Quadratic gravity in first order formalism

Enrique Alvarez, Jesus Anero and Sergio Gonzalez-Martin

Departamento de Física Teórica and Instituto de Física Teórica (IFT-UAM/CSIC),
Universidad Autónoma de Madrid,
Cantoblanco, 28049, Madrid, Spain
E-mail: enrique.alvarez@uam.es, jesusanero@gmail.com,
sergio.gonzalez.martin@uam.es

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Abstract. We consider the most general action for gravity which is quadratic in curvature. In this case first order and second order formalisms are not equivalent. This framework is a good candidate for a unitary and renormalizable theory of the gravitational field; in particular, there are no propagators falling down faster than $\frac{1}{p^2}$. The drawback is of course that the parameter space of the theory is too big, so that in many cases will be far away from a theory of gravity alone. In order to analyze this issue, the interaction between external sources was examined in some detail. We find that this interaction is conveyed mainly by propagation of the three-index connection field. At any rate the theory as it stands is in the conformal invariant phase; only when Weyl invariance is broken through the coupling to matter can an Einstein-Hilbert term (and its corresponding Planck mass scale) be generated by quantum corrections.

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1 Introduction

It is well-known that general relativity is not renormalizable (cf. [1, 2] and references therein for a general review). However, quadratic (in curvature) theories are renormalizable, albeit not unitary [3–5] — at least in the standard second order formalism — although they have been widely studied over the years [6, 7]. When considering the Palatini version of the Einstein-Hilbert lagrangian the connection and the metric are treated as independent variables and the Levi-Civita connection appears only when the equations of motion are used.

It is however the case that when more general quadratic in curvature metric-affine actions are considered in first order formalism the deterministic relationship between the affine connection and the Levi-Civita one is lost, even on shell. That is, the equations of motion do not force the connection to be the Levi-Civita one.

This is quite interesting because it looks as if we could have all the goods of quadratic lagrangians [3–5] (mainly renormalizability) without conflicting with Källen-Lehmann’s spectral theorem.
The purpose of the present paper is to explore the most general first order Weyl invariant quadratic lagrangian. By considering all possible monomials of a given symmetry the system is closed under renormalization in background field gauge. This work is a continuation of [8](cf. also [9, 10]), whose conventions we follow here.

A general issue when considering first order versus second order theories is that in general the manifold of solutions in the first order treatment is too big. This means in our case that in many situations we are not dealing with a theory of gravity. One of our aims in this paper is to analyze the properties that physical sources need to have in order to reproduce a proper gravitational potential energy between static energy-momentum sources.

Let us now summarize our general framework.

Let us start with some general remarks. An orthonormalized coframe will be characterized by $n$ differential forms

$$e^a \equiv e^a_\mu dx^\mu$$

(1.1)

$a = 1 \ldots n$ are tangent (Lorentz) indices, and $\mu, \nu \ldots$ are spacetime (Einstein) indices. They obey

$$\eta_{ab} e^a_\mu(x) e^b_\nu(x) = g_{\mu\nu}(x)$$

(1.2)

(where $\eta_{ab}$ is the flat metric). Spacetime tensors are observed in the frame as spacetime scalars, id est,

$$V^a(x) \equiv e^a_\mu(x) V^\mu(x)$$

(1.3)

The Lorentz (usually called spin) connection is defined by demanding local Lorentz invariance of derivatives of such scalars as

$$\nabla_\mu V^b \equiv \partial_\mu V^b + \omega^b_{\mu} c V^c$$

(1.4)

Physical consistency demands that the Lorentz and Einstein connections are equivalent, that is,

$$\nabla_a V^b = e^a_\rho V^\rho$$

(1.5)

In this equation we use the spin connection $\omega$ in the left hand side, and the Einstein connection, $\Gamma$ in the right hand side.

It follows that

$$\omega_{abc} = -e^c_\rho \Gamma^\rho_{ac} + \eta_{bd} \Gamma^d_{ac}$$

(1.6)

showing that Lorentz and Einstein connections are equivalent assuming knowledge of the frame field (tetrad).

The Riemann Christoffel tensor is completely analogous to the usual gauge non-abelian field strength. When the metric compatible connection is used, the main difference between the curvature tensor and the non-abelian field strength stems from the torsionless\(^\text{1}\) algebraic Bianchi identity

$$R^a_{\, b} \wedge e^b = 0$$

(1.7)

which is the origin of the symmetry between Lorentz and Einstein indices

$$R_{\alpha\beta\gamma\delta} \equiv e^\alpha_{\alpha} e^b_{\beta} R_{ab\gamma\delta} = R_{\gamma\delta\alpha\beta} \equiv e^c_{\gamma} e^d_{\delta} R_{cd\alpha\beta}$$

(1.8)

This identity does not have any analogue in a non abelian gauge theory in which these two sets of indices remain unrelated. The opposite happens with the differential Bianchi identity

$$dR^a_{\, b} + R^a_{\, c} \wedge \omega^c_{\, b} - \omega^a_{\, c} \wedge R^c_{\, b} = 0$$

(1.9)

\(^{1}\)Torsion could be easily included; we did not do it mainly for simplicity.
which still holds for non-abelian gauge theories when the gauge group is not identified with the tangent group.

The non-metricity tensor (NM) is just the covariant derivative of the metric tensor

\[ \nabla_c \eta_{ab} = -Q_{abc} \]

This vanishing characterizes the Levi-Civita connection, whose components are given by the Christoffel symbols. The symmetric piece of the connection is then precisely

\[ \omega_{ab|c} = Q_{abc} \]

The structure constants of the frame field are defined by

\[ [e_a, e_b] = \left[ e_\mu^a \tilde{\epsilon}_\mu, e_\lambda^b \tilde{\epsilon}_\lambda \right] = C_{ab}^c e_\sigma^c \tilde{\epsilon}_\sigma \equiv C_{abc}^c \]

Indeed, the vanishing of the torsion tensor

\[ de^\alpha + \omega^a_{\ d} \wedge e^b = 0 = \tilde{\epsilon}_{\rho} e^a_{\sigma} - \tilde{\epsilon}_{\sigma} e^a_{\rho} + \omega^a_{\sigma\rho} - \omega^a_{\rho\sigma} \]

yields the missing antisymmetric piece of the Lorentz connection \( \omega_{a|bc} \) (remember that the symmetric piece was determined by the non-metricity)

\[ \omega_{abc} = \omega_{a(cb)} + \omega_{a[bc]} \]

Then

\[ \omega_{[ac]b} + \omega_{[ba]c} + \omega_{[cb]a} = \frac{1}{2} (C_{cab} + C_{bca} + C_{abc}) \]

\[ \omega_{[ab]c} = -\frac{1}{2} C_{abc} \]

The general torsionless connection is then determined in terms of the non-metricity and the structure constants of the frame field. It could be thought that there is some difference between the use of the one forms

\[ \omega_{\mu a} \]

which can be thought of as gauge fields valued on the Lorentz group \( O(1,3) \) or else the three-index objects

\[ \Gamma_{\mu \alpha}^\alpha \]

We think this is not the case, owing to the fact already mentioned, that the Lorentz covariant derivative is the projection of the Einstein covariant derivative.

The gauge field \( \Gamma \in \mathfrak{gl}(n) \). There is a natural mapping between

\[ \mathfrak{gl}(n) \to \mathfrak{sl}(n) \times \mathfrak{R} \]

namely

\[ g \to \left( \hat{g} \equiv (\det g)^{-1/n} g, \quad \det g \right) \]

in such a way that any representation of \( \mathfrak{gl}(n) \) also yields a representation of \( \mathfrak{sl}(n) \times \mathfrak{R} \). The converse is also true. Consider a representation \( D_k \) of \( \mathfrak{R} \)

\[ r \to r^k \]

where \( k \in \mathbb{R} \), and a finite-dimensional representation of \( \mathfrak{gl}(n) \). This is seen to generate a representation of \( \mathfrak{gl}(n) \)

\[ g \to r^k \hat{g} \]
1.1 Analogies with a gauge theory

The fact that the Riemann tensor $R_{\mu\nu ab}$ is quite similar to the gauge field strength, $F_{\mu\nu}$, when viewed as a Lie algebra matrix has been highlighted many times. The thing reads as follows. Were we to contract in the most natural SO\(p\)\(n\) invariant way

$$R^{\mu}_{\quad \nu ab} R^{\nu}_{\quad \mu cd}$$

with

$$g^{abcd} \equiv \left( \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc} \right)$$

the result is not GL\(p\)\(n\) invariant, in spite of the fact that the field A lives in the algebra, $A \in \text{gl}(n)$. The reason is, of course, that $g^{abcd}$ is not proportional to the Killing metric of $\text{gl}(p\mid n)$. The result is only so\(p\)\(n\) invariant.

In a gauge theory with gauge group GL\(p\)\(n\) the first thing that strikes the eye is that the gauge group fails to be compact. There is then the general question as to whether any gauge theory defined with a non-compact version of a given group, is in any sense the analytic continuation of the same theory defined by standard techniques from a compact version of the same group. In [12, 13] evidence is given in the negative, at least for three-dimensional Chern-Simons theories. To be specific, the gl\(p\)\(4\)\(q\) Lie algebra generators are 6 antisymmetric $J_{[\alpha\beta]}$ that generate the so\(p\)\(4\) subgroup, nine traceless symmetric shears $T_{(\alpha\beta)}$ and one dilatation, $T$.

The algebra reads [22]

$$[J_{\alpha\beta}, J_{\gamma\delta}] = i (\delta_{\alpha\gamma} J_{\beta\delta} - \delta_{\alpha\delta} J_{\beta\gamma} - \delta_{\beta\gamma} J_{\alpha\delta} + \delta_{\beta\delta} J_{\alpha\gamma})$$

$$[J_{\alpha\beta}, T_{\gamma\delta}] = i (\delta_{\alpha\gamma} T_{\beta\delta} + \delta_{\alpha\delta} T_{\beta\gamma} - \delta_{\beta\gamma} T_{\alpha\delta} - \delta_{\beta\delta} T_{\alpha\gamma})$$

$$[T_{\alpha\beta}, T_{\gamma\delta}] = i (\delta_{\alpha\gamma} T_{\beta\delta} + \delta_{\alpha\delta} T_{\beta\gamma} + \delta_{\beta\gamma} T_{\alpha\delta} + \delta_{\beta\delta} T_{\alpha\gamma})$$

The so\(3\) \(\oplus\) so\(3\) subalgebra can be highlighted by defining

$$J_1 \equiv J_{23} \quad K_1 \equiv J_{14}$$

$$J_2 \equiv -J_{13} \quad K_2 \equiv J_{24}$$

$$J_3 \equiv J_{12} \quad K_3 \equiv J_{34}$$

as well as

$$J_1^\pm \equiv \frac{J_1 \pm K_1}{2}$$

Then the first line of the algebra collapses to

$$[J_1^+, J_2^+] = i \epsilon_{ijk} J_3^k$$

$$[J_1^+, J_3^+] = 0$$

The symmetric generators (which do not close in a subgroup) can be thought of as $\frac{n(n-1)}{2}$ non diagonal traceless matrices, plus $n-1$ diagonal traceless ones; plus the trace. An explicit representation is

$$(J_{\alpha\beta})_{\rho\sigma} \equiv \delta_{\alpha\rho} \delta_{\beta\sigma} - \delta_{\alpha\sigma} \delta_{\beta\rho}$$

$$(T_{\alpha\beta})_{\rho\sigma} \equiv \delta_{\alpha\rho} \delta_{\beta\sigma} + \delta_{\alpha\sigma} \delta_{\beta\rho}$$

$$(T_{\alpha\beta}^D)_{\rho\sigma} \equiv \delta_{\alpha\rho} \delta_{\alpha\sigma} + \delta_{\beta\sigma} \delta_{\beta\rho}$$

$$T_{\rho\sigma} \equiv \delta_{\rho\sigma}$$
It is a known fact that the Killing form of $\mathfrak{gl}(n)$ is given by

$$B(A, B) \equiv 2 \left( n \text{tr}(AB) - \text{tr}A \text{tr}B \right)$$

(1.30)

When we put indices

$$B_{abcd} \equiv 2 \left( n \delta^{bc} \delta^{ad} - \delta^{ab} \delta^{cd} \right)$$

(1.31)

The generator responsible for the group not being semisimple is just the dilatation

$$T \sim tI$$

(1.32)

because

$$B(T, T) = 0$$

(1.33)

whereas the remaining traceless generators of $\mathfrak{gl}(n)$ are responsible for non-compactness even when the algebra belongs to the first $A_n$-Cartan series.

It is well-known [12, 13, 23] that when the gauge group is non compact (which manifests itself in the Killing metric not being positive definite, actually of signature $\left( \frac{n(n-1)}{2}, \frac{(n-1)(n+2)}{2} \right)$, some analytic continuation is in order (which naively means putting all euclidean signs as +). As pointed out in those references sometimes (like in the Chern-Simons case) even that is not enough and some more elaborate physical analysis is in order.

The relationship between $A_\mu$ and $\omega_\mu$ is

$$\omega_\mu^a_b = \Gamma_\mu^a_b - e_b^\lambda \epsilon^a_\mu \epsilon^b_\lambda$$

(1.34)

The Einstein index in $\Gamma_\mu$ is not the contravariant one, but rather one of the two equivalent covariant ones. It is quite easy to check that

$$\omega_{\mu[ab]} \equiv \Gamma_{[a\mu]b} - e_b^\lambda \epsilon^a_\mu \epsilon^b_\lambda$$

$$\omega_{\mu(ab)} = \Gamma_{\mu(ab)} \equiv \frac{1}{2} \left( \Gamma_{a\mu b} + \Gamma_{b\mu a} \right)$$

(1.35)

where the antisymmetry instruction acts on the $a, b$ indices only. All these results allow to trade the $\mathfrak{so}(n)$ metric for the $\mathfrak{gl}(n)$ metric if so desired. As it stands, the theory is only invariant under an $\mathfrak{so}(n) \subset \mathfrak{gl}(n)$ subgroup, and the full gauge symmetry is broken by the kinetic energy metric.

The absence of torsion implies

$$\Gamma_{a[\mu b]} = \Gamma_{a\mu b}$$

(1.36)

Actually, the difference between an arbitrary connection and the Levi-Civita one

$$A_{\mu}^{a \beta} \equiv \Gamma_{\mu}^{a \beta} - \left\{ \frac{\alpha}{\mu \beta} \right\}$$

(1.37)

is a true tensor, so that

$$A_{\mu}^{a \beta} \equiv e_{a}^{\sigma} A_{\mu}^{a \beta} e_{\beta}^{b}$$

(1.38)

and there is a simple field redefinition between the two languages. The condition we have imposed of absence of torsion has a much simpler expression in spacetime language (where it just states that the connection is symmetric) than in the frame one, where it reads

$$\omega_{u[a\nu} - \omega_{v]a\nu} = e_{v}^{\sigma} \epsilon_{u \alpha \sigma} - e_{u}^{\sigma} \epsilon_{v \alpha \rho}$$

(1.39)
As will be seen soon, the most general lagrangian is a quite complicated one, with 16 independent coupling constants and many possible vacua to consider. In this paper we present the general setup and analyze the response of the flat vacuum to external graviton sources.

To be specific, in the second section we analyze the case of General relativity in the first order formalism. In section three we do a careful study of the independent monomia that can be written with the assumed fields and symmetries, and we find that there are indeed sixteen of them. In the fourth section a background field expansion is performed. Many unwieldy formulas are relegated to an appendix. Then we study the effect of external sources on the system, and we analyze carefully the conditions for this effect to mimic the one of General Relativity in section five. The necessity to break Weyl invariance in order to make contact with phenomenology is emphasized in section six. Finally, we end this work with some conclusions.

A word of warning. We shall still call the metric fluctuations, $h_{\mu\nu}$ graviton fluctuations and the fluctuations of the connection, $A_{\alpha\beta\gamma}$ (three-index) gauge fluctuations, in spite of the fact that both are related to the gravitational field.

2 General relativity

In order to understand the role of external sources in first order formalism, let us consider first the Einstein-Hilbert action (FOEH).

To be specific, we define the action like

$$S_{\text{FOEH}} = -\frac{1}{2\kappa^2} \int d^n x \sqrt{|g|} g^{\mu\nu} R_{\mu\nu}$$  (2.1)

On the one hand, it is well known that the classical equations of motion are equivalent to Einstein’s equations. Our aim here is to understand this from the path integral in the presence of external sources. The first question is, which sources? In principle, we are supposed to assume sources for physical fields only. This would mean to include a source for the graviton field, and not for the connection. We shall come back to that.

