1 INTRODUCTION

Let $k$ be a non-Archimedean local field of zero characteristic. Consider an increasing sequence of its finite extensions

$$ k = K_1 \subset K_2 \subset \ldots \subset K_n \subset \ldots. $$

The infinite extension

$$ K = \bigcup_{n=1}^{\infty} K_n $$

may be considered as a topological vector space over $k$ with the inductive limit topology. Its strong dual $K^*$ is the basic object of the non-Archimedean infinite-dimensional analysis initiated by the author [1]. Let us recall its main constructions and results.

Consider, for each $n$, a mapping $T_n : K \to K_n$ defined as follows. If $x \in K_\nu$, $\nu > n$, put

$$ T_n(x) = \frac{m_n}{m_\nu} \text{Tr}_{K_\nu/K_n}(x) $$

where $m_n$ is the degree of the extension $K_n/k$, $\text{Tr}_{K_\nu/K_n} : K_\nu \to K_n$ is the trace mapping. If $x \in K_n$ then, by definition, $T_n(x) = x$. The
mapping \(T_n\) is well-defined and \(T_n \circ T_\nu = T_n\) for \(\nu > n\). Below we shall often write \(T\) instead of \(T_1\).

The strong dual space \(\mathcal{K}\) can be identified with the projective limit of the sequence \(\{K_n\}\) with respect to the mappings \(\{T_n\}\), that is with the subset of the direct product \(\prod_{n=1}^\infty K_n\) consisting of those \(x = (x_1, \ldots, x_n, \ldots)\), \(x_n \in K_n\), for which \(x_n = T_n(x_\nu)\) if \(\nu > n\). The topology in \(\mathcal{K}\) is introduced by seminorms
\[
\|x\|_n = \|x_n\|, \quad n = 1, 2, \ldots,
\]
where \(\|\cdot\|\) is the extension onto \(K\) of the absolute value \(|\cdot|_1\) defined on \(k\).

The pairing between \(K\) and \(\mathcal{K}\) is defined as
\[
<x, y> = T(xy_n)
\]
where \(x \in K_n \subset K\), \(y = (y_1, \ldots, y_n, \ldots) \in \mathcal{K}\), \(y_n \in K_n\). Both spaces are separable, complete, and reflexive. Identifying an element \(x \in K\) with \((x_1, \ldots, x_n, \ldots) \in \mathcal{K}\) where \(x_n = T_n(x)\), we can view \(K\) as a dense subset of \(\mathcal{K}\). The mappings \(T_n\) can be extended to linear continuous mappings from \(\mathcal{K}\) to \(K_n\), by setting \(T_n(x) = x_n\) for any \(x = (x_1, \ldots, x_n, \ldots) \in \mathcal{K}\).

Let us consider a function on \(K\) of the form
\[
\Omega(x) = \begin{cases} 
1, & \text{if } \|x\| \leq 1; \\
0, & \text{if } \|x\| > 1.
\end{cases}
\]
\(\Omega\) is continuous and positive definite on \(K\). That results in the existence of a probability measure \(\mu\) on the Borel \(\sigma\)-algebra \(\mathcal{B}(\mathcal{K})\), such that
\[
\Omega(a) = \int_{\mathcal{K}} \chi(<a, x>) \, d\mu(x), \quad a \in K,
\]
where \(\chi\) is a rank zero additive character on \(k\). The measure \(\mu\) is Gaussian in the sense of Evans [2]. It is concentrated on the compact additive subgroup
\[
S = \{x \in \mathcal{K}: \|T_n(x)\| \leq q_n^{d_n/m_n}\|m_n\|, \ n = 1, 2, \ldots\}
\]
where \(q_n\) is the residue field cardinality for the field \(K_n\), \(d_n\) is the exponent of the different for the extension \(K_n/k\). The restriction of \(\mu\) to \(S\) coincides with the normalized Haar measure on \(S\). On the other hand, \(\mu\) is singular with respect to additive shifts by elements from \(\mathcal{K} \setminus S\).
Having the measure $\mu$, we can define a Fourier transform $\hat{f} = \mathcal{F}f$ of a complex-valued function $f \in L_1(K)$ as

$$\hat{f}(\xi) = \int_K \chi(\langle \xi, x \rangle) f(x) \, d\mu(x).$$

Let $\mathcal{E}(K) \subset L_1(K)$ be the set of “cylindrical” functions of the form $f(x) = \varphi(T_n(x))$ where $n \geq 1$, $\varphi$ is a locally constant function. The fractional differentiation operator $D^\alpha$, $\alpha > 0$, is defined on $\mathcal{E}(K)$ as $D^\alpha = \mathcal{F}^{-1} \Delta^\alpha \mathcal{F}$ where $\Delta^\alpha$ is the operator of multiplication by the function

$$\Delta^\alpha(\xi) = \begin{cases} ||\xi||^\alpha, & \text{if } ||\xi|| > 1 \\ 0, & \text{if } ||\xi|| \leq 1 \end{cases}, \quad \xi \in K.$$

