Quasi-Integrability of The KdV System

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Abstract: The quasi-integrable KdV equation has been obtained from the corresponding deformation of the Hamiltonian for the usual KdV system. Following suitable gauge-fixing, it has been found that the quasi-conservation condition is satisfied and an infinite number of anomalous conservation laws are obtained, with some containing possible conserved charges. Judicious choice of deformation of the Hamiltonian clearly leads to a conserved charge, manifesting quasi-integrability, but also creates a hierarchy of higher-derivative equations with at least one conserved charge. A particular quasi-deformation parameterization of the Hamiltonian is found to complement the same of the NLS system, following an approximate equivalence of the two systems obtained earlier, in the weak coupling limit. Single-soliton solutions for all these cases are obtained, with manifest scaling due to the quasi-integrable deformation. Finally, quasi-conservation formulation for the complex coupled generalized KdV system is obtained.

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1 Introduction

The Korteweg-de Vries (KdV) equation and the Nonlinear Schrödinger (NLS) equation are the best known 1 + 1-dimensional integrable partial differential equations (PDEs) [1]. However, their mathematical natures are different. In fact, KdV is geometrically connected to diffeomorphism group [2] whereas NLS is tagged with loop algebra [3]. Moreover the Lax representation [4] of KdV system involves second and third order monic differential operators \((L, A)\), whereas the Lax representation of NLS is given by \(2 \times 2\) matrices [1]. However, it is known that using a particular ansatz, in the suitable weak-coupling limit, the nonlinear Schrödinger equation can be approximated by the KdV equation [5, 6].

Real physical systems are characterized by finite number of degrees of freedom, prohibiting integrability of the corresponding field-theoretical models in principle. However, they are physically known to posses solitonic states, very similar in structure to integrable ones, like sine-Gordon (SG) [7]. This motivates the study of physical continuous systems with the interpretation as slightly deformed integrable models. In a recent work [8, 9], the SG model was shown to be deformable into an approximate system, characterized by a finite number of conserved charges. The corresponding connection was almost flat, yielding an anomalous zero-curvature condition. The system was thus deemed quasi-integrable (QI).

In a very interesting paper Ferreira et al [10] considered modifications of the NLSE to investigate the recently introduced concept of quasi-integrability, where they modified NLS potential of the form \(V \approx (|\psi|^2)^{2+\epsilon}\), with \(\epsilon\) being a perturbation parameter, and
showed that such models possess an infinite number of quasi-conserved charges. Recently, a connection between non-holonomic (NH) and quasi-integrable (QI) deformations, subjected to the NLS system, has been obtained [11]. This is of interest owing to the fact that QI systems retain integrability in the asymptotic limit, whereas NH deformation maintains integrability generically [8].

In this paper we obtain the quasi-integrable deformation of the KdV equation. It is known that the Lax representation of KdV involves second and third order monic differential operators \((L, A)\) and on the other hand Lax representation of NLS is given by \(2 \times 2\) matrices [1]. Unfortunately the traditional Lax representation of the KdV equation cannot be applied to obtain the quasi-integrable deformation because of the appearance of spurious terms. So we use loop algebra representation of the KdV equation to derive the quasi-integrable deformation. However, KdV being a more than second order differential system, the equation of motion does not leave room for a dynamic deformation of the Lax pair, as done for SG [8, 9] and NLS [10, 11] systems. Instead, an off-shell deformation, that of the KdV Hamiltonian has been achieved, allowing for specific deformation of the suitable Lax component, interestingly yielding a hierarchy of higher-derivative extensions of KdV which are QI in nature, including a scaled version of KdV. In the perturbative domain, as it is known that using a certain ansatz the nonlinear Schrödinger equation can be approximated by the KdV equation [5, 6], a particular type of the said QI deformation of KdV was directly mapped to QI NLS systems, extending the validity of our result. Using this connection, and from the preceding general treatment, we obtain a general structure for the quasi-integrable deformation of KdV system and recover the results similar to that of Ferreira et al. [10] for NLS system; however, through an independent approach owing to the absence of definite parity-resolution in the solution-space, unlike that in case of QI NLS system.

2 Quasi-Integrable Deformation of KdV equation

2.1 Zakharov-Sabat representation

From the Adler, Kostant and Symes (AKS) equation, it is possible to obtain a pair of coupled complex KdV equations [12], through construction of the Lax pair:

\[
A = Q \quad \text{and} \quad B = T + [S, Q], \tag{2.1}
\]

where,

\[
T = -Q_{xx} + [Q^+, [Q^-, Q^+]] - [Q^-, [Q^-, Q^+]] \quad \text{and}
\]

\[
S = Q^+_x + Q^-_x + 4c(Q^+ + Q^-), \quad c \in \mathbb{R}, \tag{2.2}
\]

with the definitions,
\[ Q = \begin{pmatrix} 0 & \bar{q} \\ -q & 0 \end{pmatrix}; \quad Q^+ = \begin{pmatrix} 0 & \bar{q} \\ 0 & 0 \end{pmatrix}, \quad Q^- = \begin{pmatrix} 0 & 0 \\ 0 & -q \end{pmatrix}, \quad \] (2.3)

\[ Q = Q^+ + Q^- \equiv \bar{q}\sigma_+ - q\sigma_. \]

where, the Pauli matrices satisfy the SU(2) algebra:

\[ [\sigma_+, \sigma_-] = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad [\sigma_3, \sigma_{\pm}] = \pm 2\sigma_{\pm}, \quad \] (2.4)

and \( q, \bar{q} \) are mutually complex-conjugate amplitudes, leading to the coupled KdV equations. It is easy to see that an \( sl(2) \)-loop algebra can be constructed on the SU(2) basis, leading to a complete gauge-group interpretation of this system.

Incorporating the generic representation of Eqs 2.3, the Lax pair takes the form,

\[ A = \bar{q}\sigma_+ - q\sigma_- \quad \text{and} \quad B = (\bar{q}q_x - q\bar{q}x)\sigma_3 - (\bar{q}xxx + 2|q|^2\bar{q}x)\sigma_+ + (q_{xxx} + 2|q|^2q)\sigma_-, \quad |q|^2 = q\bar{q}. \quad \] (2.5)

The corresponding curvature then can be evaluated as,

\[ F_{tx} = (\bar{q}_t + \bar{q}xxx + 6|q|^2\bar{q}_x)\sigma_+ - (q_t + q_{xxx} - 6|q|^2q_x)\sigma_-, \quad \] (2.6)

yielding two coupled KdV-like equations,

\[ \bar{q} + qxxx + 6|q|^2\bar{q}x = 0 \quad \text{and} \quad q_t + q_{xxx} - 6|q|^2q_x = 0, \quad \] (2.7)

under zero-curvature condition, by considering each linearly independent component of the curvature matrix. Although the above equations posses higher order non-linearity than the usual KdV system, a straight-forward choice of variables,

\[ \bar{q} = u, \quad \text{and} \quad q = 1; \quad \quad u \in \mathbb{R}, \quad \] (2.8)

immediately leads to the non-coupled (usual) KdV equation,

\[ u_t + 6uu_x + u_{xxx} = 0. \quad \] (2.9)

The other possibility: \( \bar{q} = 1, \quad q = u \) leads to a KdV equation with a negative sign to the non-linear term, which can be transformed to the ‘usual’ one through the transformation \( u \rightarrow -u \). The Lax pair corresponding to the choice in Eq. 2.8 is,

\[ A = u\sigma_+ - \sigma_- \quad \text{and} \quad B = -u_x\sigma_3 - (u_{xx} + 2u^2)\sigma_+ + 2u\sigma_-, \quad \] (2.10)

leading to the curvature,

\[ F_{tx} := A_t - B_x + [A, B] \equiv (u_t + 6uu_x + u_{xxx})\sigma_. \quad \] (2.11)
2.2 Quasi-integrable Deformation

