Anomalous Hall effect in spin-polarized
two-dimensional electron gases with
Rashba spin–orbit interaction

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Abstract. We present details of the analysis of the effect of disorder on
the intrinsic anomalous Hall conductivity (AHC) in a spin-polarized two-
dimensional electron gas with a Rashba-type spin–orbit interaction. We show
that the AHC derived by the Kubo formula vanishes when the Fermi energy is
larger than the exchange energy unless the lifetime is spin-dependent. Results
obtained by numerical integration of the general expressions of the AHC suggest
that the AHC could depend on the lifetime, in contrast to previous results.

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1. Introduction

The spin–orbit interaction (SOI) in semiconductors allows optical and electrical control of spins without any use of ferromagnets or magnetic field. Because of this the SOI has recently attracted much attention in the field of spintronics. The SOI gives rise to unusual Hall effects, such as the anomalous Hall (AH) effect [1] in ferromagnets, and the spin Hall (SH) effect in normal conductors [2]–[6]. The detailed mechanisms of these effects have been controversially discussed. For the AH effect, initially an intrinsic mechanism originating from an effective magnetic field in momentum space was proposed [7]. Extrinsic mechanisms, such as skew [8] and side-jump [9] scattering at impurities, were subsequently proposed. Most experiments have been interpreted in terms of the extrinsic mechanisms. However, it has recently been shown that the intrinsic AH effect can quantitatively explain the AH effect in ferromagnetic semiconductors [10]–[12]. The SH effect was initially predicted by assuming extrinsic scattering [2, 3]. However as for the AH effect, an intrinsic SH effect is possible. Murakami et al [5] predicted that the effective magnetic field associated with the Berry phase in the valence band induces drift of up and down spin carriers towards opposite directions in p-doped zincblende-type semiconductors. Sinova et al [6] predicted a universal SH conductivity for a two-dimensional electron gas (2DEG) with a Rashba-type SOI produced by the asymmetry of the potential.

It has been shown that the intrinsic SH effect can be strongly suppressed by disorder effects. Especially, the intrinsic SH current vanishes identically by disorder scattering of electrons in a Rashba-split 2DEG [13]–[15]. However, the Rashba-split 2DEG is believed to be rather special in this respect, since the intrinsic SH effects in other systems survive disorder scattering [16]–[19]. Since we may view the SH effect as the zero-magnetization limit of the AH effect, vertex-canceling of the intrinsic SH effect in a Rashba-split 2DEG may also take place in the AH effect. Previously, we calculated the AH conductivity (AHC) in the diffusive regime for a uniformly exchange split 2DEG with Rashba-SOI, and have shown that the AHC vanishes unless the lifetime is spin-dependent [20]. This demonstrates the strong similarity between the SH and AH effects. These results are important for a full understanding of the Hall effects [21]. It is also noted that recent analyses done by using Kubo and Boltzmann equations [22, 23] and by the kinetic-equation approach [24, 25] have deepened the understandings of the AHC for 2DEG caused by intrinsic and extrinsic origins with the effects of disorder. Some of the recent progress in the understanding of AHC have been summarized by Nunner et al [26].

In the previous work [20], however, we have presented the analytical expressions which is correct up to the second order of the SOI and valid for large values of the magnetization as well. Therefore, the analytical results were not able to give the correct limit of zero-magnetization. The purpose of the present paper is twofold. One is to present the details of the analysis of the AHC which was omitted in the previous paper, and the other is to show results calculated in a numerical integration of the general expression by including the effects of spin-dependent random potentials. The calculated results of the AHC show a non-monotonic dependence on the exchange splitting (ExS) as expected. In addition, results of the AHC calculated in a 2D space of ExS and SOI for different degrees of disorder have been found to be non-scalable, suggesting the life-time dependence of the AHC may be non-trivial.

It should be remembered, however, that the method used and the results are applicable only to a case where the Fermi level intersects both the energy dispersions split by the SOI, since they are calculated by using a Kubo formula given by a product of the retarded and advanced
Figure 1. A schematic figure of 2DEG magnetized along the z-direction. The AHC $\sigma_{yx}$ is calculated as a response of the current density $J_y$ for an electric field $E_x$ applied along the x-direction.

