Stability properties of solitary waves for fractional KdV and BBM equations

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Abstract
This paper sheds new light on the stability properties of solitary wave solutions associated with Korteweg–de Vries-type models when the dispersion is very low. Using a compact, analytic approach and asymptotic perturbation theory, we establish sufficient conditions for the existence of exponentially growing solutions to the linearized problem and so a criterium of spectral instability of solitary waves is obtained for both models. Moreover, the nonlinear stability and spectral instability of the ground state solutions for both models is obtained for some specific regimen of parameters. Via a Lyapunov strategy and a variational analysis, we obtain the stability of the blow-up of solitary waves for the critical fractional KdV equation.

The arguments presented in this investigation show promise for use in the study of the instability of traveling wave solutions of other nonlinear evolution equations.

Keywords: orbital stability, linear instability, lower dispersion models, fKdV equations, fBBM equations
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1. Introduction
This paper provides a detailed study of various stability issues associated with the dynamic of solitary wave solutions for the so-called fractional Korteweg–de Vries (fKdV) equation

\[ u_t + u^pu_x - D^\alpha u_x = 0, \quad p \in \mathbb{N}, \]  

(1.1)
where \( u = u(x,t), \ x, t \in \mathbb{R} \) represents a real valued function, and \( D^\alpha \) is defined via Fourier transform by
\[
\hat{D^\alpha f}(\xi) = |\xi|^\alpha \hat{f}(\xi), \quad \alpha \in (0, 1).
\]

The importance of the study of this model for any \( \alpha > 0 \) is due to its physical relevance and its own mathematical interest. We recall that the model \((1.1)\) contains two famous families of equations—the generalized Korteweg–de Vries for \( \alpha = 2 \) \([5, 30]\), and the generalized Benjamin–Ono equation for \( \alpha = 1 \) \([5, 9, 31]\)—and, in this case, \( D \) can be written as \( D = \mathcal{H} \partial_x \), where \( \mathcal{H} \) denotes the Hilbert transform which may be defined by \( \mathcal{H} f(\xi) = -\text{sgn}(\xi) \hat{f}(\xi) \).

For \( \alpha \geq 1 \), studies of the Cauchy problem, blow-up issues, the large-time asymptotic behavior of solutions, the stability of solitary wave solutions, breather solutions and multi-solution solutions (as well as periodic traveling wave solutions) have been the focus of intense research over the past years using a rich variety of techniques (see, for instance, Albert \([1]\), Albert and Bona \([2]\), Albert et al \([3]\), Alejo and Muñoz \([4]\), Angulo \([5]\), Angulo et al \([7]\), Benjamin \([9, 10]\), Bona \([12]\), Bona et al \([14]\), Bona and Saut \([13]\), Iorio \([24]\), Grillakis et al \([19, 20]\), Kenig et al \([28]\), Lopes \([38]\), Martel and Merle \([40–42]\), Martel and Pilod \([43]\), Muñoz \([44]\), Weinstein \([51, 52]\)).

The case \( \alpha \in (0, 1) \) has been the focus of many recent studies. The Cauchy problem, the existence of solitary wave solutions, and the stability properties of ground states and numerical simulations have been addressed by Linares et al \([36, 37]\), Frank and Lenzmann \([16]\) and Klein and Saut \([29]\).

One of the objectives of this paper is to extend the theory of Vock and Hunziker in \([49]\) regarding the stability of Schrödinger eigenvalue problems to the study of the linear instability of solitary wave solutions for the fKdV equation with a ‘lower dispersion’ (see theorems 1.2–1.3 below). In particular, we recover the linear instability results in \([25]\) for the ground state solutions (see definition 1.1 below) of \((1.1)\) with \( p = 1 \) and \( \alpha \in (\frac{1}{2}, \frac{1}{2}) \). For completeness, we show in a unified way the nonlinear stability results for the ground state solutions with \( p < 2\alpha \) and \( \alpha \in (\frac{1}{2}, 2) \) (see theorem 1.1 below).

The case \( \alpha = \frac{1}{2} \) and \( p = 1 \) in \((1.1)\), the so-called critical case for the fKdV model, remains open for a stability analysis of solitary waves. Indeed, in this case, the recent numerical simulations in Klein and Saut \([29]\) suggest the existence of the blow up of solutions for initial data close to the solitary waves, but proving such a result seems to be out of reach. Here, we will show that for this critical case, a kind of ’stability of the blow-up’ near the possible unstable ground state solutions occurs and we checked one of the conjectures emerging from the numerical findings in \([29]\) (see theorem 3.1 below).

Our linear instability approach for the solitary wave solutions of \((1.1)\) is extended to the following generalized fractional Korteweg–de Vries (gKdV) models
\[
\frac{\partial u}{\partial t} + (f(u))_x - (\mathcal{M} u)_x = 0, \quad (1.2)
\]
where \( \mathcal{M} \) is a differential or pseudo-differential operator defined as a Fourier multiplier operator
\[
\hat{\mathcal{M}} g(\xi) = \beta(\xi) \hat{g}(\xi), \quad \xi \in \mathbb{R}, \quad (1.3)
\]
and \( f \) is assumed to be a smooth nonlinear function. The symbol \( \beta \) of \( \mathcal{M} \) (representing the lower dispersion effects) is assumed to be a continuous, locally bounded, even function on \( \mathbb{R} \), satisfying the conditions
\[
a_1 |\xi|^\gamma \leq \beta(\xi) \leq a_2 (1 + |\xi|)^\alpha,
\]
for $|\xi| \geq b_0$, $0 \leq \gamma \leq \alpha < 1$, with $\beta(\xi) > b$, for all $\xi \in \mathbb{R}$ and $a_i > 0$, $i = 1, 2$. At this point of the analysis, we extend the linear instability results in Lin [34] for models (1.2) with the growth of the symbol $M$ determined by $\alpha \in (0, 1)$.

We note that in various models of fluid dynamics and mathematical physics, the symbol $\beta$ in (1.3) is not necessarily polynomial, such as in the case of the Whitham equation describing water waves in the small amplitude and long wave regime when surface tension is included ([32, 33, 50])

$$\beta(\xi) = (1 + \gamma |\xi|^2)^{1/2} \left( \frac{\tanh(\xi)}{\xi} \right)^{1/2}, \quad (1.4)$$

where $\gamma \geq 0$ measures the surface tension effects. Here $\beta$ satisfies

$$\frac{1}{2} |\xi|^{1/2} \leq \beta(\xi) \leq 2 |\xi|^{1/2}, \quad \text{for } |\xi| \text{ large.}$$

The analysis established above for the fKdV equation (1.1) was also extended to the fractional BBM (fBBM) equation [11, 36]

$$u_t + u_x + \partial_x(u^2) + D^\alpha u_t = 0. \quad (1.5)$$

In this case we show for $\alpha \in (\frac{1}{3}, 1)$ that the ground state solutions associated with the fBBM equation (see (1.40)) are spectrally unstable if the wavespeed $c$ is in an interval $(1, c_0(\alpha))$ near $c = 1$. On the other hand, we prove nonlinear stability in the cases when $\alpha \in (\frac{1}{3}, \frac{1}{2})$ and $c > c_0(\alpha)$, and when $\alpha \in [\frac{1}{2}, 1)$ and $c > 1$ is arbitrary (see theorems 1.4 and 1.5 below).

Before establishing our results more precisely, we will provide a brief summary of some known results for the fKdV model, $\alpha \in (0, 1)$, which will be useful in our exposition. We start initially with some basic information regarding the existence of solitary wave solutions for this model. A solitary wave solution for (1.1) is a solution of the form $u(x,t) = \varphi_c(x - ct)$ with

$$\lim_{|\xi| \to \infty} \varphi_c(\xi) = 0,$$

which (if they exist!) will represent a ‘perfect’ balance between the lower dispersion and the effects of nonlinearity. Where $\varphi \equiv \varphi_c$ belongs to the space $\mathcal{H}^{\alpha/2}(\mathbb{R}) \cap L^{p+2}(\mathbb{R})$ we have that

$$D^\alpha \varphi + c \varphi - \frac{1}{p+1} \varphi^{p+1} = 0. \quad (1.6)$$

The existence of solutions for (1.6), with the later specified regularity conditions, can be deduced from the concentration-compactness method for any $c > 0$ and $p \in (1, \frac{2\alpha}{1-\alpha})$ (see Weinstein [52] and Arnesen [8]). We can also see (by the so-called Pohozaev identities) that the pseudo-differential equation satisfied by the profile $\varphi$ does not admit any non-trivial solutions for the following cases:

1. for $\alpha \geq 1$ and $c < 0$ (without restrictions on power $p$);
2. for $\alpha \in (0, 1)$, $c > 0$ and $\alpha \leq \frac{p}{p+2}$.

For completeness, we will establish item (2) above (item (1) is very well known). Indeed, for $\alpha \in (0, 1)$ we have from lemma B.2 in Frank and Lenzmann [16] that $\varphi \in \mathcal{H}^{\alpha+1}(\mathbb{R})$. Then by Plancherel theorem, the following energy identity is immediate:
\[
\int_{\mathbb{R}} |D^{\alpha/2} \varphi|^2 \, dx + c \int_{\mathbb{R}} \varphi^2 \, dx - \frac{1}{p + 1} \int_{\mathbb{R}} \varphi^{p+2} \, dx = 0.
\] (1.7)

Next, since \( \varphi' \) makes sense, we have by Plancherel and integration by parts that

\[
\int_{\mathbb{R}} x \varphi' D^{\alpha} \varphi \, dx = \frac{\alpha - 1}{2} \int_{\mathbb{R}} |D^{\alpha/2} \varphi|^2 \, dx.
\] (1.8)

Thus, from (1.7) and (1.8) follows

\[
(\alpha(p + 2) - p) \int_{\mathbb{R}} |D^{\alpha/2} \varphi|^2 \, dx = c \int_{\mathbb{R}} |\varphi|^2 \, dx,
\] (1.9)

proving that no finite energy solitary waves exist when \( c > 0 \) and \( \alpha \leq \frac{p}{p+2} \) hold.

The following definition will be useful in our study (see Frank and Lenzmann [16]).

**Definition 1.1.** Let \( Q \in H^{\alpha/2}(\mathbb{R}) \) be an even and positive solution of

\[
D^{\alpha} Q + Q - Q^{p+1} = 0 \quad \text{in} \quad \mathbb{R}.
\] (1.10)

If \( Q \) solves the minimization problem

\[
J^{\alpha,p}(Q) = \inf \{ J^{\alpha,p}(v) : v \in H^{\alpha/2}(\mathbb{R}) \setminus \{0\} \}
\] (1.11)

where \( J^{\alpha,p} \) is the ‘Weinstein’ functional

\[
J^{\alpha,p}(v) = \left( \int_{\mathbb{R}} |D^\frac{\alpha}{2} v|^2 \right)^{\frac{1}{\alpha}} \left( \int_{\mathbb{R}} |v|^p \right)^{\frac{\alpha}{p+2}}.
\] (1.12)

then we say that \( Q \in H^{\alpha/2}(\mathbb{R}) \) is a ground state solution of equation (1.10). Here, \( 0 < \alpha < 2 \) and \( 0 < p < p_{\max}(\alpha) \), and where the critical exponent \( p_{\max}(\alpha) \) is defined as

\[
p_{\max}(\alpha) \equiv \begin{cases} \frac{2\alpha}{4 - \alpha}, & \text{for } 0 < \alpha < 1, \\ +\infty, & \text{for } 1 \leq \alpha < 2. \end{cases}
\] (1.13)

From Frank and Lenzmann (proposition 1.1 and theorem 2.2 in [16]) there is a unique (modulo translation) ground state solution for (1.10). For \( p < p_{\max}(\alpha) \) and \( \alpha \in (0, 1) \), it is the so-called \( H^{\alpha/2} \)-subcritical case because this condition on \( p \) is necessary for the existence of solutions for (1.10) (see the analysis above). Thus, via a scaling argument, we obtain that the equation in (1.6) has a unique ground state solution, denoted by \( Q_c \). Moreover, we have the following regularity and decay properties for \( Q_c : Q_c \in H^{\alpha+1}(\mathbb{R}) \cap C^\infty(\mathbb{R}), \)

\[
\frac{C_0}{1 + |x|^\alpha} \leq |Q_c(x)| \leq \frac{C}{1 + |x|^\alpha}, \quad |x Q_c'(x)| \leq \frac{C}{1 + |x|^\alpha}, \quad \text{for all } x \in \mathbb{R},
\] (1.14)

with some constants \( C \geq C_0 > 0 \) depending of \( \alpha, p \) and \( c \).

The study of stability properties for the solitary wave profile \( \varphi \) in (1.6) for the \( \alpha \geq 1 \) case is well developed. Indeed, briefly, there are two useful lines of exploration to study this relevant property in the vicinity of the wave \( \varphi \). First, we have a global variational characterization of solutions of (1.6) such that a profile \( \varphi \) satisfying that \( \varphi > 0 \) on \( \mathbb{R} \), \( \varphi \) even and \( \varphi' < 0 \) on \( (0, +\infty) \) can be seen as the infima of the constrained-mass energy minimizer.
\[ J = \inf \{ E(v) : v \in H^{\alpha/2}(\mathbb{R}) \text{ and } \int_{\mathbb{R}} v^2 dx = \lambda \} \]  

(1.15)

with \( E(v) \) being the conservation-energy functional

\[ E(v) = \frac{1}{2} \int_{\mathbb{R}} |D^{\alpha/2} v|^2 - \frac{2}{(p + 1)(p + 2)} v^{p + 2} dx. \]  

(1.16)

We recall, since \( \alpha \geq 1 \), that the Sobolev embedding \( H^{\alpha/2}(\mathbb{R}) \hookrightarrow L^{p+2}(\mathbb{R}) \) ensures that the functional \( E \) is well-defined for any \( p \geq 0 \) and the infimum in (1.15) will satisfy \(-\infty < J < 0\) exactly for \( p < 2\alpha \) (the so-called \( L^2 \)-subcritical case). Thus, the concentration-compactness method will work very well for obtaining both existence and stability properties of \( \varphi \). More precisely, in this case we obtain the global stability property of the nonempty set of minimizer \( \mathcal{G} \) associated with the variational problem (1.15),

\[ \mathcal{G} = \{ v \in H^{\alpha/2}(\mathbb{R}) : E(v) = J \text{ and } \int_{\mathbb{R}} v^2 dx = \lambda \}. \]  

(1.17)

Thus, via a scaling argument and from the uniqueness results of the ground state solutions \( Q_\alpha \) of (1.6) for \( 1 \leq \alpha \leq 2 \) (see remark 2.1 in [16]), we obtain for a specific choice of \( \lambda \) in (1.17) that

\[ \mathcal{G} = \{ Q_\alpha(\cdot + y) : y \in \mathbb{R} \} \equiv \Omega_{Q_\alpha}, \]  

(1.18)

where \( \Omega_{Q_\alpha} \) is called the orbit generated by \( Q_\alpha \) via the basic symmetry of translations associated with the model (1.1). For the \( p \geq 2\alpha \) case, it is well known that the profile \( \varphi \) is nonlinearly unstable (see Bona et al [14] for \( p > 2\alpha (\alpha \geq 1) \), Martel and Merle [41, 42] for \( \alpha = 2, p = 4 \), and Merle and Pilod [43] for \( \alpha = 1, p = 2 \)). We note that by using a variational approach, it is also possible to obtain the instability result in the \( L^2 \)-supercritical case \( p > 2\alpha \geq 2 \) (see Angulo, [5], ch 10).

A similar approach of stability for the orbit \( \Omega_{Q_\alpha} \) has been used recently by Linares et al in [36], in the case \( \frac{1}{2} < \alpha < 1 \) and \( p < 2\alpha \) (here the condition that \( p \in \mathbb{N} \) requires us to take \( p = 1 \)). The concentration-compactness method was applied successfully to the minimizer problem in (1.15) and again the property \(-\infty < J < 0\) is necessary for the stability result. Regarding this point, it is also worth noting that for \( \alpha = \frac{1}{4} \) and \( p = 1 \), we have \( J = 0 \). Indeed, since \( \alpha = \frac{1}{4} \) falls into the \( H^{\alpha/2} \)-subcritical range \( \alpha > \frac{1}{4} \) for \( p = 1 \), the ground state for (1.6) in case \( \alpha = \frac{1}{4} \) and \( p = 1 \) is characterized (via a scaling) as the solution of the minimization problem \( J^{1/4} \) in (1.11). Therefore we obtain from \( J^{1/4}(Q_\alpha) = \frac{1}{4} \| Q_\alpha \| \equiv \frac{1}{4} (\int_{\mathbb{R}} |Q_\alpha(x)|^2 dx)^{1/2} \) the sharp inequality

\[ \frac{1}{3} \int_{\mathbb{R}} |v|^3 dx \leq \| v \| \| Q_\alpha \| \int_{\mathbb{R}} |D^{\alpha/2} v|^2 dx. \]  

(1.19)

Thus, for the restriction \( \| v \| = \| Q_\alpha \| \) we immediately obtain that \( E(v) \geq 0 \) and \( E(Q_\alpha) = 0 \). Moreover, from (1.19) the subsequent main property follows:

\[ \text{if } \| v \| \leq \| Q_\alpha \| \text{ then } E(v) \geq 0. \]  

(1.20)

We recall that the latter result is similar to that for \( \alpha \geq 1 \) and \( 2\alpha = p \), namely, \( J^{0,2\alpha}(Q_\alpha) = 0 \), the so-called \( L^2 \)-critical case (we note that for \( \alpha \in [1, 2] \) and \( |u|^2u \), as the nonlinear part in (1.1), recently Kenig et al in [27] have proved for \( \alpha \) close to 2, solutions of negative energy \( E \) close to the ground state blow up in finite or infinite time in the energy space \( H^2(\mathbb{R}) \)). The \( \alpha = \frac{1}{4} \) case is the so-called critical case for the fKdV model (1.1).
From the recent numerical study in Klein and Saut in [29], the simulations showed a possible blow-up phenomenon of the associated solutions for (1.1) with an initial data \( u_0 \) of negative energy \( (E(u_0) < 0) \) and therefore with a mass larger than the ground state mass \( Q_c \) (\( \|Q_c\| < \|u_0\| \)) (see figure 10 in [29]). Here we will show in theorem 3.1 below that, in fact, for this regimen of \( \alpha \) we have a kind of ‘stability of the blow-up’ near to the possible unstable ground state solutions, and so will check one of the conjectures emerging from the numerical findings in [29].

