Non-alternating Hamiltonian Lie algebras in characteristic 2. I

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ABSTRACT. The classification of graded non-alternating Hamiltonian Lie algebras over perfect field of characteristic 2 is obtained. It is shown that the filtered deformations of such algebras correspond to non-alternating Hamiltonian forms with polynomial coefficients in divided powers. It is proved that graded non-alternating Hamiltonian algebras are rigid with respect to filtered deformations except for some cases when the number of variables is 2, 3, 4 or when the heights of some variables are equal to 1.

ACKNOWLEDGEMENTS. The authors thank S.M. Skryabin for sending them his papers on modular Lie algebras of Cartan type. The investigation is funded by RFBR according to research project N 18-01-00900

I. Kaplansky in the work [1] gave examples of exceptional simple Lie algebras of characteristic 2. The third series of constructed algebras, denoted as Kap 3, is similar to the construction of Block algebras [2], in which the skew-symmetric form is replaced by a non-alternating form. As it is well known, Blok algebras over a field of characteristic \( p \) are isomorphic to Hamiltonian Lie algebras over an algebra of divided powers [3], [4]. L. Lin
constructed a series of Hamiltonian Lie algebras of characteristic 2 with the simplest symmetric Poisson bracket \( \{ f, g \} = \sum \partial_i f \partial_i g \) on the algebra of divided degrees and showed that the Lie algebras Kap 3 correspond to the case when the heights of the variables are equal to 1. Over the field of zero characteristic, this bracket occurs in finite-dimensional Hamiltonian Lie superalgebras \( H(n) \). Graded non-alternating Hamiltonian Lie algebras were intensively investigated by D. Leites, S. Bouarrouj, U. Yier, M. Messaoudene, P. Grozman, A. Lebedev, I. Schepochkina in the direction of disseminating ideas and methods Lie superalgebra theory for the case of Lie algebras of even characteristic. A complex of symmetric differential forms in divided powers was constructed, which led to a more natural definition of non-alternating Hamiltonian Lie algebras within the framework of Hamiltonian formalism, an analysis of graded algebras from the point of view of Cartan prolongations was carried out, some of Volichenko algebras were considered (see [6], [7], [8]). In these papers, the class of investigated Poisson brackets on the algebra of divided powers of characteristic 2 corresponded to standard brackets with constant coefficients for Hamiltonian Lie superalgebras containing even and odd variables.

In this paper, we develop a general theory of non-alternating Hamiltonian Lie algebras over a field of characteristic 2, which makes it possible to assign these algebras to Lie algebras of Cartan type. Here, a classification of graded non-alternating Hamiltonian Lie algebras is obtained based on the constructed complete system of invariants of non-alternating symmetric bilinear forms of characteristic 2 with respect to the flag automorphism group (Theorem 1.1). Namely, two graded non-alternating Hamiltonian Lie algebras over the algebra of divided powers defined by the flag \( \mathcal{F} \) are isomorphic if and only if the invariants of the corresponding bilinear forms coincide (Theorem 5.2). In contrast to the classical Hamiltonian case, there is a large number of equivalence classes of non-alternating forms with constant coefficients for a fixed nontrivial flag, which correspond to new simple graded Lie algebras. It is shown that the filtered deformations of graded non-alternating Hamiltonian Lie algebras with certain exceptions correspond to non-alternating Hamiltonian forms with polynomial coefficients in divided powers (Theorem 2.3). Note that classical graded Hamiltonian Lie algebras can have filtered deformations which correspond to differential forms with non-polynomial coefficients [9]. Moreover, it is proved that, in contrast to the classical Hamiltonian algebras, under fairly general conditions, for example, when the heights of some variables are greater than 1, the graded
non-alternating Hamiltonian algebras are rigid with respect to filtered deformations (Theorem 3.2). A description of derivations and automorphisms of filtered (and in particular, graded) non-alternating Hamiltonian Lie algebras is given (Theorems 3.2, 5.2). All results are valid except for some cases when the heights of some variables are 1, or when the number of variables is \( n = 2, 3, \) or \( 4 \). The fact that these exceptions are not random is confirmed by the examples of simple filtered Lie algebras announced at the end of the paper. It has been shown earlier that a graded non-alternating Hamiltonian Lie algebra of four variables of height 1 is isomorphic to a classical filtered Hamiltonian Lie algebra ([10]). In Section 0 the invariant definition of a complex of symmetric differential forms in divided powers, a description of its cohomology, other necessary definitions, and preliminary results are given.

We employ the technique used in the study of modular Lie algebras of Cartan type (see [11]). The work of V.G. Kac [9], in which it was shown that all filtered deformations of graded Lie algebras of Cartan type correspond to more general forms of the same kind, exerted a significant impact on the field. In this work the results and methods of the general theory of filtered Lie algebras (Spencer cohomology, Serre’s involutivity criterion, etc.) were used for the first time in the case of characteristic \( p \). S.M. Skryabin in [12] – [16] developed another approach to the description of filtered deformations of Lie algebras of Cartan type, based on the theory of coinduced modules [17], [18]. Both approaches can be applied to the study of filtered deformations of non-alternating Hamiltonian Lie algebras. In the present paper the results and methods of S.M. Skryabin are used. Earlier, invariants of classical Hamiltonian forms with coefficients in divided power algebras (so called polynomial case) were found that determine the isomorphism class of general Hamiltonian Lie algebras, the value of the form at zero and its cohomological class [9], [19], [20]. The problem of classification of classical Hamiltonian forms was considered in [20] and was completely solved by S.M. Skryabin (see [12], [13]). We use the methods of S.M. Skryabin [13] in the study of non-alternating Hamiltonian forms and, in particular, in the construction of invariants of non-alternating symmetric forms of characteristic 2.

The ground field \( K \) is assumed to be a perfect one of characteristics 2.
Symmetric differential forms

Let $R$ be a commutative algebra over a field $K$ of characteristic $p > 0$, $E$ be a free $R$-module of rank $n$. The commutative $R$-algebra of divided powers $O(E)$ is given by generators $u^{(r)}$, $u \in E$, $r \in \mathbb{Z}$, $r \geq 0$ and relations

\begin{align*}
  u^{(0)} &= 1, \quad u^{(1)} = u, \\
  (au)^{(r)} &= a^r u^{(r)}, \quad a \in K, \\
  u^{(r)} u^{(s)} &= \binom{r + s}{r} u^{(r+s)}, \\
  (u_1 + u_2)^{(r)} &= \sum_{i=1}^{r} u_1^{(i)} u_2^{(r-i)}
\end{align*}

(see [21]).

Let $\{x_1, \ldots, x_n\}$ be the basis of $E$ over $R$. Elements $x^{(\alpha)} = x_1^{(\alpha_1)} \cdots x_n^{(\alpha_n)}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in \mathbb{Z}$, $\alpha_i \geq 0$ form the monomial basis of $O(E)$,

\[ x^{(\alpha)} \cdot x^{(\beta)} = \binom{\alpha + \beta}{\alpha} x^{(\alpha+\beta)}. \]

(0.2)

An algebra $O(E)$ with a fixed basis $\{x^{(\alpha)}\}$ is denoted by $O(n)$. The algebra $O(E)$ has the standard grading, $O(E) = \bigoplus_{i \geq 0} O_i$, $O_i = \langle x^{(\alpha)} \mid \alpha = \alpha_1 + \ldots + \alpha_n = i \rangle$, and filtration $O(E)_{(i)} = \bigoplus_{j \geq i} O_j$. The standard grading and filtration are independent on the $\{x_1\}$ basis of the module $E$. $O(E)$ is a local algebra with maximal ideal $m = O(E)_{(1)}$. The completion of $O(E)$ by the system of ideals $\{m^{(i)} = O(E)_{(i)}\}$ is denoted by $\hat{O}(E)$.

Let $V$ be a free $R$-module of rank $n$, $E = \text{Hom}_R(V, R)$. The algebra $O(E)$ is canonically isomorphic to the algebra of symmetric polynomial functions on $V$ with values in $R$ with multiplication

\[ \omega_1 \omega_r(v_1, \ldots, v_{l+r}) = \sum_{\sigma} \omega_{\sigma}(v_{\sigma(1)}, \ldots, v_{\sigma(l)}) \omega_r(v_{\sigma(l+1)}, \ldots, v_{\sigma(l+r)}), \]

(0.3)

where the sum is taken over all substitutions of $\sigma \in S_{l+r}$ such that $\sigma(1) < \ldots < \sigma(l)$, $\sigma(l+1) < \ldots < \sigma(l+r)$. Indeed, the symmetric polynomial map $\omega_r: V^r \to R$ corresponds to $\hat{\omega}_r \in S^r(V)^* = \text{Hom}_R(S^r(V), R)$,

\[ \hat{\omega}_r(v_1 \cdot \ldots \cdot v_r) = \omega_r(v_1, \ldots, v_r). \]

(0.4)
Here $S(V)$ is the symmetric bialgebra of $R$-module $V$, $S(V) = \bigoplus_{r \geq 0} S^r(V)$, with comultiplication $\Delta$, $\Delta(v) = v \otimes 1 + 1 \otimes v$, and the counit $\varepsilon$, $\varepsilon(v) = 0$, $v \in V$. Correspondence (0.3) is an isomorphism of algebras $S(V)^* = \bigoplus_{r \geq 0} S^r(V)^*$ with the multiplication $\hat{\omega}_1 \hat{\omega}_r = \hat{\omega}_1 \otimes \hat{\omega}_r \circ \Delta$ and the algebra of symmetric multilinear functions on $V$ with multiplication (0.3).

For $u \in S^1(V)^* = E = \text{Hom}_R(V,R)$ the divided powers $u^{(r)}$, $r \geq 0$ are defined as follows

$$u^{(0)} = 1, \quad u^{(r)}(v_1 \cdots v_r) = u(v_1) \cdots u(v_r),$$

for which (0.1) relations (0.1) hold. Let $\{v_i\}$ be the basis of $V$, $\{w_i\}$ be the dual basis of the module $E$. Denote by $\{w^{(\alpha)}\}$ the basis of $S(V)^*$, the dual to the basis $\{v^\alpha, \alpha = (\alpha_1, \ldots, \alpha_n), \alpha_i \geq 0\}$ of $S(V)$. Then

$$w^{(\alpha)}w^{(\beta)} = \binom{\alpha + \beta}{\alpha} w^{(\alpha+\beta)}.$$

Thus, $S(V)^*$ isomorphic to the algebra $O(E)$. In particular $u^p = 0$ for any 1-form $u \in V$.

Divided powers can be defined for any $u \in \mathfrak{m} \subset O(E)$ (or $u \in \hat{\mathfrak{m}} \subset \hat{O}(E)$) so that the (0.1) and

$$(u^{(l)})^{(r)} = \frac{(l!)^r}{(l)^r r!} u^{(lr)}, \quad (\hat{\omega}_1 \hat{\omega}_2)^{(r)} = \omega_1^r \omega_2^r,$$

are fulfilled (see [9], [14], [17]).

Using the isomorphism $\omega_r \mapsto \hat{\omega}_r$ (see (0.3)), we will identify the algebra of symmetric multilinear functions on $V$ and the algebra of divided powers $O(E) = S(V)^*$.

Let $R = K$. A derivation $D$ of algebra $O(E)$ is called special if $Du^{(r)} = u^{(r-1)}Du$, $u \in E$. The Lie algebra of all special derivations of the algebra $O(E)$ is denoted by $W(E)$.

Let $\mathcal{F}: E = E_0 \supseteq E_1 \supseteq \ldots \supseteq E_r \supseteq E_{R+1} = \{0\}$ be a flag of $E$. A basis $\{x_i\}$ of $E$ is called coordinated with $\mathcal{F}$ if $\{x_i\} \cap E_j$ is a basis of $E_j$. A subalgebra $O(\mathcal{F})$ of $O(E)$ is generated by $u^{(p)}$, $u \in E, j = 0, \ldots, r$.

Let $\{x_i\}$ be a basis of $E$ coordinated with $\mathcal{F}$. The elements $x^{(\alpha)}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $0 \leq \alpha_i < p^m$ form the basis of $O(\mathcal{F})$. Here $m_i$ is the height of variable $x_i$, $m_i = \min \{s \mid x_i \notin E_s\}$. The algebra $O(\mathcal{F})$ with such basis is denoted by $O(n,m)$, $\overline{m} = (m_1, \ldots, m_n)$. By $W(\mathcal{F})$ (resp. $W(n,m)$) is denoted
the Lie subalgebra in $W(E)$ consisting of all special derivations preserving $O(\mathcal{F})$ (resp. $O(n, \mathfrak{m})$). The partial derivations $\partial_i, \partial_i x^{(\alpha)} = x^{(\alpha - \varepsilon_i)}, \varepsilon_i = (0, \ldots, 1, \ldots, 0)$, form the basis of the free $O(\mathcal{F})$-module $W(\mathcal{F})$. The standard grading (resp. filtration) of the algebra $O(E)$ induces a standard grading (resp. filtration) of algebras $W(E)$ and $W(\mathcal{F}), W(\mathcal{F})_i = \langle x^{(\alpha)} \partial_j, |\alpha| = i+1, j = 1, \ldots, n \rangle, x^{(\alpha)} \in O(\mathcal{F}), W(\mathcal{F})_{(i)} = \langle x^{(\alpha)} \partial_j, |\alpha| \geq i+1, j = 1, \ldots, n \rangle$ (similar for $W(E)$). Let $\mathfrak{m}$ be the maximum ideal of $O(E)$, then $\mathfrak{m} \cap O(\mathcal{F})$ be the maximum ideal of $O(\mathcal{F})$, which is denoted by $\mathfrak{m}(\mathcal{F})$.

Now let $R = O(\mathcal{F}), V = W(\mathcal{F})$. Then $V^* = \text{Hom}_R(W(\mathcal{F}), R)$ is $R$-module of differential 1-forms and is denoted by $S\Omega^1(\mathcal{F})$. The $R$-algebra of the divided powers $O(V^*) = S(V)^*$ is the algebra of symmetric differential forms denoted by $S\Omega(\mathcal{F}) = \bigoplus_{i \geq 0} S\Omega^i(\mathcal{F}), S\Omega^0 = R = O(\mathcal{F}), S\Omega^1(\mathcal{F}) = \Omega^1(\mathcal{F}) = V^*$.

In what follows we assume that $K$ is a perfect field of characteristic 2.

The inner product of $D \in W(\mathcal{F})$,

$$D \omega_r(D_1, \ldots, D_{r-1}) = \omega_r(D, D_1, \ldots, D_{r-1})$$

has properties

$$D \omega(\omega_1 \omega_2) = (D \omega_1) \omega_2 + \omega_1 (D \omega_2),$$

$$D \omega^{(r)} = (D \omega) \omega^{(r-1)}.$$  

The Lie algebra $W(\mathcal{F})$ acts naturally on $S\Omega(\mathcal{F})$ by derivations, i.e.

$$D(f \omega) = D(f) \omega + f(D \omega),$$

$\omega \in s\Omega(\mathcal{F}), f \in O(\mathcal{F})$.

The following identity, known for skew-symmetric forms, is also true in $S\Omega(\mathcal{F})$

$$D_1(D_2 \omega) = [D_1, D_2] \omega + D_2 \omega(D_1 \omega). \quad (0.5)$$

Over a field of characteristic 2, we define the differential $d$: $s\Omega^r(\mathcal{F}) \rightarrow S\Omega^{r+1}(\mathcal{F})$ such that $df(D) = D(f),

$$(d\omega_r)(D_1, \ldots, D_{r+1}) = \sum_{i=1}^{r+1} D_i(\omega(D_1, \ldots, \widehat{D_i}, \ldots, D_{r+1}) +$$

$$+ \sum_{i<j} \omega([D_i, D_j], D_1, \ldots, \widehat{D_i}, \ldots, \widehat{D_j}, \ldots, D_{r+1}),$$

6
\[ d^2 = 0, \quad d\omega^{(r)} = \omega^{(r-1)}d\omega, \quad d(\omega_1\omega_2) = (d\omega_1)\omega_2 + \omega_1(d\omega_2). \]

We consider alternating forms as symmetric, such that \( \omega_r(D_1, \ldots, D_r) = 0 \) if \( D_i = D_j \) for some \( i \neq j \). Therefore, the de Rham complex \((\Omega(\mathcal{F}), d)\) is a subcomplex of the complex \((S\Omega(\mathcal{F}), d)\). The Cartan homotopy formula also holds in \( S\Omega(\mathcal{F}) \):

\[ D\omega = D_\downarrow d\omega + d(D_\downarrow \omega). \quad (0.6) \]

**Theorem 0.1** ([22]). Let \( x_i, i = 1, \ldots, n \) be the standard variables of the algebra of divided powers \( O(\mathcal{F}) = O(n, \mathfrak{m}) \) over field \( K \) of characteristics 2, \( \mathfrak{x}_i = x_i^{(2^{m_i}-1)} \), \( B(n) \) graded algebra of divided powers over \( K \) in variables \((dx_i)^{(2)}, i = 1, \ldots, n \) of degree two, \((\Omega(\mathcal{F}), d)\) de Rham complex over \( O(\mathcal{F}) \). The following statements are true:

1. The cohomology ring \( H^*(S\Omega(\mathcal{F})) \) is a tensor product of graded algebras

\[ H^*(S\Omega(\mathcal{F})) = B(n) \otimes_K H^*(\Omega(\mathcal{F})), \]

2. \( \dim H^i(S\Omega(\mathcal{F})) = \binom{n + i - 1}{i} \),

3. \( H^1(S\Omega(\mathcal{F})) = H^1(\Omega(\mathcal{F})) = \langle [\mathfrak{x}_idx_i], i = 1, \ldots, n \rangle \),

4. \( H^2(S\Omega(\mathcal{F})) = \langle [(dx_i)^{(2)}], i = 1, \ldots, n, [\mathfrak{x}_ix_jdx_idx_j], 1 \leq i < j \leq n \rangle \).

