Mass and chirality inversion of a Dirac cone pair in St"uckelberg interferometry

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We show that a St"uckelberg interferometer made of two massive Dirac cones can reveal information on band eigenstates such as the chirality and mass sign of the cones. For a given spectrum with two gapped cones, we propose several low-energy Hamiltonians differing by their eigenstates properties. The corresponding inter-band transition probability is affected by such differences in its interference fringes being shifted by a new phase of geometrical origin. This phase can be a useful bulk probe for topological band structures realized with artificial crystals.

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Introduction. Topological properties of band structure are key to the modern classification of quantum phases of matter [1, 2]. In his seminal work, Haldane has shown that a pair of gapped Dirac cones realizing a trivial insulator can be turned into a Chern insulator upon reversal of the mass sign of a single cone [3]. The resulting quantum anomalous Hall effect was recently measured in a magnetic topological insulator [4]. In addition to their mass sign, Dirac cones are also characterized as quantized vortices in the relative phase of their spinor eigenstates (corresponding to ±1 winding number or chirality) [5, 6].

Recent developments in artificial solids open the field of topological band structure engineering [7–10]. Standard solid state techniques that are used to extract topological information - such as Shubnikov-de Haas oscillations, quantum Hall measurements and Landau-level spectroscopy - are typically unavailable in these systems. On the other hand, they offer the possibility of measuring new physical observables, such as that studied in cold atoms experiments [17–25]. For instance, the long coherence time typical of cold atoms permits the study of St"uckelberg interferences in an optical lattice [20–27]. In this Letter, we show that the phase in the St"uckelberg interference pattern contains information not only on the energy bands [28], but also on geometrical quantities characterizing the band eigenstates.

In order to illustrate our findings, we first consider a toy-model St"uckelberg interferometer made of a pair of one-dimensional gapped Dirac cones a distance \(d\) apart in reciprocal space (Fig. 1a). A particle initially in the lowest band moves from negative to positive momentum \(p_x\) under the influence of a constant force and encounters the double cone structure. The two avoided crossings act as beam splitters controlled by Landau-Zener (LZ) tunneling [29]. A “flux” parameter \(\beta\) allows one to tune the relative sign of the two Dirac masses, similar to Haldane’s model [3]. The latter can be realized in an optical lattice with cold atoms, see e.g. Ref. 7, 10. At \(\beta = 0\), the two masses have the same sign and fringes are clearly seen in the final transition probability as a function of the distance between the cones (Fig. 1a). A mass inversion (induced by the parameter \(\beta\) going from 0 to \(\pi/2\)), while keeping the bulk energy bands unchanged (Fig. 1b), nevertheless leads to a \(\pi\)-shift in the St"uckelberg interference fringes (Fig. 1b), compare \(\beta = 0\) and \(\pi/2\). At the transition \(\beta = \pi/4\) (Fig. 1c), one of the Dirac cones becomes gapless and the interference contrast fades.

The basic understanding of such a \(\pi\)-shift stems from the Berry phase of band eigenstates [30, 31]. As the particle is accelerated through two crossings in succession, phase information related to band eigenstates is encoded into the probability amplitude during tunneling events. Geometrical characteristics of a gapped Dirac point, such as...
as its chirality and its mass are thus rendered observable in the interferometry thanks to non-adiabatic transitions.

In the following, we introduce several double LZ Hamiltonians corresponding to the same energy spectrum but differing by the chirality of Dirac cones, their relative mass sign and also consider different trajectories in reciprocal space. We first concentrate on a specific case, which we solve using analytical and numerical methods to show that the usual St"uckelberg interferences in the inter-band transition probability are shifted by what is shown to be a geometrical contribution. Then, we briefly consider all other cases for which we give an analytical expression of the geometrical phase shift.

**Low-energy double cone Hamiltonian.** We define a class of effective two band models featuring two distinct Dirac cones by the following two-dimensional Bloch Hamiltonian [33]:

$$H(\hat{p}) = \left(\frac{\hat{p}_x^2}{2m} - \Delta_x\right)\hat{\sigma}_x + c_y \hat{p}_y \hat{\sigma}_y + M_z(\hat{p})\hat{\sigma}_z.$$  

(1)

