Characterizing sequences for precompact group topologies

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Abstract

A precompact group topology \( \tau \) on an abelian group \( G \) is called single sequence characterized (for short, ss-characterized) if there is a sequence \( u = (u_n) \) in \( G \) such that \( \tau \) is the finest precompact group topology on \( G \) making \( u = (u_n) \) converge to zero. It is proved that a metrizable precompact abelian group \( (G, \tau) \) is ss-characterized iff it is countable. For every metrizable precompact group topology \( \tau \) on a countably infinite abelian group \( G \) there exists a group topology \( \eta \) such that \( \eta \) is strictly finer than \( \tau \) and the groups \( (G, \tau) \) and \( (G, \eta) \) have the equal Pontryagin dual groups. We give a complete description of all ss-characterized precompact abelian groups modulo countable ss-characterized groups from which we derive:

(1) No infinite pseudocompact abelian group is ss-characterized.
(2) An ss-characterized precompact abelian group is hereditarily disconnected.

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1 Introduction

1.1 Notations and terminology

Usually, all considered groups will be abelian and additive notation will be used. All considered topological groups are Hausdorff. For a subset \( A \) of a group \( G \) denote by \( \langle A \rangle \) the subgroup of \( G \) generated by \( A \). An abelian group \( G \) endowed with the discrete topology is denoted by \( G_d \).

For a topological group \( (G, \tau) \) we denote by \( \mathcal{N}(G, \tau) \) the filter of all neighborhoods of the neutral element of \( G \). The group of all continuous characters of \( (G, \tau) \) we denote by \( G^* = (G, \tau)^* \). The group \( G^* \) endowed with the compact open topology is called the Pontryagin dual of \( (G, \tau) \) and it is denoted by \( G^\wedge \). For \( x \in G \) we define a mapping \( \hat{x} : G^\wedge \to \mathbb{T} \) by \( \hat{x}(\chi) = \chi(x) \) for \( \chi \in G^\wedge \). Then \( \hat{x} \in G^{\wedge\wedge} \) for each \( x \in G \) and the mapping \( G \to G^{\wedge\wedge}, \alpha_G(x) = \hat{x}, \) is a group homomorphism, called the canonical homomorphism. The group \( G \) is called maximally almost periodic, for short a MAP-group, if \( G^* \) separates points of \( G \), i.e., \( \alpha_G \) is injective. We denote by MAP the class of all MAP-groups. A subgroup \( H \) of \( G \) is called dually closed if for every \( x \in G \setminus H \) there exists \( \chi \in G^* \) such that \( \chi(H) = \{0\} \) and \( \chi(x) \neq 0 \), and it is called dually embedded if for every \( \varphi \in H^* \) there exists \( \chi \in G^* \) such that \( \chi|_H = \varphi \).

Let us recall [10] that every precompact group topology \( \sigma \) on an abelian group \( G \) is uniquely determined by the set \( G^* \) of all continuous characters of \( (G, \sigma) \). Namely, \( \sigma \) has the form \( T_{G^*} \), where \( T_{G^*} := \sigma(G, G^*) \) is the coarsest group topology on \( G \) with respect to which all members of \( G^* \) are continuous. Conversely, if \( H \) is an arbitrary dense subgroup of the Pontryagin dual compact group \( (G_d)^\wedge \), then \( T_H := \sigma(G, H) \) is a precompact group topology on \( G \) such that \( (G, T_H)^* = H \) (see Fact 3.1).

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For a topological group \((G, \tau)\) we write \(\tau^+ := \sigma(G, \Gamma)\), where \(\Gamma = (G, \tau)^*\). Clearly \(\tau^+ \leq \tau\). The topology \(\tau^+\) is often called the Bohr modification of \(\tau\).

A subset \(A\) of a topological abelian group \(G\) is called quasi-convex if for every \(g \in G \setminus A\) there exists \(\chi \in G^*\) such that

\[
\chi(A) \subset T_+, \quad \text{but} \quad \chi(x) \notin T_+,
\]

where \(T_+\) is the image of the segment \([-\frac{1}{2}, \frac{1}{2}]\) with respect to the natural quotient map \(\mathbb{R} \to T\). If \(A\) is a subset of \(G\), then the set

\[
qc(A) := \bigcap \{Q : A \subseteq Q \text{ and } Q \text{ is quasi-convex}\}
\]

is quasi-convex in \(G\) and it is called the quasi-convex hull of \(A\). A topological group \((G, \tau)\) (as well as its topology \(\tau\)) is called locally quasi-convex if it has a basis consisting quasi-convex subsets of \((G, \tau)\). We denote by \(\text{LQC}\) the class of all locally quasi-convex groups. Clearly, every \(\text{LQC}\)-group is also \(\text{MAP}\).

For a \(\text{MAP}\) abelian topological group \((G, \tau)\) (cf. [2] 6.18) the collection \(\{\{V\} : V \in \mathcal{N}(G, \tau)\}\) is a base of a locally quasi-convex group topology \(\tau^\text{lqc}\) called the locally quasi-convex modification of \(\tau\). The topology \(\tau^\text{lqc}\) is the finest one among the locally quasi-convex group topologies on \(G\) which are coarser than \(\tau\). Note that every precompact group topology is locally quasi-convex.

If \(\tau\) and \(\eta\) are group topologies on a group \(G\), \(\eta\) is said to be (strongly) compatible with \(\tau\) if \((G, \eta)^* = (G, \tau)^*\) \((G, \eta)^\wedge = (G, \tau)^\wedge\), resp.). If \((G, \tau)\) is \(\text{MAP}\), then \(\tau^+\) and \(\tau^\text{lqc}\) are compatible with \(\tau\). Moreover, if \(\tau^+\)-compact sets are \(\tau\)-compact, then \(\tau^+\) is strongly compatible with \(\tau\).

Let \(\mathcal{G}\) be a class of topological abelian groups (for example \(\text{MAP}\) or \(\text{LQC}\)). Following [25], a topological group \((G, \mu)\) is called a Mackey group in \(\mathcal{G}\) (or a \(G\)-Mackey group) if \((G, \mu) \in \mathcal{G}\) and if \(\nu\) is a compatible group topology with \(\mu\) such that \((G, \nu) \in \mathcal{G}\), then \(\nu \leq \mu\).

For an abelian topological group \(X\) and a sequence \(u = (u_n)\) in its dual group one puts:

\[
s_u(X) := \{x \in X : u_n(x) \to 0 \text{ in } T\}. \tag{1}
\]

The subgroups of the form \((1)\) will play a crucial role in our considerations.

1.2 Main results

The article is devoted to investigation of the following notions introduced in [5, 24]:

**Definition 1.1** Let \(u = (u_n)\) be a sequence in an abelian group \(G\).

(a) [5] If there exists a precompact group topology making \(u = (u_n)\) converge to 0, we call \(u\) a TB-sequence. For a TB-sequence \(u\) there exists the finest precompact group topology \(\tau_u\) making \(u = (u_n)\) converge to 0.