**Remark 0.1.** 1. De Rham cohomology over the algebra \( O(\mathcal{F}) \) is obtained in [24] (see also [11]).

2. All definitions of this section hold for the case of infinite dimensional algebras such as \( O(E), \hat{O}(E), O(\mathcal{F}), \hat{O}(\mathcal{F}) \) for the infinite flag \( \mathcal{F} \), with natural constraints of topological or other nature. For example, for \( \hat{O}(\mathcal{F}) \) it is natural to consider continuous in \( \mathfrak{m}_i \)-adic topology \( \mathfrak{m}_i \)-adic topology special derivations, and for algebras \( O(E), \hat{O}(E) \) in Theorem 0.1 we must take into account that \( H^*(\Omega) = H^0(\Omega) = K \) (Poincaré Lemma).

The continuous isomorphism \( \sigma: \hat{O}(E) \to \hat{O}(\mathcal{F}') \) is called admissible if \( \sigma(x^{(\alpha)}) = \sigma(x)^{(\alpha)} \). If \( \sigma(O(\mathcal{F}) = O(\mathcal{F}') \), then \( \sigma \) is called an admissible isomorphism of the corresponding algebras. In this case, \( \sigma \) induces an isomorphism of Lie algebras \( W(\mathcal{F}) \) and \( W(\mathcal{F}') \), \( \sigma(D) = \sigma \circ D \circ \sigma^{-1} \) ([23]).
admissible isomorphism \( \sigma \) continues to the isomorphism \( S\Omega(\mathcal{F}) \to S\Omega(\mathcal{F}') \) by the rule
\[
(\sigma \omega)(D_1, \ldots, D_k) = \sigma(\omega^{-1}D_1\sigma, \ldots, \omega^{-1}D_k\sigma)).
\]
Moreover, \( \sigma d\omega = d\sigma \omega, \sigma(\omega_1 \omega_2) = \sigma \omega_1 \cdot \sigma \omega_2 \) and \( \sigma(\omega^{(k)}) = (\sigma \omega)^{(k)} \).

The following Lemma will be applied in section 3.

**Lemma 0.1.** Let \( \omega = \sum_{i=1}^{n} \omega_{ii}(dx_i)^{(2)} + \sum_{i<j} \omega_{ij}dx_i dx_j \) be a symmetric 2-form with coefficients from \( \hat{O}(E) \). If \( d\omega = 0 \), then \( \omega_{ii} \in K \).

**Proof.** Write \( \omega \) as \( \omega = \omega_1 + \omega_2, \omega_1 = \sum_{i=1}^{n} \omega_{ii}(dx_i)^{(2)}, \omega_2 = \sum_{i<j} \omega_{ij}dx_i dx_j \).

Obviously \( \omega_2 \in \hat{\Omega}(E) \). Let \( I \) be the ideal of \( \hat{S}\Omega(E) \) generated by \((dx_i)^{(2)}, i = 1, \ldots, n \). Here \( \hat{S}\Omega(E) \) is the algebra of \( \hat{O}(E) \)-multilinear maps from \( \hat{W}(E) \) to \( \hat{O}(E) \). Since \( d\omega_2 \in \hat{\Omega}(E), d(f(dx_i)^{(2)}) = \sum_j \partial_j f dx_j dx_i^{(2)} \in I \) and \( I \cap \hat{\Omega}(E) = 0 \), then \( d\omega_1 = d\omega_2 = 0 \). Now, \( d\omega_1 = \sum_i d\omega_{ii}(dx_i)^{(2)} = 0 \), therefore, \( d\omega_{ii} = 0 \). \( \square \)

Symmetric 2-form \( \omega \in S\Omega^2(\mathcal{F}) \) is called *non-alternate* if there exists \( D \in W(\mathcal{F}) \) such that \( \omega(D, D) \neq 0 \). Let
\[
\omega = \sum_{i=1}^{n} \omega_{ii}(dx_i)^{(2)} + \sum_{i<j} \omega_{ij}dx_i dx_j
\]
be a non-alternating 2-form. For \( j > i \) put \( \omega_{ji} = \omega_{ij} \), \( M = (\omega_{ij}) \) is the matrix of \( \omega \). The form \( \omega \) is called *nondegenerate* if \( \det \omega = \det M \) is invertible in \( O(\mathcal{F}) \). Let \( a_{ij} = \omega_{ij}(0) = \omega_{ij}(\text{mod } \mathfrak{m}) \in O(\mathcal{F})/\mathfrak{m} = K, \omega(0) = \sum_{i=1}^{n} a_{ii}(dx_i)^{(2)} + \sum_{i<j} a_{ij}dx_i dx_j \). Obviously, the form \( \omega \) is nondegenerate if and only if \( \det \omega(0) = \det M(0) \neq 0 \). Throughout what follows, we will denote \( (\omega_{ij}) = M^{-1}, (\omega_{ij}) = M^{-1}(0) \).

The closed nondegenerate non-alternating form \( \omega \in s\Omega^2(\mathcal{F}) \) is called the *non-alternating Hamiltonian* form. The corresponding Lie algebra of Hamiltonian vector fields is denoted by \( \tilde{P}(\mathcal{F}, \omega), \)
\[
\tilde{P}(\mathcal{F}, \omega) = \{ D \in W(\mathcal{F}) \mid D\omega = 0 \}.
\]
It follows from (0.6) that $D \in \tilde{P}(\mathcal{F}, \omega)$ if and only if $d(D, \omega) = 0$, i.e. $D, \omega \in Z^1(S\Omega^1(\mathcal{F})) = Z^1(\Omega(\mathcal{F})).$ By Theorem 0.1 $D, \omega = df$, $f \in O(\mathcal{F}) + \langle x_i^{(2m_i)}, i = 1, \ldots, n \rangle$. Whence

$$D = D_f = \sum_{i,j} \omega_{ij} \partial_j f \partial_i.$$ 

From the formula (0.5) we obtain that

$$[D_f, D_g], \omega = D_f(D_g, \omega) = D_f d g = d(D_f(g)).$$

Thus, the correspondence $f \mapsto D_f$ is an isomorphism of the Lie algebra $\tilde{P}(\mathcal{F}, \omega)$ and the Lie algebra $\tilde{O}(\mathcal{F})/K$ with Poisson bracket

$$\{f, g\} = D_f(g) = D_g(f) = \sum_{i,j} \omega_{ij} \partial_i f \partial_j g. \quad (0.7)$$

Let

$$P(\mathcal{F}, \omega) = \{D_f \mid f \in O(\mathcal{F})/K\}.$$ 

It follows from (0.7) that $P(\mathcal{F}, \omega)$ is the ideal of $\tilde{P}(\mathcal{F}, \omega)$ of codimension $n$. A Lie algebra $\mathcal{L}$ such that $P^{(1)}(\mathcal{F}, \omega) \subseteq \mathcal{L} \subseteq \tilde{P}(\mathcal{F}, \omega)$ will be called the non-alternating Hamiltonian Lie algebra.

Remark 0.2. Poincaré algebra $\hat{O}(\mathcal{F})$ (resp. $O(\mathcal{F})$) with Poisson bracket (0.7) is a Leibniz algebra with center $K$, but not a Lie algebra.

1 Canonical form of bilinear non-alternating symmetric forms with respect to the flag

Let $V$ be a finite-dimensional vector space over the field $K$ of an even characteristic. $b$ bilinear nondegenerate non-alternating symmetric form on $V$, $V^0$ the hyperplane consisting of isotropic vectors (i.e. $V^0 = \{v \in V \mid b(v, v) = 0\}$) and $\mathcal{F}: 0 = V_0 \subseteq V_1 \subseteq \ldots$ be a flag of $V$ such that $V_q = V$ for sufficiently large $q$. For the subspace $\mathcal{L} \subseteq V$, denote by $\mathcal{L}^\perp$ its orthogonal complement with respect to $b$. We recall canonical isomorphisms and pairings constructed in [13]. Let

$$\Phi_q \mathcal{F} = V_q/V_{q-1} \text{ for } q \geq 1.$$
\[ \Phi_q(\mathcal{F})_r = (V_q \cap V_{r-1}^\perp + V_{q-1})/V_{q-1} \text{ for } q \geq 1, r \geq 0, \]
\[ \Phi_q \Phi_r \mathcal{F} = \Phi_q(\mathcal{F})_{r-1}/\Phi_q(\mathcal{F})_r \text{ for } q, r \geq 1. \]

There are canonical isomorphisms
\[ \Phi_q \Phi_r \mathcal{F} \cong (V_q \cap V_{r-1}^\perp + V_{q-1})/(V_q \cap v_{r-1}^\perp + V_{q-1}) \cong \]
\[ \cong (V_q \cap V_{r-1}^\perp)/(V_q \cap V_{r-1}^\perp + V_{q-1} \cap V_{r-1}^\perp) = \Phi_{qr} \] (1.1)

The form \( b \) induces a non-degenerate pairing of subspaces \( U, W \subset V \).
\[ U/(U \cap W^\perp) \times W/(U^\perp \cap W) \rightarrow K, \quad (u, w) \mapsto b(u, w). \]

Let \( L_1, M_1 \subseteq V \) and \( L_0 \subseteq L_1, M_0 \subseteq M_1 \) be subspaces of \( V \). We construct a pairing \((L_1 \cap M_0^\perp) \times (L_0^\perp \cap M_1) \rightarrow K\). Find the left and right core. Since \((L \cap M^\perp)^\perp = L^\perp + M\),
\[ (L_1 \cap M_0^\perp)^\perp \cap (L_0^\perp \cap M_1) = (L_1^\perp + M_0) \cap (L_0^\perp + M_1) = \]
\[ = (L_1^\perp \cap M_1 + M_0) \cap L_0^\perp = L_1^\perp \cap M_1 + L_0^\perp \cap M_0 \]
and analogously
\[ (L_1 \cap M_0^\perp) \cap (L_0^\perp \cap M_1)^\perp = L_1 \cap M_1^\perp + L_0 \cap M_0^\perp. \]

Hence, \( b \) induces a non-degenerate bilinear pairing
\[ (L_1 \cap M_0^\perp)/(L_1 \cap M_1^\perp + L_0 \cap M_0^\perp) \times (L_0^\perp \cap M_1)/(L_1^\perp \cap M_1 + L_0^\perp \cap M_0) \rightarrow K. \]

Given isomorphisms of (1.1), we obtain non-degenerate bilinear pairings
\[ \Phi_{qr} \times \Phi_{rq} \rightarrow K, \quad q, r \geq 1. \] (1.2)

For \( q, r \geq 1 \) define
\[ \Phi^0_{qr} \] as the image of \( V_q \cap V_{r-1}^\perp \cap V^0 \) under the canonical projection on \( \Phi_{qr} \),
\[ n_{qr} = \dim \Phi_{qr}, \quad n^1_{qr} = \dim \Phi_{qr} - \dim \Phi^0_{qr}. \]

Note that \( n^1_{qr} \) can only be equal to 0 or 1.

The decomposition of \( V \) into the direct sum of its subspaces \( P \) and \( Q \) is called coordinated with the flag \( \mathcal{F} \) if \( V_j = V_j \cap P + V_j \cap Q \) for all \( j \geq 0 \). The number \( m = \min\{j \in V_j \} \) is the height of an element \( e \).
Lemma 1.1 (cf. [13]). Suppose that for some \( q, r \geq 1 \) either \( q \neq r \), or \( q = r \), \( n_{qq}^1 = 0 \). If \( u \in V_q \cap V_{r-1}^\perp \), \( v \in V_r \cap V_{q-1}^\perp \), and \( b(u,v) = 1 \) then the decomposition
\[
V = \langle u, v \rangle \oplus \langle u, v \rangle^\perp
\]
coordinated with \( \mathcal{F} \) and \( b(u,u) \cdot b(v,v) = 0 \).

Proof. (1) Assume for certainty that \( q < r \).
   (1.1) If \( j < q \), then \( j \leq q - 1 < r - 1 \) and \( V_j \subseteq V_{q-1} \), \( V_j \subseteq V_{r-1} \). Then \( V_j \subseteq u^\perp \) and \( V_j \subseteq v^\perp \), hence \( V_j \subseteq \langle u, v \rangle^\perp \).
   (1.2) If \( j \geq r \), then \( V_q, V_r \subseteq V_j \), then \( \langle u, v \rangle \subseteq V_j \) and \( V_j = V_j \cap V = V_j \cap \langle u, v \rangle + \langle u, v \rangle^\perp \cap V_j \). (1.3) Let \( q \leq j < r \). Then \( V_q \subseteq V_j \subseteq V_{r-1} \), hence \( u \in V_j \cap V_j^\perp \), i.e. \( b(u,u) = 0 \). Since \( b(u,v) = 1 \), then \( v \notin V_j \) and for any \( u' \in V_j \) such that \( b(u',v) = \alpha \) we have \( b(\alpha u + u',v) = 0 \). Let’s denote \( w = \alpha u + u' \) and get the decomposition \( u' = \alpha u + w \), i.e.
\[
V_j = \langle u \rangle + \{ w \in V_j \mid b(w,v) = 0 \} = \langle u, v \rangle \cap V_j + \langle u, v \rangle^\perp \cap V_j.
\]

(2) for \( q = r \) we should consider only cases similar to (1.1) and (1.2).
Since \( n_{qq}^1 = 0 \), the set \( V_q \cap V_{q-1}^\perp \) does not contain a vector of nonzero length and therefore for \( u, v \in v_q \cap V_{q-1}^\perp \), \( b(u,u) = 0 \) and \( b(v,v) = 0 \).

Lemma 1.2. Let \( u \in V_q \cap V_{q-1}^\perp \) be given For some \( q \geq 1 \), with \( b(u,u) = 1 \). Then decomposition
\[
V = \langle u \rangle \oplus \langle u \rangle^\perp
\]
coordinated with \( \mathcal{F} \).

Proof. If \( j < q \), then \( V_j \subseteq V_{q-1} \) and \( V_j \subseteq u^\perp \). If \( j \geq r \), then \( V_q \subseteq V_j \) and \( \langle u \rangle \subseteq V_j \). Now, \( v_j = V_j \cap V = V_j \cap (\langle u \rangle + \langle u \rangle^\perp) = \langle u \rangle + \langle u \rangle^\perp \cap V_j \). Consider triples \((V, \mathcal{F}, b)\) and call \((V, \mathcal{F}, b)\) and \((V', \mathcal{F}', b')\) equivalent if there is an isomorphism \( V \to V' \) that translates the flag \( \mathcal{F} \) onto \( \mathcal{F}' \), and the form \( b \) into \( b' \). Such isomorphism induces isomorphisms \( \Phi_{qr} \to \Phi_{qr}' \) for all \( q, r \geq 1 \). Therefore, the values \( n_{qr} \) and \( n_{qr}^1 \) are invariants of the triple \((V, \mathcal{F}, b)\).

Theorem 1.1. (1) There is coordinated with the flag \( \mathcal{F} \) a basis of the space \( V \) relative to which the matrix of non-alternating form \( b \) has the form
diag(M₀, . . . , M₀, M₁, . . . , M₁, 1_s), where s = ∑ₙ q n¹_qq, the number of matrices M₁ is ∑ₙ q<r n¹_qr, 

\[ M₀ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad M₁ = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}. \]

(2) Triples (V, F, b) and (V′, F′, b′) are equivalent if and only if they have the same invariants nqr and n¹ qr for all q, r ≥ 1.

Proof. (1) Since the form is nondegenerate, there is nqr > 0 for some 1 ≤ q ≤ r.

Step 1. Among nqr > 0 choose the one for which q < r or q = r, n¹ qr = 0. Due to the nondegeneracy of the pairing (1.2), there are u, v that satisfy the conditions of the Lemma 1.1. In this case, if nqr > 2 and n¹ rq = 1, then take v such that b(v, v) = 0. We include elements u, v in the basis. Let’s put \( W = \langle u, v \rangle ⊥ \) and define the flag \( H \) of subspaces in W, assuming \( W_j = W \cap V_j \) for \( j ≥ 0 \). Repeat Step 1 for \( W, H, \) and \( b|_W \) until nqr > 0 meet the condition.

Step 2. For the remaining \( nqq > 0, n¹ qq = 1 \) is fulfilled. Due to the nondegeneracy of the pairing (1.2), there is u that satisfies the conditions of the Lemma 1.2. Include in the basis element u. Let’s put \( W = \langle u \rangle ⊥ \) and define the flag \( H \) of subspaces in W, assuming \( W_j = W \cap V_j \) for \( j ≥ 0 \). Repeat Step 2 for \( W, H, \) and \( b|_W \).

Step 3. Enumerate the basis so that the shape matrix takes the form specified in the condition.

