Interior Structure of a Charged Spinning Black Hole in (2 + 1)-Dimensions

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Abstract

The phenomenon of mass inflation is shown to occur for a rotating black hole. We demonstrate this feature in (2 + 1) dimensions by extending the charged spinning BTZ black hole to Vaidya form. We find that the mass function diverges in a manner quantitatively similar to its static counterparts in (3 + 1), (2 + 1) and (1 + 1) dimensions.
The endpoint of gravitationally collapsing matter is a subject of long-standing interest in general relativity. Given the plausible (but unproven) hypothesis of cosmic censorship, it has been established that the spacetime exterior to a collapsing body relaxes to that of a Kerr-Newman (KN) black hole, with radiative perturbations decaying as advanced time increases according to a power law.

The question of what happens to the collapsing matter, along with everything else falling into the black hole, is somewhat more problematic. Infalling matter will either encounter a spacelike region of diverging curvature (at which point quantum gravitational effects presumably dominate) or alternatively will avoid the singularity and emerge into another universe via a 'white hole', the KN geometry being prototypical of this latter possibility. However it has been shown that the interior geometry of a KN black hole is unstable: the stress-energy associated with massless test fields diverges at a null hypersurface inside the black hole called the Cauchy horizon [1]. Any object falling into a KN black hole must eventually cross the Cauchy horizon, and so an understanding of its stability is intimately connected to the question of the final fate of the infalling matter.

Significant progress on this problem was made by Poisson and Israel [2], who demonstrated that the Cauchy horizon of the Reissner-Nordström solution forbids any evolution of spacetime beyond this horizon. Inside the black hole the mass parameter becomes unbounded due to the presence of ingoing and backscattered outgoing radiation, and the Kretschmann scalar diverges. This phenomenon was subsequently confirmed by Ori [3], who constructed an exact solution of the Einstein-Maxwell equations in a simpler model. He argued that the mass inflation singularity was too weak to forbid passage through the Cauchy horizon, since its tidal forces do not necessarily destroy any physical objects. This extensibility problem remains a subject of some controversy [4, 5].

Mass inflation has also been shown to take place in lower-dimensional analogs of the Reissner-Nordström solution, both in (1 + 1) [6, 7] and (2 + 1) dimensions [8].

As the causal structure of the Reissner-Nordström geometry is similar to that of the KN geometry, it is generally believed that the KN Cauchy horizon forms a similar obstruction to the evolution of the spacetime. However no models have appeared to date.

We present here the first exact mass-inflation solution of a charged spinning black hole. In particular, we consider the (2 + 1)-dimensional BTZ black hole geometry [9] in the context of Ori’s model, and show explicitly that it mass-inflates in a manner analogous to its Reissner-Nordström counterpart.

Let us consider Einstein’s equations with cosmological constant \( \Lambda < 0 \) in (2 + 1) dimensions:

\[
G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu},
\]

where \( 8\pi G \) is the coupling constant in (2 + 1) dimensions and the stress-energy tensor is that for the electrovacuum. The standard BTZ spinning black hole solution for \([1]\) with static electric charge has the form

\[
ds^2 = -N^2(r)dt^2 + N^{-2}dr^2 + r^2(N^\phi dt + d\phi)^2
\]

where \( N^\phi := -\frac{J}{2r} \) and

\[
N^2(r) := -\Lambda r^2 - M + \frac{J^2}{4r^2} - 4\pi Gq^2 \ln(r/r_0).
\]

The integration constants \( M \) and \( J \) are interpreted as the mass and the angular momentum of the black hole respectively [11, 12] and the constant \( q \) is the charge carried by the hole. Figure 1 shows the causal structure of the BTZ spacetime.

![Fig. 1. Penrose diagram for the BTZ solution](image_url)
when \( q = 0 \). In this case there is a singularity in
the causal structure \([4]\); however when this solution
is matched to a collapsing cloud of dust, there is again
a curvature singularity at \( r = 0 \) \([5]\). For non-zero \( q \),
\( r_\pm \) are the solutions to the transcendental equation
\( N(r_\pm) = 0 \), and there is a curvature singularity at
\( r = 0 \).

