Charm tetraquarks in a non-relativistic quark model

V.R. Debastiani, F.S. Navarra
Instituto de Física, Universidade de São Paulo, Rua do Matão Travessa R, 187, 05508-090 São Paulo, SP, Brazil
E-mail: vinicius.debastiani@gmail.com, navarra@if.usp.br

Abstract.
New charmonium states have been discovered since 2003. Although some of them can be explained as \( c\bar{c} \) states, others do not fit in the predictions of that spectrum. Exotic models have been proposed to explain these new states, regarded now as multiquark states in the charm sector. In this work we focus on two of the most discussed models: the tetraquark, which consists on a diquark-antidiquark system, and the meson molecule, which is a weakly bound state of two mesons. We have developed a non-relativistic approach to study exotic heavy hadron spectroscopy, where the four body system is considered as three subsequent two-body systems. We solve numerically the Schroedinger equation using an effective Cornell-like potential to model the two-body interaction in each step of the tetraquark calculation, and a Yukawa-like potential to describe the meson molecule.

1. Introduction

Due to the B-meson factories there has been great progress on the study of the charmonium states. These machines are \( e^+ e^- \) colliders operating on a center of mass energy of 10.600 MeV, just enough to produce \( B \) meson pairs. The decay of \( B \) produces charmonium. Over the past decade, the BABAR, BELLE and BESIII collaborations, working in different \( B \) factories, produced a large amount of data on charmonium spectra. Since 2003, new charmonium states have been continuously discovered [1, 2, 3]. While some of them, like the \( X(3943) \), can be explained as conventional \( c\bar{c} \) states, others, such as the \( X(3872) \), do not fit in the predictions of the spectrum of \( c\bar{c} \) states and are called exotic charmonium states [1, 2, 3]. Models have been proposed to explain these new states, considering them as four quark states in the charm sector \( (cq\bar{q}, q \in u, d) \). The most discussed models are the tetraquark, the meson molecule, hadro-charmonium, hibrid states and mixtures. Here, the tetraquark model consists on a diquark-antidiquark system \( (Qq - \bar{Q}\bar{q}) \), in which the diquarks interact through the exchange of colored objects, and the attraction force can be intense. In contrast, the meson molecule \( (Q\bar{q} - \bar{Q}q) \) is a weakly bound state of two mesons.
1.1. **Meson molecules**

The molecular picture has been used to understand the well established $X(3872)$, which is considered to be a $D\bar{D}^*$ molecule. Of course, the actual physical state is probably rather complex, with a short-range component of $(c\bar{c})$ nature and a long-range component with a charmed meson and an anticharmed meson almost bound by attractive Yukawa forces. The idea of molecules with hidden charm has been proposed long ago by Okun and Voloshin [4]. A molecular interpretation was proposed for some high-lying $1^{--}$ charmonium resonances [5], due to puzzling branching ratios into $D\bar{D}, D\bar{D}^*$ and $D^*\bar{D}^*$, which turned out to be due to the node structure of these states as radial excitations of the $J/\psi$ [6]. The possibility of hidden-charm meson molecules has been revisited in the 90s by Törnqvist [7], Ericson and Karl [8] and Manohar and Wise [9]. More recently, after the discovery of the $X(3872)$, the meson molecule picture was further developed by several authors [10]. The main idea is that the Yukawa interaction, that successfully binds nuclei, is not restricted to the nucleon-nucleon interaction. The exchange of light mesons also generates a potential between flavored mesons, which is sometimes attractive. Although usually weaker than the proton-neutron interaction that binds the deuteron, it is probed by much heavier particles, and thus can lead to bound states with a binding energy of a few MeV, or even a few tens of MeV, as shown by Törnqvist [7].

1.2. **Tetraquarks**

The first papers discussing the existence of tetraquark configurations were all based on the MIT bag model and were published in the seventies [11, 12]. In the beginning, only light tetraquark states were considered. Later on, Weinstein and Isgur [13, 14] extended the tetraquark picture to other quark models and tetraquarks with heavy quarks were included. After the discovery of the $X(3872)$ the multiquark picture has been revisited by Jaffe and Wilczek [15] and more recently by other authors [16, 17, 18, 19].

