Semilocal nontopological vortices in a Chern-Simons theory

by

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Abstract

We show the existence of self-dual semilocal nontopological vortices in a $\Phi^2$ Chern-Simons (C-S) theory. The model of scalar and gauge fields with a $SU(2)_{\text{global}} \times U(1)_{\text{local}}$ symmetry includes both the C-S term and an anomalous magnetic contribution. It is demonstrated here, that the vortices are stable or unstable according to whether the vector topological mass $\kappa$ is less than or greater than the mass $m$ of the scalar field. At the boundary, $\kappa = m$, there is a two-parameter family of solutions all saturating the self-dual limit. The vortex solutions continuously interpolates between a ring shaped structure and a flux tube configuration.

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1 Introduction

Topological defects are known to exist in theories with spontaneously broken symmetries. For example, Nielsen and Olesen discovered that Abrikosov type vortices (A-N-O) appear as classical solutions of an abelian Higgs model [1]. These vortices carry magnetic flux but are electrically neutral. Furthermore, for a $\Phi^4$ Higgs potential, and when the parameters are chosen to make the vector and scalar masses equal, minimum energy vortex configurations arise that satisfy first order differential equations [2, 3]. In this limit, known as the Bogomol’nyi limit, the vortices become non-interacting [4].

In recent years the charged vortex solution [5] of the Abelian Higgs model in (2+1) dimensions with a Chern-Simons term has attracted a lot of attention in the literature, because they can be considered as candidates for anyonlike objects [6]. These vortices were shown to exist even in the absence of gauge field kinetic term (Maxwell term) [24]. This theory, where the kinetic action for the gauge field is solely the Chern-Simons term is known as the pure C-S theory (P-C-S) [8, 9]. It was recently shown that the P-C-S theory with a special choice of the scalar potential $V(\Phi)$ supports self-dual topological and nontopological vortices [10, 11]. For this theory, $V(\Phi)$ is a sixth-order potential, and the corresponding Bogomol’nyi limit is obtained when the vector and scalar masses are equal.

More recently an Abelian Chern-Simons model which includes both the C-S term and an anomalous magnetic contribution, in addition to the Maxwell term, has been studied [12]. It was shown that for a special relation between the C-S mass and the anomalous magnetic coupling, the equations for the gauge fields reduce from second- to first-order differential equations, similar to those of the pure C-S theory. Furthermore, it was demonstrated that nontopological charged vortices satisfy a set of Bogomol’nyi-type equations for a quadratic potential $V(\Phi) = (m^2/2)\Phi^2$, when $m$ and the topological masses are equal. This model possess a local $U(1)$ symmetry, so we will refer to it as the local $\Phi^2$ model.

On the other hand, new interest in the study of string-like defects has arisen after the observation made by Vachasparti and Achúcarro [13] that the Nielsen-Olesen
vortex solution can be embedded into a larger theory which has a global $SU(N)$ symmetry in addition to the local $U(1)$ symmetry, these objects are known as semilocal vortices. Semilocal vortices also appear in Chern-Simons theories; Khare [14] obtained semilocal vortex solutions when the local $\Phi^6$ Abelian Higgs model of Refs. [10, 11] is extended to a semilocal one.

In this paper we consider a $SU(2)_g \otimes U(1)_l$ model, where only the overall $U(1)$ phase is gauged and $SU(2)$ is a global symmetry. We will consider a simple $\Phi^2$ scalar potential, so we are necessarily in the symmetric phase of the theory; nevertheless we shall find static minimum energy nonotopological vortex configurations. We will be interested in the stability of these semilocal vortices. We will be able to write a lower bound for the energy (Bogomol’nyi bound), when the scalar ($m$) and the topological ($\kappa$) masses are equal. The lower bound is saturated when the fields satisfy a set of first order differential equations (self-dual or Bogomol’nyi equations). The self-dual vortices are neutrally stable, but it will be shown that the vortex solutions become stable or unstable according to whether $\kappa$ is less than or greater than $m$. The semilocal $\Phi^2$ model exhibits a richer spectrum of solutions as compared with the local model. In particular the vortices can assume both a ring shaped structure (typical of the local P-C-S models) as well as a flux tube structure (typical of the A-N-O vortices).

We should also point out that the stability condition $m > \kappa$ is exactly opposite to the one obtained for the semilocal Nielsen-Olesen vortices [22].

The paper is organized as follows. In Sec. 2 we present the semilocal model and discuss their general properties. In Sec. 3 we derive the self-duality (Bogomol’nyi) equations. The derivation of this self-duality equations is much more involved than the original Bogomol’nyi derivation for the Nielsen-Olesen model, therefore we present it with some detail. Sec. 4 is devoted to the study of the cylindrically symmetric vortex solutions. First, it is shown that the local $\Phi^2$ vortex solution can be embedded into the larger semilocal string; then a generalization of the ansatz is presented, this generalized solution represents a two-parameter family of solutions all saturating the self-dual limit. Section 5 deals with stability analysis of the solutions. Sec. 6 presents a numerical study of the self-dual solitons. Concluding remarks comprise the final section.
2 Model

The model consists of a complex doublet

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix},$$

(2.1)

with only the overall phase gauged. The semilocal Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\kappa}{4} \epsilon^{\mu\nu\rho} A_\mu F_{\nu\rho} + \frac{1}{2} |D_\mu \Phi|^2 - \frac{m^2}{2} |\Phi|^2,$$

(2.2)

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, we use natural units $\hbar = c = 1$ and the Minkowski-space metric is $g_{\mu\nu} = \text{diag}(+1, -1, -1)$; $\mu = (0, 1, 2)$. The covariant derivative $D_\mu$ includes both the usual minimal coupling plus the anomalous magnetic contribution:

$$D_\mu \Phi = (\partial_\mu - ie A_\mu - ig g^{\nu\alpha} F_{\nu\alpha}) \Phi,$$

(2.3)

with $g$ the anomalous magnetic moment $[13]$. Notice that the Lagrangian Eq. (2.2) has a global $SU(2)$ symmetry and a local $U(1)$ symmetry. We should also remark that it is an specific feature of a $(2 + 1)$ dimensional world, that a Pauli-type coupling (i.e., a magnetic coupling) can be incorporated into the covariant derivative, even for spinless particles $[10, 12]$. In fact, it was demonstrated in Ref. $[17]$, that radiative corrections can induce a magnetic coupling for anyons, that is proportional to the fractional spin.

