On a Conjecture for a Hypergraph Edge Coloring Problem

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June 12, 2020

Abstract

Let $H = (M \cup J, E \cup E)$ be a hypergraph with two hypervertices $G_1$ and $G_2$ where $M = G_1 \cup G_2$ and $G_1 \cap G_2 = \emptyset$. An edge $[h, j] \in E$ in a bi-partite multigraph graph $(M \cup J, E)$ has an integer multiplicity $b_{jh}$, and a hyperedge $[g, j] \in E$, $\ell = 1, 2$, has an integer multiplicity $a_{j \ell}$. We limit ourselves to the just defined hypergraphs $H$ in this paper. It has been conjectured in [5] that $\chi(H) = \lceil \chi_f(H) \rceil$, where $\chi(H)$ and $\chi_f(H)$ are the edge chromatic number of $H$ and the fractional edge chromatic number of $H$ respectively. Motivation to study this hyperedge coloring conjecture comes from the University timetabling, and open shop scheduling with multiprocessors. We prove this conjecture in this paper.

1 The conjecture

Let $H = (M \cup J, E \cup E)$ be a hypergraph with two hypervertices $G_1$ and $G_2$ where $M = G_1 \cup G_2$ and $G_1 \cap G_2 = \emptyset$. An edge $[h, j] \in E$ in a bi-partite multigraph graph $(M \cup J, E)$ has an integer multiplicity $b_{jh}$, and a hyperedge $[g, j] \in E$, $\ell = 1, 2$, has an integer multiplicity $a_{j \ell}$. We limit ourselves to the just defined hypergraphs $H$ in this paper. It has been conjectured in [5] that $\chi(H) = \lceil \chi_f(H) \rceil$, where $\chi(H)$ and $\chi_f(H)$ are the edge chromatic number of $H$ and the fractional edge chromatic number of $H$ respectively, see [12] for more on the fractional graph theory. Observe that $G_1 = \emptyset$ or $G_2 = \emptyset$ results in the $\chi(H) = \chi_f(H) = \Delta(G_2) + \chi'(M \cup J, E)$ and $\Delta(G_1) + \chi'(M \cup J, E)$ respectively, where $\chi'(M \cup J, E) = \max\{\max_j\{\sum_b b_{jh}\}, \max_h\{\sum_j b_{jh}\}\}$ is the edge chromatic number of the bi-partite multigraph $(M \cup J, E)$, and $\Delta(G_1) = \sum_{j \in J} a_{j \ell}$ for $\ell = 1, 2$. Thus the conjecture holds in this case and we assume non-empty $G_1$ and non-empty $G_2$ from now on in the paper.

A feasible edge coloring in $H$ can be partitioned in the following four parts: part (a) includes matchings with hyperedges $(G_1, j)$ for some $j \in J$, and edges $(h, j)$ where $h \in G_2$ and $j \in J$; part (b) includes matchings with hyperedges $(G_1, j)$, and hyperedges $(G_2, jr)$ for some $j, jr \in J$; part (c) includes matchings with hyperedges $(G_2, j)$ for some $j \in J$, and edges $(h, j)$ where $h \in G_1$ and $j \in J$; and part (d) includes matchings with edges $(h, j)$ only where $h \in M$ and $j \in J$. The parts (a), (b), (c) and (d) have multiplicities $\Delta(G_1) - r, r, \Delta(G_2) - r, w$ respectively, for some $r$ and $w$. Therefore the total of $\Delta(G_1) + \Delta(G_2) - r + w$ colors are used, and the minimization of the number of colors required to color the edges of $H$ reduces to the minimization of $w - r$. Following the convention used in [4] and [5] we refer to $h \in M$ as machine $h$, and to $j \in J$ as job $j$ for convenience. It was shown in [4] that $\chi(H) > \chi_f(H)$ for some hypergraphs $H$, and in [5] that $[\chi_f(H)] + 1 \geq \chi(H)$ for each hypergraph $H$.

Let $m = |M|$ be the number of machines and $n = |J|$ be the number of jobs. Without loss of generality $J = \{1, \ldots, n\}$ and $M = \{1, \ldots, m\}$. The following integer linear program \( ILP \) with variables $r, w$, and $y_{jh}$,
x_{j\ell}$, for $j \in \mathcal{J}$, $h \in \mathcal{M}$, and $\ell = 1, 2$, and integer coefficients $b_{jh}$, $a_{j\ell}$, for $j \in \mathcal{J}$, $h \in \mathcal{M}$, and $\ell = 1, 2$, for $\ell = 1, 2$ was given in [4] and [5] to find $\chi'(H)$.

$$ILP = \min(w - r)$$

Subject to

$$\sum_j b_{jh} - (\Delta(G_2) - r) \leq \sum_j y_{jh} \leq w \quad h \in G_1$$ (1.2)

$$\sum_j b_{jh} - (\Delta(G_1) - r) \leq \sum_j y_{jh} \leq w \quad h \in G_2$$ (1.3)

$$\sum_h y_{jh} \leq w \quad j \in \mathcal{J}$$ (1.4)

$$0 \leq y_{jh} \leq b_{jh} \quad h \in \mathcal{M} \quad j \in \mathcal{J}$$ (1.5)

$$\sum_j x_{j1} = r$$ (1.6)

$$\sum_j x_{j2} = r$$ (1.7)

$$x_{j1} + x_{j2} \leq r \quad j \in \mathcal{J}$$ (1.8)

$$0 \leq x_{j\ell} \leq a_{j\ell} \quad j \in \mathcal{J} \quad \ell = 1, 2$$ (1.9)

$$\sum_{h \in G_1} (b_{jh} - y_{jh}) + a_{j2} - x_{j2} \leq \Delta(G_2) - r \quad j \in \mathcal{J}$$ (1.10)

$$\sum_{h \in G_2} (b_{jh} - y_{jh}) + a_{j1} - x_{j1} \leq \Delta(G_1) - r \quad j \in \mathcal{J}$$ (1.11)

The variable $y_{jh}$ represents the amount of $j \in \mathcal{J}$ on $h \in \mathcal{M}$ in part (d). The variable $x_{j\ell}$ represents the amount of $j \in \mathcal{J}$ on $G_{\ell}$, $\ell = 1, 2$, in part (b). The variable $w$ is the size of (d), and the variable $r$ is the size of (b). The constraints (1.2)-(1.5) guarantee that the size of part (d) does not exceed $w$. The constraints (1.6)-(1.9) guarantee that the size of part (b) equals $r$. The constraints (1.10)-(1.11) along with the left hand side inequalities in (1.2) and (1.3) guarantee that the size of part (a) does not exceed $\Delta(G_1) - r$ and that the size of part (c) does not exceed $\Delta(G_2) - r$.

Let $ILP$ be the value of optimal solution to this program, and let $LP$ be the value of optimal solution to the $LP$-relaxation of this program. The conjecture $\chi'(H) = \lceil \chi_f'(H) \rceil$ for each hypergraph $H$ is therefore equivalent to the following conjecture that we show in this paper to hold.

**Conjecture 1** $ILP = \lceil LP \rceil$.

Motivation to study this hyperedge coloring problem comes from the University timetabling studied in [7] and [11], and open shop scheduling with multiprocessors studied in [4], [5], see also [10]. The University timetabling is a generalization of the well-known class-teacher timetabling model. In the generalization an edge represent a single-period lecture given by a teacher to a class, and a hyperedge represent a single-period lecture given by a teacher to a group of classes simultaneously. One looks for a minimum number of periods in which to complete all lecture without conflicts. In the open shop scheduling with multiprocessors a set of jobs $\mathcal{J} = \{J_1, ..., J_n\}$ is scheduled on machines $\mathcal{M} = \{M_1, ..., M_m\}$. The set of machines is partitioned into two
groups $G_1$ and $G_2$. Each job consists of single-processor and multiprocessor operations. A single-processor operation requires one of the machines in $M$, and a multiprocessor operation requires all machines from the group, $G_1$ or $G_2$. Each machine can execute at most one operation at a time, and no two operations of the same job can be executed simultaneously. Any operation can be preempted at any moment though we limit preemptions to integer points for the edge coloring, and at any point for the fractional edge coloring. The makespan is to be minimized. Please see [7], [1], [4], [5], and [10] for more on the applications of the hyperedge coloring problem.

It was pointed out in [4] that the hypergraphs considered in this paper generalize bipartite multigraphs, but they do not belong to known classes, like balanced, normal or with the Kőnig-Egerváry property, see [2].

### 2 Fractional edge colorings with $[LP]$ colors

Let $(y^*, x^*, w^*, r^*)$ be an optimal solution to the LP-relaxation of $ILP$. Let $w^* = [w^*] + \lambda_{w^*}$ and $r^* = [r^*] + \lambda_{r^*}$, where $0 \leq \lambda_{w^*} < 1$ and $0 \leq \lambda_{r^*} < 1$. Consider the following linear program $lp$.

$$lp = \min r$$

Subject to

$$w - r = [w^* - r^*]$$

$$[r^*] \leq r$$

$$\sum_j b_{jh} - (\Delta(G_2) - r) \leq \sum_j y_{jh} \leq w \quad h \in G_1$$

$$\sum_j b_{jh} - (\Delta(G_1) - r) \leq \sum_j y_{jh} \leq w \quad h \in G_2$$

$$\sum_h y_{jh} \leq w \quad j \in J$$

$$0 \leq y_{jh} \leq b_{jh} \quad h \in M \quad j \in J$$

$$\sum_j x_{j1} = r$$

$$\sum_j x_{j2} = r$$

$$x_{j1} + x_{j2} \leq r \quad j \in J$$

$$0 \leq x_{j\ell} \leq a_{j\ell} \quad j \in J \quad \ell = 1, 2$$

$$\sum_{h \in G_1} (b_{jh} - y_{jh}) + a_{j2} - x_{j2} \leq \Delta(G_2) - r \quad j \in J$$

$$\sum_{h \in G_2} (b_{jh} - y_{jh}) + a_{j1} - x_{j1} \leq \Delta(G_1) - r \quad j \in J$$

Let $(y, x, r, w)$ be an optimal solution to $lp$. The solution exists since $(y^*, x^*, r^*, [w^*] + \lambda_{r^*})$ is feasible for $lp$ if $\lambda_{w^*} \leq \lambda_{r^*}$, and $(y^*, x^*, r^*, [w^*] + \lambda_{r^*})$ is feasible for $lp$ if $\lambda_{w^*} > \lambda_{r^*}$, thus $lp$ is feasible and clearly it
is also bounded. Observe that \( w^* \leq \lfloor w^* \rfloor + \lambda_r \) for \( \lambda_w \leq \lambda_r \), and \([w^*] + \lambda_r - r^* = [w^*] - [r^*] = [w^* - r^*] \) for \( \lambda_w > \lambda_r \). Thus the \( \ell p \) gives a fractional edge coloring of the hypergraph with \([LP]\) colors.

We assume without loss of generality that the solution meets the machine saturation condition, i.e. the upper and lower bounds in (2.3) and (2.4) are equal. If the machine saturation is not met by the solution for some machine \( h \), then a job \( j(h) \) with \( b_{j(h),h} = \sum_j b_{jh} + (\Delta(G_2) - r) \), \( a_{j(h)1} = a_{j(h)2} = 0 \) should be added to the instance for each such machine to make the solution meet the saturation condition. Observe that by (2.1) \( b_{j(h),h} \) is integral so the extended instance is a valid instance of the hypergraph edge coloring problem. We take \( y_{j(h),h} = \sum_j y_{jh} \) in the extended solution. Observe that \( n = |J| \geq \lfloor |G_1| + |G_2| \rfloor \) for the solutions that meet the saturation condition.

An optimal solution \((y,x,r,w)\) to \( \ell p \) that is integral is feasible for \( \ell LP \), and \( w - r = [w^* - r^*] = [LP] \). Moreover this solution is optimal for \( \ell LP \) since by definition of \( LP \)-relaxation we have \( LP \leq ILP \) for any feasible solution to \( ILP \). This proves Conjecture [1]. Therefore it suffices to prove that there is an optimal solution to \( \ell p \) that is integral. To that end, we prove the following theorem in the remainder of the paper.

**Theorem 1** The \( r \) in an optimal solution to \( \ell p \) is integral. Moreover, there is optimal solution to \( \ell p \) that is integral.

**Proof:** Let \( s = (y,x,r,w) \) be an optimal solution to \( \ell p \). Suppose for a contradiction that the \( r \) in \( s \) equals

\[
 r = \lfloor r^* \rfloor + \epsilon
\]

where \( 0 < \epsilon < 1 \). Thus by (2.1)

\[
 w = \lfloor w \rfloor + \epsilon.
\]

In the remaining sections of the paper we show that such \( s \) can not be optimal which leads to a contradiction and proves the first part of the theorem. We then show that an optimal solution that is integral can be found in polynomial time. An outline of the proof will be given at the end of the next section after we first introduce the necessary notations and definitions. \( \square \)

## 3 Preliminaries

Consider the solution \( s = (y,x,r,w) \). Let \( B_1 \) be the set of all jobs \( j \) with fractional \( x_{j1} \), and let \( B_2 \) be the set of all jobs \( j \) with fractional \( x_{j2} \). Clearly both sets are non-empty for \( \epsilon > 0 \). By (2.7) and (2.8) the fractions in \( B_\ell \) sum up to \( i_\ell + \epsilon \), where \( i_\ell \) is non-negative integer, for \( \ell = 1,2 \).

A job \( j \) is \( d \)-tight if

\[
\sum_{h} y_{jh} = w.
\]

Denote by \( D \) the set of all \( d \)-tight jobs.

