ON THE SOLUTIONS OF INFINITE SYSTEMS OF LINEAR EQUATIONS

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Abstract

New theorems about the existence of solution for a system of infinite linear equations with a Vandermonde type matrix of coefficients are proved. Some examples and applications of these results are shown. In particular, a kind of these systems is solved and applied in the field of the General Relativity Theory of Gravitation. The solution of the system is used to construct a relevant physical representation of certain static and axisymmetric solution of the Einstein vacuum equations. In addition, a newtonian representation of these relativistic solutions is recovered. It is shown as well that there exists a relation between this application and the classical Haussdorff moment problem.

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1 Introduction

We study in this work the existence of solution for an algebraical system with an infinite number of linear equations and unknowns whose matrix of coefficients is a Vandermonde matrix. This kind of system of equations broadly arises, for example, in the field of plasma physics [3] and research on this topic becomes of interest for both a theoretical point of view as well as experimental problems in plasma or nuclear physics. In this work we apply our results into the field of exact solutions in the General Relativity (GR) Theory of Gravitation where a relevant representation of some static and axially symmetric solutions of the Einstein vacuum field equations can be obtained by means of solving that kind of systems of linear equations.

In the mathematical literature already exists from the nineteenth century the interest in some cases where an infinite system of linear equations needs to be solved [9]. A method so called the finite section method is introduced in the work of Fourier (cited in [9]). This method is an approach to finding a solution of an infinite system of linear equations. A rigorous and robust analysis of the problem has been developed in [10], as well as previous works have enquired this topic [7]. Is not our aim to improve their results but to use these mathematical techniques into a theoretical physical problem. In [10] the authors study how the infinite section method works for the class of systems described by an infinite Vandermonde matrix. Here we provide the problem with alternative theorems proving the existence of solutions of such kind of systems of linear equations for a determined set of the independent terms of the system whenever the coefficients of the Vandermonde matrix fulfill some condition. These results are applied to some toy models as well as to solve an infinite system of linear equations that appears at the field of relativistic exact solutions of Einstein equations.

In [12] the aim of the work consists of looking for some object whose newtonian gravitational potential reproduces the metric function of some Weyl solutions [1]. The solution with spherical symmetry of the Einstein vacuum equations is given by the Schwarzschild space-time. The relevant metric function of the line element of this solution can be interpreted as the gravitational potential of a bar of length $2M$ ($M$ being the mass of the solution) with constant linear density $\mu = 1/2$. In [12] we proved, by carrying on with the spherical-symmetry analogy, that it is possible to construct a well-behaved linear density of a bar, amended with other characteristics, which allows us to provide a physical interpretation of some Weyl solutions.
In particular the LM solutions, with very interesting physical properties (see [11]), is developed. An alternative way to obtain a definition of the density given in [12] comes from a discrete outlook of the bar by considering it as the infinite sum of point-like particles. Hence, we can identify the infinite sum of Curzon-like potentials [5] associated to these masses placed along the bar, with the potential of the bar obtained by means of an integral involving the linear density. We make use of a limit, where the number of points of the discrete version of the bar goes to infinity in such a way that the Riemann integral sums lead to the Leibnitz integral. Hence a continuous definition of the bar is recovered. The matching between both discrete and continuous descriptions of the bar allows us to supply the two families of parameters appearing at the discrete version of the bar with a physical meaning, as well as to construct a linear density for the continuous bar.

Summarizing, we could say that determined Weyl solutions may be written as a linear combination of infinite Curzon solutions and they can be interpreted by means of an artificial object whose gravitational potential provides the metric function of the solution. Now, the unsolved question is to know whether any solution of the Weyl family can be written as an infinite sum of Curzon-like potentials bounded by a finite length along the symmetry axis. The solution of this problem is equivalent to solve an algebraical system with an infinite number of linear equations whose matrix of coefficients is a Vandermonde type and its matrix of independent terms consists of the set of Weyl coefficients. The theorems proved in this work try to shed light into this problem.

The work is organized as follows: Section 2 is devoted to introduce the mathematical problem to solve and some theorems are proved. Section 3 contains some examples and applications; first, we deal with two different types of successions for the set of independent coefficients of the linear system of equations. Second, we apply the theorems into the field of GR: when trying to write an static and axisymmetric solution of the Einstein vacuum equations as an infinite sum of Curzon solutions [5], a system of linear equations arises. In addition we recover a newtonian representation of some relativistic solutions by means of the linear density of certain object [12]. We show that a discrete version of the problem leads to a characterization of an infinite system of linear equations that can be solved from the continuous definition of the gravitational potential defined in terms of the density. The LM [12] and Erez-Rosen [4] solutions are addressed, and differences with respect to the spherical case (Schwarzschild solution) are discussed.
Section 4 is devoted to show that there exists a relationship between the previous problem and the classical Hausdorff moment problem. A section with conclusions follows this section and finally an appendix that contains some classical known results about newtonian gravity, which are used in the text, is added.

2 Existence of solution for an infinite system of linear equations

Let us consider a system of \( n \) linear equations, \( A = \mathcal{V}Y \), whose matrix of coefficients is given by a Vandermonde matrix,

\[
\mathcal{V}_n = \mathcal{V}_n(x_1, \ldots, x_n) = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
x_1 & x_2 & \cdots & x_n \\
x_1^2 & x_2^2 & \cdots & x_n^2 \\
\vdots & \vdots & \ddots & \vdots \\
x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1}
\end{pmatrix} ; \quad (\mathcal{V}_n)_{ij} = x_{i}^{j-1}
\]

where \( x_1, \ldots, x_n \) is a set of \( n \) real (or complex) numbers, associated to the Vandermonde matrix, such that \( x_i \neq x_j, \forall i \neq j \), and \( A \) represents the \((n \times 1)\)-matrix of independent terms \((A)_{i} = a_i\), and \( Y \) represents the \((n \times 1)\)-matrix of unknown variables \((Y)_{i} = y_i\).

