Computational Complexity Measures of
Multipartite Quantum Entanglement
(Extented Abstract)

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Abstract. We shed new light on entanglement measures in multipartite quantum systems by taking a computational-complexity approach toward quantifying quantum entanglement with two familiar notions—approximability and distinguishability. Built upon the formal treatment of partial separability, we measure the complexity of an entangled quantum state by determining (i) how hard to approximate it from a fixed classical state and (ii) how hard to distinguish it from all partially separable states. We further consider the Kolmogorovian-style descriptive complexity of approximation and distinction of partial entanglement.

1 Computational Aspects of Quantum Entanglement

Entanglement is one of the most puzzling notions in the theory of quantum information and computation. A typical example of an entangled quantum state is the Bell state (or the EPR pair) \((|00\rangle + |11\rangle)/\sqrt{2}\), which played a major role in, e.g., superdense coding [4] and quantum teleportation schemes [1]. Entanglement can be viewed as a physical resource and therefore can be quantified. Today, bipartite pure state entanglement is well-understood with information-theoretical notions of entanglement measures (see the survey [8]).

These measures, nevertheless, do not address computational aspects of the complexity of entangled quantum states. For example, although the Bell state is maximally entangled, it is computationally constructed from the simple classical state \(|00\rangle\) by an application of the Hadamard and the Controlled-NOT operators. Thus, if the third party gives us a quantum state which is either the Bell state or any separable state, then one can easily tell with reasonable confidence whether the given state is truly the Bell state by reversing the computation since the minimal trace distance between the Bell state and separable states is at least 1/2. This simple fact makes the aforementioned information-theoretical measures unsatisfactory from a computational point of view. We thus need different types of measures to quantify multipartite quantum entanglement.

We first need to lay down a mathematical framework for multipartite quantum entanglement and develop a useful terminology to describe a nested structure of entangled quantum states. In this paper, we mainly focus on pure quantum states in the Hilbert space \(\mathbb{C}^{2^n}\) of dimension \(2^n\). Such a state is called, analogous to a classical string, a quantum string (or qustring, for short) of length \(n\). Any qustring of length \(n\) is expressed in terms of the standard basis \(\{|s\rangle\}_{s \in \{0,1\}^n}\). Given a qustring \(|\phi\rangle\), let \(\ell(|\phi\rangle)\) denote its length. By \(\Phi_n\) we denote the collection
of all qustrings of length $n$ and set $\Phi_\infty$ to be $\bigcup_{n \in \mathbb{N}^+} \Phi_n$, where $\mathbb{N}^+ = \mathbb{N} - \{0\}$. Ensembles (or series) of qustrings of (possibly) different lengths are of particular interest. We use families of quantum circuits as a mathematical model of quantum-mechanical computation. A quantum circuit has input qubits and (possibly) ancilla qubits, where all ancilla qubits are always set to $|0\rangle$ at the beginning of computation. We fix a finite universal set of quantum gates, including the identity and the NOT gate. As a special terminology, we say that a property $P(n)$ holds for almost all (or any sufficiently large) $n \in \mathbb{N}$ if the set $\{x \in \mathbb{N} \mid P(x) \text{ does not hold} \}$ is finite. All logarithms are conventionally taken to base two.

### 2 Separability Index and Separability Distance

We begin with a technical tool to identify the entanglement structure of an arbitrary quantum state residing in a multipartite quantum system. In a bipartite quantum system, any separable state can be expressed as a tensor product $|\phi\rangle \otimes |\psi\rangle$ of two quubits $|\phi\rangle$ and $|\psi\rangle$ and thus, any other state has its two quibits entangled with a physical correlation or “bonding.” In a multipartite quantum system, however, all “separable” states may not have such a simple tensor-product form. Rather, various correlations of entangled qubits may be intertwined over different groups of entangled qubits. For example, consider the qustring $|\psi_{2n}\rangle = 2^{-n/2} \sum_{x \in \{0,1\}^n} |xx\rangle$ of length $2n$. For each $i \in \{1,2,\ldots,n\}$, the $i$th qubit and the $n + i$th qubit in $|\psi_{2n}\rangle$ are entangled. The reordering of each qubit, nevertheless, unwinds its nested correlations and sorts all the qubits in the blockwise tensor product form $|\psi'_{2n}\rangle = (|00\rangle + |11\rangle)^{\otimes n}$. Although $|\psi_{2n}\rangle$ and $|\psi'_{2n}\rangle$ are different inputs for a quantum circuit, such a reordering is done at the cost of additional $O(n)$ quantum gates. Thus, the number of those blocks represents the “degree” of the separability of the given qustring. Our first step is to introduce the appropriate terminology that can describe this “nested” bonding structure of a qustring.

We introduce the structural notion, separability index, which indicates the maximal number of entangled “blocks” that build up a target qustring of a multipartite quantum system. See [14] also for multipartite separability.

**Definition 1.** 1. For any two qustrings $|\phi\rangle$ and $|\psi\rangle$ of length $n$, we say that $|\phi\rangle$ is isotopic to $|\psi\rangle$ via a permutation $\sigma$ on $\{1,2,\ldots,n\}$ if $\sigma(|\phi\rangle) = |\psi\rangle$.

2. A qustring $|\phi\rangle$ of length $n$ is called $k$-separable if $|\phi\rangle$ is isotopic to $|\phi_1\rangle \otimes |\phi_2\rangle \otimes \cdots \otimes |\phi_k\rangle$ via a certain permutation $\sigma$ on $\{1,2,\ldots,n\}$ for a certain $k$-tuple $(|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_k\rangle)$ of qustrings of length $\geq 1$. This permutation $\sigma$ is said to achieve the $k$-separability of $|\phi\rangle$ and the isotopic state $|\phi_1\rangle \otimes |\phi_2\rangle \otimes \cdots \otimes |\phi_k\rangle$ is said to have a $k$-unnested form. The series $m = (\ell(|\phi_1\rangle), \ell(|\phi_2\rangle), \ldots, \ell(|\phi_k\rangle))$ is called a $k$-sectioning of $|\phi\rangle$ by $\sigma$.

