Magnetic virial identities and applications to blow-up for Schrödinger and wave equations

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Received 24 March 2011, in final form 15 September 2011
Published 2 December 2011
Online at stacks.iop.org/JPhysA/45/015202

Abstract
We prove blow-up results for the solution of the initial-value problem with negative energy of the focusing mass-critical and supercritical nonlinear Schrödinger and the focusing energy-subcritical nonlinear wave equations with electromagnetic potential.

PACS number: 02.30.Jr
Mathematics Subject Classification: 35J10, 35L05, 58J45

1. Introduction
In space dimension $n \geq 2$, we study the blow-up for solutions with initial negative energy of the focusing $L^2$-critical and supercritical nonlinear Schrödinger equation with magnetic potential,

$$\begin{cases}
in_t(t,x) - Hu(t,x) + |u|^{p-1}u(t,x) = 0 \\
u(0,x) = f(x).
\end{cases}$$  \hspace{2cm} (1.1)

Here the power of the nonlinearity corresponds to $1 + \frac{4}{n} \leq p < 1 + \frac{4}{n-2}$. We also consider for $n \geq 3$ the blow-up for solutions with initial negative energy of the focusing energy-subcritical nonlinear wave equation with magnetic potential,

$$\begin{cases}
utt(t,x) + Hu(t,x) - |u|^{p-1}u(t,x) = 0 \\
u(0,x) = f(x) \\
u_t(0,x) = g(x).
\end{cases}$$  \hspace{2cm} (1.2)

For the wave equation, we deal with the whole range of nonlinearities $1 < p < 1 + \frac{4}{n-2}$. In both cases, the solution is a complex-valued function $u : \mathbb{R}^{1+n} \to \mathbb{C}$.

Therefore, instead of considering the free Schrödinger Hamiltonian $H = -\Delta$, we consider electromagnetic Schrödinger Hamiltonians in the standard covariant form

$$H = -\nabla_{A}^2 + V(x),$$  \hspace{2cm} (1.3)
where
\[ \nabla_A = \nabla - iA, \quad \Delta_A = \nabla^2_A. \] (1.4)
Here \( A = (A^1, \ldots, A^n) : \mathbb{R}^n \to \mathbb{R}^n \) is the magnetic potential and \( V : \mathbb{R}^n \to \mathbb{R} \) is the electric potential.

The study of the blow-up phenomena for solutions of the free nonlinear Schrödinger equation dates back to Zakharov and Glassey (see [29] and [18], respectively). It is based on a convexity method called virial identity. For the free nonlinear wave equation, the first result was due to Levine (see [25]).

For the case of the purely electric Schrödinger equation, i.e. for Schrödinger Hamiltonians of the type \( H = \Delta - V \), blow-up results are given in [2]. If we consider first-order perturbations of the free Schrödinger Hamiltonian by dealing with the nonlinear Schrödinger equation with magnetic potential, a blow-up result for the solution of the Schrödinger equation when the magnetic potential \( A \) is of the form
\[ A = \frac{b}{2}(-y, x, 0) \] (1.5)
and was obtained by Gonçalves–Ribeiro (see [22]).

The aim of the paper is, in the case of the Schrödinger equation with magnetic potential, to generalize the previous example to any space dimension \( n \geq 2 \) and to provide new examples of potentials for which the solution of the focusing \( L^2 \)-critical and supercritical nonlinear Schrödinger equation (1.1) blows up in finite time. Concerning the focusing energy-subcritical nonlinear wave equation with magnetic potential, namely (1.2), we will prove some blow-up results, and will conclude by showing that the magnetic potential \( A \) has no influence in the blow-up.

We will start by considering the setting of our problem. First, denote by \( H^1_A(\mathbb{R}^n) \) the following Hilbert space:
\[ H^1_A = \left\{ f : f \in L^2, \int |\nabla_A f|^2 < \infty \right\}. \] (1.6)
Throughout the paper we will assume some regularity on the Hamiltonian \( H \).

(H1) The Hamiltonian \( H_A = -\nabla_A^2 \) is essentially self-adjoint on \( L^2(\mathbb{R}^n) \), with the form domain
\[ D(H_A) = H^1_A = \left\{ f : f \in L^2, \int |\nabla_A f|^2 < \infty \right\}. \]

(H2) The potential \( V \) is a perturbation of \( H_A \) in the Kato–Rellich sense, i.e. there exists a small \( \epsilon > 0 \), such that
\[ \|Vf\|_{L^2} \leq (1-\epsilon)\|H_A f\|_{L^2} + C\|f\|_{L^2}^2, \] (1.7)
for all \( f \in D(H_A) \).

(H3) The potentials \( A \) and \( V \) are assumed to be \( A \in C^2, V \in C^1 \).

Assumptions (H1) and (H2) have several consequences. First of all, they imply the self-adjointness of \( H \), by standard perturbation techniques (see e.g. [6]); hence, by the spectral theorem we can define the Schrödinger and wave propagators \( S(t) = e^{itH}, W(t) = H^{-\frac{1}{2}} e^{it\sqrt{H}} \) and the powers \( H^s \). Moreover, we can define for any \( s \) the distorted norms
\[ \|f\|_{\tilde{H}^s} = \|H^s f\|_{L^2}, \quad \|f\|_{\tilde{H}^s} = \|f\|_{L^2} + \|H^s f\|_{L^2}. \]
For the validity of (H1) and (H2), see the standard reference [6]. We do not make any attempt to optimize on the regularity assumptions that we need. Essentially, the sufficient assumptions needed can be expressed in terms of the local integrability properties of the coefficients (see [6, 24]). Clearly (H3) suffices for our purposes.
We can also consider the space $H^2_A(\mathbb{R}^n)$ by

$$H^2_A = \{ f : f \in L^2, H_A f \in L^2 \}.$$

The corresponding norm will be given by

$$\|f\|_{H^2_A} = (\|f\|^2_{L^2} + \|H_A f\|^2_{L^2})^{1/2}.$$

**Definition 1.1.** For any $n \geq 2$, the matrix-valued field $B : \mathbb{R}^n \to \mathcal{M}_{n \times n}(\mathbb{R})$ is defined by

$$B := DA - DA^t, \quad B_{ij} = \frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}.$$

We also define the trapping component of the vector field $B$ as $B_r : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$B_r = \frac{x}{|x|} B.$$

Hence, $B$ is defined in terms of the anti-symmetric gradient of $A$. In dimension $n = 3$, the previous definition identifies $B = \text{curl} A$, namely

$$B v = \text{curl} A \wedge v, \quad \forall v \in \mathbb{R}^3.$$

In particular, we have

$$B_r = \frac{x}{|x|} \wedge \text{curl} A, \quad n = 3. \quad (1.8)$$

Hence, $B_r(x)$ is the projection of $B = \text{curl} A$ on the tangential space in $x$ to the sphere of radius $|x|$, for $n = 3$. Observe also that $B_r \cdot x = 0$ for any $n \geq 2$; hence, $B_r$ is a tangential vector field in any dimension.

