Deformation Quantization of Pseudo Symplectic(Poisson) Groupoids

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Abstract

We introduce a new kind of groupoid—a pseudo étale groupoid, which provides many interesting examples of noncommutative Poisson algebras as defined by Block, Getzler, and Xu. Following the idea that symplectic and Poisson geometries are the semiclassical limits of the corresponding quantum geometries, we quantize these noncommutative Poisson manifolds in the framework of deformation quantization.

Dedicated to A. Weinstein on his 60th birthday

1 Introduction

About ten years ago, following an idea from deformation theory, Block and Getzler in [2], and Xu in [28] independently introduced the notion of a Poisson structure on an associative algebra.

Definition 1.1. A Poisson structure $\Pi$ on an associative algebra $A$ is an element in the second Hochschild cohomology $H^2(A; A)$ such that $[\Pi, \Pi]$ is 0, where $[,]$ is the Gerstenhaber bracket.

In [2] and [28], authors showed many interesting examples. However, for about ten years, there were no further results on this subject.

In Section 2, we will introduce a new kind of groupoid, a pseudo étale groupoid, generalizing the notion of an étale groupoid and show that under an extra assumption, the corresponding groupoid algebras provide a large amount of new examples of noncommutative Poisson algebras as defined in Definition 1.1.

In physics, a noncommutative Poisson algebra as a phase space of a noncommutative field theory, and the quantization of this noncommutative Poisson algebra corresponds to the quantization of the corresponding field theory. After introducing the noncommutative Poisson algebras, we study the deformation quantization of these algebras.

We make the following definition of a formal deformation quantization of a noncommutative Poisson algebra.

Definition 1.2. Let $(A, \pi)$ be a noncommutative Poisson algebra, and let $A[[\hbar]]$ be the linear space of formal power series with coefficients in $A$. A formal deformation quantization of $(A, \pi)$ is an associative product $\star$ on $A[[\hbar]]$, $c(\hbar) = a(\hbar) \star b(\hbar) \overset{def}{=} \sum_{k=0}^{\infty} \hbar^k c_k$ satisfying the following properties:

1. $c_k$ is $C[[\hbar]]$ bilinear,
2. $c_0 = a_0 \cdot b_0$, 

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3. \( a_0 \star b_0 = c_0 + \hbar \pi(a_0, b_0) + o(\hbar). \)

**Remark 1.3.** The locality assumption of a star product is very important. Without the locality condition, it is not even known whether the 2nd Hochschild cohomology of the algebra of smooth functions on a manifold is equal to the space of bivector fields. In Definition 1.2, we have to drop the locality assumption, since we do not know its corresponding noncommutative analogue. It would be interesting and useful to have a notion with which to replace the notion of locality in noncommutative geometry.

In Section 3, we show that the noncommutative Poisson algebra introduced by a pseudo étale groupoid can be formally deformation quantized. In addition to this, we also discuss various extensions of deformation quantization. Firstly, we consider existence of traces on the quantized algebras. We show that if certain modular classes of the corresponding groupoids vanish, then there is a closed star product as defined by Flato, Connes, and Sternheimer. Secondly, instead of taking \( \hbar \) as a formal parameter, we want to look at \( \hbar \) as a real number as in the following definition:

**Definition 1.4.** Let \( A \) be a \( C^* \)-algebra, and \( A_\infty \) be its dense \( * \)-subalgebra closed under holomorphic function calculus, which represents the smooth algebra. Let \( \pi \) be a Poisson structure on \( A_\infty \). Then a strict deformation quantization of \( (A, A_\infty, \pi) \) is a set \( (\star_\hbar, \ast_\hbar, \| \cdot \|_\hbar) \) parameterized by \( \hbar \), with \( \hbar \) in a closed subset \( I \) of the real line containing 0 as a non-isolated point. For each \( \hbar \), \( \star_\hbar \) is an associative product, \( \ast_\hbar \) is an involution, and \( \| \cdot \|_\hbar \) is a \( C^* \)-norm on \( A_\infty \) satisfying the following properties:

1. When \( \hbar = 0 \), the product, involution, and norm come from those on \( A \).
2. The completions of \( A_\infty \) for the various \( C^* \)-norms form a continuous field of \( C^* \)-algebras over \( I \).
3. For \( a, b \in A_\infty \), we have
   \[
   \| (a \ast_\hbar b - a \ast_0 b - i\hbar \pi(a, b)) \|_\hbar \to 0, \quad \hbar \to 0.\]

In 3.4, we discuss strict deformation quantizations in some interesting examples, which were used in our proof of Morita invariance of noncommutative tori under the \( SO(n/n[\mathbb Z]) \) action in [24].

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## 2 Noncommutative Poisson algebra

In this section, we introduce the notion of a pseudo étale groupoid, which is a generalization of an étale groupoid. Our main aim is to prove that under an extra assumption, a pseudo Poisson groupoid defines a noncommutative Poisson structure on the smooth groupoid algebra\(^2 \) (Theorem 2.24 and 2.30).

### 2.1 Pseudo Poisson groupoid

In this subsection, we introduce the main object of this paper—a pseudo Poisson groupoid.

#### 2.1.1 Pseudo étale groupoid

A pseudo étale groupoid is a generalization of an étale groupoid. The source and target maps of an étale groupoid are defined to be local diffeomorphisms, so that each element of an étale groupoid defines a local diffeomorphism on the unit space, from a neighborhood of an element’s source to a neighborhood of its target. The étale assumption of the source and target maps is so restrictive that a transformation groupoid of a group acting on a manifold is étale if and only if the group is étale (discrete). In the definition of a pseudo étale groupoid, we want to weaken this étale assumption, but still keep the property that each element of the groupoid defines an infinitesimal diffeomorphism on the unit space, such that the definition includes all transformation groupoids. A natural way to achieve this is to choose an infinitesimal bisection at each point of a groupoid so that this choice is compatible with groupoid operations.

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1. \( o(\hbar) \) stands for terms with higher order than \( \hbar \).
2. See Definition 2.23 for a smooth groupoid algebra.
Definition 2.1. Let \( D \) be a distribution on a groupoid \( G \). We say \( D \) is multiplicative if for any \( \alpha, \beta, \gamma \in G \) with \( \alpha = \beta \cdot \gamma \) and \( X \in D|_\alpha \), there are paths \( x(t), y(t) \) and \( z(t) \) satisfying the following properties:

1. \( x(t) = y(t) \cdot z(t) \), and \( \dot{x}(t) \in D|x(t), \ y(t) \in D|y(t), \ \dot{z}(t) \in D|z(t). \)
2. \( x(0) = \alpha, \ y(0) = \beta, \ z(0) = \gamma \), and \( \dot{x}(0) = X \).

An example of a multiplicative distribution is the one dimensional distribution generated by a multiplicative vector field (see [15]) on a Lie groupoid.\(^3\) We will see more interesting examples later.

Definition 2.2. An étalification of a Lie groupoid \( G \) is an integrable subbundle \( F \) of \( TG \), satisfying the following conditions:

1. \( F \) is complementary to the \( s \) and \( t \)-fibers. Therefore, \( s_* \) (and \( t_* \)) induces an isomorphism between \( F|_\gamma \) and \( T_s(\gamma)G_{|0} \) (and \( T_t(\gamma)G_{|0} \)). (\( s \) and \( t \) are the source and target maps of \( G \).)
2. \( F|_{G(0)} = TG^{(0)} \), where \( G^{(0)} \) is the unit space of \( G \).
3. \( F \) is a multiplicative distribution.

This assumption together with condition (i) implies that
\[
\dot{y}(0) = (t_*)^{-1}(X), \ \dot{z}(0) = (s_*)^{-1}(X).
\]

Definition 2.3. A pseudo étale groupoid is a Lie groupoid with a chosen étalification.

Remark 2.4. An étalification of a Lie groupoid redefines the “topology” on the groupoid, making it behave like an étale groupoid.

The following is a list of examples of pseudo étale groupoids.

Example 2.5.

1. Étale groupoid. An étale groupoid has a natural étalification, its tangent bundle.

2. Transformation groupoid \( M \times G \rightrightarrows M \). The manifold of \( M \times G \) is \( M \times G \), and the \( s, t \) maps are defined by \( s((x, \gamma)) = x, \ t((x, \gamma)) = x \cdot \gamma \) (\( \gamma \) is the action of \( \gamma \) on \( M \)). We choose an étalification to be the \( M \)-component of the tangent bundle \( T(M \times G) \).

3. Pseudo group. In [22], a pseudo group is defined as “an algebraic structure whose elements consist of selected homeomorphisms between open subsets of a space, with the composition of two transformations defined on the largest possible domain”. The “germs” of the elements of a pseudo group form a groupoid, which we will also call a pseudo group.\(^4\) An étalification of a pseudo group \( G \) can be chosen as follows. A typical element of a pseudo group is of the form \( (x, \phi, y) \) with \( x, y \) two arbitrary elements of a manifold \( M \) and \( \phi \) a germ of a local diffeomorphism from \( x \) to \( y \). For \( \gamma = (x, \phi, y) \in G \), we choose a representative \( \tilde{\gamma} \) for \( \gamma \), which defines a diffeomorphism from a neighborhood \( U_x \) of \( x \) to \( U_y \) of \( y \), and also induces a diffeomorphism from any other point in \( U_x \) to its image under \( \tilde{\gamma} \). Therefore, \( \tilde{\gamma} \) defines a local bisection near \( \gamma \). We define an étalification of \( G \) to be the collection of all germs of \( \tilde{\gamma} \).

4. Pair groupoid of \( \mathbb{T}^k \times \mathbb{T}^k \rightrightarrows \mathbb{T}^k \) (\( \mathbb{T} = \mathbb{R}/\mathbb{Z} \)). For simplicity, we describe \( \mathbb{T} \times \mathbb{T} \), which is represented by a unit square \([0, 1] \times [0, 1] \) with edges identified. The base of the pair groupoid is the diagonal interval connecting \([0, 0] \) and \([1, 1] \). The source and target maps are the projections along the \( x \) and \( y \) directions. An étalification is chosen to be the subtangent bundle in the diagonal direction.

5. Germs of bisections on a groupoid. Generally, a Lie groupoid \( G \) may not have any étalification. It is not hard to see that germs of bisections on \( G \) form a pseudo étale groupoid having a natural étalification. An étalification can be chosen in the same way as a pseudo group in Example 3.

\(^3\)In this sense, we can look at a multiplicative distribution as a generalization of a multiplicative vector field.

\(^4\)In general, a pseudo group is a not a Lie groupoid. Here, we only look at those cases which are Lie groupoids.
From the above, we have seen many interesting examples of pseudo étale groupoids. Nevertheless, as was mentioned, not every Lie groupoid admits an étalification. A counter example is given by the pair groupoid $S^2 \times S^2 \rightrightarrows S^2$.