$\hat{p} = (p_x, p_y)$ is the momentum, $m$ gives the band curvature in the $x$ direction, and $c_y > 0$ is the $y$-direction velocity. $\Delta_x \geq 0$ is the merging gap – which determines the distance $d$ between the two Dirac cones located at valleys $\vec{p} = D, D' \approx (\pm \sqrt{2m\Delta_x}, 0) - M_z(D) \text{ and } M_z(D')$ are the corresponding “masses” and $\hat{\sigma}_{x,y,z}$ are Pauli matrices operating in the pseudospin space. If the full Brillouin zone contains two and only two Dirac cones, the behaviour of the mass function $M_z(\hat{p})$ determines fully the quantum anomalous Hall state [2]. A constant mass function $M_z(\hat{p}) = M$ describes equal Dirac masses and a vanishing Chern number. A mass function with a sign inversion in between $D$ and $D'$ for $M_z(\hat{p}) = c_x p_x$, gives a non-zero Chern number in the individual bands. The Hamiltonian [1], therefore, is sufficiently general for describing different topological states. It describes Dirac cones with opposite chirality in the two valleys. A second class of Dirac cones having the same chirality can also be envisaged, as we discuss at the end of this Letter.

**Time-dependent Hamiltonian.** We now study inter-band dynamics of a particle experiencing a constant force $\vec{F} = (F_x, F_y)$ in such a system. The applied force is equivalent to a time-dependent gauge potential and thus leads to the substitution $(p_x, p_y) \to (p_x + F_xt, p_y + F_yt)$ in the Bloch Hamiltonian [1]. We distinguish two types of trajectories, one in which the two Dirac cones are on the opposite side of the full trajectory, termed “diagonal” and meaning $\vec{p} \to (F_x t, F_y t)$, and the other in which the two Dirac cones are on the same side, termed “parallel” and meaning $\vec{p} \to (F_x t, p_y = \text{const.})$, see Fig. 2.

We consider first the Hamiltonian with a constant mass function $M_z(\hat{p}) = M$ and a diagonal trajectory, as the same consideration can be generalized to other cases (see

FIG. 2: Momentum space trajectories driven by the force $\vec{F} = (F_x, F_y)$ in the vicinity of two Dirac points $D, D'$: (a) diagonal trajectory ($\vec{p} \to (F_x t, F_y t)$); (b) parallel trajectory ($\vec{p} \to (F_x t, p_y = \text{const.})$).

Table I. The time-dependent Hamiltonian is

$$H(t) = c_y F_y t \hat{\sigma}_x + M \hat{\sigma}_y + \left(\frac{F_x^2 t^2}{2m} - \Delta_x\right)\hat{\sigma}_z,$$

(2)

where we have also performed a parameter-independent rotation in the pseudospin space $(\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z) \to (\hat{\sigma}_x, \hat{\sigma}_z, \hat{\sigma}_y)$. From now on, we use units such that $2m = F_x = \hbar = 1$. The adiabatic spectrum is given by $E_t(t) = \pm \sqrt{\Delta^2 c_x^2 F_y^2 t^2 + M^2 + (t^2 - \Delta_x)^2}$. By assuming $\Delta > \Delta_x > \sqrt{\Delta_x}$, the two avoided crossings in the spectrum are located at $\pm t_0 = \pm \sqrt{\Delta_\uparrow}$. For an initial state in the lower band far from the first crossing, we seek to solve, in various limits, the probability for a particle ending up in the upper band after the second crossing. The most intuitive approach is to develop the so-called St"uckelberg theory, where the dynamics is assumed to be adiabatic except close to $t = \pm t_0$ where non-adiabatic transition occurs [28]. The adiabaticity parameter $\delta$ of the problem “$\text{energy}/(\hbar \text{-force-speed})$” is given by $\delta = \Delta^2/(4\sqrt{\Delta_\uparrow})$ with $\Delta = \sqrt{\Delta_x c_x^2 F_y^2 + M^2}$, and the St"uckelberg limit corresponds to the regime where the time separation $\sim 2t_0$ between the two tunneling events is much larger than the tunneling time $t_{LZ} \sim \max(\sqrt{\delta}, \delta)/\Delta [28]$. St"uckelberg theory. We make a linear expansion in the Hamiltonian [2] around the crossings $t = -\xi t_0$ (where $\xi = \pm 1$ corresponds to the first/second crossing) to arrive at two LZ-type Hamiltonians

$$H_\xi(t) = \left(\begin{array}{cc} -\xi 2t_0 t & -\Delta e^{i\phi_\xi} \\ -\Delta e^{-i\phi_\xi} & 2t_0 \xi \end{array}\right).$$  

(3)

