REPRESENTATIONS OF NON QUASI-SPLIT UNRAMIFIED U(4) OVER A $p$-ADIC FIELD I: REPRESENTATIONS OF NON-INTEGRAL LEVEL

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Abstract. Let $F_0$ be a non-archimedean local field of odd residual characteristic and let $G$ be the non quasi-split unramified unitary group in four variables defined over $F_0$. In this paper, we give a classification of the irreducible smooth representations of $G$ of non-integral level using the Hecke algebraic method developed by Howe and Moy.

Introduction

Let $F_0$ be a non-archimedean local field of odd residual characteristic and let $G$ be the non quasi-split unramified unitary group in four variables defined over $F_0$. Konno [6] classified the non supercuspidal representations of $G$ modulo supercuspidal representations of proper Levi subgroups of $G$, by the method which Sally and Tadić [14] used for GSp(4). Note that her result does not depend on whether $G$ is unramified over $F_0$ or not. In this paper, we give a classification of the irreducible smooth representations of $G$ of non-integral level using the Hecke algebraic approach developed by Moy [12] for GSp(4).

One of the purposes of this paper is a classification of the supercuspidal representations of $G$. Stevens [18] proved that every irreducible supercuspidal representation of a $p$-adic classical group is irreducibly induced from an open compact subgroup when $p$ is odd. Although the supercuspidal representations of $G$ are exhausted by his construction, the classification has not been completed.

The other purpose of this research is Jacquet-Langlands correspondence of $p$-adic unramified unitary groups in four variables. By the philosophy of Langlands program, there seems to be a certain correspondence of discrete series representations between $G$ and unramified $U(2, 2)$ over $F_0$. In this paper and [8], we can find similar phenomenons on supercuspidal representations of those two groups.

In [11] and [12], Moy gave a classification of the irreducible smooth representations of unramified $U(2, 1)$ and GSp(4) over $F_0$, based on the concepts of nondegenerate representations and Hecke algebra isomorphisms. A nondegenerate representation of GSp(4) is an irreducible representation $\sigma$ of an open compact subgroup which satisfies a certain cuspidality or semisimplicity condition. An important property of nondegenerate representations of GSp(4) is that every irreducible smooth representation of GSp(4) contains some nondegenerate representation. For $\sigma$ a nondegenerate representation of GSp(4), the set of equivalence classes of irreducible representations of GSp(4) which contain $\sigma$ can be identified with the set of equivalence classes of irreducible representations of the Hecke algebra $\mathcal{H}$ associated to $\sigma$. Moy described $\mathcal{H}$ as a Hecke algebra of some smaller group.
and reduced the classification of the irreducible smooth representations of GSp(4) which contain \( \sigma \) to that of a smaller group.

In [13], Moy and Prasad developed the concept of nondegenerate representations into that of unrefined minimal \( K \)-types for reductive \( p \)-adic groups. For classical groups, Stevens [17] gave an explicit construction of unrefined minimal \( K \)-types as fundamental skew strata in terms of the lattice theory by Bushnell and Kutzko [3] and Morris [9].

Throughout this paper, we use the notion of fundamental skew strata introduced by [17] for our nondegenerate representations of \( G \). We give a brief summary of this in Section 1. Let \( F \) be the unramified quadratic extension over \( F_0 \). Then \( G \) is realized as the group of isometries of an \( F/F_0 \)-hermitian form on 4-dimensional \( F \)-vector space \( V \). According to [17], a skew stratum is a 4-tuple \([\Lambda, n, r, \beta]\). A periodic lattice function \( \Lambda \) with a certain duality induces a filtration \( \{ P_k(\Lambda) \}_{k \geq 1} \) on a parahoric subgroup \( P_0(\Lambda) \) of \( G \). Integers \( n > r \geq 0 \) and an element \( \beta \) in the Lie algebra of \( G \) determine a character \( \psi \) of the group \( P_{r+1}(\Lambda) \) which is trivial on \( P_{n+1}(\Lambda) \). Writing \( e(\Lambda) \) for the period of \( \Lambda \), we refer to \( n/e(\Lambda) \) as the level of the stratum. A skew stratum \([\Lambda, n, r, \beta]\) is called fundamental if \( \beta \) is not nilpotent modulo \( p \), and semisimple if \( \beta \) is semisimple and satisfy a good property on the adjoint action. Every irreducible smooth representation of \( G \) of positive level contains a character \( \psi \) induced by a fundamental skew stratum, and the level of the fundamental skew strata contained in an irreducible smooth representation \( \pi \) of \( G \) is an invariant of \( \pi \). We refer to it as the level of \( \pi \).

In section 2, we give a generalization of a result of Moy [11] and [12]. For a skew semisimple stratum \([\Lambda, n, n-1, \beta]\) with tamely ramified algebra \( F[\beta] \) over \( F \), we construct an irreducible representation \( \rho \) of an open compact subgroup \( J \) with the following two properties:

(i) An irreducible smooth representation of \( G \) contains the skew stratum \([\Lambda, n, n-1, \beta]\) if and only if it contains \( \rho \);

(ii) Writing \( G_E \) for the \( G \)-centralizer of \( \beta \), the intertwining of \((J, \rho)\) equals to \( JG_EJ \). For \( g \in G_E \), we have \( JgJ \cap G_E = (J \cap G_E)g(J \cap G_E) \).

From the second property, we guess the Hecke algebra associated to the pair \((J, \rho)\) is isomorphic to a certain Hecke algebra of \( G_E \). It is true when \( G_E \) is compact like in many cases in [11] and [12]. But in general, to construct Hecke algebra isomorphisms, we need to know at least the structure of Hecke algebras of \( p \)-adic classical groups associated to Iwahori higher congruence subgroups.

From Section 3, we start to classify the irreducible smooth representations of \( G \). First, we prove a rigid result on the existence of fundamental skew strata, which implies that this property holds for a set of skew strata with finite \( G \)-orbits modulo \( p \). Based on this, we replace fundamental skew strata with semisimple ones, case by case.

In Sections 4 and 5, we classify representations of \( G \) of level \( n/4 \) and \( n/3 \), respectively. All of these representations are supercuspidal. The algebra \( F[\beta] \) of a semisimple stratum of level \( n/3 \) possesses a simple component whose ramification index is 3. Although we can’t apply the Moy’s construction to this case when \( p \) is 3, the method by Stevens [15] and [16] to construct supercuspidal representations from maximal compact tori does work well.

In Section 6, we give a classification of the irreducible smooth representations of \( G \) of half-integral level. First, we replace fundamental skew strata with semisimple ones. According to the form of the centralizer \( G_E \), there are three kinds of such strata. In two
cases, $G_E$ is compact and in the other case $G_E$ is isomorphic to a product of unramified $U(1, 1)$ over $F_0$ and unramified $U(1)$ over a quadratic ramified extension over $F_0$. In the non compact case, we establish a Hecke algebra isomorphism by checking the relations of elements of the corresponding Hecke algebra.

In Section 7 we compare semisimple skew strata for $G$ of non-integral level with those for unramified $U(2, 2)$ over $F_0$ in [8]. In the point of view of Hecke algebraic method, behavior of supercuspidal representations of those two groups of non-integral level are quite similar. In fact, the value of level and characteristic polynomials are same and the difference of the form of $G_E$ relates to inner forms.

1. Preliminaries

In this section, we recall the notion of fundamental skew strata for $p$-adic classical groups from [3] and [15]. We refer the reader to those papers for more details.

1.1. Filtrations. Let $F$ be a non-archimedean local field of odd residual characteristic equipped with a (possibly trivial) Galois involution $\tau$, and let $\mathfrak{o}_F$ denote the ring of integers in $F$, $\mathfrak{p}_F$ the maximal ideal in $\mathfrak{o}_F$, $k_F = \mathfrak{o}_F/\mathfrak{p}_F$ the residue field.

Let $F_0$ denote the $\tau$-fixed subfield of $F$. We denote by $\mathfrak{o}_0$, $\mathfrak{p}_0$, $k_0$ the objects for $F_0$ analogous to those above for $F$, and by $q$ the number of elements in $k_0$.

We select a uniformizer $\varpi_F$ in $F$ so that $\varpi_F^q = -\varpi_F$ if $F/F_0$ is ramified. Otherwise we take $\varpi_F$ in $F_0$.

Let $V$ be an $N$-dimensional $F$-vector space equipped with a nondegenerate hermitian form $h$ with respect to $F/F_0$. We put $A = \text{End}_F(V)$, $\tilde{G} = A^\times$ and denote by $\sigma$ the involution on $A$ induced by $h$. We also put $G = \{ g \in \tilde{G} \mid g\sigma(g) = 1 \}$, the corresponding classical group over $F_0$, $A_+ = \{ X \in A \mid X + \sigma(X) = 0 \} \simeq \text{Lie}(G)$.

Recall from [3] (2.1) that an $\mathfrak{o}_F$-lattice sequence in $V$ is a function $\Lambda$ from $\mathbb{Z}$ to the set of $\mathfrak{o}_F$-lattices in $V$ such that

(i) $\Lambda(i) \supset \Lambda(i + 1)$ for all $i \in \mathbb{Z}$;
(ii) there exists an integer $e(\Lambda)$ such that $\varpi_F \Lambda(i) = \Lambda(i + e(\Lambda))$ for all $i \in \mathbb{Z}$.

The integer $e(\Lambda)$ is called the $\mathfrak{o}_F$-period of $\Lambda$. We say that an $\mathfrak{o}_F$-lattice sequence $\Lambda$ is strict if $\Lambda(i) \supset \Lambda(i + 1)$ for all $i \in \mathbb{Z}$.

For $L$ an $\mathfrak{o}_F$-lattice in $V$, we define its dual lattice $L^\#$ by $L^\# = \{ v \in V \mid h(v, L) \subset \mathfrak{o}_F \}$. Recall from [15] Section 3 that an $\mathfrak{o}_F$-lattice sequence $\Lambda$ in $V$ is called self-dual if there exists an integer $d(\Lambda)$ such that $\Lambda(i)^\# = \Lambda(d(\Lambda) - i)$ for all $i \in \mathbb{Z}$.

Recall from [8] §1.4 that a $C$-sequence in $V$ is a self-dual $\mathfrak{o}_F$-lattice sequence $\Lambda$ in $V$ which satisfies

$C(i)$ $\Lambda(2i + 1) \supset \Lambda(2i + 2)$ for all $i \in \mathbb{Z}$,
$C(ii)$ $e(\Lambda)$ is even and $d(\Lambda)$ is odd.

We remark that this is a realization of a $C$-chain in [9] §4.3 as an $\mathfrak{o}_F$-lattice sequence.

An $\mathfrak{o}_F$-lattice sequence $\Lambda$ in $V$ induces a filtration $\{ a_n(\Lambda) \}_{n \in \mathbb{Z}}$ on $A$ by

$a_n(\Lambda) = \{ X \in A \mid X\Lambda(i) \subset \Lambda(i + n) \text{ for all } i \in \mathbb{Z} \}, \quad n \in \mathbb{Z}.$

Note that an $\mathfrak{o}_F$-lattice sequence $\Lambda$ in $V$ is self-dual if and only if $\sigma(a_n(\Lambda)) = a_n(\Lambda)$, $n \in \mathbb{Z}$. This filtration determines a kind of “valuation” $\nu_\Lambda$ on $A$ by

$\nu_\Lambda(x) = \sup\{ n \in \mathbb{Z} \mid x \in a_n(\Lambda) \}, \quad x \in A \setminus \{ 0 \},$

with the usual understanding that $\nu_\Lambda(0) = \infty$. 

Let $\Lambda$ be an $\mathfrak{o}_F$-lattice sequence in $V$. For $k \in \mathbb{Z}$, we define an $\mathfrak{o}_F$-lattice sequence $\Lambda + k$ by $(\Lambda + k)(i) = \Lambda(i + k)$, $i \in \mathbb{Z}$. Then we have $\mathfrak{a}_n(\Lambda) = \mathfrak{a}_n(\Lambda + k)$, $n \in \mathbb{Z}$. We refer to $\Lambda + k$ as a translate of $\Lambda$. For $g \in G$, we define an $\mathfrak{o}_F$-lattice sequence $g\Lambda$ by $(g\Lambda)(i) = g\Lambda(i)$, $i \in \mathbb{Z}$. Note that if $\Lambda$ is self-dual, then $\Lambda + k$ and $g\Lambda$ are also self-dual.

For $\Gamma$ an $\mathfrak{o}_F$-lattice in $A$, we define its dual $\Gamma^*$ by $\Gamma^* = \{X \in A \mid \text{tr}_{A/F_0}(X^t) \subset p_0\}$, where $\text{tr}_{A/F_0}$ denotes the composition of traces $\text{tr}_{F/F_0} \circ \text{tr}_{A/F}$. Recall from [3] (2.10) that, if $\Lambda$ is an $\mathfrak{o}_F$-lattice sequence in $V$, then we have $\mathfrak{a}_n(\Lambda)^* = \mathfrak{a}_1 - \mathfrak{a}_n(\Lambda)$, $n \in \mathbb{Z}$.

For $S$ a subset of $G$, we write $S = S \cap A\pi$. Let $\Lambda$ be a self-dual $\mathfrak{o}_F$-lattice sequence in $V$. We set $P_0(\Lambda) = G \cap \mathfrak{a}_0(\Lambda)$ and $P_n(\Lambda) = G \cap (1 + \mathfrak{a}_n(\Lambda))$, for $n \in \mathbb{Z}$, $n \geq 1$.

We fix an additive character $\psi_0$ of $F_0$ with conductor $p_0$. Let $^\wedge$ denote the Pontrjagin dual. For $x$ a real number, we write $[x]$ for the greatest integer less than or equal to $x$.

**Proposition 1.1.** Let $\Lambda$ be a self-dual $\mathfrak{o}_F$-lattice sequence in $V$ and let $n, r \in \mathbb{Z}$ satisfy $n > r \geq [n/2] \geq 0$. Then the map $x \mapsto x - 1$ induces an isomorphism $P_{r+1}(\Lambda)/P_{n+1}(\Lambda) \simeq \mathfrak{a}_{r+1}(\Lambda)/\mathfrak{a}_{n+1}(\Lambda)$ and there exists an isomorphism of finite abelian groups

$$\mathfrak{a}_{-n}(\Lambda)/\mathfrak{a}_{-r}(\Lambda) \simeq (P_{r+1}(\Lambda)/P_{n+1}(\Lambda))^\wedge; b + \mathfrak{a}_{-r}(\Lambda) \mapsto \psi_b,$$

where $\psi_b(x) = \psi_0(\text{tr}_{A/F_0}(b(x - 1)))$, $x \in P_{r+1}(\Lambda)$.

