Congruences for partition functions related to mock theta functions

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**Abstract.** Partitions associated with mock theta functions have received a great deal of attention in the literature. Recently, Choi and Kim derived several partition identities from the third and sixth order mock theta functions. In addition, three Ramanujan-type congruences were established by them. In this paper, we present some new congruences for these partition functions.

**Keywords.** Partition, t-core partition, cubic partition, mock theta function, Ramanujan-type congruence.

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**1. Introduction**

A partition of a positive integer $n$ is a finite nonincreasing sequence of positive integers whose sum equals $n$. Furthermore, a partition is called a $t$-core partition if there are no hook numbers being multiples of $t$. Let $a_t(n)$ be the number of $t$-core partitions of $n$. It is known [17] that

$$
\sum_{n=0}^{\infty} a_t(n)q^n = \frac{(q^t; q^t)_\infty}{(q; q)_\infty}.
$$

Here and in what follows, we make use of the standard $q$-series notation (cf. [18]).

$$(a)_n = (a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k),$$

$$(a)_\infty = (a; q)_\infty := \prod_{k=0}^{\infty} (1 - aq^k),$$

$$(a_1, a_2, \cdots, a_m; q)_\infty := (a_1; q)_\infty (a_2; q)_\infty \cdots (a_m; q)_\infty.$$

In his last letter to Hardy [9, pp. 220–223], Ramanujan defined 17 functions, which he called mock theta functions. Since then, there has been an intense study of partition interpretations for mock theta functions; see [2, 3, 4, 5, 6].
Recently, Choi and Kim [15] obtained the following identity related to the third order mock theta function,

\[ v(q) + v_3(q, q; q) = 2 \frac{(q^4; q^4)^3}{(q^2; q^2)^2} \]

where \( v(q) \) is the third mock theta function and \( v_3(q, q; q) \) is defined by Choi [14],

\[ v(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q; q^2)_{n+1}}, \quad v_3(q, q; q) = \sum_{n=0}^{\infty} q^n (-q; q^2)_n. \]

They also gave the following identities related to the sixth order mock theta functions,

\[ \Psi(q) + 2\Psi_-(q) = 3 \frac{q(q^6; q^6)_3}{(q; q)_6(q^2; q^2)_2}, \]

\[ 2\rho(q) + \lambda(q) = 3 \frac{(q^2; q^2)_2}{(q; q)_2(q^2; q^2)_2}, \]

where \( \Psi(q), \Psi_-(q), \rho(q) \) and \( \lambda(q) \) are the sixth order mock theta functions,

\[ \Psi(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)} (q; q^2)_n}{(-q; q)_{2n+1}}, \quad \Psi_-(q) = \sum_{n=1}^{\infty} \frac{q^n (-q; q)_{2n-2}}{(q; q^2)_n}, \]

\[ \rho(q) = \sum_{n=0}^{\infty} q^{n+1} (-q; q)_n (q; q^2)_{n+1}, \quad \lambda(q) = \sum_{n=0}^{\infty} (-1)^n q^n (q; q^2)_n. \]

Meanwhile, they studied three analogous partition functions defined by

\[ \sum_{n=0}^{\infty} b(n)q^n = \frac{(q^4; q^4)^3}{(q^2; q^2)^2} \]  \hfill (1.1)

\[ \sum_{n=0}^{\infty} c(n)q^n = \frac{q(q^6; q^6)_3}{(q; q)_6(q^2; q^2)_2} \]  \hfill (1.2)

\[ \sum_{n=0}^{\infty} d(n)q^n = \frac{(q^2; q^2)_2}{(q; q)_2(q^2; q^2)_2} \]  \hfill (1.3)

where \( b(n) \) denotes the number of partition pairs \((\lambda, \sigma)\) where \( \sigma \) is a partition into distinct even parts and \( \lambda \) is a partition into even parts of which 2-modular diagram is 2-core, and both \( c(n) \) and \( d(n) \) can be regarded as 3-core cubic partitions.

