Hyperrigid generators in $C^*$-algebras

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Abstract
In this article, we show that, if $S \in B(H)$ is irreducible and essential unitary, then \(\{S, SS^*\}\) is a hyperrigid generator for the unital $C^*$-algebra $T$ generated by $S$. We prove that, if $T$ is an operator in $B(H)$ that generates a unital $C^*$-algebra $A$ then \(\{T, TT^*\}\) is a hyperrigid generator for $A$. As a corollary it follows that, if $T \in B(H)$ is normal then \(\{T, TT^*\}\) is hyperrigid generator for the unital $C^*$-algebra generated by $T$ and if $T \in B(H)$ is unitary then \(\{T\}\) is hyperrigid generator for the $C^*$-algebra generated by $T$. We show that if $V \in B(H)$ is an isometry (not unitary) that generates the $C^*$-algebra $A$ then the minimal generating set \(\{V\}\) is not hyperrigid for $A$.

Keywords Hyperrigidity · Essential unitary operator · Unital completely positive map · Unique extension property

Mathematics Subject Classification 46L07 · 46L52 · 47A13 · 47L80

1 Introduction and preliminaries

Korovkin [13] made an assertion that, if a sequence of positive linear maps $\phi_n : C[0, 1] \to C[0, 1], n = 1, 2, 3, \ldots,$ has the property

$$\lim_{n \to \infty} ||\phi_n(f_k) - f_k|| = 0, \quad k = 0, 1, 2,$$

for the three functions $f_0(x) = 1, f_1(x) = x, f_2(x) = x^2$ then

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\[
\lim_{n \to \infty} \| \phi_n(f) - f \| = 0, \quad \forall \ f \in C[0, 1].
\]

The set \(\{1, x, x^2\}\) is called a Korovkin set or test set. Korovkin [13] showed that, the set \(\{1, x\}\) is not a Korovkin set. Therefore, the set \(\{1, x, x^2\}\) is a minimal set to satisfy the above assertion.

Arveson [4] initiated the study of noncommutative approximation theory focusing on the question: How does one determine whether a set of generators of a \(C^*\)-algebra is hyperrigid? Arveson [4] introduced a noncommutative counterpart of Korovkin set as follows:

**Definition 1** A finite or countably infinite set \(\mathcal{G}\) of generators of a \(C^*\)-algebra \(\mathcal{A}\) is said to be hyperrigid if for every faithful representation \(\mathcal{A} \subseteq \mathcal{B}(H)\) of \(\mathcal{A}\) on a Hilbert space \(H\) and every sequence of unital completely positive (UCP) maps \(\phi_n : \mathcal{B}(H) \to \mathcal{B}(H)\), \(n = 1, 2, \ldots\),

\[
\lim_{n \to \infty} \| \phi_n(g) - g \| = 0, \quad \forall \ g \in \mathcal{G} \quad \implies \quad \lim_{n \to \infty} \| \phi_n(a) - a \| = 0, \quad \forall \ a \in \mathcal{A}.
\]

Note that, a set \(\mathcal{G}\) is hyperrigid if and only if \(\mathcal{G} \cup \mathcal{G}^*\) is hyperrigid if and only if the linear span of \(\mathcal{G}\) is hyperrigid. If \(\mathcal{A}\) is unital, then \(\mathcal{G}\) is hyperrigid if and only if \(\mathcal{G} \cup \{1\}\) is hyperrigid.

The following characterization of hyperrigid operator systems due to Arveson [4] is more of a workable definition of hyperrigidity of operator systems.

**Theorem 1** [4, Theorem 2.1] Let \(\mathcal{S}\) be a separable operator system generating the \(C^*\)-algebra \(\mathcal{A} = C^*(\mathcal{S})\) then \(\mathcal{S}\) is hyperrigid if and only if every nondegenerate representation \(\pi : \mathcal{A} \to \mathcal{B}(H)\) on a separable Hilbert space, \(\pi|_{\mathcal{S}}\) has the unique extension property in the sense that the only unital completely positive (UCP) map \(\phi : \mathcal{A} \to \mathcal{B}(H)\) that satisfies \(\phi|_{\mathcal{S}} = \pi|_{\mathcal{S}}\) is \(\phi = \pi\) itself.

The interesting results of hyperrigid generators are obtained by a direct application of the above criterion.

**Theorem 2** [4, Theorem 3.3] Let \(V \in \mathcal{B}(H)\) be an isometry that generates a \(C^*\)-algebra \(\mathcal{A}\). Then \(\mathcal{G} = \{V, VV^*\}\) is hyperrigid generator for \(\mathcal{A}\).

