The Renormalization Group According to Balaban
III. Convergence

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Abstract

This is an expository account of Balaban’s approach to the renormalization group. The method is illustrated with a treatment of the the ultraviolet problem for the scalar $\phi^4$ model on toroidal lattice in dimension $d = 3$. In this third paper we demonstrate convergence of the expansion and complete the proof of a stability bound.

1 Introduction

We recall the general setup from part I [14] and part II [15]. We are studying the $\phi^4$ field theory on a toroidal lattice of the form

$$
\mathbb{T}^{-N}_M = (L^{-N}Z/L^M Z)^3
$$

The theory is scaled up to the unit lattice $\mathbb{T}^0_{M+N}$ and there the partition function has the form

$$
Z_{M,N} = \int \rho^N_0(\Phi) d\Phi
$$

where for fields $\Phi: \mathbb{T}^0_{M+N} \to \mathbb{R}$ we have the density

$$
\rho^N_0(\Phi) = \exp \left(-S^N_0(\Phi) - V^N_0(\Phi)\right)
$$

with

$$
S^N_0(\Phi) = \frac{1}{2}\|\partial \Phi\|^2 + \frac{1}{2} \mu_0^N \|\Phi\|^2
$$

$$
V^N_0(\Phi) = \varepsilon^N_0 \text{Vol}(\mathbb{T}^0_{M+N}) + \frac{1}{2} \mu_0^N \|\Phi\|^2 + \frac{1}{4} \lambda_0^N \sum_x \Phi^4(x)
$$

and very small positive coupling constants $\lambda_0^N = L^{-N} \lambda, \mu_0^N = L^{-2N} \mu$, etc. The superscript $N$ is generally omitted so we have $\lambda_0, \mu_0, etc.$.

Our goal is to show that with intelligent choices of the counter terms $\varepsilon^N_0, \mu_0^N$ the partition function $Z_{M,N}$ satisfies stability bounds which are uniform in the ultraviolet cutoff $N$ and with bulk dependence on the volume parameter $M$. The method is the renormalization group method of Balaban (I - II).

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In fact our primary goal is not the stability bounds, which are interesting but not new, but rather the illustration of Balaban’s method.

We repeatedly block average starting with \( \rho_0 \) given by (3). Given \( \rho_k(\Phi_k) \) we define for \( \Phi_{k+1} : \mathbb{T}_{M+N-k}^1 \rightarrow \mathbb{R} \) and block averaging operator \( Q[\Pi] \)

\[
\tilde{\rho}_{k+1}(\Phi_{k+1}) = \mathcal{N}_{\alpha L, \mathbb{T}_{M+N-k}^1}^{-1} \int \exp \left( -\frac{1}{2} \alpha L |\Phi_{k+1} - Q\Phi_k|^2 \right) \rho_k(\Phi_k) d\Phi_k
\]

(5)

Next we scale by

\[
\rho_{k+1}(\Phi_{k+1}) = \tilde{\rho}_{k+1}(\Phi_{k+1,L}) L^{-|\mathbb{T}_{M+N-k}^1|/2}
\]

(6)

Then for any \( k \) the partition function can be expressed as

\[
Z_{M,N} = \int \rho_k^N(\Phi_k) d\Phi_k
\]

(7)

We quote the main result on these densities from part II. It says that after \( k \) steps the density can be represented in the form

\[
\rho_k(\Phi_k) = Z_k \sum_{\Pi} d\Phi_k,\Omega^e dW_k,\Pi K_{k,\Pi} C_{k,\Pi} \chi_k(\Lambda_k) \exp \left( -S^+_{\Pi}(\Lambda_k) + E_k(\Lambda_k) + R_{k,\Pi}(\Lambda_k) + B_{k,\Pi}(\Lambda_k) \right)
\]

(8)

where

\[
d\Phi_{k,\Omega^e} = \prod_{j=0}^{k-1} \exp \left( -\frac{1}{2} \alpha L^{-(k-j-1)} |\Phi_{j+1} - Q\Phi_j|^2_{\Omega_{j+1}} \right) d\Phi_{j,\Omega_{j+1}}^{(k-j)}
\]

\[
dW_{k,\Pi} = \prod_{j=0}^{k-1} (2\pi)^{-|\Omega_{j+1}-\Lambda_{j+1}|^{(j)}/2} \exp \left( -\frac{1}{2} L^{-(k-j)} |W_{j}^{2}_{\Omega_{j+1}-\Lambda_{j+1}}| dW_{j,\Omega_{j+1}-\Lambda_{j+1}}^{(k-j)} \right)
\]

\[
K_{k,\Pi} = \prod_{j=0}^{k} \exp \left( c_j |\Omega_{j}^{(j-1)}| - S^+_{j,L}^{+,u}(\Lambda_{j-1} - \Lambda_j) + \left( \tilde{B}_{j,L}^{-(k-j)} \right)_{\Pi_j}(\Lambda_{j-1}, \Lambda_j) \right)
\]

\[
C_{k,\Pi} = \prod_{j=0}^{k} (C_{j,L}^{-(k-j)})_{\Lambda_{j-1},\Omega_j,\Lambda_j}
\]

(9)

Here

\[
\Pi = (\Lambda_0, \Omega_1, \Lambda_1, \ldots, \Omega_k, \Lambda_k)
\]

(10)

is a decreasing sequence of small field regions in \( \mathbb{T}_{M+N-k}^1 \), with \( \Omega_j, \Lambda_j \) a union of \( L^{-(k-j)} M \) cubes.

With \( \delta \Omega_j = \Omega_j - \Omega_{j+1} \) our basic variables are

\[
\Phi_{k,\Omega} = (\Phi_1, \delta \Omega_1, \Phi_2, \delta \Omega_2, \ldots, \Phi_{k-1}, \delta \Omega_{k-1}, \Phi_k, \Omega_k)
\]

(11)

where \( \Phi_j, \delta \Omega_j : (\delta \Omega_j)^{(j)} \rightarrow \mathbb{R} \). There are also variables \( \Phi_0, \Omega_1, \Phi_1, \Omega_2, \ldots, \Phi_k, \Omega_k \) which play a lesser role.

In \( d\Phi_{k,\Omega^e} \) the measure is

\[
d\Phi_{j,\Omega_{j+1}}^{(k-j)} = [L^{-(k-j)/2}] |\Omega_{j+1}^{(j)}| \prod_{x \in [\Omega_{j+1}^{(j)}]} d\Phi_j(x)
\]

(12)

---

1. \( \mathcal{N}_{\alpha,\Omega} = (2\pi/\alpha)^{\Omega/2} \) where \( |\Omega| \) is the number of elements in \( \Omega \).
2. If \( X \subset \mathbb{T}_{M+N-k}^1 \) then \( X^{(j)} \subset \mathbb{T}_{M+N-k}^{(j)} \) are the centers of \( L^j \) cubes in \( X \).
Besides our basic variables there are auxiliary variables

\[ W_{k,\Pi} = (W_{0,\Omega_1 - \Lambda_1}, \ldots, W_{k-1,\Omega_{k-1} - \Lambda_{k-1}}) \]  (13)

with \( W_{j,\Omega_{j-1} - \Lambda_{j-1}} : [\Omega_{j-1} - \Lambda_{j-1}]^{(j)} \rightarrow \mathbb{R} \). In \( dW_{k,\Pi} \) the measure \( dW_{j,\Omega_{j-1} - \Lambda_{j-1}}^{(k-j)} \) is defined as in (12).

We employ the convention that \( \Lambda_{-1}, \Omega_0 \) are the full torus \( T^{N-k}_{M+N-k} \).

The precise statement of the result is the following:

**Theorem 1.** Let \( 0 < \lambda < e^{-1} \) and \( 0 < \bar{\mu} \leq 1 \). Let \( \lambda_k = L^{-(N-k)} \lambda \) and \( \bar{\mu}_k = L^{-2(N-k)} \bar{\mu} \) be running coupling constants. Let \( L \) be sufficiently large, \( \lambda_k \) be sufficiently large (depending on \( L \)), and \( \lambda_k \) be sufficiently small (depending on \( L, M \)). Let \( \varepsilon_k, \mu_k \) be the dynamical coupling constants selected in part I. Then the representation (3), (4) holds with the following properties:

1. \( Z_k \) is the global normalization factor of part I. It satisfies \( Z_0 = 1 \) and

\[ Z_{k+1} = Z_k \cdot N^{k-1}_{-\alpha_k} \left( 2\pi \right)^{2M+2N-k} \left( \det C_k \right)^{1/2} \]  (14)

2. With \( p_k = (\log \lambda_k)^p \) and \( \alpha_k = \max \{ \lambda_k^{1/2}, \lambda_k^{1/4} \} \) the characteristic function \( \chi_k(\Lambda_\Lambda) \) enforces bounds on \( \Lambda_\Lambda \) stronger than:

\[ |\Phi_k| \leq 2p_k \alpha_k^{-1} \quad |\partial \Phi_k| \leq 3p_k \]  (15)

3. The characteristic functions \( C_{k-1,\Omega_{k-1},\Lambda_k}(\Phi_{k-1}, W_{k-1}, \Phi_k) \) enforce bounds on \( \Lambda_{k-1} - \Omega_k \) stronger than:

\[ |\Phi_{k-1}| \leq 2p_{k-1} \alpha_{k-1}^{-1} L^2 \quad |\partial \Phi_{k-1}| \leq 3p_{k-1} L^2 \]  (16)

and enforce bounds on \( \Omega_k - \Lambda_k \) stronger than:

\[ |\Phi_k| \leq 3p_{k-1} \alpha_{k-1}^{-1} L^2 \quad |\partial \Phi_k| \leq 4p_{k-1} L^2 \quad |W_{k-1}| \leq C_k p_{k-1} L^2 \]  (17)

for some constant \( C_k \). In the expression (3) this is scaled down by \( L^{-(k-j)} \).

4. The bare action is \( S_k^+(\Lambda_k) = S_k^+(\Lambda_k, \Phi_k, \phi_k, \Omega_{\Lambda_{k-1}}) \) where \( \phi_k, \Omega_{\Lambda_{k-1}} \) is a field approximately localized in \( \Lambda_{k-1} \), an enlargement of \( \Lambda_k \), and for \( \phi : T^{N-k}_{M+N-k} \rightarrow \mathbb{R} \)

\[ S_k^+(\Lambda_k, \Phi_k, \phi) = S_k^+(\Lambda_k, \Phi_k, \phi) + V_k(\Lambda_k, \phi) \]

\[ S_k^+(\Lambda_k, \Phi_k, \phi) = \frac{\alpha_k}{2} \| \Phi_k - \bar{Q}_k \phi \|_{\Lambda_k}^2 + \frac{1}{2} \| \partial \phi \|_{\Lambda_k}^2 + \frac{1}{2} \| \phi \|_{\Lambda_k}^2 \]  (18)

\[ V_k(\Lambda_k, \phi) = \varepsilon_k \text{Vol}(\Lambda_k) + \frac{1}{2} \mu_k \| \phi \|_{\Lambda_k}^2 + \frac{1}{4} \lambda_k \int_{\Lambda_k} \phi^4 \]

5. \( E_k(\Lambda_k) = E_k(\Lambda_k, \phi_k, \Omega_{\Lambda_{k-1}}) \) are the main corrections to the bare action. For \( \phi : T^{N-k}_{M+N-k} \rightarrow \mathbb{R} \) it has the local expansion

\[ E_k(\Lambda_k, \phi) = \sum_{X \subset \Lambda_k} E_k(X, \phi) \]  (19)

where \( X \) is a connected union of \( M \)-cubes. \( E_k(X, \phi) \) depends on \( \phi \) in \( X \), is analytic in a certain complex domain \( \phi \in \mathcal{R}_k(X, \epsilon) \), and satisfies there for \( \beta < \frac{1}{4} - 10\epsilon \)

\[ |E_k(X)| \leq \lambda_k^2 \epsilon^{-\kappa M(X)} \]  (20)

where \( M_d(M(X)) \) is the length of the shortest tree joining the \( M \)-cubes in \( X \).
6. \( R_{k, \Pi}(\Lambda_k) = R_{k, \Pi}(\Lambda_k, \Phi_k) \) is a tiny remainder and has the local expansion

\[
R_{k, \Pi}(\Lambda_k, \Phi_k) = \sum_{X \subset \Lambda_k} R_{k, \Pi}(X, \Phi_k)
\]  

(21)

where \( R_{k, \Pi}(X, \Phi_k) \) is analytic in a certain complex domain \( \mathcal{P}_k(X, 2\delta) \), and satisfies there for a fixed integer \( n_0 \geq 4 \):

\[
|R_{k, \Pi}(X)| \leq \lambda_k^{n_0} e^{-\kappa d_M(X)}
\]  

(22)

7. \( B_{k, \Pi}(\Lambda_k) = B_{k, \Pi}(\Lambda_k, \Phi_k, \Omega, W_{k, \Pi}) \) is the active boundary term. It has the local expansion

\[
B_{k, \Pi}(\Lambda_k) = \sum_{X \in \mathcal{P}_k(\Omega, \Phi, \Omega), X \# \Lambda_k} B_{k, \Pi}(X)
\]  

(23)

where \( X \# \Lambda_k \) means \( X \) intersects both \( \Lambda_k \) and \( \Lambda_k^c \). The function \( B_{k, \Pi}(X, \Phi_k, \Omega, W_{k, \Pi}) \) is analytic in a certain complex domain \( \mathcal{P}_k(\Omega, \Phi, \Omega) \) and

\[
|W_j| \leq B_w p_j L^5(k-j) \quad \text{on} \quad \Omega_{j+1} - \Lambda_{j+1}
\]  

(24)

and it satisfies there

\[
|B_{k, \Pi}(X)| \leq B_0 \lambda_k^2 e^{-\kappa d_M(X, \Omega)}
\]  

(25)

for some constant \( B_0 \) depending on \( L, M \).

8. \( \tilde{B}_{k, \Pi}(\Lambda_k-1, \Lambda_k) = \tilde{B}_{k, \Pi}(\Lambda_k-1, \Lambda_k, \Phi_k, \Omega, W_{k, \Pi}) \) is the inactive boundary term. It depends on the variables only in \( \Omega_1 - \Lambda_k \), is analytic in \( \mathcal{P}_k(\Omega, \Phi, \Omega) \) and

\[
|\tilde{B}_{k, \Pi}(\Lambda_k-1, \Lambda_k)| \leq B_0 \left| \Lambda_k^{(k)} - \Lambda_k^{(k)} \right| = B_0 \text{Vol}(\Lambda_k - 1 - \Lambda_k)
\]  

(26)

Also it is additive in the connected components of \( \Lambda_k^c \).

9. With \( \delta \Lambda_k-1 = \Lambda_k-1 - \Lambda_k \), the unrenormalized action is \( S_k^{+,-u}(\delta \Lambda_k-1, \Phi_k, \Omega, \phi_{k, \Omega}(\Lambda_k-1, \Omega, \Lambda_k)) \) where \( \phi_{k, \Omega}(\Lambda_k-1, \Omega, \Lambda_k) \) is a field approximately localized in \( \Lambda_k-1 \cap (\Lambda_k^c)^* \) and

\[
\begin{align*}
S_{k}^{+,-u}(\delta \Lambda_k-1, \Phi_k, \Omega, \phi) &= S_k^+(\delta \Lambda_k-1, \Phi_k, \Omega, \phi) + V_k^u(\delta \Lambda_k-1, \phi) \\
V_k^u(\Lambda, \phi) &= L^3 \varepsilon_{k-1} \text{Vol}(\Lambda) + \frac{L^2}{2} \mu_{k-1} ||\phi||_2 + \frac{1}{4} \lambda_k \int_\Lambda \phi^4
\end{align*}
\]  

(27)

Remark. The complex domains \( \mathcal{R}_k(X, \varepsilon), \mathcal{P}_k(X, 2\delta), \mathcal{P}_k(\Omega, \Phi, \Omega) \) are defined in section 3.2 of part II.

Convention: Throughout the paper \( \mathcal{O}(1) \) stands for a generic constant independent of all parameters, \( C \) stands for a generic constant possibly depending on \( L \).
2 The last step

For the rest of the paper we take \( \bar{\mu} = 1 \) and fulfill the condition by \( \lambda_k \) be sufficiently small by requiring that \( \lambda \) be sufficiently small. Then we can run the iteration all the way to \( k = N \) and we have an expression back on the original torus \( T_{M}^{-N} \). (In the terminology of paper I we are taking \( K = N \) and \( \Delta = 0 \)). At the end our running coupling constants are \( \bar{\mu}_N = \bar{\mu} = 1 \) and \( \lambda_N = \lambda \). The dynamical coupling constants are fixed to satisfy \( \varepsilon_N = 0, \mu_N = 0 \) by the choice of initial conditions \( \varepsilon_0 = \varepsilon_0^N, \mu_0 = \mu_0^N \).

The partition function is now given by \( Z_{M,N} = \int \rho_N(\Phi_N)d\Phi_N \) and substituting the expression (3) for \( \rho_N(\Phi_N) \) it is

\[
Z_{M,N} = Z_N \sum_\Pi \int d\Phi_N d\Omega^{-} dW_{N,\Pi} K_{N,\Pi} C_{N,\Pi} \chi_N(\Lambda_N) \exp \left( -S_N(\Lambda_N) + E^+_N(\Lambda_N) + R_N,\Pi(\Lambda_N) + B_N,\Pi(\Lambda_N) \right)
\]  

(28)

Here we have transferred the potential from \( S^+_N(\Lambda_N) \) to \( E^+_N(\Lambda_N) \).

The mass \( \bar{\mu}_N = \bar{\mu} = 1 \) will enable us to control the final integral over \( \Phi_N \). Before estimating this expression we manipulate it into a form which exhibits the local structure. These manipulations are similar to the general step in the iteration and they will occupy the reminder of this section. The final estimates come in the next section.

2.1 minimizers

We analyze further the integral over \( \Phi_N \). Split the integral on \( 3 \)

\[
\Omega_{N+1} \equiv \Lambda_N^{5^*}
\]

(29)

With this definition there is no new large field region \( (P_{N+1} = 0) \) and \( \Omega_{N+1} \) is a union of \( M \) blocks (rather than \( LM \) blocks as in the general step). We split \( \Phi_{N,\Omega_N} = (\Phi_N,\Omega_{N+1}, \Phi_N,\Omega_{N+1}) \) and analyze the integral over \( \Phi_{N,\Omega_{N+1}} \) in more detail. For this idea would be to take the main term in the action \( S^+_N(\Lambda_N, \Phi_N, \phi_N, \Omega(\Lambda_N^*) \) and expand around the minimum in \( \Phi_{N,\Omega_{N+1}} \).

Instead we use the minimizer for a related action better suited to \( \phi_N,\Omega(\Lambda_N^*) \). Let

\[
\tilde{\Phi}_N,\Omega(\Lambda_N^*) = \tilde{Q}^{T_N}_N,\Omega(\Lambda_N^*) \Phi_N = \left( [Q_N^{T_N} \Phi_N]|_{\Omega_1(\Lambda_N^*)}, [Q_N^{T_N} \Phi_N]|_{\Omega_2(\Lambda_N^*)} \Phi_N \right)
\]

(30)

then \( \phi_N,\Omega(\Lambda_N^*) = \phi_N,\Omega(\Lambda_N^*) (\tilde{\Phi}_N,\Omega(\Lambda_N^*)) \) is defined to be the minimizer in \( \phi \) on \( \Omega_1(\Lambda_N^*) \) for

\[
\frac{1}{2} \|a(\tilde{\Phi}_N,\Omega(\Lambda_N^*) - Q_N,\Omega(\Lambda_N^*) \phi)\|_{\Omega_1(\Lambda_N^*)}^2 + \frac{1}{2} \|\partial \phi\|^2 + \frac{1}{2} \|\bar{\mu}N\| \phi\|^2
\]

(31)

with \( \phi = Q_N^{T_N} \Phi_N \) on \( \Omega_1(\Lambda_N^*)^c \).

