Pseudo-Gap Phase, the Quantum-Critical Point and superconductivity in Copper-Oxide Metals

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Abstract

A systematic solution of a general model for Copper-Oxides reveals a line of transitions $T = T_p(x)$ for $x$, the doping away from half-filling less than a critical value, to a phase with broken time reversal and rotational symmetry but with translational symmetry preserved. The single-particle spectrum in this phase is calculated to have a gap of $d_{x^2−y^2}$ symmetry. The properties in this phase are compared to the properties of the so called "pseudo-gap phase" in the Copper-Oxides. The fluctuations towards this phase promote superconductivity which is either of the $d_{x^2−y^2}$ or the generalised s-symmetry depending on whether the projected density of states at the chemical potential is larger in the $d_{x^2−y^2}$ channel, as in the hole-doped copper-oxides, or in the s-wave channel.
A schematic phase-diagram of the Copper-Oxide (CuO) metals is shown in fig.(1). The superconducting region is surrounded by three regions with distinct properties: A region marked (III) with properties characteristic of a Fermi-liquid, a region marked (I) in which a Fermi-surface is discerned but in which the quasiparticle concept is inapplicable, and a region marked (II), the so-called pseudo-gap region, in which the concept of a Fermi-surface itself is lost.

The topology of fig.(1) around the superconducting region is that expected around a quantum critical point \([\xi]\) in itinerant Fermions. The marginal Fermi-liquid (MFL) phenomenology \([\xi]\) with which many of the unusual properties in region (I) are understood assumes a scale-invariant low energy fluctuation spectrum characteristic of a quantum-critical point at \(x = x_c\), the composition near the highest \(T_c\). A region of Fermi-liquid is then expected for \(x > x_c\) at low temperatures, as in region (III), and a region with a broken symmetry for \(x < x_c\) at low temperatures. As in heavy-Fermion compounds, \([\xi]\) a region of superconductivity is found at low temperatures peaked in the region around the quantum critical point. From this point of view, the crucial question in CuO metals is the symmetry of the phase in the region (II) below \(T_p(x)\).

A systematic theory starting with a very general model for CuO provides an answer \([\xi]\) to this question. The region (II) in Fig. (1) is derived to be a phase in which a four-fold pattern of current flows in the ground state in each unit-cell as shown in Fig. (2). Four-fold symmetry around the copper-atoms as well as time-reversal symmetry are broken while their product as well as lattice translation symmetry are preserved. This phase will be referred to as the Circulating Current (CC) Phase. The properties of this phase were not studied in Ref. (4), which was mainly concerned with the fluctuations leading to the MFL properties in region (I) and the superconductivity induced by them.

In the last few years, the properties in region (II) have become much clearer, thanks specially to Angle-Resolved Photoemission (ARPES) \([\xi]\) and thermodynamic measurements. \([\xi]\) A pseudo-gap in the single particle spectra, consistent with \(d_{x^2-y^2}\) symmetry begins to develop at \(T < T_p(x)\) which is similar to the temperature at which the specific heat coefficient \(\gamma(T)\) \([\xi]\) and the magnetic susceptibility \(\chi(T)\) \([\xi]\) begin to decrease and
the resistivity [8] and optical [9] and Raman-spectra [10] begin to drop below the MFL behavior of region (I). I calculate in this paper that the (CC) phase has a single-particle spectra with a gap of \( d_{x^2-y^2} \) symmetry, as observed, and of the right order of magnitude, and from which the other properties in region (II) follow. The principal features of the phase-diagram in fig.(1) then arise in systematic calculations from a single model. I suggest further experiments to confirm this identification. (Numerous ideas [11] already suggested for the pseudo-gap phase are discussed elsewhere [12]).