**1.2. Skew strata.**

**Definition 1.2** ([3] (3.1), [15] Definition 4.5). (i) A stratum in $A$ is a 4-tuple $[\Lambda, n, r, \beta]$ consisting of an $\mathfrak{o}_F$-lattice sequence $\Lambda$ in $V$, integers $n, r$ such that $n > r \geq 0$, and an element $\beta$ in $\mathfrak{a}_{-n}(\Lambda)$. We say that two strata $[\Lambda, n, r, \beta_i]$, $i = 1, 2$, are equivalent if $\beta_1 \equiv \beta_2$ (mod $\mathfrak{a}_{-r}(\Lambda)$).

(ii) A stratum $[\Lambda, n, r, \beta]$ in $A$ is called skew if $\Lambda$ is self-dual and $\beta \in \mathfrak{a}_{-n}(\Lambda)$.

The fraction $n/e(\Lambda)$ is called the level of the stratum. If $n > r \geq [n/2]$, then by Proposition [14] an equivalence class of skew strata $[\Lambda, n, r, \beta]$ corresponds to a character $\psi_\beta$ of $P_{r+1}(\Lambda)/P_{n+1}(\Lambda)$.

For $g \in \tilde{G}$ and $x \in A$, we write $\text{Ad}(g)(x) = gxg^{-1}$. We define the formal intertwining of a skew stratum $[\Lambda, n, r, \beta]$ to be

$$I_G[\Lambda, n, r, \beta] = \{g \in G \mid (\beta + \mathfrak{a}_{-r}(\Lambda)) \cap \text{Ad}(g)(\beta + \mathfrak{a}_{-r}(\Lambda)) \neq \emptyset\}.$$  

If $n > r \geq [n/2]$, then it is just the intertwining of the character $\psi_\beta$ of $P_{r+1}(\Lambda)$ in $G$.

For $[\Lambda, n, n-1, \beta]$ a stratum in $A$, we set $y_\beta = \varpi_F^{n/k}\beta^{e(\Lambda)/k} \in \mathfrak{a}_0(\Lambda)$, where $k = (e(\Lambda), n)$. Then the characteristic polynomial $\Phi_\beta(X)$ of $y_\beta$ lies in $\mathfrak{o}_F[X]$. We define the characteristic polynomial $\phi_\beta(X) \in k_F[X]$ of the stratum to be the reduction modulo $p_F$ of $\Phi_\beta(X)$.

**Definition 1.3** ([2] (2.3)). A stratum $[\Lambda, n, r, \beta]$ in $A$ is called fundamental if $\phi_\beta(X) \neq X^n$.

Let $\pi$ be a smooth representation of $G$ and $[\Lambda, n, r, \beta]$ a skew stratum with $n > r \geq [n/2]$. We say that $\pi$ contains $[\Lambda, n, r, \beta]$ if the restriction of $\pi$ to $P_{r+1}(\Lambda)$ contains the corresponding character $\psi_\beta$. A smooth representation $\pi$ of $G$ is called of positive level if $\pi$ has no non-zero $P_1(\Lambda)$-fixed vector, for $\Lambda$ any self-dual $\mathfrak{o}_F$-lattice sequence in $V$.

Let $\pi$ be an irreducible smooth representation of $G$ of positive level. Due to [17] Theorem 2.11, $\pi$ contains a fundamental skew stratum $[\Lambda, n, n-1, \beta]$. Let $[\Lambda, n, n-1, \beta]$ be any skew stratum contained in $\pi$. Then by the philosophy of minimal $K$-types, $[\Lambda, n, n-1, \beta]$ is fundamental if and only if the level $n/e(\Lambda)$ of the stratum is smallest.
among in those of skew strata occurring in $\pi$. Thus we can define the level of $\pi$ by the level of the fundamental skew strata contained in $\pi$.

Recall that when skew strata $[\Lambda, n, n-1, \beta]$ and $[\Lambda', n', n'-1, \beta']$ are contained in an irreducible smooth representation of $G$, there exists $g \in G$ such that

$$(\beta + a_{1-n}(\Lambda) \cap \text{Ad}(g)(\beta' + a_{1-n'}(\Lambda') \cap \text{Ad}(g)) \neq \emptyset. \quad (1.1)$$

We therefore see that for an irreducible smooth representation $\pi$ of $G$ of positive level, the characteristic polynomial of a fundamental skew stratum contained in $\pi$ depends only on $\pi$. We refer to it as the characteristic polynomial of $\pi$.

1.3. Semisimple strata.

**Definition 1.4** ([2] (1.5.5), [3] (5.1)). A stratum $[\Lambda, n, r, \beta]$ in $A$ is called simple if

(i) the algebra $E = F[\beta]$ is a field, and $\Lambda$ is an $\sigma_E$-lattice sequence;

(ii) $\nu_\Lambda(\beta) = -n$;

(iii) $\beta$ is minimal over $F$ in the sense of [2] p. 41.

**Remark 1.5.** The definition above is a special case of that in [2]. With the notion of [2], our simple stratum is a simple stratum $[\Lambda, n, r, \beta]$ with $k_0(\beta, \Lambda) = -n$.

The following criterion of the simplicity of strata is well known.

**Proposition 1.6** ([7] Proposition 1.5). Let $\Lambda$ be a strict $\sigma_F$-lattice sequence in $V$ with $e(\Lambda) = N$ and let $n$ be an integer such that $(n, N) = 1$. Suppose that $[\Lambda, n, r, \beta]$ is a fundamental stratum in $A$. Then $F[\beta]$ is a totally ramified extension of degree $N$ over $F$ and $[\Lambda, n, r, \beta]$ is simple.

Let $[\Lambda, n, r, \beta]$ be a stratum in $A$. We assume that there is a non-trivial $F$-splitting $V = V^1 \oplus V^2$ such that

(i) $\Lambda(i) = \Lambda^1(i) \oplus \Lambda^2(i)$ for all $i \in \mathbb{Z}$, where $\Lambda^j(i) = \Lambda(i) \cap V^j$, for $j = 1, 2$;

(ii) $\beta V^j \subset V^j$ for $j = 1, 2$.

For $j = 1, 2$, we write $\beta_j = \beta|_{V^j}$. By [3] (2.9), we get a stratum $[\Lambda^j, n, r, \beta_j]$ in $\text{End}_F(V^j)$.

Recall from [4] (3.6) that a stratum $[\Lambda, n, r, \beta]$ in $A$ is called split if

(iii) $\nu_{\Lambda^1}(\beta_1) = -n$ and $X$ does not divide $\phi_{\beta_1}(X)$;

(iv) either $\nu_{\Lambda^2}(\beta_2) > -n$, or else all the following conditions hold:

(a) $\nu_{\Lambda^2}(\beta_2) = -n$ and $X$ does not divide $\phi_{\beta_2}(X)$,

(b) $(\phi_{\beta_1}(X), \phi_{\beta_2}(X)) = 1$.

**Definition 1.7** ([15] Definition 4.8, [17] Definition 2.10). (i) (Inductive definition on the dimension of $V$) A stratum $[\Lambda, n, r, \beta]$ is called semisimple if it is simple, or else it is split as above and satisfies the following conditions:

(a) $[\Lambda^1, n, r, \beta_1]$ is simple;

(b) $[\Lambda^2, n_2, r, \beta_2]$ is semisimple or $\beta_2 = 0$, where $n_2 = \max\{-\nu_{\Lambda^2}(\beta_2), r + 1\}$.

(ii) A skew stratum in $A$ is called split if it is split with respect to an orthogonal $F$-splitting $V = V^1 \perp V^2$.

If a skew stratum $[\Lambda, n, r, \beta]$ is split with respect to $V = V^1 \perp V^2$, then $\Lambda^j$ is a self-dual $\sigma_F$-lattice sequence in $(V^j, h|_{V^j})$ with $d(\Lambda^j) = d(\Lambda)$, for $j = 1, 2$. 
Remark 1.8. Let \([\Lambda, n, r, \beta]\) be a skew semisimple stratum in \(A\). Then the \(G\)-centralizer \(G_E\) of \(\beta\) is a product of classical groups over extensions of \(F_0\).

We claim that if \(F/F_0\) is quadratic unramified, then \(G_E\) is a product of unramified unitary groups. It suffices to prove this in the simple case.

Suppose that \([\Lambda, n, n - 1, \beta]\) is a skew simple stratum. Since \(F/F_0\) is unramified, there is \(\varepsilon \in \mathfrak{o}_0^\times\) such that \(F = F_0[\sqrt{\varepsilon}]\). Let \(E_0\) denote the \(\sigma\)-fixed subfield of \(E\). Then \(E/E_0\) is a quadratic extension and \(G_E\) is the unitary group over \(E_0\), corresponding to the involutive algebra \((\text{End}_E(V), \sigma)\). Suppose that \(E/E_0\) is ramified. We can take a uniformizer \(\varpi_E\) of \(E\) so that \(\sigma(\varpi_E) = -\varpi_E\). Then the element \(\varpi_E\sqrt{\varepsilon} \in E_0\) is a uniformizer of \(E\). This contradicts the assumption.

1.4. Representations associated maximal tori. We recall from [15] and [16] the construction of supercuspidal representations associated to maximal compact tori.

Let \([\Lambda, n, n - 1, \beta]\) be a skew semisimple stratum such that the \(G\)-centralizer \(G_E\) of \(\beta\) is a maximal torus. Then \(G_E\) lies in \(P_0(\Lambda)\) and by [15] Theorem 4.6, we get

\[
I_G[\Lambda, n, [n/2], \beta] = G_E P_{[(n+1)/2]}(\Lambda).
\]

(1.2)

Put \(J = G_E P_{[(n+1)/2]}(\Lambda)\), \(J^1 = P_1(\Lambda_E) P_{[(n+1)/2]}(\Lambda)\), and \(H^1 = P_1(\Lambda_E) P_{[n/2]+1}(\Lambda)\), where \(P_1(\Lambda_E) = G_E \cap P_1(\Lambda)\). The character \(\psi_\beta\) of \(P_{[n/2]+1}(\Lambda)\) can extend to a character \(\theta\) of \(H^1\) since \(H^1/P_{[n/2]+1}(\Lambda)\) is abelian. Thank to [16] Proposition 4.1, there exists a unique irreducible representation \(\eta_\theta\) of \(J^1\) which contains \(\theta\). Moreover the restriction of \(\eta_\theta\) to \(H^1\) is a multiple of \(\theta\) and \(\dim \eta_\theta = [J^1 : H^1]^{1/2}\), which is a power of \(q\).

The order of the finite abelian group \(J/J^1 \cong G_E/P_1(\Lambda_E)\) is coprime to \(q\), and hence \(\eta_\theta\) can extend an irreducible representation \(\kappa_\theta\) of \(J\). For any \(\chi \in (G_E/P_1(\Lambda_E))^\wedge\), we have

\[
I_G(\chi \otimes \kappa_\theta) = I_G(\psi_\beta) = J,
\]

so that the compactly induced representation \(\text{Ind}_J^G(\chi \otimes \kappa_\theta)\) is irreducible and supercuspidal.

It is easy to check that every irreducible smooth representation of \(G\) which contains \([\Lambda, n, [n/2], \beta]\) can be constructed in this way.

We classify these representations. Let \(\theta\) and \(\theta'\) be extensions of \(\psi_\beta\) to \(H^1\). Let \(\kappa_\theta\) and \(\kappa_{\theta'}\) be as above. Let \(\chi_1\) and \(\chi_2\) be characters of \(G_E/P_1(\Lambda_E)\). Suppose that \(\text{Ind}_J^G(\chi_1 \otimes \kappa_\theta) \simeq \text{Ind}_J^G(\chi_2 \otimes \kappa_{\theta'}\)). Then there is \(g \in G\) which intertwines \(\chi_1 \otimes \kappa_\theta\) and \(\chi_2 \otimes \kappa_{\theta'}\). Thinking of the restriction to \(P_{[n/2]+1}(\Lambda)\), we obtain \(g \in I_G(\psi_\beta) = J\), and hence \(\chi_1 \otimes \kappa_\theta \simeq \chi_2 \otimes \kappa_{\theta'}\). Restricting it to \(H^1\), we get \(\theta = \theta'\). The representation \(\kappa_{\theta'}\) is isomorphic to \(\chi'' \otimes \kappa_\theta\), for some \(\chi'' \in (G_E/P_1(\Lambda_E))^\wedge\), and we get \(\chi = \chi''\).

Remark 1.9. The number of the irreducible smooth representations of \(G\) containing \([\Lambda, n, [n/2], \beta]\) equals to \([G_E : G_E \cap P_{[n/2]+1}(\Lambda)]\).

1.5. Uniqueness of lattice sequences. Let \([\Lambda, n, n - 1, \beta]\) be a skew semisimple stratum associated to an orthogonal \(F\)-splitting \(V = V_1 \perp \ldots \perp V_k\). Then \(\Lambda = \Lambda_1 \perp \ldots \perp \Lambda_k\) and \(\beta = \beta_1 + \cdots + \beta_k\), where \(\Lambda_i(j) = \Lambda(j) \cap V^i\), \(j \in \mathbb{Z}\) and \(\beta_i = \beta_i|_{V^i}\), for \(1 \leq i \leq k\). If we write \(e = e(\Lambda) = e(\Lambda^t)\) and \(k = (n, e)\), then we have \(\Phi_\beta(X) = \prod_i \Phi_i(X)\), where \(\Phi_i(X)\) is the characteristic polynomial of \(y_i = \varpi_F^{n/k} \beta_i|_{V^i}\) \(\in \text{End}_F(V^i)\).

