Green’s Function of Anyons in Calogero Model and Quantum Hydrodynamics

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We find that correlation functions at one dimension are crucially affected by the curvature of the bare single particle spectrum. Parabolic curvature leads to two closely related phenomena: the Green’s function exhibits oscillation (as a function of the coordinate), while the polarization operator acquires support in part of the frequency-momentum plane. We calculated the Green’s function using the WKB approximation for collective variables theory. Within this approach, the single particle Green’s function is related to a quantum soliton. The finite support of the polarization operator is due to periodic density waves.

Tomonaga-Luttinger liquid has been a paradigm of non-Fermi liquid systems. The bosonization technique has been a particularly convenient way to study it. Recently it was pointed out that for some one dimensional phenomena the finite curvature of conductance band is essential. Such are photo-voltaic effect, thermo-power low temperature Coulomb drag etc. The violation of particle-hole symmetry is necessary for this and alike effect to exist. Since conventional bosonization relies on the linearization of the single-particle spectrum such phenomena can hardly be approached.

In this work we address the issue of spectrum curvature for correlation functions of Calogero-Sutherland (CS) model. This model appears in various branches of physics, such as spin chains, FQHE, disordered metals, Dyson’s symmetry classes of Random Matrix Theory (RMT) . The exact eigenstates and eigenfunctions of CS model have been found, first by Forrester for integer and later on by H
c

The special values of it \( \lambda = 1, 2, 4 \) corresponds to the Dyson’s symmetry classes of Random Matrix Theory (RMT). The exact eigenstates and eigenfunctions of CS model have been found, first by Forrester for integer and later on by H
c

The Hamiltonian of CS is

\[
H = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \partial_x^2 + \left( \frac{\pi \lambda}{L} \right)^2 \sum_{i>j} \frac{\lambda(\lambda-1)}{\sin^2\left( \frac{\pi}{L}(x_i - x_j) \right)} \]

(1)

The Hamiltonian of CS is

The Hamiltonian can be rewritten as

\[
H = \left( \frac{2\pi}{L} \right)^2 \left[ \frac{\hbar^2}{2m} \sum_{j=1}^{N} (z_j \partial_{z_j})^2 + \sum_{i \neq j} \lambda(\lambda-1) |z_i - z_j|^2 \right].
\]

(2)

Ground state of the Hamiltonian is

\[
\Psi_0 = \left( \prod_{i=1}^{N} z_i \right)^{-\lambda(N-1)/2} |\Delta|^{\lambda-1} \Delta, \quad \Delta = \prod_{i<j}^N (z_i - z_j).
\]

The excited states \( \Psi_\kappa = \Psi_0 \Phi_\kappa^B \), where \( \Phi_\kappa^B \) is a symmetric function of the particle coordinates parameterized by partition \( \kappa \). The Hamiltonian

\[
H = \psi_0^{-1} H \psi_0 \text{ acting in the space of symmetric (bosonic) function. After some algebra one arrives to}
\]

\[
H = \sum_{i=1}^{N} D_i^2 + \lambda \sum_{i<j} \frac{z_i + z_j}{|z_i - z_j|} (D_i - D_j)
\]

(3)

where \( D_i = z_i \partial_x \). To prepare for second quantization one defines so called collective variables

\[
p(\theta) = \sum_{i=1}^{N} \delta(\theta - \theta_i), \quad p_k = \int_0^{2\pi} d\theta \rho e^{ik\theta} p(\theta).
\]

In terms of collective variables the Hamiltonian has a form

\[
H = \frac{1}{2} \sum_{m,n=-N}^{N} mnp_{n+m} \frac{\partial^2}{\partial p_m \partial p_n} + \frac{1}{2} \sum_{m=0}^{N} (n + m) \left[ p_np_m \frac{\partial}{\partial p_{n+m}} + p_{-n-p_{-m}} \frac{\partial}{\partial p_{-n-m}} \right]
\]

(4)

So far our calculation have been exact. Now we pass to the hydrodynamic limit. In the limit of \( N \to \infty \) the Hamiltonian eq. can be written as

\[
H = \int dx \left[ \frac{1}{2} \rho \frac{\rho}{\rho} + U[\rho]\right]
\]

(5)

Here \( x \) is a coordinate along the circle \( x = \frac{\rho}{\rho} \), the linear density \( \rho(x) = \frac{\rho}{\rho} \rho(\theta) \) etc. The potential energy of quantum liquid is given by

\[
U = \frac{\rho x^2}{6} - \frac{\lambda(\lambda-1)}{2} \rho^3 + \frac{(\lambda-1)^2}{8} \rho^3 \rho_x^2.
\]

