Mixed states for mixing neutrinos

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(Dated: April 10, 2018)

Here we discuss the description of flavor neutrinos produced or detected in processes which involve more than one neutrino. We show that in these cases flavor neutrinos cannot be separately described by pure states, but require a density matrix description. We consider explicitly the examples of $\nu_e$ and $\bar{\nu}_\mu$ production in $\mu^+$ decay and $\nu_\mu$ detection through scattering on electrons. We show that the density matrix which describes a flavor neutrino can be approximated with a density matrix of a pure state only when the differences of the neutrino masses are neglected in the interaction process. In this approximation, the pure states are the standard flavor states and one recovers the standard expression for the neutrino oscillation probability. We discuss also the more complicated case of neutrino-electron elastic scattering, in which the initial and final neutrinos do not have determined flavors, but there is a flavor dependence due to the different contributions of charged-current and neutral-current interactions.

I. INTRODUCTION

In the last years neutrino physics has proven to be a fertile ground for particle theory. The discovery \cite{1,2} of neutrino oscillations \cite{3,4,5} has recently been awarded the Nobel prize, recognizing its importance. Neutrino oscillations imply that neutrinos must have non-zero masses and that there is neutrino mixing. The standard theory of neutrino mixing and oscillations is well known (see, for example, Refs. \cite{6,7}), but there are subtle issues that require a special treatment (see, for example, the recent discussions in Refs. \cite{9,11}).

Neutrino oscillations are transitions among different neutrino flavors that can be observed at macroscopic distances from a neutrino source. Different neutrino flavors ($\nu_e$, $\nu_\mu$, $\nu_\tau$) are characterized by their production or detection in association with the corresponding charged lepton ($e$, $\mu$, $\tau$). In the standard treatment of neutrino oscillations, flavor neutrinos are described by states which are unitary superpositions of massive neutrino states and the mixing matrix is the unitary matrix that diagonalizes the mass matrix of the neutrino fields (see Refs. \cite{6,7}). However, in the description of neutrinos as excitations of quantum fields, the relation between mass and flavor states is not as simple, due to the non-existence of a canonical set of creation and annihilation operators for the flavor fields \cite{12}. This fact implies that the neutrino flavor states are phenomenological quantities that describe neutrinos created or detected in a weak interaction process as superpositions of massive neutrinos with coefficients determined by the respective interaction amplitudes \cite{12,15}. The standard neutrino flavor states are recovered in the realistic approximation of neglecting the neutrino mass differences in the interaction process.

The localization of the production and detection processes in a neutrino oscillation experiment and the associated energy-momentum uncertainties require a wave-packet description \cite{16} (see Refs. \cite{6,17,18}). It has been shown in Ref. \cite{14} that also in this case flavor neutrinos are described by states that are determined by the interaction process. However, in this paper we will avoid the complications of the wave packet description by considering the plane-wave approximation in which flavor neutrinos are described by superpositions of massive neutrino states with definite energy and momentum.

In this paper we discuss the description of flavor neutrinos produced or detected in weak interaction processes in which multiple neutrinos are involved. We will show that the different flavor neutrinos cannot be described by pure states, but require a density matrix description (note that there are other situations involving neutrinos which also require a density matrix description, such as when dealing with unpolarized beams; see, e.g., Refs. \cite{19,20}). We will derive the appropriate density matrix and will show that, under the appropriate approximations the density matrix description leads to the standard oscillation probability.

We discuss also the more complicated case of neutrino-electron elastic scattering, in which the flavors of the initial and final neutrinos are not determined, but there is a flavor dependence, because the $\nu_e$ component interacts through both charged and neutral currents whereas the $\nu_\mu$ and $\nu_\tau$ components interact only through neutral currents.

This paper is organized as follows. In Sec. \cite{11} we briefly review the derivation of the flavor states for processes involving only one flavor neutrino. In Sec. \cite{11} we present the density matrix description of the $\nu_e$ and $\nu_\mu$ produced in $\mu^+$ decay as an example of a production process in-
II. FLAVOR STATES FOR ONE-NEUTRINO PROCESSES

Flavor neutrinos states are commonly described as superpositions of neutrino states with a well-defined mass which amounts to a change of orthonormal basis using the unitary mixing matrix that diagonalizes the mass matrix of the neutrino fields. In quantum field theory, however, flavor neutrinos cannot be fundamentally described as excitations of flavor fields (due to the lack of a natural Fock space) and are basically a phenomenological concept. In this Section we give a short review of the derivation of the flavor states that describe flavor neutrinos produced or detected in weak interaction processes [6] [12] [15].