With \( \Omega_{N+1} \) we introduce \( \Omega' = (\Omega(\Lambda_N^*), \Omega_{N+1}) \) and \( \delta \Omega' = (\delta \Omega_1', \ldots, \delta \Omega_N') \). Then \( \Omega_1(\Lambda_N^*) = \Omega'_1 \) and we further split \( 30 \) as

\[
\Phi_{N,\Omega(\Lambda_N^*)} = \left( [Q_N^{T_N} \Phi_N]|_{\Omega_1^*}, \Phi_{N,\delta \Omega_1} \right)
\]

(32)

where

\[
\Phi_{N,\delta \Omega'} = \left( [Q_N^{T_N} \Phi_N]|_{\delta \Omega_1'}, \ldots, [Q_N^{T_N} \Phi_N]|_{\delta \Omega_{N-1}}, \Phi_{N,\delta \Omega_N} \right)
\]

(33)

\footnote{Let \( \tau_k = (-\log \lambda_k) \). Recall that for a union of \( M \) cubes \( X \) in \( T^k_{N+M-k} \), \( X^* \) is an enlargement by \( \lfloor \tau_k \rfloor \) layers of \( M \) cubes, and \( X^\times \) is a shrinkage of \( X \) by \( \lfloor \tau_k \rfloor \) layers of \( M \) cubes.}
We ask for the minimizer of (31) in $\Phi_{N,\Omega_{n+1}}$ and $\phi_{\Omega'}$. This is discussed in a general context in appendix A; the minimum comes at $\Phi_{N,\Omega_{n+1}} = \Psi_{N,\Omega_{n+1}}(\delta \Omega')$ and at $\phi_{\Omega'} = \phi_{N,\delta \Omega'}$ where

$$
\phi_{N,\delta \Omega'} = \phi_{N,\delta \Omega'}(\{\tilde{\Phi}_{N,\Omega_{n+1}}\}_{n+1})
$$

$$
\equiv G_{N,\delta \Omega'}\left(Q_{N,\delta \Omega'}^T a \tilde{\Phi}_{N,\delta \Omega'} + \nu_{\Omega',\Omega'}^T, Q_{N,\Phi_{N}}^T\right) 
$$

$$
\Psi_{N,\Omega_{n+1}}(\delta \Omega') \equiv [Q_{N,\Phi_{N,\delta \Omega'}}]_{n+1} 
$$

Here

$$
G_{N,\delta \Omega'} = \left[-\Delta + \tilde{\mu}_N + Q_{N,\delta \Omega'}^T a \tilde{G}_{N,\delta \Omega'}\right]^{-1} 
$$

With our choice of $N$, the mass term $\tilde{\mu}_N$ in $G_{N,\delta \Omega'}$ is now substantial. As a consequence the kernel of $G_{N,\delta \Omega'}$ has exponential decay as we will see, and so $\phi_{N,\delta \Omega'}$ is approximately localized in $\Omega_{n+1}$. We also note the identity from appendix A

$$
\phi_{N,\delta \Omega'} = \phi_{N,\Omega_{n+1}}\left(\{\tilde{\Phi}_{N,\Omega_{n+1}}\}_{n+1}, \Psi_{N,\Omega_{n+1}}(\delta \Omega')\right) 
$$

Returning to the original problem with $S_{N}(\Lambda_N, \Phi_N, \phi_{N,\Omega_{n+1}})$ we expand around the minimizer by making the translation

$$
\Phi_{N,\Omega_{n+1}} = \Psi_{N,\Omega_{n+1}}(\delta \Omega') + Z 
$$

and change the integral to an integral over $Z : \Omega_{n+1} \rightarrow \mathbb{R}$. Then by (36) and defining $Z_N = \phi_{N,\Omega_{n+1}}(0, Z)$

$$
\phi_{N,\Omega_{n+1}} = \phi_{N,\Omega_{n+1}}(\{\tilde{\Phi}_{N,\Omega_{n+1}}\}_{n+1}, \Psi_{N,\Omega_{n+1}}) 
$$

$$
= \phi_{N,\Omega_{n+1}}(\{\tilde{\Phi}_{N,\Omega_{n+1}}\}_{n+1}, \Psi_{N,\Omega_{n+1}}(\delta \Omega')) + \phi_{N,\Omega_{n+1}}(0, Z) 
$$

$$
\equiv \phi_{N,\delta \Omega'} + Z_N 
$$

Lemma 1.

$$
S_{N}(\Lambda_N, (\Phi_{N,\delta \Omega_{n}}, \Psi_{N,\Omega_{n+1}}(\delta \Omega') + Z), \phi_{N,\delta \Omega'} + Z_N) 
$$

$$
= S_{N}(\Lambda_N - \Omega_{n+1}, \Phi_{N,\delta \Omega_{n}}, \phi_{N,\delta \Omega'} + \hat{S}_N(\Omega_{n+1}, \psi_{N,\delta \Omega'} + \frac{1}{2}(Z, \Delta \Phi_{n+1}, Z)) + R_{n+1}^{N+1} \Omega_{n+1} 
$$

where $\hat{S}_N(\Omega, \phi) = \frac{1}{2}\|\phi\|_{L^2}(\Omega) + \frac{1}{2}\mu_N\|\phi\|_{L^2}(\Omega)$ and $R_{n+1}^{N+1} \Omega_{n+1}$ is a tiny term.

Remark. A "tiny term" is $O(\lambda^{-n_0})$ for our standard integer $n_0 \geq 4$. We are more precise about this when we discuss localization.

Proof. With $\Psi_N = \Psi_{N,\Omega_{n+1}}(\delta \Omega')$ we expand in $Z$ and find as in lemma 2.4 in part II:

$$
S_{N}(\Lambda_N, (\Phi_{N,\delta \Omega_{n}}, \Psi_{N}) + (0, Z), \phi_{N,\delta \Omega'} + Z_N) = S_{N+1}(\Lambda_N, (\Phi_{N,\delta \Omega_{n}}, \Psi_N), \phi_{N,\delta \Omega'}) + a_k < Z, (\Psi_N - Q_N \phi_{N,\delta \Omega'}) >_{\Lambda_N} + b_{\Lambda_N}(\partial \phi_{N,\delta \Omega'}, Z_N) + S_{N}(\Lambda_N, (0, Z), Z_N) 
$$

Here we have made a cancellation in the linear terms using again (36). The term $b_{\Lambda_N}(\partial \phi_{N,\delta \Omega'}, Z_N)$ is a boundary term localized on $\partial \Lambda_N$. It is tiny since $Z_N$ is tiny on $\partial \Lambda_N$, and so contributes to $R_{n+1}^{N+1} \Omega_{n+1}$. 

6
The second term on the right side of this equation vanishes by the definition of $\Psi_N$. As in lemma 2.5 in part II, the last term can be written

$$S_N^0(\Lambda_N, (0, Z), Z_N) = \frac{1}{2} \left\langle Z, \left[\Delta_{\Omega_N(\Lambda_N)}\right]_{\Omega_{N+1}} Z \right\rangle$$

(41)

where the omitted terms are tiny and contribute to $R^{(1)}_{II\Omega_{N+1}}$. Finally using the definition of $\Psi_N$ again we have

$$S_N(\Lambda_N, (\Phi_{N,\delta\Omega_N}, \Psi_N), \phi_{N,\delta\Omega'}) = S_N^0(\Lambda_N - \Omega_{N+1}, \Phi_{N,\delta\Omega_N}, \phi_{N,\delta\Omega'}) + \hat{S}_N(\Omega_{N+1}, \phi_{N,\delta\Omega'})$$

(42)

to complete the proof.

Actually we use a modification of (37) as in part II. The propagator $G_{N,\delta\Omega'}$ has a random walk expansion, explained in more detail later. Hence one can introduce weakening parameters $s = \{s_\square\}$ for multiscale cubes $\square$, and define a weakened version $G_{N,\delta\Omega'}(s)$. This leads to a weekend field $\phi_{N,\delta\Omega'}(s)$, hence a field $\phi_{N,\delta\Omega'}(\square)$ localized in $\square'$. Approximating $\phi_{N,\delta\Omega'}(\square)$ by $\phi_{N,\delta\Omega'}(\square)$ on $\square$ gives a localized field $\phi_{N,\delta\Omega'}^{loc}$ and hence a localized minimizer $\Psi_{N,\delta\Omega'}^{loc,\Omega_{N+1}}(\delta\Omega')$. The actual translation is then

$$\phi_{N,\delta\Omega_N} = \Psi_{N,\Omega_{N+1}}^{loc}(\delta\Omega') + Z$$

(43)

This is done for benefit of the characteristic functions. But in $S_N^0(\Lambda_N - \Omega_{N+1}) + \hat{S}_N(\Omega_{N+1})$ and in $E_N^+(\Lambda)$ we immediately undo it and return from $\Psi_{N,\Omega_{N+1}}^{loc}(\delta\Omega')$ to $\Psi_{N,\Omega_{N+1}}(\delta\Omega')$ at the cost of some more tiny terms $R^{(2)}_{II\Omega_{N+1}}, R^{(3)}_{II\Omega_{N+1}}$. We define $R^{(\leq 3)}_{II\Omega_{N+1}} = R_{N,II}(\Lambda_N) + R^{(1)}_{II\Omega_{N+1}} + \cdots + R^{(3)}_{II\Omega_{N+1}}$.

Now (28) can be written

$$Z_{M,N} = Z_N \sum_\Pi \int d\tilde{\Phi}_{N,\Omega'} dW_{N,\Pi} K_{N,\Pi} C_{N,\Pi} \exp \left(-S_N^0(\Lambda_N - \Omega_{N+1}) - \hat{S}_N(\Omega_{N+1})\right)$$

$$\int dZ \chi_N(\Lambda_N) \exp \left(-\frac{1}{2} \left\langle Z, \left[\Delta_{\Omega_N(\Lambda_N)}\right]_{\Omega_{N+1}} Z \right\rangle + E_N^+(\Lambda_N) + R^{(\leq 3)}_{II\Omega_{N+1}} + B_{N,II}(\Lambda_N)\right)$$

(44)

Here

$$d\tilde{\Phi}_{N,\Omega'} = d\Phi_{N,\Omega_{N+1}}d\Phi_{N,\Omega'}$$

(45)

and $S_N^0(\Lambda_N - \Omega_{N+1}), \hat{S}_N(\Omega_{N+1})$ are as in lemma [1] and

$$E_N^+(\Lambda_N) = E_N^+(\Lambda_N, \phi_{N,\delta\Omega'} + Z_N)$$

$$\chi_N(\Lambda_N) = \chi_N(\Lambda_N, \phi_{N,\delta\Omega_N}, \Psi_{N,\Omega_{N+1}}^{loc}(\delta\Omega') + Z)$$

(46)

and $R_{N,II}(\Lambda_N)$ and $B_{N,II}(\Lambda_N)$ also have their arguments shifted by (43).

### 2.2 fluctuation integral

The integral over $Z$ is a Gaussian integral with covariance

$$\tilde{C}_{N,\Omega'} = \left[\Delta_{N,\Omega_N(\Lambda_N)}\right]^{-1}_{\Omega_{N+1}}$$

(47)

There is now no term $aL^{-2}Q^TQ$ in the operator we are inverting, as was the case in earlier steps. To obtain an ultra local measure we want to change variables by $Z = \tilde{C}_{N,\Omega'}^{-1/2}W$ where $W : \Omega_{N+1}^N \to \mathbb{R}$. 

\[7\]
For this we need to control the operator $\hat{C}^{1/2}_{N,\Omega}$. It has the representation (Appendix C in part II with $a = 0$):

$$\hat{C}^{1/2}_{N,\Omega} = \frac{1}{\pi} \int_0^\infty \frac{dr}{\sqrt{r}} \hat{C}_{N,\Omega,r} \quad \hat{C}_{N,\Omega'} = \left[ \Delta_{N,\Omega(\Lambda_n^0)} + r \right]^{-1}_{\Omega_{n+1}}$$

And $\hat{C}_{N,\Omega,r}$ has the representation (Appendix C in part II with $a = 0$)

$$\hat{C}_{N,\Omega',r} = \left[ \frac{1}{a_N + r} + \left( \frac{a_N}{a_N + r} \right)^2 Q_N G_{N,\delta,\Omega',r} Q_N^T \right]_{\Omega_{n+1}}$$

where

$$G_{N,\delta,\Omega',r} = \left[ -\Delta + \mu_N + Q_N^T \delta Q_N \right. + \frac{a_N r}{a_N + r} \left[ Q_N^T Q_N \right]_{\Omega_{n+1}} \left. \right]_{\Omega_{n+1}}$$

The Green’s function $G_{N,\delta,\Omega',r}$ has a random walk expansion and hence there is a weakened form $G_{N,\delta,\Omega',r}(s)$. This leads to weakened forms $\hat{C}_{N,\Omega,s}(s)$, $\hat{C}^{1/2}_{N,\Omega}(s)$, therefore to $\hat{C}_{N,\Omega,r}(s^*)$, $\hat{C}^{1/2}_{N,\Omega}(s^*)$ and so to $(\hat{C}^{1/2}_{N,\Omega})^{loc}$. The actual change of variables is then

$$Z = \left( \hat{C}^{1/2}_{N,\Omega} \right)^{loc} W_N$$

This localization is for the benefit of the characteristic functions. In $E_k(\Lambda_N)$ we immediately change back to $\hat{C}^{1/2}_{N,\Omega} W_N$. The new measure in $W_N$ is replaced by the Gaussian measure $d\mu_{\Omega_{n+1}}(W_N)$ with identity covariance. Furthermore the induced determinant $\det((\hat{C}^{1/2}_{N,\Omega})^{loc})$ is changed back to $\det(\hat{C}^{1/2}_{N,\Omega})$, and then to a global determinant $\det(\Delta_{N}^{\frac{1}{2}})$, the special case in which all small field regions are the whole torus. The operator $\Delta_{N}$ has the representation $\Delta_{N} = -\Delta + \mu_{N} + Q_N^T \delta Q_N$ and $\Delta_{N}^{\frac{1}{2}}$ has a representation which is a special case of (48) - (50). All these replacements are at the cost of further tiny terms $R_{\Pi,\Omega_{n+1}}^{(1)}$, $\ldots$, $R_{\Pi,\Omega_{n+1}}^{(7)}$ and an overall volume factor $\exp(\tilde{c}_n|Q_N^{(N)}|)$. See part II for more details.

We also note that we can identify $Z_N \det(\Delta_{N}^{\frac{1}{2}})$ as the bare normalization factor $Z_{M,N}(0)$ defined with $\lambda_0 = 0$ and no counter terms (i.e. $V_0 = 0$). This is so since if we run the global renormalization group as in section 2.2 in part I with we find

$$Z_{M,N}(0) = \int \rho_N(\Phi_N) d\Phi_N = Z_N \int \exp \left( -S_N(\Phi_N, \phi_N) \right) d\Phi_N$$

$$= Z_N \int \exp \left( -\frac{1}{2} < \Phi_N, \Delta_{N} \Phi_N > \right) d\Phi_N = Z_N(\det(\Delta_{N}^{\frac{1}{2}}))$$

With these changes we find

$$Z_{M,N} = Z_{M,N}(0) \sum \prod d\tilde{\Phi}_N,\Omega = dW_N,\Pi K_N,\Pi G_{N,\Pi} \exp \left( -S_N(\Lambda_N - \Omega_{N+1}) - S_N(\Omega_{N+1}) \right)$$

$$\exp \left( \tilde{c}_n|Q^{(N)}_{\Omega_{n+1}}(\Lambda_N) \right) \int d\mu_{\Omega_{n+1}}(W_N) \chi_N(\Lambda_N) \exp \left( E_N^+ (\Lambda_N) + R_{\Pi,\Omega_{n+1}}^{(1)} + B_N,\Pi (\Lambda_N) \right)$$

The field $Z_N = a_N G_{\delta,\Omega(\Lambda_n^0)} Q_N^T Z$ has become

$$\tilde{W}_{N,\Omega} = a_N G_{N,\Omega(\Lambda_n^0)} Q_N^T (\hat{C}^{1/2}_{N,\Omega} W_N)$$

(54)
Lemma 2. We collect some estimates we need.

\[ E_N^\pm (\Lambda_N) = E_N^\pm (\Lambda_N, \phi_N, \delta \Omega') + \tilde{W}_N(\delta \Omega') \]
\[ \chi(\Lambda_N) = \chi_{\delta \Omega', \delta} \left( \Lambda_N, \Phi_N, \delta \Omega', \Psi_{N, \Omega, N+1}^{\text{loc}}(\delta \Omega') + \left( C_{N, \Omega, N+1}^{\text{loc}} \right)^{\text{loc}} W_N \right) \]

with the change of variables (51) also made where appropriate in \( R_{\Omega, \Omega_{n+1}}^{(\leq 7)} \) and \( B_{N, \Omega}(\Lambda_N) \).

2.3 estimates

We collect some estimates we need.

Lemma 3. The Green’s function \( G_{N, \delta \Omega'} \) has a random walk expansion based on multi-scale cubes for \( \delta \Omega' \), convergent in \( L^2 \) and \( L^\infty \) norms for \( M \) sufficiently large. There are constants \( C, \gamma_0 \) (depending on \( L \)) such that for \( L^{-(N-j)} \) cubes \( \Delta_y \subset \delta \Omega_j \) and \( L^{-(N-j)} \) cubes \( \Delta_{y'} \subset \delta \Omega_{j'} \):

\[ |1_{\Lambda_y} \partial G_{N, \delta \Omega'} \delta \Omega'_{1\Delta_{y'}} f| \leq CL^{-2(N-j)} e^{-\frac{1}{2} \gamma_0 d \Omega'_{(y, y')}} \|f\|_{\infty} \]
\[ L^{-\alpha(N-j)} |1_{\Lambda_y} \partial G_{N, \delta \Omega'} \delta \Omega'_{1\Delta_{y'}} f| \leq CL^{-2(N-j)} e^{-\frac{1}{2} \gamma_0 d \Omega'_{(y, y')}} \|f\|_{\infty} \]

Proof. This follows the proof of Theorem 2.2 in part II, to which we refer for details. In that theorem, specialized to the case at hand, the proof is based on local inverses for multi-scale cubes \( \square \) given by

\[ G_{N, \delta \Omega'}(\square) = \left[ -\Delta + \bar{\mu}_N + Q_T k, \delta \Omega_{(\Lambda_N)}^{(\Delta)} a \Omega_{(\Lambda_N)}^{(\Delta)} \right]^{-1} \]

Now with \( \Omega' = (\Lambda_N^+, \Omega_{k+1}) \) we modify this to \( G_{N, \delta \Omega'}(\square) \) defined in (55). The only difference is that there are no averaging operators in \( \Omega_{N+1} \). Then if \( \square \subset \Omega_{N+1} \) we have since \( \bar{\mu}_N = 1 \)

\[ G_{N, \delta \Omega'}(\square) = \left[ -\Delta + 1 \right]^{-1} \]

This satisfies the same bounds as \( G_{N, \delta \Omega'}(\square) \). Also the operator \( \mathcal{H}_N \) in lemma 31 in part I, now with no averaging operators and \( \bar{\mu}_N = 1 \), satisfies the same bounds. Thus the proof goes through as before.

Remark. The random walk expansion still has \( M \) cubes in \( \Omega_{N+1} \), unlike the general step where it had \( LM \) cubes.