By the definition of split strata, \(\Phi_i(X)\) and \(\Phi_j(X)\) are coprime modulo \(p_F\), for \(i \neq j\). Note that if \([\Lambda', n, n - 1, \beta'_i]\) is simple, then \(\Phi_i(X)\) mod \(p_F\) is a power of an irreducible polynomial in \(k_F\). If \(\beta_i = 0\), then we have \(\Phi_i(X)\) mod \(p_F = X^{N_i}\), where \(N_i = \dim_F V^i\). So we see that \(V^i\) is just the kernel of \(\Phi_i(y_\beta)\), for \(1 \leq i \leq k\). In particular, \(y_\beta\) determines the \(F\)-splitting \(V = V_1 \perp \ldots \perp V_k\) uniquely.
The algebra $E_i = F[\beta_i]$ is an extension over $F$ and $\Lambda^i$ is an $\mathfrak{so}_E$-lattice sequence in $V^i$. We write $B_i$ for the $\text{End}_F(V^i)$-centralizer of $\beta_i$. Then the involutive algebra $(B_i, \sigma)$ defines a nondegenerate hermitian form $h_{E_i}$ on the $E_i$-space $V^{i}$ up to scalar in $E_i^\times$. For $1 \leq i \leq k$, since $\mathfrak{a}_j(\Lambda) \cap B_i$ is $\sigma$ stable for all $j \in \mathbb{Z}$, the sequence $\Lambda^i$ is a self-dual $\mathfrak{so}_{E_i}$-lattice sequence in $(V^i, h_{E_i})$.

**Lemma 1.10.** Suppose that the space $(V^i, h_{E_i})$ is anisotropic for all $1 \leq i \leq k$. Then we can recover $\Lambda$ from $\beta$, $n$, $e(\Lambda)$ and $d(\Lambda)$.

**Proof.** The assumption implies that $\Lambda^i$ is a unique self-dual $\mathfrak{so}_{E_i}$-lattice sequence in $(V_i, h_{E_i})$ of $\mathfrak{so}_F$-period $e(\Lambda)$, up to translation. Recall that $\Lambda^i$ is a self-dual $\mathfrak{so}_F$-lattice sequence in $(V^i, h|_{V^i})$ with $d(\Lambda^i) = d(\Lambda)$. So $d(\Lambda)$ determines $\Lambda^i$, for all $1 \leq i \leq k$. Therefore we can recover $\Lambda$ by the equation $\Lambda = \Lambda^1 \perp \ldots \perp \Lambda^k$. \hfill \Box

1.6. **Hecke algebras.** Let $G$ be a unimodular, locally compact, totally disconnected topological group, $J$ an open compact subgroup of $G$, and $(\sigma, W)$ an irreducible smooth representation of $J$. For $g \in G$, we write $\sigma^g$ for the representation of $J^g = g^{-1}Jg$ defined by $\sigma^g(x) = \sigma(gxg^{-1})$, $x \in J^g$. We define the intertwining of $\sigma$ in $G$ by

$$I_G(\sigma) = \{ g \in G \mid \text{Hom}_{J \cap J^g}(\sigma, \sigma^g) \neq 0 \}.$$

Let $(\widetilde{\sigma}, \widetilde{W})$ denote the contragradient representation of $(\sigma, W)$. The Hecke algebra $\mathcal{H}(G//J, \sigma)$ is the set of compactly supported functions $f : G \to \text{End}_C(\widetilde{W})$ such that

$$f(kgk') = \widetilde{\sigma}(k)f(g)\widetilde{\sigma}(k'), \quad k, k' \in J, \ g \in G.$$

Let $dg$ denote the Haar measure on $G$ normalized so that the volume vol$(J)$ of $J$ is 1. Then $\mathcal{H}(G//J, \sigma)$ becomes an algebra under convolution relative to $dg$. Recall from [2] (4.1.1) that the support of $\mathcal{H}(G//J, \sigma)$ is the intertwining of $\widetilde{\sigma}$ in $G$, that is,

$$I_G(\widetilde{\sigma}) = \bigcup_{f \in \mathcal{H}(G//J, \sigma)} \text{supp}(f).$$

Since $J$ is compact, there exists a $J$-invariant, positive definite hermitian form on $\widetilde{W}$. This form induces an involution $X \mapsto \overline{X}$ on $\text{End}_C(\widetilde{W})$. For $f \in \mathcal{H}(G//J, \sigma)$, we define $f^* \in \mathcal{H}(G//J, \sigma)$ by $f^*(g) = \overline{f(g^{-1})}$, $g \in G$. Then the map $\ast : \mathcal{H}(G//J, \sigma) \to \mathcal{H}(G//J, \sigma)$ is an involution on $\mathcal{H}(G//J, \sigma)$.

Let $\text{Irr}(G)$ denote the set of equivalence classes of irreducible smooth representations of $G$ and $\text{Irr}(G)^{(J, \sigma)}$ the subset of $\text{Irr}(G)$ consisting of elements whose $\sigma$-isotypic components are not zero. Let $\text{Irr}\mathcal{H}(G//J, \sigma)$ denote the set of equivalence classes of irreducible representations of $\mathcal{H}(G//J, \sigma)$. Then, by [2] (4.2.5), there is a bijection $\text{Irr}(G)^{(J, \sigma)} \simeq \text{Irr}\mathcal{H}(G//J, \sigma)$.

## 2. A generalization of a result of Moy

Let $[\Lambda, n, n - 1, \beta]$ be a skew semisimple stratum in $A$ associated to an orthogonal $F$-splitting $V = V^1 \perp \ldots \perp V^k$. As usual, we write $\beta = \beta_1 + \cdots + \beta_k$, where $\beta_i = \beta|_{V^i}$. We have $E = F[\beta] = \bigoplus_{1 \leq i \leq k} E_i$, where $E_i = F[\beta_i]$. Throughout this section, we assume that $E_i$ is tamely ramified over $F$ for all $1 \leq i \leq k$. We put $A_i^j = \text{Hom}_F(V^j, V^i)$, for $1 \leq i, j \leq k$. When we write $B_i$ for the $A_i^i$-centralizer of $\beta_i$, the $A$-centralizer $B$ of $\beta$ equals to $\bigoplus_{1 \leq i \leq k} B_i$. Let $B^\perp$ denote the orthogonal
complement of $B$ in $A$ with respect to the pairing induced by $\text{tr}_{A/F}$, and let $B_i^\perp$ denote that of $B_i$ with respect to $\text{tr}_{A_i/F_i}$. Then we have

$$B^\perp = \bigoplus_{i \neq j} A_i^j \oplus \bigoplus_i B_i^\perp.$$  \hfill (2.3)

and $A = B \oplus B^\perp$. Note that the set $B$ and $B^\perp$ are $\sigma$-stable since $\beta \in A_-$.

**Proposition 2.1.** For $k \in \mathbb{Z}$, we have $a_k(\Lambda) = a_k(\Lambda) \cap B \oplus a_k(\Lambda) \cap B^\perp$.

**Proof.** By \cite{3} Proposition 2.9, we have $a_k(\Lambda) = \bigoplus_{i,j} a_k(\Lambda) \cap A_i^j$ and $a_k(\Lambda) \cap A_i^{i+} = a_k(\Lambda)$, for $1 \leq i \leq k$.

Suppose that $[\Lambda^i, n, n - 1, \beta_i]$ is simple. It follows from \cite{3} Remark (1.3.8) (ii) that the $(B_i, B_i)$-bimodule projection $s_i : A_i^{i+} \to B_i$ with kernel $B_i^\perp$ satisfies $s_i(a_k(\Lambda^i)) = a_k(\Lambda) \cap B$ because we are assuming that $E_i$ is tamely ramified over $F$. Hence we get $a_k(\Lambda) \cap A_i^{i+} = a_k(\Lambda) \cap B_i \oplus a_k(\Lambda) \cap B_i^\perp$. If $\beta_i = 0$, then $B_i = A_i^{i+}$. This completes the proof.  \hfill $\Box$

For $k \in \mathbb{Z}$, we abbreviate $a_k = a_k(\Lambda)$, $a_k^j = a_k(\Lambda) \cap B$ and $a_k^\perp = a_k(\Lambda) \cap B^\perp$. Define $\sigma$-stable $\mathfrak{g}_F$-lattices $\mathfrak{J}$ and $\mathfrak{J}_+$ in $A$ by

$$\mathfrak{J} = a'_n \oplus a_{[n+1]/2}^n, \quad \mathfrak{J}_+ = a'_n \oplus a_{[n/2]+1}^n,$$

and open compact subgroups $J$ and $J_+$ of $G$ by

$$J = (1 + \mathfrak{J}) \cap G, \quad J_+ = (1 + \mathfrak{J}_+) \cap G,$$  \hfill (2.5)

as in \cite{12} (4.16).

Since $J_+ \subset P_{[n/2]+1}(\Lambda)$, the quotient $J_+ / P_{n+1}(\Lambda)$ is abelian. As usual, we get an isomorphism of finite abelian groups

$$(a_{-n})_+ / (\mathfrak{J}_+)_+ \simeq (J_+ / P_{n+1}(\Lambda))^\Lambda; \quad b + (\mathfrak{J}_+)_+ \mapsto \Psi_b,$$

where

$$\Psi_b(p) = \psi_0(\text{tr}_{A/F_0}(b(p - 1))), \quad p \in J_+.$$  \hfill (2.6)

Due to Proposition 2.1 and \cite{3} (2.10), we get

$$\mathfrak{J}_+ = a_{-n}^+ \oplus a_{[n/2]}^+.$$  \hfill (2.7)

For $X \in A$, we write $\text{ad}(\beta)(X) = \beta X - X \beta$. Since $\beta \in (a_{-n})_+$, the map $\text{ad}(\beta)$ induces a quotient map $\text{ad}(\beta) : a_k^+ / a_{k+1}^+ \to a_{k-n}^+ / a_{k-n+1}^+$, for $k \in \mathbb{Z}$.

**Lemma 2.2.** For $k \in \mathbb{Z}$, the map $\text{ad}(\beta) : a_k^+ / a_{k+1}^+ \to a_{k-n}^+ / a_{k-n+1}^+$ is an isomorphism.

**Proof.** By the periodicity of the filtration $\{a_k(\Lambda)\}$, it suffices to prove that the induced map is injective for all $k \in \mathbb{Z}$.

Recall that $a_i^j = \bigoplus_{i \neq j} a_i \cap A_i^j \oplus \bigoplus_i a_i \cap B_i^\perp$, for $l \in \mathbb{Z}$. By \cite{3} §3.7 Lemma 2, $\text{ad}(\beta)$ maps $a_k \cap A_i^j$ onto $a_{k-n} \cap A_i^j$, for $i \neq j$. Thus the assertion is reduced to the simple case.

Let $[\Lambda, n, n - 1, \beta]$ be a simple stratum in $A$. Let $e$ denote the $\mathfrak{g}_E$-period of $\Lambda$ and let $V'$ be an $e$-dimensional $E$-vector space. Then there is a strict $\mathfrak{g}_E$-lattice sequence $\Lambda'$ in $V'$ of $\mathfrak{g}_E$-period $e$. Define a strict $\mathfrak{g}_E$-lattice sequence $\Lambda''$ in $V'' = V \oplus V'$ by $\Lambda''(i) = \Lambda(i) \oplus \Lambda'(i)$, $i \in \mathbb{Z}$. We confuse $\beta$ with $\beta \cdot 1_{V''}$. Then we get a simple stratum $[\Lambda'', n, n - 1, \beta]$ in $A'' = \text{End}_F(V'')$.

Let $B''$ denote the $A''$-centralizer of $\beta$. It follows from \cite{2} (1.4.9) that if $x \in a_0(\Lambda'')$ satisfies $\text{ad}(\beta)(x) \in a_{-r-n}(\Lambda'')$, for $r \geq 1$, then we have $x \in B'' \oplus a_r(\Lambda'')$. For $k \geq 0$, we see
that if \( x \in a_k(\Lambda) \cap B^\perp \) satisfies \( \text{ad}(\beta)(x) \in a_{k-n+1}(\Lambda) \), then \( x \) lies in \((B''_n + a_{k+1}(\Lambda'')) \cap B^\perp = a_{k+1}(\Lambda) \cap B^\perp \), by \[^3\] Proposition 2.9. By the periodicity of \( \{a_i(\Lambda)\}_{i \in \mathbb{Z}} \), this holds for all \( k \in \mathbb{Z} \). This completes the proof. \( \square \)

Since \( \beta \) is skew, we obtain the following corollary:

**Corollary 2.3.** For \( k \in \mathbb{Z} \), the map \( \text{ad}(\beta) : (a^+_k)/(a^+_{k+1}) \rightarrow (a^+_{k-n})/(a^+_{k-n+1}) \) is an isomorphism.

**Proposition 2.4** \(^{[12]}\) Lemma 4.4. Suppose that an element \( \gamma \in \beta + (a_{1-n})_- \) lies in \( B_- \) modulo \( (a^+_{1-n})_- \) for some integer \( k \geq 1 \). Then, there exists \( p \in P_k(\Lambda) \) such that \( \text{Ad}(p)(\gamma) \in \beta + (a'_{1-n})_- \).

**Proof.** Exactly the same as the proof of \(^{[12]}\) Lemma 4.4. \( \square \)

As an immediate corollary of the proof we have

**Corollary 2.5** \(^{[12]}\) Corollary 4.5. \( \text{Ad}(J)(\beta + a'_{1-n}) = \beta + (3^+_n)_- \).

**Proposition 2.6** \(^{[12]}\) Theorem 4.1. Let \( \pi \) be an irreducible smooth representation of \( G \). Then \( \pi \) contains \( [\Lambda, n, n-1, \beta] \) if and only if \( \pi \) contains \( (J_+, \Psi_\beta) \).

**Proof.** Since \( \Psi_\beta \) is an extension of \( \psi_\beta \) to \( J_+ \), it is obvious that if \( \pi \) contains \( \Psi_\beta \), then \( \pi \) contains \( \psi_\beta \).

Suppose that \( \pi \) contains \( [\Lambda, n, n-1, \beta] \). Then \( \pi \) contains an extension of \( \psi_\beta \) to \( P_{[n/2]+1}(\Lambda) \). This extension has the form \( \psi_\gamma \), for some \( \gamma \in \beta + (a_{1-n})_- \). By Proposition 2.4, \(^{[12]}\) replacing \( \psi_\gamma \) with a \( P_1(\Lambda) \)-extension, we may assume \( \gamma \in \beta + (a'_{1-n})_- \). Then the restriction of \( \psi_\gamma \) to \( J_+ \) is equal to \( \Psi_\beta \). This completes the proof. \( \square \)

We put \( G_E = G \cap B \).