Correctness of this definition follows from the theorem of Paley-Wiener-Schwartz type for the transform $\mathcal{F}$. The operator $D^\alpha$ is essentially self-adjoint on $L_2(K)$.

The operator $D^\alpha$ is an infinite-dimensional counterpart of the $p$-adic fractional differentiation operator introduced by Vladimirov [3] and studied extensively in [4-14]. In some respects the infinite-dimensional $D^\alpha$ resembles its finite-dimensional analogue though it possesses some new features. For example, we shall show below that the structure of its spectrum depends on arithmetic properties of the extension $K$.

Both in the finite-dimensional and infinite-dimensional cases $D^\alpha$ admits a probabilistic interpretation. Namely, $-D^\alpha$ is a generator of a cadlag Markov process, which is a $p$-adic counterpart of the symmetric stable process. In analytic terms, that corresponds to a hyper-singular integral representation of $D^\alpha f$, for a suitable class of functions $f$. In [1] such a representation was obtained for $f \in \mathcal{E}(K)$, $f(x) = \varphi(T_n(x))$, $\varphi$ locally constant:

$$(D^\alpha f)(x) = \psi(T_n(x))$$

where

$$\psi(z) = q_n^{d_{n/m}} \left[ 1 - q_n^{\alpha/m} \|m_n\|^{-m_n} \right]^{1 - q_n^{1-\alpha/m}} \|m_n\|^{-m_n} \times \left[ \|x\|^{-m_n-\alpha} \|m_n\|^{m_n+\alpha} + \frac{1 - q_n^{-1}}{q_n^{\alpha/m} - 1} q_n^{-d_n(1+\alpha/m)} \right] \times [\varphi(z - x) - \varphi(z)] \, dx, \quad (1)$$

$z \in K_n, \|z\| \leq q_n^{d_{n/m}} \|m_n\|$. 

In this paper we shall show that there exists also a representation in terms of the function \( f \) itself:

\[
(D^\alpha f)(y) = \int_{\mathcal{B}(\mathcal{K} \setminus \{0\})} [f(y) - f(x + y)] \Pi(dx)
\]

where \( \Pi \) is a measure on \( \mathcal{B}(\mathcal{K} \setminus \{0\}) \) finite outside any neighbourhood of zero.

Another measure of interest in this context is the heat measure \( \pi(t, dx) \) corresponding to the semigroup \( \exp(-tD^\alpha), t > 0 \). We show that in contrast both to the finite-dimensional case and to the similar problem for the real infinite-dimensional torus [15], \( \pi(t, dx) \) is not absolutely continuous with respect to \( \mu \), whatever is the sequence \( \{K_n\} \).

2 SPECTRUM

We shall preserve the notation \( D^\alpha \) for its closure in \( L_2(\mathcal{K}) = L_2(S, d\mu) \). It is clear from the definition that the spectrum of \( D^\alpha \) coincides with the closure in \( \mathbb{R} \) of the range of the function \( \Delta^\alpha(\xi), \xi \in \mathcal{K} \). In order to investigate the structure of the spectrum, we need some auxiliary results.

It follows from the duality theory for direct and inverse limits of locally compact groups [16] that the character group \( \mathcal{K}^* \) of the additive group of \( \mathcal{K} \) is isomorphic to \( \mathcal{K} \). The isomorphism is given by the relation

\[
\psi(y) = \chi(<a_\psi, y>), \quad y \in \mathcal{K},
\]

where \( a_\psi \in \mathcal{K} \) is an element corresponding to the character \( \psi \).

Denote \( O = \{\xi \in \mathcal{K} : \|\xi \leq 1\} \), \( O_n = O \cap K_n \).

**Lemma 1** The dual group \( S^* \) of the subgroup \( S \subset \mathcal{K} \) is isomorphic to the quotient group \( \mathcal{K}/O \).