We prove this by contradiction. Suppose that there is an étalification $\mathcal{F}$ on the pair groupoid $S^2 \times S^2 \rightrightarrows S^2$. Choose $x \in S^2$. At $(x,x)$, we fix one nonzero element $v$ in the tangent space of the diagonal $S^2$. Since $s_*$ is an isomorphism between $T_{(x,x)}S^2$ (tangent bundle of the diagonal $S^2$) and $\mathcal{F}_{(y,x)}$, we conclude that $(s_*)^{-1}(v)$ induces a nonzero element in the étalification at $(y,x)$, $\forall y \in S^2$. Obviously, at each point $(y,x)$, $y \in S^2$, ker($s_*$) and ker($t_*$) are transverse to each other and span the whole tangent space. As $\mathcal{F}_{(y,x)}$ is transverse to both fibers, it is isomorphic to ker($t_*$) by projection along ker($s_*$). By this isomorphism, $(s_*)^{-1}(v)$ is projected onto ker($t_*$), which is not equal to zero anywhere on the source fiber of $(y,x)$. This is impossible by the fact that ker($t_*$) is the tangent bundle of the $t-$fiber at $(x,x)$, which is $TS^2$, and that $S^2$ has no nowhere vanishing vector fields. Hence, we conclude that there is no étalification on the pair groupoid $S^2 \times S^2 \rightrightarrows S^2$. In general, by the same type of argument, we can easily prove the following statement.

**Lemma 2.6.** A pair groupoid $M \times M \rightrightarrows M$ has an étalification if and only if $M$ is parallelizable.

The following proposition explains the above lemma.

**Proposition 2.7.** Let $(\mathcal{G}, \mathcal{F})$ be a pseudo étale groupoid. Since $\mathcal{F}$ is transverse to $t$-fibers ($s$-fibers), locally $\mathcal{F}$ induces identifications of $t$-fibers ($s$-fibers) along paths in $\mathcal{G}^{(0)}$ (an Ehresmann connection), which defines a connection on ker($t_*|_{\mathcal{G}^{(0)}}$) (ker($s_*|_{\mathcal{G}^{(0)}}$)). This connection is flat. We call this connection the canonical connection of $(\mathcal{G}, \mathcal{F})$.

**Proof.** The existence of this connection is already explained in its statement. To see that the Ehresmann connection is flat, we notice that $\mathcal{F}$ is closed under the Lie bracket and $t_*$ is a Lie algebra homomorphism of vector fields. □

There is another description of a pseudo étale groupoid. We state it without proof$^5$.

**Proposition 2.8.** An étalification of a Lie groupoid $\mathcal{G}$ is a subtangent groupoid of $T\mathcal{G}$ which is integrable and transversal to the source and target fibers of $T\mathcal{G}$.

### 2.1.2 Poisson structure

Having introduced the notion of a pseudo étale groupoid, we will define and study symplectic (Poisson) structures on this groupoid.

**Definition 2.9.** A symplectic structure on a pseudo étale groupoid $(\mathcal{G}, \mathcal{F})$ is a closed 2-form $\omega$ on $\mathcal{G}$, satisfying the following conditions:

1. $\omega|_{\mathcal{F}}$ makes $\mathcal{F}$ into a symplectic bundle;

2. $\omega|_{\mathcal{F}}$ is invariant under $s^*, t^*$.

A pseudo étale groupoid equipped with a symplectic structure is called a pseudo symplectic groupoid.

**Definition 2.10.** A Poisson structure on a pseudo étale groupoid $(\mathcal{G}, \mathcal{F})$ is a bivector $\pi$ on $\mathcal{G}$, satisfying the following conditions:

1. $\pi|_{\mathcal{F}}$ makes $\mathcal{F}$ into a Poisson bundle, which makes $\mathcal{F}^*$ into a Lie algebroid.

2. $\pi|_{\mathcal{F}}$ is invariant under $s^*, t^*$.

A pseudo étale groupoid equipped with a Poisson structure is called a pseudo Poisson groupoid.

**Remark 2.11.** There are two reasons to call these objects “pseudo”. One is that generally they are not étale groupoid, but have similar properties. The other is that they are not the symplectic (Poisson) groupoids in the sense of Weinstein.

$^5$It is a straightforward check.
On a symplectic (Poisson) manifold, there is the famous Darboux’s theorem that locally the manifold looks like the standard symplectic (Poisson) linear space, which we recall in the framework of a pseudo étale groupoid.

**Theorem 2.12.** Let \((G, F, \omega)\) be a pseudo symplectic groupoid. For any \(\gamma \in G\), there is an integral submanifold \(U\) of \(F\) through \(\gamma\). In a suitable coordinate system on \(U\), \(\omega\) is expressed by

\[
\sum_{i=1}^{n} dx^i \wedge dy^i.
\]

**Theorem 2.13.** Let \((G, F, \pi)\) be a pseudo Poisson groupoid. For any \(\gamma \in G\), there is an integral submanifold \(U\) of \(F\) through \(\gamma\). In a suitable coordinate system of \(U\), \(\pi\) is expressed by

\[
\sum_{i} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \frac{1}{2} \sum_{i,j} \phi_{ij} \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}, \quad \phi_{ij}(0) = 0.
\]

Example 2.14. of pseudo symplectic (Poisson) groupoids

1. Transformation groupoid. A transformation groupoid with an invariant symplectic (Poisson) structure on the manifold.

2. Étale groupoid. An étale groupoid with an invariant symplectic (Poisson) structure.

3. Pseudo group. For a symplectic (Poisson) manifold, if we ask the local diffeomorphism to be symplectic (Poisson), the corresponding pseudo group forms a pseudo symplectic (Poisson) groupoid.

4. Pair groupoid \(T^k \times T^k \Rightarrow T^k\). For a pair groupoid \(T^k \times T^k \Rightarrow T^k\), we associate a constant symplectic (Poisson) structure on \(T^k\), and \(s^*\) induces a symplectic (Poisson) structure on the entire groupoid. In this way, the pair groupoid forms a pseudo symplectic (Poisson) groupoid.

5. Orientable contact manifold \((M, \alpha)\). On a contact manifold, the Reeb vector field \(R\) generates a 1-dimensional foliation on \(M\). We choose a submanifold of \(M\) which is transverse to the foliation. It is easy to check that \(d\alpha\) defines a symplectic form on the transversal, which is invariant along the foliation, and the reduced foliation groupoid to this transversal forms a pseudo symplectic groupoid.

6. Dirac manifold \((M, L)\) with a constant rank characteristic distribution. The characteristic distribution \(L \cap TM\) of a Dirac manifold, which is integrable, forms a foliation on \(M\). As in the contact case, we choose a submanifold of \(M\) which is transverse to the foliation. On this transversal, there is a natural Poisson structure invariant along the foliation. This makes the reduced foliation groupoid on the chosen transversal into a pseudo Poisson groupoid.

The last two examples are specific examples of étale groupoids with an invariant symplectic (Poisson) structure. We point them out to show a wide applicability of our results.

Given a pseudo symplectic groupoid \(G\), its étalification \(F\) forms a symplectic vector bundle on \(G\). So we can ask for a symplectic connection on the bundle.

**Definition 2.15.** A pseudo symplectic connection on a pseudo symplectic groupoid is a symplectic connection on the symplectic bundle \((F|_I, \omega)\) invariant under \(s, t\), where \(I\) denotes the leaves of \(F\).

**Lemma 2.16.** If a pseudo symplectic groupoid \(G\) is proper, then there is a pseudo symplectic connection on it.

**Proof.** A symplectic connection on \(F\) always exists by the same arguments as in Proposition 2.5.2 in [11]. So we only have to calculate its invariance under \(s, t\). When \(G\) is proper, one can choose a Haar system (see 2.1.3) on \(G\), and integrate the connection by this Haar system. It is easy to check that the integrated connection is a pseudo symplectic connection. \(\square\)

**Remark 2.17.** A similar result holds for the regular Poisson case. On a general Poisson manifold, there is usually no Poisson connection. It can be easily shown that a Poisson manifold has a Poisson connection if and only if the Poisson structure is regular.

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6The Darboux’s theorem in Poisson geometry is more complicated, which should be called linearization. See [5].
2.1.3 Groupoid algebra

The smooth groupoid algebra of a Lie groupoid consists of smooth functions on the groupoid with a convolution product, which is an important example of noncommutative differentiable manifolds. In the rest of this section, we plan to translate a symplectic (Poisson) structure on a pseudo étale groupoid to a noncommutative Poisson structure on the corresponding smooth groupoid algebra. In this subsection, we will recall the concept of a smooth groupoid algebra. There are two issues we will talk about. One is how to define a convolution product, the other is the definition of a smooth function on a Lie groupoid.

There are usually two ways to define a convolution product on a groupoid algebra. One is to use a Haar system, the other is to replace functions by half densities. The two methods yield the same algebra. For the details and the equivalence of the two versions, readers are referred to Paterson’s book [17]. In this thesis, we will define a groupoid algebra by a Haar system. We begin by recalling the definition of a Haar system.

Proposition 2.19. For any pseudo étale groupoid $G$, if the canonical connection defined in Proposition 2.7 has holonomy in $SL(k; \mathbb{R})$ ($k = \dim(\ker t_s)$), there is a transversal Haar system.

Proof. Consider the density bundle $\hat{H}$ of $\ker t_s|_{G^{(0)}}$ on $G^{(0)}$, which is flat. It is easy to see that there is a one to one correspondence between the Haar systems and the sections of $\hat{H}$. Given a section $\lambda \in \Gamma(\hat{H})$, we can check that $\Lambda(\alpha) \overset{\text{def}}{=} (\alpha^{-1})^*(\lambda(s(\alpha)))$ ($\alpha : s(\alpha) \mapsto \alpha = \alpha \cdot s(\alpha)$ while $\alpha^{-1} : \alpha \mapsto s(\alpha) = \alpha^{-1} \cdot \alpha$) defines a left invariant volume form along the $t$-fibers, which is a Haar system. For the other direction, a smooth Haar system when restricted to the unit space defines a section of $\hat{H}$. Obviously, the two maps are inverse to each other.

Since the holonomy of the canonical connection is in $SL(k; \mathbb{R})$, $\hat{H}$ is trivial and has global flat sections. We choose one global flat section, which is everywhere nonvanishing, denoted by $\lambda$. We show that $\Lambda(\alpha) \overset{\text{def}}{=} (\alpha^*)^{-1}(\lambda(s(\alpha)))$ is a transversal Haar system.

For $X \in \mathcal{F}$, locally we have

$$\mathcal{L}_X \Lambda = s^* \circ (s^{-1})^*(\mathcal{L}_X \Lambda) = s^*(\mathcal{L}_{s^*(X)}(s^{-1})^*(\Lambda)) = s^*(\mathcal{L}_{s^*(X)}(\Lambda)) = 0$$
where $\alpha^{-1} : \alpha \mapsto s(\alpha)$ is equal to $s$. \hfill $\Box$

There are many examples satisfying the condition of Proposition 2.19.

**Example 2.20.**

1. Étale groupoid. In the case of an étale groupoid, $k = \dim(\ker t_*) = 0$. The bundle $M \times G$ is a trivial $G$-bundle over $M$, which makes $\ker(t_*)|_{M}$ also a trivial bundle.

2. Transformation groupoid. $M \times G$ is a trivial $G$-bundle over $M$, which makes $\ker(t_*)|_{M}$ also a trivial bundle.

3. Pseudo groupoid. If we require homomorphisms in a pseudo group to preserve a metric, then the holonomy of the canonical connection is contained in $SO(k; \mathbb{R})$ which is in $SL(k; \mathbb{R})$.

4. Pair groupoid $\mathbb{T}^n \times \mathbb{T}^n \Rightarrow \mathbb{T}^n$. It is straightforward to check that the holonomy of the canonical connection is also trivial.