Lemma 3. The Green’s function \( G_{N, \delta \Omega'} \) has a random walk expansion convergent in the \( L^2 \) norm for \( M \) sufficiently large. It yields the bounds for all \( r \geq 0 \)

\[ \|1_{\Delta_y} G_{N, \delta \Omega'} \delta \Omega'_{1\Delta_{y'}} f\|_2 \leq CL^{-2(N-j)} e^{-\gamma_0 d \Omega'_{(y, y')}} \|f\|_2 \]

Proof. This follows the proof of lemma 3.5 in part II, to which we refer for details. Again the absence of averaging operator in \( \Omega_{N+1} \) is compensated by \( \bar{\mu}_N = 1 \).
Lemma 4.

\[ |\tilde{C}_{N,\Omega}^{1/2} f|, \quad \left| (\tilde{C}_{N,\Omega}^{1/2})^{\text{loc}} f \right| \leq C \|f\|_{\infty} \]

\[ |\delta \tilde{C}_{N,\Omega}^{1/2} f| = \left| (\tilde{C}_{N,\Omega}^{1/2} - (\tilde{C}_{N,\Omega}^{1/2})^{\text{loc}}) f \right| \leq C \|f\|_{\infty} e^{-\tau N} \]  

(60)

Proof. \( \tilde{C}_{N,\Omega}^{1/2} \) is expressed in terms of \( G_{N,\delta \Omega',r} \) in [48-50]. The results then follow from the previous lemma, as in lemma 3.6 in part II.

2.4 new characteristic functions

Since \( \phi_{N,\delta\Omega'} \) is already tiny inside \( \Omega_{N+1} \) we do not introduce any new conditions on this field. Thus \( Q_k = \emptyset \). We still need a small field expansion to remove the non-locality in the characteristic function. We therefore introduce

\[ 1 = \sum_{\Lambda_{N+1} \subset \Omega_{N+1}} \chi_{w}^{w}(\Omega_{N+1}) \chi_{N+1}^{w}(\Omega_{N+1} - R_{N+1}) \]  

(61)

where \( \chi_{N+1}^{w}(\Lambda_{N+1}) \) enforces \( |W_N| \leq \rho_{0,N} \) everywhere in \( \Lambda_{N+1} \). The new small field region is

\[ \Lambda_{N+1} = \Omega_{N+1}^{\delta_{N+1}} - R_{N+1}^{\delta_{N+1}} \quad \text{or} \quad \Lambda_{N+1} = (\Omega_{N+1}^{\delta_{N+1}})^{5^*} \cup R_{N+1}^{\delta_{N+1}} \]  

(62)

Then (61) is rewritten as

\[ 1 = \sum_{\Lambda_{N+1} \subset \Omega_{N+1}^{\delta_{N+1}}} \tilde{C}_{N+1}(\Omega_{N+1}, \Lambda_{N+1}) \chi_{N}^{w}(\Lambda_{N+1}) \]

(63)

\[ \tilde{C}_{N+1}(\Omega_{N+1}, \Lambda_{N+1}) = \sum_{\Lambda_{N+1} \subset \Omega_{N+1}^{\delta_{N+1}}} \zeta_{N+1}^{w}(\Omega_{N+1}^{\delta_{N+1}}) \chi_{N+1}^{w}(\Omega_{N+1}^{\delta_{N+1}} - \Lambda_{N+1}) \]

This is inserted under the integral signs in (68) and then the sum is taken outside the integrals.

The characteristic functions are now

\[ C_{N,\Pi}^{w} \chi_{N}(\Lambda_{N+1}) \tilde{C}_{N+1}(\Omega_{N+1}, \Lambda_{N+1}) \chi_{N+1}^{w}(\Lambda_{N+1}) = \tilde{C}_{N+1,\Pi^{+}}^{w} \chi_{N}(\Lambda_{N+1}^{*}) \chi_{N}^{w}(\Lambda_{N+1}) \]  

(64)

where with \( \Pi^{+} = (\Pi, \Omega_{N+1}, \Lambda_{N+1}) \)

\[ \tilde{C}_{N+1,\Pi^{+}}^{w} = C_{N,\Pi}^{w} \chi_{N}(\Lambda_{N+1}^{*}) \tilde{C}_{N+1}(\Omega_{N+1}, \Lambda_{N+1}) \]  

(65)

Lemma 5. \( \tilde{C}_{N+1,\Pi^{+}}^{w} \) is independent of \( W_N \) in \( \Lambda_{N+1}^{*} \) and on the support \( \tilde{C}_{N+1,\Pi^{+}}^{w} \chi_{N}^{w}(\Lambda_{N+1}) \)

\[ \chi_{N}(\Lambda_{N+1}^{*}) = 1 \]  

(66)

Proof. We must show that \( \Psi_{N,\Omega_{N+1},\delta \Omega'}^{\text{loc}}(\delta \Omega') + (\tilde{C}_{N,\Omega}^{1/2})^{\text{loc}} W_N \) is in the space \( S_N(\Box) \) for any \( M \)-cube \( \Box \in \Lambda_{N+1}^{*} \). We show separately that \( \Psi_{N,\Omega_{N+1},\delta \Omega'}^{\text{loc}}(\delta \Omega') \) and \( (\tilde{C}_{N,\Omega}^{1/2})^{\text{loc}} W_N \) are in \( S_N(\Box) \).
The functions \( \tilde{C}_{N+1}^1 \) force that \( |W_N| \leq p_0 N \) on \( \Lambda_{N+1}^{3*} \). Then by (60) we have on \( \Lambda_{N+1}^{3*} \)
\[
|\langle C_{N,\Omega}^{1/2} \rangle| W_N \rangle \leq C p_{0N} = \left( \frac{C p_{0N}}{p_N} \right) p_N
\]
and the derivative satisfies the same bound since we are on a unit lattice. By lemma 3.1 in part II it follows that for \( \square \subset \Lambda_{N+1}^{3*} \) that \( \langle C_{N,\Omega}^{1/2} \rangle| W_N \rangle \) is in \((C p_{0N}/p_N)S_N(\square)\). But for \( \lambda \) sufficiently small and \( p_0 < p \)
\[
C p_{0N} = C(- \log \lambda)^{p_0 - \frac{1}{2}} \leq \frac{1}{2}
\]
Therefore \( \langle C_{N,\Omega}^{1/2} \rangle| W_N \rangle \in \frac{1}{2} S_N(\square) \).

For the second point note that \( \phi_{\Omega,\delta} \) depends on \( \Phi_N \) on \( \delta \Omega = \Omega_N - \Omega_{N+1} \). But \( C_{N,\Omega} \) gives a bound on \( \Omega_N - \Lambda_N \) and \( \chi_N(\Lambda_N - \Lambda_{N+1}^{3*}) \) gives a bound on \( \Lambda_N - \Omega_{N+1} \). These imply \( |\Phi_N| \leq C p_{0N} \alpha_{N-1} \leq C p_{0N}^{1/2} \). Since \( \Lambda_{N+1}^{3*} \) is at least \( 2[p_N] \) layers of \( M \) blocks away from \( \Omega_{N+1} \), the estimates on \( C_{N,\Omega} \) give that on \( \Lambda_{N+1}^{3*} \) we have \( |\omega_{\Omega,\delta}| \leq e^{-\gamma} \). Then \( \Psi_{\Omega_N,\Omega_{N+1}}(\delta \Omega) = [\Omega_N \phi_{\Omega,\delta}]\Omega_{N+1} \) satisfies a similar bound as does \( \Psi_{\Omega_N,\Omega_{N+1}}(\delta \Omega) \in \frac{1}{2} S_N(\square) \). This completes the proof.

We use this result in (63). We also split the measure \( d\mu_{\Omega_{N+1}}(W_N) \) on \( \Lambda_{N+1} \) and identify
\[
dW_{N+1,\Pi} = dW_{N,\Pi} d\mu_{\Omega_{N+1}}(W_N)
\]
and the ultralocal probability measure
\[
d\mu_{\Omega_{N+1}}(W_N) = (\Lambda_{N+1}^{uN})^{-1} \chi_{\Omega_N}(\Lambda_{N+1}) d\mu_{\Omega_{N+1}}(W_N)
\]
The normalizing factor \( \Lambda_{N+1}^{uN} \) has the form \( \exp(-\xi_N(0) \text{Vol}(\Lambda_{N+1})) \). We also define \( \delta E_N^+ \) by
\[
E_N^+(X, \phi + \mathcal{W}) = E_N^+(X, \phi) + \delta E_N^+(X, \phi, \mathcal{W})
\]
Then (63) becomes
\[
Z_{M,N} = \sum_{\Pi^+} \int d\Phi_{N,\Omega} dW_{N+1,\Pi} K_{N,\Pi} \tilde{C}_{N+1,\Pi} + \exp \left( \tilde{E}_{N,\Pi} - E_N^+(\Lambda_N) \right)
\]
where we have isolated the fluctuation integral
\[
\tilde{E}_{N,\Pi} = \exp \left( \frac{\tilde{E}_N^+(\Lambda_N)}{E_N^+(\Lambda_N)} \right)
\]
2.5 localization

We next localize the expressions in the fluctuation integral.

Lemma 6. For complex \( |\Phi_{N,\delta \Omega}| \leq \lambda_N^{+ \delta} \) and \( |W_N| \leq B_{\omega p_N} \):
\[
\delta E_N^+(\Lambda_N, \phi_{N,\delta \Omega}) + \tilde{W}_N(\Omega) = \sum_{X \in D_N} (\delta E_N^+)_{\text{loc}}(X)
\]
\[
+ \sum_{X \in \pi N X \subset \Lambda_{N+1}} B_N(\Pi^+) X \neq \Lambda_{N+1}^\#
\]
where
1. The leading terms \((\delta E_N^+)^{\text{loc}}(X) = (\delta E_N^+)^{\text{loc}}(X, W_N)\) depends on \(W_N\) only in \(X\), are analytic, and satisfy
\[
|(\delta E_N^+)^{\text{loc}}(X)| \leq \mathcal{O}(1)\lambda^\beta e^{-(\kappa-\kappa_0-2)d_M(X)}
\] (75)

2. The boundary terms \(B^{(E)}_{N,1}\) depends on \(\Phi_{N,\delta \Omega}, W_N\) only in \(X\), are analytic, and satisfy
\[
|B^{(E)}_{N,1}(X)| \leq \mathcal{O}(1)\lambda^\beta e^{-(\kappa-\kappa_0-2)d_M(X, \text{mod } \Omega_{N+1}^c)}
\] (76)

3. \(\tilde{B}_{N+1,1}\) terms are bounded by \(C|\Lambda_N^{(N)} - \Lambda_{N+1}^{(N)}| = C \text{ Vol}(\Lambda_N - \Lambda_{N+1})\).

**Proof.** We are studying \(\delta E_N^+(\Lambda_N) = \sum_{X \subset \Lambda_N} \delta E_N^+(X)\) where \(\delta E_N^+(X) = \delta E_N^+(X, \phi_{N,\delta \Omega}, \hat{W}_{N,\Omega}')\). Our assumptions imply \(|\phi_{N,\delta \Omega}'| \leq C \lambda_N^{\frac{1}{2} - \delta}\) also for derivatives, and \(|\hat{W}_{N,\Omega}'| \leq C p_N\). Since \(\delta < \epsilon\) these bounds put us well inside the domain of analyticity \(R_N\) for \(\Lambda_N = \bar{\Lambda}_N\) sufficiently small. So the basic bound \(|\delta E_N^+(X)| \leq \lambda e^{-\kappa d_M(X)}\) is satisfied.

To localize we introduce weakened fields \(\phi_{N,\delta \Omega}'(s)\) and \(\hat{W}_{N,\Omega}'(s)\) based on the random walk expansions for \(G_{N,\delta \Omega}'(s)\) and \(\hat{C}_{N,\delta \Omega}'(s)\). These satisfy the same bounds. We proceed as in the proof of lemma 3.15 in part II with the following modifications. (1.) The random walk is based has \(M\)-cubes, not \(LM\)-cubes, in \(\Omega_{N+1}\). (2.) There is no reblocking, (3.) The decoupling expansion can be done for \(\phi_{N,\delta \Omega}'\) and \(W_{N,\Omega}'\) simultaneously. The result is an expansion
\[
\delta E_N^+(\Lambda_N) = \sum_{X \in \mathcal{D}_N, X \cap \Lambda_N \neq \emptyset} (\delta E_N^+)'(X, \Phi_{N,\delta \Omega}, W_N)
\] (77)

where
\[
|(\delta E_N^+)'(X)| \leq \mathcal{O}(1)\lambda^\beta e^{-(\kappa-\kappa_0-2)d_M(X)}
\] (78)

Now terms in (77) with \(X \subset \Lambda_{N+1}\) depend only on \(W_N\) in \(X\) and are identified as the terms \((\delta E_N^+)^{\text{loc}}(X)\). If \(X \neq \Lambda_N\) we add on any connected component of \(\Omega_{N+1}\) which is connected to \(X\) to get \(X^+ \in \mathcal{D}_N(\text{mod } \Omega_{N+1})\). Terms in (77) with \(X \neq 

**Lemma 7.** For complex \(|\Phi_{N,\delta \Omega}| \leq \lambda_N^{\frac{1}{2} - \delta}\) and \(|W_N| \leq B_{\epsilon}\),
\[
R^{(\leq \ell)}(\Pi, \Omega_{N+1}) = \sum_{X \in \mathcal{D}_N(X \subset \Lambda_{N+1})} R^{\text{loc}}_{N,1}(X) + \sum_{X \in \mathcal{D}_N(X \# \Lambda_{N+1})} B^{(R)}_{N,1}(X) + \tilde{B}_{N+1,1}\] (79)

Here \(R^{\text{loc}}_{N,1}(X, W_N)\) and \(B^{(R)}_{N,1}(X, \Phi_{N,\delta \Omega}, W_N)\) are strictly localized, analytic in the fields, and satisfy
\[
|R^{\text{loc}}_{N,1}(X)| \leq \mathcal{O}(1)\lambda^\beta e^{-(\kappa-\kappa_0-2)d_M(X)}
\]
\[
|B^{(R)}_{N,1}(X)| \leq \mathcal{O}(1)\lambda^\beta e^{-(\kappa-2 \kappa_0-3)d_M(X, \text{mod } \Omega_{N+1})}
\] (80)
\textbf{Proof.} The function $R_{\Pi,\Omega_{k+1}}^{(\leq 7)}$ has many parts. Consider the original term $R_{\Pi,\Omega_{k+1}}^{(0)} \equiv R_{k,\Pi}(\Lambda_N)$. After the change of variables this has the form

$$R_{N,\Pi}(\Lambda_N) = \sum_{X \in D_k, X \subset \Lambda_N} R_{N,\Pi}(X, \Phi_{N,\delta N}, \Psi_{N,\delta N+1}^{\text{loc}}(\delta \Omega') + (C^1_{N,\Omega'})^{\text{loc}} W_N) \quad (81)$$

In addition to $|\Phi_{N,\delta N}| \leq \lambda_N^{-\frac{2}{4} - \delta}$ we have $|\Psi_{N,\delta N+1}^{\text{loc}}(\delta \Omega')| \leq C \lambda_N^{-\frac{2}{4} - \delta}$ and $|(C^1_{N,\Omega'})^{\text{loc}} W_N| \leq C \rho_N \leq C \lambda_N^{-\frac{2}{4} - \delta}$. Then for $\square \subset \Lambda_N$ and local fields

$$\Phi_{N,\Omega}(\square) = \Phi_{N,\Omega}(\square) \left( \Phi_{N,\delta N}, \Psi_{N,\delta N+1}^{\text{loc}}(\delta \Omega') + (C^1_{N,\Omega'})^{\text{loc}} W_N \right) \quad (82)$$

we have the estimates $|\phi_{N,\Omega,\square}(\square)| \leq C \lambda_N^{-\frac{2}{4} - \delta}$ and $|\partial \phi_{N,\Omega,\square}(\square)| \leq C \lambda_N^{-\frac{2}{4} - \delta}$. These are the estimates give that $(\Phi_{N,\delta N}, \Psi_{N,\delta N+1}^{\text{loc}}(\delta \Omega') + (C^1_{N,\Omega'})^{\text{loc}} W_N)$ is in $CP_N(\square, \delta)$ and hence in $CP_N(\Lambda_N, \delta)$. But for $\lambda_N = \lambda$ sufficiently small $C \lambda_N^{-\frac{2}{4} - \delta} \leq \lambda_N^{-\frac{2}{4} - 2\delta}$ so this field is in $P_N(\Lambda_N, 2\delta)$. Thus we are in the analyticity domain for $R_{N,\Pi}(X)$ and can use the bound $|R_{N,\Pi}(X)| \leq \lambda e^{-\kappa \delta d_M(X)}$.

To localize we introduce weakened fields $\Psi_{N,\delta N+1}^{\text{loc}}(\delta \Omega', s)$ and $(C^1_{N,\Omega'})^{\text{loc}}(s) W_k$. These satisfy the same bounds. We proceed with a decoupling expansion as in the proof of lemma 3.16 in part II, except that the random walk has no $LM$-cubes and there is no reblocking. As in the previous lemma the result is an expansion

$$R_{N,\Pi}(\Lambda_N) = \sum_{X \in D_k, X \subset \Lambda_N \neq 0} (R_{N,\Pi})'(X, \Phi_{N,\delta N}, W_N) \quad (83)$$

where $(R_{N,\Pi})'(X)$ is strictly localized and analytic and satisfies

$$|(R_{N,\Pi})'(X)| \leq O(1) \lambda^{n_0} e^{-(\kappa - \kappa_0 - 2)d_M(X)} \quad (84)$$

Now divide the terms by $X \subset \Lambda_{N+1}$, $X \neq \Lambda_{N+1}$, and $X \subset \Lambda_{N+1}$ and get a contribution for each of the three types of terms.

The other contributions to $R_{\Pi,\Omega_{k+1}}^{(\leq 7)}$ are treated similarly, see lemma 3.16 in part II. This completes the proof.