Let us consider the decay

\[ P_I \rightarrow P_F + \bar{l}_\alpha + \nu_\alpha, \]

where \( P_I \) and \( P_F \) stand for the initial and final particles besides the produced anti-lepton \( \bar{l}_\alpha \) (with \( \alpha \in \{e, \mu, \tau\} \)) and its associated neutrino \( \nu_\alpha \), here understood as the superposition of states of massive neutrinos \( \nu_i \) (with \( i \in \{1, 2, 3\} \)) which we derive explicitly below.

The final state of such decay is given by the action of the \( S \) matrix over the initial state \( |i\rangle = |P_I\rangle \), i.e.,

\[ |f\rangle \propto (S - I)|i\rangle \]

\[ = \sum_j A^P_{\alpha,j}|\nu_j, l^+_\alpha, P_F\rangle + \ldots, \]

(2)

where we disregard the possibility of no decay and \( \ldots \) denotes all other possible channels which do not concern us here. Since the other decays contained in \( \ldots \) are orthogonal to \( |\nu_j, l^+_\alpha, P_F\rangle \) and these states are orthonormal, the coefficients \( A^P_{\alpha,j} \) (where \( P \) stands for [neutrino production]) are given by

\[ A^P_{\alpha,j} = \langle \nu_j, l^+_\alpha, P_F|S|P_I\rangle. \]

(3)

The state describing the emitted flavor neutrino \( \nu_\alpha \) is obtained by projecting the state \( |f\rangle \) over \( |l^+_\alpha, P_F\rangle \):

\[ |\nu_\alpha\rangle \propto \langle l^+_\alpha, P_F|f\rangle. \]

(4)

The resulting normalized flavor neutrino state is

\[ |\nu_\alpha\rangle = \left( \sum_k |A^P_{\alpha,k}|^2 \right)^{-1/2} \sum_j A^P_{\alpha,j}|\nu_j\rangle. \]

(5)

This state describes the neutrino produced in the decay \([\mu] \) together with the charged antilepton of the same flavor. It is different from the standard flavor state,

\[ |\nu_\alpha\rangle_{\text{std}} = \sum_i U^*_{\alpha i}|\nu_i\rangle, \]

(6)

because the coefficients that determine the superposition of the massive neutrinos are the matrix elements \([3]\) of the neutrino production process \([1]\). In order to show how they are related, we expand the \( S \) matrix up to first order in the Fermi coupling constant \( G_F \) as

\[ S \approx 1 - i \frac{G_F}{\sqrt{2}} \int d^4x J^\dagger_{\nu CC}(x) \tilde{J}^{\nu CC}(x), \]

(7)

with the weak charged current \( J^{\nu CC}(x) \) given by

\[ \tilde{J}^{\nu CC}(x) = \sum_{\alpha,k} U^*_{\alpha k} \hat{\nu}_k(x) \gamma^0 (1 - \gamma^5) \hat{l}_\alpha(x) + \hat{h}^{\nu CC}_\alpha(x), \]

(8)

where \( \hat{h}^{\nu CC}_\alpha(x) \) is the hadronic part of the weak charged current. Now, using Eq. \([3]\) we can write

\[ A^P_{\alpha,j} = U^*_{\alpha j}M^P_j, \]

(9)

where \( M^P_j \) is given by

\[ M^P_j = -i \frac{G_F}{\sqrt{2}} \int d^4x (\nu_j l^+_\alpha |\tilde{\nu}_\alpha\rangle \gamma^0 (1 - \gamma^5) \hat{l}_\alpha(x)|0\rangle \]

\[ \times J^\nu_{P_I \rightarrow P_F}(x), \]

