Qualitative properties of positive solutions of quasilinear equations with Hardy terms

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Abstract In this paper, we are concerned with the quasilinear PDE with weight
\[-\text{div} A(x, \nabla u) = |x|^a u^q(x), \quad u > 0 \quad \text{in} \ R^n,\]
where \( n \geq 3, \ q > p - 1 \) with \( p \in (1, 2] \) and \( a \in (-n, 0] \). The positive weak solution \( u \) of the quasilinear PDE is \( A \)-superharmonic and satisfies \( \inf_{R^n} u = 0 \). We can introduce an integral equation involving the Wolff potential
\[ u(x) = R(x) W_{\beta,p}( |y|^a u^q(y))(x), \quad u > 0 \quad \text{in} \ R^n, \]
which the positive solution \( u \) of the quasilinear PDE satisfies. Here \( p \in (1, 2], \ q > p - 1, \ \beta > 0 \) and \( 0 \leq -a < p \beta < n \). When \( 0 < q \leq \frac{(n+a)(p-1)}{n-p\beta} \), there does not exist any positive solution to this integral equation. When \( q > \frac{(n+a)(p-1)}{n-p\beta} \), the positive solution \( u \) of the integral equation is bounded and decays with the fast rate \( \frac{n-p\beta}{p-1} \) if and only if it is integrable (i.e. it belongs to \( L^{\frac{n-p\beta}{p-1}}(R^n) \)). On the other hand, if the bounded solution is not integrable and decays with some rate, then the rate must be the slow one \( \frac{p\beta+a}{q-p+1} \). Thus, all the properties above are still true for the quasilinear PDE. Finally, several qualitative properties for this PDE are discussed.

Keywords: Integral equations involving Wolff potential, decay rate, quasilinear equation, Hardy-Sobolev inequality, \( A \)-superharmonic function
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1 Introduction

In this paper, we are concerned with positive solutions of the following quasilinear equation with a Hardy term
\[-\text{div} A(x, \nabla u) = |x|^a u^q(x), \quad u > 0 \quad \text{in} \ R^n, \quad (1.1)\]
Here \( n \geq 3, \ q > p - 1 \) with \( p \in (1, 2], \ -a \in [0, n) \) and \( A : R^n \times R^n \rightarrow R^n \) is a vector valued mapping satisfying:
1. the mapping \( x \rightarrow A(x, \xi) \) is measurable for all \( \xi \in \mathbb{R}^n \);
2. the mapping \( \xi \rightarrow A(x, \xi) \) is continuous for a.e. \( x \in \mathbb{R}^n \).

In addition, there are constants \( 0 < \mu_1 \leq \mu_2 < \infty \) such that for a.e. \( x \in \mathbb{R}^n \), and for all \( \xi \in \mathbb{R}^n \),

\[
\begin{cases}
A(x, \xi) \cdot \xi \geq \mu_1|\xi|^p, & A(x, \xi) \leq \mu_2|\xi|^{p-1}; \\
[A(x, \xi_1) - A(x, \xi_2)] \cdot (\xi_1 - \xi_2) > 0, & \text{if } \xi_1 \neq \xi_2; \\
A(x, \lambda\xi) = \lambda|\lambda|^{p-2}A(x, \xi), & \text{if } \lambda \neq 0.
\end{cases}
\]

(1.2)

A function \( u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n) \cap C(\mathbb{R}^n) \) is called the weak solution of (1.1), if

\[
\int_{\mathbb{R}^n} A(x, \nabla u) \nabla \zeta \, dx = \int_{\mathbb{R}^n} |x|^a u^q \zeta \, dx, \quad \forall \zeta \in C_0^\infty(\mathbb{R}^n).
\]

(1.3)

In the special case \( A(x, \xi) = |\xi|^{p-2}\xi \), \( \text{div}A(x, \nabla u) \) is the usual \( p \)-Laplacian defined by \( \text{div}(|\nabla u|^{p-2}\nabla u) \). Now, (1.1) becomes

\[
-\text{div}(|\nabla u(x)|^{p-2}\nabla u(x)) = |x|^a u^q(x), \quad u > 0 \text{ in } \mathbb{R}^n.
\]

(1.4)

This equation arose in many fields such as nonlinear functional analysis, astrophysics and astronomy (cf. [2], [3] and [6]). In particular, it is essential in the study of the extremal function of the Hardy-Sobolev type inequality [1]

\[
\Lambda\left(\int_{\mathbb{R}^n} |u|^{q+1}|x|^a dx\right)^{1/(q+1)} \leq \left(\int_{\mathbb{R}^n} |\nabla u|^p dx\right)^{1/p}, \quad \forall u \in D^{1,p}(\mathbb{R}^n),
\]

(1.5)

where \( 0 \leq -a < p \), \( 1 \leq p < q + 1 = \frac{p(n+a)}{n-p} \), and \( D^{1,p}(\mathbb{R}^n) \) is the homogeneous Sobolev space. We call \( q = \frac{p(n+a)}{n-p} - 1 \) the critical exponent. Such an inequality is the special case of the Caffarelli-Kohn-Nirenberg inequality (cf. [5], [8] and [29]). Applying the symmetrization and the theories of ODE, one can obtain the sharp constant \( \Lambda \) (cf. [28], [38] and [40]). To find the corresponding extremal function, we consider the minimization problem

\[
\Lambda = \inf\{\|\nabla u\|^p; u \in D^{1,p}(\mathbb{R}^n), \int_{\mathbb{R}^n} |u|^{q+1}|x|^a dx = 1\}.
\]

Since this minimization problem is invariant under the scaling transformation, the variational methods are difficult to be used. Badiale and Tarantello found the solution by the concentration compactness principle (cf. [1]). To describe the shape of the extremal functions, we investigate the Euler-Lagrange equation (1.4). In [33], it was proved that all the extremal functions are cylindrically symmetric.

When \( p = 2 \), (1.4) becomes

\[
-\Delta u = |x|^a u^q, \quad u > 0 \text{ in } \mathbb{R}^n.
\]

(1.6)

Phan and Souplet [37] studied the existence of the positive solution and obtained the Liouville type results. Recently, [23] used an equivalent integral equation to
obtain the decay rates of the positive solutions when $|x| \to \infty$. If $a = 0$, it is associated with the study of the well known Lane-Emden equation

$$-\Delta u = u^q, \quad u > 0 \quad \text{in} \quad R^n,$$

which has been well studied (cf. [4], [9], [15] and [26]).

When $p \neq 2$, it is difficult to find an equivalent integral equation. If a positive solution $u$ is a $A$-superharmonic function and satisfies $\inf_{R^n} u = 0$, then $u$ solves another integral equation involving the Wolff potential

$$u(x) = R(x)W_{\beta,p}(|y|^p u^q(y))(x)$$

with $\beta = 1$ (cf. §4). Here $R(x)$ is double bounded. Namely, there exists $C > 0$ such that $\frac{1}{C} \leq R(x) \leq C$ for all $x \in R^n$. The definition of the $A$-superharmonic function can be found in [21] and [39].

The Wolff potential of a positive function $f \in L^1_{\text{loc}}(R^n)$ is defined as (cf. [16])

$$W_{\beta,p}(f)(x) = \int_0^\infty \left[ \frac{\int_{B_t(x)} f(y)dy}{t^{n-p\beta}} \right] \frac{dt}{t},$$

where $p > 1$, $\beta > 0$, $p\beta < n$, and $B_t(x)$ is a ball of radius $t$ centered at $x$. This potential can help us to understand many nonlinear problems (see [12], [19], [20], [21], [22], [35], [36] and [39]).

When $a = 0$ and $q$ is the critical exponent $\frac{np}{n-p} - 1$, Ma, Chen and Li [34] obtained the integrability, boundedness and the Lipschitz continuity of positive solutions of (1.7). Based on these results, paper [24] estimated the fast decay rate. This asymptotic result is also true for the Wolff type integral system (cf. [42]). Moreover, if $R(x) \equiv 1$, those positive solutions are radially symmetric and decreasing about $x_0 \in R^n$ (cf. [10]).

Furthermore, if $\beta = \alpha/2$ and $p = 2$, (1.7) (with $R(x) \equiv 1$) is reduced to an integral equation involving the Riesz potential

$$u(x) = \int_{R^n} \frac{|y|^\alpha u^q(y)dy}{|x - y|^{n-\alpha}}, \quad u > 0 \quad \text{in} \quad R^n.$$

Lu and Zhu [31] obtained the radial symmetry and the regularity of weak solutions. Moreover, if $\alpha = 2$, this integral equation is reduced to (1.6). Mancini, Fabbri and Sandeep [32] studied the sharp constant and classified the extremal functions of (1.5). If $a = 0$, the integral equation above becomes

$$u(x) = \int_{R^n} \frac{u^q(y)dy}{|x - y|^{n-\alpha}}. \quad (1.8)$$

This equation can be used to describe the extremal functions of the Hardy-Littlewood-Sobolev inequality (cf. [11], [27] and [28]). The radial symmetry of integrable solutions was proved by the method of moving planes in integral forms. Using the regularity lifting lemma by the contraction operators, Jin and Li [17] obtained the optimal integrability. Afterwards, [25] presented the fast decay rates.
In this paper, we consider that $q$ is not only the critical exponent $\frac{p(n+a)}{n-p} - 1$, but also the supercritical and the subcritical cases. Let

$$n \geq 3, \beta > 0, p \in (1,2], q > p - 1, 0 \leq -a < p\beta < n.$$  \hfill (1.9)

Write $p^* = \frac{np}{n-p\beta}$. When $\beta = 1$, $p^*$ is the Sobolev conjugate index, and the weak solution $u \in D^{1,p}(R^n)$ belongs to $L^{p^*}(R^n)$.

