Lonesum decomposable matrices

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Abstract

A lonesum matrix is a (0, 1)-matrix that is uniquely determined by its row and column sum vectors. In this paper, we introduce lonesum decomposable matrices and study their properties. We provide a necessary and sufficient condition for a matrix $A$ to be lonesum decomposable, and give a generating function for the number $D_k(m, n)$ of $m \times n$ lonesum decomposable matrices of order $k$. Moreover, by using this generating function we prove some congruences for $D_k(m, n)$ modulo a prime.

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1 Introduction

A (0, 1)-matrix (resp. vector) is a matrix (resp. vector) in which each entry is zero or one. A (0, 1)-matrix $A$ is called a lonesum matrix if $A$ is uniquely determined by its row and column sum vectors. For example, a (0, 1)-matrix with a row sum vector $t(3, 1)$ and a column sum vector $(1, 2, 1)$ is uniquely determined as the following:

$$\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 0
\end{pmatrix}.$$

Hence, the matrix $\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 0
\end{pmatrix}$ is a lonesum matrix. Because $\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}$ and $\begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 0
\end{pmatrix}$ have the same row and column sum vectors, they are not lonesum matrices. We denote by $L(m, n)$ the number of $m \times n$ lonesum matrices. For simplicity, we set $L(m, 0) = L(0, m) = 1$ for any non-negative integer $m$. It is known that lonesum matrices are related to certain combinatorial objects. For example, the number $L(m, n)$ is equal to the number of acyclic orientations of the complete bipartite graph $K_{m,n}$ ([5, Theorem 2.1]).
An \( m \times n \) \((0,1)\)-matrix \( A = (a_{ij}) \) is called a Ferrers matrix if \( A \) satisfies the condition
\[
\begin{align*}
    a_{ij} = 0 \Rightarrow a_{kj} = 0 & \quad (k \geq i), \\
    a_{ij} = 0 \Rightarrow a_{il} = 0 & \quad (l \geq j).
\end{align*}
\]
This condition means that all 1 entries of \( A \) are placed to the upper left of \( A \). For example, the matrix
\[
\begin{pmatrix}
    1 & 1 & 1 \\
    1 & 0 & 0
\end{pmatrix}
\]
is a Ferrers matrix. Ryser [12] investigated matrices that have fixed row and column sum vectors. In our setting, his result can be written as follows:

**Proposition 1.1.** Let \( A \) be a \((0,1)\)-matrix. Then, the following conditions are equivalent:

(i) \( A \) is a lonesum matrix.

(ii) \( A \) does not contain
\[
\begin{pmatrix}
    1 & 0 \\
    0 & 1
\end{pmatrix}
\] or
\[
\begin{pmatrix}
    0 & 1 \\
    1 & 0
\end{pmatrix}
\] as a submatrix.

(iii) \( A \) is obtained from a Ferrers matrix by permutations of rows and columns.

For an integer \( k \), Kaneko [10] introduced poly-Bernoulli numbers \( B_n^{(k)} \) of index \( k \) as
\[
\frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!},
\]
where \( \text{Li}_k(z) \) denotes the \( k \)-th polylogarithm, defined by \( \text{Li}_k(z) := \sum_{n=1}^{\infty} z^n / n^k \). Brewbaker [4] proved that the numbers \( L(m, n) \) are equal to the poly-Bernoulli numbers of negative indices:
\[
L(m, n) = B_n^{(-m)} \quad (m, n \geq 0).
\]
The generating function of poly-Bernoulli numbers of negative indices has been given by Kaneko [10], hence the numbers of lonesum matrices have the following generating function:
\[
\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} L(m, n) \frac{x^m y^n}{m! n!} = \frac{e^{x+y}}{e^x + e^y - e^{x+y}}.
\]
The present author, Ohno, and Yamamoto [9] introduced “weighted” lonesum matrices and a simple proof of (3) was given (see [9] Proof of Theorem 1).
For \( m \times n \) matrices \( A \) and \( B \), we write \( A \sim B \) if \( A \) is obtained from \( B \) by row or column exchanges. We call a \((0, 1)\)-matrix \( A \) is \textit{lonesum decomposable} if \( A \) satisfies the condition