The second approach for an analysis of orbital stability is that of local type; more precisely, it is a fixed solitary wave profile \( \varphi_c \) of (1.6) and we study the behavior of the flow associated with (1.1) in a neighborhood of the orbit \( \Omega_{\varphi_c} \). The main property of the energy \( E \) to be obtained in this case is as follows:

\[
\begin{align*}
\text{There are } \delta > 0 \text{ and } \beta_0 > 0 \text{ such that} \\
E(v) - E(\varphi_c) \geq \beta_0 |d(v; \Omega_{\varphi_c})|^2 \\
\text{for } d(v; \Omega_{\varphi_c}) < \delta \text{ and } F(v) = F(\varphi_c),
\end{align*}
\]

with \( F(v) = \frac{1}{2} \int v^2 \, dx \) and \( d(v; \Omega_{\varphi_c}) = \inf_{\varphi \in \Omega} \|v - \varphi_c(\cdot + y)\|_{H^2} \). So, from (1.21), the continuity of the functional \( E \) and of the flow \( t \mapsto u(t) \), we immediately obtain the stability property of \( \Omega_{\varphi_c} \) by initial perturbations in the manifold

\[
\mathcal{M} = \{ v : \int_\mathbb{R} v^2 \, dx = \int_\mathbb{R} \varphi^2_c \, dx \}.
\]

The stability for general perturbations of \( \Omega_{\varphi_c} \) can be obtained via the existence of a regular curve of solitary waves, \( c \rightarrow \varphi_c \).

Now, a way of obtaining (1.21) is to use Taylor’s theorem, and so the analysis is reduced to studying the quadratic form \( \langle \mathcal{L}_c f, f \rangle \) on the tangent space to the manifold \( \mathcal{M} \) at the point \( \varphi_c \). \( \mathcal{L}_c \) represents the second variation of the action \( S(v) = E(v) + cF(v) \) at point \( v = \varphi_c \), namely, the unbounded self-adjoint operator

\[
S''(\varphi_c) \equiv \mathcal{L}_c = D^2 + c - \varphi^2_c
\]

with domain \( \mathcal{D}(\mathcal{L}_c) = H^2(\mathbb{R}) \). Thus, it is well known that by proving the inequality

\[
\langle \mathcal{L}_c f, f \rangle \geq \beta_1 \|f\|_{H^2}^2 \quad \text{for every } f \in \mathcal{T}_{\varphi_c} \mathcal{M} \cap \text{Ker}(\mathcal{L}_c)^\perp
\]

for \( \beta_1 > 0 \) and \( \text{Ker}(\mathcal{L}_c) \) representing the kernel of \( \mathcal{L}_c \), we obtain the key inequality (1.21). The inequality in (1.24) cannot, in general, be obtained immediately, for instance, the operator \( \mathcal{L}_c \) may have a nontrivial negative eigenspace. Indeed, since \( \varphi_c \) is a positive solitary wave solution, we immediately obtain \( \langle \mathcal{L}_c \varphi_c, \varphi_c \rangle < 0 \) and so the mini-max principle implies \( n(\mathcal{L}_c) \geq 1 \) (here \( n(\mathcal{L}_c) \) denotes the Morse index of \( \mathcal{L}_c \)). The works of Benjamin [10], Weinstein [51, 52] and Grillakis et al. [19] provide a nice test that guarantees when (1.24) is satisfied. More precisely, we suppose that \( n(\mathcal{L}_c) = 1, \text{Ker}(\mathcal{L}_c) = \text{span} \{ \varphi_c \} \) and the remainder of the spectrum of \( \mathcal{L}_c \) is positive and bounded away from zero. Then, the strictly increasing property of the mapping \( c \mapsto \int_\mathbb{R} \varphi^2_c \, dx \) will imply inequality (1.24) and so the stability property of \( \Omega_{\varphi_c} \) follows from (1.21).

Next, we turn our attention to the assumption of the existence of a \( C^1 \)-mapping \( c \rightarrow \varphi_c \) of solitary waves. If we assume this condition holds for every \( c > 0 \) and by considering the new variable
\[ \phi(x) = c^{-\frac{1}{2}} \varphi_c(c^{-\frac{1}{2}}x), \]

we see that \( \phi \) will be a solution of
\[ D^p \phi + \phi - \frac{1}{p+1} \phi^{p+1} = 0. \] (1.25)

Note the independence of \( \phi \) with regards to the wave-speed \( c \). Therefore,
\[ \frac{d}{dc} \int_{\mathbb{R}} \varphi^2_c dx = \| \varphi \|^2 \frac{d}{dc} c^{\frac{1}{2}-\frac{1}{p}} = \left( \frac{2}{p} - \frac{1}{\alpha} \right) c^{\frac{1}{2}-\frac{1}{p}-1} \| \varphi \|^2. \] (1.26)

Therefore,
\[ \frac{d}{dc} \int_{\mathbb{R}} \varphi^2_c dx > 0 \iff p < 2\alpha. \] (1.27)

Thus we see that the condition in (1.27) is the same as that imposed in order to obtain a minimum of the variational problem (1.15), at least for \( \alpha > \frac{1}{2} \), and therefore it is not just a technical condition.

Next, if we consider that the curve \( c \to \varphi_c \) has sufficient regularity, then by differentiating (1.6) with regards to the variable \( c \), we obtain that
\[ L_c \left( -\frac{d}{dc} \varphi_c \right) = \varphi_c. \] (1.28)

Now, if for some \( \psi \in D(L_c) \) we have that \( L_c \psi = \varphi_c \), then from (1.28) it follows that
\[ L_c \left( \frac{d}{dc} \varphi_c + \psi \right) = 0. \]

Hence, if we suppose that \( \text{Ker}(L_c) = \left\{ \frac{d}{dc} \varphi_c \right\} \) then \( \frac{d}{dc} \varphi_c + \psi = \theta \frac{d}{dc} \varphi_c \), for \( \theta \in \mathbb{R} \), and therefore
\[ \langle \psi, \varphi_c \rangle = -\frac{1}{2} \frac{d}{dc} \int_{\mathbb{R}} \varphi^2_c dx. \]

So, we have that the condition of strictly increasing the mapping \( c \to \int_{\mathbb{R}} \varphi^2_c dx \) can be replaced by the condition:
\[ \text{if } L_c \psi = \varphi_c, \text{ then } \langle \psi, \varphi_c \rangle = \langle L_c^{-1} \varphi_c, \varphi_c \rangle < 0. \] (1.29)

Condition (1.29) is useful in situations where the existence of a family of solitary waves \( \varphi_c \) depending smoothly on \( c \) (see Albert [1]) is not clear. We recall that as \( L_c \) is a self-adjoint operator and \( \varphi_c \in \text{Ker}(L_c) \), the Fredholm solvability theorem always guarantees the existence of an element \( \psi \in D(L_c) \) such that \( L_c \psi = \varphi_c \).

Before establishing our first stability result, let us define the orbital stability for equation (1.1). If \( \varphi \) is a given solitary wave solution of (1.6), define for any \( \eta > 0 \) the set \( U_\eta \subset H^2(\mathbb{R}) \) by
\[ U_\eta = \{ v \in H^2 : \inf_{y \in \mathbb{R}} \| v - \varphi(\cdot + y) \|_{H^2} < \eta \}. \]

**Definition 1.2.** \( \varphi \) is defined as (orbitally) stable in \( H^2 \) if
(i) there is a Banach space \( Y \subset H^2 \) such that for all \( u_0 \in Y \), there is a unique solution \( u \) of (1.1) in \( C(\mathbb{R}, Y) \subset C(\mathbb{R}, H^2) \) with \( u(x, 0) = u_0 \), and
(ii) for every \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that for all \( u_0 \in U_\delta \cap Y \), the solution \( u \) of (1.1) with \( u(x, 0) = u_0 \) satisfies \( u(t) \in U_\epsilon \) for all \( t \in \mathbb{R} \).

If it is only asserted that \( u \in C((-T^*, T^*); Y) \subset C((-T^*, T^*); \mathcal{H}^2) \), where \( T^* \) is the maximal time of existence of \( u \) in \( Y \), and that (ii) holds for all \( t \in (-T^*, T^*) \), then we say that \( \varphi \) is conditionally stable in \( \mathcal{H}^2 \).

Our first theorem of orbital stability for (1.1) with a ‘lower’ dispersion (more precisely, of conditional type for \( \alpha \in (\frac{1}{2}, 1) \)—see remark 2.1 below) is as follows [1, 14, 36].

**Theorem 1.1 (Nonlinear stability of the ground state).** Let \( \frac{1}{2} < \alpha < 2 \) and \( 0 < p < p_{\text{max}}(\alpha) \). Then the ground state solution \( Q_c \) for equation (1.6) is \( \mathcal{H}^\infty(\mathbb{R}) \)-stable by the flow of equation (1.1) for all \( c > 0 \) and \( p < 2\alpha \).

Now, with regards to the stability (linear instability) properties of the solitary waves for (1.6) in the \( \alpha \in (\frac{1}{2}, 1] \) case, and \( p = 1 \) in (1.1), by considering the new variable

\[
w(x, t) = u(x + ct, t) - \varphi_c(x)
\]

into the fKdV equation and using equation (1.6) satisfied by \( \varphi_c \), one finds that \( w \) satisfies the nonlinear equation

\[
(\partial_t - c\partial_x)w + \partial_x(ww - D^2w + O(\|w\|^2)) = 0.
\]  
(1.30)

As a leading approximation for small perturbation, we replace (1.30) by its linearization around \( \varphi_c \), and hence obtain the linear equation

\[
(\partial_t - c\partial_x)w + \partial_x(ww - D^2w) = 0.
\]  
(1.31)

Since \( \varphi_c \) depends on \( x \) but not \( t \), the equation (1.31) admits treatment by the separation of variables, which leads naturally to a spectral problem. Seeking particular solutions of (1.31) of the form \( w(x, t) = e^{\lambda t}u(x) \) (so-called growing mode solution), where \( \lambda \in \mathbb{C}, u \) satisfies the linear problem

\[
(\lambda - c\partial_x)u + \partial_x(uu - D^2u) = 0.
\]  
(1.32)

We can say from (1.32) that the complex growth rate \( \lambda \) appears as a (spectral) parameter for the extended eigenvalue problem

\[
\partial_x \mathcal{L}_c u = \lambda u,
\]  
(1.33)

with \( \mathcal{L}_c \) defined in (1.23) with \( p = 1 \). If equation (1.33) has a nonzero solution \( u \in D(\mathcal{L}_c) = \mathcal{H}^s(\mathbb{R}) \) then a bootstrapping argument shows that \( u \in \mathcal{H}^s(\mathbb{R}) \) for all \( s \geq 1 \), so that (1.33) is satisfied in a classical sense. A necessary condition for the ‘stability’ of \( \varphi_c \) is that there are no points \( \lambda \) with \( \text{Re}(\lambda) > 0 \) (which would imply the existence of a solution \( u = u(x) \) of (1.33) that grows exponentially in time). If we denote by \( \sigma \) the ‘spectrum’ of \( \partial_x \mathcal{L}_c \) (namely, \( \lambda \in \sigma \) if there is a \( u \neq 0 \) satisfying (1.33)), the later discussion suggests the utility of the following definition.

**Definition 1.3 (Spectral stability and instability).** A solitary wave solution \( \varphi_c \) of the fKdV equation (1.1) is said to be spectrally stable if \( \sigma \subset \mathbb{R} \). Otherwise (i.e. if \( \sigma \) contains a point with \( \text{Re}(\lambda) > 0 \)), \( \varphi_c \) is spectrally unstable.

We recall that as (1.31) is a real Hamiltonian equation, it forces certain elementary symmetries on the spectrum of \( \sigma \). More precisely, \( \sigma \) will be symmetric with respect to the reflection in the real and imaginary axes. Therefore, it implies that exponentially growing perturbations are
always paired with exponentially decaying ones. It is the reason why, in definition 1.3, only the spectral parameter $\lambda$ satisfying $\text{Re}(\lambda) > 0$ was required.

A similar spectral problem to (1.33) for traveling wave solutions (solitary or periodic) has been the focus of many research studies in the last years (see Grillakis et al [20], Lopes [39], Lin [34], Lin and Zeng [35], Kapitula and Stefanov [25], among others).

Our spectral instability result for the fKdV equation (1.1) is as follows.

**Theorem 1.2 (Spectral instability criterium for fKdV equations).** Let $c \rightarrow \varphi_c \in H^{\alpha+1}(\mathbb{R})$ be a smooth curve of positive solitary wave solutions to equation (1.6) with $\alpha \in (\frac{1}{3}, \frac{1}{2})$, $p = 1$. The wave-speed $c$ can be considered over some nonempty interval $I$, $I \subset (0, +\infty)$. We assume that the self-adjoint operator $L_c = D^{\alpha} + c - \varphi_c$ with domain $D(L_c) = H^\alpha(\mathbb{R})$ satisfies

$$\text{Ker}(L_c) = \left[\frac{d}{dx} \varphi_c \right].$$

(1.34)

Denote by $n(L_c)$ the number (counting multiplicity) of negative eigenvalues of the operator $L_c$. Then there is a purely growing mode $e^{\lambda t}u(x)$ with $\lambda > 0$, $u \in H^s(\mathbb{R}) - \{0\}$, $s \geq 0$, to the linearized equation (1.31) if one of the following two conditions is true:

(i) $n(L_c)$ is even and $\frac{d}{dc} \langle \varphi_c, \varphi_c \rangle > 0$.

(ii) $n(L_c)$ is odd and $\frac{d}{dc} \langle \varphi_c, \varphi_c \rangle < 0$.

The proof of the instability criterium established in theorem 1.2 is based on the compactness of some specific commutators associated with the family $A^\lambda$ defined in (2.9) below, and the analytic and asymptotic perturbation theory for linear operators. These approaches can also be applied to the general model (1.2) with a linear operator $M$ of ‘lower dispersion’ under some specific conditions of the symbol $\beta$. Also, our approach can be extended to the case of periodic traveling wave solutions associated with the model (1.2) (a work in progress).

**Remark 1.1.**

(1) Conditions (i) and (ii) in theorem 1.2 are similar to that obtained in Lin [34] for the case of (1.6) with $\alpha \geq 1$. Recently, in Lin and Zeng [35], an instability index theorem was deduced, which, together with conditions (i) and (ii), implies spectral instability results. A similar situation occurs with the instability index theory established by Kapitula and Stefanov in [25].

(2) Our approach provides the existence of a nonzero solution $u \in D(L_c) = H^s(\mathbb{R})$ satisfying the eigenvalue problem (1.33), via a different approach than that given in Lin and Zeng [35], Kapitula and Stefanov [25] and Pelinovsky [46].

(3) If there is $\psi \in D(L_c)$ such that $L_c \psi = \varphi_c$, conditions (i) and (ii) in theorem 1.2 can be changed by:

(i) $n(L_c)$ is even and $\langle \psi, \varphi_c \rangle < 0$.

(ii) $n(L_c)$ is odd and $\langle \psi, \varphi_c \rangle > 0$.

(4) The former criterium (3) is very useful when we do not have a smooth curve $c \rightarrow \varphi_c$ of solitary waves.

As a consequence of theorem 1.2, we obtain the following instability result for the ground state solutions of equation (1.6) (see [25]).

**Corollary 1.1 (Spectral instability of ground state for fKdV equations).** For $\alpha \in (\frac{1}{3}, \frac{1}{2})$, $p = 1$ and $c > 0$, the ground state profiles $Q_c$ for (1.6) are spectrally unstable.
Next, we consider \( u(x,t) = \psi_c(x - ct) \) a solitary wave solution for the model (1.2). Then \( \psi_c \) satisfies
\[
\mathcal{M}\psi_c + c\psi_c - f(\psi_c) = 0,
\]
and, similar to model (1.1), we also have a linearized equation around \( \psi_c \)
\[
(\partial_t - c\partial_x)u + \partial_x (f'(\psi_c)u - Mu) = 0.
\]
In order to obtain a growing mode solution of the form \( e^{\lambda t}w(x) \), \( \Re\lambda > 0 \), function \( w \) must satisfy
\[
(\lambda - c\partial_x)w + \partial_x (f'(\psi_c)w - Mw) = 0. \tag{1.37}
\]
Then, similar to model (1.1), we obtain the following spectral instability result for the \( \text{gfKdV} \) equation (1.2), provided that the symbol \( \beta \) defining the pseudo-differential operator \( \mathcal{M} \) satisfies the following condition for \( \alpha \in (0, 1) \):
\[
\text{if } \eta(\xi) \equiv \beta(\xi) - |\xi|^\alpha \text{ then } \eta' \in L^2(\mathbb{R}). \tag{1.38}
\]

**Theorem 1.3 (Spectral instability criterium for \( \text{gfKdV} \) equation).** Let \( c \in I \subset (0, +\infty) \rightarrow \psi_c \in H^{\alpha+1}, \ 0 < \alpha < 1 \) be a smooth curve of positive solitary wave solutions to equation (1.35). We assume condition (1.38) and that the self-adjoint operator \( \mathcal{N}_c = \mathcal{M} + c - f'((\psi_c) \) with domain \( D(\mathcal{N}_c) = H^\alpha(\mathbb{R}) \) satisfies
\[
\ker(\mathcal{N}_c) = [d/dx \psi_c]. \tag{1.39}
\]
Denote by \( n(\mathcal{N}_c) \) the number (counting multiplicity) of negative eigenvalues of the operator \( \mathcal{N}_c \). Then there is a purely growing mode \( e^{\lambda t}w(x) \) with \( \lambda > 0 \), \( w \in H^s(\mathbb{R}) - \{0\} \), \( s \geq 0 \) to the linearized equation (1.36) if one of the following two conditions is true:

(i) \( n(\mathcal{N}_c) \) is even and \( \frac{d}{dc} \langle \psi_c, \psi_c \rangle > 0 \).