The Hilbert transform is defined as \( \rho^H(x) = \frac{1}{\pi} \int \rho(x') \frac{dx'}{x-x'} \). The modes of velocity operator are defined as \( v_n = \frac{1}{n} \rho \langle \rho \rangle \). It is easy to see that with these definitions particle velocity and density satisfy a
standard commutation relations one expect for any hydrodynamic theory \[ [v(x), \rho(y)] = -i \delta'(x - y). \] To reproduce the conventional bosonization theory one expands the Hamiltonian up to second order in fluctuations (due to conservation laws the velocity \( v \) and density operators \( \delta \bar{\rho} \) are of the same order) and spatial gradient (keeping only the terms that have no gradient). By doing so we see that on the conventional Luttinger model interaction enters only through sound velocity of bosonic modes.

Now we use the Hamiltonian to study effects of the spectrum curvature. We start with single particle Green’s function \( G(x, t) = -i(\Psi_A(x, t)\Psi_A(0, 0)) \), where \( \Psi \) is an anyonic (for \( \lambda \neq 1 \) field). Interesting on its own (it’s spectrum function can be measured in angle resolved tunneling experiments) Green’s function will also be helpful for study of more involved correlators. As we will see shortly, the Green’s function for CS model and for free fermions turn out somewhat similar. We consider non-interacting case first. By the Wick theorem the Green’s function can be easily evaluated. For parabolic dispersion the time ordered Green’s function \( (t > 0) \) is given by

\[
G = -\sqrt{\frac{\pi m}{2t}} e^{-\pi/4} \left[ e^{i\pi/4} \sqrt{\frac{m}{2t}} (x+vt)^2 \right] \text{Erfc}\left(\frac{m}{\sqrt{2t}} (x+vt)^2\right) + e^{-i\pi/4} \sqrt{\frac{m}{2t}} (x-vt)^2 \text{Erfc}\left(\frac{m}{\sqrt{2t}} (x-vt)^2\right)
\]

The Green’s function is a sum of left and right moving terms, multiplied by fast oscillating term. The smooth part of the Green’s function for right moving particles is plotted on Fig(1). The left tail of Green’s function is captured by linear bosonization. There is no right tail, because we deal with strictly parabolic spectrum. The oscillations are universal. The right tail will appear as soon as negative spectrum curvature is take into account, to match a linear bosonization. A transition between oscillating a power law region is model dependent.

One can get a good idea about function using a saddle point approximation for the integral over momentum.

\[
G = -i \sqrt{\frac{\pi m}{t}} e^{i\pi/4} \sqrt{\frac{m}{2t}} \theta(x-vt) + (L \leftrightarrow R)
\]

One sees that saddle point gives a dominant contribution only if the particle moves faster than the Fermi velocity. In this region the result is oscillatory function. For the particles moving slower than Fermi velocity the saddle point lies out of the region of integration and the integral is determined by its low limit. In that case we restore a standard power decay of the Green’s function. The result for particles moving vaster than Fermi velocity particle does not agree with linear bosonization prediction

\[
G(x, t) = -\frac{1}{2} \left[ \frac{e^{i\pi/4}}{\sqrt{\frac{m}{2t}} (x+vt)^2} + (R \leftrightarrow L) \right].
\]

The reason of this disagreement is simple. We have evaluated the Green’s function of free fermion with a parabolic spectrum. In metals electrons are subject to periodic potential. They form Bloch states and have periodic spectrum that has a negative curvature close to the boundary of Brillouin zone. This limits the maximally accessible velocity of the classical trajectory. The standard bosonization considers only the situation in which there is no classical trajectory available.

Indeed, within a tight-binding model the single particle spectrum is given by \( \varepsilon(p) = W (1 - \cos \frac{p a}{2W}) \), where \( a \) is a lattice constant and \( W \) is a band width. We see that integral has a stationary point at \( \frac{2\pi}{aW} \). The right hand side of this equation is a periodic function of momentum and it is smaller than one. So the saddle point contribution disappears if the velocity on the classical trajectory exceeds the one available by band structure \(( haW \ll (x-vFt)/t )\). This is a linear bosonization limit, intensively studied in the literature so far. We will consider the situation when the classical trajectory dominates the result, but the distance to the light cone is still large \((v_F \ll (x-vFt)/t)\). In this case the semiclassical approximation is justified. Close to the light cone \(( (x-vFt)/t \ll v_F) \) WKB approximation fails and other methods are needed. In all cases the absolute values of times and coordinates, measured in the Fermi energy and wave length, are large.