(10)

with the hadronic transition amplitude

\[ J^\nu_{P_I \rightarrow P_F}(x) = \langle P_F|\hat{h}_\rho(x)|P_I\rangle. \]

(11)

This decomposition allows us to expand Eq. \([5]\) as

\[ |\nu_\alpha\rangle = \left( \sum_k |U_{\alpha k}|^2 |M^P_k|^2 \right)^{-1/2} \sum_j U^*_{\alpha j}M^P_j|\nu_j\rangle. \]

(12)

If all the massive neutrinos can be considered massless or with almost degenerate masses, we have \( M^P_j \approx M^P \), \forall j, \]

and, using the unitarity of the mixing matrix, we obtain the standard states given in Eq. \([6]\).

The above construction is well-suited, for instance, for the description of \( \beta^+ \) and \( \beta^- \) decays or \( \nu_\mu \) produced in \( \pi^+ \) decays (and, with obvious modifications, \( \bar{\nu}_e \)'s produced in \( \beta^- \) decays or \( \bar{\nu}_\mu \) produced in \( \pi^- \) decays). In these cases we have only one flavor neutrino which is described by a pure state.

The flavor neutrino states allow us to give a complete picture of neutrino production, oscillations and detection in the case of processes involving a single neutrino [13]. Particularly notable is that they give the correct production rate at the neutrino source as an incoherent sum of massive neutrino states [22] [20].

In the next section we discuss how the description of flavor neutrinos must be modified in the case of reactions involving more than one neutrino.

III. MULTIPLE NEUTRINO PRODUCTION

Let us consider as an example the neutrino creation process

\[ \mu^+ \rightarrow e^+ + \nu_e + \bar{\nu}_\mu. \]

(13)
How to describe the emitted electron neutrino and muon antineutrino? One can see that the method reviewed in Sec. 4 must be extended by considering, for example, the description of the $\nu_e$. The derivation of its state would require the projection $\langle e^+, \bar{\nu}_\mu | f \rangle$ analogous to Eq. (4), that in this case is not possible, because the state $| \bar{\nu}_\mu \rangle$ is still not defined.

In order to derive the correct description of the emitted $\nu_e$ and $\bar{\nu}_\mu$ let us first consider the final state of the process (13).

$$| f \rangle \propto (S - 1) | \mu^+ \rangle = \sum_{k,j} A_{\mu, e; k, j}^P | e^+, \nu_k, \bar{\nu}_j \rangle + \ldots, \quad (14)$$

where “...” denotes other possible decay channels irrelevant for our purposes (e.g., $\mu^+ \rightarrow e^+ + \gamma$). Using the orthonormality of the states, the coefficients $A_{\mu, e; k, j}^P$ are given by

$$A_{\mu, e; k, j}^P = \langle e^+, \nu_k, \bar{\nu}_j | S | \mu^+ \rangle = U_{ek}^* U_{k\mu} M_{k, j}^P, \quad (15)$$

where

$$M_{k, j}^P = -i \frac{G_F}{\sqrt{2}} \int d^4 x \langle e^+, \nu_k, \bar{\nu}_j | \gamma^\rho (1 - \gamma^5) \bar{e}(x) \times \bar{\nu}(x) | \mu^+ \rangle. \quad (16)$$

The final state in Eq. (14) is an entangled state in which it is not possible to separate the neutrino and antineutrino components. Therefore, the neutrino and the antineutrino cannot be described by a pure state and should each separately be described by a density matrix, which is the most general description of a quantum system which may be a subsystem of a larger closed system.

The density matrix operator that describes the complete final state in Eq. (14) is

$$\hat{\rho} = | f \rangle \langle f | = N \sum_{k, j, k', j'} A_{\mu, e; k, j}^P A_{\mu, e; k', j'}^P \langle e^+, \nu_k, \bar{\nu}_j | e^+, \nu_{k'}, \bar{\nu}_{j'} \rangle, \quad (17)$$

where $N$ is a normalization coefficient determined by the condition $\text{Tr}(\hat{\rho}) = 1$, i.e.,

$$\sum_{k', j'} \langle e^+, \nu_{k'}, \bar{\nu}_{j'} | \hat{\rho} | e^+, \nu_{k'}, \bar{\nu}_{j'} \rangle = 1. \quad (18)$$