Green’s functions. It is well-known that the Streda formula [27] is necessary to give full account of the AHC. Nunner et al [26], however, have shown after detailed examination of the AHC of the Rashba-split 2DEG that the Kubo formula used in our work is sufficient for AHC when the Fermi level is above the energy gap and intersects the spin-split energy bands. Calculation of the AHC has recently been extended to d-electron systems [28], and it was shown that the contribution from the so-called Fermi sea term is rather small as compared with that from the term given by the product of the retarded and advanced Green’s functions.

In the next section, we present the model used and follow in section 3 with some details of analysis to obtain the general expression of the AHC. Since we are interested in the transport in the diffusive regime, we adopt the simple Born approximation for the self-energy, and the ladder approximation for the vertex corrections. We neglect the crossing diagrams which are responsible for the weak localization in 2D systems. The method used is essentially the same as that used in our previous works [13, 29]. Numerical results and discussion are presented in section 4, and a short summary is given in the final section.

2. Rashba split 2DEG with uniform spin polarization

We consider a system of a 2DEG magnetized along the z-direction on an xy-plane as shown in figure 1. The Hamiltonian is given by

$$H = H_0 + V,$$

where the first and second terms are the unperturbed Hamiltonian and isotropic random impurity potentials with a $\delta$-function type and are given as

$$H_0 = \begin{pmatrix} \frac{\hbar^2}{2m} k^2 & i\lambda \hbar k_- \\ -i\lambda \hbar k_+ & \frac{\hbar^2}{2m} k^2 \end{pmatrix} - \Delta_{ex} \sigma_z,$$

$$V = \begin{pmatrix} V_\uparrow & 0 \\ 0 & V_\downarrow \end{pmatrix} \sum_i \delta (r - R_i),$$

respectively. Here $\lambda$ indicates the SOI, $k_\pm = k_x \pm i k_y$, $\Delta_{ex} \sigma_z$ is the Zeeman term due to an ExS $\Delta_{ex}$ with a Pauli spin matrix $\sigma_z$, and $R_i$ indicates an atomic site. The energy dispersion of the unperturbed Hamiltonian is given as

$$E_{ks} = \frac{\hbar^2}{2m} k^2 + s \sqrt{F_k},$$

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where $s = \pm$, and $F_k = \lambda^2 \hbar^2 k^2 + \Delta_{ex}^2$. The result is shown in figure 2. The eigenfunctions are obtained as

$$
\phi_{ks} = \frac{1}{\sqrt{2L^2}} \left( 1 + s \frac{\Delta_{ex}}{\sqrt{F_k}} \right)^{1/2} e^{i\lambda \hat{\mathbf{r}} \cdot \mathbf{r}} \left( \frac{i\lambda h \Delta_{ex} + s \sqrt{F_k}}{\Delta_{ex} + s \sqrt{F_k}} \right).
$$

(5)

The space spanned by these eigenfunctions is referred to as $|ks\rangle$ space, and the usual Pauli spin space is denoted as $|k\sigma\rangle$ space, hereafter. These two spaces are related by a unitary matrix,

$$
U = \begin{pmatrix}
N_+ & N_- \\
\frac{\Delta_{ex} + \sqrt{F_k}}{i\lambda h \Delta_{ex}} & \frac{\Delta_{ex} - \sqrt{F_k}}{i\lambda h \Delta_{ex}}
\end{pmatrix},
$$

(6)

with

$$
N_{ks} = \frac{1}{\sqrt{2}} \left( 1 + s \frac{\Delta_{ex}}{\sqrt{F_k}} \right)^{1/2}.
$$

(7)

Because the analysis of the AHC is complicated due to the spin-dependent potential, we perform the analysis in $|ks\rangle$ space. However, it may be often convenient to perform manipulations in $|ks\rangle$ space, and in this case, we make use of the unitary transformation given above.

The current operators used in the Kubo formula are given as

$$
J_{x(y)} = e \frac{\partial H_0}{\partial p_{x(y)}} = e \left( bk_{x(y)} (1 + (\lambda)\sigma_{y(x)}) \right).
$$

(8)

It should be noted that the second term in the parenthesis is constant and non-diagonal in the Pauli spin space.

3. Formalism and general expression of AHC

3.1. Self-energy

In the diffusive transport regime it was convenient for us to include the SOI in the eigenstates of the Hamiltonian and treat the impurity potentials as a perturbation. We then proceeded to

\[\text{Figure 2. The energy dispersion relation for } M \parallel z \text{ calculated by equation (4).}\]
obtain the AHC as before by adopting the Born approximation for the self-energy and the ladder approximation for the current vertex determined self-consistently to satisfy the Ward identity.

