EQUATIONAL CONDITIONS IN UNIVERSAL ALGEBRAIC GEOMETRY

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Abstract. In this article, the properties of being equational noetherian, $q_\omega$ and $u_\omega$-compactness, and equational Artinian are studied from the perspective of the Zariski topology. The equational conditions on the relative free algebras of arbitrary varieties are also investigated and their relations to some logic and model theory notions are obtained.

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1. Introduction

Universal algebraic geometry is a new area of modern algebra, whose subject is basically the study of equations over an arbitrary algebraic structure \( A \). In the classical algebraic geometry \( A \) is a field. Many articles already published about algebraic geometry over groups, see [2], [3], [4], [14], and [16]. In an outstanding series of papers, Z. Sela, developed algebraic geometry over free groups to give affirmative answers for old problem of Alfred Tarski concerning elementary theory of free groups (see [18] and also [12] for the independent solution of Kharlampovich-Miyasnikov). Also in [13], a problem of Tarski about decidability of the elementary theory of free groups are solved (see also [19]). Algebraic geometry over algebraic structures is also developed for algebras other than groups, for example there are results about algebraic geometry over Lie algebras and monoids, see [10], [15], and [21]. Systematic study of universal algebraic geometry is done in a series of articles by V. Remeslennikov, A. Myasnikov and E. Daniyarova in [6], [7], [8], and [9].

In this article, we are dealing with the equational conditions in the universal algebraic geometry, i.e. different conditions relating systems of equations especially conditions about systems and sub-systems of equations over algebras. The main examples of such conditions are equational noetherian property and its variants (weak equational noetherianity, \( n \)-equational noetherianity, \( q_\omega \) and \( u_\omega \)-compactness), as well as equational Artinian property of algebras. We begin with a review of basic concepts of universal algebraic geometry and we describe the relation between the properties of being equational noetherian, \( q_\omega \) and \( u_\omega \)-compactness and compactness of algebraic sets in the Zariski topology. We also discuss the concept of a meta-compact algebra and
its relation to the meta-compactness of the sets in the Zariski topology. Then we introduce the notion of equational Artinian algebras and we show that it is related to the notion of $\omega$-cocompactness. We prove that the (non-finitely generated) relative free algebras of certain varieties, are $u_\omega$-compact and the we provide some necessary conditions for a relative free algebra to be equational noetherian. We also show that the equational noetherianity of some relatively free algebras has interesting logical implications for certain subclasses of the corresponding variety. After defining the notion of relative systems of equations, we show that ascending (descending) chain conditions on the ideals of the relative free algebra is equivalent to equational noetherian (equational Artinian) property of all elements of the corresponding variety. A set of other types of interesting equational conditions and some questions is also presented at the end of the article.

2. Basic notions

This section is devoted to a fast review of the basic concepts of the universal algebraic geometry. We suggest [5], [11] and [17] for reader who is not familiar to the universal algebra. The reader also would use [6], [7], [8], and [9], for extended exposition of the universal algebraic geometry. Our notations here are almost the same as in the above mentioned papers. Many results of this work can be stated for structures over any first order language, but for the sake of simplicity, we restrict ourself for the case of algebraic languages.

2.1. Systems of equations and algebraic sets. Suppose $\mathcal{L}$ is an arbitrary algebraic language and $A$ is a fixed algebra of type $\mathcal{L}$. The extended language will be denoted by $\mathcal{L}(A)$ and it is obtained from $\mathcal{L}$ by adding new constant symbols $a \in A$. An algebra $B$ of type $\mathcal{L}(A)$ is called $A$-algebra, if the map $a \mapsto a^B$ is an embedding of $A$ in $B$. Note that here, $a^B$ denotes the interpretation of the constant symbol $a$ in $B$. We assume that $X = \{x_1, \ldots, x_n\}$ is a finite set of variables. We denote the term algebra in the language $\mathcal{L}$ and variables from $X$ by $T_\mathcal{L}(X)$, and similarly the term algebra in the extended language $\mathcal{L}(A)$ will denoted by $T_{\mathcal{L}(A)}(X)$. For the sake of the simplicity, we define our notions in the coefficient free frame, i.e. in the language $\mathcal{L}$ and then we can extend all the definitions to the language $\mathcal{L}(A)$.

Fix an algebraic language $\mathcal{L}$ and a set of variables $X = \{x_1, \ldots, x_n\}$. An equation is a pair $(p, q)$ of the elements of the term algebra $T_\mathcal{L}(X)$. In many cases, we assume that such an equation is the same as the atomic formula $p(x_1, \ldots, x_n) \approx q(x_1, \ldots, x_n)$ or $p \approx q$ in short. Hence,
in this article the set $\text{At}_L(X)$ of atomic formulae in the language $L$ and the product algebra $T_L(X) \times T_L(X)$ are assumed to be equal.

Any subset $S \subseteq \text{At}_L(X)$ is called a system of equations in the language $L$. A system $S$ is called consistent, over an algebra $A$ if there is an element $(a_1, \ldots, a_n) \in A^n$ such that for all equations $(p \approx q) \in S$, the equality

$$p^A(a_1, \ldots, a_n) = q^A(a_1, \ldots, a_n)$$

holds. Otherwise, we say that $S$ is in-consistence over $A$. Note that, $p^A$ and $q^A$ are the corresponding term functions on $A^n$. A system of equations $S$ is called an ideal, if it corresponds to a congruence on $T_L(X)$. For an arbitrary system of equations $S$, the ideal generated by $S$, is the smallest congruence containing $S$ and it is denoted by $[S]$.

For an algebra $A$ of type $L$, an element $(a_1, \ldots, a_n) \in A^n$ will be denoted by $\overline{a}$, some times. Let $S$ be a system of equations. Then the set

$$V_A(S) = \{\overline{a} \in A^n : \forall (p \approx q) \in S, \; p^A(\overline{a}) = q^A(\overline{a})\}$$

is called an algebraic set. It is clear that for any non-empty family $\{S_i\}_{i \in I}$, we have

$$V_A(\bigcup_{i \in I} S_i) = \bigcap_{i \in I} V_A(S_i).$$

So, we define a closed set in $A^n$ to be empty set or any finite union of algebraic sets. Therefore, we obtain a topology on $A^n$, which is called Zariski topology. For a subset $Y \subseteq A^n$, its closure with respect to Zariski topology is denoted by $\overline{Y}$. We also denote by $Y^{ac}$ the smallest algebraic set containing $Y$. In general $\overline{Y} = Y^{ac}$ is not true. We say that an algebra $A$ is equational domain, if the union of two algebraic set in $A^n$ is again an algebraic set for any $n$. In this case, clearly we have the equality $\overline{Y} = Y^{ac}$. In the next section, we will prove some of our results in the case of equational domains, therefore we should mention here that there is a long list of known examples of such algebras. For details, the reader can see [9], where there are also some interesting criteria for an algebra to be equational domain.

Similarly, we can work with systems of equations with coefficients from a fixed algebra $A$. To do this, let $A$ be an algebra of type $L$ and consider a pair of terms $(p, q)$ in the extended language $L(A)$. Then we call the atomic formula $p \approx q$ an equation with coefficient from $A$. We can define a system of equations with coefficients from $A$, so for any such system $S$ and any $A$-algebra $B$, we can define the algebraic set $V_B(S)$. Other notions may also be defined in a similar manner. Note that some examples of equational domain appear in the recent case, for example, as it is proved in [9], no non-trivial group is equational.
domain in the language $\mathcal{L} = (1, -1, \cdot)$ of groups, but any non-abelian free group $F$ is equational domain in the language $\mathcal{L}(F)$.

Almost all results of the next section may also stated for the general case of $A$-algebras and systems in the extended language $\mathcal{L}(A)$, but for the sake of simplicity, we restrict ourself to the coefficient free case.

2.2. Radicals and coordinate algebras. For any set $Y \subseteq A^n$, we define

$$\text{Rad}(Y) = \{(p, q) : \forall \overline{a} \in Y, p^A(\overline{a}) = q^A(\overline{a})\}.$$ 

It is easy to see that $\text{Rad}(Y)$ is an ideal in the term algebra. Any ideal of this type is called a radical ideal. Note that any ideal in the term algebra is in fact a radical ideal. To see the reason, just note that for any ideal $R$ in the term algebra $T_\mathcal{L}(X)$, if we consider the algebra $B(R) = T_\mathcal{L}(X)/R$, then $\text{Rad}_{B(R)}(R) = R$.

