Anderson localization of Cooper pairs and Majorana fermions in an ultracold atomic Fermi gas with synthetic spin-orbit coupling

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We theoretically investigate two-particle and ground-state many-particle Anderson localizations of a spin-orbit coupled ultracold atomic Fermi gas trapped in a quasiperiodic potential and subjected to an out-of-plane Zeeman field. We solve exactly the two-particle problem in a finite length system by exact diagonalization and solve approximately the ground-state many-particle problem within the mean-field Bogoliubov-de Gennes approach. At a small Zeeman field, the localization properties of the system are similar to that of a Fermi gas with conventional s-wave interactions. As the disorder strength increases, the two-particle binding energy increases and the fermionic superfluidity of many particles disappears above a threshold. At a large Zeeman field, where the interatomic interaction behaves effectively like a p-wave interaction, the binding energy decreases with increasing disorder strength, and the resulting topological superfluidity shows a much more robust stability against disorder than the conventional s-wave superfluidity. We also analyze the localization properties of the emergent Majorana fermions in the topological phase. Our results could be experimentally examined in future cold-atom experiments, where the spin-orbit coupling can be induced artificially by using two Raman lasers, and the quasiperiodic potential can be created by using bichromatic optical lattices.

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I. INTRODUCTION

Ultracold atomic Fermi gases provide one of the most versatile platforms for realizing exotic many-body quantum states of matter, due to their unprecedented tunability and controllability [1,2]. Strongly interacting Fermi gases are easily achievable through the use of Feshbach resonances [3], which enable the realization of the crossover from Bose-Einstein condensates (BECs) to Bardeen-Cooper-Schrieffer (BCS) superfluids [2]. Low-dimensional Fermi gases are accessible by imposing optical lattice potentials [4–6], potentially allowing the observation of the elusive antiferromagnetic Néel order in the fermionic Hubbard model [7]. Two more examples are the recently synthesized spin-orbit coupled Fermi gases [8–10] and quasidisordered Fermi gases in bichromatic optical lattices [11,12], which open the ways to realize topological superfluids [13–16] and many-body localizations [11,12,17], respectively.

In this work, motivated by these rapid experimental progress, we propose to investigate Anderson localization of two and many interacting fermions in one-dimensional (1D) quasidisordered lattices [18], in the presence of a synthetic spin-orbit coupling and an out-of-plane Zeeman field. By increasing the strength of the Zeeman field above a threshold, the underlying character of the effective interatomic interaction changes from s-wave to p-wave [19], and in the absence of disorder the many-fermion system is known to experience a topological phase transition [13–15]. The main purpose of this work is to determine a rich phase diagram due to the interplay between disorder and s-wave or p-wave superfluidity.

Under sufficiently strong disorder potential, a dirty s-wave superconductor such as a superconducting NbN thin film undergoes a quantum phase transition towards a gapped insulator [20]. In condensed matter physics, such a superconductor-insulator transition has attracted a lot of interest over the past two decades [21–26]. Yet a complete understanding of the transition remains elusive. Also, the behavior of a dirty p-wave superconductor and its phase transition towards a gapped insulator is less known [27–29]. Our proposed system of disordered spin-orbit coupled atomic Fermi gases offers an unique opportunity to better understand these quantum phase transitions. In particular, it is of great interest to determine how disorder affects the topological phase transition and what is the fate of Majorana fermion edge modes, as the key feature of a topological superfluid, against disorder. Naively, we may anticipate two different phase diagrams, as schematically illustrated in Fig. 1. As the disorder strength increases, the topological phase transition line (the dashed curve) could bend to either the left- or right-hand side of the circle that indicates the initial transition point in the clean limit. In other words, with increasing disorder, an s-wave superfluid may first become a topological superfluid before it finally turns into a gapped insulator (i.e., the scenario a), or vice versa (b).