\[
A \sim \begin{pmatrix}
L_1 & O \\
L_2 & \ddots \\
& \ddots & \ddots \\
O & & & L_k
\end{pmatrix},
\]

where \( L_i (1 \leq i \leq k) \) are lonesum matrices. A lonesum matrix is clearly lonesum decomposable. Since a lonesum matrix can be obtained from a Ferrers matrix, a lonesum decomposable matrix \( A \) can be transformed as

\[
A \sim \begin{pmatrix}
F_1 & O \\
F_2 & \ddots \\
& \ddots & \ddots \\
O & & & F_k
\end{pmatrix},
\]

where \( F_i (1 \leq i \leq k) \) are Ferrers matrices with no zero rows or zero columns. We call the right-hand side of (4) the \textit{decomposition matrix} of \( A \) and \( k \) the \textit{decomposition order} of \( A \).

\textbf{Proposition 1.2.} Let \( A \) be a lonesum decomposable matrix. Then the decomposition matrix of \( A \) is uniquely determined up to the order of \( F_i (1 \leq i \leq k) \). In particular, the decomposition order of \( A \) is uniquely determined.

\textit{Proof.} For a lonesum decomposable matrix \( A = (a_{ij}) \), it follows from Proposition 1.1 that two elements \( a_{ij} = 1 \) and \( a_{i'j'} = 1 \) belong to the same Ferrers block if and only if \( a_{ij} \) and \( a_{i'j'} \) do not form a submatrix \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) or \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). Because the type of Ferrers matrix is uniquely determined, a decomposition matrix of \( A \) is also uniquely determined up to the order of its Ferrers blocks. \qed

The outline of this paper is as follows. In Section 2 we show that a \((0, 1)\)-matrix \( A \) is lonesum decomposable if and only if \( A \) does not contain certain matrices as submatrices. In Section 3 we give a generating function for the number of lonesum decomposable matrices. In Section 4 we derive some congruences for the numbers of lonesum decomposable matrices of order \( k \) by using the generating function given in Section 3.
2 Lonesum decomposable matrices

Let us define a $2 \times 3$ matrix $U$ as

$$U := \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$ 

It can be easily checked that $U$ is not lonesum decomposable. Let $\mathcal{N}$ be a set of all matrices obtained from $U$ or $tU$ by permutations of rows and columns. Namely, the elements of $\mathcal{N}$ are the following twelve matrices:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \end{pmatrix}.$$

The following is the first main result of this paper.

**Theorem 2.1.** Let $A$ be a $(0,1)$-matrix. Then, the following two conditions are equivalent.

(i) $A$ is lonesum decomposable.

(ii) $A$ does not contain an element of $\mathcal{N}$ as a submatrix.

**Proof.** It is clear that (i) $\Rightarrow$ (ii), and we show (ii) $\Rightarrow$ (i). This statement clearly holds for $0 \leq m, n \leq 2$, where $m$ and $n$ are the numbers of rows and columns of $A$, respectively. A transpose of a lonesum decomposable matrix is also lonesum decomposable, hence we only have to prove that if the statement holds for all $m \times n$ matrices, then it holds for any $m \times (n+1)$ matrix for $m, n \geq 2$.

Let $A$ be an $m \times (n+1)$ $(0,1)$-matrix not containing an element of $\mathcal{N}$. The matrix obtained by removing the $(n+1)$-st column from $A$ is $m \times n$ matrix. Hence, by the induction assumption, the matrix $A$ can be transformed as

$$A \sim \begin{pmatrix} F_1 & O & b_1 \\ \vdots & F_k & b_k \\ O & c \end{pmatrix},$$

where $F_i$ ($1 \leq i \leq k$) are Ferrers matrices with no zero rows or columns, and $b_i$ ($1 \leq i \leq k$) and $c$ are $(0,1)$-vectors. If there exist two non-zero
vectors \( \mathbf{b}_i \) and \( \mathbf{b}_j \) \((i \neq j)\), then the submatrix \( \begin{pmatrix} F_i & O & \mathbf{b}_i \\ O & F_j & \mathbf{b}_j \end{pmatrix} \) contains a matrix 
\[
\begin{pmatrix} 1 & 0 & 1 \\
0 & 1 & 1 \\
\end{pmatrix},
\]
and this contradicts the assumption that \( \mathbf{A} \) does not contain any element of \( \mathcal{N} \). Therefore, there is at most one non-zero vector in \( \mathbf{b}_i \) \((1 \leq i \leq k)\), and we can set \( \mathbf{b}_1 = \cdots = \mathbf{b}_{k-1} = \mathbf{0} \) without loss of generality.

