New Hermite–Hadamard Inequalities in Fuzzy-Interval Fractional Calculus and Related Inequalities

Muhammad Bilal Khan 1, Pshtiwan Othman Mohammed 2,*, Muhammad Aslam Noor 1 and Y. S. Hamed 3

Abstract: It is a familiar fact that inequalities have become a very popular method using fractional integrals, and that this method has been the driving force behind many studies in recent years. Many forms of inequality have been studied, resulting in the introduction of new trend in inequality theory. The aim of this paper is to use a fuzzy order relation to introduce various types of inequalities. On the fuzzy interval space, this fuzzy order relation is defined level by level. With the help of this relation, firstly, we derive some discrete Jensen and Schur inequalities for convex fuzzy interval-valued functions (convex fuzzy-IVF), and then, we present Hermite–Hadamard inequalities (HH-inequalities) for convex fuzzy-IVF via fuzzy interval Riemann–Liouville fractional integrals. These outcomes are a generalization of a number of previously known results, and many new outcomes can be deduced as a result of appropriate parameter “γ” and real valued function “Ω” selections.

We hope that our fuzzy order relations results can be used to evaluate a number of mathematical problems related to real-world applications.

Keywords: convex fuzzy interval-valued function; fuzzy fractional integral operator; Jensen inequality; Schur inequality; Hermite–Hadamard inequality; Hermite–Hadamard–Fejér inequality

1. Introduction

A real valued function \( Q : \mathbb{T} \to \mathbb{R} \) is called convex on the interval \( \mathbb{T} \) if the condition

\[
Q(\varrho \ddagger + (1 - \varrho) y) \leq \varrho Q(\ddagger) + (1 - \varrho) Q(y),
\]

holds for all \( \ddagger, y \in \mathbb{T} \) and \( \varrho \in [0, 1] \). We say that \( Q \) is concave if \( -Q \) is convex.

The concept of convexity in the sense of integral problems is a fascinating area for research. Therefore, many inequalities have been introduced as applications of convex functions such as Gagliardo–Nirenberg-type inequality [1], Ostrowski-type inequality [2], Olsen-type inequality [3], midpoint-type inequality [4] Hardy-type inequality [5], and trapezoidal-type inequality [6]. Among those, the Hermite–Hadamard inequality is an interesting outcome in convex analysis. The \( HH \)-inequality [7,8] for convex function \( Q : \mathbb{T} \to \mathbb{R} \) on an interval \( \mathbb{T} = [u, v] \) is defined in the following way:

\[
Q\left(\frac{u + v}{2}\right) \leq \frac{1}{v - u} \int_{u}^{v} Q(\ddagger) \, d\ddagger \leq \frac{Q(u) + Q(v)}{2},
\]

for all \( u, v \in \mathbb{T} \). If \( Q \) is concave, then inequality (2) is reversed. For more useful details, see [9,10] and the references therein.
In 2013, Sarika et al. [11] introduced the following fractional HH-inequality for convex function:

\[ Q\left(\frac{u + v}{2}\right) \leq \frac{\Gamma(\alpha + 1)}{2(u - v)\Gamma(\alpha)} \left[ T_{u}^{\alpha} Q(v) + T_{v}^{\alpha} Q(u) \right] \leq \frac{Q(u) + Q(v)}{2}. \] (3)

where \( Q : \mathbb{I} = [u, v] \rightarrow \mathbb{R} \) assumed to be a positive function on \([u, v]\), \( Q \in L_{1}([u, v]) \) with \( u \leq v \), and \( T_{u}^{\alpha} \) and \( T_{v}^{\alpha} \) are the left sided and right sided Riemann–Liouville fractional of order \( 0 \leq \alpha \), and, respectively, are defined as follows:

\[ T_{u}^{\alpha} Q(\tau) = \frac{1}{\Gamma(\alpha)} \int_{u}^{\tau} (\tau - \xi)^{\alpha - 1} Q(\xi) d\xi \quad (\tau > u), \] (4)

\[ T_{v}^{\alpha} Q(\tau) = \frac{1}{\Gamma(\alpha)} \int_{\tau}^{v} (\rho - \xi)^{\alpha - 1} Q(\xi) d\xi \quad (\tau < v). \] (5)

If \( \alpha = 1 \), then from Equation (3), we obtain Equation (2). We can easily say that inequality (3) is a generalization of inequality (2). Many fractional inequalities have been introduced by several authors in the view of inequality (3) for different convex and nonconvex functions.

In 1966, the concept of interval analysis was first introduced by the late American mathematician Ramon E. Moore in [9]. Since its inception, various authors in the mathematical community have paid close attention to this area of research. Interval analysis has been found to be useful in global optimization and constraint solution algorithms, according to experts. It has slowly risen in popularity over the last few decades. Scientists and engineers engaged in scientific computation have discovered that interval analysis is useful, especially in terms of accuracy, round-off error affects, and automatic validation of results. After the invention of interval analysis, the researchers working in the area of inequalities want to know whether the inequalities in the abovementioned results can be found substituted with the inclusion relation. In certain cases, the question is answered correctly. In light of this, Zhao et al. [12] arrived at the following conclusion for an interval-valued function (IVF).

Let \( Q : [u, v] \subset \mathbb{R} \rightarrow K_{C}^{+} \) be a convex IVF given by \( Q(\tau) = [Q_{L}(\tau), Q^{*}(\tau)] \) for all \( \tau \in [u, v] \), where \( Q_{L}(\tau) \) is a convex function and \( Q^{*}(\tau) \) is a concave function. If \( Q \) is Riemann integrable, then

\[ Q\left(\frac{u + v}{2}\right) \supseteq \frac{1}{v - u} (\operatorname{IR}) \int_{u}^{v} Q(\tau) d\tau \supseteq \frac{Q(u) + Q(v)}{2}. \] (6)

After that, several authors developed a strong relationship between inequalities and IVFs by means of an inclusion relation via different integral operators, as one can see in Costa [13], Costa and Roman-Flores [14], Roman-Flores et al. [15,16], and Chalco-Cano et al. [17,18], but also in more general set valued maps by Nikodem et al. [19], and Matkowski and Nikodem [20]. In particular, Zhang et al. [21] derived the new version of Jensen’s inequalities for set-valued and fuzzy set-valued functions by means of a pseudo order relation, and proved that these Jensen’s inequalities are a generalized form of Costa Jensen’s inequalities [13].

In the last two decades, in the development of pure and applied mathematics, fractional calculus has played a key role. Yet, it attains magnificent deliberation in the ongoing research work; this is because of its applications in various directions such as image processing, signal processing, physics, biology, control theory, computer networking, and fluid dynamics [22–25].

For inclusion relation, Budek [26] established the following strong relation between convex IVF and interval HH-inequality as a counter part of (6)

\[ Q\left(\frac{u + v}{2}\right) \supseteq \frac{\Gamma(\alpha + 1)}{2(u - v)} \left[ T_{u}^{\alpha} Q(v) + T_{v}^{\alpha} Q(u) \right] \supseteq \frac{Q(u) + Q(v)}{2}. \] (7)
where \( \mathcal{Q} : \mathcal{T} = [u, v] \rightarrow \mathcal{K} \mathcal{C}^+ \) is assumed to be a positive IVF on \([u, v]\), \( \mathcal{Q} \in L_1([u, v], \mathcal{K} \mathcal{C}^+) \) with \( u \leq v \), and \( I_u^\alpha \) and \( I_v^\alpha \) are the left sided and right sided Riemann–Liouville fractional integral operators of order \( 0 \leq \alpha \).

If \( \alpha = 1 \), then from (7), we obtain (6). We can easily say that inequality (7) is a generalization of inequality (6). Many fractional inequalities have been introduced by several authors in the view of inequality (7) for different convex- and nonconvex-IVFs.

Due to the vast applications of convexity and fractional \( HH \)-inequality in mathematical analysis and optimization, many authors have discussed the applications, refinements, generalizations, and extensions, see [27–38] and the references therein.

Recently, fuzzy interval analysis and fuzzy interval differential equations have been put forward to deal the ambiguity originating from insufficient data in some mathematical or computer models that find out real-world phenomena [39–50]. There are some integrals generalizations, and extensions, see [27–38] and the references therein.

Due to the vast applications of convexity and fractional \( HH \)-inequality in mathematical analysis and optimization, many authors have discussed the applications, refinements, generalizations, and extensions, see [27–38] and the references therein.

Let \( \alpha > 0 \) and \( L([u, v], \mathcal{F}_0) \) be the collection of all Lebesgue measurable fuzzy-IVFs on \([u, v]\). Then, the fuzzy interval integral \( \mathcal{Q} \in L([u, v], \mathcal{F}_0) \) with order \( \alpha > 0 \) are defined by

\[
I_u^\alpha \mathcal{Q}(\xi) = \frac{1}{\Gamma(\alpha)} \int_u^\xi (\xi - \eta)^{\alpha-1} \mathcal{Q}(\eta) d(\eta), \quad (\xi > u),
\]

and

\[
I_v^\alpha \mathcal{Q}(\xi) = \frac{1}{\Gamma(\alpha)} \int_{\xi}^v (\xi - \eta)^{\alpha-1} \mathcal{Q}(\eta) d(\eta), \quad (\xi < v),
\]

respectively, where \( \Gamma(\xi) = \int_0^\infty e^{\xi-1} u^{-\xi} d(\xi) \) is the Euler \( \gamma \) function. The fuzzy interval left and right Riemann–Liouville fractional integral \( \mathcal{T} \) based on left and right end point functions can be defined as

\[
[I_u^\alpha \mathcal{Q}(\xi)]^\gamma = \frac{1}{\Gamma(\alpha)} \int_u^\xi (\xi - \eta)^{\alpha-1} \mathcal{Q}(\eta) d(\eta) = \frac{1}{\Gamma(\alpha)} \int_u^\xi (\xi - \eta)^{\alpha-1} \mathcal{Q}_\gamma(\eta, \gamma) d(\eta), \quad (\xi > u),
\]

where

\[
I_u^\alpha \mathcal{Q}_\gamma(\xi, \gamma) = \frac{1}{\Gamma(\alpha)} \int_u^\xi (\xi - \eta)^{\alpha-1} \mathcal{Q}_\gamma(\eta, \gamma) d(\eta), \quad (\xi > u),
\]

and

\[
I_u^\alpha \mathcal{Q}_\gamma(\xi, \gamma) = \frac{1}{\Gamma(\alpha)} \int_u^\xi (\xi - \eta)^{\alpha-1} \mathcal{Q}_\gamma(\eta, \gamma) d(\eta), \quad (\xi > u).
\]

Similarly, we can define right Riemann–Liouville fractional integral \( \mathcal{Q} \) of \( \mathcal{T} \) based on left and right end point functions.

Motivated by the ongoing research works and by the importance of the concept of fuzzy convexity, in Section 2, we have discussed the preliminary notions, definitions and some relevant results. Section 3, introduces discrete Jensen and Schur inequalities for convex fuzzy-IVF by means of fuzzy order relation. Basically, these inequalities are generalizations of convex fuzzy IVFs. Section 4 derives the most important class of \( HH \)-inequalities for convex fuzzy IVFs, known as fuzzy interval fractional \( HH \)-inequalities, and establishes some related inequalities.

