Effect of stationary longitudinal structures on the growth of Tollmin-Schlichting waves

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Abstract. The paper considers the possibility of increasing of the nonstationary wave (similar to Tollmin-Schlichting (T-S) wave) growth by striped stationary structures of eigen-problem of stability of the supersonic boundary layer stability. The problem is solved within the framework of three-wave resonance interaction in local-parallel approximation. In this work the pumping wave is the stationary almost striped structures. Studies have shown that even in a stable area under the influence of longitudinal structures there is an increase of T-S waves. At sufficiently high amplitudes of stationary perturbations their effect on T-W waves is noticeable even in unstable area.

1. Introduction
It is known that the interaction of the external longitudinal vorticity with the boundary layer leads to the formation of longitudinal structures in the boundary layer. For the first time longitudinal structures were investigated in [1]. It was found that the amplitude of the boundary layer thickness, periodically changing in z-direction with the wave number β, increased linearly downstream under the influence of longitudinal external vorticity and was proportional to β. About the interaction of longitudinal or localized structures with T-S waves can be found in [2].

In supersonic flows along with longitudinal stationary structures in the boundary layer, there are stationary disturbances, excited by Mach waves which were explored in [3]. Stationary disturbances as the limit of non-stationary waves propagating down the stream were detected in [4]. There properties of three types of perturbations were investigated in detail: 1) moderately fading disturbances (similar to T-S waves), 2) fading downstream perturbations with a small longitudinal wave number, 3) strongly growing waves, propagated upstream. In the paper first of all special attention is paid to the three-dimensional perturbations of the second type, the structure of which is similar to the longitudinal structures. Their parameters, obtained in a locally parallel approximation, are close to stationary perturbations studied using parabolized stability equations.

The paper considers the amplification(deceleration) possibility of the growth of T-S waves by the striated stationary structures of the eigen-task of the supersonic boundary layer stability. Taking into account that parameters of recent disturbances are similar to parameters of striated structures excited by external turbulence, it is possible to use results on effect of eigen stationary disturbances on T-S waves in a prediction of an external turbulence influence on a laminar-turbulent transition of a boundary layer. In addition, this data can be used to control T-S waves growth, and thus the transition location.

The problem is solved within the framework of three-wave resonance interaction in a locally parallel approximation. In most of papers researches were based on Craik's theory [5] for three waves, one of
which (with the largest increment) had a frequency $\omega$, and the other two ─ $\omega/2$. However in the boundary layer, especially at supersonic speeds, may be in a quasi-resonance interaction with the main wave waves with other frequencies. In addition, low-frequency disturbances can play the role of the main wave (pumping waves). As an example, it is possible to refer on the paper [6] where the pumping wave had the frequency much less than the frequency of T-S waves. In the paper the pumping wave is the stationary almost striped structure.

2. Formulation of a task

Studies have been carried out for a boundary layer on a flat plate based on the theory of three-wave interaction. The interaction of a stationary wave with an amplitude of $V$ and two oblique non-stationary waves which are symmetric on the relation to each other with amplitudes of $G_1$ and $G_{-1}$ which parameters are proportional to $\frac{1}{2} w_{1/d} \exp \left( \frac{i}{2} \sum_{i=1}^{N} \int \alpha_i' dx \right)$, where $\theta = \beta z + \int \alpha_i' dx - \omega t$, is considered. Here $x, z$ - coordinates in the direction of the main stream and normal to it, $t$ ─ time, $\alpha'$ and $\beta$ ─ wave vector projection values $k$ on $x$ and $z$, $\omega$ ─ a frequency of a moving wave. The directions of wave vectors are shown in the figure 1.

![Figure 1. Directions of wave vectors of interacting resonant waves.](image)

In accordance with Craik's theory amplitude equations are used in a non-size form [6]:

$$
\frac{dV}{dz} = -\alpha_i^* V + S_{G_1G_{-1}} \exp(i\Delta),
$$

$$
\frac{dG_1}{dz} = -\alpha_{G_1}^* V + S_{V_{G_1G_{-1}}} \exp(i\Delta),
$$

$$
\frac{dG_{-1}}{dz} = -\alpha_{G_{-1}}^* V + S_{V_{G_{-1}G_1}} \exp(i\Delta).
$$

