STABILITY OF LODAY CONSTRUCTIONS

AYELET LINDENSTRAUSS AND BIRGIT RICHTER

Abstract. We study the question for which commutative ring spectra $A$ the tensor of a simplicial set $X$ with $A$, $X \otimes A$, is a stable invariant in the sense that it depends only on the homotopy type of $\Sigma X$. We prove several structural properties about different notions of stability, corresponding to different levels of invariance required of $X \otimes A$. We establish stability in important cases, such as complex and real periodic topological $K$-theory, $KU$ and $KO$.

1. Introduction

For any simplicial set $X$ and any commutative ring spectrum $A$ one can form the tensor of $X$ with $A$, $X \otimes A$. An important special case of this construction is topological Hochschild homology of $A$, $\text{THH}(A)$, which is $S^1 \otimes A$. In the following we will often work with commutative $R$-algebras for some commutative ring spectrum $R$. We will sometimes take coefficients in a commutative $A$-algebra $C$, which requires working with pointed simplicial sets $X$; we denote the corresponding object (whose definition we recall in Section 1.1 below) by $L^R_X(A; C)$. When $C = A$, $L^R_X(A; A)$ is just $X \otimes A$, and $L^R_{S^1}(A; C)$ is $\text{THH}^R(A; C)$.

As topological Hochschild homology is the target of a trace map from algebraic K-theory

\begin{equation}
K(A) \to \text{THH}(A)
\end{equation}

it has been calculated in many cases. Higher order topological Hochschild homology, which is $L^R_{S^1}(A; C)$, has also been determined in many important classes of examples, see for instance [3,5,13,22,24]. In [3] we develop several tools for calculating $L^R_{S^1}(A; C)$. However, if we want to determine the homotopy type of $L^R_{S^1}(A; C)$ and $X$ doesn’t happen to be a suspension, then the range of methods is much sparser.

Rognes’ redshift conjecture [1] predicts that applying algebraic K-theory raises chromatic level by $1$.

This gave rise to the question whether $L^R_X(A; C)$ is a stable invariant, i.e., whether the homotopy type of $L^R_X(A; C)$ only depends on the homotopy type of $\Sigma X$. There are positive results: $L^R_X(k(HA))$ is a stable invariant if $k$ is a field and $A$ is a commutative Hopf algebra over $k$ [2, Theorem 1.3] or if $k$ is an arbitrary commutative ring and $A$ is a smooth $k$-algebra [2, Example 2.6]. But Dundas and Tenti also show [9, §3.8] that $L^R_X(H\mathbb{Q}[t]/t^2)$ is not a stable invariant. They show that $L^R_{S^1 \times S^1 \times S^1}(H\mathbb{Q}[t]/t^2)$ and $L^R_{S^1 \times S^1}(H\mathbb{Q}[t]/t^2)$ differ and that reducing the coefficients from $H\mathbb{Q}[t]/t^2$ to $H\mathbb{Q}$ doesn’t eliminate

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this discrepancy. This also implies that $L^R_X(HQ[t]/t^2)$ and $L^R_X(HQ[t]/t^2; HQ)$ are not stable invariants because $S^Q \simeq HQ$.

Our aim is to investigate the question of stability in a systematic manner. We start by defining several different notions of stability. Instead of asking for equivalent homotopy types of $L^R_X(A; C)$ and $L^R_Y(A; C)$ if $\Sigma X \simeq \Sigma Y$ we are asking when we actually get an equivalence $L^R_X(A; C) \simeq L^R_Y(A; C)$ of augmented commutative $C$-algebras. There are intermediate notions that ask for less structure to be preserved, for instance, that the equivalence $L^R_X(A; C) \simeq L^R_Y(A; C)$ is one of commutative $R$-algebras or of $C$- or $R$-modules.

We establish that stability is preserved by several constructions such as base-change and products but we also show which procedures do not preserve stability. For instance stability is not a transitive property: if $R \to A$ and $A \to B$ satisfy stability then this does not imply that $R \to B$ has this property.

A central purpose of this paper is to establish new cases where stability holds. For instance for $\text{Thom spectra}$ $R$ that can be factored as the canonical inclusion of $R$ into a free commutative $R$-algebra spectrum followed by a map that satisfies étale descent, these maps model the local behaviour of smooth maps in the context of algebra, compare [15] Proposition E.2 (d)]. We show that really smooth maps satisfy stability. Other examples where stability holds are Thom spectra as well as $S \to KU$ and other spectra of the form $S \to Rh = (\Sigma^\infty_{+}W_h)[x^{-1}]$ considered in [5]. Using Galois descent we also obtain stability for $S \to KO$.

For calculations like that of torus homology, one often doesn’t really need stability, but the property of the suspension to decompose products is the crucial feature that one wants to have on the level of $L^R_{-}(A; C)$. Therefore we say that $R \to A \to C$ decomposes products if

$$L^R_{X \times Y}(A; C) \simeq L^R_{X \times Y \times X \times Y}(A; C)$$

for all pointed simplicial sets $X$ and $Y$. We use Greenlees’ spectral sequence [12] Lemma 3.1] in the case $C = H\kappa$ for $\kappa$ a field to show that this decomposition property is preserved under forming suitable retracts.

In Section 7 we close with some observations on stability in characteristic zero, using that rationally the suspension of pointed simply connected simplicial sets splits into a pointed sum of rational spheres and using [2] Proposition 4.2] where Berest, Ramadoss and Yeung describe the behaviour of representation homology and higher order Hochschild homology under rational equivalences.

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1.1. Definition of $L^R_X(A; C)$. We denote the category of simplicial sets by $s\text{Sets}$ and the one of pointed simplicial sets by $s\text{Sets}_*$. Let $X$ be a finite pointed simplicial set and let $R \to A \to C$ be a sequence of maps of commutative ring spectra. We assume that $R$ is a cofibrant commutative $S$-algebra and that $A$ and $C$ are cofibrant commutative $R$-algebras. The cofibrancy assumptions on $R$, $A$ and $C$ will ensure that the homotopy type of $L^R_X(A; C)$ is well-defined:

The *Loday construction with respect to $X$ of $A$ over $R$ with coefficients in C* is the simplicial commutative augmented $C$-algebra spectrum $L^R_X(A; C)$ whose $p$-simplices are

$$C \wedge \bigwedge_{x \in X \setminus \ast} A$$

where the smash products are taken over $R$. Here, $\ast$ denotes the basepoint of $X$ and we place a copy of $C$ at the basepoint. As the smash product over $R$ is the coproduct in the category of commutative $R$-algebra spectra, the simplicial structure is straightforward: Face maps $d_i$ on $X$ induce multiplication in $A$ or the $A$-action on $C$ if the basepoint is involved. The degeneracy maps $s_i$ on $X$ cause the insertion of the unit map $\eta_A: R \to A$ over all $n$-simplices which are not hit by $s_i: X_{n-1} \to X_n$. As defined, $L^R_X(A; C)$ is a simplicial commutative augmented $C$-algebra spectrum. We use the same symbol $L^R_X(A; C)$ for its geometric realization. For $C = A$ we abbreviate $L^R_X(A; A)$ by $L^R_X(A)$. 
For \( X = S^n \) we write \( \text{THH}^{[n]}(A; C) \) for \( \mathcal{L}^B_{X^n}(A; C) \) and if \( R = S \), then we omit it from the notation, so \( \text{THH}^{[n]}(A; C) = \mathcal{L}^S_{X^n}(A; C) \). For \( n = 1 \) this is the classical case of topological Hochschild homology of \( A \) with coefficients in \( C \), \( \text{THH}(A; C) \).

Note that \( \mathcal{L}^R_X(A) \) is by definition \( \text{THH}, \) VII, §2, §3 equal to \( X \otimes A \) where \( X \otimes A \) is formed in the category of commutative \( R \)-algebras.

If \( X \in \text{sSets}_- \) is an arbitrary object, then we can write it as the colimit of its finite pointed subcomplexes and the Loday construction with respect to \( X \) can then also be expressed as the colimit of the Loday construction for the finite pointed subcomplexes.

## 2. Notions of Stability

The weakest notion of stability just asks for an abstract equivalence in the stable homotopy category:

**Definition 2.1.**

1. Let \( R \to A \) be a cofibration of commutative \( S \)-algebras with \( R \) cofibrant. We call \( R \to A \) stable if for every pair of pointed simplicial sets \( X \) and \( Y \) an equivalence \( \Sigma X \simeq \Sigma Y \) implies that \( \mathcal{L}^R_X(A) \simeq \mathcal{L}^Y_Y(A) \).

2. Let \( S \to R \to A \to B \) be a sequence of cofibrations of commutative \( S \)-algebras. Then we call \( (R,A,B) \) stable, if for every pair of pointed simplicial sets \( X \) and \( Y \) an equivalence \( \Sigma X \simeq \Sigma Y \) in \( \text{sSets} \), implies that \( \mathcal{L}^R_X(A; B) \simeq \mathcal{L}^B_Y(A; B) \).

**Examples 2.2.**

- Dundas and Tenti show that for any discrete smooth \( k \)-algebra \( A \) we have that \( Hk \to HA \) is stable \([8]\) Example 2.6).
- They show, however, that \( H\mathbb{Q} \to H\mathbb{Q}[t]/t^2 \) and \( (H\mathbb{Q}, H\mathbb{Q}[t]/t^2, H\mathbb{Q}) \) are not stable.
- If \( A \) is a commutative Hopf algebra over a field \( k \), then Berest, Ramadoss and Yeung prove \([2]\) §5 that \( Hk \to HA \) and \( (Hk, HA, Hk) \) are stable by comparing higher order Hochschild homology to representation homology. For a purely homotopy-theoretic proof see \([13]\) Theorem 3.8).
- In \([3]\) we show that for any sequence of cofibrations of commutative \( S \)-algebras \( S \to A \to B \to A \) we get that
  \[
  \mathcal{L}^R_X(A) \simeq \mathcal{L}^A_{S,X}(B; A)
  \]
  as augmented commutative \( A \)-algebras and hence \( B \to A \) is stable if \( B \) is a cofibrant commutative augmented \( A \)-algebra.