We consider the two cases where (i) \( \mathbf{c} \) has 1’s and (ii) \( \mathbf{c} \) has no 1’s.

(i). The case that \( \mathbf{c} \) has 1’s.

If the vector \( \mathbf{b}_k \) has both 0’s and 1’s, then 
\[
\begin{pmatrix} F_k & \mathbf{b}_k \\ O & \mathbf{c} \end{pmatrix}
\]
contains a matrix \[
\begin{pmatrix} 1 & 1 \\
1 & 0 \\
0 & 1 \\
\end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\
1 & 1 \\
0 & 1 \\
\end{pmatrix},
\]
and this contradicts the assumption that \( \mathbf{A} \) does not contain an element of \( \mathcal{N} \). Therefore, \( \mathbf{b}_k = \mathbf{1} \) or \( \mathbf{0} \). If \( \mathbf{b}_k = \mathbf{1} \), then
\[
\begin{pmatrix} F_k & \mathbf{b}_k \\ O & \mathbf{c} \end{pmatrix} \sim \begin{pmatrix} 1 \\ \mathbf{c} \\ F_k \\ O \end{pmatrix}.
\]

Because the right-hand side of (5) is a lonesum matrix, the statement holds. If \( \mathbf{b}_k = \mathbf{0} \), then
\[
\begin{pmatrix} F_k & \mathbf{b}_k \\ O & \mathbf{c} \end{pmatrix} \sim \begin{pmatrix} F_k & \mathbf{0} \\ \mathbf{c} & O \end{pmatrix}.
\]

The right-hand side of (6) is lonesum decomposable of order 2, and hence the statement again holds.

(ii). The case that \( \mathbf{c} \) has no 1’s.

We have
\[
\begin{pmatrix} F_k & \mathbf{b}_k \\ O & \mathbf{c} \end{pmatrix} \sim \begin{pmatrix} F_k & \mathbf{b}_k \\ \mathbf{0} & O \end{pmatrix}.
\]

By Proposition 1.1 if the matrix \( \begin{pmatrix} F_k & \mathbf{b}_k \end{pmatrix} \) is not a lonesum matrix then it contains \[
\begin{pmatrix} 1 & 0 \\
0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\
1 & 0 \end{pmatrix}
\]
as a submatrix. Because \( F_k \) has no zero columns, the matrix \( \begin{pmatrix} F_k & \mathbf{b}_k \end{pmatrix} \) also contains \[
\begin{pmatrix} 1 & 1 & 0 \\
1 & 0 & 1 \\
\end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 & 1 \\
1 & 1 & 0 \end{pmatrix}
\]
as a submatrix, and this contradicts the assumption that \( \mathbf{A} \) does not contain an element of \( \mathcal{N} \). Therefore, the matrix \( \begin{pmatrix} F_k & \mathbf{b}_k \end{pmatrix} \) is a lonesum matrix and the statement also holds in this case.

For a \((0,1)\)-matrix \( \mathbf{A} \), we define \( \overline{\mathbf{A}} \) as the matrix in which the 0 and 1 entries of \( \mathbf{A} \) are inverted. If \( \mathbf{A} \) is a lonesum matrix, then \( \overline{\mathbf{A}} \) is also a lonesum matrix. However, lonesum decomposable matrices do not have this
property. For example, the matrix $V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ is lonesum decomposable, but $\overline{V} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \in \mathcal{N}$ is not lonesum decomposable. The following theorem determines when a matrix $A$ satisfies that both $A$ and $\overline{A}$ are lonesum decomposable.

**Theorem 2.2.** Let $A$ be a $(0,1)$-matrix. Then, the following conditions are equivalent.

(i) Both $A$ and $\overline{A}$ are lonesum decomposable.

(ii) $A$ is a lonesum matrix or $A \sim \begin{pmatrix} 1 \\ O \\ 1 \end{pmatrix}$, where $1$ (resp. $O$) is a matrix whose entries are all 1 (resp. 0).

