NON POWER BOUNDED GENERATORS
OF STRONGLY CONTINUOUS SEMIGROUPS

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1. Introduction

Given a Banach space, then every linear and continuous operator from the space into itself generates a
C₀-semigroup which is given by an exponential series representation. In the Banach space world of
C₀-semigroups this case of a continuous generator is thus considered to be the trivial situation. The
picture changes completely already in the considerably harmless appearing case of complete metrizable
spaces. Based on a question of Conejero [7] several relations between continuity of the generator,
uniforminity of the semigroup and validity of a series representation have recently been revealed
by Albanese, Bonet, Ricker [11 Thm. 3.3 and Prop. 3.2] and Frerick, Jordá, Kaln«es, Wengenroth [10].
In the general case of a sequentially complete locally convex space X, it seems that the only result
available so far is the generation theorem mentioned in the book [14] of Yosida, which states that a
power bounded operator is always a generator. Here, A ∈ L(X) is power bounded if all its powers
form an equicontinuous subset of L(X), i.e., if for every continuous seminorm p on X there exists a
continuous seminorm q on X and a constant M > 0 such that the estimate p(A^n x) ≤ Mq(x) holds for
all n ∈ N₀ and all x ∈ X. It is straightforward to generalize the above statement as follows.

Theorem 0. Let A ∈ L(X). Assume that there exists µ > 0 such that µA is power bounded. Then
A generates a uniformly continuous C₀-semigroup (T(t))₁≥₀ which is given by the formula
T(t) = ∑∞ₙ=₀ (tA)^n / n! for t ≥ 0 where the series converges absolutely with respect to the topology of uniform
convergence on the bounded subsets of X.

Inspired by Allan [2, p. 400] in this paper operators with the property assumed in Theorem 0 are said
to be a-bounded. Recently, Domański [8] asked, if the statement above can be improved in the sense
of a condition weaker than a-boundedness which still assures the series representation or at least the
generator property. Frerick, Jordá, Kaln«es, Wengenroth [10] characterized generation for a special
class of Fréchet spaces by a condition closely related to the notions of m-topologizable and topologizable
operators due to Żelazko [15, 16].

In this paper we consider a quantitative version of topologizability which is weaker than m-topologizability
but implies the generator property and guarantees a series representation. Also for the case of an
m-topologizable operator this result is new and improves Theorem 0. In combination with Bonet [11,
Ex. 6] the latter shows that there exist complete, non-normal spaces on which every continuous op-
erator is a generator—as in the case of a Banach space—although in general this is well-known to be
not the case, see [10, Ex. 1]. We provide examples which show that our result applies to a class of
operators which is strictly larger than those of the m-topologizable ones. Our counterexamples show
that topologizability alone in general neither is necessary nor sufficient for generation. A variation

2010 Mathematics Subject Classification: Primary 47D06; Secondary 34G10, 46E10, 46A03, 46A45.
Key words and phrases: Strongly continuous semigroup, power bounded operator, Fréchet space

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a Anna Golińska is supported by the National Science Centre (Poland) grant no. 2013/10/A/ST1/00091.
of our main result, see Theorem 2 suggests that it might be possible that an operator generates a \( C_0 \)-semigroup but that the series representation is only valid on a finite time interval and fails for large times. It is open if such \( C_0 \)-semigroups does really occur in nature.

For the theory of locally convex spaces we refer to Meise, Vogt [13] and Jarchow [11]. For basic facts about semigroups on locally convex spaces we refer to Yosida [14] and Komura [12, Section 1].

2. Notation

For the whole article let \( X \) be a sequentially complete locally convex space. We denote by \( cs(X) \) the system of all continuous seminorms on \( X \), by \( \mathcal{B} \) the collection of all bounded subsets of \( X \) and by \( L(X) \) the space of all linear and continuous maps from \( X \) into itself. We write \( L_b(X) \), if \( L(X) \) is furnished with the topology of uniform convergence on the bounded subsets of \( X \) given by the seminorms

\[ q_\beta(S) = \sup_{x \in \beta} q(Sx) \]

for \( S \in L(X) \), \( \beta \in \mathcal{B} \) and \( q \in cs(X) \). Under a \( C_0 \)-semigroup \( (T(t))_{t \geq 0} \) on \( X \) we understand a family of maps \( T(t) \in L(X) \) such that \( T(0) = \text{id}_X \), \( T(t+s) = T(t)T(s) \) for \( t, s \geq 0 \) and \( \lim_{t \to t_0} T(t)x = T(t_0)x \) for \( x \in X \) and \( t_0 \geq 0 \). \( (T(t))_{t \geq 0} \) is said to be uniformly continuous if \( T(t) \colon [0, \infty) \to L_b(X) \) is continuous. The generator \( A \colon D(A) \to X \) of a \( C_0 \)-semigroup \( (T(t))_{t \geq 0} \) is defined by

\[ Ax = \lim_{t \to 0} \frac{T(t)x-x}{t}, \quad x \in D(A) = \{x \in X \colon \lim_{t \to 0} T(t)x-x \text{ exists}\}. \]