**Proof:** We may assume (without restricting generality) that \( k = \mathbb{Q}_p \).

Let

\[
S^{(n)} = \{z \in K_n : \|z\| \leq q_n^{d_n/m_n}\|m_n\|\}, \quad n = 1, 2, \ldots .
\]

It is clear that \( S^{(n)} \) is a compact (additive) group. If \( z \in S^{(\nu)}, \nu > n \), then

\[
\|T_n(z)\| = \|m_n\| \cdot \|m_\nu\|^{-1}\|\text{Tr}_{K_\nu/K_n}(z)\|
\]
where $|m_{\nu}^{-1}z|_{\nu} \leq q_{\nu}^{d_{\nu}}$, and $| \cdot |_\nu$ is the normalized absolute value on $K_\nu$ [17]. Therefore (see Chapter 8 in [18])

$$|m_{\nu}^{-1}\text{Tr}_{K_\nu/K_n}(z)|_n \leq q_{n}^{d_{n}}$$

where $l \in \mathbb{Z}$, $e_{n\nu}(l-1) < d_{\nu} - d_{n\nu} \leq e_{n\nu}l$, $e_{n\nu}$ and $d_{n\nu}$ are the ramification index and the exponent of the different for the extension $K_\nu/K_n$. On the other hand, $d_{\nu} = e_{n\nu}d_{n} + d_{n\nu}$, so that $l = d_{n}$, and we find that $T_n(z) \in S^{(n)}$.

Hence, $T_n : S^{(\nu)} \to S^{(n)}$. It is clear that $T_n$ is a continuous homomorphism. As a result, $S$ can be identified with an inverse limit of compact groups $S^{(n)}$ with respect to the sequence of homomorphisms $T_n$. Using the auto-duality of each field $K_n$, we can identify the group dual to $S^{(n)}$ with $K_n/\Phi_n$, where $\Phi_n$ is the annihilator of $S^{(n)}$ in $K_n$. On the other hand, $\Phi_n = O_n$.

Indeed,

$$\Phi_n = \{ \xi \in K_n : \chi(T(z\xi)) = 1 \quad \forall z \in S^{(n)} \}$$

$$= \{ \xi \in K_n : |T(z\xi)|_1 \leq 1 \quad \forall z \in S^{(n)} \}.$$

If $\xi \in O_n$ then $|m_{n}^{-1}z\xi|_n \leq q_{n}^{d_{n}}$ for any $z \in S^{(n)}$, whence $\xi \in \Phi_n$ by the definition of the number $d_{n}$.

Thus $O_n \subset \Phi_n$. Conversely, suppose that $\xi \in \Phi_n \setminus O_n$, that is $\|\xi\| > 1$, $|T(z\xi)|_1 \leq 1$ for all $z \in S^{(n)}$. Let $z$ be such that $\|z\| = \|m_{n}\|q_{n}^{d_{n}/m_{n}}$. Then

$$|m_{n}^{-1}|_n \cdot |\xi|_n \cdot |z|_n = |\xi|_nq_{n}^{d_{n}} > q_{n}^{d_{n}}.$$

It follows from the properties of trace [18] that $z$ can be chosen in such a way that $|\text{Tr}_{K_n/k}(m_{n}^{-1}z\xi)|_1 > 1$, and we have a contradiction. So $\Phi_n = O_n$.

It follows from the identity

$$T(\xi T_n(z)) = T(\xi z), \quad \xi \in K_n, \quad z \in K_\nu, \quad \nu > n,$$

that the natural imbeddings $K_n/O_n \to K_\nu/O_\nu, \nu > n$, are the dual mappings to the homomorphisms $T_n : S^{(\nu)} \to S^{(n)}$. Using the duality theorem [16], we find that

$$S^* = \lim_{\to} K_n/O_n$$

where the direct limit is taken with respect to the imbeddings. The right-hand side of (4) equals $K/O$.