Defining new coordinates \( v \) and \( \theta \) as
\[
v := t + \int^r \frac{d\tilde{r}}{N^2(\tilde{r})}\quad \text{and} \quad \theta := \phi - \int^r \frac{N^\phi(\tilde{r})}{N^2(\tilde{r})} d\tilde{r}
\] (4)
the metric can be written as
\[
ds^2 = -\left[\mathcal{M}(r) - M - \frac{f^2}{4r^2}\right] dv^2 + 2dvdr - Jdv\theta + r^2d\theta^2
\] (5)
where the function \( \mathcal{M}(r) = -\Lambda r^2 + \frac{q^2}{r^2} - 4\pi G\rho^2 \ln(r/r_0) \). In this coordinate system, \( \partial_v \) and \( \partial_\theta \) are Killing vectors. As one approaches the Cauchy
horizon, \( r \) is decreasing and \( N^2 \) approaches zero from
below; thus \( v \) tends to positive infinity.

Consider next the stress-energy-momentum
tensor of null spinning dust which has the form
\[
\left[T_{\mu\nu}\right] = \begin{pmatrix}
\hat{\rho}(v, r) & 0 & -\hat{\omega}(v, r) \\
0 & 0 & 0 \\
-\hat{\omega}(v, r) & 0 & 0
\end{pmatrix}
\] (6)
where the energy density \( \hat{\rho} \) and angular momentum
density \( \hat{\omega} \) have the form
\[
\hat{\rho}(v, r) = \frac{\rho(v)}{r} + \frac{j(v)\omega(v)}{2r^3} \quad \text{and} \quad \hat{\omega}(v, r) = \frac{\omega(v)}{r}
\] (7)
as may easily be shown from the conservation laws.
Incorporating the stress-energy-momentum tensor for
the Maxwell field along with \([4]\) into Einstein’s equations \([4]\), we obtain an exact solution
\[
ds^2 = -\alpha(v, r)dv^2 + 2dvdr - j(v)dv\theta + r^2d\theta^2
\] (8)
with \( \alpha := \mathcal{M}(r, v) - m(v) - \frac{j^2(v)}{4r^2} \), where now
\( \mathcal{M}(r, v) = -\Lambda r^2 + \frac{J^2}{4r^2} - 4\pi G\rho^2 \ln(r/r_0) \equiv N^2(r, v) + m(v) \). The functions \( m(v) \) and \( j(v) \) satisfy the ordinary
differential equations
\[
\frac{dm(v)}{dv} = 16\pi G\rho(v) \quad \text{and} \quad \frac{dj(v)}{dv} = 16\pi G\omega(v).
\] (9)
When \( j = 0 \) this solution is analogous to the charged
Vaidya solution and represents a black hole which is
irradiated by an influx of null radiation \([3]\). When \( j \)
is non-zero, the black hole and the surrounding null
dust rotate. Uniform rotation takes place only when
\( j \) is a non-zero constant.

We consider next a pulse of radiation along an
outgoing null ring \( S \). For simplicity we take \( \omega(v) = 0 \),
so that \( j(v) = J = \text{constant} \). As \( S \) is a ring in \( r-\theta \)
coordinates, we denote the region enclosed by the ring
as \( I \) and its complement as region \( II \), characterized
by mass functions \( m = m_1(v_1) \), and \( m = m_2(v_2) \)
respectively. We proceed now to match two patches
of solution \([8]\), one from each region, along the ring
\( S \).

Any null ray satisfies the equation
\[
-\alpha\ddot{v}^2(\lambda) + 2\dot{v}(\lambda)\dot{\varphi}(\lambda) - J\dot{\psi}(\lambda)\dot{\varphi}(\lambda) + r^2\ddot{\varphi}(\lambda) = 0
\] (10)
where \( \lambda \) is an affine parameter and the dot denotes
derivative with respect to \( \lambda \). Without loss of generality,
we choose the parameter \( \lambda \) to be zero at the Cauchy
horizon and positive beyond that. The geodesic equations are
\[
2\ddot{v} = -\partial_\alpha \ddot{v}^2 + 2r\ddot{\varphi}, \quad 0 = \frac{d}{d\lambda} \left[ -J\ddot{\psi} + 2r\ddot{\varphi} \right]
\] (11)(12)
Equation \([12]\) can be integrated and gives \( \dot{\varphi} = \frac{J}{r^2} \dot{v} \),
where we have set a constant of the motion \( g(\partial_\theta, u) = 0 \), \( u^\mu \) being the 3-velocity of the ring. Equation \([11]\)
then becomes
\[
\frac{d}{d\lambda} \left[ \frac{2}{v} \right] = \partial_\alpha \mathcal{M}(r)
\] (13)
and the null condition \([10]\) implies
\[
2\ddot{r} = N^2(r, v)\dot{v}^2.
\] (14)