Since the KdV equation has derivatives higher than two, a dynamic interpretation of the same is not possible at the level of the equation itself. In order to employ the quasi-integrability mechanism of Ref.s [8–10, 13], the notion of potential is essential, that emerges from such an interpretation of the equation. In case of KdV system, however, a well-known Hamiltonian formulation [1] exists. In fact, the KdV equation 2.9 can be shown to emerge from two different equivalent Hamiltonians. Subjected to the order of non-linearity appearing in the Lax pair of Eq. 2.10, we opt for the following Hamiltonian,

\[ H_1[u] = \int_{-\infty}^{\infty} dx \left( \frac{1}{2} u_x^2 - u^3 \right) \quad \text{with} \quad \frac{\delta H_1[u]}{\delta u(x)} = -3u_x^2 - u_{xx}. \]  
(2.12)

This enables us to re-express the time component \((B)\) of the Lax pair as,

\[ B \equiv u_x \sigma_3 - \left[ u_{xx} - \frac{2}{3} \left( \frac{\delta H_1[u]}{\delta u(x)} + u_{xx} \right) \right] \sigma_+ + 2u \sigma_- . \]  
(2.13)

The above is a general expression to accommodate any possible deformation at the Hamiltonian level. We propose that the deformation of the system is implemented in the non-linear part of Hamiltonian for the KdV system to impart quasi-integrability, the explicit form of which will be discussed below. The corresponding curvature takes the form:

\[ F_{tx} \equiv \left[ u_t + u_{xxx} - \frac{2}{3} \partial_x \left( \frac{\delta H_1[u]}{\delta u(x)} + u_{xx} \right) \right] \sigma_+ + \mathcal{X} \sigma_3 , \]  
(2.14)

with the supposed anomaly term,

\[ \mathcal{X} = 2u_x^2 + \frac{2}{3} \left( \frac{\delta H_1[u]}{\delta u(x)} + u_{xx} \right) , \]  
(2.15)

that vanishes for undeformed system\(^1\). In the presence of this term, implementation of the deformed ‘equation of motion’ (EOM) (i.e., the KdV equation),

\[ u_t + u_{xxx} - \frac{2}{3} \partial_x \left( \frac{\delta H_1[u]}{\delta u(x)} + u_{xx} \right) \sigma_+ + 2u u_x = 0 , \]

or, \[ u_t + 6u u_x + u_{xxx} = \mathcal{X} , \]  
(2.16)

leaves the curvature non-zero.

One can construct an infinite number of quasi-conserved charges through the Abelianization procedure applied in Ref.s [8–10], through gauge-transforming the Lax components. In doing so, the anomaly \( \mathcal{X} \) prevents rotation of both of them into the same infinite dimensional Abelian subalgebra of the characteristic \( sl(2) \) loop algebra, eventually leading to

\(^1\)One can very well work with the second Hamiltonian form for the KdV system [1]:

\[ H_2[u] = - \int_{-\infty}^{\infty} dx \left( \frac{1}{2} u_x^2 \right) \times (x) , \]  

with the alternate fundamental bracket defined as \( \{ u(x), u(y) \} = [\partial_x^3 + 2 (u_x + u_{xx})] \delta(x - y) \). Then, the time component of the Lax pair will take the form:

\[ B \equiv -u_x \sigma_3 - \left[ u_{xx} - \frac{4}{3} \left( \frac{\delta H_2[u]}{\delta u(x)} \right) \right] \sigma_+ + 2u \sigma_- . \]  
Rest will follow through the replacement: \[ 2 \left( \frac{\delta H_1[u]}{\delta u(x)} + u_{xx} \right) \rightarrow \frac{4 \delta H_2[u]}{\delta u(x)} . \]
an infinite set of quasi-conservation laws characterized by $\mathcal{X}$. In doing so, to project-out the anomaly term beforehand, one can perform an initial gauge transformation, which is generically defined as,

$$ (A, B) \rightarrow U(A, B)U^{-1} + \partial_{(x,t)}UU^{-1}, \quad F_{tx} \rightarrow UF_{tx}U^{-1}, $$

(2.17)

for a constant gauge choice,

$$ U = \exp \left( -\frac{1}{2} \sigma_3 \right), $$

(2.18)

so that the second term of the first equation in Eq.s 2.17 does not contribute presently. On further imposing the deformed equation of motion, this leads to the on-shell curvature:

$$ \tilde{F}_{tx} = \frac{1}{e} \left[ u_t + u_{xxx} - \frac{2}{3} \partial_x \left( \frac{\delta H_1[u]}{\delta u(x)} + u_{xx} \right) + 2uu_x \right] \sigma_+ + \mathcal{X} \sigma_3 \equiv \mathcal{X} \sigma_3, $$

(2.19)

where $e$ being the exponential function with unit argument. The KdV equation now can be applied as a particular choice of gauge has been made. The corresponding ‘rotated’ Lax pair is,

$$ \tilde{A} = \frac{1}{e} u \sigma_+ - e \sigma_- \quad \text{and} \quad \tilde{B} = -u_x \sigma_3 - \frac{1}{e} \left[ u_{xx} - \frac{2}{3} \left( \frac{\delta H_1[u]}{\delta u(x)} + u_{xx} \right) \right] \sigma_+ + 2uu_x \sigma_-. $$

(2.20)

The $sl(2)$ loop algebra: The $SU(2)$ algebraic structure for the KdV system [12] enables the construction of an $sl(2)$ loop algebra:

$$ [b^n, F^{m+n}_{1,2}] = 2F^{m+n}_{2,1}, \quad [F^n_1, F^n_2] = \lambda b^n_{m+n}, $$

(2.21)

consistent with the definitions,

$$ b^n = \lambda^n \sigma_3, \quad F^n_1 = \frac{1}{\sqrt{2}} \lambda^n (\lambda \sigma_+ - \sigma_-) \quad \text{and} \quad F^n_2 = \frac{1}{\sqrt{2}} \lambda^n (\lambda \sigma_+ + \sigma_-), $$

(2.22)

with $\lambda$ being the spectral parameter. Such a structure is essentially same as that in Ref. [10] for quasi-integrable (QI) NLS systems. This serves as a strong connection between QI deformations of the two systems, which we will address soon.

The Gauge Transformation: On following the general approach utilized in Ref.s [8–10], we undertake another gauge transformation with respect to the operator,

$$ g = \exp \left( \sum_{n=0}^{\infty} G_n \right), \quad \text{where,} \quad G_n = a^n_1 F^n_1 + a^n_2 F^n_2. $$

(2.23)

Here the coefficients $a^n_{1,2}$ are to be chosen such that the transformed spatial component of the Lax pair $\tilde{A} = g \tilde{A} g^{-1} + g_x g^{-1}$ depends only on $b^n$s:
\[ A \equiv \sum_n \beta_n A_n, \quad (2.24) \]

making it diagonal in the \( SU(2) \) basis. On considering the BCH formula:

\[ e^X Y e^{-X} = Y + [X, Y] + \frac{1}{2!}[X, [X, Y]] + \frac{1}{3!}[X, [X, [X, Y]]] + \cdots, \]

the gauge-transformed spatial component takes the form,

\[ \tilde{A} = G_n x + \bar{A} + [G_n, \bar{A}] + \frac{1}{2!}[G_m, [G_n, \bar{A}]] + \frac{1}{3!}[G_l, [G_m, [G_n, \bar{A}]]] + \cdots, \quad (2.25) \]

where summation is understood over all integers, which are semi-positive. The first few of the individual commutators are,

\[ [G_n, \bar{A}] = \frac{1}{\sqrt{2}e} u (a_1^n - a_2^n) b^n - \frac{e}{\sqrt{2}} (a_1^n + a_2^n) b^{n+1}, \]
\[ \frac{1}{2!}[G_m, [G_n, \bar{A}]] = -\frac{1}{\sqrt{2}e} u (a_1^n - a_2^n) (a_1^m F_2^{m+n} + a_2^m F_1^{m+n}) + \frac{e}{\sqrt{2}} (a_1^n + a_2^n) \]
\[ \times (a_1^m F_2^{m+n+1} + a_2^m F_1^{m+n+1}), \]
\[ \frac{1}{3!}[G_l, [G_m, [G_n, \bar{A}]]] = \frac{1}{3\sqrt{2}} (a_1^l a_1^m - a_2^l a_2^m) \left[ \frac{u}{e} (a_1^n - a_2^n) b^{l+m+n+1} - e (a_1^n + a_2^n) b^{l+m+n+2} \right], \]
\[ \frac{1}{4!}[G_p, [G_l, [G_m, [G_n, \bar{A}]]]] = \frac{1}{6\sqrt{2}} u (a_1^n - a_2^n) (a_1^l a_1^m - a_2^l a_2^m) \left( a_1^p F_2^{p+l+m+n+1} + a_2^p F_1^{p+l+m+n+1} \right) \]
\[ - \frac{e}{6\sqrt{2}} (a_1^n + a_2^n) (a_1^l a_1^m - a_2^l a_2^m) \left( a_1^p F_2^{p+l+m+n+2} + a_2^p F_1^{p+l+m+n+2} \right), \]

\[ \vdots \quad (2.26) \]

