Finite-Blocklength Performance of Sequential Transmission over BSC with Noiseless Feedback

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Abstract—In this paper, we consider the expected blocklength of variable-length coding over the binary symmetric channel (BSC) with noiseless feedback. Horstein first proposed a simple one-phase scheme to achieve the capacity of BSC. Naghshvar et al. used a novel extrinsic Jensen-Shannon (EJS) divergence in a sequential transmission scheme that maximizes EJS (MaxEJS) and provided a non-asymptotic upper bound on the expected blocklength for MaxEJS. Simulations in this paper show that MaxEJS provides lower expected blocklengths than the original Horstein scheme, but the non-asymptotic bound of Naghshvar et al. is loose enough that lies above the simulated performance of Horstein scheme for a BSC with a small crossover probability. This paper proposes a new expression for MaxEJS expected blocklength that is a tight approximation of simulated performance. This expression is developed by exploring a genie-aided decoder (GAD) whose expected blocklength will always be larger than MaxEJS and can be approximated by two random walks. We conjecture that even with these two approximations, the expression may still be an upper bound on blocklength as suggested by the simulation results.

I. INTRODUCTION

For memoryless channels, noiseless feedback does not increase capacity [1], but variable-length codes (VLCs) using noiseless feedback achieve significantly larger error exponents than fixed length codes (FLCs) without feedback. Horstein [2] proposed an early and elegant demonstration of the error exponent benefit of VLCs with a sequential transmission scheme on the binary symmetric channel (BSC) with full feedback (FF). Numerous works have examined the benefits of VLCs under different channels with feedback, e.g. [3]–[6].

For a general discrete memoryless channel (DMC), Burnashev [7] proposed a two-phase variable-length coding scheme that can achieve the best error exponent. The first phase is called the communication phase which tries to increase the decoder’s belief about the true message. The second phase is called the confirmation phase, which seeks to verify the correctness of the codeword identified in the first phase.

Schalkwijk and Post [8], [9] verified that the Horstein scheme achieves the capacity on the BSC for a discrete set of crossover probabilities. Recently, Shayevitz and Feder [10] generalized Horstein’s approach using the framework of posterior matching and showed that any coding scheme following this principle can achieve the capacity of a general DMC with noiseless feedback. When posterior matching is applied to the BSC, it naturally reduces to the Horstein scheme.

Polyanskiy et al. developed a finite-blocklength information theory for FLCs [11] and VLCs [12], showing that for the same expected blocklength the achievable rate of a VLC with simple stop feedback (SF) is larger than the achievable rate of an FLC without feedback. The VLC advantage over FLC is significant for short expected blocklengths. An expected blocklength characterization for VLCs with FF that shows the benefit of FF over the more restrictive SF has been missing and is the focus of this paper.

Naghshvar et al. [13] used extrinsic Jensen-Shannon divergence (EJS) in an FF sequential transmission scheme, called MaxEJS, that maximizes EJS. Our simulation results show that MaxEJS scheme has lower expected blocklengths than either the scheme of [2] or the upper bounds of [5]. However, the upper bound on expected blocklength provided in [13] is larger than both.

We seek an expected blocklength characterization of MaxEJS that provides a demonstration of the benefit of FF over SF on the BSC with crossover probability $p < 1/2$ and $q = 1 − p$. This lays the groundwork for a generalization of that characterization to posterior matching for general DMCs.

This paper is organized as follows. Sec. II introduces VLC, the Horstein scheme, and MaxEJS. Sec. III presents the random walk argument of Conjecture I and some useful auxiliary results. Sec. IV demonstrates the simulation results of both schemes to support the validity of our conjecture. Finally, Sec. V concludes the entire paper.

II. VARIABLE-LENGTH CODING, HORSTEIN, MAXEJS

This section introduces VLCs and presents the Horstein scheme and MaxEJS scheme. Notation follows [13]. All logarithms are in base 2.