A job \( j \) is \( a \)-tight if

\[
\sum_{h \in G_2} (b_{jh} - y_{jh}) + a_{j1} - x_{j1} = \Delta(G_1) - r.
\]

A job \( j \) is \( c \)-tight if

\[
\sum_{h \in G_1} (b_{jh} - y_{jh}) + a_{j2} - x_{j2} = \Delta(G_2) - r.
\]


For jobs $g$ and $k$ such that $x_{g1} > 0$ and $x_{k2} > 0$ define
\[
e_r(g, k) = \begin{cases} 
\min_{j \in (B_1 \cup B_2) \setminus \{g, k\}} \left\{ r - (x_{j1} + x_{j2}), \epsilon \right\} & \text{if } (B_1 \cup B_2 \setminus \{g, k\}) \neq \emptyset; \\
\epsilon & \text{if } B_1 \cup B_2 \subseteq \{g, k\}.
\end{cases}
\]
Observe that jobs $g$ and $k$ with $e_r(g, k) > 0$ can potentially be used to obtain a solution to $lp$ with smaller $r$ since the reduction of both $x_{g1}$ and $x_{k2}$ by some small enough $\epsilon > 0$ will leave the resulting constraint (2.9) satisfied. Moreover define
\[
e_c(k) = \sum_{h \in \mathcal{G}_1} y_{kh} - \left( \sum_{h \in \mathcal{G}_1} b_{kh} + a_{k2} - x_{k2} - \Delta(G_2) + r \right),
\]
\[
e_d(g) = \sum_{h \in \mathcal{G}_2} y_{gh} - \left( \sum_{h \in \mathcal{G}_2} b_{gh} + a_{g1} - x_{g1} - \Delta(G_1) + r \right).
\]

Let $G$ be a job-machine bi-partite graph such that there is an edge between machine $h \in \mathcal{M}$ and job $j \in \mathcal{J}$ if and only if $y_{jh} > 0$. A column $I = (M_1, \epsilon_I)$ consists of a matching $M_I$ in $G$ that matches all $m$ machines in $\mathcal{M}$ with a subset of exactly $m$ jobs in $\mathcal{J}$, and its multiplicity $\epsilon_I > 0$. Let $\mathcal{J}_I$ be the set of all jobs matched in $M_I$, i.e. $\mathcal{J}_I = \{ j \in \mathcal{J} : (j, h) \in M_I \text{ for some } h \in \mathcal{M} \}$. By definition of $D$ we require that $D \subseteq \mathcal{J}_I$ for a column in $s$. By [8], see also [6], part (d) can be represented by a set of columns $d(y, w) = \{ I_1, \ldots, I_p \}$. For a set $X$ of columns let $l(X)$ denote the total multiplicity of all columns in $X$. We have $l(d(y, w)) = w$. Let $I_1 = (M_{I_1}, \epsilon_{I_1}), \ldots, I_q = (M_{I_q}, \epsilon_{I_q})$ be a subset of $q \geq 1$ columns from $d(y, w)$, the set of columns $(M_{I_1}, \lambda_1), \ldots, (M_{I_q}, \lambda_q)$, where $0 \leq \lambda_1 \leq \epsilon_{I_1}, \ldots, 0 \leq \lambda_q \leq \epsilon_{I_q}$ and $\lambda_1 + \cdots + \lambda_q = \lambda$ is called the interval of length $\lambda$ in $d(y, w)$.

Let $u_1, \ldots, u_p$ and $l_1, \ldots, l_q$ be different jobs from $\mathcal{J}$, and $I$ be a column. We say that $I$ is of type
\[
\begin{pmatrix}
*, u_1, \ldots, u_p \\
*, l_1, \ldots, l_q
\end{pmatrix}
\]
if $\{u_1, h_1\}, \ldots, \{u_p, h_p\} \subseteq M_I$ for some machines $h_1, \ldots, h_p$ in $\mathcal{G}_1$, and $\{l_1, H_1\}, \ldots, \{l_q, H_q\} \subseteq M_I$ for some machines $H_1, \ldots, H_q$ in $\mathcal{G}_2$. For convenience, we sometimes use the following notation
\[
\begin{pmatrix}
*, U \\
*, L
\end{pmatrix}
\]
where $U = \{u_1, \ldots, u_p\}$ and $L = \{l_1, \ldots, l_q\}$. By definition if $p = 0$ or $q = 0$, then the asterisk alone denotes any matching on $\mathcal{G}_1$ or $\mathcal{G}_2$ respectively. We extend this notation for convenience as follows. Let $u$ and $l$ be different jobs from $\mathcal{J}$, and $I$ be a column. We say that $I$ is of type
\[
\begin{pmatrix}
*, \bar{u} \\
*, \bar{l}
\end{pmatrix}
\]
if $u, h \notin M_I$ for any machine $h \in \mathcal{G}_1$, and $(l, H) \notin M_I$ for any machine $H \in \mathcal{G}_2$.

The outline of the proof is as follows. Sections 4 and 5 characterize those columns that cannot occur in $s$ with $\epsilon > 0$ since their presence would contradict the optimality of $r$. Section 6 shows that each job in $B_1$ must be both $a$-tight and $d$-tight, and each job in $B_2$ must be both $c$-tight and $d$-tight in $s$ using the characterization. Section 7 proves that $B_1 \cap B_2 = \emptyset$ in $s$. Section 8 shows that the product $x_{j1}x_{j2} = 0$ in $s$ except for the case where $B_1 = \{j\}$ or $B_2 = \{j\}$. Section 9 proves that the fractions in $B_1$ or $B_2$ may not sum up to $\epsilon$, $(l_1 = 0$ or $l_2 = 0)$, since if they did the $s$ would not be optimal. The proof relies on the results
of earlier sections. Section 11 proves that the fractions in $B_1$ and $B_2$ may not sum up to $i_1 + \epsilon$ and $i_2 + \epsilon$, $(i_1 > 0$ and $i_2 > 0)$, since again if they did the $s$ would not be optimal. The proof relies on the projections introduced in Section 10 and the results of earlier sections. Section 12 summarizes these results proving that the $r$ in $s$ must be integral, $\epsilon = 0$. The section also shows how to obtain an integral optimal solution for $\ell p$ using the network flow problems from Sections 9 and 11, and the integrality of $r$ and $w$.

4 Columns absent from $d(y, w) \in s$

In this section we show that for two different jobs $g$ and $k$ jobs such that $x_{g1} > 0$ and $x_{k2} > 0$ certain columns or subsets of columns must be missing from $d(y, w)$ if $\epsilon > 0$. Though these results are contingent on $\epsilon,(g, k) > 0$, we show that this condition often holds, for instance in Section 7 we show that this inequality holds for each pair $g \in B_1$ and $k \in B_2$.

Let $g$ and $k$ be two different jobs such that $x_{g1} > 0$ and $x_{k2} > 0$. A $(g, k)$-feasible semi-matching in $G$ is a set of edges $E$ of $G$ of cardinality $m = |G_1| + |G_2|$ such that

1. $E_1 = \{(j, h) \in E : h \in G_1\}$ and $E_2 = \{(j, h) \in E : h \in G_2\}$ are matchings;
2. there are $h \in M$ and $(j, h) \in E$ for each $j \in D$;
3. if $\epsilon_d(g) = 0$, then $(g, h) \notin E_2$ for any $h \in G_2$;
4. if $\epsilon_c(k) = 0$, then $(k, h) \notin E_1$ for any $h \in G_1$.

If $E$ is a matching, then a $(g, k)$-feasible semi-matching in $G$ is called a $(g, k)$-feasible matching in $G$.

We define solution $(y(E), x(g, k), r(g, k), w(g, k), \epsilon)$ for jobs $g, k$, and a $(g, k)$-feasible semi-matching $E$, where

$$
\epsilon = \begin{cases} 
\epsilon' & \text{if } \epsilon_d(g) = 0 \text{ and } \epsilon_c(k) = 0 ; \\
\min\{\epsilon', \epsilon_d(g)\} & \text{if } \epsilon_d(g) > 0 \text{ and } \epsilon_c(k) = 0 ; \\
\min\{\epsilon', \epsilon_c(k)\} & \text{if } \epsilon_d(g) = 0 \text{ and } \epsilon_c(k) > 0 ; \\
\min\{\epsilon', \epsilon_d(g), \epsilon_c(k)\} & \text{if } \epsilon_d(g) > 0 \text{ and } \epsilon_c(k) > 0 ; 
\end{cases}
$$

(4.1)

and

$$
\epsilon' = \min\{\epsilon, \epsilon_r(g, k), x_{g1}, x_{k2}, \min_{(j, h) \in E} \{y_{jh}\}, \min_{j \in J \setminus D} \{r - \sum_{h} y_{jh}\}\},
$$

(4.2)

as follows

$$
y_{jh}(E) = \begin{cases} 
y_{jh} - \epsilon & \text{if } (j, h) \in E ; \\
y_{jh} & \text{otherwise ;}
\end{cases}
$$

(4.3)

$$
x_{j1}(g, k) = \begin{cases} 
x_{j1} - \epsilon & \text{if } j = g ; \\
x_{j1} & \text{if } j \neq g ;
\end{cases}
$$

(4.4)

$$
x_{j2}(g, k) = \begin{cases} 
x_{j2} - \epsilon & \text{if } j = k ; \\
x_{j2} & \text{if } j \neq k ;
\end{cases}
$$

(4.5)

$$
r(g, k) = r - \epsilon ;
$$

(4.6)

$$
w(g, k) = w - \epsilon .
$$

(4.7)

We have the following lemma
Lemma 1 Let \( g \) and \( k \) be two different jobs such that \( x_{g1} > 0 \) and \( x_{k2} > 0 \). If \( \varepsilon_r(g,k) > 0 \), then no \((g,k)\)-feasible semi-matching \( E \) in \( G \) exists.

Proof: Suppose for a contradiction that a \((g,k)\)-feasible semi-matching \( E \) in \( G \) exists. It suffices to show that solution \( s = (y(E), x(g,k), r(g,k), w(g,k), \varepsilon) \) is feasible for \( lp \). The feasibility of \( s \) implies that \( r \) is not optimal for \( lp \) since by the lemma assumptions \( \varepsilon > 0 \) and thus \( r(g,k) < r \) which gives a contradiction. To prove the feasibility of \( s \) we observe that since both \( E_1 \) and \( E_2 \) are matchings and \( |E_1| + |E_2| = m \) we have

\[
\sum_j y_{jh}(E) = \sum_j y_{jh} - \varepsilon \quad h \in G_1
\]

and

\[
\sum_j y_{jh}(E) = \sum_j y_{jh} - \varepsilon \quad h \in G_2.
\]

Thus both (2.3) and (2.4) hold for \( s \) by (4.6) and (4.7). The (4.6) and (4.7) also imply (2.1) for \( s \). The (4.5) imply (2.7) and (2.8) for \( s \). Since \( \varepsilon_r(g,k) > 0 \), definition of \( \varepsilon \) guarantees (2.9) for \( s \). The \( E \) covers each job in \( D \) at least once, thus (2.5) holds for each job in \( D \), and by definition of \( \varepsilon \) for each job in \( J \setminus D \). Clearly, the (2.6) and (2.10) are met for \( s \) by the lemma assumptions and definition of \( \varepsilon \). Therefore it remains to show that (2.11) and (2.12) hold for \( s \). First, since both \( E_1 \) and \( E_2 \) are matchings we have

\[
\sum_{h \in G_1} y_{jh} - \varepsilon \leq \sum_{h \in G_1} y_{jh}(E) \leq \sum_{h \in G_1} y_{jh} \quad j \in J
\]

and

\[
\sum_{h \in G_2} y_{jh} - \varepsilon \leq \sum_{h \in G_2} y_{jh}(E) \leq \sum_{h \in G_2} y_{jh} \quad j \in J.
\]

Hence (2.11) and (2.12) hold for all jobs in \( J \setminus \{g,k\} \) for \( s \), and for job \( g \) provided that the slack \( \varepsilon_s(g) \) of \( g \) in (a) is positive, and for \( k \) provided that the slack \( \varepsilon_c(k) \) of \( k \) in (c) is positive. The (2.11) and (2.12) hold for \( g \) and \( k \) with no slack in (a) and (c) respectively as well since then

\[
\sum_{h \in G_1} y_{kh}(E) = \sum_{h \in G_1} y_{kh}
\]

and

\[
\sum_{h \in G_2} y_{gh}(E) = \sum_{h \in G_2} y_{gh}
\]

by the conditions (3) and (4) in definition of \( E \). Therefore \( s \) is feasible for \( lp \) and we get the required contradiction. \( \square \)

Lemma 2 Let \( g \) and \( k \) be two different jobs such that \( x_{g1} > 0 \) and \( x_{k2} > 0 \). If \( \varepsilon_r(g,k) > 0 \), then no column of type \((*,k)\) exists in \( d(y,w) \).

