Constructive Combinatorics of Dickson’s Lemma

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Abstract

We study constructively the relations between the finite cases of Dickson’s lemma. Although there are many constructive proofs of them, the novel aspect of our proofs is the extraction of a corresponding bound. We provide some new one-step unprovability results i.e., results of the form “a finite case of Dickson’s lemma does not prove in one step a stronger case of it”. Moreover, we study the infinite cases of Dickson’s lemma from the point of view of constructive reverse mathematics. We work within Bishop’s informal system of constructive mathematics BISH.

1 Introduction

1.1 The finite and infinite cases of a combinatorial theorem $\tau$

According to [10], p.391, the basic propositions of (classical) combinatorics assert, crudely speaking, that every system of a certain class possesses a large subsystem with a higher degree of organization than the original system.

The larger the subsystem is proven to be, the stronger the corresponding theorem is. Suppose that $\tau$ is a theorem of combinatorics asserting for a system $S$ in a class of systems $\Sigma$ the existence of a subsystem $I$ of $S$ that has property $P$, which generally $S$ does not. In most cases property $P$ is hereditary, i.e., if $I' \subseteq I$ and $P(I)$, then $P(I')$. If $|X|$ denotes the cardinality of a set $X$, $l \geq 1$ and $\xi$ is a cardinal strictly larger than $\aleph_0$, usually the following finite and infinite cases of $\tau$ are considered.

1. The finite case $\tau(l)$: If $|S| \geq l$, there is $I \subseteq S$ such that $|I| = l$ and $P(I)$.
2. The strong finite case $\tau^*(l)$: There is $M(l) > 0$ such that if $l \leq |S| \leq M(l)$, there is $I \subseteq S$ such that $|I| = l$ and $P(I)$.
3. The unbounded case: If $|S| \geq \aleph_0$, then $\forall \forall l \geq 1(\tau(l))$.
4. The infinite case $\tau(\aleph_0)$: If $|S| \geq \aleph_0$, there is $I \subseteq S$ such that $|I| = \aleph_0$ and $P(I)$.
5. The higher infinite case $\tau(\xi)$: If $|S| \geq \xi$, there is $I \subseteq S$ such that $|I| = \xi$ and $P(I)$.

For the constructive study of such a combinatorial theorem $\tau$ a general pattern can be described.

a. The finite case $\tau(l)$ is constructively proved, although there are finite combinatoric propositions, like Friedman’s Proposition B, which is provable only with the use of large cardinals (see [14] and [16]), or the proposition of Paris-Harrington, which is provable in second-order analysis but not in Peano arithmetic, and also lacks a constructive proof.

b. In many cases a strong case $\tau^*(l)$ is also constructively proved. To find explicitly though, a bound for the strong case $\tau^*(l)$ is usually a difficult problem, and for many well-studied combinatorial theorem, like Higman’s lemma, or Kruskal’s theorem, the extraction of a bound $M(l)$ from a constructive proof of $\tau(l)$ is, to our knowledge, not yet known.

c. The unbounded case $\forall l \geq 1(\tau(l))$ is generally constructively proved.

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1 On 2011, during a colloquium-talk at LMU, Veldman suggested to try to find such a proof.
The inductive proofs of them. It is not uncommon that non-constructive proofs inspire, or have a constructive counterpart. E.g., minimal-bad-sequence-proofs of Higman’s lemma, or of Dickson’s lemma inspired corresponding constructive (inductive) proofs of them.

e. The higher infinite case $\tau(\xi)$ is generally beyond the scope of constructive combinatorics. Often the proof of some case of $\tau$ is based on the use of a repetitive argument, that is on the repetition of the same proof-step for an appropriate number of times. In this way the power of repetition of a simple, single argument is revealed. Moreover, if a bound is extracted from the single proof-step, then a bound is extracted from the whole proof. Most of the proofs included in this paper are based on repetitive arguments. Although from such proofs we do not extract the best possible, or optimal bounds, we find them interesting because they are somehow “elementary”.

1.2 The finite and infinite cases of Dickson’s lemma

Dickson’s lemma is the simplest theorem of the form “a certain quasi-order is a well-quasi-order”, and it is connected to the the theory of Gröbner bases and the termination of Buchberger’s algorithm for finding them (see [12] and [11]). This was one of the first examples of how a well-quasi-order can be used as a technique applied to program termination (for more on this see [28]). Here we present though, the finite and infinite cases of Dickson’s lemma independently from the theory of well-quasi-orders. First we need a definition.

\textbf{Definition 1.1.} If $X, Y$ are sets, $\mathcal{F}(X, Y)$ denotes the set of functions from $X$ to $Y$. Let $k \in \mathbb{N}$ such that $k \geq 1$, $\alpha_1, \ldots, \alpha_k \in \mathcal{F}(\mathbb{N}, \mathbb{N})$, $(i, j) \in \mathbb{N}^2$ such that $i < j$, and $I \subseteq \mathbb{N}$. The pair $(i, j)$ is called a good pair of indices for $\alpha_1, \ldots, \alpha_k$, or $\alpha_1, \ldots, \alpha_k$ are called good on $(i, j)$, if $\alpha_n(i) \leq \alpha_m(j)$, for every $n \in \{1, \ldots, k\}$. We say that $\alpha_1, \ldots, \alpha_k$ are good on $I$, or $I$ is good for $\alpha_1, \ldots, \alpha_k$, if $\alpha_1, \ldots, \alpha_k$ are good on every pair of indices $(i, j) \in I^2$ such that $i < j$.

If $k \geq 1$ and $l \geq 2$, the following finite and infinite cases of Dickson’s lemma are usually considered.

1. DL$(k, l)$: If $\alpha_1, \ldots, \alpha_k \in \mathcal{F}(\mathbb{N}, \mathbb{N})$, there exists $I_l = \{i_1 < i_2 < \ldots < i_l\} \subseteq \mathbb{N}$ such that $\alpha_1, \ldots, \alpha_k$ are good on $I_l$.
2. DL$(k, \infty)$: If $\alpha_1, \ldots, \alpha_k \in \mathcal{F}(\mathbb{N}, \mathbb{N})$, there exists $I_\infty = \{i_1 < i_2 < \ldots < i_n < \beta_{n+1} < \ldots\} \subseteq \mathbb{N}$ such that $\alpha_1, \ldots, \alpha_k$ are good on $I_\infty$.
3. DL$(k, U)$: If $\alpha_1, \ldots, \alpha_k \in \mathcal{F}(U, \mathbb{N})$, where $U$ is an unbounded\footnote{That is $\forall n \in \mathbb{N} \exists m \in \mathbb{N}(m > n \land m \in U)$.} subset of $\mathbb{N}$, there exists an unbounded subset $I_U$ of $U$ such that $\alpha_1, \ldots, \alpha_k$ are good on $I_U$.

If $\Sigma = \mathcal{P}(\mathbb{N})$, and $S = n$, where $n := \{0, \ldots, n - 1\}$, or $S = \mathbb{N}$, and if $P(I)$, where $I \subseteq n$, for some $n \in \mathbb{N}$, or $I \subseteq \mathbb{N}$, is the hereditary property defined as “the sequences $\alpha_1, \ldots, \alpha_k$ are good on $I$”, then the cases DL$(k, l)$ and DL$(k, \infty)$ are special cases of a combinatorial theorem $\tau$, for which no higher infinite case is meaningful.

Note that an infinite case DL$(\infty, 2)$ of DL$(k, 2)$ does not hold; if we consider the sequence of sequences $(\alpha_n)^{\infty}_{n=1}$, where for every $n \geq 1$ the sequence $\alpha_n \in \mathcal{F}(\mathbb{N}, \mathbb{N})$ is

$$\alpha_n = (n, n - 1, \ldots, 1, n + 1, n + 2, n + 3, \ldots),$$

we cannot find a pair of indices which is good for all $\alpha_n$; If $n = \alpha_n(0)$, then $\alpha_n(n - 1) = 1$, for every $n \geq 1$. Hence, if $i < j$, then $\alpha_{j+1}(i) > 1$, while $\alpha_{j+1}(j) = 1$, i.e., $(i, j)$ cannot be a good pair for $\alpha_{j+1}$. 


This is a simple example of a finite combinatorial proposition the infinite case of which does not hold, even classically².

The original formulation of Dickson’s lemma in [13] is equivalent to DL(k, U), which, as we show in section 4, is equivalent to DL(k, ∞) and cannot be constructively accepted. On the other hand, the finite case DL(k, l) has already a short constructive history. As Veldman and Bezem say in [3], p.210, it was John Burgess who, in a letter from 1983, asked for a constructive proof of DL(2, 2), which is shown to be a consequence of the intuitionistic Ramsey theorem in [6]. In [29] Veldman gave an elementary inductive, constructive proof of DL(k, 2), independently from the intuitionistic Ramsey theorem or some special intuitionistic principle. In [11] Coquand and Persson gave a constructive proof of an inductive version of DL(k, k). In [4] a program is extracted from a classical proof of DL(2, 2), by transforming the classical proof into a constructive one through a refined version of A-translation, and the proof is implemented in MINLOG (see also [5], [24], [27]). From the program extraction-point of view Dickson’s lemma has been studied within systems like Mizar, Coq and ACL2 (see [26], [11], [19], respectively). In [17] Hertz mined two classical proofs of DL(k, 2) using the Dialectica interpretation. We refer here only to direct constructive approaches to Dickson’s lemma. Since the finite cases of Dickson’s lemma follow easily from Higman’s lemma, a constructive proof of the latter gives a constructive proof of the former (see [25]). In [2] it is shown that all finite cases of Dickson’s lemma imply Higman’s lemma for words of an alphabet with two letters.

