SYMPLECTIC ASPHERICITY, CATEGORY WEIGHT, AND CLOSED CHARACTERISTICS OF K-CONTACT MANIFOLDS

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Abstract. Let $M$ be a closed K-contact $(2n+1)$-manifold equipped with a quasi-regular K-contact structure. Rukimbira [Ruk1] proved that the Reeb vector field $\xi$ of this structure has at least $n + 1$ closed characteristics. We note that $\xi$ has at least $2n + 1$ closed characteristics provided that the space of leaves of the foliation determined by $\xi$ is symplectically aspherical.

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1. Introduction

1.1. Proposition-Definition ([BG]). Given a $(2n+1)$-dimensional manifold $M$, a contact form on $M$ is a 1-form $\alpha$ on $M$ such that the inequality $\alpha \wedge (d\alpha)^n \neq 0$ is valid everywhere on $M$. In this case there is a unique vector field $\xi$, satisfying the equalities $\alpha(\xi) = 1$ and $i_\xi(d\alpha) = 0$. This vector field is called the Reeb vector field. An integral curve of the Reeb vector field $\xi$ is called a characteristic of the contact manifold $(M, \alpha)$.

1.2. Definition (see [MRT]). A K-contact structure is a quintuple $(M, \alpha, \xi, \Phi, g)$ such that

- $(M, \alpha)$ is a contact form as in Definition 1.1;
- $\xi$ is the Reeb vector field for $\alpha$;
- $\Phi$ is an endomorphism $\Phi$ of $TM$ (a tensor of type (1,1)) such that $\Phi^2 = -\operatorname{id} + \xi \otimes \alpha$;
- $d\alpha(\Phi X, \Phi Y) = d\alpha(X, Y)$ for all $X, Y$;
- $d\alpha(\Phi X, X) > 0$ for all nonzero $X \in \ker \alpha$;

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the Reeb vector field $\xi$ is a Killing vector field with respect to the Riemannian metric defined by the formula
\[ g(X, Y) = d\alpha(\Phi X, Y) + \alpha(X)\alpha(Y). \]

Let $(M, \alpha, \xi, \Phi, g)$ be a K-contact manifold. Consider the contact cone as the Riemannian manifold $C(M) := (M \times \mathbb{R}^{>0}, t^2 g + dt^2)$. Define the almost complex structure $I$ on $C(M)$ by:
- $I(X) = \Phi(X)$ on $\text{Ker} \alpha$,
- $I(\xi) = t \frac{\partial}{\partial t}$, $I(t \frac{\partial}{\partial t}) = -\xi$, for the Killing vector field $\xi$ of $\alpha$.

The K-contact manifold $(M, \alpha, \Phi, \xi, g)$ is Sasakian if $I$ is integrable.

The theory of such structures appeared to be very important in geometry and its applications, especially after the publishing of the fundamental monograph by Boyer and Galicki [BG]. In this monograph, the authors presented a research program of studying topological properties of K-contact manifolds. Recently, there were published several works on different aspects of the topology of K-contact and Sasakian manifolds [BFMT], [MT], [MRT]. One of the early results on topological properties of K-contact manifolds was obtained by Rukimbira [Ruk1] who proved that the Reeb vector field on any K-contact manifold has at least $n+1$ closed characteristics. In this work we discuss the following question: is this estimate best possible? We show that under the additional assumption of symplectic asphericity of the space of leaves of the foliation determined by the Reeb field, the estimate can be improved to $2n+1$.

Recall that a manifold (or orbifold) $X$ with symplectic form $\omega$ is symplectically aspherical if $[\omega]|_{\pi_2(X)} = 0$; here $[\omega]$ is the de Rham cohomology class of the form $\omega$. The condition $[\omega]|_{\pi_2(X)} = 0$ appeared originally in the papers of Floer [F] and Hofer [H] in the context of Lagrangian intersection theory, where symplectic asphericity ensures the absence of non-trivial pseudo-holomorphic spheres in $X$.

Floer and Hofer used the advantages of analytic properties of such manifolds. However, it turned out that symplectically aspherical manifolds have nice and controllable homotopy properties. Later it appeared to be clear that most of these properties are based on the fact that, in the case of symplectic asphericity, the category weight of $[\omega]$ equals 2, cf. Lemma 4.4. It was an important ingredient in proving of the Arnold conjecture about symplectic fixed points [Rud2, RO] and in an improvement of the estimate of the number of closed orbits of charged particles in symplectic magnetic fields [RT]. In our paper we exploit this approach in Section 4.

In [RT] the authors expressed the hope that the technique of category weight potentially may have many applications to problems in symplectic topology, and, in a broader sense, to other non-linear analytic problems. In this paper we, in a sense, equal the hopes by applying the theory of category weight to a special problem in the topology of contact metric manifolds.

The paper is organized as follows. In Section 2 we collect preliminary information on closed characteristics of contact manifolds. In Section 3 we present the necessary information on K-contact structures. In Section 4 we prove the main theorem of manifolds with $\geq 2n+1$ closed characteristics, and in Section 5 we give some examples of such manifolds. In Section 6 we outline some of the possible direction

\[ \frac{\partial}{\partial t}. \]
of research in the area. In appendices we remind basic information on orbifolds (following [BG, MRT] and Sullivan model theory (following [FHT]).

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2. Preliminaries

The section is an extract from [Ruk1, BR].

