demonstrate that $\mathcal{S}_\mathcal{G}$ in (37) identifies all the trivial variations in $\mathbb{R}^d \times S^1$.

**Lemma V.2.** For a framework $(\mathcal{K}, \chi)$ in $\mathbb{R}^d \times S^1$, with $d \in \{2, 3\}$, it holds that $\mathcal{S}_\mathcal{G} \subseteq \ker (B_\mathcal{K} (\chi))$.

**Proof.** When $\delta_\alpha = 0$, according to Lemma IV.1, the instantaneous variation vectors $\delta$ with $\delta_p \in \text{span} \left\{ 1_n \otimes I_d \right\}$ or $\delta_p = p$ fulfill condition (33) for $\mathcal{G} = \mathcal{K}$. Furthermore, for each pair of non-coincident vertices $v_i, v_j \in \mathcal{V}$, we can verify that, for $d = 2$, it holds that

$$d_{ij} P (\hat{p}_{ij}) R (\pi/2) \hat{p}_{ij} = R (\pi/2) \hat{p}_{ij}$$

(38)

$$R (\pi/2) \hat{p}_{ij} - \hat{p}_{ij} R (\pi/2) \hat{p}_{ij} = R (\pi/2) \hat{p}_{ij}$$

(39)

since $\hat{p}_{ij} R (\pi/2) \hat{p}_{ij} = 0$, while, for $d = 3$, the following chains of equations is valid since $\hat{p}_{ij} \{ d \} \times \hat{p}_{ij} = 0$:

$$\begin{align*}
&d_{ij} P (\hat{p}_{ij}) \{ d \} \times \hat{p}_{ij} = [d]_x \hat{p}_{ij} \\
&[d]_x \hat{p}_{ij} - \hat{p}_{ij} [d]_x \hat{p}_{ij} = [d]_x \hat{p}_{ij}.
\end{align*}$$

(40)

(41)

This proves that any variation vector $\delta \in \mathcal{R}_\mathcal{G}$ satisfies condition (33) for an arbitrary $\mathcal{G}$, and thus also for $\mathcal{K}$.

**Theorem V.3.** For a non-degenerate framework $(\mathcal{K}, \chi)$ in $\mathbb{R}^d \times SO(2)$, with $d \in \{2, 3\}$ and $n \geq 3$, it holds that

$$\ker (B_\mathcal{K} (\chi)) = \mathcal{S}_\mathcal{G}, \quad \text{or equivalently,}$$

$$\text{rank (} B_\mathcal{K} (\chi) \text{)} = (d + 1)n - d - 2.$$
Hence the unique solution of (33) in correspondence to $K$ the imposed angular variation is the same for all the agents. Moreover, since the Euclidean group equipped with an omnidirectional camera to recover relative actuated aerial platforms. Assuming that each vehicle is in the 3D space. An example is given by a swarm of fully-directed edges are reported in red to distinguish them in $R_3$ and in $R_3 \times S^1$ with $d = e_3$. Examples of IBR frameworks in $R_3 \times S^1$ (Figs. 3(e), 3(f)) and in $R_3 \times S^1$ with $d = e_3$ (Figs. 3(g), 3(h)).

Proof. According to Thm. V.1 $\ker (B_K (\chi))$ contains all the instantaneous variations that are solutions of (33) in correspondence to $K$. The ones among them for which $\delta_o = 0$ and $\delta_p \neq 0$, are such that $\delta_p \in \ker (G_K (\chi_p)) = \text{span} \{ I_n \otimes I_3, p \}$ because of Thm. IV.2. Evaluating (33) for $K$ and imposing $\delta_p \neq 0$, one can observe that for each pair of directed edges involving vertices $v_i, v_j \in \mathcal{V}$, it occurs that

$$d_{ij}p (\bar{p}_{ij}) A (\bar{p}_j - \bar{p}_i) = A \bar{p}_{ij} \alpha_i$$

(44)

$$d_{ij}p (\bar{p}_{ij}) A (p_j - p_i) = A \bar{p}_{ij} \alpha_j$$

(45)

where $A = R (\pi / 2) \in SO(2)$ for $d = 2$, and $A = [d]_3 \in R^{3 \times 3}$ for $d = 3$. This implies that $\alpha_i = \alpha_j$. Moreover, since the $i$-th agent interacts with all the other agents due to the full connectivity of the complete graph, the imposed angular variation is the same for all the agents. Hence the unique solution of (33) in correspondence to $K$ with $\delta_o \neq 0$ is represented by vectors belonging to the coordinated rotations subspace. These considerations prove the thesis.