A very popular representation of the pictures above discussed can be seen in Fig. 1.

In this work, following the steps of Ref. [19], we have developed a non-relativistic approach to study exotic heavy hadron spectroscopy, where the heavy hadrons are tetraquarks. The four-body system is considered as three two-body systems. Our work consists on numerically solving the Schroedinger equation using an effective Cornell-like potential ($V(r) = -\alpha/r + \sigma r$) to model the two-body interactions among the hadron constituents. More specifically we wish to answer two questions: 1) Can the $Z(4430)$ be the first radial excitation of the $X(3872)$ ? 2) Is there a $T_{cc}(ccqq)$ ? If so with which mass ?

**Figure 1.** $X(3872)$ as a tetraquark (left panel) and as a meson molecule (right panel).
2. A non-relativistic model

2.1. The Schroedinger equation

In the charmonium states the kinetic energy is small compared to the rest energy of the constituents hence we can develop a non-relativistic approach. In order to obtain the energy spectrum we only need to solve the time independent Schroedinger equation:

\[
\left[ \frac{1}{2\mu} \left( -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} \right) + V(r) \right] y_{n,l}(r) = E_{n,l} y_{n,l}(r) \tag{1}
\]

where \( y_{n,l}(r) = r R(r) \) is the reduced radial wave function and \( \mu \equiv \frac{m_1 m_2}{m_1 + m_2} \) is the reduced mass (we use natural units \( \hbar = c = 1 \)). By solving the above eigenvalue equation, we find the interaction energy of the two body system, \( E_{n,l} \), which depends on the number of nodes and on the orbital angular momentum. We are also interested in the wave function itself, which will be used to calculate the normalization factor, the mean square radius and the spin-dependent corrections. With the Cornell potential the above equation has no analytical solution and we need to solve it numerically. We have used the program written by the authors of [20], built specially to solve Schroedinger equation for bound states.

2.2. The Cornell Potential

The Cornell Potential [21] has a Coulomb term, representing the one gluon exchange in QCD, and a linear term responsible for confinement. It is given by:

\[
V(r) = V_{Coul} + V_{conf} = \kappa_s \frac{\alpha_s}{r} + \sigma r \tag{2}
\]

In the above equation the constant \( \kappa_s \) (the color factor) is related to the color configuration of the quarks. The constant \( \alpha_s \) is the QCD analogue to the fine structure constant of the electromagnetic interactions. The constant \( \sigma \) is called the string tension.

In our calculation we first use the potential (2) to calculate the mass of the diquark, solving the two body problem with the masses of a charm and a light quark. Then, the diquark mass is calculated by summing the masses of its constituents plus the interaction energy, i.e., \( m_d = m_c + m_q + E_{cq} \), where \( E_{cq} \) is the solution of (1). Spin-dependent corrections can be included perturbatively, but their contribution is much smaller than the ones coming from a small change in the Cornell potential parameters, so they have been neglected at this stage.

The antidiquark mass is calculated in the same way (without spin-corrections it is essentially the same as the diquark mass). Then the tetraquark (treated now as a diquark-antidiquark two-body system) interaction energy is calculated, using different parameters in the potential, and the diquark masses as inputs. The final state mass is the sum of diquark and antidiquark masses and the interaction energy between them.

3. Results

3.1. \( Z(4430) \) as the first excitation of \( X(3872) \)

Relations between exotic states have been suggested in recent works on the field [3]. Here we explore the relation between the \( X(3872) \) and \( Z(4430) \), assumed to be tetraquarks. We adjust
the parameters of (2) to reproduce the mass of the $X(3872)$ and then, with all the parameters fixed, we calculate the energy of the first excitation. Using $m_c = 1.486$ GeV, $m_u = m_d = 0.323$ GeV, as suggested in [19], we find the diquark mass to be: $m_{dq} = 1.949$ GeV. Then we use this mass as input to calculate the tetraquark state with constants chosen based on a parametrization of the usual $c\bar{c}$ meson [21], where the final state energy is calculated subtracting a constant. Here we use $\Delta E = 0.312$ GeV and hence:

$$m_{4q} = m_d + m_{\bar{d}} + E_{dq} - \Delta E \quad (3)$$

For the ground state we find $m_{4q} = m_X = 3.872$ GeV, and the mass of the first radial excitation (one node) turns out to be:

$$m_{4q} = m_Z = 4.431\text{GeV} \quad (4)$$

The values of the parameters used in the calculation are listed in Table 1 and the wave functions are plotted in Fig. 2 and 3, where $R(r) = y(r)/r$ (see equation 1).

