On the existence, uniqueness and nature of Carathéodory and Filippov solutions for bimodal piecewise affine dynamical systems

L. Q. Thuan\textsuperscript{a,b}, M. K. Camlibel\textsuperscript{a,c}

\textsuperscript{a}Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, P.O. Box 800, 9700 AV Groningen, The Netherlands
\textsuperscript{b}Department of Mathematics, Quy Nhon University, 170 An Duong Vuong, Quy Nhon, Binh Dinh, Vietnam
\textsuperscript{c}Department of Electronics and Communication Engineering, Dogus University, Kadikoy 34722, Istanbul, Turkey

Abstract

In this paper, we deal with the well-posedness (in the sense of existence and uniqueness of solutions) and nature of solutions for discontinuous bimodal piecewise affine systems in a differential inclusion setting. First, we show that the conditions guaranteeing uniqueness of Filippov solutions in the context of general differential inclusions are quite restrictive when applied to bimodal piecewise affine systems. Later, we present a set of necessary and sufficient conditions for uniqueness of Filippov solutions for bimodal piecewise affine systems. We also study the so-called Zeno behavior (possibility of infinitely many switchings within a finite time interval) for Filippov solutions.

Key words: Piecewise affine systems, well-posedness, existence and uniqueness of solutions, Carathéodory solutions, Filippov solutions, one-sided Lipschitz condition.

1. Introduction

A piecewise affine dynamical system is a special type of finite-dimensional, nonlinear input-state-output systems with the distinguishing feature that the functions representing the system of differential equations and output equations are piecewise affine functions. Such systems arise in various contexts of system and control theory such as variable structure systems \cite{19}, bang-bang control \cite{12}, and linear relay systems \cite{4,20}.

Piecewise affine functions which describe the dynamics of a piecewise affine dynamical system are not necessarily continuous. As such, piecewise affine dynamical systems form a subclass of discontinuous dynamical systems (see \cite{6} as an excellent survey). An immediate consequence of discontinuous dynamics is that the existing results of mainstream smooth nonlinear systems and control theory (see e.g. \cite{16}) cannot be indiscriminately applied to piecewise affine dynamical systems. This departure from smooth systems begins already from the definition of a notion of solution. Indeed, meaning of a solution of a differential equation given by continuous functions is rather straightforward whereas it becomes a much more complicated matter in the absence of continuity.

The typical framework to deal with discontinuous dynamical systems is the framework of differential inclusions (see e.g. \cite{18}). Roughly speaking, one replaces a differential equation with discontinuous right-hand side (see e.g. Filippov’s seminal work \cite{2}) by a differential inclusion given by a set-valued mapping. There are several ways of defining a set-valued mapping (and hence a differential inclusion) for a differential equation with discontinuous right-hand side. Each of these leads to a different solution concept (see e.g. \cite{11,16} for details) such as Carathéodory, Krasovskii, Filippov, and Euler solutions. A good deal of the literature on differential inclusions is devoted to the investigation of existence and uniqueness of solutions. Typically, existence of solutions is guaranteed by less restrictive conditions than those for uniqueness.
In this paper, we focus on a particular class of piecewise affine dynamical systems, namely bimodal piecewise affine systems without external inputs. The main goal of the paper is to investigate existence, uniqueness, and nature of solutions (in the sense Carathéodory and Filippov) for this class of systems. It turns out that existence of Filippov solutions immediately follows from the existing results for general differential inclusions. However, existing conditions ensuring uniqueness for general differential inclusions are quite restrictive in the context of piecewise affine dynamical systems (see Theorems 2.6 in Section 2). Motivated by this fact, we turn our attention to tailor-made conditions for bimodal systems. The main results of this paper are a set of necessary and a set of sufficient conditions for uniqueness of Filippov solutions of bimodal systems. Furthermore, these results provide necessary and sufficient conditions for the existence and uniqueness of Filippov solutions for bimodal piecewise linear systems.

A curious phenomenon in the context of discontinuous dynamical systems is the so-called Zeno behavior (see e.g. [10, 11]) which refers to infinitely many switchings in a finite time interval. Presence of such behavior causes serious difficulties not only in analysis and design but also in simulation of such systems. As a by-product of our main results, we obtain conditions under which Zeno behavior is ruled out for bimodal piecewise affine systems.

Well-posedness of piecewise affine dynamical systems has received considerable attention in the last two decades. In [8], the authors consider bimodal piecewise linear systems. They work with what we call forward Carathéodory solutions and provide necessary and sufficient conditions for existence and uniqueness of these solutions. Forward Carathéodory solutions rule out the possibility of left accumulation points for switching instance by their very definition. In this paper, we consider more general bimodal systems, namely bimodal piecewise affine systems. Also we work not only with forward Carathéodory solutions but also with Filippov solutions. As such, the main result of [8] becomes a special case of our main results. In [20], well-posedness of linear relay systems was addressed for forward Carathéodory solutions and sufficient conditions for uniqueness were presented. A linear relay system with a single relay boils down to a bimodal piecewise affine system as studied in this paper (see Example 2.1). The results presented in this paper show that the very same conditions of [20] ensure uniqueness of Filippov solutions for this case. The paper [14] studied linear relay systems with a single relay and provided sufficient conditions for the uniqueness of Filippov solutions. Also the results of [14] can be recovered from our main results. Another related paper is [5] which considers Filippov solutions bimodal piecewise linear systems. The results of [5] can also be recovered as a special case from our main results (see Corolloary 3.5).

Zeno behavior of systems that are closely related to piecewise affine dynamical systems has been considered in [9] for a class of linear complementarity systems, in [15] for conewise linear systems, in [17] for linear relay systems with a single relay, and in [14] for continuous bimodal piecewise affine systems.

The paper is organized as follows. In Section 2, we introduce the object of the study in this paper, i.e. bimodal piecewise affine dynamical systems in a differential inclusion setting. We also present two examples of such systems and define what a solution means for such systems. This is followed by a discussion on existence of solutions as well as a discussion of the restrictiveness of conditions that guarantee uniqueness of Filippov solution of general differential inclusions when applied to bimodal systems. Section 3 presents the main results related to the uniqueness of Filippov solutions whereas we investigate Zeno behavior of bimodal systems in Section 4. In Section 5, we present the proofs of the main results. Finally, the paper closes with the conclusions in Section 6.

2. Bimodal piecewise affine systems

Consider the differential inclusion

$$\dot{x}(t) \in F(x(t))$$

with

$$F(x) = \begin{cases} 
\{A_1 x + e_1\} & \text{if } c^T x + f < 0 \\
\{A_1 x + e_1, A_2 x + e_2\} & \text{if } c^T x + f = 0 \\
\{A_2 x + e_2\} & \text{if } c^T x + f > 0 
\end{cases}$$

where \( x \in \mathbb{R}^n \), \( A_1, A_2 \in \mathbb{R}^{n \times n} \), \( e_1, e_2, c \in \mathbb{R}^n \) and \( f \in \mathbb{R} \). Also consider the convexified differential inclusion
\[
\dot{x}(t) \in G(x(t))
\]
with
\[
G(x) = \begin{cases} 
\{A_1 x + e_1\} & \text{if } c^T x + f < 0 \\
\text{conv}(\{A_1 x + e_1, A_2 x + e_2\}) & \text{if } c^T x + f = 0 \\
\{A_2 x + e_2\} & \text{if } c^T x + f > 0
\end{cases}
\]
where conv stands for the convex hull.

Throughout the paper, we call the systems of the form (1) and (2) \textit{bimodal piecewise affine systems}. In the sequel, we investigate existence and uniqueness of different kinds of solutions of bimodal piecewise affine systems.

Before elaborating on the solution concepts for these systems, we provide some examples of bimodal piecewise affine systems.

The first class of examples consists of linear systems with ideal relay elements which serve as an idealized models of Coulomb friction, bang-bang control, etc.

\textbf{Example 2.1.} (Linear relay systems \cite{[1]} \cite{[20]}) Consider the linear relay system
\[
\begin{align*}
\dot{x}(t) &= A x(t) + b u(t) \\
y(t) &= c^T x(t) \\
u(t) &\in -\text{sgn}(y(t))
\end{align*}
\]
where sgn is the set-valued relay function defined by
\[
\text{sgn}(y) = \begin{cases} 
\{-1\} & \text{if } y < 0 \\
[-1, 1] & \text{if } y = 0 \\
\{1\} & \text{if } y > 0.
\end{cases}
\]

Clearly, such a linear relay system is a bimodal piecewise affine system of the following form
\[
\dot{x}(t) \in \begin{cases} 
\{A x(t) + b\} & \text{if } c^T x(t) < 0 \\
\text{conv}(\{A x(t) + b, A x(t) - b\}) & \text{if } c^T x(t) = 0 \\
\{A x(t) - b\} & \text{if } c^T x(t) > 0.
\end{cases}
\]

Note that linear relay systems are particular cases of (2) where \( A_1 = A_2 \) and \( f = 0 \). Well-posedness of such systems were studied in \cite{[1]} and \cite{[20]}. Another example of bimodal piecewise affine systems is the following water tank system.