In this paper, we mainly study Ramanujan-type congruences for these partition functions. This paper is organized as follows. In Sect. 2, we introduce some preliminary results. In the next two sections, we will prove some Ramanujan-type congruences for \( b(n) \) and \( c(n) \), respectively. In Sect. 5, by employing \( p \)-dissection formulas of Ramanujan’s theta functions \( \psi(q) \) and \( f(-q) \) established by Cui and Gu [16] as well as \((p, k)\)-parameter representations due to Alaca and Williams [1], we show some congruences for \( d(n) \). Finally, we end this paper with several open problems.
2. Preliminaries

Let \( f(a, b) \) be Ramanujan’s general theta function given by

\[
f(a, b) = \sum_{n=-\infty}^{\infty} a^{n^2} b^{n(n+1)/2}, \quad |ab| < 1.
\]

We now introduce the following Ramanujan’s classical theta functions,

\[
\varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{f_5^2}{f_1^2} f_4^2, \tag{2.1}
\]

\[
\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{f_2}{f_1}, \tag{2.2}
\]

\[
f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} = f_1. \tag{2.3}
\]

One readily verifies

\[
\varphi(-q) = f_1^2. \tag{2.4}
\]

Here and in the sequel, we write \( f_k := (q^k; q^k)_\infty \) for positive integers \( k \) for convenience.

We first require the following 2-dissections.

**Lemma 2.1.** It holds that

\[
\begin{align*}
\frac{1}{f_1^2} &= \frac{f_3^2}{f_2 f_6} + 2q \frac{f_3^2 f_6 f_{10}}{f_2 f_8}, \tag{2.5} \\
\frac{f_3}{f_1^2} &= \frac{f_3^3 f_6}{f_2 f_{12}} + 3q \frac{f_3^2 f_6 f_{12}}{f_2^2}, \tag{2.6} \\
\frac{f_3^3}{f_1} &= \frac{f_3^2 f_6}{f_2 f_{12}} + q \frac{f_3^3 f_4}{f_4}. \tag{2.7}
\end{align*}
\]

**Proof.** Here (2.5) comes from the 2-dissection of \( \varphi(q) \) (cf. [8, p. 40, Entry 25]). For (2.6) and (2.7), see [25]. \( \square \)

The following 3-dissections are also necessary.

**Lemma 2.2.** It holds that

\[
\begin{align*}
\frac{1}{\varphi(-q)} &= \frac{\varphi^3(-q^9)}{\varphi^4(-q^3)} \left(1 + 2qw(q^3) + 4q^2 w^2(q^3)\right), \tag{2.8} \\
\frac{1}{\psi(q)} &= \frac{\psi^3(q^3)}{\psi^4(q^3)} \left(\frac{1}{w^2(q^3)} - \frac{q}{w(q^3)} + q^2\right), \tag{2.9}
\end{align*}
\]

where

\[
w(q) = \frac{f_1 f_3^2}{f_2 f_6}. \tag{2.10}
\]

Furthermore,

\[
\frac{1}{f_1^3} = \frac{f_3^3}{f_3^2} \left(P^2(q^3) + 3q P(q^3) f_3^3 + 9q^2 f_3^5\right), \tag{2.11}
\]

where

\[
P(q) = f_1 \left(\frac{\varphi^3(-q^3)}{\varphi(-q)} + 4q \frac{\psi^3(q^3)}{\psi(q)}\right), \tag{2.12}
\]
Proof. For (2.8) and (2.9), see Baruah and Ojah [7]. For (2.11), see Wang [23]. Note that Wang [23] showed

\[ P(q) = f_1 \left( 1 + 6 \sum_{n \geq 0} \left( \frac{q^{3n+1}}{1 - q^{3n+1}} - \frac{q^{3n+2}}{1 - q^{3n+2}} \right) \right). \]

We know from [22, Eqs. (3.2) and (3.5)] that

\[ 4q \psi^3(q^3) \psi(q) = 4 \sum_{n \geq 0} \left( \frac{q^{3n+1}}{1 - q^{3n+1}} - \frac{q^{3n+2}}{1 - q^{3n+2}} \right), \]

\[ \frac{\varphi^3(-q^3)}{\varphi(-q)} = 1 + 2 \sum_{n \geq 0} \left( \frac{q^{6n+1}}{1 - q^{6n+1}} + \frac{q^{6n+2}}{1 - q^{6n+2}} - \frac{q^{6n+4}}{1 - q^{6n+4}} - \frac{q^{6n+5}}{1 - q^{6n+5}} \right). \]