Note that the above isomorphism is given by the same formula (3) where this time $\psi_{a}$ is an arbitrary representative of a coset from $K/O$.  □
Let \( \varphi_a(x) = \chi(<a,x>) \), \( x \in \overline{K} \), where \( a \in K \), \( \|a\| > 1 \) or \( a = 0 \). It is known [1] that

\[
(D^\alpha \varphi_a)(x) = \|a\|^\alpha \varphi_a(x)
\]

for \( \mu \)-almost all \( x \in \overline{K} \). Note that the set of values of the function \( a \mapsto \|a\|^\alpha \) with \( \|a\| > 1 \) coincides with the set

\[
\{q_n^{-\alpha/m_n}, n, N = 1, 2, \ldots\} \cup \{q_1^{-\alpha/e_n}, n, N = 1, 2, \ldots\}
\]

where \( e_n \) is the ramification index of the extension \( K_n/k \). Denote its residue degree by \( f_n \). It is well known that the sequences \( \{f_n\}, \{e_n\} \) are non-decreasing and \( e_n f_n = m_n \).

**THEOREM 1** Let \( A \subset K \) be a complete system of representatives of cosets from \( K/O \). Then \( \{\varphi_a\}_{a \in A} \) is the orthonormal eigenbasis for the operator \( D^\alpha \) in \( L_2(S,d\mu) \). As a set, its spectrum equals the set (6) complemented with the point \( \lambda = 0 \). Each non-zero eigenvalue of \( D^\alpha \) has an infinite multiplicity. The point \( \lambda = 0 \) is an accumulation point for eigenvalues if and only if \( e_n \to \infty \).

Proof: The first statement follows from (5) and Lemma 1. The assertion about the accumulation at zero is obvious from (6).

Let us construct \( A \) as the union of an increasing family \( \{A_n\} \) of complete systems of representatives of cosets from \( K_n/O_n \). Each \( A_n \) consists of elements of the form \( a = \pi_n^{-N}(\sigma_1 + \sigma_2 \pi_n + \cdots + \sigma_{N-1} \pi_n^{N-1}) \), \( N \geq 1 \), where \( \sigma_1, \ldots, \sigma_{N-1} \) belong to the set \( F_n \) of representatives of the residue field corresponding to the field \( K_n \), \( \sigma_1 \neq 0 \), \( \pi_n \) is a prime element of \( K_n \). We have \( |a|_n = q_n^N \) so that \( \|a\| = q_1^{Nf_n/m_n} = q_1^{N/e_n} \).

Meanwhile card \( F_n = q_n = q_1^{f_n} \).

If \( e_n \to \infty \) then the same value of \( \|a\| \) corresponds to elements from infinitely many different sets \( A_n \) with different values of \( N \) (\( e_n \) is a multiple of \( e_{n-1} \) due to the chain rule for the ramification indices; see [19]). If the sequence \( \{e_n\} \) is bounded, then it must stabilize, and we obtain the same value of \( \|a\| \) for elements from infinitely many sets \( A_n \) with possibly the same \( N \). However, in this case \( f_n \to \infty \), the number of such elements (for a fixed \( N \)) is \( Nq_1^{f_n} - 1 \) (\( \to \infty \) for \( n \to \infty \)). In both cases we see that all the non-zero eigenvalues have an infinite multiplicity.

Note that the cases where \( e_n \to \infty \) or \( e_n \leq \text{const} \) both appear in important examples of infinite extensions. Let \( K \) be the maximal
unramified extension of \( k \). Then one may take for \( K_n \) the unramified extension of \( k \) of the degree \( n! \), \( n = 2, 3, \ldots \). Here \( e_n = 1 \) for all \( n \).

On the other hand, if \( K \) is the maximal abelian extension of \( k = \mathbb{Q}_p \) then a possible choice of \( K_n \) is the cyclotomic extension \( C_n = \mathbb{Q}_p(W_n) \) where \( W_l \) is the set of all roots of 1 of the degree \( l \) (see [20,21]). Writing \( n! = n'p^{h_n} \), \( (n', p) = 1 \), we see that \( e_n = (p - 1)p^{h_n - 1} \to \infty \) as \( n \to \infty \).

\[ \]

### 3 HYPER-SINGULAR INTEGRAL REPRESENTATION

The main aim of this section is the following result.

**THEOREM 2** There exists such a measure \( \Pi \) on \( B(K \setminus \{0\}) \) finite outside any neighbourhood of the origin that \( D^\alpha \) has the representation (2) on all functions \( f \in E(K) \).

In the course of proof we shall also obtain some new information about the Markov process \( X(t) \) generated by the operator \(-D^\alpha \). We assume below that \( X(0) = 0 \).

The projective limit topology on \( K \) coincides with the one given by the shift-invariant metric

\[ r(x, y) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|x - y\|_n}{1 + \|x - y\|_n}, \quad x, y \in K. \]

It is known [22] that the main notions and results regarding stochastic processes with independent increments carry over to the case of a general topological group with a shift-invariant metric.

