A Note on the Construction of Exponential Attractors in Hilbert and Banach Spaces

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Abstract

Exponential attractors describe the long-time behavior of dissipative dynamical systems. In this note, we dwell on the notions of exponential attractors for strongly continuous semigroups acting on complete metric spaces. We address the construction of the exponential attractors in two different functional space settings; one in Hilbert space, the other in Banach space. The former relies on the squeezing properties of solution trajectories, and the latter does not. We present these different approaches for the construction of exponential attractors with the three-dimensional Lagrangian-averaged Navier-Stokes system. Then, we compare those attractors obtained from two different methods and show their difference.

1 Introduction

In attempting to model phenomena in nature that change with time, the model equations generally come in as a system of partial differential equations, and nonlinearities occur in the modeling process. The obtained nonlinear system evolves in time, and exhibit gradual or rapid change as time proceeds. Typically, it has infinite dimensional aspects, the dimensions here being the number of parameters which is necessary to describe the configuration of the motion at a given instant in time.

If there are no restoring forces, the flow of quantities like density, concentration, or heat usually has a tendency to spread out, which is called dissipation. The dissipative effects are reflected in the infinite-dimensional nonlinear systems, and they define a forward regularizing flow in an adequate phase space containing an absorbing set. An absorbing set

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is a bounded set that attracts all bounded solutions in \textit{finite time}. The existence of such an absorbing set $B$ can be taken as a definition of dissipative partial differential equations.

Since all solution trajectories of dissipative systems eventually enter and stay in $B$, we may expect the existence of a set which would capture all the asymptotic dynamics. Such a set is called the \textit{global attractor} and it is the largest set that enjoys positively and negatively invariant properties under the flow. If a dynamical system possesses a global attractor, it is unique for the system. The global attractor, however, is not stable under perturbations of the underlying evolution equations and its rate of attraction of solution trajectories is arbitrarily slow. Those reasons led to the development of the concept of exponential attractors. In contrast to global attractors, exponential attractors are strongly stable, attract all solution trajectories at exponential rates, but not unique. Further, the attractors often have finite fractal dimensions and the asymptotic behavior of the given system can be approximated by a finite-dimensional dynamical system.

There are two methods to construct an exponential attractor for a dissipative dynamical system. One is to construct the attractors in Hilbert-space settings, and the other is to do it in Banach-space settings. The former implicitly relies on some squeezing properties of trajectories \cite{3}. No squeezing conditions are needed in the latter. We present these different approaches for the construction of exponential attractors with an exemplary dynamical system, the three-dimensional Lagrangian-averaged Navier-Stokes system.

The Lagrangian-averaged Navier-Stokes-\(\alpha\) equations were introduced as a turbulence closure model \cite{1} in 1998, \cite{6}. This work is based on theoretical results from \cite{7}, where the Lagrangian-averaged Navier-Stokes-\(\alpha\) equations are considered for fluids in a periodic box, with uniform rotation about the vertical axis $e_3 = (0, 0, 1)$ of angular frequency $f = 2\Omega$. In a rotating frame of reference, the equations (RLANS-\(\alpha\) equations) are given by

\begin{align*}
\frac{\partial V}{\partial t} + (U \cdot \nabla) V + V_j \nabla U_j + f e_3 \times U &= -\nabla \pi + \nu \Delta V + F \\
\nabla \cdot V &= 0 \\
V(t, x)_{|t=0} &= V(0, x) = V(0),
\end{align*}

where

\begin{align*}
V(t, x) &= (V_1, V_2, V_3) \quad \text{the actual velocity}, \\
U(t, x) &= (I - \alpha^2 \Delta)^{-1} V(t, x) \quad \text{the regularized velocity}, \\
\pi &= \frac{\rho}{\rho} \left( \frac{1}{2} |U|^2 - \frac{\alpha^2}{2} |\nabla U|^2 \right) \quad \text{a modified pressure}.
\end{align*}

\footnote{In the computation of turbulent fluid flows, capturing the small scale effects are on the larger ones has been a central issue, which is called a closure problem. One of favorite modeling tools to reach the closure is to use the mechanism of diffusion. Lagrangian-averaged modeling scheme uses the mechanism of nonlinear dispersion instead of diffusion so that it reaches closure without enhancing viscosity.}
Here \( x = (x_1, x_2, x_3) \), \( f = 2\Omega \) is the Coriolis parameter, \( F = (F_1, F_2, F_3) \) is a divergence free force, \( \nu > 0 \) is the kinetic viscosity, \( \rho \) is the fluid density, and \( p \) is the pressure. For simplicity we will assume the forcing term to be time independent; that is, \( F(x, t) \equiv F(x) \). The parameter \( \alpha \) is a length scale, below which wave activity is filtered, with \( 0 < \alpha \ll 1 \).