The trapping component $B_r$ represents an obstruction to dispersion of solutions. Some explicit examples of magnetic potentials $A$ with $B_r = 0$ in dimension 3 are given in [12, 14].

Moreover, by

$$\partial_r V = \nabla V \cdot \frac{x}{|x|},$$

we denote the radial derivative of $V$. We must mention that, if the radial derivative is decomposed as

$$\partial_r V = (\partial_r V)_+ - (\partial_r V)_-,$$

the positive part $(\partial_r V)_+$ also represents an obstruction to dispersion. In [14], $B_r$ and $(\partial_r V)_+$ are assumed to be suitably small in order to prove weak dispersive estimates. Also both components must be small in order to prove endpoint Strichartz estimates, as can be seen in [8]. Strichartz estimates for the magnetic Schrödinger equation were also obtained in [9] and [10] among others.

Our magnetic potential $A$ is assumed to satisfy the so-called Coulomb gauge condition

$$\text{div} A = 0. \quad (1.9)$$

Observe that this does not suppose a restriction, since $A$ and $A + \nabla \psi$ produce the same magnetic field $B$, for any $n \geq 2$.

We now show some examples of magnetic potentials $A$, which satisfy our assumptions and which will be considered later. The first kind of potentials are in some sense a natural generalization of the magnetic potential $A$ presented in [2, 3, 11] and [22]. Let us consider the $2 \times 2$ anti-symmetric matrix

$$\sigma := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$
For any even $n = 2k \in \mathbb{N}$, we denote by $\Omega_n$ the $n \times n$ anti-symmetric matrix generated by $k$-diagonal blocks of $\sigma$ in the following way:

$$\Omega_n := \begin{pmatrix} \sigma & 0 & \cdots & 0 \\ 0 & \sigma & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \sigma \end{pmatrix}. \quad (1.10)$$

In order to define the magnetic potential $A$, we will distinguish between the odd and even dimension. If $n \geq 3$ is an odd number, let us consider the following anti-symmetric matrix:

$$M := \begin{pmatrix} \Omega_{n-1} & 0 \\ 0 & 0 \end{pmatrix}, \quad (1.11)$$

where $\Omega_{n-1}$ is the $(n-1) \times (n-1)$-matrix defined in (1.10).

The analogous magnetic potential $A$ in even dimensions will be constructed in the following way. For $n = 2$, consider the $2 \times 2$ anti-symmetric matrix

$$M := \begin{pmatrix} \Omega_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (1.12)$$

and if $n \geq 4$ is an even number, let $0$ be the null $2 \times 2$-block and let us consider the following anti-symmetric $n \times n$-matrix:

$$M := \begin{pmatrix} \Omega_{n-2} & 0 \\ 0 & 0 \end{pmatrix}, \quad (1.13)$$

where $\Omega_{n-2}$ is the $(n-2) \times (n-2)$-matrix defined in (1.10).

Now, from the definition of $M$ we can define the following magnetic potential $A$:

$$A(x) = \frac{1}{2} M x, \quad (1.14)$$

where the expression of $M$ depends on the dimension $n \geq 2$.

**Remark 1.1.** This kind of magnetic potential $A$ generated from an anti-symmetric matrix $M$ was also shown in [13]. More precisely, if we consider the magnetic potential $A$ of the form

$$A = |x|^{-\alpha} M x, \quad 1 < \alpha < 2, \quad (1.15)$$

Strichartz estimates for the linear version of (1.1) with $V = 0$ are false in the whole range of Schrödinger admissibility. These counterexamples for the magnetic Schrödinger equation were clearly inspired by the ones produced in the electric case in [20]. More concretely, the counterexamples in [20] are based on potentials of the form

$$V(x) = (1 + |x|^2)^{-\frac{\alpha}{2}} \varphi \left( \frac{x}{|x|} \right), \quad 0 < \alpha < 2, \quad (1.16)$$

where $\varphi$ is a positive scalar function, homogeneous of degree 0, which has a non-degenerate minimum point $P \in S^{n-1}$. Moreover, it is crucial to assume there that $\varphi(P) = 0$. The main idea is to approximate $H = -\Delta + V(x)$ by a second-order Taylor expansion, with an harmonic oscillator. Then, the condition $\alpha < 2$ causes the lack of global (in time) dispersion.

The other magnetic potentials $A$ that we are going to consider must satisfy that the trapping part of the magnetic field $B$ must be identically zero,

$$B_r = 0.$$

When $n = 3$, some examples are those appearing in [14]. These are described as follows. First we consider singular potentials. Let

$$A = \frac{1}{x^2 + y^2 + z^2} (-y, x, 0) = \frac{1}{x^2 + y^2 + z^2} (x, y, z) \wedge (0, 0, 1). \quad (1.17)$$
We can check that 
\[ \nabla \cdot A = 0, \quad B = -2 \frac{z}{(x^2 + y^2 + z^2)^{3/2}} (x, y, z), \quad B_r = 0. \]

Another (more singular potential) example is as follows:
\[ A = \left( \frac{-y}{x^2 + y^2 + z^2}, \frac{x}{x^2 + y^2 + z^2}, 0 \right) = \frac{1}{x^2 + y^2} (x, y, z) \wedge (0, 0, 1). \quad (1.18) \]

Here, we have \( B = (0, 0, \delta) \), with \( \delta \) denoting Dirac’s delta function. Again we have \( B_r = 0 \).

The electric potential \( V \) will be assumed to be
\[ V \in L^r(\mathbb{R}^n) + L^{\infty}(\mathbb{R}^n), \quad (1.19) \]
for some \( r > n/2 \).