**Proposition 2.7.** \( I_G(\Psi_\beta) = JG_EJ \).

**Proof.** We see that an element \( g \in G \) lies in \( I_G(\Psi) \) if and only if \( \text{Ad}(g)(\beta + (3^+_n)_-) \cap (\beta + (3^+_n)_-) \neq \emptyset \). We have \( JG_E(\Psi)J = I_G(\Psi) \) and \( G_E \subset I_G(\Psi) \), and hence \( JG_EJ \subset I_G(\Psi) \).

Let \( g \in I_G(\Psi) \). Due to Corollary 2.5, there exists an element \( k \in JgJ \) such that \( \text{Ad}(k)(\beta + (a'_{1-n})_-) \cap (\beta + (a'_{1-n})_-) \neq \emptyset \). Take \( x, y \in (a^+_{1-n})_- \) so that \( \text{ad}(\beta)(k) = kx - yk \). If we write \( k^\perp \) for the \( B^\perp \)-component of \( k \), then we get \( \text{ad}(\beta)(k^\perp) = k^\perp x - yk^\perp \). Suppose \( k^\perp \in a_l \), for some \( l \in \mathbb{Z} \). Then we have \( \text{ad}(\beta)(k^\perp) \in a_{l-n+1} \), and hence by Lemma 2.2, \( k^\perp \in a_{l+1} \). This implies \( k^\perp = 0 \) and hence \( k \in G_E \). This completes the proof. \( \square \)

Since \( [J, J] \subset P_n(\Lambda) \subset J_+ \), we can define an alternating form \( \theta \) on \( J/J_+ \) by \( \theta(x, y) = \Psi_\beta([x, y]) \), \( x, y \in J \).

**Lemma 2.8.** The form \( \theta \) is nondegenerate.

**Proof.** Let \( 1 + x \) be an element in \( J \) such that \( \Psi_\beta([1 + x, 1 + y]) = 1 \), for all \( 1 + y \in J \).

Since \( \Psi_\beta([1 + x, 1 + y]) = \psi_0(\text{tr}_{A/F_0}(\beta(xy - yx))) = \psi_0(\text{tr}_{A/F_0}(\text{ad}(\beta)(y))) \), we obtain \( \text{ad}(\beta)(x) \in 3^+_n = (a'_{1-n} \oplus a^+_{l-(n+1)/2}) \cap A_- \). If we write \( x^\perp \) for the \( B^\perp \)-part of \( x \), then we have \( x^\perp \in a^+_{l-(n+1)/2} \) and \( \text{ad}(\beta)(x) \in a^+_{l-(n+1)/2} \). Lemma 2.2 implies that \( x^\perp \) lies in \( a^+_{l-(n+1)/2 + 1} \). So we get \( 1 + x \in J_+ \). This completes the proof. \( \square \)
It follows from Lemma 2.8 that there exists a unique irreducible representation \( \rho \) of \( J \) which contains \( \Psi_\beta \). Moreover the restriction of \( \rho \) to \( J_+ \) is a multiple of \( \Psi_\beta \) and \( \dim \rho = [J : J_+]^{1/2} \), which is a power of \( q \).

As direct consequences of Propositions 2.6 and 2.7 we get the following two propositions.

**Proposition 2.9.** An irreducible smooth representation \( \pi \) of \( G \) contains \( [\Lambda, n, n-1, \beta] \) if and only if \( \pi \) contains \( \rho \).

**Proposition 2.10.** \( I_G(\rho) = JG_EJ \).

We put \( J' = G_E \cap J \). Then we have \( J' = G_E \cap J_+ = P_\pi(\Lambda) \cap B \).

**Proposition 2.11.** For \( g \in G_E \), we have \( JgJ \cap G_E = J'gJ' \).

**Proof.** Suppose that \( [\Lambda, n, n-1, \beta] \) is simple. Let \( \Lambda' \) be a strict \( \mathfrak{a}_E \)-lattice sequence such that \( \Lambda'(Z) = \Lambda(Z) \). Then \( \Lambda' \) is also an \( \mathfrak{a}_E \)-lattice sequence. Write \( e' \) for the \( \mathfrak{a}_E \)-period of \( \Lambda' \) and \( \nu_E \) for the normalized valuation on \( E \). Thus we have \( a_n(\Lambda) = a_n'(\Lambda') \), where \( n' = -e'\nu_E(\beta) \). Due to [2] Theorem (1.6.1), we obtain \( (1 + a_n(\Lambda))x(1 + a_n(\Lambda)) \cap B = (1 + a_n(\Lambda) \cap B)x(1 + a_n(\Lambda) \cap B) \), for \( x \in B^* \).

Now the proof is exactly same as that of [18] Lemma 2.6. \( \square \)

For \( x \) in \( A \), we denote by \( x' \) its \( B \)-component and by \( x^\perp \) its \( B^\perp \)-component. The next lemma is useful to check some relations on \( \mathcal{H}(G//J, \rho) \).

**Lemma 2.12.** For \( g \in JG_EJ \), we have \( \nu_\Lambda(g^\perp) \geq \nu_\Lambda(g') + [(n+1)/2] \).

**Proof.** Put \( k = \nu_\Lambda(g) \). Then, for any element \( y \) in \( JgJ \), we have \( y \equiv g \pmod{a_{k+[(n+1)/2]}} \), so that \( y \equiv g^\perp \pmod{B + a_{k+[(n+1)/2]}} \). Therefore if \( g \in JG_EJ \), then \( g^\perp \in a_{k+[(n+1)/2]} \). In particular, we have \( \nu_\Lambda(g^\perp) \geq \nu_\Lambda(g) + [(n+1)/2] > \nu_\Lambda(g) \), and hence \( \nu_\Lambda(g') = \nu_\Lambda(g) \). This completes the proof. \( \square \)

The contragradient representation \( \tilde{\rho} \) of \( \rho \) is the unique irreducible representation of \( J \) which contains \( \Psi_\beta = \Psi_{-\beta} \). Suppose that \( G_E \) is compact. Then we have \( G_E \subset P_0(\Lambda) \) and \( G_E \) is the Iwahori subgroup of itself. The oscillator representation yields a representation \( \omega \) of \( G_E/(G_E \cap P_\Lambda(\Lambda)) \) on the space of \( \tilde{\rho} \) with the property

\[
\omega(g)\tilde{\rho}(p)\omega(g^{-1}) = \tilde{\rho}(Ad(g)(p)), \quad p \in J, \quad g \in G_E.
\] (2.9)

For \( g \in G_E \), let \( f_g \) denote the element in \( \mathcal{H}(G//J, \rho) \) such that \( f_g(g) = \omega(g) \) and \( \text{supp}(f_g) = JgJ \), and let \( e_g \) denote the element in \( \mathcal{H}(G_E//J', \Psi_\beta) \) such that \( e_g(g) = 1 \) and \( \text{supp}(e_g) = J'gJ' \). Then we obtain the following

**Theorem 2.13.** With the notations as above, suppose that \( G_E \) is compact. Then the map \( \eta : \mathcal{H}(G_E//J', \Psi_\beta) \to \mathcal{H}(G//J, \rho) \) defined by \( \eta(e_g) = f_g \) is a support preserving, \(*\)-isomorphism.

**Proof.** It is obvious that \( \eta \) is an algebra homomorphism which preserves supports. Since \( g \in G_E \) normalizes \( (J_+, \Psi_\beta) \), we have \( \rho \simeq \rho^g \). Hence the algebra \( \mathcal{H}(G//J, \rho) \) is spanned by \( f_g, \ g \in G_E \). Then Propositions 2.10 and 2.11 implies that \( \eta \) is an isomorphism. The \(*\)-preservation follows from \( e_g^* = e_{g^{-1}} \) and \( f_g^* = f_{g^{-1}} \), for \( g \in G_E \). \( \square \)
3. Fundamental strata for non quasi-split $U(4)$

3.1. Non quasi-split $U(4)$. From now on, we assume that $F/F_0$ is quadratic unramified. Let $\varepsilon$ denote a non-square unit in $\mathfrak{o}_0$. Then we have $F = F_0[\sqrt{\varepsilon}]$. We can (and do) take a common uniformizer $\varpi$ of $F$ and $F_0$.

Let $V = F^4$ denote the four dimensional $F$-space of column vectors. We put $A = M_4(F)$ and $\tilde{G} = A^\times$. Let $\{e_i\}_{1 \leq i \leq 4}$ denote the standard $F$-basis of $V$, and $E_{ij}$ the element in $A$ whose $(k,l)$ entry is $\delta_{ik}\delta_{jl}$. We put

$$H = E_{14} + \varpi E_{22} + E_{33} + E_{41} \in A$$

and define a nondegenerate $F/F_0$-hermitian form $h$ on $V$ by $h(v, w) = \langle \varpi Hw, v \rangle_F$, for $v, w \in V$. Then $h$ induces an involution $\sigma$ on $A$ by the formula $\sigma(X) = H^{-1}XH$, for $X \in A$. For $X = (X_{ij}) \in A$, we have

$$\sigma(X) = \begin{pmatrix} \overline{X}_{44} & \varpi \overline{X}_{24} & \varpi^{-1} \overline{X}_{34} & \varpi^{-1} \overline{X}_{14} \\ \overline{X}_{43} & \overline{X}_{22} & \overline{X}_{32} & \overline{X}_{12} \\ \overline{X}_{41} & \overline{X}_{21} & \overline{X}_{31} & \overline{X}_{11} \end{pmatrix}.$$  

(3.2)

We put $G = \{g \in \tilde{G} \mid \sigma(g) = g^{-1}\}$ and $A_- = \{X \in A \mid \sigma(X) = -X\} \simeq \text{Lie}(G)$. Then $G$ is the non quasi-split unramified unitary group in four variables defined over $F_0$, and $A_-$ consists of matrices of the form

$$\begin{pmatrix} Z & \varpi C & D & a\sqrt{\varepsilon} \\ M & b\sqrt{\varepsilon} & Y & -\overline{C} \\ N & -\varpi Y & c\sqrt{\varepsilon} & -D \\ d\sqrt{\varepsilon} & -\varpi M & -N & -Z \end{pmatrix}, \quad C, D, M, N, Y, Z \in F, \quad a, b, c, d \in F_0.$$  

(3.3)

3.2. A version of the existence of fundamental strata. Define $\mathfrak{o}_F$-lattices $N_0$ and $N_1$ in $V$ by

$$N_0 = \mathfrak{o}_F e_1 \oplus \mathfrak{o}_F e_2 \oplus \mathfrak{o}_F e_3 \oplus \mathfrak{o}_F e_4, \quad N_1 = \mathfrak{o}_F e_1 \oplus \mathfrak{o}_F e_2 \oplus \mathfrak{o}_F e_3 \oplus \mathfrak{p}_F e_4.$$  

Then we have

$$N_0^\# = \mathfrak{o}_F e_1 \oplus \mathfrak{p}_F^{-1} e_2 \oplus \mathfrak{o}_F e_3 \oplus \mathfrak{o}_F e_4, \quad N_1^\# = \mathfrak{p}_F^{-1} e_1 \oplus \mathfrak{p}_F^{-1} e_2 \oplus \mathfrak{o}_F e_3 \oplus \mathfrak{o}_F e_4$$

and obtain the following sequence of $\mathfrak{o}_F$-lattices in $V$:

$$\ldots \supseteq N_0 \supseteq N_1 \supseteq \varpi N_1^\# \supseteq \varpi N_0^\# \supseteq \varpi N_1 \supseteq \ldots.$$  

Recall from [10] that a self-dual $\mathfrak{o}_F$-lattice sequence $\Lambda$ in $V$ is called standard if $\Lambda(\mathbb{Z}) = \{\Lambda(i) \mid i \in \mathbb{Z}\}$ is contained in the set $\{\varpi^m N_0, \varpi^m N_1, \varpi^m N_0^\#, \varpi^m N_1^\# \mid m \in \mathbb{Z}\}$. By op. cit. Proposition 1.10, every self-dual $\mathfrak{o}_F$-lattice sequence is a $G$-conjugate of a standard one.

Let $\Lambda$ be a $C$-sequence in $V$ and $L$ an $\mathfrak{o}_F$-lattice in $V$. Since $d(\Lambda)$ is odd, we see that $L \in \Lambda(2\mathbb{Z})$ if and only if $L^\# \in \Lambda(2\mathbb{Z} + 1)$. So it is easy to observe that there are just the following 8 standard $C$-sequences $\Lambda_i$, $1 \leq i \leq 8$ in $V$, up to translation:

(I) $C$-sequences with $\Lambda(2\mathbb{Z}) \cap \Lambda(2\mathbb{Z} + 1) = \emptyset$:

$$\Lambda_1(2i) = \varpi^i N_0, \quad \Lambda_1(2i + 1) = \varpi^{i+1} N_0^\#, \quad i \in \mathbb{Z};$$

$$\Lambda_2(2i) = \varpi^i N_1, \quad \Lambda_2(2i + 1) = \varpi^{i+1} N_1^\#, \quad i \in \mathbb{Z};$$

(3.5)
\[ \Lambda_3(4i) = \varpi^i N_0, \quad \Lambda_3(4i + 1) = \varpi^i N_1, \]

(3.6)

\[ \Lambda_3(4i + 2) = \varpi^{i+1} N_1^#, \quad \Lambda_3(4i + 3) = \varpi^{i+1} N_0^#, \quad i \in \mathbb{Z}. \]

\[ \Lambda_4(6i) = \Lambda_4(6i + 1) = \varpi^i N_0, \quad \Lambda_4(6i + 2) = \varpi^i N_1, \]

(3.7)

\[ \Lambda_4(6i + 3) = \varpi^{i+1} N_1^#, \quad \Lambda_4(6i + 4) = \Lambda_4(6i + 5) = \varpi^{i+1} N_0^#, \quad i \in \mathbb{Z}; \]

(III) \( C \)-sequences with \( \Lambda(2\mathbb{Z}) = \Lambda(2\mathbb{Z} + 1) \):

\[ \Lambda_5(6i) = \Lambda_5(6i + 1) = \varpi^i N_0, \quad \Lambda_5(6i + 2) = \varpi^i N_1, \]

(3.8)

\[ \Lambda_5(6i + 3) = \varpi^{i+1} N_1^#, \quad \Lambda_5(6i + 4) = \varpi^{i+1} N_0^#, \quad i \in \mathbb{Z}. \]

\[ \Lambda_6(4i) = \Lambda_6(4i + 1) = \varpi^i N_0, \quad \Lambda_6(4i + 2) = \Lambda_6(4i + 3) = \varpi^{i+1} N_0^#, \quad i \in \mathbb{Z}; \]

(3.9)

\[ \Lambda_7(4i) = \Lambda_7(4i + 1) = \varpi^i N_1, \quad \Lambda_7(4i + 2) = \Lambda_7(4i + 3) = \varpi^{i+1} N_1^#, \quad i \in \mathbb{Z}; \]

(3.10)

\[ \Lambda_8(8i) = \Lambda_8(8i + 1) = \varpi^i N_0, \quad \Lambda_8(8i + 2) = \Lambda_8(8i + 3) = \varpi^i N_1, \]

\[ \Lambda_8(8i + 4) = \Lambda_8(8i + 5) = \varpi^{i+1} N_1^#, \quad \Lambda_8(8i + 6) = \Lambda_8(8i + 7) = \varpi^{i+1} N_0^#, \quad i \in \mathbb{Z}. \]

(3.11)

\[ \Lambda_8(8i + 4) = \Lambda_8(8i + 5) = \varpi^{i+1} N_1^#, \quad \Lambda_8(8i + 6) = \Lambda_8(8i + 7) = \varpi^{i+1} N_0^#, \quad i \in \mathbb{Z}. \]

(3.11)

**Theorem 3.1.** Let \( \pi \) be an irreducible smooth representation of \( G \) of positive level. Then \( \pi \) contains a fundamental skew stratum \( [\Lambda, n, n - 1, \beta] \) which satisfies one of the following conditions:

(i) \( \Lambda = \Lambda_i \), for some \( 1 \leq i \leq 5 \) and \( (e(\Lambda), n) = 2 \);

(ii) \( \Lambda = \Lambda_i \), for some \( 1 \leq i \leq 3 \) and \( (e(\Lambda), n) = 1 \).