The gap $\Delta e^{i\phi_\xi}$ is generally complex with magnitude $\Delta = \sqrt{\Delta_x c_x^2 F_y^2 + M^2}$ and phase $\phi_+ = \tan^{-1}[M/(c_y F_y \sqrt{\Delta_\uparrow})]$ and $\phi_- = \pi - \phi_+$, for $\xi = \pm 1$, respectively. It is crucial that the linearized Hamiltonian captures the full adiabatic spectrum $E_{\pm}(t)$ up to linear order in $t$ around the minima. In terms of the adiabaticity parameter $\delta$, the standard LZ formula $P_{\uparrow,\downarrow} = \exp(-2\pi \delta)$ gives the tunneling probability of traversing one crossing [29]. However, we are interested in the transition amplitudes where the phase information is also important. To this end, we express a general state
in terms of the adiabatic basis of Hamiltonian $\hat{H}^\text{ad} = b_+^\dagger (t) b_+ (t) + b_-^\dagger (t) b_- (t)$ or in vectorial notation $\hat{b}(t)^T = (b_+ (t), b_- (t))$, and the basic element of the theory is first to construct the scattering $N_\xi$-matrix for each time crossing [28]. The $N_\xi$-matrix basically relates an asymptotic incoming-state $|\Psi(-t_0)\rangle$ to an asymptotic outgoing-state $|\Psi(t_0)\rangle$ across the crossing, i.e., $\hat{b}(t_0) = \hat{U}(t_0) \cdot N_{\xi} \cdot \hat{U}(0^+) \cdot \hat{b}(-t_0)$ with the unitary evolution matrix $\hat{U}(t_2, t_1) = \exp(-i \hat{H} \int_{t_1}^{t_2} dt E_+ (t))$ accounting for the dynamical phase, in the asymptotic time regime $t_0 \gg \delta / \Delta$. Specifically, the time-dependent Schrödinger equation for the Hamiltonian $\hat{H}$ can be solved via Weber equation [29], giving

$$N_\xi = \left( \sqrt{1 - P_{LZ}} e^{-i(\varphi_s + \xi \varphi_t)} \right) \left( \frac{-\xi \sqrt{P_{LZ}}}{\sqrt{1 - P_{LZ}} e^{i(\varphi_s + \xi \varphi_t)}} \right).$$

Except for the adiabatically accumulated dynamical phase, the $N_\xi$-matrices encode the rest of the information for the amplitudes across a single crossing. The Stokes phase $\varphi_S = \pi/4 + \delta (\ln \delta - 1) + \arg \Gamma(1 - i \delta)$ is associated with the particle staying in the same band [28]. In addition, we find a non-perturbative correction $\varphi_\xi$-angle due to the phase of the complex gap and encapsulates information on the return trajectory. The phase shift $\varphi_\xi$ can be obtained as a line integral of the Berry connection $\phi(t)$ (in the vicinity of a Dirac cone pair seen as a vortex and an anti-vortex indicated by dots. Parallel (black) and diagonal (red) trajectories are represented. (b) Berry connection $\phi(t)(1 - \cos \theta(t))$ (in the south pole gauge) along the parallel (black) and diagonal trajectories (red) as a function of time $t/t_0$ in the Stückelberg limit $t_0 \gg t_{LZ}$. Two values of the mass $M = 0$ (higher peak) and $M \neq 0$ are considered. The geometrical phase $\varphi_\xi$ is obtained as a line integral of the Berry connection between $-t_0$ and $+t_0$ (thick lines in (a)).

The diabatic $\delta \to 0$ and adiabatic $\delta \gg 1$ limits, see Ref. [35] for similar notations. In the first case, the time evolution of the upper band amplitude $A_+(t)$ in the diabatic basis is given by

$$i\dot{A}_+(t) \approx (c_y F_y t - i M) \exp \left[ 2i \int t^t dt' (t'^2 - \Delta_s) \right]$$

with the lower band amplitude $A_-(t) \approx 1$. The equation can be integrated, with the boundary condition $A_1(-\infty) = 0$, to give $A_1(+\infty) = -2^{2/3} \pi M A_i(s) - 2^{1/3} \pi c_y F_y A_i'(s)$, with $s \equiv -2^{2/3} \Delta_s$ and $A_i(s)$ the Airy function. In the Stückelberg limit, i.e., $t_0 \gg t_{LZ}$, or equivalently, $|s| \gg 1$, we obtain $|A_1(+\infty)|^2 \approx 8 \pi \delta \sin^2(\pi/4 + (\varphi_{dy} + \varphi_{\Delta})/2)$ which agrees with the diabatic limit $(\delta \to 0)$ of expression [28] with $P_{LZ} \approx 1 - 2 \pi \delta, \varphi_{dy} \to \varphi_S \approx 4 \Delta_s^2/3$ and $\varphi_{\Delta} = 2 - 2 \tan^{-1}[c_y F_y \sqrt{\Delta_s|M|}]$.