2.1 A toy model

In order to understand properly what is going on, let us first consider an ordinary integral that shares most of the features of our path integral, namely,

$$I(j, h) \equiv \int_{-\infty}^{\infty} dx dy \ e^{-nxy - k y^2 - j x - hy}$$  (2.2)

Let us first compute $I(0, 0)$ in two different ways. We shall as usual, define the integrals by analytic continuation from the region where they are convergent. First, complete the square

$$-nxy - k y^2 = -k \left( y + \frac{n x}{2k} \right)^2 - \frac{n^2 x^2}{4k}$$  (2.3)

It follows that

$$I(0, 0) = \sqrt{\frac{\pi}{k}} \int_{-\infty}^{\infty} dx \ e^{-\frac{x^2}{4k}} = \sqrt{\frac{\pi}{k}} \sqrt{\frac{4\pi}{n^2/k}} = \frac{2\pi}{n}$$  (2.4)
A different way to proceed would be to first perform the integral over \( dx \), getting

\[
2\pi\delta(ny) = \frac{2\pi}{n} \delta(y)
\]  
(2.5)

The integral over \( dy \) is now immediate, yielding again

\[
I(0, 0) = \frac{2\pi}{n}
\]  
(2.6)

In the presence of sources, the integral over \( dx \) yields

\[
2\pi\delta(ny + j) = \frac{2\pi}{n} \delta \left( y + \frac{j}{n} \right)
\]  
(2.7)

so that

\[
I(j, h) = \frac{2\pi}{n} e^{-\frac{j\lambda + hj}{n}}
\]  
(2.8)

### 2.2 Einstein-Hilbert in first order

Let us start by analyzing the action \( S_{\text{FOEH}} \) with a graviton source

\[
S_M = -\frac{1}{2} \int d^4x \, \kappa h^{\gamma\epsilon} T_{\gamma\epsilon}
\]  
(2.9)

we expand around Minkowski spacetime as

\[
g_{\mu\nu} \equiv \eta_{\mu\nu} + \kappa h_{\mu\nu} \\
\Gamma^\alpha_{\beta\gamma} \equiv A^\alpha_{\beta\gamma}
\]  
(2.10)

this yields

\[
S_{\text{FOEH}} + S_M = \bar{S}_0 - \int d^4x \left\{ \frac{1}{2} \left( h^{\gamma\epsilon} N_{\gamma\epsilon \lambda} A^\lambda_{\alpha\beta} + A^\lambda_{\alpha\beta} N^{\lambda\beta}_{\gamma\epsilon} h^{\gamma\epsilon} \right) + \frac{1}{2} A^\lambda_{\alpha\beta} K^{\gamma\epsilon}_{\tau \lambda} A^\lambda_{\alpha\beta} + \frac{1}{2} \kappa h^{\gamma\epsilon} T_{\gamma\epsilon} \right\}
\]  
(2.11)

where

\[
\bar{S}_0 = -\frac{1}{2\kappa^2} \int d^4x \sqrt{|g|} R
\]  
(2.12)

and

\[
N_{\gamma\epsilon \lambda} = \frac{1}{2\kappa} \left\{ \frac{1}{2} \left( \eta_{\gamma\epsilon} \eta^{\alpha\beta} - \delta^{\alpha}_\gamma \delta^{\beta}_\epsilon + \eta^{\alpha\beta} \right) \nabla_\lambda - \frac{1}{4} \left( \eta_{\gamma\epsilon} \delta^{\alpha}_\lambda \nabla^{\alpha} - \delta^{\alpha}_\gamma \delta^{\beta}_\epsilon \nabla^\alpha - \delta^{\alpha}_\gamma \delta^{\beta}_\epsilon \nabla_{\gamma} + \eta_{\gamma\epsilon} \delta^{\alpha}_\lambda \nabla^{\beta} - \delta^{\alpha}_\gamma \delta^{\beta}_\epsilon \nabla_{\gamma} - \delta^{\alpha}_\gamma \delta^{\beta}_\epsilon \nabla_{\gamma} \right) \left( \eta_{\gamma\epsilon} \delta^{\alpha}_\lambda \nabla^{\beta} - \delta^{\alpha}_\gamma \delta^{\beta}_\epsilon \nabla_{\gamma} - \delta^{\alpha}_\gamma \delta^{\beta}_\epsilon \nabla_{\gamma} \right) \right\}
\]  
(2.13)

\[
K^{\gamma\epsilon \lambda}_{\tau} = \frac{1}{\kappa^2} \left\{ \frac{1}{4} \left[ \delta^{\alpha}_{\gamma} \delta^{\alpha}_{\tau} h^{\beta\epsilon} - \delta^{\beta}_{\gamma} \delta^{\beta}_{\tau} h^{\alpha\epsilon} - \delta^{\alpha}_{\gamma} \delta^{\alpha}_{\tau} h^{\beta\epsilon} - \delta^{\beta}_{\gamma} \delta^{\beta}_{\tau} h^{\alpha\epsilon} \right] + \delta^{\alpha}_{\gamma} \delta^{\alpha}_{\tau} h^{\beta\epsilon} + \delta^{\beta}_{\gamma} \delta^{\beta}_{\tau} h^{\alpha\epsilon} \right\}
\]  
(2.14)
Let us define as usual
\[ Z[T] = \int \mathcal{D}\varphi \ e^{iS[\varphi] + i \int d^nx \ T(x)\varphi(x)} \] (2.14)
so that the free energy,
\[ e^{iW} = \frac{Z[T]}{Z[0]} \] (2.15)
reads in our case
\[ e^{iW_{FO}[T_{\mu\nu}]} = \int \mathcal{D}h \mathcal{D}A e^{\{-i \int d^nx \ (\frac{1}{2} \left(h^{\gamma\nu} N_{\gamma\nu} + A^\lambda_{\alpha\beta} + k_{\alpha\beta} + \eta^{\alpha\beta} \eta_{\alpha\beta} + \frac{1}{2} A^\gamma_{\alpha\beta} A_{\gamma\alpha\beta} \right) + \frac{1}{2} \delta \gamma \epsilon \ k_{\gamma\epsilon} T_{\gamma\epsilon})\}} \] (2.16)

Our purpose in life is to derive the lowest order interaction between external sources. Notice that the quadratic graviton term
\[ \mathcal{Z} h M_h \] (2.17)
vanishes in our case.

Let us face the consequences of this fact. Integrating over \( \mathcal{D}h \) yields a Dirac delta
\[ \delta \left( \bar{N}_{\gamma\epsilon \alpha\beta} A^\lambda_{\alpha\beta} + \kappa_{\gamma\epsilon} \right) \] (2.18)
we define by \( \bar{A} \) the solution of the equation
\[ \bar{N}_{\gamma\epsilon \alpha\beta} \bar{A}^\lambda_{\alpha\beta} = -\kappa_{\gamma\epsilon} \] (2.19)
This is not an EM for any background field; it is the argument of a Dirac delta function, consequence of having integrated \( \mathcal{D}h_{\mu\nu} \) away.

Then it is clear that (modulo a jacobian independent of the sources) the integral over \( \mathcal{D}A \) yields
\[ W_{FOEH}[T_{\mu\nu}] = -\frac{1}{2} \int d^nx \ \bar{A}^\tau_{\gamma\epsilon} \bar{K}^\gamma_{\tau \lambda} A^\lambda_{\alpha\beta} + \log J \] (2.20)
and this should be proportional to \( W_{SOEH}[T_{\mu\nu}] \) (2.40).

\[ W_{SOEH}[T_{\mu\nu}] = -\int d^nx \ \frac{K^2}{4k^2} T_{\mu\nu} \left( \eta^{\rho\sigma} \eta^{\mu\nu} + \eta^{\mu\sigma} \eta^{\nu\rho} - \eta^{\mu\nu} \eta^{\rho\sigma} \right) T_{\rho\sigma} \] (2.21)

then
\[ \bar{A}^\tau_{\gamma\epsilon} \bar{K}^\gamma_{\tau \lambda} A^\lambda_{\alpha\beta} = \frac{K^2}{2k^2} T_{\mu\nu} \left( \eta^{\rho\sigma} \eta^{\mu\nu} + \eta^{\mu\sigma} \eta^{\nu\rho} - \eta^{\mu\nu} \eta^{\rho\sigma} \right) T_{\rho\sigma} \] (2.22)

Let us now determine \( \bar{A} \). In momentum space (2.19)
\[ \frac{1}{k} \left\{ \frac{1}{2} \eta_{\gamma\epsilon} \left( \eta^{\alpha\beta} k_{\gamma} A^\lambda_{\alpha\beta} - k^\alpha A^\lambda_{\alpha\beta} \right) + \frac{1}{2} \left( k_{\epsilon} A^\lambda_{\gamma\lambda} + k_{\gamma} A^\lambda_{\epsilon\lambda} - 2 k_{\lambda} A^\lambda_{\gamma\epsilon} \right) \right\} = -\kappa_{\gamma\epsilon} \] (2.23)
The integrability condition stemming from conservation of the source
\[ k_{\mu} T^{\mu\nu} = 0 \] (2.24)
necessary for maintaining gauge invariance determines uniquely
\[ \bar{A}^\lambda_{\alpha\beta} = f(k^2) \left[ k^{\lambda} \eta_{\alpha\beta} - \left( \delta^\lambda_{\alpha} k_{\beta} + \delta^\lambda_{\beta} k_{\alpha} \right) \right] \] (2.25)
Assuming, that is, that it depends on the metric and the momentum only. In this same spirit, the source must be

\[ T_{\mu\nu} = -\frac{2-n}{k^2} f(k^2)[k_\mu k_\nu - k^2 \eta_{\mu\nu}] \]  

(2.26)

With these expressions of \( \bar{A}_\alpha^\lambda \) and \( T_{\mu\nu} \), we can work out the equation (2.22)

\[ 2(2-n)(n-1) = (3-n)(2-n)^2(n-1) \]  

(2.27)

This equation admits \( n = 4 \) as a solution.

We can integrate instead over the connection perturbation, yielding

\[ e^{iW_{GF}[T_{\mu\nu}]} = \int \mathcal{D}h e^{-\frac{i}{2} \int d^nx \left( \frac{1}{4} h_{\mu\nu} N_{\mu\nu}^{\alpha\beta}(K^{-1})_{\alpha\beta} T_{\mu\nu}^{\gamma\epsilon} N_{\gamma\epsilon}^{\tau\rho} T_{\tau\rho} - \frac{1}{2} \eta_{\mu\nu} \eta_{\tau\rho} h^{\tau\rho} + \frac{i}{2} \kappa h^{\gamma\epsilon} T_{\gamma\epsilon} \right)} \]

(2.28)

where the graviton operator is

\[ N_{\mu\nu}^{\alpha\beta}(K^{-1})_{\alpha\beta} T_{\mu\nu}^{\gamma\epsilon} N_{\gamma\epsilon}^{\tau\rho} T_{\tau\rho} = \frac{1}{8} \left( \eta_{\mu\nu} \eta_{\tau\rho} + \eta_{\mu\tau} \eta_{\nu\rho} - 2 \eta_{\mu\rho} \eta_{\nu\sigma} \right) - \frac{1}{8} \left( \eta_{\mu\tau} \hat{\epsilon}_{\nu} \hat{\epsilon}_{\sigma} + \eta_{\mu\sigma} \hat{\epsilon}_{\nu} \hat{\epsilon}_{\tau} - 2 \eta_{\mu\sigma} \hat{\epsilon}_{\nu} \hat{\epsilon}_{\tau} + \eta_{\nu\sigma} \hat{\epsilon}_{\mu} \hat{\epsilon}_{\tau} + \eta_{\nu\tau} \hat{\epsilon}_{\mu} \hat{\epsilon}_{\sigma} - 2 \eta_{\nu\sigma} \hat{\epsilon}_{\mu} \hat{\epsilon}_{\tau} \right) \]

(2.29)

It is easy to check that this whole action is invariant under the gauge symmetry

\[ \delta h_{\mu\nu} = \hat{\omega}_{\mu} \xi_{\nu} + \hat{\omega}_{\nu} \xi_{\mu} \]

\[ \delta A_{\beta\gamma}^\mu = \hat{\epsilon}_\beta \xi^\gamma \]

(2.30)

that we need to fix.

Therefore, we still have the freedom to fix the gauge in a way that simplifies the computation. The gauge fixing term will be

\[ S_{gf} = \frac{1}{2} \int d^nx \frac{1}{2\kappa} \eta_{\mu\nu} \chi^\mu \chi^\nu \]

(2.31)

where the function characterizing the harmonic gauge is

\[ \chi^\nu = \hat{\epsilon}_{\mu} T_{\mu}^{\mu\nu} - \frac{1}{2} \hat{\epsilon}^{\nu} h \]

(2.32)

and in the minimal gauge, corresponding to \( \xi = 1 \), the path integral can be rewritten as

\[ e^{iW_{GF}[T_{\mu\nu}]} = \int \mathcal{D}h e^{-\frac{i}{2} \int d^nx \left( \frac{1}{4} h_{\mu\nu} D_{\mu\nu\rho\sigma} h^{\rho\sigma} + \frac{i}{2} \kappa h^{\gamma\epsilon} T_{\gamma\epsilon} \right)} \]

(2.33)

where

\[ D_{\mu\nu\rho\sigma} = \frac{1}{4} \left( \eta_{\mu\lambda} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\lambda} - \eta_{\mu\nu} \eta_{\lambda\sigma} \right) \]

(2.34)

Finally, we can integrate over \( h \)

\[ e^{iW_{GF}[T_{\mu\nu}]} = e^{\frac{i}{4} \int d^nx \left( \frac{1}{4} h_{\mu\nu} D_{\mu\nu\rho\sigma} h^{\rho\sigma} T_{\rho\sigma} \right)} \]

(2.35)

Getting the result,

\[ W_{GF}^{gf}[T_{\mu\nu}] = -\frac{\kappa^2}{4} \int d^nx \ T_{\mu\nu} D_{\mu\nu\rho\sigma} T_{\rho\sigma} \]

(2.36)

It is remarkable that the divergent part also coincides exactly off-shell [25].
2.3 Einstein-Hilbert in second order

Now, we consider the Einstein-Hilbert action in second order. In the same way that before, we perform an expansion around flat space $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$. This reads

$$S_{SOEH} + S_M = -\frac{1}{2} \int d^4x \left\{ \frac{1}{2} \left( \frac{1}{2} \partial_\lambda h^{\mu\nu} \partial_\lambda h_{\mu\nu} - \frac{1}{2} \partial_\lambda h^\lambda h_{\mu\nu} - \frac{1}{2} \partial_\lambda h^\lambda h_{\mu\nu} + \partial_\nu h^\lambda h_{\mu\nu} \right) + \kappa h^{\gamma\epsilon} T_{\gamma\epsilon} \right\}$$

(2.37)

Adding the usual harmonic gauge fixing $\partial_\mu h^\mu = \frac{1}{2} \partial_\nu h^\lambda$, and integrating by parts in (2.37) we get

$$S_{SOEH}^f + S_M = -\frac{1}{2} \int d^4x \left\{ -\frac{1}{4} \left( h^{\mu\nu} \partial^2 h_{\mu\nu} - \frac{1}{2} \partial^2 h \right) + \kappa h^{\gamma\epsilon} T_{\gamma\epsilon} \right\} =$$

$$= -\frac{1}{2} \int d^4x \left\{ -\frac{1}{2} h^{\mu\nu} \left( \eta_{\mu\lambda} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\lambda} - \eta_{\mu\nu} \eta_{\lambda\sigma} \right) \partial^2 h_{\lambda\sigma} + \kappa h^{\gamma\epsilon} T_{\gamma\epsilon} \right\}$$

(2.38)

that we can again rewrite as

$$S_{SOEH}^f + S_M = -\frac{1}{2} \int d^4x \left\{ -\frac{1}{4} h^{\alpha\beta} D_{\alpha\beta} \partial^\gamma h_{\gamma\epsilon} + \kappa h^{\gamma\epsilon} T_{\gamma\epsilon} \right\}$$

(2.39)

This is the same operator that in (2.34), so we get for the free energy

$$W_{SO}^{gl} = -\kappa^2 \frac{1}{4} \int d^4x \sqrt{|g|} T^{\mu\nu} D_{\mu\nu}^{-1} T^{\rho\sigma}$$

(2.40)

The final conclusion is that the first order formalism is equivalent to the second order one with external sources for the graviton $W_{FO}^{gl} = W_{SO}^{gl}$.

This seems the best procedure in order to compute the one loop divergences by heat kernel methods.

3 The most general quadratic action

3.1 First order versus second order

In the paper [14] a full analysis is made of first order versus second order EM and it is concluded that coincidence in the above sense (that is, once the Levi-Civita connection has been substituted in the general EM) is only found for Lanczos-Lovelock (LL) and related lagrangians.

Anticipating the notation we shall introduce in our equation (4.21) this happens when

$$\alpha_1 = \alpha_3 = -\frac{\alpha_2}{4}$$

(3.1)

It is of course well-known that quadratic LL lagrangians are trivial in four dimensions (where they reduce to the Gauss-Bonnet density), but they appear in brane-world scenarios as well as in some dark matter proposals. There are other, less restrictive, instances where the EM are also equivalent in the above sense. The starting point is the equation found in [14] giving the difference between both EM, namely

$$\Delta H_{\mu\nu} = H_{\mu\nu}^{SO} - H_{\mu\nu}^{FO} = -\frac{1}{2} \nabla_\lambda K^{\lambda}_{\mu\nu} + \frac{1}{4} g_{\lambda\mu} \nabla^\rho K^{\lambda}_{\rho\nu} + \frac{1}{4} g_{\lambda\nu} \nabla^\rho K^{\lambda}_{\mu\rho}$$

(3.2)
where
\[ H_{\mu \nu} = \frac{1}{\sqrt{|g|}} \frac{\delta S}{\delta g^{\mu \nu}} \] (3.3)
and
\[ \frac{1}{\sqrt{|g|}} \frac{\delta S}{\delta \Gamma^\mu_{\nu \lambda}} \equiv K^\lambda_{\mu \nu} = 2(\alpha_2 + \alpha_3)g_{\mu \nu} \nabla^\lambda R + (\alpha_2 + 4\alpha_1)\nabla^\lambda R_{\mu \nu} - \\
- 2(\alpha_3 + \alpha_2)\delta^\lambda_\rho \nabla_\mu R - (\alpha_1 + \alpha_2)\nabla_\mu R^\lambda_\nu \] (3.4)

It is plain that for constant curvature backgrounds the whole tensor $K^\lambda_{\mu \nu}$ vanishes and both sets of EM are equivalent.

Actually more is true. In this same reference \[14\] general lagrangians involving the metric and the Riemann tensor (but not its derivatives) have been considered.

Again, in the Levi-Civita case, the relationship between the first order and second order EM is exactly as in (3.2), and besides,
\[ K^\mu_\nu = \nabla_\nu B^\mu_\nu \] (3.6)
with
\[ B^\mu_\nu = \frac{\delta L}{\delta R^\mu_\nu} - \frac{\delta L}{\delta R^\mu_\nu\rho} \] (3.7)

### 3.2 Quadratic actions

It is worth pointing out that when the nonmetricity is non-vanishing the Riemann tensor does not enjoy the usual symmetries
\[ R^r[\Gamma]_{\mu \nu \rho \sigma} \neq R^r[\Gamma]_{\rho \sigma \mu \nu} \] (3.8)
\[ R^r[\Gamma]_{(\mu \nu)\rho \sigma} \neq 0 \] (3.9)

There are then two different traces. The one that corresponds to the Ricci tensor
\[ R^+ [\Gamma]_{\nu \sigma} \equiv g^{\nu \rho} R[\Gamma]_{\mu \nu \rho \sigma} \] (3.10)
and a different one
\[ R^- [\Gamma]_{\mu \nu} \equiv g^{\nu \rho} R[\Gamma]_{\mu \nu \rho \sigma} \] (3.11)
Neither of them is in general symmetric now. There is also an antisymmetric further trace
\[ R_{\rho \sigma} \equiv g^{\mu \nu} R[\Gamma]_{\mu \nu \rho \sigma} \] (3.12)
However, it is easy to check that there is an only scalar
\[ R^+ \equiv g^{\mu \nu} R^\mu_\nu = -R^- \equiv g^{\mu \nu} R^-_\mu_\nu, \] (3.13)
while $g^{\rho \sigma} R_{\rho \sigma} = 0$.