The equations of motion for the lagrangian in Eq. (2.2) are

$$D_\mu D^\mu \Phi = -m^2 \Phi,$$

(2.4)

$$\epsilon_{\mu\nu\alpha} \partial^\alpha [F^\alpha + \frac{g}{2e} J^\alpha] = J_\nu - \kappa F_\nu.$$

(2.5)

The last equation has been written in terms of the dual field, $F_\mu \equiv \frac{1}{2} \epsilon_{\mu\nu\rho} F^{\nu\rho}$, and the conserved matter current $J_\mu = (\rho, \vec{J})$ is given by

$$J_\mu = -\frac{ie}{2} \left[ \Phi^* D_\mu \Phi - \Phi (D_\mu \Phi)^* \right].$$

(2.6)
The energy momentum tensor is obtained by varying the curved-space form of the action with respect to the metric

\[ T_{\mu\nu} = \left(1 - \frac{g^2}{4} |\phi|^2 \right) \left(F_\mu F_\nu - \frac{1}{2} g_{\mu\nu} F_\alpha F^\alpha \right) \]

\[ + \frac{1}{2} \left( \nabla_\mu \Phi (\nabla_\nu \Phi)^* - g_{\mu\nu} \left[ \frac{1}{2} |\nabla_\lambda \Phi|^2 - \frac{m^2}{2} |\Phi|^2 \right] + H.c. \right), \tag{2.7} \]

where \( \nabla_\mu = \partial_\mu - \frac{ie}{g} A_\mu \) includes only the gauge potential contribution. Notice, that both the Chern-Simons and linear terms in \( g \) do not appear explicitly in \( T_{\mu\nu} \). This is a consequence of the fact that these terms do not make use of the space-time metric tensor \( g_{\mu\nu} \); consequently, when \( g_{\mu\nu} \) is varied to produce \( T_{\mu\nu} \) no contributions arise from these terms [9].

For the selected scalar potential the theory is always in a symmetric phase. In this case the theory possess two propagating modes in the trivial sector (excitations around the vacuum): an scalar field excitation with mass \( m \) and a massive vector mode with mass \( \kappa \). Instead, if we would consider a broken symmetry phase the theory would contain three propagating modes, because the gauge field acquire two distinct masses due to the \( P \) and \( T \) violating terms [18, 19].

### 3 The self-duality equations

There is a particular relation between the C-S mass and the anomalous magnetic moment for which the Eq. (2.5) for the gauge fields reduce from second- to first-order differential equations [16, 12, 20], similar to those of the P-C-S type [8]. To get this limit notice that if the following relation holds

\[ \kappa = -\frac{2e}{g}, \tag{3.1} \]

then it is clear that the Eq. (2.5) are solved identically if we choose the first order ansatz

\[ F_\mu = \frac{1}{\kappa} J_\mu, \tag{3.2} \]
that have the same structure as the equations of the P-C-S theory \[9\]. We will refer to the previous conditions as the P-C-S limit. However, we should notice that the explicit expression for \( J_\mu \) differs from the usual expression of the P-C-S theory, because according to Eq. (2.6) and Eq. (2.3) \( J_\mu \) receives contributions from the anomalous magnetic moment. This P-C-S equations (Eq. (3.2)) imply that any object carrying magnetic flux (\( \Phi_B \)) must also carry electric charge (\( Q \)), with the two quantities related as \( Q = -\kappa \Phi_B \). In what follows we shall work in the limit in which Eq. (3.2) and Eq. (3.1) are valid, so we consider Eq. (3.2) as the equation of motion for the gauge fields, instead of the Eq. (2.5).

In the so called Bogolmol’nyi limit all the equations of motion are known to become first order differential equations \[4\]; furthermore, it is possible to write the equations of motion as self-duality equations. In reference \[12\], it was shown that for the local model, it is possible to find Bogomol’nyi-type equations for a quadratic potential \((m^2/2)|\Phi|^2\), when \( m \) and the topological masses are equal. In that case, a rotationally invariant vortex ansatz was substituted in the expression for the energy functional, then the Bogomol’nyi equations were found in terms of the reduced number of functions that appear in the ansatz. However, the self-duality equations in terms of the original fields \( \Phi \) and \( A_\mu \) without assuming the rotational invariance were no presented there. In order to discuss the vortex solutions in the extended semilocal model we require to know the form of the self-duality equations. The derivation of this self-duality equations is not as straightforward as in the original Bogomol’nyi derivation for the Nielsen-Olesen solutions, therefore we present with some detail the deduction of the self-duality equations for the present model.

