Standard $\lambda$-lattices, rigid $C^*$ tensor categories, and (bi)modules

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Abstract

In this article, we construct a 2-shaded rigid $C^*$ multitensor category with canonical unitary dual functor directly from a standard $\lambda$-lattice. We use the notions of traceless Markov towers and lattices to define the notion of module and bimodule over standard $\lambda$-lattice(s), and we explicitly construct the associated module category and bimodule category over the corresponding 2-shaded rigid $C^*$ multitensor category.

As an example, we compute the modules and bimodules for Temperley-Lieb-Jones standard $\lambda$-lattices in terms of traceless Markov towers and lattices. Translating into the unitary 2-category of bigraded Hilbert spaces, we recover DeCommer-Yamshita’s classification of TLJ modules in terms of edge weighted graphs, and a classification of TLJ bimodules in terms of bimunitary connections on square-partite weighted graphs.

As an application, we show that every (infinite depth) subfactor planar algebra embeds into the bipartite graph planar algebra of its principal graph.

Introduction

Since Jones landmark article [Jo83], the modern theory of subfactors has developed deep connections to numerous branches of mathematics, including representation theory, category theory, knot theory, topological quantum field theory, statistical mechanics, conformal field theory, and free probability. The standard invariant of a type $\text{II}_1$ subfactor was first defined as a standard $\lambda$-lattice [Po95]. Since, it has been reinterpreted as a planar algebra [Jo99] and a $\mathcal{Q}$-system [Lo89], or unitary Frobenius algebra object, in a rigid $C^*$ tensor category [Mii03].

For a given standard $\lambda$-lattice, Jones proved in [Jo99, Thm. 4.2.1] that one can construct a subfactor planar algebra by passing through Popa’s subfactor reconstruction theorem [Po95, Thm. 3.1]. One primary motivation of this paper is to provide a construction of a 2-shaded rigid $C^*$ multitensor category directly from a standard $\lambda$-lattice without reconstructing a subfactor.

Theorem A. There is a bijective correspondence between equivalence classes of the following:

\[
\left\{ \text{Standard } \lambda\text{-lattices } A = (A_{i,j})_{0 \leq i \leq j} \right\} \cong \left\{ \text{Pairs } (\mathcal{A}, X) \text{ with } \mathcal{A} \text{ a 2-shaded rigid } C^* \text{ multitensor category with a generator } X, \text{ i.e., } 1_{\mathcal{A}} = 1^+ \oplus 1^-, 1^+, 1^- \text{ are simple and } X = 1^+ \otimes X \otimes 1^- \right\}
\]

Equivalence on the left hand side is unital $*$-isomorphism of standard $\lambda$-lattices; equivalence on the right hand side is unitary equivalence between their Cauchy completions which maps generator to generator.
Given \((A, X)\), we obtain a standard \(\lambda\)-lattice \(A\) by

\[
A_{i,j} := \begin{cases} 
\text{id}_{X^{alt\otimes 2k}} \otimes \text{End}\left(X^{alt\otimes (j-2k)}\right) & i = 2k \\
\text{id}_{X^{alt\otimes (2k+1)}} \otimes \text{End}\left(X^{alt\otimes (j-2k-1)}\right) & i = 2k + 1
\end{cases}
\]

where \(\overline{X}\) is a dual of \(X\) and

\[
X^{alt\otimes n} := X \otimes X \otimes X \otimes \cdots \text{\(n\) tensorands}
\]

and similarly for \(X^{alt\otimes n}\). The inclusion \(A_{i,j} \subset A_{i,j+1}\) sends \(x\) to \(x \otimes \text{id}\), the inclusion \(A_{i+1,j} \subset A_{i,j}\) sends \(x\) to \(x\). The Jones projections are defined using the canonical balanced evaluation and coevaluation for \(X\).

Going the other way is harder. Using [CHPS18, Def. 3.1], we construct a skeletal (when \(d > 1\)) \(W^*\) category explicitly from \(A\) whose objects are \([n, \pm]\) for \(n \geq 0\) and whose hom spaces can be identified with the algebras \(A_{i,j}\). We endow it with a tensor structure using the 2-shift map in the standard \(\lambda\)-lattice, which is a trace-preserving \(*\)-isomorphism \(S_{i,j} : A_{i,j} \to A_{i+2,j+2}\) [Bi97, Cor. 2.8]. We call this skeletal category a **planar tensor category**, and we provide a string diagram calculus to perform computations. The Cauchy completion of this planar tensor category is the target 2-shaded rigid \(C^*\) multitensor category.

Given a standard \(\lambda\)-lattice \(A\), an \(A\)-module is a Markov tower as a standard \(A\)-module. In more detail, let \(A = (A_{i,j})_{0 \leq i, j < \infty}\) be a standard \(\lambda\)-lattice with Jones projection \(\{e_i\}_{i \geq 1}\) and compatible conditional expectations. An \(A\)-module is a **Markov tower** of finite dimensional von Neumann algebras \((M_n)_{n \geq 0}\) such that \(A_{0,n} \subset M_n\) together with conditional expectations \(E_i : M_i \to M_{i-1}\) implemented by the Jones projections, which satisfy the appropriate commuting square conditions.

\[
\begin{align*}
M_0 & \subset M_1 \subset M_2 \subset \cdots \subset M_n \subset \cdots \\
A_{0,0} & \subset A_{0,1} \subset A_{0,2} \subset \cdots \subset A_{0,n} \subset \cdots \\
A_{1,0} & \subset A_{1,1} \subset A_{1,2} \subset \cdots \subset A_{1,n} \subset \cdots
\end{align*}
\]

We refer the reader to Definition 1.1.3 below for the complete definition.

We warn the reader that our definition is slightly different from the original one from [CHPS18, Def. 3.1]; our tower of algebras \((M_n)_{n \geq 0}\) does not necessarily have a Markov trace. An important difference in our construction is that we do **not** use the trace, but rather the commuting square of conditional expectations. In §2.3, by using this technique, we are able to discuss arbitrary modules over a standard \(\lambda\)-lattice instead of merely pivotal modules.

We call an \(A\)-module **standard** if \([M_i, A_{k,l}] = 0\) for \(i \leq k \leq l\). By similar techniques used to prove Theorem A above, we obtain the following theorem.

**Theorem B.** There is a bijective correspondence between equivalence classes of the following:

\[
\begin{align*}
\left\{ \text{Traceless Markov towers } M = (M_l)_{l \geq 0} \text{ with } \dim(M_0) = 1 \text{ as standard right modules over a standard } \lambda\text{-lattice } A \right\} & \cong \left\{ \text{Pairs } (M, Z) \text{ with } M \text{ an indecomposable semisimple right } A\text{-module } C^*\text{ category} \right. \\
& \left. \text{together with a choice of simple object } Z = Z < 1^+_A \right\}
\end{align*}
\]

Equivalence on the left hand side is \(*\)-isomorphism of traceless Markov towers as standard \(A\)-modules; equivalence on the right hand side is unitary \(A\)-module equivalence on Cauchy completions which maps the simple base object to simple base object.

Tracial Markov towers as standard \(A\)-modules correspond to pivotal \(A\)-module categories.
In §3, we discuss bimodules. Given two standard $\lambda$-lattices $A$ and $B$, we define an $A - B$ bimodule as a **standard Markov lattice**, which consists of a doubly indexed sequence $M = (M_{i,j})_{i,j \geq 0}$ of finite dimensional von Neumann algebras with two sequences of Jones projections $(e_i)_{i \geq 1}$ and $(f_j)_{j \geq 1}$ where the following conditions hold.

(a) $M_{i,j} \subset M_{i,j+1}$ and $M_{i,j} \subset M_{i+1,j}$ are unital inclusions.

(b) $M_{i,j} = (M_{i,j}, E_{i,j}^{M_{i,j}}, e_{i+1})_{i,j \geq 0}$ are Markov towers with the same modulus $d_0$ and $e_i \in M_{i+1,j}$ for all $i$; $M_{i,j} = (M_{i,j}, E_{i,j}^{M_{i,j}}, f_{j+1})_{j \geq 0}$ are Markov towers with the same modulus $d_1$ and $f_j \in M_{i,j+1}$ for all $j$. We call $M$ of modulus $(d_0, d_1)$.

$$M_{i+1,j} \subset M_{i+1,j+1} \cup M_{i,j} \subset M_{i,j+1}$$

(c) The commuting square condition:

$$
\begin{array}{c}
M_{i+1,j} \\
\downarrow M_{i,j}
\end{array}
\begin{array}{c}
E_{i+1,j+1}^{M_{i+1,j+1}}
\end{array}
\begin{array}{c}
\downarrow E_{i+1,j+1}^{M_{i,j+1}}
\end{array}
\begin{array}{c}
M_{i+1,j+1}
\end{array}
\begin{array}{c}
E_{i,j+1}^{M_{i,j}}
\end{array}
\begin{array}{c}
\downarrow E_{i,j+1}^{M_{i,j+1}}
\end{array}
\begin{array}{c}
M_{i,j+1}
\end{array}
$$

is a commuting square, i.e., $E_{i,j+1}^{M_{i,j}} \circ E_{i+1,j+1}^{M_{i+1,j+1}} = E_{i+1,j+1}^{M_{i+1,j+1}} \circ E_{i,j+1}^{M_{i,j}}$.

We require $A_{i,0}^{op} \subset M_{i,0}$ and $B_{0,j} \subset M_{0,j}$ for all $i,j$ with conditional expectations satisfying the appropriate commuting square conditions. Here, we take the opposite $\lambda$-lattice $A_{i,j}^{op}$ of $A_{i,j}$, so the indices for $A$ and $B$ are transposed.

\[
\begin{array}{cccccccc}
A_{3,1} & \subset & A_{3,0} & \subset & M_{3,0} & \subset & M_{3,1} & \subset & M_{3,2} & \subset & M_{3,3} & \subset & M_{3,4} & \subset & \cdots \\
A_{2,1} & \subset & A_{2,0} & \subset & M_{2,0} & \subset & M_{2,1} & \subset & M_{2,2} & \subset & M_{2,3} & \subset & M_{2,4} & \subset & \cdots \\
A_{1,1} & \subset & A_{1,0} & \subset & M_{1,0} & \subset & M_{1,1} & \subset & M_{1,2} & \subset & M_{1,3} & \subset & M_{1,4} & \subset & \cdots \\
A_{0,0} & \subset & M_{0,0} & \subset & M_{0,1} & \subset & M_{0,2} & \subset & M_{0,3} & \subset & M_{0,4} & \subset & M_{0,5} & \subset & \cdots \\
B_{0,0} & \subset & B_{0,1} & \subset & B_{0,2} & \subset & B_{0,3} & \subset & B_{0,4} & \subset & B_{0,5} & \subset & B_{0,6} & \subset & \cdots \\
B_{1,1} & \subset & B_{1,2} & \subset & B_{1,3} & \subset & B_{1,4} & \subset & B_{1,5} & \subset & B_{1,6} & \subset & B_{1,7} & \subset & \cdots \\
\end{array}
\]

We call an $A - B$ bimodule **standard** if $[M_{i,j}, A_{pq}] = 0$ for $i \leq q \leq p$; $[M_{i,j}, B_{kl}] = 0$, for $j \leq k \leq l$. Similar to the proofs of Theorems A and B above, we obtain the following theorem.

**Theorem C.** There is a bijective correspondence between equivalence classes of the following:

\[
\begin{aligned}
\{ & \text{Traceless Markov lattices } M = (M_{i,j})_{i,j \geq 0} \text{ with } \dim(M_{0,0}) = 1 \text{ as } \\
& \text{standard } A - B \text{ bimodules over } \text{standard } \lambda\text{-lattices } A, B \}
\end{aligned}
\]

\[
\{ & \text{Pairs } (\mathcal{M}, Z) \text{ with } \mathcal{M} \text{ an indecomposable semisimple } C^* A - B \text{ bimodule category together with a choice of simple object } Z = \mathbb{1}^+_A \triangleright Z \triangleright \mathbb{1}^-_B \}
\]

3
Equivalence on the left hand side is \( \ast \)-isomorphism on the traceless Markov lattice as a standard \( A - B \) bimodule; equivalence on the right hand side is unitary \( A - B \) bimodule equivalence between their Cauchy completions which maps the simple base object to simple base object.

Tracial Markov lattices as standard \( A - B \) bimodules correspond to pivotal \( A - B \) bimodule categories.

Examples  As a natural corollary from Theorem B, a Markov tower corresponds to a Temperley-Lieb-Jones (TLJ) module category. This result generalizes the pivotal module case from [CHPS18, Thm. A.]. To translate our classification into that of [DY15] which uses fair and balanced graphs, we obtain an elegant graphical version of a Markov tower using a \( W^* \) 2-subcategory \( \mathcal{C}(\Lambda, \omega) \) of bigraded Hilbert spaces \( \text{BigHilb} \) which is built from a fair and balanced graph \( (\Lambda, \omega) \). Our approach is inspired by Ocneanu’s path algebras [Oc88] [EK98] [JS97, §5.4]. The following diagram shows how these notions are related to each other in §4:

\[ \text{Markov tower } M \text{ with modulus } d \]
\[ \text{indecomposable semisimple } C^* \]
\[ \mathcal{TLJ}(d) - \text{module category } \mathcal{M} \]

\[ \text{balanced } d\text{-fair bipartite graph } (\Lambda, \omega) \]
\[ \text{2-subcategory } \mathcal{C}(K, ev_K) \text{ of } \text{BigHilb} \]

As an application, in the pivotal/tracial setting, we obtain the embedding theorem for (infinite depth) subfactor planar algebras.

Theorem D. Every (infinite depth) subfactor planar algebra embeds in the bipartite graph planar algebra of its principal graph.

By Theorem C above, a Markov lattice corresponds to a \( \mathcal{TLJ} - \mathcal{TLJ} \) bimodule category. By work-in-progress of Penneys-Peters-Snyder, pivotal \( \mathcal{TLJ} - \mathcal{TLJ} \) bimodule categories correspond to Ocneanu’s biunitary connections on associative square-partite graphs with vertex weightings. For the non-pivotal case, the weighting on the square-partite graph is the edge-weighting and we obtain the non-pivotal analog of a biunitary connection. To translate between these classifications, we use the well-known fact that a commuting square of finite dimensional von Neumann algebras gives a biunitary connection [JS97]. We then introduce a graphical version of a Markov lattice using a \( W^* \) 2-subcategory \( \mathcal{C}(\Phi) \) of \( \text{BigHilb} \) obtained from a biunitary connection \( \Phi \). It turns out that the biunitary connection \( \Phi \) corresponds to the bimodule associator of the bimodule category. The following diagram shows how these notions are related to each other in §5:

\[ \text{Markov lattice } M \text{ with modulus } (d_0, d_1) \]
\[ \text{indecomposable semisimple } C^* \]
\[ \mathcal{TLJ}(d_0) - \mathcal{TLJ}(d_1) \text{ bimodule category } \mathcal{M} \]

\[ \text{balanced } (d_0, d_1)\text{-fair square-partite graph } (\Lambda, \omega) \text{ with biunitary connection } \Phi \]
\[ \text{2-subcategory } \mathcal{C}(\Phi) \text{ of } \text{BigHilb} \]
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1 Standard λ-lattices and tensor category

1.1 Traceless Markov tower and its properties

Definition 1.1.1. Let $A \subset B$ be a unital inclusion of finite von Neumann algebras. A conditional expectation $E : M \to N$ is a positive linear map satisfying the following conditions:

(a) $E(x) = x$ for all $x \in A$,
(b) $E(axb) = aE(x)b$ for all $a, b \in A, x \in B$.

Definition 1.1.2. Let $C$ be a unital $C^*$-algebra. We call a linear functional $\text{tr} : C \to \mathbb{C}$ a trace if it satisfies the following conditions:

(a) (tracial) $\text{tr}(xy) = \text{tr}(yx)$, for all $x, y \in C$.
(b) (positive) $\text{tr}(x^*x) \geq 0$, for all $x \in C$.
(c) (faithful) $\text{tr}(x^*x) = 0$ if and only if $x = 0$.

In addition, we call $\text{tr}$ unital if $\text{tr}(1) = 1$.

Definition 1.1.3. A traceless Markov tower $M = (M_n, E_n, e_{n+1})_{n \geq 0}$ consists of a sequence $(M_n)_{n \geq 0}$ of finite dimensional von Neumann algebras, such that $M_n$ is unitally included in
Some properties of a traceless Markov tower include:

1. $[x, e_k] = 0$, for $x \in M_n$, $k \geq n + 1$.
2. The map $M_n \ni x \mapsto xe_n \in M_{n+1}$ is injective.
3. For $x \in M_{n+1}$, $d^2E_{n+1}(xe_n)$ is the unique element $y \in M_n$ such that $xe_n = ye_n$.
4. Property (3) is equivalent to (M3).
5. If $x \in M_n$ and $[x, e_n] = 0$, then $x \in M_{n-1}$. Together with (1), we have $M_{n-1} = M_n \cap \{e_n\}'$.
6. $e_nM_{n+1}e_n = M_{n-1}e_n$.

Proof.

(1) For $x \in M_n$ and $k \geq n + 1$, $E_k(x) = x, E_k(x^*) = x^*$, then

$$xe_k = E_k(x)e_k = e_kxe_k = (e_kx^*e_k)^* = (E_k(x^*)e_k)^* = (x^*e_k)^* = e_kx.$$  

(2) If $x \in M_n$ and $xe_n = 0$, then by (M3),

$$0 = E_{n+1}(xe_n) = xe_{n+1}(e_n) = d^{-2}x.$$  

Thus, $x \mapsto xe_n$ is injective.

(3) By (M4) and (2), the existence and uniqueness hold. Then by (M3),

$$E_{n+1}(xe_n) = E_{n+1}(ye_n) = ye_{n+1}(e_n) = d^{-2}y,$$

so $y = d^2E_{n+1}(xe_n)$.

(4) First, let’s prove that in this setting, $M_n \ni x \mapsto xe_n \in M_{n+1}$ is injective. If $xe_n = 0$, then

$$0 = d^2E_{n+1}(xe_n) = d^2xe_{n+1}(e_n).$$

Note that $E_{n+1}$ is faithful and $E_{n+1}(e_n) \neq 0$, so $x = 0$.

Let $x = e_n$, then we have $d^2E_{n+1}(e_n) = e_n$. Since $d^2E_{n+1}(e_n)$ and $1 \in M_n$, we have $d^2E_{n+1}(e_n) = 1$ by (2).

(5) Since $xe_n = e_nx$,

$$E_n(x)e_n = e_nx e_n = xe_ne_n = xe_n.$$  

Then by (2), $E_n(x) = x$, which implies $x \in M_{n-1}$.

(6) By (M2) and (M4).

We will explore more properties of traceless Markov towers in §4.
Remark 1.1.5. If there is a faithful normal trace on $\bigcup_{n=0}^{\infty} M_n$ and $E_n$ is the canonical faithful normal trace-preserving conditional expectation for $n = 1, 2, \cdots$, then $M$ is called a tracial Markov tower. Thus, tracial Markov towers defined in [CHPS18] are traceless Markov towers.

Example 1.1.6 (Markov tower without a trace). Let $d > 0$ such that $d^2 > 4$. There is a unique $\lambda \in (0, \frac{1}{2})$ such that $d^{-2} = \lambda(1 - \lambda)$. Then $d\lambda + d(1 - \lambda) = d$ and $\frac{1}{\lambda} + \frac{1}{d(1 - \lambda)} = d$. Let $e_{ij}$ denote the matrix units of $M_2(\mathbb{C})$, $i, j = 1, 2$, and $1 = e_{11} + e_{22} \in M_2(\mathbb{C})$.

Define $E_\lambda : M_2(\mathbb{C}) \to \mathbb{C}$ by $E_\lambda(e_{11}) = \lambda$, $E_\lambda(e_{22}) = 1 - \lambda$, and $E_\lambda(e_{12}) = E_\lambda(e_{21}) = 0$. It is clear that $E_\lambda$ is a normal faithful conditional expectation and not tracial.

Define $e_\lambda \in M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ by

$$e_\lambda = (1 - \lambda)e_{11} \otimes e_{11} + \lambda e_{22} \otimes e_{22} + \sqrt{\lambda(1 - \lambda)}(e_{12} \otimes e_{12} + e_{21} \otimes e_{21}),$$

and one can check that:

(a) $e_\lambda$ is a projection.
(b) $E_\lambda(e_\lambda) = d^{-2}(e_{11} + e_{22}) = d^{-2} \cdot 1$.
(c) $(e_\lambda \otimes 1)(1 \otimes e_\lambda - 1)(e_\lambda \otimes 1) = d^{-2}(e_\lambda \otimes 1)$ and $(e_1 - 1 \otimes 1)(e_1 - 1 \otimes 1) = d^{-2}(e_1 - 1 \otimes 1)$.

Define $\text{id} : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ to be the identity map. Let $M_n := M_2(\mathbb{C})^\otimes n$. The inclusion $M_n \subset M_{n+1}$ maps $x$ to $x \otimes \text{id}$. Jones projection $e_{2n+1} = 1^\otimes 2n \otimes e_1 - e_\lambda \in M_{2n+2}$ and $e_{2n+2} = 1^\otimes 2n+1 \otimes e_\lambda \in M_{2n+3}$, $n = 0, 1, 2, \cdots$. The conditional expectation is defined as follows:

$$E_{2n+1} = \text{id} \otimes 2n+1 \otimes E_\lambda \quad E_{2n+2} = \text{id} \otimes 2n+2 \otimes E_{1-\lambda}.$$ 

Now we build a Markov tower with modulus $d$ and without a trace:

$1 \xleftarrow{E_\lambda} M_2(\mathbb{C}) \xrightarrow{\text{id} \otimes E_{1-\lambda}} M_2(\mathbb{C})^\otimes 2 \xrightarrow{\text{id} \otimes 2 \otimes E_\lambda} M_2(\mathbb{C})^\otimes 3 \xrightarrow{\text{id} \otimes 3 \otimes E_{1-\lambda}} M_2(\mathbb{C})^\otimes 4 \xrightarrow{\cdots} \cdots$

1.2 Standard $\lambda$-lattice and its properties

Definition 1.2.1 ([Po95]). Let $A = (A_{i,j})_{0 \leq i \leq j < \infty}$ be a system of finite dimensional $C^*$ algebras with $A_{i,i} = \mathbb{C}$ with unital inclusions $A_{i,j} \subset A_{k,l}$, for $k \leq i, j \leq l$.

$$A_{0,0} \subset A_{0,1} \subset A_{0,2} \subset A_{0,3} \subset A_{0,4} \subset \cdots$$

$$A_{1,1} \subset A_{1,2} \subset A_{1,3} \subset A_{1,4} \subset \cdots$$

$$A_{2,2} \subset A_{2,3} \subset A_{2,4} \subset \cdots$$

$$A_{3,3} \subset A_{3,4} \subset \cdots$$

$$A_{4,4} \subset \cdots$$

Let $E_{i,j} : A_{i,j} \to A_{i,j-1}$ be the (horizontal) faithful normal conditional expectation, $j = 1, 2 \cdots, i = 0, \cdots, j - 1$ and $E_{i,j}^l : A_{i,j} \to A_{i+1,j}$ be the (vertical) faithful normal conditional expectation $i = 0, 1, \cdots, j = i + 1, i + 2, \cdots$. We also require that
Remark 1.2.3.

(a) (commuting square condition)

\[
\begin{array}{ccc}
A_{i,j} & \xrightarrow{E_{i,j+1}^l} & A_{i,j+1} \\
\downarrow E_{i,j}^l & & \downarrow E_{i,j+1}^l \\
A_{i+1,j} & \xrightarrow{E_{i+1,j+1}^l} & A_{i+1,j+1}
\end{array}
\]

is a commuting square, i.e., \(E_{i,j}^l \circ E_{i,j+1}^r = E_{i+1,j+1}^r \circ E_{i,j+1}^l\).

(b) (existence of Jones \(\lambda\)-projections)

There exists a sequence of Jones projections \(\{e_i\}_{i \geq 1}\) in \(\bigcup_{n=0}^{\infty} A_{0,n}\) such that

\(\text{TL1})\) \(e_i^2 = e_i = e_i^*\) for all \(i\).
\(\text{TL2})\) \(e_je_i = e_je_i\) for \(\vert i - j \vert > 1\).
\(\text{TL3})\) There is a fixed constant \(d > 0\) called the modulus such that \(e_ie_{\pm 1}e_i = d^{-2}e_i\) for all \(i\).

(b3) \(e_jxe_j = E_{i,j}^r(x)e_j\), for \(x \in A_{i,j}, i + 1 \leq j\).

(b4) \(e_jxe_i = E_{i,j}^l(x)e_i\), for \(x \in A_{i,j}, i + 1 \leq j\).

(c) (Markov conditions)

\(\text{c1})\) \(\text{dim } A_{i,j} = \text{dim } A_{i,j+1}e_j = \text{dim } A_{i+1,j+1}, \text{ for } i \leq j\).
\(\text{c2})\) \(E_{i,j+1}^l(e_j) = E_{i-1,k}^l(e_j) = d^{-2}e_i\), for \(j > i + 1, k \geq j + 1\).

Then \(A = (A_{i,j})_{0 \leq i \leq j < \infty}\) is called a \(\lambda\)-lattice of commuting squares. If there is a faithful normal trace \(\text{tr}\) on \(\bigcup_{n=0}^{\infty} A_{0,n}\) and \(E_{i,j}^r, E_{i,j}^l\) are the canonical faithful normal trace-preserving conditional expectation, then \(A\) is called a tracial \(\lambda\)-lattice.

Definition 1.2.2 ([Po95]). A \(\lambda\)-lattice \((A_{i,j})_{0 \leq i \leq j}\) is called a standard \(\lambda\)-lattice if \([A_{i,j}, A_{k,l}] = 0\) for \(i \leq j \leq k \leq l\). This condition is called the standard condition.

Remark 1.2.3. In the definition of (standard) \(\lambda\)-lattice, we may not require a trace and the conditional expectations are trace-preserving. In fact, the reader can construct an example of (standard) \(\lambda\)-lattice without a trace from Example 1.1.6 easily. We will not further discuss the traceless standard \(\lambda\)-lattices, though the following statements do NOT require the trace at all!

Remark 1.2.4. Each row \(A_i = (A_{i,j})_{j \geq i}\) is a Markov tower, \(i = 0, 1, 2, \ldots\); each column \(A_j = (A_{i,j})_{i = j}\) is a Markov tower, \(j = 1, 2, \ldots\). From Proposition 1.1.4, we have

(1) If \(x \in A_{i,j}\), \([x, e_k] = 0\) for \(k \geq j + 1; [x, e_l] = 0\) for \(1 \leq l \leq i - 1\).

(2) The map \(A_{i,j} \ni x \mapsto xe_j \in A_{i,j+1}\) is injective; the map \(A_{i,j} \ni x \mapsto xe_i \in A_{i-1,j}\) is injective.

(3) The Markov condition is equivalent to the pull-down condition:

\(\text{c1})\) \(d^2E_{i,j+1}^r(xe_j)e_j = xe_j\), for \(x \in A_{i,j+1}, j \geq i \geq 0\).
\(\text{c2})\) \(d^2E_{i-1,j}^l(xe_i)e_i = xe_i\), for \(x \in A_{i-1,j}, j \geq i \geq 1\).

The following property was proved in [Po95, Prop. 1.4] by using the trace, here we provide another proof without it.

Proposition 1.2.5. Let

\[
\begin{align*}
A_{0,0} & \subset A_{0,1} \subset A_{0,2} \subset \cdots \\
A_{1,1} & \subset A_{1,2} \subset A_{1,3} \subset \cdots \\
\bigcup & \bigcup & \bigcup \\
\end{align*}
\]
be a $\lambda$-sequence of commuting squares, and define $A_{i,j} := A_{i-1,j} \cap \{e_{i-1}\}' = A_{i,j} \cap \{e_1, \ldots, e_{i-1}\}'$, $2 \leq i \leq j$. Then $(A_{i,j})_{0 \leq i \leq j < \infty}$ is a $\lambda$-lattice of commuting squares.

Proof. We construct $A_{i,j}$ and conditional expectation $E_{i-1,j}^l : A_{i-1,j} \to A_{i,j}$ by induction on $i$, and show that Jones projections $\{e_{i+1}, \ldots, e_{j-1}\} \subset A_{i,j}$ for $i + 2 \leq j$. Suppose $A_{i-1,j}$ is constructed (or given) and $\{e_i, \ldots, e_{j-1}\} \subset A_{i-1,j}$, We define $A_{i,j} := A_{i-1,j} \cap \{e_{i-1}\}'$. Then clearly, $\{e_{i+1}, \ldots, e_{j-1}\} \subset A_{i,j}$.

According to Proposition 1.1.4(5) and (6), for each $x \in A_{i-1,j} \subset A_{i-2,j}$, there exists a $y \in A_{i,j}$ such that

$$ye_{i-1} = e_{i-1}xe_{i-1}.$$ 

By Proposition 1.1.4(2), $A_{i-1,j} \ni y \mapsto ye_{i-1} \in A_{i-2,j}$ is injective, so $y$ is unique for each given $x$. This technique is often used in this chapter. We define $E_{i-1,j}^l(x) := y$. Now we show that $E_{i-1,j}^l$ is a faithful normal conditional expectation:

(a) It is clear that $E_{i-1,j}^l$ is linear, and $E_{i-1,j}^l(1) = 1$. The ultraweak continuity/normality follows from the definition.

(b) $E_{i-1,j}^l(x^*) = E_{i-1,j}^l(x)^*$:

$$E_{i-1,j}^l(x)^*e_{i-1} = (e_{i-1}E_{i-1,j}^l(x))e_{i-1} = e_{i-1}xe_{i-1} = E_{i-1,j}^l(x^*)e_{i-1}.$$ 

(c) $E_{i-1,j}^l(ab) = aE_{i-1,j}^l(b)$ for $a, b \in A_{i,j}$: Note that $[a, e_{i-1}] = [b, e_{i-1}] = 0$, then

$$E_{i-1,j}^l(x^*)e_{i-1} = e_{i-1}axbe_{i-1} = ae_{i-1}xe_{i-1}b = aE_{i-1,j}^l(x)e_{i-1}b = aE_{i-1,j}^l(x)(xe_{i-1}) = 0.$$ 

(d) $E_{i-1,j}^l(x^*x) \geq E_{i-1,j}^l(x)^*E_{i-1,j}^l(x)$, which follows that $E_{i-1,j}^l$ is positive:

$$E_{i-1,j}^l(x)^*E_{i-1,j}^l(x)e_{i-1} = E_{i-1,j}^l(x)^*e_{i-1}xe_{i-1} = e_{i-1}xe_{i-1} \leq e_{i-1}xe_{i-1} = E_{i-1,j}^l(x^*x)e_{i-1}.$$ 

so $E_{i-1,j}^l(x^*x) \geq E_{i-1,j}^l(x)^*E_{i-1,j}^l(x)$ by applying the inductive hypothesis that $E_{i-2,j}^l$ is a positive conditional expectation and $E_{i-2,j}^l(e_{i-1}) = d^{-2} \cdot 1$.

(e) $E_{i-1,j}^l(x^*x) = 0$ if and only if $x = 0$, i.e., $E_{i-1,j}^l$ is faithful:

$$0 = E_{i-1,j}^l(x^*x)e_{i-1} = e_{i-1}xe_{i-1} = (xe_{i-1})^*(xe_{i-1}),$$ 

which follows that $xe_{i-1} = 0$. Note that $A_{i-1,j} \ni x \mapsto xe_{i-1} \in A_{i-2,j}$ is an injection, so $x = 0$.

Then define $E_{i,j+1}^r : A_{i,j+1} \to A_{i,j}$ as the restriction of $E_{i-1,j+1}^l$ on $A_{i,j+1}$, which is also a conditional expectation.