The condition of vanishing the coefficients of \( F_1^{n,2} \) leads to the order-by-order relations,
\[
\mathcal{O}(F_1^0) : \quad a_1^0 = \frac{1}{\sqrt{2}} \left( a_1^0 - a_2^0 \right) a_2^0 - \frac{1}{\sqrt{2}} \frac{u}{\epsilon \lambda} + c \epsilon \left( a_1^0 - a_2^0 \right) a_2 \left( a_1^0 + a_2^0 \right) + \epsilon \left( a_1^0 - a_2^0 \right) a_2^0,
\]

\[
\mathcal{O}(F_2^0) : \quad a_2^0 = \frac{1}{\sqrt{2}} \left( a_1^0 - a_2^0 \right) a_1^0 - \frac{1}{\sqrt{2}} \frac{u}{\epsilon \lambda} - \epsilon \left( a_1^0 - a_2^0 \right) a_1^0,
\]

\[
\mathcal{O}(F_1^1) : \quad a_1^1 = \frac{1}{\sqrt{2}} \left( a_1^1 - a_2^1 \right) a_2^1 + \left( a_1^1 - a_2^1 \right) a_1^1 - \frac{1}{\sqrt{2}} \frac{u}{\epsilon \lambda} - \epsilon \left( a_1^1 - a_2^1 \right) a_1^1,
\]

\[
\mathcal{O}(F_2^1) : \quad a_2^1 = \frac{1}{\sqrt{2}} \left( a_1^1 - a_2^1 \right) a_2^1 + \left( a_1^1 - a_2^1 \right) a_2^1 - \frac{1}{\sqrt{2}} \frac{u}{\epsilon \lambda} - \epsilon \left( a_1^1 - a_2^1 \right) a_2^1,
\]

\[
\mathcal{O}(F_1^2) : \quad a_1^2 = \frac{1}{\sqrt{2}} \left( a_1^2 - a_2^2 \right) a_2^2 + \left( a_1^2 - a_2^2 \right) a_2^2 - \frac{1}{\sqrt{2}} \frac{u}{\epsilon \lambda} - \epsilon \left( a_1^2 - a_2^2 \right) a_2^2,
\]

\[
\mathcal{O}(F_2^2) : \quad a_2^2 = \frac{1}{\sqrt{2}} \left( a_1^2 - a_2^2 \right) a_2^2 + \left( a_1^2 - a_2^2 \right) a_2^2 - \frac{1}{\sqrt{2}} \frac{u}{\epsilon \lambda} - \epsilon \left( a_1^2 - a_2^2 \right) a_2^2,
\]

\[
(2.27)
\]

satisfying which, we get the desired surviving coefficients of \(b^n\) in Eq. 2.24 as,

\[
\beta_0^A = \frac{1}{\sqrt{2}} \frac{u}{\epsilon \lambda} \left( a_1^0 - a_2^0 \right),
\]

\[
\beta_1^A = \frac{1}{\sqrt{2}} \frac{u}{\epsilon \lambda} \left( a_1^1 - a_2^1 \right) - \frac{1}{\sqrt{2}} \left( a_1^1 + a_2^1 \right) - \frac{1}{\sqrt{2}} \frac{u}{\epsilon \lambda} \left( a_1^1 a_1^0 - a_2^0 a_2^0 \right) \left( a_1^0 - a_2^0 \right),
\]

\[
\beta_2^A = \frac{1}{\sqrt{2}} \frac{u}{\epsilon \lambda} \left( a_1^2 - a_2^2 \right) - \frac{1}{\sqrt{2}} \left( a_1^2 + a_2^2 \right) - \frac{1}{\sqrt{2}} \frac{u}{\epsilon \lambda} \left[ 2 \left( a_1^1 a_1^0 - a_2^0 a_2^0 \right) \left( a_1^0 - a_2^0 \right) + \left( a_1^1 a_1^0 - a_2^0 a_2^0 \right) \right] \left( a_1^0 - a_2^0 \right)
\]

\[
\times \left( a_1^0 + a_2^0 \right) + \frac{1}{\sqrt{2}} \frac{u}{\epsilon \lambda} \left( a_1^2 a_1^0 - a_2^0 a_2^0 \right) \left( a_1^0 + a_2^0 \right) + \cdots.
\]

\[
(2.28)
\]

It is clear that all the coefficients \(a_{n,2}^n\) can completely be determined by solving Eq.s 2.27, thereby leading to complete evaluation of the coefficients \(\beta_n^A\) in Eq.s 2.28. Therefore, the rotated Lax component \(\hat{A}\) is completely known.
The same gauge transformation transforms the temporal Lax component $\tilde{B}$ to $\tilde{B} = g\bar{B}g^{-1} + g_xg^{-1}$, with corresponding commutators being,

\[
\left[ G_n, \tilde{B} \right] = 2u_xG_n - \frac{1}{\sqrt{2}} \frac{f(u)}{e} (a_1^n - a_2^n) b^n + 2eu(a_1^n + a_2^n) b^{n+1},
\]

\[
\frac{1}{2!} \left[ G_m, [G_n, \tilde{B}] \right] = \frac{1}{\sqrt{2}} \frac{f(u)}{e} (a_1^n - a_2^n) (a_1^mF_2^{m+n} + a_2^mF_1^{m+n}) - \sqrt{2}eu(a_1^n + a_2^n)
\times (a_1^mF_2^{m+n+1} + a_2^mF_1^{m+n+1}),
\]

\[
\frac{1}{3!} \left[ G_l, [G_m, [G_n, \tilde{B}]] \right] = \frac{1}{3} (a_1^n a_1^m - a_2^n a_2^m) \left[ \frac{1}{\sqrt{2}} \frac{f(u)}{e} (a_1^n - a_2^n) b^{l+m+n+1}
\right. \\
\left. - \sqrt{2}eu(a_1^n + a_2^n) b^{l+m+n+2} \right],
\]

\[
\frac{1}{4!} \left[ G_p, [G_l, [G_m, [G_n, \tilde{B}]]] \right] = -\frac{1}{6\sqrt{2}} \frac{f(u)}{e} (a_1^n - a_2^n) \left( a_1^l a_1^m - a_2^l a_2^m \right)
\times \left( a_1^p F_2^{p+l+m+n+1} + a_2^p F_1^{p+l+m+n+1} \right) + \frac{\sqrt{2}}{6} eu(a_1^n + a_2^n)
\times \left( a_1^l a_1^m - a_2^l a_2^m \right) \left( a_1^p F_2^{p+l+m+n+2} + a_2^p F_1^{p+l+m+n+2} \right);
\]

\begin{equation}
\left(2.29\right)
\end{equation}

where, $f(u) = u_{xx} - \frac{2}{3} \left( \frac{\delta H_1[u]}{\delta u(x)} + u_{xx} \right) \equiv u_{xx} - \chi + 2u^2$.

