THE GRAPH CASES OF THE RIEMANNIAN POSITIVE MASS
AND PENROSE INEQUALITIES IN ALL DIMENSIONS

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ABSTRACT. We consider complete asymptotically flat Riemannian manifolds
that are the graphs of smooth functions over $\mathbb{R}^n$. By recognizing the scalar
curvature of such manifolds as a divergence, we express the ADM mass as an
integral of the product of the scalar curvature and a nonnegative potential func-
tion, thus proving the Riemannian positive mass theorem in this case. If the
graph has convex horizons, we also prove the Riemannian Penrose inequality by
giving a lower bound to the boundary integrals using the Aleksandrov-Fenchel
inequality.

1. INTRODUCTION

The Riemannian positive mass theorem states that an asymptotically flat Rie-
mannian manifold $M^n$ with nonnegative scalar curvature has nonnegative ADM
mass, and that the ADM mass is strictly positive unless $M^n$ is isometric to flat $\mathbb{R}^n$.
The theorem was first proved in 1979 by Schoen and Yau [12, 13] for manifolds of
dimension $n \leq 7$ using minimal surface techniques. Witten [15] later proved the
theorem for spin manifolds of any dimension using spinors and the Dirac operator.

The Penrose inequality can be viewed as a generalization of the positive mass
theorem in the presence of an area outer minimizing horizon. A horizon is simply
a minimal surface, and we say that it is area outer minimizing if every other sur-
face which encloses it has greater area. When an asymptotically flat Riemannian
manifold $M^n$ with nonnegative scalar curvature contains an area outer minimizing
horizon $\Sigma$, the Riemannian Penrose inequality gives a lower bound for the ADM
mass $m$ in terms of the $(n-1)$ volume $A$ of $\Sigma$ and the volume $\omega_{n-1}$ of the unit
$(n-1)$ sphere:

$$m \geq \frac{1}{2} \left( \frac{A}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}},$$

with equality if and only if $M^n$ is isometric to a Schwarzschild metric. This in-
equality was first proved in dimension $n = 3$ by Huisken and Illmanen [8] in 1997
for the case of a single horizon using inverse mean curvature flow, a method origin-
ally proposed by Geroch [7], Jang and Wald [9]. In 1999, Bray [3] extended this
result to the general case of a horizon with multiple components using a conformal
flow of metrics. Note that in dimension three, the Riemannian Penrose inequality
is $m \geq \sqrt{A/16\pi}$. Later, Bray and Lee [6] generalized Bray’s proof for dimen-
sions $n \leq 7$, with the extra requirement that $M$ be spin for the rigidity statement.

To the author’s knowledge, there are no known proofs of the Riemannian Penrose
inequality in dimensions $n \geq 8$ other than in the spherically symmetric cases.

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The case of equality of the Riemannian Penrose inequality is attained by the Schwarzschild metric. It is conformal to $\mathbb{R}^n \backslash \{0\}$ and may be expressed as
\[
\left( \mathbb{R}^n \backslash \{0\}, \left( 1 + \frac{m}{2|x|^{n-2}} \right)^{4/(n-2)} \delta \right),
\]
where $m$ is a positive constant and $\delta$ is the flat Euclidean metric. Moreover, when $n = 3$, the Schwarzschild metric $(M^3, g) = (\mathbb{R}^3 \backslash \{0\}, (1 + m/2|x|)^4 \delta)$ can also be isometrically embedded as a rotating parabola in $\mathbb{R}^4$ as the set of points $\{(x, y, z, w)\} \subset \mathbb{R}^4$ satisfying $|x, y, z| = \frac{w^2}{8m} + 2m$.

Solving for $w$, we see that the end of the three dimensional Schwarzschild metric containing infinity, called the outer end, is the graph of the spherically symmetric function $f: \mathbb{R}^3 \backslash B_{2m}(0) \to \mathbb{R}$ given by $f(r) = \sqrt{8m(r - 2m)}$, where $r = |(x, y, z)|$. In this case, one can check directly that the ADM mass of $(M^3, g)$ is the positive constant $m$ by computing a certain boundary integral at infinity involving the function $f$.