\textbf{Example 2.2.} Consider the two tank system depicted in Figure 1.

The deviations of the water level from the bottom of the first and second tank are denoted by \( x_1 \) and \( x_2 \), respectively. Let \( u \) be the constant flow of water into the first tank. The valve \( V \) is opened if \( x_2 < 1 \) and closed if \( x_2 \geq 1 \). By defining the state \( x = \text{col}(x_1, x_2) \) and taking all involved parameters unity, one obtains the following equations describing the dynamics of the system
\[
\dot{x} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} u \\ 0 \end{bmatrix} \text{ if } x_2 - 1 < 0
\]
\[
\dot{x} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ if } x_2 - 1 \geq 0
\]

As such, this system is of the form of a bimodal piecewise affine system \cite{[1]}. 

\[3\]
We now turn our attention to formalizing what will be meant by a solution of the system (1). There are many ways of defining a solution for a differential inclusion (see e.g. [1, 6]). In this paper, we focus on Carathéodory and Filippov solution concepts.

**Definition 2.3.** An absolutely continuous function $x: \mathbb{R} \to \mathbb{R}^n$ is said to be a solution of the system (1) for the initial state $\xi$ in the sense of

- **Carathéodory** if $x(0) = \xi$ and $x$ satisfies the differential inclusion (1) for almost all $t \in \mathbb{R}$.

- **forward Carathéodory** if it is a solution in the sense of Carathéodory, and for each $t^*$ there exist $i \in \{1, 2\}$ and $\epsilon_{t^*} > 0$ such that
  \[
  \dot{x}(t) = A_i x(t) + e_i \quad \text{and} \quad (-1)^{i-1} [c^T x(t) + f] \leq 0
  \]
  for all $t \in (t^* - \epsilon_{t^*}, t^*)$.

- **backward Carathéodory** if it is a solution in the sense of Carathéodory, and for each $t^*$ there exist $i \in \{1, 2\}$ and $\epsilon_{t^*} > 0$ such that
  \[
  \dot{x}(t) = A_i x(t) + e_i \quad \text{and} \quad (-1)^{i-1} [c^T x(t) + f] \leq 0
  \]
  for all $t \in (t^*, t^* + \epsilon_{t^*})$.

- **Filippov** if $x(0) = \xi$ and $x$ satisfies the convexified differential inclusion (2) for almost all $t \in \mathbb{R}$.

Clearly, every Carathéodory solution is a Filippov solution since $F(x) \subseteq G(x)$ for all $x \in \mathbb{R}^n$. However, not every Filippov solution is a Carathéodory solution in general.

When the right hand side of (1) is single-valued and hence is Lipschitz continuous, that is the implication

\[
c^T x + f = 0 \implies A_1 x + e_1 = A_2 x + e_2
\]

holds, Carathéodory and Filippov solutions coincide. In this case, existence and uniqueness of solutions are guaranteed by the theory of ordinary differential equations.

In general, existence of solutions of the differential inclusion (1) readily follows from the theory of differential inclusions (see e.g. [2, Theorem 2.7.1]).

**Proposition 2.4.** There exists a solution of the differential inclusion (1) in the sense of Filippov for each initial state.
Definition 2.5. We say that a Filippov solution for the initial state \( \xi \) is right-unique (left-unique) if for any Filippov solution \( x' \) for the initial state \( \xi \) there exists \( \epsilon > 0 \) such that \( x(t) = x'(t) \) for all \( t \in [0, \epsilon) \) \((t \in (-\epsilon, 0))\).

The main goal of the paper is to present necessary and/or sufficient conditions for uniqueness of Filippov solutions that are tailored to bimodal piecewise affine systems of the form (1). To motivate these new conditions, we first review one of the most typical uniqueness conditions that is employed in the literature of (general) differential inclusions and discuss its limitations for bimodal piecewise affine systems.

A set-valued mapping \( H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n \) is said to be one-sided Lipschitz (see e.g. [1]) if there exists a number \( L \) such that

\[
(x_1 - x_2)^T(y_1 - y_2) \leq L\|x_1 - x_2\|^2
\]

(6) for all \( x_1, x_2 \) belonging to the domain of \( H \), \( y_1 \in H(x_1) \) and \( y_2 \in H(x_2) \).

The following theorem presents necessary and sufficient conditions for one-sided Lipschitzness of the set-valued mapping \( G \).

Theorem 2.6. The set-valued mapping \( G \) is one-sided Lipschitz if and only if there exist a vector \( g \in \mathbb{R}^n \) and a number \( \mu \geq 0 \) such that

\[
A_1 - A_2 = g e^T \quad \text{and} \quad e_1 - e_2 = f g + \mu c.
\]

Proof. For the ‘if’ part, suppose that we have \( A_1 - A_2 = g e^T, e_1 - e_2 = f g + \mu c \) where \( \mu \geq 0 \). Then, the piecewise affine function which is defined by

\[
\tilde{G}(x) = \begin{cases} A_1 x + e_1 - \frac{\mu c}{2} & \text{if } c^T x + f \leq 0 \\ A_2 x + e_2 + \frac{\mu c}{2} & \text{if } c^T x + f > 0 \end{cases}
\]

is continuous, so that it is globally Lipschitz continuous (see [7]). Observe that \( G(x) = \tilde{G}(x) + [1 - 2\lambda(x)]\frac{\mu c}{2} \) where

\[
\lambda(x) = \begin{cases} \{0\} & \text{if } c^T x + f < 0 \\ [0, 1] & \text{if } c^T x + f = 0 \\ \{1\} & \text{if } c^T x + f > 0. \end{cases}
\]

Thus, for any \( x_i \in \mathbb{R}^n \) and \( y_i \in G(x_i) \), there exists \( \tilde{y}_i \in \tilde{G}(x_i) \) such that \( y_i = \tilde{y}_i + (1 - 2\lambda_i)\frac{\mu c}{2} \) where \( \lambda_i \in \lambda(x_i) \), and then

\[
(x_1 - x_2)^T(y_1 - y_2) = (x_1 - x_2)^T [(\tilde{y}_1 - \tilde{y}_2) - \mu (\lambda_1 - \lambda_2) c].
\]

This together with the observation that \( \mu \geq 0 \) and \( (\lambda_1 - \lambda_2)(x_1 - x_2)^T c \geq 0 \) implies that

\[
(x_1 - x_2)^T(y_1 - y_2) \leq (x_1 - x_2)^T(\tilde{y}_1 - \tilde{y}_2).
\]

On the other hand, the Cauchy-Schwarz inequality and Lipschitzness of \( \tilde{G} \) implies \( (x_1 - x_2)^T(\tilde{y}_1 - \tilde{y}_2) \leq L\|x_1 - x_2\|^2 \) where \( L \) denotes a Lipschitz constant of \( G \). Thus, we have

\[
(x_1 - x_2)^T(y_1 - y_2) \leq L\|x_1 - x_2\|^2
\]

for all \( x_1, x_2 \in \mathbb{R}^n \) and \( y_i \in G(x_i) \), and hence \( G \) is one-sided Lipschitz.

For the ‘only if’ part, we suppose that \( G \) is one-sided Lipschitz. Let \( \Sigma_- = \{ x \mid c^T x + f \leq 0 \} \) and \( \Sigma_+ = \{ x \mid c^T x + f \geq 0 \} \). Let \( x_1 \in \Sigma_- \), \( x_2 \in \Sigma_+ \), and let \( \tilde{x} \) be such that \( c^T \tilde{x} + f = 0 \). For \( \alpha \in (0, 1] \), define \( x'_1 = \alpha x_1 + (1 - \alpha) \tilde{x} \) and \( x'_2 = \alpha x_2 + (1 - \alpha) \tilde{x} \). Clearly, \( x'_1 \in \Sigma_- \) and \( x'_2 \in \Sigma_+ \). Since \( G \) is one-sided Lipschitz, one has

\[
[(A_1 x'_1 + e_1) - (A_2 x'_2 + e_2)]^T(x'_1 - x'_2) \leq L\|x'_1 - x'_2\|^2,
\]

