A Better Alternative to Error Feedback for Communication-Efficient Distributed Learning

Samuel Horváth  
Visual Computing Center  
KAUST, Thuwal  
Saudi Arabia  
samuel.horvath@kaust.edu.sa

Peter Richtárik  
Visual Computing Center  
KAUST, Thuwal  
Saudi Arabia  
peter.richtarik@kaust.edu.sa

Abstract

Modern large-scale machine learning applications require stochastic optimization algorithms to be implemented on distributed computer systems. A key bottleneck of such systems is the communication overhead for exchanging information across the workers, such as stochastic gradients. Among the many techniques proposed to remedy this issue, one of the most successful is the framework of compressed communication with error feedback (EF). EF remains the only known technique that can deal with the error induced by contractive compressors which are not unbiased, such as Top-K. In this paper, we propose a new and theoretically and practically better alternative to EF for dealing with contractive compressors. In particular, we propose a construction which can transform any contractive compressor into an induced unbiased compressor. Following this transformation, existing methods able to work with unbiased compressors can be applied. We show that our approach leads to vast improvements over EF, including reduced memory requirements, better communication complexity guarantees and fewer assumptions. We further extend our results to federated learning with partial participation following an arbitrary distribution over the nodes, and demonstrate the benefits thereof. We perform several numerical experiments which validate our theoretical findings.

1 Introduction

We consider distributed optimization problems of the form

$$\min_{x \in \mathbb{R}^d} f(x) := \frac{1}{n} \sum_{i=1}^{n} f_i(x),$$

(1)

where $x \in \mathbb{R}^d$ represents the weights of a statistical model we wish to train, $n$ is the number of nodes, and $f_i : \mathbb{R}^d \to \mathbb{R}$ is a smooth differentiable loss function composed of data stored on worker $i$. In a classical distributed machine learning scenario, $f_i(x) := E_{\zeta \sim D_i} [f_\zeta(x)]$ is the expected loss of model $x$ with respect to the local data distribution $D_i$ of the form, and $f_\zeta : \mathbb{R}^d \to \mathbb{R}$ is the loss on the single data point $\zeta$. This definition allows for different distributions $D_1, \ldots, D_n$ on each node, which means that the functions $f_1, \ldots, f_n$ can have different minimizers. This framework covers

- Stochastic Optimization when either $n = 1$ or all $D_i$ are identical,
- Empirical Risk Minimization (ERM), when $f_i(x)$ can be expressed as a finite average, i.e., $f_i(x) = \frac{1}{m_i} \sum_{j=1}^{m_i} f_{ij}(x)$ for some $f_{ij} : \mathbb{R}^d \to \mathbb{R}$,
- Federated Learning (FL) [Kairouz et al., 2019] where each node represents a client.
1.1 Communication as the Bottleneck

In distributed training, model updates (or gradient vectors) have to be exchanged in each iteration. Due to the size of the communicated messages for commonly considered deep models [Alistarh et al., 2016], this represents a significant bottleneck of the whole optimization procedure. To reduce the amount of data that has to be transmitted, several strategies were proposed.

One of the most popular strategies is to incorporate local steps and communicated updates every few iterations only [Stich, 2019a, Lin et al., 2018a, Stich and Karimireddy, 2020, Karimireddy et al., 2019a, Khaled et al., 2020]. Unfortunately, despite their practical success, local methods are poorly understood and their theoretical foundations are currently lacking. Almost all existing error guarantees are dominated by a simple baseline, minibatch SGD [Woodworth et al., 2020].

In this work, we focus on another popular approach: gradient compression. In this approach, instead of transmitting the full dimensional (gradient) vector \( g \in \mathbb{R}^d \), one transmits a compressed vector \( C(g) \), where \( C : \mathbb{R}^d \to \mathbb{R}^d \) is a (possibly random) operator chosen such that \( C(g) \) can be represented using fewer bits, for instance by using limited bit representation (quantization) or by enforcing sparsity. A particularly popular class of quantization operators is based on random dithering [Goodall, 1951, Roberts, 1962]; see [Alistarh et al., 2016, Wen et al., 2017, Zhang et al., 2017, Horváth et al., 2019a, Ramezani-Kebrya et al., 2019]. Much sparser vectors can be obtained by random sparsification techniques that randomly mask the input vectors and only preserve a constant number of coordinates [Wangni et al., 2018, Konečný and Richtárik, 2018, Stich et al., 2018, Mishchenko et al., 2019b, Vogels et al., 2019]. There is also a line of work [Horváth et al., 2019a, Basu et al., 2019] in which a combination of sparsification and quantization was proposed to obtain a more aggressive effect. We will not further distinguish between sparsification and quantization approaches, and refer to all of them as compression operators hereafter.

Considering both practice and theory, compression operators can be split into two groups: biased and unbiased. For the unbiased compressors, \( C(g) \) is required to be an unbiased estimator of the update \( g \). Once this requirement is lifted, extra tricks are necessary for Distributed Compressed Stochastic Gradient Descent (DCSGD) utilizing such a compressor to work, even if the full gradient is computed by each node. Indeed, a naive approach can lead to divergence [Beznosikov et al., 2020], and Error Feedback (EF) [Seide et al., 2014, Karimireddy et al., 2019b] is the only known mechanism able to remedy the situation and lead to a convergent method.

1.2 Contributions

Our contributions can be summarized as follows:

- We provide a theoretical analysis of Compressed SGD under weak and general assumptions. If \( f \) is \( \mu \)-quasi convex (not necessarily convex) and local functions \( f_i \) are \((L, \sigma^2)\)-smooth (weaker version of \( L \)-smoothness), we obtain the rate
  \[
  O\left(\frac{\delta L r^0}{\mu T} \exp \left[ -\frac{\mu T}{4\delta L} \right] + \frac{\sigma^2}{\mu T} \right),
  \]
  where \( \delta \geq 1 \) is the parameter which bounds the second moment of the compression operator, and \( T \) is the number of iterations. This rate is strictly better than the best-known rate for Compressed SGD with EF. Moreover, the latter requires extra assumptions. In addition, our theory guarantees convergence in both iterates and functional value. For EF, the best known rates [Karimireddy et al., 2019b, Beznosikov et al., 2020] are expressed in terms of functional values only. Another practical implication of our findings is the reduction of the memory requirements by half; this is because in Compressed SGD, one does not need to store the error vector.

- We propose a construction that can transform any biased compressor into an unbiased one (Section 3). We argue that using such an induced compressor within Compressed SGD is superior, both in theory and practice, to using the original biased compressor in conjunction with EF.

- We further extend our results to the multi-node scenario and show that the resulting method, Distributed Compressed SGD (DCSGD), improves linearly with the number of nodes, which is not the case for EF. Moreover, we obtain the first convergence guarantee for partial
participation with arbitrary distributions over nodes, which plays a key role in Federated Learning.

• Finally, we provide experimental evaluation on an array of classification tasks with MNIST and CIFAR10 datasets corroborating our theoretical findings.

2 Error Feedback is not a Good Idea when Using Unbiased Compressors

In this section we first introduce the notions of unbiased and general compression operators, and then compare Distributed Compressed SGD (DCSGD) without (Algorithm 1) and with (Algorithm 2) Error Feedback.

2.1 Unbiased vs General Compression Operators

We start with the definition unbiased and general (contractive) compression operators \cite{Cordonnier,Stich,Stich2,Koloskova}.

**Definition 1** (Unbiased Compression Operator). A randomized mapping \(C : \mathbb{R}^d \rightarrow \mathbb{R}^d\) is an unbiased compression operator (unbiased compressor) if there exists \(\delta \geq 1\) such that

\[
E[C(x)] = x, \quad E[\|C(x)\|^2] \leq \delta \|x\|^2, \quad \forall x \in \mathbb{R}^d.
\]  

(2)

If this holds, we will for simplicity write \(C \in U(\delta)\).

