REDUCIBLE GALOIS REPRESENTATIONS AND THE HOMOLOGY OF GL(3, Z)

AVNER ASH AND DARRIN DOUD

Abstract. Let $\overline{F}_p$ be an algebraic closure of a finite field of characteristic $p$. Let $\rho$ be a continuous homomorphism from the absolute Galois group of $\mathbb{Q}$ to $GL(3, \overline{F}_p)$ which is isomorphic to a direct sum of a character and a two-dimensional odd irreducible representation. Under the condition that the conductor of $\rho$ is squarefree, we prove that $\rho$ is attached to a Hecke eigenclass in the homology of an arithmetic subgroup $\Gamma$ of $GL(3, \mathbb{Z})$. In addition, we prove that the coefficient module needed is, in fact, predicted by the main conjecture of [3].

1. Introduction

Fix a prime $p$ and let $\overline{F}_p$ be an algebraic closure of a finite field of characteristic $p$. Generalizations of Serre’s conjecture [13] connect the homology of an arithmetic group with a Galois representation $\rho$. When the target of $\rho$ is $GL(n, \overline{F}_p)$, a conjecture was first published in [5], which was extended in [3] and further improved in [8]. See Section 2 for definitions and a statement of the conjecture in the cases that we will require.

In this paper, we will first discuss a general method of computing Hecke operators acting on the homology of certain complexes. This should help us attach Galois representations to the boundary homology of congruence subgroups of $GL(n, \mathbb{Z})$ for any $n$. In this paper we are able to treat the case of $n = 3$. We use this method to prove the existence of eigenclasses in the homology of arithmetic groups that are attached (see section 2) to Galois representations that decompose as the sum of an irreducible two-dimensional odd Galois representation and a character.

If we let $X_n$ be the Borel-Serre bordification of the symmetric space for $GL(n, \mathbb{R})$, the representations that we consider should be attached to eigenclasses in the homology of the boundary of $X_n/\Gamma$ for $n = 3$. However, when $n > 2$, the mod $p$ topology of the boundary is quite complicated. Our method is designed to avoid the need to compute the homology of this boundary.

Our main theorem (Theorem 2.5) will show that for any $\rho : G_{\mathbb{Q}} \to GL(3, \overline{F}_p)$ with $p > 3$ prime, such that $\rho$ is a direct sum of a two-dimensional odd irreducible representation and a character, and the Serre conductor of $\rho$ is squarefree, there is a Hecke eigenclass in the homology $H_3(\Gamma_0(3, N), V \otimes \epsilon)$ that is attached to $\rho$, where $N$, $V$, and $\epsilon$ are predicted by Conjecture 2.4. In addition, for any such $\rho$, there may be several possible values for $V$; we will show that all of them yield eigenclasses with $\rho$ attached.

Date: May 14, 2012.

The first author thanks the NSA for support of this research through NSA grant H98230-09-1-0050. This manuscript is submitted for publication with the understanding that the United States government is authorized to reproduce and distribute reprints.
2. Conjectural connections between Galois representations and arithmetic homology

A Hecke pair \((\Gamma, S)\) in a group \(G\) is a subgroup \(\Gamma \subset G\) and a subsemigroup \(S \subset G\) such that \(\Gamma \subset S\) and for any \(s \in S\) both \(\Gamma \cap s^{-1} \Gamma s\) and \(s^{-1} \Gamma s \cap \Gamma\) have finite index in \(\Gamma\).

If we let \(R\) be a ring and \(M\) a right \(R[S]\)-module, then for \(s \in S\) there is a natural action of a double coset \(\Gamma s \Gamma\) on the homology \(H_i(\Gamma, M)\). We denote by \(\mathcal{H}(\Gamma, S)\) the \(R\)-algebra under convolution generated by all the double cosets \(\Gamma s \Gamma\) with \(s \in S\). We call \(\mathcal{H}(\Gamma, S)\) a Hecke algebra, and the double cosets Hecke operators. The action of the double cosets on homology makes \(H_i(\Gamma, M)\) an \(\mathcal{H}(\Gamma, S)\)-module.

We will use the following groups and semigroups in \(\text{GL}_n\).

**Definition 2.1.** Let \(N\) be a positive integer and \(p\) a prime.

1. \(S_0(n, N)^\pm\) is the semigroup of matrices \(s \in M_n(\mathbb{Z})\) such that \(\det(s)\) is relatively prime to \(pN\) and the first row of \(s\) is congruent to \((*, 0, \ldots, 0)\) modulo \(N\).
2. \(S_0(n, N)\) is the subsemigroup of \(s \in S_0(n, N)^\pm\) such that \(\det(s) > 0\).
3. \(\Gamma_0(n, N)^\pm = S_0(n, N)^\pm \cap \text{GL}(n, \mathbb{Z})\).
4. \(\Gamma_0(n, N) = S_0(n, N) \cap \text{GL}(n, \mathbb{Z})\).

In the case in which we are interested, the ring \(R\) will be the algebraic closure \(\overline{\mathbb{F}}_p\) of a finite field of order \(p\). \(S\) will be \(S_0(n, N)\), and \(\Gamma\) will be \(\Gamma_0(n, N)\). We will denote the Hecke algebra \(\mathcal{H}(\Gamma_0(n, N), S_0(n, N))\) by \(\mathcal{H}_{n,N}\). We note that \(\mathcal{H}_{n,N}\) is commutative, and contains the Hecke operators \(T(\ell, k) = \Gamma D_{\ell,k} \Gamma\) where

\[
D_{\ell,k} = \text{diag}(1, \ldots, 1, \ell, \ldots, \ell).
\]

**Definition 2.2.** Let \(V\) be an \(\mathcal{H}_{n,N}\)-module, and let \(v \in V\) be a simultaneous eigenvector of all the \(T(\ell, k)\) with \(\ell \nmid p\) and \(0 \leq k \leq n\). Denote by \(a(\ell, k)\) the eigenvalue of \(T(\ell, k)\) acting on \(v\).

We say that the Galois representation \(\rho : G_{\mathbb{Q}} \to \text{GL}(n, \overline{\mathbb{F}}_p)\) is attached to \(v\) if

\[
\det(I - \rho(Frob_\ell)X) = \sum_{k=0}^{n} (-1)^k \ell^k(k-1)/2a(\ell, k)X^k
\]

for all prime \(\ell \nmid p\) for which \(\rho\) is unramified at \(\ell\).

(Note that we use the arithmetic Frobenius, so that if \(\omega\) is the cyclotomic character, \(\omega(Frob_\ell) = \ell\).)

The \(\mathcal{H}_{n,N}\) modules that we use to find Hecke eigenvectors attached to Galois representations will be computed as the homology of admissible modules for \(S_0(n, N)\).

**Definition 2.3.** For a fixed prime \(p\), an admissible module \(M\) (cf. [2]) for a subsemigroup \(S\) of \(\text{GL}(n, \mathbb{Q})\) is a finite dimensional vector space over \(\mathbb{F}_p\) on which there exists an \(N\) such that \(S\) acts on \(M\) through reduction modulo \(N\). (In particular, the denominators in \(S\) are all prime to \(N\).)

Specifically, the admissible modules that we need will be irreducible \(\overline{\mathbb{F}}_p[\text{GL}(3, \mathbb{F}_p)]\)-modules, on which subsets of \(\text{GL}(3, \mathbb{Z})\) consisting of matrices of determinant prime to \(p\) will act through reduction modulo \(p\). Such modules are parameterized by triples \((a, b, c)\) with \(0 \leq a - b, b - c \leq p - 1\) and \(0 \leq c \leq p - 2\). The module
corresponding to the triple \((a, b, c)\) will be denoted \(F(a, b, c)\). (See Section 5 for details.)

In order to connect irreducible modules with Galois representations, we will need the mod \(p\) cyclotomic character \(\omega\), and the niveau two characters \(\omega_2\) and \(\omega'_2\) [12]. Note that a power of the cyclotomic character may be written as \(\omega^a\), with \(a\) well defined modulo \(p - 1\) (since \(\omega\) has order \(p - 1\)). Similarly, a power of \(\omega_2\) that is not a power of \(\omega\) may be written as \(\omega_2^m\) with \(m\) well defined modulo \(p^2 - 1\) and not a multiple of \(p + 1\). Given such an \(m\), with \(0 \leq m < p^2 - 1\), we may write \(m = a + bp\) with \(0 \leq a, b < p\) and \(a \neq b\) (for instance writing \(m\) in base \(p\)). In fact, we may write \(m = a + bp\) with \(0 < a - b \leq p\) (by adding \(p\) to \(a\) and subtracting one from \(b\), as necessary). Note that \(a\) and \(b\) are only well defined modulo \(p - 1\), since adding \(p - 1\) to each of \(a\) and \(b\) changes \(m\) by \(p^2 - 1\), yielding the same power of \(\omega_2\).

Finally, we recall the two different types of wild ramification for a representation \(\rho : I_p \to \text{GL}_2(\mathbb{F}_p)\) of the form

\[
\rho(I_p) \sim \begin{pmatrix} \omega^{a+1} & * \\ 0 & \omega^a \end{pmatrix},
\]

namely peu ramifié and trés ramifié, and refer the reader to [13] for definitions.

We now state a conjecture connecting certain three-dimensional Galois representations with eigenvectors in arithmetic cohomology groups. We state the conjecture only for certain representations; for a more general conjecture that applies to a much wider class of representations see [3, 5, 8].

**Conjecture 2.4.** Let \(p\) be a prime, and let \(\overline{\mathbb{F}}_p\) be an algebraic closure of \(\mathbb{F}_p\). Let \(\rho : \Gamma \to \text{GL}(3, \overline{\mathbb{F}}_p)\) be a Galois representation that is a sum of an irreducible odd two-dimensional representation and a character. Let \(N\) be the Serre conductor of \(\rho\) and \(\epsilon\) the nebentype of \(\rho\) (see [13]). Then we may choose an irreducible admissible \(F_p[\text{GL}(3, \overline{\mathbb{F}}_p)]\)-module \(V\) such that \(\rho\) is attached to a cohomology class in the \(H_{3,N}\)-module \(H_3(\Gamma_0(3, N), V \otimes \epsilon)\).

If \(\rho = \sigma \oplus \omega \psi\), where \(\psi\) has conductor prime to \(p\), we may describe the possible \(V\) in terms of the restriction of \(\sigma\) to inertia at \(p\). If

\[
\sigma|_{I_p} \sim \begin{pmatrix} \omega^{a+1} & * \\ 0 & \omega^a \end{pmatrix}
\]

we choose \(a, b, c\) modulo \(p - 1\) so that \(0 < a - b, b - c \leq p\) and \(0 \leq c \leq p - 1\), with the restriction that if \(\sigma\) is trés ramifié, then \(a - b = p\) and let \(V\) be the irreducible module \(F(a - 2, b - 1, c)\). We may also choose \(a, b, c\) so that \(0 < c - a, a - b \leq p\) and \(0 \leq b < p - 1\), with \(a - b = p\) if \(\sigma\) is trés ramifié, and let \(V = F(c - 2, a - 1, b)\).