3. The separability index of $|\phi\rangle$, denoted $\text{sind}(|\phi\rangle)$, is the maximal integer $k$ with $1 \leq k \leq n$ such that $|\phi\rangle$ is $k$-separable.

Let $\sigma$ be any permutation on $\{1,2,\ldots,n\}$ and let $|\phi\rangle$ be any qustring of length $n$. The notation $\sigma(|\phi\rangle)$ denotes the qustring that results from permuting its qubits by $\sigma$; that is, $\sigma(|\phi\rangle) = \sum_{x} a_x |x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}\rangle$ if $|\phi\rangle = \sum_{x} a_x |x_1 x_2 \cdots x_n\rangle$, where $x = x_1 x_2 \cdots x_n$ runs over all binary strings of length $n$. 

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For any indices \( n, k \in \mathbb{N}^+ \) with \( k \leq n \), let \( QS_{n,k} \) denote the set of all qustrings of length \( n \) that have separability index \( k \).

For clarity, we re-define the terms “entanglement” and “separability” using the separability indices. These terms are different from the conventional ones.

**Definition 2.** For any qustring \(|\phi\rangle\) of length \( n \), \(|\phi\rangle\) is fully entangled if its separability index equals 1 and \(|\phi\rangle\) is fully separable if it has separability index \( n \). For technicality, we call \(|\phi\rangle\) partially entangled if it is of separability index \( \leq n - 1 \). Similarly, a partially separable qustring is a qustring with separability index \( \geq 2 \).

We assume the existence of a quantum source of information; namely, a certain physical process that produces a stream of quantum systems (i.e., qustrings) of (possibly) different lengths. Such a quantum source generates an ensemble (or a series) of qustrings. Of such ensembles, we are particularly interested in the ensembles of partially entangled qustrings. For convenience, we call them entanglement ensembles.

**Definition 3.** Let \( \ell \) be any strictly increasing function from \( \mathbb{N} \) to \( \mathbb{N} \). A series \( \Xi = \{\xi_n\}_{n \in \mathbb{N}} \) is called an entanglement ensemble with size factor \( \ell \), if for every index \( n \in \mathbb{N} \), \( \xi_n \) is a partially entangled qustring of length \( \ell(n) \).

How close is a fully entangled state to its nearest partially separable state? Consider the fully entangled qustring \(|\phi_n\rangle = (|0^n\rangle + |1^n\rangle)/\sqrt{2}\) for any \( n \in \mathbb{N} \).

For comparison, let \(|\psi\rangle\) be any partially separable qustring of length \( n \). By a simple calculation, the \( L_2 \)-norm distance \(||\phi_n\rangle - |\psi\rangle||\) is shown to be at least \( \sqrt{2} - \sqrt{2} \). The Bures metric \( B(|\phi_n\rangle, |\psi\rangle) = 2(1 - F(|\phi_n\rangle, |\psi\rangle)) \), where \( F \) is the fidelity, is at least \( 2 - \sqrt{2} \) since we have \( F(|\phi_n\rangle, |\psi\rangle) \leq 1/\sqrt{2} \) using the equality \( F(|\phi_n\rangle, |\psi\rangle) = tr(\sqrt{|\phi_n\rangle\langle\phi_n| |\psi\rangle\langle\psi|}) \). The trace distance\(^2\) \( |||\phi_n\rangle\langle\phi_n| - |\psi\rangle\langle\psi|||_\tau \) is bounded below by \( 1/2 \) using the inequality \( 1 - F(|\phi_n\rangle, |\psi\rangle)^2 \leq |||\phi_n\rangle\langle\phi_n| - |\psi\rangle\langle\psi|||_\tau \), and the above bound for the fidelity.

This example motivates us to introduce the following notion of “closeness” similar to \([13]\) using the trace norm. Note that the choice of a distance measure is not essential for our study.

**Definition 4.** Let \( k, n \in \mathbb{N}^+ \), \( \delta \in [0, 1] \), and let \( \xi \) be any qustring of length \( n \).

1. The \( k \)-separability distance of \( \xi \), denoted \( sdis_k(\xi) \), is the infimum of \(||\xi\rangle\langle\xi| - |\phi\rangle\langle\phi|||_\tau \) over all \( k \)-separable qustrings \(|\phi\rangle\) of length \( n \).

2. A qustring \( \xi \) is said to be \((k, \delta)\)-close to separable states if \( sdis_k(\xi) \leq \delta \). Otherwise, \( \xi \) is \((k, \delta)\)-far from separable states.

3. Let \( k \) be any function from \( \mathbb{N} \) to \( \mathbb{N}^+ \) and let \( \delta \) be any function from \( \mathbb{N} \) to \([0, 1]\). An ensemble \( \Xi = \{\xi_n\}_{n \in \mathbb{N}} \) of qustrings is \((k, \delta)\)-close (infinitely-often \((k, \delta)\)-close, resp.) to separable states if \( \xi_n \) is \((k(n), \delta(n))\)-close to separable states for almost all \( n \in \mathbb{N} \) (for infinitely many \( n \in \mathbb{N} \), resp.). We say that \( \Xi \) is \((k, \delta)\)-far (infinitely-often \((k, \delta)\)-far, resp.) from separable states if \( \xi_n \) is \((k(n), \delta(n))\)-far from separable states for almost all \( n \in \mathbb{N} \) (for infinitely many \( n \in \mathbb{N} \), resp.).

\(^2\) There are two different definitions in the literature. Following \([11]\), we define the fidelity of two density operators \( \rho \) and \( \tau \) as \( F(\rho, \tau) = Tr(\sqrt{\sqrt{\rho}\tau\sqrt{\rho}}) \).

