Anomalies in conductance and localization length of disordered ladders

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We discuss the conditions under which an anomaly occurs in conductance and localization length of Anderson model on a lattice. Using the ladder hamiltonian and analytical calculation of average conductance we find the set of resonance conditions which complements the π-coupling rule for anomalies. We identify those anomalies that might vanish due to the symmetry of the lattice or the distribution of the disorder. In terms of the dispersion relation it is known from strictly one-dimensional model that the lowest order (i.e., the most strong) anomalies satisfy the equation $E(k) = E(3k)$. We show that the anomalies of the generalized model studied here are also the solutions of the same equation with modified dispersion relation.

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I. INTRODUCTION

Several studies have been done on the anomalous behavior of one-dimensional models of Anderson localization. The anomaly is mainly understood as a resonance effect that occurs upon commensurability of particle wavelength with lattice spacing of the background periodic potential.

The main tool to investigate the anomalous behavior has been the weak disorder expansion of Lyapunov exponent [1]. Anomaly is signaled by divergence of certain orders in the expansion. The divergences can be overcome by taking into account the degeneracies and implementing the correct perturbation theory [2]. One then finds corrections of sub-leading order and the largest correction is obtained at the center of the band. At the band edge of pure system however all orders of expansion diverge as it requires a nonanalytic dependence on disorder strength [1].

The problem of anomaly has attracted considerable interest within the mathematical community. There exist several rigorous results regarding the existence and classification of the anomalies [3–5]. Besides being mathematically subtle, anomaly is accompanied with interesting physical situations. The coupled statistical evolution of the phase and amplitude of waves along the system implies a connection between the anomaly and violation of the random phase hypothesis [6]. Namely, one obtains the result of non-degenerate perturbation theory by assuming uniform phase distribution [7]. Another property that follows from random phase assumption is log-normal distribution of conductance [8] which does not hold at the anomaly [9]. This has been considered as violation of single parameter scaling theory [10] which is one of the fundamental frameworks of localization. Moreover the violation of reflection phase randomization at the anomaly which implies a phase relation between incident and reflected waves suggests applications in designing photonic or electronic filters [11].

Perturbative calculation of Lyapunov exponents has been extended to include multiple chains [12] and next-nearest-neighbor hopping terms [13]. Similarly the breakdown of perturbation theory determines some exceptional energies. However the condition for resonance is not always a simple commensurability in such cases. In contrast to the strictly one dimensional model, commensurability of a combination of wave vectors corresponding to different transmission channels can result in the anomaly [12–14].

Then the natural question that arises is whether or not there exist a unified way of describing the anomalies in terms of fundamental properties of the system. There have been efforts to ascribe the anomaly to the symmetries of the hamiltonian but this approach has remained limited to simple models [15]. Recently a diagrammatic explanation of anomalous behavior is provided based on scattering theory methods [16]. It turns out to be a useful method in application to more complicated cases. Accordingly, the anomaly is the result of coherent interference of scattering amplitudes from different lattice points. To have such coherency, specific relation between the wavevectors of left-going and right-going waves propagating in the chain is needed. It is shown that center of the band anomaly requires $k^+ - k^- = \pm\pi$, where $k^\pm$ are the wavevectors of the right and left going waves. This also generalizes to the case of multiple coupled chains with several energy bands. The so called π-coupling, $k^+_\nu - k^-_\nu = q\pi$ with integer $q$, between different bands also results in the similar anomalies. This result shows that even though the anomalies do not exist in the density of states of pure system, the structure of the energy bands tells us where they would appear by turning on the disorder.