2. Preliminaries

In this section, we first give some definitions, preliminary notations, and results which will be helpful for further study. Then, we define new definitions and discuss some properties of convex fuzzy-IVFs.
Let $\mathcal{K}_C$ be the space of all closed and bounded intervals of $\mathbb{R}$ and $\eta \in \mathcal{K}_C$ be defined by

$$\eta = [\eta_\ast, \eta^*] = \{ \xi \in \mathbb{R} | \eta_\ast \leq \xi \leq \eta^* \}, \eta^* \in \mathbb{R}. \tag{13}$$

If $\eta_\ast = \eta^*$, then $\eta$ is said to be degenerate. In this article, all intervals will be non-degenerate intervals. If $\eta_\ast \geq 0$, then $[\eta_\ast, \eta^*]$ is called a positive interval. The set of all positive intervals is denoted by $\mathcal{K}_C^+$ and defined as $\mathcal{K}_C^+ = \{ [\eta_\ast, \eta^*] : [\eta_\ast, \eta^*] \in \mathcal{K}_C \text{ and } \eta_\ast \geq 0 \}$. Let $\phi \in \mathbb{R}$ and $\eta \phi$ be defined by

$$\phi \eta = \begin{cases} [\phi \eta_\ast, \phi \eta^*] & \text{if } \phi \geq 0, \\ [\phi \eta^*, \phi \eta_\ast] & \text{if } \phi < 0 . \end{cases} \tag{14}$$

Then, the Minkowski difference $\xi - \eta$, addition $\eta + \xi$ and $\eta \times \xi$ for $\eta, \xi \in \mathcal{K}_C$ are defined by

$$[\xi_\ast, \xi^*] - [\eta_\ast, \eta^*] = [\xi_\ast - \eta_\ast, \xi^* - \eta^*], \tag{15}$$

$$[\xi_\ast, \xi^*] + [\eta_\ast, \eta^*] = [\xi_\ast + \eta_\ast, \xi^* + \eta^*],$$

and

The inclusion “$\subseteq$” means that

$$\xi \subseteq \eta \text{ if and only if, } [\xi_\ast, \xi^*] \subseteq [\eta_\ast, \eta^*] \text{ if and only if, } \eta_\ast \leq \xi_\ast, \xi^* \leq \eta^*. \tag{16}$$

Remark 1. [47] The relation “$\leq_i$” defined on $\mathcal{K}_C$ by

$$[\nabla_\ast, \nabla^*] \leq_i [\eta_\ast, \eta^*] \text{ if and only if } \nabla_\ast \leq \eta_\ast, \nabla^* \leq \eta^*, \tag{17}$$

for all $[\nabla_\ast, \nabla^*], [\eta_\ast, \eta^*] \in \mathcal{K}_C$, is an order relation. For given $[\nabla_\ast, \nabla^*], [\eta_\ast, \eta^*] \in \mathcal{K}_C$, we say that $[\nabla_\ast, \nabla^*] \leq_i [\eta_\ast, \eta^*]$ if and only if $\nabla_\ast \leq \eta_\ast, \nabla^* \leq \eta^*$ or $\nabla_\ast \leq \eta_\ast, \nabla^* < \eta^*$.

For $[\xi_\ast, \xi^*], [\eta_\ast, \eta^*] \in \mathbb{R}_I$, the Hausdorff–Pompeiu distance between intervals $[\xi_\ast, \xi^*]$ and $[\eta_\ast, \eta^*]$ is defined by

$$d([\xi_\ast, \xi^*], [\eta_\ast, \eta^*]) = \max \{ [\xi_\ast, \xi^*], [\eta_\ast, \eta^*] \}. \tag{18}$$

It is a familiar fact that $(\mathbb{R}_I, \delta)$ is a complete metric space.

Let $\mathbb{R}$ be the set of real numbers. A fuzzy subset $A$ of $\mathbb{R}$ is characterized by a mapping $\zeta : \mathbb{R} \to [0, 1]$ called the membership function, for each fuzzy subset and $\gamma \in (0, 1]$, then $\gamma$-level sets of $\zeta$ is denoted and defined as follows $\zeta_\gamma = \{ u \in \mathbb{R} | \zeta(u) \geq \gamma \}$. If $\gamma = 0$, then $supp(\zeta) = \{ \xi \in \mathbb{R} | \zeta(\xi) > 0 \}$ is called support of $\zeta$. By $[\zeta]^0$ we define the closure of $supp(\zeta)$.

Let $\mathbb{F}(\mathbb{R})$ be the family of all fuzzy sets and $\zeta \in \mathbb{F}(\mathbb{R})$ denote the family of all nonempty sets. $\zeta \in \mathbb{F}(\mathbb{R})$ is a fuzzy set. Then, we define the following:

1. $\zeta$ is said to be normal if there exists $\xi \in \mathbb{R}$ and $\zeta(\xi) = 1$;
2. $\zeta$ is said to be upper semicontinuous on $\mathbb{R}$ if for given $\xi \in \mathbb{R}$, there exist $\varepsilon > 0$ there exist $\delta > 0$ such that $\zeta(\xi) - \zeta(y) < \varepsilon$ for all $y \in \mathbb{R}$ with $|\xi - y| < \delta$;
3. $\zeta$ is said to be fuzzy convex if $\zeta_\gamma$ is convex for every $\gamma \in [0, 1]$;
4. $\zeta$ is compactly supported if $supp(\zeta)$ is compact.

A fuzzy set is called a fuzzy number or fuzzy interval if it has properties (1)–(4). We denote by $\mathbb{F}_0$ the family of all intervals.

Let $\zeta \in \mathbb{F}_0$ be a fuzzy interval if and only if $\gamma$-levels $[\zeta]^\gamma$ is a nonempty compact convex set of $\mathbb{R}$. From these definitions, we have

$$[\zeta]^\gamma = [\zeta_\ast(\gamma), \zeta^*(\gamma)],$$

where

$$\zeta_\ast(\gamma) = \inf \{ \xi \in \mathbb{R} | \zeta(\xi) \geq \gamma \}, \zeta^*(\gamma) = \sup \{ \xi \in \mathbb{R} | \zeta(\xi) \geq \gamma \}.$$
Proposition 1. [14] If $\xi, \eta \in \mathbb{F}_0$, then relation “$\preceq$” defined on $\mathbb{F}_0$ by

$$\zeta \preceq \eta \text{ if and only if, } [\zeta]^\gamma \leq [\eta]^\gamma, \text{ for all } \gamma \in [0, 1],$$

this relation is known as partial order relation.

For $\xi, \eta \in \mathbb{F}_0$ and $\varphi \in \mathbb{R}$, the sum $\zeta + \eta$, product $\zeta \cdot \eta$, scalar product $\varphi \cdot \zeta$ and sum with scalar are defined by:

Then, for all $\gamma \in [0, 1]$, we have

$$[\zeta + \eta]^\gamma = [\zeta]^\gamma + [\eta]^\gamma,$$  \hspace{1cm} (19)

$$[\zeta \cdot \eta]^\gamma = [\zeta]^\gamma \times [\eta]^\gamma,$$  \hspace{1cm} (20)

$$[\varphi \cdot \zeta]^\gamma = \varphi \cdot [\zeta]^\gamma.$$  \hspace{1cm} (21)

For $\psi \in \mathbb{F}_0$ such that $\xi = \eta + \psi$, then by this result we have existence of Hukuhara difference of $\xi$ and $\eta$, and we say that $\psi$ is the H-difference of $\xi$ and $\eta$, and denoted by $\tilde{\xi} - \eta$. If H-difference exists, then

$$(\psi)^*(\gamma) = (\tilde{\xi} - \eta)^*(\gamma) = \xi^*(\gamma) - \eta^*(\gamma), (\psi)_*(\gamma) = (\tilde{\xi} - \eta)_*(\gamma) = \xi_*(\gamma) - \eta_*(\gamma).$$  \hspace{1cm} (23)

A partition of $[u, v]$ is any finite ordered subset $P$ having the form

$$P = \{u = \underline{t}_1 < \underline{t}_2 < \underline{t}_3 < \underline{t}_4 < \underline{t}_5 < \ldots < \underline{t}_k = v\}.$$  \hspace{1cm}

The mesh of a partition $P$ is the maximum length of the subintervals containing $P$ that is,

$$\text{mesh}(P) = \max\{\underline{t}_j - \underline{t}_{j-1} : j = 1, 2, 3 \ldots k\}.$$  \hspace{1cm}

Let $\mathcal{P}(\delta, [u, v])$ be the set of all $P \in \mathcal{P}(\delta, [u, v])$ such that mesh $(P) < \delta$. For each interval $[\underline{t}_{j-1}, \underline{t}_j]$, where $1 \leq j \leq k$, choose an arbitrary point $\eta_j$ and taking the sum

$$S(Q, P, \delta) = \sum_{j=1}^k Q(\eta_j)(\underline{t}_j - \underline{t}_{j-1}),$$  \hspace{1cm}

where $Q : [u, v] \rightarrow \mathbb{R}_I$. We call $S(Q, P, \delta)$ a Riemann sum of $Q$ corresponding to $P \in \mathcal{P}(\delta, [u, v])$.

Definition 1. [10] A function $Q : [u, v] \rightarrow \mathbb{R}_I$ is called interval Riemann integrable (IR-integrable) on $[u, v]$ if there exists $B \in \mathbb{R}_I$ such that, for each $\varepsilon$, there exists $\delta > 0$ such that

$$d(S(Q, P, \delta), B) < \varepsilon,$$

for every Riemann sum of $Q$ corresponding to $P \in \mathcal{P}(\delta, [u, v])$ and for arbitrary choice of $\eta_j \in [\underline{t}_{j-1}, \underline{t}_j]$ for $1 \leq j \leq k$. Then, we say that $B$ is the IR-integral of $Q$ on $[u, v]$ and is denoted by $B = (IR) \int_u^v Q(\underline{t})d\underline{t}$.

Moore [9] first proposed the concept of Riemann integral for IVF, and it is defined as follows:

Theorem 1. [9] If $Q : [u, v] \subset \mathbb{R} \rightarrow \mathbb{R}_I$ is an IVF on such that $Q(\underline{t}) = [Q^*, Q^+]$. Then, $Q$ is Riemann integrable over $[u, v]$ if and only if $Q_*$ and $Q^*$ are both Riemann integrable over $[u, v]$ such that

$$\left(IR\right) \int_u^v Q(\underline{t})d\underline{t} = \left[R\right] \int_u^v Q_*(u)d\underline{t} + \left(R\right) \int_u^v Q^*(u)d\underline{t}.$$  \hspace{1cm} (24)
The collection of all Riemann integrable real valued functions and Riemann integrable IVF is denoted by $\mathcal{R}_{[c, d]}$ and $\mathcal{IR}_{[c, d]}$, respectively.

**Definition 2.** [30] A fuzzy map $Q : [c, d] \subset \mathbb{R} \rightarrow \mathbb{F}$ is called fuzzy-IVF. For each $\gamma \in [0, 1]$, whose $\gamma$-levels define the family of IVFs $Q_{\gamma} : [c, d] \rightarrow K$ are given by $Q_{\gamma}(\xi) = \{Q_{\gamma}(\xi, \gamma), Q^{*}_{\gamma}(\xi, \gamma)\}$ for all $\xi \in [c, d]$. Here, for each $\gamma \in [0, 1]$, the left and right real valued functions $Q_{\gamma}(\xi, \gamma)$, $Q^{*}_{\gamma}(\xi, \gamma) : K \rightarrow \mathbb{R}$ are also called lower and upper functions of $Q$.

