LIPSCHITZ STABILITY FOR AN INVERSE SOURCE PROBLEM IN ANISOTROPIC PARABOLIC EQUATIONS WITH DYNAMIC BOUNDARY CONDITIONS

EL MUSTAPHA AIT BEN HASSI, SALAH-EDDINE CHORFI
LAHCEN MANIAR* AND OMAR OUKDACH
Cadi Ayyad University
Faculty of Sciences Semlalia
LMDP, UMMISCO (IRD-UPMC) B.P. 2390
Marrakesh, Morocco

(Communicated by Genni Fragnelli)

Abstract. In this paper, we study an inverse problem for linear parabolic system with variable diffusion coefficients subject to dynamic boundary conditions. We prove a global Lipschitz stability for the inverse problem involving a simultaneous recovery of two source terms from a single measurement and interior observations, based on a recent Carleman estimate for such problems.

1. Introduction and statement of the problem. We are interested in the inverse source problem for linear parabolic system with variable diffusion coefficients and dynamic boundary conditions in bounded domains. It consists of recovering two source terms from a single measurement of the temperature at a given time with an additional internal observation on the solution localized in a small region of the physical domain.

To introduce the problem, let $T > 0$ and $\Omega \subset \mathbb{R}^N$ a bounded domain, $N \geq 2$, with smooth boundary $\Gamma = \partial \Omega$ of class $C^2$, and outer unit normal field $\nu$ on $\Gamma$ be given. We denote $\Omega_T = (0,T) \times \Omega$, $\omega_T = (0,T) \times \omega$, $\Gamma_T = (0,T) \times \Gamma$, where $\omega \Subset \Omega$ is a nonempty open subset. Consider the following system

\[
\begin{cases}
\partial_t y - \text{div}(A(x)\nabla y) + B \cdot \nabla y + p(x)y = F(t,x), & \text{in } \Omega_T, \\
\partial_t y|_\Gamma - \text{div}_\Gamma(D(x)\nabla y|_\Gamma) + \partial^\Gamma_A y + \langle b, \nabla y|_\Gamma \rangle\Gamma + q(x)y|_\Gamma = G(t,x), & \text{on } \Gamma_T, \\
y|_\Gamma(t,x) = y|_\Gamma(t,x), & \text{on } \Gamma_T, \\
(y,y|_\Gamma)|_{t=0} = (y_0, y_1), & \Omega \times \Gamma.
\end{cases}
\]

(1)

The initial states are denoted by $(y_0, y_1) \in L^2(\Omega) \times L^2(\Gamma)$, while the source terms are $F \in L^2(\Omega_T)$ and $G \in L^2(\Gamma_T)$. All the coefficients in system (1) are assumed to be bounded,

\[
B \in L^\infty(\Omega)^N, \quad p \in L^\infty(\Omega), \quad b \in L^\infty(\Gamma)^N, \quad q \in L^\infty(\Gamma).
\]  

(2)

2020 Mathematics Subject Classification. Primary: 35R30; Secondary: 35K05.

Key words and phrases. Inverse problem, Carleman estimate, Lipschitz stability, dynamic boundary conditions, surface diffusion.

* Corresponding author: Lahcen Maniar.
We assume that the diffusion matrices $A$ and $D$ are symmetric and uniformly elliptic, i.e.,

$$A = (a_{ij})_{i,j} \in C^1(\overline{\Omega}; \mathbb{R}^{N \times N}), \quad a_{ij} = a_{ji}, \quad 1 \leq i,j \leq N, \quad (3)$$

$$D = (d_{ij})_{i,j} \in C^1(\Gamma; \mathbb{R}^{N \times N}), \quad d_{ij} = d_{ji}, \quad 1 \leq i,j \leq N, \quad (4)$$

and there exists a constant $\beta_0 > 0$ such that

$$\langle A(x)\zeta, \zeta \rangle \geq \beta_0 |\zeta|^2, \quad x \in \overline{\Omega}, \quad \zeta \in \mathbb{R}^N, \quad (5)$$

$$\langle D(x)\zeta, \zeta \rangle_{\Gamma} \geq \beta_0 |\zeta|^2_{\Gamma}, \quad x \in \Gamma, \quad \zeta \in \mathbb{R}^N, \quad (6)$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product (also denoted by “$\cdot \cdot $”) and $\langle \cdot, \cdot \rangle_{\Gamma}$ is the Riemannian inner product on $\Gamma$ as defined below. We denote by $y|_{\Gamma}$ the trace of $y$.

The conormal derivative with respect to $A$ is given by

$$\partial^A y := \left( A \nabla y \cdot \nu \right)|_{\Gamma} = \sum_{i,j=1}^N a_{ij}(x)(\partial_i y)|_{\Gamma} \nu_j. \quad (7)$$

For the identity matrix, the normal derivative is $\partial_{\nu} y := (\nabla y \cdot \nu)|_{\Gamma}$. Here, div denotes the divergence operator with respect to the space variable in $\Omega$. The boundary $\Gamma$ is considered to be a $(N-1)$-dimensional compact Riemannian submanifold, without boundary. Let $g$ be the Riemannian metric on $\Gamma$ induced by the natural embedding $\Gamma \hookrightarrow \mathbb{R}^N$. We fix a coordinate system $x = (x^j)$ and we denote by $\left( \frac{\partial}{\partial x^j} \right)$ the corresponding tangent vector field. In local coordinates, $g$ is given by $g_{ij} \equiv (g_{ij})$, $(g^{ij})$ its inverse and $|g| = \det(g_{ij})$. It is well-known that $\nabla y$ is the projection of the standard Euclidean gradient $\nabla y$ onto the tangent space on $\Gamma$, that is,

$$\nabla y = \nabla - \langle \nabla y, \nu \rangle \nu. \quad (7)$$

The divergence operator $\text{div}_\Gamma$ associated with the Riemannian metric $g$ is defined locally as follows

$$\text{div}_\Gamma(X) = \frac{1}{\sqrt{|g|}} \sum_{j=1}^{N-1} \frac{\partial}{\partial x^j} \left( \sqrt{|g|} X^j \right), \quad X = \sum_{j=1}^{N-1} \frac{\partial}{\partial x^j} X^j.$$

For any $x \in \Gamma$, the inner product and the norm on the tangent space $T_x \Gamma$ are given by

$$g(X_1, X_2) = \langle X_1, X_2 \rangle_{\Gamma} = \sum_{i,j=1}^{N-1} g_{ij} X_1^i X_2^j, \quad |X|_{\Gamma} = \langle X, X \rangle_{\Gamma}^{1/2}.$$

Then, the Laplace-Beltrami operator $\Delta_\Gamma$ associated to $g$ is given by

$$\Delta_\Gamma = \text{div}_\Gamma(\nabla y) = \frac{1}{\sqrt{|g|}} \sum_{i,j=1}^{N-1} \frac{\partial}{\partial x^j} \left( \sqrt{|g|} g^{ij} \frac{\partial}{\partial x^i} \right).$$
Since $\Gamma$ is a compact Riemannian manifold without boundary, the following divergence formula holds

$$\int_{\Gamma} (\text{div}_{\Gamma} X) z \, dS = - \int_{\Gamma} (X, \nabla_{\Gamma} z)_{\Gamma} \, dS, \quad z \in H^1(\Gamma),$$

(8)

where $X$ is any $C^1$ vector field on $\Gamma$ and $dS$ denotes the surface measure on $\Gamma$.

Dynamic boundary conditions of type (1)$_2$ appear in various mathematical models such as population dynamics, hydrodynamics and thermodynamics, including heat transfer and diffusion phenomena where the diffusion lies in the interface between a solid and a fluid as an example of non perfect contact. See [12, 26] and the references therein. The dynamic boundary conditions arise naturally as part of the physical derivation when incorporating boundary conditions into the formulation of the problem. We refer to [17] for the physical interpretation and derivations of such boundary conditions.

A number of authors have studied evolution equations with dynamic boundary conditions from different mathematical aspects, see for instance [16, 13, 17, 28, 29, 33]. The main ingredient to establish various facts in control theory as well as inverse problems is Carleman estimates, which are, roughly speaking, some inequalities estimating the solutions of PDEs in terms of associated differential operators, using large parameters and appropriate weight functions.

Recently, Maniar et al. have established a new Carleman estimate [28] in context of null controllability of system (1) with constant diffusion matrices and without drift terms, i.e., $A = dI, D = \delta I$ and $B = b = 0$, for some constants $d, \delta > 0$. The drift term case has been recently studied in [23, 24].

**Inverse Source Problem.** Let $T > 0, t_0 \in (0, T), T_0 = \frac{T + t_0}{2}$ and denote $L^2 := L^2(\Omega) \times L^2(\Gamma)$. For a given $C_0 > 0$, we introduce the set of admissible source terms as follows

$$S(C_0) := \left\{ (F, G) \in H^1(0, T; L^2) : \begin{array}{l} |F(t, x)| \leq C_0 |F(T_0, x)|, \quad \text{a.e. } (t, x) \in \Omega_T \\ |G(t, x)| \leq C_0 |G(T_0, x)|, \quad \text{a.e. } (t, x) \in \Gamma_T \end{array} \right\}. \tag{9}$$

The set $S(C_0)$ contains an interesting class of source terms which include the sources (11)-(12) below. Our purpose is to determine the couple of source terms $F = (F, G)$ in (1) belonging to $S(C_0)$, from a single measurement $Y(T_0, \cdot) = (y, y_{\Gamma})|_{t=T_0}$ and internal partial observation on the first component of the solution, namely, $y|_{(t_0, T) \times \omega}$. The case of boundary measurements $\partial_{\nu} y|_{(t_0, T) \times \gamma}$ or $y|_{(t_0, T) \times \gamma}$, for an arbitrary sub-boundary $\gamma \subset \Gamma$, requires a new boundary Carleman estimate. This will be treated in a forthcoming paper. We emphasize that $T_0$ can be arbitrary chosen in $(t_0, T)$, since it suffices to work in a time sub-interval centered at $T_0$. We mainly aim to establish a global Lipschitz stability result, for the source terms in (1) belonging to $S(C_0)$, which reads as follows

**Theorem 1.1.** Let $T > 0, t_0 \in (0, T)$ and $T_0 = \frac{T + t_0}{2}$. Consider $Y := (y, y_{\Gamma})$ the mild solution of (1) and $C_0 > 0$. Then, there exists a positive constant $C = C(\Omega, \omega, T_0, C_0, \|B\|_{\infty}, \|p\|_{\infty}, \|b\|_{\infty}, \|q\|_{\infty})$ such that, for any admissible source $F = (F, G) \in S(C_0)$, we have

$$\|(F, G)\|_{L^2^2} \leq C \left( \|Y(T_0, \cdot)\|_{H^2} + \|\partial_{\nu} y\|_{L^2(\omega_0, \tau)} \right). \tag{10}$$
In the above inverse source problem, if we only consider the single measurement of the temperature $Y(T_0, \cdot) = (y, y_T)|_{t=T_0}$ as observation, the inverse problem becomes ill-posed in the sense of Hadamard, due to compactness reasons. Hence, the additional observation $y|_{(t_0, T) \times \omega}$ is important to overcome the instability. The internal observation regions required by the Carleman estimate approach are often of the form $\mathcal{O} := (\{T_0\} \times \Omega) \cup ((t_0, T) \times \omega)$.

A new approach with quite realistic observations was introduced in [11] for a uniqueness result in an inverse parabolic problem, where the observation is taken on a single point $x_0 \in \Omega$ of the spatial domain, and any small time interval $(0, t_0)$, that is, $\mathcal{O}_0 := (0, t_0) \times \{x_0\}$. However, up to our knowledge, no stability result was proven by this method. This type of inverse source problems was studied by many researchers in the case of static boundary conditions of Dirichlet, Neumann or Robin types, see for instance [8, 18, 19, 20, 25]. For a general review on inverse parabolic problems by Carleman estimate we refer to [34], and the recent book [1] for inverse hyperbolic problems. As for coupled systems, it is worth to mention the paper [2], where a Carleman estimate with second large parameter is applied to thermoelasticity systems, and [3] which deals with an inverse source problem for a transmission problem in a heat equation.

Our result is based on a new Carleman estimate for the system (1), which extends the one proved in [28] for the system with standard Laplace and Laplace-Beltrami operators, to a general second order elliptic operators in divergence form; applied with the pioneering idea of applying such estimate to inverse problems, originally proposed by Bukhgeim and Klibanov in [8]. This new approach allows to prove uniqueness and Hölder stability result using a local Carleman estimate. In 1998, Imanuvilov and Yamamoto in [18] adapted the idea of Bukhgeim and Klibanov with global Carleman estimate proved by Fursikov and Imanuvilov in [15]. This permits to improve the Hölder stability to a global Lipschitz stability for inverse source problems with classical boundary conditions in the parabolic case. The aforementioned method has also been successfully applied to degenerate/singular parabolic equations and coupled systems, see e.g., [5, 6, 7, 9, 32]. Our results extend those for parabolic equations with static boundary conditions in [18] to the dynamic boundary condition case.