It is easy to see that a set $Y$ is algebraic if and only if $V_A(\text{Rad}(Y)) = Y$. In the general case, we have $V_A(\text{Rad}(Y)) = Y^{ac}$, see [7]. The coordinate algebra of a set $Y$ is the quotient algebra

$$\Gamma(Y) = \frac{T_\mathcal{L}(X)}{\text{Rad}(Y)}.$$ 

An arbitrary element of $\Gamma(Y)$ is denoted by $[p]_Y$. We define a function $p^Y : Y \to A$ by the rule

$$p^Y(\overline{a}) = p^A(a_1, \ldots, a_n),$$

which is a term function on $Y$. The set of all such functions will be denoted by $T(Y)$ and it is naturally an algebra of type $\mathcal{L}$. It is easy to see that the map $[p]_Y \mapsto p^Y$ is a well-defined isomorphism. So, we have $\Gamma(Y) \cong T(Y)$.

For a system of equation, we can also define the radical $\text{Rad}_A(S)$ to be $\text{Rad}(V_A(S))$. Two systems $S$ and $S'$ are called equivalent over $A$, if they have the same set of solutions in $A$, i.e. $V_A(S) = V_A(S')$. So, clearly $\text{Rad}_A(S)$ is the largest system which is equivalent to $S$. Note that $[S] \subseteq \text{Rad}_A(S)$.

One of the major problems of the universal algebraic geometry is to determine the structures of algebras which appear as the coordinate algebras. There are many necessary and sufficient conditions for an algebra to be a coordinate algebra and we will give a summary of such results in the subsection 2.4.

2.3. Equational noetherian algebras. In this article, we are dealing with equational conditions on algebras. The first and maybe the most important condition of this type can be formulated as follows.
Definition 1. An algebra $A$ is called equational noetherian, if for any system of equations $S$, there exists a finite subsystem $S_0 \subseteq S$, which is equivalent to $S$ over $A$, i.e. $V_A(S) = V_A(S_0)$.

Note that we can consider a bound on the number of variables appearing in $S$ and obtain a weaker notion of $n$-equational noetherian algebra; an algebra $A$ is $n$-equational noetherian, if for any system $S$ with at most $n$ variables, there exists a subsystem $S_0 \subseteq S$, which is equivalent to $S$ over $A$, i.e. $V_A(S) = V_A(S_0)$.

If an $A$-algebra is equational noetherian in the language $\mathcal{L}(A)$, then we call it $A$-equational noetherian. Many examples of equational noetherian algebras are introduced in [7]. Among them are noetherian rings and linear groups over noetherian rings as well as non-abelian free groups. In [7], it is proved that the next four assertions are equivalent:

i- An algebra $A$ is equational noetherian.

ii- For any system $S$, there exists a finite $S_0 \subseteq [S]$, such that $V_A(S) = V_A(S_0)$.

iii- For any $n$, the Zariski topology on $A^n$ is noetherian, i.e. any descending chain of closed subsets terminates.

iv- Any chain of coordinate algebras and epimorphisms

$$\Gamma(Y_1) \rightarrow \Gamma(Y_2) \rightarrow \Gamma(Y_3) \rightarrow \cdots$$

terminates.

So, in the case of equational noetherian algebras, any closed set in $A^n$ is equal to a minimal finite union of irreducible algebraic sets which is unique up to a permutation. Note that a set is called irreducible, if it has no proper finite covering consisting of closed sets. The following theorem is proved in [7].

Theorem 1. Let $A$ be an equational noetherian algebra. Then the following algebras are also equational noetherian:

i- any subalgebra and filter-power of $A$.

ii- any coordinate algebra over $A$.

iii- any fully residually $A$-algebra.
iv- any algebra belonging to the quasi-variety generated by $A$.

v- any algebra universally equivalent to $A$.

vi- any limit algebra over $A$.

vii- any finitely generated algebra defined by a complete atomic type in the universal theory of $A$ or in the set of quasi-identities of $A$.

We can generalize the concept of equational noetherian algebras by dropping the condition $S_0 \subseteq S$ in the definition 1. More precisely we have:

**Definition 2.** An algebra $A$ is weak equational noetherian, if for any system $S$ there exists a finite system $S_0$, equivalent to $S$ over $A$.

Equivalently, an algebra $A$ is weak equational noetherian, if and only if for any system $S$, the radical ideal $\text{Rad}_A(S)$ is finitely generated, i.e. there exists a finite $S_0 \subseteq \text{Rad}_A(S)$ such that $\text{Rad}_A(S) = \text{Rad}_A(S_0)$. We can explain the logical meaning of this equality as follows. Let $\text{QId}(A)$ be the set of quasi-identities of $A$. Then $A$ is weak equational noetherian iff, for any system of equations $S$, exist finitely many equation $p_1 \approx q_1, \ldots, p_m \approx q_m$ such that

$$\text{Rad}_A(S) = \{(p, q) : (\forall x_1 \ldots \forall x_n (\bigwedge_{i=1}^{m} p_i \approx q_i \Rightarrow p \approx q)) \in \text{QId}(A)\}.$$ 

2.4. **Unification theorems.** Many variants of the unification theorems is proved for universal algebraic geometry in [6], [7] and [8]. The main aim of this type of theorems is to determine which algebras are the coordinate algebra of an algebraic set. In this subsection, we discuss just one of these unification theorems and the reader can consult the above mentioned articles for detailed exposition of notions and proofs.

**Theorem 2.** Let $A$ and $\Gamma$ be algebras in a language $\mathcal{L}$. Suppose $A$ is equational noetherian and $\Gamma$ is finitely generated. Then the following assertions are equivalent.

i- $\Gamma$ is the coordinate algebra of some irreducible algebraic set over $A$.

ii- $\Gamma$ is a fully residually $A$-algebra. This means that for any finite subset $C \subseteq \Gamma$, there exists a homomorphism $\alpha : \Gamma \rightarrow A$, such that the restriction of $\alpha$ to $C$ is injective.
iii- $\Gamma$ embeds into some ultra-power of $A$.

iv- $\Gamma$ belongs to the universal closure of $A$, i.e. $Th_\forall(A) \subseteq Th_\forall(\Gamma)$.

v- $\Gamma$ is a limit algebra over $A$.

vi- $\Gamma$ is defined by a complete type in $Th_\forall(A)$.

There are similar theorems for the cases where $A$ is weak equational noetherian, or it is $q_\omega$-compact or $u_\omega$-compact. See [8] for a detailed discussion.

3. Compactness conditions

In this section we study some of the important equational conditions over algebras by means of the Zariski topology. The notions of $q_\omega$ and $u_\omega$-compact algebras are introduced in [8]. In the equational domain case, we will give a topological characterization of $q_\omega$ and $u_\omega$-compact algebras. We also introduce a new class of algebras which generalizes the class of equational noetherian algebras using the concept of meta-compact topological spaces.

3.1. Variants of equational conditions. Remember that an algebra $A$ is equational noetherian if and only if any system of equations is equivalent to a finite subsystem over $A$. We also defined weak equational noetherian algebras in the previous section. Both these properties are defined by applying certain conditions on the systems of equations. Clearly, one can use many different conditions to obtain new classes of algebras. Here we introduce two more examples of such classes.

**Definition 3.** An algebra $A$ is called $q_\omega$-compact if for any system of equations $S$ and any equation $p \approx q$ with $V_A(S) \subseteq V_A(p \approx q)$, there exists a finite subset $S_0 \subseteq S$ such that $V_A(S_0) \subseteq V_A(p \approx q)$. Similarly $A$ is called $u_\omega$-compact, if for any arbitrary system $S$ and any finite system $\{p_1 \approx q_1, \ldots, p_m \approx q_m\}$, the inclusion

$$V_A(S) \subseteq \bigcup_{i=1}^{m} V_A(p_i \approx q_i)$$

implies the existence of a finite subset $S_0 \subseteq S$ such that

$$V_A(S_0) \subseteq \bigcup_{i=1}^{m} V_A(p_i \approx q_i).$$
It is easy to see that any equational noetherian algebra is $u_\omega$-compact and any $u_\omega$-compact algebra is $q_\omega$-compact. The converse statements are not true and the reader may see [8] for counterexamples. It is also possible to define the \textit{weak} versions of these new classes. The unification theorem is proved for the algebras in these new classes, [8].

In the next subsection, we give topological characterization for $q_\omega$ and $u_\omega$-compact algebras. Note that, an algebra $A$ is equational noetherian if and only if for all $n$, the space $A^n$ is noetherian and this is equivalent to say that any subset of $A^n$ is compact in the Zariski topology. To see this, suppose for example every subset of $A^n$ is compact and $S$ is an arbitrary system of equations. Clearly we have

$$V_A(S) = \bigcap_{(p \approx q) \in S} V_A(p \approx q).$$

hence equivalently

$$A^n \setminus V_A(S) = \bigcup_{(p \approx q) \in S} A^n \setminus V_A(p \approx q).$$

By the compactness, there are finite number of equations $p_1 \approx q_1, \ldots, p_m \approx q_m$ in $S$, such that

$$A^n \setminus V_A(S) = \bigcup_{i=1}^m A^n \setminus V_A(p_i \approx q_i).$$

This shows that $V_A(S) = V_A(S_0)$, where $S_0 = \{p_1 \approx q_1, \ldots, p_m \approx q_m\}$. Therefore $A$ is equational noetherian. It can be easily shown that the converse is also true. So we have

**Proposition 1.** An algebra $A$ is equational noetherian if and only if, for any $n$ all subsets of $A^n$ are compact.

This proposition is our main motivation to investigate similar criteria for the case of $q_\omega$ and $u_\omega$-compact algebras.