Introduce an important index

$$s_0 := \frac{n(q-p+1)}{p\beta + a.}$$

This index $s_0$ is closely related to the invariant of the equation and the energy under the scaling transformation (see Theorem 4.3). In addition, $s_0 = p^*$ if and only if $q$ is the critical exponent $q = \frac{p(n+a)}{n-p\beta} - 1$. When $a = 0$, this critical exponent plays an important role in studying the existence of positive solutions of (1.4) (cf. Corollary II in [41]). When $q < \frac{p(n+a)}{n-p\beta} - 1$, (1.4) has not any regular solution. When $q \geq \frac{p(n+a)}{n-p\beta} - 1$, (1.4) has positive solutions. Papers [14] and [18] estimated the decay rates of solutions. Moreover, [24] obtained the fast decay rates of the $L^{p^*}(R^n)$-solutions of (1.7) when $q = \frac{p(n+a)}{n-p\beta} - 1$.

In section 2, we have a Liouville type theorem (see Theorem 2.1). In addition, we prove the following decay estimates when $|x|$ → ∞ in sections 2 and 3.

**Theorem 1.1.** If $q \in (0, \frac{(n+a)(p-1)}{n-p\beta}]$, then (1.7) does not have any positive solution for any double bounded function $R(x)$. Assume $u$ is a positive solution of (1.7) with (1.9). Then

1. $u$ is bounded and decays with the fast rate $\frac{n-p\beta}{p-1}$ if and only if $u \in L^{s_0}(R^n)$.
2. If a bounded solution $u \notin L^{s_0}(R^n)$ decays with some rate, then the rate must be the slow one $\frac{p\beta + a}{q-p+1}$.

Consider the PDEs (1.1) and (1.4). We have the following qualitative results which are proved in section 4.

**Theorem 1.2.** Let $u$ be a positive weak solution of (1.1) with (1.9). Then

1. the results in Theorem 1.1 are still true. (Now, $\beta = 1$.) Furthermore, if $u \in L^{s_0}(R^n)$, then $u \in D^{1,p}(R^n)$.
2. Assume the weak solution $u \in L^{p^*}(R^n)$. Then $\nabla u \in L^p(R^n)$ if and only if $|x|^a u^{q+1} \in L^1(R^n)$.
3. Let $\lambda \neq 0$. The scaling function $u_\lambda(x) := \lambda \theta u(\lambda x)$ is still a weak solution of (1.1) if and only if $\theta$ is the slow rate $\frac{p\beta + a}{q-p+1}$. Moreover, $\|u_\lambda\|_\eta = \|u\|_\eta$ if and only if $\eta = s_0$.

**Corollary 1.3.** If $u$ is a classical solution of (1.4) with (1.9), then the following three items are equivalent:
1. \( u \in L^\infty(\mathbb{R}^n) \);

2. \( u \) is bounded and decays with the fast rate \( \frac{n-p}{p-1} \) when \( |x| \to \infty \);

3. \( u \in \mathcal{D}^{1,p}(\mathbb{R}^n) \).

Finally, we shows that the weak solution of (1.1) cannot be defined in \( W^{1,p}(\mathbb{R}^n) \) when \( p \in [\sqrt{n}, 2] \).

2 Nonexistence and slow decay rate

In this section, we discuss the slow decay of positive solutions of (1.7). In order to estimate the decay rate, we first show that the exponent \( q \) is larger than \( \frac{(n+a)(p-1)}{n-p} \).

2.1 Nonexistence

**Theorem 2.1.** If \( q \in (0, \frac{(n+a)(p-1)}{n-p}) \), then (1.7) does not have any positive solution for any double bounded function \( R(x) \).

*Proof. Step 1.* Let 

\[
0 < q < \frac{(n+a)(p-1)}{n-p}.
\]  

(2.1)

Suppose that \( u \) solves (1.7), then we deduce a contradiction.

*Substep 1.1.* From (1.7), for \( |x| > 1 \) we can get

\[
u(x) \geq c \int_2^\infty t^{\frac{n-p}{p-1}} dt = \frac{c}{|x|^{a_0}},
\]

(2.2)

where

\[
a_0 = \frac{(n+a)(p-1)}{n-p}.
\]

(2.3)

Since \( \int_{|y|=1}^\infty \frac{|y|^a u^q|y|^a}{|y|^a} dy \geq c \) by this estimate and (2.1), we have

\[
u(x) \geq c \int_2^\infty \left( \int_{|y|=1}^\infty \frac{|y|^a u^q|y|^a}{|y|^a} dy \right)^{\frac{1}{p-1}} dt
\]

(2.3)

When \( \frac{q}{p-1} \in (0, \frac{\beta p+a}{n-p}) \), we have \( \beta p+a-q_0 \geq 0 \). Eq. (2.3) implies \( u(x) = \infty \).

It is impossible.

Next, we consider the case \( \frac{q}{p-1} \in (\frac{\beta p+a}{n-p}, \frac{n+a}{n-p}) \). Now (2.3) leads to

\[
u(x) \geq \frac{c}{|x|^{a_1}},
\]

where

\[
a_1 = \frac{q}{p-1} a_0 - \frac{\beta p}{p-1}.
\]

*Substep 1.2. Write*

\[
a_j = \frac{q}{p-1} a_{j-1} - \frac{\beta p + a}{p-1}, \quad j = 1, 2, \ldots
\]  

(2.4)
Suppose that $a_k < a_{k-1}$ for $k = 1, 2, \ldots, j - 1$. By virtue of (2.1), it follows

$$a_j - a_{j-1} = \left(\frac{q}{p-1} - 1\right)a_{j-1} - \frac{\beta p + a}{p-1} < \left(\frac{q}{p-1} - 1\right)a_0 - \frac{\beta p + a}{p-1}$$

$$= \left(\frac{q}{p-1} - 1\right)n - \frac{\beta p + a}{p-1} = (n - \beta p)\left(\frac{q}{p-1}\right)q - \frac{n + a}{p-1}$$

$$< \left(\frac{n - \beta p}{n - \beta p}\right)(n + a)(p-1) - \frac{n + a}{p-1} = 0.$$ 

Thus, $\{a_j\}_{j=0}^\infty$ is decreasing as long as (2.1) is true.

Furthermore, we claim that there must be $j_0 > 0$ such that $a_{j_0} \leq 0$. Once it is true, similar to the argument in Substep 1.1, we also get $\beta p + a - qa_{j_0-1} \geq 0$, which leads to $u(x) = \infty$. This contradicts with the fact that $u$ is a positive solution.

In fact, by (2.1) we get

$$a_j = \left(\frac{q}{p-1}\right)^j a_0 - \left[1 + \frac{q}{p-1} + \cdots + \left(\frac{q}{p-1}\right)^{j-1}\right]\frac{\beta p + a}{p-1}.$$

If $\frac{q}{p-1} = 1$, then we can find a large $j_0$ such that

$$a_{j_0} = a_0 - j_0\frac{\beta p + a}{p-1} \leq 0.$$

If $\frac{q}{p-1} \in (1, \frac{n+a}{n-\beta p})$, then $a_0 - \frac{\beta p + a}{q - p+1} < 0$. We can find a large $j_0$ such that

$$a_{j_0} = \left(\frac{q}{p-1}\right)^{j_0} a_0 - \left(\frac{q}{p-1}\right)^{j_0} - 1 \frac{\beta p + a}{p-1}$$

$$= \left(\frac{q}{p-1}\right)^{j_0}a_0 - \frac{\beta p + a}{q - p+1} + \frac{\beta p + a}{q - p+1} \leq 0.$$

If $\frac{q}{p-1} \in (0, 1)$, letting $j \to \infty$, we get

$$a_j = \left(\frac{q}{p-1}\right)^j a_0 - \frac{1 - \left(\frac{q}{p-1}\right)^j}{1 - \frac{q}{p-1}} \frac{\beta p + a}{p-1} \to \frac{\beta p + a}{q - p+1} < 0.$$

Thus, there must be $j_0$ such that $a_{j_0} \leq 0$.

Step 2. Let $q = \frac{(n+a)(p-1)}{n-\beta p}$. We deduce the contradiction if $u$ is a positive solution of (1.7).