The aim of this article is to identify conditions which guarantee that for a given \( A \in L(X) \) there exists a \( C_0 \)-semigroup \( (T(t))_{t \geq 0} \) such that \( A \) is its generator. In Section 1 we mentioned already a classical condition of this type. In the remainder we use the following three conditions which appeared in the literature in different contexts. The first condition below is the assumption of Theorem 0. In view of Allan [2, p. 400] we say that an operator is \( a \)-bounded if

\[ \exists \mu > 0 \forall p \in cs(X) \exists q \in cs(X) \forall n \in \mathbb{N}_0, x \in X : p(A^n x) \leq \mu^n q(x) \tag{1} \]

holds; this prevents confusion with the notion of a bounded operator in the sense of [13, p. 375]. The following two conditions are due to Želazko [15, 16] and arise in view of (1) by allowing that \( \mu \) is not constant but may depend on \( p, q \) or \( n \). We say that \( A \) is \( m \)-topologizable if

\[ \forall p \in cs(X) \exists q \in cs(X), \mu > 0 \forall n \in \mathbb{N}_0, x \in X : p(A^n x) \leq \mu^n q(x) \tag{2} \]

holds and we say that \( A \) is topologizable if

\[ \forall p \in cs(X) \exists q \in cs(X) \forall n \in \mathbb{N}_0 \exists \mu_n > 0 \forall x \in X : p(A^n x) \leq \mu_n q(x) \tag{3} \]

is valid. In both conditions \( \mu \) depends on \( p \), and in the sense that different configurations of \( n \) and \( q \) are possible, also on \( q \). Clearly, every \( a \)-bounded operator is \( m \)-topologizable and every \( m \)-topologizable operator is topologizable; in Theorem 1 below we employ a quantitative version of topologizability; this will make clear why in our notation only the dependency \( \mu = \mu_n \) is mentioned explicitly.

3. Generation

**Theorem 1.** Let \( X \) be a sequentially complete locally convex space and \( A \in L(X) \). Assume that

\[ \forall R > 0, p \in cs(X) \exists q \in cs(X) \forall n \in \mathbb{N}_0 \exists \mu_n > 0 \forall x \in X : p(A^n x) \leq \mu_n q(x) \text{ and } \sum_{n=0}^{\infty} \frac{\mu_n}{n!} R^n < \infty \tag{4} \]

holds. Then \( A \) generates a uniformly continuous semigroup \( (T(t))_{t \geq 0} \) which is given by the exponential series expansion

\[ (\text{EXP}) \quad T(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n \]

where the latter converges in \( L_b(X) \).
Proof. Fix $R > 0$, $p \in \text{cs}(X)$ and $B \subseteq X$ bounded. Select $q$ and $(\mu_n)_{n \in \mathbb{N}_0}$ such that (4) is satisfied. Put $K = \sup_{x \in B} q(x)$. By (4) we have $p(A^n x) \leq \mu_n q(x)$ and consequently $p(\frac{\mu_n}{n!} A^n x) \leq \frac{\mu_n}{n!} \mu_n q(x)$ for every $n \in \mathbb{N}_0$ and every $x \in X$. Thus for all $t < R$

$$\sum_{n=0}^{\infty} p_B(\frac{\mu_n}{n!} A^n x) = \sum_{n=0}^{\infty} p(\frac{\mu_n}{n!} A^n x) \leq \sup_{x \in B} q(x) \sum_{n=0}^{\infty} \frac{\mu_n}{n!} = K f(t) < \infty$$

holds, where $f(t) = \sum_{n=0}^{\infty} \frac{\mu_n t^n}{n!}$ according to (4) defines a function $f \in C([0, R))$. This implies the absolute convergence of the exponential series in $L_b(X)$.

It is clear that $T(0) = 1_{X}$ holds. Let $t$, $s \geq 0$ be given. Fix $x \in X$ and $p \in \text{cs}(X)$. Select $q$ and $(\mu_n)_{n \in \mathbb{N}_0}$ according to (4) for some $R > \max\{s, t\}$. Put

$$a_i = \frac{\mu_i}{i!} A^i, \quad b_i = \frac{\mu_i}{i!} A^i \quad \text{and} \quad c_i = \sum_{k=0}^{i} a_k b_{i-k}.$$ 

Denote by $A_n$, $B_n$ and $C_n$ the corresponding partial sums. By $A_\infty$ and $B_\infty$ we denote the limits of $A_n$ and $B_n$ in $L_b(X)$. Then

$$C_n = \sum_{i=0}^{n} \sum_{k=0}^{i} a_k b_{i-k} = \sum_{i=0}^{n} a_{n-i} B_i = A_n B_\infty + \sum_{i=0}^{n} a_{n-i} (B_i - B_\infty)$$