We recall the definition of exponential attractors (\[\II\]):

**Definition 1.1 (Exponential Attractor)** Let \( E \) be a complete metric space with a metric \( d \), \( X \) a compact subset of \( E \), and \( \{S(t)| t \geq 0\} \) the semigroup on \( X \) for the topology of \( E \). Assume that \( S(t) \) possesses a global attractor \( A \). A compact set \( M \) is called an exponential attractor for the semidynamical system \((S(t), X)\) if

(i) \( A \subseteq M \subseteq X \),

(ii) \( S(t)M \subseteq M \) for \( t \geq 0 \), (positively invariant under the flow),

(iii) the fractal dimension of \( M \) is finite, \( \text{dim}_F(M) < \infty \),

(iv) there exists positive constants \( c_0 \) and \( c_1 \) such that

\[
d_h(S(t)X, M) \leq c_0 e^{-c_1 t}, \quad \forall t \geq 0,
\]

where \( d_h \) is the Hausdorff semi-distance for the metric \( E \) defined by

\[
d_h(X, Y) = \sup_{x \in X} \inf_{y \in Y} d(x, y).
\]

2 Existence of an absorbing set

We denote \( P_L \) as the usual Leray projector and introduce an operator \( R_\alpha = (1 - \alpha^2 \Delta)^{-1} \), which is defined by \( R_\alpha v = (1 - \alpha^2 \Delta)^{-1} v \). We also define a bilinear operator \( B_\alpha \) on divergence free vector fields by

\[
B_\alpha(u, v) = P_L[(R_\alpha u \cdot \nabla)v + v_j \nabla(R_\alpha u)_j].
\]

Then Eq(1.1) can be rewritten in the form

\[
\frac{\partial V}{\partial t} + fP_LJP_L R_\alpha V + \nu AV + B_\alpha(V, V) = F,
\]

where \( A = -P_L \Delta \) is the Stokes operator and \( J \) is a rotation matrix given by

\[
J = \begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]
The system is considered subject to periodic boundary conditions in a lattice $Q = [0, 2\pi a_1] \times [0, 2\pi a_2] \times [0, 2\pi a_3]$ as well as stress-free boundary conditions in the vertical. The corresponding function spaces are Fourier-Sobolev spaces of periodic functions, $H^s$, $s \geq 0$, with the norm

$$
\| u \|_s^2 = \sum_{n \in \mathbb{Z}^3} |\hat{n}|^{2s} |u_n|^2,
$$

where $n = (n_1, n_2, n_3) \in \mathbb{Z}^3$ is a wave number and $\hat{n} = (\hat{n}_1, \hat{n}_2, \hat{n}_3)$ with $\hat{n}_j = n_j/a_j$ for $j = 1, 2, 3$. We set $a_1 = 1$ without loss of generality.

The existence of unique regular solutions for all $f$ greater than some threshold $f^*$ has been proved in [7], which has led to the existence of absorbing sets:

**Theorem 2.1** Let $0 \leq \alpha, \nu > 0$; let $a_1, a_2, a_3$ be arbitrary and fixed. Let $\beta > 5/2, \gamma > \beta + 4$, and $F$ a time-independent force such that

$$
\| F \|_{\beta-1}^2 \leq M_{\beta F}^2 \quad \text{and} \quad \| F \|_{\gamma-1}^2 \leq M_{\gamma F}^2.
$$

Let $V(0) \in B_{\gamma I}$ be initial data in a ball in $H^\gamma$. Let $\text{diam}(B_{\gamma I}) < 2\rho_{\gamma I}$ in $H^\gamma$ norm and $\text{diam}(B_{\gamma I}) < 2\rho_{\beta I}$ in $H^\beta$ norm. Then, for each $f \geq f^*(M_{\beta F}, M_{\gamma F}, \rho_{\gamma I}, \rho_{\beta I}, \nu, a_1, a_2, a_3)$, the 3D rotating Navier-Stokes-$\alpha$ equations possess an absorbing set $B_{\beta}$ in $H^\beta$; that is, there exists $t_{\beta} = t_{\beta}(\rho_{\beta I})$, such that $f \geq f^*$ and $V(0) \in B_{\gamma I}$ imply

$$
\| V(t) \|_{\beta} \leq \rho_{\beta} \quad \text{and} \quad \nu \int_{t}^{t+1} \| V(\tau) \|_{\beta+1}^2 \, d\tau \leq M_{\beta+1}^2,
$$

for all $t \geq t_{\beta}$. This absorbing set is uniform in $\alpha$ and $\rho_{\beta} = \rho_{\beta}(M_{\beta F}, \nu, a_1, a_2, a_3)$, $M_{\beta+1} = M_{\beta+1}(M_{\beta F}, \nu, a_1, a_2, a_3)$ (with no dependence on $M_{\gamma F}$, nor on $\rho_{\gamma I}$).