**Definition 2** (General Compression Operator). A (possibly) randomized mapping \(C : \mathbb{R}^d \rightarrow \mathbb{R}^d\) is a general compression operator (general compressor) if there exists \(\lambda > 0\) and \(\delta \geq 1\) such that

\[
E\left[\|\lambda C(x) - x\|^2\right] \leq \left(1 - \frac{1}{\delta}\right) \|x\|^2, \quad \forall x \in \mathbb{R}^d.
\]  

(3)

If this holds, we will for simplicity write \(C \in C(\delta)\).

To link these two definitions, we include the following simple lemma (see, e.g. \cite{Beznosik}):

**Lemma 1.** If \(C \in U(\delta)\), then \(3\) holds with \(\lambda = \frac{1}{\delta}\), i.e., \(C \in C(\delta)\). That is, \(U(\delta) \subset C(\delta)\).

Note that the opposite inclusion to that established in the above lemma does not hold. For instance, the Top-\(K\) operator belongs to \(C(\delta)\), but does not belong to \(U(\delta)\). In the next section we develop a procedure for transforming any mapping \(C : \mathbb{R}^d \rightarrow \mathbb{R}^d\) (and in particular, any general compressor) into a closely related induced unbiased compressor.
2.2 Distributed SGD with vs without Error Feedback

In the rest of this section, we compare the convergence rates for Distributed Compressed SGD (Algorithm 1) and Distributed Compressed SGD with Error Feedback (Algorithm 2). We do this comparison under standard assumptions [Karimi et al., 2016, Bottou et al., 2018, Necula et al., 2019, Gower et al., 2019, Stich, 2019b, Stich and Karimireddy, 2020], listed next.

First, we assume throughout that $f$ has a unique minimizer $x^*$, and let $f^* = f(x^*) > -\infty$.

**Assumption 1** ($\mu$-quasi convexity). $f$ is $\mu$-quasi convex, i.e.,
\[
f^* \geq f(x) + \langle \nabla f(x), x^* - x \rangle + \frac{\mu}{2} \|x^* - x\|^2, \quad \forall x \in \mathbb{R}^d.
\]

**Assumption 2** (unbiased gradient oracle). The stochastic gradient used in Algorithms 1 and 2 satisfies
\[
E \left[ g^k_i \mid x^k \right] = \nabla f_i(x^k), \quad \forall i, k.
\]

Note that this assumption implies $E \left[ \frac{1}{n} \sum_{i=1}^n g^k_i \mid x^k \right] = \nabla f(x^k)$.

**Assumption 3** ($4(L, \sigma^2)$-expected smoothness). Each function $f_i$ is $(L, \sigma^2)$-smooth, i.e., there exist constants $L > 0$ and $\sigma^2 \geq 0$ such that
\[
E \left[ \|f_i^k - f_i^\star\|^2 \right] \leq 2L(f_i(x^k) - f_i^\star) + \sigma^2,
\]

where $f_i^\star$ is the minimum functional value of $f_i$.

This assumption can be seen as a generalization of standard $L$-smoothness. For more details and discussion, see e.g. [Gower et al., 2019, Stich, 2019b, Stich and Karimireddy, 2020]. Equipped with these assumptions, we are ready to proceed with the convergence theory.

**Theorem 2** (Convergence of DCSGD in $n = 1$ case). Consider the DCSGD algorithm in the single node ($n = 1$) case. Let Assumptions 1 and 2 hold, and let $C \in \mathbb{U}(\delta)$. Then there exist stepsizes $\eta^k \leq \frac{1}{2 L}$ and weights $w^k \geq 0$ such that for all $T \geq 1$ we have
\[
E \left[ f(\bar{x}^T) - f^\star \right] + \mu E \left[ \|x^T - x^\star\|^2 \right] \leq 64\delta L r^0 \exp \left[ -\frac{\mu T}{40 L} \right] + \frac{36\sigma^2}{\mu T},
\]

where $r^0 = \|x^0 - x^\star\|^2$, $W^T = \sum_{k=0}^T w^k$, and $\text{Prob}(\bar{x}^T = x^k) = w^k/W^T$.

Note that the statistical term $\frac{36\sigma^2}{\mu T}$ does not depend on compression and matches the optimal rate for SGD [Stich, 2019b], including the constant. The other important aspect to consider is the first term. It guarantees linear convergence if $\sigma^2 = 0$, which holds for commonly used over-parameterized networks [Vaswani et al., 2019] as one can reach zero training loss. Comparing our results to the best-known result for Error Feedback [Stich and Karimireddy, 2020] used with $C \in \mathbb{U}(\delta) \subset \mathbb{C}(\delta)$, our theory allows for $10 \times$ larger stepsizes. Moreover, our convergence guarantee 1 for unbiased compressors implies convergence for both the functional values and the last iterate, rather than for functional values only. In addition, while the rate of DCSGD as captured by Theorem 2 and the rate of DCSGD with Error Feedback [Stich and Karimireddy, 2020] are the same in $O$ notation, our rate has at least 10 times better constants and does not contain any hidden polylogarithmic factors. Another practical advantage is that there is no need to store an extra vector for the error, which reduces the storage costs by a factor of two, making Algorithm 1 a viable choice for Deep Learning models with millions of parameters. Finally, one does not need to assume standard $L$-smoothness in order to prove convergence, while, one the other hand, $L$-smoothness is an important building block for proving convergence for general compressors due to the presence of error [Stich and Karimireddy, 2020, Beznosikov et al., 2020]. Putting all together, this suggests that standard DCSGD (Algorithm 1) is preferable, in theory, to DCSGD with Error Feedback (Algorithm 2) for $C \in \mathbb{U}(\delta)$.

3 Fixing Bias with Error-Compression

In the previous section, we showed that compressed DCSGD is theoretically preferable to DCSGD with Error Feedback for $C \in \mathbb{U}(\delta)$. Unfortunately, $\mathbb{C}(\delta) \not\subset \mathbb{U}(\delta)$, an example being the Top-$K$ compressor [Alistarh et al., 2018, Stich et al., 2018], which operates by keeping the top $K$
coordinates in magnitude only and setting rest to zero. This compressors belongs to \(\mathbb{C}(\frac{1}{2})\), but does not belong to \(U(\delta)\) for any \(\delta\). On the other hand, multiple unbiased alternatives to Top-\(K\) have been proposed in the literature, including gradient sparsification [Wangni et al., 2018] and adaptive random sparsification [Beznosikov et al., 2020].

### 3.1 Induced Compressor

We now propose a new way of constructing an unbiased compressor from any compressor \(C \in \mathbb{C}\). We shall argue that using this induced compressor within DCSGD is preferable, both in theory and practice, to using the original compressor within DCSGD + Error Feedback.

**Theorem 3.** For \(C_1 \in \mathbb{C}(\delta_1)\) with \(\lambda = 1\), choose \(C_2 \in U(\delta_2)\) and define the induced compressor via

\[
C(x) := C_1(x) + C_2(x - C_1(x)).
\]

The induced operator satisfies \(C \in U(\delta)\) with \(\delta = \delta_2 (1 - \frac{1}{\delta_1}) + \frac{1}{\delta_1} \).

To get some intuition about this procedure, first, recall the structure used in Error Feedback. The gradient estimator is first compressed with \(C_1(g)\) and the error \(e = g - C_1(g)\) is computed and stored in memory. For our proposed approach, instead of storing the error \(e\), we compress it with an unbiased compressor \(C_2\) and communicate both these compressed vectors. Note that this procedure results in extra variance as we do not work with the exact error, but with its unbiased estimate only. On the other hand, there is no bias. In addition, due to our construction, at least the same amount of information is sent as for plain \(C_1(g)\). The only drawback is the necessity to send two compressed vectors instead of one. Theorem 3 provides freedom in generating the induced compressor through the choice of the unbiased compressor \(C_2\). In practice, it makes sense to choose \(C_2\) with similar (or smaller) compression factor to the the compressor \(C_1\) we are transforming as this way the total communication complexity per iteration is preserved, up to the factor of two.