\(^3\) The trace norm of a linear operator \( X \) is defined as \( ||X||_\tau = \frac{1}{2} Tr(\sqrt{X^\dagger X}) \) \([11]\).
Notice that $sdis_k(|\xi\rangle) = 0$ if $|\xi\rangle$ is $k$-separable. Moreover, the $k$-separability distance is invariant to permutation; namely, $sdis_k(\sigma(|\xi\rangle)) = sdis_k(|\xi\rangle)$ for any permutation $\sigma$. The previous example shows that the entanglement ensemble $\{(|0^n\rangle + |1^n\rangle)/\sqrt{2}\}_{n \in \mathbb{N}}$ are $(2, 1/2 - \epsilon)$-far from separable states for any constant $\epsilon > 0$. Our measure also has a connection to the geometric measure (see [16] for a review).

The notion of von Neumann entropy has been proven to be useful for the characterization of entanglement of bipartite pure quantum states. The von Neumann entropy measures the mixedness of a mixed quantum state. Let $|\psi\rangle$ be any qustring of length $n$. For each $i \in \{1, \ldots, n\}$, let $H_{\geq i}$ denote the Hilbert space corresponding to the last $n - i + 1$st qubits of $|\psi\rangle$. Consider the set $S = \{S(T_{H_{\geq i}}(|\psi\rangle\langle\psi|)) \mid i = 2, 3, \ldots, n\}$, where $T_{H_{\geq i}}$ is the trace-out operator\(^5\). We define the average entropy of $|\psi\rangle$ as $E(|\psi\rangle\langle\psi|) = \frac{1}{n} \sum_{i=2}^{n} S(T_{H_{\geq i}}(|\phi\rangle\langle\phi|))$.

The following lemma then holds.

**Lemma 1.** Let $n \in \mathbb{N}^+$, $|\xi\rangle \in \Phi_n$, and $k \in \{2, 3, \ldots, n\}$. If $sdis_k(|\xi\rangle) \leq 1/e$, then $\min_{|\phi\rangle} \{E(|\xi\rangle\langle\xi|) - E(|\phi\rangle\langle\phi|)\} \leq sdis_k(|\xi\rangle)(n - \log sdis_k(|\xi\rangle))$, where the minimization is taken over all $k$-separable qustrings in $\Phi_n$.

For Lemma 1, note that $|E(|\xi\rangle\langle\xi|) - E(|\phi\rangle\langle\phi|)| \leq \frac{1}{n} \sum_{i=2}^{n} |S(T_{H_{\geq i}}(|\xi\rangle\langle\xi|)) - S(T_{H_{\geq i}}(|\phi\rangle\langle\phi|))|$. By the Fannes inequality (see, e.g., [11]), the difference $|S(T_{H_{\geq i}}(|\xi\rangle\langle\xi|)) - S(T_{H_{\geq i}}(|\phi\rangle\langle\phi|))|$ is at most $||T_{H_{\geq i}}(|\xi\rangle\langle\xi|) - T_{H_{\geq i}}(|\phi\rangle\langle\phi|)||_{tr} \log 2^{i-1} + \eta(||T_{H_{\geq i}}(|\xi\rangle\langle\xi|) - T_{H_{\geq i}}(|\phi\rangle\langle\phi|)||_{tr})$, which is bounded by $sdis_k(|\xi\rangle)(n - \log sdis_k(|\xi\rangle))$, where $\eta(\gamma) = -\gamma \log \gamma$ for $\gamma > 0$.

## 3 Entanglement Distinguishability

We measure the complexity of each entangled state $|\phi\rangle$ by determining how hard it is to distinguish $|\phi\rangle$ from all $k$-separable states. Earlier, Vedral et al. [11] recognized the importance of distinguishability for quantifying entanglement. Fuchs and van de Graaf [7] took a cryptographic approach to quantum state distinguishing problems and briefly discussed computational indistinguishability of quantum states.

Cryptography has utilized the notion of “distinguishers” as, e.g., an adversary to a pseudorandom generator. Such a distinguisher is designed to distinguish between two different distributions of strings of fixed length with reasonable confidence. Since a quantum state can be viewed as an extension of a classical distribution, we can naturally adapt this cryptographic concept into a quantum context. For a quantum circuit $C$ and a density operator $\rho$, the notation $C(\rho)$, ignoring ancilla qubits, stands for the random variable describing the measured output bit of $C$ on input $\rho$. However, for a qustring $|\phi\rangle$, $C(|\phi\rangle)$ denotes the quantum state that results from $|\phi\rangle$ by an application of $C$.

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4 The von Neumann entropy $S(\rho)$ of a density operator $\rho$ is $-\text{Tr}(\rho \log \rho)$, where the logarithm is taken to base 2. See, e.g., [11].

5 For any bipartite quantum system $\mathcal{H} \otimes \mathcal{K}$, the trace-out operator (or partial trace) $T_{\mathcal{K}}$ is the mapping defined by $T_{\mathcal{K}}(\rho) = \sum_{j=1}^{n} (I \otimes |e_j\rangle\langle e_j|)\rho(I \otimes |e_j\rangle\langle e_j|)$ for any density operator $\rho$ of $\mathcal{H} \otimes \mathcal{K}$, where $\{|e_1\rangle, \ldots, |e_n\rangle\}$ is any fixed orthonormal basis of $\mathcal{K}$.
Definition 5. Let $\epsilon \in [0, 1]$ and let $\rho$ and $\tau$ be any two density operators of the same dimension. We say that a quantum circuit $C$ $\epsilon$-distinguishes between $\rho$ and $\tau$ if $|\text{Prob}_C[C(\rho) = 1] - \text{Prob}_C[C(\tau) = 1]| \geq \epsilon$. This circuit $C$ is called an $\epsilon$-distinguisher of $\rho$ and $\tau$.

