Minimal Actuator Placement with Bounds on Control Effort

V. Tzoumas, M. A. Rahimian, G. J. Pappas, A. Jadbabaie*

Abstract—We address the problem of minimal actuator placement in a linear system subject to an average control energy bound. First, following the recent work of Olshevsky, we prove that this is NP-hard. Then, we prove that the involved control energy metric is supermodular. Afterwards, we provide an efficient algorithm that approximates up to a multiplicative factor of $O(\log n)$, where $n$ is the size of the system, any optimal actuator set that meets the same energy criteria. This is the best approximation factor one can achieve in polynomial-time, in the worst case. Next, we focus on the related problem of cardinality-constrained actuator placement for minimum control effort, where the optimal actuator set is selected so that an average input energy metric is minimized. While this is also an NP-hard problem, we use our proposed algorithm to efficiently approximate its solutions as well. Finally, we run our algorithms over large random networks to illustrate their efficiency.

Index Terms—Multi-agent Networked Systems, Input Placement, Leader Selection, Controllability Energy Metrics, Minimal Network Controllability.

I. INTRODUCTION

During the past decade, an increased interest in the analysis of large-scale systems has led to a variety of studies that range from the mapping of the human’s brain functional connectivity to the understanding of the collective behavior of animals, and the evolutionary mechanisms of complex ecological systems [1], [2], [3], [4]. At the same time, control scientists develop methods for the regulation of such complex systems, with the notable examples in [5], for the control of biological systems; [6], for the regulation of brain and neural networks; [7], for robust information spread over social networks, and [8], for load management in smart grid.

On the other hand, the large size of these systems, as well as the need for low cost control, has made the identification of a small fraction of their states, to steer them around the entire space, an important problem [9], [10], [11], [12]. This is a task of formidable complexity; indeed, it is shown in [9] that finding a small number of actuators, so that a linear system is controllable, is NP-hard. However, mere controllability is of little value if the required input energy for the desired transfers is exceedingly high, when, for example, the controllability matrix is close to singularity [13]. Therefore, by choosing input states to ensure controllability alone, one may not achieve a cost-effective control for the system.

In this paper, we address this important requirement by providing efficient approximation algorithms to actuate a small fraction of a system’s states so that a specified control energy performance over the entire state space is guaranteed. In particular, we first consider the selection of a minimal number of actuated states so that an average control energy bound, along all the directions in the state space, is satisfied. Finding such a subset of states is a challenging task; since, it involves the search for a small number of actuators that induce controllability, which constitutes a combinatorial problem that can be computationally intensive. Indeed, identifying a small number of actuated states for inducing controllability alone is NP-hard [9]. Therefore, we extend this computationally hard problem by introducing an energy performance requirement on the choice of the optimal actuator set, and we solve it with an efficient approximation algorithm.

Specifically, we first generalize the involved energy objective to an $\epsilon$-close one, which remains well-defined even for actuator sets that render the system uncontrollable. Then, we make use of this metric and relax the implicit controllability constraint from the original actuator placement problem. Notwithstanding, we prove that for small values of $\epsilon$ all solutions of this auxiliary program still render the system controllable. This fact, along with the supermodularity of the generalized objective with respect to the choice of the actuator set, leads to a polynomial-time algorithm that approximates up to a multiplicative factor of $O(\log n)$, where $n$ is the size of the system, any optimal actuator set that meets the specified energy criterion. Moreover, this is the best approximation factor one can achieve in polynomial-time, in the worst case. Hence, with this algorithm we address the open problem of minimal actuator placement subject to bounds on the control effort [9], [11], [12], [14], [15].

Relevant results are also found in [12], where the authors study the controllability of a system with respect to the smallest eigenvalue of the controllability Gramian, and they derive a lower bound on the number of actuators so that this eigenvalue is lower bounded by a fixed value. Nonetheless, they do not provide an algorithm to identify the actuators that achieve this value. Our proposed algorithm, on the other hand, selects a minimal number of actuators so that this control objective is satisfied.

Next, we consider the problem of cardinality-constrained actuator placement for minimum control effort, where the optimal actuator set is selected so that an average control energy objective around the entire state space is minimized. The most related work to this problem is [11], in which the authors assume a controllable system and consider the problem of choosing a few extra actuators in order to optimize some of the input energy metrics proposed in [16]. Their main contribution is in observing that these energy metrics are supermodular with respect to the choice of the extra actuated states. The assumption of a controllable system is necessary since these metrics depend on the inverse of the controllability Gramian, as they quantify the average control energy for

*All authors are with the Department of Electrical and Systems Engineering, University of Pennsylvania, Philadelphia, PA 19104-6228 USA (email: {vtzoumas, mohar, pappasg, jadbabai}@seas.upenn.edu). This work was supported in part by TerraSwarm, one of six centers of STARnet, a Semiconductor Research Corporation program sponsored by MARCO and DARPA, and in part by AFOSR Complex Networks Program.
steering the system around the entire state space. Nonetheless, it should be also clear that making a system controllable by first placing some actuators to ensure controllability alone, and then adding some extra ones to optimize a desired energy metric, introduces a sub-optimality that is carried over to the end result. In this paper, we follow a parallel line of work for the minimal actuator placement problem, and provide an efficient algorithm that selects all the actuated states towards the minimization of an average control energy metric without any assumptions on the controllability of the involved system.