### 3.2. $q_\omega$-compactness and Zariski topology

Let $A$ be an algebra in a language $\mathcal{L}$ and $p \approx q$ be an equation. We denote the open set $A^n \setminus V_A(p \approx q)$ by $C_A(p \approx q)$.

**Proposition 2.** Let $A$ be an equational domain. Then $A$ is $q_\omega$-compact if and only if $C_A(p \approx q)$ is compact for all $p \approx q$. 
Proof. First suppose $A$ is $q_\omega$-compact. Let $C_A(p \approx q) \subseteq \bigcup_{i \in I} C_i$, with $C_i \subseteq A^n$ open. Since $A$ is equational domain, so $C_i = A^n \setminus V_A(S_i)$ for some system $S_i$. We have

$$A^n \setminus V_A(p \approx q) \subseteq \bigcup_{i \in I} (A^n \setminus V_A(S_i)) = A^n \setminus \bigcap_{i \in I} V_A(S_i),$$

hence

$$\bigcap_{i \in I} V_A(S_i) \subseteq V_A(p \approx q).$$

This shows that

$$V_A(\bigcup_{i \in I} S_i) \subseteq V_A(p \approx q),$$

and so, there is a finite $S' \subseteq \bigcup_{i \in I} S_i$, with $V_A(S') \subseteq V_A(p \approx q)$. We have

$$S' \subseteq S_{i_1} \cup \cdots \cup S_{i_m},$$

for some $i_1, \ldots, i_m \in I$. Hence

$$\bigcap_{j=1}^m V_A(S_{i_j}) \subseteq V_A(S') \subseteq V_A(p \approx q),$$

and therefore $C_A(p \approx q) \subseteq \bigcup_{j=1}^m C_{i_j}$. This shows that $C_A(p \approx q)$ is compact. Conversely, suppose any $C_A(p \approx q)$ is compact. Let $V_A(S) \subseteq V_A(p \approx q)$. Then

$$C_A(p \approx q) \subseteq A^n \setminus V_A(S)$$

$$= A^n \setminus \bigcap_{(p \approx q) \in S} V_A(p \approx q)$$

$$= \bigcup_{(p \approx q) \in S} (A^n \setminus V_A(p \approx q)).$$

Hence

$$C_A(p \approx q) \subseteq \bigcup_{i=1}^m (A^n \setminus V_A(p_i \approx q_i)),$$

with $p_1 \approx q_1, \ldots, p_m \approx q_m \in S$. Therefore

$$V_A(p_1 \approx q_1, \ldots, p_m \approx q_m) \subseteq V_A(p \approx q),$$

and so, $A$ is $q_\omega$-compact. □

A similar result is true for $u_\omega$-compact equational domains. It can be shown that an equational domain $A$ is $u_\omega$-compact if and only if any finite intersection of sets of the form $C_A(p \approx q)$ is compact.
3.3. Meta-compact algebras. A topological space is called meta-compact if every open covering of it, has a refinement which is also a covering and every point belongs to finitely many element of the refinement. Motivating by meta-compact topological spaces, we define meta-compact algebras. Let \( S \) be a system of equations in a language \( \mathcal{L} \) and \( A \) be an algebra. We denote by \( V^*_A(S) \) the set of all points \((a_1, \ldots, a_n) \in A^n \) such that for all but finitely many equations \((p \approx q) \in S\), we have \( p^A(a_1, \ldots, a_n) = q^A(a_1, \ldots, a_n) \).

**Definition 4.** Let for any in-consistence system \( S \) over \( A \), there exists an in-consistence subsystem \( S' \subseteq S \), such that \( V^*_A(S') = A^n \). Then we call \( A \) a meta-compact algebra.

Any equational noetherian algebra is also meta-compact. This is because, if \( A \) is equational noetherian and \( S \) is an in-consistent system, then there is a finite \( S_0 \subseteq S \) such that \( V_A(S_0) = V_A(S) \) and so, \( S_0 \) is also in-consistence. But since \( S_0 \) is finite, so we have clearly \( V^*_A(S_0) = A^n \). Hence, \( A \) is meta-compact.

**Proposition 3.** Let \( A \) be meta-compact equational domain. Then for any \( n \) the space \( A^n \) is a meta-compact topological space.

**Proof.** Let \( A^n = \bigcup_{\alpha \in I} C_\alpha \) be a covering of \( A^n \), indexed by a set of ordinals \( I = \{ \alpha : \alpha \leq \kappa \} \). Since \( A \) is equational domain, every \( C_\alpha \) has the form

\[
C_\alpha = A^n \setminus V_A(S_\alpha),
\]

for some system of equations \( S_\alpha \). We have \( \bigcap_\alpha V_A(S_\alpha) = \emptyset \). Suppose \( S = \bigcup_\alpha S_\alpha \). Then \( V_A(S) = \emptyset \), and therefore there exists an in-consistence subsystem \( S' \), such that \( V^*_A(S') = A^n \). Define by transfinite induction

\[
S'_0 = S' \cap S_0,
\]
\[
S'_{\alpha^+} = (S' \cap S_{\alpha^+}) \setminus S_\alpha,
\]

and for any limit ordinal, we set

\[
S'_\lambda = (S' \cap S_\lambda) \setminus \bigcup_{\alpha < \lambda} S'_\alpha.
\]

We have clearly \( S' = \bigcup_\alpha S'_\alpha \), \( S'_\alpha \subseteq S_\alpha \), and \( S'_\alpha \cap S'_\beta = \emptyset \), for any distinct \( \alpha \) and \( \beta \). Now, let

\[
C'_\alpha = A^n \setminus V_A(S'_\alpha).
\]

We have \( C'_\alpha \subseteq C_\alpha \) and \( \bigcup_\alpha C'_\alpha = A^n \). Hence, we obtained a refinement of the given covering. Now, for \( \overline{a} \in A^n \), we have \( \overline{a} \in V^*_A(S') \), so there
are finitely many equations

\[ p_1 \approx q_1, \ldots, p_m \approx q_m \in \mathcal{S}' \]

such that \( p_i^A(\bar{a}) \neq q_i^A(\bar{a}), \) for \( 1 \leq i \leq m. \) For any \( i, \) there exists a unique \( \alpha_i \in I \) such that \((p_i \approx q_i) \in \mathcal{S}'_{\alpha_i} .\) Therefore, \( \bar{a} \) does not belong to

\[ V_A(S'_{\alpha_1}), \ldots, V_A(S'_{\alpha_m}) \]

and for other \( \alpha \in I, \) we have \( \bar{a} \in V_A(S'_{\alpha}) \) (since \( S'_{\alpha_i} \cap S'_{\alpha} = \emptyset \)). This shows that \( \bar{a} \) just belongs to \( C'_{\alpha_1}, \ldots, C'_{\alpha_m} .\) We now, proved that \( A^n \) is meta-compact. \( \square \)

4. Relatively free algebras

In this section, we study the universal algebraic geometry of the relatively free algebras. We show that for a variety \( \mathbf{V}, \) any solution of an equation over the relatively free algebras of \( \mathbf{V} \) is corresponds to an identity in \( \mathbf{V} .\) This idea, leads us to obtain some interesting results concerning classes of algebras (especially varieties) using concepts of the universal algebraic geometry.