For $R > 0$, denote $B_R(0)$ by $B_R$. Using (1.7) and the Hölder inequality, we see that for any $x \in B_R$,

$$u(x) \geq c \int_0^R \left(\int_{B_t(x)} |y|^a u^q(y) dy\right)^{\frac{1}{q}} dt$$

$$\geq cR^{-\frac{n+\beta+1}{p-1}} \left(\int_0^R \left(\int_{B_t(x)} |y|^a u^q(y) dy\right)^{\frac{1}{q}} dt\right)^{\frac{1}{q}}.$$
By exchanging the order of the integral variables, and noting $B_{R/4} \times [R/4, R]$ is the subset of the cone \( \{(y, t); t \in |x - y|, R, y \in B_R\} \), we have

\[
    u(x) \geq c R^{-\frac{n+\alpha}{p-1}} \left( \int_{B_R} |y|^\alpha u^q(y) \left( \int_{|x-y|}^R dt \right) dy \right)^{\frac{1}{q-1}}
\]

\[
    \geq c R^{-\frac{n+\alpha}{p-1}} \left( \int_{B_{R/4}} |y|^\alpha u^q(y) dy \right)^{\frac{1}{q-1}}.
\]

Therefore, we get

\[
    |x|^\alpha u^q(x) \geq c |x|^\alpha R^{\frac{n+\alpha}{p-1} - \frac{n}{p-1}} \left( \int_{B_{R/4}} |y|^\alpha u^q(y) dy \right)^{\frac{q}{p-1}}.
\]

(2.5)

Integrating on $B_{R/4}$ and using $q = \frac{(n+\alpha)(p-1)}{n-p\beta}$, we obtain

\[
    \int_{B_{R/4}} |x|^\alpha u^q(x) dx \geq c R^{\frac{n+\alpha}{p-1} - \frac{n}{p-1}} \int_{B_{R/4}} |x|^\alpha dx \left( \int_{B_{R/4}} |y|^\alpha u^q(y) dy \right)^{\frac{q}{p-1}}
\]

\[
    \geq c \left( \int_{B_{R/4}} |y|^\alpha u^q(y) dy \right)^{\frac{q}{p-1}}.
\]

Here $c$ is independent of $R$. Letting $R \to \infty$ and noting $q > p - 1$, we have

\[
    \int_{R^n} |x|^\alpha u^q(x) dx < \infty.
\]

Integrating (2.5) on $A_R = B_{R/4} \setminus B_{R/8}(0)$ yields

\[
    \int_{A_R} |x|^\alpha u^q(x) dx \geq c R^{\frac{n+\alpha}{p-1} - \frac{n}{p-1}} \int_{A_R} |x|^\alpha dx \left( \int_{B_{R/4}} |y|^\alpha u^q(y) dy \right)^{\frac{q}{p-1}}.
\]

By $q = \frac{(n+\alpha)(p-1)}{n-p\beta}$, it follows

\[
    \int_{A_R} |x|^\alpha u^q(x) dx \geq c \left( \int_{B_{R/4}} |y|^\alpha u^q(y) dy \right)^{\frac{q}{p-1}},
\]

where $c$ is independent of $R$. Letting $R \to \infty$, and noting (2.6), we obtain

\[
    \int_{R^n} |y|^\alpha u^q(y) dy = 0,
\]

which implies $u \equiv 0$. It is impossible.

\[\square\]

**2.2 Slow decay rate**

Theorem 2.1 shows that if (1.7) has the positive solution $u$, then

\[
    q > \frac{(n + \alpha)(p - 1)}{n - p\beta}.
\]

(2.7)
To investigate the decay rates of $u$, we always assume (2.7) holds hereafter. By (2.7), we can see that $\frac{\nu+p}{p-1} > \frac{p\beta+a}{q-p+1}$. Thus, we call $\frac{\nu+p}{p-1}$ the fast decay rate and $\frac{p\beta+a}{q-p+1}$ the slow one.

Let $u$ be bounded but not integrable. If it decays along $u(x) \simeq |x|^{-\theta}$ when $|x| \to \infty$, we prove that the rate $\theta$ must be the slow one $\frac{p\beta+a}{q-p+1}$.

**Theorem 2.2.** Let $r_0$ be an arbitrary given positive number in $[s_0, \infty)$. Assume $u \in L^\infty(\mathbb{R}^n) \setminus L^{r_0}(\mathbb{R}^n)$ solves (1.7) with (1.9). If

$$\lim_{|x| \to \infty} u(x)|x|^{-\theta} \in (0, \infty),$$

then $\theta$ must be the slow decay rate $\frac{p\beta+a}{q-p+1}$.

**Proof.** Step 1. Let $\theta < \frac{p\beta+a}{q-p+1}$. We claim that there does not exist $C > 0$ such that as $|x| \to \infty$,

$$u(x) \geq C|x|^{-\theta}.$$

This result shows that the decay rate of $u$ is not faster than the slow one $\frac{p\beta+a}{q-p+1}$.

If there exists $C > 0$ such that for some large $|x|$,

$$u(x) \geq C|x|^{-\theta}, \quad \theta < \frac{p\beta+a}{q-p+1},$$

by an iteration we can deduce the contradiction.

Denote $\theta$ by $b_0$. Similar to the derivation of (2.3), for $|x| > 1$ we have

$$u(x) \geq c|x|^{-b_1}, \quad b_1 = \frac{q b_0 - p \beta - a}{p - 1}.$$ 

By induction, for some large $|x|$ there holds

$$u(x) \geq c|x|^{-b_j}, \quad b_0 = \theta, \quad b_j = \frac{q b_{j-1} - p \beta - a}{p - 1}, \quad j = 1, 2, \ldots.$$

We claim that there must be $j_0$ such that $b_{j_0} < 0$, which leads to $u(x) = \infty$. In fact, similar to the proof of Theorem 2.1, there also holds

$$b_j = \left( b_0 - \frac{p \beta + a}{q - p + 1} \right) \left( \frac{q}{p - 1} \right)^j + \frac{p \beta + a}{q - p + 1}.$$ 

Noting $q > p - 1$ (which is implied by (2.7)) and $b_0 - \frac{p \beta + a}{q - p + 1} < 0$, we can find a large $j_0$ such that $b_{j_0} < 0$. It is impossible since the solution $u$ blows up.

Step 2. Let $\theta > \frac{p \beta + a}{q - p + 1}, r_0 \geq \frac{n(q-p+1)}{p \beta + a}$. If $u \not\in L^{r_0}(\mathbb{R}^n)$, we claim that there does not exist $C > 0$ such that as $|x| \to \infty$,

$$u(x) \leq C|x|^{-\theta}.$$

This result shows that the decay rate of $u$ is not faster than the slow rate $\frac{p \beta + a}{q - p + 1}$.
Suppose there exists $C > 0$ such that as $|x| \to \infty$,

$$u(x) \leq C|x|^{-\theta}, \quad \text{where} \quad \theta > \frac{p\beta + a}{q - p + 1}.$$ 

Since $u$ is bounded, for some large $R > 0$, there holds

$$\int_{R^n} u^a(x)dx = \int_{B_R(0)} u^a(x)dx + \int_{R^n \setminus B_R(0)} u^a(x)dx \leq C + C \int_{R}^\infty r^{n-r_0 \theta} \frac{dr}{r}.$$ 

By virtue of $r_0 \geq n\left(q - p + 1\right)\left(p\beta + a\right)$, we see

$$\int_{R^n} u^a(x)dx < \infty,$$

which contradicts with $u \notin L^a(R^n)$.

\[\square\]

**Remark 2.1.** Moreover, if the positive solution is radially symmetric and decreasing about the origin, then as $|x| \to \infty$

$$u(x) = O\left(|x|^{-\frac{p\beta + a}{q - p + 1}}\right).$$

In fact, when $t \in (0, |x|/2)$, in $D := B_t(x) \cap \{|y| \geq |x|\}$, there hold

$$u(y) \geq u(x), \quad \text{and} \quad |y| \leq 3|x|/2.$$ 

In addition, $|D| > \frac{1}{2} |B_t(x)| \geq c t^n$. Therefore,

$$u(x) \geq c|y|^a u^q(x) \int_0^{|x|/2} \left(\int_D dy \right)^{\frac{p - 1}{q - p + 1}} \geq c|y|^\frac{p\beta + a}{q - p + 1} u^{\frac{q}{q - p + 1}}(x).$$

This implies $u(x) \leq C|x|^{-\frac{p\beta + a}{q - p + 1}}$.

**Remark 2.2.** Consider the positive solutions $u \notin L^\infty(R^n)$. We can find a singular solution with the slow rate $\frac{p\beta + a}{q - p + 1}$. Let

$$u(x) = c|x|^{-t}.$$ 

1. If (2.7) holds, we claim that $u(x)$ solves (1.4) with

$$t = \frac{p + a}{q - p + 1}, \quad c = t^{\frac{p - 1}{q - p + 1}} [n - 1 - (p - 1)(t + 1)]^{\frac{q}{q - p + 1}}.$$ 

In fact, if writing $u(x) = U(|x|) = U(r) := cr^{-t}$, we can see that the left hand side of (1.4) is

$$-\text{div}(\nabla u)|^{p - 2} \nabla u)$$

$$= -|U'|^{p - 2}[(p - 1)U'' + \frac{n - 1}{n} U']$$

$$= \frac{c^{p - 1}p - 1}{r^{(p - 2)(t + 1) + (t + 1)}} [n - 1 - (p - 1)(t + 1)].$$

(2.8)
By virtue of (2.7), we get \( n - 1 > (p - 1)(t + 1) \), and hence the value of the result above is positive. Noting the values of \( t \) and \( c \), we see that (2.8) is equal to \( c^{q-t} \), which is the exact right hand side of (1.4). In addition, the asymptotic rate \( t \) is the slow one when \( |x| \to \infty \) and \( |x| \to 0 \), respectively.

