Quasinormal modes of Reissner-Nordström Anti-de Sitter black holes

Bin Wang\textsuperscript{a,b}, Chi-Yong Lin\textsuperscript{a,2} and Elcio Abdalla\textsuperscript{a,3}

\textsuperscript{a} Instituto De Fisica, Universidade De Sao Paulo, C.P.66.318, CEP 05315-970, Sao Paulo, Brazil
\textsuperscript{b} Department of Physics, Shanghai Teachers’ University, P. R. China

Abstract

Complex frequencies associated with quasinormal modes for large Reissner-Nordström Anti-de Sitter black holes have been computed. These frequencies have close relation to the black hole charge and do not linearly scale with the black hole temperature as in Schwarzschild Anti-de Sitter case. In terms of AdS/CFT correspondence, we found that the bigger the black hole charge is, the quicker for the approach to thermal equilibrium in the CFT. The properties of quasinormal modes for $l > 0$ have also been studied.

PACS number(s): 04.30.Nk, 04.70.Bw

\textsuperscript{1} e-mail: binwang@fma.if.usp.br
\textsuperscript{2} e-mail: lcyong@fma.if.usp.br
\textsuperscript{3} e-mail: eabdalla@fma.if.usp.br
Quasinormal modes of black holes have been an intriguing subject of discussions for the last few decades. It has become an evidence that the quasinormal ringing will dominate most processes involving perturbed black holes. This means that quasinormal modes will carry a unique fingerprint which would lead to the direct identification of the black hole existence. Detection of these quasinormal modes is expected to be realized through gravitational wave observation in the near future. In order to extract as much information as possible from gravitational wave signal, it is important that we understand exactly how the quasinormal modes behave for the parameters of black holes in different models.

For black holes in asymptotically flat spacetime, especially spherical cases, they have been studied extensively, for a review see [1]. The study for the nonspherical black holes is developing [2,3]. Considering the case when the black hole is immersed in an expanding universe, the quasinormal modes of black holes in de Sitter space have also been investigated recently [4,5]. It was found that there are qualitative differences from the asymptotically flat case, in particular the scalar field decay is always exponential, rather than a power-law tail in asymptotically flat spacetime. This result has also been uncovered by a very recent study of the Schwarzschild AdS black hole model [6]. These observations support the earlier argument by Ching et al. [7] that usual inverse power-law tails as seen in asymptotically flat black hole spacetime, are not a general feature of wave propagation in curved spacetime.

Motivated by the recent discovery of the AdS/CFT correspondence, the investigation of the quasinormal modes of AdS black holes becomes more appealing nowadays. The quasinormal frequencies of AdS black hole have direct interpretation in terms of the dual conformal field theory (CFT). In terms of the AdS/CFT correspondence [8-10], a large black hole corresponds to an approximately thermal state in the field theory, and the decay of the scalar field corresponds to the decay of perturbation of the state. After computing the scalar quasinormal modes of Schwarzschild AdS black holes in four, five and seven dimensions, Horowitz and Hubeny claimed that for large black hole both the real and the imaginary parts of the quasinormal frequencies scale linearly with the black hole temperature. The timescale for approaching to the thermal equilibrium is determined by
the imaginary part of the lowest quasinormal frequency and is proportional to the inverse of the black hole temperature. However, for a small black hole the results are no longer the same as that for a large black hole. The quasinormal frequencies do not continue to scale with temperature. Some comments on small black holes were presented therein.

The Schwarzschild AdS black hole studied in [6] is the simplest model in Anti-de Sitter space, which is determined by only two dimensionful parameters, the black hole event horizon \( r_+ \) and the AdS radius \( R \) relating to the cosmological constant by \( \Lambda = -3/R^2 \).

It is of interest to generalize the study of ref. [6] to a more general model, say Reissner-Nordström (RN) AdS black holes. Besides \( r_+ \) and \( R \), the RN AdS black hole has another parameter, the charge \( Q \). Thus it possesses richer physics to be explored. In this paper we are going to study RN AdS black hole in the hope of getting more understanding of how the quasinormal modes depend on this additional parameter and whether there is more information on AdS/CFT correspondence.