That an end of the three dimensional Schwarzschild metric can be isometrically embedded in $\mathbb{R}^4$ as the graph of a function over $\mathbb{R}^3 \backslash B_{2m}(0)$ raises the following questions: if $\Omega$ is a bounded open set in $\mathbb{R}^n$ and $f$ is a smooth function on $\mathbb{R}^n \backslash \Omega$ such that the graph of $f$ is an asymptotically flat manifold $M$ with nonnegative scalar curvature $R$ and horizon $f(\partial \Omega)$, can we prove the Penrose inequality for $M$ using elementary techniques in this setting? And if so, do we get a stronger statement than the standard Penrose inequality? We answer both questions in the affirmative, and we will begin by proving a stronger version of the Riemannian positive mass theorem for manifolds that are graphs over $\mathbb{R}^n$ by expressing $R$ as a divergence and applying the divergence theorem, giving the ADM mass as an integral over the manifold of the product of $R$ and a nonnegative potential function. In the presence of a boundary whose connected components are convex, we prove a...
stronger Penrose inequality by giving lower bounds to the boundary integrals using the Aleksandrov-Fenchel inequality. Before we state our theorems, we begin with some definitions.

**Definition 1.** \cite{1} A complete Riemannian manifold \((M^n, g)\) of dimension \(n\) is said to be **asymptotically flat** if there is a compact subset \(K \subset M^n\) such that \(M^n \setminus K\) is diffeomorphic to \(\mathbb{R}^n \setminus \{|x| \leq 1\}\), and a diffeomorphism \(\Phi : M^n \setminus K \rightarrow \mathbb{R}^n \setminus \{|x| \leq 1\}\) such that, in the coordinate chart defined by \(\Phi\), \(g = g_{ij}(x) dx^i dx^j\), where

\[
    g_{ij}(x) = \delta_{ij} + O(|x|^{-p})
\]

\[
    |x||g_{ij,k}(x)| + |x|^2|g_{ij,kl}(x)| = O(|x|^{-p})
\]

\[
    |R(g)(x)| = O(|x|^{-q})
\]

for some \(q > n\) and \(p > (n - 2)/2\).

**Remark 2.** Note that all this means is that outside a compact set, \(M^n\) is diffeomorphic to \(\mathbb{R}^n\) minus a closed ball and the metric \(g\) decays sufficiently fast to the flat metric at infinity. The constants \(p\) and \(q\) are chosen so that the ADM mass (see below) is finite.

For such an asymptotically flat manifold \((M^n, g)\), we can define its total mass (called the ADM mass):

**Definition 3.** \cite{1} The **ADM mass** \(m\) of a complete, asymptotically flat manifold \((M^n, g)\) is defined to be

\[
    m = \lim_{r \to \infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} \sum_{i,j} (g_{ij,i} - g_{ii,j})\nu_j dS_r,
\]

where \(\omega_{n-1}\) is the volume of the \(n - 1\) unit sphere, \(S_r\) is the coordinate sphere of radius \(r\), \(\nu\) is the outward unit normal to \(S_r\) and \(dS_r\) is the area element of \(S_r\) in the coordinate chart.

This definition in dimension three was originally due to Arnowitt, Deser and Misner \cite{1}. Bartnik \cite{2} showed that the ADM mass is independent of the choice of asymptotically flat coordinates.

Given a smooth function \(f : \mathbb{R}^n \rightarrow \mathbb{R}\), the graph of \(f\) is a complete Riemannian manifold. Since the graph of \(f\) with the induced metric from \(\mathbb{R}^{n+1}\) is isometric to \((M^n, g) = (\mathbb{R}^n, \delta + df \otimes df)\), we will from now on refer to \((M^n, g)\) as the graph of \(f\). For such a graph, we can rephrase the notion of asymptotic flatness in terms of the function \(f\):

**Definition 4.** Let \(f : \mathbb{R}^n \rightarrow \mathbb{R}\) be a smooth function and let \(f_i\) denote the \(i\)th partial derivative of \(f\). We say that \(f\) is **asymptotically flat** if

\[
    f_i(x) = O(|x|^{-p/2})
\]

\[
    |x||f_{ij}(x)| + |x|^2|f_{ijk}(x)| = O(|x|^{-p/2})
\]

at infinity for some \(p > (n - 2)/2\).