In the sequel we focus on the uniqueness of Filippov solutions for the system (1).
or equivalently
\[
\frac{(1-\alpha)}{\alpha} \left\{ (A_1 - A_2)\bar{x} + (e_1 - e_2) \right\} + \left\{ (A_1 x_1 + e_1) - (A_2 x_2 + e_2) \right\}^T(x_1 - x_2) \leq L||x_1 - x_2||^2.
\]
By taking sufficiently small \( \alpha \), we obtain that \([A_1 - A_2] \bar{x} + (e_1 - e_2) \right\}^T(x_1 - x_2) \leq 0 \) for all \( x_1 \in \Sigma_- \) and \( x_2 \in \Sigma_+ \). This implies that
\[
(A_1 - A_2) \bar{x} + (e_1 - e_2) \in (\Sigma_- - \Sigma_+)^o
\]
for any \( \bar{x} \) with \( c^T \bar{x} + f = 0 \). Here the notation \( S^o \) denotes the polar cone of the set \( S \) that is \( S^o = \{ y \mid x^T y \leq 0, \forall x \in S \} \). Then, we get
\[
(A_1 - A_2)(\ker c^T) + (A_1 - A_2) \bar{x} + (e_1 - e_2) \subseteq (\Sigma_- - \Sigma_+)^o = \{ ac \mid \alpha \geq 0 \}
\]
for fixed \( \bar{x} \) satisfying \( c^T \bar{x} + f = 0 \). Since the left hand side is an affine set and the right hand side is a cone, we can conclude that \([A_1 - A_2](\ker c^T) = \{ 0 \}\). Hence, \( A_1 - A_2 = gc^T \) for some \( g \). Then it follows from \( 7 \) that \( e_1 - e_2 - fg \in (\Sigma_--\Sigma_+)^o \). Note that \( \Sigma_- - \Sigma_+ = \{ x \mid c^T x \leq 0 \} \). Hence, we have \( (\Sigma_- - \Sigma_+)^o = \{ ac \mid \alpha \geq 0 \} \). This means that \( e_1 - e_2 = fg + \mu c \) for some \( \mu \geq 0 \).

Theorem 2.6 shows that we can employ one-sided Lipschitzness in order to conclude uniqueness of Filippov solutions of bimodal systems only under quite restrictive conditions. Note that these conditions are met for a linear relay system as in Example 2.1 only if \( b = \alpha c \) for a nonnegative real number \( \alpha \). Note also that the conditions of Theorem 2.6 are never met for a two-tank system as in Example 2.2. Motivated by the restrictiveness of one-sided Lipschitzness, we investigate tailor-made uniqueness conditions for bimodal systems in the sequel.

3. Uniqueness of solutions

In this section, a set of necessary and a set of sufficient conditions for right-uniqueness of solutions of the bimodal piecewise affine systems \([4]\) will be provided. These conditions are less restrictive than the conditions guaranteeing the one-sided Lipschitz condition. To do so, we need to introduce some nomenclature as follows. For a vector \( v \), we write \( v \succ 0 \) if it is nonzero and the first nonzero entry is positive. We write \( v \succeq 0 \) if either \( v = 0 \) or \( v \succ 0 \). Similarly, we write \( v \prec 0 \) when \( -v \succ 0 \) and \( v \preceq 0 \) when \( -v \succeq 0 \). The observability index of the pair \((c^T, A_1)\) is denoted by \( h_1 \), that is the largest integer such that the matrix \( \text{col}(c^T, c^T A_1, \ldots, c^T A_1^{h_1}) \) has full row rank. Note that for each \( k \geq 1 \) there exists a unique matrix \( P_k^h \in \mathbb{R}^{k \times h} \) such that
\[
\begin{bmatrix}
 c^T A_1^{h_1+1} \\
 c^T A_1^{h_1+2} \\
 \vdots \\
 c^T A_1^{h_1+k}
\end{bmatrix}
= P_k^h
\begin{bmatrix}
 c^T A_1 \\
 \vdots \\
 c^T A_1^h
\end{bmatrix}.
\]

We now present the main results concerning the well-posedness of solutions of the bimodal piecewise affine system \([4]\) in the three theorems below.

**Theorem 3.1.** Let \( h := \min\{h_1, h_2\} \). Consider the statements:
1. Every Filippov solution of \([4]\) is right-unique.
2. Every Filippov solution of \([4]\) is both a forward and backward Carathéodory solution.
3. There exist an integer \( k \) with \( 1 \leq k \leq h + 1 \) and a \((k+1) \times (k+1)\) lower triangular matrix \( M \) with positive diagonal elements such that
\[
\begin{bmatrix}
 c^T A_1 \\
 \vdots \\
 c^T A_k
\end{bmatrix} = M
\begin{bmatrix}
 c^T A_1 \\
 \vdots \\
 c^T A_k
\end{bmatrix},
\begin{bmatrix}
 f \\
 \vdots \\
 f
\end{bmatrix}
\begin{bmatrix}
 c^T e_1 \\
 \vdots \\
 c^T e_1
\end{bmatrix} \succ M
\begin{bmatrix}
 f \\
 \vdots \\
 f
\end{bmatrix}
\begin{bmatrix}
 c^T e_2 \\
 \vdots \\
 c^T e_2
\end{bmatrix}.
\]
4. There exists a \((h+1) \times (h+1)\) lower triangular matrix \(M\) with positive diagonal elements such that
\[
\begin{bmatrix}
    c^T A_1 \\
    c^T A_1^h \\
    \vdots \\
    c^T A_1^{h+1}
\end{bmatrix} = M
\begin{bmatrix}
    c^T A_2 \\
    c^T A_2^h \\
    \vdots \\
    c^T A_2^{h+1}
\end{bmatrix},
\begin{bmatrix}
    f \\
    f \\
    \vdots \\
    f
\end{bmatrix} = M
\begin{bmatrix}
    f \\
    f \\
    \vdots \\
    f
\end{bmatrix}
\]
and either \(h_1 < h_2\) and \(c^T A_1^{h_1} e_1 - p_1^T e_1^{h_1} > 0\) or \(h_1 > h_2\) and \(c^T A_2^{h_2} e_2 - p_2^T e_2^{h_2} < 0\) where \(p_i\) is uniquely determined from
\[
c^T A_i^{h_i+1} = p_i^T T_i^{h_i}, \quad i = 1, 2.
\]

5. The observability indices \(h_1\) and \(h_2\) are the same and there exists a \((h+2) \times (h+2)\) lower triangular matrix \(M\) with positive diagonal elements such that
\[
\begin{bmatrix}
    c^T A_1 \\
    c^T A_1^h \\
    \vdots \\
    c^T A_1^{h+1}
\end{bmatrix} = M
\begin{bmatrix}
    c^T A_2 \\
    c^T A_2^h \\
    \vdots \\
    c^T A_2^{h+1}
\end{bmatrix},
\begin{bmatrix}
    f \\
    f \\
    \vdots \\
    f
\end{bmatrix} = M
\begin{bmatrix}
    f \\
    f \\
    \vdots \\
    f
\end{bmatrix}.
\]

6. The following implication holds
\[
\begin{bmatrix}
    c^T A_1 \\
    c^T A_1^h \\
    \vdots \\
    c^T A_1^{h+1}
\end{bmatrix} \xi + \begin{bmatrix}
    f \\
    f \\
    \vdots \\
    f
\end{bmatrix} e_1 = 0 \Rightarrow A_1 \xi + e_1 = A_2 \xi + e_2.
\]

Then, the following implications hold:

A. \(1 \Rightarrow 3\) or \(1 \Rightarrow 5\)
B. \(1 \text{ and } 3\) \(\Rightarrow 6\)
C. \(5 \text{ and } 6\) \(\Rightarrow 2\)
D. \(5 \text{ and } 6\) \(\Rightarrow 1\)

A proof of this theorem will be presented in Section 5. Note that this theorem provides only a set of necessary and a set of sufficient conditions, but not necessary and sufficient conditions in general. The following example (see \([4\text{ Eq’s. (13) and (14)})]) illustrates the gap between the necessary and the sufficient conditions.

**Example 3.2.** Consider the binodal piecewise affine system \(1\) with \(c^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\), \(f = 0\),
\[
A_1 = A_2 = \begin{bmatrix}
    0 & 1 & 0 \\
    0 & 0 & 1 \\
    0 & 0 & 0
\end{bmatrix}, \text{ and } e_1 = -e_2 = \begin{bmatrix} 0 \\
0 \\
1
\end{bmatrix}.
\]
For this system, it can be verified that the third statement of Theorem \(3\) is satisfied with \(k = 3\). However, as it has been shown in \([4\text{], there are infinitely many Filippov solutions for the zero initial state.}

However, the third statement of Theorem \(3\) with \(k = 1\) is sufficient for right-uniqueness of Filippov solutions as stated in the following.
Theorem 3.3. If there exists a $2 \times 2$ lower triangular matrix $M$ with positive diagonal elements such that
\[
\begin{bmatrix}
      c^T A_1 \\ c^T A_2 
\end{bmatrix} = M \begin{bmatrix}
      c^T e_1 \\ c^T e_2 
\end{bmatrix},
\]
then right-uniqueness of Filippov solutions holds at any state of the state space $\mathbb{R}^n$.