(2) Assume that the listed invariants of two triples coincide. Then the dimensions of the spaces V, V′ are equal and \( n = \sum nqr \). We have bases \{e₁, . . . , eₙ\} in V and \{e′₁, . . . , e′ₙ\} in V′, as in (1). Let \( mᵢ \) (\( mᵢ′ \), respectively) be the height of the element \( eᵢ \) (\( eᵢ′ \), respectively). Let’s put for \( q ≠ r ≥ 1 \) and \( 2t = n - \sum nqq n¹ qq \)

\[ \tilde{i} = \begin{cases} i + 1, & \text{if } i = 2k - 1 \\ i - 1, & \text{if } i = 2k \end{cases} \]

\[ I_{qr} = \{ i / 1 ≤ i ≤ 2t, \ mᵢ = q, \ mᵢ = r \}, \]

\[ I′_{qr} = \{ i / 1 ≤ i ≤ 2t, \ mᵢ′ = q, \ mᵢ′ = r \}, \]
\[ I_{qq} = \{ i / 2t + 1 \leq i \leq n, \ m_i = q \}, \]
\[ I_{qq}' = \{ i / 2t + 1 \leq i \leq n, \ m_i' = q \}. \]

Classes of vectors \( e_i, i \in I_{qr} \) (possibly \( q = r \)) form the basis \( \Phi_{qr} \). So \( I_{qr} \) consists of \( n_{qr} \) indexes and the same is true for \( I_{qq}' \). Similarly, the sets \( I_{qq} \) and \( I_{qq}' \) consist of the same number of \( n_{qq} \) indexes. Note also that \( \tilde{i} \in I_{rq} \) for \( i \in I_{qr} \). All of the above allows us to construct a permutation \( \pi \) of indices \( 1, \ldots, n \), which maps each \( I_{qr} \) to \( I_{qr}' \), \( q, r \geq 1 \), and for which \( \pi \tilde{i} = \pi i \) at \( 1 \leq i \leq 2t \). Then \( m_{\pi i}' = m_i \) for all \( i \). The linear isomorphism \( V \to V' \), which translates \( e_i \) to \( e_i' \), \( 1 \leq i \leq n \), specifies the equivalence of triples \((V, \mathcal{F}, b)\) and \((V', \mathcal{F}', b')\).

\[
\text{Remark 1.1.} \quad 1. \text{From the proof of Lemma 1.1 it follows that for the pair } (u, v) \text{ corresponding to the matrix } M_1, \text{ the height of } u \text{ is less than the height of } v. \\
2. \text{In the case where } p > 2, \text{ the canonical form of a matrix of symmetric bilinear form contains no blocks } M_1, \text{ and } 1_s \text{ is replaced by a diagonal matrix (see [26], Theorem 1, Section 2, Chapter 9).} \\
\]

\[
\text{Remark 1.2.} \quad \text{The invariants mutually uniquely define a matrix of canonical form and a set of heights. For a triple } (V, \mathcal{F}, b) \text{ with invariants } n_{qr} \text{ and } n^1_{qr} (q, r \geq 1), \text{ the height set consists of three groups: } M_0, M_1 \text{ and } 1_s, \text{ respectively. For } q \leq r \text{ in the first group of pair } q, r \text{ is included in the amount of } (1 - n^1_{qr})n_{qr}. \text{ For } q < r \text{ in the second group of the pair } q, r \text{ is in the amount of } n_{qr}n^1_{qr}. \text{ The third group includes the height } q \text{ in the amount of } n_{qq}n^1_{qq}. \\
\]

In the following examples, the differential form with constant coefficients is considered as a bilinear form on the space \( V = W_{-1} \).

\[
\text{Example 1.} \quad \text{Let } n = 3, \mathcal{F}: V = V_3 = \langle \partial_1, \partial_2, \partial_3 \rangle \supset V_2 = \langle \partial_1, \partial_2 \rangle \supset V_1 = \langle \partial_1 \rangle \supset V_0 = 0 \text{ and a coherent } \mathcal{F} \text{ basis form has the form } \omega = dx_1^2 + dx_1dx_3 + dx_2dx_3. \\
\text{We have } V^0 = \langle \partial_2, \partial_3 \rangle \text{ and } V^1_1 = \langle \partial_2, \partial_1 + \partial_3 \rangle. \text{ We apply the algorithm specified in the proof of Theorem 1.1.} \\
\Phi_{11} = \frac{(V_1 \cap V^0_0)}{(V_1 \cap V_1^1 + V_0 \cap V_0^1)} = V_1, \quad V_1 \cap V_0^1 \cap V^0 = 0 \Rightarrow \Phi^0_{11} = 0 \Rightarrow n^1_{11} = 1; \\
\Phi_{12} = \frac{(V_1 \cap V^1_1)}{(V_1 \cap V^0_1 + V_0 \cap V_1^0)} = 0; \\
\Phi_{13} = \frac{(V_1 \cap V^1_2)}{(V_1 \cap V^0_3 + V_0 \cap V_1^2)} = 0; \\
\Phi_{21} = \frac{(V_2 \cap V^1_0)}{(V_2 \cap V_1^1 + V_1 \cap V^0_0)} = 0; \\
\]

13
\[ \Phi_{22} = (V_2 \cap V_1^\perp) / (V_2 \cap V_2^\perp + V_1 \cap V_1^\perp) = 0; \]
\[ \Phi_{23} = (V_2 \cap V_2^\perp) / (V_2 \cap V_3^\perp + V_1 \cap V_1^\perp) = \langle \partial_2 \rangle, \quad V_2 \cap V_2^\perp \cap V^0 = \langle \partial_2 \rangle \Rightarrow \Phi_{23} = \Phi_{21}; \]
\[ \Phi_{31} = (V_3 \cap V_0^\perp) / (V_3 \cap V_1^\perp + V_2 \cap V_0^\perp) = 0; \]
\[ \Phi_{32} = (V_3 \cap V_1^\perp) / (V_3 \cap V_2^\perp + V_2 \cap V_1^\perp) = \langle \partial_2 \rangle; \quad V_3 \cap V_1^\perp \cap V^0 = \langle \partial_2 \rangle \Rightarrow \Phi_{32} = 0 \Rightarrow n_{32} = 1; \]
\[ \Phi_{33} = (V_3 \cap V_2^\perp) / (V_3 \cap V_3^\perp + V_2 \cap V_2^\perp) = 0. \]

We see that \( n_{23} > 0 \). Since \( n_{32} = 1 \), the pair \( (\partial_2, \partial_1 + \partial_3) \) corresponds to the matrix \( M_1 \). We include these vectors in the basis. Step 1 completed.

We see that \( n_{11} > 0 \) and \( n_{11} = 1 \). Include the vector \( \partial_1 \) in the basis.

As a result, \( \omega = dx_1 dx_2 + dx_2^{(2)} + dx_3^{(2)} \). Set of heights: \( (2, 3, 1) \).

**Example 2.** Let \( \mathcal{F}: V = V_4 \supset V_3 = V_2 = V_1 \supset V_0 = 0 \) and \( V_4 / V_3 = \langle \partial_1 \rangle \), in some coordinated with the flag, the form matrix has the form of

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & B
\end{pmatrix}
\]

where \( B \) is a non-degenerate skew-symmetric matrix. Then

\[ V^0 = \langle \partial_2, \partial_3, \ldots, \partial_n \rangle \] and

\[ \Phi_{12} = (V_1 \cap V_1^\perp) / (V_1 \cap V_2^\perp + V_0 \cap V_1^\perp) = \Phi_{21} = (V_2 \cap V_0^\perp) / (V_2 \cap V_1^\perp + V_1 \cap V_0^\perp) = 0; \]

\[ \Phi_{13} = (V_1 \cap V_2^\perp) / (V_1 \cap V_3^\perp + V_0 \cap V_2^\perp) = \Phi_{31} = (V_3 \cap V_0^\perp) / (V_3 \cap V_1^\perp + V_2 \cap V_0^\perp) = 0; \]

\[ \Phi_{14} = (V_1 \cap V_3^\perp) / (V_1 \cap V_4^\perp + V_0 \cap V_3^\perp) = \Phi_{41} = (V_4 \cap V_0^\perp) / (V_4 \cap V_1^\perp + V_4 \cap V_0^\perp) = 0; \]

\[ \Phi_{23} = (V_2 \cap V_2^\perp) / (V_2 \cap V_3^\perp + V_1 \cap V_2^\perp) = \Phi_{32} = (V_3 \cap V_1^\perp) / (V_3 \cap V_2^\perp + V_2 \cap V_1^\perp) = 0; \]

\[ \Phi_{24} = (V_2 \cap V_3^\perp) / (V_2 \cap V_4^\perp + V_1 \cap V_3^\perp) = \Phi_{42} = (V_4 \cap V_1^\perp) / (V_4 \cap V_2^\perp + V_3 \cap V_1^\perp) = 0; \]

\[ \Phi_{34} = (V_3 \cap V_3^\perp) / (V_3 \cap V_4^\perp + V_2 \cap V_3^\perp) = \Phi_{43} = (V_4 \cap V_2^\perp) / (V_4 \cap V_3^\perp + V_3 \cap V_2^\perp) = 0; \]

\[ \Phi_{11} = (V_1 \cap V_0^\perp) / (V_1 \cap V_1^\perp + V_0 \cap V_0^\perp) = V_1, \quad \Phi_{11}^0 = \langle \partial_3, \ldots, \partial_n \rangle, \quad n_{11}^1 = 1; \]

\[ \Phi_{22} = (V_2 \cap V_1^\perp) / (V_2 \cap V_2^\perp + V_1 \cap V_1^\perp) = \Phi_{33} = (V_3 \cap V_2^\perp) / (V_3 \cap V_3^\perp + V_2 \cap V_2^\perp) = 0; \]

\[ \Phi_{44} = (V_4 \cap V_3^\perp) / (V_4 \cap V_4^\perp + V_3 \cap V_3^\perp) = \langle \partial_1 \rangle, \quad \Phi_{44}^0 = 0, \quad n_{22}^1 = 1; \]

As a result, \( \omega = dx_1^{(2)} + \ldots + dx_n^{(2)} \). Set of heights: \( (3, \ldots, 3, 4) \).
2 Filtered deformations of graded non-alternating Hamiltonian Lie algebras

The filtered Lie algebra \( L = L_q \supset \cdots \supset L_1 \supset L_0 \supset L_1 \supset \cdots \supset L_s \supset 0 \) is called the filtered deformation of the graded Lie algebra \( L = \text{gr} L \). Let \( \tilde{P}(1) = \{ \mathcal{F}, \omega_0 \} \subseteq L \subseteq \tilde{P}(\mathcal{F}, \omega_0) \) be the graded non-alternating Hamiltonian Lie algebra, corresponding to the form \( \omega \) with constant coefficients, \( \mathcal{F} : E = E_0 \supseteq E_1 \supseteq \cdots \supseteq E_r \supseteq E_{r+1} = 0, W = W(\mathcal{F}), V = L_0 = W_{-1} \cong W(\mathcal{F})/W(\mathcal{F})_{(0)}, E = V^* \).

According to Embedding Theorem [18] there is a minimal embedding \( \tau : (L, L_0) \to (W(\mathcal{F}'), W(\mathcal{F}'_{(0)})) \) with the minimal flag \( \mathcal{F}' = \mathcal{F}(L, L_0) \). The embedding \( \tau \) is uniquely determined up to the automorphism of the pair \( (W(\mathcal{F}'), W(\mathcal{F}'_{(0)})) \). Such automorphisms are induced by admissible automorphisms of the algebra \( O(\mathcal{F}') \). In particular, \( \mathcal{F}(L, L_{(0)}) = \mathcal{F} \). It is known that \( \mathcal{F} \subseteq \mathcal{F}' ( [18]) \). The following theorem is a special case of the theorem proved in [18], [25].

**Theorem 2.1.** Let \( L = L_{-1} + L_0 + \cdots \) be a transitive graded Lie algebra. If

1. \( L_{-1} \) is an irreducible \( L_0 \)-module,
2. \( H^1(L_0, L_{-1}) = 0, \)
3. \( \text{mtp}(L_{-1}, X(L_{(1)})) \leq \text{m}(\mathcal{F}(L_{(0)})) - n, n = \dim L_{-1}, \)

then for any filtered deformation of \( L \) of Lie algebra \( L \)

\( a) \ \mathcal{F}(L, L_0) = \mathcal{F}(L, L_{(0)}), \)
\( b) \ \text{Der} L \cong N_{W(\mathcal{F})}(\tau(L)). \)

Here \( \tau : L \to W(\mathcal{F}) \) is the minimum embedding, \( W(\mathcal{F}) \) is the \( p \)-closure of \( W(\mathcal{F}) \) in \( \text{Der} O(\mathcal{F}) \), \( X(L_{(1)}) = L_{(1)}/[L_{(1)}, L_{(1)}] \), \( \text{mtp}(Q, V) \) is the multiplicity of \( L_0 \)-module \( Q \) in \( L_0 \)-module \( V \). □

Let \( L \) be a non-alternating Hamiltonian Lie algebra, \( P^{(1)}(\mathcal{F}, \omega_0) \subseteq L \subseteq \tilde{P}(\mathcal{F}, \omega_0), \omega_0 \) be a non-degenerate non-alternating form with constant coefficients.
Proposition 2.1. Let $\overline{L}_0$ be the $p$ - closure of $L_0$ in $W_0 \cong gl(L_{-1})$, $V = W/W_{(0)} \cong L_{-1}$, $\omega_0$ non-alternating symmetric bilinear form at $E = V^*$, dual $\omega_0$, $E^0$ subspace of isotropic vectors of $E$. The following statements are true.

i. $\overline{L}_0 = \overline{L}_0(1) \oplus T$, where $T$ - torus, $T_0 \subseteq T \subseteq T_1$, $T_0 \cong \langle x_i^{(2)} + x_j^{(2)} \rangle = \langle x_i^{(2)} \rangle$, $i,j = 1, \ldots, n$, $T_1 \cong \langle x_i^{(2)} \rangle$, $i = 1, \ldots, n \rangle \subseteq \tilde{F}_0$. Here $\{x_i\}$ is the orthonormal basis of $E$ with respect to $\omega_0$.

ii. If $E_1 \not\subset E^0$, then $T = T_1$.

iii. If $n > 2$ or $n = 2$ and $T = T_1$, then $L_{-1}$ is an absolutely irreducible $\overline{L}_0$-module.

iv. If $n > 4$ or $n = 2, 3, 4$ and $T = T_1$,

\begin{enumerate}
\item $H^1(\overline{L}_0, L_{-1}) = 0$,
\item $(S^3(V^*))^T = 0$.
\item $\text{mtp}(L_{-1}, X(L_{(1)})) \leq m(\mathcal{F}(L, L_{(0)})) - n$.
\end{enumerate}

Proof. i. While studying the pair $(L_{-1}, L_0)$ we can assume that the form $\omega_0$ has the form $\omega_0 = (dx_1)^{(2)} + \ldots + (dx_n)^{(2)}$. Then $\overline{\omega}_0(x_i, x_j) = \delta_{ij}$ and $\{f, g\} = \sum \partial_i f \partial_i g$. It is easy to check that $(\text{ad}(x_i x_j))^{(2)} = \text{ad}(x_i^{(2)} + x_j^{(2)})$, which implies $i$.

ii. Let $y \in E_1$, $y \not\in E^0$. Choose the basis $\{y_i\}$ in $E$, coordinated with the flag $\mathcal{F}$, $y = y_1$. Let $m_1$ be the height of $y_1$. We have $m_1 > 1$. Therefore, $y_1^{(2)} \in L_0$. In the basis $\{y_1\}$ $\overline{\omega}_0$ looks like $\overline{\omega}_0(u, v) = \sum_{i=1}^{n} \overline{\alpha}_{ii} \partial_i u \partial_i v + \sum_{i,j} \overline{\alpha}_{ij} \partial_i u \partial_j v$, $u, v \in E$, $\overline{\alpha}_{ii}$, $\overline{\alpha}_{ij}$ looks like $\overline{\omega}_0(y_1, y_1) \neq 0$. Here $(\overline{\alpha}_{ij}) = (\omega_{ij})^{-1}$.

\begin{equation}
\{f, g\} = \sum_{i=1}^{n} \overline{\alpha}_{ii} \partial_i f \partial_i g + \sum_{i,j} \overline{\alpha}_{ij} \partial_i f \partial_j g,
\end{equation}

\begin{equation}
\overline{\alpha}_{ij} = \overline{\alpha}_{ji}. \quad \text{Therefore}, \quad \text{ad} y_1^{(2)} = \overline{\alpha}_{11} y_1 \partial_1 + \sum_{j>1} \overline{\alpha}_{1j} y_1 \partial_j, \quad \text{tr} \left( \text{ad} y_1^{(2)} \right) = \overline{\alpha}_{11} \neq 0. \quad \text{Obviously}, \quad \text{tr}_{\overline{L}_0(1) \oplus T_0} = 0. \quad \text{Therefore}, \quad y_1^{(2)} = a + t, \quad a \in \overline{L}_0(1), \quad t \in T, \quad t \not\in T_0. \quad \text{Therefore}, \quad T = T_1.
\end{equation}

iii. The statement is verified directly.
iv. If $T = T_1$, then $z = x_1^{(2)} + \ldots + x_n^{(2)} \in Z(\mathcal{L}_0)$ and $\text{ad}_z|_{L_{-1}} = \text{id}$. So $H^1(\mathcal{T}_0, L_{-1}) = 0$. Note that if $n$ is even, $z \in T_0 \subset \mathcal{T}_0$. Let $n > 3$ and $T = T_0 = \langle h_1, \ldots, h_{n-1} \rangle$, $h_i = x_i^{(2)} + x_i^{(2)}$, $\{ \varepsilon_i \}$ dual basis of $T_0$. Then the weights of $L_{-1}$ are $\alpha_1 = \varepsilon_1$, $\alpha_2 = \varepsilon_1 + \varepsilon_2$, $\ldots$, $\alpha_{n-1} = \varepsilon_n - 2 + \varepsilon_{n-1}$, $\alpha_n = \varepsilon_{n-1}$. All weights are nonzero and weight spaces are one-dimensional.