Now we prove the commuting square condition $E_{i-1,j}^l \circ E_{i-1,j+1}^r = E_{i,j+1}^r \circ E_{i-1,j+1}^l$: for $x \in A_{i-1,j+1},$

$$E_{i-1,j}^l(E_{i-1,j+1}^r(x)e_{i-1}) = e_{i-1}E_{i-1,j+1}^r(x)e_{i-1}$$ 

and

$$E_{i,j+1}^r(E_{i-1,j+1}^l(x))e_{i-1} = E_{i,j+1}^r(E_{i-1,j+1}^l(x)e_{i-1}) = E_{i-1,j+1}^l(E_{i-1,j+1}^r(x)e_{i-1}) = E_{i-1,j+1}^l(e_{i-1}xe_{i-1}) = e_{i-1}E_{i-1,j+1}^r(x)e_{i-1}.$$ 

Finally, we prove the Markov condition:

(a) $\dim A_{i,j} = \dim A_{i-1,j} \cap \{e_{i-1}\}' = \dim A_{i-1,j} \cap \{e_j\}' = \dim A_{i-1,j-1}.$

(b) $E_{i,j+1}^r(e_j) = E_{i-1,j+1}^l(e_j) = d^{-2} \cdot 1.$

(c) $E_{i-1,j}^l(e_i)e_{i-1} = e_{i-1}e_i = d^{-2}e_{i-1},$ so $E_{i-1,j}^l(e_i) = d^{-2} \cdot 1.$
Corollary 1.2.6. Let \((A_{i,j})_{0 \leq j \leq 0,1}\) be a \(\lambda\)-sequence of commuting squares. If \(A_{i,j} := \{e_1, \ldots, e_{i-1}\}\cap A_{i,j}\), for all \(2 \leq i \leq j\), then \((A_{i,j})_{0 \leq j \leq 0,1}\) is a standard \(\lambda\)-lattice if and only if \((A_{i,j})_{i \leq j, i=0,1}\) satisfies
\[
[A_{0,1}, A_{i,j}] = 0, \quad \forall 1 \leq j \\
[A_{0,i}, A_{i,j}] = 0, \quad \forall 2 \leq i \leq j.
\]

Now we define the opposite standard \(\lambda\)-lattice, which will be used in Definition 3.2.1.

Definition 1.2.7. \(A^{op} = (A_{i,j})_{0 \leq j \leq i}\) is the opposite of \(\lambda\)-lattice \(A\) if \(A_{j,i}^{op} = A_{i,j}\) as opposite algebras, \(E_{ij}^{op,l} = E_{ij}^r\), \(E_{ji}^{op,r} = E_{ij}^l\) for \(i \leq j\).

Example 1.2.8. The Temperley-Lieb-Jones algebra TLJ(d) forms a standard \(\lambda\)-lattice with the modulus \(d\) by letting \(A_{i,i} = A_{i,i+1} = \mathbb{C}\) and \(A_{i,j} = (e_{i+1}, \ldots, e_{j-1})\) for \(j - i \geq 2\), which is called a Temperley-Lieb-Jones standard \(\lambda\)-lattice.

Example 1.2.9 ([Po95]). If \(A_0 \subset A_1\) is a unital inclusion of type \(\text{II}_1\) subfactors with finite index and \(A_0 \subset A_1 \subset A_2 \subset A_3 \subset \cdots\) is the Jones tower, then \(A_{i,j} := A_i \cap A_j\) forms a standard \(\lambda\)-lattice, which is called the standard invariant of \(A_0 \subset A_1\).

1.3 The 2-shift map

In this section, we discuss an important type of \(*\)-isomorphism in a standard \(\lambda\)-lattice, so-called the 2-shift map [Bi97]. Here we provide the definition by using the conditional expectations and Jones projections instead of trace and Pimsner-Popa basis.

For \(i, k \geq 0\), define the following element of \(A_{i,i+2k,l+1} = i + 2k:\)
\[
e^i_k := d^{k(k-1)}(e_{i+1}e_{i+2} \cdots e_{i+k})(e_{i+k+1}e_{i+k} \cdots e_{i+n-k+2}) \cdots (e_{i+k+1}e_{i+k-2} \cdots e_{i+k+1}).
\]

For \(i, j, k \geq 0\), define the following element of \(A_{i,i+j+2k,l+1} = i + j + 2k:\)
\[
e^i_jk := d^{i+k+1}e^i_0e^i_1 \cdots e^i_k.
\]

Clearly, \(e_n = e^{n-1}_1 = e^{n-1}_{0,0,1}; e^i_k = e^{i-1}_{0,k}; (e^i_k)^2 = (e^i_k)^* = e^i_k\) and \(e^i_jk(e^i_jk)^* = e^i_{0,k}; (e^i_jk)^*e^i_jk = e^{i+j}_0\).

Definition 1.3.1 (Multi-step condition expectation). Define the \(k\)-step horizontal conditional expectation as \(E^i_{i,j} = E^i_{i,j+1-k} \circ E^i_{i,j+2-k} \circ \cdots \circ E^i_{i,j} : A_{i,j} \rightarrow A_{i,j-k}\) for \(k \leq j - i\) and we have \(E^i_{i,j} = E^i_{i,j};\) the \(k\)-step vertical conditional expectation as \(E^i_{i,j} = E^i_{i,k-1,j} \circ E^i_{i,k-2,j} \circ \cdots \circ E^i_{i,j} : A_{i,j} \rightarrow A_{i,k+j}\) for \(k \leq j - i\) and we have \(E^i_{i,j} = E^i_{i,j}\).

In particular, the trace is made by the composition of conditional expectations, i.e., \(E^i_{i,j-k} \circ E^i_{i,j-k} \circ \cdots \circ E^i_{i,j} = \text{tr} = E^i_{i,j-t} \circ E^i_{i,j}, \) for \(0 \leq k \leq j - i, 0 \leq t \leq j - i\).

Definition 1.3.2 (2-shift map). Define the 2-shift map \(S_{i,j} : A_{i,j} \rightarrow A_{i+2,j+2}\), \(i \leq j\) by
\[
S_{i,j}(x) := d^{2j-2i+2}E^i_{i,j+2}(e_{i+1}e_{i+2} \cdots e_j e_{i+2} \cdots e_{i+1}).
\]

Proposition 1.3.3. The followings are the properties of the 2-shift map.

1. \(S_{i,j}\) is well defined, i.e., \(S_{i,j}(x) \in A_{i+2,j+2}\) for \(x \in A_{i,j}\).
(2) \( S_{i,j} \) is a unital \( * \)-isomorphism.

(3) (commuting parallelogram) \( S_{i,j-1} \circ E_{i,j}^r = E_{i+2,j+2}^r \circ S_{i,j} \) and \( S_{i+1,j} \circ E_{i,j}^l = E_{i+2,j+2}^l \circ S_{i,j} \).

(4) \( S_{i,j+1}(x) = S_{i,j}(x) \) for \( x \in A_{i,j} \) and \( S_{i-1,j}(x) = S_{i,j}(x) \) for \( x \in A_{i,j} \).

(5) (shift) \( e_{i+1}e_{i+2} \cdots e_{j+1}x = S_{i,j}(x)e_{i+1}e_{i+2} \cdots e_{j+1} \) for \( x \in A_{i,j} \). Taking adjoints, \( xe_{j+1}e_{j} \cdots e_{i+1} = e_{j+1}e_{j} \cdots e_{i+1} S_{i,j}(x) \). In other word, \( e_{j-i}^l x = S_{i,j}(x)e_{j-i}^l \).

(6) \( S_{i,j} \) is trace-preserving.

(7) \( S_{i,j}(e_k) = e_{k+2} \), where \( i + 1 \leq k \leq j - 1 \).

Proof.

(1) Note that \( S_{i,j}(x) \in A_{i+1,j+2} \), we shall show that \( E_{i+1,j+2}^l(S_{i,j}(x)) = S_{i,j}(x) \). Since \( E_{i+1,j+2}^l(S_{i,j}(x)) - S_{i,j}(x) \in A_{i+1,j+2} \) and the map \( A_{i+1,j+2} \ni y \mapsto ye_{i+1} \in A_{i,j+2} \) is injective, we shall show that \( E_{i+1,j+2}^l(S_{i,j}(x))e_{i+1} = S_{i,j}(x)e_{i+1} \).

\[
E_{i+1,j+2}^l(S_{i,j}(x))e_{i+1} = e_{i+1}S_{i,j}(x)e_{i+1}
\]

\[
= d^{2j-2i}e_{i+1}E_{i,j+2}^l(e_{i+1}e_{i+2} \cdots e_{j}xe_{j+1}e_{j} \cdots e_{i+1})e_{i+1}
\]

\[
= d^{2j-2i}e_{i+1}S_{i,j}(x)e_{i+1} \quad \text{ (pull down)}
\]

\[
= d^{2j-2i}e_{i+1}e_{i+2} \cdots e_{j}xe_{j+1}e_{j} \cdots e_{i+1}
\]

\[
= d^{2j-2i+2}E_{i,j+2}^l(e_{i+1}e_{i+2} \cdots e_{j}xe_{j+1}e_{j} \cdots e_{i+1})e_{i+1} \quad \text{ (pull down)}
\]

\[
= S_{i,j}(x)e_{i+1}.
\]

(2) For \( x \in A_{i,j} \), we have \([x,e_{j+1}] = 0\). First, we show that \( S_{i,j} \) is a homomorphism, i.e., \( S_{i,j}(xy) = S_{i,j}(x)S_{i,j}(y) \) for \( x,y \in A_{i,j} \). Note that the map \( A_{i+2,j+2} \ni y \mapsto ye_{i+1} \in A_{i,j+2} \) is injective, we shall prove that \( S_{i,j}(xy)e_{i+1} = S_{i,j}(x)S_{i,j}(y)e_{i+1} \).

\[
S_{i,j}(x)S_{i,j}(y)e_{i+1} = d^{2j-2i+2}S_{i,j}(x)E_{i,j+2}^l(e_{i+1}e_{i+2} \cdots e_{j}ye_{j+1}e_{j} \cdots e_{i+1})e_{i+1}
\]

\[
= d^{2j-2i}S_{i,j}(x)e_{i+1}e_{i+2} \cdots e_{j}ye_{j+1}e_{j} \cdots e_{i+1} \quad \text{ (pull down)}
\]

\[
= d^{2j-2i}e_{i+1}e_{i+2} \cdots e_{j}xe_{j+1}e_{j} \cdots e_{i+1} \quad \text{ (pull down)}
\]

\[
= d^{2j-2i}e_{i+1}e_{i+2} \cdots e_{j}xe_{j+1}e_{j} \cdots e_{i+1} \quad \text{ (pull down)}
\]

\[
= S_{i,j}(x)S_{i,j}(y)e_{i+1}.
\]

Next, \( S_{i,j} \) is a \( * \)-homomorphism. Note that \( E_{i,j+2}^l \) is a \( * \)-homomorphism, we have

\[
S_{i,j}(x^*) = d^{2j-2i+2}E_{i,j+2}^l(e_{i+1}e_{i+2} \cdots e_{j}x^*e_{j+1}e_{j} \cdots e_{i+1})
\]

\[
= d^{2j-2i+2}E_{i,j+2}^l((e_{i+1}e_{i+2} \cdots e_{j}xe_{j+1}e_{j} \cdots e_{i+1})^*)
\]

\[
= d^{2j-2i+2}E_{i,j+2}^l(e_{i+1}e_{i+2} \cdots e_{j}xe_{j+1}e_{j} \cdots e_{i+1})
\]

\[
= S_{i,j}^*(x).
\]

When \( x = 1 \),

\[
e_{i+1}e_{i+2} \cdots e_{j}e_{j+1}e_{j} \cdots e_{i+1} = d^{-2}e_{i+1}e_{i+2} \cdots e_{j}e_{j-1}e_{j-1} \cdots e_{i+1}
\]

\[
= \cdots = d^{2(i-j)}e_{i+1}e_{i+2}e_{i+1} = d^{2(i-j)}e_{i+1}.
\]
Thus, \( S_{i,j}(1) = d^2 E_{i,j+2}^r(e_{i+1}) = 1 \), i.e., \( S_{i,j} \) is unital.

In order to prove that \( S_{i,j} \) is an isomorphism, we shall show \( S_{i,j} \) is injective and surjective. If \( S_{i,j}(x) = 0 \), then

\[
0 = S_{i,j}(x)e_{i+1} = d^{2j-2i}e_{i+1}e_{i+2} \cdots e_{j}xe_{j+1}e_{j} \cdots e_{i+1} = d^{2j-2i}(e_{i+1}e_{i+2} \cdots e_{j})xe_{j+1}(e_{i+1}e_{i+2} \cdots e_{j})^*,
\]

which follows that \( xe_{j+1} = 0 \). Since the map \( A_{i,j} \ni y \mapsto ye_{j+1} \in A_{i,j+1} \) is injective, we have \( x = 0 \).

Note that \( \dim A_{i,j} = \dim A_{i+1,j+1} = \dim A_{i+2,j+2} < \infty \), so the injectivity implies the surjectivity. Thus, \( S_{i,j} \) is a unital \(*\)-isomorphism.

3. For \( x \in A_{i,j}, \ E_{i,j}^{r}(x) \in A_{i,j-1} \) and \([E_{i,j}^{r}(x), e_{j}] = 0\),

\[
S_{i,j-1} \circ E_{i,j}^{r}(x) = d^{2j-2i} E_{i,j+1}^{l}(e_{i+1}e_{i+2} \cdots e_{j} E_{i,j}^{r}(x)e_{j+1}e_{j} \cdots e_{i+1}) = d^{2j-2i} E_{i,j+1}^{l}(e_{i+1}e_{i+2} \cdots e_{j} E_{i,j}^{r}(x)e_{j+1}e_{j} \cdots e_{i+1}) = d^{2j-2i} E_{i,j+1}^{l}(e_{i+1}e_{i+2} \cdots e_{j} E_{i,j}^{r}(x)e_{j} \cdots e_{i+1}),
\]

\[
E_{i+2,j+2}^{r} \circ S_{i,j}(x) = E_{i+2,j+2}^{r} \circ E_{i+1,j+2}^{r} \circ S_{i,j}(x) = E_{i+1,j+1}^{r} \circ E_{i+1,j+2}^{r} \circ S_{i,j}(x) = E_{i+1,j+1}^{r} \circ E_{i+1,j+2}^{r} \circ S_{i,j}(x).
\]

Thus, \( S_{i,j-1} \circ E_{i,j}^{l} = E_{i+2,j+2}^{r} \circ S_{i,j} \).

Note that \( \{e_{i+1}, \cdots, e_{j-1}\} \subset A_{i,j} \), we have

\[
E_{i,j+2}^{l}(ekxe_{n}) = ekE_{i,j+2}^{l}(x)e_{n} \quad \text{for all} \ k, n \in \{i + 1, \cdots, j - 1\}. \quad (*)
\]

In order to prove that \( S_{i+1,j} \circ E_{i,j}^{l} = E_{i+2}^{l} \circ S_{i,j} \), by Remark 1.2.4 (2), we shall show that
$S_{i+1,j} \circ E^l_{i,j}(x)e_{i+2} = E^l_{i+2,j+2} \circ S_{i,j}(x)e_{i+2}$ for all $x \in A_{i,j}$.

$S_{i+1,j} \circ E^l_{i,j}(x)e_{i+2} = d^{2j-2i+2}E^l_{i+1,j+2}(e_{i+1} \cdots e_j)E^l_{i,j}(x)e_{j+1} \cdots e_{i+2}e_{i+2}$

$E^l_{i+2,j+2} \circ S_{i,j}(x)e_{i+2} = d^{2j-2i+2}E^l_{i,j+2}(e_{i+1} \cdots e_jx)e_{j+1} \cdots e_{i+2}e_{i+2}$ (pull down)

Thus, $S_{i+1,j} \circ E^l_{i,j} = E^l_{i+2} \circ S_{i,j}$.

(4) This is a particular case of (3) by the property of conditional expectation.

(5) For $x \in A_{i,j}$, $[x, e_{j+1}] = 0$,

$S_{i,j}(x)e_{i+1}e_{i+2} \cdots e_{j+1} = d^{2j-2i+2}E^l_{i,j+2}(e_{i+1}e_{i+2} \cdots e_jx)e_{j+1}e_{i+1}e_{i+2} \cdots e_{j+1}$

$(i,j)$ by (commuting square)

(6) By (3) and Definition 1.3.1.

(7) Note that the map $A_{i+2,j+2} \subset A_{i+1,j+2} \ni y \mapsto ye_{i+1} \in A_{i,j+2}$ is injective, we shall prove that $S_{i,j}(e_k)e_{i+1} = e_{k+2e_{i+1}}$. For $i + 1 \leq k \leq j - 1$,

$S_{i,j}(e_k)e_{i+1} = d^{2j-2i+k+2}E^l_{i,j+2}(e_{i+1}e_{i+2} \cdots e_je_{k+1}e_j \cdots e_{i+1})e_{i+1}$

$= d^{2j-2i}e_{i+1}e_{i+2} \cdots e_je_{k+1}e_j \cdots e_{i+1}$ (pull down)

$= d^{2j-2i}e_{i+1} \cdots e_{k-1}e_k e_{k+1}e_{k+2} \cdots e_{j+1}e_j \cdots e_{k+2}e_{k+1} \cdots e_{i+1}$

$(e_t e_{t+1} e_t = d^2 e_t) (e_t e_{t+1} e_t = d^2 e_t)$

\[\square\]

**Definition 1.3.4** ($2n$-shift map). Define $S^{(n)}_{i,j} : A_{i,j} \rightarrow A_{i+2n,j+2n}$ by

$S^{(n)}_{i,j} = S_{i+2(n-1),j+2(n-1)} \circ S^{(n-1)}_{i,j} = S_{i+2(n-1),j+2(n-1)} \circ S_{i+2(n-2),j+2(n-2)} \circ \cdots \circ S_{i,j}$

to be the $2n$-shift map.
Proposition 1.3.5. The followings are the properties of the 2n-shift map.

1. $S^{(n)}_{i,j}$ is a unital $\ast$-isomorphism.

2. (commuting parallelogram) $S^{(n)}_{i,j-1} \circ E^{r,k}_{i,j} = E^{r,k}_{i+2n,j+2n} \circ S^{(n)}_{i,j}$ and $S^{(n)}_{i+1,j} \circ E^{l,k}_{i,j} = E^{l,k}_{i+2n,j+2n} \circ S^{(n)}_{i,j}$.

3. $S^{(n)}_{i,j+k}(x) = S^{(n)}_{i,j}(x)$ for $x \in A_{i,j}$ and $S^{(n)}_{i-j-k}(x) = S^{(n)}_{i,j}(x)$ for $x \in A_{i,j}$. 

4. (shift) For $x \in A_{i,j}$, $e^{i}_{j-i,n}e^{i}_{j-i,n}x = S^{(n)}_{i,j}(x)e^{i}_{j-i,n}$. By taking adjoint, $x^\ast e^{i}_{j-i,n} = e^{i\ast}_{j-i,n}S^{(n)}_{i,j}(x)$. 

5. $S^{(n)}_{i,j}$ is trace-preserving.

Proof. (1),(2),(3),(5) follow from Proposition 1.3.3.

(4) First, we show that $e^{i+2(n-1)}_{j-i,1}e^{i+2(n-2)}_{j-i,1}\cdots e^{i}_{j-i,1}x = S^{n}_{i,j}(x)e^{i+2(n-1)}_{j-i,1}e^{i+2(n-2)}_{j-i,1}\cdots e^{i}_{j-i,1}x$ for $x \in A_{i,j}$.

$S^{n}_{i,j}(x)e^{i+2(n-1)}_{j-i,1}e^{i+2(n-2)}_{j-i,1}\cdots e^{i}_{j-i,1}x = S^{(n)}_{i,j}(x)e^{i+2(n-1)}_{j-i,1}e^{i+2(n-2)}_{j-i,1}\cdots e^{i}_{j-i,1}x$.

Second, $e^{i\ast}_{j-i,n} = a^{i\ast}_{j-i,n}e^{i}_{j-i,1}e^{i}_{j-i,1}b^{i}_{j-i,n}$ with $a^{i}_{j-i,n} \in A_{i,i+2n}$ and $b^{i}_{j-i,n} \in A_{j,j+2n}$, which will be showed below in Lemma 1.5.1 and 1.5.2. Then by the standard condition, since $x \in A_{i,j}$ and $S^{(n)}(x) \in A_{i+2n,j+2n}$, we have $[S^{(n)}_{i,j}(x), a^{i}_{j-i,n}] = 0$ and $[x, b^{i}_{j-i,n}] = 0$, which follows that

$S^{(n)}_{i,j}(x)e^{i\ast}_{j-i,n} = e^{i\ast}_{j-i,n}S^{(n)}_{i,j}(x)a^{i}_{j-i,n}e^{i+2(n-1)}_{j-i,1}e^{i+2(n-2)}_{j-i,1}\cdots e^{i}_{j-i,1}b^{i}_{j-i,n}$

$= a^{i}_{j-i,n}S^{(n)}_{i,j}(x)e^{i+2(n-1)}_{j-i,1}e^{i+2(n-2)}_{j-i,1}\cdots e^{i}_{j-i,1}b^{i}_{j-i,n}$

$= a^{i}_{j-i,n}e^{i\ast}_{j-i,n}e^{i}_{j-i,1}\cdots e^{i}_{j-i,1}b^{i}_{j-i,n}x$

$= a^{i}_{j-i,n}e^{i\ast}_{j-i,n}\cdots e^{i}_{j-i,1}b^{i}_{j-i,n}x$.

$\square$

1.4 String diagram explanation

In this section, we use Temperley-Lieb-Jones (TLJ) string diagram to explain the elements in $A_{i,j}$, horizontal (right) and vertical (left) conditional expectations, the Jones projections, 2n-shift maps and their properties.

In the following sections, we will use these diagrams to do the algebraic computation and readers may interpret these diagrams directly into the algebraic computations by looking at the dictionary here.

$\lambda 1$ Element $x \in A_{i,j}$. $A_{i,j}$ is a (rectangular) box space with $j$ shaded/unshaded strands where the left $i$ strands are straight strands together with a $j-i$ box space. We set the left part of left most strand to be always unshaded; the shading on the left part of the $j-i$
box space depends on the parity of $i$:

If $2 \mid i$:

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{box.png}
\end{array}
\end{array}
\]

If $2 \nmid i$:

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{box.png}
\end{array}
\end{array}
\]

**Remark**: The reader shall understand the meaning of rectangular box and round box of an element. And the shading type of an element is the shading on the left of the round box.

(\lambda 2) Horizontal inclusion $x \in A_{i,j} \subset A_{i,j+1}$. The inclusion $A_{i,j} \subset A_{i,j+1}$ means adding one straight strand on the right and regarding the $j - i$ box space in $A_{i,j}$ as a part of the $j - i + 1$ box space in $A_{i,j+1}$ together with the straight strand, which does not change the shading type of the box space:

If $2 \mid i$:

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{box.png}
\end{array}
\end{array}
\]

If $2 \nmid i$:

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{box.png}
\end{array}
\end{array}
\]

(\lambda 3) Vertical inclusion $x \in A_{i,j} \subset A_{i-1,j}$. The inclusion $A_{i,j} \subset A_{i-1,j}$ means regarding the right most straight strand together with the original $j - i$ box space in $A_{i,j}$ as a part of the $j - i + 1$ box space in $A_{i-1,j}$, which changes the shading type of the box space:

If $2 \mid i$:

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{box.png}
\end{array}
\end{array}
\]

If $2 \nmid i$:

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{box.png}
\end{array}
\end{array}
\]

(\lambda 4) Jones projections:

\[
e_{2k+1} = d^{-1}
\]

\[
e_{2k+2} = d^{-1}
\]

\[
e_{k}^{i} = d^{-k}
\]

\[
e_{j,k}^{i} = d^{-k}
\]

\[
e_{j,k}^{i,*} = d^{-k}
\]

**Remark**: See the string diagram calculation of Jones projections in the Temperley-Lieb-Jones algebra.

(\lambda 5) Horizontal (right) conditional expectation $E_{i,j}^{r} : A_{i,j} \rightarrow A_{i,j-1}$, $x \in A_{i,j}$:

\[
E_{i,j}^{r}(x) = d^{-1}
\]

(\lambda 6) Vertical (left) conditional expectation $E_{i,j}^{l} : A_{i,j} \rightarrow A_{i+1,j}$, $x \in A_{i,j}$. The vertical (left) conditional expectation is the left conditional expectation acting on the left of the box.
space and then adding one straight strand on the left of the box space, which changes the shading type of box space:

\[
\text{If } 2 \mid i : E_{i,j}^{l}(x) = d^{-1} \bigg| \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} \bigg| \begin{array}{c}
x \\
\text{ } \\
\text{ } \\
\end{array} = d^{-1} \bigg| \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} \bigg| \begin{array}{c}
x \\
\text{ } \\
\text{ } \\
\end{array}
\]

\[
\text{If } 2 \nmid i : E_{i,j}^{l}(x) = d^{-1} \bigg| \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} \bigg| \begin{array}{c}
x \\
\text{ } \\
\text{ } \\
\end{array} = d^{-1} \bigg| \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} \bigg| \begin{array}{c}
x \\
\text{ } \\
\text{ } \\
\end{array}
\]

(\lambda 7) \ e_j x e_j = E_{i,j}^{r}(x)e_j, \text{ for } x \in A_{i,j}, i + 1 \leq j:
\ e_i x e_i = E_{i,j}^{l}(x)e_i, \text{ for } x \in A_{i,j}, i + 1 \leq j:

(\lambda 8) \text{ Commuting square of conditional expectation: For } x \in A_{i,j}, E_{i,j}^{l} \circ E_{i,j+1}^{r}(x) = E_{i+1,j+1}^{r} \circ E_{i,j+1}^{l}(x):

\[
E_{i,j}^{l} \circ E_{i,j+1}^{r}(x) = \bigg| \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\end{array} \bigg| \begin{array}{c}
x \\
\text{ } \\
\text{ } \\
\end{array} = \bigg| \begin{array}{c}
x \\
\text{ } \\
\text{ } \\
\end{array} \bigg| \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\end{array}
\]

(\lambda 9) \ E_{i,j+1}^{r}(e_j) = E_{j-1,k}^{r}(e_j) = d^{-2}1, \text{ for } j \geq i + 1, k \geq j + 1.

(\lambda 10) \text{ Conditional expectation property } E_{i,j}^{r}(axb) = aE_{i,j}^{r}(x)b, \text{ for } x \in A_{i,j}, a,b \in A_{i,j-1};
\ E_{i,j}^{l}(axb) = aE_{i,j}^{l}(x)b, \text{ for } x \in A_{i,j}, a,b \in A_{i+1,j}.
(λ11) Standard condition: For $x \in A_{i,j}$, $y \in A_{k,l}$ with $k \geq j$, then we regard $x, y$ as elements in $A_{i,l}$, $xy = yx$.

(λ12) Pull down condition
\[ d^2E^r_{i,j+1}(xe_j)e_j = xe_j, \text{ for } x \in A_{i,j+1}, j \geq i \geq 0; \]
\[ d^2E^l_{i-1,j}(xe_i)e_i = xe_i, \text{ for } x \in A_{i-1,j}, j \geq i \geq 1; \]

(λ13) 2-shift map $S_{i,j}: A_{i,j} \rightarrow A_{i+2,j+2}$: For $x \in A_{i,j}$,

(λ14) 2n-shift map $S^{(n)}_{i,j}: A_{i,j} \rightarrow A_{i+2n,j+2n}$: For $x \in A_{i,j}$,

(λ15) Commuting parallelogram:
For $x \in A_{i,j}$, $S^{(n)}_{i,j-1} \circ E^r_{i,j}(x) = E^r_{i+2n,j+2n} \circ S^{(n)}_{i,j}(x)$;
For $x \in A_{i,j}$, $S^{(n)}_{i+1,j} \circ E^l_{i,j}(x) = E^l_{i+2n,j+2n} \circ S^{(n)}_{i,j}(x)$.

\[ S^{(n)}_{i,j-1} \circ E^r_{i,j}(x) = E^r_{i+2n,j+2n} \circ S^{(n)}_{i,j}(x) \]
\[ S_{i+1,j}^{(n)} \circ E_{i,j}^{l,k}(x) = E_{i+2n,j+2n}^{l,k} \circ S_{i,j}^{(n)}(x) \]

(\(\lambda 1\)) Shift property: For \(x \in A_{i,j}, e_{j,k}^i x = S_{i,j}^{k}(x)e_{j,k}^i\).

1.5 Some useful lemmas

In this section, we are going to show some important lemmas. One can interpret the string diagram computation into algebraic computation by the above dictionary.

Lemma 1.5.1.

Lemma 1.5.2. For \(\sum_{l=1}^{n} k_{pi} = \sum_{r=1}^{m} k_{qi}, k_{pi}, k_{qi} \in \mathbb{Z}_{\geq 0}\), and \(x \in A_{i,j}\), we have:

Proof. By the above lemma.

These two lemmas are used in the proof of Proposition 1.3.5(4).

Lemma 1.5.3.

Proof.
Lemma 1.5.4 ([CHPS18]). For $x \in A_{m,n+2i+j}$, $m \leq n+2i+j$, we have:

Proof.
1.6 From standard $\lambda$-lattice to pivotal planar tensor category

1.6.1 Planar tensor category

**Definition 1.6.1.** A planar tensor category $A_0$ has the following properties.

(a) $A_0$ is a 2-shaded category with objects $[n, +], [n, -], n \in \mathbb{Z}_{\geq 0}$, where $1^+ := [0, +], 1^- := [0, -]$ are simple and the tensor unit $1_{A_0} = 1^+ \oplus 1^-$, which means $A_0$ is 2-shaded.

(b) $A_0$ is a strict tensor category. The tensor product of objects is $[m, ?] \otimes [n, ?] \rightarrow [2i + n + 1, +], [2i + n + 1, +], [2i + n, -], [2i + n, -]$, $i \in \mathbb{Z}_{\geq 0}$.

(c) Only $A_0([m, +] \rightarrow [m \pm 2i, +])$ and $A_0([m, -] \rightarrow [m \pm 2i, -])$ are non-empty, $m, i \in \mathbb{Z}_{\geq 0}$, and $A_0([m, +] \rightarrow [m, +])$ and $A_0([m, -] \rightarrow [m, -])$ are finite-dimensional $C^*$-algebras. The tensor product of morphisms should match the shading types.

(d) $A_0$ is a dagger category. There is a dagger structure $^\dagger$ such that $[n, +]^\dagger = [n, +], [n, -]^\dagger = [n, -]$ and $^\dagger id = id$. Only $A_0([m, +] \rightarrow [m \pm 2i, +])$ and $A_0([m, -] \rightarrow [m \pm 2i, -])$ are non-empty, $m, i \in \mathbb{Z}_{\geq 0}$, and $A_0([m, +] \rightarrow [m, +])$ and $A_0([m, -] \rightarrow [m, -])$ are finite-dimensional $C^*$-algebras. The tensor product of morphisms should match the shading types.

(e) $A_0$ is rigid. For $X \in A_0$, there exist

1. $ev_X : X \otimes X \rightarrow 1^i$, where $? = +$ if $X$ is unshaded on the right, i.e., $X = 1^+ \oplus X$, $? = -$ if $X$ is shaded on the right, i.e., $X = 1^- \otimes X$;
2. $coev_X : 1^i \rightarrow X \otimes X$, where $? = +$ if $X$ is unshaded on the left, $? = -$ if $X$ is shaded on the left.

such that

- $id_X \otimes ev_X \circ (coev_X \otimes id_X) = id_X$
• \((ev_X \otimes \text{id}_X) \circ (\text{id}_X \otimes \coev_X) = \text{id}_X\).

• \( ev_X := (\coev_X)\dagger \) and \( \coev_X = (\coev_X)\dagger \).

In other word, \( (\cdot) \) is a unitary dual functor, which will be discussed in §1.7.1.

**Definition 1.6.2.** We call a planar tensor category \( A_0 \) pivotal, if the left trace \( Tr_L \) and right trace \( Tr_R \) defined as follows are faithful normal tracial and equal. For \( X = [2k + 1, +] \) and \( f \in A_0(X \to X) \), since \([2k + 1, +] = [2k + 1, -]\), we define

\[
\begin{align*}
    ev_X \circ (\text{id}_X \otimes f) \circ ev_X^\dagger &=: Tr_L(f)\text{id}_{1^+}, \\
    \coev_X \circ (f \otimes \text{id}_X) \circ \coev_X &=: Tr_R(f)\text{id}_{1^-}.
\end{align*}
\]

We require that \( Tr_R(f) = Tr_L(f) \). Similar for other three cases \([2k, +], [2k, -]\) and \([2k + 1, -]\). And there exists a \( d > 0 \) such that \( ev_{[n, ?]} \circ \coev_{[n, ?]} = d^{2n} \cdot 1^\dagger \), \( ? \in \{+,-\} \).

**Remark 1.6.3.** The traces \( Tr_L, Tr_R \) are defined in the sense of Definition 1.7.6.

**Definition 1.6.4.** The 2-shaded Temperley-Lieb-Jones multitensor category \( \mathcal{TLLJ}(d) \) is a planar tensor category with the endomorphism spaces being 2-shaded Temperley-Lieb-Jones algebras with modulus \( d \), namely, \( \text{End}([n, +]) \) is a 2-shaded Temperley-Lieb-Jones algebra with \( n \) points on one side and unshaded on the left; \( \text{End}([n, -]) \) is a 2-shaded Temperley-Lieb algebra with \( n \) points on one side and shaded on the left.

**Remark 1.6.5.** The morphisms in \( A_0 \) are determined by its representation in endomorphism and its domain and range.