Hence (2.12) follows immediately by the following trivial identity

\[ \frac{x}{1 - x^2} = \frac{x}{1 - x} - \frac{x^2}{1 - x^2}. \]

\[ \square \]

Furthermore, we need

**Lemma 2.3** ([16, Theorem 2.1]). For any odd prime \( p \),

\[ \psi(q) = q^{\frac{p-3}{8}} \psi^3(q^3) + \sum_{k=0}^{p-3} q^{\frac{k^2+k}{2}} f \left( q^{\frac{2^3+2k+1}{2}}, q^{\frac{2^3-(2k+1)}{2}} \right). \]

Furthermore, we claim that for \( 0 \leq k \leq (p - 3)/2 \),

\[ \frac{k^2 + k}{2} \neq \frac{p^2 - 1}{8} \pmod{p}. \]

**Lemma 2.4** ([16, Theorem 2.2]). For any prime \( p \geq 5 \),

\[ f(-q) = (-1)^{\frac{p-1}{6}} q^{2-k+1} f(-q^3) \]

\[ + \sum_{k=-\frac{p-1}{6}}^{\frac{p-1}{6}} (-1)^k q^{3k^2+k} f \left( q^{\frac{3k^2+(6k+1)p}{2}}, q^{\frac{3k^2-(6k+1)p}{2}} \right). \]

Furthermore, we claim that for \( -(p - 1)/2 \leq k \leq (p - 1)/2 \) and \( k \neq (\pm p - 1)/6 \),

\[ \frac{3k^2 + k}{2} \neq \frac{b^2 - 1}{24} \pmod{p}. \]

Here for any prime \( p \geq 5 \),

\[ \frac{\pm p - 1}{6} := \begin{cases} \frac{p-1}{6}, & p \equiv 1 \pmod{6}, \\ -\frac{p+1}{6}, & p \equiv -1 \pmod{6}. \end{cases} \]

At last, we require the following relations due to Alaca and Williams [1].
Lemma 2.5. Let
\[ p = p(q) := \frac{\varphi^2(q) - \varphi^2(q^3)}{2\varphi^2(q^3)}, \]
and
\[ k = k(q) := \frac{\varphi^3(q^3)}{\varphi(q)}. \]
Then
\[ f_1 = 2^{-\frac{1}{2}}q^{-\frac{1}{2}}p^\frac{1}{2}p(1-p)^\frac{1}{2}(1+p)^\frac{1}{2}(1+2p)^\frac{1}{2}(2+p)^\frac{1}{2}k^\frac{1}{2}, \]
\[ f_2 = 2^{-\frac{1}{2}}q^{-\frac{1}{2}}p^\frac{1}{2}p(1-p)^\frac{1}{2}(1+p)^\frac{1}{2}(1+2p)^\frac{1}{2}(2+p)^\frac{1}{2}k^\frac{1}{2}, \]
\[ f_3 = 2^{-\frac{1}{2}}q^{-\frac{1}{2}}p^\frac{1}{2}(1-p)^\frac{1}{2}(1+p)^\frac{1}{2}(1+2p)^\frac{1}{2}(2+p)^\frac{1}{2}k^\frac{1}{2}, \]
\[ f_4 = 2^{-\frac{1}{2}}q^{-\frac{1}{2}}p^\frac{1}{2}(1-p)^\frac{1}{2}(1+p)^\frac{1}{2}(1+2p)^\frac{1}{2}(2+p)^\frac{1}{2}k^\frac{1}{2}, \]
\[ f_6 = 2^{-\frac{1}{2}}q^{-\frac{1}{2}}p^\frac{1}{2}(1-p)^\frac{1}{2}(1+p)^\frac{1}{2}(1+2p)^\frac{1}{2}(2+p)^\frac{1}{2}k^\frac{1}{2}, \]
\[ f_{12} = 2^{-\frac{1}{2}}q^{-\frac{1}{2}}p^\frac{1}{2}(1-p)^\frac{1}{2}(1+p)^\frac{1}{2}(1+2p)^\frac{1}{2}(2+p)^\frac{1}{2}k^\frac{1}{2}. \]