Now, we introduce a special type of distinguisher, which distinguishes a given ensemble of partially entangled qustrings from $k$-separable states using only polynomially-many quantum gates. Let $|\phi\rangle$ be any $k$-separable qustring of length $n$ that is isotopic to the state $|\phi_1 \otimes |\phi_2 \otimes \cdots \otimes |\phi_k\rangle$ via a permutation $\sigma$. Let $m = (\ell(|\phi_1\rangle), \ldots, \ell(|\phi_k\rangle))$ be its $k$-sectioning. For notational convenience, we write $1^m$ for $1^{\ell(|\phi_1\rangle)}01^{\ell(|\phi_2\rangle)}0 \cdots 1^{\ell(|\phi_n\rangle)}0$ whose length is exactly $n + k$. Let $1^{\sigma}$ be $1^{\sigma(1)}01^{\sigma(2)}0 \cdots 1^{\sigma(n)}0$ of length $n^2/2 + 3n/2$. Moreover, we write $1^{\sigma,m}$ for $1^{\sigma}01^m$. Note that the length of $1^{\sigma,m}$ is $n^2/2 + 5n/2 + k + 2$.

Definition 6. Let $k$ be any function from $\mathbb{N}$ to $\{0, 1\}$ and $\epsilon$ be any function from $\mathbb{N}$ to $[0, 1]$. Let $\ell$ and $s$ be any functions from $\mathbb{N}$ to $\mathbb{N}$. Assume that $\ell$ is strictly increasing. Let $\Xi = \{|\xi_n\rangle\}_{n \in \mathbb{N}}$ be an ensemble of qustrings with size factor $\ell$.

1. A family $\{D_n\}_{n \in \mathbb{N}}$ of quantum circuits with $\ell(n)^2/2 + 7\ell(n)/2 + k(n) + 2$ input qubits and (possibly) ancilla qubits is called a non-uniform entanglement $(k, \epsilon)$-distinguisher (non-uniform infinitely-often entanglement $(k, \epsilon)$-distinguisher, resp.) of $\Xi$ if, for almost all $n$’s (for infinitely many $n \in \mathbb{N}$, resp.), $D_n(\epsilon(n))$-distinguishes between $|1^{\sigma,m}\rangle|\xi_n\rangle$ and $|1^{\sigma,m}\rangle|\phi\rangle$ for any $k$-separable qustring $|\phi\rangle$ of length $\ell(n)$ and any permutation $\sigma$ that achieves the $k$-separability of $|\phi\rangle$ with $k(n)$-sectioning $m$. In particular, if we want to emphasize a pair $(\sigma, m)$, we call $D_n$ a non-uniform (infinitely-often) entanglement $\epsilon$-distinguisher with respect to $(\sigma, m)$.

2. The ensemble $\Xi$ is called non-uniformly $(k, \epsilon, s)$-distinguishable from separable states if there is a non-uniform entanglement $(k, \epsilon)$-distinguisher of $\Xi$ that has size\(^6\) at most $s(n)$. In contrast, $\Xi$ is non-uniformly $(k, \epsilon, s)$-indistinguishable from separable states if there is no $s$-size non-uniform infinitely-often entanglement $(k, \epsilon)$-distinguisher of $\Xi$. In case where $s$ is a polynomial, we simply say that $\Xi$ is non-uniformly $(k, \epsilon)$-distinguishable from separable states and non-uniformly $(k, \epsilon)$-indistinguishable from separable states, respectively. Similarly, we can define the infinitely-often version of distinguishability and indistinguishability. For readability, we often drop the word “non-uniform” if it is clear from the context.

We can also define a “uniform” entanglement distinguisher using a $P$-uniform family of quantum circuits (or equivalently, a multi-tape quantum Turing machine $\text{2TQ}$).

Obviously, any ensemble of $k$-separable qustrings is $(k, \epsilon)$-indistinguishable from separable states for any $\epsilon \geq 0$. The following lemma is an immediate consequence of Definition 5.

Lemma 2. Let $\Xi = \{|\xi_n\rangle\}_{n \in \mathbb{N}}$ be any ensemble of qustrings with size factor $\ell$.

1. Let $k, k'$ be any functions from $\mathbb{N}$ to $\{0, 1\}$, let $\epsilon, \epsilon'$ be any functions from $\mathbb{N}$ to $[0, 1]$, and let $s, s'$ be any functions from $\mathbb{N}$ to $\mathbb{N}$. Assume that $k'$.\(^6\) The size of a quantum circuit is the total number of quantum gates in it.
\[ \varepsilon \text{ and } s' \text{ majorize } k, \varepsilon', \text{ and } s, \text{ respectively. If } \Xi \text{ is (infinitely-often) } (k, \varepsilon, s)-\text{distinguishable from separable states, then } \Xi \text{ is also (infinitely-often) } (k', \varepsilon', s')-\text{distinguishable from separable states.} \]

2. Let } \sigma = \{\sigma_n\}_{n \in \mathbb{N}} \text{ be any family of permutations } \sigma_n \text{ on } \{1, 2, \ldots, \ell(n)\} \text{ for each } n. \text{ Define } \sigma(\Xi) = \{\sigma_n(\xi_n)\}_{n \in \mathbb{N}}. \text{ If } \Xi \text{ is (infinitely-often) } (k, \varepsilon, s)-\text{distinguishable from separable states, then } \sigma(\Xi) \text{ is (infinitely-often) } (k, \varepsilon, s(n) + O(n))-\text{distinguishable from separable states.} \]

Of course, there are entangled states that no quantum circuit can distinguish from separable states. For instance, if two qustrings are close to each other, then no polynomial-size quantum circuit can tell their difference. In what follows, we show that any entangled state close to separable states is indistinguishable.

