Thermodynamic Geometry of the Born-Infeld-anti-de Sitter black holes

Peng Chen

Institute of Theoretical Physics
Chinese Academy of Sciences, Beijing 100190, PRC

Abstract

Thermodynamic geometry is applied to the Born-Infeld-anti-de Sitter black hole (BIAdS) in the four dimensions, which is a nonlinear generalization of the Reissner-Norström-AdS black hole (RNAdS). We compute the Weinhold as well as the Ruppeiner scalar curvature and find that the singular points are not the same with the ones obtained using the heat capacity. Legendre-invariant metric proposed by Quevedo and the metric obtained by using the free energy as the thermodynamic potential are obtained and the corresponding scalar curvatures diverge at the Davies points.

Keywords: Born-Infeld-anti-de Sitter black hole; Thermodynamic Geometry; Phase transition.

chenpeng@itp.ac.cn
1 Introduction

Born-Infeld electrodynamics was first introduced by Born and Infeld in 1930’s to remove the divergence of the electron’s self-energy [1]. It has received renewed attention for the last fifteen years since it arises naturally in open superstrings and in D-branes [2, 3, 4]. On the other hand, thermodynamics of black holes in AdS space has also generated renewed attention due to the AdS/CFT duality [5], which relates thermodynamics of black holes in AdS space to the dual CFT in a lower dimension. Studying the phase transitions of AdS black holes is an effective way of exploring phase structures in the dual field theories. Born-Infeld-AdS black hole solution and the thermodynamic properties has been analyzed in [6]-[10].

Introducing differential geometric concepts into ordinary thermodynamics was first done by Weinhold [11]. He proposed a Riemannian metric defined as the second derivatives of internal energy $U$ with respect to entropy and other extensive quantities of a thermodynamic system. However, the geometry of this metric seems to have no physical meaning in the context of equilibrium thermodynamics. Few years later, Ruppeiner [12] introduced another metric, defined as the negative Hessian of entropy $S$ with respect to the internal energy and other extensive quantities of a thermodynamic system. In [13] it was proved that the Ruppeiner metric is conformal to the Weinhold metric with the inverse temperature as the conformal factor. The Ruppeiner geometry has its physical meaning in the fluctuation theory of equilibrium thermodynamics. Both metrics have been applied to study the geometry of the thermodynamics of ordinary systems [14]-[22]. In particular, it was found that the Ruppeiner geometry carries information of phase structure of thermodynamic system; and scalar curvature of the metric diverges at the phase transition and critical point, which shows interaction of the system. For thermodynamic systems with no statistical mechanical interactions (for example, ideal gas), the scalar curvature is zero and the Ruppeiner metric is flat. Because of the success of their applications to ordinary thermodynamic systems, they have also been used to study black hole phase structures and lots of results have been obtained for different sorts of black holes [23]-[39]. However, the results they present are sometimes contradictory. For instance, for the RN black hole: the Weinhold metric predicts phase transitions which are compatible with standard black hole thermodynamics, while the Ruppeiner curvature is flat, giving no information at all about phase transition. To overcome this inconsistency, the theory of Gometrothermodynam-
ics (GTD) was proposed recently [40, 41, 42]. It incorporates arbitrary Legendre transformations into the geometric structure of the equilibrium space in an invariant manner. In [43] other thermodynamic potentials were proposed and metrics on all these thermodynamic potentials were investigated. They showed that in general for a system with \( n \)-pairs of intensive/extensive variables, all thermodynamical potential metrics can be embedded into a flat \((n, n)\)-dimensional space. In this paper we will apply thermodynamic geometry into BIAdS black holes and see whether the information about phase transition represented by divergence of heat capacity can be reproduced.

This paper is organized as follows. In Section II we review the basics of Born-Infeld-AdS black hole and its most important thermodynamic quantities, plot the graphics of the heat capacities for fixed charge and fixed potential. In Section III we apply thermodynamic geometry to BIAdS black holes, getting the metrics and their corresponding scalar curvatures. In the last section we discuss our results and suggestions.

## 2 Basics of BIAdS black holes

First let’s consider the BIAdS action which is a \((3+1)\)-dimensional gravity coupled with nonlinear electrodynamics [6, 8, 9]

\[
S = \int d^4x \sqrt{-g} \left( \frac{R - 2\Lambda}{16\pi G} + L(F) \right),
\]

where

\[
L(F) = \frac{b^2}{4\pi G} \left( 1 - \sqrt{1 + \frac{2F}{b^2}} \right)
\]

with \( F \equiv F_{\mu\nu}F^{\mu\nu}/4 \), where the constant \( b \) is the Born-Infeld parameter, and \( \Lambda = -3/l^2 \) is the cosmological constant. Note that this Lagrangian reduces to the RNAdS one in the limit \( b^2 \to \infty \).