holds. We fix $\varepsilon > 0$. Since $A_n \rightarrow A_\infty$ in $L_1(X)$ there exists $L \in \mathbb{N}_0$ such that $p((A_n - A_\infty) B_\infty x) \leq \frac{\varepsilon}{3}$ holds for all $n \geq L$. Since $B_n \rightarrow B_\infty$ in $L_1(X)$ there exists $N \in \mathbb{N}_0$ such that $q((B_i - B_\infty) x) \leq \frac{\varepsilon}{3} f(t)^{-1}$ holds for $i \geq N$. Now we put $K = \max_{i=0, \ldots, N-1} q((B_i - B_\infty) x)$ and select $M \in \mathbb{N}_0$ such that the estimate $\sum_{i=M}^{\infty} \frac{\mu_i}{i!} \leq \frac{\varepsilon}{3} K^{-1}$ holds. For $n \geq \max\{L, N, M\}$ we compute

$$p((C_n - A_\infty B_\infty)x) = p((A_n - A_\infty) B_\infty x) + \sum_{i=0}^{n} p(a_{n-i} (B_i - B_\infty) x) \leq p((A_n - A_\infty) B_\infty x) + \sum_{i=n}^{N-1} p(a_{n-i} (B_i - B_\infty) x) + \sum_{i=n}^{\infty} p(a_{n-i} (B_i - B_\infty) x)$$

The first summand is less or equal to $\frac{\varepsilon}{3}$ since $n \geq L$. For the second summand we estimate

$$\sum_{i=N}^{n} p(a_{n-i} (B_i - B_\infty) x) \leq \sum_{i=N}^{n} \frac{\mu_{n-i}}{(n-i)!} q((B_i - B_\infty) x) \leq \frac{\varepsilon}{3} f(t)^{-1} \sum_{i=N}^{n} \frac{\mu_{n-i}}{(n-i)!} \leq \frac{\varepsilon}{3}.$$ 

Here we used (4) as at the very beginning of this proof for the first estimate. The second follows from $i \geq N$ according to our selection of $N$. For the third summand we get

$$\sum_{i=0}^{N-1} p(a_{n-i} (B_i - B_\infty) x) \leq \sum_{i=0}^{N-1} \frac{\mu_{n-i}}{(n-i)!} q((B_i - B_\infty) x) \leq K \sum_{i=0}^{N-1} \frac{\mu_{n-i}}{(n-i)!} \mu_{n-i} \leq K \sum_{i=M}^{\infty} \frac{\mu_i}{i!} \leq \frac{\varepsilon}{3}.$$ 

This shows that $C_n \rightarrow A_\infty B_\infty$ holds in $L_b(X)$. By construction $A_\infty = T(t)$ and $B_\infty = T(s)$. Moreover, $C_n = \sum_{i=0}^{n} \frac{\mu_i}{i!} A^i$ by direct computation and thus by the first part $C_n \rightarrow T(t+s)$ in $L_b(X)$. This establishes the evolution property $T(t+s) = T(t) T(s)$.

Let $p \in \text{cs}(X)$ and $B \subseteq X$ bounded be given. Select $q \in \text{cs}(X)$ and $(\mu_n)_{n \in \mathbb{N}_0}$ as in (4) for $R = 1$. Put $K = \sup_{x \in B} q(x)$ and denote by $f$ the function given by the power series in (4). Then we have

$$p_B(T(0) - T(t)) = \sup_{x \in B} p(\sum_{n=1}^{\infty} \frac{\mu_n}{n!} A^n x) \leq K \sum_{n=1}^{\infty} \frac{\mu_n}{n!} = K (f(t) - f(0))$$

for every $0 \leq t < 1$ which shows together with the evolution property that $T(\cdot)\colon [0, \infty) \rightarrow L_b(X)$ is continuous at $t = 0$. We have the following condition

$$\forall p \in \text{cs}(X), \ R > 0 \ \exists \ q \in \text{cs}(X), \ f \in C([0, R)) \ \forall \ x \in X, \ t \in [0, R): \ p(T(t)x) \leq f(t) q(x).$$

Indeed, for given $p \in \text{cs}(X)$, $R > 0$ we select $q \in \text{cs}(X)$ and $(\mu_n)_{n \in \mathbb{N}_0}$ as in (4). We define $f \in C([0, R))$ by the series in (4) and use the estimate in (4) to obtain $p(T(t)x) \leq q(x)f(t)$ as desired. Let now
Let $B \subseteq X$ bounded be given. We fix $t > 0$ and put $R = t + 1$. Then we select $q \in \text{cs}(X)$ and $f \in C([0, R])$ according to the condition above. For $0 < h < 1$ we compute

$$p_B(T(t) - T(t + h)) = \sup_{x \in B} p(T(t)(T(0) - T(h))) = f(t) \sup_{x \in B} q(T(0) - T(h)) = f(t) q_B(T(0) - T(h))$$

which converges to zero for $h \searrow 0$ by the first part. For $-t/2 < h < 0$ we compute

$$p_B(T(t) - T(t + h)) = \sup_{x \in B} p(T(t + h)(T(-h) - T(0))) = f(t + h) q_B(T(-h) - T(0))$$

which converges to zero for $h \nearrow 0$ by the first part, since $f([0, t]) \subseteq C$ is bounded, and since $0 \leq t + h < t$ holds for all $h$ under consideration. This establishes the continuity of $T(\cdot): [0, \infty) \rightarrow L_b(X)$ at every $t > 0$.