### THE MODEL

The CC phase is a mean-field solution of a general Hamiltonian in the basis of three orbitals per unit cell, \( d, p_x, p_y \): [13]

\[
H = K + H_{\text{int}}^{(1)} + H_{\text{int}}^{(2)}
\]  

\[
K = \sum_{k,\sigma} \epsilon_d n_{dk\sigma} + 2t_{pd} d_{k,\sigma}^+ (s_x(k)p_{sk\sigma} + s_y(k)p_{yk\sigma}) - 4t_{pp} s_x(k) s_y(k) p_{sk\sigma}^+ p_{yk\sigma} + \text{H.C.}
\]

Here a particular choice of the relative phases of the x and y orbitals in the unit cell has been made, \( s_{x,y}(k) = \sin(k_x a/2, k_y a/2) \) and for later, \( c_{x,y}(k) = \cos((k_x a/2, k_y a/2) \) and \( s_{xy}^2(k) = (\sin^2(k_x a/2 + \sin^2(k_y a/2)) \). I consider the local interaction on the Cu and the O orbitals,

\[
H_{\text{int}}^{(1)} = \sum_{l,\sigma} U_d n_{dl\sigma} n_{dl\neg\sigma} + U_p (n_{p_xl\sigma} n_{p_xl\neg\sigma} + n_{p_yl\sigma} n_{p_yl\neg\sigma})
\]

and the nearest neighbor interaction between the Cu and the O orbitals,

\[
H_{\text{int}}^{(2)} = 2V \sum_{kk',q,\sigma,\sigma'} c_x(q) d_{k+q\sigma}^+ d_{k\sigma} p_{sk'-\sigma}^+ p_{sk'\sigma'} + x \to y
\]

More general interactions do not change the essential results derived here. Throughout this paper (renormalised) energy difference \( \epsilon_d \) between the Cu and the O orbitals is taken zero. It is important that in CuO, \( \epsilon_d \lesssim O(t_{pd}) \). Taking it as zero simplifies the calculations presented; the principal effect of a finite \( \epsilon_d \) will be mentioned.
THE CIRCULATING CURRENT PHASE

First, I derive some results of Ref. (4) in a simpler way by a simpler treatment of the large on-site repulsions in Eq. (3) and calculate the phase diagram of the CC phase. In the limit \((U_d, U_p) \gg (t_{pd}, t_{pp})\), a good mean-field approximation \[14\] for low density of holes (or electrons) consists in replacing

\[
\begin{align*}
t_{pd} & \rightarrow \bar{t}_{pd} = t_{pd}|x| \\
t_{pp} & \rightarrow \bar{t}_{pp} = t_{pp}|x|
\end{align*}
\]

where \(|x|\) is the deviation from half-filling in the conduction band: \(x > 0\) for holes and \(x < 0\) for electrons. A more general treatment would consider separately the average occupation in the oxygen and copper orbitals and renormalize \(t_{pd}, t_{pp}\) accordingly. This is an unnecessary complication - no new physical principle is involved and small quantitative modifications are expected since the average occupations of Cu and O orbitals in CuO-metals are quite close (near optimum doping).

A mean-field (non-superconducting) order parameter is sought which does not break translational symmetry (nor the rotational symmetry of spins). This means that the mean-field Hamiltonian is just a change of the coefficients in the kinetic energy operator \(K\). All the possible mean-field decompositions of the interactions except one only change the magnitude of the coefficients while preserving the symmetry. The only interesting mean-field decomposition comes from \(H_{int}^{(2)}\) and yields the complex mean-field order parameter

\[
\text{Re}^{i\phi} \equiv V/2 \sum_{k\sigma} s_x(k) \left\langle d_{k\sigma}^+ p_{xk\sigma} \right\rangle - s_y(k) \left\langle d_{k\sigma}^+ p_{yk\sigma} \right\rangle
\]  

The mean-field Hamiltonian itself is

\[
H_{mf} = K - \text{Re}^{-i\phi} \sum_{k\sigma} \left( s_x(k) d_{k\sigma}^+ p_{xk\sigma} - s_y(k) d_{k\sigma}^+ p_{yk\sigma} \right) + \text{H.C.} + \frac{R^2}{2V}
\]  