After solving the equations of motion for the gauge field \( A_\mu \) and the gravitational field \( g_{\mu\nu} \), the BIAdS black hole solutions [6, 8, 9] can be written as

\[
ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2.
\]

Here, the metric function \( f(r) \) is given by

\[
f(r) = 1 - \frac{2M}{r} + \frac{r^2}{l^2} + \frac{2b^2r^2}{3} \left( 1 - \sqrt{1 + \frac{Q^2}{b^2r^4}} \right) + \frac{4Q^2}{3r^2} \mathcal{F} \left( \frac{1}{4}, \frac{1}{2}, \frac{5}{4}, -\frac{Q^2}{b^2r^4} \right),
\]

where

\[
\mathcal{F}(a, b, c, z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n}{(n!)^2} \frac{z^n}{n!},
\]

and \((a)_n = a(a+1)\cdots(a+n-1)\) is the Pochhammer symbol.

This expression for \( f(r) \) generalizes the RNAdS black hole metric and includes the Born-Infeld parameter \( b \) as well as the charge \( Q \). The parameter \( b \) controls the degree of nonlinearity in the electrodynamics, while \( Q \) is the electric charge of the black hole.

In Section III we will discuss the thermodynamic geometry of these black holes, focusing on the phase transition associated with the divergence of the heat capacity.
where \( F \) is a hypergeometric function. The only non-zero component with
the electric charge \( Q \) is given by

\[
F_{01} = -E = -Q/\sqrt{r^4 + Q^2/b^2}.
\]

Hereafter we only consider \( Q \geq 0 \) and \( b \geq 0 \) without any loss of
generality. In the limit \( Q \to 0 \), \( f(r) \) reduces to the Schwarzschild-anti
de Sitter black hole (SAdS) case, while in the limit \( b \to \infty \) and \( Q \neq 0 \), \( f(r) \) reduces to the RNAdS case.

We can solve the equation \( f(r) = 0 \) to get the ADM mass \( M \) which is
given by

\[
M(r_+, Q, b) = \frac{r_+}{2} + \frac{r_+^3}{2l^2} + \frac{b^2 r_+^3}{3} \left( 1 - \sqrt{1 + \frac{Q^2}{b^2 r_+^4}} \right) + \frac{2Q^2}{3r_+} F \left( \frac{1}{4}, \frac{1}{2}, \frac{5}{4}, -\frac{Q^2}{b^2 r_+^4} \right)
\]

with the outer horizon \( r = r_+ \). If we demand both \( f(r) \) and \( df(r)/dr \) vanish
at the degenerate horizon, then we get the extremal BIAdS. The radius \( r_e^2 \)
of the extremal BIAdS is given by

\[
r_e^2 = \frac{l^2}{6} \left( 1 + \frac{3}{4b^2} \right) \left[ -1 + \frac{12 \left( 1 - \frac{3}{4b^2} \right)}{b^2 l^2 \left( 1 + \frac{3}{4b^2} \right)^2 \left( b^2 Q^2 - \frac{1}{4} \right)} \right].
\]

So the condition \( bQ \geq 0.5 \) must be satisfied in order to have a real root for
\( r_e^2 \). As a result, the parameter space for the BIAdS is

\[
0.5 \leq bQ \leq \infty.
\]

We call the lower bound \( (bQ = 0.5) \) as the critical BIAdS and the upper
bound of \( b \to \infty \) as the RNAdS.

The Hawking temperature \( T_H = f'(r_+)/4\pi \) is given by

\[
T_H(r_+, Q, b) = \frac{1}{4\pi} \left( \frac{1}{r_+} + \frac{3r_+}{l^2} + 2b^2 r_+ \left[ 1 - \sqrt{1 + \frac{Q^2}{b^2 r_+^4}} \right] \right).
\]

Note that in the limit \( Q \to 0 \), \( T_H \) reduces to the SAdS case, while in the
limit of \( b \to \infty \) and \( Q \neq 0 \), \( T_H \) reduces to the RNAdS case.