In the above definition we just require an abstract weak equivalence, but one can also pose additional conditions on the equivalence \( \mathcal{L}^R_X(A; B) \simeq \mathcal{L}^B_Y(A; B) \). A strong version of stability is the following:

**Definition 2.3.**

1. Let \( R \to A \) be a cofibration of commutative \( S \)-algebras with \( R \) cofibrant. We call \( R \to A \) multiplicatively stable if for every pair of pointed simplicial sets \( X \) and \( Y \) an equivalence \( \Sigma X \simeq \Sigma Y \) in \( \text{sSets} \), implies that \( \mathcal{L}^R_X(A) \simeq \mathcal{L}^Y_Y(A) \) as commutative augmented \( A \)-algebra spectra.

2. Let \( S \to R \to A \to B \) be a sequence of cofibrations of commutative \( S \)-algebras. Then we call \( (R,A,B) \) multiplicatively stable if for every pair of pointed simplicial sets \( X \) and \( Y \) an equivalence \( \Sigma X \simeq \Sigma Y \) in \( \text{sSets} \) implies that \( \mathcal{L}^R_X(A; B) \simeq \mathcal{L}^B_Y(A; B) \) and \( \mathcal{L}^R_X(B) \simeq \mathcal{L}^B_Y(B) \) as commutative augmented \( B \)-algebras such that the diagram
  \[
  \begin{array}{ccc}
  \mathcal{L}^R_X(A; B) & \simeq & \mathcal{L}^B_Y(A; B) \\
  \mathcal{L}^R_X(B) & \simeq & \mathcal{L}^B_Y(B) \\
  \mathcal{L}^R_X(\beta) \downarrow & & \downarrow \mathcal{L}^B_Y(\beta) \\
  \end{array}
  \]
  commutes.

Of course, there is a whole hierarchy of notions of stability. Instead of asking that the equivalence \( \mathcal{L}^R_X(A) \simeq \mathcal{L}^Y_Y(A) \) is one of augmented commutative \( A \)-algebras, we could ask for one of augmented commutative \( R \)-modules or \( A \)- or just \( R \)-modules.

**Definition 2.4.** Let \( R \to A \) be a cofibration of commutative \( S \)-algebras with \( R \) cofibrant. We call \( R \to A \) \( A \)-linearly stable if for every pair of pointed simplicial sets \( X \) and \( Y \) an equivalence \( \Sigma X \simeq \Sigma Y \) in \( \text{sSets} \), implies that \( \mathcal{L}^R_X(A) \simeq \mathcal{L}^Y_Y(A) \) as \( A \)-modules. Similarly, we call \( R \to A \) \( R \)-linearly stable if for every pair of pointed simplicial sets \( X \) and \( Y \) an equivalence \( \Sigma X \simeq \Sigma Y \) in \( \text{sSets} \), gives rise to an equivalence of \( R \)-modules \( \mathcal{L}^R_X(A) \simeq \mathcal{L}^Y_Y(A) \).
Remark 2.5. If \( R \to A \) is \( A \)-linearly stable, then \((R, A, B)\) is stable because
\[
\mathcal{L}^A_\Sigma(B; A) \simeq \mathcal{L}^A_\Sigma(A) \wedge_A B.
\]
If \( R \to A \) is multiplicatively stable, then so is \( R \to A \to B \) for every cofibrant commutative \( A \)-algebra \( B \).

A converse might not be true: Even if \( B \) is faithful as an \( A \)-module, we might not know that the equivalence \( \mathcal{L}^A_\Sigma(A) \wedge_A B \simeq \mathcal{L}^B_\Sigma(A) \wedge_A B \) is of the form \( f \wedge_A B \), so we cannot deduce that \( \mathcal{L}^A_\Sigma(B) \simeq \mathcal{L}^B_\Sigma(B) \).

Let us start with several examples of multiplicative stability.

**Proposition 2.6.** If \( B \) is an augmented commutative \( A \)-algebra, then \( B \to A \) and \( A \to \mathcal{L}^A_\Sigma(B; A) \to A \) are multiplicatively stable.

**Proof.** In the augmented case \( A \to B \to A \), as an equivalence \( \Sigma X \simeq \Sigma Y \) in \( \text{sSets} \) implies that \( \mathcal{L}^A_\Sigma(B; A) \simeq \mathcal{L}^A_\Sigma(Y; A) \) as augmented commutative \( A \)-algebras, we also get that \( \mathcal{L}^B_\Sigma(A) \simeq \mathcal{L}^B_\Sigma(Y) \) as augmented commutative \( A \)-algebras by applying [3 Theorem 3.3] to the sequence of maps \( A \to A \to B \), so \( B \to A \) is multiplicatively stable.

For the second claim observe that
\[
\mathcal{L}^A_\Sigma(\mathcal{L}^A_\Sigma(B; A); A) \simeq \mathcal{L}^A_\Sigma(\mathcal{L}^A_\Sigma(B; A) \wedge_A \mathcal{L}^A_\Sigma(A); A).
\]
Observe that for all \( X \) we have that \( \mathcal{L}^A_\Sigma(A) \simeq A \) so \( A \to \mathcal{L}^A_\Sigma(B; A) \to A \) is multiplicatively stable. \( \square \)

**Theorem 2.7.** Let \( R \to A \) be a cofibration of commutative \( S \)-algebras with \( R \) cofibrant. Then \( A \to \mathcal{L}^R_\Sigma(X) \) is multiplicatively stable for all \( X \).

**Proof.** We have to show that \( \mathcal{L}^A_\Sigma(\mathcal{L}^R_\Sigma(X); A) \) only depends on the homotopy type of \( \Sigma Y \). We first identify \( \mathcal{L}^A_\Sigma(\mathcal{L}^R_\Sigma(X); A) \) with the help of [13 Remark 3.3] as
\[
\mathcal{L}^A_\Sigma(\mathcal{L}^R_\Sigma(X); A) \simeq \mathcal{L}^R_\Sigma(\mathcal{L}^R_\Sigma(X); A) \wedge \mathcal{L}^R_\Sigma(A);
\]
\[
\simeq \mathcal{L}^R_\Sigma(\mathcal{L}^R_{Y \wedge \Sigma X}; Y \wedge X; A);
\]
\[
\simeq \mathcal{L}^R_\Sigma(\Sigma Y \wedge X; A) \simeq \mathcal{L}^R_\Sigma(\Sigma(Y); X; A).
\]
As \( \Sigma(Y) \simeq \Sigma Y \vee S^1 \) for \( Y \in \text{sSets} \), this depends only on \( \Sigma Y \). \( \square \)

**Example 2.8.** Applying Theorem 2.7 to \( HF_p \) and \( \Sigma X = S^2 \) gives that the map
\[
HF_p \to \text{THH}^{[2]}(HF_p) \simeq HF_p \vee \Sigma^3 HF_p
\]
is multiplicatively stable for all primes \( p \).

As we know from the algebraic setting that smooth algebras are stable, it is natural to consider free commutative \( A \)-algebra spectra. Let \( M \) be an \( A \)-module spectrum for some commutative \( S \)-algebra \( A \).

We consider the free commutative \( A \)-algebra on \( M \),
\[
\mathbb{P}_A(M) = \bigvee_{n \geq 0} M^\wedge A^n / \Sigma_n
\]
with the usual convention that \( M^\wedge A^0 / \Sigma_0 = A \).

In the following we use several categories, so let’s fix some notation. Let \( \mathcal{U} \) denote the category of unbased (compactly generated weak Hausdorff) spaces. For a commutative ring spectrum \( R \), \( \mathcal{M}_R \) denotes the category of \( R \)-module spectra and \( \mathcal{C}_R \) denotes the category of commutative \( R \)-algebras.

**Lemma 2.9.** For every simplicial set \( X \) there is a weak equivalence of commutative \( A \)-algebras
\[
\mathcal{L}_X^A(\mathbb{P}_A(M)) \simeq \mathbb{P}_A(X_+ \wedge M).
\]

**Proof.** For the proof we use the fact that the category of commutative \( A \)-algebras is tensored over unpointed topological spaces and simplicial sets in a compatible way [10 VII 2, §3]. Note that \( \mathcal{L}_X^A(\mathbb{P}_A(M)) = X \otimes_A \mathbb{P}_A(M) \) in the notation of [10].
We have the following chain of bijections for an arbitrary commutative $A$-algebra $B$:

$$\mathcal{C}_A(X \otimes_A \mathbb{P}_A(M), B) \cong \mathcal{U}(X, \mathcal{C}_A(\mathbb{P}_A(M), B))$$

$$\cong \mathcal{U}(X, \mathcal{M}_A(M, B))$$

$$\cong \mathcal{M}_A(X_+ \wedge M, B)$$

$$\cong \mathcal{C}_A(\mathbb{P}_A(X_+ \wedge M), B)$$

where $X_+ \wedge M$ is the tensor of $X$ with $M$ in the category of $A$-modules. Hence the Yoneda lemma implies the claim. \qed

**Corollary 2.10.** In the setting above, if $\Sigma X \simeq \Sigma Y$, then $\mathcal{L}_{A}(\mathbb{P}_A(M)) \cong \mathcal{L}_{A}(\mathbb{P}_A(M))$ as commutative $A$-algebras.

**Proof.** If $\Sigma X \simeq \Sigma Y$, then $\Sigma^\infty X \simeq \Sigma^\infty Y$ and as $X_+ \wedge M = \Sigma^\infty X \wedge M$ this implies that $\mathbb{P}_A(X_+ \wedge M) \simeq \mathbb{P}_A(Y_+ \wedge M)$ as commutative $A$-algebras. \qed

The following example was also considered in [19] Lemma 5.5. A cofibration $A \to B$ of commutative $S$-algebras with $A$ cofibrant is called THH-étale if the canonical map $B \to \text{THH}^A(B)$ is a weak equivalence.