In the adiabatic limit $\delta \gg 1$ the time-dependent Schrödinger equation in the adiabatic basis $|\psi_\pm(t)\rangle$ gives

$$\dot{\psi}_+(t) \approx -(\psi_+ |\partial_t| \psi_-) e^{-i w(t)}$$

where $A_-(t) \approx 1$ and $w(t) \equiv \int t^t dt' (E_+ - E_-) - \int t^t dt' (\psi_- |i \partial_{t'}| \psi_-) + \int t^t dt' (\psi_+ |i \partial_{t'}| \psi_+)$. Following Refs. [34], the transition amplitude $A_+(\infty)$ can be obtained from the complex time crossings $E_+(t_c) = 0$, giving rise to two complex roots, $t_c$’s, lying closest to the real-time axis in the upper-half complex plane. The sum of the residue contributions leads to an interference effect and we find that the transition probability $|A_+(\infty)|^2 \approx \sin^2[(\varphi_{dy} + \varphi_S)/2]$ where $\varphi_{dy}$ is the dynamical phase introduced above, and $\varphi_S$ a gauge-invariant
It is also possible to consider a pair of gapped Dirac cones with the same chirality given by

$$H(\vec{p}) = \left(\frac{p_x^2 - p_y^2}{2m} - \Delta_s\right) \hat{\sigma}_x + \frac{p_x p_y}{m} \hat{\sigma}_y + M_z(\vec{p}) \hat{\sigma}_z,$$  \hspace{1cm} (9)

as occurs, e.g., around a single $K$ point in a twisted graphene bilayer $^{10}$. Changing the mass function and the trajectory type gives four additional cases, the phase shifts of which are given in the last two lines of Table 1.

**Conclusion.** The main result of our work is contained in Eqs. (5) and (8) with $\Delta \varphi = \varphi_g$. These equations show that a Stückelberg interferometer carries information not only on the band energy spectrum but also on coupling between bands via a geometric phase. The latter could be accessed experimentally in a double cone interferometer involving Bloch oscillations and LZ tunneling, e.g. with cold atoms in a graphene-like optical lattice as recently demonstrated with non-interacting fermions $^{11}$. The inter-band transition probability – averaged over the initial Fermi sea – can be measured in a time-of-flight experiment as the fraction of atoms that tunneled from the lower to the upper band during a single Bloch oscillation $^{19, 27}$. Alternatively, a solid state realization of Bloch-Zener oscillations with multiple passages on a single Dirac cone has been proposed in a graphene ribbon superlattice, with constructive interferences showing up in the $I - V$ characteristics as sharp current peaks $^{11}$. This can be generalized to the double cone case. In both realizations, a practical way of extracting the geometrical phase from the measured total phase of the interferometer is via the different force $F$ dependences: the dynamical phase scales as $1/F$, the Stokes phase varies slightly with $\delta \propto 1/F$, whereas the geometric phase is force independent $^{42}$.

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| Chirality   | Mass function $M_{s}(p)$ (mass sign) | Parallel trajectory | Diagonal trajectory |
|------------|-------------------------------------|---------------------|---------------------|
| opposite   | $c_{s}p_{s}$ (opposite)             | $2 \tan^{-1}[c_{s}\sqrt{\Delta_{r}/(c_{s}p_{s})}]$ | $-2 \tan^{-1}[c_{s}F_{y}\sqrt{\Delta_{r}/M}]$ |
| identical  | $M$ (identical)                     | $-2 \tan^{-1}[2p_{y}\sqrt{\Delta_{r}/M}]$ | $\pi$               |
| identical  | $c_{s}p_{s}$ (opposite)             | $\pi$               | $0$                 |

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**TABLE I:** Summary of phase shifts $\Delta \varphi$ realized for double Dirac cone interferometers differing by the chirality, relative mass sign and the trajectories. We assume well separated avoided crossings ($p_{s}^{2} \ll \Delta_{r}$ in the parallel, $F_{y} \ll 1$ in the diagonal and $c_{s} \ll 1$ in the opposite mass cases).