The above mentioned rule for resonance is obtained for the special type of tight-binding models for which the hamiltonian of the unit cell commutes with the hopping matrix (see Eq. (1)). As a result the eigenfunctions of corresponding pure hamiltonians are plane waves, $\Phi_\nu(n) = e^{in\theta} e^{in2\pi}$, with $k$-independent amplitudes $\chi_\nu$.
the results are compared

II. MODEL AND UNPERTURBED GREEN’S FUNCTION

We consider the Anderson model with next-nearest-neighbor hopping which can be viewed as a double chain (see Fig. 1) with nearest-neighbor hopping

\[ T\Phi(n+1) + U\Phi(n) + T^\dagger\Phi(n-1) = E\Phi(n), \]

where \( t \) and \( t' \) are nearest-neighbor and next-nearest-neighbor hopping integrals, respectively. As we mentioned \([U, T] \neq 0\) and the amplitudes of eigenfunctions are \( k \)-dependent

\[ \chi_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{i\frac{\pi}{4}} \end{pmatrix}, \quad \chi_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -e^{i\frac{\pi}{4}} \end{pmatrix}. \]

It is clear from (2) that these eigenfunctions are also plane wave solutions for single chain with halved lattice constant.

To avoid the matrix notation we use the single chain picture. In the real space basis the hamiltonian corresponding to Eq. (1) has the following matrix elements

\[ H_{nm} = t(\delta_{n,m+1} + \delta_{n,m-1}) + t'(\delta_{n,m+2} + \delta_{n,m-2}), \]

where \( \delta_{ij} \) is the Kronecker delta. Eigenfunctions of this hamiltonian are \( \phi^\nu(n) = e^{ik_n}\nu \) and its eigenvalues satisfy the dispersion relation

\[ E(k) = 2t \cos k + 2t' \cos 2k, \quad -\pi \leq k \leq \pi. \]

The Green’s function of the model (1) and similar models have been already discussed in the literature [17, 18]. Appropriate expression for real space matrix elements of Green’s function \( G^\nu_0 = (E - H_0 + i\eta)^{-1} \) can be obtained using the representation in the basis of eigenfunctions of \( H_0 \),

\[ G^\nu_{mn} = \frac{i}{2\pi} \int \frac{z^{|m-n|-1} dz}{E - E(k(z)) + i\eta}, \]

where the integral is taken along the unit circle \( z = e^{ik} \). With the dispersion (4) the integrand has two pairs of poles \( e^{\pm ik_1} \) and \( e^{\pm ik_2} \). Each pair corresponds to either an open or a closed channel, depending on if they lie on the unit circle or not. One pole from each pair which lies inside the circle contributes to the integral (5).

III. TRANSMISSION AND CONDUCTANCE

We use the scattering approach based on Lippmann-Schwinger equation. As usual we consider an infinite chain with a disordered segment of size \( L \) in the middle. Motion of particle is governed by full hamiltonian \( H = H_0 + V \) in the disordered part and the pure hamiltonian \( H_0 \) in two perfect leads. We will discuss the weak disorder limit so the random part of hamiltonian, \( V \), will be assumed as perturbation. Starting with an incident wave \( \Phi \) from the left in specific channel we end up with the scattering state \( \Psi \) satisfying the Lippmann-Schwinger equation \( \Psi = \Phi + G_0^+ V \Phi \). It can be rewritten as

\[ \Psi = (1 - G_0^+ V)^{-1}\Phi. \]

Away from the scattering center this state will have the asymptotic form \( \phi^\mu(n) + \sum_{\nu} r_{\mu\nu}\phi^\nu(n) \) in the left lead and \( \sum_{\nu} t_{\mu\nu}\phi^\nu(n) \) in the right lead. The + and − signs denote the restriction of summations to right and left-going waves, respectively. At distances which are not far from the scattering region the evanescent modes should also be included in the later sum [19–21]. The conductance will be obtained using the Landauer formula upon calculation of reflection \( r_{\mu\nu} \) and transmission \( t_{\mu\nu} \) amplitudes (not to be confused with hopping integrals \( t, t' \)).