(b) [26] For a precompact group topology \(\tau\) on \(G\) we say that the sequence \(u = (u_n)\) in \(G\) characterizes \(\tau\) (and \(u\) is a characterizing sequence for \((G, \tau)\)\), if \(u\) is a TB-sequence and \(\tau = \tau_u\).

(c) A precompact group \((G, \tau)\), as well as its topology \(\tau\), is called single sequence characterized (for short, ss-characterized) if there exists a sequence of elements of \(G\) which characterizes \(\tau\).

Hence, Definition [1.1] defines a correspondence

\[
\begin{equation}
u \mapsto \tau_u \tag{2}
\end{equation}
\]

between TB-sequences and precompact topologies. The necessity to understand better the nature of the correspondence [2] motivates the following general question:

**Problem 1.2** Describe all ss-characterized precompact abelian groups.

Let us note that a set-theoretic description of the topology of an ss-characterized group was obtained in [26] (see the more general Fact [1.13]):

**Fact 1.3** [26] For every TB-sequence \(u\) in an abelian group \(G\) one has \((G, \tau_u)^* = s_u((G_\delta)^*)\).

Nevertheless, till now nothing was known about the topological properties of the groups in the codomain of [2]. That is, we did not know even answers to the following questions. Which metrizable precompact groups are ss-characterized? Which other natural classes of precompact group topologies are ss-characterized? Which totally disconnected precompact or pseudocompact group topologies are in the codomain in [2]? What we can say about topological properties of ss-characterized precompact groups as being a \(k\)-space? The main goal of the article is to give, among others, complete answers to these questions (see Theorem A and Corollaries C1-C3). Also we give some applications of the obtained results (see Corollary A and Theorem 3.11). Our solution is essentially supported by an appropriate reduction of Problem [1.2] to the case of countable groups (see Theorem C).
The next theorem shows that the codomain in (2) covers all metrizable precompact topologies on countably infinite abelian groups:

**Theorem A.** A metrizable precompact group $G$ is ss-characterized if and only if $G$ is countable. Moreover, every characterizing sequence of $G$ generates a finite index subgroup of $G$.

For instance, the sequence $p = (p^n)$, where $p$ is prime, is a $TB$-sequence generating the $p$-adic topology of $\mathbb{Z}$ (see Example 1.8(b) below).

As a corollary of Theorem A we show that precompact metrizable countable groups cannot be MAP-Mackey.

**Corollary A.** Let $(G, \tau)$ be a countably infinite precompact metrizable group. Then on $G$ there exists a group topology $\eta$ which is strictly finer than $\tau$ and it is strongly compatible with $\tau$. In particular, $(G, \tau)$ is not a MAP-Mackey group.

Let us note that Corollary A is the best possible in the following sense: there are countably infinite precompact metrizable groups which are LQC-Mackey [8] (see also [15]). However, there are also countably infinite precompact metrizable groups which are not LQC-Mackey [3]: see also [25], where a wide class of (uncountable) precompact metrizable non-LQC-Mackey groups is described. Note also that $\mathbb{T}$ is a LQC-Mackey group, however it remains unknown whether $\mathbb{T}$ is a MAP-Mackey group as well.

Let $\mathfrak{u}$ be a $TB$-sequence in an abelian group $G$. Clearly, the countable subgroup $\langle \mathfrak{u} \rangle$ of $G$ generated by $\mathfrak{u}$ must play a crucial role to a solution of Problem 1.2. The first natural step in the understanding of ss-characterized topologies is to reduce the general problem to the case of countable groups:

**Problem 1.4** Describe all countable ss-characterized precompact abelian groups.

This problem is especially important in the light of Theorem A. Moreover, a study of the dual group of a countable ss-characterized group is interesting from the duality theory point of view (see Problem section in [37]).

It turns out that the reduction of Problem 1.2 to Problem 1.4 that is one of the main goals of the article (see Theorem C), is non-trivial and needs two new notions which are of independent interest.

**Definition 1.5** A subgroup $H$ of a topological abelian group $G$ is said to be $B$-embedded if every (algebraic) character $\chi : G/H \to \mathbb{T}$ is continuous in the quotient topology of $G/H$.

The choice of the term $B$-embedded is explained by the fact that the Bohr topology of $G/H$ coincides with the Bohr topology of $(G/H)_d$. Obviously, $B$-embedded subgroups are closed (see below Proposition 3.5). It should be noted that this notion is strongly related to the existing notion of an $h$-embedded subgroup ($H$ is an $h$-embedded subgroup of a topological abelian group $G$ if every homomorphism $\chi : H \to \mathbb{T}$ extends to a continuous character of $G$) [13].

Let $H$ be a subgroup of an abelian group $G$ and $\tau$ be a Hausdorff group topology on $H$. Then the set $\mathcal{N}(H, \tau)$ is a local base of a Hausdorff group topology $\tilde{\tau}$ on $G$. Clearly, $\tilde{\tau}$ is the finest group topology on $G$ in the class of all Hausdorff group topologies on $G$ extending $\tau$. (The fact that $G$ is abelian is important, in the non-abelian case this construction does not work even if $H$ is a subgroup of index 2 of $G$ [28].) In particular, every subgroup $N$ of $G$ containing $H$ is open (so clopen) in $\tilde{\tau}$.

In the above situation, if the subgroup $H$ of $G$ carries a precompact topology $\tau$, it is natural to ask whether there is any precompact topology on $G$ that extends $\tau$. Moreover, motivated by the case of Hausdorff group topologies, it is natural to ask whether a finest precompact topology $\tau$ exists in this case.

**Definition 1.6** Let $H$ be a subgroup of an abelian group $G$ and let $\tau$ be a precompact topology on $H$. A precompact topology $\tau^*$ on $G$ is called a finest precompact extension of $\tau$ on $G$ if $\tau^*|_H = \tau$ and $\tau^* \leq \tau^*$ for every precompact topology $\tau'$ on $G$ such that $\tau'|_H = \tau$.

Obviously, the finest precompact extension is unique, whenever it exists. The existence of some precompact extension was established in [29] Theorem 10.1. Moreover, a careful analysis of the proof of [29] Theorem 10.1] shows that the specific extension constructed in [29] Theorem 10.1] is actually the finest precompact extension of $\tau$. The next theorem provides an explicit description as well as other descriptions of this finest precompact extension $\tau^*$. Its proof, given in §3, contains also the crucial steps from the proof of [29] Theorem 10.1], for reader’s convenience.

**Theorem B.** Let $G$ be an abelian group and $H$ be an arbitrary subgroup of $G$. Then

(a) If $\zeta$ is a precompact topology on $H$, then $(\zeta)^+$ is the finest precompact extension of $\zeta$.

