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Processing of materials of laser scanning of roads on the basis of recursive cubic splines

E A Esharov¹ and B M Shumilov¹
¹Tomsk State University of Architecture and Building, Department of Applied Mathematics, Tomsk, 634003, Russia

E-mail: sbm@tsuab.ru

Abstract. The problem of approximation of discrete dependence \((t_i, y_i)\) by means of continuous function \(f(t)\) arises in tasks of the analysis and data processing of measurements \(y_i\) at successive time points of \(t_i\). The solution of the presented task becomes significantly complicated in cases when measurements form a continuous stream, and the results of processing are required to be used before full reception of measurements. The computational schemes based on recurrent spline of degree 3 depths 1 and 2 which provide accuracy for polynomials of the third degree were used in the work. The main results consist in elaboration of mathematical formulas and the proof of stability of the constructed computing schemes. Relevance of the work consists in possibility of application of the constructed splines, for example, for processing materials of laser scanning of roads. Advantages of the constructed splines are the piecewise cubic character of the recovering of the simulated smooth spatial curve and possibility of non-uniform sampling on the route when the quantity of spline knots are less than the number of samples of a function. Examples of calculation of parameters of computing schemes, and also results of numerical experiments are presented.

1. Introduction

Samples of experimental data are often represented as an array consisting of pairs of numbers \((t_i, y_i)\). Therefore, in order to further use them for analysis and processing, the problem of approximating a discrete dependence \(y(t_i)\) by continuous function \(f(t)\) arises. Function \(f(t)\), depending on the specific tasks, may respond to different requirements [1]:

- \(f(t)\) should pass through the points \((t_i, y_i)\), i.e. \(f(t_i) = y_i\), \(i = 1, 2, ..., n\). In this case it should be said about the interpolation function \(f(t)\) at the interior points between \(t_i\) or extrapolation outside the range that contains all the \(t_i\) – forecasting (prediction);
- \(f(t)\) should in some way (for example as determined according to analytical dependence) approximate \(y_i\), not necessarily pass through the points \((t_i, y_i)\). This is the statement of the problem of regression, which in many cases can also be called smoothing data;
- \(f(t)\) should approximate the experimental dependence \(y(t_i)\) at the given points considering that the data \((t_i, y_i)\) are obtained with an error, which expresses the measurement noise component. The function \(f(t)\) using a particular algorithm reduces the error in the data \((t_i, y_i)\). This type of problem is called filtering. Smoothing – a special case of filtering;
- \(f(t)\) is used for more efficient use of storage and data transfer. This allows the data compression. Data compression process is recoding data produced in order to reduce their storage.
The solution of the presented problems becomes significantly complicated in cases when measurements form a continuous stream, so results of processing are required to be used before full reception of these measurements. Such situation arises, for example, at laser scanning of roughness of a roadbed [2].

2. Recursive formulas of approximation compressing splines of degree 3

Let the measured process \( f(t) \) is registered in the equally spaced time points \( t_i \) for measuring steps \( \Delta t \), and \( f(t) \) allows for adequate representation in the form of an expansion in \( B \)-splines of the third degree and between neighboring nodes are \( m+1 \) measurements. Then the measured values \( y_{i,k} \) in the intervals \( [t_k, t_{k+1}] \), \( k = 0, 1, ..., \) are [3]:

\[
y_{i,k} = \frac{1}{6} \left( 1 - \frac{i}{m} \right)^3 y_{k-1} + \left[ \frac{2}{3} \left( \frac{i}{m} \right)^2 + \frac{1}{2} \left( \frac{i}{m} \right)^3 \right] y_k + \\
+ \frac{1}{2} \left[ \frac{1}{3} + \frac{i}{m} \right] \left( \frac{i}{m} \right)^2 \left( \frac{i}{m} \right)^3 y_{k+1} + \frac{1}{6} \left( \frac{i}{m} \right)^3 y_{k+2} + \xi_{i,k},
\]

where \( \xi_{i,k} \) – the measurement errors.