Finally let $p \in \text{cs}(X)$ be given and select a last time $q \in \text{cs}(X)$ and $(\mu_n)_{n \in \mathbb{N}_0}$ as in (4) for $R = 1$. For $x \in X$ and $0 < t < 1$ we have

$$p\left(\frac{(t)x - T(0)x}{t} - Ax\right) = p\left(\frac{1}{t} \sum_{n=2}^{\infty} \frac{\mu_n}{n!} A^n x\right) \leq \frac{1}{t} q(x) \sum_{n=2}^{\infty} \frac{1}{n!} \mu_n = q(x) \left(\frac{t(f(t) - f(0))}{t} - \mu_1\right)$$

which shows that the generator of $(T(t))_{t \geq 0}$ is indeed $A$.

**Theorem 2.** Let $X$ be a sequentially complete locally convex space and $A \in L(X)$. Assume that

$$\exists R > 0 \forall p \in \text{cs}(X) \exists q \in \text{cs}(X) \forall n \in \mathbb{N}_0 \exists \mu_n > 0 \forall x \in X : p(A^n x) \leq \mu_n q(x) \text{ and } \sum_{n=0}^{\infty} R^n < \infty \hspace{1cm} (5)$$

holds. Then $A$ generates a $C_0$-semigroup $(T(t))_{t \geq 0}$. The map $T(\cdot): [0, R] \rightarrow L_b(X)$ is uniformly continuous and given by the exponential series expansion (EXP) where the latter converges in $L_b(X)$.

**Proof.** The proof of Theorem 1 shows that with $R > 0$ as in (5) the formula (EXP) defines a function $T(\cdot): [0, R] \rightarrow L(X)$ which is uniformly continuous and satisfies $T(s + t) = T(s)T(t)$ whenever $s, t \geq 0$ are such that $s + t \leq R$ holds. For $t > R$ we define $T(t) = T(R)^n T(w)$ where $t = nR + w$ with $n \in \mathbb{N}_0$ and $0 \leq w < R$ and get the $C_0$-semigroup $(T(t))_{t \geq 0}$ whose generator is $A$ by the proof of Theorem 1. \qed

**Remark 3.**

(i) Minor adjustments in the proof of Theorem 1 show that if $A$ satisfies condition (4) or condition (5) then it even generates a $C_0$-group.

(ii) Let $\Gamma \subseteq \text{cs}(X)$ be a fundamental system of seminorms. Each of the conditions (1)-(5) holds already if the condition is true with $\Gamma$ instead of the set $\text{cs}(X)$ of all continuous seminorms. For the conditions (2)-(5) the equivalence holds; (1) is satisfied if and only if

$$\exists \mu > 0 \forall p \in \Gamma \exists q \in \Gamma, M \geq 0 \forall n \in \mathbb{N}_0, x \in X : p(A^n x) \leq M \mu^n q(x) \hspace{1cm} (6)$$

is valid for some, or equivalently, for all fundamental systems of seminorms $\Gamma$.

(iii) Under the assumptions of Theorem 2 the series representation (EXP) holds for $0 \leq t \leq R$. The proof does not show that it also holds for $t > R$. However, there is no concrete example of a $C_0$-semigroup with this property.

4. **Examples**

Let $B = (b_{j,k})_{j,k \in \mathbb{N}}$ be a Köthe matrix [13 Section 27], i.e., $0 \leq b_{j,k} \leq b_{j,k+1}$ holds for all $j, k \in \mathbb{N}$ and for every $j \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $b_{j,k} > 0$ holds. Let

$$
\lambda^r(B) = \{ x \in \mathbb{C}^\mathbb{N} : \forall k \in \mathbb{N} : \|x\|_k = \left(\sum_{j=1}^{\infty} |b_{j,k} x_j|^r\right)^{1/r} < \infty \},
$$

$$
\lambda^\infty(B) = \{ x \in \mathbb{C}^\mathbb{N} : \forall k \in \mathbb{N} : \|x\|_k = \sup_{j \in \mathbb{N}} b_{j,k} |x_j| < \infty \}
$$

denote the Köthe echelon spaces of order $r \in [1, \infty]$, which are Fréchet spaces with the fundamental system $\Gamma = ([\| \cdot \|_q]_{q \in \mathbb{N}})$ of seminorms. In view of the conditions explained in Section 2 and 3 we identify $q \equiv \| \cdot \|_q$, cf. also Remark 3(ii).
Our first result shows that for diagonal operators $Ax = (a_jx_j)_{j \in \mathbb{N}}$ the situation is similar to the case in a classical Banach sequence space if and only if $(a_j)_{j \in \mathbb{N}}$ is bounded and that in this case Theorem 0 is applicable.

**Proposition 4.** Let $r \in [1, \infty]$ and let $B$ be a Köthe matrix such that there exists a continuous norm on $\lambda^r(B)$. Let $(a_j)_{j \in \mathbb{N}} \subseteq \mathbb{C}$ be such that $A: \lambda^r(B) \to \lambda^r(B)$ with $Ax = (a_jx_j)_{j \in \mathbb{N}}$ is well-defined. Then the following are equivalent.

(i) The operator $A$ is a-bounded.

