Research on a Novel Flexure Hinge

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Abstract. A novel single axis flexure hinge named hyperbolic flexure hinge is introduced as new designs in two-dimensional monolithic mechanisms. Based on the Castigliano’s second theorem, closed-form equations are formulated for compliances to characterize both the active rotation and all other parasitic motions. Precision of rotation and the stress levels are also evaluated in terms of compliances. A unitary compliance-based analysis is performed by means of two-dimensional geometric parameters by which the performance of flexure hinge can be evaluated in terms of their desired flexibility, capacity of preserving the centre of rotation position and stress levels. Checked against finite element analysis, the model predictions are within 8% error margins. So a theoretical foundation for flexure hinges design is provided.

1. Introduction

The flexure hinge is an alternative solution to classical rotation joints in designs that require limited rotation capability of adjoining members, one-piece (monolithic) manufacturing, reduced weight, zero backlash and friction, no lubrication, smooth motion. The flexure hinges are incorporated in a large number of applications, including translation micro-positioning stages, piezoelectric actuators and motors, high-accuracy alignment, displacement amplifiers, robotic micro-displacement mechanisms.

The circular flexure hinges were introduced by Paros and Weisbord [1] who formulated the design equations, both exact and simplified, for calculating the compliances of single-axis and two-axes circular cutout constant cross-section flexure hinges; Smith et al. [2] introduced the elliptic cross-section flexure hinge. They expressed closed-form compliance equations for an elliptic flexure hinge by extrapolating the equations of Paros and Weisbord for circular flexure hinges. The elliptical flexure hinge was shown to span a field limited by the circular flexure hinge and the simple beam in terms of its compliance. In a recent monograph, Smith [3] presented the analytic models of leaf-type springs such as notch or two-axis flexure hinges. Lobontiu et al. [4] recently developed an analytical model of corner-filleted flexure hinges that can be implemented into planar amplification mechanisms. A corner-filleted flexure hinge ranges within a design space that is confined by the right circular flexure and the simple beam, in terms of its compliance. Xu and King [5] performed static finite element analysis and compared the performance of circular, corner-filleted and elliptic flexure hinges in terms of motion, stress levels and stiffness. Ryu and Gweon [6] modeled the motion errors that are induced by machining imperfections into a flexure hinge mechanism.

The present work introduces a new configuration: the hyperbolic flexure hinges. Based on the Castigliano’s second theorem, closed-form compliance equations are developed to characterize the flexibility, precision of rotation and the stress levels. The compliance factors equations are verified by
means of the finite element analysis. Results are in good agreement with Closed-form compliance equations with relative errors less than 8 percent.

2. Compliance equations
The formulation that follows is based on the following simplifying assumptions: the flexure hinges consist of two symmetric cutouts, as illustrated in Figure 1. The flexure hinges are modeled and analyzed as small-displacement fixed-free Euler–Bernoulli beams subjected to bending produced by forces and moments; axial loading is also considered (see Figure 2) while shearing and torsional effects are not taken into account.

![Figure 1. Main parameters of flexure hinge.](image1)

![Figure 2. Schematic of flexure hinge with loading.](image2)

A hyperbolic flexure hinge is defined by the geometric parameters indicated in Figure 1. The sub-problems will be addressed in the following, namely the in-plane compliant behavior of a flexure hinge. The in-plane compliance equations will be formulated with respect to translations along the y- and x-axes, and rotation about the z-axis. The displacement–loading relationship at the free end 1 (see Figure 2) is of the form:

\[
\begin{bmatrix}
\theta_{z1} \\
y_1 \\
x_1
\end{bmatrix}
= \begin{bmatrix}
C_{\theta_z,M_x} & C_{\theta_z,F_y} & 0 \\
C_{y,M_x} & C_{y,F_y} & 0 \\
0 & 0 & C_{x,F_x}
\end{bmatrix}
\begin{bmatrix}
M_{z1} \\
F_{y1} \\
F_{x1}
\end{bmatrix}
\tag{1}
\]

With \(C_{\theta_z,F_y}=C_{y,M_x}\), according to the reciprocity principle. The Castigliano’s second theorem is applied to evaluate the displacement vector in the left-hand side of Equation (1):

\[
\theta_{z1} = \frac{\partial U_e}{\partial M_{z1}}, y_1 = \frac{\partial U_e}{\partial F_{y1}}, x_1 = \frac{\partial U_e}{\partial F_{x1}}
\tag{2}
\]

Where

\[
U_e = \frac{1}{2} \left( \int \frac{F_x^2}{EA(x)} \, dx + \int \frac{M_z^2}{EI_z(x)} \, dx \right), F_x = F_{x1}; M_z = M_{z1} + F_{y1}x; A(x) = bt(x); I(x) = \frac{bt(x)^3}{12};
\]

The formulation that follows is based on two non-dimensional parameters \(\beta\) and \(\gamma\), that are defined in terms of the parameters shown in Figure 1, namely:

\[
\beta = \frac{t}{2c}; \quad \gamma = \frac{t}{l}
\]