(b) For a precompact topology $\tau$ on $G$ and $\zeta = \tau|_H$, the following assertions are equivalent:

   (b1) $\tau$ is the finest precompact extension of $\zeta$.

   (b2) $H$ is $B$-embedded in $(G, \tau)$.

   (b3) Let $j : H_d \to G_d$ be the identity map. Then $(G, \tau)^\wedge = (j^\wedge)^{-1}((H, \zeta)^\wedge)$.
Let $u$ be a $TB$-sequence in an abelian group $G$ and $H$ be an arbitrary subgroup containing $\langle u \rangle$. Then $u$ is a $TB$-sequence in $H$. We denote by $\tau_u(H)$ the finest precompact group topology on $H$ in which $u$ converges to zero. The next proposition is used in the proof of Theorem C.

**Proposition 1.7** Let $u$ be a $TB$-sequence in an abelian group $G$ and $H$ be an arbitrary subgroup of $G$ containing $\langle u \rangle$. Then $\tau_u$ is the maximal precompact extension of $\tau_u(H)$. In particular, $\tau_u|_H = \tau_u(H)$.

Since the maximal precompact extension is unique, Proposition 1.7 implies:

**Corollary 1.8** Let $u$ and $v$ be $TB$-sequences in an abelian group $G$ and $H$ be an arbitrary subgroup of $G$ containing $u$ and $v$. Then $\tau_u = \tau_v$ if and only if $\tau_u(H) = \tau_v(H)$.

The following theorem describes all ss-characterized precompact abelian groups by modulo countable ss-characterized groups and is the reduction principle to the countable case:

**Theorem C.** For a precompact abelian group $(G, \tau)$ the following are equivalent:

(i) $(G, \tau)$ is ss-characterized.

(ii) $(G, \tau)$ has a countable $B$-embedded ss-characterized subgroup $H$.

Compact characterized groups are finite, as the following more precise corollary shows. Let us recall first that a topological group $G$ is said to be pseudocompact if every real-valued continuous function on $G$ is bounded (so that every compact group is pseudocompact).

**Corollary C1.** A pseudocompact Hausdorff group $G$ is ss-characterized if and only if $G$ is finite.

Other two compact-like properties missed by the uncountable ss-characterized precompact groups are provided by the next two corollaries:

**Corollary C2.** Let $(G, \tau)$ be ss-characterized precompact group which is a $k$-space too. Then $G$ is countable and sequential.

According to [30], a topological group $G$ is an Arhangel’skiǐ group, if the weight $w(G)$ of $G$ does not exceed of the cardinality $|G|$ of $G$, i.e., $w(G) \leq |G|$. Compact groups are Arhangel’skiǐ groups, more generally, locally minimal groups are Arhangel’skiǐ groups [1] (it is known that locally compact groups, as well as minimal groups are locally minimal).

**Corollary C3.** A precompact Arhangel’skiǐ group is ss-characterized if and only if $G$ is metrizable (and countable).

The article is organized as follows. In Section 1 we give all necessary definitions and examples explaining our main notions. In Section 2 we prove the sufficiency in Theorem A and we deduce from it Corollary A. In §3.1 we prove Theorem B. Theorems A and C, and Corollaries C1, C2 and C3 are proved in §3.2.

### 1.3 Background on $T$- and $TB$-sequences

The question of when a given sequence in an abelian group may converge to 0 in some Hausdorff topology is of independent interest.

**Definition 1.9** [10] Let $G$ be an abelian group and let $u = (u_n)$ be a sequence in $G$. If there exists a Hausdorff group topology making $u = (u_n)$ converge to 0, then we call $u$ a $T$-sequence. For a $T$-sequence $u$ there exists the finest Hausdorff group topology $T_u$ that makes $u = (u_n)$ converge to 0.

This topology was first considered by Graev [38], and later by Protasov and Zelenyuk [10] [11]. $T$-sequences were thoroughly studied in [10]. Clearly, every $TB$-sequence is also a $T$-sequence, but the converse in general is not true. While it is quite hard to check whether a given sequence is a $T$-sequence, the case of $TB$-sequences is much easier to deal with due to a very simple criterion (see Fact 1.12).

**Remark 1.10** Let us note the following properties of topologies of the form $T_u$ discovered in [10] [11]. Let $u = (u_n)$ be an arbitrary nontrivial $T$-sequence in an abelian group $G$ (i.e., $u_n \neq 0$ for infinitely many indices). Then $(G, T_u)$ is a complete sequential (and hence a $k$-space) but not a Fréchet-Urysohn group. In particular, $(G, T_u)$ is not metrizable.

The next fact will be essentially used in the proof of Corollary A:

**Fact 1.11** [11] 2.3.12 Let $u$ be a $T$-sequence in an abelian group $G$. Then $(G, T_u)$ is not precompact. In particular, if $u$ is a $TB$-sequence in $G$, then $\tau_u \neq T_u$. 


The following criterion gives a simple dual condition on a sequence to be a TB-sequence:

**Fact 1.12** A sequence \( u \) in an abelian group \( G \) is a TB-sequence if and only if the subgroup \( s_u((G_d)^\wedge) \) is dense in the compact group \( (G_d)^\wedge \).

A good supply of examples of TB-sequences can be found in the cyclic group \( G = \mathbb{Z} \), where the criterion 1.12 works especially well, since the dense subgroups of \( T = \mathbb{Z}^\wedge \) are exactly the infinite subgroups of \( T \).

**Example 1.13** Let \( a = (a_n) \) be a sequence of positive integers and \( q_n = a_{n+1}/a_n \) for every \( n \).

(a) If \( q_n \to \infty \), then

\[
\text{(a1)} \quad |s_u(T)| = \infty. \quad \text{(This was proved by Egglestone [33]; see also [5]).}
\]

\[
\text{(a2)} \quad a \text{ is a } TB\text{-sequence by (a1) and Fact 1.12.}
\]

(b) For a prime number \( p \) let us consider the sequence \( p = (p^n) \). According to [1], the subgroup \( s_p(T) \) coincides with the \( p \)-torsion subgroup \( \mathbb{Z}(p^\infty) \) of \( T \), so it is infinite. Hence, by Fact 1.12 the sequence \( p \) is a TB-sequence (note that the sequence of ratios \( (q_n) \) now is bounded, actually, constant) and \( \tau_p \) is the \( p \)-adic topology of \( \mathbb{Z} \). Note that \( (\mathbb{Z}, \tau_p) \) is not metrizable.

(c) Let \( f = (f_n) \) be the sequence defined by \( f_0 = f_1 = 1 \) and \( f_{n+2} = f_{n+1} + f_n \) for all \( n \in \mathbb{N} \). This is the celebrated Fibonacci’s sequence. It was proved by Larcher [39] (see also [40] that \( s_f(T) = (\alpha) \), where \( \alpha \) is the positive solution of the equation \( x^2 - x - 1 = 0 \) (namely, the Golden Ratio) taken modulo 1 (i.e., as an element of \( T \)). In particular, \( s_f(T) \) is infinite, hence dense (as \( \alpha \) is irrational). So, by Fact 1.12 \( f \) is a TB-sequence. A significant generalization of this fact to recursively defined sequences in \( \mathbb{Z} \) of higher degree can be found in [6].

