Steinberg representations and harmonic cochains for split adjoint quasi-simple groups

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Abstract

Let $G$ be an adjoint quasi-simple group defined and split over a non-archimedean local field $K$. We prove that the dual of the Steinberg representation of $G$ is isomorphic to a certain space of harmonic cochains on the Bruhat-Tits building of $G$. The Steinberg representation is considered with coefficients in any commutative ring.

Introduction

Let $K$ be a non-archimedean local field. Let $G$ be the $K$-rational points of a reductive $K$-group of semi-simple rank $l$. Let $T$ be a maximal $K$-split torus in $G$ and let $P$ be a minimal parabolic $K$-subgroup of $G$ that contains $T$. There is an abuse of language because we mean the $K$-rational points of these algebraic subgroups of $G$. For a commutative ring $M$, the Steinberg representation of $G$ with coefficients in $M$ is the $M[G]$-module:

$$\text{St}(M) = \frac{C^\infty(G/P, M)}{\sum_Q C^\infty(G/Q, M)}$$

where $Q$ runs through all the parabolic subgroups of $G$ containing $P$.

In [1], A. Borel and J.-P. Serre, computed the reduced cohomology group $\tilde{H}^{l-1}(Y_t, M)$ of the topologized building $Y_t$ of the parabolic subgroups of $G$ and proved that we have an isomorphism of $M[G]$-modules:

$$\tilde{H}^{l-1}(Y_t, M) \cong \text{St}(M).$$
Then they "added" this building at infinity to the Bruhat-Tits building $X$ of $G$ to get $X$ compactified to a contractible space $Z_t = X \amalg Y_t$. Using the cohomology exact sequence of $Z_t$ mod. $Y_t$, they deduce an isomorphism of $M[G]$-modules:

$$H^l_c(X, M) \cong \tilde{H}^{l-1}(Y_t, M).$$

Thus, an isomorphism of $M[G]$-modules between the compactly supported cohomology of the Bruhat-Tits building and the Steinberg representation of $G$:

$$H^l_c(X, M) \cong St(M).$$

In case $G$ is simply connected and $M$ is the complex field $\mathbb{C}$, see A. Borel [3], if we consider $C^j(X, \mathbb{C})$ to be the space of $j$-dimensional cochains and $\delta : C^j(X, \mathbb{C}) \to C^{j-1}(X, \mathbb{C})$ the adjoint operator to the coboundary operator $d : C^j(X, \mathbb{C}) \to C^{j+1}(X, \mathbb{C})$ with respect to a suitable scalar product, we get the $l$th homology group $H_l(X, \mathbb{C})$ of this complex as the algebraic dual of the compactly supported cohomology group $H^l_c(X, \mathbb{C})$. So, with the isomorphism above, we get a $G$-equivariant $\mathbb{C}$-isomorphism:

$$H_l(X, \mathbb{C}) \cong \text{Hom}_{\mathbb{C}}(\text{St}(\mathbb{C}), \mathbb{C}).$$

A $j$-cochain $c \in C^j(X, \mathbb{C})$ is an harmonic cochain if we have $d(c) = \delta(c) = 0$. In case of chambers $j = l$, it is clear that we have $d(c) = 0$. So if we denote by $\text{Har}^l(\mathbb{C}, \mathbb{C})$ the space of the $\mathbb{C}$-valued harmonic cochains defined on the chambers of $X$, we have $\text{Har}^l(\mathbb{C}, \mathbb{C}) = Z_l(X, \mathbb{C}) = H_l(X, \mathbb{C})$, where $Z_l(X, \mathbb{C}) = \text{Ker} \delta$ is the space of the cycles at the level $l$ of the homological complex defined by $\delta$ above. Therefore

$$\text{Har}^l(\mathbb{C}, \mathbb{C}) \cong \text{Hom}_{\mathbb{C}}(\text{St}(\mathbb{C}), \mathbb{C}).$$

