Inverse boundary value problem for Schrödinger equation in two dimensions

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Abstract

We relax the regularity condition on potentials of the Schrödinger equation in uniqueness results on the inverse boundary value problem which were recently proved in [11] and [5].

Let \( \Omega \subset \mathbb{R}^2 \) be a bounded smooth domain with \( \partial \Omega = \cup_{j=0}^{K} \Sigma_j \) where \( \Sigma_j \) are smooth contours and \( \Sigma_0 \) is the external contour. Let \( \nu = (\nu_1, \nu_2) \) be the unit outer normal to \( \partial \Omega \) and let \( \frac{\partial}{\partial \nu} = \nabla \cdot \nu \).

In this domain we consider the Schrödinger equation with some potential \( q \):

\[
(\Delta + q)u = 0 \quad \text{in } \Omega.
\]

(1)

Let \( \Gamma \) be a non-empty arbitrary fixed relatively open subset of \( \partial \Omega \). Denote \( \Gamma_0 = \text{Int}(\partial \Omega \setminus \tilde{\Gamma}) \). Consider the partial Cauchy data

\[
\mathcal{C}_q = \left\{ \left( u, \frac{\partial u}{\partial \nu} \right) \mid \tilde{\Gamma} \right\} ; \quad (\Delta + q)u = 0 \quad \text{in } \Omega, \ u|_{\Gamma_0} = 0, u|_{\tilde{\Gamma}} = f.
\]

(2)

The goal of this article is to improve the regularity assumption on the potential \( q \) in the case of arbitrary subboundary \( \tilde{\Gamma} \) for the uniqueness result in the inverse problem of recovery of potential from the partial data [2]. In the case of \( \tilde{\Gamma} = \partial \Omega \), this inverse problem was formulated by Calderón in [7]. Under the assumption \( q \in C^{4+\alpha}(\Omega) \) the result was proved in Imanuvilov, Uhlmann and Yamamoto [11]. In Guillarmou and Tzou [10], the assumption on potentials was improved up to \( C^{2+\alpha}(\Omega) \).

In particular, in the two-dimensional full Cauchy data case of \( \tilde{\Gamma} = \partial \Omega \), we refer to Astala and Päivärinta [1], Blaisten [2], Brown and Uhlmann [4], Bukhgeim [5], Nachman [14]. In [2], the full Cauchy data uniquely determine the potential within \( W^1_p(\Omega) \) with \( p > 2 \). As for the related problem of recovery of the conductivity, [4] proved the uniqueness result for conductivities from \( L^\infty(\Omega) \), improving the result of [14]. We also mention that for the case of full Cauchy data a relaxed regularity assumption on potential was claimed in [5] but the proof itself is missing some details.

In three or higher dimensions, for the full Cauchy data, Sylvester and Uhlmann [16] proved the uniqueness of recovery of conductivity in \( C^2(\Omega) \), and later the regularity assumption was relaxed up to \( C^4(\Omega) \) in Päivärinta, Panchenko and Uhlmann [15] and up to \( W^2_p(\Omega) \) with \( p > 2n \) in Brown and Torres [3]. For the case of partial Cauchy data, uniqueness theorems were proved under assumption that a potential of the Schrödinger equation belongs to \( L^\infty(\Omega) \) (see Bukhgeim and Uhlmann [6], Kenig, Sjöstrand and Uhlmann [13]).

Our main result is as follows

**Theorem 1** Let \( q_1, q_2 \in C^\alpha(\Omega) \) for some \( \alpha \in (0, 1) \) if \( \tilde{\Gamma} = \partial \Omega \) and \( q_1, q_2 \in W^1_p(\Omega) \) for some \( p > 2 \) otherwise. If \( \mathcal{C}_{q_1} = \mathcal{C}_{q_2} \), then \( q_1 = q_2 \).

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The rest part of the paper is devoted to the proof of the theorem. Throughout the article, we use the following notations.

**Notations.** $i = \sqrt{-1}, x_1, x_2 \in \mathbb{R}^1, z = x_1 + ix_2, \overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. We identify $x = (x_1, x_2) \in \mathbb{R}^2$ with $z = x_1 + ix_2 \in \mathbb{C}$. $\partial_z = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2}), \partial_{\overline{z}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2}), D = \left(\frac{1}{4}\partial_{x_1}, \frac{1}{4}\partial_{x_2}\right)$. The tangential derivative on the boundary is given by $\partial_{\overline{\nu}} = \nu_2 \partial_{x_1} - \nu_1 \partial_{x_2}$, where $\nu = (\nu_1, \nu_2)$ is the unit outer normal to $\partial \Omega$.