Here $-\alpha_j^* = -\delta \alpha_j^*$ ─ degree of linear growth (fading) of the corresponding waves, $S_{k,l}$ ─ interaction coefficients, $\frac{d}{dz} = \frac{1}{2} \frac{d}{dz}$, $\alpha_j^* = \delta \alpha_j$ ─ dimensionless linear problem eigenvalues, $\text{Re} = \text{Reynolds number constructed by the thickness of the boundary layer} \delta = \sqrt{\nu_e U_e}$, $\nu_e, u_e$ ─ Kinematic viscosity and velocity at a boundary layer edge, $x$ ─ distance to the beginning of the plate, $\Delta = \int_{l=0}^{x} \alpha_i' dx'$. Detuning. Asterisk indicates complex conjugated values. Interaction coefficients $S_{k,l}$ are determined by the ratios:

$$
S_{k,l}^i = \int_{0}^{x} \left( Z_j^* \cdot M_{k,l}^j \right) dY / \int_{0}^{x} \left( Z_j^* \cdot \partial L(Z_j) / \partial \alpha_j \right) dY,
$$

where $L$ ─ linear differential operator of stability equations and $Z_j$ ─ vector-function of solving of the equations system $L(\alpha_j)Z_j = 0$ with homogeneous boundary conditions, $Z_j^*$ ─ vector-function of solving equations of conjugate operator $L^\dagger$. As dimensionless amplitude the relation of amplitude of
velocity perturbation in Y direction on boundary layer edge to the velocity of the running flow (to speed on boundary layer edge) is taken. Dann-Lin equations [7] were used to construct the solution and find eigenvalues \( \bar{\alpha}_j \) in the paper. Detailed information on the three-wave interaction for the supersonic boundary layer can be found in the [8]. All results were obtained with Mach number \( M = 2 \).

3. Results
The text figure 2 shows the dependence of the increments of stationary (a) and non-stationary (b) waves on the Reynolds number, which show that the increments of stationary waves of the first family are negative and much smaller than the increments of the second family. Increments of the second family are small on an absolute value and therefore amplitudes corresponding to them decrease slowly on longitudinal coordinate (on a Reynolds number).

![Figure 2](image)

**Figure 2.** Increments of stationary waves (a) of the first (1) and second family (2) with \( b = \beta \cdot 10^{-3} / \text{Re} = 0.15 \) and moving waves (b) with \( b = 0.075 \) and \( F = \omega V_e / u_e^2 = (0.192, 0.384, 0.576) \cdot 10^4 \)

One of the tasks of this work was to study the possibility of amplifying traveling waves like T-S waves under the influence of stationary waves. Therefore a little of the traveling waves amplitude at the beginning of the calculation area in comparison with amplitude of a stationary wave (similar to longitudinal structure) was assumed.

![Figure 3](image)

**Figure 3.** Amplitudes of moving waves at frequency \( F = 0.384 \cdot 10^{-4} \) and \( b = 0.075 \) (1), \( b = -0.075 \) (2) in the absence of an initial phase shift at the initial amplitude of the stationary wave of the first family \( V = 1.5\% \), linear amplitude (in the absence of a stationary perturbation) (3).
From the amplitude equations it is visible that at an intensive reduction of stationary wave amplitude a fading (increase) of traveling waves with increase of longitudinal coordinate tends at the linear law with growth rate $G_\alpha$. This is well demonstrated in figure 3. It is evident that, if to normalize the dependencies on the appropriate minimums, all the lines will merge.

In the case of the stationary wave of the second family, the damping of which is weak, its effect on the amplification of the moving waves is significant, figure 4. It should be noted that the amplitudes of $G_j$ and $G_{\alpha,j}$ is different at the same frequency, that is, they depend on the sign of the wave number $\beta$. From this data it can be seen that stationary perturbations can lead to an intense of non-stationary waves significantly exceeding linear growth. Other interesting fact consists that with increase in frequency of non-stationary perturbations their growth rate of amplitudes increases. Intensity of the nonlinear process directly depends on initial values of stationary waves amplitudes. Also the growth rate of amplitudes of non-stationary disturbances dependences on ratios of their initial phases, figure 5.

In figure 6 dependences of moving waves amplitudes at $F=0.384 \cdot 10^{-4}$ on Reynolds number are shown. It is visible that with increase of the initial amplitude of a stationary wave leads to an increase of the non-stationary perturbations growth rate.
4. Summary
The results of the studies show that under the influence of stationary waves there is an active increase of non-stationary (running) waves, several times higher than the linear growth.

Growth of the moving waves amplitudes dependences on its initial phase shift, the initial amplitude of the stationary wave.

With increase of the frequency of non-stationary perturbations their growth amplitudes rate amplitudes increases.

Growth of the moving waves amplitudes dependences on its spatial orientation (positive or negative wave number $\beta$).

Intense oscillation initiated by moving waves, in turn, can destroy the original stationary wave, and lead to the beginning of a laminar-turbulent transition or the turbulent spots formation.

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