If

\[
\sigma|_{I_p} \sim \begin{pmatrix} \omega_2^{a+bp} & 0 \\ 0 & \omega_2^{a+bp} \end{pmatrix}
\]

and \(0 < a - b \leq p\), we either take \(0 < a - b, b - c \leq p\) and \(0 \leq c < p - 1\) and choose \(V = F(a - 2, b - 1, c)\), or we take \(0 < c - a, a - b \leq p\) and \(0 \leq b < p - 1\) and take \(V\) to be \(F(c - 2, a - 1, b)\).

For each value of \(V\) described above, there is an \(H_{3,N}\)-eigenclass in \(H_3(\Gamma_0(3, N), V \otimes \epsilon)\) with \(\rho\) attached.

Our goal in this paper is to prove the following theorem.

**Theorem 2.5.** Let \(p > 3\). Then Conjecture 2.4 is true for representations \(\rho\) having squarefree conductor.
Note that generically, for a tamely ramified Galois representation, there will be two choices of the integers $a$ and $b$ (obtained by permuting the diagonal characters in each case). Hence, for a tamely ramified representation there will normally be four predicted weights (if $\sigma$ is wildly ramified, there will only be two weights, since we cannot permute the two characters on the diagonal). It can happen that there are additional weights. For instance, if $a - b \equiv 1 \pmod{p - 1}$, we may choose $a = b + 1$ or $a = b + p$. Unless otherwise indicated (i.e. in the *très ramifié* case), all of these weights are predicted.

In the conjecture, the two predictions of weights arise from embedding the image of $\rho$ into one of the two standard Levi subgroups

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}. $$

There is also an embedding of the image of $\rho$ into the Levi subgroup

$$\begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{pmatrix}$$

but this embedding, for $\rho$ of the type we consider, would violate a strict parity condition, and thus has no predicted weights according to the main conjecture of [3]. Theoretical considerations lead us to believe that we might find an eigenclass attached to $\rho$ in a weight predicted from this forbidden embedding in a homology group of different degree, namely, $H_3(\Gamma_0(3, N), V \otimes \epsilon)$. Proving the existence of this class is one possible future application of the techniques described in this paper.

3. Hecke actions on induced representations

For any set $A$, $\mathbb{Z}[A]$ denotes the abelian group of formal finite linear combinations of elements of $A$. If $A$ is a semigroup or group, $\mathbb{Z}[A]$ is naturally a ring. Tensor products without a subscript are to be taken over $\mathbb{Z}$.

We follow Brown’s notation for tensor products p. 55, Chap. III [6], except our conventions reverse left and right. If $G$ is a group, $H$ a subgroup of $G$, $A$ is a right $\mathbb{Z}H$-module and $B$ is a left $\mathbb{Z}H$-module and a right $\mathbb{Z}G$-module, then $A \otimes_{\mathbb{Z}H} B$ denotes the right $\mathbb{Z}G$-module where $ah \otimes_{\mathbb{Z}H} b = a \otimes_{\mathbb{Z}H} hb$ and $(a \otimes_{\mathbb{Z}H} b)g = a \otimes_{\mathbb{Z}H} bg$ for any $h \in H$ and $g \in G$. If $B = \mathbb{Z}G$ (with the obvious right action of $H$ and left action of $G$), $A \otimes_{\mathbb{Z}H} \mathbb{Z}G$ is the induced module from $H$ to $G$ of $A$.

If $M$ and $N$ are two right $\mathbb{Z}G$-modules, then $M \otimes_{\mathbb{Z}G} N$ denotes the right $\mathbb{Z}$-module where $m \otimes_{\mathbb{Z}G} n = mg \otimes_{\mathbb{Z}G} ng$. It is equal to the coinvariants of the right $\mathbb{Z}G$-module $M \otimes N$ where $(m \otimes n) = mg \otimes ng$.

Let $\Gamma \subset S \subset G$ where $G$ is a group, $\Gamma$ a subgroup, $S$ a subsemigroup and $(\Gamma, S)$ a Hecke pair. Let $X$ be a set on which $G$ acts on the right.

Remark: In general, the $S$-action will not preserve the $\Gamma$-orbits in $X$. Here is an example: Consider $\Gamma = \Gamma_0(2, 25)$ acting on $P^1(\mathbb{Z})$. Let $s = \text{diag}(2, 1)$. Note that $(5 : 1)$ and $(5 : 6) = (5 : 1) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are in the same $\Gamma$-orbit. However $(5 : 1)s = (10 : 1)$ and $(5 : 6)s = (10 : 6) = (5 : 3)$ are in different $\Gamma$ orbits.

For any $x \in X$ and any subsemigroup $T$ of $G$, write $T_x = \text{Stab}_T x$. We define the concept of an $S$-sheaf $W$ on $X$: For any $x \in X$, $W_x$ is an abelian group, and for any $g \in S, x \in X$ there is given a homomorphism $\mu(g) : W_x \rightarrow W_{sx}$ such that
\[ \mu(h) \circ \mu(g) = \mu(gh) \] for all \( g, h \in S \) and \( \mu(1) = id_{W_x} \). Then \( W = \bigoplus_{x \in X} W_x \) is a right \( \mathbb{Z} \)-module. If \( F \) is any right \( \mathbb{Z} \)-module, then \( W \otimes F \) is a right \( \mathbb{Z} \)-module with the action given by \( (w \otimes f)g = wg \otimes fg \) for any \( g \in S \).

By Corollary 5.4 p. 68 in [6], an \( S \)-sheaf \( W \) restricted to \( \Gamma \) is a direct sum of induced \( \Gamma \)-modules. For \( a \in X \), write \( X_a = a\Gamma \) for the orbit of \( a \) under \( \Gamma \). Choose a subset \( A \subset X \) such that \( X = \bigcup_{a \in A} X_a \). For each \( a \in A \) let \( W[a] = \bigoplus_{x \in X_a} W_x \subset W \). Then \( W[a] \) is naturally isomorphic as right \( \Gamma \)-module to the induced module \( W_a \otimes_{\underline{Z}_\Gamma} \mathbb{Z} \Gamma \). Note that in general \( S \) does not preserve \( W[a] \).

We abbreviate \( \otimes_{\underline{Z}_\Gamma a} \) by \( \otimes_a \). Then

\[ W \approx \bigoplus_{a \in A} W[a] \approx \bigoplus_{a \in A} W_a \otimes_a \mathbb{Z} \Gamma. \]

Let \( F \) be a right \( S \)-module. Then \( W \otimes F \) is an \( S \)-module and therefore \( W \otimes_{\Gamma} F = H_0(\Gamma, W \otimes F) \) has a natural action on it of the Hecke algebra \( \mathcal{H}(\Gamma, S) \) which we want to compute.

If \( Y \) is a set on which a group \( K \) acts on the right, let \( \mathcal{F}_c(Y/K) \) denote the set of \( K \)-invariant functions whose support is a finite number of \( K \)-orbits. If in addition a group \( H \) acts on the left, commuting with the \( K \)-action, let \( \mathcal{F}_c(H/K) \) denote the set of \( H \times K \)-invariant functions whose support is a finite number of \( H \times K \)-orbits.

Given a Hecke pair \( (\Gamma, S) \), the Hecke algebra \( \mathcal{H} = \mathcal{H}(\Gamma, S) \) will be identified with \( \mathcal{F}_c(\Gamma \backslash S/\Gamma) \), where multiplication is convolution of functions.

For any \( \Gamma \)-module \( N \), let \( \Phi \) denote the natural projection from \( N \) to the coinvariants \( N_\Gamma = H_0(\Gamma, N) \). If \( h \in \mathcal{H} \), and \( z \in W \otimes F \), writing \( T_h \) for the Hecke operator corresponding to \( h \) we have:

\[ \Phi(z)T_h = \Phi \left( \sum_{g \in \Gamma \backslash S/\Gamma} h(g)zg \right). \]

This does not depend on the choice of coset representatives. If \( h \) is the characteristic function of \( \Gamma s \Gamma = \prod s_a \Gamma \), then

\[ \Phi(z)T_h = \Phi \left( \sum \alpha z s_a \right) \]

since \( h(g) = 0 \) unless \( g \in \Gamma s \Gamma \), in which case \( g = s_a \gamma \) for some \( \alpha, \gamma \), and then \( h(s_a) = 1 \).

We will now determine the Hecke action in terms of the isomorphism of homology given by Shapiro's lemma. Associativity of tensor products gives a canonical isomorphism \( \lambda : W[a] \otimes_{\Gamma} F = (W_a \otimes_{a\Gamma} \mathbb{Z} \Gamma) \otimes_{\Gamma} F \approx W_a \otimes_{\Gamma a} F \) via \( w \otimes_{a\gamma} \gamma \otimes f \mapsto w \otimes_{\Gamma a} f \gamma^{-1} \).

