THE FLUCTUATIONS OF THE COSMIC MICROWAVE BACKGROUND FOR A COMPACT HYPERBOLIC UNIVERSE

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ABSTRACT

The fluctuations of the cosmic microwave background (CMB) are investigated for a small, open universe, i.e., one that is periodically composed of a small fundamental cell. The evolution of initial metric perturbations is computed using the first 749 eigenmodes of the fundamental cell in the framework of linear perturbation theory using a mixture of radiation and matter. The fluctuations of the CMB are investigated for various density parameters $\Omega_0$, taking into account the full Sachs-Wolfe effect. The corresponding angular power spectrum, $C_\ell$, is compared with recent experiments.

Subject headings: cosmic microwave background — cosmology: theory — large-scale structure of universe

1. INTRODUCTION

In recent years, much attention has been paid to the possibility that an open universe could have a nontrivial topology. It is supposed that the four-dimensional spacetime can be represented as the direct product $\mathbb{R} \times \mathcal{M}$, where $\mathcal{M}$ is a compact 3-manifold and the real line $\mathbb{R}$ represents time, and the spacetime has a local structure as in the Friedmann-Lemaître universe. The topology is at least constrained, but not fixed, by Einstein's theory of gravitation, because Einstein's equations deal as partial differential equations with the local geometry of spacetime, whereas the global structure of space depends on the metric as well as the topology. A nontrivial topology can lead to a spatial closure in the universe, because of the global connectivity, instead of a positive spatial curvature.

In order to manifest in observations, the topology should lead to a small universe (Ellis & Schreiber 1986), i.e., one in which light has been circled around the universe at least once at our present epoch. Such a small universe can be obtained from the universal covering space by suitable identifications under a discrete group of isometries. In the case of a flat universe, the full isometry group $\mathbb{R}^3 \times \text{SO}(3)$ leads to 17 multiconnected types of locally Euclidean spaces, of which 10 are spatially compact (see, e.g., Lachièze-Rey & Luminet 1995). The simplest case is a flat universe with a toroidal structure, i.e., the topology of a 3-torus. For such rectangular fundamental cells, the implications for the universe are studied in Sokolov (1993), Starobinsky (1993), Stevens, Scott, & Silk (1993), de Oliveira-Costa & Smoot (1995), and de Oliveira-Costa, Smoot, & Starobinsky (1996), yielding the conclusion that the dimension of the smallest toroidal structure must be at least of the order of the horizon size to be compatible with the COBE Differential Microwave Radiometer (DMR) data (Bennet et al. 1996). The other compact, orientable flat spaces are investigated in Levin, Scannapieco, & Silk 1998, showing that periodicities with half the horizon size are marginally consistent with the data. In the case of positive curvature, the elliptic topology is studied in, e.g., Petrosian & Salpeter (1968) and Sollheim (1968, 1969). However, since astrophysical data suggest that the cosmological density parameter $\Omega_0$ is subcritical (see, for example, Peebles 1998), in the following discussion only the case of negative curvature is considered. There is now evidence for a nonvanishing cosmological constant $\Lambda$, but this case will be considered in a forthcoming publication. In this paper a vanishing cosmological constant $\Lambda = 0$ is assumed.

For negative curvature there exists an infinite variety of possible fundamental cells; see, e.g., Lachièze-Rey & Luminet (1995) and Thurston (1979), or the so-called census of compact hyperbolic manifolds of the Geometry Centre at the University of Minnesota (Weeks/SnapPea). In contrast to the Euclidean case, there is no scaling freedom for the fundamental cells. As a consequence of the rigidity theorem of Mostow (Mostow 1973), the volume as well as the lengths of closed geodesics are topological invariants for a given hyperbolic 3-manifold. Thus, if the curvature scale, i.e., the density parameter $\Omega_0$ and $\Lambda$, is given, then the geometrical properties of $\mathcal{M}$ are fixed.