1.1. Recursive formulas of approximation compressing splines of degree 3 depth 1

In this case to find estimates \( \hat{y}_k \) of expansion coefficients of the cubic spline we will use a recursive filter of depth 1 with the pattern averaging width \( m+1 \):

\[
\hat{y}_k = \lambda_1 \hat{y}_{k-1} + \sum_{i=0}^{m} a_i y_{i,k+1}, \quad k = -1, 0, 1, ..., \tag{2}
\]

where the initial condition \( \hat{y}_k, k = -2 \), determined by the measured values \( y(t) \) at the initial segments, for example, by the least squares method, \( m+1 \) – quantity of measurements on each segment of spline, \( y_{i,k+1} \) – observation coming from the \((k+1)\)-th segment, the coefficients \( \lambda_1, a_i \) – free parameters of the recursive averaging algorithm.

**Theorem 1.** Let for a given \( m > 2 \) the coefficients \( \lambda_1, a_i \) of algorithm (2) satisfy the following conditions

\[
\sum_{i=0}^{m} a_i = 1 - \lambda_1, \quad \sum_{i=0}^{m} a_i \left( \frac{i}{m} \right) = 2 \lambda_1 - 1, \quad \sum_{i=0}^{m} a_i \left( \frac{i}{m} \right)^2 = \frac{2}{3} \Delta t^2, \quad \sum_{i=0}^{m} a_i \left( \frac{i}{m} \right)^3 = 6 \lambda_1. \tag{3}
\]

Then the coefficients \( \hat{y}_k \) of the 3rd degree spline approximation of depth 1, defined by the algorithm (2), are stable calculated. Spline, constructed by Theorem 1, provides accuracy up to third-degree polynomials.

**Proof.** Verify the conditions of the accuracy of third-degree polynomials [4]. For that spline, in order to exactly reproduce the measurement \( y_{i,k} \forall i, k \) for the corresponding \( y_{i,k} \) from the set of values of any polynomial with a degree not higher than 3, it is necessary and sufficient that the flowing equalities valid

\[
\begin{align*}
\sum_{i=0}^{m} a_i + \lambda_1 &= 1, \\
\sum_{i=0}^{m} a_i t_{i,k+1} + \lambda_1 t_{k+1} &= t_k, \\
\sum_{i=0}^{m} a_i t_{i,k+1}^2 + \lambda_1 t_k^2 - 2 t_k \Delta t + \frac{2 \Delta t^2}{3} &= t_k^2 - \frac{2 \Delta t^2}{3}, \\
\sum_{i=0}^{m} a_i t_{i,k+1}^3 + \lambda_1 t_k^3 - 3 t_k \Delta t^2 + 2 t_k \Delta t^3 &= t_k^3 - t_k \Delta t^3.
\end{align*}
\]
Here \( t_{i,k} = t_k + \frac{i}{m} \Delta t \), \( i = 0, m \). Performance the first of (2) is obvious, from the first equality we obtain \( \sum_{i=0}^{m} a_i = 1 - \lambda_1 \).

From the second equality, substituting \( t_k + \frac{i+m}{m} \Delta t \) for \( t_{i,k+1} \) and substituting \( t_k - \Delta t \) for \( t_{i,k} \), we obtain \( \sum_{i=0}^{m} a_i \left( t_k + \frac{i+m}{m} \Delta t \right) + \lambda_1 \left( t_k - \Delta t \right) = t_k \). Removing the brackets and taking \( t_k \) out, we have

\[
t_k \left( \sum_{i=0}^{m} a_i + \lambda_1 - 1 \right) + \sum_{i=0}^{m} a_i \frac{i}{m} \Delta t + \sum_{i=0}^{m} a_i \Delta t - \lambda_1 \Delta t = 0 .
\]

Since the first equality has been proved, the expression in the brackets is zero. Hence, we obtain

\[
\sum_{i=0}^{m} a_i \frac{i}{m} = 2\lambda_1 - 1 .
\]

Removing the brackets and taking \( \Delta t \neq 0 \), we come to the second of the conditions being proved: \( \sum_{i=0}^{m} a_i \frac{i}{m} = 2\lambda_1 - 1 .\)

The third and fourth equalities of system are similarly considered.