2.1. Definition (CLOT). Given a map $\varphi : A \to Y$, we say that a subset $U$ of $A$ is $\varphi$-categorical if it is open (in $A$) and $\varphi|U$ is null-homotopic. We define the Lusternik–Schnirelmann category $\text{cat} \varphi$ of $\varphi$ as follows:

$$\text{cat} \varphi = \min \{ k \mid A = U_0 \cup U_1 \cup \cdots \cup U_{k+1} \text{ where each } U_i \text{ is } \varphi\text{-categorical} \}.$$  

Furthermore, we define the Lusternik–Schnirelmann category $\text{cat} Y$ of a space $Y$ by setting $\text{cat} X := \text{cat} 1_X$.

Consider a smooth action on a compact Lie group $G$ on a closed smooth manifold $M$ and let $f : M \to \mathbb{R}$ be a smooth $G$-equivariant function. Clearly, if $p$ is a critical point of $f$ then every point $gp, g \in G$ is critical. So, each critical point $p$ produces a critical orbit.

2.2. Proposition. The number of critical orbits of $f$ is at least $1 + \text{cat}(M/G)$.

Proof. Consider the gradient flow of $f$ with respect to a $G$-invariant Riemannian metric on $M$. The flow induces a (gradient-like) flow $\Phi$ on $M/G$, and critical orbits of $f$ are exactly the rest points of $\Phi$. Now the result follows from [CLOT, Theorem 1.82]. One can also consult [W]. □

Let $M$ be a $(2n+1)$-dimensional compact manifold equipped with an effective $S^1$-action $a : S^1 \times M \to M$. Let $\alpha$ be a circle invariant contact form on $M$. Let $Z$ be the infinitesimal generator of the $S^1$-action. Consider the function $S = i_Z(\alpha)$ on $M$ and suppose that $S \neq 0$ everywhere on $M$.

2.3. Proposition. (i) We have $dS = -i_Zd\alpha$;
(ii) if $p$ is a critical point of $S$ then the $Z$-orbit through $p$ is a characteristic;
(iii) the Reeb vector field $\xi$ has at least $1 + \text{cat}(M/S^1)$ closed characteristics.

Proof. (i) Since $\alpha$ is circle invariant, $L_Z\alpha = 0$. So, $i_Zd\alpha + di_Z\alpha = 0$, and hence $dS = di_Z\alpha = -i_Zd\alpha$.

(ii) If $p$ is a critical point of $S$ then $dS_p = 0$, and so $i_Zd\alpha = 0$ at $p$. Since $i_\xi d\alpha = 0$, we conclude that $Z_p$ and $\xi_p$ are proportional (because $d\alpha$ has one-dimensional kernel). Hence the $Z$-orbit through $p$ is a characteristic.

(iii) Since all $Z$-orbits are closed, every critical orbit is a closed characteristic, and the result follows from Proposition 2.2. □

2.4. Lemma. Consider the 1-form $\beta = \frac{\alpha}{S}$. The form $\beta$ is contact, and

$$i_Z(d\beta) = 0.$$  

Furthermore, the orbifold $M/S^1$ possesses a symplectic differential form $\omega$. 
Proof. We have
\[ d\beta = -\frac{1}{S^2} dS \wedge \alpha + \frac{1}{S} d\alpha \]
Hence
\[ \beta \wedge (d\beta^n) = \alpha \wedge (d\alpha)^n S^{-n} \neq 0 \]
everywhere on \( M \), and hence \( \beta \) is contact. Now (we use Proposition \ref{prop:contact}(i))
\[ d\beta = -\frac{1}{S^2} dS \wedge \alpha + \frac{1}{S} d\alpha = -\frac{1}{S^2} (izd\alpha) \wedge \alpha + \frac{1}{S} d\alpha = \frac{1}{S^2} (izd\alpha) \wedge \alpha + \frac{1}{S} d\alpha \]
Hence,
\[ izd\beta = \frac{1}{S^2} (izizd\alpha + (-1)izd\alpha \wedge iz\alpha) + \frac{1}{S} izd\alpha = 0 + \frac{-1}{S^2} Sizd\alpha + \frac{1}{S} izd\alpha = 0. \]
Since \( izd\beta = 0 \), the differential form \( d\beta \) yields the differential form \( \omega \) on the orbifold \( M/S^1 \), see \cite[Proposition B]{W}. Moreover, \( \omega \) is a symplectic form, see loc. cit. \( \square \)

2.5. Theorem. The vector field \( \xi \) has at least \( n+1 \) closed characteristics.

Proof. Let \( \omega \) be the above-mentioned symplectic form. Then, because of the de Rham Theorem for orbifolds, we have a cohomology class \( [\omega] \in H^2(M/S^1; \mathbb{R}) \), and \( [\omega]^n \neq 0 \). So, the cup-length of \( M/S^1 \) is at least \( n \). Hence \( \text{cat}(M/S^1) \geq n \). Now the theorem follows directly from Proposition \ref{prop:contact}. \( \square \)

3. K-contact structures and Seifert \( S^1 \)-bundles

To construct K-contact \((2n+1)\)-manifolds with at least \((2n+1)\) closed characteristics, we use a particular type of smooth 4-orbifolds called cyclic. We refer to \cite{MRT} for a detailed description and terminology, as well as to Appendix A. Clearly, any smooth manifold yield a "trivial" orbifold structure, where each \( \Gamma_\alpha \) is trivial groups. On the other hand, generally, a smooth orbifold is not a smooth manifold, see Definition A.1(b). However, there is a method to produce a smooth orbifold from a smooth manifold. This is given by \cite[Proposition 4]{MRT} below.