As $\beta > 5/2$, the semiflow $S_{\alpha}(t)$ is compact on $H^\beta$ and we can take $B_{\beta}$ compact in $H^\beta$, modulo a small translate in time.

**Remark 2.2** Existence of unique regular solutions of the exact rotating Navier-Stokes equations ($\alpha = 0$) was established by Babin, Mahalov and Nicolaenko in [7] and [2].

### 3 Existence of exponential attractors in Hilbert spaces

The existence of exponential attractors for the system (1.1)-(1.3) in Hilbert spaces was established by Kim and Nicolaenko ([7]). The procedure and results are reproduced in this section. We first would like to point out that exponential attractors are, unlike global attractors, stable under perturbations of the underlying evolution equations. The full 3D rotating Navier-Stokes systems, including Lagrangian-averaged Navier-Stokes-$\alpha$ equations, are considered to be an $f$-singular perturbation from $f$-singular limit equations. With
manifolds that stay stable under the perturbation, we are able to talk about convergence as $f \to \infty$. See [7] for more details on this. Now, we start out by recalling the procedure with which exponential attractors are constructed in Hilbert spaces.

Let $E$ be a Hilbert space with norm $\|\cdot\|_E$ induced by the inner product $(\cdot, \cdot)_E$. Let $X$ be a compact subset of $E$ and $S : X \to X$ a Lipschitz continuous map with Lipschitz constant $L$. Then $S$ possesses a global attractor $A$ which is a compact, connected set given by

$$A = \bigcap_{n=1}^{\infty} S^n(X)$$

(Theorem 2.4.2, [5]). Exponential attractors for a map $S$ are defined as

**Definition 3.1 (Discrete Exponential Attractor)** A compact set $M$ is called an exponential attractor for $(S, X)$ if $A \subset M \subset X$ and

(i) (positively invariant) $S(M) \subset M$.

(ii) $M$ has finite fractal dimension, $\dim_F(M) < \infty$.

(iii) There exist positive constants $c_0$ and $c_1$ such that

$$d_h(S^n X, M) \leq c_0 e^{-c_1 n}, \forall n \geq 1$$

where $d_h$ is the standard Hausdorff semi-distance between two sets.

In establishing the existence of discrete exponential attractors key techniques are those based on examining the difference of two solutions and verifying the squeezing property on the underlying mapping $S$. The idea of the squeezing property is that we can split the phase space $X$ into a finite-dimensional subspace and its infinite-dimensional orthogonal complement, such that the finite-dimensional part of the solution dominates; or if not, then at least the solutions are closer together than they were at $t = 0$, which serves to dampen the effect of such ill-behaved solutions:

**Definition 3.2** Let $E$ be a Hilbert space and $X$ a subset of $E$. A map $S$ has the squeezing property in $X$ if, for some $\delta \in (0, \frac{1}{4})$, there exists an orthogonal projection $P_{N_0} = P_{N_0}^{(\delta)}$ of finite rank $N_0 = N_0(\delta)$ such that, $\forall u, v \in X$, if $\|(I - P_{N_0})(Su - Sv)\|_E \geq \|P_{N_0}(Su - Sv)\|_E$ then $\|Su - Sv\|_E \leq \delta \|u - v\|_E$.

In general, to decrease $\delta$ we need to increase the rank of the orthogonal projection $P_{N_0}$ (that is, the dimension of $P_{N_0} X$). The squeezing property guarantees the existence of discrete exponential attractors (Ch. 2, [4]):
Theorem 3.3 If $S$ has the squeezing property in $X$, then there exists an exponential attractor $\mathcal{M}$ for $(S, X)$ and, moreover,

$$d_B(\mathcal{M}) \leq N_0 \max\{1, \log\left(\frac{2L}{\delta} + 1\right)/\log\left(\frac{1}{\theta}\right)\},$$

where $\theta \in (4\delta, 1)$ arbitrary and $d_B$ is the fractal box dimension for the metric $E$.

We now turn to the continuous case. Given the semigroup $\{S(t) | t \geq 0\}$ of solution operators, we will choose a positive $t_\ast$ small enough such that $S_\ast = S(t_\ast)$ possesses the squeezing property in $X$. If $S_\ast$ is Lipschitz continuous, then the existence of a discrete exponential attractor $M_\ast$ for $(S_\ast, X)$ is guaranteed by Theorem 3.3. Next we define

$$\mathcal{M} = \bigcup_{0 \leq t \leq t_\ast} S(t)M_\ast$$

and $G : [0, T] \times \mathcal{M} \to \mathcal{M}$ as $G(t, x) = S(t)x$. If $G$ is Lipschitz, then it can be shown that $\mathcal{M}$ is a compact set with finite fractal box dimension, and $\mathcal{M}$ will be an exponential attractor for $(S(t), X)$ (Theorem 3.1, [4]). The exponential attractors for the continuous dynamical systems generated by a semigroup $\{S(t)\}_{t \geq 0}$ are unions of exponential attractors restricted by squeezing time $t_\ast$. In addition, given an estimate for $M_\ast$, it is not difficult to get an estimate for the fractal box dimension of $\mathcal{M}$ (Theorem 3.1, [4]):

