Aberrational Effects for Shadows of Black Holes

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Abstract
In this paper, we discuss how the shadow of a Kerr black hole depends on the motion of the observer. In particular, we derive an analytical formula for the boundary curve of the shadow for an observer moving with given four-velocity at given Boyer–Lindquist coordinates. We visualize the shadow for various values of parameters.

1 Introduction: What is the “Shadow”?
For an observer at Boyer–Lindquist coordinates $(r_O, \vartheta_O)$ the shadow of a black hole is defined as that region of the observer’s celestial sphere which is left dark if all light sources are distributed on a sphere with radius $r_L > r_O$. To calculate the shadow, we consider light rays, i.e. lightlike geodesics, which are sent into the past from the observer’s position. Then, the boundary curve of the shadow corresponds to the limiting case between geodesics going towards the horizon (→ darkness) and geodesics going to the sphere at $r_L$ with light sources (→ brightness). The limiting case are geodesics that asymptotically spiral towards (unstable) lightlike geodesics which fill a specific spatial region, the photon region $\mathcal{K}$, and are propagating on a sphere. Consequently, the shadow of the black hole is a mapping of this photon region $\mathcal{K}$ and not of the horizon.

Starting with the theoretical work of Bardeen who for the first time calculated the shadow of a Kerr black hole correctly [2] we now reach the time where it seems to be possible to observe the shadow of the black hole

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near Sgr A* at the center of our Galaxy. The expected image is shown by calculations of how the shadow of a black hole would look like if matter, in the form of an accretion disc, a corona, or a jet, is included in the model. These calculations are based on ray tracing and GRMHD, see e.g. [3, 4, 5, 6, 7, 8]. It is hoped that the shadow of Sgr A* will indeed be observed in the near future within the Event Horizon Telescope project, see [9], or the Black Hole Cam project.

In the following, we discuss how differently moving observers at a given position see the shadow of a Kerr black hole. A detailed discussion of a more general space-time describing a Kerr–Newman–NUT black hole with cosmological constant can be found in [1]. There, we demonstrate how the shadow is influenced by a charge, the cosmological constant or the NUT parameter. Since there one can also find a discussion of metric properties and visualizations of the photon regions of the black holes, we restrict ourselves in this proceedings volume to the Kerr case. Our construction of the shadow is a geometrical one, based on the geodesic equation and ignoring the influence of matter.

2 The Kerr Metric

The Kerr space-time is a stationary, axially symmetric type D solutions of the Einstein vacuum equations that describes a rotating black hole with mass $m$ and spin $a$. In Boyer–Lindquist coordinates $(r, \vartheta, \varphi, t)$ the Kerr metric can be written as ([10], p. 314)

$$g_{\mu\nu} \, dx^\mu \, dx^\nu = \Sigma \left( \Delta d r^2 + d \vartheta^2 \right) + \frac{1}{\Sigma} \left( (\Sigma + a \chi)^2 \sin^2 \vartheta - \Delta \chi^2 \right) \, d\varphi^2 + \frac{2}{\Sigma} \left( \Delta \chi - a (\Sigma + a \chi) \sin^2 \vartheta \right) \, dt \, d\varphi - \frac{1}{\Sigma} \left( \Delta - a^2 \sin^2 \vartheta \right) \, dt^2 \tag{1}$$

where we use the abbreviations

$$\Sigma = r^2 + (a \cos \vartheta)^2, \quad \chi = a \sin^2 \vartheta, \quad \Delta = r^2 - 2mr + a^2, \tag{2}$$

and rescaled units; hence, the speed of light and the gravitational constant are normalized ($c = 1, G = 1$). The coordinates $t$ and $r$ range over $]-\infty, \infty[$, while $\vartheta$ and $\varphi$ are standard angular coordinates on the two-sphere. Whereas the parameter $m$ for the mass of the black hole could take all values in $\mathbb{R}^+$ the absolute value of the spin parameter $a$ is bounded by $m$ since the event horizon of the Kerr black hole is at $r + \sqrt{m^2 - a^2}$.
3 Calculating the Shadows of Black Holes