Proof: If such a column \( I = (M_I, \varepsilon_I) \) exists, then \( M_I \) is \((g,k)\)-feasible semi-matching \( E \) in \( G \) which contradicts Lemma[1] \( \square \)
We now consider another forbidden configuration of columns in \( d(y,w) \). Let \( I_1 = (M_{I_1}, \epsilon_{I_1}) \) and \( I_2 = (M_{I_2}, \epsilon_{I_2}) \) be two columns. Let \( g, k, a, \) and \( b \) be four different jobs such that \( x_{g1} > 0, x_{k2} > 0, x_{a1} > 0, \) and \( x_{b2} > 0 \). Define solution \((y(I_1, I_2), x', r', w', \epsilon)\), where

\[
\epsilon = \min\{\epsilon, \epsilon_y(g,k), \epsilon_y(a,b), x_{g1}, x_{a1}, x_{b2}, x_{k2}, \epsilon_{I_1}, \epsilon_{I_2}, \min_{j \in \mathcal{J}\setminus D} \{r - \sum_h y_{jh}\}\} \tag{4.8}
\]

as follows

\[
y_{jh}(I_1, I_2) = \begin{cases} 
y_{jh} - \epsilon & \text{if } (j, h) \in M_{I_1} \text{ and } (j, h) \in M_{I_2}; \\
y_{jh} - \epsilon/2 & \text{if } (j, h) \in M_{I_1} \text{ and } (j, h) \notin M_{I_2}; \\
y_{jh} - \epsilon/2 & \text{if } (j, h) \notin M_{I_1} \text{ and } (j, h) \in M_{I_2}; \\
y_{jh} & \text{otherwise};
\end{cases} \tag{4.9}
\]

\[
x'_{j1} = \begin{cases} 
x_{j1} - \epsilon/2 & \text{if } j = g \text{ or } j = a; \\
x_{j1} & \text{otherwise};
\end{cases} \tag{4.10}
\]

\[
x'_{j2} = \begin{cases} 
x_{j2} - \epsilon/2 & \text{if } j = k \text{ or } j = b; \\
x_{j2} & \text{otherwise};
\end{cases} \tag{4.11}
\]

\[
r' = r - \epsilon; \tag{4.12}
\]

\[
w' = w - \epsilon. \tag{4.13}
\]

We have the following lemma

**Lemma 3** Let \( g, k, a, \) and \( b \) be four different jobs such that \( x_{g1} > 0, x_{k2} > 0, x_{a1} > 0, \) and \( x_{b2} > 0 \). If \( \epsilon_y(g,k) > 0 \) and \( \epsilon_y(a,b) > 0 \), then a column of type \( \left(\ast, a, b, g, k\right) \) does not exist in \( d(y,w) \) or a column of type \( \left(\ast, a, b, k, g\right) \) does not exist in \( d(y,w) \).

**Proof:** Suppose for a contradiction that a column \( I_1 \) of type \( \left(\ast, a, b, g, k\right) \) is in \( d(y,w) \) and a column \( I_2 \) of type \( \left(\ast, a, b, k, g\right) \) is in \( d(y,w) \). It suffices to show that solution \( s = (y(I_1, I_2), x', r', w', \epsilon) \) is feasible for \( lp \). The feasibility of \( s \) implies that \( r \) is not optimal for \( lp \) since by the lemma assumptions \( \epsilon > 0 \) and thus \( r' < r \) which leads to a contradiction. To prove the feasibility of \( s \) we observe that since both \( M_{I_1} \) and \( M_{I_2} \) are matchings that cover all machines we have

\[
\sum_j y_{jh}(I_1, I_2) = \sum_j y_{jh} - \epsilon \quad h \in \mathcal{G}_1 
\]

and

\[
\sum_j y_{jh}(I_1, I_2) = \sum_j y_{jh} - \epsilon \quad h \in \mathcal{G}_2.
\]

Thus both (2.3) and (2.4) hold for \( s \) by (4.12) and (4.13). The (4.12) and (4.13) also imply (2.1) for \( s \). The (4.10) and (4.11) imply (2.7) and (2.8) for \( s \) respectively. Since \( \epsilon_y(g,k) > 0 \) and \( \epsilon_y(a,b) > 0 \), definition of \( \epsilon \) guarantees (2.9) for \( s \). The \( M_{I_1} \cup M_{I_2} \) covers each job in \( D \) exactly twice, thus (2.5) holds for each job in \( D \) in \( s \), and by definition of \( \epsilon \) for each job in \( \mathcal{J} \setminus D \). Clearly, the (2.6) and (2.10) are met for \( s \) by the lemma assumptions and definition of \( \epsilon \). Therefore it remains to show that (2.11) and (2.12) hold for \( s \). First, both \( M_{I_1} \) and \( M_{I_2} \) are matchings we have
\[ \sum_{h \in G_1} y_{h} - \varepsilon \leq \sum_{h \in G_1} y_{h}(I_1, I_2) \leq \sum_{h \in G_1} y_{h} \quad j \in \mathcal{J} \]

and

\[ \sum_{g \in G_2} y_{g} - \varepsilon \leq \sum_{g \in G_2} y_{g}(I_1, I_2) \leq \sum_{g \in G_2} y_{g} \quad j \in \mathcal{J} \]

Hence (2.11) and (2.12) hold for all jobs in \( \mathcal{J} \setminus \{a, g, b, k\} \) for \( s \). They hold for \( g \) and \( a \) as well since

\[ \sum_{g \in G_2} y_{gh}(I_1, I_2) = \sum_{g \in G_2} y_{gh} - \varepsilon/2 \]

and

\[ \sum_{g \in G_2} y_{gh}(I_1, I_2) = \sum_{g \in G_2} y_{gh} - \varepsilon/2 \]

by the types of \( I_1 \) and \( I_2 \), and for \( b \) and \( k \) since

\[ \sum_{h \in G_1} y_{bh}(I_1, I_2) = \sum_{h \in G_1} y_{bh} - \varepsilon/2 \]

and

\[ \sum_{h \in G_1} y_{bh}(I_1, I_2) = \sum_{h \in G_1} y_{bh} - \varepsilon/2 \]

by the types of \( I_1 \) and \( I_2 \). Therefore \( s \) is feasible for \( lp \) and we get the required contradiction. \( \square \)

The following two corollaries follow immediately from the proof of Lemma 3.

**Corollary 1** Let \( g, k, \) and \( a \) be three different jobs such that \( x_{g1} > 0, x_{k2} > 0, \) and \( x_{a1} x_{a2} > 0 \). If \( \varepsilon_r(g,a) > 0 \) and \( \varepsilon_r(a,k) > 0 \), then a column of type \( (^{*a,g,k}) \) does not exist in \( d(y,w) \) or a column of type \( (^{*a,k,g}) \) does not exist in \( d(y,w) \).

**Corollary 2** Let \( g, k \) be two different jobs such that \( x_{g1} x_{g2} > 0, \) and \( x_{k1} x_{k2} > 0 \). If \( \varepsilon_r(g,k) > 0 \), then a column of type \( (^{*g,k}) \) does not exist in \( d(y,w) \) or a column of type \( (^{*k,g}) \) does not exist in \( d(y,w) \).

### 5 Pairs of Columns Absent from \( d(y,w) \) in \( s \)

Let \( g \) and \( k \) be two different jobs such that \( x_{g1} > 0, x_{k2} > 0 \). Let \( I_k = (M_{I_k}, \varepsilon_{I_k}) \) be a column of type \( (^{*k}) \) and \( I_g = (M_{I_g}, \varepsilon_{I_g}) \) a column of type \( (^{*g}) \). Without loss of generality we assume \( \varepsilon_{I_k} = \varepsilon_{I_g} = \varepsilon \). Let \( G(I_g, I_k) = (M_{I_g} \cup M_{I_k}) \) be a job-machine bi-partite multigraph graph, where an edge connects a machine \( h \) and a job \( j \) if and only if \( (j,h) \in M_{I_g} \cup M_{I_k} \). The degree of each machine-vertex in \( G(I_g, I_k) \) is exactly 2 and the degree of each job-vertex in \( G(I_g, I_k) \) is either 1 or 2. Thus, \( G(I_g, I_k) \) is a collection of connected components each of which is either a job-machine path or a job-machine cycle.

**Lemma 4** If \( I_k, I_g \in d(y,w) \), and \( \varepsilon_r(g,k) > 0 \), then \( I_k \) is of type \( (^{*k}) \) and \( I_g \) is of type \( (^{*k,g}) \) and both \( k \) and \( g \) belong to the same connected component of \( G(I_g, I_k) \).

**Proof:** By contradiction. We can readily verify that if
1. $k$ and $g$ are in different connected components of $G(I_g, I_k)$, or
2. $k$ or $g$ is missing from $G(I_g, I_k)$ (i.e. at least one of them is not in $D$), or
3. $k$ or $g$ is of degree 1 in $G(I_g, I_k)$ (i.e. at least one of them is not in $D$), or
4. $(k, h) \in G(I_g, I_k)$ implies $h \in G_2$, or
5. $(g, h) \in G(I_g, I_k)$ implies $h \in G_1$,

then there is a matching $M$ of cardinality $m$ in $G(I_g, I_k) \subseteq G$ that satisfies definition of $(g, k)$-feasible semi-matching in $G$. This however contradicts Lemma 1 since $I_k, I_g$ in $d(y, w)$ can be replaced by columns $I' = (M, \epsilon)$ and $I'' = ((M_{I_k} \cup M_{I_g}) \setminus (M, \epsilon))$ resulting into another feasible solution to $\ell p$ with the same value $r$ of objective function but with a $(g, k)$-feasible semi-matching $M$. □

6. **The $a$-, $c$-, and $d$-tightness in $s$**

We show that each job in $B_1$ is both $a$-tight and $d$-tight, and each job in $B_2$ is both $c$-tight and $d$-tight. We begin by showing the $a$- and $c$- tightness.

**Lemma 5** Each job $g \in B_1$ is $a$-tight and

$$\sum_{h \in G_2} y_{gh} < w, \quad (6.1)$$

and each job $k \in B_2$ is $c$-tight and

$$\sum_{h \in G_1} y_{kh} < w. \quad (6.2)$$

**Proof:** By (2.1), (2.5), and (2.12) at least one of the following two inequalities

$$a_{g_1} - x_{g_1} + \sum_{h \in G_2} (b_{gh} - y_{gh}) < \Delta(G_1) - r \quad (6.3)$$

or

$$\sum_{h \in G_2} y_{gh} < w, \quad (6.4)$$

holds for any job $g \in B_1$. Likewise, by (2.1), (2.5), and (2.11) at least one of the following two inequalities

$$a_{k_2} - x_{k_2} + \sum_{h \in G_1} (b_{kh} - y_{kh}) < \Delta(G_2) - r \quad (6.5)$$

or

$$\sum_{h \in G_1} y_{kh} < w, \quad (6.6)$$

holds for any job $k \in B_2$.

Let $l$ be a job with the largest sum $x_{i_1} + x_{i_2}$ among the jobs $i \in B_1 \cup B_2$. Suppose $l \in B_1$ in the proof. If $l \in B_2 \setminus B_1$ the proof proceeds in a similar way and thus will be omitted.

We prove the lemma for any $k \in B_2$ first. For any $k \in B_2$ there is $g \in B_1$ such that $\epsilon_r(g, k) > 0$. Simply take a job $g \in B_1$ with the largest $x_{i_1} + x_{i_2}$ among the jobs $i \in B_1 \setminus \{k\}$, or take $g = k$ if $B_1 = \{k\}$. It suffices
to show that there is no $i \in (B_1 \cup B_2) \setminus \{g, k\}$ with $x_{i1} + x_{i2} = r$. Suppose for a contradiction that $x_{i1} + x_{i2} = r$ for some $i \in (B_1 \cup B_2) \setminus \{g, k\}$. If $g \neq k$, then, since $l \in B_1$, we have $x_{j1} + x_{j2} = r$ for some $j \in \{g, k\}$. Therefore $\{k, i, g\} \subseteq B_1 \cup B_2$ and we get a contradiction by (2.7) and (2.8). If $g = k$, then $B_1 = \{k\}$. Thus, $l = k$ and $x_{k1} + x_{k2} = r$ which implies $B_1 \cup B_2 = \{i, k\}$. Since $k \in B_1 \cap B_2$ we have $i \in B_1 \cap B_2$ which gives contradiction since $i \notin B_1$.

Suppose for a contradiction that $k$ is not $c$-tight, i.e. (6.5) holds for $k$. We have $\varepsilon_c(k) > 0$. If (6.3) holds for $g$, then $\varepsilon_a(g) > 0$. Thus $\min(\varepsilon_a(g), \varepsilon_c(k)) > 0$ which contradicts Lemma 1 since then any column from $d(y,w)$ is a $(g,k)$-feasible matching. If (6.3) does not hold for $g$, then $\varepsilon_a(g) = 0$ and $\varepsilon_c(k) > 0$. Take any column $I$ of type $(\vec{x},\vec{\varepsilon})$ from $d(y,w)$. This column exists since (6.4) holds for $g$. This however contradicts Lemma 1 since $I$ is a $(g,k)$-feasible matching. Therefore, $k$ is $c$-tight and thus by (6.6) the condition (6.2) holds for $k \in B_2$. This proves the lemma for any $k \in B_2$.

We now prove the lemma for any $g \in B_1$. We begin with $g = l$. There is $k \in B_2$ such that $\varepsilon_c(g,k) > 0$. Simply take a job $k \in B_2$ with the largest $x_{k1} + x_{k2}$ among the jobs in $B_2 \setminus \{g\}$, or $k = g$ if $B_2 = \{g\}$. Again, it suffices to show that there is no $i \in (B_1 \cup B_2) \setminus \{g, k\}$ such that $x_{i1} + x_{i2} = r$. Suppose for a contradiction that $x_{i1} + x_{i2} = r$ for some $i \in (B_1 \cup B_2) \setminus \{g, k\}$. Then, $x_{i1} + x_{i2} = r$ implies $B_1 \cup B_2 = \{i, g\}$ by (2.7) and (2.8). Hence $k = g$, which implies $B_2 = \{g\}$ and $B_1 = \{i, g\}$. Thus, $x_{i2}$ is integer and by (2.8) $x_{i2} = \lceil x_{i2} \rceil + \varepsilon$. This implies integral $x_{i1}$ and thus $g \notin B_1$ which gives contradiction.

Suppose for contradiction that $l$ is not $a$-tight, i.e. (6.3) holds for $l$. We have $\varepsilon_a(l) > 0$. If (6.5) holds for $k$, then $\varepsilon_c(k) > 0$. Thus $\min(\varepsilon_a(l), \varepsilon_c(k)) > 0$ which contradicts Lemma 1 since then any column from $d(y,w)$ is a $(g,k)$-feasible matching. If (6.3) does not hold for $k$, then $\varepsilon_a(l) = 0$ and $\varepsilon_c(k) > 0$. Take any column $I$ of type $(\vec{x},\vec{\varepsilon})$ from $d(y,w)$. This column exists since (6.6) holds for $k$. This contradicts Lemma 1 since $I$ is a $(g,k)$-feasible matching. Therefore, $l$ is $a$-tight and thus by (6.4) the condition (6.1) holds for $l \in B_1$. This proves the lemma for $l \in B_1$.