The thickness of a hyperbolic flexure hinge is:

\[
t(x) = \left( t^2 + 4c(c + t)(1 - 2 \frac{x^2}{l^2}) \right)^{\frac{1}{2}}
\tag{3}
\]

The closed-form in-plane compliance expressions for a hyperbola flexure hinge are

\[
C_{\theta_z,M_z} = \frac{12}{Ebt^2} \frac{\beta}{\gamma(1 + \beta)}
\tag{4}
\]
\[
C_{\theta_x, F_y} = -\frac{t}{2\gamma} C_{\theta_x, M_Z} 
\]
\[
C_{y,F_y} = \frac{3}{2Eb(1+\beta)(1+2\beta)^2} \beta \times \left[ \frac{\beta + \sqrt{1+2\beta}}{1+\beta - \sqrt{1+2\beta}} \right] 
\]
\[
C_{x,F_x} = \frac{1}{E_b \gamma \sqrt{1+2\beta}} \log \left( 1 + \sqrt{1+2\beta} \right) \frac{\beta}{\beta} 
\]

3. Precision of rotation

The relative rotation of two mechanical members that are connected by a conventional rotation joint is produced along an axis that passes through the geometric center of the joint, which is fixed, provided one member is also fixed. In the case of a symmetric flexure hinge, the center of rotation (the geometric symmetry center of the flexure) is no longer fixed since the flexure is subjected to deformation.

The displacement of the rotation center of a flexure hinge (point 2 in Figure 2) is evaluated by applying three 'dummy' loads, \( F_{x2}, F_{x2}, \) and \( F_z2 \), in addition to the actual load vector made up of \( F_{x1}, F_{y1}, F_{z1}, M_{y1}, \) and \( M_{z1} \). The Castigliano’s second theorem is again applied to find the displacements of the rotation center.

The matrix-form equation that relates deformations to load at the flexure mid-point is similar to Equation (1), namely:

\[
\begin{align*}
\begin{bmatrix}
  x_2 \\
y_2 \\
z_2
\end{bmatrix} &= \begin{bmatrix}
  C'_{x,F_x} & 0 & 0 & 0 & 0 \\
  0 & C'_{y,F_y} & 0 & 0 & C'_{y,M_Z} \\
  0 & 0 & C'_{z,F_Z} & C'_{z,M_Y} & 0
\end{bmatrix} \begin{bmatrix}
  F_{x1} \\
  F_{y1} \\
  F_{z1} \\
  M_{y1} \\
  M_{z1}
\end{bmatrix}
\end{align*}
\]

The compliances of Equation (8) define the offset of the rotation center and are calculated similarly to the ‘full’ compliance factors that were previously derived. For hyperbolic flexure hinges, the compliance closed-form equations are:

\[
C'_{x,F_x} = \frac{1}{2Eb \sqrt{1+2\beta}} \beta \log \left( 1 + \beta \right) 
\]

\[
C'_{y,F_y} = \frac{3}{2Eb (1+2\beta)^3} \beta \times \log \left( 1 + \beta \right) 
\]

\[
C'_{y,M_Z} = -\frac{3}{Eb \gamma (1+2\beta)^2} \beta \frac{\beta^2}{(1+2\beta)^2} 
\]

\[
C'_{z,F_Z} = \frac{3t^2}{4Eb^3 (1+2\beta)^3} \beta \times \log \left( 1 + \beta + \frac{\beta}{\sqrt{1+2\beta}} \right) 
\]

\[
C'_{z,M_Y} = \frac{3t}{Eb^2 (1+2\beta)^2} \beta 
\]

4. Stress considerations

The stresses can be expressed as, only the in-plane bending and axial effects are considered, Lobontiu et al. [3] mentioned the stress concentration factors in bending \( k_b \) and axial \( k_a \) are specified.
\[ \sigma_{\text{max}} = 6 \frac{k_b}{b t^2} (M_{z1} + l F_{y1}) + \frac{k_a}{b t} F_{x1} \]  

(14)

In order to express the load of Equation (14) in terms of displacement, Equation (1) can be written in terms of stiffness as

\[
\begin{bmatrix}
M_{z1} \\
F_{y1} \\
F_{x1}
\end{bmatrix} =
\begin{bmatrix}
K_{\theta_z,M_z} & K_{\theta_z,F_y} & 0 \\
K_{y,M_z} & K_{y,F_y} & 0 \\
0 & 0 & K_{x,F_x}
\end{bmatrix}
\begin{bmatrix}
\theta_{z1} \\
y_1 \\
x_1
\end{bmatrix}
\]  

(15)

Where the stiffness matrix can be determined by inverting the compliance matrix in Equation (1). The stiffness factors are:

\[
\begin{align*}
K_{\theta_z,M_z} &= \frac{C_{y,F_y}}{C_{\theta_z,M_z} C_{y,F_y} - C_y^2 M_z} \\
K_{y,M_z} &= \frac{C_y M_z}{C_{\theta_z,M_z} C_{y,F_y} - C_y^2 M_z} \\
K_{y,F_y} &= \frac{C_{\theta_z,M_z}}{C_{\theta_z,M_z} C_{y,F_y} - C_y^2 M_z} \\
K_{x,F_x} &= \frac{1}{C_{x,F_x}}
\end{align*}
\]  

