Minimal hypersurfaces in $\mathbb{S}^5$
with vanishing Gauss–Kronecker curvature

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Abstract. We give a local description of complete minimal hypersurfaces in $\mathbb{S}^5$ with zero Gauss–Kronecker curvature, zero 3-mean curvature and nowhere-zero second fundamental form.

Keywords: minimal hypersurfaces, Gauss–Kronecker curvature, complete hypersurfaces.

Introduction

In order to study hypersurfaces in spheres, consider the unit sphere $\mathbb{S}^{n+1}$ with the usual metric, and let $g: M^n \hookrightarrow \mathbb{S}^{n+1}$ be an isometric immersion. In this case, the second fundamental form is determined by $n$ continuous functions $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ called principal curvatures. The elementary symmetric functions of the principal curvatures define the mean curvature $H = (\lambda_1 + \lambda_2 + \cdots + \lambda_n)/n$, the Gauss–Kronecker curvature $K = \lambda_1 \lambda_2 \cdots \lambda_n$ and the scalar curvature $R = \sum_{i \neq j} \lambda_i \lambda_j + n(n-1)$. In particular, if $H = 0$, the hypersurface is minimal. An important open question that motivates the study of minimal hypersurfaces in $\mathbb{S}^{n+1}$ is the Chern conjecture [1].

Let $M^n$, $n \geq 3$, be a closed minimal hypersurface of constant scalar curvature $R$ in $\mathbb{S}^{n+1}$, and let $R_n$ be the set of possible values of $R$. Is $R_n$ a discrete set of real numbers?

This question was answered affirmatively in the case $n = 3$ by Almeida and Brito [2] by proving that $R_3 = \{0, 3, 6\}$.

A hypersurface in $\mathbb{S}^{n+1}$ is said to be isoparametric if the principal curvatures are constant and have constant multiplicity. Such hypersurfaces were classified by Cartan [3] in the case $n \leq 3$, and Almeida, Brito and Sousa [4] proved the following theorem.

Let $M^3$ be a closed hypersurface in $\mathbb{S}^4$. Suppose that two of the three functions $H$, $K$, $R$ are constant. Then either $M^3$ is an isoparametric hypersurface in $\mathbb{S}^4$ or $H = K = 0$.

In [5] de Almeida and Brito classified compact minimal hypersurfaces $M^3$ in $\mathbb{S}^4$ with zero Gauss–Kronecker curvature and non-vanishing second fundamental form

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on $M^3$. They also gave infinitely many non-isoparametric examples of such hypersurfaces. Later, Ramanathan [6] classified compact minimal hypersurfaces $M^3$ in $S^4$ with zero Gauss–Kronecker curvature without the condition on the second fundamental form. In [7] Hasanis, Halilaj and Vlachos classified complete minimal immersions $g: M^3 \hookrightarrow S^4$ with Gauss–Kronecker curvature identically zero under certain assumptions on the second fundamental form. In [8] Asperti, Chaves and Sousa showed that the infimum of the absolute value of the Gauss–Kronecker curvature of a complete minimal hypersurface in a 4-dimensional space form vanishes under assumptions on the Ricci curvature. They also classified complete minimal hypersurfaces with constant Gauss–Kronecker curvature in 4-dimensional space forms.

It is important to consider what happens in the case of hypersurfaces in $S^5$ with three of the four symmetric functions of the principal curvatures being constant. In this direction Lusala, Scherfner and Sousa proved in [9] that a closed minimal Willmore hypersurface of constant scalar curvature in $S^5$ must be isoparametric. A hypersurface is a Willmore hypersurface if it is a critical point of the Willmore functional

$$
\int_M (nS - H^2)^{n/2} \text{vol},
$$

where $S$ is the squared norm of the second fundamental form and vol is the volume form.

Let $g: M^2 \hookrightarrow S^4$ be a minimal immersion of a 2-dimensional manifold $M^2$. Dajczer and Gromoll [10] proved that if $g$ has nowhere vanishing normal curvature, then the polar map of $g$ is everywhere regular and provides a minimal hypersurface in $S^4$ with Gauss–Kronecker curvature identically zero.

In this paper we consider minimal immersions $g: M^2 \hookrightarrow S^5$, where $M^2$ is a complete 2-dimensional manifold, and give an example of a minimal hypersurface in $S^5$ with zero Gauss–Kronecker curvature (Theorem 2.2). This example is a 2-spherical local bundle over a minimal surface in $S^5$. Theorem 3.3 is a local study of complete minimal hypersurfaces in $S^5$ with zero Gauss–Kronecker curvature, zero 3-mean curvature and nowhere-zero second fundamental form.

Finally, writing $K$ and $K_N$ (see (1.12)) for the Gauss and scalar normal curvatures respectively, we show in Proposition 2.1 that $g$ is superminimal if and only if

$$(K - 1)^2 - \frac{1}{4}K_N = 0.$$

§ 1. Preliminaries

1.1. Isometric immersions in a Euclidean sphere. In this section we shall introduce the structure equations of an isometric immersion in a sphere using the moving frame formalism. For a detailed description of these techniques see [11], Ch. 2 and [12], Ch. 2.

Let $g: M^n \hookrightarrow S^{n+p}$ be an isometric immersion of an $n$-manifold $M^n$ in the $(n+p)$-sphere. Let $\mathcal{B} = \{e_1, e_2, \ldots, e_{n+p}\}$ be an orthonormal frame on $S^{n+p}$ adapted to this immersion in the sense that $\{e_1, \ldots, e_n\}$ spans $g_*(TM)$ while $\{e_{n+1}, \ldots, e_{n+p}\}$ spans $(g_*(TM))^\perp$. Let $\mathcal{B}^* = \{\omega^1, \omega^2, \ldots, \omega^{n+p}\}$ be its dual coframe.
Thus, the curvature form $\Omega^i_j$ with

$$\Omega^i_j = -\omega^i_k \wedge \omega^j_k - \omega^i_j \wedge \omega^k_j + 1\omega^i \wedge \omega^j,$$

$$\Omega^\alpha_j = -\omega^\alpha_k \wedge \omega^j_k - \omega^\alpha_j \wedge \omega^k_j,$$

$$\Omega^\alpha_\beta = -\omega^\alpha_i \wedge \omega^j_\beta - \omega^\alpha_\beta \wedge \omega^j_\gamma$$

are the usual identification $T\mathbb{S}^{n+p} \cong \mathbb{R}^{n+p+1}$. By making the usual identification $T\mathbb{R}^{n+p+1} \cong \mathbb{R}^{n+p+1}$, we may use $g$ to complete the frame $B$ to an orthogonal frame $\{g, e_1, e_2, \ldots, e_{n+p}\}$ on $\mathbb{R}^{n+p+1}$.