The proposed 1D spin-orbit coupled Fermi system in quasidisordered lattices can be easily realized in cold-atom laboratory, where the synthetic spin-orbit coupling and disorder potential can be created by using Raman laser beams and bichromatic optical lattices [30], respectively. By using the self-consistent Bogoliubov-de Gennes mean-field theory, we calculate the superfluid density (under periodic boundary condition) and the Majorana fermion edge modes (with open boundary condition) of the system at zero temperature. A vanishingly small superfluid density indicates the transition to a gapped insulator, while the existence of Majorana fermions signals the topological phase transition. This allows us to determine the whole phase diagram (see Fig. 4), the main result of this work. To understand the phase diagram, we exactly solve the two-fermion problem and address the Anderson localization of a Cooper pair in its ground state [31]. To utilize and better understand the robustness of Majorana fermions with respect to disorder, we also consider the soliton-induced Majorana fermions [32,33] and investigate the evolution of their energies and wave functions as a function of the disorder.
strong. Our results on the Anderson localization of Majorana fermions may be useful for manipulating these exotic non-Abelian quasiparticles in a realistic noisy environment, for the purpose of performing fault-tolerant quantum information processing [34].

II. MODEL HAMILTONIAN

We start by considering a 1D disordered spin-orbit coupled Fermi gas in a lattice with \( L \) sites. The system can be described by the fermionic model Hamiltonian,

\[
\mathcal{H} = -t \sum_{i=1}^{L} \sum_{\sigma} (c_{i+1,\sigma}^\dagger c_{i,\sigma} + \text{H.c.}) + \sum_{i,\sigma} V_i n_{i\sigma} + \mathcal{H}_R + \mathcal{H}_Z - U \sum_i n_{i\uparrow} n_{i\downarrow},
\]

(1)

where \( c_{i\sigma} \) is the annihilation operator with spin \( \sigma \in \{\uparrow, \downarrow\} \) at site \( i \), \( n_{i\sigma} = c_{i\sigma}^\dagger c_{i\sigma} \) is the local number operator, \( t \) is the tunneling strength between neighboring lattice sites, \( V_i = V \cos(2\pi \beta i + \phi) \) with an irrational number \( \beta \), and phase offset \( \phi \) describes the quasirandom disorder potential induced by bichromatic (i.e., additional incommensurate) lattices [11,12], and \( U > 0 \) in the last term represents the on-site attractive interaction. For definiteness, we shall take \( \beta = (\sqrt{5} - 1)/2 \) and \( \phi = 0 \). We assume a periodic boundary condition such that \( c_{L+1,\sigma} = c_{1,\sigma} \), unless specified otherwise. Finally, the spin-orbit term with Rashba-type coupling \( \mathcal{H}_R \) and the Zeeman energy term \( \mathcal{H}_Z \) are given by

\[
\mathcal{H}_R = \frac{\lambda}{2} \sum_{i=1}^{L} (c_{i+1,\uparrow}^\dagger c_{i,\uparrow} - c_{i+1,\downarrow}^\dagger c_{i,\downarrow} + \text{H.c.}),
\]

(2)

\[
\mathcal{H}_Z = h_z \sum_{i=1}^{L} (c_{i\uparrow}^\dagger c_{i\uparrow} - c_{i\downarrow}^\dagger c_{i\downarrow}),
\]

(3)

respectively. Here \( \lambda \) is the spin-orbit coupling strength and \( h_z \) is the Zeeman field.

We note that, in the absence of \( \mathcal{H}_R \) and \( \mathcal{H}_Z \), the model Hamiltonian Eq. (1), known as the Aubry-André-Harper model [35,36], has been experimentally explored [11]. In the noninteracting limit (\( U = 0 \)), all single-particle states become localized at the same critical disorder strength \( V = 2t \) [36]. For an initial charge density-wave state, its many-body localization at arbitrary \( U \) has been demonstrated [11]. The possibility of having the spin-orbit and Zeeman terms in the model Hamiltonian was recently proposed and derived by Zhou and co-workers [30]. The Anderson localization of a single atom in the noninteracting limit (\( U = 0 \)) was studied. It was found that the spin-orbit coupling can lead to a nonpure spectrum (i.e., coexistence of extended and localized states) and the appearance of mobility edges. Here, we are interested in the localization of the many-body ground state and the interplay between disorder and conventional/unconventional superfluidity.