(d) Let us show that for every non-zero polynomial \( P(x) \in \mathbb{Z}[x] \), the sequence of integers \( u = (P(n))_{n \in \mathbb{N}} \) is not a T-sequence in \( \mathbb{Z} \). The proof goes by induction on the degree \( \deg(P) \) of \( P(x) \). If \( \deg(P) = 0 \), then \( P(x) = A \neq 0 \) and hence \( u_n = A \neq 0 \) for every \( n \in \mathbb{N} \). Clearly, \( u \) is not a T-sequence. Assume that for every polynomial \( P(x) \in \mathbb{Z}[x] \) with \( \deg(P) \leq m \) the sequence \( u = (P(n)) \) is not a T-sequence. Let \( P_0(x) \in \mathbb{Z}[x] \) with \( \deg(P_0) \leq m + 1 \). Assume that \( \tau \) is a Hausdorff group topology on \( \mathbb{Z} \) with \( u = (P_0(n))_{n \in \mathbb{N}} \). Then also \( v_n := u_{n+1} - u_n = P_0(n+1) - P_0(n) \neq 0 \) in \( \mathbb{Z} \). Thus \( v \) is a T-sequence defined by the polynomial \( P(x) := P_0(x+1) - P_0(x) \) of degree \( m \), a contradiction. (The case of the sequence \( (n^2)_{n \in \mathbb{N}} \) was settled in a different way in [15].)

It is worth mentioning that the subgroups of the form \( (\mathbb{Z}, \tau) \) of the torus \( T \) characterize the so called topologically torsion elements. Using the sequence \( p = (p^n) \) from Example 1.13(b), Braconnier [10] and Vilenkin [44] defined topologically p-torsion elements for an arbitrary locally compact abelian groups, namely the elements of the subgroup \( t_p(G) := \{ x \in G : p^n x \to 0 \} \) of \( G \). Using the sequence \( u = (n!) \), Robertson (see [11]) defined topologically torsion elements by \( t_u(G) := \{ x \in G : n! x \to 0 \} \). As noticed in [22], the notion of topologically \( u \)-torsion element can be extended to any sequences \( u \) of integers and an arbitrary topological abelian group \( G \) by letting

\[
t_u(G) := \{ x \in G : u_n x \to 0 \} \tag{3}
\]

(for sequences with \( u_n \) this can be found already in [27], Chapter 4, Notes)). The subgroups of \( T \) of this form where studied by Borel [3], who proved that every countable subgroup of \( T \) has the form \( t_u(T) \) for an appropriate \( u \). Later on, this theorem was reproved in [12] (with a gap in the case of torsion subgroups), where the sequence \( u \) ensuring \( H = t_u(G) \) was called a characterizing sequence for \( H \).

**Remark 1.14** For a sequences \( u \) of integers and a topological group \( G \), a non-torsion element \( x \in G \) generates a subgroup \( \langle x \rangle \) algebraically isomorphic to \( \mathbb{Z} \). Since \( x \in t_u(G) \) precisely when \( x \in t_u(\langle x \rangle) \), this shows that \( u \) is a T-sequence in \( \mathbb{Z} \) whenever \( t_u(G) \) contains non-torsion elements \( x \). Therefore, by Example 1.13(d) \( t_u(G) \) contains only torsion elements whenever \( u_n = P(n) \), for some non-zero polynomial \( P(x) \in \mathbb{Z}[x] \) (this was announced without proof in [23], Example 2.10(a)).

The possibility to extend the fact that all countable subgroups of \( T \) admit a characterizing sequence to countable subgroups of arbitrary compact metrizable groups was very briefly mentioned in [12] without saying explicitly what a characterized subgroup and a characterizing sequence must be in the general case. The necessity to change the pattern \( t_u(X) \) used in the case \( X = T \) was pointed out in [26] (see also [23]). Actually, it was proved in [19] that if all cyclic subgroups of a locally compact group \( G \) are intersections of subgroups of the form \( \langle \mathbb{Z}, \tau \rangle \), then \( G \cong T \), thereby clarifying the fact that the pattern \( \mathbb{Z} \) cannot be used for a reasonable definition of characterized subgroup (for example, if \( X \) is the group of \( p \)-adic integers for some prime \( p \), then \( t_u(X) \) coincides with either \( X \) or \( \{0\} \) [23], Example 4.11]).

Motivated by the situation described above, the following notion was proposed in [26], making use the subgroups of the form \( s_u(X) \) of a topological abelian group \( X \):
Definition 1.15 [24] Let $G$ be a subgroup of a topological abelian group $H$. We say that $H$ is characterized, if there exists a sequence $u = (u_n)_{n \in \mathbb{N}}$ in $G^\wedge$ such that $H = s_u(G)$. In such a case, we say that $u$ characterizes $H$.

The importance of the notion of characterized subgroup is obvious also from the following fact:

Fact 1.16 [24] Let $u = (u_n)$ be a $TB$-sequence in an infinite abelian group $G$ and $H = s_u((G_d)^\wedge)$. Then for every group topology $\tau$ such that $\tau_u \leq \tau \leq \tau_u$ one has $(G, \tau)^* = H$. In particular, the topology $\tau$ is compatible with $\tau_u$ and the sequence $u = (u_n)$ characterizes $\tau_u$.

Let us point out that in the special case $T^\wedge = \mathbb{Z}$, one has $s_u(T) = \{x \in T : u_n x \to 0 \text{ in } T\} = t_u(T)$, for every sequence $u$ of integers.

The following theorem, which was proved by Kunen and the first named author [24] and by Beiglböck, Steineder, and Winkler [7] independently and almost simultaneously, will play a crucial role in our considerations:

Theorem 1.17 [24] Let $H$ be a countable subgroup of a compact metrizable abelian group $G$. Then $H$ is a characterized subgroup.

It follows easily from Definition 1.15 that every characterized subgroup $H$ is a countable intersection of $F_\sigma$-sets and hence it is a Borel set. In particular, characterized subgroups of compact metrizable groups can either be countable or has size $\mathfrak{c}$.

2 Characterizing sequences of the metrizable countable precompact groups

We start from the proof of the sufficiency in Theorem A:

Theorem 2.1 Let $G$ be a countable abelian group. Then every metrizable precompact group topology on $G$ has a characterizing sequence.