Since the Vandermonde matrix is invertible the solution of the above mentioned system of linear equations is unique for any dimension \( n \) of the system. The question is whether we also can obtain solutions of that system if its dimension \( n \) goes to infinity. The finite section method [10] works as follows: we consider the first \( n \) equations and \( n \) unknowns and we solve this finite system, where the rest of the terms have been neglected. As \( n \) grows larger these solutions are expected to approximate a solution of the infinite system. In what follows we present a theorem that prove the convergence of this procedure if the sets of independent terms and Vandermonde parameters fulfill some particular conditions.

**Theorem 1: A sufficient condition**

The solution of a system of infinite linear-equations with a Vandermonde matrix of coefficients exists if the Vandermonde parameters \( \{x_i\} \) fulfill the conditions \( x_i \neq x_j, \forall i \neq j \), \( x_i = 1 \) for some \( i \), and the set \( \{a_i\} \) of indepen-
dent terms of the system is a bounded, monotonic succession (and therefore convergent) with $a_i \neq 0 \quad \forall i$.

**Proof:**

If we write explicitly the unknown variables in terms of the inverse of the Vandermonde matrix we obtain the following expression:

$$y_k = \sum_{i=1}^{\infty} b_i^{(k)} a_{i-1}, \quad k = 1..\infty \tag{2}$$

where $b_i^{(k)} \equiv (\mathcal{V}_{\infty})^{-1}_{ki}$ represents the $(ki)$-coefficient of the inverse Vandermonde matrix of infinite dimension.

The expression (2) for the unknown variables is a series whose convergence is the point we want to prove for any value of $k$. We limit ourself to consider the independent terms of the system (the matrix $A$) in such a way that the set $\{a_i\}$ be a convergent succession with $a_i \neq 0 \quad \forall i$ . For this case we make use of the Dirichlet-Abel theorem that holds the convergence of the series (2) if the series $\sum_{i=1}^{\infty} b_i^{(k)}$ converges as well. The coefficients $b_i^{(k)}$ can be written as follows

$$b_i^{(k)} = \frac{v_i^{(k)}}{f_k},$$

where $v_i^{(k)}$ are the coefficients of the following $n - 1$ degree polynomial:

$$F_j(z) = \prod_{i=1, i \neq j}^{n} (z - x_i) = \sum_{i=1}^{n} v_i^{(j)} z^{i-1},$$

and

$$f_j(x_1, \ldots, x_n) \equiv f_j = F_j(x_j).$$

We can prove that every series $\sum_{i=1}^{\infty} b_i^{(k)}$ is convergent if a suitable choice of the Vandermonde parameters is considered; in fact, it is easy to see that the notation used leads to the following sum $S^{(k)}$ of those series:

$$S^{(k)} \equiv \sum_{i=1}^{\infty} b_i^{(k)} = \frac{F_k(z = 1)}{f_k}, \tag{3}$$
i.e., the sum of those series is just the quotient of the values of the polynomial $F_k(z)$ at the points $z = 1$ and $z = x_k$ respectively. If we consider the Vandermonde parameters such that one of them is equal to 1, i.e., $x_i = 1$, for $i = i_0$, then we can get a convergent sum because the partial sum of order $n$ of the series (3) are

$$S_n^{(k)} = \sum_{i=1}^{n} a_i b_i^{(k)} = \prod_{i=1, i \neq k}^{n} \left( \frac{1-x_i}{x_k-x_i} \right) = \begin{cases} 1, & k = i_0 : x_k = 1 \\ 0, & k \neq i_0 : x_k \neq 1 \end{cases}, \quad \text{for} \quad k \neq i_0,$$

(4)

and consequently, the series (2) are convergent for any $k$ since the sums are bounded $S^{(k)} = \lim_{n \to \infty} S_n^{(k)} \neq \pm \infty$.

In this theorem we have forced the succession $\{a_i\}$ to possess no-zero terms. Nevertheless, it is clear that this condition can be reduced to impose that the succession $\{a_i\}$ contains a finite number of zero terms since the convergence of the series (3) is unchanged in that case. On the contrary, the special case with a finite number of terms $a_i$ different to zero needs an alternative treatment. In particular the case with $\{a_i\} = \{\delta_j\}_{j=1}^{\infty}$ is solved in [10].

As a consequence of the previous theorem the solution of the infinite system of linear equations is given by the following expression:

$$y_k = \sum_{i=1}^{\infty} v_i^{(k)} a_{i-1}. \quad \text{(5)}$$

In the following theorem we introduce an algebraical equation equivalent to the system of Vandermonde linear equations. The solution of this new equation for some toy models, both for the cases with a finite or infinite number of unknowns, is obtained in the following section.

**Theorem 2: A necessary and sufficient condition**

The solution of a system of $n$ (\forall n) linear equations, $A = \mathcal{V}_n Y$, whose matrix of coefficients is given by a Vandermonde matrix, exists iff the following homogeneous differences equation is fulfilled:

$$\sum_{i=1}^{n+1} v_i a_{i-1} = 0, \quad \text{(6)}$$
where \( v_i \) are the coefficients of the following \( n \) degree polynomial:

\[
F(z) = \prod_{i=1}^{n} (z - x_i) = \sum_{i=1}^{n+1} v_i z^{i-1} .
\] (7)

Proof:

A) The necessary condition: Let us suppose that given a succession \( \{a_i\} \) it can be written in the form \( A = V_n Y \) for some sets of parameters \( \{y_i\} \), and \( \{x_i\} \), or equivalently that a succession \( \{y_i\} \) verifies the condition (2), i.e. a solution of the system of infinite linear-equations exists. Therefore, we can multiply each file \( i \) of the system by \( v_i \) and consider the sum of all the files:

\[
v_1 a_0 + v_2 a_1 + \ldots + v_n a_{n-1} = y_1 \left( \sum_{j=1}^{n} v_j x_1^{j-1} \right) + \ldots + y_n \left( \sum_{j=1}^{n} v_j x_n^{j-1} \right) .
\] (8)

By taking into account that \( \left( \sum_{j=1}^{n} v_j x_a^{j-1} \right) = -x_a^n \) since \( F(x_a) = 0 \) for all \( a = 1..n \), then we can write the following expression

\[
\sum_{i=1}^{n} v_i a_{i-1} = - \sum_{i=1}^{n} x_i^n y_i ,
\] (9)

and we conclude this part of the proof by means of the starting condition \( a_k = \sum_{j=1}^{n} x_j^k y_j \) and the following fact: \( v_{n+1} = 1 \).

B) The sufficient condition: Let us consider the following differences equation for the succession \( \{a_k\} \):

\[
a_{k+n} v_{n+1} + \ldots + v_3 a_{k+2} + v_2 a_{k+1} + v_1 a_k = 0\, ,
\] (10)

or equivalently, we can afford solving the following equation: \( p(\lambda) a_k = 0 \) where \( p(\lambda) = \lambda^n v_{n+1} + \lambda^{n-1} v_n + \ldots + \lambda v_2 + v_1 \) is the characteristic polynomial of the differences equation (10). The general solution of this equation is the following:

\[
a_k = \sum_{i=1}^{n} p_i^k \left( \beta_{i_1} + k \beta_{i_2} + k^2 \beta_{i_3} + \ldots + k^{m_i} \beta_{i_{m_i-1}} \right) ,
\] (11)
where \( p_i \) are the roots of the characteristic polynomial \( p(\lambda) \) with multiplicity \( m_i \) for \( i = 1..n \) and \( \beta_j \) are arbitrary constants. Hence, we conclude this part of the proof since the roots \( p_i \) are the Vandermonde coefficients \( x_i \), and we take \( \beta_{i_2} = \beta_{i_3} = \ldots = \beta_{i_{m_i-1}} = 0 \), as well as \( \beta_{i_1} = y_i \) fulfilling the initial condition \( a_0 = \sum_{i=1}^{\infty} y_i \).

\[ \square \]

Theorem 3: Corollary

With a set of Vandermonde parameters given by Theorem 1, \( \{x_i\} \) fulfilling the conditions \( x_i \neq x_j, \forall i \neq j, x_i = 1 \) for some \( i = i_0 \) any convergent succession \( \{a_n\} \) of positive terms verifies the following homogeneous differences equation \( (n \to \infty) \):

\[ \sum_{i=1}^{n+1} v_i a_{i-1} = 0, \quad (12) \]

where \( v_i \) are the coefficients of the following \( n \) degree polynomial:

\[ F(z) = \prod_{i=1}^{n} (z - x_i) = \sum_{i=1}^{n+1} v_i z^{i-1}. \quad (13) \]

Proof: It is a consequence of Theorem 1 and Theorem 2.

3 Some examples and applications

3.1 Toy models

A) A constant succession

This kind of succession \( \{a_k = a\} \) verifies the following differences equation

\[ a_{k+1} - a_k = 0, \quad (14) \]

and therefore \( (15) \) implies that \( v_1 = -1, v_2 = 1, v_i = 0, \forall i \neq 1, 2 \). One solution of the equation \( (14) \) is obtained from the characteristic polynomial:

\[ (\lambda - 1)a_k = 0 \iff a_k = c, \quad (15) \]
where \( c \) is an arbitrary constant. Hence, for the case of a constant succession, one particular selection of the sets of coefficients \( \{x_i\} \) and \( \{c_i\} \) that verify the condition (27) are the following

\[
x_{i_0} = 1, \quad c_{i_0} = c = a, \quad x_i = c_i = 0 \quad \forall i \neq i_0.
\] (16)

This selection is a degenerate case of the Theorem 1 since \( x_i = x_j, \forall j \neq i_0 \), and therefore the matrix of the system is not a Vandermonde type matrix. In fact, the determinant of the matrix of the system is non zero only for \( n = 1 \).

We give now an alternative argument to get the previous result (16): By virtue of Theorem 2 the condition (6) is equivalent to the existence of the solution (5). For the case of a constant succession we find from (6) that

\[
a \sum_{i=1}^{\infty} v_i = 0 \quad \text{which is always true for any set } \{x_i\} \text{ if some point } x_{i_0} \text{ is equal to } 1, \text{ since } F(z = 1) = \sum_{i=1}^{n} v_i (z = 1)^{i-1} = 0 \text{ (see the equation (7)). In fact, we know that (see the equation (4) in Theorem 1)}
\]

\[
\frac{1}{f_k} \sum_{i=1}^{n} v_i^{(k)} = \delta_{k,i_0}.
\] (17)

Let us take for example \( \{x_k = k\} \) \((i_0 = 1, x_{i_0} = 1)\) and we find the following coefficients (5) \( \{c_i\} \):

\[
c_k = \lim_{n \to \infty} \frac{a(-1)^{k-1}(n-k+1)!}{(k-1)!n!} = \delta_{k1}.
\] (19)

\[1\text{ The following relations have been used}
\]

\[
f_k = (n-k)!(1)^{n-k}(k-1)! , \quad \sum_{i=1}^{n} v_i^{(k)} = (n-1)!(1)^{n-1}
\] (18)