III. LOCALIZATION OF A SINGLE COOPER PAIR

Before discussing the localization of the coherent (superfluid) state of many Cooper pairs, it is instructive to understand the localization of a single Cooper pair [31]. For this purpose, we numerically exactly solve the two-fermion problem:

\[
(\mathcal{H} - E) \sum_{i,j} \left[ \psi_{ij}^\dagger c_{i\uparrow}^\dagger c_{j\uparrow} + \psi_{ij}^\dagger c_{i\downarrow}^\dagger c_{j\downarrow} + \psi_{ij}^\dagger c_{i\uparrow} c_{j\uparrow} \right] |0\rangle = 0,
\]

(4)

where \( \psi_{ij}^\sigma \) is the two-particle wave-function and \( E \) is the energy. It is understood that for the same spin \( \sigma = \sigma' \) at the same site \( i = j \), \( \psi_{ij}^\sigma \) = 0 because of Pauli exclusion principle. For numerical stability, we approximate the irrational number \( \beta \) as the limit of a continued fraction, \( \beta \approx F_{l+1}/F_l \), where \( F_l \) are Fibonacci numbers and \( l \) is a sufficiently large integer. We minimize the finite-size effect by taking the length of the lattice \( L = F_l \) [31]. To characterize the Anderson localization of the pair, we use the inverse participation ratio (IPR) [37]:

\[
\alpha_{\text{IPR}} = \sum_{i,j} (|\psi_{ij}^\dagger|^4 + |\psi_{ij}^\downarrow|^4 + |\psi_{ij}^\uparrow|^4).
\]

(5)

For an extended state, \( \alpha_{\text{IPR}} \propto 1/L^2 \) decreases to zero in the thermodynamic limit. While for a localized state, it saturates to a finite value.

We have calculated \( \alpha_{\text{IPR}} \) for the ground state of the pair at a moderate interaction \( U = 4.5t \) and spin-orbit coupling \( \lambda = 2t \), as reported in Fig. 2(a). The results do not depend on the different realization of disorder, which is characterized by the phase offset \( \phi \) [31]. In our calculations, we therefore focus on a particular disorder realization with \( \phi = 0 \). As the disorder strength \( V \) increase, there is a sharp increase in the IPR. We then identify the critical strength \( V_c \) as the inflection point of the calculated curve \( \alpha_{\text{IPR}}(V) \). By repeating calculations at different lengths of lattices (up to \( L = 233 \)), we eliminate the finite-size dependence and determine the phase boundary \( V_c \) in the thermodynamic limit, as shown by solid circles in the figure. It is interesting that with increasing Zeeman field, the threshold \( V_c \) increases. This might be understood from the fact that a finite Zeeman field plays a role of pair breaker and the resulting weakly bound pair is easier to move in disordered potential than a tightly bound pair [31]. In Fig. 2(b) we present the binding energy \( E_B = 2\epsilon_1 - E \) of the pair as a function of the disorder strength at different Zeeman fields. Here \( \epsilon_1 \) is the ground state energy for a single
fermion. Indeed, with increasing Zeeman field, the binding energy decreases quickly. As a result, at sufficiently large Zeeman field, a Cooper pair may break into two fermions. This pair-breaking effect is mostly evident at $h_z = 2.5t$, where the pair ceases to exist at $V \gtrsim 1.2t$. In this part of parameter space [i.e., the top-right part of Fig. 2(a)], we observe the Anderson localization of a single fermion instead of a Cooper pair [30].

It is also worth noting that the binding energy shows very different dependence on the disorder strength at small and large Zeeman fields. While the disorder enhances the binding energy at low Zeeman field, it breaks pair at high field. These two distinct behaviors may be attributed to the different effective interatomic interactions. For a spin-orbit coupled Fermi gas, with increasing Zeeman field, the underlying interaction between atoms changes from $s$-wave like to $p$-wave like with increasing the Zeeman field above a threshold [19]. For the parameter used in Fig. 2(b), this transition occurs at about $h_z \sim 1.8t$.

IV. MEAN-FIELD PHASE DIAGRAM AND ANDERSON LOCALIZATION OF MAJORANA FERMIONS

We now turn to consider the ground state of many fermions by using mean-field Bogoliubov-de Gennes theory, which is known to capture the qualitative physics in low dimensions [22,38,39]. In particular, for the exactly solvable Gaudin-Yang model of a 1D two-component Fermi gas, by comparing the mean-field result with the exact solution it was found that the mean-field approach provides a reasonable description in the weak-coupling limit [39]. Here we anticipate that the mean-field theory still works in 1D even in the presence of disorder and spin-orbit coupling. On the other hand, with ultracold atoms, it is possible to induce directly superfluid pairing by using a Raman laser in proximity to a molecular BEC [40–42]. In that case, the mean-field Hamiltonian, shown in Eq. (6) below, provides an exact starting point for theoretical description.