Note that there is a quantity associated with the solutions of equations (1.1) and (1.2). We define the energy for the nonlinear magnetic Schrödinger equation as
\[ E_V(t) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^n} V(x)|u(x)|^2 \, dx - \frac{1}{p+1} \int_{\mathbb{R}^n} |u(x)|^{p+1} \, dx. \quad (1.20) \]

The energy associated with the nonlinear magnetic wave equation is the following:
\[ E_W(t) = \frac{1}{2} \int_{\mathbb{R}^n} (|u(x)|^2 + |
abla^2 u(x)|^2 \, dx + V(x)|u(x)|^2 \, dx) - \frac{1}{p+1} \int_{\mathbb{R}^n} |u(x)|^{p+1} \, dx. \quad (1.21) \]

It is well known that if we consider equation (1.1) with initial datum \( f \in H_1^1(\mathbb{R}^n) \) and equation (1.2) with initial data \( (f, g) \in (H_1^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)) \), these initial-value problems are energy-subcritical. Therefore, there exists a unique local solution of (1.1) and (1.2) defined in a maximal time of existence. For the existence theorems, we refer to section 2 below.

Let us now study the Cauchy problem (1.1) for a given initial datum \( f \in \Sigma \), where \( \Sigma \) is the space defined in (1.22).
\[ \Sigma := \{ f \in H_1^1(\mathbb{R}^n) : |x|f \in L^2(\mathbb{R}^n) \}. \quad (1.22) \]

In the Schrödinger case, we will prove that under some assumptions made on the potentials, the unique local solution \( u \in \mathcal{C}((-T_\star, T_\star^\ast); H_1^1(\mathbb{R}^n)) \) of the Cauchy problem (1.1) in the maximal time of existence \((-T_\star, T_\star^\ast)\) blows up in finite time.

This will be based on a convexity method called virial identity. The setting of the free nonlinear Schrödinger equation is due to Zakharov and Glasey (see [29, 18]). When instead of considering the free Hamiltonian we deal with the electromagnetic Hamiltonian \( H \), the virial identity differs from the one performed in the free case. This can be seen in [14], where virial identities for the linear version of equations (1.1) and (1.2) are presented. A 3D version of the virial identity for the magnetic Schrödinger equation appears in [21, 22]. Here we give the statement of the theorems for the Schrödinger and wave equations, with the proofs presented in section 4.

For the wave equation, we will proceed following Levine (see [25]).

Now we are in a condition to state the theorems for the blow-up for the solution of the focusing \( L^2 \)-critical and supercritical nonlinear Schrödinger equation with magnetic potential and the focusing energy-supercritical nonlinear wave equation with magnetic potential. The hypotheses concerning the local existence are included in the statements (see section 2). The statement for the Schrödinger case is the following.

**Theorem 1.1.** Let \( n \geq 2 \) and consider the Cauchy problem (1.1) for \( 1 + \frac{4}{n} \leq p < 1 + \frac{4}{n-2} \). Let \( f \in \Sigma \) such that \( E_V(0) < 0 \). Assume (H1), (H2) and that \( A \) is of the form (1.14) or \( A \) and \( V \) satisfy the assumptions of theorem 2.1 and \( B_r = 0 \). Moreover, assume that
(i) \( V + \frac{1}{2} rV_r \geq 0 \).
Then, the unique solution \( u \in C((-T_*, T^*); \Sigma) \) of the focusing equation (1.1) blows up in finite time.

For the wave equation, we are able to prove the following result.

**Theorem 1.2.** Let \( n \geq 3 \) and consider the Cauchy problem (1.2) for \( 1 < p < 1 + \frac{4}{n-2} \). Here \( f \in H^1_1(\mathbb{R}^n) \), \( g \in L^2(\mathbb{R}^n) \). Let \( E_0(0) < 0 \). Assume (H1), (H2) and that A and V satisfy the assumptions of theorem 2.3 and further that

(i) \( V \geq 0 \).

Then, the unique solution \( u \in C([0, T^*); H^1_1(\mathbb{R}^n)) \), \( u_t \in C([0, T^*); L^2(\mathbb{R}^n)) \) of the focusing equation (1.2) blows up in finite time.

The rest of the paper is organized as follows. In section 2, we state and give the proofs of the theorems for the local existence of solution for equations (1.1) and (1.2). Section 3 is devoted to the virial identities for Schrödinger and wave equations and in section 4 we give the proofs of the main theorems of the paper, theorems 1.1 and 1.2.

**2. Local existence results**

In this section, we state the theorems for the local existence of solution for equations (1.1) and (1.2). Therefore, we consider the initial-value problem for the focusing nonlinear Schrödinger equation with magnetic potential (1.1). The nonlinearity will be assumed to be energy-subcritical, namely \( 1 < p < 1 + \frac{4}{n-2} \). Concerning the wave equation, we deal with the initial-value problem for the focusing energy-subcritical nonlinear wave equation with magnetic potential (1.2). It can be proved that, for the Schrödinger equation, whenever the nonlinearity \( p \) is energy-subcritical, then for \( f \in H^1_1 \), there exists a unique local solution of the Cauchy problem (1.1). The proof of the result relies on the Strichartz estimates that are satisfied by the solution \( u \) of the linear version of equation (1.1). We will distinguish between the two kinds of magnetic potentials \( A \) that we presented above. For the magnetic potentials \( A \) of the form (1.14), the homogeneous and inhomogeneous Strichartz estimates appear in [3] for the case \( n = 3 \) and these estimates can be generalized in the same way to the case \( n \geq 2 \). For the second kind of magnetic potentials, namely the ones for which \( B_1 = 0 \), the Strichartz estimates that we use depend on the dimensions \( n = 2 \) and \( n \geq 3 \). In 2D, we can apply the homogeneous Strichartz estimates from [7] and derive the nonhomogeneous ones by a standard \( TT^* \)-argument. The potentials \( A \) and \( V \) should be small in order for the estimates to be valid. For \( n \geq 3 \), the estimates considered are those appearing in [8]. We refer to the case \( n \geq 3 \) by including the precise estimates.