This gives

$$N = \left( \sum_{k, j} | A_{\mu, e; k, j}^P |^2 \right)^{-1} = \left( \sum_{k, j} | U_{ek}^* |^2 | U_{k\mu} |^2 | M_{k, j}^P |^2 \right)^{-1}. \quad (19)$$

The $\nu_e$ and $\nu_{\mu}$ are separately described by the partial traces over the other degrees of freedom of the complete system:

$$\hat{\rho}_{\nu_e} = \sum_j \langle e^+, \bar{\nu}_j | \hat{\rho} | e^+, \bar{\nu}_j \rangle = N \sum_{k, k', j} A_{\mu, e; k, j}^P A_{\mu, e; k', j}^P \langle \nu_k | \langle \nu_{k'} |, \quad (20)$$

$$\hat{\rho}_{\bar{\nu}_\mu} = \sum_k \langle e^+, \nu_k | \hat{\rho} | e^+, \nu_k \rangle = N \sum_{k, j, j'} A_{\mu, e; k, j}^P A_{\mu, e; k, j'}^P \langle \bar{\nu}_j | \langle \bar{\nu}_{j'} |, \quad (21)$$

Using Eq. (15), we obtain

$$\hat{\rho}_{\nu_e} = N \sum_j | U_{\mu j} |^2 \sum_{k, k'} U_{ek}^* U_{k\mu} M_{k, j}^P M_{k', j}^P \langle \nu_k | \langle \nu_{k'} |, \quad (22)$$

$$\hat{\rho}_{\bar{\nu}_\mu} = N \sum_k | U_{\mu j} |^2 \sum_{j, j'} U_{\nu j}^* U_{j\mu} M_{k, j}^P M_{k, j'}^P \langle \bar{\nu}_j | \langle \bar{\nu}_{j'} |. \quad (23)$$

These density matrices describe separately the $\nu_e$ and $\bar{\nu}_\mu$ and one can see that, because of the dependence of the interaction matrix elements on $j$ in Eq. (22) and on $k$ in Eq. (23), they are not density matrices of pure states (as one can also verify by checking that $\text{Tr}(\hat{\rho}_{\nu_e}^2) < 1$ and $\text{Tr}(\hat{\rho}_{\bar{\nu}_\mu}^2) < 1$). Hence, the $\nu_e$ and $\bar{\nu}_\mu$ cannot be described by pure states. It is interesting to note that while the complete density matrix in Eq. (17) allows us to correctly calculate the decay rate for the process (13) as an incoherent sum over massive neutrino states, the density matrices in Eqs. (22) and (23) do not. This can be understood as a consequence of information loss due to taking the partial trace of the complete density matrix.

The $\nu_e$ and $\bar{\nu}_\mu$ can be approximately described by pure states in experiments which are not sensitive to the dependence of $M_{k, j}^P$ on the neutrino masses, where it is possible to approximate $M_{k, j}^P \approx M^P$, $\forall j, k$. Taking into account that in this case $N^{-1} \approx |M|^2$, we obtain

$$\hat{\rho}_{\nu_e} \approx \left( \sum_k U_{ek}^* | \nu_k \rangle \langle \nu_k | \right), \quad (24)$$

$$\hat{\rho}_{\bar{\nu}_\mu} \approx \left( \sum_j U_{\nu j}^* | \bar{\nu}_j \rangle \langle \bar{\nu}_j | \right), \quad (25)$$

which are the density matrices associated with the standard flavor states given in Eq. (6). Therefore, in this approximation we recover the standard description of the neutrino flavor states.