Proof. Let $K = (G_d)^\wedge$ and $\tau$ be a metrizable precompact group topology on $G$. Then $H = (G, \tau)^*$ is a dense countable subgroup of $K$ and $G$ can be identified with $K^\wedge$ up to isomorphism. By Theorem 1.17 the subgroup $H$ admits a sequence $u = (u_n)$ in $G$ such that

$$H = s_u(K).$$

Since $H$ is dense in $K$, $u$ is a $TB$-sequence by Fact 1.12. By Fact 1.16 $(G, \tau_u)^* = (G, \tau)^*$. Since precompact topologies uniquely determined by the set of all continuous characters, we have $\tau_u = \tau$. So $u = (u_n)$ characterizes $\tau$. \(\Box\)

The necessity of Theorem A will be proved in §3.2.

Remark 2.2 It is relevant to note that, if $G$ is a non-metrizable countable precompact abelian $ss$-characterized group, then $w(G) = \mathfrak{c}$. Indeed, it is well-known that $w(G) = \text{card}(G^\wedge)$. Since $G$ is not metrizable, $w(G) > \aleph_0$. So, by Fact 1.16 $H = s_u((G_d)^\wedge)$ is uncountable. As it was noticed after Theorem 1.17 $H$ has size $\mathfrak{c}$. Thus $w(G) = \mathfrak{c}$.

The following fact and its proof were kindly communicated to us by L. Aussenhofer:

Proposition 2.3 (L. Aussenhofer) Let $G$ be a Hausdorff locally quasi-convex group with discrete Pontryagin dual $G^\wedge$. Then $G$ is precompact.

Proof. Since $G$ is a Hausdorff locally quasi-convex group, $\alpha_G$ is an injective and open as mapping from $G$ onto $\alpha_G(G)$ by [2, Proposition 6.10]. Since $G^\wedge$ is discrete, its compact subsets are finite and hence equicontinuous. Thus, $\alpha_G$ is continuous by [2, Proposition 5.10]. Therefore $\alpha_G$ is an embedding. So $G$ can be identified with a subgroup of the compact group $G^\wedge$ and hence $G$ is precompact. \(\Box\)

As an immediate consequence we obtain the following nice and surprising characterization of precompactness for the metrizable abelian groups:

Corollary 2.4 Let $G$ be a metrizable abelian group. Then the following are equivalent:

(i) $G$ is precompact.

(ii) $G$ is a locally quasi-convex group with discrete Pontryagin dual $G^\wedge$.

Proof. (i) \(\implies\) (ii) Let $G$ be precompact. Then its completion $K$ is a metrizable compact group. So the algebraic isomorphism between $G^\wedge$ and $K^\wedge$ is also topological [2, 13]. Hence $G^\wedge$ is discrete. $G$ as a subgroup of the locally quasi-convex group $K$ is locally quasi-convex group itself.

(ii) \(\implies\) (i) is true by Proposition 2.3. \(\Box\)

The following assertion is of independent interest and prepares the proof of Corollary A.
Proposition 2.5 Let $G$ be a countably infinite abelian group and $u = (u_n)$ be a $T_B$-sequence in $G$ such that $(G, \tau_u)$ is precompact metrizable. Then

(a) $(G, \bar{T_u})^\wedge$ is discrete.

(b) $\tau_u \neq \bar{T_u}$.

(c) $\tau_u = (\bar{T_u})_{lqc}$, in particular, $\bar{T_u}$ is not locally quasi-convex.

Proof. (a) By Fact 1.16 we have that $(G, \tau_u)^* = (G, \bar{T_u})^*$. Since $(G, \tau_u)$ is precompact metrizable, $(G, \tau_u)^* = (G, \bar{T_u})^*$ is countable. By [29, Theorem 10.1] for the detailed routine verification of this fact.

(b) follows from Fact 1.11.

(c) From (a) and the compatibility of $(\bar{T_u})_{lqc}$ with $\bar{T_u}$ we get that $(G, (\bar{T_u})_{lqc})^\wedge$ is discrete. From this by Proposition 2.3 we get that $(\bar{T_u})_{lqc}$ is precompact. This implies that $(\bar{T_u})_{lqc} = \tau_u$, as $\tau_u = \bar{T_u}^\wedge$. □

Proof of Corollary A. By Theorem 2.1 there exists a $T_B$-sequence $u = (u_n)$ in $G$ such that $\tau = \tau_u$. As $u = (u_n)$ is also a $T$-sequence, we can consider the Hausdorff group topology $\eta = \bar{T_u}$ on $G$.

Clearly, $\tau \leq \eta$. The topology $\eta$ is compatible with $\tau$ because of $\eta^+ = \tau$ by Fact 1.16. We have $\tau \neq \eta$ by Proposition 2.3(b). By Proposition 2.3(a), $(G, \eta)^\wedge$ is discrete. This implies that $(G, \tau)^\wedge$ is discrete as well, and so $\eta$ is strongly compatible with $\tau$. □

3 Characterizing sequences of arbitrary precompact abelian groups

3.1 The finest precompact extension of a precompact group topology

For the use in the forthcoming proof of Theorem B, we recall here some well known facts from [10]. For a discrete group $X$ and a dense subgroup $L$ of the compact dual $(X_d)^\wedge$ we let $T_{L,X}$ (or simply, $T_L$, when no confusion is possible) denote the weak topology $\sigma(X, L)$ of $X$ induced by the subgroup $L$.

Fact 3.1 (a) The correspondence $L \mapsto T_{L,X}$ between dense subgroups $L$ of $(X_d)^\wedge$ and precompact topologies on $X$ is bijective and monotone.

(b) Let $Y$ be a subgroup of $X$ with inclusion $\iota: Y \hookrightarrow X$, and let $L$ and $M$ be dense subgroups of $(Y_d)^\wedge$ and $(X_d)^\wedge$ respectively. Then the inclusion $(Y, T_{L,Y}) \to (X, T_{M,X})$ is continuous (resp., an embedding) if and only if $\iota^*(M) \subseteq L$ (resp., $\iota^*(M) = L$).

Item (a) comes from [10]. To verify (b), it suffices to note, that since both $T_{L,Y}$ and $T_{M,X}$ are weak topologies, the continuity of $(Y, T_{L,Y}) \to (X, T_{M,X})$ is equivalent to the fact that $\chi \circ \iota = \chi |_Y$ is continuous whenever $\chi \in (X, T_{M,X})^\wedge = M$, i.e., $\chi \circ \iota = \iota^*(\chi) \in L = (Y, T_{L,Y})^\wedge$ if $\chi \in M$. This simply means $\iota^*(M) \subseteq L$, as put in (b). The version in brackets is verified similarly (see also the proof of [28] Theorem 10.1) for the detailed routine verification of this fact.

Proof of Theorem B. We split the verification of (a), namely the equality $(\bar{\zeta})^+ = \zeta^+$, in two steps.

(1) The topology $(\bar{\zeta})^+$ is precompact and $(\bar{\zeta})^+|_H = \zeta$.