The extraction of a bound for DL(k, l) i.e., the mining of a number $M_{α_1,...,α_k}(l) > 0$ out of a proof of DL(k, l) such that $i_1 < \ldots < i_l$ is good for $α_1,...,α_k$ and $i_l \leq M_{α_1,...,α_k}(l)$ is, surprisingly, not well-studied (neither constructively nor classically). An exception to this is the work [3], where with the use of the finite pigeonhole principle a strong case of DL(2, 2) is shown. It doesn’t seem possible though, to generalize this result to a method to prove strong cases of DL(k, 2), for $k > 2$.

The main results of this paper are the following.

1. Proposition 2.3, a strong case DL*(1, l) of DL(1, l), for every $l \geq 3$.
2. Proposition 2.6, a strong case DL*(2, 2) of DL(2, 2).
3. Proposition 2.7, a strong case DL*(2, l) of DL(2, l), for every $l \geq 3$.
4. We explain how our proof of Proposition 2.7 generates a proof of a strong case DL*(k, l) of DL(k, l), where $k > 2$ and $l \geq 3$, and how the latter together with the proof of Proposition 2.6 generate a proof of a strong case DL*(k + 1, 2) of DL(k + 1, 2).
5. Theorem 3.2, a positive formulation of the non-existence of an one step-proof of DL(2, 2) from DL(1, l).
6. Theorem 3.4, a positive formulation of the non-existence of an one step-proof of DL(3, 2) from DL(2, 2).
7. Propositions 4.2 and 4.6 which express the constructive equivalence between DL(1, ∞) and LPO.

Results 5 and 6 are technically the more involved and are, as far as we know, together with result 7, new. They are motivated by Corollaries 3.3 and 3.5, respectively, which were conceived first.

We work within Bishop’s informal system of constructive mathematics BISH (see [7], [8], [9]). A formal system that corresponds to BISH is CZF (see [1]) together with the principle of dependent choices (DC), or Myhill’s system CST (see [20]). For a recent reconstruction of Bishop’s set theory within BISH see [21], [22], [23].

2 Strong finite cases of Dickson’s lemma

The strong form DL*(1, 2) of DL(1, 2), although trivial, is essential to the description of a bound in all other strong cases DL*(k, l) of DL(k, l) presented here.

²A deeper example is related to van der Waerden’s theorem. According to it, if $N$ is partitioned into two classes, then at least one of them contains arbitrarily long arithmetic progressions. But that does not imply that an infinite arithmetic progression in one of them exists (see [15], p.69).
Proposition 2.1 (DL*(1, 2)). \( \forall n \in \mathbb{N} \forall \alpha \in \mathbb{F}(\mathbb{N}, \mathbb{N})(\alpha(0) \leq n \Rightarrow \exists i < \alpha(0) + 1(\alpha(i) \leq \alpha(i + 1))) \).

Proof. If \( n = 0 \), then \( \alpha(0) = 0 \), and \( i = 0 \) is the required index. Next we suppose that \( \forall \alpha \in \mathbb{F}(\mathbb{N}, \mathbb{N})(\alpha(0) \leq n \Rightarrow \exists i < \alpha(0) + 1(\alpha(i) \leq \alpha(i + 1))) \) and we show that \( \forall \alpha \in \mathbb{F}(\mathbb{N}, \mathbb{N})(\alpha(0) \leq n + 1 \Rightarrow \exists i < \alpha(0) + 1(\alpha(i) \leq \alpha(i + 1))) \).

Let \( \alpha \in \mathbb{F}(\mathbb{N}, \mathbb{N}) \) such that \( \alpha(0) \leq n + 1 \). If \( \alpha(0) \leq n \), we use the inductive hypothesis. If \( \alpha(0) = n + 1 \), then if \( \alpha(0) \leq \alpha(1) \), we get \( i = 0 \). If \( \alpha(0) > \alpha(1) \), then \( \alpha(1) \leq n \). By the inductive hypothesis on the sequence \( \alpha^* \), where \( \alpha^*(n) = \alpha(n + 1) \), for every \( n \in \mathbb{N} \), there is \( j < \alpha^*(0) + 1 = \alpha(1) + 1 \) such that \( \alpha^*(j) \leq \alpha^*(j + 1) \) i.e., \( \alpha(j + 1) \leq \alpha(j + 2) \), and \( i = j + 1 < (n + 1) + 1 = \alpha(0) + 1 \).

If \( \alpha \in \mathbb{F}(\mathbb{N}, \mathbb{N}) \), we use the notation \( M_\alpha(1, 2) := \alpha(0) + 1 \) for the bound of \( \text{DL}^*(1, 2) \) that corresponds to \( \alpha \). It is immediate to see that \( M_\alpha(1, 2) \) is an optimal bound for \( \text{DL}(1, 2) \). The first part of the next simple corollary of \( \text{DL}^*(1, 2) \) expresses that for each sequence \( \alpha \) we can find a good pair \((i, j)\) for \( \alpha \) such that \((j - i)\) is arbitrary large. For its last part recall that the lexicographic ordering \( <_{\text{lex}} \) on \( \mathbb{N} \) is defined by \((n_1, m_1) <_{\text{lex}} (n_2, m_2) \Leftrightarrow (n_1 < n_2) \lor (n_1 = n_2 \land m_1 < m_2) \), for every \( n_1, n_2, m_1, m_2 \in \mathbb{N} \).

Corollary 2.2. (i) For every \( \alpha \in \mathbb{F}(\mathbb{N}, \mathbb{N}) \) and \( n > 0 \)

\[ \exists i \in \mathbb{N} \left( i \leq \sum_{j=0}^{n-1} \alpha(j) \land \alpha(i) \leq \alpha(i + n) \right). \]

Moreover, the bound \( \sum_{j=0}^{n-1} \alpha(j) \) is the best possible i.e., there exists a sequence \( \alpha \) such that \( \alpha(i) > \alpha(i + n) \), for every \( i < M < \sum_{j=0}^{n-1} \alpha(j) \).

(ii) If \( n \in \mathbb{N} \), there is no sequence \( \alpha \in \mathbb{F}(\mathbb{N}, \mathbb{N}) \) such that \( \forall k \in \mathbb{N}(\alpha(k) < \alpha(k + 1) \land \alpha(k) < n) \).

(iii) There exists no function \( f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \) such that

\[ f(n_1, m_1) < f(n_2, m_2) \iff (n_1, m_1) <_{\text{lex}} (n_2, m_2), \]

for every \( n_1, n_2, m_1, m_2 \in \mathbb{N} \).

Proof. (i) If \( \alpha \in \mathbb{F}(\mathbb{N}, \mathbb{N}) \) and \( n > 0 \), we consider the sequence \( \beta \in \mathbb{F}(\mathbb{N}, \mathbb{N}) \) defined by

\[ \beta(m) = \sum_{j<n} \alpha(m + j), \]

for every \( m \in \mathbb{N} \). By \( \text{DL}^*(1, 2) \) there exists \( i \leq \beta(0) = \sum_{j=0}^{n-1} \alpha(j) \) such that

\[ \beta(i) \leq \beta(i + 1) \iff \sum_{j<n} \alpha(i + j) \leq \sum_{j<n} \alpha(i + 1 + j) \iff \alpha(i) \leq \alpha(i + n). \]

In order to show the optimality of the specified bound consider, for an arbitrary \( n > 0 \), any infinite sequence \( \alpha \) extending the finite sequence \( 1, 0, \ldots, 0 \). Clearly, \( \sum_{j=0}^{n-1} \alpha(j) = 1 \) and \( \alpha(0) > \alpha(0 + n) \), while \( \alpha(1) \leq \alpha(1 + n) \).

(ii) Suppose that such a sequence \( \alpha \) exists, and consider any infinite extension of the finite sequence \( \beta(0) = n_0 > \beta(1) = \alpha(n_0) > \beta(2) = \alpha(n_0 - 1) > \ldots > \beta(n_0 + 1) = \alpha(0) \). By \( \text{DL}^*(1, 2) \) there exists \( i < \beta(0) + 1 = n_0 + 1 \) such that \( \beta(i) \leq \beta(i + 1) \), which contradicts the supposed strict monotonicity of \( \alpha \).

(iii) Suppose that such a function \( f \) exists. By the definition of \( <_{\text{lex}} \) we get \((0, 0) <_{\text{lex}} (0, 1) <_{\text{lex}} (0, 2) <_{\text{lex}} \ldots <_{\text{lex}} (0, n) <_{\text{lex}} \ldots <_{\text{lex}} (1, 0) \), while by the supposed property of \( f \) we have \( f(0, 0) < f(0, 1) < f(0, 2) < \ldots < f(0, n) < \ldots < f(1, 0) \), which is impossible by (ii).

