CONVERGENCE OF THE FREE ENERGY FOR SPHERICAL SPIN GLASSES

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Abstract. We prove that the free energy of any spherical mixed $p$-spin model converges as the dimension $N$ tends to infinity. While the convergence is a consequence of the Parisi formula, the proof we give is independent of the formula and uses the well-known Guerra-Toninelli interpolation method. The latter was invented for models with Ising spins to prove that the free energy is super-additive and therefore (normalized by $N$) converges. In the spherical case, however, the configuration space is not a product space and the interpolation cannot be applied directly. We first relate the free energy on the sphere of dimension $N + M$ to a free energy defined on the product of spheres in dimensions $N$ and $M$ to which we then apply the interpolation method. This yields an approximate super-additivity which is sufficient to prove the convergence.

1. Introduction

In this work we consider the spherical mixed $p$-spin spin glass models. The limit of their free energy as the dimension tends to infinity is given by the celebrated Parisi formula [15, 16] or its representation by Crisanti and Sommers [8]. The formula was proved by Talagrand [20] after a breakthrough by Guerra [9] for models with even interactions and later extended to general mixtures by Chen [4], using the Aizenman-Sims-Starr representation [1] and ultrametricity [11, 12, 13].

Our goal in this note is only to prove the convergence of the free energy, but without relying on the heavy machinery which was developed to prove the Parisi formula. One of our main motivations comes from recent works [5, 6, 18, 19] on the Thouless-Anderson-Palmer (TAP) approach [22]. In [18] we proved a generalized TAP representation for the free energy of the spherical models and that for the maximal multi-samplable overlap the correction term in the representation coincides with the classical Onsager correction. Importantly, the proof of those results was independent of the Parisi formula, but for the latter result on the Onsager correction we had to assume that the free energy converges. In another work [19], we used the TAP representation to compute the free energy of the spherical pure $p$-spin models from the generalized TAP representation of [18], also there assuming the convergence of the free energy. Our main result in the current paper fills the gap and removes those assumptions from [18, 19] without appealing to the Parisi formula.

For models with Ising spins, defined on the hyper-cube $\Sigma_N = \{\pm 1\}^N$, the convergence of the free energy was proved by Guerra and Toninelli [10] who invented a simple, yet ingenious, interpolation technique to show that the (unnormalized) free energy is superadditive. The convergence immediately follows from superadditivity by invoking Fekete’s Lemma. The argument of [10] exploits the fact that the configuration space $\Sigma_{N+M} = \Sigma_N \times \Sigma_M$ in dimension $N + M$ is equal to the product of the configuration space in dimensions $N$ and $M$. In the spherical setting, this is no longer the case, and the method of [10] cannot be adapted directly. In our proof we therefore first relate the free energy in dimension $N + M$ to another free energy defined on the product of the spherical
configuration space in dimensions $N$ and $M$ (using the Hamiltonian in dimension $N + M$), to which we will be able to apply the Guerra-Toninelli interpolation technique.

The spherical mixed $p$-spin spin glass model is defined as follows. Suppose that $\gamma_p \geq 0$ is a sequence such that $\sum_{p=1}^{\infty} \gamma_p^p (1 + \epsilon)^p < \infty$ for some numbers $C$. The mixed $p$-spin Hamiltonian $H_N$ corresponding to the mixture $\xi(t) = \sum_{p=1}^{\infty} \gamma_p^p t^p$ is the random function on the sphere

$$ S_N := \{ \sigma = (\sigma_1, \ldots, \sigma_N) \in \mathbb{R}^N : \| \sigma \| = \sqrt{N} \}, $$

given by

$$ H_N(\sigma) = \sum_{p=1}^{\infty} \gamma_p N^{-\frac{p}{p+1}} \sum_{i_1, \ldots, i_p=1}^{N} J_{i_1, \ldots, i_p} \sigma_{i_1} \cdots \sigma_{i_p}, $$

where $J_{i_1, \ldots, i_p}$ are i.i.d. standard normal variables. An easy calculation shows that the covariance function of the centered Gaussian field $H_N(\sigma)$ is

$$ \mathbb{E} H_N(\sigma) H_N(\sigma') = N \xi(\mathbf{R}(\sigma, \sigma')) $$

where $\mathbf{R}(\sigma, \sigma') := \frac{1}{N} \sigma \cdot \sigma' := \frac{1}{N} \sum_{i \leq N} \sigma_i \sigma_i'$ is called the overlap of $\sigma$ and $\sigma'$.

The free energy is defined by

$$ F_N := \frac{1}{N} \mathbb{E} \log \int_{S_N} e^{H_N(\sigma)} d\mu_N(\sigma), $$

where $\mu_N$ is the uniform measure on $S_N$. The following is our main result.

**Theorem 1.** $F_N$ converges as $N \to \infty$.

At the very last step of the proof of Theorem 1 after we apply the Guerra-Toninelli interpolation, we will need to invoke Talagrand’s positivity principle in order to restrict to overlap values in $[0, 1]$ (on which $\xi(t)$ is convex). The positivity principle applies to general mixtures, if we add a certain perturbation to the Hamiltonian.

We will give the precise definition of the perturbed Hamiltonian $\bar{H}_N(\sigma)$ and its associated free energy $\bar{F}_N$, which satisfies

$$ \lim_{N \to \infty} | \bar{F}_N - F_N | = 0, $$

in a moment. Before that, we state the following approximate superadditivity of $\bar{F}_N$ and observe how the convergence of $\bar{F}_N$ follows from it.