To complete the proof assume $B_1 \setminus \{l\} \neq \emptyset$ in the remainder of the proof. Observe also that for $l$ we have

$$\sum_{h \in G_l} y_{hl} < w, \quad (6.7)$$

for if

$$\sum_{h \in G_l} y_{hl} = w, \quad (6.8)$$

then any column $I$ in $d(y,w)$ is of type $(\vec{x},\vec{\varepsilon})$. Since we already have shown that the lemma holds for any $k \in B_2$, there is $I$ of type $(\vec{x},\vec{\varepsilon})$, i.e. of type $(\vec{x},\vec{\varepsilon})$. This contradicts Lemma 2 since we observe that $\varepsilon_c(l,k)$ for $k \in B_2$ with the largest $x_{k1} + x_{k2}$.

Consider any $g \in B_1 \setminus \{l\}$. Observe that if $x_{i1} + x_{i2} = r$, then $x_{i2} > 0$. Otherwise $B_1 = \{l\}$ and we get a contradiction. There is $k$ such that $\varepsilon_c(g,k) > 0$. Simply take $k = l$, if $x_{i1} + x_{i2} = r$, or any $k$ from $B_2$ if $x_{i1} + x_{i2} < r$. It suffices to show that there is no $i \in (B_1 \cup B_2) \setminus \{g, k\}$ that satisfies $x_{i1} + x_{i2} = r$. Suppose for a contradiction that $x_{i1} + x_{i2} = r$ for some $i \in (B_1 \cup B_2) \setminus \{g, k\}$. Then $x_{i1} + x_{i2} = r$. Since $k \neq g$, we have $\{k, i, g\} \subseteq B_1 \cup B_2$ and thus we get a contradiction by (2.7) and (2.8).

Suppose for a contradiction that $g$ is not $a$-tight, i.e. (6.3) holds for $g$. We have $\varepsilon_a(g) > 0$. If (6.5) holds for $k$, then $\varepsilon_c(k) > 0$. Thus $\min(\varepsilon_a(g), \varepsilon_c(k)) > 0$ which contradicts Lemma 1 since then any column from $d(y,w)$ is a $(g,k)$-feasible matching. If (6.5) does not hold for $k$, then $\varepsilon_a(g) > 0$ and $\varepsilon_c(k) = 0$. Take any column $I$ of type $(\vec{x},\vec{\varepsilon})$ from $d(y,w)$. Such column exists for $k \in B_2$ since (6.6) holds in this case, it also exists for $k = l$ (and $l \notin B_2$) by (6.7). This contradicts Lemma 1 since $I$ is a $(g,k)$-feasible matching. Therefore the lemma holds for each $g \in B_1$. □
We now prove $d$-tightness for each job in $B_1 \cup B_2$.

**Theorem 2** Each job in $B_1 \cup B_2$ is $d$-tight.

**Proof:** By (6.1) in Lemma 5 there is a column $I_g$ of type $\left(\frac{\epsilon}{\epsilon^g} \right)$ in $d(y,w)$ for each $g \in B_1$. By (6.2) in Lemma 5 there is a column $I_k$ of type $\left(\frac{\epsilon^k}{\epsilon} \right)$ in $d(y,w)$ for each $k \in B_2$.

Consider job $g$ with the largest $x_{i1} + x_{i2}$ among the jobs $i \in B_1 \cup B_2$. Suppose $g \in B_1$. If $g \in B_2 \setminus B_1$, then the proof proceeds in a similar way and thus will be omitted. Take any $k \in B_2 \setminus \{g\}$ or $k = g$ if $B_2 = \{g\}$. Observe that by our choice of $g$, if $x_{i1} + x_{i2} = r$ for some $i \in (B_1 \cup B_2) \setminus \{g,k\}$, then $x_{i1} + x_{i2} = r$. Therefore $\{k,i,g\} \subseteq B_1 \cup B_2$ which leads to a contradiction by (2.7) and (2.8) if $k \neq g$. Otherwise, if $k = g$, then by (2.8) $B_1 \cup B_2 = \{i,g\}$ and $g \in B_1 \cap B_2$. Thus $i \in B_1 \cap B_2$ and we get contradiction since $i \notin B_2$. Thus $\epsilon_i(g,k) > 0$.

If $k$ is not $d$-tight, then there is a column $I$ of type $\left(\frac{\epsilon^k}{\epsilon} \right)$ in $d(y,w)$. Thus, if $I \neq I_g$, then we get a contradiction with Lemma 3 applied to $I$ and $I_g$. Otherwise, if $I = I_g$, then $I$ is of type of type $\left(\frac{\epsilon}{\epsilon^g} \right)$ which contradicts Lemma 2. Similarly, if $g$ is not $d$-tight, then there is a column $I$ of type $\left(\frac{\epsilon}{\epsilon^g} \right)$ in $d(y,w)$. Thus, if $I \neq I_k$, then we get a contradiction with Lemma 3 applied to $I_k$ and $I$. Otherwise, if $I = I_k$, then $I$ is of type of type $\left(\frac{\epsilon^k}{\epsilon} \right)$ which contradicts Lemma 2. Therefore the lemma holds for each job in $\{g\} \cup B_2$.

Moreover, there is a column $I_g'$ of type $\left(\frac{\epsilon}{\epsilon^g} \right)$. Otherwise all columns in $d(y,w)$ are of type $\left(\frac{\epsilon^k}{\epsilon} \right)$ and thus $I_k$ is of type of type $\left(\frac{\epsilon^k}{\epsilon} \right)$ for any $k \in B_2$ which contradicts Lemma 2.

It remains to prove the lemma for each $a \in B_1 \setminus \{g\}$ whenever $B_1 \setminus \{g\} \neq \emptyset$. Observe that if $x_{g1} + x_{g2} = r$, then $x_{g2} > 0$. Otherwise $B_1 = \{g\}$ and we get a contradiction. Take a job $k = g$, if $x_{g1} + x_{g2} = r$, or any job $k \in B_2$, if $x_{g1} + x_{g2} < r$. We have $\epsilon_i(a,k) > 0$. This holds since there is no $i \in (B_1 \cup B_2) \setminus \{a,k\}$ that meets $x_{i1} + x_{i2} = r$. Suppose for a contradiction that $x_{i1} + x_{i2} = r$ for some $i \in (B_1 \cup B_2) \setminus \{a,k\}$. Then $x_{k1} + x_{k2} = r$. Since $a \neq k$, we have $\{k,i,a\} \subseteq B_1 \cup B_2$ which leads to a contradiction by (2.7) and (2.8).

Thus if $a$ is not $d$-tight, then there is a column $I$ of type $\left(\frac{\epsilon}{\epsilon^g} \right)$ in $d(y,w)$. Then, if $\epsilon_i(a,k) > 0$ for $k \in B_2$, we have either $I \neq I_k$ which leads a contradiction with Lemma 3 applied to $I_k$ and $I$ or $I = I_k$ which implies that $I$ is of type $\left(\frac{\epsilon}{\epsilon^g} \right)$ which contradicts Lemma 2. If $\epsilon_i(a,k) > 0$ for $k \notin B_2$, then $k = g$. Thus, if $I \neq I_g'$, then we get a contradiction with Lemma 3 applied to $I$ and $I_g'$. Otherwise, if $I = I_g'$, then $I$ is of type of type $\left(\frac{\epsilon}{\epsilon^g} \right)$ which contradicts Lemma 2.

For $j \in B_1 \cup B_2$ define

$$
\alpha_j = \sum_{h \in G_1} y_{jh} \quad \text{and} \quad \beta_j = \sum_{h \in G_2} y_{jh}.
$$

The following two lemmas relate the fractions of $x_{j1}, x_{j2}, \alpha_j$, and $\beta_j$ for $j \in B_1 \cup B_2$. The lemmas follow from Lemmas 5 and Theorem 2 and will prove useful in the remainder of the paper.

**Lemma 6** For $g \in B_1$, let

$$
\lambda_g = \left[\frac{\alpha_g}{\beta_g} \right] + \lambda_g, \quad \beta_g = \left[\frac{\beta_g}{\beta_g} \right] + \lambda_g, \quad \text{and} \quad \alpha_g = \left[\frac{\alpha_g}{\alpha_g} \right] + \omega_g
$$

where $0 \leq \lambda_g, \omega_g < 1$, $0 < \epsilon_g < 1$ for $g \in B_1$. Then, $\omega_g = \epsilon_g$, and $\lambda_g = \varepsilon - \epsilon_g$ for $\varepsilon \geq \epsilon_g$, and $\lambda_g = 1 - (\epsilon_g - \varepsilon)$ for $\varepsilon < \epsilon_g$. 

12
Proof: By Lemma 5, $g \in B_1$ is $a$-tight, i.e.

$$a_{g1} - x_{g1} + \sum_{h \in G_2} b_{gh} - \beta_g = \Delta(G_1) - r,$$

by Theorem 2, $g$ is $d$-tight, i.e.

$$\alpha_g + \beta_g = w.$$  \hspace{1cm} (6.9)

The two imply $\omega_g = \varepsilon_g$ since $w - r$ is integral. By (6.9), $\lambda_g + \omega_g - \varepsilon = 0$ or 1. Thus $\lambda_g + \epsilon_g - \varepsilon = 0$ or 1. Therefore, $\lambda_g = \epsilon - \epsilon_g$ for $\epsilon \geq \epsilon_g$, and $\lambda_g = 1 - (\epsilon_g - \epsilon)$ for $\epsilon < \epsilon_g$. \hfill \Box

Lemma 7 For $j \in B_2$, let

$$x_{k2} = \lfloor x_{k2} \rfloor + \lambda_k \quad \text{and} \quad \beta_k = \lfloor \beta_k \rfloor + \lambda_k \quad \text{and} \quad \alpha_k = \lfloor \alpha_k \rfloor + \omega_k$$

where $0 \leq \lambda_k$, $\omega_k < 1$, $0 < \epsilon_k < 1$ for a job $k \in B_2$. Then, $\lambda_k = \epsilon$, and $\omega_k = \epsilon - \epsilon_k$ for $\epsilon \geq \epsilon_k$, and $\lambda_k = 1 - (\epsilon_k - \epsilon)$ for $\epsilon < \epsilon_k$.

Proof: The proof is similar to the proof of Lemma 6 and will be omitted. \hfill \Box

7 The Absence of Crossing Jobs in $s$

Each job $k \in B_1 \cap B_2$ is called crossing. We call a job $a \in B_1 \cup B_2$ an $e$-crossing job, if it meets the following conditions:

- $0 < x_{a2}$ and $0 < x_{a1}$;
- both $B_1 \setminus \{a\}$ and $B_2 \setminus \{a\}$ are not empty.

We have the following.

Theorem 3 Each crossing job is $e$-crossing.

Proof: Suppose for a contradiction that $a$ is crossing but not $e$-crossing. By Theorem 2, $a$ is $d$-tight and thus

$$\sum_{h \in G_2} y_{ah} + \sum_{h \in G_1} y_{ah} = w.$$  \hspace{1cm} (7.1)

By Lemma 5, $a$ is both $a$-tight and $c$-tight, thus

$$a_{a1} - x_{a1} + \sum_{h \in G_2} (b_{ah} - y_{ah}) = \Delta(G_1) - r$$  \hspace{1cm} (7.2)

and

$$a_{a2} - x_{a2} + \sum_{h \in G_1} (b_{ah} - y_{ah}) = \Delta(G_2) - r.$$  \hspace{1cm} (7.3)

By summing up (7.1), (7.2), and (7.3) side by side we obtain
\[ a_{a1} + a_{a2} + \sum_h b_{ah} - \Delta(G_1) - \Delta(G_2) + r - w = -r + x_{a1} + x_{a2}. \]

Since \( a \) is not \( e \)-crossing, \( B_1 \setminus \{a\} = \emptyset \) or \( B_2 \setminus \{a\} = \emptyset \). Thus, \( x_{a1} = |x_{a1}| + \epsilon \) or \( x_{a2} = |x_{a2}| + \epsilon \). Therefore, the left hand side of (7.4) is integral but its right hand side is fractional since both \( x_{a1} \) and \( x_{a2} \) are fractional. This leads to contradiction and thus the theorem holds. \( \square \)

**Theorem 4** For each \( e \)-crossing job \( a \) we have \( x_{a1} + x_{a2} < r \).

**Proof:** By contradiction. Let \( a \) be \( e \)-crossing with \( x_{a1} + x_{a2} = r \). Let \( g \in B_1 \setminus \{a\} \) and \( k \in B_2 \setminus \{a\} \). By Theorem 2 and Lemma 5 there are columns \( I_k \) of type \( (e^*, e) \) and \( I_g \) of type \( (e^g, e) \) in \( d(y, w) \). By Theorem 2 \( I_k \) is either of type \( (e^*, e) \) or of type \( (e^g, e) \), and \( I_g \) is either of type \( (e^g, e) \) or of type \( (e^*, e) \). Suppose that \( I_k \) or \( I_g \) is of type \( (e^g, e) \), then \( g \neq k \). Since \( a \) is \( e \)-crossing, by Theorem 2 this column, say \( I_k \), is other of type \( (e^g, e) \) or of type \( (e^*, e) \). The former is of type \( (e^*, e) \) and the latter of type \( (e^g, e) \). Since \( g \neq k \), \( a \) is the only job \( i \) with \( x_{i1} \) or \( x_{i2} = r \). Thus \( e_i(a, k) > 0 \) and \( e_i(g, a) > 0 \). Therefore we get a contradiction with Lemma 2 which implies that \( I_g \) is of type \( (e^g, e) \) and \( I_k \) is of type \( (e^*, e) \) (observe that we may now have \( g = k \)). Since \( a \) is \( e \)-crossing, by Theorem 2 we have \( I_g \) of type \( (e^a, e) \) or of type \( (e^g, e) \), and \( I_k \) is of type \( (e^*, e) \) or of type \( (e^g, e) \). The \( I_g \) of type \( (e^g, e) \) is of type \( (e^g, e) \), and the \( I_k \) of type \( (e^*, e) \) is of type \( (e^g, e) \). Moreover, if \( g \neq k \), then \( a \) is the only job \( i \) with \( x_{i1} + x_{i2} = r \), and if \( k = g \), then \( x_{k1} + x_{k2} = r \) or \( a \) is the only job \( i \) with \( x_{i1} + x_{i2} = r \). Thus \( e_i(a, k) > 0 \) and \( e_i(g, a) > 0 \). Therefore, \( I_g \) being of type \( (e^a, e) \) or \( I_k \) being of type \( (e^g, e) \) contradicts Lemma 2. Thus it remains to consider \( I_g \) of type \( (e^a, e) \) and \( I_k \) is of type \( (e^g, e) \). This leads to a contradiction by Corollaries 1 and 2 since \( e_i(g, a) > 0 \) and \( e_i(a, k) > 0 \). The last two inequalities clearly hold if \( a \) is the only job \( i \) with \( x_{i1} + x_{i2} = r \), otherwise \( g = k \) and \( k \) is the other job \( i \) with \( x_{i1} + x_{i2} = r \). \( \square \)

The following corollary follows immediately from the proof of Theorem 4 since the assumption \( x_{i1} + x_{i2} < r \) for each \( i \in B_1 \cup B_2 \) implies \( e_i(g, a) > 0 \) for each \( g \in B_1 \) and \( k \in B_2 \).