There is a canonical isomorphism \( \phi : A_0([m, +], [m + 2i, +]) \to A_0([m + i, ?] \to [m + i, ?]) \) by Frobenius reciprocity, where \( ? = + \) if \( i \) is even and \( ? = - \) if \( i \) is odd.

\[
\begin{align*}
    \phi : \begin{array}{c}
            \quad \\
            x \\
            \quad \quad \\
            m \\
        \end{array} & \mapsto \begin{array}{c}
            \quad \\
            x \\
            \quad \quad \\
            m + i \\
        \end{array} \\
    \phi^{-1} : \begin{array}{c}
            \quad \\
            x \\
            \quad \quad \\
            m + i \\
        \end{array} & \mapsto \begin{array}{c}
            \quad \\
            x \\
            \quad \quad \\
            m \\
        \end{array}
\end{align*}
\]

For morphism \( x \in A([m, ?] \to [n, ?]) \), we can write a triple \( (\phi(x); [m, ?], [n, ?]) \) to represent \( x \), where \( \phi(x) \in \text{End}([m + 2n, ?]) \), which is called the endomorphism representation part of \( x \). In the following context, we simply write \( x \) instead of \( \phi(x) \) in the triple \( (x; [m, ?], [n, ?]) \).

1.6.2 From standard \( \lambda \)-lattice to pivotal planar tensor category

We regard the elements in algebra \( A_{i,j} \) as endomorphisms in the category and the idea in Remark 1.6.5 gives us the way to represent the morphism by using its corresponding endomorphism, source and target, then we can construct a pivotal planar tensor category from a given standard \( \lambda \)-lattice.

**Definition 1.6.6.** Let \( A = (A_{i,j})_{0 \leq i \leq j} \) be a standard \( \lambda \)-lattice. We define a planar tensor category \( A_0 \) from \( A \) as follows.

(a) The objects of \( A_0 \) are the symbols \([n, +], [n, -]\) for \( n \in \mathbb{Z}_{\geq 0}\).

(b) Given \( n \geq 0 \), define \( A_0([n, +] \to [n, +]) := A_{0,n} \) and \( A_0([n, -] \to [n, -]) := A_{1,n+1} \). Define \( 1 := [0, +] \oplus [0, -] \).

(c) The identity morphism in \( A_0([n, +] \to [n, +]) \) is \( 1_{A_{0,n}} \) and in \( A_0([n, -] \to [n, -]) \) is \( 1_{A_{1,n+1}} \).
(d) For \((x; [n, +], [n + 2k, +])\) (or \((x; [n + 2k, +], [n, +])\)), we define the dagger structure as \((x; [n, +], [n + 2k, +])^\dagger := (x^*; [n + 2k, +], [n, +])\), where \(x^* \in A_{0, n+k}\); for \((x; [n, -], [n + 2k, -])\) (or \((x; [n + 2k, -], [n, -])\)), we define \((x; [n, -], [n + 2k, -])^\dagger := (x^*; [n + 2k, -], [n, -])\), where \(x, x^* \in A_{1, n+k+1}\).

(e) We define composition in six cases.

(C1) \((y; [n + 2i, +], [n + 2i + 2j, +]) \circ (x; [n, +], [n + 2i, +]) = (d^i E_{0,n+2i+j}^r(y x e^i_{j,i}); [n, +], [n + 2i + 2j, +])\), where \(x \in A_{0, n+i}, y \in A_{0, n+2i+j}\) and \(d^i E_{0, n+2i+j}^r(x e^i_{j,i}) \in A_{0, n+i+j}\).

(C2) \((y; [n + 2i + 2j, +], [n + 2i + 2j, +]) \circ (x; [n, +], [n + 2i + 2j, +]) = (d^i E_{0, n+2i+j}^r(x e^{n,s}_{j,i}); [n, +], [n + 2i + 2j, +])\), where \(x \in A_{0, n+i+j}, y \in A_{0, n+2i+j}\) and \(d^i E_{0, n+2i+j}^r(x e^{n,s}_{j,i}) \in A_{0, n+i}\).

(C3) \((y; [n, +], [n + 2i + 2j, +]) \circ (x; [n + 2i, +], [n + 2i + 2j, +]) = (d^j y e^{n,s}_{j,i}; [n + 2i + 2j, +])\), where \(x \in A_{0, n+i+j}\) and \(d^j y e^{n,s}_{j,i} x \in A_{0, n+2i+j}\).

(C4) \((y; [n + 2i, -], [n + 2i + 2j, -]) \circ (x; [n, -], [n + 2i, -]) = (d^i E_{1, n+2i+j+1}^r(y x e^{n+1}_{j,i}); [n, +], [n + 2i + 2j, +])\), where \(x \in A_{1, n+i+1}, y \in A_{1, n+2i+j+1}\) and \(d^i E_{1, n+2i+j+1}^r(x e^{n+1}_{j,i}) \in A_{1, n+i+j+1}\).

(C5) \((y; [n + 2i + 2j, -], [n + 2i, -]) \circ (x; [n, -], [n + 2i + 2j, -]) = (d^i E_{1, n+2i+j+1}^r(x e^{n+1,s}_{j,i}); [n, -], [n + 2i + 2j, -])\), where \(x \in A_{1, n+i+j+1}, y \in A_{1, n+2i+j+1}\) and \(d^i E_{1, n+2i+j+1}^r(x e^{n+1,s}_{j,i}) \in A_{1, n+i+j+1}\).

(C6) \((y; [n, -], [n + 2i + 2j, -]) \circ (x; [n + 2i, -], [n + 2i + 2j, -]) = (d^j y e^{n+1,s}_{j,i}; [n + 2i + 2j, -])\), \(y e^{n+1,s}_{j,i} x \in A_{1, n+2i+j+1}\).

If \(x \in A_0([n + 2i, -] \to [n, -])\) and \(y \in A_0([n, -] \to [n + 2i + 2j, -])\), we define
\[
y \circ x := d^j y e^{n+1,s}_{j,i} x \in A_{1, n+2i+j+1} = A_0([n + 2i + 2j, -] \to [n + 2i + 2j, -]).
\]

We define the composition \(x^\dagger \circ y^\dagger := (y \circ x)^\dagger\), which defines composition \(A_0([n + 2i + 2j, -] \to [n, -]) \otimes A_0([n, -] \to [n + 2i, -]) \to A_0([n + 2i + 2j, -] \to [n + 2i, -]).\)

It has been proved in [CHPS18, §3.4] that the composition and dagger structure are well defined as Markov tower, and \(A_0\) is a \(C^*\) category.

Before we define the tensor product of morphisms, we use the string diagrams to explain the composition. The box space in the following diagram is always the endomorphism representation of corresponding morphism.

The string diagram of case (C4) comes from the string diagram of case (C1) by adding a straight strand on the leftmost of the diagram and change the shading. In the same way, we obtain (C5) from (C2) and (C6) from (C3).

Now we define the tensor product of morphisms.

![Diagram](image-url)
**Definition 1.6.7.** $x \otimes 1$ and $1 \otimes y$, $x, y \in \text{Hom}(A_0)$:

First, we define $x \otimes 1$ as

\[
\begin{array}{c|c}
  x & x \otimes 1_j \\
  \hline
  (x; [m, +], [m + 2i, +]), i \leq j & (x_e^{m+n}_{-i, j}; [m + j, +], [m + 2i + j, +]) \\
  (x; [m, +], [m + 2i, +]), i > j & (x_e^{m+n}_{-i, j}; [m + j, +], [m + 2i + j, +]) \\
  (x; [m, -], [m + 2i, -]), i \leq j & (x_e^{m-n}_{-i, j}; [m + j, -], [m + 2i + j, -]) \\
  (x; [m, -], [m + 2i, -]), i > j & (x_e^{m-n}_{-i, j}; [m + j, -], [m + 2i + j, -]) \\
\end{array}
\]

Because of the shading, we define $1 \otimes y$ as:

\[
\begin{array}{c|c|c}
  y & 1_{2i} \otimes y & 1_{2i+1} \otimes y \\
  \hline
  (y; [n, +], [n \pm 2j, +]) & (S_{0,n\pm j}^{(1)}(y); [n + 2i, +], [n + 2i \pm 2j, +]) & 0 \\
  (y; [n, -], [n \pm 2j, -]) & 0 & (S_{1,n\pm j+1}^{(1)}(y); [n + 2i, -], [n + 2i \pm 2j, -]) \\
\end{array}
\]

**Proposition 1.6.8.** For $x, y \in \text{Hom}(A_0)$, $(x \otimes 1) \circ (1 \otimes y) = (1 \otimes y) \circ (x \otimes 1)$.

**Proof.** Here, we check the case $(x; [m, +], [m + 2i, +])$ and $(y; [n, +], [n + 2j, +])$, where $2 \mid m$ (or $(y; [n, -], [n + 2j, -])$ if $2 \nmid m$) and $n + j \geq i$. We shall prove that

\[
(x; [m, +], [m + 2i, +]) \otimes (1; [n + 2j, +], [n + 2j, +]) \circ ((1; [m, +], [m, +]) \otimes (y; [n, +], [n + 2j, +])) = ((1; [m + 2i, +], [m + 2i, +]) \otimes (y; [n, +], [n + 2j, +])) \circ ((x; [m, +], [m + 2i, +]) \otimes (1; [n, +], [n + 2j, +]))
\]

First, they both in $A_0([m + n, +] \rightarrow [m + n + 2i + 2j, +])$.

The right hand side:

\[
((1; [m+2i, +], [m+2i, +]) \otimes (y; [n, +], [n+2j, +])) \circ ((x; [m, +], [m+2i, +]) \otimes (1; [n, +], [n+2j, +])):
\]

The left hand side:

\[
((x; [m, +], [m+2i, +]) \otimes (1; [n+2j, +], [n+2j, +])) \circ ((1; [m, +], [m, +]) \otimes (y; [n, +], [n+2j, +])):
\]

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(1) If $i \leq j$,

List the formulas used in above equalities:

①: uses $(\lambda_{16})$;
②: uses $(\lambda_{11})$ and $(\lambda_{15})$;
③: Jones projection property;
④: uses $(\lambda_{10})$;
⑤: uses Lemma 1.5.3.
(2) If $i > j$,

\[\text{List of the formulas used in above equalities:} \]
\[\text{1: uses Lemma 1.5.3;} \quad \text{2: uses (}\lambda_{10})\text{;} \]
\[\text{3: uses (}\lambda_{16})\text{ and Lemma 1.5.2;} \quad \text{4: uses (}\lambda_{11})\text{ and (}\lambda_{15})\text{;} \]
\[\text{5: Jones projection property.}\]

Therefore, $(x \otimes 1) \circ (1 \otimes y) = (1 \otimes y) \circ (x \otimes 1)$ in this case. The remaining cases are left to the reader. \hfill \Box

**Definition 1.6.9** (tensor product of morphisms). Define $x \otimes y := (x \otimes 1) \circ (1 \otimes y)$.

We need to prove that the tensor product defined above is functorial and associative.

**Proposition 1.6.10.** Tensor product is associative and strict, i.e., for $x, y, z \in \text{Hom}(A_0)$, $(x \otimes y) \otimes z = x \otimes (y \otimes z)$.

**Proof.** Here, we check the case $(x;[m, +],[m+2i,+]), (y;[n, +],[n+2j,+])$ and $(z;[l, -],[l+2k,-])$, where $2 \mid m, 2 \nmid n$ and $n + j \geq i, l + k \geq i + j$. Then $(x \otimes y) \otimes z, x \otimes (y \otimes z) \in A_0([m+n+l, +] \rightarrow [m+n+l+2i+2j+2k, +])$.

By Proposition 1.6.8, the endomorphism representation parts of $x \otimes y$ and $y \otimes z$ are defined in this way:
Then $(x \otimes y) \otimes z$:

List of the formulas used in above equalities:

1: Jones projection property; 2: uses Lemma 1.5.4; 3: Jones projection property.

And $x \otimes (y \otimes z)$:

List of the formulas used in above equalities:

1: uses $(\lambda 11)$; 2: uses $(\lambda 10)$; 3: Jones projection property.

Therefore, $(x \otimes y) \otimes z = x \otimes (y \otimes z)$ in this case. Readers can check the rest of the cases by using the string diagram dictionary and the lemmas.

Proposition 1.6.11. For $x, y \in \text{Hom}(\mathcal{A}_0)$, $(x \circ y) \otimes 1 = (x \otimes 1) \circ (y \otimes 1)$ and $1 \otimes (x \circ y) = (1 \otimes x) \circ (1 \otimes y)$. 
Proof. By our construction, \(1 \otimes (x \circ y) = (1 \otimes x) \circ (1 \otimes y)\) only uses the fact that the shift map is a \(\ast\)-homomorphism.

As for \((x \circ y) \otimes 1 = (x \otimes 1) \circ (y \otimes 1)\), we check the case \((x; [m, +], [m + 2i, +])\) and \((y; [m + 2i], [m + 2i + 2j, +])\), where \(n \geq i + j\). Then \((x \circ y) \otimes 1_n, (x \otimes 1_n) \circ (y \otimes 1_n) \in A_0([m + n, +] \to [m + n + 2i + 2j, +])\). Next, let us compare their endomorphism representation parts.

\((x \circ y) \otimes 1_n:\)
\[
\begin{array}{c}
\begin{array}{c}
\text{(1)} \quad d^{i-j}
\end{array}
\end{array}
\]

where only the straight strands are allowed in the blank.

List of the formulas used in above equalities:

1: Jones projection property;  
2: uses Lemma 1.5.4;  
3: uses Lemma 1.5.3 and Jones projection property.

Therefore, \((x \circ y) \otimes 1 = (x \otimes 1) \circ (y \otimes 1)\) in this case. Readers can check the rest of the cases by using the string diagram dictionary and the lemmas.
Proposition 1.6.12. The tensor product is functorial. For \( x, y, z, w \in \text{Hom}(A_0) \), \((x \circ y) \otimes (z \circ w) = (x \otimes z) \circ (y \otimes w)\).

**Proof.** Based on Proposition 1.6.8 and Proposition 1.6.11, we have

\[
(x \circ y) \otimes (z \circ w) = ((x \circ y) \otimes 1) \circ (1 \otimes (z \circ w))
\]

\[
= ((x \otimes 1) \circ (y \otimes 1)) \circ ((1 \otimes z) \circ (1 \otimes w))
\]

\[
= (x \otimes 1) \circ ((y \otimes 1) \circ (1 \otimes z)) \circ (1 \otimes w)
\]

\[
= (x \otimes 1) \circ ((1 \otimes z) \circ (y \otimes 1)) \circ (1 \otimes w)
\]

\[
= ((x \otimes 1) \circ (1 \otimes z)) \circ ((y \otimes 1) \circ (1 \otimes w))
\]

\[
= (x \otimes z) \circ (y \otimes w).
\]

Therefore, the tensor product in Definition 1.6.9 is well-defined.

Next, we show that \( A_0 \) has a pivotal structure.

**Definition 1.6.13** (ev and coev). Note that \([n, \pm] \otimes [n, \pm] = [2n; \pm]; [n, +] \otimes [n, +] = [2n, +]\) if \(2 \mid n\) and \([2n, -] \otimes [n, -] = [2n, -]\) if \(2 \mid n\) and \([2n, +] \otimes [2n, +]\) if \(2 \notmid n\).

Define

\[
\text{coev}_{[n,+]} : 1^+ \to [2n,+] = [n,+] \otimes [n,+]
\]

as \( \text{coev}_{[n,+]} = (d^n; [0, +], [2n, +]) \).

\[
\text{ev}_{[n,+]} : [2n,+] \otimes [n,+] = [2n,?] \to 1^+
\]

as \( \text{ev}_{[n,+]} = (d^n; [2n,?],[0,?]), \ ? = +, \text{ if } 2 \mid n \).

\[
\text{coev}_{[n,-]} : 1^- \to [2n,-] = [n,-] \otimes [n,-]
\]

as \( \text{coev}_{[n,-]} = (d^n; [0,-],[2n,-]) \).

\[
\text{ev}_{[n,-]} : [2n,-] \otimes [n,-] = [2n,?] \to 1^-
\]

as \( \text{ev}_{[n,-]} = (d^n; [2n,?],[0,?]), \ ? = -, \text{ if } 2 \mid n \).

**Proposition 1.6.14.** \( A_0 \) is rigid.

**Proof.** First we prove that

\[
(id_{[n,+]} \otimes \text{ev}_{[n,+]} \circ \text{coev}_{[n,+]} \otimes id_{[n,+]} = id_{[n,+]})
\]

Note that \( id_{[n,+]} \otimes ev_{[n,+]} = (S(n)(d^n); [2n + n, +], [0 + n, +]) = (d^n; [3n, +], [n, +]) \) and \( coev_{[n,+]} \otimes id_{[n,+]} = (d^n; e_{0,n}^0, [0 + n, +], [2n + n, +]) = (d^n; e_{0,n}^0, [n, +], [3n, +]). \)

Then by the composition case (C2), where \( i = 0, j = n \),

\[
(id_{[n,+]} \otimes ev_{[n,+]} \circ (coev_{[n,+]} \otimes id_{[n,+]} = (d^n; [3n, +], [n, +]) \circ (d^n; e_{0,n}^0, [n, +], [3n, +])
\]

\[
= (d^n; E_{0,n}^{0,n+2n}(d^n; e_{0,n}^0, [n, +], [n + 2i, +])
\]

\[
= (d^n; E_{0,3n}^{0,n}(e_{0,n}^0, [n, +], [n, +])
\]

\[
= (1; [n, +], [n, +]) = id_{[n,+]}.
\]

The other three cases are left to the reader. Therefore, \( A_0 \) is rigid.

**Proposition 1.6.15.** \( A_0 \) has a pivotal structure.
Proof. First, we prove that the right trace $\text{Tr}_R$ is a normal faithful trace. Let $X = [n,+]$. Given $(f; [n],[n])$, $f \otimes \text{id}_{[n,+]} = (f; [2n,],[2n])$, then

$$\text{Tr}_R(f) = \text{coev}_{[n,+]}^\dagger \circ (f \otimes \text{id}_{[n,+]}) \circ \text{coev}_{[n,+]}$$

$$= (d^n; [2n,+],[0,+]) \circ (f; [2n,],[2n]) \circ (d^n; [0,],[2n])$$

$$= (d^n; [2n,],[0,]+) \circ (d^n E_{0,2n}^{r,n}(f \cdot d^n e_{0,n}^0; [0,],[2n]))$$

$$= (d^n; [2n,],[0,]+) \circ (f; [0,],[2n])$$

$$= (d^n E_{0,2n}^{r,n}(f e_{0,n}^0; [0,],[0,]+))$$

$$= (\text{tr}(f); [0,],[0,]+).$$

The third equality uses (C1), where $n = 0, i = n, j = 0$; the forth equality uses (A10); the fifth equality uses (C2), where $n = i = 0, j = n$.

The case $X = [n,-]$ is left to the reader.

Next, we prove that the left trace $\text{Tr}_L$ is a normal faithful trace. Let $X = [2n,+]$. Given $(f; [2n,],[2n,])$, $\text{id}_{[2n,+]} \otimes f = (S_{0,2n}^{(n)}(f); [4n,],[4n])$, then

$$\text{Tr}_L(f) = \text{ev}_{[2n,+]} \circ (\text{id}_{[2n,+]} \otimes f) \circ \text{ev}_{[2n,]}^\dagger$$

$$= (d^{2n}; [4n,],[0,]+) \circ (S_{0,2n}^{(n)}(f); [4n,],[4n]) \circ (d^{2n};[0,],[4n])$$

$$= (d^{2n}; [4n,],[0,]+) \circ (d^{2n} E_{0,4n}^{r,2n}(S_{0,2n}^{(n)}(f) \cdot d^{2n} e_{0,2n}^0); [0,],[4n])$$

$$= (d^{2n} \cdot d^n E_{0,2n}^{r,2n}(S_{0,2n}^{(n)}(f) e_{0,2n}^0) e_{0,2n}^0; [0,],[0,]+)$$

$$= (\text{tr}(f); [0,],[0,]+).$$

The last equality: since $e_{0,2n}^0 = 1$ and $E_{0,2n}^{r,2n} \circ E_{0,4n}^{r,2n} = \text{tr} = E_{2n,4n}^{r,2n} \circ E_{0,4n}^{r,2n}$, $S_{0,2n}^{(n)}(f) \in A_{2n,4n}$ and $S_{0,2n}^{(n)}$ is trace-preserving, then

$$d^{4n} \cdot d^n E_{0,2n}^{r,2n}(E_{0,4n}^{r,2n}(S_{0,2n}^{(n)}(f) e_{0,2n}^0) e_{0,2n}^0) = d^{4n} \text{tr}(S_{0,2n}^{(n)}(f) e_{0,2n}^0)$$

$$= d^{4n} E_{2n,4n}^{r,2n} \circ E_{0,4n}^{r,2n}(S_{0,2n}^{(n)}(f) e_{0,2n}^0)$$

(by (A10))

$$= E_{2n,4n}^{r,2n}(S_{0,2n}^{(n)}(f))$$

(by Prop 1.3.5(2))

$$= E_{0,2n}^{r,2n}(f) = \text{tr}(f).$$

The cases $X = [2n+1,],[n,-]$ are left to the reader.

Therefore, $\text{Tr}_R = \text{Tr}_L$ are the trace, so $A_0$ has a pivotal structure.

Moreover, by the composition case (C2), where $i = n = 0, j = n,$

$$\text{ev}_{[n,+]} \circ \text{coev}_{[n,+]} = (d^n; [2n,],[0,]+) \circ (d^n; [0,],[2n])$$

$$= (d^n E_{0,2n}^{r,n}(d^{2n} e_{0,n}^0; [0,],[0,]+))$$

$$= (d^{2n}; [0,],[0,]+) = d^{2n} \cdot 1^+.$$
1.7 From 2-shaded rigid C* multitenor category to standard $\lambda$-lattice

In this section, we show the relation between the 2-shaded rigid C* multitensor category and planar tensor category, and give the construction from the category to standard $\lambda$-lattice.

1.7.1 Rigid C* multitenor category

In this subsection, we are going to review the unitary dual functors in a rigid C* (multi)tensor category $C$ [Pe18].

Definition 1.7.1. Recall that every object $c \in C$ is dualizable, i.e., there is an object $\overline{c} \in C$ together with morphisms $ev_c \in C(\overline{c} \otimes c \to 1_C)$ and $coev_c \in C(1_C \to c \otimes \overline{c})$ satisfying the zig-zag condition:

$$(id_c \otimes ev_c) \circ (coev_c \otimes id_c) = id_c$$

$$(ev_c \otimes id_c) \circ (id_{\overline{c}} \otimes coev_c) = id_{\overline{c}}.$$  

We also require that every object $c \in C$ admits a predual object $\underline{c}$ such that $\overline{\underline{c}} \cong c$.

Definition 1.7.2. A choice of dual for every object in $C$ assembles into a dual functor $(\cdots) : C \to C_{mop}$, which is a tensor functor with a canonical tensorator $\nu_{a,b}$. To be precise, for a morphism $f \in C(a \to b)$, define

$$f := (ev_b \otimes id_a) \circ (id_{\overline{b}} \otimes f \otimes id_{\overline{a}}) \circ (id_a \otimes coev_b) : \overline{b} \to \overline{a}.$$  

The tensorator $\nu_{a,b} : \overline{a} \otimes \overline{b} \to \overline{b} \otimes \overline{a}$ is defined as

$$\nu_{a,b} := (ev_a \otimes id_{\overline{b} \otimes \overline{a}}) \circ (id_{\overline{a}} \otimes ev_b \otimes id_a \otimes id_{\overline{b} \otimes \overline{a}}) \circ (id_{\overline{a} \otimes \overline{b}} \otimes coev_{b \otimes a}).$$

Note that $\nu$ is completely determined by $ev$ and $coev$.

Proposition 1.7.3. Any two dual functors $(\cdots)_1$ and $(\cdots)_2$ are equivalent up to a unique natural isomorphism. Define $\zeta : (\cdots)_2 \to (\cdots)_1$ as follows: for $c \in C$,

$$\zeta_c := (ev_c^2 \otimes id_{\overline{c}_1}) \circ (id_{\overline{c}_2} \otimes coev_c^1).$$

Then we have $\zeta(f)_2 = \zeta_a \circ f \circ \zeta_b^{-1} = \zeta(f)_1$ for all $f \in C(a \to b)$.  

Definition 1.7.4. [EGNO15] A pivotal structure on a rigid monoidal category $C$ is a pair $(\bar{\cdot}, \varphi)$, where $\bar{\cdot}$ is a dual functor and $\varphi : \text{id} \Rightarrow \bar{\cdot}$ is a monoidal natural isomorphism. To be precise, for all $a, b \in C$, the following diagram commutes:

\[
\begin{array}{ccc}
a \otimes b & \xrightarrow{\varphi_a \otimes \varphi_b} & a \otimes b \\
\varphi_{a \otimes b} \downarrow & & \downarrow \varphi_{a \otimes b} \\
\bar{a} \otimes \bar{b} & \xrightarrow{\bar{\varphi}_{\bar{a} \otimes \bar{b}}} & \bar{b} \otimes \bar{a}
\end{array}
\]

Definition 1.7.5 (Pivotal trace). Let $1_C = \bigoplus_{i=1}^r 1_i$ be a decomposition into simples. For $c \in C$ and $f \in C(c \to c)$, define the left/right pivotal traces $\text{tr}_L^c$ and $\text{tr}_R^c : C(c \to c) \to C(1_C \to 1_C) \cong M_r(\mathbb{C})$ by

\[
\text{tr}_L^c(f) := ev_c \circ (\text{id}_c \otimes f) \circ (\text{id}_c \otimes \varphi_c^{-1}) \circ \text{coev}_c \\
\text{tr}_R^c(f) := ev_c \circ (\varphi_c \otimes \text{id}_c) \circ (f \otimes \text{id}_c) \circ \text{coev}_c
\]

The traces are tracial and non-degenerate.

Definition 1.7.6. Let $p_i \in C(1_C \to 1_C)$ be the projection onto $1_i, i = 1, 2, \ldots, r$. We define the $M_r(\mathbb{C})$-valued traces $\text{Tr}_L^c$ and $\text{Tr}_R^c$ by the formulas:

\[
(\text{Tr}_L^c(f))_{i,j} \text{id}_{1_j} := \text{tr}_L^c(p_i \otimes f \otimes p_j) \\
(\text{Tr}_R^c(f))_{i,j} \text{id}_{1_i} := \text{tr}_R^c(p_i \otimes f \otimes p_j)
\]

Note that $\text{Tr}_L^c$ and $\text{Tr}_R^c$ are tracial, and $\text{Tr}_L^c(\bar{f}) = \text{Tr}_R^c(f)^T$ for all $f \in C(c \to c)$.

We call the pivotal structure $(\bar{\cdot}, \varphi)$ spherical, if $\text{Tr}_L^c(f) = \text{Tr}_R^c(f)$, for all $c \in C$, $f \in C(c \to c)$.

Definition 1.7.7. For each $c \in C$, define $\text{Dim}_L^c, \text{Dim}_R^c \in M_r(\mathbb{C})$ by

\[
\text{Dim}_L^c(c) := \text{Tr}_L^c(\text{id}_c) \\
\text{Dim}_R^c(c) := \text{Tr}_R^c(\text{id}_c)
\]

If $c$ is simple, then $\text{Dim}_L^c(c), \text{Dim}_R^c(c)$ have only one non-zero entry, which we denote $\dim_l(c), \dim_r(c)$ respectively.

If the pivotal structure $(\bar{\cdot}, \varphi)$ is spherical, $\text{Dim}_L^c(c) = \text{Dim}_R^c(c) := \text{Dim}(c)$ for all object $c$.

Definition 1.7.8. A dagger structure on a $\mathbb{C}$-linear category is a collection of anti-linear maps $\dagger : C(c \to d) \to C(d \to c)$ for all $c, d \in C$ such that $(f \circ g)^\dagger = g^\dagger \circ f^\dagger$ and $(f^\dagger)^\dagger = f$. A morphism $f : C(a \to b)$ is called unitary if $f^\dagger = f^{-1}$.

A dagger (multi)tensor category is a (multi)tensor category equipped with a dagger structure so that $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$ for all morphisms $f, g$, and all associator and unitors are unitary.

Definition 1.7.9. A functor between dagger categories $F : C \to D$ is called a dagger functor if $F(f^\dagger) = F(f)^\dagger$ for all $f \in \text{Hom}(C)$. 

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Definition 1.7.10 (Rigid $C^*$ (multi)tensor category). A $C^*$ category is a dagger category which is Cauchy complete and each endomorphism algebra is a $C^*$-algebra, where the dagger structure is compatible with the $*$-structure.

A $C^*$ (multi)tensor category is a dagger (multi)tensor category whose underlying dagger category is $C^*$.

A rigid $C^*$ (multi)tensor category is a $C^*$ (multi)tensor category equipped with a dual functor. It is known that a rigid $C^*$ multitensor category is Cauchy complete if and only if it is semisimple [LR96].

Proposition 1.7.11 (Unitary dual functor). Fix a dual functor $(\cdot)$ on a rigid $C^*$ (multi)tensor category $C$, the followings are equivalent:

(1) $(\cdot)$ is a unitary dual functor, i.e., for all $a, b \in C$, $f \in C(a \to b)$, the tensorator $\nu_{a,b}$ is unitary and $f^\dagger = f^\dagger$.

(2) Defining $\varphi_c := (\text{coev}_c^\dagger \otimes \text{id}_c) \circ (\text{id}_c \otimes \text{coev}_c)$ is a pivotal structure $\varphi : \text{id} \Rightarrow (\cdot)$.

Proof. [Se11], see also [Pe18, Prop. 3.9].

Definition 1.7.12. Two unitary dual functors are called unitary equivalent, if the canonical natural transformation $\zeta$ from Proposition 1.7.3 is unitary, i.e., $\zeta_c$ is unitary for all $c \in C$.

Proposition 1.7.13. For a unitary dual functor $(\cdot)$, the left/right pivotal traces have alternate formulas:

$$\text{tr}_L^c(f) = \text{ev}_c \circ (\text{id}_c \otimes f) \circ \text{ev}_c^\dagger$$

$$\text{tr}_R^c(f) = \text{coev}_c^\dagger \circ (f \otimes \text{id}_c) \circ \text{coev}_c.$$

Theorem 1.7.14 ([BDH14] [Pe18, Prop. 3.24]). For a rigid $C^*$ (multi)tensor category $C$, there exists a unique unitary dual functor whose induced pivotal structure is spherical up to unitary equivalence. In other words, the pivotal structure can be trivial, so that $\text{ev}_c = \text{coev}_c^\dagger$ and $\text{coev}_c = \text{ev}_c^\dagger$ for all $c \in C$.

1.7.2 2-shaded rigid $C^*$ multitensor category with a choice of generator and planar tensor category

Let $\mathcal{A}$ be a 2-shaded rigid $C^*$ multitensor category together with $1 = 1^+ \oplus 1^-$, where $1^+, 1^-$ are simple, and a generator $X = 1^+ \otimes X \otimes 1^-$. Here, the generating means for any simple object $P$, it is a direct summand of $X_{\text{alt}}^\otimes n$ or $X_{\text{alt}}^\otimes n$ (defined below) for some $n \in \mathbb{Z}_{\geq 0}$.

Let $(\cdot)$ be a unitary dual functor that induced a spherical pivotal structure $\varphi$. Note that only $(+, -)$ entry of $\text{Dim}(X)$ is non-zero and we denote this number as $d_X$ to be the modulus of category $C$.

Construction 1.7.15. We construct a planar tensor category $\mathcal{A}_0$ from $\mathcal{A}$. By MacLane’s coherence theorem, $\mathcal{A}$ is unitary equivalent to a strict tensor category with the above properties and the dual functor is strict, WLOG, we also denote it as $\mathcal{A}$. Construct the pivotal planar tensor category $\mathcal{A}_0$ as follows:

(a) Objects: Define $[0, +] := 1^+$, $[0, -] := 1^-$, and

$$[n, +] := [n - 1, +] \otimes X^? = (\cdots (X \otimes X) \otimes X) \otimes \cdots) \otimes X^? =: X_{\text{alt}}^\otimes n,$$

$$n \text{ tensorands}$$
where $X^? = \overline{X}$ if $n$ is even and $X$ if $n$ is odd, and

$$[n, -] := [n - 1, -] \otimes X^? = (\cdots (\overline{X} \otimes X) \otimes \overline{X}) \otimes \cdots \otimes X^? =: \overline{X}^{alt \otimes n},$$

where $X^? = X$ if $n$ is even and $\overline{X}$ if $n$ is odd, for $n \in \mathbb{Z}_{\geq 0}$.

(b) Morphisms: $A_0$ is the full subcategory of $A$ with above objects.

(c) Duality: The dual functor is unitary as a dual functor on the subcategory, which also induces a spherical pivotal structure on the subcategory.

