Agnostic Learning of a Single Neuron with Gradient Descent

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Abstract
We consider the problem of learning the best-fitting single neuron as measured by the expected squared loss $E_{(x,y) \sim D}[(\sigma(w^\top x) - y)^2]$ over an unknown joint distribution of the features and labels by using gradient descent on the empirical risk induced by a set of i.i.d. samples $S \sim D^n$. The activation function $\sigma$ is an arbitrary Lipschitz and non-decreasing function, making the optimization problem nonconvex and nonsmooth in general, and covers typical neural network activation functions and inverse link functions in the generalized linear model setting. In the agnostic PAC learning setting, where no assumption on the relationship between the labels $y$ and the features $x$ is made, if the population risk minimizer $v$ has risk $\text{OPT}$, we show that gradient descent achieves population risk $O(\text{OPT}^{1/2} + \varepsilon)$ in polynomial time and sample complexity. When labels take the form $y = \sigma(v^\top x) + \xi$ for zero-mean sub-Gaussian noise $\xi$, we show that gradient descent achieves population risk $\text{OPT} + \varepsilon$. Our sample complexity and runtime guarantees are (almost) dimension independent, and when $\sigma$ is strictly increasing and Lipschitz, require no distributional assumptions beyond boundedness. For ReLU, we show the same results under a nondegeneracy assumption for the marginal distribution of the features. To the best of our knowledge, this is the first result for agnostic learning of a single neuron using gradient descent.

1 Introduction

In this paper, we describe the properties of gradient descent for learning the best possible single neuron that captures the relationship between a set of features $x \in \mathbb{R}^d$ and labels $y \in \mathbb{R}$ as measured by the expected squared loss over some unknown joint distribution $(x,y) \sim D$. In particular, for a given activation function $\sigma : \mathbb{R} \to \mathbb{R}$, we define the population risk $F(w)$ associated with a set of weights $w$ as

$$F(w) := (1/2)E_{(x,y) \sim D}[(\sigma(w^\top x) - y)^2].$$

(1.1)

The activation function is assumed to be non-decreasing and Lipschitz, and includes nearly all activation functions used in neural networks such as the rectified linear unit (ReLU), sigmoid, tanh, and so on. In the agnostic PAC learning setting of Kearns et al. (1994), no structural assumption is made regarding the relationship of the features and the labels, and so the best-fitting neuron could, in the worst case, have nontrivial population risk. Concretely, if we denote

$$v := \arg\min_{\|w\|_2 \leq 1} F(w), \quad \text{OPT} := F(v),$$

(1.2)

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then the goal of a learning algorithm is to (efficiently) return weights $w$ such that the population risk $F(w)$ is close to the best possible risk $\text{OPT}$. The agnostic learning framework stands in contrast to the \textit{realizable} PAC learning setting, where one assumes $\text{OPT} = 0$, so that there is some $v$ such that the labels are given as $y = \sigma(v^\top x)$.

The learning algorithm we use in this paper is vanilla gradient descent. We assume we have access to a set of i.i.d. samples $\{(x_i, y_i)\}_{i=1}^n \sim D^n$, and we run gradient descent with a fixed step size on the empirical risk $\hat{F}(w) = (1/2n) \sum_{i=1}^n (\sigma(w^\top x_i) - y_i)^2$ induced by the empirical distribution of the samples.

Surprisingly little is known about gradient descent-trained neural networks in the agnostic PAC learning framework. We are aware of two works in the \textit{improper} agnostic learning setting, where the goal is to return a hypothesis $h \in \mathcal{H}$ that achieves population risk close to $\hat{\text{OPT}}$, where $\hat{\text{OPT}}$ is the smallest possible population risk achieved by a disjoint set of hypotheses $\hat{\mathcal{H}}$ (Allen-Zhu et al. (2019); Allen-Zhu and Li (2019)). Another work considered the random features setting where only the final layer of the network is trained and the marginal distribution over the features is uniform on the unit sphere (Vempala and Wilmes (2019)). But none of these address the simplest possible neural network: that of a single neuron $x \mapsto \sigma(w^\top x)$. We believe a full characterization of what we can (or cannot) guarantee for gradient descent in the single neuron setting will help us understand what is possible in the more complicated deep neural network setting. Indeed, two of the most common hurdles in the analysis of deep neural networks trained by gradient descent—nonconvexity and nonsmoothness—are also present in the case of the single neuron. We hope that our analysis in this relatively simple setup will be suggestive of what is possible in more complicated neural network models.

Our main contributions can be summarized as follows.

1) \textbf{Agnostic setting.} Without any assumptions on the relationship between $y$ and $x$, and assuming only boundedness of the marginal distributions on $x$ and $y$, we show that for any $\varepsilon > 0$, gradient descent finds a point $w_t$ with population risk $O(\text{OPT}^{1/2}) + \varepsilon$ when $\sigma$ is strictly increasing and Lipschitz. We can show the same result for ReLU when the marginal distribution of $x$ satisfies a marginal spread condition (Assumption 3.2). The sample complexity is of the order $O(\varepsilon^{-4})$ and runtime of order $O(\varepsilon^{-2})$, with both complexities independent of the input dimension.

2) \textbf{Noisy teacher network setting.} When $y = \sigma(v^\top x) + \xi$, where $\xi|x$ is mean zero and sub-Gaussian (and possibly dependent on $x$), we demonstrate that gradient descent finds $w_t$ satisfying $F(w_t) \leq \text{OPT} + \varepsilon$ for activation functions that are strictly increasing and Lipschitz assuming only boundedness of the marginal distribution over $x$. The same result holds for ReLU under a marginal spread assumption given below in Assumption 3.2. The runtime and sample complexity is of order $O(\varepsilon^{-2})$, with logarithmic dependence on the input dimension. When the noise is bounded, our guarantees are dimension independent. If we further know $\xi \equiv 0$, i.e. the learning problem is in the realizable rather than agnostic setting, we can improve the complexity guarantees from $O(\varepsilon^{-2})$ to $O(\varepsilon^{-1})$ by using online stochastic gradient descent.

2 Related work

Below, we provide a high-level summary of related literature in the agnostic learning and teacher network settings. Detailed comparisons with the most related works will appear after we present our
main theorems in Sections 3 and 4. In Appendix A, we provide tables that describe the assumptions and complexity guarantees of our work in comparison to related works.

**Agnostic learning:** The simplest version of the agnostic regression problem is that of finding a hypothesis that matches the performance of the best linear predictor. In our setting, this corresponds to $\sigma$ being the identity function. This problem is essentially completely characterized: Shamir (2015) showed that any algorithm that returns a linear predictor $v$ has risk $OPT + \Omega(\epsilon^{-2} \land d\epsilon^{-1})$ when the labels satisfy $|y| \leq 1$ and the features are supported on the unit ball, matching upper bounds proved by Srebro et al. (2010) using mirror descent.

When $\sigma$ is not the identity, related works are scarce. The only work on agnostic learning of a single neuron that we are aware of is Goel et al. (2019), where the authors considered the problem of learning a single ReLU when the features are standard $d$-dimensional Gaussians. In this setting, they showed learning up to risk $OPT + \varepsilon$ in polynomial time is as hard as the problem of learning sparse parities with noise, long believed to be computationally intractable. By reducing the problem of learning a ReLU to one of learning a halfspace, they use an algorithm of Awasthi et al. (2017) to show learnability up to $O(OPT^{2/3}) + \varepsilon$. In a related but incomparable set of results, Allen-Zhu et al. (2019) and Allen-Zhu and Li (2019) studied improper agnostic learnability for neural networks in the multilayer setting when the labels are generated by some multilayer network with a smooth activation function and the hypothesis class is a deep ReLU network. Vempala and Wilmes (2019) studied agnostic learning of a one-hidden-layer neural network when the first layer is fixed at its (random) initial values and the second layer is trained.

**Teacher network:** The literature refers to the case of $y = \sigma(v^\top x) + \xi$ for some possible mean zero noise $\xi$ variously as the “noisy teacher network” or “generalized linear model” (GLM) setting, and is related to the probabilistic concepts model introduced by Kearns and Schapire (1994). In the GLM setting, $\sigma$ plays the role of the inverse link function; in the case of logistic regression, $\sigma$ is the sigmoid.

The results in the teacher network setting can be broadly characterized by (1) whether they cover arbitrary distributions over the features and (2) the presence of noise (or lackthereof). The GLMTron algorithm proposed by Kakade et al. (2011), itself a modification of the Isotron algorithm of Kalai and Sastry (2009), is known to learn a noisy teacher network up to risk $OPT + \varepsilon$ for any $L$-Lipschitz and non-decreasing $\sigma$ and any distribution with bounded marginals over $x$. Mei et al. (2018) showed that regularized gradient descent learns the noisy teacher network under a smoothness assumption of the activation function for a large class of distributions. Foster et al. (2018) provided a meta-algorithm for translating $\varepsilon$-stationary points of the empirical risk to minimal points of the population risk under certain conditions, and showed that such conditions are satisfied by regularized gradient descent. A recent work by Mukherjee and Muthukumar (2020) develops a modified SGD algorithm for learning a ReLU with bounded noise on distributions where the features are bounded.

Of course, any guarantee that holds for a neural network with a single fully connected hidden layer of arbitrary width holds for the single neuron, so in a sense our work connects to a larger body of work on the analysis of gradient descent used for learning neural networks. The majority of such works are restricted to particular distributions of the feature set, whether it is Gaussian or uniform distributions (Soltanolkotabi, 2017; Tian, 2017; Soltanolkotabi et al., 2019; Zhang et al., 2019; Goel et al., 2018; Cao and Gu, 2019). Du et al. (2018) showed that in the noiseless (a.k.a. realizable) setting, a single neuron can be learned with SGD if the feature distribution satisfies a certain subspace eigenvalue property. Yehudai and Shamir (2020) studied the properties of learning a single neuron for a variety of increasing and Lipschitz activation functions using gradient descent,
as we do in this paper, although their analysis was restricted to the noiseless setting.

3 Agnostic setting

We begin our analysis by assuming there is no \textit{a priori} relationship between \(x\) and \(y\), and so the population risk \(\text{OPT}\) of the population risk minimizer \(v\) defined in (1.2) may, in general, be a large quantity. If \(\text{OPT} = 0\), then \(\sigma(v^\top x) = y\) a.s., and so we are in the realizable PAC learning setting. In this case, we can use a modified proof technique to improve our guarantee from \(O(\text{OPT}^{1/2}) + \varepsilon\) to \(\text{OPT} + \varepsilon\), with sample and runtime complexity of order \(O(\varepsilon^{-1})\) by using online stochastic gradient descent; see Appendix B for the complete theorems and proofs in this setting. In what follows, we will therefore assume without loss of generality that \(0 < \text{OPT} \leq 1\).

The gradient descent method we use in this paper is as follows. We assume we have samples \(\{(x_i, y_i)\}_{i=1}^n \overset{i.i.d.}{\sim} \mathcal{D}^n\), and define the empirical risk for weight \(w\) by

\[
\hat{F}(w) = \frac{1}{2n} \sum_{i=1}^n (\sigma(w^\top x_i) - y_i)^2.
\]

We perform full-batch gradient updates on the empirical risk using a fixed step size \(\eta\),

\[
w_{t+1} = w_t - \eta \nabla \hat{F}(w_t) = w_t - \frac{\eta}{n} \sum_{i=1}^n (\sigma(w_i^\top x_i) - y_i)\sigma'(w_i^\top x_i).
\]

After running \(T\) updates, the algorithm outputs \(w_T = \arg\min_{0 \leq t \leq T} \hat{F}(w_t)\).

We begin by describing one set of activation functions under consideration in this paper.

\textbf{Assumption 3.1.} (a) \(\sigma\) is continuous, non-decreasing, and differentiable almost everywhere.

(b) For any \(\rho > 0\), there exists \(\gamma > 0\) such that \(\inf_{|z| \leq \rho} \sigma'(z) \geq \gamma > 0\). If \(\sigma\) is not differentiable at \(z \in [-\rho, \rho]\), assume that every subgradient \(g\) on the interval satisfies \(g(z) \geq \gamma\).

(c) \(\sigma\) is \(L\)-Lipschitz, i.e. \(|\sigma(z_1) - \sigma(z_2)| \leq L|z_1 - z_2|\) for all \(z_1, z_2\).