3.1. Proposition. Let \( X \) be a smooth oriented 4-manifold with embedded closed surfaces \( D_i \) intersecting transversely. Given integers \( m_i > 1 \) such that \( \gcd(m_i, m_j) = 1 \) if \( D_i \cap D_j \neq \emptyset \). Then there is a smooth orbifold \((X, U)\) with isotropy surfaces \( D_i \) of multiplicities \( m_i \).

3.2. Definition. Let \((X, U)\) be a cyclic, oriented \( n \)-dimensional orbifold. A Seifert \( S^1 \)-bundle over \((X, U)\) is an oriented \((n+1)\)-dimensional manifold \( M \) equipped with a smooth \( S^1 \)-action and a continuous map
\[ \pi : M \to \text{quotient} \to M/S^1 \cong X \]
with the following property: for every orbifold chart \((\tilde{U}, \phi, \mathbb{Z}_m)\), there is a commutative diagram
\[
\begin{array}{ccc}
(S^1 \times \tilde{U})/\mathbb{Z}_m & \cong & \pi^{-1}(U) \\
\pi \downarrow & & \pi \downarrow \\
\tilde{U}/\mathbb{Z}_m & \cong & U
\end{array}
\]
where the action of \( \mathbb{Z}_m \) on \( S^1 \) is by multiplication by \( \zeta_m := e^{2\pi i/m} \) and the top diffeomorphism is \( S^1 \)-equivariant.
3.3. Remark. We cohere the orientations of $X$ and $M$ as follows. Note that $S^1$ has a canonical (counterclockwise) orientation. Now, an orientation of $X$ gives an oriented orbifold chart $U$. We equip $S^1 \times U$ with the product orientation. This gives us a local orientation on $M$, and so we get the desired orientation on $M$ (since $M$ is orientable because of the construction).

Suppose in the following that $(X,U)$ is a 4-dimensional orbifold and $\pi : M \to X$ is a Seifert $S^1$-bundle over $X$. According to the normal form of the $\mathbb{Z}_m$-action given in (A.1), the open subset $\pi^{-1}(U) \cong (S^1 \times \tilde{U})/\mathbb{Z}_m$ is parametrized by $(u,z_1,z_2) \in S^1 \times \mathbb{C}^2$, modulo the $\mathbb{Z}_m$-action $\xi \cdot (u,z_1,z_2) = (\xi u, \xi^{j_1} z_1, \xi^{j_2} z_2)$, for some integers $j_1, j_2$, where $\xi = e^{2\pi i/m}$. The $S^1$-action on $M$ is given by $s \cdot (u,z_1,z_2) = (su,z_1,z_2)$, so $\mathbb{Z}_m \subset S^1$ is the isotropy group and the exponents $j_1, j_2$ are determined by the $S^1$-action.

Following [MRT], define an orbit invariant of the Seifert $S^1$-bundle to be a finite set of triples $\{(D_i,m_i,j_i)\}$ where (see Definitions A.2 and A.3)

- $D_i$ is an isotropy surface in $X$;
- $m_i \in \mathbb{Z}^+$ is the multiplicities $D_i$;
- $j_i \in \mathbb{Z}^+$,

and the local model around a point $p \in D_i^\circ = D_i - \bigcup_{i \neq j} (D_i \cap D_j)$ is of the form $(S^1 \times \tilde{U})/\mathbb{Z}_m$, with action

$$\xi \cdot (u,z_1,z_2) = (\xi u, \xi^{j_1} z_1, \xi^{j_2} z_2), \quad D_i = \{z_2 = 0\}.$$ 

If the orbifold is smooth, then for a point $p \in D_i \cap D_j$, the local model is of the form $(S^1 \times \tilde{U})/\mathbb{Z}_m$, with the action

$$\xi \cdot (u,z_1,z_2) = (\xi u, \xi^{j_i} z_1, \xi^{j_2} z_2), \quad D_i = \{z_2 = 0\}, \quad D_j = \{z_1 = 0\}.$$ 

3.4. Definition. For a Seifert $S^1$-bundle $\pi : M \to X$, define its first Chern class as follows. Let $\mu = \mathbb{Z}_{m(X)}$, where $m(X) = \text{lcm}\{m(x) \mid x \in X\}$. Consider the circle bundle $M/\mu \to X$ and its Chern class $c_1(M/\mu) \in H^2(X;\mathbb{Z})$. Define

$$c_1(M/X) = \frac{1}{m(X)} c_1(M/\mu) \in H^2(X;\mathbb{Q}).$$

The next proposition is crucial in our constructions, since it shows that the orbit invariants determine the Seifert $S^1$-bundle globally, when $X$ is a smooth orbifold.

3.5. Proposition. Let $(X,U)$ be an oriented 4-manifold, and $D_i \subset X$ be closed surfaces in $X$ that intersect transversely. Let $m_i > 1$ be such that $\text{gcd}(m_i, m_j) = 1$ if $D_i \cap D_j \neq \emptyset$. Let $0 < j_i < m_i$ with $\text{gcd}(j_i, m_i) = 1$ for every $i$. Let $0 < b_i < m_i$ be such that $j_i b_i = 1 \mod m_i$. Finally, let $B$ be a complex line bundle over $X$. Then there exists a Seifert $S^1$-bundle $f : M \to X$ with the orbit invariant $\{(D_i,m_i,j_i)\}$ and the first Chern class

$$c_1(M/X) = c_1(B) + \sum_i \frac{b_i}{m_i} [D_i].$$

Proof. See [MRT Proposition 14].