**Proposition 2.** For any $N$ and $M$,

$$ (N + M) \bar{F}_{N+M} \geq N \bar{F}_N + M \bar{F}_M - C_{N,M}, $$

for some numbers $C_{N,M}$ such that

$$ \limsup_{M \to \infty} \limsup_{N \to \infty} \frac{C_{N,M}}{M} = 0. $$

**Proof of Theorem 1.** In light of (1.4), it is enough to show that $\bar{F}_N$ converges. By induction on $k$, for any $N'$, $M$ and $k$,

$$ (N' + kM) \bar{F}_{N'+kM} \geq N' \bar{F}_{N'} + kM \bar{F}_M - \sum_{i=0}^{k-1} C_{N'+iM,M}. $$

Let $\delta > 0$ be an arbitrary number. Choose some large $M$ such that

$$ \bar{F}_M > \limsup_{N \to \infty} \bar{F}_N - \delta \quad \text{and} \quad \limsup_{N \to \infty} \frac{C_{N,M}}{M} < \delta. $$

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2For mixtures such that $\xi(t)$ is convex on $[-1, 1]$ this step is not required and one can work with the original Hamiltonian without adding a perturbation.
Given some \( N \), let \( N' \in \{0, 1, \ldots, M - 1\} \) and \( k \geq 0 \) be the integers such that \( N = N' + kM \). Then by dividing both sides of the inequality above by \( N \) and taking limits we obtain that with \( M \) fixed,
\[
\lim_{N \to \infty} \inf \tilde{F}_N \geq \frac{C_{N,M}}{M} \geq \limsup_{N \to \infty} \tilde{F}_N - 2\delta.
\]
Since \( \delta > 0 \) is arbitrary, \( \tilde{F}_N \) converges and the theorem follows.

We now turn to the definition of the perturbed Hamiltonian \( \tilde{H}_N(\sigma) \), which we take from Section 3.2 of [14]. Let \( H_{N,p}(\sigma) \) denote the pure \( p \)-spin Hamiltonian with mixture \( \xi(t) = t^p \). For \( p \geq 1 \), let \( g_{N,p}(\sigma) \) be a sequence of Hamiltonians such that \( g_{N,p}(\sigma) = \frac{1}{\sqrt{N}} H_{N,p}(\sigma) \) in distribution. Let \( x_p \) be i.i.d. random variables uniform on \([1, 2]\). Assume that \( g_{N,p}(\sigma) \) and \( x_p \) are independent of each other and everything else. Set \( s_N = N^c \) for some \( c \in (1/4, 1/2) \), which we now fix once and for all.

Finally, define
\[
g_N(\sigma) = \sum_{p=1}^{\infty} 2^{-p} x_p g_{N,p}(\sigma) \quad \text{and} \quad \tilde{H}_N(\sigma) = H_N(\sigma) + s_N g_N(\sigma).
\]

We define the free energy \( \tilde{F}_N \) from Proposition 2 by
\[
\tilde{F}_N := \frac{1}{N} \mathbb{E} \log \int_{S_N} e^{\tilde{H}_N(\sigma)} d\mu_N(\sigma),
\]
where the expectation is also w.r.t. the randomness of the uniform variables \( x_p \). The choice of \( s_N \) as above implies [14], see [14].

The proof of Proposition 2 will consist of two steps, stated in the lemmas below. The first will be to relate the free energy in dimension \( N + M \) to another free energy defined on the product space \( S_N \times S_M \subset S_{N+M} \). By an abuse of notation we write \( \tilde{H}_{N+M}(\rho, \tau) \) for \( \tilde{H}_{N+M}(\rho, \tau) \) where \((\rho, \tau) \in S_{N+M} \) denotes the vector obtained by concatenating \( \rho \in S_N \) and \( \tau \in S_M \).

**Lemma 3.** For any \( N \) and \( M \),
\[
(N + M) \tilde{F}_{N+M} \geq \mathbb{E} \log \int_{S_N \times S_M} e^{\tilde{H}_{N+M}(\rho, \tau)} d\mu_N \times \mu_M(\rho, \tau) - C'_{N,M},
\]
for some numbers \( C'_{N,M} \) such that
\[
\limsup_{M \to \infty} \limsup_{N \to \infty} \frac{C'_{N,M}}{\sqrt{M}} = C(\xi)
\]
where \( C(\xi) \) is a constant which depends only on \( \xi \).

We remark that the perturbation has no role in the proof the lemma above, and it still holds also if we work with the unperturbed Hamiltonian \( H_N \) and free energy \( F_N \). The free energy in the right-hand side of [14] is defined on a product space. We will therefore be able to apply to it the Guerra-Toninelli interpolation and obtain the following lemma. Its proof is where we will use Talagrand’s positivity principle.

**Lemma 4.** For any \( N \) and \( M \),
\[
\mathbb{E} \log \int_{S_N \times S_M} e^{\tilde{H}_{N+M}(\rho, \tau)} d\mu_N \times \mu_M(\rho, \tau) \geq N \tilde{F}_N + M \tilde{F}_M - C_{N,M},
\]
for some numbers \( C_{N,M} \) as in Proposition 2.

Proposition 2 directly follows from the two lemmas. The rest of the paper consists of the proof of Lemmas 3 and 4 in Sections 2 and 3, respectively.
2. Proof of Lemma 3

Fix some integers \( N, M \geq 1 \). We will denote by \( \nu_d \) the \( d - 1 \) dimensional Hausdorff measure. In principle, it depends on the dimension of the ambient space, but we will omit this from the notation. By definition,

\[
(N + M)\tilde{F}_{N + M} = \mathbb{E} \log \left( \frac{1}{\nu_{N + M}(S_{N + M})} \int_{S_{N + M}} e^{\tilde{H}_{N + M}(\sigma)} d\nu_{N + M}(\sigma) \right).
\]

Similarly to (1.1), we will use the notation

\[
S_N(r) := \{ \sigma = (\sigma_1, \ldots, \sigma_N) \in \mathbb{R}^N : \| \sigma \| = r \}
\]

for the sphere of radius \( r \) in \( \mathbb{R}^N \). Define the function

\[
\eta(r) = \sqrt{N + M - r^2}.
\]

For two vectors \( \rho \in \mathbb{R}^N \) and \( \tau \in \mathbb{R}^M \), we will denote by \( (\rho, \tau) \in \mathbb{R}^{N + M} \) the vector obtained by concatenating \( \rho \) and \( \tau \). Given some \( \sigma \in S_{N + M} \), whenever we write \( \sigma = (\rho, \tau) \) it should be understood that \( \rho \) and \( \tau \) are the projections of \( \sigma \) to \( \mathbb{R}^N \) and \( \mathbb{R}^M \).