### 3.2 Benefits of Induced Compressor

In the light of the results in Section 2, we argue that one should always prefer unbiased compressors to biased ones as long as their variances \(\delta\) and communication complexities are the same, e.g., Rand-\(K\) over Top-\(K\). Contrary to the theory, greedy compressors are often observed to perform better due to their lower empirical variance [Beznosikov et al., 2020].

These considerations give a practical significance to Theorem 3 as we demonstrate on the following example. Let us consider two compressors—one biased \(C_1 \in \mathbb{C}(\delta_1)\) and one unbiased \(C_2 \in U(\delta_2)\), such that \(\delta_1 = \delta_2 = \delta\), having identical communication complexity, e.g., Top-\(K\) and Rand-\(K\). The induced compressor

\[
C_3(x) := C_1(x) + C_2(x - C_1(x))
\]

belongs to \(U(\delta_3)\), where

\[
\delta_3 = \delta - \left(1 - \frac{1}{\delta}\right) < \delta.
\]

While the size of the transmitted message is doubled, one can use Algorithm 2 since \(C_3\) is unbiased, which provides at least \(10\times\) better convergence guarantees to Algorithm 1.

Based on the construction of the induced compressor, one might suspect that we need extra memory as “the error” \(e = g - C_1(g)\) needs to be stored, but during computation only. This is not an issue as compressors for DNNs are always applied layer-wise [Dutta et al., 2019], and hence the size of the extra memory is negligible. It does not help EF, as the error needs to be stored at any time for each layer.

### 4 Extensions

We now develop several extensions of Algorithm 1 relevant to distributed optimization in general, and to Federated Learning in particular. This is all possible due to the simplicity of our approach. Note that in the case of Error Feedback, these extensions have either not been obtained yet, or similarly to Section 2, the results are worse when compared to our derived bounds for unbiased compressors.
4.1 Multi-node scenario

We begin with the case of general \( n \geq 1 \). The following theorem provides the convergence rate of Algorithm 1.

**Theorem 4** (Convergence of DCSGD in \( n \geq 1 \) case). Consider the DCSGD algorithm in the multiple nodes \( (n \geq 1) \) case. Let Assumptions 1–3 hold, and \( C \in \mathbb{U}(\delta) \). Then there exist stepizes \( \eta^k \leq \frac{1}{2\delta n L} \) and weights \( w^k \geq 0 \) such that for all \( T \geq 1 \) we have

\[
E \left[ f(\bar{x}^T) - f^* \right] + \mu E \left[ \|x^T - x^*\|^2 \right] \leq 64\delta_n L r^0 \exp \left[ -\frac{\mu T}{4\delta_n L} \right] + \frac{36(\sigma^2 + D)}{\mu T},
\]

where \( r^0, W^T, \bar{x}^T \) are defined in Theorem 2. \( D = \frac{2L}{n} \sum_{i=1}^{n} (f_i(x^*) - f_\delta) \) and \( \delta_n = \frac{\delta - 1}{n} + 1 \).

Inspecting the convergence rate, observe that Theorem 2 arises as a special case of Theorem 4 for \( n = 1 \). Similar arguments and comments can be made as those we have made in the discussion after Theorem 2. However, now we need to make a comparison with the complexity results of Beznosikov et al. [2020], who analyzed Algorithm 2 in the \( n > 1 \) case. In addition, the multi-node scenario reduces the effect of the variance constant \( \delta \) by a factor of \( 1/n \), which is not the case for EF.

4.2 Partial Participation with Arbitrary Distribution over Nodes

In this section, we extend our results to a variant of DCSGD utilizing partial participation, which is of key relevance to Federated Learning. In this framework, only a subset of all nodes communicates to the master node in each communication round. In this work, we consider a very general partial participation framework: we assume that the subset of participating clients is determined by a fixed but otherwise arbitrary random set-valued mapping \( S \) (a “sampling”) with values in \( 2^{[n]} \), where \( [n] = \{1, 2, \ldots, n\} \). To the best of our knowledge, this is the first partial participation result where an arbitrary distribution over the nodes is considered.

On the other hand, this is not the first work which makes use of the arbitrary sampling paradigm; this was used before in other contexts, e.g., for obtaining importance sampling guarantees for coordinate descent [Qu et al., 2015], primal-dual methods [Chambolle et al., 2018], and variance reduction [Horváth and Richtárik, 2019].

Note that the sampling \( S \) is uniquely defined by assigning probabilities to all \( 2^n \) subsets of \( [n] \). With each sampling \( S \) we associate a probability matrix \( P \in \mathbb{R}^{n \times n} \) defined by \( P_{ij} := \text{Prob}(\{i, j\} \subseteq S) \). The probability vector associated with \( S \) is the vector composed of the diagonal entries of \( P \): \( p = (p_1, \ldots, p_n) \in \mathbb{R}^n \), where \( p_i := \text{Prob}(i \in S) \). We say that \( S \) is proper if \( p_i > 0 \) for all \( i \). It is easy to show that \( b := E |S| = \text{Trace}(P) = \sum_{i=1}^{n} p_i \), and hence \( b \) can be seen as the expected number of clients participating in each communication round.

There are two algorithmic changes due to this extension: line 4 of Algorithm 1 does not iterate over every node, only over nodes \( i \in S^k \), where \( S^k \sim S \), and the aggregation step in line 9 is adjusted to lead to an unbiased estimator of the gradient, which gives \( \Delta_k = \sum_{i \in S^k} \frac{1}{n p_i} \Delta_i \).

To prove convergence, we exploit the following lemma.

**Lemma 5** (Lemma 1, Horváth and Richtárik [2019]). Let \( \zeta_1, \zeta_2, \ldots, \zeta_n \) be vectors in \( \mathbb{R}^d \) and let \( \bar{\zeta} := \frac{1}{n} \sum_{i=1}^{n} \zeta_i \) be their average. Let \( S \) be a proper sampling. Then there exists \( v \in \mathbb{R}^n \) such

\[
P - vv^T \preceq \text{Diag} (p_1 v_1, p_2 v_2, \ldots, p_n v_n).
\]

Moreover,

\[
E \left[ \left\| \sum_{i \in S} \frac{\zeta_i}{p_i} - \bar{\zeta} \right\|^2 \right] \leq \frac{1}{n^2} \sum_{i=1}^{n} \frac{v_i}{p_i} ||\zeta_i||^2,
\]

where \( S \sim S \) and the expectation is taken over sampling \( S \).

The following theorem establishes the convergence rate for Algorithm 1 with partial participation.

**Theorem 6.** Let Assumptions 1–3 hold and \( C \in \mathbb{U}(\delta) \), then there exist stepizes \( \eta^k \leq \frac{1}{2\delta L} \) and weights \( w^k \geq 0 \) such that

\[
E \left[ f(\bar{x}^T) - f^* \right] + \mu E \left[ \|x^T - x^*\|^2 \right] \leq 64\delta_2 L r^0 \exp \left[ -\frac{\mu T}{4\delta_2 L} \right] + \frac{36(\sigma^2 + D)}{\mu T},
\]
where \( r_0, W^T, \bar{x}^T \) are defined in Theorem 2, \( D \) in Theorem 4, and \( \delta_S = \frac{\max_{i \in [n]} \{v_i / p_i + 1\} - 1}{n} + 1 \).