**Proof.**

**First Step.**

Let $\Phi = \varphi + i\psi$ be a holomorphic function on $\Omega$ such that $\varphi, \psi$ are real-valued and

$$\Phi \in C^2(\overline{\Omega}), \quad \operatorname{Im}\Phi|_{\Gamma_0} = 0.$$  \hspace{1cm} (3)

Denote by $\mathcal{H}$ the set of the critical points of the function $\Phi$. Suppose that this set is not empty, each critical point is nondegenerate, $\mathcal{H} \cap \overline{\Omega}_0 = \emptyset$ and

$$\text{mes}(\mathcal{J}) = 0, \quad \mathcal{J} = \{x; \partial_{\overline{\nu}} \psi(x) = 0, x \in \overline{\Gamma}\}. \hspace{1cm} (4)$$

Here $\overline{\nu}$ is an unit tangential vector to $\partial \Omega$. Consider the operator $L_q(x, D) = - \sum_{j=1}^2 (D_j + \tau i \varphi_{x_j})^2 + q$. It is known (see [12] Proposition 2.5) that there exists a constant $\tau_0$ such that for $|\tau| \geq \tau_0$ and any $f \in L^2(\Omega)$, there exists a solution to the boundary value problem

$$L_q(x, D)u = f \quad \text{in} \quad \Omega, \quad u|_{\Gamma_0} = 0$$  \hspace{1cm} (5)

such that

$$\|u\|_{H^1(\Omega)} \leq C |\tau|^{1/2} \|f\|_{L^2(\Omega)}. \hspace{1cm} (6)$$

Moreover if $f/\partial_{\overline{\nu}} \in L^2(\Omega)$, then for any $|\tau| \geq \tau_0$ there exists a solution to the boundary value problem (5) such that

$$\|u\|_{H^1(\Omega)} \leq C \|f/\partial_{\overline{\nu}} \Phi\|_{L^2(\Omega)}. \hspace{1cm} (7)$$

The constants $C$ in (6) and (7) are independent of $\tau$. Here and henceforth we set

$$\|u\|_{H^1(\Omega)} = (\|u\|_{H^1(\Omega)}^2 + |\tau|^2 \|u\|^2_{L^2(\Omega)})^{1/2}.$$  

**Second Step.**

Here we will construct complex geometrical optics solutions. Henceforth by $o_{L^2(\Omega)}(\frac{1}{|\tau|})$, we mean a function $f(\epsilon, \tau, \cdot) \in L^2(\Omega)$ such that $\lim_{|\tau| \to \infty} |\tau|^{-1/2} \|f(\epsilon, \tau, \cdot)\|_{L^2(\Omega)} = 0$ for all small $\epsilon > 0$, and by $o(\frac{1}{|\tau|})$, we mean $a(\epsilon, \tau)$ such that $\lim_{|\tau| \to \infty} |\tau|^{-1} a(\epsilon, \tau) = 0$ for all small $\epsilon > 0$.

Let $\{q_{1, \epsilon}\}_{\epsilon \in (0, 1)}$ be a sequence of smooth functions converging to $q_1$ in $W^1_p(\Omega)$ or $C^0(\overline{\Omega})$ (depending on the assumption on the regularity of $q_1$) such that $q_{1, \epsilon} = q_1$ on $\mathcal{H}$. Let $p_\epsilon$ be the complex geometrical optics solution to the Schrödinger operator $\Delta + q_{1, \epsilon}$ which we constructed in [11]. The function $p_\epsilon$ can be written in the form:

$$p_\epsilon(x) = e^{\tau \Phi}(a + a_{0, \epsilon}/\tau) + e^{\tau \Phi}(a + b_{1, \epsilon}/\tau)$$

$$- \frac{4}{\tau} \left(e^{\tau \Phi}(a_{0, \epsilon} - M_{1, \epsilon}) + e^{\tau \Phi}(\partial_{\overline{\nu}}^{-1}(a_{0, \epsilon} - M_{1, \epsilon})) + e^{\tau \Phi}(a_{0, \epsilon} - M_{3, \epsilon})\right), \hspace{1cm} \text{as} \ \tau \to +\infty,$$  \hspace{1cm} (8)

where $a \in C^6(\overline{\Omega})$ is some holomorphic function on $\Omega$ such that $\operatorname{Re} a|_{\Gamma_0} = 0$. The operators $\partial_{\overline{\nu}}^{-1}$ and $\partial_{\overline{\nu}}^{-1}$ are given by

$$\partial_{\overline{\nu}}^{-1} = \frac{1}{\pi} \int_{\Omega} \frac{g(z, \zeta)}{\zeta - z} d\zeta d\xi_1, \quad \partial_{\overline{\nu}}^{-1} = \frac{1}{\pi} \int_{\Omega} \frac{g(z, \zeta)}{\zeta - z} d\zeta d\xi_1.$$  