Let us now write the most general quadratic action, made with 16 Weyl scalars.\(^2\)

\(^2\)We only consider parity-conserving operators, therefore terms like
\[ \epsilon^{\mu \rho \sigma} R^\pm_{\mu \nu \rho \sigma} R^\pm_{\mu \nu \rho \sigma}, \epsilon^{\mu \rho \sigma} R^\pm_{\mu \nu \rho \sigma} R^\pm_{\mu \nu \rho \sigma} \] are excluded.
There are then six independent quadratic scalar operators that can be built out of two Riemann tensors which are all of the general form

\[ \mathcal{O}_I = R^\mu_{\nu\rho\sigma} (D_1)^{\nu\rho\sigma\nu'}_{\mu\mu'} R^{\mu'}_{\nu'\rho'\sigma'} \]

for \( I = 1 \ldots 6 \), where

\[
\begin{align*}
(D_1)^{\nu\rho\sigma\nu'}_{\mu\mu'} &= g_{\mu\nu'} g^{\nu\rho} g^{\sigma\nu'} \\
(D_2)^{\nu\rho\sigma\nu'}_{\mu\mu'} &= g_{\mu\nu'} g^{\nu\rho} g^{\sigma\nu'} \\
(D_3)^{\nu\rho\sigma\nu'}_{\mu\mu'} &= \delta^\nu_{\mu} \delta^\rho_{\mu'} g^{\sigma\nu'} \\
(D_4)^{\nu\rho\sigma\nu'}_{\mu\mu'} &= \delta^\nu_{\mu} \delta^\rho_{\mu'} g^{\sigma\nu'} \\
(D_5)^{\nu\rho\sigma\nu'}_{\mu\mu'} &= \delta^\rho_{\mu} \delta^\nu_{\mu'} g^{\sigma\nu'} \\
(D_6)^{\nu\rho\sigma\nu'}_{\mu\mu'} &= \delta^\rho_{\mu} \delta^\nu_{\mu'} g^{\sigma\nu'}
\end{align*}
\]

We follow the Landau-Lifshitz spacelike conventions, in particular

\[ R^\mu_{\nu\rho\sigma} = \partial^\mu \Gamma^\nu_{\rho\sigma} - \partial^\nu \Gamma^\mu_{\rho\sigma} + \Gamma^\mu_{\lambda\rho} \Gamma^{\lambda}_{\nu\sigma} - \Gamma^\mu_{\lambda\sigma} \Gamma^{\lambda}_{\nu\rho} \]

(3.16)

A remarkable fact is that under Weyl rescaling

\[ g_{\mu\nu} \rightarrow \Omega^2(x) g_{\mu\nu} \]

(3.17)

assuming the connection remains inert, all operators transform as

\[ \mathcal{O}_I \rightarrow \Omega^{4-I} \mathcal{O}_I \]

(3.18)

so that all these operators remain Weyl invariant when integrated in four dimensions. This means that the most general quadratic action is Weyl invariant in this sense.

There are then 9 Weyl scalar operators that can be formed with the three different traces, (3.10), (3.11) and (3.12)

\[
\begin{align*}
(D_7)^{\nu\rho\sigma\nu'}_{\mu\mu'} &= \delta^\rho_{\mu} \delta^\nu_{\mu'} g^{\sigma\nu'} \\
(D_8)^{\nu\rho\sigma\nu'}_{\mu\mu'} &= \delta^\rho_{\mu} \delta^\nu_{\mu'} g^{\sigma\nu'} \\
(D_9)^{\nu\rho\sigma\nu'}_{\mu\mu'} &= g_{\mu\nu'} g^{\nu\rho} g^{\sigma\nu'} \\
(D_{10})^{\nu\rho\sigma\nu'}_{\mu\mu'} &= \delta^\rho_{\mu} \sigma^\nu_{\mu'} g^{\rho\nu'} \\
(D_{11})^{\nu\rho\sigma\nu'}_{\mu\mu'} &= \delta^\nu_{\mu'} \sigma^\rho_{\mu'} g^{\sigma\nu'} \\
(D_{12})^{\nu\rho\sigma\nu'}_{\mu\mu'} &= \delta^\nu_{\mu'} \sigma^\rho_{\mu'} g^{\sigma\nu'} \\
(D_{13})^{\nu\rho\sigma\nu'}_{\mu\mu'} &= \delta^\rho_{\mu} \delta^\nu_{\mu'} g^{\rho\nu'} \\
(D_{14})^{\nu\rho\sigma\nu'}_{\mu\mu'} &= \delta^\rho_{\mu} \delta^\nu_{\mu'} g^{\rho\nu'} \\
(D_{15})^{\nu\rho\sigma\nu'}_{\mu\mu'} &= \delta^\nu_{\mu'} \delta^\rho_{\mu'} g^{\rho\nu'}
\end{align*}
\]

Finally there is only one independent curvature scalar operator

\[ \mathcal{O}_{16} = \sqrt{|g|} R^2 \]

(3.19)
which also admits the canonical form (3.15) with
\[ (D_{16})^{\nu \rho \sigma \rho' \sigma'}_{\mu \mu'} = \delta^\rho_{\mu'} \delta^{\rho'}_{\mu} g^{\nu \sigma} g^{\nu' \sigma'} \] (3.20)

The most general Weyl invariant lagrangian is then a sum of these sixteen operators with arbitrary coefficients
\[ \mathcal{L} = \sum_{I=1}^{16} g_I O_I \] (3.21)
where \( g_I \) are arbitrary, generically non-vanishing, dimensionless coupling constants. This lagrangian is expected to be renormalizable by power counting. Unitarity may be an issue and has to be analyzed in detail.

Finally, we can write the most general quadratic action as
\[ S = \int d^n x \sqrt{|g|} \mathcal{L} (g_{\mu \nu}, A^\nu_\mu) = \int d^n x \sqrt{|g|} \sum_{I=1}^{16} g_I R^\mu_{\nu \rho \sigma} (D_I)^{\nu' \rho' \sigma' \mu'} R^{\mu'}_{\nu' \rho' \sigma'} \] (3.22)

The mass dimension of all coupling constants is
\[ [g_I] = n - 4 \] (3.23)

4 Background field expansion

The general background field expansion reads
\[ g_{\mu \nu} = \bar{g}_{\mu \nu} + \kappa h_{\mu \nu} \]
\[ \Gamma^\mu_{\nu \rho} = \bar{\Gamma}^\mu_{\nu \rho} + B^\mu_{\nu \rho} = \left\{ \begin{array}{c} \mu \\ \nu \\ \rho \end{array} \right\} + A^\mu_{\nu \rho} \] (4.1)

that is, we can assume without loss of generality that the background connection \( \bar{\Gamma}^\mu_{\nu \rho} \) is the Levi-Civita connection corresponding to the metric \( \bar{g}_{\mu \nu} \). The tensor \( A^\mu_{\nu \rho} \) contains all the relevant information on the non-metricity of the connection.\(^3\) The constant \( \kappa \) has mass dimension
\[ [\kappa] = 1 - \frac{n}{2} \] (4.3)
adequate for the kinetic energy of the field \( h_{\mu \nu} \) to be canonically normalized (that is, \( [h_{\mu \nu}] = \frac{n}{2} - 1 \)). This means that in spite of the fact that
\[ \nabla_{\rho} g_{\mu \nu} \neq 0 \] (4.4)

\(^3\)We use a torsionless connection, i.e. \( A^\mu_{\nu \rho} = A^\mu_{\rho \nu} \) (for an analysis of quadratic theories with torsion see [15, 16]).

Observe that this is a perfectly acceptable expansion. Were we to allow torsion, \( A^\lambda_{\alpha \beta} - A^\lambda_{\beta \alpha} = T^\lambda_{\alpha \beta} \), the connection tadpole will read
\[ \delta S |_{g_{\mu \nu} = \bar{g}_{\mu \nu}} = \int d^n x \sqrt{|\bar{g}|} \sum_{I=1}^{16} g_I R^\mu_{\nu \rho \sigma} (D_I)^{\nu' \rho' \sigma' \mu'} \times \]
\[ \times \left\{ \delta^\rho_{\mu'} \left( \delta^{\rho'}_{\nu'} \delta^{\sigma'}_{\rho'} \nabla_{\nu'} - \delta^{\rho'}_{\nu'} \delta^{\sigma'}_{\rho} \nabla_{\sigma'} \right) + \delta^\sigma_{\nu'} \left( \delta^{\rho'}_{\nu'} \delta^{\rho}_{\sigma'} \nabla_{\rho} - \delta^{\rho'}_{\nu} \delta^{\rho}_{\sigma'} \nabla_{\sigma'} \right) \right\} A^{\nu'}_{\rho' \mu'} \] (4.2)

Therefore the torsionless choice is allowed as it corresponds to the case where the second term is zero.
which prevents integration by parts
\[ \int \sqrt{|g|} d^n x \nabla_\mu V^\mu \neq 0 \] (4.5)

we can always write
\[ \int \sqrt{g} d^n x \nabla_\mu V^\mu = \int \sqrt{g} d^n x \left( \nabla_\mu V^\mu + A_{\lambda \mu}^\lambda V^\mu \right) = \int \sqrt{g} d^n x A_{\lambda \mu}^\lambda V^\mu \] (4.6)

Therefore integration by parts is still possible at the price of introducing potential terms involving the field \( A_{\mu \nu}^\lambda \). We shall then continue using the notation
\[ d(vol) = \sqrt{g} d^n x \] (4.7)

when appropriate. Let us define
\[ A^\lambda \equiv g^{\mu \nu} A_{\mu \nu}^\lambda \]
\[ A_\sigma \equiv A_{\sigma \lambda}^\lambda \] (4.8)

Please note that
\[ A^\lambda \neq g^{\lambda \mu} A_\mu \] (4.9)

It is also natural to define a field strength and a quadratic term
\[ F_{\nu \rho \sigma}^\mu \equiv \nabla_\rho A_{\nu \sigma}^\mu - \nabla_\sigma A_{\nu \rho}^\mu \] (4.10)
\[ O_{\nu \rho \sigma}^\mu \equiv A_{\lambda \rho}^\mu A_{\nu \sigma}^\lambda - A_{\nu \sigma}^\lambda A_{\lambda \rho}^\mu \] (4.11)

but there is an extra symmetry, similar to the usual algebraic Bianchi identity
\[ F_{\nu \rho \sigma}^\mu + F_{\sigma \nu \rho}^\mu + F_{\rho \sigma \nu}^\mu = 0 \] (4.12)
\[ O_{\nu \rho \sigma}^\mu + O_{\sigma \nu \rho}^\mu + O_{\rho \sigma \nu}^\mu = 0 \] (4.13)

In this way the Riemann tensor reads
\[ R_{\nu \rho \sigma}^\mu = \tilde{R}_{\nu \rho \sigma}^\mu + F_{\nu \rho \sigma}^\mu + O_{\nu \rho \sigma}^\mu \] (4.14)

where the first term is just the contribution of the background; the second is linear in the connection fluctuations, and the third is quadratic in the same quantities.

We can define three different traces for \( F_{\nu \rho \sigma}^\mu \) and \( O_{\nu \rho \sigma}^\mu \), in a manner identical to the way we did it for Riemann’s tensor
\[ F_{\nu \rho \sigma}^\mu \]
\[ O_{\nu \rho \sigma}^\mu \] (4.15)
It should be noted that these objects are not symmetric in general
\[ F_{\nu\sigma}^+ \neq F_{\sigma\nu}^+ \] (4.16)
although certainly
\[ F_{\rho\sigma} = -F_{\sigma\rho} \] (4.17)

The corresponding scalars read
\[
\begin{align*}
F^+ &\equiv g^{\nu\sigma} F_{\nu\sigma}^+ = g^{\nu\sigma} \left( \nabla_\lambda A^\lambda_{\nu\sigma} - \nabla_\sigma A^\lambda_{\nu\lambda} \right) = g^{\nu\sigma} \nabla_\lambda A^\lambda_{\nu\sigma} - \nabla_\lambda A^\lambda \\
F^- &\equiv F^{-\lambda}_\nu = \nabla^\lambda A_\nu - g^{\nu\sigma} \nabla_\lambda A^\lambda_{\nu\sigma} = -F^+ \\
O^+ &\equiv g^{\nu\sigma} O_{\nu\sigma}^+ = A_\alpha A^\alpha - g^{\nu\sigma} A^\alpha_{\nu\alpha} A^\alpha_{\nu\lambda} \\
O^- &\equiv O^{-\lambda}_\nu = g^{\nu\sigma} A^\alpha_{\nu\alpha} A^\alpha_{\nu\lambda} - A_\lambda A^\lambda = -O^+ 
\end{align*}
\] (4.18)

Now, we take our action (3.22).
It is clear that when expanding around these background fields, i.e. when the connection
is the Levi-Civita one, there are many relationships with the preceding operators, to wit
\[
\begin{align*}
O_1 &= 2O_2 = -O_3 = -2O_4 = 2O_5 = O_6 \\
O_7 &= O_8 = O_9 = O_{10} = -O_{12} = -O_{13} \\
O_{11} &= O_{14} = O_{15} = 0 
\end{align*}
\] (4.19)
The sixteen constants collapse to only three:
\[
\begin{align*}
\alpha_1 &= g_1 + \frac{1}{2}g_2 - g_3 - \frac{1}{2}g_4 + \frac{1}{2}g_5 + g_6 \\
\alpha_2 &= g_7 + g_8 + g_9 + g_{10} - g_{12} - g_{13} \\
\alpha_3 &= g_{16} 
\end{align*}
\] (4.20)
so the lowest order in the expansion of the action reduces to
\[
S_0 = \int d^n x \sqrt{|g|} \left( \alpha_1 \overline{R}_{\mu\nu\rho\sigma} \overline{R}^{\mu\nu\rho\sigma} + \alpha_2 \overline{R}_{\mu\nu} \overline{R}^{\mu\nu} + \alpha_3 \overline{R}^2 \right) 
\] (4.21)
The equations of motion are given by the vanishing of the tadpoles. For the metric, this reads
\[
\begin{align*}
\delta S_{g_{\mu\nu}=\delta g_{\mu\nu}} &= \int d^n x \kappa \sqrt{|g|} \left\{ \frac{1}{2} g^{\alpha\beta} \overline{\nabla} \cdot 2\alpha_1 \overline{R}_{\mu\nu\rho} \overline{R}^{\mu\nu\rho\beta} - 2q_1 \overline{R}_{\mu}^{\alpha} \overline{R}^{\mu\beta} - \\
&\quad - 2q_2 \overline{R}^{\alpha\beta\gamma} \overline{R}_{\mu\nu} - 2\alpha_3 \overline{R}^{\alpha\beta} \overline{R} \right\} \delta g_{\alpha\beta} 
\end{align*}
\] (4.22)
where we define two more combinations of \( g \) constants
\[
\begin{align*}
q_1 &= g_7 + g_8 - \frac{1}{2}g_{12} - \frac{1}{2}g_{13} \\
q_2 &= -g_9 - g_{10} + \frac{1}{2}g_{12} + \frac{1}{2}g_{13} 
\end{align*}
\] (4.23)
We have relegated most general formulas to the appendix B. It is immediate to check that the EM are identically satisfied for any Riemannian maximally symmetric, constant curvature manifold, where

$$R_{\alpha\beta\gamma\delta} = -\frac{2\lambda}{(n-1)(n-2)} (\bar{g}_{\alpha\gamma} \bar{g}_{\beta\delta} - \bar{g}_{\alpha\delta} \bar{g}_{\beta\gamma}) = \pm \frac{1}{L^2} (\bar{g}_{\alpha\gamma} \bar{g}_{\beta\delta} - \bar{g}_{\alpha\delta} \bar{g}_{\beta\gamma})$$  \hspace{1cm} (4.24)

With our conventions, the scalar curvature is related to the cosmological constant through

$$R = -\frac{2n}{n-2} \lambda$$ \hspace{1cm} (4.25)

This means that a priori both de Sitter (positive cosmological constant, but negative curvature) and anti de Sitter (negative cosmological constant and positive curvature) are possible vacua for our quadratic theories. In the following expansions we have restricted ourselves to negative values for the cosmological constant (which is the sphere $S^n$ with our conventions) for definiteness. The problem to find more solutions to the above equations is of course an interesting although daunting task.