Consider the mapping \( \phi : S_{N + M} \rightarrow \mathbb{R}, \sigma = (\rho, \tau) \mapsto \| \tau \| \) and note that

\[
\phi^{-1}(r) = S_N(\eta(r)) \times S_M(r).
\]

For \( \sigma = (\rho, \tau) \in S_{N + M} \) such that \( 0 < \| \tau \| < \sqrt{N + M} \), the Jacobian of the differential \( D_{\sigma} \phi : T_{\sigma}S_{N + M} \rightarrow T_{\| \tau \|} \mathbb{R} \) is equal to\footnote{Where on \( S_{N + M} \) and \( \phi^{-1}(r) \) we assume the Riemannian structure induced by \( \mathbb{R}^{N + M} \) which is compatible with the Hausdorff measures on them.}

\[
\sqrt{N + M - \| \tau \|^2} = \frac{\eta(\| \tau \|)}{\sqrt{N + M}}.
\]

For an interval \( I \subset [0, \sqrt{N + M}] \), consider the sub-manifold

\[
S^I_{N + M} := \phi^{-1}(I) = \{ \sigma = (\rho, \tau) \in S_{N + M} : \| \tau \| \in I \}.
\]

By the coarea formula,

\[
\int_{S^I_{N + M}} e^{\tilde{H}_{N + M}(\sigma)} d\nu_{N + M}(\sigma) = \int_I \frac{\sqrt{N + M}}{\eta(r)} X_{N, M}(r) dr,
\]

where we define

\[
X_{N, M}(r) := \int_{S_N(\eta(r)) \times S_M(r)} e^{\tilde{H}_{N + M}(\rho, \tau)} d\nu_N \times \nu_M(\rho, \tau).
\]

Applying the same argument to the constant function identically equal to 1 over \( S_{N + M} \), we have that

\[
\nu_{N + M}(S^I_{N + M}) = \int_{S^I_{N + M}} d\nu_{N + M}(\sigma)
\]

\[
= \int_I \frac{\sqrt{N + M}}{\eta(r)} \left( \frac{\eta(r)}{\sqrt{N}} \right)^{N - 1} \left( \frac{r}{\sqrt{M}} \right)^{M - 1} \nu_N(S_N) \nu_M(S_M) dr.
\]
Using the above and (2.1), we have that
\[
(N + M) \tilde{F}_{N + M} \geq \mathbb{E} \log \left( \frac{\nu_{N + M}(S^I_{N + M})}{\nu_{N + M}(S_{N + M})} \int_{S_{N + M}} e^{\tilde{H}_{N + M}(\sigma)} d\nu_{N + M}(\sigma) \right) + \mathbb{E} \log \frac{\nu_{N + M}(S^I_{N + M})}{\nu_{N + M}(S_{N + M})} + \mathbb{E} \log \int_{S_{N + M}} \frac{X_{N, M}(r)}{\nu_{N + M}(S_{N + M})} dr.
\]
(2.3)

From now on, we will work with the interval
\[
I = I^+ \cup I^- := [\sqrt{M} - a, \sqrt{M}] \cup [\sqrt{M}, \sqrt{M} + a],
\]
where \(a \in (0, 1)\) is some fixed number, the value of which will not be important. We will show that the first term in (2.3) converges to a constant and prove a lower bound for the second term, as \(N \to \infty\) first and then \(M \to \infty\).

The following quite elementary lemma is usually attributed to Poincaré [17].

**Lemma 5** (Poincaré’s limit). Fix an integer \(M \geq 1\). Suppose that \(\sigma = (\rho, \tau)\) is a random point uniformly distributed on \(S_{N + M}\). As \(N \to \infty\), the marginal distribution of \(\tau\) weakly converges to the standard Gaussian distribution on \(\mathbb{R}^M\).

Let \(W_M \in \mathbb{R}^M\) be a random vector of i.i.d. standard Gaussian variables. By the lemma,
\[
\lim_{N \to \infty} \frac{\nu_{N + M}(S^I_{N + M})}{\nu_{N + M}(S_{N + M})} = \lim_{N \to \infty} \mu_{N + M}(S^I_{N + M}) = \mathbb{P} \left( \|W_M\|^2 - M - a^2 \in [-2a\sqrt{M}, 2a\sqrt{M}] \right).
\]

By the central limit theorem, for some constant \(b > 0\),
\[
\lim_{M \to \infty} \lim_{N \to \infty} \frac{\nu_{N + M}(S^I_{N + M})}{\nu_{N + M}(S_{N + M})} = \mathbb{P} \left( W_1^2 \in [-\sqrt{2a}, \sqrt{2a}] \right) = b.
\]

Similarly, for the intervals \(I^+\) and \(I^-\) as defined in (2.3) we have that
\[
\lim_{M \to \infty} \lim_{N \to \infty} \frac{\nu_{N + M}(S^I_{N + M})}{\nu_{N + M}(S_{N + M})} = \frac{b}{2}.
\]

Define the two sets
\[
D_{N, M}^\pm = \left\{ (\rho, \tau) \in S_N \times S_M : \pm \frac{d}{dt} \tilde{H}_{N + M}(\rho, \tau + t \frac{\tau}{\|\tau\|}) \geq 0 \right\}.
\]