Let \(S = (S_1, \ldots, S_d)\) denote the compression of the \(d\)-shift to the complement of a homogeneous ideal \(I\) of \(C[z_1, \ldots, z_d]\). Following the remark above, Kennedy and Shalit [11, Theorem 4.12] proved that, if homogeneous ideals are sufficiently non-trivial then \(S\) is essentially normal if and only if it is hyperrigid as the generating set of a \(C^*\)-algebra.

The main purpose of this paper is to find the minimal hyperrigid generators for certain class of \(C^*\)-algebras.

Here, we recall the necessary definitions, conventions and notations. Let \(H\) be a separable complex Hilbert space and let \(\mathcal{B}(H)\) be the set of all bounded linear operators on \(H\). An operator system \(\mathcal{S}\) in a \(C^*\)-algebra \(\mathcal{A}\) is a self-adjoint linear subspace of \(\mathcal{A}\) containing the identity of \(\mathcal{A}\). An operator algebra \(\mathcal{A}_0\) in a \(C^*\)-algebra \(\mathcal{A}\) is a unital subalgebra of \(\mathcal{A}\). Given a linear map \(\phi\) from a \(C^*\)-algebra \(\mathcal{A}\) into a \(C^*\)-
algebra $B$ we can define a family of maps $\phi_n : M_n(A) \to M_n(B)$ given by $\phi_n([a_{ij}]) = [\phi(a_{ij})]$, $n \in \mathbb{N}$. We say that $\phi$ is completely positive (CP) if $\phi_n$ is positive for all $n \geq 1$, and that $\phi$ is unital completely positive (UCP) if in addition $\phi(1) = 1$.

**Definition 2** Let $S$ be an operator system that generates a $C^*$-algebra $A$. A unital completely positive map $\phi : S \to B(H)$ is said to have the unique extension property if it has a unique extension to a UCP map $\bar{\phi} : A \to B(H)$.

The boundary representations of $A$ for $S$, which were introduced by Arveson [2], are precisely the irreducible representations $\pi : A \to B(H)$ with the property that the restriction $\pi|_S$ has the unique extension property.

Arveson [4] attempted to prove the non-commutative analogue of Saskin’s theorem [16] using theory of noncommutative Choquet boundary for unital completely positive maps on $C^*$-algebras and noncommutative counterpart of the Korovkin’s set which is the hyperrigid set. Arveson [4] proved that if the separable operator system is hyperrigid in the $C^*$-algebra then every irreducible representation of $C^*$-algebra is a boundary representation for the operator system. The converse to this result is called hyperrigidity conjecture: that is, if every irreducible representation of a $C^*$-algebra is a boundary representation for a separable operator system then the operator system is hyperrigid.

Arveson [4] showed that the hyperrigidity conjecture is true for $C^*$-algebras with countable spectrum. Kleski [12] established the hyperrigidity conjecture for all type-I $C^*$-algebras with additional assumptions on the co-domain. Davidson and Kennedy [8] proved the conjecture for function systems. The hyperrigidity conjecture is still open for general $C^*$-algebras.

Namboodiri, Pramod, Shankar and Vijayarajan [14] approached the hyperrigidity conjecture with weaker notions. They got the partial answers. Shankar and Vijayarajan [18] examined the tensor product of hyperrigid operator systems. Clouatre [6] studied the hyperrigidity conjecture using states and the notion of unperforated pairs. Clouatre and Hartz [7] determined hyperrigidity for certain analogues of the disc algebra. Dor-on and Salomon [9] and Salomon [17] examined the hyperrigid generators of the graph $C^*$-algebras. Katsoulis and Ramsey [10] gave sufficient and necessary conditions for tensor algebra to be hyperrigid.

### 2 Essential unitary and hyperrigidity

Let $B(H)$ be the algebra of bounded linear operators on a separable complex Hilbert space $H$ and $\mathcal{K}(H)$ ideal of compact operators on $H$. Let $\pi : B(H) \to B(H)/\mathcal{K}(H)$ be the natural surjection onto the Calkin algebra $B(H)/\mathcal{K}(H)$. The operator $T \in B(H)$ is called essentially normal if $\pi(T)$ is normal in the Calkin algebra, or equivalently, $T^*T - TT^*$ is compact. The operator $S \in B(H)$ is called essentially unitary if $\pi(S)$ is unitary in the Calkin algebra, or equivalently, $I - S^*S$ and $I - SS^*$ are compact. The above definitions can be found in Ref. [5].