On the other hand, the connection tadpole reads

$$\delta S|_{g_{\mu\nu} = \bar{g}_{\mu\nu}} = \int d^n x \sqrt{|\bar{g}|} \left\{ 4\alpha_1 \bar{R}^\alpha_{\lambda} \bar{\nabla}^\alpha \bar{\nabla}^\rho + 2q_1 \left( \bar{R}^\alpha_{\lambda} \bar{\nabla}^\lambda - \delta^\lambda_{\alpha} \bar{R}^\mu_{\lambda} \bar{\nabla}^\mu \right) + 2q_2 \left( \bar{R}^\beta_{\lambda} \bar{\nabla}^\alpha - \delta^\alpha_{\beta} \bar{R}^\mu_{\lambda} \bar{\nabla}^\mu \right) + 2\alpha_3 \left( \bar{g}^\alpha_{\beta} \bar{R} \bar{\nabla}^\alpha_{\lambda} - \delta^\alpha_{\beta} \bar{R} \bar{\nabla}^\alpha_{\lambda} \right) \right\} \delta \Gamma_{\alpha\beta}$$  \hspace{1cm} (4.26)

The quadratic term in the expansion can be written as

$$S^{(2)} = \int \sqrt{|\bar{g}|} d^n x \left\{ \frac{1}{2} h_{\mu\nu} M^{\mu\nu\rho\sigma} h_{\rho\sigma} + h_{\mu\nu} N^{\mu\nu\rho\sigma} A^\rho_{\sigma} \lambda + \frac{1}{2} A_{\mu \nu} K^{\mu\nu\rho\sigma} A^\rho_{\sigma} \right\}$$  \hspace{1cm} (4.27)

Here the total mass dimension of the operators reads as follows.

$$[M] = 2$$  \hspace{1cm} (4.28)

But $M$ is proportional to $g \kappa^2$ (where $g$ is a generic coupling constant) times some background squared; there is then no room for derivatives (momenta)

$$M \sim \frac{g \kappa^2}{L^4}$$  \hspace{1cm} (4.29)

where we have assumed that the background curvature is $\sim L^{-2}$. On the other hand,

$$[N] = \frac{n}{2}$$  \hspace{1cm} (4.30)

$N$ is proportional to $g \kappa$; so that the rest has mass dimension 3, namely one background field plus one momentum. That is

$$N \sim \frac{g \kappa \kappa}{L^2}$$  \hspace{1cm} (4.31)

Finally,

$$[K] = n - 2$$  \hspace{1cm} (4.32)
This is proportional to $g$ only; so that the rest has dimension 2. There are terms with one background, and also terms with two derivatives. In the ultraviolet ($kL \gg 1$, $k \to \infty$)

$$K \sim g k^2 \quad (4.33)$$

To be specific, the different operators appearing are

\begin{align*}
M^{\alpha\beta\gamma\epsilon} &= \kappa^2 \left\{ \left(\frac{1}{4} g^{\alpha\beta\gamma\epsilon} - \frac{1}{2} g^\gamma g^\epsilon\right) \left(\alpha_1 \, R_{\mu\nu\rho\sigma} \, R^{\mu\nu\rho\sigma} + \alpha_2 \, R_{\mu\nu} \, R^{\mu\nu} + \alpha_3 \, R^2 \right) + \\
&\quad -g^\gamma g^\epsilon \left(2\alpha_1 \, R_{\mu\nu} \, R^{\mu\nu} + 2q_1 \, R_{\mu} \, R^{\mu} + 2q_2 \, R_{\mu\nu\rho\sigma} \, R^{\mu\nu\rho\sigma} + 2\alpha_3 \, R\Delta \, R \right) + \\
&\quad +2 \left(2\alpha_1 + g_1 + \frac{1}{2}g_2\right) \, R_{\mu\nu\rho\sigma} \, R^{\mu\nu\rho\sigma} \, R^{\mu\nu} + 4q_2 \, R^{\mu\nu\rho\sigma} \, R^{\mu\nu} \cdot R^{\mu\nu} + \\
&\quad +4 \alpha_3 \, R^\gamma \, R^{\delta\epsilon} + 2 \left(g_9 + g_{10}\right) \, R_{\rho\gamma} \, R^{\rho\gamma} - 2 \left(g_5 + \frac{1}{2}g_6\right) \, R_{\mu\nu} \, R^{\mu\nu} \cdot R^{\mu\nu} + 2g_6 \, R_{\mu} \, R^{\mu} \, R^{\mu} + \\
&\quad +2 \left(2g_7 + g_8 - g_9\right) \, R_{\mu\nu} \, R^{\mu\nu} \cdot R^{\mu\nu} + 2\left(2g_7 + g_8 - g_9\right) \, R^{\mu\nu} \, R^{\mu\nu} + 2g_10 \, R^3 \, R^3 \right) \right\} + \\
&\quad + \{\alpha \leftrightarrow \beta\} + \{\gamma \leftrightarrow \epsilon\} + \{\alpha \beta \leftrightarrow \gamma\epsilon\} \quad (4.34)
\end{align*}

where $\{\alpha \leftrightarrow \beta\} + \{\gamma \leftrightarrow \epsilon\}$ stands for the symmetrization under the exchange of $\alpha, \beta$ and $\gamma, \epsilon$ respectively and $\{\alpha \beta \leftrightarrow \gamma\epsilon\}$ refers to the symmetrization under the interchange of $\alpha, \beta$ and $\gamma, \epsilon$.

The mixed graviton-connection piece reads, still in a somewhat symbolic way, where we indicate explicitly the graviton, whereas the connection is implicit

\begin{align*}
N_{\gamma\epsilon \lambda}^{\alpha\beta} A_{\alpha\beta}^\lambda &= \kappa \bar{g}_{\gamma\epsilon} \sum_{I=1}^{I=16} g_1 R_{\mu\nu} \left( D_I \right)_{\mu\nu} \delta^\rho_{\beta} \delta^\gamma_{\alpha} \lambda \delta^{\mu\nu} \sigma \, \delta^{\alpha\beta} \sigma \, \delta^{\beta\gamma} \sigma \, \delta^{\gamma\epsilon} \sigma \, \delta^{\epsilon\alpha} \sigma \, A_{\alpha\beta}^\lambda + \\
&\quad + 2g_9 \kappa \left\{ -\delta^{\mu\nu} \delta^{\rho\sigma} \delta^{\gamma\epsilon} \right\} + 2g_{10} \kappa \left\{ -\delta^{\mu\nu} \delta^{\rho\sigma} \delta^{\gamma\epsilon} \right\} + 2g_{11} \kappa \left\{ -\delta^{\mu\nu} \delta^{\rho\sigma} \delta^{\gamma\epsilon} \right\} + \\
&\quad + 2g_{12} \kappa \left\{ -\delta^{\mu\nu} \delta^{\rho\sigma} \delta^{\gamma\epsilon} \right\} + 2g_{13} \kappa \left\{ -\delta^{\mu\nu} \delta^{\rho\sigma} \delta^{\gamma\epsilon} \right\} + 2g_{14} \kappa \left\{ -\delta^{\mu\nu} \delta^{\rho\sigma} \delta^{\gamma\epsilon} \right\} + \\
&\quad + 2g_{15} \kappa \left\{ -\delta^{\mu\nu} \delta^{\rho\sigma} \delta^{\gamma\epsilon} \right\} + \{\alpha \leftrightarrow \beta\} + \{\gamma \leftrightarrow \epsilon\} \quad (4.35)
\end{align*}
Finally, the operator relating the connection fluctuations reads

\[
K_{\lambda}^{\alpha\beta\gamma\tau} = \left[ 8\alpha_1 g^{\alpha\beta} \bar{R}_{\lambda} \gamma^\tau \delta_\epsilon + 4q_{10} \delta_\epsilon \left( \delta_\lambda \bar{R}^{\tau\epsilon} - \delta_\lambda \bar{R}^\tau \right) + \
+ 4q_2 g^{\alpha\beta} \left( \bar{g}^{\gamma\tau} \bar{R}_{\lambda} \epsilon - \bar{g}^{\gamma\tau} \bar{R}_{\lambda} \beta \right) + 4\alpha_3 \delta_\epsilon \left( \delta_\lambda \bar{g}^{\tau\epsilon} \bar{R} - \delta_\lambda \bar{g}^{\tau} \right) \right] + \
+ 2\nabla^\epsilon \nabla_{\beta} \left[ \bar{g}_{\lambda\tau} \bar{g}^{\alpha\tau} \left( 2g_1 + 2g_2 - g_9 \right) + \delta_\lambda \delta_\epsilon \left( 2g_3 + g_4 + g_6 - g_{10} \right) + \
+ \delta_\lambda \delta_\epsilon \left( -g_8 + 2g_{11} + g_{14} - g_{16} \right) + \
+ 2\nabla_{\epsilon} \nabla_{\lambda} \left[ \bar{g}^{\alpha\tau} \delta_\beta \left( 2g_5 + g_6 - g_{12} + g_{13} \right) + \delta_\lambda \bar{g}^{\gamma\tau} \left( -g_{13} + g_{15} + 2g_{16} \right) \right] + \
+ 2\nabla_{\epsilon} \nabla_{\lambda} \left[ \bar{g}^{\alpha\tau} \delta_\beta \left( 2g_7 + 2g_8 + g_{13} - g_{15} \right) + \bar{g}^{\alpha\tau} \delta_\lambda \left( 2g_{10} + g_{12} + g_{13} \right) \right] + \
+ 2\nabla^\alpha \nabla_{\beta} \left[ 2g_{\lambda\tau} \bar{g}^{\gamma\tau} g_9 + \delta_\lambda \delta_\epsilon \left( g_{12} + g_{15} \right) + \
+ 2A_{\alpha\beta} \Box A_{\epsilon} \left[ \bar{g}_{\lambda\tau} \bar{g}^{\alpha\tau} \bar{g}^{\gamma\tau} \left( -2g_4 - g_2 \right) + \bar{g}_{\lambda\tau} \bar{g}^{\alpha\tau} \bar{g}^{\gamma\tau} g_9 + \bar{g}^{\alpha\tau} \delta_\lambda \delta_\epsilon \left( -2g_3 - g_4 - g_5 \right) + \
+ \bar{g}^{\alpha\tau} \delta_\lambda \delta_\epsilon \left( -2g_7 - 2g_{11} - g_{14} \right) - \delta_\lambda \delta_\epsilon \bar{g}^{\tau} \left( g_{12} + g_{15} \right) \right] + \
+ \{ \alpha \leftrightarrow \beta \} + \{ \gamma \leftrightarrow \epsilon \} + \{ \lambda \alpha \beta \leftrightarrow \tau \gamma \epsilon \} \right] (4.36)
\]

with \( \{ \alpha \leftrightarrow \beta \} + \{ \gamma \leftrightarrow \epsilon \} \) defined before and \( \{ \lambda \alpha \beta \leftrightarrow \tau \gamma \epsilon \} \) referring to the symmetrization under the interchange of \( \lambda, \alpha, \beta \) and \( \tau, \gamma, \epsilon \).

5 Interaction between external sources

We have already mentioned the enormity of the theory space we have been considering. Our main interest, however, is to find a theory describing the gravitational interaction. Let us now discuss our general strategy in order to determine the correct physical effect of external sources. What we want is to characterize the physical sources that interact gravitationally in our theory.

To begin with, assume that we introduce two external sources of dimension \( [T] = 1 + \frac{3}{2} \), one coupled to the graviton

\[
\int d(\text{vol}) \ T_{\mu
u} h^{\mu\nu} \tag{5.1}
\]

In order for this term to be gauge invariant under linearized gauge transformations

\[
\delta h_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \tag{5.2}
\]

The source needs to be symmetric and background-covariantly conserved

\[
T_{\mu
u} = T_{\nu\mu} \\
\nabla_\mu T^{\mu\nu} = 0 \tag{5.3}
\]

We could also introduce another source coupled to the connection with dimension \( [J] = n - 1 \)

\[
\int d(\text{vol}) \ J^{\mu\nu\lambda} A_{\mu\nu\lambda} \tag{5.4}
\]

where the source is got to be symmetric in the last two indices \( J_{\alpha\beta\gamma} = J_{\alpha\gamma\beta} \). Gauge invariance now means that

\[
\mathcal{L}(\xi) J_{\alpha\beta\gamma} = 0 \tag{5.5}
\]
Let us think about the relationship between the response to a graviton source $T_{\mu\nu}$ which we denote by $h_{\mu\nu}$ and the response to a connection source $J_{\alpha\beta\gamma}$ which we denote by $A_{\alpha\beta\gamma}$.

In GR the graviton fluctuation is given in the linear approximation by

$$H_{\mu\nu} = \int \Delta^{GR}_{\mu\nu} T_{\rho\sigma}$$

(5.6)

If the connection were Levi-Civita and the theory were formulated in second order, then the relationship between the responses to both sources would read

$$A_{\mu\nu\rho} = -\frac{1}{2} \left\{ - \nabla_\mu H_{\nu\rho} + \nabla_\nu H_{\mu\rho} + \nabla_\rho H_{\mu\nu} \right\} = -\frac{1}{2} \int d^nx \left\{ - \nabla_\mu \Delta^{GR}_{\nu\rho} \alpha^\beta (x, y) + \nabla_\nu \Delta^{GR}_{\mu\rho} \alpha^\beta (x, y) + \nabla_\rho \Delta^{GR}_{\mu\nu} (x, y) \right\} T_{\alpha\beta}(y)$$

(5.7)

We expect that this is related in some limit to (always in the linear approximation)

$$\langle A_{\mu\nu\rho} \rangle = \frac{\delta W[J]}{\delta J_{\mu\nu\rho}} = K^{-1}_{\mu\nu\rho} \alpha^\beta\gamma J_{\alpha\beta\gamma}$$

(5.8)

In this limit

$$J_{\alpha\beta\gamma} = \frac{1}{2} K_{\alpha\beta\gamma} \int d^nx \left\{ - \nabla_\mu \Delta^{GR}_{\nu\rho} \alpha^\beta (x, y) + \nabla_\nu \Delta^{GR}_{\mu\rho} \alpha^\beta (x, y) + \nabla_\rho \Delta^{GR}_{\mu\nu} (x, y) \right\} T_{\alpha\beta}(y)$$

(5.9)

Then we should recover the GR result for the free energy (at least in the lowest order approximation), namely

$$W[T] = C \int \frac{d^4k}{k^2} \left( |T_{\mu\nu}(k)T^{\mu\nu}(k)| - \frac{1}{2} |T(k)|^2 \right)$$

(5.10)

This presumably yields a general idea of what is what we should expect in the first order case.

The gaussian path integral yields for the free energy the result (up to an additive constant)

$$W[J_{\alpha\beta\gamma}, T_{\mu\nu}] = -\log Z[J_{\alpha\beta\gamma}, T_{\mu\nu}] = \int \sqrt{g} \, d^nx \left\{ -\frac{1}{2} T_{\mu\nu} (M^{-1})^\nu\rho_{\mu\rho} T_{\rho\sigma} + \frac{1}{2} \left( N^{\alpha\beta}_{\mu\nu\lambda} (M^{-1})^\rho_{\alpha\beta} T_{\rho\sigma} - J_{\mu\nu\lambda} \right) \left( K^{\mu\nu\lambda\alpha\beta\gamma} - N_{uv}^{\mu\nu \lambda} (M^{-1})^{\alpha\beta\gamma}_{uvwx} N_{\alpha\beta\gamma} \right) \right\}$$

Please note that the strength of the interaction between external graviton sources is always a contact one

$$\langle TT \rangle \sim \frac{L^4}{g \kappa^2}$$

(5.12)

The mixing between the graviton source and the connection source, on the other hand, is ultralocal

$$\langle TJ \rangle \sim \frac{L^2}{g \kappa k}$$

(5.13)

Finally, the interaction between connection sources allows a long-range potential

$$\langle JJ \rangle \sim \frac{1}{g \kappa^2}$$

(5.14)
5.1 Flat background

Let us now work out in turn with some detail the structure of the fluctuations around a flat background (it is not then a background gauge calculation, which should be background independent). Assume then

\[ \bar{g}_{\mu\nu} = \eta_{\mu\nu} \quad (5.15) \]

so that the whole contribution to \( M_{\mu\nu\rho\sigma} \) comes from the gauge fixing term (in case we choose to gauge fix the graviton piece). This has the following problem. We have only four gauge parameters, whereas there are ten components in the graviton field. The mismatch means that there are undamped components in the graviton field. For example, with our gauge fixing, only the graviton trace, \( h \) gets a kinetic term quadratic in derivatives. Besides, the mixing graviton/ connection also vanishes in this background, \( \bar{N} = 0 \). Defining the traceless component of the graviton field

\[ h^T_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{n} \, h \, \eta_{\mu\nu} \quad (5.16) \]

the path integral over \( D h^T_{\mu\nu} \) is not bounded (it would put restrictions on the source, \( \delta (T^T_{\mu\nu}) \)), and so is the total functional integral.

This should be contrasted with what was derived earlier for the Einstein-Hilbert case, where the integration over graviton fluctuations yields a delta-function that defines the connection \( \bar{A} \) in terms of the external source. The main difference is that in the Einstein-Hilbert case the off-diagonal graviton-gauge term \( h \bar{N} A \) did not vanish when in a flat background, so that the path integral could be interpreted as a Dirac delta by analytic continuation. Here what happens is that this same term \( \bar{N} \) does vanish in a flat background.

It is however still possible to define the theory in Minkowski space assuming that gravitation is defined by the three-index field \( A_{\mu\nu\lambda} \) exclusively and normalizing the path integral accordingly, \( \text{id est} \)

\[ e^{i \mathcal{W}} = \frac{\mathcal{Z}[J_{\mu\nu\lambda}]}{\mathcal{Z}[0]} = \left\langle \frac{D h_{\mu\nu} \, DA_{\mu\nu\lambda} \, e^{iS[h_{\alpha\beta\gamma}, A_{\mu\nu\lambda}; J_{\alpha\beta\gamma}]} }{\mathcal{Z}[h_{\alpha\beta\gamma}, A_{\mu\nu\lambda}]} \right\rangle \quad (5.17) \]

This is more or less equivalent to consider that all the graviton dynamics is to be obtained as a consequence of the dynamics of the three-index field \( A_{\mu\nu\lambda} \), considered as a composite field of sorts.