For the case \( S = [n] \) with probability 1, one can show that Lemma 5 holds with \( v = 0 \), and hence we exactly recover the results of Theorem 4. In addition, we can quantify the slowdown factor with respect to full participation regime (Theorem 4), which is \( \max_{i \in [n]} \frac{v_i}{p_i} \). While in our framework we assume the distribution \( S \) to be fixed, using results of Eichner et al. [2019], one could extend this result to a block-cyclic structure with each block having an arbitrary distribution \( S_j \).

Note that in all the previous theorems, we can only guarantee a sublinear \( O(1/T) \) convergence rate. Linear rate is obtained in the special case when \( \sigma^2 = 0 \) (in which case \( D = 0 \)). This is satisfied if there is no noise at the optimum, which is the case for over-parameterized models. Furthermore, linear rate can be obtained using compression of gradient differences, as pioneered in the DIANA algorithm [Mishchenko et al., 2019a]. Both of these scenarios were already considered in [Horváth et al., 2019b] for the framework of Theorem 4 and full participation. These results can be easily extended to partial participation using our proof technique for Theorem 6. Note that this reduction is not possible for Error Feedback as the analysis of the DIANA algorithm is heavily dependent on the unbiasedness property. This points to another advantage of the induced compressor framework introduced in Section 3.

4.3 Acceleration

As the last comparison, we discuss the combination of compression and acceleration/momentum. This setting is very important to consider as essentially all state-of-the-art methods for training deep learning models, including Adam [Kingma and Ba, 2015, Reddi et al., 2018], rely on the use of momentum in one form or another. One can treat the unbiased compressed gradient as a stochastic gradient [Gorbunov et al., 2020] and the theory for momentum SGD [Yang et al., 2016, Gadat et al., 2018, Loizou and Richtárik, 2017] would be applicable with an extra smoothness assumption. Moreover, it is possible to remove the variance caused by stochasticity and obtain linear convergence with an accelerated rate [Li et al., 2020]. Similarly to our previous discussion, both of these techniques are heavily dependent on the unbiasedness property. It is an intriguing question, but out of the scope of the paper, to investigate the combined effect of momentum and Error Feedback and see whether these techniques are compatible theoretically.

5 Experiments

In this section, we compare Algorithms 1 and 2 for several compression operators. If method contains “ + EF ”, it means that EF is applied, thus Algorithm 2 is applied. Otherwise, Algorithm 1 is displayed. To be fair, we always compare methods with the same communication complexity per iteration. We report the number of epochs (passes over the dataset) with respect to training loss, testing loss,
We do an evaluation on 2 datasets – MNIST and CIFAR10. For MNIST, we consider a small neural network model with QSGD [Alistarh et al., 2016] and natural dithering [Horváth et al., 2019a] (with the infinity norm and one level) as compressors. This agrees with our theoretical findings. In addition, for sparsification techniques such as Random Sparsification or Gradient Sparsification [Wangni et al., 2018], we observed that when sparsity is set to be 10%, Algorithm 1 converges for all the selected values of step-sizes, but Algorithm 2 diverges and a smaller step-size needs to be used. This is an important observation as many practical works [Li et al., 2014, Wei et al., 2015, Aji and Heafield, 2018] use sparsification techniques mentioned in this section, but proposed to use EF, while our work shows that using unbiasedness property leads not only to better convergence but also to memory savings.

Figure 2: Comparison of different sparsification techniques with and without usage of Error Feedback on MNIST with 2 FC layers (top) and CIFAR10 with VGG11 (bottom). $K = 5\% \times d$, for Induced compressor $C_1$ is Top-$K/2$ and $C_2$ is Rand-$K/2$ (Wangni et al.).
while Error Feedback fixes divergence, it is still significantly dominated by unbiased non-uniform sparsification.

As observed in the first experiment, adding EF has a negative impact on the convergence of TernGrad.

5.4 Effect of Acceleration/Momentum

As the next experiment, we look at the effect of momentum on DCSGD with and without EF, which is set to 0.9. We consider the same setup as in the previous subsections. Based on our discussion on acceleration, we know that unbiased compressors are compatible with momentum and one can obtain theoretical guarantees, while for biased compressors with EF, this is not clear. Figure 3 suggests that both of the proposed techniques can outperform Top-K with EF, as can be seen for CIFAR10 with VGG11. Moreover, they do not require extra memory to store the error vector. In addition, our unbiased induced compressor further improves over Rand-K (Wagni et al.). Finally, Top-K without EF suffers a significant decrease in performance, which stresses the necessity of error correction.

5.5 Failure of DCSGD with biased Top-1

In this experiment, we present example considered in [Beznosikov et al., 2020], which was used as a counterexample to show that some form of error correction is needed in order for biased compressors to work/provably converge. In addition, we run experiments on their construction and show that while Error Feedback fixes divergence, it is still significantly dominated by unbiased non-uniform sparsification (NU Rand-1), which works by only keeping one non-zero coordinate sampled with probability equal to $|x|/\sum_{i=1}^d |x_i|$, where $|x|$ denotes element-wise absolute value, as can be seen in Figure 4.
Example from Beznosikov et al. [2020]. Consider \( n = d = 3 \) and define the following smooth and strongly convex quadratic functions
\[
    f_1(x) = \langle a, x \rangle^2 + \frac{1}{4} \|x\|^2, \quad f_2(x) = \langle b, x \rangle^2 + \frac{1}{4} \|x\|^2, \quad f_3(x) = \langle c, x \rangle^2 + \frac{1}{4} \|x\|^2,
\]
where \( a = (-3, 2, 2), b = (2, -3, 2), c = (2, 2, -3) \). Then, with the initial point \( x^0 = (t, t, t), \, t > 0 \)
\[
    \nabla f_1(x^0) = \frac{t}{2}(-11, 9, 9), \quad \nabla f_2(x^0) = \frac{t}{2}(9, -11, 9), \quad \nabla f_3(x^0) = \frac{t}{2}(9, 9, -11).
\]
Using the Top-1 compressor, we get
\[
    C(\nabla f_1(x^0)) = \frac{t}{2}(-11, 0, 0), \quad C(\nabla f_2(x^0)) = \frac{t}{2}(0, -11, 0), \quad C(\nabla f_3(x^0)) = \frac{t}{2}(0, 0, -11).
\]
The next iterate of DCGD is
\[
    x^1 = x^0 - \eta \frac{3}{3} \sum_{i=1}^3 C(\nabla f_i(x^0)) = \left( 1 + \frac{11\eta}{6} \right) x^0.
\]
Repeated application gives \( x^k = \left( 1 + \frac{11\eta}{6} \right)^k x^0 \), which diverges exponentially fast to \( +\infty \) since \( \eta > 0 \).

Initialization. In our experiments, we use the starting point \( x^0 = (1, 1, 1) \) and choose step size \( \frac{1}{L} \), where \( L \) is the smoothness parameter of \( f = \frac{1}{3}(f_1 + f_2 + f_3) \). Note that zero vector \( x^* = (0, 0, 0) \) is the unique minimizer of \( f \).

6 Conclusion

In this paper, we argue that if compressed communication is required for distributed training due to communication overhead, it is better to use unbiased compressors. We show that this leads to strictly better convergence guarantees with fewer assumptions. In addition, we propose a new construction for transforming any compressor into an unbiased one using a compressed EF-like approach. Besides theoretical superiority, usage of unbiased compressors enjoys lower memory requirements. Our theoretical findings are corroborated with empirical evaluation.

References

Alham Fikri Aji and Kenneth Heafield. Sparse communication for distributed gradient descent. *Proceedings of the 2017 Conference on Empirical Methods in Natural Language Processing*, 2017.

Dan Alistarh, Jerry Li, Ryota Tomioka, and Milan Vojnovic. QSGD: Randomized quantization for communication-optimal stochastic gradient descent. *arXiv preprint arXiv:1610.02132*, 2016.