For applications, if we limit ourselves to the particular but important case where the source terms in (1) are given by

\begin{align}
F(t, x) &= f(x)r(t, x) \quad \text{for all } (t, x) \in \Omega_T, \\
G(t, x) &= g(x)\tilde{r}(t, x) \quad \text{for all } (t, x) \in \Gamma_T,
\end{align}

uniqueness and stability results can be established as a direct consequence of our Lipschitz stability result, where the inverse source problem is to determine the couple of spacewise dependent sources $(f, g)$ by the same measurements, provided that the couple of functions $(r, \tilde{r})$ is known and satisfying some positivity assumption. The couple of functions $(f, g)$ models one special but important case of spatial distributions of source terms arising in several fields of applications such as biology, population dynamics, chemistry, etc.

The rest of the paper is organized as follows: in Section 2, the well-posedness of system (1) is discussed, and a special attention is paid to the regularity results, since the Bukhgeim-Klibanov method requires some regularity on the time derivative of the solution, and then we present the Carleman estimate relevant to the system
Finally, in Section 3, we apply the Carleman estimate to prove the Lipschitz stability result.

2. General framework.

2.1. Functional setting. We denote the Lebesgue measure on Ω and the surface measure on Γ by dx and dS, respectively. We will use the following real spaces

\[ L^2 := L^2(\Omega, dx) \times L^2(\Gamma, dS), \quad L_T^2 := L^2(\Omega_t) \times L^2(\Gamma_t). \]

\( L^2 \) is a real Hilbert space with the corresponding scalar product given by

\[ \langle (y, y_\Gamma), (z, z_\Gamma) \rangle_{L^2} = \langle y, z \rangle_{L^2(\Omega)} + \langle y_\Gamma, z_\Gamma \rangle_{L^2(\Gamma)}. \]

Analogously to \( H^k(\Omega) \) and \( H^k(\Gamma) \), the usual second order Sobolev spaces over Ω and Γ, we consider

\[ \mathbb{H}^k := \{(y, y_\Gamma) \in H^k(\Omega) \times H^k(\Gamma) : y|_\Gamma = y_\Gamma \}, \]

with the standard norm induced by \( H^k(\Omega) \times H^k(\Gamma) \). Recall that \( \|y\|_{L^2(\Gamma)} \) + \( \|\nabla y\|_{L^2(\Gamma)} \) defines an equivalent norm on \( H^1(\Gamma) \). Moreover, \( \|y\|_{L^2(\Gamma)} \) + \( \|\Delta y\|_{L^2(\Gamma)} \) yields an equivalent norm on \( H^2(\Gamma) \).

For the regularity of the solution, we introduce the following spaces

\[ E_1(t_0, t_1) := H^1(t_0, t_1; \mathbb{H}^2) \cap L^2(t_0, t_1; \mathbb{H}^2) \] for \( t_1 > t_0 \) in \( \mathbb{R} \),

\[ E_2(t_0, t_1) := H^1(t_0, t_1; \mathbb{H}^2) \cap H^2(t_0, t_1; L^2) \] for \( t_1 > t_0 \) in \( \mathbb{R} \). In particular,

\[ E_1 := E_1(0, T) \quad \text{and} \quad E_2 := E_2(0, T). \]

For the known part \((r, \tilde{r})\) of the source term \((F, G)\) in the special form \((11)-(12)\), we use the following space

\[ C^{1,0} := C^{1,0}([0, T] \times \overline{\Omega}) \times C^{1,0}([0, T] \times \Gamma), \]

where \( C^{1,0}([0, T] \times E) = \{y = y(t, x) | y, \partial_y \in C([0, T] \times E)\} \) for \( E = \overline{\Omega} \) or \( \Gamma \).

We conclude by recalling an important regularity result that we will use in the sequel. Since Ω is assumed to be of class \( C^2 \) and \( A \in C^1(\overline{\Omega}; \mathbb{R}^{N \times N}) \), the elliptic regularity states that: if \( y \in H^1(\Omega) \) is such that \( \text{div}(A\nabla y) \in L^2(\Omega) \) and the trace \( y|_\Gamma \in H^2(\Gamma) \), then \( y \in H^2(\Omega) \), see for instance [21, Theorem 9.3.3]. A similar regularity result holds for the elliptic operator on Γ. If \( u \in H^1(\Gamma) \) and \( \text{div}_\Gamma(D\nabla u) \in L^2(\Gamma) \), then \( u \in H^2(\Gamma) \), see e.g., [31, Proposition 1.6].

2.2. Wellposedness and time regularity of the solution. In this section, we mainly borrow our terminology from [30].

The system (1) can be written in the following abstract form

\[ (ACP) \begin{cases} \partial_t Y = AY + F, & 0 < t < T, \\ Y(0) = Y_0 = (y_0, y_1), \end{cases} \]

where \( Y := (y, y_\Gamma), \quad F = (F, G) \) and the linear operator

\[ A: D(A) \subset L^2 \rightarrow L^2 \]

given by

\[ A = \begin{pmatrix} \text{div}(A\nabla) - B \cdot \nabla - p \\ -\partial_{\nu} A \end{pmatrix} \begin{pmatrix} \partial_t \nabla & 0 \\ \text{div}_\Gamma(D\nabla_\Gamma) - \langle b, \nabla_\Gamma \rangle_\Gamma - q \end{pmatrix}, \quad D(A) = \mathbb{H}^2. \quad (13) \]
Following [28], we introduce the densely defined bilinear form given by
\[
a((y, y_T), (z, z_T)) = \int_{\Omega} [A(x)\nabla y \cdot \nabla z + (B(x) \cdot \nabla y)z + pyz] \, dx
+ \int_{\Gamma} [(D(x)\nabla y_T, \nabla z_T)_{\Gamma} + (b(x), \nabla y_T)_{\Gamma}z_T + qy_Tz_T] \, dS,
\]
with form domain \( D(a) = H^1 \) on the Hilbert space \( L^2 \). For a real number \( \mu \), we denote by \( a + \mu \) the following bilinear form
\[
(a + \mu)((y, y_T), (z, z_T)) = a((y, y_T), (z, z_T)) + \mu((y, y_T), (z, z_T))_{L^2}.
\]
By virtue of (5)-(6) and using Cauchy-Schwarz inequality, there is a constant \( \mu \in \mathbb{R} \) such that
\[
a((y, y_T), (y, y_T)) + \mu \|(y, y_T)\|_{L^2}^2 \geq \frac{\beta_0}{2} \|(y, y_T)\|_{H^1}^2 \quad \text{for all } (y, y_T) \in H^1.
\]
Therefore, following [30] one can check that the form \( a + \mu \) is densely defined, accretive, continuous and closed. Then, we can associate with the form \( a \) an operator \( \overline{A} \) given by
\[
D(\overline{A}) := \{(y, y_T) \in H^1, \text{ there exists } (w, w_T) \in L^2 \text{ such that }
\]
\[
a((y, y_T), (z, z_T)) = \langle (w, w_T), (z, z_T) \rangle_{L^2} \text{ for all } (z, z_T) \in H^1, \quad (14)
\]
\[
\overline{A}(y, y_T) := (w, w_T) \quad \text{for all } (y, y_T) \in D(\overline{A}). \quad (15)
\]
It follows from [30, Theorem 1.52] that the operator \( \overline{A} \) generates an analytic \( C_0 \)-semigroup on \( L^2 \).

The generation result for the operator \( A \) in the case of constant diffusion matrices and without drift terms was proved in [28, Proposition 2.6], using a Lemma by Miranville-Zelik [29]. We refer to [24] for a complete study of the well-posedness in the presence of drift terms. Here we prove the result in a slightly different way based on the elliptic regularity stated in Section 2.

**Proposition 1.** The operator \( A \) generates an analytic \( C_0 \)-semigroup \( (e^{tA})_{t \geq 0} \) on \( L^2 \).

**Proof.** It suffices to prove that \( A = \overline{A} \). Let \( (y, y_T) \in D(A) \) and \( (z, z_T) \in H^1 \). Using integration by parts and divergence formula (8), we obtain
\[
a((y, y_T), (z, z_T)) = \int_{\Omega} [-\text{div}(A(x)\nabla y)z + (B(x) \cdot \nabla y)z + pyz] \, dx
+ \int_{\Gamma} [-\text{div}_\Gamma(D(x)\nabla y_T)z_T + \partial_x^\Gamma b(x)\nabla y_Tz_T + qy_Tz_T] \, dS
= \langle -A(y, y_T), (z, z_T) \rangle_{L^2}.
\]
Then, \( D(A) \subseteq D(\overline{A}) \) and \( \overline{A}(y, y_T) = A(y, y_T) \) for all \( (y, y_T) \in D(A) \), i.e., the operator \( \overline{A} \) is an extension of \( A \). In order to prove the converse, let \( (y, y_T) \in D(\overline{A}) \). The above calculation implies that \( \text{div}(A\nabla y) \in L^2(\Omega) \) and \( \text{div}_\Gamma(D\nabla y_T) \in L^2(\Gamma) \). Since \( y_T \in H^1(\Gamma) \), the elliptic regularity on \( \Gamma \) yields that \( y_T \in H^2(\Gamma) \), and by the same argument for the operator on \( \Omega \), we have \( y \in H^2(\Omega) \). Hence, \( D(\overline{A}) \subseteq D(A) \).

Finally, \( A = \overline{A} \) and \( A \) generates an analytic \( C_0 \)-semigroup on \( L^2 \). \( \square \)

In the sequel, we adopt the following notions of solutions.

**Definition 2.1.** Let \( (F, G) \in L^2_\Gamma \) and \( Y_0 \in L^2 \).
Lemma 2.3.
The purpose of the following lemma is to establish
similar properties with the conormal derivative instead of the normal derivative.

Since $A$ generates an analytic $C_0$-semigroup on $L^2$, the following regularity result holds. See for instance Theorem 3.1 and Proposition 3.8 in [4].

**Proposition 2.** Let $F \in L^2(\Omega_T)$ and $G \in L^2(\Gamma_T)$.

(i) For all $Y_0 \in H^1$, there exists a unique strong solution of (1) such that

\[ Y := (y, y_t) \in H^1(0, T; H^2) \cap L^2(0, T; L^2). \]

(ii) For all $Y_0 \in L^2$, there exists a unique mild solution of (1) $Y := (y, y_t) \in C([0, T]; L^2)$ such that for all $\tau \in (0, T),
\[ Y = \mathbb{E}_1(\tau, Y_0) := H^1(\tau, T; H^2) \cap L^2(\tau, T; L^2). \]

Moreover, if $\mathcal{F} = (F, G) \in H^1(0, T; L^2)$, then for all $\tau \in (0, T)$, we have
\[ Y \in \mathbb{E}_2(\tau, T) := H^1(\tau, T; H^2) \cap H^2(\tau, T; L^2). \]

2.3. **Carleman estimate.** To state and prove our Carleman estimate, we need a weight function with special properties. The existence of such function is proved in [15].

**Lemma 2.2.** Let $\omega' \Subset \Omega$ be a nonempty open subset. Then there is a function $\eta^0 \in C^2(\Omega)$ such that

\[ \eta^0 > 0 \quad \text{in } \Omega, \quad \eta^0 = 0 \quad \text{on } \Gamma, \quad |\nabla \eta^0| > 0 \quad \text{in } \overline{\Omega \setminus \omega'}. \]

Moreover, the identity $|\nabla \eta^0|^2 = |\nabla \Gamma \eta^0|^2 + |\partial_\nu \eta^0|^2$ on $\Gamma$ implies

\[ \nabla \Gamma \eta^0 = 0, \quad |\nabla \eta^0| = |\partial_\nu \eta^0|, \quad \partial_\nu \eta^0 \leq -c < 0 \quad \text{on } \Gamma \quad (16) \]

for some constant $c > 0$.

**Remark 1.** The identity $(\partial_\nu \psi)^2 = |\nabla \psi|^2 - |\nabla \Gamma \psi|^2$ and the property $\partial_\nu \eta^0 < -c < 0$ have played important roles in the proof of Carleman estimate with standard Laplacians in [28]. Since we deal with general elliptic second order operators, we need similar properties with the conormal derivative instead of the normal derivative. This is the purpose of the following lemma.