2. We can find a double bounded function \( R(x) \) such that \( u(x) = c|x|^{-t} \) also solves (1.7). Here \( t = \frac{p^\beta q}{q-p+1} \). In fact, write

\[
W_1 = \int_0^{|x|/2} \left( \int_{B_t(x)} \frac{|y| u^q(y)dy}{t^{n-p}} \right)^{\frac{1}{q-t}} dt,
\]

\[
W_2 = \int_{|x|/2}^\infty \left( \int_{B_t(x)} \frac{|y| u^q(y)dy}{t^{n-p}} \right)^{\frac{1}{q-t}} dt.
\]

If \( t \in (0, |x|/2) \), we have \( |x|/2 < |y| < 3|x|/2 \) for \( y \in B_t(x) \). Thus,

\[
c_1 |x|^{a-qt} \leq \int_{B_t(x)} |y| u^q(y)dy \leq c_2 |x|^{a-qt},
\]

where \( 0 < c_1 \leq c_2 \), and hence

\[
c_1 |x|^{\frac{p^\beta + a - qt}{p-1}} \leq W_1 \leq c_2 |x|^{\frac{p^\beta + a - qt}{p-1}}.
\]

If \( t \geq |x|/2 \), we have \( B_t(x) \subset B(0, |x| + t) \subset B(0, 3t) \). Thus, by (2.7),

\[
\int_{B_t(x)} u^q(y)dy \leq C \int_{B_{3t}(0)} |y|^{a-qt}dy \leq C t^{a-qt}.
\]

Therefore,

\[
W_2 \leq C |x|^{\frac{p^\beta + a - qt}{p-1}}.
\]

Noticing \( t = \frac{p^\beta q}{q-p+1} \), and combining the estimates of \( W_1 \) and \( W_2 \), we obtain

\[
c_1 (W_1 + W_2) \leq u(x) \leq c_2 (W_1 + W_2).
\]

Setting

\[
R(x) = u(x)[W_1 + W_2]^{-1},
\]

we know that \( R(x) \) is double bounded and \( u(x) \) solves (1.7).

3 Integability and fast decay rate

3.1 Integability

**Theorem 3.1.** Assume \( u \in L^n(R^n) \) solves (1.7) with (1.9), where \( s_0 = \frac{n(q-p+1)}{p^\beta + a} \). Then

\[
u \in L^s(R^n), \quad \forall s > \frac{n(p-1)}{n-\beta p}.
\]

In addition, the lower bound \( \frac{n-\beta p}{n(p-1)} \) of \( s \) is optimal.
Proof. Step 1. For $A > 0$, set

$$u_A(x) = w(x), \quad \text{if } u(x) > A \text{ or } |x| > A;$$

$$u_A(x) = 0, \quad \text{otherwise},$$

and $u_B(x) = u(x) - u_A(x)$. Let $\sigma$ satisfy

$$\frac{2 - p}{s_0} < \frac{1}{\sigma} < \frac{2 - p}{s_0} + \frac{n - \beta p}{n}. \quad \text{(3.2)}$$

For $g \in L^\sigma(\mathbb{R}^n)$, define operators $T$ and $S$,

$$T g(x) := R(x) \int_0^\infty \left( \frac{\int_{B_t(x)} |y|^a u^q(y) dy}{t^{n-\beta p}} \right)^{\frac{p}{p-1}} \int_{B_t(x)} |y|^a u_A^{q-1}(y) g(y) dy dt \quad t$$

$$S g(x) := \int_0^\infty \left( \frac{\int_{B_t(x)} |y|^a u_A^{q-1}(y) g(y) dy}{t^{n-\beta p}} \right)^{\frac{p}{p-1}} \int_{B_t(x)} |y|^a u_B^q(y) dy dt \quad t$$

and write

$$F(x) := R(x) \int_0^\infty \left( \frac{\int_{B_t(x)} |y|^a u^q(y) dy}{t^{n-\beta p}} \right)^{\frac{p}{p-1}} \int_{B_t(x)} |y|^a u_A^{q-1}(y) g(y) dy dt \quad t$$

Clearly, $u$ is a solution of the following equation

$$g = Tg + F.$$

Step 2. $T$ is a contraction map from $L^\sigma(\mathbb{R}^n)$ into itself.

In fact, by the Hölder inequality, there holds $|Tg| \leq C u^{2-p} |Sg|^{p-1}$. Therefore, we get

$$\|Tg\|_\sigma \leq C \|u\|_{s_0}^{2-p} \|Sg\|_\tau^{p-1} \quad \text{(3.3)}$$

where $\tau > 0$ satisfies

$$\frac{1}{\sigma} = \frac{2 - p}{s_0} + \frac{p - 1}{\tau}. \quad \text{(3.4)}$$

By (3.2) and (3.4), we get

$$0 < \frac{p - 1}{\tau} < 1 - \frac{\beta p}{n}. \quad \text{(3.5)}$$

Therefore, we can use the weighted Hardy-Littlewood-Sobolev inequality and the Wolff type inequality to obtain

$$\|Sg\|_\tau \leq C \|u_A^{q-1} g\|_{s_0}^{\frac{p-1}{n(p-1)+\tau(p+\alpha)}}. \quad \text{(3.6)}$$

Since (3.4) and $s_0 = \frac{n(q-p+1)}{p\beta + a}$ lead to $\frac{p-1}{\tau} - 1 = \frac{2-1}{s_0} - \frac{\beta p}{n}$, it follows from (3.5) and the Hölder inequality that $\|Sg\|_{s_0}^{p-1} \leq C \|u_A\|_{s_0}^{q-1} \|g\|_\sigma$. Inserting this into (3.3) yields

$$\|Tg\|_\sigma \leq C \|u\|_{s_0}^{2-p} \|u_A\|_{s_0}^{q-1} \|g\|_\sigma. \quad \text{(3.7)}$$
By virtue of $u \in L^s(R^n)$, $C\|u\|_{s_0}^{2-p}\|u_A\|_{s_0^{-1}} \leq \frac{1}{2}$ when $A$ is sufficiently large. Then $T$ is a shrinking operator. Noticing that $T$ is linear, we know that $T$ is a contraction map from $L^s(R^n)$ to itself as long as $\sigma$ satisfies (3.2).

**Step 3.** Estimating $F$ to lift the regularity.

Similar to (3.3) and (3.6), for all $\sigma$ satisfying (3.2), there holds
\[
\|F\|_{\sigma} \leq C\|u\|_{s_0}^{2-p}\|u_B\|_{\alpha(s_0+\gamma)}
\]
where $\tau$ satisfies (3.5). Noting $u \in L^s(R^n)$ and the definition of $u_B$, we see that $F \in L^\sigma(R^n)$ as long as $\sigma$ satisfies (3.2).

**Step 4.** Extend the interval from (3.2).

Let $\frac{1}{s} \in (0, \frac{n-\beta p}{n(p-1)})$. (3.8)

Thus, we can use the weighted Hardy-Littlewood-Sobolev inequality and the Wolff type inequality to deduce that
\[
\|u\|_s \leq C\|u\|^{\frac{1}{s}}_{\|u\|^{1/p}_q} \leq C\|u\|^{\frac{1}{n(p-1)+\gamma(s_0+\gamma)}}_{(p-1)+\gamma(s_0+\gamma)}.
\]
(3.9)

Noting (3.2), from (3.9) we see that $\|u\|_s < \infty$ as long as $s$ satisfies
\[
\frac{2-p}{s_0} < \frac{n(p-1)+s(\beta p + a)}{nsq} \leq \frac{2-p}{s_0} + \frac{n-\beta p}{n}.
\]
(3.10)

Next, we will prove that

Eq. (3.10) is true as long as (3.8) holds. (3.11)

First, $(p-1)(q-1) > 0$ leads to $q(2-p) < q - (p-1)$. Thus, $\frac{2-p}{q(p-1)} < \frac{1}{q}$.

Multiplying by $\frac{p\beta + a}{n}$ yields
\[
\frac{2-p}{s_0} < \frac{\beta p + a}{nq}.
\]
(3.12)

Second, (2.7) shows $n+a \geq \frac{q(p\beta + a)}{q-p+1}$. Hence, $n-p\beta \geq \frac{(p-1)(p\beta + a)}{q(p-1)}$. Multiplying by $(q-1)$ yields $(n-p\beta)(q-1) \geq (p\beta + a)(1 - \frac{q(2-p)}{q(p-1)})$ or
\[
q(2-p)(p\beta + a) \leq q(n-p\beta) - (p\beta + a) \geq n - p\beta.
\]

Multiplying by $\frac{1}{n(p-1)}$, we get
\[
\frac{2-p}{s_0} + \frac{n-p\beta}{n} \leq \frac{p\beta + a}{nq} \leq \frac{q}{n(p-1)} \geq \frac{n-p\beta}{n(p-1)}.
\]
(3.13)
By using (3.12) and (3.13), we can see (3.11).