There is a natural identification $TM_x \cong T(\mathbb{S}^{n+p})|_{g(x)}$, and so we shall denote the vector fields $\tilde{e}_j$ on $M$ such that $e_j = g \cdot \tilde{e}_j$, $j = 1, \ldots, n$, simply by $e_j$. We also write $\omega^j_i$ for the forms $g^* \omega^j_i$ on $M$. With these identifications we may write

$$dg = \omega^j_i \otimes e_j.$$

In the above equations and throughout the text we use the Einstein convention for summation over repeated indices, meaning that

$$dg = \omega^j_i \otimes e_j = \sum_{j=1}^n \omega^j_i \otimes e_j.$$

Since $\{g, e_1, e_2, \ldots, e_{n+p}\}$ is an orthonormal frame, it is immediate that $\langle dg, e_A \rangle = -\langle g, de_A \rangle$ and $\langle de_A, e_B \rangle = -\langle e_A, de_B \rangle$, where $1 \leq A, B \leq n+p$, and we can write down the structure equations of $g(M^n)$:

$$\begin{cases}
  dg = \omega^j_i \otimes e_j, \\
  de_i = -\omega^i_j \otimes g + \omega^j_i \otimes e_j + \omega^\beta_i \otimes e_\beta, \\
  de_\alpha = \omega^j_i \otimes e_j + \omega^\beta_i \otimes \epsilon_\beta, \\
  dw^i = -\omega^j_i \wedge \omega^j, \\
  dw^\alpha = -\omega^\alpha_i \wedge \omega^i,
\end{cases}$$

with $\omega^A_B + \omega^B_A = 0$, where $1 \leq i, j \leq n$, $n+1 \leq \alpha, \beta \leq n+p$ and $1 \leq A, B \leq n+p$. As in the preceding statement, throughout all this work we shall use small Latin letters $i, j, \ldots$ (resp. Greek letters $\alpha, \beta, \ldots$) for indices which take values in the range $\{1, \ldots, n\}$ (resp. $\{n+1, \ldots, n+p\}$). By differentiating the connection forms $\omega^A_B$ and using the fact that the sectional curvature of $\mathbb{S}^{n+p}$ is constant, $c = 1$, we obtain the Gauss, Codazzi and Ricci equations ([12], p. 27) respectively:

$$\begin{cases}
  dw^i_j = -\omega^i_k \wedge \omega^j_k - \omega^i_j \wedge \omega^k_j + 1\omega^i \wedge \omega^j, \\
  dw^\alpha_j = -\omega^\alpha_k \wedge \omega^j_k - \omega^\alpha_j \wedge \omega^k_j, \\
  dw^\alpha_\beta = -\omega^\alpha_i \wedge \omega^j_\beta - \omega^\alpha_\beta \wedge \omega^j_\gamma,
\end{cases}$$

Thus, the curvature form $\Omega^i_j$ and the normal curvature form $\Omega^\alpha_\beta$ are

$$\begin{cases}
  \Omega^i_j = -\omega^i_k \wedge \omega^j_k + 1\omega^i \wedge \omega^j, \\
  \Omega^\alpha_j = -\omega^\alpha_k \wedge \omega^j_k,
\end{cases}$$

$$\begin{cases}
  \Omega^\alpha_\beta = -\omega^\alpha_i \wedge \omega^j_\beta - \omega^\alpha_\beta \wedge \omega^j_\gamma,
\end{cases}$$

We restrict these forms to $M^n$. Then

$$\omega^\alpha = 0.$$
Since
\[ 0 = d\omega^\alpha = -\omega^\alpha_i \wedge \omega^i, \]
bvby Cartan’s lemma we may write
\[ \omega^\alpha_i = h^\alpha_{ij} \omega^j, \quad h^\alpha_{ij} = h^\alpha_{ji}. \] (1.4)

We call
\[ B = h^\alpha_{ij} \omega^i \otimes \omega^j \otimes e_\alpha \] (1.5)
the second fundamental form of the immersed manifold \( M^n \). The mean curvature vector of \( M^n \) is given by
\[ H = \frac{1}{n} \sum_i h^\alpha_{ii} e_\alpha. \] (1.6)

An immersion is said to be \textit{minimal} if its mean curvature vanishes identically, that is, if \( \sum_i h^\alpha_{ii} = 0 \) for all \( \alpha \).

\begin{enumerate}
\item \textbf{1.2. Hypersurfaces.} In the special case of hypersurfaces in \( S^5 \) (\( n = 4, p = 1 \)), by taking a local orthonormal frame such that
\[ h^5_{ij} = \lambda_i \delta_j^i, \]
we can write the second fundamental form as
\[ B = \lambda_i \omega^i \otimes \omega^i \otimes e_5, \]
and the equation (1.4) becomes
\[ \omega^5_i = \lambda_i \omega^i. \] (1.7)

The equations (1.1) take the form
\[
\begin{aligned}
dg &= \omega^i \otimes e_i, \\
de_i &= -\omega^i \otimes g + \omega^j_i \otimes e_j + \lambda_i \omega^i \otimes e_5, \\
de_5 &= -\lambda_i \omega^i \otimes e_i, \\
d\omega^i &= -\omega^j_i \wedge \omega^j, \quad \omega^j_i + \omega^i_j = 0, \\
d\omega^5 &= 0,
\end{aligned}
\] (1.8)

and the Gauss and Codazzi equations (1.2) become
\[
\begin{aligned}
d\omega^i_j &= -\omega^i_k \wedge \omega^j_k + (1 + \lambda_i \lambda_j) \omega^i \wedge \omega^j, \\
d\omega^5_i &= -\lambda_k \omega^i_k \wedge \omega^5.
\end{aligned}
\] (1.9)

The \textit{r}-mean curvature \( H_r \) of an immersion \( g: M^4 \hookrightarrow S^5 \) with principal curvatures \( \lambda_1, \lambda_2, \lambda_3 \) and \( \lambda_4 \) is given by the formula
\[ H_r = \left[ \binom{n}{k} \right]^{-1} \sum_{i_1 < i_2 < \cdots < i_r} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_r}. \]
Thus,
\[
\begin{align*}
H_1 &= \frac{1}{4}(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4), \\
H_2 &= \frac{1}{6}(\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_1 \lambda_4 + \lambda_2 \lambda_3 + \lambda_2 \lambda_4 + \lambda_3 \lambda_4), \\
H_3 &= \frac{1}{4}(\lambda_1 \lambda_2 \lambda_3 + \lambda_1 \lambda_2 \lambda_4 + \lambda_1 \lambda_3 \lambda_4 + \lambda_2 \lambda_3 \lambda_4), \\
H_4 &= \lambda_1 \lambda_2 \lambda_3 \lambda_4. 
\end{align*}
\] (1.10)

Note that \(H_1\) (resp. \(H_4\)) is the \textit{mean curvature} (resp. the \textit{Gauss–Kronecker curvature}) of the hypersurface.