In the present work, we consider $G$ to be a split quasi-simple adjoint group. For any commutative ring $M$ and for any $M$-module $L$ on which we assume $G$ acts linearly, we define $\text{Har}^l(M, L)$ to be the space of $L$-valued harmonic cochains on the pointed chambers of the Bruhat-Tits building, where a pointed chamber means a chamber with a distinguished special vertex. The notion of harmonic cochains we use here is the same as above in case the group $G$ is also simply connected, otherwise since we are considering pointed chambers of the building there is an orientation property that our cochains should also satisfy. Using a result we have proved in our preceding paper [2] that gives the Steinberg representation of $G$ in terms of the parahoric subgroups of $G$, we prove explicitly that we have a canonical $M[G]$-isomorphism

$$\text{Har}^l(M, L) \cong \text{Hom}_M(\text{St}(M), L).$$

First, we give a very brief introduction to the Bruhat-Tits building to fix our notations. Then we recall the results obtained in [2], giving an expression of the Steinberg representation in terms of parahoric subgroups, we will also reformulate this result in way it becomes easier to see the link to the harmonic cochains. Finally, we introduce the space of harmonic cochains on the building and prove the isomorphism between this space and the dual of the Steinberg representation of $G$. 

2
1 Bruhat-Tits buildings

Notations Let $K$ be a non-archimedean local field, that is a complete field with respect to a discrete valuation $\omega$. We assume $\omega$ to have the value group $\omega(K^*) = \mathbb{Z}$.

We consider $G$ to be the group of $K$-rational points of an adjoint quasi-simple algebraic group defined and split over $K$. Let $T$ be a maximal split torus in $G$, $N = N_G(T)$ be the normalizer of $T$ in $G$ and $W = N/T$ be the Weyl group of $G$ relative to $T$.

The group of characters and the group of cocharacters of $T$ are respectively the free abelian groups $X^*(T) = \text{Hom}(T, GL_1)$ and $X_*(T) = \text{Hom}(GL_1, T)$.

There is a perfect duality over $\mathbb{Z}$

$$\langle \cdot, \cdot \rangle : X_*(T) \times X^*(T) \to \mathbb{Z} \cong X^*(GL_1)$$

with $\langle \lambda, \chi \rangle$ given by $\chi \circ \lambda(x) = x^{(\lambda, \chi)}$ for any $x \in GL_1(K)$.

Let $V = X_*(T) \otimes \mathbb{R}$ and identify its dual space $V^*$ with $X^*(T) \otimes \mathbb{R}$. Denote by $\Phi = \Phi(T, G) \subseteq X^*(T)$ the root system of $G$ relative to $T$. By the above duality, any root $\alpha$ induces a linear form $\alpha : V \to \mathbb{R}$. To every root $\alpha \in \Phi$ corresponds a coroot $\alpha^\vee \in V$, and a convolution $s_\alpha$ that acts on $V$ by

$$s_\alpha(x) = x - \langle x, \alpha \rangle \alpha^\vee.$$

This convolution $s_\alpha$ is the orthogonal reflection with respect to the hyperplane $H_\alpha = \text{Ker} \alpha$.

On the other side, we can see that the group $N$ acts on $X_*(T)$ by conjugations. This clearly induces an action of $W$ on $V$ by linear automorphisms. We can identify $W$ with the Weyl group $W(\Phi)$ of the root system $\Phi$, that is the subgroup of $GL(V)$ generated by all the reflections $s_\alpha$, $\alpha \in \Phi$.

Let $\Delta = \{1, 2, \ldots, l\}$ and let $D = \{\alpha_i; i \in \Delta\}$ be a basis of simple roots in $\Phi$. For any $i \in \Delta$, denote $s_i = s_{\alpha_i}$. Consider $S = \{s_i; i \in \Delta\}$. The pair $(W, S)$ is a finite Coxeter system.

Denote by $\Phi^\vee$ the coroot system dual to the root system $\Phi$. Denote by $Q(\Phi^\vee)$ (resp. $P(\Phi^\vee)$) the associated coroot lattice (resp. coweight lattice). Since we have assumed $G$ of adjoint type we have $X_*(T) = P(\Phi^\vee)$. 

3
The fundamental apartment  Let $A_0$ be the natural affine space under $V$. Denote by $\text{Aff}(A_0)$ the group of affine automorphisms of $A_0$. For $v \in V$, denote by $\tau(v)$ the translation of $A_0$ by the vector $v$. We have

\[ \text{Aff}(A_0) = V \rtimes GL(V). \]

There is a unique homomorphism

\[ \nu : T \rightarrow X_*(T) = P(\Phi^\vee) \subseteq V \]

such that $\langle \nu(t), \chi \rangle = -\omega(\chi(t))$ for any $t \in T$ and any $\chi \in X^*(T)$. In our situation this homomorphism is surjective.

