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Higher-order asymptotic solution for the fatigue crack growth problem based on continuum damage mechanics

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Abstract

In the paper an analytical solution of the nonlinear eigenvalue problem arising from the fatigue crack growth problem in a damaged medium in coupled formulation is obtained. In order to evaluate the mechanical behavior in the vicinity of a growing fatigue crack for plane strain and plane stress conditions of mode I the asymptotic governing equations are derived and analyzed by the light of Continuum Damage Mechanics. It is shown that the growing fatigue crack problem can be reduced to the nonlinear eigenvalue problem. The perturbation technique for solving the nonlinear eigenvalue problem is used. The method allows to find the analytical formula expressing the eigenvalue as the function of parameters of the damage evolution law. The eigenvalues of the nonlinear eigenvalue problem are fully determined by the exponents of the damage evolution law. The higher-order asymptotic expansions of the angular functions determining the stress and continuity fields in the neighborhood of the crack tip are given. The asymptotic expansions of the angular functions permit to find the closed-form solution for the problem. The higher order stress, strain and continuity asymptotic fields in the vicinity of the fatigue growing crack are either obtained, in which analytical expressions of the higher order exponents and angular distribution functions (eigenfunctions) of the near tip stress and continuity fields are derived.

Keywords: fatigue crack growth, damage accumulation, asymptotic solution, higher-order asymptotic crack-tip fields, perturbation method.

1. Introduction

The asymptotic analysis of the near crack tip stress field is an integral part of fracture mechanics analysis for both linear elastic media and nonlinear media. In nonlinear fracture mechanics the eigenfunction expansion method results in nonlinear eigenvalue problems (Murakami (2012); Stepanova (2008)). The most commonly encountered approaches to tackle the nonlinear eigenvalue problem are the Runge-Kutta method and FEM approach. However, the numerical integration of the equations realized by the Runge-Kutta method in conjunction with the shooting method generally becomes multiparametric. Thus, the numeric results still require further verification. That is the reason why much attention is currently given to analytical methods of solution of nonlinear fracture mechanics problems as

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a whole and, specifically, to methods of solution of nonlinear eigenvalue problems (Stepanova (2008, 2009)). One of promising approaches is the perturbation theory technique applied to a wide variety of static and dynamic solid mechanics problems. In this contribution an analytical solution of the nonlinear eigenvalue problem arising from the fatigue crack growth problem in a damaged medium in coupled formulation is obtained. The original statement of the problem considered here is proposed by Zhao and Zhang (1995) where it is noted that further theoretical efforts are wanted in the respects of appropriate determination of higher ordered asymptotic studies. The present study continues the analysis of Zhao and Zhang (1995); Stepanova and Igonin (2013) and gives an attempt to construct the higher-order asymptotic expansions of stresses, strains and continuity in the vicinity of the growing fatigue crack and to derive the analytical expressions of the higher-order crack-tip fields.

2. Basic formulation

Consider mode I deformations around a fatigue growing crack in a damaged material. We will neglect the anisotropic effects induced by damage. Under assumption of linear elasticity the constitutive equation of damaged materials is

\[ \varepsilon_{ij} = (1 + \nu)\sigma_{ij} / (E\psi) - v\sigma_{kk}\delta_{ij} / (E\psi), \]

where \( \varepsilon_{ij} \) are infinitesimal strains, \( \sigma_{ij} \) are stresses, \( E \) is Young’s modulus, \( \nu \) is Poisson’s ratio of undamaged material, \( \psi \) is a scalar internal variable that characterizes damage \( (0 \leq \psi \leq 1) \), where \( \psi = 1 \) and \( \psi = 0 \) signify the initial undamaged state and the final completely damaged state, respectively, Murakami (2012); Kuna (2013). Following Zhao and Zhang (1995) we assume that the cumulative damage evolution equation has the form

\[ d\psi / dN = \left\{ -c \left( \sigma_c / \psi \right)^m \psi^{m-n}, \right. \quad \text{for } \sigma_c > \sigma_{th}\psi; \quad \left. 0 \right. \quad \text{for } \sigma_c < \sigma_{th}\psi, \]

where \( N \) is the number of cycles, \( c, m, n, \sigma_{th} \) are positive material parameters. The constitutive equations for plane stress conditions are given as

\[ \varepsilon_{rr} = (\sigma_{rr} - \nu\sigma_{\theta \theta}) / (E\psi), \quad \varepsilon_{\theta \theta} = (\sigma_{\theta \theta} - \nu\sigma_{rr}) / (E\psi), \quad \varepsilon_{r\theta} = (1 + \nu)\sigma_{r\theta} / (E\psi). \]