4.1. Relatively free algebras. One of the major tools in the universal algebraic geometry is the relatively free algebra of a given variety over a given set of variables. We can discuss this notion in more general framework of pre-varieties. A class \( \mathbf{V} \) of algebras of type \( \mathcal{L} \) is a pre-variety, if it is closed under the operations of taking subalgebra and arbitrary direct product. For an arbitrary algebra \( A, \) we denote the set of all congruences of \( A \) by \( \text{Cong}(A) .\) If \( \mathbf{V} \) is a pre-variety and \( X \) is a set of variables, we can define an ideal of the term algebra \( T_L(X) \) by

\[
R_V(X) = \bigcap\{ R \in \text{Cong}(T_L(X)) : \frac{T_L(X)}{R} \in \mathbf{V} \}.
\]

So, \( R_V(X) \) is the smallest congruence in the term algebra such that the corresponding quotient belongs to \( \mathbf{V} .\) The quotient algebra

\[
F_V(X) = \frac{T_L(X)}{R_V(X)}
\]

is called the relative free algebra over \( X \) in \( \mathbf{V} .\) It is a member of \( \mathbf{V} \) and it can be characterized by the universal mapping property: it is generated by the set \( \overline{X} = \{ x/R_V(X) : x \in X \} \) and any map from \( \overline{X} \) to an algebra \( A \in \mathbf{V} \) extends uniquely to a homomorphism from \( F_V(X) \) to \( A. \) It can be easily seen that \( |X| = |\overline{X}| .\) More details on the universal mapping property can be find in \([5]\). We also have a logical characterization of \( R_V(X) .\)
Lemma 1. Let $V$ be a pre-variety and $X$ be a set of variables. Then
$$RV(X) = \{(p,q) : V \models (\forall x_1 \ldots \forall x_n p \approx q)\}.$$  

Proof. We prove the assertion for finite $X$ and by a small modification, it can be proved for infinite set of variables. Note that if $p$ and $q$ are terms with variables $x_1, \ldots, x_n$, then $V \models (\forall x_1 \ldots \forall x_n p \approx q)$ means that for all $A \in V$ and all $a_1, \ldots, a_n \in A$, we have the equality
$$p^A(a_1, \ldots, a_n) = q^A(a_1, \ldots, a_n).$$

To prove the lemma, assume that
$$K = \{(p,q) : V \models (\forall x_1 \ldots \forall x_n p \approx q)\}.$$  

Let $R$ be an ideal in $T_L(X)$ such that $T_L(X)/R \in V$ and let $(p,q) \in K$. If we let $A = T_L(X)/R$, then
$$p^A(x_1/R, \ldots, x_n/R) = q^A(x_1/R, \ldots, x_n/R),$$
where $x/R$ denotes the class containing $x$. This equality is equivalent to $p/R = q/R$, so $(p,q) \in R$. This proves that $K \subseteq RV$.

To see the inverse inclusion, let $F = T_L(X)/K$, which is generated by the set $X^* = \{x/K : x \in X\}$. We show that $F$ has the universal mapping property with respect to the set $X^*$ and the pre-variety $V$.  

Let $\alpha : X^* \to A$ be any map, where $A \in V$. Define $\alpha_0 : X \to A$ by $\alpha_0(x) = \alpha(x/K)$. We know that there exists a homomorphism $\alpha'_0 : T_L(X) \to A$, extending $\alpha_0$. It is easy to see that for all term $p \in T_L(X)$, we have $\alpha'_0(p) = p^A(\alpha(x_1/K), \ldots, \alpha(x_n/K))$. This shows that for $(p,q) \in K$, we have $\alpha'_0(p) = \alpha'_0(q)$, and hence $(p,q) \in \ker \alpha'_0$. Therefore we have a homomorphism $\alpha' : F \to A$ such that
$$\alpha'(t/K) = \alpha'_0(t).$$

Clearly, $\alpha'$ coincides with $\alpha$ over $X^*$. We show that $\alpha'$ is unique. Let $h : F \to A$ be another homomorphism such that $h$ coincides with $\alpha$ over $X^*$. Using induction on the complexity of the term $t = f(t_1, \ldots, t_m)$, we have
$$h(t/K) = h\left(\frac{f(t_1, \ldots, t_m)}{K}\right) = f^A(h(t_1/K), \ldots, h(t_m/K)) = f^A(\alpha'(t_1/K), \ldots, \alpha'(t_m/K)) = \alpha'\left(\frac{f(t_1, \ldots, t_m)}{K}\right) = \alpha'(t/K).$$

This argument shows that $F$ is free relative to $V$ and hence it belongs to $V$. Therefore $RV(X) \subseteq K$. \hfill $\square$
We give another interpretation of this lemma using the terminology of the equational logic of Tarski. Remember that a congruence of the term algebra $T(x_1, x_2, \ldots)$ is called fully invariant, if it is invariant under any endomorphism of the term algebra. For any set $\Sigma$ of identities, $\Theta_{fi}(\Sigma)$ denotes the fully invariant closure of $\Sigma$. As in [5], this set is equal to the deductive closure of $\Sigma$, i.e.

$$\Theta_{fi}(\Sigma) = D(\Sigma),$$

in the sense of [5]. Now, let $V$ be a variety of algebras in the language $L$. Let $\Sigma$ be a set of identities for $V$ with variables from $X$. Then the above lemmas says that $R_V(X) = D(\Sigma)$. The next result will be used frequently in the subsequence parts of this article.

**Corollary 1.** Let $V$ be a pre-variety of algebras in a language $L$ and $X$ be a set. Let $F = F_V(X)$ and $p \approx q$ be an equation with $n$ indeterminate. Then $(\overline{t}_1, \ldots, \overline{t}_n) \in F^n$ is a solution of $p \approx q$, if and only if

$$\forall x_1 \ldots \forall x_m \; p(t_1, \ldots, t_n) \approx q(t_1, \ldots, t_n)$$

is an identity in $V$. Here $x_1, \ldots, x_m$ are the variables appearing in the terms $t_1, \ldots, t_n$.

**Proof.** Suppose $(\overline{t}_1, \ldots, \overline{t}_n) \in F^n$ is a solution of $p \approx q$. then we have

$$p^F(\overline{t}_1, \ldots, \overline{t}_n) = q^F(\overline{t}_1, \ldots, \overline{t}_n),$$

and therefore

$$\frac{p(t_1, \ldots, t_n)}{R_V(X)} = \frac{q(t_1, \ldots, t_n)}{R_V(X)}.$$

This shows that $(p(t_1, \ldots, t_n), q(t_1, \ldots, t_n)) \in R_V(X)$ and hence by the above lemma

$$V \models \forall x_1 \ldots \forall x_m p(t_1, \ldots, t_n) \approx q(t_1, \ldots, t_n).$$

The converse statement can be proved similarly. \qed

4.2. $u_\omega$-compactness of certain relatively free algebras. We now prove that for certain varieties $V$ and infinite sets $X$, the relatively free algebra $F_V(X)$ is $u_\omega$-compact. A quite similar argument can be applied for finite $X$ with $|X| = n$ using $u_\omega^n$-compactness instead of $u_\omega$-compactness. Before stating the next theorem, remember that if $t = t(x_1, \ldots, x_n)$ is a term in a language $L$ and $A$ is an algebra of type $L$, then there exists correspondingly a term function $t^A : A^n \to A$. In some important situations, these term functions are surjective (see examples below).
**Theorem 3.** Let $\mathcal{L}$ be an algebraic language without constants and let $A$ be an algebra of type $\mathcal{L}$. Suppose all term functions over $A$ are surjective and $\mathbf{V} = \text{Var}(A)$, the variety of $\mathcal{L}$-algebras generated by $A$. If $X$ is an infinite set, then $\mathcal{F}_\mathbf{V}(X)$ is $u_\omega$-compact.

**Proof.** We give a proof for $q_\omega$-compactness here and the main proof is completely similar. Suppose $F = \mathcal{F}_\mathbf{V}(X)$. An arbitrary element of $F$ is a class $t/R_\mathbf{V}(X)$ and during the proof, we will denote it as $\overline{t}$. Suppose $S = \{p_i \approx q_i : i \in I\} \subseteq \mathcal{A}_\mathcal{L}(x_1, \ldots, x_n)$ is a system of equations. Suppose also $p \approx q \in \mathcal{A}_\mathcal{L}(x_1, \ldots, x_n)$ and $\mathcal{V}_F(S) \subseteq \mathcal{V}_F(p \approx q)$. By the corollary 1 of the previous subsection, for all terms $t_1, \ldots, t_n$, the assumption

$$\mathbf{V} \models \bigwedge_{i \in I} \forall x_1 \ldots \forall x_m p_i(t_1, \ldots, t_n) \approx q_i(t_1, \ldots, t_n),$$

implies

$$\mathbf{V} \models \forall x_1 \ldots \forall x_m p(t_1, \ldots, t_n) \approx q(t_1, \ldots, t_n),$$

where $x_1, \ldots, x_m$ are variables appearing in $t_1, \ldots, t_n$.