**Proof.** It is clear that (ii) $\Rightarrow$ (i), and we only have to prove that (i) $\Rightarrow$ (ii). Assume that $A$ and $\overline{A}$ are both lonesum decomposable, and let $k$ be the decomposition order of $A$. When $k = 0$ or 1, $A$ is a lonesum matrix. When $k = 2$, the matrix $A$ satisfies that

$$A \sim \begin{pmatrix} L_1 \\ O \\ L_2 \end{pmatrix},$$

where $L_1$ and $L_2$ are non-zero lonesum matrices. If $L_1$ or $L_2$ has 0’s, then the matrix $\overline{A}$ contains an element of $\mathcal{N}$ as a submatrix, and $\overline{A}$ is not lonesum decomposable. Therefore, $L_1 = 1$ and $L_2 = 1$. When $k \geq 3$, the matrix $A$ contains a $3 \times 3$ submatrix $W$ satisfying $W \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. This matrix contains $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and this contradicts the condition that $\overline{A}$ is lonesum decomposable. As a consequence, either $A$ is a lonesum matrix or $A \sim \begin{pmatrix} 1 \\ O \\ 1 \end{pmatrix}$. $\square$

3 Generating function of lonesum decomposable matrices

For a positive integer $k$, let $D_k(m,n)$ denote the number of $m \times n$ lonesum decomposable matrices of decomposition order $k$. For simplicity, we set $D_k(m,0) = D_k(0,m) = 0$ for $k \geq 1$ and $m \geq 0$, and $D_0(m,n) = 1$.
for \((m, n) \in \mathbb{Z}_{\geq 0}^2\). Moreover, we define \(D(m, n) := \sum_{k=0}^{\infty} D_k(m, n)\) for \((m, n) \in \mathbb{Z}_{\geq 0}^2\). This means that \(D(m, n)\) is the number of all \(m \times n\) lonesum decomposable matrices. We can see that \(D_k(m, n) = 0\) for \(k > \min(m, n)\) and \(L(m, n) = D_0(m, n) + D_1(m, n)\). We present tables showing \(D_1(m, n), D_2(m, n),\) and \(D(m, n)\) at the end of this paper.

The generating functions for \(D_k\) and \(D\) are given as follows:

**Theorem 3.1.** The following equations hold:

\[
\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} D_k(m, n) \frac{x^m y^n}{m! n!} = \frac{e^{x+y}}{k!} \left( \frac{1}{e^x + e^y - e^{x+y}} - 1 \right)^k \quad (k \geq 0). \tag{8}
\]

\[
\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} D(m, n) \frac{x^m y^n}{m! n!} = \exp \left( \frac{1}{e^x + e^y - e^{x+y}} + x + y - 1 \right) . \tag{9}
\]

**Proof.** Let \(\hat{L}(m, n)\) be the number of \(m \times n\) lonesum matrices with no zero rows or columns. Here, we set \(\hat{L}(0, 0) = 1\) and \(\hat{L}(m, 0) = \hat{L}(0, m) = 0\) for \(m > 0\). Benyi and Hajnal [3, Theorem 3] mentioned that the generating function of \(\hat{L}(m, n)\) is given by

\[
\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \hat{L}(m, n) \frac{x^m y^n}{m! n!} = \frac{1}{e^x + e^y - e^{x+y}}. \tag{10}
\]

By definition, it holds that

\[
L(m, n) = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} \hat{L}(i, j), \tag{11}
\]

and we can also obtain the generating function (34) of \(L(m, n)\) from (10).

We note that multiplying the generating function (34) by \(e^{x+y}\) means that it allows the lonesum matrices to have zero columns or zero rows.

Let \(\hat{D}_k(m, n)\) be the number of \(m \times n\) lonesum decomposable matrices of order \(k\) with no zero rows and columns. We set \(\hat{D}_0(m, n) = 0\) if \((m, n) \neq (0, 0)\) and \(= 1\) if \((m, n) = (0, 0)\). When \(k = 1\), we have \(\hat{D}_1(m, n) = \hat{L}(m, n)\) if \((m, n) \neq (0, 0)\) and \(= 0\) if \((m, n) = (0, 0)\). Therefore, we have

\[
\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \hat{D}_1(m, n) \frac{x^m y^n}{m! n!} = \frac{1}{e^x + e^y - e^{x+y}} - 1.
\]

In general, the generating function of \(\hat{D}_k\) can be given by

\[
\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \hat{D}_k(m, n) \frac{x^m y^n}{m! n!} = \frac{1}{k!} \left( \frac{1}{e^x + e^y - e^{x+y}} - 1 \right)^k \quad (k \geq 0). \tag{12}
\]
The generating function of $D_k$ can be obtained by multiplying (12) by $e^{x+y}$, hence we obtain (8). Equation (9) follows immediately from (8).