One manifestation of the nontrivial topology is in multiple images of objects at sufficient distances determined by the discrete group of isometries $\Gamma$. However, for a realistic value of $\Omega_0 \approx 0.2$–$0.3$, the distance to the nearest mirror images is larger than the range of survey galaxy catalogs, roughly 200–600 Mpc. Quasars are more distant, but the quasar phenomenon is probably too short-lived to be able to observe multiple quasar images, because the distances and thus the look-back times of the images are in general all distinct (see Lehoucq, Luminet, & Uzan 1999 and references therein). Another manifestation of the topology can be seen in pairs of circles of correlated microwave radiation coming from the surface of last scattering (SLS) (Cornish, Spergel, & Starkman 1998a; Weeks 1998). The identical temperatures at points identified according to $\Gamma$ originate at the SLS having a redshift of roughly $z \approx 1200$. Unfortunately, in the case of $\Omega_0 \approx 0.2$–$0.3$, the main contribution of the CMB does not come from the SLS, which corresponds to the naive Sachs-Wolfe effect (Sachs & Wolfe 1967), but instead comes from much nearer regions, $z \ll 1200$, corresponding to the integrated Sachs-Wolfe effect (see, e.g., Gouda, Sugiyama, & Sasaki 1991; Kamionkowski & Spergel 1994; Cornish, Spergel, & Starkman 1998b, and below). In the case of a flat universe where the integrated Sachs-Wolfe effect is absent and the CMB is due to the SLS, the pairs of circles allow determination of the isometry group by deter-
mining the generators of the group $\Gamma$. For $\Omega_0 < 1$, it is necessary to compute the CMB for given examples of compact models to obtain ideas of the expected structure of the CMB. For two compact models, the expected CMB is computed using the method of images in Bond, Pogosyan, & Souradeep 1998. This method requires the computation of the group elements of $G$, which is not an easy task since these groups are not free, i.e., there are relations among the generators of the group such that not all products of generators yield new group elements. Since the method using images requires only the distinct group elements, much attention has to be paid to it. Nevertheless, Bond et al. (1998) find that only for $\Omega_0 \simeq 0.8$ is a CMB obtained that is in accord with the COBE measurements. An alternative for calculating the CMB demands computation of the eigenmodes of the considered 3-manifold. In the case of the so-called Thurston manifold, the first 14 eigenmodes are computed using the boundary-element method from Inoue (1999), and the statistic of expansion coefficients of the eigenmodes is investigated, showing pseudorandom behavior, where the term “pseudo” reflects the fact that the coefficients are determined by the eigenmodes and not by a genuine random process.

Interestingly, the possible volumes of compact hyperbolic 3-manifolds are bounded from below, which means that there exists a hyperbolic 3-manifold with minimal volume. It has been suggested that the creation probability of the universe increases dramatically with decreasing volume (e.g., Atkatz & Pagels 1982). Thus, from a cosmological point of view, the most interesting hyperbolic 3-manifolds are those with volumes near to the volume of the smallest hyperbolic 3-manifold. Unfortunately, the smallest hyperbolic 3-manifold is unknown. The two smallest known ones have volumes $\text{vol}(\mathcal{M}) \simeq 0.98139R^3$ (Thurston 1982) and $\text{vol}(\mathcal{M}) \simeq 0.94272R^3$ (Weeks 1985; Matveev & Fomenko 1988), respectively, where $R$ is the curvature radius of the universal covering space. Another possibility is to consider not only manifolds, but instead also to allow orbifolds as possible models for the universe. The difference is that orbifolds can possess points that do not locally look like the usual $\mathbb{R}^3$. Orbifolds can possess rotation elements in their group of isometries. Around the axis of a given rotation element, the space must be identified with respect to the discrete angle of the rotation element, which does not happen in the usual $\mathbb{R}^3$. However, as long as there is no elaborated quantum cosmology that describes the way in which the topology and topological defects of the universe develop, orbifolds are as good as manifolds for a model of the cosmos. In this paper, an orbifold with volume $\text{vol}(\mathcal{M}) \simeq 0.7173068R^3$ is chosen. For this system, the fluctuations of the CMB are computed using the eigenmodes. The initial scalar metric perturbations are expanded in terms of the eigenmodes of the orbifold such that the modes develop independently in the framework of linear perturbation theory, assuming adiabatic evolution. From the metric perturbations, the fluctuations of the CMB can then be computed by the Sachs-Wolfe effect (Sachs & Wolfe 1967).

The next section describes the selected orbifold. Section 3 outlines the procedure for the computation of the CMB. Section 4 describes the properties of the CMB depending on the density parameter $\Omega_0$. Finally, the angular power spectrum $C_\ell$ of the fluctuations is compared with experimental data from COBE, Saskatoon, and QMAP.