Note that by the unbounded case \( \forall i \geq 2(\text{DL}(1, i)) \) we get that \( \forall \alpha \in \mathbb{F}(\mathbb{N}, \mathbb{N}) \forall n > 0 \exists i, j \in \mathbb{N}(j - i \geq n \land \alpha(i) \leq \alpha(j)) \), since by \( \text{DL}(1, n + 1) \) there exist \( i_1 < \ldots < i_{n+1} \), such that \( \alpha(i_1) \leq \ldots \leq \alpha(i_{n+1}) \), therefore \( i_{n+1} - i_1 \geq n \), and \( \alpha(i_1) \leq \alpha(i_{n+1}) \). By Corollary 2.2(i) though, we “strongly” know that the distance between the elements of the good pair is exactly \( n \).

*For \( n = 1 \) we get \( \sum_{j=0}^{n-1} \alpha(j) = \alpha(0) \), the optimal bound of \( \text{DL}^*(1, 2) \).
Proposition 2.3 (DL∗(1, l)). If l ≥ 3 and α ∈ ℘(N, N), there exist i1, i2, . . . , iℓ, and Mα(1, l) ∈ N such that

\[ i_1 < i_2 < \ldots < i_\ell \leq M_\alpha(1, l) \quad \text{and} \quad \alpha(i_1) \leq \alpha(i_2) \leq \ldots \leq \alpha(i_\ell), \]

where

\[ M_\alpha(1, l) = \sum_{j=1}^{N} M_j, \]

\[ N = \alpha(i^{(1)}) + 2, \]

\[ M_1 = M_\alpha(1, l-1), \]

\[ M_{j+1} = M_\alpha(j+1, l-1), \]

for every j ∈ {1, . . . , N−1}, and Mα(1, l − 1) is the bound according to DL∗(1, l − 1) on α, α(j) is the tail of α starting from the index Mj, Mα(j)(1, l − 1) is the bound according to DL∗(1, l − 1) on the sequence α(j), and i(j) is the index determined by the application of DL∗(1, l − 1) on α.

Proof. Suppose first that l = 3. If we apply DL∗(1, 2) on α, we get an index i(j) ≤ α(0), such that α(i(1)) ≤ α(i(1) + 1). We write M1 = Mα(1, 2) = α(0) + 1. If we apply DL∗(1, 2) on the tail α(1) of α starting from M1, i.e., α(1)(n) = α(M1 + n), for every n ∈ N, then we get an index i(2) ≤ α(1)(0) = α(M1), such that α(i(1)(i(2))) ≤ α(i(1)(i(2) + 1)). We write M2 = α(M1) + 1. Repeating these steps N = α(i(1)) + 2 number of times we get indices i(1) < i(2) < . . . < i(N), such that the application of DL∗(1, 2) on α(i(1)), α(i(2)), . . . , α(i(N)) gives the existence of an index i(k), where k ≤ α(i(1)), such that α(i(k)) ≤ α(i(k+1)). By the definition of the indices i(1) < i(2) < . . . < i(N) we conclude that

\[ \alpha(i(k)) \leq \alpha(i(k+1)) \leq \alpha(i(k+1) + 1). \]

The initial segment of α required to find the indices i(k), i(k+1), and i(k+1) + 1 is Mα(1, 3) = \sum_{i=1}^{N} M_i, where M1 = α(0) + 1, and for every i ∈ {1, . . . , N−1} we have that Mj+1 = α(Mj) + 1.

If l > 3, we show that

\[ DL^*(1, l) \rightarrow DL^*(1, l + 1) \]

by repeating N number of times the application of DL∗(1, l) on the corresponding tails of α, exactly as in the l = 3 case. In this way we get indices i(1) < i(2) < . . . < i(N), such that the application of DL∗(1, 2) on α(i(1)), α(i(2)), . . . , α(i(N)) gives the existence of an index i(k), such that α(i(k)) ≤ α(i(k+1)). By the definition of the indices i(1) < i(2) < . . . < i(N) we conclude that

\[ \alpha(i(k)) \leq \alpha(i(k+1)) \leq \alpha(i(k+1) + 1) \leq \ldots \leq \alpha(i(k+1)). \]

Within the above proof the rightmost pair of the indices on which α weakly increases is a pair of consecutive numbers. Generally, these indices are not consecutive. E.g.,

\[ \alpha(n) = \begin{cases} 
0 & \text{if } n = 2k \\
1 & \text{if } n = 2k + 1.
\end{cases} \]

doesn’t weakly increase on any triad of consecutive numbers.

Definition 2.4. Let A be an inhabited set and n ≥ 1. A coloring of A with n colors, or an n-coloring of A, is a function χ : A → n. If a1, a2 ∈ A, the set \{a1, a2\} is called a monochromatic pair under χ if χ(a1) = χ(a2). A subset B of A is called monochromatic under χ, if every two elements of B form a monochromatic pair. The notation PH(n, m, l), where n ∈ N and m, l ∈ N ∪ {N}, expresses that if χ is an n-coloring of a sequence of A of length m, then this sequence contains a monochromatic subsequence B of length l.
Consequently, the case \( \text{PH}(2, \mathbb{N}, l) \) of the pigeonhole principle, where \( l \in \mathbb{N} \) and \( l \geq 2 \), says that if \( \chi \) is a 2-coloring of \( \{ \alpha_n : n \in \mathbb{N} \} \subseteq A \), then \( \{ \alpha_n : n \in \mathbb{N} \} \) has a monochromatic subsequence of length \( l \).

**Proposition 2.5.** \( \forall l \geq 2 (\text{DL}(1, l) \rightarrow \text{PH}(2, \mathbb{N}, l)) \).

**Proof.** Suppose that \( l \geq 2, \alpha \in \mathbb{F}\{\mathbb{N}, \mathbb{N}\} \) and \( \chi \) is a 2-coloring of \( \{ \alpha_n : n \in \mathbb{N} \} \). By \( \text{DL}(1, l) \) on \( \chi \circ \alpha : \mathbb{N} \rightarrow \mathbb{2} \) there are indices \( i_1 < i_2 < \ldots < i_l \), such that \( \chi(\alpha_{i_1}) \leq \chi(\alpha_{i_2}) \leq \ldots \leq \chi(\alpha_{i_l}) \). If \( \chi(\alpha_{i_1}) = 0 \), therefore \( \chi(\alpha_{i_1}) = \chi(\alpha_{i_2}) = \ldots = \chi(\alpha_{i_l}) = 0 \), the sequence \( \alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_l} \) is a monochromatic subsequence of \( \alpha \) of length \( l \). If \( \chi(\alpha_{i_1}) = 1 \), then we repeat the previous step on the tail \( \alpha_{i_1+1}, \alpha_{i_1+2}, \ldots \), of \( \alpha \). By \( \text{DL}(1, l) \), there are indices \( n_1 < n_2 < \ldots < n_l \), such that \( \chi(\alpha_{i_1+n_1}) \leq \chi(\alpha_{i_1+n_2}) \leq \ldots \leq \chi(\alpha_{i_1+n_l}) \).

If \( \chi(\alpha_{i_1+n_l}) = 0 \), then we get a monochromatic subsequence of \( \alpha \) of length \( l \). If \( \chi(\alpha_{i_1+n_l}) = 1 \), we repeat the same procedure. It suffices to repeat the above steps at most \( l \) number of times to find a monochromatic subsequence of \( \alpha \) of length \( l \).

It is easy to provide a bound for \( \text{PH}(2, \mathbb{N}, l) \) based on the bounds determined by \( \text{DL}^*(1, l) \) on the sequences considered in the previous proof.

**Proposition 2.6 (DL*\((2, 2)\)).** If \( \alpha, \beta \in \mathbb{F}\{\mathbb{N}, \mathbb{N}\} \), there exist \( i, j \) and \( \text{M}_{\alpha, \beta}(2, 2) \in \mathbb{N} \) such that

\[
i < j < \text{M}_{\alpha, \beta}(2, 2), \quad \text{and} \quad \alpha(i) \leq \alpha(j) \land \beta(i) \leq \beta(j),
\]

where

\[
\text{M}_{\alpha, \beta}(2, 2) = \sum_{j=1}^{K} M_{j},
\]

\[
1 \leq K \leq N,
\]

\[
N = \alpha(i^{(1)}) + 2,
\]

\[
M_1 = M_\alpha(1,3),
\]

\[
\beta(i^{(1)}) \leq 1 \rightarrow K = 1,
\]

\[
\beta(i^{(1)}) \geq 2 \rightarrow M_2 = M_\alpha(1, \beta(i^{(1)}) + 1),
\]

\( i^{(1)} \) is the first index of the application of \( \text{DL}^*(1, 3) \) on \( \alpha \) and \( \alpha^{(1)} \) is the tail of \( \alpha \) starting from index \( M_1 \). If \( 1 \leq j \leq N - 1 \), then

\[
\beta(i^{(j+1)}) \leq \beta(i^{(j)}) - 1 \rightarrow K = j + 1,
\]

\[
\beta(i^{(j+1)}) \geq \beta(i^{(j)}) \rightarrow M_{j+1} = M_\alpha(1, \beta(i^{(j+1)}) + 1),
\]

and \( i^{(j+1)} \) is the first index of the application of \( \text{DL}^*(1, \beta(i^{(j)}) + 1) \) on \( \alpha^{(j)} \), where \( \alpha^{(j)} \) is the tail of \( \alpha \) starting from index \( M_j \).