The equilibrium equations \( r\sigma_{r\theta} + \sigma_{r\theta,\theta} + (\sigma_{rr} - \sigma_{\theta \theta}) = 0, \ r\sigma_{r\theta} + \sigma_{r\theta,\theta} + 2\sigma_{\theta \theta} = 0 \) can be satisfied by introducing the Airy stress function \( \chi(r, \theta) = \sigma_{\theta \theta}(r, \theta) = \chi, \sigma_{rr}(r, \theta) = \chi, \sigma_{rr}, f_\theta/\psi = \chi, r + \chi, r, r^2 \), \( \sigma_{r\theta}(r, \theta) = -(\chi, r_j). \) The compatibility equation is

\[ 2r_\theta(\sigma_{r\theta}) = \varepsilon_{rr,\theta} - r_\theta^{\theta\theta,\theta} + r(\varepsilon_{r\theta})_\theta. \]

The solution of Eqs. (1) – (3) should satisfy the traditional traction free boundary conditions on crack surfaces \( \sigma_{\theta \theta}(r, \theta = \pm \pi) = 0, \sigma_{r\theta}(r, \theta = \pm \pi) = 0. \)

3. Asymptotic solution. Eigenvalues and eigenfunctions

The Airy stress function \( \chi(r, \theta) \) and the continuity parameter \( \psi(r, \theta) \) in the damage accumulation process zone (as \( r \to 0 \)) are assumed in the separated form as follows

\[ \chi(r, \theta) = \sum_{k=0}^\infty \alpha_k r^{k+2} f_k(\theta), \quad \psi(r, \theta) = \sum_{k=0}^\infty \beta_k r^{k+2} g_k(\theta), \]

where \( f_k(\theta), g_k(\theta), k = 0, 1, \ldots \) are the eigenfunctions that define the angular dependence; \( \alpha_k, \beta_k \) are constants depending on the geometry and loading conditions, \( \lambda_k, \mu_k \) are the eigenvalues that govern the behavior of stresses and continuity at the tip of a crack. The asymptotic expansions of the stress components take the form \( \sigma_{ij}(r, \theta) = \sum_{k=0}^\infty \alpha_k r^{k+2} \sigma_{ij}(k, \theta), \)

where \( \sigma_{ij}(\theta) = (\lambda_k + 2)(\lambda_k + 1)f_k(\theta), \) \( \sigma_{r\theta}(k, \theta) = (\lambda_k + 2)\chi_k(\theta), \sigma_{r\theta}(\theta) = -(\lambda_k + 1)\chi(k, \theta), k = 0, 1, \ldots \) The constitutive equations (2) and the asymptotic expansions introduced allow us to develop the asymptotic expansions for the strain components near the crack tip as \( r \to 0 \)

\[ \varepsilon_{rr} = \frac{1}{E} \sum_{j=0}^\infty \frac{\alpha_j r^{j+2} \varepsilon_{rr}(j, \theta), \quad \varepsilon_{\theta \theta} = \frac{1}{E} \sum_{j=0}^\infty \frac{\alpha_j r^{j+2} \varepsilon_{\theta \theta}(j, \theta), \quad \varepsilon_{r\theta} = \frac{1 + \nu}{E} \sum_{j=0}^\infty \frac{\alpha_j r^{j+2} \varepsilon_{r\theta}(j, \theta), \]

(5)
where the coefficients of the first and second terms of the asymptotic expansions are defined by the formulae

\[
\begin{align*}
E_{rr}^{(0)} &= (c_0 f_0 + f_1')(g_0)^{-1}, \\
E_{00}^{(0)} &= (a_0 f_0 - v f_0')(g_0)^{-1}, \\
E_{rr}^{(1)} &= -\frac{c_1 f_1 + f_1'}{f_0} - \frac{E_{rr}^{(0)} g_1}{g_0} + \frac{E_{00}^{(0)} g_1}{g_0}, \\
E_{00}^{(1)} &= -\frac{a_1 f_1 - v f_0'}{f_0} - \frac{E_{rr}^{(0)} g_1}{g_0}, \\
E_{rr}^{(1)} &= -\frac{(a_1 + 1) f_1'}{f_0} + \frac{E_{00}^{(0)} g_1}{g_0}, \\
E_{00}^{(1)} &= -\frac{(a_0 + 1) f_0'}{f_0} + \frac{E_{rr}^{(0)} g_1}{g_0},
\end{align*}
\]

where

\[
c_0 = (\lambda_0 + 2)[1 - v(\lambda_0 + 1)], \quad a_0 = (\lambda_0 + 2)(\lambda_0 + 1 - v), \quad c_1 = (\lambda_1 + 2)[1 - v(\lambda_1 + 1)], \quad a_1 = (\lambda_1 + 2)(\lambda_1 + 1 - v).
\]

The compatibility condition (3) and the asymptotic expansions of the strain components (5) permit us to derive the ordinary differential equations for the \(k\)th term of the asymptotic expansions (5)

\[
2(1 + v)(\lambda_k - \mu_0 + 1)E_{rr,k}^{(k)} = E_{rr,0}^{(k)} - (\lambda_k - \mu_0)E_{rr,0}^{(k)} + (\lambda_k - \mu_0 + 1)(\lambda_k - \mu_0)E_{rr,0}^{(k)}.
\]

