The Maximum Distance Problem and Minimum Spanning Trees

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Under what conditions can a subset $E$ of $\mathbb{R}^n$ be covered by a rectifiable curve?

1. In $\mathbb{R}^2$: Jones, Peter (1990). "Rectifiable sets and the Traveling Salesman Problem". Inventiones Mathematicae. 102: 1–15.

2. In $\mathbb{R}^n$: Okikiolu, Kate (1992). "Characterization of subsets of rectifiable curves in $\mathbb{R}^n$". Journal of the London Mathematical Society. 46 (2): 336–348.

Both papers used what are now called Jones’ beta numbers and gave quantitative conditions on $E$ such that it may be covered by a rectifiable curve.
Beta Numbers

Let $E \subset \mathbb{R}^2$ and let $Q$ be a cube. The Beta number of $E$ within the cube $Q$ is given by

$$\beta_E(Q) := \inf \sup_{y \in E \cap Q} \frac{\text{dist}_\infty(y, L)}{l(Q)}.$$ 

Consider the sum of $\beta_E(Q)$ over the family of dyadic cubes $\Delta$:

$$\beta(E) := \text{diam } E + \sum_{Q \in \Delta} \beta_E(Q)^2 l(Q).$$
Theorem

There exists a number $C$ such that if $\beta(E) < \infty$ then $E \subset \mathbb{R}^2$ can be covered by a rectifiable curve of length no more than $C\beta(E)$. Conversely, if $E$ can be covered by a rectifiable curve $\Gamma$, then $\beta(E) < CH^1(\Gamma)$. 
Although, for example, a cube cannot be covered by a rectifiable curve, for any $\epsilon > 0$ there are many curves whose $\epsilon$-neighborhoods $B(\Gamma, \epsilon)$ can cover $E$. 
The Maximum Distance Problem

Let $E \subset \mathbb{R}^n$ be bounded and $\epsilon > 0$. Find the minimizers of

$$\lambda(E, \epsilon) := \inf \{ \mathcal{H}^1(\Gamma) : \Gamma \text{ is compact, connected and } B(\Gamma, \epsilon) \supset E \}.$$

A Naive Heuristic

1. Cover $E$ with some number of closed $\epsilon$ balls with centers $X$.
2. Find a Steiner tree $S_X$ spanning $X$.

Main Question

How close is $\mathcal{H}^1(S_X)$ to $\lambda(E, \epsilon)$?
Let $E \subset \mathbb{R}^n$ be compact and $\epsilon > 0$, we define the $(\epsilon, k)$-minimal Steiner length of $E$ to be

$$\sigma_k(E, \delta) := \inf\{\mathcal{H}^1(S_X) : X \in \mathcal{N}_k(E, \epsilon)\},$$

where $\mathcal{N}_k(E, \epsilon)$ is the family of all $k$-point sets whose $\epsilon$-balls cover $E$ and $S_X$ is any Steiner tree over $X$.

Figure: $X_1 \in \mathcal{N}_4(E, \epsilon)$

Figure: $X_2 \in \mathcal{N}_4(E, \epsilon)$
$(\epsilon, k)$-minimal Steiner length

Figure: $S_{X_1}$ for $X_1 \in \mathcal{N}_4(E, \epsilon)$.

Figure: $(\epsilon, 4)$-minimal Steiner tree
$(\epsilon, k)$-minimal Steiner length

Figure: $S_{X_1}$ for $X_1 \in N_4(E, \epsilon)$.

Figure: $(\epsilon, 4)$-minimal Steiner tree
How close are $\sigma_k(E, \delta)$ and $\lambda(E, \epsilon)$?

1. $\sigma_k(E, \epsilon) \neq \lambda(E, \epsilon)$ for any $k \in \mathbb{N}$.

\[ C \]

\[ E = B(C, \epsilon) \]

\[ B(P, \epsilon) \nsubseteq E \]
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Modified question

What if we allow an arbitrary number of points? i.e.

$$\sigma(E, \varepsilon) := \inf\{H^1(S_X) : X \in \mathcal{N}(E, \varepsilon) \text{ where } \mathcal{N}(E, \varepsilon) = \bigcup_{i=1}^{\infty} \mathcal{N}_k(E, \varepsilon)\}$$
The Case of $E = B(L, \epsilon)$

Figure: Partition $L$ into $N$ subintervals of length $L/N$. 
The Case of $E = B(L, \epsilon)$

Figure: We must lift each circle up and down by $\delta := \delta(L, N, \epsilon)$. 