1.3. \textbf{Immersed surfaces in} \(S^5\). In the case of immersed surfaces in \(S^5\) \((n = 2, p = 3)\), equations (1.3) and (1.4) yield that the Gaussian curvature of \(M^2\) is given by
\[
K = \Omega^1_2(e_1, e_2) = d\omega^1_2(e_1, e_2) = 1 + \sum_{\alpha = 3}^5 \frac{|h^\alpha_{11} h^\alpha_{12}|}{h^\alpha_{21} h^\alpha_{22}}. 
\] (1.11)

The \textit{scalar normal curvature} \(K_N\) of the immersion \(g\) is the square of the length of the normal curvature form:
\[
K_N = \sum_{i,j,\alpha,\beta} (R^\alpha_{ij})^2, \quad R^\alpha_{ij} = \sum_k \frac{|h^\alpha_{ki} h^\alpha_{kj}|}{|h^\alpha_{ki} h^\alpha_{kj}|}. 
\] (1.12)

Let \(S\) be the square of the length of the second fundamental form,
\[
S = \sum (h^\alpha_{ij})^2.
\]

Note that if \(g\) is minimal, then
\[
K = 1 - \sum_{\alpha} \left[ (h^\alpha_{11})^2 + (h^\alpha_{12})^2 \right], \quad K_N = 8 \sum (h^\alpha_{11} h^\beta_{12} - h^\alpha_{12} h^\beta_{11})^2,
\]
that is,
\[
K = 1 - \frac{1}{2} S, \quad K_N = 4 \left[ (R^3_{412})^2 + (R^3_{512})^2 + (R^4_{512})^2 \right].
\] (1.13)

The \textit{curvature ellipse} \(E_p\) of \(g\) at \(p\) is the image of the unit circle under the second fundamental form \(B\) of \(g\) at \(p\):
\[
E_p = \{B_p(X, X) \in (T_p M)^\perp : X \in T_p M, \, ||X|| = 1 \}.
\]

When \(X = \cos \theta e_1 + \sin \theta e_2\), it is easy to see that
\[
B_p(X, X) = H_p + [e_3 \quad e_4 \quad e_5] \cdot \begin{bmatrix} (h^3_{11} - h^3_{22})/2 & h^3_{12} \\ (h^4_{11} - h^4_{22})/2 & h^4_{12} \\ (h^5_{11} - h^5_{22})/2 & h^5_{12} \end{bmatrix} \cdot \begin{bmatrix} \cos 2\theta \\ \sin 2\theta \end{bmatrix}. 
\] (1.14)

If \(g\) is minimal, then
\[
B_p(X, X) = [e_3 \quad e_4 \quad e_5] \cdot \begin{bmatrix} h^3_{11} & h^3_{12} \\ h^4_{11} & h^4_{12} \\ h^5_{11} & h^5_{12} \end{bmatrix} \cdot \begin{bmatrix} \cos 2\theta \\ \sin 2\theta \end{bmatrix}. 
\] (1.15)
Observe that saying that $K_N \neq 0$ is equivalent to saying that the rank of the matrix $(h^{\alpha}_{ij})_{3\times 2}$ in (1.15) is 2 and $B_p(T_pM)$ is 2-dimensional.

The map $g$ is superminimal if it is minimal and the curvature ellipse is always a circle. This is equivalent to saying that

$$\|B_{11}\| = \|B_{12}\|, \quad \langle B_{11}, B_{12} \rangle = 0, \quad \text{where } B_{ij} := B(e_i, e_j). \quad (1.16)$$

§ 2. Example of a minimal hypersurface with vanishing Gauss–Kronecker curvature

In this section we shall give an example of a minimal hypersurface in $S^5$ with vanishing Gauss–Kronecker and 3-mean curvatures. This example is a 2-spherical local bundle over a minimal surface in $S^5$.

Before constructing the example, we give a result on immersed superminimal surfaces with vanishing scalar normal curvature.

**Proposition 2.1.** Let $M^2$ be a complete surface and $g: M^2 \hookrightarrow S^5$ a superminimal immersion. If the scalar normal curvature $K_N$ vanishes, then $M^2$ is compact and $g(M^2)$ is a totally geodesic sphere in $S^5$.

**Proof.** It is easy to see from § 1 that, for such immersions,

$$(K - 1)^2 - \frac{1}{4}K_N = (\|B_{11}\|^2 + \|B_{12}\|^2)^2 - 4 \sum_{\alpha < \beta} \begin{vmatrix} h^{\alpha}_{11} & h^{\alpha}_{12} \\ h^{\beta}_{11} & h^{\beta}_{12} \end{vmatrix} = (\|B_{11}\|^2 + \|B_{12}\|^2)^2 - 4(\|B_{11}\|^2\|B_{12}\|^2 - \langle B_{11}, B_{12} \rangle^2) = (\|B_{11}\|^2 - \|B_{12}\|^2)^2 + 4\langle B_{11}, B_{12} \rangle^2.$$

Therefore it follows from (1.16) that $g$ is superminimal if and only if

$$(K - 1)^2 - \frac{1}{4}K_N = 0. \quad (2.1)$$

We now assume that $g: M^2 \hookrightarrow S^5$ is superminimal and $K_N = 0$. Thus, the vectors $B_{11}$ and $B_{12}$ are orthogonal and linearly dependent, whence $B \equiv 0$ and $K = 1$. Therefore, if $M^2$ is complete, it is compact by the Bonnet-Myers theorem and, since $B \equiv 0$, the set $g(M^2)$ is a totally geodesic sphere in $S^5$. □

Now, to begin the construction of the example, let $g: M^2 \hookrightarrow S^5$ be an isometric immersion and consider the unit normal bundle $N \subset (g_* (TM))^\perp$ of the immersion $g$. Then

$$N = \{(p, V) \in M^2 \times \mathbb{R}^6: \|V\| = 1, \ V \perp \mathbb{R} \cdot g(p) \oplus g_*(T_pM)\}.$$

Denote the projections to the first and second factors by $\pi_1: N \rightarrow M^2$ and $x_g: N \rightarrow S^5$ respectively.

Consider

$$N_*(p) = N(p) \setminus \{B_p(T_pM)^\perp \cap g_*(T_pM)^\perp\}. \quad (2.2)$$

We have three possibilities.
(a) If \( K_N \neq 0 \), that is, if the rank of the matrix \( (h^\alpha_{1j})_{3 \times 2} \) is 2, then \( N_*(p) \) is a 2-sphere without two antipodal points, given by the orthogonal complement of \( B_p(T_pM) \) in \( g_*(T_pM)^\perp \).

(b) If the rank of the matrix \( (h^\alpha_{1j})_{3 \times 2} \) is 1, then \( N_*(p) \) is a 2-sphere without a great circle.

(c) If the rank of the matrix \( (h^\alpha_{1j})_{3 \times 2} \) is 0, that is, if \( g(M) \) is totally geodesic, then \( N_*(p) = \emptyset \).

Theorem 2.2. Let \( g : M^2 \hookrightarrow S^5 \) be a minimal immersion with nowhere-zero second fundamental form. Then there is an open subset \( N_* \) of \( N \) such that \( x_g : N_* \rightarrow S^5 \) is an immersion of the 4-dimensional manifold \( N_* \). Furthermore, \( x_g : N_* \rightarrow S^5 \) is an immersed minimal hypersurface of \( S^5 \) with zero Gauss–Kronecker curvature and zero 3-mean curvature.