The RN black hole solution of Einstein’s equations in free space with a negative cosmological constant \( \Lambda = -3/R^2 \) is given by

\[
\text{d}s^2 = -h \text{d}t^2 + h^{-1} \text{d}r^2 + r^2 \text{d}\Omega^2, \quad A = Q/\text{r} \text{d}t, \tag{1}
\]

with

\[
h = 1 - \frac{r_+}{r} - \frac{r_+^3}{R^2r} - \frac{Q^2}{r_+r} + \frac{Q^2}{r^2} + \frac{r^2}{R^2}. \tag{2}
\]

The asymptotic form of this spacetime is AdS. There is an outer horizon located at \( r = r_+ \). The mass of the black hole is

\[
M = \frac{1}{2}(r_+ + \frac{r_+^3}{R^2} + \frac{Q^2}{r_+}). \tag{3}
\]

The Hawking temperature is given by the expression

\[
T_H = \frac{1 - \frac{Q^2}{r_+^2} + \frac{3r_+^2}{R^2}}{4\pi r_+} \tag{4}
\]

and the potential by

\[
\phi = \frac{Q}{r_+} \tag{5}
\]
In the extreme case $r_+, Q$ satisfy the relation

$$1 - \frac{Q^2}{r_+^2} + \frac{3r_+^2}{R^2} = 0. \quad (6)$$

In the following discussions, we will concentrate our attention on the large black hole with $r_+ \gg R$. We will not consider the case of the small black hole here, partly because it is unstable, having negative specific heat [11] and partly because it is not of direct interest for the AdS/CFT correspondence [6].

Let us consider a massless scalar field $\Phi$ in the RN AdS spacetime, obeying the wave equation

$$\Box \Phi = 0 \quad (7)$$

where $\Box = g^{\alpha\beta} \nabla_\alpha \nabla_\beta$ is the d’Alembertian operator. If we decompose the scalar field according to

$$\Phi = \sum_{l,m} \frac{1}{r} \psi_l(t, r) Y_{lm}(\theta, \phi) \quad (8)$$

then each wave function $\psi_l(r)$ satisfies the equation

$$-\frac{\partial^2 \psi_l}{\partial t^2} + \frac{\partial^2 \psi_l}{\partial r^*_2} = \mathcal{U}_l \psi_l \quad (9)$$

where

$$\mathcal{U}_l = h \left[ \frac{l(l+1)}{r^2} + \frac{1}{r} \frac{dh}{dr} \right] = h \left[ \frac{l(l+1)}{r^2} + \frac{r_+^2 + Q^2}{r^3} \right] - \frac{2Q^2}{r^4} + \frac{2}{R^2} \quad (10)$$

and $r_*$ here is the tortoise coordinate defined by $r_* = \int \frac{dr}{h}$.

The potential $\mathcal{U}$ has the same characteristic as that in Schwarzschild AdS black hole. It is positive and vanishes at the horizon, however it diverges at $r = \infty$, which requires that $\Phi$ vanishes at infinity. This is the boundary condition to be satisfied by the wave equation for the scalar field in AdS space.

Quasinormal modes of AdS space are defined to be modes with only ingoing wave near the horizon. There only exists a discrete set of complex quasinormal frequencies. The quasinormal modes behave like $e^{-i\omega(t+r_*)}$ near the horizon in RN AdS background. We
thus introduce the ingoing Eddington coordinates by setting \( v = t + r* \). The metric (1) can be rewritten as

\[
\mathrm{d}s^2 = -h \mathrm{d}v^2 + 2 \mathrm{d}v \mathrm{d}r + r^2 \left( \mathrm{d}\theta^2 + \sin^2 \theta \mathrm{d}\phi^2 \right)
\]  (11)

We separate the scalar field in a product form as

\[
\Phi = \frac{1}{r} \psi(r) Y(\theta, \phi) e^{-i\omega v}
\]  (12)

The minimally-coupled scalar wave equation (7) may thereby be reduced to an ordinary, second order, linear differential equation with the radial terms yielding

\[
h(r) \frac{d^2 \psi(r)}{dr^2} + (h'(r) - 2i\omega) \frac{d\psi(r)}{dr} - V(r) \psi(r) = 0,
\]  (13)

where the potential function is given by

\[
V(r) = \frac{h'(r) r}{r} + \frac{l(l + 1)}{r^2}
\]

\[
= \frac{1}{r} \left( \frac{r_+}{r^2} + \frac{r_+^3}{R^2 r^2} + \frac{Q^2}{r_+ r^2} \right) + \frac{2r^2}{r^3} + \frac{2r}{R^2} + \frac{l(l + 1)}{r^2}.
\]  (14)

Note that by setting \( Q^2 = 0, R = 1 \), Eqs.(13,14) go back to the 4D Schwarzschild AdS black hole case addressed in [6]. In the following discussion we adopt \( R = 1 \).