[Diagram showing circles being lifted up and down by \( \delta \)]
The Case of $E = B(L, \epsilon)$

Figure: Construct a polygonal curve $C_N$ of length $(N + 1)2\delta + L$ spanning points $X_N$ whose $\epsilon$ balls cover $E$. 
The Case of $E = B(L, \varepsilon)$

Figure: Increase $N$ and see that $\delta$ is smaller.
The Case of $E = B(L, \epsilon)$

We have that
\[
L \leq \mathcal{H}^1(S^X_N) \leq \mathcal{H}^1(C_N)
= (N + 1)2\delta + L.
\]

If we can show that $\mathcal{H}^1(C_N) \to L$ as $N \to 0$ we get
\[
\sigma(E, \epsilon) = L = \lambda(E, \epsilon).
\]
The Case of $E = B(L, \epsilon)$

\[
\frac{x^2}{\epsilon} \\
0 \quad \frac{L}{2N} \quad \frac{L}{N}
\]
The Case of $E = B(L, \epsilon)$

\[ \delta = \left( \frac{L}{2N} \right)^2 \frac{2}{\epsilon} = CN^{-2} \]
The Case of $E = B(L, \epsilon)$

We then have

$$\delta = \left(\frac{L}{2N}\right)^2 = CN^{-2}$$

implying

$$\mathcal{H}^1(C_N) = (N + 1)2\delta + L = C\frac{(N + 1)}{N^2} + L \to L$$

as $N \to \infty$.

∴ For $E = B(L, \epsilon)$ we can instead minimize over Steiner trees which span points whose $\epsilon$ balls cover $E$. 
Lemma 3.5 (A, Krishnamoorthy, Vixie; arXiv:2004.07323)

Let $\epsilon > 0$ and let $\Gamma \subset \mathbb{R}^2$ be a rectifiable curve. For any $\delta > 0$, there exists a finite point set $X = \{x_i\}_{i=1}^N$ and another rectifiable curve $\Gamma_*$ containing $X$ such that

$$B(X, \epsilon) \supset B(\Gamma, \epsilon) \quad \text{and} \quad \mathcal{H}^1(\Gamma_*) \leq \mathcal{H}^1(\Gamma) + \delta.$$ 

Theorem 3.7 (Miranda Jr., Paolini, Stepanov; Calc. Var. 27, 287–309 (2006))

Let $\Gamma \subset \mathbb{R}^2$ be a rectifiable curve and with $F_E(\Gamma) > 0$. Then for each $\delta > 0$ there exists a compact connected $\Gamma_*$ such that

$$F_E(\Gamma_*) < F_E(\Gamma) \quad \text{and} \quad \mathcal{H}^1(\Gamma_*) \leq \mathcal{H}^1(\Gamma) + \delta.$$
The General Case in $\mathbb{R}^2$

Proof of Lemma.

1. Parameterize $\Gamma$ with a Lipschitz map $\gamma : I \rightarrow \mathbb{R}^2$ s.t. $L(\gamma) \leq 2\mathcal{H}^1(\Gamma)$.

2. Partition the domain $I$ s.t. a large fraction of $I$ is made up of good pieces ($G_i$) and a small fraction of bad pieces ($B_j$).

3. Cover the neighborhoods of the $\gamma(G_i)$ with endpoints of prongs (as in the case of a line segment) and cover the neighborhoods of $\gamma(B_j)$ with endpoints of spokes.
Proof of Theorem.

1. Work only in the image of $\gamma$.

2. Use Egorof’s theorem to get uniform estimates on a Beta number to decompose $\Gamma$ into a good pieces and a bad piece.

3. Construct vertical lines and small circles around good pieces and construct larger circles around the bad pieces.
Future Work

- $\sigma_k(E, \epsilon)$ for fixed $k \in \mathbb{N}$ computationally and theoretically.

- Configuration spaces of $\mathcal{N}_k(E, \epsilon)$ and how they change as a function of $k$ and $\epsilon$.

- Minimal spanning trees over centers of random covers of $E$.
Thank You!