**Remark 2.** If $Q : [c, d] \subset \mathbb{R} \rightarrow \mathbb{F}$ is a fuzzy-IVF, then $Q(\xi)$ is called continuous function at $\xi \in K$, if for each $\gamma \in [0, 1]$, both left and right real valued functions $Q_{\gamma}(\xi, \gamma)$ and $Q^{*}_{\gamma}(\xi, \gamma)$ are continuous at $\xi \in K$.

The following conclusion can be drawn from the above literature review, see [10,14,30]:

**Definition 3.** Let $Q : [c, d] \subset \mathbb{R} \rightarrow \mathbb{F}$ is called fuzzy-IVF. The fuzzy Riemann integral of $Q$ over $[c, d]$, denoted by $(\text{FR}) \int_{c}^{d} Q(\xi)d\xi$, is defined level by level

$$
\left(\text{FR}\right) \int_{c}^{d} Q_{\gamma}(\xi)d\xi = \left(\text{IR}\right) \int_{c}^{d} Q_{\gamma}(\xi)d\xi = \left\{ \int_{c}^{d} Q_{\gamma}(\xi)d\xi : Q_{\gamma}(\xi, \gamma) \in \mathcal{R}_{[c, d]} \right\},
$$

for all $\gamma \in [0, 1]$, where $\mathcal{R}_{[c, d]}$ contains the family of left and right functions of IVFs. $Q$ is $(\text{FR})$-integrable over $[c, d]$ if $(\text{FR}) \int_{c}^{d} Q_{\gamma}(\xi)d\xi \in \mathbb{F}$. Note that, if left and right real valued functions are Lebesgue-integrable, then $Q$ is fuzzy Aumann-integrable over $[c, d]$, denoted by $(\text{FA}) \int_{c}^{d} Q(\xi)d\xi$, see [30].

**Theorem 2.** Let $Q : [c, d] \subset \mathbb{R} \rightarrow \mathbb{F}$ be a fuzzy-IVF, whose $\gamma$-levels obtain the collection of IVFs $Q_{\gamma} : [c, d] \rightarrow K$ are defined by $Q_{\gamma}(\xi, \gamma) = \{Q_{\gamma}(\xi, \gamma), Q^{*}_{\gamma}(\xi, \gamma)\}$ for all $\xi \in [c, d]$ and for all $\gamma \in [0, 1]$. Then, $Q$ is $(\text{FR})$-integrable over $[c, d]$ if and only if $Q_{\gamma}(\xi, \gamma)$ and $Q^{*}_{\gamma}(\xi, \gamma)$ both are $R$-integrable over $[c, d]$. Moreover, if $Q$ is $(\text{FR})$-integrable over $[c, d]$, then

$$
\left(\text{FR}\right) \int_{c}^{d} Q_{\gamma}(\xi)d\xi = \left(\text{IR}\right) \int_{c}^{d} Q_{\gamma}(\xi)d\xi = \left(\text{IR}\right) \int_{c}^{d} Q_{\gamma}(\xi)d\xi,
$$

for all $\gamma \in [0, 1]$. For each $\gamma \in [0, 1]$, $\mathcal{FR}_{[c, d], \gamma}$ and $\mathcal{R}_{[c, d], \gamma}$ denote the collection of all $(\text{FR})$-integrable fuzzy-IVFs and, $R$-integrable left and right functions over $[c, d]$.

**Definition 4.** [4] A real valued function $Q : [u, v] \rightarrow \mathbb{R}$ is called P-convex function if

$$Q(qx + (1-q)y) \leq Q(x) + Q(y),$$

for all $x, y \in [u, v]$, $q \in [0, 1]$. If (27) is reversed, then $Q$ is called P-concave.

**Definition 5.** [6] The fuzzy-IVF $Q : [u, v] \rightarrow \mathbb{F}$ is called fuzzy-IVF on $[u, v]$ if

$$Q(qx + (1-q)y) \leq qQ(x) + (1-q)Q(y),$$

for all $x, y \in [u, v]$, $q \in [0, 1]$, where $Q(\xi) \geq 0$ for all $\xi \in [u, v]$. If (28) is reversed, then $Q$ is called concave fuzzy-IVF on $[u, v]$. $Q$ is affine if and only if it is both convex and concave fuzzy-IVF.

**Remark 3.** If $Q_{\gamma}(\xi, \gamma) = Q^{*}_{\gamma}(\xi, \gamma)$ and $\gamma = 1$, then we obtain the inequality (1).
3. Fuzzy-Interval Jensen and Schur Inequalities

In this section, the discrete Jensen and Schur inequalities for convex fuzzy-IVF are proposed. Firstly, we give the following results connected with Jensen inequality for convex fuzzy-IVF by means of fuzzy order relation.

**Theorem 3.** (Jensen inequality for convex fuzzy-IVF) Let \( \forall j \in [u, v], (j = 1, 2, 3, \ldots, k, k \geq 2) \) and \( Q : [u, v] \rightarrow \mathbb{R}^k \) be a convex fuzzy-IVF, whose \( \gamma \)-levels define the family of IVFs \( Q_\gamma : [u, v] \subset \mathbb{R} \rightarrow K_{\mathbb{C}^+} \) are given by \( Q_\gamma(\ddagger) = [Q_\gamma(\ddagger, \gamma), Q_\gamma^*(\ddagger, \gamma)] \) for all \( \ddagger \in [u, v] \) and for all \( \gamma \in (0, 1] \). Then,

\[
Q \left( \frac{1}{W_k} \sum_{j=1}^{k} \gamma_j u_j \right) \leq \frac{1}{W_{k-1}} \sum_{j=1}^{k-1} \gamma_j Q(u_j),
\]

where \( W_k = \sum_{j=1}^{k} \gamma_j \). If \( Q \) is concave, then inequality (29) is reversed.

**Proof.** When \( k = 2 \) then inequality (29) is true. Consider inequality (28) is true for \( k = n - 1 \), then

\[
Q \left( \frac{1}{W_n} \sum_{j=1}^{n} \gamma_j u_j \right) \leq \frac{1}{W_{n-1}} \sum_{j=1}^{n-1} \gamma_j Q(u_j).
\]

Now, let us prove that inequality (29) holds for \( k = n \).

\[
Q \left( \frac{1}{W_n} \sum_{j=1}^{n} \gamma_j u_j \right) = Q \left( \frac{1}{W_n} \sum_{j=1}^{n-2} \gamma_j u_j + \frac{\gamma_j u_j}{W_n} \left( \frac{\gamma_j u_j}{W_n} \frac{1}{n-1} \sum_{j=1}^{n-1} \gamma_j u_j \right) \right)
\]

Therefore, for each \( \gamma \in [0, 1] \), we have

\[
Q_\gamma \left( \frac{1}{W_n} \sum_{j=1}^{n} \gamma_j u_j, \gamma \right) \leq \frac{1}{W_{n-1}} \sum_{j=1}^{n-2} \gamma_j Q_\gamma(u_j, \gamma) + \frac{\gamma_j u_j}{W_n} \left( \frac{\gamma_j u_j}{W_n} \frac{1}{n-1} \sum_{j=1}^{n-1} \gamma_j u_j \right)
\]

Similarly, for \( Q_\gamma^* (\ddagger, \gamma) \), we have

\[
Q_\gamma^* \left( \frac{1}{W_n} \sum_{j=1}^{n} \gamma_j u_j, \gamma \right) \leq \frac{1}{W_{n-1}} \sum_{j=1}^{n-2} \gamma_j Q_\gamma^*(u_j, \gamma).
\]

From (30) and (31), we have

\[
\left[ Q_\gamma \left( \frac{1}{W_n} \sum_{j=1}^{n} \gamma_j u_j, \gamma \right) , \frac{1}{W_n} \sum_{j=1}^{n} \gamma_j u_j, \gamma \right] \leq \frac{1}{W_{n-1}} \sum_{j=1}^{n-2} \gamma_j Q_\gamma^*(u_j, \gamma),
\]

that is,

\[
Q \left( \frac{1}{W_n} \sum_{j=1}^{n} \gamma_j u_j \right) \leq \frac{1}{W_{n-1}} \sum_{j=1}^{n-2} \gamma_j Q(u_j),
\]

and the theorem has been proved. \( \square \)

If \( \geq 1 = \geq 2 = \geq 3 = \ldots = \geq k = 1 \), then Theorem 3 reduces to the following result:
Corollary 1. Let $u_j \in [u, v], (j = 1, 2, 3, \ldots, k, k \geq 2)$ and $Q : [u, v] \rightarrow \mathbb{F}_0$ be a convex fuzzy-IVF, whose $\gamma$-levels define the family of IVFs $Q_\gamma : [u, v] \subset \mathbb{R} \rightarrow K_{\mathbb{C}^+}$ are given by $Q_\gamma(\xi) = [Q_\gamma(\xi, \gamma), Q_\gamma^*(\xi, \gamma)]$ for all $\xi \in [u, v]$ and for all $\gamma \in (0, 1)$. Then,

$$Q\left(\frac{1}{W_k} \sum_{j=1}^{n} \alpha_j u_j\right) \leq \sum_{j=1}^{n} \frac{1}{k} Q(u_j).$$

(32)

If $Q$ is a concave fuzzy-IVF, then inequality (32) is reversed.

Theorem 4. (Schur inequality for convex fuzzy-IVF) Let $Q : [u, v] \rightarrow \mathbb{F}_0$ be a convex fuzzy-IVF, whose $\gamma$-levels define the family of IVFs $Q_\gamma : [u, v] \subset \mathbb{R} \rightarrow K_{\mathbb{C}^+}$ are given by $Q_\gamma(\xi) = [Q_\gamma(\xi, \gamma), Q_\gamma^*(\xi, \gamma)]$ for all $\xi \in [u, v]$ and for all $\gamma \in (0, 1)$. Then,

$$Q(u_1) + Q(u_2) + \cdots + Q(u_k) \leq \frac{1}{k} \left(\sum_{j=1}^{k} Q(u_j)\right).$$

(33)

If $Q$ is concave, then inequality (33) is reversed.

Proof. Let $u_1, u_2, u_3 \in [u, v]$ and $u_3 - u_1 > 0$. Consider $\varrho = \frac{u_3 - u_2}{u_3 - u_1}$, then $u_2 = \varrho u_1 + (1 - \varrho) u_3$. Since $Q$ is a convex fuzzy-IVF, then, by hypothesis, we have

$$Q(u_2) \leq \frac{u_3 - u_2}{u_3 - u_1} Q(u_1) + \frac{u_2 - u_1}{u_3 - u_1} Q(u_3).$$

Therefore, for each $\gamma \in [0, 1]$, we have

$$Q_\gamma(u_2, \gamma) \leq \frac{u_3 - u_2}{u_3 - u_1} Q_\gamma(u_1, \gamma) + \frac{u_2 - u_1}{u_3 - u_1} Q_\gamma(u_3, \gamma).$$

(34)

From (34), we have

$$(u_3 - u_1) Q_\gamma(u_2, \gamma) \leq (u_3 - u_2) Q_\gamma(u_1, \gamma) + (u_2 - u_1) Q_\gamma(u_3, \gamma).$$

(35)

Similarly, for $Q^*(\cdot, \cdot, \gamma)$, we have

$$(u_3 - u_1) Q^*(u_2, \gamma) \leq (u_3 - u_2) Q^*(u_1, \gamma) + (u_2 - u_1) Q^*(u_3, \gamma).$$

(36)

From (35) and (36), we have

$$[(u_3 - u_1) Q_\gamma(u_2, \gamma), (u_3 - u_1) Q^*(u_2, \gamma)] \leq [(u_3 - u_2) Q_\gamma(u_1, \gamma) + (u_2 - u_1) Q_\gamma(u_3, \gamma), (u_3 - u_2) Q^*(u_1, \gamma) + (u_2 - u_1) Q^*(u_3, \gamma)].$$

That is

$$(u_3 - u_1) Q(u_2) \leq (u_3 - u_2) Q(u_1) + (u_2 - u_1) Q(u_3).$$

Hence, the result has been proved. □

Now we obtain a refinement of Schur inequality for convex fuzzy-IVF is given in the following result.