By the use of Eqs. (6) from Eqs. (7) one can obtain equations for the functions \(f_0(\theta)\) and \(f_1(\theta)\)

\[
\begin{align*}
f_0^{IV} &= 2f_1^{IV} + (b_1 + G)f_0'' - b_2 E f_0' - (c_0 G + b_3) f_0 = 0, \\
f_1^{IV} &= 2f_1^{IV} + (b_1 + G)f_1'' - b_2 E f_1' - (c_1 G + b_6) f_1 = g_1' \left[ 4(c_0 f_0 + f_0') \bar{E} - 2b_7 f_0'' - 2(c_0 f_0' + f_0''') \right] g_0^2 + \\
&+ g_1 \left[ 4b_7 f_0'' \bar{E}^2 - 2b_7 f_0' - (c_0 f_0' + f_0''') E - 2(c_0 f_0' + f_0'') (G + \bar{E}^2) \right] g_0 + \\
&+ g_1 \left[ (\lambda_k - \mu_0)(c_0 f_0 + f_0') + (\lambda_k - \mu_0 + 1)(\lambda_k - \mu_0)(a_0 f_0 - v f_0') \right] g_0^2 - (c_0 f_0 + f_0'') g_1',
\end{align*}
\]

where the following notations are adopted \(\bar{E} = g_0^0/g_0\), \(G = 2 \bar{E}^2 - g_0''/g_0\)

\[
b_1 = 2(1 + v)(\lambda_0 - \mu_0 + 1)(\lambda_0 + 1 - \lambda_0 - \mu_0)\left[ 1 - v(\lambda_0 - \mu_0 + 1) \right] + c_0, \quad b_2 = 2(1 + v)(\lambda_0 - \mu_0 + 1)(\lambda_0 + 1 - \lambda_0 - \mu_0) + 2c_0, \\
b_3 = (\lambda_0 - \mu_0)(a_0 \lambda_0 - \mu_0 + 1 - c_1), \quad b_4 = 2(1 + v)(\lambda_1 - \mu_0 + 1)(\lambda_1 + 1 - \lambda_1 - \mu_0)\left[ 1 + v(\lambda_1 - \mu_0 + 1) \right] + c_1, \\
b_5 = 2(b_7 + c_1), \quad b_6 = a_0(\lambda_1 - \mu_0 + 1)(\lambda_1 - \mu_0) - c_1(\lambda_1 - \mu_0), \quad b_7 = (1 + v)(\lambda_1 - \mu_0 + 1)(\lambda_1 + 1).
\]

The left-hand side of the damage evolution law (1) can be represented as \(d\psi/dN = (d\psi/dl)(dl/dN)\), where \(l\) is the current crack length, \(dl/dN\) is velocity of crack growth. Taking into account the asymptotic expansion for the continuity parameter (4) and the equalities \(dr/dl = -\cos \theta, \, d\theta/dl = \sin \theta/r\) (Zhao and Zhang (1995)) the left-hand side of the damage evolution equation can be rewritten as

\[
(d\psi/dl)(dl/dN) = (dl/dN) \sum_{k=0}^{\infty} \left[ \left( d\beta_k/dl \right) \nu^m g_k + \beta_d \nu^{m-1} (g_k') \sin \theta - \mu_k g_k \cos \theta \right].
\]

The right-hand side of the damage evolution equation (1) can be presented in the form

\[
\begin{align*}
-\frac{\sigma_{e}^{(k)}}{\beta_{0}'} m &\lambda_{0} - \nu_{0} \left( \frac{\sigma_{e}^{(k)}}{g_{0}'} \right)^{m} \left[ 1 + \nu^{m_{1} - \mu_{0}} \left( \frac{m^{(1)} \sigma_{e}^{(k)}}{\sigma_{e}^{(0)}} - n \frac{g_{1}'}{g_{0}'} \right) + \\
&+ \nu^{m_{2} - \mu_{0}} \left( \frac{m^{(2)} \sigma_{e}^{(k)}}{\sigma_{e}^{(0)}} - n \frac{g_{2}'}{g_{0}'} + m_{0} + 1 \frac{g_{1}'}{2g_{0}'} \right) \right] \sigma_{e}^{(1)} \left( \frac{\sigma_{e}^{(0)}}{\sigma_{e}^{(0)}} \right)^{2} + \ldots
\end{align*}
\]

where \(\sigma_{e}^{(k)}(\theta), \, k = 0, 1, 2\) are the coefficients of the asymptotic expansion of the Von Mises equivalent stress:

\[
\sigma_{e}(r, \theta) = \alpha_{0} e^{\sigma_{e}^{(0)}(r, \theta) + \alpha_{1} e^{\sigma_{e}^{(1)}(r, \theta)} + \alpha_{2} e^{\sigma_{e}^{(2)}(r, \theta)} + \ldots}, \quad \sigma_{e}^{(0)} = \frac{\sqrt{(\sigma_{e}^{(0)}r_{rr})^{2} + (\sigma_{e}^{(0)}r_{r})^{2} - (\sigma_{e}^{(0)}r_{\theta})^{2} + 3(\sigma_{e}^{(0)})^{2}}}{2}, \\
\sigma_{e}^{(1)} = \left[ 2\sigma_{e}^{(0)}r_{rr} + 2\sigma_{e}^{(0)}r_{r} + 2\sigma_{e}^{(0)}r_{\theta} + 2\sigma_{e}^{(0)}r_{\theta} - 2\sigma_{e}^{(0)}r_{\theta} - 2\sigma_{e}^{(0)}r_{\theta} + 6\sigma_{e}^{(0)}r_{\theta} \right] / (2\sigma_{e}^{(0)}).
\]

Comparison of Eq. (10) and Eq. (11) allows to find \(dl/dN = c_{a} g_{0}^{m}/\beta_{0}^{m+1}\) and equalities \(\alpha_{1}/\beta_{1} = c_{a} / g_{0}^{m}\). Asymptotic analysis of the damage evolution law permits to derive ordinary differential equations

\[
\begin{align*}
g_{0}' \sin \theta - \mu_{0} g_{0} \cos \theta &= -\left( \frac{\sigma_{e}^{(0)}}{g_{0}'} \right)^{m} \frac{\sigma_{e}^{(1)}}{\sigma_{e}^{(0)}} \left( \frac{\sigma_{e}^{(0)}}{\sigma_{e}^{(0)}} \right) - \frac{g_{1}'}{g_{0}'}, \\
g_{1}' \sin \theta - \mu_{1} g_{1} \cos \theta &= -\left( \frac{\sigma_{e}^{(0)}}{g_{0}'} \right)^{m} \frac{\sigma_{e}^{(1)}}{\sigma_{e}^{(0)}} \left( \frac{\sigma_{e}^{(0)}}{\sigma_{e}^{(0)}} \right) - \frac{g_{1}'}{g_{0}'}.
\end{align*}
\]
4. Nonlinear eigenvalue problem. Numerical solution

Thus the system of ordinary differential equations for the functions \( f_0(\theta) \) and \( g_0(\theta) \) is obtained

\[
f_0' - 2E\theta f_0''' + (b_1 + G) f_0'' - b_2 E f_0' + (c_0 G + b_3) f_0 = 0, \quad g_0' \sin \theta - \mu_0 g_0 \cos \theta = -\left(\frac{\sigma}{\epsilon}\right)^m g_0^n.
\]  

(13)

The solution of Eqs. (13) should satisfy the traction free boundary conditions and the continuity requirement \( f_0(\theta = \pm \pi) = 0, g_0(\theta = \pm \pi) = 0 \). Due to the symmetry one can consider the upper half-plane and formulate the symmetry conditions \( f_0'(\theta = 0) = 0, f_0'''(\theta = 0) = 0, g_0'(\theta = 0) = 0 \). The first equation of Eqs. (13) is homogeneous in \( f_0(\theta) \) and it is imposed that \( f_0(\theta = 0) = 1 \). Thus the nonlinear eigenvalue problem is formulated: it is necessary to find the eigenvalue \( \mu_0 \) corresponding to the nontrivial solution of Eqs. (13) which satisfy the boundary conditions. Eqs. (13) can be integrated numerically. The two-point boundary value problem formulated is reduced to the initial problem with the following initial conditions \( f_0(0) = 1, f_0'(0) = 0, f_0'''(0) = 0, g_0(0) = \left(\frac{\sigma(0)^0}{\epsilon}\right)^m / \mu_0^{1/(n+1)} \), \( g_0'(0) = 0 \). Unknowns \( \mu_0 \) and \( f_0'''(0) = 0 \) are sought so that the boundary conditions at \( \theta = \pi \) may be satisfied. Through numerical analysis shows that the traditional traction free boundary conditions should be modified and replaced by the requirements \( f_0(\theta = \theta_d) = 0, f_0'(\theta = \theta_d) = 0, g_0(\theta = \theta_d) = 0 \), where \( \theta_d \) is the angle separating the active damage accumulation zone (process zone) \( 0 \leq \theta \leq \theta_d \) and the completely damaged zone \( \theta_d \leq \theta \leq \pi \). It turns out that for all values of material constants \( m, n \) the angle \( \theta_d \) is equal to \( \theta_d = \pi / 2 \). It should be noted that the numerical solution of Eqs. (13) obtained by the Runge–Kutta method in conjunction with the shooting method which becomes multiparametric since three parameters \( \mu, f'''(0) = A \) and \( \theta_d \) are required to choose as part of the numerical solution needs to be additionally justified. The numerical results still require further investigations and verification (especially for the case when the system includes the singular disturbed as \( \theta \to 0 \) equation). The approach proposed here enables to overcome difficulties caused by the singular disturbed equation of Eqs. (13).