Proof. Let \( \{e_1, e_2, e_3, e_4, e_5\} \) be a frame adapted to \( g \) on an open set \( U \subset M^2 \), and let \( W \) be a coordinate neighbourhood of \( S^2 \). Parametrize \( \pi_1^{-1}(U) \) locally by the set \( U \times W \) using the map \( y : U \times W \rightarrow N \) given by

\[
(p, \theta, \varphi) \mapsto (p, \sin \varphi \cos \theta e_3 + \sin \varphi \sin \theta e_4 + \cos \varphi e_5), \quad 0 \leq \theta < 2\pi, \quad 0 < \varphi < \pi.
\]

Then \( x : U \times W \rightarrow S^5 \) given by

\[
x(p, \theta, \varphi) = \sin \varphi \cos \theta e_3 + \sin \varphi \sin \theta e_4 + \cos \varphi e_5
\]

is a local representation of \( x_g \). Thus,

\[
dx = \sin \varphi \cos \theta de_3 + \sin \varphi \sin \theta de_4 + \cos \varphi de_5
\]

\[
+ \cos \varphi \cos \theta d\varphi e_3 + \cos \varphi \sin \theta d\varphi e_4 - \sin \varphi d\varphi e_5
\]

\[
- \sin \varphi \sin \theta d\theta e_3 + \sin \varphi \cos \theta d\theta e_4.
\]

The structure equations adapted to \( g \) then yield

\[
dx = (\sin \varphi \cos \omega^1_3 + \sin \varphi \sin \omega^1_4 + \cos \varphi \omega^1_5) \otimes e_1
\]

\[
+ (\sin \varphi \cos \omega^2_3 + \sin \varphi \sin \omega^2_4 + \cos \varphi \omega^2_5) \otimes e_2
\]

\[
+ (- \sin \varphi \sin \omega^3_3 + \cos \varphi \cos \theta d\varphi + \cos \varphi \omega^3_4 - \sin \varphi \sin \omega^3_5) \otimes e_3
\]

\[
+ (\sin \varphi \cos \theta d\theta + \cos \varphi \sin \theta d\varphi + \cos \varphi \omega^4_4 + \sin \varphi \cos \omega^4_3) \otimes e_4
\]

\[
+ (- \sin \varphi d\varphi + \sin \varphi \cos \omega^5_3 + \sin \varphi \sin \omega^5_4) \otimes e_5.
\]
From (1.4) and (2.4), we obtain the first fundamental form of $x$:

$$
\text{d}s^2_x = \langle \text{d}x, \text{d}x \rangle = [\omega^1 \omega^2] \otimes \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \cdot [\omega^1 \omega^2] + (-\sin \varphi \sin \theta \, d\theta + \cos \varphi \cos \theta \, d\varphi + \cos \varphi \omega_5^3 - \sin \varphi \sin \theta \omega_3^4)^2
$$

$$+ (\sin \varphi \cos \theta \, d\theta + \cos \varphi \sin \theta \, d\varphi + \cos \varphi \omega_5^4 + \sin \varphi \cos \theta \omega_3^4)^2$$

$$+ (-\sin \varphi \, d\varphi + \sin \varphi \cos \theta \omega_3^5 + \sin \varphi \sin \theta \omega_2^5)^2,$$

where

$$c_{ij} = \sum_k a_{ik} a_{kj}, \quad a_{ij} = \sin \varphi \cos \theta h_{ij}^3 + \sin \varphi \sin \theta h_{ij}^4 + \cos \varphi h_{ij}^5.$$

Since $g$ is minimal, we have

$$C = A^2 = \begin{bmatrix} a_{11}^2 + a_{12}^2 & 0 \\ 0 & a_{11}^2 + a_{12}^2 \end{bmatrix} = \begin{bmatrix} -\det A & 0 \\ 0 & -\det A \end{bmatrix}.$$

Thus,

$$\text{d}s^2_x = -\det A ((\omega^1)^2 + (\omega^2)^2)$$

$$+ (-\sin \varphi \sin \theta \, d\theta + \cos \varphi \cos \theta \, d\varphi + \cos \varphi \omega_5^3 - \sin \varphi \sin \theta \omega_3^4)^2$$

$$+ (\sin \varphi \cos \theta \, d\theta + \cos \varphi \sin \theta \, d\varphi + \cos \varphi \omega_5^4 + \sin \varphi \cos \theta \omega_3^4)^2$$

$$+ (-\sin \varphi \, d\varphi + \sin \varphi \cos \theta \omega_3^5 + \sin \varphi \sin \theta \omega_2^5)^2.$$  (2.5)

The quadratic form $-\det A ((\omega^1)^2 + (\omega^2)^2)$ is positive definite if and only if $\det C \neq 0$.

Since

$$\det C = \left( (\sin \varphi \cos \theta h_{11}^3 + \sin \varphi \sin \theta h_{11}^4 + \cos \varphi h_{11}^5)^2 + (\sin \varphi \cos \theta h_{12}^3 + \sin \varphi \sin \theta h_{12}^4 + \cos \varphi h_{12}^5)^2 \right)^2,$$

we get

$$\det C = 0 \iff \begin{bmatrix} h_{11}^3 & h_{11}^4 & h_{11}^5 \\ h_{12}^3 & h_{12}^4 & h_{12}^5 \end{bmatrix} \cdot \begin{bmatrix} \sin \varphi \cos \theta \\ \sin \varphi \sin \theta \\ \cos \varphi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$  (2.6)

Since the second fundamental form is non-zero by hypothesis, there are two possibilities at each point $p$: the rank of $(h_{ij}^\alpha)_{3 \times 2}$ is either 2 or 1. We claim that in both cases $\det C \neq 0$ for all points $(p, \theta, \varphi) \in N_*(p)$.

(i) Suppose that $\text{rank}((h_{ij}^\alpha)_{3 \times 2}) = 2$. In this case, we may choose a particular orthonormal local frame $\{e_1, e_2, e_3, e_4, e_5\}$ such that the matrix $(h_{ij}^\alpha)_{3 \times 2}$ takes the form

$$\begin{bmatrix} \alpha & \beta \\ 0 & \gamma \\ 0 & 0 \end{bmatrix}.$$
where $\alpha \gamma \neq 0$. In fact, starting from an orthonormal frame $\{e_1, e_2, e_3, e_4, e_5\}$ with a generic matrix $\left(h_{ij}^\alpha\right)$, we define
\[
\tilde{e}_3 = h_{11}^3 e_3 + h_{11}^4 e_4 + h_{11}^5 e_5, \\
\tilde{e}_4 = h_{12}^3 e_3 + h_{12}^4 e_4 + h_{12}^5 e_5.
\]
One can apply the Gram–Schmidt process to these two vectors and obtain vectors $\{\tilde{e}_3, \tilde{e}_4\}$. Then we choose $\tilde{e}_5$ such that $\{e_1, e_2, \tilde{e}_3, \tilde{e}_4, \tilde{e}_5\}$ is orthonormal. This is the required basis. It is easy to verify that $\alpha = \|\tilde{e}_3\|$, $\alpha \beta = \langle \tilde{e}_3, \tilde{e}_4 \rangle$ and $\beta^2 + \gamma^2 = \|\tilde{e}_4\|^2$.