**Remark 3.2.** Ju and Seo [8] studied generating functions for the number of matrices not including various $2 \times 2$ matrices. Theorem 3.1 gives a similar result on matrices that do not include the elements of $N$.

It is known that the numbers $L(m, n)$ (or the poly-Bernoulli numbers of negative indices) satisfy a recurrence relation (e.g. [2, Prop. 14.3 and 14.4]). Our numbers $D_k(m, n)$ also satisfy a recurrence relation.

**Proposition 3.3.** For $k \geq 1$ and $m, n \geq 0$, we have

$$D_k(m + 1, n) = D_k(m, n) + \sum_{l=0}^{n-1} \binom{n}{l} ((k - 1)D_k(m, l) + D_{k-1}(m, l) + D_k(m, l + 1)).$$

**Proof.** Let $G_k(x, y) := \frac{e^{x+y}}{k!} \left( \frac{1}{e^x + e^y - e^{x+y}} - 1 \right)^k$. By direct calculations, we can verify that

$$\frac{\partial}{\partial x} G_k = G_k + (e^y - 1) \left( (k - 1)G_k + G_{k-1} + \frac{\partial}{\partial y} G_k \right). \quad (13)$$

By comparing the coefficients of both sides of (13), we obtain the proposition. □

To conclude this section, we give a relation between $D_k(m, n)$ and the poly-Bernoulli polynomials. For any integers $k_1, \ldots, k_r$, we define the multi-poly Bernoulli(-star) polynomials $B_{n,*,\ldots,*}^{(k_1,\ldots,k_r)}(x)$ by

$$e^{-xt} \frac{Li_{k_1,\ldots,k_r}^{*}(1 - e^{-t})}{1 - e^{-t}} = \sum_{n=0}^{\infty} B_{n,*,\ldots,*}^{(k_1,\ldots,k_r)}(x) t^n \frac{n!}{n!}, \quad (14)$$

where

$$Li_{k_1,\ldots,k_r}^{*}(z) := \sum_{1 \leq m_1 \leq \cdots \leq m_r} \frac{z^{m_r}}{m_1^{k_1} \cdots m_r^{k_r}}.$$ These polynomials have been introduced by Imatomi [7, §6], but they were defined there with $e^{-xt}$ replaced by $e^{xt}$ in (14). When $r = 1$, the polynomial $B_{n,*}^{(k)}(x)$ coincides with the $n$-th poly-Bernoulli polynomial $B_n^{(k)}(x)$ defined by

$$e^{-xt} \frac{Li_k(1 - e^{-t})}{1 - e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k)}(x) t^n \frac{n!}{n!}.$$ (see e.g., Coppo-Candelpergher [5]).
Proposition 3.4. For integers \( k, m, n \geq 0 \), we have

\[
D_k(m, n) = \frac{(-1)^k}{k!} \left( 1 + \sum_{i=1}^{k} \binom{k}{i} (-1)^i B_{n, \ldots, 0, -m}^{i-1}(i - 1) \right).
\]

Proof. For an integer \( i \geq 1 \), we have

\[
e^{x+y} \left( \frac{1}{e^x + e^y - e^{x+y}} \right)^i = e^{x+y} e^{-iy} \sum_{l_1, \ldots, l_i \geq 0} e^{(l_1 + \cdots + l_i)x} (1 - e^{-y})^{l_1 + \cdots + l_i} \frac{1}{1 - e^{-y}}
\]

\[
= e^{-(i-1)y} \sum_{m=0}^{\infty} \sum_{l_1, \ldots, l_i \geq 0} (1 - e^{-y})^{l_1 + \cdots + l_i + 1} \frac{1}{(l_1 + \cdots + l_i + 1)^m} \frac{1}{1 - e^{-y}} \frac{m!}{x^m}
\]

\[
= e^{-(i-1)y} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\text{Li}_0^{*}(0, -m)(1 - e^{-y}) \frac{x^m}{m!}}{1 - e^{-y}} \frac{m!}{n!}.
\]