**Lemma 2.3.** Let $\psi$ be any smooth function.

(i) The following identity holds

\[ (\partial_\nu^* \psi)^2 - (A \nabla \psi \cdot \nu)^2 = |A^\Delta \psi|^2 \left( |A^\Delta \nabla \psi|^2 - |A^\Delta \nabla \Gamma \psi|^2 \right). \quad (17) \]

(ii) Let $c$ be the same constant in (16). Then

\[ \partial_\nu^* \eta^0 \leq \beta_0 \partial_\nu \eta^0 \leq -c \beta_0 < 0. \quad (18) \]
Proof. (i) Using the identity (7), we obtain
\[ A \nabla \psi = A \nabla_G \psi + (\partial_\nu \psi) A \nu. \] (19)
\[ \nabla \psi = \nabla_G \psi + (\partial_\nu \psi) \nu. \] (20)

Composing (19) by \( \nu \) yields the following identity
\[ \partial_\nu A \psi = A \nabla_G \psi \cdot \nu + (\partial_\nu \psi)(A \nu \cdot \nu). \]

By taking the scalar product of (19) and (20) with multiplication of the resulting identity by \((A \nu \cdot \nu)\), we infer that
\[ (A \nu \cdot \nu)(A \nabla \psi \cdot \nabla \psi - A \nabla_G \psi \cdot \nabla_G \psi) = (\partial_\nu \psi)^2 (A \nu \cdot \nu)^2 + 2(\partial_\nu \psi)(A \nu \cdot \nu) A \nabla_G \psi \cdot \nu, \]
where we used the symmetry of \(A\). Completing the square yields the result.

(ii) Since \(\eta^0|_{\Gamma} = 0\), we obtain
\[ \nabla \eta^0 = (\partial_\nu \eta^0) \nu \quad \text{on } \Gamma. \] (21)

Then, \(\partial_\nu A \eta^0 = (A \nu \cdot \nu) \partial_\nu \eta^0\). Hence, by virtue of (5) and (16), we have
\[ \partial_\nu A \eta^0 \leq \beta_0 \partial_\nu \eta^0 \leq -c\beta_0 < 0. \]

\[ \square \]

Remark 2. In the isotropic case, i.e., \(A(x) = I\), since \(\nabla_G \psi \cdot \nu = 0\), the identity (17) is simply the same as \((\partial_\nu \psi)^2 = |\nabla \psi|^2 - |\nabla_G \psi|^2\).

We introduce the following weight functions
\[ \alpha(t, x) = \frac{2\lambda \|\eta^0\|_{\infty} - e^{\lambda \eta^0(x)}}{t(T-t)} \quad \text{and} \quad \xi(t, x) = \frac{e^{\lambda \eta^0(x)}}{t(T-t)}. \]

for all \((t, x) \in \overline{\Omega}_T\), and \(\lambda \geq 1\) is a parameter (to fix later) which depends only on \(\Omega\) and \(\omega\). Note that \(\alpha\) and \(\xi\) are of class \(C^2\), strictly positive on \(\overline{\Omega}_T\) and blow up as \(t \rightarrow 0\) and as \(t \rightarrow T\), and we have
\[ |\partial_t \alpha| \leq CT\xi^2 \quad \text{and} \quad \xi \leq T^2\xi^2. \]

Furthermore,
\[ \nabla_G \alpha = 0 \quad \text{and} \quad \nabla_G \xi = 0 \quad \text{on } \Gamma. \]

We notice that, for fixed \(x \in \Omega\), \(\alpha(\cdot, x)\) attains the minimum in \((0, T)\) at \(\frac{T}{2}\).

For \((z, z_T) \in \mathbb{E}_1\), we set
\[ Lz = \partial_t z - \text{div}(A(x)\nabla z) + B(x) \cdot \nabla z + p(x)z, \quad (t, x) \in \Omega_T, \]
and
\[ L_T(z_T, z) = \partial_t z_T - \text{div}_T(D(x)\nabla_T z_T) + \partial_\nu \nu z + \langle b(x), \nabla_T z_T \rangle + q(x)z_T, \quad (t, x) \in \Gamma_T. \]

The following lemma is the key tool to prove the main result on global Lipschitz stability in our inverse source problem.

Lemma 2.4 (Carleman estimate). Let \(T > 0\), \(\omega' \subseteq \Omega\) be nonempty and open subset. Consider \(\eta^0\), \(\alpha\) and \(\xi\) as above with respect to a nonempty open set \(\omega' \subseteq \omega\). Then there are three positive constants \(\lambda_1, s_1 \geq 1\) and \(C > 0\) such that, for any \(\lambda \geq \lambda_1\) and \(s \geq s_1\), the following inequality holds
\[ \int_{\Omega_T} \left( \frac{1}{s} |\partial_t \xi|^2 + |\text{div}(A \nabla z)|^2 \right) + s\lambda^2 \xi |\nabla z|^2 + s^3 \lambda^4 \xi^2 |z|^2 \right) e^{-2s\alpha} \, dx \, dt + \]
\[
\int_{\Gamma_T} \left( \frac{1}{\xi^3} (|\partial_t z|^2 + |\text{div}(D \nabla z)|^2) + s\lambda |\nabla \nabla z|^2 + s^3 \lambda^3 |z_T|^2 + s\lambda |\partial_t^2 z|^2 \right) e^{-2s\alpha} dS dt \\
\leq Cs^3 \lambda^4 \int_{\omega_T} e^{-2s\alpha} \xi^3 |z|^2 dx dt + C \int_{\Omega_T} e^{-2s\alpha} |Lz|^2 dx dt + C \int_{\Gamma_T} e^{-2s\alpha} |L_T(z_T, z)|^2 dS dt
\]

(24)

for all \((z, z_T) \in \mathbb{E}_1\). Given \(K > 0\), the constant \(C = C(K)\) can be chosen independently of all potentials \(p\) and \(q\) such that \(||p|| \leq K\).

This Carleman estimate extends the one obtained in [28, Lemma 3.2]. Since the proof is slightly different, we only have to revisit some terms by expanding the computation.

**Proof.** It suffices to prove the inequality (24) for 

\[L_0 z = \partial_t z - \text{div}(A(x) \nabla z) \quad \text{and} \quad L_{0, \Gamma}(z_T, z) = \partial_t z_T - \text{div}_\Gamma(D(x) \nabla z_T) + \partial_t^A z,\]

since lower order terms with bounded coefficients do not influence the Carleman estimate. Taking into account (3) and (5), we denote \(A_0 := \sup_{x \in \Omega} \langle A(x) \zeta, \zeta \rangle\).

Then, we have

\[\beta_0 |\zeta|^2 \leq \langle A(x) \zeta, \zeta \rangle \leq A_0 |\zeta|^2, \quad x \in \overline{\Omega}, \quad \zeta \in \mathbb{R}^N.\]

(25)

**Step 1. Conjugate operators.**

Let \(z \in C^\infty((0, T] \times \overline{\Omega}), \lambda \geq \lambda_1 \geq 1\) and \(s \geq s_1 \geq 1\) be given. Set

\[\psi := e^{-s\alpha} z, \quad f := e^{-s\alpha} L_0 z, \quad g := e^{-s\alpha} L_{0, \Gamma}(z_T, z), \quad \sigma := A(\cdot) \nabla \eta_0 \cdot \nabla \eta_0.\]

By definition and by (5) there exists a positive constant \(C_1 > 0\) such that

\[\beta_0 |\nabla \eta_0|^2 \leq \sigma(x) \leq C_1, \quad x \in \overline{\Omega}.\]

(26)

The corresponding conjugate operators of \(L_0\) and \(L_{0, \Gamma}\) are given by

\[M \psi := e^{-s\alpha} L_0 (e^{s\alpha} \psi) = e^{-s\alpha} L_0 z, \quad \psi := e^{-s\alpha} L_{0, \Gamma}(e^{s\alpha} \psi_T), \quad e^{-s\alpha} L_{0, \Gamma}(z_T, z).\]

For the sake of simplicity, we will write \(z\) and \(\psi\) instead of \(z_T\) and \(\psi_T\) on \(\Gamma_T\). First, we determine the problem fulfilled by \(\psi\) by expanding the spatial derivatives of \(\alpha\) and using the symmetry of \(A\). We have

\[\nabla \alpha = -\nabla \xi = -\lambda \xi \nabla \eta_0,\]

\[\text{div}(A(x) \nabla \alpha) = -\lambda^2 \xi \sigma - \lambda \xi \text{div}(A(x) \nabla \eta_0),\]

\[\partial_t \psi = e^{-s\alpha} \partial_t z - s \psi \partial_t \alpha,\]

\[\nabla \psi = e^{-s\alpha} \nabla z + s \lambda \xi \nabla \eta_0,\]

\[\text{div}(A(x) \nabla \psi) = e^{-s\alpha} \text{div}(A(x) \nabla z) + 2 s \lambda \xi A(x) \nabla \eta_0 \cdot \nabla \psi - s^2 \lambda^2 \xi^2 \psi \sigma + s \lambda \xi \psi \text{div}(A(x) \nabla \eta_0) + s \lambda^2 \xi \psi \sigma.\]

(28)

Regrouping the previous formulae we obtain the following evolution equation

\[\partial_t \psi - \text{div}(A(x) \nabla \psi) = f - 2 s \lambda \xi A(x) \nabla \eta_0 \cdot \nabla \psi - s \lambda^2 \xi \psi \sigma + s^2 \lambda^2 \xi^2 \psi \sigma - s \lambda \xi \psi \text{div}(A(x) \nabla \eta_0) - s \psi \partial_t \alpha.\]

(29)

Similarly, on \(\Gamma_T\) we obtain

\[\partial_t \psi - \text{div}_\Gamma(D(x) \nabla \psi) + \partial_t^A \psi = g - s \psi \partial_t \alpha + s \lambda \xi \partial_t^A \eta_0.\]

(30)
Extending the corresponding decomposition in [28], we rewrite the equations (29) and (30) as

\[ M_1 \psi + M_2 \psi = \tilde{f} \text{ in } \Omega_T, \quad N_1 \psi + N_2 \psi = g \text{ on } \Gamma_T, \quad (31) \]

where

\[ M_1 \psi = 2s\lambda^2 \xi \psi + 2s\lambda \xi A(x) \nabla \eta^0 \cdot \nabla \psi + \partial_\nu \psi = M_{1,1} \psi + M_{1,2} \psi + M_{1,3} \psi, \]
\[ M_2 \psi = -s^2 \lambda^2 \xi^2 \psi \sigma - \text{div}(A(x) \nabla \psi) + s\psi \partial_\nu \alpha = M_{2,1} \psi + M_{2,2} \psi + M_{2,3} \psi, \]
\[ N_1 \psi = \partial_\nu \psi - s\lambda \xi \xi \partial_\nu^A \eta^0 = N_{1,1} \psi + N_{1,2} \psi, \]
\[ N_2 \psi = -\text{div}_T(D(x) \nabla \psi) + s\psi \partial_\nu \alpha + \partial_\nu^A \psi = N_{2,1} \psi + N_{2,2} \psi + N_{2,3} \psi, \]
\[ \tilde{f} = f - s\lambda \xi \psi \text{div}(A(x) \nabla \eta^0) + s\lambda^2 \psi \sigma. \]

By taking \( \| \cdot \|_{L^2(\Omega_T)} \) and \( \| \cdot \|_{L^2(\Gamma_T)} \) in the equations (31) and adding the resulting identities, we obtain

\[ \| \tilde{f} \|^2_{L^2(\Omega_T)} + \| g \|^2_{L^2(\Gamma_T)} = \| M_1 \|^2_{L^2(\Omega_T)} + \| M_2 \|^2_{L^2(\Omega_T)} + \| N_1 \|^2_{L^2(\Gamma_T)} \]
\[ + \| N_2 \|^2_{L^2(\Gamma_T)} + 2 \sum_{i,j=1}^N \langle M_{i,1} \psi, M_{2,j} \psi \rangle_{L^2(\Omega_T)} + 2 \sum_{i,j=1}^N \langle N_{i,1} \psi, N_{2,j} \psi \rangle_{L^2(\Gamma_T)}. \]

**Step 2. Estimating the mixed terms from below.** We will use the following estimates on \( \Omega \) in the sequel,

\[ |\nabla \alpha| \leq C \lambda, \quad |\partial_\nu \alpha| \leq C \xi^2, \quad |\partial_\nu \xi| \leq C \xi^2. \]

**Step 2a.** The first term is negative

\[ \langle M_{1,1} \psi, M_{2,1} \psi \rangle_{L^2(\Omega_T)} = -2s^3 \lambda^4 \int_{\Omega_T} \sigma^2 \xi^3 \psi^2 \, dx \, dt. \]