For such a frame, the equivalence (2.6) reduces to
\[
\det C = 0 \iff \begin{cases} 
\alpha \sin \varphi \cos \theta = 0, \\
\beta \sin \varphi \cos \theta + \gamma \sin \varphi \sin \theta = 0.
\end{cases}
\]
Since $\alpha \neq 0$ and $0 < \varphi < \pi$, we have $\det C \neq 0$ for all points $(p, \theta, \varphi)$ of the set $N_\ast (p)$ defined in (2.2).

(ii) Suppose that rank($\left(h_{ij}^\alpha\right)_{3 \times 2}$) = 1. We may choose a particular orthonormal local frame $\{e_1, e_2, e_3, e_4, e_5\}$ such that the matrix $\left(h_{ij}^\alpha(p)\right)_{3 \times 2}$ at $p$ takes the form
\[
\begin{bmatrix} 
\alpha & \beta \\
0 & 0 \\
0 & 0
\end{bmatrix},
\]
where $\alpha^2 + \beta^2 \neq 0$. In this case, the system (2.6) takes the form
\[
\det C = 0 \iff \begin{cases} 
\alpha \sin \varphi \cos \theta = 0, \\
\beta \sin \varphi \cos \theta = 0.
\end{cases}
\]
Since $\alpha^2 + \beta^2 \neq 0$ and $0 < \varphi < \pi$, we have $\det C \neq 0$ if and only if $\theta$ is different from $\pi/2$ and $3\pi/2$. Then $\det C \neq 0$ for all points $(p, \theta, \varphi)$ of the set $N_\ast (p)$ defined in (2.2).

Let
\[
w = a e_1 + b e_2 + c \frac{\partial}{\partial \theta} + d \frac{\partial}{\partial \varphi}, \quad a, b, c, d \in \mathbb{R},
\]
be a vector in $T_pU \times T_pW$. The 1-forms $\omega_\alpha^\beta$ are the connection forms of the normal bundle of $g$ and $\omega_\beta^\alpha(\partial/\partial \theta)$, $\omega_\beta^\alpha(\partial/\partial \varphi)$ are equal to zero. Suppose that
\[
dx_{(p, \theta, \varphi)}(w) = 0.
\]
Since $d\mathbf{s}_x^2(w) = 0$ and $\det C \neq 0$, we obtain from (2.5) that $a = b = 0$ while $c$ and $d$ must satisfy
\[
\begin{bmatrix} 
-\sin \varphi \sin \theta & \cos \varphi \cos \theta \\
\sin \varphi \cos \theta & \cos \varphi \sin \theta \\
0 & -\sin \varphi
\end{bmatrix} \begin{bmatrix} c \\
d
\end{bmatrix} = \begin{bmatrix} 0 \\
0 \\
0
\end{bmatrix}.
\]
Since $0 < \varphi < \pi$, the matrix of this system has rank 2, whence necessarily $c = d = 0$. Thus,
\[
dx_{(p, \theta, \varphi)}(w) = 0 \Rightarrow w = 0.
\]
Therefore $x_g$ is an immersion.
Let \( \{e_1, e_2, \partial/\partial \theta, \partial/\partial \varphi, g \circ \pi_1\} \) be an orthogonal frame adapted to the immersion \( x_g \). Note that \( g \circ \pi_1(p, \theta, \varphi) \) is normal to \( T_{x(p, \theta, \varphi)}N_* \) and tangent to \( S^5 \). The second fundamental form of \( x_g \) is given by
\[
\Pi_{x_g} = -\langle dx, d(g \circ \pi_1) \rangle = -\langle dx, \omega^1 \otimes e_1 + \omega^2 \otimes e_2 \rangle
= a_{11} \omega^1 \otimes \omega^1 + 2a_{12} \omega^1 \otimes \omega^2 + a_{22} \omega^2 \otimes \omega^2.
\]
Since \( g \) is minimal, the trace of the matrix \( A = (a_{ij}) \) is zero. Thus the trace of the matrix of \( \Pi_{x_g} \) is also zero. Therefore \( x_g : N_* \to S^5 \) is a minimal immersion. Since the rank of \( A \) is 2, the 3-mean curvature and 4-mean curvature vanish. □

\[\] 

\textbf{§ 3. Minimal hypersurfaces in } S^5 \textit{ with zero Gauss–Kronecker curvature and zero 3-mean curvature}

In this section we shall study the local structure of complete minimal hypersurfaces in \( S^5 \) with zero Gauss–Kronecker curvature, zero 3-mean curvature and nowhere-zero second fundamental form. First we need the following two lemmas, which will be used in the particular case when \( W^5 = S^5 \).

\textbf{Lemma 3.1.} Let \( g : M^4 \to W^5 \) be an immersed hypersurface in a 5-dimensional manifold \( W^5 \) and let \( \{e_1, e_2, e_3, e_4, e_5\} \) be an orthonormal local frame adapted to \( g \) with dual coframe \( \{\omega^1, \omega^2, \omega^3, \omega^4, \omega^5\} \). If \( \{e_1, e_2, \bar{e}_3, \bar{e}_4, e_5\} \) is another orthonormal frame with dual coframe \( \{\bar{\omega}^1, \bar{\omega}^2, \bar{\omega}^3, \bar{\omega}^4, \bar{\omega}^5\} \) such that
\[
\begin{bmatrix}
\bar{e}_3 \\
\bar{e}_4
\end{bmatrix} = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix} \begin{bmatrix}
e_3 \\
e_4
\end{bmatrix},
\]
then
\[
\begin{bmatrix}
\bar{\omega}^3 \\
\bar{\omega}^4
\end{bmatrix} = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix} \begin{bmatrix}
\omega^3 \\
\omega^4
\end{bmatrix},
\]
\[
\begin{bmatrix}
\bar{\omega}^1 \\
\bar{\omega}^2
\end{bmatrix} = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix} \begin{bmatrix}
\omega^1 \\
\omega^2
\end{bmatrix},
\]
\[
\bar{\omega}^2 = \omega^2, \quad \bar{\omega}^3 = \omega^3 + d\theta, \quad \bar{\omega}^5 = \omega^5.
\]

\textbf{Proof.} Taking the exterior differential of
\[
\begin{cases}
\omega^3 = \cos \theta \bar{\omega}^3 + \sin \theta \bar{\omega}^4, \\
\omega^4 = -\sin \theta \bar{\omega}^3 + \cos \theta \bar{\omega}^4,
\end{cases}
\]
and using (1.8), we obtain
\[
d\bar{\omega}^3 = -(\cos \theta \omega^3 - \sin \theta \omega^4) \wedge \omega^1 - (\cos \theta \omega^3 - \sin \theta \omega^4) \wedge \omega^2 - (\omega^4 + d\theta) \wedge \bar{\omega}^4,
\]
\[
d\bar{\omega}^4 = -(\sin \theta \omega^3 + \cos \theta \omega^4) \wedge \omega^1 - (\sin \theta \omega^3 + \cos \theta \omega^4) \wedge \omega^2 - (\omega^4 + d\theta) \wedge \bar{\omega}^3.
\]
The lemma is proved. □
Lemma 3.2. Under the hypotheses above, define functions \( f_3, f_4, g_3, g_4 \) by the formulae
\[
\begin{align*}
\omega_2' &= f_3 \omega^1 + g_3 \omega^2, \\
\omega_4' &= f_4 \omega^1 + g_4 \omega^2.
\end{align*}
\]
(3.1)