Denote

\[
\tilde{y}_k = \sum_{i=0}^{m} a_i y_{i,k+1} .
\]

Then (2) can be rewritten in the following form

\[
\hat{y}_k = \lambda_1 \hat{y}_{k-1} + \tilde{y}_k .
\]

Equation (6) is a 1st order linear inhomogeneous differential equation, with the following form of characteristic equation

\[
z - \lambda_1 = 0 .
\]

The equation root equals \( \lambda_1 \), and it is known [5] that condition (4), i.e. \( |\lambda_1| < 1 \), makes the solution of the equation (6) be stable to the initial condition \( \hat{y}_{-2} \). Theorem 1 is proved.

1.2. Recursive formulas of approximating compression spline of degree 3 depth 2

Unlike to the previous section, to find the estimates \( \hat{y}_k \) of expansion coefficients of the cubic spline will be used the recursive filter of depth 2, with predetermined averaging width \( m + 1 \):

\[
\hat{y}_k = \lambda_1 \hat{y}_{k-1} + \lambda_2 \hat{y}_{k-2} + \sum_{i=0}^{m} a_i y_{i,k+1} , \quad k = 1, 2, ..., \tag{7}
\]

where \( m + 1 \) – the quantity of measurements at each spline segment, \( y_{i,k+1} \) are the observations coming from the \( (k+1) \)-th segment and the coefficients \( \lambda_1, \lambda_2, a_i \) – free parameters of the recursive averaging algorithm. Here, the initial conditions \( \hat{y}_{-i} \), \( k = -3, -2, -1, 0 \), are determined from the measured values \( y(t) \) at the initial segments, for example, by the least squares method.

Theorem 2. Let for a given \( m \) coefficients \( \lambda_1, \lambda_2, a_i \) of algorithm (7) satisfy the following conditions

\[
\sum_{i=0}^{m} a_i = 1 - \lambda_1 - \lambda_2 , \quad \sum_{i=0}^{m} a_i \frac{i}{m} = 2\lambda_1 + 3\lambda_2 - 1 ,
\]

\[
\sum_{i=0}^{m} a_i \left( \frac{i}{m} \right)^2 = \frac{2-11\lambda_1 - 26\lambda_2}{3} , \quad \sum_{i=0}^{m} a_i \left( \frac{i}{m} \right)^3 = 6\lambda_1 + 24\lambda_2 .
\]

\[
\lambda_1^2 + 4\lambda_2 > 0 ; \quad |\lambda_1| + |\lambda_2| < 1 . \tag{9}
\]
Then the coefficients \( \hat{y}_k \) of the spline approximation of degree 3 depth 2, defined by the algorithm (7), are stable calculated. Spline, constructed under the conditions of Theorem 2, provides accuracy to third-degree polynomials.

**Proof** is carried out similarly, beginning with the conditions of the accuracy for third-degree polynomials:

\[
\begin{align*}
\sum_{i=0}^{m} a_i + \lambda_1 + \lambda_2 &= 1, \\
\sum_{i=0}^{m} d_i t_{i,k+1} + \lambda_1 t_{k-1} + \lambda_2 t_{k-2} &= t_k, \\
\sum_{i=0}^{m} d_i t_{i,k+1}^2 + \lambda_1 (t_k^2 - 2t_k \Delta t + \frac{2}{3} \Delta t^2) + \lambda_2 (t_k^2 - 4t_k \Delta t + \frac{11}{3} \Delta t^2) &= t_k^2 - \frac{\Delta t^2}{3}, \\
\sum_{i=0}^{m} d_i t_{i,k+1}^3 + \lambda_1 (t_k^3 - 3t_k^2 \Delta t + 2t_k \Delta t^2) + \lambda_2 (t_k^3 - 6t_k^2 \Delta t + 11t_k \Delta t - 6\Delta t^3) &= t_k^3 - t_k \Delta t^2.
\end{align*}
\]