3. Congruences for \( b(n) \)

Theorem 3.1. For \( n \geq 0, \alpha \geq 1, \) and prime \( p \geq 5, \) we have
\[ b\left(p^{2\alpha}n + \frac{(3j+p)p^{2\alpha-1} - 1}{3}\right) \equiv 0 \pmod{2}, \quad (3.1) \]
where \( j = 1, 2, \ldots, p-1. \)

Proof. In light of (1.1), we derive that
\[ \sum_{n=0}^{\infty} b(n)q^n = \frac{f_3^4}{f_2} \equiv f_8 \pmod{2}. \]

Applying Lemma 2.4, we deduce that, for any prime \( p \geq 5, \)
\[ \sum_{n=0}^{\infty} b\left(p n + \frac{p^2 - 1}{3}\right) q^n \equiv (-1)^{\frac{p-1}{8}}f(-q^{8p}) \pmod{2}, \]
and
\[ \sum_{n=0}^{\infty} b\left(p^2 n + \frac{p^2 - 1}{3}\right) q^n \equiv (-1)^{\frac{p-1}{8}}f(-q^8) \pmod{2}. \]

Moreover,
\[ \sum_{n=0}^{\infty} b\left(p^3 n + \frac{p^4 - 1}{3}\right) q^n \equiv f(-q^{8p}) \pmod{2}. \]

Hence, by induction on \( \alpha, \) we derive that, for \( \alpha \geq 1, \)
\[ \sum_{n=0}^{\infty} b\left(p^{2\alpha}n + \frac{p^{2\alpha} - 1}{3}\right) q^n \equiv (-1)^{\alpha\left(\frac{p-1}{8}\right)}f(-q^{8p}) \pmod{2}. \]

This immediately leads to
\[ b\left(p^{2\alpha}(pn + j) + \frac{p^{2\alpha} - 1}{3}\right) \equiv 0 \pmod{2}, \]
for \( j = 1, 2, \ldots, p-1. \) We complete the proof. \( \square \)
Remark 3.1. When studying 1-shell totally symmetric plane partition function \( f(n) \) (which is different to Ramanujan’s theta function \( f(-q) \) given in Sect. 2) introduced by Blecher [10], Hirschhorn and Sellers [19] proved that, for \( n \geq 1 \),
\[
\begin{align*}
f(3n - 2) &= h(n),
\end{align*}
\]
with
\[
\sum_{n=0}^{\infty} h(2n+1)q^n = \frac{f_3^3}{f_1}.
\]

A couple of congruences modulo powers of 2 and 5 for \( h(n) \) have been obtained subsequently; see [13, 24, 26]. We see from (1.1) that
\[
b(2n) = h(2n+1).
\]
One therefore may obtain some congruences for \( b(n) \) as well. For example,
\[
b(8n+6) \equiv 0 \pmod{4}.
\]

4. Congruences for \( c(n) \)

Theorem 4.1. For \( n \geq 0 \), we have
\[
c(27n + 24) \equiv 0 \pmod{9}. \tag{4.1}
\]

Proof. We see from (1.2) and Lemma 2.2 that
\[
\begin{align*}
\sum_{n=0}^{\infty} c(n)q^n &= \frac{Qf_6^3}{\varphi(-q)\psi(q)} \\
&= qf_6^3 \frac{\varphi^3(-q^9)\psi^3(q^9)}{\varphi^4(-q^9)\psi^4(q^9)} \left(1 + 2qw(q^3) + 4q^2w^2(q^3)\right) \\
&\times \left(\frac{1}{w^2(q^3)} - \frac{q}{w(q^3)} + q^2\right). \tag{4.2}
\end{align*}
\]