**Definition 2.1.** Let \( n \geq 3 \). A measurable function \( V(x) \) is said to be in the Kato class \( K_n \)

\[
\lim_{r \downarrow 0} \sup_{x \in \mathbb{R}^n} \int_{|x-y| \leq r} \frac{|V(y)|}{|x-y|^{n-2}} \, dy = 0.
\]

We shall usually omit the reference to the space dimension and write \( K \) instead of \( K_n \). The Kato norm is defined as

\[
\|V\|_K = \sup_{x \in \mathbb{R}^n} \int_{|x-y| \leq r} \frac{|V(y)|}{|x-y|^{n-2}} \, dy.
\]

The final notation we require is the radial-tangential norm

\[
\|f\|_{L^p(S^r)} := \sup_{|x|=r} \int_0^r |f|^p \, dr.
\]

The result is the following (see [8] theorem 1.1).
Theorem 2.1. Let $n \geq 3$. Given $A, V \in C^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$, assume (H1), (H2). Moreover, assume that
\[ \|V\|_k < \frac{\pi^2}{\Gamma\left(\frac{n}{2} - 1\right)} \]  (2.1)
and
\[ \sum_{j \in \mathbb{Z}} 2^j \sup_{x \in C_j} |A| + \sum_{j \in \mathbb{Z}} 2^j \sup_{x \in C_j} |V| < \infty, \]  (2.2)
where $C_j = \{x : 2^j \leq |x| \leq 2^{j+1}\}$ and the Coulomb gauge condition
\[ \nabla \cdot A = 0. \]  (2.3)
Finally, when $n = 3$, we assume that for some $M > 0$,
\[ \frac{(M + \frac{1}{2})^2}{M} \|x\|^2 B_2 \|_{L^\infty} + (2M + 1) \|x\|^2 (\partial_0 V)_+ \|_{L^2 L^\infty(S_n)} < \frac{1}{2}, \]  (2.4)
while for $n \geq 4$ we assume that
\[ \|x\|^2 B_2 (x) \|_{L^\infty} + 2 \|x\| \|x\|^2 (\partial_0 V)_+ \|_{L^\infty} < \frac{2}{3} (n - 1)(n - 3). \]  (2.5)
Then, for any Schrödinger admissible couple $(p, q)$, the following Strichartz estimates hold:
\[ \|e^{itH} \|_{L^p L^q} \leq C \|\psi\|_{L^2}, \quad \frac{2}{p} = \frac{n}{2} - \frac{n}{q}, \quad p \geq 2, \quad p \neq 2 \text{ if } n = 3. \]  (2.6)
In dimension $n = 3$, we have the endpoint estimate
\[ \|D^2 e^{itH} \|_{L^p L^q} \lesssim \|H^3 \|_{L^2}. \]  (2.7)

Remark 2.1. By a standard TT"-argument, for admissible couples $(p, q), (\tilde{p}, \tilde{q})$ as in (2.6), there follows the inhomogeneous estimate
\[ \left\| \int_0^t e^{i(t-s)H} F(s) \, ds \right\|_{L^p L^q} \lesssim \|F\|_{L^p L^q}. \]  (2.8)

On the other hand, we cannot justify the above estimate using (2.7) in the 3D-endpoint case $(p, q) = (2, 6)$. Incidentally estimates (2.6) and (2.8) are enough to prove the following local well-posedness result.

Now, we give the statement of the local existence result for the Schrödinger equation in dimension $n \geq 3$.

Theorem 2.2. Let $A$ be of the form (1.14) or $A$ and $V$ satisfy the assumptions of theorem 2.1. Let $1 < p < 1 + \frac{1}{n-2}$. Given $f \in H^1_\lambda(\mathbb{R}^n)$, there exists a unique maximal solution $u \in \mathcal{C}((-T_*, T^*); H^1_\lambda(\mathbb{R}^n))$ of (1.1) such that $u(0) = f$. It holds that if $T_* < \infty$ or $T^* < \infty$, then $\|u\|_{H^1_\lambda} \to \infty$ as $t \downarrow -T_*$ or $\|u\|_{H^1_\lambda} \to \infty$ as $t \uparrow T^*$, respectively. In addition, the conservation of the mass and the energy hold, that is, for all $t \in (-T_*, T^*)$,
\[ \int_{\mathbb{R}^n} |u(t, x)|^2 \, dx = \int_{\mathbb{R}^n} |f(x)|^2 \, dx \]  (2.9)
\[ E_Q(t) = E_Q(0). \]  (2.10)
Moreover, if $f \in H^1_\lambda(\mathbb{R}^n)$, then $u \in \mathcal{C}((-T_*, T^*); H^1_\lambda(\mathbb{R}^n)) \cap \mathcal{C}((-T_*, T^*); L^2(\mathbb{R}^n)).$
Proof. The proof of the local existence is straightforward by just applying the corresponding Strichartz estimates depending on the magnetic potential $A$ we are considering. The conservation of the mass follows by multiplying equation (1.1) by $\bar{u}$, integrating by parts and taking the resulting imaginary part. The conservation of the energy comes from multiplying equation (1.1) by $\bar{u}_t$, integrating by parts and taking the resulting real part. All the computations are justified for the solutions $u \in H^2_A(\mathbb{R}^n)$. Then, proceeding by a density argument in order to obtain the result for $u \in H^1_A(\mathbb{R}^n)$. The proof of the persistence of the solution in $H^2_A(\mathbb{R}^n)$ for a given $f \in H^2_A(\mathbb{R}^n)$ can be found for $n = 3$ in [3].

Remark 2.2. We also want to mention the existence result appearing in [3] for the 3D case. We consider a constant magnetic field that, without loss of generality, can be assumed to be $B = (0, 0, b)$, (2.11)
for some $b \in \mathbb{R} - \{0\}$. Therefore, up to a gauge transform, the magnetic potential $A$ can be chosen in the following way:

$A = \frac{b}{2}(-y, x, 0)$. (2.12)

Remark 2.3. Note that in [11], general nonlinearities of the type $g(x, u)$ are considered. The version we present here is corresponding to power-type nonlinearities.

For the case of $A \equiv 0$, the initial-value problem for the nonlinear Schrödinger equation has been studied in the past by, among others, Ginibre and Velo (see [15–17]), Kato [23] or Cazenave and Weissler (see [4, 5]). The methods are of a perturbative nature and rely basically on sharp dispersive properties of the linear equation.

Concerning the nonlinear magnetic wave equation (1.2), a local existence result can be proved. As in the Schrödinger case, the result is based on the Strichartz estimates for the solution $u$ of the linear version of equation (1.2). The estimates that we are going to consider appear in [14]. We include the statement of the theorem.