In order to obtain the self-duality equations of motion let us consider the energy density \((T_{00})\) in the static limit written in terms of the magnetic \( B = \frac{1}{2}\epsilon_{ij}F^{ij} \) and electric \( E_i = F^{0i} \) fields, this yields:

\[
T_{00} = \frac{1}{2} \left(1 - \frac{e^2}{\kappa^2} |\Phi|^2 \right) \left(B^2 + |E|^2 \right) + \frac{1}{2} \left[ |\nabla_0 \Phi|^2 + (\nabla_i \Phi)^* \nabla_i \Phi \right] + \frac{1}{2} m^2 |\Phi|^2. \tag{3.3}
\]

We recall that the covariant derivative \( \nabla_\mu = \partial_\mu - ieA_\mu \) includes only the gauge potential contribution. Now, we notice that we can exploit the \( \mu = 0 \) component of
the equation Eq. (2.6) together with the C-S equations of motion Eq. (3.2) to express

\[ A_0 = \frac{\kappa}{e^2|\Phi|^2} \left[ 1 - \frac{e^2}{\kappa^2} |\Phi|^2 \right] B. \]

(3.4)

From this equation we can observe, that for a time-independent vortex solution, the \( A_0 \) component cannot be set to zero otherwise the magnetic flux would vanish. Using the above equation, the first and the third terms in Eq. (3.3) can be added together to obtain the simplifying result

\[ \frac{1}{2} \left( 1 - \frac{e^2}{\kappa^2} |\Phi|^2 \right) B^2 + \frac{1}{2} |\nabla_0 \Phi|^2 = \frac{1}{2} \frac{\kappa^2}{e^2|\Phi|^2} \left( 1 - \frac{e^2}{\kappa^2} |\Phi|^2 \right) B^2. \]

(3.5)

Likewise, from the \( \mu = i \) components of Eq. (3.2) and Eq. (2.6) we can express the electric field in terms of \( \nabla_i \Phi \):

\[ E_i = \frac{1}{\kappa \left( 1 - \frac{e^2}{\kappa^2} |\Phi|^2 \right)} \left( -\frac{ie}{2} \right) \epsilon_{ij} \left[ \Phi^* \nabla_j \Phi - \Phi (\nabla_j \Phi)^* \right]. \]

(3.6)

In order to simplify the second and fourth terms of the energy density Eq. (3.3), it is convenient to define a new covariant derivative \( \tilde{D}_i \) according to the relation

\[ \tilde{D}_i = \nabla_i + i HF_i = \partial_i - i e A_i + i HF_i, \]

(3.7)

where we have introduced the auxiliary function \( H \) defined as

\[ H = -\frac{\kappa}{e|\Phi|^2} \left[ 1 - \frac{e^2 |\Phi|^2}{\kappa^2} - \sqrt{1 - \frac{e^2 |\Phi|^2}{\kappa^2}} \right]. \]

(3.8)

Using the previous results (Eq. (3.6) and Eq. (3.7)), we can now find that the second and fourth terms in Eq. (3.3) add together to give the result

\[ \left( 1 - \frac{e^2}{\kappa^2} |\Phi|^2 \right) |\tilde{E}|^2 + (\nabla_i \Phi)^* \nabla_i \Phi = (\tilde{D}_i \Phi)^* \tilde{D}_i \Phi, \]

(3.9)

where summation over latin indices is from \( i = 1 \) to \( i = 2 \).

Employing the results of Eq. (3.5) and Eq. (3.9) we can write down the energy

\[ E = \int T_{00} d^2 x \]

as
\[ E = \int d^2x \left( \frac{\kappa^2}{2e^2|\Phi|^2} \left( 1 - \frac{e^2|\Phi|^2}{\kappa^2} \right) B^2 + \frac{1}{2} \left( \bar{D}_i \Phi \right)^* \bar{D}_i \Phi + \frac{1}{2} m^2|\Phi|^2 \right) . \] (3.10)

The energy written in this form is similar to the expression that appears in the Nielsen-Olesen model. Thus, starting from Eq. (3.10) we can follow the usual Bogomol’nyi-type arguments in order to obtain the self-dual limit. The energy may then be rewritten, after an integration by parts, as

\[ E = \frac{1}{2} \int d^2x \left[ \kappa^2 \left( 1 - \frac{e^2|\Phi|^2}{\kappa^2} \right) \left( B + \frac{e^2|\Phi|^2}{\left( 1 - \frac{e^2|\Phi|^2}{\kappa^2} \right)^{1/2}} \right)^2 + \left( \bar{D}_1 + i\bar{D}_2 \right) \Phi \right] \]

\[ + \int d^2x \left[ (m^2 - \kappa^2)|\Phi|^2 \right] \pm \frac{\kappa^2}{e} \oint_{r=\infty} d\vec{l} \cdot \vec{\Omega} \pm \frac{i}{2} \oint_{r=\infty} d\vec{l} \cdot \vec{\Lambda}, \] (3.11)

where the vectors \( \vec{\Omega} \) and \( \vec{\Lambda} \) are defined according to the relations:

\[ \Omega_i = \left[ 1 - \frac{e^2|\Phi|^2}{\kappa^2} \right]^{1/2} A_i, \]

\[ \Lambda_i = \frac{\kappa^2}{e|\Phi|^2} \left( 1 - \left[ 1 - \frac{e^2|\Phi|^2}{\kappa^2} \right]^{1/2} \right) \left( \Phi^* \partial_i \Phi - \Phi \partial_i \Phi^* \right). \] (3.12)

For any nontopological soliton the asymptotic conditions are such that \( \Phi \to 0 \) at spatial infinity. Thus, the line integral of \( \vec{\Lambda} \) in Eq. (3.11) vanishes, whereas the line integral of \( \vec{\Omega} \) yields the magnetic flux

\[ \oint_{r=\infty} d\vec{l} \cdot \vec{\Omega} \to \oint_{r=\infty} d\vec{l} \cdot \vec{A} \equiv \Phi_B. \] (3.13)