For diagonal disorder, which will be considered here, the matrix elements of the perturbation are

\[ V_{nm} = \begin{cases} w\varepsilon_n\delta_{nm}, & 1 \leq n, m \leq L, \\ 0 & \text{otherwise}, \end{cases} \]
where \( w \) is the disorder strength and \( \varepsilon_{n} \) are uncorrelated random variables with symmetric distribution (for simplicity) and finite moments. Up to fourth order of disorder strength from Eq. (6) we have

\[
\psi^i(r) = e^{i r k_i} + w \sum_{n=1}^{L} G_{0rn}\varepsilon_n e^{i n k_i} \\
+ w^2 \sum_{n,m=1}^{L} G_{0rn}G_{0mn}\varepsilon_n\varepsilon_m e^{i n k_i} \\
+ w^3 \sum_{l,m,n=1}^{L} G_{0rl}G_{0lm}G_{0mn}\varepsilon_l\varepsilon_m\varepsilon_n e^{i l k_l} \\
+ w^4 \sum_{f,l,m,n=1}^{L} G_{0rf}G_{0lf}G_{0lm}G_{0mn}\varepsilon_f\varepsilon_l\varepsilon_m\varepsilon_n e^{i f k_f},
\]

where \( r = L + 1, L + 2 \).

### A. Two open channels

For hopping ratios \( \frac{|t|}{2} < 4 \), there are four Fermi points at a given energy in the interval \(-2t' - \frac{t^2}{4} < E < 2t' - 2t \) which two of them have positive group velocity (i.e., positive slope in Fig. 1) and contribute to the Green’s function (5)

\[
G_{0mn}^+ = -i \left( \frac{e^{i k_1|m-n|}}{v_1} + \frac{e^{i k_2|m-n|}}{v_2} \right),
\]

where \( k_i \) are real roots of \( E - E(k) = 0 \) with positive group velocities \( v_i = \frac{\partial E}{\partial k} \).

Suppose now the incident wave from left is \( \phi_i \). The scattering state would then have the form \( \psi^i(n) = t_{11} e^{i k_1 n} + t_{12} e^{i k_2 n} \) in the right lead. To obtain two amplitudes \( t_{11}, t_{12} \) in this expression we need to know \( \psi^i \) at two lattice points say \( \psi^i(L + 1) \) and \( \psi^i(L + 2) \),

\[
\begin{pmatrix}
- \left( t_{11} \right) \\
- \left( t_{12} \right)
\end{pmatrix} = \begin{pmatrix}
\psi^{i(L + 1)} \\
\psi^{i(L + 2)}
\end{pmatrix}^{-1} \begin{pmatrix}
\psi^{i(L + 1)} \\
\psi^{i(L + 2)}
\end{pmatrix}.
\]

From (10), (8) we get

\[
t_{ij} = \delta_{ij} - i \left( \frac{w}{v_j} \right) \sum_{n=1}^{L} \varepsilon_n e^{i (n k_i - n k_j)} - i \left( \frac{w}{v_j} \right) \sum_{l,m=1}^{L} G_{0mn}^+ \varepsilon_l e^{i (n k_i - n k_l)} \\
- i \left( \frac{w}{v_j} \right) \sum_{f,l,m,n=1}^{L} G_{0rf}G_{0lf}G_{0lm}G_{0mn}^+ e^{i (n k_i - n k_f)}.
\]

Conductance is given now in terms of transmission amplitudes

\[
g = \frac{2e^2}{h} \sum_{i,j=1}^{2} \rho_i v_j |t_{ij}|^2,
\]

where \( \rho_i = \left| \frac{\partial E}{\partial k} \right|^{-1} \) is density of states. Four ensemble averaged transmission probabilities up to fourth order of disorder strength are obtained as follows

\[
\langle |t_{11}|^2 \rangle = 1 - \left( \frac{\varepsilon_2^2}{v_1^2} \right) (1 + v^2)Lw^2 + \frac{1}{2v_1^4} (1 + v^2)^2 (1 + 2v)(\varepsilon_4^2) Lw^4 + \frac{1}{v_1^4} (3v^2 + 2v + 1)(\varepsilon_2^2)^2 L(L - 1)w^4 \\
+ \frac{2(\varepsilon_2^2)^2}{v_1^4} \left( v^2 C(1, k_1 - 3k_2) + v^3 C(1, k_1 + 3k_2) + 3v^2 C(1, 2k_1 + 2k_2) + v C(1, 3k_1 - k_2) \\
+ 3v C(1, 3k_1 + k_2) + C(1, 4k_1) + (2v^3 + 3v^2) C(1, 2k_2) + (v^3 + v) C(1, k_1 - k_2) \\
+ (v^3 + 4v^2 + 3v) C(1, k_1 + k_2) + (2v^2 + 2v + 1) C(1, 2k_1) \right) w^4,
\]