**B) A geometric succession**

This kind of succession \( \{a_k = c \eta^k\} \) verifies the following differences equation

\[
a_{k+1} - a_k \eta = 0 ,
\] (20)

and therefore (6) implies that \( v_1 = -\eta, v_2 = 1, v_i = 0, \forall i \neq 1, 2 \). One solution of the equation (20) is obtained from the characteristic polynomial:

\[
(\lambda - \eta)a_k = 0 \iff a_k = \eta^k c ,
\] (21)
where $c$ is an arbitrary constant, and hence, for the case of a geometric succession suitable sets of coefficients $\{x_i\}$ and $\{c_i\}$ that verify the condition (27) are the following

$$x_{i_0} = \eta, \quad c_{i_0} = c = a_0, \quad x_i = c_i = 0 \quad \forall i \neq i_0. \quad (22)$$

If we alternatively use the equation (5) to obtain the set $\{c_i\}$ we find the following conclusion if we take $x_{i_0} = \eta$:

$$c_k = \lim_{n \to \infty} \sum_{i=1}^{n} v^{(k)}_i \eta^{i-1} c \frac{f_k}{f_k} = c \delta_{ki_0}. \quad (23)$$

It is clear that $\eta = 1$ ends up with the previous case.

### 3.2 Representations of relativistic solutions

#### 3.2.1 Static and axisymmetric vacuum solutions.

As is known, the line element of a static and axisymmetric vacuum space-time is represented in Weyl form as follows

$$ds^2 = -e^{2\Psi} dt^2 + e^{-2\Psi} \left[ e^{2\gamma} \left( d\rho^2 + dz^2 \right) + \rho^2 d\varphi^2 \right], \quad (24)$$

where $\Psi$ and $\gamma$ are functions of the cylindrical coordinates $\rho$ and $z$ alone. The metric function $\Psi$ is a solution of the Laplace’s equation ($\Delta \Psi = 0$), and the other metric function $\gamma$ satisfies a system of differential equations whose integrability condition is just the equation for the function $\Psi$. The Weyl family of solutions with a good asymptotical behaviour is given in spherical coordinates $\{r, \theta\}$ as the series ($r \equiv \sqrt{\rho^2 + z^2}$, $\cos \theta = z/r$)

$$\Psi = \sum_{k=0}^{\infty} a_k \frac{r^{k+1} P_k(\cos \theta)}{r^{k+1}}. \quad (25)$$

The Weyl series (25) could be rewritten as the following linear superposition of Curzon solutions:

$$\Psi = \sum_{k=0}^{\infty} \frac{a_k}{r^{k+1}} P_k(\cos \theta) = \sum_{k=1}^{\infty} \frac{c_k}{\sqrt{\rho^2 + (z - z_k)^2}}, \quad (26)$$

where the Curzon solution [5] corresponds to a point-like particle with mass $c_k$ located at the point $z_k$ on the $Z$ axis. The equation (26) for the metric
function $\Psi$ requires the following relation between Weyl coefficients and the parameters $c_i, z_i$:

$$a_k = \sum_{i=1}^{\infty} z_i^k c_i, \forall k.$$  \hfill (27)

The expression (27) for $a_k$ can be seen as an algebraical condition written in matrix form as follows:

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \ldots \\ z_1 & z_2 & z_3 & \ldots \\ z_1^2 & z_2^2 & z_3^2 & \ldots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix}. \hfill (28)$$

The question previously formulated is whether any set of coefficients $\{a_k\}$ can be written in that form (27), or equivalently we want to know whether every solution of the Weyl family could be written in the form (26). Hence we are looking for sets of values $\{z_k\}, \{c_k\}$ corresponding to certain set of known values $\{a_k\}$. Equation (28) represents an infinite system of linear equations that is invertible at any order if the matrix of coefficients possesses a Vandermonde determinant which is different from zero iff $z_i \neq z_j, \forall i \neq j$. In the previous section we have proved some theorems that show the existence of a convergent solution of this system with an infinite number of linear equations: a suitable choice of the Vandermonde parameters $\{z_k\}$, if the coefficients $\{a_k\}$ generate a convergent succession with a finite number of null terms, is a sufficient condition.

The toy models that we have previously studied correspond to a particular set of Weyl coefficients. From those results, we already can state that the solution of the Weyl family whose coefficients $\{a_k\}$ are of all them equal to a constant, can be represented by a single point-like particle of mass $c$ displaced from the origin of coordinates. Let us note that this is a known result because a single point-like mass at the position $z = 1$ is described with the following expression:

$$\Psi = \frac{-c}{\sqrt{\rho^2 + (z-1)^2}}, \hfill (29)$$

and this metric function can be written in the Weyl form, by means of the decomposition of that function in terms of Legendre polynomials series, as...
follows \((\omega \equiv \cos \theta)\):

\[
\Psi = \sum_{k=0}^{\infty} \frac{a_0}{r^{k+1}} P_k(\omega) , \quad a_0 = -c .
\]

Obviously, the trivial case of a single point-like particle of mass \(c\) located at the origin of coordinates (the original Curzon solution), i.e., \(\Psi = \frac{-c}{\sqrt{\rho^2 + z^2}} = \frac{a_0}{r} P_0(\omega)\) corresponds to the set of coefficients \(\{c_1 \equiv c = -a_0, c_i = 0, \forall i \neq 1\}\), \(\{z_1 = 0\}\).

In addition, from the example B) in section 3.1 we can say that the solution of the Weyl family whose coefficients are given by a geometric succession \(\{a_k = c \eta^k\}\) is represented by a single point-like particle of mass \(c\) located at the point \(z = \eta\).

The requirement imposed on the sets of coefficients \(\{a_k\}, \{c_k\}, \{z_k\}\) by equation (27) supply us with a gauge of freedom for the selection of the coefficients since we are dealing with two sets of arbitrary parameters in the second series of the equation (26) instead of the unique set of Weyl coefficients. In the following subsection we address the resolution of the equation (27) by means of an infinite set of coefficients \(\{z_i\}\), for a class of static and axisymmetric relativistic solutions.