In what follows, we consider only those configurations that fulfill the condition \( |\Phi| \leq \kappa/e \). Consequently, from Eq. (3.11) we can conclude that the energy is bounded below; for a fixed value of the magnetic flux, the lower bound is given by \( E \geq \frac{e^2}{e} \Phi_B \) provided that the potential is chosen as a \( \frac{m^2}{2}|\Phi|^2 \) with the critical value \( m = \kappa \), i.e. when the scalar and the topological masses are equal. Therefore, in this limit we
are necessarily in the symmetric phase of the theory. From Eq. (3.11) we see that the lower bound for the energy

\[ E = \frac{\kappa^2}{e} |\Phi_B| = \frac{\kappa}{e} |Q|, \]  

(3.14)
is saturated when the following self-duality equations are satisfied:

\[ \tilde{D}_1 \Phi = \pm i \tilde{D}_2 \Phi, \]  

(3.15)

\[ B = \pm \frac{e |\Phi|^2}{\left[ 1 - \frac{e^2 |\Phi|^2}{\kappa^2} \right]^{1/2}}, \]  

(3.16)

where the upper (lower) sign corresponds to positive (negative) value of the magnetic flux. We should remark that these self-duality are valid both for the local as well that for the semilocal \(\Phi^2\) model. The second of these equations implies that the magnetic field vanish whenever \(\Phi\) does. For the local \(\Phi^2\) model [12] the finiteness energy condition forces the scalar field to vanish both at the center of the vortex and also at spatial infinity; consequently for the local model the magnetic flux of the vortices lies in a ring, rather than being concentrated at the center as in the A-N-O solution. As we shall see below, for the present semilocal model \(\Phi\) does not vanish at the origin; furthermore, the vortices can assume both a ring shape as well as a flux tube form.

With regard to the self-duality equations (Eq. (3.15) and Eq. (3.16)), we note that they are similar to those found in other models, but there are some important differences. First, we have to point out that in order to derive the self-dual limit it was essential to introduce a new covariant derivative \(\tilde{D}_\mu\) (Eq. (3.7)) that is different from the original \(D_\mu\) (Eq. (2.3)) that appears in the Lagrangian. Of particular interest is to compare the present results with those of the Nielsen-Olesen (A-N-O) and the pure Chern-Simons (P-C-S) models. (i) Similar self-duality equations arise both in the local [2] and semilocal [13, 22] A-N-O models for a potential of the form \((|\Phi|^2 - v^2)^2\) when the scalar and vector masses are equal. For the self-dual A-N-O model the Eq. (3) is similar, but with the normal covariant derivative \(D_\mu = \partial_\mu - ieA_\mu\) instead
of $\tilde{D}_\mu$; furthermore, Eq. (3.16) is replaced by one of the form $B \sim v^2 - |\Phi|^2$, and the magnetic field is maximum at the center of the vortex. (ii) Self-dual soliton solutions have also been found in a local [10, 11] and semilocal [14] versions of the P-C-S with no Maxwell term and no magnetic moment contribution for a sixth order potential of the form $|\Phi|^2(|\Phi|^2 - v^2)^2$, when the parameters are chosen to make the vector and scalar masses equal. Again the form of Eq. (3.15) holds, but with the normal covariant derivative $D_\mu = \partial_\mu - ieA_\mu$; whereas, Eq. (3.16) is now replaced by one of the form $B \sim |\Phi|^2(v^2 - |\Phi|^2)$; for this kind of vortices the magnetic flux is localized within a ring around the center of the vortex.

4 Vortex solutions

The local $\Phi^2$ vortex solution of reference [12] can be embedded in one of the components of $\Phi$. Explicitly for the semilocal model a static rotationally symmetric configuration of vorticity $n$ can be written in in terms of the following ansatz:

$$
\tilde{A}(\tilde{\rho}) = -\theta \frac{a(\rho) - n}{e\rho}, \quad A_0(\tilde{\rho}) = \frac{\kappa}{e} h(\rho),
\Phi(\tilde{\rho}) = \frac{\kappa}{e} f(\rho) \exp(-in\theta) \begin{pmatrix} 1 \\ 0 \end{pmatrix},
$$

(4.1)

where ($\rho, \theta$) are the polar coordinates. If the previous ansatz is substituted into the self-duality equations of motion (Eq. (3.15) and Eq. (3.16)) the equations reduce exactly to those of the local $\Phi^2$ vortices [12], and hence we have self-dual nontopological solitons also in the present semilocal model. That this is the case, may be seen by noting that if the lower component of $\Phi$ is set to zero the resulting model is exactly the local $\Phi^2$ model. This means, that a solution of the local $\Phi^2$ model is automatically a solution of the semilocal model. Most of the properties of the semilocal string are identical to those of the local vortex: namely they represent charged flux tubes that carry fractional spin as well as magnetization. However, there may be a difference: the stability of the semilocal vortex may be different from the stability of the local
one. Indeed, if one holds the lower component of $\Phi$ to zero and perturbs only the upper component, it is exactly the same as perturbing the local $\Phi^2$ solution that is known to be stable when $m > \kappa$ [21]. However, we can also perturb the lower component of $\Phi$, so we should check whether the semilocal vortex remains stable for $m > \kappa$.