Here, we will have the following assumptions to proceed. Let $S$ be an irreducible and essential unitary but not unitary operator in $B(H)$ and let $\mathcal{G} = \{S, SS^*\}$. Let $S$ be
an operator system generated by \( \mathcal{G} \). Let \( \mathcal{T} = \mathcal{C}^*(\mathcal{G}) \) be the unital \( \mathcal{C}^* \)-algebra generated by \( \mathcal{G} \). The unital \( \mathcal{C}^* \)-algebra \( \mathcal{T} \) contains the compact operators \( \mathcal{K}(\mathcal{H}) \).

A representation \( \rho : \mathcal{T} \to \mathcal{B}(\mathcal{H}) \) is said to be singular representation if it annihilates the compact operators \( \mathcal{K}(\mathcal{H}) \).

**Proposition 1** Suppose that \( S \) is irreducible and essential unitary (not unitary) and \( \mathcal{G} = \{ S, SS^* \} \). Let \( \mathcal{S} \) be a operator system generated by \( \mathcal{G} \) and \( \mathcal{T} = \mathcal{C}^*(\mathcal{G}) \). Then the identity representation of \( \mathcal{T} \) is a boundary representation for \( \mathcal{S} \).

**Proof** Since \( S \) is irreducible and essential unitary, then unital \( \mathcal{C}^* \)-algebra generated by \( \mathcal{G} \) contains the compact operators, that is, \( \mathcal{K}(\mathcal{H}) \subseteq \mathcal{T} = \mathcal{C}^*(\mathcal{G}) \). The operator system \( \mathcal{S} \subset \mathcal{T} \) is irreducible and contains the identity operator. By our assumption, \( 0 \neq K = I - SS^* \in \mathcal{S} \) is a compact operator, we have \( ||K - K|| < ||K|| \). Therefore, the quotient map \( q : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H}) \) is not completely isometric on \( \mathcal{S} \). Hence by boundary theorem of Arveson [3, Theorem 2.1.1], identity representation of \( \mathcal{T} \) is a boundary representation for \( \mathcal{S} \).

**Theorem 3** Let \( S \) be an irreducible and essential unitary (not unitary) and \( \mathcal{G} = \{ S, SS^* \} \). Let \( \mathcal{T} = \mathcal{C}^*(\mathcal{G}) \) be the unital \( \mathcal{C}^* \)-algebra generated by \( \mathcal{G} \). Then \( \mathcal{G} \) is a hyperrigid generator for \( \mathcal{T} \).

**Proof** Let \( \mathcal{S} \) be the operator system generated by \( \mathcal{G} \). Note that \( \mathcal{G} \) is hyperrigid if and only if \( \mathcal{S} \) is hyperrigid. By Theorem 1, it suffices to show that for every nondegenerate representation of \( \mathcal{T} \) restricted to \( \mathcal{S} \) has the unique extension property.

Let \( \rho : \mathcal{T} \to \mathcal{B}(\mathcal{H}) \) be a singular representation. Let \( \phi : \mathcal{T} \to \mathcal{B}(\mathcal{K}) \) be a unital completely positive extension of \( \rho|_\mathcal{S} \) and let \( \pi : \mathcal{T} \to \mathcal{B}(\mathcal{K}) \) be a Stinespring dilation of \( \phi \). Since \( \phi(SS^*) = \rho(SS^*) \) implies \( \phi(SS^*) = \phi(S)\phi(S^*) \). By [1, Theorem 1.5] (see also [7, Lemma 3.2]), we have \( \mathcal{H} \) is coinvariant for \( \pi(S) \). Since \( S \) is essential unitary, it follows from [15, Theorem 2.2] that \( \mathcal{H} \) is also invariant for \( \pi(S) \). Therefore, \( \mathcal{H} \) reduces \( \pi(S) \), and thus reduces \( \pi(\mathcal{T}) \). Hence \( \rho|_\mathcal{S} \) has the unique extension property.

By Proposition 1, the restriction of the identity representation of \( \mathcal{T} \) to \( \mathcal{S} \) has the unique extension property. Using [7, Lemma 3.3] every nondegenerate representation of \( \mathcal{T} \) splits as the direct sum of a multiple of the identity representation and a singular nondegenerate representation and by [4, Proposition 4.4] the unique extension property passes to direct sums. Hence every nondegenerate representation of \( \mathcal{T} \) restricted to \( \mathcal{S} \) has the unique extension property.