We can now give the precise statement of our first theorem:

**Theorem 5** (Positive mass theorem for graphs over \(\mathbb{R}^n\)). Let \((M^n, g)\) be the graph of a smooth asymptotically flat function \(f : \mathbb{R}^n \rightarrow \mathbb{R}\) with the induced metric from \(\mathbb{R}^{n+1}\). Let \(R\) be the scalar curvature and \(m\) the ADM mass of \((M^n, g)\). Let \(\nabla f\)
denote the gradient of $f$ in the flat metric and $|\nabla f|$ its norm with respect to the flat metric. Let $dV_g$ denote the volume form on $(M^n, g)$. Then

$$m = \frac{1}{2(n-1)\omega_{n-1}} \int_{M^n} \frac{1}{\sqrt{1 + |\nabla f|^2}} dV_g.$$  

In particular, $R \geq 0$ implies $m \geq 0$.

Now let $\Omega$ be a bounded open set in $\mathbb{R}^n$ with $\Sigma = \partial \Omega$ and $f$ a smooth function on $\mathbb{R}^n \setminus \Omega$. If $f(\Sigma)$ is contained in a level set of $f$, then the mean curvature $H$ of $f(\Sigma)$ in $(M^n, g)$ and the mean curvature $H_0$ with respect to the flat metric $\delta$ are related by

$$H = \frac{1}{\sqrt{1 + |\nabla f|^2}} H_0.$$  

Thus if $|\nabla f(x)| \to \infty$ as $x \to \Sigma$, then $f(\Sigma)$ is a horizon in $(M^n, g)$.

**Theorem 6.** Let $\Omega$ be a bounded and open (but not necessarily connected) set in $\mathbb{R}^n$ and $\Sigma = \partial \Omega$. Let $f : \mathbb{R}^n \setminus \Omega \to \mathbb{R}$ be a smooth asymptotically flat function such that each connected component of $f(\Sigma)$ is in a level set of $f$ and $|\nabla f(x)| \to \infty$ as $x \to \Sigma$. Let $(M^n, g)$ be the graph of $f$ with the induced metric from $\mathbb{R}^n \setminus \Omega \times \mathbb{R}$ and ADM mass $m$. Let $H_0$ be the mean curvature of $\Sigma$ in $(\mathbb{R}^n \setminus \Omega, \delta)$. Then

$$m = \frac{1}{2(n-1)\omega_{n-1}} \int_{\Sigma} H_0 d\Sigma + \frac{1}{2(n-1)\omega_{n-1}} \int_{M^n} \frac{1}{\sqrt{1 + |\nabla f|^2}} dV_g.$$  

Let $\Omega_i$ be the connected components of $\Omega$, $i = 1, \ldots, k$, and let $\Sigma_i = \partial \Omega_i$. If we assume that each $\Omega_i$ is convex, then we have the following Penrose inequality:

**Corollary 7** (Penrose inequality for graphs on $\mathbb{R}^n$ with convex boundaries). With the same hypotheses as in Theorem 6 and the additional assumption that each connected component $\Omega_i$ of $\Omega$ is convex, then

$$m \geq \sum_{i=1}^{k} \frac{1}{2} \left( \frac{|\Sigma_i|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} + \frac{1}{2(n-1)\omega_{n-1}} \int_{M^n} \frac{1}{\sqrt{1 + |\nabla f|^2}} dV_g.$$  

In particular,

$$R \geq 0 \text{ implies } m \geq \sum_{i=1}^{k} \frac{1}{2} \left( \frac{|\Sigma_i|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}.$$  

**Remark 8.** Since

$$\sum_{i=1}^{k} \frac{1}{2} \left( \frac{|\Sigma_i|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} \geq \frac{1}{2} \left( \frac{\sum_{i=1}^{k} |\Sigma_i|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} = \frac{1}{2} \left( \frac{A}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}},$$

Corollary 7 is a stronger statement than (1) on top of the fact that the lower bound for the ADM mass involves a nonnegative integral when the scalar curvature is nonnegative.

2. **Positive Mass Theorem for Graphs over $\mathbb{R}^n$**

Let $(M^n, g) = (\mathbb{R}^n, \delta + df \otimes df)$ be the graph of a smooth asymptotically flat function $f : \mathbb{R}^n \to \mathbb{R}$. Since $g_{ij} = \delta_{ij} + f_i f_j$, the inverse of $g_{ij}$ is

$$g^{ij} = \delta^{ij} - \frac{f^i f^j}{1 + |\nabla f|^2},$$
where the norm of $\nabla f$ is taken with respect to the flat metric $\delta$ on $\mathbb{R}^n$. We first compute the Christoffel symbols $\Gamma^k_{ij}$ of $(M^n, g)$:

$$\Gamma^k_{ij} = \frac{1}{2} g^{km} (g_{im,j} + g_{jm,i} - g_{ij,m})$$

$$= \frac{1}{2} \left( \delta^{km} - \frac{f^k f^m}{1 + |\nabla f|^2} \right) (f_{ij} f_m + f_i f_{jm} + f_j f_{im} - f_{im} f_j - f_i f_{jm})$$

$$= \frac{1}{2} \left( \delta^{km} - \frac{f^k f^m}{1 + |\nabla f|^2} \right) 2 f_{ij} f_m$$

$$= f_{ij} f^k - \frac{f_{ij} f^k |\nabla f|^2}{1 + |\nabla f|^2}$$

$$= \frac{f_{ij} f^k}{1 + |\nabla f|^2}.$$

**Remark 9.** Since the indices are raised and lowered using the flat metric on $\mathbb{R}^n$, it will be notationally more convenient from now on to write everything as lower indices, with the implicit assumption that any repeated indices are being summed over as usual.

With the above remark in mind, we have

$$\Gamma^k_{ij} = \frac{f_{ij} f_k}{1 + |\nabla f|^2}$$

$$\Gamma^k_{ij,k} = \frac{f_{ij,k} f_k}{1 + |\nabla f|^2} + \frac{f_{ij} f_{kk} f_k}{1 + |\nabla f|^2} - \frac{2 f_{ij} f_{kl} f_k f_l}{(1 + |\nabla f|^2)^2}.$$

We can now compute the scalar curvature $R$ of $(M^n, g)$ using the coordinate expression for scalar curvature:

$$R = g^{ij} (\Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^l_{ij} \Gamma^k_{kl} - \Gamma^l_{ik} \Gamma^k_{jl})$$

$$= \left( \delta_{ij} - \frac{f_{ij}}{1 + |\nabla f|^2} \right) \left( \frac{f_{ij} f_k}{1 + |\nabla f|^2} + \frac{f_{ij} f_{kk} f_k}{1 + |\nabla f|^2} - \frac{2 f_{ij} f_{kl} f_k f_l}{(1 + |\nabla f|^2)^2} - \frac{f_{ij,k} f_k}{1 + |\nabla f|^2} \right)$$

$$- \frac{f_{ik} f_{jk}}{1 + |\nabla f|^2} + \frac{2 f_{ik} f_{jl} f_k f_l}{(1 + |\nabla f|^2)^2} + \frac{f_{ij} f_{kl} f_k f_l}{(1 + |\nabla f|^2)^2} - \frac{f_{ik} f_{jl} f_k f_l}{(1 + |\nabla f|^2)^2}$$

$$= \left( \delta_{ij} - \frac{f_{ij}}{1 + |\nabla f|^2} \right) \left( \frac{f_{ij} f_{kk}}{1 + |\nabla f|^2} - \frac{f_{ik} f_{jk}}{1 + |\nabla f|^2} - \frac{f_{ij} f_{kl} f_k f_l}{(1 + |\nabla f|^2)^2} + \frac{f_{ik} f_{jl} f_k f_l}{(1 + |\nabla f|^2)^2} \right)$$

$$- \frac{f_{ij} f_j}{1 + |\nabla f|^2} \left( f_{ij} f_{kk} - f_{ik} f_{jk} \right) - \frac{f_{ik} f_{jl}}{1 + |\nabla f|^2} \left( f_{ij} f_{kk} - f_{ik} f_{jk} \right) - \frac{f_{ij} f_{jk} f_{kl}}{(1 + |\nabla f|^2)^2} \left( f_{ij} f_{kl} - f_{ik} f_{jl} \right).$$

By symmetry, the last term in the last expression is 0. After relabeling the indices, the expression for the scalar curvature is

$$R = \frac{1}{1 + |\nabla f|^2} \left( f_{ii} f_{jj} - f_{ij} f_{ij} - \frac{2 f_{ij} f_k}{1 + |\nabla f|^2} (f_{ii} f_{jk} - f_{ij} f_{ik}) \right).$$

Let us denote by $\nabla \cdot$ the divergence operator on $(\mathbb{R}^n, \delta)$. The key observation we need to prove Theorem 5 is the following lemma:
Lemma 10. The scalar curvature $R$ of the graph $(\mathbb{R}^n, \delta + df \otimes df)$ satisfies

$$R = \nabla \cdot \left( \frac{1}{1 + |\nabla f|^2} (f_{ii}f_j - f_{ij}f_i) \partial_j \right).$$