\[
\langle |t_{12}|^2 \rangle = \langle |t_{11}|^2 \rangle |_{i \rightarrow 2, j \rightarrow 1},
\]

\[
\langle |t_{12}|^2 \rangle = \langle |t_{12}|^2 \rangle |_{i \rightarrow 2, j \rightarrow 1},
\]

where \( v = \frac{w}{v_j} \) and \( C(a, \theta) = \sum_{n=1}^{L} a^{m-n} \cos[(m-n)\theta] \). Summations in (13-16) with \( a = 1 \) can be written in the
following closed form

\[ C(1, \theta) = \sum_{n < m}^{L} \cos[(m - n)\theta] \]

\[ = \frac{1 - \cos L\theta}{2(1 - \cos \theta)} - \frac{1}{2}L. \tag{17} \]

For \( \theta \neq 2n\pi \) the second term dominates and the sum behaves linearly with size at \( L \to \infty \) but this is not the case if \( \theta = 2n\pi \) for which the dependence on \( L \) is quadratic.

At the limit \( L \to \infty \) the applicability of perturbation theory is determined by the leading terms which are proportional to \( w^2L \) and \( (w^2L)^2 \) in the second and fourth orders, respectively. It can be easily seen that the leading term in higher orders is proportional to \( (w^2L)^{2n} \) where \( n \) is order of perturbation. We need to decrease the disorder strength \( w \) as we are increasing the \( L \), such that

\[ \frac{w^2L(\varepsilon^2)}{\min\{v_1^2, v_2^2\}} \ll 1. \tag{18} \]

Consequently the terms proportional to \( w^4L \) in the fourth order will vanish in such limit. Therefor the summations in each transmission probability will give a finite contribution to the fourth order term only if \( \theta = 2n\pi \), with \( n \) being an integer. This leads a narrow peak at some special energies which is called anomaly.

According to dispersion relation (4) and depending on the ratio \( \frac{L}{\pi} \), combinations which can satisfy the condition \( \theta = 2n\pi \) are

\[ \theta = 4k_1, 2k_2, 4k_2, k_1 + k_2, 2(k_1 + k_2) \]

\[ , 3k_1 + k_2, k_1 + 3k_2, 3k_1 - k_2, \tag{19} \]

\[ \left| \frac{\varepsilon^2}{v_1} \right| \ll 1 \quad \text{and} \quad \frac{w^2L}{\min\{v_1^2, v_2^2\}} \ll 1 \]

where \( k_{1,2} = \mp \arccos \left[ \frac{1}{4}(-\frac{L}{\pi} \pm \sqrt{4\frac{L}{\pi} + (\frac{L}{\pi})^2 + 8}) \right] \). For \( |\frac{L}{\pi}| < \frac{L}{\pi} \) those terms which lead resonance inside the interval \(-2t' - \frac{L}{4\pi} < E < 2t' - 2t \) are \( \theta = 4k_1, 3k_1 + k_2, 2k_1 + 3k_2, 3k_1 - k_2 \). The rest of them get resonant at the edges \( E = -2t' - \frac{L}{4\pi} \) or \( E = 2t' - 2t \).

By inserting the transmission probabilities in conductance (Eq. (12)), we see that the terms containing \( C(1, 3k_1 - k_2) \) cancel each other. So one of the above four terms vanishes and there will be three peaks in conductance with the relative heights of \( \frac{A_{1k_1 + 3k_2}}{A_{4k_1}} = 4v, \frac{A_{3k_1 - k_2}}{A_{4k_1}} = 4v^3 \).