A. Variable-Length Coding

Consider the problem of sequential transmission over a DMC with full noiseless feedback as depicted in Fig. 1. The
DMC is described by the finite input set \( \mathcal{X} \), finite output set \( \mathcal{Y} \), and a collection of conditional probabilities \( P(Y|X) \). Without loss of generality, we assume that

\[
\mathcal{X} = \{0, 1, \ldots, |\mathcal{X}| - 1\}, \quad \mathcal{Y} = \{0, 1, \ldots, |\mathcal{Y}| - 1\}.
\]

Let \( C = \max_{p_X} I(X;Y) \) denote the capacity of the DMC and define \( C_1 \) and \( C_2 \) as follows:

\[
C_1 = \max_{x, x'} D\left( P(Y|X = x) \| P(Y|X = x') \right)
\]

\[
C_2 = \max_{y \in \mathcal{Y}} \min_{x \in \mathcal{X}} P(Y = y|X = x)
\]

where \( D(P|Q) = \sum_{x} p(x) \log \frac{p(x)}{q(x)} \) denotes the Kullback-Leibler (KL) divergence between two probability distributions \( P \) and \( Q \).

The transmitter in Fig. 2 wishes to communicate a message \( \Theta \) to the receiver, where the message is uniformly distributed over the set

\[
\Omega = \{1, 2, \ldots, M\}.
\]

At time \( t \), the transmitter produces channel input \( X_t \) based on the message point \( \Theta \) and received output sequence \( Y^{t-1} = (Y_0, Y_1, \ldots, Y_{t-1}) \). That is,

\[
X_t = \epsilon_t(\Theta, Y^{t-1}), \quad t = 0, 1, \ldots, \tau - 1
\]

where \( \epsilon_t : \Omega \times \mathcal{Y} \rightarrow \mathcal{X} \) is the encoding function. In particular, \( X_0 = \epsilon_t(\Theta, \emptyset) \) for \( t = 0 \).

Let \( \tau \) denote the total number of transmissions until the decoder makes a final selection of message \( \Theta \), where \( \tau \) is the (variable) blocklength of the code. After observing \( \tau \) channel outputs, \( Y_0, Y_1, \ldots, Y_{\tau-1} \), the receiver makes a guess about message \( \Theta \) as

\[
\hat{\Theta} = d(Y^{\tau-1}),
\]

for some decoding function \( d : \mathcal{Y}^\tau \rightarrow \Omega \). The probability of error is

\[
P_e = \Pr(\hat{\Theta} \neq \Theta).
\]

For a fixed DMC and for a given \( \epsilon > 0 \), a sequential transmission scheme is defined by an encoding function as in (5) and the corresponding receiver is defined by a decoding function as in (6) and a rule governing the stopping time \( \tau_e \) such that the probability of error satisfies \( P_e \leq \epsilon \) and the expected number of channel uses \( E[\tau_e] \) is minimized.

As noted in (13), the posteriors about the message,

\[
\rho_i(t) = \{\rho_i(1), \rho_i(2), \ldots, \rho_i(M)\}
\]

form a sufficient statistic of \( Y^t \) for \( \Theta \) so that FF is equivalent to feedback of \( \rho(t) \). Assuming equally likely messages, \( \rho_i(0) = \Pr(\Theta = i) = \frac{1}{M} \) denotes the receiver’s initial belief of \( \Theta = i \).

Given \( Y = y_i \), the receiver’s posteriors \( \rho(t+1) \) are updated according to the Bayes’ rule: for \( i \in \Omega \)

\[
\rho_i(t+1) = \frac{P(Y = y_i|X = \gamma(i))\rho_i(t)}{\sum_{j=1}^{M} P(Y = y_i|X = \gamma(i))\rho_j(t)},
\]

where \( \gamma : \Omega \rightarrow \mathcal{X} \) denotes the adaptive encoding function, which makes the explicit dependence on \( Y^t \) of \( \epsilon_t \) implicit. The current adapted version of \( \gamma \) is available both at transmitter and receiver, and the transmitter sends the bit produced by \( \gamma \) that corresponds to the true message.