Theorem 3.4 Let $S(t_\ast)$ have the squeezing property in $X$ for some time $t_\ast > 0$ and let $\mathcal{M}$ be an exponential attractor for $(S(t), X)$ and $G(t_\ast, x) = S(t_\ast)x$ for $x \in X$, $t \geq 0$. If $G(t_\ast, \cdot)$ is Lipschitz in $X$ with Lipschitz constant $L_\ast$, then

$$d_B(\mathcal{M}) \leq d_B(M_\ast) + 1.$$ 

Furthermore,

$$d_h(S(t)X, \mathcal{M}) \leq cL_\ast \exp\left(\frac{-(\ln 8)t}{t_\ast}\right)$$

for all $t \geq 0$, where $c$ is a positive constant.

Now we follow the above procedure to establish the existence of an exponential attractor in $L^2$ for the 3D RNS-$\alpha$ equations. We do this for all $f$ that allow the existence of a global attractor. Assume that $F$ is time-independent and smooth and that $f \geq f^*$ as in Th 2.1. Let $S_\alpha(t)$ be the semiflow for solutions of the 3D RNS $\alpha$-equations and let $B_\beta, \beta > 5/2$, be the compact absorbing set obtained in Theorem 2.1. Set

$$X_{\alpha, \beta} = \bigcup_{t \geq t_\beta(B_\beta) + \frac{1}{\alpha^4}} S_\alpha(t)B_\beta |\cdot|,$$
where the closure is taken in $L^2$-topology and $\lambda_1$ denotes the first eigenvalue of the Stokes operator. Then $X_{\alpha,\beta}$ is a bounded subset of $B_\beta$, compact in $H^s, 0 \leq s < \beta$, and positively invariant under $S_\alpha(t)$ such that, for all $V_\alpha(0) \in X_{\alpha,\beta}$,

$$||S_\alpha(t)V_\alpha(0)||_{H^s} \leq \rho_{\alpha,\beta}, \quad \forall t \geq 0,$$

where $\rho_{\alpha,\beta}$ is the uniform bound obtained in Th 2.1. In particular, there exist absolute bounds $\rho_{\alpha,s} = \rho_{\alpha,s}(M_\beta F, \nu, a_1, a_2, a_3)$ such that $||S_\alpha(t)V_\alpha(0)||_{H^s} \leq \rho_{\alpha,s} \leq \rho_{\alpha,\beta}$ for $0 \leq s < \beta$.

We can deduce that the underlying semigroup $S_\alpha(t)$ is uniformly compact for large $t$ so that it possesses a unique global attractor $\mathcal{A}$ in $H^s$ for $0 \leq s < \beta$, (Theorem 1.1, [9]). Moreover, it can be proved that $\mathcal{A}$ lies in $H^\beta$ for $\beta > 5/2$.

We consider the solution operator $S_\alpha(t)$ as a map from $X_{\alpha,\beta}$ into $X_{\alpha,\beta}$. We only need to show that there exists a squeezing time $t_*$ such that the discrete operator $S_* = S_\alpha(t_*)$ has the squeezing property in $L^2$-topology. To achieve it we first examine the difference between two solutions, $V_a$ and $V_b$, of 3D RNS-$\alpha$ equations in $X_{\alpha,\beta}$. Let $W = V_a - V_b$ and $W' = \frac{V_a + V_b}{2}$. Then $W$ satisfies the equation

$$\frac{\partial W}{\partial t} + \nu AW + fM R_\alpha W = - [B_\alpha(W', W) + B_\alpha(W, W')]$$  \hspace{1cm} (3.1)

$$W(0) = V_a(0) - V_b(0).$$  \hspace{1cm} (3.2)