We note that if \(\sigma\) is strictly increasing and continuous, then \(\sigma\) satisfies Assumption 3.1(b) since its derivative is never zero. In particular, the assumption covers the typical activation functions in neural networks like leaky ReLU, softplus, sigmoid, tanh, etc., but excludes ReLU. Yehudai and Shamir (2020) recently showed that when \(\sigma\) is ReLU, there exists a distribution \(\mathcal{D}\) supported on the unit ball and unit length target neuron \(v\) such that \textit{even in the realizable case of} \(y = \sigma(v^\top x)\), if the weights are initialized randomly using a product distribution, then there exists a constant \(c_0\) such that with high probability, \(F(w_t) \geq c_0 > 0\) throughout the trajectory of gradient descent. This suggests that gradient-based methods for learning ReLUs are likely to fail without additional assumptions. Because of this, they introduced the following marginal spread assumption to allow for convergence guarantees.

\textbf{Assumption 3.2.} There exist constants \(\alpha, \beta > 0\) such that the following holds. For \(w \neq u\), denote by \(\mathcal{D}_{w,u}\) the marginal distribution of \(\mathcal{D}\) on \(\text{span}(w, u)\), viewed as a distribution over \(\mathbb{R}^2\), and let \(p_{w,u}\) be its density function. Then \(\inf_{z \in \mathbb{R}^2, \|z\| \leq \alpha} p_{w,u}(z) \geq \beta\).

This assumption covers, for instance, standard Gaussian distributions and centered uniform distributions with \(\alpha, \beta = O(1)\), and holds for any distribution mixed with some Gaussian or uniform noise. We note that a similar assumption was used in recent work by Diakonikolas et al. (2020)
on learning halfspaces with Massart noise. We will use this assumption for all of our results when \( \sigma \) is ReLU. Additionally, although the ReLU is not differentiable at the origin, we will denote by \( \sigma'(0) \) its subgradient, with the convention that \( \sigma'(0) = 1 \). Such a convention is consistent with the implementation of ReLUs in modern deep learning software packages.

With the above in hand, we can describe our main theorem.

**Theorem 3.3.** Suppose the marginals of \( \mathcal{D} \) satisfy \( \|x\|_2 \leq B_X \) a.s. and \( |y| \leq B_Y \) a.s. Let \( a := (|\sigma(B_X)| + B_Y)^2 \). Assume gradient descent is initialized at \( w_0 = 0 \) and fix a step size \( \eta \leq (1/4)L^{-2}B_X^{-2} \). If \( \sigma \) satisfies Assumption 3.1, let \( \gamma \) the constant corresponding to \( \rho = 2B_X \). For any \( \delta > 0 \), with probability at least \( 1 - \delta \), gradient descent run for \( T = \lceil \eta^{-1}L^{-1}B_X^{-1}[\text{OPT}^{1/2} + an^{-1/2}\log^{1/2}(2/\delta)]^{-1} \rceil \) iterations finds weights \( w_t, t < T \), such that

\[
F(w_t) \leq C_1 \text{OPT}^{1/2} + C_2 n^{-1/4} + C_3 n^{-1/2},
\]

where \( C_1 = O(\gamma^{-1}L^2B_X) \), \( C_2 = O(C_1 a^{1/2}\log^{1/4}(1/\delta)) \), \( C_3 = O(L^3B_X^2\log^{1/2}(1/\delta)) \).

When \( \sigma \) is ReLU, further assume that \( \mathcal{D}_x \) satisfies Assumption 3.2 for constants \( \alpha, \beta > 0 \), and let \( \nu = \alpha^4\beta/8\sqrt{2} \). Then (3.2) holds by replacing \( C_i \) with \( \tilde{C}_i \), where \( \tilde{C}_1 = O(B_X\nu^{-1}) \), \( \tilde{C}_2 = O(a^{1/2}\nu^{-1}B_X\log^{1/4}(1/\delta)) \), and \( \tilde{C}_3 = O(B_X^2\nu^{-1}\log^{1/2}(1/\delta)) \).

In comparison to recent work, Goel et al. (2019) considered the agnostic setting for the ReLU activation when the marginal distribution over \( x \) is a standard Gaussian and showed that learning up to risk \( \text{OPT} + \varepsilon \) is as hard as learning sparse parities with noise, long believed to be computationally intractable. By using an approximation algorithm of Awasthi et al. (2017), they were able to show that one can learn up to \( O(\text{OPT}^{2/3}) + \varepsilon \) with \( O(\text{poly}(d, \varepsilon^{-1})) \) runtime and sample complexity. By contrast, we use gradient descent to learn up to a (weaker) risk of \( O(\text{OPT}^{1/2}) + \varepsilon \), but for any joint distribution with bounded marginals when \( \sigma \) satisfies Assumption 3.1. In the case of ReLU, our guarantee holds for the class of distributions over \( x \) with finite support that satisfy the marginal spread condition of Assumption 3.2, and for all activation functions we consider, the runtime and sample complexity guarantees do not have (explicit) dependence on the dimension. (We note that for some distributions, the \( B_X \) term may hide an implicit dependence on \( d \); more detailed comments on this are given in Appendix A.) Moreover, we shall see in the next section that if the data is known to come from a noisy teacher network, the guarantees of gradient descent improve from \( O(\text{OPT}^{1/2}) + \varepsilon \) to \( \text{OPT} + \varepsilon \).

In the remainder of this section we will prove Theorem 3.3. Our proof relies upon the following two auxiliary errors for the true risk \( F \):

\[
G(w) := (1/2)\mathbb{E}_{(x,y) \sim \mathcal{D}} \left[ \left( \sigma(w^\top x) - \sigma(v^\top x) \right)^2 \right],
\]

\[
H(w) := (1/2)\mathbb{E}_{(x,y) \sim \mathcal{D}} \left[ \left( \sigma(w^\top x) - \sigma(v^\top x) \right)^2 \sigma'(w^\top x) \right].
\]

We will denote the corresponding empirical risks by \( \hat{G}(w) \) and \( \hat{H}(w) \). We first note that \( G \) trivially upper bounds \( F \): this follows by a simple application of Young’s inequality and, when \( \mathbb{E}[y|x] = \sigma(v^\top x) \), by using iterated expectations.

**Claim 3.4.** For any joint distribution \( \mathcal{D} \), for any vector \( u \), and any continuous activation function \( \sigma \),

\[
F(u) \leq 2G(u) + 2F(v).
\]
If additionally we know that \( \mathbb{E}[y|x] = \sigma(v^\top x) \), we have \( F(u) = G(u) + F(v) \).

To see that \( H \) is an upper bound for \( G \), it is easy to see that if \( \inf_{z \in \mathbb{R}} \sigma'(z) \geq \gamma > 0 \), then \( H(w) \leq \varepsilon \) implies \( G(w) \leq \gamma^{-1} \varepsilon \). However, the only typical activation function that is covered by such an assumption is the leaky ReLU. Fortunately, when \( \sigma \) satisfies Assumption 3.1, or when \( \sigma \) is ReLU and \( \mathcal{D} \) satisfies Assumption 3.2, Lemma 3.5 below shows that \( H \) is still an upper bound for \( G \). The proof is left for Appendix B.

**Lemma 3.5.** If \( \sigma \) satisfies Assumption 3.1, \( \|x\|_2 \leq B \) a.s., and \( \|w\|_2 \leq W \), then for \( \gamma \) corresponding to \( \rho = WB \), \( H(w) \leq \varepsilon \) implies \( G(w) \leq \gamma^{-1} \varepsilon \). If \( \sigma \) is ReLU and \( \mathcal{D} \) satisfies Assumption 3.2 for some constants \( \alpha, \beta > 0 \), and for some \( \varepsilon > 0 \), \( H(w) \leq \beta \alpha^4 \varepsilon / 8 \sqrt{2} \), then \( \|w - v\|_2 \leq 1 \) implies \( G(w) \leq \varepsilon \) holds.

We can now focus on showing that gradient descent finds a point where \( H(w_t) \) is small. In Lemma 3.6 below, we show that \( \hat{H}(w_t) \) is a natural quantity of the gradient descent algorithm that in a sense tells us how good of a direction the gradient is pointing at time \( t \), and that \( \hat{H}(w_t) \) can be as small as \( O(\hat{F}(v)^{1/2}) \). Our proof technique is similar to that of Kakade et al. (2011), who studied the GLMTron algorithm in the (non-agnostic) noisy teacher network setup.

**Lemma 3.6.** Suppose that \( \|x\|_2 \leq B_X \) a.s. under \( \mathcal{D}_x \). Suppose \( \sigma \) is non-decreasing and \( L \)-Lipschitz. Assume \( \hat{F}(v) \in (0, 1) \). Gradient descent run with fixed step size \( \eta \leq (1/4)L^{-2}B_X^{-2} \) from initialization \( w_0 = 0 \) finds weights \( w_t \) satisfying \( \hat{H}(w_t) \leq 2L^2B_X\sqrt{\hat{F}(v)} \) within \( T = \lceil \eta^{-1}L^{-1}B_X^{-1}\hat{F}(v)^{-1/2} \rceil \) iterations, with \( \|w_t - v\|_2 \leq 1 \) for each \( t = 0, \ldots, T - 1 \).

Before beginning the proof, we first note the following simple fact, which allows for us to connect terms that appear in the gradient to the squared loss.

**Fact 3.7.** If \( \sigma \) is non-decreasing and \( L \)-Lipschitz, then for any \( z_1, z_2 \) in the domain of \( \sigma \),

\[
(\sigma(z_1) - \sigma(z_2))(z_1 - z_2) \geq \frac{L^{-1}1}{2}(\sigma(z_1) - \sigma(z_2))^2.
\]

**Proof of Lemma 3.6.** The proof comes from the following induction statement. We claim that for every \( t \in \mathbb{N} \), either (a) \( \hat{H}(w_\tau) \leq 2L^2B_X\hat{F}(v)^{1/2} \) for some \( \tau < t \), or (b) \( \|w_t - v\|_2^2 \leq \|w_{t-1} - v\|_2^2 - \eta L B_X \hat{F}(v)^{1/2} \). If this claim is true, then at every iteration of gradient descent, we either have \( \hat{H}(w_\tau) \leq 2L^2B_X\hat{F}(v)^{1/2} \) or \( \|w_t - v\|_2^2 \leq \|w_{t-1} - v\|_2^2 - \eta L B_X \hat{F}(v)^{1/2} \). Since \( \|w_0 - v\|_2^2 = 1 \), this means there can be at most \( 1/(\eta L B_X \hat{F}(v)^{1/2}) = \eta^{-1}L^{-1}B_X^{-1}\hat{F}(v)^{-1/2} \) iterations until we reach \( \hat{H}(w_t) \leq 2L^2B_X\hat{F}(v) \). This shows the induction statement implies the theorem.