As a special case (assuming that $t_i = x_i$, note that since $X$ is infinite, there are enough variables), we have

$$\mathbf{V} \models \bigwedge_{i \in I} \forall x_1 \ldots \forall x_n p_i \approx q_i \Rightarrow \mathbf{V} \models \forall x_1 \ldots \forall x_n p \approx q.$$

Since $\mathbf{V} = \text{Var}(A)$, so we have also

$$A \models \bigwedge_{i \in I} \forall x_1 \ldots \forall x_n p_i \approx q_i \Rightarrow A \models \forall x_1 \ldots \forall x_n p \approx q.$$

Equivalently,

$$A \models (\bigwedge_{i \in I} \forall x_1 \ldots \forall x_n p_i \approx q_i \Rightarrow \forall x_1 \ldots \forall x_n p \approx q).$$

Therefore, if $\text{Th}(A)$ denotes the first order theory of $A$, then

$$\text{Th}(A) + (\bigwedge_{i \in I} \forall x_1 \ldots \forall x_n p_i \approx q_i) \models \forall x_1 \ldots \forall x_n p \approx q,$$

and by the compactness of the first order logic, there exists a finite subset $I_0 \subseteq I$ such that we have also

$$\text{Th}(A) + (\bigwedge_{i \in I_0} \forall x_1 \ldots \forall x_n p_i \approx q_i) \models \forall x_1 \ldots \forall x_n p \approx q.$$

This shows that

$$A \models \bigwedge_{i \in I_0} \forall x_1 \ldots \forall x_n p_i \approx q_i \Rightarrow A \models \forall x_1 \ldots \forall x_n p \approx q.$$
Now, assume that $t_1, \ldots, t_n$ are arbitrary terms with variables $x_1, \ldots, x_m$. Since any $t_i^A$ is surjective function, so the assumption

$$A \models \bigwedge_{i \in I_0} \forall x_1 \ldots \forall x_m p_i(t_1, \ldots, t_n) \approx q_i(t_1, \ldots, t_n)$$

implies

$$A \models \forall x_1 \ldots \forall x_m p(t_1, \ldots, t_n) \approx q(t_1, \ldots, t_n).$$

This shows that if for all $i \in I_0$, the formula $p_i(t_1, \ldots, t_n) \approx q_i(t_1, \ldots, t_n)$ is an identity in $A$, then so is $p(t_1, \ldots, t_n) \approx q(t_1, \ldots, t_n)$. In the other words, the assumption

$$V \models \bigwedge_{i \in I_0} \forall x_1 \ldots \forall x_m p_i(t_1, \ldots, t_n) \approx q_i(t_1, \ldots, t_n),$$

implies

$$V \models \forall x_1 \ldots \forall x_m p(t_1, \ldots, t_n) \approx q(t_1, \ldots, t_n).$$

Now, suppose that $S_0 = \{ p_i \approx q_i : i \in I_0 \}$. Then we have $V_F(S_0) \subseteq V_F(p \approx q)$.

We will give examples of algebras satisfying the requirements of our theorem.

**Example 1.** Let $\mathcal{L} = (+, \times)$ be a language consisting of two binary functional symbols and $K$ be an algebraically closed field. Then clearly $K$ is an $\mathcal{L}$-algebra and any term function over $K$ is surjective. Let $V = \text{Var}_\mathcal{L}(K)$, the variety of $\mathcal{L}$-algebras generated by $K$ (note that this is completely different from the variety of rings generated by $K$). Now by the above theorem, for any infinite set $X$, the $\mathcal{L}$-algebra $F_V(X)$ is $u_\omega$-compact.

**Example 2.** Remember that a radicable group is group $G$ such that for any natural number $n$ and any $g \in G$, there exists an element $h \in G$ such that $h^n = g$. There are many examples of such groups, any divisible abelian group is radicable and any rational exponential group is also radicable. Let $\mathcal{L} = (\cdot)$ be the language of semigroups and $G$ be a radicable group. Then clearly $G$ is an algebra of type $\mathcal{L}$ and any term function over $G$ is surjective. Let $V = \text{Var}_\mathcal{L}(G)$ be the variety of semigroups generated by $G$. Then for any infinite set $X$, the infinitely generated relatively free semigroup $F_V(X)$ is $u_\omega$-compact.

As we mentioned before, it is also possible to prove a version of the theorem 3, for finite sets $X$. 
Theorem 4. Let $\mathcal{L}$ be an algebraic language without constants and let $A$ be an algebra of type $\mathcal{L}$. Let $X$ be a finite set of cardinality $n$ and suppose all term functions of at most $n$ variables over $A$ are surjective and $V = \text{Var}(A)$. Then $F_{V}(X)$ is $\omega$-compact.

Let $A$ be an arbitrary algebra. It is easy to check that $A$ is relatively free in some variety if and only if it is relatively free in the variety $\text{Var}(A)$. Benjamin Steinberg gave a general note about such algebras to the second author via Mathoverflow (www.mathoverflow.net): a finitely generated algebra $A$ is relatively free if and only if it has a generating set $X$ such that any map $X \to A$ extends to an endomorphism. By this fact, we have the next corollary.

Corollary 2. Let $\mathcal{L}$ be an algebraic language without constants and let $A$ be a finitely generated $\mathcal{L}$-algebra having a generating set $X$, such that any map $X \to A$ extends to an endomorphism. Suppose $|X| = n$ and any $n$-variable term function over $A$ is surjective. Then $A$ is $\omega$-compact.

The next example is also a result of a communication of the second author and Anton Klyachko in the above mentioned website.

Example 3. Let $G$ be a radicable group. A semigroup polynomial function over $G$ is a map $f : G^{n} \to G$ of the form

$$f(x_{1}, \ldots, x_{n}) = x_{j_{1}}^{\alpha_{1}} \cdots x_{j_{k}}^{{\alpha_{k}}},$$

where $1 \leq k$ and $1 \leq j_{1}, \ldots, j_{k} \leq n$ and all $\alpha_{i}$ are positive integers. Denote the set of all such functions by $P_{G}^{sg}(x_{1}, \ldots, x_{n})$. This set is a semigroup with the operation

$$(fg)(x_{1}, \ldots, x_{n}) = f(x_{1}, \ldots, x_{n})g(x_{1}, \ldots, x_{n}).$$

We show that

$$P_{G}^{sg}(x_{1}, \ldots, x_{n}) = F_{V}(X),$$

where $V = \text{Var}_{L}(G)$ and $X = \{x_{1}, \ldots, x_{n}\}$. First, note that

$$P_{G}^{sg}(x_{1}, \ldots, x_{n}) \leq G^{G^{n}},$$

and hence it belongs to $V$. Let $f_{i} : G^{n} \to G$ be the $i$-th projection map. Clearly, $P_{G}^{sg}(x_{1}, \ldots, x_{n})$ is generated by the set $X' = \{f_{1}, \ldots, f_{n}\}$. Let $\varphi : X' \to P_{G}^{sg}(x_{1}, \ldots, x_{n})$ be a function. We show that it extends to an endomorphism

$$\varphi'(f_{j_{1}}^{\alpha_{1}} \cdots f_{j_{k}}^{\alpha_{k}}) = \varphi(f_{j_{1}})^{\alpha_{1}} \cdots \varphi(f_{j_{k}})^{\alpha_{k}}.$$ 

It is enough to prove that $\varphi'$ is well-defined. So, let $w$ and $w'$ be two semigroup word such that $w(f_{1}, \ldots, f_{n}) = w'(f_{1}, \ldots, f_{n})$. Then the
maps

\((a_1, \ldots, a_n) \mapsto w(a_1, \ldots, a_n), \text{ and } (a_1, \ldots, a_n) \mapsto w'(a_1, \ldots, a_n)\)

are equal and hence \(w(x_1, \ldots, x_n) = w'(x_1, \ldots, x_n)\) is an identity of the semigroup \(G\). This shows that \(\varphi'\) is well-defined and the assertion proved. Therefore \(P_{G}^{sg}(x_1, \ldots, x_n)\) is \(u_{\omega}\)-compact.

4.3. Finitely axiomatizable classes. We can apply the property of being equational noetherian for certain relatively free algebras, to obtain finite bases of axioms (consisting of identities) for some classes of algebras. In the next theorem, \(Ucl(A)\) is used for the universal closure of \(A\), i.e. \(Ucl(A) = \text{Mod}(\text{Th}_{\forall}(A))\).