A stopping rule of primary interest stops at \( \tau_e \) which is the first time one of the posteriors becomes greater than \( 1 - \epsilon \), i.e.,

\[
\tau_e = \min_{t \in [1, M]} \left( \max_{\rho_i(t) - 1} \right) \geq 1 - \epsilon.
\]

Obviously, any coding scheme that satisfies (10) automatically meets the requirement

\[
P_e = \left( 1 - \max_{t \in [1, M]} \rho_i(\tau_e) \right) \leq \epsilon.
\]

B. The Horstein Scheme

Under the framework introduced in Sec. [II-A], the Horstein scheme assigns at time \( t \) to each possible message \( i \) in \( \Omega \) a sub-interval of the unit interval length \( \rho_1(t) \) starting at \( \sum_{j=1}^{t-1} \rho_j(t) \). The encoding rule transmits a 0 if the sub-interval corresponding to \( \Theta \) lies entirely in the lower half of the unit interval. Conversely, the encoding rule transmits a 1 if the sub-interval corresponding to \( \Theta \) lies entirely in the upper half of the unit interval. When the sub-interval corresponding to \( \Theta \) contains the midpoint of the unit interval, the encoder sends either 1 or 0 with probability equal to the fraction of the portions of subinterval above or below the midpoint, which leads to a randomized encoding.

Using Bayes’ rule, when the Horstein decoder receives a 0, it expands the lower half of the unit interval by \( 2 \eta \) and compresses the upper half of the unit interval by \( 2 \rho \). This expansion and compression is reversed when a 1 is received. Note that when a sub-interval contains the midpoint, one portion is compressed and the remaining part is expanded. For asymptotic analysis, this sub-optimal encoding is not consequential, but for finite \( M \) it matters. The MaxEJS algorithm addresses this suboptimality.

C. The MaxEJS scheme

The Jensen-Shannon (JS) divergence is defined for multiple probability distributions. That is, given \( M \) probability distributions \( P_1, P_2, \ldots, P_M \) over a set \( \mathcal{Y} \) and a vector of prior weights \( \rho = [\rho_1, \rho_2, \ldots, \rho_M] \), where \( \rho \in [0,1]^M \) and \( \sum_{i=1}^{M} \rho_i = 1 \), the JS divergence is defined as

\[
JS(\rho; P_1, \ldots, P_M) \triangleq \sum_{i=1}^{M} \rho_i D \left( P_i \| \sum_{j=1}^{M} \rho_j P_j \right)
\]

\[
= H \left( \sum_{i=1}^{M} \rho_i P_i \right) - \sum_{i=1}^{M} \rho_i H(P_i).
\]

In [13], the extrinsic Jensen-Shannon (EJS) divergence is defined as follows: for \( M \) distributions \( P_1, P_2, \ldots, P_M \) and for an \( M \)-dimensional weight vector \( \rho \):

\[
EJS(\rho; P_1, \ldots, P_M) \triangleq \sum_{i=1}^{M} \rho_i D \left( P_i \| \sum_{j \neq i}^{M} \frac{\rho_j}{1-\rho_i} P_j \right),
\]

(14)
Algorithm 1 MaxEJS Algorithm to compute $\gamma$

1: $S_0 = \{1, 2, \ldots, M\}$ and $S_1 = \emptyset$;
2: $r_0 = 1, r_1 = 0, \rho_{\min} = 0$, and $\delta = 1$;
3: while $\rho_{\min} < \delta$ do
4: \hspace{1em} $k = \operatorname{argmin}_{i \in S_0} \rho_i(t)$;
5: \hspace{1em} $r_0 = r_0 - \rho_k(t)$ and $r_1 = r_1 + \rho_k(t)$;
6: if $r_0 < r_1$ then
7: \hspace{2em} Swap $S_0$ and $S_1$;
8: \hspace{2em} Swap $r_0$ and $r_1$;
9: end if
10: $\delta = r_0 - r_1$;
11: $\rho_{\min} = \min_{i \in S_0} \rho_i(t)$;
12: end while
13: for $i = 1, 2, \ldots, M$ do
14: \hspace{1em} $\gamma(i) = \begin{cases} 0, & \text{if } i \in S_0; \\ 1, & \text{if } i \in S_1; \end{cases}$
15: end for

where $\rho_i < 1$ for all $i \in \{1, \ldots, M\}$, and as

$$EJS(\rho; P_1, \ldots, P_M) \triangleq \max_{j \neq i} D(P_j || P_j),$$

when $\rho_i = 1$ for some $i \in \{1, \ldots, M\}$.