An element $t \in T$ acts on $A_0$ by the translation $\tau(\nu(t))$ :

\[ tx := \tau(\nu(t))(x) = x + \nu(t), \quad x \in A_0, \]

so if we put $T_0 = \text{Ker} \nu$, this clearly induces an action of the so-called extended affine Weyl group $\tilde{W}_a := N/T_0$ on $A_0$. This group is an extension of the finite group $W$ by $T/T_0$ :

\[ \tilde{W}_a = N/T_0 = \frac{T}{T_0} \rtimes W \cong P(\Phi^\vee) \rtimes W \subseteq V \rtimes GL(V) = \text{Aff}(A_0). \]

We deduce an action of $N$ on $A_0$ by affine automorphisms that comes from the action of $T$ by translations on $A_0$ and the linear action of $W$ on $V$.

For any root $\alpha \in \Phi$ and any $r \in \mathbb{Z}$, let $H_{\alpha,r}$ be the hyperplane in $A_0$ defined by

\[ H_{\alpha,r} = \{ x \in A_0; \langle x, \alpha \rangle - r = 0 \}. \]

Let $s_{\alpha,r}$ be the orthogonal reflection with respect to $H_{\alpha,r}$. We have

\[ s_{\alpha,r} = \tau(r\alpha^\vee) \circ s_{\alpha}. \]  

The hyperplanes $H_{\alpha,r}$ define a structure of an affine Coxeter complex on $A_0$. Let $W_a$ be the associated affine Weyl group. It is a subgroup of the group $\text{Aff}(A_0)$ generated by the reflections $s_{\alpha,r}$ with respect to the hyperplanes $H_{\alpha,r}$. We have

\[ W_a \subseteq \text{Aff}(A_0) = V \rtimes GL(V). \]

In fact, $W_a$ is the semi-direct product of $Q(\Phi^\vee)$ and $W$ (see [1 Ch.VI,§ 2.1,Prop. 1])

\[ W_a = Q(\Phi^\vee) \rtimes W \subseteq P(\Phi^\vee) \rtimes W = \tilde{W}_a. \]

The Coxeter complex $A_0$ is the fundamental apartment of the Bruhat-Tits building.
The fundamental chamber  Let \( \tilde{\alpha} \) be the highest root in \( \Phi \). The fundamental chamber \( C_0 \) of the Bruhat-Tits building is the chamber with the bounding walls 

\[
H_{\alpha_i} = H_{\alpha_i,0}, \ldots, H_{\alpha_i} = H_{\alpha_i,0} \text{ and } H_{\tilde{\alpha},1}.
\]

It is the intersection in \( A_0 \) of the open half spaces 

\[
\langle x, \alpha_i \rangle > 0 \quad 1 \leq i \leq l \quad \text{and} \quad \langle x, \tilde{\alpha} \rangle < 1.
\]

Denote \( s_i = s_{\alpha_i} = s_{\alpha_i,0} \) for any \( i \), \( 1 \leq i \leq l \), and \( s_0 = s_{\tilde{\alpha},1} \). The set \( S_a = \{s_0, s_1, \ldots, s_l\} \) generates the affine Weyl group \( W_a \). The pair \( (W_a, S_a) \) is an affine Coxeter system and the topological closure \( \overline{C_0} \) of \( C_0 \) is a fundamental domain for the action of \( W_a \) on \( A_0 \).

The Bruhat-Tits building The Bruhat-Tits building \( X \) associated to \( G \) is defined as the quotient 

\[
X = \frac{G \times A_0}{\sim}
\]

where \( \sim \) is a certain equivalence relation on \( G \times A_0 \), see [2] or any reference on Bruhat-Tits buildings. The group \( G \) acts transitively on the chambers (the simplices of maximal dimension) of \( X \).

2 The Steinberg representation and the Iwahori subgroup

Let \( M \) be a commutative ring on which we assume \( G \) acts trivially. For a closed subgroup \( H \) of \( G \), denote by \( C^\infty(G/H, M) \) (resp. \( C^\infty_c(G/H, M) \)) the space of \( M \)-valued locally constant functions on \( G/H \) (resp. those which moreover are compactly supported). The action of the group \( G \) on the quotient \( G/H \) by left translations induces an action of \( G \) on the spaces \( C^\infty(G/H, M) \) and \( C^\infty_c(G/H, M) \).