**Theorem 5.** Let \(A\) be an algebra of type \(L\) and \(V = \text{Var}(A)\). Let \(F_{V}(X)\) be equational noetherian for all finite \(X\). Suppose \(W\) is a subclass of \(Ucl(A)\) axiomatized by a set of identities \(\Sigma \subseteq \text{At}_{L}(x_1, \ldots, x_n)\). Then there exists a finite subset \(\Sigma_0 \subseteq \Sigma\) which axiomatizes \(W\), i.e.

\[ W = \{ B \in Ucl(A) : B \models \Sigma_0 \}. \]

**Proof.** Suppose \(\Sigma = \{ p_i \approx q_i : i \in I \}\), so we have

\[ W = \{ B \in Ucl(A) : B \models \bigwedge_{i \in I} \forall x_1 \ldots \forall x_n p_i \approx q_i \}. \]

Let \(X = \{ x_1, \ldots, x_n \}\) and \(F = F_{V}(X)\). We can consider \(\Sigma\) as a system of equations over \(F\) and since \(F\) is equational noetherian, so there exists a finite \(\Sigma_0 \subseteq \Sigma\) such that \(V_F(\Sigma) = V_F(\Sigma_0)\). Let \(I_0\) be the corresponding set of indices, i.e.

\[ \Sigma_0 = \{ p_i \approx q_i : i \in I_0 \}. \]

Now, using the corollary 1 in 4.1, for any \(t_1, \ldots, t_n\), we have

\[ V \models \bigwedge_{i \in I_0} \forall x_1 \ldots \forall x_n p_i(t_1, \ldots, t_n) \approx q_i(t_1, \ldots, t_n), \]

if and only if

\[ V \models \bigwedge_{i \in I} \forall x_1 \ldots \forall x_n p_i(t_1, \ldots, t_n) \approx q_i(t_1, \ldots, t_n). \]

Since \(V = \text{Var}(A)\), so as a special case we have

\[ A \models (\bigwedge_{i \in I_0} \forall x_1 \ldots \forall x_n p_i \approx q_i) \Rightarrow (\bigwedge_{i \in I} \forall x_1 \ldots \forall x_n p_i \approx q_i). \]

This shows that for any \(j \in I\), the universal sentence

\[ \bigwedge_{i \in I_0} \forall x_1 \ldots \forall x_n p_i \approx q_i \Rightarrow \forall x_1 \ldots \forall x_n p_j \approx q_j \]
belongs to $\text{Th}_\forall(A)$. Therefore

$$\text{Th}_\forall(A) + \left( \bigwedge_{i \in I_0} \forall x_1 \ldots \forall x_n p_i \approx q_i \right) \models \forall x_1 \ldots \forall x_n p_j \approx q_j,$$

and hence

$$W = \{ B \in \text{Ucl}(A) : \models \bigwedge_{i \in I_0} \forall x_1 \ldots \forall x_n p_i \approx q_i \},$$

therefore $\Sigma_0$ is a set of axioms for $W$ inside $\text{Ucl}(A)$.

In the next example, we use the fact that every variety is generated by any of its infinitely generated relatively free elements. We also use the fact that the free group of the rank two, contains a free group of infinite rank as a subgroup.

**Example 4.** Let $V$ be the variety of all groups. Clearly $V = \text{Var}(F_2)$, where $F_2$ is the free group of rank two. Let $n \geq 1003$ be an odd number and consider the following set of group identities

$$\Sigma = \{ [x^{p_i n}, y^{p_i n}]^n \approx 1 : p = \text{prime} \}.$$

Let $W_1$ be the variety of groups axiomatized by $\Sigma$. Then, as Adian proves in [1], $W_1$ is not finitely based, i.e. it is impossible to axiomatize it using a finite set of identities. Now, suppose

$$W = \{ B \in \text{Ucl}(F_2) : \models \Sigma \} = W_1 \cap \text{Ucl}(F_2).$$

Since for any finite $X$, the free group $F(X)$ is equational noetherian, so by the above theorem $W$ can be axiomatized by a finite subset of $\Sigma$. This means that there are prime numbers $p_1, \ldots, p_m$ such that

$$\text{Th}_\forall(F_2) + \left( \bigwedge_{i=1}^m \forall x \forall y [x^{p_i n}, y^{p_i n}]^n \approx 1 \right) \vdash \Sigma.$$

Hence, although $\Sigma$ is independent over $\text{Id}(F_2)$, it is not so over $\text{Th}_\forall(F_2)$.

Note that for any class of algebras, say $K$, its universal closure equals $SP_u(K)$, where $S$ and $P_u$ are the class operations of taking subalgebras and ultra-products, [3]. Specially, for an algebra $A$, we have $\text{Ucl}(A) = SP_u(A)$ so, any element of $\text{Ucl}(A)$ is a subalgebra of some ultra-power of $A$. If $A$ is finite, then by the theorem of Loś, we have $P_u(A) = \{ A \}$ and hence

$$\text{Ucl}(A) = \{ B : B \leq A \}.$$

This shows that our theorem is trivial in the case of finite algebras. But, if $A$ is infinite, then Lowenheim-Skolem’s theorem implies that $P_u(A)$ is a proper class and so $\text{Ucl}(A)$ may have infinitely many subclasses axiomatized by sets of identities.
5. EQUATIONAL ARTINIAN ALGEBRAS

We say that an algebra $A$ is *equational Artinian* if for any $n$, the Zariski topology on $A^n$ is Artinian, i.e. every ascending chain of closed sets terminates. In this section, we investigate some properties of equational Artinian algebras.

5.1. $\omega$-cocompactness and Artinian topological spaces. A topological space is called Artinian, if any ascending chain of closed sets terminate. As we know, a space is noetherian if and only if every subset of that space is compact. We obtain an analogue for this fact in this subsection.

**Definition 5.** A topological space $M$ is $\omega$-cocompact if and only if for any countable open covering $M = \bigcup_{i=1}^{\infty} C_i$, there exists $m \geq 1$ such that for all $j_1, j_2 \geq m$, we have

$$\bigcup_{i=j_1}^{\infty} C_i = \bigcup_{i=j_2}^{\infty} C_i.$$ 

**Proposition 4.** If $M$ is Artinian, then every subset of $M$ is $\omega$-cocompact. The converse is also true.

*Proof.* We first show that for an Artinian topological space $M$, every $N \subseteq M$ is also Artinian. Suppose

$$Y'_1 \subseteq Y'_2 \subseteq Y'_3 \subseteq \cdots$$

is a chain of closed sets in $N$. We have $Y'_i = Y_i \cap N$, for all $i$, where $Y_i$ is closed in $M$. Let $V_i = Y'_1 \cup \cdots \cup Y'_i$. Then $V_i$ is closed in $M$ and we have

$$V_1 \subseteq V_2 \subseteq V_3 \subseteq \cdots.$$ 

So, there is $m$ such that $V_m = V_{m+1} = V_{m+2} = \cdots$. This shows that

$$Y_1 \cup \cdots \cup Y_m = Y'_1 \cup \cdots \cup Y'_m \cup Y'_{m+1} = \cdots,$$

and hence

$$Y'_1 \cup \cdots \cup Y'_m = Y'_1 \cup \cdots \cup Y'_m \cup Y''_{m+1} = \cdots,$$

which implies that $Y'_m = Y'_{m+1} = \cdots$. This shows that $N$ is also Artinian. Now, suppose $M$ is Artinian and $M = \bigcup_{i=1}^{\infty} C_i$. We have $C_i = M \setminus D_i$, with $D_i$ closed. Then $\bigcap_{i=1}^{\infty} D_i = \emptyset$. Define

$$Y_m = \bigcap_{i=m}^{\infty} D_i.$$
Then $Y_1 \subseteq Y_2 \subseteq Y_3 \subseteq \cdots$ is a chain of closed sets and so there exists $m$ with $Y_m = Y_{m+1} = \cdots$. Hence, for $j_1, j_2 \geq m$ we have

$$\bigcap_{i=j_1}^{\infty} D_i = \bigcap_{i=j_2}^{\infty} D_i,$$

and so

$$\bigcup_{i=j_1}^{\infty} C_i = \bigcup_{i=j_2}^{\infty} C_i.$$

Hence $M$ is $\omega$-cocompact. Conversely, suppose $M$ is $\omega$-cocompact and $Y_1 \subseteq Y_2 \subseteq Y_3 \subseteq \cdots$ is a chain of closed subsets of $M$. Then we have

$$M = \bigcup_{i=1}^{\infty} Y_i^c \cup M,$$

where the superscript $^c$ denotes the complement. So, there exists $m$ such that for all $j_1, j_2 \geq m$,

$$\bigcup_{i=j_1}^{\infty} Y_i^c = \bigcup_{i=j_2}^{\infty} Y_i^c.$$

This implies that

$$\bigcap_{i=m}^{\infty} Y_i = \bigcap_{i=m+1}^{\infty} Y_i = \cdots.$$

Therefore, we have

$$Y_m = \bigcap_{i=m+1}^{\infty} Y_i = \bigcap_{i=m+2}^{\infty} Y_i.$$

Similarly, we have

$$Y_{m+1} = \bigcap_{i=m+2}^{\infty} Y_i = \cdots.$$

Hence $Y_m = Y_{m+1} = \cdots$, and so $M$ is Artinian.

\[ \square \]

5.2. Equational Artinian algebras. By the previous subsection, we know that an algebra $A$ is equational Artinian, if and only if, for all $n$, any subset of $A^n$ is $\omega$-cocompact. Is there an equational condition, which is equivalent to being equational Artinian? A partial answer is given in the next theorem. Our effort to find such a good equational condition was unsuccessful. While there are many equational conditions which can be applied for algebras to introduce new classes (see for example the last section), however we didn’t find a suitable condition equivalent to the property of being equational Artinian.
Let $A$ be an algebra and $S$ be a system of equation. We say that $S$ is $A$-stable, if for any proper finite subset $S' \subseteq S$, we have $V_A(S) = V_A(S \setminus S')$.