In this case we can easily invert the \( K_{\alpha_{\lambda} \beta_{\gamma}} (K^{-1})_{\mu_{\nu} \rho_{\sigma}} \) operator by imposing

\[ K_{\lambda\mu\sigma} (K^{-1})_{\nu\rho\tau} = \frac{1}{2} \left( \delta^\lambda_\alpha \delta^\mu_\beta \delta^\nu_\gamma + \delta^\mu_\alpha \delta^\nu_\beta \delta^\lambda_\gamma + \delta^\mu_\alpha \delta^\lambda_\beta \delta^\nu_\gamma + \delta^\nu_\alpha \delta^\lambda_\beta \delta^\mu_\gamma \right) \quad (5.18) \]

although the answer is a bit cumbersome, namely

\[
(K^{-1})_{\alpha\beta\gamma}^\lambda \tau = \frac{1}{k^2} \left( \beta_1 \eta_{\alpha\beta} \eta_{\gamma\tau} k^\lambda k^\tau + \beta_2 \eta_{\alpha\gamma} \eta_{\beta\tau} k^\lambda k^\tau + \beta_3 \delta_\alpha^\lambda \delta_\beta^\gamma \eta_{\tau \epsilon} k^\lambda k^\tau \right. \\
+ \beta_5 \delta_\alpha^\tau \delta_\beta^\lambda \eta_{\gamma \epsilon} k^\lambda k^\tau + \beta_6 \eta_{\alpha\beta} \eta_{\gamma\tau} k^\lambda k^\tau + \beta_7 \eta_{\alpha\gamma} \eta_{\beta\tau} k^\lambda k^\tau + \beta_8 \delta_\alpha^\lambda \delta_\beta^\gamma \eta_{\tau \epsilon} k^\lambda k^\tau \\
+ \beta_9 \delta_\alpha^\tau \delta_\beta^\lambda \eta_{\gamma \epsilon} k^\lambda k^\tau + \beta_10 \delta_\alpha^\lambda \delta_\beta^\gamma \eta_{\tau \epsilon} k^\lambda k^\tau + \beta_11 \delta_\alpha^\lambda \delta_\beta^\gamma \eta_{\tau \epsilon} k^\lambda k^\tau + \beta_12 \eta_{\alpha\beta} \eta_{\gamma\tau} k^\lambda k^\tau + \\
+ \beta_13 \delta_\alpha^\lambda \delta_\beta^\gamma \eta_{\tau \epsilon} k^\lambda k^\tau + \beta_14 \eta_{\alpha\beta} \eta_{\gamma\tau} k^\lambda k^\tau + \beta_15 \delta_\alpha^\lambda \delta_\beta^\gamma \eta_{\tau \epsilon} k^\lambda k^\tau + \beta_16 \delta_\alpha^\lambda \delta_\beta^\gamma \eta_{\tau \epsilon} k^\lambda k^\tau \right) \]

\[ - 20 - \]

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where the coefficients $\beta_i$ are complicated functions of the coupling constants $g_i$, whose explicit expression is not very illuminating. What we want in the end is, of course, to recover General Relativity (GR), again, in the lowest order approximation. Therefore, we should be able to fulfill that, at the lowest order, 

$$J^{\alpha\beta\gamma}(K^{-1})_{\alpha\beta\gamma\mu
u\rho}J^{\mu\nu\rho} = T^{\mu\nu} \frac{1}{2k^2} (\eta_{\mu\nu}\eta_{\sigma\tau} + \eta_{\mu\sigma}\eta_{\nu\tau} - \eta_{\mu\nu}\eta_{\sigma\tau}) T^{\sigma\tau} =$$

$$= \frac{T^{a} T^{b} b}{k^2} - \frac{T^{a} T^{b} b}{2k^2}$$

Equation (5.20) then reduces to 

$$\frac{\beta_1 J^{\alpha\beta\gamma} J^{\alpha\beta\gamma}}{k^2} + \frac{\beta_2 J^{\alpha\beta\gamma} J^{\alpha\beta\gamma}}{k^2} + \frac{\beta_3 J^{\alpha\beta\gamma} J^{\alpha\beta\gamma}}{k^2} + \frac{\beta_3 J^{\alpha\beta\gamma} J^{\alpha\beta\gamma}}{k^2} = \frac{T^{a} T^{b} b}{k^2} - \frac{T^{a} T^{b} b}{2k^2}$$

Assuming, as we did earlier when dealing with the Einstein Hilbert term in the first order formalism, that all physical quantities must be expressed in momentum space in terms of the basic quantities $\eta_{\mu\nu}$ and $k_\alpha$, 

$$J^{\alpha\beta\gamma} = Ak_{\alpha} \eta_{\beta\gamma} + B(k_{\beta} \eta_{\alpha\gamma} + k_{\gamma} \eta_{\alpha\beta})$$

as well as 

$$T_{\alpha\beta} = t(k^2 \eta_{\alpha\beta} - k_\alpha k_\beta)$$

the preceding equation reduces to 

$$(\beta_1 + \beta_2)(nA + 2B)^2 + \beta_3 (A^2 + 2AB(n + 1) + B^2(n + 3)) +$$

$$+ (\beta_4 + \beta_5)(A + (n + 1)B)(nA + 2B) = -t^2 \frac{(n - 1)(n - 3)}{2}$$

which has a huge space of solutions.

Alternatively, one may guess a different ansatz of the type 

$$J^{\alpha\beta\gamma} = Aj_\alpha T_{\beta\gamma} + B(j_{\beta} T_{\alpha\gamma} + j_{\gamma} T_{\alpha\beta})$$

where $j_\alpha$ is some conserved vector: $k_\alpha j^\alpha = 0$. This ansatz illuminates other physical possibilities. In that case the left hand side of (5.20) reads 

$$\frac{(A^2(\beta_3 + \beta_5) + 2AB(2\beta_2 + \beta_3 + \beta_4 + \beta_5) + B^2(4\beta_1 + 2\beta_2 + \beta_3 + 2\beta_4 + 3\beta_5))j^a j^b T_{a}^{c} T_{b}^{c} +}{k^2} +$$

$$\frac{j^2(A^2j_1 + B^2j_3 + ABj_4)T_{a}^{a} T_{b}^{b}}{k^2} + \frac{j^2(A^2j_2 + 2ABj_5 + B^2(2\beta_2 + \beta_5))T_{ab} T_{ab}}{k^2} +$$

$$+ \frac{(A^2j_4 + 2B^2(\beta_3 + \beta_4) + AB(4\beta_1 + 2\beta_3 + \beta_4))j^a j^b T_{a}^{c} T_{b}^{c}}{k^2}$$
There is no general solution with this ansatz (i.e. without constraints on the $\beta_i$). However, if we allow that constraints, we could just set $B = 0$ so the previous reduces to

$$\frac{A^2 j^2 \beta_2 T_{ab} T^{ab}}{k^2} + \frac{A^2 (\beta_3 + \beta_5) j^a j^b T_{a} T_{bc}}{k^2} + \frac{A^2 j^2 \beta_1 T_{a} T^{ab}}{k^2} + \frac{A^2 \beta_4 j^a j^b T_{ab} T^{c}}{k^2} \quad (5.27)$$

Therefore, by making $\beta_4 = 0$, $\beta_3 = -\beta_5$ and $\beta_1 = \frac{1}{2} \beta_2$ we would achieve the desired result.

This choice, although not unique, proves that the connection sources can be related to the usual ones so we can recover the classical tests of GR again for this ansatz.

5.2 Curved background

It is however possible to assume a constant curvature background with cosmological constant $\lambda$ (mass dimension 2). Recall the behavior of the different operators in the UV ($k \to \infty$)

$$M \sim \frac{g\kappa^2}{L^4}$$
$$N \sim \frac{g\kappa k}{L^2}$$
$$K \sim gk^2$$

$$N M^{-1} \sim gk^2 \quad (5.28)$$

The direct coupling between two graviton energy-momentum sources is proportional to $M^{-1} \sim \lambda^{-2}$, so that it is a contact interaction. The other coupling of two energy-momentum sources is proportional to

$$(N M^{-1} T) (K - N M^{-1} N)^{-1} (N M^{-1} T) \sim k^0 \quad (5.29)$$

so that it is again a contact interaction. It is then unavoidable to introduce connection sources in order to obtain a non-trivial potential. Indeed the coupling between two such sources is proportional to

$$J (K - N M^{-1} N)^{-1} J \sim \frac{1}{r} \quad (5.30)$$

There is some mixing between the two sources

$$(N M^{-1} T) (K - N M^{-1} N)^{-1} J \sim \frac{1}{r^2} \quad (5.31)$$

At this point it seems that we can dispose of the graviton source altogether.

To be specific, the graviton EM collapses to

$$\kappa \sqrt{|g|} \left( \frac{n}{2} - 2 \right) \left( \frac{n - 1}{L^4} \right) \{2 \alpha_1 + (n - 1) \alpha_2 + n(n - 1) \alpha_3 \} g^{\alpha \beta} = 0 \quad (5.32)$$

For $n \neq 4$, this is just a constraint on the coupling constants, that reduces the number of independent parameters of the most general lagrangian to 15.

The connection EM collapses to

$$\int d^4x \sqrt{|g|} \frac{2}{L^2} \{2 \alpha_1 + (n - 1) \alpha_2 + n(n - 1) \alpha_3 \} \left( \delta^{\alpha \beta} \nabla_{\lambda} - \delta^{\alpha \beta} \nabla_{\lambda} \right) \delta \Gamma_{\alpha \beta} = 0 \quad (5.33)$$

which is identically zero because it is a total derivative.
The operator relating the graviton fluctuations in the action (4.27), reads

$$M_{\alpha\beta\gamma\epsilon} = c_1 \bar{g}_{\alpha\beta} \bar{g}_{\gamma\epsilon} + c_2 (\bar{g}_{\alpha\gamma} \bar{g}_{\beta\epsilon} + \bar{g}_{\alpha\epsilon} \bar{g}_{\beta\gamma})$$

(5.34)

where

$$c_1 = \frac{\kappa^2}{L^4} \left\{ 2\alpha_1 - (n-1) \left[ 2 - \frac{n}{4} \right] (2\alpha_1 + (n-1)\alpha_2 + n(n-1)\alpha_3) \right\}$$

$$c_2 = \frac{\kappa^2}{L^4} \left\{ \alpha_5 + (n-1) \left[ 2 - \frac{n}{4} \right] (2\alpha_1 + (n-1)\alpha_2 + n(n-1)\alpha_3) \right\}$$

$$\alpha_4 = -(g_{3} + g_4 - g_6) + (n-2)(g_9 + g_{10}) - (n-1)(g_{12} + g_{13}) + (n-1)^2 g_{16}$$

$$\alpha_5 = (g_3 + g_4 + g_9 + g_{10}) + (n-1)(g_1 + g_2 + g_5 + g_{12} + g_{13}) + (n-2)g_6 + (n-1)^2(g_7 + g_8)$$

while the other two are,

$$N_{\lambda}^{\alpha\beta} \gamma_{\epsilon} = \frac{\kappa}{L^2} (\bar{g}_{\alpha\beta} \bar{g}_{\gamma\epsilon} \bar{g}_{\lambda\tau} + \bar{g}_{\alpha\gamma} \bar{g}_{\beta\epsilon} \bar{g}_{\lambda\tau} + \bar{g}_{\alpha\epsilon} \bar{g}_{\beta\gamma} \bar{g}_{\lambda\tau})$$

(5.37)

$$K_{\lambda}^{\alpha\beta\gamma\epsilon} = \frac{4}{L^2} \left\{ 2\alpha_1 + (n-1)\alpha_2 + n(n-1)\alpha_3 \left( \bar{g}^{\gamma\epsilon} \delta_{\lambda}^{\beta} \delta_{\tau}^{\alpha} - \bar{g}^{\beta\gamma} \delta_{\tau}^{\alpha} \delta_{\lambda}^{\epsilon} \right) + 2 \bar{g}_{\alpha\beta} \bar{g}_{\gamma\epsilon} \left[ g_{\alpha\beta} (g_{1} + g_{2} - g_{6}) + \delta_{\lambda}^{\beta} \delta_{\tau}^{\alpha} (2g_{3} + g_{4} - g_{6} - g_{10}) + \delta_{\lambda}^{\alpha} \delta_{\tau}^{\beta} (-g_{8} + 2g_{11} + g_{14} - g_{16}) \right] + 2 \bar{g}_{\alpha\beta} \bar{g}_{\gamma\epsilon} \left[ g_{\alpha\gamma} \delta_{\lambda}^{\beta} (2g_{5} + g_{6} - g_{12} - g_{13}) + \delta_{\lambda}^{\alpha} \delta_{\tau}^{\beta} (g_{13} + g_{15} + 2g_{16}) \right] + 2 \bar{g}_{\alpha\beta} \bar{g}_{\gamma\epsilon} \left[ g_{\alpha\epsilon} \delta_{\lambda}^{\beta} (2g_{7} + g_{8} - g_{13} - g_{15}) + \delta_{\lambda}^{\alpha} \delta_{\tau}^{\beta} (2g_{10} + g_{12} + g_{13}) \right] + 2 \bar{g}_{\alpha\beta} \bar{g}_{\gamma\epsilon} \left[ 2g_{\lambda\tau} \bar{g}_{\gamma\epsilon} \bar{g}_{\alpha\beta} + \delta_{\lambda}^{\alpha} \delta_{\tau}^{\beta} (g_{12} + g_{15}) \right] + 2 \bar{g}_{\alpha\beta} \bar{g}_{\gamma\epsilon} \left[ -g_{\lambda\tau} \bar{g}_{\gamma\epsilon} \bar{g}_{\alpha\beta} + \delta_{\lambda}^{\alpha} \delta_{\tau}^{\beta} (-2g_{3} - g_{4} - g_{5}) \right] + \cdots \right\}$$

(5.36)
+ g^{\alpha \gamma} \delta_\lambda^\alpha \delta_\gamma^\beta \left( -g^r - 2g_{11} - g_{14} - \delta_\lambda^\alpha \delta_\gamma^\beta \left( g_{12} + g_{15} \right) \right) + \\
+ \{ \alpha \leftrightarrow \beta \} + \{ \gamma \leftrightarrow \epsilon \} + \{ \lambda \alpha \beta \leftrightarrow \tau \gamma \epsilon \} 
\end{equation}

Let us work out the zero modes of the operator \( M \).
\begin{equation}
M_{\alpha \beta \gamma \epsilon} \left( \nabla^\gamma \xi^\epsilon + \nabla^\epsilon \xi^\gamma \right) = 2c_1 g_{\alpha \beta} \nabla_\lambda \xi^\lambda + 2c_2 \left( \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha \right) = 0 
\end{equation}

It is easy to see that, by taking the trace, consistency demands that
\begin{equation}
2nc_1 = -4c_2 
\end{equation}

The equations of motion put a constraint on the \( g_i \) constants of the Lagrangian that implies that this equality is satisfied for every \( n \). When this condition is satisfied, the conformal Killing vectors of the manifold are zero modes of \( M \). For example, in the case of the sphere \( S_n \) the \( \frac{(n+1)(n+2)}{2} \) conformal Killing vectors close the Lie algebra of \( O(1, n + 1) \).

As said before, due to the equations of motion \( 2nc_1 = -4c_2 \) and we need to add the gauge fixing term. It is enough to invert the operator to add to the lagrangian
\begin{equation}
L_{gf}[h] = -\frac{1}{2\rho} \sqrt{\bar{g}} C_\mu C^\mu 
\end{equation}

where \( C_\mu C^\mu = \bar{\nabla}^\mu h_\mu \bar{\nabla}_\mu h^\rho \). The inverse operator \( M^{-1} \) is then obtained by imposing
\begin{equation}
M_{\alpha \beta \gamma \epsilon} (M^{-1})^{\gamma \epsilon \sigma} = \frac{1}{2} \left( \delta_\alpha^\rho \delta_\beta^\sigma + \delta_\alpha^\sigma \delta_\beta^\rho \right) 
\end{equation}

This reads
\begin{equation}
\begin{aligned}
(M^{-1})^{\alpha \beta \gamma \epsilon} &= - \frac{\square + \rho c_1}{4c_2 \left( \square + \rho \left( 2c_1 + c_2 \right) \right)} g^{\alpha \beta} g^{\gamma \epsilon} + \frac{1}{4c_2} \left( g^{\alpha \gamma} g^{\beta \epsilon} + \bar{g}^{\beta \gamma} g^{\alpha \epsilon} \right) \\
&= \frac{\square + \rho c_1}{16c_1} g^{\alpha \beta} g^{\gamma \epsilon} - \frac{1}{8c_1} \left( g^{\alpha \gamma} g^{\beta \epsilon} + \bar{g}^{\beta \gamma} g^{\alpha \epsilon} \right)
\end{aligned}
\end{equation}

The other term needed to find the free energy (5.11) is
\begin{equation}
\left( (K - NM^{-1} N)^{-1} \right)^{\lambda \alpha \beta \gamma \epsilon} = \beta_1(s) g_{\alpha \beta \gamma \epsilon} g^{\lambda \tau} + \beta_2(s) g_{\alpha \gamma} g_{\beta \epsilon} g^{\lambda \tau} + \beta_3(s) \delta_\alpha^\lambda \delta_\gamma^\tau g_{\beta \epsilon} + \beta_4(s) \delta_\alpha^\lambda \delta_\beta^\tau g_{\gamma \epsilon} + \\
+ \beta_5(s) \delta_\alpha^\lambda \delta_\gamma^\epsilon g_{\beta \tau} + \beta_6(s) g_{\alpha \beta} g_{\gamma \epsilon} s^l s^\tau + \beta_7(s) g_{\alpha \gamma} g_{\beta \epsilon} s^l s^\tau + \beta_8(s) \delta_\alpha^\lambda s_{\gamma \epsilon} \tau s^l + \\
+ \beta_9(s) \delta_\alpha^\epsilon \delta_\beta^\gamma s_{\alpha \beta} s^l \tau + \beta_{10}(s) \delta_\alpha^\gamma \delta_\beta^\epsilon s_{\alpha \beta} s^l \tau + \beta_{11}(s) \delta_\epsilon^\gamma \delta_\alpha^\beta s_{\gamma \epsilon} s^l \tau + \beta_{12}(s) g_{\gamma \epsilon} g^{\lambda \tau} s_{\alpha \beta} \tau s^l + \\
+ \beta_{13}(s) \delta_\alpha^\epsilon \delta_\gamma^\beta s_{\alpha \beta} s^l \tau + \beta_{14}(s) g_{\epsilon \beta} g^{\lambda \tau} s_{\alpha \gamma} s^l \tau + \beta_{15}(s) \delta_\alpha^\lambda \delta_\gamma^\epsilon s_{\beta \tau} s^l \tau + \beta_{16}(s) \delta_\alpha^\lambda \delta_\epsilon^\gamma s_{\beta \tau} s^l \tau + \\
+ \beta_{17}(s) g_{\epsilon \beta} s_{\alpha \beta} s^l \tau \tau + \beta_{18}(s) \delta_\alpha^\lambda s_{\beta \gamma} s_{\epsilon \tau} s^l \tau + \beta_{19}(s) \delta_\alpha^\epsilon \delta_\gamma^\beta s_{\alpha \beta} s^l \tau + \\
+ \beta_{20}(s) \delta_\alpha^\epsilon \delta_\gamma^\beta s_{\alpha \beta} s^l \tau + \{ \alpha \leftrightarrow \beta \} + \{ \gamma \leftrightarrow \epsilon \} + \{ \lambda \alpha \beta \leftrightarrow \tau \gamma \epsilon \}
\end{equation}

This will look as the inverse obtained earlier for \( K^{-1} \) in flat space (5.19), but instead of constants, there will be a set of 22 functions determined by a system of ordinary differential equations similar to the ones solved for simple (but similar) models in the appendix D. The explicit expressions for those differential equations is even less illuminating than the flat space expressions so we refrain from considering them further. As an example, and in terms of the arc-length, \( s \), and its derivative, \( s_\mu \),
\begin{equation}
\left( (K - NM^{-1} N)^{-1} \right)^{\lambda \alpha \beta \gamma \epsilon} = \beta_1(s) g_{\alpha \beta \gamma \epsilon} g^{\lambda \tau} + \beta_2(s) g_{\alpha \gamma} g_{\beta \epsilon} g^{\lambda \tau} + \beta_3(s) \delta_\alpha^\lambda \delta_\gamma^\tau g_{\beta \epsilon} + \beta_4(s) \delta_\alpha^\lambda \delta_\beta^\tau g_{\gamma \epsilon} + \\
+ \beta_5(s) \delta_\alpha^\lambda \delta_\gamma^\epsilon g_{\beta \tau} + \beta_6(s) g_{\alpha \beta} g_{\gamma \epsilon} s^l s^\tau + \beta_7(s) g_{\alpha \gamma} g_{\beta \epsilon} s^l s^\tau + \beta_8(s) \delta_\alpha^\lambda s_{\gamma \epsilon} \tau s^l + \\
+ \beta_9(s) \delta_\alpha^\epsilon \delta_\beta^\gamma s_{\alpha \beta} s^l \tau + \beta_{10}(s) \delta_\alpha^\gamma \delta_\beta^\epsilon s_{\alpha \beta} s^l \tau + \beta_{11}(s) \delta_\epsilon^\gamma \delta_\alpha^\beta s_{\gamma \epsilon} s^l \tau + \beta_{12}(s) g_{\gamma \epsilon} g^{\lambda \tau} s_{\alpha \beta} \tau s^l + \\
+ \beta_{13}(s) \delta_\alpha^\epsilon \delta_\gamma^\beta s_{\alpha \beta} s^l \tau + \beta_{14}(s) g_{\epsilon \beta} g^{\lambda \tau} s_{\alpha \gamma} s^l \tau + \beta_{15}(s) \delta_\alpha^\lambda \delta_\gamma^\epsilon s_{\beta \tau} s^l \tau + \beta_{16}(s) \delta_\alpha^\lambda \delta_\epsilon^\gamma s_{\beta \tau} s^l \tau + \\
+ \beta_{17}(s) g_{\epsilon \beta} s_{\alpha \beta} s^l \tau \tau + \beta_{18}(s) \delta_\alpha^\lambda s_{\beta \gamma} s_{\epsilon \tau} s^l \tau + \beta_{19}(s) \delta_\alpha^\epsilon \delta_\gamma^\beta s_{\alpha \beta} s^l \tau + \\
+ \beta_{20}(s) \delta_\alpha^\epsilon \delta_\gamma^\beta s_{\alpha \beta} s^l \tau + \{ \alpha \leftrightarrow \beta \} + \{ \gamma \leftrightarrow \epsilon \} + \{ \lambda \alpha \beta \leftrightarrow \tau \gamma \epsilon \}
\end{equation}
6 Dynamical generation of the Einstein-Hilbert term

The theory so far considered is always in the conformal phase; it is Weyl invariant. This is the symmetry that prevents the appearance of a cosmological constant on the theory and ensures that all counterterms must be inside our list of quadratic operators.