Let \( P \) be the Borel subgroup of \( G \) that corresponds to the basis \( D \) of the root system \( \Phi \). For any \( i \in \Delta \), let \( P_i = P \coprod P s_i P \) be the parabolic subgroup of \( G \) generated by \( P \) and the reflection \( s_i \). The Steinberg representation of \( G \) is the \( M[G] \)-module 

\[
\text{St}(M) = \frac{C^\infty(G/P_i, M)}{\sum_{i \in \Delta} C^\infty(G/P_i, M)}.
\]

Now, let \( B \) be the Iwahori subgroup of \( G \) corresponding to \( P \). Recall from [2] Th. 3.4] that \( C^\infty(G/P, M) \) is generated as an \( M[G] \)-module by the characteristic function \( \chi_{BP} \) of
the open subset $BP/P \subseteq G/P$, and then that we have a surjective $M[G]$-homomorphism

$$\Theta : C_c^\infty (G/B, M) \longrightarrow C_c^\infty (G/P, M)$$

defined by $\Theta (\varphi ) = \sum_{g \in G/B} \varphi (g) g \cdot \chi_{BP}$.

For any $i \in \Delta$, let $B_i = B \coprod Bs_i B$ be the parahoric subgroup of $G$ that corresponds to the parabolic $P_i$. Let $\{ \varpi_i ; i \in \Delta \}$ be the fundamental coweights with respect to the simple basis $D$ and, by the surjective homomorphism (1), take $t_i \in T$ such that $\nu (t_i) = \varpi_i$. Computing the kernel of $\Theta$, cf. [loc. cit., Th. 4.1 and Cor. 4.2], we have:

**Proposition 2.1.** We have a canonical isomorphism of $M[G]$-modules:

$$St(M) \cong \frac{C_c^\infty (G/B, M)}{R + \sum_{i \in \Delta} C_c^\infty (G/B_i, M)}$$

where $R$ is the $M[G]$-submodule of $C_c^\infty (G/B, M)$ generated by the functions $\chi_{Bt_iB} - \chi_B$, $1 \leq i \leq l$.

Under the action of $G$ on the Bruhat-Tits building $X$, the Iwahori $B$ is the pointwise stabilizer of the fundamental chamber $C_0$. Let $B_0 = B \coprod Bs_0 B$ be the parahoric subgroup of $G$ generated by $B$ and the reflection $s_0$. The parahoric subgroups $B_i$, $0 \leq i \leq l$, are the pointwise stabilizers of the $l + 1$ codimension 1 faces of $C_0$.

We would like to reformulate the isomorphism in this proposition in such way the connection of the Steinberg representation to harmonic cochains on the Bruhat-Tits building looks more clear.

Denote by $l(w)$ the length of an element $w$ of the Coxeter group $W_a$ with respect to the set $S_a = \{ s_0, s_1, \ldots, s_l \}$ and recall that we can look at the linear Weyl group $W$ as the subgroup of $W_a$ generated by the subset $S = \{ s_1, \ldots, s_l \}$ of $S_a$.

**Lemma 2.1.** Let $g \in G$. For any $w \in W_a$ (resp. $w \in W$), we have

$$\chi_{BgB} - (-1)^{l(w)} \chi_{BgwB} \in \sum_{i=0}^l C_c^\infty (G/B_i, M) \quad \text{(resp.} \quad \sum_{i=1}^l C_c^\infty (G/B_i, M) \text{)}.$$

**Proof.** Let $u_1, \ldots, u_d \in S_a$ (resp. $\in S$) such that $w = u_1 \cdots u_d$ is a reduced expression in $W_a$ (resp. in $W$). We have

$$\chi_{BgB} - (-1)^d \chi_{BgwB} = \sum_{i=1}^d (-1)^{i-1} (\chi_{Bgu_1\cdots u_{i-1}B} + \chi_{Bgu_1\cdots u_iB}).$$
For any $i$, if $u_i$ is the reflection $s_j$ then $\chi_{Bgu_1\cdots u_{i-1}B} + \chi_{Bgu_1\cdots u_iB} \in C_c^\infty(G/B_j, M)$. □

Since we have assumed $G$ to be split quasi-simple, its root system $\Phi$ is reduced and irreducible. Thus, the Dynkin diagram of the root system $\Phi$ is one of the types described in [5], this classification is summarized in [loc. cit., Planches I-IX].