In the mean-field framework, we decouple the interaction term $-U \sum_i n_i \hat{S}_i$ into the pairing term and the Hartree-Fock term, $\mathcal{H}_{\Delta} = \sum_i (\Delta \hat{c}_{i\uparrow}^\dagger \hat{c}_{i\downarrow} + \Delta^* \hat{c}_{i\downarrow}^\dagger \hat{c}_{i\uparrow})$ and $\mathcal{H}_{HF} = \sum_i (-U (n_i \uparrow) n_i \downarrow + U (n_i \downarrow) n_i \uparrow)$, which yields an effective quadratic Hamiltonian,

$$\mathcal{H}_{\text{eff}} = -t \sum_{\sigma} (\hat{c}_{i+1,\sigma}^\dagger e_{i,\sigma} + \text{H.c.}) + \sum (V_i - \mu_{i\sigma}) n_{i\sigma}$$

$$+ \mathcal{H}_R + \mathcal{H}_Z + \sum_i (\Delta^c_i \hat{c}_{i\uparrow}^\dagger + \text{H.c.}) + E_0,$$

where $E_0 = \sum_i (|\Delta_i|^2 / U + U \langle n_i \rangle \langle n_i \rangle)$ is a constant and $\mu_{i\sigma} = \mu + U \langle n_{i\sigma} \rangle$ is the local chemical potential that incorporates the site-dependent Hartree shift. We diagonalize the effective Hamiltonian by using the standard Bogoliubov transformation, which leads to

$$\begin{bmatrix}
\hat{K}^+ & \hat{\Lambda}^i & 0 & \hat{\Delta} \\
\hat{\Lambda} & \hat{K}^- & -\Delta & 0 \\
0 & -\hat{\Lambda}^* & -\hat{K}^*_+ & -\hat{\Delta}^* \\
\hat{\Lambda}^* & 0 & -\hat{\Lambda}^- & -\hat{K}^-_+
\end{bmatrix}
\begin{bmatrix}
\hat{u}_{i\uparrow}^{(n)} \\
\hat{u}_{i\downarrow}^{(n)} \\
\hat{v}_{i\uparrow}^{(n)} \\
\hat{v}_{i\downarrow}^{(n)}
\end{bmatrix}
= E_n
\begin{bmatrix}
\hat{u}_{i\uparrow}^{(n)} \\
\hat{u}_{i\downarrow}^{(n)} \\
\hat{v}_{i\uparrow}^{(n)} \\
\hat{v}_{i\downarrow}^{(n)}
\end{bmatrix},$$

where $u_{i\sigma}^{(n)}$ and $v_{i\sigma}^{(n)}$ are the Bogoliubov quasiparticle wave functions and $E_n$ is the associated quasiparticle energy. The operators $\hat{K} = \hat{K} \pm h_z$, $\hat{\Lambda}$, and $\hat{\Delta}$, respectively, are given by

$$\hat{K}_\pm = \hat{K} \pm h_z, \quad \hat{\Lambda}_\pm = \hat{\Lambda} \pm \hat{\Delta}, \quad \hat{\Delta} = \sqrt{\Delta^2 - \hat{\Lambda}^2},$$

and similarly for $\hat{v}_{i\sigma}^{(n)}$. The local pairing gap and local density are determined by the self-consistency conditions,

$$\Delta = -U \sum_i \left[ u_{i\uparrow}^{(n)} u_{i\downarrow}^{(n)*} f(E_n) + u_{i\downarrow}^{(n)} u_{i\uparrow}^{(n)*} f(-E_n) \right],$$

$$\langle n_{i\sigma} \rangle = \sum_n \left[ |u_{i\sigma}^{(n)}|^2 f(E_n) + |v_{i\sigma}^{(n)}|^2 f(-E_n) \right].$$

where $f(E_n) = 1/(e^{E_n/k_B T} + 1)$ is the Fermi-Dirac distribution function. Throughout the paper, we focus on the zero-temperature case where $f(E_n)$ becomes a step function.