Theorem V.4 (Condition for IBR, Thm. III.6 in [41]). A non-degenerate framework $(G, \chi)$ in $R^d \times S^1$, with $d \in \{2, 3\}$ and $n \geq 3$, is IBR if and only if rank $(B_G (\chi)) = (d + 1)n - d - 2$.

Proof. The thesis can be proved from Def. III.11 and Thms. III.4, V.3 as in Thm. IV.3.

In Fig. 3 are reported some examples of IBR and IBF frameworks in $R^d \times S^1$ for $d \in \{2, 3\}$. Here, the local reference frames are omitted for sake of simplicity, while the bi-directed edges are reported in red to distinguish them from the directed ones in blue.

VI. BEARING RIGIDITY THEORY IN $SE(3)$

This section is devoted to (non-degenerate) formations wherein the state of each agent belongs to the Special Euclidean group $SE(3)$ that allows to describe roto-translations in the 3D space. An example is given by a swarm of fully-actuated aerial platforms. Assuming that each vehicle is equipped with an omnidirectional camera to recover relative bearing measurements, we aim at describing the bearing rigidity properties of such multi-agent systems within the context introduced in Sec. III. starting from the results in [45].

A. Formation Description

Consider a team of $n$ agents $(n \geq 3)$ whose state is given by an element of $SE(3)$, i.e., of the Cartesian product $R^3 \times SO(3)$ where $SO(3) = \{ R \in R^{3 \times 3} \mid RR^T = I_3, \det(R) = +1 \}$ describes the rotations in 3D space. According to the rigid body model, we can associate to each $i$-th agent, $i \in \{1 \ldots n\}$, a local reference frame $\mathcal{F}_i$. As in Sec. V-A, the origin $O_i$ of $(\mathcal{F}_i)$ is supposed to coincide with the agent com, while the x-axis is selected parallel to the focal axis of the agent on-board camera. The controllable state $\chi_i$ of the $i$-th agent then corresponds to the pair $(p_i, R_i)$ where the vector $p_i \in R^3$ identifies the position of $O_i$ in $\mathcal{F}_w$ and the matrix $R_i \in SO(3)$ defines the orientation of $\mathcal{F}_i$ w.r.t. $\mathcal{F}_w$.

Remark 5. It can be proved that Special Orthogonal group $SO(3)$, containing $3 \times 3$ rotation matrices, is not isomorphic to the 2-sphere $S^2$, but $S^2 = SO(3) \setminus SO(2)$. Intuitively, the 2-sphere can be parametrized by using two angles, while a rotation in 3D space involves three dofs and can thus be associates to a triplet of angles (Euler angles representation).

The considered $n$-agents formation can be modeled as a framework $(\mathcal{G}, \chi)$ in $D = SE(3)$, where $\mathcal{G}$ is a directed graph according to the motivations illustrated in Sec. V-B about the bearing measurements features. As regards the configuration, trivially it holds that $\chi = \{(p_1, R_1) \ldots (p_n, R_n)\} \in SE(3)^n$. We can then identify the position vector $p = [p_1^T \ldots p_n^T]^T \in R^{3n}$, and the $3n \times 3$ orientation matrix $R_s = [R_1^T \ldots R_n^T]^T \in SO(3)^n$, that stack all the agent position vectors and rotation matrices, respectively.

Analogously to the $R^d \times S^1$ case, the bearing measurement of the $j$-th agent w.r.t. the $i$-th agent (i.e., the one associated to the edge $e_k = (v_i, v_j) \in \mathcal{E}$) can be expressed as

$$b_k = b_{ij} = R_i^T \bar{p}_{ij} \in S^2,$$

(46)

where $\bar{p}_{ij} \in R^3$ and $R_i \in SO(3)$ have the same meaning provided in Sec. V-A and it results $\mathcal{M} = S^2$. For this reason, similarly to (25) and according to Def. III.8 the bearing rigidity function results to be

$$b_{ij} (\chi) = \text{diag}(d_{ij}R_i^T) \bar{E}^T \bar{p} \in S^{2m}.$$
The thesis can be proved following the same reasoning where to be of the addition, the time derivative of a matrix $R_i \in SO(3)$ results to be $\dot{R}_i = [\omega_i] \times R_i$, where $\omega_i \in \mathbb{R}^3$ is the angular velocity of the $i$-th agent expressed in the global inertial frame $\mathcal{F}_W$.