5. Artificial small parameter method

The artificial small parameter method is based on the introducing the small parameter \( \varepsilon = \mu_0 - \mu_0^{(0)} \), where \( \mu_0^{(0)} \) refers to the “undisturbed” linear problem and \( \varepsilon \) is the deviation on account of the nonlinearity. Furthermore \( \lambda \), the exponents \( n, m \), the angular functions \( f_0(\theta) \), and \( g_0(\theta) \) are represented as power series

\[
\begin{align*}
\mu_0 = & \mu_0^{(0)} + \varepsilon, \quad \lambda_0 = \sum_{j=0}^{\infty} \varepsilon^j \lambda_0^{(j)}, \\
n = & \sum_{j=0}^{\infty} \varepsilon^j n_j, \quad m = \sum_{j=0}^{\infty} \varepsilon^j m_j, \quad f_0(\theta) = \sum_{j=0}^{\infty} \varepsilon^j f_0^{(j)}(\theta), \quad g_0(\theta) = \sum_{j=0}^{\infty} \varepsilon^j g_0^{(j)}(\theta),
\end{align*}
\]

(14)

where \( n_0, m_0, \lambda_0^{(0)}, f_0^{(0)}(\theta), g_0^{(0)}(\theta) \) are referred to the linear solution (linear damage evolution law). Introducing (14) into the first equation of Eqs. (13) and collecting terms of equal power in \( \varepsilon \) the following set of differential equations is obtained

\[
\begin{align*}
\varepsilon^0 : \quad & \left(f_0^{(0)}\right)' - 2E_0f_0^{(0)}' + (b_1 + G_0)f_0^{(0)}' - b_2 E_0 f_0^{(0)} + (c_0 G_0 + b_3) f_0^{(0)} = 0, \\
\varepsilon^k : \quad & \left(f_0^{(k)}\right)' - 2E_0f_0^{(k)}' + (b_1 + G_0)f_0^{(k)}' - b_2 E_0 f_0^{(k)} + (c_0 G_0 + b_3) f_0^{(k)} = \\
& \sum_{j=0}^{k-1} 2E f_0^{(k-j-1)} - (b_{j+1}) + G_{j+1}) \left(f_0^{(k-j-1)} + \frac{\sum_{j=0}^{k-1} b_{j+1} E_{k-j}}{c_0 G_{k-j}} f_0^{(k-j)} \right),
\end{align*}
\]

(15)

where \( k \geq 1 \), \( E, G, b_j, c_0 \) are coefficients of the asymptotic expansions of functions \( \tilde{E}, \tilde{G} \) and \( b_k (9) \), \( a_0, c_0 \) represented as power series either:

\[
\begin{align*}
\tilde{E} = & \sum_{j=0}^{\infty} \varepsilon^j E_j, \quad \tilde{G} = \sum_{j=0}^{\infty} \varepsilon^j G_j, \quad b_k = \sum_{j=0}^{\infty} \varepsilon^j b_{k}^{(j)}, \quad a_0 = \sum_{j=0}^{\infty} \varepsilon^j a_{0}^{(j)}, \quad c_0 = \sum_{j=0}^{\infty} \varepsilon^j c_{0}^{(j)}.
\end{align*}
\]

(17)

In view of (14) the coefficients of the asymptotic expansions (17) take the form

\[
E_j = \frac{1}{E_0^{(0)}} \sum_{j=0}^{p-1} \left(g_0^{(j)}\right)'_{t_{j-1}}, \quad G_j = 2\sum_{j=0}^{p-1} E_j E_{j-1} - \frac{1}{E_0^{(0)}} \sum_{j=0}^{p-1} \left(g_0^{(j)}\right)'_{t_{j-1}}.
\]
The coefficients of the asymptotic expansions \( b_p \) \((p = 1, 2, 3)\) are determined by the following formulae

\[
b_1^{(j)} = 2(1 + \nu) \sum_{l=0}^{j} \eta_l^{(j)} \xi_0^{(j-l)} - \left(\lambda_0^{(j)} - \mu_0^{(j)}\right) - \nu \sum_{l=0}^{j} \eta_l^{(j)} \left(\lambda_0^{(j-l)} - \mu_0^{(j-l)}\right) + c_l^{(j)},
\]

\[
b_2^{(j)} = 2(1 + \nu) \sum_{l=0}^{j} \eta_l^{(j)} \xi_0^{(j-l)} + 2c_l^{(j)}, \quad b_3^{(j)} = \sum_{l=0}^{j} \left(\epsilon_l^{(j-k)} \eta_l^{(j)} - c_l^{(j)}\right) \left(\lambda_0^{(j-l)} - \mu_0^{(j-l)}\right),
\]