With $\gamma : \Omega \to X$ the encoding function used at time $t$ over a DMC $P(Y | X)$, the EJS measures the expected reduction in uncertainty in the form of

$$EJS(\rho(t), \gamma) \triangleq EJS(\rho(t); P_{\gamma(1)}, \ldots, P_{\gamma(M)}).$$

In [13], the authors proposed a deterministic variable-length coding scheme called MaxEJS, which determines $\gamma$ through a greedy maximization of the EJS divergence at each time $t$. In the setting of BSC with noiseless feedback, the MaxEJS scheme reduces to Algorithm 1 to compute the encoding function $\gamma$. Intuitively, Algorithm 1 partitions the message set $\Omega$ into two subsets $\gamma^{-1}(0)$ and $\gamma^{-1}(1)$ with approximately equal probability when all posterior probabilities $\{\rho_i(t)\}_{i \in \Omega}$ are small. However as soon as $\max_{i \in \Omega} \rho_i(t) > 1/2$, $\gamma^{-1}(0)$ and $\gamma^{-1}(1)$ will have probability $(\max_{i \in \Omega} \rho_i(t), 1 - \max_{i \in \Omega} \rho_i(t))$.

Naghshvar et al. [13] provide an upper bound on $E[\tau_e]$ using MaxEJS scheme as follows:

$$E[\tau_e] \leq \frac{\log M + \log \frac{M}{2}}{C} + \frac{\log \frac{1}{\epsilon} + 1}{C_1} + \frac{6(4C_2)^2}{CC_1},$$

where $C_1$ and $C_2$ are defined in (2) and (3), respectively.

III. TWO RANDOM WALKS WITH A GENIE

This section introduces a genie-aided decoder (GAD) whose expected blocklength is strictly larger than that of the true MaxEJS decoder, but is easier to analyze. An exact characterization of this GAD would produce an upper bound on the expected blocklength of MaxEJS. Two approximations facilitate two applications of Wald’s equality to produce an approximation of the expected blocklength $E[\tau^*_e]$ of the GAD as follows:

$$E[\tau^*_e] \approx \frac{\log M}{C} + \frac{\log \frac{1 - \epsilon}{C_1}}{C_1}.$$
always impossible in this phase, therefore $\gamma^{-1}(0)$ and $\gamma^{-1}(1)$ remain impossible in this phase. As shown in Fig. [2] that confirmation phase could go back to communication phase again and also can enter the final termination state, denoted by $\Pi^*_1$ or $\Pi^*_2$ depending on whether the final most likely message $\Theta = \Theta$ is true or not, as soon as $\max_{i \in \Omega} \rho_i(t) \geq 1 - \epsilon$.

B. The Genie-Aided Decoder

As shown in [11], the actual MaxEJS decoder stops when any message $i$ attains $\rho_i(t) \geq 1 - \epsilon$. The GAD knows the true message $\Theta$ and implements a more restrictive stopping rule

$$\tau'_* = \min \{ t : \rho_\Theta(t) \geq 1 - \epsilon \}.$$  

(22)

Because MaxEJS stops when the GAD stops or earlier,

$$E[\tau_*] \leq E[\tau'_*].$$  

(23)

For the GAD, we need not consider states $\Pi^*_1$ and $\Pi^*_2$, we are only interested in the expected time until $\rho_\Theta(t) > 1/2$, and then the expected additional time until $\rho_\Theta(t) \geq 1 - \epsilon$.

1) The first random walk - the communication phase: Let us consider the expected number of transmissions for the GAD until the first time $\rho_\Theta(t) > 1/2$. To analyze this case, we will make the approximation that two sets $\gamma^{-1}(0)$ and $\gamma^{-1}(1)$ are equally likely until the first time $\rho_\Theta(t) > 1/2$.