Theorem 2.3. Let $n \geq 3$. Assume (H1), (H2) and either

$$\| |x|^3 B \|_{L^2 \rightarrow (S_t)} + \| |x|^2 (\partial_t V) \|_{L^2 \rightarrow (S_t)} \leq \frac{1}{2},$$

(2.13)

for $n = 3$, or

$$|B_r(x)| \leq \frac{C_1}{|x|^3}, \quad |(\partial_r V) \| \leq \frac{C_2}{|x|^3}, \quad C_1^2 + 2C_2 \leq \frac{2}{3} (n - 1) (n - 3),$$

(2.14)

for $n \geq 4$. Moreover, assume that

$$|B(x)| \leq \frac{C}{(1 + |x|)^{2+m}}, \quad |V(x)| \leq \frac{C}{(1 + |x|)^{2+m}},$$

(2.15)

for some $C > 0$ and some $\delta > 0$. Then, for any non-endpoint admissible couple $(p, q)$, the following Strichartz estimates hold:

$$\| u \|_{L^p H^2} \lesssim \| f \|_{X^1} + \| g \|_{L^2},$$

(2.16)

$$\frac{2}{p} + \frac{n - 1}{q} = \frac{n - 1}{2}, \quad 2 \leq p \leq \infty, \quad \frac{2(n - 1)}{n - 3} \geq q \geq 2, \quad q \neq \infty.$$  \hspace{1cm} (2.17)

Remark 2.4. We consider the analog of remark 2.1 for the case of the wave equation.

Relying on the last result, the following theorem for the local existence can be proved.
Theorem 2.4. Let $A$ and $V$ satisfy the assumptions of theorem 2.3. Let $1 < p < 1 + \frac{4}{n-2}$. Given $f \in H^1_0(\mathbb{R}^n)$, $g \in L^2(\mathbb{R}^n)$, there exists a unique maximal solution $u \in C([0, T^*); H^1_0(\mathbb{R}^n))$, $u_t \in C([0, T^*); L^2(\mathbb{R}^n))$ of (1.2) such that $u(0) = f$, $u_t(0) = g$. In addition, the conservation of the energy holds, that is, for all $t \in [0, T^*)$,

$$E_W(t) = E_W(0). \quad (2.18)$$

Proof. The proof of the local existence follows by a fixed-point argument just applying the Strichartz estimates given above (see [14], theorem 1.13). The conservation of the energy follows by multiplying equation (1.2) by $\bar{u}_t$, integrating by parts and taking the real part. All of the computations are justified for the solutions $u \in H^1_0(\mathbb{R}^n)$. Then by a density argument, we obtain the result for the solutions $u \in H^1_0(\mathbb{R}^n)$.

In addition, for $L^2$-subcritical nonlinearities, i.e. $1 < p < 1 + \frac{4}{n}$, the solution of the Schrödinger equation (1.1) is in fact global in time. The theorem reads as follows.

Theorem 2.5. Let $1 < p < 1 + \frac{4}{n}$ and $f \in H^1_0(\mathbb{R}^n)$. Let $u \in C((-T_*, T^*); H^1_0(\mathbb{R}^n))$ be a local solution of (1.1), and maximal time of existence. Then $T_* = T^* = \infty$.

Proof. It can be derived directly from the conservation of the energy.

3. Magnetic virial identities

In this section, we present the virial identities for the magnetic Schrödinger equation and the magnetic wave equation in any dimension. For the free nonlinear Schrödinger equation, the identity is due to Zakharov [29] and Glassey [18]. When the free equation is perturbed by a electromagnetic potential, Fanelli and Vega (see [14]) performed the corresponding virial identities for the linear Schrödinger and linear wave equations. The proof is based on the standard technique of Morawetz multipliers, introduced in [26] for the Klein–Gordon equation. For the Schrödinger equation in 3D and magnetic potential $A$ of the form (1.5), a virial identity appears in [21, 22]. Theorem 3.1 gives the virial identity for Schrödinger, while in theorem 3.2 the corresponding one for the wave equation appears. The identities are the same as those of [14], just adding the term corresponding to the nonlinear contribution. We must say that although for the wave equation we do not use the virial identity in order to prove the blow-up of the solution, we will include it for the sake of completeness.

Theorem 3.1 (Virial for magnetic nonlinear Schrödinger). Let $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a radial, real-valued multiplier, $\phi = \phi(|x|)$, and let

$$\Theta_S(t) = \int_{\mathbb{R}^n} \phi|u|^2 \, dx. \quad (3.1)$$

Then, for any solution $u$ of the magnetic nonlinear Schrödinger equation (1.1) with initial datum $f \in L^2$, $H_A f \in L^2$, the following virial-type identities hold:

$$\dot{\Theta}_S(t) = 2\Im \int_{\mathbb{R}^n} \bar{u}(t, x) \nabla_A u(t, x) \cdot \nabla \phi(x) \, dx, \quad (3.2)$$

$$\ddot{\Theta}_S(t) = 4 \int_{\mathbb{R}^n} \nabla_A u \nabla^2 \phi \nabla_A u \, dx - \int_{\mathbb{R}^n} |u|^2 \Delta^2 \phi \, dx - 2 \int_{\mathbb{R}^n} \phi' V_j |u|^2 \, dx$$

$$+ 4\Im \int_{\mathbb{R}^n} u \phi' B_z \cdot \nabla_A u \, dx - 2 \frac{(p-1)}{(p+1)} \int_{\mathbb{R}^n} |u|^{p+1} \Delta \phi \, dx, \quad (3.3)$$

with $\Theta_S(t)$ and $\dot{\Theta}_S(t)$ defined as in (3.1).
where
\[(D^2\phi)_{jk} = \frac{\partial^2}{\partial x^j \partial x^k} \phi \quad \text{and} \quad \Delta^2 \phi = \Delta(\Delta \phi),\]
for \(j, k = 1, \ldots, n\), are respectively the Hessian matrix and the bi-Laplacian of \(\phi\).