The general form of the final temporal component is,

\[
\tilde{B} = \sum_n \left[ \beta_n b^n + \varphi_1^n F_1^m + \varphi_2^n F_2^m \right],
\]

\begin{equation}
\left(2.30\right)
\end{equation}

wherein, a few of the coefficients are,
\[ \beta_0^B = -u_x - \frac{1}{\sqrt{2}} \frac{f(u)}{e} (a_1^0 - a_2^0), \]
\[ \beta_1^B = -\frac{1}{\sqrt{2}} \frac{f(u)}{e} (a_1^1 - a_2^1) + \sqrt{2} e u (a_1^0 + a_2^0), \]
\[ \beta_2^B = -\frac{1}{\sqrt{2}} \frac{f(u)}{e} (a_1^2 - a_2^2) + \sqrt{2} e u (a_1^1 + a_2^1) + \frac{1}{3\sqrt{2}} \frac{f(u)}{e} [2(a_1^0 a_1^1 - a_2^0 a_2^1)(a_1^1 - a_2^1) \]
\[ + (a_1^0 a_2^0 - a_2^0 a_1^0)(a_1^1 - a_2^1)] - \frac{\sqrt{2}}{3} e u (a_1^0 a_1^0 - a_2^0 a_2^0)(a_1^1 + a_2^1) + \cdots, \] 
\[ \vdots \] 
\[ \varphi_0^1 = a_1^0 t - \left( \frac{1}{\sqrt{2}} \frac{f(u)}{e \lambda} + \sqrt{2} e u \right) + 2u_x a_1^0, \]
\[ \varphi_1^1 = a_1^1 t + 2u_x a_1^1 + \frac{1}{\sqrt{2}} \frac{f(u)}{e} [(a_1^0 - a_2^0) a_2^1 + (a_1^1 - a_2^1) a_2^0] - \sqrt{2} e u (a_1^0 + a_2^0) a_2^0, \]
\[ \varphi_2^1 = a_1^2 t + 2u_x a_1^2 + \frac{1}{\sqrt{2}} \frac{f(u)}{e} [(a_1^0 - a_2^0) a_2^2 + (a_1^1 - a_2^1) a_2^1 + (a_1^2 - a_2^2) a_2^0] \]
\[ - \sqrt{2} e u [(a_1^1 + a_2^1) a_2^0 + (a_1^0 + a_2^0) a_2^1] + \frac{1}{3\sqrt{2}} e u (a_1^0 a_1^0 - a_2^0 a_2^0)(a_1^1 + a_2^1) a_2^0 \]
\[ + \frac{1}{6\sqrt{2}} \frac{f(u)}{e} [2(a_1^0 a_1^0 - a_2^0 a_2^0)(a_2^0 - a_2^0) a_2^1 + (a_1^0 a_1^0 - a_2^0 a_2^0) a_1^1] \]
\[ \left\{ (a_2^1 - a_1^1) a_2^0 + (a_2^0 - a_1^0) a_2^1 \right\}] + \cdots, \] 
\[ \vdots \]
\[ \varphi_0^2 = a_2^0 t - \left( \frac{1}{\sqrt{2}} \frac{f(u)}{e \lambda} - \sqrt{2} e u \right) + 2u_x a_2^0, \]
\[ \varphi_1^2 = a_2^1 t + 2u_x a_2^1 + \frac{1}{\sqrt{2}} \frac{f(u)}{e} [(a_1^0 - a_2^0) a_2^1 + (a_1^1 - a_2^1) a_2^0] - \sqrt{2} e u (a_1^0 + a_2^0) a_2^0, \]
\[ \varphi_2^2 = a_2^2 t + 2u_x a_2^2 + \frac{1}{\sqrt{2}} \frac{f(u)}{e} [(a_1^0 - a_2^0) a_2^2 + (a_1^1 - a_2^1) a_2^1 + (a_1^2 - a_2^2) a_2^0] \]
\[ - \sqrt{2} e u [(a_1^1 + a_2^1) a_2^0 + (a_1^0 + a_2^0) a_2^1] + \frac{1}{3\sqrt{2}} e u (a_1^0 a_1^0 - a_2^0 a_2^0)(a_1^1 + a_2^1) a_2^0 \]
\[ + \frac{1}{6\sqrt{2}} \frac{f(u)}{e} [2(a_1^0 a_1^0 - a_2^0 a_2^0)(a_2^0 - a_2^0) a_2^1 + (a_1^0 a_1^0 - a_2^0 a_2^0) a_1^1] \]
\[ \times \left\{ (a_2^1 - a_1^1) a_2^0 + (a_2^0 - a_1^0) a_2^1 \right\}] + \cdots, \] 
\[ \vdots \] 

Finally, upon the second transformation, the curvature takes the form,
\[ F_{tx} := g F_{tx} g^{-1} = \hat{A}_t - \hat{B}_x + \left[ \hat{A}, \hat{B} \right] \]
\[ = X \left[ b^0 - 2 (a^0_1 F_2^0 + a^0_2 F_1^0) - (a^0_1 a^0_1 - a^0_1 a^0_2) b^{m+n+1} \right] + \frac{2}{3} (a^1_1 a^m_1 - a^1_2 a^m_2) \left( a^l_1 F_2^{l+m+n+1} + a^l_2 F_1^{l+m+n+1} \right) + \frac{1}{6} (a^{ar{n}}_1 a^m_1 - a^{ar{n}}_2 a^m_2) \left( a^l_1 a^l_1 - a^l_2 a^l_2 \right) b^{p+l+m+n+2} - \frac{1}{15} (a^1_1 a^1_1 - a^2_2 a^m_2) \left( a^l_1 a^l_1 - a^l_2 a^l_2 \right) \left( a^q F_2^{q+p+l+m+n+2} + a^q F_1^{q+p+l+m+n+2} \right) - \frac{1}{90} (a^1_1 a^m_1 - a^2_2 a^m_2) \left( a^l_1 a^l_1 - a^l_2 a^l_2 \right) \left( a^r a^q F_1^{q+p+l+m+n+2} + a^r a^q F_2^{q+p+l+m+n+2} \right) + \ldots \]
\[ := X \sum_n (f^n_0 b^n + f^n_1 F_1^n + f^n_2 F_2^n), \quad (2.34) \]

with coefficients,

\[ f^0_0 = 1, \]
\[ f^0_1 = -(a_1^0 a_1^0 - a_2^0 a_2^0), \]
\[ f^2_0 = -2 (a_1^0 a_1^1 - a_2^0 a_2^1) + \frac{1}{6} (a_1^0 a_1^0 - a_2^0 a_2^0)^2, \]
\[ f^0_0 = -2 (a_1^0 a_1^2 - a_2^0 a_2^2) - (a_1^0 a_1^0 - a_2^0 a_2^0) + \frac{2}{3} (a_1^0 a_1^0 - a_2^0 a_2^0) \left( a_1^0 a_1^0 - a_2^0 a_2^0 \right) - \frac{1}{60} (a_1^0 a_1^0 - a_2^0 a_2^0)^2 \right\}, \quad (2.35) \]
\[ f^0_2 = -2 a_2^0, \]
\[ f^1_0 = 2 \left[ -a_2^0 + \frac{1}{3} (a_1^0 a_1^0 - a_2^0 a_2^0) a_2^0 \right], \]
\[ f^1_2 = 2 \left[ -a_2^0 + \frac{2}{3} (a_1^0 a_1^1 - a_2^0 a_2^1) a_2^0 + \frac{1}{3} (a_1^0 a_1^0 - a_2^0 a_2^0) a_2^0 \right] - \frac{1}{30} (a_1^0 a_1^0 - a_2^0 a_2^0)^2 a_2^0 \right\}], \quad (2.36) \]

### 2.3 Quasi-conservation

In order to demonstrate the deviation from integrability, based on the QI deformation, it is pertinent to evaluate quantities which would have represent conservation or have themselves be conserved for the undeformed system. For brevity, contributions of zeroth order in the spectral parameter \( \lambda \), which has effectively been the order of expansion following the loop algebra defined in Eq. 2.22, are considered. With this motive, we first single-out the available relations from the expressions of different coefficients of that order, obtained in above. On subtracting the zeroth order terms in Eq.s 2.27, one finds,
\[ a^0_{-x} + \frac{1}{\sqrt{2e}} u (a^0_x)^2 + \sqrt{2e} = 0, \quad \text{where} \quad a^0_- := a^0_1 - a^0_0, \quad (2.37) \]

which can exactly be solved for a given solution \( u(x) \). Further, on considering expressions of \( \tilde{A}, \tilde{B} \) and \( \tilde{F}_{tx} \) from Eq.s 2.24, 2.30 and 2.34 in the zeroth order in the spectral parameter \( \lambda \), the definition of the curvature leads to,

\[
\frac{1}{\sqrt{2e}} (ua^0_x)_t + u_{xx} + \frac{1}{\sqrt{2e}} (f(u)a^0_x)_x = \mathcal{X} \quad \text{and} \\
\varphi^0_- x + \sqrt{2e} \varphi^0_- a^0_0 = 2 \mathcal{X} a^0_0; \quad \text{where} \quad \varphi^0_- := \varphi^1_0 - \varphi^2_0, \quad (2.38)
\]

wherein the expressions from Eq.s 2.28, 2.31, 2.32, 2.33, 2.35 and 2.36 have respectively been utilized. Finally, from Eq.s 2.32 and 2.33 one finds,

\[ a^0_- t = \varphi^0_- + 2 \sqrt{2e} u - 2u_x a^0_- \quad (2.39) \]

From Eq.s 2.37, 2.38 and 2.39, the system can completely be solved up to the zeroth order.