Proof. Due to (9), for each state $\xi$ satisfying $c^T \xi + f = 0$ at least one of the inequalities $c^T A_1 \xi + c^T e_1 > 0$ or $c^T A_2 \xi + c^T e_2 < 0$ is satisfied. By [2, Theorem 2.10.2], every Filippov solution is right-unique. ■

Further, the third statement of Theorem 3.1 with $k = 2$ is sufficient for right-uniqueness of Filippov solutions for some initial states.

Theorem 3.4. If there exists $3 \times 3$ lower triangular matrix $M$ with positive diagonal elements such that
\[
\begin{bmatrix}
      c^T A_1 \\ c^T A_2 \\ c^T A_3 
\end{bmatrix} = M \begin{bmatrix}
      c^T A_1 e_1 \\ c^T A_2 e_2 \\ c^T A_3 e_3 
\end{bmatrix},
\]
then for the system (1), right-uniqueness of Filippov solutions holds at all states from $\mathbb{R}^n \setminus \Omega$ where $\Omega$ is the set of all $\xi \in \mathbb{R}^n$ such that
\[
\begin{bmatrix}
      c^T A_1 \\ c^T A_2 \\ c^T A_3 
\end{bmatrix} \xi + \begin{bmatrix}
      f^T e_1 \\ f^T A_1 e_2 \\ f^T A_2 e_3 
\end{bmatrix} = 0 \text{ and } (-1)^k (c^T A_k^2 \xi + c^T A_k e_k) < 0
\]
for some $k,j \in \{1,2\}, k \neq j$.

A proof of this theorem will be given in Section 5.

Theorem 3.3 and Theorem 3.4 present two particular cases under which the third statement of Theorem 3.1 becomes sufficient as well as necessary for right-uniqueness of Filippov solutions for bimodal systems. Another interesting particular case occurs when there are no affine terms in the dynamics, that is when $e_1 = e_2 = 0$ and $f = 0$. In this case, one can state necessary and sufficient conditions (see also [5]) as in the following.

Corollary 3.5. Consider the system (1) with $e_1 = e_2 = 0$ and $f = 0$. Then, every Filippov solution of (1) is unique if and only if the following statements hold:

1. $h_1 = h_2$.
2. There exists an $(h_1 + 1) \times (h_1 + 1)$ lower triangular matrix $M$ with positive diagonal elements such that
   \[
   \begin{bmatrix}
      c^T A_1 \\ c^T A_2 \\ \vdots \\ c^T A_{h_1} 
   \end{bmatrix} = M \begin{bmatrix}
      c^T A_1 \\ c^T A_2 \\ \vdots \\ c^T A_{h_1} 
   \end{bmatrix}.
   \]
3. The following implication holds
   \[
   \begin{bmatrix}
      c^T A_1 \\ c^T A_2 \\ \vdots \\ c^T A_{h_1} 
   \end{bmatrix} x = 0 \Rightarrow A_1 x = A_2 x.
   \]

Proof. Note that the third and fourth statements of Theorem 3.1 never holds as $e_1 = e_2 = 0$ and $f = 0$. Then, the first statement of Theorem 3.1 holds if and only if the fifth and the sixth hold. ■
4. Switching behavior

In this section, we investigate mode switching behavior of bimodal systems. We say that a time instant \( t^* \in \mathbb{R} \) is a \textit{non-switching time} for a Filippov solution \( x \) if there exist an interval \((t^* - \epsilon, t^* + \epsilon)\) and an index \( i \) with \( i \in \{1, 2\} \) such that

\[
\dot{x}(t) = A_i x(t) + e_i \quad \text{and} \quad (-1)^{i-1} [c^T x(t) + f] \leq 0
\]

for all \( t \in (t^* - \epsilon, t^* + \epsilon) \). We say that a time instant \( t^* \in \mathbb{R} \) is a \textit{switching time} for a Filippov solution \( x \) if \( t^* \) is not a non-switching time for the same solution.

The distribution of the switching times along a solution is an important issue for various reasons. For instance, the so-called event-driven simulation methods (see e.g. [3]) may fail if the switching times accumulate around a point. This type of phenomenon is known as Zeno behavior in hybrid systems literature. We say that a time instant \( t^* \in \mathbb{R} \) is a \textit{left/right Zeno time} for a Filippov solution \( x \) if for each \( \epsilon > 0 \) the interval \((t^* - \epsilon, t^* + \epsilon)/)(t^* - \epsilon, t^*)\) contains a switching time for the same solution.

A Filippov solution will be called \textit{left (right) Zeno-free} if there exists no left (right) Zeno time for it. Also, we say that the system (1) is \textit{left (right) Zeno-free} if all its Filippov solutions are left (right) Zeno-free.

Further, we say that the system (1) is \textit{Zeno-free} if it is both is left and right Zeno-free.

With these preparations, we can state the following sufficient condition for Zeno-freeness.

**Theorem 4.1.** Suppose that the observability indices \( h_1 \) and \( h_2 \) are the same. Let \( h = h_1 = h_2 \). If there exists a \((h + 2) \times (h + 2)\) lower triangular matrix \( M \) with positive diagonal elements such that

\[
\begin{bmatrix}
    c^T \\
    c^T A_1 \\
    \vdots \\
    c^T A_1^{h+1}
\end{bmatrix} = M
\begin{bmatrix}
    c^T \\
    c^T A_2 \\
    \vdots \\
    c^T A_2^{h+1}
\end{bmatrix},
\]

\[
\begin{bmatrix}
    f \\
    c^T e_1 \\
    \vdots \\
    c^T A_1^{h} e_1
\end{bmatrix} = M
\begin{bmatrix}
    f \\
    c^T e_2 \\
    \vdots \\
    c^T A_2^{h} e_2
\end{bmatrix}
\]

and the implication

\[
\begin{bmatrix}
    c^T A_1 \\
    \vdots \\
    c^T A_1^{h+1}
\end{bmatrix} \xi + \begin{bmatrix}
    f \\
    c^T e_1 \\
    \vdots \\
    c^T A_1^{h} e_1
\end{bmatrix} = 0 \Rightarrow A_1 \xi + e_1 = A_2 \xi + e_2
\]

holds, then the system (1) is Zeno-free.

**Proof.** Note that the system (1) is Zeno-free if and only if any Filippov solution is both forward and backward Carathéodory solution. As such, the claim follows from Theorem 3.1.C. \( \blacksquare \)

5. Proofs

5.1. Proof of Theorem 3.1

To prove Theorem 3.1, we first consider affine systems and introduce some notations. An affine system \( \Sigma(A, e, c^T, f) \) is given by

\[
\begin{align*}
\dot{x} &= Ax + e \quad \text{(11a)} \\
y &= c^T x + f \quad \text{(11b)}
\end{align*}
\]

where \( x \in \mathbb{R}^n \) is the state, \( y \in \mathbb{R} \) is the output, and all involved matrices are of appropriate sizes. By \( x(t; \xi) \) and \( y(t; \xi) \), we denote the state and the output of the system (11) for the initial state \( \xi \), respectively. We
define the sets

\[
W^-_\Sigma := \{ \xi | \exists \epsilon > 0 \text{ such that } y(t; \xi) < 0, \forall t \in (0, \epsilon) \}, \\
W^i_\Sigma := \{ \xi | \exists \epsilon > 0 \text{ such that } y(t; \xi) = 0, \forall t \in (0, \epsilon) \}, \\
W^+_\Sigma := \{ \xi | \exists \epsilon > 0 \text{ such that } y(t; \xi) > 0, \forall t \in (0, \epsilon) \}.
\]

Since \( y(t; \xi) \) is a real-analytic function for each initial state \( \xi \), its sign on a small time interval \((0, \epsilon)\) is completely determined by the values of its derivatives at \( t = 0 \), i.e. \( y^{(k)}(0; \xi) \) for \( k = 0, 1, \ldots \). Note that \( y^{(0)}(0; \xi) = c^T \xi + f \) and \( y^{(k)}(0; \xi) = c^T A^k \xi + c^T A^{k-1} e \) for \( k \geq 1 \). Let \( \nu \) denote the observability index of the pair \((c^T, A)\). It can be verified that \( c^T A^\nu \xi + c^T A^{\nu-1} e = 0 \) for all \( \nu + 1 \geq \ell \geq 1 \) implies \( c^T A^\nu \xi + c^T A^{\nu-1} e = 0 \) for all \( \ell \geq 1 \). Together with the analyticity of the output \( y(t; \xi) \), this observation leads to the following immediate characterizations of the \( \mathcal{W} \)-sets:

\[
\begin{align*}
(W^-_\Sigma \cup W^0_\Sigma \cup W^+_\Sigma) \backslash \{0\} &= 0, \\
\begin{bmatrix} W^-_\Sigma \cup W^0_\Sigma \cup W^+_\Sigma \end{bmatrix} &= 0,
\end{align*}
\]

Note also that

\[
W^-_\Sigma \cup W^0_\Sigma \cup W^+_\Sigma = \mathbb{R}^n. 
\]

In the sequel, we often use the \( \mathcal{W} \)-sets corresponding to the two modes of the system \((1)\). For brevity, we denote \( \Sigma_i = \Sigma_i(A_i, e_i, c^T, f) \) and define

\[
W^0_i := W^0_{\Sigma_i}, \quad W^-_i := W^-_{\Sigma_i}, \quad W_i := W^0_{\Sigma_i} \cup W^-_{\Sigma_i}, \\
W^+_i := W^0_{\Sigma_i}, \quad W^+_i := W^+_{\Sigma_i} \cup W^0_{\Sigma_i}. 
\]

We also define

\[
T_i^k := \begin{bmatrix} c^T A_i \\ c^T A_i^2 \\ \vdots \\ c^T A_i^k \end{bmatrix} \text{ and } e_i^k := \begin{bmatrix} f \\ c^T e_i \\ \vdots \\ c^T A_i e_i \end{bmatrix}
\]

for \( k \geq 1 \). With these preparations, we are ready to prove Theorem 3.1.

