RIESZ $s$-EQUILIBRIUM MEASURES ON $d$-DIMENSIONAL FRACTAL SETS AS $s$ APPROACHES $d$

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ABSTRACT. Let $A$ be a compact set in $\mathbb{R}^p$ of Hausdorff dimension $d$. For $s \in (0, d)$, the Riesz $s$-equilibrium measure $\mu^{s,A}$ is the unique Borel probability measure with support in $A$ that minimizes

$$I_s(\mu) := \iint \frac{1}{|x-y|^s} d\mu(y) d\mu(x)$$

over all such probability measures. In this paper we show that if $A$ is a strictly self-similar $d$-fractal, then $\mu^{s,A}$ converges in the weak-star topology to normalized $d$-dimensional Hausdorff measure restricted to $A$ as $s$ approaches $d$ from below.

keywords: Riesz potential, equilibrium measure, fractal

1. Introduction

Let $A$ be a compact subset of $\mathbb{R}^p$ with positive $d$-dimensional Hausdorff measure. Let $\mathcal{M}(A)$ denote the (unsigned) Radon measures supported on $A$ and $\mathcal{M}_1(A) \subset \mathcal{M}(A)$ the probability measures in $\mathcal{M}(A)$. Recall (cf. [8, 5, 9]) that for $s \in (0, d)$ the Riesz $s$-energy of a measure $\mu \in \mathcal{M}(A)$ is

$$I_s(\mu) := \iint \frac{1}{|x-y|^s} d\mu(y) d\mu(x),$$

and that there is a unique measure $\mu^{s,A} \in \mathcal{M}_1(A)$ called the equilibrium measure with the property that $I_s(\mu^{s,A}) < I_s(\nu)$ for all $\nu \in \mathcal{M}_1(A) \setminus \{\mu^{s,A}\}$. The $s$-potential of a measure $\mu$ at a point $x$ is

$$U^s_x(\mu) := \int \frac{1}{|x-y|^s} d\mu(y),$$

and for any measure $\mu$ with finite $s$-energy

$$I_s(\mu) = \int U^s_x d\mu.$$ 

For $s \geq d$, $I_s(\mu) = \infty$ for all non-trivial $\mu \in \mathcal{M}(A)$. We shall denote the $d$-dimensional Hausdorff measure as $\mathcal{H}^d$ and the restriction of a measure $\mu$ to a set $E$ as $\mu_E$ e.g. $\mathcal{H}^d_A := \mathcal{H}^d(\cdot \cap A)$. The closed ball of radius $r$ centered at $x$ is denoted $B(x, r)$.

The study of equilibrium measures arises naturally in electrostatics. Specifically one may consider $\mu^{s,A}$ as the positive charge distribution on $A$ that minimizes a generalized electrostatic energy mediated by the kernel $|x-y|^{-s}$ where, in the classical electrostatic or Newtonian setting $s = d - 2$. In the case of the interval $A = [-1, 1]$ ($d = 1$) it is known $\mu^{s[-1,1]}$ is absolutely continuous with respect to the one-dimensional Lebesgue measure and the Radon-Nikodym derivative of $\mu^{s[-1,1]}$ is $c_s (1 - x^2)^{s/2}$ where $c_s$ is chosen to make the measure of unit mass. One can see $\mu^{s[-1,1]}$ converges in the weak-star topology on $\mathcal{M}(A)$ to $\mathcal{H}^d_A/\mathcal{H}^d(\cdot \cap A)$ as $s \uparrow 1$. More generally it is shown in [2] that this convergence occurs for certain $d$-rectifiable sets.

In this paper we prove the same result for any compact self-similar fractal $A \subset \mathbb{R}^p$ satisfying

$$A = \bigcup_{i=1}^N \varphi_i(A),$$

where the union is disjoint and the maps $\varphi_1, \ldots, \varphi_N$ satisfy $|\varphi_i(x)| = L_i|x|$ for all $x \in \mathbb{R}^p$ and where $L_i \in (0, 1)$. We refer to such sets as strictly self-similar $d$-fractals. In [10] Moran shows for strictly self-similar $d$-fractals
the Hausdorff dimension is also the unique value of \( d \) that satisfies the equation
\[
\sum_{i=1}^{N} l_{i}^{d} = 1,
\]
and that \( \mathcal{H}^{d}(A) \in (0, \infty) \). Moran shows this results for fractals satisfying the broader open set condition (cf. [3]), however we use the strict separation in the proofs of the following results.

Given a Borel measure \( \mu \), let \( \Theta_{d}(\mu, x) := \mu(B(x, r))/r^{d} \) denote the average \( d \)-density of \( \mu \) over a radius \( r \) about \( x \). The limit as \( r \downarrow 0 \),
\[
\Theta_{d}(\mu, x) := \lim_{r \downarrow 0} \Theta_{d}(\mu, x),
\]
when it exists, is the classical point density of \( \mu \) at \( x \). It is a consequence of a result of Preiss [12] (also cf. [9]) that if \( A \) is a strictly self-similar \( d \)-fractal, then at \( \mathcal{H}_{A}^{d} \)-a.e. \( x \in A \) the point densities \( \Theta_{d}(\mathcal{H}_{A}^{d}, x) \) do not exist. However, Bedford and Fisher in [11] consider the following normalized averaging integral:
\[
D_{a}^{d}(\mu, x) := \lim_{\epsilon \downarrow 0} \frac{1}{|\ln \epsilon|} \int_{\epsilon}^{1} \frac{1}{r} \Theta_{d}(\mu, x) dr,
\]
which they call an order-two density of \( \mu \) at \( x \). It is known (cf. [4] [11] [14]) that for a class of sets including strictly self-similar \( d \)-fractals \( D_{a}^{d}(\mathcal{H}_{A}^{d}, x) \) is positive, finite and constant \( \mathcal{H}_{A}^{d} \)-a.e. We shall denote this \( \mathcal{H}_{A}^{d} \)-a.e. constant as \( D_{a}^{d}(A) \).