(ii) The operator $A$ is $m$-topologizable.

(iii) The sequence $(a_j)_{j \in \mathbb{N}}$ is bounded.

If (i)–(iii) are satisfied then $A$ generates a $C_0$-semigroup by Theorem 0.

**Proof.** Let $1 \leq r < \infty$. The case $r = \infty$ is similar. Since $\lambda^r(B)$ has a continuous norm, there exists $k_0 \in \mathbb{N}$ such that $b_{j,k_0} > 0$ holds for $j \in \mathbb{N}$. W.l.o.g. we may assume $k_0 = 1$ and get that $b_{j,k} > 0$ for all $k$ and $j \in \mathbb{N}$. Now we prove the equivalences.

(i)⇒(ii) This is true in general.

(ii)⇒(iii) Let $A$ satisfy (2) and assume that $(a_j)_{j \in \mathbb{N}}$ is unbounded. Let $p \in \mathbb{N}$ be given. Select $q \in \mathbb{N}$ and $\mu \geq 0$ as in (2). As $(a_j)_{j \in \mathbb{N}}$ is unbounded we can select $j_0 \in \mathbb{N}$ with $|a_{j_0}| > \mu$. Put $x = (b_{j_0,j})_{j \in \mathbb{N}}$. By (2) we have

$$
\sup_{n \in \mathbb{N}_0} \|\mu^{-n} A^n x\|_p \leq \|x\|_q < \infty.
$$

On the other hand we compute

$$
\sup_{n \in \mathbb{N}_0} \|\mu^{-n} A^n x\|_p = \sup_{n \in \mathbb{N}_0} \mu^{-n} \left( \sum_{j=1}^{\infty} |b_{j,q}^n a_j^n x_j|^r \right)^{1/r} = \sup_{n \in \mathbb{N}_0} b_{j_0,q} \left( \frac{2n}{p} \right)^n = \infty
$$

which gives the desired contradiction.

(iii)⇒(i) Let $(a_j)_{j \in \mathbb{N}}$ be bounded. Put $\mu := \sup_{j \in \mathbb{N}} |a_j|$. Let $p \equiv \| \cdot \|_p$ be given. Select $q = p$ and $M = 1$. We estimate

$$
\|A^n x\|_p = \left( \sum_{j=1}^{\infty} |b_{j,p}^n a_j^n x_j|^r \right)^{1/r} \leq \left( \sum_{j=1}^{\infty} |b_{j,q}^n \lambda^n x_j|^r \right)^{1/r} \leq \mu^n \|x\|_q
$$

for $x \in \lambda^r(B)$ and $n \in \mathbb{N}_0$ which shows (i). □

Proposition 4 applies in particular to diagonal operators on power series spaces [13, Section 29]. Let us mention that many classical Fréchet function spaces allow for a sequence space representation within this class of spaces.

For $r \in [1, \infty)$ Bonet, Ricker [5] Prop. 5.5] showed that there exists an unbounded sequence $(\alpha_j)_{j \in \mathbb{N}} \subseteq \mathbb{C}$ such that $A: \lambda^r(J) \to \lambda^r(B)$, $Ax = (a_jx_j)_{j \in \mathbb{N}}$, is well-defined if and only if there exists an infinite subset $J \subseteq \mathbb{N}$ such that the sectional subspace $\lambda^r(J,B) = \{ x\chi_J : x \in \lambda^r(J,B) \}$ see [5] p. 481] for details, is Schwartz. Using [5] Prop. 2.2] it follows that the latter is for instance the case if $\lambda^r(B)$ is Montel. This exhibits a large class of sequence spaces on which Theorem 0 does not even apply to every diagonal operator.

Next, we give an example of an operator $A$ on a Köthe echelon space which is not a-bounded but $m$-topologizable and thus satisfies the assumptions of Theorem 1. In particular this shows that the statements in Proposition 4 are not equivalent for every Köthe matrix.

**Example 5.** Let $B = (b_{j,k})_{j,k \in \mathbb{N}}$ be given by $b_{j,k} = 1$ for $j \leq k$ and zero otherwise. Then, $\lambda^1(B) = \lambda^\infty(B) = \omega$ is the space of all sequences endowed with the topology of pointwise convergence. Let $A: \omega \to \omega$ be defined via $Ax = (jx_j)_{j \in \mathbb{N}}$ which gives rise to a linear and continuous operator. Then $A$ is not a-bounded, but $m$-topologizable and a generator by Theorem 1.

**Proof.** Firstly, we observe that $A \in L(X)$ holds since for each $k$ and each $x \in \omega$ the estimate $\|Ax\|_k \leq \|x\|_{k+1}$ is valid. Assume that $A$ is a-bounded. Select $\mu > 0$ as in (1). Take $p > \mu$ and put $p \equiv \| \cdot \|_p$. Select $q \equiv \| \cdot \|_q$ and $M \geq 0$ as in (1). Put $x = (1, 1, \ldots)$. By (6) the estimate $\|A^n x\|_p \leq M \mu^n \|x\|_m$
holds for every \( n \in \mathbb{N}_0 \) and thus \( \sup_{n \in \mathbb{N}_0} \mu^{-n} \| A^n x \|_p < \infty \) holds. We compute
\[
\sup_{n \in \mathbb{N}_0} \mu^{-n} \| A^n x \|_p = \sup_{n \in \mathbb{N}_0} \mu^{-n} \max_{j=1, \ldots, p} | j^n \cdot 1 | = \sup_{n \in \mathbb{N}_0} \left( \frac{\mu}{\mu} \right)^n = \infty
\]
since \( p/\mu > 1 \). Contradiction.