**Proof.** We show that

\[
\forall l \geq 2 (\text{DL}^*(1, l) \rightarrow \text{DL}^*(2, 2)),
\]

hence by Proposition 2.3 we get a proof of \( \text{DL}^*(2, 2) \). Applying \( \text{DL}^*(1, 3) \) on \( \alpha \) we find indices \( i_1^{(1)} < i_2^{(1)} < i_3^{(1)} \), for which \( \alpha(i_1^{(1)}) \leq \alpha(i_2^{(1)}) \leq \alpha(i_3^{(1)}) \), based on the initial segment of \( \alpha \) of length \( M_1 = M_\alpha(1, 3) \). We also consider the finite sequence \( \beta(i_1^{(1)}), \beta(i_2^{(1)}), \beta(i_3^{(1)}) \).

Suppose that \( \beta(i_1^{(1)}) \leq 1 \). If we form the sequence \( \gamma(0) = \beta(i_1^{(1)}), \gamma(1) = \beta(i_2^{(1)}), \gamma(2) = \beta(i_3^{(1)}) \) and extend it in any way we like, then, by \( \text{DL}^*(1, 2) \) there exists \( j \leq \gamma(0) \leq 1 \), such that \( \gamma(j) \leq \gamma(j+1) \leftrightarrow \beta(i_1^{(j+1)}) \leq \beta(i_2^{(j+1)}) \), while \( \alpha(i_1^{(j+1)}) \leq \alpha(i_1^{(j+1)}) \) also holds. Hence, in case \( \beta(i_1) \leq 1 \), we can find a pair of indices for which \( \text{DL}^*(2, 2) \) is satisfied, and then trivially \( K = 1 \).

If \( \beta(i_1^{(1)}) = \mu \geq 2 \), we consider the tail \( \alpha^{(1)} \) of \( \alpha \) which starts from index \( M_1 \). By \( \text{DL}^*(1, \mu + 1) \) on \( \alpha^{(1)} \) we find a finite sequence of indices \( i_1^{(2)} < i_2^{(2)} < \ldots < i_{\mu+1}^{(2)} \), for which \( i_1^{(1)} < i_2^{(1)} < i_3^{(1)} < \ldots < i_{\mu+1}^{(2)} \).
\( i_1^{(2)} < i_2^{(2)} < \ldots < i_{\mu+1}^{(2)} \), such that \( \alpha^{(1)}(i_1^{(2)}) \leq \alpha^{(1)}(i_2^{(2)}) \leq \ldots \leq \alpha^{(1)}(i_{\mu+1}^{(2)}) \). Of course, \( \alpha \) also weakly increases on these indices. Considering \( \beta(i_1^{(2)}) \) we work as follows:

If \( \beta(i_1^{(2)}) \leq \mu - 1 \), then we can find the required pair of indices using \( DL^*(1, 2) \). If \( \beta(i_1^{(2)}) \geq \mu = \beta(i_1^{(1)}) \), we repeat the previous step working with the tail \( \alpha^{(2)} \) of \( \alpha \) which starts from index \( M_2 = M_{\alpha^{(1)}(1, \beta(i_1^{(1)}) + 1)} \).

If we are at step \( j \), where \( 1 \leq j \leq N - 1 \), we find index \( i_1^{(j+1)} \), which is the first index of the application of \( DL^*(1, \beta(i_1^{(j)}) + 1) \) on \( \alpha^{(j)} \), the tail of \( \alpha \) starting from the index \( M_j \).

If \( \beta(i_1^{(j+1)}) \leq \beta(i_1^{(j)}) - 1 \), then, by \( DL^*(1, 2) \), the required pair of indices is found, and \( K = j + 1 \).

If \( \beta(i_1^{(j+1)}) \geq \beta(i_1^{(j)}) \), we repeat the procedure at most \( N = \alpha(i_1^{(1)}) + 2 \) number of times. Then indices \( i_1^{(1)} < i_2^{(1)} < \ldots < i_1^{(N)} \) will have been constructed for which, by the previous constructions, we have that

\[
\beta(i_1^{(1)}) \leq \beta(i_1^{(2)}) \leq \ldots \leq \beta(i_1^{(N)}).
\]

Applying \( DL^*(1, 2) \) on any extension of the finite sequence \( \alpha(i_1^{(1)}) , \alpha(i_2^{(2)}) , \ldots , \alpha(i_1^{(N)}) \), we find a pair of indices on which \( \alpha \) weakly increases. Since \( \beta \) already weakly increases on them, we have found the required pair based on an initial segment of \( \alpha, \beta \) of length at most \( M = \sum_{j=1}^{K} M_j \).

**Proposition 2.7 (DL*'(2,l), l \geq 3)**. If \( l \geq 3 \) and \( \alpha, \beta \in F(N, N) \), there exist \( i_1, i_2, \ldots, i_l \), and \( M_{\alpha, \beta}(2,l) \in N \) such that

\[
i_1 < i_2 < \ldots < i_l \leq M_{\alpha, \beta}(2,l) \quad \text{and} \quad i_1 < i_2 < \ldots < i_l \leq M_{\alpha, \beta}(2,l)
\]

where

\[
M_{\alpha, \beta}(2,l) = \sum_{k=1}^{K} M^{(k)},
\]

\[
1 \leq K \leq \Lambda,
\]

\[
\Lambda = \alpha(i_1^{(1)}) + 1,
\]

and \( i_1^{(1)} \) is the first index determined by the application of \( DL^*(1, l) \) on the sequence \( \alpha^*(n) = \alpha(i_n) \), where the indices \( i_n \) are formed as follows: \( i_1 \) is the first component of the common good pair resulting from the application of \( DL^*(2, 2) \) on \( \alpha, \beta \) requiring the initial segment of \( \alpha, \beta \) of length \( M_1 = M_{\alpha, \beta}(2,2) \), and \( i_{n+1} \) is the first component of the common good pair resulting from the application of \( DL^*(2, 2) \) on \( \alpha^{(n)}, \beta^{(n)} \), which are the tails of \( \alpha, \beta \) starting from index \( M_n \). Moreover,

\[
M^{(1)} = \sum_{j=1}^{N_1} M^{(j)}_1,
\]

\[
M^{(1)}_1 = M_{\alpha, \beta}(2,l),
\]

\[
M^{(1)}_j = M_{\alpha^{(j)}, \beta^{(j)}}(2,l),
\]

while

\[
\beta(i_1^{(1)}) \leq 1 \rightarrow K = 1,
\]

\[
\beta(i_1^{(1)}) \geq 2 \rightarrow M^{(2)} = \sum_{j=1}^{N_2} M^{(2)}_j,
\]

where \( N_2 = M_{\alpha, \beta}(1, 1, \beta(i_2^{(1)}) + 1) \), \( \alpha^{*(1)} \) is the tail of \( \alpha^* \) starting from index \( i_{N_1} \),

\[
M^{(2)}_1 = M_{\alpha^{(1)}, \beta^{(1)}}(2,l), M^{(2)}_{j+1} = M_{\alpha^{(j)}, \beta^{(j)}}(2,l),
\]

where \( \alpha^{(1)}, \beta^{(1)} \) are the tails of \( \alpha, \beta \) starting from index \( M^{(1)}_1 \) and \( \alpha^{(j)}, \beta^{(j)} \) are the tails of \( \alpha, \beta \), respectively, starting from index \( M^{(2)}_j \). If \( 2 \leq k \leq \Lambda - 1 \), then \( M^{(k)} \) is defined through \( M^{(k)}_j \)'s, \( 1 \leq j \leq N_k \), in a similar way.
Proof. For simplicity we show here only the case \( l = 3 \). Applying \( \text{DL}^*(2, 2) \) on \( \alpha, \beta \) using their initial segment of length \( M_1^{(1)} = M_{\alpha, \beta}(2, 2) \) we find a common good pair of indices \((i_1, j_1)\) for them. Then we apply \( \text{DL}^*(2, 2) \) on \( \alpha^{(1)}, \beta^{(1)} \), the tails of \( \alpha, \beta \) starting from index \( M_1 \), using the initial segment of them of length \( M_2^{(1)} = M_{\alpha^{(1)}, \beta^{(1)}}(2, 2) \), and we find a common good pair of indices \((i_2, j_2)\) for them. We repeat this procedure enough number of times so that the sequences \( \alpha^*(n) = \alpha(i_n), \beta^*(n) = \beta(i_n) \) reach a common good pair of indices \((i_s, i_t)\) for them. Then \((i_s, i_t, j_t)\) is the required good triplet for \( \alpha, \beta \). In order to find the pair \((i_s, i_t)\) we need to repeat the initial procedure so many times so that for the sequences \( \alpha^*, \beta^* \) we can find a common good pair of indices. It is clear that \( M, \Lambda \) and \( i^{(1)} \) as defined above for the case \( l = 3 \) determine the bound which corresponds to this proof. \( \square \)