In this paper we examine the limiting case as \( s \uparrow d \) of the Riesz potential and energy of a measure \( \mu \) by considering the following normalized \( d \)-energy and \( d \)-potential:
\[
\tilde{I}_{d}(\mu) := \lim_{s \uparrow d} (d - s) I_{s}(\mu) \quad \tilde{U}_{d}^{\mu}(x) := \lim_{s \uparrow d} (d - s) U_{s}^{\mu}(x),
\]
when they exist. In [15], Zähle provides conditions on a measure \( \mu \) for which \( D_{a}^{d}(\mu, \cdot) \) and \( \tilde{U}_{d}^{\mu} \) agree. (cf. [6] for generalizations to other averaging schemes.) We use this result to prove that the limit \( \tilde{I}_{d}(\mu) \) exists for all \( \mu \in \mathcal{M}(A) \), that this normalized energy gives rise to a minimization problem with a unique solution and use this minimization problem to study the behavior of the equilibrium measures \( \mu^{+A} \) as \( s \uparrow d \).

The study of Riesz potentials on fractals is also examined in [16] [17] by Zähle in the context of harmonic analysis on fractals. In [13], Putinar considers a different normalization for the Riesz \( d \)-potential in his work on inverse moment problems.

1.1. Results.

**Theorem 1.1.** Let \( A \) be a strictly self-similar \( d \)-fractal and let \( \lambda^{d} := \mathcal{H}_{A}^{d}/\mathcal{H}^{d}(A) \), then

1. The limit \( \tilde{I}_{d}(\mu) \) exists for all \( \mu \in \mathcal{M}(A) \) and
\[
\tilde{I}_{d}(\mu) = \begin{cases} 
D_{a}^{d}(A) \frac{d\mu}{d\mathcal{H}_{A}^{d}}^{2} & \text{if } \mu \ll \mathcal{H}_{A}^{d}, \\
\infty & \text{otherwise}.
\end{cases}
\]

2. If \( \tilde{I}_{d}(\mu) < \infty \), then the limit \( \tilde{U}_{d}^{\mu} \) equals \( \frac{d\mu}{d\mathcal{H}_{A}^{d}} \) \( \mu \)-a.e. and
\[
\tilde{I}_{d}(\mu) = \int \tilde{U}_{d}^{\mu} d\mu.
\]

3. \( \tilde{I}_{d}(\lambda^{d}) < \tilde{I}_{d}(\nu) \) for all \( \nu \in \mathcal{M}_{1}(A) \backslash \{\lambda^{d}\} \).

**Theorem 1.2.** Let \( A \) be a strictly self-similar \( d \)-fractal, then there is a constant \( K \) depending only on \( A \), so that for any \( s \in (0, d) \), \( \mu^{+A}(B(x, r)) \leq Kr^{s} \) for \( \mu^{+A} \)-a.a. \( x \in A \) and \( r > 0 \).

A bound similar to that in Theorem [12] is presented in [9] Ch. 8. This result differs in that the constant \( K \) does not depend on \( s \).

**Theorem 1.3.** Let \( A \) be a strictly self-similar \( d \)-fractal and let \( \lambda^{d} := \mathcal{H}_{A}^{d}/\mathcal{H}^{d}(A) \), then \( \mu^{+A} \) converges in the weak-star topology on \( \mathcal{M}(A) \) to \( \lambda^{d} \) as \( s \uparrow d \).
2. The Existence of a Unique Minimizer of $\tilde{I}_d$

A set $A$ is said to be Ahlfors $d$-regular if there are constants $0 < C_1, C_2 < \infty$ depending only on $A$ so that for all $x \in A$ and all $r \in (0, \text{diam} A)$

$$C_1 r^d < \mathcal{H}_A^d(B(x, r)) < C_2 r^d.$$  

The proof of Proposition 2.1 is given by Hutchinson in [7, §5.3].

**Proposition 2.1.** If $A$ is a strictly self-similar $d$-fractal, then $A$ is Ahlfors $d$-regular.

The potential $U^\mu_s(x)$ of a finite Borel measure $\mu$ at a point $x$ has the following useful expression in terms of densities: (cf. [9])

$$U^\mu_s(x) = \int_{\mathbb{R}^p} \frac{1}{|x - y|^s} d\mu(y) = \int_0^\infty \mu(y : |x - y|^{-s} \geq t) dt = \int_0^\infty \mu(y : |x - y| \leq r^{-1/s}) dt = s \int_0^\infty \frac{\mu(B(x, r))}{r^{d+1}} dr = s \int_0^\infty \Theta^s_d(\mu, x) \frac{1}{r^{d-(d-s)}} dr,$$

where the second to last equality results from a change of variables replacing $r^{-1/s}$ with $r$. Note that for all $R > 0$

$$\lim_{s \uparrow d} (d - s) s \int_R^\infty \Theta^s_d(\mu, x) \frac{1}{r^{d-(d-s)}} dr = 0.$$  

From this we conclude that if $\tilde{U}^\mu_s(x)$ exists, then

$$\tilde{U}^\mu_d(x) = \lim_{s \uparrow d} (d - s) s \int_0^R \Theta^s_d(\mu, x) \frac{1}{r^{d-(d-s)}} dr,$$

for any $R > 0$.

The relationship between the order-two density and the limiting potential is examined by Zähle in the context of stochastic differential equations in [15] and also by Hinz, in [6]. We include a proof of this relationship from [6].