Now let \( h \in \mathcal{H} \) acts on \( W \otimes_{\Gamma} F = \bigoplus_{a \in A} W[a] \otimes_{\Gamma} F \approx \bigoplus_{a \in A} W_a \otimes_{\Gamma a} F \). We seek a formula for \( T_h \) on an element in \( \bigoplus_{a \in A} W_a \otimes_{\Gamma a} F \). We know that if \( w \in W_a, \gamma \in \Gamma \) and \( f \in F \),

\[ \Phi(w \otimes_{a \gamma} \gamma \otimes f)T_h = (w \otimes_{a \gamma} \gamma \otimes f)T_h = \sum_{a \in A, \gamma \in \Gamma \backslash S/\Gamma} h(g)((w \otimes_{a \gamma})g \otimes \Gamma f)g. \]

Compute \( (w \otimes_{a \gamma})g \in W \) as follows:

\[ (w \otimes_{a \gamma})g = ((w \otimes 1)\gamma)g = w \gamma g. \]

Write \( a \gamma g = b(a, \gamma, g)\delta(a, \gamma, g) \) with \( b(a, \gamma, g) \in A, \delta(a, \gamma, g) \in \Gamma \). Then \( w \in W_a \) implies that \( w \gamma g \in W_{b(a, \gamma, g)}\delta(a, \gamma, g) \) which we can write as

\[ (w \gamma g\delta(a, \gamma, g)^{-1} \otimes_{b(a, \gamma, g)} 1)\delta(a, \gamma, g) = w \gamma g\delta(a, \gamma, g)^{-1} \otimes_{b(a, \gamma, g)} \delta(a, \gamma, g). \]
Thus
\[ \Phi(w \otimes_a \gamma \otimes f)|T_h = \sum_{a,g} h(g)w^\gamma g\delta(a, \gamma, g)^{-1} \otimes_{b(a, \gamma, g)} \delta(a, \gamma, g) \otimes_{\Gamma} fg. \]

If \( z = \sum_a w^a \otimes_a \gamma^a \otimes f^a \), we obtain:
\[ \Phi(z)|T_h = \sum_{g \in \Gamma \backslash S/\Gamma} \sum_a h(g)w^a \gamma^a g\delta(a, \gamma^a, g)^{-1} \otimes_{b(a, \gamma^a, g)} \delta(a, \gamma^a, g) \otimes_{\Gamma} f^ag. \]

We interchange the order of summation, and then given a fixed \( a \), we can choose the representatives \( g \) of the double cosets in a way that depends on \( a \), without changing the value of the sum. If \( ag \in X_b \) then we choose \( g \) so that \( ag = b \). Call such a choice \( g_{ab} \). In other words, we always have
\[ ag_{ab} = b. \]

Since we want to compute \( T_h \) on \( \oplus_a W_a \otimes_{\Gamma_a} F \) via \( \lambda \), without loss of generality we may take \( \gamma^a = 1 \) for all \( a \). Then \( b(a, 1, g_{ab}) = b \) and \( \delta(a, 1, g_{ab}) = 1 \). We obtain
\[ \Phi \left( \sum_a w^a \otimes_a 1 \otimes f^a \right) |T_h = \Phi \left( \sum_a \sum_{b \in \Gamma \backslash S/\Gamma} \sum_{g_{ab} \in \Gamma : ag = b} h(g_{ab})w^ag_{ab} \otimes b \otimes_{\Gamma} f^ag_{ab} \right). \]

Via the isomorphism \( \lambda \), this corresponds to
\[ \left( \sum_a w^a \otimes_{\Gamma_a} f^a \right) |T_h = \sum_a \sum_{b \in \Gamma \backslash S/\Gamma} \sum_{g_{ab} \in \Gamma : ag = b} h(g)w^ag \otimes_{\Gamma_a} f^ag. \]

Call this Formula (1).

Define \( h_{ab}(g) = h(g) \) if \( ag = b \) and \( = 0 \) otherwise. Clearly, \( h_{ab} \in \mathcal{F}(\Gamma_a \backslash S/\Gamma_b) \).

Corresponding to \( h_{ab} \) is a Hecke operator \( T_{ab} \). It maps \( \Gamma_a \)-homology to \( \Gamma_b \)-homology.

**Theorem 3.1.** Let \( W \) be an \( S \)-sheaf on the \( G \)-set \( X \) and \( F \) a right \( S \)-module. Let \( A \) be a set of representatives of the \( \Gamma \)-orbits of \( X \). Let
\[ \lambda : W \otimes_{\Gamma} F \to \oplus_a W_a \otimes_{\Gamma_a} F \]
be the natural isomorphism described above. Then \( T_h \) on the left is equivariant to the matrix \( T_{ab} \) on the right.

If \( F \) is a resolution of \( Z \) by projective \( \mathbb{Z}[\Gamma] \)-modules which are also \( S \)-modules, then \( \lambda \) induces the isomorphism on homology given by Shapiro’s lemma and we have
\[ H_q(\Gamma_a, W) \approx \oplus_{a \in A} H_q(\Gamma_a, W_a), \]
and again \( T_h \) on the left is equivariant to the matrix \( T_{ab} \) on the right.

**Proof.** Without loss of generality, \( h \) is the characteristic function of \( \Gamma a \Gamma = \prod s_a \Gamma \).

We fix \( a, b \). If \( \alpha \in X_b \), we choose \( s_\alpha = g_{ab} \).

Then the term in Formula (1) corresponding to \( a, b \) is
\[ \Theta_{ab} := \sum_{s_\alpha : \alpha \in X_b} h(s_\alpha)w^as_\alpha \otimes_{\Gamma_b} f^as_\alpha. \]

We must show that
\[ \Theta_{ab} = (w^a \otimes_{\Gamma_a} f^a)|T_{ab}. \]
To compute \( |T_{ab} \), write \( \Gamma_a s_\Gamma = \prod t \Gamma_b \). Then
\[ (w^a \otimes_{\Gamma_a} f^a)|T_{ab} = \sum_{t} h_{ab}(t)w^at \otimes_{\Gamma_b} f^at. \]
Now $h_{ab}(t) = 0$ unless $at = b$. So if $h_{ab}(t) \neq 0$, since $t \in \Gamma s \Gamma$, we have $t = s_\alpha \gamma$ for some $\alpha, \gamma$, and $b = at = as_\alpha \gamma = b \gamma$ (since $s_\alpha = g_{ab}$). It follows that $\gamma \in \Gamma_b$.

In other words, $h_{ab}(t) = 0$ unless $t \in \Gamma_b$ for some $\alpha$ for which $as_\alpha = b$, and in this case $h_{ab}(t) = h(t)$. Therefore

$$(\omega^a \otimes \Gamma_a f^a)|T_{ab} = \sum_{t \in \Gamma_a \Gamma_b \Gamma_s \alpha \in X_b} h(t) \omega^a t \otimes \Gamma_a f^a t = \sum_{s_\alpha \alpha \in X_b} h(s_\alpha) \omega^a s_\alpha \otimes \Gamma_b f^a s_\alpha,$$

which equals $\Theta_{ab}$.

If $F$ is a complex, the action of the Hecke operators on $W \otimes \Gamma F$ and $\oplus_a W_a \otimes \Gamma_a F$ commutes with the boundary maps in $F$. Therefore $T_h$ on $H_q(\Gamma, W)$ and $T_{ab}$ on $\oplus_a A H_q(\Gamma_a, W_a)$ are equivariant with respect to $\lambda$. \hfill \Box

4. Preparing to Compute Hecke Operators in $GL_3$

In this section we determine the $g_{ab}$’s that we need to study reducible 3-dimensional Galois representations.

Let $P_0$ be the stabilizer of the line spanned by $(1, 0, \ldots, 0)$ in affine $n$-space, on which $GL(n)$ acts on the right. Note that the elements of $P_0$ are characterized by the fact that all entries in the top row except for the first are zero.

We set

$$U_0 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & 1 & \cdots & 0 \\ * & 0 & \cdots & 1 \end{pmatrix}, \quad L_0^1 = \begin{pmatrix} * & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \quad L_0^2 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & * \end{pmatrix}.$$  

For $g \in P_0$, we define $\psi_0^1(g) \in GL(1)$ and $\psi_0^2(g) \in GL(n - 1)$ by

$$g = \begin{pmatrix} \psi_0^1(g) & 0 \\ * & \psi_0^2(g) \end{pmatrix}.$$  

We set

$$g_x = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix},$$

and we define

$$P_x = g_x^{-1} P_0 g_x, \quad U_x = g_x^{-1} U_0 g_x, L_x^1 = g_x^{-1} L_0^1 g_x, L_x^2 = g_x^{-1} L_0^2 g_x.$$  

For $s \in P_x$, we set $\psi_x^1(s) = \psi_0^1(g_x s g_x^{-1}).$

We have the following theorem (the steps of the proof are identical to those in [1, Theorem 7], after transposing and replacing $d$ by $-d$):

**Theorem 4.1.** Let $d$ be a positive divisor of $N$, and assume $\gcd(d, N/d) = 1$. Then

1. $U_d L_d^1 \cap \Gamma_0(n, N) = U_d \cap \Gamma_0(n, N)$.
2. If $s \in P_d \cap \Gamma_0(n, N)^\pm$, then $\psi_d^1(s) \equiv s_{11} \pmod{d}$ and $\psi_d^2(s)_{11} \equiv s_{11} \pmod{N/d}$.
3. $\psi_d^2(P_d \cap \Gamma_0(n, N)^\pm) \subseteq \Gamma_0(n - 1, N/d)^\pm$.
4. $\psi_d^2$ induces an exact sequence

$$1 \to U_d \cap \Gamma_0(n, N) \to P_d \cap \Gamma_0(n, N) \to \Gamma_0(n - 1, N/d)^\pm \to 1.$$  

In order to compute Hecke operators with respect to the Hecke pair $(\Gamma_0(3, N), \Gamma_0(3, N))$, we use the coset representatives described in the next theorem.
Theorem 4.2. We have the following coset decompositions of double cosets
\[ \Gamma_0(3, N)s\Gamma_0(3, N) = \bigcup_{g \in C} g\Gamma_0(3, N). \]

(1) For \( s = \text{diag}(1, 1, \ell) \), with \( \ell \) prime and \((\ell, N) = 1\),
\[ C = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & \ell & 0 \end{pmatrix}, \begin{pmatrix} \ell & 0 & 0 \\ 0 & 1 & 0 \\ b & c & \ell \end{pmatrix} : 0 \leq a, b, c \leq \ell - 1 \right\} \]

(2) For \( s = \text{diag}(1, \ell, \ell) \), with \( \ell \) prime and \((\ell, N) = 1\),
\[ C = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & \ell & 0 \\ b & 0 & \ell \end{pmatrix}, \begin{pmatrix} \ell & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{pmatrix} : 0 \leq a, b, c \leq \ell - 1 \right\} \]

Proof. One checks that the given elements are all in the double coset, and that none are in the same coset of \( \Gamma_0(3, N) \). Because the cardinality of \( C \) is equal to the number of cosets of \( \Gamma_0(3, N) \) in the double coset, they must form a complete set of coset representatives.

The next theorem is adapted to the following situation: Using the notations of Section 3, let \( G = \text{GL}(3, \mathbb{Q}) \), \( X = \mathbb{P}^2(\mathbb{Q}) \), \( \Gamma = \Gamma_0(3, N) \). We assume \( N \) is squarefree, in which case, as proved in [1], the \( \Gamma \)-orbits of \( X \) may be represented by the set
\[ A = \{(1 : d : 0) \mid d > 0, \ d|N\}. \]

Also, any \( s \) as in Theorem 4.2 takes each \( \Gamma \)-orbit to itself. For each \( s \) the following theorem gives a \( \gamma \) such that \((1 : d : 0)s\gamma = (1 : d : 0)\). Thus \( g_{ab} = 0 \) unless \( a = b \) and then \( g_{aa} = s\gamma \), where \( a = (1 : d : 0) \). The theorem also gives the values of \( \psi_i^0, i = 1, 2 \) which we will need to compute the action of \( \Gamma \) on the \( S \)-sheaves we deal with in Sections 8 and 9.