Given $A_0$ to be a pivotal planar tensor category, then its Cauchy completion $\overline{A}_0$ is a Cauchy completed 2-shaded rigid $C^*$ multitensor category with a generator $[1, +]$ and a canonical unitary dual functor $\overline{\circ}$.

**Proposition 1.7.16.** Suppose $A_0$ is a pivotal planar tensor category constructed from $(A, X)$, then there is a unitary equivalence between $(\overline{A}_0, [1, +])$ and the Cauchy completion of $(A, X)$ with respect to their unitary dual functors.

**Remark 1.7.17.** Suppose $A, B$ are two 2-shaded rigid $C^*$ multitensor categories with generator $X$ and $Y$ respectively and $A_0, B_0$ are corresponding pivotal planar tensor categories. Then $A_0$ and $B_0$ are unitary equivalent if and only if the Cauchy completions of $A$ and $B$ are unitary equivalent which maps generator to generator.

**Remark 1.7.18.** The planar tensor category $A_0$ is not Cauchy complete, i.e., additive complete and idempotent complete. In fact, as for skeletalness, strictness and Cauchy complete, most tensor categories can require at most two of them. Vec($G$) is an exception.

### 1.7.3 From planar tensor category to standard $\lambda$-lattice

**Construction 1.7.19.** Let $A_0$ be a pivotal planar tensor category with modulus $d$. Define $A_{0,j} = \text{End}([j, +]), A_{1,j} = \text{id}_{[i, +]} \otimes \text{End}([j - 1, -]), j \in \mathbb{Z}_{\geq 0}$, so that $A_{0,0} = A_{1,1} = C$. In general, for $i \leq j$, define

$$A_{i,j} = \begin{cases} \text{id}_{[i, +]} \otimes \text{End}([j - i, +]) & 2 \mid i \\ \text{id}_{[i, +]} \otimes \text{End}([j - i, -]) & 2 \nmid i. \end{cases}$$

Then we check $A = (A_{i,j})_{i,j \geq 0}$ to be a standard $\lambda$-lattice.

(a) The vertical inclusion $A_{i+1,j} \subset A_{i,j}$ is clear. The right inclusion: the right inclusion send $x \in A_{i,j}$ to $x \otimes \text{id}_{[1, +]} \in A_{i,j+1}$, where $? = +$ if $2 \mid j$ and $? = -$ if $2 \nmid j$.

(b) Horizontal conditional expectation: Define $E^r_{i,j} : A_{i,j} \to A_{i,j-1}$ by

$$E^r_{i,2k}(x) = d^{-1}(\text{id}_{[2k-1, +]} \otimes \text{ev}_{[1, +]}) \circ (x \otimes [1, +]) \circ (\text{id}_{[2k-1, +]} \otimes \text{coev}_{[1, +]}),$$

$$E^r_{i,2k+1}(x) = d^{-1}(\text{id}_{[2k, +]} \otimes \text{ev}_{[1, -]}) \circ (x \otimes [1, -]) \circ (\text{id}_{[2k, +]} \otimes \text{coev}_{[1, -]}).$$

(c) Vertical conditional expectation: Define $E^d_{i,j} : A_{i,j} \to A_{i+1,j}$ by

$$E^d_{2k}(x) = d^{-1}(\text{id}_{[2k+2, +]} \otimes \text{ev}_{[1, +]}) \circ (\text{id}_{[2, +]} \otimes x) \circ (\text{id}_{[2k+2, +]} \otimes \text{coev}_{[1, +]}),$$

$$E^d_{2k+1}(x) = d^{-1}(\text{id}_{[2k+3, +]} \otimes \text{ev}_{[1, -]}) \circ (\text{id}_{[2, +]} \otimes x) \circ (\text{id}_{[2k+3, +]} \otimes \text{coev}_{[1, -]}).$$

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(d) Jones projection: the \( n \)-th Jones projection is defined as
\[
e_{2k+1} = d^{-1} \cdot \text{id}_{[2k,+]} \otimes (\text{coev}_{[1,+]} \circ \text{ev}_{[1,+]}) \in A_{i,2k+2}
\]
\[
e_{2k+2} = d^{-1} \cdot \text{id}_{[2k+1,+]} \otimes (\text{coev}_{[1,+] \circ \text{ev}_{[1,-]}}) \in A_{i,2k+3}.
\]

The check that \( A = (A_{i,j})_{i \geq j \geq 0} \) satisfies Definition 1.2.1(a), (b), (c) and standard condition is left to the reader. In particular, \( e_n e_{n \pm 1} e_n = d^{-2} e_n \), \( E^r_{i,j+1}(e_j) = E^l_{j-1,i}(e_j) = d^{-2} 1 \).

Note that the dual functor is unitary and we divide the loop parameter, the composition of these conditional expectations is actually a unital trace on \( A \).

Remark 1.7.20. The idea of drawing the string diagram explanation in §1.4 comes from here.

In this section, the class of unitary equivalent pairs \((A,X)\) with \( A \) a 2-shaded rigid \( C^* \) multitensor category and \( X \) a generator induces the class of isomorphic pivotal planar tensor categories; in §1.6, the class of isomorphic pivotal planar tensor categories is one to one corresponding to the class of isomorphic standard \( \lambda \)-lattices.

Combining above discussion, we can deduce the equivalence between standard \( \lambda \)-lattice \( A \) and pair 2-shaded rigid \( C^* \) multitensor category with a generator \((A,X)\).

Theorem 1.7.21. There is a bijective correspondence between equivalence classes of the following:

\[
\left\{ \text{Standard } \lambda \text{-lattices } A = (A_{i,j})_{0 \leq i \leq j} \right\} \cong \left\{ \text{Pairs } (A,X) \text{ with } A \text{ a 2-shaded rigid } C^* \text{ multitensor category with a generator } X, \text{i.e., } 1_A = 1^+ \oplus 1^-, 1^+, 1^- \right\}
\]

Equivalence on the left hand side is unital \(*\)-isomorphism of standard \( \lambda \)-lattices; equivalence on the right hand side is unitary equivalence between their Cauchy completions which maps generator to generator.

2 Markov towers as standard right module over standard \( \lambda \)-lattice and module categories

Now we move to the module case. One motivation that regards a Markov tower as a right module over a standard \( \lambda \)-lattice is to answer the question in [CHPS18, Rmk. 3.34].

2.1 Markov tower as a standard right module over standard \( \lambda \)-lattice

Definition 2.1.1.

\[
M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n \subset \cdots \\
\cup \cup \cup \cup \\
A_{0,0} \subset A_{0,1} \subset A_{0,2} \subset \cdots \subset A_{0,n} \subset \cdots \\
\cup \cup \cup \cup \\
A_{1,1} \subset A_{1,2} \subset \cdots \subset A_{1,n} \subset \cdots
\]

Let \( A = (A_{i,j})_{0 \leq i \leq j < \infty} \) be a standard \( \lambda \)-lattice with Jones projection \( \{e_i\}_{i \geq 1} \) and compatible conditional expectations. Let \( M = (M_n, e_n)_{n \geq 0} \) be a Markov tower with conditional expectation \( E_i : M_i \to M_{i-1}, i \geq 1 \). (\( M \) and \( A \) share the same Jones projections) We call a Markov tower \( M \) a standard right \( A \)-module, if it satisfies the following three conditions.
(a) \( A_{0,i} \subset M_i \) is a unital inclusion, \( i = 0, 1, 2, \ldots \).
(b) \( E_i|_{A_{0,i}} = E_{0,i}^r \), \( i = 1, 2, \ldots \).
(c) (standard condition) \([M_i, A_{k,l}] = 0\) for \( i \leq k \leq l \).

In the rest of this Chapter, we only consider the Markov tower with \( \dim(M_0) = 1 \) unless stated.

### 2.2 String diagram explanation

We now introduce the diagrammatic explanation of the element, conditional expectation, Jones projection and their relations in a Markov tower with the same spirit in §1.4.

**MT1** Element \( x \in M_n \):

**MT2** Vertical inclusion \( x \in A_{0,n} \subset M_n \):

**MT3** Horizontal inclusion \( x \in M_n \subset M_{n+1} \):

**MT4** Jones projections:

\[
e_{2i+1} = d^{-1} x_i x_i \in M_{2i+2} \quad e_{2i+2} = d^{-1} x_{2i+1} x_{2i+1} \in M_{2i+3}
\]

**MT5** Conditional expectation \( E_n : M_n \to M_{n-1} \) and \( E_n|_{A_{0,n}} = E_{0,n}^r \):

\[
E_n(x) = d^{-1} x_{n-1}, x \in M_n \quad E_n(x) = E_{0,n}^r(x) = d^{-1} x_{n-1}, x \in A_{0,n}
\]

**MT6** Pull down condition: For \( x \in M_{n+1} \), \( x e_n = dE_{n+1}(x e_n) e_n \).
(MT7) Standard condition: For \( f \in M_i, x \in A_{k,i} \) with \( k \geq i \), then we regard \( \phi, x \) as elements in \( M_i, fx = xf \).

![Diagram](image)

2.3 From Markov tower as a standard module to planar module category

2.3.1 Planar module category over planar tensor category

**Definition 2.3.1.** Let \( A_0 \) be a planar tensor category defined in §1.6.1. Let \( M_0 \) be an indecomposable semisimple \( C^* \) right \( A_0 \)–module category with following properties:

(a) Object: The objects of \( M_0 \) are \([n] = [n]_{M_0}, n \in \mathbb{Z}_{\geq 0}\), where \([0]\) is simple.

(b) The tensor product of objects are

\[ [m]_{M_0} \otimes [n,+]_{A_0} = [m+n]_{M_0}, \quad [m]_{M_0} \otimes [n,-]_{A_0} = 0. \]

(c) Only \( M_0([n] \to [n \pm 2i]) \) is non-empty, \( n, i \in \mathbb{Z}_{\geq 0} \). The module product of morphism in \( \text{Hom}(M_0) \) and \( \text{Hom}(A_0) \) should match the shading types.

(d) \( M_0 \) is a strict right \( A_0 \)–module category, i.e., the module associator is identity. For \( x_1, x_2 \in A_0 \) and \( f \in M_0 \),

\[ (f \triangleleft x_1) \triangleleft x_2 = f \triangleleft (x_1 \otimes x_2). \]

(e) \( M_0 \) is a \( C^* \) category with a natural dagger structure such that \( \triangleleft \) is a dagger functor, i.e., for \( x \in \text{Hom}(A_0) \) and \( f \in \text{Hom}(M_0) \),

\[ (f \triangleleft x)^\dagger = f^\dagger \triangleleft x^\dagger. \]

Such module category is called a **planar module category**.

**Remark 2.3.2.** Similar to Remark 1.6.5, the morphisms in \( M_0 \) is determined by its representation as an endomorphism and its domain and range.

There is a canonical isomorphism \( \phi : M_0([m] \to [m+2i]) \to M_0([m+i] \to [m+i]) \) by using the rigid structure on \( A_0 \).

\[
\phi : \begin{vmatrix}
\hline
m \\
\hline
\end{vmatrix} \xrightarrow{m+i}
\begin{vmatrix}
\hline
m \\
\hline
\end{vmatrix}
\]

\[
\phi^{-1} : \begin{vmatrix}
\hline
m \\
\hline
\end{vmatrix} \xleftarrow{m+i}
\begin{vmatrix}
\hline
m \\
\hline
\end{vmatrix}
\]

For morphism \( x \in M_0([m], [n]) \), we can write a triple \((\phi(x); [m], [n])\) to represent \( x \), where \( \phi(x) \in \text{End}([m+n]) \), which is called the **endomorphism representation part** of \( x \). In the following context, we simply write \( x \) instead of \( \phi(x) \) in the triple \((x; [m], [n])\).

2.3.2 From Markov tower as a standard module to planar module category

Define the multi-step conditional expectation \( E^m_n = E_{n-m+1} \circ \cdots \circ E_n \), for \( m \leq n \). Similar to Definition 1.6.6, we may regard the elements in \( M_n \) as endomorphisms in the category, we can construct a planar module category from a given Markov tower as a standard module over a standard \( \lambda \)-lattice.
**Definition 2.3.3.** Let \( M = (M_n)_{n \geq 0} \) be a Markov tower as a standard right module over standard \( \lambda \)-lattice \( A = (A_{i,j}) \) with \( \dim(M_0) = 1 \). We define a planar module category \( \mathcal{M}_0 \) from \( M \) as follows.

(a) The objects of \( \mathcal{M}_0 \) are the symbols \([n]\) for \( n \in \mathbb{Z}_{\geq 0} \).

(b) Given \( n \geq 0 \), define \( \mathcal{M}_0([n] \to [n]) := M_n \).

(c) The identity morphism in \( \mathcal{M}_0([n] \to [n]) \) is \( 1_{M_n} \).

(d) For \( (f;[m],[n]) \) with \( 2 \mid m+n \), we define \( (f;[m],[n])^\dagger := (f^*;[n],[m]) \), where \( f, f^* \in M_{\frac{m+n}{2}} \).

(e) We define composition in three cases.

\[
\begin{align*}
(C1) \quad & (g;[n+2i],[n+2i+2j]) \circ (f;[n],[n+2i]) = (d^iE_{n+2i}\circ gfe_{j,i};[n],[n+2i+2j]), & f \in M_{n+i}, g \in M_{n+2i+j} \text{ and } d^iE_{n+2i+j}(gfe_{j,i}) \in M_{n+i+j}, \\
(C2) \quad & (g;[n+2i+2j],[n+2i]) \circ (f;[n],[n+2i+2j]) = (d^{i+j}E_{n+2i+j}\circ gfe_{j,i}^*;[n],[n+2i]), & f \in M_{n+i+j}, g \in M_{n+2i+j} \text{ and } d^{i+j}E_{n+2i+j}(gfe_{j,i}^*) \in M_{n+i}, \\
(C3) \quad & (g;[n],[n+2i+2j]) \circ (f;[n+2i],[n]) = (d^ige_{j,i}^*f;[n+2i],[n+2i+2j]), & f \in M_{n+i}, g \in M_{n+2i+j} \text{ and } d^ige_{j,i}^*f \in M_{n+2i+j}.
\end{align*}
\]

For the other cases, we can use the dagger structure \( f^\dagger \circ g^\dagger := (g \circ f)^\dagger \) to define.

Similarly, the composition and the dagger structure are well defined, and \( \mathcal{M}_0 \) is \( C^* \) [CHPS18, §3.4].

![Diagram](https://example.com/diagram.png)

**Remark 2.3.4.** Readers can observe the similarity between the diagrammatic explanation of elements in \( M_n \) and \( A_{i,n} \), difference only appears on the leftmost. Moreover, the similar version of Lemma 1.5.3 and Lemma 1.5.4 is also true for Markov tower case.

Now we define the module action of morphisms.

**Definition 2.3.5.** \( f \triangleleft 1 \) and \( 1 \triangleleft x \), \( f \in \text{Hom}(\mathcal{M}_0) \) and \( x \in \text{Hom}(A_0) \). The idea is the same as in Definition 1.6.7.

First, we define \( f \triangleleft 1 \) as

\[
\begin{array}{c|c}
\hat{f} & f \triangleleft 1_j \\
\hline
(f;[m],[m+2i], i \leq j) & (fe_{i,j}^*;[m+j],[m+2i+j]) \\
(f;[m],[m+2i], i > j) & (fe_{i,j}^*;[m+j],[m+2i+j])
\end{array}
\]

The definition of \( 1 \triangleleft x \) will be the same as \( 1 \otimes x \) by using the 2-shift maps in Definition 1.6.7.
The proof of following propositions are the same as in Proposition 1.6.8, 1.6.10 and 1.6.11.

**Proposition 2.3.6.** For \( f \in \text{Hom}(\mathcal{M}_0), \ x \in \text{Hom}(\mathcal{A}_0), \) \( (f \triangleleft 1) \circ (1 \triangleleft x) = (1 \triangleleft x) \circ (f \triangleleft 1). \)

**Definition 2.3.7.** Define \( f \triangleleft x := (f \triangleleft 1) \circ (1 \triangleleft x). \)

The following propositions guarantee the module action defined above is well-defined.

**Proposition 2.3.8.** For \( f \in \text{Hom}(\mathcal{M}_0), \ x, y \in \text{Hom}(\mathcal{A}_0), \) \( (f \triangleleft x) \triangleleft y = f \triangleleft (x \otimes y). \)

**Proposition 2.3.9.** For \( f, g \in \text{Hom}(\mathcal{M}_0), \) \( (f \circ g) \triangleleft 1 = (f \triangleleft 1) \circ (g \triangleleft 1) \) and \( 1 \triangleleft (x \otimes y) = (1 \triangleleft x) \circ (1 \triangleleft y). \)

### 2.4 Indecomposable semisimple \( \mathcal{C}^* \) \( \mathcal{A} \)−module categories and planar \( \mathcal{A}_0 \)−module categories

#### 2.4.1 Indecomposable semisimple \( \mathcal{C}^* \) \( \mathcal{A} \)−module category

Let \( \mathcal{A} \) be a 2-shaded rigid \( \mathcal{C}^* \) multitensor category with a generator \( X = 1^+ \otimes X \otimes 1^- \) with a canonical unitary dual functor \( (\cdot)^\dagger \). Let \( \mathcal{M} \) be a Cauchy complete indecomposable semisimple \( \mathcal{C}^* \) \( \mathcal{A} \)−module category. Note that there is a natural dagger structure on \( \mathcal{M} \), and the module action \( \triangleleft \) is a dagger functor, namely, for morphism \( f \in \text{Hom}(\mathcal{M}) \) and \( x \in \text{Hom}(\mathcal{A}), \)

\[
(f \triangleleft x) \dagger = f \dagger \triangleleft x \dagger.
\]

We call a module category \( \mathcal{M} \) **indecomposable** if for any two simple objects \( P, Q \in \mathcal{M}, \ Q \) is a direct summand of \( P \triangleleft X^{alt \otimes n} \) if \( P = P \triangleleft 1^+ (P \triangleleft X^{alt \otimes n} \) if \( P = P \triangleleft 1^- \) for some \( n \in \mathbb{Z}_{\geq 0} \).

**Construction 2.4.1.** Let \( \mathcal{A}_0 \) be a planar tensor category obtained from \((\mathcal{A}, X)\) via the construction in §1.7.2. By MacLane's coherence theorem, \( \mathcal{M}_\mathcal{A} \) is unitary equivalent to a strict one, i.e., \( \mathcal{M} \) and \( \mathcal{A} \) are strict and the right module associator is trivial. Then \( \mathcal{M} \) is also a strict right \( \mathcal{A}_0 \)−module category.

We construct the planar \( \mathcal{A}_0 \)−module category \( \mathcal{M}_0 \) as follows:

(a) **Objects:** Pick a simple object \( Z = Z \triangleleft 1^+ \in \mathcal{M}, \) define \([0] := Z, \) and

\[
[n + 1] := [n] \triangleleft [1, \dagger],
\]

where \([1, \dagger] = [1, +] \) if \( 2 \mid n \) and \([1, \dagger] = [1, -] \) if \( 2 \nmid n. \)

(b) **Morphisms:** \( \mathcal{M}_0 \) is a full subcategory of \( \mathcal{M} \) with above objects.

Given \( \mathcal{M}_0 \) to be a planar \( \mathcal{A}_0 \)−module category, then its Cauchy completion \( \widehat{\mathcal{M}}_0 \) is an \( \widehat{\mathcal{A}}_0 \)−module, compatible with the dagger structure. The proof is left to the reader as an exercise.

**Remark 2.4.2.** Suppose \( \mathcal{M}_0 \) is a planar \( \mathcal{A}_0 \)−module category constructed from \((\mathcal{M}, Z)\) over \((\mathcal{A}, X), \) then there is a unitary equivalence between \( \mathcal{M} \) as \( \mathcal{A} \)−module and \( \widehat{\mathcal{M}}_0 \) as \( \mathcal{A}_0 \)−module, which sends base object to base object.
2.4.2 From planar module category to Markov tower as a standard module over a standard $\lambda$-lattice

**Construction 2.4.3.** Let $\mathcal{M}_0$ be a planar $\mathcal{A}_0$-module category with modulus $d$ and $A$ is a standard $\lambda$-lattice constructed from $\mathcal{A}_0$ as in §1.7.3. Define $M_j = \text{End}([j])$, $j \in \mathbb{Z}_{\geq 0}$. Then we check $M = (M_j)_{j \geq 0}$ to be a Markov tower as a standard $A$-module.

(a) The horizontal inclusion $M_j \subset M_{j+1}$ sends $x \in M_j$ to $x \triangleleft \text{id}_{[1,n]} \in M_{j+1}$, where $n = +$ if $2 \mid j$ and $n = -$ if $2 \nmid j$. The vertical inclusion $A_{0,j} \subset M_j$ sends $x \in A_{0,j}$ to $\text{id}_{[0]} \triangleleft x \in M_j$.

(b) Conditional expectation: Define $E_j^M: M_j \to M_{j-1}$ by

$$E_j^M(x) = d^{-1}((\text{id}_{[2k-1]} \otimes \text{ev}_{[1,-]}) \circ (x \triangleleft [1,+] \otimes (\text{id}_{[2k-1]} \otimes \text{coev}_{[1,+]})),
E_{2k+1}^M(x) = d^{-1}((\text{id}_{[2k]} \otimes \text{ev}_{[1,-]}) \circ (x \triangleleft [1,-] \otimes (\text{id}_{[2k]} \otimes \text{coev}_{[1,-]})).$$

(c) Jones projections: the same Jones projections in $A$ and identify $e_n \in A_{0,n+1}$ with $1 \triangleleft e_n \in M_{n+1}$.

The check that $M$ is a Markov tower and a standard $A$-module is left to the reader. In particular, we have $E_{n+1}(e_n) = d^{-2} \cdot 1$.

In this section, we show that the class of unitary equivalent pairs $(\mathcal{M}, Z)$ with $\mathcal{M}$ an indecomposable right $\mathcal{A}$-module category and $Z$ a simple base point induces the equivalent class of planar module categories; according to §2.3.2, the class of equivalent planar module categories is one to one corresponding to the class of isomorphic Markov towers as standard module over isomorphic standard $\lambda$-lattices.

Combining above discussion, we can deduce the equivalence between $(\mathcal{M}, Z)$ as $\mathcal{A}$-module category and Markov tower $M$ as standard $A$-module.

**Theorem 2.4.4.** There is a bijective correspondence between equivalence classes of the following:

$$\left\{ \begin{array}{l}
\text{Traceless Markov tower } M = (M_i)_{i \geq 0} \text{ with } \dim(M_0) = 1 \text{ as a standard right module over a standard } \lambda\text{-lattice } A \\
\text{Pairs } (\mathcal{M}, Z) \text{ with } \mathcal{M} \text{ an indecomposable semisimple } C^* \text{ right } \mathcal{A}\text{-module category together with a choice of simple object } Z = Z \triangleleft 1^+_A
\end{array} \right\} \cong \mathbb{R}$$

Equivalence on the left hand side is $*$-isomorphism of traceless Markov towers as standard $A$-modules; equivalence on the right hand side is unitary $A$-module equivalence on their Cauchy completions which maps the simple base object to simple base object.

**Corollary 2.4.5.** Any Markov tower $M$ with modulus $d$ and $\dim(M_0) = 1$ is naturally a standard right TLJ$(d)$-module, where TLJ$(d)$ is a Temperley-Lieb-Jones standard $\lambda$-lattice as in Example 1.2.8, which corresponds to an indecomposable semisimple $C^*$ right TLJ$(d)$-module category with a simple base object.

**Remark 2.4.6.** The tracial case will be discussed in §6.1.

3 Markov lattices as standard bimodule over two standard $\lambda$-lattices and bimodule categories

In this chapter, we extend the discussion into the bimodule case. We give the notion Markov lattices and Markov lattices as bimodule over two standard $\lambda$-lattices, by using the similar method, which correspond to bimodule categories.
3.1 Markov lattice and basic properties

**Definition 3.1.1 (Markov lattice).** A tuple \( M = (M_{i,j}, E_{i,j}^{M,l}, E_{i,j}^{M,r}, e_i, f_j)_{i,j \geq 0} \) is called a Markov lattice if the following conditions hold.

\[
M_{i+1,j} \subset M_{i+1,j+1} \quad \bigcup \quad M_{i,j} \subset M_{i,j+1}
\]

(a) \( M_{i,j} \subset M_{i+1,j} \) and \( M_{i,j} \subset M_{i+1,j} \) are unital inclusions.
(b) \( M_{i,j} = (M_{i,j}, E_{i,j}^{M,l}, e_i)_i, j \geq 0 \) are Markov towers with the same modulus \( d_0 \) and \( e_i \in M_{i+1,j}, j \geq 0 \) for all \( j \);
(c) The commuting square condition:

\[
\begin{align*}
\begin{array}{c}
M_{i+1,j} \\
M_{i,j}
\end{array}
\begin{array}{c}
E_{i+1,j+1}^{M,r} \\
E_{i,j}^{M,l}
\end{array}
\begin{array}{c}
\downarrow \\
\downarrow
\end{array}
\begin{array}{c}
M_{i+1,j+1} \\
M_{i,j+1}
\end{array}
\end{align*}
\]

is a commuting square, i.e., \( E_{i+1,j+1}^{M,r} \circ E_{i,j}^{M,l} = E_{i+1,j+1}^{M,l} \circ E_{i,j+1}^{M,r} \).

Here are some properties of Markov lattice.

**Proposition 3.1.2.** Let \( M = (M_{i,j}, E_{i,j}^{M,l}, E_{i,j}^{M,r}, e_i, f_j)_{i,j \geq 0} \) be a Markov lattice.

(1) \( E_{i,j+1}^{M,l} \circ f_j = f_j \) for each \( i, j = 1, 2, 3, \cdots \).

(2) \( [f_j, e_i] = 0 \) for each \( i, j = 1, 2, 3, \cdots \).

**Proof.**

(1) Note that \( e_i \in M_{i+1,j} \subset M_{i+1,j+1} \) and \( E_{i+1,j+1}^{M,r} : M_{i+1,j+1} \to M_{i+1,j} \) is a conditional expectation, we have \( E_{i+1,j+1}^{M,l} \circ e_i = e_i \). Similarly, \( E_{i,j+1}^{M,l} \circ f_j = f_j \).

(2) By Proposition 1.1.4(1).

**Remark 3.1.3.** If there is a faithful normal trace on \( \bigcup_{i,j \geq 0} M_{i,j} \) and \( E_{i,j}^{M,r}, E_{i,j}^{M,l} \) are the canonical faithful normal trace-preserving conditional expectations for \( i, j = 0, 1, 2, \cdots \), then \( M \) is called a tracial Markov lattice.

In the rest of this Chapter, we only consider the traceless Markov lattice with \( \dim(M_{0,0}) = 1 \) unless stated.
3.2 Markov lattice as a standard bimodule over two standard \( \lambda \)-lattices

**Definition 3.2.1** (Markov lattice as a standard bimodule over two standard \( \lambda \)-lattices).

\[
\begin{array}{cccccccc}
A_{3,1} & \subset & A_{3,0} & \subset & M_{3,0} & \subset & M_{3,1} & \subset & M_{3,2} & \subset & M_{3,3} & \subset \\
A_{2,1} & \subset & A_{2,0} & \subset & M_{2,0} & \subset & M_{2,1} & \subset & M_{2,2} & \subset & M_{2,3} & \subset \\
A_{1,1} & \subset & A_{1,0} & \subset & M_{1,0} & \subset & M_{1,1} & \subset & M_{1,2} & \subset & M_{1,3} & \subset \\
A_{0,0} & \subset & M_{0,0} & \subset & M_{0,1} & \subset & M_{0,2} & \subset & M_{0,3} & \subset & M_{0,4} & \subset \\
B_{0,0} & \subset & B_{0,1} & \subset & B_{0,2} & \subset & B_{0,3} & \subset & B_{0,4} & \subset & B_{0,5} & \subset \\
B_{1,1} & \subset & B_{1,2} & \subset & B_{1,3} & \subset & B_{1,4} & \subset & B_{1,5} & \subset & B_{1,6} & \subset \\
\end{array}
\]

Let \( A^{\text{op}} = (A_{i,j})_{0 \leq j \leq i \leq \infty} \) \( B = (B_{i,j})_{0 \leq i \leq j \leq \infty} \) be two standard \( \lambda \)-lattices with Jones projection \( e_i \in A_{i+1,j} \), \( f_j \in B_{i,j+1} \) respectively and compatible conditional expectations. Here, \( A \) and \( B \) share the same Jones projections \( e_i \). \( B \) and \( M \) share the same Jones projections \( f_j \).

**Remark 3.2.2.** The standard condition implies that \([A_{p,q},B_{k,l}] = 0\) for all \( q \leq p, k \leq l \) since \( A_{p,q} \subset A_{p,0} \subset M_{p,0} \) and \( B_{k,l} \subset B_{0,l} \subset M_{0,l} \). Moreover, \( E_{i,j}^{M_r}|_{A_{k,l}} = \text{id}, E_{i,j}^{M_l}|_{B_{k,l}} = \text{id} \). In particular, we have \( E_{i,j}^{M_r}(e_k) = e_k \), \( E_{i,j}^{M_l}(f_l) = f_l \) for Jones projections.

3.3 String diagram explanation

We now provide the string diagram explanation of the element, conditional expectation, Jones projection and their relations in a Markov lattice with the same spirit in §2.2.

(ML1) Element \( x \in M_{i,j} \):

![String diagram for element x](image)

(ML2) Horizontal inclusion \( x \in M_{i,j} \subset M_{i,j+1} \) and \( x \in A_{i,0} \subset M_{i,j} \):

![String diagram for horizontal inclusion x](image)
(ML3) Vertical inclusion \( x \in M_{i,j} \subset M_{i+1,j} \) and \( x \in B_{0,j} \subset M_{i,j} \):

- \( x \in M_{i,j} \)
- \( x \in B_{0,j} \)

(ML4) Horizontal conditional expectation \( E_{i,j}^{M,r} : M_{i,j} \to M_{i,j-1} \) and \( E_{i,j}^{M,r}|_{B_{0,j}} = E_{0,j}^{B,r} \):

\[
E_{i,j}^{M,r}(x) = d_{1}^{-1} x, \quad x \in M_{i,j} \quad E_{i,j}^{M,r}(x) = d_{1}^{-1} x, \quad x \in B_{0,j}
\]

(ML5) Vertical conditional expectation \( E_{i,j}^{M,l} : M_{i,j} \to M_{i-1,j} \) and \( E_{i,j}^{M,l}|_{A_{i,0}} = E_{i,0}^{A,l} \):

\[
E_{i,j}^{M,l}(x) = d_{0}^{-1} x, \quad x \in M_{i,j} \quad E_{i,j}^{M,l}(x) = d_{0}^{-1} x, \quad x \in A_{i,0}
\]

(ML6) Commuting square of conditional expectation \( E_{i,j+1}^{M,r} \circ E_{i,j}^{M,l} = E_{i,j+1}^{M,l} \circ E_{i+1,j+1}^{M,r} : M_{i+1,j+1} \to M_{i,j}, \ x \in M_{i+1,j+1} \):

\[
E_{i,j+1}^{M,r} \circ E_{i,j}^{M,l}(x) = E_{i,j+1}^{M,l} \circ E_{i+1,j+1}^{M,r}(x) = d_{0}^{-1} d_{1}^{-1} x
\]

(ML7) Horizontal Jones projections \( f_{j} \in M_{i,j+1} \) and vertical Jones projections \( e_{i} \in M_{i+1,j} \):

\[
f_{2j+1} = d_{1}^{-1} \quad f_{2j+2} = d_{1}^{-1} \quad e_{2i+1} = d_{0}^{-1} \quad e_{2i+2} = d_{0}^{-1}
\]

(ML8) Standard condition:

- \([M_{i,j}, A_{p,q}] = 0\) for \( i \leq q \leq p \). For \( g \in M_{i,j}, x \in A_{p,q} \), regard them as elements in \( M_{p,j} \), then \( gx = xg \);
- \([M_{i,j}, B_{k,l}] = 0\), for \( j \leq k \leq l \). For \( g \in M_{i,j}, y \in B_{k,l} \), regard them as elements in \( M_{i,l} \), then \( gy = yg \).
3.4 From Markov lattice as standard bimodule to planar bimodule category

3.4.1 Planar bimodule category

Let $\mathcal{A}_0$ and $\mathcal{B}_0$ be planar tensor categories. Let $\mathcal{M}_0$ be a $\mathbb{C}^*$ $\mathcal{A}_0 - \mathcal{B}_0$ bimodule category with following properties:

(a) Object: The objects of $\mathcal{M}_0$ are $[m,n] = [m,n]_{\mathcal{M}_0}$, $m, n \in \mathbb{Z}_{\geq 0}$, where $[0,0] := 1_{\mathcal{M}_0}$ is simple.