**Proposition 1.** Let } k \text{ be any function from } \mathbb{N} \text{ to } \mathbb{N}^+ \text{ and let } \ell \text{ be any function from } \mathbb{N} \text{ to } \mathbb{N}. \text{ Any entanglement ensemble } \Xi = \{\xi_i\}_{i \in \mathbb{N}} \text{ is } (k(n), sdis_k(n)(\xi_n)) + \delta-\text{indistinguishable from separable states for any constant } \delta > 0. \]

Proposition 1 is proven by the inequality } \|\text{Prob}_C[C(|1^m, m\rangle^\otimes \langle 1^m, m\rangle^\otimes |\xi_i\rangle_n)] = 1 - \text{Prob}_C[C(|1^m, m\rangle^\otimes \langle 1^m, m\rangle^\otimes |\phi\rangle_n)] \| \leq \|\xi_i \langle \xi_i| - |\phi\rangle \langle \phi|\|_\text{tr} \text{ for any } k\text{-separable state } |\phi\rangle, \text{ which follows from the fact that } \|\rho - \sigma\|_\text{tr} = \max_P \{\text{Tr}(P(\rho - \sigma))\}, \text{ where the maximization is taken over all positive semidefinite contractive matrices } P. \]

We note that, for every qustring } |\xi\rangle \in \Phi_n, \text{ there exists a positive operator-valued measure } W \text{ such that } \max_\psi \{\langle \xi|W|\xi\rangle - \langle \psi|W|\psi\rangle\} \geq sdis_k(n)(\xi)^2, \text{ where the maximization is taken over all } k\text{-separable qustrings in } \Phi_n. \text{ Such a } W \text{ is given, for example, as } W = I - \langle \xi|\xi\rangle. \text{ Lemma 4 will present its special case.} \]

How do we construct our distinguisher? A basic way is to combine all distinguishers built with respect to different pairs of permutations and sectionings. Suppose that we have } s\text{-size entanglement distinguishers with respect to permutations } \sigma \text{ and } k(n)\text{-sectionings } m \text{ targeting the same entanglement ensemble } \Xi \text{ with size factor } \ell(n). \text{ Although the number of such pairs } (\sigma, m) \text{ may be nearly } \ell(n)! \cdot \left(\frac{\ell(n)}{k(n)}\right)^{-1}, \text{ the following lemma shows that it is possible to build a } O(s)\text{-size distinguisher that works for all permutation-sectioning pairs.} \]

**Lemma 3.** Let } \Xi \text{ be any entanglement ensemble with size factor } \ell. \text{ Let } s \text{ be any strictly increasing function from } \mathbb{N} \text{ to } \mathbb{N}. \text{ If, for every } n \in \mathbb{N}, \text{ every permutation } \sigma \text{ on } \{1, \ldots, \ell(n)\}, \text{ and every } k(n)\text{-sectioning } m, \text{ there exists an } s(n)\text{-size } \varepsilon\text{-distinguisher of } \Xi \text{ with respect to } (\sigma, m), \text{ then there exists an } O(s(n)^c)\text{-size } (k, \varepsilon)\text{-distinguisher of } \Xi, \text{ where } c \text{ is an absolute positive constant.} \]

### 4 Entanglement Approximability

What types of entangled states are easily distinguishable from separable states? We first claim that any entangled state that is computationally “constructed” from the classical state } |0^m\rangle \text{ is distinguishable. The precise definition of constructibility is given as follows.} \]

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7 For any two functions } f, g \text{ from } \mathbb{N} \text{ to } \mathbb{R}, \text{ we say that } f \text{ majorizes } g \text{ if } g(n) < f(n) \text{ for every } n \in \mathbb{N}. \]

8 A square matrix } A \text{ is contractive if } \|A\| \leq 1, \text{ where } \|A\| = \sum_{|\phi\rangle \neq 0} \frac{\|A|\phi\rangle\|}{\||\phi\rangle\|}. \]
Lemma 4. Let \( H \) be any function from \( \mathbb{N} \) to \( \mathbb{N} \). An ensemble \( \Xi = \{ |\xi_n\rangle \}_{n \in \mathbb{N}} \) of qustrings with size factor \( \ell(n) \) is non-uniformly \( s \)-size constructible if there exists a non-uniform family \( \{ C_n \}_{n \in \mathbb{N}} \) of quantum circuits of size at most \( s(n) \) having \( \ell(n) \) input qubits and no ancilla qubit such that, for every \( n \), \( C_n |0^{\ell(n)}\rangle = |\xi_n\rangle \), where \( C_n |0^{\ell(n)}\rangle \) denotes the qustring obtained after the computation of \( C_n \) on input \( |0^{\ell(n)}\rangle \). This family \( \{ C_n \}_{n \in \mathbb{N}} \) is called a non-uniform \( s \)-size constructor of \( \Xi \).

Consider a partially entangled qustring \( |\xi\rangle \) of length \( n \) with \( \delta = \text{dis}_k(|\xi\rangle) > 0 \). If \( |\xi\rangle \) is computationally constructed from \( |0^n\rangle \), then we can easily determine whether a quantum state given from the third party is exactly \( |\xi\rangle \) by reversing the construction process to test whether it returns to \( |0^n\rangle \). This induces a distinguisher \( D \). This is seen as follows. For any \( k \)-separable state \( |\phi\rangle \), we have
\[
|\text{Prob}_D[D(|\xi\rangle) = 1] - \text{Prob}_D[D(|\phi\rangle) = 1]| \geq \delta^2 \text{ since } \text{Prob}_D[D(|\phi\rangle) = 1] = \mathcal{F}(|\xi\rangle, |\phi\rangle)^2,
\]
which is bounded above by \( 1 - \delta^2 \). Therefore, we obtain:

**Lemma 4.** Let \( \ell \) and \( s \) be any functions from \( \mathbb{N} \) to \( \mathbb{N} \) and \( k \) be any function from \( \mathbb{N} \) to \( \mathbb{N}^+ \). Assume that \( \ell \) is strictly increasing. For any ensemble \( \Xi = \{ |\xi_n\rangle \}_{n \in \mathbb{N}} \) of qustrings of size factor \( \ell \), if \( \Xi \) is \( s \)-size constructible, then it is \((k(n), \text{dis}_k(n)(|\xi_n\rangle^2, O(s(n)))\)-distinguishable from separable states.