Dan Alistarh, Torsten Hoeffer, Mikael Johansson, Nikola Konstantinov, Sarit Khirirat, and Cédric Renggli. The convergence of sparsified gradient methods. In *Advances in Neural Information Processing Systems*, pages 5973–5983, 2018.
Debraj Basu, Deepesh Data, Can Karakus, and Suhas Diggavi. Qsparse-local-SGD: Distributed SGD with quantization, sparsification and local computations. In Advances in Neural Information Processing Systems, pages 14668–14679, 2019.

Aleksandr Beznosikov, Samuel Horvath, Peter Richtárik, and Mher Safaryan. On biased compression for distributed learning. arXiv preprint arXiv:2002.12410, 2020.

Léon Bottou, Frank E Curtis, and Jorge Nocedal. Optimization methods for large-scale machine learning. Siam Review, 60(2):223–311, 2018.

Antonin Chambolle, Matthias J Ehrhardt, Peter Richtárik, and Carola-Bibiane Schonlieb. Stochastic primal-dual hybrid gradient algorithm with arbitrary sampling and imaging applications. SIAM Journal on Optimization, 28(4):2783–2808, 2018.

Jean-Baptiste Cordonnier. Convex optimization using sparsified stochastic gradient descent with memory. Technical report, 2018.

Aritra Dutta, El Houcine Bergou, Ahmed M Abdelmoniem, Chen-Yu Ho, Atal Narayan Sahu, Marco Canini, and Panos Kalnis. On the discrepancy between the theoretical analysis and practical implementations of compressed communication for distributed deep learning. arXiv preprint arXiv:1911.08250, 2019.

Hubert Eichner, Tomer Koren, H Brendan McMahan, Nathan Srebro, and Kunal Talwar. Semi-cyclic stochastic gradient descent. arXiv preprint arXiv:1904.10120, 2019.

Sébastien Gadat, Fabien Panloup, Sofiane Saadane, et al. Stochastic heavy ball. Electronic Journal of Statistics, 12(1):461–529, 2018.

WM Goodall. Television by pulse code modulation. Bell System Technical Journal, 30(1):33–49, 1951.

Eduard Gorbunov, Filip Hanzely, and Peter Richtárik. A unified theory of sgd: Variance reduction, sampling, quantization and coordinate descent. In The 23rd International Conference on Artificial Intelligence and Statistics, 2020.

Robert Mansel Gower, Nicolas Loizou, Xun Qian, Alibek Sailanbayev, Egor Shulgin, and Peter Richtárik. SGD: General analysis and improved rates. Proceedings of the 36th International Conference on Machine Learning, Long Beach, California, 2019.

Benjamin Grimmer. Convergence rates for deterministic and stochastic subgradient methods without Lipschitz continuity. SIAM Journal on Optimization, 29(2):1350–1365, 2019.

Kaiming He, Xiangyu Zhang, Shaoqing Ren, and Jian Sun. Deep residual learning for image recognition. In Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition, pages 770–778, 2016.

Samuel Horváth and Peter Richtárik. Nonconvex variance reduced optimization with arbitrary sampling. Proceedings of the 36th International Conference on Machine Learning, 2019.

Samuel Horváth, Chen-Yu Ho, L’udovit Horváth, Atal Narayan Sahu, Marco Canini, and Peter Richtárik. Natural compression for distributed deep learning. arXiv preprint arXiv:1905.10988, 2019a.

Samuel Horváth, Dmitry Kovalev, Konstantin Mishchenko, Sebastian Stich, and Peter Richtárik. Stochastic distributed learning with gradient quantization and variance reduction. arXiv preprint arXiv:1904.05115, 2019b.

Kevin Hsieh, Aaron Harlap, Nandita Vijaykumar, Dimitris Konomis, Gregory R Ganger, Phillip B Gibbons, and Onur Mutlu. Gaia: Geo-distributed machine learning approaching LAN speeds. In 14th Symposium on Networked Systems Design and Implementation, pages 629–647, 2017.

Peter Kairouz, H Brendan McMahan, Brendan Avent, Aurélien Bellet, Mehdi Bennis, Arjun Nitin Bhagoji, Keith Bonawitz, Zachary Charles, Graham Cormode, Rachel Cummings, et al. Advances and open problems in federated learning. arXiv preprint arXiv:1912.04977, 2019.
Hamed Karimi, Julie Nutini, and Mark Schmidt. Linear convergence of gradient and proximal-gradient methods under the polyak-łojasiewicz condition. In Joint European Conference on Machine Learning and Knowledge Discovery in Databases, pages 795–811. Springer, 2016.

Sai Praneeth Karimireddy, Satyen Kale, Mehryar Mohri, Sashank J Reddi, Sebastian U Stich, and Ananda Theertha Suresh. Scaffold: Stochastic controlled averaging for on-device federated learning. arXiv preprint arXiv:1910.06378, 2019a.

Sai Praneeth Karimireddy, Quentin Rebjock, Sebastian U Stich, and Martin Jaggi. Error feedback fixes signSGD and other gradient compression schemes. arXiv preprint arXiv:1901.09847, 2019b.

Ahmed Khaled, Konstantin Mishchenko, and Peter Richtárik. Tighter theory for local SGD on identical and heterogeneous data. In The 23rd International Conference on Artificial Intelligence and Statistics (AISTATS 2020), 2020.

Diederik P Kingma and Jimmy Ba. Adam: A method for stochastic optimization. Published as a conference paper at the 3rd International Conference for Learning Representations, San Diego, 2015.

Anastasia Koloskova, Sebastian U Stich, and Martin Jaggi. Decentralized stochastic optimization and gossip algorithms with compressed communication. arXiv preprint arXiv:1902.00340, 2019.

Jakub Konečný and Peter Richtárik. Randomized distributed mean estimation: Accuracy vs. communication. Frontiers in Applied Mathematics and Statistics, 4:62, 2018.

Simon Lacoste-Julien, Mark Schmidt, and Francis Bach. A simpler approach to obtaining an $O(1/t)$ convergence rate for the projected stochastic subgradient method. arXiv preprint arXiv:1212.2002, 2012.

Mu Li, David G Andersen, Jun Woo Park, Alexander J Smola, Amr Ahmed, Vanja Josifovski, James Long, Eugene J Shekita, and Bor-Yiing Su. Scaling distributed machine learning with the parameter server. In 11th {USENIX} Symposium on Operating Systems Design and Implementation ({OSDI} 14), pages 583–598, 2014.

Zhize Li, Dmitry Kovalev, Xun Qian, and Peter Richtárik. Acceleration for compressed gradient descent in distributed and federated optimization. arXiv preprint arXiv:2002.11364, 2020.

Hyeontaek Lim, David G Andersen, and Michael Kaminsky. 3LC: Lightweight and effective traffic compression for distributed machine learning. arXiv preprint arXiv:1802.07389, 2018.

Tao Lin, Sebastian U Stich, Kumar Kshitij Patel, and Martin Jaggi. Don’t use large mini-batches, use local SGD. arXiv preprint arXiv:1808.07217, 2018a.

Yujun Lin, Song Han, Huizi Mao, Yu Wang, and William J Dally. Deep gradient compression: Reducing the communication bandwidth for distributed training. ICLR 2018 - International Conference on Learning Representations, 2018b.

Nicolas Loizou and Peter Richtárik. Momentum and stochastic momentum for stochastic gradient, Newton, proximal point and subspace descent methods. arXiv preprint arXiv:1712.09677, 2017.

Konstantin Mishchenko, Eduard Gorbunov, Martin Takáč, and Peter Richtárik. Distributed learning with compressed gradient differences. arXiv preprint arXiv:1901.09269, 2019a.

Konstantin Mishchenko, Filip Hanzely, and Peter Richtárik. 99% of parallel optimization is inevitably a waste of time. arXiv preprint arXiv:1901.09437, 2019b.