The roots of $\mathcal{T}_0$ are the sums of different weights of $L_{-1}$. If $n > 4$, all root spaces are one-dimensional, the roots are different from the weights of $L_{-1}$.

If $n = 4$, then the roots are different from the weights $L_{-1}$, the root spaces are two-dimensional. From the decomposition of the complex $C^*(\mathcal{T}_0, L_{-1})$ into weight spaces with respect to $T$ we obtain

$$H^1(\mathcal{T}_0, L_{-1}) = H^1_0(\mathcal{T}_0, L_{-1}) = H^1(C^*_0(\mathcal{T}_0, L_{-1})).$$

Since all weights of $L_{-1}$ are nonzero and distinct from roots of $\mathcal{T}_0$, $C^*_0(\mathcal{T}_0, L_{-1}) = 0$ and $H^1(\mathcal{T}_0, L_{-1}) = 0$. Note that for $n = 4$, $z \in T_0 \subset \mathcal{T}_0$ and thus $H^1(\mathcal{T}_0, L_{-1}) = 0$.

Similarly, for $\varphi \in (S^3(V)^*)^T$ we have $\varphi(\mu_1, \mu_2, \mu_3) = 0$, if $\mu_1 + \mu_2 + \mu_3 = 0$, $\mu_i \in \{ \alpha_1, \ldots, \alpha_n \}$. The weights of module $V = L_{-1}$ satisfy the only relation $\alpha_1 + \ldots + \alpha_n = 0$. Thus, for $n > 3$, $\varphi = 0$. Let $T = T_1$. Then $T = \langle h_1, \ldots, h_n \rangle$. Weights of $L_{-1}$ are $\varepsilon_1, \ldots, \varepsilon_n$, where $\{ \varepsilon_i \}$ is the dual basis of $T^*$. Therefore, $\varphi = 0$. In particular, it is true for $n = 2$ and 3. The statement (c) is proved by direct computations analogous to the computation of $X(L_{(1)})$ for the classical Hamiltonian Lie algebra in [21] (see [21], Chapter 3, Proposition 1).

**Theorem 2.2.** Let $L = L_{-1} + L_0 + \ldots$ be a graded non-alternating Hamiltonian Lie algebra, $P^{(1)}(\mathcal{F}, \omega_0) \subseteq L \subseteq \mathcal{P}(\mathcal{F}, \omega_0)$, $\mathcal{L}$ filtered deformation of $L$. For $n = 2, 3$, assume that $E_1 \not\subset E^0$, where $E^0$ is the subspace of isotropic vectors with respect to the form $\omega_0$ on the $E$ dual to the form $\omega$. For $n = 4$, assume that $\mathcal{F}$ is a nontrivial flag. Then

i. $\mathcal{F}(\mathcal{L}, \mathcal{L}_0) = \mathcal{F}(L, L(0)) = \mathcal{F}$;

ii. $\text{Der}\mathcal{L} \cong N_{W(\mathcal{F})}(\tau(\mathcal{L}))$,

where $\tau: \mathcal{L} \rightarrow W(\mathcal{F})$ is the minimal embedding.

**Proof.** According to [21] the conditions of the theorem [21] are satisfied for the Lie algebra $\mathcal{T} = L_{-1} + \mathcal{T}_0 + L_1 + \ldots$. Hence, for any filtered deformation $\mathcal{F}$ of the Lie algebra $\mathcal{T}$

$$\mathcal{F} (\mathcal{F}, \mathcal{F}_0) = \mathcal{F}(L, \mathcal{L}(0)) = \mathcal{F}, \text{ Der} \mathcal{F} \cong N_{W(\mathcal{F})}(\tau(\mathcal{F})),$$

(2.1)
where \( \overline{\tau} \) is the minimal embedding of \( \overline{\mathcal{L}} \) in \( W(\mathcal{F}) \).

Let now \( \mathcal{L} \) be a filtered deformation of the Lie algebra \( L \), \( \tau: \mathcal{L} \to W(\mathcal{F}') \) minimal embedding, \( \mathcal{F}' = \mathcal{F}(\mathcal{L}, \mathcal{L}_0) \). Identify \( \mathcal{L} \) and \( \tau(L) \).

Let \( \mathcal{L}_0 = L_0 + \langle a_i^2, \ i = 1, \ldots, s \rangle, \ a_i \in L_0, \ l_i \in \mathcal{L}_0, \ l_i \equiv a_i(\text{mod } W(1)) \).

As to \( W(\mathcal{F}')(0) \) is a \( p \)-subalgebra of \( \text{Der} O(\mathcal{F}') \), \( l_i^2 \equiv a_i^2(\text{mod } W(1)) \). Hence, \( \mathcal{L} = \mathcal{L}_0 + \langle l_i^2, \ i = 1, \ldots, s \rangle \) is a filtered deformation of the Lie algebra \( \overline{\mathcal{L}} \).

Thus, \( \mathcal{F}' \geq \mathfrak{F}(\mathcal{L}, \mathcal{L}_0) \) (see [18]). On the other hand, \( \tau: \mathcal{L} \to W(\mathcal{F}) \) is embedding, where \( \tau: \mathcal{L} \to W(\mathcal{F}) \) is the minimal embedding of \( L \), and \( \mathcal{F}' \leq \mathcal{F} \). Thus \( \mathcal{F}' = \mathcal{F} \) and we can assume that \( \tau = \overline{\tau} \).

Identifying \( \mathcal{L} \) and \( \text{ad} \mathcal{L} \subset \text{Der} \mathcal{L} \), we get \( [D, \text{ad}(l^2)] = \text{ad}[D(l), l] \).

Thus, \( \mathcal{L} \) may be extended up to derivation of the Lie algebra \( \overline{\mathcal{L}} \) in such a way that \( D(l_i) = [D(l_i), l_i] \). From here we obtain, using (2.1), that \( \text{Der} \mathcal{L} \cong N_{W(\mathcal{F})}(\tau(\mathcal{L})) \).

The proof of the following theorem is similar to the proof of Theorem 8.1 [14] (or Theorem 7.1 [16]) for the classical Hamiltonian Lie algebras based on the theory of truncated coinduced modules [17], [18], [11] and the theory of modular pairs of Lie-Cartan (see [14] and [16] for detailed presentation).

**Theorem 2.3.** Under the conditions of theorem 2.2, there exists a unique up to an admissible automorphism of the algebra \( W(\mathcal{F}) \) a non-alternating Hamiltonian form \( \omega \) with coefficients from \( O(\mathcal{F}) \), such that \( \omega(0) = \omega_0 \) and \( P^{(1)}(\mathcal{F}, \omega) \subseteq \mathcal{L} \subseteq \overline{\mathcal{F}}(\mathcal{F}, \omega) \).

**Proof.** We note only the changes in the proof of Theorem 8.1 [14] that need to be done to extend Theorem 8.1 [14] to the case of filtered deformations of graded non-alternating Hamiltonian Lie algebras. First, we will consider the Lie algebra \( \overline{\mathcal{L}} \) as in the proof of the theorem 2.2 which is a filtered deformation of the graded Lie algebra \( \overline{\mathcal{L}} \) (see [2.1], \( \mathcal{L} \) subset \( \overline{\mathcal{L}} \)). Second, in the proof of theorem 8.1 [14] we need to replace \( \overline{\mathcal{P}} = \Lambda^2 V/\mathcal{L}_0 \cdot \Lambda^2 V \) with \( \overline{\mathcal{P}} = S^2 V/\overline{\mathcal{L}_0} \cdot S^2 V \).

All the information necessary to apply the proof of Theorem 8.1 [14] in the case of filtered deformation of a non-alternating Hamiltonian Lie algebra is given in [2.1].

As a result, we obtain that

\[ P^{(1)}(\mathcal{F}, \omega) \subseteq \overline{\mathcal{F}}(\mathcal{F}, \omega), \]

\( \omega(0) = \omega_0 \), \( \omega \) is a non-alternating Hamiltonian form on \( W(\mathcal{F}) \) with values in some invertible module of the de Rham coefficients of the pair \((O(\mathcal{F}), W(\mathcal{F}))\).
It is easy to check that any invertible module of the coefficients of the de
Rham $P$ of the pair $(O(\mathcal{F}), W(\mathcal{F}))$ is isomorphic to the submodule of
$O(\mathcal{F})u \subset \hat{O}(E)$, $u = \exp f$, $f \in (x_i^{p m_i})$, $i = 1, \ldots, n$, where \{\{x_i\}\} is a
basis of $E$, coordinated with the flag $\mathcal{F}$, $m_i$ is the height of $x_i$. However,
Lemma 0.1 implies that $f = 0$. Hence, $P = O(\mathcal{F})$. \hfill \Box

3 Non-alternating Hamiltonian forms with
non-constant coefficients

Let $m(\mathcal{F})^{(j)}$ be the standard filtration of $O(\mathcal{F})$. Thus, $m(\mathcal{F})^{(1)} = m(\mathcal{F})$.
Denote by $G(\mathcal{F})$ the group of admissible automorphisms of the algebra
$O(\mathcal{F})$. Let $G'(\mathcal{F})$ be a subgroup of automorphisms $\sigma \in G(\mathcal{F})$ such that
$\sigma f - f \in m(\mathcal{F})^2$ for all $f \in O(\mathcal{F})$. The group $G(\mathcal{F})$ preserves ideals $(\mathcal{F})^{(k)}$.
Define the filtration of corresponding groups by normal subgroups, $j \geq 0$

$$G(\mathcal{F})_j = \{ \sigma \in G(\mathcal{F}) \mid \sigma f - f \in m(\mathcal{F})^{(j+1)} \ \forall f \in m(\mathcal{F})^{(l)} , \ l \geq 0 \}, \ j \geq 0$$

$$G'(\mathcal{F})_j = G(\mathcal{F})_j \cap G'(\mathcal{F})$$

and the following Lie subalgebras of $W(\mathcal{F})$

$$g'(\mathcal{F}) = m(\mathcal{F})^2 W(\mathcal{F}), \quad g'(\mathcal{F})_j = (m(\mathcal{F})^2 \cap m(\mathcal{F})^{(j+1)}) W(\mathcal{F})$$

Put

$$z_i = x_i^{2m_i}, \quad \langle x_1, \ldots, x_n \rangle = E, \quad m_i = \min \{ j \mid x_i \notin E \}.$$

Note that the group $G'(\mathcal{F})$ acts trivially in $H^*(\Omega(\mathcal{F})$ (this follows from
the triviality of the action on $H^1(\Omega(\mathcal{F})) \equiv \tilde{\text{tilde}}O(\mathcal{F})/O(\mathcal{F})$).

**Proposition 3.1** (13). For $j \geq 1$ for a given $\sigma \in G'(\mathcal{F})_j$ there exists the
unique $D \in g'(\mathcal{F})_j$ such that

$$\sigma x = x + Dx \quad \forall x \in E,$$

$$\sigma - id - D)(m(\mathcal{F})^{(l)}) \subseteq m(\mathcal{F})^{(j+l+1)} \quad l \geq 0.$$ (3.1)

**Corollary 3.1** (see 13). Let $\psi \in m(\mathcal{F})^{(r)}S\Omega^k(\mathcal{F}), \ \sigma \in G'(\mathcal{F})_j$, where
$j \geq 1$. Then $\sigma \psi - \psi \in m(\mathcal{F})^{(j+r)}S\Omega^k(\mathcal{F})$. Moreover, if $D \in g'(\mathcal{F})_j$ is
related to $\sigma$ by (3.1), $\sigma \psi - \psi - D\psi \in m(\mathcal{F})^{(j+r+1)}S\Omega^k(\mathcal{F})$.  

19
For the non-alternating Hamiltonian form $\omega \in S\Omega^2(\mathcal{F})$, we define the isomorphism of $O(\mathcal{F})$-modules $i_\omega: W(\mathcal{F}) \to S\Omega^1(\mathcal{F})$, $i_\omega(D) = D\omega$.

In the future, we use abbreviations $G'_j = G'(\mathcal{F})_j$, $S\Omega = S\Omega(\mathcal{F})$, $\mathfrak{m} = \text{mathfrak{m}}(\mathcal{F})$, etc.

Write non-alternating Hamiltonian form $\omega$ in the form $\omega = \sum a_i dx_i^{(2)} + \sum_{i<j} a_{ij} dx_i dx_j + d\varphi + \sum_{i<j} b_{ij} dz_i dz_j$, where $a_{ij}, b_{ij} \in K$, $\varphi \in \mathfrak{m}^2 S\Omega^1$. We assume that $\omega(0)$ is given to canonical form.

We identify $H^2(\Omega)$ with the subspace $\langle dz_i dz_j \rangle \subset H^2(S\Omega)$ and define the semi-linear map $\lambda: H^2(\Omega) \to O(\mathcal{F})$ assuming $\lambda(b^2 dz_i dz_j) = bx_i^{(2m-1)} x_j^{(2m-1)}$.

**Lemma 3.1.** (1) An automorphism $\sigma \in G'$ acts identically on the cohomological class of the form $\omega$ if and only if for all $i$, such that $\omega(0)$ contains $dx_i^{(2)}$, runs in $\sigma x_i = x_i + f$ where $f \in \mathfrak{m}^2$ and $f$ does not contain monomials of the form $\lambda(dz_i dz_j)$.

(2) Let $\{dx_i^{(2)}, dz_i dz_j\}$ be the basis of $H^2(S\Omega)$. The automorphism $\sigma \in G'$ can act not identically only on the elements $dx_i^{(2)}$.

If $\sigma x_i = x_i + bx_i^{(2ms-1)} x_r^{(2mr-1)}$, then $\sigma dx_i^{(2)} = dx_i^{(2)} + d\varphi + b^2 dz_s dz_r$, where $\varphi = bx_i^{(2ms-1)} x_r^{(2mr-1)} dx_i$.

**Proof.** According to theorem 0.1 $H^2(S\Omega) = H^2(\Omega) \oplus \langle dx_1^{(2)}, \ldots, dx_n^{(2)} \rangle$. On $H^2(\Omega)$, the automorphism $\sigma \in G'$ acts trivially.

Let $\sigma \in G'_j$, $j \geq 1$, $\sigma x = x + DX$ for $x \in E$, $D \in g'_j$. Let $DX = \sum a_\alpha x^{(\alpha)}$. Then $\sigma(dx)^{(2)} = (\sigma dx)^{(2)} = (\sigma(dx)^{(2)} = (dx + D\sigma x)^{(2)} = (dx + D\sigma x)^{(2)} = (dx)^{(2)} + d\sigma x^{(2)} + (dx)^{(2)} + dDx + (dx)^{(2)} + dDx + (dx)^{(2)} = (dx)^{(2)} + d(Dx)^{(2)} + (dx)^{(2)} + dDx$.

$+ dDx + d\varphi + \sum a_{\sigma r} dz_s dz_r = (dx)^{(2)} + d\varphi + \eta$, where $a_{\sigma r} = a^2_{\alpha}$ is coefficient square at $x^{(\alpha)} = x_r^{(2ms-1)} x_r^{(2mr-1)}$ in $dx$. Consequently, the cohomology class of the form $(dx)^{(2)}$ changes only if $DX$ has a summand $a_\alpha x_s^{(2ms-1)} x_r^{(2mr-1)} = \lambda(a_{\sigma r} dz_s dz_r)$. For $DX = b\lambda(dz_s dz_r)$ we get $\sigma(dx)^{(2)} = (dx)^{(2)} + d(Dx)^{(2)} + b^2 dz_s dz_r$. \qed
Corollary 3.2. For $\sigma \in G'$ such that $\sigma x_i = x_i + bx_i(2^r)x_j(2^r)$, we have $\sigma dx_i(2) = dx_i(2) + d\varphi + b^2 x_s^{(2r+1)} x_j^{(2^r+1)} dx_s dx_j$, where $\varphi = bx_i(2^r)x_j(2^r) dx_i$. Here $0 \leq r < m_s$, $0 \leq t < m_j$.

Lemma 3.2. Let $\omega \in Z^k(S\Omega)$, $k = 1, 2$ and $\sigma \in G'$, $D \in G'$, where $j \geq 1$, are bound by the condition (3.1). If $\omega \in Z^1(S\Omega)$, then there is $\varphi \in m$ such that $\sigma \varphi - \varphi = d\varphi$ and $\varphi - D\omega \in m^{(j+2)}$. If $\omega \in Z^2(S\Omega)$, then $\sigma \varphi - \varphi = d\varphi + \eta$, where $\varphi \in S\Omega^1$, $\eta \in H^2(\Omega)$, $\lambda(\eta) \in m^{(j+1)}$, and $\varphi - D\omega \in m^{(j+2)}$.

Proof. Consider first the action of $\sigma$ on $\omega = d\psi$, where $\psi \in S\Omega^{k-1}$. Correcting $\psi$ on coboundary, we can assume that $\psi \in mS\Omega^{k-1}$. Assuming $\varphi = \sigma \psi - \psi - d(D\omega)$, we get $\sigma \psi - d\psi = d(\sigma \psi - \psi) = d\varphi$. Since $D\omega = D_{ij}d\psi = D\psi - d(D\omega)$, by corollary 3.1 $\varphi - D\omega \in m^{(j+2)}$.