5.1.1. Proof of Theorem 3.1

Right-uniqueness of every Filippov solution of the system \((1)\) implies that

\[
W^-_1 \cap W_2 = \emptyset \quad \text{and} \quad W_1 \cap W^+_2 = \emptyset.
\]  

In view of \((12)\) and \((14)\), therefore, we have

\[
\begin{align*}
T_1^{h_1+1} \xi + e_1^{h_1+1} &< 0 \quad \Rightarrow \quad T_2^{h_2+1} \xi + e_2^{h_2+1} < 0, \\
T_2^{h_2+1} \xi + e_2^{h_2+1} &> 0 \quad \Rightarrow \quad T_1^{h_1+1} \xi + e_1^{h_1+1} > 0,
\end{align*}
\]

as necessary conditions for right-uniqueness of every Filippov solution. Note that a consequence of the first implication is that

\[
T_1^h \xi + e_1^h \not< 0 \quad \Rightarrow \quad T_2^h \xi + e_2^h \not< 0.
\]

In order to formulate this condition in terms of the parameters of the system \((1)\), we invoke the following lemma which was proven in \((13)\).

Lemma 5.1. Let \( P_i \) be an \( m \times n \) matrix of full row rank and \( q_i \) be an \( n \)-vector for \( i = 1, 2 \). Then, the following statements are equivalent:

10
1. \( P_1 x < q_1 \) implies \( P_2 x \leq q_2 \).

2. \( P_1 x < q_1 \) implies \( P_2 x < q_2 \).

3. Either

\[ P_1 = MP_2 \quad \text{and} \quad q_1 = Mq_2 \]

for some \( m \times m \) lower triangular matrix \( M \) with positive diagonal elements, or there exist \( \ell \) with \( 1 \leq \ell \leq m \) and \( \ell \times \ell \) lower triangular matrix \( M \) with positive diagonal elements such that

\[ P_{1[k]} = MP_{2[k]} \quad \text{and} \quad q_{1[k]} < Mq_{2[k]} \]

where the notation \( \cdot[k] \) denotes the first \( k \) rows of a matrix/vector.

Note that \( T^h_1 \) is of full row rank for \( i = 1, 2 \) since \( h = \min\{h_1, h_2\} \). Then, it follows from (17) and Lemma 5.1 that either

\[ T^h_1 = MT^h_2 \quad \text{and} \quad e^h_1 = Me^h_2 \]  \hspace{1cm} (18)

for some \((h + 1) \times (h + 1)\) lower triangular matrix \( M \) with positive diagonal elements, or there exist \( \ell \) with \( 1 \leq \ell \leq h + 1 \) and \( \ell \times \ell \) lower triangular matrix \( M \) with positive diagonal elements such that

\[ T^{\ell-1}_1 = MT^{\ell-1}_2 \quad \text{and} \quad e^{\ell-1}_1 > Me^{\ell-1}_2. \]  \hspace{1cm} (19)

Suppose first that the latter holds. Since \( T^0_1 = e^T = T^0_2 \) and \( e^0_1 = f = e^0_2 \), the case \( \ell = 1 \) is impossible. Then, we get

\[ T^k_1 = MT^k_2 \quad \text{and} \quad e^k_1 > Me^k_2 \]

for some \( k \) with \( 1 \leq k \leq h \) and \((k + 1) \times (k + 1)\) lower triangular matrix \( M \) with positive diagonal elements. This is nothing but the statement [3]. Thus, it remains to show that (18) also implies the statement [3] or [4] or [5]. To do so, we consider two cases: \( h_1 = h_2 \) and \( h_1 \neq h_2 \).

For the case \( h_1 = h_2 \), we prove that (18) implies statement [5]. Note first that the first part of statement [4] holds due to (18). For the second part, we only need to prove for \( h_1 < h_2 \). For the case \( h_1 > h_2 \), the proof is similar.

Since \( h_1 < h_2 \) and \( T^h_{22} \) is of full row rank, there exists \( \xi \) such that \( T^h_2 \xi + e^h_2 = 0 \) and \( T^h_2 \xi + e^h_2 > 0 \). Together with (16b) and (18), this implies

\[ T^h_1 \xi + e^h_1 = 0 \quad \text{and} \quad T^h_1 \xi + e^h_1 > 0. \]

This immediately implies \( c^T A^h_1 e_1 = p^T e^h_1 > 0 \).

For the case \( h_1 = h_2 = h \), we prove that (18) implies the statement [3] or [5]. To this end, first note that it suffices to show that either

\[ T^{h+1}_1 = \hat{M} T^{h+1}_2 \quad \text{and} \quad e^{h+1}_1 > \hat{M} e^{h+1}_2 \]  \hspace{1cm} (20)

or

\[ T^{h+1}_1 = \hat{M} T^{h+1}_2 \quad \text{and} \quad e^{h+1}_1 = \hat{M} e^{h+1}_2 \]  \hspace{1cm} (21)

for some \((h + 2) \times (h + 2)\) lower triangular matrix \( \hat{M} \) with positive diagonal elements. Indeed, the former would imply the statement [3] and the latter [5]. In what follows, we will construct a matrix \( \hat{M} \) which will satisfy one of these two conditions. To do so, note that

\[ c^T A^{h+1}_1 = p_i^T T^h_i \quad \text{with} \quad i = 1, 2 \]  \hspace{1cm} (22)

for some \( p_1, p_2 \in \mathbb{R}^{h+1} \) as \( h \) is the observability index of both \((c^T, A_1)\) and \((c^T, A_2)\). Define

\[ q^T = p_1^T M - \alpha p_2^T \quad \text{and} \quad \hat{M} = \begin{bmatrix} M & 0 \\ q^T & \alpha \end{bmatrix} \]
for some $\alpha$. It follows from \(18\) and \(22\) that $T_1^{h+1} = \hat{M}T_2^{h+1}$. Therefore, it remains to show that we can choose $\alpha > 0$ such that either $e_1^{h+1} > M e_2^{h+1}$ or $e_1^{h+1} = M e_2^{h+1}$. Since $e_1^h = Me_2^h$, $e_1^{h+1} > M e_2^{h+1}$ holds if and only if $c^T A_1^h e_1 > q^T e_2 + \alpha c^T A_2^h e_2$ and $e_1^{h+1} = M e_2^{h+1}$ holds if and only if $c^T A_1^h e_1 = q^T e_2 + \alpha c^T A_2^h e_2$.

As such, it is enough to show that we can choose $\alpha > 0$ such that $c^T A_1^h e_1 > q^T e_2 + \alpha c^T A_2^h e_2$. By using the definition of $q$, we see that the last inequality is equivalent to

$$
(c^T A_1^h e_1 - p_2^T e_2^h) \geq \alpha (c^T A_2^h e_2 - p_2^T e_2^h).
$$

(23)

Since $T_1^h$ is of full column rank, there exists $\xi_0$ such that $T_1^h \xi_0 + e_1^h = 0$. It follows from \(18\) that $T_2^h \xi_0 + e_2^h = 0$. By using \(22\), we can rewrite \(23\) as

$$(c^T A_1^h e_1 + c^T A_1^{h+1} \xi_0) \geq \alpha (c^T A_2^h e_2 + c^T A_2^{h+1} \xi_0).$$

(24)

Define $\rho_i = c^T A_i^h e_1 + c^T A_i^{h+1} \xi_0$ for $i = 1, 2$ and observe that there exists $\alpha > 0$ satisfying the above inequality unless $(\rho_1 \leq 0$ and $\rho_2 > 0)$ or $(\rho_1 < 0$ and $\rho_2 \geq 0)$. However, neither of these two cases can occur due to the definition of $\xi_0$ and \(16\) with $\xi = \xi_0$.