To find the complex values of \( \omega \) such that (13) has a solution with \( \psi \) finite at the horizon \( r = r_+ \), and vanishing at infinity, we have to count on the numerical calculations. By using the numerical method suggested in [6] to compute the quasinormal modes, we will expand the solution in power series about the horizon and impose the boundary condition that the solution vanishes at infinity. Adopting the new variable \( x = 1/r \), (13) can be reexpressed as

\[
s(x) \frac{d^2 \psi(x)}{dx^2} + t(x) \frac{d}{x - x_+} \psi(x) + u(x) \frac{\psi(x)}{(x - x_+)^2} = 0
\]  (15)

where the coefficient functions are

\[
s(x) = \frac{r_0 x^5 - x^4 - x^2 - Q^2 x^6}{x - x_+}
\]  (16)

\[
t(x) = 3r_0 x^4 - 2x^3 - 4Q^2 x^5 - 2i\omega x^2
\]  (17)

\[
u(x) = (x - x_+)V(x)
\]  (18)
and the parameter $r_0 = \frac{1 + x_+^2 + Q^2 x_+^4}{x_+^3}$. As done in [6], we can expand $s, t$ and $u$ about the horizon $x = x_+$ in the form $s(x) = \sum s_n(x - x_+)^n$ etc. The first terms are $s_0 = 2x_+^2 \kappa, t_0 = 2x_+^2(\kappa - i\omega)$ and $u_0 = 0$, where $\kappa$ is the surface gravity and has the form $\kappa = (x_+ + 3/x_+ - Q^2 x_+^3)/2$. The solution of (15) can then be expressed as a power series

$$\psi(x) = \sum_{n=0}^{\infty} a_n(x - x_+)^n$$

(19)

Figure 1: Lines from the top to the bottom correspond to $Q^2 = 10^7, 10^6, 10^5$ etc.

Substituting (19) into (15) and equating coefficients of $(x - x_+)^n$ for each $n$, we have the recursion relations for $a_n$

$$a_n = -\frac{1}{P_n} \sum_{k=0}^{n-1} [k(k-1)s_{n-k} + kt_{n-k} + u_{n-k}]a_k$$

(20)

where

$$P_n = n(n-1)s_0 + nt_0 = 2x_+^2 n(n\kappa - i\omega).$$

(21)

This relation has the same form as that of the Schwarzschild AdS case, but with an additional parameter $Q$ herein.
In order to find the quasinormal modes for the AdS spacetime, we must select solutions which satisfy the boundary condition \( \psi = 0 \) as \( r \to \infty \) \((x \to 0)\). Thus we need to look for the zeros of Eq(19) at \( x = 0 \) in complex \( \omega \) planes. For a given \( l, x_+, Q \) the algorithm to find these frequencies follows the simple steps: (i) truncate (19) at a number \( N \) of terms, construct the polynomial equation of \( \omega \) using (20,21) so that (19) reduces to \( \sum_{n=0}^{N} a_n(\omega)(-x_+)^n = 0 \), (ii) find roots of interest of this function, (iii) increase \( N \) until these roots become constant within the desired precision. Since the problem is to find numerical solutions of a polynomial equation, it becomes easy to use a built-in Mathematica function to locate zeros of \( \sum_{n=0}^{N} a_n(\omega)(-x_+)^n \) directly. This procedure may reduce tedious trials and make the numerical calculations neat.

We decompose the quasinormal frequencies into real and imaginary parts in the form

\[
\omega = \omega_r - i\omega_i
\] (22)

which makes \( \omega_i \) positive for all quasinormal frequencies. For large black holes \((r_+ \gg R)\), the relation of the values of the lowest quasinomal mode frequencies for \( l = 0 \) and selected values of \( r_+ \) for different charge \( Q \) are exhibited in fig.1 and fig.2. The dots represent the
Figure 3: Lines from the top to the bottom correspond to $Q^2 = 10^4, 10^5, 10^6, 10^7$ etc.

lowest modes. In fig.1, lines from the top to the bottom correspond to $Q^2 = 10^7, 10^6, 10^5$ etc. However the lines shown in fig.2 are corresponding to $Q^2 = 10, 10^2, ..10^7$ from the top to the bottom, respectively. It is easy to see that with an additional parameter, the charge $Q$, neither the real nor the imaginary part of the frequency is a linear function of $r_+$ as found in Schwarzschild AdS case [6]. The bigger the charge $Q$ is, the larger is the deviation from the linear relation we observe.