Ion Necoara, Yu Nesterov, and François Glineur. Linear convergence of first order methods for non-strongly convex optimization. Mathematical Programming, 175(1-2):69–107, 2019.

Zheng Qu, Peter Richtárik, and Tong Zhang. Quartz: Randomized dual coordinate ascent with arbitrary sampling. In Advances in Neural Information Processing Systems, pages 865–873, 2015.

Ali Ramezani-Kebrya, Fartash Faghri, and Daniel M Roy. NUQSGD: Improved communication efficiency for data-parallel SGD via nonuniform quantization. arXiv preprint arXiv:1908.06077, 2019.
Sashank J Reddi, Satyen Kale, and Sanjiv Kumar. On the convergence of Adam and beyond. *ICLR 2018 - International Conference on Learning Representations*, 2018.

Peter Richtárik and Martin Takáč. Parallel coordinate descent methods for big data optimization. *Mathematical Programming*, 156(1-2):433–484, 2016.

Lawrence Roberts. Picture coding using pseudo-random noise. *IRE Transactions on Information Theory*, 8(2):145–154, 1962.

Frank Seide, Hao Fu, Jasha Droppo, Gang Li, and Dong Yu. 1-bit stochastic gradient descent and its application to data-parallel distributed training of speech dnns. In *Fifteenth Annual Conference of the International Speech Communication Association*, 2014.

Karen Simonyan and Andrew Zisserman. Very deep convolutional networks for large-scale image recognition. *ICLR 2015 - International Conference on Learning Representations*, 2015.

Sebastian U Stich. Local SGD converges fast and communicates little. *ICLR 2019 - International Conference on Learning Representations*, 2019a.

Sebastian U Stich. Unified optimal analysis of the (stochastic) gradient method. *arXiv preprint arXiv:1907.04232*, 2019b.

Sebastian U Stich and Sai Praneeth Karimireddy. The error-feedback framework: Better rates for SGD with delayed gradients and compressed communication. *ICLR 2020 - International Conference on Learning Representations*, 2020.

Sebastian U Stich, Jean-Baptiste Cordonnier, and Martin Jaggi. Sparsified SGD with memory. In *Advances in Neural Information Processing Systems*, pages 4447–4458, 2018.

Sharan Vaswani, Francis Bach, and Mark Schmidt. Fast and faster convergence of SGD for over-parameterized models and an accelerated perceptron. *Proceedings of the 22nd International Conference on Artificial Intelligence and Statistics (AISTATS) 2019*, Naha, Okinawa, Japan, 2019.

Thijs Vogels, Sai Praneeth Karimireddy, and Martin Jaggi. PowerSGD: Practical low-rank gradient compression for distributed optimization. In *Advances in Neural Information Processing Systems*, pages 14236–14245, 2019.

Jianqiao Wangni, Jialei Wang, Ji Liu, and Tong Zhang. Gradient sparsification for communication-efficient distributed optimization. In *Advances in Neural Information Processing Systems*, pages 1299–1309, 2018.

Jinliang Wei, Wei Dai, Aurick Qiao, Qirong Ho, Henggang Cui, Gregory R Ganger, Phillip B Gibbons, Garth A Gibson, and Eric P Xing. Managed communication and consistency for fast data-parallel iterative analytics. In *Proceedings of the Sixth ACM Symposium on Cloud Computing*, pages 381–394, 2015.

Wei Wen, Cong Xu, Feng Yan, Chunpeng Wu, Yandan Wang, Yiran Chen, and Hai Li. Terngrad: Ternary gradients to reduce communication in distributed deep learning. In *Advances in Neural Information Processing Systems*, pages 1509–1519, 2017.

Blake Woodworth, Kumar Kshitij Patel, Sebastian U Stich, Zhen Dai, Brian Bullins, H Brendan McMahan, Ohad Shamir, and Nathan Srebro. Is local SGD better than minibatch SGD? *arXiv preprint arXiv:2002.07839*, 2020.

Tianbao Yang, Qihang Lin, and Zhe Li. Unified convergence analysis of stochastic momentum methods for convex and non-convex optimization. *arXiv preprint arXiv:1604.03257*, 2016.

Hantian Zhang, Jerry Li, Kaan Kara, Dan Alistarh, Ji Liu, and Ce Zhang. Zipml: Training linear models with end-to-end low precision, and a little bit of deep learning. In *Proceedings of the 34th International Conference on Machine Learning-Volume 70*, pages 4035–4043. JMLR. org, 2017.
Appendix

A Extra Experiments

In this section, we include extra experiments which complement the figures in the main paper. Figure 5 corresponds to the same settings as Figure 1. Analogously, Figure 6 corresponds to Figure 2 and Figure 7 to Figure 3. Essentially, the same can be concluded as we argue in the main paper.

Figure 5: Algorithm 1 vs. Algorithm 2

(a) MNIST with 2 FC layers + TernGrad

(b) CIFAR10 with ResNet18 + Natural Compression

(c) CIFAR10 with VGG11 + TernGrad

(d) CIFAR10 with VGG11 + Natural Compression
Figure 6: Comparison of different sparsification techniques w/ and w/o usage of Error Feedback on MNIST with 2 FC layers. $K = 5\% \times d$, for Induced compressor $C_1$ is Top-$K/2$ and $C_2$ is Rand-$K/2$ (Wangni et al.).

Figure 7: Comparison of different sparsification techniques with momentum and w/ and w/o usage of Error Feedback on CIFAR10 with VGG11.

B Proofs

B.1 Proof of Lemma 1

We follow (2), which holds for $C \in \mathbb{U}(\delta)$.

$$
E \left[ \frac{1}{\delta} C(x) - x \right]^2 = \frac{1}{\delta^2} E \left[ \|C(x)\|^2 \right] - 2 \frac{1}{\delta} \langle E [C(x)], x \rangle - \|x\|^2 \\
\leq \left( 1 - \frac{2}{\delta} + 1 \right) \|x\|^2 \\
= \left( 1 - \frac{1}{\delta} \right) \|x\|^2 ,
$$

which concludes the proof.

B.2 Proof of Theorem 2

For the case $n = 1$, Algorithm 1 is reduced to $f_1 = f$, thus the update

$$
x^{k+1} = x^k - C(g^k) .
$$

We start with

$$
E \left[ \|x^{k+1} - x^*\|^2 | x^k \right] = \|x^k - x^*\|^2 - \eta^k E \left[ \langle C(g^k), x^k - x^* \rangle | x^k \right] + (\eta^k)^2 E \left[ \|C(g^k)\|^2 | x^k \right] \\
\leq \|x^k - x^*\|^2 - \eta^k \langle \nabla f(x^k), x^k - x^* \rangle + (\eta^k)^2 \delta E \left[ \|g^k\|^2 | x^k \right] \\
\leq \|x^k - x^*\|^2 - \eta^k \langle \nabla f(x^k), x^k - x^* \rangle + (\eta^k)^2 \delta (2L(f(x^k) - f^*) + \sigma^2) \\
\leq (1 - \mu \eta^k) \|x^k - x^*\|^2 - 2 \eta^k (1 - \eta^k \delta L) (f(x^k) - f^*) + (\eta^k)^2 \delta \sigma^2 .
$$

Taking full expectation and $\eta^k \leq \frac{1}{2 \delta L}$, we obtain

$$
E \left[ \|x^{k+1} - x^*\|^2 \right] \leq (1 - \mu \eta^k) E \left[ \|x^k - x^*\|^2 \right] - \eta^k E \left[ f(x^k) - f^* \right] + (\eta^k)^2 \delta \sigma^2 .
$$
The rest of the analysis is closely related to the one of [Stich 2019b] with an extra adjustments such that this analysis is able to accommodate compression represented by parameter $\delta$. We would like to point out that similar results to [Stich 2019b] were also present in [Lacoste-Julien et al., 2012, Stich et al., 2018, Grimmer, 2019].