Theorem 5. Let $\xi_j \in \mathbb{R}^+, u_j \in [u, v], (j = 1, 2, 3, \ldots, k, k \geq 2)$ and $Q : [u, v] \rightarrow \mathbb{F}_0$ be a convex fuzzy-IVF, whose $\gamma$-levels define the family of IVFs $Q_\gamma : [u, v] \subset \mathbb{R} \rightarrow K_{\mathbb{C}^+}$ are given by $Q_\gamma(\xi) = [Q_\gamma(\xi, \gamma), Q_\gamma^*(\xi, \gamma)]$ for all $\xi \in [u, v]$ and for all $\gamma \in (0, 1)$. If $(L, U) \subseteq [u, v]$, then

$$\sum_{j=1}^{k} \frac{\xi_j}{W_k} Q(u_j) \leq \sum_{j=1}^{k} \left(\frac{U - u_j}{U - L}\right) \left(\frac{\xi_j}{W_k}\right) Q(L, \gamma) + \left(\frac{u_j - L}{U - L}\right) \left(\frac{\xi_j}{W_k}\right) Q(U, \gamma).$$

(37)
where \( W_k = \sum_{j=1}^{k} \exists_j \). If \( Q \) is concave, then inequality (37) is reversed.

**Proof.** Consider \( u_1, u_j = u_2, (j = 1, 2, 3, \ldots, k) \), \( U = u_3 \). Then, by hypothesis, we have

\[
Q(u_j) \leq \frac{U - u_j}{U - L} Q(L, \gamma) + \frac{u_j - L}{U - L} Q(U, \gamma).
\]

Therefore, for each \( \gamma \in [0, 1] \), we have

\[
Q_*(u_j, \gamma) \leq \frac{U - u_j}{U - L} Q_*(L, \gamma) + \frac{u_j - L}{U - L} Q_*(U, \gamma).
\]

Above inequality can be written as,

\[
\sum_{j=1}^{k} \frac{\exists_j}{W_k} Q_*(u_j, \gamma) \leq \left( \frac{U - u_j}{U - L} \right) \left( \frac{\exists_j}{W_k} \right) Q_*(L, \gamma) + \left( \frac{u_j - L}{U - L} \right) \left( \frac{\exists_j}{W_k} \right) Q_*(U, \gamma). \tag{38}
\]

Taking sum of all inequalities (38) for \( j = 1, 2, 3, \ldots k \), we have

\[
\sum_{j=1}^{k} \frac{\exists_j}{W_k} Q_*(u_j, \gamma) \leq \sum_{j=1}^{k} \left( \left( \frac{U - u_j}{U - L} \right) \left( \frac{\exists_j}{W_k} \right) Q_*(L, \gamma) \right) + \left( \frac{u_j - L}{U - L} \right) \left( \frac{\exists_j}{W_k} \right) Q_*(U, \gamma). \tag{39}
\]

Similarly, for \( Q^* (\exists, \gamma) \), we have

\[
\sum_{j=1}^{k} \frac{\exists_j}{W_k} Q^*(u_j, \gamma) \leq \sum_{j=1}^{k} \left( \left( \frac{U - u_j}{U - L} \right) \left( \frac{\exists_j}{W_k} \right) Q^*(L, \gamma) \right) + \left( \frac{u_j - L}{U - L} \right) \left( \frac{\exists_j}{W_k} \right) Q^*(U, \gamma). \tag{40}
\]

From (39) and (40), we have

\[
\sum_{j=1}^{k} \frac{\exists_j}{W_k} [Q(u_j)]^? \leq \left[ \sum_{j=1}^{k} \frac{\exists_j}{W_k} Q_*(u_j, \gamma) \right] + \left[ \sum_{j=1}^{k} \frac{\exists_j}{W_k} Q^*(u_j, \gamma) \right] \leq \left[ \sum_{j=1}^{k} \frac{\exists_j}{W_k} Q_* (L, \gamma) \right] + \left[ \sum_{j=1}^{k} \frac{\exists_j}{W_k} Q^* (U, \gamma) \right] = \frac{\sum_{j=1}^{k} \frac{\exists_j}{W_k}}{(U - u)} Q(L, \gamma) + \frac{\sum_{j=1}^{k} \frac{\exists_j}{W_k}}{(U - u)} Q(U, \gamma).
\]

Thus,

\[
\sum_{j=1}^{k} \frac{\exists_j}{W_k} Q(u_j) \leq \sum_{j=1}^{k} \left( \frac{U - u_j}{U - L} \right) \left( \frac{\exists_j}{W_k} \right) Q(L) + \left( \frac{u_j - L}{U - L} \right) \left( \frac{\exists_j}{W_k} \right) Q(U),
\]

this completes the proof. \( \square \)

We now consider some special cases of Theorems 3 and 5.

If \( Q_*(\exists, \gamma) = Q_*(\exists, \gamma) \) with \( \gamma = 1 \), then Theorems 3 and 5 reduce to the following results:

**Corollary 2.** (Jensen inequality for convex function) Let \( \exists_j \in \mathbb{R}^+, u_j \in [u, v], (j = 1, 2, 3, \ldots, k, k \geq 2) \) and let \( Q : [u, v] \rightarrow \mathbb{R}^+ \) be a non-negative real-valued function. If \( Q \) is a convex function, then

\[
Q\left( \frac{1}{W_k} \sum_{j=1}^{k} \exists_j u_j \right) \leq \frac{1}{W_k} \sum_{j=1}^{k} \frac{\exists_j}{W_k} Q(u_j), \tag{41}
\]

where \( W_k = \sum_{j=1}^{k} \exists_j \). If \( Q \) is concave function, then inequality (41) is reversed.
Corollary 3. Let \( \exists j \in \mathbb{R}^+ \), \( u_j \in [u, v] \), \( j = 1, 2, \ldots, k \), \( k \geq 2 \), and \( Q : [u, v] \rightarrow \mathbb{R}^+ \) be a non-negative real-valued function. If \( Q \) is a convex function and \( u_1, u_2, \ldots, u_j \in (L, U) \subseteq [u, v] \), then
\[
\sum_{j=1}^{k} \frac{Q(u_j)}{W_k} = \sum_{j=1}^{k} \left( \frac{U - u_j}{U - L} \right) \left( \frac{U - u_j}{W_k} \right) Q(L) + \left( \frac{U - L}{U - L} \right) \left( \frac{U - u_j}{W_k} \right) Q(U),
\]
where \( W_k = \sum_{j=1}^{k} \exists j \). If \( Q \) is a concave function, then inequality (42) is reversed.

4. Fuzzy-Interval Fractional Hermite–Hadamard Inequalities

In this section, we shall continue with the following fractional HH-inequality for convex fuzzy-IVFs, and we also give fractional HH–Fejér inequality for convex fuzzy-IVF through fuzzy order relation. In what follows, we denote by \( L([u, v], F_0) \) the family of Lebesgue measurable fuzzy-IVFs.

Theorem 6. Let \( Q : [u, v] \rightarrow F_0 \) be a convex fuzzy-IVF on \([u, v]\), whose \( \gamma \)-levels define the family of IVFs \( \mathcal{Q}_\gamma : [u, v] \rightarrow \mathbb{K}_C^+ \) are given by \( \mathcal{Q}_\gamma (\cdot) = [Q_{\gamma, \delta, \gamma} (\cdot, \gamma)] \) for all \( \delta \in [u, v] \) and all \( \gamma \in [0, 1] \). If \( Q \in L([u, v], F_0) \), then
\[
Q \left( \frac{u + v}{2} \right) \leq \frac{\Gamma(\alpha + 1)}{2(\nu - u)^\alpha} \left[ T_{\alpha}^u \mathcal{Q}(\nu) + T_{\alpha}^\nu \mathcal{Q}(u) \right] \leq \frac{Q(u) + Q(v)}{2}.
\]

If \( Q(\exists) \) is concave fuzzy-IVF, then
\[
Q \left( \frac{u + v}{2} \right) \geq \frac{\Gamma(\alpha + 1)}{2(\nu - u)^\alpha} \left[ T_{\alpha}^u \mathcal{Q}(\nu) + T_{\alpha}^\nu \mathcal{Q}(u) \right] \geq \frac{Q(u) + Q(v)}{2}.
\]

Proof. Let \( Q : [u, v] \rightarrow F_0 \) be a convex fuzzy-IVF. Then, by hypothesis, we have
\[
2Q \left( \frac{u + v}{2} \right) \leq Q(\nu u + (1 - \nu)v) + Q((1 - \nu)u + \nu v).
\]

Therefore, for every \( \gamma \in [0, 1] \), we have
\[
2Q \left( \frac{u + v}{2}, \gamma \right) \leq Q_{\gamma}(\nu u + (1 - \nu)v) + Q_{\gamma}((1 - \nu)u + \nu v, \gamma).
\]

Multiplying both sides by \( \nu^{\alpha - 1} \) and integrating the obtained result with respect to \( \nu \) over \((0, 1)\), we have
\[
2 \int_0^1 \nu^{\alpha - 1} Q_{\gamma}(\nu u + (1 - \nu)v, \nu) d\nu \leq \int_0^1 \nu^{\alpha - 1} Q_{\gamma}((1 - \nu)u + \nu v, \nu) d\nu.
\]

Let \( \exists = \nu u + (1 - \nu)v \) and \( y = (1 - \nu)u + \nu v \). Then, we have
\[
\frac{2}{\alpha} \alpha Q_{\gamma} \left( \frac{u + v}{2}, \gamma \right) \leq \frac{1}{(v - u)^\alpha} \int_u^v (v - y)^{\alpha - 1} Q_{\gamma}(y, \gamma) dy + \frac{1}{(v - u)^\alpha} \int_u^v (\exists - u)^{\alpha - 1} Q_{\gamma}(\exists, \gamma) dz \leq \frac{\Gamma(\alpha)}{(v - u)^\alpha} \left[ T_{\alpha}^u \mathcal{Q}(\nu, \gamma) + T_{\alpha}^\nu \mathcal{Q}(u, \gamma) \right].
\]
From (45) and (46), we have

\[
\frac{2}{\alpha} Q\left(\frac{u + v}{2}\right) \leq \frac{\Gamma(\alpha)}{(v-u)\alpha} \left[ I_u^{\alpha+} Q_+(v, \gamma) + I_{v/u}^{\alpha+} Q_+(u, \gamma) \right] + \left[ I_{v/u}^{\alpha-} Q^*(v, \gamma) + I_u^{\alpha-} Q^*(u, \gamma) \right].
\]

That is

\[
\frac{2}{\alpha} Q\left(\frac{u + v}{2}\right) \leq \frac{\Gamma(\alpha)}{(v-u)\alpha} \left[ I_u^{\alpha+} Q(v) + I_{v/u}^{\alpha+} Q(u) \right].
\]

In a similar way as above, we have

\[
\frac{\Gamma(\alpha)}{(v-u)^2} \left[ I_u^{\alpha+} Q(v) + I_{v/u}^{\alpha+} Q(u) \right] \leq \frac{Q(u) + Q(v)}{2}.
\]

Combining (47) and (48), we have

\[
Q\left(\frac{u + v}{2}\right) \leq \frac{\Gamma(\alpha + 1)}{2(v-u)^2} \left[ I_u^{\alpha+} Q(v) + I_{v/u}^{\alpha+} Q(u) \right] \leq \frac{Q(u) + Q(v)}{2}.
\]

Hence, the required result. \(\square\)

Remark 4. From Theorem 6 we clearly see that

Let \(\alpha = 1\). Then, Theorem 6 reduces to the result for convex-IVF given in [31]:

\[
Q\left(\frac{u + v}{2}\right) \leq \frac{1}{v-u} (FR) \int_u^v Q(\xi) d\xi \leq \frac{Q(u) + Q(v)}{2}.
\]

If \(Q_+ (\xi, \gamma) = Q^+ (\xi, \gamma)\) and \(\gamma = 1\), then from Theorem 6 we get inequality (3).