The formulation of \( \text{DL}^*(3, 2) \) has a complexity similar to that of the formulation of \( \text{DL}^*(2, 3) \), while its proof follows the pattern of the proof of \( \text{DL}^*(2, 2) \). If \( \alpha, \beta, \gamma \) are given sequences, then applying \( \text{DL}^*(2, 3) \) on \( \alpha, \beta \) using their initial segment of length \( M_1 = M_{\alpha, \beta}(2, 3) \) we find indices \( i_1^{(1)} < i_2^{(1)} < i_3^{(1)} \leq M_1 \), such that both \( \alpha \) and \( \beta \) weakly increase on them. If \( \gamma(i_1^{(1)}) \leq 1 \), we are done, while if not, we apply \( \text{DL}^*(2, \mu + 1) \) on \( \alpha^{(1)}, \beta^{(1)} \), the tails of \( \alpha, \beta \) starting from the index \( M_1 \), where \( \mu = \gamma(i_1^{(1)}) \). Let \( i_1^{(2)} \) be the first index of this application. If \( \gamma(i_1^{(2)}) \leq \mu - 1 \) we stop, while if \( \gamma(i_1^{(2)}) \geq \mu \) we repeat the procedure. At any step, either we have found the required pair, or the sequence \( \gamma(i_1^{(1)}) \leq \gamma(i_1^{(2)}) \leq \ldots \), is formed. Our algorithm of finding the required pair terminates with bound \( M = M_{\alpha, \beta, \gamma}(3, 2) \), where \( M \) is the bound within which sequences \( \alpha(i_1^{(1)}), \alpha(i_1^{(2)}), \ldots \), and \( \beta(i_1^{(1)}), \beta(i_1^{(2)}), \ldots \), have a common good pair of indices. Consequently, this is a good pair for \( \gamma \) too. To determine \( M \) we work in a completely similar way to the determination of \( M_{\alpha, \beta}(2, 3) \).

If \( k > 3 \), the formulations of \( \text{DL}^*(k, 2) \), and of \( \text{DL}^*(k, l) \), for every \( l \geq 3 \), are similar to the formulations of \( \text{DL}^*(3, 2) \) and of \( \text{DL}^*(3, k) \), respectively. The general proofs

\[
\text{DL}^*(k, 2) \to \text{DL}^*(k, l),
\]

and

\[
\forall_l \geq 2(\text{DL}^*(k, l)) \to \text{DL}^*(k + 1, 2)
\]

are similar to the proofs of Propositions 2.4, 2.7 and the proof of \( \text{DL}^*(3, 2) \), respectively. Although we avoid here the cumbersome details of the general case, we may conclude the following regarding our proof of \( \text{DL}^*(k, l) \):

1. It is based on two simple repetitive arguments, a “horizontal” one, found in the proof of the implication \( \text{DL}^*(k, l) \to \text{DL}^*(k, l + 1) \), and a “vertical” one, found in the proof of the implication \( \forall_l \geq 2(\text{DL}^*(k, l)) \to \text{DL}^*(k + 1, 2) \). Both arguments depend on the simplest case \( \text{DL}^*(1, 2) \), something which is not the case in other constructive proofs of the finite cases of Dickson’s lemma (e.g., like the ones in \([29], [3]\)).

2. It provides a method to extract a bound \( M_{\alpha_1, \ldots, \alpha_k}(k, l) \) for \( \text{DL}(k, l) \).

3. Our proof of \( \forall_l \geq 2(\text{DL}^*(k, l)) \to \text{DL}^*(k + 1, 2) \) is the constructive analogue of the constructively non-accepted proof

\[
\text{DL}(k, \infty) \to \text{DL}(k + 1, l),
\]

according to which one first applies the case \( \text{DL}(k, \infty) \) on \( \alpha_1, \ldots, \alpha_k \) to determine some \( I_\infty \subseteq \mathbb{N} \), which is good for \( \alpha_1, \ldots, \alpha_k \), and then applies \( \text{DL}(1, l) \) on the subsequence of \( \alpha_{k+1} \) determined by \( I_\infty \). Here we replaced \( \text{DL}(k, \infty) \) by \( \forall_l \geq 2(\text{DL}^*(k, l)) \).

### 3 One-step unprovability results

The results included in this section are, as far as we know new, and they are motivated by our intuition that it is not possible to prove \( \text{DL}(k + 1, 2) \) from a finite number of cases \( \text{DL}(k, l) \) i.e., from “less information” than \( \forall_l \geq 2(\text{DL}(k, l)) \). First we show that no single case \( \text{DL}(1, l) \) proves \( \text{DL}(2, 2) \) “directly in one step”. We give a simple example to explain what we mean: if we define

\[
(n_1, n_2) \leq (m_1, m_2) \iff n_1 \leq m_1 \land n_2 \leq m_2,
\]
for every \(n_1, n_2, m_1, m_2 \in \mathbb{N}\), then
\[
\#_{f \in \mathbb{F}[\mathbb{N}, \mathbb{N}]} \forall_{n_1, n_2, m_1, m_2 \in \mathbb{N}} (f(n_1, n_2) \leq f(m_1, m_2) \rightarrow (n_1, n_2) \leq (m_1, m_2)),
\]
since, if there was such a function \(f\), then \(f(0, 1) > f(1, 0) > f(0, 1)\). From this we conclude that 
\(DL(1, 2)\) doesn’t prove \(DL(2, 2)\) in one step, since if there was such a function \(f\) and \(\alpha, \beta\) are given sequences, by \(DL(1, 2)\) on \((f(\alpha(n), \beta(n)))\) there are indices \(i < j\) such that
\[
f(\alpha(i), \beta(i)) \leq f(\alpha(j), \beta(j))
\]
hence
\[
(\alpha(i), \beta(i)) \leq (\alpha(j), \beta(j)).
\]
A positive version of the above negation is the following, constructively stronger, formula:
\[
\forall_{f \in \mathbb{F}[\mathbb{N}, \mathbb{N}]} \exists_{n_1, n_2, m_1, m_2 \in \mathbb{N}} (f(n_1, n_2) \leq f(m_1, m_2) \land (n_1, n_2) \not\in (m_1, m_2)).
\]
Next we prove constructively a strong form of this positive version, for arbitrary \(l > 1\), concluding that no single case \(DL(1, l)\) can prove \(DL(2, 2)\) in one step. In this way a “meta-mathematical” question leads to a positive mathematical fact. First we show the following lemma.

**Lemma 3.1.** Let \(M \in \mathbb{N}, l > 1\) and \(\alpha, \beta \in \mathbb{F}(\mathbb{N}, \mathbb{N})\).
\[
\exists_{n_1 < n_2 < \ldots < n_l} (\alpha(n_1) = \ldots = \alpha(n_l) < M \lor 
\beta(n_1) = \ldots = \beta(n_l) < M) \lor 
\exists_{m, n \in \mathbb{N}} (M \leq \alpha(n) \leq \beta(m) \lor M \leq \beta(n) \leq \alpha(m)).
\]

**Proof.** The number \(K = (l-1)M + 1\) is the bound on the length of a sequence colored with the \(M\) colors of \(\{0, \ldots, M-1\}\) in order to have a monochromatic subsequence of length \(l\) (this simple case of the finite pigeonhole principle has an immediate inductive proof within BISH). If all the first \(K\)-terms of \(\alpha\) are strictly smaller than \(M\), or all the first \(K\)-terms of \(\beta\) are strictly smaller than \(M\), then the conclusion follows immediately. Suppose that not all the first \(K\)-terms of \(\alpha\) and not all the first \(K\)-terms of \(\beta\) are strictly smaller than \(M\). The use of the principle of the excluded middle here is unproblematic as the related property is decidable. Hence, there are \(n_1, m_1 < K\) such that \(\alpha(n_1), \beta(m_1) \geq M\). We repeat the previous step on the tails \(\alpha^{(1)}, \beta^{(1)}\) of \(\alpha, \beta\) starting from \(\alpha(K + 1), \beta(K + 1)\), respectively. Then again either the first \(K\)-terms of \(\alpha^{(1)}\) are strictly smaller than \(M\), or the first \(K\)-terms of \(\beta^{(1)}\) are strictly smaller than \(M\). If not there are numbers \(n_2, m_2\) such that \(K < n_2, m_2 < 2K\) and \(\alpha(n_2), \beta(m_2) \geq M\). We repeat this procedure at most \(\Lambda = (\alpha(n_1) + 1)\)-number of times. If the first disjunct has not been proved, applying \(DL^*(1, 2)\) on the sequence
\[
\gamma(0) = \alpha(n_1), \quad \gamma(1) = \beta(m_1), \quad \gamma(2) = \alpha(n_2), \quad \gamma(3) = \beta(m_2), \ldots,
\]
we get an index \(i < \Lambda\) such that \(M \leq \alpha(n_i) \leq \beta(m_i) \lor M \leq \beta(n_i) \leq \alpha(n_{i+1})\).  