We first rewrite the previous inequality to the form

$$r^{k+1} \leq (1 - a\eta) r^k - b\eta^k s^k + (\eta^k)^2 \alpha c,$$

where $r^k = E \left[ \|x^k - x^*\|^2 \right]$, $s^k = E \left[ f(x^k) - f^* \right]$, $a = \mu$, $b = 1$, $c = \sigma^2$ and $\alpha = \delta$.

We proceed with lemmas that establish a convergence guarantee for every recursion of type (10).

**Lemma 7.** Let $\{r^k\}_{k \geq 0}$, $\{s^k\}_{k \geq 0}$ be as in (10) for $a > 0$ and for constant stepsizes $\eta^k \equiv \eta := \frac{1}{a d}$, $\forall k \geq 0$. Then it holds for all $T \geq 0$:

$$r^T \leq r^0 \exp \left[-\frac{a T}{a d}\right] + \frac{c}{a d}.$$

**Proof.** This follows by relaxing (10) using $E \left[ f(x^k) - f^* \right] \geq 0$, and unrolling the recursion

$$r^T \leq (1 - a\eta)r_{T-1} + c\eta^2 \leq (1 - a\eta)^T r^0 + c\eta^2 \sum_{k=0}^{T-1} (1 - a\eta)^k \leq (1 - a\eta)^T r^0 + \frac{c\eta^T}{a}. \quad (11)$$

Let $\{r^k\}_{k \geq 0}$, $\{s^k\}_{k \geq 0}$ as in (10) for $a > 0$ and for decreasing stepsizes $\eta^k := \frac{2}{a a(\kappa + k)}$, $\forall k \geq 0$, with parameter $\kappa := \frac{2 a}{a}$, and weights $w^k := (k + k)$. Then

$$\frac{b}{W^T} \sum_{k=0}^{T} s^k w^k + ar_{T+1} \leq \frac{2 a a k^2 r^0}{T^2} + \frac{2 c}{a T},$$

where $W^T := \sum_{k=0}^{T} w^k$.

**Proof.** We start by re-arranging (10) and multiplying both sides with $w^k$

$$bs^k w^k \leq \frac{w^k (1 - a\eta^k)r^k}{\eta^k} - \frac{w^k r_{k+1}}{\eta^k} + c\alpha \eta^k w^k$$

$$= a a/(\kappa + k)(\kappa + k - 2) r^k - a a/(\kappa + k)^2 r_{k+1} + k a$$

$$\leq a a/(\kappa + t - 1)^2 r^k - a a/(\kappa + k)^2 r_{k+1} + k a,$$

where the equality follows from the definition of $\eta^k$ and $w^k$ and the inequality from $(\kappa + k)(\kappa + k - 2) = (k + k - 1)^2 - 1 = (k + k - 1)^2$. Again we have a telescoping sum:

$$\frac{b}{W^T} \sum_{k=0}^{T} s^k w^k + a a/(\kappa + T)^2 r_{T+1} \leq \frac{a a k^2 r^0}{W^T} + \frac{c(T + 1)}{a W^T},$$

with

- $W^T = \sum_{k=0}^{T} w^k = \sum_{k=0}^{T} (k + k) = \frac{(2k + T)(T+1)}{2} \geq \frac{T(T+1)}{2} \geq \frac{T^2}{2},$
- and $W^T = \frac{(2k + T)(T+1)}{2} \leq \frac{2(2k + T)(1 + T)}{2} \leq (k + T)^2$ for $k = \frac{2 a}{a} \geq 1$.

By applying these two estimates we conclude the proof.

The convergence can be obtained as the combination of these two lemmas.
Lemma 9. Let \( \{r^k\}_{k \geq 0}, \{s^k\}_{k \geq 0} \) as in (10), a > 0. Then there exists stepsizes \( \eta^k \leq \frac{1}{\alpha d} \) and weighs \( w^k \geq 0 \), \( W^T := \sum_{k=0}^{T} w^k \), such that
\[
\frac{b}{W^T} \sum_{k=0}^{T} s^k w^k + ar^{T+1} \leq 32 \alpha dr_0 \exp\left[-\frac{aT}{2 \alpha d}\right] + 36c \alpha \frac{T}{aT}.
\]

Proof of Lemma 9. For integer \( T \geq 0 \), we choose stepsizes and weights as follows

- if \( T \leq \frac{d}{a} \), \( \eta^k = \frac{1}{\alpha d}, \quad w^k = \left(1 - a \eta^k\right)^{(k+1)} = \left(1 - \frac{a}{\alpha d}\right)^{-(k+1)} \),
- if \( T > \frac{d}{a} \) and \( k < t_0 \), \( \eta^k = \frac{1}{\alpha d}, \quad w^k = 0 \),
- if \( T > \frac{d}{a} \) and \( k \geq t_0 \), \( \eta^k = \frac{2 \alpha d}{\alpha a(k + k - t_0)^2}, \quad w^k = \left(\alpha a(k + k - t_0)^2\right)^{-1} \),

for \( k = \frac{2 d}{a} \) and \( t_0 = \left\lceil \frac{T}{2} \right\rceil \). We will now show that these choices imply the claimed result.
We start with the case \( T \leq \frac{d}{a} \). For this case, the choice \( \eta = \frac{1}{\alpha d} \) gives
\[
\frac{b}{W^T} \sum_{k=0}^{T} s^k w^k + ar^{T+1} \leq \frac{r_0}{\eta} + c a \eta 
\leq \frac{r_0}{\eta} \exp\left[-a \eta(T + 1)\right] + c a \eta 
\leq \alpha d r_0 \exp\left[-\frac{aT}{2 \alpha d}\right] + \frac{c}{aT}.
\]
If \( T > \frac{d}{a} \), then we obtain from Lemma 7 that
\[
3r_0 \leq r_0 \exp\left[-\frac{aT}{2 \alpha d}\right] + \frac{c}{aT}.
\]
From Lemma 8, we have for the second half of the iterates:
\[
\frac{b}{W^T} \sum_{k=0}^{T} s^k w^k + ar^{T+1} = \frac{b}{W^T} \sum_{k=t_0}^{T} s^k w^k + ar^{T+1} \leq \frac{8 \alpha \sigma^2 r_0 T^2}{T^2} + \frac{4c}{aT}.
\]
Now we observe that the restart condition \( r^{t_0} \) satisfies:
\[
\frac{\alpha \sigma^2 r^{t_0}}{T^2} = \frac{\alpha \sigma^2 r_0 \exp\left(-\frac{aT}{2 \alpha d}\right)}{T^2} + \frac{\kappa^2 c}{dT^2} \leq 4 \alpha a \sigma \exp\left[-\frac{aT}{2 \alpha d}\right] + \frac{4c}{aT},
\]
because \( T > \frac{d}{a} \). These conclude the proof.

Having these general convergence lemmas for the recursion of the form (10), the proof of the theorem follows directly from Lemmas 7 and 9 with \( a = \mu, b = 1, c = \sigma^2, d = 2L \) and \( \alpha = \delta \). It is easy to check that condition \( \eta^k \leq \frac{1}{\alpha d} = \frac{1}{2 \delta^2 \tau} \) is satisfied.