Defining a function \( R \) such that \( 2\pi R \) is the perimeter of the ring \( S \) and \( z(\lambda) := 2R(\lambda)\dot{\varphi}(\lambda) \) it
is straightforward to show that \([13]\) and \([14]\) yield the matching conditions
\[
m_i(v_i(\lambda)) = \mathcal{M}(R(\lambda)) + R(\lambda)\mathcal{M}'(R(\lambda)) - \dot{z}_i(\lambda)
\] (15)
\[
v_i(\lambda) = 2\int^\lambda R(\zeta)/z_i(\zeta) \, d\zeta,
\] (16)
\[
z_i(\lambda) = R(\lambda) \left[ Z_i + \int_0^\lambda \mathcal{M}'(R(\zeta)) \, d\zeta \right].
\] (17)
where the subscript \( i \) has a value either ‘1’ or ‘2’ to
denote quantities defined in the respective regions. The
terms \( Z_i \) in equation \([17]\) are integration constants.
Given the boundary function \( R \), equations \([15]\) to
\([17]\) determine the evolution of the spacetime.

We define the “mass” of the ring as
\[
\Delta m(\lambda) := m_2(\lambda) - m_1(\lambda) = (Z_1 - Z_2) \dot{R}(\lambda)
\] (18)
Moreover, we define a constant \( M := m_1(\lambda) + \delta m(\lambda) \) as the final mass of the black hole observed in region I after it has absorbed all the ingoing radiation. The term \( \delta m \) is interpreted as the mass of the flux of ingoing radiation. Because the Cauchy horizon corresponds to the limit \( v_1 \to \infty \), we expect that
\[
\lim_{\lambda \to 0^-} \dot{v}_1(\lambda) = \frac{2}{Z_1} = \infty
\]
implying \( Z_1 = 0 \). Since \( \dot{R}(\lambda) \) is expected to be negative inside the black hole the sign of \( Z_2 \) must be positive in order to have a positive-energy ring \( S \).

Equation (17) can be written as
\[
z_i(\lambda) = R(\lambda) \left[ Z_i - 2 k_o \lambda \right], \tag{19}
\]
where \( k_o \) is defined as
\[
k_o := -\frac{1}{2} M'(R(\epsilon)) \tag{20}\]
By the Mean Value Theorem, the small constant \( \epsilon \in (\lambda_0, 0) \). Since \( N^r(r,v) - m(v) \), the slope of \( \mathcal{M} \) must be negative at Cauchy horizon (see Figure 2), and so \( k_o \) is positive definite.

Equation (14) yields an approximation
\[
\dot{R}(\lambda) \approx -\frac{1}{2} \left[ M - m_1(v_1(\lambda)) \right] \dot{v}_1(\lambda) = \frac{1}{2 k_o \lambda} \delta m(\lambda) \tag{21}
\]
when \( |\lambda| \ll 1 \). On the other hand, since
\[
v_i(\lambda) = 2 \int_\lambda^\lambda \frac{R(\zeta)}{z_i(\zeta)} \, d\zeta = \int_\lambda^\lambda \frac{2}{Z_i - 2 k_o \zeta} \, d\zeta,
\]
when the magnitude of \( \lambda \) is small, \( v_i \) can be approximated as
\[
v_1(\lambda) = -\frac{1}{k_o} \ln |\lambda| \quad \text{and} \quad v_2(\lambda) \approx \frac{2}{Z_2} \lambda. \tag{22}
\]
As a result, we obtain
\[
m_2(v_2) \approx M - h \left[ 1 + \frac{1}{k_o v_2} \right] k_o \ln \left[ \frac{Z_2 v_2}{2} \right]^{-p}, \tag{23}
\]
where a power law fall off \( \delta m(\lambda) \sim h v_1^{-p} \) has been assumed \([13]\). Thus \( m_2 \) diverges to positive infinity as \( v_2 \to 0^- \), since \( Z_2 \) and \( k_o \) are positive.

We have shown that, at least in \((2+1)\) dimensions, all the generic features of spherically symmetric mass inflation are preserved when the black hole has angular momentum, as is expected to be the case for \((3+1)\) dimensional black holes \([5]\). We have checked the tidal forces associated with the mass inflation singularity and have found that they produce a bounded distortion of a \((2+1)\) dimensional physical object. Furthermore, it is straightforward to check that the Kretschmann scalar is finite at the singularity, in contrast to the \((3+1)\)-dimensional case. This suggests that classical continuation of the spacetime beyond the mass-inflation singularity is not forbidden as Ori has suggested \([3]\); indeed, the \((2+1)\) dimensional model presented here may permit an explicit construction of such a continuation.

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