The imaginary and real parts of the quasinormal frequencies relate to the damping time scale ($\tau_1 = 1/\omega_i$) and oscillation time scale ($\tau_2 = 1/\omega_r$), respectively. From fig.1 we learn that as $Q$ increase, $\omega_i$ increases as well, which corresponds to the decrease of the damping time scale. According to the AdS/CFT correspondence, this means that for big $Q$, it is quicker for the quasinormal ringing to settle down to the thermal equilibrium. Fig.2 tells us that the bigger charge $Q$ leads to the smaller $\omega_r$, which means that the frequency of the oscillation becomes small as $Q$ increases. Therefore from fig.1 and fig.2 we can have a picture that if we perturbe a RN AdS black hole with high charge, the surrounding geometry will not “ring” as much and long as that of the black hole with
small $Q$. It is easy for the perturbation on the highly charged AdS black hole background to return to thermal equilibrium. This is the new physics brought by the additional parameter $Q$ in RN AdS black hole.

From (4), we learn that the temperature of the large charged AdS black hole does not scale linearly with the event horizon as that in Schwarzschild AdS black hole case. The behavior of the temperature is shown in fig.3. The linear relation between $T$ and $r_+$ is broken as $Q$ increases. The relations between the real and the imaginary parts of the frequency and the temperature for large RN AdS black hole are displayed in fig.4,5. Again we see that in contrast to the results in [6], the relations are no longer linear when the additional parameter $Q$ is taken into account.

We have so far discussed only the lowest quasinormal modes with $l = 0$. Increasing $l$, we obtain the surprising effect of increasing the damping time scale ($\omega_i$ decreases), and decreasing the oscillation time scale ($\omega_r$ increases) as in the case of Schwarzschild AdS black holes. As pointed out in [6], here we may also meet the problem of possible negative $\omega_i$ as it continuously decreases with $l$. However from fig.6, it looks that for the
large black hole, the problem is not as serious as that shown in [6] for small Schwarzschild AdS case. Despite the similar behavior, once again the additional parameter $Q$ led us to further new properties. As fig.6 shows, different values of $Q$ do not change the qualitative characteristic of decreasing $\omega_i$ with $l$. Their decreasing rates are the same and do not depend on the value $Q$. However it is clear that the higher is the charge of AdS black hole, the later we will confront the tough question of $\omega_i$ as $l \to \infty$. The dependence of $\omega_r$ on $l$ for different values of $Q$ are exhibited in fig.7. Lines from upper to the bottom are $Q^2 = 10^3, 10^4...10^6$, respectively. As shown for the case of the imaginary part, the lines of different charges are parallel.

In our numerical calculations, we found that we need for a large number $N$ of terms in the partial sum to reduce the relative error in the computation of quasinormal frequencies as $r_+$ decreases and $Q$ increases. However when the charge $Q$ increases to nearly the extreme value satisfying (6), we cannot accurately determine the mode, no matter how large is the value of $N$ that we adopt. The numerical convergence problem can be attributed to the method we adopted to compute the quasinormal modes. In the expansion of the
solution of the differential equation into power series, we have the convergence radius

\[ L = \lim_{n \to \infty} \left| \frac{\psi_n}{\psi_{n+1}} \right| \sim \left| \frac{3 + x_+^2 - Q^2 x_+^4}{9 - 3x_+ + 4x_+^2 - x_+^3 - Q^2(5x_+^4 - x_+^5)} \right| \]  

(23)

To expand the solution in a power series about the horizon, we need \( L > x_+ \) to ensure that the expansion is valid. In the range \( 0 < Q^2 < \frac{9 - 3x_+ + 4x_+^2 - x_+^3}{5x_+^4 - x_+^5} \), \( L \) increases from \( L_0 = \frac{(3 + x_+^2)/(9 - 3x_+ + 4x_+^2 - x_+^3)}{5x_+^4 - x_+^5} \) to infinity, where \( L_0 > x_+ \) for a big black hole, which means that \( L > x_+ \) always hold in this range. However for \( Q^2 > \frac{9 - 3x_+ + 4x_+^2 - x_+^3}{5x_+^4 - x_+^5} \), \( L \) continuously decreases with increasing values of \( Q \). Finally when \( Q^2 \to \frac{3 + x_+^2}{x_+^4} = 3r_+^4 + r_+^2 \), which is the exact extreme value, \( L \to 0 < x_+ \). Therefore although the numerical approach is very efficient for determining the quasinormal modes for Schwarzschild AdS black hole and lowly charged AdS black hole, it breaks down for the nearly extreme RN AdS black hole case.