Now consider, for example, the electron neutrino, described by $\hat{\rho}_{\nu_e}$, as the initial state of an oscillation experiment. It propagates freely and then it can be detected with a different flavor, for instance via the process

$$\nu_\mu + D_I \rightarrow \mu^- + D_F, \quad (26)$$
where $D_I$ and $D_F$ are the initial and final states of the other particles involved in the detection. We first apply a spatio-temporal translation, $U(t = T, \vec{x} = \vec{L}) = \exp \left(-i \vec{p}^0 T + i \vec{p} \cdot \vec{L} \right)$ to the density matrix $\hat{\rho}_{\nu_e}$, which gives

$$\begin{align*}
\hat{\rho}_{\nu_e}(T, \vec{L}) &= U(T, \vec{L}) \hat{\rho}_{\nu_e} U^\dagger(T, \vec{L}) \\
&= N \sum \left[U_{\mu j}\right]^2 \sum_{k,k'} U_{\nu e k} U_{\nu e k'}^\dagger M_{k,j}^P M_{k',j}^{P*} \\
&\quad \times \exp \left[-i (E_k - E_{k'}) T \right] + i (\vec{p}_k - \vec{p}_{k'}) \cdot \vec{L} \right] |\nu_k\rangle \langle \nu_{k'}|.
\end{align*}$$

(27)

In the relativistic approximation, where $T \simeq |\vec{L}|$, considering for simplicity all the massive neutrinos propagating in the direction of $\vec{L}$ (see the discussion in Section 8.1.3 of Ref. [6]) and using Eq. (5.7) of Ref. [15], we obtain

$$\begin{align*}
\hat{\rho}_{\nu_e}(T, \vec{L}) &= U(T, \vec{L}) \hat{\rho}_{\nu_e} U^\dagger(T, \vec{L}) \\
&= N \sum \left[U_{\mu j}\right]^2 \sum_{k,k'} U_{\nu e k} U_{\nu e k'}^\dagger M_{k,j}^P M_{k',j}^{P*} \\
&\quad \times \exp \left(-\frac{i \Delta m^2_{kk'} |\vec{L}|}{2E} \right) |\nu_k\rangle \langle \nu_{k'}|,
\end{align*}$$

(28)

where $\Delta m^2_{kk'} = m_k^2 - m_{k'}^2$, and $E$ is the neutrino energy in the massless approximation.

IV. DETECTION PROCESSES

It is interesting to study detection processes for flavor neutrinos where there is more than one neutrino involved with the approach described above for production processes. There are subtle differences which we discuss in this Section.

Let us consider as an example the “inverse muon decay” neutrino detection process

$$\nu_\mu + e^- \rightarrow \mu^- + \nu_e.$$  

(32)

Although this process can be used to detect muon neutrinos $[27,29]$ it is not used in practice for neutrino oscillation experiments, because the neutrino energy threshold is high (about 10.92 GeV) and the cross section is about one thousand times smaller than that of $\nu_\mu$ charged-current scattering on neutrons. However, at least in principle one can ask which is the correct description of the detected $\nu_\mu$, taking into account that the $\nu_e$ in the final state is a superposition of massive neutrinos which is not known \textit{a priori}.

Since the final neutrino is a superposition of orthogonal massive neutrinos, the cross section of the process (32) is the incoherent sum of the cross sections with the different massive neutrinos in the final state:

$$\sigma(\nu_\mu + e^- \rightarrow \mu^- + \nu_e) = \sum_j \sigma(\nu_\mu + e^- \rightarrow \mu^- + \nu_j).$$  

(33)

Therefore, the detected $\nu_j$ must be described by a density matrix, which allows us to describe the incoherent sum in Eq. (33). We start by considering the separate processes

$$\nu_\mu + e^- \rightarrow \mu^- + \nu_j.$$  

(34)

The corresponding initial states are given by

$$|i_{\nu_j} \rangle \propto (S^I - I) |\mu^-, \nu_j\rangle = \sum_k A_{\mu e; k, j}^D |\nu_k, e^-\rangle + \ldots ,$$  

(35)

with

$$A_{\mu e; k, j}^D = \langle \nu_k, e^- |S^I |\mu^-, \nu_j\rangle = U^*_{\mu k} U_{e j} M^D_{k,j},$$  

(36)

where

$$M^D_{k,j} = \frac{G_F}{\sqrt{2}} \int d^4x \langle \nu_k, e^- |\bar{\psi}(x) \gamma^\mu (1 - \gamma^5) |\mu^- \rangle \langle \mu^- | \nu_j\rangle.$$  

(37)