By a unitary transformation

\[
\begin{align*}
d_{k\sigma} & \rightarrow d_{k\sigma}, \quad p_{xk\sigma} \rightarrow p_{xk\sigma} e^{i\theta_x}, \quad p_{yk\sigma} \rightarrow p_{yk\sigma} e^{i\theta_y} \\
4t^2_{pd,xy} & = (2\bar{t}_{pd} \pm \cos \phi)^2 + R^2 \sin^2 \phi
\end{align*}
\]

\[
\tan \theta_{x,y} = \pm \cos \phi / (2\bar{t}_{pd} \pm \cos \phi)
\]
the phase difference is transferred completely to the $p_x - p_y$ bond, so that $H_{mf} \rightarrow H_\theta$,

$$H_\theta = \sum_{k\sigma} 2 \left[ d_{k\sigma}^\dagger \left( \hat{t}_{pdx} s_x(k) p_{x\sigma} + \hat{t}_{pdy} s_y(k) p_{y\sigma} \right) \right] - 4\bar{t}_{pp} e^{i\theta} s_x(k) s_y(k) p_{x\sigma} p_{y\sigma} + H.C. + R^2/2V,$$

(10)

where $\theta = \theta_x - \theta_y$. It is apparent that only the phase difference of the two $p - d$ bonds in the unit cell is gauge-invariant. With the most general interaction terms, the gauge-invariant phase is the change in phase going around the plaquette formed by a Cu-atom and any two of its oxygen neighbors, which differs from the above only in the value of $\theta$.

The energy of each of the three bands obtained by diagonalising Eq. (7) is changed by a finite $(R, \phi)$. But for any $k$ the trace of the change in energy of the three bands is zero. The change in energy can therefore be expressed purely in terms of the change in energy $\delta \epsilon_{ck}$ of the filled part of the (hole) conduction band. So the mean-field values $R_0$ and $\phi_0$ are determined by minimizing

$$\frac{R^2}{2V} - 2 \sum_k \delta \epsilon_{ck} (\phi, R) f(\epsilon_{ck}).$$

(11)

$\delta \epsilon_{ck}$ has a complicated expression. To $O(R^2)$ and leading order in $(\bar{t}_{pp}/t_{pd})$ it is

$$\delta \epsilon_{ck} \approx \left( \frac{R}{2t_{pd}} \right)^2 \left( \bar{t}_{pd} s_{xy}^2(1 - 1/4\cos^2 \phi) + \frac{4(\bar{t}_{pp} s_x^2(k) s_y^2(k))}{s_{xy}^2(k)} (1 - \frac{1}{2} \cos^2 \phi) \right).$$

(12)

From Eq. (11) and (12) $R_0 \neq 0$ at $T = 0$ for

$$\frac{2|\chi|\bar{t}_{pd}}{V} < \sum_{k<k_F} \left( s_{xy}^2(k) + \frac{8\bar{t}_{pp} s_x^2(k) s_y^2(k)}{t_{pd}} \right),$$

(13)

and $\phi_0$ is $\pi/2$ or $-\pi/2$. The symmetry of the transition is therefore of the Ising variety.