Taking the inner product with $2W$ yields

$$\frac{d}{dt} |W|^2 + 2\nu \|W\|^2 \leq 2 \{ |< B_\alpha(W', W), W >| + |< B_\alpha(W, W'), W >| \},$$  \hspace{1cm} (3.3)

where $B_\alpha(u, v) = (R_\alpha u \cdot \nabla)v + v_j \nabla(R_\alpha u)_j$. Estimating the right hand side of (3.3) and using Young’s inequality yield

$$\frac{d}{dt} |W|^2 + \nu \|W\|^2 \leq \frac{K_1}{\nu^3} |W|^2,$$  \hspace{1cm} (3.4)

where $K_1 = c_1^4 \rho_\nu^4$ with $c_1$ a constant. Letting $\lambda(t) = \frac{\|W(t)\|^2}{|W(0)|^2},$ (3.4) becomes

$$\frac{d}{dt} [\ln|W(t)|^2] \leq -\nu \lambda(t) + \frac{K_1}{\nu^3}$$

so that

$$|W(t)|^2 \leq \delta(t)|W(0)|^2$$  \hspace{1cm} (3.5)

with

$$\delta(t) = \exp \left( -\nu \int_0^t \lambda(s) ds + \frac{K_1}{\nu^3} t \right).$$
Next, we need to find a time $t_*$ such that the estimate for $\delta(t_*)$ allows squeezing. Thus it is essential to bound $\int_0^{t_*} \lambda(s) ds$, and following the exact line of section 6.1 in [10] we obtain

$$t_* = \frac{c_3^2 \nu^{3/2}}{c_2 K_2 K_3^3}, \tag{3.6}$$

where $K_2 = c_2 \rho V$ and $K_3^2 = \frac{27 c_4^4}{2\nu} \rho_V^6 + \frac{2}{\nu H_1} \rho_H$ with $c_2$ and $c_3$ constants. Furthermore,

$$\int_0^{t_*} \lambda(t) dt \geq c_4 \lambda N_0^{1/2} \frac{\nu^{3/2}}{K_2 K_3},$$

where $c_4 = \frac{1}{2} [1 - \exp(-c_2^3/c_2)] > 0$, so that

$$\delta(t_*) \leq \exp \left( -\frac{c_4}{c_2} \lambda N_0^{1/2} \frac{\nu^{5/2}}{K_2 K_3^3} + \frac{c_5 \rho_V^3}{\nu^{3/2} K_3} \right), \tag{3.7}$$

where $c_5 = \frac{27}{16} c_4^4 c_3^2 c_2^3$. By the definition of $K_3$ there exists a constant $\tilde{c} > 0$ such that

$$K_3 \leq \tilde{c} \left( \frac{\rho_V^3}{\nu^{3/2}} + \nu^{1/2} \lambda_1^{1/2} \rho_H \right).$$

Choosing $N_0$ such that

$$N_0 \geq \tilde{c}^{3/2} \max \left\{ \frac{1}{\lambda_1^{3/4}} \left( \frac{\rho_H \rho_V}{\nu^{3/2}} \right)^{3/2}, \frac{\rho_V^6}{\lambda_1^{3/2} \rho_6} \right\},$$

gives $\delta(t_*) < \frac{1}{5}$. Under the above condition of $N_0$, the following Lemma assures the existence of an exponential attractor $\mathcal{M}_0$ for $(S_*, X_{\alpha,\beta})$ for $f \geq f_*$ (Ch 3, [4]; Proposition 2.2.7, [10]):

**Lemma 3.5** Let $t_* > 0$ be given and $u, v \in X$. Define

$$\lambda_* = \frac{||w_*||^2}{||w||^2},$$

where $w_* = S_* u - S_* v$. Then $S_*$ possesses the squeezing property in $X$, if there exists $\delta \in (0, 1/4)$ and $N_0 = N_0(\delta) \in N$, such that $\lambda_* > \frac{1}{2} \lambda N_0^{1/2}$ implies that $|S_* u - S_* v| < \delta |u - v|$, for all $u, v \in X$.

Furthermore, the Lipschitz constant for $S_*$ on $X_{\alpha,\beta}$ is estimated as

$$L_* = \delta(t_*) \leq \exp \left( \frac{c_5 \rho_V^3}{\nu^{3/2} K_3} \right).$$
and hence

\[ d_h(S_\alpha(t)X_{\alpha,\beta}, \mathcal{M}_0^*) \leq cL_\ast \left( (\delta(t_\ast))^{1/t_\ast} \right)^t \]

\[ \leq cL_\ast \left( e^{-\ln 8} \right)^{1/t_\ast} \]

\[ = c_{\alpha F} e^{-\delta_{\alpha F} t_\ast} , \]

where \( c_{\alpha F} = cL_\ast \) and \( \delta_{\alpha F} = \frac{\ln 8}{t_\ast} \).