As the geodesic equation in the Kerr space-time has four constants of motion—the Lagrangian $L$, the energy and the $z$-component of angular momentum

$$ E = -\frac{\partial L}{\partial \dot{t}} = -g_{\phi t} \dot{\phi} - g_{tt} \dot{t}, \quad L_z = \frac{\partial L}{\partial \dot{\phi}} = g_{\phi t} \dot{\phi} + g_{\phi t} \dot{t}, $$ (3)

plus the Carter constant $K$ [11]—the lightlike geodesics ($L = 0$) are given by four separated equations of motion

$$ \Sigma \dot{t} = \frac{\chi (L_z - E \chi)}{\sin^2 \vartheta} + \frac{\left( \Sigma + a \chi \right) \left( \left( \Sigma + a \chi \right) E - a L_z \right)}{\Delta}, $$ (4a)

$$ \Sigma \dot{\phi} = \frac{L_z - E \chi}{\sin^2 \vartheta} + \frac{a \left( \left( \Sigma + a \chi \right) E - a L_z \right)}{\Delta}, $$ (4b)

$$ \Sigma^2 \dot{\vartheta}^2 = K - \left( \frac{\chi E - L_z}{\sin^2 \vartheta} \right)^2 =: \Theta(\vartheta), $$ (4c)

$$ \Sigma^2 \dot{r}^2 = \left( \left( \Sigma + a \chi \right) E - a L_z \right)^2 - \Delta K =: R(r). $$ (4d)

The existence of the photon region $\mathcal{K}$, i.e. the region filled with lightlike geodesics staying on a sphere $r = \text{constant}$, is crucial for calculating the shadow because it will give us a parametrization of the shadow’s boundary curve. By (4d), the sphere conditions $\dot{r} = 0$ and $\ddot{r} = 0$ imply that $R(r) = 0$ and $R'(r) = 0$. Hence,

$$ K_E = \frac{\left( \left( \Sigma + a \chi \right) - a L_E \right)^2}{\Delta}, \quad K_E = \frac{2r \left( \left( \Sigma + a \chi \right) - a L_E \right)}{r - m}, $$ (5)

where $L_E = L_{\phi}$ and $K_E = K_{E_{\phi}}$. Solving for these constants gives the expressions

$$ K_E = \frac{4r^2 \Delta}{(r - m)^2}, \quad a L_E = \left( \Sigma + a \chi \right) - \frac{2r \Delta}{r - m}, $$ (6)

for spherical light rays. Since $0 \leq \Sigma^2 \dot{\vartheta}^2$ we find by (4c) an inequality characterizing the photon region

$$ \mathcal{K}: \left( 2r \Delta - \Sigma (r - m) \right)^2 \leq 4a^2 r^2 \Delta \sin^2 \vartheta. $$ (7)

Through each point $(r, \vartheta)$ of $\mathcal{K}$ there is a light ray propagating on a sphere. Plots and a detailed discussion of the photon region for different space-times can be found in [11, Fig. 3–5].
4 Viewing the Shadows of Black Holes

For determining the shadow of a black hole we consider an observer at Boyer-Lindquist coordinates \((r_0, \vartheta_0)\) and assume, for the sake of simplicity, that the light sources are distributed on a sphere with radius \(r_L > r_0\).

Lightlike geodesics reaching the observer can be divided into two types of orbits. There are geodesics which passed the sphere with light sources and there are those coming from the horizon. Thus, our observer would see brightness in the direction of light rays of the first type and darkness for the other ones. The boundary curve of the shadow is therefore given by lightlike geodesics that spiraled from one of the unstable spherical light orbits of the photon region \(\mathcal{K}\).