By integration by parts and (27), we obtain

\[ \langle M_{1,2} \psi, M_{2,1} \psi \rangle_{L^2(\Omega_T)} = -s^3 \lambda^3 \int_{\Omega_T} \xi^3 \sigma A(x) \nabla \eta^0 \cdot \nabla (\psi^2) \, dx \, dt. \]
\[ = s^3 \lambda^3 \int_{\Omega_T} \text{div}(\xi^3 \sigma A(x) \nabla \eta^0) \psi^2 \, dx \, dt - s^3 \lambda^3 \int_{\Gamma_T} \xi^3 \sigma \partial_\nu^A \eta^0 \psi^2 \, dS \, dt \]
\[ = 3s^3 \lambda^4 \int_{\Omega_T} \sigma^2 \xi^3 \psi^2 \, dx \, dt + s^3 \lambda^3 \int_{\Omega_T} \xi^3 (\nabla (\sigma \cdot A(x) \nabla \eta^0)) \psi^2 \, dx \, dt \]
\[ + s^3 \lambda^3 \int_{\Omega_T} \xi^3 \sigma \text{div}(A(x) \nabla \eta^0) \psi^2 \, dx \, dt - s^3 \lambda^3 \int_{\Gamma_T} \xi^3 \sigma \partial_\nu^A \eta^0 \psi^2 \, dS \, dt. \]

Using the fact that \( \nabla \eta^0 \neq 0 \) on \( \Omega \setminus \Omega' \), (26) and (18), we obtain

\[ \langle M_{1,1} \psi, M_{2,1} \psi \rangle_{L^2(\Omega_T)} + \langle M_{1,2} \psi, M_{2,1} \psi \rangle_{L^2(\Omega_T)} \]
\[ \geq C s^3 \lambda^4 \int_{\Omega_T} \xi^3 \psi^2 \, dx \, dt - C s^3 \lambda^4 \int_{(0,T) \times \Omega'} \xi^3 \psi^2 \, dx \, dt - C s^3 \lambda^3 \int_{\Gamma_T} \xi^3 \sigma \partial_\nu \eta^0 \psi^2 \, dS \, dt \]
\[ \geq C s^3 \lambda^4 \int_{\Omega_T} \xi^3 \psi^2 \, dx \, dt - C s^3 \lambda^4 \int_{(0,T) \times \Omega'} \xi^3 \psi^2 \, dx \, dt + C s^3 \lambda^3 \int_{\Gamma_T} \xi^3 \psi^2 \, dS \, dt. \]

After integrating by parts in time and using (26) with (33), we obtain

\[ \langle M_{1,3} \psi, M_{2,1} \psi \rangle_{L^2(\Omega_T)} = -\frac{1}{2} s^2 \lambda^2 \int_{\Omega_T} \sigma \xi^2 \partial_\nu (\psi^2) \, dx \, dt = s^2 \lambda^2 \int_{\Omega_T} \sigma \partial_\nu \xi \psi^2 \, dx \, dt \]
\[
\geq -Cs^2 \lambda^2 \int_{\Omega_T} \xi^3 \psi^2 \, dx \, dt,
\]
since \(\psi\) vanishes at \(t = 0\) and \(t = T\).

**Step 2b.** Integration by parts and (5) yield
\[
\langle M_{1,1}, M_{2,2} \rangle_{L^2(\Omega_T)} = -2s\lambda^2 \int_{\Omega_T} \sigma \xi \psi \text{div}(A(x) \nabla \psi) \, dx \, dt
\]
\[
= 2s\lambda^2 \int_{\Omega_T} \nabla(\sigma \xi \psi) \cdot A(x) \nabla \psi \, dx \, dt - 2s\lambda^2 \int_{\Gamma_T} \xi \psi \partial_n^A \psi \, dS \, dt
\]
\[
= 2s\lambda^2 \int_{\Omega_T} \sigma \xi A(x) \nabla \psi \cdot \nabla \psi \, dx \, dt + 2s\lambda^2 \int_{\Omega_T} \xi \psi \nabla \sigma \cdot A(x) \nabla \psi \, dx \, dt
\]
\[
+ 2s\lambda^3 \int_{\Omega_T} \sigma \xi \psi \nabla \eta^0 \cdot A(x) \nabla \psi \, dx \, dt - 2s\lambda^2 \int_{\Gamma_T} \xi \psi \partial_n^A \psi \, dS \, dt
\]
\[
\geq 2s\lambda^2 \int_{\Omega_T} \sigma \xi A(x) \nabla \psi \cdot \nabla \psi \, dx \, dt - Cs^2 \lambda^4 \int_{\Omega_T} \xi^2 \psi^2 \, dx \, dt
\]
\[
- C \int_{\Omega_T} (s\xi + \lambda^2) |\nabla \psi|^2 \, dx \, dt - 2s\lambda^2 \int_{\Gamma_T} \sigma \xi \psi \partial_n^A \psi \, dS \, dt,
\]
where we employed Cauchy-Schwarz inequality for the terms in the middle as in [28] and (26). Using integration by parts and \(\partial_i(a_{ij} \partial_j \psi) = a_{ij} \partial_i \partial_j \psi + \partial_i(a_{ij}) \partial_j \psi\), with help of (27) the next addend becomes

\[
\langle M_{1,2}, M_{2,2} \rangle_{L^2(\Omega_T)} = -2s\lambda \int_{\Omega_T} \xi (\nabla \eta^0 \cdot A(x) \nabla \psi) \text{div}(A(x) \nabla \psi) \, dx \, dt
\]
\[
= -2s\lambda \int_{\Omega_T} \sum_{i,j=1}^N \sum_{k,l=1}^N \xi a_{ij} a_{kl} (\partial_k \eta^0) (\partial_l \psi) (\partial_i \partial_j \psi) \, dx \, dt
\]
\[
- 2s\lambda \int_{\Omega_T} \xi \sum_{i,j=1}^N \sum_{k,l=1}^N a_{kl} (\partial_k \eta^0) (\partial_i \psi) (a_{ij}) \partial_l \psi \, dx \, dt
\]
\[
= 2s\lambda^2 \int_{\Omega_T} \sum_{i,j=1}^N \sum_{k,l=1}^N (\partial_i \eta^0) \xi a_{ij} a_{kl} (\partial_k \eta^0) (\partial_l \psi) (\partial_j \psi) \, dx \, dt
\]
\[
+ 2s\lambda \int_{\Omega_T} \sum_{i,j=1}^N \sum_{k,l=1}^N \xi \partial_i (a_{ij} a_{kl} \partial_k \eta^0) (\partial_l \psi) (\partial_j \psi) \, dx \, dt
\]
\[
+ 2s\lambda \int_{\Omega_T} \sum_{i,j=1}^N \sum_{k,l=1}^N \xi a_{ij} a_{kl} \partial_k \eta^0 (\partial_i \partial_l \psi) (\partial_j \psi) \, dx \, dt
\]
\[
- 2s\lambda \int_{\Gamma_T} \sum_{k,l=1}^N \xi a_{kl} (\partial_k \eta^0) (\partial_l \psi) \partial_n^A \psi \, dS \, dt
\]
\[
- 2s\lambda \int_{\Omega_T} \xi \sum_{i,j=1}^N \sum_{k,l=1}^N a_{kl} (\partial_k \eta^0) (\partial_i \psi) (a_{ij}) \partial_j \psi \, dx \, dt
\]
\[
= 2s\lambda^2 \int_{\Omega_T} \xi |\nabla \eta^0 \cdot A(x) \nabla \psi|^2 \, dx \, dt.
\]
\[ + 2s\lambda \int_{\Omega_T} \sum_{i,j=1}^{N} \sum_{k,l=1}^{N} \xi \partial_i(a_{ij}a_{kl}\partial_k\eta^0)(\partial_l\psi)(\partial_j\psi) \, dx \, dt \]

\[ + 2s\lambda \int_{\Omega_T} \sum_{i,j=1}^{N} \sum_{k,l=1}^{N} \xi a_{ij}a_{kl}(\partial_k\eta^0)(\partial_l\psi)(\partial_j\psi) \, dx \, dt \]

\[ - 2s\lambda \int_{\Gamma_T} \xi(\partial_{\nu\eta^0})(\partial_{\nu\psi})^2 \, dS \, dt \]

\[ - 2s\lambda \int_{\Omega_T} \xi \sum_{i,j=1}^{N} \sum_{k,l=1}^{N} a_{kl}(\partial_k\eta^0)(\partial_l\psi)(\partial_i(a_{ij})\partial_j\psi) \, dx \, dt \]

\[ = D_1 + D_2 + D_3 + D_4 + D_5. \]

Observe that the first term \( D_1 \) is nonnegative. Similarly to previous integration by parts, we obtain

\[ D_3 = s\lambda \int_{\Omega_T} \sum_{i,j=1}^{N} \sum_{k,l=1}^{N} \xi a_{ij}a_{kl}(\partial_k\eta^0)(\partial_l(\partial_i\psi)(\partial_j\psi)) \, dx \, dt \]

\[ = -s\lambda^2 \int_{\Omega_T} \xi \sum_{i,j=1}^{N} a_{ij}(\partial_i\psi)(\partial_j\psi) \, dx \, dt \]

\[ - s\lambda \int_{\Omega_T} \xi \sum_{i,j=1}^{N} \sum_{k,l=1}^{N} \partial_l(a_{ij}a_{kl}\partial_k\eta^0)(\partial_l\psi)(\partial_j\psi) \, dx \, dt \]

\[ + s\lambda \int_{\Gamma_T} \xi(\partial_{\nu\eta^0})(\partial_{\nu\psi})^2 \, dS \, dt \]

\[ = -s\lambda^2 \int_{\Omega_T} \sigma \xi A(x)\nabla\psi \cdot \nabla\psi \, dx \, dt \]

\[ - s\lambda \int_{\Omega_T} \xi \sum_{i,j=1}^{N} \sum_{k,l=1}^{N} \partial_l(a_{ij}a_{kl}\partial_k\eta^0)(\partial_l\psi)(\partial_j\psi) \, dx \, dt \]

\[ + s\lambda \int_{\Gamma_T} \xi(\partial_{\nu\eta^0})(A(x)\nu \cdot \nu)(A(x)\nabla\psi \cdot \nabla\psi) \, dS \, dt, \]

where we employed \( \partial_{\nu\eta^0} = (\partial_{\nu\eta^0})(A\nu \cdot \nu) \), since \( \eta^0|_{\Gamma} = 0 \). Then, using the symmetry of \( A \) and (21), we obtain

\[ \langle M_{1,2}\psi, M_{2,2}\psi \rangle_{L^2(\Omega_T)} \]

\[ \geq -2s\lambda \int_{\Gamma_T} \xi(\partial_{\nu\eta^0})(\partial_{\nu\psi})^2 \, dS \, dt + s\lambda \int_{\Gamma_T} \xi(\partial_{\nu\eta^0})(A(x)\nu \cdot \nu)(A(x)\nabla\psi \cdot \nabla\psi) \, dS \, dt \]

\[ - C\lambda \int_{\Omega_T} \xi |\nabla\psi|^2 \, dx \, dt - s\lambda^2 \int_{\Omega_T} \sigma \xi A(x)\nabla\psi \cdot \nabla\psi \, dx \, dt \]

\[ \geq -s\lambda \int_{\Gamma_T} \xi(\partial_{\nu\eta^0})(\partial_{\nu\psi})^2 \, dS \, dt + s\lambda \int_{\Gamma_T} \xi(\partial_{\nu\eta^0})(A(x)\nu \cdot \nu)(A(x)\nabla\psi \cdot \nabla\psi) \, dS \, dt \]

\[ - C\lambda \int_{\Omega_T} \xi |\nabla\psi|^2 \, dx \, dt - s\lambda^2 \int_{\Omega_T} \sigma \xi A(x)\nabla\psi \cdot \nabla\psi \, dx \, dt \]
Combining this with (35), we derive

\[-s\lambda \int_{\Gamma_T} \xi (\partial_s \eta^0)(\partial_s^3 \psi)^2 \, dS \, dt.\]  

(35)

Next we estimate \( J \) with help of (17) and (25) as follows

\[ J = s\lambda \int_{\Gamma_T} \xi (\partial_s \eta^0)[- (\partial_s^4 \psi)^2 + |A^2 \nu|^2 |A^2 \nabla \psi|^2] \, dS \, dt \]

\[ = s\lambda \int_{\Gamma_T} \xi (\partial_s \eta^0)[|A^2 \nu|^2 |A^2 \nabla \psi|^2 - (A \nabla \psi \cdot \nu)^2] \, dS \, dt \]

\[ \geq s\lambda \int_{\Gamma_T} \xi (\partial_s \eta^0)|A^2 \nabla \psi|^2 \, dS \, dt \]

\[ \geq C s\lambda \int_{\Gamma_T} \xi (\partial_s \eta^0)|\nabla \psi|^2 \, dS \, dt. \]

Combining this with (35), we derive

\[ \langle M_{1,2} \psi, M_{2,2} \psi \rangle_{L^2(\Omega_T)} \geq -s\lambda \int_{\Gamma_T} \xi (\partial_s \eta^0)(\partial_s^4 \psi)^2 \, dS \, dt \]

\[ + C s\lambda \int_{\Gamma_T} \xi (\partial_s \eta^0)|\nabla \psi|^2 \, dS \, dt \]

\[ - C s\lambda \int_{\Gamma_T} \xi |\nabla \psi|^2 \, dx \, dt - s\lambda^2 \int_{\Omega_T} \sigma \xi A(x) \nabla \psi \cdot \nabla \psi \, dx \, dt. \]

The last term cancels with the one from (34). Integration by parts once again, we obtain

\[ \langle M_{1,3} \psi, M_{2,2} \psi \rangle_{L^2(\Omega_T)} = - \int_{\Omega_T} \partial_t \psi \text{div}(A(x) \nabla \psi) \, dx \, dt \]

\[ = \int_{\Omega_T} A(x) \nabla \psi \cdot \partial_t (\nabla \psi) - \int_{\Gamma_T} \partial_t \psi \partial_s^4 \psi \, dS \, dt \]

\[ = \frac{1}{2} \int_{\Omega_T} \frac{d}{dt} (A(x) \nabla \psi \cdot \nabla \psi) \, dx \, dt - \int_{\Gamma_T} \partial_t \psi \partial_s^4 \psi \, dS \, dt \]

\[ = - \int_{\Gamma_T} \partial_t \psi \partial_s^4 \psi \, dS \, dt, \]  

(36)

where we used the symmetry of \( A \) and the fact that \( \nabla \psi \) vanishes at \( t = 0 \) and \( t = T \).