Before proceeding to study the stability of the semilocal vortices we notice, following Hindmarsh [22], that the ansatz Eq. (4.1) can be generalized. In fact, the most general ansatz which maintains the cylindrical symmetry is

$$\vec{A}(\vec{\rho}) = -\hat{\theta} \frac{a(\rho) - \frac{n}{e}}{e \rho}, \quad A_0(\vec{\rho}) = \frac{\kappa}{e} h(\rho),$$

$$\Phi(\vec{\rho}) = \frac{\kappa}{e} \left( \frac{f(\rho) \exp(-in\theta)}{F(\rho) \exp(-im\theta)} \right).$$

(4.2)

The same as in reference [22] it is sufficient to examine only the case $m = 0$; this is equivalent to add a cylindrically symmetrical perturbation to the lower component of the ansatz Eq. (4.1). With this ansatz the self-duality equations of motion Eq. (3.15) and Eq. (3.16) become

$$\frac{a'}{r} = \pm \frac{f^2 + F^2}{h},$$

$$f' = \pm \frac{af}{rh} \pm \frac{nF^2}{r(f^2 + F^2)} \left( \frac{1}{h} - \frac{1}{h^2} \right),$$

$$F' = \pm \frac{(a - n)F}{rh} \pm \frac{nf^2F}{r(f^2 + F^2)} \left( \frac{1}{h} - \frac{1}{h^2} \right),$$

(4.3)

where we have introduced the dimensionless variable $r = \kappa \rho$, primes denote differentiation with respect to $r$ and we make use of Eq. (3.14) and Eq. (3.16) to solve for the $\mu = 0$ component of the gauge field that yields

$$h = \left( 1 - f^2 - F^2 \right)^{\frac{1}{2}}.$$ 

(4.4)
The boundary conditions are selected in such a way that the solution Eq. (4.2) is non-singular at the origin and gives rise to a finite energy solution; then, the problem is to solve Eq. (4.3) subject to the following boundary conditions:

\[
\begin{align*}
    a &= n, \quad f = 0, \quad F' = 0, \quad \text{at} \quad r = 0, \\
    a &\to -\alpha, \quad f \to 0, \quad F \to 0, \quad \text{as} \quad r \to \infty.
\end{align*}
\]  

(4.5)

Notice that at spatial infinity the value \(a(\infty) = -\alpha\) is not constrained. Consequently, we shall see that for nontopological solitons the magnetic flux is not quantized, but rather it is a continuous parameter describing the solution.

The system of differential equations Eq. (4.3) looks rather involved, however the system can be simplified considerably if we notice that the equations for \(f\) and \(F\) are not independent. In fact it can be easily demonstrated, that subject to the boundary conditions Eq. (4.5), \(f\) and \(F\) are related according to the relation

\[
    F = \pm \left( \frac{r_0}{r} \right)^{\pm n} f, 
\]

(4.6)

where \(r_0\) is an arbitrary parameter. Exploiting this result, the three differential equations (Eq. (4.3)) reduce to two; it is convenient to write them in terms of the functions \(a(r)\) and \(h(r)\), this brings the equations to the form

\[
\begin{align*}
    a' &= r \frac{(h^2 - 1)}{h}, \\
    h' &= \frac{(h^2 - 1)}{r h^2} \left[ a - \frac{n}{1 + \left( \frac{r}{r_0} \right)^{2n}} \right].
\end{align*}
\]

(4.7)

In what follows, we select the signs (upper signs in all the previous equations) corresponding to positive magnetic flux \((n > 0)\). The equations for \(n < 0\) are obtained with the replacement \(a \to -a, f \to f, F \to F\) and \(h \to -h\). This previous system of equations is more amenable to be treated numerically and various properties of the solutions can be discerned by general considerations. In particular the large distance behavior of the solutions yields

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\[ a = -\alpha + \frac{C}{(\alpha - 1) r^{2\alpha - 2}} + \mathcal{O}(r^{-4\alpha + 4}), \]

\[ h = 1 - \frac{C}{r^{2\alpha}} + \mathcal{O}(r^{-4\alpha}), \quad (4.8) \]

where \( C \) is a constant. From these asymptotic expressions we see that the magnetic field, \( B = a' / r \), falls off like \( r^{-2\alpha} \). Likewise the electric field \( |\vec{E}| \propto h' \) falls off like \( r^{-(2\alpha + 1)} \). We should remark that although the theory includes gauge fields with mass \( \kappa \), the magnetic field departs from the usual \( e^{-\kappa r} \) asymptotic behavior and becomes a power law. From the asymptotic expression for \( a(r) \) we notice that there should be lower bound on the values of \( \alpha \). Indeed, \( \alpha > 1 \) so the second term in the first of Eq. (4.8) is subleading compared with the first. For small \( r \) a power-series solution gives

\[ a = n + \frac{(h_0^2 - 1)}{2h_0} r^2 + \mathcal{O}(r^4), \]

\[ h = h_0 + \frac{(h_0^2 - 1)}{2h_0^2} \left[ \frac{(h_0^2 - 1)}{2h_0} + \frac{1}{r_0^2 \delta_{n,1}} \right] r^2 + \mathcal{O}(r^4). \quad (4.9) \]

The constant \( h_0 \) is not determined by the behavior of the fields near the origin, but it is instead a parameter of the vortex solutions. The value of \( h_0 \) may be restricted by requiring the proper behavior (Eq. (4.5)) as \( r \to \infty \).

Once that the boundary conditions are known, the topological numbers of the soliton can be explicitly computed. With the ansatz Eq. (4.2) the magnetic field is \( B = a' / r \), and using the boundary conditions Eq. (4.5) the magnetic flux yields

\[ \Phi_B = \int d^2 x B = \frac{2\pi}{e} (a(0) - a(\infty)) = \frac{2\pi}{e} (n + \alpha), \quad (4.10) \]

notice that for nontopological solitons the magnetic flux is not quantized. The solutions are also characterized by the charge \( Q \), spin \( S \) (which is general fractional) and magnetic moment \( M \):
\[ Q = -\kappa \Phi_B = \frac{2\pi \kappa}{e} [n + \alpha], \quad S = \frac{\pi \kappa}{e^2} (\alpha^2 - n^2), \]

\[ M = -\frac{\pi}{e} \int_0^\infty r^2 \frac{dh}{dr} dr. \]  (4.11)

Notice that the magnetic flux, the charge and the spin can be calculated explicitly, because they depend only on the boundary conditions. Whereas, the magnetic moment depends on the structure factor of the vortex configuration.