One can construct \( n \) number of quasi-continuity expression for the present system, by considering the expressions in Eq.s 2.24 and 2.30 followed by the final one in Eq. 2.34. Then, the co-efficients of \( b^n \) in the definition of curvature leads to the of the quasi-continuity forms,

\[ \Gamma^n := \beta^n_A - \beta^n_B x \equiv \mathcal{X} f^n_0, \quad (2.40) \]

which vanish for the undeformed KdV system as \( \mathcal{X} = 0 \). From the zeroth order \( (n = 0) \) expressions in Eq.s 2.28 and 2.31,

\[ \Gamma^0 = \beta^0_A - \beta^0_B x = \frac{1}{\sqrt{2e}} (ua^0_x)_t + u_{xx} + \frac{1}{\sqrt{2e}} (f(u)a^0_x)_x \equiv \mathcal{X}, \quad (2.41) \]

with the final equality coming from the first of Eq. 2.38. At higher orders, it can be shown that the RHS is a pure function of the anomaly \( \mathcal{X} \) that vanishes for the undeformed system. Therefore, the anomaly function obtained is solely responsible for the deviation of the system from integrability, as was observed in Ref. [7–10]. It will be shown later, by considering a particular form of deformation, that this deviation is infinitesimal in magnitude. For the mean-time, from an order-by-order verification, it is easy to foresee that,

\[ \Gamma^n \equiv \mathcal{X} f^n_0, \quad (2.42) \]

with coefficients \( f^n_0 \) are given in Eq. 2.35.

Following the treatment for QI NLS system in Ref. [10], on the basis of the weak equivalence of NLS and KdV systems [5], the zeroth order quasi-conserved charge is,

\[ Q^0 := \int_x \beta^0_0 \equiv \frac{1}{\sqrt{2e}} \int_x ua^0_0, \quad (2.43) \]

which needs to be conserved for quasi-integrability. This is consistent with the definition of the quasi-continuity expression in Eq. 2.41 [10]. From the first of Eq.s 2.38,
\[
\frac{dQ^0}{dt} = \frac{1}{\sqrt{2}e} \int_x (u a^0) \frac{d}{dt} = \int_x \left[ \mathcal{X} - u_{xx} - \frac{1}{\sqrt{2}e} (f(u)a^0)_x \right] \equiv \int_x \mathcal{X}, \tag{2.44}
\]

modulo vanishing total derivatives of functions of the amplitude \( u \), that and its derivatives can safely be assumed to vanish asymptotically. In general,

\[
\frac{dQ^n}{dt} = \int_x \mathcal{X} f^n_0 \equiv \int_x \Gamma^n := \Lambda^n. \tag{2.45}
\]

The RHS above is not zero in general, but the integral can vanish, following judicious choice of \( \mathcal{X} \). Such a result will physically imply that the asymptotically the system is integrable, which will correspond to the scattering states of the same. For a localized solution \( u \), it is safe to assume that it attains at least a constant value (mostly 0) in such scattering states, and its derivatives duly vanish. This amounts for the ‘asymptotic integrability’ of the QI systems \([8–10]\). In the present case, where the QI deformation is obtained through that of the Hamiltonian \( H_1[u] \), a suitable choice for the same can yield the desired result. The integrand itself can also exactly be zero, for particular value of \( n \), provided corresponding \( f^n_0 \) vanishes. This can very much be possible for particular KdV amplitude \( u \), leading to a particular value of \( f^n_0 \) for which the space-integral of \( \Gamma^n \) vanishes.

### 3 Comparison with QI NLS System and Definite Results

In order to understand the possibility of explicit quasi-integrability of KdV system, we now try to utilize the \( \mathbb{Z}_2 \) symmetry of \( sl(2) \) loop algebra, as explicated in Ref. \([10]\). We start with equivalently re-defining the generators as,

\[
[b^n, F_{1,2}^n] = 2F_{2,1}^{n+n}, \quad [F_1^n, F_2^n] = \delta b^{n+n}, \quad \text{where,}
\]

\[
b^n = \lambda^n \sigma_3, \quad F_1^n = \frac{1}{\sqrt{2}} \lambda^n (\delta \sigma_+ - \sigma_-) \quad \text{and} \quad F_2^n = \frac{1}{\sqrt{2}} \lambda^n (\delta \sigma_+ + \sigma_-), \tag{3.1}
\]

with \( \delta \) is a parameter, to be identified later. In case of NLS system, it was the sign of the coefficient of the non-linear term in the EOM \([10]\). Under this definition, instead of the constant gauge choice of Eq. 2.18, we choose a new one,

\[
\mathcal{U} = \exp \left[ \tanh^{-1} \left( \frac{\delta + u}{\delta - u} \right) \right] b^0. \tag{3.2}
\]

This leads to a new spatial Lax component,

\[
\hat{A} = \left[ \tanh^{-1} \left( \frac{\delta + u}{\delta - u} \right) \right] b^0 - 2i \sqrt{\frac{u}{2\delta}} F_2^0. \tag{3.3}
\]

The crucial difference between the components \( \hat{A} \) of Eq. 2.20 or \( \hat{A} \) of Eq. 2.24 and \( \hat{A} \) is that only the last one is a definite eigenfunction of the \( \mathbb{Z}_2 \) transformation mentioned above. It is a combination of the order 2 automorphism of \( sl(2) \) loop algebra:

\[
\Sigma(b^n) = -b^n, \quad \Sigma(F_1^n) = -F_1^n \quad \text{and} \quad \Sigma(F_2^n) = F_2^n, \tag{3.4}
\]
and parity:

\[ \mathcal{P} : (\tilde{x}, \tilde{t}) \rightarrow (-\tilde{x}, -\tilde{t}) \; \text{with} \; \tilde{x} = x - x_0 \; \text{and} \; \tilde{t} = t - t_0 \]  (3.5)

about a particular point \((x_0, t_0)\) in space-time, which can very well be chosen to the origin. These transformations are mutually commute, as they work in two different spaces (i. e., group and coordinate subspaces). Thus,

\[ \Omega (\vec{A}) = \vec{A}, \quad \Omega = \Sigma \mathcal{P}, \]  (3.6)

for \(u\) being parity-even. It is sensible enough to assume so as KdV equation is parity-invariant to begin with, and the quasi-modified one in Eq. 2.16 is also the same, especially subjected to the explicit form of deformation to be introduced in the next section\(^2\). More intuitively, as QI systems support single-soliton structures of the undeformed systems, the well-known bright and dark soliton solutions of standard KdV are parity-even. Therefore, it is sensible to consider \(u\) as such.