Now let $\omega = (dx(2))_k$, $x \in E$. We use notations from the proof of the Lemma 3.1. For $Dx = \sum a\omega x^{(a)} \in m^2 \cap m^{(j+1)}$ we have $\sigma dx(2) = (dx(2))_k + d\varphi + \eta$, where $\lambda(\eta) = \lambda(\sum a_i dz_i d\bar{z}_i) \in m^{(j+1)}$, and $\varphi - D\omega \in m^{(j+2)}S\Omega^{k-1}$.

Show that for $\omega \in Z^1(S\Omega)$ the Lemma is true. For $f \in m$

$$\sigma d(f(2)) - d(f(2)) = d((\sigma f)(2) - f(2)) = dh,$$

where $h = (\sigma f - f)(2) + f(\sigma f - f) \in O(\mathcal{F})$, because $\sigma f - f \in m^2$. Herewith

$$h - D_{ij}d(f(2)) = h - Df(2) = h - fDf = f(\sigma f - f - Df) + (\sigma f - f)(2).$$

In view of (3.1), $f(\sigma f - f - Df) \in m^{(j+2)}$ and $(\sigma f - f)(2) \in m^{(j+2)}$, so that the right side of 3.2 is known to be $m^{(j+2)}$, that is, for $d(f(2))$ the Lemma is true. But any 1-cocycle comparable modulo coboundaries with the appropriate cocycle $fDf$. So the Lemma is true for all $\omega \in Z^1(S\Omega)$.

It remains to prove that the Lemma is true for $\omega_1 \omega_2 \in Z^1(S\Omega) \cdot Z^1(S\Omega)$. Let $\sigma \omega_i - \omega_i = d\varphi_i$. Then $\sigma (\omega_1 \omega_2) - \omega_1 \omega_2 = (\sigma \omega_1 - \omega_1)(\sigma \omega_2 - \omega_2) + \omega_1(\sigma \omega_2 - \omega_2) + (\sigma \omega_1 - \omega_1)\omega_2 = d\varphi_1 d\varphi_2 + \omega_1 d\varphi_2 + d\varphi_1 \omega_2 = d(\varphi_1 d\varphi_2) + d(\omega_1 \varphi_2 + \varphi_1 \omega_2) = d\varphi$ and $\varphi_1 d\varphi_2 + \omega_1 \varphi_2 + \varphi_1 \omega_2 = D_{ij}(\omega_1 \omega_2) = \varphi d\varphi_2 + \omega_1 d\varphi_2 + \varphi_1 \omega_2$. As in corollary 3.1 $d\varphi_i \in m^{(j)S\Omega}$, then $\varphi - D\omega_2 \omega_1 \omega_2 \in m^{(j+1)}S\Omega + m^{(j+2)}S\Omega^{1} \subseteq m^{(j+2)}S\Omega^{1}$. It follows from the Theorem 0.1 that $Z^2(S\Omega)$ is spanned by $(dx_i(2)) (i = 1, \ldots, n)$, $Z^1(S\Omega) \cdot Z^1(S\Omega)$, and $B^2(S\Omega)$. So, the Lemma is proven. \qed
Corollary 3.3. If $\omega \in S\Omega^2$ is a closed form, $\sigma \in G_j$, $j \geq 1$, then $\sigma \omega = \omega = d\varphi + \eta$ for a suitable form $\varphi \in m^{(j+1)} \Omega^1$ and $\eta \in H^2(\Omega)$, $\lambda(\eta) \in m^{(j+1)}$.

For the non-alternating Hamiltonian form $\omega = \omega(0) + d\varphi + \eta$, where $\varphi \in m^{(2)} \Omega^1$ and $\eta \in H^2(\Omega)$ we introduce the set of indices $I = \{i \mid a_{ii} \neq 0\}$. We also need the set

$$\tilde{m}^{(j)} \Omega^1 = \langle T \rangle,$$

where

$$T = \{x^{(\alpha)} dx_k \mid \alpha \geq j, k = 1, \ldots, n\} \setminus \{x_r^{(2)} x_s^{(2)} dx_i, x_r^{(2)} dx_i, i, q \in I\}.$$

Lemma 3.3. Let $j \geq 1$ and $\varphi = x_r^{(2)} x_s^{(2)} dx_i \in m^{(j+1)} \Omega^1$, $i \in I$. If $r, s \notin I$, or $l, t > 0$, or $r \notin I$, $t > 0$, or $s \notin I$, $l > 0$, then $\varphi = d\psi + \tilde{\varphi}$, where $\tilde{\varphi} \in \tilde{m}^{(j+1)} \Omega^1$. For $i = r$ and $m_i > 1$ if $l > 0$ or $t > 1$, or $s \notin I$, then $x_r^{(2)} x_s^{(2)} dx_i = d\psi + \tilde{\varphi}$, where $\tilde{\varphi} \in \tilde{m}^{(j+1)} \Omega^1$.

Proof. If $i \neq r, s$, then

$$d(x_r^{(2)} x_s^{(2)} dx_i) = x_r^{(2)} x_s^{(2)} dx_i dx_r + x_r^{(2)} x_s^{(2)} dx_r dx_i =$$

$$= d(x_r^{(2)} x_s^{(2)} dx_i) + x_r^{(2)} x_s^{(2)} dx_i.$$  

Therefore,

$$x_r^{(2)} x_s^{(2)} dx_i = d\psi + x_r^{(2)} x_s^{(2)} dx_i.$$  

Obviously, $x_r^{(2)} x_s^{(2)} dx_i + x_r^{(2)} x_s^{(2)} dx_i \in \tilde{m}^{(j+1)} \Omega^1$, except for $r \in I$, $l = 0$ or $s \in I$, $t = 0$.

If $m_i > 1$, then $d(x_r^{(2)} x_s^{(2)} dx_i) = x_r^{(2)} x_s^{(2)} dx_i dx_r + x_r^{(2)} x_s^{(2)} dx_i dx_s = d(x_r^{(2)} x_s^{(2)} dx_i)$. Therefore, $x_r^{(2)} x_s^{(2)} dx_i = d\psi + x_r^{(2)} x_s^{(2)} dx_i$. Obviously,

$$x_r^{(2)} x_s^{(2)} dx_i \in \tilde{m}^{(j+1)} \Omega^1$$

for $l > 0$ or $t > 1$, or $s \notin I$. \hfill $\Box$

Lemma 3.4. If $h \in m^{(2)}$, then $(dh)^{(2)} = d\varphi$, where $\varphi \in m^{(2)} \Omega^1$, then and only then, when $h$ does not contain monomials $\lambda(dz_r dz_s)$ and monomials of the form $x_q x^{(2)}_j$, where $q \in I$, $m_q = 1$ or $q, j \in I$, $m_q > 1$, $t = 0$.  

22
Proof. Since \( h \in \mathfrak{m}^2 \), \( h = \sum \beta_j g_j \) is a linear combination of decomposable monomials \( g_j \). Then

\[
(dh)^{(2)} = \left( \sum \beta_j dg_j \right)^{(2)} = \sum_{j<k} \beta_j \beta_k dg_j dg_k + \sum_j \beta_j^2 (dg_j)^{(2)} = \sum_{j<k} d(\beta_j \beta_k g_j g_k) + \sum_j \beta_j^2 (dg_j)^{(2)} = d\psi + \sum_j \beta_j^2 (dg_j)^{(2)},
\]

where \( \psi \in \tilde{\mathfrak{m}}^{(2)} S\Omega^1 \) because the form \( \xi \notin \tilde{\mathfrak{m}}^{(2)} S\Omega^1 \) must contain \( x_r^{(2)} x_s^{(2)} dx_i \) which is the product of two indecomposable elements and \( dx \), or \( x_q^{(2)} dx_i \) which is a product of an indecomposable element and \( dx \). If a monomial \( g \) is \( y_1 y_2 \) and one of the monomials \( y_1 \in \mathfrak{m}^2 \) or \( y_2 \in \mathfrak{m}^2 \) decompose, then

\[
(dg)^{(2)} = y_1 d(y_2) + d(y_1) y_2 = d(y_1) d(y_2) = 0. \tag{2}
\]

Let \( g = x_r^{(2k)} x_s^{(2^t)} \), then

\[
(dg)^{(2)}((x_r^{(2k)}))^{(2)} dx_r dx_q = d\varphi_1 \text{ if } t < m_s - 1, \text{ or } (dg)^{(2)} = d\varphi_2 \text{ if } k < m_r - 1. \tag{3}
\]

Here \( \varphi_1 = x_r^{(2k+1)} x_s^{(2^{t+1})} dx_r \), \( \varphi_2 = x_r^{(2k+1)} x_s^{(2^{t+1})} dx_s \). By Lemma \ref{lem:3.3} \( \varphi_1 + df_1 \in \tilde{\mathfrak{m}}^{(2)} S\Omega^1 \) if \( r \notin I \) or \( k > 0 \), or \( m_r > 1 \), \( s \notin I \), or \( m_r > 1 \), \( t > 0 \) and \( \varphi_2 + df_2 \in \tilde{\mathfrak{m}}^{(2)} S\Omega^1 \) if \( s \notin I \) or \( t > 0 \), or \( m_s > 1 \), \( r \notin I \), or \( m_s > 1 \), \( k > 0 \). Thus, if \( h \) does not contain monomials of the form \( x_r^{(2m_r-1)} x_s^{(2m_s-1)} = \lambda dz_r dz_s \) and monomials of the form \( x_q x_j^{(2t)} \), where \( q \in I \), \( m_q = 1 \) or \( m_q > 1 \), \( t = 0, q, j \in I \), then \( \varphi \in \tilde{\mathfrak{m}}^{(2)} S\Omega^1 \). The sufficiency is proved.

Now, let \( \sigma \in G' \), \( \sigma x_i = x_i + h \), then \( \sigma(dx)^{(2)} = dx_i^{(2)} + dhdx_i + (dh)^{(2)} \) (see proof of Lemma \ref{lem:3.1}). If \( h \) contains \( \lambda dz_r dz_s \), then the Lemma \ref{lem:3.1} \((dh)^{(2)} \) contains \( dz_r dz_s \). If \( h \) contains \( x_q x_j^{(2t)} \), where \( q \in I \), \( m_q = 1 \) or \( q, j \in I \), \( m_q > 1 \), \( t = 0 \), then from corollary \ref{cor:3.2} and the reasoning above, we obtain that \( (dh)^{(2)} \) contains \( x_q x_j^{(2t+1)} dx_q dx_j = d\psi \), where \( \psi = x_q x_j^{(2t+1)} dx_q \), or \( \psi = x_q^{(2)} x_j dx_j \), and \( \varphi \notin \tilde{\mathfrak{m}}^{(2)} S\Omega^1 \). That is, the necessity is proved. \( \square \)

**Proposition 3.2.** Let \( \omega, \omega' \) be two non-alternating Hamiltonian forms, and \( \omega' - \omega = d\varphi \), where \( \varphi \in \tilde{\mathfrak{m}}^{(j+1)} S\Omega^1, j \geq 1 \). Then, there exists \( \sigma \in G'_j \) such that \( \omega' - \sigma \omega = d\tilde{\varphi} \) for some \( \tilde{\varphi} \in \tilde{\mathfrak{m}}^{(j+2)} S\Omega^1 \).

**Proof.** Let \( \varphi = \varphi_1 + \varphi_2 \), where \( \varphi_1 \in (\mathfrak{m}^2 \cap \tilde{\mathfrak{m}}^{(j+1)}) S\Omega^1, \varphi_2 \in \tilde{\mathfrak{m}}^{(j+1)} S\Omega^1 \), \( \varphi_2 \notin \mathfrak{m}^2 S\Omega^1 \).

Then \( \varphi_2 \) consists of \( \beta x_k^{(2)} dx_s \) and \( t > 1 \) terms. Correcting \( \varphi_2 \) on coboundary, \( \varphi_2 = d(\sum \beta x_k^{(2)} x_s) - \sum x_s d(\beta x_k^{(2)}) = dg + \tilde{\varphi}_2 \) and \( \tilde{\varphi}_2 \in (\mathfrak{m}^2 \cap \tilde{\mathfrak{m}}^{(j+1)}) S\Omega^1 \),
we can assume that \( \varphi \in (m^2 \cap \bar{m}^{(j+1)})SO^1 \). Since \( i_\omega \) is an isomorphism of \( O \) - modules, there is \( D \in (m^2 \cap m^{(j+1)})W = g'_j \) such that \( D \omega = \varphi \). According to \( \text{[3.1]} \) there is \( \sigma \in G'_j \) such that \( \text{[3.1]} \) holds.

Denote \( Dx_r = h_r \in m^2 \cap m^{(j+1)} \). The form \( \omega \) is divided into \( \omega(0) \) and summands of the form \( dz_kdz_r \) and \( dfdx_k, f \in m^2 \). Since

\[
D_dz_kdz_r = x^{(2m_k-1)}_k x^{(2m_r-1)}_r h_kdz_r + x^{(2m_k-1)}_k x^{(2m_r-1)}_r (2m_r - 1)h_rdx_k
\]

where each term can be decomposed into the product of the four elements and \( dx \) and \( D_dfdx_k = h_kdf + D(f)dx_k \), where each summand can be decomposed into the product of three elements and \( dx \), then \( D_j(\omega - \omega(0)) \in \bar{m}^{(j+1)}SO^1 \) as the form \( \xi \notin \bar{m}^{(j+1)}SO^1 \) must contain \( x^{(2)}_r x^{(2)}_s dx_i \) which is the product of two irreducible elements and \( dx \), or \( x^{(2)}_i dx_i \) which is the product of an irreducible element and \( dx \). Thus, \( D_{j\omega} \in \bar{m}^{(j+1)}SO^1 \).

Then if \( dx_i^{(2)} \) is included in \( \omega(0) \), then \( h_i \) does not contain monomials of the form \( x^{(2m_r-1)}_r x^{(2m_s-1)}_s = \lambda(dz_rdz_s) \). By corollary \( \text{[3.3]} \) and Lemma \( \text{[3.1]} \) \( \sigma \omega - \omega = d\psi \) for some \( \psi \in m^{(j+1)}SO^1 \).

Now prove that \( \psi \in \bar{m}^{(j+1)}SO^1 \). To check \( dz_kdz_r \), we use the formula

\[
\sigma d(x^{(2m_k)}_k) - d(x^{(2m_k)}_k) = d((\sigma x_k)(2m_k) - x^{(2m_k)}_k) = d((x_k + h_k)(2m_k) - x^{(2m_k)}_k) =
\]

\[
d \sum_{l=1}^{2m_k} x^{(2m_k-1)}_k h^{(l)}_k = d \varphi_k
\]

where \( \varphi_k \in O(\mathcal{F}) \), since \( h_k \in m^2 \). We find that \( \sigma(dz_kdz_r) - dz_kdz_r = d(\varphi_k dz_r + \varphi_r dz_k + \varphi_r dz_k) \) and \( \varphi_k dz_r + \varphi_r dz_k + \varphi_r dz_k = \psi_1 \) and each term can be decomposed into the product of three elements, and \( dx \), i.e. \( \psi_1 \in \bar{m}^{(j+1)}SO^1 \).

Consider \( dfdx_k, f \in m^2 \). Let \( \psi_2 = \sigma(fdx_k) - f dx_k \). Then \( d\psi_2 = \sigma(dfdx_k) - dfdx_k \). By virtue of the Lemma \( \text{[3.3]} \) it is sufficient to check whether \( \psi_2 \) contains monomials of the form \( x_q x^{(2)}_i dx_i \) and \( x_q(2)dx_i \) for \( i, q \in I \). Since \( \sigma(fdx_k) - f dx_k = (\sigma f)(\sigma dx_k - dx_k) + (\sigma f - f)dx_k = (\sigma f)dh_k + (\sigma f - f)dx_k \in m^2 SO^1 \), then \( \psi_2 \) does not contain summands of the form \( x^{(2)}_q dx_i \). If \( (\sigma f)dh_k \) contains \( x_q x^{(2)}_i dx_i \), then there are two cases: 1) \( dh_k \) contains \( x^{(2)}_i dx_i \) and \( \sigma f \) contains \( x_q \), and therefore, \( f \) contains \( x_q \); 2) \( dh_k \) contains \( x_q dx_i \) and \( \sigma f \) contains \( x^{(2)}_q \). Then \( h_k \) contains \( x_q x_i \), \( q \neq I \) and \( \sigma f \) contains \( x^{(2)}_q \), hence \( f \) contains \( x^{(2)}_q \) and \( t > 0 \). In this case \( (\sigma f)dh_k = x^{(2)}_q d(x_q x_i) + \ldots =
\]

\begin{align*}
x_qx_s^{(2)}dx_i + x_i x_s^{(2)}dx_q + \ldots = \\
dg + x_qx_s^{(2t-1)}dx_s + \ldots \text{ and } x_qx_s^{(2t-1)}dx_s \in \mathfrak{m}^{(2)}S\Omega^1 \text{ since } t > 0. \text{ If } (\sigma f - f)dx_i \text{ contains } x_qx_s^{(2t)}dx_i, \text{ then } \sigma f - f \text{ contains the product of two indecomposable elements } x_qx_s^{(2t)}, \text{ which is only possible when } f \text{ contains } x_k^{(r)}.
\end{align*}

Now
\begin{align*}
\sigma x_k^{(r)} - x_k^{(r)} = (x_k + h_k)^{(r)} - x_k^{(r)} = \sum_{s=1}^r x_k^{(r-s)}h_k^{(s)}.
\end{align*}

Therefore, if \(\sigma x_k^{(r)} - x_k^{(r)}\) contains \(x_qx_s^{(2t)}\), then \(f\) contains \(x_q\), but \(f \in \mathfrak{m}^{(2)}\). Thus, we can assume that \(\psi_2 \in \mathfrak{m}^{(j+1)}S\Omega^1\).