The density matrix operator that describes the initial state in the process (32) is then

$$\rho^D = \frac{1}{3} \sum_j |i_{\nu_j}\rangle \langle i_{\nu_j}|$$

$$= N^D \sum_{j, k, k'} A_{\mu e; k, j}^D A_{\mu e; k', j}^D |\nu_k, e^-\rangle \langle \nu_{k'}, e^-|,$$  

(38)
where $N^D$ is the normalization coefficient given by

$$N^D = \left( \sum_{k,j} |A^D_{\mu,e,k,j}|^2 \right)^{-1}. \quad (39)$$

The normalized density matrix that describes the detected $\nu_\mu$ is given by the trace over the initial electron state:

$$\rho^D_{\nu_\mu} = \langle e^- | \hat{\rho}^D | e^- \rangle = N^D \sum_{j,k,k'} A^D_{\mu,e,k,j} A^{D*}_{\mu,e,k',j} |\nu_k\rangle \langle \nu_{k'}|$$

$$= N^D \sum_j [U_{ej}]^2 \sum_{k,k'} U_{j\mu k} M^D_{k,j} M^{D*}_{k',j} |\nu_k\rangle \langle \nu_{k'}|. \quad (40)$$

If reaction (32) is used in a neutrino oscillation experiment with initial $\nu_e$'s produced by the decay of $\mu^+$ (for example, in a Neutrino Factory) and described by the density matrix $^{28}$, we can calculate an oscillation probability associated with the reaction by $^1$

$$P_{e-\mu} = \text{Tr} \left[ \hat{\rho}_{\nu_e} (T, \hat{L}) \hat{\rho}^D_{\nu_\mu} \right]. \quad (41)$$

We omit the lengthy explicit expression of the probability resulting from Eq. (41), that can be calculated straightforwardly, but note that it can be shown that $P_{e-\mu} \leq 1$. We also emphasize that in the usual approximations in which the differences of the neutrino masses are negligible in the production and detection processes we have $M^F \approx 0$ and $M^{D*} \approx M^D$, $\forall j,k$. In this approximation we recover the standard expression in Eq. (31) for the oscillation probability.

V. NEUTRINO-ELECTRON ELASTIC SCATTERING

Neutrinos can also be detected with the neutrino-electron elastic scattering (ES) process

$$\nu + e^- \rightarrow \nu + e^-. \quad (42)$$

This is a more complicated case, because it is not a pure charged-current interaction in which a leptonic flavor is selected. However, there is a flavor dependence, due to the fact that $\nu_\mu$'s and $\nu_e$'s interact only through neutral currents, whereas $\nu_\tau$'s interact through both charged and neutral currents. For example, in water Cherenkov solar neutrino experiments information on solar neutrino oscillations is obtained by observing the ES reaction (42) induced by solar neutrinos, taking into account that the cross section $\sigma^ES_{\nu_e}(E)$ of $\nu_e$'s is about six times larger than the cross section $\sigma^ES_{\nu_\mu}(E)$ of $\nu_\mu$'s and $\nu_\tau$'s. In these experiments, the rate of ES events in a detector is calculated as

$$R^{ES} = N_e \int dE \phi_{\nu_e}(E) \left[ P_{\nu_e \rightarrow \nu_e} \sigma^ES_{\nu_e}(E) + \sum_{\alpha=\nu_\tau,\mu} P_{\nu_\mu \rightarrow \nu_\alpha} \sigma^ES_{\nu_\alpha}(E) \right], \quad (43)$$

where $N_e$ is the number of electrons in the detector, $\phi_{\nu_e}(E)$ is the solar $\nu_e$ flux, and $P_{\nu_\mu \rightarrow \nu_\alpha}$ is the probability of $\nu_\mu \rightarrow \nu_\alpha$ oscillations from the center of the Sun to the detector. In the following we present a schematic calculation of $R^{ES}$ which takes into account the neutrino masses in the interaction process and we show that it reduces to the expression in Eq. (43) only when the difference of the neutrino masses is neglected in the interaction process.