2. THE GEOMETRIC MODEL

The orbifold used in this paper is obtained from a Kleinian group, which yields a pentahedron as a fundamental cell, which in turn is symmetric along an intersection plane. The pentahedron is divided by this intersection plane into two equal tetrahedra. Thus, the eigenmodes can be computed by desymmetrizing the pentahedron (for more details, see Aurich & Marklof 1996). A group-symmetry consideration shows that the eigenmodes of the pentahedron obeying periodic boundary conditions decompose into two symmetry classes, one having Dirichlet boundary conditions, i.e., $\psi = 0$, at the surface of the tetrahedron, and the other having Neumann boundary conditions, i.e., a vanishing normal derivative $\partial \psi / \partial n = 0$. Using the tetrahedron with Dirichlet and Neumann boundary conditions facilitates the numerical computation of the eigenmodes. In the nomenclature of Lannér (1950), Maclachlan & Reid (1989), and Maclachlan (1996), this tetrahedron is called $T_3$. It has a volume of $\text{vol}(\mathcal{M}) \simeq 0.3586534R^3$ and is defined by the dihedral angles

$$\angle BC = \frac{\pi}{2}, \quad \angle CA = \frac{\pi}{3}, \quad \angle AB = \frac{\pi}{4},$$

$$\angle DA = \frac{\pi}{2}, \quad \angle DB = \frac{\pi}{3}, \quad \angle DC = \frac{\pi}{5},$$

where $A$, $B$, $C$, and $D$ are the four corner points. For the tetrahedron $T_3$, the first 749 eigenmodes corresponding to Dirichlet boundary conditions have been computed using the boundary-element method as described in Aurich & Marklof (1996). It is worthwhile to note that there are only nine compact tetrahedra in hyperbolic space and that $T_3$ is the only compact tetrahedron whose generating group is not arithmetic (Maclachlan & Reid 1989). Furthermore, the smallest tetrahedron, $T_3$, has a volume $\text{vol}(\mathcal{M}) \simeq 0.03588506R^3$, roughly 10 times smaller than the volume of $T_3$.

The CMB depends on the position of the observer within the fundamental cell. The computations are carried out in the so-called upper-half space,

$$\mathcal{H}_3 = \{(x, y, z) \in \mathbb{R}^3 | z > 0\},$$

with the hyperbolic metric

$$ds^2 = \frac{1}{z^2} (dx^2 + dy^2 + dz^2),$$

yielding a constant curvature of $-1$. In this model for hyperbolic space, the tetrahedral cell is oriented such that the corner points are approximately at $A \simeq (0.4348, 0, 0.2537)$, $B \simeq (0.3978, 0.6889, 0.2869)$, $C \simeq (0, 0, 0.2824)$, and $D \simeq (0, 0, 2.3829)$. The observer is situated at $(0.15, 0.2, 0.5)$, which lies well within the fundamental cell.