Now we show that \( A \) is m-topologizable. Let \( p \equiv \| \cdot \|_p \) be given. Select \( q = p \) and \( \mu = p > 0 \). Then for \( n \in \mathbb{N}_0 \) and \( x \in \omega \) the estimate
\[
\| A^n x \|_p = \max_{j=1, \ldots, p} | j^n x_j | \leq p^n \max_{j=1, \ldots, p} | x_j | = \mu^n \| x \|_q
\]
is true. Theorem 4 can be applied since \( q \) is satisfied with \( (\mu_n)_{n \in \mathbb{N}_0} \equiv (\mu_{n,p,R})_{n \in \mathbb{N}_0} \equiv p \).

A combination of Proposition 4 and Example 5 even gives a characterization of those Köthe spaces which have a continuous norm. In particular, Corollary 6 shows that m-topologizable operators which are not a-bounded do exist on a large class of Fréchet spaces.

**Corollary 6.** Let \( B \) be a Köthe matrix and let \( r \in [1, \infty) \) be fixed. Then \( \lambda^r(B) \) has a continuous norm if and only if every m-topologizable diagonal operator on \( \lambda^r(B) \) is a-bounded.

**Proof.** “⇒” This is Proposition 4.

“⇐” The following arises from an inspection of the proof of a classical result due to Bessage, Pełczyński [3], see Bonet, Perez Carreras [6] Thm. 2.6.13. Under the assumption that \( \lambda^r(B) \) has no continuous norm, the latter proof shows that
\[
T : \omega \to \lambda^r(B), (a_n)_{n \in \mathbb{N}} \mapsto \sum_{n=1}^{\infty} a_n x_n
\]
is an isomorphism whenever the sequence \( (x_n)_{n \in \mathbb{N}} \) is selected such that \( x_n \in \ker \| \cdot \|_n \setminus \ker \| \cdot \|_{n+1} \) for any \( n \in \mathbb{N} \). Here, w.l.o.g. we assume that \( \ker \| \cdot \|_{n+1} \subset \ker \| \cdot \|_n \) is a strict subspace. Taking \( \ker \| \cdot \|_n = \lambda^r(\{ j : b_{n,j} = 0 \}, B) \) into account it follows that we can select \( x_n \) to be \( j_n \)-th unit vector with \( j_n \) such that \( b_{n,j_n} = 0 \) and \( b_{n+1,j_n} > 0 \). This selection produces a sequence \( (x_n)_{n \in \mathbb{N}} \) of vectors with pairwise disjoint supports. With this selection the image of \( T \) is exactly the sectional subspace \( \lambda^r(J, B) \) with \( J = \{ j_n : n \in \mathbb{N} \} \). We define the operator \( A : \lambda^r(B) \to \lambda^r(B) \) via \( Ay = (a_jy_j)_{j \in \mathbb{N}} \) with
\[
a_j = \begin{cases} jx_j & \text{for } j \in J, \\ 0 & \text{otherwise}, \end{cases}
\]
and conclude from Example 5 and the above that \( A \) is not a-bounded but m-topologizable.

We conclude this section with an example of an operator which is not m-topologizable but which nevertheless satisfies condition (i) so that Theorem 4 can still be applied.

**Example 7.** Let \( B = (b_{j,k})_{j,k \in \mathbb{N}} \) be given by \( b_{j,k} = j^k \). Then, \( \lambda^1(B) = \lambda^\infty(B) = s \) is the space of rapidly decreasing sequences. Let \( A : s \to s \) be defined via \( Ax = (\log j \cdot x_j)_{j \in \mathbb{N}} \) which gives rise to a linear and continuous operator. Then the following statements are true.

(i) Condition (i) holds and \( A \) thus generates a uniformly continuous \( C_0 \)-semigroup by Theorem 4.

(ii) In condition (i), w.r.t. \( \Gamma = (\| \cdot \|_q)_{q \in \mathbb{N}} \), the constants \( \mu_n > 0 \) depend necessarily on \( R > 0 \).

(iii) The operator \( A \) is topologizable but not m-topologizable.