Then
\[
\begin{bmatrix}
\tilde{f}_3 \\
\tilde{f}_4
\end{bmatrix}
=
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
f_3 \\
g_3
\end{bmatrix}
\]
and
\[
\begin{align*}
\tilde{f}_3^2 + \tilde{f}_4^2 &= f_3^2 + f_4^2, \\
\tilde{g}_3^2 + \tilde{g}_4^2 &= g_3^2 + g_4^2, \\
\tilde{f}_3 \tilde{g}_3 + \tilde{f}_4 \tilde{g}_4 &= f_3 g_3 + f_4 g_4.
\end{align*}
\]

Proof. Since
\[
\begin{align*}
e_3 &= \cos \theta \tilde{e}_3 + \sin \theta \tilde{e}_4, \\
e_4 &= -\sin \theta \tilde{e}_3 + \cos \theta \tilde{e}_4,
\end{align*}
\]
it follows from (3.1) that
\[
(\nabla e_2) = \omega_2' \otimes e_1 + \omega_3' \otimes e_3 + \omega_4' \otimes e_4
\]
\[
= \omega_2' \otimes e_1 + [(\cos \theta f_3 - \sin \theta f_4) \omega^1 + (\cos \theta g_3 - \sin \theta g_4) \omega^2] \otimes \tilde{e}_3
\]
\[
+ [(\sin \theta f_3 + \cos \theta f_4) \omega^1 + (\sin \theta g_3 + \cos \theta g_4) \omega^2 t] \otimes \tilde{e}_4,
\]
where \( \nabla \) is the Levi-Civita connection of \( M^4 \). □

Theorem 3.3. Let \( g: M^4 \to S^5 \) be a complete oriented minimal immersed hypersurface in \( S^5 \) with zero Gauss–Kronecker curvature and zero 3-mean curvature. If the square \( S \) of the length of the second fundamental form is nowhere zero, then one can find a minimal immersion \( \tilde{\eta}: \mathbb{V}^2 \to S^5 \) of some 2-dimensional surface \( \mathbb{V}^2 \) and a local isometry \( \tau: M^4 \to N^5 \), where \( M^4 = M^4 \setminus \{g^{-1}(B_\pi(p)(T_\pi(p)\mathbb{V}) \perp \tilde{\eta}_*(T_\pi(p)\mathbb{V}) \perp) \} \), such that \( \eta|_{M^4} = \tilde{\eta} \circ \pi \circ \tau \) and \( x_\tau \circ \tau = g|_{M^4} \).

Proof. Let \( B = \{e_1, e_2, e_3, e_4, e_5\} \) be an orthonormal frame on \( S^5 \) adapted to the immersion \( g: M^4 \to S^5 \) in the sense that \( \{e_1, \ldots, e_4\} \) spans \( g_*(TM) \), \( e_5 \) determines a global Gauss map \( \eta: M^4 \to S^5 \) and, as described in (1.7),
\[
\omega_i = \lambda_i \omega, \quad i = 1, 2, 3, 4.
\]
By hypothesis and (1.10), exactly two of the numbers \( \lambda_i \) are equal to zero and the sum of the other two is zero. Hence we may assume that
\[
\lambda = \lambda_1 = -\lambda_2 > 0, \quad \lambda_3 = \lambda_4 = 0
\]
and
\[
\omega_1 = \lambda \omega^1, \quad \omega_2 = -\lambda \omega^2, \quad \omega_3 = \omega_4 = 0.
\]
(3.2)
In this case we have two directions and one plane well defined. Since \( S > 0 \) on \( M^4 \), we see that
\[
D_p := \{v \in T_pM^4: B_p(v, w) = 0 \forall w \in T_pM^4\}
\]
is a 2-dimensional distribution.
We now proceed to show that \( \mathcal{D}_p \) is involutive, its integral surfaces \( \mathcal{F}^2 \) are totally geodesic in \( M^4 \), and the surfaces \( g(\mathcal{F}^2) \) are totally geodesic in \( S^5 \). It follows from (3.2) and the Codazzi equations in (1.9) that
\[
\left\{
\begin{array}{l}
\omega_1^3 \wedge \omega^1 - \omega_2^3 \wedge \omega^2 = 0, \\
\omega_1^4 \wedge \omega^1 - \omega_2^4 \wedge \omega^2 = 0.
\end{array}
\right.
\] (3.3)

Let \( \mathcal{I}(\omega^1, \omega^2) \) be the ideal generated by \( \omega^1 \) and \( \omega^2 \). By Cartan’s lemma, \( \omega_1^3, \omega_2^3, \omega_1^4 \) and \( \omega_2^4 \) belong to the ideal \( \mathcal{I}(\omega^1, \omega^2) \). It follows from (1.8) that
\[
d\omega^i \in \mathcal{I}(\omega^1, \omega^2), \quad i = 1, 2, \quad \text{or} \quad \omega^i([e_3, e_4]) = 0, \quad i = 1, 2.
\]

Therefore, \( \mathcal{D}_p \) is involutive. Then, by Frobenius’ theorem, there is a unique maximal connected integral surface \( \mathcal{F}^2 \) of \( \mathcal{D}_p \) through \( p \). Since \( g: M^4 \to S^5 \) is an immersion of a complete manifold, it follows from a theorem of Ferus [13] that \( \mathcal{F}^2 \) is complete and totally geodesic in \( M^4 \). We easily see from (3.3) that
\[
\left\{
\begin{array}{l}
\omega^3_2(e_1) + \omega^3_1(e_2) = 0, \\
\omega^4_2(e_1) + \omega^4_1(e_2) = 0, \\
\omega^i_j(e_k) = 0, \quad i = 1, 2, \quad j, k = 3, 4.
\end{array}
\right.
\] (3.4)

Substituting (3.2) and (3.4) into (1.8), we obtain
\[
\begin{align*}
\text{de}_3(e_3) &= -g + \omega^3_3(e_3)e_4, \\
\text{de}_3(e_4) &= \omega^3_3(e_4)e_3, \\
\text{de}_4(e_3) &= \omega^4_3(e_3)e_3, \\
\text{de}_4(e_4) &= -g + \omega^4_3(e_4)e_3.
\end{align*}
\] (3.5)