(ii) \( n(\mathcal{N}_c) \) is odd and \( \frac{d}{dc} \langle \psi_c, \psi_c \rangle < 0 \).

The proof of theorem 1.3 follows the same lines of that for theorem 1.2, but because of the generality of the symbol associated with the operator \( \mathcal{M} \), some points in the analysis need to be treated carefully. We note that the spectral instability approach in Kapitula and Stefanov [25] was applied to the general model (1.2) and they consider the condition-convergence of the specific integral (see proposition 17 in [25])
\[
\int_{-\infty}^{\infty} \frac{|\beta'(\xi)|}{(c + \beta(\xi))^2} d\xi < \infty, \quad \text{with } c > 0,
\]
for applying the Hamiltonian–Krein index theory. Our condition (1.38) in theorem 1.3 was not considered by them.

The analysis established above for the \( \text{fKdV} \) equation (1.1) can be extended to the \( \text{fBBM} \) equation (1.5) for \( \alpha \in (\frac{1}{3}, 1) \). In section 3 below, we show the following stability properties associated with the ground state solutions \( \Phi_c \) satisfying
\[
D^\alpha \Phi_c + \left( 1 - \frac{1}{c} \right) \Phi_c - \frac{1}{c} \Phi_c^2 = 0, \quad c > 1. \tag{1.40}
\]

**Theorem 1.4 (Nonlinear stability of the ground state for the \( \text{fBBM} \)).** Let \( \frac{1}{3} < \alpha < 1 \) and \( c > 1 \). Then, the ground state solution \( \Phi_c \) of (1.40) is \( H^2(\mathbb{R}) \)-stable by the flow of equa-
tion (1.5) provided $\alpha \in \left(\frac{1}{2}, 1\right)$ and $c > 1$, and for $\alpha \in \left(\frac{1}{2}, \frac{1}{3}\right)$ and $c > c_0$. Here $c_0$ is given by

$$c_0 = \frac{2 + \sqrt{2(3\alpha - 1)}}{6\alpha} > 1.$$ 

Our nonlinear stability results for the fBBM equation extends and complements those in Linares et al [36] in the sense that we show the stability of the orbit $\Omega_{\Phi_c} = \{\Phi_c(\cdot + y) : y \in \mathbb{R}\}$ for $\alpha = \frac{1}{2}$ and $c > 1$, and for $\frac{1}{4} < \alpha < \frac{1}{2}$ with the specific restriction on the wave velocity $c$. It also confirms the numerical simulations in Klein and Saut [29] regarding the stability of the solitary waves in this regimen of $\alpha$’s. Similar to the fKdV equation, the statement of orbital stability in theorem 1.4 is a conditional one (see remark 4.1 below).

**Theorem 1.5 (Spectral instability of ground state for fBBM equations).** For $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$ and $c \in (1, c_0)$, the ground state profiles $\Phi_c$ for (1.40) are spectrally unstable.

Our paper is organized as follows. In section 1 we prove theorem 1.1 and present the proof of the spectral instability criterium in theorem 1.2 for the fKdV model (1.1) and the criterium for the general dispersive equation (1.2) in theorem 1.3. In section 2, we prove our ‘stability of the blow-up’ for the critical fKdV equation (1.1) ($\alpha = \frac{1}{2}$, $p = 1$). In the final section 3, we prove the results of nonlinear stability and linear instability for the fBBM equation (1.5) (theorems 1.4 and 1.5).

**Notation.** We will denote $|\cdot|_p$ as the norm in Lebesgue space $L^p(\mathbb{R}), 1 \leq p \leq \infty$ and $\|\cdot\|_s$ as the norm in Sobolev space $H^s(\mathbb{R})$, $s \in \mathbb{R}$. For $X, Y$ Banach spaces, $B(X, Y)$ represents the set of bounded linear operators $F : X \to Y$, $[A, B] = AB - BA$ represents the commutator of the operators $A$ and $B$, and $\rho(A)$ will represent the resolvent of the linear operator $A$.

2. Nonlinear stability and spectral instability for the fKdV equation

This section is devoted to showing theorems 1.1 and 1.2 established in the introduction. The proof of the nonlinear stability of the ground state is a consequence of the results of Grillakis et al [19] and Frank and Lenzmann [16]. For the linear instability result we extend the theory of Vock and Hunziker in [49] regarding the stability of Schrödinger eigenvalue problems to the study of the spectral instability of solitary wave solutions for the fKdV type model in (1.1) with a ‘lower dispersion’. In particular, we recover the spectral instability results in [25] for the ground state solutions associated with equation (1.6) with $p = 1$ and $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$. Our analysis is also extended to the general lower-dispersion models (1.2).

2.1. Nonlinear stability of ground state for the fKdV equation

In the following, we show theorem 1.1, and we recall that in the literature, the result of stability has been shown using different methods [1, 14, 36]. Here, for completeness, we show this in a unified way for $\alpha \in \left(\frac{1}{4}, 2\right)$.

**Proof of theorem 1.1.** Let $Q$ be the ground state solution for (1.10), namely, $Q = Q(|x|) > 0$, which satisfies

$$D^\alpha Q + Q - Q^2 = 0,$$  \hspace{1cm} (2.1)

and a minimum for the functional $J^{m,1}$ in (1.11). Thus we obtain that the self-adjoint operator,

$$L_1 = D^\alpha + 1 - 2Q$$  \hspace{1cm} (2.2)
satisfies the so-called nondegeneracy property, namely, \( \text{Ker}(L_1) = [d/d\eta Q] \). Moreover, since \( \langle L_1 Q, Q \rangle \leq 0 \) and for \( \eta \in C_0^\infty(\mathbb{R}) \)
\[ \langle L_1 \eta, \eta \rangle \geq 0, \quad \text{for all } \eta \perp Q^2, \]
we have \( n(L_1) = 1 \) (see [16]). Now, \( R = \alpha Q + xQ' \in L^2(\mathbb{R}) \) (see (1.14))
follows \( L_1 R = -\alpha Q \)
(at least in the distributional sense). Thus a bootstrapping argument shows that \( R \in H^{\alpha+1}(\mathbb{R}) \)
and so \( R \in D(L_1) = H^\alpha(\mathbb{R}) \).

Next, for any real number \( \theta \neq 0 \), define the dilation operator \( T_\theta \) by \( (T_\theta f)(x) = f(\theta x) \). Then, via elementary scaling \( Q_c(x) = 2cQ(c^{1/\alpha}x) \) and the relation \( D^\alpha(T_\theta f)(x) = \theta^\alpha D^\alpha f(\theta x) \),
we can show that for \( \theta = c^{1/\alpha} \) we obtain that \( Q_c \)
satisfies
\[ D^\alpha Q_c + cQ_c - \frac{1}{2} Q_c^2 = 0, \quad (2.3) \]
and so, we obtain its linearized operator
\[ L_c = D^\alpha + c - Q_c. \quad (2.4) \]

Now, the relation \( L_c = c T_{\theta c} L_{\theta c}^{-1} \) implies with \( \theta = c^{1/\alpha} \) that \( \text{spec}(L_c) = \{ cr : r \in \text{spec}(L_1) \} \)
and therefore \( L_c \) and \( L_1 \) have the ‘same structure’. Thus, \( \psi \) is an eigenfunction of \( L_c \) with eigenvalue \( \lambda \) if and only if \( T_\theta \psi \) is an eigenfunction of \( L_c \) with eigenvalue \( c \lambda \). Then, we conclude immediately that \( n(L_c) = 1 \) and \( \text{Ker}(L_c) = [\frac{Q_c}{c}] \). Thus, since \( R_c = \alpha Q_c + xQ'_c \in D(L_c) \)
with \( L_c R_c = -\alpha Q_c \) (where we used \( D^\alpha(xQ') = \alpha D^\alpha Q + xD^\alpha Q' \)) we obtain
\[ \langle L_c^{-1} Q_c, Q_c \rangle = -\frac{1}{\alpha c} \langle R_c, Q_c \rangle = \|Q_c\|^2 \left[ \frac{1}{2\alpha c} - \frac{1}{c} \right] < 0, \quad (2.5) \]
where we use integration by parts \( (xQ_c^2(x) \to 0 \text{ as } |x| \to +\infty) \) and \( \alpha > \frac{1}{2} \). Hence, from the regularity properties of the curve \( c \to Q_c \) (see proof of corollary 1.1 below) and from the Lyapunov property of the energy \( E \) in (1.21), we complete the proof.

\textbf{Remark 2.1.} The statement in theorem 1.1 deserves to be clarified, at least with reference to some points, regarding the Cauchy problem.

(1) For \( 2 > \alpha > 1 \), the solutions of the Cauchy problem are global in \( H^\frac{\alpha}{2}(\mathbb{R}) \) and so the stability result is not conditional (see definition 1.2). Indeed, by using a similar strategy to that in the proof of theorem 1 in Kenig \textit{et al} [27] we obtain local well-posedness of the model in (1.1) for all initial data \( u_0 \in H^\frac{\alpha}{2}(\mathbb{R}) \) for \( p < 2\alpha \) (the case of the critical-nonlinearity \( |u|^{2\alpha}u \) was studied in [27]). Moreover, the conservation of the energy \( E \) in (1.16) and the charge \( F(u) = \int_{\mathbb{R}} u^2 dx \) by the flow of (1.1), together with an application of the Gagliardo–Nirenberg type inequality (see (1.12))
\[ \int_{\mathbb{R}} |u|^{p+2} \leq C_{\alpha,p} \left( \int_{\mathbb{R}} |D^\frac{\alpha}{2} u|^2 \right)^\frac{\alpha}{2} \left( \int_{\mathbb{R}} |u|^2 \right)^\frac{\alpha-1}{2}, \quad (2.6) \]
gives us exactly the ‘a priori’ estimative for \( p < 2\alpha \),
\[ \|D^\frac{\alpha}{2} u\|^2 \leq E(u_0) + C_{\alpha,p} \|D^\frac{\alpha}{2} u\|^\frac{\alpha}{2} \|u\|^\alpha \leq C(|u_0|^{2\alpha} + \frac{D^\frac{\alpha}{2}}{2\alpha} \|u_0\|^{p+2} + \frac{D^2}{2\alpha} \|u\|^2) \quad (2.7) \]
with \( \beta = \frac{p}{\alpha} (\alpha - 1) + 2 \) and \( r = \frac{2\alpha}{\alpha - p} \). Thus, the global-posedness of the initial valued problem for (1.1) follows in the energy space \( H^2(\mathbb{R}) \).

(2) Herr et al [21] showed that for \( \alpha \in (1, 2) \), the solutions of the Cauchy problem for (1.1) with \( p = 1 \) are globally well-posed in \( L^2(\mathbb{R}) \).

(3) The \( \alpha = 1 \) case \( (p = 1) \), that is, the Benjamin–Ono equation was shown to be globally well-posed in \( H^0(\mathbb{R}) \) for \( s \geq 0 \) by Ionescu and Kenig [23].

(4) The \( \alpha \in (\frac{1}{2}, 1) \) case is more delicate with regards to the local and global well-posedness problem. Saut [47] proved that (1.1) admits global weak solutions (without uniqueness) in the space \( L^\infty(\mathbb{R}; H^{-2}(\mathbb{R})) \) and global weak solutions in \( L^\infty(\mathbb{R}; L^2(\mathbb{R})) \cap L^2_{loc}(\mathbb{R}; H^2_{loc}(\mathbb{R})) \) in Ginibre and Velo [17, 18].

(5) The best known result of local well-posedness for (1.1) has been established by Linares et al [36] in \( H^s(\mathbb{R}) \), for \( s > s_0 \equiv \frac{1}{2} - \frac{3\alpha}{4} > \frac{3}{4} \), for \( \alpha \in (0, 1) \), which does not allow the globalization of the solution using conservation laws.

(6) Proving local well-posedness in \( H^2(\mathbb{R}) \) in the \( \alpha \in [\frac{1}{2}, 1) \) case, which would imply global well-posedness by using the conserved quantities \( E \) and \( F \), is still an open problem.

(7) Therefore, the statement of stability in theorem 1.1 for \( \alpha \in (\frac{1}{2}, 1) \) is of a conditional type by definition 1.2, where we have used \( Y = H^0(\mathbb{R}), s > s_0 \geq \frac{1}{2}, and so for all \( \epsilon > 0 \) there is a \( \delta > 0 \) such that if \( u_0 \in H^s(\mathbb{R}) \cap U_{\delta}, then u(t) \in U_{\epsilon}, for all \( t \in (-T_{\epsilon}, T_{\epsilon}) \), where \( T_{\epsilon} \) is the maximal time of existence of \( u \) satisfying \( u(0) = u_0 \).

2.2. Spectral instability criterion for the fKdV equation

In order to illustrate the strategy for obtaining a growing mode solution of (1.31) with the form \( w(x, t) = e^{\lambda t}u(x) \) and \( \text{Re} \lambda > 0 \), we can see the eigenvalue problem (1.32) for \( \lambda \) and \( u \) rewritten in the form

\[
cu + \frac{c\partial_x}{\lambda - c\partial_x}(\varphi, u - D^\alpha u) = 0. \tag{2.8}
\]

Here, the expression \( \frac{\partial}{\partial_x} \) is a notation for the well-defined linear operator \( \partial_x (\lambda - c\partial_x)^{-1} \) and \( \varphi \) is any positive solitary wave solution for (1.6). Thus, if we consider the following family of closed linear operators \( A^\lambda : H^0(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \), with \( \text{Re} \lambda > 0 \),

\[
A^\lambda \varphi \equiv cu + \frac{c\partial_x}{\lambda - c\partial_x}(\varphi, u - D^\alpha u), \tag{2.9}
\]

it follows immediately that the solution of the eigenvalue problem (2.8) is reduced to finding \( \lambda \in \mathbb{C} \) with \( \text{Re} \lambda > 0 \) such that the operator \( A^\lambda \) possesses a nontrivial kernel. Now, we note from the analyticity of the resolvent associated with the operator \( \partial_x, \lambda \in S \rightarrow (\lambda - c\partial_x)^{-1} \), for \( S = \{ z \in \mathbb{C} : \text{Re} z > 0 \} \), and we obtain that the mapping \( \lambda \in S \rightarrow A^\lambda \) represents an analytical family of operators of type-A (see Kato [26]), namely,

(1) \( D(A^\lambda) = H^0(\mathbb{R}) \) for all \( \lambda \in S \),

(2) for \( u \in H^s(\mathbb{R}), \lambda \in S \rightarrow A^\lambda u \) is analytic in the topology of \( L^2(\mathbb{R}) \).

Therefore, from classical analytic perturbation theory, all discrete eigenvalues of \( A^\lambda \) (\( \text{Re} (\lambda) > 0 \)) will be stable: for \( \eta \) in the discrete spectrum of \( A^\lambda \), there is \( \delta > 0 \) such that for \( \lambda_0 \in B(\lambda; \delta) \), \( A^{\lambda_0} \) has \( \eta(\lambda_0) \) eigenvalues close to \( \eta \) with total algebraic multiplicity equal to that of \( \eta \).
In our approach, we will find a growing mode solution for \( \lambda > 0 \). Indeed, since we have that
\[
A^\lambda \longrightarrow \mathcal{L}_c \equiv D^\alpha + c - \varphi_c \quad \text{as} \quad \lambda \to 0^+,
\]
strongly in \( L^2(\mathbb{R}) \) (see proposition 2.1 below), we will use similar asymptotic perturbation arguments as Vock and Hunziker in [49] and Lin in [34] (see also Hislop and Sigal in [22]) for obtaining our criterion established in theorem 1.2. In our analysis, it will prove decisive to count the number of eigenvalues of \( A^\lambda \) (for \( \lambda \) small) in the left-half plane (for \( \lambda \) large, there are no growing modes—see lemma 2.3 below), and so we will need to know how the zero eigenvalue of \( \mathcal{L}_c \) will be perturbed. To this end, we obtain a moving kernel formula (see lemma 2.6 below) which will decide whether zero jumps to the left or right. Thus we obtain conditions (i) and (ii) in theorem 1.2.

As the structure of the proof of theorem 1.2 follows some ideas used by Lin in [34] for the case \( \alpha \geq 1 \), we will only indicate the new basic differences due to the structure of the operator \( A^\lambda \) defined in (2.9) for \( \alpha \in (0, 1) \).

### 2.2.1. Stability of the discrete spectrum of \( \mathcal{L}_c \), with \( \alpha \in (0, 1) \).

In this section we study the behavior of the family \( A^\lambda \) by depending on \( \lambda \). In particular, we show that every discrete eigenvalue of the limiting operator \( \mathcal{L}_c = D^\alpha + c - \varphi_c \) is stable with respect to the family \( A^\lambda \) for small positive \( \lambda \). Our first result concerns the strong convergence of \( A^\lambda \).