In the following, we will always assume that there exists a transversal Haar system on a pseudo étale groupoid, and define a convolution product on $C^\infty_c(G)$,

$$f \circ g(\alpha) \overset{\text{def}}{=} \int_{\alpha = \beta \cdot \gamma} f(\beta)g(\gamma)d\lambda.$$

It is not hard to check that $\circ$ is associative.

Now we discuss the definition of a smooth function on a Lie groupoid. The problem comes from the possibility that a Lie groupoid may not be a Hausdorff manifold. When a Lie groupoid is Hausdorff, compactly supported smooth functions on it form an associative algebra under the convolution product. But in a non-Hausdorff case, the sum of two compactly supported smooth functions may not be smooth. Therefore, we have to enlarge the set of compactly supported smooth functions to make it closed under summation and convolution. There are two definitions of a smooth groupoid algebra. One can be found in Connes’ book [9], the other is used in Crainic and Moerdijk’s work (see [7], [8]). The two definitions agree.

**Definition 2.21.** (2.11, [7]) Let $X$ be a topological space. A sheaf $\mathcal{A} \in \text{Sh}(X)$ is called $c$-soft if there is a Hausdorff open covering $\mathcal{U}$ of $X$ such that $\mathcal{A}|_U \in \text{Sh}(U)$ is $c$-soft (see [7] and [8]) for all $U \in \mathcal{U}$. In this case, define $\Gamma_c(U; \mathcal{A})$ as the image of the map

$$\bigoplus_U \Gamma_c(U; \mathcal{A}) \to \Gamma(X_{\text{dis}}; \mathcal{A}),$$

where $\Gamma(X_{\text{dis}}; \mathcal{A}) = \{ u : X \to \bigsqcup_{x \in X} A_x : u(x) \in A_x, \forall x \in A_x \}$ ($X_{\text{dis}}$ is $X$, considered with the discrete topology) and $\Gamma_c(U; \mathcal{A}) \to \Gamma(X_{\text{dis}}; \mathcal{A})$, $s \mapsto \bar{s}$ is given by

$$\bar{s}(x) = \text{germ}_x(s) \text{ for } x \in U, \text{ and } 0 \text{ otherwise}.$$

**Definition 2.22.** (2.14, [7]) If $M$ is a manifold, not necessarily Hausdorff, we define $C^\infty_c(M) \overset{\text{def}}{=} \Gamma_c(M; C^\infty_M)$ where $C^\infty_M$ is the sheaf of smooth functions on $M$. From the Mayer-Vietoris sequence, we have an alternative description of $C^\infty_c(M)$, as the cokernel of:

$$\bigoplus_{U \in \mathcal{U}} C^\infty_c(U \cap V) \to \bigoplus_U C^\infty_c(U) \quad (U \in \mathcal{U}),$$

where $\mathcal{U}$ is a Hausdorff open covering of $M$.  

Definition 2.23. We define the smooth groupoid algebra of \( \mathcal{G} \) to be \( C^\infty_c(\mathcal{G}) \) defined by Definition 2.22.

If \( \mathcal{G} \) is pseudo étale, the étalification makes groupoid operations on \( \mathcal{G} \) into local diffeomorphisms, i.e. the left multiplication of \( \alpha \in \mathcal{G} \) on \( \beta \in \mathcal{G} \) naturally identifies \( \beta \)'s s-fiber with \( (\alpha \cdot \beta)'s \) s-fiber, and simultaneously the étalification maps \( \mathcal{F}_{|\beta} \) to \( \mathcal{F}_{|\alpha \cdot \beta} \) isomorphically. Hence, the left multiplication of \( \alpha \) defines a local diffeomorphism. With this observation, the same argument as in [11] shows that \( C^\infty_c(\mathcal{G}) \) is closed under the convolution product \( \circ \). This algebra is what we will work with in the next subsection.

2.2 Noncommutative Poisson Structure

For a pseudo symplectic (Poisson) groupoid, we define

\[
\Pi(f,g)(\alpha) \overset{\text{def}}{=} \int_{\beta \gamma = \alpha} \pi(\alpha)((t^*)^{-1}(df(\beta)), (s^*)^{-1}(dg(\gamma))), \quad \forall f, g \in C^\infty_c(\mathcal{G}),
\]

where \((t^*)^{-1}(df(\beta)) \ (or \ (s^*)^{-1}(dg(\gamma))) \) means that we first restrict \( f \) (or \( g \)) to the integrated submanifold near \( \beta \) (or \( \gamma \)), then calculate the differential there to form an element in \( \mathcal{F}_{|\beta}^{*} \) (or \( \mathcal{F}_{|\gamma}^{*} \)), and finally push the element forward to \( \mathcal{F}_{|\alpha}^{*} \) by \((t^*)^{-1} \ (or \ (s^*)^{-1}) \). The main objective of this section is to see when formula \( \Pi \) defines a noncommutative Poisson structure on the groupoid algebra \( C^\infty_c(\mathcal{G}) \).

2.2.1 Regular Poisson case

The main result of this subsection is the following theorem, which generalizes Proposition 2.3 in [11].

Theorem 2.24. A pseudo regular Poisson\(^7\) (e.g. pseudo symplectic) groupoid \((\mathcal{G}, \mathcal{F}, \omega) \) (or \((\mathcal{G}, \mathcal{F}, \pi)\)) with a given invariant connection \( \nabla \) naturally defines a noncommutative Poisson structure on \( C^\infty_c(\mathcal{G}) \).

Remark 2.25. Here, We will use the invariant connection \( \nabla \) to construct a “coboundary” of \([\Pi, \Pi]\). We believe that the existence of a Poisson structure on a groupoid algebra should imply the existence of a kind of invariant “connection” in a generalized sense. However, we do not know how to define this.

Proof. To prove that \( \Pi \) is a noncommutative Poisson structure, we have to show that \( \Pi \) defines a Hochschild 2-cocycle with \([\Pi, \Pi]\) being a 3-coboundary. We start with proving the following lemma.

Lemma 2.26. When restricted to the leaves of \( \mathcal{F} \), \( d \) has the following formula: \( \forall f, g, \in C^\infty_c(\mathcal{G}) \):

\[
d(f \circ g)(\alpha) = \int_{\beta \gamma = \alpha} (t^*)^{-1}(df(\beta))g(\gamma) + \int_{\alpha = \beta \gamma} f(\beta)(s^*)^{-1}(dg(\gamma)),
\]

where \( \circ \) stands for the convolution of groupoid algebra.

Proof. We have to use the multiplicativity of \( \mathcal{F} \). Suppose that \( x \in \mathcal{F}_{|\alpha} \). Then the multiplicative assumption on \( \mathcal{F} \) provides \( x(t), y(t), z(t), \) satisfying

1. \( x(0) = \alpha, \ y(0) = \beta, \ z(0) = \gamma, \)
2. \( \dot{x}(t) \in \mathcal{F}_{|x(t)}, \ \dot{y}(t) \in \mathcal{F}_{|y(t)}, \ \dot{z}(t) \in \mathcal{F}_{|z(t)}, \)
3. \( \dot{x}(0) = x, \ \dot{y}(0) = (t^*)^{-1}(x), \ \dot{z}(0) = (s^*)^{-1}(x). \)

Therefore, \( x(f \circ g)(\alpha) = \)

\[
= \left. \frac{d}{dt} \right|_{t=0} f \circ g(x(t)) \\
= \left. \frac{d}{dt} \right|_{t=0} \int_{y(t)=x(t)} f(y(t))g(z(t)) \\
= \int_{y(t)=x(t)} \frac{d}{dt} \left|_{t=0} (f(y(t))g(z(t))) \right. \\
= \int_{y(t)=x(t)} \left( (t^*)^{-1} f(\beta)(g(\gamma)) + f(\beta)((s^*)^{-1} x)g(\gamma) \right) \\
= \int_{\alpha = \beta \gamma} (t^*)^{-1} (df(\beta))(x)g(\gamma) + f(\beta)(s^*)^{-1}(dg(\gamma)),
\]

where in the third equality we have used the fact that our Haar system is transversal, otherwise there would be one more term like \( f(\gamma)g(\beta)\frac{d}{dt}(x(\pi(t))) \).

\(^7\)Here, “regular” means that \( \pi \) has constant rank.
Lemma 2.27. \(\Pi\) satisfies the cycle condition,

\[ f \circ \Pi(g, h) - \Pi(f \circ g, h) + \Pi(f, g \circ h) - \Pi(f, g) \circ h = 0, \quad \forall f, g, h \in C_c^\infty(\mathcal{G}). \]

**Proof.** By Lemma 2.27 we have

\[
\Pi(f \circ g, h)(\alpha) = \int_{\beta \gamma = \alpha} \pi(\alpha)((t^*)^{-1}(d(f \circ g)(\beta)), (s^*)^{-1}(dh(\gamma))) = \int_{\beta \gamma = \alpha} \pi(\alpha)((t^*)^{-1}(\int_{\xi \eta = \beta} \pi(\alpha)((t^*)^{-1}(df(\xi)) \circ (s^*)^{-1}(dh(\eta)))) + \int_{\xi \eta = \beta} f(\xi)(s^*)^{-1}(dg(\eta)), (s^*)^{-1}(dh(\gamma))
\]

\[= \int_{\beta \gamma = \alpha} \pi(\alpha)((t^*)^{-1}(df(\xi)) \circ (s^*)^{-1}(dh(\eta)) = f(\Pi(g, h))(\alpha) + \int_{\beta \gamma = \alpha} \pi(\alpha)((t^*)^{-1}(df(\xi)) \circ (s^*)^{-1}(dh(\gamma))). \quad \square \]

Similarly,

\[
\Pi(f, g \circ h)(\alpha) = \Pi(f, g) \circ h(\alpha) + \int_{\beta \gamma = \alpha} \pi(\alpha)((t^*)^{-1}(df(\xi)) \circ (s^*)^{-1}(dh(\gamma))).
\]

To prove Theorem 2.28 we still need to show that \([\Pi, \Pi]\) is a 3-coboundary. To prove this, we use an invariant Poisson connection \(\nabla\) on \((\mathcal{G}, \mathcal{F})\) to define \(P_2\) as follows:

\[ P_2(f, g)(\alpha) \overset{df}{=} \int_{\alpha \beta \gamma} <\pi \otimes \pi(\alpha), (t^*)^{-1}(\nabla^2 f(\beta)) \otimes (s^*)^{-1}(\nabla^2 g(\gamma)) >, \]

where we restrict \(f, g\) to leaves of \(\mathcal{F}\) to construct \(\nabla^2 f, \nabla^2 g\), and pair the tensor \(\pi \otimes \pi\) with an element \(\alpha \otimes \beta \in S^2(T^*M) \otimes S^2(T^*M)\) by the following formula

\[\pi^i \pi^k \alpha_{ik} \beta_{jl}.\]

It is easy to see that \(P_2\) is skew-symmetric.