We first define the one-particle Green's function (resolvent) as

$$G_0 = \left[ zI - H_0 \right]^{-1},$$

with a matrix element for spin $\sigma = \uparrow (\sigma) \text{ or } \downarrow (- \sigma)$

$$g_\sigma = \left[ z - \epsilon_k + \sigma \Delta_{eq} \right]^{-1},$$

where $\epsilon_k = \hbar^2 k^2 / 2m$, and $z = \epsilon \pm i\eta$, with an infinitesimally small positive value $\eta$. The plus and minus signs in $z$ indicate that the Green's functions are retarded (R) and advanced (A), respectively. By inserting the Hamiltonian equation (2), the unperturbed Green's function is obtained to be

$$G_0 = \begin{pmatrix} f_\uparrow & 0 \\ 0 & f_\downarrow \end{pmatrix} + \lambda \hbar f_2 \left( -k_x \sigma_y + k_y \sigma_x \right),$$

where

$$f_2 \equiv \frac{1}{g_\uparrow^{-1}g_\downarrow^{-1} - \lambda^2 \hbar^2 k^2},$$

$$f_\uparrow \equiv f_2 g_\downarrow^{-1},$$

$$f_\downarrow \equiv f_2 g_\uparrow^{-1}.$$

In $|ks\rangle$ space, the unperturbed Green's function is diagonal as

$$G_0 = \begin{pmatrix} g_{k\uparrow} & 0 \\ 0 & g_{k\downarrow} \end{pmatrix},$$

with

$$g_{ks} = \left[ z - \epsilon_k - s\sqrt{F_k} \right]^{-1}.$$

The self-energy $\Sigma$ caused by the random impurity potential is given as

$$\Sigma = \langle V \rangle_{\text{imp}} + \langle VG_0 V \rangle_{\text{imp}},$$

where $\langle \cdots \rangle_{\text{imp}}$ indicates an average over impurity distribution. The corresponding diagrams are shown in figure 3. It should be noted that the first term in the self-energy is usually neglected since it gives only an energy shift, however, it is treated explicitly as it gives spin-dependent potentials.

**Figure 3.** Self-energy $\Sigma$ in the Born approximation.
The first term of the self-energy is \( \langle V \rangle_{\text{imp}} = \langle ks | V | k's' \rangle \), which is first evaluated in \( |ks \rangle \) space and transformed into \( |k\sigma \rangle \) space, giving

\[
\langle \langle k | V | k' \rangle \rangle_{\text{imp}} = n (V 1 - \delta V \sigma_z) \delta_{kk'},
\]

where \( n \) is the impurity concentration and \( \delta V \) is defined via \( V_{\uparrow(\downarrow)} = V - (\pm) \delta V \). The first term in the parenthesis may be neglected since it gives a constant energy shift.

The second term of the self-energy is calculated similarly. Keeping only the first order terms in \( n \), the expression of the term in \( |k\sigma \rangle \) space is given as

\[
\langle \langle k | V G_0 V | k' \rangle \rangle_{\text{imp}} = n^2 \left( \frac{\hbar}{2 \tau_{\uparrow}} \right) \delta_{kk'},
\]

As shown in figure 2, we assume that the Fermi energy satisfies \( \epsilon_F \geq \Delta_{\text{ex}} \), i.e., the Fermi energy cuts the two energy branches. For this situation, the integral in equation (18) gives

\[
\langle \langle k | V G_0 V | k' \rangle \rangle_{\text{imp}} = \pm i \left( \begin{array}{cc} \hbar \tau_{\uparrow} & 0 \\ 0 & \hbar \tau_{\downarrow} \end{array} \right) \delta_{kk'},
\]

where \( \tau_{\uparrow(\downarrow)} \) is the lifetime of up (down) spin electrons, and is defined via \( (\hbar/2 \tau_{\uparrow(\downarrow)}) = \pi n DV_{\uparrow(\downarrow)}^2 \), where \( D = m/2 \pi \hbar^2 \) is the density of states per spin of 2DEG, and the sign \( - (+) \) corresponds to the retarded (advanced) Green’s function. Summing up these results, self-energy is given as

\[
\Sigma = -n \delta V \sigma_z \delta_{kk'} + i \left( \begin{array}{cc} \hbar \tau_{\uparrow} & 0 \\ 0 & \hbar \tau_{\downarrow} \end{array} \right) \delta_{kk'},
\]

in \( |k\sigma \rangle \) space. Thus, the self-energy is independent of the wavevector and is diagonal in the Pauli spin space.