From this formula and the binomial expansion, we obtain that

\[
\frac{e^{x+y}}{k!} \left( \frac{1}{e^x + e^y - e^{x+y}} - 1 \right)^k = \frac{(-1)^k}{k!} \left( e^{x+y} + \sum_{i=1}^{k} \binom{k}{i} (-1)^i \frac{e^{x+y}}{(e^x + e^y - e^{x+y})^i} \right)
\]

\[
= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^k}{k!} \left( 1 + \sum_{i=1}^{k} \binom{k}{i} (-1)^i B_{n, \ldots, 0, -m}^{i-1}(i - 1) \right) \frac{x^m x^n}{m! n!},
\]

and this proves the proposition.

Remark 3.5. Kaneko, Sakurai, and Tsumura \[11\] introduced a sequence \( \mathcal{B}_{m}^{(l)}(n) \) as

\[
\mathcal{B}_{m}^{(l)}(n) := \sum_{j=0}^{n} \binom{n}{j} B_{m, \ldots, 0, -m}^{l-j}(n) \quad (l, m, n \in \mathbb{Z}_{\geq 0}),
\]
where \([\binom{n}{j}]\) are the Stirling numbers of the first kind. They proved that this sequence has the following simple generating function:

\[
\sum_{i=0}^{\infty} \sum_{m=0}^{\infty} A_{m}^{(i)}(n) \frac{x^{i} y^{m}}{i! m!} = \frac{n! e^{x+y}}{(e^{x}+e^{y}-e^{x+y})^{n+1}}.
\] (15)

By using this formula, we can also give an expression for \(D_{k}(m, n)\) in terms of poly-Bernoulli polynomials:

\[
D_{k}(m, n) = \frac{(-1)^{k}}{k!} \left[ 1 + \sum_{i=0}^{k-1} \frac{(-1)^{i+1}}{i!} \binom{k}{i+1} \sum_{j=0}^{i} \left[ \binom{i}{j} \right] B_{n}^{(-m-j)}(i) \right].
\] (16)

## 4 Congruences for \(D_{k}(m, n)\)

It is known that the numbers of \(m \times n\) lonesum matrices (or poly-Bernoulli numbers of negative indices) have the following expression:

\[
L(m, n) = \sum_{j=0}^{\min(m,n)} (j!)^{2} \left\{ \frac{m+1}{j+1} \right\} \left\{ \frac{n+1}{j+1} \right\},
\]

where \(\left\{ \frac{m}{j} \right\}\) are the Stirling numbers of the second kind (see e.g., [1] [2]). We note that \(\left\{ \frac{m}{j} \right\} = 0\) for \(j > m \geq 1\). The following proposition says that the numbers \(D_{k}(m, n)\) also have a similar expression.

**Proposition 4.1.** For integers \(k \geq 1\) and \(m, n \geq 0\) we have

\[
D_{k}(m, n) = \frac{1}{k!} \sum_{j=0}^{\min(m,n)} \binom{j-1}{k-1} (j!)^{2} \left\{ \frac{m+1}{j+1} \right\} \left\{ \frac{n+1}{j+1} \right\}.
\] (17)

**Proof.** The generating function for \(D_{k}(m, n)\) can be transformed as

\[
\frac{e^{x+y}}{k!} \left[ \frac{1}{e^{x}+e^{y}-e^{x+y}} - 1 \right]^{k} = \frac{e^{x+y}}{k!} \left[ \frac{1}{e^{x}+e^{y}-e^{x+y}} - 1 \right]^{k} = \frac{e^{x+y}}{k!} \sum_{j=k}^{\infty} \frac{(j-1)}{(k-1)} (e^{x}-1)^{j+1} (e^{y}-1)^{j}
\]

\[
= \frac{1}{k!} \sum_{j=k}^{\infty} \frac{(j-1)}{(j+1)^{2}} \frac{d}{dx} (e^{x}-1)^{j+1} \frac{d}{dy} (e^{y}-1)^{j+1}.
\]
Because 
\[ (e^z - 1)^m = m! \sum_{n=m}^{\infty} \binom{n}{m} \frac{z^n}{n!}, \]
we have
\[
\frac{e^{x+y}}{k!} \left( \frac{1}{e^x + e^y - e^{x+y}} - 1 \right)^k
= \frac{1}{k!} \sum_{j=k}^{\infty} \binom{j-1}{k-1} (j!)^2 \sum_{i=1}^{\infty} \sum_{m=j}^{\infty} \binom{l+1}{j+1} \binom{m+1}{j+1} \frac{x^iy^m}{l!m!}.
\]
Therefore, we obtain (17).}