Proof. This is a direct calculation:

$$\nabla \cdot \left( \frac{1}{1 + |\nabla f|^2} (f_{ii}f_j - f_{ij}f_i) \partial_j \right)
= \frac{1}{1 + |\nabla f|^2} (f_{iij}f_j + f_{iij}f_i - f_{ijj}f_i) - \frac{2f_{jk}f_k}{(1 + |\nabla f|^2)^2} (f_{ii}f_j - f_{ij}f_i)
= \frac{1}{1 + |\nabla f|^2} \left( f_{ii}f_j - f_{ij}f_i - \frac{2f_{jk}f_k}{1 + |\nabla f|^2} (f_{ii}f_{jk} - f_{ij}f_{ik}) \right)
= R$$

by (2).

We are now in the position to prove Theorem 5:

Proof of Theorem 5. By definition, the ADM mass of $(M^n, g) = (\mathbb{R}^n, \delta + df \otimes df)$ is

$$m = \lim_{r \to \infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} (g_{iij,i} - g_{ii,j}) \nu_j dS_r
= \lim_{r \to \infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} (f_{iij}f_j + f_{ijj}f_i - 2f_{ijj}f_i) \nu_j dS_r
= \lim_{r \to \infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} (f_{ii}f_j - f_{ij}f_i) \nu_j dS_r.$$

By the asymptotic flatness assumption, the function $1/(1 + |\nabla f|^2)$ goes to 1 at infinity. Hence we can alternately write the mass as

$$m = \lim_{r \to \infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} \frac{1}{1 + |\nabla f|^2} (f_{ii}f_j - f_{ij}f_i) \nu_j dS_r.$$

Now apply the divergence theorem in $(\mathbb{R}^n, \delta)$ and use Lemma 10 to get

$$m = \frac{1}{2(n-1)\omega_{n-1}} \int_{\mathbb{R}^n} \nabla \cdot \left( \frac{1}{1 + |\nabla f|^2} (f_{ii}f_j - f_{ij}f_i) \partial_j \right) dV_{\delta}
= \frac{1}{2(n-1)\omega_{n-1}} \int_{\mathbb{R}^n} RdV_{\delta}
= \frac{1}{2(n-1)\omega_{n-1}} \int_{M^n} R \frac{1}{\sqrt{1 + |\nabla f|^2}} dV_g$$

since

$$dV_g = \sqrt{\det g} dV_{\delta} = \sqrt{1 + |\nabla f|^2} dV_{\delta}.$$

Remark 11. If $(M^n, g)$ is the graph of a smooth spherically symmetric function $f = f(r)$ on $\mathbb{R}^n$, then it turns out that the ADM mass of $(M^n, g)$ is nonnegative even
without the nonnegative scalar curvature assumption. To see this, let \( f_r = \partial f / \partial r \) denote the radial derivative of \( f \). By the chain rule, the derivatives of \( f \) satisfy

\[
\begin{align*}
\nabla_i f &= f_r x_i r \\
\nabla_i \nabla_j f &= f_{rr} x_i x_j r^2 + f_r \left( \frac{\delta_{ij}}{r} - \frac{x_i x_j}{r^3} \right).
\end{align*}
\]

The ADM mass of \((M^n, g)\) is

\[
\begin{align*}
\cdot \cdot \cdot & = \lim_{r \to \infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} (g_{ij,i} - g_{ii,j}) \nu_j dS_r \\
& = \lim_{r \to \infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} (f_{ii} f_j - f_{ij} f_i) \nu_j dS_r \\
& = \lim_{r \to \infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} f_{rr} x_i^2 x_j^2 r^2 + f_r^2 \left( \frac{x_i^2}{r^3} - \frac{x_j^2}{r^3} \right) - f_{rr} \frac{x_i^2 x_j^2}{r^2} dS_r \\
& = \lim_{r \to \infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} \frac{2f_r^2}{r} \geq 0.
\end{align*}
\]

A consequence of this fact and Theorem 5 is that there are no spherically symmetric asymptotically flat smooth functions on \( \mathbb{R}^n \) whose graphs have negative scalar curvature everywhere.