By using the Dyson equation, the normalized Green’s function \( \tilde{G}^z = (G_0^{-1} - \Sigma)^{-1} \) is given as

\[
\tilde{G} = \left( \begin{array}{cc} \tilde{f}_{\uparrow} & 0 \\ 0 & \tilde{f}_{\downarrow} \end{array} \right) + \lambda h \tilde{f}_x (-k_x \sigma_y + k_y \sigma_x),
\]

where

\[
\tilde{f}_2 = \frac{1}{\tilde{g}_-^{-1} \tilde{g}_+^{-1} - \hbar^2 h^2 k^2}, \quad \tilde{f}_x = \tilde{f}_2 \tilde{g}_-^{-1}, \quad \tilde{f}_{\downarrow} = \tilde{f}_2 \tilde{g}_+^{-1}
\]

with

\[
\tilde{g}_\sigma = \left[ z - \epsilon_k + \sigma \Delta \pm i \frac{\hbar}{2 \tau_{\sigma}} \right]^{-1},
\]

where \( \Delta = \Delta_{\text{ex}} + n \delta V \). In the following, the retarded and advanced Green’s functions will be denoted as \( \tilde{G}^{R(A)} \) or \( \tilde{g}^{R(A)} \).
3.2. Kubo formula and vertex corrections

In the Kubo formalism the AHC is given as,

$$\sigma_{yx} = \frac{\hbar}{2\pi L^2} \text{Tr} \langle J_y G^R J_x G^A \rangle_{\text{imp}}.$$  \hspace{1cm} (22)

In order to calculate the average of the product of two Green’s function, The AHC is rewritten as

$$\sigma_{yx} = \frac{\hbar}{2\pi L^2} \text{Tr} J_y \langle G^R \rangle_{\text{imp}} J_x \langle G^A \rangle_{\text{imp}} + \text{vc},$$  \hspace{1cm} (23)

where vc is the ‘vertex correction’ to AHC, determined self-consistently to satisfy the current conservation law. Since the self-energy is determined in the Born approximation, the approximation to vc is the ladder approximation. In this approximation, the Hall conductivity is given as

$$\sigma_{yx} \sim \frac{\hbar}{2\pi L^2} \text{Tr} \left( J_y \tilde{G}^R J_x \tilde{G}^A + J_y \tilde{G}^R \langle V \Gamma V \rangle_{\text{imp}} \tilde{G}^A \right).$$  \hspace{1cm} (24)

where $\Gamma$ is the vertex function and is determined from

$$\Gamma = \tilde{G}^R J_y \tilde{G}^A + \tilde{G}^R \langle V \Gamma V \rangle_{\text{imp}} \tilde{G}^A,$$  \hspace{1cm} (25)

(see figure 4), and $\langle G^{(A)} \rangle_{\text{imp}} \sim \tilde{G}^{(A)}$. Since we have adopted the simple Born approximation for the self-energy, the bare Green’s function $G_0$ without the effect of disorder is used to evaluate the self-energy. The present approximation gives the Drude formula for the longitudinal conductivity for systems with weak isotropic random potentials [29].

By using the current operator $J_x$ equation (8) and Green’s functions $\tilde{G}^{(A)}$ equation (21), equation (25) is given as

$$\Gamma = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} + \frac{n V \uparrow V \downarrow}{L^2} \sum_k \begin{pmatrix} 0 & \tilde{f}^R \tilde{B} \tilde{f}^A \\ \tilde{f}^R \tilde{B}^* \tilde{f}^A & 0 \end{pmatrix},$$  \hspace{1cm} (26)

where

$$B = \frac{A}{1 - (n V \uparrow V \downarrow / L^2) \sum_k \tilde{f}^R \tilde{f}^A},$$  \hspace{1cm} (27)

$$A = \sum_k i e \lambda \left[ b \hbar k_x^2 \left( \tilde{f}_{\uparrow \downarrow}^R \tilde{f}_{\uparrow \downarrow}^A + \tilde{f}_{\uparrow \downarrow}^R \tilde{f}_{\uparrow \downarrow}^A \right) + \tilde{f}_{\uparrow \downarrow}^R \tilde{f}_{\uparrow \downarrow}^A \right].$$  \hspace{1cm} (28)
and
\[ \Gamma_0 = \sum_k \tilde{G}_R J_s \tilde{G}_A = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}. \] (29)