A similar actuator placement problem is studied in [12] for stable systems. Nevertheless, its authors propose a heuristic actuator placement procedure that does not constrain the number of available actuators and does not optimize their control energy objective. Our proposed algorithm selects a cardinality-constrained actuator set that minimizes a control energy metric, even for unstable systems.

The remainder of this paper is organized as follows. The formulation and model for the actuator placement problems are set forth in Section III, where the corresponding integer optimization programs are stated. In Sections III and IV we discuss our main results, including the intractability of these problems, as well as the supermodularity of the involved control energy metrics with respect to the choice of the actuator sets. Then, we provide efficient approximation algorithms for their solution that guarantee a specified control energy performance over the entire state space. Finally, in Section V we illustrate our analytical findings using an integrator chain network and test their efficiency over large random networks that are commonly used to model real-world networked systems. Our simulation results further stress the importance of selecting the actuators of a system for efficient control energy performance and not for mere controllability. Section VI concludes the paper.

II. PROBLEM FORMULATION

A. Notation

We denote the set of natural numbers \{1, 2, \ldots\} as \mathbb{N}, the set of real numbers as \mathbb{R}, and let \([n] \equiv \{1, 2, \ldots, n\}\) for all \(n \in \mathbb{N}\). Also, given a set \(\mathcal{X}\), we denote as \(|\mathcal{X}|\) its cardinality. Matrices are represented by capital letters and vectors by lower-case letters. For a matrix \(A\), \(A^T\) is its transpose and \(A_{ij}\) is its element located at the \(i\)-th row and \(j\)-th column. If \(A\) is positive semi-definite or positive definite, we write \(A \succeq 0\) and \(A \succ 0\), respectively. Moreover, for \(i \in [n]\), let \(\delta_{i}^{(j)}\) be an \(n \times n\) matrix with a single non-zero element: \(\delta_{i}^{(j)} = 1\), while \(\delta_{i}^{(j)} = 0\), for \(j, k \neq i\). Furthermore, denote as \(I\) the identity matrix, whose dimension is inferred from the context. Additionally, for \(\delta \in \mathbb{R}^n\), let \(\text{diag}(\delta)\) denote an \(n \times n\) diagonal matrix such that \(\text{diag}(\delta)_{ii} = \delta_i\) for all \(i \in [n]\).

B. Actuator Placement Model

Consider a linear system of \(n\) states, \(x_1, x_2, \ldots, x_n\), whose evolution is described by

\[
\dot{x}(t) = Ax(t) + Bu(t), \quad t > t_0, \tag{1}
\]

where \(t_0 \in \mathbb{R}\) is fixed, \(x \equiv \{x_1, x_2, \ldots, x_n\}\), \(\dot{x}(t) \equiv dx/dt\), while \(u\) is the corresponding input vector. The matrices \(A\) and \(B\) are of appropriate dimension. We equivalently refer to \(\mathbb{R}\) as a network of \(n\) nodes, \(1, 2, \ldots, n\), which we associate with the states \(x_1, x_2, \ldots, x_n\), respectively. Moreover, we denote their collection as \(\mathcal{V} \equiv [n]\).

Henceforth, \(A\) is given while \(B\) is a diagonal zero-one matrix that we design so that \(1\) satisfies a specified control energy criterion over the entire state space.

Assumption 1. \(B = \text{diag}(\delta)\), where \(\delta \in \{0, 1\}^n\).

Specifically, if \(\delta_i = 1\), state \(x_i\) may receive an input, while if \(\delta_i = 0\), it receives none.

Definition 1 (Actuator Set, Actuator). Given a \(\delta \in \{0, 1\}^n\), let \(\Delta \equiv \{i : i \in \mathcal{V} \text{ and } \delta_i = 1\}\); then, \(\Delta\) is called an actuator set and each \(i \in \Delta\) an actuator.

C. Controllability and Related Average Energy Metrics

We consider the notion of controllability and relate it to the problems of this paper, that is, the minimal actuator placement for constrained control energy and the cardinality-constrained actuator placement for minimum control effort.

System \(1\) is controllable — equivalently, \((A, B)\) is controllable — if for any finite \(t_1 > t_0\) and any initial state \(x_0 \equiv x(t_0)\) it can be steered to any other state \(x_1 \equiv x(t_1)\) by some input \(u(t)\) defined over \([t_0, t_1]\). Moreover, for general matrices \(A\) and \(B\), the controllability condition is equivalent to the matrix

\[
W \equiv \int_{t_0}^{t_1} e^{At}BB^Te^{A^Tt} dt, \tag{2}
\]

being positive definite for any \(t_1 > t_0\) [13]. Therefore, we refer to \(W\) as the \textit{controllability matrix} of \(1\).