3. Numerical examples and results of the experiments

3.1. Numerical examples

3.1.1. Case \( p = 1 \). When \( m < 2 \) solution does not exist, because the quantity of parameters is less than the amount of conditions (3). For \( m = 2 \) the measuring points are conveniently placed at the nodes of the spline and at one point midway between the nodes of the spline. Then the values \( \frac{i}{m} = 0, 1/2, 1 \), where \( i \) – numbers of measurements between the nodes of the spline. As a result, we have 4 unknown values \( \lambda_1, a_0, a_1, a_2 \) to determine which is required to solve a system of linear algebraic equations of the form

\[
\begin{align*}
a_0 + a_1 + a_2 + \lambda_1 &= 1, \\
\frac{1}{2} a_1 + a_2 - 2\lambda_1 &= -1, \\
\frac{1}{4} a_1 + a_2 + \frac{11}{3} \lambda_1 &= \frac{2}{3}, \\
\frac{1}{8} a_1 + a_2 - 6\lambda_1 &= 0.
\end{align*}
\]

The determinant of the matrix of the system is \(-25/8\), so there is a unique solution

\[
a_0 = \frac{271}{75}, a_1 = \frac{-296}{75}, a_2 = \frac{91}{75}, \lambda_1 = \frac{3}{25}.
\]

Because the absolute value of \( \lambda_1 = 3/25 \) is \(<1\), the obtained solution fully satisfies the conditions of Theorem 1. The next version – when the measurement points are located at the nodes of the spline and other 2 points for measurement are spaced evenly between the nodes of the spline. If the calculation of spline coefficients takes the points corresponding \( \frac{i}{m} = 1/3, 2/3, 1 \) (shift of averaging pattern), the only solution exists:

\[
a_0 = \frac{721}{96}, a_1 = \frac{-127}{12}, a_2 = \frac{1123}{288}, \lambda_1 = \frac{25}{144}.
\]
Since in this case $\lambda_1 = \frac{25}{144} < 1$, the resulting solution fully satisfies the conditions of Theorem 1. However, it is not entirely comfortable with the point of view of prediction properties of the resulting spline solution. Nevertheless, there is an acceptable option, corresponding to the points \( i = 0, \frac{1}{3}, \frac{2}{3}, \frac{3}{3} \), in which the only solution exists, \( a_0 = \frac{1123}{182}, a_1 = -\frac{732}{91}, a_2 = \frac{39}{14}, \lambda_1 = \frac{8}{91} \), and it completely satisfies the conditions of Theorem 1.

Let now \( m = 3 \). As a result, we have 5 unknowns $\lambda_1, a_0, a_1, a_2, a_3$, to determine which is required to solve the under determined system of 4 linear algebraic equations, depending on the parameter $\lambda_1$,
\[
\begin{align*}
    a_0 + a_1 + a_2 + a_3 &= 1 - \lambda_1, \\
    \frac{1}{3}a_1 + \frac{2}{3}a_2 + a_3 &= 2\lambda_1 - 1, \\
    \frac{1}{9}a_1 + \frac{4}{9}a_2 + a_3 &= \frac{2 - 11\lambda_1}{3}, \\
    \frac{1}{27}a_1 + \frac{8}{27}a_2 + a_3 &= 6\lambda_1.
\end{align*}
\]

The determinant of the matrix of the system is equal to \( 4/243 \), so there is a total solution \( a_0 = -72\lambda_1 + \frac{25}{2}, a_1 = \frac{363}{2}\lambda_1 - 24, a_2 = -156\lambda_1 + \frac{33}{2}, a_3 = \frac{91}{2}\lambda_1 - 4 \).