The shape of the shadow depends on the observer’s state of motion. At Boyer–Lindquist coordinates \((r_0, \vartheta_0)\), we choose an orthonormal tetrad adapted to the symmetry of the space-time ([10], p. 307)

\[
e_0 = \frac{(\Sigma + a\chi)\partial_t + \chi a\partial_\varphi}{\sqrt{\Sigma \Delta}} \big|_{(r_0, \vartheta_0)}, \quad e_1 = \frac{\sqrt{\frac{1}{\Sigma}} \partial_{\vartheta}}{\big|_{(r_0, \vartheta_0)}},
\]

\[
e_2 = -\frac{\partial_\varphi + \chi \partial_t}{\sqrt{\Sigma \sin \vartheta}} \big|_{(r_0, \vartheta_0)}, \quad e_3 = -\frac{\sqrt{\frac{\Delta}{\Sigma}} \partial_r}{\big|_{(r_0, \vartheta_0)}}.
\]

(8)

It is chosen such that \(e_0 \pm e_3\) are tangential to the principal null congruences of our metric. Here, \(e_0\) is interpreted as the four-velocity of an observer at \((r_0, \vartheta_0)\) because it is a timelike vector; \(e_3\) points into the direction towards the center of the black hole. An observer with this tetrad is called a standard observer in the following.

If another observer at \((r_0, \vartheta_0)\) moves with velocity \(v = (v_1, v_2, v_3)\), \(|v| < 1 = c\), with respect to our standard observer, we have to modify the tetrad. The four-velocity of the moving observer is

\[
\tilde{e}_0 = \frac{v_1 e_1 + v_2 e_2 + v_3 e_3 + e_0}{\sqrt{1 - v^2}}.
\]

(9a)
Figure 1: The direction of each light ray reaching the observer is given by the celestial coordinates $\theta$ and $\psi$ (Eq. (11)) of their tangents, see figure on the left. These points $(\theta, \psi)$ on the celestial sphere (black ball) can be identified with points in the plane (red ball) by stereographic projection, see figure on the right. The dashed (red) circles mark the celestial equator $\theta = \pi/2$ respectively its projection.

From $\tilde{e}_0, e_1, e_2, e_3$ we find an orthonormal tetrad $\tilde{e}_0, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ with the Gram–Schmidt procedure by adding $e_3, e_1, e_2$—in this order—successively to $\tilde{e}_0$

$$
\tilde{e}_1 = \frac{(1 - v_2^2)e_1 + v_1(v_2e_2 + e_0)}{\sqrt{1 - v_2^2} \sqrt{1 - v_1^2 - v_2^2}},
$$

$$
\tilde{e}_2 = \frac{e_2 + v_2e_0}{\sqrt{1 - v_2^2}},
$$

$$
\tilde{e}_3 = \frac{(1 - v_1^2 - v_2^2)e_3 + v_3(v_1e_1 + v_2e_2 + e_0)}{\sqrt{1 - v_1^2 - v_2^2} \sqrt{1 - v^2}}.
$$

Note that $\tilde{e}_i = e_i$ if $v_i = 0$, i.e., for $v = 0$ this procedure recovers the tetrad $e_0, e_1, e_2, e_3$. As before, the spacelike vector $\tilde{e}_3$ corresponds to the direction towards the black hole. The interpretation of $\tilde{e}_1$ and $\tilde{e}_2$ becomes clear if we introduce celestial coordinates, see (11) and Fig. [1] Then, $\tilde{e}_1$ and $\tilde{e}_2$ point into the north–south respectively the west–east direction.

We can now describe the tangent vector of a light ray $\lambda(s)$ by Boyer–Lindquist coordinates

$$
\dot{\lambda} = \dot{r}\partial_r + \dot{\theta}\partial_\theta + \dot{\varphi}\partial_\varphi + \dot{t}\partial_t
$$