To verify $(\bar{\zeta})^+|_H = \zeta$, note that $H$ is dually closed and dually embedded in $(G, \bar{\zeta})$, being open in $(G, \bar{\zeta})$. Hence

$$(H, \zeta)^* = \{\chi H : \chi \in (G, \bar{\zeta})^*\}.$$  

From the last equality and Fact 3.1(b), we get that $(\bar{\zeta})^+|_H = \zeta$. Since the subgroup $H$ is dually closed (i.e., $(\bar{\zeta})^+\text{-closed}$), the equality $(\bar{\zeta})^+|_H = \zeta$ implies that the $(\bar{\zeta})^+\text{-closure of }0 \in G$ coincides with its $\zeta$-closure in $H$, namely $\{0\}$. So, $(G, \bar{\zeta})$ is MAP, hence $(\bar{\zeta})^+$ is precompact.

(2) Let $\tau'$ be a precompact topology on $G$ with $\tau'|_H = \zeta$. Then $\tau' \leq (\bar{\zeta})^+.$

By Fact 3.1(a), it suffices to show that

$$(G, \tau')^* \subseteq (G, \bar{\zeta})^*.$$  

So, fix $\chi \in (G, \tau')^*$. As $\tau'|_H = \zeta$, we have also that $\chi|_H \in (H, \zeta)^*$. Since $H$ is open in $(G, \bar{\zeta})$ from $\chi|_H \in (H, \zeta)^*$ and $\bar{\zeta}|_H = \zeta$ we get easily that $\chi \in (G, \bar{\zeta})^*$ and $[11]$ is proved.

From (1) and (2) we get that the topology $(\bar{\zeta})^+$ is indeed the finest precompact extension of $\zeta$.

(b) The equivalence $(b_1) \iff (b_4)$ immediately follows from (a).

For the remaining part of the proof of (b), we need to clarify first the properties of the duals $N = (H, \zeta)^*$ and $Q = (G, \tau)^*$ in view of Fact 3.1. By item (a) of that fact, $N$ and $Q$ are dense subgroups of the compact groups $(H_d)^\wedge$.
and \((G_d)^{\wedge}\), respectively. Let \(j : H_d \rightarrow G_d\) be the inclusion and \(j^\wedge : G_d^\wedge \rightarrow H_d^\wedge\) be the corresponding adjoint continuous surjective homomorphism. By Fact 3.1(b), the embedding \((H, \zeta) \rightarrow (G, \tau)\) yields

\[ j^\wedge(Q) = N. \]  

(5)

Hence, the subgroup \(A = (j^\wedge)^{-1}(N)\) of \(G_d^\wedge\) contains \(Q\) and obviously satisfies \(j^\wedge(A) = N\). So, by Fact 3.1(b), the precompact topology \(T_A := T_{A,G}\) on \(H\) the original topology \(\zeta = T_N\), i.e., \(T_A\) is an extension of \(\zeta\). Moreover, the choice of \(A\) implies that \(A\) is the largest subgroup of \(G_d^\wedge\) with \(j^\wedge(A) = N\). Therefore, \(T_A\) is the finest precompact topology on \(G\) inducing on \(H\) the original topology \(\zeta = T_N\), i.e.,

\[ T_A = \zeta^*. \]

(6)

At this point we can prove the remaining equivalence of (b).

\((b_1) \Leftrightarrow (b_2)\). In view of (5), it suffices to note that \((b_1)\) means \(\tau = \zeta^*\), while \((b_2)\) means \(\tau = T_A\).

To prove \((b_3) \Leftrightarrow (b_2)\), let \(\pi : G \rightarrow G/H\) be the canonical map. The continuous characters of the quotient \(G/H\), when equipped with the quotient topology of \(\tau\), are precisely those coming from the factorization via \(\pi\), of the \(\tau\)-continuous characters of \(G\) vanishing on \(H\), i.e., \(\pi^\wedge((G/H)^*) = H^\perp \cap Q\). Therefore, \(H\) is \(B\)-embedded in \(G\) precisely when \(H^\perp \subseteq Q\). By (5), \(H^\perp \subseteq Q\) is equivalent to \(Q = A\), i.e., \(\tau = T_A\). As mentioned above, this is precisely \((b_3)\). \(\Box\)

The quasi-component \(Q(G)\) of a topological group \(G\) is the intersection of all clopen sets of \(G\) containing the neutral element of \(G\). One can prove that \(Q(G)\) is always a normal subgroup of \(G\) in the general case when \(G\) need not be abelian \((20)\). A topological group \(G\) is

- **totally disconnected** when \(Q(G)\) is trivial;
- **hereditarily disconnected** when the connected component \(c(G)\) of \(G\) is trivial.

Since the quasi-component obviously contains the connected component, totally disconnected groups are hereditarily disconnected.

**Proposition 3.2** Let \(H\) be a \(B\)-embedded subgroup of a precompact abelian group \(G\). Then

(a) if \(H\) is of infinite index, then \(w(G/H) = 2^{[G/H]}\);

(b) if \(H\) is of infinite index, then \(w(G) \geq w(G/H) \geq \mathfrak{c}\);

(c) if \(G\) is metrizable, then \(G/H\) is finite;

(d) all subgroups containing \(H\) are \(B\)-embedded as well.

(e) the subgroup \(H\) contains the quasi-connected component \(Q(G)\) of \(G\).

**Proof.** (a) is a well-known property of the Bohr topology and follows from Kakutani’s theorem \(|\text{Hom}(X, \mathbb{T})| = 2^{|X|}\) for an infinite discrete abelian group \(X\).

(b) and (c) follow immediately from (a).

(d) follows from the well-known property of the Bohr topology (every subgroup of \((G_d)^{\wedge}\) is closed).

(e) It is well-known that abelian groups equipped with their Bohr topology are zero-dimensional \((22)\). Since the quasi-connected component of a zero-dimensional group is trivial, we conclude that \(Q(G/H)\) is trivial. Since the Bohr topology is a Hausdorff topology, the quasi-connected component \(Q(G)\) of \(G\) is still clopen, we can then deduce that \(H = \ker h\) is an intersection of clopen sets in \(G\), hence \(Q(G) \subseteq H\). \(\Box\)

Since countable groups are hereditarily disconnected (actually, zero-dimensional), from Proposition 3.2(e) and Theorem B we deduce

**Corollary 3.3** Every characterized precompact group is hereditarily disconnected.

### 3.2 Proofs of Theorems A, C and Corollaries C1, C2, C3

We start this section from the proof of Proposition 1.7.

**Proof of Proposition 1.7.** By the definition of \(\tau_u(H)\), we have \(\tau_u|H \leq \tau_u(H)\). Let \(\tau^*\) be the finest precompact extension of \(\tau_u(H)\) that exists by Theorem B. Clearly, \(u \rightarrow 0\) in \(\tau^*\). Thus, by definition, \(\tau^* \leq \tau_u\) and hence \(\tau_u|H = \tau_u(H) \leq \tau_u|H\). So \(\tau_u|H = \tau_u(H)\). This also means that \(\tau_u\) is an extension of \(\tau_u(H)\) and hence \(\tau_u \leq \tau^*\). So \(\tau^* = \tau_u\). \(\Box\)

The following lemma is a folklore fact.