**Proposition 2.11.** If $A \to B$ is THH-étale, then for all connected pointed $X$ the canonical map $B \to \mathcal{L}_{A}^B(B)$ is an equivalence. Hence, as this map is a map of augmented commutative $B$-algebras, $\mathcal{L}_{A}^B(B) \cong \mathcal{L}_{A}^B(B)$ for any pair of connected simplicial sets $X$ and $Y$.

**Proof.** The proof is by induction on the top dimension of a non-degenerate simplex in a finite connected simplicial set, and then by taking colimits in the infinite case. A connected 0-dimensional simplicial set consists of a point, where there is nothing to prove. Any 1-dimensional connected finite simplicial set is homotopy equivalent to a wedge of circles, so if $X \simeq S^1 \vee S^1 \vee \cdots \vee S^1$ and $B \simeq \mathcal{L}_{A}^B$, $\mathcal{L}_{A}^B(B) \cong B \wedge B \wedge \cdots \wedge B \simeq B$. Once we know the result for simplicial sets of dimension $\leq n - 1$, if we get a simplicial set $X$ with a finite number of non-degenerate $n$-cells we proceed by induction on the number of those. As in the proof of Proposition 8-4 in [3], using the homotopy invariance of the construction and subdivision, if needed, we can assume that $X$ can be constructed by adding a new non-degenerate simplex with an embedded boundary to a simplicial set homotopy equivalent to $X$ with one non-degenerate $n$-cell deleted, for which the proposition holds by the induction on the number of non-degenerate $n$-cells. By the inductive hypothesis it also holds for the embedded boundary $\partial \Delta^n$, and since the new simplex being added is homotopy equivalent to a point, the proposition holds for it. By the connectivity and by homotopy invariance we can also assume that the basepoint of $X$ is contained in the boundary of the new simplex being attached, so the identifications of all three Loday constructions with $B$ are compatible. Then $\mathcal{L}_{A}^B(B) \cong B \wedge B \simeq B$. \qed

**Remark 2.12.** Examples of THH-étale maps $A \to B$ are Galois extensions in the sense of [21] but also étale maps in the sense of Lurie [17] Definition 7.5.1.4. For a careful discussion of these notions and for comparison results see [18].

3. Inheritance properties and descent

With the assumption of multiplicative stability we get a descent result:

**Theorem 3.1.** If $R \to A \to B$ is multiplicatively stable, then $A \to B$ is multiplicatively stable.

**Proof.** Let’s assume that $\Sigma X \simeq \Sigma Y$ in $\text{sSets}$. Then by assumption we get that $\mathcal{L}_R(X) \simeq \mathcal{L}_R(Y)$ and $\mathcal{L}_R(A; B) \simeq \mathcal{L}_R(A; B)$ as commutative augmented $B$-algebras, compatibly with the module structure of the former over the latter. The Juggling Lemma [3] Lemma 3.1] yields an equivalence of augmented commutative $B$-algebras

$$\mathcal{L}_A^B(B) \simeq B \wedge \mathcal{L}_R(A; B) \mathcal{L}_R(B)$$

$$\mathcal{L}_A^B(B) \simeq B \wedge \mathcal{L}_R(A; B) \mathcal{L}_R(B).$$

Our assumptions guarantee that therefore $\mathcal{L}_A^B(B) \simeq \mathcal{L}_A^B(B)$ as commutative augmented $B$-algebras. \qed

One can upgrade this slightly and introduce coefficients:

**Corollary 3.2.** If $S \to R \to A \to B \to C$ is a sequence of cofibrations of commutative $A$-algebras and both $R \to A \to C$ and $R \to B \to C$ are multiplicatively stable, then $A \to B \to C$ is multiplicatively stable as well.
Lemma 3.3. Let $A \leftarrow R \rightarrow B$ be cofibrations of commutative $S$-algebras with $R$ cofibrant, then $\mathcal{L}_X^A(A \wedge_R B) \cong A \wedge_R \mathcal{L}_X^B(B)$ as simplicial commutative augmented $A \wedge_R B$-algebras and hence on realizations as commutative augmented $A \wedge_R B$-algebras.

Proof. There is a direct isomorphism sending $A \wedge_R (B \wedge_R \ldots \wedge_R B)$ to $(A \wedge_R B) \wedge_A \ldots \wedge_A (A \wedge_R B)$ and this isomorphism is compatible with the multiplication. \hfill \Box

This implies that stability is closed under base-change:

Proposition 3.4. Let $A$ and $E$ be cofibrant commutative $R$-algebra spectra. If $R \rightarrow A$ is $R$-linearly stable, then so is $E \rightarrow E \wedge_R A$. If $R \rightarrow A$ is multiplicatively stable, then so is $E \rightarrow E \wedge_R A$.

Proof. Assume that $\Sigma X \simeq \Sigma Y$ in $\text{sSets}_s$. Then by assumption $\mathcal{L}_X^R(A) \simeq \mathcal{L}_Y^R(A)$ as $R$-modules or as augmented commutative $A$-algebras. But then also $E \wedge_R \mathcal{L}_X^R(A) \simeq E \wedge_R \mathcal{L}_Y^R(A)$ and by Lemma 3.3 this implies $\mathcal{L}_X^E(E \wedge_R A) \simeq \mathcal{L}_Y^E(E \wedge_R A).

Remark 3.5. Note that the above implication cannot be upgraded to an equivalence: starting with the assumption that $\mathcal{L}_X^E(E \wedge_R A) \simeq \mathcal{L}_Y^E(E \wedge_R A)$, we get $E \wedge_R \mathcal{L}_X^E(A) \simeq E \wedge_R \mathcal{L}_Y^E(A)$. Even if $\mathcal{L}_X^E(A)$ and $\mathcal{L}_Y^E(A)$ are $E$-local in the category of $R$-modules, however, we don’t know that the weak equivalence $E \wedge_R \mathcal{L}_X^E(A) \simeq E \wedge_R \mathcal{L}_Y^E(A)$ is of the form $E \wedge_R f$ (or a zigzag of such maps), but for the $E$-local Whitehead Theorem [3, Lemma 1.2] we have to have a map and not just an abstract isomorphism of $\text{E}_n$-homology groups.

Smashing with a fixed commutative $R$-algebra preserves stability:

Lemma 3.6. Let $A$, $B$ and $C$ be cofibrant commutative $R$-algebras. Then there is an equivalence of commutative augmented $C \wedge_R B$-algebras

$$\mathcal{L}_X^{C \wedge_R A}(C \wedge_R B) \simeq C \wedge_R \mathcal{L}_X^A(B).$$

Hence if $f : A \rightarrow B$ is multiplicatively stable, then so is $C \wedge_R f : C \wedge_R A \rightarrow C \wedge_R B$.

Proof. The equivalence

$$\mathcal{L}_X^{C \wedge_R A}(C \wedge_R B) \simeq C \wedge_R \mathcal{L}_X^A(B)$$

is based on the equivalence

$$(C \wedge_R B) \wedge_{(C \wedge_R A)} (C \wedge_R B) \simeq C \wedge_R (B \wedge_A B).$$

\hfill \Box

Proposition 3.7. Let $R$ be a commutative ring and let $a \in R$ be a regular element. Then $HR \rightarrow HR/a$ is multiplicatively stable.

Proof. We consider the pushout $HR \wedge_{HR[t]}^L HR$ where the right algebra map $R[t] \rightarrow R$ sends $t$ to zero and the left algebra map sends $t$ to $a$. Note that with respect to both of these maps $HR[t]$ is an augmented commutative $HR$-algebra spectrum. The Künneth spectral sequence for $\pi_*(HR \wedge_{HR[t]}^L HR)$ has as its $E^2$-term $\text{Tor}_r^R(R, R)$. We take the standard free $R[t]$ resolution

$$0 \rightarrow R[t] \rightarrow R[t] \rightarrow R \rightarrow 0$$

of $R$. Applying $(\sim) \otimes_R[t] R$ yields

$$0 \rightarrow R[t] \otimes_R[t] R \xrightarrow{t \otimes \text{id}} R[t] \otimes_R[t] R \xrightarrow{\text{id} \otimes a} R[t] \otimes_R[t] R \xrightarrow{a} R.$$

Note, that the regularity of $a$ is needed to ensure injectivity on the left hand side.

We apply Lemma 3.3 and choose a cofibrant model of $HR$ as a commutative $HR[t]$-algebra and obtain $\mathcal{L}_X^{HR}(HR/a) \simeq \mathcal{L}_X^{HR}(HR \wedge_{HR[t]}^L HR) \simeq HR \wedge_{HR[t]}^L \mathcal{L}_X^{HR}(HR)$ where the right $HR[t]$-module structure of $\mathcal{L}_X^{HR}(HR)$ factors through the augmentation map sending $t$ to $0$. Assume that $\Sigma X \simeq \Sigma Y$ in $\text{sSets}_s$. By Proposition 3.4 we have that $HR[t] \rightarrow HR$ is multiplicatively stable, so $\mathcal{L}_X^{HR[t]}(HR) \simeq \mathcal{L}_Y^{HR[t]}(HR)$ as commutative augmented $HR$-algebras. This yields
an equivalence of commutative augmented $HR \wedge_{HR[t]} HR \simeq HR/a$-algebras between $\mathcal{L}^R_X(HR/a)$ and $\mathcal{L}^R_Y(HR/a)$.

**Remark 3.8.** The above result can be used for calculating torus homology for instance for $H\mathbb{Z} \to H\mathbb{Z}/p\mathbb{Z}$ for every prime $p$. We know the homotopy type of $\mathcal{L}^R_{S^2}(H\mathbb{Z}/p\mathbb{Z})$ by [8] Proposition 5.32 for all $k$ and therefore we get the homotopy type of $\mathcal{L}^R_{S^2}(H\mathbb{Z}/p\mathbb{Z})$ as smash products over $H\mathbb{Z}/p\mathbb{Z}$ of $(n/k)$ copies of $\mathcal{L}^R_{S^2}(H\mathbb{Z}/p\mathbb{Z})$ for $1 \leq k \leq n$.