It remains to check \(\omega(0)\). We have
\begin{align*}
\sigma(a_l l^{-1}dx_{l-1}dx_l + dx_l^{(2)}) - (a_l l^{-1}dx_{l-1}dx_l + dx_l^{(2)}) = \\
a_l l^{-1}(dh_{l-1}dx_l + h_{l-1}dx_l) + a_l l^{-1}dh_{l-1}dx_l + d(h_{l-1}dx_l) + (dh_l)^{(2)} = \\
d(D(a_l l^{-1}dx_{l-1}dx_l + dx_l^{(2)})) + a_l l^{-1}dh_{l-1}dx_l + (dh_l)^{(2)}.
\end{align*}

From \(D_\omega(0) \in \tilde{\mathfrak{m}}^{(j+1)}S\Omega^1\) it follows that
\begin{align*}
D_\omega(a_l l^{-1}dx_{l-1}dx_l + dx_l^{(2)}) \in \tilde{\mathfrak{m}}^{(j+1)}S\Omega^1.
\end{align*}

Since each term in \(h_{l-1}dh_l\) can be decomposed into a product of three elements and \(dx, h_{l-1}dh_l \in \tilde{\mathfrak{m}}^{(j+1)}S\Omega^1\). Thus, it is sufficient to investigate \((dh_l)^{(2)}\). But due to Lemma 3.4 \((dh_l)^{(2)} = d\psi_3\), where \(\psi_3 \in \tilde{\mathfrak{m}}^{(j+1)}S\Omega^1\), since \(h_l\) does not contain monomials of the form \(\lambda(dz_r dz_s), \ x_qx_s^{(2t)}\). For \(s, k \notin I\) we obtain
\begin{align*}
\sigma dx_sd x_k - dx_sd x_k = d(D_\omega dx_sd x_k) + +d(h_sd h_k)
\end{align*}

and \(D_\omega dx_sd x_k + h_sd h_k \in \tilde{\mathfrak{m}}^{(j+1)}S\Omega^1\).

Lemma 3.2 and the above guarantees that \(\omega - D_\omega = \psi - \varphi \in \tilde{\mathfrak{m}}^{(j+2)}S\Omega^1\). Also, \(\omega' - \sigma \omega = \omega' - \omega - (\sigma \omega - \omega) = d(\varphi - \psi)\).

**Lemma 3.5.** If \(\omega\) is a non-alternating Hamiltonian form and \(\omega = \omega(0) + d\varphi + \eta\), where \(\varphi \in \tilde{\mathfrak{m}}^{(2)}S\Omega^1\) and \(\eta \in H^2(\Omega)\), for any \(\sigma \in G'\) occurs \(\sigma(d\varphi + \eta) - (d\varphi + \eta) = d\varphi\), where \(\psi \in \tilde{\mathfrak{m}}^{(2)}S\Omega^1\).

**Proof.** The proof of 3.2 shows that for any \(\sigma \in G', j \geq 1, \sigma(dz_k dz_r) - dz_k dz_r = d\psi_1\) is executed, where \(\psi_1 \in \tilde{\mathfrak{m}}^{(j+1)}S\Omega^1\), and \(\sigma(df dx_k) - df dx_k = d\psi_2\), where \(\psi_2 \in \tilde{\mathfrak{m}}^{(j+1)}S\Omega^1\). Take \(j = 1\).
Select two properties of \( \omega = \omega(0) + d\varphi + \eta \), where \( \varphi \in \mathfrak{m}^{(2)}S\Omega^1 \) and \( \eta \in H^2(\Omega) \)

there exists \( i \in I \) such that \( m_i > 1 \),

(3.3)

\[
m_i = 1 \quad \text{for any} \quad i \in I \quad \text{and} \quad \eta = \sum_{s,j \notin I} b_{sj} dz_s dz_j.
\]

(3.4)

Let us consider the automorphism \( \sigma \in G' \) such that

\[
\sigma x_i = x_i + b x_s^{(2m_s-1)} x_k^{(2m_k-1)}.
\]

Since the form \( \omega(0) \) is one of canonical form, only two variants of \( dx_i^{(2)} \) are possible

\[
\omega_1(0) = \ldots + dx_1^{(2)} + \ldots,
\]

\[
\omega_2(0) = \ldots + dx_{i-1} dx_i + dx_1^{(2)} + \ldots.
\]

By Lemma 3.1 we have

\[
\sigma \omega_1(0) = \omega_1(0) + d( b x_s^{(2m_s-1)} x_k^{(2m_k-1)} dx_i ) + b^2 dz_s dz_k,
\]

\[
\sigma \omega_2(0) = \omega_2(0) + d( b x_s^{(2m_s-1)} x_k^{(2m_k-1)} ) (dx_{i-1} + dx_i) + b^2 dz_s dz_k.
\]

If in the second case we apply automorphism \( \sigma_1 \), which translates \( x_{i-1} \) to \( x_{i-1} + b x_s^{(2m_s-1)} x_k^{(2m_k-1)} \), we get

\[
\sigma_1 \sigma \omega_2(0) = \omega_2(0) + d( b x_s^{(2m_s-1)} x_k^{(2m_k-1)} ) dx_{i-1} + b^2 dz_s dz_k.
\]

Thus, only the summand of the form remains \( d( x_s^{(2)} x_k^{(2)} dx_{i-1} ) \), \( i - 1 \in I \).

Let \( d( b x_s^{(2m_s-1)} x_k^{(2m_k-1)} dx_i ) = d\varphi \), where \( i \in I \). Note that \( \varphi \notin \tilde{\mathfrak{m}}^{(2)}S\Omega^1 \) and prove the following theorem.

**Theorem 3.1.** Let \( \omega \) be a non-alternating Hamiltonian form, \( \omega = \omega(0) + d\psi + \eta \), where \( \eta \in H^2(\Omega) \), \( \psi \in \mathfrak{m}^{(j+1)}S\Omega^1 \), \( j \geq 1 \).

(1) If \( (3.3) \) holds, then there exists an automorphism \( \sigma \in G' \) such that \( \sigma \omega = \omega(0) + d\tilde{\psi} + \eta \), where \( \tilde{\psi} \in \tilde{\mathfrak{m}}^{(j+1)}S\Omega^1 \).

(2) If \( m_r = 1 \) for all \( r \in I \), then there is an automorphism \( \sigma \in G' \) such that

\[
\sigma \omega = \omega(0) + d\tilde{\psi} + \psi + \sum_{q,i \in I} \sum_{s \notin I} b_{qsti}^{2m_s-1} dz_i dz_s + \sum_{q,i \in I} b_{iq} 2 dz_i dz_q,
\]

where \( \tilde{\psi} \in \tilde{\mathfrak{m}}^{(j+1)}S\Omega^1 \), \( b_{qsti} x_q x_s^{(2)} dx_i \) or \( b_{iq} x_q x_i dx_q \) is included in \( \psi \).
Proof. Let \( \varphi = x_i^{(2r)} x_s^{(2q)} \, dx_i, \, I \in I, \, 0 \leq r < m_i, \, 0 \leq t < m_s \) and \( \varphi_1 = x_q^{(2)} \, dx_i, \, q \in I \). Except for \( l \in I, \, r = 0 \) \((s \in I, \, t = 0) \) and \( m_i > 1, \, i = l, \, r = 0, \, t \leq 1, \, s \in I \) Lemma 3.3 guarantees that \( d\varphi = d\tilde{\varphi}, \) where \( \tilde{\varphi} \in \tilde{m}^{(j+1)} \Omega^1 \). Therefore, consider \( \varphi = x_q x_s^{(2)} \, dx_i \), where \( i, q \in I \). By virtue of the Lemma 3.5 it is sufficient to follow only the action of \( \sigma \in G' \) on \( \omega(0) \).

(1) Let \( m_q > 1 \). Take \( \sigma \in G'_j \) such that \( \sigma x_i = x_i + x_q x_s^{(2)} \) and if \( \bar{a}_{ii} = a_{i+i+1} \), then \( \sigma x_{i+1} = x_{i+1} + x_q x_s^{(2)} \). Note that for \( D \in g_j' \) associated with \( \sigma \) by the formula (3.1), \( D \omega(0) = x_q x_s^{(2)} \, dx_i \). By corollary 3.2 \( \sigma \omega(0) = \omega(0) + d\varphi + x_q x_s^{(2)} \, dx_i + x_q x_s^{(2)} \, dx_s \). By the corollary 3.2 we obtain

\[
\sigma_1 \sigma_2 \omega(0) = \omega(0) + d(x_q x_s^{(2)} \, dx_i) + d(x_q x_s^{(2)} \, dx_s) + d\tilde{\psi} =
\]

\[
\omega(0) + d\varphi + d(x_q x_s^{(2)} \, dx_i) + d\tilde{\psi}
\]

where \( \tilde{\psi} \in \tilde{m}^{(2)} \Omega^1 \) by Lemma 3.5. Now

\[
d(x_q x_s^{(2)} \, dx_k) = x_s^{(2)} \, dx_kdxq + x_q x_s^{(2-1)} \, dx_kdxs =
\]

\[
d(x_k x_s^{(2)} \, dx_q) + x_k x_q x_s^{(2-1)} \, dx_s
\]

where \( x_k x_q x_s^{(2-1)} \, dx_s \in \tilde{m}^{(j+1)} \Omega^1 \), except for \( t = 0, \, s \in I \). That is \( \varphi = x_s x_q \, dx_i \), where \( s \in I \). The previous case holds for \( x_k x_s^{(2)} \, dx_q \). If \( k = s \), then \( d(x_k x_s^{(2)} \, dx_q) = x_s^{(2)} \, dx_d dx_s = d(x_s^{(2+1)} \, dx_q) \), and \( x_s^{(2+1)} \, dx_q \in \tilde{m}^{(j+1)} \Omega^1 \), except for \( t = 0 \).

Now consider \( \varphi = x_s x_q \, dx_i \), where \( i, q, s \in I \). If \( m_q > 1 \), then, as we have already found out, \( \varphi \) can be replaced with \( x_s x_q^{(2)} \, dx_s \). If \( m_q > 1 \), then similarly \( x_s x_q^{(2)} \, dx_s \) is replaced by \( x_s^{(2)} x_q^{(2)} \, dx_s \). If \( m_s = 1 \), then replace \( x_s x_q^{(2)} \, dx_s \) with \( x_s x_q^{(2)} \, dx_d \) and \( d(x_s x_q^{(2)} \, dx_d) = x_s^{(2)} \, dx_d dx_s = d(x_q^{(3)} \, dx_q) \), where \( x_q^{(3)} \, dx_q \in \tilde{m}^{(j+1)} \Omega^1 \). If \( m_q = m_s = 1 \) but there is \( k \in I \) with \( m_k > 1 \) then replace \( x_s x_q \, dx_k \) with \( x_s x_q \, dx_k \) and \( d(x_s x_q \, dx_k) = x_s dx_d dx_q dx_k + x_q dx_d dx_k = d(x_k x_s dx_q + x_k x_q dx_k) \). The case \( \varphi = x_k x_s dx_q \), where \( m_k > 1 \) has already been considered.

27
The remaining case is \( \varphi_1 = x_q^{(2)} dx_i, q, i \in I \). Then \( m_q > 1 \) and \( d\varphi = x_q dx, dx_q = d(x_q x, dx_q) \), which reduces to the previous case.

(2) Let \( m_r = 1 \) for all \( r \in I \). Since \( d\varphi \neq d\tilde{\varphi} \) for \( \tilde{\varphi} \in \tilde{m}^{(2)}SO^1 \), by virtue of the \( \text{(3.2)} \) the degree of \( d\varphi \) cannot be raised or lowered by transformations that do not change the cohomology class \( \omega \). The only \( \sigma \in G' \) that does not lower the degree of \( bd\varphi \) results in \( b^2_s x_s^{(2+1-1)} dx_q dx_s \) (see corollary \( \text{(3.2)} \)).

If \( s \notin I \), then using the chain of transformations, change \( d(x_q x_q^{(2)} dx_i) \) by \( b^{2m_s-i+1} dz_q dz_s \). If \( s \in I \), then by Lemma \( \text{(3.1)} \) there is \( \sigma \in G' \) such that \( \sigma \omega(0) = \omega(0) + d(x_s x_q dx_i) + b^2 dz_q dz_s \).

As a result, we have a transformation \( \sigma \in G' \) that translates \( \omega \) to \( \omega(0) + d\tilde{\psi} + \eta \) in the case of (1) and translates \( \omega \) to \( \omega(0) + d\tilde{\psi} + \eta + \sum \text{cd} dz_q dz_s \) in the case of (2). Now, if necessary, using Lemma \( \text{(3.2)} \) raise the degree \( \tilde{\psi} \) to \( j + 1 \).

**Lemma 3.6.** Let \( \omega, \omega' \) be two non-alternating Hamiltonian forms, and \( \omega - \omega = d\varphi + \eta \), where \( \varphi \in \tilde{m}^{(j+1)}SO^1, j \geq 1, \eta \in H^2(\Omega) \) and \( \lambda(\eta) \in m^{(j+1)} \). Then if for \( \omega \) and \( \omega' \) \( \text{(3.3)} \) or \( \text{(3.4)} \) holds, there exists \( \sigma \in G'_j \) such that \( \omega' - \sigma \omega = d\tilde{\varphi} + \tilde{\eta} \) for some \( \tilde{\varphi} \in \tilde{m}^{(j+2)}SO^1 \) and \( \lambda(\tilde{\eta}) \in m^{(j+2)} \).

**Proof.** We fix \( i \) from the property \( \text{(3.3)} \) in the first case and \( i \in I \) if \( \text{(3.4)} \) is holds. Since \( i_\omega \) is an isomorphism of \( O - \) modules, there is \( D \in g'_j \) such that \( D_{i_\omega} = \lambda(\eta) d i_i \). Proposition \( \text{(3.1)} \) clause guarantees the existence of \( \sigma_1 \in G'_j \) such that \( \text{(3.1)} \) is satisfied. By Lemma \( \text{(3.2)} \) \( \sigma_1 \omega - \omega = d\tilde{\psi} + \nu \) for some \( \tilde{\psi} \in SO^1 \) and \( \nu \in H^2(\Omega) \) such that \( \lambda(\nu) = \tilde{m}^{(j+1)} \), with \( \lambda(\eta) - \lambda(\nu) \in \text{im}^{(j+2)} \). By theorem \( \text{(3.1)} \) we can assume that \( \sigma_1 \omega - \omega = d\tilde{\psi} + \nu \), where \( \tilde{\psi} \in \tilde{m}^{(j+1)}SO^1 \). Then from Proposition \( \text{(3.2)} \) there is \( \sigma_2 \in G'_j \) such that \( \sigma_2 \sigma_1 \omega - \sigma_1 \omega = d\tilde{\psi}_1 \) where \( \tilde{\psi}_1 \in \tilde{m}^{(j+2)}SO^1 \) and \( \tilde{\psi}_1 = D_{j_\omega} = \tilde{\psi}_1 - \tilde{\psi} - \varphi \in \tilde{m}^{(j+2)}SO^1 \), by Lemma \( \text{(3.2)} \). We obtain that \( \omega' - \sigma_2 \sigma_1 \omega = (\omega' - \omega) - (\sigma_2 \sigma_1 \omega - \sigma_1 \omega) = d(\varphi - \tilde{\psi} - \tilde{\psi}_1) + \eta - \nu = d\tilde{\varphi} + \tilde{\eta} \), where \( \lambda(\tilde{\eta}) \in m^{(j+2)} \) and \( \tilde{\varphi} \in \tilde{m}^{(j+2)}SO^1 \).

**Corollary 3.4.** Let \( \omega, \omega' \) be two non-alternating Hamiltonian forms, with \( \omega' - \omega = d\varphi + \eta \), where \( \varphi \in m^{(j+1)}SO^1, j \geq 1, \eta \in H^2(\Omega) \) and \( \lambda(\eta) \in m^{(j+1)} \). Then if for \( \omega \) \( \text{(3.3)} \) holds, there exists \( \sigma \in G'_j \) such that \( \omega' - \sigma \omega = d\tilde{\varphi} + \tilde{\eta} \) for some \( \tilde{\varphi} \in \tilde{m}^{(j+2)}SO^1 \) and \( \lambda(\tilde{\eta}) \in m^{(j+2)} \).