Many fully entangled quantum states used in the literature are polynomial-size constructible. For instance, the entanglement ensemble \( \{(|0^n\rangle + |1^n\rangle)/\sqrt{2} \}_{n \in \mathbb{N}} \) is \( O(n) \)-size constructible and its 2-separability distance is at least \( 1/2 \). Thus, it is \((2, 1/4, O(n))\)-distinguishable from separable states.

We further relax the computability requirement for partially entangled states. Below, we introduce quantum states that can be “approximated” rather than “constructed.”

**Definition 7.** Let \( s \) be any function from \( \mathbb{N} \) to \( \mathbb{N} \). An ensemble \( \Xi = \{ |\xi_n\rangle \}_{n \in \mathbb{N}} \) of qustrings with size factor \( \ell(n) \) is said to be non-uniformly \((\epsilon, s)\)-approximable (non-uniformly infinitely-often \((\epsilon, s)\)-approximable, resp.) if there exists a non-uniform family \( \{ C_n \}_{n \in \mathbb{N}} \) of quantum circuits of size at most \( s(n) \) having \( \ell(n) \) input qubits and \( p(n) \) ancilla qubits \((p(n) \geq 0)\) such that, for almost all \( n \in \mathbb{N} \) (for infinitely many \( n \in \mathbb{N} \), resp.),
\[
\left| \text{Tr}_{\mathcal{H}_{n}}(C_n |0^{p(n)+\ell(n)}\rangle\langle 0^{p(n)+\ell(n)}|C_n^\dagger) - |\xi_n\rangle\langle \xi_n| \right|_{\text{tr}} \leq \epsilon(n),
\]
where \( \mathcal{H}_{n} \) refers to the Hilbert space corresponding to the \( p(n) \) ancilla qubits of \( C_n \). The family \( \{ C_n \}_{n \in \mathbb{N}} \) is called a non-uniform \( \epsilon \)-approximator (non-uniform infinitely-often \( \epsilon \)-approximator, resp.) of \( \Xi \). In particular, if \( \Xi \) is non-uniformly (infinitely-often) \((\epsilon, s)\)-approximable for a certain polynomial \( s \), then we simply say that \( \Xi \) is non-uniformly (infinitely-often) \( \epsilon \)-approximable.

The “uniform” version of approximability can be defined using a \( \mathbf{P} \)-uniform family of quantum circuits or a multi-tape quantum Turing machine. As seen before, we drop the phrase “non-uniform” in the above definition for simplicity unless otherwise stated. Clearly, any \((\epsilon, s)\)-constructible quantum state is \((\epsilon, s)\)-approximable.
The following lemma shows that any ensemble of qustrings has an exponential-size approximator; however, there exists an ensemble that is not approximated by any polynomial-size approximators.

**Lemma 5.** 1. Let \( \epsilon \) be any function from \( \mathbb{N} \) to \( (0,1] \). Any ensemble of qustrings with size factor \( n \) has a non-uniform \((\epsilon, s)\)-approximator, where \( s(n) = n^{2^{2n}} \log^2 \frac{n^2 2^{2n}}{\epsilon(n)} \).

2. For each constant \( \epsilon > 0 \), there exists an entanglement ensemble that is not \((\epsilon, n^{O(1)})\)-approximable.

Lemma 5.1 follows from the Solovay-Kitaev theorem (see [11]). Lemma 5.2 uses the result in [9] that there exists a quantum state that is not approximated by any polynomial-size quantum circuits together with the fact that there is always an entangled state close to each separable state.

A role of approximators is to build distinguishers. We can show that approximability implies distinguishability if the target entanglement ensemble is far from separable states.

**Proposition 2.** Let \( k \) be any function from \( \mathbb{N} \) to \( \mathbb{N} - \{0, 1\} \) and \( \epsilon, \delta \) be any functions from \( \mathbb{N} \) to \( [0,1] \) such that \( \delta(n) > \epsilon(n) + \sqrt{\epsilon(n)} \) for all \( n \). For any \((\text{infinitely-often}) \) \((\epsilon, s)\)-approximable entanglement ensemble, if it is \((k, \delta)\)-far from separable states, then it is \((\text{infinitely-often}) \) \((k, \epsilon', O(s(n)))\)-distinguishable from separable states, where \( \epsilon'(n) = \frac{(\delta(n) - \epsilon(n))^2 - \epsilon(n)}{2} \).

The proof of Proposition 2 is based on the fact that any \((\epsilon, s)\)-approximable entanglement ensemble \( \Xi = \{|\xi_n\rangle\}_{n \in \mathbb{N}} \) can be distinguished from separable states by use of the Controlled-SWAP operator (see [11]). Let \( D_n \) be the circuit that runs an \((\epsilon(n), s(n))\)-approximator \( C_n \) and then carries out the C-SWAP procedure (first apply the Hadamard \( H \) to the controlled bit \( |0\rangle \), then Controlled-SWAT, and finally \( H \)) and outputs the complement of the controlled bit. Let \( |\psi\rangle \) be any qustring of length \( \ell(n) \). It follows that \( \text{Prob}_{D_n}[D_n(|\psi\rangle) = 1] = 1/2 + Tr(\rho(|\psi\rangle\langle\psi|))/2 \), where \( \rho = Tr_{H_N}(C_n(|0\rangle\langle0|_N^m)) \) for some appropriate \( m \). On one hand, we have \( \text{Prob}_{D_n}[D_n(|\xi_n\rangle) = 1] = 1 - \epsilon(n)/2 \). On the other hand, if \( |\psi\rangle \) is \( k(n)\)-separable and \((k(n), \delta(n))\)-far from separable states, then \( \text{Prob}_{D_n}[D_n(|\xi_n\rangle) = 1] < 1 - \delta(n) - \epsilon(n))^2/2 \). Therefore, \( |\text{Prob}_{D_n}[D_n(|\psi\rangle) = 1] - \text{Prob}_{D_n}[D_n(|\xi_n\rangle) = 1]| \) is greater than \( \epsilon'(n) \). Note that Proposition 2 also holds for the uniform case.