We note that in defining the operators $\hat{K}$ and $\hat{\Lambda}$, a phase twist $\theta$ is explicitly added, which corresponds to introduce a constant vector potential $\theta = \theta$ across the system [43]. Physically,
with the periodic boundary condition where our 1D system is wrapped to be a ring, the vector potential can be simply viewed as the Aharonov-Bohm (AB) phase \( \theta = 2\pi \Phi/\Phi_0 \) with a magnetic-field-induced AB flux penetrating the ring \( \Phi \) and the flux quantum \( \Phi_0 = \hbar c/e \) [44]. This implementation is convenient for the calculation of the superfluid stiffness \( D_s \). Indeed, with the vector potential, \( D_s \) is given by the change in the free energy \( F = \langle \mathcal{H}_{\text{eff}} \rangle + \mu N \) [26]:

\[
D_s = \frac{1}{L} \frac{\partial^2 F(\theta)}{\partial \theta^2} .
\]

It is necessary to use such a formal expression in the presence of disorder, since the decoupling between longitudinal and transverse electromagnetic responses does not hold and the standard approach of calculating \( D_s \) through the BCS response function fails [26].

For the localization of many fermions in their ground state, we use a vanishingly small superfluid stiffness to characterize a gapped insulator phase. Alternatively, we may also determine the onset of the localization phase through the calculation of the IPR of Bogoliubov quasiparticle wave functions, as in the case of two fermions. However, the use of exponentially small superfluid stiffness turns out to be a more physical criterion (see the Appendix for more discussion). On the other hand, as we mentioned earlier, the system features a topological superfluid at relatively large Zeeman fields. While the existence of a nontrivial topology of the superfluid can be revealed by some topological invariants [15], numerically it is actually more convenient to be identified from the appearance of two Majorana fermion modes at the edges, if one imposes an open boundary condition.

In the left panel of Fig. 3, we report the superfluid stiffness as a function of the disorder strength and of the Zeeman field. The corresponding results for the energy of low-lying quasiparticle modes are shown in the right panel of the figure.

At a given Zeeman field [Fig. 3(a)], the superfluid stiffness vanishes exponentially above a critical value \( V_c \), signifying the transition towards a gapped insulator. The energy of low-lying modes exhibits different dependence on disorder strength at low and high Zeeman fields [Fig. 3(b)]. At a small Zeeman field \( h_z = 0.9t \), the energy gap \( (E_{\text{gap}} = 2E_1) \) is always nonzero and increases monotonically with disorder strength, as anticipated for a disordered conventional s-wave superfluid [22]. In contrast, at a large Zeeman field \( h_z = 1.8t \), there are two zero-energy modes that correspond to the Majorana fermions localized at the two edges. By increasing the disorder strength above a threshold \( V_c^* \), the zero-energy modes cease to exist [28,29], suggesting the loss of the nontrivial topological property of the superfluid. The two critical disorder strengths, \( V_c \) and \( V_c^* \), do not necessarily take the same value.

By collecting the two critical disorder strengths at different Zeeman fields, we arrive at our main result, the phase diagram of a spin-orbit coupled Fermi gas in bichromatic optical lattices at \( U = 4.5t \) and \( \lambda = 2t \). The color bar indicates the superfluid stiffness. A vanishingly small superfluid stiffness (i.e., the red dotted line) determines the phase transition to a normal state (\( V_c \)), due to the Anderson localization of the ground state of many particles. The blue squares with dashed line (\( V_c^* \)) enclose the phase space, where the superfluid is topologically nontrivial and Majorana fermions exist at the two open edges. We use an average filling factor \( n = N/L = 44/89 \), as in Fig. 3, except for the four points (green empty squares), which show the critical disorder strength \( V_c \) at a smaller filling factor \( n = 22/89 \).

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FIG. 3. Left panel: Evolution of the superfluid stiffness as a function of the disorder strength (a) or Zeeman field strength (c). Right panel: The lowest or the second lowest quasiparticle excitation energy as a function of the disorder strength (b) or Zeeman field strength (d), under an open boundary condition. Here we take \( L = 89 \) and set \( U = 4.5t \) and \( \lambda = 2t \). Moreover, we use an average (quarter) filling factor \( n = N/L = 44/89 \).