Let \(\alpha = 1 = \gamma\) and \(Q_+ (\xi, \gamma) = Q^+ (\xi, \gamma)\). Then, from Theorem 6 we obtain inequality (2).

Example 1. Let \(\alpha = \frac{1}{2}\), \(\xi \in [2, 3]\), and the fuzzy-IVF \(Q: [u, v] = [2, 3] \to \mathbb{F}_0\), defined by

\[
Q(\xi)(\theta) = \begin{cases} \frac{\theta}{2 - \frac{1}{2}} & \theta \in [0, 2 - \frac{1}{2}] \\ \frac{2(2 - \frac{1}{2}) - \theta}{2 - \frac{1}{2}} & \theta \in \left(2 - \frac{1}{2}, 2\left(2 - \frac{1}{2}\right)\right) \\ 0 & \text{otherwise,} \end{cases}
\]

Then, for each \(\gamma \in [0, 1]\), we have \(Q_{\gamma}(\xi) = [\gamma \left(2 - \frac{1}{2}\right), (2 - \gamma)\left(2 - \frac{1}{2}\right)]\). Since left and right end point functions \(Q_+ (\xi, \gamma) = (2 - \frac{1}{2}), Q^+ (\xi, \gamma) = (2 - \gamma)\left(2 - \frac{1}{2}\right)\) are convex functions for each \(\gamma \in [0, 1]\), then \(Q(\xi)\) is convex fuzzy-IVF. We clearly see that \(Q \in L([u, v], \mathbb{F}_0)\) and

\[
Q_+ \left(\frac{u + v}{2}, \gamma\right) = Q_+ \left(\frac{3}{2}, \gamma\right) = \gamma \left(2 - \frac{1}{2}\right) = \frac{4 - \sqrt{10}}{2}.
\]

\[
Q^+ \left(\frac{u + v}{2}, \gamma\right) = Q^+ \left(\frac{5}{2}, \gamma\right) = (2 - \gamma)\left(2 - \frac{1}{2}\right) = \gamma \left(4 - \frac{\sqrt{10}}{2}\right).
\]

\[
\frac{Q_+(u, \gamma) + Q_+(v, \gamma)}{2} = \gamma \left(2 - \frac{1}{2}\right) = \frac{4 - \sqrt{10}}{2}.
\]

\[
\frac{Q^+(u, \gamma) + Q^+(v, \gamma)}{2} = (2 - \gamma)\left(2 - \frac{1}{2}\right) = \gamma \left(4 - \frac{\sqrt{10}}{2}\right).
\]
Note that
\[
\frac{\Gamma(\alpha+1)}{2^{(1-n)^{-}}} \left[ I_{v}^{\alpha} Q_{v}(v, \gamma) + I_{v}^{\alpha} Q_{v}(u, \gamma) \right] \\
= \frac{\Gamma(\frac{1}{2})}{\sqrt{\pi}} \int_{\frac{1}{2}} I_{\frac{1}{2}}^{\frac{1}{2}} (3-\frac{1}{2}) \cdot \gamma \left(2-\frac{1}{2}\right) \frac{d\gamma}{d\gamma} \\
+ \frac{\Gamma(\frac{1}{2})}{\sqrt{\pi}} \int_{\frac{1}{2}} I_{\frac{1}{2}}^{\frac{1}{2}} (\frac{1}{2}) \cdot \gamma \left(2-\frac{1}{2}\right) \frac{d\gamma}{d\gamma} \\
= \frac{1}{4} \gamma \left[ \frac{7393}{10000} + \frac{9291}{10000} \right] \\
= \gamma \frac{4294}{10000}.
\]

Therefore
\[
\left[ \gamma \frac{4-\sqrt{10}}{2}, (2-\gamma) \frac{4-\sqrt{10}}{2} \right] \leq \left[ \gamma \frac{8447}{20000}, (2-\gamma) \frac{8447}{20000} \right] \leq \left[ \gamma \left(4-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \right), (2-\gamma) \left(4-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \right) \right],
\]
and Theorem 6 is verified.

Now, we derive fractional HH–Fejér inequality for convex fuzzy-IVF, which generalize the classical fractional HH and HH–Fejér inequality. Firstly, we give the following result connected with the right part of the classical HH–Fejér inequality for convex fuzzy-IVF through fuzzy order relation, which is known as second fuzzy fractional HH–Fejér inequality.

**Theorem 7.** Second fuzzy fractional HH–Fejér inequality Let \( Q : [u, v] \to \mathbb{R}_0 \) be a convex fuzzy-IVF with \( u < v \), whose \( \gamma \)-levels define the family of IVFs \( Q_{\gamma} : [u, v] \subset \mathbb{R} \to \mathbb{K}_{\mathbb{C}}^{+} \) are given by \( Q_{\gamma}(\xi) = [Q_{\gamma}(\xi, \gamma), Q_{\gamma}(\xi, \gamma)] \) for all \( \xi \in [u, v] \) and all \( \gamma \in [0, 1] \). If \( Q \in L([u, v], \mathbb{R}) \) and \( \Omega : [u, v] \to \mathbb{R}, \Omega(\xi) \geq 0 \), symmetric with respect to \( \frac{u+v}{2} \), then
\[
[I_{u}^{\alpha} Q \Omega(v) + I_{v}^{\alpha} Q \Omega(u)] \leq \frac{Q(u) + Q(v)}{2} [I_{u}^{\alpha} Q (\Omega(v) + I_{v}^{\alpha} Q \Omega(u)],
\]
If \( Q \) is concave fuzzy-IVF, then inequality (49) is reversed.

**Proof.** Let \( Q \) be a convex fuzzy-IVF and \( Q_{\alpha}^{-1} \Omega((1-\bar{\theta})u + (1-\bar{\theta})v) \geq 0 \). Then, for each \( \gamma \in [0, 1] \), we have
\[
e^{\alpha-1} Q_{\alpha}(u, \gamma) \Omega((1-\bar{\theta})u + (1-\bar{\theta})v) \leq e^{\alpha-1} Q_{\alpha}(u, \gamma) \Omega(u) + (1-\bar{\theta}) \Omega(v) + (1-\bar{\theta}) \Omega(u),
(50)
\]
and
\[
e^{\alpha-1} Q_{\alpha}((1-\bar{\theta})u + (1-\bar{\theta})v) \Omega((1-\bar{\theta})u + (1-\bar{\theta})v) \leq e^{\alpha-1} Q_{\alpha}(u, \gamma) \Omega((1-\bar{\theta})u + (1-\bar{\theta})v) + (1-\bar{\theta}) Q_{\alpha}(v, \gamma) \Omega((1-\bar{\theta})u + (1-\bar{\theta})v).
(51)
\]
After adding (50) and (51), and integrating over \([0, 1]\), we get
\[
\int_{0}^{1} e^{\alpha-1} Q_{\alpha}(u, \gamma) \Omega((1-\bar{\theta})u + (1-\bar{\theta})v) d\gamma \\
+ \int_{0}^{1} e^{\alpha-1} Q_{\alpha}(v, \gamma) \Omega((1-\bar{\theta})u + (1-\bar{\theta})v) d\gamma \\
\leq \int_{0}^{1} e^{\alpha-1} Q_{\alpha}(u, \gamma) \Omega(u) + e^{\alpha-1} Q_{\alpha}(v, \gamma) \Omega(v) + e^{\alpha-1} Q_{\alpha}(u, \gamma) \Omega(v) + e^{\alpha-1} Q_{\alpha}(v, \gamma) \Omega(u) d\gamma,
\]
and
\[
= Q_{\alpha}(u, \gamma) \int_{0}^{1} e^{\alpha-1} Q_{\alpha}((1-\bar{\theta})u + (1-\bar{\theta})v) d\gamma + Q_{\alpha}(v, \gamma) \int_{0}^{1} e^{\alpha-1} Q_{\alpha}((1-\bar{\theta})u + (1-\bar{\theta})v) d\gamma.
\]
Since $\Omega$ is symmetric, then
\[
\begin{align*}
&= [Q(u, \gamma) + Q(v, \gamma)] \int_0^1 \epsilon^{a-1} \Omega((1 - \epsilon)u + \epsilon v) \, d\epsilon.
&= \frac{Q(u, \gamma) + Q(v, \gamma)}{2} \frac{\Gamma(a)}{(v-u)^a} [T_{u^+}^a \Omega(v) + T_{v^-}^a \Omega(u)].
\end{align*}
\]
Since
\[
\begin{align*}
&\int_0^1 \epsilon^{a-1} Q_u((1 - \epsilon)u + \epsilon v) \Omega((1 - \epsilon)u + \epsilon v) \, d\epsilon \\
&= \frac{1}{(v-u)^a} \int_u^v (\tau - u)^{a-1} Q_u(u - v - \tau, \gamma) \Omega(\tau) d\tau \\
&= \frac{1}{(v-u)^a} \int_u^v (\tau - u)^{a-1} Q_u(\tau, \gamma) \Omega(\tau) d\tau
\]
(53)
Then, from (53), we have
\[
\begin{align*}
&\leq \frac{\Gamma(a)}{(v-u)^a} [T_{u^+}^a Q_u(v) + T_{v^-}^a Q_u(u)].
&\leq \frac{\Gamma(a)}{(v-u)^a} [T_{u^+}^a Q^*(v) + T_{v^-}^a Q^*(u)].
\end{align*}
\]
Similarly, for $Q^*(\subseteq, \gamma)$, we have
\[
\begin{align*}
&\leq \frac{\Gamma(a)}{(v-u)^a} [T_{u^+}^a Q^*(v) + T_{v^-}^a Q^*(u)].
&\leq \frac{\Gamma(a)}{(v-u)^a} [T_{u^+}^a \Omega(v) + T_{v^-}^a \Omega(u)].
\end{align*}
\]
From (54) and (55), we have
\[
\begin{align*}
&\leq \frac{\Gamma(a)}{(v-u)^a} \left[ \frac{T_{u^+}^a Q^*(v) + T_{v^-}^a Q^*(u)}{2} \right] \left[ T_{u^+}^a Q_u(v) + T_{v^-}^a Q_u(u) \right]
&\leq \frac{\Gamma(a)}{(v-u)^a} \left[ \frac{Q^*(u, \gamma) + Q^*(v, \gamma)}{2} \right] \left[ T_{u^+}^a \Omega(v) + T_{v^-}^a \Omega(u) \right].
\end{align*}
\]
That is
\[
\begin{align*}
&\frac{\Gamma(a)}{(v-u)^a} [T_{u^+}^a Q^*(v) + T_{v^-}^a Q^*(u)] \leq \frac{Q(u) + Q(v)}{2} [T_{u^+}^a \Omega(v) + T_{v^-}^a \Omega(u)].
\end{align*}
\]
Now, we obtain the following result connected with left part of classical $HH$–Fejér inequality for convex fuzzy-IVF through fuzzy order relation, which is known as first fuzzy fractional $HH$–Fejér inequality.