### 3.2.2 A Newtonian representation of the Weyl solutions.

In [12] a description of some relativistic solutions by means of a singular Newtonian source is developed. The argument used is the equivalence between the newtonian gravitational potential \(\Phi\) and the metric function \(\Psi\) of the relativistic solution. An object so called *dumbbell* is constructed to describe a class of static and axisymmetric solutions. This object consists of a bar of length \(2M\) with determined linear density (\(M\) being the mass of the relativistic solution) and a ball at each end of the bar. The density of the bar is used to describe physical properties and relevant characteristics of the relativistic solution. In the Appendix we address a brief review about Newtonian Gravity contents. Some formulae that we shall use in what follows are introduced.

The aim of this section is approximating a bar, endowed with a continuous line density, with a series of point-like masses. The corresponding gravitational potential could be written as a linear combination of Curzon
solutions. In other words, we want to search for a set of coefficients \( \{c_i\} \) in such a way that a linear superposition of Curzon solutions with respective masses \( c_i \) placed in points \( z_i \) along the Z-axis provides a potential \( \Psi \) like (26), as well as the relation (27) could be satisfied.

We are constrained by the fact that our discrete representation of the bar is requested to recover the linear density of this object. The set of coefficients \( \{z_i\} \) should provide us with a continuous distribution of points along the \( Z \)-axis between both ends of the bar. The transition from the integral expression (A4) to a Riemann integral sum is made by means of a limit that makes the width of the sub-intervals of integration goes to zero. Consequently the number of points of the partition on the interval \([-L, L]\) tends to infinity in such a way that we handle with the following homogeneous distribution of \( 2n \) points:

\[
z_i = \pm i \frac{L}{n}, \quad i = 1 \ldots n.
\]

Within this selection of coefficients \( \{z_i\} \), the solution of the equations (27) provide us with a set of coefficients \( \{c_i\} \) representing the masses located at each point \( z_i \). The discrete consideration of the bar, as well as the implementation of the condition (26) allows us to identify the coefficients \( \{c_k\} \) and supply them with a physical significance from the gravitational potential of the bar (A4). The integral appearing at this equation can be given as a limit (Leibnitz notation) of the Riemann integral sums:

\[
\Phi(\vec{x}) = -\int_{-L}^{L} \frac{\mu(z')}{\sqrt{\rho^2 + (z - z')^2}} dz' = \lim_{\ell \rightarrow 0} \sum_i K(\varphi, f) \Delta z_i ,
\]

where \( f(z') \equiv -\frac{\mu(z')}{\sqrt{\rho^2 + (z - z')^2}} \), \( \varphi \) denotes a partition of the integration interval, \( \Delta z_i \) is the width between two adjoining points of the partition, \( \ell_p \) being the symbol to define the generic width of the partition, and \( K(\varphi, f) \) represents the maximum or minimum of the function \( f \) in each sub-interval. In fact, the required identification between the Newtonian potential (A4) and the relativistic metric function \( \Psi \) (26) leads to the following interpretation of the quantities \( c_i / \Delta z_i \): they must be the maximum or minimum values of the density \( \mu(z) \) in each sub-interval, and hence the following relation could be fulfilled:

\[
\lim_{n \rightarrow \infty} \sum_{i=1}^{n} \frac{c_i / \Delta z_i}{\sqrt{\rho^2 + (z - z_i)^2}} \Delta z_i = -\int_{-L}^{L} \frac{\mu(z')}{\sqrt{\rho^2 + (z - z')^2}} dz'.
\]
Therefore, an infinite set of point-like particles with respective
masses defines a collection of Curzon potential whose masses are
infinitesimal quantities characterized by the following way:
\[ \mu(z_i) \equiv -\frac{c_i}{\Delta z_i} \quad \text{and} \quad \lim_{n \to \infty, \ell \to 0} \mu(z_i) = \mu(z)\delta(z - z_i) \quad \text{(34)} \]

Hence the density of the dumbbell bar can be defined as the continuous limit of a discrete function that is defined at each point \( z_i \) of the bar with the value \(-c_i/\Delta z_i\).

**A) The Schwarzschild solution.**

We are going to analyze thoroughly the case of spherical symmetry. For this case we can prove the following theorem:

**Theorem**

With the partition of the interval \([-L,L]\) given by \((31)\) the Weyl coefficients \(\{a_k\}\) of the Schwarzschild solution fulfill the condition \((27)\), when \(n\) goes to infinity, if \(c_i = -\frac{L}{2n}, \forall i\).

Proof:

Accordingly to \((27)\) we have to prove the following condition:

\[ a_{schw}^{k} = \frac{1}{2} \lim_{n \to \infty} \left( \frac{L}{n} \right)^{k+1} \left[ \sum_{i=1}^{n} i^k + \sum_{i=1}^{n} (-i)^k \right] \quad \text{(35)} \]

Firstly let us note that the first equation in \((27)\) \((k = 0)\) implies the following constraint:

\[ \sum_{i=1}^{\infty} c_i = a_0^{schw} \Leftrightarrow \lim_{n \to \infty} \sum_{i=1}^{2n} c_i = -M \quad \text{(36)} \]

which is verified by the formulated coefficients \(c_i\) if we take \(L = M\). Secondly, it is also clear from \((35)\) that odd coefficients \(a_{2k+1}^{schw}\) are zero because of the equatorial symmetry \((|z_i| = iL/n)\). And finally, in the general case \(k = 2j\) we have

\[ a_{2j}^{schw} = -\lim_{n \to \infty} \left( \frac{L}{n} \right)^{2j+1} \sum_{i=1}^{n} i^{2j} = \]
\[ = -\lim_{n \to \infty} \left( \frac{L}{n} \right)^{2j+1} \left[ \frac{n^{2j+1}}{2j+1} + \frac{1}{2} n^{2j} + O(n^{2j-1}) \right] \quad \text{(37)} \]
where $O(n)^{2j-1}$ denotes all terms with powers of $n$ less than $2j$. Hence equation (37) shows that Weyl coefficients for the Schwarzschild solution are
\[ a_{2j}^{schw} = -\frac{L^{2j+1}}{2j+1} \]
for all $j \geq 1$, and so we recover from the continuous limit (34) the known result for the density of the bar in this case:
\[ -\frac{L}{2n} \equiv c_i : \mu(z_i) = -\lim_{n \to \infty} \frac{c_i}{L/n} \iff \mu(z_i) = \frac{1}{2} , \quad \forall i , \quad (38) \]
where we have used $\triangle z_i \equiv dz = z_{i+1} - z_i = \frac{L}{n}$.