FIG. 4. Phase diagram of a 1D spin-orbit coupled Fermi gas in bichromatic optical lattices at \( U = 4.5t \) and \( \lambda = 2t \). The color bar indicates the superfluid stiffness. A vanishingly small superfluid stiffness (i.e., the red dotted line) determines the phase transition to a normal state (\( V_c \)), due to the Anderson localization of the ground state of many particles. The blue squares with dashed line (\( V_c^* \)) enclose the phase space, where the superfluid is topologically nontrivial and Majorana fermions exist at the two open edges. We use an average filling factor \( n = N/L = 44/89 \), as in Fig. 3, except for the four points (green empty squares), which show the critical disorder strength \( V_c \) at a smaller filling factor \( n = 22/89 \).
We now examine in more detail the localization of Majorana fermions, which gives the critical strength $V_c^*$. Experimentally, it seems convenient to create and manipulate Majorana fermions by engineering solitons via the phase-imprinting technique [45,46]. At the site of each soliton, there exist two soliton-induced Majorana fermions [32,33]. In Fig. 5 we plot the energy and wave functions of low-lying quasiparticle modes in the presence of a dark soliton at the center of the topological superfluid and under an open boundary condition [47]. The wave functions of Majorana fermions are essentially not affected by a weak or moderately strong disorder [Fig. 5(b)] and are well localized at the edges or at the site of soliton. In particular, the soliton-induced Majorana fermions are clearly revealed by the finite oscillation in the middle of the lattice [32,33]. Towards the critical strength $V_c^*$ [Fig. 5(c)], however, the wave functions spread over the whole lattice sites. Majorana fermions disappear, although the quasiparticle states are still extensive.

V. CONCLUSIONS

We have proposed to study the Anderson localization of a spin-orbit coupled Fermi gas in one-dimensional quasidissordered lattices. In the absence of disorder, the system features a conventional $s$-wave-like superfluid and topological $p$-wave-like superfluid at small and large Zeeman fields [19], respectively. By tuning the Zeeman field, we have investigated how these superfluid states lose their superfluidity with increasing disorder strength. We have found that topological superfluids (and hence the hosted Majorana fermions) are very robust against disorder and they lose their nontrivial topological properties before finally turn into gapped insulators. We have complemented our many-body study by considering the localization of a single Copper pair. The result indicates that the localization of pairs and the loss of coherence between pairs occurs simultaneously.

In the near future, it is of interest to consider Anderson localization of the spin-orbit coupled Fermi gas system in two-dimensional disordered lattices [12]. The strong quantum phase fluctuations near topological and localization transitions could be addressed by using a zero-temperature Gaussian fluctuation theory [48–50].

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APPENDIX: THE IPR OF BOGOLIUBOV QUASIPARTICLE WAVE FUNCTIONS

In this Appendix, we examine the mean IPR of Bogoliubov quasiparticle wave functions, which can be calculated by using the expression

$$d_{\text{IPR}}^{(\text{bog})} = \frac{1}{4L} \sum \sum |\psi_{\sigma}^{(n)}|^2 + |\psi_{\sigma}^{(n)}|^2.$$  \hfill (A1)

A rapid increase in the mean IPR is a useful indicator of Anderson localization of the Bogoliubov quasiparticles. We anticipate that Anderson localization of quasiparticles should lead to a vanishingly small superfluid stiffness and hence the loss of superfluidity. However, the reverse may not be true. The loss of superfluidity may appear well before the Anderson localization of quasiparticles. In that case, there exists an immediate phase with both extended quasiparticle wave functions and zero superfluid stiffness.

In Fig. 6 we examine this possibility by plotting the evolution of the superfluid stiffness and the mean IPR of Bogoliubov quasiparticles, as a function of the disorder strength.

![Figure 6](image_url)
It is readily seen that a decrease of the superfluid stiffness is generally associated with an increase in the mean IPR. However, the length of the system that we considered ($L = 89$) is not large enough to yield a very sharp transition in IPR, and therefore we are not able to identify the existence of a possible immediate phase. A further check with a much larger system size, i.e., $L \sim 1000$, would be desirable. Unfortunately, with increasing length, the calculation of the superfluid stiffness becomes increasingly difficult and practically we are not able to reach such a large system size.

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