**Theorem 8.** (First fuzzy fractional $HH$–Fejér inequality) Let $Q : [u, v] \to F_0$ be a convex fuzzy-IVF with $u < v$, whose $\gamma$-levels define the family of IVFs $Q_\gamma : [u, v] \to K_{C^+}$ are given by $Q_\gamma(\tau) = [Q_\gamma(\tau, \gamma)]$ for all $\tau \in [u, v]$ and for all $\gamma \in [0, 1]$. If $Q \in L([u, v], F_0)$ and $\Omega : [u, v] \to K_{C^+}$, symmetric with respect to $\frac{u+v}{2}$, then
\[
Q\left(\frac{u+v}{2}\right) [T_{u^+}^a \Omega(v) + T_{v^-}^a \Omega(u)] \leq \frac{Q(u) + Q(v)}{2} [T_{u^+}^a \Omega(v) + T_{v^-}^a \Omega(u)].
\]
(56)
If $Q$ is concave fuzzy-IVF, then inequality (56) is reversed.

**Proof.** Since $Q$ is a convex fuzzy-IVF, then for $\gamma \in [0, 1]$, we have
\[
Q_\gamma(\frac{u+v}{2}, \gamma) \leq \frac{1}{2} (Q_\gamma(eu + (1 - \epsilon)v, \gamma) + Q_\gamma((1 - \epsilon)u + \epsilon v, \gamma)).
\]
(57)
Since $\Omega(\varepsilon u + (1-\varepsilon)v) = \Omega((1-\varepsilon)u + \varepsilon v)$, then by multiplying (57) by $\varepsilon^a \Omega((1-\varepsilon)u + \varepsilon v)$ and integrating it with respect to $\varepsilon$ over $[0, 1]$, we obtain

\[
\begin{split}
&\frac{\Gamma(a) \omega^a}{(v-u)^a} \int_u^v Q_s(\alpha + (1-\alpha)v, \gamma) \Omega((1-\alpha)u + \varepsilon v) d\varepsilon \\
&\leq \frac{1}{\Gamma(a) \omega} \left[ T_u^a \Omega(v) + T_v^a \Omega(u) \right] \hspace{1cm} (58)
\end{split}
\]

Let $\varepsilon = (1-\varepsilon)u + \varepsilon v$. Then, we have

\[
\begin{split}
&\frac{1}{(v-u)^a} \int_u^v Q_s(\alpha + (1-\alpha)v, \gamma) \Omega((1-\alpha)u + \varepsilon v) d\varepsilon \\
= &\frac{1}{(v-u)^a} \int_u^v Q_s((1-\varepsilon)u + \varepsilon v) \Omega((1-\varepsilon)u + \varepsilon v) d\varepsilon \\
&\quad + \frac{1}{(v-u)^a} \int_u^v Q_s((1-\varepsilon)u + \varepsilon v) \Omega((1-\varepsilon)u + \varepsilon v) d\varepsilon \\
= &\frac{1}{(v-u)^a} \int_u^v Q_s((1-\varepsilon)u + \varepsilon v) \Omega((1-\varepsilon)u + \varepsilon v) d\varepsilon \\
&\quad + \frac{1}{(v-u)^a} \int_u^v Q_s((1-\varepsilon)u + \varepsilon v) \Omega((1-\varepsilon)u + \varepsilon v) d\varepsilon \\
= &\frac{1}{(v-u)^a} \left[ T_u^a \Omega(v) + T_v^a \Omega(u) \right] .
\end{split}
\]

Then, from (59), we have

\[
\begin{split}
&\frac{\Gamma(a) \omega^a}{(v-u)^a} Q_s(\alpha + (1-\alpha)v, \gamma) \left[ T_u^a \Omega(v) + T_v^a \Omega(u) \right] \\
\leq &\frac{1}{\Gamma(a) \omega} \left[ T_u^a \Omega(v) + T_v^a \Omega(u) \right] .
\end{split}
\]

Similarly, for $Q^*(\varepsilon, \gamma)$, we have

\[
\begin{split}
&\frac{\Gamma(a) \omega^a}{(v-u)^a} Q^*(\alpha + (1-\alpha)v, \gamma) \left[ T_u^a \Omega(v) + T_v^a \Omega(u) \right] \\
\leq &\frac{1}{\Gamma(a) \omega} \left[ T_u^a \Omega(v) + T_v^a \Omega(u) \right] .
\end{split}
\]

From (60) and (61), we have

\[
\begin{split}
&\frac{\Gamma(a) \omega^a}{(v-u)^a} \left[ Q_s(\alpha + (1-\alpha)v, \gamma), Q^*(\alpha + (1-\alpha)v, \gamma) \right] \left[ T_u^a \Omega(v) + T_v^a \Omega(u) \right] \\
\leq &\frac{1}{\Gamma(a) \omega} \left[ T_u^a \Omega(v) + T_v^a \Omega(u) \right] .
\end{split}
\]

that is

\[
Q \left( \frac{\mu + \nu}{2} \right) \left[ T_u^a \Omega(v) + T_v^a \Omega(u) \right] \leq \left[ T_u^a \Omega(v) + T_v^a \Omega(u) \right] .
\]

This completes the proof. \(\square\)

**Remark 5.** If $\Omega(\varepsilon) = 1$ then from Theorem 7 and Theorem 8, we get Theorem 3.

Let $\alpha = 1$. Then, we obtain the following $HH$–Fejér inequality for convex fuzzy-IVF, which is also new one.

\[
Q \left( \frac{\mu + \nu}{2} \right) \leq \frac{1}{\Gamma(a) \omega} \left( FR \right) \int_u^v Q(\varepsilon)W(\varepsilon) d\varepsilon \leq \frac{Q(u) + Q(v)}{2} .
\]

If $Q_s(\varepsilon, \gamma) = Q^*(\varepsilon, \gamma)$ and $\Omega(\varepsilon) = \alpha = 1 = \gamma$, then from Theorems 7 and 8, we get the classical $HH$–inequality (2).

If $Q_s(\varepsilon, \gamma) = Q^*(\varepsilon, \gamma)$ and $\alpha = 1$, then from Theorems 7 and 8, we obtain the classical $HH$–Fejér inequality [50].
Example 2. We consider the fuzzy-IVF \( Q : [0, 2] \rightarrow \mathbb{F}_0 \) defined by,
\[
Q(\gamma)(\sigma) = \begin{cases} 
\frac{\sigma}{2(\gamma - \sqrt{\frac{3}{4}})}, & \sigma \in [0, 2 - \sqrt{\frac{3}{4}}], \\
\frac{2(\gamma - \sqrt{\frac{3}{4}})}{\sqrt{\frac{3}{4}} - \sigma}, & \sigma \in (2 - \sqrt{\frac{3}{4}}, 2(2 - \sqrt{\frac{3}{4}})], \\
0, & \text{otherwise.}
\end{cases}
\]

Then, for each \( \gamma \in [0, 1] \), we have \( Q_\gamma(\gamma) = [\gamma(2 - \sqrt{\frac{3}{4}}), (2 - \gamma)(2 - \sqrt{\frac{3}{4}})] \). Since end point functions \( Q_\gamma(\gamma), Q^*(\gamma, \gamma) \) are convex functions for each \( \gamma \in [0, 1] \), then \( Q(\gamma) \) is convex fuzzy-IVF. If
\[
\Omega(\gamma) = \begin{cases} 
\sqrt{\frac{3}{4}}, & \sigma \in [0, 1], \\
\frac{2(\gamma - \sqrt{\frac{3}{4}})}{\sqrt{\frac{3}{4}} - \sigma}, & \sigma \in (1, 2],
\end{cases}
\]
then \( \Omega(2 - \gamma) = \Omega(\gamma) \geq 0 \), for all \( \gamma \in [0, 2] \). Since \( Q_\gamma(\gamma, \gamma) = \gamma(2 - \sqrt{\frac{3}{4}}) \) and \( Q^*(\gamma, \gamma) = (2 - \gamma)(2 - \sqrt{\frac{3}{4}}) \). If \( \alpha = \frac{1}{2} \), then we compute the following:
\[
\frac{Q_\gamma(u, \gamma) + Q^*(\gamma, \gamma)}{2} \left[ T^a_\gamma, \Omega(v) + T^a_\gamma, \Omega(u) \right] = \sqrt{\pi} \gamma \left( \frac{4 - \sqrt{\frac{3}{4}}}{2} \right),
\]
\[
\frac{Q^* (u, \gamma) + Q^*(\gamma, \gamma)}{2} \left[ T^a_\gamma, \Omega(v) + T^a_\gamma, \Omega(u) \right] = \sqrt{\pi} (2 - \gamma) \left( \frac{4 - \sqrt{\frac{3}{4}}}{2} \right),
\]
and
\[
T^a_\gamma, Q_\gamma(\gamma) + T^a_\gamma, Q^*(\gamma, \gamma) = \frac{1}{\sqrt{\pi}} (2\pi + \frac{4 - \sqrt{\frac{3}{4}}}{2}),
\]
\[
T^a_\gamma, Q^*(\gamma, \gamma) = \frac{1}{\sqrt{\pi}} (2\pi + \frac{4 - \sqrt{\frac{3}{4}}}{2}).
\]
From (62) and (63), we have
\[
\frac{1}{\sqrt{\pi}} \left[ \gamma \left( 2\pi + \frac{4 - \sqrt{\frac{3}{4}}}{3} \right), (2 - \gamma) \left( 2\pi + \frac{4 - \sqrt{\frac{3}{4}}}{3} \right) \right] \leq \frac{1}{\sqrt{\pi}} \left[ \gamma \left( \frac{4 - \sqrt{\frac{3}{4}}}{2} \right), (2 - \gamma) \left( \frac{4 - \sqrt{\frac{3}{4}}}{2} \right) \right],
\]
for each \( \gamma \in [0, 1] \). Hence, Theorem 8 is verified.

From Theorem 8, we have
\[
Q_\gamma(\gamma, \gamma) \left[ T^a_\gamma, \Omega(v) + T^a_\gamma, \Omega(u) \right] = \gamma \sqrt{\pi},
\]
\[
Q^* (\gamma, \gamma) \left[ T^a_\gamma, \Omega(v) + T^a_\gamma, \Omega(u) \right] = (2 - \gamma) \sqrt{\pi}.
\]
From (63) and (64), we have
\[
\sqrt{\pi} [\gamma, (2 - \gamma)] \leq \frac{1}{\sqrt{\pi}} \left[ \gamma \left( 2\pi + \frac{4 - \sqrt{\frac{3}{4}}}{3} \right), (2 - \gamma) \left( 2\pi + \frac{4 - \sqrt{\frac{3}{4}}}{3} \right) \right],
\]
for each \( \gamma \in [0, 1] \).