5 Vortex stability

In the self-dual limit the soliton energy is proportional to its charge (Eq. (3.14)). This result is somehow similar the one obtained for the Q-balls [23]; although in our model the soliton solutions are time-independent, whereas the Q-balls are necessarily time-dependent. For nontopological solutions the soliton charge is of the same type (Noether-charge) as that carried by the elementary excitations of the theory. Consequently, we should check whether the soliton is stable against emission of elementary particles. As we shall see at the self-dual point \((m = \kappa)\) the vortices are neutrally stable. For other values of the parameters the semilocal \(\Phi^2\) vortices are stable when \(m > \kappa\) and are unstable otherwise. Notice that this situation is reversed as compared to the results obtained for the semilocal Nielsen-Olesen (A-N-O) vortices. Indeed, Hindmarsh [22] demonstrated that the A-N-O vortices are stable when the mass of the scalar particle \((m)\) is smaller that the mass of the vector field \((m_v)\). We shall explain the reason of this difference at the end of the section.

As stated above, the self-dual vortices are neutrally stable. This fact can be easily demonstrated on account of the relation between the energy and the charge (Eq. (3.14)): \(E = \kappa|Q|/e\). First, we recall that the mass of the elementary excitations of the theory (scalar particles) is \(m\) and the charge \(e\). Because of the charge conservation a decaying soliton should radiate \(Q/e\) “quanta” of the scalar particles. Therefore, the energy of the elementary excitations is \(E = mQ/e\). This indicates that the vortices are at the threshold of stability against decay to the elementary
excitations, due to the fact that the ratio \( \frac{E_n}{\mathcal{E}} = \frac{\kappa}{m} \) is equal to one at the critical point \( m = \kappa \).

In order to discuss the soliton stability we need to discuss the existence of two sums rules that can be derived from Eq. (4.7). To obtain these sum rules we follow the reasoning introduced by Khare [24]. The first sum rule can be established by multiplying the first of Eq. (4.7) by \( r \), integrating over \( dr \) and using the asymptotic values of \( a \), this yields

\[
\alpha + n = \int_0^\infty \frac{r(1-h^2)}{h}dr.
\]  

(5.1)

For the second sum rule we multiply the first Eq. (4.7) by \( a \), after integrating and using the second of Eq. (4.7) we obtain

\[
\alpha^2 - n^2 = 2n \int_0^\infty \frac{1}{1 + \left( \frac{r}{r_0} \right)^2} \frac{da}{dr} dr + 2 \int_0^\infty (1-h^2) r dr.
\]  

(5.2)

From these two sum rules we an derive an upper limit for \( \alpha \). On account of Eq. (4.4) we see that \( 0 < h < 1 \); thus we can combine Eq. (5.1) and Eq. (5.2) to obtain

\[
\alpha + n = \int_0^\infty \frac{r(1-h^2)}{h}dr > \int_0^\infty r \left( 1 - h^2 \right) dr,
\]

\[
= \frac{1}{2} \left( \alpha^2 - n^2 \right) - n \int_0^\infty \frac{1}{1 + \left( \frac{r}{r_0} \right)^2} \frac{da}{dr} dr.
\]  

(5.3)

From Eq. (4.7) we notice that for the positive flux solution \( da/dr \) is negative; then, the previous equation yields the upper limit \( \alpha < n + 2 \). As mentioned in the previous section, \( \alpha \) is also bounded below: \( \alpha > 1 \). Consequently, we find the bounds \( 1 < \alpha < n + 2 \). This result should be compared with the bounds that were found for the local \( \Phi^2 \) model: \( n < \alpha < n + 2 \) [21]. In the limit \( r_0 \to 0 \) Eq. (5.2) implies the lower limit \( n < \alpha \), that is the same bound that was found found for the local vortex [21].

We now take up the discussion of the the vortex stability for solution away from the self-dual point \((m \neq \kappa)\). We shall follow [25] and [26] and consider the effect of perturbing the potential around the self-dual point. We write the potential as
\[ V(|\Phi|) = \frac{(\kappa + \epsilon)^2}{2} |\Phi|^2, \]  

that corresponds to a scalar mass \( m = \kappa + \epsilon \), with \( |\epsilon| \ll 1 \). The functions \( h, f, F \) and \( a \) will be modified from their self-dual configurations. We write them as:

\[
\begin{align*}
  a(r) &= a_{sd}(r) + \epsilon a_1(r) + \mathcal{O}(\epsilon^2), \\
  f(r) &= f_{sd}(r) + \epsilon f_1(r) + \mathcal{O}(\epsilon^2), \\
  F(r) &= F_{sd}(r) + \epsilon F_1(r) + \mathcal{O}(\epsilon^2), \\
  h(r) &= h_{sd}(r) + \epsilon h_1(r) + \mathcal{O}(\epsilon^2).
\end{align*}
\]  

From now on, we use the subindex \( sd \) to denote the values of the quantities in the self-dual point. Therefore, the functions \( a_{sd}(r) \) and \( h_{sd}(r) \) are solutions of the differential equations Eq. (4.7) and they satisfy the sum rules Eq. (5.1) and Eq. (5.2). On substituting these expressions into the ansatz Eq. (4.2) the resulting expressions for the energy functional Eq. (3.11), to \( O(\epsilon) \), can be recast in the form

\[
E_n = \frac{\kappa}{\epsilon} |Q| + \epsilon \frac{2\pi \kappa}{\epsilon^2} \int_0^\infty r \left(1 - h_{sd}^2\right) dr. 
\]  