**Step 2c.** Using (33), we estimate

\[ \langle M_{1,1} \psi, M_{2,3} \psi \rangle_{L^2(\Omega_T)} = 2 s^2 \lambda^2 \int_{\Omega_T} \sigma \xi (\partial_t \alpha) \psi^2 \, dx \, dt \geq -C s^2 \lambda^2 \int_{\Omega_T} \xi^3 \psi^2 \, dx \, dt. \]

Integration by parts, (27) and (33) imply

\[ \langle M_{1,2} \psi, M_{2,3} \psi \rangle_{L^2(\Omega_T)} = s^2 \lambda \int_{\Omega_T} (\partial_t \alpha) \xi A(x) \nabla \eta^0 \cdot \nabla (\psi^2) \, dx \, dt \]

\[ = s^2 \lambda \int_{\Gamma_T} (\partial_t \alpha) \xi \partial_s^4 \eta^0 \psi^2 \, dS \, dt - s^2 \lambda \int_{\Omega_T} \text{div}(\xi \partial_t \alpha A(x) \nabla \eta^0) \psi^2 \, dx \, dt \]

\[ = s^2 \lambda \int_{\Gamma_T} (\partial_t \alpha) \xi \partial_s^4 \eta^0 \psi^2 \, dS \, dt - s^2 \lambda \int_{\Omega_T} \nabla (\partial_t \alpha) \cdot A(x) \nabla \eta^0 \xi \psi^2 \, dx \, dt \]

\[ - s^2 \lambda^2 \int_{\Omega_T} (\partial_t \alpha) \xi \sigma \psi^2 \, dx \, dt - s^2 \lambda \int_{\Omega_T} (\partial_t \alpha) \xi \text{div}(A(x) \nabla \eta^0) \psi^2 \, dx \, dt \]
\[ \geq -Cs^2 \lambda \int_{\Gamma_T} \xi^3 \psi^2 \, dS \, dt - Cs^2 \lambda^2 \int_{\Omega_T} \xi^3 \psi^2 \, dx \, dt. \]

Since \( \psi(0) = \psi(T) = 0 \) and \( |\partial_t^2 \alpha| \leq C \xi^3 \), integration by parts yields

\[ (M_{1,3}, M_{2,3})_{L^2(\Omega_T)} = \frac{s}{2} \int_{\Omega_T} \partial_t \alpha \partial_t (\psi^2) \, dx \, dt = \frac{s}{2} \int_{\Omega_T} \partial_t^2 \alpha \psi^2 \, dx \, dt \]

(37)

\[ \geq -Cs \int_{\Omega_T} \xi^3 \psi^2 \, dx \, dt. \]

**Step 2d. Estimating boundary terms.** For the boundary terms \( N_1 \) and \( N_2 \), we will use the divergence formula \( (8) \). Using \( (8) \), we have

\[ \langle N_{1,1}, N_{2,1} \rangle_{L^2(\Gamma_T)} = - \int_{\Gamma_T} \text{div}_T (D(x) \nabla \psi) \partial_t \psi \, dS \, dt \]

\[ = \int_{\Gamma_T} \langle D(x) \nabla \psi, \partial_t (\nabla \psi) \rangle \, dS \, dt \]

\[ = \frac{1}{2} \int_{\Gamma_T} \frac{d}{dt} \langle D(x) \nabla \psi, \nabla \psi \rangle \, dS \, dt = 0, \]

by means of \( \psi(0) = \psi(T) = 0 \). Since \( \xi(t, \cdot) \) is constant on \( \Gamma \), \( (8) \) and \( (6) \) yield

\[ \langle N_{1,2}, N_{2,2} \rangle_{L^2(\Gamma_T)} = s \lambda \int_{\Gamma_T} \partial_t^3 \eta^0 \xi \psi \partial_t (D(x) \nabla \psi) \, dS \, dt \]

\[ = s \lambda \int_{\Gamma_T} \xi \psi (D(x) \nabla \psi + \nabla (\partial_t A \eta^0 \xi)) \, dS \, dt \]

\[ = s \lambda \int_{\Gamma_T} \xi \psi D(x) \nabla \psi \, dS \, dt - s \lambda \int_{\Gamma_T} (\partial_t A \eta^0 \xi) \psi \, dS \, dt \]

\[ \geq -s \lambda \int_{\Gamma_T} \xi \psi D(x) \nabla \psi \, dS \, dt - s \lambda \int_{\Gamma_T} (\partial_t A \eta^0 \xi) \psi \, dS \, dt \]

\[ \geq -s \lambda \int_{\Gamma_T} \xi \psi D(x) \nabla \psi \, dS \, dt - C \lambda \int_{\Gamma_T} \partial_t \eta^0 \xi \, dS \, dt, \]

where we employed \( (18) \). The next terms are estimated by

\[ \langle N_{1,1}, N_{2,2} \rangle_{L^2(\Gamma_T)} = \frac{s}{2} \int_{\Gamma_T} \partial_t \alpha \partial_t (\psi^2) \, dS \, dt \]

\[ \geq -Cs \int_{\Gamma_T} \xi^3 \psi^2 \, dS \, dt, \]

and by \( (33) \) we have

\[ \langle N_{1,2}, N_{2,2} \rangle_{L^2(\Gamma_T)} = -s \lambda \int_{\Gamma_T} \partial_t (\psi^2) \, dS \, dt \]

\[ \geq -Cs^2 \lambda \int_{\Gamma_T} \xi^3 \psi^2 \, dS \, dt. \]

Finally, the term

\[ \langle N_{1,1}, N_{2,3} \rangle_{L^2(\Gamma_T)} = \int_{\Gamma_T} \partial_t \psi \partial_t \eta \psi \, dS \, dt, \]

cancels with the one from \( (36) \), and

\[ \langle N_{1,2}, N_{2,3} \rangle_{L^2(\Gamma_T)} = -s \lambda \int_{\Gamma_T} \xi \partial_t \eta \partial_t \psi \, dS \, dt. \]
Step 3. The transformed estimate. By regrouping final estimates in the previous steps and increasing $\lambda_1$ and $s_1$ to absorb lower order terms, we derive

$$\begin{align*}
\sum_{i,j=1}^{N} \langle M_{1,i}\psi, M_{2,j}\psi \rangle_{L^2(\Omega_T)} + \sum_{i,j=1}^{N} \langle N_{1,i}\psi, N_{2,j}\psi \rangle_{L^2(\Gamma_T)} \\
\geq C s^3 \lambda^4 \int_{\Omega_T} \xi^3 \psi^2 \, dx \, dt - C s^3 \lambda^4 \int_{(0,T) \times \omega'} \xi^3 \psi^2 \, dx \, dt \\
+ C s^3 \lambda^3 \int_{\Gamma_T} \xi |\nabla \psi|^2 \, dx \, dt + C s \lambda^2 \int_{\Omega_T} \xi |\nabla \psi|^2 \, dx \, dt \\
- C s \lambda^2 \int_{(0,T) \times \omega'} \xi |\nabla \psi|^2 \, dx \, dt - C s \lambda \int_{\Gamma_T} \xi |\nabla \psi|^2 \, dx \, dt \\
- s \lambda \int_{\Gamma_T} \xi |\psi| \langle \partial_\nu \eta^0 \rangle D(x) \nabla \psi \rangle \, dx \, dt + C s \lambda \int_{\Gamma_T} \xi |\nabla \psi|^2 \, dx \, dt \\
- s \lambda \int_{\Gamma_T} \xi |\nabla (\partial_\nu^A \eta^0) \rangle D(x) \nabla \psi \rangle \, dx \, dt.
\end{align*}$$

We combine this estimate with (32) and absorb lower order terms resulting from $\tilde{f}$ and $\tilde{g}$ to left-hand side by increasing $\lambda_1$ and $s_1$. Using (16) and (18), we deduce

$$\begin{align*}
&\|M_1\psi\|_{L^2(\Omega_T)}^2 + \|M_2\psi\|_{L^2(\Omega_T)}^2 + \|N_1\psi\|_{L^2(\Gamma_T)}^2 + \|N_2\psi\|_{L^2(\Gamma_T)}^2 \\
&\quad + s \lambda^4 \int_{\Omega_T} \xi^3 \psi^2 \, dx \, dt + s \lambda^2 \int_{\Omega_T} \xi |\nabla \psi|^2 \, dx \, dt + s^3 \lambda^3 \int_{\Gamma_T} \xi^3 \psi^2 \, dx \, dt \\
&\quad + s \lambda \int_{\Gamma_T} \xi |\nabla \psi|^2 \, dx \, dt \\
&\leq C \int_{\Omega_T} e^{-2s\alpha} |\partial_t z - \text{div}(A(x) \nabla z)|^2 \, dx \, dt \\
&\quad + C \int_{\Gamma_T} e^{-2s\alpha} |\partial_t z - \text{div}_\Gamma(D(x) \nabla \Gamma z)| + \partial_\nu^A |z|^2 \, dS \, dt \\
&\quad + C s^3 \lambda^4 \int_{\omega'_T} \xi^3 \psi^2 \, dx \, dt + C s \lambda^2 \int_{\omega'_T} \xi |\nabla \psi|^2 \, dx \, dt \\
&\quad + C s \lambda^2 \int_{\Gamma_T} (\partial_\nu \eta^0)^2 \xi |\partial_\nu \lambda^0| \, dS \, dt + C s \lambda \int_{\Gamma_T} \xi |\partial_\nu \psi| \, dS \, dt \\
&\quad + C s \lambda \int_{\Gamma_T} \xi |\psi| \phi_{\Gamma} (\partial_\nu \psi)^0 \rangle |D(x) \nabla \psi \rangle \, dx \, dt + C s \lambda \int_{\Gamma_T} \xi |\partial_\nu \psi| \, dS \, dt.
\end{align*}$$

(38)

By Young's inequality, $I_1$ can be estimated by

$$\begin{align*}
I_1 &\leq C \int_{\Gamma_T} s^2 \lambda^3 \xi^3 \psi^2 \, dx \, dt + \lambda \int_{\Gamma_T} \xi |\partial_\nu \psi|^2 \, dx \, dt \\
&\leq C \left(s^2 \lambda^3 \int_{\Gamma_T} \xi^3 \psi^2 \, dx \, dt + \lambda \int_{\Gamma_T} \xi |\partial_\nu \psi|^2 \, dx \, dt \right).
\end{align*}$$

(39)
Choosing $s_1$ large enough, we can then control (39) by the left-hand side of (38). In similar way, one can absorb $I_4$ and also $I_3$, since

$$I_3 \leq C \left( \int_{\Gamma_T} \xi |\nabla_T \psi|^2 \, dS \, dt + s^2 \lambda^2 \int_{\Gamma_T} \xi \psi^2 \, dS \, dt \right).$$

(40)

Using the ellipticity of $D$ with divergence formula (8) and the fact that $\xi(t, \cdot)$ is constant on $\Gamma$, the integral $I_2$ can be bounded by

$$I_2 \leq Cs\lambda \int_{\Gamma_T} \xi |\nabla_T \psi|^2 \, dS \, dt \leq Cs\lambda \int_0^T \xi \int_{\Gamma} \langle D(x) \nabla_T \psi, \nabla_T \psi \rangle \, dS \, dt$$

$$\leq C \int_0^T \int_{\Gamma} (s^{-1/2} \xi^{-1/2} |\text{div}_T(D(x) \nabla_T \psi)|)(s^{3/2} \lambda \xi^{3/2} |\psi|) \, dS \, dt$$

$$\leq s^{-1} \int_{\Gamma_T} \xi^{-1} |\text{div}_T(D(x) \nabla_T \psi)|^2 \, dS \, dt + Cs^3 \lambda^2 \int_{\Gamma_T} \xi \psi^2 \, dS \, dt.$$  