**Step 5.** We claim that $n - \beta p$ is optimal. In fact, for sufficiently large $|x|$, from (1.7) we deduce that

$$u(x) \geq c \int_{2|x|}^{4|x|} \left( \int_{B_t(0)} |y|^{n-\beta p} dy \right)^{\frac{1}{p-1}} \frac{dt}{t} \geq c |x|^{\frac{\beta p - n}{p-1}}. \tag{3.14}$$

If $\frac{1}{s} \geq \frac{n - \beta p}{n(p-1)}$, then for some large constant $d > 0$,

$$\|u\|^s_{L^s(R^n \setminus B_d(0))} \geq c \int_d^{\infty} r^{n-s \frac{\beta p - n}{p-1}} dr = \infty.$$

Theorem 3.1 is proved. \(\square\)

**Theorem 3.2.** Assume $u \in L^s(R^n)$ solve (1.7) with (1.9). Then $u$ is bounded in $R^n$.

**Proof.** In view of (1.7),

$$u(x) \leq C \left( \int_0^1 \left( \int_{B_t(x)} |y|^{n-\beta p} dy \right)^{\frac{1}{p-1}} \frac{dt}{t} \right. \left. + \int_1^{\infty} \left( \int_{B_t(x)} |y|^{n-\beta p} dy \right)^{\frac{1}{p-1}} \frac{dt}{t} \right) \tag{3.15}$$

:= C(H_1 + H_2).

By Hölder’s inequality, for any $l > 1$ satisfying $n + \frac{al}{l-1} > 0$, we have

$$\int_{B_t(x)} |y|^{n-\beta p} dy \leq C\|u^q\|_{l} \left( \int_{B_t(x)} |y|^{\frac{al}{l-1}} dy \right)^{1/l}. \tag{3.16}$$

When $t \geq |x|/2$,

$$\int_{B_t(x)} |y|^{\frac{al}{l-1}} dy \leq \int_{B_{|x|/2}(0)} |y|^{\frac{al}{l-1}} dy \leq C t^{n + \frac{al}{l-1}}.$$

When $t < |x|/2$, $|y| > |y-x|$ for all $y \in B_t(x)$. Hence

$$\int_{B_t(x)} |y|^{\frac{al}{l-1}} dy \leq \int_{B_t(x)} |y-x|^{\frac{al}{l-1}} dy \leq C t^{n + \frac{al}{l-1}}.$$

Substituting these estimates into (3.15), we get

$$\int_{B_t(x)} |y|^{n-\beta p} dy \leq C\|u^q\|_{l} t^{(1-1/l)n + a}. \tag{3.17}$$
Take $l$ sufficiently large such that $ql > \frac{u(p-1)}{n-p\beta}$ and $p\beta + a - n/l > 0$. According to Theorem 3.1, $\|u\|_t < \infty$. Therefore,

$$H_1 \leq C \int_{0}^{1} \left( \frac{t^{(1-1/t)n+a}}{t^{n-\beta p}} \right)^{\frac{1}{p-1}} dt \leq C \int_{0}^{1} t^{\frac{\beta p + a - n/l}{p-1}} dt \leq C.$$

If $z \in B_{\delta}(x)$, then $B_{t}(x) \subset B_{1+\delta}(z)$. For $\delta \in (0, 1)$ and $z \in B_{\delta}(x)$,

$$H_2 = \int_{1}^{\infty} \frac{\int_{B_{t}(x)} |y|^n u^q(y) dy}{t^{n-\beta p}} \frac{1}{t} dt \leq \int_{1}^{\infty} \frac{\int_{B_{1+\delta}(z)} |y|^n u^q(y) dy}{(t + \delta)^{n-\beta p}} \frac{1}{t} dt \leq (1 + \delta)^{\frac{n-\beta p}{p-1} + 1} \int_{1+\delta}^{\infty} \frac{\int_{B_{t}(z)} |y|^n u^q(y) dy}{t^{n-\beta p}} \frac{1}{t} dt \leq Cu(z).$$

Combining the estimates of $H_1$ and $H_2$, we have $u(x) \leq C + Cu(z)$ for $z \in B_{\delta}(x)$, where $\delta \in (0, 1)$. Integrating on $B_{\delta}(x)$, we get

$$|B_{\delta}(x)|u(x) \leq C + C \int_{B_{\delta}(x)} u(z) dz \leq C + C\|u\|_{\infty} |B_{\delta}(x)|^{1-\frac{1}{n}} \leq C.$$

This shows $u$ is bounded in $R^n$. Theorem 3.2 is proved.

### 3.2 Fast decay rate

**Theorem 3.3.** Assume $u \in L^n(R^n)$ solves (1.7) with (1.9). Then

$$\lim_{|x| \to \infty} u(x) = 0. \quad (3.17)$$

**Proof.** Take $x_0 \in R^n$. By Theorem 3.2, $\|u\|_{\infty} < \infty$. Thus, $\forall \varepsilon > 0$, there exists $\delta \in (0, 1)$ such that

$$\int_{0}^{\delta} \left[ \int_{B_{t}(x_0)} |z|^n u^q(z) dz \right]^{\frac{1}{p-1}} dt \leq C\|u\|_{\infty} \int_{0}^{\delta} t^{\frac{\beta p + a - n/l}{p-1}} dt < \varepsilon.$$

On the other hand, similar to the derivation of (3.10), as $|x - x_0| < \delta$,

$$\int_{\delta}^{\infty} \left[ \int_{B_{t}(x_0)} |z|^n u^q(z) dz \right]^{\frac{1}{p-1}} dt \leq Cu(x).$$

Combining these estimates, we get

$$u(x_0) < \varepsilon + Cu(x), \quad \text{for} \quad |x - x_0| < \delta.$$
Since $u \in L^{s_0}(\mathbb{R}^n)$, there holds $\lim_{|x_0| \to \infty} \int_{B_1(x_0)} u^{s_0}(x) dx = 0$. Thus, we have

$$u^{s_0}(x_0) = |B_5(x_0)|^{-1} \int_{B_5(x_0)} u^{s_0}(x_0) dx \leq C \varepsilon^{s_0} + C |B_5(x_0)|^{-1} \int_{B_5(x_0)} u^{s_0}(x) dx \to 0$$

(3.18)

when $|x_0| \to \infty$ and $\varepsilon \to 0$. Thus, (3.17) is proved.

**Theorem 3.4.** Assume $u \in L^{s_0}(\mathbb{R}^n)$ solves (1.7) with (1.9). Then we can find $c > 0$ such that

$$u(x) \geq c |x|^{-\beta_p - n/p - 1}$$

when $|x| \to \infty$.

**Proof.** Clearly, $\int_{B_2(0) \setminus B_1(0)} u^p(y) v^q(y) dy \geq c > 0$. It follows that

$$u(x) \geq c \int_{|x| + 2}^{\infty} \frac{\int_{B_2(0) \setminus B_1(0)} |y|^a u^q(y) dy}{t^{n-a}} dt \geq c |x|^{-\beta_p - n/p - 1}.$$

Theorem 3.4 is proved.

**Theorem 3.5.** Assume $u \in L^{s_0}(\mathbb{R}^n)$ solves (1.7) with (1.9). Then we can find $C > 0$ such that

$$u(x) \leq C |x|^{-\beta_p - n/p - 1}$$

when $|x| \to \infty$.

**Proof.** Take a cutting-off function $\psi(x) \in C_0^\infty(B_2 \setminus B_1)$ satisfying

$$0 \leq \psi(x) \leq 1, \quad \text{for} \quad 1 \leq |x| \leq 2;$$

$$\psi(x) = 1, \quad \text{for} \quad \frac{5}{4} \leq |x| \leq \frac{7}{4};$$

For any $\rho > 0$, set $\psi_\rho(x) = \psi(\frac{x}{\rho})$. Define

$$h(x) = u(x)|x|^{(n+a)/q} \psi_\rho(x).$$

Then, one of the following two cases holds:

1. There exists a positive constant $C$ (independent of $\rho$) such that

$$h(x) \leq C, \quad \forall x; \quad (3.19)$$

2. There exists an increasing sequence $\{\rho_j\}_{j=1}^\infty$ satisfying $\lim_{j \to \infty} \rho_j = \infty$, such that as $x_{\rho_j} \in B_{2\rho_j} \setminus B_{\rho_j}$,

$$\lim_{j \to \infty} h(x_{\rho_j}) = \infty. \quad (3.20)$$

**Step 1.** If (3.19) is true, then for large $|x|$,

$$u(x) \leq C |x|^{-(n+a)/q}. \quad (3.21)$$
When $t \in (0, |x|/2)$, $y \in B_t(x)$ implies $|x|/2 \leq |y| \leq 3|x|/2$, which leads to $u^q(y) \leq C|x|^{-(n+a)}$. In addition, $|y| \geq |x|/2 \geq |y-x|$ leads to

$$\int_{B_t(x)} |y|^a dy \leq \int_{B_t(x)} |y-x|^a dy \leq Ct^{n+a}.$$ 

Thus, we have

$$\int_0^{|x|/2} \left( \int_{B_t(x)} |y|^a u^q(y) dy \right)^{\frac{1}{p+a}} dt \leq \frac{C}{|x|^{\frac{n+a}{p+a}}} \int_0^{|x|/2} \frac{dt}{t^{\frac{n+a}{p+a}}} \leq \frac{C}{|x|^{\frac{n+a}{p+a}}}. \tag{3.22}$$