From Theorems 9 and 10, we obtain some fuzzy-interval fractional integral inequalities related to fuzzy-interval fractional HH-inequalities.

Theorem 9. Let \( Q, P : [u, v] \rightarrow \mathbb{F}_0 \) be two convex fuzzy-IVFs on \([u, v]\), whose \( \gamma \)-levels \( Q_\gamma, P_\gamma : [u, v] \subset \mathbb{R} \rightarrow \mathbb{K}_+ \) are defined by \( Q_\gamma(\gamma) = [Q_\gamma(\gamma, \gamma), Q^*(\gamma, \gamma)] \) and \( P_\gamma(\gamma) = [P_\gamma(\gamma, \gamma), P^*(\gamma, \gamma)] \), for all \( \gamma \in [0, 1] \). If \( Q, P \) and \( Q \times P \in L([u, v], \mathbb{F}_0) \), then
\[
\Delta(u, v) = Q(\gamma) \times P(\gamma) + Q^*(\gamma) \times P^*(\gamma),
\]
where \( \Delta(\gamma, u, v) = Q(u, v) \times P(u, v) \), for all \( \gamma \in [0, 1] \).
Proof. Since $Q, P$ both are convex fuzzy-IVFs, then for each $\gamma \in [0, 1]$ we have

$$Q_*(qu + (1 - e)v, \gamma) \leq eQ_*(u, \gamma) + (1 - e)Q_*(v, \gamma).$$

and

$$P_*(qu + (1 - e)v, \gamma) \leq eP_*(u, \gamma) + (1 - e)P_*(v, \gamma).$$

From the definition of convex fuzzy-IVFs it follows that $\tilde{0} \preceq Q(\tilde{\cdot})$ and $\tilde{0} \preceq P(\tilde{\cdot})$, so

$$Q_*(qu + (1 - e)v, \gamma) \times P((qu + (1 - e)v, \gamma)
\leq \left( eQ_*(u, \gamma) + (1 - e)Q_*(v, \gamma) \right) \left( eP_*(u, \gamma) + (1 - e)P_*(v, \gamma) \right)$$

$$= e^2Q_*(u, \gamma) \times P_*(u, \gamma) + (1 - e)^2Q_*(v, \gamma) \times P_*(v, \gamma)
+ e(1 - e)Q_*(u, \gamma) \times P_*(v, \gamma) + e(1 - e)Q_*(v, \gamma) \times P_*(u, \gamma).$$

Adding (65) and (66), we have

$$Q_*(1-e)u + Qu, \gamma) \times P((1-e)u + Qu, \gamma)
\leq (1 - e)^2Q_*(u, \gamma) \times P_*(u, \gamma) + e^2Q_*(v, \gamma) \times P_*(v, \gamma)
+ e(1 - e)Q_*(u, \gamma) \times P_*(v, \gamma) + e(1 - e)Q_*(v, \gamma) \times P_*(u, \gamma).$$

Adding (65) and (66), we have

$$Q_*(1-e)u + Qu, \gamma) \times P((1-e)u + Qu, \gamma)
\leq e^2 + (1 - e)^2 \left[ Q_*(u, \gamma) \times P_*(u, \gamma) + Q_*(v, \gamma) \times P_*(v, \gamma) \right]
+ 2e(1 - e)Q_*(u, \gamma) \times P_*(v, \gamma) + Q_*(v, \gamma) \times P_*(u, \gamma).$$

Taking multiplication of (67) by $e^{a-1}$ and integrating the obtained result with respect to $e$ over (0,1), we have

$$\int_0^1 e^{a-1}Q_*(qu + (1 - e)v, \gamma) \times P_*(qu + (1 - e)v, \gamma)
+ e^{a-1}Q_*(((1-e)u + Qu, \gamma) \times P_*(((1-e)u + Qu, \gamma)\leq \Delta_*((u, v), \gamma) \int_0^1 e^{a-1}e((1-e)d e.$$

It follows that,

$$\frac{\Gamma(a)}{(v - u)^a} \left[ I_{u}^a \Delta_* Q_*(v, \gamma) \times P_*(v, \gamma) + I_{v}^a \Delta_* Q_*(u, \gamma) \times P_*(u, \gamma) \right]\leq \frac{2}{a} \left( \frac{1}{2} - \frac{a}{(a+1)(a+2)} \right) \Delta_*((u, v), \gamma) + \frac{2}{a} \left( \frac{a}{(a+1)(a+2)} \right) \nabla_*((u, v), \gamma).$$

Similarly, for $Q^*(\tilde{\cdot})$, we have

$$\frac{\Gamma(a)}{(v - u)^a} \left[ I_{u}^a Q^*(v, \gamma) \times P^*(v, \gamma) + I_{v}^a Q^*(u, \gamma) \times P^*(u, \gamma) \right].$$

$$\leq \frac{2}{a} \left( \frac{1}{2} - \frac{a}{(a+1)(a+2)} \right) \Delta^*((u, v), \gamma) + \frac{2}{a} \left( \frac{a}{(a+1)(a+2)} \right) \nabla^*((u, v), \gamma).$$

From (68) and (69), we have

$$\frac{\Gamma(a)}{(v - u)^a} \left[ I_{u}^a Q_*(v, \gamma) \times P_*(v, \gamma) + I_{v}^a Q_*(u, \gamma) \times P_*(u, \gamma) \right]$$
\begin{align*}
\leq & \frac{2}{a} \left( \frac{1}{2} - \frac{\alpha}{(\alpha + 1)(\alpha + 2)} \right) [\Delta_s((u, v), \gamma), \Delta^s((u, v), \gamma)] + \frac{2}{a} \left( \frac{\alpha}{(\alpha + 1)(\alpha + 2)} \right) [\nabla_s((u, v), \gamma), \nabla^s((u, v), \gamma)].
\end{align*}

That is
\begin{align*}
\frac{\Gamma(\alpha + 1)}{2(\nu - u)^x} \left[ I_{u^+}^\alpha Q(\nu) \times \mathcal{P}(\nu) + I_{u^-}^\alpha Q(u) \times \mathcal{P}(u) \right] & = \left( \frac{1}{2} - \frac{\alpha}{(\alpha + 1)(\alpha + 2)} \right) \Delta(u, v) + \left( \frac{\alpha}{(\alpha + 1)(\alpha + 2)} \right) \nabla(u, v).
\end{align*}

and the theorem has been established. \( \square \)

**Example 3.** Let \([u, v] = [0, 2], \alpha = \frac{1}{2}, \ \mathcal{Q}(\uparrow) = [\uparrow, 2\uparrow], \text{ and } \mathcal{P}(\uparrow) = [2\uparrow, 3\uparrow]. \)

\[
\mathcal{Q}(\uparrow)(\theta) = \begin{cases}
\frac{\theta}{4\uparrow} & \theta \in [0, \uparrow]
\\
\frac{\theta - \theta}{4\uparrow} & \theta \in (z, 2z]
\\
0 & \text{otherwise},
\end{cases}
\]

\[
\mathcal{P}(\uparrow)(\theta) = \begin{cases}
\frac{\theta}{4\uparrow} & \theta \in [0, 2\uparrow]
\\
\frac{\theta - \theta}{4\uparrow} & \theta \in (2z, 4z]
\\
0 & \text{otherwise}.
\end{cases}
\]

Then, for each \( \gamma \in [0, 1], \) we have \( \mathcal{Q}_\gamma(\uparrow) = [\gamma, (2 - \gamma)\uparrow] \) and \( \mathcal{P}_\gamma(\uparrow) = [2\gamma, 2(2 - \gamma)\uparrow]. \) Since left and right end point functions \( \mathcal{Q}_\gamma(\uparrow, \gamma) = \gamma \uparrow, \mathcal{Q}_\gamma^*(\uparrow, \gamma) = (2 - \gamma)\uparrow, \mathcal{P}_\gamma(\uparrow, \gamma) = 2\gamma \uparrow \) and \( \mathcal{P}_\gamma^*(\uparrow, \gamma) = 2(2 - \gamma)\uparrow \) are convex functions for each \( \gamma \in [0, 1], \) then \( \mathcal{Q}_\gamma(\uparrow) \) and \( \mathcal{P}_\gamma(\uparrow) \) both are convex fuzzy-IVF. We clearly see that \( \mathcal{Q}_\gamma(\uparrow) \times \mathcal{P}_\gamma(\uparrow) \in L([u, v], \mathbb{F}_0) \) and

\begin{align*}
& \frac{\Gamma(1 + a)}{2(\nu - u)^x} \left[ I_{u^+}^\alpha \mathcal{Q}_\gamma(\nu) \times \mathcal{P}_\gamma(\nu) + I_{u^-}^\alpha \mathcal{Q}_\gamma(u) \times \mathcal{P}_\gamma(u) \right] \\
& = \frac{\Gamma(\frac{1}{2})}{2 \sqrt{\pi}} \int_0^2 (2 - \nu)^{\frac{\gamma}{4\uparrow}} \left( 2\gamma^2 + \frac{1}{2}\right) d\nu + \frac{\Gamma(\frac{1}{2})}{2 \sqrt{\pi}} \int_0^2 (\nu)^{\frac{\gamma}{4\uparrow}} \left( 2\gamma^2 + \frac{1}{2}\right) d\nu \\
& \approx 2.9332\gamma^2,
\end{align*}

Note that
\[
\left( \frac{1}{2} - \frac{\alpha}{(\alpha + 1)(\alpha + 2)} \right) \Delta_s((u, v), \gamma) = \left[ \mathcal{Q}_\gamma(u) \times \mathcal{P}_\gamma(u) + \mathcal{Q}_\gamma(v) \times \mathcal{P}_\gamma(v) \right] = \frac{11}{32} \cdot 8\gamma^2,
\]
\[
\left( \frac{1}{2} - \frac{\alpha}{(\alpha + 1)(\alpha + 2)} \right) \Delta^s((u, v), \gamma) = \left[ \mathcal{Q}_\gamma^*(u) \times \mathcal{P}_\gamma^*(u) + \mathcal{Q}_\gamma^*(v) \times \mathcal{P}_\gamma^*(v) \right] = \frac{13}{32} \cdot 8(2 - \gamma)^2,
\]
\[
\left( \frac{\alpha}{(\alpha + 1)(\alpha + 2)} \right) \nabla_s((u, v), \gamma) = \left[ \mathcal{Q}_\gamma(u) \times \mathcal{P}_\gamma(u) + \mathcal{Q}_\gamma(v) \times \mathcal{P}_\gamma(u) \right] = \frac{7}{2\uparrow}(0),
\]
\[
\left( \frac{\alpha}{(\alpha + 1)(\alpha + 2)} \right) \nabla^s((u, v), \gamma) = \left[ \mathcal{Q}_\gamma^*(u) \times \mathcal{P}_\gamma^*(u) + \mathcal{Q}_\gamma^*(v) \times \mathcal{P}_\gamma^*(u) \right] = \frac{7}{2\uparrow}(0).
\]