**Corollary 3.9.** For every commutative ring $R$ and every regular element $a \in R$ the square-zero extension

$$HR/a \to HR/a \cup \Sigma HR/a$$

is multiplicatively stable. In particular, for every commutative ring $R$, $HR \to HR \cup \Sigma HR$ is multiplicatively stable.

**Proof.** As $HR \to HR/a$ is multiplicatively stable we get by Lemma 3.3 that

$$\mathcal{L}^R_X(HR/a \wedge_{HR} HR/a) \simeq HR/a \wedge_{HR} \mathcal{L}^R_X(HR/a)$$

as augmented commutative $HR/a \wedge_{HR} HR/a$-algebras and hence $HR/a \to HR/a \wedge_{HR} HR/a$ is multiplicatively stable. The Künneth spectral sequence yields that $\pi_*(HR/a \wedge_{HR} HR/a) \cong \Lambda_{R/a}(x)$ with $[x] = 1$. By [8] Proposition 2.1 this implies that

$$HR/a \wedge_{HR} HR/a \simeq HR/a \cup \Sigma HR/a$$

as a commutative augmented $HR/a$-algebra.

Considering the regular element $t \in R[t]$ gives that $HR \to HR \cup \Sigma HR$ is multiplicatively stable.

Stability is inherited by Loday constructions.

**Proposition 3.10.** If $R \to A$ is multiplicatively stable, then so is $R \to \mathcal{L}^R_Z(A)$ for any $Z$.

**Proof.** Assume that $\Sigma X \simeq \Sigma Y$. As $\Sigma(X \times Z) \simeq \Sigma X \vee \Sigma Z \vee \Sigma X \wedge Z$ we get that

$$\Sigma(X \times Z) \simeq \Sigma X \vee \Sigma Z \vee \Sigma X \wedge Z \simeq \Sigma Y \vee \Sigma Z \vee \Sigma Y \wedge Z \simeq \Sigma(Y \times Z)$$

and thus, as $R \to A$ is multiplicatively stable

$$\mathcal{L}^R_X(\mathcal{L}^R_Z(A)) \simeq \mathcal{L}^R_X \wedge_{\mathcal{L}^R_Z(A)} \mathcal{L}^R_Y.$$
Proof. If $\Sigma X \simeq \Sigma Y$ in sSets, then $\mathcal{L}_X^R(B) \simeq \mathcal{L}_Y^R(B)$ and $\mathcal{L}_Y^R(C) \simeq \mathcal{L}_X^R(C)$ by assumption and these equivalence are of commutative augmented $B$- and $C$-algebras, so in particular of commutative augmented $R$-algebras. Note that $\mathcal{L}_X^R(R) \simeq R$ for all pointed $X$. Hence by Lemma 3.12 we obtain

$$\mathcal{L}_X^R(B \wedge_R C) \simeq \mathcal{L}_Y^R(B) \wedge_R \mathcal{L}_Y^R(C) \simeq \mathcal{L}_X^R(B) \wedge_R \mathcal{L}_X^R(C) \simeq \mathcal{L}_Y^R(B \wedge_R C)$$

and this is an equivalence of commutative augmented $B \wedge_R C$-algebras.

Example 3.15. We know from Proposition 3.7 that $HR \rightarrow HR/a$ is multiplicatively stable for every commutative ring $R$ and every regular element $a \in R$. Corollary 3.14 implies that $HR \rightarrow HR/a \wedge_{HR} HR/a$ is multiplicatively stable and as before we know that $HR/a \wedge_{HR} HR/a \simeq HR/a \vee \Sigma HR/a$, so $HR \rightarrow HR/a \vee \Sigma HR/a$ is multiplicatively stable. For instance $HZ \rightarrow HZ/p \vee \Sigma HZ/p$ is multiplicatively stable for all primes $p$.

Example 3.16. Taking the coproduct (with a cofibrant model of $HZ[t]$ as a commutative $HZ$-algebra)

$$\begin{align*}
HZ & \rightarrow HZ/p \\
HZ[t] & \rightarrow HZ/t \wedge_{HZ} HZ/p \simeq HZ/p[t]
\end{align*}$$

shows that $HZ \rightarrow HZ/p[t]$ is multiplicatively stable.

Corollary 3.17. Let $R$ be a commutative ring and let $(a_1, \ldots, a_n)$ be a regular sequence in $R$, then $HR \rightarrow HR/(a_1, \ldots, a_n)$ is multiplicatively stable.

Proof. We use induction. We have shown in Proposition 3.7 that $HR \rightarrow HR/a_1$ is multiplicatively stable, so we can inductively assume that $HR \rightarrow HR/(a_1, \ldots, a_{n-1})$ is multiplicatively stable. We use the fact that the coproduct $HR/(a_1, \ldots, a_{n-1}) \wedge_{HR} HR/a_n$ of

$$\begin{align*}
HR & \rightarrow HR/(a_1, \ldots, a_{n-1}) \\
HR/a_n
\end{align*}$$

is $HR/(a_1, \ldots, a_n)$, and then by Corollary 3.14 the claim follows.

This identification of the coproduct be proven using the Künneth spectral sequence

$$\text{Tor}^R(R/(a_1, \ldots, a_{n-1}), R/a_n) \Rightarrow \pi_*(HR/(a_1, \ldots, a_{n-1}) \wedge_{HR} HR/a_n).$$

The Tor can be calculated by tensoring the standard free resolution $0 \rightarrow R \overset{a_n}{\rightarrow} R$ of $R/a_n$ with $R/(a_1, \ldots, a_{n-1})$ to obtain

$$0 \rightarrow R/(a_1, \ldots, a_{n-1}) \otimes_R R \overset{\text{id} \otimes a_n}{\rightarrow} R/(a_1, \ldots, a_{n-1}) \otimes_R R.$$

Since multiplication by $a_n$ is injective on $R/(a_1, \ldots, a_{n-1})$, the $E^2$ term of the spectral sequence consists only of $R/(a_1, \ldots, a_n)$ and we are done. \hfill \Box

Proposition 3.18. If $S \rightarrow A$ and $S \rightarrow B$ are cofibrations of commutative $S$-algebras and if $A$ and $B$ are (multiplicatively) stable, then if $X$ and $Y$ are connected and $\Sigma X \simeq \Sigma Y$, then

$$\mathcal{L}_X^S(A \times B) \simeq \mathcal{L}_Y^S(A \times B)$$

as commutative $S$-algebras.

Proof. This follows from Proposition 8.4] because $\mathcal{L}_X^S(A \times B) \simeq \mathcal{L}_Y^S(A) \times \mathcal{L}_Y^S(B)$ as commutative $S$-algebras. \hfill \Box

The following notion is investigated in [19][13].

Definition 3.19. Let $R \rightarrow A \rightarrow B$ be a sequence of cofibrations of commutative $S$-algebras with $R$ cofibrant. Then this sequence satisfies étale descent if for all connected $X$ the canonical map

$$\mathcal{L}_X^R(A) \wedge_A B \rightarrow \mathcal{L}_X^R(B)$$

is an equivalence.
If \( R \to A \to B \) satisfies étale descent and if \( X \) is not connected, so for example \( X = X_1 \sqcup X_2 \) with \( X_i \) connected for \( i = 1, 2 \), then the formula becomes
\[
\mathcal{L}_X^R(B) = \mathcal{L}_{X_1 \sqcup X_2}^R(B) \simeq \mathcal{L}_{X_1}^R(B) \wedge_R \mathcal{L}_{X_2}^R(B) \simeq \mathcal{L}_X^R(A) \wedge_A B \wedge_R \mathcal{L}_X^R(A) \wedge_A B.
\]

The property of satisfying étale descent is closed under smashing with a fixed commutative \( S \)-algebra:

**Lemma 3.20.** If \( R \to A \to B \) satisfies étale descent and if \( C \) is a cofibrant commutative \( R \)-algebra, then \( C \to C \wedge_R A \to C \wedge_R B \) satisfies étale descent.

**Proof.** We know from Lemma 3.11 that \( \mathcal{L}_C^R(C \wedge_R A) \simeq C \wedge_R \mathcal{L}_C^R(A) \). Therefore an exchange of pushouts yields
\[
\mathcal{L}_X^R(C \wedge_R A) \wedge_{(C \wedge_R A)} (C \wedge_R B) \simeq (C \wedge_R \mathcal{L}_C^R(A)) \wedge_{(C \wedge_R A)} (C \wedge_R B)
\]
\[
\simeq (C \wedge_C C) \wedge_{R \wedge_R R} (\mathcal{L}_C^R(A) \wedge_A B)
\]
\[
\simeq C \wedge_R \mathcal{L}_X^R(B) \simeq \mathcal{L}_X^R(C \wedge_R B).
\]

\( \Box \)

In the case of étale descent we can extend stable maps and get maps that are stable for connected \( X \):

**Proposition 3.21.** Let \( R \to A \to B \) be a sequence of cofibrations of commutative \( S \)-algebras with \( R \) cofibrant. If \( R \to A \) is multiplicatively stable and if \( R \to A \to B \) satisfies étale descent, then if \( \Sigma X \simeq \Sigma Y \) in \( \text{sSets} \), for connected \( X \) and \( Y \) we can conclude that there is a weak equivalence of augmented commutative \( B \)-algebras
\[
\mathcal{L}_X^R(B) \simeq \mathcal{L}_Y^R(B).
\]

**Proof.** As \( X \) and \( Y \) are connected and as \( R \to A \) is stable, the equivalence \( \Sigma X \simeq \Sigma Y \) in \( \text{sSets} \) implies that \( \mathcal{L}_X^R(A) \simeq \mathcal{L}_Y^R(A) \) and with étale descent we can upgrade this to
\[
\mathcal{L}_X^R(B) \simeq \mathcal{L}_X^R(A) \wedge_A B \simeq \mathcal{L}_Y^R(A) \wedge_A B \simeq \mathcal{L}_Y^R(B).
\]

\( \Box \)

**Remark 3.22.** We know that \( HQ \to HQ[t] \) is stable and as \( Q[t]/t^2 \) and \( Q[t] \) are commutative augmented \( \mathbb{Q} \)-algebras, we also know that \( HQ[t]/t^2 \to HQ \) and \( HQ[t] \to HQ \) are stable, but since \( HQ \to HQ[t]/t^2 \) and \( HQ \to HQ[t]/t^2 \to HQ \) are not stable, we won’t have general descent results. For instance in the diagram
\[
\begin{array}{ccc}
\mathbb{Q}[t]/t^2 & \xrightarrow{\eta} & \mathbb{Q} \\
\downarrow \cong & & \downarrow \cong \\
\mathbb{Q} & \xrightarrow{\epsilon} & \mathbb{Q}
\end{array}
\]
the maps \( H(\epsilon) \) and the identity on \( HQ \) are (even multiplicatively) stable, but \( H\eta \) isn’t.