For $|\chi| << 1$, Eq.(13) is satisfied only for $x$ less than critical doping $|\chi_c|$, 

$$|\chi_c| \approx \frac{1}{2} \left( \frac{V}{t_{pd}} \right) (0.25 + 0.5t_{pp}/t_{pd}).$$

(14)

$|\chi_c|$ defines the quantum critical points for both electron and hole dopings. An estimate of $\theta_0 \equiv R_0/(2\bar{t}_{pd})$ obtained by expanding Eq. (12) to $O(R^4)$ is

$$\theta_0^2 = O \left( \frac{t_{pd}^2}{2[V(t_{pp} + t_{pd})]} (\chi_c - x) \right).$$

(15)
For $\epsilon_d \neq 0$, $t_{pd}^2$ is replaced by $t_{pd}^2 + O(\epsilon_d^2)$. With $V, t_{pp}, t_{pd},$ and $\epsilon_d$ of similar magnitude, as for CuO compounds, $x_{ec}, x_{hc}$ are therefore about 0.2. The line of transitions at finite temperatures varies with $x$ as

$$T_p/E_F \sim |x_c - x|^{1/2}$$

This is to be identified with $T_p(x)$ of fig.(1). The competition with antiferromagnetism near $x = 0$ is not treated here.

The energy of the (transverse) fluctuations in $\phi$ about $\phi_0$ at long wavelengths is estimated from Eq. (12) to be

$$\Omega_0^0 = O \left( \theta_0^2 (t_{pp} + t_{pd}) / 4 \right).$$

The variation of $\Omega_0^0$ with $q$ is estimated to be slow, of $O(qa)^2$.

The eigenvectors of the states in the conduction band to leading order in $\theta$ and $t_{pp}/t_{pd}$ are

$$|c_{k\theta\sigma} > = \frac{1}{N_k} \left[ |c_{ko\sigma} > + i 2 \sqrt{2} \theta_0 \frac{t_{pp}}{t_{pd}} s_x(k) s_y(k) |a_{ko\sigma} > \right]$$

where $|a_{ko} >$ and $|c_{ko} >$ are the nonbonding and conduction band states of $k$ for $\theta = 0$ and $N_k$ is the normalization factor. The term proportional to $\theta_0$ leads to a difference in the phase between the $|p_{xk\sigma} >$ and $|p_{yk}\sigma_\sigma >$ terms in Eq. (18) producing a time-reversal breaking state. Similarly expressions can be derived for the other two bands. The three together correspond to a current carrying state with pattern shown in Fig. (2). The wave function of states with $\phi_0 = -\pi/2$ is (18) with $-i\theta_0$ in the second term. Note that states with $\phi_0 = \pi/2$ and $-\pi/2$ are not orthogonal. But the two ground states formed from the Hartree-Fock product of such states are orthogonal.

The CC phase breaks a rotational invariance of the Cu-O lattice (besides the time-reversal invariance). So lattice defects, such as interstitials, dislocations or grain boundaries couple as external fields do to a $d = 2$ Ising order parameter. No thermodynamically sharp transition is therefore possible at $T_p(x)$. 

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ANISOTROPIC GAP IN THE NORMAL STATE

In the mean-field approximation above, the energy of states in the conduction band shift in a k-dependent fashion, Eq. (12) with nothing special happening at the chemical potential. This is because the source of the instability considered is interband transitions. This solution will now be shown to be modified by the fluctuations which lead to scattering among the conduction band states, (18), near the chemical potential. The fluctuations to be considered are transverse, i.e. of $\phi$, whose energy $\Omega_0$ at long wave-lengths is given by Eq.(17). The Hamiltonian for such fluctuations, generated by operators $\delta \phi_q, \delta \phi_q^+$, is quite generally obtained from the deviation of the starting Hamiltonian from $H_{mf}$.

$$H_{\text{fluct}} = \sum_q \Omega_0^q \delta \phi_q^+ \delta \phi_q + \sum_{kk',\sigma} g(k,k') c_{k',\theta \sigma}^+ c_{k,\theta \sigma} \left( \delta \phi_{k-k'} + \delta \phi_{k'-k}^+ \right) + \text{H.C.} \tag{19}$$

The coupling function to zeroth order in $t_{pp}/t_{pd}$ is calculated to be

$$g(k,k') = \theta_0 t_{pd} [s_x(k)s_x(k') - s_y(k)s_y(k')]/s_{xy}(k). \tag{20}$$

For $k \to k'$, this is $\sim (\cos(k_x a) - \cos(k_y a))$.