Given these premises, we can identify the instantaneous variation vector as $\delta = [\delta_p \; \delta_o]^\top \in \mathbb{R}^{6n}$, where
\[
\delta_p = [p_1^\top \ldots p_n^\top]^\top \in \mathbb{R}^{3n}, \\
\delta_o = [\omega_1^\top \ldots \omega_n^\top]^\top \in \mathbb{R}^{3n}.
\]

It is thus possible to prove that the bearing rigidity matrix, belonging to $\mathbb{R}^{3m \times 3n}$, is
\[
B_\mathcal{G}(\chi) = \begin{bmatrix} \text{diag}(d_i, R_i^T P(p_{ij})E^\top \delta - \text{diag}(R_i^T, p_{ij})E_o^\top) \end{bmatrix},
\]

since the derivative of the bearing $\delta_{ij} = d_i R_i^T p(p_{ij}) (\dot{p}_i - \dot{p}_j) - R_i^T [\dot{p}_{ij}] \times \omega_i$.

Let recall that the bearing matrix $B_\mathcal{G}(\chi)$ of a given framework $(\mathcal{G}, \chi)$ allows to characterize its infinitesimal rigidity properties, in detail the null space of $B_\mathcal{G}(\chi)$ corresponds to the space of the infinitesimal variations of $\chi$. A preliminary result about these variations is given in the next theorem, that particularize the thesis of Thm. V.1 for the $SE(3)$ case.

**Theorem VI.1** (Cor. 1 in [45]). Given a non-degenerate framework $(\mathcal{G}, \chi)$ in $SE(3)$ with $n \geq 3$, any infinitesimal variation $\delta = [\delta_p \; \delta_o]^\top \in \mathbb{R}^{6n}$ satisfies the condition
\[
G_\mathcal{G}(\chi) \delta_p = -F_\mathcal{G}(\chi) \delta_o
\]

with
\[
G_\mathcal{G}(\chi) = \text{diag}(d_i, R_i^T P(p_{ij})E^\top),
\]
\[
F_\mathcal{G}(\chi) = -\text{diag}(p_{ij}),
\]

where $E \in \mathbb{R}^{d \times n \times m}$ and $E_o \in \mathbb{R}^{d \times n \times m}$ are derived from the incidence matrix of the graph $\mathcal{G}$.

**Proof.** The thesis can be proved following the same reasoning carried out in the proof of Thm. VI.1.

The considerations inferred in Sec. VII about $G_\mathcal{G}(\chi)$ and the bearing rigidity matrix $B_\mathcal{K}(\chi)$ defined in Sec. VII are still valid and also the fact that the trivial variations satisfy (52) in correspondence of $\mathcal{K}$. Hence, the translation, uniform scaling and coordinated rotation are trivial variations also for a framework $(\mathcal{G}, \chi)$ in $SE(3)$, however the concept of coordinated rotation has to be redefined since the agents orientation is no longer controllable only via a single angle.

First, note that the angular velocity of each agent can be an arbitrary vector in $\mathbb{R}^3$, i.e., it arises from the linear combination of the unit vectors $e_h$, $h = 1, 2, 3$ that identify the axes of the frame $\mathcal{F}_W$. Hence, we can distinguish three basic coordinate rotations such that all the agents are rotated in the same way of the whole formation and such a rotation is around $e_h$, $h = 1, 2, 3$. Each coordinated rotation of the framework can thus be expressed as a suitable sequence of basic coordinated rotations, hence dim $(\mathcal{R}_\mathcal{K}) = 3$ where $\mathcal{R}_\mathcal{K}$ is the the coordinated rotation subspace. More specifically, it can be proved that
\[
\mathcal{R}_\mathcal{K} = \text{span} \left\{ \left\{ \begin{bmatrix} I_d & [e_h]_x \end{bmatrix} p \right\} \middle| p_1 = 1, p_2 = 0 \right\},
\]
\[
\mathcal{R}_\mathcal{K} = \text{span} \left\{ \left\{ \begin{bmatrix} I_d & [e_h]_x \end{bmatrix} p \right\} \middle| p_1 = 1, p_2 = 0 \right\},
\]

having dim $(\mathcal{S}_2) = 7$, includes all the instantaneous variation vectors related to translation, dilation and coordinated rotation of the framework, and the next theorems can be stated.

**Theorem VI.2.** For a framework $(\mathcal{K}, \chi)$ in $SE(3)$, it holds that $\mathcal{S}_2 \subseteq \ker (B_\mathcal{K}(\chi))$.

**Proof.** Applying Thm. III.4, the variation vectors having $\delta_o = 0$ and $\delta_p = 1_n \otimes I_3$ or $\delta_p$ satisfies condition (52) in correspondence of $\mathcal{K}$ associated to $G$. Moreover, for each pair of non coincident vertices $v_i, v_j \in \mathcal{V}$, it holds that
\[
d_i R_i^T p(p_{ij}) [e_h]_x p_{ij} = -[p_{ij}]_x e_h
\]

since $p_{ij} [e_h]_x p_{ij} = 0$, proving that every $\delta \in \mathcal{R}_\mathcal{K}$ satisfies condition (52), for any graph $\mathcal{G}$, and then also for $\mathcal{K}$.