Theorem 4.3. Let \( \ell \) be a prime not dividing \( N \), let \( d \) be a divisor of \( N \) with \((d, N/d) = 1\) and let \( s \) be a matrix of the form
\[ s = \begin{pmatrix} \ell_1 & 0 & 0 \\ a & \ell_2 & 0 \\ b & c & \ell_3 \end{pmatrix} \]
that is in one of the sets \( C \) in Theorem 4.2. Then there exists a \( \gamma \in \Gamma_0(3, N) \) of the form
\[ \gamma = \begin{pmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{pmatrix} \]
such that \( s\gamma \in P_d = g_d^{-1}P_0g_d \), and for \( x = g_d s \gamma g_d^{-1} \in P_0 \) we have

(1) If \( \ell_1 = \ell_2 \) and \( a = 0 \), then \( x_{11} = \ell_1 \) and
\[ \psi_0^2(x) = \begin{pmatrix} \ell_2 \\ c - bd & \ell_3 \end{pmatrix}, \]

(2) If \( \ell_1 = \ell, \ell_2 = 1 \) and \( a = 0 \), then \( x_{11} = 1 \) and
\[ \psi_0^2(x) = \begin{pmatrix} \ell \\ -bd + c\ell & \ell_3 \end{pmatrix}. \]
(3) If $\ell_1 = 1$, $\ell_2 = \ell$, and $\ell \nmid ad + 1$, then $x_{11} = 1$ and
\[
\psi^g_0(x) = \begin{pmatrix}
\ell & 0 \\
-b\ell d + c(ad + 1) & \ell_3
\end{pmatrix}.
\]
(4) If $\ell_1 = 1$, $\ell_2 = \ell$, and $\ell \nmid ad + 1$, then $x_{11} = \ell$ and
\[
\psi^g_0(x) = \begin{pmatrix}
1 & 0 \\
-b\ell d + c(ad + 1) & \ell_3
\end{pmatrix}.
\]
We give the proof of this theorem in the appendix.

5. Irreducible representations

Let $B_m$ be the Borel subgroup of $\text{GL}(m)$ consisting of upper triangular matrices, and let $T_m$ be the maximal torus of diagonal matrices. An algebraic weight with respect to the pair $(B_m,T_m)$ is an $m$-tuple of integers $(a_1, \ldots, a_m)$ which represents the map $\text{diag}(t_1, \ldots, t_m) \mapsto t_1^{a_1} \cdots t_m^{a_m}$. This weight is dominant if $a_1 \geq a_2 \geq \cdots \geq a_m$.

A dominant weight is said to be $p$-restricted if $0 \leq a_i - a_{i+1} \leq p - 1$ for $1 \leq i < m$ and $0 \leq a_m \leq p - 2$. For any $p$-restricted weight there exists a unique (up to isomorphism) irreducible right $\mathbb{F}_p[\text{GL}(m, \bar{\mathbb{F}}_p)]$-module $F(a_1, \ldots, a_m)$ with highest weight $(a_1, \ldots, a_m)$ which remains irreducible when restricted to $\mathbb{F}_p[\text{GL}(m, \mathbb{Z}/p\mathbb{Z})]$, and all such irreducible $\mathbb{F}_p[\text{GL}(m, \mathbb{Z}/p\mathbb{Z})]$-modules occur this way [7, p. 412]. These modules are, in fact, absolutely irreducible [11, Corollary II.2.9]. We will relax the condition that $0 \leq a_m \leq p - 2$, and allow $a_m$ to be arbitrary, stipulating that we can adjust the entire $m$-tuple by adding the same multiple of $p - 1$ to each entry without changing the corresponding module. This is equivalent to tensoring with the $(p - 1)$ power of the determinant, which changes the module over $\mathbb{F}_p[\text{GL}(m, \bar{\mathbb{F}}_p)]$, but not over $\mathbb{F}_p[\text{GL}(m, \mathbb{Z}/p\mathbb{Z})]$.

In a similar way, all the irreducible $\mathbb{F}_p[\text{GL}(1, \mathbb{Z}/p\mathbb{Z}) \times \text{GL}(m-1, \mathbb{Z}/p\mathbb{Z})]$-modules are classified by $m$-tuples $(a_1, \ldots, a_m)$ such that $(a_2, \ldots, a_m)$ is $p$-restricted for $\text{GL}(m-1)$ and $a_1$ is considered modulo $p - 1$. We will denote the $\mathbb{F}_p[\text{GL}(1, \mathbb{Z}/p\mathbb{Z}) \times \text{GL}(m-1, \mathbb{Z}/p\mathbb{Z})]$-module corresponding to $(a_1, \ldots, a_m)$ by $M(a_1; a_2, \ldots, a_m)$. We note that this module is just $F(a_2, \ldots, a_m)$ as an $\mathbb{F}_p[\text{GL}(1, \mathbb{Z}/p\mathbb{Z}) \times \text{GL}(m-1, \mathbb{Z}/p\mathbb{Z})]$-module, with $\text{GL}(1, \mathbb{Z}/p\mathbb{Z})$ acting as scalars via the $a_1$-power map.

Note that we will also consider any $\mathbb{GL}(m, \mathbb{Z}/p\mathbb{Z})$-module as a $\mathbb{GL}(m, \mathbb{Z}/p\mathbb{Z})$-module via reduction modulo $p$. In this way, we make any $\mathbb{GL}(m, \mathbb{Z}/p\mathbb{Z})$-module into an $S_0(m,N)$-module. Note that if $N$ and $p$ are relatively prime, then the image of $S_0(m,N)$ under reduction modulo $p$ is all of $\mathbb{GL}(m, \mathbb{Z}/p\mathbb{Z})$.

Given a $\mathbb{GL}(m, \mathbb{Z}/p\mathbb{Z})$-module $E$, we will denote the action of $s \in \mathbb{GL}(m, \mathbb{Z}/p\mathbb{Z})$ by $e|s$. We denote by $E^s$ the module with the same underlying abelian group as $E$, but with $s \in \mathbb{GL}(m, \mathbb{Z}/p\mathbb{Z})$ acting via $e|s = e|g_s g^{-1}_x$. We note that if $E = F(a_1, \ldots, a_m)$ is an irreducible module, then $E^s$ is also isomorphic to $F(a_1, \ldots, a_m)$, since it is irreducible of the same dimension, with the same highest weight viewed as an $\mathbb{F}_p[\text{GL}(m, \bar{\mathbb{F}}_p)]$-module (since conjugation by $g_x$ does not introduce any new torus characters). For a Dirichlet character $\chi$ of conductor $N$, we denote by $E^\chi$ the $S_0(n,N)$ module with the same underlying abelian group as $E$ but with the action of $s$ given by $e|\chi(s) = \chi(s)e|g_s g^{-1}_x$, where $\chi(s)$ is defined as $\chi(s_1)$.

**Theorem 5.1.** Let $n \geq 2$, let $(a_1, \ldots, a_n)$ be a $p$-restricted weight, and set $F = F(a_1, \ldots, a_n)$. Let $\chi$ be a Dirichlet character of conductor $N$, which factors as
\( \chi_0 \chi_1 \), where \( \chi_0 \) has conductor \( d \), and \( \chi_1 \) has conductor \( N/d \) with \( (d, N/d) = 1 \). Then

1. The module of invariants \( F^{U_d}(\mathbb{Z}/p) \) is isomorphic to \( M(a_n; a_1, \ldots, a_{n-1}) \).

2. The module \( F^{U_d \cap S_0(n, N)} \) considered as a \( P_d \cap S_0(n, N) \)-module is isomorphic to \( (F^{U_d})^d \).

3. The action of \( P_d \cap S_0(n, N) \) on the module \( F^{U_d \cap S_0(n, N)} \) is given by

\[
e[s] = \chi_0(\psi_1^1(s))\psi_1^1(s))^a\chi_1(\psi_2^1(s))\psi_2^1(s)),
\]

where the vertical bar on the right denotes the action of \( GL(n-1, \mathbb{Z}/p) \) on \( F(a_1, \ldots, a_{n-1}) \).

Proof. (1) Set \( U = U_0(\mathbb{Z}/p) \), and \( L = P_0/U = L_0(\mathbb{Z}/p) \times L_0(\mathbb{Z}/p) \). Let \( A \) be the outer automorphism of \( GL(n) \) given by \( A(g) = g^{-1} \). The contragredient of \( F \), or \( F^\vee \) is a \( GL(n, \mathbb{Z}/p) \)-module with the same underlying abelian group as \( F \), but with the action given by \( f^\vee g = f(A(g)). \) We note that as \( GL(n, \mathbb{Z}/p) \)-modules, the contragredient and the dual, \( \text{Hom}(F, \mathbb{F}_p) \), are isomorphic.

As \( L \)-modules, we have that \( F^U \cong ((F^\vee)^{A(U)})^\vee \). Then by [10, Proposition 5.10] (see also [9, Lemma 2.5]), we see that

\[
F^U \cong (F(-a_n, \ldots, -a_1)^{(A(U))})^\vee \cong M(-a_n; -a_{n-1}, \ldots, -a_1)^\vee \cong M(a_n; a_1, \ldots, a_{n-1}).
\]

(2) Since \( N \) and \( p \) are relatively prime, \( U_d \cap S_0(n, N) \) and \( U_d \) have the same image in \( GL(n, \mathbb{Z}/p) \), so we need only consider \( F^{U_d} \). We see that \( F^{U_d} = ((F^{-d})^{U_0})^d \). However, as described above, \( F^{-d} \cong F \cong F(a_1, \ldots, a_n) \). Hence, by part (1), we find that \( F^{U_d} \cong M(a_n; a_1, \ldots, a_{n-1})^d \).

(3) We identify \( F^{U_d \cap S_0(n, N)} \) with \( M(a_n; a_1, \ldots, a_{n-1})^d \). Then, for \( e \in F^U \), we have

\[
e[d]s = \chi(s)e\chi_1(\psi_2^1(s))\chi_1(\psi_2^1(s)).
\]

since \( e \in F^U \cong M(a_n; a_1, \ldots, a_{n-1}) \) as a \( GL(1) \times GL(n-1) \)-module and

\[
\chi(s) = \chi(s_{11}) = \chi_0(s_{11})\chi_1(s_{11}) = \chi_0(\psi_1(s))\chi_1(\psi_2(s))
\]

by Theorem 4.1(2).