Let \( \nu(t, \Gamma) \) be a Poisson random measure corresponding to the process \( X(t) \). Here \( \Gamma \in B_0 = \bigcup_{\gamma > 0} B_\gamma \),

\[ B_\gamma = \{ \Gamma \subset B(K), \text{dist} (\Gamma, 0) \geq \gamma \}. \]

For any \( \Gamma \in B_0 \) we can define a stochastically continuous process \( X_\Gamma(t) \), the sum of all jumps of the process \( X(\tau) \) for \( \tau \in [0, t] \) belonging to \( \Gamma \). In a standard way [23] we find that

\[ E \chi(\langle \lambda, X_\Gamma(t) \rangle) = \exp \left\{ \int_{\Gamma} (\chi(\langle \lambda, x \rangle) - 1) \Pi(t, dx) \right\}, \quad \lambda \in K, \]

where \( E \) denotes the expectation, \( \Pi(t, \cdot) = E\nu(t, \cdot) \).
Let \( \lambda \in K_n \). Consider the set 
\[ V_{\delta,n} = \{ x \in \overline{K} : \| T_n(x) \| \geq \delta \}, \quad 0 < \delta \leq 1. \]

Let us look at the equality (7) with \( \Gamma = V_{\delta,n} \), \( \delta \leq \| \lambda \|^{-1} \). If \( x \in V_{\delta,n} \) then 
\[ r(x, 0) \geq 2^{-n} \frac{\| x \|_n}{1 + \| x \|_n} \geq 2^{-n-1} \delta, \]
whence \( V_{\delta,n} \in B_0 \). If \( \delta \leq \| \lambda \|^{-1} \), \( x \not\in V_{\delta,n} \), then 
\[ \| < \lambda, x > \| = \| T(\lambda T_n(x)) \| \leq 1 \]
which implies that the integral in the right-hand side of (7) coincides with the one taken over \( \overline{K} \).

On the other hand, almost surely 
\[ \| X(t) - X_{V_{\delta,n}}(t) \|_n < \delta. \] (8)

Indeed, let \( t_0 \) be the first exit time of the process \( X(t) - X_{V_{\delta,n}}(t) \) from the set \( \overline{K} \setminus V_{\delta,n} \). Suppose that \( t_0 < \infty \) with a non-zero probability. Then 
\[ \| X(t_0 - 0) - X_{V_{\delta,n}}(t_0 - 0) \|_n < \delta, \] (9)
\[ \| X(t_0 + 0) - X_{V_{\delta,n}}(t_0 + 0) \|_n \geq \delta. \] (10)

If \( \| X(t_0 + 0) - X(t_0 - 0) \|_n < \delta \) then \( X_{V_{\delta,n}}(t_0 + 0) = X_{V_{\delta,n}}(t_0 - 0) \), so that 
\[ \| [X(t_0 + 0) - X_{V_{\delta,n}}(t_0 + 0)] - [X(t_0 - 0) - X_{V_{\delta,n}}(t_0 - 0)] \|_n < \delta. \] (11)

If, on the contrary, \( \| X(t_0 + 0) - X(t_0 - 0) \|_n \geq \delta \), then 
\[ X_{V_{\delta,n}}(t_0 + 0) - X_{V_{\delta,n}}(t_0 - 0) = X(t_0 + 0) - X(t_0 - 0), \]
the expression in the left-hand side of (11) equals zero, and the inequality (11) is valid too. In both cases it contradicts (9), (10). Thus \( t_0 = \infty \) almost surely, and the inequality (8) has been proved. We have come to the following formula of Lévy-Khinchin type.

**LEMMA 2**  For any \( \lambda \in K, \ t > 0 \)
\[ E\chi(< \lambda, X(t) >) = \exp \left\{ \int_{\overline{K}} [\chi(< \lambda, x >) - 1] \Pi(t, dx) \right\}. \] (12)

**REMARK**  Lemma 2 can serve as a base for developing the theory of stochastic integrals and stochastic differential equations over \( \overline{K} \). In fact, the techniques and results of [24] carry over to this case virtually unchanged.
Both sides of (12) can be calculated explicitly if we use the heat measure $\pi(t, dx)$ (see [1]). We have

$$E\chi(<\lambda, X(t)>) = \int \chi(<\lambda, x>)\pi(t, dx) = \rho_\alpha(\|\lambda\|, t)$$

where

$$\rho_\alpha(s, t) = \begin{cases} e^{-ts^\alpha}, & \text{if } s > 1 \\ 1, & \text{if } s \leq 1, \end{cases} \quad (13)$$