3.6. Theorem. Let $(X,U)$ be a symplectic (almost Kähler) cyclic orbifold. Let $\omega \in H^2(X;\mathbb{Q})$ be a symplectic form on $X$ and $\pi : M \to X$ be the Seifert $S^1$-bundle with $c_1(M/X) = [\omega]$. Then there exists a $K$-contact structure $(M,\alpha,\xi,\Phi,g)$ such that $\pi^* (\omega) = d\alpha$. 

Proof. See [MRT Proposition 14].
4. Category weight and the main theorem

4.1. Definition (Rud1). Let $Y$ be a Hausdorff paracompact space, let $R$ be a commutative ring, and let $u \in H^q(Y; R)$ be an arbitrary element. We define the category weight of $u$ (denoted by $\text{wgt} u$) by setting

$$\text{wgt} u = \sup \left\{ k \mid \varphi^* u = 0 \text{ for every map } \varphi : A \to Y \text{ with } \text{cat } \varphi < k \right\}$$

where $A$ runs over all Hausdorff paracompact spaces.

4.2. Lemma. If $u \neq 0 \in H^*(Y; R)$ then $\text{cat } Y \geq \text{wgt } u$.

Proof. This follows directly from Definition 4.1 of $\text{wgt}$.

4.3. Lemma. For all $u, v \in H^*(Y; R)$ we have $\text{wgt}(u \lrcorner v) \geq \text{wgt } u + \text{wgt } v$.

Proof. See [RT, Theorem 1.5(v)].

Given a space $Y$ and an element $u \in H^n(X; R)$, the notation $u|_{\pi_2(Y)} = 0$ means that

$$\langle u, h(a) \rangle = 0 \text{ for every } a \in \pi_2(Y)$$

where $h : \pi_2(Y) \to H_n(Y)$ is the Hurewicz homomorphism and $\langle -, - \rangle$ is the Kronecker pairing.

4.4. Lemma. Let $Y$ be a finite CW-space, and let $u \in H^*(Y; R)$ be a cohomology class such that $u|\pi_2(Y) = 0$. Then $\text{wgt } u \geq 2$.

Proof. See [RT, Lemma 2.1(ii)].

Given a symplectic orbifold $(X, \omega)$, we say that $(X, \omega)$ is symplectically aspherical if $[\omega]|\pi_2(X) = 0$.

4.5. Theorem. Let $(X, \omega), \dim X = 2n$ be a symplectic cyclic orbifold with $[\omega] \in H^2(X; \mathbb{Q})$, and let $\pi : M \to X$ be a Seifert $S^1$-bundle with $c_1(M/X) = [\omega]$. Assume that $(X, \omega)$ is symplectically aspherical. Then the total space $M$ of the Seifert fibration admits a K-contact structure with at least $2n + 1$ closed characteristics.

Proof. It follows from Theorem 3.6 that $M$ admits a K-contact structure. So, it suffices to prove that $\xi$ possesses at least $2n + 1$ closed characteristics, where $\xi$ is the Reeb vector field of the K-contact structure. Since $X = M/S^1$, and because of Proposition 2.3 it suffices to prove that $\text{cat } X \geq 2n$. (Note that in this case $\text{cat } X = 2n$, because $\text{cat } Y + 1 \leq \dim Y$ for all connected $Y$).

We have $\text{wgt } [\omega] \geq 2$ by Lemma 4.4. Hence, by Lemma 4.3

$$\text{wgt } [\omega]^n \geq n \text{wgt } [\omega] \geq 2n.$$ 

Since $[\omega]^n \neq 0$ because of the symplecticity of the form $\omega$, we conclude that $\text{cat } X \geq \text{wgt } [\omega]^n \geq 2n$ by Lemma 4.2.
5. Examples of K-contact manifolds with at least $2n + 1$ closed characteristics

We present examples of K-contact manifolds with at least $2n + 1$ closed characteristics as total spaces of Seifert fibrations over symplectic orbifolds. First, a big source of K-contact manifolds is given by [BG, Theorem 6.1.26]. Such manifolds can be described as the total spaces of so-called Boothby–Wang fibrations. These are circle bundles over symplectic manifolds $(X, \omega)$ with integral symplectic form $\omega$. The first Chern class of such bundle is equal to the integral lift of the cohomology class $[\omega]$. If one takes a symplectically aspherical base, one obtains an example of a K-contact manifold with at least $2n + 1$ closed characteristics.

To get a K-contact manifold which do not appear as a Boothby–Wang fibration, we can take any symplectically aspherical 4-manifold with a collection of codimension 2 symplectic submanifolds which intersect transversely and choose the orbit invariant, according to Proposition 3.5 and Theorem 3.6.

5.1. Example. Consider the Kodaira–Thurston manifold. It is defined as a nilmanifold of the form $N/\Gamma$ where $N = N_3 \times \mathbb{R}$, $N_3$ is a 3-dimensional Heisenberg group and $\Gamma = \Gamma_3 \times \mathbb{Z}$, $\Gamma_3$ denotes a co-compact lattice in $N_3$. It is easy to see and well known that $N/\Gamma$ is a symplectic manifold, and that $N/\Gamma$ is a symplectic fiber bundle over $\mathbb{T}^2$ with a symplectic fiber $F = \mathbb{T}^2$. Take, for example, the orbit invariant $(D, m, j)$ where $D = F$, $m$ is a chosen integer $m$, and $j < m$ with $\gcd(j, m) = 1$. It produced a Seifert $S^1$-bundle over a smooth orbifold $N/\Gamma$ determined by the cohomology class $[\omega]$ of the symplectic form $\omega$. The existence of the structure of the smooth orbifold determined by the data $(D, m, j)$ follows from Proposition 3.1. By Proposition 3.5 there is a Seifert bundle determined by $[\omega]$. By Theorem 3.6 there is a quasi-regular non-regular K-contact structure on the total space $M$ of this Seifert $S^1$-bundle over a smooth orbifold $N/\Gamma$ determined by $[\omega]$. The latter follows, since $N/\Gamma$ is aspherical.