6. Points in general position

Let \( K \) be an infinite field, \( V \) an \( n \)-dimensional \( K \)-vector space, and denote the projective space by \( \mathbb{P} = \mathbb{P}(V) \). Fix a basis of \( V \). Given points \( a_1, \ldots, a_r \in \mathbb{P} \), we say that the points are in general position if any subset of them spans a linear space of maximal possible dimension. We define a simplicial complex \( Y^g = Y^g(K) \) as follows. The vertices of \( Y^g \) are points in \( \mathbb{P} \), and are acted upon by \( GL_n(K) \) on the right. The \( p \)-simplices of \( Y^g \) are spanned by \( (p + 1) \)-tuples of vertices that are in general position. We let \( X^g \) denote the chain complex of oriented chains on \( Y^g \). The augmentation is the map \( \varepsilon : X^0 \to \mathbb{Z} \) that sends each vertex to 1. If \( x_0, \ldots, x_p \) are vertices spanning a \( p \)-simplex \( \Delta \) in \( Y^g \), we denote by \( \bar{x} = (x_0, \ldots, x_p) \) the chain supported on \( \Delta \) with coefficient 1. It is antisymmetric in the arguments. Then \( X^g_\Delta \) is generated over \( \mathbb{Z} \) by these basic chains \( \bar{x} \).
Theorem 6.1. If $K$ is infinite, then $X^g$ is an acyclic resolution of $\mathbb{Z}$ by $\text{GL}(n, K)$-modules.

Proof. Let $x_i = (x_{i0}, \ldots, x_{ip})$ be a basic chain supported on a $p$-simplex for each $i$, and suppose that $z = \sum_i c_i x_i$ is a cycle, i.e. a $p$-chain that is taken to zero by the boundary map.

Since $K$ is infinite, we may choose a point $y \in \mathbb{P}$ such that for each $i$, the set \{ $x_{i0}, x_{i1}, \ldots, x_{ip}, y$ \} is in general position. Let $y_i = (x_{i0}, x_{i1}, \ldots, x_{ip}, y)$.

Then

$$\partial \sum_i c_i y_i = \sum_i c_i \partial y_i$$

$$= \sum_i \sum_{j=0}^{p} (-1)^{j+1} c_i (x_{i0}, \ldots, \hat{x}_{ij}, \ldots, x_{ip}, y) + (-1)^{p+2} \sum_i c_i (x_1, \ldots, x_p, \hat{y})$$

$$= 0 \pm z,$$

where the double sum is 0 since $z$ is a cycle. Adjusting the sign, we see that every cycle is a boundary. \hfill \Box

7. A spliced sharbly complex

In this section, $K$ is an infinite field. We are going to splice the complex $X^g$ with the sharbly complex. Recall the Steinberg module $St_n$ and the sharbly complex $Sh^n$ for an $n$-dimensional $K$-vector space $V$ as described for example in [4]. If $v_1, \ldots, v_{n+k}$ are vectors in $V$, denote by $[v_1, \ldots, v_{n+k}]$ the basic $k$-sharbly. It is antisymmetric in the arguments, it doesn’t change if any argument is multiplied by a non-zero element of $K$, and it vanishes if the arguments do not span $V$. An element $g \in \text{GL}(n, K)$ acts on a basic $k$-sharbly by $[v_1, \ldots, v_{n+k}] g = [v_1 g, \ldots, v_{n+k} g]$. The $\mathbb{Z}$-span of the basic $k$-sharblies, subject to these relations, is by definition $Sh^n_k(V)$. The boundary map $Sh^n_{k+1}(V) \rightarrow Sh^n_k(V)$ is given by $[v_1, \ldots, v_{n+k+1}] \mapsto \sum_i (-1)^i [v_1, \ldots, \hat{v}_i, \ldots, v_{n+k+1}]$.

The module $St_n(V)$, which is free as a $\mathbb{Z}$-module, is isomorphic to a $\text{GL}(n, K)$-module through the cokernel of the boundary map $Sh^n_0(V) \rightarrow Sh^n_0(V)$. For $[v_1, \ldots, v_n] \in Sh^n_0(V)$, we denote the image of $[v_1, \ldots, v_n]$ in the cokernel $St_n(V)$ by $\{v_1, \ldots, v_n\}$.

If $K = \mathbb{Q}$, Borel-Serre duality, as improved by Brown [6, X.3.6], gives us Hecke-equivariant isomorphisms $H_1(\Gamma, St_n \otimes M) \approx H^{n(n-1)/2}(-\Gamma, M)$ for any subgroup of finite index $\Gamma \subset \text{GL}(n, \mathbb{Z})$ and any $S$-module $M$ on which $(n+1)!$ acts invertibly.

We know that

$$\cdots \rightarrow Sh^n_1(V) \rightarrow Sh^n_{i-1}(V) \rightarrow Sh^n_i(V) \rightarrow St_n(V) \rightarrow 0$$

is an exact sequence of $\text{GL}(n, K)$-modules.

Sending $(v_1, \ldots, v_{n+k})$ to $[v_1, \ldots, v_{n+k}]$ defines an injective map of $\text{GL}(n, K)$-modules $\iota : X^n_{n+k-1} \rightarrow Sh_k$. Note that $\iota$ induces an isomorphism $X^n_{n-1} \approx Sh_0$.

We now set $n = 3$ and $V = K^3$, dropping the $n$ and the $V$ from the notation $Sh^n_0(V)$. Define a new complex $X$ of $\text{GL}(3, K)$-modules as follows. For $i \geq 2$, $X_i = Sh_{i-2}$ and the boundary map $X_{i+1} \rightarrow X_i$ is the same as in the sharbly complex. We define $X_0 = \mathbb{Z}^9$ with the same augmentation map $\varepsilon : X_0 \rightarrow \mathbb{Z}$. It remains to define $X_1$ and the boundary maps $X_2 \rightarrow X_1 \rightarrow X_0$.

Let $\mathbb{P}^d$ denote the set of planes in $K^3$. We set $X_1 = \bigoplus_{H \in \mathbb{P}^d} St_2(H)$. An element $g \in \text{GL}(3, K)$ acts on $\{a, b\} \in X_1$ by sending it to $\{ag, bg\}$. 

REDUCIBLE GALOIS REPRESENTATIONS AND THE HOMOLOGY OF $\text{GL}(3, \mathbb{Z})$
Note that $X_2$ is generated freely over $\mathbb{Z}$ by “generic sharblies”, i.e. \([a, b, c]\) such that the determinant of the matrix with rows \([a, b, c]\) is nonzero. Define the boundary map $\partial_2 : X_2 \to X_1$ by \([a, b, c] \mapsto \{a\} + \{b, c\} + \{c, a\}\) for any generic \([a, b, c]\). It is well-defined and is $GL(3, K)$-equivariant. We define the boundary map $\partial_1 : X_1 \to X_0$ by $\{a\} \mapsto (b) - (a)$.

**Lemma 7.1.** Let $R$ be the submodule of $X^g_1$ generated by
\[
\{(a, b) + (b, c) + (c, a) \mid H \in \mathbb{P}^*, \ a, b, c \in H, \ (a, b, c) \in X^g_2(H)\}.
\]
For $H \in \mathbb{P}^*$ and $a, b \in H$, let $\psi(a, b) = \{a, b\} \in St_2(H)$. Then
(a) The map $\psi : X^g_1 \to X_1$ induces an isomorphism of $GL(3, K)$-modules $\phi : X^g_1/R \to X_1$.

(b) The boundary map in the generic complex $X^g_1 \to X^g_0 = X_0$ induces the $GL(3, K)$-equivariant boundary map $\partial_1 : X_1 \to X_0$ after identifying $X_1$ with $X^g_1/R$ via $\phi$.

**Proof.** (a) The map $\psi$ is clearly $GL(3, K)$-equivariant and surjective. If $t \in X^g_1$, we can write $t = \sum_{H \in \mathbb{P}^*} t_1$, where $t_H$ is supported on symbols \((u, v)\) with $u, v \in H$. Then $\psi(t) = 0$ if and only if $\psi_H(t) = 0$ for all $H$. For each $H$, $St_2(H)$ is the cokernel of the boundary map $\beta : Sh_1(H) \to Sh_0(H)$. The image of $\beta$ is generated by the basic relations \((u, v) + (v, w) + (w, u)\) where \((u, v, w)\) runs over triples in $H$ such that $u, v, w$ generate pairwise distinct lines. Therefore the kernel of $\psi$ is exactly $R$.

(b) The boundary map $\partial_1 : X_1 \to X_0$ in $X^g$ sends $(a, b)$ to $(b) - (a)$. This contains $R$ in its kernel, and induces a map $X_1/R \to X_0$ which clearly becomes $\partial_1$ after the identification via $\phi$. It is obviously $GL(3, K)$-equivariant. \hfill $\square$

**Lemma 7.2.** The sequence of $\mathbb{Z}[GL(3, K)]$-modules
\[
\cdots \to X_i \to X_{i-1} \to \cdots \to X_2 \to X_1 \to X_0 \to \mathbb{Z} \to 0
\]
is exact.

**Proof.** Since the sharblies sequence is exact and the augmentation map is surjective, the only nodes which need checking are those at $X_i$, $i = 0, 1, 2$. Denote the boundary map from $X_i \to X_{i-1}$ by $\partial_i$. Denote the boundary maps in $X^g$ by $\partial^g$. We identify $X^g_{k+2}$ with its image in $Sh_k$ via $\iota$, under which identification (1) $\partial^g_{k+2} = \partial_k|X^g_{k+2}$ and (2) $X^g_3$ and $Sh_0$ are isomorphic.

Node at $X_0$: Clearly $\varepsilon \circ \partial_1 = 0$. Let $x \in X_0$. If $\varepsilon(x) = 0$, there exists $y \in X^g_1$ with $\partial_1^g(y) = x$ because $X^g$ is exact. Then $\partial_1^g(\psi y) = x$.

Node at $X_1$: Clearly $\partial_1 \circ \partial_1 = 0$. Let $y \in X_1$ such that $\partial_1(y) = 0$. Choose $y' \in X^g_1$ such that $y = \psi y'$. Then $\partial_1^g(y') = 0$. Hence there exists $z \in X^g_2 = Sh_0 = X_2$ with $\partial_1^g(z) = y'$ because $X^g$ is exact. Then $\partial_1(z) = y$.

Node at $X_2$: Clearly $\partial_2 \circ \partial_2 = 0$. Now suppose $x \in X_2$ such that $\partial_2(x) = 0$. Then $\partial_2^g(x) \in R$. We will show (*) for every $x \in R$ there exists $\tilde{r} \in X_3 = Sh_1$ such that $r = \partial_2^g(\partial_3(\tilde{r})) \in X^g_1$. Then $\partial_2^g(x - \partial_3(\partial_2^g(x))) = 0$. Therefore there exists $t \in X^g_3 \subset X_3$ such that $\partial_3(t) = \partial_2^g(t) = x - \partial_3(\partial_2^g(x))$. Then $\partial_3(t + \partial_2^g(x)) = x$.