Employing Lemma 2.2, we deduce that
\[
\begin{align*}
\sum_{n=0}^{\infty} c(3n)q^n &= \frac{3q\varphi^3(-q^3)\psi^3(q^3)}{f_1^2\varphi(-q)\psi(q)} \\
&= \frac{3q\varphi^3(-q^9)\psi^3(q^9)f_3^3}{\varphi(-q^9)\psi(q^9)f_3^2} \left(P^2(q^3) + 3qP(q^3)f_3^3 + 9q^2f_6^2\right) \\
&\times \left(1 + 2qw(q^3) + 4q^2w^2(q^3)\right) \left(\frac{1}{w^2(q^3)} - \frac{q}{w(q^3)} + q^2\right). \tag{4.2}
\end{align*}
\]

Extracting terms involving \( q^{3n+2} \) and replacing \( q^3 \) by \( q \) in (4.2), it follows that
\[
\sum_{n=0}^{\infty} c(9n + 6)q^n = 12f_2^2f_6^2f_{10}f_{12} + 135q^2f_3^2f_6^2f_{10}f_{12} + 72q^3f_3^3f_6^2f_{10}f_{12} + 192q^3f_6^3f_{10}f_{12}f_{14}.
\]

Hence,
\[
\begin{align*}
\sum_{n=0}^{\infty} c(9n + 6)q^n &\equiv 3f_2^2f_{10}f_6^2 + 3q^3f_6^3f_{14}f_6 \pmod{9} \\
&\equiv 3f_2^2 \left(\frac{f_3^{16}}{f_6} + q^3\frac{f_6^{18}}{f_6^8}\right) \pmod{9}.
\end{align*}
\]
Noting that \( f_2^2/f_1 \) contains no terms of the form \( q^{3n+2} \), we have
\[
\sum_{n=0}^{\infty} c(27n + 24)q^n \equiv 0 \pmod{9}.
\]
It therefore ends the proof. \( \square \)

**Theorem 4.2.** For \( n \geq 0 \), we have
\[
c(45n + t) \equiv 0 \pmod{5},
\]
where \( t = 9 \) and 18.

**Proof.** Referring to (4.2), we have
\[
\sum_{n=0}^{\infty} c(9n)q^n = 45q f_2 f_1^{30} f_3^{18} + 90q^2 f_2^3 f_6^6 f_1^{12} f_2 + 288q^3 f_2^{15} f_3^9 f_6^6 f_1^9 f_2^2.
\]
Hence,
\[
\sum_{n=0}^{\infty} c(9n)q^n \equiv 3q^3 f_1 f_2 f_5 f_1^{10} \pmod{5}.
\]
Since \( f_1 \) contains no terms of the form \( q^{5n+3} \) and \( q^{5n+4} \), we have
\[
c(9(5n + 1)) = c(45n + 9) \equiv 0 \pmod{5},
\]
and
\[
c(9(5n + 2)) = c(45n + 18) \equiv 0 \pmod{5}.
\]
This yields that (4.3). \( \square \)

**Corollary 4.3.** For \( n \geq 0 \), we have
\[
c(45n + t) \equiv 0 \pmod{15},
\]
where \( t = 9 \) and 18.

**Proof.** We know from [15, Theorem 4.2] that
\[
c(3n) \equiv 0 \pmod{3}.
\]
In fact, it is a direct consequence of (4.2). Hence, Corollary 4.3 follows by Theorem 4.2. \( \square \)

**5. Congruences for \( d(n) \)**

**Theorem 5.1.** For \( n \geq 0 \), \( \alpha \geq 1 \), and prime \( p \geq 3 \),
\[
d\left(2p^{2\alpha} + \frac{(8j + p)p^{2\alpha-1} - 1}{4}\right) \equiv 0 \pmod{2},
\]
where \( j = 1, 2, \ldots, p - 1 \).