**Proof.** The proof of (3.3) comes from the one performed in [14] for the linear version of equation (1.1). More precisely, instead of considering the electric potential \(V\), we only apply the results in [14] for the potential created adding the nonlinear contribution, namely
\[\tilde{V} = V - |u|^{p-1}.\] (3.4)

Therefore, recalling that for the linear equation,
\[\dot{\Theta}_S(t) = 2\Im \int \bar{u} \nabla_A u \cdot \nabla \phi \, dx,\] (3.5)
clearly this term is not affected by the nonlinear part. Now, if we consider the second derivative of \(\dot{\Theta}_S\), for the linear version of the equation, we obtain
\[\ddot{\Theta}_S(t) = 4 \int_{\mathbb{R}^n} \nabla_A u D^2 \phi \nabla_A u \, dx - \int_{\mathbb{R}^n} |u|^2 \Delta^2 \phi \, dx\]
\[\quad - 2 \int_{\mathbb{R}^n} \phi' V_r |u|^2 \, dx + 4 \Im \int_{\mathbb{R}^n} u \phi' B_t \cdot \nabla_A \bar{u} \, dx.\] (3.6)

**Remark 3.1.** As is pointed out in [14], in the computations which lead us to (3.6), the highest order term in \(u\) that appears is of the form
\[\int \nabla_A^2 u \nabla \phi \cdot \nabla_A u;\]
this makes sense since \(f \in L^2, H_A f \in L^2\), which implies \(H_A e^{itf} f \in L^2\), and by interpolation \(\nabla_A e^{itf} f \in L^2\).

The main difference appears in the term involving \(V\). We include the details in order to be clear. We proceed as follows only considering the potential \(\tilde{V}\). Since \(V \in C^1\), it holds
\[- \int \phi' \tilde{V} |u|^2 = \int \nabla \phi \cdot \nabla \tilde{V} |u|^2 = \int \phi' \tilde{V} |u|^2 + \int \nabla \phi \cdot \nabla (|u|^{p-1}) |u|^2.\] (3.7)

Integrating by parts, noting that the unique solution \(u\) of (1.1) stays in \(H^1_A(\mathbb{R}^n)\) for all time, we obtain
\[\int \nabla \phi \cdot \nabla (|u|^{p-1}) |u|^2 = - \int \Delta \phi |u|^{p+1} - \int |u|^{p-1} \nabla \phi \cdot \nabla (|u|^2)\]
\[= - \int \Delta \phi |u|^{p+1} - 2\Re \int |u|^{p+1} \nabla \phi \cdot \nabla u\bar{u}\]
\[= - \int \Delta \phi |u|^{p+1} - \frac{2}{p+1} \int \nabla \phi \cdot \nabla (|u|^{p+1})\]
\[= - \frac{p-1}{p+1} \int \Delta \phi |u|^{p+1}.\] (3.8)

**Remark 3.2.** Note that all the computations above are justified since \(u \in H^1_A(\mathbb{R}^n)\), only applying Sobolev embedding.

The result follows directly from (3.6)–(3.8). \(\square\)
Theorem 3.2 (Virial for magnetic nonlinear Schrödinger). Let \(\phi, \Psi : \mathbb{R}^n \to \mathbb{R}\) be two radial, real-valued multipliers, and let

\[
\Theta_W(t) = \int_{\mathbb{R}^n} \left( \phi |u|^2 + \phi |\nabla_A u|^2 - \frac{1}{2} (\Delta \phi) |u|^2 \right) \, dx + \int_{\mathbb{R}^n} |u|^2 \phi \, V \, dx + \int_{\mathbb{R}^n} |u|^2 \Psi \, dx.
\]

(3.9)

Then, for any solution \(u\) of the magnetic nonlinear wave equation (1.2) with initial data \(f, g \in L^2, H_A f, H_A g \in L^2\), the following virial-type identity holds:

\[
\dot{\Theta}_W(t) = 2 \int_{\mathbb{R}^n} \nabla_A u \nabla^2 \phi \overline{\nabla_A u} \, dx - \frac{1}{2} \int_{\mathbb{R}^n} |u|^2 \Delta^2 \phi \, dx
\]

\[
+ 2 \int_{\mathbb{R}^n} |u_t|^2 \phi \, dx - 2 \int_{\mathbb{R}^n} |\nabla_A u|^2 \Psi \, dx + \int_{\mathbb{R}^n} |u|^2 \Delta \Psi \, dx
\]

\[
- \int_{\mathbb{R}^n} \phi V_r |u|^2 \, dx - 2 \int_{\mathbb{R}^n} \Psi |u|^2 \, dx + 2 \int_{\mathbb{R}^n} u \phi B_x \cdot \overline{\nabla_A u} \, dx
\]

\[
+ \int_{\mathbb{R}^n} \left( 2 \Psi - \frac{p-1}{p+1} \Delta \phi \right) |u|^{p+1} \, dx.
\]

(3.10)

Proof. We can argue as in the case of the Schrödinger equation only considering the analog to remark 3.1. Recall the expression for the second derivative of \(\Theta_W\) for the linear version of (1.2) that appears in [14]. It reads as follows:

\[
\dot{\Theta}_W(t) = 2 \int_{\mathbb{R}^n} \nabla_A u \nabla^2 \phi \overline{\nabla_A u} \, dx - \frac{1}{2} \int_{\mathbb{R}^n} |u|^2 \Delta^2 \phi \, dx
\]

\[
+ 2 \int_{\mathbb{R}^n} |u_t|^2 \phi \, dx - 2 \int_{\mathbb{R}^n} |\nabla_A u|^2 \Psi \, dx + \int_{\mathbb{R}^n} |u|^2 \Delta \Psi \, dx
\]

\[
- \int_{\mathbb{R}^n} \phi V_r |u|^2 \, dx - 2 \int_{\mathbb{R}^n} \Psi |u|^2 \, dx + 2 \int_{\mathbb{R}^n} u \phi B_x \cdot \overline{\nabla_A u} \, dx.
\]

(3.11)

Now, by considering the potential

\[\tilde{V} = V - |u|^{p-1},\]

(3.12)

and proceeding as in the Schrödinger case for the nonlinear part, the result follows easily. \(\square\)

Remark 3.3. Note that \(B_x\) only involves the terms with \(\phi\). This will be useful in section 4.

We give two corollaries of the previous theorems.