Proof of (*): Let $H \in \mathbb{P}^*$, $a, b, c \in H$, $(a, b, c) \in X^g_2(H)$. (The last condition just means that $a, b, c$ are pairwise distinct.) It suffices to let $r = (a, b) + (b, c) + (c, a)$ and find $\tilde{r}$. Pick $v \not\in H$. Set $\tilde{r} = [a, b, c, v]$. Then $\partial_3(\tilde{r}) = [b, c, v] - [a, c, v] + [a, b, v]$ since $[a, b, c] = 0$ in $Sh_0$. Hence $\partial_3(\partial_3(\tilde{r})) = r$. \hfill $\square$
A straightforward attempt to generalize this lemma for \( n > 3 \) doesn’t work. The reason the lemma works for \( n = 3 \) is that any tuple of pairwise distinct lines in a plane is generic.

We now specialize to the case in which \( K = \mathbb{Q} \), and we let \( \Gamma \) be a subgroup of finite index in \( \text{SL}(3, \mathbb{Z}) \). If \( M \) is any \( S \)-module, \( X_1 \otimes M = \bigoplus_{H \in P^*} \text{St}_2(H) \otimes M \). If \( H \) is a fixed plane, let \( \text{P}_H \) be its stabilizer in \( \text{GL}(3) \) and \( \text{U}_H \) the unipotent radical of \( \text{P}_H \). Since \( \text{SL}(3, \mathbb{Z}) \) acts transitively on \( P^* \), \( \Gamma \) has a finite number of orbits in \( P^* \), represented by the planes in a finite set, say \( \mathbb{H}(\Gamma) \). It follows from [6, Corollary III.5.4] that \( X_1 \otimes M \) is isomorphic as \( S \)-module to a finite sum of induced modules \( \bigoplus_{H \in \mathbb{H}(\Gamma)} \text{Ind}^{\Gamma}_{\text{P}_H \cap \Gamma} \text{St}_2(H) \otimes M \), where \( \text{U}_H \cap \Gamma \) acts trivially on \( \text{St}_2(H) \).

8. The spectral sequence for \( \text{GL}_3 \)

Now set \( K = \mathbb{Q} \). Let \( F \) be a resolution of \( \mathbb{Z} \) by \( \mathbb{Z}[S] \)-modules which are free as \( \mathbb{Z}[\Gamma] \)-modules. Let \( W = X \otimes M \) with the diagonal \( S \)-action and form the double complex \( W \otimes F \). To compute the homology, we use the spectral sequence from [6, VII.5.(5.3)]. All the differentials on the \( E_i \) page, \( i \geq 1 \) are Hecke-equivariant because they are all induced by the differential on the double complex, which is an \( S \)-module. From now on, we assume that \( 6 \) is invertible on \( M \).

Since \( X \) is a resolution of \( \mathbb{Z} \) by free \( \mathbb{Z} \)-modules, we see that \( W \) is a resolution of \( M \). Therefore there is a weak equivalence from \( W \) to the chain complex consisting of \( M \) concentrated in dimension 0, so that by [6, VII.5.2], \( H^*(\Gamma, W) \approx H^*(\Gamma, M) \).

Hence, in the spectral sequence, we have

\[ E^1_{jq} = H_q(\Gamma, X_j \otimes M) \Rightarrow H_{j+q}(\Gamma, M). \]

For each \( j \neq 1 \), the \( j \)-chains \( X_j \) are isomorphic to a direct sum of induced representations

\[ X_j \approx \bigoplus_{\sigma \in C_j} \text{Ind}^{\Gamma}_{\Gamma_\sigma} M_\sigma \]

where \( C_0 \) is the set of vertices in \( Y_0^3 \); if \( j \geq 2 \), \( C_j \) is the set of basic sharblies \([v_1, \ldots, v_{j+1}]\) with \( v_1, \ldots, v_{j+1} \) vectors in \( \mathbb{Q}^3 \) that span \( \mathbb{Q}^3 \); \( \Gamma_\sigma \) is the stabilizer of \( \sigma \), \( \varepsilon_\sigma \) is the orientation character recording how \( \Gamma_\sigma \) acts on \( \sigma \); and \( M_\sigma = M \otimes \varepsilon_\sigma \).

As in [6, VII.7] we have from Shapiro’s lemma:

\[ E^1_{j} = \bigoplus_{\sigma \in C_j} H_q(\Gamma_\sigma, M_\sigma) \]

if \( j \neq 1 \). We see easily that \( H_q(\Gamma_\sigma, M_\sigma) = 0 \) in the following cases:

1. For \( q > 3 \) and \( \sigma \in C_0 \), since the virtual cohomological dimension of the stabilizer \( P \) of a 0-cycle is 3 and any torsion element of \( P \) has order invertible on \( M \).
2. For \( q \geq 1 \) and \( \sigma \in C_j \) with \( j > 1 \), since the stabilizer of a basic sharbly is a finite subgroup of \( \text{GL}(3, \mathbb{Z}) \), hence of order invertible in \( M \) (see [6, Corollary III.10.2]).

The column with \( j = 1 \) will be treated later.

We then have the \( E^1 \) page of our spectral sequence as follows, where \( H_i(\sigma) := H_i(\Gamma_\sigma, M_\sigma) \).
where \( U \) is a Galois representation that is a sum of three characters.

\[ \sigma \in \mathbb{G} \]

The homological dimension of \( U \) also occurs in \( \mathbb{S} \), which 6 acts invertibly. Also, the homological dimension of \( U \) is 2. Therefore \( j \in \mathbb{S} \), occurring in \( H \), and all systems of Hecke eigenvalues appearing in them are attached to Galois representations that are sums of characters.

**Theorem 8.1.** Assume that \( M \) is an admissible \( \mathbb{F}_p[S] \)-module with \( p > 3 \). Then the \( E_{10}^2 \) and \( E_{13}^2 \) terms of the spectral sequence are finite dimensional \( \mathbb{F}_p \)-vector spaces and all systems of Hecke eigenvalues appearing in them are attached to Galois representations that are sums of characters.

**Proof.** Since the sharbly complex is a resolution of \( \mathcal{S} := \mathcal{S} \mathcal{T} \mathcal{S} (\mathbb{Q}^3) \), and 6 acts invertibly on \( M \), \( E_{10}^2 \approx H_2(\Gamma, \mathbb{S} \mathcal{T} \mathcal{S} \mathcal{M}) \), cf. Corollary 8 in [4]. Borel-Serre duality then gives an isomorphism of Hecke-modules \( E_{10}^2 \approx H^1(\Gamma, \mathbb{M}) \), which is a finite dimensional \( \mathbb{F}_p \)-vector space. By [2, Theorem 4.1.5], \( H^1(\Gamma, \mathbb{M}) \) is a sum of generalized Hecke eigenspaces, and any eigenclass appearing in \( H^1(\Gamma, \mathbb{M}) \) has an attached Galois representation that is a sum of three characters.

We now consider \( E_{13}^1 \). Recall that there is a finite set of planes \( \mathbb{H} (\Gamma) \) such that

\[ X_1 \otimes M \approx \bigoplus_{H \in \mathcal{H}(\Gamma)} \text{Ind}_{\Gamma}^{\mathbb{H}(\Gamma)} \mathcal{S} \mathcal{T} \mathcal{M} \]

where \( U_H \cap \Gamma \) acts trivially on \( \mathcal{S} \mathcal{T} \mathcal{M} \).

We define the \( S \)-sheaf of \( \mathbb{F}^+ \) by \( H \mapsto \mathcal{S} \mathcal{T} \mathcal{M} \). We then use Theorem 3.1. By Shapiro’s lemma, \( E_{13}^1 \approx \bigoplus_{H \in \mathcal{H}(\Gamma)} H_3(\Gamma, \mathcal{S} \mathcal{T} \mathcal{M}) \).

Let \( \Gamma_H \) denote \( (\mathcal{P} \cap \Gamma)/(\mathcal{U} \cap \Gamma) \). It is isomorphic to a congruence subgroup of \( \mathbb{GL}(2, \mathbb{Z}) \). We have the Hochschild-Serre spectral sequence

\[ E_{0q}^2 = H_j(\Gamma_H, \mathcal{U} \cap \Gamma, \mathcal{S} \mathcal{T} \mathcal{M}) \implies H_{j+q}(\mathcal{P} \cap \Gamma, \mathcal{S} \mathcal{T} \mathcal{M}). \]

Because \( U_H \cap \Gamma \) acts trivially on \( \mathcal{S} \mathcal{T} \mathcal{M} \), we have \( H_q(\mathcal{U} \cap \Gamma, \mathcal{S} \mathcal{T} \mathcal{M} \otimes M) = \mathcal{S} \mathcal{T} \mathcal{M} \otimes C(\mathcal{H}, q) \) where \( C(\mathcal{H}, q) = H_q(\mathcal{U} \cap \Gamma, M) \) is an admissible \( \Gamma \)-module on which 6 acts invertibly. Also, the homological dimension of \( U_H \cap \Gamma \) is 2. Therefore the only nonzero term in the \( E^2 \) page of the Hochschild-Serre spectral sequence when \( j + q = 3 \) occurs when \( j = 1, q = 2 \). So any packet of Hecke eigenvalues occurring in

\[ H_3(\mathcal{P} \cap \Gamma, \mathcal{S} \mathcal{T} \mathcal{M} \otimes M) \]

also occurs in

\[ H_1(\Gamma_H, \mathcal{S} \mathcal{T} \mathcal{M} \otimes C(\mathcal{H}, 2)). \]
Reducible Galois Representations and the Homology of $GL(3, \mathbb{Z})$

By Borel-Serre duality, this is isomorphic to $H^0(\Gamma_H, C(H, 2))$ and by [2, Theorem 4.1.4], this is a sum of generalized Hecke eigenspaces, and any system of Hecke eigenvalues occurring here has as attached Galois representation a sum of two characters. At this point we also see that $E_{13}^1$ is finite dimensional over $\mathbb{F}_p$.

Now we use Theorem 3.1 to compute the Hecke operators on $E_{13}^1$, following the same outline as that used below for $E_{03}^1$. This proves that $E_{13}^1$ is a sum of generalized Hecke eigenspaces, and any system of Hecke eigenvalues occurring here has as attached Galois representation a sum of three characters. 