**Proof.** From (1.3), one can see
\[
\sum_{n=0}^{\infty} d(n)q^n = f_3^3 f_3^3 f_1 f_2 \equiv f_6 f_3^3 f_3 f_1 \pmod{2}.
\]
With the help of (2.6), we have
\[
\sum_{n=0}^{\infty} d(n)q^n \equiv f_6 f_3^4 f_6 f_2 f_1^2 f_2 + 3q f_3^2 f_6 f_2 f_1^2 f_2 \pmod{2}.
\]
Hence,
\[ \sum_{n=0}^{\infty} d(2n)q^n \equiv f_2^6 f_6^3 \equiv \psi(q) \pmod{2}. \]

Invoking Lemma 2.3, for any odd prime \( p \), we derive that
\[ \sum_{n=0}^{\infty} d \left( 2 \left( pn + \frac{p^2 - 1}{8} \right) \right) q^n \equiv \psi(q^p) \pmod{2}, \]
and
\[ \sum_{n=0}^{\infty} d \left( 2 \left( p^2 n + \frac{p^2 - 1}{8} \right) \right) q^n \equiv \psi(q) \pmod{2}. \]
Furthermore,
\[ \sum_{n=0}^{\infty} d \left( 2p^3 n + \frac{p^4 - 1}{4} \right) q^n \equiv \psi(q^p) \pmod{2}. \]

It therefore follows by induction on \( \alpha \) that for \( \alpha \geq 1, \)
\[ \sum_{n=0}^{\infty} d \left( 2p^{2\alpha-1} n + \frac{p^{2\alpha} - 1}{4} \right) q^n \equiv \psi(q^p) \pmod{2}. \]
Thus, for \( j = 1, 2, \ldots, p - 1, \)
\[ d \left( 2p^{2\alpha-1}(pm + j) + \frac{p^{2\alpha} - 1}{4} \right) \equiv 0 \pmod{2}, \]
which is the desired result. \( \square \)

**Theorem 5.2.** For \( n \geq 0, \alpha \geq 1, \) and prime \( p \geq 5, \) we have
\[ d \left( 6p^{2\alpha} n + \frac{(24j + p)p^{2\alpha-1} - 1}{4} \right) \equiv 0 \pmod{3}, \] where \( j = 1, 2, \ldots, p - 1. \)

**Proof.** It follows by (2.8) and (2.9) that
\[
\sum_{n=0}^{\infty} d(n) = \frac{f_3^6}{\varphi(q^3)\psi(q)} \\
= f_3^6 \frac{\varphi^3(-q^3)\psi^3(q^3) \left( 1 + 2qw(q^3) + 4q^2w^2(q^3) \right)}{\varphi^4(-q^3)\psi^4(q^3)} \times \left( \frac{1}{w^2(q^3)} - \frac{q}{w(q^3)} + q^2 \right). \tag{5.3}
\]
So we get
\[
\sum_{n=0}^{\infty} d(3n)q^n = f_1^3 \frac{\varphi^3(-q^3)\psi^3(q^3) \left( 1 + 2qw(q^3) + 4q^2w^2(q^3) \right)}{\varphi^4(-q^3)\psi^4(q^3)} \times \left( \frac{1}{w^2(q^3)} - 2qw(q) \right) \\
= \frac{1}{f_2^2 f_6^3} \left( \frac{f_1^3}{f_3^6} \right)^3 - 2q\frac{f_6^6}{f_2^3}. 
\]
Based on (2.7), we derive that
\[
\sum_{n=0}^{\infty} d(6n)q^n = \frac{f_2^3 f_3^3}{f_8^3 f_6^3} + 3q \frac{f_2^3 f_6^3}{f_2^3 f_6^3} \equiv \frac{f_2^3 f_3^3}{f_8^3 f_6^3} \equiv f_1 \pmod{3}.
\]

Invoking Lemma 2.4, we arrive at that, for any prime \( p \geq 5, \)
\[
\sum_{n=0}^{\infty} d \left( 6 \left( pn + \frac{p^2 - 1}{24} \right) \right) q^n \equiv (-1)^{\frac{p-1}{2}} f(-q^p) \pmod{3},
\]
and
\[
\sum_{n=0}^{\infty} d \left( 6 \left( p^2n + \frac{p^2 - 1}{24} \right) \right) q^n \equiv (-1)^\frac{p-1}{2} f(-q) \pmod{3}.
\]