| Object   | $\kappa_s$ | $\alpha_s$ | $\sigma$ | $m_1$ | $m_2$ | $n$ | $l$ | $E_{int}$ | $\Delta E$ | $E_{total}$ | $<r^2>$ $^{1/2}$ |
|----------|------------|------------|----------|-------|-------|-----|-----|-----------|-----------|-------------|----------------|
| diquark  | $-2/3$     | 0.30       | 0.015    | 1.486 | 0.323 | 0   | 0   | 0.140     | 0         | 1.949       | 1.538         |
| tetraquark | $-4/3$     | 0.34       | 0.180    | 1.949 | 1.949 | 0   | 0   | 0.286     | 0.312     | 3.872       | 0.379         |
| tetraquark | $-4/3$     | 0.34       | 0.180    | 1.949 | 1.949 | 1   | 0   | 0.845     | 0.312     | 4.431       | 0.756         |

This result strongly favors the assignment proposed first in [17] and then in [3] and also in [22].

**Figure 2.** Radial squared normalized wave function $|R(r)|^2$ and reduced radial squared normalized wave function $|y(r)|^2$ for $X(3872)$ as the ground state in the tetraquark model.
3.2. The meson molecule model

The meson molecule, shown in Fig. 1, is a weakly bound state of two mesons \((Q\bar{q} - \bar{Q}q)\), usually described as \(D^0 - \bar{D}^{*0}\). Its mass can be calculated in the same way as in the tetraquark model, solving Eq. 1 for the interaction of the quarks inside the meson, and then for the interaction between the two mesons inside the molecule. However, instead of this three steps approach, here we already adopt the experimentally measured meson masses \(m_{D^0} = 1.864 \text{ GeV}\) and \(m_{\bar{D}^{*0}} = 2.008 \text{ GeV}\) in the molecule calculation.

The interaction between mesons bound in a molecule can be described by a Yukawa potential, representing the exchange of light mesons like the pion. Because of the exponential decrease in this binding force, the meson molecule is expected to have a weaker coupling and to be bigger in size compared to the tetraquark. Also, it may not be able to support radial excitations, which would destroy the bound state. The Yukawa potential adopted is the following:

\[
V_Y = -g^2e^{-kr}/r
\]  

First we choose \(k = m_{\pi^\pm} = 0.139 \text{ GeV}\) and \(g^2 = 1\). Solving Eq. 1, we obtain negative interaction energies for the ground state and its first radial excitation, as shown in Table 2. The wavefunctions are shown in Figs. 4 and 5:

Table 2. Parameters and results for the \(D^0 - \bar{D}^{*0}\) meson molecule bound by the Yukawa potential with \(k = m_{\pi^\pm} = 0.139 \text{ GeV}\). Units are GeV for energies and masses \((c = 1)\) and fermi \((\text{fm})\) for the radius.

| Object   | \(g^2\) | \(k\)  | \(m_{D^0}\) | \(m_{\bar{D}^{*0}}\) | \(n\) | \(l\) | \(E_{\text{int}}\) | \(E_{\text{total}}\) | \(\langle r^2 \rangle^{1/2}\) |
|----------|---------|--------|-------------|-----------------|------|-----|--------------|----------------|------------------|
| molecule | 1       | 0.139  | 1.864       | 2.008           | 0    | 0   | -0.358       | 3.514           | 0.360            |
| molecule | 1       | 0.139  | 1.864       | 2.008           | 1    | 0   | -0.029       | 3.843           | 1.585            |
If we choose $k = 1$, we obtain a much more weakly bound ground state, as shown in Table 3:

| Object | $g^2$ | $k$ | $m_{D^0}$ | $m_{D^{*0}}$ | $n$ | $l$ | $E_{int}$ | $E_{total}$ | $<r^2>^{1/2}$ |
|--------|-------|-----|-----------|-------------|----|----|-----------|-------------|--------------|
| molecule | 1     | 1   | 1.864     | 2.008       | 0  | 0  | −0.007    | 3.865       | 1.373        |
| molecule | 1     | 1   | 1.864     | 2.008       | 1  | 0  | $4 \times 10^{-6}$ | 3.872       | 110.189      |

Figure 4. Radial squared normalized wave function $|R(r)|^2$ and reduced radial squared normalized wave function $|y(r)|^2$ for the $D^0 - \bar{D}^{*0}$ meson molecule bound by the Yukawa potential with $k = m_{\pi \pm} = 0.139$ GeV.