Moreover for some $\bar{x} \in \mathcal{H}$, we assume that $a(\bar{x}) \neq 0$ and $a(x) = 0$ for $x \in \mathcal{H} \setminus \{\bar{x}\}$, and the polynomials $M_{1, \epsilon}(z)$ and $M_{3, \epsilon} \zeta)$ satisfy

$$\partial_{\overline{\nu}}^0(\partial_{\overline{\nu}}^{-1}(a_{0, \epsilon} - M_{1, \epsilon}))(x) = 0, \quad \partial_{\overline{\nu}}^0(\partial_{\overline{\nu}}^{-1}(a_{0, \epsilon} - M_{3, \epsilon}))(x) = 0, \quad x \in \mathcal{H},$$  

where $\partial_{\overline{\nu}}^0$ denotes the tangential derivative on the boundary.
\(a_0, a_1 \in C^0(\Omega)\) are holomorphic functions such that
\[
(a_0, a_1 + \overline{a_1})|_{\Gamma_0} = \frac{\partial z^{-1}(a_0, 1) - M_1}{4 \partial \Phi} + \frac{\partial z^{-1}(\overline{a_0}, 1) - M_3}{4 \partial \Phi}.
\]

We look for a solution \(u_1\) in the form \(u_1 = p_\epsilon + m_\epsilon\). Consider the equation
\[
L_{q_1}(x, D)u_1 = L_{q_1}(x, D)(p_\epsilon + m_\epsilon) + (q_1 - q_1)(p_\epsilon + m_\epsilon) = L_{q_1}(x, D)m_\epsilon + (q_1 - q_1)p_\epsilon = 0.
\]

By (7) there exists a solution to the boundary value problem
\[
L_{q_1}(x, D)m_\epsilon + (q_1 - q_1)p_\epsilon = 0 \quad \text{in} \quad \Omega, \quad m_\epsilon|_{\Gamma_0} = 0
\]
such that
\[
\|m_\epsilon\|_{H^1(\Omega)} \leq C(\epsilon) \quad \forall \tau > \tau_0(\epsilon),
\]
where \(C(\epsilon)\) is independent of \(\tau\) and
\[
C(\epsilon) \to 0 \quad \text{as} \quad \epsilon \to 0.
\]

Since the Cauchy data (2) for potentials \(q_1\) and \(q_2\), are equal, there exists a solution \(u_2\) to the Schrödinger equation with the potential \(q_2\) such that \(u_1 = u_2\) on \(\partial \Omega\) and \(\frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu}\) on \(\Gamma\). Setting \(u = u_1 - u_2\), we obtain
\[
(\Delta + q_2)u = (q_2 - q_1)u_1 \quad \text{in} \quad \Omega, \quad u|_{\partial \Omega} = \frac{\partial u}{\partial \nu}|_{\partial \Omega} = 0.
\]

In a way similar to the construction of \(u_1\), we construct the complex geometrical optics solution \(v\) for the Schrödinger equation with the potential \(q_2\). The construction of \(v\) repeats the corresponding steps of the construction of \(u_1\). The only difference is that instead of \(q_1\) and \(\tau\), we use \(q_2\) and \(\tau\) respectively. We provide details of the construction of \(v\) for the sake of completeness.