**Lemma 2.28.**

\[\delta P_2 + [\Pi, \Pi] = 0.\]

**Proof.** Since \(\nabla\) is an invariant Poisson connection, we have

\[\nabla \pi = 0,\]

and

\[d < \pi, (t^*)^{-1}df \wedge (s^*)^{-1}dg > = < \pi, (t^*)^{-1}(\nabla^2 f(\alpha)) \otimes (s^*)^{-1}dg > + (t^*)^{-1}df \otimes (s^*)^{-1}(\nabla^2 g(\beta)) >.\]

Using the above formulas, we calculate

\[
\Pi(\Pi(f, g), h)(\alpha) = \int_{\alpha \beta \gamma} \pi(\alpha)((t^*)^{-1}(d(\Pi(g, f)))(\beta), (s^*)^{-1}(h(\gamma))) = \int_{\alpha \beta \gamma} \pi(\alpha)((t^*)^{-1}(d < \pi(\alpha), (t^*) \circ df(\xi) \wedge (s^*) \circ (s^*)^{-1}(dh(\eta)) >, (s^*)^{-1}(h(\gamma)))
\]

\[= \int_{\alpha \beta \gamma} \pi(\alpha)((t^*)^{-1}(df(\xi)) \otimes (s^*)^{-1}(\nabla^2 g(\eta)) \otimes (s^*)^{-1}(dh(\gamma)) >,\]

\[
\Pi(f, \Pi(g, h))(\alpha) = \int_{\alpha \beta \gamma} \pi(\alpha)((t^*)^{-1}(df(\xi)) \otimes (s^*)^{-1}(\nabla^2 g(\eta)) \otimes (s^*)^{-1}(dh(\gamma)) >.\]

On the other hand, using the formula \(\nabla^2 f(\circ g)(\alpha) =\)

\[\int_{\alpha \beta \gamma} (t^*)^{-1}(\nabla^2 f(\beta))(\xi) \circ (s^*)^{-1}(dg(\eta)) + f(\beta)(s^*)^{-1}(\nabla^2 g(\xi)),\]

we can show

\[f \circ P_2(g, h)(\alpha) - P_2(f \circ g, h)(\alpha) = \int_{\alpha \beta \gamma} \pi(\alpha), (t^*)^{-1}(\nabla^2 f(\beta)) g(\xi)(s^*)^{-1}(dh(\eta)) + (t^*)^{-1}(df(\xi))(s^*)^{-1}(dh(\eta)) >,\]
and also,
\[-P_2(f, g \circ h)(\alpha) + P_2(f, g) \circ h(\alpha)\]
\[= \int_{s=\beta} \int_{t=\xi \eta} <\pi \otimes \pi(\alpha), (t^*)^{-1}(df(\beta)) \otimes (t^*)^{-1}(dg(\xi)) \otimes (s^*)^{-1}(dh(\eta)) + (t^*)^{-1}(df(\beta))g(\xi) \otimes (s^*)^{-1}(\nabla^2 dh(\eta)) > .\]

The equality we need for this lemma easily follows from the above calculation. \(\square\)

We have finished the proof of Theorem 2.24. We see that the above proof strongly depends on the existence of an invariant connection. Hence it cannot deal with general pseudo Poisson groupoids. We will look at the Poisson cases in the next subsection.

### 2.2.2 General Poisson case

In [13], Kontsevich proved his famous formality theorem. As a corollary, he showed that every Poisson manifold can be deformation quantized. In this subsection, we discuss an “equivariant formality theorem” with a groupoid action.

Let \((\mathcal{G}, \mathcal{F})\) be a proper pseudo étale groupoid. On the unit space \(\mathcal{G}^{(0)}\) of \(\mathcal{G}\), a differentiable manifold, there are two differential graded algebras which are defined as follows:

1. Let \(\mathcal{T}^{*}_{poly}\) be the space of multi-vector fields on \(\mathcal{G}^{(0)}\). \((\mathcal{T}^{*}_{poly}, 0, [\cdot, \cdot])\) defines a differential graded Lie algebra (DGLA), where 0 is the 0 differential, and \([\cdot, \cdot]\) is the Schouten-Nijenhuis bracket.

2. Let \(\mathcal{D}^{*}_{poly}\) be the subspace of \(Hom_{\mathbb{C}}(\mathcal{C}^{\infty}(\mathcal{G}^{(0)}), \mathcal{C}^{\infty}(\mathcal{G}^{(0)}))\) consisting of multi-differential operators. \((\mathcal{D}^{*}_{poly}, \delta, [\cdot, \cdot])\) also defines a DGLA, where \(\delta\) is the Hochschild differential and \([\cdot, \cdot]\) is the Gerstenhaber bracket.

Kontsevich, in [13], proved a formality theorem that the above two DGLAs are quasi-isomorphic. When a pseudo étale groupoid \(\mathcal{G}\) is proper, we can consider the corresponding \(\mathcal{G}\) invariant sub DGLA. By integration, it is easy to check that Kontsevich’s proof of formality theorem still works in the invariant case. So we have the following theorem.

**Theorem 2.29.** If \(\mathcal{G}\) is proper, \(((\mathcal{T}^{*}_{poly})^{\mathcal{G}}, 0, [\cdot, \cdot])\) is quasi-isomorphic to \(((\mathcal{D}^{*}_{poly})^{\mathcal{G}}, \delta, [\cdot, \cdot])\), where \((\mathcal{T}^{*}_{poly})^{\mathcal{G}}\) and \((\mathcal{D}^{*}_{poly})^{\mathcal{G}}\) are the corresponding \(\mathcal{G}\) invariant subspace.

**Proof:** Following the proof of Theorem 4.6.2 in [13], we globalize the local formality theorem on \(\mathbb{R}^n\) by the following steps

\[
\mathcal{T}_{poly}(\mathcal{G}^{(0)})[1]_{formal} \xrightarrow{1} \Gamma(\mathcal{T}_{aff} \rightarrow T[1]\mathcal{G}^{(0)})_{formal} \xrightarrow{3} \Gamma(\mathcal{D}_{aff} \rightarrow T[1]\mathcal{G}^{(0)})_{formal} \xrightarrow{\delta} \mathcal{D}_{poly}(\mathcal{G}^{(0)})[1]_{formal}.
\]

In the above maps, 2 is a fiberwise map which is invariant under \(\mathcal{G}\) action. We have to integrate maps 1 and 3 by \(\mathcal{G}\) to make them \(\mathcal{G}\) invariant, where we use the properness assumption of \(\mathcal{G}\). In this way, we obtain a quasi-isomorphism between the invariant parts of the two \(L_\infty\) algebras. \(\square\)

**Theorem 2.30.** Let \((\mathcal{G}, \mathcal{F}, \pi)\) be a proper pseudo Poisson groupoid. \(\Pi\) in [7] defines a Poisson structure on \((\mathcal{C}_{c}^{\infty}(\mathcal{G}), \circ)\).

**Proof.** The different part of the proof from Theorem 2.24 is to show that \([\Pi, \Pi]\) is a 3-coboundary. We will use the above equivariant formality Theorem 2.29 to construct a 2-cochain whose coboundary is \([\Pi, \Pi]\).

We look at \(\pi\) restricted to the unit space \(\mathcal{G}^{(0)}\). As \(\pi\) is a Poisson structure, there is a Hochschild 2-cocycle \(P_2\) on \(\mathcal{G}^{(0)}\) with

\[\left[\pi, \pi\right] = \delta P_2.\]  \(\text{(2)}\)

Furthermore, because \(\pi\) is invariant under \(\mathcal{G}\) and \(\mathcal{G}\) is proper, we can integrate \(P_2\) along \(\mathcal{G}\) orbits to make it invariant.

Then because \(s, t\) defines a local diffeomorphism between the leaves of \(\mathcal{F}\) and \(\mathcal{G}_0\), we can pull back \(P_2\) to the entire \(\mathcal{G}\), which is still denoted by \(P_2\). We define

\[
\hat{P}_2(f, g)(\alpha) \overset{def}{=} \int_{\alpha=\beta \gamma} P_2(\alpha)((t^*)^{-1} f(\beta), (s^*)^{-1} g(\gamma)),
\]
where \( f, g \) are smooth compactly supported functions on \( \mathcal{G} \).

From equality (2), it is straightforward to check that
\[
\delta \hat{P}_2 = [\Pi, \Pi]. \quad \square
\]

**Remark 2.31.** The assumption of Theorem 2.29 can be weakened. Dolgushev, in \[10\], showed a new way to prove the global formality theorem from Kontsevich’s local formula by a Fedosov (see \[11\]) type resolution. Dolgushev proved that, with a connection, there is a global quasi-isomorphism between the two \( L_\infty \)–algebras. To get an equivariant or invariant quasi-isomorphism, we only need to require that there is an invariant connection. Following his methods without repeating the details, we can easily show that if there is an invariant connection, then \( \Pi \) defines a Poisson structure on \( C^\infty_c(\mathcal{G}) \).

**Remark 2.32.** If \( \Pi \) is the first term in a deformation quantization of a noncommutative algebra, then we can show that \([\Pi, \Pi]\) must be a 3-coboundary as a Hochschild cocycle. This statement is closely related to Theorem 2.29.

In conclusion, in this section we have introduced a new class of Lie groupoid—pseudo symplectic (Poisson) groupoid, which provides many new examples of noncommutative Poisson algebras. This is the object we will quantize in the next chapter.

### 3 Quantization of a Pseudo Étale Groupoid

In this section, we consider quantization of pseudo étale groupoids. In 3.1, we construct a star product on a noncommutative Poisson manifold introduced in Theorem 2.30 and 2.31. In 3.2 and 3.4, we discuss two extensions of quantization problems, closed star products and strict deformation quantizations. In 3.3, we demonstrate our quantization methods on a typical example, a transformation groupoid.

#### 3.1 Quantization

In this section, using Fedosov’s machinery, we construct a formal deformation quantization of the noncommutative Poisson algebra defined in the last section.

If we forgot about the groupoid structure, a pseudo Poisson groupoid is a Poisson manifold. We know how to quantize a Poisson manifold from \[13\] and \[6\]. However, the quantization of this manifold is of less interest, since it does not contain the information of the groupoid operations. Instead, we should consider a quantization of the noncommutative Poisson algebra, which is more complicated than the commutative Poisson algebra of smooth functions on a Poisson manifold. In 3.1.1, we explain each step of our construction carefully for a pseudo symplectic groupoid, and in 3.1.2, we will deal with the Poisson case.

#### 3.1.1 Pseudo symplectic groupoid

Our construction of a quantization of the noncommutative Poisson algebra is divided into the following steps:

1. We use Fedosov’s method to construct a resolution of a Weyl algebra bundle and define a quantization map forgetting about the groupoid operations,
2. We prove that the quantization map is compatible with the groupoid convolution product,
3. we construct a quantization of a groupoid algebra.

**Quantization map**

There are three steps in Fedosov’s construction of a deformation quantization of a symplectic manifold. We follow them to construct a quantization map for a pseudo symplectic groupoid.

1. A Weyl algebra bundle \( \mathcal{W} \) on \( \mathcal{G} \);
2. A flat connection on \( \mathcal{W} \) of the form \(-\delta + \partial + \frac{i}{\hbar} [r, ]\);
3. Quantization map \( Q : C^\infty_c(\mathcal{G}) \to \mathcal{W} \);
Step 1. We start with introducing the Weyl algebra bundle $W$ on $\mathcal{G}$. We associate to every symplectic vector space $(V, \omega)$ a Weyl algebra over $\mathbb{C}$ with a unit, which consists of formal series

$$\sum_{i \geq 0, \ |\alpha| \geq 0} h^i a_{i,\alpha} y^\alpha. \quad (3)$$

The multiplication $\circ$ is defined as

$$a(h, y) \circ b(h, y) = \exp(-\frac{i}{\hbar} \omega^{ij} \frac{\partial}{\partial y^j}) a(h, y) b(h, y) \big|_{z=y}$$

$$= \sum_{k=0}^{\infty} (-\frac{i}{\hbar})^k \frac{1}{k!} \omega^{i_1 j_1} \cdots \omega^{i_k j_k} \frac{\partial^k a}{\partial y^{i_1} \cdots \partial y^{i_k}} \frac{\partial^k b}{\partial y^{j_1} \cdots \partial y^{j_k}},$$

where $y^i$ is the $i$-th coordinate on the vector space and $\omega^{ij}$ is the inverse of the symplectic form $\omega$. It is easy to check that the multiplication $\circ$ defined above is associative and independent of choice of basis.