**Proposition 2.2.** Let $\mu$ be a finite Borel measure with support in $\mathbb{R}^p$, $x \in \text{supp} \mu$, $d \in (0, p]$. If $D^2_d(\mu, x)$ exists and is finite, then

$$\tilde{U}^\mu_d(x) = dD^2_d(\mu, x).$$

**Proof.** One may verify that the function $k_\varepsilon(t) := \varepsilon^2 \chi_{0,1}(t) t^{p-1} \log t$ is an approximate identity in the following sense: If $f : \mathbb{R} \rightarrow \mathbb{R}$ is right continuous at 0 and is bounded on $(0, 1)$, then

$$\lim_{\varepsilon \downarrow 0} \int_0^\infty k_\varepsilon(t)f(t) dt = f(0).$$

Define the following function:

$$f(t) := \begin{cases} \frac{1}{\log t} \int_t^1 \frac{1}{r} \Theta^s_d(\mu, x) dr & \text{when } t > 0 \\ D^2_d(\mu, x) & \text{when } t = 0 \end{cases}$$
If $D^2_d(\mu, \chi)$ exists and is finite, then $f$ is right-continuous at 0 and bounded on $(0, 1)$ thus

$$D^2_d(\mu, \chi) = \lim_{\varepsilon \downarrow 0} \int_0^\infty k_\varepsilon(t)f(t)dt$$

$$= \lim_{\varepsilon \downarrow 0} \varepsilon^2 \int_0^1 t^{1-\varepsilon} \int_0^1 \frac{\chi_{[e,1]}(r)}{r} \Theta_\alpha'(\mu, \chi)drdt$$

$$= \lim_{\varepsilon \downarrow 0} \varepsilon^2 \int_0^1 \frac{1}{r} \Theta_\alpha'(\mu, \chi) \int_1^r r^{-1}dtdr$$

$$= \lim_{\varepsilon \downarrow 0} \varepsilon \int_0^1 \frac{1}{r} \Theta_\alpha'(\mu, \chi) r^\varepsilon dr$$

$$= \lim_{s \uparrow (d-s)} \int_0^1 \Theta_\alpha'(\mu, \chi) \frac{1}{r^{1/d-s}} dr = \frac{1}{d} \Upsilon^d_\alpha(\chi).$$

\[\square\]

**Lemma 2.3.** Let $A$ be a strictly self-similar $d$-fractal and let $\mu \in M(A)$. If $\mu = \mu^c + \mu^1$ is the Lebesgue decomposition of $\mu$ with respect to $\mathcal{H}^d_A$, then

1. $\Upsilon^d_\alpha(x) = \infty$ for $\mu^1$-a.a. $x$.  
2. $\Upsilon^d_\alpha(x) = dD^2_d(A) \frac{d\mu}{d\mathcal{H}^d_A}(x)$ for $\mu^c$-a.a. $x$.

**Proof.** The Radon-Nikodým theorem ensures that for $\mu^1$-a.a. $x$,

$$\lim_{s \uparrow 0} \frac{\mu^1(B(x, r))}{\mathcal{H}^d_A(B(x, r))} = \infty.$$  

For such an $x$, let $M \in \mathbb{R}$ be arbitrary and $R > 0$ such that for all $r \in (0, R)$ we have $\mu^1(B(x, r))/\mathcal{H}^d_A(B(x, r)) > M$. Then it follows that

$$\liminf_{s \downarrow (d-s)} \int_0^R \frac{\mu^1(B(x, r))}{r^{d-s}} dr \geq \left( \inf_{r \in (0, R)} \frac{\mu^1(B(x, r))}{\mathcal{H}^d_A(B(x, r))} \right) \liminf_{s \downarrow (d-s)} \int_0^R \frac{\mathcal{H}^d_A(B(x, r))}{r^{d-s}} dr$$

$$\geq M \lim_{s \downarrow (d-s)} s \int_0^R \frac{d\mu^1}{d\mathcal{H}^d_A}(x) dr.$$  

where $C_1$ is the lower bound from the Ahlfors $d$-regularity of $A$. $M$ is arbitrary, and this proves the first claim.

To prove the second claim we begin with the following equality for an arbitrary $R > 0$:

$$\int_0^R \frac{d\mu^1}{d\mathcal{H}^d_A}(x) dr = \int_0^R \frac{d\mu^1}{d\mathcal{H}^d_A}(x) dr + (d-s) \int_0^R \frac{\mu^1(B(x, r))}{\mathcal{H}^d_A(B(x, r))} \mathcal{H}^d_A(B(x, r)) dr + (d-s) \int_0^R \left( \frac{\mu^1(B(x, r))}{\mathcal{H}^d_A(B(x, r))} - \frac{d\mu^1}{d\mathcal{H}^d_A}(x) \right) \mathcal{H}^d_A(B(x, r)) dr.$$

By Proposition 2.2 the limit as $s \uparrow d$ of the first summand in (1) is $\frac{d\mu^1}{d\mathcal{H}^d_A}(x)dD^2_d(A)$ for $\mathcal{H}^d_A$-a.a. x. The absolute value of the limit superior of the second summand in (1) is bounded for $\mathcal{H}^d_A$-a.a. x by

$$\sup_{r \in (0, R)} \left( \frac{\mu^1(B(x, r))}{\mathcal{H}^d_A(B(x, r))} - \frac{d\mu^1}{d\mathcal{H}^d_A}(x) \right) dD^2_d(A),$$

which can be made arbitrarily small by choosing $R$ sufficiently small. Thus the limit as $s \uparrow d$ of (1) exists $\mathcal{H}^d_A$-a.e. and hence $\Upsilon^d_\alpha(x)$ does as well.