We begin with the proof by supposing the induction hypothesis holds for \( t \), and want to consider the case \( t + 1 \). If (a) holds, then we are done. So now consider the case that for every \( \tau \leq t \), we have \( \hat{H}(w_\tau) \geq 2L^2B_X\hat{F}(v)^{1/2} \). Since (a) does not hold, \( \|w_\tau - v\|_2^2 \leq \|w_{\tau-1} - v\|_2^2 - \eta L B_X \hat{F}(v)^{1/2} \) holds for each \( \tau = 1, \ldots, t \), and so \( \|w_0 - v\|_2 = 1 \) implies

\[
\|w_\tau - v\|_2 \leq 1 \quad \forall \tau \leq t.
\]

(3.4)
We can therefore bound
\[ \langle \nabla \hat{F}(w_t), w_t - v \rangle = \frac{1}{n} \sum_{i=1}^{n} \left( \sigma(w_i^\top x_i) - \sigma(v^\top x_i) \right) \sigma'(w_i^\top x_i)(w_i^\top x_i - v^\top x_i) \]
\[ + \langle (1/n) \sum_{i=1}^{n} \left( \sigma(v^\top x_i) - y_i \right) \sigma'(w_i^\top x_i)x_i, w_t - v \rangle \]
\[ \geq (1/Ln) \sum_{i=1}^{n} \left( \sigma(w_i^\top x_i) - \sigma(v^\top x_i) \right)^2 \sigma'(w_i^\top x_i) \]
\[ - \|w_t - v\|_2 \left\| (1/n) \sum_{i=1}^{n} \left( \sigma(v^\top x_i) - y_i \right) \sigma'(w_i^\top x_i)x_i \right\|_2 \]
\[ \geq 2L^{-1} \bar{H}(w_t) - LB_X \hat{F}(v)^{1/2} \]  
(3.6)

In the first inequality, we have used Fact 3.7 and that \( \sigma'(z) \geq 0 \) for the first term. For the second term, we use Cauchy–Schwarz. The last inequality is a consequence of (3.4), Cauchy–Schwarz, and that \( \sigma'(z) \leq L \) and \( \|x\|_2 \leq B_X \). As for the gradient upper bound at \( w_t \), we have
\[ \|\nabla \hat{F}(w_t)\|_2^2 \leq 2 \left\| \frac{1}{n} \sum_{i=1}^{n} \left( \sigma(w_i^\top x_i) - \sigma(v^\top x_i) \right) \sigma'(w_i^\top x_i)x_i \right\|_2^2 \]
\[ + 2 \left\| \frac{1}{n} \sum_{i=1}^{n} \left( \sigma(v^\top x_i) - y_i \right) \sigma'(w_i^\top x_i)x_i \right\|_2^2 \]
\[ \leq 4B_X^2 L \bar{H}(w_t) + 4L^2 B_X^2 \hat{F}(v). \]  
(3.7)

The first inequality uses Young’s inequality, and the second uses Jensen’s inequality and that \( \sigma \) is \( L \)-Lipschitz and \( \|x\|_2 \leq B_X \). Putting (3.6) and (3.7) together, the choice of \( \eta \) ensures
\[ \|w_t - v\|_2^2 - \|w_{t+1} - v\|_2^2 \geq 2\eta \left( 2L^{-1} \bar{H}(w_t) - LB_X \hat{F}(v)^{1/2} \right) \]
\[ - \eta^2 \left( 4B_X^2 L \bar{H}(w_t) + 4L^2 B_X^2 \hat{F}(v) \right) \]
\[ \geq \eta \left( 3L^{-1} \bar{H}(w_t) - 3LB_X \left( \hat{F}(v) \vee \hat{F}(v)^{1/2} \right) \right) \]
\[ \geq \eta LB_X \hat{F}(v)^{1/2} \]  
(3.8)

The last line comes from the induction hypothesis that \( \bar{H}(w_t) \geq 2L^2 B_X \hat{F}(v)^{1/2} \) and since \( \hat{F}(v) \in (0, 1) \). This completes the proof.

Since the auxiliary error \( \bar{H} \) is controlled by \( \hat{F}(v)^{1/2} \), we need to bound \( \hat{F}(v)^{1/2} \), which we can do by demonstrating a bound on \( \hat{F}(v) \). Since the marginals of \( \mathcal{D} \) are bounded, Lemma 3.8 below shows that \( \hat{F}(v) \) concentrates around \( F(v) = \text{OPT} \) at rate \( n^{-1/2} \) by Hoeffding’s inequality; for completeness, the proof is given in Appendix E.

**Lemma 3.8.** If \( \|x\|_2 \leq B_X \) and \( |y| \leq B_Y \) a.s. under \( \mathcal{D}_x \) and \( \mathcal{D}_y \) respectively, and if \( \sigma \) is non-decreasing, then for \( a := (|\sigma(B_X)| + B_Y)^2 \) and \( \|v\|_2 \leq 1 \), we have with probability at least \( 1 - \delta \),
\[ |\hat{F}(v) - \text{OPT}| \leq 3a \sqrt{n^{-1} \log(2/\delta)}. \]
The final ingredient to the proof is translating the bounds for the empirical risk to one for the population risk. Since \( D_x \) is bounded and since we showed in Lemma 3.6 that \( \| w_t - v \|_2 \leq 1 \) throughout the gradient descent trajectory, we can use standard properties of Rademacher complexity to translate the training loss bound to one for the test loss. The proof for Lemma 3.9 can be found in Appendix E.

**Lemma 3.9.** For training set \( S \sim D^n \), let \( \mathcal{R}_S(\mathcal{G}) \) denote the empirical Rademacher complexity of a class of functions \( \mathcal{G} \), and suppose \( \sigma \) is \( L \)-Lipschitz. Suppose \( \| x \|_2 \leq B_X \) a.s. Denote \( \ell(w; x) := (1/2) \left( \sigma(w^\top x) - \sigma(v^\top x) \right)^2 \) and 

\[
\mathcal{G} := \{ x \mapsto w^\top x : \| w - v \|_2 \leq 1, \| v \|_2 = 1 \},
\]

Then

\[
\mathcal{R}(\ell \circ \sigma \circ \mathcal{G}) = \mathbb{E}_{S \sim D^n} \mathcal{R}_S(\ell \circ \sigma \circ \mathcal{G}) \leq 2L^3B_X^2/\sqrt{n}.
\]

With Lemmas 3.6, 3.8 and 3.9 in hand, the bound for the population risk follows in a straightforward manner.

**Proof of Theorem 3.3.** By Lemma 3.6, there exists some \( w_t \) with \( t < T \) and \( \| w_t - v \|_2 \leq 1 \), such that \( \bar{H}(w_t) \leq 2L^2B_X\hat{F}(v)^{1/2} \). For \( \sigma \) satisfying Assumption 3.1, Lemmas 3.5 and 3.8 imply that

\[
\hat{G}(w_t) \leq 2\gamma^{-1}L^2B_X \left( \text{OPT}^{1/2} + (3a)^{1/2}n^{-1/4}\log^{1/4}(2/\delta) \right).
\]

Since \( \| w - v \|_2 \leq 1 \) implies \( \ell(w; x) = (1/2)(\sigma(w^\top x) - \sigma(v^\top x))^2 \leq L^2B_X^2/2 \), standard results from Rademacher complexity imply (e.g. Theorem 26.5 of Shalev-Shwartz and Ben-David (2014)) that with probability at least \( 1 - \delta \),

\[
G(w_t) \leq \hat{G}(w_t) + \mathbb{E}_{S \sim D^n} \mathcal{R}_S(\ell \circ \sigma \circ \mathcal{G}) + 2L^2B_X^2 \sqrt{2\log(4/\delta) / n},
\]

where \( \ell \) is the loss and \( \mathcal{G} \) is the function class defined in Lemma 3.9. For the second term above, Lemma 3.9 and rescaling \( \delta \) yields that

\[
G(w_t) \leq 2L^3B_X^2 \sqrt{\frac{1}{n}} + 2L^2B_X^2 \sqrt{\frac{2\log(8/\delta)}{n}} + 2\gamma^{-1}L^2B_X \left( \text{OPT}^{1/2} + \frac{\sqrt{3a}\log^{1/4}(2/\delta)}{n^{1/4}} \right).
\]

This shows that \( G(w_t) \leq O(\text{OPT}^{1/2} + n^{-1/4}) \). By Claim 3.4,

\[
F(w_t) \leq 2G(w_t) + 2F(v) \leq O(\text{OPT}^{1/2} + n^{-1/4}) + O(\text{OPT}) = O(\text{OPT}^{1/2} + n^{-1/4}),
\]

completing the proof when \( \sigma \) is strictly increasing.

When \( \sigma \) is ReLU, the proof has one technical difference. Although Lemma 3.9 applies to the loss function \( \ell(w; x) = (1/2) \left( \sigma(w^\top x) - \sigma(v^\top x) \right)^2 \), the same results hold for the loss function \( \tilde{\ell}(w; x) = \ell(w; x)\sigma'(v^\top x) \) for ReLU, since \( \nabla \sigma'(w^\top x) \equiv 0 \) a.e. and so \( \tilde{\ell} \) is still \( B_X \)-Lipschitz. We thus
have
\[ \mathbb{E}_{S \sim \mathcal{D}^n} \mathcal{R}_S \left( \hat{\ell} \circ \sigma \circ \mathcal{G} \right) \leq \frac{2B_X^2}{\sqrt{n}}. \quad (3.10) \]
With this in hand, the proof is essentially identical: By Lemmas 3.6 and 3.8,
\[ \hat{H}(w_t) \leq 2B_X \hat{F}(v)^{1/2} \leq 2L^2 B_X \left( \text{OPT}^{1/2} + \frac{\sqrt{3}a \log^{1/4}(2/\delta)}{n^{1/4}} \right), \quad (3.11) \]
so that we have
\[ H(w_t) \leq 2B_X \left( \text{OPT}^{1/2} + \frac{\sqrt{3}a \log^{1/4}(2/\delta)}{n^{1/4}} \right) + \frac{2B_X^2}{\sqrt{n}} + 2B_X^2 \sqrt{\frac{\log(4/\delta)}{n}}. \quad (3.12) \]
Since \( \mathcal{D}_x \) satisfies Assumption 3.2 and \( \|w_t - v\|_2 \leq 1 \), Lemma 3.5 yields \( G(w_t) \leq 8\sqrt{2}a^{-4} \beta^{-1} H(w_t) \). Then Claim 3.4 completes the proof. \( \square \)

**Remark 3.10.** An inspection of the proof of Theorem 3.3 shows that when \( \sigma \) satisfies Assumption 3.1, any initialization with \( \|w_0 - v\|_2 \) bounded by a universal constant will suffice. In particular, if we use Gaussian initialization \( w_0 \sim N(0, \tau^2 I_d) \) for \( \tau^2 = O(1/d) \), then by concentration of the chi-squared distribution the theorem holds with (exponentially) high probability over the random initialization. For ReLU, initialization at the origin greatly simplifies the proof since Lemma 3.6 shows that \( \|w_t - v\|_2 \leq \|w_0 - v\|_2 \) for all \( t \). When \( w_0 = 0 \), this implies that \( \|w_t - v\|_2 \leq 1 \) throughout the trajectory of gradient descent, and thus allows for an easy application of Lemma 3.5. For isotropic Gaussian initialization, one can show that with probability approaching \( 1/2 \) that \( \|w_0 - v\|_2 < 1 \) provided its variance satisfies \( \tau^2 = O(1/d) \) (see e.g. Lemma 5.1 of Yehudai and Shamir (2020)). In this case, the theorem will hold with constant probability over the random initialization.

## 4 Noisy teacher network setting

We now assume the joint distribution of \((x, y) \sim \mathcal{D}\) is given by a target neuron \( v \) (with \( \|v\|_2 \leq 1 \)) plus zero-mean and \( s \)-sub-Gaussian noise,
\[ y|x \sim \sigma(v^\top x) + \xi, \quad \mathbb{E}[\xi|x] = 0. \]

We assume throughout this section that \( \xi \neq 0 \); we deal with the realizable setting separately (and achieve improved sample complexity) in Appendix D. We note that this is precisely the setup of the generalized linear model with (inverse) link function \( \sigma \). We further note that we only assume that \( \mathbb{E}[y|x] = \sigma(v^\top x) \), i.e., the noise is not assumed to be independent of the features \( x \), and thus falls into the probabilistic concept learning model of Kearns and Schapire (1994).

With the additional structural assumption of a noisy teacher, we can improve the agnostic result from \( O(\text{OPT}^{1/2}) + \varepsilon \) to exactly \( \text{OPT} + \varepsilon \), as well as improve the order of the sample complexity from \( \varepsilon^{-4} \) to \( \varepsilon^{-2} \). The key difference with the agnostic proof is that when trying to show the gradient points in a good direction as in (3.5), since we know \( \mathbb{E}[y|x] = \sigma(v^\top x) \), the average of terms of the form \( a_i(v^\top x_i) - y_i \) will concentrate around zero provided the \( |a_i| \) are bounded. This allows for us to improve the lower bound from \( \langle \nabla \hat{F}(w_t), w_t - v \rangle \geq \hat{H}(w) - \hat{F}(v)^{1/2} \) to one of the form \( \geq \hat{H}(w) - \varepsilon \). The full proof of Theorem 4.1 is given in Appendix C.
Theorem 4.1. Suppose $D_x$ satisfies $\|x\|_2 \leq B_X$ a.s. and that $E[y|x] = \sigma(v^\top x)$ for some $\|v\|_2 \leq 1$. Assume that $\sigma(v^\top x) - y$ is $s$-sub-Gaussian. Assume gradient descent is initialized at $w_0 = 0$ and fix a step size $\eta \leq (1/4)L^2B_X^{-2}$. If $\sigma$ satisfies Assumption 3.1, let $\gamma$ be the constant corresponding to $\rho = 2B_X$. There exists an absolute constant $c_0 > 0$ such that for any $\delta > 0$, with probability at least $1 - \delta$, gradient descent run for $T = \eta^{-1}\sqrt{n}/(c_0LB_x\sqrt{\log(4d/\delta)})$ finds weights $w_t$, $t < T$, satisfying

$$F(w_t) \leq F(v) + C_1n^{-1/2} + C_2n^{-1/2}\sqrt{\log(8/\delta)} + C_3n^{-1/2}\sqrt{\log(4d/\delta)},$$

where $C_1 = 4L^3B_X^2$, $C_2 = 2\sqrt{2}L^2B_X\sqrt{2}$, and $C_3 = 4c_0\gamma^{-1}L^2sB_X$. When $\sigma$ is ReLU, further assume that $D_x$ satisfies Assumption 3.2 for constants $\alpha, \beta > 0$, and let $\nu = \alpha^4\beta/8\sqrt{2}$. Then (4.1) holds for $C_1 = B^2X\nu^{-1}$, $C_2 = 2C_1$, and $C_3 = 4c_0s\nu^{-1}B_X$.