3. COMPUTATION OF THE CMB

In order to compute the evolution of the initial scalar metric perturbations, it is necessary to define a background model that describes the homogeneous, isotropic spacetime without any perturbations. The perturbations are assumed to be sufficiently small that the linear perturbation theory is applicable. Let us set the speed of light equal to $c = 1$ and define the conformal time $\eta$ by $d\eta = dt$, where $a$ is the scale factor. The background metric is chosen to be the
Friedmann-Lemaître-Robertson-Walker metric, 
\[ ds^2 = a^2(\eta)(d\eta^2 - \gamma_{ij}dx^idx^j) , \]
with
\[ \gamma_{ij} = \delta_{ij}[1 - \frac{1}{4}(x^2 + y^2 + z^2)]^{-2} \]
for the case of negative curvature. The spatial part, \( \gamma_{ij} \), corresponds to the unit-ball model used in hyperbolic geometry, with the difference that the ball here has a radius of 2 instead of 1. The Einstein equations, expressed in conformal time \( \eta \), reduce for the background metric to the time-space equation
\[ a^2 - a^2 = \frac{8\pi G}{3} T^0_\alpha a^4 , \]
and to the trace of the Einstein equations,
\[ a' - a = \frac{4\pi G}{3} T^\mu_\mu a^3 , \]
where \( a' \equiv da/d\eta \), \( T^\mu_\mu \) is the energy-momentum tensor, and \( G \) denotes Newton’s gravitational constant. In the following discussion, a model with conventional relativistic hydrodynamic matter behaving as a perfect fluid is assumed. The energy-momentum tensor, which is then diagonal, is described in terms of the energy density \( \epsilon \), the pressure \( p \), and the 4-velocity \( u^\mu \) as
\[ T^\mu_\mu = (\epsilon + p)u^\mu u_\mu - p\delta^\mu_\mu . \]
Assuming a two-component model containing radiation with energy density \( \epsilon_r \propto a^{-4} \) and cold dark matter with energy density \( \epsilon_m \propto a^{-3} \), the pressure perturbation \( \delta p \) for a vanishing entropy perturbation \( \delta S = 0 \) is given by
\[ \delta p = \left( \frac{\delta \rho}{\delta \epsilon} \right)_{\delta S = 0} \delta \epsilon \equiv c_s^2 \delta \epsilon , \]
where \( c_s \) can be interpreted as the sound velocity. From this it follows, with \( \epsilon = \epsilon_m + \epsilon_r \) and \( p = \frac{4}{3}\epsilon_r \) (see, e.g., Mukhanov, Feldman, & Brandenberger 1992), that
\[ c_s^2 = \frac{1}{3} \left( \frac{9}{4}\epsilon_m/\epsilon_r \right) . \]
For this two-component model, equations (1) and (2) are solved by
\[ a(\eta) = \frac{2a_{eq}}{\tilde{\eta}^2} (\tilde{\eta} \sinh \eta + \cosh \eta - 1) , \]
\[ \tilde{\eta} = \sqrt{\frac{2a_{eq}}{2\pi G \epsilon_r a^4}} , \]
where \( a_{eq} \) is the scale factor at the time of equal matter and radiation density, i.e., \( \epsilon_m = \epsilon_r \). In the gauge-invariant formalism (Bardeen 1980; Kodama & Sasaki 1984; Mukhanov et al. 1992) the perturbed metric can be written as
\[ ds^2 = a^2(\eta)[(1 + 2\Phi)d\eta^2 - (1 - 2\Psi)\gamma_{ij}dx^idx^j] . \]
For a diagonal energy-momentum tensor, as assumed above, one gets \( \Phi = \Psi \), which can be considered to be a generalized Newtonian gravitational potential. Assuming vanishing entropy perturbations, \( \delta S = 0 \), the gauge-invariant formalism for the evolution of the metric perturbation \( \Phi \) gives in first-order perturbation theory (Bardeen 1980; Kodama & Sasaki 1984; Mukhanov et al. 1992)
\[ \Phi'' + 3\dot{H}(1 + c_s^2)\Phi' - c_s^2 \Delta \Phi + [2\dot{H}^2 + (1 + 3c_s^2)(\dot{H}^2 + 2)]\Phi = 0 . \]
The prime denotes differentiation with respect to \( \eta \) and \( \dot{H} \equiv a'/a \). The Laplace-Beltrami operator of the hyperbolic space is denoted by \( \Delta \). Expanding the metric perturbation \( \Phi \) with respect to the eigenmodes \( \psi_n(x) \) of the orbifold, i.e.,
\[ \Phi(\eta, x) = \sum_{n=1}^{\infty} f_n(\eta)\psi_n(x) , \]
yields for \( f_n(\eta) \) the differential equation
\[ f''_n(\eta) + 3\dot{H}(1 + c_s^2)f'_n(\eta) + [c_s^2 E_n + 2\dot{H}' + (1 + 3c_s^2)(\dot{H}^2 + 1)]f_n(\eta) = 0 . \]
Here \( E_n \) denotes the eigenvalue corresponding to \( \psi_n \), i.e., \( (\Lambda + E_n)\psi_n = 0 \), with Dirichlet and Neumann boundary conditions, respectively. The wavenumber is given by \( k_n = (E_n - 1)^{1/2} \). The computation of the time evolution of \( f(\eta, x) \) is now reduced to the integration of the ordinary differential equation (4), which can easily be done numerically. It is worthwhile to note that the expansion runs only over discrete levels. At this point enters the compact nature of the model of the universe assumed here.
Now one is left to define the initial conditions of \( f_n(\eta) \) for some initial time \( \eta_i \). Inflationary models suggest a scale-invariant, so-called Harrison-Zeldovich spectrum for the density perturbation modes \( \delta_\eta \) up to logarithmic corrections (see Mukhanov et al. 1992 and references therein). Since the corrections depend on the details of the inflationary model, we assume here
\[ f_0(\eta) = \frac{\alpha}{k^{1/2}} \quad \text{and} \quad f'_0(\eta) = 0 , \]
which carries over to a Harrison-Zeldovich spectrum. The constant \( \alpha \) is later fixed such that the order of the fluctuations \( \delta T/T \) are in agreement with the COBE results. The value of \( \eta_i = 0.001 \) is used, which is well within the radiation-dominated epoch such that the transition to the matter-dominated epoch is included in the computations. All Dirichlet eigenmodes with \( k_n < k_{max} = 55 \), i.e., with \( E_n < 3026 \), are taken into account (\( n_{max} = 749 \)).
It is worthwhile to emphasize that the initial coefficients \( f_0(\eta) \) are given by equation (5). Thus, \( f_0(\eta) \) is not considered as a random variable that could be, e.g., distributed as a Gaussian. Therefore, the randomness in \( \Phi \) is solely due to the properties of the eigenmodes of the considered orbifold. The eigenmodes are square-normalized with respect to the volume of the orbifold using the hyperbolic metric. The only freedom is a factor of \( \pm 1 \), where the sign is used that comes out from the boundary-element calculations.
From the metric perturbations, \( \Phi \), the temperature fluctuations \( \delta T/T \) are computed by the Sachs-Wolfe effect (Sachs & Wolfe 1967),
\[ \frac{\delta T}{T} = \frac{1}{3} \Phi(x_{\text{SLS}}, \eta) + 2 \int_{x_{\text{SLS}}}^x d\eta \frac{\partial \Phi(\eta, x(\eta))}{\partial \eta} , \]
where \( x_{\text{SLS}} \) denotes the time of last scattering, assumed to be at \( z \approx 1200 \), and \( \eta_0 \) is the present time. The factor \( \frac{1}{3} \) in the first term, corresponding to the naive Sachs-Wolfe effect (NSW), is justified in the models considered below because
the time of decoupling occurs well within the matter-dominated epoch. The second term is the integrated Sachs-Wolfe effect (ISW). Note that equation (6) is valid only on scales that are large compared to the horizon at the time \( \eta_{\text{SLS}} \). In Table 1 the angle \( \Theta_H \) under which the horizon appears is shown, together with the angle \( \Theta_k \), which is the angle under which the highest eigenmode fluctuation appears. Since \( \Theta_k \) is at least roughly 2 times larger than \( \Theta_H \), equation (6) can be used for all considered eigenmodes.