For a measure $\mu \in M(A)$, let

$$\tilde{\mathcal{H}}^d(\mu) := \liminf_{s \downarrow 0} \int_0^1 \frac{1}{|x-y|^d} d\mu(y)d\mu(x).$$

**Proposition 2.4.** Let $A$ be a strictly self-similar $d$-fractal. If $\tilde{\mathcal{H}}^d(\mu) < \infty$ for $\mu \in M(A)$, then $\mu \ll \mathcal{H}^d_A$ and $\frac{d\mu}{d\mathcal{H}^d_A} \in L^2(\mathcal{H}^d_A)$.  


Proof. Let $\mu \in \mathcal{M}(A)$ so that $I_\delta(\mu) < \infty$, then by Fatou’s lemma
\[
\liminf_{s \downarrow 0} (d-s)U_s^\mu d\mu \leq I_\delta(\mu) < \infty.
\]
This implies that $\liminf_{s \downarrow 0} (d-s)U_s^\mu$ is finite $\mu$-a.e. and, by the first claim in Lemma 2.3 $\mu \ll \mathcal{H}_s^d$. By the second claim in Lemma 2.3 and the previous equation
\[
\int \left( \frac{d\mu}{d\mathcal{H}_s^d} \right)^2 d\mathcal{H}_s^d = \int \left( \frac{d\mu}{d\mathcal{H}_s^d} \right) d\mu = \int \frac{1}{dD^2_s(A)} U_s^\mu d\mu < \infty.
\]
\end{proof}

2.1. Proof of Theorem 1.1. With the preceding results we may now prove Theorem 1.1.

Proof. Let $\mu \in \mathcal{M}(A)$ so that $I_\delta(\mu) < \infty$, then $\mu \ll \mathcal{H}_s^d$ and $d\mu/d\mathcal{H}_s^d \in L^2(\mathcal{H}_s^d)$. The maximal function of $\mu$ with respect to $\mathcal{H}_s^d$ is
\[
M_{\mathcal{H}_s^d}\mu(x) := \sup_{r > 0} \frac{\mu(B(x, r))}{\mathcal{H}_s^d(B(x, r))} = \sup_{r > 0} \frac{1}{\mathcal{H}_s^d(B(x, r))} \int_{B(x, r)} d\mu.
\]
The maximal function is bounded on $L^2(\mathcal{H}_s^d)$ and so $M_{\mathcal{H}_s^d}\mu \in L^2(\mathcal{H}_s^d)$. We shall use this to provide a $\mu$-integrable bound for $(d-s)U_s^\mu$ that is independent of $s$ and appeal to dominated convergence. We begin with the point-wise bound
\[
(d-s) \int \frac{1}{|x-y|^s} d\mu(y) = (d-s) s \int_0^\infty \frac{\mu(B(x, r))}{\mathcal{H}_s^d(B(x, r))} \frac{\mathcal{H}_s^d(B(x, r))}{r^{s+1}} dr
\]
\[
\leq M_{\mathcal{H}_s^d}\mu(x)(d-s)s \int_0^{\text{diam } A} \frac{\mathcal{H}_s^d(B(x, r))}{r^{s+1}} dr + \int_0^{\text{diam } A} \frac{\mathcal{H}_s^d(B(x, r))}{r^{s+1}} dr
\]
\[
\leq M_{\mathcal{H}_s^d}\mu(x) \left[ (d-s)s \int_0^{\text{diam } A} C_2 r^d dr + (d-s)s \int_0^{\text{diam } A} \frac{1}{r^{s+1}} dr \right],
\]
where $C_2$ is the constant in the upper bound of the Ahlfors $d$-regularity of $A$. The quantity in brackets in (2) may be maximized over $s \in (0, d)$ and we denote this maximum by $K$. Then, by the Cauchy-Schwarz inequality,
\[
\int K M_{\mathcal{H}_s^d}\mu d\mu < K \int \left( M_{\mathcal{H}_s^d}\mu \right) d\mathcal{H}_s^d < K \left\| M_{\mathcal{H}_s^d}\mu \right\|_{L^2(\mathcal{H}_s^d)} \left\| \frac{d\mu}{d\mathcal{H}_s^d} \right\|_{L^2(\mathcal{H}_s^d)} < \infty.
\]
By dominated convergence the second claim follows. The first claim follows from the second and from Lemma 2.3 and Proposition 2.4.

The final claim of the theorem follows from a straightforward Hilbert space argument. Let $\nu$ denote the finite measure $dD^2_s(A)^{-1} \mathcal{H}_s^d$. By Proposition 2.4 the set of measures with finite normalized $d$-energy is identified with the non-negative cone in $L^2(\nu)$ (denoted $L^2(\nu)_+$) via the map $\mu \mapsto d\mu/d\nu$. Under this map we have $I_\delta(\mu) = \|d\mu/d\nu\|_{L^2}^2$. A measure $\mu$ of finite $d$-energy is a probability measure if and only if $\|d\mu/d\nu\|_{1,v} = 1$. We seek a unique non-negative function $f$ that minimizes $\| \cdot \|_{L^2,\nu}$ subject to the constraint $\|f\|_{1,v} = 1$. The non-negative constant function $1/\nu(\mathbb{R}^d)$ satisfies the constraint $\|1/\nu(\mathbb{R}^d)\|_{1,v} = 1$. Let $f \in L^2(\nu)_+$ such that $\|f\|_{1,v} = 1$ and $\|f\|_{2,v} \leq \|1/\nu(\mathbb{R}^d)\|_{2,v}$, then
\[
\frac{1}{\nu(\mathbb{R}^d)} = \left\| f \right\|_{L^2,\nu} \leq \left( f, \frac{1}{\nu(\mathbb{R}^d)} \right)_v \leq \left\| f \right\|_{L^2,\nu} \left\| \frac{1}{\nu(\mathbb{R}^d)} \right\|_{L^2,\nu} \leq \frac{1}{\nu(\mathbb{R}^d)} \left\| f \right\|_{L^2,\nu} = \frac{1}{\nu(\mathbb{R}^d)}.
\]
Thus
\[
\left\| f \right\|_{L^2,\nu} = \left\| f \right\|_{L^2,\nu} \left\| \frac{1}{\nu(\mathbb{R}^d)} \right\|_{L^2,\nu}.
\]
From the Cauchy-Schwarz inequality $f = 1/\nu(\mathbb{R}^d)$ $\nu$-a.e. By the identification above the measure $\lambda^d := \mathcal{H}_s^d/\mathcal{H}^d(A) \in \mathcal{M}_1(A)$, uniquely minimizes $I_d$ over $\mathcal{M}_1(A)$. \qed
3. The Weak-Star Convergence and Bound on the Growth of \( \mu^s \)