(b) The module tensor product of objects are

$$[i,+]_{\mathcal{A}_0} \triangleright [m,n]_{\mathcal{M}_0} = [m+i,n]_{\mathcal{M}_0}, \quad [i,-]_{\mathcal{A}_0} \triangleright [m,n]_{\mathcal{M}_0} = \text{none}$$

$$[m,n]_{\mathcal{M}_0} \triangleright [j,+]_{\mathcal{B}_0} = [i,n+j]_{\mathcal{M}_0}, \quad [m,n]_{\mathcal{M}_0} \triangleright [j,-]_{\mathcal{B}_0} = \text{none}$$

$$([i,+]_{\mathcal{A}_0} \triangleright [m,n]_{\mathcal{M}_0}) \triangleright [j,+]_{\mathcal{B}_0} = [m+i,n+j]_{\mathcal{M}_0} = [i,+]_{\mathcal{A}_0} \triangleright ([m,n]_{\mathcal{M}_0} \triangleright [j,+]_{\mathcal{B}_0})$$

(c) Only $\mathcal{M}_0([m,n] \rightarrow [m \pm 2i, n \pm 2j])$ is non-empty, $m, n, i, j \in \mathbb{Z}_{\geq 0}$. The module tensor product of morphisms in $\text{Hom}(\mathcal{A}_0)$, $\text{Hom}(\mathcal{M}_0)$ and $\text{Hom}(\mathcal{B}_0)$ should match the shading types.

(d) $\mathcal{M}_0$ is a strict $\mathcal{A}_0 - \mathcal{B}_0$ bimodule category, i.e., the left/right module associator and bimodule associator are trivial. For $x,x_1,x_2 \in \text{Hom}(\mathcal{A}_0)$, $g \in \text{Hom}(\mathcal{M}_0)$ and $y,y_1,y_2 \in \text{Hom}(\mathcal{B}_0)$,

$$x_2 \triangleright (x_1 \triangleright g) = (x_2 \otimes x_1) \triangleright g \quad (g \triangleright y_1) \triangleright y_2 = g \triangleright (y_1 \otimes y_2)$$

$$x \triangleright g \triangleright y = x \triangleright (g \triangleright y)$$

(e) $\mathcal{M}_0$ is a $\mathbb{C}^*$ category with a natural dagger structure such that $\triangleright$ and $\triangleright$ are dagger functors, i.e., for $x \in \text{Hom}(\mathcal{A}_0)$, $g \in \text{Hom}(\mathcal{M}_0)$ and $y \in \text{Hom}(\mathcal{B}_0)$,

$$(x \triangleright g \triangleright y)^\dagger = x^\dagger \triangleright g^\dagger \triangleright y^\dagger$$

Such bimodule category is called a planar bimodule category.

**Remark 3.4.1.** As in Remark 2.3.2, the morphisms in $\mathcal{M}_0$ is determined by its representation as an endomorphism and its domain and range.

There is a canonical isomorphism $\phi : \mathcal{M}_0([m,n] \rightarrow [m + 2i, n + 2j]) \rightarrow \mathcal{M}_0([m + i, n + j] \rightarrow [m+i, n+j])$ by using the rigid structure on $\mathcal{A}_0$ and $\mathcal{B}_0$.

![Diagram](image)

**Remark 3.4.2.** Let $\mathcal{M}_0$ and $\mathcal{N}_0$ be planar bimodule categories over the same planar tensor category. If they are unitary monoidal equivalent, then they are unitary isomorphic.

### 3.4.2 From Markov lattice as standard bimodule to planar bimodule category

Use the similar notion as we define the planar module category in Definition 2.3.3.

Define the multi-step conditional expectations $E^{d,i}_{m,n} := E^{M,i}_{m-i+1,n} \circ \cdots \circ E^{M,i}_{m,n}$ and $E^{r,k}_{m,n} := E^{M,r}_{m,n-k+1} \circ \cdots \circ E^{M,r}_{m,n}$.

**Definition 3.4.3.** Let $A, B$ be standard $\lambda$-lattices and $M = (M_{m,n})_{m,n \geq 0}$ be a Markov lattice as a standard $A - B$ bimodule with $\dim(M_{0,0}) = 1$. We define a planar bimodule category $\mathcal{M}_0$ from $M$ as follows.

44
(a) The objects of $\mathcal{M}$ are the symbols $[m, n]$ for $m, n \in \mathbb{Z}_{\geq 0}$.

(b) Given $m, n \geq 0$, define $\mathcal{M}_0([m, n] \rightarrow [m, n]) := M_{m,n}$.

(c) The identity morphism in $\mathcal{M}_0([m, n] \rightarrow [m, n])$ is $1_{[m,m]}$.

(d) For $(f_1, n_1) \in \mathcal{M}_0([m, n] \rightarrow [m, n])$ with $2 \mid m_1 + m_2$ and $2 \mid n_1 + n_2$, define $(f_1, n_1), (f_2, n_2) \in (f_1^*(m_1, n_1), (m_1, n_1))$, where $f, f^* \in M_{m_1+m_2, n_1+n_2}$.

(e) Define the composition in nine cases.

C11 (h) $\mathcal{M}_0([m, n] \rightarrow [m, n])$ o $\mathcal{M}_0([m, n] \rightarrow [m, n])$ = $\mathcal{M}_0([m, n] \rightarrow [m, n])$ = $\mathcal{M}_0([m, n] \rightarrow [m, n])$

C12 (h) $\mathcal{M}_0([m, n] \rightarrow [m, n])$ o $\mathcal{M}_0([m, n] \rightarrow [m, n])$ = $\mathcal{M}_0([m, n] \rightarrow [m, n])$

C13 (h) $\mathcal{M}_0([m, n] \rightarrow [m, n])$ o $\mathcal{M}_0([m, n] \rightarrow [m, n])$ = $\mathcal{M}_0([m, n] \rightarrow [m, n])$

C21 (h) $\mathcal{M}_0([m, n] \rightarrow [m, n])$ o $\mathcal{M}_0([m, n] \rightarrow [m, n])$ = $\mathcal{M}_0([m, n] \rightarrow [m, n])$

C22 (h) $\mathcal{M}_0([m, n] \rightarrow [m, n])$ o $\mathcal{M}_0([m, n] \rightarrow [m, n])$ = $\mathcal{M}_0([m, n] \rightarrow [m, n])$

C31 (h) $\mathcal{M}_0([m, n] \rightarrow [m, n])$ o $\mathcal{M}_0([m, n] \rightarrow [m, n])$ = $\mathcal{M}_0([m, n] \rightarrow [m, n])$

C32 (h) $\mathcal{M}_0([m, n] \rightarrow [m, n])$ o $\mathcal{M}_0([m, n] \rightarrow [m, n])$ = $\mathcal{M}_0([m, n] \rightarrow [m, n])$

C33 (h) $\mathcal{M}_0([m, n] \rightarrow [m, n])$ o $\mathcal{M}_0([m, n] \rightarrow [m, n])$ = $\mathcal{M}_0([m, n] \rightarrow [m, n])$

For the other cases, we can use the dagger structure $g^\dagger \circ h^\dagger := (h \circ g)^\dagger$ to define.

Similarly, we use the string diagrams to explain the composition.
Proposition 3.4.6. The composition is well-defined, and \( \mathcal{M}_0 \) is a \( C^* \) category as before.

**Remark 3.4.4.** The composition is well-defined, because of the commuting square of left/right conditional expectation condition and Proposition 3.1.2.

The definition of \( x \uparrow \downarrow 1 \) and \( 1 \downarrow \uparrow y \) for \( x \in \text{Hom}(\mathcal{A}_0) \) and \( y \in \text{Hom}(\mathcal{B}_0) \) are the same as in Definition 3.4.5. \( 1 \uparrow g \downarrow 1 \), \( x \uparrow \downarrow 1 \) and \( 1 \downarrow y \) will be the same as \( x \otimes 1 \) and \( 1 \otimes y \) in Definition 1.6.7 by using the shift maps.

![Diagram](C12) \quad ![Diagram](C23) \quad ![Diagram](C31)

The proof of the following propositions are the same as in the Markov tower case with the fact in Remark 3.2.2. To be precise, the diagrammatic proof can be split as left-hand-side and right-hand-side independently, and the proof on each side is the same as the Markov tower case.

**Proposition 3.4.6.** \( \mathcal{M}_0 \) is a left \( \mathcal{A}_0 \)-module. That is,

1. For \( g \in \text{Hom}(\mathcal{M}_0) \), \( x \in \text{Hom}(\mathcal{A}_0) \), \( (1 \downarrow 1) \circ (x \downarrow 1) = (x \downarrow 1) \circ (1 \downarrow 1) \).
2. For \( g \in \text{Hom}(\mathcal{A}_0) \), \( x_1, x_2 \in \text{Hom}(\mathcal{A}_0) \), \( x_2 \triangleright (x_1 \triangleright g) = (x_2 \otimes x_1) \triangleright g \).
3. For \( g_1, g_2 \in \text{Hom}(\mathcal{M}_0) \), \( x_1, x_2 \in \text{Hom}(\mathcal{A}_0) \), \( 1 \triangleright (g_1 \circ g_2) = (1 \triangleright g_1) \circ (1 \triangleright g_2) \) and \( (x_1 \circ x_2) \triangleright 1 = (x_1 \triangleright 1) \circ (x_2 \triangleright 1) \).

**Proposition 3.4.7.** Similarly, \( \mathcal{M}_0 \) is a right \( \mathcal{B}_0 \)-module. That is,
Remark 3.5.1. Now let $M(b) = \langle M, id \rangle$, objects: Define $A\in\mathcal{M}_0$, $(a)\in\mathcal{M}$.

Proposition 3.4.8. $M_0$ is a $A_0 - B_0$ bimodule. That is, for $g\in\mathcal{M}$, $x \in \mathcal{A}_0$, $y \in \mathcal{B}_0$, $(x \triangleright y) = (x^{-1} \triangleleft y^{-1})$.

Definition 3.4.9. Define $x \triangleright g \triangleleft y := (x \triangleright 1) \circ (1 \triangleleft y) \circ (1 \triangleright g \triangleleft 1)$.

3.5 Indecomposable semisimple $C^* A - B$ bimodules and planar $A_0 - B_0$ bimodule categories

3.5.1 Indecomposable semisimple $C^* A - B$ bimodule category

Let $A$ and $B$ be 2-shaded rigid $C^*$ multitensor categories with generators $X = 1_A^+ \otimes X \otimes 1_A^-$ and $Y = 1_B^+ \otimes Y \otimes 1_B^-$. Let $\mathcal{M}$ be a Cauchy complete indecomposable semisimple $C^* A - B$ bimodule category. Note that there is a natural dagger structure on and $Y$ module actions are dagger functors, i.e., for morphism $g \in \mathcal{M}$, $x \in \mathcal{A}_0$ and $y \in \mathcal{B}_0$.

We call $\mathcal{M}$ indecomposable if for any two simple objects $P,Q \in \mathcal{M}$ (WLOG, $P = 1_A^+ \triangleright P < 1_B^-$), $Q$ is a direct summand of $(X^{alt\otimes m} \triangleright P) < Y^{alt\otimes n}$ for some $m,n \in \mathbb{Z}_{\geq 0}$.

Let $\mathcal{A}_0, \mathcal{B}_0$ be planar tensor categories constructed from $(A,X)$ and $(B,Y)$ respectively. By MacLane’s coherence theorem, $\mathcal{A}\mathcal{B}$ is unitary equivalent to a strict one, i.e., $A,B$ are strict, the right/left module associators and the bimodule associator are trivial. This strict category is also a strict $\mathcal{A}_0 - \mathcal{B}_0$ bimodule category. WLOG, we also denote it as $\mathcal{M}$.

Pick a simple object $Z = 1_A^+ \triangleright Z < 1_B^-$ in $\mathcal{M}$, then we construct a planar $\mathcal{A}_0 - \mathcal{B}_0$ bimodule category $\mathcal{M}_0$ as follows:

(a) Objects: Define $[0,0] := Z$, and

$$[m + 1,0] := [1,?]_A \triangleright [m,0], \quad [m,n+1] := [m,n] \triangleleft [1,?]_B,$$

where $[1,?]_A = [1,+]_A$ if $2 \mid m$ and $[1,?]_A = [1,-]_A$ if $2 \nmid m$; $[1,?]_B = [1,+]_B$ if $2 \mid n$ and $[1,?]_B = [1,-]_B$ if $2 \nmid n$.

(b) $\mathcal{M}_0$ is a full subcategory of $\mathcal{M}$ with above objects.

Given $\mathcal{M}_0$ to be a planar $\mathcal{A}_0 - \mathcal{B}_0$ bimodule category, for the similar reason, its Cauchy completion $\mathcal{M}_0$ is a $\mathcal{A}_0 - \mathcal{B}_0$ bimodule category, compatible with the dagger structure.

Remark 3.5.1. Suppose $\mathcal{M}_0$ is a planar $\mathcal{A}_0 - \mathcal{B}_0$ bimodule category constructed from $\mathcal{M}$ over $(A,X)$ and $(B,Y)$, then there is a unitary equivalence between $\mathcal{M}$ as $A - B$ bimodule category and $\mathcal{M}_0$ as $\mathcal{A}_0 - \mathcal{B}_0$ bimodule category, which maps base object to base object.

3.5.2 From planar bimodule to Markov lattice as standard bimodule

Construction 3.5.2. Now let $M_{i,j} = \text{End}([i,j])$, $i,j \in \mathbb{Z}_{\geq 0}$. After identifying $f \in M_{i,j}$ with $id_{[1,?]_A} \triangleright f \in M_{i+1,j}$ and $f \triangleleft id_{[1,?]_B} \in M_{i,j+1}$ and identifying $x \in A_{i,0} = \text{End}([i,+]_A)$ with $x \triangleleft id_{[0,j]} \in M_{i,j}$ and $y \in B_{0,j} = \text{End}([j,+]_B)$ with $id_{[1,0]} \triangleleft y \in M_{i,j}$. It is easy to show that $M = (M_{i,j})_{i,j \geq 0}$ is a Markov lattice as a standard $A - B$ bimodule with modulus $(d_0, d_1)$.
Similar to the module case, combining above discussion, we have the following theorem.

**Theorem 3.5.3.** There is a bijective correspondence between equivalence classes of the following:

\[
\begin{align*}
\left\{ \text{Traceless Markov lattice } M = (M_{i,j})_{i,j \geq 0} \text{ with } \dim(M_{0,0}) = 1 \right\} & \cong \left\{ \text{Pairs } (\mathcal{M}, Z) \text{ with } \mathcal{M} \text{ an indecomposable semisimple } C^*-A-B \text{ bimodule category together with a choice of simple object } Z = 1_A^A \triangleright Z \triangleleft 1_B^B \right\}
\end{align*}
\]

Equivalence on the left hand side is the $*$-isomorphism on the traceless Markov lattice as standard $A-B$ bimodule; the equivalence on the right hand side is the unitary $A-B$ bimodule equivalence between their Cauchy completions which maps the simple base object to simple base object.

**Corollary 3.5.4.** Any Markov lattice $M$ with modulus $(d_0, d_1)$ and $\dim(M_{0,0}) = 1$ is naturally a standard $\text{TLJ}(d_0) - \text{TLJ}(d_1)$ bimodule, which corresponds to an indecomposable semisimple $C^*\text{TLJ}(d_0) - \text{TLJ}(d_1)$ bimodule category with a simple base object.

**Remark 3.5.5.** The tracial case will be discussed in §6.3.

4 Markov towers, bigraded Hilbert spaces, and balanced fair graphs

In this Chapter, as an application, we are going to classify all indecomposable semisimple $\mathcal{TLJ}$–modules (see Corollary 2.4.5) to get Markov tower, which are also the same as balanced $d$-fair bipartite graphs [DY15]. We will explain exactly how these two classifications agree by directly constructing the correspondence passing through the 2-category $\text{BigHilb}$ [FP19]. Although this is known [DY15, FP19], we explain in detail here so that we are able to do the bimodules in §5 below.

4.1 Balanced $d$-fair bipartite graph

In [DY15], the authors classify unshaded unoriented $\mathcal{TLJ}(d)$–modules in terms of the combinatorial data of fair and balanced graphs. This classification was generalized to $\mathcal{TLJ}(\Gamma)$–modules in [FP19], where $\mathcal{TLJ}(\Gamma)$ is a generalized Temperley-Lieb-Jones category associated to a weighted bidirected graph $\Gamma$. We will be interested in the special case of 2-shaded $\mathcal{TLJ}(d)$–modules.

**Notation 4.1.1.** Let $\Lambda$ be a graph where $V(\Lambda)$ is the set of vertices and $E(\Lambda)$ is the set of edges. Let $s, t : E(\Lambda) \to V(\Lambda)$ be the source and target functions respectively.

**Definition 4.1.2.** Let $\Lambda$ be a bipartite graph with vertices $V(\Lambda) = V_0 \sqcup V_1$ and $\{ e | s(e), t(e) \in V_i \} = \emptyset$, $i = 0, 1$. Let $\omega : E(\Lambda) \to (0, \infty)$ be the weighting on the edges of graph [FP19].

We call $(\Lambda, \omega)$ a $d$-fair graph if for each $P \in V_0$, $Q \in V_1$,

\[
\sum_{\{ e | s(e) = P \}} w(e) = \sum_{\{ e | s(e) = Q \}} w(e) = d.
\]

We call $(\Lambda, \omega)$ a balanced graph if there exists an involution $(\tau)$ on $E(\Lambda)$ that switches sources and targets for each $e \in E(\Lambda)$ and

\[
\omega(e)\omega(\tau) = 1.
\]
Proposition 4.1.3. Suppose \((\Lambda, \omega)\) is a balanced \(d\)-fair bipartite graph. Then the graph is locally finite, i.e., the number of edges coming in or out of any vertex is uniformly bounded:

\[
\# \{ e : s(e) = P \} = \# \{ e : t(e) = P \} \leq d^2 \quad \text{for any vertex } P.
\]

Proof. Suppose \(P\) has \(N\) edges, then there exists an edge \(e_0 : P \to Q\) such that \(\omega(e_0) \leq \frac{d}{N}\) and hence \(\omega(e_0) = 1\) \(\omega(e) \geq \frac{N}{d}\). Note that

\[
d = \sum_{\{ e : s(e) = Q \}} \omega(e) \geq \frac{N}{d},
\]

which follows that \(N \leq d^2 < \infty\). \(\square\)

Definition 4.1.4. We call \(\theta : (\Lambda, \omega) \to (\Lambda', \omega')\) an isomorphism of edge-weighted graphs if \(\theta\) is a graph isomorphism and \(\omega'(\theta(e)) = \omega(e)\) for each \(e \in E(\Lambda)\).

4.2 \textbf{BigHilb} and 2-subcategory \(C(K, ev_K)\)

Definition 4.2.1. Let \(U, V\) be countable sets. Define a category \(\text{Hilb}_{f}^{U \times V}\) as follows:

(a) Object: \(U \times V\)–bigraded Hilbert spaces

\[
H = \bigoplus_{u \in U} \bigoplus_{v \in V} H_{uv},
\]

where \(H_{uv}\) is finite dimensional for each pair \((u, v)\), and only finite many \(H_{uv}\) is non-trivial for each fixed \(u \in U\) or each fixed \(v \in V\).

(b) Morphism: The morphisms are defined as uniformly bounded operators

\[
f = \bigoplus_{u \in U} \bigoplus_{v \in V} f_{uv} : H \to G,
\]

where \(f_{uv} : H_{uv} \to G_{uv}\) are morphisms in \(\text{Hilb}_f\), the category of finitely dimensional Hilbert spaces. Uniformly boundedness means

\[
\sup_{u \in U, v \in V} \|f_{uv}\| < \infty.
\]

(c) The composition: For morphisms \(f, g\), define the composition entry-wisely as

\[
g \circ f := \bigoplus_{u \in U} \bigoplus_{v \in V} g_{uv} \circ f_{uv}.
\]

(d) The identity morphism: Define the identity morphism \(\text{id}_H : H \to H\) as

\[
\text{id}_H := \bigoplus_{u \in U} \bigoplus_{v \in V} \text{id}_{H_{uv}},
\]

where \(\text{id}_{H_{uv}} = \text{id}_{H_{uv}}\) is the identity map on \(H_{uv}\).

Definition 4.2.2. Let \(\text{BigHilb}\) be a dagger 2-category defined as follows:
(a) Object: Countable sets.
(b) For objects $U, V$, $\text{Hom}(U, V) = \text{Hilb}_f^{U \times V}$.
(c) The composition of 1-morphisms: For 1-morphisms $H : U \to V$, $G : V \to W$, the composition of $U, V$ denoted by $\otimes$ is defined as

$$G \circ H = H \otimes G := \bigoplus_{u \in U} \bigoplus_{v \in V} H_{uv} \otimes G_{vu} : U \to W,$$

where the $\otimes$ on the right hand side is the tensor product of Hilbert spaces. The operator is analogous to matrix multiplication, the product is replaced by tensor product and the sum is replaced by direct sum. Clearly, $(H \otimes G) \otimes L = H \otimes (G \otimes L)$.

(d) The identity 1-morphism: For an object $U$, the identity 1-morphism $\mathbb{C}[U] \in \text{Hom}(U, U)$ is defined as

$$\mathbb{C}[U] := \bigoplus_{u,v \in U} \delta_{u=v} \cdot \mathbb{C}.$$

(e) The dual 1-morphism: For 1-morphism $H = \bigoplus_{u \in U} H_u : U \to V$, define its dual as

$$\overline{H} := \bigoplus_{v \in V} \overline{H}_{uv} : V \to U,$$

where $\overline{H}_{uv} := \overline{H_{uv}}$ and $\overline{H_{uv}}$ is the complex conjugate Hilbert space of $H_{uv}$.

(f) Tensor product of 2-morphisms. Let $H_1, H_2 : U \to V$, $G_1, G_2 : V \to W$, and $f : H_1 \to H_2$, $g : G_1 \to G_2$, define $f \otimes g$ as

$$(f \otimes g)_{uv} := \bigoplus_{v \in V} f_{uv} \otimes g_{vw} : \bigoplus_{v \in V} H_{1,uv} \otimes G_{1,vw} \to \bigoplus_{v \in V} H_{2,uv} \otimes G_{2,vw}.$$

Clearly, $(f \otimes g) \otimes h = f \otimes (g \otimes h)$.

(g) Dagger structure: For a 2-morphism $f = \bigoplus_{u,v} f_{uv} : H \to G$, define its adjoint $f^\dagger := \bigoplus_{u,v} f_{uv}^\ast : G \to H$, where $f_{uv}^\ast$ is the adjoint of $f_{uv}$ as a bounded linear map. Clearly, $(f^\dagger)^\dagger = f$.

**Definition 4.2.3.** We call a 1-morphism $H : U \to V$ **dualizable**, if there exist evaluation and coevaluation 2-morphisms $\text{ev}_H : \overline{H} \otimes H \to \mathbb{C}[V]$ and $\text{coev}_H : \mathbb{C}[U] \to H \otimes \overline{H}$ meeting the **zigzag condition**:

$$(\text{id}_H \otimes \text{ev}_H) \circ (\text{coev}_H \otimes \text{id}_H) = \text{id}_H$$

$$(\text{ev}_H \otimes \text{id}_{\overline{H}}) \circ (\text{id}_{\mathbb{C}} \otimes \text{coev}_H) = \text{id}_{\mathbb{C}}.$$

We are going to discuss the evaluation and coevaluation $\text{ev}_H$ and $\text{coev}_H$ in more details.

**Definition 4.2.4.** Note that $\text{ev}_{H,uv} : \bigoplus_{u \in U} \overline{H}_{uv} \otimes H_{uv} = (\overline{H} \otimes H)_{uv} \to (\mathbb{C}[V])_{uv} = \delta_{u=v} \cdot \mathbb{C}$, only $\text{ev}_{H,uv}$ is nonzero for $v \in V$. Let $C_{H,uv} : \overline{H}_{uv} \otimes H_{uv} = \overline{H_{uv}} \otimes H_{uv} \to \mathbb{C}$ such that $\text{ev}_{H,uv} = \bigoplus_{u \in U} C_{H,uv}$. Similarly, only $\text{coev}_{H,uv} : \mathbb{C} \to (H \otimes \overline{H})_{uv} = \bigoplus_{v \in V} H_{uv} \otimes \overline{H}_{uv}$ is nonzero for $u \in U$. Let $D_{H,uv} : \mathbb{C} \to H_{uv} \otimes \overline{H}_{uv} = H_{uv} \otimes \overline{H}_{uv}$ such that $\text{coev}_{H,uv} = \bigoplus_{v \in V} D_{H,uv}$. 
Then
\[
\id_{H,uv} = \left( (\id_H \otimes \ev_H) \circ (\coev_H \otimes \id_H) \right)_{uv}
\]
\[
= (\id_H \otimes \ev_H)_{uv} \circ (\coev_H \otimes \id_H)_{uv}
\]
\[
= \left( \bigoplus_{w \in V} \id_{H,uw} \otimes \ev_{H,wv} \right) \circ \left( \bigoplus_{t \in U} \coev_{H,u} \otimes \id_{H,uv} \right)
\]
\[
= (\id_{H,uv} \otimes \ev_{H,vv}) \circ (\coev_{H,uu} \otimes \id_{H,uv})
\]
\[
= \left( \bigoplus_{w \in V} \id_{H,uw} \otimes \ev_{H,wv} \right) \circ \left( \bigoplus_{t \in U} \coev_{H,ut} \otimes \id_{H,uv} \right)
\]
\[
= (\id_{H,uv} \otimes \ev_{H,vv}) \circ (\coev_{H,uu} \otimes \id_{H,uv})
\]
for \( u \in U, \ v \in V \). Similarly,

\[
\id_{\overline{\Pi},vu} = (\ev_{H,vv} \otimes \id_{\overline{\Pi},vu}) \circ (\id_{\overline{\Pi},vu} \otimes \coev_{H,uu}) = (\id_{H,vu} \otimes \id_{\overline{\Pi},vu}) \circ (\id_{\overline{\Pi},vu} \otimes \id_{H,uv}),
\]

for \( v \in V, u \in U \).

**Remark 4.2.5.** \( \ev_{H} \) and \( \coev_{H} \) are completely determined by \( C_{H,uv} \) and \( D_{H,uv} \).

**Definition 4.2.6.** Let \( \mathcal{C}(K,\ev_{K}) = \mathcal{C}(K,\ev_{K},\coev_{K}) \) be a 2-subcategory of \( \text{BigHilb} \) with a 1-morphism generator \( K : V_0 \to V_1 \) and distinguished 2-morphisms evaluation and coevaluation \( \ev_{K}, \coev_{K} \). We require that

(a) \( K \) is dualizable.

(b) The evaluation and coevaluation for the dual \( \overline{K} \):

\[
\ev_{\overline{K}} := (\coev_{K})^{\dagger} \quad \text{and} \quad \coev_{\overline{K}} := (\ev_{K})^{\dagger}.
\]

(c) They satisfy the \( d-\text{fairness condition} \), namely,

\[
\ev_{\overline{K}} \circ \coev_{K} = d \cdot \id_{\mathcal{C}(V_0)} \quad \ev_{K} \circ \coev_{\overline{K}} = d \cdot \id_{\mathcal{C}(V_1)}.
\]

In other words,

\[
C_{\overline{K},uv} = (D_{K,uv})^{\dagger} \quad D_{\overline{K},vu} = (C_{K,vu})^{\dagger},
\]

and

For each \( P \in V_0 \),

\[
\sum_{Q \in V_1} C_{\overline{K},PQ} \circ D_{K,PQ} = d \cdot \id_{\mathcal{C}}
\]

For each \( Q \in V_1 \),

\[
\sum_{P \in V_0} C_{K,QP} \circ D_{\overline{K},QP} = d \cdot \id_{\mathcal{C}},
\]

Here, the 1-morphism generator means all the 1-morphism is Cauchy generated by \( K \) and \( \overline{K} \).

**Remark 4.2.7.** \( \coev_{K}, \ev_{\overline{K}} \) and \( \coev_{\overline{K}} \) are determined by \( \ev_{K} \) in \( \mathcal{C}(K,\ev_{K}) \).

**Proposition 4.2.8.** The followings are some properties of \( \mathcal{C}(K,\ev_{K}) \).

(1) Let \( V = V_0 \sqcup V_1 \), then all the 1-morphisms in \( \mathcal{C}(K,\ev_{K}) \), including \( K, \overline{K} \), can be regarded as \( V \times V - \text{bigraded Hilbert spaces} \). So we can regard \( \mathcal{C}(K,\ev_{K}) \) as a 2-category with one object \( V \). Then all the 2-morphisms can be regarded as \( V \times V - \text{bigraded uniformly bounded operators} \).

If \( (P,Q) \notin V_0 \times V_1 \), then \( K_{PQ} = \overline{K}_{QP} = 0 \), which follows that \( C_{K,QP} = D_{K,PQ} = 0 \). The zigzag condition between them still hold.
(2) All the 1-morphisms in $\mathcal{C}(K, ev_K)$ are dualizable.

(3) $\sup_{P \in V_0, Q \in V_1} \dim(K_{PQ}) < \infty$. In fact, we will see $\sup_{P \in V_0, Q \in V_1} \dim(K_{PQ}) \leq d^2$ in the next section §4.3 together with Proposition 4.1.3.

(4) There exist standard spherical evaluation and coevaluation in 2-morphisms:

$$\text{ev}^{st}_K : K \otimes K \to C^{[V_1]}$$

$$\text{coev}^{st}_K : C^{[V_0]} \to K \otimes K$$

$$\text{ev}^{st}_K := (\text{coev}_K)^\dagger$$

$$\text{coev}^{st}_K := (\text{ev}_K)^\dagger.$$

In more details, Let $\{\epsilon_i\}_{i=1}^k$ be the orthonormal basis (ONB) of $K_{PQ}$ and $\{\epsilon_i^*\}$ be the dual basis of $K_{PQ}$, $P \in V_0$, $Q \in V_1$ then

$$C^{st}_{K,PQ} : K_{PQ} \otimes K_{PQ} = K_{PQ} \otimes K_{PQ} \to C$$

$$C^{st}_{K,PQ} := (D^{st}_{K,PQ})^\dagger$$

$$D^{st}_{K,PQ} : C \to K_{PQ} \otimes K_{PQ} = K_{PQ} \otimes K_{PQ}$$

are defined as

$$C^{st}_{K,PQ} : \epsilon_i^* \otimes \epsilon_j \mapsto \delta_{i,j}$$

$$D^{st}_{K,PQ} : 1 \mapsto \sum_{i=1}^k \epsilon_i \otimes \epsilon_i^*.$$

Note that $\text{ev}^{st}_K$ and $\text{coev}^{st}_K$ are well-defined 2-morphisms because of (3), and the definitions of $\text{ev}^{st}_K$ and $\text{coev}^{st}_K$ do not depend on the choice of ONB on each $K_{PQ}$ and they also meet the zigzag condition.

Notation 4.2.9. Now, we use the graphic calculus to describe $\mathcal{C}(K, ev_K)$. The idea is from the graphical calculus for 2-Hilb [RV16]. However, in their paper, they only care about the case when $ev = ev^{st}$ and $coev = coev^{st}$, which is not necessarily true in our context.

First we provide the single object version:

(1) For $P \in V_0, Q \in V_1$, $C^{st}_{K,PQ}, D^{st}_{K,PQ}, C^{st}_{K,PQ}$ and $D^{st}_{K,PQ}$.

(2) Rigidity:

$$P \quad Q = P \quad Q = P \quad Q = P \quad Q = P$$

(3) $d$-fairness. For $P \in V$,

$$\sum_{Q \in V} P \quad Q \quad = d \cdot P$$

Then the graphical calculus version: In the $n$-category setting, $n$-morphisms are $n$-morphisms are used to label codimension $n$ cells of an $n$-manifold. So here, 0-morphisms in $\text{BigHilb}$ label regions of the plane, 1-morphisms label strings from left to right, and 2-morphisms label tickets (including ev and coev) from bottom to top. Shading is just shorthand for the labelling. The unshaded region indicates the object $V_0$ and the shaded region indicates $V_1$. 52
(1) coev\textsubscript{K}, ev\textsubscript{K}, coev\textsubscript{st}\textsubscript{K} and ev\textsubscript{st}\textsubscript{K}.



(2) Rigidity:



(3) d-fairness:



(4) Dagger structure on ev and ev\textsubscript{st}.



4.3 The 2-subcategory of BigHilb generated by a balanced d-fair bipartite graph

In this section, we show the relation between 2-categories \(C(K, ev\textsubscript{K})\) and \(d\)-fair bipartite graphs \((\Lambda, \omega)\). Then we may regard the generator \(K\) as a Hilb-enriched graph, and the edge-weighting \(\omega\) giving the interesting dual pair.