In a multicomponent model, in which the components possess different sound velocities, the time evolution generates an entropy perturbation even if one starts with the same initial entropy. The correction is not strictly zero. However, the corrections are negligible as long as the wavelengths of the eigenmodes are much larger than the horizon. The highest considered eigenmode \( (k_{\text{max}} = 55) \) enters the horizon at the conformal time \( \eta = 2\pi/k_{\text{max}} \approx 0.11 \), and all other eigenmodes correspondingly later. This is to be compared with \( \eta_{\text{SLS}} \approx 0.063 \) and \( \eta_{\text{SLS}} \approx 0.033 \) for \( \Omega_0 = 0.2 \) and \( \Omega_0 = 0.6 \), respectively. Thus, entropy perturbations can only slightly alter the ISW.

4. PROPERTIES OF THE CMB

The CMB is computed for different densities \( \Omega_0 \) with respect to the Hubble constant \( h = 0.6 \) in units of 100 km s^{-1} Mpc^{-1}. The radiation density, \( \epsilon_r \), is chosen to be in agreement with the present background radiation temperature of \( T = 2.728 \) K. In Figures 1 and 2 the fluctuations of the CMB are shown for \( \Omega_0 = 0.3 \) and 0.6, respectively. The monopole and dipole contributions are subtracted such that the first nonvanishing multipole is the quadrupole, in accordance with the usual representation of the CMB fluctuations.

A quantitative measure of the scale of the fluctuations is provided by the angular power spectrum \( C_l \), defined by

\[
C_l = \frac{1}{2l+1} \sum_{m=-l}^{l} |a_{lm}|^2 ,
\]

where \( a_{lm} \) are the expansion coefficients of \( \delta T \) with respect to the spherical harmonics \( Y_l^m(\theta, \phi) \). In the following discussion, the angular power spectrum \( C_l \) is compared with the data measured for \( l < 30 \) by COBE (Tegmark 1996), around \( l \sim 100 \) by QMAP (de Oliveira-Costa et al. 1998), and above \( l \sim 80 \) by the Saskatoon experiment (Netterfield et al. 1997). These experiments provide evidence that the angular power spectrum increases up to a maximum around \( l \simeq 200 \).