We first note that although (4.1) contains a $\log(d)$ term, this term can be removed if we assume that the noise is bounded rather than sub-Gaussian; details for this are given in Appendix C. As mentioned previously, if we are in the realizable setting, i.e. $\xi = 0$, we can improve the sample and runtime complexity to $\varepsilon^{-1}$ by using online SGD and a martingale Bernstein inequality. For details on the realizable case, see Appendix D.

In comparison with recent literature, Kakade et al. (2011) proposed GLMTron to show learnability of the noisy teacher network for any non-decreasing and Lipschitz activation $\sigma$ when the noise is bounded. (A close inspection of the proof shows that subgaussian noise can be handled with the same norm sub-Gaussian concentration that we use for our results.) In GLMTron, updates take the form $w_{t+1} = w_t - \eta\tilde{g}_t$ where $\tilde{g}_t = \sigma(w_t^\top x) - y$, while in gradient descent, the updates take the form $w_{t+1} = w_t - \eta g_t$ where $g_t = \tilde{g}_t\sigma'(w_t^\top x)$. Intuitively, when the weights are in a bounded region and $\sigma$ is strictly increasing and Lipschitz, then the derivative satisfies $\sigma'(w_t^\top x) \in [\gamma, L]$ and so the additional $\sigma'$ factor should not substantially affect the algorithm. For ReLU this is more complicated as the gradient could in the worst case be zero in a large region of the input space, preventing effective learnability using gradient-based optimization, as was demonstrated in the negative result of Yehudai and Shamir (2020). For this reason, a type of nondegeneracy condition like our Assumption 3.2 is natural for gradient descent on ReLUs.

In terms of other results for ReLU, recent work by Mukherjee and Muthukumar (2020) introduced another modified version of SGD, where updates now take the form $w_{t+1} = w_t - \eta\tilde{g}_t$, where $\tilde{g}_t = \tilde{g}_t\sigma'(y > \theta)$, where $\theta$ is an upper bound for the noise term. Using this modified SGD, they showed learnability of the ReLU in the noisy teacher network setting with bounded noise under the nondegeneracy condition that the matrix $E_{x}[xx^\top 1(v^\top x \geq 0)]$ is positive definite. A similar assumption was used by Du et al. (2018) in the realizable setting.

Our GLM result is also comparable to recent work by Foster et al. (2018), where the authors provide a meta-algorithm for translating guarantees for $\varepsilon$-stationary points of the empirical risk to guarantees for the population risk under Polyak–Lojasiewicz-like (PL-like) conditions on the population risk, provided the algorithm can guarantee that the weights remain bounded (see their Proposition 3). By considering GLMs with bounded, strictly increasing, Lipschitz activations, they show the PL-type condition holds, and that any algorithm that can find a stationary point of an $\ell^2$-regularized empirical risk objective is guaranteed a population risk bound. In contrast, our result concretely shows that vanilla gradient descent learns the GLM, even in the ReLU setting.
5 Conclusion and remaining open problems

In this work, we showed that gradient descent can achieve $O(\OPT^{1/2}) + \varepsilon$ population risk in the agnostic setting, and $\OPT + \varepsilon$ in the noisy teacher network and realizable settings, for the most common activation functions used in practice. Is it possible to show stronger results for gradient descent, or are there distributions for which gradient descent cannot learn better than $O(\OPT^{1/2})$ without further assumptions? This question remains for neural networks with one or more hidden layers as well. Additionally, we focused on the regression problem with real-valued label outputs. Understanding the properties of gradient descent for the agnostic learning of halfspaces generated by single neurons remains an interesting open problem.

A Detailed comparisons with related work

Table 1: Comparison of results in the agnostic setting

| Algorithm                      | Activations | Pop. risk      | $\mathcal{D}_x$          | Sample Complexity |
|--------------------------------|-------------|----------------|---------------------------|-------------------|
| Halfspace reduction (Goel et al., 2019) | ReLU        | $O(\OPT^{2/3})$ | standard Gaussian         | $O(\text{poly}(d, \varepsilon^{-1}))$ |
| Gradient Descent (This paper)   | strictly increasing + Lipschitz | $O(\OPT^{1/2})$ | bounded                   | $O(\varepsilon^{-4})$ |
| Gradient Descent (This paper)   | ReLU        | $O(\OPT^{1/2})$ | bounded + marginal spread | $O(\varepsilon^{-4})$ |

Here, we describe comparisons of our results to those in the literature and give detailed comments on the specific rates we achieve. In Table 1, we compare our agnostic learning result with that of Goel et al. (2019). We note the guarantees for the population risk in the fourth column, the marginal distributions over $x$ for which the bounds hold in the fifth column, and the sample complexity required to reach the specified level of risk plus some $\varepsilon > 0$ in the final column. Our results in this setting come from Theorem 3.3. The Big-O notation hides constant that may depend on the parameters of the distribution or activation function, but does not hide explicit dependence on the dimension $d$. However, the parameters of the distribution itself may have implicit dependence on the dimension. In particular, for bounded distributions that satisfy $\|x\|_2 \leq B_X$, the $O()$ hides multiplicative factors that depend on $B_X$. This means that if $B_X$ depends on $d$, so will our bounds. For non-ReLU, the worst-case activation functions under consideration in Assumption 3.1 (e.g. the sigmoid) can have $\gamma \sim \exp(-B_X)$, making the runtime and sample complexity depend on $\gamma^{-1} \sim \exp(B_X)$, in which case it is better to assume that $B_X$ is a constant independent of the dimension.

In Table 2, we provide comparisons of our noisy teacher network (also known as the generalized linear model or the probabilistic concepts model) results. Our results in this setting come from Theorem 4.1. The complexity column here denotes the sample complexity required to reach population risk $\OPT + \varepsilon$. The subspace eigenvalue assumption given by Mukherjee and Muthukumar (2020) is
Table 2: Comparison of results in the noisy teacher network setting

| Algorithm                                      | Activations       | $D_x$             | Sample Complexity |
|------------------------------------------------|-------------------|-------------------|-------------------|
| GLMTron (Kakade et al., 2011)                  | increasing        | bounded           | $O(\varepsilon^{-2})$ |
|                                                | + Lipschitz       |                   |                   |
| Modified Stochastic Gradient Descent           | ReLU              | bounded           | $O(\log(1/\varepsilon))$ |
| (Mukherjee and Muthukumar, 2020)               |                   | + subspace eigenvalue |                   |
| Meta-algorithm (Foster et al., 2018)           | strictly increasing| bounded           | $O(\varepsilon^{-2} \wedge d\varepsilon^{-1})$ |
|                                                | + Lipschitz       |                   |                   |
|                                                | + $\sigma'$ Lipschitz |             |                   |
| Gradient Descent (Mei et al., 2018)            | strictly increasing| centered          | $O(d\varepsilon^{-1})$ |
|                                                | + diff'ble        | + sub-Gaussian    |                   |
|                                                | + Lipschitz       |                   |                   |
|                                                | + $\sigma'$ Lipschitz |             |                   |
|                                                | + $\sigma''$ Lipschitz |         |                   |
| Gradient Descent (This paper)                  | strictly increasing| bounded           | $O(\varepsilon^{-2})$ |
|                                                | + Lipschitz       |                   |                   |
| Gradient Descent (This paper)                  | ReLU              | bounded           | $O(\varepsilon^{-2})$ |
|                                                | + marginal spread |                   |                   |

that $\mathbb{E}[xx^\top 1 (v^\top x \geq 0)] > 0$. Of course, any result that holds for the agnostic setting also holds in the generalized linear model setting, but for all results we consider, the population risk guarantee is strictly worse than what is achieved in the noisy teacher network setting.

Finally, in Table 3, we provide comparisons with results in the realizable setting. (Our results in this setting are given in Theorem D.1 in Appendix D.) For G.D. and S.G.D., the complexity column denotes the sample complexity required to reach population risk $\varepsilon$. For G.D. or gradient flow on the population risk (‘Pop. G.D.’), it refers to the runtime complexity only as there are no samples in this setting. For Du et al. (2018), the subspace eigenvalue assumption is that for any $w$ and for the target neuron $v$, it holds that $\mathbb{E}[xx^\top 1 (w^\top x \geq 0, v^\top x \geq)] > 0$. This is a nondegeneracy assumption that is related to the marginal spread condition given in Assumption 3.2, in the sense that it allows for one to show that $H$ is an upper bound for $G$. Finally, we note that any result in the agnostic or noisy teacher network settings applies in the realizable setting as well.

B Proof of Lemma 3.5

To prove Lemma 3.5, we use the following result of Yehudai and Shamir (2020).

Lemma B.1 (Lemma B.1, Yehudai and Shamir). Under Assumption 3.2, for any two vectors
Table 3: Comparison of results in the realizable setting

| Algorithm                        | Activations     | $D_x$            | Sample Complexity |
|----------------------------------|-----------------|------------------|-------------------|
| Stochastic Gradient Descent      | ReLU            | bounded          | $O(\log(1/\varepsilon))$ |
| (Du et al., 2018)                |                 | + subspace eigenvalue | |
| Projected Regularized Gradient Descent (Soltanolkotabi, 2017) | ReLU            | standard Gaussian | $O(\log(1/\varepsilon))$ |
| Population Gradient Descent      | leaky ReLU      | bounded          | $O(\log(1/\varepsilon))$ |
| (Yehudai and Shamir, 2020)       |                 | $+ \mathbb{E}[xx^\top] > 0$ | |
| Population Gradient Descent      | inf$_{0 < z < \alpha} \sigma'(z) > 0$ | bounded          | $O(\log(1/\varepsilon))$ |
| (Yehudai and Shamir, 2020)       |                 | + Lipschitz      | |
| Population Gradient Flow         | ReLU            | marginal spread  | $O(\log(1/\varepsilon))$ |
| (Yehudai and Shamir, 2020)       |                 | + spherical symmetry | |
| Stochastic Gradient Descent      | inf$_{0 < z < \alpha} \sigma'(z) > 0$ | bounded          | $O(\varepsilon^{-2})$ |
| (Yehudai and Shamir, 2020)       |                 | + Lipschitz      | |
| + Stochastic Gradient Descent    | strictly increasing | bounded          | $O(\varepsilon^{-1})$ |
| (This paper)                     |                 | + Lipschitz      | |
| Population Gradient Descent      | ReLU            | bounded          | $O(\varepsilon^{-1})$ |
| + Stochastic Gradient Descent    |                 | + marginal spread | |
| (This paper)                     |                 |                 | |

$a, b \in \mathbb{R}^2$ satisfying $\theta(a, b) \leq \pi - \delta$ for $\delta \in (0, \pi]$, it holds that

$$
\inf_{u \in \mathbb{R}^2: \|u\|=1} \int(u^\top y)^2 1(a^\top y \geq 0, b^\top y \geq 0, \|y\| \leq \alpha)dy \geq \frac{\alpha^4}{8\sqrt{2}} \sin^3(\delta/4).
$$

**Proof of Lemma 3.5.** By assumption,

$$
H(w) = (1/2)\mathbb{E} \left[ (\sigma(w_i^\top x) - \sigma(v^\top x))^2 \sigma'(w_i^\top x) \right] \leq \varepsilon.
$$

We first consider the case when $\sigma$ satisfies Assumption 3.1. Since the term in the expectation is nonnegative, restricting the integral to a smaller set only decreases its value, so that

$$
(1/2)\mathbb{E} \left[ (\sigma(w_i^\top x) - \sigma(v^\top x))^2 \sigma'(w_i^\top x) 1(|w_i^\top x| \leq \rho) \right] \leq \varepsilon. \quad (B.1)
$$
For $\rho = BW$, since $\|w\|_2 \leq W$, the inclusion $\{\|x\|_2 \leq \rho/W\} \subset \{|w^\top x| \leq \rho\}$ holds. We thus have

$$\frac{\gamma}{2} \mathbb{E} \left[ \left( \sigma(w^\top x) - \sigma(v^\top x) \right)^2 \right] \leq (1/2) \mathbb{E} \left[ \left( \sigma(w^\top x) - \sigma(v^\top x) \right)^2 \sigma'(w^\top x) \mathbbm{1}(\|x\|_2 \leq \rho/4) \right] \leq \varepsilon. $$

Dividing both sides by $\gamma$ completes this part of the proof.