Corollary 3.3. Let \(u\) be a solution of the magnetic nonlinear Schrödinger (1.1) with \(f \in L^2, H_A f \in L^2\). Then the variance

\[
Q(t) = \int_{\mathbb{R}^n} |x|^2 |u|^2 \, dx
\]

satisfies the identities

\[\dot{Q}(t) = 4 \Im \int_{\mathbb{R}^n} \overline{\dot{u}}(t, x) \nabla_A u(t, x) \cdot x \, dx.\]

(3.13)

\[
\ddot{Q}(t) = 8 \int_{\mathbb{R}^n} |\nabla_A u|^2 \, dx - 4 \int_{\mathbb{R}^n} |x| V_r |u|^2 \, dx
\]

\[
+ 8 \Im \int_{\mathbb{R}^n} |x| B_x \cdot \overline{\nabla_A u} \, dx - 4 \frac{n(p-1)}{(p+1)} \int_{\mathbb{R}^n} |u|^{p+1} \, dx.
\]

(3.14)
Corollary 3.4. Let $u$ be a solution of the magnetic nonlinear wave equation (1.2) with $f, g \in L^2$, $H_A f, H_A g \in L^2$. Then the quantity

$$Q(t) = \int_{\mathbb{R}^n} (|x|^2 |u|^2 + |\nabla_A u|^2 + |u|^2 V) - (n - 1)|u|^2 \, dx$$

satisfies the identity

$$\ddot{Q}(t) = 2 \int_{\mathbb{R}^n} |u_t|^2 + |\nabla_A u|^2 \, dx - 2 \int_{\mathbb{R}^n} |x| V_x |u|^2 \, dx - 2 \int_{\mathbb{R}^n} V |u|^2 \, dx$$
$$+ 4 \int_{\mathbb{R}^n} |x| u B_x \cdot \nabla_A u \, dx + 2 \left(1 - \frac{n}{p} - 1\right) \int_{\mathbb{R}^n} |u|^{p+1} \, dx.$$  \hspace{1cm} (3.15)

The proofs of the corollaries are immediate applications of identities (3.3) and (3.10) with the choice $\phi = |x|^2$, $\Psi \equiv 1$.

Let us now consider the virial identity (3.14) particularized to the case of $L^2$-critical or supercritical nonlinearity, namely $1 + \frac{2}{n} \leq p < 1 + \frac{4}{n-2}$. It reads as follows:

$$\ddot{Q}(t) = 8 \int_{\mathbb{R}^n} |\nabla_A u|^2 \, dx - 4 \int_{\mathbb{R}^n} |x| V_x |u|^2 \, dx$$
$$+ 8 \int_{\mathbb{R}^n} |x| u B_x \cdot \nabla_A u \, dx + 4(1 + \frac{n}{2} - 1) \int_{\mathbb{R}^n} |u|^{2+1} \, dx.$$  \hspace{1cm} (3.16)

The last quantity can be expressed in terms of energy (1.20). Hence,

$$\ddot{Q}(t) = 16 E_2(t) - 8 \int_{\mathbb{R}^n} V |u|^2 \, dx$$
$$- 4 \int_{\mathbb{R}^n} |x| V_x |u|^2 \, dx + 8 \int_{\mathbb{R}^n} |x| u B_x \cdot \nabla_A u \, dx$$
$$+ \frac{16 - 4n(p - 1)}{p + 1} \int_{\mathbb{R}^n} |u|^{p+1} \, dx,$$  \hspace{1cm} (3.17)

and since the energy is conserved, $E_2(t) = E_2(0)$ for all times $t \in (-T_*, T^*)$, it holds

$$\ddot{Q}(t) = 16 E_2(0) - 8 \int_{\mathbb{R}^n} V |u|^2 \, dx$$
$$- 4 \int_{\mathbb{R}^n} |x| V_x |u|^2 \, dx + 8 \int_{\mathbb{R}^n} |x| u B_x \cdot \nabla_A u \, dx$$
$$+ \frac{16 - 4n(p - 1)}{p + 1} \int_{\mathbb{R}^n} |u|^{p+1} \, dx.$$  \hspace{1cm} (3.18)

Now, since we have $1 + \frac{2}{n} \leq p < 1 + \frac{4}{n-2}$, the last term in (3.18) is clearly negative. Therefore, it can be expected that, given $f \in \Sigma$ such that the corresponding energy $E_2(0) < 0$, the solution $u(t)$ of (1.1) blows up in finite time.

4. Proofs of theorems 1.1 and 1.2

In this section, we give the proofs of theorems 1.1 and 1.2. We will prove that, under some assumptions for the magnetic potential $A$ and the electric potential $V$, the local solution of the Cauchy problems (1.1) and (1.2) with initial data of negative energy actually blows up in finite time. For the 3D Schrödinger equation, a blow-up result is given by Gonçalves–Ribeiro in [22]. In some sense, our result includes a natural generalization to higher dimensions. During the proof we will use the virial identity given by (3.14), with mass-critical and supercritical nonlinearity, written in terms of the energy, namely (3.17).
Proof of theorem 1.1. We will argue by contradiction showing that there exists a finite time $T^+$ satisfying $Q(T^+) < 0$. This contradicts the fact that $Q$ is nonnegative for all times. Recall that if we start with an initial datum $f \in \Sigma$, the solution $u$ is a $\Sigma$-valued continuous function. The proof can be found for the case $n = 3$ in [22] (see the third step of the proof of theorem 1.2).

Let $f \in \Sigma$ such that $E_3(0) < 0$. Let us assume (i).

Now, recalling (3.17), we obtain that

$$
\dot{Q}(t) = 16E_3(t) - 8 \int_{\mathbb{R}^6} V|u|^2 \, dx \\
- 4 \int_{\mathbb{R}^6} |x|V_x|u|^2 \, dx + 83 \int_{\mathbb{R}^6} |x|u_{B_r} \cdot \nabla A \dot{u} \, dx \\
+ \frac{16}{p+1} \int_{\mathbb{R}^6} |u|^{p+1} \, dx.
$$

(4.1)

Let us now expand the first term of the previous equality. We have that

$$
\int_{\mathbb{R}^6} |\nabla A u|^2 \, dx = \int_{\mathbb{R}^6} |\nabla u|^2 \, dx + \int_{\mathbb{R}^6} |A|^2 |u|^2 \, dx - 2|\int_{\mathbb{R}^6} \nabla \tilde{u} \cdot Au \, dx.
$$

(4.2)

Note that for the magnetic potentials $A$ given (1.14), it holds

$$
-2A(x) = |x|B_r(x).
$$

(4.3)

Therefore, we obtain

$$
\exists \int_{\mathbb{R}^6} |x|u_{B_r} \cdot \nabla A \dot{u} \, dx = -2\Im \int_{\mathbb{R}^6} \nabla \tilde{u} \cdot Au \, dx - 2 \int_{\mathbb{R}^6} |A|^2 |u|^2 \, dx.
$$

(4.4)

Hence, from (4.1) particularized to the case $1 + \frac{4}{n} \leq p < 1 + \frac{4}{n-2}$, (4.2) and (4.4), we obtain

$$
\dot{Q}(t) \leq 8 \int_{\mathbb{R}^6} |\nabla u|^2 \, dx - 8 \int_{\mathbb{R}^6} |A|^2 |u|^2 \, dx \\
+ 8 \int_{\mathbb{R}^6} V|u|^2 \, dx - \frac{16}{p+1} \int_{\mathbb{R}^6} |u|^{p+1} \, dx \leq CE_3(0),
$$

(4.5)

for some positive constant $C$.