By using this expression, we give some congruences for \( D_k(m, n) \) modulo a prime. We first recall the following lemma in order to prove them. All of the formulas are deduced from the well-known identities
\[
\binom{m}{k} = \binom{m-1}{k-1} + k \binom{m-1}{k}, \quad \binom{m}{k} = \frac{1}{k!} \sum_{n=1}^{k} (-1)^{k-n} \binom{k}{n} n^m,
\]
and we omit their proofs.

**Lemma 4.2.** Let \( p \) be a prime.

(i) For positive integers \( m \) and \( m' \) with \( m \equiv m' \pmod{p-1} \) and \( 0 \leq i \leq p \), we have \( \binom{m}{i} \equiv \binom{m'}{i} \pmod{p} \).

(ii) \( \binom{p}{i} \equiv 0 \pmod{p} \) for \( 2 \leq i \leq p-1 \).

(iii) \( \binom{m}{2} = 2^{m-1} - 1 \) for \( m \geq 1 \).

**Theorem 4.3.** Let \( k, m, m', n, \) and \( n' \) be positive integers. For any prime \( p \), the following congruences hold:

(i) If \( k \geq p \), then \( D_k(m, n) \equiv 0 \pmod{p} \). (18)

(ii) If \( m \equiv m' \) and \( n \equiv n' \pmod{p-1} \), then \( D_k(m, n) \equiv D_k(m', n') \pmod{p} \). (19)
(iii) If \( p > k \), then
\[
D_k(p - 1, n) \equiv \begin{cases} 
0 & (n \not\equiv 0 \pmod{p - 1}) \\
\frac{(-1)^{k-1}}{(k-1)!} & (n \equiv 0 \pmod{p - 1}) 
\end{cases} \pmod{p}. \quad (20)
\]

(iv)
\[
D_k(p, n) \equiv \begin{cases} 
2^{n - 1} & (k = 1) \\
0 & (k \geq 2) 
\end{cases} \pmod{p}. \quad (21)
\]

**Proof.** (i) If \( k \geq p \), then \( (j!)^2/k! \equiv 0 \pmod{p} \) in (17), and this proves that \( D_k(m, n) \equiv 0 \pmod{p} \).

(ii) By (i), when \( k \geq p \) both sides of (19) vanish modulo \( p \) and the congruence holds. We may assume that \( p > k \). By the symmetric property \( D_k(m, n) = D_k(n, m) \), we only have to show that \( D_k(m + p - 1, n) \equiv D_k(m, n) \pmod{p} \). By Proposition 4.1, we have
\[
D_k(m + p - 1, n) = \sum_{j=k}^{\min(m+p-1,n)} \frac{(j - 1)}{k - 1} \left( \frac{j!}{k!} \right)^2 \left\{ m + p \right\} \left\{ n + 1 \right\} \left\{ j + 1 \right\} (\mod p). \quad (22)
\]
The terms for \( j \geq m + 1 \) in (22) vanish modulo \( p \). In fact, if \( m + 1 \leq j \leq p - 1 \) then \( \left\{ \frac{m+p}{j+1} \right\} \equiv \left\{ \frac{m+1}{j+1} \right\} \equiv 0 \pmod{p} \) by Lemma 4.2 (i), and if \( j \geq p \) then \( j! \equiv 0 \pmod{p} \). Consequently, we have
\[
D_k(m + p - 1, n) \equiv \sum_{j=k}^{\min(m,n)} \frac{(j - 1)}{k - 1} \left( \frac{j!}{k!} \right)^2 \left\{ m + 1 \right\} \left\{ n + 1 \right\} \left\{ j + 1 \right\} \pmod{p},
\]
and this is equal to \( D_k(m, n) \).