$s \geq 0, \ t > 0$. It follows from the definitions that $\Pi(t, \cdot)$ is symmetric with respect to the reflection $x \mapsto -x$. Therefore

$$\int K [\chi(<\lambda, x> - 1)\Pi(t, dx) = \begin{cases} -t\|\lambda\|^\alpha, & \text{if } \|\lambda\| > 1 \\ 0, & \text{if } \|\lambda\| \leq 1, \end{cases} \quad (14)$$

**LEMMA 3** Let $M_n$ be a compact subset of $K_n \setminus \{0\}$, $M = T_n^{-1}(M_n)$. Then

$$\Pi(t, M) = -t \int_{\eta \in K_n: \|\eta\| > 1} \|\eta\|^{\alpha}w_n(\eta) \, d\eta \quad (15)$$

where $w_n(\eta)$ is the inverse Fourier transform of the function $y \mapsto q_n^{-dn/2} \omega_M^{(n)}(m_n y)$, $\omega_M^{(n)}$ is the indicator of the set $M_n$ in $K_n$, and $dx$ is the normalized additive Haar measure on $K_n$.

**Proof:** Let $\omega_M$ be the indicator of the set $M$ in $\overline{K}$. Then $\omega_M(x) = \omega_M^{(n)}(T_n(x))$, $x \in \overline{K}$,

$$\omega_M^{(n)}(\xi) = \int_{K_n} \chi(\xi \eta)w_n(\eta) \, d\eta, \quad \xi \in K_n.$$ 

Since

$$\int_{K_n} w_n(\eta) \, d\eta = \omega_M^{(n)}(0) = 0,$$

we get

$$\omega_M(x) = \int_{K_n} [\chi(\eta T_n(x)) - 1]w_n(\eta) \, d\eta.$$ 

Integrating with respect to $\Pi(t, dx)$ and using (14) we come to (15).
It follows from Lemma 3 that $\Pi(t, dx) = t \Pi(1, dx)$. We shall write $\Pi(dx)$ instead of $\Pi(1, dx)$.

Proof of Theorem 2: Let $f(x) = \varphi(T_n(x))$, $x \in K$, where $\varphi$ is locally constant and in addition $\text{supp } \varphi$ is compact, $0 \notin \text{supp } \varphi$. It follows from Lemma 3 that

$$\int f(x) \Pi(dx) = -\int_{K_n} \Delta^\alpha(\eta) \psi(\eta) \, d\eta$$

where $\psi$ is the inverse Fourier transform of the function $y \mapsto q_n^{-dn/2} \varphi(m_n y)$.

The right-hand side of (16) is an entire function with respect to $\alpha$. Assuming temporarily $\text{Re } \alpha < -1$, we can use the Plancherel formula with subsequent analytic continuation (see [1]). As a result we find that for $\alpha > 0$

$$\int f(x) \Pi(dx) = -q_n^{d_n \alpha/m_n} \frac{1 - q_n^{\alpha/m_n}}{1 - q_n^{-1-\alpha/m_n}} \int_{x \in K_n, |x| \leq q_n^{d_n}} |x|^{-1-\alpha/m_n}$$

$$+ \frac{1 - q_n^{1/m_n}}{q_n^{\alpha/m_n} - 1} q_n^{-d_n(1+\alpha/m_n)} \varphi(m_n x) \, dx. \quad (17)$$

An obvious approximation argument shows that (17) is valid for any $f \in \mathcal{E}(K)$. Comparing (17) with (1) we obtain (2). \qed

4 HEAT MEASURE

Recall that the heat measure $\pi(t, dx)$ corresponding to the operator $-D^\alpha$ is defined by the formula

$$\int \chi(<\lambda, x>) \pi(t, dx) = \rho_\alpha(\|\lambda\|, t), \quad \lambda \in K, \ t > 0,$$

where $\rho_\alpha$ is given by (13).

THEOREM 3 For each $t > 0$ the measure $\pi(t, \cdot)$ is not absolutely continuous with respect to $\mu$.