5.1.2. Proof of Theorem 3.1.C

On the one hand, right-uniqueness of Filippov solutions (statement \(\dagger\)) necessitates that the implication

$$\dot{x}_i = A_i x_i + e_i, \quad x_i(0) = \xi \implies x_i(t) = \hat{x}_i(t) \quad \text{for all } t \geq 0$$

(25)

holds for all $\xi \in \mathcal{W}_i^0 \cap \mathcal{W}_2^0$. On the other hand, it follows from statement \(\ddagger\) and \(\|\) that $\mathcal{W}_1^0 = \mathcal{W}_2^0 = \{\xi \mid T_1^{h+1} \xi + e_1^{h+1} = 0\}$. As such, statement \(\ddagger\) immediately follows from \(25\).

5.1.3. Proof of Theorem 3.1.C

Note that both statements \(\dagger\) and \(\ddagger\) are invariant under time-reversal. As such, it is enough to show that the forward Carathéodory property holds for every Filippov solution.

It follows from statement \(\ddagger\) that $\mathcal{W}^0 := \mathcal{W}_1^0 = \mathcal{W}_2^0$. We claim first that forward Carathéodory property holds for every Filippov solution with the initial state $\xi \in \mathcal{W}^0$. To this end, note that statement \(\ddagger\) implies that

$$\xi \in \mathcal{W}^0 \implies A_1 \xi + e_1 = A_2 \xi + e_2.$$  

(26)

Note also that the implication

$$\xi \in \mathcal{W}_i^0 \implies A_i \xi + e_i \in \mathcal{W}_i^0$$

(27)

readily holds from the very definition of the sets $\mathcal{W}_i^0$ for $i = 1, 2$. Together with \(26\), this invariance property yields

$$\dot{x}_i = A_i x_i + e_i, \quad x_i(0) = \xi \implies x_i(t) = \hat{x}_i(t) \quad \text{for all } t \in \mathbb{R}$$

(28)

for all $\xi \in \mathcal{W}^0$. Let $x^*$ satisfy $\dot{x}^* = A_1 x^* + e_1$ with $x^*(0) = \xi$. Also let $x$ be a Filippov solution of the system with $x(0) = \xi$. Then, there exists a function $\lambda : \mathbb{R} \to [0, 1]$ such that

$$\dot{x}(t) = \lambda(t) [A_1 x(t) + e_1] + (1 - \lambda(t)) [A_2 x(t) + e_2]$$

(29)

for almost all $t \in \mathbb{R}$. It follows from \(28\) that

$$\dot{x}^*(t) = \lambda(t) [A_1 x^*(t) + e_1] + (1 - \lambda(t)) [A_2 x^*(t) + e_2]$$

(30)

for all $t \in \mathbb{R}$. Define $A(t) := \lambda(t) A_1 + (1 - \lambda(t)) A_2$. Then, we get

$$\frac{d}{dt} (||x^*(t) - x(t)||^2) = 2 \langle x^*(t) - x(t), \dot{x}^*(t) - \dot{x}(t) \rangle$$

$$= 2 \langle x^*(t) - x(t), A(t)(x^*(t) - x(t)) \rangle \leq \alpha ||x^*(t) - x(t)||^2$$

12
where $\alpha := 2 \max \{\|\lambda A_1 + (1 - \lambda) A_2\| \mid \lambda \in [0, 1]\}$. Since $x^+(0) - x(0) = 0$, the last inequality readily implies that $x(t) = x^+(t)$ for all $t \in \mathbb{R}$. In other words, $x^+$ is the unique Filippov solution for the initial state $\xi$. It follows from (27) that $x^+(t) \in \mathcal{W}^0$ for all $t \in \mathbb{R}$. Hence, $x^+$ is a forward Carathéodory solution and statement 2 holds for every Filippov solution with the initial state $\xi \in \mathcal{W}^0$.

Next, we claim that if $x$ is a Filippov solution for the initial state $\xi$ and $t^* \in \mathbb{R}$ is such that $x(t^*) \in \mathcal{W}^0$ then $x(t) \in \mathcal{W}^0$ for all $t \in \mathbb{R}$. To see this, note that $\tilde{x}(t) = x(t + t^*)$ is a Filippov solution for the initial state $x(t^*) \in \mathcal{W}^0$. As such, the above argument yields $x(t) = \tilde{x}(t-t^*) \in \mathcal{W}^0$ for all $t \in \mathbb{R}$.

Therefore, it remains to show that forward Carathéodory property holds for every Filippov solution $x$ with the property that

$$x(t) \notin \mathcal{W}^0$$

for all $t \in \mathbb{R}$. Let $x$ be such a Filippov solution and let $t^* \in \mathbb{R}$. If $c^T x(t^*) + f \neq 0$, it follows from statement 5 and continuity of $x$ that there exists $\varepsilon_{t^*} > 0$ such that (31) holds. Suppose that $c^T x(t^*) + f = 0$. It follows from (31) and (12) that there exists an integer $k$ with $0 \leq q < h$ such that

$$c^T A_1^k x(t^*) + c^T A_1^{q-1} e_1 = 0 \quad \text{for all } k = 1, 2, \ldots, q$$

$$c^T A_1^q x(t^*) + c^T A_1^q e_1 
eq 0.$$  (32)

Now, suppose that

$$c^T A_1^q x(t^*) + c^T A_1^q e_1 > 0.$$  (33)

It follows from statement 5 that

$$c^T A_2^k x(t^*) + c^T A_2^{q-1} e_2 = 0 \quad \text{for all } k = 1, 2, \ldots, q$$

$$c^T A_2^q x(t^*) + c^T A_2^q e_2 > 0.$$  (34)

Since $x$ is continuous, there must exist $\varepsilon_{t^*} > 0$ such that

$$c^T A_1^q x(t) + c^T A_1^q e_1 > 0$$
$$c^T A_2^q x(t) + c^T A_2^q e_2 > 0$$

for all $t \in (t^*, t^* + \varepsilon_{t^*})$.

For $\lambda \in [0, 1]$, we define $A(\lambda) := \lambda A_1 + (1 - \lambda) A_2$, $e(\lambda) := \lambda e_1 + (1 - \lambda) e_2$. We also define

$$\mathbb{G}_0 = \mathbb{H}_0 := \{1\},$$

$$\mathbb{G}_k := \{A_i G' \mid G' \in \mathbb{G}_{k-1}, i = 1, 2\} \text{ for } k \geq 1,$$

$$\mathbb{H}_k := \{A(\lambda) H' \mid \lambda \in [0, 1] \text{ and } H' \in \mathbb{H}_{k-1}\} \text{ for } k \geq 1.$$  (35)

For each $k \geq 1$ and $(G, H) \in \mathbb{G}_k \times \mathbb{H}_k$ of the form

$G = A_{i_k} A_{i_{k-1}} \cdots A_{i_1}$, $H = A(\lambda_k) A(\lambda_{k-1}) \cdots A(\lambda_1)$,

we define

$$e_G := c^T A_{i_k} A_{i_{k-1}} \cdots A_{i_2} e_{i_1},$$
$$e_H := c^T A(\lambda_k) A(\lambda_{k-1}) \cdots A(\lambda_2) e(\lambda_1)$$

with the convention that $e_I := f$ for $k = 0$. Finally, define $\mathcal{G}_k := \{(G, e_G) \mid G \in \mathbb{G}_k\}$ and $\mathcal{H}_k := \{(H, e_H) \mid H \in \mathbb{H}_k\}$.

Note that $\mathbb{G}_k \subseteq \mathbb{H}_k$ and $\mathcal{G}_k \subseteq \mathcal{H}_k$ for all $k \geq 0$. Also note that conv($\mathcal{G}_k$) $\subseteq$ conv($\mathcal{H}_k$) and that $\mathcal{H}_k$ $\subseteq$ conv($\mathcal{H}_k$). Hence, we have