The controllability of a linear system is of interest because it is related to the solution of the following minimum-energy transfer problem

\[
\text{minimize } \int_{t_0}^{t_1} u(t)^T u(t) dt \tag{3}
\]

subject to

\[
\dot{x}(t) = Ax(t) + Bu(t), \quad t_0 < t \leq t_1, \quad x(t_0) = x_0, x(t_1) = x_1, \]

where \(A\) and \(B\) are any matrices of appropriate dimension.

In particular, if for the given \(A\) and \(B\) \((1)\) is controllable the resulting minimum control energy is given by

\[
(x_1 - e^{A^T t_0} x_0)^T W^{-1} (x_1 - e^{A^T t_0} x_0) \tag{4}
\]

where \(\tau = t_1 - t_0\) [16]. Thereby, the states that belong to the eigenspace of the smallest eigenvalues of \(2\) require higher energies of control input [13]. Extending this observation along all the directions of transfers in the state space, we infer that the closer \(W\) is to singularity the larger the expected input energy required for these transfers to be achieved [16]. For example, consider the case where \(W\) is singular, i.e. when there exists at least one direction along which system \(1\) cannot be steered [13]. Then, the corresponding minimum control energy along this direction is infinity.
This motivates the consideration of control energy metrics that quantify the average steering energy along all the directions in the state space, as the $\tr(W^{-1})$ [16]; indeed, this metric is well-defined only for controllable systems ($W$ must be invertible) and is directly related to [4]. In this paper, we aim to select a small number of actuators for system (1) so that $\tr(W^{-1})$ either meets a specified upper bound or is minimized.

Per Assumption [1], further properties for the controllability matrix are due: For any actuator set $\Delta$, let $W_\Delta \equiv W$; then,

$$W_\Delta = \sum_{i=1}^{n} \delta_i W_i,$$  \hspace{1cm} (5)

where $W_i = \int_{0}^{t_1} e^{At} I e^{AT} dt$ for any $i \in [n]$. This follows from [2] and the fact that $BB^T = B = \sum_{i=1}^{n} \delta_i I(i)$ for $B = \text{diag}(\delta)$. Finally, for any $\Delta_1 \subseteq \Delta_2 \subseteq \mathcal{V}$, (5) and $W_1, W_2, \ldots , W_n \geq 0$ imply $W_{\Delta_1} \preceq W_{\Delta_2}$.

D. Actuator Placement Problems

We consider the selection of a small number of actuators for system (1) so that $\tr(W^{-1})$ either satisfies an upper bound or is minimized. The challenge is in doing so with as few actuators as possible. This is an important improvement over the existing literature where the goal of actuator placement problems has either been to ensure controllability alone [9] or the weaker property of structural controllability [13], [12]. Other relevant results consider the task of leader-selection [20], [21], where the leaders are the actuated states and are chosen so to minimize a mean-square convergence error of the remaining states.

Furthermore, the most relevant work to our study is [11] since its authors consider the minimization of $\tr(W^{-1})$; nevertheless, their results rely on a pre-existing actuator set that renders (1) controllable although this set is not selected for the minimization of this energy metric. One of our contributions is in achieving optimal actuator placement for minimum average control effort without assuming controllability beforehand. Also, the authors of [12] adopt a similar framework for actuator placement but focus on deriving an upper bound for the smallest eigenvalue of $W$ with respect to the number of actuators and a lower bound for the required number actuators so that this eigenvalue takes a specified value. In addition, they are motivated by the inequality $\tr(W^{-1}) \geq n/\tr(W)$, and consider the maximization of $\tr(W)$; however, their techniques cannot be applied when minimizing the $\tr(W^{-1})$, while the maximization of $\tr(W)$ may not ensure controllability [12].

We next provide the exact statements of our actuator placement problems, while their solution analysis follows in Sections III and IV. We first consider the problem

$$\text{minimize} \quad \left\| \Delta \right\| \quad \text{subject to} \quad \tr(W_{\Delta}^{-1}) \leq E,$$  \hspace{1cm} (I)

for some constant $E$. Its domain is $\{\Delta : \Delta \subseteq \mathcal{V} \text{ and } (A, B(\Delta)) \text{ is controllable}\}$ since the controllability matrix $W(\Delta)$ must be invertible. This also implies that (I) is NP-hard [9]. It looks for a minimal solution, thereby it requests for every $r \leq n$ if there exist a $\Delta$ such that $|\Delta| \leq r$ and $(A, B(\Delta))$ is controllable.