The proofs of Theorems 1.2 and 1.3 rely on the following classical results from Potential Theory (cf. [8, 5]). Let \( \mathcal{E}_\alpha \) denote the set of signed Radon measures with finite total variation such that \( \mu \in \mathcal{E}_\alpha \) if and only if \( I_s(\mu) < \infty \). The set \( \mathcal{E}_\alpha \) is a vector space and when combined with the following positive-definite bilinear form

\[
I_s(\mu, \nu) := \int \frac{1}{|x-y|^s} d\mu(y) d\nu(x),
\]

is a pre-Hilbert space. Further, the minimality of the \( s \)-energy of \( \mu^tA \) implies \( U_s^\mu = I_s(\mu^tA) \) \( \mu^tA \)-a.e.

We shall also use the Principle of Descent: Let \( \{\mu_n^\infty\}_{n=1}^\infty \subset M(A) \) be a sequence of measures converging in the weak-star topology on \( M(A) \) to \( \psi \) (we shall denote such weak-star convergence with a starred arrow, i.e. \( \mu_n \overset{*}{\to} \psi \)) then for \( s \in (0, d) \)

\[
I_s(\psi) \leq \liminf_{n \to \infty} I_s(\mu_n).
\]

**Lemma 3.1.** Let \( A \) be a compact set for which there is a \( C > 0 \) such that \( I_d(\mu) > C \) for all \( \mu \in M_1(A) \), then

\[
\lim_{t \downarrow d} I_t(\mu^tA) = \infty.
\]

**Proof.** Without loss of generality we shall assume that \( \text{diam} A \leq 1 \), then for \( 0 < s < t < d \) and any measure \( \mu \in M(A) \), \( I_s(\mu) \leq I_t(\mu) \). For sake of contradiction, assume that the claim does not hold, then there is sequence \( \{\mu_n\}_{n=1}^\infty \) increasing to \( d \) so that

\[
\lim_{n \to \infty} I_{\mu_n}(\mu^{\mu_n}A) = L < \infty.
\]

Let \( \psi \) be a weak-star cluster point of \( \{\mu_n^{\mu_n}A\}_{n=1}^\infty \) (hence a probability measure), and let \( \{s_n\}_{n=1}^\infty \subset (s, \infty) \) so that \( \mu^{\mu_n}A \overset{*}{\to} \psi \).

For any \( s \in (0, d) \) we have

\[
(d - s)I_s(\psi) \leq (d - s) \liminf_{n \to \infty} I_s(\mu^{\mu_n}A) \leq (d - s) \liminf_{n \to \infty} I_{\mu_n}(\mu^{\mu_n}A) \leq (d - s)L.
\]

Letting \( s \uparrow d \) implies \( \tilde{I}_d(\psi) = 0 \), which is a contradiction. \( \square \)

**Lemma 3.2.** Let \( A \) be a compact set for which there is a \( C > 0 \) such that \( I_d(\mu) > C \) for all \( \mu \in M_1(A) \), then

\[
\lim_{t \downarrow d} \sup_{y \in A} \text{dist}(y, \text{supp} \mu^tA) = 0.
\]

**Proof.** Let \( s \in (0, d) \) and \( \delta = \sup_{y \in A} \text{dist}(y, \text{supp} \mu^tA) \). We consider the possibility that \( \delta > 0 \). Pick \( y' \in A \) so that \( \text{dist}(y', \text{supp} \mu^tA) > \delta/2 \). Let \( \nu = H^t_{\mu^tA}(B(\delta/4)) \). For \( \beta \in [0, 1] \) we have \( (1 - \beta)\mu^tA + \beta \nu \in M_1(A) \). Arguments similar to those used in the proof of Lemma 3.3 show that \( I_s(\nu) < \infty \) for all \( s \in (0, d) \). Define the function

\[
f(\beta) := I_s((1 - \beta)\mu^tA + \beta \nu) = (1 - \beta)^2 I_s(\mu^tA) + \beta^2 I_s(\nu) + 2\beta(1 - \beta)I_s(\mu^tA, \nu),
\]

Differentiating gives

\[
\frac{1}{2} \frac{df}{d\beta} = \beta \left[ I_s(\mu^tA - v) - I_s(\mu^tA, \nu) \right] \quad \text{and} \quad \frac{1}{2} \frac{d^2f}{d\beta^2} = \left[ I_s(\mu^tA - v) \right].
\]

Because \( I_s(\cdot, \cdot) \) is positive definite, \( I_s(\mu^tA - v) > 0 \). Because \( \mu^tA \) is the unique minimizer of \( I_s \), \( f \) cannot have a minimum for any \( \beta > 0 \), hence \( I_s(\mu^tA) - I_s(\mu^tA, \nu) \leq 0 \). We obtain

\[
I_s(\mu^tA) \leq I_s(\mu^tA, \nu) \leq \frac{1}{(\delta/4)^s}, \quad \text{and hence} \quad \delta \leq \frac{4}{I_s(\mu^tA)^{1/s}}.
\]

By Lemma 3.1 \( \delta \downarrow 0 \) as \( s \uparrow d \). \( \square \)
3.1. **Proof of Theorem 1.2.** The next lemma is straightforward and its proof, which is included for completeness, employs common techniques and ideas presented by e.g. Hutchinson in [7]. For the rest of the paper we shall order our maps \( \{\varphi_1, \ldots, \varphi_N\} \) so that the scaling factors satisfy \( L_1 \leq L_2 \leq \ldots \leq L_N \).