Now we summarize the results:

**Theorem 3.6** Let \( F \) be a smooth, time-independent force and let \( a = (a_1, a_2, a_3) \) be a domain size parameter. For \( f \geq f_\ast \) as in Th 2.1 let \( X_{\alpha,\beta} \) be the positively invariant set from (3.1). Then \( \{ S_\alpha(t) | t \geq 0 \} \) restricted to \( X_{\alpha,\beta} \) admits an exponential attractor \( \mathcal{M}_0 \) in \( L^2 \). Moreover, the rate of convergence to the exponential attractor is given by

\[ d_h(S_\alpha(t)X_{\alpha,\beta}, \mathcal{M}_0) \leq c_{\alpha F} e^{-\delta_{\alpha F} t_\ast} , \]

where \( c_{\alpha F}, \delta_{\alpha F} \) are constants, which only depend on \( \nu, a, \rho_{\alpha H}, \rho_{\alpha V} \) and are independent of the angular frequency \( f \geq f_0 \) and \( \alpha > 0 \).

**Remark 3.7** \( \mathcal{M}_0 \) is bounded in \( H^\gamma \) and attracts all orbits in the \( L^2 \)-norm topology. It is compact in the space \( H^{\gamma_0} \) if \( 0 \leq \gamma < \beta \).

### 4 Existence of exponential attractors in Banach spaces

Since the Hilbert space is also a Banach space, we may construct an exponential attractor using the method developed by Le Dung and Nicolaenko in [3], which doesn’t require the squeezing properties of trajectories. Let \( \mathcal{L}(E) \) be the space of bounded linear maps from \( E \) into itself. For a given positive real \( \lambda \) we denote by \( \mathcal{L}_\lambda(E) \) the set of maps \( L \in \mathcal{L}(E) \) such that \( L \) can be decomposed as \( L = K + C \) with \( K \) compact and \( \| C \| < \lambda \). Here \( \| C \| \) denotes the norm of the operator \( C \). The following theorems were established in [3].

**Theorem 4.1** If there exists \( \lambda \in (0, 1) \) such that \( D_xS(x) \in \mathcal{L}_\lambda(E) \) for all \( x \in X \) then the discrete dynamical system \( \{ S^n \}_{n=1}^\infty \) possesses an exponential attractor.

Once the existence of exponential attractors for the discrete case is proved the result for the continuous case follows in a standard way (see Ch. 3, [4]). Define \( S_\ast \) as the map induced by Poincaré sections of a Lipschitz continuous semiflow \( S(t), \ t \geq 0 \) at the time \( t = t_\ast \) for some \( t_\ast > 0 \); that is, \( S_\ast := S(t_\ast) \). Let \( \{ S^n_\ast \}_{n \geq 0} \) be the discrete semigroup generated by \( S_\ast \). Then
Theorem 4.2 Let $X$ be a compact absorbing set for a continuous semiflow $S(t)$. Suppose that there is $t^* > 0$ such that $S_{t^*} = S(t^*)$ satisfies the condition of Theorem 4.1. Assume further that the map $G(x, t) = S(t)x$ is Lipschitz from $[0, T] \times X$ into $X$ for any $T > 0$. Then the flow $\{S(t)\}_{t \geq 0}$ admits an exponential attractor $\mathcal{M}$ as well as a unique global attractor $A$.

Theorem 4.1 and 4.2 were already proved by Temam ([9]) and J. Hale ([5]) for the global attractor.

Theorem 4.3 Let $F$ be a smooth, time-independent force. The 3D RNS-$\alpha$ equations possess for $f > f^*$, where $f^*$ is defined in Theorem 2.1, a global compact attractor $A_\beta$ in the topology of $H^\beta$, $\beta > 5/2$, as well as exponential attractors $\mathcal{M}_\beta$ in the absorbing set $B_\beta$ established in Theorem 2.1. Both fractal dimensions and rates of exponential attraction are uniform in $\alpha$.

Proof. We place ourselves in the context of the compact absorbing ball $B_\beta$ of Theorem 2.1, which is absorbing in the $H^\beta$-topology the initial set $B_{\gamma I}$ in $H^\gamma$. We prove in the below Lemma 4.5 that the map $F(v, t) = S(t)v$ is Lipschitz from $[0, T] \times B_\beta$ into $B_\beta$ for any $T > 0$, as well as the uniform Fréchet Differentiability of $S_\alpha(t)v$ with respect to $v \in B_\beta$, $0 \leq t \leq T$. Then, all assumptions in Theorem 4.1 and 4.2 are satisfied uniformly in $\alpha$, $0 \leq \alpha \leq \alpha_M$, and the result follows.

For Lemma 4.5, we first need the simple corollary of Lemma 5.1 and 5.2, and the inequality (6.6) in [7].

Corollary 4.4 For any $V$ in $H^\beta$, $W$ in $H^{\beta+1}$, one has
\begin{align*}
(i) \quad & \|B_\alpha(V, W)\|_\beta \leq C\beta\|V\|_\beta \|W\|_{\beta+1}, \\
(ii) \quad & |\langle B_\alpha(V, W), A^2W \rangle| \leq C\beta'\|V\|_\beta\|W\|^2_\beta.
\end{align*}

Proof. (i) was obtained in [7], the inequality (6.6). (ii) can be derived along the exact lines of Lemma 5.1 and 5.2 in [7].