(41)

The second addend in (41) can be absorbed by the left-hand side of (38) by choosing $\lambda_1$ sufficiently large. Thus, we arrive at

$$\|M_1 \psi\|_{L^2(\Omega_T)}^2 + \|M_2 \psi\|_{L^2(\Omega_T)}^2 + \|N_1 \psi\|_{L^2(\Gamma_T)}^2 + \|N_2 \psi\|_{L^2(\Gamma_T)}^2$$

$$+ s^3 \lambda^4 \int_{\Omega_T} \xi^3 \psi^2 \, dx \, dt + s\lambda^2 \int_{\Omega_T} \xi |\nabla \psi|^2 \, dx \, dt + s^3 \lambda^3 \int_{\Gamma_T} \xi^3 \psi^2 \, dS \, dt$$

$$+ s\lambda \int_{\Gamma_T} \xi (\text{div}_T \psi)^2 + (\partial^A \psi)^2) \, dS \, dt$$

$$\leq C \int_{\Gamma_T} e^{-2s\alpha} |\partial_t \psi - \text{div}(A(x) \nabla z)|^2 \, dx \, dt$$

$$+ C \int_{\Gamma_T} e^{-2s\alpha} |\partial_t \psi - \text{div}_T(D(x) \nabla_T z_T) + \partial^A \psi|^2 \, dS \, dt + Cs^3 \lambda^4 \int_{\omega_T} \xi^3 \psi^2 \, dx \, dt$$

$$+ Cs^2 \lambda^2 \int_{\omega_T} \xi |\nabla \psi|^2 \, dx \, dt + s^{-1} \int_{\Gamma_T} \xi^{-1} |\text{div}_T(D(x) \nabla_T \psi)|^2 \, dS \, dt.$$  

(42)

To transmit the last term in (42) to the left, we observe first that

$$-\text{div}_T(D(x) \nabla_T \psi) = N_2 \psi - s \psi \partial_A \nu - \partial^A \psi.$$  

Combined with (33), this identity yields

$$I := s^{-1} \int_{\Gamma_T} \xi^{-1} |\text{div}_T(D(x) \nabla_T \psi)|^2 \, dS \, dt$$

$$\leq \frac{1}{2} \|N_2 \psi\|_{L^2(\Gamma_T)}^2 + Cs \int_{\Gamma_T} \xi^3 \psi^2 \, dS \, dt + C \int_{\Gamma_T} \xi (\partial^A \psi)^2 \, dS \, dt,$$  

(43)

for sufficiently large $s_1$. Choosing $\lambda_1$ and $s_1$ large enough so that (42) becomes

$$\|M_1 \psi\|_{L^2(\Omega_T)}^2 + \|M_2 \psi\|_{L^2(\Omega_T)}^2 + \|N_1 \psi\|_{L^2(\Gamma_T)}^2 + \|N_2 \psi\|_{L^2(\Gamma_T)}^2$$

$$+ s^3 \lambda^4 \int_{\Omega_T} \xi^3 \psi^2 \, dx \, dt + s\lambda^2 \int_{\Omega_T} \xi |\nabla \psi|^2 \, dx \, dt + s^3 \lambda^3 \int_{\Gamma_T} \xi^3 \psi^2 \, dS \, dt$$

$$+ s\lambda \int_{\Gamma_T} \xi (\text{div}_T \psi)^2 \, dS \, dt + s\lambda \int_{\Gamma_T} \xi (\partial^A \psi)^2 \, dS \, dt$$

$$\leq C \int_{\Omega_T} e^{-2s\alpha} |\partial_t \psi - \text{div}(A(x) \nabla z)|^2 \, dx \, dt$$

$$+ C \int_{\Gamma_T} e^{-2s\alpha} |\partial_t \psi - \text{div}_T(D(x) \nabla_T z_T) + \partial^A \psi|^2 \, dS \, dt.$$
we may apply Carleman estimate to (45) to obtain
\[ + C_s^3 \lambda^4 \int_{\omega_T^0} \xi^3 \psi^2 \, dx \, dt + C_s \lambda^2 \int_{\omega_T^0} \xi|\nabla \psi|^2 \, dx \, dt. \] (44)

The rest of the proof follows from the same strategy as in [28].

Remark 3. By means of the transformation \( t' = T^{-1}(T - t_0)t + t_0, \) the Carleman estimate (24) remains true replacing \( t(T - t) \) by \( (t - t_0)(T - t) \) in the weight functions \( \alpha \) and \( \xi \) defined by (22), and integrating on \( (t_0, T) \) instead of \((0, T), \) for \( t_0 \in (0, T). \) In that case, we adopt the same notation for \( \alpha \) and \( \xi, \) and we further denote
\[ \Omega_{t_0, T} := (t_0, T) \times \Omega, \quad \Gamma_{t_0, T} := (t_0, T) \times \Gamma, \quad \omega_{t_0, T} := (t_0, T) \times \omega. \]

3. Global Lipschitz stability for an inverse source problem. The object of this section is to recover the source term \( F = (F, G) \) in (1) belonging to \( S(C_0) \) defined in (9), from a single measurement \( Y(T_0, \cdot) = (y, y_T)|_{t=T_0} \) and some extra partial observation on the first component of the solution \( y|_{\omega_{t_0, T}}. \) We notice here that the set of admissible source terms \( S(C_0) \) is necessarily involved, since the uniqueness for inverse source problems falls into default in the general case (see e.g., [20, Commentary 6.6]).

Proof of Theorem 1.1. Following Remark 3, we may apply Carleman estimate (24) on the interval \((t_0, T)\) instead of \((0, T).\) Throughout the proof, \( C \) will denote a generic constant which is independent of \( Y. \) It may vary even from line to line. The terms appearing in (10) are well defined, indeed, as mentioned in Section 2, we have then \( Y := (y, y_T) \in \mathbb{E}_1(t_0, T). \) The functions \( z = \partial_t y \) and \( z_T = \partial_t y_T, \) where \((y, y_T)\) is the solution of (1), are solutions of the system
\[
\begin{aligned}
&\partial_t z - \text{div}(A(x)\nabla z) + B(x) \cdot \nabla z + p(x)z = F_t(t, x), \\
&\partial_t z_T - \text{div}(D(x)\nabla_T z_T) + \partial_t^3 z + (b(x), \nabla_T z_T)_T + q(x)z_T = G_t(t, x),
\end{aligned}
\] (45)
and we have
\[
\begin{aligned}
z(T_0) &= -\text{div}(A\nabla y(T_0)) + B \cdot \nabla y(T_0) + py(T_0) = F(T_0), \\
\partial_t y(T_0) &= \text{div}(D\nabla y(T_0)) + \partial_t^3 y(T_0) + (b, \nabla y(T_0))_T + qy(T_0) = G(T_0).
\end{aligned}
\] (46)
(47)
Since \((F, G) \in H^1(0, T; L^2),\) by Proposition 2 we have \((z, z_T) \in \mathbb{E}_1(t_0, T).\) Hence, we may apply Carleman estimate to (45) to obtain
\[
\int_{\Omega_{t_0, T}} \left( \frac{1}{s^2} |\partial_t z|^2 + s^3 \lambda^3 |z|^2 \right) e^{-2s\alpha} \, dx \, dt + \int_{\Gamma_{t_0, T}} \left( \frac{1}{s^2} |\partial_t z_T|^2 + s^3 \lambda^3 |z_T|^2 \right) e^{-2s\alpha} \, dS \, dt
\leq C_s^3 \lambda^4 \int_{\omega_{t_0, T}} e^{-2s\alpha} |z|^2 \, dx \, dt + C \int_{\Omega_{t_0, T}} e^{-2s\alpha} |F_t|^2 \, dx \, dt + C \int_{\Gamma_{t_0, T}} e^{-2s\alpha} |G_t|^2 \, dS \, dt
\]
(48)
for any \( s > 0 \) large enough. Since \( F = (F, G) \in S(C_0), \) we have
\[
\int_{\Omega_{t_0, T}} \left( \frac{1}{s^2} |\partial_t z|^2 + s^3 \lambda^3 |z|^2 \right) e^{-2s\alpha} \, dx \, dt + \int_{\Gamma_{t_0, T}} \left( \frac{1}{s^2} |\partial_t z_T|^2 + s^3 \lambda^3 |z_T|^2 \right) e^{-2s\alpha} \, dS \, dt
\leq C_s^3 \lambda^4 \int_{\omega_{t_0, T}} e^{-2s\alpha} |z|^2 \, dx \, dt + C \int_{\Omega_{t_0, T}} e^{-2s\alpha} |F(T_0, x)|^2 \, dx \, dt
+ C \int_{\Gamma_{t_0, T}} e^{-2s\alpha} |G(T_0, x)|^2 \, dS \, dt
\]
(49)
From (46)-(47), to estimate the term
\[ \int_{\Omega} |F(T_0, x)|^2 e^{-2s\alpha(T_0, x)} \, dx + \int_{\Gamma} |G(T_0, x)|^2 e^{-2s\alpha(T_0, x)} \, dS, \]
we have to estimate the term
\[ \int_{\Omega} |z(T_0, x)|^2 e^{-2s\alpha(T_0, x)} \, dx + \int_{\Gamma} |z_T(T_0, x)|^2 e^{-2s\alpha(T_0, x)} \, dS. \]

Fix \( x \in \Omega \) and take \( H(t) = \int_{\Omega} |z(t, x)|^2 e^{-2s\alpha(t, x)} \, dx \) for \( t \in (0, T) \).

Since \( \partial_t(z^2 e^{-2s\alpha}) = (2\partial_t z - 2s\partial_t \alpha z^2) e^{-2s\alpha} \in L^2(\Omega_T) \) and \( \lim_{t \to t_0} e^{-2s\alpha(t, x)} = 0 \) for \( x \) in \( \Omega \), we can differentiate \( H \) under the integral sign. We further have
\[
\int_{\Omega} |z(T_0, x)|^2 e^{-2s\alpha(T_0, x)} \, dx = \int_{t_0}^{T_0} \frac{\partial}{\partial t} \left( \int_{\Omega} |z(t, x)|^2 e^{-2s\alpha(t, x)} \, dx \right) \, dt
\]
\[
= \int_{t_0}^{T_0} \int_{\Omega} (2\partial_t y(t, x) \partial_t^2 y(t, x) - 2s(\partial_t \alpha)|\partial_t y(t, x)|^2) e^{-2s\alpha} \, dx \, dt
\]
\[
\leq \int_{t_0}^{T_0} \int_{\Gamma_{t_0, T}} (2|\partial_t y(t, x)| |\partial_t^2 y(t, x)| + C s \xi^2 |\partial_t y(t, x)|^2) e^{-2s\alpha} \, dS \, dt,
\]
where we employed \( |\partial_t \alpha| \leq C \xi^2 \). On the other hand, we have
\[
2|\partial_t y(t, x)| |\partial_t^2 y(t, x)| = 2 \cdot \frac{1}{s \sqrt{\xi}} |\partial_t^2 y(t, x)| s \sqrt{\xi} |\partial_t y(t, x)|
\]
\[
\leq \frac{1}{s^2 \xi} |\partial_t^2 y(t, x)|^2 + s^2 \xi |\partial_t y(t, x)|^2
\]
\[
\leq C \left( \frac{1}{s^2 \xi} |\partial_t^2 y(t, x)|^2 + s^2 \lambda^4 \xi \right) |\partial_t y(t, x)|^2
\]
for large \( \lambda \), using \( \xi \leq C \xi^2 \). Hence,
\[
\int_{\Omega} |z(T_0, x)|^2 e^{-2s\alpha(T_0, x)} \, dx \leq C \int_{t_0}^{T_0} \left( \frac{1}{s^2 \xi} |\partial_t^2 y|^2 + s^2 \lambda^4 \xi \right) e^{-2s\alpha} \, dx \, dt.
\]