**Proof.** For \( p \in \mathbb{N} \) and \( x \in s \) we compute
\[
\| Ax \|_p = \sup_{j \in \mathbb{N}} \log j |x_j| j^p = \sup_{j \in \mathbb{N}} j^{-1} \log j |x_j| j^{p+1} \leq e^{-1} \| x \|_{p+1}
\]
which shows \( A \in L(s) \).
(i) Let \( p \in \mathbb{N} \) and \( R > 0 \) be given. Select \( q \in \mathbb{N} \) such that \( q - p > R \). For given \( n \in \mathbb{N} \) put \( \mu_n = (n/(q-p))^n e^{-n} \). For \( x \in s \) we have
\[
\|A^n x\|_p = \sup_{j \in \mathbb{N}} (\log j)^n |x_j^p| = \sup_{j \in \mathbb{N}} (\log j)^n j^{-(q-p)} \|x\|_q \leq \left( \frac{n}{q-p} \right)^n e^{-n} \|x\|_q = \mu_n \|x\|_q
\]
and by the Cauchy root test the series
\[
\sum_{n=0}^{\infty} \frac{\mu_n}{n!} R^n = \sum_{n=0}^{\infty} \frac{n^n}{e^n (q-p)^n} R^n
\]
converges.

(ii) Let \( p \in \mathbb{N} \). We assume that we can select \( q \in \mathbb{N} \) and \( (\mu_n)_{n \in \mathbb{N}_0} \) such that \( \|A^n x\|_p \leq \mu_n \|x\|_q \) holds for every \( x \in s \) and such that \( \sum_{n=0}^{\infty} \frac{\mu_n}{n!} R^n \) is convergent for every \( R > 0 \). For \( j \in \mathbb{N} \) we denote by \( e_j \) the \( j \)-th unit vector and compute
\[
\|A^n e_j\|_p = (\log j)^n j^p = (\log j)^n j^{p-q} \|e_j\|_q
\]
which yields
\[
\mu_n \geq \sup_{j \in \mathbb{N}} (\log j)^n j^{p-q}.
\]
Computations show that for every fixed \( n \in \mathbb{N}_0 \) with \( (q-p)n \) the supremum over \( j \in \mathbb{N} \) is attained for \( j = e^{n/(q-p)} \) and is equal to \( (n/(q-p))^n e^{-n} \). We get
\[
\limsup_{n \to \infty} \left( \frac{\mu_n}{n!} \right)^{1/n} = \lim_{n \to \infty} \left( \frac{\mu_n}{(q-p)n} \right)^{1/n} = \lim_{n \to \infty} \left( (q-p)^n e^n n!/e^n (q-p)^n (q-p)n! \right)^{1/n} = \frac{1}{q-p}
\]
that is the radius of convergence of the power series above is less or equal to \( q-p \). Contradiction.

(iii) Condition (i) implies that \( A \) is topologizable. Assume that \( A \) is \( m \)-topologizable. Let \( p \in \mathbb{N} \) be given. We select \( q \in \mathbb{N} \) and \( \mu > 0 \) according to (2). Then for \( n \in \mathbb{N}_0 \) and \( x \in X \) the estimate \( \|A^n x\|_p \leq \mu^n \|x\|_q \) holds. The estimates and computations performed in the proof of (ii) show that
\[
\mu^n \geq \left( \frac{n}{q-p} \right)^n e^{-n}
\]
holds for infinitely many \( n \in \mathbb{N}_0 \) which is not possible with a finite \( \mu > 0 \). Contradiction.

5. Counterexamples

In this final section we show that topologizability alone, i.e., without any growth control on the \( \mu_n \) is neither sufficient nor necessary for the generation of a \( C_0 \)-semigroup. To establish our first example we need the following lemma which is kind of a counterpart of [9, II.2.3].

**Lemma 8.** Let \( X, Y \) be Fréchet spaces such that \( Y \subseteq X \) holds with continuous inclusion map. Let \( A \in L(X) \) be such that \( B := A|Y \in L(Y) \) and \( B \) generates a \( C_0 \)-group \( (S(t))_{t \geq 0} \). If \( A \) generates a \( C_0 \)-semigroup \( (T(t))_{t \geq 0} \), then \( T(t)|_Y = S(t) \) is valid for all \( t \geq 0 \).

**Proof.** Take \( y \in Y \), \( t > 0 \), \( \delta > 0 \) and define the function \( \varphi : [0, t] \to X \), \( \varphi(s) := T(s)S(t-s)y \). We show that \( \varphi \) is differentiable and that we have \( \varphi' = 0 \) for all \( s \). Fix \( s \in [0, t] \) and take \( (h_n)_{n \in \mathbb{N}} \subseteq \mathbb{R} \), \( h_n \to 0 \). Then
\[
(\varphi(s) - \varphi(s + h_n))/h_n = h_n^{-1} (T(s)S(t-s)y - T(s+h_n)S(t-s-h_n)y)
\]
\[
= T(s)h_n^{-1} (S(t-s)y - S(t-s-h_n)y)
\]
\[
+ h_n^{-1} (S(t-s-h_n)y - T(h_n)S(t-s-h_n)y).
\]
For \( n \to \infty \) the first summand in the bracket on the right hand side converges to \( BS(t-s)y \) in \( Y \) and hence in \( X \). For the second summand we define the sequence \( (A_n)_{n \in \mathbb{N}} \in L(X) \), \( A_n x = h_n^{-1} (id_X - T(h_n))x \) for \( x \in X \), which converges pointwise to \(-A \). By the Banach-Steinhaus Theorem [11, 11.1.3] we get that \( A_n \to -A \) converges uniformly on the precompact subsets of \( X \). The set