### 3.3. Anomalous Hall conductivity

In order to clarify the role of the non-vertex part and vertex part, we give their explicit expressions as,

\[ \sigma_{yx} \equiv \sigma_{yx}^{nv} + \sigma_{yx}^v, \]

\[ \sigma_{yx}^{nv} = \frac{\hbar}{2\pi L^2} \mathrm{Tr} J_y \tilde{G}_R^R J_s \tilde{G}_A^A, \] (30)

\[ \sigma_{yx}^v = \frac{\hbar}{2\pi L^2} \mathrm{Tr} J_y \tilde{G}_R^R \langle V \Gamma V \rangle_{\text{imp}} \tilde{G}_A^A. \] (31)

By using the current operators equation (8), the Green’s function in equation (11) and the vertex function \( \Gamma_1 \), we obtain,

\[ \sigma_{yx}^{nv} = \frac{\hbar}{2\pi L^2} \mathrm{ie}^2 \lambda^2 \sum_k \left[ \tilde{f}_k^R \tilde{f}_k^A - \tilde{f}_k^R \tilde{f}_k^A + 2bhk_y^2 \left\{ \left( \tilde{f}_k^R - \tilde{f}_k^R \right) \tilde{f}_k^A - \tilde{f}_k^R \left( \tilde{f}_k^A - \tilde{f}_k^A \right) \right\} \right], \] (32)

\[ \sigma_{yx}^v = \frac{\hbar}{2\pi L^2} \frac{nV_\uparrow V_\downarrow}{L^2} e\lambda \sum_k \left[ \tilde{f}_k^R B \tilde{f}_k^A + \tilde{f}_k^R B^* \tilde{f}_k^A + bhk_y^2 \right. \]
\[ \times \left\{ \left( \tilde{f}_k^R B + \tilde{f}_k^R B^* \right) \tilde{f}_k^A + \tilde{f}_k^R \left( B \tilde{f}_k^A + B^* \tilde{f}_k^A \right) \right\} \]
\[ = \frac{\hbar}{2\pi L^2} \frac{nV_\uparrow V_\downarrow}{L^2} \mathrm{Im} [2BA]. \] (33)

Using these expressions, we first prove that the AHC vanishes identically unless the lifetime is spin-dependent. By expressing the Green’s function in \( |ks\rangle \) space as

\[ \tilde{G} = \begin{pmatrix} \tilde{G}_{++} & \tilde{G}_{+-} \\ \tilde{G}_{-+} & \tilde{G}_{--} \end{pmatrix}, \] (34)

the AHC is given as

\[ \sigma_{yx} = 2e^2 D \sqrt{\tau_\uparrow \tau_\downarrow} \left[ i \left( \lambda + \lambda' \right) P + \lambda'' \left( \lambda + Q \right) \right]. \] (35)

with

\[ P = \frac{nV_\uparrow V_\downarrow}{4L^2} \sum_k bk \left( \tilde{G}_{++}^R \tilde{G}_{--}^A - \tilde{G}_{++}^R \tilde{G}_{--}^A + \tilde{G}_{++}^R \tilde{G}_{--}^A - \tilde{G}_{++}^R \tilde{G}_{--}^A \right), \] (36)

\[ Q = \frac{nV_\uparrow V_\downarrow}{4L^2} \sum_k bk \left( \tilde{G}_{++}^R \tilde{G}_{--}^A - \tilde{G}_{++}^R \tilde{G}_{--}^A + \tilde{G}_{++}^R \tilde{G}_{--}^A - \tilde{G}_{++}^R \tilde{G}_{--}^A \right). \] (37)