**Proposition 2.1.** For \( \lambda > 0 \), the operator \( A^\lambda \) converges to \( \mathcal{L}_c \) strongly in \( L^2(\mathbb{R}) \) when \( \lambda \to 0^+ \).

**Proof.** For \( \lambda > 0 \) and \( v \in D(A^\lambda) = D(\mathcal{L}_c) \) we have the relation
\[
(A^\lambda - \mathcal{L}_c)v = \frac{\lambda}{\lambda^2 + c^2} (\varphi_c - D^\alpha)v.
\]
Thus, by Plancherel the dominated convergence theorem follows
\[
\| (A^\lambda - \mathcal{L}_c)v \|^2 = \int_{\mathbb{R}} \frac{\lambda^2}{\lambda^2 + c^2} \left| \hat{\varphi}_c(\xi) - \hat{D}^\alpha(\xi) \right|^2 d\xi \to 0
\]
when \( \lambda \to 0^+ \). \( \square \)

Next, we localized the essential spectrum of \( A^\lambda \), \( \sigma_{\text{ess}}(A^\lambda) \). We will see that the set is situated in the right-hand side of the complex-plane and away from the imaginary axis. We begin with the following two basic definitions relating to the \( \sigma_{\text{ess}}(A^\lambda) \) (see Hislop and Sigal [22]).

**Definition 2.1 (Zhislin spectrum \( Z(A^\lambda) \)).** A Zhislin sequence for \( A^\lambda \) and \( z \in \mathbb{R} \) is a sequence
\[
\{u_n\} \subset H^\alpha(\mathbb{R}), \quad \|u_n\| = 1, \quad \text{supp} \ u_n \subset \{x : |x| \geq n\}
\]
and \( \|(A^\lambda - z)u_n\| \to 0 \) as \( n \to +\infty \).

The set of all \( z \) such that a Zhislin sequence exists for \( A^\lambda \) and \( z \) is denoted by \( Z(A^\lambda) \).
Remark 2.2. Every Zhislin sequence \( \{u_n\} \) necessarily converges weakly to zero in \( L^2(\mathbb{R}) \).

Definition 2.2 (Weyl spectrum \( W(\mathcal{A}^\lambda) \)). A Weyl sequence for \( \mathcal{A}^\lambda \) and \( z \in \mathbb{R} \) is a sequence
\[
\{u_n\} \subset H^\alpha(\mathbb{R}), \quad \|u_n\| = 1, \quad u_n \to 0 \text{ weakly in } L^2(\mathbb{R}),
\]
and \( \|(\mathcal{A}^\lambda - z)u_n\| \to 0 \) as \( n \to +\infty \).

The set of all \( z \) such that a Weyl sequence exists for \( \mathcal{A}^\lambda \) and \( z \) is denoted by \( W(\mathcal{A}^\lambda) \).

From the last two definitions we have the following result (see [22]).

Proposition 2.2. \( Z(\mathcal{A}^\lambda) \subset W(\mathcal{A}^\lambda) \subset \sigma_{\text{ess}}(\mathcal{A}^\lambda) \) and \( \partial(\sigma_{\text{ess}}(\mathcal{A}^\lambda)) \subset W(\mathcal{A}^\lambda) \).

Our main result regarding the \( \sigma_{\text{ess}}(\mathcal{A}^\lambda) \) is as follows.

Proposition 2.3. For any \( \lambda > 0 \), we have
\[
\sigma_{\text{ess}}(\mathcal{A}^\lambda) \subset \{z : \Re z \geq \frac{1}{2c}\}. \tag{2.10}
\]

The idea of the proof of proposition 2.3 will be to see \( W(\mathcal{A}^\lambda) = Z(\mathcal{A}^\lambda) \) and it will be based on the following two lemmas.

Lemma 2.1. For any \( \lambda > 0 \), we have
\[
Z(\mathcal{A}^\lambda) \subset \{z : \Re z \geq \frac{1}{2c}\}. \tag{2.11}
\]

Proof. Let \( z \in Z(\mathcal{A}^\lambda) \) and suppose \( \Re z < \frac{1}{4c} \). It follows immediately from the Fourier transform that for any \( u \in H^\alpha(\mathbb{R}) \) we have
\[
I_0(u) \equiv \Re \langle -\frac{c\partial_x}{\lambda - c\partial_x}D^\alpha u, u \rangle \geq 0.
\]
Then, for any sequence \( \{u_n\} \subset H^\alpha(\mathbb{R}), \|u_n\| = 1, \) and satisfying \( \text{supp } u_n \subset \{x : |x| \geq n\} \), we have from the following trivial estimative for any \( c \) and \( \lambda \)
\[
\left\| \frac{c\partial_x}{\lambda + c\partial_x} u_n \right\| \leq \|u_n\| = 1,
\]
that for \( n \) large,
\[
\Re \langle (\mathcal{A}^\lambda - z)u_n, u_n \rangle \geq I_0(u_n) + c - \Re z + \Re \langle \frac{c\partial_x}{\lambda + c\partial_x}(u_n\varphi_c), u_n \rangle
\]
\[
\geq c - \Re z - \Re \langle u_n\varphi_c, \frac{c\partial_x}{\lambda + c\partial_x} u_n \rangle
\]
\[
\geq c - \Re z - \sup_{|x| \geq n} |\varphi_c(x)| \geq c - \frac{1}{4c} - \frac{1}{2}c = \frac{1}{4}c.
\]
Then, since \( |\Re \langle (\mathcal{A}^\lambda - z)u_n, u_n \rangle| \leq \|(\mathcal{A}^\lambda - z)u_n\| \), for all \( n \), and \( (\mathcal{A}^\lambda - z)u_n \to 0 \) as \( n \to +\infty \), we obtain a contradiction. \( \square \)

The next lemma extends lemma 2.3 in [34] to the case \( \alpha \in (0, 1) \).
Lemma 2.2. Given \( \lambda > 0 \), let \( \zeta \in C_0^\infty(\mathbb{R}) \) be a cut-off function such that \( \zeta([0,\rho_0]) = 1 \), for some \( \rho_0 \). Define \( \zeta_d(x) = \zeta(x/d) \), \( d > 0 \). Then, for each \( d \), the operator \( \zeta_d(\mathcal{L} - z)^{-1} \) is compact for some \( z \in \rho(\mathcal{L}) \), and there exists \( C(d) \) as \( d \to 0 \) such that for any \( u \in C_0^\infty(\mathbb{R}) \),

\[
\| [\mathcal{L}, \zeta_d] u \| \leq C(d) (\| \mathcal{L} u \| + \| u \|).
\]  

(2.12)

Proof. Initially we prove that for sufficiently large \( k > 0 \), \( -k \in \rho(\mathcal{L}) \). Indeed, for \( \lambda > 0 \) we write \( \mathcal{L} = D^\omega + c + \mathcal{K} \), where

\[
\mathcal{K} = \frac{c}{\lambda - c \partial_x} \partial_x - \frac{\lambda}{\lambda - c \partial_x} D^\omega : L^2(\mathbb{R}) \to L^2(\mathbb{R})
\]  

(2.13)

is a bounded operator because the symbols of \( \partial_x (\lambda - c \partial_x)^{-1} \) and \( (\lambda - c \partial_x)^{-1} D^\omega \) are bounded (here we have \( \alpha < 1 \)). Thus, since \( \mathcal{L} = D^\omega + c \) is a nonnegative self-adjoint operator and \( \mathcal{K} \) is a \( \lambda \)-bounded operator with relative \( \lambda \)-bound equal to zero, we have that \( -k \in \rho(\mathcal{L}) \) for all \( k > 0 \) and

\[
\| \mathcal{K}(\mathcal{L} + k)^{-1} u \| \leq C_{\lambda, \varepsilon} \| (\mathcal{L} + k)^{-1} u \| \leq C_{\lambda, \varepsilon} \frac{1}{k} \| u \|.
\]

Therefore, the relation \( \mathcal{L} + k = [1 + \mathcal{K}(\mathcal{L} + k)^{-1}] (\mathcal{L} + k) \) implies that for \( k \) large, \( z = -k \in \rho(\mathcal{L}) \). Now, the compactness of \( \zeta_d(\mathcal{L} - z)^{-1} \) follows from the relation

\[
\| D^\omega (\mathcal{L} + k)^{-1} f_k \| \leq \| [1 + \mathcal{K}(\mathcal{L} + k)^{-1}]^{-1} f_k \| \leq M \| f_k \|
\]

for all \( L^2(\mathbb{R}) \)-bounded sequence \( \{ f_k \} \), the local compactness of \( H^\omega(\mathbb{R}) \hookrightarrow L^2(\mathbb{R}) \) and a Cantor diagonalization argument, which implies that \( (\mathcal{L} + k)^{-1} f_k \to f \) in \( L^2_{loc}(\mathbb{R}) \) and so the sequence \( \zeta_d(\mathcal{L} - z)^{-1} f_k \) is convergent.

To show the commutator estimate in (2.12), we note initially that the graph norm of \( \mathcal{L} \) appearing at the right-hand side of (2.12) is equivalent to the \( \| \cdot \|_{u_0} \)-norm (it follows immediately from the relations \( \mathcal{L} = D^\omega + c + \mathcal{K} \) and \( (\mathcal{L} + k) = [1 + \mathcal{K}(\mathcal{L} + k)^{-1}]^{-1} (\mathcal{L} + k) \)).

Now, it is not difficult to see that \( \mathcal{E} = \frac{\lambda}{\lambda - c \partial_x} \in B(L^2(\mathbb{R})) \) and we have the equality

\[
[\mathcal{L}, \zeta_d] = (1 - \mathcal{E}) [D^\omega, \zeta_d] + [\mathcal{E}, \zeta_d] (\varphi_c - D^\omega).
\]  

(2.14)

Next, we estimate every term at the right-hand side of (2.14). First, from the relation

\[
[\mathcal{E}, \zeta_d] (\varphi_c - D^\omega) = \frac{1}{\mathcal{E}} [\mathcal{E}[c \partial_x, \zeta_d]] - \frac{1}{\lambda - c \partial_x} (\varphi_c - D^\omega)
\]

\[
= \frac{c}{\lambda d} \mathcal{E} (\zeta'(x/d) \mathcal{E}(\varphi_c - D^\omega))
\]

we obtain

\[
\| [\mathcal{E}, \zeta_d] (\varphi_c - D^\omega) u \| \leq \frac{c}{\lambda d} \| \zeta' \|_{\infty} \| \varphi_c u - D^\omega u \|
\]

\[
\leq \frac{C_0}{\lambda d} (\| u \| + \| D^\omega u \|).
\]  

(2.15)

Now, since \( 1 - \mathcal{E} \in B(L^2(\mathbb{R})) \), we obtain from theorem 3.3 in Murray [45] the estimative for \( \alpha \in (0, 1) \).
\begin{equation}
\| (1 - \mathcal{E}^\lambda) [D^\alpha, \zeta_d] u \| \leq \| [D^\alpha, \zeta_d] u \| \leq K(\alpha) \| D^\alpha \zeta_d \| \| u \| \leq \frac{C_1}{d_0} \| D^\alpha \zeta \|_\infty \| u \|,
\end{equation}

where \( \| \cdot \|_\ast \) is the BMO norm and we use the identity \( [D^\alpha, \zeta_d](x) = \frac{d}{dx} (D^\alpha \zeta)(x/d) \) and the classical embedding \( L^\infty(\mathbb{R}) \hookrightarrow \text{BMO}(\| f \|_\ast \leq 2 \| f \|_\infty) \). Therefore, from (2.14)–(2.16) we obtain the right-hand side in (2.12). It completes the proof of the lemma.

**Proof of proposition 2.3.** By using theorem 10.12 in [22] and lemma 2.2 above, we have for any \( \lambda > 0 \) that \( W(\mathcal{A}^\lambda) = Z(\mathcal{A}^\lambda) \). Therefore, lemma 2.1 and propositions 2.2 imply 2.3.

Indeed, suppose that \( z \in \sigma_{\text{ess}}(\mathcal{A}^\lambda) \) and \( \text{Re } z < \frac{1}{2} c \). Then \( z \in \mathbb{C} - W(\mathcal{A}^\lambda) \subset \mathbb{C} - \partial(\sigma_{\text{ess}}(\mathcal{A}^\lambda)) \). Therefore, \( z \in \text{Int}(\sigma_{\text{ess}}(\mathcal{A}^\lambda)) \). So, if we consider that \( C_z \) is the maximal non-empty open connected component of \( \text{Int}(\sigma_{\text{ess}}(\mathcal{A}^\lambda)) \) containing point \( z \), we see that \( \partial C_z \cap \{ z : \text{Re } z < \frac{1}{2} c \} \neq \emptyset \). Therefore, since \( \partial C_z \subset \partial(\sigma_{\text{ess}}(\mathcal{A}^\lambda)) \) we obtain \( \partial(\sigma_{\text{ess}}(\mathcal{A}^\lambda)) \cap \{ z : \text{Re } z < \frac{1}{2} c \} \neq \emptyset \) and so \( W(\mathcal{A}^\lambda) \cap \{ z : \text{Re } z < \frac{1}{2} c \} \neq \emptyset \), which is a contradiction. This completes the proof.

Next, we study the behavior of \( \mathcal{A}^\lambda \) near infinity. We will show the non-existence of growing modes at the left-hand side of the complex-plane for large \( \lambda \) (so, since the eigenvalues of \( \mathcal{A}^\lambda \) appear in conjugate pairs, there are no growing modes for large \( \lambda \)).

**Lemma 2.3.** There exists \( \Lambda > 0 \), such that for \( \lambda > \Lambda \), \( \mathcal{A}^\lambda \) has no eigenvalues in \( \{ z : \text{Re } z \leq 0 \} \).

The proof of lemma 2.3 is the same as that of lemma 4.1 in [34] which still works for \( \alpha \in (0, 1) \). Next, we study the behavior of \( \mathcal{A}^\lambda \) for small positive \( \lambda \) and it is the more delicate part of the theory. The next two results are at the heart of Vock and Hunziker theory obtaining that every discrete eigenvalue of the limiting operator \( \mathcal{L}_c = D^\alpha + c - \varphi_c \) is stable with respect to the family \( \mathcal{A}^\lambda \) for small positive \( \lambda \) (see ch 19 in Hislop and Sigal [22]). The following result extends to those of Lin in [34] for the \( \alpha \in (0, 1) \) case.

**Lemma 2.4.** Given \( F \in \mathcal{C}^\infty_0(\mathbb{R}) \), consider any sequence \( \lambda_n \to 0^+ \) and \( \{ u_n \} \subset H^\alpha(\mathbb{R}) \) satisfying
\begin{equation}
\| \mathcal{A}^{\lambda_n} u_n \| + \| u_n \| \leq M_1 < \infty
\end{equation}
for some constant \( M_1 \), Then if \( w = \lim_{n \to \infty} u_n = 0 \), we have
\begin{equation}
\lim_{n \to \infty} \| F u_n \| = 0
\end{equation}
and
\begin{equation}
\lim_{n \to \infty} \| [\mathcal{A}^{\lambda_n}, F] u_n \| = 0.
\end{equation}

**Proof.** The convergence in (2.18) is immediate. Next, from the relation \( \mathcal{A}^{\lambda_n} = D^\alpha + c + \mathcal{K}^{\lambda_n} \), with \( \mathcal{K}^{\lambda_n} \) defined in (2.13), we have
\begin{equation}
[\mathcal{A}^{\lambda_n}, F] u_n = [D^\alpha, F] u_n + [\mathcal{K}^{\lambda_n}, F] u_n.
\end{equation}

Now, we show that every commutator at the right-hand side of (2.20) goes to zero for \( n \to \infty \).

(1) \( [D^\alpha, F] u_n \to 0 \) as \( n \to \infty \) for \( \mathcal{C}_\alpha \equiv [D^\alpha, F] \) we have that \( \mathcal{C}_\alpha : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) is a compact operator for every \( \alpha \in (0, 1) \). Indeed, from theorem 3.3 in [45], we know that \( \mathcal{C}_\alpha \) is a bounded operator on \( L^2(\mathbb{R}) \). With regards to the compactness property, we have from relation (3.18) in [45] the representation formula
\( \mathcal{E}_\alpha = ik(\alpha) \int_0^\infty [F, P_t] t^{-\alpha} \frac{dt}{t}, \quad \alpha \in (0, 1), \)  
(2.21)

where \( P_t = (1 - t^2 \partial_x^2)^{-1} \) and \( k(\alpha) = (2i/\pi) \sin(\pi\alpha/2) \). Now, since the symbol associated with the operator \( P_t, p_t(\xi) = \frac{1}{1 + t^2 \xi^2} \) satisfies \( \frac{d}{d\xi} p_t(\xi) \to 0 \) when \( |\xi| \to \infty \), by theorem C in [15] we obtain that the commutator \([F, P_t] : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) is compact. Therefore \( \mathcal{E}_\alpha \) is compact because it is the limit of a sequence of compact operators. It completes this item.