3. Penrose Inequality for Graphs over \( \mathbb{R}^n \)

Let \( \Omega \) be a bounded open set in \( \mathbb{R}^n \) and \( \Sigma = \partial \Omega \). If \( f : \mathbb{R}^n \setminus \Omega \to \mathbb{R} \) is a smooth asymptotically flat function such that each connected component of \( f(\Sigma) \) is in a level of \( f \) and \( |\nabla f(x)| \to \infty \) as \( x \to \Sigma \), then the graph of \( f \), \((M^n, g) = (\mathbb{R}^n \setminus \Omega, \delta + df \otimes df)\), is an asymptotically flat manifold with area outer minimizing horizon \( \Sigma \). In this setting, proving Theorem 5 is a matter of keeping track of the extra boundary term when we apply the divergence theorem in the proof of Theorem 5. As before, any repeated indices are being summed over.

**Proof of Theorem 5** As in the proof of Theorem 5, we can write the mass of \((M^n, g)\) as

\[
m = \lim_{r \to \infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} \frac{1}{1 + |\nabla f|^2} (f_{ii} f_j - f_{ij} f_i) \nu_j dS_r.
\]
The difference here is that when we apply the divergence theorem, we get an extra boundary integral:

\[
m = \lim_{r \to \infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} \frac{1}{1 + |\nabla f|^2} (f_{ij}f_j - f_{ij}f_i) \nu_j dS_r
\]

\[
= \frac{1}{2(n-1)\omega_{n-1}} \int_{\mathbb{R}^n \setminus \Omega} \nabla \cdot \left( \frac{1}{1 + |\nabla f|^2} (f_{ij}f_j - f_{ij}f_i) \partial_j \right) dV_f
\]

\[
- \frac{1}{2(n-1)\omega_{n-1}} \int_{\Sigma} \frac{1}{1 + |\nabla f|^2} (f_{ij}f_j - f_{ij}f_i) \nu_j d\Sigma
\]

\[
= \frac{1}{2(n-1)\omega_{n-1}} \int_{M^n} R \frac{1}{\sqrt{1 + |\nabla f|^2}} dV_g
\]

\[
- \frac{1}{2(n-1)\omega_{n-1}} \int_{\Sigma} \frac{1}{1 + |\nabla f|^2} (f_{ij}f_j - f_{ij}f_i) \nu_j d\Sigma.
\]

The outward normal to \( \Sigma \) is \( \nu = -\nabla f / |\nabla f| \). Let \( \Delta f \) be the Laplacian of \( f \) in \((M^n, g)\) and \( \Delta_\Sigma f \) the Laplacian of \( f \) along \( \Sigma \). Let \( H^f \) denote the Hessian of \( f \) and \( H_0 \) the mean curvature of \( \Sigma \) with respect to the flat metric. We will use the following well known formula to relate the two Laplacians:

\[
\Delta f = \Delta_\Sigma f + H^f(\nu, \nu) + H_0 \cdot \nu(f)
\]

\[
= \frac{1}{|\nabla f|^2} H^f \left( \nabla f, \frac{\nabla f}{|\nabla f|} \right) + H_0 |\nabla f|,
\]

where \( \Delta_\Sigma f = 0 \) since \( f \) is constant on \( \Sigma \). Now

\[
- \frac{1}{1 + |\nabla f|^2} (f_{ij}f_j - f_{ij}f_i) \nu_j
\]

\[
= \frac{1}{1 + |\nabla f|^2} \left[ (\Delta f)|\nabla f| - H^f \left( \nabla f, \frac{\nabla f}{|\nabla f|} \right) \right]
\]

\[
= \frac{1}{1 + |\nabla f|^2} \left[ \left( \frac{1}{|\nabla f|^2} H^f \left( \nabla f, \frac{\nabla f}{|\nabla f|} \right) + H_0 |\nabla f| \right) |\nabla f| - H^f \left( \nabla f, \frac{\nabla f}{|\nabla f|} \right) \right]
\]

\[
= \frac{|\nabla f|^2}{1 + |\nabla f|^2} H_0.
\]

Therefore,

\[
m = \frac{1}{2(n-1)\omega_{n-1}} \int_{M^n} R \frac{1}{\sqrt{1 + |\nabla f|^2}} dV_g + \frac{1}{2(n-1)\omega_{n-1}} \int_{\Sigma} \frac{|\nabla f|^2}{1 + |\nabla f|^2} H_0 d\Sigma
\]

\[
= \frac{1}{2(n-1)\omega_{n-1}} \int_{M^n} R \frac{1}{\sqrt{1 + |\nabla f|^2}} dV_g + \frac{1}{2(n-1)\omega_{n-1}} \int_{\Sigma} H_0 d\Sigma.
\]