**Proof.** Let \( \pi \) be an irreducible smooth representation of \( G \) of positive level. Thanks to \( [5] \) Proposition 3.1.1, \( \pi \) contains a fundamental skew stratum \( [\Lambda, n, n - 1, \beta] \) such that \( \Lambda \) is a \( C \)-sequence and \( (e(\Lambda), n) = 2 \). After \( G \)-conjugation, we may assume \( \Lambda \) is one of \( \Lambda_i \), \( 1 \leq i \leq 8 \). If \( \Lambda = \Lambda_i \), for \( 1 \leq i \leq 5 \), then there is nothing left to prove.

Suppose \( \Lambda = \Lambda_i \), for \( 6 \leq i \leq 8 \). Put \( \Lambda' = \Lambda_{i-5} \). Then \( \Lambda \) is the double of \( \Lambda' \), whence \( \mathfrak{a}_k(\Lambda') = \mathfrak{a}_{2k-1}(\Lambda) = \mathfrak{a}_{2k}(\Lambda), k \in \mathbb{Z} \). Since skew strata \( [\Lambda, n, n - 1, \beta] \) and \( [\Lambda', n/2, n/2 - 1, \beta] \) correspond to the same character \( \psi_{\beta} \) of the group \( P_n(A) = P_{n/2}(\Lambda') \), \( \pi \) contains a fundamental skew stratum \( [\Lambda', n/2, n/2 - 1, \beta] \), which satisfies the condition (ii). \( \square \)
We list up $\Lambda$ and $n$ of the fundamental strata $[\Lambda, n, n - 1, \beta]$ satisfying one of the conditions in Theorem 3.1.

| $\Lambda$ | $e(\Lambda)$ | $d(\Lambda)$ | $n$  | $n/e(\Lambda)$ |
|----------|--------------|---------------|------|----------------|
| $\Lambda_1$ | 2            | -1            | $2m$ | $m$            |
| $\Lambda_2$ | 2            | -1            | $2m$ | $m$            |
| $\Lambda_3$ | 4            | -1            | $4m - 2$ | $m - 1/2$         |
| $\Lambda_4$ | 6            | -1            | $6m - 2$ | $m - 1/3$         |
| $\Lambda_5$ | 6            | -3            | $6m - 2$ | $m - 1/3$         |
| $\Lambda_2$ | 2            | -1            | $2m - 1$ | $m - 1/2$         |
| $\Lambda_3$ | 4            | -1            | $4m - 1$ | $m - 1/4$         |

(3.12)

3.3. Filtrations. We give an explicit description of the filtrations on $A$ induces by standard $C$-sequences $\Lambda_i$, for $1 \leq i \leq 5$. Since $\{a_k(\Lambda_i)\}_{k \in \mathbb{Z}}$ is periodic, it suffices to describe $a_k(\Lambda_i)$, $0 \leq k \leq e(\Lambda_i) - 1$.

The sequences $\Lambda_1$ and $\Lambda_2$ correspond to the standard filtrations of maximal compact subgroups of $G$:

$$a_0(\Lambda_1) = \begin{pmatrix} o_F & p_F & o_F & o_F \\ o_F & o_F & o_F & o_F \\ o_F & p_F & o_F & o_F \\ o_F & p_F & p_F & o_F \end{pmatrix}, \quad a_1(\Lambda_1) = \begin{pmatrix} p_F & p_F & p_F & p_F \\ o_F & p_F & o_F & o_F \\ p_F & p_F & p_F & p_F \\ p_F & p_F & p_F & p_F \end{pmatrix}; \quad (3.13)$$

$$a_0(\Lambda_2) = \begin{pmatrix} o_F & o_F & o_F & p_F^1 \\ o_F & o_F & o_F & p_F^1 \\ p_F & p_F & o_F & o_F \\ p_F & p_F & p_F & o_F \end{pmatrix}, \quad a_1(\Lambda_2) = \begin{pmatrix} p_F & p_F & o_F & o_F \\ p_F & p_F & o_F & o_F \\ p_F & p_F & p_F & p_F \\ p_F^2 & p_F^2 & p_F & p_F \end{pmatrix}; \quad (3.14)$$

The sequence $\Lambda_3$ corresponds to the standard filtration of the Iwahori subgroup of $G$:

$$a_0(\Lambda_3) = \begin{pmatrix} o_F & p_F & o_F & o_F \\ o_F & o_F & o_F & o_F \\ p_F & p_F & o_F & o_F \\ p_F & p_F & p_F & o_F \end{pmatrix}, \quad a_1(\Lambda_3) = \begin{pmatrix} p_F & p_F & o_F & o_F \\ o_F & p_F & o_F & o_F \\ p_F & p_F & o_F & o_F \\ p_F & p_F & p_F & o_F \end{pmatrix},$$

$$a_2(\Lambda_3) = \begin{pmatrix} p_F & p_F & p_F & o_F \\ p_F & p_F & p_F & o_F \\ p_F & p_F & p_F & p_F \\ p_F & p_F & p_F & p_F \end{pmatrix}, \quad a_3(\Lambda_3) = \begin{pmatrix} p_F & p_F & p_F & p_F \\ p_F & p_F & p_F & p_F \\ p_F & p_F & p_F & p_F \\ p_F & p_F & p_F & p_F \end{pmatrix}; \quad (3.15)$$

Sequences $\Lambda_4$ and $\Lambda_5$ give non-standard filtrations of the Iwahori subgroup of $G$:

$$a_0(\Lambda_4) = a_0(\Lambda_5) = a_0(\Lambda_3), \quad a_1(\Lambda_4) = a_1(\Lambda_5) = a_1(\Lambda_3),$$
Let $n$ be a positive integer such that $(n, 4) = 1$. In this section, we give a classification of the irreducible smooth representations of $G$ of level $n/4$. Theorem 3.1 says that such a representation contains a fundamental skew stratum $[\Lambda_3, n, [n/2], \beta]$. By Proposition 1.6 $[\Lambda_3, n, n-1, \beta]$ is simple and $E = F[\beta]$ is a totally ramified extension of degree 4 over $F$.

**Proposition 4.1.** Let $[\Lambda_3, n, [n/2], \beta]$ and $[\Lambda_3, n, [n/2], \gamma]$ be fundamental skew strata contained in some irreducible smooth representation of $G$. Then $(P_{[n/2]+1}(\Lambda_3), \psi_\gamma)$ is a $P_0(\Lambda_3)$-conjugate of $(P_{[n/2]+1}(\Lambda_3), \psi_\beta)$.

**Proof.** By assumption, there is $g \in G$ such that

$$(\beta + a_{[n/2]}(\Lambda_3)) \cap \text{Ad}(g)(\gamma + a_{[n/2]}(\Lambda_3)) \neq \emptyset.$$  

Take $\delta \in (\beta + a_{[n/2]}(\Lambda_3)) \cap \text{Ad}(g)(\gamma + a_{[n/2]}(\Lambda_3))$. Then we get simple strata $[\Lambda_3, n, [n/2], \delta]$ and $[g\Lambda_3, n, [n/2], \delta]$. Recall that $d(\Lambda_3) = d(g\Lambda_3)$. By Lemma 1.10, we have $\Lambda_3 = g\Lambda_3$ and hence $g \in P_0(\Lambda_3)$. This completes the proof. \hfill \square

By Proposition 4.1, the set of equivalence classes of level $n/4$ representations of $G$ is the disjoint union $\bigcup_\beta \text{Irr}(G)(P_{[n/2]+1}(\Lambda_3), \psi_\beta)$, where $\beta$ runs over the $P_0(\Lambda_3)$-conjugacy classes of elements in $a_{[n/2]}(\Lambda_3) \cap a_{[n/2]}(\Lambda_3)$ such that $[\Lambda_3, n, n-1, \beta]$ is fundamental. For each $\beta$, the set $\text{Irr}(G)(P_{[n/2]+1}(\Lambda_3), \psi_\beta)$ is classified via the result in 4.1.4.

**5. Representations of level $n/3$**

Let $n$ be a positive integer such that $(n, 3) = 1$. In this section, we classify the irreducible smooth representations of $G$ of level $n/3$. It follows from Theorem 3.1 that every irreducible smooth representation of $G$ of level $n/3$ contains a fundamental skew stratum $[\Lambda, 2n, n, \beta]$ such that $\Lambda = \Lambda_4$ or $\Lambda_5$. 

\[
\begin{align*}
\begin{pmatrix}
p_F & p_F & p_F & p_F \\
p_F & p_F & o_F & o_F \\
p_F & p_F & p_F & p_F \\
p_F & p_F & p_F & p_F
\end{pmatrix},
\begin{pmatrix}
p_F & p_F & p_F & p_F \\
p_F & p_F & o_F & o_F \\
p_F & p_F & p_F & p_F \\
p_F & p_F & p_F & p_F
\end{pmatrix},
\begin{pmatrix}
p_F & p_F & p_F & p_F \\
p_F & p_F & p_F & p_F \\
p_F & p_F & p_F & p_F \\
p_F & p_F & p_F & p_F
\end{pmatrix},
\begin{pmatrix}
p_F & p_F & p_F & p_F \\
p_F & p_F & p_F & p_F \\
p_F & p_F & p_F & p_F \\
p_F & p_F & p_F & p_F
\end{pmatrix},
\begin{pmatrix}
p_F & p_F & p_F & p_F \\
p_F & p_F & p_F & p_F \\
p_F & p_F & p_F & p_F \\
p_F & p_F & p_F & p_F
\end{pmatrix}.
\end{align*}
\]

\[5.\text{Representations of level } n/3\]
Let $\Lambda = \Lambda_4$ and let $[\Lambda, 2n, n, \beta]$ be a fundamental skew stratum. The characteristic polynomial $\phi_\beta(X)$ depends only on the coset $\beta + a_{1-2n}(\Lambda)_-$. By (3.16), we can take $\beta \in a_{-2n}(\Lambda)_-$ to be a band matrix modulo $a_{1-2n}(\Lambda)_-$, so we have $\phi_\beta(X) = (X - a\sqrt{\varepsilon})^3 X$, for $a \in k^*_F$.

Using Hensel’s Lemma, we can lift this to $\Phi_\beta(X) = f_a(X)f_0(X)$ where $f_a(X)$, $f_0(X)$ are monic, $f_a(X) \mod p_F = (X - a\sqrt{\varepsilon})^3$ and $f_0(X) \mod p_F = X$. As in the proof of [17] Theorem 4.4, when we put $V^o = \ker f_a(y_\beta)$, $V^o = \ker f_0(y_\beta)$, the skew stratum $[\Lambda, 2n, n, \beta]$ is split with respect to the $F$-splitting $V = V^o \perp V^0$.

For $b \in \{a, 0\}$, we write $\Lambda^b(i) = \Lambda(i) \cap V^b$, $i \in \mathbb{Z}$, and $\beta_b = |\beta|_{V^b}$. Since $X$ does not divide $\phi_{\beta_b}(X) = (X - a\sqrt{\varepsilon})^3$, it follows from [3] Proposition 3.5 that $\beta_b \Lambda^b(i) = \Lambda^b(i - 2n)$, for $i \in \mathbb{Z}$. Since $(2n, e(\Lambda)) = 2$, we obtain $[\Lambda^b(i) : \Lambda^b(i + 1)] = [\Lambda^b(i + 2) : \Lambda^b(i + 3)]$, for $i \in \mathbb{Z}$.

Recall $\Lambda(0) = \Lambda(1)$. So we have $\Lambda^a(2i) = \Lambda^a(2i + 1)$, for $i \in \mathbb{Z}$. Since $\Lambda(2) \neq \Lambda(3)$, we get $\Lambda^a(2) \neq \Lambda^a(3)$. This implies $\Lambda^a(3) = \omega \Lambda^a(2)$ and $\Lambda^a(3) = \ldots = \Lambda^a(1) = \Lambda^a(2)$ because $\dim_F V^0 = 1$ and $e(\Lambda^0) = 6$.

**Lemma 5.1.** Let $[\Lambda_4, 2n, n, \beta]$ be a fundamental skew stratum. Then the space $(V^0, h|_{V^0})$ represents 1.

**Proof.** The dual lattice of $\Lambda^0(0)$ with respect to $(V^0, h|_{V^0})$ is $\Lambda^0(-1) = \Lambda^0(0)$. Since $\dim_F V^0 = 1$ and $F$ is unramified over $F_0$, this implies that the form $f|_{V^0}$ represents 1. This completes the proof. □

We can apply the construction of a splitting to $\Lambda_5$. The proof of the next lemma is similar to that of Lemma 5.1.

**Lemma 5.2.** Let $[\Lambda_5, 2n, n, \beta]$ be a fundamental skew stratum. Then the space $(V^0, h|_{V^0})$ does not represent 1.