(2) \([\mathcal{K}, F] u_n \to 0 \) as \( n \to \infty \): We begin by studying the commutator

\[
[\mathcal{E}^\lambda D^\alpha, F] u_n = \mathcal{E}^\lambda D^\alpha (F u_n) - F \mathcal{E}^\lambda D^\alpha u_n
\]
(2.22)

with \( \mathcal{E}^\lambda = \frac{\lambda}{\lambda - i\partial_x} \). Indeed, since \( F u_n \to 0 \) on \( L^2(\mathbb{R}) \) by (2.18) and \( \mathcal{E}^\lambda D^\alpha : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) is a bounded operator for \( \alpha \in (0, 1) \), we immediately find that the first term on the right-hand side of (2.22) goes to zero. Now, consider \( \delta > 0 \) such that \( \delta + \alpha < 1 \). Then it is not difficult to see that \( \{\mathcal{E}^\lambda D^\alpha u_n\} \) is a bounded sequence in \( H^\delta(\mathbb{R}) \). Therefore, by the local compact embedding of \( H^\delta(\mathbb{R}) \) in \( L^2(\mathbb{R}) \) and a Cantor diagonalization argument, we have \( \mathcal{E}^\lambda D^\alpha u_n \to 0 \) in \( L^2_{\text{loc}}(\mathbb{R}) \). Thus, \( F \mathcal{E}^\lambda D^\alpha u_n \to 0 \) in \( L^2(\mathbb{R}) \).

Now, we study the commutator

\[
\left[ \frac{c\partial_x}{\lambda - c\partial_x} \varphi_c, F \right] u_n = \frac{c\partial_x}{\lambda - c\partial_x} (\varphi_c F u_n) - F \frac{c\partial_x}{\lambda - c\partial_x} (\varphi_c u_n).
\]
(2.23)

Thus, since \( \varphi_c F u_n \to 0 \) and \( \frac{c\partial_x}{\lambda - c\partial_x} \in B(L^2(\mathbb{R})) \) we obtain immediately that the first term of the right-hand side of (2.23) goes to zero. Next, we consider

\[
w_n = \frac{c\partial_x}{\lambda - c\partial_x} (\varphi_c u_n) \equiv \mathcal{P}(\varphi_c u_n).
\]

We shall see that \( \{w_n\} \) is bounded in \( H^\alpha(\mathbb{R}) \). Indeed, since \( \mathcal{P} \in B(H^\alpha(\mathbb{R})) \cap B(L^2(\mathbb{R})) \) (since \( D^\alpha \mathcal{P} \equiv \mathcal{P} D^\alpha \); we obtain

\[
\|w_n\|_{H^\alpha} \leq \|\varphi_c u_n\|_{H^\alpha} \leq \|D^\alpha \varphi_c u_n\| + \|\varphi_c D^\alpha u_n\| + \|\varphi_c u_n\| \\
\leq C_0 \|\alpha^{\alpha} \varphi_c \|_{L^\infty} \|u_n\| + \|\alpha^{\alpha} \|_{L^\infty} \|u_n\| \\
\leq C_1 \|\varphi_c\|_{H^{\alpha}}\|u_n\|_{H^\alpha} \leq M_2,
\]
(2.24)

where we used theorem 3.3 in [45] and the embedding \( L^\infty(\mathbb{R}) \hookrightarrow \text{BMO} \). Therefore, \( w_n \to f \) in \( H^\alpha(\mathbb{R}) \). Next, since \( \mathcal{P}(\varphi_c u_n) \to 0 \) in \( L^2(\mathbb{R}) \), we obtain that \( f \equiv 0 \). Then, by the local compact embedding \( H^\alpha(\mathbb{R}) \hookrightarrow L^2(\mathbb{R}) \) we finally obtain that \( F w_n \to 0 \) as \( n \to \infty \). It completes this item and the proof of the lemma.

**Remark 2.3.** We note that lemma 2.4 implies that the commutator operator \([A^\lambda, F] \) is compact for \( F \in C_0^\infty(\mathbb{R}) \).

The next lemma represents a crucial piece in the asymptotic perturbation theory.

**Lemma 2.5.** Let \( z \in \mathbb{C} \) with \( \text{Re} \, z \leq \frac{1}{2} c \), then there is \( n > 0 \) such that for all \( u \in C_0^\infty(|x| \geq n) \), we have

\[
\|(A^\lambda - z)u\| \geq \frac{1}{4} c \|u\|,
\]
(2.25)

when \( \lambda \) is sufficiently small.
Proof. Let \( n > 0 \) such that \( \max_{|x| \geq n} |\varphi_c(x)| \leq \frac{c}{4} \). Then for \( u \in C_0^\infty(|x| \geq n) \), it follows from the proof of lemma 2.1 above and the bounded property of the operator \( \frac{\partial}{\partial x} + c \cdot \varphi_c \) on \( L^2(\mathbb{R}) \), that
\[
\Re((A^\lambda - z)u, u) \geq (c - \Re z)\|u\|^2 - \Re(u_n \varphi_c, \frac{\partial}{\partial x} + c \varphi_c)\|u\|^2 \geq (c - \Re z - \max_{|x| \geq n} |\varphi_c(x)|)\|u\|^2 \geq \frac{1}{4} c\|u\|^2.
\]
Then, since \( |\Re((A^\lambda - z)u, u)| \leq \|(A^\lambda - z)u\||\|u\| \), we complete the proof. 

Thus, from lemmas 2.4 and 2.5 above we can apply the asymptotic perturbation theory in [49] (see also, definition 19.5, theorem 19.12 and lemmas 19.13 and 19.14 in ch 19 of Hislop and Sigal [22]) to the continuous family of closed operators \( A^\lambda \) to obtain the eigenvalue perturbations of \( A^0 \equiv \mathcal{L}_c = \mathcal{D}^\alpha + c - \varphi_c \) to \( A^\lambda \) with a small \( \lambda \) positive. More precisely, we have the following stability result for every discrete eigenvalue of \( \mathcal{L}_c \) with \( \alpha \in (0, 1) \).

**Theorem 2.1.** Each discrete eigenvalue \( \gamma \) of \( \mathcal{L}_c \) with \( \gamma \leq \frac{1}{4} c \) is stable with respect to the family \( A^\lambda \) in the sense that there exists \( \lambda_1, \delta > 0 \), such that for \( \lambda \in (0, \lambda_1) \), we have
(i) \( A_\lambda(\gamma) = \{ z : 0 < |z - \gamma| < \delta \} \subset \Delta_b \), where \( \Delta_b \) is called the region of boundedness for the family \( \{ A^\lambda \} \) and is defined by \( \Delta_b \equiv \{ z : R_\lambda(z) = (A^\lambda - z)^{-1} \text{ exists and is uniformly bounded for } \lambda \in (0, \lambda_1) \} \).

(ii) Let \( \Gamma \) be a simple closed curve about \( \gamma \) such that \( \Gamma \subset A_\delta(\gamma) \subset \rho(\mathcal{L}_c) \cap \rho(A^\lambda) \), for all \( \lambda \) small, and define the associated Riesz projector for \( A^\lambda \)
\[
P_\lambda = \frac{1}{2\pi i} \oint_{\Gamma} R_\lambda(z)dz.
\]
Then,
\[
\lim_{\lambda \to 0^+} \|P_\lambda - P_{\gamma}\| = 0,
\]
where \( P_{\gamma} \) is the Riesz projector for \( \mathcal{L}_c \) and \( \gamma \).

**Remark 2.4.** It follows from theorem 2.1 that for all \( 0 < \lambda \ll 1 \), the operators \( A^\lambda \) have discrete spectra inside the domain determined by \( \Gamma \) with total algebraic multiplicity equal to that of \( \gamma \), because from (2.26) we obtain that \( \dim(\text{Im}P_\lambda) = \dim(\text{Im}P_{\gamma}) \) for \( \lambda \) small. In order to simplify the notation, we write \( \dim(P_\lambda) \) to refer to \( \dim(\text{Im}P_\lambda) \).

**Proof.** As the proof follows the same lines as that of theorem 19.12 in [22], for convenience, we provide the main points of the analysis. The proof of item (i) follows from the property of \( \gamma \) as an isolated point of the spectrum of \( \mathcal{L}_c \) and lemma 2.5 (see lemma 19.14 in [22]).

Regarding item (ii), from proposition 2.1 and the property \( \rho(\mathcal{L}_c) \cap \Delta_b \neq \emptyset \), we obtain from Kato [26] the following strong resolvent convergence
\[
\lim_{\lambda \to 0^+} R_\lambda(z)u = (\mathcal{L}_c - z)^{-1}u, \quad \text{for all } u \in C_0^\infty(\mathbb{R}).
\]
Hence the Riesz projections \( P_\lambda \) satisfy \( \lim_{\lambda \to 0^+} P_\lambda u = P_{\gamma}u \), and therefore we obtain from the principle of the non-expansion of the spectrum, the inequality \( \dim(P_\lambda) \geq \dim(P_{\gamma}) \) (see lem-
ma 1.23 in Kato [26, p 438]. Now, as $P_\gamma$ is self-adjoint, we have $\lim_{\lambda \to 0^+} P_\lambda^* u = P_0 u$, and so, by using lemma 1.24 in Kato [26], the two convergences of the Riesz projectors and the local compactness lemma 2.4, we have the inequality $\dim(P_\lambda) \leq \dim(P_0)$, for all $0 < \lambda \ll 1$, and so the norm convergence of the projections in (2.26). It completes the theorem.

2.2.2. The moving kernel formula. In this section we study the perturbation of the eigenvalue $\gamma = 0$ associated with $\mathcal{L}_c$ with respect to the family $A^\lambda$ for small $\lambda > 0$. For this purpose, we derive a moving kernel formula in the same spirit as in Lin [34] in order to determine when the zero eigenvalue will jump to the right or the left. We will see that it depends on the sign of $b_\lambda$.

By hypotheses, we have that $\ker(\mathcal{L}_c) = \left[ \frac{d}{d\lambda} \varphi_c \right]$. Then we obtain that $\dim(P_0) = 1$, with $P_0$ being the Riesz projector associated with $\gamma = 0$ and $\mathcal{L}_c$. Therefore, from theorem 2.1, one has $\dim(P_\lambda) = 1$ for all $0 < \lambda \ll 1$. So, we obtain that $A^\lambda$ has exactly one spectral point in the disc $B(0; \epsilon) = \{ \mu \in \mathbb{C} : |\mu| < \epsilon \}$, with small $\epsilon$, and it is non degenerate (simple).

Moreover, since the eigenvalues of $A^\lambda$ appear in conjugate pairs, we have that there is only one real eigenvalue $b_\lambda$ of $A^\lambda$ inside $B(0; \epsilon)$. We note that from the analytic property of $A^\lambda$ with regards to $\lambda$ and from zero being a simple eigenvalue for $\mathcal{L}_c$, we have that the mapping $\lambda \to b_\lambda$ is analytic around zero.

The idea in the next result is to determine the sign of $b_\lambda$ for small $\lambda$.

**Lemma 2.6.** Let $c > 0$. Assume that $\ker(\mathcal{L}_c) = \left[ \frac{d}{d\lambda} \varphi_c \right]$. For small enough $\lambda > 0$, let $b_\lambda \in \mathbb{R}$ be the only eigenvalue of $A^\lambda$ near the origin. Then,

$$\lim_{\lambda \to 0^+} \frac{b_\lambda}{\lambda^2} = -\frac{1}{\|\varphi_c\|^2} \frac{dF}{dc}$$

with $F(c) = \frac{1}{2} \langle \varphi_c, \varphi_c \rangle$. Therefore, for $\frac{dF}{dc} > 0$ we obtain $b_\lambda < 0$ and for $\frac{dF}{dc} < 0$ we obtain $b_\lambda > 0$.

At this point in our theory, we will use the existence of a smooth curve of solitary wave solutions to equation (1.6), $c \mapsto \varphi_c \in H^{\alpha+1}(\mathbb{R})$ (see remark 2.5 below for the other version of the limit appearing in (2.27)).

**Proof.** As the proof follows the same lines as that of (4.7) in Lin [34], for convenience, we provide the main points of the analysis. From theorem 2.1, we see that for small enough $\lambda > 0$, there exists $u_\lambda \in D(A^\lambda)$ such that

$$\int_\mathbb{R} \varphi_c(x) |u_\lambda(x)|^2 dx = 1,$$

and $(A^\lambda - b_\lambda)u_\lambda = 0$, $b_\lambda \in \mathbb{R}$ and $\lim_{\lambda \to 0^+} b_\lambda = 0$. Next, it is not difficult to see that for $\text{Re } z \leq \frac{1}{2}$ and $u, v$ satisfying $(A^\lambda - z)u = v$, we have

$$\|u\|_{L^2} \leq M \int_\mathbb{R} |\varphi_c(x)||u(x)|^2 dx + \int_\mathbb{R} |v(x)|^2 dx,$$

with $M$ independent of $\lambda$. Thus, we immediately obtain that $\|u_\lambda\|_{L^2} \leq C$, for some constant $C > 0$ which does not depend on $\lambda > 0$. Therefore, $u_\lambda \rightarrow u_0$ in $H^2$ as $\lambda \rightarrow 0^+$. It is not difficult to see that $u_0 \neq 0$ and $\int \varphi_c(x) |u_\lambda - u_0|^2 dx \rightarrow 0$ as $\lambda \rightarrow 0^+$ (because
\( \varphi_\epsilon(x) \to 0 \) and the renormalization condition in (2.28)). Moreover, the relation 
\( (A^\lambda - b_\lambda)u_\lambda - u_0 = b_\lambda u_0 + (L_\epsilon - A^\lambda)u_0 \) implies from (2.28) and proposition 2.1 
that \( u_\lambda \to u_0 \) in \( H^\frac{3}{2} \). Then, \( A^\lambda u_\lambda \to L_\epsilon u_0 = 0 \) as \( \lambda \to 0^+ \). By hypothesis we have 
\( u_0 = \theta \frac{d}{dx} \varphi_\epsilon = \varphi'_\epsilon \) (without loss of generality we can assume \( \theta = 1 \)). So, \( u_\lambda \to \varphi'_\epsilon \) in \( H^\frac{3}{2} \).

The relation, 
\[
\frac{b_\lambda}{\lambda} (u_\lambda, \varphi'_\epsilon) = \frac{1}{c} \langle (\varphi_\epsilon - D^\alpha)u_\lambda, \frac{c\partial_x}{\lambda - c\partial_x} \varphi_\epsilon \rangle \to -\frac{1}{c} \langle (\varphi_\epsilon - D^\alpha)\varphi'_\epsilon, \varphi_\epsilon \rangle = \langle \varphi'_\epsilon, \varphi_\epsilon \rangle = 0,
\]
implies that \( \frac{b_\lambda}{\lambda} \to 0 \) as \( \lambda \to 0^+ \).

Next, we calculate \( \lim_{\lambda \to 0^+} \frac{b_\lambda}{\lambda^2} \). We write \( u_\lambda = c_\lambda \varphi'_\epsilon + \lambda \psi_\lambda \), with \( c_\lambda = \langle u_\lambda, \varphi'_\epsilon \rangle / \langle \varphi'_\epsilon, \varphi'_\epsilon \rangle \). Then, \( (\psi_\lambda, \varphi'_\epsilon) = 0 \) and \( c_\lambda \to 1 \) as \( \lambda \to 0^+ \). Following the same strategy as [34], we can show that \( \psi_\lambda \to \psi_0 \) in \( H^\frac{3}{2} \) with \( \psi_0 \neq 0 \) satisfying \( L_\epsilon \psi_0 = \varphi_\epsilon \) and \( \langle \psi_0, \varphi'_\epsilon \rangle = 0 \). Since \( L_\epsilon (\frac{d}{dx} \varphi_\epsilon) = -\varphi_\epsilon \) it follows that 
\[
\psi_0 = -\frac{d}{dx} \varphi_\epsilon + d_0 \varphi'_\epsilon, \quad d_0 = \frac{\langle \frac{d}{dx} \varphi_\epsilon, \varphi'_\epsilon \rangle}{\| \varphi'_\epsilon \|^2}.
\]

Similar to that in [34], we can rewrite \( u_\lambda = \tau_\lambda \varphi'_\epsilon + \lambda \psi_\lambda \), with \( \tau_\lambda \to 1 \), \( \tau_\lambda \to -\frac{d}{dx} \varphi_\epsilon \) in \( H^\frac{3}{2} \), as \( \lambda \to 0^+ \). Moreover, the equality 
\[
\frac{b_\lambda}{\lambda^2} (u_\lambda, \varphi'_\epsilon) = -\frac{\tau_\lambda}{c} \langle \frac{c\partial_x}{\lambda - c\partial_x} \varphi'_\epsilon, \varphi_\epsilon \rangle - \frac{1}{c} \langle \frac{c\partial_x}{\lambda - c\partial_x} (\varphi_\epsilon - D^\alpha)\psi_\lambda, \varphi_\epsilon \rangle
\]
implies the limit 
\[
\frac{b_\lambda}{\lambda^2} (u_\lambda, \varphi'_\epsilon) \to -\frac{1}{c} \langle (\varphi_\epsilon - D^\alpha) \frac{d}{dx} \varphi_\epsilon, \varphi_\epsilon \rangle = -\langle \frac{d}{dx} \varphi_\epsilon, \varphi_\epsilon \rangle.
\]

Therefore, 
\[
\lim_{\lambda \to 0^+} \frac{b_\lambda}{\lambda^2} = -\frac{1}{\| \varphi'_\epsilon \|^2} \langle \frac{d}{dx} \varphi_\epsilon, \varphi_\epsilon \rangle,
\]
and we complete the lemma. \( \square \)

**Remark 2.5.** In the proof of lemma 2.6, the existence of the curve of solitary waves 
\( c \to \varphi_\epsilon \in H^{\alpha+1}(\mathbb{R}) \) was used exactly to obtain the relation \( L_\epsilon (\frac{d}{dx} \varphi_\epsilon) = -\varphi_\epsilon \). Thus, it is not 
difficult to see that we can change the hypothesis on the curve by the existence of \( \psi \in D(L_\epsilon) \) 
such that \( L_\epsilon \psi = \varphi_\epsilon \), with \( \varphi_\epsilon \) being a positive solution for (1.6). Therefore, 
supposing that for 
\( L = D^\alpha + c - \varphi \) we have \( \ker(L) = \left[ \frac{d}{dx} \varphi \right] \), then relation (2.27) can be rewritten as 
\[
\lim_{\lambda \to 0^+} \frac{b_\lambda}{\lambda^2} = -\frac{1}{\| \varphi'_\epsilon \|^2} \langle \psi, \varphi_\epsilon \rangle.
\]

**Proof. (Theorem 1.2: spectral instability criterion for fKdV equations).** The 
proof follows the same lines as that in Lin [34] by using the fact that the mapping \( \lambda \in \mathbb{R} \to A^\lambda \) 
represents an analytical family of operators of type-A, theorem 2.1, lemma 2.6, proposition 
2.3 and lemma 2.3 above. So, there exists \( \lambda > 0 \) and \( 0 \neq u \in H^\alpha(\mathbb{R}) \) such that 
\( A^\lambda u = 0 \) and therefore \( e^{\lambda t}u(x) \) is a purely growing mode solution to (1.31).
Proof. (Corollary 1.1: spectral instability of ground state for fKdV equations). From the analysis in the proof of theorem 1.1, it follows that the self-adjoint operator $\mathcal{L}_c = D^\alpha + c - Q_c$ satisfies $n(\mathcal{L}_c) = 1$ and $\text{Ker}(\mathcal{L}_c) = \{ \frac{1}{\alpha}Q_c \}$. Now, the curve $c \to V(c) = Q_c \in H^\alpha(\mathbb{R})$ is at least of $C^1$-class. Indeed, we know that $Q_c(x) = 2cQ(e^{1/\alpha}x)$, for $Q$ being the ground state associated with (1.10), then from the relation

$$V'(c) = 2Q(e^{1/\alpha}x) + \frac{2}{\alpha}e^{1/\alpha}xQ'(e^{1/\alpha}x)$$

and $R = cQ + xQ' \in H^{\alpha+1}(\mathbb{R})$ (seeproof of theorem 1.1) we obtain $V'(c) \in H^{\alpha+1}(\mathbb{R}) \subset H^\alpha(\mathbb{R})$.