This implies that
\[
\omega^k(\nabla_{e_j} e_i) = 0, \quad k = 1, 2, \quad i, j = 3, 4,
\]
where \( \nabla \) is the Levi-Civita connection on \( S^5 \). Therefore, \( g(\mathcal{F}^2) \) is totally geodesic in \( S^5 \). Since \( \mathcal{F}^2 \) is complete and \( g|_{\mathcal{F}^2}: \mathcal{F}^2 \to S^5 \) is an isometric immersion, we see that \( g(\mathcal{F}^2) \) is a unit 2-sphere \( S^2 \) in \( S^5 \). Thus, we have that \( g|_{\mathcal{F}^2}: \mathcal{F}^2 \to S^2 \) is a covering map (see [14], p. 146, Proposition 6.16). Thus, \( \mathcal{F}^2 \) is a 2-sphere in \( M^4 \) and \( g|_{\mathcal{F}^2}: \mathcal{F}^2 \to S^2 \) is a diffeomorphism (see [14], p. 141). Therefore, the maximal integral surface \( \mathcal{F}^2 \) is regular (see [16], p. 98) and it follows from a theorem of Palais (see [16]) that the quotient space
\[
\mathbb{V}^2 = M^4 \setminus \mathcal{F}^2
\]
can be endowed with the structure of a 2-dimensional differentiable manifold such that \( \pi: M^4 \to \mathbb{V}^2 \) is a submersion. The Gauss map \( \eta: M^4 \to S^5 \) induces a smooth map \( \tilde{\eta}: \mathbb{V}^2 \to S^5 \) such that \( \tilde{\eta} \circ \pi = \eta \).
In fact, from (3.2), we have that \( \nabla_{e_3} \eta \) and \( \nabla_{e_4} \eta \) vanish. Thus, \( \eta \) is constant along the integral surfaces \( \mathcal{F}^2 \) and \( \tilde{\eta} \) is well defined. Let \( S \) be a smooth transversal surface to the leaf \( \mathcal{F}^2 \) of \( \mathcal{D}_p \) through a point \( p \in M^4 \) such that \( T_p S = \text{span}\{e_1|_p, e_2|_p\} \) and \( T_p \mathcal{F} = \text{span}\{e_3|_p, e_4|_p\} \). Since \( \pi: M^4 \to \mathbb{V}^2 \) is a submersion, we have

\[
\text{span}\{d\pi_p(e_1|_p), d\pi_p(e_2|_p)\} = T_{\pi(p)}\mathbb{V}^2.
\]

The third formula given in (1.8) implies that

\[
d\tilde{\eta}_{\pi(p)}(d\pi_p(e_1|_p)) = d(\tilde{\eta} \circ \pi)_p(e_1|_p) = d\eta_{p}(e_1|_p) = -\lambda(p)e_1|_p,
\]

\[
d\tilde{\eta}_{\pi(p)}(d\pi_p(e_2|_p)) = \lambda(p)e_2|_p.
\]

Thus, the first fundamental form of \( \tilde{\eta} \) is given by

\[
ds_{\tilde{\eta}}^2 = \lambda^2(\omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2).
\]

Since \( \lambda(p) > 0 \) for all \( p \in M^4 \), the map \( \tilde{\eta}: \mathbb{V}^2 \to S^5 \) is an immersion. We now claim that \( \tilde{\eta} \) is a minimal immersion. Let \( \{X_1, X_2\} = \left\{d\pi_p\left(\frac{1}{\lambda}e_1|_p\right), d\pi_p\left(\frac{1}{\lambda}e_2|_p\right)\right\} \) be an orthonormal basis of \( T_{\pi(p)}\mathbb{V}^2 \) and let \( \{X_3, X_4, X_5\} \) be an orthogonal frame in the normal bundle of \( \tilde{\eta} \) such that

\[
X_3 \circ \pi|_S = e_3|_S, \quad X_4 \circ \pi|_S = e_4|_S, \quad X_5 \circ \pi|_S = g|_S.
\]

The second fundamental form \( B_{\pi(p)} \) of \( \tilde{\eta} \) is determined by the bilinear forms

\[
\Pi_{\tilde{\eta}}^\alpha := -\langle d\tilde{\eta}, dX_\alpha|_{\pi(p)} \rangle, \quad \alpha = 3, 4, 5.
\]

We easily see from (1.8) that

\[
\Pi_{\tilde{\eta}}^3(X_i \otimes X_j) = -\frac{1}{\lambda^2} \langle d\eta, de_3\rangle(e_i|_p \otimes e_j|_p)
\]

\[
= \frac{1}{\lambda}(\omega^1 \otimes \omega^3_1 - \omega^2 \otimes \omega^3_2)(e_i|_p \otimes e_j|_p),
\]

\[
\Pi_{\tilde{\eta}}^4(X_i \otimes X_j) = \frac{1}{\lambda}(\omega^1 \otimes \omega^4_1 - \omega^2 \otimes \omega^4_2)(e_i|_p \otimes e_j|_p),
\]

\[
\Pi_{\tilde{\eta}}^5(X_i \otimes X_j) = \frac{1}{\lambda}(\omega^1 \otimes \omega^1 - \omega^2 \otimes \omega^2)(e_i|_p \otimes e_j|_p), \quad i, j = 1, 2.
\]

Denote by \( \tilde{A}^\alpha \) the shape operators of \( \tilde{\eta} \) at \( \pi(p) \). Then

\[
\tilde{A}^3 = \frac{1}{\lambda(p)}\begin{bmatrix}
\omega^3_1(e_1) & \omega^3_1(e_2) \\
-\omega^3_2(e_1) & -\omega^3_2(e_2)
\end{bmatrix}, \quad \tilde{A}^4 = \frac{1}{\lambda(p)}\begin{bmatrix}
\omega^4_1(e_1) & \omega^4_1(e_2) \\
-\omega^4_2(e_1) & -\omega^4_2(e_2)
\end{bmatrix},
\]