5.2. Remark. Many more examples can be obtained by taking solvmanifolds $G/\Gamma$, that is, homogeneous spaces of solvable Lie groups $G$ over lattices $\Gamma \subset G$ (see, for example [TO], Chapter 3).

5.3. Example. Let $(X, \omega)$ be a closed symplectic manifold. Assume that the class $[\omega]$ is integral, and let $h$ denote its integral lift. Then, by a famous theorem of Donaldson [D], for $N$ large enough, the Poincaré dual of $Nh$ in $H_{2n-2}$ can be realized by a closed symplectic submanifold $D^{2n-2} \subset X^{2n}$. If one takes $(X, \omega)$ to be 4-dimensional and symplectically aspherical, and a surface $D \subset X$ guaranteed by Donaldson’s theorem, then one obtains the necessary Seifert bundle choosing the orbit invariant $(D, m, j < m)$ in the same manner as in the previous example. The total space $M$ of this Seifert bundle admits a K-contact structure with at least $5 = 2 \cdot 2 + 1$ closed characteristics.

6. Remarks on Rukimbira’s theorem

In [Ruk1, Theorem 2] Rukimbira formulated a theorem that any closed manifold which carries a Sasakian structure with finite number of closed characteristics must have $b_1 = 0$. In [BG] (Theorem 7.4.8) this theorem is formulated for K-contact...
manifolds (not necessarily Sasakian), however, with a small inaccuracy: Rukimbira’s proof goes through for K-contact manifolds (not necessarily Sasakian), except that the final result must be $b_1 \leq 1$.

### 6.1. Theorem (Ruk1)

Let $(M, \alpha, \xi, \Phi, g)$ be a closed K-contact manifold where $\alpha$ is a circle invariant form. Suppose that $\xi$ has only finite number of closed characteristics. Then $b_1(M) \leq 1$. If, in addition, $M$ is Sasakian then $b_1(M) = 0$.

**Proof.** For the convenience of the reader we give here a sketch (following Rukimbira). As in Section 2, consider the function $S = i_\xi(\alpha)$. By Proposition 2.2, critical circles of $S$ are closed characteristics of $\xi$. Consider the set $F_\xi$ consisting of periodic points of $\xi$. It is proved in Rukimbira [Ruk1, page 354] that $F_\xi$ is a union of closed characteristics and each component of $F_\xi$ is a totally geodesic odd-dimensional submanifold of $M$. Take a connected component of $F_\xi$ and call it $N$. Following Rukimbira [Ruk1, page 354], compute the Hessian $\text{Hess}_S$ of $S$ on vectors orthogonal to $N$. Note that the calculation does not use the integrability condition, and is valid not only for Sasakian manifolds, but for arbitrary K-contact manifolds as well. It turns out that for each critical point, $x \in N$, the Hessian, $d^2S_x$, of $S$ is non-degenerate in directions normal to $N$ at $x$, ($S$ is a clean function in terminology [GS]) and each of its critical points has even index [Ruk1, Prop. 2]). Again, the proof is based only on computing the Hessian, and is valid for any K-contact manifold.

By assumption, $\xi$ has only finite number of closed characteristics, and so $M$ has a finite number of critical $Z$-circles. Now, writing down the Morse inequalities as in [Ruk1, §5], one obtains

$$\dim H^1(M, \mathbb{R}) \leq 1.$$ 

Finally, the first Betti number of any Sasakian manifold is even (the zero case included), [1], and so $b_1(M) = 0$ provided that $M$ is Sasakian. □

### 6.2. Corollary

Let $(M, \alpha, \xi, \Phi, g)$ be a closed K-contact manifold where $\alpha$ is a circle invariant form. If $b_1(M) > 1$, then the Reeb vector field of the K-contact structure has infinite number of closed characteristics.

### 6.3. Question

If a closed K-contact $(2n+1)$-manifold admits exactly $n+1$ closed characteristics, then it is finitely covered by the sphere $S^{2n+1}$, Rukimbira [Ruk3, Ruk4]. Can one characterize $(2n+1)$-manifolds which admit exactly $2n+1$ closed characteristics?

### 6.4. Question

In view of Theorem 6.1, we pose the following question: Are there examples of K-contact manifolds with $b_1(M) = 1$ and with finite number of closed characteristics?

If we remove the requirement of finiteness of closed characteristics, one can easily produce examples of K-contact closed manifolds with any prescribed $b_1(M)$ (and, moreover, any finitely presented $\pi_1(M)$), using the following two results.

### 6.5. Theorem (G)

Any finitely presented group $\Gamma$ can be realized as a fundamental group of a closed symplectic 4-dimensional manifold.

### 6.6. Theorem (HT)

Let $(X, \omega)$ be a closed symplectic manifold. If

$$S^1 \to P \to X$$

then
is a Boothby–Wang fibration with the Chern class equal to $[\omega]$, then the total space $M$ of the fiber bundle

$$S^3 \to P \times _{S^1} S^3 \to X$$

associated with $P \to X$ by the Hopf action of $S^1$ on $S^3$ admits a K-contact structure.

Now, we see that $M$ is a compact K-contact manifold with $\pi_1(M) = \pi_1(X)$. In particular, $b_1(M) = b_1(X)$, with any prescribed $b_1(X)$ because of Theorem 6.5.