where \( k \) is a number of keeping terms, \( a_0^{(j)} \) and \( c_0^{(j)} \) are determined by the formulae \( a_0^{(j)} = \sum_{l=0}^{j} \xi_0^{(j-l)} - \nu \zeta_0^{(j-l)} \), \( c_0^{(j)} = \nu \zeta_0^{(j)} - \nu \zeta_0^{(j-l)} \), \( \lambda_0 - \mu_0 + 1 = \eta_0 = \sum_{j=0}^{k} \epsilon_j \xi_0^{(j)}\), \( \lambda_0 + 1 = \xi_0 = \sum_{j=0}^{k} \epsilon_j \xi_0^{(j)}\), \( \lambda_0 + 2 = \zeta_0 = \sum_{j=0}^{k} \epsilon_j \xi_0^{(j)}\).

Introducing (14) into the second equation of (13) and collecting terms of equal power in \( \varepsilon \) one can deduce the following set of differential equations

\[
\begin{align*}
\varrho^0 : (g_0^{(0)})' \sin \theta - \mu_0^{(0)} g_0^{(0)} \cos \theta &= -\sigma_0^{(0)} / g_0^{(0)}, \\
\varrho^1 : (g_0^{(1)})' \sin \theta - \mu_0^{(1)} g_0^{(1)} \cos \theta &= g_0^{(2)} \cos \theta + \epsilon_0^{(1)} \bigl(\frac{\Omega - \Lambda}{g_0^{(0)}}\bigr), \quad \Omega(\theta) = m_1 \ln g_0^{(0)} + \frac{g_0^{(1)}}{g_0^{(0)}}, \quad \Lambda(\theta) = m_1 \ln \sigma_0^{(0)} + \frac{\sigma_0^{(1)}}{\sigma_0^{(0)}}, \\
\varrho^2 : (g_0^{(2)})' \sin \theta - \mu_0^{(2)} g_0^{(2)} \cos \theta &= g_0^{(3)} \cos \theta + \epsilon_0^{(2)} \bigl(\frac{\Omega - \Lambda}{g_0^{(0)}}\bigr) - \frac{\sigma_0^{(0)}}{g_0^{(0)}} \frac{\Omega^2}{2} - \left(\frac{m_2 \ln g_0^{(0)} + m_1 g_0^{(1)} / g_0^{(0)} + g_0^{(2)} / g_0^{(0)} + \left(\frac{g_0^{(1)}}{g_0^{(0)}}\right)^2}{2}\right) / g_0^{(0)} + \\
&- \frac{\sigma_0^{(0)}}{g_0^{(0)}} \left(\frac{\Lambda^2}{2} + m_2 \ln \sigma_0^{(0)} + m_1 \sigma_0^{(1)} / \sigma_0^{(0)} + \sigma_0^{(2)} \right) / \sigma_0^{(0)} - \left(\frac{\sigma_0^{(1)}}{\sigma_0^{(0)}}\right)^2 / \sigma_0^{(0)} .
\end{align*}
\]

Thus the system of equations for the functions \( f_0^{(k)}(\theta) \) and \( g_0^{(k)}(\theta) \), \( k = 0, 1, 2, 3, 4, \ldots \) is obtained. The equations are solved successively. The solution of Eqs. (15), (18) \( f_0^{(0)}(\theta) \) and \( g_0^{(0)}(\theta) \) should satisfy the traction free boundary conditions, the symmetry requirements and regularity condition

\[
f_0^{(0)}(\pi) = 0, \quad (f_0^{(0)})'(\pi) = 0, \quad (f_0^{(0)})'(0) = 0, \quad (f_0^{(0)})''(0) = 0, \quad (g_0^{(0)})'(0) = 0, \quad g_0^{(0)}(0) = \left(\frac{\sigma_0^{(0)}(0)}{\mu_0^{(0)}}\right)^{1/2} .
\]

Integration of the nonlinear system of equations (15), (18) together with the boundary conditions (21) and consideration of the solution result in the need to introduce the completely damaged zone \( \pi/2 \leq \theta \leq \pi \) inside of which all the stress components and the continuity parameter are equal to zero. In the active damage accumulation zone \( 0 \leq \theta \leq \pi/2 \) the solution is defined as \( f_0^{(0)}(\theta) = (\cos \theta)^{1/2}, \quad g_0^{(0)}(\theta) = \cos \theta, \quad \mu_0^{(0)} = 1 \). Eqs. (16), (19), (20) determining the angular distributions of the higher order terms of the asymptotic expansions (14) for the \( k \)th terms can be represented in the following form (Stepanova and Igonin (2013))