\[ \square \]

B) The Erez-Rosen vs the M-Q$^{(1)}$ solution.

These are both two-parameters non spherical symmetric solutions of Weyl family. One of the parameter $M$ represents the mass and the other $q$ or $q_2$ denoting the dimensionless quadrupole moment for M-Q$^{(1)}$ or ER solutions respectively, and related by $q_2 = \frac{15}{2} q$ [2]. In [8] a comparison between both solutions were done and different conclusions were obtained regarding the behaviour of gyroscope precessing in circular orbits into these gravitational fields. The Weyl coefficients of the ER solution are known [2]:
\[ a_{2k}^{ER} = -\frac{M^{2k+1}}{2k+1} \left( 1 + q_2 \frac{2k}{2k+3} \right) , a_{2k+1}^{ER} = 0 , \quad (39) \]
and a representation of this solution by means of a bar with linear density
\[ \mu^{ER}(X) = \frac{1}{2} \left( 1 - \frac{q_2}{2} \right) + \frac{3}{4} q_2 X^2 , \quad X \equiv \frac{z}{M} \in [-1,1] , \quad (40) \]
is carried out in [12]. We can solve the equations (27) for the set of coefficients \{c_i\} assuming that $|z_i| = iL/n$ in analogy with the spherical case (Schwarzschild’s solution). It is easy to prove that
\[ c_i = -\left( 1 - \frac{q_2}{2} \right) \frac{L}{2n} - \frac{3}{4} q_2 \left( \frac{L}{n} \right)^3 i^2 , \quad (41) \]
since these quantities verifies $(L = M)$
\[ a_{2k} = \lim_{n \to \infty} \sum_{k=1}^{n} z_i^{2k} c_i = \]

\[ ^{2} \text{These factors are given by the Bernoulli numbers (see [6] for details).} \]
\[- \frac{1}{2} \lim_{n \to \infty} \left[ \left(1 - \frac{q_2}{2} \right) \left( \frac{L}{n} \right)^{2k+1} \sum_{i=1}^{n} \left( \pm i \right)^{2k} - \frac{3}{2} \frac{q_2}{M^2} \left( \frac{L}{n} \right)^{2k+3} \sum_{i=1}^{n} \left( \pm i \right)^{2k+2} \right] =
\] \[- \left(1 - \frac{q_2}{2}\right) \frac{L^{2k+1}}{2k+1} - \frac{3}{2} \frac{q_2}{M^2} \frac{L^{2k+3}}{2k+3} = a_{2k}^{ER}. \quad (42)\]

In addition we see that these selection of coefficients \(\{z_i\}, \{c_i\}\) allows us to recover the density (40) by satisfying the condition (34):
\[\frac{c_i}{\triangle z_i} = \left(1 - \frac{q_2}{2}\right) - 3 \frac{q_2}{4} \left( \frac{iL}{nM} \right)^2 = -\mu(X_i). \quad (43)\]

In contrast with the ER solution, the M-Q\(^{(1)}\) solution is represented by a dumbbell consisting of a bar of length 2\(L\) and linear density \(\mu^{MQ^{(1)}}(X) = \frac{1}{2} \left( 1 - \frac{15}{8} q \right) + \frac{15}{16} q X^2\), \quad (44)\)
and a point-like particle at each end of the dumbbell with respective mass \(\nu = \frac{5}{8} q M\).

We can solve the system of equations (27) for the unknown set of coefficients \(\{c_i\}\) in a similar way used for the Schwarzschild solution (35-37). We suppose again an homogeneous distribution of points \(\{z_i\}\) recovering the bar from one end to another (31). The M-Q\(^{(1)}\) solution is a subclass of the LM solution (12), and for all these solutions we can make use of the following decomposition of the coefficients \(\{a_k\}\):
\[a_{2k}^{LM} = - \frac{g}{2k+1} \frac{M^{2k+2j+1}}{2k+2j+1} \frac{H_j}{M^{2j}} - M^{2k+1} H, \quad (45)\]
where \(H\) and \(H_j\) are well defined coefficients (in particular \(H_j/2\) are the coefficients of the density \(\mu(X)\) (12)). Therefore, it is easy to prove that the solution of (27) for this case must be obtained by splitting the expression

\[a_{2k} = - \frac{M^{2k+1}}{2k+1} \left( 1 + \frac{5k(k+2)}{2k+3} \right) \quad (\text{In particular, the Weyl coefficients of the M-Q}^{(1)}\text{ solution are})\]
\[a_{2k} = - \frac{M^{2k+1}}{2k+1} \left( 1 - \frac{15q}{8} \right) - \frac{M^{2k+3}}{2k+3} \left( \frac{15q}{8M^2} \right) - M^{2k+1} \frac{5}{4} q. \quad (\text{and they can be decomposed as follows:})\]

\[
16
\]
in two parts since it provides two kind of successions for \( \{a_n\} \). On the one hand, with respect to the second term in (45) we must consider one point-like mass located at each end of the bar. The general term of the succession associated to the second part of (45) \( \{-M^{2k+1} H\} = \{\eta^{2k} b\} \) provides a geometric type succession with rate \( \eta = M \) and initial term \( a_0 = b = -M H \), and therefore the parameter \(-b/2\) acquires the meaning of being the mass of each point-like particle located at both ends of the bar.