For \(r \in (0, \sqrt{N + M})\), consider the mapping
\[
S_N(\sqrt{N}) \times S_M(\sqrt{M}) \to S_N(\eta(r)) \times S_M(r),
\]
(2.7)

\[
f_r : (\rho, \tau) \mapsto \left( \frac{\eta(r)}{\sqrt{N}} \rho, \frac{r}{\sqrt{M}} \tau \right).
\]

Define the subsets
\[
D_{N, M}^\pm(\eta) := f_r(D_{N, M}^\pm) \subset S_N(\eta(r)) \times S_M(r)
\]
and variables
\[
X_{N, M}^\pm(\eta) := \int_{D_{N, M}^\pm(\eta)} e^{\tilde{H}_{N + M}(\rho, \tau)} d\nu_N \times \nu_M(\rho, \tau)
\]
and
\[
Y_{N, M}^\pm := \int_{D_{N, M}^\pm(\eta)} \frac{X_{N, M}^\pm(\eta)}{\nu_{N + M}(S^I_{N + M})} dr.
\]
To lower bound the second term in (2.3), we will prove a lower bound for $Y_{N,M}^+ \lor Y_{N,M}^-$ (where we denote by $\lor$ the maximum of $a, b \in \mathbb{R}$). The main estimates we will use are in the following lemma which we prove below.

**Lemma 6.** There exist some positive constants $A = A(\xi)$ and $B = B(\xi)$ depending only on $\xi$ and random variables $L^{(1)} = L_{N,M}^{(1)}$ and $L^{(2)} = L_{N,M}^{(2)}$ such that:

1. For any $t > A$,
   \[
P\left(L^{(i)} \geq (N + M)^{\frac{2}{3} + t}\right) \leq \exp\left(-\frac{N + M}{B}(t - A)^2\right).
   \]

2. For any $r > \sqrt{M}$,
   \[
   X_{N,M}^+(r) \geq \left(\frac{\eta(r)}{\sqrt{N}}\right)^{N-1} \left(\frac{r}{\sqrt{M}}\right)^{M-1} X_{N,M}^+(\sqrt{M}) e^{-L^{(1)} |\eta(r)| - \sqrt{N(-L^{(2)} |r - \sqrt{M}|)^2}}.
   \]

3. For any $r < \sqrt{M}$,
   \[
   X_{N,M}^-(r) \geq \left(\frac{\eta(r)}{\sqrt{N}}\right)^{N-1} \left(\frac{r}{\sqrt{M}}\right)^{M-1} X_{N,M}^-(\sqrt{M}) e^{-L^{(1)} |\eta(r)| - \sqrt{N(-L^{(2)} |r - \sqrt{M}|)^2}}.
   \]

For large $N$, using that for fixed $x \in \mathbb{R}$, $\sqrt{N + x} = \sqrt{N} + \frac{x}{2\sqrt{N}} + O(N^{-3/2})$, one obtains that

\[
\sup_{r \in \mathbb{R}} \left(L^{(1)} |\eta(r)| - \sqrt{N} + L^{(2)} |r - \sqrt{M}|^2\right) \leq a^2 + 2a\sqrt{M} + L^{(1)} + a^2 L^{(2)}
\]

\[
\leq 3\sqrt{\frac{M}{N}} L^{(1)} + L^{(2)}.
\]

Denote the ratio from (2.3) by

\[
T_{N,M} := \int_{\mathbb{R}} \frac{\sqrt{N-M} X_{N,M}(r) dr}{\nu_{N,M}(S_{N,M})}.
\]

Since $X_{N,M}(r) \geq X_{N,M}^\pm(r)$, using (2.3), (2.6) and (2.2) we have that

\[
T_{N,M} \geq \frac{1}{4} Y_{N,M}^\pm,
\]

assuming that $M \geq M_0(\xi)$ and $N \geq N_0(M, \xi)$, for appropriate constants $M_0(\xi)$ and $N_0(M, \xi)$.

On the event that $X_{N,M}^\pm(\sqrt{M}) \geq \frac{1}{2} X_{N,M}(\sqrt{M})$, using Part 2 of the lemma,

\[
Y_{N,M}^+ \geq \frac{1}{2} \frac{X_{N,M}(\sqrt{M})}{\nu_{N,M}(S_{N,M})} \exp\left(-3\sqrt{\frac{M}{N}} L^{(1)} - L^{(2)}\right).
\]

Respectively, using Part 3 of the lemma, on the event that $X_{N,M}^\pm(\sqrt{M}) \geq \frac{1}{2} X_{N,M}(\sqrt{M})$, the same bound holds for $Y_{N,M}^-$. Since $X_{N,M}(\sqrt{M}) = X_{N,M}^+(\sqrt{M}) + X_{N,M}^-(\sqrt{M})$, deterministically,

\[
X_{N,M}(\sqrt{M}) \lor X_{N,M}^-(\sqrt{M}) \geq \frac{1}{2} X_{N,M}(\sqrt{M})
\]

and

\[
T_{N,M} \geq \frac{1}{4} \frac{X_{N,M}(\sqrt{M})}{\nu_{N,M}(S_{N,M})} \exp\left(-3\sqrt{\frac{M}{N}} L^{(1)} - L^{(2)}\right).
\]
For \(N\) and \(M\) as above, we therefore have that
\[
\mathbb{E} \log T_{N,M} \geq -\log 8 + \mathbb{E} \log \frac{X_{N,M}(\sqrt{M})}{\nu_N(S_N)\nu_M(S_M)} + \mathbb{E} \left(-3\sqrt{\frac{M}{N}}L^{(1)} - L^{(2)}\right).
\]

Note that the middle term above is equal to
\[
\mathbb{E} \log \int_{S_N \times S_M} e^{H_{N,M}(\rho, \tau)} d\mu_N \times \mu_M(\rho, \tau).
\]

Hence, by combining the above with (2.3) and (2.5), to complete the proof it remains to show that
\[
\limsup_{M \to \infty} \limsup_{N \to \infty} \frac{1}{\sqrt{M}} \mathbb{E} \left(3\sqrt{\frac{M}{N}}L^{(1)} + L^{(2)}\right) \leq C(\xi),
\]
for some constant \(C(\xi)\).