Figure 5. Radial squared normalized wave function $|R(r)|^2$ and reduced radial squared normalized wave function $|y(r)|^2$ for the first radial excitation of the $D^0 - \bar{D}^{*0}$ meson molecule in Yukawa-like potential with $k = m_{\pi \pm} = 0.139$ GeV.
Now the first radial excitation has an unrealistically large radius with negligible interaction energy. This is an indication that this excited state cannot exist, supporting our hypothesis that the meson molecule is not able to hold radial excitations. The wavefunctions are shown in Figs. 6, 7 and 8. The $|y(r)|^2$ wavefunction actually has a wide range. A zoom around the origin is shown in Fig. 7 and the whole range is shown in 8.

**Figure 6.** Radial squared normalized wave function $|R(r)|^2$ and reduced radial squared normalized wave function $|y(r)|^2$ for the $D^0 - \bar{D}^{*0}$ meson molecule bound by the Yukawa potential with $k = 1$ GeV.

**Figure 7.** Radial squared normalized wave function $|R(r)|^2$ and zoom around the origin of the reduced radial squared normalized wave function $|y(r)|^2$ for the first radial excitation of the $D^0 - \bar{D}^{*0}$ meson molecule bound by the Yukawa potential with $k = 1$ GeV.
Figure 8. Complete range of the reduced radial squared normalized wave function $|y(r)|^2$ for the first radial excitation of the $D^0 - \bar{D}^0$ meson molecule in Yukawa-like potential with $k = 1$ GeV.

3.3. The $T_{cc}$ tetraquark

In this subsection we present our results for the tetraquark configuration known as $T_{cc}$ [23]. In this configuration, instead of a diquark composed by a charm and a light quark, we consider a diquark as being composed by two charm quarks, whereas the two light antiquarks compose the antidiquark, forming a $cc - \bar{q}\bar{q}$ tetraquark structure, where $q \in u, d$, (or $c\bar{c} - q\bar{q}$). Using the same parameters as in [19], we calculate the $T_{cc}$ ground state energy and the first radial excitation ($n = 1$). In these calculations we have adopted $\Delta E = 0$. Parameters and results are listed in Table 4 and wavefunctions in Figs. 9 and 10.

Table 4. Parameters and results for $T_{cc}$ tetraquark model in ground state tetraquark and its first radial excitation ($n = 1$). Units are GeV$^2$ for $\sigma$, GeV for energies and masses ($c = 1$) and fermi (fm) for the radius.

| Object     | $\kappa_s$ | $\alpha_s$ | $\sigma$ | $m_1$ | $m_2$ | $n$ | $l$ | $E_{int}$ | $E_{total}$ | $<r^2>^{1/2}$ |
|------------|------------|-------------|----------|-------|-------|-----|-----|-----------|-------------|-------------|
| diquark cc | $-2/3$     | $0.3$      | $0.015$  | $1.486$ | $1.486$ | $0$  | $0$  | $0.072$   | $3.044$     | $0.990$     |
| antidiquark $\bar{d}\bar{d}$ | $-2/3$     | $0.3$      | $0.015$  | $0.323$ | $0.323$ | $0$  | $0$  | $0.178$   | $0.824$     | $1.864$     |
| tetraquark | $-4/3$     | $0.3$      | $0.030$  | $3.044$ | $0.824$  | $0$  | $0$  | $0.072$   | $3.940$     | $0.749$     |
| tetraquark | $-4/3$     | $0.3$      | $0.030$  | $3.044$ | $0.824$  | $1$  | $0$  | $0.278$   | $4.146$     | $1.539$     |
Figure 9. Normalized squared reduced radial wavefunction $|y(r)|^2$ and normalized squared radial wavefunction $|R(r)|^2$ for the $T_{cc}$ ground state.