For a pseudo symplectic groupoid $(\mathcal{G}, \mathcal{F}, \omega)$, at each point $\gamma \in \mathcal{G}$, $(\mathcal{F}|_{\gamma}, \omega|_{\gamma})$ is a symplectic vector space. Therefore, by the above construction, we can define a Weyl algebra bundle by associating a Weyl algebra $W_\gamma$ to $(\mathcal{F}|_{\gamma}, \omega|_{\gamma})$.

On a Weyl algebra, we prescribe a grading to the variables by setting $deg(y^i) = 1$ and $deg(h) = 2$. By virtue of this grading, $W$ becomes an $\mathbb{N}$ graded bundle.

Here, to define a deformed groupoid algebra, we need to integrate a Weyl algebra valued function. To make this integration well-defined, we need to define an $\mathbb{R}^n$ translation invariant topology on the Weyl algebra to make it a locally convex topological space. From [3], the elements $h^k y^\alpha$ form a basis of $W_\gamma$. Therefore, as a vector space, $W_\gamma$ can be identified with $\mathbb{C}^N$. On $\mathbb{C}^N$, we can choose the compact open topology. The induced compact open topology defines a topology on $W_\gamma$.

**Proposition 3.1.** A Weyl algebra with the above topology is a complete locally convex topological algebra.

**Proof.** “Completeness” comes from the completeness of the space $\{ f : \mathbb{N} \to \mathbb{C} \}$ with the compact open topology. The continuity of the multiplication is from the observation that a coefficient of the multiplication is determined by a finite number of coefficients of the two original elements. The local convexity is a straightforward check. We know that all finite dimensional vector spaces with the compact open topology are locally convex spaces. The same arguments work for an infinite dimensional vector space. □

It is easy to check that the compact open topology on a Weyl algebra is invariant under the $GL(n, \mathbb{C})$ action on the symplectic vector space. With the above topology of Weyl algebra, the Weyl algebra bundle becomes a topological bundle.

**Remark 3.2.** The topology we give here is not a $C^*-$norm topology. One may define a $C^*-$norm on the Weyl algebra, but it is very hard to make the later induction steps continuous in $C^*-$norm. The above defined topology makes the induction continuous, but is not a $C^*-$norm topology. There is a well-known topology—$\hbar$-adic topology on the Weyl algebra. But it does not work in our construction, because $\hbar$-adic topology does not make a Weyl algebra into a locally convex topological space.

Step 2. The next step in our construction is to find a resolution of the Weyl algebra bundle. We show that there is a flat connection on $\wedge^* \mathcal{F}^* \otimes W$. For this purpose, we introduce two operators $\delta$ and $\delta^*$ on $\wedge^* \mathcal{F}^* \otimes W$, defined by

$$\delta a = dx^i \wedge \frac{\partial a}{\partial y^i}, \quad \delta^* a = y^i (\frac{\partial}{\partial x^i}) a.$$

$\delta$ ($\delta^*$) is a degree decreasing (increasing) operator, satisfying the following properties.

**Proposition 3.3.** The operators $\delta$ and $\delta^*$ do not depend on the choices of local coordinates and satisfy:

1. $\delta^2 = (\delta^*)^2 = 0$.

2. On a monomial $y^{i_1} y^{i_2} \cdots y^{i_r} dx^{j_1} \wedge dx^{j_2} \cdots \wedge dx^{j_k}$, we have $\delta \delta^* + \delta^* \delta = (p+q)i\hbar d$. Generally, for $a \in \wedge^* \mathcal{F} \otimes W$,

$$a = \delta \delta^{-1} a + \delta^{-1} \delta a + a_{00},$$

where $\delta^{-1}$ is defined by

$$\delta^{-1} a_{pq} = \frac{1}{p+q} \delta^* a_{pq}, \quad p+q > 0,$$

$$\delta^{-1} a_{00} = 0,$$

and in which $a_{pq}$ is the homogeneous part of $a$ with degree $p$ in $y$ and degree $q$ in $dx$. 

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In the following, we assume that there is an invariant pseudo symplectic connection $\partial$ on $\mathcal{G}$. For convenience, we will omit the words “invariant pseudo”, and simply say “symplectic connection”.

A symplectic connection $\partial$ also defines a connection on the Weyl algebra bundle and the tensor bundle $\wedge^q \mathcal{F}^* \otimes \mathcal{W}|_I$, $^8$ which can be expressed as

$$\partial a = dx^i \wedge \partial_i a,$$

where $\partial_i a$ is a covariant derivative. It is not difficult to check that $\partial$ has the following properties:

**Proposition 3.4.**

1. $\partial(a \circ b) = \partial a \circ b + (-1)^q a \circ \partial b$, for $a \in \wedge^q \mathcal{F}^* \otimes \mathcal{W}$.

2. for any scalar form $\phi \in \wedge^q \mathcal{F}^*$, $\partial(\phi \wedge a) = d\phi \wedge a + (-1)^q \phi \wedge \partial a$.

3. $\partial \delta a + \delta \partial a = 0$.

4. $\partial^2 a = \partial(\partial a) = \frac{i}{\hbar} [R, a]$ where

$$R = \frac{1}{4} R_{ijkl} y^i y^j dx^k \wedge dx^l,$$

is the curvature of the symplectic connection.$^9$

In a Darboux chart, the connection can be written as

$$\partial a = da + \frac{i}{\hbar} [\Gamma, a],$$

where $\Gamma$ is a local 1-form with values in $\mathcal{W}$, and $d = dx^i \wedge \frac{\partial}{\partial x^i}$ is the exterior differential with respect to $x$.

To find an abelian connection, we will consider a connection on $\mathcal{W}$ with a more general form,

$$Da = \partial a + \frac{i}{\hbar} [\gamma, a] = da + \frac{i}{\hbar} [(\Gamma + \gamma), a],$$

where $\gamma$ is a section of $\mathcal{W} \otimes \wedge^1 \mathcal{F}^*$. Here, we only consider the case where $\gamma_0|_{y=0} = 0$. For the new connection, we call the following 2-form

$$\Omega = R + \partial \gamma + \frac{i}{\hbar} \gamma^2,$$

the curvature form of $D$.

**Proposition 3.5.**

1. (Bianchi identity)

$$D \Omega = \partial \Omega + \frac{i}{\hbar} [\gamma, \Omega] = 0.$$

2. for any section $a \in \wedge^q \mathcal{F}^* \otimes \mathcal{W}$, we have

$$D^2 a = \frac{i}{\hbar} [\Omega, a].$$

**Proof.** The proof is the same as the proof of Lemma 5.1.5 of [11]. $\Box$

After this long preparation, we are able to define the key notion in Fedosov’s method.

**Definition 3.6.** A connection $D$ on the bundle $\mathcal{W}$ is called abelian if for any section $a \in \wedge^q \mathcal{F}^* \otimes \mathcal{W}$,

$$D^2 a = \frac{i}{\hbar} [\Omega, a] = 0.$$

**Remark 3.7.** Here, we follow Fedosov using the term “abelian connection”. This is also known as the “Fedosov connection” in the literature.

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$^8$I is a leave of $\mathcal{F}$. For convenience, in the following we will omit “$\wedge$”.

$^9$\[ , \] is defined as $[a, b] = a \circ b - (-1)^{deg(a)deg(b)} b \circ a$. 

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The main theorem in this step is

**Theorem 3.8.** There exists one and only one abelian connection of the form

\[ D = -\delta + \partial + \frac{i}{\hbar} [ r, ] , \]

with

\[ \text{deg}(r) \geq 2, \quad \delta^{-1}r = 0, \]

and \( r \) is invariant under \( s^*, t^* \) maps.

**Proof.** The key idea of the proof is iteration. The steps are similar to the proof of Theorem 5.2.2 in [11]. Here, there are two more things new from [11]. One is that \( r \) is invariant under the \( s, t \) maps. The reason for this is that we have required all the data in the construction to be invariant under \( s, t \) maps, i.e. the étalification, the 2-form and the symplectic connection are invariant under the \( s, t \) maps, and also the iteration steps are \( s, t \) invariant. Therefore, it is straightforward to check that \( r \) is also \( s, t \) invariant. The other is that all the steps of the proof in [11] are continuous with respect to the topology we defined in 6.1, which follows from the degree increasing iteration procedure (see Lemma 5.2.3 in [11]). \(\square\)

**Step 3.** For \( D \), we consider the set of flat sections \( W_D \) defined \( \{ a \in W : Da = 0 \} \). It has the following important property.

**Proposition 3.9.** For any \( a_0 \in C^\infty(G)[\hbar] \), there exists a unique section \( a \in W_D \) such that \( \sigma(a) = a_0 \). (\( \sigma(a) \) means the projection onto the center: \( \sigma(a) = a(x, 0, h) \)).

**Proof.** Follows from Theorem 5.2.4 of [11]. \(\square\)

By Proposition 3.9 we define a quantization map \( Q : C^\infty(G)[\hbar] \to W_D \) to be the inverse of \( \sigma \) in the above proposition.

**Groupoid operation**

In the above steps, almost everything is analogous to Fedosov’s quantization of a symplectic manifold. But it only provides us with a formal deformation quantization of the commutative algebra \( C^\infty_c(G)[\hbar] \) with the pointwise multiplication. In the following, we will use this construction to quantize the groupoid \( C^\infty_c(G)[\hbar] \) with the convolution product. Our strategy to define the deformed groupoid algebra is the following:

1. Define a new multiplication \( * \) on sections of Weyl algebra bundles, which corresponds to the convolution.
2. Prove that \( (C^\infty_c(\wedge^* F^* \otimes W), *) \) is associative and \( D \) acts as a derivation.

**Remark 3.10.** We use \( * \) and \( \ast \) for different meanings. \( * \) is a new product on smooth sections of Weyl algebra bundle, and \( \ast \) is a star product on the groupoid algebra.

**Step 1.** We introduce a new algebraic product \( * \) on the sections of \( \wedge^* F^* \otimes W \). For \( f, g \in C^\infty_c(\wedge^* F^* \otimes W) \),

\[ f * g(\gamma) \overset{def}{=} \int_{\alpha, \beta = \gamma} t^*(f(\alpha)) \circ s^*(g(\beta))d\lambda^\gamma. \]

**Remark 3.11.** In the construction of \( \wedge^* F^* \otimes W \), we required everything to be invariant under the \( s, t \) maps. Therefore, for \( \gamma \in G \), both \( s^* : \wedge^* F^* \otimes W|_{\gamma} \to \wedge^* F^* \otimes W|_{\gamma} \) and \( t^* : \wedge^* F^* \otimes W|_{\gamma} \to \wedge^* F^* \otimes W|_{\gamma} \) are isomorphisms of Weyl algebras. In this way, under \( t^* \), \( f(\alpha) \in \wedge^* F^* \otimes W|_{\alpha} \) is mapped into \( \wedge^* F^* \otimes W|_{\gamma} \), and so is \( g(\beta) \) by \( s^* \). So the \( * \) in the above formula is well-defined, and when there is no confusion, we will drop \( s^*, t^* \) and \( \lambda^\gamma \).