9. Reducible Galois Representations

We continue to assume that $6$ is invertible on $M$. We note that for $\sigma \in C_0$, the orientation character is trivial, so that $M_\sigma = M$. For any $d\mid N, d > 0$, we have taken $(1 : d : 0)$ (with stabilizer $P_d$) as a representative of its orbit in $C_0$. Therefore, $E_{03}^1$ contains $H_3(P_1 \cap \Gamma_0(3, N), M)$ as a direct summand. We will find the system of Hecke eigenvalues in which we are interested in this summand.

Let $\rho : G_Q \rightarrow GL_3(\mathbb{F}_p)$ be a Galois representation that can be written as a direct sum $\rho = \sigma \oplus \psi$, where $\sigma : G_Q \rightarrow GL_3(\mathbb{F}_p)$ is an odd, irreducible, two-dimensional representation, and $\psi : G_Q \rightarrow GL_1(\mathbb{F}_p)$ is a character. Let $N_1$ be the Serre conductor of $\sigma$, and let $d$ be the conductor of $\psi$, and assume that $N_1d$ is squarefree. We will show that $\rho = \sigma \oplus \psi$ is attached to a Hecke eigenclass in $E_{3,0}^1$ with $\Gamma = \Gamma_0(3, N)$, and coefficient module as predicted by Conjecture 2.4.

We set $\tau = \sigma \otimes \omega^{-1}$, so that $\rho = (\tau \otimes \omega) \oplus \psi$. Assume that the predicted weight of $\tau$ in Serre’s conjecture is $F(a, b)$, with $0 \leq a - b \leq p - 1$ and $0 \leq b < p - 1$. Then if $\tau$ is ordinary, we have

$$\tau|_{I_p} \sim \begin{pmatrix} \omega^{a+1} & * \\ 0 & \omega^b \end{pmatrix}$$

and if $\tau$ is supersingular, we have

$$\tau|_{I_p} \sim \begin{pmatrix} \omega_2^{(a+1)+bp} & 0 \\ 0 & \omega_2^{(a+1)+bp} \end{pmatrix}.$$  

Note that $\tau$ has Serre conductor $N_1$ (the same as $\sigma$), and we may factor $\det(\tau) = \omega^{(a+b+1)}\chi_1$, where the conductor of $\chi_1$ divides $N_1$. We may also factor $\psi = \omega^c\chi_0$, where $\chi_0$ has conductor $d$ and $0 \leq c < p - 1$. We will denote by $\lambda_\ell$ the trace of $\tau(Frob_\ell)$. We note then that the trace of $\rho(Frob_\ell)$ is equal to $\ell\lambda_\ell + \chi_0(\ell)\ell^c$ and its cotrace (the coefficient of $X^2$ in $\det(I - \rho(Frob_\ell))$) is equal to $\ell(\chi_1(\ell)\ell^{a+b+2} + \chi_0(\ell)\ell^c\lambda_\ell)$. Examining Conjecture 2.4, we see that the predicted weight of $\rho$ is $F(a, b, c)$ (where, if necessary we add $p - 1$ to both $a$ and $b$ so that $(a, b, c)$ will be $p$-restricted, and note that by our conventions this change does not change the module $F(a, b)$). In addition, the nebentype of $\rho$ is $\chi_0\chi_1$, and the level of $\rho$ is $N = N_1d$.

We know by Serre’s conjecture, which is now a theorem, that $\tau$ is attached to a Hecke eigenclass in $H_1(\Gamma_0(2, N_1), F(a, b)_{\chi_1})$. Since $\tau$ is absolutely irreducible, it is attached to a cusp form, and by Eichler-Shimura it is also attached to a Hecke eigenclass $f$ in

$$H_1(\Gamma_0(2, N_1)^\pm, F(a, b)_{\chi_1}).$$
Denote by $M$ the module $F(a, b, c)_\chi$. We now consider the Hochschild-Serre spectral sequence for the homology of the exact sequence

$$1 \to U_d \cap \Gamma_0(3, N) \to P_d \cap \Gamma_0(3, N) \xrightarrow{\psi^2} \Gamma_0(2, N/d) \xrightarrow{\chi} 1$$

with coefficients in $M$.

This spectral sequence degenerates at $E^2$ because it is only two columns thick, and we have

$$E^2_{pq} = H_q(U_d \cap \Gamma_0(3, N), M).$$

The desired Hecke eigenclass stems from $H_1(\Gamma_0(2, N/d) \to H_2(U_d \cap \Gamma_0(3, N), M))$.

Note that since $U_d \cap \Gamma_0(3, N)$ is an abelian group of rank 2, we have $U_0(\Gamma_0(3, N), M) = M_{U_0(\Gamma_0(3, N), M)}$.

By Theorem 5.1, $M_{U_0(\Gamma_0(3, N), M)} \cong M(c; a, b)$ as a $GL_2$-module, with an additional action of $GL_1$ through the $c$-power map. Hence, we are interested in finding a certain class in $H_1(\Gamma_0(2, N/d) \to H_2(U_0(\Gamma_0(3, N), M))$, where $P_d$ acts on $F(a, b)_\chi$, (and hence on the cohomology) as described in Theorem 5.1.

We have chosen $f \in H_1(\Gamma_0(2, N_1) \to F(a, b)_\chi)$ to correspond to $\tau$, so that for the two dimensional Hecke operator $T_\ell = T(\ell, 1)$, we have $f \mid T_\ell = \lambda_\ell f$, where $\lambda_\ell = \text{Tr}(\tau(Frob))$. We now examine how the three-dimensional Hecke operators $T(\ell, 1)$ and $T(\ell, 2)$ act on $f$.

For each prime $\ell \nmid pN$ and for each coset representative in $T(\ell, 1)$, we apply Theorem 4.3 to translate the coset representatives $s$ by an element $\gamma \in \Gamma_0(3, N)$ into $P_d$ and then let $x = g_d s \gamma g_d^{-1} \in P_0$, and obtain:

If $s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha_1 & \alpha_2 & \ell \end{pmatrix}$ then (by case 1 of Theorem 4.3),

$$\psi^1_d(s \gamma) = \psi^1_0(x) = 1 \quad \text{and} \quad \psi^2_d(s \gamma) = \psi^2_0(x) = \begin{pmatrix} 1 \\ \alpha_2 - \alpha_1 d \\ \ell \end{pmatrix}.$$

If $s = \begin{pmatrix} 1 & 0 & 0 \\ \alpha_0 & \ell & 0 \\ 0 & 0 & 1 \end{pmatrix}$ with $\ell \nmid \alpha_0 d + 1$, then (by case 3 of Theorem 4.3)

$$\psi^1_0(x) = 1 \quad \text{and} \quad \psi^2_0(x) = \begin{pmatrix} \ell \\ 0 \\ 1 \end{pmatrix}.$$

If $s = \begin{pmatrix} 1 & 0 & 0 \\ \alpha_0 & \ell & 0 \\ 0 & 0 & 1 \end{pmatrix}$ with $\ell | \alpha_0 d + 1$, then (by case 4 of Theorem 4.3)

$$\psi^1_0(x) = \ell \quad \text{and} \quad \psi^2_0(x) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

If $s = \begin{pmatrix} \ell & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ then (by case 2 of Theorem 4.3)

$$\psi^1_0(x) = 1 \quad \text{and} \quad \psi^2_0(x) = \begin{pmatrix} \ell \\ 0 \\ 1 \end{pmatrix}.$
Define the $S$-sheaf on $C_0$ by $y \mapsto H_1(P_y \cap \Gamma_0(3, N))$, where $P_y$ is the stabilizer of $y$. Use Theorem 3.1 and the fact that $N$ is assumed to be squarefree. Then by Section 4, we know that $s$ preserves the $\Gamma$-orbits of $C_0$ and $T(\ell, 1) = \oplus T_{yy}$ where $y$ runs through a set of representatives of the $\Gamma$-orbits. Since $f$ is supported on the orbit of $(1 : d : 0)$ and the $g_{yy}$ are just the $s_{r\gamma}$, we obtain from Theorems 3.1 and 5.1(3),

$$f|T(\ell, 1) = \sum_{\alpha_1, \alpha_2} f\bigg|_{\chi_1} \left( \begin{array}{ccc} 1 & 0 & 0 \\ \alpha_2 - \alpha_1 d & \ell \\ \ell \end{array} \right) + \chi_0(\ell)\ell^c f\bigg|_{\chi_1} I + f\bigg|_{\chi_1} \left( \begin{array}{ccc} \ell & 0 & 0 \\ 0 & 1 \\ 0 \end{array} \right)$$

$$= \ell \left( f\bigg|_{T_2} - f\bigg|_{\chi_1} \left( \begin{array}{ccc} \ell & 0 & 0 \\ 0 & 1 \\ 0 \end{array} \right) \right) + (\ell - 1) f\bigg|_{\chi_1} \left( \begin{array}{ccc} \ell & 0 & 0 \\ 0 & 1 \\ 0 \end{array} \right) + \chi_0(\ell)\ell^c f$$

$$= (\ell\lambda_\ell + \chi_0(\ell)\ell^c)f.$$

Similarly, for $T(\ell, 2)$:

If $s = \left( \begin{array}{ccc} 1 & 0 & 0 \\ \alpha_0 & \ell & 0 \\ \alpha_1 & 0 & \ell \end{array} \right)$ with $\ell \nmid \alpha_0 d + 1$, then (by case 3 of Theorem 4.3)

$$\psi^1_0(x) = 1 \text{ and } \psi^2_0(x) = \left( \begin{array}{ccc} \ell & 0 \\ -\alpha_1 d & \ell \end{array} \right).$$

If $s = \left( \begin{array}{ccc} 1 & 0 & 0 \\ \alpha_0 & \ell & 0 \\ \alpha_1 & 0 & \ell \end{array} \right)$ with $\ell|\alpha_0 d + 1$, then (by case 4 of Theorem 4.3)

$$\psi^1_0(x) = \ell \text{ and } \psi^2_0(x) = \left( \begin{array}{ccc} 1 & 0 \\ -\alpha_1 d & \ell \end{array} \right).$$

If $s = \left( \begin{array}{ccc} \ell & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \alpha_2 & \ell \end{array} \right)$ then (by case 2 of Theorem 4.3)

$$\psi^1_0(x) = 1 \text{ and } \psi^2_0(x) = \left( \begin{array}{ccc} \ell & 0 \\ \alpha_2 \ell & \ell \end{array} \right).$$

If $s = \left( \begin{array}{ccc} \ell & 0 & 0 \\ 0 & \ell & 0 \\ 0 & 0 & 1 \end{array} \right)$ then (by case 1 of Theorem 4.3)

$$\psi^1_0(x) = \ell \text{ and } \psi^2_0(x) = \left( \begin{array}{ccc} \ell & 0 \\ 0 & 1 \end{array} \right).$$
Hence,

\[ f \mid_{T(\ell, 2)} = \sum_{\alpha_0, \alpha_1 : \alpha_0 \neq \alpha_1 \neq \alpha_0} f \bigg|_{\chi_1} \begin{pmatrix} \ell & 0 \\ 0 & \ell \end{pmatrix} + \sum_{\alpha_1} \chi_0(\ell) \ell^c f \bigg|_{\chi_1} \begin{pmatrix} 1 & 0 \\ -\alpha_1 d & \ell \end{pmatrix} + \sum_{\alpha_2} f \bigg|_{\chi_1} \begin{pmatrix} \ell & 0 \\ 0 & \ell \end{pmatrix} + \chi_0(\ell) \ell^c f \bigg|_{\chi_1} \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} \]

\[ = \ell^2 f \bigg|_{\chi_1} \begin{pmatrix} \ell & 0 \\ 0 & \ell \end{pmatrix} + \chi_0(\ell) \ell^c f \bigg|_{T\ell} = (\chi_1(\ell) \ell^{a+b+2} + \chi_0(\ell) \ell^c) f. \]

Therefore \( f \) is attached to \( \omega_\tau \oplus \omega^c \chi_0 = \rho \), which has predicted weight \( F(a, b, c) \).