For the next result we recall that the analyticity domain for $B_{N,\Pi}$ is $|W_j| \leq |B_{N} p_j L^5(N-j)$ on $\Omega_{j+1} - \Lambda_{j+1}$ for $j = 1, \ldots, N - 1$ and $\Phi_{k,\Omega}$ in

$$P_{N,\Omega} = \bigcap_{j=1}^{N-1} \left[ \frac{P_{\Omega}}{\delta (\Omega_j, \delta)} \right]_{L=(N-j)} \cap P_N(\Omega_N - \Omega_{N+1}^2, \delta) \cap \bigcap_{j=1}^{N-1} \left[ P_{\Omega}(\Omega_{N} - \Omega_{N+1}^2, \delta) \right] \quad (85)$$

We modify it to a complex domain for

$$\Phi_{N,\delta \Omega}^+ \equiv (\Phi_{1,\delta \Omega_1}, \ldots, \Phi_{N,\delta \Omega_N}) \quad (86)$$

which is

$$\bar{P}_{N,\Omega}^+ = \bigcap_{j=1}^{N-1} \left[ \frac{P_{\Omega}}{\delta (\Omega_j, \delta)} \right]_{L=(N-j)} \cap P_N(\Omega_N - \Omega_{N+1}^2, \delta) \cap \bigcap_{j=1}^{N-1} \left[ P_{\Omega}(\Omega_{N} - \Omega_{N+1}^2, \delta) \right] \quad (87)$$

This is contained in the domain $|\Phi_{N,\delta \Omega_j}| \leq \lambda N^{-\frac{2}{4} - \delta}$ used in lemma 6 and lemma 7 but still is large enough to contain the domain specified by the characteristic functions.
Lemma 8. For \( \Phi_{N\beta}^{\Omega^+} \in \tilde{P}_N, \Omega \) and \(|W_j| \leq B_{w\beta}L_\beta^{(N-j)}\)

\[
B_{N,\Pi}(\Lambda_N) = \sum_{X \in D_N(\mod \Omega_{N+1}), X \# \Lambda_{N+1}} \frac{B_{N,\Pi}^{(B)}(X)}{\Lambda_N} + \tilde{B}_{N+1,\Pi^+} \text{ terms}
\]

(88)

Here \(B_{N,\Pi}^{(B)}(X, \Phi_{N\beta}^{\Omega^+}, W_{N+1,\Pi^+}, W_{N,\lambda_{N+1}})\) is strictly local in the fields, analytic, and satisfies

\[
|B_{N,\Pi}^{(B)}(X)| \leq \lambda_n^\alpha e^{-\kappa_n \delta} d_M(X, \mod \Omega_{N+1})
\]

(89)

Proof. We are studying

\[
B_{N,\Pi}(\Lambda_N) = \sum_{X \in D_N(\mod \Omega_{N+1}), X \# \Lambda_{N+1}} B_{N,\Pi} X, \Phi_{N\beta}^{\Omega^+}, \Psi_{N,\Omega_{N+1}}^{loc}(\delta \Omega') + (C_{N,\Omega}^{1/2})^{\text{loc}} W_N, W_{N,\Pi^+} \]

(90)

We claim that under our assumptions the field \(\Phi_{N\beta}^{\Omega^+}, \Psi_{N,\Omega_{N+1}}^{loc}(\delta \Omega') + (C_{N,\Omega}^{1/2})^{\text{loc}} W_N\) is in the domain \(\tilde{P}_N, \Omega\). This is a statement about the fields \(\phi_{N,\Omega}(\square)\) defined in (82). If \(\square \subseteq (\Omega_{N+1}^*)^c\) the statement is inherited from the definition of \(\tilde{P}_N, \Omega\). On the other hand if \(\square \subseteq \Omega_{N+1}^c\), then, as in the proof of the previous lemma, the bounds \(|\Phi_{N\beta}^{\Omega}| \leq \lambda_n^\alpha \delta\) and \(|W_N| \leq p_n\) imply that the field is in \(\tilde{P}_N(\square, 2\delta)\) and hence in \(\tilde{P}_N(\Omega_{N+1}^c, 2\delta)\) as required. Hence the claim is verified (twice in \(\Omega_{N+1} - \Omega_{N+1}^c\)). Thus we are in the analyticity domain for \(B_{N,\Pi}\) and have the bound

\[
|B_{N,\Pi}(X)| \leq B_0 \lambda^\beta e^{-\kappa_{\beta} \delta} d_M(X, \mod \Omega_{N+1})
\]

(91)

Terms with \(X \subset \Omega_{N+1}^c\) are already localized and qualify as \(\tilde{B}_{N+1,\Pi^+}\) terms. The remaining terms have \(X \cap \Omega_{N+1} \neq \emptyset\). For these we localize by introducing \(\Psi_{N,\Omega_{N+1}}^{loc}(\Omega', s)\) and \((C_{N,\Omega}^{1/2})^{\text{loc}} W_N\) and making a decoupling expansion. This follows the proof of lemma 3.17 in part II, except that the random walk has no \(LM\)-cubes and there is no reblocking. After adding on appropriate connected components of \(\Omega_{N+1}^c\) the result is

\[
B_{N,\Pi}(\Lambda_N) = \sum_{X \# \Lambda_N, X \cap \Omega_{N+1}^c \neq \emptyset} (B_{N,\Pi}^{(B)}(X, \Phi_{N\beta}^{\Omega^+}, W_{N+1,\Pi^+}, W_{N,\lambda_{N+1}}) + \tilde{B}_{N+1,\Pi^+} \text{ terms}
\]

(92)

where now the sum is over \(X \in D_N(\mod \Omega_{N+1}^c)\) and where

\[
|(B_{N,\Pi}^{(B)}(X)| \leq O(1)B_0 \lambda^\beta e^{-\kappa_{\beta} \delta} d_M(X, \mod \Omega_{N+1}^c)
\]

(93)

Terms with \(X \subset \Lambda_{N+1}^c\) are \(\tilde{B}_{N+1,\Pi^+}\) terms. Terms with \(X \# \Lambda_{N+1}\) are the terms \(B_{N,\Pi}^{(B)}(X)\). The stronger bound with \(\lambda_n^\alpha\) is obtained from the separation of \(\Lambda_{N+1} + \Lambda_{N+1}^c\). This completes the proof.

Now all the active boundary terms can be combined into a single boundary term

\[
B_{N,\Pi}^{loc}(X) = B_{N,\Pi}^{(B)}(X) + B_{N,\Pi}^{(R)}(X) + B_{N,\Pi}^{(B)}(X)
\]

(94)

analytic in \(\tilde{P}_N, \Omega^+ \times \{|W_j| \leq B_{w\beta}L_\beta^{(N-j)}\}\) and satisfying the various stated bounds. All the inactive boundary terms \(\tilde{B}_{N+1,\Pi^+}\) terms are combined into a single term \(\tilde{B}_{N+1,\Pi^+}(\Lambda_N, \Lambda_{N+1})\) analytic in the same domain satisfying there

\[
|\tilde{B}_{N+1,\Pi^+}(\Lambda_N, \Lambda_{N+1})| \leq C|\Lambda_{N+1}^{(N)} - \Lambda_{N+1}^{(N)}|
\]

(95)
The fluctuation integral is then

\[ \tilde{\Xi}_{N,\Pi^+} = \exp \left( -\varepsilon_N^{(N)}\Vol(\Lambda_{N+1}) + E_N^+(\Lambda_{N+1}) + \hat{B}_{N+1,\Pi^+}(\Lambda_N,\Lambda_{N+1}) \right) \int d\mu_{N+1}^*(W_N) \exp \left( (\delta E_N^+)_{\text{loc}}(\Lambda_{N+1}) + R_{N,\Pi^+}^\text{loc}(\Lambda_{N+1}) + B_{N,\Pi^+}^\text{loc}(\Lambda_{N+1}) \right) \] (96)

2.6 cluster expansion

A cluster expansion is now carried out as in section 3.14 in part II, and in the resulting localization expansion we identify leading, tiny, and boundary terms. We find

\[ \int d\mu_{N+1}^*(W_N) \exp \left( (\delta E_N^+)_{\text{loc}}(\Lambda_{N+1}) + R_{N,\Pi^+}^\text{loc}(\Lambda_{N+1}) + B_{N,\Pi^+}^\text{loc}(\Lambda_{N+1}) \right) = \exp \left( E_N^*(\Lambda_{N+1}) + R_{N,\Pi^+}^*(\Lambda_{N+1}) + B_{N,\Pi^+}^*(\Lambda_{N+1}) \right) \] (97)

Here \( E_N^*(\Lambda_{N+1}), R_{N,\Pi^+}^*(\Lambda_{N+1}) \) have local expansions like \( E_N^*(\Lambda_{N+1}) = \sum_{X \subset \Lambda_{N+1}} E_N^*(X) \) with \( X \in \mathcal{D}_N \). These are are independent of all fields and satisfy

\[ |E_N^*(X)| \leq O(1)\lambda^d e^{-(\kappa-6\kappa_0-6)d_M(X)} \]
\[ |R_{N,\Pi^+}^*(X)| \leq O(1)\lambda^{6\alpha} e^{-(\kappa-6\kappa_0-6)d_M(X)} \] (98)

Also \( B_{N,\Pi^+}^*(\Lambda_{N+1}) = \sum_{X \#_{\Lambda_{N+1}}} B_{N,\Pi^+}^*(X) \) with \( X \in \mathcal{D}_N(\mod \Omega_{N+1}) \). Here \( B_{N,\Pi^+}^*(X) \) is a function of \( (\Phi_{N,\Omega^+}, W_{N+1,\Pi^+}) \), is analytic in \( \mathcal{P}_{N,\Omega^+} \times \{ |W_j| \leq B_0 p_j L^{(N-j)} \} \), and satisfies there

\[ |B_{N,\Pi^+}^*(X)| \leq O(1)B_0 \lambda^d e^{-(\kappa-6\kappa_0-6)d_M(X, \mod \Omega_{N+1})} \] (99)

Now insert (97) back into (96), and then (98) back into (72), and obtain

\[ Z_{M,N} = Z_{M,N}(0) \sum \int d\Phi_{N,\Omega^+} dW_{N+1,\Pi^+} K_{N+1,\Pi^+} \hat{\xi}_{N+1,\Pi^+} \exp \left( (\hat{\xi}_N^{(N)})^\text{loc}(\Lambda_{N+1}) - \varepsilon_N^{(N)}\Vol(\Lambda_{N+1}) - S_N^*(\Lambda_N - \Omega_{N+1}) - \hat{S}_N^*(\Omega_{N+1}) + E_N^+ + E_N^*(\Lambda_{N+1}) + R_{N,\Pi^+}^*(\Lambda_{N+1}) + B_{N,\Pi^+}^*(\Lambda_{N+1}) + \hat{B}_{N+1,\Pi^+}(\Lambda_N,\Lambda_{N+1}) \right) \] (100)

2.7 final localization

We would like to write the action in the final small field region \( \Lambda_{N+1}^c \) as a sum over pieces concentrated in the various connected components. However there is still some dependence on the field \( \phi_{N,\Omega^+} \) which is defined all over \( \Lambda_{N+1} \) (but is tiny there) and penetrates into \( \Lambda_{N+1}^c \). This gives weak coupling between the connected components of \( \Lambda_{N+1}^c \). We have to exhibit this coupling in a form we can use.

The terms we have to consider are

\[ S_N^*(\Lambda_N - \Omega_{N+1}) + \hat{S}_N^*(\Omega_{N+1}) = S_N^*(\Lambda_N - \Omega_{N+1}) + \hat{S}_N^*(\Omega_{N+1} - \Omega_{N+1}^2) + \hat{S}_N^*(\Omega_{N+1}) \] (101)

and

\[ E_N^+(\Lambda_N) = E_N^+(\Lambda_N - \Omega_{N+1}^2) + E_N^+(\Omega_{N+1}^2) \] (102)

We first deal with the last two terms which contain \( \Lambda_{N+1} \).
Lemma 9. For $|\phi_{N,\delta \Omega_n}| \leq \lambda^{-\frac{r}{2}}$:

$$E_N^+(\Omega_{N+1}^0, \phi_{N,\delta \Omega}) - E_N^+(\Omega_{N+1}^0, 0) = \sum_{Y \cap \Omega_{N+1}^0 \neq \emptyset, Y \cap \delta \Omega_n \neq \emptyset} R_{N,\Pi}^{*,(1)}(Y, \Phi_{N,\delta \Omega_n})$$

$$\tilde{S}_N(\Omega_{N+1}^0, \phi_{N,\delta \Omega}) = \sum_{Y \cap \Omega_{N+1}^0 \neq \emptyset, Y \cap \delta \Omega_n \neq \emptyset} R_{N,\Pi}^{*,(2)}(Y, \Phi_{N,\delta \Omega_n})$$

where the terms on the right are strictly localized and satisfy

$$|R_{N,\Pi}^{*,(i)}(Y)| \leq \lambda^{n_0} e^{-(\kappa - \kappa_0 - 2)dt(Y)} \quad i = 1, 2$$

Proof. This follows the proof of lemma 3.22 in part II. Consider $E_N^+(\Omega_{N+1}^0)$. Since $\Omega_{N+1}^0$ and $\Omega_{N+1}$ are separated by $[r_N]$ layers of $M$-cubes we have $|\phi_{N,\delta \Omega}| \leq e^{-rN/2}$ on $\Omega_{N+1}^0$. Then we can write for $X \subset \Omega_{N+1}^0$

$$E_N^+(X, \phi_{N,\delta \Omega}) - E_N^+(X, 0) = \frac{1}{2\pi i} \int_{|t| = e^{1/2}} \frac{dt}{t(t-1)} E_N(X, t\phi_{N,\delta \Omega})$$

and obtain the bound

$$|E_N^+(X, \phi_{N,\delta \Omega}) - E_N^+(X, 0)| \leq e^{-rN/2} e^{-\kappa dt(X)}$$

Next replace $\phi_{N,\delta \Omega}$ by the weakened version $\phi_{N,\delta \Omega}(s)$, and then $E_N^+(X, \phi_{N,\delta \Omega}(s)) - E_N^+(X, 0)$ has the same bound. Then $E_N^+(X, \phi_{N,\delta \Omega}) - E_N^+(X, 0) = \sum_{Y \subset X} E_N^+(X, Y)$ where $Y$ is connected and

$$E_N^+(X, Y, \Phi_{N,\delta \Omega_n}) = \int ds_{Y - X} \frac{d}{ds_{Y - X}} [E_N^+(X, \phi_{N,\delta \Omega}(s)) - E_N^+(X, 0)]_{s_Y = 0, s_X = 1}$$

which is strictly localized in $Y$. In the random walk expansion for $\phi_{N,\delta \Omega}(s)$ here only paths which start in $X$ and finish in $\delta \Omega_N$ contribute. In addition the condition $s_{Y - X} = 0$ imposes that only paths in $Y$ contribute. Thus paths must intersect $Y \cap \delta \Omega_N$. If $Y \cap \delta \Omega = \emptyset$ then $\phi_{N,\delta \Omega}(s) = 0$ and so $E_N^+(X, Y) = 0$. Thus we can assume $Y \cap \delta \Omega \neq \emptyset$.

If we now define

$$R_{N,\Pi}^{*,(1)}(Y) = \sum_{X \subset Y \cap \Omega_{N+1}^0} E_N^+(X, Y)$$

then we have (103). Using Cauchy bounds we obtain the stated bound on $R_{N,\Pi}^{*,(1)}(Y)$ by the usual analysis. The analysis of $\tilde{S}_N(\Omega_{N+1}^0)$ is similar. This completes the proof.

Recall that the field $\phi_{N,\delta \Omega}$ is defined in (38) in terms of $\Omega' = (\Omega(\Lambda_n), \Omega_{N+1})$ and $G_{N,\delta \Omega'}$ defined in (35). We modify this to a more local field by introducing

$$\Omega'' = \Omega(\Lambda_n, \Omega_{N+1}, \Omega_{N+1}^0) = \Omega' \cap \Omega((\Omega_{N+1}^0)^c)$$

and the Green’s function

$$G_{N,\Omega''} = \left[ -\Delta + \bar{\mu} + Q_{N,\Omega'}^T \mu \Omega'' \right]^{-1}$$

The field $\phi_{N,\delta \Omega''}$ is defined just as $\phi_{N,\delta \Omega'}$ in (38) but with $G_{N,\delta \Omega'}$ replaced by $G_{N,\Omega''}$ (and still with vanishing field in $\Omega_{N+1}$). The field $\phi_{N,\delta \Omega''}$ is localized in a region slightly larger than $\Lambda_n \cap \Omega_{N+1}^0 = \Lambda_n \cap (\Omega_{N+1}^0)^2$ and hence outside of $\Lambda_{N+1}$.
Lemma 10. For $|\Phi_{N,\delta \Omega}| \leq \lambda^{-\frac{1}{4}} - \delta$

$$E_N^+(\Lambda_N - \Omega_{N+1}^2, \phi_{N,\delta \Omega}) = E_N^+(\Lambda_N - \Omega_{N+1}^2, \phi_{N,\Omega'}) + \sum_{Y \cap (\Lambda_N - \Omega_{N+1}^2) \neq \emptyset} R_{N,\Pi^+}^{\ast,(3)}(Y, \Phi_{N,\delta \Omega_{N+1}})$$

$$S_N^+(\Lambda_N - \Omega_{N+1}^2, \phi_{N,\delta \Omega}) = S_N^+(\Lambda_N - \Omega_{N+1}^2, \phi_{N,\Omega'}) + \sum_{Y \cap (\Lambda_N - \Omega_{N+1}^2) \neq \emptyset} R_{N,\Pi^+}^{\ast,(4)}(Y, \Phi_{N,\delta \Omega_{N+1}})$$

$$\tilde{S}_N(\Omega_{N+1} - \Omega_{N+1}^2, \phi_{N,\delta \Omega}) = \tilde{S}_N(\Omega_{N+1}^2, \phi_{N,\Omega'}) + \sum_{Y \cap (\Omega_{N+1} - \Omega_{N+1}^2) \neq \emptyset} R_{N,\Pi^+}^{\ast,(5)}(Y, \Phi_{N,\delta \Omega_{N+1}})$$

(111)

where the terms on the right are strictly localized and satisfy

$$|R_{N,\Pi^+}^{\ast,(i)}(Y)| \leq C(1)\lambda^{n_0} e^{-(\kappa_0 - 2)d_M(Y)} \quad i = 3, 4, 5$$

(112)

Remark. The leading terms on the right are bounded but not small. We do not need them for large field bounds, but have made a point to localize them in a way that preserves positivity so they can be estimated in the exponential.

Proof. Consider $E_N(\Lambda_N - \Omega_{N+1}^2)$. First note that on $\Lambda_N - \Omega_{N+1}^2$ we have $|\phi_{N,\delta \Omega}|, |\phi_{N,\Omega'}| \leq C\lambda^{-\frac{1}{4}} - \delta$. Therefore with either field $|E_N(X)| \leq \lambda^3 e^{-\kappa d_M(X)}$ and since $V_N(\emptyset, \Phi) = \lambda_N \int \phi^4$ we have $|V_N(\emptyset)| \leq CM^3\lambda^{-\delta}$. Therefore $E_N^+(X) = E_N(X) - V_N(X)$ satisfies for $X \subset \Lambda_N - \Omega_{N+1}^2$

$$|E_N^+(X)| \leq CM^3\lambda_{N}^{-\delta} e^{-\kappa d_M(X)}$$

(113)

But also we have on $\Lambda_N - \Omega_{N+1}^2$

$$|\phi_{N,\delta \Omega} - \phi_{N,\Omega'}| \leq e^{-r_N/2}$$

(114)

This holds since $G_{N,\delta \Omega}$ and $G_{N,\Omega'}$ have random walk expansions differing only in $\Omega_{N+1}^2$ which is at least $|r_N|$ layers of $M$-cubes away from $\Lambda_N - \Omega_{N+1}^2$. Hence we can write

$$E_N^+(X, \phi_{N,\delta \Omega}) - E_N^+(X, \phi_{N,\delta \Omega'}) = \frac{1}{2\pi i} \int_{|t|=e^{r_N/2}} \frac{dt}{t(t-1)} E_N^+(X, \phi_{N,\delta \Omega'} + t(\phi_{N,\delta \Omega} - \phi_{N,\Omega'}))$$

(115)

and have the estimate

$$|E_N^+(X, \phi_{N,\delta \Omega}) - E_N^+(X, \phi_{N,\Omega'})| \leq e^{-r_N/2} CM^3\lambda_{N}^{-\delta} e^{-\kappa d_M(X)} \leq \lambda_{N+1}^{n_0+1} e^{-\kappa d_M(X)}$$

(116)

The localization now proceeds more or less as in lemma 3.22 in part II and gives the representation [111] with the bound [112]. The terms $S_N^+, \tilde{S}_N$ are treated similarly.

2.8 summary

We rearrange all these terms and insert them into [110]. First we write

$$E_N^+(\Omega_{N+1}^2, 0) = E_N^+(\Omega_{N+1}^2 - \Lambda_{N+1}, 0) + E_N^+(\Lambda_{N+1}, 0) + \sum_{X \subset \Omega_{N+1}^2, X \neq \Lambda_{N+1}} E_N^+(X, 0)$$

(117)
The first term is absorbed into \( \tilde{B}_{N+1, \Pi}^+ (\Lambda_N, \Lambda_{N+1}) \). We also write

\[
E_N^+ (\Lambda_N - \Omega_{N+1}^2, \phi_N, \Omega') = E_N (\Lambda_N - \Omega_{N+1}^2, \phi_N, \Omega') - V_N (\Lambda_N - \Omega_{N+1}^2, \phi_N, \Omega')
\]

and the first term here is absorbed into \( \tilde{B}_{N+1, \Pi}^+ (\Lambda_N, \Lambda_{N+1}) \).