Thus,

$$\frac{d}{dc} \langle Q_c, Q_c \rangle = 4\|Q\|^2 \frac{d}{dc} e^{2\frac{t}{\alpha}} = 4\left(2 - \frac{1}{\alpha}\right) e^{1 - \frac{t}{\alpha}} \|Q\|^2 < 0 \quad (2.32)$$

where we have used that $\alpha < \frac{1}{2}$ (for $\alpha = \frac{1}{2}$, we obtain the equality $\frac{d}{dc} \langle Q_c, Q_c \rangle = 0$ for all $c$). Hence, condition (ii) in theorem 1.2 can be applied and therefore we complete the proof. \(\square\)

Proof. (Theorem 1.3: spectral instability criterium for gfKdV equations). The proof follows the same lines as that established for the spectral instability criterium for the fKdV equation, but the strategy showing the basic lemma 2.2 and the compactness lemma 2.4 associated with the family of linear operators $V^\lambda : H^\alpha(\mathbb{R}) \to L^2(\mathbb{R})$, with $\text{Re} \lambda > 0$,

$$V^\lambda v \equiv cv + \frac{c\partial_x}{\lambda - c\partial_x} (\varphi_\epsilon v - Mv), \quad (2.33)$$

needs to be changed. Indeed, with regards to lemma 2.2 for $A^\lambda$, we change the relation (2.14) by

$$[V^\lambda, \zeta_d] = (1 - E^\lambda)[D^\alpha, \zeta_d] + [E^\lambda, \zeta_d](\varphi_\epsilon - M) + (1 - E^\lambda)[M - D^\alpha, \zeta_d]. \quad (2.34)$$

The estimative required for the two first terms in the right-hand side of (2.34) is equal to that in (2.15) and (2.16). For the third term, we use the condition in (1.38) for $\eta(\xi) = \beta(\xi) - |\xi|^\alpha$. Indeed, it is not difficult to see that the kernel

$$K_r(x, y) = (x - y)\eta(x - y)\zeta'_d(r(x - y) + y) \quad r \in [0, 1], \quad (2.35)$$

where ‘\(\eta\)’ represents the inverse Fourier transform of $\eta$, which satisfies

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |K_r(x, y)|^2 dx dy = \int_{\mathbb{R}} \int_{\mathbb{R}} |x\eta(x)|^2 |\zeta'_d(y)|^2 dx dy = \|\eta'\|^2 \||\zeta'_d\|^2 = \frac{1}{d} \|\eta'\|^2 \||\zeta'_d\|^2. \quad (2.36)$$

Therefore,

$$\|\|M - D^\alpha, \zeta_d|u\|^2 \leq \int_{\mathbb{R}} \int_{\mathbb{R}} |\zeta'_d(r(y)dy)|^2 dx \leq \int_{\mathbb{R}} \int_{\mathbb{R}} u(y)K_r(x, y)dy dr^2 dx \leq \int_{0}^{1} \int_{\mathbb{R}} u(y)K_r(x, y)dy |dr|^2 dx \leq \frac{1}{d} \|\eta'\|^2 \||\zeta'_d\|^2 \|u\|^2. \quad (2.37)$$
Therefore, \( \| (1 - \mathcal{E}) [M - D^\alpha, \zeta_0] u \| \leq \frac{1}{\pi^2} \| \eta' \| \| \zeta' \| \| u \| \), which completes the estimative.

Now, with regards to lemma 2.4 for \( \mathcal{V}^\lambda \), we change the relation (2.20) by

\[
[\mathcal{V}^\lambda, F] = [D^\alpha, F] + [W^\lambda, F] + [M - D^\alpha, F]
\]

(2.38)

with

\[
W^\lambda = \frac{c \partial_x}{\lambda - c \partial_x} \varphi_c - \frac{\lambda}{\lambda - c \partial_x} M : L^2(\mathbb{R}) \to L^2(\mathbb{R}).
\]

The estimative required for the first two terms in the right-hand side of (2.38) is equal to that in the proof of lemma 2.4. For the third term, we use the condition in (1.38). Indeed, since \( \eta \) is bounded and continuous over \( \mathbb{R} \) and \( \eta'(x) \to 0 \), as \(|x| \to \infty\), (because \( \eta' \in L^2(\mathbb{R}) \)), it follows from theorem C in Cordes [15] that the commutator \( [M - D^\alpha, F] : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) is compact. It completes the proof of the theorem.

\[ \square \]

3. ‘Stability of the blow-up’ for the critical fKdV equation

In this section we obtain information regarding the large-time asymptotic behaviour of solutions for the critical fKdV equation:

\[
u_t + u u_x - D^{1/2} u_x = 0
\]

(3.1)
on \( \mathbb{R} \). As we saw in the last section, the orbit generated by ground state solutions \( \varphi \) associated with equation (1.6) for \( \frac{1}{2} < \alpha < 2 \) and \( 1 \leq p < p_{\text{max}}(\alpha) \) are nonlinearly stable in \( H^{\frac{3}{2}}(\mathbb{R}) \) by the flow of equation (1.1) for \( p < 2\alpha \) (see theorem 1.1). Moreover, the general spectral instability criterium, theorem 1.2, shows the spectral instability of \( \varphi \) for \( \frac{1}{3} < \alpha < \frac{1}{2} \) and \( p = 1 \).

Now, from the proof of these later results we can see that the behavior of the solutions \( \varphi \) for \( \alpha = \frac{1}{2} \) and \( p = 1 \) is unclear, essentially because the expression

\[
\frac{d}{dc} \langle \varphi, \varphi \rangle = 4 \left( 2 - \frac{1}{\alpha} \right) c^{1-\frac{2}{\alpha}} \| \varphi \|^2
\]

(3.2)
is exactly zero for \( \alpha = \frac{1}{2} \).

Recently, Saut and Klein in [29] provided a detailed numerical study pertaining to the dynamics of the fKdV model (1.1) with \( 0 < \alpha < 1 \) and \( p = 1 \). For the specific case of \( \alpha = \frac{1}{2} \) and with initial data \( \varphi \) of negative energy \( (\mathcal{E}(\varphi) < 0) \) and with a mass larger that the solitary wave mass \( M(\varphi) < \| \varphi \|^2 \) the simulations show a possible blow-up phenomenon of the associated solution (see figure 10 in [29]). Moreover, the peak which appears to blow-up eventually becomes more and more compressed laterally, grows in height and propagates faster with a profile of a dynamically rescaled solitary wave. Here we will show that in fact we have a kind of ‘stability of the blow-up’ near to the possible unstable ground state solutions for equation (3.1).

The strategy we use to demonstrate our ‘stability’ result follows that used by Angulo et al in [6] (see also Angulo [5]) to study the critical case in the model (1.1) for \( \alpha \geq 1 \), namely, \( p = 2\alpha, p \in \mathbb{N} \). Thus, we consider that \( \mathcal{L}_c \) is the linear, self-adjoint, closed, unbounded operator defined on \( H^{1/2}(\mathbb{R}) \) by

\[
\mathcal{L}_c = D^{1/2} + c - Q_c
\]

(3.3)
where $Q_c$ is the ground-state solution associated with (1.6). Therefore, from [16] we have the following properties:

1. $L_c$ has a single negative eigenvalue which is simple, with eigenfunction $\chi_c > 0$, the zero eigenvalue is simple with eigenfunction $Q'_c$, and the remainder of the spectrum of $L_c$ is positive and bounded away from zero.

2. The curve $c \rightarrow Q_c$ is $C^1$ with values in $H^2(\mathbb{R})$.

Next, we consider the conserved energy functional $E$ in (1.16) with $\alpha = \frac{1}{2}$ and $p = 1$. Therefore, from (1.7)–(1.9) we have that $E(Q_c) = 0$. Moreover, from (1.19), the a priori estimative follows

$$\|D^{1/4}u(t)\|^{2} \left[ 1 - \frac{\|u_0\|}{\|Q_c\|} \right] \leq 2E(u_0).$$  \hspace{1cm} (3.4)

Thus, if we consider $E(u_0) \leq 0$ then necessarily we have the condition $\|Q_c\| \leq \|u_0\|$.

Now, we introduce the auxiliary functions

$$\psi(x,t) = \mu(t)^{-\frac{1}{2}} u(\mu(t)^{-1} x,t)$$  \hspace{1cm} (3.5)

where

$$\mu(t) = \frac{\|D^{1/4}u(t)\|^{4}}{\|D^{1/2}Q_c\|^{2}}.$$  \hspace{1cm} (3.6)

$\mu(0) = 1$ and $0 \leq t < t^*$ with $t^*$ the maximal time of existence of the solution of (3.1) under consideration, if the solution is global, $t^* = +\infty$ (see remark 2.1). Note that unless $u$ is the zero-solution, $\mu(t) \in (0, \infty)$ for $0 < t < t^*$. The normalization $\mu(0) = 1$ is a temporary one made to simplify the presentation of the argument, and it can be dispensed of (see [6]). By using $E$ defined in (1.16), it is easy to check that the function $\psi$ verifies the identities

\begin{align*}
(\text{i}) & \quad \|\psi(t)\| = \|u(t)\| = \|u_0\|, \\
(\text{ii}) & \quad \langle \psi(t), D^{1/2}\psi(t) \rangle = \langle Q_c, D^{1/2}Q_c \rangle, \\
(\text{iii}) & \quad E(\psi(t)) = \frac{1}{\mu(t)^{\frac{1}{2}}} E(u(t)).
\end{align*}  \hspace{1cm} (3.7)

Since the stability considered here is with respect to form, i.e. up to translation in space, we introduce the pseudo-metric

$$\rho_c(\psi(t), Q_c)^2 = \inf_{r \in \mathbb{R}} \{ \|D^{1/4}(\psi(\cdot + r, t) - D^{1/4}Q_c(\cdot))\|^2 + c\|\psi(\cdot + r, t) - Q_c(\cdot)\|^2 \}$$

on $H^{1/4}(\mathbb{R})$. Define the set $\mathcal{K}$ to be

$$\mathcal{K} = \{ u_0 : u_0 \in H^s(\mathbb{R}) \text{ and } E(u_0) \leq 0 \} \subset H^s(\mathbb{R}), \quad s > \frac{21}{16}.$$  \hspace{1cm} (3.8)

We recall that the condition $s > \frac{21}{16}$ ensures that the Cauchy problem for (3.1) is local well-posedness in $H^s(\mathbb{R})$ (see [37]). Of course, proving well-posedness in $H^{\frac{3}{2}}(\mathbb{R})$ in the general case $\alpha \in [\frac{1}{2}, 1)$ is still an open problem (see remark 2.1 above).

The next theorem is a stability result which belongs to the spatial structure of the solutions of (3.1) in the critical case.

**Theorem 3.1.** Let $Q_c$ be the ground state profile for (1.6). Then, for any $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that if $u_0 \in \mathcal{K}$ with $\rho_c(u_0, Q_c) < \delta$ and $u$ is the solution of (3.1) corresponding to the initial value $u_0$ then $u \in C([0, t^*); H^{1/4}(\mathbb{R}))$ and
\[
\inf_{r \in \mathbb{R}} \left\{ c \| u(\cdot, t) - \mu(t)^{\frac{1}{4}} Q_c(\mu(t)(\cdot - r)) \|^2 + \frac{1}{\mu(t)^{\frac{1}{4}}} \| D^{1/4} u(\cdot, t) - \mu(t)^{\frac{1}{4}} D^{1/4} Q_c(\mu(t)(\cdot - r)) \|^2 \right\} < \epsilon
\]  

(3.9)

for all \( t \in [0, t^*] \), where \( t^* \) is the maximal existence time for the solution \( u \), and \( \mu \) is as in (3.6).

**Proof.** Suppose at the outset that \( \mu(0) = 1 \). The proof is based on the time-dependent functional

\[
B_t[u] = \frac{1}{\mu(t)^{\frac{1}{4}}} E(u(t)) + \frac{c}{2} \left( \frac{\| u(t) \|}{\| Q_c \|} \right)^{2k} (\| u(t) \|^2 - \| Q_c \|^2)
\]

where \( k \in \mathbb{N} \) will be chosen later. From the definition of \( B_t \), it is clear that if \( u \) is a solution of (3.1) then \( B_t[u] = B_t[u_0] \). Using (3.5)–(3.7), we may write \( B_t[u] \) in terms of \( \psi \) thusly:

\[
B_t[\psi] = E(\psi(t)) + \frac{c}{2} \left( \frac{\| \psi(t) \|}{\| Q_c \|} \right)^{2k} (\| \psi(t) \|^2 - \| Q_c \|^2)
\]

(3.10)

where the explicit dependence on \( \mu \) disappears. As will be argued presently, if it is established that, modulo translations, the inequalities

\[
(i) \quad \Delta \tilde{B}_t \leq c_0 \| u_0 - Q_c \| \quad \text{and}
\]

\[
(ii) \quad \Delta \tilde{B}_t \geq c_1 \| \psi(t) - Q_c \|^2 - c_2 \| \psi(t) - Q_c \|^3 - \sum_{j=1}^{2k} c_{k,j} \| \psi(t) - Q_c \|^{j/2},
\]

(3.11)

(3.12)

hold for \( \Delta \tilde{B}_t = \tilde{B}_t[\psi] - \tilde{B}_t[Q_c] \), where \( c_i, c_{k,j} \) are fixed constants, then the result in theorem 3.1 follows in a well-known form. Hence, attention is turned to establishing these bounds. The upper bound (3.11) is a straightforward consequence of \( E(u_0) \leq 0 \) and \( E(Q_c) = 0 \) (note that \( \| u_0 \| = \| Q_c \| \geq 0 \)), where the constant \( c_0 \) depends on \( \| Q_c \| \) (and on an upper bound for the choice of \( \delta \)). To prove (3.12), consider the perturbation of the ground state \( Q_c \)

\[
\psi(\cdot + \gamma, t) = Q_c(\cdot + a(t, \cdot), t)
\]

(3.13)

where \( a \) is a real function and \( \gamma = \gamma(t) \) minimizes the functional

\[
\Pi(\gamma) = \| D^{1/4} \psi(\cdot + \gamma, t) - D^{1/4} Q_c(\cdot) \|^2 + c \| \psi(\cdot + \gamma, t) - Q_\cdot(\cdot) \|^2.
\]

Using the representation (3.13), one calculates that

\[
\Delta \tilde{B}_t = \tilde{B}_t[Q_c + a] - \tilde{B}_t[Q_c]
\]

\[
= E(Q_c + a) - E(Q_c) + \frac{\eta}{2} \left( \frac{\| Q_c + a \|}{\| Q_c \|} \right)^{2k} (\| Q_c + a \|^2 - \| Q_c \|^2)
\]

\[
\geq \frac{1}{2} \langle L_c a, a \rangle + \frac{2kc}{\| Q_c \|} (\| a \|^2 - c_2(c) \| a \|^{3/4}) - \sum_{j=1}^{2k} c_{k,j}(c) \| a \|^{j/4}.  
\]

(3.14)
The inequality in (3.14) is obtained using the definition (3.3) of $\mathcal{L}_c$, the Cauchy–Schwartz inequality and interpolation (see (1.19)).  