**Lemma 3.3.** Let \( A \) be a strictly self-similar \( d \)-fractal then, for each \( x \in A \) and \( r > 0 \) there is a subset \( A' \subset A \) so that

1. \( B(x, r) \cap A \subset A' \).
2. \( A' = \varphi(A) \) for some similitude \( \varphi \).
3. \( \text{diam } A' < \text{Wr where } \text{Wr depends only on the set } A \).

**Proof.** Choose \( x \in A \) and \( r > 0 \). Let \( \tilde{K} = \min_{i \in \{1, \ldots, N\}} \{ \text{dist}(\varphi_i(A), A \setminus \varphi_i(A)) \} \). If \( r \geq L_1 \tilde{K} \), let \( A' = A \) and then trivially \( A \cap B(x, r) \subset A' \) and \( \text{diam } A' < r(2 \text{ diam } A)/(L_1 \tilde{K}) \).

We now consider the case when \( r < L_1 \tilde{K} \). Because the images of \( A \) under each \( \varphi_i \) are disjoint, we may assign to every \( y \in A \) a unique infinite sequence \( \{j_1, j_2 \ldots\} \subset \{1, \ldots, N\} \) so that \( y = \cap_{i=1}^{\infty} \varphi_{j_i}(\varphi_{j_{i-1}}(\ldots \varphi_{j_1}(A) \ldots)) \). If \( i_1, i_2, \ldots \) is the sequence identifying \( x \), let \( M \) be the smallest natural number so that \( L_{i_1} L_{i_2} \ldots L_{i_M} \tilde{K} < r \) (note that \( M \geq 2 \)), then

\[
\text{r} \leq L_{i_1} L_{i_2} \ldots L_{i_{M-1}} \tilde{K} < \frac{r}{L_{i_M}} < \frac{r}{L_1}.
\]

Let \( A' = \varphi_{i_{M-1}}(\varphi_{i_{M-2}}(\ldots \varphi_{i_1}(A) \ldots)) \), hence \( \text{diam } A' = L_{i_1} L_{i_2} \ldots L_{i_{M-1}} \text{ diam } A < r \text{ diam } A/(L_1 \tilde{K}) \). To complete the proof we shall show \( B(x, r) \cap A \subset A' \).

Choose \( y \in B(x, r) \cap A \). If \( y = x \), then \( y \in A' \), otherwise let \( \{j_1, j_2 \ldots\} \) be the sequence identifying \( y \in A \) and \( m \) the smallest natural number so that \( j_m \neq i_m \). We have that

\[
L_{i_1} L_{i_2} \ldots L_{i_{M-1}} \tilde{K} \leq \text{dist}(x, y) \leq r \leq L_{i_1} L_{i_2} \ldots L_{i_{M-1}} \tilde{K},
\]

from which we conclude \( m \geq M \) forcing \( y \in \varphi_{i_{M-1}}(\varphi_{i_{M-2}}(\ldots \varphi_{i_1}(A) \ldots)) = A' \).

The claimed constant \( W \) is \( 2(\text{ diam } A)/(L_1 \tilde{K}) \). \( \square \)

The remaining proofs will make use of the following fact regarding the behavior of equilibrium measures on scaled sets: If \( B' = \varphi(B) \) where \( \varphi \) is a similitude with a scale factor of \( L \), then for any Borel set \( E \subset B' \),

\[
\mu^{sB'}(E) = \mu^{sB}(\varphi^{-1}(E)) \text{ and } I_s(\mu^{sB'}) = L^{-s}I_s(\mu^{sB}).
\]

This follows from scaling properties of the Riesz kernel.

**Proof of Theorem 1.2** Without loss of generality assume \( \text{diam } A \leq 1 \). Let \( x \in A \) and \( r \in (0, \text{diam } A/4) \), then

\[
I_s(\mu^sA) = I_s \left( \mu^s_{B(x,r)} + \mu^s_{A \setminus B(x,r)} \right) \geq I_s \left( \mu^s_{B(x,r)} \right) + I_s \left( \mu^s_{A \setminus B(x,r)} \right).
\]

By Lemma 3.3, there is an \( s_0 \in (0, d) \) so that \( \mu^s A \setminus B(x, \text{diam } A/4) > 0 \) for all \( s \in (s_0, d) \). Note that the choice of \( s_0 \) depends only on \( A \) and not on \( x \). First, consider the case \( s \in (s_0, d) \). If \( \mu^s A \setminus B(x, r) = 0 \), then the claim is trivially proven. Assume \( \mu^s A \setminus B(x, r) > 0 \). We normalize the measures on the right hand side of (3) to be probability measures and obtain

\[
I_s \left( \mu^s_{B(x,r)} \right) + I_s \left( \mu^s_{A \setminus B(x,r)} \right) = \mu^s A \setminus B(x, r) \cdot I_s \left( \frac{\mu^s_{B(x,r)}}{\mu^s A \setminus B(x, r)} \right) + (1 - \mu^s A \setminus B(x, r)) \cdot I_s \left( \frac{\mu^s_{A \setminus B(x,r)}}{1 - \mu^s A \setminus B(x, r)} \right).
\]