Let $\tilde{\alpha} = \sum_{i=1}^{l} n_i \alpha_i$ be the highest root of $\Phi$. From [5] Ch.VI, §2.2, Cor. of Prop. 5], we know that the $l+1$ vertices $v_i^0$ of the fundamental chamber $C_0$ are $v_0^0 = 0$ and:

$$v_i^0 = \varpi_i/n_i \quad \text{for } 1 \leq i \leq l.$$ 

To each vertex $v_i^0$ of the fundamental chamber $C_0$ we give the label $i$. This gives a labeling of the chamber and then of the whole building $X$.

Denote by $J$ the subset of $\Delta = \{1, 2, \ldots, l\}$ given by $n_i = 1$. Notice that, except for a group of type $A_l$, the coroot $\tilde{\alpha}^\vee$ dual to the highest root is equal to some fundamental coweight $\varpi_{i_0}$, $i_0 \in \Delta - J$, that induces a special automorphism on $X$, i.e. an automorphism of $X$ that preserves labels. So, from (2), we get

$$\tau(\varpi_{i_0}) = \tau(\tilde{\alpha}^\vee) = s_{\tilde{\alpha}, 1}s_{\tilde{\alpha}}.$$ 

**Theorem 2.1.** Assume $G$ is not of type $A_l$. We have a canonical isomorphism of $M[G]$-modules:

$$\text{St}(M) \cong \frac{C_c^\infty(G/B, M)}{R' + \sum_{i=0}^{l} C_c^\infty(G/B_i, M)}$$

where $R'$ is the $M[G]$-submodule of $C_c^\infty(G/B, M)$ generated by the functions $\chi_{Bt_iB} - \chi_B$, $i \in J$.

**Proof.** From Proposition 2.1 we need to prove the equality

$$R + \sum_{i=1}^{l} C_c^\infty(G/B_i, M) = R' + \sum_{i=0}^{l} C_c^\infty(G/B_i, M).$$

Let us prove that the left hand side is contained in the right hand side. Let $i \in \Delta - J$. Then $t_i$ acts on $X$ as a special automorphism. So, the chamber $t_iC_0$ is a chamber of the apartment $A_0$ that is of the same type as $C_0$, the same type means that any vertex $t_i v_j^0$ of the chamber $t_i C_0$ has the same label $j$ of $v_j^0$. Therefore, there is $w \in W_a$ such that $t_i C_0 = w C_0$. This means that $\chi_{Bt_iB} = \chi_{BwB}$ and $w$ is of even length. From Lemma 2.1 we get

$$\chi_{Bt_iB} - \chi_B = (-1)^{l(w)} \chi_B - \chi_B = 0 \mod \sum_{i=0}^{l} C_c^\infty(G/B_i, M).$$
Therefore, $\chi_{Bt_iB} - \chi_B \in \sum_{i=0}^l C^\infty_c (G/B_i, M)$.

Now, let us prove the other inclusion. Again from Lemma 2.1 we have

$$\chi_{B_0} = \chi_{B_{s_0}B} + \chi_B = \chi_{B_{s_0}s_\alpha B} + \chi_B = (-1)^{(s_\alpha)} \chi_{B_{s_0}s_\alpha B} + \chi_B \mod \sum_{i=1}^l C^\infty_c (G/B_i, M).$$

As we have seen, there is an $i_0 \in \Delta - J$ such that $Bt_{i_0}B = Bs_0s_\alpha B$ and $s_\alpha \in W$ being a reflection it is of odd length. Therefore,

$$\chi_{B_0} = -\chi_{Bt_{i_0}B} + \chi_B \mod \sum_{i=1}^l C^\infty_c (G/B_i, M),$$

and this finishes the proof. \hfill $\square$

**Remark 2.1.** In case $G$ is adjoint simply connected group, so of type $E_8$, $F_4$ or $G_2$, the subset $J$ of $\Delta$ is empty, and therefore the $M[G]$-submodule $R'$ is trivial. The theorem above gives an isomorphism of $M[G]$-modules:

$$St(M) \cong \frac{C^\infty_c (G/B, M)}{\sum_{i=0}^l C^\infty_c (G/B_i, M)}.$$

### 3 Steinberg representation and harmonic cochains

Recall that the vertex $v_0^0$ of $C_0$ is a special vertex and that every chamber of the building has at least one special vertex.