**Theorem 6.** Let $A$ be an equational domain which is weak equational noetherian. Suppose for any system $S$, there exists a finite subset $S_0$ such that $S \setminus S_0$ is $A$-stable. Then $A$ is equational Artinian. The converse is true if in addition, the language $\mathcal{L}$ is countable.

**Proof.** Suppose for any system $S$, there exists a finite subset $S_0$ such that $S \setminus S_0$ is $A$-stable. We show that $A^n$ is $\omega$-cocompact. Let $A = \bigcup_{i=1}^{\infty} C_i$ be an open covering. We have $C_i = A^n \setminus V_A(S_i)$ for some finite $S_i$. So, $\bigcap_{i=1}^{\infty} V_A(S_i) = \emptyset$ and hence $V_A(\bigcup_{i=1}^{\infty} S_i) = \emptyset$. Suppose $S = \bigcup_{i=1}^{\infty} S_i$. Then, there exists a finite subset $S_0 \subseteq S$ such that $S \setminus S_0$ is $A$-stable. Since $S_0$ is finite, so there exists $m$ such that for all $j \geq m$, the set $V_A(\bigcup_{i=j}^{\infty} S_i)$ does not depend on $j$. Hence for all $j_1, j_2 \geq m$, we have

$$\bigcap_{i=j_1}^{\infty} V_A(S_i) = \bigcap_{i=j_2}^{\infty} V_A(S_i).$$

This is equivalent to

$$\bigcup_{i=j_1}^{\infty} C_i = \bigcup_{i=j_2}^{\infty} C_i,$$

and hence $A^n$ is $\omega$-cocompact. Now, suppose the language $\mathcal{L}$ is countable and $A$ is equational Artinian. Let $S$ be a system of equations. We index $S$ by natural numbers as $S = \{p_i \approx q_i\}_{i=1}^{\infty}$. Suppose $C = A^n \setminus V_A(S)$. We have

$$C = \bigcup_{i=1}^{\infty} (A^n \setminus V_A(p_i \approx q_i)),$$

so by our assumption on $\omega$-cocompactness of $A$, there exists $m$ such that for all $j_1, j_2 \geq m$,

$$\bigcup_{i=j_1}^{\infty} (A^n \setminus V_A(p_i \approx q_i)) = \bigcup_{i=j_2}^{\infty} (A^n \setminus V_A(p_i \approx q_i)).$$

Let $S_0 = \{p_1 \approx q_1, \ldots, p_m \approx q_m\}$. Therefore for any $j$, we have

$$V_A(S \setminus S_0) = V_A(S \setminus (S_0 + p_m \approx q_m + \cdots + p_{m+j} \approx q_{m+j})),$$

so $S \setminus S_0$ is $A$-stable. □
5.3. **Relative systems of equations.** Let $V$ be a variety of algebras. In the next subsection, we will show that the relative free algebra $F_V(X)$ has descending chain condition on its ideals, if and only if every element of $V$ is equational Artinian. We will do this in a more general context. A similar result obtained for the property of being equational noetherian by the second author in [20] and our method in this subsection is completely similar to [20].

In the sequel, we assume that $A$ is an algebra containing a trivial subalgebra. Suppose $V$ is a pre-variety of $A$-algebras. As before, let $X$ be a finite set of variables. Suppose $R_V(X)$ is the smallest $A$-congruence with the property $T_{\mathcal{L}(A)}(X)/R_V(X) \in V$. Let

$$F_V(X) = \frac{T_{\mathcal{L}(A)}(X)}{R_V(X)}.$$ 

As before, we denote an arbitrary element of $F_V(X)$ by $\overline{t}$, where $t$ is a term in $\mathcal{L}(A)$. Note that if $\mathcal{L}$ is the language of groups and $V$ is the variety of all groups, then $F_V(X) = F(X)$, the free group with the basis $X$. If $A$ is a group and $V$ is the class of all $A$-groups, then $F_V(X) = A \ast F(X)$, the free product of $A$ and the free group $F(X)$.

Suppose now, $B \in V$ and $(b_1, \ldots, b_n) \in B^n$. We know that there exists a homomorphism $\varphi : F_V(X) \to B$ such that

$$\varphi(\overline{p}) = p^B(b_1, \ldots, b_n).$$

Therefore, if $\overline{p}_1 = \overline{p}_2$, then $p_1^B(b_1, \ldots, b_n) = p_2^B(b_1, \ldots, b_n)$. This shows that the following definition has no ambiguity.

**Definition 6.** A $V$-equation is an expression of the form $\overline{p} \approx \overline{q}$, where $p$ and $q$ are terms in the language $\mathcal{L}(A)$. If $B$ is an $A$-algebra and $(b_1, \ldots, b_n)$ is an element of $B^n$, we say that $(b_1, \ldots, b_n)$ is a solution of $\overline{p} \approx \overline{q}$, if $p^B(b_1, \ldots, b_n) = q^B(b_1, \ldots, b_n)$.

Let $S$ be a system of $V$-equations. The set of all solutions of elements of $S$, will be denoted by $V_B^V(S)$. The following observation shows that this is an ordinary algebraic set. Let $S'$ be the set of all equations $p \approx q$ such that $\overline{p} \approx \overline{q}$ belongs to $S$. Then it can be easily verified that

$$V_B^V(S) = V_B(S').$$

Therefore, in the sequel we will denote the algebraic set $V_B^V(S)$ by the same notation $V_B(S)$. The Zariski topology arising from algebraic sets relative to the pre-variety $V$ is the same as the ordinary Zariski topology. If $Y \subseteq B^n$, we define

$$\text{Rad}_B^V(Y) = \{\overline{p} \approx \overline{q} : \forall \overline{b} \in Y \ p^B(b_1, \ldots, b_n) = q^B(b_1, \ldots, b_n)\}.$$
The quotient algebra
\[ \Gamma_V(Y) = \frac{F_V(X)}{\text{Rad}_B(Y)} \]
is the \( V \)-coordinate algebra of \( Y \). Again, it is easy to see that \( \Gamma_V(Y) \cong \Gamma(Y) \).

5.4. Chain conditions on the ideals of relatively free algebras.
An algebra \( B \) will be called noetherian (Artinian), if any ascending (descending) chain of ideals in \( B \) terminates. In the case of \( A \)-algebras, we restrict ourself to \( A \)-ideals. A congruence \( R \) in \( B \) is called \( A \)-ideal, if for all \( a_1, a_2 \in A \), the assumption \((a_1, a_2) \in R\) implies \( a_1 = a_2 \). In [20], the following theorem is proved.

**Theorem 7.** Let \( \mathfrak{V} \) be a variety of algebras of type \( \mathcal{L} \) and \( A \in \mathfrak{V} \) containing a trivial subalgebra. Let \( V = \mathfrak{V}_A \) be the class of elements of \( \mathfrak{V} \) which are \( A \)-algebra. Then the relatively free algebra \( F_V(X) \) is noetherian if and only if every \( B \in V \) is \( A \)-equationally noetherian.

Note that an \( A \)-algebra \( B \) is called \( A \)-equational noetherian (\( A \)-equational Artinian), if it is equational noetherian (equational Artinian) as an algebra of type \( \mathcal{L}(A) \).

We are now ready to prove the analogue of the above theorem for the property of being Artinian.

**Theorem 8.** Let \( \mathfrak{V} \) be a variety of algebras of type \( \mathcal{L} \) and \( A \in \mathfrak{V} \) containing a trivial subalgebra. Let \( V = \mathfrak{V}_A \) be the class of elements of \( \mathfrak{V} \) which are \( A \)-algebra. Then the relatively free algebra \( F_V(X) \) is Artinian, if and only if every \( B \in V \) is \( A \)-equationally Artinian.