**Construction 4.3.1.** First, we construct a \(W^*\) 2-subcategory \(C(\Lambda, \omega)\) of BigHilb from a balanced \(d\)-fair bipartite graph \((\Lambda, \omega)\) as follows:

(a) Object is \(V = V(\Lambda) = V_0 \sqcup V_1\), which is a countable set.
(b) The 1-morphism generator \(K = K_\Lambda\): At \((P, Q) \in V_0 \times V_1\), \(K_{PQ}\) is the Hilbert space with ONB \(\{|e\rangle : e \in E(\Lambda), s(e) = P, t(e) = Q\}\) and other entries are 0. The uniform boundedness condition follows from Proposition 4.1.3.

As for the dual 1-morphism \(\overline{K}\), at entry \((Q, P) \in V_1 \times V_0\), \(\overline{K}_{QP}\) is the Hilbert space with ONB \(\{|e\rangle : e \in E(\Lambda), s(e) = Q, t(e) = P\} = \{\overline{|e\rangle} : e \in E(\Lambda), s(e) = P, t(e) = Q\}\), where \(\overline{\cdot}\) is the involution of edge.

So we may regard \(K\) as a Hilb-enriched graph.
(c) All the 1-morphisms are Cauchy generated by \(K\) and \(\overline{K}\).
(d) 2-morphisms are \(V \times V\)-bigraded uniformly bounded operators between those 1-morphisms.
(e) The edge-weighting gives the distinguished evaluation and coevaluation ev and coev. Note that \(K_{PQ}\) is a Hilbert space with orthonormal basis \(\{|e\rangle : e \in E(\Lambda), s(e) = P, t(e) = Q\}\),
then \( \{|e\} : e \in E(\Lambda), s(e) = P, t(e) = Q \rangle \) is an orthonormal basis for \( \overline{K}_{QP} \). Define

\[
C_{\overline{K}, PQ} : K_{PQ} \otimes \overline{K}_{QP} \to \mathbb{C} \quad \text{by} \quad |e\rangle \otimes |\bar{e}\rangle \mapsto \delta_{e=\bar{e}} w(e) \frac{1}{2}, \quad e : P \to Q
\]

\[
D_{K, PQ} : \mathbb{C} \to K_{PQ} \otimes \overline{K}_{QP} \quad \text{by} \quad 1 \mapsto \sum_{e \in P \to Q} w(e) \frac{1}{2} |e\rangle \otimes |\bar{e}\rangle = \sum_{e \in P \to Q} w(\bar{e}) \frac{1}{2} |\bar{e}\rangle \otimes |e\rangle.
\]

\[
C_{K, PQ} : \overline{K}_{QP} \otimes K_{PQ} \to \mathbb{C} \quad \text{by} \quad |e\rangle \otimes |\bar{e}\rangle \mapsto \delta_{e=\bar{e}} w(e) \frac{1}{2}, \quad e : Q \to P
\]

\[
D_{\overline{K}, PQ} : \mathbb{C} \to \overline{K}_{QP} \otimes K_{PQ} \quad \text{by} \quad 1 \mapsto \sum_{e \in Q \to P} w(e) \frac{1}{2} |e\rangle \otimes |\bar{e}\rangle = \sum_{e \in Q \to P} w(\bar{e}) \frac{1}{2} |\bar{e}\rangle \otimes |e\rangle.
\]

**Proposition 4.3.2.** \( C(\Lambda, \omega) \) satisfies the condition in Definition 4.2.6.

**Proof.** We shall prove that \( C(\Lambda, \omega) \) is rigid and \( d \)-fair.

(a) **Rigidity:** For each \( P, Q \in V, \ e : P \to Q, \)

\[
(C_{\overline{K}, PQ} \otimes \text{id}_{K, PQ}) \circ (\text{id}_{K, PQ} \otimes D_{\overline{K}, PQ})(|e\rangle \otimes 1) = (C_{\overline{K}, PQ} \otimes \text{id}_{K, PQ})(|e\rangle \otimes \sum_{e \in P \to Q} w(e) \frac{1}{2} |\bar{e}\rangle \otimes |e\rangle) = w(e) \frac{1}{2} w(\bar{e}) \frac{1}{2} |e\rangle = |e\rangle,
\]

\[
(\text{id}_{K, PQ} \otimes C_{K, PQ}) \circ (D_{K, PQ} \otimes \text{id}_{K, PQ})(1 \otimes |e\rangle) = (\text{id}_{K, PQ} \otimes C_{K, PQ})(\sum_{e \in P \to Q} w(e) \frac{1}{2} |e\rangle \otimes |\bar{e}\rangle \otimes |e\rangle) = w(e) \frac{1}{2} w(\bar{e}) \frac{1}{2} |e\rangle = |e\rangle.
\]

(b) **\( d \)-fairness:**

\[
\sum_{Q \in V_1} C_{\overline{K}, PQ} \circ D_{K, PQ}(1) = \sum_{Q \in V_1} C_{\overline{K}, PQ} \left( \sum_{e \in P \to Q} w(e) \frac{1}{2} |e\rangle \otimes |\bar{e}\rangle \right) = \sum_{\{e|s(e) = P\}} w(e) \frac{1}{2} w(e) \frac{1}{2} = d, \]

\[
\sum_{P \in V_0} C_{K, PQ} \circ D_{\overline{K}, PQ}(1) = \sum_{P \in V_0} C_{K, PQ} \left( \sum_{e \in Q \to P} w(e) \frac{1}{2} |e\rangle \otimes |\bar{e}\rangle \right) = \sum_{\{e|s(e) = Q\}} w(e) \frac{1}{2} w(e) \frac{1}{2} = d.
\]

\[\square\]

**Remark 4.3.3.** Suppose \( \theta : (\Lambda, \omega) \to (\Lambda', \omega') \) is an isomorphism of edge-weighted graphs (see Definition 4.1.4). We construct a unitary equivalence between \( C(\Lambda, \omega) \) and \( C(\Lambda', \omega') \). For the 1-morphism generators \( K_{\Lambda} \) and \( K_{\Lambda'} \), we have

\[
K_{\Lambda, PQ} \cong K_{\Lambda', \theta(P)\theta(Q)}
\]

as finite dimensional Hilbert spaces, via the bijection of ONBs given by \( |e\rangle \mapsto |\theta(e)\rangle \). Denote by \( u_\theta : K_{\Lambda} \to K_{\Lambda'} \) this unitary isomorphism.

As for the evaluation \( \text{ev}_{K_{\Lambda}} \) and \( \text{ev}_{K_{\Lambda'}} \), we look at \( C_{K_{\Lambda}, PQ} \) and \( C_{K_{\Lambda'}, \theta(P)\theta(Q)} \) (see Definition 4.2.4). Note that \( C_{K_{\Lambda'}, \theta(P)\theta(Q)} : \overline{K}_{\Lambda', \theta(Q)\theta(P)} \otimes K_{\Lambda', \theta(P)\theta(Q)} \to \mathbb{C} \) by

\[
|\theta(e)\rangle \otimes |\theta'(e)\rangle \mapsto \delta_{\theta(e)=\theta'(e)} \omega'(\theta(e)) = \delta_{e=e'} \omega(e), \quad \forall e : Q \to P \in E(\Lambda).
\]

We have

\[
C_{K_{\Lambda'}, \theta(P)\theta(Q)} = C_{K_{\Lambda}, PQ} \circ (\overline{u_\theta}_{QP} \otimes u_\theta_{PQ}).
\]

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In other words, 

\[ \text{ev}_{K'} = \text{ev}_{K} \circ (\overline{u_\theta} \otimes u_\theta^\dagger). \]

Therefore, \( C(\Lambda, \omega) \) and \( C(\Lambda', \omega') \) are unitary equivalent up to the unitary 2-morphism \( u_\theta \).

Next, start with a 2-category \( C(K, \text{ev}_K) \), we construct a balanced \( d \)-fair bipartite graph \( (\Lambda, \omega) \).

**Definition 4.3.4.** For \( P \in V_0 \), \( Q \in V_1 \), let \( v_{PQ} : K_{PQ} \to K_{PQ} \) be the canonical dual map that \( \xi \mapsto \xi^* \) and \( v_{PQ}^\dagger : K_{PQ} \to K_{PQ} \) defined by \( \xi^* \to \xi^{**} = \xi \). Then \( v_{PQ} \circ v_{PQ} = \text{id}_{K_{PQ}} \) and \( v_{PQ}^\dagger \circ v_{PQ} = \text{id}_{K_{PQ}} \).

**Proposition 4.3.5.** Here are some properties for \( \varphi_K \) and \( \varphi_{\Lambda} \).

1. \( \varphi_K \circ \varphi_{\Lambda} = \text{id}_{K_{PQ}} \).
2. \( \sum_{Q \in V_1} \text{Tr}(\varphi_{K,PQ} \circ \varphi_K^\dagger) = \sum_{P \in V_0} \text{Tr}(\varphi_{\Lambda,QP} \circ \varphi_{\Lambda,QP}^\dagger) = d. \)

**Proof.** See [DY15, Prop. 1.8], [FP19, Prop. 3.10].

**Construction 4.3.6.** Define the graph \( \Lambda \) to be \( V(\Lambda) := V \) and the number of edges from \( P \in V_0 \) to \( Q \in V_1 \) to be \( \text{dim} K_{PQ} \). Define edge-weighting function \( \omega : E(\Lambda) \to (0, \infty) \) as the multiset \( \{ \omega(e) \}_{e : P \to Q} := \{ \text{eigenvalues of } \varphi_{K,PQ} \circ \varphi_K^\dagger \} \) and \( \{ \omega(e) \}_{e : Q \to P} := \{ \text{eigenvalues of } \varphi_{\Lambda,QP} \circ \varphi_{\Lambda,QP}^\dagger \}. \)

From above Proposition 4.3.5, \( (\Lambda, \omega) \) is a \( d \)-fair and balanced bipartite graph. To be precise, (1) gives the balance condition and (2) gives the \( d \)-fairness. In fact,

\[
\varphi_{K,PQ} \circ \varphi_K^\dagger = (\text{id}_{K_{PQ}} \otimes C_{K,QP}^{\text{st}}) \circ (D_{K_{PQ}} \otimes \text{id}_{K_{PQ}}) \circ (C_{K_{PQ}} \otimes \text{id}_{K_{PQ}}) \circ (D_{K_{PQ}} \otimes \text{id}_{K_{PQ}}) \circ (C_{K_{PQ}} \otimes \text{id}_{K_{PQ}}) \circ (D_{K_{PQ}} \otimes \text{id}_{K_{PQ}}).
\]

**Remark 4.3.7.** For a given 2-category \( C(K, \text{ev}_K) \), let \( (\Lambda, \omega) \) be the balanced \( d \)-fair bipartite graph obtained from Construction 4.3.6. When we construct the 1-morphism generator \( K = K_\Lambda \) in \( C(\Lambda, \omega) \) from the bipartite graph \( \Lambda \), we secretly make a choice of ONB for each \( (\Lambda_\Lambda)_{PQ} \), so there is a unitary 2-morphism \( \alpha : K \to K_\Lambda \) such that \( \text{ev}_K = \text{ev}_{K_\Lambda} \circ (\overline{\alpha} \otimes \alpha) \). Therefore, \( C(K, \text{ev}_K) \) and \( C(\Lambda, \omega) \) are unitary equivalent up to a unitary 2-morphism \( \alpha \).
4.4 From $\mathcal{C}(K,\text{ev}_K)$ to Markov tower

**Construction 4.4.1.** Here, we are going to build a tower of algebra from the 2-category $\mathcal{C}(K,\text{ev}_K)$ discussed above with a chosen point, say $P_0 \in V_0$. Let $\mathcal{C}^{[P_0]}$ be a 1-morphism with all the entry being 0 except $(\mathcal{C}^{[P_0]})_{P_0} = \mathbb{C}$.

Note that $\mathcal{C}^{[P_0]} \otimes K^{\text{alt} \otimes n}$ is a 1-morphism for each $n \in \mathbb{Z}_{\geq 0}$.

Let $M_n = \text{End} \left( \mathcal{C}^{[P_0]} \otimes K^{\text{alt} \otimes n} \right)$ and identify $M_n \ni x$ with $x \otimes \text{id}_K \in M_{n+1}$, where $K^? = K$ if $2 \mid n$, $K^? = \mathbb{K}$ if $2 \nmid n$. We use the graphical calculus to show $M = (M_n)_{n \geq 0}$ is a Markov tower.

1. Element $x \in M_n$:

2. Inclusion $x \in M_n \subset M_{n+1}$:

3. Conditional expectation $E_{n+1} : M_{n+1} \to M_n$, $x \in M_n$:

Here, the choice of the duality pair $(\text{coev}_K, (\text{coev}_K)^\dagger)$ or $(\text{ev}_K, (\text{ev}_K)^\dagger)$ depends on the shading.

4. Jones projection $e_n \in M_{n+1}$:

5. The pull down property is true automatically in this setting. See the diagram 2.2(MT6).

4.5 More properties of Markov tower

Here, we are going to explore more properties of Markov tower. The tracial version has been proved in [GHJ89, Thm. 4.1.4, Thm. 4.6.3] [CHPS18, Prop. 3.4]. For convenience, here we will prove those properties for the traceless case.
Lemma 4.5.1. Suppose $A \subset B$ is a unital inclusion of finite dimensional $C^*$-algebras and $E : B \to A$ is a faithful conditional expectation. Then there is an orthonormal basis $\{u_i\}_{i \in I}$ such that $\sum_{i \in I} u_i E(u_i^* x) = x$ for all $x \in B$, where $|I| < \infty$.

Proof. Regard $B$ as a right $A$-module equipped with an $A$-valued inner product $\langle x|y\rangle_A := E(x^* y)$. Note that $A$ and $B$ are finite dimensional, so $B$ is a finitely generated projective Hilbert $A$-module. By [FL02, Thm. 4.1] [KW00, Lemma. 1.7], there exists an orthonormal basis $\{u_i\}_{i \in I} \subset B$ such that $x = \sum_{i \in I} u_i (u_i^* x)$ for all $x \in B$ and $|I| < \infty$. □

Proposition 4.5.2. (1) $X_{n+1} := M_n e_n M_n$ is a 2-sided ideal of $M_{n+1}$ and hence $M_{n+1}$ splits as a direct sum of von Neumann algebras $X_{n+1} \oplus Y_{n+1}$. We also define $Y_0 = M_0$, $Y_1 = M_1$ so that $X_0 = X_1 = 0$. $X_{n+1}$ is called the old stuff and $Y_{n+1}$ is called the new stuff.

(2) $X_{n+1}$ is isomorphic to $M_n \otimes_{M_{n-1}} M_n$, which is the basic construction from $E_n : M_n \to M_{n-1}$. Denote this isomorphism as $\phi$. Here, $M_n \otimes_{M_{n-1}} M_n$ is a $*$-algebra with multiplication $(x_1 \otimes y_1)(x_2 \otimes y_2) = x_1 E_n(y_1 x_2) \otimes y_2$ and adjoint $(x \otimes y)^* = y^* \otimes x^*$.

(3) If $y \in Y_{n+1}$ and $x \in X_n$, then $yx = 0$ in $M_{n+1}$. Hence $E_{n+1}(Y_{n+1}) \subset Y_n$, which means the new stuff comes from the old stuff.

(4) If $Y_n = 0$, then $Y_k = 0$ for all $k \geq n$.

Proof. (1) Note that $M_{n+1} e_n = M_n e_n$, then $M_{n+1} M_n e_n M_n \subset M_{n+1} e_n M_n = M_n e_n M_n$ and $M_n e_n M_{n+1} = (M_{n+1} M_n e_n M_n)^* \subset (M_n e_n M_n)^* = M_n e_n M_n$.

(2) See Watatani index theory [Wa90, §1] with Lemma 4.5.1.

(3) Note that as a finite dimensional von Neumann algebra, $M_{n+1} = \bigoplus_i M_{n+1} p_i$, where $p_i$ are the minimum central projections. So if $y \in Y_{n+1}$, then $y = \sum_j m_j p_j$, where $[p_j, e_n] = 0$.

For $ae_{n-1} b \in X_n$ and $m_j p_j \in Y_{n+1}$, by Jones projection property,

$$m_j p_j a e_{n-1} b = d^{-2} m_j p_j a e_{n-1} e_n e_{n-1} b = d^{-2} m_j a e_{n-1} p_j e_n e_{n-1} b = 0,$$

so $yx = 0$ for any $x \in X_n$, $y \in Y_{n+1}$.

Let $X_n = \bigoplus_k M_n q_k$, where $q_k$ are the minimum central projections. For any $y \in Y_{n+1}$, $q_k E_{n+1}(y) = E_{n+1}(q_k y) = 0$ for all $k$, which implies that $E_{n+1}(y) \in Y_n$.

(4) By (3) and faithfulness of $E_n$. □

4.6 From Markov tower to $C(\Lambda, \omega)$

Now we are able to extract the so-called principal graph data from the Markov tower, which is similar to the classical tracial Markov tower [Oc88] [JS97, §4.2].

If $A$ is a finite dimensional $C^*$-algebra, we write $\pi(A)$ to be the set of minimal central projections of $A$. If $A \subset B$ is a unital inclusion of finite dimensional $C^*$-algebras, then the inclusion matrix is the $\pi(A) \times \pi(B)$ matrix, with $(p, q)$-th entry being $(\dim_{C^*}(pq A'pq \cap pq Bpq))^\frac{1}{2}$. If $A \subset B \subset B_1$ is a basic construction, then the inclusion matrix of $B \subset B_1$ is the transpose of the inclusion matrix of $A \subset B$ [GHJ89, §2] [JS97].

The inclusion matrix of $A \subset B$ can be described as the Bratteli diagram of $A \subset B$, whose vertices are the minimal central projections and the number of edges between $p$ and $q$ is the $(p, q)$-th entry.

The Bratteli diagram $\Delta$ of the Markov tower $M = (M_n)_{n \geq 0}$ contains all the Bratteli diagram $\Delta_n$ of $M_n \subset M_{n+1}$. Then by the property of inclusion matrix of basic construction and

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Proposition 4.5.2(2), the Bratteli diagram for $M_n \subset M_{n+1}$ contains the reflection of the Bratteli diagram of $M_{n-1} \subset M_n$ and new part, which is called the **principal part**. A vertex in the new part is called a **new vertex**, otherwise, called an **old vertex**. The reflected vertex from a new vertex is called a **new old vertex**. Moreover, for a new vertex $p \in Y_n$, denote $p'$ to be the new old vertex of $p$ in $M_{n+2}$.

The **principal graph** $\Lambda$ contains the new part in the Bratteli diagram $\Delta$, so its vertices are new vertices. To be precise, $V(\Lambda)$ contains all the minimal central projections in the new stuff. By Proposition 4.5.2(4), the new stuff comes from the old new stuff, then for $p, q \in \Lambda$, $E(\Lambda)$ contains all the edges between $p$ and $q$.

It is clear that both the Bratteli diagram and the principal graph are bipartite. We can also use the principal graph to construct the Bratteli diagram by doing the reflection at each level.

Let us then compute the edge weighting $w : E(\Lambda) \to (0, \infty)$. Before that, we first give a lemma:

**Lemma 4.6.1.** The follows are some properties for the relative commutant in $\text{BigHilb}$:

1. Let $H_1, H_2, \ldots, H_n, G_1, G_2, \ldots, G_n$ be finite dimensional Hilbert spaces. We identify $B(H_i)$ with $B(H_i) \otimes \text{id}_{G_i}$ and $B(G_i)$ with $\text{id}_{H_i} \otimes B(G_i)$ as subalgebras in $B(\bigoplus_{i=1}^n H_i \otimes G_i)$ for each $i = 1, \ldots, n$, then the relative commutant

$$\bigcap_{i=1}^n B(H_i)' \cap B\left(\bigoplus_{i=1}^n H_i \otimes G_i\right) = \bigoplus_{i=1}^n B(G_i). \quad (*)$$

2. Let $H$ be a 1-morphism in $\text{BigHilb}$, then the center $Z(\text{End}(H))$ is the linear span of all the direct summands of $\text{id}_H$.

3. Let $G$ be another 1-morphism in $\text{BigHilb}$ such that $H \otimes G$ is nondegenerate, i.e., for each nonzero $H_{pq}$, there is a nonzero $G_{qr}$ and vice versa. We identify $\text{End}(H)$ with $\text{End}(H) \otimes \text{id}_G$ and $\text{End}(G)$ with $\text{id}_H \otimes \text{End}(G)$ as subalgebras in $\text{End}(H \otimes G)$. Then the relative commutant

$$\text{End}(H)' \cap \text{End}(H \otimes G) = Z(\text{End}(H)) \otimes \text{End}(G).$$

4. Moreover, if $H_{pq}$ is nonzero only when $p = p_0 \in V$, then the relative commutant can be represented as

$$\text{End}(H)' \cap \text{End}(H \otimes G) = \text{id}_H \otimes \text{End}(G).$$

**Warning:** the tensor product in (1) is the tensor product of Hilbert spaces and bounded operators; the tensor product in (3) and (4) is the tensor product of 1-morphisms/2-morphisms in $\text{BigHilb}$, see Definition 4.2.2.

**Proof.**
(1) ⊃ is clear. We show ⊂.

For \( f \in B(\bigoplus_{i=1}^{n} H_{i} \otimes G_{i}) \), \( f = \bigoplus_{j=1}^{n} f_{i,j} \), where \( f_{i,j} \in B(H_{i} \otimes G_{i}, H_{j} \otimes G_{j}) \). We shall prove that \( f_{i,j} = 0 \) for \( i \neq j \) and \( f_{i,i} \in \text{id}_{H_{i}} \otimes B(G_{i}) \) if \( f \in \text{LHS of equation } (*) \). Let \( x_{i} \in B(H_{i}) \), then

\[
f(x_{i} \otimes \text{id}_{G_{i}}) = \bigoplus_{j=1}^{n} f_{i,j}(x_{i} \otimes \text{id}_{G_{i}}) = \bigoplus_{k=1}^{n} (x_{i} \otimes \text{id}_{G_{i}}) f_{k,i} = (x_{i} \otimes \text{id}_{G_{i}}) f,
\]

which implies that \( f_{i,j}(x_{i} \otimes \text{id}_{G_{i}}) = (x_{i} \otimes \text{id}_{G_{i}}) f_{k,i} = 0 \) for \( k \neq i, j \neq i \) and \( f_{i,i}(x_{i} \otimes \text{id}_{G_{i}}) = (x_{i} \otimes \text{id}_{G_{i}}) f_{i,i} \).

From the first half, if we choose \( x_{i} = \text{id}_{H_{i}} \), we obtain \( f_{i,j} = f_{k,i} = 0, j \neq i, k \neq i \); from the second half, a well-known statement that \( B(H_{i})' \cap B(H_{i} \otimes G_{i}) = B(G_{i}) \), so that \( f_{i,i} \in \text{id}_{H_{i}} \otimes G_{i} \).

(2) Clear, see Definition 4.2.1(d).

(3) ⊃ is clear. We show ⊂.

For \( f \in \text{End}(H)' \cap \text{End}(H \otimes G) \), we shall prove that \( f_{pq} \in \bigoplus_{r \in V} \text{id}_{H_{r}} \otimes B(H_{rq}) \).

Note that

\[
(\text{End}(H \otimes G))_{pq} = \text{End}((H \otimes G)_{pq}) = B \left( \bigoplus_{r \in V} H_{pr} \otimes G_{rq} \right)
\]

For \( f \in \text{End}(H)' \cap \text{End}(H \otimes G) \), \( f_{pq} \) commute with \( B(H_{pr} \otimes \text{id}_{G_{rq}}) \) for all \( r \in V \). By (1), we have \( f_{pq} \in \bigoplus_{r \in V} \text{id}_{H_{pr}} \otimes B(H_{rq}) \). Together with (2), we prove this statement.

(4) From (3), for \( f \in \text{End}(H)' \cap \text{End}(H \otimes G) \),

\[
f = \bigoplus_{q \in V} \text{id}_{H_{pq}} \otimes g^{(q)},
\]

where \( g^{(q)} \in \text{End}(G) \).

Now we define \( g \in \text{End}(G) \) by \( g_{ij} := g_{ij}^{(q)} \). Then \( f = \text{id}_{H} \otimes g \).

By §4.3, we are able to construct a \( W^{*} \) 2-subcategory \( \mathcal{C}(\Lambda) \) without providing the distinguished evaluation and coevaluation given by the edge weighting, though we still have the canonical evaluation and coevaluation denoted by \( \text{ev}^{\text{st}} \) and \( \text{coev}^{\text{st}} \), which are drawn in green below. We denote the generators by \( K = K_{\Lambda} \) and \( \overline{K} \). From Construction 4.4.1, let \( N_{n} := \text{End}(\mathcal{C}^{[pq]} \otimes K^{\text{alt} \otimes n}) \).

**Notation 4.6.2. and Observation** Denote \( \Lambda_{n} \) to be the subgraph of \( \Lambda \) with vertices depth \( \leq n \) and the corresponding Hilb-enriched graph to be \( K_{n} := K_{\Lambda_{n}} \) and \( \overline{K}_{n} \) the dual space in the sense of Construction 4.3.1. As a convention, \( p_{0} \) is of depth 0. Observe that

\[
N_{n} = \text{End}(K_{1} \otimes \overline{K}_{2} \otimes K_{3} \otimes \overline{K}_{4} \otimes \cdots \otimes \overline{K}_{n}^{2} K_{n}^{2} K_{n}).
\]

where \( K_{n}^{2} = K_{n} \) if \( 2 \nmid n \), \( K_{n}^{2} = \overline{K}_{n} \) if \( 2 \mid n \).
Example 4.6.3. Let us take $A_5$ graph for example. We label the vertices as follows.

Then

$$K_1 = \begin{bmatrix}
0 & 0 & 0 & C & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}$$

$$K_2 = \begin{bmatrix}
0 & 0 & 0 & C & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
C & C & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}$$

$$K_3 = \begin{bmatrix}
0 & 0 & 0 & C & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}$$

$$K_4 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
C & C & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}$$

$$K_5 = \begin{bmatrix}
0 & 0 & 0 & C & 0 \\
0 & 0 & 0 & C & 0 \\
0 & 0 & 0 & 0 & C \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}$$

$$K_1 \otimes K_2 \otimes K_3 = \begin{bmatrix}
0 & 0 & C & C \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}$$

$$K_1 \otimes K_2 \otimes K_3 \otimes K_4 = \begin{bmatrix}
C & C & C & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}$$

For this example, observe that $\text{End}(K_1 \otimes K_2 \otimes \cdots \otimes K_n^2)$ is the semisimple quotient of $TLJ_n(\sqrt{3})$.

One can regard $\Lambda_n$ as the subgraph of the Bratteli diagram between depth $n - 1$ and $n$, and $K_n$ is the Hilb-enriched graph of $\Lambda_n$. The entry $(i, j)$ in $K_1 \otimes K_2 \otimes \cdots \otimes K_n^2$ indicates the number of paths from the vertex $p_i$ at depth 0 to the vertex $p_j$ at depth $n$. Note that the base point is a single vertex $p_1$, so entry only at $(1, j)$ can be nonzero.

Proposition 4.6.4.

$$N_{n-1} \cap N_{n+1} = \begin{cases}
\text{id}_{K_1 \otimes K_2 \otimes \cdots \otimes K_{2k-1}} \otimes \text{End}(K_{2k} \otimes K_{2k+1}) & n = 2k \\
\text{id}_{K_1 \otimes K_2 \otimes \cdots \otimes K_{2k}} \otimes \text{End}(K_{2k+1} \otimes K_{2k+2}) & n = 2k + 1.
\end{cases}$$

Proof. Note that $K_1 \otimes K_2 \otimes \cdots \otimes K_n^2$ satisfies the condition in Lemma 4.6.1(3) and (4). □

The idea is to transport the Jones projections from the Markov tower $(M_n)$ to the endomorphism algebras $(N_n)$ in order to obtain the edge weighting $\omega$. Let $\psi_n : M_n \to N_n$ be a $*$-algebra isomorphism for each $n \geq 0$ with $\psi_{n+1}|_{M_n} = \psi_n$.

Let us consider the image of Jones projection $\psi(e_n) \in N_{n+1}$. Note that $e_n \in M_{n-1} \cap M_{n+1}$, so $\psi(e_n) \in N_{n-1} \cap N_{n+1}$.
Proposition 4.6.5. WLOG, let \( n = 2k \). There exists a projection \( \varepsilon_{2k} \in \text{End}(\overline{K}_{2k} \otimes K_{2k+1}) \) such that \( \psi(\varepsilon_{2k}) = \text{id}_{K_1 \otimes K_2 \otimes \cdots \otimes K_{2k-1} \otimes \varepsilon_{2k}} \).

Proof. By proposition 4.6.4, there exists \( \varepsilon_{2k} \in \text{End}(\overline{K}_{2k} \otimes K_{2k+1}) \) such that \( \psi(\varepsilon_{2k}) = \text{id}_{K_1 \otimes K_2 \otimes \cdots \otimes K_{2k-1} \otimes \varepsilon_{2k}} \). Note that \( \varepsilon_{2k} \) is a projection, so is \( \varepsilon_{2k} \).

Lemma 4.6.6. Let \( H \) be a Hilbert space and \( p \neq 0 \) be a projection on \( H \). Suppose \( pfp \in \mathbb{C}p \) for all \( f \in \mathcal{B}(H) \), then \( p = r^*r \), where \( r : H \to \mathbb{C} \) and \( rr^* = 1 \).

Similarly, let \( H \) be a 1-morphism in \( \text{BigHilb} \) and \( p \neq 0 \) be a projection on \( H \). Suppose \( pfp \in \mathbb{C}p \) for all \( f \in \text{End}(H) \), then \( p = r^*r \), where \( r : H \to \mathbb{C} \) and \( rr^* = \mathbb{C} \).

Proof. For the Hilbert space case: Note that \( \text{Im}(fp) \) can be any subspace of \( H \) and \( \text{Im}(p(fp)) = \text{Im}(p) \), so \( \text{Im}(p) \) does not depend on the input, i.e., \( p \) facts through \( \mathbb{C} \). Let \( r : H \to \mathbb{C} \) and \( p = r^*r \) with \( rr^* = 1 \), since \( p^* = p = p^p \).

The similar argument on 1-morphisms in \( \text{BigHilb} \).

As we see the construction of Jones projection in Construction 4.4.1(4), we shall prove that the Jones projection splits into two pieces.

By Proposition 1.1.4(6), \( e_nM_{n+1}e_n = M_{n-1}e_n \), so \( \psi(e_n)N_{n+1}\psi(e_n) = N_{n-1}e_n \). WLOG, let \( n = 2k \). For each \( f \in \text{End}(\overline{K}_{2k} \otimes K_{2k+1}) \), \( \text{id}_{K_1 \otimes K_2 \otimes \cdots \otimes K_{2k-1} \otimes f} \in \mathbb{C} = \mathbb{C}+ \mathbb{C} \).

By Lemma 4.6.6, there exists \( r_{2k} : \overline{K}_{2k} \otimes K_{2k+1} \to \mathbb{C} | V_{1,2k-1} \) such that

\[ e_{2k} = r_{2k}^\dagger r_{2k} \quad \text{and} \quad r_{2k}r_{2k}^\dagger = \mathbb{C} | V_{1,2k-1} \],

where \( V_{1,2k-1} \) contains all the simple objects in \( \Lambda_{2k-1} \) with odd depth.

Similarly, we can define \( \varepsilon_{2k+1} \in \text{End}(\mathcal{K} \otimes \overline{K}) \) corresponding to Jones projection \( e_{2k+1} \) and there exists \( r_{2k+1} : K_{2k+1} \otimes \overline{K}_{2k+2} \to \mathbb{C} | V_{0,2k} \) such that

\[ e_{2k+1} = r_{2k+1}^\dagger r_{2k+1} \quad \text{and} \quad r_{2k+1}r_{2k+1}^\dagger = \mathbb{C} | V_{0,2k} \],

where \( V_{0,2k} \) contains all the simple objects in \( \Lambda_{2k} \) with even depth.