For morphisms that are faithful Galois extensions and satisfy étale descent, we obtain a descent result for multiplicative stability:

**Theorem 3.23.** Let \( A \to B \) be a faithful Galois extension with finite Galois group \( G \) and assume that \( A \to B \) satisfies étale descent. Assume that \( \Sigma X \simeq \Sigma Y \) for connected \( X \) and \( Y \) implies that there is a \( G \)-equivariant equivalence \( \mathcal{L}_X^A(B) \simeq \mathcal{L}_Y^A(B) \) as commutative \( B \)-algebras. Then also \( \mathcal{L}_X^A(A) \simeq \mathcal{L}_Y^A(A) \) as commutative \( A \)-algebras.

**Proof.** The base-change result for Galois extensions [21, Lemma 7.1.1] applied to the diagram
\[
\begin{array}{ccc}
A & \xrightarrow{\eta} & B \\
\downarrow \cong & & \downarrow \cong \\
\mathcal{L}_X^A(A) & \xrightarrow{\wedge_A} & \mathcal{L}_X^A(A)
\end{array}
\]
yields that \( \mathcal{L}_X^A(A) \to B \wedge_A \mathcal{L}_X^A(A) \) is a \( G \)-Galois extension and by étale descent there is an equivalence of augmented commutative \( B \)-algebras \( B \wedge_A \mathcal{L}_X^A(A) \simeq \mathcal{L}_X^B(B) \) which is \( G \)-equivariant where on the left hand side the only non-trivial \( G \)-action is on the \( B \)-factor and on the right hand side \( G \) acts on \( \mathcal{L}_X^B(B) \) by naturality in \( B \). Hence we get a chain of \( G \)-equivariant equivalences of commutative \( B \)-algebras
\[
B \wedge_A \mathcal{L}_X^A(A) \simeq \mathcal{L}_X^B(B) \simeq \mathcal{L}_Y^A(B) \simeq B \wedge_A \mathcal{L}_Y^A(A).
\]
Taking $G$-homotopy fixed points then gives an equivalence of augmented commutative $A$-algebras

\[
\mathcal{L}_X^R(A) \simeq \mathcal{L}_X^S(B)^{hG} \simeq \mathcal{L}_Y^S(B)^{hG} \simeq \mathcal{L}_Y^R(A).
\]

There exist several definitions of smoothness in the literature (see for instance [21, 19]) using $\text{THH}$, étaleness and TAQ-étaleness. Using the local behaviour of smooth commutative $k$-algebras [16, Appendix E, Proposition E.2 (d)] as a template we suggest the following variant.

**Definition 3.24.** We call a map of cofibrant $S$-algebras $\varphi: R \to A$ really smooth if it can be factored as

\[
R \xrightarrow{i_R} \mathbb{P}_R(X) \xrightarrow{f} A
\]

where $i_R$ is the canonical inclusion, $X$ is an $R$-module, and $R \xrightarrow{i_R} \mathbb{P}_R(X) \xrightarrow{f} A$ satisfies étale descent.

Combining Proposition 3.21 and Corollary 2.10 we get:

**Proposition 3.25.** If $R \to A$ is really smooth then $\Sigma X \simeq \Sigma Y$ for connected $X$ and $Y$ implies

\[
\mathcal{L}_X^R(A) \simeq \mathcal{L}_Y^R(A)
\]

as commutative $R$-algebras.

The notion of being really smooth is transitive and closed under base change.

**Lemma 3.26.**

- If $\varphi: R \to A$ and $\psi: A \to B$ are really smooth, then so is $\psi \circ \varphi: R \to B$.
- If $\varphi: R \to A$ is really smooth and if $C$ is a cofibrant commutative $R$-algebra, then $C \to C \wedge_A B$ is really smooth.

**Proof.** To prove the transitivity, we take the two given factorizations $\varphi = R \xrightarrow{i_R} \mathbb{P}_R(X) \xrightarrow{f} A$ and $\psi = A \xrightarrow{i_A} \mathbb{P}_A(Y) \xrightarrow{g} B$ and combine them to give

\[
R \xrightarrow{i_R} \mathbb{P}_R(X \vee Y) \simeq \mathbb{P}_R(X) \wedge_R \mathbb{P}_R(Y) \xrightarrow{f \wedge \text{id}} A \wedge_R \mathbb{P}_R(Y) \simeq \mathbb{P}_A(Y) \xrightarrow{g} B
\]

So we have to show that for general maps $f: D \to A$, $g: A \to B$, $h: B \to C$ of commutative $R$-algebras:

1. If $f$ satisfies étale descent, then so does $f \wedge_R \text{id}_C$ for every commutative $R$-algebra $C$.
2. If $g$ and $h$ satisfy étale descent, then so does $h \circ g$.

For (1) let $X$ be connected. As $\mathcal{L}_X^R(\_)$ commutes with pushouts (see Lemma 3.12), we get that $\mathcal{L}_X^R(A \wedge_R C) \simeq \mathcal{L}_X^R(A) \wedge_R \mathcal{L}_X^R(C)$. As $f$ satisfies étale descent,

\[
\mathcal{L}_X^R(A) \wedge_R \mathcal{L}_X^R(C) \simeq A \wedge_D \mathcal{L}_X^R(D) \wedge_R \mathcal{L}_X^R(C) \simeq A \wedge_D \mathcal{L}_X^R(D \wedge_R C)
\]

and this in turn is equivalent to $(A \wedge_R C) \wedge_{(D \wedge_R C)} \mathcal{L}_X^R(D \wedge_R C)$.

The proof of (2) is straightforward because

\[
C \wedge_A \mathcal{L}_X^R(A) \simeq C \wedge_B (B \wedge_A \mathcal{L}_X^R(A)) \simeq C \wedge_B \mathcal{L}_X^R(B) \simeq \mathcal{L}_X^R(C).
\]

For the claim about base change consider the diagram

\[
\begin{array}{ccc}
R & \xrightarrow{i_R} & \mathbb{P}_R(X) \\
\eta \downarrow & & \downarrow f \\
C & \longrightarrow & C \wedge_R \mathbb{P}_R(X).
\end{array}
\]

Adjunction gives that $C \wedge_R \mathbb{P}_R(X) \simeq \mathbb{P}_C(C \wedge_R X)$. As $R \to \mathbb{P}_R(X) \to C$ satisfies étale descent we obtain with Lemma 3.20 that $C \to C \wedge_R \mathbb{P}_R(X) \to C \wedge_R A$ satisfies étale descent. □
4. Truncated polynomial algebras

Note that we know that the square zero extensions $H^p_2 \to H^p_2 \vee \Sigma^3 H^p_2$ (Example 2.8) and $H^p_2 \to H^p_2 \vee \Sigma H^p_2$ (Corollary 3.9) are multiplicatively stable. However, if place the module $H^p_2$ in degree zero, then the following result shows that the square zero extension $H^p_2 \to H^p_2 \vee H^p_2 \cong H^p_2[t]/t^2$ is not multiplicatively stable for odd primes $p$. The proof is a direct adaptation of [9, §3.8].

**Theorem 4.1.** Let $p$ be an odd prime. Then $(H^p_2, H^p_2[t]/t^2, H^p_2)$ is not stable.

**Corollary 4.2.** The commutative $H^p_2$-algebra $H^p_2[t]/t^2$ is neither multiplicatively stable nor $H^p_2[t]/t^2$-linearly stable over $H^p_2$.

**Proof.** If it were, then this would imply that $(H^p_2, H^p_2[t]/t^2, H^p_2)$ is stable. □

**Remark 4.3.** In [14, Theorem 4.18] we extend Theorem 4.1 to $F_2$.

**Proof of Theorem 4.2.** We know that

$\pi_* L^H_{S^1 \vee S^1 \vee S^2}(H^p_2[t]/t^2; H^p_2) \cong \pi_* L^H_{S^1}(H^p_2[t]/t^2; H^p_2)^{\otimes 2} \otimes \pi_* L^H_{S^2}(H^p_2[t]/t^2; H^p_2)$

and by [9] we know what the tensor factors are:

$\pi_* L^H_{S^1}(H^p_2[t]/t^2; H^p_2) \cong HH^p_{2}(F_2[t]/t^2; F_2) \cong \Lambda_{F_2}(\varepsilon t) \otimes \pi_* \Gamma_{F_2}(\varphi^0 t)$

and

$\pi_* L^H_{S^2}(H^p_2[t]/t^2; H^p_2) \cong HH^p_{2}(F_2[t]/t^2; F_2) \cong \Gamma_{F_2}(\varphi^0 t) \otimes \bigotimes_k (\Lambda_{F_2}(\varphi^k t) \otimes \Gamma_{F_2}(\varphi^0 \varphi^k t))$.