On the other hand, with respect to the first part of the Weyl coefficients (45) is easy to prove, in similar way to the applied for Schwarzschild solution (37) that the corresponding coefficients \( \{c_i\} \) of the MQ\(^{(1)} \) solution fulfilling the equations (27) are

\[
c_i = -\frac{1}{2} \sum_{j=0}^{1} \left( \frac{L}{n} \right)^{2j+1} i^{2j} \frac{H_j}{M^{2j}}, \quad H_0 = 1 - \frac{15}{8} q, \quad H_1 = \frac{15}{8} q. \quad (46)
\]

Before concluding this section it should be pointed the consistency between formulae derived from Theorems 1 and 2, and these results obtained above. Let us consider the notation \( \{x_i = z_i/L\} \) and hence the coefficients \( \{a_i\} \) of the theorems transforms into \( \{a_i/L^i\} \). The equation (5) in previous section allows us to calculate the coefficients \( \{c_i\} \) for a given set of \( \{z_i\} \) and \( \{a_i\} \). By taking into account that \( \{a_i\} \) represent the newtonian multipole moments of the object whose density has been defined, we can put expression (A5) into (5) to obtain:

\[
c_k = \frac{1}{f_k} \sum_{i=1}^{\infty} v_i^{(k)} a_{i-1} = -\int_{-1}^{1} L \mu(Lx) \left[ \frac{1}{f_k} \sum_{i=1}^{n} v_i^{(k)} (Lx)^{i-1} \right] dx
\]

4An alternative argument comes from the direct identification between the gravitational potential of two particles of mass \( m \) situated at distances \(-L\) and \( L \) respectively along the Z axis, \( \Psi = \Phi = -\frac{m}{\sqrt{\rho^2 + (z+L)^2}} - \frac{m}{\sqrt{\rho^2 + (z-L)^2}} \), with the corresponding metric function by means of performing a power series expansion as follows: \( \Phi = -\frac{m}{\sqrt{\rho^2 + (z+L)^2}} - \frac{m}{\sqrt{\rho^2 + (z-L)^2}} = -\sum_{n=0}^{\infty} \frac{P_{2n}(\omega)}{\gamma^{2n+1}} (2mL^{2n}) \), and therefore the Weyl coefficients related to this potential of two particles are \( a_{2n} = -2mL^{2n} \). Let us remind that the condition (27) is fulfilled and so, \( a_0 = \sum_{i=1}^{\infty} z_i c_i = -2m \). That is to say, the sum of the coefficients \( \{c_i\} \) corresponding to the geometric succession in (45) provides the mass of both balls except for sign.
= - \int_{-L}^{L} \mu(z) \left[ \prod_{j=1, j \neq k}^{n} \frac{z - z_j}{z_j - z_k} \right] dz , \quad (47)

and consequently, by taking into account the equation (4), since \( z \in [-L, L] \) and the Vandermonde parameters \( \{z_i\} \) generate an homogeneous distribution of points from \( z = -L \) to \( z = L \), then we have

\[ c_k = - \lim_{\nu \to 0} \sum_{i=1}^{\infty} \mu(z_i)^{max} \triangle z_i \delta(z_i - z_k) = -\mu(z_k) \triangle z_k . \quad (48) \]

4 Relation with the Hausdorff Moment Problem

In [12] newtonian representations for a class of solutions of the Weyl family are obtained by means of an artificial object so called dumbbell. We must remind that not every solution of the Weyl family can be identified with the potential of a bar, and so the search for the object that is able to describe other solutions may be a matter of consideration for future works. Furthermore we might consider extended newtonian objects, rather than singular sources, to obtain the gravitational potential for each particular case.

We have seen in previous sections that this newtonian representation can be connected with a description of those solutions by means of an infinite sum of Curzon solutions. In fact, Theorem 1 and Theoreme 2 provide conditions to be fulfilled by the Weyl coefficients in order to write the metric function of the solution \( \Psi \) like equation (26).

We should point out that it is possible to obtain general results about the existence of an even density with prescribed moments like those of equation (A5):

\[ M_{2k} = \int_{-1}^{1} z^{2k} \mu(z) dz = \int_{0}^{1} w^k \left[ \frac{\mu(\sqrt{w})}{\sqrt{w}} \right] dw , \quad (49) \]

since Hausdorff [13] proved a set of necessary and sufficient conditions for the existence of a positive function \( f \) with prescribed half-range moments \( b_n \) in the sense of equation (49), \( b_n = \int_{0}^{1} w^n f(w) dw \), that involves the following inequality conditions:

\[ 0 \leq \sum_{j=0}^{k} (-1)^{j+k} \binom{k}{j} b_{n+j} , \quad \forall n, k \geq 0 . \quad (50) \]
We are referring to the classical problem in analysis called the Hausdorff Moment Problem \[14\]. This condition is equivalent to say that the succession of moments is completely monotonic. We want to show that the classical problem of Haussdorf is a (continuous) integral version of the discrete problem outlined by the equation (27), and the inequality conditions (50) established by Haussdorf can be recovered by the equation (6) of the Theorem 2. It is known that the Weyl coefficients are the newtonian multipole moments (A5) of a relativistic solution (see [12] and references therein). Hence we can write the following expression:

\[
\int_{-1}^{1} x^{2k} \mu(x) dx = 2 \int_{0}^{1} x^{2k} \mu(x) dx , \quad a_{2k+1} = 0 ,
\]

if we deal with an even linear density \(\mu(x)\) that is describing the solution, as is the case for the LM solutions [12]. Theorem 2 states that the existence of solution for the infinite system of equations (27) is equivalent to solve the equation (6) (in the limit \(n\) going to infinity). It is easy to see that if equation (6) is fulfilled then the following expression is true as well:

\[
\sum_{j=1}^{n+1} v_j a_{j-1+k} = 0 , \quad k, n .
\]