**Corollary 3** If \( x_{i1} + x_{i2} < r \) for each \( i \in B_1 \cup B_2 \), then no job is \( e \)-crossing.

We are now ready to prove two main results of this section.

**Theorem 5** No crossing job exists.

**Proof:** By contradiction. Suppose \( a \) is a crossing job. Take a crossing job with the largest \( x_{a1} + x_{a2} \). By Theorem 3 \( a \) is \( e \)-crossing, and by Theorem 4 \( x_{a1} + x_{a2} < r \). By Corollary 3 \( x_{i1} + x_{i2} = r \) for some \( i \in B_1 \cup B_2 \). Thus \( i \neq a \). By Theorem 4 \( j \) is not \( e \)-crossing. Thus \( (x_{j1} = 0 \text{ or } x_{j2} = 0) \) which implies \( (B_1 = \{i\} \text{ or } B_2 = \{i\}) \). This leads to contradiction since \( a \in B_1 \cap B_2 \) and \( a \neq i \). \( \square \)

**Theorem 6** For each \( g \in B_1 \) and \( k \in B_2 \), \( e_i(g, k) > 0 \).

**Proof:** Suppose for a contradiction that \( e_i(g, k) = 0 \) for some \( g \in B_1 \) and \( k \in B_2 \). By Theorem 5 \( g \neq k \). Then \( r = x_{j1} + x_{j2} \) for some \( j \in (B_1 \cup B_2) \setminus \{g, k\} \). By Theorem 5 \( j \) is not crossing thus \( \{j, g\} \subseteq B_1 \) and \( j \notin B_2 \), or \( \{j, k\} \subseteq B_2 \) and \( j \notin B_1 \). Suppose the former, the proof for the latter is similar and thus will be omitted. We have \( x_{j2} \) integral. However, by Theorem 4 \( j \) is not \( e \)-crossing. Hence \( x_{j2} = 0 \). Thus \( r = x_{j1} \) and \( B_1 = \{j\} \) which gives a contradiction. \( \square \)
8 Characterization of $d(\mathbf{y}, \mathbf{w})$ in $s$

We give a characterization of $d(\mathbf{y}, \mathbf{w})$ that will be used in the remainder of the proof.

**Lemma 8** For each $g \in B_1$ and $k \in B_2$, any column $I$ in $d(\mathbf{y}, \mathbf{w})$ is either of type $(^{*}_{g,k})$ or of type $(^{*}_{s,k})$ or of type $(^{*}_{s,g})$. Moreover, for each $g \in B_1$ and $k \in B_2$, there is $I_k$ of type $(^{*}_{s,k})$, and there is $I_g$ of type $(^{*}_{s,g})$ in $d(\mathbf{y}, \mathbf{w})$. Finally, if there is $i \in B_1 \cup B_2$ such that $x_{i1} + x_{i2} = r$, then either $B_1 = \{i\}$ or $B_2 = \{i\}$.

**Proof:** Let $g \in B_1$ and $k \in B_2$. By Lemma 5 and Theorem 2 there are columns $I_k$ of type $(^{*}_{s,k})$ and $I_g$ of type $(^{*}_{s,g})$ in $d(\mathbf{y}, \mathbf{w})$. By Theorem 2 $I_k$ is either of type $(^{*}_{s,k})$ or of type $(^{*}_{s,g})$, and $I_g$ is either of type $(^{*}_{s,g})$ or of type $(^{*}_{s,k})$. By Theorem 6 we have $\varepsilon_r(g, k) > 0$ and thus by Lemma 2 neither $I_k$ nor $I_g$ is of type $(^{*}_{s,k})$. This proves the first part of the lemma.

If there is $i \in B_1 \cup B_2$ such that $x_{i1} + x_{i2} = r$, then by Theorem 5 the $i$ is not crossing. Hence either $i \in B_1 \setminus B_2$ or $i \in B_2 \setminus B_1$. By Theorem 4 $i$ is not $e$-crossing thus either $B_1 = \{i\}$ or $B_2 = \{i\}$. \hfill $\Box$

**Theorem 7** If there is a job $j$ such that $x_{j1}x_{j2} > 0$, then $B_1 = \{j\}$ or $B_2 = \{j\}$.

**Proof:** Let $x_{j1}x_{j2} > 0$ for a job $j$. Without loss of generality let $j$ be a job with the largest value of $x_{j1} + x_{j2}$ among jobs with $x_{j1}x_{j2} > 0$. Suppose for a contradiction that $B_1 \setminus \{j\} \neq \emptyset$ and $B_2 \setminus \{j\} \neq \emptyset$. Thus if $j \in B_1 \cup B_2$, then $j$ is $e$-crossing. By Theorem 4 $x_{j1} + x_{j2} < r$. Let $g \in B_1 \setminus \{j\}$ and $k \in B_2 \setminus \{j\}$. By Theorem 6 we have $\varepsilon_r(g, j) > 0$ and $\varepsilon_r(j, k) > 0$. Thus by Corollary 1 column of type $(^{*}_{s,j,k})$ does not exist in $d(\mathbf{y}, \mathbf{w})$ or column of type $(^{*}_{s,j,k})$ does not exist in $d(\mathbf{y}, \mathbf{w})$. Suppose the former holds, then by Lemma 8 a column of type $(^{*}_{s,j})$ exists in $d(\mathbf{y}, \mathbf{w})$ which contradicts Lemma 2. For the latter, by Lemma 8 a column of type $(^{*}_{s,j})$ exists in $d(\mathbf{y}, \mathbf{w})$ which contradicts Lemma 2.

If $j \notin B_1 \cup B_2$, then both $x_{j1}$ and $x_{j2}$ are integral. Thus $x_{j1} + x_{j2} < r$. Let $g \in B_1 \setminus \{j\}$ and $k \in B_2 \setminus \{j\}$. We have $\varepsilon_r(g, j) > 0$ and $\varepsilon_r(j, k) > 0$. To prove the former inequality we observe that by our choice of job $j$ for any job $i \in B_1 \cup B_2$ different from $g$ and $j$, and such that $x_{i1} + x_{i2} = r$ must be either $r = x_{i1}$ or $r = x_{i2}$. Otherwise $x_{i1}x_{i2} > 0$, thus $i$ would have been chosen instead of $j$. The proof of the latter inequality follows by a similar argument. Thus by Corollary 1 column of type $(^{*}_{s,j,k})$ does not exist in $d(\mathbf{y}, \mathbf{w})$ or column of type $(^{*}_{s,j,k})$ does not exist in $d(\mathbf{y}, \mathbf{w})$. Suppose the former holds, then by Lemma 8 a column of type $(^{*}_{s,g,k})$ exists in $d(\mathbf{y}, \mathbf{w})$. This column is either of type $(^{*}_{s,j})$ or of type $(^{*}_{s,j,k})$ which implies that the column is of type $(^{*}_{s,j})$. This however contradicts Lemma 2. For the latter, we prove in a similar fashion that a column of type $(^{*}_{s,j})$ exists in $d(\mathbf{y}, \mathbf{w})$ which contradicts Lemma 2. Therefore we get a contradiction which proves the theorem. \hfill $\Box$

8.1 The overlap

An overlap of $B_1$ is a column $I = (M_1, e) \in d(\mathbf{y}, \mathbf{w})$ that matches at least two different jobs from $B_1$ with machines in $G_1$. Similarly, an overlap of $B_2$ is a column $I = (M_1, e) \in d(\mathbf{y}, \mathbf{w})$ that matches at least two different jobs from $B_2$ with machines in $G_2$.

**Lemma 9** An overlap of $B_1$ and an overlap of $B_2$ do not occur simultaneously.
Proof: Suppose for contradiction that both overlaps occur simultaneously. Then there are different jobs \(a\) and \(g\) both from \(B_1\) done on \(G_1\) in a column \(I_{a,g} \in d(y,w)\) of type \(\left(\ast^{*,a,g}\right)\), and different jobs \(b\) and \(k\) both from \(B_2\) done on \(G_2\) in a column \(I_{b,k} \in d(y,w)\) of type \(\left(\ast^{*,b,k}\right)\). By Lemma \([8]\) \(I_{a,g}\) is of type \(\left(\ast^{*,a,b,k,g}\right)\) and \(I_{b,k}\) is of type \(\left(\ast^{*,a,b,k,g}\right)\). This, by Theorem \([6]\) contradicts Lemma \([3]\) (by Theorem \([5]\) there are no crossing jobs thus all four jobs \(a, g, b,\) and \(k\) are different). This proves the lemma. \(\square\)

9 Integral optimal solution to \(\ell p\) for \(\sum_{j \in B_1} \epsilon_j = \epsilon\) or \(\sum_{j \in B_2} \epsilon_j = \epsilon\).

In this section we prove that an integral optimal solution for \(\ell p\) exists if \(\epsilon > 0\) and \(\sum_{j \in B_1} \epsilon_j = \epsilon\) or \(\sum_{j \in B_2} \epsilon_j = \epsilon\). We first prove this assuming \(\sum_{j \in B_2} \epsilon_j = \epsilon\) in Section 9 throughout this section. The proof for \(\sum_{j \in B_2} \epsilon_j = \epsilon\) proceeds in a similar fashion and thus will be omitted.

Consider the following network flow problem \(F\) with variables \(t_{jh}\) for \(j \in J\) and \(h \in G_2\), and \(z_{jh}\) for \(j \in J\) and \(h \in G_1\). The \(r, w,\) and \(x_{jf}\) for \(j \in J\) and \(\ell = 1, 2\) in \(F\) are constants obtained from the solution \(s = (y, x, r, w)\).

\[
F = \max \sum_{j \in B_1} \sum_{h \in G_2} t_{jh}
\]

Subject to

\[
\sum_j t_{jh} = \lfloor w \rfloor \quad h \in G_2
\]

(9.1)

\[
\sum_{h \in G_2} b_{jh} + a_{j1} - \Delta(G_1) + \lfloor r \rfloor - [x_{j1}] \leq \sum_{h \in G_2} t_{jh} \quad j \in J \setminus B_1
\]

(9.2)

\[
\sum_{h \in G_2} b_{jh} + a_{j1} - \Delta(G_1) + \lfloor r \rfloor - [x_{j1}] \leq \sum_{h \in G_2} t_{jh} \leq \sum_{h \in G_2} b_{jh} + a_{j1} - \Delta(G_1) + \lfloor r \rfloor - [x_{j1}] \quad j \in B_1
\]

(9.3)

\[
\sum_{j} z_{jh} = \lfloor w \rfloor \quad h \in G_1
\]

(9.4)

\[
\sum_{h \in G_1} b_{jh} + a_{j2} - \Delta(G_2) + \lfloor r \rfloor - [x_{j2}] \leq \sum_{h \in G_1} z_{jh} \quad j \in J
\]

(9.5)

\[
0 \leq t_{jh} \leq b_{jh} \quad h \in M \quad j \in G_2
\]

(9.6)

\[
0 \leq z_{jh} \leq b_{jh} \quad h \in M \quad j \in G_1
\]

(9.7)

\[
\sum_{h \in G_1} z_{jh} + \sum_{h \in G_2} t_{jh} \leq \lfloor w \rfloor \quad j \in J
\]

(9.8)

**Lemma 10** There is a feasible solution to \(F\) with value

\[
\sum_{j \in B_1} \sum_{h \in G_2} b_{jh} - \sum_{j \in J \setminus B_1} (a_{j1} - x_{j1}) - (|B_1| - 1)(\Delta(G_1) - \lfloor r \rfloor) - \epsilon.
\]

(9.9)
Proof: For s, consider the set \( Y_j \) of all columns of type \( \left( \frac{y}{1} \right) \) in \( d(y, w) \) for \( j \in B_2 \). By Lemma 7, \( l(Y_j) = \beta_j = \lceil \beta_j \rceil + \epsilon_j \). If there is no overlap of \( B_2 \) or \( \sum_{j \in B_2} |\beta_j| > 0 \), then take an interval \( Y \subseteq \bigcup_{j \in B_2} Y_j \) such that \( l(Y) = \epsilon, l(Y \cap Y_j) \geq \epsilon_j \) for \( j \in B_2 \). Otherwise, if there is overlap of \( B_2 \) and \( \sum_{j \in B_2} |\beta_j| = 0 \), then take an interval \( Y \subseteq (\bigcup_{j \in B_2} Y_j) \cup Z \) such that \( l(Y) = \epsilon, l(Y \cap Y_j) \geq \epsilon_j \) for \( j \in B_2 \). Here the \( Z \) is the set of all columns of type \( \left( \frac{y}{1} \right) \) in \( d(y, w) \). In order for such \( Y \) to exist we show that \( l((\bigcup_{j \in B_2} Y_j) \cup Z) \geq 1 \). By Lemma 9, there is no overlap of \( B_2 \), thus \( l(\bigcup_{j \in B_2} W_j) = \epsilon + i \) for some integer \( i \geq 0 \), where \( W_j \) is the set of all columns of type \( \left( \frac{y}{1} \right) \) in \( d(y, w) \) for \( j \in B_1 \). Thus \( l(d(y, w)) = (\bigcup_{j \in B_1} W_j) \) is integral since \( l(d(y, w)) = w, \) and positive. However \( d(y, w) \setminus (\bigcup_{j \in B_2} W_j) = (\bigcup_{j \in B_2} Y_j) \cup Z \) by Theorem 2 and Lemma 8. This proves \( l((\bigcup_{j \in B_2} Y_j) \cup Z) \geq 1 \), and the required \( Y \) exists.