It is important to notice that Eq(15) has the same form as the generalized spheroidal wave equations which also arises both in the quantum scattering theory of nonrelativistic
electrons from polar molecules and ions and in the theory of radiation process involving black hole in asymptotically flat spacetime. Similar convergence problem also appeared and hindered the earlier study of the quasinormal modes for extreme charged black hole and extreme rotating black hole [11,12].

Besides problems related to the method, is there some deep physics to account for the similar no convergence problem in determining quasinormal modes for extreme black holes both in asymptotically flat and AdS spaces? We know that the frequencies and damping times of the quasinomal modes are entirely fixed by the black hole, and are independent of the initial perturbation. It has been shown that there is a second order phase transition in the extreme limit of black holes [14-16]. This result has been further promoted in a recent study for the charged AdS black holes [11,17]. Because of the phase transition, the fluctuation of thermodynamic quantities become tremendous. It is hard to expect that we can obtain the fixed quansinormal frequencies to characterize the thermalization timescale in the strong coupled CFT on the extreme RN AdS background.
We speculate that the black hole phase transition maybe the candidate physical reason behind this problem.

In summary, we have computed the scalar quasinormal modes of large RN AdS black hole. These modes govern the late time decay of a minimally coupled scalar field, such as the dilaton. Compared to the Schwarzschild AdS black hole, we found that the additional parameter $Q$ in RN AdS black hole has brought in some new properties. We observed that these modes no longer linearly scale with the black hole temperature. By AdS/CFT correspondence, we can interpretate the decay of the quasinormal modes to the time scale approaching thermal equilibrium in CFT. We learnt that perturbations are associated with the black hole charge. The larger the charge of the RN AdS black hole, the sooner it returns to the thermal equilibrium. The dependence of the quasinormal frequencies on nonzero angular momentum $l$ has also been discussed. Due to the convergence problem, the method we adopted cannot be extended directly to study of the extreme black holes cases. Further refinement of the numerical method of solving the differential equation (13) is called for. Based upon some critical phenomena uncovered, we have given some speculation of the possible physical reason behind the problem for the extreme black holes.

ACKNOWLEDGMENT: This work was partially supported by Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) and Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq). B. Wang would like to acknowledge the support given by Shanghai Science and Technology Commission.

References

[1] K. D. Kokkotas, B. G. Schmidt, gr-qc/9909058 and references therein

[2] W. Krivan, Phys. Rev. D 60, 101501 (1999)

[3] S. Hod, gr-qc/9902072

[4] P. R. Brady, C. M. Chambers, W. G. Laarakkers and E. Poisson, Phys. Rev. D 60, 064003 (1999)
[5] P. R. Brady, C. M. Chambers, W. Krivan and P. Laguna, Phys. Rev. D 55, 7538 (1997)

[6] G. T. Horowitz and V. E. Hubeny, hep-th/9909036; G. T. Horowitz, hep-th/9910082

[7] E. S. C. Ching, P. T. Leung, W. M. Suen and K. Young, Phys. Rev. D 52, 2118 (1995)

[8] J. M. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998)

[9] E. Witten, Adv. Theor. Math. Phys. 2, 253 (1998)

[10] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B428, 105 (1998)

[11] A. Chamblin, R. Emparan, C. V. Johnson and R. C. Myers, Phys. Rev. D 60, 064018 (1999)

[12] E. W. Leaver, J. Math. Phys. 27, 1238 (1986)

[13] E. W. Leaver, Phys. Rev. D 41, 2986 (1990)

[14] C. O. Lousto, Phys. Rev. D51, 1733 (1995)

[15] O. Kaburaki, Phys. Lett. A 217, 316 (1996)

[16] R. K. Su, R. G. Cai and P. K. N. Yu, Phys. Rev. D 50, 2932 (1994); ibid 48, 3473 (1993); ibid 52, 6186 (1995)

B. Wang, J. M. Zhu, Mod. Phys. Lett. A 10, 1269 (1995)

[17] A. Chamblin, R. Emparan, C. V. Johnson and R. C. Myers, Phys.Rev. D60, 104026 (1999)