Then, using Eqs. (5) and (8), the heat capacity \( C(r_+, Q, b) = (dM/dT_H)_Q \)
for fixed-charge is

\[
C_Q = \frac{2\pi r_+^2 \left[ 1 + \frac{Q^2}{b^2 r_+^4} \right] \left[ 3r_+^4 + l^2 \left( r_+^2 + 2b^2 \left( 1 - \sqrt{1 + \frac{Q^2}{b^2 r_+^4}} \right) r_+^4 \right) \right]}{3 \left[ 1 + \frac{Q^2}{b^2 r_+^4} \right]^2 + l^2 \left[ -\sqrt{1 + \frac{Q^2}{b^2 r_+^4} + 2Q^2 - 2b^2 \left( 1 - \sqrt{1 + \frac{Q^2}{b^2 r_+^4}} \right) r_+^4} \right]}.\]
Note that in the limit $Q \to 0$, the heat capacity $C_Q$ reduces to the SAdS case as
\begin{equation}
C_Q^{SAdS}(r_+) = 2\pi r_+^2 \left( \frac{3r_+^2 + l^2}{3r_+^2 - l^2} \right),
\end{equation}
(10)
In the limit of $b^2 \to \infty$, $C_Q$ reduces to the RNAdS case as
\begin{equation}
C_Q^{RNAdS}(r_+, Q) = 2\pi r_+^2 \left[ \frac{3r_+^4 + l^2(r_+^2 - Q^2)}{3r_+^4 + l^2(-r_+^2 + 3Q^2)} \right],
\end{equation}
(11)
According to the well-known Area_Entropy formula, the Bekenstein-Hawking entropy of the BIAdS black hole is
\begin{equation}
S = \pi r_+^2.
\end{equation}
(12)
By the first law of thermodynamics,
\begin{equation}
dM = TdS + \Phi dQ
\end{equation}
(13)
we can get the electric potential as
\begin{equation}
\Phi = \frac{Q}{r_+} \mathcal{F} \left( \frac{1}{4}, \frac{1}{2}, \frac{5}{4}, -\frac{Q^2}{b^2 r_+^4} \right).
\end{equation}
(14)
The heat capacity for fixed potential is
\begin{equation}
C_\Phi = -\frac{2\pi r_+ \left( r_+ \left( \frac{3}{r_+^2} - 2b^2 \left( \frac{\sqrt{Q^2/b^2 r_+^4} + 1 - 1}{\sqrt{Q^2/b^2 r_+^4} + 1} \right) + \frac{1}{r_+} \right) \right)}{r_+^4 \sqrt{\frac{Q^2}{b^2 r_+^4} + 1} \mathcal{F} \left( \frac{1}{4}, \frac{1}{2}, \frac{5}{4}, -\frac{Q^2}{b^2 r_+^4} \right) + 2b^2 \left( \frac{\sqrt{Q^2/b^2 r_+^4} + 1 - 1}{\sqrt{Q^2/b^2 r_+^4} + 1} \right) - \frac{4Q^2}{r_+^4 \sqrt{\frac{Q^2}{b^2 r_+^4} + 1} + 3} + \frac{1}{r_+^2}}.
\end{equation}
(15)
Here we plot $C_Q$ and $C_\Phi$ as follows.
Figure 1: The heat capacity for fixed charge $C_Q$ vs $r_+$ with $b = 0.5, l = 10, Q = 1$. At $r_+ = 1.5307, 5.4781, C_Q$ diverges.

Figure 2: The heat capacity for fixed charge $C_Q$ vs $r_+$ with $b = 1, l = 10, Q = 1$. (a) At $r_+ = 1.7685/6, 5.4774/5, C_Q$ diverges. (b) At $r_+ = 0.850492, C_Q$ vanishes.

Figure 3: The heat capacity for fixed potential $C_\Phi$ vs $r_+$ with $b = 0.5, l = 10, Q = 1$. At $r_+ = 5.6835/6, C_\Phi$ diverges.
Figure 4: The heat capacity for fixed potential $C_\Phi$ vs $r_+$ with $b = 1, l = 10, Q = 1$. (a) At $r_+ = 5.6834/5$, $C_\Phi$ diverges. (b) $C_\Phi$ diverges at 0.8009/11, vanishes at 0.8504/6.

We may also introduce analogously charge capacitances at fixed temperature or entropy, they are given by

$$
\tilde{C}_T \equiv \frac{\partial Q}{\partial \Phi} |_{T} = \frac{\partial Q}{\partial \Phi} |_{T}, \tilde{C}_S \equiv \frac{\partial Q}{\partial \Phi} |_{S} = \frac{\partial Q}{\partial \Phi} |_{S}
$$

It turns out that $\tilde{C}_S$ is positive definite (for $b = 0.5, 1; l = 10, Q = 1$), and there are no singular points associated with it, so we will skip it in the following discussion. $\tilde{C}_T$ is plotted as follows.