(iii) By (ii), we only have to consider the cases with \( 1 \leq n \leq p - 1 \). By Proposition 4.1, we have
\[
D_k(p - 1, n) = \sum_{j=k}^{\min(p-1,n)} \frac{(j - 1)}{k - 1} \left( \frac{j!}{k!} \right)^2 \left\{ p \right\} \left\{ n + 1 \right\} \left\{ j + 1 \right\}. \quad (23)
\]
If \( n \leq p - 2 \), then \( \left\{ \frac{p}{j+1} \right\} \equiv 0 \pmod{p} \) by Lemma 4.2 (ii), and \( D_k(p - 1, n) \equiv 0 \pmod{p} \). If \( n = p - 1 \), then only the term for \( j = p - 1 \) in (23) remains, and
\[
D_k(p - 1, n) \equiv \frac{(p - 2)}{(k - 1)} \left( \begin{pmatrix} p - 1 \\ k - 1 \end{pmatrix} \right)^2 \equiv \frac{(-1)^{k-1}}{(k-1)!} \pmod{p}.
\]
The final equivalence is derived from the congruence \((p-2)\binom{k-1}{k-1} \equiv (-1)^{k-1} k\) and Wilson’s theorem, which states that \((p-1)! \equiv -1 \pmod{p}\).

\((iv)\) By Proposition 4.1, we have
\[
D_k(p, n) = \sum_{j=k}^{\min(p,n)} (j-1)\binom{j-1}{k-1} \binom{p+1}{j+1} \binom{n+1}{j+1}. \tag{24}
\]

When \(j = p\), we have \((j!)^2/k! \equiv 0 \pmod{p}\). When \(2 \leq j \leq p-1\), we have \(p+1\binom{p}{j+1} \equiv 0 \pmod{p}\) because of Lemma 4.2 (ii). Therefore, the congruence \(D_k(p, n) \equiv 0 \pmod{p}\) holds for \(k \geq 2\).

If \(k = 1\), then the term for \(j = 1\) in (24) remains, and \(D_k(p, n) \equiv \binom{p+1}{2} \binom{n+1}{2} = (2^p - 1)(2^n - 1) \equiv 2^n - 1 \pmod{p}\) by Lemma 4.2 (iii) and Fermat’s little theorem.

\[\square\]

| \(m\) | \(n\) | 0 | 1 | 2 | 3 | 4 | 5 |
|-----|-----|---|---|---|---|---|---|
| 0   | 0   | 0 | 0 | 0 | 0 | 0 |
| 1   | 0   | 1 | 3 | 7 | 15 | 31 |
| 2   | 0   | 3 | 13 | 45 | 145 | 453 |
| 3   | 0   | 7 | 45 | 229 | 1065 | 4717 |
| 4   | 0   | 15 | 145 | 1065 | 6901 | 41505 |
| 5   | 0   | 31 | 453 | 4717 | 41505 | 329461 |

Table 1: \(D_1(m, n)\)

| \(m\) | \(n\) | 0 | 1 | 2 | 3 | 4 | 5 |
|-----|-----|---|---|---|---|---|---|
| 0   | 0   | 0 | 0 | 0 | 0 | 0 |
| 1   | 0   | 0 | 0 | 0 | 0 | 0 |
| 2   | 0   | 0 | 2 | 12 | 50 | 180 |
| 3   | 0   | 0 | 12 | 108 | 660 | 3420 |
| 4   | 0   | 0 | 50 | 660 | 5714 | 40860 |
| 5   | 0   | 0 | 180 | 3420 | 40860 | 391500 |

Table 2: \(D_2(m, n)\)

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| m \ n | 0   | 1   | 2   | 3   | 4   | 5   |
|-------|-----|-----|-----|-----|-----|-----|
| 0     | 1   | 1   | 1   | 1   | 1   | 1   |
| 1     | 1   | 2   | 4   | 8   | 16  | 32  |
| 2     | 1   | 4   | 16  | 58  | 196 | 634 |
| 3     | 1   | 8   | 58  | 344 | 1786| 8528|
| 4     | 1   | 16  | 196 | 1786| 13528| 90946|
| 5     | 1   | 32  | 634 | 8528| 90446| 833432|

Table 3: \( D(m, n) \)

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