**Example 1** Let \( \mathcal{H} \) be a Hilbert space having an orthonormal basis \( \{ e_n : n \geq 0 \} \). The unilateral shift \( S \) is defined by \( Se_n = e_{n+1} \). The \( \mathcal{C}^* \)-algebra \( \mathcal{T} \) generated by \( S \) is called the Toeplitz \( \mathcal{C}^* \)-algebra. Observe that \( I - SS^* \) and \( I - SS^* \) are compact, therefore \( S \) is essential unitary. Also, \( S \) is irreducible. The Toeplitz \( \mathcal{C}^* \)-algebra \( \mathcal{T} \) contains the compact operators \( \mathcal{K}(\mathcal{H}) \). We know that the set \( \{ S, SS^* \} \) also generates the Toeplitz \( \mathcal{C}^* \)-algebra \( \mathcal{T} \). Hence, by Theorem 3, the set \( \{ S, SS^* \} \) is hyperrigid generator for Toeplitz \( \mathcal{C}^* \)-algebra \( \mathcal{T} \).
The main purpose of this section is to find the hyperrigid generators for the $C^*$-algebras generated by a single operator.

**Theorem 4** Let $T$ be an operator in $B(H)$ that generate a unital $C^*$-algebra $A$ and let $\mathcal{G} = \{T, T^*T, TT^*\}$. Then $\mathcal{G}$ is a hyperrigid generators for unital $C^*$-algebra $A$.

**Proof** Let $\mathcal{S}$ be the operator system generated by $\mathcal{G}$. By Theorem 1, it suffices to show that for every nondegenerate representation $\pi$ of $A$, $\pi|_{\mathcal{S}}$ has the unique extension property.

Let $\pi : A \rightarrow B(H)$ be a nondegenerate representation. Let $\phi : A \rightarrow B(H)$ be a UCP map satisfying $\phi(T) = \pi(T), \phi(T^*T) = \pi(T^*T)$ and $\phi(TT^*) = \pi(TT^*)$. We have to show that $\phi = \pi$ on $A$.

Using Stinespring theorem, we can express $\phi$ in the form

$$\phi(S) = V^*\sigma(S)V, \quad \forall \quad S \in A.$$ 

Where $\sigma$ is a representation of $A$ on a Hilbert space $K$, $V : H \rightarrow K$ is an isometry, and which is minimal in the sense that $\overline{\sigma(A)VH} = K$.

Since $\phi(T) = \pi(T)$ and $\phi(T^*T) = \pi(T^*T)$, by [1, Theorem 1.5] we have $VH$ is invariant for $\sigma(T)$.

$$VV^*\sigma(T)(1 - VV^*)\sigma(T)^*VV^* = VV^*\sigma(T)\sigma(T)^*VV^* - VV^*\sigma(T)VV^*\sigma(T)^*VV^* = VV^*\sigma(TT^*)VV^* - V\pi(T)\pi(T)^*V^* = V\pi(TT^*)V^* - V\pi(TT^*)V^* = 0.$$ 

Hence $(1 - VV^*)\sigma(T)^*VV^* = 0$, we conclude that $VH$ is invariant for $\sigma(T)^*$. Since $A$ is generated by $T$ it follows that $\sigma(A)VH \subseteq VH$. By minimality we must have $VH = K$, which implies that $V$ is unitary and therefore $\phi(S) = V^{-1}\sigma(S)V$ is a representation. Since $\phi$ agrees with $\pi$ on a generating set, therefore $\phi = \pi$ on $A$.

**Corollary 1** Let $T$ be a normal operator in $B(H)$ that generate a unital $C^*$-algebra $A$ and let $\mathcal{G} = \{T, TT^*\}$. Then $\mathcal{G}$ is hyperrigid generator for unital $C^*$-algebra $A$.

**Corollary 2** Let $T$ be an unitary operator in $B(H)$ that generate a $C^*$-algebra $A$ and let $\mathcal{G} = \{T\}$. Then $\mathcal{G}$ is hyperrigid generator for $C^*$-algebra $A$.

**Proposition 2** Let $V \in B(H)$ be an isometry (not unitary) that generates a $C^*$-algebra $A$. Then

(i) $\mathcal{G} = \{V, VV^*\}$ is hyperrigid generator for $A$.

(ii) The smaller generating set $\mathcal{G}_0 = \{V\}$ is not hyperrigid.

**Proof** (i) follows from the Theorem 2. Now we will prove (ii), let $\mathcal{S}$ be the operator system generated by $\mathcal{G}_0$. Let $Id$ denote the identity representation of a $C^*$-algebra $A$. Let $V^*Id(\cdot)V$ be a completely positive map on the $C^*$-algebra $A$. We have $V^*Id|_S = Id|_S$, but
This implies that \( \text{Id} \) representation restricted to \( S \) has two UCP map extensions \( V^*\text{Id}(V^*) \) and \( \text{Id} \). Therefore the nondegenerate representation \( \text{Id}|_S \) does not have unique extension property. Using the Theorem 1, \( S \) is not hyperrigid operator system in a \( C^* \)-algebra \( \mathcal{A} \). This will imply that \( G_0 \) is not hyperrigid in \( \mathcal{A} \).

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