Figure 5: The charge capacitance at fixed temperature $\tilde{C}_T$ vs $r_+$ with $b = 0.5, l = 10, Q = 1$. At $r_+ = 5.6835/6$, $\tilde{C}_T$ diverges. $\tilde{C}_T$ vanishes at 1.5307, 5.4781.
Figure 6: The charge capacitance at fixed temperature $\tilde{C}_T$ vs $r_+$ with $b = 1, l = 10, Q = 1$. At $r_+ = 0.8009/11, 5.6834/5$, $\tilde{C}_T$ diverges. $\tilde{C}_T$ vanishes at $1.7685/6, 5.4774/5$.

From Figure 1 and Figure 5 (both are for $b = 0.5$), we see clearly that when $C_Q$ diverges (at $r_+ = 1.5307, 5.4781$), $\tilde{C}_T$ vanishes. From Figure 2 and Figure 6 (both are for $b = 1$), we see clearly that when $C_Q$ diverges (at $r_+ = 1.7685/6, 5.4774/5$), $\tilde{C}_T$ vanishes. It is consistent with the observation in [43] that the vanishing of a certain capacity is always associated with the divergence of another.

3 Thermodynamic geometry of BIAdS black holes

Study of black hole phase transitions from the point of view of thermodynamic geometry has been quite active recently. In this section, we will apply thermodynamic geometry to BIAdS black holes.

3.1 The Weinhold metric

The Weinhold metric is defined as [11]

$$dS_W^2 = g_{ij}^W dX^i dX^j$$

(17)

where

$$g_{ij}^W = \frac{\partial^2 M(X^k)}{\partial X^i \partial X^j}, \quad \text{and} \quad X^i \equiv X^i(S, N^a)$$

(18)
Here $N^a$'s are all other extensive variables of the system. In this case, $N^a$ would be the charge $Q$. We can substitute $r_+ = \sqrt{S}/\sqrt{\pi}$ into (5) and get the mass as a function of $S$ and $Q$ (Note that in this paper $l$ and $b$ will be fixed at some value). At this stage $dS^2_{\mu\nu}$ can be calculated straightforward. As the metric and corresponding scalar curvature are very complicated, we will not present them here. Instead we plot the scalar curvature as follows.

Figure 7: The Weinhold curvature $R^W$ vs $r_+$ with $b = 0.5, l = 10, Q = 1$. At $r_+ = 0, 5.6835/6, R^W$ diverges.

Figure 8: The Weinhold curvature $R^W$ vs $r_+$ with $b = 1, l = 10, Q = 1$. At $r_+ = 0.8010/11, 5.6834/5, R^W$ diverges.
For BIAdS black holes, the Weinhold geometry is curved, signaling interaction for this thermodynamic system. There are singular points, but they are not consistent with the ones of the heat capacity for fixed charge.

### 3.2 The Ruppeiner metric

The Ruppeiner metric is given by [12]

\[
dS_R^2 = g^R_{ij} dX^i dX^j
\]

where,

\[
g^R_{ij} = -\frac{\partial^2 S(X^k)}{\partial X^i \partial X^j}, \quad \text{and} \quad X^i \equiv X^i(M, N^a)
\]

For BIAdS black hole \( N^a = Q \).

In order to find \( g^R_{ij} \) it is desirable to express \( S \) in terms of \( M \) and \( Q \). However from (5) we see that \( M \) is expressed as a function of \( r_+(S = \pi r_+^2) \) and \( Q \), which is a hypergeometric function and is invertible. Fortunately, it was proved that Ruppeiner metric and Weinhold metric are related with each other by a conformal factor [13]

\[
dS_R^2 = \frac{1}{T} dS_W^2
\]

where \( T \) is the temperature of the system. In this case this would correspond to the Hawking temperature of the BIAdS black hole. So we can use the Weinhold metric and the relation of the two metrics to get the Ruppeiner metric. We plot the Ruppeiner scalar curvature as follows.

![Figure 9: The Ruppeiner curvature \( R^R \) vs \( r_+ \) with \( b = 0.5, l = 10, Q = 1 \). At \( r_+ = 0.56835/6 \), \( R^R \) diverges. At \( r_+ = 1.4590/2 \), it vanishes.](image)
Like the Weinhold geometry, the Ruppeiner geometry is also curved, signaling interaction for this thermodynamic system. There are singular points, but they are not consistent with the ones of the heat capacity for fixed charge (except one singular point at \( r_+ = 0.8504/6 \), at which \( C_Q \) vanishes.).