This symmetry is not to be found at low energies, however; which means that it must be broken at some scale. Once this happens, both a cosmological constant and an Einstein-Hilbert term in the lagrangian are not forbidden anymore. Several scenarios for this breaking can be proposed; may be the simplest possibility [7, 24] is through interaction with a minimally coupled scalar sector

$$L_s = \sqrt{|g|} \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right)$$

(6.1)

Quantum corrections will include a term

$$\Delta L = C_\epsilon \epsilon R \phi^2$$

(6.2)

Were the scalar field to get a nonvanishing vacuum expectation value

$$\langle \phi \rangle = v$$

(6.3)

the counterterm implies an Einstein-Hilbert term

$$L_{EH} = M^2 \sqrt{|g|} R$$

(6.4)

The Planck scale $M$ is arbitrary, because it comes about through renormalization; nevertheless the only scale present in the problem to begin with is precisely the symmetry breaking one, $v$.

7 Conclusions

When considering quadratic in the Riemann tensor gravity theories in the first order formalism, quartic propagators never appear. The ensuing theory naively appears to be both renormalizable and unitary.

In order to laid out the terrain for future work we have considered all operators with the postulated symmetries and appropriate dimension, with arbitrary coupling constants in front of them. Even if we put some of them equal to zero in the classical lagrangian, quantum effects will generate all the different operators. This makes a grand total of sixteen free coupling constants, which fall naturally into three different groups.

Implicit in this general framework is that we have to give a physical interpretation not only to the spacetime metric, but also to the connection field (which behaves entirely as a complicated gauge field). It is clear that the theory space is much greater in the first order formulation than in the second order one. One of the first tasks we tackled was to analyze the equations of motion in order to examine what relationship is there between both formulations.

It is precisely this gauge field (id est, the variation of the connection) that encodes all information on the gravitational field. It is not compulsory to think that there are physical external sources for it, although we have examined this possibility as well. At any rate, we
have determined the conditions under which external sources yield a gravitational potential between external energy-momentum sources compatible with the observed one.

The interaction between two external graviton sources has been analyzed both in first-order Einstein-Hilbert and in our quadratic theories. In order for this general approach to be of any physical interest, the theory should generate a mass scale (Newton’s constant) through quantum effects. Do not forget that our general framework is Weyl invariant and, correspondingly, all coupling constants are dimensionless as long as the theory remains in the conformal phase. It is then only natural that this process would be related to the spontaneous breaking of Weyl (conformal) symmetry through matter effects, as we suggested earlier in the text, at least in the asymptotically free branch [3–5] of the theory. Then the Einstein-Hilbert term

\[ S_{EH} = M_p^2 \int d^4x \sqrt{|g|} g^{\mu\nu} R_{\mu\nu} \]  

which is not Weyl invariant could be generated by quantum corrections. Were the breaking explicit, it could of course spoil the renormalizability of the theory. But it is known that some theories, like QCD, can dynamically generate a mass scale (like \( \Lambda_{QCD} \)) while preventing Einstein-Hilbert-like terms to appear in the lagrangian. These terms would then appear in effective low energy theories in terms of different dynamical fields. Indeed, in [11] the related conjecture was put forward that the spin 2 ghost that appears in the (second order) quadratic Stelle [3–5] lagrangians does not appear in the physical spectrum. Similar ideas have been put forward in a related context in [24].

If the confining scale of our theory in this sector is \( \Lambda_{QG} \), this means that the theory would be strongly coupled in the infrared; but then General Relativity would be an adequate effective theory, playing a somewhat similar relationship with the full theory as chiral effective theories play with respect to QCD.

It has been suggested [26] that the ultraviolet completion of some theories involve a mechanism dubbed as classicalization. The main idea is that instead of a strong coupling phase, the ultraviolet regime involves a high multiplicity of quanta. Owing to this high occupation number, the classical approximation is enough to describe this phase.

These process is suggested by the usual (Schwarzschild) black hole physics and the consequent area law for the entropy. It is not known to what extent they apply to theories quadratic in curvature. There is no Birkhoff theorem that applies there, and there are now three asymptotic families of spherically symmetric solutions [27]( in the second order formulation). One of them, that can be matched to an asymptotically flat solution at spatial infinity without encountering a horizon. Another one that contains both Schwarzschild and non-Schwarzschild black holes. Finally, a third family which is nonsingular and corresponds to vacuum solutions.

These facts shed doubts on whether the classicalization mechanism would apply to our theory. At any rate, this problem deserves further thought.

We have only begun to scratch the surface of this beautiful framework. There remains in particular, to understand the spin content of the three-index gauge field as well as to compute quantum corrections and check explicitly that everything works according to our expectations. This computation is not altogether trivial owing to the appearance of non-minimal operators, which need a special treatment.

It is plain that this whole approach is related to the age-old question as to what are the fundamental variables in gravitation; the metric or the connection. Work is in progress in this and related matters.

\[ – 26 – \]
A The variation of the Levi Civita is not the Levi-Civita of the variation

The fact that there are two different Ricci tensors for a general connection, $R^+_{\mu \nu}$ and $R^-_{\mu \nu}$, implies the at first sight surprising fact that it is not the same thing to first put the action on shell (that is, assume the connection is a Levi-Civita one) and then compute its variation or doing things in the opposite order, that is compute the general variation, an then putting the variation on Leci-Civita shell.

Here we would like to point out that for the Einstein-Hilbert term, this two operations do in fact commute. We define the action like

$$S_{EH}^+ = \int d^nx \sqrt{|g|} g^{\mu \nu} R^+_{\mu \nu} = \int d^nx \sqrt{|g|} g^{\mu \nu} R^+_{\mu \lambda \nu}$$

(A.1)

and

$$S_{EH}^- = \int d^nx \sqrt{|g|} g^{\mu \nu} R^-_{\mu \nu} = \int d^nx \sqrt{|g|} g^{\mu \nu} g^{\rho \sigma} R_{\rho \sigma \mu \nu} = \int d^nx \sqrt{|g|} g^{\mu \nu} R^-_{\mu \lambda \nu}$$

(A.2)

Therefore, doing Levi-Civita first means that $S_{EH}^+ = - S_{EH}^-$. On the other hand, if we perform the background field expansion first in $S_{EH}^-$$\delta S_{EH}^- = \int d^nx \sqrt{|g|} \left\{ \frac{1}{2} g_{\alpha \beta} \partial^\gamma \left[ \frac{1}{2} h_{\gamma \epsilon} h_{\epsilon \delta} \right] - \int d^nx \sqrt{|g|} \left( \delta_\mu^\gamma \delta_\nu^\lambda \nabla_\lambda - \delta_\nu^\lambda \nabla_\lambda \right) \right\} A^\lambda_{\alpha \beta}$

(A.3)
doing now Levi-Civita $\bar{R}_{\mu\nu}^+ = -R_{\mu\nu}^- \gamma \bar{R}^+ = -\bar{R}^-$ we get

$$
\delta S^{|_{\bar{g}_{\mu\nu}=\bar{g}_{\mu\nu}}} = - \delta S^{|_{g_{\mu\nu}=\bar{g}_{\mu\nu}}} \tag{A.4}
$$

To conclude, in first order Einstein-Hilbert, these two operations do in fact commute.

B Details on the background expansion

The equation of motion for the graviton and the gauge field read respectively

$${\frac{\delta S}{\delta g_{\alpha\beta}}}{|_{g_{\mu\nu}=\bar{g}_{\mu\nu}}} = \kappa\sqrt{|g|} \left\{ \frac{1}{2} g_{\alpha\beta} \bar{\mathcal{L}} + g_1 \left\{ \bar{R}^\alpha_{\nu\rho\sigma} \bar{R}^{\beta\mu\nu\rho\sigma} - \bar{R}^\alpha_{\mu\rho\sigma} \bar{R}^{\beta\mu\rho\sigma} - \bar{R}^\alpha_{\mu\nu\rho} \bar{R}^{\beta\mu\nu\rho} - \bar{R}^\alpha_{\mu\nu\rho\sigma} \bar{R}^{\beta\mu\nu\rho\sigma} \right\} - 
- g_2 \left\{ \bar{R}^\alpha_{\nu\rho\sigma} \bar{R}^{\beta\mu\nu\rho\sigma} - \bar{R}^\alpha_{\mu\rho\sigma} \bar{R}^{\beta\mu\rho\sigma} - \bar{R}^\alpha_{\mu\nu\rho} \bar{R}^{\beta\mu\nu\rho} - \bar{R}^\alpha_{\mu\nu\rho\sigma} \bar{R}^{\beta\mu\nu\rho\sigma} \right\} - 
- g_3 \left\{ \bar{R}^\alpha_{\nu\rho\sigma} \bar{R}^{\beta\mu\nu\rho\sigma} + \bar{R}^\alpha_{\mu\nu\rho\sigma} \bar{R}^{\beta\mu\nu\rho\sigma} \right\} - g_4 \left\{ \bar{R}^\alpha_{\nu\rho\sigma} \bar{R}^{\beta\mu\nu\rho\sigma} + \bar{R}^\alpha_{\mu\nu\rho\sigma} \bar{R}^{\beta\mu\nu\rho\sigma} \right\} - 
- g_5 \left\{ \bar{R}^\alpha_{\nu\rho\sigma} \bar{R}^{\beta\mu\nu\rho\sigma} + \bar{R}^\alpha_{\mu\nu\rho\sigma} \bar{R}^{\beta\mu\nu\rho\sigma} \right\} - g_6 \left\{ \bar{R}^\alpha_{\nu\rho\sigma} \bar{R}^{\beta\mu\nu\rho\sigma} + \bar{R}^\alpha_{\mu\nu\rho\sigma} \bar{R}^{\beta\mu\nu\rho\sigma} \right\} - 
- g_7 \left\{ \bar{R}^\alpha_{\nu\rho\sigma} \bar{R}^{\beta\mu\nu\rho\sigma} + \bar{R}^\alpha_{\mu\nu\rho\sigma} \bar{R}^{\beta\mu\nu\rho\sigma} \right\} - g_8 \left\{ \bar{R}^\alpha_{\nu\rho\sigma} \bar{R}^{\beta\mu\nu\rho\sigma} + \bar{R}^\alpha_{\mu\nu\rho\sigma} \bar{R}^{\beta\mu\nu\rho\sigma} \right\} - 
- g_9 \left\{ \bar{R}^\alpha_{\nu\rho\sigma} \bar{R}^{\beta\mu\nu\rho\sigma} + \bar{R}^\alpha_{\mu\nu\rho\sigma} \bar{R}^{\beta\mu\nu\rho\sigma} \right\} - g_{10} \left\{ \bar{R}^\alpha_{\nu\rho\sigma} \bar{R}^{\beta\mu\nu\rho\sigma} + \bar{R}^\alpha_{\mu\nu\rho\sigma} \bar{R}^{\beta\mu\nu\rho\sigma} \right\} - 
- g_{11} \left\{ \bar{R}^\alpha_{\nu\rho\sigma} \bar{R}^{\beta\mu\nu\rho\sigma} + \bar{R}^\alpha_{\mu\nu\rho\sigma} \bar{R}^{\beta\mu\nu\rho\sigma} \right\} - g_{12} \left\{ \bar{R}^\alpha_{\nu\rho\sigma} \bar{R}^{\beta\mu\nu\rho\sigma} + \bar{R}^\alpha_{\mu\nu\rho\sigma} \bar{R}^{\beta\mu\nu\rho\sigma} \right\} - 
- g_{13} \left\{ \bar{R}^\alpha_{\nu\rho\sigma} \bar{R}^{\beta\mu\nu\rho\sigma} + \bar{R}^\alpha_{\mu\nu\rho\sigma} \bar{R}^{\beta\mu\nu\rho\sigma} \right\} - g_{14} \left\{ \bar{R}^\alpha_{\nu\rho\sigma} \bar{R}^{\beta\mu\nu\rho\sigma} + \bar{R}^\alpha_{\mu\nu\rho\sigma} \bar{R}^{\beta\mu\nu\rho\sigma} \right\} - 
- g_{15} \left\{ \bar{R}^\alpha_{\nu\rho\sigma} \bar{R}^{\beta\mu\nu\rho\sigma} + \bar{R}^\alpha_{\mu\nu\rho\sigma} \bar{R}^{\beta\mu\nu\rho\sigma} \right\} - g_{16} \left\{ \bar{R}^\alpha_{\nu\rho\sigma} \bar{R}^{\beta\mu\nu\rho\sigma} + \bar{R}^\alpha_{\mu\nu\rho\sigma} \bar{R}^{\beta\mu\nu\rho\sigma} \right\} \right\} + \{ \alpha \leftrightarrow \beta \} \tag{B.1}
$$

$$
\delta S \left. \right|_{g_{\alpha\beta}=\bar{g}_{\alpha\beta}} = \sqrt{|g|} \left\{ 2g_1 \left( \nabla_\mu \bar{R}^{\alpha\beta}_{\lambda} - \nabla_\sigma \bar{R}^{\alpha\beta}_{\lambda} \right) + 2g_2 \left( \nabla_\mu \bar{R}^{\alpha\beta}_{\lambda} - \nabla_\sigma \bar{R}^{\alpha\beta}_{\lambda} \right) + 
+ 2g_3 \left( \nabla_\mu \bar{R}^{\alpha\beta}_{\lambda} - \nabla_\sigma \bar{R}^{\alpha\beta}_{\lambda} \right) + 
+ g_4 \left( \nabla_\mu \bar{R}^{\alpha\beta}_{\lambda} - \nabla_\sigma \bar{R}^{\alpha\beta}_{\lambda} \right) \right\} + 
+ 2g_5 \left( \nabla_\mu \bar{R}^{\alpha\beta}_{\lambda} - \nabla_\sigma \bar{R}^{\alpha\beta}_{\lambda} \right) + 
+ 2g_6 \left( \nabla_\mu \bar{R}^{\alpha\beta}_{\lambda} - \nabla_\sigma \bar{R}^{\alpha\beta}_{\lambda} \right) + 
+ 2g_7 \left( \nabla_\mu \bar{R}^{\alpha\beta}_{\lambda} - \nabla_\sigma \bar{R}^{\alpha\beta}_{\lambda} \right) + 
+ 2g_8 \left( \nabla_\mu \bar{R}^{\alpha\beta}_{\lambda} - \nabla_\sigma \bar{R}^{\alpha\beta}_{\lambda} \right) + 
+ 2g_9 \left( \nabla_\mu \bar{R}^{\alpha\beta}_{\lambda} - \nabla_\sigma \bar{R}^{\alpha\beta}_{\lambda} \right) + 
+ 2g_{10} \left( \nabla_\mu \bar{R}^{\alpha\beta}_{\lambda} - \nabla_\sigma \bar{R}^{\alpha\beta}_{\lambda} \right) + 
+ 2g_{11} \left( \nabla_\mu \bar{R}^{\alpha\beta}_{\lambda} - \nabla_\sigma \bar{R}^{\alpha\beta}_{\lambda} \right) + 
+ 2g_{12} \left( \nabla_\mu \bar{R}^{\alpha\beta}_{\lambda} - \nabla_\sigma \bar{R}^{\alpha\beta}_{\lambda} \right) + 
+ 2g_{13} \left( \nabla_\mu \bar{R}^{\alpha\beta}_{\lambda} - \nabla_\sigma \bar{R}^{\alpha\beta}_{\lambda} \right) \right\} \tag{B.1}
$$

\[\text{– 28 –}\]
The quadratic operator relating graviton-graviton fluctuations is

\[ M^{\alpha\beta\gamma} = \kappa^2 \left\{ g_{14} \left\{ \left( \frac{1}{2} g^{\alpha\beta} g^{\gamma} - \frac{1}{2} g^{\alpha\gamma} g^{\beta} \right) R_{\mu
u\rho\sigma} R^{\mu\nu\rho\sigma} + g^{\gamma} \left\{ R^{\alpha}_{\nu\rho} R^{\beta\mu\rho\sigma} - R_{\mu\rho} \right\} \right\} + g_{15} \left\{ \left( \frac{1}{2} g^{\alpha\beta} g^{\gamma} - \frac{1}{2} g^{\alpha\gamma} g^{\beta} \right) \right\} + g_{16} \left\{ \alpha \leftrightarrow \beta \right\} \right\} \]