Everywhere in what follows assume that for \( \omega \in SO^2 \) the transformation \( \sigma \in G' \) from the theorem \( \text{3.1} \) is performed.
Proposition 3.3. (1) If \( (3.3) \) is satisfied for a non-alternating Hamiltonian form \( \omega \), then its orbit with respect to \( G'_j \), \( j \geq 1 \) consists of the forms

\[
\{ \omega + d\varphi + \eta \mid \varphi \in m^{(j+1)}S\Omega^1, \eta \in H^2(\Omega), \lambda(\eta) \in m^{(j+1)} \}.
\]

(2) If \( (3.4) \) is satisfied for a non-alternating Hamiltonian form \( \omega \), then in its orbit with respect to \( G'_j \), \( j \geq 1 \) lie the forms

\[
\{ \omega + d\varphi + \eta \mid \varphi \in \tilde{m}^{(j+1)}S\Omega^1, \eta \in H^2(\Omega), \lambda(\eta) \in m^{(j+1)}, \text{ and } (3.4) \text{ holds } \}.
\]

(3) in the orbit of a non-alternating Hamiltonian form \( \omega \) with respect to \( G'_j \), \( j \geq 1 \) lie the forms \( \{ \omega + d\varphi \mid \varphi \in \tilde{m}^{(j+1)s}\Omega^1 \} \).

Proof. The orbit \( G'_j \omega \) is contained in the set of forms specified in (1) by virtue of the Corollary 3.3. Prove that for any non-alternating Hamiltonian form \( \omega \in S\Omega^2 \) and an integer \( k \geq 1 \)

\[
\{ \omega + d\varphi + \eta \mid \varphi \in m^{(k+1)}S\Omega^1, \eta \in H^2(\Omega), \lambda(\eta) \in m^{(k+1)} \} \subseteq G'_k \omega \quad (3.5)
\]

by induction on \( k \) from above. For \( k \) large enough \( m^{(k+1)} = 0 \) and the set on the left of the inclusion \( (3.5) \) consists of one form \( \omega \), so everything is obvious. Let \( (3.5) \) be proved for \( k = j + 1 \), where \( j \geq 1 \). Let \( \varphi \in m^{(j+1)}S\Omega^1 \) and \( \lambda(\eta) \in m^{(j+1)} \). By corollary 3.3 applied to \( \omega \) and \( \omega' = \omega + d\varphi + \eta \), there exists \( \sigma \in G'_j \), \( \tilde{\varphi} \in \tilde{m}^{(j+2)}S\Omega^1 \) and \( \tilde{\eta} \in H^2(\Omega) \), \( \lambda(\tilde{\eta}) \in m^{(j+2)} \) such that \( \omega' - \sigma \omega = d\tilde{\varphi} + \tilde{\eta} \). By induction assumption \( \omega' \in G'_{j+1} \sigma \omega \subseteq G'_j \omega \).

The case (2) is proved similarly if we replace the corollary 3.3 with the Lemma 3.6 and take \( \varphi \in \tilde{m}^{(j+1)}S\Omega^1 \). The case (3) is also proved similarly if we replace the corollary 3.4 with the proposition 3.2 take \( \varphi \in \tilde{m}^{(j+1)}S\Omega^1 \) and \( \eta = 0 \). \( \square \)

Proposition 3.4. Let for non-alternating Hamiltonian forms \( \omega, \omega' \) \( (3.3) \) be fulfilled. Then \( \omega \) and \( \omega' \) are conjugated with respect to \( G' \) if and only if their images coincide in \( S\Omega^2/\text{ms}\Omega^2 \).

Proof. Since the mapping \( S\Omega^2 \to S\Omega^2/\text{ms}\Omega^2 \) equivariant with respect to \( G' \), and \( G' \) acts trivially in the space \( S\Omega^2/\text{ms}\Omega^2 \), then the images of conjugated forms in the space are the same.

Now, let \( \omega, \omega' \) have the same initial terms. In particular, \( \omega' - \omega = d\psi + \eta \), where \( \psi \in \text{ms}\Omega^1 \) and \( \eta \in H^2(\Omega) \). Write \( \psi = \sum_{i,j=1}^{n} b_{ij}x_idx_j + \varphi \), where
\(\varphi \in m^{(2)}S\Omega^1, \ b_{ij} \in K\). Then \(d\psi \equiv \sum_{i,j=1}^n b_{ij}dx_i dx_j \pmod{mS\Omega^2}\). Since the images \(\omega, \omega'\) in \(S\Omega^2/mS\Omega^2\) coincide, \(b_{ij} = b_{ji}\). Since \(\varphi = \psi + d \left( \sum_{i<j} b_{ij}x_i x_j \right) + \sum b_{ij}x_i x_j, \ \omega' - \omega = d\varphi + \eta\), but already \(\varphi \in m^{(2)}s\Omega^1\). It remains to use the case (1) of Proposition 3.3 with the value \(j = 1\).

**Theorem 3.2.** Let \(\omega\) be a non-alternating Hamiltonian form,

\[\omega = \omega(0) + d\varphi + \sum_{i<j} b_{ij}dz_i dz_j,\]

where \(\varphi \in \tilde{m}^{(2)}S\Omega^1\) and \(b_{ij} \in K\). Then

1. if \(3.3\) or \(3.4\) is satisfied, then \(\omega\) is conjugated with respect to \(G'\) to the form \(\omega(0)\).
2. \(\omega\) is conjugated with respect to \(G'\) to the form \(\omega(0) + \sum_{i \in I, j \notin I} b_{ij}dz_i dz_j + \sum_{i<j \in I} b_{ij}dz_i dz_j\).

The theorem follows from Proposition 3.3.

Let \(\omega = \omega(0) + \sum_{i<j} b_{ij}dz_i dz_j\), where \(b_{ij} \in K\) and \(\omega(0)\) have a canonical form, \(3.3\) and \(3.4\) do not hold, that is \(m_i = 1\) for all \(i \in I\) and \(b_{sr} = 0\) for \(s, r \notin I\).

**Remark 3.1.** 1. If \(\omega\) has a summand \(dx_i dx_j, i, j \notin I\), with \(m_i < m_j\) and summands \(b_{si}dz_idz_i + b_{sj}dz_j dz_j\), where \(b_{si}, b_{sj} \neq 0, s \in I\), one can get rid of the last term by replacing the variables \(x_i = x_i + (b_{sj}/b_{si})x_j\), where \(\tilde{b}_{si} = b_{si}\) and \(\tilde{b}_{sj} = b_{sj}\). This replacement does not change the canonical form of \(\omega(0)\).

2. Form \(\omega = \omega(0) + \sum_{i \in I} b_{ij}dz_i dz_j\), where \(j \notin I\), is conjugate to the form \(\omega = \omega(0) + c_{sj}dz_s dz_j\) with respect to \(G'\), where \(s = \min\{i \mid i \in I, b_{ij} \neq 0\}\) and \(c_{sj} = \sum b_{ij}\). Make a transformation \(\sigma\), such that for corresponding

\[D \in g' \ (\text{see 3.1}), \ D\omega = \lambda \left( \sum_{i \neq s} b_{ij}dz_i dz_j \right) dx_s + \sum_i \lambda(b_{ij}dz_s dz_j) dx_i, \ i \in I.\]
3. Form \( \omega = \omega(0) + bdz_jdz_s + bdz_jdz_s \), where \( i, j, s \in I \), is conjugate to the form \( \omega = \omega(0) + bdz_jdz_j \) with respect to \( G' \). Make the transformation \( \sigma \in G' \) such that for corresponding \( D \in g' \) (3.1) \( D_\omega = \lambda(bdz_jdz_s)dx_j + \lambda(bdz_jdz_s)dx_1 + \lambda(bdz_jdz_j)dx_s \).

In all cases, terms of the form \( d\varphi \) are not essential by virtue of the theorem 3.2

4. The simplicity of graded Lie algebras

We write the form in canonical form

\[
\omega = dx_1dx_2 + \ldots + dx_{2r-1}dx_{2r} + \varepsilon_1dx_2^{(2)} + \ldots + \varepsilon_{2r-1}dx_{2r}^{(2)} + dx_{2r+1}^{(2)} + \ldots + dx_n^{(2)},
\]

where \( r \) is an integer satisfying the inequality \( 0 \leq r \leq n/2 \), and \( \varepsilon_j = 0 \) or \( \varepsilon_j = 1 \).

Recall that \( P(n, \overline{m}, \omega) \) is identified with the space \( O(n, \overline{m})/K \) equipped with the Poisson bracket \( \{f, g\} = \sum_{i,j=1}^{n} \pi_{ij} \partial_i f \partial_j g \).

**Theorem 4.1.** If there is \( x_i \) such that \( m_i > 1 \) and \( \overline{a}_{ii} \neq 0 \), then \( P(n, \overline{m}, \omega) \) is a simple Lie algebra of dimension \( 2^m - 1 \). If \( m_i = 1 \) for all \( i \) such that \( \overline{a}_{ii} \neq 0 \), then for \( n > 3 \) or for \( n = 2, 3, \overline{m} \neq \overline{T}, [P(n, \overline{m}, \omega), P(n, \overline{m}, \omega)] \) is a simple Lie algebra of dimension \( 2^m - 2 \).

**Proof.** Denote \( P(n, \overline{m}, \omega) = L \). Let \( L = L_{-1} + L_0 + \ldots + L_r \) be the standard grading of \( L \). Let \( y_i \in L \) be such that \( ad y_i = \partial_i, i = 1, \ldots, n \). Suppose that \( I \) is a nonzero ideal in \( L \) and \( 0 \neq f \in I \). By commuting \( f \) with \( y_i \), we obtain that \( I \cap L_{-1} \neq 0 \). Now \( L_0 \)-module \( L_{-1} \) irreducibility implies that \( L_{-1} \subseteq I \).

Hence we obtain, \( L_i \subset I \) for \( i < r \).

Let \( \delta = (2^{m_1} - 1, \ldots, 2^{m_n} - 1) \). If \( \overline{a}_{kk} \neq 0 \), then \( \{x^{(2)}_k, x^{(\delta)}_k \} = x^{(\delta)} \in I \). We obtain that if there is \( i \) such that \( m_i > 1 \) and \( \overline{a}_{ii} \neq 0 \), then \( I = L \), i.e. the algebra is simple.

Suppose that \( m_i = 1 \) if \( \overline{a}_{ii} \neq 0 \). Then \( x^{(2)}_i \notin L \). Let’s check whether the element \( x^{(\delta)} \) is in \( I \). To do this, we prove that for any monomials \( f = x^{(\alpha)}_s g = x^{(\beta)}_k, \partial_s f \partial_k g + \partial_k f \partial_s g \neq x^{(\delta)} \).

\[
\partial_s f \partial_k g = x^{(\alpha - \varepsilon_s)}_s x^{(\beta - \varepsilon_k)}_k = \prod_{j \neq s, k} \left( \frac{\alpha_j + \beta_j}{\alpha_j} \right) \left( \frac{\alpha_k + \beta_k - 1}{\alpha_k} \right) \left( \frac{\alpha_s + \beta_s - 1}{\alpha_s - 1} \right) x^{(\delta)},
\]

(4.2)
that is \( \alpha_j + \beta_j = 2^{m_j} - 1, \alpha_k + \beta_k - 1 = 2^{m_k} - 1 \) and \( \alpha_s + \beta_s - 1 = 2^{m_s} - 1 \). Obviously, \( \binom{2^{m_j} - 1}{\alpha_j} = \binom{2^{m_k} - 1}{\alpha_k} = \binom{2^{m_s} - 1}{\alpha_s - 1} = 1 \). Therefore, 
\[
\partial_s f \partial_k g + \partial_k f \partial_s g = (1 + 1)x^{(\delta)} = 0.
\]
From here we get that
\[
\{ f, g \} = \sum_{s,j=1}^n \alpha_{s,j} \partial_s f \partial_j g = \sum_{\alpha_{i,j} \neq 0} \partial_s f \partial_s g,
\]
and since \( m_i = 1 \), either \( \partial_s f = 0 \), or \( \partial_s g = 0 \), or \( x_i^2 = 0 \) occurs. Thus, \( x^{(\delta)} \notin I \). So \( L \) is not a simple algebra and \( [L, L] = \langle x^{(\alpha)}, \alpha \neq \delta \rangle \).

Let now \( I \) is a non-zero ideal of \( [L, L] \). We have \( L_i \subset I \) for \( i < r - 1 \). If \( n > 3 \), then there are different \( x_i x_k, x_{i+1} x_k \in I \), where \( i \) is odd. It follows from (4.1) that
\[
\{ x^{(\delta - \epsilon_i)}, x_{k} x_i \} = x_{i} x^{(\delta - \epsilon_i)} + x_{i+1} x^{(\delta - \epsilon_{i+1})},
\]
\[
\{ x^{(\delta - \epsilon_i)}, x_{k} x_{i+1} \} = x_{i+1} x^{(\delta - \epsilon_{i+1})} + x_{i+1} x^{(\delta - \epsilon_i)}
\]
and \( \alpha_{i}, \alpha_{i+1,i+1}, \alpha_{i,i+1} \) are not simultaneously equal zero and one. Thus, \( I = [L, L] \).

If \( n = 3 \) and \( \overline{m} \neq \overline{1} \), then \( x_1^{(\delta_1)} x_2^{(\delta_2)} = \{ x_1^{(\delta_1)}, x_2^{(\delta_2)} x_3 \}, x_1^{(\delta_1-1)} x_2^{(\delta_2)} x_3 = \{ x_1^{(\delta_1-1)} x_2^{(\delta_2)}, x_3, x_1 x_2 \} \) and
\[
x_1^{(\delta_1)} x_2^{(\delta_2-1)} x_3 = \{ x_1^{(\delta_1)} x_2^{(\delta_2-1)} x_3, x_1 x_2 \} + a_{22} x_1^{(\delta_1-1)} x_2^{(\delta_2)} x_3
\]
. That is \( I = [L, L] \).

If \( n = 2 \) and \( m_2 \neq 1 \), then
\[
x_2^{(\delta_2)} = \{ x_1 x_2^{(\delta_2-1)}, x_2^{(2)} \}
\]
and \( x_1 x_2^{(\delta_2-1)} = \{ x_1 x_2^{(\delta_2-1)}, x_2 x_2 \} + x_2^{(\delta_2)} \). That is \( I = [L, L] \). Hence, \( [L, L] \) is a simple algebra of dimension \( 2^m - 2 \).

\[\square\]

5 The invariance of the standard filtration

Let \( P^{(1)}(n, m, \omega) \subseteq L \subseteq \tilde{P}(n, m, \omega) \), i.e. \( L \) be a non-alternating Hamiltonian Lie algebra, \( \{ L_i \} \) standard filtering and \( L = \text{gr} L = \bigoplus L_i \) its associated graded Lie algebra, which is a non-alternating Hamiltonian Lie algebra

32
corresponding to the form $\omega(0)$. $\tilde{P}(n, m, \omega)$ is identified with the algebra $O(\mathfrak{F})/K$, and $P(n, m, \omega)$ is identified with the algebra $O(\mathfrak{F})/K$ equipped with Poisson bracket. In what follows we assume that

\begin{equation}
\begin{aligned}
n &> 4 \\
n &> 2, 3 \text{ and } m_i > 1, m_j > 1 \text{ for } i \neq j, \text{ or } \\
n &> 4 \text{ and } m_i > 1, m_j > 1, m_k > 1 \text{ for various } i, j, k.
\end{aligned}
\end{equation}

Further, in the graded case, we will use the dual form $\varpi(x, y)$ on $E$, which coincides with the Poisson bracket $\{x, y\}$.

Similarly [16] we introduce an invariant set

$$\mathfrak{N}(\mathfrak{L}) = \{D \in \mathfrak{L} \mid (\text{ad } D)(\text{ad } D') \text{ nilpotent for any } D' \in \mathfrak{L}\}.$$ 

Next we need a number of lemmas proved for the classical Hamiltonian case in [16], Chapter III.

**Lemma 5.1.** For $\mathfrak{N}(\mathfrak{L})$ the following inclusions $\mathfrak{L}_2 \subset \mathfrak{N}(\mathfrak{L}) \subset \mathfrak{L}_0$ are fulfilled.

**Proof.** If $D \in \mathfrak{L}_2$ and $D' \in \mathfrak{L}$, then $(\text{ad } D)(\text{ad } D') \mathfrak{L}_j \subset \mathfrak{L}_{j+1}$ for all $j$, whence the nilpotence of the operator $(\text{ad } D)(\text{ad } D')$ follows, that is $\mathfrak{L}_2 \subset \mathfrak{N}(\mathfrak{L})$.

Suppose now that $D \notin \mathfrak{L}_0$. For $D' \in \mathfrak{L}_1$ and $j \geq -1$, the operator $(\text{ad } D)(\text{ad } D')$ induces a linear transformation of the space $L_j = \mathfrak{L}_j/\mathfrak{L}_{j+1}$, which is nilpotent if only $D \in \mathfrak{N}(\mathfrak{L})$. Therefore, when checking the inclusion of $\mathfrak{N}(\mathfrak{L}) \subset \mathfrak{L}_0$, we can proceed to the associated graded algebra.

Thus, it is enough to show that for each $0 \neq D \in L_{-1}$ there exist $D' \in \mathfrak{L}_1$ and $D'' \in \mathfrak{L}_j$ for suitable $j$, that $[D, [D', D'']] = \lambda D''$, where $\lambda \neq 0$, i.e. the operator $(\text{ad } D)(\text{ad } D')$ is nilpotent on $L_j$. Let $D = x$, where $0 \neq x \in E$.