Let $v_i^0$ be a special vertex of $C_0$, this means that $i \in J$ and that $t_i$ is a non-special automorphism of $X$. Let $w_0$ be the longest element in $W$ and $w_i$ be the longest element in the Weyl group of the root system of linear combinations of the simple roots $\alpha_j, j \neq i$. Then, see [5] Ch. VI, §2.3, Prop. 6], we have $t_i w_i w_0 C_0 = C_0$.

Denote by $\hat{X}^l$ the set of pointed chambers of $X$. A pointed chamber of $X$ is a pair $(C, v)$ where $C$ is a chamber and $v$ is a vertex of $C$ which is special. The map which to $gB$ associates the pointed chamber $g(C_0, v_0^0)$ gives a bijection

$$G/B \sim \hat{X}^l.$$  \hspace{1cm} (4)

There is a natural ordering on the vertices of a pointed chamber. Indeed, we have

$$(C_0, v_0^0) = (v_0^0, v_1^0, \ldots, v_l^0),$$
which corresponds to the ordering of the vertices of the extended Dynkin diagram, and if we choose to distinguish another special vertex in \( C_0 \) then the ordering on the vertices of \( C_0 \) will be the one that corresponds to the ordering of the vertices of the extended Dynkin diagram we get when applying the automorphism of the Dynkin graph that takes 0 to the label of the new special vertex we have chosen. We have:

**Lemma 3.1.** Let \( \sigma_i \) be the permutation of the set \( \{0, 1, \ldots, l\} \) such that

\[
t_i w_i w_0 (v_0^\circ, v_1^\circ, \ldots, v_l^\circ) = (v_{\sigma_i(0)}^\circ, v_{\sigma_i(1)}^\circ, \ldots, v_{\sigma_i(l)}^\circ),
\]

then

\[
\text{sign}(\sigma_i) = (-1)^{l(w_i w_0)}.
\]

**Proof.** For any \( k \in \{0, 1, \ldots, l\} \), we have

\[
v_{\sigma_i(k)}^\circ = t_i w_i w_0 (v_k^\circ) = w_i w_0 (v_k^\circ) + \omega_i = w_i w_0 (v_k^\circ) + v_i^\circ,
\]

thus \( w_i w_0 (v_k^\circ) = v_{\sigma_i(k)}^\circ - v_i^\circ \). So if we compute the determinant of the linear automorphism \( w_i w_0 \) of the vector space \( V \) in the basis \( (v_0^\circ, v_2^\circ, \ldots v_l^\circ) \), we get

\[
det(w_i w_0) = det(v_{\sigma_i(1)}^\circ - v_i^\circ, v_{\sigma_i(2)}^\circ - v_i^\circ, \ldots, v_{\sigma_i(j-1)}^\circ - v_i^\circ, -v_i^\circ, v_{\sigma_i(j+1)}^\circ - v_i^\circ, \ldots, v_{\sigma_i(l)}^\circ - v_i^\circ),
\]

where \( j \in \{1, 2, \ldots, l\} \) is such that \( \sigma_i(j) = 0 \). By subtracting the \( j \)th vector \(-v_i^\circ\) from the other vectors of the determinant, we get

\[
det(w_i w_0) = -det(v_{\sigma_i(1)}^\circ, v_{\sigma_i(2)}^\circ, \ldots, v_{\sigma_i(j-1)}^\circ, v_i^\circ, v_{\sigma_i(j+1)}^\circ, \ldots, v_{\sigma_i(l)}^\circ)
\]

\[
= -det(v_{\tau\sigma_i(1)}^\circ, v_{\tau\sigma_i(2)}^\circ, \ldots, v_{\tau\sigma_i(j-1)}^\circ, v_{\tau\sigma_i(j)}^\circ, v_{\tau\sigma_i(j+1)}^\circ, \ldots, v_{\tau\sigma_i(l)}^\circ)
\]

where \( \tau = (0 \ i) \) is the transposition that interchanges 0 and \( i \). Therefore,

\[
det(w_i w_0) = -\text{sign}(\tau \sigma_i) = \text{sign}(\sigma_i),
\]

and it is clear that \( det(w_i w_0) = (-1)^{l(w_i w_0)} \). \( \Box \)

Denote by \( \hat{X}^{l-1} \) the set of all codimension one simplicies of \( X \) that are ordered sets of \( l \) vertices \( \eta = (v_0, \ldots, v_i, \ldots, v_l) \) such that \( v_i \) is an omitted vertex from a pointed chamber \( C = (v_0, \ldots, v_i, \ldots, v_l) \in \hat{X}^l \). We write \( \eta < C \).