From Part 1 of Lemma 6 and the tail formula,
\[
\mathbb{E}L^{(1)} \leq \sqrt{N + MA} + \int_{\sqrt{N + MA}}^{\infty} \exp \left(-\frac{N + M}{B} \left(t - \sqrt{N + M} - A\right)^2\right) dt = \sqrt{N + MA} + \sqrt{\frac{\pi B}{2}}.
\]

Similarly,
\[
\mathbb{E}L^{(2)} \leq A + \frac{1}{2} \sqrt{\frac{\pi B}{N + M}}.
\]

This proves (2.8) and completes the proof. It remains to prove Lemma 6.

2.1. Proof of Lemma 6. Let \(L^{(1)} = L_{N,M}^{(1)}\) be the Lipschitz constant of \(H_{N,M}(\sigma)\) over the ball of radius \(\sqrt{N + M}\),
\[
L^{(1)} := \max_{\|\sigma\| \leq \sqrt{N + M}} \|\nabla H_{N,M}(\sigma)\| = \max_{\|u\| = 1} \max_{\|\sigma\| \leq \sqrt{N + M}} u \cdot \nabla H_{N,M}(\sigma).
\]

Let \(L^{(2)} = L_{N,M}^{(2)}\) be the maximal directional second order derivative of \(H_{N,M}(\sigma)\) over the same ball,
\[
L^{(2)} := \max_{\|u\| = 1} \max_{\|\sigma\| \leq \sqrt{N + M}} u^T (\nabla^2 H_{N,M}(\sigma)) u.
\]

Suppose that \((\rho, \tau) \in D_{N,M}^+\) and let \(r > \sqrt{M}\). Since
\[
\|\frac{r}{\sqrt{M}} \tau - \tau\| = |r - \sqrt{M}|
\]
and
\[
\|\frac{\eta(r)}{\sqrt{N}} \rho - \rho\| = |\eta(r) - \sqrt{N}|,
\]
by Taylor’s approximation,
\[
H_{N,M}(\rho, \frac{r}{\sqrt{M}} \tau) \geq H_{N,M}(\rho, \tau) - L^{(2)}|r - \sqrt{M}|^2
\]
and
\[
\hat{H}_{N,M}(\frac{\eta(r)}{\sqrt{N}} \rho, \frac{r}{\sqrt{M}} \tau) \geq \hat{H}_{N,M}(\rho, \frac{r}{\sqrt{M}} \tau) - L^{(1)}|\eta(r) - \sqrt{N}|.
\]
Hence,
\[ \bar{H}_{N,M}(f(\rho, \tau)) = \bar{H}_{N,M}(\frac{\eta(r)}{\sqrt{N}} \rho, \frac{r}{\sqrt{M}} \tau) \]
\[ \geq \bar{H}_{N,M}(\rho, \tau) - L^{(1)}|\eta(r)| - L^{(2)}|r - \sqrt{M}|^2. \]

Clearly,
\[ \frac{\nu_N(D^+_{N,M}(r))}{\nu_N(D^+_{N,M}(\sqrt{M}))} = \left( \frac{\eta(r)}{\sqrt{N}} \right)^{N-1} \left( \frac{r}{\sqrt{M}} \right)^{M-1}. \]

Therefore,
\[ X^+_{N,M}(r) = \int_{D^+_{N,M}(r)} e^{\bar{H}_{N,M}(\rho, \tau)} d\nu_N \times \nu_M(\rho, \tau) \]
\[ = \frac{\nu_N(D^+_{N,M}(r))}{\nu_N(D^+_{N,M}(\sqrt{M}))} \int_{D^+_{N,M}(\sqrt{M})} e^{\bar{H}_{N,M}(f(\rho, \tau))} d\nu_N \times \nu_M(\rho, \tau) \]
\[ \geq \left( \frac{\eta(r)}{\sqrt{N}} \right)^{N-1} \left( \frac{r}{\sqrt{M}} \right)^{M-1} X^+_{N,M}(\sqrt{M}) e^{-L^{(1)}|\eta(r)| - L^{(2)}|r - \sqrt{M}|^2}. \]

This proves Part 2 of the lemma. Part 3 follows by a similar argument.

For the rest of the proof we will work conditional on the uniform random variables \( x_p \). We will prove the bounds in Part 1 with some constants \( A = A(\xi) \) and \( B = B(\xi) \) independent of the values of \( x_p \), which of course gives the same bounds unconditionally. Note that under the conditioning, \( \bar{H}_{N,M}(\sigma) \) is a Gaussian process.

From the proof of [2, Lemma 58], one can see that for some constant \( A = A(\xi) \) that only depends on \( \xi \),
\[ \mathbb{E}L^{(1)} \leq \sqrt{N + MA}, \]
\[ \mathbb{E}L^{(2)} \leq A. \]

Note that
\[ \mathbb{E}(u \cdot \nabla \bar{H}_{N,M}(\sigma))^2 = \frac{d}{dt} \bigg|_{t=0}^{d} \mathbb{E} \left( \bar{H}_{N,M}(\sigma + tu) \bar{H}_{N,M}(\sigma + su) \right) \]
\[ = (N + M) \frac{d}{ds} \bigg|_{s=0}^{d} \mathbb{E}_{N,M}(R(\sigma + tu, \sigma + su)) \]
where, recalling the definition [15], we define
\[ \xi^x_N(t) = \xi(t) + \frac{2r}{N} \sum_{p=1}^{\infty} 4^{-p} x_p^2 t^p. \]

From this one can easily check that for any \( u \) and \( \sigma \) as above,
\[ \mathbb{E}(u \cdot \nabla \bar{H}_{N,M}(\sigma))^2 \leq B, \]
for some constant \( B = B(\xi) \). By a similar argument, for such \( u \) and \( \sigma \),
\[ \mathbb{E}(u^T (\nabla^2 \bar{H}_{N,M}(\sigma)) u) \leq \frac{B}{N + M}, \]
where we may need to increase the constant \( B = B(\xi) \).