Consider now the renormalisation of the phase-fluctuation energy through Eq.(19). For small $q$

$$\Omega^2(q) = \Omega_q^{02} - 2\Omega_q^{0\alpha^2} \theta_0^2 (1 - q^2/8) \sum (\cos(k_x a) - \cos(k_y a)/s_{xy})^2 / (\epsilon_{k+q} - \epsilon_k), \tag{21}$$

with the restriction on the sum that $k < k_f$ and $k+q > k_f$. The self-energy of the fermions at $k = k_f$ and energy $\omega$ is $\sim \ln|\omega - \Omega^0_q|$ suggesting an instability if the fluctuation energy at long wavelength approaches 0. The conclusion from Eq.(21) and Eq. (17) is that the phase fluctuations can be unstable over a substantial part of momentum-space. A way to restore stability is by a condensation of the phase fluctuations and a corresponding modification of the electronic energy.

The condition for stability is derived by the ansatz that $< \delta \phi_q >$ and $< c_{k+q,\theta \sigma}^+ c_{k,\theta \sigma} >$ are finite for a range of $q$ around zero [16]. Eq. (19) then leads to new eigenstates which are annihilated/created by $\gamma_{k\sigma}, \gamma_{k\sigma}^+$:

$$\gamma_{k\sigma} = (c_{k,\theta \sigma} + \sum_{q \neq 0} u_{k,k+q} c_{k+q,\theta \sigma})/M_k, \tag{22}$$
where $M_k$ is the normalisation. It will turn out that $u_{k,k+q} \neq 0$ only for states with $|\langle \epsilon(k), \epsilon(k+q) \rangle - \mu | \lesssim \Omega_0^0$. For such states momentum is not a good quantum number; I label the new states by $k$ to indicate that the average momentum of such a state is $k$. In this sense, the problem retains translational invariance only on the average.

From Eq. (19),

$$< \delta \phi_q > = -\frac{1}{\Omega_0^0} \sum_k g(k, k + q) < c_{k+q,\theta\sigma}^+ c_{k\theta\sigma} > .$$

(23)

I use the Brillouin-Wigner (B-W) self-consistent approximation to get:

$$< c_{k+q,\theta\sigma}^+ c_{k\theta\sigma} > = g(k, k + q) < \delta \phi_q > / (E_k - E_{k+q}),$$

(24)

where $E_k$s are the new one-particle eigenvalues to be determined. The B-W approximation is exact in the limit that the number of states $(k + q)$ coupled to a given state $k$ is very large compared to 1, as here.

Combining Eqs.(23) and (24) yields the self-consistency equations

$$1 = \Omega_0^0 \sum_k |g(k, k + q)|^2 / (E_{k+q} - E_k),$$

(25)

$$\Phi(k, k + q) = g(k, k + q) / \Omega_0^0 \sum_{k',\sigma'} g(k', k' - q) / (E_{k' - q} - E_k') \Phi(k', k' - q),$$

(26)

where

$$\Phi(k, k + q) = (E_{k+q} - E_k) < c_{k+q,\theta\sigma}^+ c_{k\theta\sigma} > .$$

(27)

In Eqs. (23)-(26), the sum is restricted to states $k$ etc. such that $\epsilon(k)$ is within about $\Omega_0^0$ of the chemical $\mu$, as required by the retarded nature of the Fermion-Boson interaction. The restriction on the sums is more clearly seen if Eq.(19) is used to generate an effective frequency dependent Fermion vertex

$$\sum_{k,k',\sigma,\sigma'} g(k, k + q) g(k' - q, k') D_\phi(q, \omega) c_{k+q\theta\sigma}^+ c_{k\theta\sigma} c_{k'-q\theta'\sigma'}^+ c_{k'\theta'\sigma'.}$$

(28)

where $D_\phi(q, \omega)$ is the propagator for the fluctuations which on the energy shell may be approximated by $-1/\Omega_0^0$ with a cut-off such that $\epsilon(k')$, $\epsilon(k' + q)$ are both within about $\Omega_0^0$. 
of \( \mu \). A quadratic hamiltonian obtained by taking the expectation value of \(< c_{k+q,\sigma}^+ c_{k,\sigma} >\) etc. yields (26) using the B-W approximation.