(10)
and by celestial coordinates $\theta$ and $\psi$ for our moving observer

$$\dot{\lambda} = \sigma(-\tilde{e}_0 + \sin \theta \cos \psi \tilde{e}_1 + \sin \theta \sin \psi \tilde{e}_2 + \cos \theta \tilde{e}_3) \quad (11)$$

where $\theta = 0$ corresponds to the direction towards the black hole. For the tetrad (9) we observe the following dependencies regarding (8):

$$\tilde{e}_0 = k_{0r} \partial_r + k_{0\theta} \partial_\theta + k_{0\varphi} \partial_\varphi + k_{0t} \partial_t,$$

$$\tilde{e}_1 = k_{1r} \partial_r + k_{1\theta} \partial_\theta + k_{1\varphi} \partial_\varphi + k_{1t} \partial_t,$$

$$\tilde{e}_2 = k_{2r} \partial_r + k_{2\theta} \partial_\theta + k_{2\varphi} \partial_\varphi + k_{2t} \partial_t,$$

$$\tilde{e}_3 = k_{3r} \partial_r + k_{3\theta} \partial_\theta + k_{3\varphi} \partial_\varphi + k_{3t} \partial_t. \quad (12)$$

Hence

$$\dot{\lambda} = \sigma((-k_{0r} + k_{3r} \cos \theta) \partial_r + (-k_{0\theta} + k_{1\theta} \sin \theta \cos \psi + k_{3\theta} \cos \theta) \partial_\theta + (-k_{0\varphi} + k_{1\varphi} \sin \theta \cos \psi + k_{3\varphi} \cos \theta) \partial_\varphi + (-k_{0t} + k_{1t} \sin \theta \cos \psi + k_{3t} \cos \theta) \partial_t). \quad (13)$$

Comparing coefficients of $\partial_r, \partial_\theta, \text{ and } \partial_\varphi$ in (10) and (13) yields

$$\dot{r} = \sigma(-k_{0r} + k_{3r} \cos \theta), \quad (14)$$

$$\dot{\theta} = \sigma(-k_{0\theta} + k_{1\theta} \sin \theta \cos \psi + k_{3\theta} \cos \theta), \quad (15)$$

$$\dot{\varphi} = \sigma(-k_{0\varphi} + k_{1\varphi} \sin \theta \cos \psi + k_{3\varphi} \cos \theta). \quad (16)$$

These equations can be solved easily for $\cos \theta$ and $\sin \psi$ (using $k_{1\theta} \sin \theta \cos \psi = \frac{1}{\sigma} \dot{\gamma} + k_{0\varphi} - k_{3\varphi} \cos \theta$),

$$\cos \theta = \frac{1}{\sigma} \frac{\dot{r} + k_{0r}}{k_{3r}}, \quad (17a)$$

$$\sin \psi = \frac{k_{3r}(\frac{1}{\sigma} \dot{\varphi} + k_{0\varphi} - \frac{k_{1\varphi}}{k_{1\theta}}(\frac{1}{\sigma} \dot{\theta} + k_{0\theta})) - (k_{3\varphi} - \frac{k_{3\theta}}{k_{1\theta}} k_{1\varphi} \frac{1}{\sigma} \dot{r} + k_{0r})}{k_{2\varphi} \sqrt{k_{3r}^2 - (\frac{1}{\sigma} \dot{r} + k_{0r})^2}} \quad (17b)$$

where $\dot{\varphi}, \dot{\theta}$ and $\dot{r}$ have to be substituted from the equations of motion (1b), (1c) and (1d); since $\dot{r}$ and $\dot{\theta}$ are given as quadratic expressions, the signs have to be chosen consistently.

The remaining scalar factor $\sigma$ can be calculated analogously to $\alpha$ in (11) by Eq. (20)]. At first, express $\tilde{e}_0$ (9a) in terms of the tetrad $\{\partial_r, \partial_\theta, \partial_\varphi, \partial_t\}$

$$\tilde{e}_0 = \frac{1}{\sqrt{\Sigma} \sqrt{1 - v^2}} \left( \frac{(\Sigma + a \chi) \partial_t + a \partial_\varphi}{\sqrt{\Delta}} + v_1 \partial_\theta - v_2 \frac{\partial_\varphi + \chi \partial_t}{\sin \theta} - v_3 \sqrt{\Delta} \partial_r \right). \quad (18)$$
As \( \sigma = g(\hat{\lambda}, \hat{v}_0) \), see (11) we get \( \sigma \) from (1), (10), and (18),

\[
\sigma = \frac{1}{\sqrt{\Sigma \sqrt{1 - v^2}}} \left( \frac{aL_z - (\Sigma + a\chi)E}{\sqrt{\Delta}} + v_1 \Sigma \hat{\theta} - v_2 \frac{L_z - \chi E}{\sin \vartheta} - v_3 \frac{\Sigma}{\sqrt{\Delta}} \hat{r} \right)_{(r_O, \vartheta_O)}
\]