The bounds as in Part 1 of the lemma therefore follow from the Borell-TIS inequality [3, 7].
3. Proof of Lemma 7

Recall the definition of the perturbed Hamiltonian

\[
H_N^x(\sigma) = H_N(\sigma) + s_N g_N^x(\sigma),
\]

\[
g_N^x(\sigma) = \sum_{p=1}^{\infty} 2^{-p} x_p g_{N,p}(\sigma).
\]

Here \(x = (x_p)_{p \geq 1}\) are uniform variables in \([1,2]\), which in the current proof we include in the notation to make the dependence on \(x\) explicit. Let \(y = (y_p)_{p \geq 1}\) be an independent copy of \(x\). Define

\[
H_{N,M}^{x,y}(\rho, \tau) = H_{N+M}(\rho, \tau) + s_N g_N^x(\rho) + s_M g_M^y(\tau).
\]

**Lemma 7.** Let

\[
\mathcal{A}_{N,M} = \mathbb{E} \log \int_{S_N \times S_M} e^{H_{N+M}(\rho, \tau)} d\mu_N \times \mu_M(\rho, \tau)
- \mathbb{E} \log \int_{S_N \times S_M} e^{\tilde{H}_{N+M}^{x,y}(\rho, \tau)} d\mu_N \times \mu_M(\rho, \tau).
\]

Then,

\[
\limsup_{M \to \infty} \limsup_{N \to \infty} \frac{1}{M} |\mathcal{A}_{N,M}| = 0.
\]

**Proof.** For \((\rho^1, \tau^1)\) and \((\rho^2, \tau^2)\) in \(S_N \times S_M\) denote

\[
R_{1,2}^1 = \frac{1}{N} \rho^1 \cdot \rho^2, \quad R_{1,2}^2 = \frac{1}{M} \tau^1 \cdot \tau^2,
\]

\[
R_{1,2} = \frac{N}{N+M} R_{1,2}^1 + \frac{M}{N+M} R_{1,2}^2.
\]

Then,

\[
\mathbb{E} (H_{N+M}(\rho^1, \tau^1) H_{N+M}(\rho^2, \tau^2)) = (N+M) \xi(R_{1,2}) + \eta_{N+M}^x(R_{1,2})
\]

and

\[
\mathbb{E} (\tilde{H}_{N,M}(\rho^1, \tau^1) \tilde{H}_{N,M}(\rho^2, \tau^2))
= (N+M) \xi(R_{1,2}) + \eta_N^x(R_{1,2}) + \eta_M^y(R_{1,2}),
\]

where we define

\[
\eta_N^x(t) := s_N^2 \sum_{p=1}^{\infty} 4^{-p} t^2 x_p.
\]

The difference of the two covariance functions above can be bounded by

\[
|\eta_{N+M}^x(R_{1,2}) - \eta_N^x(R_{1,2})| \leq |\eta_{N+M}^x(R_{1,2}) - \eta_N^x(R_{1,2})| + |\eta_N^x(R_{1,2}) - \eta_{N+M}^x(R_{1,2})| + |\eta_M^y(R_{1,2})|.
\]

For any \(x_p, y_p \in [1,2]\) we have the following. The first term in the right-hand side of (3.5) is bounded by

\[
\frac{d}{dt} \eta_{N+M}^x(1) \cdot |R_{1,2} - R_{1,2}^1| \leq \frac{2M}{N+M} s_N^2 \sum_{p=1}^{\infty} 4^{1-p} p.
\]

The middle term is bounded by

\[
(s_N^2 - s_N^2) \sum_{p=1}^{\infty} 4^{1-p}.
\]
And the last term is bounded by
\[ s_M^2 \sum_{p=1}^{\infty} A^{1-p}. \]

For large \( N \), the sum of all three above is bounded by
\[ CM(N + M)^{2c-1} + CM^{2c}, \]
or some constant \( C > 0 \), where \( c \in (1/4, 1/2) \) is the constant such that \( s_N = N^c \).

By an interpolation argument using Gaussian integration by parts, this easily implies that
\[ |A_{N,M}| \leq CM(N + M)^{2c-1} + CM^{2c}, \]
from which the lemma follows. Here we skip the details on Gaussian integration by parts as this is a standard application and since it will be used in a more complicated situation below where we give a full explanation.

To complete the proof of Lemma 4 we will show that
\[ \mathbb{E} \log \int_{S_N \times S_M} e^{H_{N,M}^{x,y}(\rho, \tau)} d\mu_N \times \mu_M(\rho, \tau) \geq N\bar{F}_N + M\bar{F}_M - C_{N,M}, \]
for some \( C_{N,M} > 0 \) as in the statement of the lemma. To prove this we will now use the Guerra-Toninelli interpolation technique [10]. Unlike the original argument of [10], here we include in the Hamiltonians the perturbation terms in order to be able to invoke Talagrand’s positivity principle at the end of the proof. The latter will be required for mixtures including odd interactions or, more precisely, mixtures such that \( \xi \) is not on \([-1,1]\) (see Footnote 4 below).

Suppose that \( H_{N+M}(\rho, \tau), H_N(\rho), H_M(\tau), g_N^x(\rho) \) and \( g_M^y(\tau) \) are defined on the same probability space such that they are all independent of each other, conditionally and unconditionally on the uniform independent variables \( x \) and \( y \). We will denote integration w.r.t. the randomness of the uniform variables \( x \) and \( y \) by \( \mathbb{E}_u \) and integration w.r.t. to all Gaussian variables in the definition of the Hamiltonians by \( \mathbb{E}_g \).