Eqs. (26), (27) have been obtained from the "direct" contraction of Eq (22). The exchange contraction yields a similar equation with a coefficient which, in the relevant channel, is \((-1/4)\) of that on the right side of Eq (26).

For an (approximate) solution of Eqs. (25), (26), note first that since \( q \to 0 \), \( g(k, k + q) \sim (\cos(k_x a) - \cos(k_y a)) \), for \( q \to 0 \), \( \Phi(k, k + q) \sim (\cos(k_x a) - \cos(k_y a)) \) also. Second, \( E(k) \approx \epsilon(k) \) for \( |\epsilon(k) - \mu| \gtrless \Omega_0^0 \). A consistent solution for \( k \) near \( k_f \) is then

\[
E_k = \epsilon_k + D(k), \quad k \gtrsim k_f
\]

\[
E_k = \epsilon_k - D(k), \quad k \lesssim k_f
\]

\[
D(k) = D_0(\cos(k_x a) - \cos(k_y a))^2, \quad (29)
\]

as may be verified by substitution. The value of \( D_0 \) is evaluated to be of \( (\theta_0^2 \tilde{t}_{pd} \rho(0)) \).

The gap in the one particle spectrum at the chemical potential \( D(k) \) has a \( d_{x^2-y^2} \) symmetry and a magnitude which can be estimated from Eqs. (15) to be \( x(x_c - x)t_{pd}^2(t_{pd} + t_{pp})/(Vt_{pp}) \) which at say \( (x_c - x) \approx .05 \), and \( t_{pd} \sim V \sim t_{pp} \sim 1\text{eV} \) is \( \approx 20\text{meV} \). This has the right order of magnitude \[6]. The differences of (29) from D-wave BCS single-particle spectrum are significant and have observable consequences discussed below.

The leading decrease in energy due to the modification of the single-particle spectrum is \( \sim \theta_0^2 \). Therefore, the nature of the transition remains unchanged, at least in mean-field theory; only quantitative changes are introduced in the conditions Eqs. (16) and (17) for the occurrence of the CC phase.

**PROPERTIES IN THE CIRCULATING CURRENT PHASE**

The single particle density of states in the CC phase calculated from Eq. (29) is

\[
\rho_{cc}(\omega) = \rho(0) \frac{2}{\pi} \text{Arcsin} \left( |\frac{\omega}{D}|^{1/2} \right), \quad |\frac{\omega}{D}| \leq 1 \\
= \rho(0) |\frac{\omega}{D}| \quad (30)
\]

This increases as \( |\omega/D|^{1/2} \) for \( |\omega/D| < < 1 \) and unlike the d-wave superconductors, (which have a logarithmic singularity at \( \omega = \Delta \), the superconducting gap), \( \rho_{cc}(\omega) \) is less than
the "normal" density of states \( \rho(0) \) at all \( |\omega| < D \).