Figures 1 and 2 show fluctuations on finer scales with decreasing density \( \Omega_0 \). This is due to the cutoff in the wave-numbers \( k < k_{\text{max}} \). For a genuine Harrison-Zeldovich spectrum having no cutoff, there are fluctuations on all scales. However, the amplitudes belonging to the different scales depend on \( \Omega_0 \) by the specific form of decay of \( f_\eta(\eta) \) via the ISW. The decay of \( f_\eta(\eta) \) is faster for smaller densities \( \Omega_0 \) with increasing eigenvalue \( E_m \), as shown in Figure 3 for \( \Omega_0 = 0.3 \). An estimate of the angular contributions that are maximally taken into account in the angular power spectrum \( C_l \) for a given \( k_{\text{max}} \) can be obtained as follows. The smallest wavelength of the eigenmodes, \( l_{\text{min}} \), determines the finest scale of the fluctuations on the SLS. The angle \( \Theta_k \)

| \( \Omega_0 \) | \( \Theta_H \) (deg) | \( l_H \) | \( \eta_0 - \eta_{\text{SLS}} \) | \( \Theta_k \) (deg) | \( l_k \) | rms NSW | rms ISW |
|---|---|---|---|---|---|---|---|
| 0.2 | 0.46 | 391 | 2.7527 | 0.8 | 215 | 0.066 | 0.106 |
| 0.3 | 0.60 | 300 | 2.3214 | 1.3 | 139 | 0.066 | 0.079 |
| 0.4 | 0.73 | 247 | 1.9863 | 1.8 | 98 | 0.067 | 0.064 |
| 0.6 | 0.94 | 191 | 1.4409 | 3.3 | 55 | 0.070 | 0.044 |
| 0.8 | 1.12 | 161 | 0.9322 | 6.1 | 30 | 0.069 | 0.026 |

**TABLE 1**

**QUANTITIES FOR VARIOUS VALUES OF \( \Omega_0 \)**

*Note.—Col. (2): Angle \( \Theta_H \) under which the horizon at the SLS is seen. Col. (3): Distance to the SLS. Col. (5): Angle due to the cutoff \( k_{\text{max}} \). Col. (6): Corresponding \( l_k \). Cols. (7)–(8): rms of the NSW and ISW contributions, respectively, in equation (6), for \( x = 1 \).*
under which the smallest scale fluctuations are seen is given by

$$\tan \frac{\Theta_k}{2} = \frac{\tanh \left( \lambda_{\text{min}}/2 \right)}{\sinh (\eta_0 - \eta_{\text{SLS}})}.$$ 

Since the $l$th spherical harmonics has along the “circumference” $2l$ zeros in $[-180^\circ, 180^\circ]$, one finds the rule of thumb that a given $\Theta$ corresponds roughly to

$$l \approx \frac{180^\circ}{\Theta}.$$ 

In Table 1 the distance $\eta_0 - \eta_{\text{SLS}}$, the angle $\Theta_k$ under which $\lambda_{\text{min}}$ appears, and the corresponding angular contribution $l_k \equiv 180^\circ/\Theta_k$ are given for several values of $\Omega_0$.

To get an impression of the fluctuations as seen with the $10^\circ$ resolution of COBE, a Gaussian smoothing of Figure 1, i.e., for $\Omega_0 = 0.3$, is presented in Figure 4 with that resolution.

Figures 5–8 show the angular power spectrum for $\Omega_0 = 0.2$, $0.3$, $0.4$, and $0.6$, where the abscissa shows $[l(l + 1)C_l/2\pi]^{1/2}$ in $\mu$K. The data for the compact hyperbolic models are shown up to roughly $l_k$. A reasonable agreement is observed for $\Omega_0 \approx 0.3$–$0.4$. In the case of $\Omega_0 = 0.2$, $C_l$ increases too fast with increasing $l$ in comparison with the experimental data. For $\Omega_0 = 0.4$, one observes a saturation above $l \approx 40$. Then the further increase of the $C_l$ must come from processes, such as acoustic oscillations, that become important on scales of the horizon size at the SLS around $l_K = 200$–$300$ (see Table 1). The necessary contributions are not considered here, but the results obtained from simply connected models should then apply, since the corresponding scales are small in comparison with the size of the fundamental cell. Further effects, such as reionization, gravitational lensing, and the Sunyaev-Zeldovich effect, influence the fluctuations on a scale of the order of $\Theta \approx 1^\circ$ and thus the values of $C_l$ for correspondingly large $l$. More important are the low multipoles, since they have ruled out a toroidal structure in the case of a flat universe, at least for periodicities significantly below the horizon size (Stevens et al. 1993; de Oliveira-Costa & Smoot 1995; Levin et al. 1998). In the flat case, the first multipoles were too small in comparison with the multipoles around $l \approx 20$. Such an effect is absent in the hyperbolic case, and thus a “small”
universe is not ruled out. This result is at variance with Bond et al. (1998), who obtained agreement with COBE only for very high densities, $\Omega_0 \approx 0.8$.