Consider first the case of $n > 2$. If $\{x, y\} = 1$ and there are $x' \neq y' \in \langle x \rangle^\perp$ such that $\{x', y'\} = 1$, $\{x', x'\} = 0$, then in this case $D' = xx'y'$, $D'' = x'$ and $\{x, xx'y', x'\} = \{x, xx'\} = x'$. Otherwise, we have different vectors $x', y' \in \langle x \rangle^\perp$ of unit length. Suppose $D' = xx'y'$, $D'' = x' + y'$, then $\{x, xx'y', x' + y'\} = \{xx'y' + xx', x'\} = x' + y'$. If $\{x, x\} = 0$, then take $y \in E$ such that $\{x, y\} = 1$. For $n > 3$ take different $x', y' \in \langle x, y \rangle^\perp$ and if $\{x', y'\} = 1$, $\{x', x'\} = 0$, then $D' = yx'$, $D'' = x'$ and $\{x, yy'y', x'\} = \{x, yx'\} = x'$ if $\{x', x'\} = \{y, y'\} = 1$ then $D' = yx'y'$, $D'' = x' + y'$ and $\{x, yy'y', x' + y'\} = \{x, yy' + yx'\} = x' + y'$. For $n = 3$ we have $z \in \langle x, y \rangle^\perp$. If $m_z > 1$, then $\{x, yz^{(2)}, z\} = \{x, yz\} = z$. If $m_z = 1$, then
$m_y > 1$ and $\{x, \{xy^{(2)}, x\}\} = \{x, xy\} = x$. The remaining case is $n = 2$. If $\{x, x\} = 0$, $\{x, y\} = 1$, then $\{x, \{xy^{(2)}, x\}\} = \{x, xy\} = x$. If $\{x, x\} = 1$, then $\{x, \{x^{(3)}, x\}\} = \{x, x^{(2)}\} = x$.

Lemma 5.2 (16). Suppose that $\mathcal{M} \subset \mathcal{L}$ is a subalgebra such that $\mathcal{M} + \mathcal{L}_0 = \mathcal{L}$ and $\mathcal{M} \supset \mathcal{L}_2$. Then $\mathcal{M}$ contains a nonzero ideal of the algebra $\mathcal{L}$. □

Lemma 5.3. Let $V$ be a vector space, $U$ its own nonzero subspace, $G$ Lie algebra of linear transformations of $V$ and $N_G(U)$ subalgebra of those linear transformations from $G$, with respect to which $U$ is stable. Let’s put $l = \text{codim}_V U$ and $t = \text{codim}_G N_G(U)$.

(1) Let $G = s(V, b)$ be the Lie algebra of transformations preserving the nondegenerate non-alternating symmetric bilinear form $b$. Then $l + t \geq n$, and equality is achieved only at $l = 1$.

(2) Let $G = s(V, b)^{(1)}$. Then $l + t > n$ except for the following cases

a. if $l = 1$ and $U \cap U^{\perp} = 0$, then $l + t = n$;

b. if $l = 1$ and $U^{\perp}$ is isotropic, then $l + t = n - 1$;

c. if $l = 2$, $n = 3$ and $U$ is isotropic, then $l + t = n$;

d. if $l = 2$, $n = 5$ and $\dim U \cap U^{\perp} = 2$, then $l + t = n$;

e. if $l = 2$, $n = 4$ and $U$ is totally isotropic, then $l + t = n - 1$;

f. if $l = 3$, $n = 6$ and $U$ is totally isotropic, then $l + t = n$.

Proof. Let’s put $r = \dim U \cap U^{\perp}$. Then $r \leq \dim U = n - l$ and $r \leq \dim U^{\perp} = l$. For $A \in \text{gl}(V)$, denote by $b_A$ the bilinear form of $V$ defined by the rule $b_A(u, v) = b(Au, v)$, where $u, v \in V$. Matching $A \mapsto b_A$ gives a linear isomorphism $\text{gl}(V)$ to the space of all bilinear forms on $V$. The transformation $A$ preserves the form $b$, that is $A \in s(V, b)$, if and only if $b_A$ is symmetric, from where $s(V, b) \cong s(V)$ follows, where $s(V)$ is the space of all symmetric bilinear forms on $V$. Since $b$ is non-degenerate, the vector $u \in V$ lies in $U$ if and only if $b(u, v) = 0$ for all $v \in U^{\perp}$. Hence, $U$ is stable with respect to $A$ if and only if $b(Au, v) = 0$, i.e. when $b_A(u, v) = 0$, for all $u \in U$ and $v \in U^{\perp}$. We denote by $T$ a subspace in $s(V)$ consisting of those bilinear forms with respect to which $U$ and $U^{\perp}$ are orthogonal to each other.
Then \( N_{s(V,b)}(U) \cong T \) and we see that in the case of (1) \( t = \text{codim}_{s(V)} T = l(n - l) - \left( \begin{array}{c} r \\ 2 \end{array} \right) \). Thus,

\[
 l + t - n = (l - 1)(n - l) - \frac{1}{2}r(r - 1) .
\]

(5.2)

Using inequalities \( r \leq l \) and \( r \leq n - l \), we obtain \( r(r - 1) \leq (l - 1)(n - l) \) and \( l + t - n \geq \frac{1}{2}(l - 1)(n - l) \). The last expression is positive if \( l > 1 \). If \( l = 1 \), then \( r = 1 \) or \( r = 0 \), and the right part of the formula (5.2) vanishes.

Now let \( G = s(V,b)^{(1)} \cong s(V)^{(1)} = sk(V) \), where \( sk(V) \) is the space of all skew-symmetric bilinear forms on \( V \). We denote by \( T' \) a subspace in \( sk(V) \) consisting of those bilinear forms with respect to which \( U \) and \( U^\perp \) are orthogonal to each other. Then \( t = \text{codim}_{sk(V)} T' = l(n - l) - \left( \begin{array}{c} r \\ 2 \end{array} \right) - r \) and

\[
 l + t - n = (l - 1)(n - l) - \frac{1}{2}r(r + 1) .
\]

(5.3)

Using the inequality \( r \leq l \) and \( r \leq n - l \), we get \( r(r + 1) \leq (l - 1)(n - l) + 2(n - l) \) and \( l + t - n \geq \frac{1}{2}(l - 1)(n - l) - (n - l) = \frac{1}{2}(l - 3)(n - l) \). The last expression is positive if \( l > 3 \). If \( l = 1 \), then at \( r = 0 \) the formula (5.3) vanishes, and at \( r = 1 \) it takes the value \(-1\). If \( l = 2 \), the formula (5.3) is \( l + t - n = (n - 2) - \frac{1}{2}r(r + 1) \). At \( r = 0 \) we get \( l + t - n > 0 \). For \( r = 1 \) we get \( l + t - n = n - 3 \) and, since \( n \geq l + r = 3 \), the expression is positive except for the case \( n = 3 \). At \( r = 2 \) we get \( l + t - n = n - 5 \) and, since \( n \geq 4 \), the expression is positive except for the case \( n = 4, 5 \). If \( l = 3 \), the formula (5.3) is \( l + t - n = 2(n - 3) - \frac{1}{2}r(r + 1) \). For \( r = 0, r = 1 \) or \( r = 2 \) we get \( l + t - n > 0 \). For \( r = 3 \) we get \( l + t - n = 2n - 12 \) and, since \( n \geq 6 \), the expression is positive except for \( n = 6 \).

The following theorem corresponds to Proposition 1.4 of Chapter III, [16] for the classical Hamiltonian case.

**Theorem 5.1.** Let \([5.1]\) conditions be met. Then \( \mathcal{L}_0 \) is the only subalgebra of the smallest codimension among all subalgebras of the Lie algebra \( \mathcal{L} \) containing \( \mathfrak{h}(\mathcal{L}) \) but not containing nonzero ideals of the algebra \( \mathcal{L} \). In particular, \( \mathcal{L}_0 \) is an invariant subalgebra in \( \mathcal{L} \).

**Proof.** Let \( \mathcal{M} \subset \mathcal{L} \) be a subalgebra that contains \( \mathfrak{h}(\mathcal{L}) \), and therefore \( \mathcal{L}_2 \), but does not contain nonzero ideals of the algebra \( \mathcal{L} \). Denote \( M = \text{gr} \mathcal{M} = \)
Assume that \( \mathcal{M} \neq \mathcal{L}_0 \). Since \( \dim L_i M = \dim_x \mathcal{M} \), it is sufficient to show that

\[
\dim L_i M = \sum \dim(L_i/M_i) > n = \dim_x \mathcal{L}_0 = \dim L_{-1}.
\]

By virtue of Lemma 5.2, \( \mathcal{M} + \mathcal{L}_0 \neq \mathcal{L} \), where \( M_{-1} \neq L_{-1} \). Put \( l_i = \dim(L_i/M_i) \).

If \( n = 2 \), then \( m_i > 1 \) for all \( i \) and \( L_0 = s(L_{-1}, \omega(0)) \). Then by Lemma 5.3, (case 1) \( l_{-1} + l_0 \geq n \), the inequality being strict if only \( l_{-1} > 1 \). Let \( x \notin M_{-1} \), \( x \in E \), then \( x^{(3)} \notin M_1 \) or \( y^{(2)}x \notin M_1 \) for some \( y \in M_{-1} \), hence \( l_{-1} + l_0 + l_1 > n \).

Suppose \( n > 2 \). Denote \( L_0 = s(L_{-1}, \omega(0)) \). Then \( l_{-1} + l_0 \geq l_{-1} + + \dim L_0/(L_0 \cap M_0) > n \) is fulfilled except for the cases listed in the Lemma 5.3. Denote \( U = M_{-1} \). Suppose first that \( l_{-1} = 1 \) and \( x \notin U \). The induced bilinear form on \( U' = U \cap \langle x \rangle^\perp \) is nondegenerate and \( n - 2 \leq \dim U' \leq n - 1 \).

If \( n \geq 6 \) or \( n = 5 \), \( \dim U' = n - 1 \) then there are \( x_1, y_1, x_2, y_2 \in U' \), the subspace \( \langle x_1, y_1 \rangle \) is orthogonal to the subspace \( \langle x_2, y_2 \rangle \) and \( \{x_i, y_i\} = 1 \) or \( \{x_i, x_i\} = \{y_i, y_i\} = 1, i = 1, 2 \). Then \( x_1y_1x \) and \( x_2y_2x \) are linearly independent modulo \( M_1 \) and do not belong to \( M_1 \). If \( n = 5 \) and \( \dim U' = 3 \) then there are \( x_1, y_1, x_2 \in U' \), the subspace \( \langle x_1, y_1 \rangle \) is orthogonal to the subspace \( \langle x_2 \rangle \), \( \{x_i, y_i\} = 1 \) or \( \{x_i, x_i\} = \{y_i, y_i\} = 1 \) and \( \{x_2, x_2\} = 1 \). Then \( x_1y_1x \) and \( x_2y_2x \) are linearly independent modulo \( M_1 \). If \( n = 4 \), then there are three elements of height greater than 1. Then for \( x_1, y_1 \in U' \) we have \( m_{x_1} > 1 \). Hence \( x_1^{(2)} \) and \( x_1y_1x \) are linearly independent modulo \( M_1 \).

So \( l_{-1} = 2 \) or \( l_{-1} = 3 \). Let us first consider the case \( n = 6 \), and \( U \) is totally isotropic and has a dimension of 3. Then we have elements \( x_1, x_2, x_3 \in U \) and \( y_1, y_2, y_3 \in E \) such that \( \{x_i, y_j\} = \delta_{ij}, i, j = 1, 2, 3 \). Hence \( y_1y_2y_3 \notin M_1 \).

Now let \( n = 5 \) and \( \dim U \cap U^\perp = 2 \). We have \( x_1, x_2, x_3 \in U \) and \( y_1, y_2 \in E \) such that \( \{x_i, y_j\} = \delta_{ij}, i = 1, 2, 3, j = 1, 2 \). Hence \( y_1y_2x \notin M_1 \). In the case of \( n = 4 \) we have \( U \) is totally isotropic and has dimension 2. Let \( x_1, x_2 \in U \) and \( y_1, y_2 \in E \) be such that \( \{x_i, y_j\} = \delta_{ij}, i, j = 1, 2 \). Then \( m_{y_1} > 1 \) and \( y_1^{(3)} \) and \( y_1^{(2)} \) are linearly independent modulo \( M_1 \) elements. The remaining case is \( n = 3 \) and \( U \) isotropic and one-dimensional. Let \( x_1 \in U \) and \( y_1, y_2 \in E \) be such that \( \{x_i, y_j\} = \delta_{ij}, j = 1, 2 \). Then \( m_{y_1} > 1 \) and \( y_1^{(3)} \notin M_1 \).

Thus, in each case, \( l_{-1} + l_0 + l_1 > n \), i.e. \( \dim_x \mathcal{M} > \dim_x \mathcal{L}_0 \).

Since the natural filtration of \( \mathcal{L} \) is defined uniquely by subalgebra \( \mathcal{L}_0 \) the following statement is obvious.

**Corollary 5.1.** If conditions 5.1 hold, then the natural filtration of \( P(n, \mathfrak{m}, \omega) \) is invariant.
Let \( P^{(1)}(\mathcal{F}, \omega) \subseteq \mathcal{L} \subseteq \tilde{P}(\mathcal{F}, \omega) \) and \( P^{(1)}(\mathcal{F}', \omega') \subseteq \mathcal{L}' \subseteq \tilde{P}'(\mathcal{F}', \omega') \). If \( \varphi: \mathcal{L} \to \mathcal{L}' \) is an isomorphism, then the theorem 5.1 implies that \( \varphi(\mathcal{L}_0) = \mathcal{L}_0' \), and the embedding theorem \([18]\) implies that \( \mathcal{F} = \mathfrak{mathscr{F}}' \) and \( \varphi \) is induced by the automorphism \( \mathcal{O}(\mathcal{F}) \to \mathcal{O}(\mathcal{F}') \), which is also denoted by \( \varphi \). According to Theorem 2.3 \( \varphi(\omega) = \omega' \). Now let \( \omega = \omega(0), \omega' = \omega'(0) \), i.e. \( \mathcal{L} = \operatorname{gr} \mathcal{L}, \mathcal{L}' = \operatorname{gr} \mathcal{L}' \) be graded Lie algebras. Then \( \operatorname{gr} \varphi \) is the homogeneous isomorphism induced by the linear isomorphism of the spaces \( E \) and \( E' \), \( \operatorname{gr} \varphi(\mathcal{F}) = \mathcal{F}' \). Since \( L_0 \)-module \( L_{-1} \) is absolutely irreducible, it follows from Schur’s Lemma that \( \operatorname{gr} \varphi(\omega(0)) = \omega'(0) \).

**Theorem 5.2.** Non-alternating Hamiltonian Lie algebras \( \mathcal{L} \) and \( \mathcal{L}' \) are isomorphic if and only if there is an admissible isomorphism \( \varphi: \mathcal{O}(\mathcal{F}) \to \mathcal{O}(\mathcal{F}') \) such that \( \varphi(\omega) = \omega' \). In particular,

\[
\text{Aut}(\mathcal{L}) \cong \{ \varphi \in \text{Aut}(\mathcal{O}(\mathcal{F})) \mid \varphi \text{ is admissible and } \varphi(\omega) = \lambda \omega, \lambda \in K \}.
\]

If the graded non-alternating Hamiltonian Lie algebras are isomorphic, then there is an admissible linear isomorphism \( \varphi \), such that \( \varphi(\mathcal{F}) = \mathcal{F}' \) and the corresponding non-alternating bilinear forms have the same invariants.

The following examples show that in exceptional cases nontrivial filtered deformations of non-alternating Hamiltonian Lie algebras are possible. In particular, the constructed algebras are new simple filtered Lie algebras of characteristic 2. The authors plan to consider exceptional cases in future publications. Below we announce some results on non-alternating Hamiltonian Lie algebras corresponding to the forms

\[
\begin{align*}
\omega_1 &= dx_1 dx_2 + dx_3^{(2)} + a x_1 x_2 dx_1 dx_2 + b x_1 x_3 dx_1 dx_3 + c x_2 x_3 dx_2 dx_3, \\
\omega_2 &= dx_1^{(2)} + dx_2^{(2)} + dx_3^{(2)} + a x_1 x_2 dx_1 dx_2 + b x_1 x_3 dx_1 dx_3 + c x_2 x_3 dx_2 dx_3, \\
\omega_3 &= dx_1 dx_2 + dx_2^{(2)} + dx_3^{(2)} + a x_1 x_2 dx_1 dx_2 + b x_1 x_3 dx_1 dx_3 + c x_2 x_3 dx_2 dx_3.
\end{align*}
\]

**Theorem 5.3.** If either \( b \neq 0 \) or \( c \neq 0 \) or \( m_3 > 1 \) then \( P(3, m, \omega_1) \) is a simple Lie algebra of dimension \( 2^m - 1 \). If \( b = c = 0, m_3 = 1 \) and \( (m_1, m_2) \neq (1, 1) \), \([P(3, m, \omega_1), P(3, m, \omega_1)]\) is a simple Lie algebra of dimension \( 2^m - 2 \).

**Theorem 5.4.** \( P(3, m, \omega_2) \) is a simple Lie algebra of dimension \( 2^m - 1 \) for \((a, b, c) \neq (t, t, t), t \in K \) or for \( m \neq 0 \).
Theorem 5.5. If either \( (m_1, m_3) \neq (1, 1) \), or \( b \neq 0 \) or \( a \neq c \), then \( P(3, m, \omega_3) \) is a simple Lie algebra of dimension \( 2^m - 1 \). If \( a = c, b = 0, (m_1, m_3) = (1, 1) \) and \( m_2 \neq 1 \), then \([P(3, m, \omega_3), P(3, m, \omega_3)]\) is a simple Lie algebra of dimension \( 2^m - 2 \).
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