Recall from Proposition 1 that any entanglement ensemble close to separable states is indistinguishable from separable states. Conversely, we claim a general result that any entangled state that is far from separable states has exponential-size distinguishers by combining Proposition 2 with Lemma 5.1 as well as the fact that \( n^{2^{2n}} \log^2 \frac{n^2 2^{2n}}{\epsilon(n)} = O(2^{2n}) \).

**Corollary 1.** Let \( k \) be any function from \( \mathbb{N} \) to \( \mathbb{N} - \{0, 1\} \) and \( \epsilon, \delta \) be any functions from \( \mathbb{N} \) to \( [0,1] \) with \( \delta(n) > \epsilon(n) + \sqrt{\epsilon(n)} \) for any \( n \). Every entanglement ensemble that is \((k, \delta)\)-far from separable states is \((k, \epsilon', O(2^{2n}))\)-distinguishable from separable states, where \( \epsilon'(n) = \frac{(\delta(n) - \epsilon(n))^2 - \epsilon(n)}{2} \).

Under the uniformity condition, we can show that distinguishability does not always imply approximability. To see this, consider the entanglement ensemble...
\[ \Xi = \{(0^n) + (-1)^{f(1^n)}|1^n\}/\sqrt{2}\}_{n \in \mathbb{N}}, \text{ where } f \text{ is any recursive function from } \{1\}^* \text{ to } \mathbb{N}, \text{ which is not computable by any } \mathsf{P}\text{-uniform family of exponential-size Boolean circuits. This } \Xi \text{ can be uniformly } (n,1/\sqrt{2},n^{O(1)})\text{-distinguishable but not uniformly } (1/\sqrt{2},n^{O(1)})\text{-approximable; otherwise, we can build from an approximator of } \Xi \text{ a family of exponential-size Boolean circuits that compute } f. \text{ Therefore, we obtain: }

**Proposition 3.** There exists an entanglement ensemble of size factor \( n \) that is uniformly \((n,1/\sqrt{2},n^{O(1)})\)-approximable from separable states and not uniformly \((1/\sqrt{2},n^{O(1)})\)-approximable.

## 5 Descriptive Complexity of Entanglement

The recent work of Vitányi [15] and Berthiaume et al. [3] brought in the notion of quantum Kolmogorov complexity to measure the descriptive (or algorithmic) complexity of quantum states. In particular, Vitányi measured the minimal size of a classical program that approximates a target quantum state. We modify Vitányi’s notion to accommodate the approximability of partially entangled qudits using quantum circuits of bounded size.

Let us fix an appropriate universal deterministic Turing machine \( M_U \) and let \( C(x|y) \) denote the Kolmogorov complexity of \( x \) conditional to \( y \) with respect to \( M_U \); that is, the minimal nonnegative integer \(|p|\) such that \( p \) is a classical program that produces \( x \) from \( y \) (i.e., \( M_U(p,y) = x \) in finite time). Abbreviate \( C(x|\lambda) = C(x) \). By identifying a quantum circuit \( D \) with its encoding\(^9\) \(|D\rangle\), we succinctly write \( C(D) \).

**Definition 9.** Let \( s \) be any function from \( \mathbb{N} \) to \( \mathbb{N} \) and let \(|\xi\rangle\) be any qudrit of length \( n \). The \( s \)-size bounded approximating complexity of \(|\xi\rangle\), denoted \( \text{QCA}^s(|\xi\rangle) \), is the infimum of \( C(D) - \log F(|\xi\rangle,\rho)^2 \) such that \( D \) is a quantum circuit of size at most \( s(n) \) with \( \ell \) inputs \( (\ell \geq n) \) and \( \rho = \text{Tr}_H(|\phi\rangle\langle\phi|) \), where \(|\phi\rangle = D|0^n\rangle \) and \( H \) is the Hilbert space associated with the last \( \ell - n \) qubits of \( D \). Its conditional version \( \text{QCA}^s(|\xi\rangle|\psi\rangle) \) is defined by \( C(D(\ell(|\xi\rangle)|\psi\rangle)) - \log F(|\xi\rangle,|\psi\rangle,\sigma)^2 \), where \(|\psi\rangle = D(|\xi\rangle|0^n\rangle \) and \( \sigma = \text{Tr}_H(|\psi\rangle\langle\psi|) \).

More generally, we can define \( \text{QCA}^s(\sigma) \) for any density operator \( \sigma \). Similar to [15], \( \text{QCA}^s(|\xi\rangle) \) is bounded above by \( 2n + c \) for any \(|\xi\rangle \in \Phi_n \) (by considering a quantum circuit \( C \) that outputs \(|x\rangle \) satisfying \( F(|\xi\rangle,|x\rangle)^2 \geq 2^{-n} \) if \( s(n) \geq n \).

We prove in the following lemma that any uniformly approximable entanglement ensemble has small approximating complexity. This lemma comes from the inequality \( \|\rho - |\xi\rangle\langle\xi|\|_1 \leq \sqrt{1 - F(\rho,|\xi\rangle)}^2 \).