A more refined question comes from the fact that the full description of fundamental groups of symplectically aspherical manifolds is still an open question. Therefore, it is not easy to describe Seifert bundles (and even principal circle bundles) over symplectically aspherical orbifolds (manifolds) with $b_1(M) \leq 1$. Partial results were obtained in [IKRT, KRT]. Below we give other examples. To do this, we need the Sullivan model theory. For the convenience of the reader, we expose the Sullivan theory in Appendix B.

6.7. Example. Here we recall some facts about symmetric spaces [He] and lattices in Lie groups [R]. Let $G$ be a simple non-compact Lie group and $K$ be a maximal compact subgroup of $G$. The homogeneous space $X := G/K$ is an irreducible symmetric space. In particular, if one considers the case of Hermitian symmetric space $X$, then it admits a Hermitian metric $h$, which is Kähler, $G$-invariant, and whose real part is a Riemannian metric on $X$ as a Riemannian symmetric space. The list of such spaces is given in [He, Table II in Chapter IX]. For example, the Hermitian symmetric spaces with classical $G$ are:

$$SU(p,q)/S(U_p \times U_q), \ SO^*(2n)/U(n),$$

$$SO_0(p,q)/(SO(p) \times SO(q)), \ Sp(n,\mathbb{R})/U(n).$$

Hence, $h = g + i\omega$, where $\omega$ is a $G$-invariant Kähler (hence, symplectic) form. There exists a co-compact lattice $\Gamma$ in $G$ which acts freely and properly on $G/K$. The latter is well known: on one hand, since $G$ is simple, it admits a co-compact lattice [R, Chapter 14], and on the other hand, since $K$ is a maximal compact subgroup, any such lattice admits a finite index sublattice acting freely on $G/K$ (Selberg’s lemma). Therefore, $X = \Gamma\backslash G/K$ is a compact symplectic aspherical manifold. The total space $M$ of the Boothby-Wang fibration determined by $[\omega]$ is a K-contact manifold. Note that $\Gamma \subset G$ is a lattice in a simple Lie group $G$, therefore, $[\Gamma,\Gamma] = \Gamma$. Hence $b_1(X) = 0$, since $\pi_1(X) \cong \Gamma$. It turns out that $b_1(M) = 0$ as well. The latter can be seen, for example, as follows. Since $b_1(X) = 0$, the minimal model $(AV,d)$ of $X$ does not have any generator of degree one, which is a cocycle. Since $M$ is a principal circle bundle over $X$, the relative Sullivan model of this fiber bundle has the form

$$\Lambda V \to (AV \otimes \Lambda(t), D) \to (\Lambda(t),0), \ (6.1)$$

where $(AV,d)$ is the minimal model of $X$, and (over $\mathbb{Q}$)

$$D| AV = d, D|t = v, |v| = 2, [v] = [\omega] \in H^2(X;\mathbb{Q}) \cong H^2(AV,d).$$

We have $[\omega] = [\omega]$ by [FHT, Example 4 in Section 15]. Observe that the relative Sullivan model of the form $(AV,d)$ cannot create elements of degree 1 which are cocycles. Hence, $b_1(M) = 0$, as required.

Note that the manifold $M$ is actually Sasakian, since $X$ is Kähler.
6.8. Question. Are there Seifert fibrations over symplectically aspherical manifolds which are K-contact, non-Sasakian, and admit K-contact structures with finite number of closed characteristics?

A. Orbifolds

A.1. Definition ([BG], [MRT]). (a) An $n$-dimensional (differentiable) orbifold is a pair $(X, U)$ where $X$ is a space and $U$ is an atlas $U = \{(U_\alpha, \phi_\alpha \Gamma_\alpha)\}$. Here $U_\alpha \subset \mathbb{R}^n$, $U_\alpha \subset X$, $\Gamma_\alpha$ is a finite subgroup of $GL(n, \mathbb{R})$, and $\phi_\alpha : U_\alpha \to U_\alpha$ is a $\Gamma_\alpha$-invariant map which induces a homeomorphism $\tilde{U}_\alpha/\Gamma_\alpha \cong U_\alpha$ onto an open set $U_\alpha$ of $X$. (Here $\Gamma_\alpha$ acts linearly on $\tilde{U}$ and trivially on $U$.) There is also a condition of compatibility of charts: for each point $p \in U_\alpha \cap U_\beta$ there is some $U_\gamma \subset U_\alpha \cap U_\beta$ with $p \in U_\gamma$, monomorphisms $\gamma_\alpha : U_\gamma \to U_\alpha$, $\gamma_\beta : U_\gamma \to U_\beta$, and open embeddings $f_\gamma : U_\gamma \to \tilde{U}_\alpha$, $f_\gamma : U_\gamma \to \tilde{U}_\beta$, which satisfy $\gamma_\alpha(g)(f_\gamma(x)) = f_\gamma(g(x))$ and $\gamma_\beta(g)(f_\gamma(x)) = f_\gamma(g(x))$, for $g \in \Gamma_\gamma$.

For brevity, sometimes we will write $X$ instead of $(X, U)$ if there is no danger of confusion.

(b) We call $x \in X$ a smooth point if a neighborhood of $x$ is homeomorphic to a ball in $\mathbb{R}^n$, and singular otherwise. We say that an orbifold $(X, U)$ is smooth if all its points are smooth. This is equivalent to say that $X$ is a topological manifold.

As the groups $\Gamma_\alpha$ are finite, we can arrange (after conjugations if necessary) that $\Gamma_\alpha \subset O(n)$.