For ReLU, denote the event

$$K_{w,v} := \{w^\top x \geq 0, v^\top x \geq 0\},$$

and define $\zeta := \beta \alpha^4 / 8\sqrt{2}$. Since $H(w) = \mathbb{E}[\sigma(w^\top x) - \sigma(v^\top x)]^2 (w^\top x \geq 0) \leq \zeta \varepsilon$, it holds that

$$\mathbb{E} \left[ \left( \sigma(w^\top x) - \sigma(v^\top x) \right)^2 \mathbbm{1}(K_{w,v}) \right] \leq \zeta \varepsilon. \quad \text{(B.2)}$$

Using a proof similar to that of Yehudai and Shamir (2020), we have

$$\mathbb{E}_{x \sim \mathcal{D}} \left[ (w^\top x - v^\top x)^2 \mathbbm{1}(K_{w,v}) \right] = \|w - v\|_2^2 \mathbb{E}_{x \sim \mathcal{D}} \left[ \left( \frac{w - v}{\|w - v\|_2} \right)^\top x \right]^2 \mathbbm{1}(K_{w,v})$$

$$\geq \|w - v\|_2^2 \inf_{u \in \text{span}(w,v), \|u\|_2 = 1} \mathbb{E}_x \left[ \mathbbm{1}(u^\top x)^2 \mathbbm{1}(K_{w,v}) \right]$$

$$= \|w - v\|_2^2 \inf_{u \in \mathbb{R}^2, \|u\|_2 = 1} \mathbb{E}_{y \sim \mathcal{D}_{w,v}} \left[ (u^\top y)^2 \mathbbm{1}(\tilde{w}^\top y \geq 0, \tilde{v}^\top y \geq 0) \right]$$

$$\geq \|w - v\|_2^2 \inf_{u \in \mathbb{R}^2, \|u\|_2 = 1} \int (u^\top y)^2 \mathbbm{1}(\tilde{w}^\top y \geq 0, \tilde{v}^\top y \geq 0, \|y\|_2 \leq \alpha) p_{w,v}(y) dy$$

$$\geq \beta \|w - v\|_2^2 \inf_{u \in \mathbb{R}^2, \|u\|_2 = 1} \int (u^\top y)^2 \mathbbm{1}(\tilde{w}^\top y \geq 0, \tilde{v}^\top y \geq 0, \|y\|_2 \leq \alpha) dy. \quad \text{(B.3)}$$

By assumption, $\|w - v\|_2 \leq 1$. Since

$$1 \geq \|w - v\|_2^2 = \|w\|_2 (\|w\|_2 - 2 \cos \theta(w,v)) + 1,$$

we must have either $w = 0$ or $\theta(w,v) \in [0, \pi/2]$. To see that $w = 0$ is impossible, suppose for the contradiction that $w = 0$ and so $H(w) = H(0) \leq \zeta \varepsilon$. Let $z$ be any vector orthogonal to $v$, so that $\theta(v,z) = \pi/2$. Then,

$$\zeta \varepsilon \geq H(0)$$

$$= \mathbb{E}_{x \sim \mathcal{D}} \left[ (v^\top x)^2 \mathbbm{1}(v^\top x \geq 0) \right]$$

$$= \mathbb{E}_{y \sim \mathcal{D}_{0,v}} \left[ (\tilde{v}^\top y)^2 \mathbbm{1}(\tilde{v}^\top y \geq 0) \right]$$

$$\geq \inf_{u: \|u\|_2 = 1} \int (u^\top x)^2 \mathbbm{1}(u^\top x \geq 0, z^\top x \geq 0, \|y\|_2 \leq \alpha) p_{0,v}(y) dy$$

$$\geq \beta \inf_{u: \|u\|_2 = 1} \int (u^\top x)^2 \mathbbm{1}(u^\top x \geq 0, z^\top x \geq 0, \|y\|_2 \leq \alpha) dy$$

$$\geq \frac{\beta \alpha^4}{8\sqrt{2}}. \quad \text{(B.4)}$$
The last line follows by using Lemma B.1. By the choice of $\zeta$, this is impossible. This shows that $
abla_{w,v} \leq \pi/2$. We can therefore apply Lemma B.1 to (B.3) to get

$$\zeta \geq \beta \|w - v\|_2^2 \inf_{u \in \mathbb{R}^2, \|u\|_2 = 1} \int (u^T y)^2 \mathbb{1}(\hat{w}^T y \geq 0, \hat{v}^T y \geq 0, \|y\|_2 \leq \alpha) dy$$

$$\geq \frac{\beta \alpha^4}{8\sqrt{2}} \|w - v\|_2^2$$

$$= \zeta \beta^2 \|w - v\|_2^2.$$

This shows that $\|w - v\|_2^2 \leq B^{-2}\varepsilon$. Since $\sigma$ is $1$-Lipschitz, Hölder’s inequality and $\mathbb{E} \|x\|_2^2 \leq B^2$ imply that $G(w) \leq \varepsilon$. $\blacksquare$

C Noisy teacher network proofs

As in the agnostic case, we have a key lemma that shows $\hat{H}$ is small when we run gradient descent for a sufficiently large time.

**Lemma C.1.** Suppose that $\|x\|_2 \leq B_X$ a.s. under $\mathcal{D}_x$. Let $\sigma$ be non-decreasing and $L$-Lipschitz. Suppose that the bound

$$\|(1/n) \sum_{i=1}^n (\sigma(v^T x_i) - y_i) \alpha_i x_i\|_2 \leq K \leq 1,$$

(C.1)

holds for scalars satisfying $\alpha_i \in [0,L]$. Then gradient descent run with fixed step size $\eta \leq (1/4) L^{-2} B_X^{-2}$ from initialization $w_0 = 0$ finds weights $w_t$ satisfying $\hat{H}(w_t) \leq 4L K$ within $T = \lceil \eta^{-1} K^{-1} \rceil$ iterations, with $\|w_t - v\|_2 \leq 1$ for each $t = 0, \ldots, T - 1$.

**Proof.** The key to the proof of the lemma comes from the following induction statement. We claim that for every $t \in \mathbb{N}$, either (a) $\hat{H}(w_t) \leq 4L K$ for some $\tau < t$, or (b) $\|w_t - v\|_2^2 \leq \|w_{t-1} - v\|_2^2 - \eta K$. If the induction hypothesis holds, we know that at every iteration of gradient descent, we either have $\hat{H}(w_t) \leq 4L K$ or $\|w_t - v\|_2^2 \leq 1 - \eta K$. Since $\|w_0 - v\|_2^2 = 1$, this means there can be at most $1/(\eta K) = \eta^{-1} K^{-1}$ iterations until we reach $\hat{H}(w_t) \leq 4L K$. This shows the induction statement implies the theorem.

We begin with the proof by supposing the induction hypothesis holds for $t$, and want to consider the case $t + 1$. If (a) holds, then we are done. So now consider the case that for every $\tau \leq t$, we have $\hat{H}(w_\tau) > 4L K$. Since (a) does not hold, $\|w_\tau - v\|_2^2 \leq \|w_{\tau-1} - v\|_2^2 - \eta K$ holds for each $\tau = 1, \ldots, t$. Since $\|w_0 - v\|_2 = 1$, this implies

$$\|w_\tau - v\|_2 \leq 1 \forall \tau \leq t.$$  

(C.2)
We can therefore bound
\[
\langle \nabla \hat{F}(w_t), w_t - v \rangle = \left\langle \frac{1}{n} \sum_{i=1}^{n} \left( \sigma(w_t^\top x_i) - y_i \right) \sigma'(w_t^\top x_i) x_i, w_t - v \right\rangle \\
= \frac{1}{n} \sum_{i=1}^{n} \left( \sigma(w_t^\top x_i) - \sigma(v^\top x_i) \right) \sigma'(w_t^\top x_i)(w_t^\top x_i - v^\top x_i) \\
+ \left\langle \frac{1}{n} \sum_{i=1}^{n} \left( \sigma(v^\top x_i) - y_i \right) \sigma'(w_t^\top x_i) x_i, w_t - v \right\rangle \\
\geq \frac{L^{-1}}{n} \sum_{i=1}^{n} \left( \sigma(w_t^\top x_i) - \sigma(v^\top x_i) \right)^2 \sigma'(w_t^\top x_i) - K \|w_t - v\|_2 \\
\geq 2L^{-1} \hat{H}(w_t) - K. \tag{C.3}
\]

In the first inequality, we have used Fact 3.7 for the first term. For the second term, we use (C.1) and that \( \alpha_i := \sigma'(w_t^\top x_i) \in [0, L] \). The last inequality uses (C.2).

For the gradient upper bound, we have
\[
\|\nabla \hat{F}(w_t)\|_2^2 = \left\| \frac{1}{n} \sum_{i=1}^{n} \left( \sigma(w_t^\top x_i) - \sigma(v^\top x_i) \right) \sigma'(w_t^\top x_i) x_i \right\|_2^2 \\
\leq 2 \left\| \frac{1}{n} \sum_{i=1}^{n} \left( \sigma(w_t^\top x_i) - \sigma(v^\top x_i) \right) \sigma'(w_t^\top x_i) x_i \right\|_2^2 \\
+ 2 \left\| \frac{1}{n} \sum_{i=1}^{n} \left( \sigma(v^\top x_i) - y_i \right) \sigma'(w_t^\top x_i) x_i \right\|_2^2 \\
\leq \frac{2LB_X^2}{n} \sum_{i=1}^{n} \left( \sigma(w_t^\top x_i) - \sigma(v^\top x_i) \right)^2 \sigma'(w_t^\top x_i) + 2K^2 \\
= 4LB_X^2 \hat{H}(w_t) + 2K^2. \tag{C.4}
\]

The first inequality uses Young’s inequality. The second uses that \( \sigma'(z) \leq L \) and that \( \|x\|_2 \leq B_X \) a.s. and (C.1).

Putting (C.3) and (C.4) together, the choice of \( \eta \leq (1/4)L^{-2}B_X^2 \) gives us
\[
\|w_t - v\|_2^2 - \|w_{t+1} - v\|_2^2 = 2\eta \left\langle \nabla \hat{F}(w_t), w_t - v \right\rangle - \eta^2 \left\| \nabla \hat{F}(w_t) \right\|_2^2 \\
\geq 2\eta L^{-1} \hat{H}(w_t) - K - \eta^2 \left( 4LB_X^2 \hat{H}(w_t) + 2K^2 \right) \\
\geq \eta L^{-1} \hat{H}(w_t) - 3\eta K.
\]

In particular, this implies
\[
\|w_{t+1} - v\|_2^2 \leq \|w_t - v\|_2^2 + 3\eta K - \eta L^{-1} \hat{H}(w_t) \tag{C.5}
\]

Since \( \hat{H}(w_t) \geq 4KL \), this completes the induction. The base case follows easily since \( \|w_0 - v\|_2 = 1 \) allows for us to deduce the desired bound on \( \|w_t - v\|_2^2 \) using (C.5). \( \square \)
To prove a concrete bound on the $K$ term of Lemma C.1, we will need the following definition of norm sub-Gaussian random vectors.