It is clear that the proof also works if \(M = 0\), and that \(n_1, n_2, \ldots, n_l, n, m \leq B = K(n_1 + 1)\) i.e., \(B\) is an extracted bound. If \(\phi_1, \ldots, \phi_n\) are formulas, then \(\bigwedge_{i=1}^{n} \phi_i \) \((\bigvee_{i=1}^{m} \phi_i)\) denotes the conjunction (disjunction) of \(\phi_1, \ldots, \phi_n\).

**Theorem 3.2.** If \(l > 1\) and \(m \in \mathbb{N}\), then
\[
\forall_{f \in \mathbb{F}[\mathbb{N}, \mathbb{N}]} \exists_{i_1, j_1, \ldots, i_l, j_l \in \mathbb{N}} \left( \bigwedge_{s=1}^{l-1} (m \leq i_s) \land \bigwedge_{s=1}^{l} (m \leq j_s) \land \bigwedge_{r=1}^{l-1} [f(i_r, j_r) \leq f(i_{r+1}, j_{r+1})] \land \bigwedge_{1 \leq r < s \leq l} (i_r, j_r) \not\in (i_s, j_s) \right).
\]
Proof. First we show this for the cases \( l = 2, 3 \) and then we prove that the case \( l - 2 \) implies the case \( l \), for every \( l > 3 \).

If \( l = 2 \), then fixing \( m \) and applying \( DL^*(1, 2) \) on the sequence

\[
\alpha(0) = f(m + 1, m), \quad \alpha(1) = f(m, m + 1),
\]

\[
\alpha(2) = f(m + 2, m), \quad \alpha(3) = f(m, m + 2), \ldots,
\]

i.e.,

\[
\alpha(2n) = f(m + (n + 1), m),
\]

\[
\alpha(2n + 1) = f(m, m + (n + 1)),
\]

we get \( i < \alpha(0) + 1 \) such that \( \alpha(i) \leq \alpha(i + 1) \). If \( i = 2k \), for some \( k \in \mathbb{N} \), then

\[
f(m + k + 1, m) \leq f(m, m + k + 1),
\]

and if \( i = 2k + 1 \), for some \( k \in \mathbb{N} \), then

\[
f(m, m + k + 1) \leq f(m + k + 1, m)
\]

while

\[
(m + k + 1, m) \not\leq (m, m + k + 1),
\]

\[
(m, m + k + 1) \not\leq (m + k + 2, m).
\]

If \( l = 3 \), we apply Lemma 3.1 on

\[
M = f(m + 1, m + 1),
\]

\[
l = 3,
\]

\[
\alpha(n) = f(m, m + n + 1),
\]

\[
\beta(n) = f(m + n + 1, m).
\]

If there are \( n_1 < n_2 < n_3 \) such that \( f(m, m + n_3 + 1) = f(m, m + n_2 + 1) = f(m, m + n_1 + 1) < M \), then

\[
(m, m + n_3 + 1) \not\leq (m, m + n_2 + 1) \land
\]

\[
(m, m + n_3 + 1) \not\leq (m, m + n_1 + 1) \land
\]

\[
(m, m + n_2 + 1) \not\leq (m, m + n_1 + 1).
\]

If there are \( n_1 < n_2 < n_3 \) such that \( \beta(n_3) = \beta(n_2) = \beta(n_1) < M \), we work similarly. Next we suppose that there exist indices \( i, j \) such that

\[
f(m + 1, m + 1) \leq f(m, m + i + 1) \leq f(m + j + 1, m).
\]

Again we conclude that

\[
(m + 1, m + 1) \not\leq (m, m + i + 1) \land
\]

\[
(m + 1, m + 1) \not\leq (m + j + 1, m) \land
\]

\[
(m + i + 1) \not\leq (m + j + 1, m).
\]

If there exist indices \( i, j \) such that \( f(m + 1, m + 1) \leq f(m + i + 1, m) \leq f(m, m + j + 1) \), we work similarly.

For the inductive step we fix \( f \) and we suppose that there exist \( i_1, j_1, i_2, j_2, \ldots, i_{l-2}, j_{l-2} \) such that

\[
m + 1 \leq i_1, j_1, i_2, j_2, \ldots, i_{l-2}, j_{l-2} \land
\]

\[
\bigwedge_{r=1}^{l-3} [f(i_r, j_r) \leq f(i_{r+1}, j_{r+1})] \land
\]

\[
\bigwedge_{r=1}^{l-3} \left[ f(i_r, j_r) \leq f(i_{r+1}, j_{r+1}) \right]
\]

\[
10
\]
Applying Lemma 3.1 on
\[ M = f(i_{l-2}, j_{l-2}), \]
\[ \alpha(n) = f(m, m + n + 1), \]
\[ \beta(n) = f(m + n + 1, m), \]
and working as in case \( l = 3 \), we reach the required conclusion for \( f \). Note that if \( 1 \leq r \leq l - 2 \), then \( (i_r, j_r) \not\in (m, m + i + 1) \) and \( (i_r, j_r) \not\in (m + j + 1, m) \), since by our hypothesis \( m + 1 \leq i_r, j_r \).

Next corollary is an immediate consequence of Theorem 3.2 (the condition \( m \leq i_1, j_1, i_2, j_2, \ldots, i_l, j_l \) in Theorem 3.2 which shows that many such \( l \)-tuples of natural numbers can be found, is not necessary to its proof).

**Corollary 3.3.** If \( l > 2 \), then
\[
\#_{f \in \mathbb{F}(\mathbb{N}^2, \mathbb{N})} \forall i_1, j_1, \ldots, i_{l+1}, j_{l+1} \in \mathbb{N} \left( \bigwedge_{r=1}^{l-1} [f(i_r, j_r) \leq f(i_{r+1}, j_{r+1})] \rightarrow \right.

\left. \bigvee_{1 \leq r < s \leq l} (i_r, j_r) \leq (i_s, j_s) \right).
\]

**Theorem 3.4.**
\[
\forall f_1, f_2 \in \mathbb{F}(\mathbb{N}^2, \mathbb{N}) \exists n_1, n_2, n_3, m_1, m_2, m_3 \in \mathbb{N} \left( f_1(n_1, n_2, n_3) \leq f_1(m_1, m_2, m_3) \land \right.

\left. f_2(n_1, n_2, n_3) \leq f_2(m_1, m_2, m_3) \land \right.

\left. (n_1, n_2, n_3) \not\in (m_1, m_2, m_3) \right).
\]

**Proof.** We suppose first that \( f_1(1, 0, 0) = f_2(1, 0, 0) = 0 \). Then \( f_1(1, 0, 0) \leq f_1(0, m_2, m_3) \), \( f_2(1, 0, 0) \leq f_2(0, m_2, m_3) \) and \((1, 0, 0) \not\in (0, m_2, m_3)\), for every \( m_2, m_3 \in \mathbb{N} \).

Next we suppose that \( f_1(1, 0, 0) = 0 \) and \( f_2(1, 0, 0) = l_2 > 0 \). Clearly, if there are \( m_2, m_3 \in \mathbb{N} \) such that \( f_2(0, m_2, m_3) \geq l_2 \), then \((1, 0, 0) \) and \((0, m_2, m_3) \) are the required triplets. Taking \( L = l_2 + 1 \) and \( m > 0 \) and applying Theorem 3.2 on \( L, m \) and the function \( f(i, j) = f_1(0, i, j) \) we find indices \( i_1, j_1, \ldots, i_L, j_L \geq m \) such that
\[
\bigwedge_{r=1}^{l_2} [f_1(0, i_r, j_r) \leq f_1(0, i_{r+1}, j_{r+1})] \land
\]

\[
\bigwedge_{r=1}^{l_2} [f_2(0, i_r, j_r) \leq f_2(0, i_{r+1}, j_{r+1})] \land
\]
Since the indices determined by Theorem 3.2 were larger than \( f \), we repeat the basic proof-step starting from \( f \) and \( m \). Hence by the previous basic proof-step there exist

\[
\begin{align*}
(i_r, j_r) &\neq (i_s, j_s).
\end{align*}
\]

Clearly \((i_r, j_r)\) and \((i_s, j_s)\) are the required triplets. Note that both of them are non-zero triplets, since the indices determined by Theorem were larger than \( m \) and \( m > 0 \).

We call the previous two cases the basic proof-step, and the arguments used for them work for any fixed non-zero triplet \((k_1, k_2, k_3)\) for which \( f_1(k_1, k_2, k_3) = f_2(k_1, k_2, k_3) = 0 \) and \( f_2(k_1, k_2, k_3) = l_2 > 0 \). If, for example, \( k_2 > 0 \), we consider the function \( f(n, m) = f_1(n, 0, m) \).