By Lemma 3.3 we may find a set \( A' \subset A \) so that \( B(x, r) \cap A \subset A' \), \( \text{diam } A' < \text{Wr} \) and \( A' \) is a scaling of \( A \). The right hand side of (4) is bounded below by

\[
\mu^s A \setminus B(x, r) \cdot I_s(\mu^sA') + (1 - \mu^s A \setminus B(x, r)) \cdot I_s(\mu^sA) \geq I_s(\mu^sA) \left[ \mu^s A \setminus B(x, r) \left( \frac{\text{diam } A'}{\text{diam } A} \right)^{-s} + (1 - \mu^s A \setminus B(x, r)) \right]
\]

\[
> I_s(\mu^sA) \left[ \mu^s A \setminus B(x, r) \left( \frac{\text{Wr}}{\text{diam } A} \right)^{-s} + (1 - \mu^s A \setminus B(x, r)) \right]
\]

Combining (3) and (4) and dividing by \( I_s(\mu^sA) \) gives the following:

\[
1 \geq \mu^s A \setminus B(x, r) \left( \frac{\text{Wr}}{\text{diam } A} \right)^{-s} + 1 - 2 \mu^s A \setminus B(x, r) + \mu^s A \setminus B(x, r)^2,
\]
hence
\[ 2\mu^{sA}(B(x,r)) \geq \mu^{sA}(B(x,r))^2 \left( \frac{W_r}{\text{diam} A} \right)^{-s} + 1 , \quad \text{and thus} \quad \mu^{sA}(B(x,r)) \leq 2 \left( \frac{W_r}{\text{diam} A} \right)^s r^s . \]

Let \( K_1 \) be the maximum of \( 2(W_r/\text{diam} A)^{-s} \) over \( s \in [0,d] \) and \( K_2 \) the maximum of \( (4/\text{diam} A)^s \) over \( s \in [0,d] \) and \( K_2 := \max\{K_1, K_2\} \), then \( \mu^{sA}(B(x,r)) < K_2 r^s \) for all \( x \in A, r > 0 \) and \( s \in (s_0,d) \).

For \( s \in (0,s_0] \) we have the bound (cf. [9, Ch. 8]) \( \mu^{sA}(B(x,r)) \leq U^{sA}_r(x) r^s = I_s(\mu^{sA})r^s \) for \( \mu^{sA} \)-a.e. \( x \). Because \( \text{diam} A \leq 1 \), \( I_s(\mu^{sA}) \leq I_0(\mu^{s_{0A}}) \) for all \( s \in (0,s_0] \). Let \( K = \max\{K_1, 2I_0(\mu^{s_{0A}})\} \), then \( \mu^{sA}(B(x,r)) < K r^s \) for \( \mu^{sA} \)-a.e. \( x \in A \) and \( r > 0 \).

\[ \square \]

3.2. Proof of Theorem 1.3.

Proof of Theorem [2]. Let \( f : A \to \mathbb{R} \) be continuous. Since \( A \) is compact \( f \) is uniformly continuous on \( A \). Fix \( \varepsilon > 0 \) and let \( \delta > 0 \) so that \( f(A \cap B(x, \delta)) \subset (f(x) - \varepsilon, f(x) + \varepsilon) \) for all \( x \in A \). Let \( M \) be a natural number high enough so that \( L^d_X \) diam \( A \leq \delta \).

Let \( a \) be a multi-index of length \( M \) taking values in \( \{1, \ldots, N\}^M \). If \( a = (i_1, \ldots, i_M) \), then we denote \( \varphi_{i_1}(\varphi_{i_2}(\cdots(\varphi_i \cdots)) \cdots) \) by \( \varphi_A \). Let \( x \) be any point in \( A \). For any \( \nu \in \mathcal{M}(A) \) we may write
\[ \int f d\nu = \sum_a \int f d\nu_A = \sum_a \nu(\varphi_{x}(\cdots(\varphi_i(x) \cdots)) \cdots) = \sum_a \int (f - \nu(\varphi_{x}(\cdots(\varphi_i(x) \cdots)) \cdots)) d\nu_A . \]

It follows that
\[ \int f d\nu = \sum_a \nu(\varphi_{x}(\cdots(\varphi_i(x) \cdots)) \cdots) = \sum_a \int (f - \nu(\varphi_{x}(\cdots(\varphi_i(x) \cdots)) \cdots)) d\nu_A . \]

As in the proof of Lemma [2], let \( \tilde{K} = \min_{\nu \in \mathcal{M}(A) \setminus \{0\}} \{|\nu(x) - \nu(y)\}| \). If \( a \) and \( a' \) are different multi-indices of length \( M \), then \( |\nu(x) - \nu(A) - \nu'(y) - \nu'(A)| \geq L^{-1} \tilde{K} \). By Lemma [2], there is an \( s_0 < d \) so that for all \( s \in (s_0,d) \), we have \( \text{supp} \nu \leq \text{diam} \nu \leq L^{-1} \tilde{K} \). From this we conclude \( \mu^{sA}(\nu_A(A)) > 0 \) for any multi-index \( a \) of length \( M \) and any \( s \in (s_0,d) \). For such a choice of \( s \) we have
\[ I_s(\mu^{sA}) > \sum_a I_s(\mu_{\phi_A}(A)) = \sum_a \left( \frac{\phi_{x}(\cdots(\varphi_i(x) \cdots)) \cdots)}{\mu^{sA}(\phi_A)} \right)^2 I_s \left( \frac{\phi_{x}(\cdots(\varphi_i(x) \cdots)) \cdots)}{\mu^{sA}(\phi_A)} \right) \geq \sum_a \mu^{sA}(\phi_A)^2 I_s \left( \mu^{sA}(\phi_A) \right) \right) . \]