A.2. Definition. The orbifold is orientable if $\Gamma_\alpha \subset SO(n)$ for all $\alpha$ and the embeddings $f_\gamma$ preserve orientation. Note that for any point $x \in X$, we can always arrange a chart $\phi : \tilde{U} \to U$ with $\tilde{U} \subset \mathbb{R}^n$ being a ball centered at $0$ and $\phi(0) = x$, and $\tilde{U}/\Gamma \cong U$, with $\Gamma \subset SO(n)$. In this case, we call $\Gamma$ the isotropy group at $x$. We call an orbifold cyclic if all its isotropy groups are cyclic groups $\Gamma \cong \mathbb{Z}_m$, and $m = m(x)$ is the order of the isotropy at $x$. We call $x \in X$ a regular point if $m(x) = 1$, otherwise we call it a (non-trivial) isotropy point. Clearly a regular point is smooth, but not the converse.

From here and till the end of the section, let $X$ be a 4-dimensional closed orientable cyclic orbifold. Take $x \in X$ and a chart $\phi : \tilde{U} \to U$ around $x$. Let $\Gamma = \mathbb{Z}_m \subset SO(4)$ be the isotropy group. Then $U$ is homeomorphic to an open neighborhood of $0 \in \mathbb{R}^4/\mathbb{Z}_m$. A matrix of finite order in $SO(4)$ is conjugate to a diagonal matrix in $U(2)$ of the type

$$\begin{pmatrix} \exp(2\pi ij_1/m), \exp(2\pi ij_2/m) \end{pmatrix} = (\xi^{j_1}, \xi^{j_2}),$$

where $\xi = e^{2\pi i/m}$. Therefore we can suppose that $\tilde{U} \subset \mathbb{C}^2$ and $\Gamma = \mathbb{Z}_m = \langle \xi \rangle \subset U(2)$ acts on $\tilde{U}$ as

$$\xi \cdot (z_1, z_2) := (\xi^{j_1} z_1, \xi^{j_2} z_2).$$

Here $j_1, j_2$ are defined modulo $m$. As the action is effective, we have $\gcd(j_1, j_2, m) = 1$.

A.3. Definition. We say that $D \subset X$ is an isotropy surface of multiplicity $m$ if

- $D$ is closed;
• there is a dense open subset \( D^o \subset D \) which is a surface;

• \( m(x) = m \) for all \( x \in D^o \).

A.4. Proposition. Let \( X \) be as above and \( x \in X \) with local model \( \mathbb{C}^2/\mathbb{Z}_m \). Then there are at most two isotropy surfaces \( D_i \), with multiplicity \( m_i \cdot m \), through \( x \). If there are two such surfaces \( D_1, D_2 \), then they intersect transversely and \( \gcd(m_1, m_1) = 1 \). The point is smooth if and only if \( m_1 m_2 = m \).

Proof. See [MRT, Proposition 2]. \( \square \)

Now we want to describe the local models for the \( \mathbb{Z}_m \)-action in two cases used in this work. For all other possibilities we refer to [MRT]. For an action given by \( [\xi] \), we set \( m_1 := \gcd(j_1, m_1), m_2 := \gcd(j_2, m_1) \). Note that \( \gcd(m_1, m_2) = 1 \), so we can write \( m_1 m_2 d = m_1 \), for some integer \( d \). Put \( j_1 = m_1 e_1, j_2 = m_2 e_2 \). Clearly \( c_1 = m_2 d \) and \( c_2 = m_2 d \) with \( d = \gcd(c_1, c_2) \).

In what follows we consider the following cases:

(a) There are two isotropy surfaces and \( x \) is a smooth point, \( m_1, m_2 > 1, d = 1 \). Let us see that the action is equivalent to the product of one action on each factor \( \mathbb{C} \). In this case \( c_2 = m_1 \) and \( c_1 = m_2 \). So \( \gcd(c_1, c_2) = 1 \) and \( m = c_1 c_2 \). The action is given by

\[
\xi \cdot (z_1, z_2) := (\exp(2\pi i e_1 / c_1) z_1, \exp(2\pi i c_2 / c_2) z_2).
\]

We see that

\[
\xi^{c_1} \cdot (z_1, z_2) = (z_1, \exp(2\pi i e_1 / c_2) z_2),
\]

\[
\xi^{c_2} \cdot (z_1, z_2) = (\exp(2\pi i e_2 / c_1) z_1, z_2),
\]

so, the surfaces \( D_1 = \{(z_1, 0)\} \) and \( D_2 = \{(0, z_2)\} \) have isotropy groups \( \langle \xi^{c_1} \rangle = \mathbb{Z}_{m_1} \) and \( \langle \xi^{c_2} \rangle = \mathbb{Z}_{m_2} \), respectively. In this case \( m = m_1 m_2, d = 1 \).

Note that \( \mathbb{Z}_m = \langle \xi^{c_1} \rangle \times \langle \xi^{c_2} \rangle \) if and only if \( d = \gcd(c_1, c_2) = 1 \). In this case the action of \( \mathbb{Z}_m \) decomposes as the product of the actions of \( \mathbb{Z}_{m_2} \) and \( \mathbb{Z}_{m_1} \) on each of the factors \( \mathbb{C} \). The quotient space is \( \mathbb{C}^2/\mathbb{Z}_m \cong \mathbb{C}/\mathbb{Z}_{m_2} \times \mathbb{C}/\mathbb{Z}_{m_1} \), which is homeomorphic to \( \mathbb{C} \times \mathbb{C} \), and hence \( x \) is a smooth point.