Finally, we treat the case \( f_1(1,0,0) = l_1 > 0 \) and \( f_2(1,0,0) = l_2 > 0 \). Without loss of generality we assume that \( l_1 \leq l_2 \). We consider the functions

\[
g_i(k_1, k_2, k_3) = f_i(k_1, k_2, k_3) - l_1,
\]

where \( \sim \) is the modified subtraction and \( i \in \{1,2\} \). Clearly, \( g_1(1,0,0) = 0 \) and \( g_2(1,0,0) = l_2 - l_1 \geq 0 \), hence by the previous basic proof-step there exist

\[
(n_1, n_2, n_3), (m_1, m_2, m_3) \neq (0,0,0)
\]

such that

\[
\begin{align*}
\bigwedge_{i=1}^{2} (g_i(n_1, n_2, n_3) - l_1) \leq (g_i(m_1, m_2, m_3) - l_1) \wedge (n_1, n_2, n_3) \neq (m_1, m_2, m_3).
\end{align*}
\]

First let \( f_i(n_1, n_2, n_3) \geq l_1 \), for every \( i \in \{1,2\} \), and we consider the following cases:

If \( f_i(n_1, n_2, n_3) > l_1 \), for every \( i \in \{1,2\} \), then \( f_i(m_1, m_2, m_3) \geq l_1 \), and hence

\[
\begin{align*}
\bigwedge_{i=1}^{2} (f_i(n_1, n_2, n_3) \leq f_i(m_1, m_2, m_3)) \wedge (n_1, n_2, n_3) \neq (m_1, m_2, m_3).
\end{align*}
\]

If \( f_1(n_1, n_2, n_3) = l_1 \) and \( f_1(m_1, m_2, m_3) < l_1 \), we repeat the previous basic proof-step starting from the two values \( f_2(m_1, m_2, m_3) \) and \( f_1(m_1, m_2, m_3) < l_1 \). If \( f_2(n_1, n_2, n_3) \geq l_1 \), then if \( f_2(n_1, n_2, n_3) > l_1 \), then \((n_1, n_2, n_3), (m_1, m_2, m_3)\) is the required pair of triplets, while if \( f_2(n_1, n_2, n_3) = l_1 \), we consider two cases: If \( f_2(m_1, m_2, m_3) < l_1 \), then we repeat the basic proof-step starting from the inequality \( f_1(m_1, m_2, m_3) \) and \( f_2(m_1, m_2, m_3) < l_1 \). If \( f_2(m_1, m_2, m_3) \geq l_1 \), then \((n_1, n_2, n_3), (m_1, m_2, m_3)\) is the required pair of triplets. If \( f_1(n_1, n_2, n_3) < l_1 \) or \( f_2(n_1, n_2, n_3) < l_1 \), we repeat the basic proof-step starting from \( f_1(n_1, n_2, n_3) \) and \( f_2(n_1, n_2, n_3) \). In each case either we find the required pair of triplets, or we find a starting triplet on which \( f_1 \) or \( f_2 \) has less value than at the starting triplet of the previous step. If we repeat the above steps at most \( l_1 \) number of times, we reach a basic proof-step, where \( f_1 \) or \( f_2 \) has on some non-zero triplet the value 0.

**Corollary 3.5.**

\[
\#f_1, f_2 \in \mathbb{P}[\mathbb{N}, \mathbb{N}] \forall n_1, n_2, n_3, m_1, m_2, m_3 \in \mathbb{N}
\]

\[
f_1(n_1, n_2, n_3) \leq f_1(m_1, m_2, m_3) \wedge
\]

\[\text{Classically this case has a simpler proof. Given functions } f_1, f_2 \text{ either one of them is 0 on some non-zero triplet, or not. In the latter case let } \Lambda_1 = \min\{f_1(n_1, n_2, n_3) \mid (n_1, n_2, n_3) \neq (0,0,0)\} \text{ and } \Lambda = \min\{\Lambda_1, \Lambda_2\}. \text{ If we consider the functions } g_1(n_1, n_2, n_3) = f_1(n_1, n_2, n_3) - \Lambda, \text{ there is a triplet on which one of them takes the value 0.}
\]

\[\text{It is easy to extract a bound from this proof considering the bound of Theorem 3.3. Note also that the whole argument can be rephrased as an inductive one over the minimum of the values of } f_1, f_2 \text{ on a non-zero triplet.}\]
\[ f_2(n_1, n_2, n_3) \leq f_2(m_1, m_2, m_3) \Rightarrow (n_1, n_2, n_3) \leq (m_1, m_2, m_3). \]

The above immediate consequence of Theorem can be interpreted as a mathematical formulation of the expression “DL(2, 2) doesn’t prove DL(3, 2) in one step”. If there were such functions \( f_1, f_2 \) and \( \alpha_1, \alpha_2, \alpha_3 \in F(\mathbb{N}^3, \mathbb{N}) \) are given, then applying DL(2, 2) on
\[
(f_1(\beta(n)))_n, \quad (f_2(\beta(n)))_n,
\]
where, for each \( n \in \mathbb{N}, \)
\[
\beta(n) = (\alpha_1(n), \alpha_2(n), \alpha_3(n)),
\]
we would get indices \( i < j \) such that
\[
f_1(\alpha_1(i), \alpha_2(i), \alpha_3(i)) \leq f_1(\alpha_1(j), \alpha_2(j), \alpha_3(j)) \land
f_2(\alpha_1(i), \alpha_2(i), \alpha_3(i)) \leq f_2(\alpha_1(j), \alpha_2(j), \alpha_3(j))
\]
which would imply
\[
(\alpha_1(i), \alpha_2(i), \alpha_3(i)) \leq (\alpha_1(j), \alpha_2(j), \alpha_3(j)).
\]

4 On the infinite cases of Dickson’s lemma

In this section we study the infinite cases of Dickson’s lemma from the point of view of constructive reverse mathematics (for more information on this subject see ). First we show the equivalence between the various infinite cases of Dickson’s lemma.

**Proposition 4.1.** If \( k > 1 \), the following are equivalent.
1. DL(1, \( \infty \)).
2. DL(k, \( \infty \)).
3. DL(1, U).
4. DL(k, U).

**Proof.** \( (i) \Rightarrow (ii) \) DL(1, \( \infty \)) is the first step in the inductive proof of DL(k, \( \infty \)). It is also used in the proof of the inductive step \( DL(k, \infty) \Leftrightarrow DL(k + 1, \infty) \). If \( \alpha_1, \alpha_2, \ldots, \alpha_{k+1} \in F(\mathbb{N}, \mathbb{N}), \) by DL(k, \( \infty \)), there is a sequence \( i_1 < i_2 < i_3 < \ldots, \) such that \( \alpha_n(i_1) \leq \alpha_n(i_2) \leq \alpha_m(i_3) \leq \ldots, \) for every \( m \in \{1, 2, \ldots, k\}. \)
If we apply DL(1, \( \infty \)) on the sequence \( \alpha_{m+1}(i_1), \alpha_{m+1}(i_2), \alpha_{m+1}(i_3), \ldots \), we get a weakly increasing subsequence of it. By hypothesis, the sequences \( \alpha_1, \alpha_2, \ldots, \alpha_k \) weakly increase on its indices too.

The implication \( (ii) \Rightarrow (i) \) is trivial.
Next we show that \( (i) \Rightarrow (iii) \). With the use of the principle of dependent choices DC a sequence \( s_0 < s_1 < \ldots < s_n < s_{n+1} < \ldots \), of elements of \( U \) is constructed. By DL(1, \( \infty \)) on the sequence \( \alpha^*, \)
where \( \alpha^*(n) = \alpha(s_n), \) for every \( n \in \mathbb{N}, \) a subsequence \( (k(n))_{n \in \mathbb{N}} \) is formed on which \( \alpha \) is good. But then \( \alpha \) is also good on \( M = \{ k(n) : n \in \mathbb{N} \}, \) and \( M \) is an unbounded subset of \( U. \)

The equivalence \( (iii) \Leftrightarrow (iv) \) is shown as the equivalence \( (i) \Leftrightarrow (ii). \)
Finally we show that \( (iii) \Rightarrow (i) \). If we take \( U = \mathbb{N}, \) then by DL(1, U) there exists \( M \) unbounded subset of \( \mathbb{N} \) such that \( i < j \Rightarrow \alpha(i) \leq \alpha(j), \) for every \( i, j \in M. \) With the use of DC a sequence \( m_0 < m_1 < \ldots < m_n < m_{n+1} < \ldots \) is formed in \( M \) such that \( \alpha(m_0) \leq \alpha(m_1) \leq \ldots \leq \alpha(m_n) \leq \alpha(m_{n+1}) \leq \ldots \).

In contrast to \( \forall l \geq 2(DL(1, l)), \) the infinite case DL(1, \( \infty \)) is not constructively acceptable. In Veldman gave a Brouwerian counterexample to DL(1, \( \infty \)). Here we show its constructive equivalence to LPO, which is the following formula
\[
\forall n \in F(\mathbb{N}, 2) \left( \exists n \in \mathbb{N}(\alpha(n) = 1) \lor \forall n \in \mathbb{N}(\alpha(n) = 0) \right).
\]

LPO is only classically true and a taboo for all varieties of constructive mathematics. Next we show that DL(1, \( \infty \)) implies LPO.
Proposition 4.2. DL(1, ∞) → LPO.