$$\text{conv}(\mathcal{H}_k) = \text{conv}(\mathcal{G}_k)$$

for every $k \geq 0$.  (36)
Now, we claim that for each $\ell \in \{0, 1, \ldots, q\}$ and each $G \in \mathcal{G}_\ell$
\[c^T G x(t^*) + e_G = 0.\] (40)
To prove this claim, we make an induction on $\ell$. The case $\ell = 0$ is evident. Suppose that the claim holds for all $\ell$ with $0 \leq \ell < p \leq q$. Let $G \in \mathcal{G}_p$. In the cases $G = A^1_p$ and $G = A^2_p$, the claim readily follows from (32) and (35), respectively. Otherwise, $G = A^{k_1}_{q} A^{k_2}_{q-1} G'$ for some $i \in \{1, 2\}$, $2 \leq k_1 + k_2 \leq p$, and $G' \in \mathcal{G}_{p-k_1-k_2}$. Then, it follows from statement 5 that there exists a positive number $\alpha_G$ such that
\[c^T G x(t^*) + e_G = \alpha(c^T \bar{G} x(t^*) + e_{\bar{G}})\] (41)
where $\bar{G} = A^{k_1+k_2}_p G'$. By repeating the same argument, one can prove the existence of a positive number $\alpha_G$ such that
\[c^T G x(t^*) + e_G = \alpha(c^T A^i_p x(t^*) + c^T A^{i-1}_p e_i).\] (42)
Then, the claim follows from either (32) or (35). The very same argument employed in the last step of the above induction yields
\[c^T G x(t^*) + e_G > 0\] (43)
for all $G \in \mathcal{G}_{q+1}$. Since $x$ is continuous and $\mathcal{G}_{q+1}$ is a finite set, we can conclude that there exists a positive number $\epsilon^*$ such that
\[c^T G x(t) + e_G > 0\] (44)
for all $G \in \mathcal{G}_{q+1}$ and for all $t \in (t^*, t^* + \epsilon^*)$.
From (39), we further get
\[c^T H x(t^*) + e_H = 0\] (45)
for all $H \in \mathbb{H}_\ell$ and $\ell = 0, 1, \ldots, q$, and also that
\[c^T H x(t) + e_H > 0\] (46)
for all $H \in \mathbb{H}_{q+1}$ and for all $t \in (t^*, t^* + \epsilon^*)$.
Let $H \in \mathbb{H}_q$. Note that
\[c^T H \dot{x}(t) \in \{c^T H' x(t) + e_H' \mid H' \in \mathbb{H}_{q+1}\}.\] (47)
Also note that
\[c^T H x(t) + e_H = c^T H x(t^*) + e_H + \int_{t^*}^t c^T H \dot{x}(s) \, ds.\] (48)
Then, it follows from (43), (46), and (47) that
\[c^T H x(t) + e_H > 0\] (49)
for all $t \in (t^*, t^* + \epsilon^*)$. The very same argument can be repeated for $H \in \mathbb{H}_\ell$ with $\ell = q-1, \ldots, 1, 0$ to obtain
\[c^T H x(t) + e_H > 0\] (50)
for all $H \in \mathbb{H}_\ell$, $\ell = 0, 1, \ldots, q$, and $t \in (t^*, t^* + \epsilon^*)$. In particular, we obtain
\[c^T x(t) + f > 0\] (51)
for all $t \in (t^*, t^* + \epsilon^*)$ with the choice of $\ell = 0$. Therefore, $x$ is a forward Carathéodory solution.
The case
\[c^T A^p_1 x(t^*) + c^T A^p_1 e_1 < 0\] (52)
can be proven by using the above arguments in a similar fashion.
5.1.4. Proof of Theorem 3.1.D

As we have just shown, statements 5 and 6 imply that every Filippov solution is a forward Carathéodory solution. Then, it is enough to show that every forward Carathéodory solution is right-unique.

Let \( x_1, x_2 \) be two forward Carathéodory solutions with \( x_1(0) = x_2(0) = \xi \). Then, there exists \( \epsilon > 0 \) such that

\[
\dot{x}_1(t) = A_i x(t) + e_i, \quad (-1)^i [c^T x(t) + f] > 0 \tag{53a}
\]

\[
\dot{x}_2(t) = A_j x(t) + e_j, \quad (-1)^j [c^T x(t) + f] > 0 \tag{53b}
\]

for all \( t \in [0, \epsilon) \). In case \( i = j \), we have readily \( x_1(t) = x_2(t) \) for all \( t \in [0, \epsilon) \). In case \( i \neq j \), we can assume, without loss generality, that \( i = 1 \) and \( j = 2 \). Then, it follows from (53) that \( \xi \in W_1 \) and \( \xi \in W_2 \). In other words, we have

\[
(-1)^i \begin{bmatrix}
  c^T A_i \\
  \vdots \\
  c^T A_i^{h+1}
\end{bmatrix} \xi + \begin{bmatrix}
  f \\
  e_i \\
  e_i
\end{bmatrix} \geq 0 \tag{54}
\]

for \( i \in \{1, 2\} \). Then, it follows from statement 5 that

\[
\begin{bmatrix}
  c^T A_i \\
  \vdots \\
  c^T A_i^{h+1}
\end{bmatrix} \xi + \begin{bmatrix}
  f \\
  e_i \\
  e_i
\end{bmatrix} = 0 \tag{55}
\]

for \( i \in \{1, 2\} \) and hence \( \xi \in W^0 = W_1^0 = W_2^0 \). As proven in 5.1.3, this means that \( x_1(t) = x_2(t) = x^*(t) \) for all \( t \in \mathbb{R} \) where \( x^* = A_1 x_e + e_1 \) with \( x^*(0) = \xi \).

5.2. Proof of Theorem 3.4

Since \( M \) is a lower triangular matrix with positive diagonal elements, it can be partitioned as

\[
M = \begin{bmatrix}
  M_{11} & 0 & m_{13} \\
  \ast & M_{22} & 0 \\
  \ast & \ast & m_{33}
\end{bmatrix}
\]

where \( m_{ii} > 0 \) for \( i = 1, \ldots, 3 \).

It follows from (10) that either

\[
\begin{bmatrix}
  c^T A_1 \\
  e^T A_1
\end{bmatrix} = M_1 \begin{bmatrix}
  c^T A_2 \\
  e^T A_2
\end{bmatrix}, \quad \begin{bmatrix}
  f \\
  e_1
\end{bmatrix} \geq M_1 \begin{bmatrix}
  f \\
  e_2
\end{bmatrix} \tag{56}
\]

or

\[
\begin{bmatrix}
  c^T A_1 \\
  e^T A_1
\end{bmatrix} = M_1 \begin{bmatrix}
  c^T A_2 \\
  e^T A_2
\end{bmatrix}, \quad \begin{bmatrix}
  e^T A_1 \\
  e^T A_1, e_1
\end{bmatrix} = M_1 \begin{bmatrix}
  e^T A_2 \\
  e^T A_2, e_2
\end{bmatrix} \tag{57}
\]

holds. The claim follows readily for the former case due to Theorem 3.3. Therefore, it remains to prove the claim when (57) holds.

Let \( \xi \in \mathbb{R}^3 \). Due to (10), (57) and the definition of \( \Omega \), only the following cases are possible:

1. \( c^T \xi + f \neq 0 \).
2. \( c^T \xi + f = 0, (c^T A_1 \xi + c^T e_1)(c^T A_2 \xi + c^T e_2) > 0 \).
3. \( c^T \xi + f = c^T A_1 \xi + c^T e_1 = c^T A_2 \xi + c^T e_2 = 0 \) and

\[
(c^T A_1^2 \xi + c^T A_1 e_1)(c^T A_2^2 \xi + c^T A_2 e_2) > 0.
\]
4. $c^T \xi + f = c^T A_1 \xi + c^T e_1 = c^T A_2 \xi + c^T e_2 = 0$ and 
\[ c^T A_1^2 \xi + c^T A_1 e_1 > 0, c^T A_2^2 \xi + c^T A_2 e_2 < 0. \] (58)

By similar arguments to those in the proof of Theorem 3.1.1, one can conclude the right-uniqueness of Filippov solutions for the initial state $\xi$ for the first three above-mentioned cases.

For the last case, we claim that if $x$ is a Filippov solution with $x(0) = \xi$ then there exists $\epsilon > 0$ such that 
\[ c^T x(t) + f = 0 \] (59a) 
\[ c^T A_1 x(t) + c^T e_1 = c^T A_2 x(t) + c^T e_2 = 0 \] (59b) 
for all $t \in [0, \epsilon)$. Note that the conditions [59] would imply that $x$ satisfies the differential inclusion 
\[ \dot{x} \in \begin{cases} 
\{ A_1 x + e_1 \} & \text{if } \dot{c}^T x + \tilde{f} < 0 \\
\text{conv}\{ A_1 x + e_1, A_2 x + e_2 \} & \text{if } \dot{c}^T x + \tilde{f} = 0 \\
\{ A_2 x + e_2 \} & \text{if } \dot{c}^T x + \tilde{f} > 0 
\end{cases} \] (60)
on the interval $[0, \epsilon)$ where $\tilde{c}^T = c^T A_1$ and $\tilde{f} = c^T e_1$. As such, right-uniqueness of $x$ would follow from [58] and [2] Theorem 2.10.2 since $\dot{c}^T x + f = 0$ and $c^T A_1 \xi + c^T e_1 > 0$.