We collect field independent terms in \( \Lambda_{N+1} \) by defining

\[
E_N^0 (\Lambda_{N+1}) = -\epsilon_N^0 \text{Vol} (\Lambda_{N+1}) + E^+_N (\Lambda_{N+1}, 0) + E^#_N (\Lambda_{N+1})
\]

Each of these is expressed as a sum over polymers \( X \subset \Lambda_{N+1} \) and we have

\[
|E_N^0 (X)| \leq O(1) \lambda^2 e^{-(\kappa-6\kappa_0-6)d_M (X)}
\]

We collect boundary terms by defining

\[
B^*_{N, \Pi} (\Lambda_{N+1}) = B^#_{N, \Pi} (\Lambda_{N+1}) + \sum_{i=1}^5 \sum_{X \# \Lambda_{N+1}} R^*_{N, \Pi} (X) + \sum_{X \subset \Omega_{N+1}^c \cap \Lambda_{N+1}} E_N^+ (X, 0)
\]

Each of these can be expressed as sum over polymers \( X \subset \Lambda_{N+1} \) and \( X \cap \Lambda_N \neq \emptyset \) can be absorbed into \( \tilde{B}_{N+1, \Pi}^+ (\Lambda_N, \Lambda_{N+1}) \).

\[
|B^*_{N, \Pi} (X)| \leq O(1) B_0 \lambda^2 e^{-(\kappa-7\kappa_0-7)d_M (Y, \mod \Lambda_{N+1})}
\]

It is convenient to make a further adjustment here. Each \( X \in \mathcal{D}_N (\mod \Omega_{N+1}^c) \) determines a \( Y \in \mathcal{D}_N (\mod \Lambda_{N+1}^c) \) by adjoining connected components of \( \Lambda_{N+1}^c \) as in part H of lemma 3.15 in part II. Hence \( B^*_{N, \Pi} (\Lambda_{N+1}) \) is a sum of such \( X \), and on the domain of \( B^#_{N, \Pi} \) and satisfies

\[
|B^*_{N, \Pi} (X)| \leq O(1) B_0 \lambda^2 e^{-(\kappa-7\kappa_0-7)d_M (Y, \mod \Lambda_{N+1})}
\]

The remaining terms coming from \( R^*_{N, \Pi} (X) \) have \( X \subset \Lambda_{N+1}^c \) and \( X \cap \Lambda_N = \emptyset \) can be absorbed into \( \tilde{B}_{N+1, \Pi}^+ (\Lambda_N, \Lambda_{N+1}) \). With all these additions it still satisfies the bound

\[
|\tilde{B}_{N+1, \Pi}^+ (\Lambda_N, \Lambda_{N+1})| \leq C |\Lambda_N^{(N)} - \Lambda_N^{(N+1)}|
\]

Finally we define (with \( S_N, \tilde{S}_N, V_N \) are evaluated at \( \phi_N, \Omega' \))

\[
\tilde{K}_{N+1, \Pi} = K_{N+1, \Pi} \exp \left( \tilde{c}_N |\Omega_{N+1}^{(N)}| - S_N (\Lambda_N - \Omega_{N+1}^2) - \tilde{S}_N (\Omega_{N+1} - \Omega_{N+1}^2) - V_N (\Lambda_N - \Omega_{N+1}^2) + \tilde{B}_{N+1, \Pi}^+ (\Lambda_N, \Lambda_{N+1}) \right)
\]

Altogether then (100) has become

\[
Z_{M,N} = Z_{M,N} (0) \sum_{\Pi^+} \int d\tilde{\phi}_{N, \Omega'} dW_{N+1, \Pi}^+ \tilde{K}_{N+1, \Pi}^+ \tilde{C}_{N+1, \Pi}^+ 
\]

\[
\exp \left( E_N^+ (\Lambda_{N+1}) + R^#_{N, \Pi} (\Lambda_{N+1}) + B^*_{N, \Pi} (\Lambda_{N+1}) \right)
\]
3 large field bounds

3.1 rearrangement

There are some further rearrangements before commencing with the final bounds. We start by making some Mayer expansions in \[127\]. Since \( E_{n+1}(\Lambda_{n+1}) = \sum_{X \subseteq \Lambda_{n+1}} E_{n+1}^+(X) \) and the same for the tiny terms we have

\[
\exp \left( E_{n+1}^+(\Lambda_{n+1}) + R_{n, \Pi^+}(\Lambda_{n+1}) \right) = \prod_{X \subseteq \Lambda_{n+1}} \left( I^+(X) + 1 \right) = \sum_{\{X\}} \prod_{\alpha} I^+(X) \tag{128}
\]

where for \( X \in \mathcal{D}_n \)

\[
I^+(X) = \exp \left( E_n^+(X) + R_{n, \Pi^+}(X) \right) - 1 \tag{129}
\]

and the sum is now over collections of distinct \( \{X_\alpha\} \) in \( \Lambda_{n+1} \). We also have that \( B_{n+1, \Pi^+}(\Lambda_{n+1}) = \sum_{Y \neq \Lambda_{n+1}} B_{n+1, \Pi^+}(Y) \) and then

\[
\exp \left( B_{n+1, \Pi^+}(\Lambda_{n+1}) \right) = \prod_{Y \neq \Lambda_{n+1}} \left( J^+(Y) + 1 \right) = \prod_{\sigma} \prod_{Y} J^+(Y) \tag{130}
\]

where for \( Y \in \mathcal{D}_n \operatorname{mod} \Lambda_{n+1} \)

\[
J^+(Y) = \exp \left( B_{n+1, \Pi^+}(Y) \right) - 1 \tag{131}
\]

and the sum is now over collections of distinct \( \{Y_\sigma\} \) which cross \( \Lambda_{n+1} \).

Classify the terms in the sum over \( \Pi^+ = (\Lambda_0, \Omega_1, \Lambda_1, \ldots, \Omega_{n+1}, \Lambda_{n+1}) \) by \( \Theta \equiv \Lambda_{n+1}^c \), a union of \( M \) cubes. We write \( \sum_{\Pi^+} \cdots = \sum_{\Theta} \sum_{\Pi^+ : \Lambda_{n+1}^c \cap \Theta = \emptyset} \cdots \). The sum over \( \Theta \) is written as a sum over its connected components \( \{\Theta_\gamma\} \). The sums over \( \{X_\alpha\}, \{Y_\sigma\} \) only depend on \( \Theta = \cup_\gamma \Theta_\gamma \) and so can come outside the sum over \( \Pi^+ \). Then we have

\[
Z_{m,n} = Z_{m,n}(0) \sum_{\{\Theta_\gamma\}} \sum_{\{Y_\sigma\}} \sum_{\{X_\alpha\}} \prod_{\Lambda_{n+1}^c = \Theta} \mathcal{L}_{\Pi^+} \left( \{X_\alpha\}, \{Y_\sigma\} \right) \tag{132}
\]

where

\[
\mathcal{L}_{\Pi^+} \left( \{X_\alpha\}, \{Y_\sigma\} \right) = \int d\Phi_n^+ \cdot dW_{n+1, \Pi^+} \cdot \tilde{K}_{n+1, \Pi^+} \cdot \tilde{c}_{n+1, \Pi^+} \prod_{\sigma} \prod_{\alpha} J^+(Y_\sigma) \prod_{\alpha} I^+(X_\alpha) \tag{133}
\]

The sum in \[132\] is over disjoint \( \Theta_\gamma \), over distinct \( \{X_\alpha\} \) satisfying \( X_\alpha \subset \Theta^c \) and over distinct \( \{Y_\sigma\} \) satisfying \( Y_\sigma \# \Theta \) and \( Y_\sigma \in \mathcal{D}_n \operatorname{mod} \Theta \).

Let \( U \) be the union of \( \{\Theta_\gamma\}, \{Y_\sigma\}, \{X_\alpha\} \) and let \( \{U_\ell\} \) be the connected components of \( U \), where now we say \( X, Y \) are connected if they have a cube \( \Box \) in common. See figure \[\Box\]. We write the sum as

\[
\sum_{\{\Theta_\gamma\}, \{Y_\sigma\}, \{X_\alpha\}} = \sum_{\{\Theta_\gamma\}, \{Y_\sigma\}, \{X_\alpha\} \to U} \sum_{\{U_\ell\}} \sum_{\{\Theta_\gamma\}, \{Y_\sigma\}, \{X_\alpha\} \to U_\ell} \tag{134}
\]

The statement that \( \tilde{B}_{k, \Pi}(\Lambda_{k-1}, \Lambda_k^c) \) is additive in the connected components of \( \Lambda_k^c \) means that

\[
\tilde{B}_{k, \Pi}(\Lambda_{k-1}, \Lambda_k^c) = \sum_{\gamma} \tilde{B}_{k, \Pi \cap \Theta_\gamma}(\Lambda_{k-1} \cap \Theta_\gamma, \Lambda_k^c \cap \Theta_\gamma) \tag{135}
\]
Thanks to the change of fields in lemma 10 a similar decomposition holds for the other contributions to $\tilde{K}_{N+1,\Pi^+}$. Thus $\tilde{K}_{N+1,\Pi^+} = \prod_{\gamma} \tilde{K}_{N+1,\Pi^+ \cap \Theta_{\gamma}}$ and hence

$$\tilde{K}_{N+1,\Pi^+} = \prod_{\ell} \tilde{K}_{N+1,\Pi^+ \cap U_{\ell}} \quad (136)$$

The same holds for $\tilde{C}_{N+1,\Pi^+}$ and thus

$$\sum_{\Pi^+:\Lambda_{N+1}=\Theta} \mathcal{L}_{\Pi^+}(\{X_{\alpha}\},\{Y_{\sigma}\}) = \sum_{\Pi^+:\Lambda_{N+1}=\Theta} \prod_{\ell} \mathcal{L}_{\Pi^+ \cap U_{\ell}}(\{X_{\alpha}\} \cap U_{\ell},\{Y_{\sigma}\} \cap U_{\ell})$$

$$= \prod_{\ell} \sum_{\Pi^+:\Lambda_{N+1}=\Theta \cap U_{\ell}} \mathcal{L}_{\Pi^+ \cap U_{\ell}}(\{X_{\alpha}\} \cap U_{\ell},\{Y_{\sigma}\} \cap U_{\ell}) \quad (137)$$

Combining (134) and (137) yields

$$Z_{M,N} = Z_{M,N}(0) \sum_{\{U_{\ell}\}} \prod_{\ell} \mathcal{K}(U_{\ell}) \quad (138)$$

where the sum is over disjoint connected $\{U_{\ell}\}$ and where for connected $U$

$$\mathcal{K}(U) = \sum_{\{\Theta_{\gamma}\},\{Y_{\sigma}\},\{X_{\alpha}\} \rightarrow U} \sum_{\Pi^+:\Lambda_{N+1}=\Theta} \mathcal{L}_{\Pi^+}(\{X_{\alpha}\},\{Y_{\sigma}\}) \quad (139)$$

$\mathcal{K}(U)$ is invariant under $M$-lattice symmetries. Our goal is to get a good bound on $\mathcal{K}(U)$ so we can exponentiate the expansion (138).
3.2 first bounds

The characteristic functions put us in the analyticity domains for the various functions. Thus we can use the bound \[|\mathcal{J}_{\Pi^+}(X)| \leq \mathcal{O}(1)\lambda^j e^{-(\kappa - 6\kappa_0 - 6)d_M(X)}\] on \(E_{N+1}(X)\) and the stronger bound \[|\mathcal{J}_{\Pi^+}(Y)| \leq \mathcal{O}(1)B_0\lambda^j e^{-(\kappa - 7\kappa_0 - 7)d_M(Y, \text{mod } \Lambda_{N+1})}\] on \(B_{N+1,\Pi^+}(Y)\) gives

\[|\mathcal{J}_{\Pi^+}(Y)| \leq \mathcal{O}(1)B_0\lambda^j e^{-(\kappa - 7\kappa_0 - 7)d_M(Y, \text{mod } \Lambda_{N+1})}\] (141)

Also recall that

\[K_{N,\Pi} = \prod_{j=0}^{N} \exp \left( \epsilon_j |(\Omega_j^*)^{(j-1)}| - S_{j,L,-(n-j)}^+ (\Lambda_{j-1} - \Lambda_j) + (\tilde{B}_{j,L,-(n-j)})_{\Pi_j} (\Lambda_{j-1} - \Lambda_j) \right)\] (142)

By (26) we have \(|\tilde{B}_{N,\Pi}(\Lambda_{N-1}, \Lambda_N)| \leq B_0|\Lambda_{N-1}^{(N)} - \Lambda_N^{(N)}|\). The right side is scale invariant, so the scaled characteristic functions imply that

\[|(\tilde{B}_{j,L,-(n-j)})_{\Pi_j} (\Lambda_{j-1} - \Lambda_j)| \leq B_0|\Lambda_{j-1}^{(j)} - \Lambda_j^{(j)}|\] (143)

The function \(K_{N,\Pi}^{N+1,\Pi^+}\) has \(K_{N,\Pi}\) and a similar factor for the last step, see (120). Here we use \(|\tilde{B}_{N,\Pi}^{N+1,\Pi^+}(\Lambda_N, \Lambda_{N+1})| \leq B_0|\Lambda_N^{(N)} - \Lambda_{N+1}^{(N)}|\). We also use positivity \(V_N(\Box, \phi) = \lambda \int_\mathbb{C} \phi^4\) to estimate

\[\exp \left( - S_N^+(\Lambda_N - \Omega_{N+1}) + \tilde{S}_N(\Omega_{N+1} - \Omega_{N+1}^{(N)}) - V_N(\Lambda_N - \Omega_{N+1}^{(N)}) \right) \leq 1\] (144)

Now with \(\kappa' = \kappa - 7\kappa_0 - 7\)

\[\mathcal{K}(U) \leq \sum_{\left\{ \Theta_{\ast}, \{Y_{\ast}\}, \{X_{\ast}\} \right\} \rightarrow U} \mathcal{K}'(\Theta) \prod_{\alpha} \mathcal{O}(1)\lambda^j e^{-(\kappa' d_M(X_n))} \prod_{\sigma} \lambda^j \mathcal{O}(1)B_0 e^{-(\kappa' d_M(Y_\sigma, \text{mod } \Theta))}\] (145)

where

\[\mathcal{K}'(\Theta) = \sum_{\Pi^+ : \Lambda_{N+1}^{(N)}} \int d\Phi_{N,\Omega^*} \quad dW_{N+1,\Pi^+} \quad \tilde{C}_{N+1,\Pi^+}\]

\[\prod_{j=0}^{N} \exp \left( \epsilon_j |(\Omega_j^*)^{(j-1)}| - S_{j,L,-(n-j)}^+ (\Lambda_{j-1} - \Lambda_j) + B_0|\Lambda_{j-1}^{(j)} - \Lambda_j^{(j)}| \right)\] (146)

Next we take a closer look at the characteristic functions which we can write as

\[\tilde{C}_{N+1,\Pi^+} = \prod_{j=1}^{N-1} \left( C_{j+1,L,-(n-j-1)}^+ \right)_{\Lambda_j, \Omega_j, \Lambda_{j+1}} \chi_N(\Lambda_N - \Lambda_{N+1}^{(N)}\Omega_{N+1}(\Omega_{N+1}, \Omega_{N+1}^{(N)})\] (147)

Recall that \(C_{j+1,\Lambda_j, \Omega_{j+1}, \Lambda_{j+1}} = (C_j^0(j, L-1)_{\Lambda_j, \Omega_j, \Lambda_{j+1}, \Lambda_{j+1}} \right)_{\Lambda_j, \Omega_{j+1}, \Lambda_{j+1}, \Omega_{j+1}} \chi_N(\Lambda_N - \Lambda_{N+1}^{(N)}\Omega_{N+1}(\Omega_{N+1}, \Omega_{N+1}^{(N)})\) where \(\Lambda_j\) is a union of \(M\) cubes and \(\Omega_{j+1}, \Lambda_{j+1}\) are unions of \(LM\) cubes

\[C_j^0(j, L+1)_{\Lambda_j, \Omega_{j+1}} = C_j^0(j, \Omega_{j+1}) \chi_j(\Lambda_j - \Lambda_{j+1}^{(j)})(\Omega_{j+1} - \Lambda_{j+1}^{(j)}\Omega_{j+1}^{(j)}\Omega_{j+1})\] (148)
Then characteristic functions in a modified form. We define
for \( \Phi, \zeta \) we retain the large field functions \( \chi \) on regions of \( L \). Then we have

\[
\Pi_N(\Omega, q) = \prod_{j=1}^N \chi_j^{0}(\Omega_j - (Q_j + R_j)) \chi_j^{w}(R_j + 1)
\]

where the sum is over unions of \( LM \) cubes \( P_{j+1}, Q_{j+1}, R_{j+1} \). Hence if \( \Omega_{j+1}, \Lambda_{j+1} \) are unions of \( L^{-(N-j-1)}M \) cubes as in (137), then \( (C_{j+1,L^{-(N-j-1)}}(\Lambda_{j+1}, \Lambda_{j+1}) = (C_{j+1,L^{-(N-j-1)}}(\Lambda_{j+1}, \Lambda_{j+1}) \) depends on

\[
(C_{j,L^{-(N-j)}}(\Lambda_{j+1}, \Lambda_{j+1}) = \sum_{P_{j+1}, Q_{j+1} \subset \Lambda_{j+1}} \zeta_j^{0}(P_{j+1}) \zeta_j^{w}(R_{j+1})
\]

where now the sum is over unions of \( L^{-(N-j-1)}M \) cubes \( P_{j+1}, Q_{j+1}, R_{j+1} \). The last step is treated similarly.

The sum over \( \Pi^{+} = (\Lambda_0, \Omega_1, \Lambda_1, \ldots, \Omega_{N+1}, \Lambda_{N+1}) \) can now be written as a sum over regions \( \{P_j, Q_j, R_j\}_{j=0}^{N+1} \) with the convention that \( P_0, R_0, Q_{N+1}, R_{N+1} = \emptyset \). Each of these is a union of \( L^{-(N-j)}M \) cubes in \( T_{N} \) (except \( R_{N+1} \) is still \( M \) cubes). They determine \( \Omega_j, \Lambda_j \) recursively by the following rules

\[
\Omega_j = (\Lambda_j)^{5s} \cup P_j^{5s}, \quad \Lambda_j = (\Omega_j^{c})^{5s} \cup Q_j^{5s} \cup R_j^{5s}
\]

Now we drop the characteristic functions (i.e., estimate them by one) with the following exceptions. We retain the large field functions \( \zeta_j^{0}(P_{j+1}) \zeta_j^{0}(Q_{j+1}) \zeta_j^{w}(R_{j+1}) \). We also keep some small field characteristic functions in a modified form. We define for \( \Phi_k \) on \( T_{N-M-k} \)

\[
\tilde{\chi}_k(X, \Phi_k) = \prod_{x \in X^{N-M-k}} \chi(\Phi_k(x)) \leq \lambda_k^{\frac{1}{2} - \delta}
\]

Then \( \tilde{\chi}_j(L^{-(N-j)}(\Omega_j, \Phi_j)) = \tilde{\chi}_j(L^{N-j}\Omega_j, \Phi_j) \) enforces \( |\Phi_j| \leq \lambda_j^{\frac{1}{2} - \delta} L^{\frac{1}{2}(N-j)} \) on \( \Omega_j \). We know that this is implied by the following characteristic functions, see (222) in part II. Similarly we can introduce \( \tilde{\chi}_0(L^{-(N-j)}(\Lambda_0 - \Omega_1, \Phi_0)) \). Everything else is dropped.

Then we have

\[
\kappa'(\Theta) \leq \sum_{\{P_j, Q_j, R_j\}_{j=0}^{N+1}} \int d\tilde{\theta}, d\tilde{\omega}, dW_{N+1, \Pi^{+}} \tilde{\chi}_0(L^{-(N-j)}(\Omega_0 - \Omega_1) \prod_{j=1}^N \tilde{\chi}_j(L^{-(N-j)}(\Omega_j, \Phi_j))
\]

\[
\prod_{j=0}^N \zeta_j^{0}(P_{j+1}) \zeta_j^{0}(Q_{j+1}) \zeta_j^{w}(R_{j+1})
\]

\[
\prod_{j=0}^N \exp \left( \zeta_j^{0}(\Omega_j) + \tilde{\chi}_{N+1}(\Omega_{N+1}) + B_0|\Lambda_0 - \Lambda(N)| \right)
\]

\[
\exp \left( \sum_{j=1}^N |\Lambda_j - \Lambda_{j-1}| + B_0|\Lambda_0 - \Lambda(N)| \right)
\]

\[
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\]
3.3 small factors

We continue the estimate on $K'(\Theta)$. Recall that

\[ d\Phi_{N,\Omega} = d\Phi_{N,\Omega} \prod_{j=0}^{N-1} \exp\left(-\frac{1}{2} aL^{-(N-j-1)}|\Phi_{j+1} - Q\Phi_j|^2_{\Omega_{j+1}}\right) d\Phi_{N-j}^{(N-j)} \tag{154} \]

We split each exponential into two factors $\exp\left(-\frac{1}{4} aL^{-(N-j-1)}|\Phi_{j+1} - Q\Phi_j|^2_{P_{j+1}}\right)$. The first is estimated to give small factors and the second is integrated over. Since $P_{j+1} \subset \Omega_{j+1}$ the first is smaller than $\exp\left(-\frac{1}{4} aL^{-(N-j-1)}|\Phi_{j+1} - Q\Phi_j|^2_{P_{j+1}}\right)$.