**Proposition 5.3.** Let $[\Lambda_4, 2n, n, \beta]$ and $[\Lambda_5, 2n, n, \gamma]$ be fundamental skew strata. Then there are no irreducible smooth representations of $G$ which contain both of them.

**Proof.** Suppose there is an irreducible smooth representation of $G$ which contain both of them. Then there is $g \in G$ such that $(\beta + a_{-n}(\Lambda_4)_- \cap \text{Ad}(g)(\gamma + a_{-n}(\Lambda_5)_-)$ is non-empty. Let $\delta$ be an element of $(\beta + a_{-n}(\Lambda_4)_- \cap \text{Ad}(g)(\gamma + a_{-n}(\Lambda_5)_-)$. Then we obtain fundamental skew strata $[\Lambda_4, 2n, n, \delta]$ and $[\Lambda_5, 2n, n, \text{Ad}(g^{-1})\delta]$

Let $V = V^a \perp V^0$ denote the $F$-splitting obtained by applying the above construction to $[\Lambda_4, 2n, n, \delta]$, and $V = W^a \perp W^0$ the same object for $[\Lambda_5, 2n, n, \text{Ad}(g^{-1})\delta]$. Then we have $W^0 = g^{-1}V^0$. This contradicts Lemmas 5.1 and 5.2. This completes the proof. □

Let $[\Lambda, 2n, n, \beta]$ be a fundamental skew stratum such that $\Lambda = \Lambda_4$ or $\Lambda_5$. We put $\Lambda'(i) = \Lambda^a(2i)$, for $i \in \mathbb{Z}$. Then $\Lambda'$ is a strict $\mathfrak{o}_F$-lattice sequence in $V^a$ of period 3. Proposition 1.6 says that the stratum $[\Lambda', n, n - 1, \beta_0]$ is simple and $E_a = F[\beta_0]$ is a totally ramified extension of degree 3 over $F$. It is easy to observe that the stratum $[\Lambda_4, 2n, 2n - 1, \beta_0]$ is also simple.

The equation $\phi_{\beta_0}(X) = X$ implies $\beta_0 \in a_{1-2n}(\Lambda^0)_-$. We can replace $\beta_0$ with 0. So $[\Lambda, 2n, n, \beta]$ is a skew semisimple stratum in $\mathfrak{a}$ such that the $G$-centralizer $G_E$ of $\beta$ is a maximal compact torus.

**Proposition 5.4.** Let $\Lambda$ be $\Lambda_4$ or $\Lambda_5$. Let $[\Lambda, 2n, n, \beta]$ and $[\Lambda, 2n, n, \gamma]$ be fundamental skew strata occurring in some irreducible smooth representation of $G$. Then $(P_{n+1}(\Lambda), \psi_\gamma)$ is a $P_0(\Lambda)$-conjugate of $(P_{n+1}(\Lambda), \psi_\beta)$.
Proof. Note that Lemma 1.10 holds for any fundamental skew stratum $[\Lambda, 2n, n, \beta]$ of this section because $[\Lambda^\circ, 2n, n, \beta_0]$ is maximal simple and $\dim_F V^0 = 1$. This is exactly as in the proof of Proposition 4.1.

By Proposition 5.3 and 5.4, the set of equivalence classes of irreducible smooth representations of $G$ of level $n/3$ is the disjoint union $\bigcup_\beta \text{Irr}(G)(P_{n+1}(\Lambda_4), \beta) \cup \bigcup_\gamma \text{Irr}(G)(P_{n+1}(\Lambda_5), \gamma)$, where $\beta$ (respectively $\gamma$) runs over the $P_0(\Lambda_4)$ (respectively $P_0(\Lambda_5)$)-conjugacy classes of elements in $a_{-2n}(\Lambda_4)/a_{-n}(\Lambda_4)$ (respectively $a_{-2n}(\Lambda_5)/a_{-n}(\Lambda_5)$) which generate a fundamental skew stratum. Each set in this union is classified by the results in §1.4.

Remark 5.5. The restriction $[\Lambda, 2n, 2n - 1, \beta]$ is not always semisimple. But (1.2) holds since $\dim_F V^0 = 1$, so we can apply the arguments in §1.4.

6. Representations of half-integral level

6.1. Semisimplification of skew strata. Let $m$ be a positive integer. By Theorem 3.1, an irreducible representation $\pi$ of $G$ of level $m - 1/2$ contains a fundamental skew stratum $[\Lambda, n, n - 1, \beta]$ which satisfies one of the following conditions:

(6-i) $\Lambda = \Lambda_3$ and $n = 4m - 2$;
(6-ii) $\Lambda = \Lambda_1$ and $n = 2m - 1$;
(6-iii) $\Lambda = \Lambda_2$ and $n = 2m - 1$.

We also consider skew strata $[\Lambda, n, n - 1, \beta]$ with one of the following conditions:

(6-ii') $\Lambda = \Lambda_4$ and $n = 6m - 3$;
(6-iii') $\Lambda = \Lambda_5$ and $n = 6m - 3$.

Let $[\Lambda_3, 4m - 2, 4m - 3, \beta]$ be a fundamental skew stratum. Up to equivalence of skew strata, we can choose $\beta \in a_{-4m}(\Lambda_3)$ as follows:

$$\beta = -m \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & Y & 0 \\ 0 & -y & 0 & 0 \\ \varphi d \sqrt{3} & 0 & 0 & 0 \end{array} \right), Y \in \mathfrak{o}_F, \ a, d \in \mathfrak{o}_0. \quad (6.1)$$

Then we have $\phi_\beta(X) = (X - ad\varepsilon)^2(X + \sqrt{3}Y \sqrt{3})^2 \pmod{\mathfrak{p}_F}$. Since we are assuming $[\Lambda_3, 4m - 2, 4m - 3, \beta]$ is fundamental, we have $Y \sqrt{3} \in \mathfrak{o}_0^\times$ or $ad \in \mathfrak{o}_0^\times$. We decompose this case into the following three cases:

(6-ia) $ad \varepsilon \equiv -Y \sqrt{3} \pmod{\mathfrak{p}_F}$;
(6-ib) $ad \varepsilon \neq -Y \sqrt{3} \pmod{\mathfrak{p}_F}$ and $ad \in \mathfrak{o}_0^\times$;
(6-ic) $ad \in \mathfrak{p}_0$.

Proposition 6.1. With the notation as above, suppose that an irreducible smooth representation $\pi$ of $G$ contains a fundamental skew stratum $[\Lambda_3, 4m - 2, 4m - 3, \beta]$ of type (6-ic). Then $\pi$ contains a fundamental skew strata of type (6-ii') or (6-iii').

Proof. If $a \in \mathfrak{p}_0$, then we have $\beta + a_{3-4m}(\Lambda_3) \subset a_{3-6m}(\Lambda_4)$. Similarly, if $d \in \mathfrak{p}_0$, then $\beta + a_{3-4m}(\Lambda_3) \subset a_{3-6m}(\Lambda_5)$. Since the level of $\pi$ is $m - 1/2$, the lemma follows immediately. □
Let \([\Lambda_1, 2m - 1, 2m - 2, \beta]\) be a fundamental skew stratum. We may assume that 
\(\beta \in a_{1-2m}(\Lambda_1)\) has the following form:

\[
\beta = \omega^{-m} \begin{pmatrix}
0 & \varpi C & 0 & 0 \\
M & 0 & Y & -\overline{C} \\
0 & -\omega Y & 0 & 0 \\
0 & -\omega M & 0 & 0
\end{pmatrix}, \ C, M, Y \in \mathfrak{o}_F.
\] (6.2)

Then we have \(\phi_\beta(X) = X^2(X + Y\sqrt{-CM - \overline{CM}})^2\).

**Proposition 6.2.** Suppose that an irreducible smooth representation \(\pi\) of \(G\) contains 
a skew stratum \([\Lambda, n, n - 1, \beta]\) of type (6-ii) (respectively (6-iii)). Then \(\pi\) contains a 
fundamental skew stratum of type (6-ii') (respectively (6-iii')).

**Proof.** We may replace \([\Lambda_1, 2m - 1, 2m - 2, \beta]\) with \([\Lambda_1, 2m - 1, 2m - 2, \text{Ad}(g)\beta]\), for 
\(g \in P_0(\Lambda_1)\). It is easy to observe that we may assume \(M \equiv 0 \pmod{pF}\) after \(P_0(\Lambda)\)-conjugation. Then we get 
\(\beta + a_{2-2m}(\Lambda_1) \subset a_{3-6m}(\Lambda_4)\). This implies that \(\pi\) contains some 
fundamental skew stratum \([\Lambda_4, 6m - 3, 6m - 4, \gamma]\).

We can treat the case (6-iii) in a similar fashion. \(\square\)

Due to Propositions 6.1 and 6.2, an irreducible smooth representation of \(G\) of level 
m\(- 1/2\) contains a fundamental skew stratum of type (6-ia), (6-ib), (6-ii'), or (6-iii').

**6.2. Case (6-ia).** Let \([\Lambda, n, n - 1, \beta]\) be a fundamental skew stratum of type (6-ia). 
Replacing \(\beta\) with an element in \(\beta + a_{1-n}(\Lambda)_\ast\), we can assume that 
\(ad\varepsilon = -Y\overline{\gamma}\).

Since \(\beta^2 = \varpi^{-2m+2}ad\varepsilon\), the algebra \(E = F[\beta]\) is a totally ramified extension of degree 2 
over \(F\), and \([\Lambda, n, n - 1, \beta]\) is a skew simple stratum in \(A\). Note that \(E\) is tamely ramified 
over \(F\) since we are assuming that \(F\) is of odd residual characteristic.

**Proposition 6.3.** The group \(G_E\) is isomorphic to the anisotropic unitary group in two 
variables over \(E_0\).

**Proof.** The group \(G_E\) is the unramified unitary in two variables defined by a hermitian 
form \((V, h_E)\) induced by the involutive algebra \((B, \sigma)\). By [1] Section 5, we may choose 
\(h_E\) so that for any \(\mathfrak{o}_E\)-lattice \(L\) in \(V\), the dual lattice of \(L\) with respect to 
\(h_E\) equals to that with respect to \(h\). Then \(\Lambda\) is a strict self-dual \(\mathfrak{o}_E\)-lattice in 
\((V, h_E)\) of period 2 with \(d(\Lambda) = -1\). The assertion follows from [8] Lemma 1.11. \(\square\)

Proposition 6.3 implies that the group \(G_E\) is compact. So Theorem 2.13 gives a classification 
of the irreducible smooth representations of \(G\) which contain \([\Lambda, n, n - 1, \beta]\).

**6.3. Case (6-ib).** Let \([\Lambda, n, n - 1, \beta]\) be a fundamental skew stratum of type (6-ib). Put 
\(V^1 = Fe_1 + Fe_4\) and \(V^2 = Fe_2 + Fe_4\). Then the stratum \([\Lambda, n, n - 1, \beta]\) is split with 
respect to the \(F\)-splitting \(V = V^1 \perp V^2\).

We use the notation in \([2]\) Since \(\beta_1^2 = \varpi^{-1-2m+2}ad\varepsilon \cdot 1_{V^1}\), the stratum \([\Lambda^1, n, n - 1, \beta_1]\) is simple. If \(Y \in \mathfrak{o}_F^\ast\), then \([\Lambda^2, n, n - 1, \beta_2]\) is also simple. Otherwise, we can assume that 
\(Y = 0\). Consequently, \([\Lambda, n, n - 1, \beta]\) is skew semisimple.

**Proposition 6.4.** The group \(G_E\) is compact.

**Proof.** If \(Y \in \mathfrak{o}_F^\ast\), then \(G_E\) is isomorphic to \(U(1)(E_1/E_1,0) \times U(1)(E_2/E_2,0)\). If not, then 
\(G_E\) is isomorphic to \(U(1)(E_1/E_1,0) \times U(2)(F/F_0)\), where \(U(2)\) denote the anisotropic 
unitary group in two variables. \(\square\)
Recall that we are assuming $F$ is of odd residual characteristic. Then $E_i$ is tamely ramified over $F$, for $i = 1, 2$. Therefore the irreducible smooth representations of $G$ containing $[\Lambda, n, n - 1, \beta]$ are classified by Theorem 2.13.

6.4. Case $(\mathcal{G}$-ii'). Let $[\Lambda, n, n - 1, \beta]$ be a fundamental skew stratum of type $(\mathcal{G}$-ii'). Up to equivalence class of skew strata, we may assume that $\beta \in a_n(\Lambda)_-$ has the following form:

$$\beta = \varpi^{-m} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & Y & 0 \\ 0 & -\varpi Y & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ Y \in \mathfrak{o}_F.$$  \hfill (6.3)

Because the stratum is fundamental, we have $Y \in \mathfrak{o}_F^\times$. As in the previous section, when we define an orthogonal $F$-splitting $V = V_1 \perp V_2$ by $V^1 = F e_1 \oplus F e_4$ and $V^2 = F e_2 \oplus F e_3$, the skew stratum $[\Lambda, n, n - 1, \beta]$ is split with respect to $V = V^1 \perp V^2$.

We use the notation in Section 2. Then $\beta_1 = 0$ and $E_2 = F[\beta_2]$ is a totally ramified extension of degree 2 over $F$, and hence $E_i$ is tamely ramified over $F$, for $i = 1, 2$.

We shall apply the construction in Section 2. Since $n$ is odd, we have $J = J_+$ and $\rho = \Psi_\beta$. We abbreviate $\Psi = \Psi_\beta$.

**Theorem 6.5.** With the notation as above, there exists a support-preserving, $\ast$-isomorphism $\eta : \mathcal{H}(G_E//J', \Psi) \simeq \mathcal{H}(G//J, \Psi)$.

**Remark 6.6.** The isomorphism in Theorem 6.5 induces a bijection from $\text{Irr}(G_E)^{(J', \Psi)}$ to $\text{Irr}(G)^{(J, \Psi)}$. The character $\Psi$ extends to a character of $G_E$, so we get a bijection from $\text{Irr}(G_E)^{(J', 1)}$ to $\text{Irr}(G)^{(J, 1)}$. Since the centers of $G$ and $G_E$ are both compact, this bijection maps a supercuspidal representation of $G_E$ to that of $G$.