The rest of this paper is devoted to study of asymptotic properties of Calogero model in the semiclassical regime. We consider the Green’s function first. Anyon operator is represented via boson operators as

\[
\Psi_A(x, t) \simeq \exp \left( i \int_{-\infty}^{t} dx (v(x, t) + \pi \lambda \rho(x, t)) \right)
\]

This definition is consistent with the commutation relations

\[
\Psi_A(x)\Psi_A(x') = e^{-i\pi \lambda \text{sgn}(x'-x)} \Psi_A(x')\Psi_A(x).
\]

The Green’s function is given by a functional integral

\[
G_A(x', t') = -\frac{i}{Z} \int \mathcal{D}\rho \mathcal{D}v e^{iS} \Psi_A(x', t') \Psi_A(0, 0).
\]

FIG. 1: single particle Green’s function (of right moving electrons) as a function of \( x \) for the finite band-width, \( v = 1, m = 1, t = 1 \)
The hydrodynamic action corresponding to the Hamiltonian eq. [5] is given by

\[ S = \int dx dt \left[ -v \partial_x^{-1} \partial_t \rho - \frac{1}{2} \rho v^2 - U[\rho] \right] \tag{10} \]

To calculate the Green’s function we use steepest decent method, known also as WKB. The action (10) has massive direction and low energy directions in the configuration space (a constant of shift \( x \rightarrow x + a, t \rightarrow t + b \) does not change the value of the action on any solution). The first may be integrated out by saddle point method, the latter need to be taken into account exactly. The saddle point equation is

\[ \rho_t + \partial_x (\rho v) = -\delta(x - x') \delta(t - t') + \delta(x) \delta(t) \tag{11} \]

\[ v_t + vv_x + \partial_x \left( \frac{\delta U}{\delta \rho} \right) = -\pi \lambda \delta(x - x') \delta(t - t') + \pi \lambda \delta(x) \delta(t) \tag{12} \]

As was suggested earlier [2, 17] the elementary particle of fermionic theory corresponds to the quantum soliton of the boson theory [19]. In the limit of large \( \lambda \) we look for solution \( 0 < t < t' \) in a form

\[ \rho(x, t) = 1 + \rho_s(x - vt - X(t)), \quad v(x, t) = v_s(x - vt - X(t)), \tag{13} \]

where \( X(t) \) is a global variable. It corresponds to the motion of the soliton as a whole. This is a zero mode of the system. The soliton’s density and velocity are given by [2]

\[ \rho_s(x) = \frac{B}{\pi} \frac{1}{x^2 + B^2}, \quad v_s(x) = \frac{Bv}{\pi} \frac{1}{x^2 + B^2 + \frac{B^2}{\pi}}. \tag{14} \]

Here \( B = \lambda \pi \xi^2 / (v^2 - \pi^2 \lambda^2), \quad v = x'/t' \). The soliton exists, provided \( B > 0 \), i.e. its velocity exceeds the sound velocity \( s = \pi \lambda \rho_0 \). It is very similar to the saddle point contribution for the ordinary integral of free fermions where saddle point contribution appeared only for supersonic trajectories. As velocity increases \( v \gg s \) the size of soliton \( B \) goes to zero and the particle is more localized. By integrating over infinitesimal region around \( x = 0, x = x' \) over \( x \) we see that the solution (14) indeed has a right singularity. The total number of electrons carried by the soliton \( \int \rho(x) dx = 1 \) is quantized to one. The velocity of soliton is not quantized. Then we exponential accuracy we find that Green’s function of anyon excitation in CS model is given by

\[ G_A(x', t') \simeq e^{iS_{cl}} \int_{X(0)=0}^{X(t')=0} DX e^{\frac{1}{2} \int_{t}^{t'} dt' \partial_x^2 + \frac{1}{2} \int_{t}^{t'} dt' \partial_x}. \tag{15} \]

Here the value of the action on classical trajectory \( S_{cl} = t' (v^2/2 - sv) = \frac{v^2}{2v} - sx' \). Performing the integration over zero mode find

\[ G_A(x, t) \simeq \sqrt{\frac{m}{t}} e^{ip_F \lambda \pi i (x-st)^2 m/2t} \theta(x-st) + (R \mapsto L). \tag{16} \]

where \( p_F = \pi \rho_0 \) is Fermi momentum. Equation (10) is the central result of this work. It shows that far from light cone the anyon’s propagator oscillates as a function of \( x \) with the period \( \sqrt{t} \), c.f. the Green’s function of free electrons eq. (7).

The soliton that determines the Green’s function is also responsible for the finite support of polarization operator [21] \( \Pi(q, t) = -i (\partial \rho(q, t) \partial (-q, 0)) \) This, as it is was pointed out in Ref. [3], is necessary for various physical phenomena. Several attempts had been in the literature to account for it [21, 22]. Here we show how broadening of the support can be understood within quantum hydrodynamics.