Additionally, Problem (I) is feasible for certain values of $E$. In particular, for any $\Delta$ such that $(A, B(\Delta))$ is controllable, $0 \preceq W_{\Delta}$, i.e. $\tr(W_{\Delta}^{-1}) \leq \tr(W_{\Delta}^{-1})$ since for any $\Delta$ (5) implies $W_{\Delta} \preceq W_{\Delta}$ [22]; thus, (I) is feasible for

$$E \geq \tr(W_{\Delta}^{-1}).$$  \hspace{1cm} (6)

Moreover, (I) is a generalized version of the minimal controllability problem of [9] so that its solution not only ensures controllability but also satisfies a guarantee in terms of an average control energy metric; indeed, for $E \to \infty$ we recover the problem of [9].

We next consider the problem

$$\text{minimize} \quad \tr(W_{\Delta}^{-1}) \quad \text{subject to} \quad |\Delta| \leq r,$$  \hspace{1cm} (II)

where the goal is to find at most $r$ actuated states so that an average control energy metric is minimized. Its domain is $\{\Delta : \Delta \subseteq \mathcal{V}, |\Delta| \leq r \text{ and } (A, B(\Delta)) \text{ is controllable}\}$; this implies the NP-hardness of (II) as well [9].

Because (I) and (II) are NP-hard, we need to identify efficient approximation algorithms for their general solution; this is the subject of Sections III and IV. In particular, in Section III we consider Problem (I) and provide for it a best-approximation algorithm. To this end, we first define an auxiliary program which ignores the controllability constraint of (I).

We present a polynomial-time best-approximation algorithm for Problem (I). To this end, we first generalize the involved energy metric to an $\epsilon$-close one that remains well-defined even when the controllability matrix is not invertible. Next, we relax (I) by introducing a new program that makes use of this metric and circumvents the restrictive controllability constraint of (I). Moreover, we prove that for certain values of $\epsilon$ all solutions of this auxiliary problem render the system controllable. This fact, along with the supermodularity property of the generalized metric that we establish, leads to our proposed approximation algorithm. The discussion of its efficiency ends the analysis of (I).
A. An $\epsilon$-close Auxiliary Problem

Consider the following approximation to (I)
\[
\begin{align*}
\text{minimize} & \quad |\Delta| \\
\text{subject to} & \quad \text{tr}(W_{\Delta} + \epsilon I)^{-1} \leq E, \\
\end{align*}
\]
where $\epsilon$ satisfies $0 < \epsilon \leq 1/E$. In contrast to (I), the domain of this problem consists of all subsets of $V$ since $W_{(i)} + \epsilon I$ is always invertible.

The $\epsilon$-closeness is evident since for any $\Delta$ such that $(A, B(\Delta))$ is controllable $\text{tr}(W_{\Delta} + \epsilon I)^{-1} \to \text{tr}(W_{\Delta}^{-1})$ as $\epsilon \to 0$. Notice that we can take $\epsilon \to 0$ since we assume any positive $\epsilon \leq 1/E$.

Taking into consideration the NP-hardness of (I), and that (I) tends to (I) as $\epsilon \to 0$, we expect that this auxiliary problem is NP-hard as well; this is proved in the following paragraphs.

B. Intractability of Problem (I)

Consider a $\Delta \subseteq V$ and let $\lambda_{\Delta}$ be the smallest positive eigenvalue of $W_{\Delta}$. Moreover, for any $r \in [n]$, set $\lambda_r \equiv \min_{\Delta \subseteq V, |\Delta| \leq r}\{\lambda_{\Delta}\}$ and then
\[
\lambda \equiv \min_{r \in [n]}\{\lambda_r\}. \tag{7}
\]

We have the following intractability result.

**Theorem 1** (Intractability). Problem (I) is NP-hard for any $\epsilon$ such that $0 < \epsilon < 1/E = (\lambda + \epsilon)/n$.

**Proof:** First consider the following lemma.

**Lemma 1** (An $\epsilon$-Equivalence for Controllability). Fix any $\epsilon$ such that $0 < \epsilon < \lambda/(n-1)$. Then, for any $\Delta \subseteq V$, \(\text{tr}(W_{\Delta} + \epsilon I)^{-1} \leq n/(\lambda + \epsilon)\) if and only if $(A, B(\Delta))$ is controllable.

To see why Lemma 1 holds true, denote as $\lambda_1, \lambda_2, \ldots, \lambda_n$ the eigenvalues of $W_\Delta$ given some $\Delta \subseteq V$. Moreover, since $\epsilon < \lambda/(n-1)$, it follows that $n/(\lambda + \epsilon) < 1/\epsilon$.

Consider a $\Delta$ such that $(A, B(\Delta))$ is controllable, if any. Then,
\[
\text{tr}(W_{\Delta} + \epsilon I)^{-1} = \frac{1}{\lambda_i} + \frac{1}{\lambda + \epsilon} \leq \frac{n}{\lambda + \epsilon}.
\]

Moreover, consider a $\Delta$ such that $(A, B(\Delta))$ is not controllable, if any. Also, let $k$ be the corresponding number of non-zero eigenvalues of $W_\Delta$. Then,
\[
\text{tr}(W_{\Delta} + \epsilon I)^{-1} = \frac{k}{\lambda_i} + \frac{n-k}{\lambda + \epsilon} > \frac{n}{\lambda + \epsilon},
\]
which is true since for any $\Delta$ such that $(A, B(\Delta))$ is not controllable $k \leq n-1$. This completes the proof of Lemma 1.