B. One open and one closed channel

In the rest of the energy band \( 2t' - 2t < E < 2t' + 2t \) there is one pair of fermi points when \( \frac{|\varepsilon^2|}{v_1} \ll 1 \). One of the poles that contribute to the Green’s function integral is on the real axis which corresponds to \( k = i\kappa + \pi \) (positive side) or \( k = i\kappa \) (negative side) and the other one is on the unit circle. The formers have real contribution to the Green’s function

\[ G_{\theta mn}^r = -i \frac{e^{\kappa|m-n|}u}{v} + (\pm 1)^{|m-n|}e^{-\kappa|m-n|}u, \tag{20} \]

where \( \kappa > 0 \) since the pole is inside the unit circle and \( u = -2\frac{\partial E}{\partial x}\big|_{x = \pm e^{-\kappa}} \).

In this case the wave function in the right lead is a combination of a propagating and an evanescent mode. Transmission occurs through one channel with ensemble averaged probability

\[ \langle |t_{11}|^2 \rangle = 1 - \frac{(\varepsilon^2)}{v_1}Lw^2 + \left( \frac{1}{v_1} - \frac{3}{v_1^2u^2} \right)(\varepsilon^4)Lw^4 + \left( \frac{2}{v_1}L(L - 1) - \frac{8}{v_1^2u^2}C(-e^{-\kappa}, k_1) - \frac{4}{v_1^2u^2}C(e^{-2\kappa}, 0) \right) \]

\[ - \frac{4}{v_1^2u^2}C(e^{-2\kappa}, 2k_1) + \frac{2}{v_1^2}(\frac{1}{v_1^2} - \frac{1}{u^2})C(1, 2k_1) + \frac{2}{v_1^2}C(1, 4k_1) - \frac{4}{v_1^2u}S(1, 2k_1) - \frac{4}{v_1^2u}S(-e^{-\kappa}, k_1) \]

\[ - \frac{4}{v_1^2u}S(-e^{-\kappa}, 3k_1) \] \varepsilon^2 \right)^2 = w^4, \tag{21} \]

where \( S(a, \theta) = \sum_{n < m} L a^{m-n} \sin[(m - n)\theta] \). The asymptotic behavior of summations like \( S(1, \theta), S(\pm e^{-\kappa}, \theta) \) and \( C(\pm e^{-\kappa}, \theta) \) should be determined in order to obtain the limiting value of (21) for long chain. Unlike the sum of cosines, \( S(1, \theta) \) can not give a quadratic dependence on size

\[ S(1, \theta) = \sum_{n < m} L \sin[(m - n)\theta] \]

\[ = -\frac{\sin L\theta}{2(1 - \cos \theta)} + \frac{\sin \theta}{2(1 - \cos \theta)}L. \tag{22} \]

Other sums with \( a = \pm e^{-\kappa} \) and \( \kappa > 0 \) at most will have
the following value
\[ S(\pm e^{-\kappa}, \theta), C(\pm e^{-\kappa}, \theta) \leq \sum_{n<m}^{L} (e^{-\kappa})^{m-n} \]
\[ \Rightarrow L \to \infty \quad \frac{1}{1 - e^{-\kappa} L}, \quad (23) \]

Therefor the only possibility to get \( L^2 \), again comes from \( C(1, 2n\pi) \), otherwise we will have linear or oscillatory asymptotic behavior. Moreover the only term which satisfy this condition is \( C(1, 4k_1) \) with \( 4k_1 = -2\pi \) at \( E = -2t' \) which results in the enhancement of transmission with the amount of \( \frac{1}{4}(\varepsilon^2)^2(Lw^2)^2 \).

C. Multi-channel case

The generalization of the model (3) for the next nearest neighbors is also straightforward only by replacing the green’s function with
\[ G_{0mn}^+ = -i \sum_{\nu} \frac{e^{ik_{\nu}[m-n]}}{v_{\nu}}. \quad (24) \]

Then different resonant wave vector combinations are expected to be found.