B.3 Proof of Theorem 3

We have to show that our new compression is unbiased and has bounded variance. We start with the first property with \( \lambda = 1 \).
\[
E[C_1(x) + C_1(x - C_1(x))] = E_{C_1} [E_{C_2} [C_1(x) + C_2(x - C_1(x))|C_1(x)]] = E_{C_1} [C_1(x) + x - C_1(x)] = x,
\]
where the first equality follows from tower property and the second from unbiasedness of $C_2$. For the second property, we also use tower property

$$
E \left[ \| C_1 (x) - x + C_2 (x - C_1 (x)) \|^2 \right] = E_{C_1} \left[ E_{C_2} \left[ \| C_1 (x) - x + C_2 (x - C_1 (x)) \|^2 \mid C_1 (x) \right] \right]
$$

$$
\leq (\delta_2 - 1) E_{C_1} \left[ \| C_1 (x) - x \|^2 \right]
$$

$$
\leq (\delta_2 - 1) \left( 1 - \frac{1}{\delta_1} \right) \| x \|^2,
$$

where the first and second inequalities follow directly from (2) and (3).

**B.4 Proof of Theorem 4**

Similarly to the proof of Theorem 2, we use the update of Algorithm 1 to bound the following quantity

$$
E \left[ \| x^{k+1} - x^* \|^2 \mid x^k \right] = \| x^k - x^* \|^2 - \frac{n}{\eta_k} \sum_{i=1}^{n} E \left[ \langle C(g_i^k), x^k - x^* \rangle \mid x^k \right] +
$$

$$
\left( \frac{\eta_k}{n} \right)^2 E \left[ \left\| \sum_{i=1}^{n} C(g_i^k) \right\|^2 \mid x^k \right]
$$

$$
\leq \| x^k - x^* \|^2 - \eta_k \left\langle \nabla f(x^k), x^k - x^* \right\rangle +
$$

$$
\left( \frac{\eta_k^2}{n^2} \right) E \left[ \sum_{i=1}^{n} \left\| C(g_i^k) - g_i^k \right\|^2 + n \left\| g_i^k \right\|^2 \mid x^k \right]
$$

$$
\leq \| x^k - x^* \|^2 - \eta_k \left\langle \nabla f(x^k), x^k - x^* \right\rangle +
$$

$$
(\eta_k)^2 \left( \frac{\delta - 1}{n} + 1 \right) \left( 2L(f(x^k) - f^*) + \sigma^2 + \frac{1}{n} \sum_{i=1}^{n} (f_i(x^* - f^*)) \right)
$$

$$
\leq (1 - \mu \eta_k) \| x^k - x^* \|^2 - 2\eta_k \left( 1 - \eta_k \delta_n \right) \left( f(x^k) - f^* \right) + (\eta_k)^2 \delta_n (\sigma^2 + D).
$$

Taking full expectation and $\eta_k \leq \frac{1}{2\delta_n L}$, we obtain

$$
E \left[ \| x^{k+1} - x^* \|^2 \right] \leq (1 - \mu \eta_k) E \left[ \| x^k - x^* \|^2 \right] - \eta_k E \left[ f(x^k) - f^* \right] + (\eta_k)^2 \delta_n (\sigma^2 + D).
$$

The rest of the analysis is identical to the proof of Theorem 2 with only difference $c = \sigma^2 + D$ and $\delta_n$ instead of $\delta$.

**B.5 Proof of Lemma 5**  Horváth and Richtarik, 2018 [Horváth and Richtárik, 2019]

For the first part of the claim, it was shown that $P - pp^\top$ is positive semidefinite [Richtárik and Takáč, 2016], thus we can bound $P - pp^\top \preceq n \text{Diag} (P - pp^\top) = \text{Diag} (p \circ v)$, where $v_i = n(1 - p_i)$, which implies that (8) holds for this choice of $v$.

For the second part of the claim, let $1_{i \in S} = 1$ if $i \in S$ and $1_{i \notin S} = 0$ otherwise. Likewise, let $1_{i,j \in S} = 1$ if $i, j \in S$ and $1_{i,j \notin S} = 0$ otherwise. Note that $E [1_{i \in S}] = p_i$ and $E [1_{i,j \in S}] = p_{ij}$. Next, let us compute the mean of $X := \sum_{i \in S} \frac{\zeta_i}{np_i}$:

$$
E [X] = E \left[ \sum_{i \in S} \frac{\zeta_i}{np_i} \right] = E \left[ \sum_{i=1}^{n} \frac{\zeta_i}{np_i} 1_{i \in S} \right] = \sum_{i=1}^{n} \frac{\zeta_i}{np_i} E [1_{i \in S}] = \frac{1}{n} \sum_{i=1}^{n} \zeta_i = \bar{\zeta}.
$$

(12)
Let $A = [a_1, \ldots, a_n] \in \mathbb{R}^{d \times n}$, where $a_i = \frac{\zeta_i}{p_i}$, and let $e$ be the vector of all ones in $\mathbb{R}^n$. We now write the variance of $X$ in a form which will be convenient to establish a bound:

$$
E \left[ \|X - E[X]\|^2 \right] = E \left[ \|X\|^2 \right] - \|E[X]\|^2
$$

$$
= E \left[ \left\| \sum_{i \in S} \frac{\zeta_i}{np_i} \right\|^2 \right] - \|\zeta\|^2
$$

$$
= E \left[ \sum_{i,j} \frac{\zeta_i^T \zeta_j}{np_i np_j} 1_{i,j \in S} \right] - \|\zeta\|^2
$$

$$
= \sum_{i,j} p_{ij} \frac{\zeta_i^T \zeta_j}{np_i np_j} - \sum_{i,j} \frac{\zeta_i^T \zeta_j}{n^2}
$$

$$
= \frac{1}{n^2} \sum_{i,j} (p_{ij} - p_i p_j) a_i^T a_j
$$

$$
= \frac{1}{n^2} e^T ((P - pp^T) \circ A^T A) e. \quad (13)
$$

Since by assumption we have $P - pp^T \preceq \text{Diag} (p \circ v)$, we can further bound

$$
e^T ((P - pp^T) \circ A^T A) e \leq e^T (\text{Diag} (p \circ v) \circ A^T A) e = \sum_{i=1}^n p_i v_i |a_i|^2.
$$

To obtain (9), it remains to combine this with (13).

### B.6 Proof of Theorem 6

Similarly to the proof of Theorem 2 we use the update of Algorithm 1 to bound the following quantity

$$
E \left[ \|x^{k+1} - x^*\|^2 |x^k\right] = \|x^k - x^*\|^2 - \eta^k \sum_{i=1}^n E \left[ \sum_{i \in S^k} \frac{1}{np_i} C(g_i), x^k - x^* \right] |x^k| +
$$

$$
E \left[ \left\| \sum_{i \in S^k} \frac{\eta^k}{np_i} C(g_i^k) \right\|^2 |x^k\right]
$$

$$\leq \|x^k - x^*\|^2 - \eta^k \langle \nabla f(x^k), x^k - x^* \rangle +
$$

$$\leq \eta^k \|\nabla f(x^k)\|_2 \|x^k - x^*\|_2 - \eta^k \langle \nabla f(x^k), x^k - x^* \rangle +
$$

$$\leq \frac{\eta^k}{n} \sum_{i=1}^n \left( \frac{\delta v_i}{np_i} + \frac{\delta - 1}{n} + 1 \right) \|g_i\|^2 |x^k|
$$

$$\leq (1 - \mu \eta^k) \|x^k - x^*\|^2 - 2\eta^k (1 - \eta^k \delta v) (f(x^k) - f^*) + (\eta^k)^2 \delta v (\sigma^2 + D).
$$

Taking full expectation and $\eta^k \leq \frac{1}{2L}$, we obtain

$$
E \left[ \|x^{k+1} - x^*\|^2 \right] \leq (1 - \mu \eta^k) E \left[ \|x^k - x^*\|^2 \right] - \eta^k E \left[ f(x^k) - f^* \right] + (\eta^k)^2 \delta v (\sigma^2 + D).
$$

The rest of the analysis is identical to the proof of Theorem 2 with only difference $c = \sigma^2 + D$ and $\delta v$ instead of $\delta$. 

19