Let \(\{q_2, \epsilon\}_{\epsilon \in (0, 1)}\) be a sequence of smooth functions converging to sufficiently close to \(q_2\) in \(W_p^1(\Omega)\) or \(C^0(\Omega)\) such that \(q_2, \epsilon = q_2\) on \(\mathcal{H}\). Let \(\overline{\tilde{p}}_\epsilon\) be the complex geometrical optics solution to the Schrödinger operator \(\Delta + q_2, \epsilon\) constructed in [11]:
\[
\overline{\tilde{p}}_\epsilon(x) = e^{-\tau \Phi}(a + b_0, \epsilon/\tau) + e^{-\tau \Phi}(a + b_1, \epsilon/\tau)
\]
\[
+ \left(e^{-\tau \Phi}(\partial z^{-1}(q_2, \epsilon) - M_2, \epsilon) + e^{-\tau \Phi}(\partial z^{-1}(\overline{q}_2, \epsilon) - M_4, \epsilon)\right) + e^{-\tau \Phi\sigma_{L^2(\Omega)}(\overline{1}, \epsilon)},
\]
where \(M_2, \epsilon(x)\) and \(M_4, \epsilon(x)\) satisfy
\[
\partial_z^2(\partial z^{-1}(q_2, \epsilon) - M_2, \epsilon)(x) = 0, \quad \partial_z^2(\partial z^{-1}(\overline{q}_2, \epsilon) - M_4, \epsilon)(x) = 0, \quad x \in \mathcal{H}.
\]
and \(b_0, \epsilon, b_1, \epsilon\) are holomorphic functions such that
\[
(b_0, \epsilon + \overline{b_1, \epsilon})|_{\Gamma_0} = \frac{\partial z^{-1}(\overline{q}_2, \epsilon) - M_2, \epsilon}{4 \partial \Phi} - \frac{\partial z^{-1}(\overline{q}_2, \epsilon) - M_4, \epsilon}{4 \partial \Phi}.
\]

We look for a solution \(v\) in the form \(v = \overline{\tilde{p}}_\epsilon + \overline{m}_\epsilon\). Consider the operator
\[
L_{q_2}(x, D)v = L_{q_2}(x, D)(\overline{\tilde{p}}_\epsilon + \overline{m}_\epsilon) + (q_2 - q_2, \epsilon)(\overline{\tilde{p}}_\epsilon + \overline{m}_\epsilon) = L_{q_2}(x, D)\overline{m}_\epsilon + (q_2 - q_2, \epsilon)\overline{p}_\epsilon = 0.
\]

By (7) there exists a solution to the boundary value problem
\[
L_{q_2}(x, D)\overline{m}_\epsilon + (q_2 - q_2, \epsilon)\overline{p}_\epsilon = 0 \quad \text{in} \quad \Omega, \quad \overline{m}_\epsilon|_{\Gamma_0} = 0
\]
such that
\[
\|\overline{m}_\epsilon\|_{H^1(\Omega)} \leq C(\epsilon) \quad \forall \tau > \tau_0(\epsilon),
\]
where \(C(\epsilon)\) is independent of \(\tau\) and
\[
C(\epsilon) \to 0 \quad \text{as} \quad \epsilon \to 0.
\]
Computing the remaining terms, we have:
\[
\int_{\Omega} qu_1 \, dx = 0.
\] (13)

By (9) and (12)
\[
0 = \int_{\Omega} qu_1 \, dx = \int_{\Omega} qp_{i} \, dx + K(\epsilon, \tau),
\] (14)
where
\[
\lim_{\tau \to +\infty} \tau |K(\epsilon, \tau)| \leq C(\epsilon), \quad C(\epsilon) \to 0 \quad \text{as} \quad \epsilon \to 0.
\] (15)

From (14), (15) and the explicit formulae (5), (11) for the construction of complex geometrical optics solutions, we have
\[
\int_{\Omega} q(a^2 + \tau^2) \, dx = 0.
\]

Computing the remaining terms, we have:
\[
K(\epsilon, \tau) + \frac{1}{\tau} \int_{\Omega} q(a_{0,\epsilon} + b_{0,\epsilon}) + a(a_{1,\epsilon} + b_{1,\epsilon}) \, dx + \int_{\Omega} q(a\rho e^{2\tau i \psi} + a\rho e^{-2\tau i \psi}) \, dx \\
+ \frac{1}{4\tau} \int_{\Omega} a \frac{\partial^{-1}(q_{2,\epsilon}) - M_{2,\epsilon}}{\partial \phi} + a \frac{\partial^{-1}(q_{2,\epsilon}) - M_{2,\epsilon}}{\partial \phi} \, dx \\
- \frac{1}{4\tau} \int_{\Omega} a \frac{\partial^{-1}(q_{1,\epsilon}) - M_{1,\epsilon}}{\partial \phi} + a \frac{\partial^{-1}(q_{1,\epsilon}) - M_{1,\epsilon}}{\partial \phi} \, dx \\
+ o(1) = 0 \quad \text{as} \quad \tau \to +\infty.
\] (16)