**Theorem VI.3.** For a non-degenerate framework $(\mathcal{K}, \chi)$ in $SE(3)$, with $n \geq 3$, it holds that
\[
\ker (B_\mathcal{K}(\chi)) = \mathcal{S}_1, \text{ or equivalently,}
\]
\[
\ker (B_\mathcal{K}(\chi)) = \mathcal{S}_1, \text{ or equivalently,}
\]

**Proof.** The thesis can be proved analogously to Thm. VII.3.

**Theorem VI.4** (Condition for IBR, Thm. 3 in [45]). A non-degenerate framework $(\mathcal{G}, \chi)$ in $SE(3)$, with $n \geq 3$, is IBR if and only if $\text{rank} (B_{\mathcal{G}}(\chi)) = 6n - 7$.

As in the previous case, this set represents a pure coordinated rotation for $(\mathcal{G}, \chi)$ only when the framework com coincides with the origin of $\mathcal{F}_W$. 

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Fig. 4: Examples of IBF frameworks (Figs. 4(a) 4(b) 4(c) 4(d)) and IBR frameworks (Figs. 4(e) 4(f) 4(g) 4(h) in $SE(3)$).
Proof. The proof derives from the application of Def. [III.11] and Thms. [III.4] [VI.3] following the same reasoning carried out in the proof of Thm. [IV.3].

Fig. 4 shows some examples of IBF and IBF frameworks in $SE(3)$.

VIII. ON THE DEGENERATE FORMATIONS

In this section we briefly discuss the degenerate formations case, focusing on the bearing-preserving variations set. In particular, we show why for these multi-agent groups the definition of infinitesimal bearing rigidity becomes meaningless and the related results are not relevant.

According to Def. [III.2] a $n$-agent formation ($n \geq 3$) is degenerate if all the agents are collinear, i.e., for any $k$-th component of the position vectors ($k \in \{1 \ldots d\}$) it exists $c \in \mathbb{R}$ such that $p^0_i = cp^i_j$ for each pair $(v_i, v_j)$ of agents in the group. Under this hypothesis, it can be observed that the shape uniqueness is guaranteed for a larger set of infinitesimal variations w.r.t. that described in the previous sections. Although this statement is valid independently from the space $D$ of interest, in the following we distinguish between the three cases previously treated.

a) $D = \mathbb{R}^d$: we first recall that for a formation composed of $n \geq 3$ agents controllable in the metric space $D = \mathbb{R}^d$ ($d \geq 2$) and aligned along a certain direction identified by the vector $v$ in $\mathbb{S}^{d-1}$ the bearing measurements are collinear, namely it holds $\mathbf{b}_D(\chi) = \text{diag}(b_1 \ldots b_m)(\mathbf{1}_n \otimes v)$ with $b_i \in \mathbb{R}$ for $i \in \{1 \ldots m\}$. Given these premises, one can realize that the bearing measurements are preserved despite the displacement of any agent along the direction specified by $v$ and the translation of the whole formation in the subspace $\mathcal{W}$ of $\mathbb{R}^d$ orthogonal to $v$. Hence the trivial variation set coincides with $\mathcal{S}^d_l = \text{span}\{\mathbf{1}_n \otimes v, \mathbf{1}_n \otimes \mathbf{W}\}$ where $\mathbf{W} \in \mathbb{R}^{d \times (d-1)}$ is a matrix whose columns represent a basis for $\mathcal{W}$. Trivially, $\mathcal{S}^d_l$ has dimension $n + d - 1 > d + 1$. In particular, $\mathcal{D} = \mathbb{R}^d \times \mathbb{S}^1$: for a formation acting in $\mathcal{D} = \mathbb{R}^d \times \mathbb{S}^1$, the bearing measurements are retrieved in the local agents frame and the agents are assumed to have a (controllable) rdof. To analyze the degenerate situation in which all the agents are aligned along the direction identified by $v \in \mathbb{S}^{d-1}$, it is necessary to distinguish between the following cases:

(i) $d = 2$ or $d = 3$ and $d \neq v$, (ii) $d = 3$ and $d = v$. For a degenerate formation satisfies conditions (i), the bearing measurements are preserved when the whole agents group translate along any direction in the $(d-1)$-dimensional subspace $\mathcal{W} \subseteq \mathbb{R}^d$ orthogonal to $v$, when a coordinated rotation is performed according to the definition given in Sec. IV and also when any agent move along the alignment direction. As a consequence the trivial variation set $\mathcal{S}^d_l$ is spanned by $n + (d - 1) + 1 = n + d$ elements. In case (ii), the dimension of $\mathcal{S}^d_l$ increases since the formation is not required to perform a coordinated rotation to preserve the bearings: also the rotation of any agent around the axis identified by $\mathbf{d} = v$ ensures the measurements maintenance. Hence, we get $|\mathcal{S}^d_l| = 2n + (d - 1)$. Note that in both cases (i) and (ii) the trivial variation set has dimension grater w.r.t. non-degenerate case for which $|\mathcal{S}^i_l| = d + 2$.

c) $\mathcal{D} = SE(3)$: when the space of interest is $\mathcal{D} = \mathbb{R}^3 \times SO(3) = SE(3)$, we figure out that for the degenerate case in which the agents are aligned along the direction identified by $v \in \mathbb{S}^{d-1}$, the bearings are preserved when any agent translates or rotates along the direction specified by $v$ and when the whole formation performs a translation or a coordinated rotation around any direction in the subspace $\mathcal{W} \subseteq \mathbb{R}^d$ orthogonal to $v$. The trivial variation set has thus dimension $2n + 2(d - 1) > 2d + 1$.

To validate the provided observations, in Tab. I we report a basis for $\mathcal{S}_d$ and for $\mathcal{S}^d_l$ in correspondence to the different metric spaces evaluated in this work. For sake of simplicity, the corresponding formations are supposed to be composed by $n = 3$ agents all equally oriented, i.e., such that $\alpha_i = 0$ or $\mathbf{R}_i = \mathbf{I}_3$ for $i \in \{1 \ldots 3\}$ according to the adopted rotation representation. Moreover for the degenerate cases, we assume that the agents are all aligned along the $x$-axis of $\mathcal{F}_W$, namely $v = \mathbf{e}_1$ and in particular $\mathbf{p}_i = \mathbf{e}_1 i$, $i \in \{1 \ldots 3\}$, while in the non-degenerate cases they are placed in order to form a triangle on the plane identified by the $x$ and $y$-axis of $\mathcal{F}_W$. As previously stated, we can observe that for every metric space $\mathcal{D}$ it occurs that $|\mathcal{S}_d^l| > |\mathcal{S}_d|$.

VIII. CONCLUSIONS

This work focuses on the bearing rigidity theory applied to multi-agent systems whose elements are characterized by a certain number of both tdofs and rdofs. As original contribution, we propose a unified framework for the definition of the main rigidity properties without accounting for the specific controllable agents state domain. Moreover, we summarize the existing results about bearing rigidity theory for frameworks embedded in $\mathbb{R}^d, \mathbb{R}^d \times \mathbb{S}^1, d \geq 2$ and in $SE(3)$. For each case, the principal definitions are provided and the infinitesimal rigidity property is investigated by deriving a necessary and sufficient condition based on the rigidity matrix rank. To recap the main results, Fig. 4 illustrates the studied scenarios with particular regard to the agent model and the available measurements, while Tab. I sums up the theoretical aspects of the bearing rigidity theory for the different metric spaces.

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TABLE II: Summary of the principal notions related to the bearing rigidity theory in the different domains accounted in Sec. IV-VI.

| $D$ | $S_r$ | $S^d_r$ | $\chi_i$ | $M$ | $b_{ij}$ | $I$ | $\delta$ | IBR condition |
|-----|-------|--------|---------|-----|--------|-----|---------|---------------|
| $d = 2$ | $\mathbb{R}^d$ | | | | | | | |
| | | | | | | | | |
| $d = 3$ | $\mathbb{R}^d \times S^1$ | | | | | | | |
| | | | | | | | | |
| $d = 2$ | $\mathbb{R}^3 \times SO(3)$ | | | | | | | |
| | | | | | | | | |
| $d = 3$ | $\mathbb{R}^3 \times SO(3)$ | | | | | | | |

TABLE I: Non-degenerate vs. degenerate: $n = 3$, $\alpha_i = 0$, $\mathbf{R}_i = \mathbf{I}_3$ for $i \in \{1 \ldots 3\}$ and $\mathbf{v} = \mathbf{e}_1$, $\mathbf{p}_i = i\mathbf{e}_1$, $i \in \{1 \ldots 3\}$ for the degenerate case.

TABLE II: Summary of the principal notions related to the bearing rigidity theory in the different domains accounted in Sec. [IV VI].
Fig. 5: Bearing measurements in the different domains accounted in Sec. [IV]-[VI]

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