Let \( Y_jh \) be the set of columns \( I \in Y \) such that \( (j, h) \in M_1 \), set \( \gamma_{j} := l(Z_{jh}) \). Informally, \( \gamma_{j} \) is the amount of \( j \) done on \( h \) in the interval \( Y \). We define a truncated solution as follows \( z^*_{j} := y_{j} - \gamma_{j} \) for \( h \in G_1 \), and \( t^*_{j} := y_{j} - \gamma_{j} \) for \( h \in G_2 \). By Theorem 2 each \( j \in B_2 \) is \( d \)-tight thus

\[
\sum_{h \in G_1} \gamma_{j} + \sum_{h \in G_2} \gamma_{j} = \epsilon \quad j \in B_2
\]

and

\[
\sum_{h \in G_2} \gamma_{j} = \eta_{j} \geq \epsilon_{j} \quad j \in B_2.
\]

We prove that this truncated solution is feasible for \( \mathcal{F} \) and meets (9.9).

We first prove the following lemma.

Lemma 11 If \( \sum_{j \in B_2} \epsilon_{j} = \epsilon \), then truncated solution meets (9.5).

Proof: We have the following for the truncated solution.

\[
\sum_{h \in G_1} z^*_{jh} = \sum_{h \in G_1} y_{j} - (\epsilon - \eta_{j}) \quad j \in B_2.
\]

By Lemma 5, each \( j \in B_2 \) is \( \epsilon \)-tight. Thus we get

\[
\sum_{h \in G_1} y_{j} = \sum_{h \in G_1} b_{jh} + a_{j} - \Delta(G_2) - \lfloor x_{j} \rfloor + \lfloor r \rfloor + \epsilon - \epsilon_{j} \quad j \in B_2.
\]

Therefore by (9.12) and (9.13) we get

\[
\sum_{h \in G_1} z^*_{jh} + (\epsilon_{j} - \eta_{j}) = \sum_{h \in G_1} b_{jh} + a_{j} - \Delta(G_2) - \lfloor x_{j} \rfloor + \lfloor r \rfloor \quad j \in B_2,
\]

and by (9.11)

\[
\sum_{h \in G_1} z^*_{jh} \geq \sum_{h \in G_1} b_{jh} + a_{j} - \Delta(G_2) - \lfloor x_{j} \rfloor + \lfloor r \rfloor \quad j \in B_2,
\]

which proves (9.5) holds for the truncated solution \( t^* \) and \( z^* \).

\[\square\]

Let \( t^* \) and \( z^* \) be a solution of Lemma 11. The \( t^* \) and \( z^* \) clearly meet (9.1), (9.4), (9.6), (9.7), (9.8). By Lemma 11 (9.5) holds. Then (9.2) also holds for \( t^* \) and \( z^* \). To show that we observe that by feasibility of \( s = (y, x, w) \) we have

\[
\sum_{h \in G_2} b_{jh} + a_{j} = x_{j} - \Delta(G_1) + r \leq \sum_{h \in G_2} (y_{j} - t^*_{jh}) + \sum_{h \in G_2} t^*_{jh} \quad j \in \mathcal{F} \setminus B_1,
\]
Since for $t^*$ we have
\[ 0 \leq \sum_{h \in G_2} (y_{jh} - t_{jh}^\ast) \leq \epsilon \quad j \in \mathcal{J}, \]
and $x_{j1}$ is integral for $\mathcal{J} \setminus B_1$ the $t^*$ satisfies the (9.2).

To prove (9.3) we observe that by Lemma 5 each $j \in B_1$ is $a$-tight and thus
\[ \sum_{h \in G_2} b_{jh} + a_{j1} - x_{j1} - \Delta(\mathcal{G}_1) + r = \sum_{h \in G_2} (y_{jh} - t_{jh}^\ast) + \sum_{h \in G_2} t_{jh}^\ast \quad j \in B_1. \]  
(9.14)

By Theorem 2 $j \in B_1$ is $d$-tight. Thus by (9.14) and (9.15)
\[ \epsilon = \sum_{h \in G_2} (y_{jh} - t_{jh}^\ast), \]
(9.15)
for $j \in B_1$.

Thus by (9.14) and (9.15)
\[ \sum_{h \in G_2} b_{jh} + a_{j1} - x_{j1} - \Delta(\mathcal{G}_1) + r - [x_{j1}] + \epsilon - \epsilon_j = \sum_{h \in G_2} t_{jh}^\ast \quad j \in B_1. \]
Hence (9.3) is met by the truncated solution $t^\ast$ and $z^\ast$. Therefore the truncated solution $t^\ast$ and $z^\ast$ is feasible for $\mathcal{F}^\ast$.

To prove the lower bound on the value of objective function we observe that by (9.14) and (9.15)
\[ \sum_{h \in G_2} b_{jh} + a_{j1} - x_{j1} - \Delta(\mathcal{G}_1) + r + \epsilon - \epsilon = \sum_{h \in G_2} t_{jh}^\ast \quad j \in B_1. \]  
(9.16)

Summing up (9.16) side by side over all $j \in B_1$ we get by (2.7) for $(y, x, r, w)$
\[ \sum_{j \in B_1} \sum_{h \in G_2} (b_{jh} + a_{j1}) - (r - c) - |B_1| \Delta(\mathcal{G}_1) - [r]) = \sum_{j \in B_1} \sum_{h \in G_2} t_{jh}^\ast, \]
where $c = \sum_{j \in \mathcal{J} \setminus B_1} x_{j1}$ is integral by definition of $B_1$. Thus
\[ \sum_{j \in B_1} \sum_{h \in G_2} b_{jh} + \Delta(\mathcal{G}_1) - [r] - \sum_{j \in \mathcal{J} \setminus B_1} (a_{j1} - x_{j1}) - |B_1| \Delta(\mathcal{G}_1) - [r]) - \epsilon = \sum_{j \in B_1} \sum_{h \in G_2} t_{jh}^\ast \]
and
\[ \sum_{j \in B_1} \sum_{h \in G_2} b_{jh} - \sum_{j \in \mathcal{J} \setminus B_1} (a_{j1} - x_{j1}) - (|B_1| - 1) \Delta(\mathcal{G}_1) - [r]) - \epsilon = \sum_{j \in B_1} \sum_{h \in G_2} t_{jh}^\ast \]
as required. 
\[ \square \]

**Lemma 12** If $\sum_{j \in B_1} \epsilon_j = \epsilon$, then
\[ F = \sum_{j \in B_1} \sum_{h \in G_2} b_{jh} + \sum_{j \in B_1} (a_{j1} - [x_{j1}]) - |B_1| \Delta(\mathcal{G}_1) - [r]) \]
(9.17)
and
\[ \sum_{h \in G_2} t_{jh} = \sum_{h \in G_2} b_{jh} + a_{j1} - [x_{j1}] - \Delta(\mathcal{G}_1) + [r] \quad j \in B_1. \]  
(9.18)
Proof: By (9.16)
\[ \sum_{h \in G_2} b_{jh} + a_{j1} - \Delta(G_1) + [r] - [x_{j1}] - \epsilon_j = \sum_{h \in G_2} t_{jh}^* \quad j \in B_1, \]  
(9.19)
summing up side by side for \( j \in B_1 \) and taking \( \sum_{j \in B_1} \epsilon_j = \epsilon \) we get
\[ \sum_{j \in B_1} \sum_{h \in G_2} b_{jh} + \sum_{j \in B_1} (a_{j1} - [x_{j1}]) - |B_1|(|\Delta(G_1) - [r]) - \epsilon = \sum_{j \in B_1} \sum_{h \in G_2} t_{jh}^*, \]  
(9.20)
for the truncated solution \( t^* \) and \( z^* \), which by Lemma 10 is feasible for \( F \). Let \( t \) and \( z \) be an optimal solution for \( F \). Since all upper and lower bounds in \( F \) are integral, we may assume both \( t \) and \( z \) integral by the Integral Circulation Theorem, see [9]. Thus by (9.20)
\[ \sum_{j \in B_1} \sum_{h \in G_2} b_{jh} + \sum_{j \in B_1} (a_{j1} - [x_{j1}]) - |B_1|(|\Delta(G_1) - [r]) \leq \sum_{j \in B_1} \sum_{h \in G_2} t_{jh}, \]  
(9.21)
and the upper bounds the in (9.3) give
\[ \sum_{j \in B_1} \sum_{h \in G_2} b_{jh} + \sum_{j \in B_1} (a_{j1} - [x_{j1}]) - |B_1|(|\Delta(G_1) - [r]) \geq \sum_{j \in B_1} \sum_{h \in G_2} t_{jh}. \]  
(9.22)
Hence by (9.21) and (9.22) we get
\[ \sum_{j \in B_1} \sum_{h \in G_2} b_{jh} + \sum_{j \in B_1} (a_{j1} - [x_{j1}]) - |B_1|(|\Delta(G_1) - [r]) = \sum_{j \in B_1} \sum_{h \in G_2} t_{jh} = F, \]
which proves (9.17) in the lemma. Finally, in order to reach this optimal value all upper bounds in (9.3) must be reached, which proves (9.18).
\( \square \)

**Theorem 8** For \( \sum_{j \in B_1} \epsilon_j = \epsilon \), an optimal solution to \( F \) can be extended to an integral feasible solution to \( \ell P \) with \( \ell P = [r^*] < r \).

**Proof:** Let \( t \) and \( z \) be an optimal solution to \( F \). This solution exists since by Lemma 10 there is a feasible solution to \( F \). Since all upper and lower bounds in \( F \) are integral, we may assume both \( t \) and \( z \) integral by the Integral Circulation Theorem, see [9]. Thus by Lemma 10
\[ \sum_{j \in B_1} \sum_{h \in G_2} t_{jh} \geq \sum_{j \in B_1} \sum_{h \in G_2} b_{jh} - \sum_{j \in F \setminus B_1} (a_{j1} - x_{j1}) - (|B_1| - 1)(|\Delta(G_1) - [r]). \]  
(9.23)
For the partial solution \((t, z), r' = [r], w' = [w])\ we have: (9.8) implies (2.5), (9.6) and (9.7) imply (2.6), (9.1) implies (2.4), and (9.4) implies (2.3). Let us now extend the solution \((t, z), r' = [r], w' = [w])\ by setting \( x^*_{j2} := [x_{j2}] \) for \( j \in B_2 \) and \( x^*_{j2} := x_{j2} \) for \( j \in F \setminus B_2 \). Since \( \sum_{j \in B_2} \epsilon_j = \epsilon \), (2.8) is met by this extension. Clearly (2.10) is also met for \( \ell = 2 \). By (9.5) we have
\[ \sum_{h \in G_1} b_{jh} + a_{j2} - x_{j2} - \Delta(G_2) + [r] \leq \sum_{h \in G_1} z_{jh} \quad j \in F \setminus B_2. \]
Also, since \( |r - x_{j2}| = |r| - |x_{j2}| \) for \( j \in B_2 \) we have
\[ \sum_{h \in G_1} b_{jh} + a_{j2} - x^*_{j2} - \Delta(G_2) + [r] \leq \sum_{h \in G_1} z_{jh} \]
for \( j \in B_2 \) by (9.5), and thus (2.11) is met for the extended solution \((t, z), r' = [r], w' = [w]\), and \( x^*_j \) for \( j \in \mathcal{J} \).

We now extend this solution further by setting

\[
x^*_{1j} := \sum_{h \in \mathcal{G}_2} b_{jh} + a_{1j} - \Delta(\mathcal{G}_1) + [r] - \sum_{h \in \mathcal{G}_2} t_{jh}
\]

(9.24)

for \( j \in B_1 \) and \( x^*_{1j} := x_{1j} \) for \( j \in \mathcal{J} \setminus B_1 \). To prove that (2.12) is met for the extended solution \((t, z), r' = [r], w' = [w]\), and \( x^*_{j2}, x^*_{j1} \) for \( j \in \mathcal{J} \) we need to show that

\[
\sum_{h \in \mathcal{G}_2} b_{jh} + a_{1j} - x^*_j - \Delta(\mathcal{G}_1) + [r] \leq \sum_{h \in \mathcal{G}_2} t_{jh}
\]

(9.25)

for each \( j \in \mathcal{J} \). By the definition (9.24) this holds for \( j \in B_1 \). For \( j \in \mathcal{J} \setminus B_1 \) we have \( x_{1j} \) integral and thus (9.25) holds since (7.2) holds. Thus (2.12) is met for the extended solution \((t, z), r' = [r], w' = [w]\), and \( x^*_{j2}, x^*_{j1} \) for \( j \in \mathcal{J} \). Moreover \( a_{1j} \geq x^*_{1j} \geq 0 \) for each \( j \) and thus (2.10) holds for \( \ell = 1 \) in this extended solution. It suffices to prove this for \( j \in B_1 \).