Denote by \( M[\hat{X}^l] \) the free \( M \)-module generated by the set of the pointed chambers of \( X \) and let \( L \) be an \( M \)-module on which we assume \( G \) acts linearly.

**Definition 3.1.** Let \( h : M[\hat{X}^l] \to L \) be an \( M \)-homomorphism. We say that \( h \) is a harmonic cochain on \( X \) if it satisfies the following properties.
(HC1) Let $C = (v_0, v_1, \ldots, v_l) \in \hat{X}^l$. Let $\sigma$ be a permutation of $\{0, 1, \ldots, l\}$ such that $v_{\sigma(0)}$ is a special vertex and that $C_\sigma = (v_{\sigma(0)}, v_{\sigma(1)}, \ldots, v_{\sigma(l)}) \in \hat{X}^l$. Then

$$h(C) = (-1)^{\text{sign}(\sigma)} h(C_\sigma)$$

(HC2) Let $\eta \in \hat{X}^{l-1}$ be a codimension one simplex. Let $B(\eta) = \{ C \in \hat{X}^l | \eta < C \}$, then

$$\sum_{C \in B(\eta)} h(C) = 0.$$

Denote by $\text{Har}^l(M, L)$ the set of harmonic cochains.

The action of $G$ on $\text{Har}^l(M, L)$ is induced from its natural action on $\text{Hom}_M(M[\hat{X}^l], L)$, namely

$$(g, h)(C) = gh(g^{-1}C)$$

for any $h \in \text{Har}^l(M, L)$, any $g \in G$ and any $C \in \hat{X}^l$.

Remark 3.1. In case of groups that are adjoint and simply connected, so of type $E_8$, $F_4$ and $G_2$, there is no non-special automorphism and therefore the first property (HC1) of harmonic cochains is voided.

To prove the main theorem we need the following lemma

Lemma 3.2. Let $g \in G$. For any $w \in W$, we have :

$$\chi_{BgP} - (-1)^{l(w)} \chi_{B gwP} \in \sum_{i=1}^l C^\infty (G/P_i, M).$$

Proof. The same arguments as in the proof of Lemma 2.1.

Theorem 3.1. We have an isomorphism of $M[G]$-modules

$$\text{Har}^l(M, L) \cong \text{Hom}_M(\text{St}(M), L).$$

Proof. Consider the map

$$\mathcal{H} : \text{Hom}_M(\text{St}(M), L) \longrightarrow \text{Hom}_M(M[\hat{X}^l], L)$$

which to $\varphi \in \text{Hom}_M(\text{St}(M), L)$ associates $h_\varphi$ defined by $h_\varphi(g(C_0, v_0^0)) = \varphi(g \chi_{BP})$ for any $g \in G$. Let us show that $h_\varphi = \mathcal{H}(\varphi)$ is a harmonic cochain.
(HC1) Let \( v_i^0 \) be a special vertex of \( C_0 \), this means that \( v_i^0 = t_i v_0^0 \) with \( i \in J \). Since \( t_i w_i w_0 \) normalizes \( B \) we have
\[
\mathfrak{h}_\varphi(C_0, v_i^0) = \mathfrak{h}_\varphi(t_i w_i w_0(C_0, v_0^0)) = \varphi(t_i w_i w_0 \chi_{BP}) = \varphi(\chi_{B t_i w_i w_0 P}) ,
\]
and by Lemma 3.2 and since \( t_i \in P \), we have
\[
\varphi(\chi_{B t_i w_i w_0 P}) = (-1)^{l(w_i w_0)} \varphi(\chi_{B t_i P}) = (-1)^{l(w_i w_0)} \varphi(\chi_{BP}) = (-1)^{l(w_i w_0)} \mathfrak{h}_\varphi(C_0, v_i^0) .
\]
Now apply Lemma 3.1.