Proof: Let us fix $N \geq 1$ and consider the set

$$M = \{ x \in K : \|T_n(x)\| \leq q_n^{d_n/m_n - N/f_n} \|m_n\|, \ n = 1, 2, \ldots \}$$
We shall show that \( \mu(M) = 0 \) whereas \( \pi(t, M) \neq 0 \).

Denote
\[
M_n = \left\{ x \in \mathcal{K} : \|T_n(x)\| \leq q_n^{d_n/m_n-N/n}m_n \right\}, \quad n = 1, 2, \ldots .
\]
It is clear that \( M = \bigcap_{n=1}^{\infty} M_n \). Repeating the arguments from the proof of Lemma 1, we see that \( M_\nu \subset M_n \) if \( \nu > n \). Thus
\[
\pi(t, M) = \lim_{n \to \infty} \pi(t, M_n), \quad \mu(M) = \lim_{n \to \infty} \mu(M_n).
\]

It follows from the integration formula for cylindrical functions [1] that
\[
\mu(M_n) = q_n^{-d_n}m_n \int_{K_n, \|z\| \leq q_n^{d_n/m_n-N/n}m_n} dz = q_n^{-d_n}m_n^{-1} \int_{K_n, \|z\| \leq q_n^{d_n-N/n}m_n} dz = q_n^{-N/n}e_n = q_n^{-N_n}e_n, \quad (18)
\]
so that \( \mu(M_n) = q_1^{-N_n} \to 0 \) for \( n \to \infty \). Thus \( \mu(M) = 0 \).

In a similar way (see [1])
\[
\pi(t, M_n) = |m_n|^{-1} \int_{K_n, \|z\| \leq q_n^{d_n-N/n}m_n} \Gamma^{(n)}(m_n^{-1}z, t) \ dz \quad (19)
\]
where \( \Gamma^{(n)}_\alpha \) is a fundamental solution of the Cauchy problem for the equation over \( K_n \) of the form \( \partial u/\partial t + \partial^\alpha_n u = 0 \). Here \( \partial^\alpha_n \) is a pseudo-differential operator over \( K_n \) with the symbol \( \Delta^\alpha_n(\xi) \). It is clear that
\[
\Gamma^{(n)}_\alpha(\zeta, t) = q_n^{-d_n/2} \tilde{\rho}_\alpha(\zeta, t)
\]
where tilde means the local field Fourier transform:
\[
\tilde{u}(\zeta) = q_n^{-d_n/2} \int_{K_n} \chi \circ \text{Tr}_{K_n/k}(z\zeta)u(z) \ dz,
\]
for a complex-valued function \( u \) over \( K_n \) (sufficient conditions for the existence of \( \tilde{u} \) and the validity of the inversion formula
\[
u(z) = q_n^{-d_n/2} \int_{K_n} \chi \circ \text{Tr}_{K_n/k}(-z\zeta)\tilde{u}(\zeta) \ d\zeta,
\]
are well known).

Using the Plancherel formula we can rewrite (19) in the form
\[
\pi(t, M_n) = \int_{K_n} \tilde{\Gamma}^{(n)}_\alpha(x, t)\tilde{\beta}_n(x) \ dx
\]
where
\[
\beta_n(\zeta) = \begin{cases} 
1, & \text{if } |\zeta|_n \leq q_n^{d_n-Nn} \\
0, & \text{if } |\zeta|_n > q_n^{d_n-Nn}.
\end{cases}
\]

We have \( \tilde{\Gamma}_\alpha^{(n)} = q_n^{-d_n/2} \rho_\alpha \),
\[
\tilde{\beta}_n(x) = \begin{cases} 
q_n^{d_n-Nn}, & \text{if } |x|_n \leq q_n^{Nn} \\
0, & \text{if } |\zeta|_n > q_n^{Nn},
\end{cases}
\]
(see e.g. [11]), so that
\[
\pi(t, M_n) = q_n^{-Nn} \int_{|x|_n \leq q_n^{Nn}} \rho_\alpha (\|x\|, t) \, dx \geq q_n^{-Nn} \int_{|x|_n = q_n^{Nn}} \rho_\alpha (|x|_n^{1/m_n}, t) \, dx = (1 - q_n^{-1}) \exp (-t q_n^{\alpha N/f_n}) \geq (1 - q_1^{-1}) \exp (-t q_1^{\alpha N}).
\]

Hence, \( \pi(t, M) > 0. \)

ACKNOWLEDGEMENT This work was supported in part by the Ukrainian Fund for Fundamental Research (Grant 1.4/62).

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