Furthermore, we have
\[
\sum_{n=0}^{\infty} d \left( 6 \left( p^2n + \frac{p^2 - 1}{24} \right) + \frac{p^2 - 1}{24} \right) q^n \equiv f(-q^p) \pmod{3}.
\]
Namely,
\[
\sum_{n=0}^{\infty} d \left( 6p^3n + \frac{p^4 - 1}{4} \right) q^n \equiv f(-q^p) \pmod{3}.
\]

Thus, by induction on \( \alpha, \) we derive that, for \( \alpha \geq 1, \)
\[
\sum_{n=0}^{\infty} d \left( 6p^{2\alpha-1}n + \frac{p^{2\alpha} - 1}{4} \right) q^n \equiv (-1)^\alpha \left( \frac{p^{2\alpha} - 1}{4} \right) f(-q^p) \pmod{3}.
\]
This yields that, for \( j = 1, 2, \cdots, p-1, \)
\[
d \left( 6p^{2\alpha-1}(pn + j) + \frac{p^{2\alpha} - 1}{4} \right) \equiv 0 \pmod{3},
\]
which implies (5.2). \( \square \)

**Theorem 5.3.** For \( n \geq 0, \alpha \geq 1, \) and prime \( p \geq 5, \)
\[
d \left( 6p^{2\alpha}n + \frac{(24j + 9p)p^{2\alpha-1} - 1}{4} \right) \equiv 0 \pmod{9},
\]
where \( j = 1, 2, \cdots, p-1. \)

**Proof.** Extracting terms involving \( q^{3n+2} \) and replace \( q^3 \) by \( q \) in (5.3), then we derive that
\[
\sum_{n=0}^{\infty} d(3n + 2)q^n = \frac{3f_3^3 f_6^3}{\varphi(-q)\psi(q)f_2^3} = \frac{3f_3^3 f_6^3}{f_1 f_2^3} \quad (5.5)
\]
It follows by (2.7) that,
\[
\sum_{n=0}^{\infty} d(3n + 2)q^n = \frac{3f_3^3 f_6^3}{f_1 f_2^3} = \frac{3f_3^3 f_6^3}{f_2 f_{12}} + 3q \frac{f_3 f_6}{f_2 f_4}.
\]
Hence,
\[
\sum_{n=0}^{\infty} d(6n + 2)q^n = \frac{3f_3^3 f_6^3}{f_6^3} \equiv 3f_9 \pmod{9}.
\]
In view of Lemma 2.4, for any prime $p \geq 5$, we deduce that
\[
\sum_{n=0}^{\infty} d \left( 6 \left( pn + \frac{3(p^2 - 1)}{8} \right) + 2 \right) q^n \equiv 3(-1)^{\frac{p-1}{6}} f(-q^p) \quad (\text{mod } 9),
\]
and
\[
\sum_{n=0}^{\infty} d \left( 6 \left( p^2n + \frac{3(p^2 - 1)}{8} \right) + 2 \right) q^n \equiv 3(-1)^{\frac{p-1}{6}} f(-q^p) \quad (\text{mod } 9).
\]
Moreover,
\[
\sum_{n=0}^{\infty} d \left( 6 \left( p^3n + \frac{3(p^4 - 1)}{8} \right) + 2 \right) q^n \equiv 3f(-q^p) \quad (\text{mod } 9).
\]
Hence, by induction on $\alpha \geq 1$, we arrive at,
\[
\sum_{n=0}^{\infty} d \left( 6 \left( p^{2\alpha-1}n + \frac{3(p^{2\alpha} - 1)}{8} \right) + 2 \right) q^n \equiv 3(-1)^{\alpha(\frac{p-1}{6})} f(-q^p) \quad (\text{mod } 9),
\]
which implies that for $j = 1, 2, \cdots, p-1$,
\[
d \left( 6 \left( p^{2\alpha-1}(pm + j) + \frac{3(p^{2\alpha} - 1)}{8} \right) + 2 \right) \equiv 0 \quad (\text{mod } 9).
\]
This leads to (5.4).

\[\square\]