On the other hand, Theorem 3.2 and $n+a > 0$ imply

$$\int_{B_t(0)} |y|^a u^q(y) dy \leq C\|u\|_q^q \int_{B_t(0)} |y|^a dy < \infty.$$ 

Noting (2.7), by the Hölder inequality and Theorem 3.1, we get

$$\int_{R^n \setminus B_t(0)} |y|^a u^q(y) dy \leq C\|u^q\|_{\kappa'} \left( \int_{R^n \setminus B_t(0)} |y|^{ak} dy \right)^{1/k} < \infty,$$

where $\frac{1}{k} = \frac{n-a}{n} \text{ and } \frac{1}{k'} = 1 - \frac{1}{k}$ with $\epsilon > 0$ sufficiently small. Combining two estimates above yields

$$\int_{R^n} |y|^a u^q(y) dy < \infty. \tag{3.23}$$

Then

$$\int_{|x|/2}^\infty \left( \int_{B_t(x)} |y|^a u^q(y) dy \right)^{\frac{1}{p+a}} dt \leq C \int_{|x|/2}^\infty \frac{dt}{t^{\frac{n+a}{p+a}}} \leq C|x|^{-\frac{n-a}{p+a}}.$$ 

Combining this result with (3.22), we obtain

$$u(x) = R(x)W_{\beta,p}(u^q)(x) \leq C|x|^{-\frac{n-\beta p}{p-1}}.$$ 

Theorem 3.3 is proved in the case of (1).

Step 2. We prove case (2) does not happen.

Let $x_\rho$ be the maximum point of $h(x)$ in $B_{2\rho} \setminus B_\rho$. It follows from (3.20) that

$$u(x_\rho) = \frac{h(x_\rho)}{\psi_{p_j}(x_\rho)} \geq \frac{c}{\rho_j^{n/q}} \tag{3.24}$$

For convenience, we denote $\rho_j$ by $\rho$.

We also obtain that $\psi_{p_j}(x_\rho) > \delta$ for some $\delta > 0$ (independent of $\rho$). The details of the proof can be seen in [24]. Therefore, by the smoothness of $\psi$, we can find a suitably small positive constant $\sigma \in (0,1/2)$, such that $\psi_{p_j}(y) > \delta/2$ for $|y - x_\rho| < \sigma|x_\rho|$. Hence, by $h(y) \leq h(x_\rho)$, we get

$$u(y) \leq C \frac{u(x_\rho)}{\psi_{p_j}(y)} \leq C(\delta)u(x_\rho), \text{ as } |y - x_\rho| < \sigma|x_\rho|.$$ 

(3.25)
Clearly,

\[
\begin{align*}
  u(x_\rho) & \leq C\left[ \int_0^{\sigma|x_\rho|} \left( \frac{\int_{B_t(x_\rho)} |y|^q u^q(y) dy}{t^{n-\beta p}} \right)^{\frac{1}{p-1}} dt \right] \\
  & + \int_{\sigma|x_\rho|}^{\infty} \left( \frac{\int_{B_t(x_\rho)} |y|^q u^q(y) dy}{t^{n-\beta p}} \right)^{\frac{1}{p-1}} dt \\
  & := C(J_1 + J_2).
\end{align*}
\] (3.26)

From (3.23), it follows

\[
J_2 \leq C \int_\sigma^{\infty} t^{-\frac{n-\beta p}{p-1}} dt \leq C|x_\rho|^{-\frac{n-\beta p}{p-1}}. \quad (3.27)
\]

Using (3.25), we obtain that, for \( r \in (0, \sigma|x_\rho|) \),

\[
\begin{align*}
  J_1 & \leq C u(x_\rho) \left[ \int_0^{r} \left( \frac{\int_{B_t(x_\rho)} |y|^q u^{q+1}(y) dy}{t^{n-\beta p}} \right)^{\frac{1}{p-1}} dt \right] \\
  & + \int_{r}^{\sigma|x_\rho|} \left( \frac{\int_{B_t(x_\rho)} |y|^q u^{q+1}(y) dy}{t^{n-\beta p}} \right)^{\frac{1}{p-1}} dt \\
  & := C u(x_\rho)(J_{11} + J_{12}).
\end{align*}
\] (3.28)

In view of \( \sigma \in (0, 1/2) \), \( |y|^a \leq C|x_\rho|^a \). According to Theorem 3.3, for any \( \varepsilon \in (0, 1) \), there holds

\[
J_{11} \leq C \left\| w \left( \frac{\mu^{q+1}}{L^{q+1}(B_{\varepsilon|x_\rho|}(x_\rho))} \right)^{\frac{1}{k'}} \right\| \int_0^{R} t^{-\frac{n-\beta p}{p-1}} dt \leq C \varepsilon|x_\rho|^\mu
\]
as long as \( \rho \) is sufficiently large. On the other hand, by Hölder’s inequality and Theorem 3.1,

\[
\int_{B_t(x_\rho)} |y|^a u^{q+1}(y) dy \leq \|u^{q+1}\|_{\kappa'} \left( \int_{B_t(x_\rho)} |y|^{ak} dy \right)^{\frac{1}{k'}} \leq Ct^{n/k + a},
\]

where \( \frac{1}{(q+1)k'} = \frac{\alpha - \beta p - \varepsilon}{n(p-1)} \) with \( \varepsilon > 0 \) sufficiently small. Hence,

\[
J_{12} \leq C \int_{r}^{\sigma|x_\rho|} t^{[\beta p - \alpha(n-\beta p) - \varepsilon]/(p-1)} dt.
\]

By virtue of (2.7), \( \beta p - \frac{a+p-1}{p}(n-p\beta - \varepsilon) + \alpha < 0 \) as long as \( \varepsilon \) is sufficiently small. Therefore, if \( \rho \) is sufficiently large and \( r \) is chosen suitably large, then

\[
J_{12} \leq \varepsilon.
\]

Substituting the estimates of \( J_{11} \) and \( J_{12} \) into (3.28), we obtain

\[
J_1 \leq C\varepsilon u(x_\rho)
\]
when \( \rho \) is sufficiently large. Inserting this result and (3.27) into (3.26), and choosing \( \varepsilon \) sufficiently small, we get

\[
u(x_\rho) \leq C |x_\rho|^{-\frac{n-\beta p}{p-1}}.
\]

By (3.25), we obtain that as \(|x - x_\rho| < \sigma |x_\rho|\),

\[
u(x) \leq C u(x_\rho) \leq C |x_\rho|^{-\frac{n-\beta p}{p-1}} \leq C |x|^{-\frac{n-\beta p}{p-1}}.
\]

Since \( \rho \) is arbitrary, the result above still holds for all \( x \) as long as \(|x| \) is large. This result contradicts (3.20) if we notice (2.7). Thus, case (2) does not happen.

**Proof of Theorem 1.1.** By the argument in section 2, we only need to prove item 1 of Theorem 1.1.

By Theorems 3.2, 3.4 and 3.5, we see that if \( u \in L^s_0(\mathbb{R}^n) \), then \( u \) is bounded and decays with the fast rate. On the contrary, if \( u \) is bounded and decays with the fast rate, we have

\[
\int_{\mathbb{R}^n} u^a dx = \int_{B_\sigma(0)} u^a dx + \int_{\mathbb{R}^n \setminus B_\sigma(0)} u^a dx \leq C + C \int_{\mathbb{R}^n} r^{n-\frac{n-\beta p}{p-1}} \frac{\log(1/r)}{r}.
\]

Noting (2.7), we see that \( u \in L^s(R^n) \).

### 4 Results on PDE

#### 4.1 Integral equation

**Theorem 4.1.** Let \( u \) be a positive weak solution of (1.1) satisfying \( \inf_{\mathbb{R}^n} u = 0 \). Then there exists a positive function \( R(x) \) such that

\[
u(x) = R(x)W_{1,p}(|y|^a u^q(y))(x), \quad \text{in} \quad \mathbb{R}^n. \tag{4.1}
\]

Moreover, there exist positive constants \( C_i, (i = 1, 2) \) such that

\[
C_1 \leq R(x) \leq C_2. \tag{4.2}
\]

**Proof.** Since \( u \) is a positive weak solution of (1.1), it is a \( \mathcal{A} \)-superharmonic function. According to Corollary 4.13 in [21], by \( \inf_{\mathbb{R}^n} u = 0 \) we can find two positive constants \( C_1 \) and \( C_2 \) such that

\[
C_1 W_{1,p}(u^p)(x) \leq u(x) \leq C_2 W_{1,p}(u^p)(x), \quad x \in \mathbb{R}^n. \tag{4.3}
\]

Set

\[
R(x) = \frac{u(x)}{W_{1,p}(u^p)(x)}.
\]

Then the solution \( u \) of (1.4) satisfies (4.1). At the same time, (4.3) leads to (4.2). \( \square \)
As a corollary of Theorems 1.1 and 4.1, we can see the following results.