Since \( f \) is not attached to a direct sum of characters, by Theorem 8.1 it not only appears in \( E_{0,3} \), but also survives into \( E_{\infty,3} \), and hence appears in \( H_3(\Gamma_0(3, N), \chi) \).

We have thus proved the following theorem:

**Theorem 9.1.** Let \( p > 3 \) and let \( \rho : \mathbb{G}_\mathbb{Q} \to \text{GL}(3, \mathbb{F}_p) \) be a Galois representation of squarefree conductor \( N \) and nebentype \( \chi \) that decomposes as a sum of a character and an irreducible odd two-dimensional Galois representation \( \tau \). Then \( \rho \) is attached to a cohomology eigenclass in \( H_3(\Gamma_0(3, N), V_\chi) \), where \( V \) is the first of the two weights predicted by conjecture 2.4.

We note that the second of the two weights predicted by Conjecture 2.4 can also be shown to work. Let \( ^t\rho^{-1} \otimes \omega^2 \) be the twisted contragredient of \( \rho \). This representation will still be a sum of a two-dimensional, odd, irreducible representation and a character, and as such, will be attached to an eigenclass for its first predicted weight of Conjecture 2.4, by Theorem 9.1. Since Conjecture 2.4 is compatible with duality according to the prescription in [5, Proposition 2.8], we find that \( \rho \) is attached to an eigenclass in the dual of this weight. A simple computation shows that this dual is exactly the second predicted weight for \( \rho \). Hence, \( \rho \) is attached to eigenclasses in both of the weights described in Conjecture 2.4, concluding the proof that Conjecture 2.4 is true for representations of squarefree conductor.

10. Appendix: Proof of Theorem 4.3

**Proof.** To prove case 1, we take \( \gamma = I \), since \( s \) is already in \( P_d \). Hence \( x = g_d s g_d^{-1} \), and we find that \( x_{11} = \ell_1 \), and \( \psi_0^2(x) = \begin{pmatrix} \ell_2 & 0 \\ c - bd & \ell_3 \end{pmatrix} \).

For the other cases, note that a matrix is in \( P_0 g_d \) if and only if it is of the form

\[
\begin{pmatrix}
  r & r d & 0 \\
  * & * & * \\
  * & * & *
\end{pmatrix}.
\]

Since we want

\[
g_d s \gamma = \begin{pmatrix}
  A(\ell_1 + ad) + C \ell_2 d & B(\ell_1 + ad) + D \ell_2 d & 0 \\
  * & * & 0 \\
  * & * & *
\end{pmatrix} \in P_0 g_d,
\]
we see that we must have \( B(\ell_1 + ad) + D\ell_2d = d(A(\ell_1 + ad) + C\ell_2d) \). Writing this as a matrix equation, we have that

\[
(\ell_1 + ad, d\ell_2) \begin{pmatrix} A & B \\ C & D \end{pmatrix} = r(1, d)
\]

for some \( r \). We will assume that \( r > 0 \) (changing the signs of \( A, B, C, D \) if needed).

Since the matrix \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) must have determinant 1, we get

\[
(\ell_1 + ad, d\ell_2) = r(1, d) \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}.
\]

We now specialize to case 2, in which \( \ell_1 = \ell \) is prime, \( \ell_2 = 1 \) and \( a = 0 \). Then we have

\[
(\ell, d) = r(1, d) \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}.
\]

Since \((\ell, d)\) and \((1, d)\) are both primitive, we must have \( r = 1 \).

Multiplying, we see that

\[
\ell = D - Cd, \text{ and } d = -B + Ad
\]

Solving for \( A \) and \( D \) (and using that \( B = kN \) for some \( k \in \mathbb{Z} \) since \( \gamma \in \Gamma_0(N) \)) we obtain

\[
A = 1 + \frac{kN}{d}, \quad B = kN, \quad D = \ell + Cd,
\]

and

\[
1 = AD - BC = \left(1 + \frac{kN}{d}\right)(\ell + Cd) - kNC = \left(\frac{kN}{d} + 1\right)\ell + Cd
\]

Now, using the Chinese Remainder Theorem to choose \( C \) so that

\[
Cd \equiv 1 \pmod{\ell} \text{ and } Cd \equiv 1 - \ell \pmod{N/d}
\]

we may choose

\[
k = \frac{-Cd - \ell + 1}{(N/d)\ell}
\]

and we have our desired

\[
\gamma = \begin{pmatrix} A & kN \\ C & D \end{pmatrix}.
\]

Note that \( x_{11} = r \), and by determinant considerations, we must have \( x_{11}x_{22} = \ell_1\ell_2 \). Hence, in case 2, \( x_{11} = 1 \), \( x_{22} = \ell \), and a quick calculation shows that

\[
\psi_0^2(x) = \begin{pmatrix} \ell & 0 \\ c\ell - bd & \ell_3 \end{pmatrix}.
\]

In case 3, we proceed similarly. We have

\[
(1 + ad, \ell d) = r(1, d) \begin{pmatrix} D & -B \\ -C & A \end{pmatrix},
\]

with both \((1 + ad, \ell d)\) and \((1, d)\) primitive, so that \( r = 1 \). Hence \( ad + 1 = D - Cd \) and \( \ell d = -B + Ad \). We set \( B = kN \) and solve for \( A = k(N/d) + \ell \) and \( D = (ad + 1) + Cd \).

Now, using the fact that we want

\[
1 = AD - BC = ((ad + 1)N/d)k + C(\ell d) + (ad + 1)\ell,
\]
and the fact that \((ad + 1)N/d, \ell d) = 1\), we see that integers \(C\) and \(k\) exist, so \(\gamma\) exists. We compute \(x\) and obtain \(\psi_0^1(x) = r = 1\) and \(\psi_0^2(x) = \begin{pmatrix} \frac{\ell}{c(ad + 1) - bd} & 0 \\ \ell \end{pmatrix} \).

Finally, for case 4, we again proceed in a similar fashion. We find that \((1 + ad, \ell d) = 1\) and \(\ell (D - Cd) = (1 + ad)\) and \(\ell (B + Ad) = \ell d\).

Under the assumption that \(\gamma\) exists, \(r = \ell\) (since \((1, d)\) is primitive, but the GCD \((1 + ad, \ell d) = \ell\)). Hence, computing \(x\) using the values for \(A, B, C, D\) below, we find that \(\psi_0(x) = \ell\) and \(\psi_0^2(x) = \begin{pmatrix} 1 \\ c \ell \end{pmatrix}

To show the existence of \(\gamma\), we note that \(\ell (D - Cd) = (1 + ad)\) and \(\ell (B + Ad) = \ell d\).

Setting \(B = kN\) and solving, we find that

\[
\begin{align*}
A &= 1 + k \frac{N}{d}, \\
B &= kN, \\
D &= \frac{1 + ad}{\ell} + Cd.
\end{align*}
\]

Then

\[
1 = AD - BC = \frac{1 + ad}{\ell} + k \left( \frac{N}{d} \left( \frac{1 + ad}{\ell} \right) + Cd. \right)
\]

Since \(d\) and \(\frac{N}{d} \left( \frac{1 + ad}{\ell} \right)\) are relatively prime, a solution exists for \(k\) and \(C\). \(\square\)

REFERENCES

[1] A. Ash, Direct sums of mod \(p\) characters of \(\text{Gal} (\overline{\mathbb{Q}}/\mathbb{Q})\) and the homology of \(GL(n, \mathbb{Z})\), to appear Comm. in Alg.

[2] A. Ash, Galois representations attached to mod \(p\) cohomology of \(GL(n, \mathbb{Z})\), Duke Math. J. 65 (1992) 235-255.

[3] A. Ash, D. Doud, and D. Pollack, Galois representations with conjectural connections to arithmetic cohomology, Duke Math. J. 112 (2002) 521-579.

[4] A. Ash, P. Gunnells, M. McConnell, Resolutions of the Steinberg module for \(GL(n)\), J. of Algebra 249 (2012) 380-390.

[5] A. Ash and W. Sinnott, An analogue of Serre’s conjecture for Galois representations and Hecke eigenclasses in the mod \(p\) cohomology of \(GL(n, \mathbb{Z})\), Duke Math. J. 106 (2000) 1-24.

[6] K. Brown, Cohomology of Groups, Springer, New York 1982.

[7] S. Doty and G. Walker, The composition factors of \(F_p[x_1, x_2, x_3]\) as a \(GL(3, p)\)-module, J. Algebra, 147 (1992), 411-441.

[8] F. Herzig, The weight in a Serre-type conjecture for tame \(n\)-dimensional Galois representations, Duke Math. J. 149 (2009) 37-116.

[9] F. Herzig, A Satake isomorphism in characteristic \(p\), Compositio Math. 147 (2011) 263-283.

[10] J. Humphreys, Modular representations of finite groups of Lie type, Cambridge University Press, Cambridge, 2005.

[11] J. C. Jantzen, Representations of Algebraic Groups, Second Edition, American Mathematical Society, Providence, 2003.

[12] J.-P. Serre, Propriétés galoisiennes des points d’ordre fini des courbes elliptiques, Invent. Math. 15 (1972) 259-331.

[13] J.-P. Serre, Sur les représentations modulaires de degré 2 de \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\), Duke Math. J. 54 (1987) 179-200.

Boston College, Chestnut Hill, MA 02445
E-mail address: Avner.Ash@bc.edu

Brigham Young University, Provo, UT 84602
E-mail address: doud@math.byu.edu