\[
\begin{align*}
(\varphi(s) - \varphi(s + h_n))/h_n &= h_n^{-1} (T(s)S(t-s)y - T(s+h_n)S(t-s-h_n)y) \\
&= T(s)h_n^{-1} (S(t-s)y - S(t-s-h_n)y) \\
&+ h_n^{-1} (S(t-s-h_n)y - T(h_n)S(t-s-h_n)y). 
\end{align*}
\]
It was shown in [4, Ex. 3] that the latter is topologizable. For the convenience of the reader, we are prepared to show that the differentiation operator on \( \|S(t)y\| = \sup_{|z|<\varepsilon} |f^n(z)| \leq \sup_{|z|<\varepsilon} \frac{n!}{2\pi i} \int_{|w|=1} \frac{f(w)}{(w-z)^{n+1}} |dw| \leq \frac{n!}{(s-q)^{n+1}} \|f\|_s \)

for any \( q < s < 1 \) and \( f \in H(\mathbb{D}) \).

To show that \( A \) does not generate a \( C_0 \)-semigroup we use Lemma \( \text{[8]} \). Let \( X = H(\mathbb{D}) \) and \( Y = H(\mathbb{C}) \) be the space of all holomorphic functions on the complex plane with the topology given by seminorms \( \|f\|_q = \sup_{|z|<\varepsilon} |f(z)| \), \( 0 < q < \infty \). We identify \( f \in H(\mathbb{C}) \) with its restriction \( f|\mathbb{D} \in H(\mathbb{D}) \). As before, the Cauchy integral formula gives \( \|A^n f\|_q = \|f^{(n)}\|_q \leq \frac{n!}{(s-q)^{n+1}} \|f\|_s \) for any \( q < s < 1 \) and \( f \in H(\mathbb{D}) \). Hence, for arbitrary \( 0 < q < \infty \) and \( R > 0 \) we take \( s > R + q \), put \( \mu_n = \frac{n!}{(s-q)^n} \) and get that \( A \) satisfies the assumptions of Theorem \( \text{[1]} \). Thus, \( A \) generates a strongly continuous group on \( H(\mathbb{C}) \), cf. Remark \( \text{[3](ii)} \). However, \( A \) does not extend to \( H(\mathbb{D}) \). Indeed, take the sequence \( f_k(z) = \sum_{n=0}^k x^n \) converging to \( f(z) = \frac{z}{e^z-1} \) in \( H(\mathbb{D}) \) and consider

\[
\lim_{k \to \infty} T(1)f_k(\frac{\varepsilon}{2}) = \lim_{k \to \infty} \sum_{n=0}^k \frac{n!}{2\pi i} \int_{|w|=1} \frac{f(w)}{(w-z)^{n+1}} |dw| \geq k^n \frac{\varepsilon}{2^{k+1}} = \lim_{k \to \infty} (\frac{\varepsilon}{2})^k 2^k = \infty.
\]

Hence, Lemma \( \text{[8]} \) implies that \( A \) does not generate a \( C_0 \)-semigroup on \( H(\mathbb{D}) \).

Finally we show that the differentiation operator in \( C^\infty(\mathbb{R}) \) is not topologizable. It is well known that the latter generates the shift semigroup, cf. \( \text{[10]} \) Ex. 1]

Example 10. The differentiation operator \( Af = f' \) generates a \( C_0 \)-semigroup on the space \( C^\infty(\mathbb{R}) \) of smooth functions but is not topologizable.

Proof. The topology of \( C^\infty(\mathbb{R}) \) is given by the seminorms \( \|f\|_{K,p} = \sup_{x\in K, \alpha\leq p} |f^{(\alpha)}(x)| \) for \( K \subseteq \mathbb{R} \) compact and \( p \in \mathbb{N} \). It is well-known that the translation semigroup \( (T(t)f)(x) = f(t+x) \) is strongly continuous and that it is generated by \( A \).

Assume that \( A \) is topologizable. Fix the seminorm \( \|\cdot\|_{K,p} \) with \( K = [0,2\pi] \) and \( p \in \mathbb{N} \). Then there exists a seminorm \( \|\cdot\|_{L,q} \) and a sequence \( \{\mu_n\}_{n\in\mathbb{N}} \subseteq (0,\infty) \) such that \( \|A^nf\|_{K,p} \leq \mu_n \|f\|_{L,q} \) holds for every \( n \in \mathbb{N}_0 \) and \( f \in C^\infty(\mathbb{R}) \). Put \( f_k(x) = \sin(kx) \), \( k \in \mathbb{N} \) and \( n = q - p + 1 \). Then for all \( k \in \mathbb{N} \) we have \( k^{n+p+1} = \|A^nf_k\|_{[0,2\pi],p} \leq \mu_n \|f\|_{L,q} = \mu_n k^q \).

Contradiction.
Acknowledgements. The authors would like to thank J. Bonet for pointing out the statement of Corollary 6 to them. In addition they would like to thank L. Frerick, E. Jordá and J. Wengenroth for valuable comments in the context of the conditions used in [10] and [15, 16].

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