**Definition C.2.** A random vector $z \in \mathbb{R}^d$ is said to be *norm sub-Gaussian with parameter $s > 0$* if

$$\mathbb{P}(\|z - \mathbb{E}z\| \geq t) \leq 2 \exp(-t^2/2s^2).$$

A Hoeffding-type inequality for norm-subgaussian vectors was recently shown by Jin et al. Jin et al. (2019) (Lemma 6).

**Lemma C.3.** Suppose $z_1, \ldots, z_n \in \mathbb{R}^d$ are random vectors with filtration $\mathcal{F}_t := \sigma(z_1, \ldots, z_t)$ such that $z_i|\mathcal{F}_{i-1}$ is a zero-mean norm sub-Gaussian vector with parameter $s_i \in \mathbb{R}$ for each $i$. Then, there exists an absolute constant $c > 0$ such that for any $\delta > 0$, with probability at least $1 - \delta$,

$$\left\| \frac{1}{n} \sum_{i=1}^{n} z_i \right\| \leq c \sqrt{\log(2d/\delta) \sum_{i=1}^{n} s_i^2}.$$

Using this, we can show that if $\xi_i := \sigma(v^T x_i) - y_i$ is $s$ sub-Gaussian, then we have the appropriate norm sub-Gaussian bound to get a $n^{-1/2}$ rate for the bound on $K$. We note that if we make the stronger assumption that $\xi_i$ is bounded a.s., we can get rid of the $\log(d)$ dependence by using concentration of bounded random variables in a Hilbert space (e.g. Pinelis and Sakhanenko (1986), Corollary 2).

**Lemma C.4.** Suppose that $\|x\|_2 \leq B_X$ a.s. under $\mathcal{D}_x$, and let $\sigma$ be any continuous function. Assume $\xi_i := \sigma(v^T x_i) - y_i$ is $s$-subgaussian and satisfies $\mathbb{E}[\xi_i|x_i] = 0$. Then there exists an absolute constant $c_0 > 0$ such that for constants $\alpha_i \in [0, L]$, with probability at least $1 - \delta$, we have

$$\mathbb{E}[\xi_i]|x_i| \leq c_0LB_Xs\sqrt{n^{-1}\log(2d/\delta)}.$$

**Proof of Lemma C.4.** Define $z_i := (\sigma(v^T x_i) - y_i) \alpha_i x_i$. Using iterated expectations, we see that $\mathbb{E}[z_i] = 0$. Since $\sigma(v^T x_i) - y_i$ is $s$-sub-Gaussian and $\|\alpha_i x_i\|_2 \leq LB_X$, it follows from the definition that $z_i$ is norm sub-Gaussian with parameter $LB_Xs$ for each $i$. By Lemma C.3, we have with probability at least $1 - \delta$,

$$\left\| \sum_{i=1}^{n} z_i \right\| \leq c \sqrt{\log(2d/\delta) L^2 B_X^2 ns^2}.$$

Dividing each side by $n$ proves the lemma.

**Proof of Theorem 4.1.** By Lemmas C.1 and C.4, there exists some $w_t$, $t < T$ and $\|w_t - v\|_2 \leq 1$, such that

$$\hat{H}(w_t) \leq 4LK \leq 4c_0L^2 B_X s \sqrt{\frac{\log(2d/\delta)}{n}}.$$

Using Lemma 3.5, since $\|w_t\|_2 \leq 2$, this implies

$$\hat{G}(w_t) \leq 4c_0\gamma^{-1} L^2 B_X s \sqrt{\frac{\log(2d/\delta)}{n}}.$$
Since \( \|w - v\|_2 \leq 1 \) implies \( G(w) \leq L^2B_x^2/2 \), standard results from Rademacher complexity imply (e.g., Theorem 26.5 of Shalev-Shwartz and Ben-David (2014)) that with probability at least \( 1 - \delta \),

\[
G(w_t) \leq \tilde{G}(w_t) + \mathbb{E}_{S \sim \mathcal{D}^n} R_S(\ell \circ \sigma \circ \mathcal{G}) + 2L^2B_x^2 \sqrt{\frac{2\log(4/\delta)}{n}},
\]

where \( \ell(w;x) = (1/2)(\sigma(w^Tx) - \sigma(v^Tx))^2 \) and \( \mathcal{G} \) are from Lemma 3.9. For the second term above, Lemma 3.9 and rescaling \( \delta \) yields that

\[
G(w_t) \leq \frac{2L^2B_x^2}{\sqrt{n}} + \frac{2L^2B_x^2 \sqrt{2\log(8/\delta)}}{\sqrt{n}} + \frac{4c_0\gamma^{-1}L^2B_x s \sqrt{\log(4d/\delta)}}{\sqrt{n}}.
\]

Then Claim 3.4 completes the proof for strictly increasing \( \sigma \).

When \( \sigma \) is ReLU, the proof follows the same argument given in the proof of Theorem 3.3. Denoting the loss function \( \ell(w;x) = (1/2)(\sigma(w^Tx) - \sigma(v^Tx))^2 \sigma'(w^Tx) \), we have

\[
\mathbb{E}_{S \sim \mathcal{D}^n} R_S(\ell \circ \sigma \circ \mathcal{G}) \leq \frac{2B_x^2}{\sqrt{n}}. \tag{C.6}
\]

By Lemmas C.1 and C.4, there exists some \( w_t, t < T \) and \( \|w_t - v\|_2 \leq 1 \), such that

\[
\tilde{H}(w_t) \leq 4L \leq 4c_0L^2B_x s \sqrt{\frac{\log(2d/\delta)}{n}}. \tag{C.7}
\]

Using standard results from Rademacher complexity,

\[
H(w_t) \leq \tilde{H}(w_t) + \mathbb{E}_{S \sim \mathcal{D}^n} R_S(\ell \circ \sigma \circ \mathcal{G}) + 2B_x^2 \sqrt{\frac{2\log(4/\delta)}{n}}.
\]

By (C.6), this means

\[
H(w_t) \leq \frac{2B_x^2}{\sqrt{n}} + \frac{2B_x^2 \sqrt{2\log(8/\delta)}}{\sqrt{n}} + \frac{4c_0B_x s \sqrt{\log(4d/\delta)}}{\sqrt{n}}.
\]

Since \( \mathcal{D} \) satisfies Assumption 3.2 and \( \|w_t - v\|_2 \leq 1 \), Lemma 3.5 shows that \( G(w_t) \leq 8\sqrt{2}\alpha^{-4}\beta^{-1}H(w_t) \). Then Claim 3.4 translates the bound for \( G(w_t) \) into one for \( F(w_t) \).

\[
\square
\]

## D Realizable setting

In this section we assume \( y = \sigma(v^Tx) \) a.s. for some \( \|v\|_2 \leq 1 \). For purpose of comparison with Yehudai and Shamir (2020), we provide analyses for two settings in the realizable case: first, gradient descent on the population loss,

\[
w_{t+1} = w_t - \eta \nabla F(w_t), \tag{D.1}
\]

where we return \( w_{t^*} := \text{argmin}_{0 \leq t < T} F(w_t) \). The second setting is using independent samples \( x_t \sim \mathcal{D} \) with online SGD, where we compute unbiased estimates of the population risk \( F_t(w_t) := (1/2)(\sigma(w_t^Tx_t) - \sigma(v^Tx_t))^2 \) and update the weights by

\[
w_{t+1} = w_t - \eta \nabla F_t(w_t). \tag{D.2}
\]
For SGD, we output \( w_{t^*} = \text{argmin}_{0 \leq t < T} F_t(w_t) \).

We summarize our results in the realizable case in Theorem D.1.

**Theorem D.1.** Suppose \( \|x\|_2 \leq B \) a.s. and \( \sigma \) is non-decreasing and \( L \)-Lipschitz. Let \( \eta \leq L^{-2}B^{-2} \) be the step size.

(a) Let \( \sigma \) satisfy Assumption 3.1, and let \( \gamma \) be the constant corresponding to \( \rho = 4B \). For any initialization satisfying \( \|w_0\|_2 \leq 2 \), if we run gradient descent on the population risk \( T = \lceil 2\varepsilon^{-1}L\eta^{-1}\gamma^{-1}\|w_0 - v\|_2^2 \rfloor \) iterations, then there exists \( t < T \) such that \( F(w_t) \leq \varepsilon \). For stochastic gradient descent, for any \( \delta > 0 \), running SGD for \( \tilde{T} = 6T\log(1/\delta) \) guarantees there exists \( w_t, t < T \), such that w.p. at least \( 1 - \delta \), \( F(w_t) \leq \varepsilon \).

(b) Let \( \sigma \) be ReLU and further assume that \( \mathcal{D} \) satisfies Assumption 3.2 for constants \( \alpha, \beta > 0 \) and that \( w_0 = 0 \). Let \( \nu = \alpha^4/8\sqrt{2} \). If we run gradient descent on the population risk \( T = \lceil 2\varepsilon^{-1}L\eta^{-1}\nu^{-1}\|w_0 - v\|_2^2 \rfloor \) iterations, then there exists \( t < T \) such that \( F(w_t) \leq \varepsilon \). For stochastic gradient descent, for any \( \delta > 0 \), running SGD for \( \tilde{T} = 6T\log(1/\delta) \) guarantees there exists \( w_t, t < T \), such that w.p. at least \( 1 - \delta \), \( F(w_t) \leq \varepsilon \).

A few remarks on the above theorem: first, in comparison with our noisy neuron result in Theorem 4.1, we are able to achieve \( \text{OPT} + \varepsilon = \varepsilon \) population risk with sample complexity and runtime of order \( \varepsilon^{-1} \) rather than \( \varepsilon^{-2} \) using the same assumptions by invoking a martingale Bernstein inequality rather than Hoeffding. Second, although Theorem D.1 requires the distribution to be bounded almost surely, we show in Section D.1 below that for GD on the population loss, we can accommodate essentially any distribution with finite expected squared norm.

In comparison with recent works, Yehudai and Shamir (2020) used the marginal spread assumption to show that with probability 1/2, a single neuron in the realizable setting can be learned using gradient-based optimization with random initialization for a collection of activation functions including softplus, sigmoid, tanh, and ReLU; under the additional assumption of spherical symmetry, they showed that this can be improved to a high probability guarantee for the ReLU activation. In each case, they proved linear convergence, i.e., \( \log(1/\varepsilon) \) rate. In comparison, our results for the non-ReLU activations requires only boundedness of the distributions and holds with high probability over random initializations, with an \( \varepsilon^{-1} \) rate, and our results for ReLU use the same marginal spread assumption. Our proof technique differs in that we do not require the angle \( \theta(w_t, v) \) between the weights in the GD trajectory and the target neuron be decreasing; this method was also used for showing learnability of the ReLU in the realizable setting by Brutzkus and Globerson (2017). Yehudai and Shamir (2020) pointed out that angle monotonicity fails to hold even when the distribution is a non-centered Gaussian, suggesting that proofs based on angle monotonicity will not translate to more general distributions. Indeed, our proofs in the agnostic and noisy teacher network setting use essentially the same proof technique as the realizable case, and our results hold without relying on angle monotonicity.

### D.1 Gradient descent on population loss

The key lemma for the proof is as follows.