\[ + 2g_{16} \left\{ \left( \frac{1}{2} g^{\alpha\beta} g^{\gamma} - \frac{1}{2} g^{\alpha\gamma} g^{\beta} \right) \right\} \]
\[-g^{\gamma\epsilon}\left\{-R^\alpha_{\sigma} R^{\beta\sigma} - R^\mu_{\alpha\beta}\sigma R_{\mu\sigma} + R^\mu_{\sigma\gamma} R^{\alpha\beta\sigma} + R^\mu_{\gamma\mu} R^{\alpha\beta}\right\} + \\
+2\left\{-R^\alpha_{\sigma\epsilon} R^{\beta\sigma} - R^\mu_{\alpha\beta}\sigma R_{\mu\sigma} - R^\mu_{\alpha\beta}\sigma R_{\mu\sigma} + R^\mu_{\sigma\gamma} R^{\alpha\beta\sigma} + R^\mu_{\gamma\mu} R^{\alpha\beta}\right\}\right\} + \\
+g_{10}\left\{\left(\frac{1}{4}g^{\alpha\beta} g^{\gamma\epsilon} - \frac{1}{2}g^{\alpha\gamma} g^{\beta\epsilon}\right) R^\mu_{\gamma\mu} R^{\mu\nu} - g^{\gamma\epsilon}\left\{R^\alpha_{\sigma\epsilon} R^{\beta\sigma} + R^\mu_{\sigma\gamma} R^{\alpha\beta\sigma}\right\} + \\
+2\left\{R^\alpha_{\gamma\mu} R^{\beta\sigma} + R^\alpha_{\sigma\gamma} R^{\beta\sigma} + R^\alpha_{\gamma\mu} R^{\beta\sigma}\right\}\right\} + \\
+g_{11}\left\{\left(\frac{1}{4}g^{\alpha\beta} g^{\gamma\epsilon} - \frac{1}{2}g^{\alpha\gamma} g^{\beta\epsilon}\right) R^\mu_{\gamma\mu} R^{\mu\nu} - g^{\gamma\epsilon}\left\{R^\alpha_{\sigma\epsilon} R^{\beta\sigma} + R^\mu_{\sigma\gamma} R^{\alpha\beta\sigma}\right\} + \\
+2\left\{R^\alpha_{\gamma\mu} R^{\beta\sigma} + R^\alpha_{\sigma\gamma} R^{\beta\sigma} + R^\alpha_{\gamma\mu} R^{\beta\sigma}\right\}\right\} + \\
+g_{12}\left\{\left(\frac{1}{4}g^{\alpha\beta} g^{\gamma\epsilon} - \frac{1}{2}g^{\alpha\gamma} g^{\beta\epsilon}\right) R^\mu_{\gamma\mu} R^{\mu\nu} - g^{\gamma\epsilon}\left\{R^\alpha_{\sigma\epsilon} R^{\beta\sigma} + R^\mu_{\sigma\gamma} R^{\alpha\beta\sigma}\right\} + \\
+2\left\{R^\alpha_{\gamma\mu} R^{\beta\sigma} + R^\alpha_{\sigma\gamma} R^{\beta\sigma} + R^\alpha_{\gamma\mu} R^{\beta\sigma}\right\}\right\} + \\
+g_{13}\left\{\left(\frac{1}{4}g^{\alpha\beta} g^{\gamma\epsilon} - \frac{1}{2}g^{\alpha\gamma} g^{\beta\epsilon}\right) R^\mu_{\gamma\mu} R^{\mu\nu} - g^{\gamma\epsilon}\left\{R^\alpha_{\sigma\epsilon} R^{\beta\sigma} + R^\mu_{\sigma\gamma} R^{\alpha\beta\sigma}\right\} + \\
+2\left\{R^\alpha_{\gamma\mu} R^{\beta\sigma} + R^\alpha_{\sigma\gamma} R^{\beta\sigma} + R^\alpha_{\gamma\mu} R^{\beta\sigma}\right\}\right\} + \\
+g_{14}\left\{\left(\frac{1}{4}g^{\alpha\beta} g^{\gamma\epsilon} - \frac{1}{2}g^{\alpha\gamma} g^{\beta\epsilon}\right) R^\mu_{\gamma\mu} R^{\mu\nu} - g^{\gamma\epsilon}\left\{R^\alpha_{\sigma\epsilon} R^{\beta\sigma} + R^\mu_{\sigma\gamma} R^{\alpha\beta\sigma}\right\} + \\
+2\left\{R^\alpha_{\gamma\mu} R^{\beta\sigma} + R^\alpha_{\sigma\gamma} R^{\beta\sigma} + R^\alpha_{\gamma\mu} R^{\beta\sigma}\right\}\right\} + \\
+g_{15}\left\{\left(\frac{1}{4}g^{\alpha\beta} g^{\gamma\epsilon} - \frac{1}{2}g^{\alpha\gamma} g^{\beta\epsilon}\right) R^\mu_{\gamma\mu} R^{\mu\nu} - g^{\gamma\epsilon}\left\{R^\alpha_{\sigma\epsilon} R^{\beta\sigma} + R^\mu_{\sigma\gamma} R^{\alpha\beta\sigma}\right\} + \\
+2\left\{R^\alpha_{\gamma\mu} R^{\beta\sigma} + R^\alpha_{\sigma\gamma} R^{\beta\sigma} + R^\alpha_{\gamma\mu} R^{\beta\sigma}\right\}\right\} + \\
+g_{16}\left\{\left(\frac{1}{4}g^{\alpha\beta} g^{\gamma\epsilon} - \frac{1}{2}g^{\alpha\gamma} g^{\beta\epsilon}\right) R^\mu_{\gamma\mu} R^{\mu\nu} - g^{\gamma\epsilon}\left\{R^\alpha_{\sigma\epsilon} R^{\beta\sigma} + R^\mu_{\sigma\gamma} R^{\alpha\beta\sigma}\right\} + \\
+2\left\{R^\alpha_{\gamma\mu} R^{\beta\sigma} + R^\alpha_{\sigma\gamma} R^{\beta\sigma} + R^\alpha_{\gamma\mu} R^{\beta\sigma}\right\}\right\}\right\}\right\}\right\} + \{\alpha \leftrightarrow \beta\} + \{\gamma \leftrightarrow \epsilon\}

(B.3)

The mixing term between graviton and gauge fluctuations reads

\[ N^\alpha_{\lambda\gamma\epsilon} = \kappa g_{\gamma\epsilon} \sum_{l=1}^{16} g_1 R^{\mu\nu} (D_l)^{\nu\rho\sigma\sigma}_{\mu\mu} (\delta^{\beta\gamma\epsilon}_{\rho\sigma} - \delta^{\beta\gamma}_{\rho\sigma} \nabla^\rho) \]

\[ \kappa \left\{ 2g_1 \left\{ g_{\gamma\lambda} \nabla^\rho - g_{\gamma\sigma} \nabla^\rho - g_{\gamma\epsilon} \nabla^\rho + \delta^\beta_{\gamma\sigma} \nabla^\rho + R^\lambda_{\epsilon} \nabla^\gamma - \delta^\gamma_{\rho\sigma} \nabla^\rho - \delta^\gamma_{\rho\sigma} \nabla^\rho + \delta^\gamma_{\rho\sigma} \nabla^\rho + \delta^\gamma_{\rho\sigma} \nabla^\rho \right\} + \\
+2g_2 \left\{ g_{\gamma\lambda} R^\epsilon_{\alpha\beta\sigma} \nabla^\rho - g_{\gamma\sigma} R^\epsilon_{\alpha\beta\sigma} \nabla^\rho - \delta^\gamma_{\rho\sigma} R^\lambda_{\rho\sigma} \nabla^\rho + \delta^\gamma_{\rho\sigma} R^\lambda_{\rho\sigma} \nabla^\rho + \delta^\gamma_{\rho\sigma} R^\lambda_{\rho\sigma} \nabla^\rho + \delta^\gamma_{\rho\sigma} R^\lambda_{\rho\sigma} \nabla^\rho + \right\} \} \]
\[-\mathcal{R}_\lambda^{\alpha^\beta} \nabla_\gamma + \delta^{\alpha^\beta}_\gamma \mathcal{R}_\lambda^\gamma - \delta^{\alpha^\beta}_\gamma \mathcal{R}_\lambda^\gamma - \mathcal{R}_\lambda^{\alpha^\beta} \nabla_\gamma + \mathcal{R}_\lambda^{\alpha^\beta} \nabla_\gamma \right\} + \\
+ 2g_3 \left\{ -\mathcal{R}_\lambda^{\alpha^\beta} \nabla_\gamma + \delta^{\alpha^\beta}_\gamma \mathcal{R}_\lambda^\gamma - \mathcal{R}_\lambda^{\alpha^\beta} \nabla_\gamma \right\} + \\
+ 2g_4 \left\{ -\mathcal{R}_\lambda^{\alpha^\beta} \nabla_\gamma + \delta^{\alpha^\beta}_\gamma \mathcal{R}_\lambda^\gamma - \mathcal{R}_\lambda^{\alpha^\beta} \nabla_\gamma \right\} + \\
+ 2g_5 \left\{ -\mathcal{R}_\lambda^{\alpha^\beta} \nabla_\gamma + \delta^{\alpha^\beta}_\gamma \mathcal{R}_\lambda^\gamma - \mathcal{R}_\lambda^{\alpha^\beta} \nabla_\gamma \right\} + \\
+ 2g_6 \left\{ -\mathcal{R}_\lambda^{\alpha^\beta} \nabla_\gamma + \delta^{\alpha^\beta}_\gamma \mathcal{R}_\lambda^\gamma - \mathcal{R}_\lambda^{\alpha^\beta} \nabla_\gamma \right\} + \\
+ 2g_7 \left\{ -\mathcal{R}_\lambda^{\alpha^\beta} \nabla_\gamma + \delta^{\alpha^\beta}_\gamma \mathcal{R}_\lambda^\gamma - \mathcal{R}_\lambda^{\alpha^\beta} \nabla_\gamma \right\} + \\
+ 2g_8 \left\{ -\mathcal{R}_\lambda^{\alpha^\beta} \nabla_\gamma + \delta^{\alpha^\beta}_\gamma \mathcal{R}_\lambda^\gamma - \mathcal{R}_\lambda^{\alpha^\beta} \nabla_\gamma \right\} + \\
+ 2g_9 \left\{ g^{\lambda\gamma} \mathcal{R}_\lambda^{\alpha^\beta} \nabla_\gamma - \mathcal{R}_\lambda^{\alpha^\beta} \nabla_\gamma \right\} + \\
+ 2g_{10} \left\{ g^{\lambda\gamma} \mathcal{R}_\lambda^{\alpha^\beta} \nabla_\gamma - \mathcal{R}_\lambda^{\alpha^\beta} \nabla_\gamma \right\} + \\
+ 2g_{11} \left\{ g^{\lambda\gamma} \mathcal{R}_\lambda^{\alpha^\beta} \nabla_\gamma - \mathcal{R}_\lambda^{\alpha^\beta} \nabla_\gamma \right\} + \\
+ 2g_{12} \left\{ g^{\lambda\gamma} \mathcal{R}_\lambda^{\alpha^\beta} \nabla_\gamma - \mathcal{R}_\lambda^{\alpha^\beta} \nabla_\gamma \right\} + \\
+ 2g_{13} \left\{ g^{\lambda\gamma} \mathcal{R}_\lambda^{\alpha^\beta} \nabla_\gamma - \mathcal{R}_\lambda^{\alpha^\beta} \nabla_\gamma \right\} + \\
+ 2g_{14} \left\{ g^{\lambda\gamma} \mathcal{R}_\lambda^{\alpha^\beta} \nabla_\gamma - \mathcal{R}_\lambda^{\alpha^\beta} \nabla_\gamma \right\} + \\
+ 2g_{15} \left\{ g^{\lambda\gamma} \mathcal{R}_\lambda^{\alpha^\beta} \nabla_\gamma - \mathcal{R}_\lambda^{\alpha^\beta} \nabla_\gamma \right\} + \\
+ 2g_{16} \left\{ g^{\lambda\gamma} \mathcal{R}_\lambda^{\alpha^\beta} \nabla_\gamma - \mathcal{R}_\lambda^{\alpha^\beta} \nabla_\gamma \right\} + \\
+ \{ \alpha \leftrightarrow \beta \} + \{ \gamma \leftrightarrow \epsilon \}

\text{(B.4)}

The quadratic term involving gauge fluctuations with themselves reads

\[
\mathcal{L}^{\alpha^\beta\gamma^\delta}_{\tau} = 2g_1 \left\{ 2\delta^\alpha_{\tau} \left( \mathcal{R}_\lambda^{\beta\gamma^\delta} - \mathcal{R}_\lambda^{\gamma^\delta\beta} \right) + 2 \left( \nabla^\beta g^\alpha\gamma^\delta - \nabla^\gamma^\delta g^\alpha\gamma^\delta - \nabla^\delta g^\alpha\gamma^\delta - \nabla^\gamma g^\alpha\gamma^\delta \right) \right\} + \\
+ 2g_2 \left\{ 2\delta^\alpha_{\tau} \left( \mathcal{R}_\lambda^{\beta^\gamma^\delta} - \mathcal{R}_\lambda^{\gamma^\delta^\beta} \right) + 2 \left( \nabla^\beta g^\alpha\gamma^\delta - \nabla^\gamma^\delta g^\alpha\gamma^\delta - \nabla^\delta g^\alpha\gamma^\delta - \nabla^\gamma g^\alpha\gamma^\delta \right) \right\} + \\
+ 2g_3 \left\{ 2\delta^\alpha_{\tau} \left( \mathcal{R}_\lambda^{\beta^\gamma^\delta} - \mathcal{R}_\lambda^{\gamma^\delta^\beta} \right) + 2 \left( \nabla^\beta g^\alpha\gamma^\delta - \nabla^\gamma^\delta g^\alpha\gamma^\delta - \nabla^\delta g^\alpha\gamma^\delta - \nabla^\gamma g^\alpha\gamma^\delta \right) \right\} + \\
+ 2g_4 \left\{ 2\delta^\alpha_{\tau} \left( \mathcal{R}_\lambda^{\beta^\gamma^\delta} - \mathcal{R}_\lambda^{\gamma^\delta^\beta} \right) + 2 \left( \nabla^\beta g^\alpha\gamma^\delta - \nabla^\gamma^\delta g^\alpha\gamma^\delta - \nabla^\delta g^\alpha\gamma^\delta - \nabla^\gamma g^\alpha\gamma^\delta \right) \right\} + \\
+ 2g_5 \left\{ 2\delta^\alpha_{\tau} \left( \mathcal{R}_\lambda^{\beta^\gamma^\delta} - \mathcal{R}_\lambda^{\gamma^\delta^\beta} \right) + 2 \left( \nabla^\beta g^\alpha\gamma^\delta - \nabla^\gamma^\delta g^\alpha\gamma^\delta - \nabla^\delta g^\alpha\gamma^\delta - \nabla^\gamma g^\alpha\gamma^\delta \right) \right\} + \\
+ \{ \alpha \leftrightarrow \beta \} + \{ \gamma \leftrightarrow \epsilon \}
\]
\[ + 2g_6 \left\{ 2\delta^\alpha_\gamma \left( \tilde{R}^{\beta\gamma}_{\lambda\tau} - \tilde{R}^{\alpha\beta}_{\lambda\tau} \right) + \right. \]
\[ + \left. \left( \tilde{\nabla}_\nu \tilde{\nabla}_\lambda \delta^\alpha_\gamma g^{\gamma\nu} - \tilde{\nabla}_\lambda \tilde{\nabla}_\nu g^{\gamma\nu} \right) g^{\beta\epsilon} + \tilde{\nabla}_\nu \tilde{\nabla}_\lambda \delta^\alpha_\gamma g^{\gamma\nu} - \tilde{\nabla}_\lambda \tilde{\nabla}_\nu \delta^\alpha_\gamma \right) \right) + \]
\[ + 2g_7 \left\{ 2\delta^\alpha_\gamma \right. \left( \delta^\beta_\lambda \tilde{R}^{\epsilon\gamma}_{\tau\nu} - \delta^\beta_\tau \tilde{R}^{\gamma\epsilon}_{\nu\lambda} \right) + \]
\[ + \left. \left( \tilde{\nabla}_\lambda \tilde{\nabla}_\beta \delta^\alpha_\gamma g^{\gamma\nu} - \tilde{\nabla}_\lambda \tilde{\nabla}_\nu g^{\gamma\nu} \left( \delta^\beta_\tau \tilde{R}^{\gamma\epsilon}_{\nu\lambda} - \tilde{\nabla}_\nu \delta^\beta_\tau \right) g^{\gamma\tau} \right) \right) + \]
\[ + 2g_8 \left\{ 2\delta^\alpha_\gamma \right. \left( \delta^\beta_\lambda \tilde{R}^{\gamma\epsilon}_{\tau\nu} - \delta^\beta_\tau \tilde{R}^{\gamma\epsilon}_{\nu\lambda} \right) + \]
\[ + \left. \left( \tilde{\nabla}_\lambda \tilde{\nabla}_\beta \delta^\alpha_\gamma g^{\gamma\nu} - \tilde{\nabla}_\lambda \tilde{\nabla}_\nu g^{\gamma\nu} \left( \delta^\beta_\tau \tilde{R}^{\gamma\epsilon}_{\nu\lambda} - \tilde{\nabla}_\nu \delta^\beta_\tau \right) g^{\gamma\tau} \right) \right) + \]
\[ + 2g_9 \left\{ 2\delta^\alpha_\gamma \right. \left( \tilde{g}^{\beta} \tilde{R}^{\epsilon\gamma}_{\lambda} - \tilde{g}^{\gamma} \tilde{R}^{\alpha\beta}_{\lambda} \right) + \]
\[ + \left. \left( \tilde{\nabla}_\lambda \tilde{\nabla}_\beta \delta^\alpha_\gamma g^{\gamma\nu} - \tilde{\nabla}_\lambda \tilde{\nabla}_\nu g^{\gamma\nu} \left( \delta^\beta_\tau \tilde{R}^{\gamma\epsilon}_{\nu\lambda} - \tilde{\nabla}_\nu \delta^\beta_\tau \right) g^{\gamma\tau} \right) \right) + \]
\[ + 2g_{10} \left\{ 2\delta^\alpha_\gamma \right. \left( \tilde{g}^{\beta} \tilde{R}^{\epsilon\gamma}_{\lambda} - \tilde{g}^{\gamma} \tilde{R}^{\alpha\beta}_{\lambda} \right) + \]
\[ + \left. \left( \tilde{\nabla}_\lambda \tilde{\nabla}_\beta \delta^\alpha_\gamma g^{\gamma\nu} - \tilde{\nabla}_\lambda \tilde{\nabla}_\nu g^{\gamma\nu} \left( \delta^\beta_\tau \tilde{R}^{\gamma\epsilon}_{\nu\lambda} - \tilde{\nabla}_\nu \delta^\beta_\tau \right) g^{\gamma\tau} \right) \right) + \]
\[ + 2g_{11} \left\{ 2\delta^\alpha_\gamma \right. \left( \tilde{g}^{\beta} \tilde{R}^{\epsilon\gamma}_{\lambda} - \tilde{g}^{\gamma} \tilde{R}^{\alpha\beta}_{\lambda} \right) + \]
\[ + \left. \left( \tilde{\nabla}_\lambda \tilde{\nabla}_\beta \delta^\alpha_\gamma g^{\gamma\nu} - \tilde{\nabla}_\lambda \tilde{\nabla}_\nu g^{\gamma\nu} \left( \delta^\beta_\tau \tilde{R}^{\gamma\epsilon}_{\nu\lambda} - \tilde{\nabla}_\nu \delta^\beta_\tau \right) g^{\gamma\tau} \right) \right) + \]
\[ + 2g_{12} \left\{ 2\delta^\alpha_\gamma \right. \left( \tilde{g}^{\beta} \tilde{R}^{\epsilon\gamma}_{\lambda} - \tilde{g}^{\gamma} \tilde{R}^{\alpha\beta}_{\lambda} \right) + \]
\[ + \left. \left( \tilde{\nabla}_\lambda \tilde{\nabla}_\beta \delta^\alpha_\gamma g^{\gamma\nu} - \tilde{\nabla}_\lambda \tilde{\nabla}_\nu g^{\gamma\nu} \left( \delta^\beta_\tau \tilde{R}^{\gamma\epsilon}_{\nu\lambda} - \tilde{\nabla}_\nu \delta^\beta_\tau \right) g^{\gamma\tau} \right) \right) + \]
\[ + 2g_{13} \left\{ 2\delta^\alpha_\gamma \right. \left( \tilde{g}^{\beta} \tilde{R}^{\epsilon\gamma}_{\lambda} - \tilde{g}^{\gamma} \tilde{R}^{\alpha\beta}_{\lambda} \right) + \]
\[ + \left. \left( \tilde{\nabla}_\lambda \tilde{\nabla}_\beta \delta^\alpha_\gamma g^{\gamma\nu} - \tilde{\nabla}_\lambda \tilde{\nabla}_\nu g^{\gamma\nu} \left( \delta^\beta_\tau \tilde{R}^{\gamma\epsilon}_{\nu\lambda} - \tilde{\nabla}_\nu \delta^\beta_\tau \right) g^{\gamma\tau} \right) \right) + \]
\[ + 2g_{14} \left\{ 2\delta^\alpha_\gamma \right. \left( \tilde{g}^{\beta} \tilde{R}^{\epsilon\gamma}_{\lambda} - \tilde{g}^{\gamma} \tilde{R}^{\alpha\beta}_{\lambda} \right) + \]
\[ + \left. \left( \tilde{\nabla}_\lambda \tilde{\nabla}_\beta \delta^\alpha_\gamma g^{\gamma\nu} - \tilde{\nabla}_\lambda \tilde{\nabla}_\nu g^{\gamma\nu} \left( \delta^\beta_\tau \tilde{R}^{\gamma\epsilon}_{\nu\lambda} - \tilde{\nabla}_\nu \delta^\beta_\tau \right) g^{\gamma\tau} \right) \right) + \]
\[ + 2g_{15} \left\{ 2\delta^\alpha_\gamma \right. \left( \tilde{g}^{\beta} \tilde{R}^{\epsilon\gamma}_{\lambda} - \tilde{g}^{\gamma} \tilde{R}^{\alpha\beta}_{\lambda} \right) + \]
\[ + \left. \left( \tilde{\nabla}_\lambda \tilde{\nabla}_\beta \delta^\alpha_\gamma g^{\gamma\nu} - \tilde{\nabla}_\lambda \tilde{\nabla}_\nu g^{\gamma\nu} \left( \delta^\beta_\tau \tilde{R}^{\gamma\epsilon}_{\nu\lambda} - \tilde{\nabla}_\nu \delta^\beta_\tau \right) g^{\gamma\tau} \right) \right) + \]
\[ + 2g_{16} \left\{ 2\delta^\alpha_\gamma \right. \left( \tilde{g}^{\beta} \tilde{R}^{\epsilon\gamma}_{\lambda} - \tilde{g}^{\gamma} \tilde{R}^{\alpha\beta}_{\lambda} \right) + \]
\[ + \left. \left( \tilde{\nabla}_\lambda \tilde{\nabla}_\beta \delta^\alpha_\gamma g^{\gamma\nu} - \tilde{\nabla}_\lambda \tilde{\nabla}_\nu g^{\gamma\nu} \left( \delta^\beta_\tau \tilde{R}^{\gamma\epsilon}_{\nu\lambda} - \tilde{\nabla}_\nu \delta^\beta_\tau \right) g^{\gamma\tau} \right) \right) + \]
\[ + \{ \alpha \leftrightarrow \beta, \gamma \leftrightarrow \epsilon, \lambda \alpha \beta \leftrightarrow \tau \gamma \epsilon \} \right) \]