Taking \( V_\uparrow = V_\downarrow \), i.e., \( \tau_\uparrow = \tau_\downarrow = \tau \), we get \( P = 0, Q = -\lambda \), resulting in \( \sigma_{yx} = 0 \). This is quite similar to the SH conductivity [13]. Thus, the result presented in our previous work [20, 30] that the AHC vanishes unless the lifetime is spin-dependent, holds in general.
We can derive the explicit expression of $\sigma_{yx}$ for $\tau_\uparrow \neq \tau_\downarrow$ $\delta V \neq 0$ by expanding the general expression up to the second order of $\lambda$, which gives

$$\sigma_{yx}^{\text{nv}} = -4e^2D\lambda^2 \frac{1}{4\Delta_{\text{ex}}^2 + (\Sigma_\uparrow + \Sigma_\downarrow)^2} \left\{ (\epsilon_F + \Delta) \Sigma_\downarrow \tau_\downarrow - (\epsilon_F - \Delta) \Sigma_\uparrow \tau_\uparrow \right\},$$

$$\sigma_{yx}^v = 4e^2D\sqrt{\tau_\uparrow \tau_\downarrow} \lambda^2 \left\{ (\epsilon_F + \Delta) \frac{\sqrt{\tau_\downarrow}}{\tau_\downarrow} - (\epsilon_F - \Delta) \frac{\sqrt{\tau_\uparrow}}{\tau_\uparrow} \right\}^2$$

$$\times \left\{ \Delta \left[ \Sigma_\uparrow + \Sigma_\downarrow + (\sqrt{\Sigma_\uparrow} - \sqrt{\Sigma_\downarrow})^2 \right] \right\} \left\{ 4\Delta^2 + (\sqrt{\Sigma_\uparrow} - \sqrt{\Sigma_\downarrow})^4 \right\},$$

where $\Sigma_{\uparrow(\downarrow)} = \hbar/(2\tau_{\uparrow(\downarrow)})$ is the spin-dependent self-energy. To simplify the expressions, they can be expanded further in terms of $\delta \tau/\tau \equiv \delta$ up to the second order of $\delta$. Here $\delta$ is defined via $\tau_{\uparrow(\downarrow)} = \tau + (-)\delta \tau$. The resultant expressions are

$$\sigma_{yx}^{\text{nv}} = -2e^2D\tau\lambda^2 \frac{\Delta \Sigma}{\Delta^2 + \Sigma^2} \left\{ 1 + 2\frac{\epsilon_F}{\Delta} \delta + \frac{2\Delta^2}{\Delta^2 + \Sigma^2} \delta^2 \right\},$$

$$\sigma_{yx}^v = 2e^2D\tau\lambda^2 \frac{\Delta \Sigma}{\Delta^2 + \Sigma^2} \times \left\{ 1 + 2\frac{\epsilon_F}{\Delta} \delta + \left( \frac{\epsilon_F^2}{\Delta^2} + \frac{2\Delta^2}{\Delta^2 + \Sigma^2} \right) \delta^2 \right\}.$$

Thus, we obtain,

$$\sigma_{yx} = \sigma_{yx}^{\text{nv}} + \sigma_{yx}^v = 2e^2D\tau\lambda^2 \frac{\epsilon_F^2\Sigma}{\Delta (\Delta^2 + \Sigma^2)} \delta^2 \sim \frac{e^2}{\hbar} \frac{\epsilon_F \Delta_{\text{SO}}^2}{2\Delta_{\text{ex}}^2} \delta^2,$$

where $\lambda \hbar k_F \equiv \Delta_{\text{SO}}$. We find that the AHC is finite in the second order of $\delta$. In the derivation of equation (42), we have assumed $\Delta_{\text{ex}} \gg n\delta V$, $\Sigma$, and hence we have neglected $n\delta V$ and $\Sigma$.

Based on the above, there are several points for discussion. Equation (42) shows that $\sigma_{yx}$ is independent of the lifetime $\tau$ and depends only on its spin-dependence. The lifetime independence of $\sigma_{yx}$ is characteristic of the AHC in the side-jump mechanism. For example, Crepieux and Bruno [31] have derived the following expression for the AHC in the side-jump mechanism: $\sigma_{yx}^{SO} = -e\lambda_0^2(n_\uparrow - n_\downarrow)/2\hbar$, where $\lambda_0$ is the SOI caused by impurities and $n_{\uparrow,\downarrow}$ are the number of carriers of up and down spin states.