Notice that the charge (Eq. (4.11)) is modified as compared to its self-dual value, because the asymptotic value \( \alpha \) changes \( \alpha = \alpha_{sd} + \epsilon \alpha_1 \), where \( \alpha_1 \) is unknown. Nevertheless, when one consider the ratio of the soliton energy to the energy of the elementary excitations \( (E = mQ/e) \) we obtain that to order \( \epsilon \), \( \alpha_1 \) does not contribute:

\[
\frac{E_n}{E} = 1 - \epsilon \frac{2\pi}{eQ_{sd}} \left[(n + \alpha_{sd}) - \int_0^\infty r \left(1 - h_{sd}^2\right) dr\right].
\]  

We notice that on account of the sum rule Eq. (5.1) the quantity inside the square brackets is positive (see Eq. (5.3)). Thus, we conclude that when \( \epsilon > 0 \) \((m > \kappa)\) the soliton solution is stable against dissociation into free scalar particles. Whereas, the soliton is unstable for \( \epsilon < 0 \) \((m < \kappa)\).

As stated before, the stability condition is exactly opposite to that obtained for the semilocal A-N-O vortices [22]. The reason is traced to the fact that, although the
semilocal A-N-O vortices are not strictly topological, they inherit some topological properties of the local model; e.g. the boundary conditions are not modified if we vary the parameters of the model and the magnetic flux is quantized. Instead, in the $\Phi^2$ model both the local and semilocal vortices are of nontopological nature. Using a variational procedure Hindmarsh [22] found that the A-N-O vortex energy increases above the Bogomol’nyi limit when $m > m_v$; but stability requires that the Bogomol’nyi limit must be saturated a condition that contradicts the possibility of having finite value for the magnetic flux. In the case of the semilocal $\Phi^2$ vortices Eq. (5.6) seems to suggest that the soliton energy also increases above the Bogomol’nyi limit when $m > \kappa$; but this is not necessarily true, because now the charge is not quantized and therefore its value is also modified. Furthermore, for nontopological solitons the relevant quantity is the ratio $E_n/\mathcal{E}$ of the vortex energy to the energy of the elementary excitations. For values of the parameters such that $m > \kappa$ the mass of the elementary excitation increases and the net effect is that the ratio $E_n/\mathcal{E}$ decreases (Eq. (5.7)), so it renders the soliton stable.

6 Numerical solutions

To complete the analysis of the previous sections we present the numerical solutions of the self-duality equations (Eq. (4.7)) with the boundary conditions given in Eq. (4.5). The solutions depend on two parameters $r_0$ and $h_0$ (see Eq. (4.6) and Eq. (4.9)) that define the regular solutions. Regular solutions exists in certain region of the parameter space ($h_0$ vs $r_0$); first we recall that $0 < h_0 \leq 1$. Furthermore, the values of $r_0$ and $h_0$ should be selected in such a way that the boundary conditions Eq. (1.5) at $r = \infty$ can be met. In this regard our results may be summarized in Fig. 1, where we show that there are three regions in parameter space. In region (III) there are no acceptable solutions consistent with the boundary conditions at infinity (Eq. (4.8)). Both in regions (I) and (II) there are acceptable vortex solutions, but the properties of these solutions vary from one region to the other. In region (I) the vortices have a flux tube structure with the magnetic field peaked at the origin; whereas in region (II) the vortices have a ring structure, in this case the maximum of the magnetic field
is not at the center of the vortex. We explain these results below, in what follows we consider the case of isolated vortex i.e. $n = 1$.

The origin of the boundary between regions (I) and (II) can be easily understood from the expression for the asymptotic behavior of the fields at small $r$ Eq. (4.3). The field $h(r)$ starts at the value $h_0$ at the origin and should always approach the value $h \to 1$ at spatial infinity. According to Eq. (4.9) the slope of $h(r)$ at the origin can either positive or negative. Looking at Eq. (4.9) it is convenient to define the critical value $h_c$ as

$$
 h_c = \frac{\sqrt{1 + r_0^4} - 1}{r_0^2}.
$$

(6.1)

The boundary between regions (I) and (II) corresponds to the equation $h_0 = h_c(r_0)$. In region (I) we have $h_0 < h_c$, the slope of $h(r)$ at the origin is positive and the function increases monotonically from $h_o$ at $r = 0$ to the value $h = 1$ at spatial infinity. Instead, in region (II) $h_0 > h_c$ the function $h(r)$ decreases from $h_0$ until it reaches a minimum value $h_{\text{min}}$, and from this point it increases to the value $h(\infty) \to 1$. It is clear that this kind of solution will be acceptable if $h_{\text{min}} > 0$, otherwise the equations become singular, see Eq. (4.7). The boundary between regions (II) and (III) represents the points at which $h_{\text{min}} = 0$; the values at this boundary are found numerically. In Fig. 2 we plot $h(r)$ for a selected $r_0 = 1$ and several values of the parameter $h_0$. For the selected value of $r_0 = 1$ and looking at Fig. 1, we expect to have acceptable solutions for any value of $h_0$; however, the behavior of the solutions should change as $h_0$ cross the point $h_c = 0.4142$. For $h_0 < h_c$ the solution increases monotonically, whereas for $h_0 > h_c$ the solution develops a minimum away from the origin. These properties of the solutions are verified in the plots presented in Figs. 1 and 2. The corresponding solutions for the function $a(r)$ are presented in Fig. 3. As expected from Eq. (4.7) the function $a(r)$ decreases monotonically from $a(0) = 1$ to the value $a = -\alpha$ at $r = \infty$.