It is possible in principle to define a density matrix which describes the neutrino detected in the ES process (42) following a method similar to that presented in Sec. IV, but such a density matrix is not useful to obtain the rate $R^{ES}$, where the oscillation probability and the cross section are not factorized. Therefore, we calculate directly $R^{ES}$ considering a neutrino with energy $E$ coming from the Sun, which is described by the state $|\nu_S(E)\rangle = \sum_k A_{\nu_e \rightarrow \nu_k}(E)|\nu_k(E)\rangle$, (44)

where $A_{\nu_e \rightarrow \nu_k}$ is the amplitude of $\nu_e \rightarrow \nu_k$ transitions from the center of the Sun to the detector. Since the final neutrino is a superposition of orthogonal massive neutrinos, the cross section of the process (42) is, similarly to that of process (32), the incoherent sum of the cross sections of the processes

$$\nu + e^- \rightarrow \nu_j + e^- \quad (45)$$

The rate of ES events in a detector is given by

$$R^{ES} = N_e \int dE \phi_{\nu_e}(E) \sigma_S(E), \quad (46)$$

with

$$\sigma_S(E) = \int dPS \sum_j |\langle \nu_j, e^- | (S - I) | \nu_S(E), e^- \rangle|^2, \quad (47)$$

where the integration over $dPS$ represents schematically the integration over the phase space. In this case, we must consider an expansion of the $S$ matrix which contains, in addition to the charged-current weak interactions already considered in Eq. (7), also neutral-current interactions:

$$S \approx 1 - i \frac{G_F}{\sqrt{2}} \int d^4x \left[ j^{\mu}_{\text{CC}}(x) \hat{J}^{\rho}_{\text{CC}}(x) + j^{\mu}_{\text{NC}}(x) \hat{J}^{\rho}_{\text{NC}}(x) \right], \quad (48)$$

$^1$ See the similar treatment in Ref. $^{29}$. Note that the set $\{\hat{\rho}^D_{\nu_\mu}, 1 - \hat{\rho}^D_{\nu_\mu}\}$ can be considered as a discrete unsharp positive operator-valued measure (POVM); see Ref. $^{30}$. 


with the relevant part of the weak neutral current being
\[
\tilde{j}_{NC}^\nu(x) = \frac{1}{2} \sum_k \tilde{\nu}_k(x) \gamma_\rho \left( 1 - \gamma^5 \right) \nu_k(x) \\
+ \tilde{\pi}(x) \gamma_\rho \left( g_\rho^V - \gamma^5 g_\rho^A \right) \bar{\nu}(x),
\]
(49)
where \( g_\rho^V = -1/2 + 2 \sin^2 \vartheta_W \) and \( g_\rho^A = -1/2 \), and \( \vartheta_W \) is the weak mixing angle. Then, \( \sigma_{NC}(E) \) is given by
\[
\sigma_{NC}(E) = \int dP \sum_j | \sum_k A_{\nu_e \rightarrow \nu_j} \nu_j \nu_k |^2 \times \left[ U_{e j}^* U_{c k} M_{NC} \right] |^2,
\]
with the charged-current (CC) and neutral-current (NC) matrix elements
\[
M_{CC}^{j,k}(E) = -i G^F \int d^4 |x| \tilde{\pi}(x) \gamma_\rho \left( 1 - \gamma^5 \right) \nu_j(x) \\
\times \tilde{\nu}_k(x) \gamma_\rho \left( 1 - \gamma^5 \right) \bar{\nu}_k(x) \nu_j(x) \\
M_{NC}^{j,k}(E) = -i G^F \int d^4 |x| \tilde{\pi}(x) \gamma_\rho \left( g_\rho^V - \gamma^5 g_\rho^A \right) \bar{\nu}_k(x) \\
\times \tilde{\nu}_j(x) \gamma_\rho \left( 1 - \gamma^5 \right) \bar{\nu}_j(x) \nu_k(x),
\]
(50)
After some algebra, we obtain
\[
\sigma = \int dP \left[ \sum_{k,j} A_{\nu_j \rightarrow \nu_j} A_{\nu_j \rightarrow \nu_j} \nu_j \nu_k |^2 \times \left[ U_{e j}^* U_{c k} M_{CC}^{j,k} |^2 + \sum_j P_{\nu_j \rightarrow \nu_j} |M_{NC}^{j,j}|^2 \\
+ 2 \operatorname{Re} \sum_{k,j} A_{\nu_j \rightarrow \nu_j} A_{\nu_j \rightarrow \nu_j} \nu_j \nu_k |^2 \right],
\]
(53)
where \( P_{\nu_j \rightarrow \nu_j} = |A_{\nu_j \rightarrow \nu_j}|^2 \) is the probability of \( \nu \rightarrow \nu_j \) transitions from the center of the Sun to the detector and we omitted for simplicity the energy dependence. Equation
\[
\sigma(\nu) \approx \int dP \left[ \sum_{\alpha=\mu,\tau} P_{\nu_\alpha \rightarrow \nu_j} |M_{CC}^{\alpha,j}(E) + M_{NC}^{j,j}(E)|^2 \\
+ \sum_{\alpha=\mu,\tau} P_{\nu_\alpha \rightarrow \nu_j} |M_{NC}^{j,j}(E)|^2 \right],
\]
(54)
where we took into account that \( A_{\nu_j \rightarrow \nu_j} = \sum_k A_{\nu_j \rightarrow \nu_j} U_{e k} \) and the unitarity relations \( \sum_k P_{\nu_j \rightarrow \nu_j} = \sum_k P_{\nu_j \rightarrow \nu_k} = 1 \). When the squared moduli of the interaction matrix elements are integrated
over the phase space, they give the cross sections in Eq. (43):
\[
\sigma_{NC}(E) = \sum_{\alpha=\mu,\tau} P_{\nu_\alpha \rightarrow \nu_j} \sigma_{ES}^{\alpha,j}(E) + \sum_{\alpha=\mu,\tau} P_{\nu_\alpha \rightarrow \nu_j} \sigma_{ES}^{\alpha,\nu_j}(E).
\]
(55)
Hence, the expression (43) used to calculate the rate of ES events in water Cherenkov solar neutrino experiments is correct under the usual approximation of neglecting the differences of the neutrino masses in the interaction.