The use of \(sl(2)\) symmetry was motivated \[10\] by determining parity properties of the integrands \(\Gamma^n\), as the vanishing of its integral over space (Eq. 2.45) ensures conservation of the corresponding charge \(Q^n\), rendering the deformed system QI. For this purpose, the kernel and image subspaces of \(sl(2)\) were considered, with generators \(b^n\) being the semi-simple element that splits the loop algebra into them. To practically utilize this, another gauge transformation can be performed with respect to,

\[ \mathfrak{g} = \exp \sum_{n=1}^{\infty} G^{-n}, \quad G^{-n} = \eta_1^{-n} F_1^{-n} + \eta_2^{-n} F_2^{-n}, \]  (3.7)

which leads to,

\[ \vec{A} \rightarrow \tilde{\vec{A}} = \mathfrak{g} \vec{A} \mathfrak{g}^{-1} + \mathfrak{g}_x \mathfrak{g}^{-1}, \]  (3.8)

with the goal of obtaining a \(\mathfrak{g}\) with definite transformation under \(\Omega\). For this purpose, following Ref. \[10\], terms with different powers of spectral parameter \(\lambda\) of \(\vec{A} = \sum_m \vec{A}^{(m)}\) are compared with those in the RHS of Eq. 3.8 as,

\[ \tilde{\vec{A}}^{(0)} = \vec{A}, \]
\[ \tilde{\vec{A}}^{(-1)} = [G^{-1}, \vec{A}] + \partial_x G^{-1}, \]
\[ \tilde{\vec{A}}^{(-2)} = [G^{-2}, \vec{A}] + \frac{1}{2!} [G^{-1}, [G^{-1}, \vec{A}]] + \frac{1}{2!} [G^{-1}, \partial_x G^{-1}] + \partial_x G^{-2}, \]
\[ \vdots \]  (3.9)

However, unlike that for QI NLS system \[10\], the spatial Lax component is entirely \(\mathcal{O}(\lambda^0)\), barring separation of commutators with \(b^0\) from those with \(F^0_2\) of \(\vec{A}\) on the basis of powers

\(^2\)Practically it amounts to having \(\chi_x\) odd in derivatives, which it is.
of the spectral parameter. As $b^0$ splits the subspaces, the above fact does not allow to separately obtain $\Omega$-transformations of the group elements $G^{-n}$ of $g$. This essentially obstructs the exact determination of parity properties of $G^{-n}$, unlike that in Ref. [10]. More directly, and decisively, the spatial KdV Lax operator itself cannot even be confined to the Kernel subspace, by demanding it to be an $\Omega$-eigenstate beforehand. It is confirmed by the first of the Eq.s 3.9, that requires vanishing of the coefficient of $b^0$ in $\tilde{A}$ itself, which requires the inconsistency $u = 0!$, as can be seen from Eq. 3.3.

Although surely the KdV and very expectedly the QI KdV are parity-invariant equations, the very nature of the Lax formulation of the same (Eq. 2.10) prohibits a sub-algebraic separation, as they are strictly $O(\lambda^0)$. A direct approach, as in the previous section, can lead to explicit quasi-integrability, through brute-force evaluation of the coefficients $f^n_0$'s and/or through judicious choice of the Hamiltonian $H_1[u]$, as discussed following Eq. 2.45. The anomaly function $\mathcal{X}$ for QI NLS [10] is explicitly parity-odd, resulting into vanishing of its integral, yielding a conserved charge. For the present case, however, the very definition of the anomaly through deformation of the Hamiltonian provides a different opportunity. If that deformation yields an anomaly $\mathcal{X}$ which is a total derivative of a function of $u$ and its derivatives, by virtue of Eq. 2.44, the charge $Q^0$ will be conserved embodying quasi-integrability. For an extended (deformed) Hamiltonian,

$$H_1[u] \to H_1[u] = \int_{-\infty}^{\infty} \left[ \frac{1}{2} u_x^2 - u^3 + \epsilon F(u) \right], \quad (3.10)$$

wherein $\epsilon$ is the deformation parameter, we consider the example $F(u) = \frac{3}{4} uu_{xx}$. This will immediately yield $\mathcal{X} = \epsilon u_{xx}$ (Eq. 2.15) yielding a conserved $Q^0$, irrespective of the parity or any other properties of $u$. From Eq. 2.16, or from the definition of EOM with the Hamiltonian,

$$u_t = \left[ \frac{\delta H[u]}{\delta u} \right]_x,$$

the QI KdV equation takes the form,

$$u_t + 6uu_x + (1 - \epsilon)u_{xxx} = 0, \quad (3.11)$$

which is essentially a scaling of the undeformed system, supporting similar solutions, including solitons. This is expected as the choice for $F(u)$ is nothing but a total derivative away from the first term in $H_1[u]$, modulo $\epsilon$. Therefore it will yield eventually a completely integrable system, with single soliton solutions of the form,

$$u = \frac{c}{2} \text{sech}^2 \left[ \frac{c}{4(1 - \epsilon)} (x - ct - x_0 + ct_0) \right], \quad c > 0, \quad (3.12)$$

with speed $c$. Non-trivially and more importantly, however, this provides an opportunity to construct a hierarchy of higher-derivative extensions of KdV, with different choices of $F(u)$. For demonstration, we consider the following two:
\[ F(u) = -\frac{3}{2m} \epsilon u_x^m \quad \text{and} \quad F(u) = \frac{3}{4} \epsilon uu^{(2n)}, \quad \text{with} \quad m = 3, 4, \cdots; \quad n = 1, 2, \cdots, \] (3.13)

where \( m \) is ordinary power and \( n \) is the order of space derivatives, leading to the higher-derivative EOMs,

\[
\begin{align*}
& u_t + 6uu_x + u_{xxx} = \epsilon (u_x)^{m-1} \\
& u_t + 6uu_x + u_{xxx} = \epsilon u^{(2n+1)},
\end{align*}
\] (3.14)

respectively. A numerous other possibilities are there and we aspire to analyze such QI systems in the future. It would be interesting study their solitonic structures, as solutions like that in Eq. 3.12 do not satisfy them.

Thus far, although KdV is parity-symmetric, the particular algebraic structure of the system does not allow for a direct utility of that property to be incorporated into the corresponding QI analysis\(^3\). However, the proposed deformation of the Hamiltonian can be utilized to obtain a hierarchy of more extensive QI systems, with \( \mathcal{X} \) itself being a total space-derivative, ensuring at least one conserved charge (\( Q^0 \)). The fact that at present it is not sure whether these systems support solitonic structures, or even definite solutions, motivates further studies. In the next section, we will consider a perturbation approximation of the QI deformation for KdV, utilizing a known weak-coupling mapping to NLS system.

## 4 Infinitesimal QI Deformation: Weak-coupling NLS Analogy

Now we consider infinitesimal deformation of the KdV Hamiltonian \( H_1[u] \) in the spirit of Ref.s [8–10]. Therein, the non-linear term in \( u \) can be deformed as \( u \to u^{1+\epsilon}, \quad 0 < \epsilon \ll 1, \) to yield,

\[
H_1^{\text{Def}}[u] = \int_{-\infty}^{\infty} dx \left( \frac{1}{2} u_x^2 - u^{3+3\epsilon} \right).
\] (4.1)

This leads to the expression of the anomaly term,

\[
\mathcal{X} \approx -2\epsilon u^2 (1 + 3 \log(u)),
\] (4.2)

which is \( O(\epsilon) \) as expected, as \( u \) is finite in general. It is clear that the above anomaly does not posses a definite parity and will not conform to a conserved charge (\( Q^0 \) at least). However, in this particular case, the expectation of quasi-integrability arise from the asymptotic behavior of the QI KdV amplitude \( u \) [8, 9], and thereby that of the anomaly. This is

\(^3\) One may directly construct a \( \eta \) with parity-odd \( \eta_1^{(-n)} \) and parity-even \( \eta_2^{(-n)} \), or vice versa. However, such an approach is analytically not practical. However, in principle, it confirms that parity-odd \( f_1^0 \)s can be constructed, as in Ref. [10], which shall lead to quasi-conservation, as \( \mathcal{X} \) is usually parity-even for a parity-even \( u \), as seen in the discussion preceding immediately. But the very fact that a parity-definite spatial Lax component has not been consistently obtained, makes this possibility very bleak.
because, following Eq. 2.16, it is the derivative of $X$ that appears in the EOM and a localized $u$, especially under weak coupling, corresponds to $u_x \to 0$ for $|x| \to \infty$. As claimed in subsection 2.3, this clearly makes RHS of Eq. 2.45 to vanish in the same limit, approaching integrability, as expected.