Figure 10. Normalized squared reduced radial wavefunction $|y(r)|^2$ and normalized squared radial wavefunction $|R(r)|^2$ for the first radial excitation of the $T_{cc}$ ($n = 1$).

4. Conclusions

As it could be seen, in quark models, as the one used here, it is difficult to justify the choice of some parameters. Finding reliable theoretical justification for those choices is an important task yet to be done. In spite of the limitations of our approach, our results are rather suggestive. We can say that we could give preliminary answers to the questions raised in the introduction. It is very interesting to observe how, fixing the parameters so as to reproduce the $X(3872)$ mass, we can obtain the first radial excitation with the mass so close to the $Z(4430)$ mass. Assuming that the $X(3872)$ is a meson molecule interacting through a Yukawa potential, we can reproduce its mass. It remains to be checked that the radius of the $X(3872)$ is larger than 1 fm. Moreover it seems quite clear that the meson molecule does not have radial excitations. Finally, we find that the $T_{cc}$ mass is very close to estimates made with other methods, such as in Ref. [23].
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6. References

[1] For a comprehensive review on charmonium physics, see N. Brambilla, et al, Eur. Phys. J. C 71, 1534 (2011).
[2] For a review on exotic charmonium states, see M. Nielsen, F.S. Navarra, S.H. Lee, Phys. Rept. 497, 41 (2010).
[3] For a review on charged exotic charmonium states, see M. Nielsen and F.S. Navarra, Mod. Phys. Lett. A 29, 1430005 (2014).
[4] M. B. Voloshin and L. B. Okun, JETP Lett. 23, 333 (1976) [Pisma Zh. Eksp. Teor. Fiz. 23, 369 (1976)].
[5] A. De Rujula, H. Georgi and S. L. Glashow, Phys. Rev. Lett. 38, 317 (1977).
[6] A. Le Yaouanc, L. Oliver, O. Pene and J. C. Raynal, Phys. Lett. B 71, 397 (1977); 72, 57 (1977); E. Eichten, K. Gottfried, T. Kinoshita, K. D. Lane and T. M. Yan, Phys. Rev. D 21, 203 (1980).
[7] N. A. Tornqvist, Z. Phys. C 61, 525 (1994).
[8] T. E. O. Ericson and G. Karl, Phys. Lett. B 309, 426 (1993).
[9] A. V. Manohar and M. B. Wise, Nucl. Phys. B 399, 17 (1993).
[10] E. S. Swanson, Phys. Rept. 429, 243 (2006).
[11] R.L. Jaffe, Phys. Rev. D 15, 281 (1977).
[12] R.L. Jaffe, Phys. Rev. D 15, 267 (1977).
[13] J.D. Weinstein and N. Isgur, Phys. Rev. D 27, 588 (1983).
[14] J.D. Weinstein and N. Isgur, Phys. Rev. D 41, 2236 (1990).
[15] R.L. Jaffe, F. Wilczek, Phys. Rev. Lett. 91, 232003 (2003).
[16] L. Maiani, F. Piccinini, A.D. Polosa and V. Riquer, Phys. Rev. D 71, 014028 (2005).
[17] L. Maiani, A.D. Polosa and V. Riquer, arXiv:0708.3997, and New Journal of Physics, 10 073004 (2008).
[18] J. Vijande, A. Valcarce, J.M. Richard and N. Barnea, Few Body Syst. 45, 99 (2009).
[19] S. Patel, M. Shah and P.C. Vinodkumar, Eur. Phys. J. A 50, 131 (2014).
[20] W. Lucha, F.F. Schoeberl, Int. J. Mod. Phys. C 10, 607 (1999).
[21] E. Eichten, K. Gottfried, T. Kinoshita, K.D. Lane, T.M. Yan, Phys Rev. D 21, 203 (1980).
[22] F.S. Navarra, M. Nielsen and J.M. Richard, J. Phys. Conf. Ser. 348, 012007 (2012).
[23] F.S. Navarra, M. Nielsen and S. H. Lee, Phys. Lett. B 649, 166 (2007).