IV. LOCALIZATION LENGTH

In earlier publication [13] we have discussed the perturbative calculation of the localization length for the model (1). There we assumed perturbative solutions
\[ \frac{\Psi_n}{\Psi_n} = e^{ik_n B_n w + C_n w^2 + \cdots} \]
with the growth rate
\[ \gamma(E) = \frac{1}{\xi} = w(B) + w^2 \langle C \rangle + \cdots. \quad (25) \]

We showed that the correlation function \( \langle B_n B_m \rangle \) has poles corresponding to anomalous energies as well as band edges of pure hamiltonian. As an example in the interval \( |f| < \frac{4\sqrt{2}}{3} \), there are four poles that correspond to anomalous energies on the real \( k \) axis
\[ k = \frac{\pi}{2}, \quad \arccos \left( \sqrt{\frac{2}{3}} \cos \left[ \frac{u}{3} + n \frac{\pi}{3} \right] \right), \quad n = 0, 1, 2. \quad (26) \]
where \( \cos u = -t/[2(\tfrac{3}{4})^{\frac{3}{2}}] \) and \( 0 < u < \pi \). In relation to the resonance conditions that was obtained in the previous section these wave vectors satisfy the equations \( 4k = 2\pi, 3k + k' = 0, 3k + k' = \pm 2\pi, 3k - k' = \pm 2\pi \), respectively, where \( k' \) is the conjugate wave vector to \( k \) (see Fig. 2).

The above relations between \( k \) and \( k' \) together with \( E(k) = E(k') \) lead the equation
\[ E(k) = E(3k), \quad (27) \]
for the poles. This equation was first obtained in 1D Anderson model with only nearest neighbor hopping [2].

V. SYMMETRIC LADDER MODEL AND INHOMOGENEOUS DISORDER

The ladder model with symmetric unit cell hamiltonian and diagonal hopping matrix
\[ U = \begin{pmatrix} 0 & t \\ t' & 0 \end{pmatrix}, \quad T = \begin{pmatrix} t' & 0 \\ 0 & t \end{pmatrix}, \quad (28) \]

exhibits anomalies resulted from intra-band and inter-band \( \pi \)-coupling [16]. In a later numerical study [22] it shown that extra anomalies appear by taking different widths of disorder in two chains. These new anomalies were also attributed to the \( \pi \)-coupling of bands but at two different energies. We show that they can be described only by the coupling of waves in a single energy that we obtained in the section III.

We consider the following random potential which is studied numerically in the reference [22]
\[ \hat{V}_n = \begin{pmatrix} w\xi_n & 0 \\ 0 & w\eta_n \end{pmatrix}, \quad (29) \]
where \( n \) indicates a column with two atoms. Eigenfunctions of unperturbed hamiltonian are
\[ \Phi_1(n) = e^{ik_1 n} \chi_1, \quad \Phi_2(n) = e^{ik_2 n} \chi_2; \quad (30) \]
each of which corresponds to a transmitting channel and \( \chi_1 = \frac{1}{\sqrt{2}} (\begin{pmatrix} 1 \\ i \end{pmatrix}), \chi_2 = \frac{1}{\sqrt{2}} (\begin{pmatrix} 1 \\ -1 \end{pmatrix}) \). In this basis the green’s function is given by
\[ \hat{G}_{0mn}^+ = -i e^{ik_{\nu}[m-n]} \frac{v_1}{v_1} \chi_1 \chi_1 + i e^{ik_{\nu}[m-n]} \frac{v_2}{v_2} \chi_2 \chi_2. \quad (31) \]

Transmission coefficients can be obtained in the similar way that we did in the section III by generalizing Eq. (8) to the matrix form. We do not give the full expressions of them and only look for the missing resonant terms arising from \( \theta = 3k_1 \pm k_2, k_1 + k_2 \). Such terms come from a fourth order term like (say in \( |t_{11}|^2 \))
\[ \langle 1 | \hat{G}_{0mn}^+ \hat{V}_n \hat{G}_{0mn}^+ \hat{V}_n | 1 \rangle + c.c. \quad (32) \]
from which we can get a term proportional to
\[ \langle \varepsilon^2_n - \eta^2_n \rangle \langle \varepsilon^2_m - \eta^2_m \rangle \cos [(3k_1 + k_2)[m-n]] \quad (33) \]
after ensemble averaging it is proportional to \( \langle \varepsilon^2 \rangle - \langle \eta^2 \rangle \rangle^2 \) which will disappear if we have \( \langle \varepsilon^2 \rangle = \langle \eta^2 \rangle \).