Proof. We prove that if \( \alpha \in \mathbb{F}(\mathbb{N}, 2) \), then \( \exists n \in \mathbb{N}(\alpha(n) = 1) \lor \forall n \in \mathbb{N}(\alpha(n) = 0) \), which is trivially equivalent to the original formulation of LPO. Applying DL(1, ∞) on \( \alpha \) we get a sequence of indices \( i_1 < i_2 < i_3 < \ldots \), such that \( \alpha(i_1) \leq \alpha(i_2) \leq \alpha(i_3) \leq \ldots \). Note that if \( \alpha(i_1) = 1 \), then \( \alpha(i_n) = 1 \), for each \( n \geq 1 \). Through \( \alpha \) we define a sequence \( \beta \in \mathbb{F}(\mathbb{N}, 2) \) by

\[
\beta(n) = \begin{cases} 
1 & \text{if } \forall m \leq n(\alpha(m) = 1) \\
0 & \text{if } \exists m \leq n(\alpha(m) = 0).
\end{cases}
\]

By DL(1, ∞) on \( \beta \), a sequence of indices \( j_1 < j_2 < j_3 < \ldots \), is formed such that \( \beta(j_1) \leq \beta(j_2) \leq \beta(j_3) \leq \ldots \). If \( \beta(j_1) = 0 \), then \( \exists m \leq j_1(\alpha(m) = 0) \), and the conclusion of LPO is reached. If \( \beta(j_1) = 1 \), then again \( \beta(j_m) = 1 \), for each \( m \in \mathbb{N} \). In that case we show that \( \forall n \in \mathbb{N}(\alpha(n) = 1) \). Consider a fixed \( n \in \mathbb{N} \). Then we can find \( i_k > n \) and \( j_l > k \). Since \( \beta(j_l) = 1, \forall m \leq j_l(\alpha(m) = 1) \). But \( k < j_l \) implies that \( n < i_k < i_{j_l} \), therefore \( \alpha(n) = 1 \).

In [24], p.148, Ratiu asked whether DL(\( k, U \)) implies LPO. By Propositions 4.1 and 4.2 we get an affirmative answer to this.

Proposition 4.3. If \( P(n) \) is a decidable predicate on \( \mathbb{N} \), then LPO \( \rightarrow [\forall n \in \mathbb{N}(P(n)) \lor \exists n \in \mathbb{N}(\neg P(n))] \).

Proof. If we define

\[
\alpha(n) = \begin{cases} 
1 & \text{if } \neg P(n) \\
0 & \text{if } P(n)
\end{cases}
\]

then LPO on \( \alpha \) is exactly \( \forall n \in \mathbb{N}(P(n)) \lor \exists n \in \mathbb{N}(\neg P(n)) \).

Definition 4.4. If \( i \in \mathbb{N} \) and \( \alpha \in \mathbb{F}(\mathbb{N}, \mathbb{N}) \), we call \( i \) a peak for \( \alpha \), Peak_\( \alpha \)(i), if and only if \( \forall n > i(\alpha(i) > \alpha(n)) \).

Proposition 4.5. If \( \alpha \in \mathbb{F}(\mathbb{N}, \mathbb{N}) \), then

\[
\text{LPO } \rightarrow \forall i \in \mathbb{N}\left(\text{Peak}_\alpha(i) \lor \exists n > i(\alpha(i) \leq \alpha(n))\right).
\]

Proof. If \( N_{>i} = \{n \in \mathbb{N} : n > i\} \) and \( e : \mathbb{N} \rightarrow N_{>i} \) is the bijection defined by \( e(n) = (n+1) + i \), for every \( n \in \mathbb{N} \), then for the decidable predicate

\[
P_i(n) \leftrightarrow \alpha(e(n)) < \alpha(i) \leftrightarrow \alpha((n+1) + i) < \alpha(i),
\]

Proposition 4.3 gives

\[
\forall n \in \mathbb{N}(\alpha((n+1) + i) < \alpha(i)) \lor \exists n \in \mathbb{N}(\alpha((n+1) + i) \geq \alpha(i)).
\]

Therefore, either \( i \) is a peak for \( \alpha \), or there is an index after \( i \) of at least the same value as \( i \) under \( \alpha \), which is exactly what we need to prove.

Proposition 4.6. LPO \( \rightarrow \) DL(1, ∞).

Proof. Through the previous decidability of Peak_\( \alpha \)(i) we define a sequence \( \beta \in \mathbb{F}(\mathbb{N}, 2) \) by

\[
\beta(n) = \begin{cases} 
0 & \text{if } \exists m > n(\alpha(n) \leq \alpha(m)) \\
1 & \text{if } \text{Peak}_\alpha(n)
\end{cases}
\]

By LPO, if \( \forall n \in \mathbb{N}(\beta(n) = 0) \leftrightarrow \forall n \in \mathbb{N}\exists m > n(\alpha(n) \leq \alpha(m)) \), then, since 0 is positively not a peak for \( \alpha \), \( \exists n_1 > 0(\alpha(0) \leq \alpha(n_1)) \). Similarly, \( \exists n_2 > n_1(\alpha(n_1) \leq \alpha(n_2)) \), and so on. By DC a sequence \( 0 = n_0 < n_1 < n_2 < \ldots \), is constructed such that \( \alpha(n_0) \leq \alpha(n_1) \leq \alpha(n_2) \leq \ldots \). If \( \exists n \in \mathbb{N}(\beta(n) = 1) \leftrightarrow \exists n \in \mathbb{N}(\text{Peak}_\alpha(n)) \), and if we consider the tail of \( \alpha \)

\[
\alpha(n+1), \alpha(n+2), \alpha(n+3), \ldots,
\]
then
\[ \alpha(j) \in \{0, 1, \ldots, \alpha(n) - 1\} , \]
for every \( j \geq n + 1 \). Since this tail of \( \alpha \) is a new sequence, then either it has positively no picks, and the previous case is applied, or there is some index \( n + m + 1 \) which is a peak for the sequence \( \alpha(n + 1), \alpha(n + 2), \alpha(n + 3), \ldots \). Since \( \alpha(n + m + 1) \in \{0, 1, \ldots, \alpha(n) - 1\} \), then \( \alpha(j) \in \{0, 1, \ldots, \alpha(n) - 2\} \), for every \( j > n + m + 1 \). After at most \( \alpha(n) - 1 \) number of steps we will have found a tail of \( \alpha \) with no peaks. If we apply then the argument of the first case, we reach our conclusion.

In analogy to Proposition 2.5 we show that \( DL(1, \infty) \) implies Stolzenberg’s principle \( PH(2, N, N) \).

**Proposition 4.7.** \( DL(1, \infty) \rightarrow PH(2, N, N) \).

**Proof.** Suppose that \( \alpha \in F(N, N) \) and that \( \chi \) is a \( 2 \)-coloring of \( \{\alpha_n : n \in N\} \). By \( DL(1, \infty) \) on \( \chi \circ \alpha : N \rightarrow 2 \) there are indices \( i_1 < i_2 < i_3 < \ldots \), such that \( \chi(\alpha_i) \leq \chi(\alpha_{i_2}) \leq \chi(\alpha_{i_3}) \leq \ldots \). Since \( DL(1, \infty) \rightarrow LPO \), either all terms of \( [\chi(\alpha_{i_n})]_n \) are 0, or there is a term \( \alpha_{i_n} \) such that \( \chi(\alpha_{i_n}) = 1 \). In the first case the sequence \( (\alpha_{i_n})_n \) itself is monochromatic, while in the second the tail \( \alpha_n, \alpha_{n+1}, \alpha_{n+2}, \ldots \) of \( \alpha \) is monochromatic.

### 5 Concluding remarks

The extraction of a bound \( M_{\alpha_1, \ldots, \alpha_k}(l) \) from our proof of \( DL(k, l) \) resembles the extraction of a term out of a proof in the field of program extraction. It is an example of term extracted in an informal system of mathematics, like BISH.

The following open questions, or tasks need to be addressed in future work.

1. To study further these terms \( M_{\alpha_1, \ldots, \alpha_k}(l) \), since by Berger’s constructive proof in [2] of Higman’s lemma for words of an alphabet with two letters by the finite cases of Dickson’s lemma, a bound for this case of Higman’s lemma can be formulated.

2. Results like Proposition 2.3 have already been implemented in MINLOG. The implementation forced the inductive formulation of appropriate lemmas that cover the repetitive arguments used in the informal proofs. It will be interesting to codify formally the more complex repetitive arguments found in the rest constructive proofs presented here.

3. To extend the tools found in the proofs of Theorems 3.2 and 3.3 in order to prove these results in complete generality.

4. To extend our study of the finite and infinite cases of Dickson’s lemma to a similar study of the finite and infinite cases of combinatorial theorems like Higman’s lemma, or Kruskal’s theorem.

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