If we calculate the coefficients \(v_j\) corresponding to the Vandermonde parameters \(z_i = iL/n\), then the following inequalities are obtained:

\[
v_i \leq \binom{n}{i} (-1)^{n+i} ,
\]

and then we can hold the following relation:

\[
0 = \sum_{j=0}^{n} v_{j+1} a_{j+k} \leq \sum_{j=0}^{n} (-1)^{n+j+1} \binom{n}{j+1} a_{j+k} , \quad \forall n, k \geq 0 .
\]

This expression reproduces the Hausdorff condition (50) for the existence of solution of the classical moment problem. Therefore we can conclude that Theorem 2 is an alternative statement for the existence requirements of the classical Haussdorf problem.
5 Conclusions

In GR the static and axially symmetric solutions of the Einstein vacuum equations are described by a single metric function, which is characterized by a set of Weyl coefficients \( \{a_i\} \). For some solutions this metric function can be obtained as the gravitational potential of a newtonian object \[12\]. A representation of the solution arises from the physical characteristics of the object, in particular its linear density, and physical properties of the relativistic solution can be described in terms of this density \[12\], \[11\]. In this work, we have shown that this linear density can be connected with a discrete description of the object leading to an infinite sum of Curzon potentials. We deal with an infinite set of point particles located, in an homogeneous distribution, along the symmetry axis into a finite length.

The convergence of this problem is equivalent to the existence of a solution for an infinite system of linear equations with a Vandermonde type of coefficients matrix. The matrix of independent terms is given by the set of Weyl coefficients and the unknows variables are the masses of the point particles. We have proved some theorems that allow us to guarantee the existence of solution for this kind of infinite systems. In particular, the solution of this problem for the LM and Erez- Rosen solutions is proved, by a suitable selection of the Vandermonde parameters which represent the positions of the point particles along the axis.

Finally, we have proved that a relationship exists between the classical Haussdorf moment problem and the existence of a newtonian representation of relativistic solutions by means of an object with a linear density. In particular the condition held by the Theorem 2 for the existence of solution of an infinite system of linear equations is equivalent to the condition established by the Haussdorf moment problem.

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The Newtonian gravitational potential of a mass distribution with density $\rho(\vec{z})$, given by the following solution of the Poisson equation

$$\Phi(\vec{x}) = -\int_V \frac{1}{R} \mu(\vec{z}) d^3 \vec{z}, \quad (A1)$$

where we have used units in which the gravitational constant $G = c = 1$, the integral is extended to the volume of the source, $\vec{z}$ is the vector that gives the position of a generic point inside the source, and $R$ is the distance between that point and any exterior point $P$ defined by its position vector $\vec{x}$. Let us now make an expansion of this potential in a power series of the inverse of the distance from the origin to the point $\vec{P}$ ($r \equiv |\vec{x}|$) by means of a Taylor expansion of the term $1/R$ around the origin of coordinates, where $R \equiv \sqrt{(x^i - \hat{z}^i)(x_i - \hat{z}_i)}$. For the case of an axially symmetric mass distribution, this multipole development leads to a Newtonian potential with the same form as equation (25) but the Weyl coefficients $\{a_k\}$ being replaced by $-M^{NG}_k$, which are parameters that denotes the massive multipole moment of order $k$ which can be defined by means of an integral expression extended to the volume of the source,

$$M^{NG}_k = 2\pi \int \int \hat{z}^{k+2} \mu(\hat{r}, \hat{\theta}) P_k(\cos \hat{\theta}) \sin \hat{\theta} d\hat{\theta} d\hat{r}, \quad (A2)$$

$\hat{r} \equiv |\vec{z}|$ representing the radius of the integration point and $\hat{\theta}$ the corresponding polar angle.

There is a well-established framework in Newtonian Gravity (NG) for handling distributional line-sources like a bar of length $2L$ centered and located along the $Z$ axis. Therefore we can consider an object described by a line singularity on the $Z$ axis with the following linear density:

$$\mu(\vec{z}) = \frac{1}{2\pi} \frac{\delta(\hat{\rho})}{\hat{\rho}} \mu(\vec{z}), \quad (A3)$$
for some non-negative function $\mu(\hat{z})$, $\delta(\hat{\rho})$ being the Dirac’s function $\delta(\hat{\rho} - \hat{\rho}_0)$ at $\hat{\rho}_0 = 0$ and where $\{\hat{z}\} \equiv \{\hat{\rho}, \hat{z}\}$ are cylindrical coordinates. Consequently, from equation (A1) the gravitational potential of such a mass distribution is the following:

$$
\Phi(\vec{x}) = -\int_V \frac{1}{R} \mu(\vec{\hat{z}}) d^3\vec{\hat{z}} = -\int_{-L}^{L} \frac{\mu(\hat{z})}{\sqrt{\rho^2 + (z - \hat{z})^2}} d\hat{z} , \quad (A4)
$$

where the position vector $\vec{x}$ is given by coordinates $(\rho, z)$, and $\vec{\hat{z}}$ is located along the $Z$ axis.

According to the equation (A2) the Newtonian multipole moments of this object (if the function $\mu(\hat{z})$ is even in $\hat{z}$) are as follows:

$$
M_{2k}^{NG} = \int_{-L}^{L} \hat{z}^{2k} \mu(\hat{z}) d\hat{z} = L^{2k+1} \int_{-1}^{1} X^{2k} \mu(LX) dX . \quad (A5)
$$

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