Now for the proof of Theorem 1 from $\epsilon < 1/E$ and $E = n/(\lambda + \epsilon)$ we get that $\epsilon < \lambda/(n-1)$. Hence, from Lemma 1 we have that there exists an actuator set $\Delta$ that makes system (I) controllable if and only if for this actuator set $\text{tr}(W_{\Delta} + \epsilon I)^{-1} \leq n/(\lambda + \epsilon)$. However, the former decision problem is proved to be NP-hard [4].

C. Approximation Algorithm for Problem (I)

We first prove that all solutions of (I) for $0 < \epsilon \leq 1/E$ render the system controllable, notwithstanding that no controllability constraint is imposed by this program on the choice of the actuator sets. Moreover, we show that the involved $\epsilon$-close energy metric is supermodular with respect to the choice of actuator sets and then we present our approximation algorithm, followed by a discussion of its efficiency which ends this subsection.

**Proposition 1.** Consider a constant $\omega > 0$, $\epsilon$ such that $0 < \epsilon < 1/\omega$, and any $\Delta \subseteq V$: If $\text{tr}(W_{\Delta} + \epsilon I)^{-1} \leq \omega$, then $(A, B(\Delta))$ is controllable.

**Proof:** Assume that $(A, B(\Delta))$ is not controllable and let $k$ be the corresponding number of non-zero eigenvalues of $W_{\Delta}$ which we denote as $\lambda_1, \lambda_2, \ldots, \lambda_k$; therefore, $k \leq n-1$. Then,
\[
\text{tr}(W_{\Delta} + \epsilon I)^{-1} = \frac{k}{\lambda_i} + \frac{n-k}{\epsilon} > \frac{1}{\epsilon} \geq \omega,
\]
and since $\epsilon \leq 1/\omega$ we have a contradiction.

Note that $\omega$ is chosen independently of the parameters of system (I). Therefore, the absence of the controllability constraint in Problem (I) for $0 < \epsilon \leq 1/E$ is fictitious; nonetheless, it obviates the necessity of considering only actuator sets that render the system controllable.

The next proposition is also essential and suggest an efficient approximation algorithm for solving (I).

**Proposition 2** (Supermodularity). The function $\text{tr}(W_{\Delta} + \epsilon I)^{-1}$ is supermodular with respect to the choice of $\Delta$.

**Proof:** Recall that $\text{tr}(W_{\Delta} + \epsilon I)^{-1}$ is supermodular if and only if $-\text{tr}(W_{\Delta} + \epsilon I)^{-1}$ is submodular, and that a function $h : V \to \mathbb{R}$ is submodular if and only if for any $a \in V$ the function $h_a : V \setminus \{a\} \to \mathbb{R}$, $h_a(\Delta) \equiv h(\Delta \cup \{a\}) - h(\Delta)$, is a non-increasing set function. In other words, if and only if for any $\Delta_1 \subseteq \Delta_2 \subseteq V \setminus \{a\}$ it holds true that $h_a(\Delta_1) \geq h_a(\Delta_2)$.

In case, $h_a(\Delta) = -\text{tr}(W_{\Delta(a)} + \epsilon I)^{-1} + \text{tr}(W_{\Delta} + \epsilon I)^{-1}$. Therefore, take any $\Delta_1 \subseteq \Delta_2 \subseteq V \setminus \{a\}$ and denote accordingly $D \equiv \Delta_2 \setminus \Delta_1$. Then, we aim to prove
\[
-\text{tr}(W_{\Delta_1(a)} + \epsilon I)^{-1} + \text{tr}(W_{\Delta_1} + \epsilon I)^{-1} \geq -\text{tr}(W_{\Delta_1(a) \cup D(a)} + \epsilon I)^{-1} + \text{tr}(W_{\Delta_1(a) \cup D} + \epsilon I)^{-1}.
\]
To this end, and for $z \in [0, 1]$, set $f(z) \equiv \text{tr}(W_{\Delta_1} + zW_{\Delta} + W_a + \epsilon I)$ and $g(z) \equiv \text{tr}(W_{\Delta_1} + zW_{\Delta} + \epsilon I)$. After some manipulations the above inequality can be written as $f(1) - f(0) \geq g(1) - g(0)$.