Eq. (30) should be compared with the single particle density of states measured in tunneling [17] and with the specific heat \( C_v \) and magnetic susceptibility \( \chi \). The former shows, (just as Eq (30)), a diminution in the single particle density of states for low energies at \( T \lesssim T_p(x) \) but show no rise above \( \rho(0) \) at finite energies until \( T \lesssim T_c(x) \), when the characteristic superconducting density of states appears. \( C_v \) in the CC phase is predicted \( \sim T^{3/2} \) and \( \chi \sim T^{1/2} \) for \( T << T_p(x) \). Due to the intervention of superconductivity, it is hard to test these power laws accurately. In the measured range \( T \chi/C_v \) is nearly independent of temperature as predicted here. \( \chi(T) \), measured more accurately than \( C_v(T) \), can be fit to \( T^{1/2} \). One can deduce the continuation to the \( T \)-dependence below \( T_c \) by invoking conservation of entropy on the \( C_v(T) \) measurements. \( C_v(T) \sim T^{3/2} \) for \( T << T_p(x) \) is then not inconsistent while \( C_v(T) \sim T \) clearly is.

To obtain the spectral function \( A(k, \omega) \) measured by ARPES [5] one needs besides Eq. (29), the self-energy of the single particle states (and the ‘coherence factors’). The situation is in some ways similar to but in an important way different from the corresponding calculation for a d-wave superconductor [18] (and completely different from s-wave superconductors). [13] In both cases, the bare polarizability \( \chi_0(q, \omega) \) (calculated from a single particle-hole bubble) is zero for \( \omega < D(q) \), (\( \Delta(q) \) for the superconductors). The lowest energy single particle scattering for momentum \( q \) occurs by an intermediate one-particle state near the zero of the gap. Therefore the threshold for single-particle scattering of a state at \( q \) is also \( D(q) \). For \( \omega \gtrsim D(q) \), \( \chi_0(q) \) with \( E_k 's \) given by Eq. (29) is proportional to \( \omega / D_{\text{ef}} \). So the renormalized \( \chi(q, \omega) \) for \( \omega > D(q) \) is expected to be similar to the normal state above \( T_p(x) \), i.e. of the marginal Fermi-liquid form. The single particle self-energy needs to be calculated in detail but its general form is evident: For \( (\omega, T) << D(q) \) it is exponentially small, but for \( (\omega, T) \gtrsim D(q) \) it returns to the value \( \Sigma_n(\omega, q, T) \) without the pseudo-gap. So

\[
\text{Im}\Sigma(\omega, q, T) \approx \text{sech} \left( \frac{D(q, T)}{(\omega^2 + \pi^2 T^2)^{1/2}} \right) \text{Im}\Sigma_n(\omega, q, T). \tag{31}
\]

The spectral function at the Fermi-wave-vectors \( \hat{k}_f \) defined by \( E(\hat{k}_f) = \pm D(\hat{k}_f) \) calculated using Eq. (29) and (31) and the marginal Fermi-liquid form for \( \Sigma_n \) is plotted in fig.(3) for a few temperatures [20]. A pseudogap in the direction \( \hat{k}_f \) appears below \( T \approx D(\hat{k}_f) \)
producing the illusion of ‘Fermi-arcs’ shrinking as temperature decreases. The lineshape in fig. (3) at low energies, in the pseudo-gap region, fits the experimental curves well within the experimental resolution. A prediction following from the results of the previous section is that \(D \sim (x - x_c)\) and also \(\sim (T_p(x) - T)\).

The single particle spectrum in \(d = 2\) d-wave superconductor only differs from the above because of a singularity in \(\chi_0(q, \omega)\) for \(\omega \approx 2\Delta\) and \(q \approx 2k_f\) in the directions connecting the maximums of the gap. This leads to a pole in \(\text{Re}\Sigma(\omega, q)\) and therefore a sharp peak in \(A(\omega, q)\) at \(q\) on the Fermi-surface close to the maximum of the gap followed by a continuum starting at \(\Delta\). No such sharp feature is to be expected in the pseudo-gap region because no singularity in \(\chi_0(q, \omega)\) exists.