Now consider \( u_{2k} := d(\text{id}_{\overline{K}_{2k+1}} \circ (r_{2k}^\dagger \otimes \text{id}_K)) \in \text{End}(\overline{K}_{2k}) \). Note that \( e_{2k}e_{2k+1}e_{2k} = d^{-2}e_{2k} \) and \( e_{2k+1}e_{2k}e_{2k+1} = d^{-2}e_{2k+1} \), we have \( u_{2k}^\dagger u_{2k} = \text{id}_{\overline{K}_{2k}} \) and \( u_{2k}u_{2k}^\dagger = \text{id}_{\overline{K}_{2k+2}} \), so \( u_{2k} \) is a unitary.
For adjacent simple objects $p, q \in \Lambda$ with $p$ at depth $n$ and $q$ at depth $n + 1$, we shall compute the edge weighting on the edges $e : p \to q$ and $e : q \to p$. WLOG, $n = 2k$. Define $\varphi_{2k}$ and $\varphi_{2k+1}$ as follows:

\[ \varphi_{2k} = d^{\frac{1}{2}} \quad \varphi_{2k+1} = d^{\frac{1}{2}} \]

and we have the following properties:

1. $\varphi_{2k+1} \circ \varphi_{2k} = \text{id}$.
2. $\text{Tr}(\varphi_{2k} \circ \varphi_{2k}) = \text{Tr}(r_{2k}^\dagger r_{2k}) = d \text{Tr}(r_{2k} r_{2k}^\dagger) = d$.
3. $\text{Tr}(\varphi_{2k+1} \circ \varphi_{2k+1}) = \text{Tr}(u_{2k} r_{2k+1}^\dagger r_{2k+1} u_{2k}) = \text{Tr}(r_{2k+1} r_{2k+1}^\dagger) = d$.

Definition 4.6.7. Define the edge-weighting function $\omega$ as the multiset:

\[
\{ \omega(e) \}_{e : p \to q} := \{ \text{eigenvalues of } (\varphi_{2k} \circ \varphi_{2k})_{pq} \} \\
\{ \omega(e) \}_{e : q \to p} := \{ \text{eigenvalues of } (\varphi_{2k+1} \circ \varphi_{2k+1})_{pq} \}
\]

Combining Construction 4.3.6 and our definition with properties for $\varphi_{2k}, \varphi_{2k+1}$, the edge weighting $\omega$ we obtained for bipartite graph $\Lambda$ is $d$-fair and balanced.

4.7 $\mathcal{C}(K, ev_K)$ and $\text{End}_{\text{id}}^1(\mathcal{M}, F)$

In this section, $\mathcal{TLJ}(d)$ means the 2-shaded pivotal rigid $\mathbb{C}^*$ multitensor category from Definition 1.6.4 with endomorphism spaces the Temperley-Lieb algebras and simple generator $X = 1^+ \otimes X \otimes 1^-$. We have already seen the ways to construct a Markov tower from $\mathcal{C}(K, ev_K)$ in this chapter or from $\mathcal{M}$ in §2 with a simple base point $Z$, where $\mathcal{M}$ is an indecomposable semisimple $\mathbb{C}^*$ multitensor category. In this section, we will show their relation to each other.

Definition 4.7.1 (Endofunctor monoidal category). Define $\text{End}_{\text{id}}^1(\mathcal{M})$ to be a $\mathbb{C}^*$ tensor category as follows:

(a) Objects: The objects are all the dagger endofunctors of $\mathcal{M}$.
(b) Morphisms: The morphisms are the uniformly bounded natural transformations between these dagger endofunctors which compatible with the dagger structure.
(c) Tensor structure: The tensor product is given by the composition of endofunctors, i.e., $F_1 \otimes F_2 := F_2 \circ F_1$ for endofunctors $F_1, F_2$.

Definition 4.7.2. Define $F := - \triangleleft X$, $\overline{F} := - \triangleleft \overline{X}$, which are endofunctors of $\mathcal{M}$. Note that $F$ and $\overline{F}$ are adjoint functors, with unit $\text{ev}_F$ and counit $\text{coev}_F$ induced by $\text{ev}_X$ and $\text{coev}_X$.

Define $\text{End}_{\text{id}}^1(\mathcal{M}, F)$ to be the full category Cauchy generated by $F$ and $\overline{F}$. Since the generators are dualizable, the category is rigid. We warn the reader that $\text{End}_{\text{id}}^1(\mathcal{M}, F)$ will only be multitensor $(\dim(\text{End}(\text{id}_\mathcal{M})) < \infty)$ when $\mathcal{M}$ is finitely semisimple. Moreover, the dual functor on $\text{End}_{\text{id}}^1(\mathcal{M}, F)$ given by $\text{ev}_F$ and $\text{coev}_F$ is not a unitary dual functor.

We can give an alternative description of $\text{End}_{\text{id}}^1(\mathcal{M}, F)$ using the following remark.
Remark 4.7.3. Let \( \mathcal{A} \) be a 2-shaded rigid C*-multitensor category with generator \( X \). The follows are equivalent [GMPPS18]:

1. \( \mathcal{M} \) is an indecomposable semisimple C*-right \( \mathcal{A} \)-module category;
2. there is a faithful dagger tensor functor \( \phi : \mathcal{A} \to \text{End}^1(\mathcal{M}) \), where \( \text{End}^1(\mathcal{M}) \) is a tensor category with all the dagger endofunctors being objects and uniformly bounded natural transformations being morphisms.

We see that under this equivalence, \( \text{End}^1(\mathcal{M}, F) := \phi(\mathcal{A}) \) is the C*-category Cauchy tensor generated by the image of the tensor functor \( T_{\mathcal{J}} \to \mathcal{J} \), where \( F = - \otimes X \). Then \( \text{End}^1(\mathcal{M}, F) \) is clearly a rigid C*-tensor category.

As the end of this chapter, we are going to show that the tensor category \( \text{End}^1(\mathcal{M}, F) \) and 2-category \( \mathcal{C}(K, \text{ev}_K) \) are unitarily equivalent.

Construction 4.7.4. We construct \( \mathcal{C}(K, \text{ev}_K) \) from \( \text{End}^1(\mathcal{M}, F) \) functorially.

(a) Object: Let \( V_0 \) be a set of representatives of all isomorphism classes of simple objects \( P \in \mathcal{M} \) such that \( P = P \otimes 1^+ \) and \( V_1 \) a set of representatives of all isomorphism classes of simple objects \( Q \in \mathcal{M} \) such that \( Q = Q \otimes 1^- \). Then the object is the set \( V = V_0 \sqcup V_1 \).

(b) 1-morphism: Let \( G \in \text{End}^1(\mathcal{M}, F) \) be an object with adjoint \( \overline{G} \). Define the \( V \times V \)-bigraded Hilbert space \( H_G \) by

\[ H_{G,PQ} := \text{Hom}(Q, G(P)), \]

with inner product \( \langle f | g \rangle_{G,PQ} \) for \( f, g \in \text{Hom}(Q, G(P)) \) defined by

\[ f^\dagger \circ g = \langle f | g \rangle_{G,PQ} \cdot \text{id}_Q, \]

since \( Q \) is simple and \( f^\dagger \circ g \in \text{End}(Q) \cong \mathbb{C} \cdot \text{id}_Q \). Note that \( \text{Hom}(Q, G(P)) \cong \text{Hom}(\overline{G}(Q), P) \) is a natural isomorphism, so \( H_{\overline{G},QP} \) and \( H_{G,PQ} \) are dual Hilbert spaces.

(c) Composition of 1-morphisms:

Proposition 4.7.5. For \( G_1, G_2 \in \text{End}^1(\mathcal{M}, F) \), we have \( H_{G_1 \circ G_2} \cong H_{G_1} \circ H_{G_2} \) as \( V \times V \)-bigraded Hilbert spaces, i.e.,

\[ H_{G_1 \circ G_2,PQ} \cong (H_{G_1} \circ H_{G_2})_{PQ} = (H_{G_2} \otimes H_{G_1})_{PQ} = \bigoplus_R H_{G_2,PR} \otimes H_{G_1,RQ}. \]

is a unitary isomorphism between Hilbert spaces for each pair \( (P,Q) \in V \times V \).

Proof. Note that the direct sum contains finite many components. For each nonzero component with respect to \( R \), define \( \theta_R : H_{G_2,PR} \otimes H_{G_1,RP} \to H_{G_1 \circ G_2,PQ} \) by

\[ \theta_R(f_2 \otimes f_1) := G_1(f_2) \circ f_1. \]

First, we prove that \( \theta_R \) is an isometry, i.e.,

\[ \langle \theta(f_2 \otimes f_1) | \theta(g_2 \otimes g_1) \rangle_{G_1 \circ G_2,PQ} = \langle f_2 \otimes f_1 | g_2 \otimes g_1 \rangle = \langle f_2 | g_2 \rangle_{G_2,PR} \cdot \langle f_1 | g_1 \rangle_{G_1,RQ}. \]
for \( f_2, g_2 \in H_{G_2, PR} \), \( f_1, g_1 \in H_{G_1, RQ} \).

\[
LHS = \langle G_1(f_2) \circ f_1 | G_1(g_2) \circ g_1 \rangle_{G_1 \circ G_2, PQ} \\
= (G_1(f_2) \circ f_1)^{\dagger} \circ (G_1(g_2) \circ g_1) \\
= f_1^{\dagger} \circ G_1(f_2) \circ g_1 \\
= f_1^{\dagger} \circ G_1((f_2 | g_2)_{G_2, PR} \cdot \text{id}_{R}) \circ g_1 \\
= (f_2 | g_2)_{G_2, PR} \cdot f_1^{\dagger} \circ \text{id}_{G_1(R)} \circ g_1 \\
= (f_2 | g_2)_{G_2, PR} \cdot f_1^{\dagger} \circ g_1 \\
= RHS.
\]

It follows that \( \bigoplus_R \theta_R : \bigoplus_R H_{G_2, PR} \otimes H_{G_1, RQ} \to H_{G_1 \circ G_2, PQ} \) is an isometry.

Note that for a semisimple rigid \( C^* \) category,

\[
dim H_{G_1 \circ G_2, PQ} = \dim \text{Hom}(Q, G_1 \circ G_2(P)) \\
= \dim \text{Hom}(\overline{G_1}(Q), G_2(P)) \\
= \dim \bigoplus_R \text{Hom}(\overline{G_1}(Q), R) \otimes \text{Hom}(R, G_2(P)) \\
= \dim \bigoplus_R \text{Hom}(Q, G_1(R)) \otimes \text{Hom}(R, G_2(P)) \\
= \dim \bigoplus_R H_{G_1, RQ} \otimes H_{G_2, PR} \\
= \dim \bigoplus_R H_{G_2, PR} \otimes H_{G_1, RQ}.
\]

Note that \( \bigoplus_R \theta_R \) is an isometry and hence injective, so \( \bigoplus_R \theta_R : \bigoplus_R H_{G_2, PR} \otimes H_{G_1, RQ} \to H_{G_1 \circ G_2, PQ} \) is a bijection and hence a unitary.

It follows that

\[
H_{G_1 \circ G_2} \circ H_{G_3} \cong H_{G_1 \circ G_2 \circ G_3} \cong H_{G_1} \circ H_{G_2 \circ G_3}
\]

as \( V \times V \)-bigraded Hilbert space.

(d) 1-morphism generator: Define \( K := H_F \) and \( \overline{K} := H_{FP} \). It is clear that \( \mathcal{C}^{\lvert V \rvert} = H_{I^+} \) and \( \mathcal{C}^{\lvert V \rvert} = H_{I^-} \).

(e) 2-morphism: The 2-morphism of \( \mathcal{C}(K) \) is the morphism of \( \text{End}_{\overline{M}}(M, F) \). Let \( \alpha : G_1 \to G_2 \) be a uniformly bounded natural transformation. Then \( \alpha(P) : G_1(P) \to G_2(P) \) and hence \( \alpha_{PQ} : \alpha_{PQ} \circ - : H_{G_1, PQ} \to \text{Hom}(Q, G_1(P)) \to \text{Hom}(Q, G_2(P)) = H_{G_2, PQ} \) is a uniformly bounded linear map.

(f) Composition of 2-morphisms: Let \( \alpha_1 : G_1 \to G_2, \alpha_2 : G_2 \to G_3 \) be uniformly bounded natural transformations. Then \( G_1(P) \xrightarrow{\alpha_1(P)} G_2(P) \xrightarrow{\alpha_2} G_3(P) \), then

\[
(\alpha_2 \circ \alpha_1)_{PQ} = (\alpha_2 \circ \alpha_1)_{PQ} = \alpha_2 \circ (\alpha_1 \circ \alpha_1)_{PQ} = \alpha_2 \circ \alpha_1_{PQ} \circ \alpha_1_{PQ} : H_{G_1, PQ} \to H_{G_2, PQ} \to H_{G_3, PQ}.
\]

(g) Tensor product of 2-morphisms: Let \( \alpha_1 : G_1 \to G_2, \alpha_2 : G_3 \to G_4 \) be uniformly bounded natural transformation. Then \( \alpha_1 \otimes \alpha_2 : G_3 \circ G_1 = G_1 \otimes G_3 \to G_2 \otimes G_4 = G_4 \circ G_2 \) defined as

\[
\begin{align*}
G_1 \otimes G_1 & \to G_3 \otimes G_2 \\
G_4 \otimes G_1 & \to G_4 \otimes G_2
\end{align*}
\]

\[
\begin{align*}
G_1 \otimes G_3 & \to G_2 \otimes G_3 \\
G_1 \otimes G_4 & \to G_2 \otimes G_4
\end{align*}
\]

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Clearly, the tensor product is strict.

(h) ev\_K and coev\_K: Define ev\_K to be the unit of adjoint pair \((F, \overline{F})\) and coev\_K to be the counit of \((F, \overline{F})\). Note that the duality is a property, not an extra structure. The dual functor is generated by the duality of generator, which is not necessarily a unitary dual functor.

**Definition 4.7.6** ([EGNO15, Def. 7.2.1]). Let \(\mathcal{M}\) and \(\mathcal{N}\) be two semisimple C\(^*\) module category categories over a semisimple rigid C\(^*\) (multi)tensor category \(\mathcal{C}\). A \(\mathcal{C}\)-module functor from \(\mathcal{M}\) to \(\mathcal{N}\) consists of a functor \(\psi:\mathcal{M} \rightarrow \mathcal{N}\) and a natural isomorphism \(s_{X,M}: \psi(M \triangleleft X) \rightarrow \psi(M) \triangleleft X\) for all \(X \in \mathcal{C}\), \(M \in \mathcal{M}\) which satisfies the pentagon equation.

We call that \(\mathcal{M}\) and \(\mathcal{N}\) are \(\mathcal{C}\)-module equivalent if \(\psi\) is an equivalence of categories.

Let \(\mathcal{C} = TL\mathcal{J}(d)\). Now we discuss the relation between the equivalence on \(TL\mathcal{J}(d)\)-module category and the equivalence on \(\text{End}^{\dagger}(\mathcal{M}, F)\), where \(F = - \triangleleft X\), and the corresponding 2-category \(\mathcal{C}(K, \text{ev}_K)\).

**Remark 4.7.7.** Let \(\mathcal{M}\) be an indecomposable semisimple \(TL\mathcal{J}(d)\)-module \(C^*\) categories and \((\psi, s) : \mathcal{M} \rightarrow \mathcal{M}\) is an \(TL\mathcal{J}(d)\)-module equivalence. Then \(\psi \in \text{End}^{\dagger}(\mathcal{M})\) is an object. Since \(TL\mathcal{J}(d)\) is generated by \(X, s_{-,-}\) in above Definition 4.7.6 is determined by \(s_{X,-}\). Note that \(s_{X,-}: \psi(F(-)) = \psi(- \triangleleft X) \rightarrow \psi(-) \triangleleft X = F(\psi(-))\) is a unitary natural isomorphism. Note that as an equivalence, \(\psi\) maps simple objects in \(\mathcal{M}\) to simple objects. Then we have

\[
H_{F,\psi(P)\psi(Q)} = \text{Hom}(\psi(Q), F(\psi(P))) \overset{\sim}{\longrightarrow} \text{Hom}(\psi(Q), F(P)) \cong \text{Hom}(Q, F(P)) = H_{F,PQ}.
\]

It follows that the 1-morphism generator \(K = H_F\) indexed by \(V\) and \(H_F\) indexed by \(\psi(V)\) are unitary equivalent.

Comparing the discussion here with Remark 4.3.3, the \(TL\mathcal{J}(d)\)-module equivalence corresponds to the unitary equivalence on \(\mathcal{C}(K, \text{ev}_K)\), which corresponds to isomorphism of edge-weighted graphs \((\Lambda, \omega)\).

**Theorem 4.7.8.** There is a bijective correspondence between equivalence classes of the following:

\[
\left\{ \text{Indecomposable semisimple C}^*\text{ categories } \mathcal{M} \right\} \cong \left\{ \text{W* 2-subcategories } \mathcal{C}(\Lambda, \omega) \text{ of BigHilb, where } \Lambda \text{ is a balanced } d\text{-fair bipartite graph with edge-weighting } \omega \right\}
\]

Equivalence on the left hand side is unitary equivalence; equivalence on the right hand side is isomorphism of edge-weighted graphs.

**Proof.** We can prove this correspondence for the version with base point by passing through the Markov tower. According to Construction 4.7.4, the correspondence holds without fixing the base point. As for the equivalence, see Remark 4.7.7.

**Remark 4.7.9.** Given a semisimple C\(^*\) category \(\mathcal{C}\), similar to Construction 4.7.4, we get a dagger tensor functor from \(\text{End}^{\dagger}(\mathcal{C})\) to the tensor category \(\text{Hilb}_{\text{Irr}(\mathcal{C}) \times \text{Irr}(\mathcal{C})}\), which is the endomorphism tensor category of the object \(\text{Irr}(\mathcal{C})\) in \(\text{BigHilb}\). One should view this as a concrete version of \(\text{End}^{\dagger}(\mathcal{C})\). Note that dualizable endofunctors always map to dualizable 1-morphisms.
5 Markov lattices and biunitary connections

5.1 Balanced \((d_0, d_1)\)-fair square-partite graph

**Definition 5.1.1.** Let Λ be an oriented square-partite graph with vertices \(V(\Gamma) = V_{00} \sqcup V_{01} \sqcup V_{10} \sqcup V_{11}\).

\[
\begin{array}{ccc}
V_{10} & \Omega_0 & V_{11} \\
\Lambda_0 & \Lambda_0 & \Lambda_0 \\
V_{00} & \Omega_1 & V_{01}
\end{array}
\]

We call that \(\Gamma\) associative if for any two vertices on opposite corners of \(\Gamma\), there are the same number of length 2 paths going either way around \(\Gamma\). In more details,

- for any \(P \in V_{00}\) and \(R \in V_{11}\), there are the same number of length 2 paths from \(P\) to \(R\) (or \(R\) to \(P\)) through vertices \(Q \in V_{01}\) and through vertices \(S \in V_{01}\);
- for any \(Q \in V_{01}\) and \(S \in V_{10}\), there are the same number of length 2 paths from \(Q\) to \(S\) (or \(S\) to \(Q\)) through vertices \(P \in V_{00}\) and through vertices \(R \in V_{11}\).

Let \(\omega : E(\Gamma) \to (0, \infty)\) be a weighting on the edges of graph \(\Gamma\). Let \(\Lambda_i\) denote the full subgraph of \(\Gamma\) on \(V_{0i} \sqcup V_{1i}\), \(i = 0, 1\); let \(\Omega_i\) denote the full subgraph of \(\Gamma\) on \(V_{0i} \sqcup V_{1i}\), \(i = 0, 1\). Then \(\Lambda_0, \Lambda_1, \Omega_0, \Omega_1\) are oriented bipartite graphs.

We call \((\Gamma, \omega)\) a balanced \((d_0, d_1)\)-fair square-partite graph if \(\Lambda_0, \Lambda_1\) are balanced \(d_0\)-fair bipartite graphs and \(\Omega_0, \Omega_1\) are balanced \(d_1\)-fair bipartite graphs.

**Remark 5.1.2.** We can define the edge-weighting preserving graph isomorphism literally the same as in Definition 4.1.4 for balanced \((d_0, d_1)\)-square partite graph.

5.2 2-subcategory \(\mathcal{C}(K_0, K_1, L_0, L_1, ev)\) of \(\text{BigHilb}\) and biunitary connection \(\Phi\)

**Definition 5.2.1.** Let \(\mathcal{C}(K_0, K_1, L_0, L_1, ev)\) be a \(W^*\) 2-subcategory of \(\text{BigHilb}\) with four 1-morphism generators \(K_i : V_{0i} \to V_{1i}\), \(L_i : V_{0i} \to V_{1i}\), \(i = 0, 1\) and a chosen evaluation and coevaluation for each generator. We require that

(a) \(K_i, L_i\) are dualizable, \(i = 0, 1\).

(b) The evaluation and coevaluation for the dual:

\[
\begin{align*}
\text{ev}_? := (\text{coev}_?)^\dagger & \quad \text{and} \quad \text{coev}_? := (\text{ev}_?)^\dagger,
\end{align*}
\]

where \(? = K_i, L_i\), \(i = 0, 1\).

(c) They satisfy the \((d_0, d_1)\)-fairness condition, namely,

\[
\begin{align*}
\text{ev}_{K_0} \circ \text{coev}_{K_0} = d_0 \cdot \text{id}_{C_{V_{00}}} & \quad & \text{ev}_{K_0} \circ \text{coev}_{K_0} = d_0 \cdot \text{id}_{C_{V_{00}}}, \\
\text{ev}_{K_1} \circ \text{coev}_{K_1} = d_0 \cdot \text{id}_{C_{V_{01}}} & \quad & \text{ev}_{K_1} \circ \text{coev}_{K_1} = d_0 \cdot \text{id}_{C_{V_{01}}}, \\
\text{ev}_{L_0} \circ \text{coev}_{L_0} = d_1 \cdot \text{id}_{C_{V_{00}}} & \quad & \text{ev}_{L_0} \circ \text{coev}_{L_0} = d_1 \cdot \text{id}_{C_{V_{00}}}, \\
\text{ev}_{L_1} \circ \text{coev}_{L_1} = d_1 \cdot \text{id}_{C_{V_{01}}} & \quad & \text{ev}_{L_1} \circ \text{coev}_{L_1} = d_1 \cdot \text{id}_{C_{V_{01}}}.
\end{align*}
\]

**Notation 5.2.2.** Now, we provide the graphical calculus to describe \(\mathcal{C}(K_0, K_1, L_0, L_1, ev)\). The white region indicates the object \(V_{00}\), the lightest gray for \(V_{10}\), the medium gray for \(V_{11}\) and the darkest gray for \(V_{01}\); the black edge indicates \(K_0, K_1\) and red for \(L_0, L_1\), so white and medium gray, lightest gray and darkest gray will not be adjacent.
Remark 5.2.3. Similar to the discussion in §4.3, from a given balanced \((d_0, d_1)\)-fair square-partite graph \((\Gamma, \omega)\), we can construct a 2-subcategory \(\mathcal{C}(\Gamma, \omega)\) of \(\text{BigHilb}\); on the other hand, if we start with \(\mathcal{C}(K_0, K_1, L_0, L_1, ev)\), we can obtain the \((\Gamma, \omega)\). Moreover, \(\mathcal{C}(K_0, K_1, L_0, L_1, ev)\) and \(\mathcal{C}(\Gamma, \omega)\) are unitary equivalent.

Similar to the discussion in Remark 4.3.3, the edge-weighting preserving graph automorphism will result in the unitary equivalence on \(\mathcal{C}(\Gamma, \omega)\).

In the rest of this section, we define a special 2-morphism \(\Phi\) in \(\mathcal{C}(K_0, K_1, L_0, L_1, ev)\), called biunitary connection.

Definition 5.2.4 (Biunitary connection). A biunitary connection \(\Phi : K_0 \otimes L_1 \rightarrow L_0 \otimes K_1\) is a 2-morphism which is a vertical unitary and a horizontal unitary, as defined as follows. Here is the graphical calculus of \(\Phi\).

1. The biunitary connection \(\Phi\):

![Graphical calculus of \(\Phi\)](image)

2. Vertical unitary: \(\Phi^\dagger \circ \Phi = \text{id}_{K_0} \otimes \text{id}_{L_1}\) and \(\Phi \circ \Phi^\dagger = \text{id}_{L_0} \otimes \text{id}_{K_1}\).

3. Horizontal unitary:

\[
\begin{align*}
\left(\text{id}_{L_0} \otimes \text{ev}_{K_1} \otimes \text{id}_{L_0}\right) \circ \left(\Phi \otimes \Phi^\dagger\right) \circ \left(\text{id}_{K_0} \otimes \text{coev}_{L_1} \otimes \text{id}_{K_0}\right) &= \text{coev}_{L_0} \circ \text{ev}_{K_0} \\
\left(\text{id}_{K_1} \otimes \text{ev}_{L_0} \otimes \text{id}_{K_1}\right) \circ \left(\Phi^\dagger \otimes \Phi\right) \circ \left(\text{id}_{L_1} \otimes \text{coev}_{K_0} \otimes \text{id}_{L_1}\right) &= \text{coev}_{K_1} \circ \text{ev}_{L_1}.
\end{align*}
\]

Here \(\Phi^\dagger\) is defined as the dual of \(\Phi\) in the sense of Definition 1.7.2.
**Definition 5.2.5.** $C(K_0, K_1, L_0, L_1, ev)$ equipped with a biunitary connection $\Phi$ is written as $C(K_0, K_1, L_0, L_1, ev; \Phi)$ or simply $C(\Phi)$.

**Remark 5.2.6.** The existence of $\Phi$ implies that
\[
\dim(K_0 \otimes L_1)_{uv} = \dim(L_0 \otimes K_1)_{uv}
\]
and
\[
\dim(K_0 \otimes L_0)_{uv} = \dim(L_1 \otimes K_1)_{uv},
\]
for each pair $(u,v) \in V \times V$. In other word, the corresponding square-partite graph is associative.

We are going to discuss some properties of biunitary connection.

**Definition 5.2.7** (Rotation by 90°). Define the rotation by 90° to be
\[
\Phi^r := (\id_{K_0} \otimes \id_{L_0} \otimes ev_{K_1}) \circ (\id_{K_0} \otimes \Phi \otimes \id_{K_1}) \circ (\cov_{K_0} \otimes \id_{L_1} \otimes \id_{K_1}).
\]
Similarly,
\[
\Phi^{r^2} := (\id_{L_1} \otimes \id_{K_0} \otimes ev_{L_0}) \circ (\id_{L_1} \otimes \Phi^r \otimes \id_{L_0}) \circ (\cov_{L_1} \otimes \id_{K_1} \otimes \id_{L_0}) = \Phi.
\]

**Remark 5.2.8.** Here are some properties for biunitary connections and rotation.

(1) The group $\langle r, \dagger \rangle = \langle r, \dagger | r^4 = \dagger^2 = \id, r \dagger = \dagger r^3 \rangle$ for the biunitary connection is isomorphic to the dihedral group $D_4$.

(2) $\Phi$ is a biunitary connection if and only if $\Phi^g$ is both vertical unitary and horizontal unitary, where $g \in \langle r, \dagger \rangle$.

**Definition 5.2.9** ([RV16, §4]). We call biunitary connections $\Phi : K_0 \otimes L_1 \rightarrow L_0 \otimes K_1$ and $\Phi' : K'_0 \otimes L'_1 \rightarrow L'_0 \otimes K'_1$ gauge equivalent, if there exist unitaries $u_1 : K'_0 \rightarrow K_0, u_2 : L_0 \rightarrow L'_0, u_3 : K_1 \rightarrow K'_1$ and $u_4 : L'_1 \rightarrow L_1$ such that $\Phi_2 = (u_2 \otimes u_3) \circ \Phi_1 \circ (u_1 \otimes u_4)$.

**Notation 5.2.10.** and Observation

Observe that once we know the color of region and the color of edge, the biunitary connection in the circle is determined. So we can simplify the graphical calculus of biunitary connection as follows.
Moreover, if the color of the leftmost region and the color of each edge are determined, then the color of the rest of the regions will be determined. The 4 colors on the leftmost region and 2 colors on the edge (8 cases) can represent all $\Phi^g, g \in \langle r, \dagger \rangle$.

Here are the simplified graphical calculus of vertical unitarity and horizontal unitarity. In the following context, We require that the leftmost regions in the uncolored equality have the same color.

$$\begin{array}{c}
= \\
\end{array}$$

Proposition 5.2.11. Here are some properties that will be used in the next section and the proof is left to the reader.

(1)

$$\begin{array}{c}
= \\
\end{array}$$

(2) For 2-morphism $x \in \text{End}(F \otimes K_0 \otimes L_1)$, where $F$ is a proper 2-morphism, we have

$$\begin{array}{c}
= \\
\end{array}$$

5.3 From $C(\Phi)$ to Markov lattice

Construction 5.3.1. Here we are going to construct a Markov lattice from the 2-category $C(\Phi)$ discussed above with a chosen point, say $P_0 \in V_{00}$. Let $C^{(P_0)}$ be a 1-morphism with all the entry being 0 except $(C^{(P_0)})_{P_0P_0} = \mathbb{C}$.

Note that $C^{(P_0)} \otimes K_0^{\text{alt}\otimes i} \otimes L_j^{\text{alt}\otimes j}$ is a 1-morphism for each $i, j \in \mathbb{Z}_{\geq 0}$.

Let $M_{i,j} = \text{End} \left( C^{(P_0)} \otimes K_0^{\text{alt}\otimes i} \otimes L_j^{\text{alt}\otimes j} \right)$, where $L_1 = L_0$ if $2 \mid i$ and $L_1 = L_1$ if $2 \nmid j$. We use the graphical calculus to show $M = (M_{i,j})_{i,j \geq 0}$ is a Markov lattice.
(1) Element $x \in M_{i,j}$:

(2) Horizontal inclusion $x \in M_{i,j} \subset M_{i,j+1}$:

(3) Vertical inclusion $x \in M_{i,j} \subset M_{i+1,j}$:

(4) Horizontal conditional expectation $E_{i,j}^{M,r} : M_{i,j} \rightarrow M_{i,j-1}, \ x \in M_{i,j}$:

(5) Vertical conditional expectation $E_{i,j}^{M,l} : M_{i,j} \rightarrow M_{i-1,j}, \ x \in M_{i,j}$:
(6) Commuting square of conditional expectations \( E_{i-1,j}^{M,r} \circ E_{i-1,j-1}^{M,l} = E_{i,j}^{M,l} \circ E_{i,j}^{M,r} : M_{i,j} \to M_{i-1,j-1}, x \in M_{i,j} \):

\[
E_{i-1,j}^{M,r} \circ E_{i-1,j-1}^{M,l}(x) = d_0^{-1}d_1^{-1}
\]

(7) Vertical Jones projections \( e_i \in M_{i+1,j} \) and horizontal Jones projection \( f_j \in M_{i,j+1} \):

(8) It is clear that \( M_j = (M_{i,j}, E_{i,j}^{M,l}, e_i)_{i \geq 0} \) are Markov towers with the same modulus \( d_0 \) and \( e_i \in M_{i+1,j} \) for all \( i, i, j = 0, 1, 2, \cdots \); \( M_i = (M_{i,j}, E_{i,j}^{M,r}, f_j)_{j \geq 0} \) are Markov towers with the same modulus \( d_1 \) and \( f_j \in M_{i,j+1} \) for all \( j \).

**Remark 5.3.2.** A gauge equivalence \( \Phi \sim \Phi' \) will result in an isomorphism of the corresponding Markov lattices.

5.4 From Markov lattice to \( \mathcal{C}(\Gamma, \omega; \Phi) \)

First, we are going to explore more properties of Markov lattice.

**Proposition 5.4.1.**

(a) \( X_{i+1,j+1} := \langle e_i, f_j \rangle \) is a 2-sided ideal of \( M_{i+1,j+1} \) and hence \( M_{i+1,j+1} \) can split as a direct sum of von Neumann algebras \( X_{i+1,j+1} \oplus Y_{i+1,j+1} \). We also define \( Y_{0,0} = M_{0,0}, Y_{1,0} = M_{1,0}, Y_{0,1} = M_{0,1}, Y_{1,1} = M_{1,1} \) so that \( X_{0,0} = X_{1,0} = X_{0,1} = X_{1,1} = 0 \). \( X_{i+1,j+1} \) is called the old stuff and \( Y_{i+1,j+1} \) is called the new stuff.

(b) If \( y \in Y_{i+1,j+1} \) and \( x \in X_{i+1,j} \), then \( yx = 0 \) in \( M_{i+1,j+1} \). Hence \( E_{i+1,j+1}^{M,r}(Y_{i+1,j+1}) \subseteq Y_{i+1,j} \) and \( E_{i+1,j+1}^l(Y_{i+1,j+1}) \subseteq Y_{i,j+1} \), which means the new stuff comes from the old new stuff.

(c) If \( Y_{i,j} = 0 \), then \( Y_{k,l} = 0 \) for all \( k \geq i, l \geq j \).

**Proof.** Similar to Proposition 4.5.2. \( \square \)

Now we are going to construct \( \mathcal{C}(\Gamma, \omega; \Phi) \) from a given Markov lattice \( M \).

**Construction 5.4.2.** The square partite graph and the edge weighting \((\Gamma, \omega)\):

From Markov lattice \( M \), since each row and column is a Markov tower, we can obtain a Bratteli diagram \( \Delta \) as in §4.6 (which can be viewed as a ‘lattice-partite’ graph). After taking only the new vertices in \( \Delta \cap Y_{i,j} \) and the edges between them, we obtain the principal graph \( \Gamma_0 \).
because of Proposition 5.4.1(2). Here, $\Gamma_0$ is not necessary a square-partite graph, so we have to do some identification.