Note that for \( b = 0.5 \), the Weinhold metric, the Ruppeiner metric all diverge at the same points \( (r_+ = 0, 5.6835/6) \), but they are inconsistent with the result of heat capacity for fixed charge \( (r_+ = 1.5307, 5.4781) \); for \( b = 1 \), \( C_\Phi \), the Weinhold metric, the Ruppeiner metric all diverge at the same points \( (r_+ = 0.8010/1, 5.6834/5) \), but they are different from the divergencies of heat capacity for fixed charge \( (r_+ = 1.7685/6, 5.4774/5) \). It is consistent with the
observation in [33] that whilst the Ruppeiner and Weinhold metrics indeed reveal the signals of black hole phase transitions associated with divergence of heat capacity with fixed electric potential or angular velocity, they are insensitive to the Davies curve [44] where the heat capacity with fixed charge and/or angular momentum diverges.

### 3.3 The Quevedo metric

Weinhold and Ruppeiner metrics are supposed to give a direct relationship between curvature singularities and divergencies of the heat capacity. Unfortunately, the singular points they present are not consistent with the ones of the heat capacity. In fact, the results they present are sometimes contradictory. For instance, for the RN black hole: the Weinhold metric predicts phase transitions which are compatible with standard black hole thermodynamics, while the Ruppeiner curvature is flat, giving no information at all about phase transition. To solve this problem, in [27] a generalized Ruppeiner metric was proposed with variables $(M, Q)$ replaced by $(M - \Phi Q, \Phi)$. After the replacement, the new metric gives the correct singular point. Another approach to solve the puzzle was given by Quevedo, who proposed a new metric, which is Legendre-invariant. The Legendre-invariant metric reproduces the corresponding phase transition structure. There are many Legendre-invariant metrics that we can use, in this case, we will use this one:

$$g^Q_{ab} = (SM_S + QM_Q) \begin{pmatrix} -M_{SS} & 0 \\ 0 & M_{QQ} \end{pmatrix}.$$  \hspace{1cm} (22)

The corresponding metric and curvature are computed and the curvature is plotted as follows.
Comparing Figure 1, 2 with Figure 11, 12 we can see that the singular points of the Legendre-invariant metric are consistent with the result of heat capacity for fixed charge, giving the same singularities.

3.4 The free-energy metric

In [43], a new thermodynamical metric was introduced based on the Hessian matrix of several free energies. The authors demonstrated that the divergence
of the heat capacity corresponds to the curvature singularities of this new metric. Let’s consider the Helmholtz free energy $F = M - TS$, as a function of $T$ and $Q$, which satisfies

$$dF = -SdT + \Phi dQ$$

The corresponding metric is given by

$$ds^2(F) = -dTdS + d\Phi dQ$$

It is not always convenient to use natural variables $(T, Q)$ to construct the metric, in this case we will use $(r_+, Q)$ variables. The metric can be obtained straightforward. The Ricci scalar for the free-energy metric is plotted as follows

![Figure 13: The curvature obtained using free-energy metric $R^F$ vs $r_+$ with $b = 0.5, l = 10, Q = 1$. At $r_+ = 0, 1.5307, 5.4781, R^F$ diverges. At $r_+ = 8.1644/5$, it vanishes.](image)
Like the Quevedo metric, the free-energy metric reproduces the singular-
ities of $C_Q$. We can also use the potential

$$\bar{F} = M - \Phi Q$$

However, thermodynamical potentials $(U, \bar{U})$ are called a conjugate pair
if they satisfy

$$U + \bar{U} = 2M - TS - \sum_{i=1}^{n} \mu_i N_i.$$  \hspace{1cm} (26)

Their associated metrics are negative of each other [43], i.e.

$$ds^2(U) = -ds^2(\bar{U}).$$  \hspace{1cm} (27)

So the potential $\bar{F}$ gives the same information about phase structure. For
the same reason, the potential

$$\bar{M} = M - TS - \Phi Q$$

is conjugate to the potential $M$ (leading to the Weinhold metric), providing
nothing new.

4 Summary and discussions

In summary, we have analyzed the thermodynamic geometry of the 4-dimensional
BIAdS. The Weinhold, Ruppeiner and a Legendre-invariant metric are ob-
tained, their scalar curvatures are computed. The Weinhold geometry is
curved for the BIAdS and the corresponding curvature diverges at some points, but these points are not the ones at which the heat capacity for fixed charge diverges or vanishes. The Ruppeiner metric diverges at the same points as the Weinhold (except two points in the $b = 1$ case) and it gives two more singularities, one of which is the zero point of heat capacity for fixed charge. For the Legendre-invariant Quevedo metric and the metric obtained using the free energy as the thermodynamic potential, they present the same divergent points with heat capacity. The totality of curvature singularities of all metrics is exactly the same as the totality of capacity divergent points.

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