Similarly, we have
\[
2|\partial_t y_T(t, x)| |\partial_t^2 y_T(t, x)| = 2 \cdot \frac{1}{s \sqrt{\xi}} |\partial_t^2 y_T(t, x)| s \sqrt{\xi} |\partial_t y_T(t, x)|
\]
\[
\leq \frac{1}{s^2 \xi} |\partial_t^2 y_T(t, x)|^2 + s^2 \xi |\partial_t y_T(t, x)|^2
\]
\[
\leq C \left( \frac{1}{s^2 \xi} |\partial_t^2 y_T(t, x)|^2 + s^2 \lambda^3 \xi \right) |\partial_t y_T(t, x)|^2
\]
and
\[
\int_{\Gamma} |z_T(T_0, x)|^2 e^{-2s\alpha(T_0, x)} \, dS \leq C \int_{t_0}^{T_0} \left( \frac{1}{s^2 \xi} |\partial_t^2 y_T|^2 + s^2 \lambda^3 \xi \right) e^{-2s\alpha} \, dS \, dt
\]
for large \( \lambda \). Adding inequalities (50), (51) and applying (49), we obtain
\[
\int_{\Omega} |z(T_0, x)|^2 e^{-2s\alpha(T_0, x)} \, dx + \int_{\Gamma} |z_T(T_0, x)|^2 e^{-2s\alpha(T_0, x)} \, dS
\]
Combining estimates (53) and (54), we obtain
\begin{equation}
\leq C \int_{\Omega_{T_0},T} \left( \frac{1}{s^2 \xi} |\partial^2_t y|^2 + s^2 \lambda^4 \xi^3 |\partial_t y|^2 \right) e^{-2s\alpha} \, dx \, dt \\
+ C \int_{\Gamma_{T_0},T} \left( \frac{1}{s^2 \xi} |\partial^2_t y|^2 + s^2 \lambda^4 \xi^3 |\partial_t y|^2 \right) e^{-2s\alpha} \, dS \, dt \\
\leq \frac{C}{s} \int_{\Omega_{T_0},T} e^{-2s\alpha} |F(T_0,x)|^2 \, dx \, dt + \frac{C}{s} \int_{\Gamma_{T_0},T} e^{-2s\alpha} |G(T_0,x)|^2 \, dS \, dt \\
+ Cs^2 \lambda^4 \int_{\Omega_{T_0},T} e^{-2s\alpha} \xi^3 |\partial_t y|^2 \, dx \, dt. \tag{52}
\end{equation}

Since the coefficients of $A$ are bounded, $B \in L^\infty(\Omega)^N$ and $p \in L^\infty(\Omega)$, we obtain
\begin{equation}
\int_{\Omega} \left( |\text{div}(A \nabla y(T_0,\cdot))|^2 + |B \cdot \nabla y(T_0,\cdot)|^2 + p^2 |y(T_0,\cdot)|^2 \right) e^{-2s\alpha(T_0,\cdot)} \, dx \\
\leq C \|y(T_0,\cdot)\|^2_{H^2(\Omega)}. \tag{53}
\end{equation}

Analogously, we have
\begin{equation}
\int_{\Gamma} \left( |\text{div}(D \nabla y(T_0,\cdot))|^2 + |b \cdot \nabla y(T_0,\cdot)|^2 + q^2 |y(T_0,\cdot)|^2 \right) e^{-2s\alpha(T_0,\cdot)} \, dS \\
+ \int_{\Gamma} \left| \partial^A_t y(T_0, x) \right|^2 e^{-2s\alpha(T_0,x)} \, dS \\
\leq C \left( \|y(T_0,\cdot)\|^2_{H^2(\Gamma)} + \|\partial^A_t y(T_0,\cdot)\|^2_{L^2(\Gamma)} \right) \\
\leq C \left( \|y(T_0,\cdot)\|^2_{H^2(\Gamma)} + \|y(T_0,\cdot)\|^2_{H^2(\Omega)} \right), \tag{54}
\end{equation}

using $\|\partial^A_t y(T_0,\cdot)\|_{L^2(\Gamma)} \leq C \|y(T_0,\cdot)\|_{H^{2}(\Omega)}$, for some positive constant $C > 0$, which holds by trace theorem since $A \in C(\overline{\Omega}; \mathbb{R}^{N \times N})$ (see Chapter 1, Theorem 8.3 in [27]).

Combining estimates (53) and (54), we obtain
\begin{equation}
\int_{\Omega} \left( |\text{div}(A \nabla y(T_0,\cdot))|^2 + |B \cdot \nabla y(T_0,\cdot)|^2 + p^2 |y(T_0,\cdot)|^2 \right) e^{-2s\alpha(T_0,\cdot)} \, dx \\
+ \int_{\Gamma} \left| \partial^A_t y(T_0, x) \right|^2 e^{-2s\alpha(T_0,x)} \, dS \\
+ \int_{\Gamma} \left( |\text{div}(D \nabla y(T_0,\cdot))|^2 + |b \cdot \nabla y(T_0,\cdot)|^2 + q^2 |y(T_0,\cdot)|^2 \right) e^{-2s\alpha(T_0,\cdot)} \, dS \\
\leq C \left( \|y(T_0,\cdot)\|^2_{H^2(\Omega)} + \|y(T_0,\cdot)\|^2_{H^2(\Gamma)} \right) = C \|Y(T_0,\cdot)\|^2_{H^2}. \tag{55}
\end{equation}

Using (52) and (46)-(47), we deduce
\begin{equation}
\int_{\Omega} e^{-2s\alpha(T_0,x)} |F(T_0,x)|^2 \, dx + \int_{\Gamma} e^{-2s\alpha(T_0,x)} |G(T_0,x)|^2 \, dS \\
\leq \frac{C}{s} \int_{\Omega_{T_0},T} e^{-2s\alpha(t,x)} |F(T_0,x)|^2 \, dx \, dt + \frac{C}{s} \int_{\Gamma_{T_0},T} e^{-2s\alpha(t,x)} |G(T_0,x)|^2 \, dS \, dt \\
+ Cs^2 \lambda^4 \int_{\Omega_{T_0},T} e^{-2s\alpha(t,x)} \xi^3 |\partial_t y|^2 \, dx \, dt + C \|Y(T_0,\cdot)\|^2_{H^2}. \tag{56}
\end{equation}

Since $\alpha(t,x) \geq \alpha(T_0,x)$, for all $(t, x) \in \overline{\Omega}_{T_0,T}$, we have
\begin{equation}
\int_{\Omega_{T_0},T} e^{-2s\alpha(t,x)} |F(T_0,x)|^2 \, dx \, dt + \int_{\Gamma_{T_0},T} e^{-2s\alpha(t,x)} |G(T_0,x)|^2 \, dS \, dt
\end{equation}
\[
\leq (T - t_0) \left( \int_{\Omega} e^{-2s\alpha(T_0, x)} |F(T_0, x)|^2 \, dx + \int_{\Gamma} e^{-2s\alpha(T_0, x)} |G(T_0, x)|^2 \, dS \right).
\]

From (56) and (57), we obtain
\[
\int_{\Omega} e^{-2s\alpha(T_0, x)} |F(T_0, x)|^2 \, dx + \int_{\Gamma} e^{-2s\alpha(T_0, x)} |G(T_0, x)|^2 \, dS
\leq C \left( \int_{\Omega} e^{-2s\alpha(T_0, x)} |F(T_0, x)|^2 \, dx + \int_{\Gamma} e^{-2s\alpha(T_0, x)} |G(T_0, x)|^2 \, dS \right)
+ C \|Y(T_0, \cdot)\|_{\mathcal{H}^2}^2 + C s^2 \lambda^4 \int_{\omega_{t_0, T}} e^{-2s\alpha(t, x)} \xi^3 |\partial_t y|^2 \, dx \, dt,
\]
and then
\[
\left( 1 - \frac{C}{s} \right) \left( \int_{\Omega} e^{-2s\alpha(T_0, x)} |F(T_0, x)|^2 \, dx + \int_{\Gamma} e^{-2s\alpha(T_0, x)} |G(T_0, x)|^2 \, dS \right)
\leq C \|Y(T_0, \cdot)\|_{\mathcal{H}^2}^2 + C s^2 \lambda^4 \int_{\omega_{t_0, T}} e^{-2s\alpha(t, x)} \xi^3 |\partial_t y|^2 \, dx \, dt.
\]

Since \( F = (F, G) \in \mathcal{S}(C_0) \), depending on \( t \geq T_0 \) or \( t \leq T_0 \), we have
\[
|F(t, x)| \leq |F(T_0, x)| + \int_{T_0}^t F_r(\tau, x) \, d\tau \leq C |F(T_0, x)|, \quad \forall (t, x) \in \Omega_T, \quad (59)
\]
\[
|G(t, x)| \leq |G(T_0, x)| + \int_{T_0}^t G_r(\tau, x) \, d\tau \leq C |G(T_0, x)|, \quad \forall (t, x) \in \Gamma_T. \quad (60)
\]

Using (59)-(60) with (58), we derive
\[
\left( 1 - \frac{C}{s} \right) \left( \int_{\Omega_T} e^{-2s\alpha(T_0, x)} |F(t, x)|^2 \, dt \, dx + \int_{\Gamma_T} e^{-2s\alpha(T_0, x)} |G(t, x)|^2 \, dS \, dt \right)
\leq C \|Y(T_0, \cdot)\|_{\mathcal{H}^2}^2 + C s^2 \lambda^4 \int_{\omega_{t_0, T}} e^{-2s\alpha(t, x)} \xi^3 |\partial_t y|^2 \, dx \, dt.
\]

The functions \( x \mapsto e^{-2s\alpha(T_0, x)} \) and \( (t, x) \mapsto e^{-2s\alpha(t, x)} \xi^3(t, x) \) are bounded on \( \Omega \) and \( \Omega_{t_0, T} \), respectively. Fixing \( \lambda, s > 0 \) sufficiently large, we obtain
\[
\|\left( F, G \right)\|_{L^2_{t/x}}^2 \leq C \left( \|Y(T_0, \cdot)\|_{\mathcal{H}^2}^2 + \|\partial_t y\|_{L^2_{\omega_{t_0, T}}}^2 \right).
\]

Thus, the proof of Theorem 1.1 is complete. \( \square \)

**Remark 4.** We emphasize that in our inverse problem, the cases \( T_0 = 0 \) or \( T_0 = T \), where \( T_0 \) is the time of observation, are not considered. In fact, the weight functions used in Carleman estimate blow up as \( t \to 0 \) and as \( t \to T \), and in the proof we used the boundedness of these functions away from 0 and \( T \). Then, it would be of much interest to prove similar results in these cases.

### 3.1. Uniqueness and stability in a particular case.

A particular but interesting case of inverse source problems is when the source terms in (1) are given by
\[
F(t, x) = f(x)r(t, x) \quad \text{for all } (t, x) \in \Omega_T, \quad (61)
\]
\[
G(t, x) = g(x)\tilde{r}(t, x) \quad \text{for all } (t, x) \in \Gamma_T. \quad (62)
\]
Here, the inverse source problem is to determine the couple of \( x \)-dependent sources \((f, g)\), by means of a single measurement \( Y(T_0, \cdot) = (y, y_T)|_{t = T_0} \) and a partial observation \( y|_{\omega_0, \tau} \), provided that the couple of \((t, x)\)-dependent functions \((r, \tilde{r})\) belonging to \( C^{1,0} \) are known and satisfying

\[
\begin{align*}
    r(T_0, x) &\neq 0, \quad x \in \overline{\Omega}, \\
    \tilde{r}(T_0, x) &\neq 0, \quad x \in \Gamma.
\end{align*}
\]

**Remark 5.** Under assumptions \((63)-(64)\), one can check that the source term \((F, G)\) in \((61)-(62)\) belongs to the set of admissible sources \( \mathcal{S}(C_0) \), for some positive constant \( C_0 = C(r, \tilde{r}) \). In fact, by \((63)\) we have

\[
|r(T_0, x)| \geq r_0 > 0 \quad \text{for all} \quad x \in \overline{\Omega},
\]

where \( r_0 = \min_{\overline{\Omega}} |r(T_0, \cdot)| \). Hence,

\[
|F_t(t, x)| = |f(x) r_t(t, x)| \leq (\sup_{\overline{\Omega}} |r_t|)|f(x)| \leq \frac{1}{r_0} (\sup_{\overline{\Omega}} |r_t|)|f(x)||r(T_0, x)| = C_1 |F(T_0, x)|.
\]

In a similar way, we obtain \( |G_t(t, x)| \leq C_2 |G(T_0, x)| \).

Thus, as a consequence of Theorem 1.1 we have the following.

**Proposition 3.** Let \( Y_0 \in L^2 \). Assume that \((r, \tilde{r}) \in C^{1,0}\) satisfies \((63)-(64)\). Then, there exists a constant \( C = C(\Omega, \omega, T, t_0, C_0, \|B\|_\infty, \|p\|_\infty, \|b\|_\infty, \|q\|_\infty, \|r, \tilde{r}\| > 0 \) such that for all \((f_1, g_1), (f_2, g_2) \in L^2\),

\[
\|\tilde{(f_1 - f_2, g_1 - g_2)}\|_{L^2} \leq C \left( \|Y_1(T_0, \cdot) - Y_2(T_0, \cdot)\|_{W^2} + \|\partial_t y_1 - \partial_t y_2\|_{L^2(\omega_0, \tau)} \right),
\]

where \( Y_1 \) and \( Y_2 \) are the mild solutions of \((1)\) respectively associated to \((f_1 r, g_1 \tilde{r})\) and \((f_2 r, g_2 \tilde{r})\).