(HC2) Let \( \eta \in \hat{X}^{l-1} \). We can assume that \( \eta = (v_0^0, v_1^0, \ldots, v_l^0) \) is a face of the pointed fundamental chamber \( \eta = (C_0, v_0^0) \). Recall from [2] that \( B_i P_i = B_i P = \prod_{b \in B_i/B} b BP \), therefore
\[
\sum_{C \in B(\eta)} \mathfrak{h}_\varphi(C) = \sum_{b \in B_i/B} \mathfrak{h}_\varphi(b(C_0, v_0^0)) = \sum_{b \in B_i/B} \varphi(b \chi_{BP}) = \varphi(\chi_{B_i P_i}) = 0 .
\]

Now, consider the map
\[
\Psi : \text{Har}^l(M, L) \longrightarrow \text{Hom}_M(C_c^\infty(G/B, M), L)
\]
which to \( \mathfrak{h} \in \text{Har}^l(M, L) \) associates \( \psi_\mathfrak{h} \) defined by \( \psi_\mathfrak{h}(g \chi_B) = \mathfrak{h}(g(C_0, v_0^0)) \). Let us show that \( \psi_\mathfrak{h} \) vanishes on the \( M[G] \)-submodule \( R' + \sum_{i=0}^l C_c^\infty(G/B_i, M) \) of \( C_c^\infty(G/B, M) \). First, since \( \mathfrak{h} \) is harmonic, from (CH2) we deduce that for any \( i, 0 \leq i \leq l \), we have
\[
\psi_\mathfrak{h}(\chi_{B_i}) = \sum_{b \in B_i/B} \psi_\mathfrak{h}(b \chi_B) = \sum_{b \in B_i/B} \mathfrak{h}(b(C_0, v_0^0)) = \sum_{C \in B(\eta)} \mathfrak{h}(C) = 0 ,
\]
where \( \eta = (v_0^0, \ldots, v_{i-1}^0, v_i^0, v_{i+1}^0, \ldots, v_l^0) \), so \( \psi_\mathfrak{h} \) vanishes on \( \sum_{i=0}^l C_c^\infty(G/B_i, M) \). In case \( t_i \) is a non-special automorphism of \( X \), we have
\[
\chi_{B t_i B} = \chi_{B t_i w_i w_0 w_i w_0 B} = t_i w_i w_0 \chi_{B w_0 w_i B} ,
\]
therefore,
\[
\psi_\mathfrak{h}(\chi_{B t_i B} - \chi_B) = \psi_\mathfrak{h}((-1)^{l(w_i w_0)} t_i w_i w_0 \chi_B - \chi_B) = (-1)^{l(w_i w_0)} \mathfrak{h}(t_i w_i w_0(C_0, v_0^0)) - \mathfrak{h}(C_0, v_0^0) .
\]
Let \( \sigma_i \) be the permutation of \( \{0, 1, \ldots, l\} \) such that
\[
t_i w_i w_0(v_0^0, v_1^0, \ldots, v_l^0) = (v_{\sigma_i(0)}^0, v_{\sigma_i(1)}^0, \ldots, v_{\sigma_i(l)}^0) .
\]
Since \( \mathfrak{h} \) is harmonic and as so satisfy the property (HC1), we have
\[
\mathfrak{h}(t_i w_i w_0(C_0, v_0^0)) = (-1)^{\text{sign}(\sigma_i)} \mathfrak{h}(C_0, v_0^0) ,
\]

Therefore,

\[ \psi_h(\chi_{Bt_B} - \chi_B) = (-1)^l(\omega \omega_i)(-1)^{\text{sign}(\sigma_i)} h(C_0, v_0^c) - h(C_0, v_0^c), \]

and from Lemma 3.1 we deduce that

\[ \psi_h(\chi_{Bt_B} - \chi_B) = 0. \]

Finally, if we denote by \( \Theta^* \) the dual homomorphism of \( \Theta \), by Theorem 2.1 we have an \( M[G] \)-homomorphism

\[ \Phi = \Theta^* \circ \Psi : \text{Har}^f(M, L) \rightarrow \text{Hom}_M(\text{St}(M), L) \]

which sends a harmonic cochain \( h \) to \( \varphi_h \) defined by \( \varphi_h(g \chi_{BP}) = h(g(C_0, v_0^c)) \) for any \( g \in G \).

It is easy to prove that \( \Phi \) and \( H \) are inverse of each other.

\[ \square \]

**Remark 3.2.** In case \( G = \text{PGL}_{l+1}(K) \), so of type \( A_l \), the isomorphism in the theorem above is established in [1].

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