In Fig. 4 we show profiles for the magnetic field $B = a'/r$ as a function of $r$ for several values of $h_0$ and $r_0 = 1$. It is useful to notice that using Eq. (3.16), Eq. (4.2) and Eq. (4.4) the magnetic field can be written in terms of the field $h(r)$ as
\[ B(r) = \frac{\kappa^2}{e} \frac{(1 - h^2(r))}{h(r)}. \] (6.2)

Thus, as \( h_0 \) increases the magnetic field at the origin \( (B \propto (1 - h_0)/h_0) \) decreases. Furthermore, as explained above, in region \( (I) \) we have \( h_0 < h_c \) and the field \( h(r) \) increases monotonically with \( r \); consequently the magnetic field decreases monotonically and the vortex has a flux tube structure. However as the value of \( h_0 \) increases above \( h_c \) the magnetic field smoothly diminishes at the origin and develops a maximum away from the origin. Therefore in region \( (II) \) the magnetic field of the vortex is localized within a ring around the normal core. We recall that from the Chern-Simon equation (Eq. (3.2)) the charge distribution is proportional to the magnetic field.

In Fig. 5 we show the electric field \( (E \propto h') \) for the values of the parameters mentioned above. We notice that in region \( (I) \) the electric field has a maximum at finite \( r \) (the minimum is reached at the boundaries), however for parameters in region \( (II) \) the field develops also a minimum away from the origin.

The vortex solutions are characterized by the quantum numbers that were calculated in section 4, namely magnetic flux (Eq. (4.10)), charge and spin (Eq. (4.11)). These quantities as well as the soliton energy (Eq. (3.14)) depend on the asymptotic value of the field \( a(\infty) \rightarrow -\alpha \). The value of \( \alpha \) is in general a function of \( r_0 \) and \( h_0 \). We recall from section 5 that \( \alpha \) is bounded \( 1 < \alpha < 3 \) (for \( n = 1 \)). Fig. 6 shows the behavior of \( \alpha \) as a function of \( h_0 \) for a fixed value of \( r_0 = 1 \); we find that \( \alpha \) approaches the upper limit \( \alpha \rightarrow 3 \) as \( h_0 \rightarrow 1 \). Instead, in Fig. 7 we present the plot of \( \alpha \) as a function of \( r_0 \) for \( h_0 = 0.5 \); in this case as \( r_0 \rightarrow \infty \) we notice that \( \alpha \) approaches the lower limit \( \alpha \rightarrow 1 \). For \( r_0 < 0.69 \) there is no vortex solution, in agreement with the results of Fig 1.
7 Conclusions

By extending the local $\Phi^2$ model of reference [12] to a $SU(2) \otimes U(1)$ semilocal model we found time independent charged vortex solutions. It was first necessary to derive the self-duality equations of motion (Eq. (3.15)) in terms of the original scalar ($\Phi$) and vector fields ($A_\mu$). This self-dual limit occurs for a simple $\Phi^2$ potential, when the scalar and the vector topological masses are equal. Although it is not possible to find exact solutions of the self-duality equations, many of the elementary properties of the vortices can be uncovered. The self-dual vortices are neutrally stable, but including small perturbations away from the self-dual point it was demonstrated that the vortices are stable or unstable according to whether the vector topological mass $\kappa$ is less than or greater than the mass $m$ of the scalar field. It is interesting to remark that this stability condition is opposite to the one obtained for the semilocal A-N-O vortices [22].

The properties of these vortices are indeed remarkable. Both the local [1] and semilocal [13, 22] A-N-O vortices are known to have a flux tube structure with the magnetic field peaked at the center of the vortex. For the charged vortices of the local P-C-S models the magnetic field is concentrated within a ring surrounding the center of the vortex with the magnetic field vanishing at this point [10, 11, 19]. Instead in the semilocal versions of the $\Phi^6$ P-C-S model [14] the magnetic field is nonzero at the center of the vortex. In the present semilocal model the vortex solution can have both a ring shaped structure or a flux tube structure. The solution continuously interpolates between the two configuration as we move from region (I) to region (II) in parameter space (Fig. 1).

There are several points to explore. In particular the study of the complete description of the multisoliton solutions and the behavior away from the self dual limit deserve special attention. Of particular interest would be to study the properties of Chern-Simons vortices upon quantization, because they can be considered as candidates for anyonlike objects in quasiplanar systems.
FIGURE CAPTIONS

FIG 1. The crossover lines separating the three regions in parameter space for the $n = 1$ vortex solution. The boundary between region (I) and (II) is obtained from $h_0 = h_c(r_0)$ where $h_c$ is given in Eq. (5.1). In region (III) there are no acceptable soliton solution, because $h_{min}$ becomes negative.

FIG 2. A plot of $h(r)$ as a function of $r$ for a fixed $r_0 = 1$ and values of the parameter $h_0$ of 0.25, 0.40, 0.50, 0.75, 0.95. For a vortex of vorticity $n = 1$.

FIG 3. A plot of $a(r)$ as a function of $r$ for a fixed $r_0 = 1$ and values of the parameter $h_0$ of 0.40, 0.75, 0.95. For a vortex of vorticity $n = 1$.

FIG 4. The magnetic field in units of $\kappa^2/e$ for the $n = 1$ vortex solution, a fixed $r_0 = 1$ and values of the parameter $h_0$ of 0.25, 0.40, 0.50, 0.75, 0.95.

FIG 5. The electric field in units of $\kappa/e$ for the $n = 1$ vortex solution, a fixed $r_0 = 1$ and values of the parameter $h_0$ of 0.25, 0.40, 0.50, 0.75, 0.95.

FIG 6. Behavior of $\alpha$ as function of $h_0$ for $r_0 = 1$ and $n = 1$.

FIG 7. Behavior of $\alpha$ as function of $r_0$ for $h_0 = 0.5$ and $n = 1$. 
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