If we consider a magnetic potential $A$ for which $B_r = 0$, we have that

$$
\dot{Q}(t) \leq 16E_3(0)
$$

(4.6)

trivially holds. Integrating relations (4.5) or (4.6) with respect to time twice, we obtain

$$
Q(t) \leq CE_3(0) + 4\Im \int_{\mathbb{R}^6} \tilde{u}(0,x) \nabla A\tilde{u}(0,x) \cdot x \, dx + Q(0),
$$

(4.7)

for some positive constant $C$. Now since $Q(0)$ is finite and $E_3(0) < 0$, there would be a finite time $T^+$ such that $Q(T^+) < 0$. This however is in contradiction with the fact that by definition $Q(t) \geq 0$ for all times. This proves that the solution stops to exist in a finite time. \hfill \square

Now, we will prove blow-up for the solution of the focusing energy-subcritical nonlinear magnetic wave equation (1.2). It will be shown that the magnetic potential $A$ has no influence on the blow-up phenomena, that is, it seems that only the electric potential $V$ plays a role.

Proof of theorem 1.2. We will proceed following the proof performed by Levine for the free nonlinear wave equation (see [25]). Similar arguments have been performed in [1, 19, 27] and [28] among others. Let us consider

$$
F(t) = \|u\|_{L^2}^2 = (u, u).
$$

(4.8)

We will show that if $f$ and $g$ are chosen correctly, then $F(t)$ goes to infinity in finite time. Therefore, suppose that we can find $\alpha > 0$ and initial data $f$ and $g$ such that
Therefore, we have to show that $F(t) \to -\infty$, \( \forall t \geq 0, \) if $u(t) \geq 0$. Since the graph of a concave function must lie below any tangent line, Condition (4) follows, and therefore $F(t)$ would be concave. Thus, if $F(0) \neq 0$, we will have for all $t$, for which $u(t)$ exists,

$$F^{-\alpha}(t) \leq F^{-\alpha}(0) - at F'(0) F^{-\alpha-1}(0),$$

since the graph of a concave function must lie below any tangent line. Condition (b) will follow if $f$ and $g$ have the same sign, since

$$(F^{-\alpha})'(0) = -\alpha F^{-\alpha-1}(0) F'(0) = -2\alpha F(0)^{-\alpha-1} \mathfrak{H}(u_t(0), u(0)).$$

Hence, (4.11) is equivalent to having

$$F^\alpha(t) \geq F^{\alpha+1}(0) [F(0) - at F'(0)]^{-1},$$

and therefore as $t \to T(\leq F(0)/aF'(0))$, from below (if $F'(0) < 0$), we see that $F(t) \to \infty$. Therefore, we have to show that

$$FF'' - (\alpha + 1)(F')^2 \geq 0.$$  

Let us start by computing the derivatives of $F$. We obtain

$$F'(t) = 2\mathfrak{H}(u_t, u)$$

and

$$F''(t) = 2(u_t, u_t) + 2\mathfrak{H}(u_{tt}, u).$$

Therefore, if we denote by $Q(t) = FF'' - (\alpha + 1)(F')^2$, we have

$$Q(t) = 4(\alpha + 1) \mathfrak{H}(u_t, u) + 2(\mathfrak{H}(u_t, u) - (2\alpha + 1)(u_t, u_t)) F - (\alpha + 1) (2\mathfrak{H}(u_t, u))^2$$

$$= 4(\alpha + 1) \left\{ \left( \int |u_t|^2 \, dx \right) \left( \int |u|^2 \, dx \right) - \mathfrak{H} \int u_t \bar{u} \, dx \right\}$$

$$+ 2F \mathfrak{H} \int u_t \bar{u} \, dx - (2\alpha + 1) \int |u_t|^2 \, dx.$$  

The first term on the right is positive by Cauchy–Schwarz, so we need only to show that $H(t) \geq 0$, where

$$H(t) = \mathfrak{H} \int u_t \bar{u} \, dx - (2\alpha + 1) \int |u_t|^2 \, dx.$$  

From (1.2), we have that

$$u_t = \nabla^2 u - Vu + |u|^{p-1} u.$$  

(4.19)
then, by substituting this into (4.18) we arrive at
\[ H(t) = \Re \int |u_t|^2\,dx - (2\alpha + 1) \int |u|^2\,dx \]
(4.20)
\[ = \Re \left\{ \int \nabla_x^2 u \bar{u} \,dx - \int V |u|^2 \,dx + \int |u|^{p+1} \,dx \right\} - (2\alpha + 1) \int |u|^2\,dx \]
\[ = -\int |\nabla_x u|^2 \,dx - \int V |u|^2 \,dx + \int |u|^{p+1} \,dx - (2\alpha + 1) \int |u|^2\,dx. \]
(4.21)
Thus, if we choose \( \alpha \) so that
\[ 2(2\alpha + 1) = p + 1, \]
(4.22)
we have
\[ H(t) = -(p + 1)E_W(t) + 2\alpha \int (|\nabla_x u|^2 + V |u|^2) \,dx. \]
(4.23)
Now, by (i) and the conservation of the energy, we have that
\[ H(t) \geq -(p + 1)E_W(t) = -(p + 1)E_W(0). \]
(4.24)
Now, suppose that \( E_W(0) < 0 \). Since \( \alpha = \frac{p+1}{2} > 0 \), \( H(t) \geq 0 \) and hence \( (F(t)^{-\alpha})'' \leq 0 \).
Also, we can choose \( f \) and \( g \) such that
\[ (F^{-\alpha})'(0) = -\alpha F^{-\alpha-1}(0)F'(0) = -2\alpha F(0)^{-\alpha-1}\Re\langle u_t(0), u(0) \rangle < 0. \]
(4.25)
For such initial data, \( F(t) \) goes to infinity in finite time.

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