\[
\begin{align*}
(p_1 f_0^{(k)})'' + p_2 (f_0^{(k)})''' + p_3 (f_0^{(k)})' + p_4 f_0^{(k)} + p_5 g_0^{(k)} + p_6 (g_0^{(k)})' + p_7 g_0^{(k)} &= F_1^{(k)}, \\
q_1 (f_0^{(k)})'' + q_2 (f_0^{(k)})''' + q_3 (f_0^{(k)})' + q_4 f_0^{(k)} + q_5 g_0^{(k)} + q_6 (g_0^{(k)})' + q_7 g_0^{(k)} &= F_2^{(k)},
\end{align*}
\]

where the coefficients \( p_j, q_j \) and \( F_1^{(k)} \) are given in Stepanova and Igonin (2013). The boundary conditions follow from the traction free boundary conditions at \( \theta = \pi/2 \) and the symmetry conditions ahead the crack tip

\[
f_0^{(k)}(\pi/2) = 0, \quad (f_0^{(k)})'(\pi/2) = 0, \quad g_0^{(k)}(\pi/2) = 0, \quad (f_0^{(k)})'(0) = 0, \quad (f_0^{(k)})''(0) = 0, \quad (g_0^{(k)})'(0) = 0.
\]

Eqs. (22), (23) can be reduced to the first order system of equations. For this purpose the notations are introduced

\[
\begin{align*}
\varphi_1 = f_0^{(k)}, \quad \varphi_2 = (f_0^{(k)})', \quad \varphi_3 = (f_0^{(k)})'', \quad \varphi_4 = (f_0^{(k)})''', \quad \varphi_5 = g_0^{(k)}, \quad \varphi_6 = (g_0^{(k)})'.
\end{align*}
\]
Eqs. (22), (23) take the form

\[
\begin{align*}
\varphi_1' - \varphi_2 &= 0, \\
\varphi_2' - \varphi_3 &= 0, \\
\varphi_3' - \varphi_4 &= 0, \\
\varphi_4' + e_1 \varphi_1 + e_2 \varphi_2 + e_3 \varphi_3 + e_4 \varphi_4 + e_5 \varphi_5 + e_6 \varphi_6 &= H_1, \\
\varphi_5' - \varphi_6 &= 0, \\
\varphi_6' + d_1 \varphi_1 + d_2 \varphi_2 + d_3 \varphi_3 + d_4 \varphi_4 + d_5 \varphi_5 + d_6 \varphi_6 &= H_2,
\end{align*}
\]

where the shorthand notations are employed:

\[
\begin{align*}
e_1 &= (p_4 q_5 - p_4 q_4)/q_5, & e_2 &= (p_3 q_5 - p_3 q_4)/q_5, & e_3 &= (p_2 q_5 - p_2 q_4)/q_5, \\
e_4 &= (p_4 q_3 - p_3 q_1)/q_5, & e_5 &= (p_3 q_5 - p_3 q_7)/q_5, & e_6 &= (p_2 q_5 - p_2 q_6)/q_5, \\
d_1 &= q_4/q_5, & d_2 &= q_5/q_5, & d_3 &= q_2/q_5, & d_4 &= q_1/q_5, & d_5 &= q_7/q_5, & d_6 &= q_6/q_5, \\
H_1 &= F_1^1 - p_5 F_2^2/q_5, & H_2 &= F_2^2/q_5.
\end{align*}
\]

The solution of the system derived should satisfy the boundary conditions \(\varphi_1(\theta = \pi/2) = 0, \varphi_2(\theta = \pi/2) = 0, \varphi_3(\theta = \pi/2) = 0, \varphi_4(\theta = 0) = 0, \varphi_5(\theta = 0) = 0\).

Since the homogeneous \(k\)-order problem (22) – (24) is the same as the zeroth-order problem and since the latter has a nontrivial solution, the inhomogeneous first-order problem has a solution only if solvability conditions are satisfied. To determine the solvability conditions we need first to determine the adjoint problem. To this end we multiply the equations for the \(\varphi_k\) by functions \(\psi_k\) respectively. Then we add the resulting equations, integrate the result by parts from \(\theta = 0\) to \(\theta = \pi/2\) to transfer the derivatives from the \(\varphi_k(\theta)\) to \(\psi_k(\theta)\) and obtain the adjoint system of equations

\[
\begin{align*}
\psi_1' &= e_1 \psi_4 + d_1 \psi_6, \\
\psi_2' &= -\psi_1 + e_2 \psi_4 + d_2 \psi_6, \\
\psi_3' &= -\psi_2 + e_3 \psi_4 + d_3 \psi_6, \\
\psi_4' &= -\psi_3 + e_4 \psi_4 + d_4 \psi_6, \\
\psi_5' &= e_5 \psi_4 + d_5 \psi_6, \\
\psi_6' &= -\psi_5 + e_6 \psi_4 + d_6 \psi_6
\end{align*}
\]

and boundary conditions of the adjoint problem \(\psi_1(\theta = \pi/2) = 0, \psi_2(\theta = \pi/2) = 0, \psi_3(\theta = \pi/2) = 0, \psi_4(\theta = 0) = 0, \psi_5(\theta = 0) = 0, \psi_6(\theta = 0) = 0\). Having defined the adjoint homogeneous problem we return to the inhomogeneous problem to determine the solvability condition and we obtain

\[
\int_0^{\pi/2} (H_1 \psi_4 + H_2 \psi_6) d\theta = 0.
\]