4.1 Relation with NLS system

The choice of such a deformation arise from the similar amplitude deformation to attain QI Non-linear Schrödinger equation [10]. These two systems can be mapped at the solution level under the weak-coupling approximation, expressed as [5],

$$u_{\text{NLS}} = \varepsilon \left( \varphi e^{i\theta} + \bar{\varphi} e^{-i\theta} \right) + \frac{\varepsilon^2}{k_0^2} \left( \varphi^2 e^{2i\theta} + \bar{\varphi}^2 e^{-2i\theta} \right) - 2 \frac{\varepsilon^2}{k_0^2} |\varphi|^2; \quad \text{where,} \quad (4.3)$$

$$\theta = k_0 x + \omega_0 t, \quad 0 < \varepsilon \ll 1, \quad \omega_0 = k_0^3 \neq 0.$$ 

Substituting the above mapping in the KdV equation 2.9, and comparing terms of $O(\varepsilon^3)$ and with phase $e^{i\theta}$, one arrives at the NLS equation,

$$\varphi_T + i3k_0 \varphi_{XX} + i \frac{6}{k_0^2} |\varphi|^2 \varphi = 0, \quad (4.4)$$

and its complex conjugate,

$$\bar{\varphi}_T - i3k_0 \bar{\varphi}_{XX} - i \frac{6}{k_0^2} |\varphi|^2 \bar{\varphi} = 0, \quad (4.5)$$

for phase $e^{-i\theta}$, with respect to the new coordinates,

$$T = \varepsilon^2 t \quad \text{and} \quad X = \varepsilon \left( x + 3k_0^2 t \right). \quad (4.6)$$

In Eq.s 4.4 and 4.5, the ‘time’-derivative term comes from that of the KdV, the second derivative term comes from the third derivative term of the same, and the non-linear term comes from its counterpart in KdV. Such direct correspondence, though approximate, motivates enough for adopting the present approach to obtain QI KdV system. However, one should expect that the present deformation works as long as the mapping of Eq. 4.3 persists, implying another infinitesimal parameter $\varepsilon$.

From Eq. 2.16, the proposed QI deformation of the KdV amplitude directly leads to the infinitesimally deformed KdV equation,

$$u_t + u_{xxx} + (6 + 10\varepsilon + 12\varepsilon \log(u)) u u_x = 0, \quad (4.7)$$

by considering terms up to $O(\varepsilon)$. As $u$ is $O(\varepsilon)$, the third term in the above bracket is sub-dominant than the second, and hence, can be dropped out. Then it can straight-forwardly be shown that the proposed KdV deformation invariably leads to QI NLS system. From Eq.s 4.3 and 4.7, the modified NLS equation is obtained as$^4$,

$^4$In the map of Eq. 4.3 [5], the non-linear terms of KdV and NLS systems map exclusively to each-other. Therefore, any scaling of the one in the KdV equation, like that in Eq. 4.7 when the logarithm is neglected, corresponds to the same scaling of the similar term in the NLS equation.
\[
\varphi_T + i3k_0\varphi_{XX} + i \left(1 + \frac{5}{3}\epsilon\right) \frac{6}{k_0} |\varphi|^2 \varphi = 0.
\] (4.8)

It is easy to see that,

\[
\left(1 + \frac{5}{3}\epsilon\right) |\varphi|^2 \approx \frac{\delta}{\delta |\varphi|^2} V(|\varphi|) \quad \text{with,}
\]

\[
V(|\varphi|) = \frac{1}{2} |\varphi|^{4(1+\tilde{\epsilon})}, \quad \text{where,} \quad \tilde{\epsilon} = \frac{5}{3}\epsilon,
\] (4.9)

following the physical fact that the density $|\varphi|^2$ is sufficiently small in the weak-coupling limit. Therefore, the particular deformation the KdV Hamiltonian in Eq. 4.1, under the weak-coupling approximate mapping of Ref. [5], leads to the QI deformation of NLS system, given by Eq. 4.9, which of the same form as given in Ref. [10]. This intuitively ensures that as long as the mapping prevails, the deformation of Eq. 4.1 leads to QI KdV.

Therefore, the justification of constructing a QI KdV system in the lines of the QI NLS system [10] is quite valid. The construction of the $sl(2)$ loop algebra in Eq. 2.22 being essentially being the same is thus justified, with consistent QI charges obtained in Eq. 2.45.

**Specific Solutions:** The one-soliton solution for the quasi-deformed KdV equation in Eq. 4.7, after dropping-out the logarithm, has the form,

\[
 u = \frac{c}{2} \text{sech}^2 \left[\frac{1}{2} \sqrt{c \beta} (x - x_0 - \beta c(t - t_0))\right], \quad \text{where} \quad \beta = 1 + \frac{5}{3}\epsilon, \quad c > 0,
\] (4.10)

which is again the standard one, with parameter/variable scaling, as the approximate deformation amounts for the scaling of the non-linear term. The corresponding one-soliton solution for the NLS system of Eq. 4.8 is,

\[
\varphi(X, T) = K \text{sech} \left[\Lambda_1 K \left(\Lambda_2 \tilde{X} - V \tilde{T}\right)\right] \exp \left[i \frac{\Lambda_2}{2} \Lambda_2 V \tilde{X} + i \frac{1}{4} \left(\Lambda_2^2 K^2 - V^2\right) \tilde{T}\right];
\] (4.11)

where \( \Lambda_1 = i \sqrt{\frac{3\beta}{k_0}} \), \( \Lambda_2 = -\frac{i}{\sqrt{3k_0}} \), \( \tilde{X} = X - X_0 \), \( \tilde{T} = T - T_0 \), and \( (K, V, X_0, T_0) \in \mathbb{R}_+ \otimes \mathbb{R} \otimes \mathbb{R} \otimes \mathbb{R} \),

which again displays similar parameter/variable scaling. The incorporation of the first non-trivial (local) contribution in the KdV sector, by bringing the logarithm in Eq. 4.7 back, is of the form \( v^2 v_x \), with \( u = 1 + v \), \( v \in \mathbb{R}_- \). For small and positive \( u \), \( v \ll 1 \) is relatively large in magnitude. However, such an addition of mKdV-type non-linearity may not lead to any conservation in general. However, by introducing a \textit{simultaneous} deformation of the type discussed in section 3, namely \( F(u) \equiv F(v) \sim v^4 \), such a system can be reduced to KdV again, in terms of \( v \) now. Equivalently, for a simultaneous \( F(u) \equiv F(v) \sim v^3 \), in can become a ‘pure’ mKdV system in \( v \), with well-known solitonic profiles. However,
it should be re-stressed that the preceding discussion is valid only under weak-coupling approximation, that validates the KdV-NLS mapping of Ref. [5], further allowing \( u \) to be small (but not tending to zero) owing to the ‘weak’ non-linearity.

4.2 Connection with Non-Holonomic Deformation

It is fruitful to compare the results obtained thus far with those of NH deformation of KdV and NLS systems. The NH deformation is practically obtained through extending the Lax components with local terms negative in power of the spectral parameter \( \lambda \). This induces an inhomogeneous extension to the original PDE, with additional local constraints imposed-on the deformation parameters, obtained though retention of the zero-curvature condition. This leaves the deformed system still integrable. The NH deformation had been well-analyzed for KdV and coupled complex KdV systems [12], from both loop-algebraic and AKNS approaches, and it has recently been shown in case of NLS systems [11] that the NH deformation is essentially different from QI deformation, as the latter leaves the system non-integrable. The QI deformation is done usually at the level of functions of the variable [8–10, 13] or functionals, as in the present case for KdV, that deforms the Lax component itself without extending it spectrally. This necessarily yields a non-zero curvature, whose vanishing leads to that of the deformation itself. Therefore, both the deformations are fundamentally different. However, in the approximate regime, such as in the present section, the QI deformation is perturbative in nature, with asymptotic regaining of integrability. This is so as the \( \log(u) \) term is relatively negligible at that limit, aided by the weak coupling, yielding a KdV system with constant parametric scaling. This is same as was observed for NLS system in Ref. [11].