A suitable lower bound on the quadratic form $\mathcal{L}_c$ is the next order of business. Initially, since the ground state solution $u(x, t) = Q_c(x - ct)$ is globally defined, we have from the continuous dependence theory for the model (3.1) in $H^s(\mathbb{R})$, $s > \frac{11}{16}$, that for $t$ in some interval of time $[0, T]$, the inf $\Pi(\gamma)$ is attained in $\gamma = \gamma(t)$ for $t \in [0, T]$ (see lemmas 6.2 and 6.3 in Angulo et al [6]). Hence, using that $Q_c$ satisfies equation $D^{1/2}Q_c + cQ_c - \frac{1}{2}Q_c^2 = 0$ we obtain

$$\frac{d}{dr} \Pi_{0}(r)_{\mid r = \gamma} = 2 \int_{\mathbb{R}} [cQ_c(x) + D^{1/2}Q_c(x)]a(x, t)dx = 2 \int_{\mathbb{R}} Q_c(x)Q_c'(x)a(x, t)dx,$$

which gives us the following compatibility relation on $a$, namely,

$$\int_{\mathbb{R}} Q_c(x)Q_c'(x)a(x, t)dx = 0 \tag{3.15}$$

for all $t$ in an interval of time $[0, T]$.  

The issue of obtaining the lower bound (3.12) for the right-hand side of inequality (3.14) is addressed in the next few lemmas.

**Lemma 3.1.** Let $\mathcal{L}_c = D^{1/2} + c - Q_c$. Then there exists $\sigma < 0$ such that if $\tilde{h} = Q_c - \sigma D^{1/2}Q_c$, then

$$\min_{\langle f, h \rangle = 0, \|f\| = 1} \langle \mathcal{L}_cf, f \rangle = 0.$$

**Proof.** For any given value $\sigma$, define the function $f_0$ by

$$f_0(x) = -\frac{1}{c}Q_c(x) - \frac{2 + 2\sigma}{c}aQ_c'(x).$$

Then using the relation $D^{1/2}(\sigma Q_c) = \frac{1}{2}D^{1/2}Q_c + \sigma D^{1/2}Q_c$, we obtain that $\mathcal{L}_cf_0 = Q_c - \sigma D^{1/2}Q_c = \tilde{h}$ and, consequently, that

$$\langle f_0, \mathcal{L}_cf_0 \rangle = \langle f_0, \tilde{h} \rangle = \left(\|Q_c\|^2 + \frac{1}{2\sigma}\|D^{1/4}Q_c\|^2\right)\sigma - \frac{1}{2}\|D^{1/4}Q_c\|^2\sigma^2.$$

It is thus obvious that for small negative values of $\sigma$, it is possible to have both

$$\langle \tilde{h}, \chi_c \rangle = \int \chi_cQ_c dx - \sigma \int \chi_cD^{1/2}Q_c dx \neq 0$$

and

$$\langle \mathcal{L}_c^{-1}\tilde{h}, \tilde{h} \rangle = \langle f_0, \tilde{h} \rangle < 0. \tag{3.16}$$

Since $\text{Ker}(\mathcal{L}_c) = [Q_c']$ and $\tilde{h} \in (\text{Ker}(\mathcal{L}_c))^\perp$, it follows from Weinstein [51]) (see also lemma 6.4 in [5]) that

$$\theta = \min\left\{ \langle \mathcal{L}_cf, f \rangle : \|f\| = 1 \text{ and } \langle f, \tilde{h} \rangle = 0 \right\} = 0. \tag{3.17}$$
The proof of the existence of the minimum in (3.17) follows the same ideas as in lemma 6.7 in Angulo et al [6]. This completes the proof of the lemma.

Lemma 3.2. If \( \tilde{h} = Q_c - \sigma D^{1/2}Q_c \) with \( \sigma < 0 \) chosen as in the last lemma, then
\[
\inf \left\{ \langle L_c f, f \rangle : \|f\| = 1, \langle f, \tilde{h} \rangle = 0, f \perp Q_c^* \right\} \equiv \nu > 0.
\] (3.18)

Proof. Because of lemma 3.1, it follows that \( \nu \geq 0 \). Suppose that \( \nu = 0 \). Then, we can guarantee the existence of a function \( f^* \) such that \( \|f^*\| = 1, \langle f^*, \tilde{h} \rangle = 0, \langle f^*, Q_c^* \rangle = 0 \) and \( \langle L_c f^*, f^* \rangle = 0 \). Therefore, at least one non-trivial critical point \((f^*, \tau, \theta, \nu)\) exists for the Lagrange multiplier problem
\[
\begin{aligned}
L_c f &= \tau f + \theta \tilde{h} + \nu Q_c^* \\\text{subject to} \\\|f\| = 1, \langle f, Q_c^* \rangle = 0 \text{ and } \langle f, \tilde{h} \rangle = 0.
\end{aligned}
\] (3.19)

Using the fact that \( \langle L_c f^*, f^* \rangle = 0 \), it can be easily seen that (3.19) implies \( \tau = 0 \). Moreover, since \( L_c Q_c^* = 0 \), we have that \( \langle L_c f^*, Q_c^* \rangle = \nu \int (Q_c^*)^2 Q_c^2 \ dx = 0 \), which implies \( \nu = 0 \). It is thereby concluded that
\[
L_c f = \theta \tilde{h}
\]
has nontrivial solutions \((f^*, \theta)\) satisfying the constraints. But if \( f \) is the auxiliary function arising in the proof of lemma 3.1, we have that \( L_c f_0 = \tilde{h} \) and so \( L_c (f^* - \theta f_0) = 0 \). Then \( f^* - \theta f_0 \in \text{Ker}(L_c) \). It follows from (3.16) that \( \langle f_0, \tilde{h} \rangle \neq 0 \), and so \( \theta = 0 \). Therefore, for some non-zero \( \lambda \in \mathbb{R} \), it is true that \( f^* = \lambda Q_c^* \), which is a contradiction since such a function cannot be orthogonal to \( Q_c^* \). Therefore, the minimum in (3.18) is positive and the proof of the lemma is complete.

We note that from (3.18) and from the specific form of \( L_c \), we have that if \( f \in H^{1/2}(\mathbb{R}) \) satisfies \( \langle f, \tilde{h} \rangle = 0 \) and \( \langle f, Q_c^* \rangle = 0 \), then
\[
\langle L_c f, f \rangle = \int |D^{1/4}f(x)|^2 + (c - Q_c(x))|f(x)|^2 \ dx \geq \beta_0 \|f\|_2^2, \quad \beta_0 > 0.
\] (3.20)

Continuation of the proof of theorem 3.1. Attention is now turned to estimating the term \( \frac{1}{2} \langle L_c a, a \rangle + \frac{2\kappa}{10} \|a, Q_c\|^2 \) in (3.14), where \( a \) satisfies the compatibility relation (3.15). We continue to carry over the notation from lemmas 3.1 and 3.2. In particular, \( \sigma \) is chosen so that the conclusions of lemma 3.1 are valid. Define \( a_\parallel \) and \( a_\perp \) to be
\[
a_\parallel = \frac{\langle a, \tilde{h} \rangle}{\|\tilde{h}\|^2} \tilde{h} \quad \text{and} \quad a_\perp = a - a_\parallel.
\]
It follows from the properties of \( a \) and \( \tilde{h} = Q_c - \sigma D^{1/2}Q_c \) that \( \langle a_\perp, \tilde{h} \rangle = 0, \int Q_c^* a_\perp \ dx = 0 \). Without loss of generality, take \( \langle a, \tilde{h} \rangle < 0 \). Thus, from lemma 3.2, the Cauchy–Schwarz inequality and from the properties of \( a, a_\perp, a_\parallel \) and \( \tilde{h} \), it follows that
\[
\begin{aligned}
\langle L_c a_\perp, a_\perp \rangle &\geq D_1 \|a_\perp\|^2, \quad \langle L_c a_\parallel, a_\parallel \rangle = \frac{\|a_\parallel\|^2}{\|\tilde{h}\|^2} \langle \tilde{h}, L_c \tilde{h} \rangle, \\
\langle L_c a_\parallel, a_\perp \rangle = \frac{\langle a, \tilde{h} \rangle}{\|\tilde{h}\|^2} \langle L_c \tilde{h}, a_\perp \rangle &\geq -D_2 \|a_\perp\| \|a_\parallel\|
\end{aligned}
\] (3.21)
for some positive constants $D_1$ and $D_2$. Identity (ii) in (3.7) implies $-2 \langle a, D^2 Q_c \rangle = \|D^{3/2}a\|^2$. Thus, from the Cauchy–Schwarz inequality we obtain (remember, $\sigma$ and $\langle a, \mathring{h} \rangle$ are both negative)

$$
\frac{2kc}{\|Q_c\|^2} \langle a, Q_c \rangle^2 \geq \frac{2kc}{\|Q_c\|^2} \left( \langle a, \mathring{h} \rangle^2 - \sigma \langle a, \mathring{h} \rangle \|D^{1/4}a\|^2 \right) \geq \frac{2kc}{\|Q_c\|^2} \|\mathring{h}\|^2 \|a\|^2 + 2kc\sigma D_3 \|a\|^2, \tag{3.22}
$$

with $D_3 > 0$. We choose $\theta > 0$ so that $D_1 - \theta D_2 \equiv D_4 > 0$. By Young’s inequality, $\|a_\perp\|\|a_\parallel\| \leq \theta \|a_\perp\|^2 + \frac{1}{\theta} \|a_\parallel\|^2$. Finally, fix $k$ in such a way that

$$
\frac{2kc}{\|Q_c\|^2} \langle \mathring{h}, \mathring{h} \rangle = \frac{2kc}{\|Q_c\|^2} \|\mathring{h}\|^2 - \frac{D_2}{\theta} \equiv D_5 > 0.
$$

With these choices, it follows from (3.21) and (3.22) that

$$
\frac{1}{2} \langle \mathcal{L}_c, a \rangle + \frac{2kc}{\|Q_c\|^2} \langle a, Q_c \rangle^2 \geq D_5 \|a_\parallel\|^2 + D_4 \|a_\perp\|^2 + 2kc\sigma D_3 \|a\|^2 + \frac{2kc}{\|Q_c\|^2} \|\mathring{h}\|^2 + \frac{D_2}{\theta} \|a\|^3, \tag{3.23}
$$

for some positive constants $D'$ and $D''$. With (3.23) in hand, it follows easily from the specific form of the operator $\mathcal{L}_c$ (see (3.20)) that

$$
\frac{1}{2} \langle \mathcal{L}_c a, a \rangle + \frac{2kc}{\|Q_c\|^2} \langle a, Q_c \rangle^2 \geq \tilde{D}_1 \|a\|^2 - \tilde{D}_2 \|a\|^3, \tag{3.24}
$$

with $\tilde{D}_1, \tilde{D}_2 > 0$. Finally, using (3.24) in conjunction with (3.14), we obtain

$$
\Delta \tilde{B}_t \geq \tilde{D}_1 \|a\|^2 - \tilde{D}_2 \|a\|^3 - c_2(e) \|a\|^3 - \sum_{j=1}^{2k} c_{k,j} \|a\|^j \tag{3.25}
$$

where $c_0, c_1, c_{k,j}$ are positive constants which depend only on $e$.

Now we are in a position to complete theorem 3.1. Suppose first that $u_0$ lies in the set $\mathcal{K}$ of ‘nonpositive-energy’ initial values and suppose $\|u_0 - Q_c\|_\parallel = \delta$. Then at least for $t \in [0, T]$, it follows from (3.11) and (3.12) that

$$
q(\rho_\epsilon(\psi(t), Q_c)) \leq \Delta \tilde{B}_t \leq c_0 \delta \tag{3.26}
$$

where $q(x) = c_0 x^2 - c_1 x^3 - \sum_{j=1}^{2k} c_{k,j} x^{j+2}$. Since $\|a(t)\|^2 = \rho_\epsilon(\psi(t), Q_c)^2$ is a continuous function of $t \in [0, t^*)$, it follows from the inequality

$$
q(\rho_\epsilon(\psi(0), Q_c)) \leq c_0 \delta \tag{3.27}
$$

and (3.25), that given $\epsilon > 0$, then for all $t \in [0, T]$,

$$
\rho_\epsilon(\psi(t), Q_c) \leq \epsilon,
$$

947
provided that \( \delta \) is chosen small enough at the outset. To complete the proof, we need to show that inequality (3.27) is still true for \( t \in [0, t^*] \). This part is shown using a method similar to that of the proof of theorem 6.1 in Angulo et al [6]. Therefore, the stability in theorem 3.1 is established if \( \mu(0) = 1 \). The general case, wherein the initial data is not necessarily such that \( \mu(0) = 1 \), requires a little more of work, and therefore we refer the reader to reference [6]. This completes the proof of theorem 3.1.

3.1. Behaviour of the stability parameters for the critical-fKdV equation

In the proof of theorem 3.1, we employ the fact that there is a specific choice of the translation parameter \( \gamma = \gamma(t) \) such that

\[
\|D^{1/4} \psi(\cdot + \gamma, t) - D^{1/4} Q_c(\cdot)\| + c\|\psi(\cdot + \gamma, t) - Q_c\| \leq \epsilon
\]  

(3.28)

for all \( t < t^* \), where \( \psi \) is the rescaled version of the solution \( u \) of (3.1) defined in (3.5). Moreover, a choice of \( \gamma \) for which (3.28) holds may be determined via the orthogonality condition in (3.15). By application of the implicit-function theorem as in lemma 4.2 in [6], it is obtained that as long as \( \psi \) satisfies (3.28), there is a unique, continuously differentiable choice of the value \( \gamma(t) \) that achieves (3.15) provided that the initial data \( u_0 \in H^s(\mathbb{R}) \) for \( s \) is sufficiently large and the profile \( Q_c \in H^n(\mathbb{R}) \) for \( n \) is sufficiently large (at least for \( n \geq 3 \)). Moreover, with the hypothesis of sufficient regularity for the initial data \( u_0 \), we can see that \( \mu \) defined in (3.6) belongs to the class \( C^4([0, t^*) : \mathbb{R}) \).

Thus, by following the line of argumentation in lemma 4.3 in [6], we obtain the relation between the translation and dilation parameters involved in our stability result in theorem 3.1.

**Theorem 3.2.** Let \( Q_c \) be the ground state profile for (3.1) such that \( Q_c \in H^n(\mathbb{R}), n \geq 3 \). Then, for any \( \epsilon > 0 \) there is a \( \delta = \delta(\epsilon) \) such that if \( u_0 \in H^s(\mathbb{R}) \cap X \), with sufficiently large \( s \) and \( \|u_0 - Q_c\|_1 < \delta \), then there exists a \( C^\ell \)-mapping \( \gamma : [0, t^*) \rightarrow \mathbb{R} \) such that

(i) \( \|\psi(\cdot + \gamma(t), t) - Q_c\|_1 \leq \epsilon \) for \( t \in [0, t^*) \),

(ii) for all \( t \in [0, t^*) \),

\[
|\gamma(t) - c\mu(t)\int_0^t \sqrt{\mu(s)}\,ds| \leq C\epsilon\mu(t)
\]

\[
\left(\int_0^t \sqrt{\mu(s)}\,ds + \int_0^t |\mu'(s)|\,ds\right)
\]

where \( C \) depends only on \( Q_c \).

**Remark 3.1.** The statement in theorem 3.2 deserves to be clarified, at least with reference to some points, and its relation with the nonlinear stability result established in theorem 1.1 above for \( \alpha \in (\frac{1}{2}, 2) \), \( p < p_{\text{max}}(\alpha) \) and \( p < 2\alpha \).

1. The regularity required for the initial data \( u_0 \) is given to ensure that the associated solution \( u \) satisfies, in a classical sense, equation (3.1).

2. A similar analysis may be made to obtain the behavior of the translation parameter involved in the nonlinear stability result in theorem 1.1. In this case, \( \mu(t) \equiv 1 \) for all \( t \) and so for \( \gamma = \gamma(t) \) such that

\[
\|u(\cdot + \gamma, t) - Q_c\|_2 \leq \epsilon,
\]  

(3.29)
satisfies for all $t \in [0, t^*)$.

\[
\left| \gamma(t) - ct \right| \leq Cct
\]

where $C$ depends only on $Q_c$.

4. Nonlinear stability and spectral instability for the fBBM equation

This section is devoted to the fBBM equation

\[
u_t + u_x + \partial_x(u^2) + D^\alpha u_t = 0,
\]

(4.1)

for $\alpha \in \left(\frac{1}{3}, 1\right)$. As the structure of the analysis is similar to that used for the fKdV, we will only indicate the new basic differences.

Consider a solitary wave solution $u(x, t) = \psi_c(x - ct)$, $c > 1$, of the fBBM equation (4.1). Then the profile $\psi_c$ satisfies the equation

\[
D^\alpha \psi_c + \left(1 - \frac{1}{c}\right) \psi_c - \frac{1}{c} \psi_c^2 = 0.
\]

(4.2)

Therefore, we obtain the following Pohozaev identity

\[
(3\alpha - 1) \int_{\mathbb{R}} |D^{\alpha/2} \psi_c|^2 \, dx = \left(1 - \frac{1}{c}\right) \int_{\mathbb{R}} |\psi_c|^2 \, dx,
\]

(4.3)

proving that no finite energy solitary waves exist when $c > 1$ and $\alpha \leq \frac{1}{3}$ hold.