Note that in Eq. (44) we described the neutrinos coming from the Sun as coherent superpositions of massive neutrinos. If solar neutrinos arrive at the Earth as incoherent sums of the mass eigenstates because of the separation of the corresponding wave packets, the rate of ES events is obtained by summing incoherently the different massive neutrino contributions. This is equivalent to neglect the interference terms in Eq. (53), but Eq. (54) is obtained anyhow in the usual approximation of neglecting the differences of the neutrino masses in the interaction.

VI. CONCLUSION

Neutrino oscillations is one of the most interesting phenomena in modern fundamental physics. It was proposed about 60 years ago [1–3], and it has been observed about 20 years ago [10–12]. Its standard theory is well known (see, for example, Refs. [9–11]), but it is also well known that it is an approximation and several subtle issues have been discussed in the literature (see, for example, the recent discussions in Refs. [9–11]). A particular subtle problem is the description of flavor neutrinos [11–16].

In this paper we discussed how to describe flavor neutrinos produced or detected in processes which involve more than one neutrino. We have shown that in these cases flavor neutrinos cannot be described by pure states, but require a density matrix description. The density matrices can be approximated with density matrices of pure states only when the differences of the neutrino masses are neglected in the interaction process. In this approximation, the pure states are the standard flavor states and one recovers the standard expression for the neutrino oscillation probability.

We also discussed the more complicated case of neutrino-electron elastic scattering, in which the flavors of the initial and a final neutrino are not determined, but there is a flavor dependence due to different contributions of charged-current and neutral-current interactions. In this case it is not useful to define a density matrix which describes the detected neutrino, because the oscillation probability and the cross section are not factorized in the detection rate. As an example, we calculated the rate of neutrino-electron elastic scattering events in a solar neutrino experiment and we have shown that the usual expression in which the rate is given by the sum of the \( \nu_e \) and \( \nu_{\mu,\tau} \) contributions is obtained only when
the difference of the neutrino masses is neglected in the interaction process.

Let us finally note that, although for simplicity we considered as examples processes in which there are only two neutrinos, the formalism can be extended in a straightforward way to the more complicated case of interactions involving more than two neutrinos.

ACKNOWLEDGMENTS

G. C. was fully supported by São Paulo Research Foundation (FAPESP) under grant 2016/08025-0.

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