We have verified this result numerically by computing the localization length (inverse of small Lyapunov
exponent) of symmetric ladder model using the transfer matrix method. Figure 2 shows the obtained results for localization length of ladder with hopping integrals $t = t' = 1$. Two different configurations of disorder are considered, $W_x = 0.1, W_\eta = 0.3$ and $W_x = W_\eta = 0.2236$ where $W_x, W_\eta$ are widths of uniformly distributed potential on each chain. Both configurations are chosen to have equal overall variance $(W_x^2 + W_\eta^2)/24$ and consequently equal localization length up to second order of perturbation. The difference in the localization lengths is of fourth order which is apparent in the small peaks corresponding to the couplings $3k_1 + k_2 = 2\pi$, $k_1 + 3k_2 = 2\pi$ in the case with inhomogeneous disorder.

VI. CONCLUSION

We conclude that the $\pi$-coupling of energy bands is an instance of wider forms of couplings leading the anomaly. Although these couplings are necessary but not enough conditions for the appearance of the resonances. Some couplings may not result in resonance due to the following reasons (i) symmetry of distribution of values of random potential. As an example, the $k = \frac{\pi}{3}$ anomaly in strictly one-dimensional Anderson model that requires asymmetric distribution of disorder [1] (ii) symmetry of the lattice, such as in the symmetric ladder (Eq. (28)) where the $k_1 + 3k_2$ coupling is absent but appears in the asymmetric model (Eq. (1)) (iii) spatial symmetry of distribution of random potential, such as in the symmetric ladder where $k_1 + 3k_2$ coupling shows up by introducing an inhomogeneous disorder.

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References

[1] B. Derrida and E. Gardner, J. Phys. (Paris) 45, 1283 (1984) and references therein.
[2] M. Kappus and F. Wegner, Z. Phys. B 45, 1521 (1981).
[3] A. Bovier, A. Klein, J. Stat. Phys. 51, 501 (1988).
[4] M. Campanino, A. Klein, Commun. Math. Phys. 130, 441 (1990).
[5] H. Schulz-Baldes, Operator Theory: Advances and Applications, 174, 159172 (Birkhauser, 2007) and references therein.
[6] C. J. Lambert, M. F. Thorpe, Phys. Rev. B 26, 4742 (1982).
[7] A. Douglas Stone, Douglas C. Allan, and J. D. Joannopoulos, Phys. Rev. B 27, 836 (2009).
[8] P.W. Anderson, D. J. Thouless, E. Abrahams, and D. S. Fisher, Phys. Rev. B 22, 3519 (1980).
[9] H. Schomerus and M. Titov, Phys. Rev. B 67, 100201(R) (2003).
[10] E. Abrahams, P.W. Anderson, D.C. Licciardello, and T.V. Ramakrishnan, Phys. Rev. Lett. 42, 673 (1979).
[11] M. Titov and H. Schomerus, Phys. Rev. Lett 95, 126602 (2005).
[12] H. Schulz Baldes, Geometric And Functional Analysis 14, 1089-1117 (2004).
[13] R. Sepehrinia, Phys. Rev. B 82, 045118 (2010).
[14] Hong-Yi Xie, Vladimir E. Kravtsov, Markus Muller, Phys. Rev. B 86, 014205 (2012).
[15] L. I. Deych, M.V. Erementchouk, A. A. Lisyansky, and B. L. Altshuler, Phys. Rev. Lett 91, 096601 (2003).
[16] Luca Alloatti, J. Phys.: Condens. Matter 21, 045503 (2009).
[17] A. A. Bahurmuz and P. D. Loly, J. Math. Phys. 22, 564-568 (1981).
[18] William A. Schwalm and Mizuho K. Schwalm, Phys. Rev. B 37, 9524-9542 (1988).
[19] P. F. Bagwell, Phys. Rev. B, 41, 10354 (1990).
[20] J. Heinrichs, Phys. Rev. B 68, 155403 (2003).
[21] V. Gasparian and A. Suzuki, J. Phys.: Condens. Matter 21, 045302 (2009).
[22] B. P. Nguyen and K. Kim, J. Phys.: Condens. Matter 24, 135303 (2012).