(19)

where \( \hat{\varphi}, \hat{\theta} \), and \( \hat{r} \) have to be substituted from (4b), (4c), and (4d) as above.

With this expression, (17) indeed describes the boundary curve of the black hole’s shadow for a moving observer. The boundary represents lightlike geodesics which, if you think of sending them from the observer’s position into the past, reach the photon region asymptotically. Each such geodesic must have constants of motion

\[
K_E = \frac{4r^2 \Delta}{(r - m)^2} \bigg|_{r=r_p}, \quad aL_E = \left( \Sigma + a\chi \right) - \frac{2r \Delta}{r - m} \bigg|_{r=r_p}
\]

(20)
given by (6) where \( r_p \) is the radius coordinate of the limiting spherical lightlike geodesic. For \( a > 0 \), this radius coordinate \( r_p \) is—as in (11)—extremal where the boundary of the exterior photon region intersects the cone \( \vartheta = \vartheta_O \). Hence, the extremal values are the values of \( r \) where (7) holds with equality. Substituting \( K_E \) and \( L_E \) in (17) and (19) by the expressions (20) provides the shadow’s boundary curve \((\theta(r_p), \psi(r_p))\) where \( r_p \) runs between the extremal values.

If \( a = 0 \), then it is not possible to parametrize the boundary curve by \( r_p \) because the right hand side of (7) is zero, so it determines a unique \( r_p \). By (20) this results in an unique value for \( K_E \) which, when inserted into (17), gives the shadow’s boundary curve in the form \((\psi(L_E), \theta(L_E))\). Here, \( L_E \) ranges between the extremal values determined by (4c) for \( \Sigma^2 \hat{\theta}^2 = 0 \).

5 Plots of Black Hole’s Shadows

As described before, we used our analytical parameter representation (17) with (19) and (20) to calculate the boundary curve of the shadow as seen by an observer moving with four-velocity \( \hat{v}_0 \). The results in Figs. 3 are visualized via stereographic projection from the celestial sphere onto a plane, as illustrated in Fig. 1. Standard Cartesian coordinates in this plane are given by

\[
x(r_p) = -2 \tan \left( \frac{\theta(r_p)}{2} \right) \sin \left( \psi(r_p) \right),
\]

\[
y(r_p) = -2 \tan \left( \frac{\theta(r_p)}{2} \right) \cos \left( \psi(r_p) \right).
\]

(21)
All plots of shadows shown in Fig. 3 belong to Kerr black holes where each subfigure combines the pictures for four spin values, see legend in Fig. 2.

In principle, the shadows for moving observers \( (v \neq 0) \) are calculable from the shadow seen by the standard observer \( (v = 0) \) with the help of Penrose’s aberration formula \([12]\)

\[
\tan \frac{\tilde{\alpha}}{2} = \sqrt{\frac{c - v}{c + v}} \tan \frac{\alpha}{2}. \tag{22}
\]

But for applying this formula, one may need to make coordinate transformations since the angles \( \alpha \) and \( \tilde{\alpha} \) have to be measured against the direction of the motion. Hence, no transformations are needed if the observer moves in radial direction. Then, the shadow is magnified if the observer moves away from the black hole, and demagnified if the observer moves towards the black hole. In this case, our formula \([17a]\) reduces to the following common variant of Penrose’s aberration formula \([22]\)

\[
\cos \tilde{\theta} = \frac{v + \cos \theta}{1 + v \cos \theta}. \tag{23}
\]

Penrose emphasized in his article \([12]\) that the aberration formula maps circles on the celestial sphere onto circles. Thus, the shadow of a non-rotating black hole \( (a = 0) \) is always circular, independent of the observer’s motion. Consequently, our pictures of the shadow are then always circular, because the stereographic projection \([21]\) maps circles onto circles, too.