Then, since \([r] \geq [r] - [x_{1j}]\), \( x^*_{1j} \geq 0 \) by (9.24) and the right hand side inequality of (9.3). Moreover, \( a_{1j} \geq [x_{1j}] \). Thus by the left hand side inequality of (9.3)

\[
\sum_{h \in \mathcal{G}_2} b_{jh} - \Delta(\mathcal{G}_1) + [r] \leq \sum_{h \in \mathcal{G}_2} t_{jh}
\]

and by (9.24)

\[
x^*_{1j} = \sum_{h \in \mathcal{G}_2} b_{jh} - \Delta(\mathcal{G}_1) + [r] - \sum_{h \in \mathcal{G}_2} t_{jh} + a_{1j} \leq a_{1j}.
\]

Therefore (2.10) holds for \( \ell = 1 \) for \( j \in B_1 \). For \( j \in \mathcal{J} \setminus B_1 \) the (2.10) for \( \ell = 1 \) in the extended solution \((t, z), r' = [r], w' = [w]\), and \( x^*_{j2}, x^*_{j1} \) follows from (2.10) for \( \ell = 1 \) in the solution \((y, x, r, w)\).

By definition of the extended solution \((t, z), r' = [r], w' = [w]\), and \( x^*_{j2}, x^*_{j1} \) for \( j \in \mathcal{J} \), and since by Theorem 5 there are no crossing jobs we have

\[
x^*_{1j} + x^*_{j2} \leq [r]
\]

(9.26)

for \( j \in \mathcal{J} \setminus B_1 \). We now need to show this inequality for \( j \in B_1 \). For these jobs by the left hand side inequality of (9.3), and by (9.24) we get \( x^*_{1j} - [r] + [r] - [x_{1j}] \leq 0 \). Thus \( x^*_{1j} \leq [x_{1j}] \) for each job \( j \in B_1 \). This unfortunately does not guarantee (9.26) for \( j \in B_1 \). However, we either have \([x_{1j}] + x_{2j} \leq [r] \) for each \( j \in B_1 \), in which case (9.26) holds for \( j \in B_1 \), or \([x_{1j}] + x_{2j} > [r] \) for some \( k \in B_1 \). The latter implies \( \sum_{j \in B_1} \epsilon_j = \epsilon \), which by Lemma 12 implies

\[
\sum_{h \in \mathcal{G}_2} t_{jh} = \sum_{h \in \mathcal{G}_2} b_{jh} + a_{1j} - \Delta(\mathcal{G}_1) + [r] - [x_{1j}] \quad j \in B_1
\]

in the optimal solution \( t \) and \( z \) to \( \mathcal{F} \). Thus by definition (9.24), \( x^*_{1j} = [x_{1j}] \) for \( j \in B_1 \). Since by Theorem 5 there are no crossing jobs the (9.26) is satisfied. Hence it remain to prove that if \([x_{1k}] + x_{2k} > [r] \) for some \( k \in B_1 \), then \( \sum_{j \in B_1} \epsilon_j = \epsilon \). For contradiction assume \([x_{1k}] + x_{2k} > [r] \) for some \( k \in B_1 \) and \( \sum_{j \in B_1} \epsilon_j > \epsilon \). If \( x_{1j}x_{2j} = 0 \) for each \( j \in \mathcal{J} \), then \( x_{2j} = 0 \). Thus \([x_{1k}] > [r] \) which implies \( \sum_{j \in B_1} \epsilon_j = \epsilon \) and gives contradiction. Otherwise, if \( x_{1i}x_{2i} > 0 \) for some \( i \in \mathcal{J} \), then by Theorem 7 we have \( B_1 = \{i\} \) or \( B_2 = \{i\} \). If \( B_1 = \{i\} \), then \( \sum_{j \in B_1} \epsilon_j = \epsilon \) which gives contradiction. Hence \( B_2 = \{i\} \) and \( x_{2j} = 0 \) for each \( j \in B_1 \).
Since by Theorem 5 there are no crossing jobs and \( x_{i1} \) is integral and positive. Thus \( x_{i1} \geq 1 \), and \( i \neq k \).

By (2.7) \( \sum_{j} x_{j1} = \sum_{j \neq i} x_{j1} + x_{i1} = r \). Hence \( \sum_{j \neq i} x_{j1} \leq r - 1 \) which gives \( x_{k1} \leq r - 1 \). Since \( x_{k2} = 0 \), we get \( x_{k1} + 1 + x_{k2} \leq r \). Thus \( [x_{k1} + x_{k2}] \leq [r] \) which again gives contradiction. This proves that if \( [x_{k1} + x_{k2}] > [r] \) for some \( k \in B_1 \), then \( \sum_{j \in B_1} \epsilon_j = \epsilon \) as required. Hence (2.9) holds for the extended solution \((t, z), r' = [r], w' = [w])\), and \( x_{j2}, x_{j1}' \).

Finally we need to prove that (2.7) holds for an extended solution. By (9.24) and (9.23)

\[
\sum_{j} x_{j1}' \leq [r]
\]  
(9.27)

for the extended solution \((t, z, [r], [w])\), and \( x_{j2}', x_{j1}' \) for \( j \in J \). This solution satisfies all constraints (2.6) - (2.12) of \( \ell p \). To complete the proof it suffices to modify the extension \( x_{j1}' \) for \( j \in J \) in order to ensure the equality in (9.27) to satisfy (2.7), and to keep other constraint (2.3) - (2.6) and (2.8) - (2.12) of \( \ell p \) satisfied.

If \( \sum_{j} x_{j1}' < [r] \), then take a \( j \in B_1 \) with a positive \( d_j = \min([x_{j1}'] - x_{j1}' - [r] - x_{j1}' - x_{j2}] \). Recall that by Theorem 5 \( x_{j2} \) is integral for each \( j \in B_1 \). Such \( j \) exists. To prove this existence define \( X = \{ j \in B_1 : [x_{j1}'] = x_{j1}' \} \) and \( Y = \{ j \in B_1 : x_{j1}' = [x_{j1}'] \} \). By definition (9.23) and (9.3) we have \( B_1 = X \cup Y \), and since

\[
\sum_{j} x_{j1}' < [r] < \sum_{j} x_{j1}'
\]  
(9.28)

we have \( Y \neq \emptyset \). Suppose for a contradiction that for each \( j \in Y \) we have \( [r] = x_{j1}' + x_{j2} \). Thus we have

\[
\sum_{j} x_{j1}' = \sum_{j \in J \setminus B_1} x_{j1} + \sum_{j \in X} [x_{j1}] + \sum_{j \in Y} [x_{j1}] < [r].
\]

Since for each \( j \in Y \) we have \( [r] = [x_{j1}] + x_{j2} \), we obtain

\[
\sum_{j \in J \setminus B_1} x_{j1} + \sum_{j \in X} [x_{j1}] + |Y||[r] - \sum_{j \in Y} x_{j2} < [r],
\]
and by (9.28) the set \( Y \) is not empty. Since \( \sum_{j \in Y} x_{j2} \leq [r] \) by (2.8) we get

\[
\sum_{j \in J \setminus B_1} x_{j1} + \sum_{j \in X} [x_{j1}] + |Y||[r] < 2[r],
\]

and thus \( |Y| \leq 1 \), and since \( Y \) is not empty we have \( |Y| = 1 \). However

\[
[r] = \sum_{j} x_{j1}' = \sum_{j \in X} [x_{j1}] + \sum_{j \in B_1} \epsilon_j,
\]

where

\[
\lfloor \sum_{j \in B_1} \epsilon_j \rfloor \leq |B_1| - 1.
\]

Thus

\[
[r] = \sum_{j} x_{j1}' \leq \sum_{j \in J \setminus B_1} x_{j1} + \sum_{j \in B_1} [x_{j1}] + |B_1| - 1 = \sum_{j \in J \setminus B_1} x_{j1} + \sum_{j \in X} [x_{j1}] + \sum_{j \in Y} [x_{j1}].
\]
since \(|Y| = 1\) which contradicts (9) and proves that \(j \in Y\) with \(d_j = 1\) exists. Set \(d := \min\{\min_{d_j > 0}(d_j), |r| - \sum_j x_{x_j}^j\} = 1\). Then, set \(x_{x_j}^j := x_{x_j}^j + 1\) for some \(j \in Y\) with \(d_j = 1\). We have \(x_{x_j}^j \leq \min\{x_{x_j}^j, |r| - x_{x_j}^j\}\) and \(\sum_j x_{x_j}^j \leq |r|\) for the new extended solution, which ensures that all constraints (2.3)-(2.6) and (2.8)-(2.12) of \(\ell_p\) are met in the new extended solution. Since \(d = 1\) the \(\sum_j x_{x_j}^j\) gets closer to but does not exceed \(|r|\). Therefore by (9.28) we finally reach an extended solution \(t, z, x_{x_j}^j\) for \(j \in J\) that meets all (2.3)-(2.12) of \(\ell_p\). The solution is integral with \(w' = \lfloor w \rfloor\), and \(r' = \lfloor r' \rfloor\) which proves the lemma.

\[\square\]

10 The Projection

Consider the following system \(S\) that defines the set of feasible solutions to the \(LP\)-relaxation of \(ILP\),

\[
\sum_j b_{jh} - (\Delta(G_2) - r) \leq \sum_j y_{jh} \leq w \quad h \in G_1
\]  
(10.1)

\[
\sum_j b_{jh} - (\Delta(G_1) - r) \leq \sum_j y_{jh} \leq w \quad h \in G_2
\]  
(10.2)

\[
\sum_h y_{jh} \leq w \quad j \in J
\]  
(10.3)

\[
0 \leq y_{jh} \leq b_{jh} \quad h \in M \quad j \in J
\]  
(10.4)

\[
\sum_j x_{j1} = r
\]  
(10.5)

\[
\sum_j x_{j2} = r
\]  
(10.6)

\[
x_{j1} + x_{j2} \leq r \quad j \in J
\]  
(10.7)

\[
0 \leq x_{j\ell} \leq a_{j\ell} \quad j \in J \quad \ell = 1, 2
\]  
(10.8)

\[
\sum_{h \in G_1} (b_{jh} - y_{jh}) + a_{j2} - x_{j2} \leq \Delta(G_2) - r \quad j \in J
\]  
(10.9)

\[
\sum_{h \in G_2} (b_{jh} - y_{jh}) + a_{j1} - x_{j1} \leq \Delta(G_1) - r \quad j \in J
\]  
(10.10)

Now consider the system \(S_r\) obtained from \(S\) by dropping (10.5) and (10.6) and adding the constraints (10.19), (10.20), and (10.21). We use \(a_{j1} = \sum_{h \in G_2}(b_{jh} - y_{jh}) + a_{j1} - \Delta(G_1)\) and \(a_{j2} = \sum_{h \in G_1}(b_{jh} - y_{jh}) + a_{j2} - \Delta(G_2)\) for \(j \in J\) for convenience.

\[
\sum_j b_{jh} - (\Delta(G_2) - r) \leq \sum_j y_{jh} \leq w \quad h \in G_1
\]  
(10.11)

\[
\sum_j b_{jh} - (\Delta(G_1) - r) \leq \sum_j y_{jh} \leq w \quad h \in G_2
\]  
(10.12)

\[
\sum_h y_{jh} \leq w \quad j \in J
\]  
(10.13)
\begin{align*}
0 & \leq y_{jh} \leq b_{jh} \quad h \in M \quad j \in \mathcal{J} \quad (10.14) \\
x_{j1} + x_{j2} & \leq r \quad j \in \mathcal{J} \quad (10.15) \\
0 & \leq x_{j\ell} \leq a_{j\ell} \quad j \in \mathcal{J} \quad \ell = 1, 2 \quad (10.16) \\
\sum_{h \in G_1} (b_{jh} - y_{jh}) + a_{j2} - x_{j2} \leq \Delta(G_2) - r \quad j \in \mathcal{J} \quad (10.17) \\
\sum_{h \in G_2} (b_{jh} - y_{jh}) + a_{j1} - x_{j1} \leq \Delta(G_1) - r \quad j \in \mathcal{J} \quad (10.18) \\
\sum_j \alpha_{j1} + (n - 1)r & \leq 0 \quad j \in \mathcal{J} \quad (10.19) \\
\sum_j \alpha_{j2} + (n - 1)r & \leq 0 \quad j \in \mathcal{J} \quad (10.20) \\
0 & \leq r \leq \min\{\Delta(G_1), \Delta(G_2)\} \quad (10.21)
\end{align*}

Finally consider the following projection on \( y, w, r \).