**Lemma 3.4** Let \((G, \tau)\) be a precompact abelian group and \(H\) be an arbitrary closed subgroup of \(G\). Then \(H\) is dually closed and dually embedded in \((G, \tau)\).
Proof. Let $G$ be the compact completion of $G$. It is well-known that the closure $\bar{H}$ of $H$ is dually closed and dually embedded in $\bar{G}$. Since $\bar{H}$ and $H$ as well as $\bar{G}$ and $G$ have the same set of continuous characters, $H$ is dually closed and dually embedded in $(G, \tau)$. □

As an immediate corollary of Theorem B, Proposition 3.5 and Proposition 3.2(d) we obtain:

**Proposition 3.5** Let $u$ be a TB-sequence in an abelian group $G$ and $H$ be an arbitrary subgroup of $G$ containing $\langle u \rangle$. Then

1. $H$ is dually closed and dually embedded in $(G, \tau_u)$;
2. the quotient topology on $G/H$ is the Bohr topology, i.e., $G/H = ((G/H)_d)^+$.

**Corollary 3.6** Let $u$ be a TB-sequence in an abelian group $G$ and $\tau$ be a group topology on $G$ such that $\tau_u \leq \tau$. Then every subgroup $H$ containing $\langle u \rangle$ is dually closed in $\tau$.

Taking the trivial sequence $u = \{0\}$ and $H = \{0\}$ in Proposition 3.5 we obtain a proof of the following well-known fact (cf. [14, 2.1]):

**Corollary 3.7** Let $G$ be an abelian group. Then every subgroup of $G^+$ is dually closed and dually embedded.

**Proof of Theorem C.** (i) $\Rightarrow$ (ii) Let $u$ be a TB-sequence in $G$ which characterizes $\tau$. Putting $H = \langle u \rangle$ assertion (ii) follows from (i) by Theorem B and Proposition 1.7.

(ii) $\Rightarrow$ (i) Let $H$ be a countable $B$-embedded subgroup of $G$ and let $u$ be a TB-sequence such that $(H, \tau|_H) = (H, \tau_u(H))$. Let us show that $\tau = \tau_u$.

The hypothesis (ii) means that $\tau = (\tau_u(H))^\ast$ is the finest precompact extension of $\tau_u(H)$. Then Proposition 1.7 implies that $\tau = \tau_u$. □

**Proof of Theorem A.** Assume that $G$ is ss-characterized. According to Theorem B, $G$ has a countable $B$-embedded subgroup $H$. By Proposition 3.2(c), $H$ must have finite index in $G$. Thus $G$ is countable as well.

Conversely, if $G$ is countable, then $G$ is ss-characterized by Theorem 2.1.

Let $u$ be a sequence which characterizes $G$. By Proposition 3.5 $G/\langle u \rangle$ carries the Bohr topology and it is metrizable. By Proposition 3.2(c), $\langle u \rangle$ must have finite index. □

**Proof of Corollary C1.** Assume that $G$ is ss-characterized. According to Theorem C, $G$ has a countable $B$-embedded subgroup $H$. So the group $G/H$ carries the Bohr topology. Since $G/H$ is also pseudocompact (as a quotient of $G$), it follows that $G/H$ is finite [17]. Hence $G$ is countable. Therefore $G$ is a countable pseudocompact group and hence it must be finite (as infinite pseudocompact groups are uncountable [33]).

If $G$ is finite it is characterized by the trivial sequence $u = (0)$. □

**Lemma 3.8** Let $X$ be a Hausdorff countable space. Then $X$ is a $k$-space if and only if it is sequential.

Proof. It is well known that every sequential space is a $k$-space [35, 3.3.20].

Let $X$ be a $k$-space and $K$ a compact subset of $X$. Being countable, $K$ is metrizable [35, 3.1.21]. Thus $X$ is sequential by Lemma 1.5 of [14]. □

**Proof of Corollary C2.** Let $(G, \tau)$ be ss-characterized precompact group, for which $(G, \tau)$ is a $k$-space. Take a sequence $u$ in $G$ which characterizes $\tau$ and set $H = \langle u \rangle$. Then the quotient group $(G/H, \tau/H)$ carries the Bohr topology by Theorem C. Thus $(G/H, \tau/H)$ has no infinite compact subsets. From this, since $(G/H, \tau/H)$ is a $k$-space as well, we get that $(G/H, \tau/H)$ is discrete. Therefore, $(G/H, \tau/H)$ is a discrete precompact group. Hence, $G/H$ is finite. Since $H$ is countable, we get that $G$ is countable as well. By Lemma 3.8 $G$ is sequential. □

**Proof of Corollary C3.** Metrizable precompact groups have countable weight, so they are Arhangel’skii groups.

Now assume that $G$ is an uncountable ss-characterized precompact group. By Theorem C, there exists a $B$-embedded countable subgroup $H$ of $G$. Then $|G/H| = |G|$, so by Proposition 3.2 $w(G/H) = 2^{|G/H|} = 2^{|G|} > |G|$. Since $w(G) \geq w(G/H)$, this proves that $w(G) > |G|$, i.e., $G$ is not an Arhangel’skii group. □
3.3 Sequential completeness

According to [31], a topological group $G$ is said to be \textit{sequentially complete}, if every Cauchy sequence in $G$ is convergent (or, equivalently, when $G$ is sequentially closed in its two-sided completion). For basic properties of sequentially complete groups see [31, 32].

By Remark [1.10] for every $T$-sequence $u$ in an infinite abelian group $G$ the group $(G, \tau_u)$ is sequential and complete. By Corollary C1, if $G$ is an infinite abelian group and $u$ is a $TB$-sequence in $G$, then the group $(G, \tau_u)$ cannot be complete. So it is natural to ask whether the group $(G, \tau_u)$ is sequentially complete. In spite of the property to be sequential complete is not a three space property in general [11], we can prove the following.

**Proposition 3.9** Let $H$ be a closed subgroup of a topological abelian group $G$. If $G/H$ has no non-trivial convergent sequences, then $G$ is sequentially complete if and only if $H$ is sequentially complete.

**Proof.** Assume that $H$ is sequentially complete. Let $\bar{G}$ be the completion of $G, \bar{H} = \cl_G(H)$ and let $q : \bar{G} \to \bar{G}/\bar{H}$ be the quotient map. To check that $G$ is sequentially closed in $\bar{G}$ pick a sequence $v = (v_n)$ in $G$ converging to an element $\bar{g} \in \bar{G}$. Then $q(v)$ converges to $q(\bar{g})$. By hypothesis, $q(v)$ is trivial. Hence there exists $k \in \mathbb{N}$ such that $q(v_n) = q(v_k)$ for all $n \geq k$. Hence $h_n := v_n - v_k \in H$ for all $n \geq k$. As $h_n \in G$ as well, we deduce that $h_n \in G \cap H = H$. As $h_n$ converges to $\bar{g} - v_k$ in $\bar{H}$ and $H$ is sequentially complete, we have $\bar{g} - v_k \in H$. Thus $\bar{g} \in G$. Therefore $G$ is sequentially closed in $\bar{G}$.