Lemma 4.5 The semiflow $S_\alpha(t)v$ is Lipschitz from $[0, T] \times B_\beta$ into $B_\beta$ for any $T$ fixed, $T > 0$, and it is uniformly Fréchet differentiable with respect to $v \in B_\beta$, $0 \leq t \leq T$; the above properties are uniform in $\alpha$.

Proof. We closely follow the methodology of Temam [9], Ch VI, section 8, in the context of our semiflow $S_\alpha(t)v$ in $B_\beta$, $\beta > 5/2$.

Let $V, \tilde{V} \in B_\beta$ satisfy the equations:
\begin{align*}
\frac{\partial V}{\partial t} + \nu AV + f_P L P_L R_\alpha V + B_\alpha(V, V) &= F \quad V(0) = V^0 \\
\frac{\partial \tilde{V}}{\partial t} + \nu A\tilde{V} + f_P L P_L R_\alpha \tilde{V} + B_\alpha(\tilde{V}, \tilde{V}) &= F \quad \tilde{V}(0) = \tilde{V}^0.
\end{align*}
(i) First we show a Lipschitz property of the semiflow $S_{\alpha}(t) : V(0) \to V(t)$. We set $W(t) = \tilde{V}(t) - V(t)$ and $W^0 = \tilde{V}^0 - V^0$. The difference $W$ satisfies the equation

$$\frac{\partial W}{\partial t} + \nu AW + fP_LJP_L\alpha W + B_{\alpha}(\tilde{V}, W) + B_{\alpha}(W, V) = 0,$$

(4.1)

$$W(0) = W^0 = \tilde{V}^0 - V^0.$$

Taking $H^\beta$-inner product (4.1) with $W$ and using Corollary 4.3 yields

$$\frac{1}{2} \frac{d}{dt} \|W\|^2_{\beta} + \nu \|W\|^2_{\beta+1} \leq c_1 \|\tilde{V}\|_{\beta} \|W\|^2_{\beta} + c_2 \|W\|^2_{\beta} \|V\|_{\beta+1}.$$  

(4.3)

By Gronwall’s lemma

$$\|W(t)\|^2_{\beta} \leq \|W(0)\|^2_{\beta} e^{\int_0^T 2G_{\beta}(\tau) \, d\tau}$$

(4.4)

where $G_{\beta}(t) = [c_1 \|\tilde{V}\|_{\beta} + c_2 \|V\|_{\beta+1}]$.

This shows the Lipschitz continuity of the semiflow $S_{\alpha}(t)$ with Lipschitz constant $C = \exp(\int_0^T 2G_{\beta}(\tau) \, d\tau)^{1/2}$.

(ii) Now we show that the Fréchet differentiability of the semiflow $S_{\alpha}(t)$. Consider the linearized equations of 3D RNS-$\alpha$ equations

$$\frac{\partial Z}{\partial t} + \nu AZ + fP_LJP_L Z + B_{\alpha}(V, Z) + B_{\alpha}(Z, V) = 0,$$

(4.5)

$$Z(0) = Z^0 = \tilde{V}^0 - V^0.$$  

(4.6)

Let $\varphi(t) = \tilde{V}(t) - V(t) - Z(t) = W(t) - Z(t)$. Clearly, $\varphi$ satisfies

$$\frac{\partial \varphi}{\partial t} + \nu A\varphi + fP_LJP_L\varphi + B_{\alpha}(V, \varphi) + B_{\alpha}(\varphi, V) + B_{\alpha}(W, W) = 0, \quad \varphi(0) = 0.$$  

(4.7)

This is the exact equations for higher order error $\varphi(t)$.

Take $H^\beta$-inner product (4.7) with $\varphi$ and use Corollary 4.3 to get

$$\frac{1}{2} \frac{d}{dt} \|\varphi\|^2_{\beta} + \nu \|\varphi\|^2_{\beta+1} \leq c_1 \|V\|_{\beta} \|\varphi\|^2_{\beta} + c_2 \|V\|_{\beta+1} \|\varphi\|^2_{\beta} + \|W\|_{\beta} \|\varphi\|_{\beta} \|\varphi\|_{\beta+1}$$

$$\leq \|\varphi\|^2_{\beta} G_{\beta}(t) + \frac{\nu}{2} \|\varphi\|^2_{\beta} + \frac{1}{2\nu} \|W\|^2_{\beta} \|\varphi\|^2_{\beta+1}$$

(4.8)