Define $S_0 = \log \rho_\Theta(n)$, where $\Theta$ is the true message. Using the approximation that two sets $\gamma^{-1}(0)$ and $\gamma^{-1}(1)$ are equally likely, according to Bayes’ rule

$$S_{n+1} = S_n + X_n, \quad X_n = \begin{cases} \log 2q, & w.p. \ q \\ \log 2p, & w.p. \ p \end{cases}$$  

(24)

with $S_0 = \log \rho_\Theta(0) = -\log M$. This is a random walk with step mean of $E[X] = C$. Let $J_1$ be the stopping time when $S_{J_1} \geq \log \frac{1}{2}$ for the first time. Noting that $E[J_1] < \infty$ and $E[S_{J_1}] < 0$, by Wald’s equality, we have

$$E[J_1] = \frac{E[S_{J_1}] - S_0}{E[X]} < \frac{\log M}{C},$$  

(25)

where the inequality is due to the fact that $S_{J_1}$ may overshoot from $\log \frac{1}{2}$ but will never exceed 0.

Lemma 1 below ensures that $\gamma^{-1}(0)$ and $\gamma^{-1}(1)$ are always equally likely for the first $k - 1$ transmissions.

**Lemma 1:** For $M = 2^k$, equally likely $\gamma^{-1}(0)$ and $\gamma^{-1}(1)$ are always possible in the first $k - 1$ transmissions. For $0 \leq t \leq k$, after the $t$-th transmission, $\rho(t)$ is the expansion terms of $\frac{1}{M}(2p + 2q)^{t-1}$ repeated $2^{k-t}$ times so that the values of $\rho(t)$ can be divided into two sets of equal probability.

**Proof:** Before the 1st transmission, clearly the lemma holds. Assume that $\gamma^{-1}(0)$ contains the expansion terms of $\frac{1}{M}(2p + 2q)^{t-1}$ and repeats through them and so is $\gamma^{-1}(1)$ in the $(t - 1)$-th round. Therefore, after the next transmission and update the posteriors, one can simply collect all expansion terms of $\frac{1}{M}(2p + 2q)^{t-1}(2q)$ and $\frac{1}{M}(2p + 2q)^{t-1}(2p)$ so that

$$1 = \frac{1}{M}(2p + 2q)^{t-1}(2q) + \frac{1}{M}(2p + 2q)^{t-1}(2p) = \frac{1}{M}(2p + 2q)^t.$$  

(26)

This construction cannot proceed once the $\frac{M}{2^t}(= 2^{k-1})$ terms in $\gamma^{-1}(0)$ constitute an entire expansion, which is exactly $\frac{1}{M}(2p + 2q)^{k-1}$ in the $(k - 1)$-th round. Thus, after the $k$-th transmission, $\rho(k)$ is

$$\frac{1}{M}(2p + 2q)^{k-1}(2q) + \frac{1}{M}(2p + 2q)^{k-1}(2p) = (p + q)^k,$$  

(27)

in which the maximal term $q^k$ only appears once. ■

2) The second random walk: After we have arrived at the stopping time $J_1$, we know that $\rho_\Theta(t) > 1/2$ and we seek the expected number of additional transmissions until $\rho_\Theta(t) \geq 1 - \epsilon$. We can upper bound this number by considering the expected number of transmissions to achieve $\rho_\Theta(t) \geq 1 - \epsilon$ from a starting position of $\rho_\Theta(J_1) = 1/2$ so that

$$A_0 = \log \frac{1 - \rho_\Theta(J_1)}{\rho_\Theta(J_1)} = 0$$  

(28)

Since $\rho_\Theta(J_1) \geq 1/2$, $\pi_0(t) = \rho_\Theta(t)$ for $t \geq J_1$. According to Bayes’ rule, we have

$$\rho_\Theta(t + 1) = \begin{cases} \frac{q\pi_0(t)}{p\pi_0(t) + q(1 - \rho_\Theta(t))}, & w.p. \ q \\ \frac{p\pi_0(t)}{p\pi_0(t) + q(1 - \rho_\Theta(t))}, & w.p. \ p \end{cases}$$  