In order to show the increasing significance of the ISW with decreasing $\Omega_0$, Table 1 shows the rms of the two terms in equation (6) using $\alpha = 1$ in equation (5). The rms value of the NSW is nearly constant, because the fluctuations are determined by equation (5), i.e., by $\alpha$ independently of the distance to the SLS. In contrast, the ISW diminishes toward $\Omega_0 = 1$. For $\Omega_0 = 1$, no ISW contribution occurs because of $d\Phi/d\eta = 0$ in this case. The Figure 9 shows the contributions to the $C_l$ spectrum separately for the NSW and the ISW, as well as both contributions together as observed in nature. The case of $\Omega_0 = 0.4$ is shown with $\alpha = 1$. The importance of the eigenmodes with respect to the contribution to the ISW is determined by two competing effects. On the one hand, higher eigenmodes are oscillating faster, such that their contribution to the integral is less important. However, the derivative $f(\eta)$ also determines the significance, and it is this derivative that increases with increasing eigenvalues, as can be inferred from Figure 3. Numerically, both effects seem to cancel, such that the ISW is nearly constant with respect to $C_l$. At $\Omega_0 = 0.3$, the NSW and the ISW are of equal significance, and for lower values of $\Omega_0$ the ISW dominates. This can lead to difficulties in detecting paired circles.

It is interesting to note that the large-scale power is caused not only by the lowest eigenmodes but also by the large-scale structure generated by the isometry group describing the fundamental cell. To stress that fact, Figure 10 shows the NSW contribution of the lowest eigenmode, having an eigenvalue $E_1 \approx 93.11$, which has only one maximum within the tetrahedral cell. One can clearly observe large circles of different radii having a scale larger than the wavelength of the eigenmode. The $C_l$ spectrum, shown in Figure 11, has a periodic structure with different maxima corresponding to the circles of different radii and to the structure formed by the arrangement of the circles itself. Only the wide last maximum around $l = 60$ corresponds to
the peak expected from the wavelength of the lowest eigen-
mode. If the peaks due to the circles are pronounced enough
in comparison with the ISW, there could survive a periodic
structure in the full $C_l$ spectrum that yields some informa-
tion about the isometry group.

In conclusion, small hyperbolic universes are not ruled
out, in contrast to the flat case. A reasonable agreement
with the experimental data is provided by the hyperbolic
models with densities around $\Omega_0 \approx 0.3 - 0.4$. In this work, an
orbifold instead of a manifold is investigated; however, the
statistical properties of the eigenmodes are expected to be
the same for orbifolds and manifolds. Thus, since the
volume of the considered pentahedron is of the same order
as of the Weeks and Thurston manifolds, the comoving
wavenumbers $k_n$ of these models are comparable, since their
mean behavior is determined by Weyl's law. (Weyl's law,
which counts the mean number of eigenvalues below a
given energy, is derived in Aurich & Marklof 1996 for
general orbifolds.) In these models no supercurvature
modes are expected; these are absent in the considered pen-
tahedron and could, if present, alter the angular power spec-
trum. The individual details of the sky maps would differ
between the models, but the angular power spectra should
show the same behavior. However, the details of the maps
already depend on the location of the observer within the
given fundamental cell. After all, such a small universe has a
uniform CMB because in all directions the fluctuations of the
same fundamental cell contribute to the CMB and thus
circumvent the horizon problem. In addition, the Machian
paradox is solved in a new manner by the nontrivial topol-
gy. A future work will include the cosmological constant
in order to investigate its effect on the structure of the CMB.

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FIG. 10.—NSW contribution of the lowest eigenmode ($E_l \approx 93.11$) to the CMB, shown for the case of $\Omega_0 = 0.2$