Therefore, it is enough to prove the existence of $\epsilon > 0$ such that the conditions [59] are satisfied. To do so, we first note that one can find positive numbers $\delta$ and $\epsilon$ such that 
\[ c^T A_1^2 x(t) + c^T A_1 e_1 > \delta, \quad c^T A_2^2 x(t) + c^T A_2 e_2 < -\delta \] (61) 
for all $t \in [0, \epsilon)$ due to [58], $x(0) = \xi$ and the continuity of $x$. To show that the conditions [59] are satisfied with this choice of $\epsilon > 0$, consider the functions $V_1, V_2 : \mathbb{R}^n \to \mathbb{R}$ defined by 
\[ V_1(z) = \begin{cases} 
(c^T z + f)(c^T A_1^2 z + c^T A_1 e_1) & \text{if } c^T z + f < 0 \\
(c^T z + f)(c^T A_2^2 z + c^T A_2 e_2) & \text{if } c^T z + f \geq 0, 
\end{cases} \]
\[ V_2(z) = \begin{cases} 
c^T A_1 z + c^T e_1 & \text{if } c^T z + f \leq 0 \\
\lambda_2(c^T A_2 z + c^T e_2) & \text{if } c^T z + f > 0.
\end{cases} \]

Also consider $V(t) := \frac{1}{2} V_2^2(x(t)) - V_1(x(t))$. From [61], we get 
\[ V(t) \geq \delta |c^T x(t) + f| + \frac{1}{2} V_2^2(x(t)) \geq 0 \] (62) 
for all $t \in [0, \epsilon)$. Note that the relation 
\[ \dot{V}(t) \leq \lambda V(t) \] (63) 
for almost all $t \in [0, \epsilon)$ and for some $\lambda > 0$ would imply $V(t) = 0$ for all $t \in [0, \epsilon)$ since $V(0) = 0$. Together with [62] and the definition of $V_2$, this would imply that the conditions [59] are satisfied. As such, it is enough to show [63] in order to complete the proof.

Define $r := \max \{ ||x(t)|| \mid t \in [0, \epsilon) \}$ and $B(0, r) := \{ x \mid ||x|| \leq r \}$. Note that $V_1$ is readily continuous and also that $V_2$ is continuous due to [67]. Furthermore, both $V_1$ and $V_2$ are Lipschitz on $B(0, r)$. Together with the absolute continuity of $x$, this implies that $V_1(x(\cdot))$ and $V_2(x(\cdot))$ are even absolutely continuous on $[0, \epsilon]$.

Let $\Lambda$ be the set of all $t \in [0, \epsilon)$ for which $x(\cdot), V_1(x(\cdot)), V_2(x(\cdot))$ are differentiable. Due to the absolute continuity of these functions, the set $[0, \epsilon] \setminus \Lambda$ is of measure zero. Thus, it suffices to prove that [63] holds for all $t \in \Lambda$.

To do so, let $t^* \in \Lambda$. First, suppose that $c^T x(t^*) + f \neq 0$. In this case, we get 
\[ \dot{V}(t^*) \leq L |c^T x(t^*) + f| \leq \frac{L}{2} V(t^*) \] (64)
where \( L := \max \{|c^TA_i^3x(t) + c^TA_i^2e_i|: t \in [0, \varepsilon], i = 1, 2\} \). In other words, (63) holds for \( t^* \).

Now, suppose that \( c^T x(t^*) + f = 0 \). The Taylor expansion of \( x \) around \( t^* \) yields
\[
c^T x(t^* + \tau) + f = \tau c^T \dot{x}(t^*) + c^T o(\tau)
\]
for all \( \tau \) sufficiently close to \( 0 \) where \( \lim_{\tau \to 0} o(\tau)/\tau = 0 \). It follows from (65) that the set of the one-sided derivatives of \( V_1(x(\cdot)) \) at \( t^* \) is equal to
\[
\{c^T \dot{x}(t^*)(c^TA_i^2x(t^*) + c^TA_i e_i) | i = 1, 2\}.
\]
Since \( V_1(x(\cdot)) \) is differentiable at \( t^* \), this set must be a singleton. In view of (61), this can happen only if \( c^T \dot{x}(t^*) = 0 \). This means that \( V_1(x(t^*)) = 0 \). Since \( x \) is a Filippov solution, we get
\[
0 \in \text{conv}\{c^TA_1x(t^*) + c^Te_1, c^TA_2x(t^*) + c^Te_2\}.
\]
By post-multiplying the first equation in (57) by \( x(t^*) \) and adding to the second, we obtain
\[
c^T A_1x(t^*) + c^T e_1 = m_{22}(c^T A_2x(t^*) + c^T e_2).
\]
From (67) and (68), we get
\[
c^T A_1x(t^*) + c^T e_1 = c^T A_2x(t^*) + c^T e_2 = 0.
\]

From the definition of \( V_2 \), this yields \( V_2(x(t^*)) = 0 \) and further
\[
\frac{d}{dt}(V_2^2(x(t^*))) = 0, \quad \dot{V}(t^*) = \frac{d}{dt}(V_2^2(x(t^*))) - \frac{d}{dt}V_1(x(t^*)) = 0.
\]
Clearly, (63) holds for \( t^* \). \( \blacksquare \)

6. Conclusions

We studied existence, uniqueness and nature of solutions Carathéodory and Filippov solutions for bimodal (possibly discontinuous) piecewise affine systems in a differential inclusion setting. First, we showed that the typical conditions that are employed in the context of general differential inclusions in order to guarantee uniqueness of Filippov are quite restrictive in the context of piecewise affine systems. Then, we presented a set of necessary and a set of sufficient conditions that ensure uniqueness of Filippov solutions for bimodal piecewise affine systems. By investigating the relationships between Carathéodory and Filippov under the presented condition, we provide conditions that rule out the so-called Zeno behavior. Possible extensions of the main results of this paper to general piecewise affine dynamical systems with external inputs emerge as future research directions.

References

[1] A. Bacciotti. On several notions of generalized solutions for discontinuous differential equations and their relationships. Internal Report 19, Dipartimento di Matematica, Politecnico di Torino, Trieste, 2003.

[2] A.F. Filippov. Differential Equations With Discontinuous Right Hand Sides. Mathematics and its applications, Prentice-Hall, Dordrecht, The Netherlands, 1988.

[3] A.J. van der Schaft and J.M. Schumacher. The complementarity-slackness class of hybrid systems. Mathematics of Control, Signals, and Systems, 9:266–301, 1996.

[4] A.Y. Pogromsky, W.P.M.H. Heemels, and H. Nijmeijer. On solution concepts and well-posedness of linear relay systems. Automatica, 39:2139–2147, 2003.

[5] M.K. Camlibel. Well-posed bimodal piecewise linear systems do not exhibit Zeno behavior. In Proceedings of 17th IFAC World Congress on Automatic Control, Seoul, Korea, 2008.

[6] J. Cortes. Discontinuous Dynamical Systems: A tutorial on solutions, nonsmooth analysis, and stability. IEEE Control Systems Magazine, 28(3):36–73, 2008.
[7] F. Facchinei and J-S. Pang. *Finite dimensional variational inequalities and complementarity problems*, volume 1. Springer, 2002.

[8] J. Imura and A.J. van der Schaft. Characterization of well-posedness of piecewise linear systems. *IEEE Transactions on Automatic Control*, 45(9):1600–1619, 2000.

[9] J. Shen and J-S. Pang. Linear complementarity systems: Zeno states. *SIAM Journal on Control and Optimization*, 44:1040–1066, 2005.

[10] J. Zhang, K.H. Johansson, J. Lygeros, and S.S. Sastry. Zeno hybrid systems. *International Journal of Robust and Nonlinear Control*, 11:435–451, 2001.

[11] K.H. Johansson, M. Egersted, J. Lygeros, and S.S. Sastry. On the regularization of Zeno hybrid automata. *Systems and Control Letters*, 38:141–150, 1999.

[12] J.P. LaSalle. Time optimal control systems. *Proceedings of the National Academy of Sciences of the United States of America*, 45(4):573–577, 1959.

[13] L.Q. Thuan and M.K. Camlibel. On well-posedness of piecewise affine bimodal dynamical systems. In *Proceedings of the 19th International Symposium on Mathematical Theory of Networks and Systems*, pages 503–510, Budapest, Hungary, 2010.

[14] L.Q. Thuan and M.K. Camlibel. Continuous piecewise affine dynamical systems do not exhibit zeno behavior. *IEEE Transactions on Automatic Control*, 56(8):1932–1936, 2011.

[15] M.K. Camlibel, J-S. Pang, and J. Shen. Conewise linear systems: non-Zenoness and observability. *SIAM Journal on Control and Optimization*, 45(5):1769–1800, 2006.

[16] H. Nijmeijer and A.J. van der Schaft. *Nonlinear Dynamical Control Systems*. Springer, 2010.

[17] J.M. Schumacher. Time-scaling symmetry and Zeno solutions. *Automatica*, 45(5):1237–1242, 2009.

[18] G.V. Smirnov. *Introduction to the Theory of Differential Inclusions*. American Mathematical Society, 2002.

[19] V.I. Utkin. *Sliding Modes in Control Optimization*. Springer, Berlin, 1992.

[20] Y.J. Lootsma, A.J. van der Schaft, and M.K. Camlibel. Uniqueness of solutions of relay systems. *Automatica*, 35(3):467–478, 1999.