C Metric from connection

The problem of determining the metric structure of the space-time manifold out of free-falling observations has been in the forefront of research at least since the pioneering work of Weyl, on the mathematical side, and Ehlers, Pirani and Schild, on the physics side confer [17] and references therein.

There are several aspects. First of all, the connection (without any use of the metric) determines uniquely the parallel propagator along a given curve. To be specific, this is given
by a Wilson line, a path ordered exponential

$$g^\alpha{}_{\beta'}(x,x') \equiv P \left[ \exp \int_i^f \Gamma_{\mu}^\alpha{}_{\beta'} \, dt \right]^\alpha_{\beta'}$$  \hspace{1cm} (C.1)

where

$$\Gamma_\mu \equiv (\Gamma_\mu)^\alpha_{\beta'}$$  \hspace{1cm} (C.2)

and the integral is done through a curve

$$x^\mu = x^\mu(\tau)$$  \hspace{1cm} (C.3)

where

$$x^\mu(\tau_i) = x^\mu$$  \hspace{1cm} (C.4)

$$x^\mu(\tau_f) = x'^\mu$$  \hspace{1cm} (C.5)

Nevertheless, not every connection is metric-compatible; that is, it is not always possible to find a metric such that the given connection (even assumed to be torsion-free) is the Levi-Civita one stemming from the metric itself.

The condition for that to be true can be clearly stated using the Christoffel’s symbols of first kind, namely

$$\tilde{\nabla}_\mu \{ \{ \delta; \beta \lambda \} + \{ \beta; \lambda \delta \} \} = \tilde{\nabla}_\lambda \{ \{ \delta; \beta \mu \} + \{ \beta; \delta \mu \} \}$$  \hspace{1cm} (C.6)

which expresses the obvious fact that

$$\tilde{\nabla}_\mu \nabla^\delta \nabla_\beta \nabla_\lambda = \tilde{\nabla}_\lambda \nabla^\delta \nabla_\mu \nabla_\beta$$  \hspace{1cm} (C.7)

In order to determine the generated metric in such cases as it exists, (that is, when the integrability condition is fulfilled), there is the linear system of partial differential equations

$$\tilde{\nabla}_\lambda \nabla^\delta \nabla_\beta = g_\alpha^\delta \Gamma^\alpha_{\beta \lambda} + g_\alpha^\delta \Gamma^\alpha_{\lambda \delta}$$  \hspace{1cm} (C.8)

whose trace implies

$$\nabla^\beta \tilde{\nabla}_\lambda \nabla^\delta = 2 \Gamma^\beta_{\beta \lambda}$$  \hspace{1cm} (C.9)

The integrability conditions for such a system are precisely as above, namely

$$\tilde{\nabla}_\mu \left( g_{\delta \alpha} \Gamma^\alpha_{\beta \lambda} + g_{\beta \alpha} \Gamma^\alpha_{\lambda \delta} \right) = \tilde{\nabla}_\lambda \left( g_{\delta \alpha} \Gamma^\alpha_{\beta \mu} + g_{\alpha \beta} \Gamma^\alpha_{\delta \mu} \right)$$  \hspace{1cm} (C.10)

At the linearized level, assuming

$$g_{\alpha \beta} \equiv \eta_{\alpha \beta} + \kappa h_{\alpha \beta}$$

$$\Gamma_{\alpha \beta \gamma} = O(\kappa)$$  \hspace{1cm} (C.11)

The integrability condition reads

$$\tilde{\nabla}_\mu \left( \Gamma_{\delta \beta \lambda} + \Gamma_{\beta \delta \lambda} \right) = \tilde{\nabla}_\lambda \left( \Gamma_{\delta \beta \mu} + \Gamma_{\beta \delta \mu} \right)$$  \hspace{1cm} (C.12)
This can be written in a suggestive way as
\[ \partial_{\mu} \Gamma_{\delta \beta \lambda} - \partial_{\lambda} \Gamma_{\delta \beta \mu} = \partial_{\lambda} \Gamma_{\beta \delta \mu} - \partial_{\mu} \Gamma_{\beta \delta \lambda} \] (C.13)
or introducing the one-forms
\[ \chi_{\alpha \beta} \equiv \Gamma_{\alpha \beta \lambda} dx^{\lambda} \] (C.14)
this is equivalent to a certain one-form to be closed, that is,
\[ d \chi_{(\alpha \beta)} = 0 \] (C.15)
It is always possible to write the connection as
\[ \Gamma_{\alpha \beta \lambda} \equiv \frac{1}{4} \left( \Gamma_{\alpha \beta \lambda}^{+} + \Gamma_{\alpha \beta \lambda}^{-} \right) \] (C.16)
where
\[ \begin{align*}
\Gamma_{\alpha \beta \lambda}^{+} & \equiv \Gamma_{\alpha \beta \lambda} + \Gamma_{\beta \alpha \lambda} + \Gamma_{\alpha \lambda \beta} + \Gamma_{\beta \lambda \alpha} \\
\Gamma_{\alpha \beta \lambda}^{-} & \equiv \Gamma_{\alpha \beta \lambda} + \Gamma_{\alpha \lambda \beta} - \Gamma_{\beta \alpha \lambda} - \Gamma_{\beta \lambda \alpha}
\end{align*} \] (C.17)
The preceding identity then implies
\[ \Gamma_{\alpha \beta \lambda}^{+} = \partial_{\lambda} \phi_{\alpha \beta} \] (C.18)
Once this condition is fulfilled, the solution is given by the solution of the first order linear differential equation
\[ \partial_{\lambda} h_{\delta \beta} = \Gamma_{\delta \beta \lambda} + \Gamma_{\beta \delta \lambda} = \frac{1}{2} \partial_{\lambda} \phi_{\delta \beta} \] (C.19)
It follows that
\[ h_{\delta \beta} = \frac{1}{2} \phi_{\delta \beta} + C \] (C.20)
This also shows that
\[ \Gamma_{\alpha \beta \lambda}^{-} \] (C.21)
is pure gauge, and the physical metric is independent of it.

\section*{D Constant curvature spaces}

For constant curvature spaces the n-dimensional Riemann tensor obeys
\[ R_{\mu \nu \rho \sigma} = -\frac{2\lambda}{(n-1)(n-2)} \left( g_{\mu \rho} g_{\nu \sigma} - g_{\mu \sigma} g_{\nu \rho} \right) \equiv \pm \frac{1}{L^2} \left( g_{\mu \rho} g_{\nu \sigma} - g_{\mu \sigma} g_{\nu \rho} \right) \] (D.1)
where \( x^\mu, \mu = 1, \ldots n \). It is useful to work with the Synge’s \cite{18} world function \( \Omega(x, y) \) which is defined as
\[ \Omega(x, x') = \frac{1}{2} \int_{0}^{1} g_{\mu \nu}(z) t^\mu t^\nu d\lambda \equiv \frac{1}{2} s_{x, x'}^2 \] (D.2)
where \( x \) and \( x' \) are two points close enough so that there is a unique geodesic joining them \( \gamma \), parametrized by an affine parameter \( \lambda \) such that
\[ \begin{align*}
\gamma(0) &= x \\
\gamma(1) &= x'
\end{align*} \] (D.3)
The only advantage of the world function over the arc is that the former is always real (although sometimes negative) even in pseudoriemannian spaces. This is not an issue on Riemannian spaces (like the sphere) though, in which case is actually simpler to work with the arc length, $s$.

The basic equation that determines the world function in general is

$$g^{\mu\nu} \partial_\mu \Omega \partial_\nu \Omega \equiv \Omega^\mu \Omega_\mu = 2 \Omega$$ \hspace{1cm} (D.4)

this just because

$$\Omega = \frac{1}{2} s^2 \implies \Omega_\mu = ss_\mu \implies g^{\mu\nu} \Omega_\mu \Omega_\nu = s^2 = 2\Omega$$ \hspace{1cm} (D.5)

It follows that

$$\Omega_{\mu\nu} \Omega^\mu \equiv \nabla_\nu \nabla_\mu \Omega \cdot \nabla^\mu \Omega = \Omega_\nu$$ \hspace{1cm} (D.6)

It is also the case that

$$[\nabla_\lambda, \nabla_\nu] \ s_\mu = R_{\lambda\nu\rho\sigma} s^\rho = \pm \frac{1}{L^2} (g_{\lambda\mu} s_\nu - s_\lambda g_{\nu\mu})$$ \hspace{1cm} (D.7)

(where $s_\mu \equiv \frac{\Omega_\mu}{s}$). Please note that in this equation all indices are covariant ones.

Invariant tensors can be expanded in outer products of $s_\mu$ (or $\Omega_\mu$) and $g_{\mu\nu}$, with coefficients that depend on $s$ only. For example [19, 20]

$$s_{\mu\nu} = \frac{1}{L \tan \frac{s}{L}} (g_{\mu\nu} - s_\mu s_\nu)$$ \hspace{1cm} (D.8)

which implies

$$\Box s = \frac{n - 1}{L \tan \frac{s}{L}}$$ \hspace{1cm} (D.9)

Let us work explicitly a couple of examples (in the case of the n-sphere $S_n$, to be specific).

**Example 1.** The inverse of the d’Alembertian, $G \equiv \Box^{-1}$

$$\Box G(s) = \delta^n(x)$$ \hspace{1cm} (D.10)

The ODE to be solved is

$$G''(s) + \frac{n - 1}{L \tan \frac{s}{L}} G'(s) = \delta^n(x)$$ \hspace{1cm} (D.11)

When $s \neq 0$

$$L \tan \frac{s}{L} G''(s) + (n - 1) L G'(s) = 0$$ \hspace{1cm} (D.12)

When $s \sim 0$

$$G(s) \sim s^{2-n}$$ \hspace{1cm} (D.13)

which is the correct behavior for a Dirac delta singularity.

The exact solution reads

$$G(s) = C_1 + C_2 \cos \frac{s}{L} \ _2F_1 \left( \frac{1}{2}, \frac{n}{2}; \frac{3}{2}, \cos^2 \frac{s}{L} \right)$$ \hspace{1cm} (D.14)
Example 2. Let us now compute a vector Green’s function. In order to do that, it is best to first compute the inverse of the second power of the d’Alembertian, $G_2 ≡ □^{-2}$. Let us start with

$$ (□δ_μ^ν + a∇_μ∇_ν) \ G_ν^σ(s) = δ_μ^σ \ δ^σ(x) \quad (D.15) $$

Please notice that on the sphere there is no zero mode even for $a = -1$, because

$$ ▽_λ ▽_μ ▽_σ ▽_λ = - \frac{1}{L^2} (n-1) ▽_μ \quad (D.16) $$

We shall need

$$ □ G_2(s) = \delta^σ(x) \quad (D.17) $$

which is equivalent to

$$ G^{IV}_2(s) + \frac{2(n-1)}{L \tan \frac{s}{L}} G^{II}_2(s) + \frac{(n-1)((n-1) \cos^2 \frac{s}{L} - 2)}{L^2 \sin^2 \frac{s}{L}} (n-1) G_2(s) + \frac{(n-1)(3-n) \cos \frac{s}{L}}{L^3 \sin^3 \frac{s}{L}} G_2'(s) = 0 \quad (D.18) $$

This can be rewritten as

$$ □(□ G_2(s)) = \left( \frac{d^2}{ds^2} + \frac{n-1}{L \tan \frac{s}{L}} \right) \frac{d^2}{ds^2} G_2(s) + \frac{n-1}{L \tan \frac{s}{L}} \frac{d}{ds} G_2(s) = \delta(x) \quad (D.19) $$

And using the result (D.14) we get that the general solution must have the form

$$ G_2(s) = G(s) + h(s) \quad (D.20) $$

Where $h(s)$ is the solution of the equation:

$$ \left( \frac{d^2}{ds^2} h(s) + \frac{n-1}{L \tan \frac{s}{L}} \frac{d}{ds} h(s) \right) = G(s) \quad (D.21) $$

We can illustrate how to get the solution of (D.15) by studying the case when $L \to \infty$ (flat space). In that case the general solution is given by

$$ G_2(s) = c_1 \frac{s^{2-n}}{2-n} + c_2 \frac{s^{4-n}}{4-n} + c_3 \frac{s^2}{2} + c_4 \quad (D.22) $$

Then, for $\lambda = -\frac{a}{1+a}$ we can obtain our solution

$$ G_μ^ν(s) = (□δ_μ^ν + aδ_μ^ν) \ G_2(s) = -\lambda \left[ c_1 ns^{-n} + c_2(n-2)s^{2-n} \right] s_μ s^ν + \left[ \lambda c_1 s^{-n} + c_2(2 + \lambda)s^{2-n} + (n + \lambda)c_3 \right] \delta^ν_μ \quad (D.23) $$

Finally, in the case of $n = 4$, we need $c_1 = 0$ to recover the correct behaviour when $s \to 0$, and since we can set $c_3 = 0$ as it enters as an additive constant we get the desired tensor Green’s function as

$$ G_μ^ν(s) = c_2 \frac{2 + a}{1+a} \left( \frac{1}{s^2} δ^ν_μ + \frac{2a}{a+2} \frac{s^ν s_μ}{s^2} \right) \quad (D.24) $$
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