### 3.1.2. Case p = 2.

If \( m < 1 \) there is no solution, since the quantity of parameters is less than the amount of conditions (8). In the case of \( m = 2 \) points of measurement are conveniently placed at the nodes of the spline and at one point midway between the nodes of the spline. Then the values \( \frac{i}{m} = 0, \frac{1}{2}, 1 \). Thus result we have five unknowns $\lambda_1, \lambda_2, a_0, a_1, a_2$, to determine which is required to solve a system of linear algebraic equations of the form
\[
\begin{align*}
    a_0 + a_1 + a_2 + \lambda_1 + \lambda_2 &= 1, \\
    \frac{1}{2}a_1 + a_2 + \lambda_1 &= 2, \\
    \frac{1}{4}a_1 + a_2 + \frac{2}{3}\lambda_1 - \lambda_2 &= \frac{11}{3}, \\
    \frac{1}{8}a_1 + a_2 &= -6.
\end{align*}
\]

The determinant of the matrix of the system is equal to \( 3/8 \), so there is a common solution, depending on the parameter $\lambda_2$, \( a_0 = \frac{1}{3} + \frac{\lambda_2}{3}, a_1 = -\frac{8\lambda_2}{3}, a_2 = \frac{19 + \lambda_2}{3}, \lambda_1 = -3 + \lambda_2 \).

Graphs of received coefficients values depending on $\lambda_2$ at $|\lambda_2|<1$ shown in Figure 1.
Thus, the solutions satisfying the conditions of the Theorem 2 do not exist.

In the case $m = 3$ we have 6 unknown values $\lambda_1, \lambda_2, a_0, a_1, a_2, a_3$, to determine which is a general solution, which depends on the parameters $\lambda_1$ and $\lambda_2$:

\[
\begin{align*}
    a_0 &= -4 - \frac{3}{2} \lambda_1 + 2 \lambda_2, \\
    a_1 &= \frac{33}{2} + 6 \lambda_1 - \frac{15}{2} \lambda_2, \\
    a_2 &= -24 - \frac{15}{2} \lambda_1 + 6 \lambda_2, \\
    a_3 &= \frac{25}{2} + 2 \lambda_1 - \frac{3}{2} \lambda_2.
\end{align*}
\]

For example, in the case of $\lambda_1 = \lambda_2 = 0$ is obtained one of the possible local approximation formulas:

\[
\hat{y}_k = \frac{1}{2} \left( -8 y_{k-1} + 33 y_{k-\frac{3}{2}} - 48 y_{k-\frac{5}{2}} + 25 y_k \right).
\]

3.2. The results of the experiments

3.2.1. The case $p = 1$. Using the obtained above stable recursive formulas for spline prediction a series of numerical experiments on the approximation of monomials of degree less than or equal to the degree of the spline 3 was done.

1) It was found that at $\sigma = 0$ these formulas provide a zero error of approximation of functions $f(t) = t^l, \ l = 0, 1, 2, 3$. Thus received plots are practically identical.

2) The numerical experiments were carried out with functions interfered with normally distributed noise with zero mean and variance $\sigma^2, \ \sigma = 1, 5, 10, 15$. It turned out that for $\sigma \leq 9$ properties of accurately reproducing polynomials of sufficiently high degree met, for $\sigma \geq 10$ filters cannot repay so large error.

3) At the last part of the experiments was considered on approximating polynomial of degree $3 + l$, $l > 1$, i.e. greater than the degree of the spline estimates. The resulting error is called monospline and in most cases has a characteristic graph. As known, monosplines are the main error term for the approximation of sufficiently smooth functions. For $l = 1$ the result is shown in Figure 2.
From comparison of the received results with [1] it is possible to draw the following conclusion: increase of degree of the filter favorably affects an error of the received predicting values on condition of smoothness of approximated dependence.

References
[1] Shumilov B M, Esharov E A and Arkabaev N K 2010 Numerical Analysis and Applications 3 186–198
[2] Shumilov B M, Baigulov A N and Abdykalyk kyzy Zh 2014 Vestnik of Tomsk State University of Architecture and Building 1 142-152 (In Russian)
[3] De Boor C 1978 A Practical Guide to Splines (Springer-Verlag: New York)
[4] Shumilov B M 1996 Recurrent approximation by splines Russian Math. (Izvestiya VUZ. Matematika) 40 78–80
[5] Gel’fond A O 1971 Calculus of finite differences (Hindustan Publ. Corp.)