If $G$ is sequentially complete, then $H$, as a closed subgroup of $G$, is sequentially complete as well. $\square$

**Corollary 3.10** Let $u$ be a $TB$-sequences in an abelian group $G$. Then $(G, \tau_u)$ is sequentially complete if and only if the countable subgroup $(\langle u \rangle, \tau_u|_{\langle u \rangle})$ is sequentially complete.

We are not aware whether such groups can be sequentially complete (see Problem 4.1).

3.4 $T$-sequences and local quasi-convexity

Let $u$ be a $TB$-sequence in a group $G$. Since $\tau_u$ is locally quasi-convex, we have $\tau_u \leq (\tau_u)_{LC}$. Now let $u$ and $v$ be $TB$-sequences in an infinite abelian group $G$. By Fact [1.10] if $\tau_u = \tau_v$, then $\tau_u = \tau_v$. The next theorem gives a particular answer to the question whether the converse assertion is true.

**Theorem 3.11** Let $u$ and $v$ be $TB$-sequences in an infinite abelian group $G$. Assume that

1. $\tau_u$ and $\tau_v$ are locally quasi-convex;
2. $\tau_u = \tau_v$.

Then $\tau_u = \tau_v$.

**Proof.** Set $H = \langle u \rangle + \langle v \rangle$. If $H$ is finite, the theorem is trivial. Let $H$ be countably infinite. Since $H$ is open in both the topologies $\tau_u$ and $\tau_v$ and since $\tau_u(H) = \tau_v(H)$ by Corollary [1.8], we can assume that $G$ is countably infinite.

Let $i_u : G \rightarrow (G, \tau_u)$ and $i_v : G \rightarrow (G, \tau_v)$ be the identity maps. Then, by item 2 and Fact [1.10] $i_u^\wedge(G) = i_v^\wedge(G) := H$ as a subgroup of the compact metrizable group $(G_u)^\wedge$. By [36], $(G, \tau_u)^\wedge$ and $(G, \tau_v)^\wedge$ are Polish groups. So, the Borel subgroup $H$ of $(G_u)^\wedge$ admits two finer Polish group topologies. By the uniqueness of Polish group topology, we obtain that $(G, \tau_u)^\wedge = (G, \tau_v)^\wedge$ topologically. Hence

$$(G, \tau_u)^\wedge = (G, \tau_v)^\wedge$$

topologically and $\alpha_G(\tau_u) = \alpha_G(\tau_v)$ as algebraic homomorphisms.

Since $\tau_u$ and $\tau_v$ are locally quasi-convex and $k$-spaces by Remark [1.10], $\alpha_G(\tau_u)$ and $\alpha_G(\tau_v)$ are embedding by [2] 5.12 and 6.10. Thus $\tau_u = \tau_v$. $\square$

4 Final remarks and problems

Clearly, if a non-trivial $TB$-sequence $u$ in an infinite abelian group $G$ is such that $(\langle u \rangle, \tau_u)$ is metrizable, the countably infinite group $(\langle u \rangle, \tau_u)$ is not sequentially complete. By Proposition [5.5] $(\langle u \rangle, \tau_u)$ is closed in $(G, \tau_u)$. So $(G, \tau_u)$ is not sequentially complete by Proposition [3.9]. Therefore, the following question is of independent interest.

**Question 4.1** (a) Is there countably infinite sequentially complete precompact ss-characterized group?

(b) In particular, if $u$ is the sequence $\langle n! \rangle$ in $\mathbb{Z}$, is $(\mathbb{Z}, \tau_u)$ sequentially complete?
The assumption on \( T_u \) and \( T_v \) to be locally quasi-convex in Theorem 3.11 is essential. As it was noticed in [37], in general, the converse assertion is not true, i.e., from the equality \( \tau_u = \tau_v \), it does not follow that \( T_u = T_v \). More precisely, in that example \( v \) converges to zero in \( T_u \), but \( v \) is not a null-sequence in \( T_u \). So \( \tau_u \) may have essentially more converging sequences than \( T_u \). We end this discussion by the following questions.

**Question 4.2** Let \( u \) be a \( TB \)-sequence in an infinite abelian group \( G \). Do the groups \((G, \tau_u)\) and \((G, (T_u)_{lqc})\) have the same null sequences (or compact subsets)?

**Question 4.3** Let \( u \) be a \( TB \)-sequence in an infinite abelian group \( G \) such that \((G, \tau_u)\) is not metrizable. Does there exist a \( LCQ \)-group topology \( \tau \) on \( G \) such that \( \tau_u < \tau < (T_u)_{lqc} \) (cf. Proposition 2.3)?

**Question 4.4** Let \( u \) and \( v \) be \( TB \)-sequences in an infinite abelian group \( G \) such that \( \tau_u = \tau_v \). Is \((T_u)_{lqc} = (T_v)_{lqc}\)?

The next problem should be compared with Proposition 3.2 and Corollary 3.3.

**Question 4.5** Prove or disprove that every \( ss \)-characterized precompact group is totally disconnected (zero-dimensional).

Let us note that the difference between totally disconnected and hereditarily disconnected groups is very subtle. While always the implications

$$
\text{zero-dimensional} \implies \text{totally disconnected} \implies \text{hereditarily disconnected}
$$

hold true, the inverse implications may fail in general (but hold true in locally compact or in countably compact groups [22]). A careful analysis of the short argument outlined in front of Corollary 3.3 shows that \( Q(Q(G)) = 0 \) for every \( ss \)-characterized precompact group. This is stronger than the current statement of that corollary, since the ordinal chain

$$
Q(G) \supseteq Q(Q(G)) \supseteq Q(Q(Q(G))) \supseteq \ldots
$$

has as intersection the connected component \( c(G) \) of the group \( G \) (note that for every ordinal \( \alpha \) one may find even a pseudocompact abelian group \( G_\alpha \) for which the ordinal chain [7] has length exactly \( \alpha \) [21]). Hence, in some sense the actual result \( Q(Q(G)) = 0 \) gives a good evidence that \( ss \)-characterized precompact group may be totally disconnected.

The next questions are motivated by Corollary C2 and by the fact that the topologies determined by a non-trivial \( T \)-sequence are always sequential, but never Fréchet-Urysohn (see Remark 3.10).

**Question 4.6** Let \((G, \tau)\) be a countable precompact \( ss \)-characterized group. Can \((G, \tau)\) be sequential, but not Fréchet-Urysohn? Can \((G, \tau)\) be Fréchet-Urysohn, but not metrizable?

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