Torus homology is the total complex of the bicomplex for $L^H_{S^1 \times S^1}(F_2[t]/t^2; F_2)$ as in [9]. In the bicomplex in bidegree $(n, m)$, we have the term

$L^H_{[n] \times [m]}(F_2[t]/t^2; F_2) \cong F_2 \otimes_{F_2} (F_2[t]/t^2)^{(n+1)(m+1) - 1} \cong (F_2[t]/t^2)^{(n+1)(m+1) - 1}.$

In total degree one we have contributions from $(0, 1)$ and $(1, 0)$ that we call $y_1$ and $y_1$ as in [9, §3.8].

Everything is a cycle here and these elements correspond to $\otimes$ and $1 \otimes t$.

From now on we suppress the tensor signs from the notation and we denote the generators by matrices.

In total degree two there are three possibilities $(0, 2)$, $(1, 1)$ and $(2, 0)$. There are the classes $y_1$ in bidegree $(0, 2)$, and $y_1$ in bidegree $(2, 0)$ corresponding to the standard Hochschild generators $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 1 & t \end{pmatrix}$.

In bidegree $(1, 1)$ there are the following possibilities for non-degenerate cycles:

$\begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix}, \begin{pmatrix} 1 & t \\ 1 & t \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ t & t \end{pmatrix}$, and $\begin{pmatrix} 1 & 1 \\ 1 & t \end{pmatrix}$.

As we are working over $F_p$ for an odd prime $p$, 2 is invertible. The boundary of $\begin{pmatrix} 1 & 1 \\ 1 & t \end{pmatrix}$ is $\begin{pmatrix} 1 & t \\ 1 & 1 \end{pmatrix}$, the boundary of $\begin{pmatrix} 1 & 1 \\ 1 & t \end{pmatrix}$ is $\begin{pmatrix} 1 & 1 \\ t & 1 \end{pmatrix}$. Finally, we identify $\begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix}$ as the boundary of $\begin{pmatrix} 1 & 1 \\ t & t \end{pmatrix}$.

The element $\begin{pmatrix} 1 & 1 \\ 1 & t \end{pmatrix}$ ensures that $\begin{pmatrix} 1 & 1 \\ t & 1 \end{pmatrix}$ is homologous to $\begin{pmatrix} 1 & 1 \\ 1 & t \end{pmatrix}$, so we are left with the generator in bidegree $(1, 1)$ given by $\begin{pmatrix} 1 & 1 \\ 1 & t \end{pmatrix}$.

So we get (at most) a 3-dimensional vector space in total degree 2.

In $\pi_* L^H_{S^1 \vee S^1 \vee S^2}(H^p_2[t]/t^2; H^p_2)$, however, we get the generators $\varphi^0 t \otimes 1 \otimes 1$, $1 \otimes \varphi^0 t \otimes 1$, $\varepsilon t \otimes \varepsilon t \otimes 1$ and $1 \otimes 1 \otimes \varphi^0 \varepsilon t$, so we have a 4-dimensional vector space. □
Remark 4.4. For odd primes 2 is invertible and this reduces the number of generators in total degree 2 to 3 in the torus homology of \( F_p[t]/t^2 \) over \( F_p \) with \( F_p \)-coefficients. For \( p = 2 \) one can check that there is an extra class coming from \( \begin{pmatrix} 1 & 1 \\ t & 1 \end{pmatrix} \) which is homologous to \( \begin{pmatrix} 1 & 1 \\ t & t \end{pmatrix} \) and to \( \begin{pmatrix} 1 & t \\ 1 & 1 \end{pmatrix} \) so together with the class \( \begin{pmatrix} 1 & 1 \\ t & 1 \end{pmatrix} \) this gives two generators in bidegree \((1, 1)\) and the ones in \((2, 0)\) and \((0, 2)\) giving a total of dimension 4. As \( F_2[t]/t^2 \) is a commutative Hopf algebra over \( F_2 \), we know that \( F_2 \to F_2[t]/t^2 \) and \( F_2 \to F_2[t]/t^2 \to F_2 \) are stable.

We can model \( S[t]/t^n \) as \( S \sma \Pi_+ \) for the commutative pointed monoid \( \Pi_+ = \{+, 1, t, \ldots, t^{n-1}\} \). In [14] Theorem 4.18 we generalize the result from Theorem 4.1 to all \( 2 \leq n < p \).

Corollary 4.5. For every \( n \geq 2 \) the map \( S \to S[t]/t^n \) is not multiplicatively stable.

Proof. If it were multiplicatively stable, then by Lemma 3.3 \( H\mathbb{F}_p \to H\mathbb{F}_p[t]/t^n \) would be as well. For \( n = 2 \) this contradicts the result above. For higher \( n \), there is a prime \( p \) with \( p > n \), and then [14] Theorem 4.18 yields that \( H\mathbb{F}_p \to H\mathbb{F}_p[t]/t^n \) isn’t multiplicatively stable.

Remark 4.6. Neither stability nor multiplicative stability are transitive: for every commutative ring \( k \) the map \( k \to k[t] \) is smooth, hence (multiplicatively) stable and \( k[t] \to k[t]/t^2 \) is stable by Proposition 3.7 but for \( k = Q \) Dundas and Tenti show [3] that \( Q \to Q[t]/t^2 \) is not stable and for \( k = \mathbb{F}_p \) the know that \( \mathbb{F}_p \to \mathbb{F}_p[t]/t^2 \) is not multiplicatively stable.

Proposition 4.7. Let \( k \) be a field and let \( \Pi_+ \) be a pointed commutative monoid. If \( S \to Hk \) and \( Hk \to Hk[\Pi_+] \) are multiplicatively stable, then \( \Sigma X \simeq \Sigma Y \) in \( \text{sSets} \) implies that \( \mathcal{L}^S_X(Hk[\Pi_+]) \simeq \mathcal{L}^S_Y(Hk[\Pi_+]) \) as augmented commutative Hopf-algebras.

Proof. This follows from the splitting of \( \mathcal{L}^S_X(Hk[\Pi_+]) \) as a commutative augmented \( Hk \)-algebra [15] Theorem 7.1] as

\[
\mathcal{L}^S_X(Hk[\Pi_+]) \simeq \mathcal{L}^S_X(Hk) \wedge_{Hk} \mathcal{L}^H_X(Hk[\Pi_+]).
\]

It is important to know whether \( S \to Hk \) is \( Hk \)-linearly stable, because if it is, then for all \( Hk \)-linearly stable \( Hk \to HA \) that satisfy a splitting formula as in [14], such as polynomial algebras, we would get that \( S \to HA \) is \( Hk \)-linearly stable.

Of course, \( S \to H\mathbb{Q} \) is multiplicatively stable because \( S\mathbb{Q} \simeq H\mathbb{Q} \) and \( H\mathbb{Q} \sma H\mathbb{Q} \simeq H\mathbb{Q} \). We do not know whether \( S \to H\mathbb{F}_p \) is stable. We can express \( H\mathbb{F}_p \) as a Thom spectrum, but this Thom spectrum structure comes from a double loop map, so it is not of the form needed for Corollary 4.4. So we leave this as an open question:

Is \( H\mathbb{F}_p \) stable?

We close with a family of examples that show that several of the Juggling Formulas from [3] cannot be generalized to arbitrary pointed simplicial sets because that would contradict certain non-stability results.

Let \( k \) be a field. The case \( k[t] = R \to k[t]/t^m = R/t^m \to k = R/t \) for \( m \geq 2 \) is special in the sense that the quotient \( k[t]/t^m \) itself is a commutative augmented \( k \)-algebra, so we can combine our splitting result for higher order Shukla homology [3 Proposition 7.5] with the Juggling Formula [3 Theorem 3.3].

We have [3] Theorem 7.6):

\[
\text{THH}^{n,k}(k[t]/t^m; k) \simeq \text{THH}^{n,k}(k[t]; k) \wedge_{Hk} \text{THH}^{n,k,Hk[k]}(k[t]/t^m; k)
\]

for all \( n \geq 1 \) and for all \( m \geq 2 \). In this special case we can get the following \( Hk \)-version of this result:

Theorem 4.8. Let \( k \) be a field and let \( m \) be greater or equal to 2. Then for all \( n \geq 1 \)

\[
\text{HH}^{n,k}(k[t]/t^m; k) \simeq \text{HH}^{n,k}(k[t]; k) \wedge_{Hk} \text{THH}^{n,k,Hk[k]}(k[t]/t^m; k).
\]

Proof. Consider the diagram

\[
\begin{array}{ccc}
Hk & \xrightarrow{\text{THH}^{n-1,Hk[k]}(k[t]; k)} & \text{HH}^{n,k}(k[t]; k) \\
\text{THH}^{n,Hk[k]}(k[t]/t^m; k) & \xrightarrow{\text{THH}^{n-1,Hk[k]/t^m}(k)} & \text{HH}^{n,k}(k[t]/t^m; k).
\end{array}
\]
Remark 5.2. Nima Rasekh, Bruno Stonek, and Gabriel Valenzuela [20, Theorem 4.1] generalize Schlichtkrull’s calculation to generalized Thom spectra, or $\text{MSpin}$ spectra to a map of $\text{THH}^{[n-1], n\text{i}(k)}(k)$. This yields that the outer rectangle is also a homotopy pushout square and this was the claim. □

Remark 4.9. Note that there cannot be a version of (4.2) and (4.3) for arbitrary connected $X$: We know that $Hk[t] \to Hk[t]/t^2 \to Hk$ is multiplicatively stable for all fields $k$ and we know that $Hk \to Hk[t] \to Hk$ is stable. But for any odd prime $p$ we know that $HF_p \to HF_p[t]/t^2 \to HF_p$ is not stable and that there is an actual discrepancy between

$$\pi_2(\mathcal{L}^{HF_p}_{S^1 \times S^1}(HF_p[t]/t^2; HF_p)) \neq \pi_2(\mathcal{L}^{HF_p}_{S^1 \times S^1 \vee S^2}(HF_p[t]/t^2; HF_p)),$$

so there cannot be an equivalence between $\mathcal{L}^{HF_p}_{S^1 \times S^1}(HF_p[t]/t^2; HF_p)$ and $\mathcal{L}^{HF_p}_{S^1 \times S^1 \vee S^2}(HF_p[t]/t^2; HF_p)$.