**Lemma 11.** For $\Phi_j : \mathbb{T}_M^{-(N-j)} \to \mathbb{R}$ and $\Phi_{j+1} : \mathbb{T}_M^{-(N-j-1)} \to \mathbb{R}$ and $P_{j+1}$ a union of $L^{-(N-j-1)}M$ cubes in $\mathbb{T}_M^{N}$:

\[ \exp\left(-\frac{1}{4} aL^{-(N-j-1)}|\Phi_{j+1} - Q\Phi_j|^2_{P_{j+1}}\right) \zeta_j^q(P_{j+1}, \Phi_{j+1}) \leq \exp\left(-\frac{1}{4} aLp_j^2 M^{-3}|P_{j+1}|\right) \tag{155} \]

**Proof.** It suffices to prove this back on the lattice where the term was born. We scale up by $L^{N-j}$ and claim that for $\Phi_j : \mathbb{T}_M^{0} \to \mathbb{R}$ and $\Phi_{j+1} : \mathbb{T}_M^{0} \to \mathbb{R}$ and $P_{j+1}$ a union of $LM$-cubes in $\mathbb{T}_M^{N}$:

\[ \exp\left(-\frac{1}{4} aL|\Phi_{j+1} - Q\Phi_j|^2_{P_{j+1}}\right) \zeta_j^q(P_{j+1}, \Phi_{j+1}) \leq \exp\left(-\frac{1}{4} aLp_j^2 M^{-3}|P_{j+1}|\right) \tag{156} \]

Keep in mind that $|P_{j+1}|$ is invariant under scaling. The left side of (156) can be written:

\[ \prod_{\Box \subset P_{j+1}} \exp\left(-\frac{1}{4} aL|\Phi_{j+1} - Q\Phi_j|^2_{\Box}\right) \zeta_j^q(\Box) \tag{157} \]

where the product is over the $LM$ cubes. The characteristic function $\zeta_j^q(\Box)$ enforces that there is at least one point in $\Box$ such that $|\Phi_{j+1} - Q\Phi_j| \geq p_j$. Therefore

\[ \exp\left(-\frac{1}{4} aL|\Phi_{j+1} - Q\Phi_j|^2_{\Box}\right) \zeta_j^q(\Box) \leq \exp\left(-\frac{1}{4} aLp_j^2\right) \tag{158} \]

The result now follows since the number of $L$-cubes in $P_{j+1}$ is $|P_{j+1}^{(j+1)}|$ so the number of $LM$ cubes in $P_{j+1}$ is $M^{-3}|P_{j+1}^{(j+1)}|$. This completes the proof.

Next consider

\[ dW_{N, \Omega} = \prod_{j=0}^{N} \frac{(2\pi)^{-|\Omega_j+1-L_{j+1}|^2}}{2} \exp\left(-\frac{1}{2} L^{-(N-j)}|W_j|^2_{\Omega_j+1-L_{j+1}}\right) dW_{N-j}^{(N-j)} \tag{159} \]

We break the exponent into two pieces $\exp(-\frac{1}{4} L^{-(N-j)}|W_j|^2_{\Omega_j+1-L_{j+1}})$. The first gives small factors and the second gives convergence of the integral. Since $R_{j+1} \subset \Omega_{j+1} - L_{j+1}$ the first is smaller than $\exp(-\frac{1}{4} L^{-(N-j)}|W_j|^2_{R_{j+1}})$.

**Lemma 12.**

\[ \exp\left(-\frac{1}{4} L^{-(N-j)}|W_j|^2_{R_{j+1}}\right) \zeta_j^w(L^{-(N-j)}(R_{j+1})) \leq \exp\left(-\frac{1}{4} p_{0j}^2 M^{-3}|P_{j+1}^{(j+1)}|\right) \tag{160} \]
Proof. Scaled up to a unit lattice this says
\[
\exp \left( -\frac{1}{4} |W_j|^2 |R_{j+1}| \right) \zeta_j^w (R_{j+1}) \leq \exp \left( -\frac{1}{4} P_0^2 M^{-3} |R_{j+1}| \right)
\]  
(161)

The left side can be written:
\[
\prod_{\square \subset R_{j+1}} \exp \left( -\frac{1}{4} |W_j|^2 \right) \zeta_j^w (\square)
\]  
(162)

where the product is over the \(LM\) cubes. The characteristic function \(\zeta_j^w (\square)\) enforces that there is at least on point in \(\square\) such that \(|W_j| \geq P_{0,j}\). Therefore
\[
\exp \left( -\frac{1}{4} |W_j|^2 \right) \zeta_j^w (\square) \leq \exp \left( -\frac{1}{4} P_{0,j}^2 \right)
\]  
(163)

and the result follows as before.

Next we extract the small factors from the action. We first note some bounds on the potential which are independent of field size. We have in general for \(\lambda > 0\)
\[
V(\Lambda; \epsilon, \mu, \lambda) \equiv \epsilon \text{ Vol}(\Lambda) + \frac{1}{2} \mu \int_\Lambda \phi^2 + \frac{1}{4} \lambda \int_\Lambda \phi^4
\]
\[
\geq -|\epsilon| \text{ Vol}(\Lambda) - \frac{1}{2} |\mu| \int_\Lambda \phi^2 + \frac{1}{4} \lambda \int_\Lambda \phi^4
\]
\[
\geq - (|\epsilon| + \frac{1}{4} |\mu| \lambda^{-1}) \text{ Vol}(\Lambda)
\]  
(164)

The last step follows since \(-\frac{1}{2} |\mu| x^2 + \frac{1}{4} \lambda x^4\) has the minimum value \(-\frac{1}{4} \mu x^2\).

Note also that
\[
\prod_{j=0}^{N-1} \zeta_{j+1,L^{-n-j}} (Q_{j+1}) = \prod_{j=0}^{N} \zeta_{j,L^{-n-j}} (Q_j) = \prod_{j=0}^{N} \zeta_{j,L^{-n-j}} (Q_j)
\]  
(165)

Thus we can use the following estimate:

**Lemma 13.** Assume the small field bounds in (153). Then there is a constant \(c_2\) (depending on \(L\)) such that for \(j = 0, \ldots, N\) and \(\delta \Lambda_{j-1} = \Lambda_{j-1} - \Lambda_j\):
\[
\exp \left( -S_{j,L^{-n-j}}^+(\Lambda_{j-1} - \Lambda_j) \right) \zeta_{j,L^{-n-j}} (Q_j) \leq \exp \left( C |\lambda_j^{\beta} \delta \Lambda_{j-1} | - c_2 P_j^2 M^{-3} |Q_j^j| \right)
\]  
(166)

**Remark.** This estimate is more involved because the action \(S_{j,L^{-n-j}}^+(\Lambda_{j-1} - \Lambda_j)\) is a function of \(\phi_j, \Omega(\Lambda_{j-1}, \Omega_j, \Lambda_j)\), but \(\zeta_j (Q_j)\) expressed in term of a different field, namely \(\phi_j, \Omega (\square)\). We need to make a connection and we do it via the fundamental fields. For this the lemma 20 in the appendix will be important.

**Proof.** (A.) The bound scales up to
\[
\exp \left( -S_{j,L^{-n-j}}^+(\Lambda_{j-1} - \Lambda_j) \right) \zeta_j (Q_j) \leq \exp \left( C |\lambda_j^{\beta} \delta \Lambda_{j-1} | - c_2 P_j^2 M^{-3} |Q_j^j| \right)
\]  
(167)

which is what we prove. Now we are on \(\mathbb{T}_{M+N-j}^{-1}\) and \(\Lambda_{j-1}\) is a union of \(L^{-1} M\) cubes and \(\Lambda_j, Q_j\) are unions of \(M\) cubes.

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Split the quartic term in the potential in half and write
\[
S_j^{+,u}(\delta \Lambda_{j-1}) = S_j^*(\delta \Lambda_{j-1}) + V_j^u(\delta \Lambda_{j-1}) = \hat{S}_j(\delta \Lambda_{j-1}) + V_j(\delta \Lambda_{j-1}, L^3 \varepsilon_{j-1}, L^2 \mu_{j-1}, \frac{1}{2} \lambda_j) \tag{168}
\]
where
\[
\hat{S}_j(X) \equiv S_j^*(X) + \frac{1}{8} \lambda_j \int_X \phi_j^4 \Omega(\Lambda_{j-1}, \Omega_j) \tag{169}
\]
is now non-negative. Then since \(\varepsilon_{j-1} \leq O(1) \lambda_j^g\) and \(\mu_{j-1} \leq O(1) \lambda_j^{g+3}\) we have by (163)
\[
\exp\left(-S_j^{+,u}(\delta \Lambda_{j-1})\right) \leq \exp\left(-\hat{S}_j(\delta \Lambda_{j-1})\right) \exp\left(C \lambda_j^g \text{Vol}(\delta \Lambda_{j-1})\right)
\leq \exp\left(-\hat{S}_j(Q_j^*)\right) \exp\left(C \lambda_j^g |\delta \Lambda_{j-1}^{(j)}|\right) \tag{170}
\]
The second inequality follows since \(Q_j^s \subset \Lambda_j^s\) and \(Q_j^{s*} \subset \Omega_j^s \subset \Lambda_{j-1} \subset \Lambda_j\) and hence \(Q_j^s \subset \Lambda_j_{j-1} - \Lambda_j\). The bound (167) is now reduced to
\[
\exp\left(-\hat{S}_j(Q_j^*)\right) \zeta_j(Q_j) \leq \exp\left(-c_2 P_j^2 M^{-3} |Q_j^{(j)}|\right) \tag{171}
\]
(B.) Let \(R_1 = 2R + 1\) where \(R\) is the parameter which enters the definition of \(\phi_j \Omega(\square)\), see section 3.1.5 in part II.
\[
\hat{S}_j(Q_j^*) = \sum_{\square \subset Q_j^s} \hat{S}_j(\square) = \sum_{\square \subset Q_j^s, \square \sim R_1} \sum_{\square'} (2R_1 + 1)^{-3} \hat{S}_j(\square')
\geq \sum_{\square \subset Q_j} \sum_{\square \sim R_1} (2R_1 + 1)^{-3} \hat{S}_j(\square) = \sum_{\square \subset Q_j} (2R_1 + 1)^{-3} \hat{S}_j(\square \sim R_1) \tag{172}
\]
Here \(\square, \square'\) are \(M\) cubes, and we use that \(\square \subset Q_j\) and \(\square' \subset \square \sim R_1\) imply \(\square' \subset Q_j^s\) and \(\square \sim R_1 \subset \square'\), so we are summing over a smaller set in the fourth expression as opposed to the third expression.

Now (171) follows if we can show for \(\square \subset Q_j\) with \(R_2 = 2R_1 + 1\)
\[
\exp\left(-R_2^{-3} \hat{S}_j(\square \sim R_1)\right) \zeta_j(\square) \leq \exp(-c_2 P_j^2) \tag{173}
\]
(C.) There is a constant \(c_1\) (small, depending on \(L\)) such that if \(|\partial \Phi_j| \leq c_1 p_j\) and \(|\Phi_j| \leq c_1 \alpha_j^{-1} p_j\) on \(\square \sim R_1\) then on \(\square\)
\[
|\Phi_j - Q_j \phi_j \Omega_j(\square)| \leq p_j
\]
\[
|\partial \phi_j \Omega_j(\square)| \leq p_j
\]
\[
|\phi_j \Omega_j(\square)| \geq \alpha_j^{-1} p_j \tag{174}
\]
This follows by a slight variation of lemma 3.1 in part II, and needs \(\square\) well inside \(\Omega_j\) which we have since \(Q_j \subset \Omega_j^s\). This implies that \(\chi_j(\square) = 1\) and hence \(\zeta_j(\square) = 1 - \chi_j(\square) = 0\) and so the inequality (173) holds. Thus we can restrict attention to fields such that either \(|\partial \Phi_j| \geq c_1 p_j\) or \(|\Phi_j| \geq c_1 \alpha_j^{-1} p_j\) hold somewhere in \(\square \sim R_1\).

(D.) If \(|\partial \Phi_j| \geq c_1 p_j\) for some bond in \(\square \sim R_1\), then by lemma [20] in appendix [20] there is a constant \(c_0 = O(1)\) so with \(\phi = \phi_j \Omega(\Lambda_{j-1}, \Omega_j, \Lambda_j)\)
\[
\hat{S}_j(\square \sim R_1) \geq \frac{a_j}{2} |\Phi_j - Q_j \phi_j|^2_{\square \sim R_1} + \frac{1}{2} |\partial \phi|_{\square \sim R_1}^2 + \frac{1}{2} \mu_j |\phi|_{\square \sim R_1}^2
\geq c_0 \left( |\partial \Phi_j|^2_{\square \sim R_1} + \mu_j |\Phi_j|^2_{\square \sim R_1} \right) \geq c_0^2 c_1^2 p_j^2 \tag{175}
\]
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This is sufficient to prove (173) if \( c_2 \leq c_0^2 R_2^{-3} \).

(E.) On the other hand suppose \( |\partial \Phi_j| \leq c_1 p_j \) everywhere in \( \square \sim R_1 \), and that \( |\Phi_j| \geq c_1 \alpha_j^{-1} p_j \) at some point in \( \square \sim R_1 \). If also \( \alpha_j \equiv \max \{ \mu_j^+, \lambda_j^+ \} = \mu_j^+ \) then \( |\Phi_j| \geq c_1 \mu_j^{-2} p_j \) at some point. Again using (175) we have the sufficient bound

\[
\hat{S}_j (\square \sim R_1) \geq c_0^2 \mu_j \| \Phi_j \|^2_{\square \sim R_1} \geq c_0^2 \mu_j^{2} p_j^2
\]  

(176)

(F.) We are now reduced to the case \( |\partial \Phi_j| \leq c_1 p_j \) everywhere in \( \square \sim R_1 \), \( |\Phi_j| \geq c_1 \alpha_j^{-1} p_j \) at some point in \( \square \sim R_1 \), and \( \alpha_j = \lambda_j^+ \). We want to show that the field

\[
\phi_j, \Omega_{(j-1, \Omega_j, \lambda_j)} (\square \sim R_1) = \phi_j, \Omega_{(j-1, \Omega_j, \lambda_j)} (\hat{Q}_j^{T} - 1, \Omega_{(j-1, \Omega_j, \lambda_j)} (\Phi_j - \Phi_j(y)) \right) (x) + \Phi_j(y) - \left[ \mu_j G_j, \Omega_{(j-1, \Omega_j, \lambda_j)} \right] (1) \phi_j(y)
\]

(178)

See the proof of lemma 3.1 in part II for a similar identity. The first term is bounded by \( C c_1 p_j \) by the bound on \( \partial \Phi_j \). The last term is bounded by \( \mu_j C |\Phi_j(y)| \) and since \( \mu_j^+ \leq \alpha_j = \lambda_j^+ \) this is bounded by \( \lambda_j^+ C \lambda_j^{-\frac{1}{2}} - \delta \). Here again we are using small field bounds. Thus we have for \( x \) in a neighborhood of \( \Delta y \),

\[
|\phi_j, \Omega_{(j-1, \Omega_j, \lambda_j)}(x) - \Phi_j(y)| \leq C p_j
\]  

(179)

Now if \( |\Phi_j(y)| \geq c_1 \lambda_j^{-\frac{1}{2}} p_j \) at some point \( y \) in \( \square \sim R_1 \), then for \( \lambda_j \leq \lambda \) sufficiently small, the last inequality implies that \( |\phi_j, \Omega_{(j-1, \Omega_j, \lambda_j)}| \geq \frac{1}{2} c_1 \lambda_j^{-\frac{1}{2}} p_j \) at all points in some unit square \( x \in \Delta y \). Then we get the small factor from the potential in \( \hat{S}_j (\square \sim R_1) \):

\[
\hat{S}_j (\square \sim R_1) \geq \frac{1}{8} \lambda_j \int_{\square \sim R_1} \phi_j^{2} \Omega_{(j-1, \Omega_j, \lambda_j)} \geq \frac{1}{8} \lambda_j \int_{\Delta y} \phi_j^{2} \Omega_{(j-1, \Omega_j, \lambda_j)} \geq \frac{1}{128} c_1^2 p_j^2
\]  

(180)

This is sufficient for (173) if \( c_2 \leq \frac{1}{128} c_1^2 R_2^{-3} \). Thus (173) is established.

(G.) A remark on the case \( j = 1 \). In this case it is not \( \phi_1, \Omega_{(0, \Omega_1, \lambda_1)} \) we are considering, but a modification \( \phi_1, \Omega_{(0, \Omega_1, \lambda_1)} \) where \( \Omega_{(1, 1_1)} = \Omega_1 \cap \Omega_1 \). The only \( \Phi_0 \) dependence just comes from a term \( \Delta_{\Omega_1, \Omega_2} \Phi_0 \), and hence from \( \Phi_0 \) near \( \partial \Omega_1 \). Here we do have a small field bound and so the above argument goes through.

(H.) \( j = 0 \) is a special case. In this case it suffices to show on \( T_0^{m+1} \) that for an \( M \)-cube \( \square \subset Q_0 \)

\[
\exp \left( -\frac{1}{2} \| \partial \Phi_0 \|^2_{\square} - \frac{1}{2} \bar{\mu}_0 \| \Phi_0 \|^2_{\square} - \frac{1}{4} \lambda_0 \int_{\square} \Phi_0^2 \right) \leq \exp(-c_2 p_0^2)
\]  

(181)
where \( \zeta_0(\square) \) enforces that either \( |\partial\Phi_0| \geq p_0 \) or \( |\Phi_0| \geq \alpha_0^{-1} p_0 \) at some point in \( \square \). Splitting into the two cases \( \alpha_0 = \hat{\rho}_0^\pm \) and \( \alpha_0 = \lambda_0^\pm \) this follows directly. (Actually \( \alpha_0 = \lambda_0^\pm \) will always hold for \( N \) sufficiently large.) This completes the proof.

**Remarks.**

1. The last lemma works as well with \( \exp(-\frac{1}{2}S^{+,\alpha}_j(\Lambda_j - \Lambda_j)) \) rather than \( \exp(-S^{+,\alpha}_j(\Lambda_j - \Lambda_j)) \). Then we would have an extra factor \( \exp(-\frac{1}{2}S^{+,\alpha}_j(\Lambda_j - \Lambda_j)) \) to use for the convergence of the integrals. We do not need to do this since we still have small field characteristic functions to enforce the convergence. However the small field characteristic functions are not available for \( j = 0 \), so in this case we do make the split and have a factor \( \exp(-\frac{1}{2}S^+_0(\Lambda_0^j)) = \exp(-\frac{1}{2}S^+_0(\Lambda_0^j)) \) left over.

2. We collect the small factors generated by the previous three lemmas. With a further shift of indices and taking account that \( P_0, R_0, P_{N+1}, Q_{N+1} = \emptyset \) they are

\[
\prod_{j=0}^{N+1} \exp \left( -\frac{1}{4} a \lambda_0^{2j-1} M^{-3} |P_j^{(j)}| - c_2 p_j^2 M^{-3} |Q_j^{(j)}| - \frac{1}{4} p_0^2 \lambda_0^{2j-1} M^{-3} |R_j^{(j)}| \right) \quad (182)
\]

Assuming \( c_2 \leq \frac{1}{4} a L \) and using \( p_j \geq p_{0,j} \) and that \( p_{0,j} - 1 \geq p_{0,j} \) this is bounded by

\[
\prod_{j=0}^{N+1} \exp \left( -c_2 p_j^2 M^{-3} (|P_j^{(j)}| + |Q_j^{(j)}| + |R_j^{(j)}|) \right) \quad (183)
\]

There is also the factor \( \exp(C\lambda^j_j|\delta\Lambda_j^{(j-1)}|) \leq \exp(C\lambda^j_j|\Lambda_j^{(j-1)}|) \) from lemma 13.