**Lemma 6.** Let \( s \) be any function from \( \mathbb{N} \) to \( [0,1) \). Let \( \Xi = \{|\xi_n\rangle\}_{n \in \mathbb{N}} \) be any entanglement ensemble. If \( \Xi \) is uniformly \((\epsilon,s)\)-approximable, then there exists an absolute constant \( c \geq 0 \) such that \( \text{QCA}^s(|\xi_n\rangle|1^n\rangle) \leq c - \log(1 - \epsilon(n)) \) for all \( n \in \mathbb{N} \). In particular, if \( \epsilon(n) \) is upper-bounded by a certain constant, then \( \text{QCA}^s(|\xi_n\rangle|1^n\rangle) \leq d \) for some absolute constant \( d \geq 0 \).\(^9\)

\(^9\) The notation \(|D\rangle\) for a quantum circuit \( D \) denotes a fixed effective encoding of \( D \) such that the size of this coding is not smaller than the number of gates in \( D \).
In connection to distinguishability, Sipser defined the notion of distinguishing complexity, which measures the minimal size of a program that distinguishes a target classical string from all other strings. Translating this distinguishing complexity into a quantum context, we introduce the $k$-separability distinguishing complexity of a partially entangled state.

**Definition 10.** Let $s$ be any function from $\mathbb{N}$ to $\mathbb{N}$ and let $k \in \mathbb{N} - \{0, 1\}$. For any qustring $|\xi\rangle$ of length $n$, the $s$-size bounded $k$-separability distinguishing complexity of $|\xi\rangle$, denoted sQCD$_s^k(|\xi\rangle)$, is defined to be the infimum of $C(D|k) - \log \epsilon$ for any quantum circuit $D$ of size at most $s(n)$ with $n^2/2 + 7n/2 + k + 2$ inputs and (possibly) ancilla qubits such that $D$ $\epsilon$-distinguishes between $|1_{\sigma,m}\rangle|\xi\rangle$ and $|1_{\sigma,m}\rangle|\phi\rangle$ for any $k$-separable qustring $|\phi\rangle$ of length $n$ and any permutation $\sigma$ that achieves the $k$-separability of $|\phi\rangle$ with $k$-sectioning $m$. For convenience, we define sQCD$_s^{k,0}(|\xi\rangle)$ similarly by requiring conditions (i) and (ii) to hold only for the fixed pair $(\sigma, m)$. The conditional version sQCD$_s^{k,\ell}(|\xi\rangle|\xi\rangle)$ is defined by $C(D|k, \ell(|\xi\rangle)) - \log \epsilon$, where $D$ takes $|1_{\sigma,m}\rangle|\psi\rangle|\xi\rangle$ as input.

It is important to note that if $|\xi\rangle$ is $k$-separable then sQCD$_s^k(|\xi\rangle)$ is not defined since $\epsilon$ becomes zero. The next lemma follows immediately from Definition 10.

**Lemma 7.** Let $k \geq 2$ and let $|\xi\rangle$ be any qustring.
1. sQCD$_s^{\sigma,m}(|\xi\rangle) \leq$ sQCD$_s^k(|\xi\rangle)$ for any permutation $\sigma$ and $k$-sectioning $m$.
2. sQCD$_s^{k+1}(|\xi\rangle) \leq$ sQCD$_s^k(|\xi\rangle)$ if $k \leq \ell(|\xi\rangle) - 1$.
3. Let $k, s$ be any functions from $\mathbb{N}$ to $\mathbb{N}$ with $k(n) \geq 2$ for all $n$. Let $\epsilon$ be any function from $\mathbb{N}$ to $(0, 1]$. If an ensemble $\Xi = \{|\xi_n\rangle\}_{n \in \mathbb{N}}$ is uniformly $(k, \epsilon, s)$-distinguishable from separable states, then there exists a constant $c \geq 0$ such that sQCD$_s^{k(n)}(|\xi_n\rangle||1^n\rangle) \leq c - \log \epsilon(n)$ for all $n \in \mathbb{N}$. In particular, if $\epsilon(n)$ is bounded above by a certain constant, then sQCD$_s^{k(n)}(|\xi_n\rangle||1^n\rangle) \leq d$ for some absolute constant $d \geq 0$.

Note that if sQCD$_s^k(|\xi\rangle) = C(D|k) - \log \epsilon$ as in Definition 10 then $D$ is a $(k, \epsilon, s)$-distinguishable of $|\xi\rangle$. By (the proof of) Proposition 4, $\epsilon$ cannot be less than or equal to $sdis_k(|\xi\rangle)$. This gives a lower bound of separability distinguishing complexity.

**Proposition 4.** For any qustring $|\xi\rangle$ and any integer $k$ with $2 \leq k \leq \ell(|\xi\rangle)$, if $sdis_k(|\xi\rangle) > 0$, then sQCD$^k(|\xi\rangle) > -\log sdis_k(|\xi\rangle)$.

At length, we exhibit two upper bounds of separability distinguishing complexity, which follow from Lemma 4 and Proposition 2. Note that Proposition 5 requires a calculation slightly different from Proposition 2.

**Proposition 5.** Let $\Xi = \{|\xi_n\rangle\}_{n \in \mathbb{N}}$ be any entanglement ensemble with size factor $\ell$. Let $k, s$ be any functions from $\mathbb{N}$ to $\mathbb{N}$ with $2 \leq k(n) \leq \ell(n)$ for all $n$.
1. If $\Xi$ is $s$-size constructible, then there exist a constant $c \geq 0$ and a function $s'(n) \in O(s(n))$ such that sQCD$^k(|\xi_n\rangle) \leq QCA^s(|\xi_n\rangle) - 2 \log sdis_{k(n)}(|\xi_n\rangle) + c$ for all $n \in \mathbb{N}$.
2. If $\Xi$ is $(\epsilon, s)$-approximable and $sdis_{k(n)}(|\xi_n\rangle) > 2\sqrt{\epsilon(n)}$ for all $n$, then there exist a constant $c \geq 0$ and a function $s'(n) \in O(s(n))$ such that, for all $n$'s, sQCD$^k(|\xi_n\rangle) \leq QCA^s(|\xi_n\rangle) - \log sdis_{k(n)}(|\xi_n\rangle) - \log \left(\frac{sdis_{k(n)}(|\xi_n\rangle) - 2\sqrt{\epsilon(n)}}{1 - \epsilon(n)^2}\right) + c$. 
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