**Theorem 5.4.** For $n \geq 0$, we have
\[
d(45n + t) \equiv 0 \pmod{5},
\]
where $t = 17$ and 35.

**Proof.** From (5.5), we have
\[
\sum_{n=0}^{\infty} d(3n + 2)q^n = \frac{3f_3^3f_6^3}{\varphi(-q)\varphi(q)f_2^3}.
\]
Again by (2.8), (2.9) and (2.11), we have
\[
\sum_{n=0}^{\infty} d(9n + 8)q^n = f_2 \cdot H,
\]
where
\[
H = \left( \frac{9f_3^3f_4^6f_6^9}{f_1^2f_2^{12}f_4^{18}} + \frac{9f_3^3f_4^6f_6^{18}}{f_1^2f_2^{14}f_4^{18}} \right) + q \left( \frac{27f_3^6f_6^9}{f_1^2f_2^{14}f_4^{18}} - \frac{18f_4^3f_6^3}{f_2^{16}f_4^{18}} \right)
\]
\[
+ q^2 \left( \frac{36f_3^6f_4^{18}}{f_1^2f_2^{18}f_4^6} + \frac{72f_3^3f_6^3f_4^6}{f_1^2f_2^{18}f_4^6} + \frac{108f_6^3}{f_2^{16}f_4^3} \right)
\]
\[
- q^3 \frac{72f_3^6f_4^{12}}{f_1^2f_2^{18}f_4^6} + q^4 \frac{144f_3^3f_4^{12}}{f_1^2f_2^{18}f_4^4}.
\]

We next show a surprising congruence.

**Lemma 5.5.** It holds that
\[
H \equiv 3f_3^3 f_5 f_6^1 \pmod{5}.
\]
Proof of Lemma 5.5. To prove (5.7), it suffices to show
\[ H - 3 \frac{f_3^{15}}{f_1^7 f_2^{10}} \equiv 0 \pmod{5}, \]
or equivalently,
\[ \left( H - 3 \frac{f_3^{15}}{f_1^7 f_2^{10}} \right) \frac{f_1^5 f_3 f_4 f_6}{f_2^4 f_1^{12}} \equiv 0 \pmod{5}, \]
since \( \frac{f_3 f_4 f_6}{f_2 f_1^{12}} \) is invertible in the ring \( \mathbb{Z}/5\mathbb{Z}[[q]] \). According to Lemma 2.5, it becomes
\[ \frac{15p^2(1-p)(1+p)^5(2+p)^2(2+5p+12p^2+5p^3+2p^4)k^8}{32q^2(1+2p)} \equiv 0 \pmod{5}. \]
Lemma 5.5 follows obviously. \( \Box \)

We know from Lemma 5.5 that
\[ \sum_{n=0}^{\infty} d(9n+8)q^n \equiv 3f_2 f_3^{15} f_4 f_6 f_1^{10} \pmod{5}. \]
Since \( f_2 = (q^2:q^2)_\infty \) contains no terms of the form \( q^{5n+1} \) and \( q^{5n+3} \), we have
\[ d(9(5n+1)+8) = d(45n+17) \equiv 0 \pmod{5}, \]
and
\[ d(9(5n+3)+8) = d(45n+35) \equiv 0 \pmod{5}, \]
which leads to Theorem 5.4. \( \Box \)

Corollary 5.6. For \( n \geq 0 \), we have
\[ d(45n+t) \equiv 0 \pmod{15}, \]
where \( t = 17 \) and 35.

Proof. Again, we know from [15, Theorem 4.2] that
\[ d(3n+2) \equiv 0 \pmod{3}. \]
It indeed follows directly from (5.5). We therefore prove Corollary 5.6 by Theorem 5.4. \( \Box \)

6. Final remarks

We end this paper by raising the following congruences.

Question 6.1. We have
\[ c(45n+21) \equiv 0 \pmod{5}, \]
\[ c(63n+t) \equiv 0 \pmod{7}, \]
where \( t = 30, 48 \) and 57.

Question 6.2. We have
\[ d(45n+41) \equiv 0 \pmod{5}, \]
\[ d(63n+t) \equiv 0 \pmod{7}, \]
where \( t = 32, 50 \) and 59.
All these congruences have been verified by the authors using an algorithm due to Radu and Sellers [21]. However, since the modular form proofs are very routine and tedious, we here want to ask if there exist elementary proofs of these congruences.

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