**Remark 6.** If the known parts \( r \) and \( \tilde{r} \) of the source terms in \((61)-(62)\) are assumed to be positive as in [32, Theorem III.2.2.], Proposition 3 follows directly from Theorem 1.1. In our case, we assume that \( r \) and \( \tilde{r} \) are only positive on \( T_0 \) which makes the situation a bit more complicated.

**Proof.** Let \( f = f_1 - f_2 \), \( g = g_1 - g_2 \) and \( Y = Y_1 - Y_2 \) the corresponding solution. By Remark 5 we have \( \mathcal{F} = (f r, g \tilde{r}) \in \mathcal{S}(C_0) \), for some \( C_0 = C(r, \tilde{r}) > 0 \). Hence, Theorem 1.1 implies that there exists \( C = C(\Omega, \omega, T, t_0, \|a\|_\infty, \|b\|_\infty, \|r, \tilde{r}\| > 0 \) such that

\[
\|\tilde{(f r, g \tilde{r})}\|_{L^2} \leq C \left( \|Y(T_0, \cdot)\|_{W^2} + \|\partial_t y_1\|_{\omega_0, \tau} \right).
\]

To obtain \((66)\), it suffices to prove that there exist \( c_0 > 0 \) and \( \tau > 0 \) such that \( (T_0 - \tau, T_0 + \tau) \subset (0, T) \) and,

\[
\begin{align*}
    \forall (t, x) \in (T_0 - \tau, T_0 + \tau) \times \overline{\Omega}: \quad |r(t, x)| &\geq c_0 > 0, \\
    \forall (t, x) \in (T_0 - \tau, T_0 + \tau) \times \Gamma: \quad |\tilde{r}(t, x)| &\geq c_0 > 0.
\end{align*}
\]

By \((65)\) we have \( |r(T_0, \cdot)| \geq r_0 \) on \( \overline{\Omega} \). Using the uniform continuity of \( r \) on \([0, T] \times \overline{\Omega} \), there exists \( \tau_1 > 0 \) such that

\[
|t_1 - t_2| + \|x_1 - x_2\| \leq \tau_1 \quad \text{implies} \quad |r(t_2, x_2)| - \frac{r_0}{2} < |r(t_1, x_1)|.
\]
We can choose \( \tau \) such that \( (T_0 - \tau, T_0 + \tau) \subset (0, T) \). It follows that, for all \((t, x) \in (T_0 - \tau_1, T_0 + \tau_1) \times \Omega \), we have

\[
|r(t, x)| > |r(T_0, x)| - \frac{r_0}{2} \geq \frac{r_0}{2}.
\]

Similarly, there exists \( \tau_2 > 0 \) such that \( (T_0 - \tau_2, T_0 + \tau_2) \subset (0, T) \) and, for all \((t, x) \in (T_0 - \tau_2, T_0 + \tau_2) \times \Gamma \), we have \( |\tilde{r}(t, x)| \geq \frac{r_0}{2} \). Let \( \tau = \min(\tau_1, \tau_2) \) and \( c_0 = \frac{r_0}{2} \). Then,

\[
\forall (t, x) \in (T_0 - \tau, T_0 + \tau) \times \Omega: |r(t, x)| \geq c_0 > 0,
\]

\[
\forall (t, x) \in (T_0 - \tau, T_0 + \tau) \times \Gamma: |\tilde{r}(t, x)| \geq c_0 > 0.
\]

Furthermore,

\[
\forall (t, x) \in (T_0 - \tau, T_0 + \tau) \times \Omega: |f(x)| \leq \frac{1}{c_0} |f(x)| |r(t, x)|,
\]

\[
\forall (t, x) \in (T_0 - \tau, T_0 + \tau) \times \Gamma: |g(x)| \leq \frac{1}{c_0} |g(x)| |\tilde{r}(t, x)|.
\]

By integrating the previous inequalities, we obtain

\[
\|(f, g)\|_{L_2}^2 = \int_{\Omega} |f(x)|^2 \, dx + \int_{\Gamma} |g(x)|^2 \, dS \\
\leq \frac{1}{2r_0^2} \left( \int_{T_0 - \tau}^{T_0 + \tau} \int_{\Omega} |f(x)r(t, x)|^2 \, dx \, dt + \int_{T_0 - \tau}^{T_0 + \tau} \int_{\Gamma} |g(x)\tilde{r}(t, x)|^2 \, dS \, dt \right) \\
\leq \frac{1}{2r_0^2} \left( \int_{\Omega_T} |f(x)r(t, x)|^2 \, dx \, dt + \int_{\Gamma_T} |g(x)\tilde{r}(t, x)|^2 \, dS \, dt \right) \\
= \frac{1}{2r_0^2} \|(f_r, g_{\tilde{r}})\|_{L_2}^2.
\]

Inequality (67) allows to conclude. \( \square \)

As an application of Proposition 3, we derive the following uniqueness result.

**Corollary 1.** Assume that \( Y_1 := (y_1, y_{1,\Gamma}) \) and \( Y_2 := (y_2, y_{2,\Gamma}) \) are the mild solutions of (1) respectively associated to \( F_1 := (f_1, \gamma_{1,\Gamma}) \) and \( F_2 := (f_2, \gamma_{2,\Gamma}) \), where \((r, \tilde{r}) \in C^{1,0}\) satisfying (63)-(64). If \( Y_1(T_0, \cdot) = Y_2(T_0, \cdot) \) and \( \partial_t y_1 = \partial_t y_2 \) in \( \omega_{t_0, T} \), then \( f_1 \equiv f_2 \) in \( \Omega \) and \( y_1 \equiv y_2 \) in \( \Gamma \).

**Remark 7.** We dealt with the case when the time of observation \( T_0 \) is in the interval \((t_0, T)\), since better regularity results hold than \((0, T)\). However, one can obtain the same stability result when the observation is taken in \((0, t_0)\).

**Acknowledgments.** The authors would like to thank anonymous referees for invaluable comments which led to this improved version.

**REFERENCES**

[1] M. Bellassoued and M. Yamamoto, *Carleman Estimates and Applications to Inverse Problems for Hyperbolic Systems*, Springer, Tokyo, 2017.

[2] M. Bellassoued and M. Yamamoto, *Carleman estimate with second large parameter for second order hyperbolic operators in a Riemannian manifold and applications in thermoelasticity cases*, *Appl. Anal.*, 91 (2012), 35–67.

[3] M. Bellassoued and M. Yamamoto, *Inverse source problem for a transmission problem for a parabolic equation*, *J. Inverse Ill-Posed Probl.*, 14 (2006), 47–56.
LIPSCHITZ STABILITY FOR AN INVERSE SOURCE PROBLEM

1. [A. Bensoussan, G. Da Prato, M. C. Delfour and S. K. Mitter, Representation and Control of Infinite Dimensional Systems, 2nd edition, Birkhäuser Boston, Inc., Boston, MA, 2007, 159–165.]

2. [I. Boutaayamou, G. Fragnelli and L. Maniar, Inverse problems for parabolic equations with interior degeneracy and Neumann boundary conditions, J. Inverse Ill-Posed Probl., 24 (2016), 275–292.]

3. [I. Boutaayamou, G. Fragnelli and L. Maniar, Lipschitz stability for linear parabolic systems with interior degeneracy, Electron. J. Differential Equations, (2014), No. 167, 26 pp.]

4. [I. Boutaayamou, A. Hajij and L. Maniar, Lipschitz stability for degenerate parabolic systems, Electron. J. Differential Equations, (2014), No. 149, 15 pp.]

5. [A. L. Bukhgeim and M. V. Klibanov, Global uniqueness of class of multidimensional inverse problems, Soviet Math. Dokl., 24 (1981), 244–247.]

6. [P. Cannarsa, J. Tort and M. Yamamoto, Determination of source terms in a degenerate parabolic equation, Inverse Problems, 26 (2010), 105003, 20 pp.]

7. [D. Chae, O. Y. Imanuvilov and S. M. Kim, Exact controllability for semilinear parabolic equations with Neumann boundary conditions, J. Dynam. Control Systems, 2 (1996), 449–483.]

8. [M. Cristofol and L. Roques, On the determination of the nonlinearity from localized measurements in a reaction-diffusion equation, Nonlinearity, 23 (2010), 675–686.]

9. [J. Z. Farkas and P. Hinow, Physiologically structured populations with diffusion and dynamic boundary conditions, Math. Biosci. Eng., 8 (2011), 503–513.]

10. [A. Favini, J. A. Goldstein, G. R. Goldstein and S. Romanelli, The heat equation with generalized Wentzell boundary condition, J. Evol. Equ., 2 (2002), 1–19.]

11. [E. Fernández-Cara and S. Guerrero, Global Carleman inequalities for parabolic systems and applications to controllability, SIAM J. Control Optim., 45 (2006), 1395–1446.]

12. [A. V. Fursikov and O. Y. Imanuvilov, Carleman estimates for the wave equation, Lecture Notes in Mathematics, Vol. 1, Grundlehren der mathematischen Wissenschaften, Springer-Verlag, Berlin, 1996.]

13. [G. C. Gal and L. Tebou, Carleman inequalities for wave equations with oscillatory boundary conditions and application, SIAM J. Control Optim., 55 (2017), 324–364.]

14. [G. R. Goldstein, Derivation and physical interpretation of general boundary conditions, Adv. Differential Equations, 11 (2006), 457–480.]

15. [O. Y. Imanuvilov and M. Yamamoto, Lipschitz stability in inverse parabolic problems by the Carleman estimate, Inverse Problems, 14 (1998), 1229–1245.]

16. [V. Isakov, Inverse Problems for Partial Differential Equations, 2nd edition, Appl. Math. Sci. 127, Springer, New York, 2006.]

17. [V. Isakov, Inverse Source Problems, Mathematical Surveys and Monographs, Vol. 34, Amer. Math. Soc., Providence, RI, 1990.]

18. [J. Jost, Partial Differential Equations, 2nd edition, Springer, New York, 2007.]

19. [J. Jost, Riemannian Geometry and Geometric Analysis, 5th edition, Springer-Verlag, Berlin, 2008.]

20. [A. Khoutaibi and L. Maniar, Null controllability for a heat equation with dynamic boundary conditions and drift terms, Evol. Equ. Control Theory, 9 (2019), 535–559.]

21. [A. Khoutaibi, L. Maniar, D. Mugnolo and A. Rhandi, Parabolic equations with dynamic boundary conditions and drift terms, preprint, 2019, arXiv:1909.02377.]

22. [M. V. Klibanov, Inverse problems and Carleman estimates, Inverse Problems, 8 (1992), 575–596.]

23. [R. E. Langer, A problem in diffusion or in the flow of heat for a solid in contact with a fluid, Tohoku Math. J., 35 (1932), 260–275.]

24. [J.-L. Lions and E. Magenes, Non-Homogeneous Boundary Value Problems and Applications, Vol. 1, Grundlagen der mathematischen Wissenschaften, Springer-Verlag, New York-Heidelberg, 1972.]

25. [L. Maniar, M. Meyries and R. Schnaubelt, Null controllability for parabolic equations with dynamic boundary conditions, Evol. Equ. Control Theory, 6 (2017), 381–407.]

26. [A. Miranville and S. Zelik, Exponential attractors for the Cahn-Hilliard equation with dynamic boundary conditions, Math. Methods Appl. Sci., 28 (2005), 709–735.]

27. [E. M. Ouhabaz, Analysis of Heat Equations on Domains, LMS Monographs Series, 31, Princeton University Press, Princeton, NJ, 2005, 284 pp.]

28. [M. E. Taylor, Partial Differential Equations. I. Basic Theory, 2nd edition, Applied Mathematical Sciences, Vol. 115, Springer, New York, 2011.]
[32] J. Vancostenoble, Lipschitz stability in inverse source problems for singular parabolic equations, *Comm. Partial Differential Equations*, **36** (2011), 1287–1317.

[33] J. L. Vazquez and E. Vitillaro, Heat equation with dynamical boundary conditions of reactive-diffusive type, *J. Differential Equations*, **250** (2011), 2143–2161.

[34] M. Yamamoto, Carleman estimates for parabolic equations and applications, *Inverse Problems*, **25** (2009), 123013, 75 pp.

Received February 2020; revised June 2020.

E-mail address: m.benhassi@gmail.com
E-mail address: chorphi@gmail.com
E-mail address: maniar@uca.ma
E-mail address: omar.oukdach@gmail.com