### 3.4 final integrals

The remaining integral over \( W_N^{(N)} \) in (183) is

\[
\int \prod_{j=0}^{N} (2\pi)^{-2|\Omega_{j+1} - \Lambda_{j+1}|^{(j)}/2} \exp \left( -\frac{1}{4} L^{-(N-j)} |W_j^{(j)}_{\Omega_{j+1} - \Lambda_j^{(j)}}| \right) dW_j^{(N-j)}_{\Omega_{j+1} - \Lambda_j^{(j)}} = \prod_{j=0}^{N} 2^{|\Omega_{j+1} - \Lambda_j^{(j)}|^{(j)}/2} \quad (184)
\]

The last line follows by the change of variables \( W_j \rightarrow \sqrt{2} W_j \) which takes us back to a probability measure.

Thus the remaining integrals in \( K'(\Theta) \) are bounded by

\[
\int \prod_{j=0}^{N} d\Phi_j^{(N-j)}_{\Omega_{j+1}} \exp \left( -\frac{1}{4} a L^{-(N-j)} |\Phi_j - Q\Phi_j - |\Omega_{j+1}|^{2} \right) \exp \left( -S^+_0, L^{-(N-j)}(\Lambda_0^j) - \chi_0, L, L^{-(N-j)}(\Lambda_0^j) \right) \quad (185)
\]

We do the integrals for \( j = N, N-1, \ldots, 2, 1 \) in that order. In each case we first scale up by \( N-j \) so that \( \Omega_j \) is a union of \( M \) cubes and \( \Omega_{j+1} \) is a union of \( LM \) cubes in \( T_{N+M-j} \), and \( \Phi_j \) is a function on \( (\Omega_{j+1}^{(j)}) \subset T_{N+M-j} \). Split the integral over \( \Phi_j_{\Omega_{j+1}} \) into an integral over \( \Phi_j_{\Omega_{j}} \) and an integral over \( \Phi_j_{\Omega_j} \). The first integral is

\[
\int d\Phi_j_{\Omega_j} \chi_j(\delta\Omega_j, \Phi_j) = 2\lambda_j^{\frac{1}{4} - \delta} |(\delta\Omega_j)^{(j)}| = \exp \left( \mathcal{O}(1)(-\log \lambda_j)(|\delta\Omega_j^{(j)}|) \right) \quad (186)
\]
The second integral is
\[
\int d\Phi_{j,\Omega_c} \exp \left( -\frac{1}{4} a |\Phi_j - Q\Phi_{j-1}|^2_{\Omega_c} \right) = \mathcal{N} \left( \frac{1}{2} a, (\Omega_c^j) \right) = \exp \left( \mathcal{O}(1)|\Omega_c^j| \right) \tag{187}
\]
These combine to give a bound \( \exp \left( \mathcal{O}(1)(-\log \lambda_j)|\Omega_c^j| \right) \)
For \( j = 0 \) we scale up by \( LN \) and then split the integral over \( \Phi_{0,\Omega_0} \) into integrals over \( \Phi_{0,\Lambda_0-\Omega_1} \) and \( \Phi_{0,\Lambda_0} \). For the first we have as before
\[
\int d\Phi_{0,\Lambda_0-\Omega_1} \tilde{\chi}_0(\Lambda_0 - \Omega_1, \Phi_0) \leq \exp \left( \mathcal{O}(1)(-\log \lambda_0)|\Lambda_0^0 - \Omega_1^0| \right) \leq \exp \left( \mathcal{O}(1)(-\log \lambda_0)|\Omega_1^0| \right) \tag{188}
\]
For the second we have
\[
\int d\Phi_{0,\Lambda_0} \exp \left( -\frac{1}{4} S^+_{\Omega_0} (\Lambda_0^c, \Phi_0) \right) \leq \int d\Phi_{0,\Lambda_0} \exp \left( -\frac{1}{4} \tilde{\mu}_0 \|\Phi_0\|^2_{\Lambda_0^c} \right) = \mathcal{N} \left( \frac{1}{2} \tilde{\mu}_0, (\Lambda_0^c) \right) \leq \exp \left( \mathcal{O}(1)(-\log \tilde{\mu}_0)|\Lambda_0^c| \right) \leq \exp \left( \mathcal{O}(1)(-\log \lambda_0)|\Lambda_0^c| \right) \tag{189}
\]
The last step follows since
\[
-\log \tilde{\mu}_0 = 2N \log L \leq 2(-\log \lambda + N \log L) = 2(-\log \lambda_0) \tag{190}
\]
Putting this together the integrals (185) are bounded by
\[
\exp \left( \mathcal{O}(1)(-\log \lambda_0)|\Lambda_0^c| + \sum_{j=0}^N C(-\log \lambda_j)|\Omega_c^{j+1}| \right) \tag{191}
\]
But \(|\Omega_c^{j+1}| = L^3|\Omega_c^j|^{j+1}| \leq L^3|\Lambda_c^{j+1}|^{j+1}| \) and \(-\log \lambda_j = -\log \lambda_{j+1} + \log L \leq -2\log \lambda_{j+1} \). Hence the above expression can be bounded by
\[
\exp \left( \sum_{j=0}^{N+1} C(-\log \lambda_j)|\Lambda_c^j| \right) \tag{192}
\]
A factor of this form also bounds the right side of (184). It also bounds contribution from factors like \( \exp(c_j|\Omega_c^{j-1}|) \) and \( \exp(B_0|\Lambda_0^j - \Lambda_0^j|_M) \) in (153).
Combining these estimates with (183) we have finally
\[
|\mathcal{K}'(\Theta)| \leq \sum_{\{P_j, Q_j, R_j\} : \Lambda_{N+1} = \emptyset} \exp \left( \sum_{j=0}^{N+1} C(-\log \lambda_j)|\Lambda_c^j| - c_2 p_{\emptyset,j}^2 M^{-3}\left(|P_j^j| + |Q_j^j| + |R_j^j| \right) \right) \tag{193}
\]
Here \( \Omega_j, \lambda_j \) are defined from \( P_j, Q_j, R_j \) by (151) and \( P_0, R_0, Q_{N+1}, P_{N+1} = \emptyset \). At this point all the fields are gone.

### 3.5 convergence

We estimate the last sum. This analysis is more or less model independent, we follow [2].
Let us return to the general step in the analysis. We have \( \Lambda_c^j \) defined by sequences \( \{P_j, Q_j, R_j\}_{j=0}^k \) which are unions of \( L^{-(k-j)} M \) cubes in \( T_{M+N-k} \). Let \( C_j \) be the set of all \( L^{-(k-j)} M \) cubes in \( P_j \cup Q_j \cup R_j \). The number of elements in this set is the same as the number of \( M \) cubes when \( P_j, Q_j, R_j \) are scaled by \( L^{k-j} \) up to \( T_{M+N-j} \). In this case \( |P_j^j| \) is the number of unit cubes so
\[
|C_j| = M^{-3}\left(|P_j^j| + |Q_j^j| + |R_j^j| \right) \tag{194}
\]
Lemma 14. 
\[ \text{Vol}(\Lambda^c_k) = \nu((\Lambda^c_k)^{(k)}) \leq O(1)(Mr_k)^3 \left( |C_0| + \ldots + |C_k| \right) \]

(195)

Proof. We first claim that \( \Lambda^c_k \) can be covered by all of the following

- \(|C_0|\) cubes of width \( \leq M \left( L^{-k}(1 + 22|r_0|) + 22L^{-(k-1)}|r_1| + \ldots + 22L^{-1}|r_{k-1}| + 22|r_k| \right) \)
- \(|C_1|\) cubes of width \( \leq M \left( L^{-(k-1)}(1 + 22|r_1|) + 22L^{-(k-2)}|r_2| + \ldots + 22L^{-1}|r_{k-1}| + 22|r_k| \right) \)
  
- \( \ldots \)
- \(|C_{k-1}|\) cubes of width \( \leq M \left( L^{-1}(1 + 22|r_{k-1}|) + 22|r_k| \right) \)
- \(|C_k|\) cubes of width \( \leq M \left( 1 + 22|r_k| \right) \)

(196)

The proof is by induction on \( k \), just as in the proof of the main theorem in part II. First we show the statement is true for \( k = 0 \). We have \( \Lambda_0^c = Q_0^* \) and \( C_0 \) is all \( M \) cubes \( \square \) in \( Q_0 \). Then

\[ \Lambda_0^c = \bigcup_{\square \subseteq C_0} \square^* \]

(197)

Hence \( \Lambda_0^c \) is covered by \(|C_0|\) cubes of width \( M(1 + 10|r_0|) \leq M(1 + 22|r_0|) \) as required.

Now assume it is true for \( k \) and we prove it for \( k + 1 \). Before scaling \( \Lambda_{k+1}^c \), a union of \( LM \) cubes in \( T_{M+N-k}^{-k} \), and is generated by

\[ \Lambda_{k+1}^c = (\Lambda_{k+1}^c)^{10*} \cup P_{k+1}^{10*} \cup Q_{k+1}^{5*} \cup R_{k+1}^{5*} \]

(198)

The covering of \( \Lambda_{k+1}^c \) is also a covering of the smaller set \( \Lambda_{k+1}^c \). Each cube in this covering is enlarged to a cube which is a union of standard \( LM \) cubes (adding less than \( 2LM \) to the width) and then further enlarged with by adding \( 10|r_{k+1}| \) layers of \( LM \) cubes. The overall enlargement is less than \( 22LM[r_{k+1}] \). Thus we have a covering of \((\Lambda_{k+1}^c)^{10*}\) by

- \(|C_0|\) cubes of width \( \leq M \left( L^{-k}(1 + 22|r_0|) + 22L^{-(k-1)}|r_1| + \ldots + 22L|r_{k-1}| + 22L|r_{k+1}| \right) \)
- \(|C_1|\) cubes of width \( \leq M \left( L^{-(k-1)}(1 + 22|r_1|) + 22L^{-(k-2)}|r_2| + \ldots + 22L|r_{k-1}| + 22L|r_{k+1}| \right) \)
  
- \( \ldots \)
- \(|C_{k-1}|\) cubes of width \( \leq M \left( L^{-1}(1 + 22|r_{k-1}|) + 22L|r_{k+1}| \right) \)
- \(|C_k|\) cubes of width \( \leq M \left( (1 + 22|r_k|) + 22L|r_{k+1}| \right) \)

(199)

In addition if \( C_{k+1} \) is the \( LM \) cubes in \( P_{k+1} \cup Q_{k+1} \cup R_{k+1} \) we can cover \( P_{k+1}^{10*} \cup Q_{k+1}^{5*} \cup R_{k+1}^{5*} \) by

\[ |C_{k+1}| \text{ cubes of width} \leq LM(1 + 22|r_{k+1}|) \]

(200)

The actual \( \Lambda_{k+1}^c \) in \( T_{N+M-k-1}^{-k} \) is obtained by scaling down by \( L^{-1} \). Thus it is covered by the cubes in \( 199 \) and \( 200 \) with widths scaled down by \( L^{-1} \). This is the claim for \( k + 1 \). Thus \( 196 \) is established.
It follows that
\[
\text{Vol}(\Lambda_k^c) \leq M^3 \left( L^{-k}(1 + 22|r_0|) + 22L^{-(k-1)}|r_1| + \cdots + 22L^{-1}|r_{k-1}| + 22|r_k| \right)^3 |C_0| \\
+ M^3 \left( L^{-(k-1)}(1 + 22|r_1|) + 22L^{-(k-2)}|r_2| + \cdots + 22L^{-1}|r_{k-1}| + 22|r_k| \right)^3 |C_1| \\
+ \cdots \\
+ M^3 \left( L^{-1}(1 + 22|r_{k-1}|) + 22|r_k| \right)^3 |C_{k-1}| \\
+ M^3(1 + 22|r_k|)^3|C_k| 
\]

(201)

However
\[
r_{k-j} \leq (1+j \log L)r_k 
\]

(202)

So the first term is bounded by
\[
M^3r_k^3 \left( \sum_{j=0}^{k} L^{-j}(1 + j \log L) \right)^3 |C_0| \leq O(1)M^3r_k^3|C_0| 
\]

(203)

The other sums in the other terms are even smaller and so \( \text{Vol}(\Lambda_k^c) \leq O(1)M^3r_k^3\left(|C_0| + \ldots |C_k|\right) \).

**Lemma 15.**
\[
\mathcal{K}'(\Theta) \leq \lambda^n e^{-\kappa'|\Theta|_M} 
\]

(204)

**Proof.** In (193) we use the last result to estimate
\[
\sum_{j=0}^{N+1} (-\log \lambda_j)|(\Lambda_j^{(j)}) \leq O(1) \sum_{j=0}^{N+1} (-\log \lambda_j)(Mr_j)^3 \sum_{i=0}^{j} |C_i| \\
= O(1) \sum_{i=0}^{N+1} M^3|C_i| \sum_{j=i}^{N+1} (-\log \lambda_j)r_j^3 \leq O(1) \sum_{i=0}^{N+1} M^3(-\log \lambda_i)^{3r+2}|C_i| 
\]

(205)

In the last step we use \( r_j^3 = (-\log \lambda_j)^{3r} \) and \( -\log \lambda_j \leq -\log \lambda_i \) and \( N - i \leq (N-i) \log L - \log \lambda = -\log \lambda_i \).

Also in (193) replace \( M^{-3} \sum_{j=0}^{N+1} |P_{j,i}^{(j)}| + |Q_{j,i}^{(j)}| + |R_{j,i}^{(j)}| \) by \( \sum_{j=0}^{N+1} |C_j| \) and then split it into three equal pieces. Then since \( p_{0,j} = (-\log \lambda_j)^{p_0} \) we have
\[
|\mathcal{K}'(\Theta)| \leq \sum_{\{P_j,Q_j,R_j\} \sim \Lambda_{k+1}^{c}} \exp \left( \sum_{j=0}^{N+1} \left( CM^3(-\log \lambda_i)^{3r+2} - \frac{1}{3} c_2(-\log \lambda_j)^{2p_0} \right) |C_j| \right) \\
\exp \left( \sum_{j=0}^{N+1} \frac{1}{3} c_2 p_0^2 |C_j| \right) \exp \left( \sum_{j=0}^{N+1} -\frac{1}{3} c_2 p_0^2 |C_j| \right) 
\]

(206)

We can assume \( 3r+2 < 2p_0 \). Then for \( -\log \lambda \) and hence \( -\log \lambda_j \) sufficiently large, the first exponential is bounded by one. The second exponential is bounded using (195) again. With a new constant \( c_2' \) it is less than
\[
\exp \left( \sum_{j=0}^{N+1} -\frac{1}{3} c_2 p_0^2 |C_j| \right) \leq \exp \left( -\frac{1}{3} c_2 p_0^2 \sum_{j=0}^{N+1} |C_j| \right) \leq \exp \left( -c_2 p_0^2 (Mr_N)^{-3}|(\Lambda_{k+1}^{c})| \right) \\
= \exp \left( -c_2' (-\log \lambda)^{2p_0-3r}|\Theta|_M \right) \leq \lambda^n e^{-\kappa'|\Theta|_M} 
\]

(207)
In the last we use \(|\Theta|_M \geq 1\) and assume \(\frac{1}{2}c_2(-\log \lambda)^{2p_0 - 3r} \geq \kappa\).

Now we have

\[
|\mathcal{K}'(\Theta)| \leq \lambda^{n_0} e^{-\kappa|\Theta|_M} \sum_{\{P_j, Q_j, R_j\} : \lambda^{n_0} |\mathcal{C}_j|} \exp \left( - \frac{1}{3} c_2 p_0^2 \sum_{j=0}^{N+1} |\mathcal{C}_j| \right)
\]

(208)

Now drop all conditions on \(P_j, Q_j, R_j\) except that they are unions of \(L^{-(N-j)}M\) cubes \(\square\) contained in \(\Theta\). Each sum is estimated separately. For \(P_j\) we use \(|\mathcal{C}_j| \geq |P_j|_{L^{N-j}M}\) and estimate

\[
\sum_{P_j \subset \Theta} \exp \left( - \frac{1}{3} c_2 p_0^2 |P_j|_{L^{N-j}M} \right) = \prod_{\square \subset \Theta} \left( 1 + e^{-\frac{1}{3} c_2 p_0^2} \right) \leq \prod_{\square \subset \Theta} \left( 1 + \lambda^{n_0} \right)
\]

(209)

\[
\leq \prod_{\square \subset \Theta} e^{\lambda^{n_0}} \leq \exp \left( \lambda^{n_0} |\Theta|_{L^{N-j}M} \right)
\]

The estimates on \(Q_j, R_j\) are the same. \((R_{N+1} has \(M\)-cubes, not \(LM\) cubes, but this only improves things.) Our bound becomes

\[
|\mathcal{K}'(\Theta)| \leq \lambda^{n_0} e^{-\kappa|\Theta|_M} \exp \left( \sum_{j=0}^{N+1} 3\lambda^{n_0} |\Theta|_{L^{N-j}M} \right)
\]

(210)

But \(|\Theta|_{L^{N-j}M} = L^{3(N-j)}|\Theta|_M\) and \(\lambda^{n_0} = L^{-(N-j)n_0} \lambda^{n_0}\). Since \(n_0 \geq 4\) the sum in the exponential is bounded by

\[
3\lambda^{n_0} |\Theta|_M \sum_{j=0}^{N+1} L^{-(N-j)(n_0-3)} \leq |\Theta|_M
\]

(211)

Since \(\kappa - 1 > \kappa'\) this yields the desired bound \(\lambda^{n_0} \exp(-\kappa'|\Theta|_M)\).

**Remark.** The sum in (208) factors over the connected components \(\{\Theta_\gamma\}\) of \(\Theta\). The bound (209) holds separately for each factor. Using also \(|\Theta_\gamma|_M \geq d_M(\Theta_\gamma)\) we have

\[
\mathcal{K}'(\Theta) \leq \prod_{\gamma} \lambda^{n_0} e^{-\kappa'|\Theta_\gamma|_M}
\]

(212)

### 3.6 the stability bound

We return to the estimate on \(\mathcal{K}(U)\) where \(U \in \mathcal{D}_M\) is a connected union of \(M\) cubes in \(T_M^{-N}\). Substitute the bound on \(\mathcal{K}'(\Theta)\) into the bound (145) on \(\mathcal{K}(U)\) and find

\[
|\mathcal{K}(U)| \leq \sum_{\{\Theta_\gamma, \{Y_\alpha\}, \{X_\alpha\} \rightarrow U} \prod_{\gamma} \prod_{\alpha} \prod_{\sigma} O(1) \lambda^\beta e^{-\kappa'd_M(\Theta_\gamma)} O(1) \lambda^\beta e^{-\kappa'd_M(X_\alpha)} O(1) \lambda^\beta e^{-\kappa'd_M(Y_\sigma, \text{mod } \Theta)}
\]

(213)

**Lemma 16.**

\[
\sum_{\ell}(d_M(\Theta_\ell) + 1) + \sum_{\alpha}(d_M(X_\alpha) + 1) + \sum_{\sigma}(d_M(Y_\sigma, \text{mod } \Theta) + 1) \geq d_M(U)
\]

(214)

**Proof.** Let \(\tau_\sigma\) be a minimal tree intersecting every cube in \(Y_\sigma \cap \Theta^c\) of length \(\ell(\tau_\sigma) = M d_M(Y_\sigma, \text{mod } \Theta)\), let \(\gamma_\gamma\) be a minimal tree intersecting every cube in \(\Theta_\gamma\) of length \(\ell(\tau_\gamma) = M d_M(\Theta_\gamma)\), and let \(\tau_\alpha\) be a minimal tree intersecting every cube in \(X_\alpha\) of length \(\ell(\tau_\alpha) = M d_M(X_\alpha)\). Also consider the graph consisting of pairs from \(\{\Theta_\gamma\}, \{X_\alpha\}, \{Y_\sigma\}\) which intersect. Consider a subgraph which is a spanning
tree. For every pair in the spanning tree take a cube $\square$ in the intersection and introduce a line between the two points in $\square$ which are vertices of the trees. This line has length $\leq M$ and the number of lines is less than the number of elements in $\{\Theta_\gamma\}, \{Y_\sigma\}, \{X_\alpha\}$. Now the tree $\tau$ formed from $\tau_\sigma, \tau_\gamma, \tau_\alpha$ and the connecting lines has length

$$\ell(\tau) \leq M \left( \sum_\gamma (d_M(\Theta_\gamma) + 1) + \sum_\sigma (d_M(Y_\sigma, mod\Theta) + 1) \right)$$

But $\tau$ spans all cubes in $U$ (cubes in $Y_\sigma \cap \Theta$ are included because of $\Theta$, not $Y_\sigma$) and so $\ell(\tau) \geq M d_M(U)$ which gives the result.