The expression for $\sigma_{yx}$ in equation (42) is odd with respect to $\Delta_{\text{ex}}$ as expected, but it gives a divergence of $\sigma_{yx}$ at a limit $\Delta_{\text{ex}} = 0$. The wrong limit for $\sigma_{yx}$ is due to the approximations that $\Delta_{\text{ex}} \gg n\delta V$, $\Sigma$ in the derivation of the expression. We will show later that the limit is correctly given by integrating the general expressions in equations (32) and (33).

We have proved that $\sigma_{yx} = 0$ in general, provided $V_\uparrow = V_\downarrow$. However, this is not a completely general statement. We have used an implicit condition that $\epsilon_F \geq \Delta_{\text{ex}}$ in the analysis. When $\epsilon_F < \Delta_{\text{ex}}$, the AHC is nonzero even for $V_\uparrow = V_\downarrow$ [26]. Onoda et al [32], have formulated AHC in the Keldysh formalism and presented numerical results using the T-matrix approximation. They have shown that $\sigma_{yx} \sim 0$ for $\epsilon_F > \Delta_{\text{ex}}$ subject to a condition that $V_\uparrow = V_\downarrow$. Their results seem to be consistent with the present results, except that a small spin dependence appears in the self-energy in their self-consistent calculations.
Figure 5. Calculated results of the Hall conductivity as functions of $\Delta_{\text{ex}}$ for various values of $\Delta_{\text{SO}}$. Solid lines and symbols are the results obtained from the analytical expressions and numerical integrations, respectively. The value of $V/\epsilon_F = 1.0$.

4. Numerical results

We have performed a numerical integration over the momentum in the general expression shown in equations (32) and (33), which are valid for $\epsilon_F \geq \Delta_{\text{ex}}$. We compare the results with those obtained by the analytical expressions given by equations (40) and (41) to show the parameter regions where the analytical expressions are valid. Hereafter, we take $\epsilon_F$ as the unit of energy.

In the numerical integration over momentum, the cut-off momentum $k_c$ must be set. Examining the convergence of the integration for several values of $k_c$ for the largest Fermi surface, we have chosen $k_c/k_F = 5$, and performed the integration for $|k_x, k_y| < k_F$ with a mesh of $10^{-3}k_F$.

When $V_\uparrow = V_\downarrow$, i.e., $\tau_\uparrow = \tau_\downarrow$, $\delta V = 0$, the numerical results show that $\sigma_{yx}$ and $\sigma_{yx}$ have opposite signs to give $\sigma_{yx} = 0$ as expected. When $\Delta_{\text{SO}}/\epsilon_F = 0.1$, the analytical results agree with the numerical ones for $\Delta_{\text{ex}}/\epsilon_F > 0.3$ within 10%. This suggests that the analytical result given by equations (40) and (41) or equation (42) may be valid for $\Delta_{\text{ex}}/\Delta_{\text{SO}} > 3$. For a fixed
Figure 6. Calculated results of the Hall conductivity as functions of $\Delta_{\text{SO}}$ for various values of $\Delta_{\text{ex}}$. Solid lines and symbols are the results obtained from the analytical expressions and numerical integrations, respectively. The value of $V/\epsilon_F = 1.0$.

value of $\Delta_{\text{SO}}$, $\sigma_{nx}^{nv}$ and $\sigma_{vy}^{v}$ show non-monotonic dependence on $\Delta_{\text{ex}}$, suggesting a competitive dependence of $\sigma_{nx}^{nv}$ and $\sigma_{vy}^{v}$ on $\Delta_{\text{SO}}$ and $\Delta_{\text{ex}}$.

Now we show the calculated results of AHC for $\tau_1 \neq \tau_1$, where $\delta V/V = 0.1$ which corresponds to $\delta = 0.2$. Figure 5 shows the calculated results of $\sigma_{nx}^{nv}$, $\sigma_{yx}^{v}$ and $\sigma_{yx}^{x}$ as functions of $\Delta_{\text{ex}}/\epsilon_F$ for $\Delta_{\text{SO}}/\epsilon_F = 0.1$, 0.3 and 0.5. The solid curves and symbols are results obtained in analytical integration and numerical integration, respectively. Figure 6 are those of $\sigma_{yx}^{nv}$, $\sigma_{yx}^{v}$ and $\sigma_{yx}^{x}$ as functions of $\Delta_{\text{SO}}/\epsilon_F$ for $\Delta_{\text{ex}}/\epsilon_F = 0.1$, 0.3 and 0.5. As shown in figure 5, the agreement between analytical and numerical results is fairly good for $\Delta_{\text{ex}}/\epsilon_F > 0.3$ when $\Delta_{\text{SO}}/\epsilon_F = 0.1$. As the value of $\Delta_{\text{SO}}/\epsilon_F$ increases, the value of $\Delta_{\text{ex}}/\epsilon_F$ for which the analytical expressions are valid increases. In figure 6, the agreement between the analytical and numerical results is good for smaller values of $\Delta_{\text{SO}}/\epsilon_F$. This is reasonable since the analytical expressions have been obtained by expanding the general expressions in terms of $\Delta_{\text{SO}}/\epsilon_F$ up to the second order.