To prove this, one it suffices to prove that $df/dz \geq dg/dz$ for any $z \in [0, 1)$. Denote $L_1(z) \equiv W_{\Delta_1} + zW_{\Delta} + W_a + \epsilon I$ and $L_2(z) \equiv W_{\Delta_1} + zW_{\Delta} + \epsilon I$. Then, the $df/dz \geq dg/dz$
Consequently, tr$(L(z)^{-1}W_DL(z)^{-1}) \leq$ tr$(L_2(z)^{-1}W_DL_1(z)^{-1})$, (8)
where we used the fact that for any $A \succ 0$, $B \succeq 0$ and $z \in (0,1)$, \[ \frac{1}{z} \text{tr}((A + zB)^{-1}) = -\text{tr}((A + zB)^{-1} B (A + zB)^{-1}). \]
To show that this holds, first observe that $L_1(z) \succeq L_2(z)$. This implies $L_2(z)^{-1} \succeq L_1(z)^{-1}$, and as a result $L_2(z)^{-2} \succeq L_1(z)^{-2}$ [22]. Hence, $W_1^{1/2}L_2(z)^{-2}W_1^{1/2} \succeq W_1^{1/2}L_1(z)^{-2}W_1^{1/2}$ which gives
\[ \text{tr}(W_1^{1/2}L_2(z)^{-2}W_1^{1/2}) \geq \text{tr}(W_1^{1/2}L_1(z)^{-2}W_1^{1/2}). \]

Finally, the cycle property of trace yields inequality (8). Consequently, $tr(W_1 + \epsilon I)^{-1}$ is supermodular.

Therefore, Theorem 1 is in agreement with the general hardness of the class of minimum set-covering problems subject to submodular constraints. Inspired by this literature [23], [24], [25], we have the following efficient approximation algorithm for Problem (1), and as we show by the end of this section, for Problem (1) as well.

Algorithm 1 Approximation Algorithm for the Problem (1).

**Input:** Bound $E$, parameter $\epsilon \leq 1/E$, matrices $W_1, W_2, \ldots, W_n$.

**Output:** Actuator set $\Delta$
\[
\Delta \leftarrow \emptyset \quad \text{while } \text{tr}(W_\Delta + \epsilon I)^{-1} > E \text{ do } \quad \text{end while}
\]
\[
a_i \leftarrow \text{argmax}_{a_i \in V \setminus \Delta} \{\text{tr}(W_\Delta + \epsilon I)^{-1} - \text{tr}(W_{\Delta \cup \{a_i\}} + \epsilon I)^{-1}\}
\]
\[
\Delta \leftarrow \Delta \cup \{a_i\}
\]

Regarding the quality of Algorithm 1 the following is true.

**Theorem 2** (A Submodular Set Coverage Optimization). Denote as $A^*$ a solution to Problem (1) and as $\Delta$ the selected set by Algorithm 1. Then,
\[
(A, B(\Delta)) \text{ is controllable,} \quad (9)
\]
\[
\text{tr}(W_\Delta + \epsilon I)^{-1} \leq E, \quad (10)
\]
\[
\frac{|\Delta|}{|\Delta^*|} \leq 1 + \log \frac{n\epsilon^{-1} - \text{tr}(W_\Delta + \epsilon I)^{-1}}{E - \text{tr}(W_\Delta + \epsilon I)^{-1}} \equiv F, \quad (11)
\]
\[
F \equiv O(\log n + \log \epsilon^{-1} + \log \frac{1}{E - \text{tr}(W_\Delta + \epsilon I)^{-1}}), \quad (12)
\]

**Proof:** We first prove (10), (11) and (12), and then (9).
First, let $\Delta_0, \Delta_1, \ldots$ be the sequence of sets selected by Algorithm 1 and $\lambda$ the smallest index such that $\text{tr}(W_{\Delta_\lambda} + \epsilon I)^{-1} \leq E$. Therefore, $\Delta_\lambda$ is the set that Algorithm 1 returns, and this proves (10).

Moreover, from [24], since for any $\Delta \subseteq V$, $h(\Delta) \equiv -\text{tr}(W_\Delta + \epsilon I)^{-1} + n\epsilon^{-1}$ is a non-negative, non-decreasing, and submodular function (cf. Proposition 2), it is guaranteed for Algorithm 1 that
\[
\frac{l}{|\Delta^*|} \leq 1 + \log \frac{h(\Delta) - h(\emptyset)}{h(\emptyset) - h(\Delta_{\lambda-1})} \leq 1 + \log \frac{n\epsilon^{-1} - \text{tr}(W_\Delta + \epsilon I)^{-1}}{\text{tr}(W_{\Delta_{\lambda-1}} + \epsilon I)^{-1} - \text{tr}(W_\Delta + \epsilon I)^{-1}}.
\]

Now, $l$ is the first time that $\text{tr}(W_{\Delta_l} + \epsilon I)^{-1} \leq E$, and a result $\text{tr}(W_{\Delta_{l-1}} + \epsilon I)^{-1} > E$. This implies (11).

Moreover, observe that $0 < \text{tr}(W_\Delta + \epsilon I)^{-1} < \text{tr}(W_\Delta^{-1})$ so that from (11) we get $F \leq 1 + \log [n\epsilon^{-1}E - \text{tr}(W_\Delta^{-1})]$, which in turn implies (12).