By considering the new variable $v(x, t) = u(x + ct, t) - \psi_c(x)$, it follows from (4.1) that

\[
(\partial_t - c\partial_x)(v + D^\alpha v) + \partial_x(v + 2\psi_c v + O(\|v\|^2)) = 0.
\]

(4.4)

The equation

\[
(\partial_t - c\partial_x)(v + D^\alpha v) + \partial_x(2\psi_c v + v) = 0,
\]

(4.5)

represents the linearized equation for (4.1) around $\psi_c$. So, we will provide sufficient conditions for obtaining that the solution $v \equiv 0$ is unstable by the linear flow of (4.5). More precisely, we are interested in finding a growing mode solution of (4.5) with the form $v(x, t) = e^{\lambda t}u(x)$ and $\text{Re}\lambda > 0$. Thus, we obtain that $u$ satisfies the following non-local differential equation,

\[
D^\alpha u + u + \frac{\partial_x}{\lambda - c \partial_x}(u + 2\psi_c u) = 0.
\]

(4.6)

This motivates us to define the following family of closed linear operators $B^\lambda : H^\alpha(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$, $\text{Re}\lambda > 0$, given by

\[
B^\lambda v \equiv (D^\alpha + 1)v + \frac{\partial_x}{\lambda - c \partial_x}(v + 2\psi_c v).
\]

(4.7)

Next, we consider the unbounded self-adjoint operator $L_0 : H^\alpha(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ associated with (4.2)

\[
L_0 = D^\alpha + \left(1 - \frac{1}{c}\right) - \frac{2}{c} \psi_c.
\]

(4.8)

and so $\psi'_c \in \text{Ker}(L_0)$. 

949
Our first result concerns the behavior of $B^\lambda$ by depending on $\lambda$.

**Proposition 4.1.** For $\lambda > 0$, the operator $B^\lambda$ converges to $L_0$ strongly in $L^2(\mathbb{R})$ when $\lambda \to 0^+$, and converges to $D^\alpha + 1$ strongly in $L^2(\mathbb{R})$ when $\lambda \to \infty$.

**Proof.** Similar to that of proposition 2.1. \qed

Next, we localized the essential spectrum of $B^\lambda$, $\sigma_{\text{ess}}(B^\lambda)$.

**Proposition 4.2.** For any $\lambda > 0$, we have

$$\sigma_{\text{ess}}(B^\lambda) \subset \{ z : \Re z \geq \frac{1}{2} \left(1 - \frac{1}{\epsilon}\right) \}. \quad (4.9)$$

The idea of the proof of proposition 4.2 is the same as proposition 1 in Lin [34]. The next lemma is similar to lemma 2.2 above.

**Lemma 4.1.** Given $\lambda > 0$, let $\zeta \in C_0^\infty(\mathbb{R})$ be a cut-off function such that $\zeta_{|\{|x| \leq R_0\}} = 1$, for some $R_0$. Define $\zeta_d(x) = \zeta(x/d)$, $d > 0$. Then, for each $d$, the operator $\zeta_d(B^\lambda - z)^{-1}$ is compact for some $z \in \rho(B^\lambda)$, and there exists $C(d) \to 0$ as $d \to \infty$ such that for any $u \in C_0^\infty(\mathbb{R})$,

$$\| [B^\lambda, \zeta_d]u \| \leq C(d)(\|B^\lambda u\| + \|u\|). \quad (4.10)$$

Next, we study the behavior of $B^\lambda$ near infinity. The next result shows the non-existence of growing modes at the left-hand side of the complex-plane for large $\lambda$ (see Lin [34]), so, since the eigenvalues of $B^\lambda$ appear in conjugate pairs, there are no growing modes for large $\lambda$.

**Lemma 4.2.** There exists $\Lambda > 0$, such that for $\lambda > \Lambda$, $B^\lambda$ has no eigenvalues in $\{ z : \Re z \leq 0 \}$.

Next, we study the behavior of $B^\lambda$ for small positive $\lambda$. Its result extends those of Lin in [34] for the case of the fBBM equation (4.1).

**Lemma 4.3.** Given $F \in C_0^\infty(\mathbb{R})$, consider any sequence $\lambda_n \to 0^+$ and $\{u_n\} \subset H^\alpha(\mathbb{R})$ satisfying

$$\|B^{\lambda_n}u_n\| + \|u_n\| \leq M_1 < \infty \quad (4.11)$$

for some constant $M_1$. Then if $w - \lim_{n \to \infty} u_n = 0$, we have

$$\lim_{n \to \infty} \|Fu_n\| = 0 \quad (4.12)$$

and

$$\lim_{n \to \infty} \|[B^{\lambda_n}, F]u_n\| = 0. \quad (4.13)$$

**Proof.** The proof in this case is immediate. First, $[B^{\lambda}, F] = [D^{\alpha}, F] + [G^{\lambda}, F]$ with

$$G^{\lambda} = \frac{\partial_x}{\lambda - c\partial_x}(1 + 2\psi_x).$$

Now, from the proof of lemma 2.4 above we have that $[D^{\alpha}, F]$ is a compact operator. Next, the convergence
(3^{\lambda_n}, F|u_n \to 0, \quad \text{as} \ n \to \infty,

is proved by following the same ideas in the proof of lemma 2.5 in [34]. This completes the lemma.

The next lemma is basic in the stability theory (see also ch 19 in Hislop and Sigal [22]), and its proof follows from the estimate

\[ \Re \langle (B^\lambda - z)u, u \rangle \geq \frac{1}{4} (1 - \frac{1}{c}) \|u\|^2. \]

**Lemma 4.4.** Let \( z \in \mathbb{C} \) with \( \Re z \leq \frac{1}{2} (1 - \frac{1}{c}) \), then there is \( n > 0 \) such that for all \( u \in C_\infty^b(\{x| n\}) \), we have

\[ \| (B^\lambda - z)u \| \geq \frac{1}{4} c\|u\|, \quad (4.14) \]

when \( \lambda \) is sufficiently small.

Thus, from lemmas 4.3 and 4.4 above, we obtain the following stability result for every discrete eigenvalue of \( \mathcal{L}_0 \) with \( \alpha \in (0, 1) \).

**Theorem 4.1.** Each discrete eigenvalue \( \kappa_0 \) of \( \mathcal{L}_0 \) with \( \kappa_0 \leq \frac{1}{2} (1 - \frac{1}{c}) \) is stable with respect to the family \( B^\lambda \).

Next, we establish the moving kernel formula for the fBBM equations.

**Lemma 4.5.** Let \( c > 1 \). Assume that \( \ker(\mathcal{L}_0) = \left[ \frac{d}{dx} \psi_c \right] \). For small enough \( \lambda > 0 \), let \( \kappa_\lambda \in \mathbb{R} \) be the only eigenvalue of \( B^\lambda \) near the origin. Then,

\[ \lim_{\lambda \to 0^+} \frac{\kappa_\lambda}{\lambda^2} = -\frac{1}{c} \frac{dM}{dc} \quad (4.15) \]

with \( M(c) = \frac{1}{2} \langle (D^\alpha + 1) \psi_c, \psi_c \rangle \). Therefore, for \( \frac{dM}{dc} > 0 \) we obtain \( \kappa_\lambda < 0 \) and for \( \frac{dM}{dc} < 0 \) we obtain \( \kappa_\lambda > 0 \).

**Proof.** The proof is similar to that of lemma 2.7 in Lin [34].

The spectral instability result for the fBBM equation (4.1) is as follows.

**Theorem 4.2 (Spectral instability criterium for fBBM equations).** Let \( c \to \psi_c \in H^{\alpha + 1}(\mathbb{R}) \) be a smooth curve of a positive solitary wave solution to equation (4.2) with \( \alpha \in (\frac{1}{2}, \frac{1}{2}) \), \( p = 1 \). The wave-speed \( c \) can be considered over some nonempty interval \( I, I \subset (1, +\infty) \). We assume that the self-adjoint operator \( \mathcal{L}_0 = D^\alpha + \left( 1 - \frac{1}{c} \right) - \frac{2}{c} \psi_c \) with domain \( D(\mathcal{L}_0) = H^\alpha(\mathbb{R}) \) satisfies

\[ \ker(\mathcal{L}_0) = \left[ \frac{d}{dx} \psi_c \right]. \quad (4.16) \]

Denote by \( n(\mathcal{L}_0) \) the number (counting multiplicity) of negative eigenvalues of the operator \( \mathcal{L}_0 \). Then there is a purely growing mode \( e^{\lambda t}u(x) \) with \( \lambda > 0 \), \( u \in H^s(\mathbb{R}) - \{0\} \), \( s \geq 0 \), to the linearized equation (4.5) if one of the following two conditions is true:
(i) $n(L_0)$ is even and $\frac{d}{dc}M(c) > 0$;
(ii) $n(L_0)$ is odd and $\frac{d}{dc}M(c) < 0$.

where $M(c) = \frac{1}{2}(D^\alpha + 1)\psi_c, \psi_c$.

Next, we show the nonlinear stability and spectral instability theorems for the fBBM established in the introduction (theorems 1.4 and 1.5) for the ground state solutions associated with equation (4.2). First, we study the sign of the quantity $\frac{d}{dc}M(c)$.

**Lemma 4.6.** Let $c > 1$ and $\alpha \in (\frac{1}{2}, 1)$. Then for any solution $\psi_c$ of (4.2) we have

$$2M(c) = \left[ -\frac{1}{3\alpha - 1}e^{\frac{2}{3\alpha} - 1}c(1 - 1)^{3 - \frac{2}{3\alpha}} + e^{\frac{2}{3\alpha}}c(1 - 1)^{3 - \frac{2}{3\alpha}} \right] \|\Psi\|, \quad (4.17)$$

where $\Psi$ satisfies $D^\alpha\Psi + \Psi - \psi^2 = 0$. Thus, we obtain that

$$\frac{d}{dc}M(c) = \begin{cases} < 0 & \text{for } \alpha \in (\frac{1}{2}, 1) \text{ and } c > 1 \\
> 0 & \text{for } \alpha \in (\frac{1}{2}, \frac{1}{3}) \text{ and } c > 0 \\
< 0 & \text{for } \alpha \in (\frac{1}{3}, \frac{1}{2}) \text{ and } 1 < c < c_0. \end{cases} \quad (4.18)$$

where $c_0 > 1$ is the larger positive root of the polynomial $q(c) = 6\alpha^2c^2 - 4\alpha c + 1 - \alpha$, and it is given in (4.22) below.

**Proof.** Initially, suppose that $\psi_c$ is a solution of (4.2) with $c > 1$ then for the scaling $\Psi(x) = a\psi_c(bx)$ with

$$a = \frac{1}{c - 1}, \quad \text{and} \quad b = \left( \frac{c}{c - 1} \right)^{1/\alpha} \quad (4.19)$$

we obtain that the profile $\Psi$ satisfies $D^\alpha\Psi + \Psi - \psi^2 = 0$. Thus, we obtain that

$$(D^\alpha + 1)\psi_c(x) = \frac{1}{ab^\alpha}D^\alpha\Psi(x/b) + \frac{1}{a}\Psi(x/b).$$

Therefore, from the relation $(3\alpha - 1)\int |D^\alpha\Psi(x)|^2dx = \int |\Psi(x)|^2dx$ we obtain

$$2M(c) = \left[ -\frac{1}{3\alpha - 1}e^{\frac{2}{3\alpha} - 1}c(1 - 1)^{3 - \frac{2}{3\alpha}} + e^{\frac{2}{3\alpha}}c(1 - 1)^{3 - \frac{2}{3\alpha}} \right] \|\Psi\|^2 \equiv p(c)\|\Psi\|^2. \quad (4.20)$$

Next, we determine the sign of the derivative of $p(c)$ defined in (4.20). A simple calculation shows that

$$p'(c) = \frac{e^{\frac{2}{3\alpha}}c(1 - 1)^{\frac{2}{3\alpha}}}{(c - 1)^{\frac{2}{3\alpha}}} \left[ \frac{1 - \alpha}{\alpha} \frac{1}{3\alpha - 1} \frac{(c - 1)^2}{c^2} + \frac{2}{\alpha} - 1 \right]. \quad (4.21)$$

Thus, it follows immediately from (4.21) that for $c > 1$ and $\alpha \in (\frac{1}{2}, 1)$ we have $p'(c) > 0$. Next, it is not difficult to see that $p''(c) > 0$ for every $c > 1$. Moreover, since $\alpha < \frac{1}{2}$ and

$$\lim_{c \to 1^+} \frac{e^{\frac{2}{3\alpha}}c(1 - 1)^{\frac{2}{3\alpha}}}{(c - 1)^{\frac{2}{3\alpha}}} = +\infty$$
we have $\lim_{c \to +\infty} p'(c) = -\infty$. Now, for $c \to +\infty$

$$p'(c) \approx c \left( \frac{1 - \alpha}{\alpha} - \frac{1}{3\alpha - 1} + \frac{2\alpha + 1}{\alpha} \right)$$

and for $1 > \alpha > \frac{1}{2}$, we get that $\lim_{c \to +\infty} p'(c) = +\infty$. Therefore, there is a unique point $c_0 > 1$ such that $p'(c_0) = 0$. Thus, we obtain for $\alpha \in (\frac{1}{2}, \frac{1}{2})$ that $p'(c) < 0$ for $c \in (1, c_0)$ and $p'(c) > 0$ for $c \in (c_0, +\infty)$.

Now, to determine $c_0$, we have that $p'(c) = 0$ if and only if $q(c) = 6\alpha^2c^2 - 4\alpha + 1 - \alpha = 0$. Since $q(1) = (3\alpha - 1)(2\alpha - 1) < 0$, we have that the real zeros of $q$ and $c_0$ satisfy $r_0 < 1 < c_0$. The exact value of $c_0$ is given by

$$c_0 = \frac{2 + \sqrt{2(3\alpha - 1)}}{6\alpha}. \tag{4.22}$$

This completes the proof.

Proof of theorem 1.4. For $\alpha \in (\frac{1}{4}, 1)$, the scaling $Q(x) = a\Phi(x)$ with $a$ and $b$ defined in (4.19) implies that the ground state $Q$ satisfies $D^\alpha Q + Q - Q^2 = 0$. Therefore, from [16] it follows that the self-adjoint operator $\mathcal{L} = D^\alpha + 1 - 2Q$ satisfies $\text{Ker}(\mathcal{L}) = [\frac{d}{dx} Q]$ and $n(\mathcal{L}) = 1$.

Thus, using a similar analysis to that in the proof of theorem 1.1 above, we conclude that for $\beta_0 = D^\alpha + \left(1 - \frac{1}{\alpha}\right) - \frac{2}{\alpha} \Phi$, satisfies $\text{Ker}(\beta_0) = [\frac{d}{dx} \Phi]$ and $n(\beta_0) = 1$.

Now, for $M(c) = \frac{1}{2}((D^\alpha + 1)\Phi, \Phi)$ we have from lemma 4.6 that $M'(c) > 0$ exactly for $\alpha \in [\frac{1}{2}, 1)$ and $c > 1$, and for $\alpha \in (\frac{1}{4}, \frac{1}{2})$ and $c > c_0$. Hence, from the theory of Grillakis et al [19] we complete the proof.

As a consequence of theorem 4.2, we obtain the proof of theorem 1.5.

Proof of theorem 1.5. From the proof of theorem 1.4 we have that the self-adjoint operator $\beta_0 = D^\alpha + \left(1 - \frac{1}{\alpha}\right) - \frac{2}{\alpha} \Phi$, satisfies $\text{Ker}(\beta_0) = [\frac{d}{dx} \Phi]$ and $n(\beta_0) = 1$. Moreover, lemma 4.6 establishes that $\frac{d}{dx} M(c) < 0$ for $\alpha \in (\frac{1}{4}, \frac{1}{2})$ and $c \in (1, c_0)$, where $c_0$ is given in (4.22). It completes the theorem.

Remark 4.1. Next we provide the following observations with regards to the stability theorems 1.4 and 1.5.

(1) Similar to the fKdV case, the statement of stability in theorem 1.4 is a conditional one in the sense that for all $\epsilon > 0$, there is a $\delta > 0$ such that if $u_0 \in H^s(\mathbb{R}) \cap U_\delta$, for $s > \frac{3}{2} - \alpha$, then $u(t) \in U_\epsilon$, for all $t \in (-T_\epsilon, T_\epsilon)$, where $T_\epsilon$ is the maximal time of existence of $u$ satisfying $u(0) = u_0$. We recall that the best known result of local well-posedness for the fBBM model (4.1) is given in [37] for the initial data in $H^s(\mathbb{R})$, $s > \frac{3}{2} - \alpha$ and $\alpha \in (0, 1)$, which does not allow a globalization of the solution using conservation laws.

(2) We recall that [36] showed the existence and stability of solitary wave solutions for the fBBM by considering the minimization problem

$$I_q = \inf \left\{ \int_{\mathbb{R}} u^2 + |D^{\alpha/2} u|^2 \, dx : u \in H^2(\mathbb{R}) \text{ and } \int_{\mathbb{R}} \frac{u^2}{2} + \frac{u^3}{3} \, dx = q \right\}.$$

953
For $\alpha \in (\frac{1}{3}, \frac{1}{2})$, a critical value constraint $q_0 = q_0(\alpha)$ was established in such a way that for $q > q_0$, the set of ground state solutions associated with the variational problem above will be stable in $H^2(\mathbb{R})$. From our analysis in lemma 4.6 and theorem 1.4, we note that the critical value constraint $q_0$ can be determined explicitly in terms of the threshold value $c_0$ in (4.22).

(3) Our orbital stability and spectral instability results in theorems 1.4 and 1.5 for the fBBM equation show a scenario similar to that known for the generalized BBM equation (GBBM)

$$u_t + uu_x + u^p u_x - u_{txt} = 0.$$ 

Indeed, the critical exponent for the stability of solitary wave solutions for the GBBM is $p = 4$, though the explanation for instability when $p \geq 4$ is different. In fact, from Souganidis and Strauss [48], solitary waves of the GBBM of arbitrary positive velocity are stable when $p < 4$, but when $p \geq 4$, there exists $c_* = c_*(p)$ such that the solitary waves of velocity $c < c_*$ are unstable (nonlinearly) while those of velocity $c > c_*$ are nonlinearly stable.

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