**Theorem 4.2.** If \( q \in (0, \frac{(n+a)(n-1)}{n-p}) \), (1.1) has no positive solution. Let \( u \) be a positive weak solution of (1.1) with (1.9). Then

1. \( u \) is bounded and decays with the fast rate \( \frac{n-p}{p-1} \) if and only if \( u \in L^s(R^n) \) with \( \beta = 1 \).
2. If \( u \in L^\infty(R^n) \setminus L^s(R^n) \) decays with some rate, then the rate must be the slow one \( \frac{p+a+q-n}{q-p+1} \).

In particular, for the positive solutions of \( p \)-Laplace equation (1.4), we also have the same conclusions.

### 4.2 Finite energy solutions and integrable solutions

The following theorem shows that the integrable solutions satisfy the invariant property of the system and the norm under the scaling \( u_{\lambda}(x) = \lambda^{\theta} u(\lambda x) \) with \( \lambda > 0 \).

**Theorem 4.3.** Assume the scaling \( u_{\lambda}(x) \) is still a solution of (1.7) with some new double bounded function \( R(x) \), if and only if \( \theta = \frac{p^2+a}{q-p+1} \). Moreover, \( \|u_{\lambda}\|_\eta = \|u\|_\eta \) if and only if \( \eta = s_0 \).

**Remark 4.1.** Let \( a = 0 \). If \( q \) is equal to the critical exponent \( p^* - 1 \), then \( \eta = q + 1 \).

**Proof.** Clearly,

\[
R(\lambda x) \lambda^{\theta} \int_0^\infty \left( \frac{\int_{B_t(\lambda x)} |y|^a u^q(y)dy}{t^{n-p^2}} \right)^{\frac{1}{p-1}} dt = R(\lambda x) \lambda^{\theta} \int_0^\infty \left( \frac{\int_{B_t(\lambda x)} |y|^a u^q(\lambda y)\lambda^n dy}{t^{n-p^2}} \right)^{\frac{1}{p-1}} dt 
\]

Here \( R(\lambda x) \) is a new double bounded function. \( u_{\lambda}(x) \) is still a weak solution of (1.7), if and only if the exponent of \( \lambda \) is equal to zero. Clearly, \( \theta + \frac{p^2+a+q}{p-1} = 0 \) implies \( \theta = \frac{p^2+a}{q-p+1} \).

In addition,

\[
\int_{R^n} u_{\lambda}(x) dx = \lambda^{\eta} \int_{R^n} u(\lambda x) dx = \lambda^{\eta-n} \int_{R^n} u(\lambda x) dx.
\]

The energy is invariant if and only if \( \eta = \frac{\eta}{\theta} \). Inserting the value of \( \theta \) we obtain \( \eta = \frac{n(a-p+1)}{p^2+a} \).

According to Theorem 4.1, the positive weak solution \( u \) of (1.1) also solves (1.7). Hence, the result above holds for (1.1). In particular, the corresponding
result is still true for \(1.4\). In fact, we can see it by a formal calculation. If denoting \(y = \lambda x\), then we have

\[
|x|^a u^q_\lambda(x) = \lambda^{q_\theta-a} |y|^a u(y)
\]

\[
= -\lambda^{q_\theta-a} div_y[|\nabla_y u(y)|^{p-2}\nabla_y u(y)]
\]

\[
= -\lambda^{q_\theta-a-(\theta+1)(p-1)-1} div_x[|\nabla_x u(x)|^{p-2}\nabla_x u(x)].
\]

Therefore, \(u_\lambda\) solves \(1.3\) if and only if \(q\theta-a-(\theta+1)(p-1)-1 = 0\), which implies \(\theta = \frac{p+a}{q-p+1}\). On the other hand, \(\|u_\lambda\|_\eta = \|u\|_\eta\) if and only if \(\eta = \frac{a}{\theta} = \frac{a(q-p+1)}{p-a}\).

By the following theorem, we can introduce another solution—the finite energy solution.

**Theorem 4.4.** Let \(u \in L^p(R^n)\) be a weak solution of \(1.1\). Then \(\nabla u \in L^p(R^n)\) if and only if \(|x|^a u^{q+1}(x) \in L^1(R^n)\). Here \(p^* = \frac{np}{n-p}\).

**Proof.** Choose a smooth function \(\zeta(x)\) satisfying

\[
\begin{cases}
\zeta(x) = 1, & \text{for } |x| \leq 1; \\
\zeta(x) \in [0,1], & \text{for } |x| \in [1,2]; \\
\zeta(x) = 0, & \text{for } |x| \geq 2.
\end{cases}
\]

Take the test function in \(1.3\) as

\[
\zeta_R(x) = \zeta(\frac{x}{R}).
\]

Thus for \(D := B_{3R}\), there holds

\[
\int_D A(x, \nabla u) \nabla u \zeta_R^p dx + p \int_D u^{p-1} \zeta_R A(x, \nabla u) \nabla \zeta_R dx = \int_D |x|^a u^{q+1} \zeta_R^p dx. \tag{4.5}
\]

**Necessity.** If \(\nabla u \in L^p(R^n)\), we claim \(|x|^a u^{q+1} \in L^1(R^n)\). In fact, by using \(1.2\) and the Young inequality, we get

\[
| \int_D u^{p-1} \zeta_R A(x, \nabla u) \nabla \zeta_R dx |
\]

\[
\leq C \left( \int_D |\nabla u|^p \zeta_R^p dx \right)^{1-1/p} \left( \int_D u^{p^*} dx \right)^{1/p^*} \left( \int_D |\nabla \zeta_R|^n dx \right)^{1/n} \tag{4.6}
\]

\[
\leq \delta \int_D |\nabla u|^p \zeta_R^p dx + C(\int_D u^{p^*} dx)^{p/p^*} \left( \int_D |\nabla \zeta_R|^n dx \right)^{p/n}
\]

for any \(\delta \in (0,1/3)\). Here \(C > 0\) is independent of \(R\). Inserting this into \(4.5\) and using \(1.2\), we obtain

\[
\int_D |x|^a u^{q+1} \zeta_R^p dx \leq C \int_D |\nabla u|^p \zeta_R^p dx + C(\int_D u^{p^*} dx)^{p/p^*} \left( \int_D |\nabla \zeta_R|^n dx \right)^{p/n}.
\]
Letting $R \to \infty$ and noting
\[
\lim_{R \to \infty} \int_{D} |\nabla \zeta R|^n dx < \infty,
\] (4.7)
we get
\[
\int_{R^n} |x|^a u^{q + 1} dx < \infty.
\]

**Sufficiency.** If $u \in L^{p^*}(R^n)$ and $|x|^a u^{q + 1} \in L^1(R^n)$, we claim $\nabla u \in L^p(R^n)$. Inserting (4.6) into (4.5), and taking $\delta$ sufficiently small, we deduce from (1.2) that
\[
\int_{D} |\nabla u|^p \zeta R^p dx \leq C \int_{D} |x|^a u^{q + 1} \zeta R^p dx + C(\int_{D} u^{p^*} dx)^{p/p'} (\int_{D} |\nabla \zeta R|^n dx)^{p/n}.
\]
Letting $R \to \infty$ and using (4.7) we also obtain
\[
\nabla u \in L^p(R^n).
\]

The proof of Theorem 4.4 is complete. \(\square\)

According to Theorem 4.4, we call a positive weak solution $u$ is the **finite energy solution** of (1.1), if $u \in L^{p^*}(R^n)$ and $|x|^a u^{q + 1} \in L^1(R^n)$. By the results in section 4, we also call a positive weak solution $u$ is the **integrable solution** of (1.1), if $u \in L^{s_0}(R^n)$. When $q$ is equal to the critical exponent $\frac{p(n+a)}{n} - 1$, $p^* = s_0$ and hence the finite energy solution is an integrable solution. On the contrary, we have the following result (Theorem 4.5). This result, together with Theorem 4.4 implies that
\[
u \in L^{s_0}(R^n) \Rightarrow u \in D^{1,p}(R^n)
\] (4.8)
as long as $u$ is a positive weak solution.

**Theorem 4.5.** The integrable solution is also a finite energy solution.

**Proof.** If $u$ is an integrable solution, Theorem 4.1 shows that $u$ solves (1.7). According to Theorems 3.1 and 3.2, we have $u \in L^s(R^n)$ for all $\frac{1}{s} \in (0, \frac{n-p}{n(p-1)})$. Therefore, $u \in L^{p^*}(R^n)$. In addition, by the H"{o}lder inequality, we obtain
\[
\int_{B_1(0)} |x|^a u^{q + 1}(x) dx \leq \|u\|^{q + 1}_\infty \int_{B_1(0)} |x|^a dx < \infty,
\]
and
\[
\int_{R^n \setminus B_1(0)} |x|^a u^{q + 1}(x) dx \leq (\int_{R^n \setminus B_1(0)} |x|^{a_k} dx)^{1/k} \|u^{q + 1}\|_{k'} < \infty
\]
by taking $\frac{1}{s} = \frac{n-p}{n}$ and $\frac{1}{s'} = 1 - \frac{1}{s}$ with $\epsilon > 0$ sufficiently small. Thus, $|x|^a u^{q + 1} \in L^1(R^n)$. Namely, $u$ is a finite energy solution. \(\square\)
Combining Theorems 4.2–4.5, we complete the proof of Theorem 1.2. Next, we prove Corollary 1.3. The following result is needed.