On the other hand, since $0 < \epsilon \leq 1/E$ and $\text{tr}(W_{\Delta_l} + \epsilon I)^{-1} \leq E$, Proposition 1 is in effect, i.e. (9) holds true.

Therefore, the polynomial-time Algorithm 1 returns a set of actuators that meets the corresponding control energy bound of Problem (1) while it renders system (1) controllable. Moreover, the cardinality of this set is up to a multiplicative factor of $F$ from the minimum cardinality actuator sets that meet the same control energy bound. Next, we elaborate on the dependence of this factor on $n$, $\epsilon$ and $E$ using (12), while in Section III-E we finalize our treatment of Problem (1) by employing Algorithm 1 to approximate its solutions.

**D. Quality of Approximation of Algorithm 1 for Problem (1)**

The result in (12) was expected from a design perspective: Increasing the network size $n$ or improving the accuracy by decreasing $\epsilon$, as well as demanding a better energy guarantee by decreasing $E$ should all push the cardinality of the selected actuator set upwards. Also, $\log \epsilon^{-1}$ is the design cost for circumventing the difficulty to satisfy controllability constraint of (1) [9], i.e. for assuming no pre-existing actuators that renders (1) controllable and choosing all the actuators towards the satisfaction of an energy performance criterion.

Furthermore, per (12) and with $E - \text{tr}(W_\Delta^{-1}) + \epsilon$ fixed, the cardinality of the actuator set that Algorithm 1 returns is up to a multiplicative factor of $O(\log n)$ from the minimum cardinality actuator sets that meet the same performance criterion; this is the best achievable bound in polynomial-time for the set covering problem in the worst case [26], while (1) is a generalization of it (cf. Theorem 1 and [9]). Thus, Algorithm 1 is a best-approximation algorithm for (1).

It is worth highlighting that by considering (1) for $\epsilon$ and $E$ such that $0 < \epsilon < 1/E = (\lambda + \epsilon)/n$, then per Theorem 1 we recover the results of [9], where it is proven that the minimum number of actuators for controllability alone can be approximated in polynomial-time only up to a multiplicative factor of $O(\log n)$.

**E. Approximation Algorithm for Problem (1)**

We present an efficient approximation algorithm for Problem (1) that is based on Algorithm 1. Let $\Delta$ be the actuator set returned by Algorithm 1 that is, $(A, B(\Delta))$ is controllable and $\text{tr}(W_{\Delta_l} + \epsilon I)^{-1} \leq E$. Moreover, denote as $\lambda_1(W_\Delta), \lambda_2(W_\Delta), \ldots, \lambda_n(W_\Delta)$ the eigenvalues of $W_\Delta$ and as $\lambda_n(W_\Delta)$ the smallest one. Finally, consider a positive $\epsilon$ such that
Theorem 3. Denote as $\Delta^*$ a solution to Problem 1 and as $\Delta$ the selected set by Algorithm 2. Then,

$$\frac{A(B(\Delta))}{2} \text{ is controllable,}$$

$$\text{tr}(W_{\Delta}^{-1}) \leq (1 + c)E,$$

$$\frac{1}{|\Delta|} \leq F,$$

$$F = O\left(\log n + \log \frac{E}{c} + \log \frac{1}{E - \text{tr}(W_{\Delta}^{-1})}\right).$$

Proof: We only prove the three last statements of the theorem, as the first follows from Theorem 2. First, when Algorithm 2 exits the while loop, and after the following if statement, $\text{tr}(W_{\Delta}^{-1}) - \text{tr}(W_{\Delta} + \epsilon I)^{-1} \leq cE$, and since $\text{tr}(W_{\Delta} + \epsilon I)^{-1} \leq E$, this implies (17).

To show (18), consider any solution $\Delta^*$ to Problem 1 and any solution $\Delta^*$ to Problem 1. Then, $|\Delta^*| \geq |\Delta^*|$; to see this, note that for any $\Delta^*$, $\text{tr}(W_{\Delta}^{-1}) - \text{tr}(W_{\Delta} + \epsilon I)^{-1} \leq cE$ since $\epsilon > 0$, i.e., $\Delta^*$ is a candidate solution to Problem 1 because it satisfies all of its constraints. Therefore, $|\Delta^*| \geq |\Delta^*|$, and as a result $|\Delta^*| \leq |\Delta^*| \leq F$ per (11).

Finally, note that (17) holds true when $\epsilon$ is of the same order as $n/(c\lambda^2_m(W_{\Delta}E))$, while $1/\lambda_m(W_{\Delta}) < \text{tr}(W_{\Delta}^{-1}) = O(E)$. Therefore, $\log \epsilon^{-1} = O(\log n + \log E/c)$ and this proves (19).