**Lemma 13** Let \( P \) be the polyhedron that consists of feasible solutions to \( S_r \). Then the projection of \( P \) on \( y, w, r \), denoted by \( Q \), is the set of solutions to the following system of inequalities \( Q \):

\begin{align*}
\sum_j b_{jh} - (\Delta(G_2) - r) & \leq \sum_j y_{jh} \leq w \quad h \in G_1 \quad (10.22) \\
\sum_j b_{jh} - (\Delta(G_1) - r) & \leq \sum_j y_{jh} \leq w \quad h \in G_2 \quad (10.23) \\
\sum_h y_{jh} & \leq w \quad j \in \mathcal{J} \quad (10.24) \\
0 & \leq y_{jh} \leq b_{jh} \quad h \in M \quad j \in \mathcal{J} \quad (10.25) \\
\sum_{h \in G_2} (b_{jh} - y_{jh}) + a_{j1} - \Delta(G_1) & \leq 0 \quad j \in \mathcal{J} \quad (10.26) \\
\sum_{h \in G_1} (b_{jh} - y_{jh}) + a_{j2} - \Delta(G_2) & \leq 0 \quad j \in \mathcal{J} \quad (10.27) \\
\sum_{h \in G_2} (b_{jh} - y_{jh}) + r - \Delta(G_1) & \leq 0 \quad j \in \mathcal{J} \quad (10.28) \\
\sum_{h \in G_1} (b_{jh} - y_{jh}) + r - \Delta(G_2) & \leq 0 \quad j \in \mathcal{J} \quad (10.29) \\
\sum_{h} (b_{jh} - y_{jh}) + a_{j1} + a_{j2} - \Delta(G_1) - \Delta(G_2) + r & \leq 0 \quad j \in \mathcal{J} \quad (10.30) \\
\sum_j \alpha_{j1} + (n - 1)r & \leq 0 \quad j \in \mathcal{J} \quad (10.31)
\end{align*}
\[ \sum_j \alpha_j + (n - 1)r \leq 0 \quad j \in \mathcal{J} \]  

(10.32)

\[ 0 \leq r \leq \min(\Delta(G_1), \Delta(G_2)) \]  

(10.33)

**Proof:** By Fourier-Motzkin elimination, see [[11]], of variables \( x_j \) from the system \( S_r \).

We summarize the results of this section in the following theorem and lemma.

**Theorem 9** Let \( (y, r, w) \) be feasible for \( Q \). There exists \( x \) such that the solution \( (y, x, w, r) \) is feasible for \( S_r \).

**Proof:** Let \( s = (y, r, w) \) be a feasible solution for \( Q \). By Lemma [[13]] there exist \( x = (x_j) \), where \( j \in \mathcal{J} \) and \( \ell = 1, 2 \), such that \( s = (y, x, w, r) \) is feasible for \( S_r \). Let \( X \) be the set of all such \( x \). Take \( x \in X \) with minimum distance \( d = |r - \sum_j x_j| + |r - \sum_j x_j^2| \). We show that \( d = 0 \) for \( x \). Suppose that \( r < \sum_j x_j \) or \( r < \sum_j x_j^2 \). Let \( r < \sum_j x_j^1 \). If there is \( k \) such that \( x_{k1} + r < x_{k1} \), then set \( x_{k1} \leftarrow x_{k1} - \alpha \) where \( \alpha = \min\{x_{k1} - (a_k + r), \sum_j x_j^1 - r\} \). The new solution is in \( X \) and reduces \( d \) which gives a contradiction. Thus we have \( x_j = x_j^1 \) for each \( j \). Therefore \( \sum_j \alpha_j + nr = \sum_j x_j^1 \leq r \) by the constraint (10.31) which contradicts this case assumption. The proof for \( r < \sum_j x_j^2 \) is similar. Therefore we have \( r \geq \sum_j x_j^1 \) and \( r \geq \sum_j x_j^2 \) for the \( x \). Suppose that \( r > \sum_j x_j^1 \) or \( r > \sum_j x_j^2 \). If there is \( k \) such that \( x_{k1} + x_{k2} < r \) and \( x_{k1} < a_{k1} \) or \( x_{k2} < a_{k2} \), then set \( x_{k1} + \lambda \), where \( \lambda = \min\{r - (x_{k1} + x_{k2}), a_{k1} - x_{k1}, d\} \) provided \( x_{k1} < a_{k1} \). Otherwise, if \( x_{k1} = a_{k1} \) and \( x_{k2} < a_{k2} \), set \( x_{k2} + \lambda \), where \( \lambda = \min\{r - (x_{k1} + x_{k2}), a_{k2} - x_{k2}, d\} \). The new solution is in \( X \) but has smaller \( d \) which gives a contradiction. Thus we have \( x_j = x_j^1 = r \) or \( x_j = x_j^2 = r \) for each \( j \). We have at least one \( j \) with \( x_j^1 + x_j^2 = r \). Otherwise, \( r > \min(\Delta(G_1), \Delta(G_2)) \) which contradicts (10.33). On the other hand, we can have at most one \( j \) with \( x_j^1 + x_j^2 = r \). Otherwise, \( \sum_j (x_j^1 + x_j^2) \geq 2r \) and since \( r \geq \sum_j x_j^1 \) and \( r \geq \sum_j x_j^2 \), we get \( r = \sum_j x_j^1 \) and \( r = \sum_j x_j^2 \) which contradicts the assumption. Therefore there is exactly one \( j \) such that \( x_j^1 + x_j^2 = r \), and \( x_{k1} = a_{k1} \), and \( x_{k2} = a_{k2} \) for \( k \in \mathcal{J} \setminus \{j\} \). Hence \( \Delta(G_1) - a_j + x_j^1 < r \) or \( \Delta(G_2) - a_j + x_j^2 < r \). Since \( \Delta(G_1) - a_j + x_j^1 \leq r \) and \( \Delta(G_2) - a_j + x_j^2 \leq r \), we have \( \Delta(G_1) + \Delta(G_2) - a_j + x_j^1 + x_j^2 < 2r \). Hence \( \Delta(G_1) + \Delta(G_2) - a_j + x_j^1 < r \) since \( x_j^1 + x_j^2 = r \). However by (10.30) and (10.25) we have \( a_j + a_j + r \leq \Delta(G_1) + \Delta(G_2) \) which gives a contradiction. Thus we have \( d = 0 \) and the solution is feasible for \( S \).

We have the following lemma.

**Lemma 14** If \( (y, x, r, w) \) is feasible for \( S_r \), then \( (y, r, w) \) is feasible for \( Q \).

**Proof:** If \( (y, x, r, w) \) is feasible for \( S_r \), then it is also feasible for \( S_r \). Observe that (10.5), (10.6), and (10.8) in \( S \) imply (10.21) in \( S_r \), the (10.9) in \( S \) implies (10.19) in \( S_r \), and the (10.10) in \( S \) implies (10.20) in \( S_r \). Finally, by Lemma [[13]] the \( (y, r, w) \) is feasible for \( Q \).

The system \( Q \) is a network flow model with lower and upper bounds on the arcs for fixed \( w \) and \( r \).
11 Integral Optimal Solution to $\ell p$ for $\sum_{j \in B_i} \varepsilon_j > \varepsilon$ for $i = 1, 2$

Consider $s$ with $\sum_{j \in B_i} \varepsilon_j > \varepsilon$ for $i = 1, 2$, by Lemma 9, overlap of $B_1$ and of $B_2$ does not occur simultaneously. Without loss of generality let us assume no overlap of $B_2$.

Consider the set $Y_j$ of all columns of type $\left(c^*_{x_j}\right)$ in $d(y, w)$ for $j \in B_2$. By Lemma 7, $l(Y_j) = \beta_j = |\beta_j| + \varepsilon_j$. Take an interval $T$ for Theorem 10. The constraints (10.26) and (10.27): For the amount of $j$ done on $h$ in the interval $Y$. We define a truncated solution as follows $\varepsilon^*_j := y_jh - y_jh$ for $h \in G_1$, and $t^*_j := y_jh - x_jh$ for $h \in G_2$, and $[r], [w]$. Thus

$$\sum_{h \in G_1} \gamma_jh + \sum_{h \in G_2} \gamma_jh \leq \varepsilon \quad j \in J$$

Theorem 10 For $\sum_{j \in B_i} \varepsilon_j > \varepsilon$ for $i = 1, 2$, there is a feasible integral solution to $\ell p$ with $lp = \lfloor r \rfloor < r$.

Proof: The constraints (10.26) and (10.27): For $s$ we have

$$\sum_{h \in G_1} b_jh + a_j2 - \Delta(G_2) + r - x_j2 \leq \sum_{h \in G_1} y_jh \quad j \in J$$

$$\sum_{h \in G_2} b_jh + a_j1 - \Delta(G_1) + r - x_j1 \leq \sum_{h \in G_2} y_jh \quad j \in J.$$ 

If $r - x_{j1} \geq \varepsilon$ and $r - x_{j2} \geq \varepsilon$ for each $j \in J$, then $\sum_{h \in G_1} y_jh - (r - x_j2) \leq \sum_{h \in G_1} t_jh$ and $\sum_{h \in G_2} y_jh - (r - x_j1) \leq \sum_{h \in G_2} t_jh$ for each $j$, Hence (10.26) and (10.27) hold for the truncated solution. Otherwise, if $r - x_{j1} < \varepsilon$ or $r - x_{j2} < \varepsilon$ for some $j \in J$, then $[r] \leq x_{j1}$ or $[r] \leq x_{j2}$ for some $j$. This implies $\sum_{j \in B_i} \varepsilon_j = \varepsilon$ or $\sum_{j \in B_2} \varepsilon_j = \varepsilon$ which contradicts the theorem’s assumption.

The constraints (10.28) and (10.29): For $s$ we have

$$\sum_{h \in G_2} b_jh + r - \Delta(G_1) + a_j1 - x_j1 \leq \sum_{h \in G_2} y_jh \quad j \in J$$

$$\sum_{h \in G_1} b_jh + r - \Delta(G_2) + a_j2 - x_j2 \leq \sum_{h \in G_1} y_jh \quad j \in J.$$ 

By constraint (2.10) and definition of the truncated solution

$$\sum_{h \in G_1} b_jh + [r] - \Delta(G_2) \leq \sum_{h \in G_1} y_jh - \varepsilon \leq \sum_{h \in G_1} t_jh \quad j \in J$$

Hence (10.28) and (10.29) hold.

The constraints (10.30): For $s$, by (2.11) and (2.12) we have

$$\sum_{h \in G_1} (b_jh - y_jh) + a_j2 - x_j2 \leq \Delta(G_2) - r$$

and

$$\sum_{h \in G_2} (b_jh - y_jh) + a_j1 - x_j1 \leq \Delta(G_1) - r,$$
by summing up the two side by side we get
\[ \sum_h (b_{jh} - y_{jh}) + a_{j1} + a_{j2} - x_{j1} - x_{j2} \leq \Delta(G_1) + \Delta(G_2) - 2r \]
or
\[ \sum_h b_{jh} + a_{j1} + a_{j2} - \Delta(G_1) - \Delta(G_2) + |r| \leq \sum_h y_{jh} - r + x_{j1} + x_{j2} - \epsilon. \]
Since \( \sum_h y_{jh} - \epsilon \leq \sum_h t_{jh} \), we have
\[ \sum_h y_{jh} - r + x_{j1} + x_{j2} - \epsilon \leq \sum_{h \in G_1} t_{jh} - r + x_{j1} + x_{j2} \]
But \(-r + x_{j1} + x_{j2} \leq 0\) by the constraint (2.9) and thus we get
\[ \sum_h b_{jh} + a_{j1} + a_{j2} - \Delta(G_1) - \Delta(G_2) + |r| \leq \sum_h t_{jh} \]
which proves that (10.30) holds. The constraints (10.22) - (10.25) are satisfied by definition of truncated solution. Finally, since \(|G_1| \leq n - 1\) and \(|G_2| \leq n - 1\) we have the constraints (10.31) and (10.32) satisfied by the truncated solution. Observe that \(|G_1| > n - 1\) or \(|G_2| > n - 1\) implies \(|G_1| = 0\) or \(|G_2| = 0\) since \(|G_1| + |G_2| \leq n\). This however contradicts the saturation.

Therefore the truncated solution \((y^* = (z^*, t^*), [r], [w])\) is feasible for \(Q\), and by Theorem 9 there exists \(x^*\) such that \((y^* = (z^*, t^*), x^*, [r], [w])\) is feasible for \(S\). Moreover \([r^*] \leq [r]\) and \([w] - [r] = [w^* - r^*]\) since \(s = (y, x, r, w)\) is feasible for \(\ell p\). Thus the solution \((y^* = (z^*, t^*), x^*, [r], [w])\) is feasible for \(\ell p\) and \(lp = [r]\). For a feasible solution to \(Q\) with integral \([w]\) and \([r]\) all lower and upper bounds in the network \(Q\) are integral thus we can find in polynomial time an integral \(y^*\). Finally for given integral and fixed \([r], [w]\) and \(y^*\) the \(S\) becomes a network flow model with integral lower and upper bounds on the flows. Thus we can find in polynomial time an integral \(x^*\) such that the integer solution \((y^*, x^*, [r], [w])\) is feasible for \(lp\) and \(lp = [r]. \)

\[ \square \]

12 The proof of the conjecture

We are now ready to prove Theorem 1 which proves the conjecture.

**Proof:** For contradiction suppose the optimal value for \(\ell p\) is fractional, \(lp = r = [r^*] + \epsilon\), where \(\epsilon > 0\). By Theorem 8 there is a feasible integral solution to \(\ell p\) with \(lp = [r^*]\) for \(\sum_{j \in B_1} \epsilon_j = \epsilon\) or \(\sum_{j \in B_2} \epsilon_j = \epsilon\). By Theorem 10 there is a feasible integral solution to \(\ell p\) with \(lp = [r^*]\) for \(\sum_{j \in B_1} \epsilon_j > \epsilon\) and \(\sum_{j \in B_2} \epsilon_j > \epsilon\). Thus there is a feasible integral solution for \(\ell p\) with \([r^*] < r\). Hence there is a feasible solution to \(\ell p\) which is smaller than optimal \(r\) which gives contradiction and proves the first part of the theorem. Thus optimal \(s\) has both \(r\) and \(w\) integer. The \(s\) is feasible for \(S\) and thus it is feasible for \(Q\) by Lemma 14. For a feasible solution to \(Q\) with integral \(w\) and \(r\) all lower and upper bounds in the network \(Q\) are integral thus we can find in polynomial time an integral \(y\). Finally for given integral and fixed \(r, w\) and \(y\) the \(S\) becomes a network with integral lower and upper bounds on the flows. Thus we can find in polynomial time an integral \(x\) such that the integer solution \((y, x, r, w)\) is feasible for \(lp\) and \(lp = r. \)

\[ \square \]
Acknowledgements

The author is grateful to Dominic de Werra for his insightful comments.

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