\( \square \)

Let us denote by \( \Omega_i, i = 1, \ldots, k \) the connected components of the bounded open set \( \Omega \). In the case that each \( \Omega_i \) is convex, it turns out we can obtain a lower bound for the boundary integral in Theorem 6. To do this, we will need the following lemma, which is a special case of the Aleksandrov-Fenchel inequality [10].

**Lemma 12.** If \( \Sigma \) is a convex surface in \( \mathbb{R}^n \) with mean curvature \( H_0 \) and area \( |\Sigma| \), then

\[
\frac{1}{2(n-1)\omega_{n-1}} \int_{\Sigma} H_0 \geq \frac{1}{2} \left( \frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-2}{n}}.
\]
Proof. Let $\Sigma \subset \mathbb{R}^n$ be a convex surface with principal curvatures $\kappa_1, \ldots, \kappa_{n-1}$. Let

$$\sigma_j(\kappa_1, \ldots, \kappa_{n-1}) = \binom{n-1}{j}^{-1} \sum_{1 \leq i_1 < \cdots < i_j \leq n-1} \kappa_{i_1} \cdots \kappa_{i_j}$$

be the $j$th normalized elementary symmetric functions in $\kappa_1, \ldots, \kappa_{n-1}$ for $j = 1, \ldots, n-1$. In particular,

$$\sigma_0(\kappa_1, \ldots, \kappa_{n-1}) = 1$$
$$\sigma_1(\kappa_1, \ldots, \kappa_{n-1}) = \frac{1}{n-1} \sum_{i=1}^{n-1} \kappa_i = \frac{1}{n-1} H_0$$
$$\sigma_{n-1}(\kappa_1, \ldots, \kappa_{n-1}) = \prod_{i=1}^{n-1} \kappa_i.$$

The $k$th quermassintegral $V_k$ of $\Sigma$ is defined to be

$$V_k = \int_{\Sigma} \sigma_k(\kappa_1, \ldots, \kappa_{n-1}).$$

A special case of the Aleksandrov-Fenchel inequality states that, for $0 \leq i < j < k \leq n-1$,

$$V_{j}^{k-1} \geq V_{i}^{k-j} V_{j-i}^{k-i}.$$

Taking $i = 0$, $j = 1$, $k = n-1$,

$$(3) \quad V_{1}^{n-1} \geq V_{0}^{n-2} V_{n-1}.$$

Now

$$V_0 = \int_{\Sigma} \sigma_0(\kappa_1, \ldots, \kappa_{n-1}) = |\Sigma|$$
$$V_1 = \int_{\Sigma} \sigma_1(\kappa_1, \ldots, \kappa_{n-1}) = \frac{1}{n-1} \int_{\Sigma} H_0$$
$$V_{n-1} = \int_{\Sigma} \sigma_{n-1}(\kappa_1, \ldots, \kappa_{n-1}) = \omega_{n-1}.$$

Thus $(3)$ becomes

$$\left( \frac{1}{n-1} \int_{\Sigma} H_0 \right)^{n-1} \geq |\Sigma|^{n-2} \omega_{n-1}$$
$$\frac{1}{n-1} \int_{\Sigma} H_0 \geq |\Sigma|^{-\frac{n-2}{n-1}} \omega_{n-1}^{\frac{1}{n-1}}$$
$$\frac{1}{2(n-1)\omega_{n-1}} \int_{\Sigma} H_0 \geq \frac{1}{2} \left( \frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}$$

as claimed. \qed

Now Corollary 7 follows directly from Theorem 6 and Lemma 12.
4. Discussions and Acknowledgements

We point out that the technique in this paper can be used to study the mass of an asymptotically flat manifold that can be isometrically embedded as a graph in Minkowski space, and a second paper in this direction is currently under progress. In this setting, zero area singularities can arise. The standard sample of such manifolds is the Schwarzschild metric with $m < 0$. For more on this topic, we refer the readers to [4,5].

We also point out that recently Schwartz [14] proved a volumetric Penrose inequality for conformally flat manifolds in all dimensions $n \geq 3$.

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