**Theorem 4.6.** If a classical solution \( u \) of (1.4) belongs to \( D^{1,p}(\mathbb{R}^n) \), then \( q \) is the critical exponent \( p_* - 1 \), and 
\[
\| \nabla u \|_p = \| |x|^a u \|_p \]
Here \( p_* = \frac{p(n + a)}{n - p} \).

**Proof.** Write \( B = B_R(0) \). First, multiplying by \( u \) and integrating on \( B \), we have
\[
\int_B |x|^a u^{q+1} dx = \int_B |\nabla u|^p dx - \int_{\partial B} |\nabla u|^{p-2} u \partial_\nu u ds. 
\]
(4.9)

Here \( \nu \) is the unit outward normal vector to \( \partial B \). By virtue of \( u \in D^{1,p}(\mathbb{R}^n) \), we can find \( R_j \to \infty \) such that
\[
R_j \int_{\partial B_j} |\nabla u|^{p-2} \partial_\nu u ds \to 0,
\]
where \( B_j = B(0, R_j) \). Hence, when \( R_j \to \infty \),
\[
| \int_{\partial B_j} |\nabla u|^{p-2} \partial_\nu u ds | 
\leq (\int_{\partial B_j} |\nabla u|^p ds)^{1 - \frac{1}{p}} (\int_{\partial B_j} u^p ds)^{\frac{1}{p}} |\partial B_j|^{\frac{n}{p}}
\leq C(R_j \int_{\partial B_j} |\nabla u|^p ds)^{1 - \frac{1}{p}} (R_j \int_{\partial B_j} u^p ds)^{\frac{1}{p}} R_j^{\frac{n}{n-1} - \frac{1}{p} - \frac{n}{n+p}}
\to 0.
\]

Inserting this into (4.9) with \( R = R_j \to \infty \), we get
\[
\| \nabla u \|_p = \| |x|^a u^{q+1} \|_1. 
\]
(4.10)

Multiplying the equation with \( (x \cdot \nabla u) \) and integrating on \( B \), we obtain
\[
\int_B |\nabla u|^{p-2} \nabla u \nabla (x \cdot \nabla u) dx - \int_{\partial B} |\nabla u|^{p-2} \partial_\nu u (x \cdot \nabla u) ds = \int_B |x|^a u^p (x \cdot \nabla u) dx.
\]

Noting
\[
\nabla u \nabla (x \cdot \nabla u) = |\nabla u|^2 + \frac{1}{2} x \cdot \nabla (|\nabla u|^2)
\]
and \( x = |x| \nu \), we have
\[
\int_B |\nabla u|^p dx + \frac{1}{p} \int_B x \cdot \nabla (|\nabla u|^p) dx - R \int_{\partial B} |\nabla u|^{p-2} |\partial_\nu u|^2 ds
\]
\[
= \frac{1}{q+1} \int_B |x|^a x \cdot \nabla u^{q+1} dx.
\]
Integrating by parts, we get
\[
(1 - \frac{n}{p}) \int_B |\nabla u|^p dx + \frac{R}{p} \int_{\partial B} |\nabla u|^p ds - R \int_{\partial B} |\nabla u|^{p-2} |\partial_\nu u|^2 ds
\]
\[
= \frac{R^{1+a}}{q+1} \int_{\partial B} u^{q+1} ds - \frac{n + a}{q+1} \int_B |x|^a u^{q+1} dx. 
\]
(4.11)
According to Theorem 4.4, \( u \in \mathcal{D}^{1,p}(\mathbb{R}^n) \) implies \( |x|^a u^{q+1} \in L^1(\mathbb{R}^n) \) and \( \nabla u \in L^p(\mathbb{R}^n) \). Therefore, we can find \( R_j \to \infty \), such that

\[
R_j \int_{\partial B_{R_j}} (|x|^a u^{q+1} + |\nabla u|^p) \, ds \to 0.
\]

Let \( R = R_j \to \infty \) in (4.11). By means of the result above, we deduce that

\[
(1 - \frac{n}{p}) \int_{\mathbb{R}^n} |\nabla u|^p \, dx = -\frac{n + a}{q + 1} \int_{\mathbb{R}^n} |x|^a u^{q+1} \, dx.
\]

Combining with (4.10), we get \( q = p^* - 1 \). Inserting this result into (4.10), we complete the proof of Theorem 4.6. \( \Box \)

**Remark 4.2.** Theorem 4.6 shows that if \( q \) is not equal to the critical exponent \( p^* - 1 \), then there does not exist any classical solution in \( \mathcal{D}^{1,p}(\mathbb{R}^n) \). In view of (4.8), there does not exist any classical solution in \( L^{s_0}(\mathbb{R}^n) \).

**Proof of Corollary 1.3.**

**Step 1.** Item 1 \( \Leftrightarrow \) item 2:

First the positive solution \( u \) of (1.4) is a \( \mathcal{A} \)-superharmonic function. In addition, the integrability and the decay property of \( u \) in items 1 and 2 ensure \( \inf_{\mathbb{R}^n} u = 0 \). Similar to the proof of Theorem 4.1, we know that \( u \) solves (1.7) with \( \beta = 1 \). According to the argument in section 3, we can see the equivalence easily.

**Step 2.** Item 1 \( \Leftrightarrow \) item 3:

If \( u \in L^{s_0}(\mathbb{R}^n) \), by Theorem 4.5, we get \( u \in L^{p^*}(\mathbb{R}^n) \) and \( |x|^a u^{q+1} \in L^1(\mathbb{R}^n) \). Thus, \( \nabla u \in L^p(\mathbb{R}^n) \) from Theorem 4.4. Therefore, \( u \in \mathcal{D}^{1,p}(\mathbb{R}^n) \).

On the contrary, if \( u \in \mathcal{D}^{1,p}(\mathbb{R}^n) \), then Theorem 4.6 shows \( q + 1 = p^* \). Thus, \( s_0 = p^* \), and hence \( u \in \mathcal{D}^{1,p}(\mathbb{R}^n) \) implies \( u \in L^{s_0}(\mathbb{R}^n) \).

**4.3 Weak solutions in \( \mathcal{D}^{1,p}(\mathbb{R}^n) \) instead of \( W^{1,p}(\mathbb{R}^n) \)**

Now, we explain that the weak bounded solution of (1.1) can not be defined in the space \( L^p(\mathbb{R}^n) \) if \( n \leq p^2 \).

If \( u \in L^p(\mathbb{R}^n) \) is a positive bounded solution of (1.1), then \( \inf_{\mathbb{R}^n} u = 0 \), and hence \( u \) solves (1.7).

When \( u \) is not integrable, according to the argument of Theorem 2.2, there exists \( R > 0 \) such that as \( |x| > R \),

\[
u(x) \geq c|x|^{-\frac{p^* + a}{q + 1}} - \epsilon
\]

with sufficiently small \( \epsilon > 0 \). Thus,

\[
\int_{\mathbb{R}^n} u^p \, dx \geq c \int_{\mathbb{R}^n \setminus B_R(0)} \frac{dx}{|x|^{(q+1)p^*}}.
\]
In view of $n \leq p^2$, we get
\[
\frac{(n + a)(p - 1)}{n - p} \geq \frac{p(p + a)}{n} + (p - 1).
\]

Therefore, (2.7) implies $q > \frac{p(p + a)}{n} + p - 1$. Hence, we obtain easily
\[
n \geq p\left(\frac{p + a}{q - p + 1} + \epsilon\right)
\]
as long as $\epsilon$ is suitably small. Hence, $\|u\|_p = \infty$. It is impossible.

When $u$ is integrable, according to Theorems 3.4, there exists $R > 0$ such that as $|x| > R$,
\[
u(x) \geq c|x|^{-\frac{n+p}{p-1}}.
\]

Therefore, by $n \leq p^2$, we have $n \geq p\frac{n+p}{p-1}$, and hence
\[
\int_{R^n} u^p(x)dx \geq c\int_{R^n \setminus B_R(0)} \frac{dx}{|x|^{\frac{n+p}{p-1}}} = \infty.
\]

This also contradicts with $u \in L^p(R^n)$.

**Remark 4.3.** In particular, if $q$ is equal to the critical exponent $\frac{p(n+a)}{n-p} - 1$, then $p^* = s_0$. By the Sobolev inequality, $u \in W^{1,p}(R^n)$ implies that $u$ is integrable. According to the argument above, we know that $u \notin L^p(R^n)$. Thus, we can only assume that the weak solution $u \in D^{1,p}(R^n)$ instead of $W^{1,p}(R^n)$ as long as $n \leq p^2$.

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