5. Thom spectra and topological $K$-theory

Christian Schlichtkrull gives a closed formula for the Landay construction on Thom spectra [22, Theorem 1.1]: Let $f: W \to BF_{hI}$ be an $E_{\infty}$-map with $W$ group-like and let $T(f)$ denote the corresponding Thom spectrum. Then for any $T(f)$-module spectrum $M$ one has

$$(5.1) \quad \mathcal{L}^S_X(T(f); M) \simeq M \wedge \Omega^\infty(EW \wedge X)_+$$

where $EW$ is the Omega spectrum associated to $W$ (i.e., $W \simeq \Omega^\infty EW$). If $M$ is a commutative $T(f)$-algebra spectrum, then the above equivalence is one of commutative $T(f)$-algebras. For $M = T(f)$ the equivalence also respects the augmentation.

An immediate consequence of Schlichtkrull’s result is the following:

**Corollary 5.1.** If $T(f)$ is a Thom spectrum as above, then $S \to T(f)$ is multiplicatively stable.

**Proof.** If $\Sigma X \simeq \Sigma Y$ in $\text{sSets}_{s_+}$, then on the level of spectra we obtain

$$\Sigma(EW \wedge X) \simeq EW \wedge \Sigma X \simeq EW \wedge \Sigma Y \simeq \Sigma(EW \wedge Y),$$

but here suspension is invertible, hence $EW \wedge X \simeq EW \wedge Y$ and therefore

$$\mathcal{L}^S_X(T(f)) \simeq \mathcal{L}^S_Y(T(f)).$$

An equivalence of spectra induces an equivalence of infinite loop spaces and the $T(f)$-algebra structure on $T(f) \wedge \Omega^\infty(EW \wedge X)_+$ just comes from the one on $T(f)$ and the infinite loop structure on $\Omega^\infty(EW \wedge X)$. This gives the multiplicativity of the stability. □

The case of the suspension spectrum of an abelian topological group is a special case where we take $f: G \to BF_{hI}$ to be the trivial map. Then $T(f) \simeq \Sigma^\infty_+(G)$. Other examples are $MU$, $MO$, $MSO$, $MSp$ or $MSpin$.

**Remark 5.2.** Nima Rasekh, Bruno Stonek, and Gabriel Valenzuela [20, Theorem 4.11] generalize Schlichtkrull’s calculation to generalized Thom spectra, i.e., Thom spectra that are formed with respect to a map of $E_{\infty}$-groups $f: G \to \text{Pic}(R)$ for some commutative ring spectrum $R$. They note (see [20, Remark 4.14]) that this implies stability for such Thom spectra.

**Remark 5.3.** Note that by Corollary 4.3, spherical abelian monoid rings are not stable in general, whereas spherical abelian group rings are.

Bruno Stonek calculates higher $\text{THH}$ of periodic complex topological $K$-theory, $KU$, and he determines topological André-Quillen homology of $KU$ [24]. He uses Smith’s description of $KU$ as the Bott localization of $\Sigma^\infty_+ CP^\infty$. The latter is a Thom spectrum because $CP^\infty = BU(1)$ can be realized as a topological abelian group.

**Theorem 5.4.** If $X$ and $Y$ are connected and $\Sigma X \simeq \Sigma Y$ in $\text{sSets}_{s_+}$, then

$$\mathcal{L}^S_X(KU) \simeq \mathcal{L}^S_Y(KU)$$

as commutative augmented $KU$-algebra spectra.
Proof. Let \( \beta \) denote the Bott element. Stonek uses Snaith’s identification of \( KU \) as \( \Sigma_+^\infty \mathbb{C}P^\infty[\beta^{-1}] \) to prove [21] Corollary 4.12 that there is a zigzag of equivalences

\[
\text{THH}(KU) \cong \text{THH}(\Sigma_+^\infty \mathbb{C}P^\infty[\beta^{-1}]) \cong \text{THH}(\Sigma_+^\infty \mathbb{C}P^\infty \wedge \Sigma_+^\infty \mathbb{C}P^\infty[\beta^{-1}]) \\
\cong \left( \text{THH}(\Sigma_+^\infty \mathbb{C}P^\infty)[\beta^{-1}] \right).
\]

The same argument yields that for any connected \( X \) the localization of \( \mathcal{L}_X^S(\Sigma_+^\infty \mathbb{C}P^\infty) \) at \( \beta \) is equivalent to \( \mathcal{L}_X^{S_p}(\Sigma_+^\infty \mathbb{C}P^\infty[\beta^{-1}]) = \mathcal{L}_X^S(KU) \).

The localization map \( \Sigma_+^\infty \mathbb{C}P^\infty \rightarrow \Sigma_+^\infty \mathbb{C}P^\infty[\beta^{-1}] \) satisfies étale descent, and therefore the composite \( S \rightarrow \Sigma_+^\infty \mathbb{C}P^\infty \rightarrow \Sigma_+^\infty \mathbb{C}P^\infty[\beta^{-1}] \) identifies \( KU \) as an étale extension of a Thom spectrum. By Proposition 3.22 we obtain multiplicative stability for connected simplicial sets. \( \square \)

**Corollary 5.5.** If \( X \) and \( Y \) are connected simplicial sets with \( \Sigma X \simeq \Sigma Y \) then \( \mathcal{L}_X^S(KO) \simeq \mathcal{L}_Y^S(KO) \) as commutative \( KO \)-algebras.

**Proof.** Rognes shows [21, §5.3] that the complexification map \( KO \rightarrow KU \) is a faithful \( C_2 \)-Galois extension of commutative ring spectra and Mathew [18, Example 4.6] deduces from [6, Example 5.9] that it satisfies étale descent. Schlichtkrull’s equivalence from [5.1] is natural hence it preserves the \( C_2 \)-action. Therefore the result follows from Theorem 3.23. \( \square \)

In [5] Hood Chatham, Jeremy Hahn, and Allen Yuan construct interesting examples of \( E_\infty \)-ring spectra. For a prime \( p \) they consider the infinite loop space

\[
W_h = \Omega^\infty \Sigma^{2\nu(h)} BP(h)
\]

where \( \nu(h) = \frac{h+1}{p-1} \). This is the \( 2\nu(h) \)th space of the Omega spectrum for the \( h \)-truncated Brown-Peterson spectrum \( BP(h) \); these spaces were extensively studied by Steve Wilson [26]. On the suspension spectrum of \( W_h \) they invert the generator \( x \) of the bottom non-trivial homotopy group \( \pi_2(h)(W_h) \cong \mathbb{Z}(p) \) and obtain an \( E_\infty \)-ring spectrum

\[
R_h := (\Sigma_+^\infty W_h)[x^{-1}]
\]

which has remarkable features [5, Theorem 1.13]: \( R_h \) has torsion-free homotopy groups that vanish in odd degrees, it is Landweber exact, and its Morava-K(n) localization \( L_{K(n)}R_h \) vanishes if and only if \( n > h + 1 \), so \( R_h \) is of chromatic height \( n + 1 \). As \( W_0 \) is \( \mathbb{C}P^\infty \), this recovers Snaith’s construction in this special case, but there are many more interesting examples. For all of these proofs, the above obtain.

**Theorem 5.6.** If \( X \) and \( Y \) are connected and \( \Sigma X \simeq \Sigma Y \) in \( \text{sSets}_+ \), then \( \mathcal{L}_X^S(R_h) \simeq \mathcal{L}_Y^S(R_h) \) as commutative augmented \( R_h \)-algebra spectra.

### 6. The Greenlees spectral sequence

Let \( k \) be a field and let \( A \rightarrow B \) be a morphism of connective commutative \( S \)-algebras with an augmentation to \( Hk \) satisfying some mild finiteness assumption. Then by [12, Lemma 3.1] there is a spectral sequence

\[
E^2_{s,t} = \pi_s(B \wedge_A Hk) \otimes_k \pi_t(A) \Rightarrow \pi_{s+t}(B).
\]

Let \( p \) be an odd prime. We consider the cofibration \( S^1 \vee S^1 \hookrightarrow S^1 \times S^1 \rightarrow S^2 \) and the associated pushout diagram

\[
\begin{array}{ccc}
\mathcal{L}_{S^1 \vee S^1}^R(HF_p[t]/t^2; HF_p) & \longrightarrow & \mathcal{L}_{S^1 \times S^1}^R(HF_p[t]/t^2; HF_p) \\
\downarrow & & \downarrow \\
HF_p & \longrightarrow & \mathcal{L}_{S^1}^R(HF_p[t]/t^2; HF_p).
\end{array}
\]

Here, \( R \) can be \( S \) or \( HF_p \). For \( R = S \) we obtain a Greenlees spectral sequence

\[
(6.1) \quad \pi_s(\mathcal{L}_{S^1}^S(HF_p[t]/t^2; HF_p)) \otimes_p \pi_t(\mathcal{L}_{S^1 \times S^1}^S(HF_p[t]/t^2; HF_p)) \Rightarrow \pi_{s+t}(\mathcal{L}_{S^1 \times S^1}^S(HF_p[t]/t^2; HF_p))
\]

whereas for \( R = HF_p \) the spectral sequence is

\[
(6.2) \quad \pi_s(\mathcal{L}_{S^1}^{HF_p}(HF_p[t]/t^2; HF_p)) \otimes_p \pi_t(\mathcal{L}_{S^1 \times S^1}^{HF_p}(HF_p[t]/t^2; HF_p)) \Rightarrow \pi_{s+t}(\mathcal{L}_{S^1 \times S^1}^{HF_p}(HF_p[t]/t^2; HF_p)).
\]
In [5.1] every term \( L^S_X(\mathbb{F}_p[t]/t^2; H\mathbb{F}_p) \) splits as
\[
L^S_X(\mathbb{F}_p[t]/t^2; H\mathbb{F}_p) \cong L^{H\mathbb{F}_p}_X(\mathbb{F}_p[t]/t^2; H\mathbb{F}_p)
\]
naturally in \( X \), and going from working over \( S \) to working over \( H\mathbb{F}_p \) simply collapses the \( L^S_X(\mathbb{F}_p[t]/t^2; H\mathbb{F}_p) \) to \( L^{H\mathbb{F}_p}_X(\mathbb{F}_p[t]/t^2; H\mathbb{F}_p) \cong H\mathbb{F}_p \). Therefore we get a surjection of the spectral sequence of [6.1] onto the one of [6.2], and if all the spectral sequence differentials vanish on the former, they have to vanish on the latter too. But we know that the rank of \( \pi_2(L^{H\mathbb{F}_p}_{S^1\times S^1}(\mathbb{F}_p[t]/t^2; H\mathbb{F}_p)) \) is less than the rank of the \( E^2 \)-term in total degree 2, hence there has to be a non-trivial differential in [6.2] and hence also in [6.1]. This implies the following result.