(16)

Substituting Equation (15) and (16) into Equation (14) results in

\[
\sigma_{\text{max}} = 6 \frac{k_b}{b t^2} \left[ (K_{\theta_z,M_z} + l K_{y,M_z}) \theta_{z1} + K_{y,F_y} + l K_{y,F_y} \right] y_1 + \frac{k_a}{b t} K_{x,F_x} x_1
\]  

(17)

Equation (16) and (17) can be utilized in evaluating the maximum normal stress at one end of the hyperbolic flexure hinge when the stress concentration factors are specified.

5. Verification of the compliance closed-form equations

The compliance closed-form expressions were checked by the finite element method. The ANSYS finite element software was utilized to calculate the compliance factors for several hyperbolic flexure designs under static loading. The procedure consisted of applying a unit load at one end of the flexure, Figure 3, and reading the displacements at either point 1 or 2 (Figure 2), which, in this case, were identical to the corresponding compliance factors, as expressed in Equation (1), respectively. The constant-valued material (an aluminum alloy was considered) and geometric data were:

\[
E = 70 \times 10^9 \text{N/m}^2; \quad \mu = 0.25; \quad b = 0.01 \text{m}; \quad F_{x1} = F_{y1} = 1 \text{N}; \quad M_{z1} = M_{z1} = 1 \text{Nm};
\]

Tables 1 show that the relative errors between the analytic model predictions and the finite element simulations were less than 8%.

6. Analysis of the results

By inspecting the compliance equations of hyperbolic flexure hinges, the following conclusions can be drawn:

All compliance factors of the hyperbolic flexure hinges vary inversely proportional with the Young’s modulus \( E \) and the flexure depth \( b \); There are two compliance factors, namely \( C_{x,F_x}, C_{y,F_y} \), that do not explicitly depend on the thickness parameter \( t \) for either of the flexures; they only depend indirectly on \( t \), through the non-dimensional parameters \( \bar{\mu} \) and \( \bar{\theta} \).

All the compliance factors, for the hyperbolic flexures, depend non-linearly on the non-dimensional parameters \( \beta \) and \( \gamma \); a compliance factor increases when \( \beta \) increases and \( \gamma \) decreases (Figure 4 illustrates this remark for a particular case, but this is valid for all other cases).

The general conclusions that were formulated in the previous sub-section are also valid for the compliance factors that describe the precision of rotation.
Figure 3. Finite element model.

Figure 4. Plot of $C_{\theta_z,M_z}$ compliance with t=0.5mm.

### Tables 1. Finite element and analytical results for the compliance factors.

| $C_{\theta_z,M_z}$ (%N$^2$m$^{-1}$) | t(m) | l(m) | c(m) | Analytic | FEM | Error(%) |
|-----------------------------------|------|------|------|----------|-----|----------|
| 0.0003                            | 0.002| 0.0008| 0.158| 0.165    |     | 4.242    |
| 0.0005                            | 0.002| 0.001 | 0.043| 0.043    |     | 0        |
| 0.0008                            | 0.0025| 0.002 | 0.011| 0.012    |     | 8.333    |
| $C_{\theta_z,F_y}$ (%N$^2$m$^{-1}$) | 0.0003| 0.002 | 0.0008| 0.158 | 0.165 | 4.242    |
| 0.0005                            | 0.002 | 0.001 | 0.432| 0.430    |     | 0.460    |
| 0.0008                            | 0.0025| 0.002 | 0.137| 0.141    |     | 2.837    |
| $C_{y,F_y}$ (%N$^2$m$^{-1}$)      | 0.0003| 0.002 | 0.0008 | 0.145 | 0.141 | 2.759    |
| 0.0005                            | 0.002 | 0.001 | 0.041| 0.042    |     | 2.380    |
| 0.0008                            | 0.0025| 0.002 | 0.016| 0.017    |     | 5.882    |
| $C_{x,F_y}$ (%N$^2$m$^{-1}$)      | 0.0003| 0.002 | 0.0008| 6.074 | 6.085 | 0.181    |
| 0.0005                            | 0.002 | 0.001 | 4.210| 4.332    |     | 2.816    |
| 0.0008                            | 0.0025| 0.002 | 2.940| 3.011    |     | 2.358    |

### 7. Conclusion

The paper introduces the hyperbolic flexure hinges as new designs in two-dimensional monolithic mechanisms. Closed-form compliance equations are developed to characterize the flexibility, precision of rotation and stress levels. The compliance factors equations are verified by means of the finite element analysis. Results are in good agreement with Closed-form compliance equations with relative errors less than 8 percent. So the paper gives some theoretic foundation for flexure hinges design.

### References

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