**Remark 3.12.** When integrating a Weyl algebra valued function \( f \) along a t-fiber, we have to use the compact open topology on a Weyl algebra defined in 3.1.

**Step 2.** We will prove that \( * \) is associative and \( D \) acts as a derivation. Before doing this, we first recall Lemma 2.2.6.

When restricted to the leaves of \( F \), \( d \) has the following formula: \( \forall f, g \in C^\infty_c(G) \),

\[ d(f \ast g)(\alpha) = \int_{\alpha = \beta \gamma} (t^*)^{-1}(df(\beta))g(\gamma) + f(\beta)(s^*)^{-1}d(g)(\gamma). \]
Lemma 3.13. * is an associative product on $\Gamma(\Lambda^* F^* \otimes W)$, and the connection $D$ is a derivation on $(\Gamma(\Lambda^* F^* \otimes W), *)$, i.e.

$$D(f * g) = Df * g + (-1)^{\deg f} f * Dg$$

Proof. For the associativity of $*$, we compute

$$f * (g * h)(\alpha) = \int_{G^\alpha} f(\alpha) \circ (g \circ h)(\beta^{-1}) d\lambda(\beta)$$

From the formulas of $D$, to prove $D$ is a derivation it is sufficient to show that $\partial$ and $[r, \cdot]$ are derivations, respectively.

1. $\partial(f * g) = (\partial f) * g + (-1)^{\deg f} f * (\partial g)$,
2. $[r, f * g] = [r, f] * g + (-1)^{\deg f} f * [r, g]$.

By partition of unity, we write $f, g$ to be of the form

$$f(\alpha) = \sum_{i, I, \tau} h^i f_{i, I, \tau}(\alpha) y^I d\tau, \quad g(\alpha) = \sum_{j, J, \upsilon} h^j g_{j, J, \upsilon}(\alpha) y^J d\upsilon.$$ 

For the proof of 1,

$$\partial(f * g)(\alpha) = \partial(\int_{\beta=\alpha} f(\beta) \circ g(\gamma) d\lambda(\gamma))$$

By Lemma 3.24 and the derivation property of $\partial$ in Proposition 3.34,

$$= \sum_{i, I, \tau, j, J, \upsilon} (\partial f_{i, I, \tau}) \circ g_{j, J, \upsilon}(\alpha) + f_{i, I, \tau} \circ (\partial g_{j, J, \upsilon})(\alpha) (h^I y^J d\tau) + (h^I y^J d\upsilon)$$

The proof for $[r, \cdot]$ is much easier than the proof for $D$, which can be derived from the invariance of $r$ under $s^*, t^*$ (see Theorem 3.32) and associativity of $\circ$. □

Quantization of a groupoid algebra

From Lemma 3.13, we know that sections of $W_D$ are closed under convolution $*$, and form a new associative algebra. Therefore, we can define a formal deformation quantization of the groupoid algebra $C^\infty_c(\mathcal{G})[[\hbar]]$ to be

$$f * g(\gamma) = \sigma(\int_{\alpha, \beta=\gamma} t^*(\mathbb{Q}(f)(\alpha)) \circ s^*(\mathbb{Q}(g)(\beta)) d\lambda^\gamma)$$

which can also be written as $\sigma(\mathbb{Q}(f) * \mathbb{Q}(g))$. 

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Remark 3.15. One can also consider the classification of the constructed algebra, either up to isomorphism as in [17] or Morita equivalence as in [16]. Here, we follow the definition of a formal deformation quantization in [11], in which there is no concern about the involution on $C^\infty(G)[[\hbar]]$. One can follow Neuumaier’s work [16] to define an involution on the deformed algebra. Furthermore, one can also look for a positive (Hermitian) deformation quantization of a groupoid algebra. We leave all this for future study.

3.1.2 Pseudo Poisson groupoid

We have already seen a deformation quantization of a pseudo symplectic groupoid. In this subsection, we will work on a pseudo Poisson groupoid. The spirit of the construction is similar to the symplectic case. However, there are two major differences. One is that we will replace the Weyl algebra by the Kontsevich algebra constructed by Kontsevich in [13] for a Poisson structure on $\mathbb{R}^n$. However, there are two major differences. One is that we will replace the Weyl algebra by the Kontsevich algebra constructed by Kontsevich in [13] for a Poisson structure on $\mathbb{R}^n$, the other is that on a general pseudo Poisson groupoid there is no Poisson connection. To solve this difficulty, we will use a structure other than a connection —a “quasi-connection”. To lift this to the Kontsevich algebra bundle, we will apply Kontsevich’s local formality theorem. Since Theorem 2.30 has already assumed the condition that $G$ is proper, which in the case of pseudo symplectic (or regular Poisson) groupoids is stronger than the existence of an invariant connection. The idea of using Fedosov’s method to construct a deformation quantization for a general Poisson manifold comes from Cattaneo, Felder and Tomassini in [6]. As most of the steps are similar to those in the symplectic case, we will be a little bit sketchy. Readers are referred to [6] and [13] for details.

Essentially, there are four steps in our construction of a formal deformation quantization of a pseudo Poisson groupoid.

1. the Kontsevich algebra $\mathcal{R}$;
2. the quantization bundle;
3. the abelian connection;
4. the quantization map $\mathcal{Q}$.

Step 1. Let us start by reviewing Kontsevich’s star product and formality theorem on $\mathbb{R}^n$. Kontsevich’s formality theorem constructs a quasi-isomorphism between the DGLA of the Hochschild complex of the algebra of polynomials and the DGLA of multi-vector fields. Kontsevich defines a sequence of operators $U_j$ to be a multi-linear symmetric function of $j$ arguments of multi-vector fields, with values in the multi-differential operators $C^\infty(\mathbb{R}^n)^{\otimes r} \to C^\infty(\mathbb{R}^n)$, where $r = \sum_k m_k - 2j + 2$. The maps are $GL(d, \mathbb{R})$-invariant and satisfy the following famous equalities:

Theorem 3.16. ([13], Theorem 3.1 of [16]) Let $\ell_{ij} \in \Gamma(\mathbb{R}^n, \wedge^{m_j} T^* \mathbb{R}^n)$, $j = 1, \ldots, s$ be multi-vector fields.

Let $\epsilon_{ij} = (-1)^{(m_i+\cdots+m_{i-1})m_i+(m_i+\cdots+m_{i-1}+m_{i+1}+\cdots+m_{j-1})m_j}$. Then, for any functions $f_0, \ldots, f_m$,\n
$$\sum_{i=0}^{s} \sum_{k=0}^{m} \sum_{l=0}^{m-k} (-1)^{k(i+1)+m} \sum_{\sigma \in S_{i+1}} \epsilon(\sigma) U_l(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(l)}) U_i(\alpha_{\sigma(l+1)}, \ldots, \alpha_{\sigma(s)}) U_s-1(\alpha_{i_1}, \ldots, \alpha_{i_{l+k}}, \alpha_{j_1}, \ldots, \alpha_{j_{m-k}}, f_0 \otimes \cdots \otimes f_{i_1} \otimes \cdots \otimes U_{s-1}(\alpha_{i_1}, \ldots, \alpha_{i_{l+k}}, f_0 \otimes \cdots \otimes f_{i_{l+k}}) \otimes f_{j_1} \otimes \cdots \otimes f_{j_{m-k}}) = \sum_{i<j} \epsilon_{ij} U_{s-1}(\alpha_{i_1}, \cdots, \alpha_{i_{l+k}}, \alpha_{j_1}, \ldots, \alpha_{j_{m-k}}, f_0 \otimes \cdots \otimes f_{j_{m-k}}).$$
To formally deformation quantize a Poisson manifold and a pseudo Poisson groupoid, we will only use some special cases of the above theorem. We follow the notation in [6].

\[
P(\pi) = \sum_{j=0}^{\infty} \frac{\hbar^j}{j!} U_j(\pi, \ldots, \pi)
\]

\[
A(\xi, \pi) = \sum_{j=0}^{\infty} \frac{\hbar^j}{j!} U_{j+1}(\xi, \pi, \ldots, \pi)
\]

\[
F(\xi, \eta, \pi) = \sum_{j=0}^{\infty} \frac{\hbar^j}{j!} U_{j+2}(\xi, \eta, \pi, \ldots, \pi).
\]

$\xi, \eta$ are vector fields on $\mathbb{R}^n$, and $\pi$ is a Poisson bivector.

Let $\mathcal{U}$ be the set of polynomial maps from $\Gamma(\mathbb{R}^n, \wedge^2 \mathbb{R}^n)$ to the space of multi-differential operators on $\mathbb{R}^n$, with differential $\delta$ defined by:

\[
\delta S(\xi_1, \ldots, \xi_{p+1}, \pi) = -\sum_{i=1}^{p+1} \frac{d}{dt} |_{t=0} S(\xi_1, \ldots, \hat{\xi}_i, \ldots, \xi_{p+1}, (\Phi^t_\xi)_* \pi) + \sum_{i < j} (-1)^{i+j} S([\xi_i, \xi_j], \xi_1, \ldots, \hat{\xi}_i, \ldots, \hat{\xi}_j, \ldots, \xi_{p+1}, \pi),
\]

where $(\Phi^t_\xi)_*$ stands for the flow generated by the vector field $\xi$.

The formality theorem (Theorem 3.16) implies that $P(\pi)$, $A(\xi, \pi)$, and $F(\xi, \eta, \pi)$ satisfy the following equations.

**Lemma 3.17.** (Corollary 3.2, [6]) Theorem 3.16 implies the following:

(i) $P(\pi) \cdot (A(\xi, \pi) \otimes 1 + 1 \otimes A(\xi, \pi)) = A(\xi, \pi) \cdot P(\pi) = \delta P(\xi, \pi)$.

(ii) $P(\pi) \cdot (F(\xi, \eta, \pi) \otimes 1 - 1 \otimes F(\xi, \eta, \pi)) = A(\xi, \pi) \cdot A(\eta, \pi) + A(\eta, \pi) \cdot A(\xi, \pi) = \delta A(\xi, \eta, \pi)$.

(iii) $-A(\xi, \pi) \cdot F(\eta, \zeta, \pi) - A(\eta, \pi) \cdot F(\zeta, \xi, \pi) - A(\zeta, \pi) \cdot F(\xi, \eta, \pi) = \delta F(\xi, \eta, \zeta, \pi)$.

With Lemma 3.17, we define a $\ast$ product on $(\mathbb{R}^n, \pi)$,

\[
f \ast g \overset{\text{def}}{=} P(\pi)(f \otimes g), \quad f, g \in \mathbb{R}[[y^1, \ldots, y^n]][[\hbar]],
\]

which is associative. We call $\mathbb{R}[[y^1, \ldots, y^n]][[\hbar]]$, with the $\ast$ product the Kontsevich algebra $\mathfrak{K}$.

**Remark 3.18.** Here, we work with $\mathbb{R}$–algebras. One can generalize this to $\mathbb{C}$–algebras without any extra effort. Also, one can define a topology as in the last section.