We put $B' = P_0(\Lambda) \cap G_E$. Then $B'$ is the Iwahori subgroup of $G_E$ and normalizes $J' = P_n(\Lambda) \cap G_E$. We define elements $s_1$ and $s_2$ in $G_E$ by

$$s_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \ s_2 = \begin{pmatrix} 0 & 0 & 0 & \varpi^{-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \varpi & 0 & 0 & 0 \end{pmatrix}.$$  

Put $S = \{s_1, s_2\}$ and $W' = \langle S \rangle$. Then we have a Bruhat decomposition $G_E = B'W'B'$.

**Lemma 6.7.** Let $t$ be a non negative integer. Then

\begin{enumerate}
\item[(i)] $[J'(s_1 s_2)^tJ' : J'] = [J'(s_1 s_2)^t s_1 J' : J'] = [J'(s_2 s_1)^t J' : J'] = q^{2t}$, $[J'(s_2 s_1)^t s_2 J' : J'] = q^{2(t+1)}$.
\item[(ii)] $[J(s_1 s_2)^t J : J] = [J(s_2 s_1)^t J : J] = [J(s_1 s_2)^t s_1 J : J] = q^t$, $[J(s_2 s_1)^t s_2 J : J] = q^{t+1}$.
\end{enumerate}

**Proof.** Let $g \in G$. Since $[JgJ : J] = [J : J \cap gJg^{-1}] = [\mathfrak{z}_- : \mathfrak{z}_- \cap g\mathfrak{z}_-g^{-1}]$, we can compute $[JgJ : J]$ by the description of $\{a_k(\Lambda)\}_{k \in \mathbb{Z}}$. \hfill \qedsymbol

For $\mu \in F$ and $\nu \in F^\times$, we set

$$u(\mu) = \begin{pmatrix} 1 & 0 & 0 & \mu \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ w(\mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \mu & 0 & 0 & 1 \end{pmatrix}, \ h(\nu) = \begin{pmatrix} \nu & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \nu^{-1} \end{pmatrix}.$$
For $g \in G_E$, let $e_g$ denote the element in $\mathcal{H}(G_{E}//J', \Psi)$ such that $e_g(g) = 1$ and $\text{supp}(e_g) = J'gJ'$.

**Theorem 6.8.** Suppose $m \geq 2$. Then the algebra $\mathcal{H}(G'//J', \Psi)$ is generated by the elements $e_g, g \in B' \cup S$. These elements are subject to the following relations:

(i) $e_k = \Psi(k)e_1$, $k \in J'$,
(ii) $e_k * e_{k'} = e_{kk'}$, $k, k' \in B'$,
(iii) $e_k * e_s = e_s * e_{sk_s}$, $s \in S$, $k \in B' \cap sB's$,
(iv) $e_s * e_{s_1} = e_1$,
\[ e_{s_2} * e_{s_2} = [J's_2J' : J'] \sum_{x \in \mathfrak{o}_0 / \mathfrak{p}_0^2} \varepsilon_{u(x^2 - 2x \sqrt{\varepsilon})} e_u, \]
(v) For $\mu \in \mathfrak{o}_0^x \setminus \mathfrak{e}$,
\[ e_{s_1} * e_{\varepsilon}(\mu) * e_{s_1} = e_{u(\mu^-)} * e_{s_1} * e_{h(\mu)} * e_{u(\mu^-)}, \]
\[ e_{s_2} * e_{\varepsilon}(\mu) * e_{s_2} = [J's_2J' : J'] e_u(\varepsilon(\mu^-)) * e_{s_2} * e_{h(-\mu^-)} * e_{\varepsilon}(\mu^-). \]

**Proof.** The proof is very similar to that of [4] Chapter 3 Theorem 2.1. \(\square\)

For $g \in G_E$, we denote by $f_g$ the element in $\mathcal{H}(G//J, \Psi)$ such that $f_g(g) = 1$ and $\text{supp}(f_g) = JgJ$.

**Theorem 6.9.** Suppose $m \geq 2$. Then the algebra $\mathcal{H}(G//J, \Psi)$ is generated by $f_g, g \in B' \cup S$ and satisfies the following relations:

(i) $f_k = \Psi(k)f_1$, $k \in J'$,
(ii) $f_k * f_{k'} = f_{kk'}$, $k, k' \in B'$,
(iii) $f_k * f_s = f_s * f_{sk_s}$, $s \in S$, $k \in B' \cap sB's$,
(iv) $f_{s_1} * f_{s_1} = f_1$,
\[ f_{s_2} * f_{s_2} = [J's_2J' : J'] \sum_{x \in \mathfrak{o}_0 / \mathfrak{p}_0^2} \varepsilon_{u(x^2 - 2x \sqrt{\varepsilon})} f_u, \]
(v) For $\mu \in \mathfrak{o}_0^x \setminus \mathfrak{e}$,
\[ f_{s_1} * f_{\varepsilon}(\mu) * f_{s_1} = f_{u(\mu^-)} * f_{s_1} * f_{h(\mu)} * f_{u(\mu^-)}, \]
\[ f_{s_2} * f_{\varepsilon}(\mu) * f_{s_2} = q^4 f_u(\varepsilon(\mu^-)) * f_{s_2} * f_{h(-\mu^-)} * f_{\varepsilon}(\mu^-). \]

**Proof.** Recall that if $x, y \in G_E$ satisfy $[JxJ : J][JyJ : J] = [JxyJ : J]$, then we have $JxJyJ = JxyJ$ and $f_x * f_y = f_{xy}$.

By Proposition 2.7, $\mathcal{H}(G//J, \Psi)$ is linearly spanned by $f_g, g \in G_E$. For $g \in G_E$, we write $g = b_1w wb_2$ where $b_1, b_2 \in B'$ and $w \in W'$. Then we have $f_g = f_{b_1} * f_w * f_{b_2}$ because $B'$ normalizes $J$. Let $w = s_{i_1}s_{i_2} \cdots s_{i_t}$ be a minimal expression for $w$ with $s_{i_j} \in S$. It follows from Lemma 6.7 that $f_w = f_{s_{i_1}} * f_{s_{i_2}} * \cdots * f_{s_{i_t}}$. Therefore $\mathcal{H}(G//J, \Psi)$ is generated by $f_g, g \in B' \cup S$.

Relations (i), (ii), (iii) are trivial. Since $[J_sJ : J] = 1$, the relations (iv) and (v) on $s_1$ are obvious. We shall prove relations (iv) and (v) on $s_2$ in the case when $m = 2k$, $k \geq 1$. The case when $m = 2k + 1, k \geq 1$ can be treated in a similar fashion.

We will abbreviate $s = s_2$. We can choose a common system of representatives for $J/J \cap sJs$ and $J \cap sJs \setminus J$ to be

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
\omega^k A & 1 & 0 & 0 \\
\omega^k B & 0 & 1 & 0 \\
\omega^m \sqrt{\varepsilon} - \omega^m (\omega xA + BB)/2 & -\omega^{k+1} A & -\omega^k B & 1
\end{pmatrix},
\]

where $A, B \in \mathfrak{o}_F / \mathfrak{p}_F$ and $a \in \mathfrak{o}_0 / \mathfrak{p}_0^2$. Note that $x(a, A, B)$ lies in the kernel of $\Psi$. 

(iv) For \( g \in G \), let \( \delta_g \) denote the unit point mass at \( g \). As in the proof of the relation (c) in \cite{11} Theorem 2.7, we obtain

\[
J_s J : J \sum_{x \in J \cap \mathcal{J} \setminus J} \Psi^{-1}(x) f_1 * \delta_{sx v} * f_1.
\]

Recall that for \( g \in G \),

\[
J_s J : J \sum_{x \in J \cap \mathcal{J} \setminus J} \Psi^{-1}(x) f_1 * \delta_{sx v} * f_1 = \begin{cases} [JGJ : J]^{-1} f_g, & \text{if } g \in I_G(\Psi), \\ 0, & \text{if } g \notin I_G(\Psi). \end{cases}
\]

Observe that the \( B \)-component of \( sx(a, A, B)s \) lies in \( a_0(\Lambda) \). By Lemma \ref{2.12}, if \( sx(a, A, B)s \in I_G(\Psi) = JG_EJ \), then the \( B^\perp \)-component of \( sx(a, A, B)s \) lies in \( a_{[n/2]+1}(\Lambda) \). This implies \( A \equiv B \equiv 0 \).

So we obtain

\[
f_s * f_s = [J_s J : J] \sum_{a \in a_0 / \mathcal{P}} f_s * \delta_{sx(0,0,0)s} * f_s = [J_s J : J] \sum_{a \in a_0 / \mathcal{P}} f_s * \delta_{u(o^{m-2a})} * f_s = [J_s J : J] \sum_{a \in a_0 / \mathcal{P}} f_s(sx) * f_s.
\]

(v) Let \( \mu \in a_0^\times \sqrt{\varepsilon} \). Put \( u = u(o\mu) \in B' \). As in the proof of the relation (d) in \cite{11} Theorem 2.7, we get

\[
f_s * f_s = [J_s J : J] \sum_{x \in J \cap \mathcal{J} \setminus J} \Psi^{-1}(x) f_1 * \delta_{sxus} * f_1.
\]

Put \( x = x(0, A, B), \nu = \mu + o^{m-2a(\varepsilon)}, v = u(o\nu), \) and \( h(-\nu^{-1}) \). Then we have

\[
sx(a, A, B)us = sxu(o\nu)s = sxusvshv = [sx, v]vshv(hv)^{-1}x(hv).
\]

Since \( hv \in B' \), the element \((hv)^{-1}x(hv)\) lies in the kernel of \( \Psi \). Since \( v \in P_4(\Lambda) \) and \( sxu \in P_{6k-5}(\Lambda) \), we have \([sxu, v] \in P_{6k-1}(\Lambda) = P_{[n/2]+1}(\Lambda) \). Observe that \([sxu, v]\) is equivalent to

\[
1 + \begin{pmatrix}
-o^{m-1}BB\nu^{-1}/2 & 0 & 0 & o^{2m-3}(BB)^2\nu^{-2}/4 \\
0 & -o^{m-1}AB\nu^{-1} & -o^{m-1}AB\nu^{-1} & 0 \\
0 & -o^{m-1}AB\nu^{-1} & -o^{m-1}BB\nu^{-1} & 0 \\
0 & 0 & 0 & -o^{m-1}BB\nu^{-1}/2
\end{pmatrix}
\]

modulo \( a_{n+1}(\Lambda) + B^\perp \). Since \( P_{[n/2]+1}(\Lambda)/P_{n+1}(\Lambda) \) is abelian, there is an element \( p(B) \in P_{[n/2]+1}(\Lambda) \) which is equivalent to

\[
1 + \begin{pmatrix}
-o^{m-1}BB\nu^{-1}/2 & 0 & 0 & o^{2m-3}(BB)^2\nu^{-2}/4 \\
0 & 0 & 0 & 0 \\
0 & 0 & -o^{m-1}BB\nu^{-1} & 0 \\
0 & 0 & 0 & -o^{m-1}BB\nu^{-1}/2
\end{pmatrix}
\]
modulo $a_{n+1}(\Lambda)$. Thus we have $[sx_s,v]p(B)^{-1} \in P_{[n/2]+1}(\Lambda)$ and $[sx_s,v]p(B)^{-1} \equiv 1 \pmod{a_n(\Lambda) + B^\perp}$. This implies $[sx_s,v]p(B)^{-1}$ lies in $J$. So we have

$$f_1 \ast \delta_{sx(a,A,B)us} \ast f_1 = \Psi^{-1}([sx_s,v]p(B)^{-1})f_1 \ast \delta_{p(B)usv} \ast f_1 = \psi_0(2(ABY - \overline{ABY})\nu^{-1})f_1 \ast \delta_{p(B)usv} \ast f_1 = \psi_0(tr_{F/F_0}(2ABY \nu^{-1}))f_1 \ast \delta_{p(B)usv} \ast f_1.$$  

We therefore have

$$f_s \ast f_u \ast f_s = [JsJ : J] \sum_{a \in \mathfrak{o}/p_0^2, A,B \in \mathfrak{o}/p} \psi_0(tr_{F/F_0}(2ABY \nu^{-1}))f_1 \ast \delta_{p(B)usv} \ast f_1 = [JsJ : J] \sum_{a \in \mathfrak{o}/p_0^2} f_1 \ast \delta_{usv} \ast f_1 = q^2 [JsJ : J] \sum_{a \in \mathfrak{o}/p_0^2} f_1 \ast \delta_{usv} \ast f_1 = q^2 \sum_{a \in \mathfrak{o}/p_0^2} f_{usv} = q^4 f_{u(\omega \mu^{-1})} \ast f_{s_2} \ast f_{h(-\mu^{-1})} \ast f_{\mu(\omega \mu^{-1})}.$$  

**Remark 6.10.** If $m = 1$, the algebra $H(G_E/J, \Psi)$ is generated by the elements $e_g$, $g \in B' \cup S$. In this case, these elements are subject to the following relations:

(i) $e_k = \Psi(k)e_1$, $k \in J'$,

(ii) $e_k \ast e_{k'} = e_{kk'}$, $k, k' \in B'$,

(iii) $e_k \ast e_s = e_s \ast e_{sk}$, $s \in S$, $k \in B' \cap sB'S$,

(iv) $e_{s_1} \ast e_{s_1} = e_1$,

$$e_{s_2} \ast e_{s_2} = (\sum_{y \in \mathfrak{o}/p_0, \gamma / p_0 \in \mathfrak{o}/p_0, \gamma \neq 0} e_{s_2} \ast e_{h(-y^{-1})} + q^2 e_1)(\sum_{x \in \mathfrak{o}/p_0, x \in \mathfrak{e}_0} e_{u(x \sqrt{\gamma})},$$

(v) For $\mu \in \mathfrak{m}_0 \sqrt{\mathfrak{e}}$, $e_{s_1} \ast e_{\mu(\mu^{-1})} \ast e_{s_1} \ast e_{h(\mu)} \ast e_{\mu(\mu^{-1})}$.

We can easily see that the analogue of Theorem 6.5 holds as well. We omit the details.

By Theorems 6.8 and 6.9, the map

$$\eta(e_{s_1}) = f_{s_1}, \eta(e_{s_2}) = q^{-2}f_{s_2}, \eta(e_b) = f_b, b \in B'$$

induces an algebra homomorphism $\eta$. Observe that $\eta(e_g) = (\text{vol}(J'gJ')/\text{vol}(JgJ))^{1/2} f_g$, for $g \in G_E$. Thus Propositions 2.7 and 2.11 imply that $\eta$ is a bijection. The $*$-preservation of $\eta$ follows from the relations $e_g^* = e_g^{-1}$ and $f_g^* = f_{g^{-1}}$, for $g \in G_E$.