Since the functions $q_j$ are not supposed to be from $C^2(\Omega)$, we can not directly use the stationary phase argument (e.g., Evans [5]). Consider two cases. Assume that $q \in W^1_p(\Omega)$ with $p > 2$. We have
\[
\int_{\Omega} q \text{Re}(a\rho e^{2\tau i \psi}) \, dx = \int_{\Omega} q \text{Re}(a\rho e^{2\tau i \psi}) \, dx + \int_{\Omega} (q - q_\epsilon) \text{Re}(a\rho e^{2\tau i \psi}) \, dx.
\] (17)

We set $q_\epsilon = q_1 - q_2$. Taking into account that $q_j, \epsilon = q_j$ on $\mathcal{H}, j = 1, 2$, (4) and using the stationary phase argument, similar to (11), we compute
\[
\int_{\Omega} q_\epsilon(a\rho e^{2\tau i \psi} + a\rho e^{-2\tau i \psi}) \, dx = \frac{2\pi q |a|^2(\tilde{x}) \text{Re} e^{2\tau i \text{Im} \phi(\tilde{x})}}{\tau |(\text{det Im } \Phi''(\tilde{x}))|} + o\left(\frac{1}{\tau}\right) \quad \text{as} \quad \tau \to +\infty.
\] (18)

For the second integral in (17) we obtain
\[
\int_{\Omega} (q - q_\epsilon) (a\rho e^{2\tau i \psi} + a\rho e^{-2\tau i \psi}) \, dx = \int_{\Omega} (q - q_\epsilon) \left( a\rho \frac{(\nabla \psi, \nabla) e^{2\tau i \psi}}{2\tau i |\nabla \psi|^2} - a\rho \frac{(\nabla \psi, \nabla) e^{-2\tau i \psi}}{2\tau i |\nabla \psi|^2} \right) \, dx \\
= \int_{\partial \Omega} (q - q_\epsilon) \left( a\rho \frac{(\nabla \psi, \nu) e^{2\tau i \psi}}{2\tau i |\nabla \psi|^2} - a\rho \frac{(\nabla \psi, \nu) e^{-2\tau i \psi}}{2\tau i |\nabla \psi|^2} \right) \, d\sigma \\
- \frac{1}{2\tau i} \int_{\Omega} e^{2\tau i \psi} \text{div} \left( (q - q_\epsilon) a\rho \frac{\nabla \psi}{|\nabla \psi|^2} \right) - e^{-2\tau i \psi} \text{div} \left( (q - q_\epsilon) a\rho \frac{\nabla \psi}{|\nabla \psi|^2} \right) \, dx.
\] (19)

Since $\psi|_{\Gamma_o} = 0$ we have
\[
\int_{\partial \Omega} (q - q_\epsilon) a\rho \left( \frac{(\nabla \psi, \nu) e^{2\tau i \psi}}{2\tau i |\nabla \psi|^2} - \frac{(\nabla \psi, \nu) e^{-2\tau i \psi}}{2\tau i |\nabla \psi|^2} \right) \, d\sigma = \int_{\partial \Omega} (q - q_\epsilon) a\rho \left( (\nabla \psi, \nu) (e^{2\tau i \psi} - e^{-2\tau i \psi}) \right) d\sigma.
\] (18)
By [4] and Proposition 2.4 in [11] we have that
\[ \int_{\partial \Omega} (q - q_c) a^2 \left( \frac{(\nabla \psi, \nu)e^{2r\psi}}{2\tau |\nabla \psi|^2} - \frac{(\nabla \psi, \nu)e^{-2r\psi}}{2\tau |\nabla \psi|^2} \right) d\sigma = o\left(\frac{1}{\tau}\right) \quad \text{as } \tau \to +\infty. \]

The last integral over \( \Omega \) in formula (19) is \( o\left(\frac{1}{\tau}\right) \) and so
\[ \int_{\Omega} (q - q_c)(ae^{2r\psi} + ae^{-2r\psi}) dx = o\left(\frac{1}{\tau}\right) \quad \text{as } \tau \to +\infty. \tag{20} \]

Taking into account that \( \psi(\tilde{x}) \neq 0 \) and using (20), (20) we have from (10) that
\[ \frac{2\pi |q(a^2)(\tilde{x})|}{|\det \Phi'(\tilde{x})|^2} + \bar{C}(\epsilon) = 0, \tag{21} \]

where \( \bar{C}(\epsilon) \to +0 \) as \( \epsilon \to 0 \). Hence
\[ q(\tilde{x}) = 0 \quad \text{if } a(\tilde{x}) \neq 0 \text{ and } a(x) = 0 \text{ for } x \in \mathcal{H} \setminus \{\tilde{x}\}. \tag{22} \]

Since a point \( \hat{x} \) can be chosen arbitrarily close to any given point in \( \Omega \) (see [11]), we have \( q \equiv 0 \), that is, the proof of the theorem is completed if \( q_1, q_2 \in W^1_p(\Omega) \).