**Step 2.** Parallel to the construction of the Weyl algebra bundle, we define a Kontsevich algebra bundle. The construction is a little more involved than the Weyl algebra bundle, because the Kontsevich algebras at different points are generally not isomorphic. Therefore, we must use some formal geometry.

Let $(\mathcal{G}, \mathcal{F}, \pi)$ be a proper pseudo Poisson groupoid, and at each point $\gamma$ of $\mathcal{G}$, let $L_\gamma$ be the leaf of $\mathcal{F}$ through $\gamma$ with rank $n$. With the help of formal geometry, we can define a vector bundle $E$ on $\mathcal{G}$, with fiber $\mathbb{R}[[y^1, \ldots, y^n]][[\hbar]](\mathfrak{K})$. By the same arguments as in [6], namely the triviality of an infinite jet bundle, one can find a family $(\phi_\gamma)_{\gamma \in \mathcal{G}}$ of infinite jets at the zero of local diffeomorphisms $\phi_\gamma : (\mathbb{R}^n, 0) \to (L_\gamma, \gamma)$ such that $\phi_\gamma(0) = \gamma$. This family of infinite jets at zero is called a quasi-connection. Because $\phi_\gamma$ is a local diffeomorphism, it pulls back the Poisson structure $\pi$ to $\mathbb{R}^n$. Therefore, at fiber $E_\gamma$, using Kontsevich’s product, one can define a Kontsevich algebra $\mathfrak{K}_\gamma$. In this way, we obtain a Kontsevich algebra bundle $\mathfrak{K}$. Here, because $\mathcal{G}$ is proper, we can integrate $\phi_\gamma$ along $\mathcal{G}$ to get a $s, t$ invariant quasi-connection.

**Step 3.** As in the pseudo symplectic groupoid case, we consider $\wedge^* \mathcal{F}^* \otimes \mathfrak{K}$ in which $\wedge^* \mathcal{F}^*$ denotes differential forms along $\mathcal{F}$. We introduce a connection $D : \Gamma(\mathfrak{K}) \to \Gamma(\mathfrak{K})$ by

\[
(Df)_\gamma = dx f + A_x^M f,
\]

where the $dx f$ is the de Rham differential of $f$, viewed as a function on $\mathcal{G}$, and $A_x^M$ is defined in [6]. From the invariance of $\phi$ under the $s, t$ maps, $D$ is also invariant under $s, t$. Furthermore, we have the following proposition for $D$.

**Proposition 3.19.** Let $F^\mathfrak{K}$ be a $\mathfrak{K}$ valued two form on $E$, defined by

\[
F^\mathfrak{K} = F((\hat{\xi}_\gamma), (\hat{\eta}_\gamma), \pi_\gamma).
\]

where $\hat{\xi}_\gamma$ and $\hat{\eta}_\gamma$ are lifted vectors on $E$ as in [6]. Then for any $f, g \in \Gamma(\mathfrak{K})$,
1. \(D(f \circ g) = Df \circ g + f \circ Dg,\)
2. \(D^2 f = F^g \circ f - f \circ F^g,\)
3. \(DF^g = 0.\)

**Remark 3.20.** Here, the \(\circ\) between \(F^g\) and \(f\) is the fiberwise multiplication on \(\mathcal{G}\). It is easy to check that the multiplication is well defined.

**Proof.** Same as the proof of Proposition 4.2 in [6].

Using \(D\), with the same arguments of [6], we have the following theorem:

**Theorem 3.21.** There exists an abelian connection \(\bar{D}\) of the form \(D + [r, f] = D_0 + \hbar D_1 + \hbar^2 D_2 + \cdots\) on \(\mathcal{G}\), so that there is an isomorphism \(Q : C_c^\infty(\mathcal{G})[[\hbar]] \to H^0(\mathcal{G}, D)\).

**Proof.** The proof follows [6].

**Remark 3.22.** Here, by integration along \(\mathcal{G}\), the construction of \(r\) and \(Q\) can be made invariant under the maps \(s, t\).

**Step 4.** On \(\mathcal{G}\), we introduce a new algebraic structure, for \(f, g \in \mathcal{G}\),

\[f \ast g(\alpha) \overset{def}{=} \int_{\beta, \gamma = \alpha} t^* (f(\beta)) \circ s^*(g(\gamma)).\]

Similar to [5] we have the following properties of \(\ast\).

**Proposition 3.23.**
1. \((f \ast g) \ast h = f \ast (g \ast h),\)
2. \(D(f \ast g) = Df \ast g + f \ast Dg,\)
3. \([r, f \ast g] = [r, f] \ast g + (-1)^{deg(f)} f \ast [r, g].\)

From Proposition 3.23, we know that the kernel of \(\bar{D}\) is closed under \(\ast\). Therefore, the deformed groupoid algebra \(\mathcal{G}\) is defined as \((C_c^\infty(\mathcal{G})[[\hbar]], \ast),\) for \(f, g \in C_c^\infty(\mathcal{G})[[\hbar]],\)

\[f \ast g(\alpha) \overset{def}{=} Q^{-1} \left( \int_{\beta, \gamma = \alpha} t^* (Q(f)(\beta)) \circ s^*(Q(g)(\gamma)) \right).\]

It is easy to check that \(\ast\) is associative from the first equality of Proposition 3.23. To finish the proof that \(\ast\) defines a deformation quantization of a groupoid algebra, we still need to show that the linearization of \(\ast\) is the noncommutative Poisson structure \(\Pi\) defined in Section 2.

As \(Q^{-1}\) is evaluation at \(y = 0\), which is independent of integration along \(s\) and \(t\) fibers,

\[f \ast g(\alpha) = \int_{\alpha = \beta, \gamma} Q^{-1} (t^* (Q(f)(\beta)) \circ s^*(Q(g)(\gamma))).\]

It is easy to check that \(Q\) is invariant under the \(s, t\) maps, so

\[f \ast g(\alpha) = \int_{\alpha = \beta, \gamma} Q^{-1} (Q(t^* (f(\beta))) \circ Q(s^*(g(\gamma))).\]

By this expression, and the fact that \(\circ\) is a formal deformation quantization of the pointwise multiplication, we have

\[f \ast g(\alpha) = \int_{\alpha = \beta, \gamma} \hbar \pi(t^* (f(\gamma)), s^*(g(\beta))) + o(\hbar)\]

\[= \hbar \Pi(f, g)(\alpha) + o(\hbar) \quad \square\]

In summary, we have proved the following theorem.

**Theorem 3.24.** For a proper pseudo Poisson groupoid, there always a formal deformation quantization of the noncommutative Poisson algebra defined in Section 2.

**Corollary 3.25.** A Poisson orbifold groupoid can be formally deformation quantized by the methods used in this section.
3.2 A trace formula

Traces on an algebra play an important role in understanding the algebra. The number of traces and the value of a trace provide a large amount of information about an algebra. In the study of the quantum index theorem, we will consider the value of a trace on a quantized algebra valued projection matrix, which requires the study of properties of traces on a quantized groupoid algebra. In this section, we will show a construction which defines a natural trace on a deformation quantized groupoid algebra. In \[23\], we will study the Hochschild and cyclic homology of a quantized groupoid algebra which consists of generalized traces.

A trace of a formal algebra \(A[[\hbar]]\) is defined as a \(C[[\hbar]](\mathbb{R}[[\hbar]])\) valued linear functional on \(A[[\hbar]]\), vanishing on commutators. Traces on a deformation quantization of a symplectic (Poisson) manifold have been well studied. It is shown that on a quantized symplectic manifold there is a unique trace up to normalization. Furthermore, following an idea from noncommutative geometry, Connes, Flato and Sternheimer introduced the notion of a closed \(\star\)–product. The existence of a closed \(\star\)–product on a symplectic manifold was first proved by Omori, Maeda and Yoshioka, and on a Poisson manifold the corresponding result is due to the work of Felder and Shoikhet in \[12\].

In the following, we define and prove the existence of a closed deformation quantization of a pseudo symplectic (Poisson) groupoid with an invariant measure.

**Definition 3.26.** A formal deformation quantization \((C^\infty_c(G)[[\hbar]],\star)\) of a pseudo symplectic (Poisson) groupoid \(G\) is closed if there is a groupoid invariant volume form \(\Omega\) on \(G(0)\), such that

\[ T_{\Omega} \overset{\text{def}}{=} \int_{G(0)} f|_{G(0)} \Omega \]

is a trace on the deformed algebra\(^{11}\).

The main result of this section is the following theorem.

**Theorem 3.27.** Let \(G\) be a pseudo symplectic (Poisson) groupoid with an invariant connection\(^{12}\), and \(\Omega\) an invariant volume form on the unit space \(G(0)\), satisfying \(\text{div}_\Omega \pi = 0\). Then

\[ \int_{G(0)} f \Omega \]

defines a trace on the deformation quantized groupoid algebra \((C^\infty_c(G)[[\hbar]],\star)\).

**Remark 3.28.** The \(\star\) product might be different from the original construction in Section 3.1.

**Remark 3.29.** For the symplectic case, one has a natural choice of \(\Omega\) which is \(\Omega^{\frac{1}{2}}\dim(G(0))\). To prove that it defines a trace, one can first follow the method in \[11\] to find a closed star product. Then the following arguments show that \(\Omega^{\frac{1}{2}}\dim(G(0))\) defines a trace.

**Remark 3.30.** For a pseudo Poisson groupoid, we can define three “modular classes”. One is the modular class of the foliation generated by \(\mathcal{F}\), which is 0 if and only if there exists a transversal Haar system; one is the modular class of the groupoid, which vanishes if and only if there is an invariant Haar system; the other is the modular class of the Poisson structure, which is zero if and only if there is a volume \(\Omega\) on \(G(0)\) with \(\text{div}_\Omega \pi = 0\). Therefore, in Theorem 3.27 we are working with a groupoid whose all three modular classes are trivial.

**Remark 3.31.** We will show in \[23\] that a quantized pseudo symplectic (Poisson) groupoid usually has more than one trace, the number of which is determined by the 0-th Hochschild homology of the quantized groupoid. Formula (5) provides one example of traces. It would be interesting to find formulas of other traces on a deformation quantized pseudo symplectic (Poisson) groupoid.

\(^{10}\)Usually, if a groupoid has a symmetric measure, then the same formula defines a trace on \(C^\infty_c(G)\). Here, we require a stronger condition –“invariant measure” for later use.

\(^{11}\)It is easy to check that the same formula also defines a trace on the groupoid algebra.

\(^{12}\)If \(G\) is a general pseudo Poisson groupoid, then instead of the existence of an invariant connection, we require \(G\) to be proper.
Proof. We will work on the Poisson case.

As Ω is invariant under groupoid action, we may define an n–form sΩ(= tΩ) on ℓ, n = dim(ℓG0).