**Lemma D.2.** Consider gradient descent on the population risk given in (D.1). Let \( w_0 \) be the initial point of gradient descent and assume \( \|w_0\|_2 \leq 2 \). Suppose that \( \mathcal{D} \) satisfies \( \mathbb{E}_x[\|x\|_2^2] \leq B^2 \). Let \( \sigma \) be
non-decreasing and $L$-Lipschitz. Assume the step size satisfies $\eta \leq L^{-2}B^{-2}$. Then for any $T \in \mathbb{N}$, we have for all $t = 0, \ldots, T - 1$, $\|w_t - v\|_2 \leq \|w_0 - v\|_2$, and

$$\|w_0 - v\|_2^2 - \|w_T - v\|_2^2 \geq \eta L^{-1} \sum_{t=0}^{T-1} H(w_t).$$

Proof. We begin with the identity, for $t < T$,

$$\|w_t - v\|_2^2 - \|w_{t+1} - v\|_2^2 = 2\langle \nabla F(w_t), w_t - v \rangle - \eta^2 \|\nabla F(w_t)\|_2^2. \quad \text{(D.3)}$$

First, we have

$$\|\nabla F(w_t)\|_2 \leq \mathbb{E}_x \left( \| (\sigma(w_i^\top x) - \sigma(v^\top x))\sigma'(w_i^\top x)x \|_2 \right) \leq \sqrt{\mathbb{E}_x \left[ \sigma'(w_i^\top x)(\sigma(w_i^\top x) - \sigma(v^\top x))^2 \right]} \sqrt{\mathbb{E}_x \sigma'(w_i^\top x) \|x\|_2^2} \leq B\sqrt{L} \mathbb{E}_x \left[ \sigma'(w_i^\top x)(\sigma(w_i^\top x) - \sigma(v^\top x))^2 \right]. \quad \text{(D.4)}$$

The first inequality is by Jensen. The second inequality uses that $\sigma'(z) \geq 0$ and Hölder, and the third inequality uses that $\sigma$ is $L$-Lipschitz and that $\mathbb{E}||x||_2^2 \leq B^2$. We therefore have the gradient upper bound

$$\|\nabla F(w_t)\|_2^2 \leq 2B^2 LH(w_t). \quad \text{(D.5)}$$

For the inner product term of (D.3), since $\sigma'(z) \geq 0$, we can use Fact 3.7 to get

$$\langle \nabla F(w_t), w_t - v \rangle \geq L^{-1} \mathbb{E}_x \left[ \left( \sigma(w_i^\top x) - \sigma(v^\top x) \right)^2 \sigma'(w_i^\top x) \right] = 2L^{-1} H(w_t). \quad \text{(D.5)}$$

Putting (D.5) and (D.4) into (D.3), we get

$$\|w_t - v\|_2^2 - \|w_{t+1} - v\|_2^2 \geq 4\eta L^{-1} H(w_t) - 2\eta^2 B^2 LH(w_t) \geq 2\eta L^{-1} H(w_t),$$

where we have used $\eta \leq L^{-2}B^{-2}$. Telescoping the above over $t < T$ gives

$$\|w_0 - v\|_2^2 - \|w_T - v\|_2^2 \geq 2\eta L^{-1} \sum_{t=0}^{T-1} H(w_t).$$

Dividing each side by $\eta T$ shows the desired bound. \hfill \Box

We will show that if $\sigma$ satisfies Assumption 3.1, then Lemma D.2 allows for a population risk bound for essentially any distribution with $\mathbb{E}||x||_2^2 \leq B^2$. In particular, we consider distributions with finite expected norm squared and the possible types of tail bounds for the norm.

Assumption D.3. (a) Bounded distributions: there exists $B > 0$ such that $||x||_2 \leq B$ a.s.

(b) Exponential tails: there exist $a_0, C_e > 0$ such that $\mathbb{P}(||x||_2^2 \geq a) \leq C_e \exp(-a)$ holds for all $a \geq a_0$.

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(c) Polynomial tails: there exist \( a_0, C_p > 0 \) and \( \beta > 1 \) such that \( \mathbb{P}(\|x\|^2_2 \geq b) \leq C_p a^{-\beta} \) holds for all \( a \geq a_0 \).

If either (a), (b), or (c) holds, there exists \( B > 0 \) such that \( \mathbb{E}\|x\|^2_2 \leq B^2 \). One can verify that for (b), taking \( B^2 = 2(a_0 \vee C_e) \) suffices, and for (c), \( B^2 = 2(a_0 \vee C_p^{1/\beta} / (1 - \beta)) \) suffices. In fact, any distribution that satisfies \( \mathbb{E}\|x\|^2_2 < \infty \) cannot have a tail bound of the form \( \mathbb{P}(\|x\|^2_2 \geq a) = \Omega(a^{-1}) \), since in this case we would have

\[
\mathbb{E}\|x\|^2_2 = \int_0^\infty \mathbb{P}(\|x\|^2_2 > t) dt \geq C \int_{a_0}^\infty t^{-1} dt = \infty.
\]

So the polynomial tail assumption (c) is tight up to logarithmic factors for distributions with finite \( \mathbb{E}\|x\|^2_2 \).

**Theorem D.4.** Let \( \mathbb{E}\|x\|^2_2 \leq B^2 \) and assume \( \mathcal{D} \) satisfies one of the conditions in Assumption D.3. Let \( \sigma \) satisfy Assumption 3.1.

(a) Under Assumption D.3a, let \( \gamma \) be the constant corresponding to \( \rho = 4B \) in Assumption 3.1. Running gradient descent for \( T = [2\varepsilon^{-1}LN^{-1}\gamma^{-1} \|w_0 - v\|_2^2] \) guarantees there exists \( t \in [T - 1] \) such that \( F(w_t) \leq \varepsilon \).

(b) Under Assumption D.3b, let \( \gamma \) be the constant corresponding to \( \rho = 4\sqrt{\log(18C_e/\varepsilon)} \). Running gradient descent for \( T = [2\varepsilon^{-1}LN^{-1}\gamma^{-1} \|w_0 - v\|_2^2] \) guarantees there exists \( t \in [T - 1] \) such that \( F(w_t) \leq \varepsilon \).

(c) Under Assumption D.3c, let \( \gamma \) be the constant corresponding to \( \rho = 4(18C_p/\varepsilon(\beta - 1))^{1-\beta}/2 \). Running gradient descent for \( T = [2\varepsilon^{-1}LN^{-1}\gamma^{-1} \|w_0 - v\|_2^2] \) guarantees there exists \( t \in [T - 1] \) such that \( F(w_t) \leq \varepsilon \).

**Proof.** First, note that the conditions of Lemma D.2 hold, so that we have for all \( t = 0, \ldots, T - 1, \|w_t\|_2 \leq 4 \) and

\[
\eta \sum_{t=0}^{T-1} H(w_t) \leq L \|w_0 - v\|^2_2 - L \|w_T - v\|^2_2. \tag{D.6}
\]

By taking \( T = \zeta^{-1}L\varepsilon^{-1}N^{-1}\|w_0 - v\|^2_2 \) for arbitrary \( \zeta > 0 \), (D.6) implies that there exists \( t \in [T - 1] \) such that

\[
H(w_t) = \mathbb{E} \left[ (\sigma(w_t^\top x) - \sigma(v^\top x))^2 \sigma'(w_t^\top x) \right] \leq \frac{L \|w_0 - v\|^2_2}{\eta T} \leq \zeta \varepsilon. \tag{D.7}
\]

It therefore suffices to bound \( F(w_t) \) in terms of the left hand side of (D.7). We will do so by using the distributional assumptions given in Assumption D.3 and by choosing \( \zeta \) appropriately.

We begin by noting that (D.7) implies, for any \( \rho > 0 \),

\[
\mathbb{E} \left[ (\sigma(w_t^\top x) - \sigma(v^\top x))^2 \sigma'(w_t^\top x) \mathbb{1}(|w_t^\top x| \leq \rho) \right] \leq \zeta \varepsilon. \tag{D.8}
\]

For any \( \rho > 0 \), since \( \|w_t\|_2 \leq 4 \), the inclusion

\[
\left\{ \|x\|_2 \leq \rho/4 \right\} \subset \left\{ |w_t^\top x| \leq \rho \right\}, \tag{D.9}
\]
holds. Under Assumption D.3a, by taking $\rho = 4B$ and letting $\gamma$ be the corresponding constant from Assumption 3.1, eqs. (D.8) and (D.9) imply

$$
\gamma \mathbb{E} \left[ \left( \sigma(w^*_t x) - \sigma(v^*_x) \right)^2 \right] \leq \mathbb{E} \left[ \left( \sigma(w^*_t x) - \sigma(v^*_x) \right)^2 \sigma'(w^*_t x) \mathbb{1}(\|x\|_2 \leq \rho/4) \right] \leq \zeta \varepsilon.
$$

By taking $\zeta = \gamma/2$, this implies $F(w_t) \leq \varepsilon$.

Under Assumption D.3b, by taking $\rho = 4\sqrt{a_0}$, we get

$$
\mathbb{E} \left[ \|x\|_2^2 \mathbb{1}(\|x\|_2^2 > \rho^2/4^2) \right] = \int_{a_0}^\infty \mathbb{P}(\|x\|_2^2 > t)dt \\
\leq C_\varepsilon \exp(-a_0).
$$

Note that Assumption D.3b holds if we take $a_0$ larger. We can therefore let $a_0$ be large enough so that $a_0 \geq \log(18C_\varepsilon/\varepsilon)$, so that then

$$
\mathbb{E} \left[ \|x\|_2^2 \mathbb{1}(\|x\|_2^2 > \rho^2/4^2) \right] \leq \varepsilon/18.
$$

Similarly, under Assumption D.3c, we can let $\gamma$ be the constant corresponding to $\rho = 4\sqrt{a_0}$ and take $a_0 \geq (\varepsilon(\beta - 1)/18C_p)^{1/(1-\beta)}$ so that

$$
\mathbb{E} \left[ \|x\|_2^2 \mathbb{1}(\|x\|_2^2 > \rho^2/4^2) \right] = \int_{a_0}^\infty \mathbb{P}(\|x\|_2^2 > t)dt \\
\leq C_p \frac{\exp(-a_0)}{\beta - 1} \\
\leq \varepsilon/18.
$$

and so (D.11) holds as well under Assumption D.3c. We can therefore bound

$$
\mathbb{E} \left[ \left( \sigma(w^*_t x) - \sigma(v^*_x) \right)^2 \mathbb{1}(\|x\|_2^2 > \rho^2/4^2) \right] \leq \mathbb{E} \left[ \|w_t - v\|_2^2 \|x\|_2^2 \mathbb{1}(\|x\|_2^2 > \rho^2/4^2) \right] \\
\leq \|w_0 - v\|_2^2 \mathbb{E} \left[ \|x\|_2^2 \mathbb{1}(\|x\|_2^2 > \rho^2/4^2) \right] \\
\leq \|w_0 - v\|_2^2 \varepsilon/18 \\
\leq \varepsilon/2.
$$

The first inequality uses that $\sigma$ is 1-Lipschitz and Cauchy–Schwarz. The second inequality uses (D.6). The third inequality uses (D.11). The final inequality uses that $\|w_0 - v\|_2 \leq \|w_0\|_2 + \|v\|_2 \leq 3$. 

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We can then guarantee

\[2\gamma F(w_0) = \gamma \mathbb{E}\left[ \left( \sigma(w_i^\top x) - \sigma(v^\top x) \right)^2 \right] \]
\[= \mathbb{E}\left[ \left( \sigma(w_i^\top x) - \sigma(v^\top x) \right)^2 \mathbb{1}(|w_i^\top x| \leq \rho) \right] + \gamma \mathbb{E}\left[ \left( \sigma(w_i^\top x) - \sigma(v^\top x) \right)^2 \mathbb{1}(|w_i^\top x| > \rho) \right] \]
\[\leq \mathbb{E}\left[ \left( \sigma(w_i^\top x) - \sigma(v^\top x) \right)^2 \sigma'(w_i^\top x) \mathbb{1}(|w_i^\top x| \leq \rho) \right] + \gamma \mathbb{E}\left[ \left( \sigma(w_i^\top x) - \sigma(v^\top x) \right)^2 \mathbb{1}(\|x\|_2^2 > \rho^2/4) \right] \]
\[\leq \zeta \epsilon + \gamma \epsilon/2 \]
\[\leq \gamma \epsilon. \]

The first inequality follows since Assumption 3.1 implies \(\sigma'(z) \mathbb{1}(|z| \leq \rho) \geq \gamma \mathbb{1}(|z| \leq \rho)\) and by (D.9). The second inequality uses (D.8) and (D.12). The final inequality takes \(\zeta = \gamma/2\).

\[\square\]

### D.2 Stochastic gradient descent proofs

We consider the online version of stochastic gradient descent, where we sample independent samples \(x_t \sim \mathcal{D}\) at each step and compute stochastic gradient updates \(g_t\), such that

\[g_t = \left( \sigma(w_i^\top x_t) - \sigma(v^\top x_t) \right) \sigma'(w_i^\top x_t)x_t, \quad w_{t+1} = w_t - \eta g_t.\]

As in the gradient descent case, we have a key lemma that relates the distance of the weights at iteration \(t\) from the optimal \(v\) with the distance from initialization and the cumulative loss.