**Proof.** First, suppose that \( F_V(X) \) is Artinian and \( B \in V \). Let
\[ Y_1 \subseteq Y_2 \subseteq Y_3 \subseteq \cdots \]
be a chain of algebraic sets in \( B^n \). Then we have
\[ \text{Rad}(Y_1) \supseteq \text{Rad}(Y_2) \supseteq \text{Rad}(Y_3) \supseteq \cdots , \]
which is a chain of \( A \)-ideals in \( F_V(X) \). So, it terminates and hence, there is \( m \) such that
\[ \text{Rad}(Y_m) = \text{Rad}(Y_{m+1}) = \cdots . \]
This implies that \( Y_m = Y_{m+1} = \cdots \) and therefore \( B \) is equational Artinian. Now, suppose \( B \) is equational Artinian for all \( B \in V \). If \( R \) is an \( A \)-ideal in \( F_V(X) \), we put \( B(R) = F_V(X)/R \), which is belong to \( V \). It is easy to see that \((\overline{p}, \overline{q}) \in R \) if and only if
\[ (\overline{x}_1/R, \ldots, \overline{x}_n/R) \in V_B(R)(\overline{p} \approx \overline{q}). \]
Now, let
\[ R_1 \supseteq R_2 \supseteq R_3 \supseteq \cdots \]
be a proper descending chain of \( A \)-ideals in \( F_V(X) \). Note that in the same time, \( R_i \) is a system of \( V \)-equations. Suppose \( (\overline{p}_i, \overline{q}_i) \in R_i \setminus R_{i+1} \). Suppose also that \( T_i \) is the \( A \)-ideal generated by the set \( R_i+1 + (\overline{p}_i, \overline{q}_i) \). Then we have

\[ R_i+1 \subsetneq T_i \subsetneq R_i. \]

Suppose \( B_i = B(R_i) \) and \( B = \prod_i B_i \). So \( B \in V \). Hence it is equational Artinian. Note that, since \( A \) contains trivial algebra, so \( B_i \leq B \). Hence

\[ U = (\overline{x}_1/R_i+1, \ldots, \overline{x}_n/R_i+1) \in V_{B_i+1}(R_i+1) \subseteq V_B(R_i+1). \]

But, \( U \) does not belong to \( V_B(T_i) \), since otherwise, \( U \in V_B(\overline{x}_i \approx \overline{q}_i) \) which implies that \( (\overline{p}_i, \overline{q}_i) \in R_{i+1} \). Therefore, we have

\[ V_B(R_i) \subseteq V_B(T_i) \subsetneq V_B(R_{i+1}), \]

so the chain \( V_B(R_1) \subsetneq V_B(R_2) \subsetneq V_B(R_3) \subsetneq \cdots \) is a proper chain of algebraic sets, which is a contradiction.

\[ \square \]

5.5. **Hilbert’s basis theorem.** A universal algebraic version of the Hilbert’s basis theorem is given in [20]. In this subsection, we discuss briefly some results on this item. Suppose \( L \) is an algebraic language and \( \mathcal{Y} \) is a variety of algebras of type \( L \). Let \( A \in \mathcal{Y} \) and \( V = \mathcal{Y}_A \) be the class of all elements of \( \mathcal{Y} \) which are \( A \)-algebra. If \( A \) has maximal property on its ideals, is the algebra \( F_V(X) \) noetherian?

**Example 5.** Let \( L = (0, 1, +, \times) \) be the language of unital rings and \( \mathcal{Y} \) be the variety of all commutative rings with unity element. Let \( A \in \mathcal{Y} \) and \( V = \mathcal{Y}_A \). If \( X = \{x_1, \ldots, x_n\} \), then \( F_V(X) = A[x_1, \ldots, x_n] \) and hence Hilbert’s basis theorem is valid in this case.

**Example 6.** Let \( L = (e, -1, \cdot) \) be the language of groups. Let \( \mathcal{Y} \) be the variety of groups. Let \( A \) be any group and \( V = \mathcal{Y}_A \). Then \( F_V(X) = A * F(X) \). We show that \( F_V(X) \) is not noetherian even if \( A \) has maximal property on its normal subgroups (max-n). Consider the Baumslag-Solitar group

\[ B_{m,n} = \langle a, t : t^m a t^{-1} = a^n \rangle, \]

where \( m, n \geq 1 \) and \( m \neq n \). Then, as is proved in [2], this group is not equationally noetherian. Let \( B = A * B_{m,n} \). Then \( B \) is an \( A \)-group which is not \( A \)-equationally noetherian. So, by the theorem 7 of the previous subsection, \( A * F(X) \) is not noetherian, Hilbert’s basis theorem fails.
Example 7. Let \( \mathfrak{Y} \) be the variety of abelian groups and \( A \in \mathfrak{Y} \) be finitely generated. Suppose \( V = \mathfrak{Y}_A \). Then it is easy to see that \( F_V(X) = A \times F_{ab}(X) \), where \( F_{ab}(X) \) is the free abelian group generated by \( X \). So, \( F_V(X) = A \times \mathbb{Z}^n \). As a \( \mathbb{Z} \)-module, clearly \( A \times \mathbb{Z}^n \) is noetherian, so Hilbert’s basis theorem is true for any finitely generated abelian group \( A \) in the variety of abelian groups. As a result, every abelian group \( B \) containing \( A \) is \( A \)-equationally noetherian.

As we mentioned above, if \( A \leq B \) and \( B \) is not equationally noetherian, then it is also not \( A \)-equationally noetherian. So, let \( \mathfrak{Y} \) be a variety of algebras and \( A \in \mathfrak{Y} \). Let \( V = \mathfrak{Y}_A \). If there exists an element \( B \in \mathfrak{Y} \) which is not equationally noetherian, then by our theorem, \( F_V(X) \) is not noetherian, so we never have a version of Hilbert’s basis theorem for the variety \( \mathfrak{Y} \).

Example 8. Let \( \mathfrak{Y} \) be the variety of nilpotent groups of class at most \( c \). If \( A \in \mathfrak{Y} \) and \( V = \mathfrak{Y}_A \) and \( B \in \mathfrak{Y} \) is not finitely generated, then by [16], \( B \) is not equationally noetherian and hence \( F_V(X) \) is not noetherian.

6. Other types of equational conditions

During this article, we saw many cases of equational conditions in the universal algebraic geometry. It is easy to find new kinds of equational conditions. Suppose \( P \) is an equational condition in a language \( \mathcal{L} \). A system \( S \) is called a \( P \)-system if, \( S \) satisfies the condition \( P \). An algebra \( A \) is called a \( P \)-algebra, if any system of equations over \( A \) is equivalent to a \( P \)-subsystem. It will be interesting and important, if one can find conditions \( P \) such that any algebra is \( P \)-algebra. In this final section, we give an example of this kind of conditions.

Definition 7. Let \( A \) be an algebra of type \( \mathcal{L} \). A system \( S \) is called \( A \)-independent if for any finite subset \( S_0 \subseteq S \), there exists an equation \((p \approx q) \in S_0 \), such that \( V_A(S_0) \not\subseteq V_A(S_0 \setminus p \approx q) \).

Theorem 9. For any algebra \( A \) and any system \( S \), there exists an \( A \)-independent subsystem \( S' \) equivalent to \( S \) over \( A \).

Proof. Suppose \( A \) and \( S \) are given and \( \mathcal{F} \) is the collection of all \( A \)-independent subsets of \( S \). Using Zorn’s lemma, we show that \( \mathcal{F} \) has a maximal element. Let \( \{T_\alpha\}_\alpha \subseteq \mathcal{F} \) be a chain. Let \( T = \bigcup T_\alpha \) and \( T^0 \subseteq T \) be finite. Then \( T^0 \subseteq T_\alpha \) for some \( \alpha \). Hence, there exists \((p \approx q) \in T^0 \), with \( V_A(T^0) \not\subseteq V_A(T^0 \setminus p \approx q) \).

This shows that \( T \) is \( A \)-independent and so the chain has upper bound. Therefore \( \mathcal{F} \) has a maximal element \( S' \). Suppose \( V_A(S) \not\subseteq V_A(S') \). So,
there exists an element $\overline{a} \in V_A(S') \setminus V_A(S)$, and hence there is an equation $(p_1 \approx q_1) \in S$, such that $p_1(\overline{a}) \neq q_1(\overline{a})$. Let $S'' = S' + (p_1 \approx q_1)$. We show that $S'' \in \mathcal{F}$. Let $T_0 \subseteq S''$ be finite. If $T_0 \subseteq S'$, then there is $(p \approx q) \in T_0$ with
\[
V_A(T_0) \not\subseteq V_A(T_0 \setminus p \approx q).
\]
If $(p_1 \approx q_1) \in T_0$, then $T_0 \setminus p_1 \approx q_1 \subseteq S'$, and hence
\[
\overline{a} \in V_A(T_0 \setminus p_1 \approx q_1).
\]
But, since $p_1(\overline{a}) \neq q_1(\overline{a})$, so $\overline{a}$ does not belong to $V_A(T_0)$. Therefore
\[
V_A(T_0) \not\subseteq V_A(T_0 \setminus p_1 \approx q_1),
\]
and hence $S'' \in \mathcal{F}$, which is impossible. Hence we must have $V_A(S) = V_A(S')$.

\[\square\]

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