\[
\tilde{A}^5 = \frac{1}{\lambda(p)}\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}.
\]
Hence \( \tilde{\eta} \) is a minimal immersion if and only if
\[
\begin{cases}
\omega_3^1(e_1) - \omega_3^2(e_2) = 0,
\omega_4^1(e_1) - \omega_4^2(e_2) = 0.
\end{cases}
\tag{3.6}
\]
By taking the exterior differential of (3.2) we have
\[
d\omega_i^5 = d\lambda_i \wedge \omega_i - \lambda_i \omega_i^k \wedge \omega_k.
\]
It follows from the Codazzi equations given in (1.9) that
\[
d\lambda_i \wedge \omega_i + (\lambda_k - \lambda_i) \omega_i^k \wedge \omega_k = 0, \quad i = 1, 2.
\tag{3.7}
\]
Evaluating these equations on \( e_k \otimes e_i \), we obtain
\[
\begin{cases}
e_1[\lambda] - 2\lambda \omega_2^1(e_2) = 0,
e_2[\lambda] + 2\lambda \omega_2^1(e_1) = 0,
e_3[\lambda] + \lambda \omega_3^2(e_1) = 0, \quad e_4[\lambda] + \lambda \omega_3^2(e_2) = 0,
2\omega_2^1(e_3) + \omega_3^2(e_1) = 0, \quad 2\omega_2^1(e_4) + \omega_3^2(e_1) = 0.
\end{cases}
\tag{3.8}
\]
Therefore, the equations (3.9) imply that the condition (3.6) is satisfied. Thus, introducing the notation
\[
f_3 := \omega_2^3(e_1), \quad f_4 := \omega_2^4(e_1),
g_3 := \omega_3^3(e_2), \quad g_4 := \omega_3^4(e_2),
\tag{3.11}
\]
and using (3.1), (3.4) and (3.6), we may write
\[
\tilde{A}^3 = \frac{1}{\lambda(p)} \begin{bmatrix}
g_3 & f_3 \\
f_3 & g_3
\end{bmatrix}, \quad \tilde{A}^4 = \frac{1}{\lambda(p)} \begin{bmatrix}
g_4 & f_4 \\
f_4 & g_4
\end{bmatrix},
\]
\[
\tilde{A}^5 = \frac{1}{\lambda(p)} \begin{bmatrix}1 & 0 \\0 & -1
\end{bmatrix}.
\]
From (3.4), (3.8), (3.9) and (3.10), we have the skew-symmetric matrices
\[
(\omega^i_j(e_1)) = \begin{bmatrix}0 & -\frac{1}{2}e_2[\ln \lambda] & -g_3 & -g_4 \\
* & 0 & -f_3 & -f_4 \\
* & * & 0 & \omega_3^2(e_1) \\
* & * & * & 0
\end{bmatrix},
\]
\[
(\omega^i_j(e_2)) = \begin{bmatrix}0 & \frac{1}{2}e_1[\ln \lambda] & f_3 & f_4 \\
* & 0 & -g_3 & -g_4 \\
* & * & 0 & \omega_3^2(e_2) \\
* & * & * & 0
\end{bmatrix},
\]
\[
(\omega^i_j(e_3)) = \begin{bmatrix}0 & \frac{1}{2}f_3 & 0 & 0 \\
* & 0 & 0 & 0 \\
* & * & \omega_3^2(e_3) & 0 \\
* & * & * & 0
\end{bmatrix},
(\omega^i_j(e_4)) = \begin{bmatrix}0 & \frac{1}{2}f_4 & 0 & 0 \\
* & 0 & 0 & 0 \\
* & * & 0 & \omega_3^2(e_4) \\
* & * & * & 0
\end{bmatrix},
\]
and the relations
\[ \begin{align*}
    e_3[\lambda] &= \lambda g_3, \\
    e_4[\lambda] &= \lambda g_4.
\end{align*} \tag{3.12} \]

Using the equations given in (1.13), we have that \( K \) and \( K_N \) of the minimal immersion \( \tilde{\eta}: \mathbb{V}^2 \to S^5 \) are given by
\[ \begin{align*}
    K(\pi(p)) &= 1 - \frac{1}{\lambda^2}(1 + f_3^2 + f_4^2 + g_3^2 + g_4^2), \tag{3.13} \\
    R_{412}^3 &= \frac{2}{\lambda^2}(f_3 g_4 - f_4 g_3), \quad R_{512}^3 = -\frac{2f_3}{\lambda^2}, \quad R_{512}^4 = -\frac{2f_4}{\lambda^2}, \tag{3.14} \\
    K_N(\pi(p)) &= \frac{16}{\lambda^4}(f_3^2 + f_4^2 + (f_3 g_4 - f_4 g_3)^2). \tag{3.15}
\end{align*} \]

By a straightforward calculation it follows that
\[ \begin{align*}
    [e_1, e_2] &= -\frac{1}{2}e_2[[\ln \lambda]e_1 + \frac{1}{2}e_1[[\ln \lambda]e_2 + 2f_3 e_3 + 2f_4 e_4, \\
    [e_3, e_1] &= g_3 e_1 + \frac{f_3}{2} e_2 + \omega_4^3(e_1) e_4, \\
    [e_3, e_2] &= -\frac{f_3}{2} e_1 + g_3 e_2 + \omega_4^3(e_2) e_4, \\
    [e_4, e_1] &= g_4 e_1 + \frac{f_4}{2} e_2 - \omega_4^3(e_1) e_3, \\
    [e_4, e_2] &= -\frac{f_4}{2} e_1 + g_4 e_2 - \omega_4^3(e_2) e_3, \\
    [e_3, e_4] &= \omega_4^3(e_3) e_3 + \omega_4^3(e_4) e_4.
\end{align*} \tag{3.16} \]

Applying the relation between the bracket operation on vector fields and the exterior differentiation of 1-forms, we obtain the derivatives of the functions \( f_i \) and \( g_i \):
\[ \begin{align*}
    e_1[f_3] &= -e_2[g_3] - \omega_4^3(e_1)f_4 - \omega_4^3(e_2)g_4, \\
    e_2[f_3] &= e_1[g_3] + \omega_4^3(e_1)g_4 - \omega_4^3(e_2)f_4, \\
    e_1[f_4] &= -e_2[g_4] + \omega_4^3(e_1)f_3 + \omega_4^3(e_2)g_3, \\
    e_2[f_4] &= e_1[g_4] - \omega_4^3(e_1)g_3 + \omega_4^3(e_2)f_3, \\
    e_3[f_3] &= 2f_3 g_3 - \omega_4^3(e_3)f_4, \\
    e_3[g_3] &= g_3^2 - f_3^2 + 1 - \omega_4^3(e_3)g_4, \\
    e_4[f_3] &= f_3 g_4 + f_4 g_3 - \omega_4^3(e_4)f_4, \\
    e_4[g_3] &= g_3 g_4 - f_3 f_4 - \omega_4^3(e_4)g_4.
\end{align*} \tag{3.17} \]

By a straightforward calculation, using (3.18) and (3.19), we have
\[ \begin{align*}
    e_3[R_{412}^3] &= R_{512}^4, \\
    e_4[R_{412}^3] &= -R_{512}^3, \\
    e_3[R_{512}^3] &= -\omega_4^3(e_3)R_{512}^4, \\
    e_4[R_{512}^3] &= R_{412}^3 - \omega_4^3(e_4)R_{512}^4, \\
    e_3[R_{512}^4] &= -R_{412}^3 + \omega_4^3(e_3)R_{512}^3, \\
    e_4[R_{512}^4] &= \omega_4^3(e_4)R_{512}^3.
\end{align*} \tag{3.20} \]
It follows from these equations that
\[ e_3[K] = 0, \quad e_4[K] = 0, \quad e_3[K_N] = 0, \quad e_4[K_N] = 0. \]

Therefore, the functions \( K \) and \( K_N \) are constant on \( F^2 \). This is in accordance with the fact that \( \tilde{\eta} \) is well defined.

By Theorem 2.2, \( x_{\tilde{\eta}} : N_* \to S^5 \) is an immersed minimal hypersurface in \( S^5 \) with zero Gauss–Kronecker curvature and zero 3-mean curvature.

Define the map \( \tau : M^4_* \to N_* \) by \( \tau(p) = (\pi(p), g(p)) \). Then \( x_{\tilde{\eta}} \circ \tau(p) = g(p) \).
Since \( \pi = \pi_1 \circ \tau \), we have \( \eta|_{M^4_*} = \tilde{\eta} \circ \pi_1 \circ \tau \). Finally, the metric on \( N_* \) is induced by \( x_{\tilde{\eta}} \), whence \( \tau \) must be a local isometry. \( \square \)

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