**Lemma D.5.** Assume that \(\sigma\) is non-decreasing and \(L\)-Lipschitz, and that \(\mathcal{D}\) satisfies \(\|x\|_2 \leq B\) a.s. Assume the initialization satisfies \(\|w_0\|_2 \leq 2\). Let \(T \in \mathbb{N}\) and run stochastic gradient descent for \(T - 1\) iterations at a fixed learning rate \(\eta\) satisfying \(\eta \leq L^{-2}B^{-2}\). Then with probability one over \(\mathcal{D}\), we have \(\|w_t - v\|_2 \leq \|w_0 - v\|_2\) for all \(t < T\), and

\[\|w_0 - v\|_2^2 - \|w_T - v\|_2^2 \geq 2\eta L^{-1} \sum_{t=0}^{T-1} H_t,\]

where \(H_t := \frac{1}{2} \left( \sigma(w_i^\top x_t) - \sigma(v^\top x_t) \right)^2 \sigma'(w_i^\top x_t)\).

**Proof.** We begin with the decomposition

\[\|w_t - v\|_2^2 - \|w_{t+1} - v\|_2^2 = 2\eta \left( g_t, w_t - v \right) - \eta^2 \|g_t\|_2^2. \quad (D.13)\]

By Assumption 3.1, since \(\|x\|_2 \leq B\) a.s. it holds with probability one that

\[\|g_t\|_2^2 = \left\| \left( \sigma(w_i^\top x_t) - \sigma(v^\top x_t) \right) \sigma'(w_i^\top x_t)x_t \right\|_2^2 \leq 2LB^2H_t. \quad (D.14)\]
By Fact 3.7, since \( \sigma'(z) \geq 0 \), we have with probability one,
\[
\langle g_t, w_t - v \rangle = \left( \sigma(w_t^\top x_t) - \sigma(v^\top x_t) \right) \sigma'(w_t^\top x_t)(w_t^\top x_t - v^\top x_t) \\
\geq L^{-1} \left( \sigma(w_t^\top x_t) - \sigma(v^\top x_t) \right)^2 \sigma'(w_t^\top x_t) \\
= 2L^{-1}H_t.
\] (D.15)

Putting (D.14) and (D.15) into (D.13), we get
\[
\|w_t - v\|^2_2 - \|w_{t+1} - v\|^2_2 \\
\geq 4\eta L^{-1}H_t - 2\eta^2 LB^2H_t \\
\geq 2\eta L^{-1}H_t,
\]
by taking \( \eta \leq L^{-2}B^{-2} \). Telescoping over \( t < T \) gives the desired bound.

We now want to translate the bound on the empirical error to that of the true error. For this we use a martingale Bernstein inequality of Beygelzimer et al. (2011). A similar analysis of SGD was used by Ji and Telgarsky (2019) for a one-hidden-layer ReLU network.

**Lemma D.6** (Beygelzimer et al. (2011), Theorem 1). Let \( \{Y_t\} \) be a martingale adapted to the filtration \( \mathcal{F}_t \), and let \( Y_0 = 0 \). Let \( \{D_t\} \) be the corresponding martingale difference sequence. Define the sequence of conditional variance
\[
V_t := \sum_{k=1}^{t} \mathbb{E}[D_k^2|\mathcal{F}_{k-1}],
\]
and assume that \( D_t \leq R \) almost surely. Then for any \( \delta \in (0,1) \), with probability greater than \( 1 - \delta \),
\[
Y_t \leq R \log(1/\delta) + (e - 2)V_t/R.
\]

**Lemma D.7.** Suppose that \( \|x\|_2 \leq B \) a.s., and let \( \sigma \) be non-decreasing and \( L \)-Lipschitz. Assume that the trajectory of SGD satisfies \( \|w_t - v\|_2 \leq \|w_0 - v\|_2 \) for all \( t \) a.s. We then have with probability at least \( 1 - \delta \),
\[
\frac{1}{T} \sum_{t=0}^{T-1} H(w_t) \leq \frac{4}{T} \sum_{t=0}^{T-1} H_t + \frac{2}{T} B^2 \|w_0 - v\|_2^2 \log(1/\delta).
\]

**Proof.** Let \( \mathcal{F}_t = \sigma(x_0, \ldots, x_t) \) be the \( \sigma \)-algebra generated by the first \( t + 1 \) draws from \( \mathcal{D} \). Then the random variable \( G_t := \sum_{\tau=0}^{t} (H(w_{\tau}) - H_{\tau}) \) is a martingale with respect to the filtration \( \mathcal{F}_t \) with martingale difference sequence \( D_t := H(w_t) - H_t \). We need bounds on \( D_t \) and on \( \mathbb{E}[D_t^2 | \mathcal{F}_{t-1}] \) in order to apply Lemma D.6.

Since \( \sigma \) is \( L \)-Lipschitz and \( \|x\|_2 \leq B \) a.s., with probability one we have
\[
D_t \leq H(w_t) \leq \frac{1}{2} L^3 B^2 \|w_t - v\|_2^2 \leq \frac{1}{2} L^3 B^2 \|w_0 - v\|_2^2.
\] (D.16)
We then can use (D.17) to bound the squared increments,

\[
\mathbb{E}[H^2_t | \mathcal{F}_{t-1}] = \frac{1}{4} \mathbb{E} \left[ \left( \sigma(w^*_t x_t) - \sigma(v^*_t x_t) \right)^4 \sigma'(w^*_t x_t)^2 | \mathcal{F}_{t-1} \right] \\
\leq \frac{1}{4} L^3 B^2 \|w_t - v\|_2^2 \mathbb{E}_x \left[ (\sigma(w_t x_t) - \sigma(v_t x_t))^2 \sigma'(w_t x_t) | \mathcal{F}_{t-1} \right] \\
\leq \frac{1}{2} L^3 B^2 \|w_0 - v\|_2^2 H(w_t). \tag{D.17}
\]

The last inequality uses the assumption that \(\|w_t - v\|_2 \leq \|w_0 - v\|_2\) a.s. Similarly,

\[
\mathbb{E}[D^2_t | \mathcal{F}_{t-1}] = H(w_t)^2 - 2H(w_t)\mathbb{E}[H_t | \mathcal{F}_{t-1}] + \mathbb{E}[H^2_t | \mathcal{F}_{t-1}] \\
= -H(w_t)^2 + \mathbb{E}[H^2_t | \mathcal{F}_{t-1}] \\
\leq \frac{1}{2} L^3 B^2 \|w_0 - v\|_2^2 H(w_t). \tag{D.18}
\]

In the first inequality, we have used \(\|x\|_2^2 \leq B^2\) a.s. and \(L\)-Lipschitzness of \(\sigma\). For the second, we use the assumption that \(\|w_t - v\|_2 \leq \|w_0 - v\|_2\) together with the fact that \(\mathbb{E}_x[H_t | \mathcal{F}_{t-1}] = H(w_t)\). We then can use (D.17) to bound the squared increments,

\[
\mathbb{E}[D^2_t | \mathcal{F}_{t-1}] = H(w_t)^2 - 2H(w_t)\mathbb{E}[H_t | \mathcal{F}_{t-1}] + \mathbb{E}[H^2_t | \mathcal{F}_{t-1}] \\
= -H(w_t)^2 + \mathbb{E}[H^2_t | \mathcal{F}_{t-1}] \\
\leq \frac{1}{2} L^3 B^2 \|w_0 - v\|_2^2 H(w_t). \tag{D.18}
\]

This allows us to bound

\[
V_T := \sum_{t=0}^{T-1} \mathbb{E}[D^2_t | \mathcal{F}_{t-1}] \leq \frac{1}{2} B^2 L^2 \|w_0 - v\|_2^2 \sum_{t=0}^{T-1} H(w_t).
\]

Since \(D_t \leq H(w_t) \leq (1/2)L^2 B^2 \|w_0 - v\|_2^2\) a.s. by (D.16), Lemma D.6 implies that with probability at least \(1 - \delta\), we have

\[
\sum_{t=0}^{T-1} (H(w_t) - H_t) \leq (\exp(1) - 2) \sum_{t=0}^{T-1} H(w_t) + \frac{1}{2} L^3 B^2 \|w_0 - v\|_2^2 \log(1/\delta),
\]

and using that \((1 - \exp(1) + 2)^{-1} \leq 4\), we divide each side by \(T\) and get

\[
\frac{1}{T} \sum_{t=0}^{T-1} H(w_t) \leq \frac{4}{T} \sum_{t=0}^{T-1} H_t + \frac{2}{T} L^2 B^2 \|w_0 - v\|_2^2 \log(1/\delta). \tag{D.19}
\]

With the above in hand, we can prove Theorem D.1 in the SGD setting.

**Proof of Theorem D.1, SGD.** By the assumptions in the theorem, Lemma D.5 holds, so that we have for any \(t = 0, \ldots, T - 1\), \(\|w_t\|_2 \leq 4\) and

\[
\|w_t - v\|_2^2 + 2\eta L^{-1} \sum_{\tau=0}^{t-1} H_\tau \leq \|w_0 - v\|_2^2. \tag{D.20}
\]

This implies two key properties: first, we have \(\|w_t - v\|_2 \leq \|w_0 - v\|_2\) holds for all \(t = 0, \ldots, T - 1\).
This allows us to apply Lemma D.7, yielding
\[
\frac{1}{T} \sum_{t=0}^{T-1} H(w_t) \leq \frac{4}{T} \sum_{t=1}^{T} H_t + \frac{2}{T} L^2 B^2 \|w_0 - v\|_2^2 \log(1/\delta).
\] (D.21)

Dividing both sides of (D.20) by \(\eta TL^{-1}\) yields
\[
\min_{t<T} H(w_t) \leq \frac{1}{T} \sum_{t=0}^{T-1} H(w_t) \leq \frac{L \|w_0 - v\|_2^2}{\eta T} + \frac{2}{T} L^2 B^2 \|w_0 - v\|_2^2 \log(1/\delta).
\]

For arbitrary \(\zeta > 0\), taking \(T = \lceil 2e^{-1} \zeta^{-1} L^3 B^2 \|w_0 - v\|_2^2 \log(1/\delta) \rceil\) shows there exists \(T\) such that \(H(w_t) \leq \zeta \epsilon\). When \(\sigma\) satisfies Assumption 3.1, since \(\|w_t\|_2 \leq 4\) for all \(t\), it holds that \(H(w_t) \geq \gamma F(w_t)\), so that \(\zeta = \gamma\) furnishes the desired bound.

When \(\sigma\) is ReLU and \(D\) satisfies Assumption 3.2, we note that Lemma D.5 implies \(\|w_t - v\|_2 \leq \|w_0 - v\|_2\) a.s. Thus taking \(\zeta = \alpha^4 \beta / 8\sqrt{2}\) and using Lemma 3.5 completes the proof. 

### E Remaining proofs

**Proof of Lemma 3.8.** Since \(\sigma\) is non-decreasing, \(|\sigma(v^\top x) + y| \leq |\sigma(B_X)| + B_Y\). In particular, each summand defining \(\hat{F}(v)\) is a random variable with absolute value at most \(a\). As \(\mathbb{E}[\hat{F}(v)] = F(v) = \text{OPT}\), Hoeffding’s inequality implies the lemma. 

**Proof of Lemma 3.9.** The bound \(\mathcal{R}(G) \leq 2 \max_i \|x_i\|_2 / \sqrt{n}\) follows since \(\|w\|_2 \leq 2\) holds on \(G\) with standard results Rademacher complexity theory (e.g. Sec. 26.2 of Shalev-Shwartz and Ben-David (2014)); this shows \(\mathcal{R}(G) \leq 2B_X / \sqrt{n}\). Using the contraction property of the Rademacher complexity, this implies \(\mathcal{R}(\sigma \circ G) \leq 2B_X L / \sqrt{n}\). Finally, note that if \(\|w - v\|_2 \leq 1\) and \(\|x\|_2 \leq B_X\), we have
\[
\|\nabla \ell(w; x)\| = \|\left(\sigma(w^\top x) - \sigma(v^\top x)\right) \sigma'(w^\top x)x\| \leq L^2 \|w - v\| \|x\| \leq L^2 B_X.
\] (E.1)

In particular, \(\ell\) is \(L^2 B_X\) Lipschitz. The result follows.

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