Because of the invariance assumption, it is easy to check that on the integrated submanifold Lγ, the divergence of π with respect to sΩ is still 0. For Ω, following the same method of [12], we can construct a family of L∞–morphisms

\[ U_L : [T^*_{poly}(L_G)]_{div} \to [D^*_{poly}(L_G)]_{cycl}, \]

where \([T^*_{poly}(L_G)]_{div}\) is the subalgebra of the DGLA of multi-vector fields whose elements have 0 divergence and \([D^*_{poly}(L_G)]_{cycl}\) is the DGLA of cyclic cochain complex (see [12]). From this map, a divergence 0 Poisson structure defines a product ◦ on \(C^\infty_c(L_G)[[h]]\) such that for any three smooth functions \(f, g, h\) with compact supports, we have:

\[ \int_{L_\gamma} (f \circ h)h\Omega = \int_{L_\gamma} (g \circ h)f\Omega. \]  

(6)

Using ◦ and following the same arguments as in the last section, we can easily prove that

\[ f \ast g(\alpha) = Q^{-1} (\int_{\beta=\alpha} t^*(Q(f)(\beta)) \circ s^*(Q(g)(\gamma))d\lambda(\alpha)) \]

defines a star product on the groupoid algebra \(C^\infty_c(\ell)[[h]]\). In the following, we prove that ◦ defines a trace on this star product.

First, we know that \(Q^{-1}\) is evaluation at \(y = 0\), so there is no difference if we put \(Q^{-1}\) inside the integral. Then \(Tr_\Omega(f \ast g) = \int_{G_0} f \ast g\Omega\) can be written as

\[ \int_{G_0} \Omega \int_{x=\alpha^{-1}} t^*(Q(f)(\alpha)) \circ s^*(Q(g)(\alpha^{-1}))d\lambda. \]

If we set \(g(\alpha) \overset{def}{=} g(\alpha^{-1})\), by the invariance of \(Q\) under groupoid operations, e.g. \(s^*, t^*, \) and inverse maps, the trace can again be written using ◦,

\[ \int_{G_0} \Omega \int_{x=\alpha^{-1}} t^*(f \circ g(\alpha))d\lambda. \]

13

Since \(F\) is transverse to the t-fiber, \(t^*(\Omega)\) and \(\lambda^x\) together forms a Borel measure \(\hat{\Omega}\) on \(G\). The above can be summarized as

\[ \int_{G_0} f \ast g\Omega = \int f \circ \hat{\Omega}. \]

On the other hand, according to the invariance of the Haar system, one can choose a transversal submanifold V to the foliation of F, and by Fubini’s theorem, the integral \(f \ast g\Omega\) can be written as

\[ \int_V \lambda_V \int_{L_\gamma} f \circ \hat{\Omega}. \]

By setting \(h = 1\) in equation (9), we have

\[ \int_{L_\gamma} f \circ \hat{\Omega} = \int_{L_\gamma} f \hat{\Omega}. \]

Therefore,

\[ \int f \circ \hat{\Omega} = \int_V \lambda_V \int_{L_\gamma} f \hat{\Omega}, \]

which is equal to

\[ \int_{G_0} \Omega \int_{x=\alpha^{-1}} f(\alpha)g(\alpha^{-1})d\lambda = Tr_\Omega(f \circ g). \]

The last line is our definition of \(Tr_\Omega(f \circ g)\), which is the same as \(Tr_\Omega(g \circ f)\). Therefore, what we have shown is that \(Tr_\Omega\) is also a trace on the deformed groupoid algebra. □

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13Here, it is safe to discard \(t^*\), because \(f \ast g\) is a formal function and not a section.

14Here, one might need to work in local charts and use partition of unity.
3.3 Example: transformation groupoids

Let $\mathcal{F}$ be the $M$ component of the tangent bundle of $T(M \times G)$. It is not difficult to check that $\mathcal{F}$ forms an étalification on $M \times G$. At each $t$–fiber, we fix the Haar measure $\lambda$ of $G$, which is constant along $\mathcal{F}$. It is easy to check that $\pi$ on $\mathcal{F}$ makes the transformation groupoid $M \times G \rightrightarrows M$ into a pseudo Poisson groupoid, which induces a noncommutative Poisson structure on its groupoid algebra $\mathcal{C}_c^\infty(G, \mathcal{C}_c^\infty(M))$, i.e. $\forall f, g \in \mathcal{C}_c(G, \mathcal{C}_c^\infty(M))$

$$\Pi(f, g)(x, \alpha) = \int_G \{f(\beta \cdot x, \alpha \beta^{-1}), g(x, \beta)\}d\lambda(\beta).$$

If the $G$ action is proper, $M \times G \rightrightarrows M$ is also proper, and has a quasi-connection. Therefore, $M \times G \rightrightarrows M$ can be formally deformation quantized.

$$f \ast g(m, \gamma) = \mathcal{Q}^{-1}\{\int_{\alpha, \beta = \gamma} t^*(\mathcal{Q}(f)(\beta \cdot m, \alpha)) \circ s^*(\mathcal{Q}(g)(m, \beta))\}.\lambda.$$  

One can directly check the following facts about the above quantization, which we will not prove.

1. The nontrivial part of the above star product is on the $M$ components.
2. The construction of the star product is invariant under the $G$ action.
3. $\mathcal{Q}^{-1}$ commutes with integration along $G$.

Using the above facts, we can rewrite the star product as

$$f \ast g(m, \gamma) = \int_{\alpha, \beta = \gamma} t^*(\mathcal{Q}(f)(\beta \cdot m, \alpha)) \circ s^*(\mathcal{Q}(g)(m, \beta))\lambda = \int_{\alpha, \beta = \gamma} \mathcal{Q}^{-1}(\mathcal{Q}(\mathcal{Q}(f)(\beta \cdot m, \alpha)) \circ \mathcal{Q}(g)(m, \beta))\lambda = \int_{\alpha, \beta = \gamma} \beta^*(f(\alpha)) \ast g(\beta)(m)\lambda.$$

In the second step, to get rid of $t^*$, $s^*$, we have used the $G$ invariance of the star product.

It is easy to check that the above product agrees with the multiplication on the crossed product algebra $\mathcal{C}_c^\infty(M)[[\hbar]] \rtimes G$.

**Corollary 3.32.** The formal deformation quantization of a transformation groupoid is the crossed product algebra of a formal deformation quantization of a Poisson manifold with the $G$ action.

**Remark 3.33.** We may straightforward check that the crossed product algebra is a formal deformation quantization of the noncommutative Poisson algebra defined by the crossed Poisson structure in the sense of Definition 1.2. However, our method in this paper provides a geometrical construction of this quantization.

Furthermore, when $G$ is unimodular, which means $G$ has a bi-invariant measure and $M$ has a $G$ invariant volume form $\Omega$, it is easy to check that

$$\text{Tr}_\Omega(f) = \int d\Omega(x) \int d\lambda(\alpha)f(x, \alpha)$$

defines a trace on $\mathcal{C}_c(G, \mathcal{C}_c^\infty(M))$ and also on $\mathcal{C}_c^\infty(M)[[\hbar]] \rtimes G$.

**Corollary 3.34.** If $G$ is unimodular and $M$ has a $G$-invariant measure $\Omega$, then $M \times G \rightrightarrows M$ has a closed deformation quantization with trace $\text{Tr}_\Omega$.

**Remark 3.35.** If the $G$ action is free and proper, then $M/G$ is again a Poisson manifold. By the results in [12], the $C^*$–completion of the crossed product algebra $\mathcal{C}_c^\infty(M) \rtimes G$ is Morita equivalent to the $C^*$–completion of the algebra $\mathcal{C}_c^\infty(M/G)$ of smooth compactly supported functions on the quotient. One natural question is whether this Morita equivalence still holds for the quantized algebras. In the future work, we will prove that in the case when $G$ is finite and $M$ is compact, the formal deformation quantizations of $M \times G \rightrightarrows M$ are Morita equivalent to the corresponding formal deformation quantizations of $M/G$ in the sense defined by Bursztyn and Waldman (see 4).
3.4 Strict deformation quantization

In previous sections of this section, when talking about deformation quantization, we looked at $\hbar$ as a formal parameter. However, sometimes we do want $\hbar$ to take some value (even very large). This idea has inspired Rieffel to introduce a notion of a strict (deformation) quantization (see Definition 1.4). In this section, we will look at the application of Rieffel’s method of strict deformation quantization [21] in strictly quantizing pseudo étale groupoids.

In his lecture notes [21], Rieffel considered a strict deformation quantization of a Poisson manifold with $\mathbb{R}^n$ action. Although he was looking at Poisson manifolds, he was already aware that his method might be able to quantize some noncommutative Poisson algebras with $\mathbb{R}^n$ action. Notations in the next theorem are explained in the remark which follows.

**Theorem 3.36.** ([21], Theorem 9.3) Let $\alpha$ be an action of $\mathbb{R}^n$ on a $C^*$-algebra $A$, and let $J$ be a skew-symmetric operator on $\mathbb{R}^n$. Let $J$ define a Poisson bracket, $\{ \cdot, \cdot \}$, on $A^\infty$. Then $A^\infty$ with the deformed products $\star_{\hbar J}$, involutions $\ast_{\hbar J}$, and $C^*$-norms $\| \cdot \|_\hbar$ as defined in Chapter 4 of [21], provides a strict deformation quantization of $A$ in the direction of $\frac{1}{2\pi} \{ \cdot, \cdot \}$.

**Remark 3.37.**
1. $A^\infty$ stands for the smooth algebra defined by the $\mathbb{R}^n$ action.
2. By “$J$ define a Poisson bracket”, we mean that for some basis of $\mathbb{R}^n$ and the corresponding derivations $\partial_1, \cdots, \partial_n$ on $A^\infty$ given by the action, the Poisson structure is defined by

$$\{ f, g \} \overset{\text{def}}{=} \sum_{j,k} J_{jk} \partial_j(f) \partial_k(g).$$

In the case of a pseudo étale groupoid, if the noncommutative Poisson structure defined by the formula (1) can be obtained from an $\mathbb{R}^n$ action as in Theorem 3.36, we can strictly deformation quantize the groupoid algebra. We illustrate this by the following example.

**Example 3.38.** ([24]) We consider a constant Dirac structure $\Gamma$ on an $n-$torus $\mathbb{T}^n$. As explained in Example 6 of 2.14, when we choose a rational transversal subtorus $M$, the reduced foliation groupoid is a pseudo Poisson groupoid. Therefore, we can consider the noncommutative Poisson structure defined on the groupoid algebra. The reduced foliation groupoid algebra can be identified with $C(M) \rtimes \mathbb{Z}^k$. Since $\mathbb{R}^{n-k}$ is the universal covering of $M$, the constant Poisson structure $\pi$ on $M$ can be lifted onto $\mathbb{R}^{n-k}$. From the identification that $M = \mathbb{R}^{n-k} / \mathbb{Z}^{n-k}$, $\mathbb{R}^{n-k}$ acts on $C(M)$ by translation commuting with the $\mathbb{Z}^k$ action; therefore, there is a well defined $\mathbb{R}^{n-k}$ action on the reduced foliation groupoid algebra. It is not difficult to check that the noncommutative Poisson structure defined by formula (1) can be obtained from the $\mathbb{R}^{n-k}$ action. Hence, the conditions of Theorem 3.36 are satisfied, and we can strictly deformation quantize the reduced foliation groupoid algebra. This is an equivalent definition of a quantization of a constant Dirac structure on an $n$-torus defined in [24].

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\(^{15}\)Recently, in [14], Li has given a “universal” way to strictly quantize Poisson manifolds. His methods can surely be used to strictly quantize a pseudo Poisson groupoid, but we will not discuss it here.

\(^{16}\)One can construct more than one groupoid from a foliation. Here, we consider the fundamental groupoid of a foliation.
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