Therefore, we have
\[
\left( \frac{1}{2} - \frac{\alpha}{(\alpha + 1)(\alpha + 2)} \right) \Delta((u, v), \gamma) + \left( \frac{\alpha}{(\alpha + 1)(\alpha + 2)} \right) \nabla((u, v), \gamma) \approx \begin{cases}
\frac{11}{32} [8\gamma^2, 8(2 - \gamma)^2] + \frac{7}{2\uparrow} [0, 0] & \text{if } \gamma \in [0, 1]
\\
[2.9332\gamma^2, 2.9332(2 - \gamma)^2] & \text{if } \gamma \in [0, 1].
\end{cases}
\]

It follows that
\[
[2.9332\gamma^2, 2.9332(2 - \gamma)^2] \leq \frac{1}{l}[2.9332\gamma^2, 2.9332(2 - \gamma)^2],
\]

and Theorem 9 has been demonstrated.
Theorem 10. Let $Q, P : [u, v] \to \mathbb{F}_0$ be two convex fuzzy-IVFs, whose $\gamma$-levels define the family of IVFs $Q_\gamma, P_\gamma : [u, v] \to \mathbb{R} \to \mathcal{K}_{\mathcal{C}^+}$ are given by $Q_\gamma(\xi) = \left[Q(\xi, \gamma), Q^*(\xi, \gamma)\right]$ and $P_\gamma(\xi) = \left[P(\xi, \gamma), P^*(\xi, \gamma)\right]$ for all $\xi \in [u, v]$ and for all $\gamma \in [0, 1]$. If $\mathcal{Q} \times \mathcal{P} \in L([u, v], \mathbb{F}_0)$, then

$$
\begin{align*}
\frac{1}{2} \left[ Q^* \left( \frac{u + v}{2} \right) \right] \times P_\gamma \left( \frac{u + v}{2} \right) & \leq \frac{1}{4(v - u)} \left[ \int_{u}^{v} Q \left( u, \gamma \right) \times P \left( u, \gamma \right) du + \int_{u}^{v} Q \left( v, \gamma \right) \times P \left( v, \gamma \right) dv \right] \\
& + \frac{1}{2 \alpha} \left[ \frac{1}{2} - \frac{\alpha}{\alpha + 1} \right] \left[ \int \Delta_\gamma(u, \gamma) \times P \left( u, \gamma \right) du + \int \Delta_\gamma(u, \gamma) \times P \left( v, \gamma \right) dv \right] \\
& + \frac{1}{2 \alpha^2} \left( \frac{1}{2} - \frac{\alpha}{\alpha + 1} \right) \left[ \int \Delta_\gamma(u, \gamma) \times P \left( u, \gamma \right) du + \int \Delta_\gamma(u, \gamma) \times P \left( v, \gamma \right) dv \right] \\
& + \frac{1}{2 \alpha} \left( \frac{1}{2} - \frac{\alpha}{\alpha + 1} \right) \left[ \int \Delta_\gamma(u, \gamma) \times P \left( u, \gamma \right) du + \int \Delta_\gamma(u, \gamma) \times P \left( v, \gamma \right) dv \right] \\
& + \frac{1}{2 \alpha} \left( \frac{1}{2} - \frac{\alpha}{\alpha + 1} \right) \left[ \int \Delta_\gamma(u, \gamma) \times P \left( u, \gamma \right) du + \int \Delta_\gamma(u, \gamma) \times P \left( v, \gamma \right) dv \right].
\end{align*}
$$

where $\Delta_\gamma(u, v) = Q(u) \times P(u) \neq Q(v) \times P(v)$, $\nabla(u, v) = Q(u) \times P(v) \neq Q(v) \times P(u)$, and $\Delta_\gamma(u, v) = \left[\Delta_\gamma(u, \gamma), \Delta_\gamma(v, \gamma)\right]$ and $\nabla_\gamma(u, v) = \left[\nabla_\gamma(u, \gamma), \nabla_\gamma(v, \gamma)\right]$.

Proof. Consider $Q, P : [u, v] \to \mathbb{F}_0$ are convex fuzzy-IVFs. Then, by hypothesis, for each $\gamma \in [0, 1]$, we have

$$
\begin{align*}
& \frac{1}{2} Q^* \left( \frac{u + v}{2} \right) \times P_\gamma \left( \frac{u + v}{2} \right) \\
& \leq \frac{1}{4(v - u)} \left[ \int_{u}^{v} Q \left( u, \gamma \right) \times P \left( u, \gamma \right) du + \int_{u}^{v} Q \left( v, \gamma \right) \times P \left( v, \gamma \right) dv \right] \\
& + \frac{1}{2 \alpha} \left[ \frac{1}{2} - \frac{\alpha}{\alpha + 1} \right] \left[ \int \Delta_\gamma(u, \gamma) \times P \left( u, \gamma \right) du + \int \Delta_\gamma(u, \gamma) \times P \left( v, \gamma \right) dv \right] \\
& + \frac{1}{2 \alpha^2} \left( \frac{1}{2} - \frac{\alpha}{\alpha + 1} \right) \left[ \int \Delta_\gamma(u, \gamma) \times P \left( u, \gamma \right) du + \int \Delta_\gamma(u, \gamma) \times P \left( v, \gamma \right) dv \right] \\
& + \frac{1}{2 \alpha} \left( \frac{1}{2} - \frac{\alpha}{\alpha + 1} \right) \left[ \int \Delta_\gamma(u, \gamma) \times P \left( u, \gamma \right) du + \int \Delta_\gamma(u, \gamma) \times P \left( v, \gamma \right) dv \right].
\end{align*}
$$

Taking multiplication of (67) with $e^{a-1}$ and integrating over $(0, 1)$, we get

$$
\begin{align*}
& \frac{1}{2} Q^* \left( \frac{u + v}{2} \right) \times P_\gamma \left( \frac{u + v}{2} \right) \\
& \leq \frac{1}{4(v - u)} \left[ \int_{u}^{v} Q \left( u, \gamma \right) \times P \left( u, \gamma \right) du + \int_{u}^{v} Q \left( v, \gamma \right) \times P \left( v, \gamma \right) dv \right] \\
& + \frac{1}{2 \alpha} \left[ \frac{1}{2} - \frac{\alpha}{\alpha + 1} \right] \left[ \int \Delta_\gamma(u, \gamma) \times P \left( u, \gamma \right) du + \int \Delta_\gamma(u, \gamma) \times P \left( v, \gamma \right) dv \right] \\
& + \frac{1}{2 \alpha^2} \left( \frac{1}{2} - \frac{\alpha}{\alpha + 1} \right) \left[ \int \Delta_\gamma(u, \gamma) \times P \left( u, \gamma \right) du + \int \Delta_\gamma(u, \gamma) \times P \left( v, \gamma \right) dv \right] \\
& + \frac{1}{2 \alpha} \left( \frac{1}{2} - \frac{\alpha}{\alpha + 1} \right) \left[ \int \Delta_\gamma(u, \gamma) \times P \left( u, \gamma \right) du + \int \Delta_\gamma(u, \gamma) \times P \left( v, \gamma \right) dv \right].
\end{align*}
$$

Similarly, for $Q^*(\xi, \gamma)$, we have

$$
\begin{align*}
& \frac{1}{2} Q^* \left( \frac{u + v}{2} \right) \times P_\gamma \left( \frac{u + v}{2} \right) \\
& \leq \frac{1}{4(v - u)} \left[ \int_{u}^{v} Q \left( u, \gamma \right) \times P \left( u, \gamma \right) du + \int_{u}^{v} Q \left( v, \gamma \right) \times P \left( v, \gamma \right) dv \right] \\
& + \frac{1}{2 \alpha} \left[ \frac{1}{2} - \frac{\alpha}{\alpha + 1} \right] \left[ \int \Delta_\gamma(u, \gamma) \times P \left( u, \gamma \right) du + \int \Delta_\gamma(u, \gamma) \times P \left( v, \gamma \right) dv \right] \\
& + \frac{1}{2 \alpha^2} \left( \frac{1}{2} - \frac{\alpha}{\alpha + 1} \right) \left[ \int \Delta_\gamma(u, \gamma) \times P \left( u, \gamma \right) du + \int \Delta_\gamma(u, \gamma) \times P \left( v, \gamma \right) dv \right] \\
& + \frac{1}{2 \alpha} \left( \frac{1}{2} - \frac{\alpha}{\alpha + 1} \right) \left[ \int \Delta_\gamma(u, \gamma) \times P \left( u, \gamma \right) du + \int \Delta_\gamma(u, \gamma) \times P \left( v, \gamma \right) dv \right].
\end{align*}
$$

From (71) and (72), we have

$$
\begin{align*}
& \frac{1}{2} \left[ Q^* \left( \frac{u + v}{2} \right) \times P_\gamma \left( \frac{u + v}{2} \right) \right] \\
& \leq \frac{1}{4(v - u)} \left[ \int_{u}^{v} Q \left( u, \gamma \right) \times P \left( u, \gamma \right) du + \int_{u}^{v} Q \left( v, \gamma \right) \times P \left( v, \gamma \right) dv \right] \\
& + \frac{1}{2 \alpha} \left[ \frac{1}{2} - \frac{\alpha}{\alpha + 1} \right] \left[ \int \Delta_\gamma(u, \gamma) \times P \left( u, \gamma \right) du + \int \Delta_\gamma(u, \gamma) \times P \left( v, \gamma \right) dv \right] \\
& + \frac{1}{2 \alpha^2} \left( \frac{1}{2} - \frac{\alpha}{\alpha + 1} \right) \left[ \int \Delta_\gamma(u, \gamma) \times P \left( u, \gamma \right) du + \int \Delta_\gamma(u, \gamma) \times P \left( v, \gamma \right) dv \right] \\
& + \frac{1}{2 \alpha} \left( \frac{1}{2} - \frac{\alpha}{\alpha + 1} \right) \left[ \int \Delta_\gamma(u, \gamma) \times P \left( u, \gamma \right) du + \int \Delta_\gamma(u, \gamma) \times P \left( v, \gamma \right) dv \right].
\end{align*}
$$
\[
\frac{1}{2} \mathcal{Q}(\frac{u+v}{2}) \bar{\mathcal{P}}(\frac{u+v}{2}) \leq \frac{\Gamma(a+1)}{4(v-a)} \left[ \mathcal{I}_a^u \mathcal{Q}(v) \bar{\mathcal{P}}(v) + \mathcal{I}_v^a \mathcal{Q}(u) \bar{\mathcal{P}}(u) \right] + \frac{1}{2a} \left( 1 - \frac{a}{(a+1)(a+2)} \right) \nabla(u,v) + \frac{1}{2a} \left( \frac{a}{(a+1)(a+2)} \right) \Delta(u,v).
\]

Hence, the required result. □

5. Conclusions

In this study, we established the Jensen, Schur, HH-inequalities, and Fejér inequalities for convex fuzzy-IVF by means of fuzzy order relation and fuzzy-interval Riemann–Liouville fractional integrals. Our results generalize some well-known inequalities. In the future, we intend to use different classes of convex and nonconvex to study the interval-valued and fuzzy-IVFs and some applications in direction interval optimization and fuzzy interval optimization theory as future research directions.

In our final view, we believe that our work can be generalized to other models of fractional calculus such as Atangana–Baleanue and Prabhakar fractional operators with Mittag–Liffler functions in their kernels. We have left this consideration as an open problem for the researchers who are interested in this field. The interested researchers can proceed as done in references [34,35].

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