Note that, from (4.3) and (4.4),

$$\int_0^T \|W(\tau)\|^2_{\beta+1} \, d\tau \leq \frac{1}{\nu} \|W^0\|^2_{\beta} + \frac{1}{\nu} \|W\|^2_{\beta} \int_0^T 2G_{\beta}(\tau) \, d\tau$$

$$\leq \frac{1}{\nu} \|W^0\|^2_{\beta} \left[ 1 + e^{\int_0^T 2G_{\beta}(\tau) \, d\tau} \right] \equiv \frac{1}{\nu} \|W^0\|^2_{\beta} H(T).$$

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Substituting this into (4.8) we obtain
\[ \|\varphi\|_\beta^2 \leq \frac{1}{\nu^2} \|W^0\|_\beta^4 H(T)e^{\int_0^T 4G(\tau)\,d\tau}. \] (4.9)

This impiles
\[ \|\tilde{V}(t) - V(t) - Z(t)\|_\beta^2 \leq \frac{1}{\nu^2} \|W^0\|_\beta^4 H(T)e^{\int_0^T 4G(\tau)\,d\tau}, \] (4.10)
and this shows the Fréchet differentiability of the semiflow \( S_\alpha(t) \). ■

**Remark** The exponential attractor \( \mathcal{M}_\beta \) lies in the absorbing ball \( B_\beta \), which is absorbing the initial ball \( B_{\gamma I} \) in the topology of \( H^\beta \). In that sense, \( \mathcal{M}_\beta \) is called an “\( H^\beta - H^{\gamma} \)” exponential attractor, following the usage from damped Hyperbolic PDE's ([4]). Technically, the global attractor \( A_\beta \) is unique in \( H^\beta \) in this “\( H^\beta - H^{\gamma} \)” sense.

The question arises as to whether \( A_\beta \) is the global attractor for more general initial data in \( H^\beta \). The following shows that this is true in a sense established in the proof below.

**Corollary 4.6** The compact global attractor \( A_\beta \) attracts trajectories with initial data in \( H^\beta, \beta > 5/2 \), with fractal dimension uniform in \( \alpha \).

**Proof.** Let \( \epsilon \) be given. Then for every \( V_s(0) \) in some arbitrary \( B_{\gamma I} \subset H^\gamma, \gamma > \beta + 4, \beta > 5/2 \), there exists a time \( T \) such that:
\[ d_{h,\beta}(V_s(T), A_\beta) \leq \epsilon/2, \text{ in } H^\beta, \] (4.11)
where \( d_{h,\beta}(x, y) = \inf_{y \in Y} \|x - y\|_\beta \); this follows from \( A_\beta \) being a global compact attractor in the “\( H^\beta - H^{\gamma} \)” sense. We then take any trajectory in some initial ball \( \tilde{B}_\beta \) in \( H^\beta \) of radius \( \tilde{M}_\beta \), exactly as in Theorem 6.5 and Corollary 6.6 in [7]. For such \( V(t) \) with \( V(0) \) in \( \tilde{B}_\beta \), we carefully follow the proof of Theorem 6.5 and Corollary 6.6 in [7], with \( 0 \leq t \leq T \), \( T \) given in (4.11), and we can construct \( \eta = 1/c_0 \epsilon \) to obtain
\[ \|V_s(t) - V(t)\| \leq \frac{C_0 \eta}{2} = \frac{\epsilon}{2} \] (4.12)
on \( 0 \leq t \leq T \). This is actually “shadowing” of the \( V(t) \) trajectory by a \( V_s(t) \)-trajectory in \( H^\beta \) on \( [0, T] \). Finally,
\[ d_{h,\beta}(V(T), A_\beta) \leq d_{h,\beta}(V_s(T), A_\beta) + \|V_s(T) - V(T)\|_\beta \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \] ■

Of course, the rate of attraction is not uniform in the initial data as it is to be expected for a topological global attractor; such an attractor may attract at a arbitrary slow rate.
5 Concluding remark

We constructed exponential attractors in two different functional settings. The exponential attractor in Hilbert spaces (Theorem 3.6 in the section 3), \( M_\alpha \) lies in the absorbing ball \( B_\beta \) obtained in the section 2 (Theorem 2.1) and attracts solution trajectories in \( L^2 \)-topology. On the other hand, the exponential attractors \( M_\beta \) also lies in the same \( B_\beta \) and attracts solution trajectories in the topology of \( H^\beta \). Two questions arise:

- Do they present the same or similar asymptotic dynamics for the given system?
- On the intersection of exponential attractors. Exponential attractors are not unique. Each exponential attractor possesses unique global attractor, which is a minimal compact attracting sets. Consequently, the intersection of any two compact attracting sets is still attracting (besides being obviously compact), for it contains the global attractor. Accordingly, if \( E_1 \) and \( E_2 \) are two exponential attractors, then their intersection is certainly an attracting set. What is expected to change is the attraction rate. This leads to the question, “Is the intersection \( E_1 \cap E_2 \) still an exponential attractor? 

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