We see that $\sigma_{yx}^{nv} < 0$, while $\sigma_{yx}^{v} > 0$ for the parameter values used, and that $|\sigma_{yx}^{v}| > |\sigma_{yx}^{nv}|$, resulting in positive $\sigma_{yx}$. Since the extrema of $\sigma_{yx}^{v}$ are more dominant than those of $\sigma_{yx}^{nv}$, a peak structure appears in $\sigma_{yx}$ for both $\Delta_{\text{ex}}/\epsilon_F$ and $\Delta_{\text{SO}}/\epsilon_F$ dependence. The value of $\sigma_{yx}$ tends to zero.
Figure 7. Calculated results of the AHC in $\Delta_{ex} - \Delta_{SO}$ space for (a) $n = 0.1$ and (b) $n = 0.01$, where $\delta V/V = 0.1$, and $V/\epsilon_F = 1.0$.

as $\Delta_{ex}/\epsilon_F \to 0$. Nevertheless, $\sigma_{yx}$ is not strictly zero when $\Delta_{ex}/\epsilon_F = 0$. This is because the first term of equation (20) gives rise to a spin-dependent self-energy since we chose $V_\uparrow \neq V_\downarrow$ with $\delta V/V = 0.1$. The spin-dependent self-energy may be an effective exchange splitting caused by the spin-dependent random potentials. Since the spin-dependent random potential depends on the exchange splitting of the matrix, a self-consistent treatment of the spin-dependent self-energy may be desired.

In order to clarify the peak structure of $\sigma_{yx}$ in $\Delta_{SO} - \Delta_{ex}$ space, we calculate $\sigma_{yx}$ for the parameter region for with $0 \leq \Delta_{ex}/\epsilon_F$ and $\Delta_{SO}/\epsilon_F \leq 0.1$. The calculated results are shown in figures 7(a) and (b) for $n = 0.1$ and 0.01, respectively. The results show that a peak appears at small values of $\Delta_{SO}$ and $\Delta_{ex}$. As mentioned above, $\sigma_{yx}$ is finite even when $\Delta_{ex} = 0$ due to an effective exchange splitting caused by the spin-dependent potential. In this case, the peak appears at $\Delta_{SO} \sim \Sigma$, i.e., at $\Sigma/\epsilon_F = 0.025$ for $n = 0.1$, and at $\Sigma/\epsilon_F = 0.0025$ for $n = 0.01$. The results indicate that a large AH effect may arise due to the large spin-dependent random potential in a non-magnetic matrix, and that the AH effect reduces with increasing exchange splitting. Furthermore, the AHC depends on the impurity concentration, i.e. on the lifetime, especially
for small values of $\Delta_{SO}$ and $\Delta_{ex}$. This is similar to the AHC due to skew scattering caused by extrinsic SOI. The similarity, however, should be carefully examined in more detail because the present model has dealt with the intrinsic SOI, rather than the extrinsic SOI, and treated $V_\sigma$ and $\Delta_{ex}$ independently.

5. Summary

We have formulated the AHC in the diffusive transport regime using the Kubo formula for the case in which the Fermi level intersects the spin-split two energy bands, i.e. $\epsilon_F \geq \Delta_{ex}$. We have presented a detailed derivation and general expression of the AHC calculated in the Born and ladder approximations, and have proven that the AHC vanishes in general when the lifetime is spin-independent. By performing a numerical integration of the general expression of the AHC, we have derived the correct dependence of the AHC on $\Delta_{SO}$ and $\Delta_{ex}$ in the zero-magnetization limit. In a more general parameter regime including a case where $\epsilon_F < \Delta_{ex}$, more analyses may be desirable for the effect of spin-dependent random potentials.

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