SUPERCONDUCTIVITY

I discuss here the superconducting instability in the quantum fluctuation regime (Region I in Fig. (1)) in which \(\theta_0 = 0\) and effective electron-electron interaction is through fluctuation in \(\theta\). (The calculation in the CC phase is similar with the added complication that one must do pairing calculations in a gapped electronic structure.) The coupling of the \(\theta\) fluctuations to the Fermions has the same functional form as \(g(k, k')\) and leads to an effective pairing interaction in the spin-singlet channel:

\[
H_{\text{pair}} = \sum_{k,k'} |V(k, k')|^2 D_\theta(k - k', \omega) c_{k',\sigma}^\dagger c_{-k, -\sigma} c_{-k', \sigma} c_{k, \sigma} \tag{32}
\]

Here \(D_\theta\) is the propagator of the \(\theta\) fluctuations, which is essentially momentum independent and has a frequency dependence of a scale-invariant form. We may approximate it below its cut-off frequency by \(-(\omega_c)^{-1}\). The coupling function in Eq (32) is then expanded in a separable form in lattice-harmonics as follows:

\[
V_{\text{pair}}(k, k') = -V^2/(4\omega_c) \sum_i g_i \eta_i(k) \eta_i(k') \tag{33}
\]

where \(i\) are the generalised \(s, d_{x^2-y^2}\), and \(d_{xy}\) channels:

\[
\eta_s(k) = s^2_{xy}(k), g_s = 1/2
\]

\[
\eta_{d_{x^2-y^2}}(k) = 1/2[\cos(k_x a) - \cos(k_y a)], g_{d_{x^2-y^2}} = 1/2
\]

\[
\eta_{d_{xy}}(k) = s_x(k)s_y(k), g_{d_{xy}} = -2. \tag{34}
\]
The methods of Ref. (3), may be used now to deduce that pairing is possible only generalized s-wave and the \( d_{x^2-y^2} \) channels. The choice between the channels is determined by the band-structure near the Fermi-surface. In the hole-doped cuprates \( d_{x^2-y^2} \) is favored because in this symmetry the gap occurs over regions of Fermi-surface where the bands-disperse the least (because of the proximity to the Van Hove critical point) thereby reducing the energy optimally. If \( \mu \) is far away from the van Hove singularity, as in electron-doped cuprates [21] the projected generalised s-wave density of states is expected to be larger than the projected \( d_{x^2-y^2} \) density of states and s-wave pairing is favored.

**CONCLUDING REMARKS**

Although, I have presented a systematic theory in agreement with the principal experimental results, one can be confident of the applicability of the theory only if the structure in Fig. (2) is observed. I have suggested elastic polarized neutron scattering to observe the orbitals moments in Fig. (2). Another way to test Fig. (2) is through the polarization dependence of ARPES in single-domain samples. Some other predictions are mentioned in the text.

Improvements could be made in the theory, especially the derivation of the gap in the Circulating-current phase. Note that the conditions (26,27) are equivalent to requiring that the phase-fluctuation modes have zero-energy as \( q \to 0 \), which appears to give an \( xy \)-symmetry character to the circulating current state. If true, no specific heat anomaly is found even in the pure limit (except for three-dimensional couplings). The ansatz, Eq.(22), by which the problem has been diagonalised may be looked upon as admixing states with \( \phi_0 = -\pi/2 \) in to states with \( \phi_0 = \pi/2 \). It is possible that a better treatment would express the one-particle states as topological excitation of fermions bound to co-moving current fluctuation. These matters require further investigation.

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FIGURE CAPTIONS

Fig. (1): Generic phase-diagram of the cuprates for hole-doping. Not shown is a low temperature "insulating phase" in region II due to disorder.

Fig. (2): Current pattern predicted in phase II of Fig. (1).

Fig. (3): The spectral function at the Fermi-wave-vectors as a function of energy below the chemical potential normalized to the gap in the direction of that wave-vector $D(\hat{k}_\ell)$ for temperatures $T = nD(\hat{k}_\ell)/(4\pi)$ for $n = 1, 2, 3, 4, 5$. 
\[ \pi D(\hat{k}_F) A(\omega, \hat{k}_F) \]