Fourth Step.
Now let \( q \in C^a(\mathcal{H}) \) with some \( a \in (0,1) \) and \( \partial \Omega = \bar{\Gamma} \).

We recall the following classical result of Hörmander [9]. Consider the "oscillatory integral operator"
\[ T_\tau f(x) = \int_{\Omega} e^{-r\psi(x, y)} a(x, y) f(y) dy, \]

where \( \psi \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^2) \) and \( a(\cdot, \cdot) \in C^\infty_0(\mathbb{R}^2 \times \mathbb{R}^2) \). We introduce the following matrix
\[ H_\psi = \{\partial^2_{x, y} \psi\}. \]

**Theorem 2** Suppose that \( \det H_\psi \neq 0 \) on \( \text{supp } a \). Then
\[ \|T_\tau\|_{L^2 \to L^2} \leq C/\tau. \]

Consider our holomorphic function \( \Phi(x, y) = (x_1 + ix_2 - (y_1 + iy_2))^2 + i \). We set \( \psi(x, y) = 2(x_1 - y_1)(x_2 - y_2) - 1 \). Then
\[ H_\psi(x, y) = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} \]

and \( \det H_\psi(x, y) = -4 \). Then the condition in Theorem 2 holds true.

We set \( a(x, y) = \chi(x) \chi(y) \) where \( \chi \in C^\infty_0(\mathbb{R}^n) \) and \( \chi|_{\Omega} \equiv 1 \). Then, by Theorem 2 there exists a constant \( C \) independent of \( \tau \) such that
\[ \|T_\tau\|_{L^2 \to L^2} + \|T_{-\tau}\|_{L^2 \to L^2} \leq C/\tau. \tag{23} \]

Setting \( f = (q - q_c)a^2 \chi_{\Omega} \) by (23) we have
\[ \|T_\tau f\|_{L^2(\Omega)} + \|T_{-\tau} f\|_{L^2(\Omega)} \leq C(\epsilon)/\tau, \quad C(\epsilon) \to 0 \quad \text{as } \epsilon \to +0. \tag{24} \]

Therefore, by (24), in the ball \( B(\tilde{x}, \delta) \equiv \{x; |x - \tilde{x}| < \delta\} \), there exists a sequence of points \( y(\tau) \) such that
\[ |(T_\tau f(y(\tau)))| + |(T_{-\tau} f(y(\tau)))| \leq \frac{C\epsilon}{\tau \delta^2}. \tag{25} \]
Let $y(\tau) = (y_1(\tau), y_2(\tau)) \to \hat{y}(\epsilon)$ as $\tau \to +\infty$. By the stationary phase argument taking into account that
\[\psi(\tilde{x}, x) = -1,\]
we have
\[
\int _\Omega (q_e - (q_e - q)(y(\tau))) \text{Re}\left\{ a \overline{e}^{-2\tau i \psi(y(\tau), x)} \right\} dx = \frac{2\pi (q|a|^2)(\hat{y}(\epsilon)) \text{Re} e^{2\tau i}}{\tau} + o\left( \frac{1}{\tau} \right).
\] (26)

From (16), (26), (25) we obtain
\[
2\pi (q|a|^2)(\hat{y}(\epsilon)) \text{Re} e^{2\tau i} + \tilde{C}(\epsilon) = 0,
\] (27)
where $\lim_{\tau \to +\infty} |\tilde{C}(\epsilon)| \to +0$ as $\epsilon \to 0$. Therefore as $\epsilon$ goes to zero, we have
\[q(\hat{x}) = 0.\]

Here $\hat{x} \in B(\tilde{x}, \delta)$ such that $\hat{y}(\epsilon) \to \hat{x}$ as $\epsilon \to +0$. Since $\delta > 0$ and $\tilde{x}$ are chosen arbitrarily, we conclude that $q \equiv 0$ in $\Omega$. Thus the proof of the theorem is completed. $\square$

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