Define on \( S_N \times S_M \) an interpolating Hamiltonian in \( t \in [0, 1] \),
\[ H_t(\rho, \tau) = \sqrt{t} H_{N+M}(\rho, \tau) \sqrt{1-t} (H_N(\rho) + H_M(\tau)) + s_N g_N^x(\rho) + s_M g_M^y(\tau). \]

Define the partition function
\[ Z_t = \int_{S_N \times S_M} e^{H_t(\rho, \tau)} d\mu_N \times \mu_M(\rho, \tau) \]
and free energy
\[ \varphi(t) = \mathbb{E}_u \mathbb{E}_g \log Z_t. \]

Let \( G_t \) denote the corresponding Gibbs measure on \( S_N \times S_M \) with density
\[ \frac{dG_t}{d\mu_N \times \mu_M}(\rho, \tau) = \frac{\exp H_t(\rho, \tau)}{Z_t}. \]

Note that \( \varphi(0) = N\bar{F}_N + M\bar{F}_M \) and \( \varphi(1) \) is equal to the left-hand side of (3.6). To complete the proof of the lemma it remains to show that
\[ \varphi(1) - \varphi(0) \geq -C_{N,M}, \]
for some \( C_{N,M} \) as above.
Using Gaussian integration by parts, one has that (see e.g. the proof of [13] Lemma 1.1)
\[
\varphi'(t) = \frac{1}{2} \mathbb{E}_u \mathbb{E}_g \left( \frac{1}{\sqrt{t}} H_{N+M}(\rho, \tau) - \frac{1}{\sqrt{1-t}} (H_N(\rho) + H_M(\tau)) \right)_t
\]
\[
= - \frac{1}{2} \mathbb{E}_u \mathbb{E}_g \langle U_{N,M} \rangle_t,
\]
where
\[
U_{N,M} := (N + M) \left( \xi(R_{1,2}) - \frac{N}{N + M} \xi(R_{1,2}^1) - \frac{M}{N + M} \xi(R_{1,2}^2) \right),
\]
(\cdot)_t denotes averaging of \((\rho^1, \tau^1)\) and \((\rho^2, \tau^2)\) with respect to \(G_t^{\otimes 2}\) and we use the notation from [13]. To prove (3.9) we will show that
\[
\limsup_{M \to \infty} \limsup_{N \to \infty} \sup_{t \in [0,1]} \frac{1}{M} \mathbb{E}_u \mathbb{E}_g \langle U_{N,M} \rangle_t \leq 0.
\]
Define
\[
\hat{H}_t(\rho) = \int H_t(\rho, \tau) d\mu_N(\tau)
\]
and let
\[
d\hat{G}_t(\rho) = \frac{\exp \hat{H}_t(\rho)}{\int \exp \hat{H}_t(\rho') d\mu_N}
\]
be the corresponding Gibbs measure. Clearly,
\[
G_t^{\otimes 2}(R_{1,2}^1) = \hat{G}_t^{\otimes 2}(R_{1,2}^1).
\]
Note that for any \(t\), we may write
\[
\hat{H}_t(\rho) = \hat{H}_t(\rho) + s_N g_N(\rho),
\]
for some Hamiltonian \(\hat{H}_t(\rho)\) which is independent of \(g_N(\rho)\). Hence, by Talagrand’s positivity principle [21], see Theorem 3.4 in [14],
\[
\limsup_{N \to \infty} \sup_{M,t} \mathbb{E}_u \mathbb{E}_g G_t^{\otimes 2}(R_{1,2}^1 \leq -\epsilon_N)
\]
\[
= \lim_{N \to \infty} \sup_{M,t} \mathbb{E}_u \mathbb{E}_g \hat{G}_t^{\otimes 2}(R_{1,2}^1 \leq -\epsilon_N) = 0,
\]
for some non-increasing sequence \(\epsilon_N \to 0\).

By a similar argument, (with the same sequence \(\epsilon_M\))
\[
\limsup_{M \to \infty} \sup_{N,t} \mathbb{E}_u \mathbb{E}_g G_t^{\otimes 2}(R_{1,2}^2 \leq -\epsilon_M) = 0.
\]
And thus, (with \(a \land b\) denoting the minimum of \(a\) and \(b\))
\[
\limsup_{M \to \infty} \sup_{N,t} \mathbb{E}_u \mathbb{E}_g G_t^{\otimes 2}(R_{1,2}^1 \land R_{1,2}^2 \leq -\epsilon_M) = 0.
\]
Note that for any choice of \(R_{1,2}^1, R_{1,2}^2 \in [-1,1],
\[
|U_{N,M}| \leq M \xi(R_{1,2}) + N \xi(R_{1,2}) - \xi(R_{1,2}^1) + M \xi(R_{1,2}^2)
\]
\[
\leq 2M(\xi(1) + \xi'(1)).
\]
Hence,
\[
(3.10) \quad \limsup_{M \to \infty} \limsup_{N \to \infty} \frac{1}{M} \sup_{t \in [0,1]} \mathbb{E}_u \mathbb{E}_g \langle U_{N,M} \cdot 1\{R_{1,2}^1 \land R_{1,2}^2 \leq -\epsilon_M\} \rangle_t = 0.
\]

\footnote{Note that for if the mixture is an even function \(\xi(t) = \xi(-t)\), then \(\xi(t)\) is convex on \([-1,1]\) and (3.10) follows immediately. The rest of the proof deals with arbitrary \(\xi(t)\) which are in general only convex on \([0,1]\).}
Define
\[ R'_+ = R'_{1,2} \vee 0, \quad R'_+ = R'_{1,2} \vee 0, \]
\[ R_+ = \frac{N}{N + M} R'_+ + \frac{M}{N + M} R'^2_+, \]
\[ R'_+ = \frac{N}{N + M} R'_{1,2} + \frac{M}{N + M} R'^2_+, \]
and
\[ U^+_{N, M} := (N + M) \left( \xi(R_+) - \frac{N}{N + M} \xi(R'_+) - \frac{M}{N + M} \xi(R'^2_+) \right). \]