**Theorem 6.1.** For every odd prime \( p \), \((S, \mathbb{F}_p[t]/t^2, H\mathbb{F}_p)\) is not stable.

With the results of [14, §4] the above result can be generalized to \( \mathbb{F}_p[t]/t^n \) for \( 2 \leq n < p \).

Instead of stability we can consider the following property of Loday constructions.

**Definition 6.2.** Let \( R \) be a cofibrant commutative ring spectrum and let \( R \rightarrow A \rightarrow C \) be a sequence of cofibrations of commutative \( R \)-algebras. We say that \( R \rightarrow A \rightarrow C \) decomposes products if for all pointed connected simplicial sets \( X \) and \( Y \)
\[
L^R_{X \times Y}(A; C) \cong L^R_{Y \times X \times Y}(A; C)
\]
Note that the right hand side is equivalent to \( L^R_X(A; C) \wedge_C L^R_Y(A; C) \wedge_C L^R_{X \wedge Y}(A; C) \).

**Proposition 6.3.** Let \( R \rightarrow A \rightarrow B \rightarrow A \rightarrow Hk \) be a sequence of commutative \( S \)-algebras that turns \( B \) into an augmented commutative \( A \)-algebra. Assume that \( k \) is a field.
If \( R \rightarrow B \rightarrow Hk \) decomposes products then so does \( R \rightarrow A \rightarrow Hk \).

**Proof.** The naturality of the Loday construction ensures that the vertical compositions in the diagram
\[
\begin{array}{ccc}
L^R_{Y \times X}(A; Hk) & \rightarrow & L^R_{X \times Y}(A; Hk) \\
\downarrow & & \downarrow \\
L^R_{Y \times X}(B; Hk) & \rightarrow & L^R_{X \times Y}(B; Hk)
\end{array}
\]
are the identity. Therefore the spectral sequence
\[
\pi_*(L^R_{X \times Y}(A; Hk)) \otimes_k \pi_*(L^R_{Y \times X}(A; Hk)) \Rightarrow \pi_*(L^R_{X \times Y}(A; Hk))
\]
is a direct summand of the one for \( B \). So if the spectral sequence for \( A \) had a non-trivial differential, then the one for \( B \) also had to have one, but as \( B \) decomposes products, this cannot happen. \( \square \)

Note that this gives an additive splitting, but we can’t rule out multiplicative extensions.
If \( B \) does not decompose products, then this does not imply that \( A \) doesn’t either. A concrete counterexample is \( S \rightarrow H\mathbb{Q} \rightarrow H\mathbb{Q}[t]/t^2 \rightarrow H\mathbb{Q} \). Here, \( S \rightarrow H\mathbb{Q}[t]/t^2 \) does not decompose products, but \( S \rightarrow H\mathbb{Q} \) is even multiplicatively stable.

7. Rational Equivalence

The starting point for this section is the following result:

**Proposition 7.1.** (Berest, Ramadoss, and Yeung [2]) If \( k \) is a field of characteristic zero and \( A \) is a commutative Hopf algebra over \( k \), then any rational equivalence \( f: X \rightarrow Y \) between simply connected spaces induces a weak equivalence
\[
f_*: L^H_X(HA) \simeq L^H_Y(HA).
\]
If \( f: X \rightarrow Y \) is a rational equivalence between simply connected pointed spaces then \( f \) induces a weak equivalence
\[
f_*: L^H_X(HA; Hk) \simeq L^H_Y(HA; Hk).
\]
Proof. This follows from [2] Theorem 1.3 (a)] which says that for such \( k \) and \( A \) and any unbased simplicial set \( X \),

\[
\pi_* \mathcal{L}_X^Hk(HA) \cong \mathcal{R}_*(\Sigma(X_+), A)
\]

where \( \mathcal{R} \) is representation homology, and from [2] Proposition 4.2], which says that rational equivalences between simply connected spaces induce isomorphisms on representation homology for such \( k \) and \( A \). In the pointed setting, [2] Theorem 1.3 (b)] applies to give the equivalence

\[
\pi_* \mathcal{L}_X^Hk(HA; Hk) \cong \mathcal{R}_*(\Sigma X, A; k)
\]

We can extend Proposition [7.1 to augmented commutative finitely generated \( k \)-algebras:

**Proposition 7.2.** If \( k \) is a field of characteristic zero and \( A \) is a finitely generated augmented commutative \( k \)-algebra, then any rational equivalence \( f : X \to Y \) of simply connected spaces induces a weak equivalence

\[
f_* : \mathcal{L}_X^Hk(HA; Hk) \simeq \mathcal{L}_Y^Hk(HA; Hk).
\]

Proof. Let \( A \) be generated by \( a_1, a_2, \ldots, a_\ell \) as a commutative \( k \)-algebra, let \( \varepsilon : A \to k \) be its augmentation, and let \( \eta : k \to A \) be the unit map. We denote by \( I \) the augmentation ideal, \( I = \ker \varepsilon \). Then for all \( 1 \leq i \leq \ell \), \( a_i - \eta(\varepsilon(a_i)) \in I \), so we can define a surjection of augmented commutative \( k \)-algebras

\[
\varphi: k[x_1, \ldots, x_\ell] \to A, \quad \varphi(x_i) = a_i - \eta(\varepsilon(a_i)) \quad \text{for all } 1 \leq i \leq \ell.
\]

Here we consider the augmentation of \( k[x_1, \ldots, x_\ell] \) that sends every \( x_i \) to zero, so that its augmentation ideal is \( (x_1, \ldots, x_\ell) \).

Since \( k \) is a field, \( k[x_1, \ldots, x_\ell] \) is Noetherian so we can find finitely many polynomials \( f_1, \ldots, f_m \) in the \( x_i \) to generate \( \ker \varphi \). Since \( \varepsilon \circ \varphi \) is the augmentation of \( k[x_1, \ldots, x_\ell] \), we get that for all \( 1 \leq j \leq m, f_j \) is an element in \( (x_1, \ldots, x_\ell) \). Hence, we can define another map of augmented commutative \( k \)-algebras

\[
\psi: k[u_1, \ldots, u_m] \to k[x_1, \ldots, x_\ell], \quad \psi(u_i) = f_j \quad \text{for all } 1 \leq j \leq m,
\]

which maps \( k[u_1, \ldots, u_m] \) onto \( \ker \varphi \). The augmentation of \( k[u_1, \ldots, u_m] \) is again the standard one. We express \( A \) as a pushout of commutative augmented \( k \)-algebras

\[
k[u_1, \ldots, u_m] \xrightarrow{\psi} k[x_1, \ldots, x_\ell] \xrightarrow{\varepsilon} A
\]

where all entries except \( A \) are known to be commutative Hopf algebras over \( k \). So for them, \( f \) induces a weak equivalence \( \mathcal{L}_X^Hk(H(-); Hk) \to \mathcal{L}_Y^Hk(H(-); Hk) \). Since both \( \mathcal{L}_X^Hk(H(-); Hk) \) and \( \mathcal{L}_Y^Hk(H(-); Hk) \) send pushouts of commutative \( k \)-algebras to homotopy pushouts of augmented commutative \( Hk \)-algebras, \( f \) also induces a weak equivalence on the pushout. □

Let \( X \) be a pointed simply connected simplicial set. Then rationally

\[
\Sigma X_\mathbb{Q} \simeq \bigvee_{i \in I} S_{\mathbb{Q}}^{k_i}
\]

for some indexing set \( I \) and some \( k_i \geq 2 \) (see for instance [11] Theorem 24.5]). In particular,

\[
\bigvee_{i \in I} S_{\mathbb{Q}}^{k_i} \simeq \Sigma \left( \bigvee_{i \in I} S_{\mathbb{Q}}^{k_i-1} \right).
\]

So with the help of [2] Theorem 1.3] we obtain:

**Theorem 7.3.** For every pointed simply connected \( X \), every field of characteristic zero \( k \) and every commutative Hopf-algebra \( A \) over \( k \), for a suitable indexing set \( I \) and integers \( k_i \geq 2 \) we get

\[
\pi_* \mathcal{L}_X^k(\Sigma X_\mathbb{Q}; k) \cong \pi_* \mathcal{L}_X^k(V_{i \in I}, S_{\mathbb{Q}}^{k_i-1}; A; k).
\]

For simply-connected spaces and \( k \) and \( A \) as above we know by [2] Proposition 4.2] that the homotopy type of the Loday construction only depends on the rational homotopy type of the suspension, so we can discard the rationalization in the above statement. This yields, for instance:
Example 7.4. Let $X = \mathbb{C}P^n$ for some $n \geq 1$. Then for every field of characteristic zero $k$ and every commutative Hopf-algebra $A$ over $k$, as $\Sigma \mathbb{C}P^n \simeq \Sigma \mathbb{C}P^1 \otimes \cdots \otimes \mathbb{C}P^1$, we obtain
\[
\pi_* \mathcal{L}^k_{\mathbb{C}P^n}(A; k) \cong \pi_* \mathcal{L}^k_{\mathbb{C}P^1}(A; k).
\]

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