The numerical solution of the adjoint problem is found by the use of Mathematica 6.0. Introducing \(H_1\) and \(H_2\) into the solvability condition one can find the equality \(n_1 - m_1 = -1\). By applying the solvability condition to the inhomogeneous equations for \(f_0^{(k)}\) and \(g_0^{(k)}\) one can find \(n_k - m_k = (-1)^k\). Hence the asymptotic expansion \(n - m = \sum_{j=1}^\infty (-1)^j e^j\) is valid. One can easily conclude that \(n - m = -e/(1 + e)\). Eliminating the artificial small parameter \(e = \mu_0 - \mu_0^{(0)}\) one can obtain the analytical expression for the eigenvalue of the nonlinear eigenvalue problem considered \(n - m = (1 - \mu_0)/\mu_0\) or \(\mu_0 = 1/(1 + n - m)\). It results in the analytical solution of the problem \(f_0(\theta) = \cos^m \theta/((\lambda_0 + 2)(\lambda_0 + 1)), g_0(\theta) = \cos^m \theta\). The angular distributions of the stress components and the continuity parameter are determined by the formulae \(\sigma_{rr}^{(0)} = k \sin^2 \theta (\cos \theta)^{\mu_k}, \sigma_{\theta\theta}^{(0)} = k (\cos \theta)^{\mu_0 + 2}, \sigma_{\theta r}^{(0)} = k \sin \theta (\cos \theta)^{\mu_0 + 1}, g_0 = k \sin \theta (\cos \theta)^{\mu_0}\). The similar procedure has been used for the boundary value problems for functions \(f_1(\theta), g_1(\theta)\) (by the use of the approach described Eqs. (8) and (12) have been analyzed) and \(f_k(\theta), g_k(\theta)\). The technique allows to find the solution in the closed form

\[
f_k(\theta) = k (\cos \theta)^{\mu_k}/((\mu_k + 2)(\mu_k + 1)), \quad g_k(\theta) = \kappa^{m/(n+1)} (\cos \theta)^{\mu_k}.
\]

Finally, the angular distributions of the stress components and the continuity parameter are determined by

\[
\begin{align*}
\sigma_{rr}^{(k)}(\theta) &= k \sin^2 \theta (\cos \theta)^{\mu_k}, & \sigma_{\theta\theta}^{(k)}(\theta) &= k (\cos \theta)^{\mu_0 + 2}, & \sigma_{\theta r}^{(k)}(\theta) &= k \sin \theta (\cos \theta)^{\mu_0 + 1}, \\
g_0^{(k)}(\theta) &= k^{m/(n+1)} (\cos \theta)^{\mu_k}, & \mu_k = (k + 1)\mu_0.
\end{align*}
\]
One can conclude that the higher order asymptotic expansions in the vicinity of the fatigue growing crack are valid:

\[
\chi(r, \theta) = \sum_{k=0}^{\infty} \alpha_k r^{k+2} \left( \frac{\cos \theta}{\lambda_k + 2} \right)^{k+2}, \quad \psi(r, \theta) = \sum_{j=0}^{\infty} \beta_k r^{\mu} (\cos \theta)^{\mu},
\]

\[
\lambda_0 = \frac{1}{1 + n - m}, \quad \lambda_k = (k + 1) \lambda_0, \quad \mu_k = \lambda_k.
\]

6. Conclusions

In order to evaluate the mechanical behavior in the vicinity of a growing fatigue crack for plane stress and plane strain conditions of mode I the asymptotic governing equations are derived and analyzed by the light of Continuum Damage Mechanics. It is shown that the growing fatigue crack problem can be reduced to the nonlinear eigenvalue problem. The perturbation theory method for solving the nonlinear eigenvalue problem has been used. It is elucidated that the perturbation method based on the artificial small parameter which quantitatively describes the nearness of the eigenvalue of the "disturbed" nonlinear problem and the eigenvalue of the "undisturbed" linear problem allows to obtain the closed-form solution. It is shown that the eigenvalues of the nonlinear eigenvalue problem are fully determined by the exponents of the damage evolution law. In the paper the higher order asymptotic expansions of the angular functions determining the stress and continuity fields in the neighborhood of the crack tip are given. The asymptotic expansions of the angular functions permit to find the closed-form solution for the problem considered. The representation (25) shows that the perturbation technique effectively constitutes the closed-form solution of the nonlinear eigenvalue problem. Finally, it is shown that the method of the artificial small parameter can be used to great effect in nonlinear eigenvalue problems arising in nonlinear fracture mechanics.

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