Write
\[
|U_{N, M} - U^+_{N, M}| \\
\leq \left| (N + M) \xi(R'_+) + M \xi(R'^2_+) - (N + M) \xi(R_{1,2}) - M \xi(R'^2_{1,2}) \right| \\
+ \left| (N + M) \xi(R_+) - N \xi(R'_+) \right| - M \xi(R_{1,2}).
\]

Suppose that \( R_{1,2} \wedge R'^2_{1,2} \geq -\epsilon_M \). Then (3.11) is bounded by \( 2\epsilon_M M' \xi'(1) \) and (3.12) is bounded by \( \epsilon_M \max_{s, r} |\frac{d}{ds} h(s, r)| \) where we define
\[ h(s, r) := (N + M) \xi(\frac{N}{N + M} s + \frac{M}{N + M} r) - N \xi(s). \]

Note that
\[ \left| \frac{d}{ds} h(s, r) \right| = \left| \frac{d}{ds} \left( \frac{N}{N + M} s + \frac{M}{N + M} r \right) - N \xi'(s) \right| \leq 2M \xi''(1). \]

Hence, on the event that \( R_{1,2} \wedge R'^2_{1,2} \geq -\epsilon_M \),
\[ U_{N, M} \leq U^+_{N, M} + 2\epsilon_M M' (\xi'(1) + \xi''(1)). \]

Lastly, since \( \xi \) in convex on \([0, 1]\),
\[ U^+_{N, M} \leq 0. \]

By combining the two inequalities we obtain that
\[ \limsup_{M \to \infty} \limsup_{N \to \infty} \frac{1}{M} \sup_{t \in [0, 1]} \mathbb{E}_u \mathbb{E}_g \left( D_{N, M} \cdot 1 \{ R_{1,2} \wedge R'^2_{1,2} \geq -\epsilon_M \} \right)_t \leq 0, \]
which together with (3.10) proves (3.9) and completes the proof. \( \square \)

REFERENCES
[1] M. Aizenman, R. Sims, and S. L. Starr. Extended variational principle for the Sherrington-Kirkpatrick spin-glass model. Phys. Rev. B, 68:214403, Dec 2003.
[2] G. Ben Arous, E. Subag, and O. Zeitouni. Geometry and temperature chaos in mixed spherical spin glasses at low temperature: the perturbative regime. Comm. Pure Appl. Math., 73(8):1732–1828, 2020.
[3] C. Borell. The Brunn-Minkowski inequality in Gauss space. Invent. Math., 30(2):207–216, 1975.
[4] W.-K. Chen. The Aizenman-Sims-Starr scheme and Parisi formula for mixed p-spin spherical models. Electron. J. Probab., 18:no. 94, 14, 2013.
[5] W.-K. Chen, D. Panchenko, and E. Subag. The generalized TAP free energy. to appear in CPAM. arXiv:1812.05066.
[6] W.-K. Chen, D. Panchenko, and E. Subag. The generalized TAP free energy II. Comm. Math. Phys., 381(1):257–291, 2021.
[7] B. S. Ciruel’son, I. A. Ibragimov, and V. N. Sudakov. Norms of Gaussian sample functions. In Proceedings of the Third Japan-USSR Symposium on Probability Theory (Tashkent, 1975), pages 20–41. Lecture Notes in Math., Vol. 550. Springer, Berlin, 1976.
[8] A. Crisanti and H.-J. Sommers. The spherical p-spin interaction spin glass model: the statics. Zeitschrift für Physik B Condensed Matter, 87(3):341–354, 1992.
[9] F. Guerra. Broken replica symmetry bounds in the mean field spin glass model. Comm. Math. Phys., 233(1):1–12, 2003.
[10] F. Guerra and F. L. Toninelli. The thermodynamic limit in mean field spin glass models. *Comm. Math. Phys.*, 230(1):71–79, 2002.

[11] M. Mézard, G. Parisi, N. Sourlas, G. Toulouse, and M. Virasoro. Nature of the spin-glass phase. *Phys. Rev. Lett.*, 52:1156–1159, Mar 1984.

[12] M. Mézard, G. Parisi, N. Sourlas, G. Toulouse, and M. Virasoro. Replica symmetry breaking and the nature of the spin glass phase. *J. Physique*, 45(5):843–854, 1984.

[13] D. Panchenko. The Parisi ultrametricity conjecture. *Ann. of Math.* (2), 177(1):383–393, 2013.

[14] D. Panchenko. The Sherrington-Kirkpatrick model. Springer Monographs in Mathematics. Springer, 2013.

[15] G. Parisi. Infinite number of order parameters for spin-glasses. *Phys. Rev. Lett.*, 43:1754–1756, 1979.

[16] G. Parisi. A sequence of approximated solutions to the s-k model for spin glasses. *Journal of Physics A: Mathematical and General*, 13(4):L115, 1980.

[17] H. Poincaré. *Calcul des probabilités*. Les Grands Classiques Gauthier-Villars. [Gauthier-Villars Great Classics]. Éditions Jacques Gabay, Sceaux, 1987. Reprint of the second (1912) edition.

[18] E. Subag. Free energy landscapes in spherical spin glasses. *arXiv:1804.10576*, 2018.

[19] E. Subag. The free energy of spherical pure $p$-spin models – computation from the TAP approach. *arXiv:2101.04352*, 2021.

[20] M. Talagrand. Free energy of the spherical mean field model. *Probab. Theory Related Fields*, 134(3):339–382, 2006.

[21] Michel Talagrand. *Spin glasses: a challenge for mathematicians*, volume 46 of *Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, 2003. Cavity and mean field models.

[22] D. J. Thouless, P. W. Anderson, and R. G. Palmer. Solution of ‘solvable model of a spin glass’. *Philosophical Magazine*, 35(3):593–601, 1977.

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