Fig. 3 shows several pictures of shadows for differently moving observers in Kerr space-times. The results for the standard observer \( (v = 0, e_\mu = e_\mu) \) are shown in the left plot in the first row of Fig. 3. The right plot is exemplary and belongs to an observer moving with velocity \( v = (\frac{2}{10}, -\frac{2}{10}, -\frac{1}{10}) \). In each of the lower rows we vary only one component \( v_i \) of \( v \); in the following we write \( v_i \) as abbreviation for those observer velocities \( v \) with \( v_i \neq 0 \) and \( v_j = 0, j \neq i \). Due to our definition of the tetrad \( e_\mu \) in \([9]\) and of the observer’s four-velocity \( \tilde{e}_0 \) in \([9a]\) the observer moves in \( \vartheta \) direction if \( v = v_1 \),

\[
\begin{array}{cccc}
\text{Legend for the different spins } a \text{ used for calculating the black hole’s shadows shown in Fig. 3.}
\end{array}
\]

|          | 0.0 | 0.4 | 0.8 | 1.0 |
|----------|-----|-----|-----|-----|
| \( a = 0 \) |     |     |     |     |
| \( a = \frac{2a_{\text{max}}}{3} \) |     |     |     |     |
| \( a = \frac{4a_{\text{max}}}{3} \) |     |     |     |     |
| \( a = a_{\text{max}} \) |     |     |     |     |
Figure 3: Aberrational effects on the shadows of Kerr black holes. The subfigures show the stereographic projection of the shadow for observers moving with various velocities $v = (v_1, v_2, v_3)$ which are noted beneath the plots ($r_O = 5m$, $\vartheta_O = \frac{\pi}{2}$). If $v \neq 0$, the projected direction of the observer’s motion is marked by a (green) star. Each plot combines the silhouettes for four different spins ($a = \kappa \cdot a_{\text{max}}$ where $a_{\text{max}} = m$), see Fig. 2 for the corresponding legend.
and in $r$ i.e. radial direction if $v = v_3$. For $v = v_2$ the motion is in $\varphi$ direction.

Since the last row of Fig. 3 shows the plots of shadows seen by a radially moving observer ($v = v_3$), the shadows are magnified if the observer moves away from the black hole ($v_3$ negative), and demagnified, if the observer moves towards the black hole ($v_3$ positive), as mentioned before.

For velocities $v = v_1$ or $v = v_2$ the shadow is shifted in the direction of the observer’s motion with bigger effects for higher velocities. Also the size of the shadow is affected. But all these aberrational changes are explainable if one relates the direction of the observer’s motion to the spin of the black hole and to the equatorial plane as symmetry plane.

Furthermore, the shadow is symmetric with respect to a horizontal axis as long as the observer does not move in $\vartheta$ direction because $\sin \psi$, see (17b), depends on $\dot{\vartheta}$ which is given by a quadratic expression, see (4c). Hence, the different signs of $\dot{\vartheta}$ yield different solutions of (17) for the points $(\theta, \psi)$ and $(\theta, \pi - \psi)$. Without a $\vartheta$ component in the velocity, the symmetry of the shadow is not affected even if the observer is not in the equatorial plane, i.e. $\vartheta_O \neq \pi / 2$.

All in all, the shadows shown in Fig. 3 are calculated for relatively fast moving observers ($v = 0.3c$ up to $v = 0.9c$). Thus, the aberrational influence for the future observations of the shadow of Sgr A* within the Event Horizon Telescope or the Black Hole Cam project is expected to be very small since our solar system orbits the galactic center with roughly $250 \text{ km/s} \approx 1 \times 10^{-4} c$, see [13]. Nevertheless, the study of aberrational effects are of interest from a fundamental point of view.

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