For the new vertices $p_1 \in \Gamma_0 \cap Y_{i,j}$ and $p_2 \in \Gamma_0 \cap Y_{i+2,j-2}$, as in §4.6, let $p'_1$ be the new old vertex of $p_1$ in $M_{i+2,j}$ and $p'_2$ be the new old vertex of $p_2$ in $M_{i+2,j}$. We identify $p_1$ with $p_2$ if $p'_2 \in M_{i+2,j}p'_1$ (or equivalently $p'_1 \in M_{i+2,j}p'_2$).

For the pairs of new vertices $p_1 \in \Gamma_0 \cap Y_{i,j}$ and $q_1 \in \Gamma_0 \cap Y_{i+1,j}$, and the pairs of new vertices $p_2 \in \Gamma_0 \cap Y_{i+2,j-2}$ and $q_2 \in \Gamma_0 \cap Y_{i+3,j-2}$, suppose $p_1$ and $p_2$ are identified in $M_{i+2,j}$, $q_1$ and $q_2$ are identified in $M_{i+3,j}$, above sense, then the numbers of edges between $p_1, q_1$ and $p_2, q_2$ are equal, since they both equal to

$$(\dim C(p'_1q'_1M_{i+2,j}p'_q_1 \cap p'_1q'_1M_{i+3,j}p'_q_1))^{\frac{1}{2}},$$

see the discussion in §4.6. Then we can also identify the edges between $p_1, q_1$ and $p_2, q_2$. Similar statement for $p_1 \in \Gamma_0 \cap Y_{i,j}$ and $r_1 \in \Gamma_0 \cap Y_{i+1,j}$, and the pairs of new vertices $p_2 \in \Gamma_0 \cap Y_{i+2,j-2}$ and $r_2 \in \Gamma_0 \cap Y_{i+2,j-1}$. After above identification as well as the edges between those identified vertices (see following example), we obtain a graph $\Gamma$, which is a square-partite graph.

Then $V_{i,j} \subset V(\Gamma)$ contains all the vertices in $V(\Gamma_0) \cap M_{i+2m,j+2n}$, $i, j = 0, 1, m, n \in \mathbb{Z}_{\geq 0}$.

The edge-weighting $\omega$ can be obtained the same way as in §4.6.

**Example 5.4.3.** Here we provide an example to see the difference between the square-partite graph and the principal graph of a Markov lattice. In the diagram below, if $p_1$ is at depth zero, then $p_2$ is at depth 2 of the principal graph. Therefore, as a new vertex, $p_2$ will appear in two places $M_{0,2}$ and $M_{2,0}$, but their reflections/new old vertices coincide in $M_{2,2}$.

![Diagram](image)

**Remark 5.4.4.** Suppose vertex $q \in V_{i0}$ is at depth $2n$ of the principal graph, then $q$ will first appear in $M_{2i,2n-2i}$, $i = 0, 1, \cdots, n$; if $q \in V_{10}$ is at depth $2n + 1$, then $q$ will first appear in $M_{2i+1,2n-2i}$, $i = 0, 1, \cdots, n$; if $q \in V_{01}$ is at depth $2n + 1$, then $q$ will first appear in $M_{2i,2n+1-2i}$, $i = 0, 1, \cdots, n$; if $q \in V_{11}$ is at depth $2n+2$, then $q$ will first appear in $M_{2i+1,2n+1-2i}$, $i = 0, 1, \cdots, n$.

Next, we compute the biunitary connection $\Phi$.

**Notation 5.4.5. and Observation** We choose $p_0 \in V_{00}$ as the base point, which is at depth 0. Similar to Observation 4.6.2, denote $\Lambda_0, n$ to be the subgraph of $\Lambda_0$ with vertices depth $\leq i$, similar definition for $\Omega_0, n$, $\Lambda_1, n$ and $\Omega_1, n$, see Definition 5.1.1. The corresponding Hilb-enriched
graphs are $K_{i,n} := K_{\Lambda_{i,n}}$, $L_{i,n} := L_{\Omega_{i,n}}$. From Construction 5.3.1, $N_{i,j} := \text{End}(C|_{i,n} \otimes K_0^{\text{alt}} \otimes L_{1,n}^{\text{alt}})$. WLOG, let $2 \nmid i$. Observe that

$$N_{i,j} = \text{End}(K_{0,1} \otimes \mathcal{K}_{0,2} \otimes \cdots \otimes K_{0,i} \otimes L_{1,i+1} \otimes \mathcal{L}_{1,i+2} \otimes \cdots \otimes L_{1,j+1}^2),$$

where $L_{1,j}^2 = L_{1,j}$ if $2 \nmid j$, $L_{1,j}^2 = \overline{L}_{1,j}$ if $2 \mid j$.

**Example 5.4.6.** Following Example 5.4.3,

we have

$$K_{0,1} = \begin{bmatrix} 0 & 0 & \mathbb{C} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \overline{K}_{0,2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbb{C} & \mathbb{C} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad K_{0,3} = \begin{bmatrix} 0 & 0 & \mathbb{C} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K_{1,4} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbb{C} & \mathbb{C} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbb{C} & 0 \end{bmatrix}, \quad L_{0,1} = \begin{bmatrix} 0 & 0 & 0 & 0 & \mathbb{C} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \overline{L}_{1,3} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbb{C} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K_{0,1} \otimes \overline{K}_{0,2} \otimes L_{0,3} = \begin{bmatrix} 0 & 0 & 0 & 0 & \mathbb{C}^6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cong K_{0,1} \otimes L_{1,2} \otimes \overline{K}_{1,3} \cong L_{0,1} \otimes K_{1,2} \otimes \overline{K}_{1,3}$$

Similar to Example 4.6.3, the entry $(i,j)$ in $N_{m,n}$ indicates number of paths from the vertex $p_i$ at depth 0 to the vertex $p_j$ at depth $m + n$. Note that the base point is a single vertex $p_1$, so only at entry $(1,j)$ can be nonzero.
Remark 5.4.7. Any automorphism of $M_n(\mathbb{C})$ is inner. To be precise, if $\alpha \in \text{Aut}(M_n(\mathbb{C}))$, then there exists a unitary $u \in M_n(\mathbb{C})$, such that $\alpha(x) = u x u^* = \text{Ad}(u)(x)$, for any $x \in M_n(\mathbb{C})$. Moreover, this unitary $u$ is unique up to a unit scalar. Indeed, if $u x u^* = u_i x u_i^*$ for all $x \in M_n(\mathbb{C})$, then $x(u^* u_i) = (u^* u_i) x$, which implies that $u^* u_i$ is in the center of $M_n(\mathbb{C})$. Thus, $u^* u_i = a \in \mathbb{C}$ with $|a| = 1$ and hence $u_i = au$.

As a corollary, for 1-morphisms $H, G$, if $\alpha : \text{End}(H) \cong \text{End}(G)$ is a $*$-isomorphism, then there exists a unitary 2-morphism $u : H \to G$ such that $\alpha = \text{Ad}(u)$.

Warning: the unitary $u$ is obtained by taking a unitary $u_{i,j}$ in each entry. Thus any two choices of implementing unitary $u = (u_{i,j})$ and $v = (v_{i,j})$ differ by a matrix of scalars $(a_{i,j})$ which may be distinct. Hence the unitary $u$ is unique up to a matrix of scalars.

Construction 5.4.8. The biunitary connection $\Phi$: The construction (for the tracial case) has been written in [JS97, §5.5] in the language of path algebras. For convenience, we will construct it here using our language.

From Construction 5.4.2 and Remark 5.2.3, the 2-category $\mathcal{C}(\Gamma, \omega)$ can be constructed.

In order to obtain the biunitary connection $\Phi$, we shall compute it componentwise, which is similar to the idea to compute the edge-weighting in §4.6. The goal is to compute $\Psi_{pr} : (K_0 \otimes L_1)_{pr} = \bigoplus_{q \in V_1} K_{0, pq} \otimes L_{1, qr} \rightarrow \bigoplus_{s \in V_0} L_{0, ps} \otimes K_{1, sr} = (L_0 \otimes K_1)_{pr}$ for each pair $(p, r) \in V_0 \times V_1$.

Suppose $p$ is at depth $2n$ of the principal graph and $r$ is at depth $2n + 2$. By Remark 5.4.4, $p$ first appear in $M_{0,2n}$ and $r$ first appears in $M_{1,2n+1}$.

Consider two path models $M_{0,0} \subset M_{0,1} \subset \cdots \subset M_{0,2n} \subset M_{0,2n+1} \subset M_{1,2n+1}$ and $M_{0,0} \subset M_{0,1} \subset \cdots \subset M_{0,2n} \subset M_{1,2n} \subset M_{1,2n+1}$.

Similar to Proposition 4.6.4, we have

$$\begin{align*}
N_{0,2n} &\cap N_{1,2n+1} = \text{id}_{K_{0,2n} \otimes K_{1,2n+1}} \otimes \text{End}(K_{0,2n} \otimes L_{1,2n+1}) \quad \text{for the first model} \\
N_{0,2n} &\cap N_{1,2n+1} = \text{id}_{K_{0,2n} \otimes K_{1,2n+1}} \otimes \text{End}(L_{0,2n} \otimes K_{1,2n+1}) \quad \text{for the second model}.
\end{align*}$$

Let $\psi : M_{1,2n+1} \rightarrow N_{1,2n+1}$ denote the $*$-isomorphism onto the first model and $\psi' : M_{1,2n+1} \rightarrow N_{1,2n+1}$ denote the $*$-isomorphism onto the second model, then

$$\begin{align*}
\psi : M_{0,2n} \cap M_{1,2n+1} &\rightarrow N_{0,2n} \cap N_{1,2n+1} \cong \text{End}(K_{0,2n+1} \otimes L_{1,2n+1}) \\
\psi' : M_{0,2n} \cap M_{1,2n+1} &\rightarrow N_{0,2n} \cap N_{1,2n+1} \cong \text{End}(L_{0,2n} \otimes K_{1,2n+1})
\end{align*}$$

are $*$-isomorphisms. Then $\psi' \circ \psi^{-1} : \text{End}(K_{0,2n+1} \otimes L_{1,2n+1}) \rightarrow \text{End}(L_{0,2n} \otimes K_{1,2n+1})$ is a $*$-isomorphism between two 1-morphisms. By Remark 5.4.7, their exists a unique unitary $u$ up to a matrix of scalars such that $\psi' \circ \psi^{-1} = \text{Ad}(u)$. We define $\Phi_{pr} := u_{pr}$.

Similar to Remark 4.3.7, we secretly make a choice of ONB when we construct the generators $K_i, L_j$ from the square-partite graph $\Gamma$, $i, j = 0, 1$. Different choice results in multiplying a unitary on each generator. Combining Definition 5.2.9 of gauge equivalence and above discussion, the biunitary connection $\Phi$ we construct here is unique up to gauge equivalence.

5.5 $\mathcal{C}(\Phi)$ and $\text{End}_0^0(\mathcal{M}, F, G)$

We have already seen the method to construct a Markov lattice from $\mathcal{C}(\Phi)$ above or from $\mathcal{M}$ in §3 with a simple base point, where $\mathcal{M}$ is an indecomposable semisimple $C^* \mathcal{A} - \mathcal{B}$ bimodule category. In this section, by using the similar technique as in §4.7, we will show their relation to each other.
Definition 5.5.1. Suppose $\mathcal{M}$ is an indecomposable semisimple $C^*$ $\mathcal{T}\mathcal{L}\mathcal{J}(d_0) - \mathcal{T}\mathcal{L}\mathcal{J}(d_1)$ bimodule category, where $X = 1^+ \otimes X \otimes 1^-$, $Y = 1^+ \otimes Y \otimes 1^-$ are the generators of $\mathcal{T}\mathcal{L}\mathcal{J}(d_0)$ and $\mathcal{T}\mathcal{L}\mathcal{J}(d_1)$ respectively. Define $F = X \rhd -$, $\overline{F} = X \rhd -$, $G = - \rhd Y$, $\overline{G} = - \rhd \overline{Y}$, which are endofunctors on $\mathcal{M}$. Note that $(F, \overline{F})$ and $(G, \overline{G})$ are adjoint pairs, with unit $e_{F, \overline{F}}$, $e_{G, \overline{G}}$ induced by $ev_X$, $ev_Y$ and counit $coev_F$, $coev_G$ induced by $coev_X$, $coev_Y$.

Define $\text{End}_{01}(\mathcal{M}, F, G)$ to be the full subcategory of $\text{End}^1(\mathcal{M})$ Cauchy tensor generated by $F, \overline{F}, G, \overline{G}$, so it is a rigid $C^*$ tensor category.

We warn the reader that $\text{End}_{01}(\mathcal{M}, F, G)$ will only be multitensor ($\dim(\text{End}(\text{id}_\mathcal{M})) < \infty$) when $\mathcal{M}$ is finitely semisimple.

Definition 5.5.2 (Biunitary connection in $\text{End}_{01}^1(\mathcal{M}, F, G)$). Note that the bimodule associator $\alpha_{X,-Y} : (X \rhd -) \triangleleft Y \to X \rhd (- \triangleleft Y)$ is a unitary, which induces a natural isomorphism $\Phi_{F,G} : F \otimes G \to G \otimes F$, where $F \otimes G := G \circ F$. Then $\Phi_{G,F} : G \otimes \overline{F} \to \overline{F} \otimes G$ is equal to the 90° rotation $\Phi_{F,G}$ defined as follows:

$$\Phi_{F,G} := (id_{\overline{F}} \otimes id_G \otimes ev_F) \circ (id_{\overline{F}} \otimes \Phi_{F,G} \otimes id_{\overline{F}}) \circ (coev_F \otimes id_G \otimes id_{\overline{F}}).$$

It is easy to show that $\Phi_{F,G}$ is vertical and horizontal unitary and so is $\Phi_{G,F}$.

Similar to §4.7, we will show that the tensor category $\text{End}_{01}^1(\mathcal{M}, F, G)$ and 2-category $\mathcal{C}(\Phi)$ are unitarily equivalent.

Construction 5.5.3. We construct $\mathcal{C}(\Phi)$ from $\text{End}_{01}^1(\mathcal{M}, F, G)$ functorially.

(a) Let $V_{00}$ be a set of representatives of all simple objects $P \in \mathcal{M}$ such that $P = 1^+ \triangleright P \triangleleft 1^+$; $V_{10}$ be the set of representatives of all simple objects $Q \in \mathcal{M}$ such that $Q = 1^+ \triangleright Q \triangleleft 1^+$; $V_{11}$ be the set of representatives of all simple objects $R \in \mathcal{M}$ such that $R = 1^+ \triangleright R \triangleleft 1^+$; $V_{01}$ be the set of representatives of all simple objects $S \in \mathcal{M}$ such that $S = 1^+ \triangleright S \triangleleft 1^-$.

Then the objects are the sets $V_{i,j}$, $i, j = 0, 1$ and their union $V = V_{00} \cup V_{01} \cup V_{11} \cup V_{10}$.

(b) 1-morphism: The 1-morphism of $\mathcal{C}(\Phi)$ is the object of $\text{End}_{01}^1(\mathcal{M}, F, G)$. The way to construct the corresponding $V \times V$-bigraded Hilbert space from an endofunctor is the same as in Construction 4.7.4. The same for the dual 1-morphism and tensor structure/composition.

(c) 2-morphism: The 2-morphism of $\mathcal{C}(\Phi)$ is the morphism of $\text{End}_{01}^1(\mathcal{M}, F, G)$.

(d) 1-morphism generator: Define

\begin{align*}
K_0 := H_{J^+} \otimes H_\mathcal{F} & \quad \overline{K}_0 = H_{J^+} \otimes H_\overline{\mathcal{F}} & \quad K_1 := H_{J^-} \otimes H_\mathcal{F} & \quad \overline{K}_1 = H_{J^-} \otimes H_\overline{\mathcal{F}} \\
L_0 := H_{I^+} \otimes H_\mathcal{G} & \quad \overline{L}_0 = H_{I^+} \otimes H_\overline{\mathcal{G}} & \quad L_1 := H_{I^-} \otimes H_\mathcal{G} & \quad \overline{L}_1 = H_{I^-} \otimes H_\overline{\mathcal{G}}
\end{align*}

(e) $ev$ and $coev$. The same as in Construction 4.7.4(h).

(f) Biunitary connection: $\Phi : K_0 \otimes L_1 \to L_0 \otimes K_1$ is defined as $\Phi_{F,G} : F \otimes G \to G \otimes F$. The check that $\Phi$ is vertical and horizontal unitary is left to the reader.

Construction 5.5.4. For the convenience to the reader, we also provide the construction from $\mathcal{C}(\Phi)$ to $\text{End}_{01}^1(\mathcal{M}, F, G)$:

(a) Object: The object are the 1-morphisms in $\mathcal{C}(\Phi)$. In particular, the generator $F = K_0 \oplus K_1$, $\overline{F} = \overline{K}_0 \oplus \overline{K}_1$, $G = L_0 \oplus L_1$ and $\overline{G} = \overline{L}_0 \oplus \overline{L}_1$; the unit $I^+ = 1^+ \triangleright - = \mathbb{C}V_{00} \cup V_{01}$, $I^- = 1^- \triangleright - = \mathbb{C}V_{10} \cup V_{11}$, $J^+ = - \triangleleft 1^+ = \mathbb{C}V_{00} \cup V_{01}$ and $J^- = - \triangleleft 1^- = \mathbb{C}V_{10} \cup V_{11}$.

(b) Morphism: The morphisms are the 2-morphisms in $\mathcal{C}(\Phi)$.

(c) The associator: Note that $F \otimes G = (K_0 \oplus K_1) \otimes (L_0 \oplus L_1) = K_0 \otimes L_1$ and $G \otimes F = (L_0 \oplus L_1) \otimes (K_0 \oplus K_1) = L_0 \otimes K_1$, the associator $\Phi_{F,G} : F \otimes G \to G \otimes F$ is defined as the biunitary connection $\Phi : K_0 \otimes L_1 \to L_0 \otimes K_1$. All the 8 cases of associators are defined as $\Phi^g$, where $g \in \{r, \overline{r}, \dagger, \overline{\dagger}\}$. 75
Theorem 5.5.5. There is a bijective correspondence between equivalence classes of the following:

\[
\begin{align*}
\text{Indecomposable semisimple } & C^* \text{-bimodule categories } \mathcal{M} \\
\mathcal{T}\mathcal{L}\mathcal{J}(d_0) - \mathcal{T}\mathcal{L}\mathcal{J}(d_1) \\
\mathcal{W}^* & 2\text{-subcategories } C(\Gamma, \omega; \Phi) \text{ of } \text{BigHilb},
\end{align*}
\]

where \( \Gamma \) is a balanced \((d_0, d_1)\)-fair square partite graph with edge-weighting \( \omega \) and \( \Phi \) a biunitary connection.

Equivalence on the left hand side is unitary equivalence; equivalence on the right hand side is isomorphism on the edge-weighted square-partite graph and gauge equivalence on biunitary connection.

Proof. We can prove this correspondence for the version with base point by using the Markov lattice. According to Construction 5.5.3, the correspondence holds without fixing the base point.

As for the equivalence, combining Remark 5.2.3, Definition 5.2.9 and the last paragraph in Construction 5.4.8, the isomorphism on the edge-weighted graph \((\Gamma, \omega)\) and gauge equivalence on \(\Phi\) corresponds to the unitary equivalence on \(C(\Phi)\), which corresponds to the unitary equivalence on \(\mathcal{T}\mathcal{L}\mathcal{J}(d_0) - \mathcal{T}\mathcal{L}\mathcal{J}(d_1)\) bimodule category \(\mathcal{M}\) based on Construction 5.4.8 and Remark 4.7.7.

\(\square\)

6 The tracial case

In this chapter, we finally discuss the tracial/pivotal case for (bi)module categories. As an application, we prove the module embedding theorem for (infinite depth) graph planar algebra.

6.1 Tracial Markov towers and pivotal module categories

Definition 6.1.1. [Sc13] Let \(C\) be a rigid C* (multi)tensor category with the canonical spherical unitary dual functor. We call \(\mathcal{M}\) a semisimple pivotal C* \(\mathcal{C}\)-module category, if there exists a pivotal trace \(\text{tr}_{\mathcal{M}}\) compatible with the spherical structure on \(\mathcal{C}\), i.e.,

\[
\text{tr}_{\mathcal{M}}(m \triangleleft c, c \in \mathcal{C}) \in \mathbb{C},
\]

for all \(f \in \text{End}(m \triangleleft c)\), where \(m \in \mathcal{M}, \ c \in \mathcal{C}\).

Remark 6.1.2. If \(f \in \text{End}(c), \ c \in \mathcal{C}\) and \(m \in \mathcal{M}\),

\[
\text{tr}_{\mathcal{M}}(m \triangleleft < f) = \text{tr}_{\mathcal{M}}((\text{id}_m \triangleleft \text{coev}_c) \circ (f \triangleleft \text{id}_c) \circ (\text{id}_m \triangleleft \text{coev}_c)),
\]

for all \(f \in \text{End}(m \triangleleft c)\), where \(m \in \mathcal{M}, \ c \in \mathcal{C}\).

Remark 6.1.3. [Sc13, §4.1] If \(\mathcal{C}\) is fusion and \(\mathcal{M}\) is indecomposable, then the pivotal trace \(\text{tr}_{\mathcal{M}}\) is unique up to scalar.

Definition 6.1.4 (Tracial Markov tower). We call \(M\) a tracial Markov tower if \(M\) a Markov tower equipped with a unital trace \(\text{tr}\) on \(\bigcup_{n \geq 0} M_n\) and the conditional expectation \(E_n\) are trace-preserving, i.e.,

\[
\text{tr} \circ E_n = \text{tr}
\]
on \(M_n\), \(n \geq 0\).
Definition 6.1.5. We call $M$ a tracial standard $A$–module, where $A$ is a standard $\lambda$-lattice, if $\text{tr}_M|A = \text{tr}_A$ and $M$ is a standard $A$–module, see Definition 2.1.1.

Let $A$ be a standard $\lambda$-lattice. If we start with a tracial standard $A$–module $M$, combining the construction in §2.3 and the proof in proposition 1.6.15, we are able to construct a pivotal planar $A_0$–module category. Furthermore, from this pivotal planar $A_0$–module category, we can construct an indecomposable semisimple pivotal $C^* A$–module category with a choice of simple base object. The following is the theorem.

Theorem 6.1.6. There is a bijective correspondence between equivalence classes of the following:

\[
\begin{cases}
\text{Tracial Markov towers } M = (M_i)_{i \geq 0} \text{ with } \dim(M_0) = 1 \text{ as standard right modules over a standard } \lambda\text{-lattice } A \\
\text{standard right modules over a standard } \lambda\text{-lattice } A
\end{cases}
\text{ } \quad \leftrightarrow \quad \begin{cases}
\text{Pairs } (\mathcal{M}, Z) \text{ with } \mathcal{M} \text{ an indecomposable semisimple pivotal } C^* \text{ right } A\text{–module category together with a choice of simple base object } Z = Z < 1^+_A
\end{cases}
\]

Equivalence on the left hand side is trace-preserving *-isomorphism on the tracial Markov tower as standard $A$–module; equivalence on the right hand side is the pivotal unitary $A$–module equivalence on their Cauchy completions which maps simple base object to simple base object.

Let us look at the balanced $d$-fair bipartite graph $(\Lambda, \omega)$ from the tracial Markov tower $M$. Since the evaluation and coevaluation are compatible with the trace, the edge-weighting comes from a vertex-weighting, see [JP19, Prop. 6.8]. To be precise,

Definition 6.1.7 (Vertex weighting). Let $\Lambda$ be a bipartite graph. Let $\nu : V(\Lambda) \to (0, \infty)$ be a weighting on the vertices of $\Lambda$ which satisfies the Frobenius-Perron condition: for each $P \in V(\Lambda),
\sum_{\{Q \in V(\Lambda) : P, Q \text{ adjacent}\}} \nu(Q) = d \cdot \nu(P).

In the sum on the left hand side, $\nu(Q)$ has number of edges between $P \to Q$ copies.

From an undirected bipartite graph, one can obtain a directed graph with involution [HP17, Def. 2.20]. Then for $e : P \to Q$, define $w(e) := \frac{\nu(P)}{\nu(Q)}$. The $d$-fairness and balance condition in Definition 4.1.2 follows automatically.

Remark 6.1.8. Suppose $\mathcal{M}$ is an indecomposable semisimple $C^*$ pivotal $A$–module category with fusion/principal graph $\Lambda$ whose vertices are simple objects of $\mathcal{M}$. We can define the vertex weighting for simple object $P$ as $\nu(P) := \text{Tr}_P(\text{id}_P)$.

Remark 6.1.9. Note that $\mathcal{M}$ being a pivotal $A$–module is equivalent to the dagger tensor functor $A \to \text{End}^\dagger(\mathcal{M})$ being pivotal [GMPPS18, Thm. 3.70], so that its essential image $\text{End}^\dagger_0(\mathcal{M}, F)$ has a unitary pivotal structure from the pivotal structure in $\mathcal{A}$, where $F = - \triangleleft X$ is the generator. We also denote the corresponding 2-subcategory of $\text{BigHilb}$ as $\mathcal{C}(K, \phi)$ or $\mathcal{C}(\Lambda, \nu)$.

6.2 The module embedding theorem

Jones’ planar algebra, as a form of standard invariant, is a method to construct and classify finite index type II$_1$ subfactors. The module embedding theorem has been known to Vaughan Jones since he first defined the graph planar algebra [Jo00]. The proof for finite depth case appears in [JP10, CHPS18, GMPPS18]. Many nontrivial examples of subfactors have been
constructed inspired by this theorem, including the Extended Haagerup subfactor and its relatives [BPMS12, GMPPS18].

In our setting, the bipartite graph $\Lambda$ can be infinite depth. We refer the reader to [Bu10] for the definition of the infinite depth bipartite graph planar algebra.

**Theorem 6.2.1.** The planar algebra constructed from $\text{End}^1_0(M, F)$ with generator $F$ mentioned in Remark 6.1.9 is isomorphic to the graph planar algebra of bipartite graph $\Lambda$, where $M$ is an indecomposable semisimple pivotal $C^*$-module category, $A$ is a 2-shaded rigid $C^*$ multitensor category with generator $X = 1^+ \otimes X \otimes 1^-$, $\Lambda$ is the (possibly infinite) fusion graph for $M$ with respect to the generator $X$, where the vertex weighting $\nu$ on $\Lambda$ comes from the trace $\text{Tr}^M$ as in Remark 6.1.8.

**Proof.** Here we provide the sketch of the proof. From the unitary pivotal dagger functor $A \to \text{End}^1(M)$, we obtain a rigid $C^*$ tensor category $\text{End}^1_0(M, F)$ with pivotal structure with generator $F = -\triangleleft X$ in the sense of §4.7.

According to §4.7 and §4.3, from $\text{End}^1_0(M, F)$, we can construct the 2-category $C(\Lambda, \nu)$ discussed in Remark 6.1.9 with its generating Hilb-enriched graph $\Lambda$, which is equivalent information. Similar to [GMPPS18, §3.5.3], the planar algebra of $C(\Lambda, \nu)$ with generator $\Lambda$ is $\ast$-isomorphic to the graph planar algebra $G_\bullet$ (in the sense of Burstein [Bu10]) of the fusion graph $\Lambda$ with vertex weighting $\nu$, which corresponds to $F$ in the sense of Remark 6.1.8.

Note that there is a well-know correspondence between [Gh11, DGG14, Pe18]:

\[
\begin{cases}
\text{Subfactor planar algebras } P_\bullet \\
\text{algebras } \mathcal{P}_\bullet
\end{cases}
\cong
\begin{cases}
\text{Pairs } (A, X) \text{ with } A \text{ a 2-shaded rigid } C^* \text{ multitensor category with a generator } X, \text{ i.e., } 1_A = 1^+ \oplus 1^- , 1^+, 1^- \\
\text{are simple and } X = 1^+ \otimes X \otimes 1^-
\end{cases}
\]

Finally, the pivotal dagger tensor functor $A \to \text{End}^1_0(M, F)$ gives a planar algebra embedding from the subfactor planar algebra $A_\bullet$ to the graph planar algebra $G_\bullet$ of its principal graph.

If we choose $M = A_+ = 1^+ \otimes A \otimes 1^+$ to be the $A$–module category, we obtain the module embedding theorem:

**Corollary 6.2.2.** Every subfactor planar algebra $P_\bullet$ embeds into the graph planar algebra of its principal graph.

### 6.3 Tracial Markov lattices and pivotal bimodule categories

**Definition 6.3.1.** Let $\mathcal{C}, \mathcal{D}$ be rigid $C^*$ (multi)tensor categories with canonical unitary dual functors respectively. We call $M$ a semisimple pivotal $C^* \mathcal{C} - \mathcal{D}$ bimodule category, if there exists a pivotal trace $\text{tr}^M$ compatible with the spherical structures in $\mathcal{C}$ and $\mathcal{D}$, i.e.,

\[
\begin{align*}
\text{tr}^M_{a \triangleright m}(f) &= \text{tr}^M_m((\text{ev}_a \triangleright \text{id}_m) \circ (\text{id}_\mathcal{D} \triangleright f) \circ (\text{ev}_a \triangleright \text{id}_m)) \\
\text{tr}^M_{m \triangleleft b}(f) &= \text{tr}^M_m((\text{id}_m \triangleleft \text{coev}_b) \circ (f \triangleleft \text{id}_\mathcal{D}) \circ (\text{id}_m \triangleleft \text{coev}_b)),
\end{align*}
\]

for $f \in \text{End}(a \triangleright m \triangleleft b)$, where $m \in M$, $a \in \mathcal{C}$, $b \in \mathcal{D}$.

**Definition 6.3.2.** (Tracial Markov lattice) We call $M$ a tracial Markov lattice if $M$ is a Markov lattice equipped with a unital trace $\text{tr}$ on $\bigcup_{i,j \geq 0} M_{i,j}$ and the conditional expectation $E_{i,j}^M, E_{i,j}^{M,r}$ are trace-preserving, i.e.,

\[
\text{tr} \circ E_{i,j}^M = \text{tr}, \quad \text{tr} \circ E_{i,j}^{M,r} = \text{tr}
\]
on $M_{i,j}$, $i, j \geq 0$.
Definition 6.3.3. We call $M$ a tracial standard $A - B$ bimodule, where $A, B$ are standard $\lambda$-lattices, if $\text{tr}_M|_A = \text{tr}_A$, $\text{tr}_M|_B = \text{tr}_B$ and $M$ is a standard $A - B$ bimodule, see Definition 3.2.1.

Similar to Theorem 6.1.6, we have the following theorem:

Theorem 6.3.4. There is a bijective correspondence between equivalence classes of the following:

$$\begin{align*}
\{ \text{Tracial Markov lattice } M = (M_{i,j})_{i,j \geq 0} \text{ with } \dim(M_{0,0}) = 1 \text{ as a standard } A - B \text{ bimodule over standard } \lambda \text{-lattices } A, B \} & \cong \{ \text{Pairs } (\mathcal{M}, Z) \text{ with } \mathcal{M} \text{ an indecomposable semisimple } \mathbb{C}^* \text{ pivotal } A - B \text{ bimodule category together with a choice of simple base object } Z = 1^+_A \triangleright Z \triangleleft 1^+_B \} \\
\end{align*}$$

Equivalence on the left hand side is the trace-preserving $*$-isomorphism on the tracial Markov lattice as standard $A - B$ bimodule; equivalence on the right hand side is the pivotal unitary $A - B$ bimodule equivalence between their Cauchy completions which maps the simple base object to simple base object.

Let us look at the balanced $(d_0, d_1)$-fair square-partite graph $(\Lambda, \omega)$ from the tracial Markov lattice $M$. Similar to the tracial Markov tower case, the edge-weighting comes from the vertex-weighting. To be precise,

$$\begin{align*}
\text{For } P \in V_{00} \sqcup V_{01}, & \quad \sum_{\{e: P \to Q \in V_{10} \sqcup V_{11}\}} \nu(Q) = d_0 \cdot \nu(P) \\
\text{For } P \in V_{00} \sqcup V_{01}, & \quad \sum_{\{e: P \to Q \in V_{01} \sqcup V_{11}\}} \nu(Q) = d_1 \cdot \nu(P).
\end{align*}$$

Remark 6.3.5. As for the biunitary connection, the computation does not change at all. In fact, the biunitary connection is independent of the pivotal structure, see Proposition 5.2.11(2) and §5.5. This now agrees with the usual definition of biunitary connection for the tracial/pivotal case discussed in [JS97, EK98, MPPS12, MP14].

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