We shall use the notation \( L_\alpha \) to denote \( L_{i_1}L_{i_2} \cdots L_{i_M} \). By appealing to the scaling properties of the Riesz energy, the above becomes
\[ I_s(\mu^{sA}) > \sum_a \mu^{sA}(\phi_A)^2 L_\alpha \frac{I_s(\mu^{sA})}{L_\alpha} . \]

Let \( \psi \) be any weak-star cluster point of \( \mu^{sA} \) as \( s \to d \) and let \( \{s_n\}_{n=1}^{\infty} \to d \) be a sequence so that \( \mu^{s_nA} \to \psi \) and hence so that \( \mu^{s_nA}(\phi_A(A)) \to 0 \) converges in \( [0,1]^{N^d} \). Then
\[ 1 = \lim_{n \to \infty} \frac{1}{L_\alpha^{d-s}} \geq \lim_{n \to \infty} \sum_a \frac{\mu^{s_nA}(\phi_A(A))^2}{L_\alpha^{d-s}} = \sum_a \frac{\lim_{n \to \infty} \mu^{s_nA}(\phi_A(A))^2}{L_\alpha^{d}} . \]

We then have that
\[ 1 = \sum_a \lim_{n \to \infty} \mu^{s_nA}(\phi_A(A)) = \sum_a \lim_{n \to \infty} \frac{\mu^{s_nA}(\phi_A(A))}{\sqrt{L_\alpha^{d}}} \sqrt{L_\alpha^{d}} \leq \sum_a \frac{\lim_{n \to \infty} \mu^{s_nA}(\phi_A(A))^2}{L_\alpha^{d}} \sqrt{L_\alpha^{d}} = 1 . \]

Note that the sum over \( a \) of \( L_\alpha^{d} \) is one because the sum over \( i \in 1, \ldots, N \) of \( L_i^{d} \) is one. From this we conclude
\[ \lim_{\alpha \to \infty} \mu^{s_A}(\phi_A(A)) = L_\alpha^{d} \]
for every multi-index \( a \) of length \( M \). Because \( L_i^{d} = L_{s_{0A}}^{d} \), we have that
\[ \lim_{n \to \infty} \sum_a f(\phi_A(x)) \mu^{s_nA}(\phi_A(A)) = \sum_a f(\phi_A(x)) \mu^{s_nA}(\phi_A(A)) . \]
and so

$$\lim_{n \to \infty} \left| \int f \mu^{s \to A} - \int f d\lambda^d \right| < 2\varepsilon.$$  

The choice of $\varepsilon$ in (6) was arbitrary as was the choice of the continuous function $f$ and so $\lambda^d = \psi$ for any weak-star cluster point $\psi$, and hence $\mu^{s \to A} \rightharpoonup \lambda^d$ as $s \uparrow d$. □

**REFERENCES**

[1] T. Bedford and A. M. Fisher. Analogues of the Lebesgue density theorem for fractal sets of reals and integers. *Proc. London Math. Soc. (3)*, 64(1):95–124, 1992.

[2] M. Calef and D. Hardin. Riesz $s$-equilibrium measures on $d$-rectifiable sets as $s$ approaches $d$. *Potential Analysis*, 30(4):385–401, May 2009.

[3] K. J. Falconer. *Fractal geometry*. John Wiley & Sons Ltd., Chichester, 1990. Mathematical foundations and applications.

[4] K. J. Falconer. Wavelet transforms and order-two densities of fractals. *J. Statist. Phys.*, 67(3-4):781–793, 1992.

[5] B. Fuglede. On the theory of potentials in locally compact spaces. *Acta Math.*, 103:139–215, 1960.

[6] M. Hinz. Average densities and limits of potentials. Master’s thesis, Universität Jena, Jena, 2005.

[7] J. E. Hutchinson. Fractals and self-similarity. *Indiana Univ. Math. J.*, 30(5):713–747, 1981.

[8] N. S. Landkof. *Foundations of Modern Potential Theory*. Springer-Verlag, New York, 1973.

[9] P. Mattila. *Geometry of Sets and Measures in Euclidian Spaces*. Cambridge University Press, Cambridge, UK, 1995.

[10] P. A. P. Moran. Additive functions of intervals and Hausdorff measure. *Proc. Cambridge Philos. Soc.*, 42:15–23, 1946.

[11] N. Patzschke and M. Zähle. Fractional differentiation in the self-affine case. IV. Random measures. *Stochastics Stochastics Rep.*, 49(1-2):87–98, 1994.

[12] D. Preiss. Geometry of measures in $\mathbb{R}^n$: distribution, rectifiability, and densities. *Ann. of Math. (2)*, 125(3):537–643, 1987.

[13] M. Putinar. A renormalized Riesz potential and applications. In *Advances in constructive approximation: Vanderbilt 2003*, Mod. Methods Math., pages 433–465. Nashboro Press, Brentwood, TN, 2004.

[14] M. Zähle. The average density of self-conformal measures. *J. London Math. Soc. (2)*, 63(3):721–734, 2001.

[15] M. Zähle. Forward integrals and stochastic differential equations. In *Seminar on Stochastic Analysis, Random Fields and Applications, III (Ascona, 1999)*, volume 52 of *Progr. Probab.*, pages 293–302. Birkhäuser, Basel, 2002.

[16] M. Zähle. Riesz potentials and Liouville operators on fractals. *Potential Anal.*, 2:193–208, 2004.

[17] M. Zähle. Harmonic calculus on fractals—a measure geometric approach. II. *Trans. Amer. Math. Soc.*, 357(9):3407–3423 (electronic), 2005.

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