In the same asymptotic limit, this ‘weak’ QI deformation can be identified with a particular version of NH deformation, having local coefficients that satisfy order-by-order relations in power of the now-small QI-parameter \( \epsilon \), that are identified with constraints. This is intuitively supported, as the QI systems show asymptotic integrability supporting single and multiple soliton solutions. It will be interesting to identify such systems with order-by-order relations (‘constraints’) by evaluating asymptotic form of the exact solution for QI KdV and other systems.

5 Deformation of General Complex Coupled KdV Equations

On considering the Lax pair for coupled complex KdV system 2.1, the QI deformation can be accommodated as,

\[
B = (\bar{q}q_x - q\bar{q}_x) \sigma_3 - \left[ \bar{q}_{xx} - \frac{2}{3} q \left( \frac{\delta H_1[q]}{\delta q} + \bar{q}_{xx} \right) \right] \sigma_+ + \left[ q_{xx} + \frac{2}{3} \bar{q} \left( \frac{\delta H_1[\bar{q}]}{\delta \bar{q}} + q_{xx} \right) \right] \sigma_- .
\]

This leads to the curvature,
\[
F_{tx} = \left[ \dot{q}_t + q_{xxx} - \frac{2}{3} \left( q \frac{\delta H_1[q]}{\delta q} + q \dot{q}_{xx} \right)_x + 2|q|^2 \dot{q}_x - 2q_x q^2 \right] \sigma_+ \\
- \left[ q_t + q_{xxx} + \frac{2}{3} \left( q \frac{\delta H_1[q]}{\delta q} + \dot{q} q_{xx} \right)_x - 2|q|^2 q_x + 2q_x q^2 \right] \sigma_- \\
+ X_c \sigma_3, \tag{5.2}
\]

where, the anomaly function now reads as,
\[
X_c \equiv \frac{2}{3} q^2 \left( \frac{\delta H_1[q]}{\delta q} + q_{xx} \right) + \frac{2}{3} q^2 \left( \frac{\delta H_1[q]}{\delta q} + \dot{q} q_{xx} \right). \tag{5.3}
\]

A gauge-transformation, similar to Eq. 2.18, leads to the on-shell curvature upon using the deformed equation of motion as,
\[
\bar{F}_{tx} = \frac{1}{e} \left[ \dot{q}_t + q_{xxx} - \frac{2}{3} \left( q \frac{\delta H_1[q]}{\delta q} + q \dot{q}_{xx} \right)_x + 2|q|^2 \dot{q}_x - 2q_x q^2 \right] \sigma_+ \\
- e \left[ q_t + q_{xxx} + \frac{2}{3} \left( \dot{q} \frac{\delta H_1[q]}{\delta q} + \dot{q} q_{xx} \right)_x - 2|q|^2 q_x + 2q_x q^2 \right] \sigma_- \\
+ X_c \sigma_3 \\
\equiv X_c \sigma_3, \tag{5.4}
\]

along-with the corresponding Lax pair:
\[
\bar{A} = \frac{1}{e} \dot{q} \sigma_+ - e q \sigma_- \quad \text{and} \\
\bar{B} = (q q_x - \dot{q} q_x) \sigma_3 - \frac{1}{e} q_{xx} - \frac{2}{3} \dot{q} \left( \frac{\delta H_1[q]}{\delta q} + \dot{q} q_{xx} \right) \sigma_+ \\
+ e \left[ q_{xx} + \frac{2}{3} \dot{q} \left( \frac{\delta H_1[q]}{\delta q} + q_{xx} \right) \right] \sigma_- . \tag{5.5}
\]

On comparing Eq.s 5.4 and 5.5 with Eq.s 2.19 and 2.20 respectively for the real KdV system, it is easy to obtain the replacements,
\[
\mathcal{X} \rightarrow \mathcal{X}_c; \\
\frac{u}{e} \rightarrow \frac{\bar{q}}{e} \quad \text{and} \quad e \rightarrow e \bar{q} \quad \text{in} \quad \bar{A}; \\
u_x \rightarrow q \bar{q}_x - \bar{q} q_x, \quad \frac{1}{e} f(u) \rightarrow \frac{1}{e} \bar{F}(q, \bar{q}) \quad \text{and} \quad 2e u \rightarrow e F(q, \bar{q}) \quad \text{in} \quad \bar{B}; \tag{5.6}
\]

wherein,
\[
\bar{F}(q, \bar{q}) := \bar{q}_{xx} - \frac{2}{3} \dot{q} \left( \frac{\delta H_1[q]}{\delta q} + \dot{q} q_{xx} \right) \quad \text{and} \quad F(q, \bar{q}) := q_{xx} + \frac{2}{3} \dot{q} \left( \frac{\delta H_1[q]}{\delta q} + q_{xx} \right). \tag{5.7}
\]
for the complex coupled system. These replacements can directly be mapped in the equations of subsection 2.3, and the corresponding generalized results can be obtained. Without going into the details of the calculations, we can write down the extended versions of equations 2.37, 2.38 and 2.39 respectively as,

\[
a^0_0 - x + \bar{q} \sqrt{2e} (a^0_0)^2 + \sqrt{2eq} = 0,
\]

\[
\frac{1}{\sqrt{2e}} \bar{q} a^0_0 (q a^0_0)_t + (q q_{xx} - q \bar{q}_{xx}) + \frac{1}{\sqrt{2e}} (\bar{F} a^0_0)_x = \mathcal{X}_c,
\]

\[
\varphi^0_0 x + \frac{\sqrt{2}}{e} q a^0_0 \varphi^0_0 = 2 \mathcal{X}_c a^0_0 \quad \text{and}
\]

\[
a^0_0 t = \varphi^0_0 + 2 \sqrt{2eq} F + 2 (q \bar{q}_x - \bar{q} q_x) a^0_0.
\] (5.8)

This enables us directly to obtain the quasi-conservation equations,

\[
\frac{dR^n}{dt} \equiv \int_x \mathcal{X}_c f^n_0,
\] (5.9)

where \(R^n\) is the corresponding anomalous charge of the \(n\)-th order. In general, the corresponding \(\bar{F}_{tx}\) coefficients will have the same algebraic form as those in Eqs. 2.35 and 2.36, as no direct \(u\)-dependence (\(q\) and \(\bar{q}\) dependence in this case) appear therein. However, the explicit expressions will be extended following the replacements of Eqs. 5.7, though it is always possible to obtain a pair of \((q, \bar{q})\) yielding \(f^n_0 = 0\) for a particular value of \(n\). This ensures the quasi-integrability of complex coupled KdV system, as per subsection 2.3 and section 4.

On a more specific note, following the treatment in Sec. 3, the expression of the anomaly in Eq. 5.3 leaves even a wider choice of deformations of the two Hamiltonians \(\bar{H}_1[q]\) and \(H_1[q]\), to construct a \(\mathcal{X}_c\) as a total space-derivative. Then, as the second of Eqs. 5.8 ensures that \(\frac{dQ^0_0}{dt} = \int_x \mathcal{X}_c\) as before, one can again construct a scaled coupled complex KdV system, as well as a tower of QI, higher-derivative extended KdV models, which can very well be of a wider interest.

6 Conclusions

It is seen that a quasi-integrable deformation of the KdV system is indeed possible, provided the loop-algebraic generalization [12] has been considered. Further, as the KdV equation does not represent ‘dynamics’, in the sense of neither Galilean (like NLS) nor Lorentz (like SG) systems, the deformation has to be performed at an off-shell level (i.e., without using the EOM). The available Hamiltonian formulation of KdV system comes to rescue in this respect, wherein both local extensions of the Hamiltonian density and power-deformation of the corresponding amplitude have been adopted. The prior allows for constructing scaled KdV, with single-soliton profile, as well as families of higher-derivative extensions to the same, with at least one conserved charge. The latter is intuitively allowed, following the weak correspondence between KdV and NLS systems, and the compatibility of the present
deformation with that of QI NLS system has been obtained, followed by corresponding one-soliton solutions with variable scaling. It will be interesting to obtain and analyze particular stable solutions to this deformed KdV and higher derivative systems, and to study their behavior with those from QI NLS system when the weak correspondence is valid. We aspire to imply the same for complex coupled KdV formalism.

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