**Lemma 17.**

$$|K(U)| \leq O(1) \lambda^{\beta/2} e^{-(\kappa' - \kappa_0 - 1)d_M(U)}$$

**Proof.** The previous result enables us to extract a factor $e^{-(\kappa' - \kappa_0)d_M(U)}$ from the sum. Furthermore since at least one of $\{\Theta_\gamma\}, \{X_\alpha\}, \{Y_\sigma\}$ must be nonempty, we can pull out an overall factor of $\lambda^{\beta/2}$. Now drop all restrictions on $\Theta_\gamma, X_\alpha, Y_\sigma$ except that they are contained in $U$ and for $\Theta = \cup_\gamma \Theta_\gamma$ that $Y_\sigma \# \Theta$ and $Y_\sigma \in D_N (mod\Theta)$ Then

$$|K(U)| \leq \lambda^{\beta/2} e^{-(\kappa' - \kappa_0)d_M(U)} \sum_{\{\Theta_\gamma\} \text{ in } U} \prod_\gamma \mathcal{O}(1) \lambda^{\beta/2} e^{-\kappa_0 d_M(\Theta_\gamma)}$$

$$\left( \sum_{\{Y_\sigma\} \text{ in } U} \prod_\sigma \mathcal{O}(1) \lambda^{\beta/2} e^{-\kappa_0 d_M(Y_\sigma, mod\Theta)} \right) \left( \sum_{\{X_\alpha\} \text{ in } U} \prod_\alpha \mathcal{O}(1) \lambda^{\beta/2} e^{-\kappa_0 d_M(X_\alpha)} \right)$$

Using $\sum_{X \subseteq U} \exp(-\kappa_0 d_M(X)) \leq \mathcal{O}(1)|U|_M$ we have the estimate

$$\sum_{\{X_\alpha\} \text{ in } U} \prod_\alpha \mathcal{O}(1) \lambda^{\beta/2} e^{-\kappa_0 d_M(X_\alpha)} \leq \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{(X_1, \ldots, X_n)} \prod_{i=1}^n \mathcal{O}(1) \lambda^{\beta/2} e^{-\kappa_0 d_M(X_i)}$$

$$\leq \sum_{N=0}^{\infty} \frac{1}{N!} \left( \mathcal{O}(1) \lambda^{\beta/2} |U|_M \right)^N$$

$$= \exp \left( \mathcal{O}(1) \lambda^{\beta/2} |U|_M \right)$$

Here the second sum is over sequences of polymers $(X_1, \ldots, X_n)$. The sum over $\{Y_\sigma\}$ is estimated similarly now using

$$e^{-\kappa_0 d_M(Y_\sigma, mod\Theta)} \leq \sum_{\square \subseteq U - \Theta} \sum_{Y \subseteq \square, Y \in D_U (mod\Theta)} e^{-\kappa_0 d_M(Y, mod\Theta)}$$

$$\leq \mathcal{O}(1)|U - \Theta|_M \leq \mathcal{O}(1)|U|_M$$

Finally the sum over $\{\Theta_\gamma\}$ is estimated just as the sum over $\{X_\alpha\}$. Thus we have

$$|K(U)| \leq \lambda^{\beta/2} \exp \left( - (\kappa' - \kappa_0)d_M(U) + \mathcal{O}(1)\lambda^{\beta/2} |U|_M \right)$$

This is sufficient since $|U|_M \leq \mathcal{O}(1)(d_M(U) + O(1))$ and $\lambda$ is small.

We are now ready to prove the main result:
Theorem 2. Let \( \bar{\mu} = 1 \) and let \( \lambda \) be sufficiently small. Then there is a choice of counterterms \( \varepsilon_{N}^{0}, \mu_{N}^{0} \) such that

\[
Z_{M,N} = Z_{M,N}(0) \exp \left( \sum_{X} H(X) \right)
\]

where the sum is over connected unions of \( M \) cubes \( X \subset T_{M}^{-N} \) and

\[
|H(X)| \leq \mathcal{O}(1) \lambda^{\beta/2} e^{-\kappa_{0}d_{M}(X)}
\]

Proof. Recall that

\[
Z_{M,N} = Z_{M,N}(0) \sum_{\{U_{\ell}\}} \prod_{\ell} K(U_{\ell})
\]

Since \( K(U) \) satisfies the bound (216), and since \( \mathcal{O}(1) \lambda^{\beta/2} \) is small, by a standard theorem (Appendix B, part I) we can exponentiate the sum to the form (221) with

\[
|H(X)| \leq \mathcal{O}(1) \lambda^{\beta/2} e^{-\kappa' - 4\kappa_{0} - 4} d_{M}(X)
\]

Since we can assume \( \kappa' - 4\kappa_{0} - 4 = \kappa - 11\kappa_{0} - 11 \geq \kappa_{0} \) we have the result.

As a corollary we have the stability bound. Earlier versions can be found in [17], [16], [1], [2], [13].

Corollary 1. (stability)

\[
\exp \left( -\lambda^{\eta} \text{Vol}(T_{M}) \right) \leq \frac{Z_{M,N}}{Z_{M,N}(0)} \leq \exp \left( \lambda^{\eta} \text{Vol}(T_{M}) \right)
\]

for some \( \eta > 0 \) independent \( M, N \).

Proof. This follows with \( \eta = \beta/2 \) since (with \( \text{Vol}(T_{M}) = \text{Vol}(T_{M}^{-N}) \))

\[
| \sum_{X \subset T_{M}^{-N}} H(X) | \leq \mathcal{O}(1) \lambda^{\beta/2} |T_{M}^{-N}_{M} | = \mathcal{O}(1) M^{-3} \lambda^{\beta/2} \text{Vol}(T_{M}) \leq \lambda^{\beta/2} \text{Vol}(T_{M})
\]

Remark. The analysis can be adapted to treat correlation functions as well. See particularly [12] for an indication of how this would go.

A minimizers

For a sequence of small field regions \( \Omega = (\Omega_{1}, \ldots, \Omega_{k}) \) and fields \( \Phi_{k} : \Omega = (\Phi_{1,\delta \Omega_{1}}, \ldots, \Phi_{k-1,\delta \Omega_{k-1}}, \Phi_{k,\Omega_{k}}) \) in \( T_{m+N-k}^{-k} \) as in (11) we consider the action

\[
S(\Omega_{1}, \Phi_{k} : \Omega, \phi_{k} : \Omega) = \frac{1}{2} \| a^{1/2} (\Phi_{k} : \Omega - Q_{k} \phi_{k} : \Omega) \|_{\Omega_{1}}^{2} + \frac{1}{2} \left\langle \phi_{k} : \Omega, (-\Delta + \bar{\mu}) \phi_{k} : \Omega \right\rangle
\]

where

\[
Q_{k} \phi : \Omega = ([Q_{1} \phi]_{\delta \Omega_{1}}, \ldots, [Q_{k-1} \phi]_{\delta \Omega_{k-1}}, [Q_{k} \phi]_{\Omega_{k}})
\]

and \( \phi_{k} : \Omega \) is the minimizer in \( \phi_{\Omega_{k}} \) of

\[
S(\Omega_{1}, \Phi_{k} : \Omega, \phi) = \frac{1}{2} \| a^{1/2} (\Phi_{k} : \Omega - Q_{k} \phi : \Omega) \|_{\Omega_{1}}^{2} + \frac{1}{2} \left\langle \phi, (-\Delta + \bar{\mu}) \phi \right\rangle
\]
defined for \( \phi : T_{m+k-n-k}^k \rightarrow \mathbb{R} \).

Given a new region \( \Omega_{k+1} \subset \Omega_k \) we want to find the minimizer of \( S(\Omega_1, \Phi_{k,\Omega}, \phi_{k,\Omega}) \) in \( \Phi_{k,\Omega_{k+1}} \) with all other variables fixed. With \( \Omega^+ = (\Omega_1, \ldots, \Omega_{k+1}) \) and \( \delta \Omega_k = \Omega_k - \Omega_{k+1} \) these are

\[
\Phi_{k,\delta \Omega} = (\Phi_{1,\Omega_1}, \ldots, \Phi_{k,\delta \Omega_k})
\]

This is the same as the minimizer of \( S(\Omega_1, \Phi_{k,\Omega}, \phi) = S(\Omega_1, \Phi_{k,\delta \Omega}^+, \Phi_{k,\Omega_{k+1}}, \phi) \) in both \( \Phi_{k,\Omega_{k+1}} \) and \( \phi_{k,\Omega} \). The solution depends on the Green’s function

\[
G_{k,\delta \Omega} = \left[ -\Delta + \bar{\mu}_k + Q^T_{k,\delta \Omega} a Q_{k,\delta \Omega} \right]^{-1} \Omega_i
\]

where

\[
Q_{k,\delta \Omega} \phi = \left( [Q_1(\phi)]_{\Omega_1}, \ldots, [Q_k(\phi)]_{\Omega_k} \right)
\]

**Lemma 18.**

1. Given \( \Phi_{k,\delta \Omega}^+ \) The unique minimum of \( S(\Omega_1, \Phi_{k,\Omega}, \phi) \) in \( \Phi_{k,\Omega_{k+1}} \) and \( \phi_{k,\Omega} \) comes at

\[
\phi_{k,\Omega} = \phi_{k,\delta \Omega}^+ = \phi_{k,\delta \Omega}^+ (\phi_{k,\Omega}^+, \Phi_{k,\delta \Omega}^+) = G_{k,\delta \Omega} \left( Q^T_{k,\delta \Omega} a \Phi_{k,\delta \Omega} + [\Delta]_{\Omega_1,\Omega_i} \phi_{k,\Omega}^0 \right)
\]

and at

\[
\Phi_{k,\Omega_{k+1}} = \Psi_{k,\Omega_{k+1}} (\delta \Omega^+) = [Q_k(\phi)]_{\Omega_{k+1}}
\]

2. We have the identity

\[
\phi_{k,\delta \Omega}^+ = \phi_{k,\Omega} (\phi_{k,\Omega}^+, \Phi_{k,\delta \Omega}^+, \Psi_{k,\Omega_{k+1}}(\delta \Omega^+))
\]

**Remark.** This is only useful for \( \bar{\mu}_k \) has a substantial size and so can take the place of the missing averaging operator in \( \Omega_{k+1} \) in \( G_{k,\delta \Omega}^+ \).

**Proof.** The variational equations for minimizing \( S(\Omega_1, \Phi_{k,\Omega}, \phi) \) are

\[
\Phi_{k,\Omega_{k+1}} - Q_k(\phi) = 0
\]

\[
\left[ -\Delta + \bar{\mu}_k + Q^T_{k,\Omega} a Q_{k,\Omega} \right]_{\Omega_1} \phi_{k,\Omega} = Q^T_{k,\Omega} a \Phi_{k,\Omega} + [\Delta]_{\Omega_1,\Omega_i} \phi_{k,\Omega}^0 \Omega_i
\]

Substituting \( \Phi_{k,\Omega_{k+1}} = Q_k(\phi) \) into the second equation and canceling a term \( a_k Q^T_{k} Q_k(\phi) \) on each side it becomes

\[
\left[ -\Delta + \bar{\mu}_k + Q^T_{k,\delta \Omega} a Q_{k,\delta \Omega} \right]_{\Omega_1} \phi_{k,\Omega} = Q^T_{k,\delta \Omega} a \Phi_{k,\delta \Omega} + [\Delta]_{\Omega_1,\Omega_i} \phi_{k,\Omega}^0 \Omega_i
\]

with the solution \( \phi_{k,\Omega} = \phi_{k,\delta \Omega}^+ \) defined by (233). With this \( \phi \) the minimum in \( \Phi_{k,\Omega_{k+1}} \) is \( \Psi_{k,\Omega_{k+1}} = [Q_k(\phi)]_{\Omega_{k+1}} \) as claimed.

Before the substitution, the solution of the second equation in (236) is \( \phi_{k,\Omega} (\phi_{k,\Omega}, \Phi_{k,\Omega}) \). At the minimum \( \Phi_{k,\Omega_{k+1}} = \Psi_{k,\Omega_{k+1}} \) it becomes \( \phi_{k,\Omega} (\phi_{k,\Omega}, \Phi_{k,\delta \Omega}^+, \Psi_{k,\Omega_{k+1}}) \). Hence this is another representation for the minimizer \( \phi_{k,\delta \Omega}^+ \).
B a resummation operation

Suppose $\Omega, \Lambda$ are unions of $M$ cubes with $\Lambda \subset \Omega$, and we have an expression

$$\sum_{X \in \mathcal{D}_k(\text{mod } \Omega'), X \cap \Lambda \neq \emptyset} B(X)$$

with

$$|B(X)| \leq B_0 e^{-\kappa d_M(X, \text{mod } \Omega')}$$

for some constant $B_0$. We want to write it as a similar sum with $\Omega$ replaced $\Lambda$ everywhere. Every such $X$ determines a $Y \in \mathcal{D}_k(\text{mod } \Lambda^c)$ with $Y \cap \Lambda \neq \emptyset$ by taking the union with any connected component of $\Lambda^c$ connected to $X$, written $X \rightarrow Y$. We define

$$B'(Y) = \sum_{X \in \mathcal{D}_k(\text{mod } \Omega'), X \cap \Lambda \neq \emptyset, X \rightarrow Y} B(X)$$

and then

$$\sum_{X \in \mathcal{D}_k(\text{mod } \Omega'), X \cap \Lambda \neq \emptyset} B(X) = \sum_{Y \in \mathcal{D}_k(\text{mod } \Lambda^c), Y \cap \Lambda \neq \emptyset} B'(Y)$$

Lemma 19.

$$|B'(Y)| \leq \mathcal{O}(1) B_0 e^{-(\kappa - \kappa_0 - 1) d_M(Y, \text{mod } \Lambda^c)}$$

Proof. We first claim that

$$d_M(Y, \text{mod } \Lambda^c) \leq d_M(X, \text{mod } \Omega')$$

Indeed let $\tau$ be a minimal tree joining the cubes in $X \cap \Omega$ of length $\ell(\tau) = M d_M(X)$. Then $\tau$ is also a tree joining the cubes in $Y \cap \Lambda$ since $Y \cap \Lambda = X \cap \Lambda \subset X \cap \Omega$. Hence $M d_M(Y, \text{mod } \Lambda^c) \leq \ell(\tau)$ and hence the result.

Then we have

$$|B'(Y)| \leq \sum_{X \in \mathcal{D}_k(\text{mod } \Lambda^c), X \cap (Y \cap \Lambda) \neq \emptyset} B_0 e^{-\kappa d_M(X, \text{mod } \Omega')}$$

$$\leq B_0 e^{-(\kappa - \kappa_0) d_M(Y, \text{mod } \Lambda^c)} \sum_{X \in \mathcal{D}_k(\text{mod } \Omega'), X \cap (Y \cap \Lambda) \neq \emptyset} e^{-\kappa_0 d_M(X, \text{mod } \Omega')}$$

$$\leq \mathcal{O}(1) B_0 e^{-(\kappa - \kappa_0) d_M(Y, \text{mod } \Lambda^c)} |Y \cap \Lambda|_M$$

Since $Y \cap \Lambda \subset \Omega$ the last step follows by lemma E.3 in part II. The result now follows by

$$|Y \cap \Lambda|_M \leq \mathcal{O}(1)(d_M(Y \cap \Lambda) + 1) = \mathcal{O}(1)(d_M(Y, \text{mod } \Lambda^c) + 1) \leq \mathcal{O}(1) e^{d_M(Y, \text{mod } \Lambda^c)}$$

C a bound below

Let $\Phi : T^0_{N+k} \rightarrow \mathbb{R}$ and $\phi : T^k_{N+k} \rightarrow \mathbb{R}$, and let $X$ be a union of unit blocks in $T^{-k}_{N+k}$. For the following result we employ Neumann boundary conditions: only bonds contained in $X$ contribute.

Lemma 20. [7] There is a constant $c_0 = \mathcal{O}(1)$ such that for $0 \leq \mu \leq 1$

$$\frac{1}{2} \|\Phi - Q_k \phi\|^2_X + \frac{1}{2} \|\nabla \phi\|^2_X + \frac{1}{2} \mu \|\phi\|^2_X \geq c_0 \left( \|\nabla \Phi\|^2_X + \mu \|\Phi\|^2_X \right)$$

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Proof. We have for \( y \in X \cap T_{N+M-k}^0 \)
\[
|\Phi(y)| \leq |\Phi(y) - (Q_k \phi)(y)| + |(Q_k \phi)(y)|
\] (247)
which yields \( \|\Phi\|_X \leq \|\Phi - Q_k \phi\|_X + \|Q_k \phi\|_X \) and hence
\[
\|\Phi\|^2_X \leq 2 \left( \|\Phi - Q_k \phi\|^2_X + \|\phi\|^2_X \right)
\] (248)
This gives half the result.
We also need a bound on \( \|\partial \phi\|_X^2 \). For a bond \( <y, y+e_{\mu}> \) in \( X \cap T_{N+M-k}^0 \)
\[
|\partial_{\mu} \Phi(y)| = |\Phi(y + e_{\mu}) - \Phi(y)| \leq |\Phi(y + e_{\mu}) - (Q_k \phi)(y + e_{\mu})| + |(Q_k \phi)(y + e_{\mu}) - (Q_k \phi)(y)| + |(Q_k \phi)(y) - \Phi(y)|
\] (249)
The middle term is written as
\[
(Q_k \phi)(y + e_{\mu}) - (Q_k \phi)(y) = \int_{\Delta_y} dx \left( \phi(x + e_{\mu}) - \phi(x) \right) = \int_0^1 dz \int_{\Delta_y} dx \left( \partial_{\mu} \phi \right)(x + ze_{\mu})
\] (250)
where \( \Delta_y \) the unit cube centered on \( y \) and \( z \in L^{-k}Z \). Therefore
\[
|(Q_k \phi)(y + e_{\mu}) - (Q_k \phi)(y)| \leq \int_0^1 dz \|\partial_{\mu} \phi \|_{\Delta_y \cup (\Delta_y + e_{\mu})} \leq \|\partial_{\mu} \phi\|_{\Delta_y \cup (\Delta_y + e_{\mu})}
\] (251)
This leads to
\[
\sum_{<y, y+e_{\mu}> \in X} |(Q_k \phi)(y + e_{\mu}) - (Q_k \phi)(y)|^2 \leq \sum_{<y, y+e_{\mu}> \in X} \|\partial_{\mu} \phi\|^2_{\Delta_y \cup (\Delta_y + e_{\mu})} \leq 2\|\partial \phi\|^2_X
\] (252)
Using this in (249) yields
\[
\|\partial \phi\|^2_X \leq O(1) \left( \|\Phi - Q_k \phi\|^2_X + \|\phi\|^2_X \right)
\] (253)
to complete the proof.

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