**6.5. Case (6-iii').** Let $[\Lambda, n, n-1, \beta]$ be a fundamental skew stratum of type (6-iii'). Put

$$t = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & \omega & 0 & 0 \\ \omega & 0 & 0 & 0 \end{pmatrix}. \quad (6.6)$$

Then $t$ is a similitude on $(V, f)$, and hence we can consider the action of $t$ on the set of skew strata in $A$. Observe that $t\Lambda$ is a translate of $\Lambda_t$. So the stratum $[\Lambda, n, n-1, \beta]$ is a $t$-conjugate of a stratum of type (6-ii'). Replacing the objects in case (6-ii') with the
t-conjugate of them, we get a classification of the irreducible smooth representations of $G$ which contain $[\Lambda, n, n-1, \beta]$.

6.6. Intertwining problems. In this section, we consider the condition when an irreducible smooth representation $\pi$ of $G$ contains two skew skew strata of type (6-ia), (6-ib), (6-ii'), or (6-iii').

**Proposition 6.11.** Let $[\Lambda_3, 4m - 2, 4m - 3, \beta]$ be a skew stratum of type (6-ia) and let $[\Lambda, n, n-1, \gamma]$ be a skew stratum of type (6-ia), (6-ib), (6-ii'), or (6-iii'). Suppose that there is an irreducible smooth representation of $G$ which contains both of them. Then $[\Lambda, n, n-1, \gamma]$ is also of type (6-ia) and $(P_n(\Lambda), \psi_\gamma)$ is a $P_0(\Lambda_3)$-conjugate of $(P_{3m-2}(\Lambda_3), \psi_\beta)$.

**Proof.** The assumption implies that $\phi_\beta(X) = \phi_\gamma(X)$, and hence $[\Lambda, n, n-1, \beta]$ is also of type (6-ia). By the uniqueness of level, we have $n = 4m - 2$. Take $\beta$ and $\gamma$ as in (6.1). Then it is easy to see that $\gamma$ is a $P_0(\Lambda_3)$-conjugate of $\beta$. \hfill $\square$

**Proposition 6.12.** Let $[\Lambda_3, 4m - 2, 4m - 3, \beta]$ be a skew stratum of type (6-ib) and let $[\Lambda, n, n-1, \gamma]$ be a skew stratum of type (6-ia), (6-ib), (6-ii'), or (6-iii'). Suppose that there is an irreducible smooth representation of $G$ which contains both of them. Then $[\Lambda, n, n-1, \gamma]$ is also of type (6-ib) and $(P_n(\Lambda), \psi_\gamma)$ is a $P_0(\Lambda_3)$-conjugate of $(P_{3m-2}(\Lambda_3), \psi_\beta)$.

**Proof.** Since $\phi_\beta(X) = (X - ad\varepsilon)^2(X + Y\overline{Y})^2$ (mod $p_F$) and $ad\varepsilon \not\equiv -Y\overline{Y}$ (mod $p_F$), the stratum $[\Lambda, n, n-1, \gamma]$ is of type (6-ib), (6-ii'), or (6-iii'). By the assumption, we have $m - 1/2 = n/e(\Lambda)$, so we can take $\gamma$ to be

$$\gamma = \omega^{-m} \begin{pmatrix} 0 & 0 & 0 & b\sqrt{\varepsilon} \\ 0 & 0 & Z & 0 \\ 0 & -\omega \overline{Z} & 0 & 0 \\ \omega c \sqrt{\varepsilon} & 0 & 0 & 0 \end{pmatrix}, \quad Z \in o_F, \quad b, c \in o_0. \quad (6.7)$$

By the uniqueness of characteristic polynomial, we obtain $(X - ad\varepsilon)^2(X + Y\overline{Y})^2 = (X - b\varepsilon)^2(X + Z\overline{Z})^2$ (mod $p_F$).

Put $V^1 = Fe_1 \oplus Fe_4$ and $V^2 = Fe_2 \oplus Fe_3$. Then the strata $[\Lambda_3, 4m - 2, 4m - 3, \beta]$ and $[\Lambda, n, n-1, \gamma]$ are split with respect to $V = V^1 \bot V^2$.

Put $\mathcal{M} = A^{11} \oplus A^{22}$. By Proposition 2.11, there is an element $g \in G$ such that $(\beta + a_{3-4m}(\Lambda_3) \cap \mathcal{M}) \cap \text{Ad}(g)(\gamma + a_{1-n}(\Lambda) \cap \mathcal{M}) \neq \emptyset$. Take $\delta$ in this intersection. Then we get two skew strata $[\Lambda_3, n, n-1, \delta]$ and $[\Lambda, n, n-1, \text{Ad}(g^{-1})(\delta)]$.

By Hensel’s lemma, there are monic polynomials $f_1(X), f_2(X)$ in $o_F[X]$ such that $f_1(X) \equiv (X - ad\varepsilon)^2$ (mod $p_F$), $f_2(X) \equiv (X + Y\overline{Y})^2$ (mod $p_F$), and $\Phi_\delta(X) = f_1(X) \cdot f_2(X)$. Because we take $\delta$ in $\mathcal{M}$, we get ker $f_1(y_6) = V^1$ and ker $f_2(y_6) = V^2$.

Similarly, there are monic polynomials $g_1(X), g_2(X)$ in $o_F[X]$ such that $g_1(X) \equiv (X - b\varepsilon)^2$ (mod $p_F$), $g_2(X) \equiv (X + Z\overline{Z})^2$ (mod $p_F$), and $\Phi_{\text{Ad}(g^{-1})(\delta)} = g_1(X) \cdot g_2(X)$. We get ker $g_1(y_6) = gV^1$ and ker $g_2(y_6) = gV^2$.

Since $\Phi_\delta(X) = \Phi_{\text{Ad}(g^{-1})(\delta)}(X)$, we see that $f_1(X) = g_1(X)$ or $g_2(X)$, and hence $V^1 = \text{ker } g_1(y_6)$ or $\text{ker } g_2(y_6)$. Since $V^1$ is isotropic and $V^2$ is anisotropic, we have $V^1 = gV^1$ and $V^2 = gV^2$. So we conclude $g \in G \cap \mathcal{M}$.

As in the proof of the uniqueness of the characteristic polynomial, we have $ad\varepsilon \equiv b\varepsilon$ and $Y\overline{Y} \equiv Z\overline{Z}$ modulo $p_F$. This implies that $[\Lambda, n, n-1, \gamma]$ is also of type (6-ib) and a $P_0(\Lambda_3)$-conjugate of $[\Lambda_3, 4m - 2, 4m - 3, \beta]$. \hfill $\square$
Remark 6.13. Let $n = 6m - 3$ and let $\beta$ be as in (6.3):

(i) We claim that a skew stratum $[\Lambda_4, n, n - 1, \beta]$ is not $G$-conjugate of $[\Lambda_5, n, n - 1, \beta]$. Let $\Lambda$ be a self-dual $\mathfrak{o}_F$-lattice sequence. From the group of $P_n(\Lambda)$, we get $a_n(\Lambda)_{\mathfrak{a}}$ as the image of Cayley map $x \mapsto (1 + x)(1 - x)^{-1}$. Since $F/F_0$ is unramified, we get $a_n(\Lambda) = a_n(\Lambda)_{\mathfrak{a}} \oplus \sqrt{c}a_n(\Lambda)_{\mathfrak{a}}$. So we can recover $a_n(\Lambda)$ and $a_{1-n}(\Lambda) = a_n(\Lambda)^*_{\mathfrak{a}}$ from $P_n(\Lambda)$.

Return to the skew stratum $[\Lambda_4, n, n - 1, \beta]$. By the periodicity of $\Lambda$, we can recover $a_3(\Lambda_4)$ and $a_4(\Lambda_4)$. Observe that $\Lambda_4(\mathbb{Z})$ consists of all $\mathfrak{o}_F$-lattices of the form $a_4(\Lambda_4) \cdot L$, where $L$ is an $\mathfrak{o}_F$-lattice in $V$ and $k = 3, 4$. The sequence $\Lambda_5$ has the same property.

Suppose that there is $g \in G$ so that $P_n(\Lambda_4) = Ad(g)(P_n(\Lambda_5))$. Then we have $\Lambda_5(\mathbb{Z}) = \Lambda_4(\mathbb{Z}) = g\Lambda_4(\mathbb{Z})$. This implies $g \in P_0(\Lambda_5)$ and hence $P_n(\Lambda_4) = P_n(\Lambda_5)$. This contradict the fact $a_n(\Lambda_4) \neq a_n(\Lambda_5)$.

(ii) By Proposition 6.13 if an irreducible smooth representation $\pi$ of $G$ contains $[\Lambda_3, 4m - 2, 4m - 3, \beta]$, then $\pi$ contains both of $[\Lambda_4, n, n - 1, \beta]$ and $[\Lambda_5, n, n - 1, \beta]$.

7. Comparison

There is another unramified unitary group in four variables defined over $F_0$ denoted by $U(2, 2)$, which is quasi-split and an inner form of the non quasi-split $U(4)$. In this section, we compare the irreducible smooth representations of non quasi-split $U(4)$ with $U(2, 2)$ of non-integral level.

Let $G$ denote the non quasi-split $U(4)$ or unramified $U(2, 2)$ defined over $F_0$. The results on $U(2, 2)$ analogous to this paper can be found in [8].

We list up the characteristic polynomials $\phi(\beta)(X)$ and the form of the groups $G_E$ for semisimple skew strata $[\Lambda, m, m - 1, \beta]$ of $G$ of non-integral level in this paper and [8]. The level of a fundamental skew stratum of $G$ should be $n, n/2, n/3$ or $n/4$, for some positive integer $n$.

(i) level $n/4$: The characteristic polynomial has the form $\phi(\beta)(X) = (X - a)^4$, for $a \in k_0^\times$. The algebra $E = F[\beta]$ is a totally ramified extension of degree 4 over $F$ and $G_E$ is isomorphic to the unramified unitary group $U(1)(E/E_0)$.

(ii) level $n/3$: The characteristic polynomial is of the form $\phi(\beta)(X) = (X - a\sqrt{\epsilon})^3X$, for $a \in k_0^\times$. The algebra $E = F[\beta]$ is isomorphic to $E_1 \oplus F$, where $E_1$ is a totally ramified extension of degree 3 over $F$. The group $G_E$ is isomorphic to a product of unramified unitary groups $U(1)(E_1/E_{1,0}) \times U(1)(F/F_0)$.

(iii) level $n/2$: The characteristic polynomial $\phi(\beta)(X)$ has one of the following form:

(iii-a) $(X - a)^4$, for $a \in k_0^\times$.

(iii-b) $(X - a)^2(X - b)^2$, for $a, b \in k_0^\times$ such that $a \neq b$.

(iii-c) $(X - a)^2X$, for $a \in k_0^\times$.

Case (iii-a): The stratum is simple and $E$ is a quadratic ramified extension over $F$. If $G = U(2, 2)$, then $G_E$ is isomorphic to $U(1, 1)(E/E_0)$. If $G$ is not quasi-split, then $G_E$ is isomorphic to $U(2)(E/E_0)$.

Case (iii-b): The algebra $E$ is isomorphic to $E_1 \oplus E_2$, where $E_i$ is a quadratic ramified extension over $F$, for $i = 1, 2$. The group $G_E$ is isomorphic to $U(1)(E_1/E_{1,0}) \times U(1)(E_2/E_{2,0})$.

Case (iii-c): The algebra $E$ is isomorphic to $E_1 \oplus F$, where $E_1$ is a quadratic ramified extension over $F$. The group $G_E$ is isomorphic to $U(1)(E_1/E_{1,0}) \times U(1)(F/F_0)$ or $U(1)(E_1/E_{1,0}) \times U(2)(F/F_0)$. 


Remark 7.1. The difference is only case (iii-a). We can see that the set of the irreducible supercuspidal representations of $U(1,1)(E/E_0)$ is very close to that of $U(2)(E/E_0)$ by establishing Hecke algebra isomorphisms for those groups.

When $G$ is not quasi-split, an irreducible smooth representation of $G$ of non-integral level contains one of skew semisimple strata listed above. For $U(2,2)$, we need to consider semisimple (but not skew semisimple), skew strata of the following type:

(iii-d) The level is half-integral and the characteristic polynomial $\phi_{\beta}(X)$ has the form $(X - \lambda)^2(X - \overline{\lambda})^2$, for $\lambda \in k_F^\times$, $\lambda \neq \overline{\lambda}$. In this case, $E = E_1 \oplus E_2$, where $E_i$ is quadratic ramified for $i = 1, 2$, and $G_E$ is isomorphic to $GL_1(E_i)$.

Remark 7.2. A stratum of case (iii-d) is called $G$-split in [17]. It follows from the proof of [17] Theorem 3.6 that if an irreducible smooth representation $\pi$ of $G = U(2,2)(F/F_0)$ contains a stratum of type (iii-d), then there is a parabolic subgroup $P$ of $G$ whose Levi component is isomorphic to $GL_2(F)$, such that the Jacquet module of $\pi$ relative to $P$ is not zero. This difference yields from the fact that the non quasi-split $U(4)$ has no parabolic subgroups of such type.

We close this paper with a table of skew semisimple stratum $[\Lambda, n, r, \beta]$ for the non quasi-split $U(4)$ we have considered.

| $\Lambda$ | $n$ | $r$ | $\beta$ | section |
|-----------|-----|-----|---------|---------|
| $A_3$     | $(n, 4) = 1$ | $n/2$ | fundamental | § 4     |
| $A_4$     | $(n, 6) = 2$ | $n/2$ | fundamental | § 5     |
| $A_5$     | $(n, 6) = 2$ | $n/2$ | fundamental | § 5     |
| $A_3$     | $(n, 4) = 2$ | $n - 1$ | $[6.1]$, $ad\varepsilon = -YY$ | § 6.2   |
| $A_3$     | $(n, 4) = 2$ | $n - 1$ | $[6.1]$, $ad\varepsilon \neq -YY$, $ad \neq 0$ | § 6.3   |
| $A_4$     | $(n, 6) = 3$ | $n - 1$ | $[6.3]$ | § 6.4   |
| $A_5$     | $(n, 6) = 3$ | $n - 1$ | $[6.3]$ | § 6.5   |

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