(29)

Define

$$A_n = \log \frac{1 - \rho_\Theta(n)}{\rho_\Theta(n)}.$$  

(30)

Substituting this into (29) yields

$$A_n = A_{n-1} + Z_n, \quad Z_n = \begin{cases} \log \frac{q}{p}, & w.p. \ q \\ \log \frac{p}{q}, & w.p. \ p \end{cases}$$  

(31)

with $E[Z] = -C_1$. Eq. (31) holds with equality when $\rho_\Theta(t) > 1/2$. Here we make the approximation that once we have achieved $\rho_\Theta(t) > 1/2$, (31) holds approximately even if $\rho_\Theta(t)$ drops below 1/2 for a while.

The stopping rule in (19) can be expressed as $\min \{ t : A_t \leq -\log \frac{1 - \epsilon}{\epsilon} \}$. Thus, consider the stopping time $J_2$ which is the first $n$ for which $A_n \leq -\log \frac{1 - \epsilon}{\epsilon}$. Again using Wald’s equality we have

$$E[J_2] = \frac{E[A_{J_2}]}{E[Z]} \approx \frac{\log \frac{1 - \epsilon}{\epsilon}}{C_1}.$$  

(32)

where $J_2$ is the stopping time for the second random walk.

In summary, combining (24), (25), and (32),

$$E[\tau_*] \leq E[\tau'_*] \approx E[J_1] + E[J_2] \approx \frac{\log M}{C} + \frac{\log \frac{1 - \epsilon}{\epsilon}}{C_1}.$$  

(33)

IV. Simulation Results

We consider the BSC with crossover probability $p = 0.05$ and $\epsilon = 10^{-3}$. Thus it can be computed that

$$C = 0.7136, \quad C_1 = 3.8231, \quad C_2 = 19.$$  

(34)

Clearly, this setting satisfies the technical conditions in [13]. Thus, from (17),

$$E[\tau_*] \leq \frac{\log M + \log \log M + 3.32}{0.7136} + 2.87 + 12702.89.$$  

(35)
For example, in order to achieve a rate of much better than the Horstein scheme on rate performance.

As expected, the MaxEJS scheme is only requires an average blocklength of 35 bits. Also, our conjectured lower bound on rate appears to be tight.

In terms of rate, which is $k/C$, the ideal blocklength shown in the plot is simply $k/C$ where $k = \log M$ denotes the information length.

It can be seen that our approximation in (18) appears to be a tight upper bound as conjectured in (19). Meanwhile, one can observe that both Horstein and MaxEJS schemes have the same slope $1/C$ and a nearly constant excess blocklength to the ideal blocklength.

In terms of rate, which is $k/\bar{\tau}_\epsilon$ with $\bar{\tau}_\epsilon$ being the average blocklength, Fig. 4 demonstrates the rate performance vs. average blocklength. As expected, the MaxEJS scheme is much better than the Horstein scheme on rate performance. For example, in order to achieve a rate of $0.9C$, MaxEJS scheme only requires an average blocklength of 35 bits. Also, our conjectured lower bound on rate appears to be tight.

Following Polyanskiy, Williamson et al. [5] derived the following variable-length feedback (VLF) upper bound.

**Theorem 1 (VLF upper bound, [5]):** Given $\epsilon > 0$, for a BSC with crossover probability $p$ and noiseless feedback,

$$E[\tau_\epsilon] \leq \frac{\log \frac{M-1}{\epsilon} + \log 2(1-p)}{C}. \quad (36)$$

We can see that the MaxEJS scheme has a significantly lower average blocklength and thus a significantly higher rate that the VLF bound, indicating the benefit that full feedback can provide over stop feedback.

**V. Conclusion**

In this paper, we conjectured a much tighter upper bound on the expected blocklength using MaxEJS scheme by Naghshvar et al using a genie-aided decoder and two random walks. Future work will mainly focus on two unsolved questions: (i) how to analyze the communication phase in which equal partition is not possible. (ii) how to analyze the case when true message falls back to the communication phase.

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