The polarization operator is determined by neutral particle hole excitations. In lineal bosonization they are acoustic phonons. In its nonlinear version they are periodic density [2, 23]. The saddle point equation on periodic waves reduces to

\[ \frac{v^2}{X^2} \left( \frac{1}{\rho^2} - 1 \right) + \pi^2 \rho^2 + 2 \pi \rho H - \frac{1}{2} \rho_{xx} + \frac{1}{4} \rho^2 = C, \tag{17} \]

where \( C \) is an arbitrary constant. Solving this equation one finds

\[ \rho_{B,i}(x) = 1 - \frac{1}{l} + \frac{1}{l} \sinh \left( \frac{2B}{\rho} \right) \frac{1}{\cosh \left( \frac{2B}{\rho} \right) - \cos \left( \frac{2\pi}{\rho} \right) \frac{2B}{\rho}}. \tag{18} \]

Here velocity of the wave

\[ v = s \left( 1 - \frac{1}{l} \right) \sqrt{1 + \frac{4}{l} \left( 1 - \frac{1}{l} \right) n_B \left( \frac{2B}{l} \right)}. \tag{19} \]

where \( n_B \) is Bose distribution function. In a momentum representation

\[ \rho_{B,i}(q) = e^{-B|q|/\pi} \sum_n \delta(ql - 2\pi n). \tag{20} \]

From eq. (19,20) we find that in a long length limit

\[ q = \frac{2\pi n}{l}, \quad v \simeq s + \frac{2\pi \lambda}{l} \left[ n_B \left( \frac{1}{2} \right) - \frac{1}{2} \right]. \tag{21} \]

If the width of the picks is bigger than the distance between them (\( B \gg l \) one recovers a dispersion of anharmonic acoustical phonons \( v(q) = s - \lambda q^2/2m \). This excitations determine the lower edge of polarization operator support. In standard bosonization only those are taken into account [24].

For the waves with period longer than \( l \gg B \) there is only singe pick (soliton) in that length. One can assume that since other pick are far apart one can study its motion regardless of others. It contributes to the energy \( E = \frac{m v^2}{2} \). By recalling [25] the soliton dispersion \( (\epsilon(q) = sq + q^4/2m, \quad q > 0) \) we obtain an upper support of polarization operator.
A bosonic and a fermionic descriptions are dual \[14\]. In fermionic language excitations are characterized by partition \(\kappa\). In bosonic notations a single period neutral excitations are described by the parameters \(B\) and \(l\), or alternatively by \(n\) and \(n_B\). The partition \(\kappa = \{q\}\) corresponds to taking the topmost electrons and moving it \(q\) steps up and corresponds to the diluted gas of solitons. The conjugated partition \(\kappa = 1^q\) that describes moving the top \(q\) electrons by one step up corresponds to the anharmonic acoustic phonons.

To summarize: the saddle point trajectory that corresponds to the soliton solution of non linear hydrodynamic equation is dominant for anyon Green’s function. By evaluation the value of the action on this trajectory we have found an explicit formula for the Green’s function of anyon.

Due to the infinite number of periodic density waves a polarization operator acquires a finite support in frequency momentum plane. Of which the lower edge of excitations is given by dispersive acoustic phonons (such excitation can be accounted by linearizing an action but accounting phonon dispersion even for generic interaction). The upper edge is determined by a superposition of non interacting solitons. As the model became non integrable and soliton is destroyed the upper part of supports is expected to smear.

Note added: As this work was nearly completed such investigation had been performed in Ref. \[20\]. It was indeed found that for generic interaction the upper boundary of the support becomes smeared, while the lower boundary is unchanged.

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[20] for free electrons the polarization operator is given by
\[ \Pi(\omega, q) = \frac{1}{\pi q} \ln \left( \frac{(\omega + q + \frac{2q}{\omega})(\omega - q - \frac{2q}{\omega})}{(\omega - q + \frac{2q}{\omega})(\omega + q - \frac{2q}{\omega})} \right) \]
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\[ \Pi_{\text{lin}}(\omega, q) = \frac{2\omega(q)\Pi(q)}{(\omega - \omega(q) + i\delta\Im\omega)(\omega + \omega(q) + i\delta\Im\omega)}, \]
(23)

It has a line \(\omega = \omega(q)\) as a support on \(\omega, q\) plane
[25] There is a difference between the soliton that was discussed in the context of Green’s function and corresponds to the particle added to the system while here it represent the particle excited from the Fermi surface. We corrected the energy of the former by hand since the particle conservation is beyond the hydrodynamics picture.
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