(b) There is exactly one isotropy surface and \( x \) is a smooth point. In this case \( m_2 = 1 \) and \( m_1 = m \). As \( d = 1 \), this is basically as case (b). The action is

\[
\xi \cdot (z_1, z_2) = (z_1, \exp(2\pi i j_2 / m) z_2). \]

There is only one surface \( D_1 = \{(z_1, 0)\} \) with non-trivial isotropy \( m \), and all its points have the same isotropy. The quotient \( \mathbb{C}^2/\mathbb{Z}_m \cong \mathbb{C} \times (\mathbb{C}/\mathbb{Z}_m) \) is topologically smooth.

In case (a), we can change the generator \( \xi = e^{2\pi i / m} \) of \( \mathbb{Z}_m \) to \( \xi' = \xi^k \) for \( k \) such that \( ke_i \equiv 1 \pmod{m_i}, i = 1, 2 \), so that

\[
\xi' \cdot (z_1, z_2) = \left( \exp \left( \frac{2\pi i}{m_2} \right) z_1, \exp \left( \frac{2\pi i}{m_1} \right) z_2 \right).
\]

With this new generator, the action has model \( \mathbb{C}^2 \) with the action

\[
\xi \cdot (z_1, z_2) = (\xi^{m_1} z_1, \xi^{m_2} z_2), \quad \xi = e^{2\pi i / m}.
\]

Similar remark applies to case (b).
B. Sullivan models

We use Sullivan models of fibrations as a tool of calculating cohomology in some examples. In the sequel our notation follows [FHT]. In this section we denote by $\mathbb{K}$ an arbitrary field of characteristic zero. We consider the category of commutative graded differential algebras (or, in the terminology of [FHT], cochain algebras). If $(A, d)$ is a cochain algebra with a grading $A = \oplus_p A^p$, the degree $p$ of $a \in A^p$ is denoted by $|a|$.

Given a graded vector space $V$, consider the algebra $\Lambda V = S(V^\text{even}) \otimes A(V^\text{odd})$, that is, $\Lambda V$ denotes a free algebra which is a tensor product of a symmetric algebra over the vector space $V^\text{even}$ of elements of even degrees, and an exterior algebra over the vector space $V^\text{odd}$ of elements of odd degrees.

We will use the following notation:
- by $\Lambda^\leq p$ and $\Lambda^> p$ are denoted the subalgebras generated by elements of degree $\leq p$ and of degree $> p$, respectively;
- if $v \in V$ is a generator, $\Lambda v$ denotes the subalgebra generated by $v \in V$,
- $\Lambda^p V = \langle v_1, \ldots, v_p \rangle$, $\Lambda^{\geq q} V = \oplus_{\geq q} \Lambda^1 V$, $\Lambda^+ V = \Lambda^{\geq 1} V$.

B.1. Definition. A Sullivan algebra is a commutative graded differential algebra of the form $(\Lambda V, d)$, where
- $V = \oplus_{p \geq 1} V^p$;
- $V$ admits an increasing filtration
  \[ V(0) \subset V(1) \subset \cdots \subset V = \bigcup_{k=0}^{\infty} V(k) \]
  such that $d = 0$ on $V(0)$ and $d : V(k) \to \Lambda V(k - 1), k \geq 1$.

B.2. Definition. A Sullivan algebra $(\Lambda V, d)$ is called minimal, if
\[ \text{Im } d \subset \Lambda^+ V \cdot \Lambda^+ V. \]

B.3. Definition. A Sullivan model of a commutative graded differential algebra $(A, d_A)$ is a morphism
\[ m : (\Lambda V, d) \to (A, d_A) \]
inducing an isomorphism $m^* : H^*(\Lambda V, d) \to H^*(A, d_A)$.

If $X$ is a CW-complex, there is a cochain algebra $(A_{PL}(X), d_A)$ of polynomial differential forms. For a smooth manifold $X$ we take a smooth triangulation of $X$ and as the model of $X$ the Sullivan model of $A_{PL}(X)$. If it is minimal, it is called the Sullivan minimal model of $X$. It is well known (see [FHT], Proposition 12.1) that any commutative cochain algebra $(A, d)$ satisfying $H^0(A, d) = \mathbb{K}$ admits a Sullivan model.

B.4. Definition. A relative Sullivan algebra is a graded commutative differential algebra of the form $(B \otimes \Lambda V, d)$ such that
- $(B, d) = (B \otimes 1, d), H^0(B) = \mathbb{K}$,
- $1 \otimes V = V = \oplus_{p \geq 1} V^p$,
- $V = \bigcup_{k=0}^{\infty} V(k)$, where $V(0) \subset V(1) \subset \cdots$,
- $d : V(0) \to B, d : V(k) \to B \otimes \Lambda V(k - 1), k \geq 1$.
Relative Sullivan algebras are models of fibrations. Let \( p : X \to Y \) be a Serre fibration with the homotopy fiber \( F \). Choose Sullivan models
\[
m_Y : (\Lambda V_Y, d) \to (\Lambda^P L(Y), d_Y), \quad m : (\Lambda V, d) \to (\Lambda^P L(F), d).
\]
There is a commutative diagram of cochain algebra morphisms
\[
A^P L(Y) \longrightarrow A^P L(X) \longrightarrow A^P L(F)
\]
\[
m_Y \uparrow \quad m \uparrow \quad m \uparrow
\]
\[
(\Lambda V_Y, d) \longrightarrow (\Lambda V \otimes \Lambda V, d) \longrightarrow (\Lambda V, d)
\]
in which \( m_Y, m, m \) are all Sullivan models (see \[FHT\], Propositions 15.5 and 15.6)).

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