Per (19) and with $E$ and $E - \text{tr}(W_{\Delta}^{-1})$ fixed, the cardinality of the actuator set that Algorithm 2 returns is up to a multiplicative factor of $O(\log n)$ from the minimum cardinality actuator sets that meet the same energy bound; this is the best achievable bound in polynomial-time for the set covering problem in the worst case [26], while (1) is a generalization of it [9]. Thus, Algorithm 2 is a best-approximation algorithm for (1).

IV. MINIMUM ENERGY CONTROL BY A CARDINALITY-CONSTRAINED ACTUATOR SET

We present an approximation algorithm for Problem (1) following a parallel line of thought as in Section III. First, we circumvent the restrictive controllability constraint of (1) using the $\epsilon$-close generalized energy metric defined in Section III. Then, we propose an efficient approximation algorithm for its solution that makes use of Algorithm 2; this algorithm returns an actuator set that always renders (1) controllable while it guarantees a value for (1) that is provably close to its optimal one. We end the analysis of (1) by explicating further the efficiency of this procedure.

A. An $\epsilon$-close Auxiliary Problem

For $\epsilon > 0$ consider the following approximation to (11)

$$\begin{align*}
\text{minimize} & \quad \text{tr}(W_{\Delta} + \epsilon I)^{-1} \\
\text{subject to} & \quad |\Delta| \leq r.
\end{align*}$$

(II')

In contrast to (11), the domain of this problem consists of all subsets of $Y$ since $W_{\Delta} + \epsilon I$ is always invertible. Moreover, its objective is $\epsilon$-close to that of Problem (11).
Taking into consideration the NP-hardness of Problem (II), and that (II) tends to (I) as $\epsilon \to 0$, we expect that this auxiliary program is NP-hard as well; this is proved in the following paragraphs.

B. Intractability of Problem (II)

Let $\lambda_r = \min_{\Delta \subseteq V, |\Delta| \leq r} \{ \lambda_\Delta \}$ for any positive integer $r \leq n$, where $\lambda_\Delta$ is the smallest positive eigenvalue of $W_\Delta$. We have the following intractability result.

**Theorem 4** (Intractability). Problem (II) is NP-hard for any $\epsilon$ such that $0 < \epsilon < \lambda_r/(n-1)$.

**Proof:** First consider the following lemma.

**Lemma 2** (An $\epsilon$-Equivalence for Controllability). Fix $r$ and an $\epsilon$ such that $0 < \epsilon < \lambda_r/(n-1)$. Then, for any $\Delta \subseteq V$ such that $|\Delta| \leq r$, $\operatorname{tr}(W_\Delta + \epsilon I)^{-1} \leq n/(\lambda_r + \epsilon)$ if and only if $(A, B(\Delta))$ is controllable.

Its proof is similar to that of Lemma 1 using $\lambda_r$ instead of $\lambda$. Then, the proof of Theorem 4 is parallel to that of Theorem 1.

This is in accordance with the hardness of the class of supermodular function minimization problems, as per Proposition 2 the objective $\operatorname{tr}(W_\Delta + \epsilon I)^{-1}$ is supermodular. The approximation algorithms used in that literature however [23], [24], [25], fail to provide an efficient solution algorithm for (II) — for completeness, we discuss this direction in the Appendix A.

In the next subsection we propose an efficient approximation algorithm for (II) that makes use of Algorithm 2.

C. Approximation Algorithm for Problem (II)

We provide an efficient approximation algorithm for Problem (II) that is based on Algorithm 2. In particular, since (II) finds an actuator set that minimizes $\operatorname{tr}(W_{(i)}^{-1})$, and any solution to (I) satisfies $\operatorname{tr}(W_{(i)}^{-1}) \leq E$, one may repeatedly execute Algorithm 2 for decreasing values of $E$ as long as the returned actuator sets are at most $r$ and $E$ satisfies the feasibility constraint $E \geq \operatorname{tr}(W_{(i)}^{-1})$ (cf. Section II-D). Therefore, for solving (II) we propose a bisection-type execution of Algorithm 2 with respect to $E$.

To this end, we also need an upper bound for the value of (II). Let $\Delta_c$ be a small actuator set that renders system (I) controllable; it is found in polynomial time using Algorithm 2 for large $E$ or the procedure proposed in [9]. Then, for any $r \geq |\Delta_c|$, $\operatorname{tr}(W_{(i)}^{-1})$ upper bounds the value of (II) since $\operatorname{tr}(W_{(i)}^{-1})$ is monotone.

Thus, having a lower and upper bound for the value of (II), we implement Algorithm 3 for approximating the solutions of (II); we consider only the non-trivial case where $r < n$ and denote the set that Algorithm 2 returns as $[\text{Algorithm } 2(E, c, a)]$ for given $E$, $c$ and $a$.

In the worst case, when we first enter the while loop the if condition is not satisfied, and as a result $E$ is set to a greater value. This process continues until the if condition is satisfied for the first time from which point and on the algorithm converges up to the accuracy level $a$ to the smallest value $E$ of $E$ such that $|\Delta| \leq r$; specifically, $|E - E| \leq a'/2$ due to the mechanics of the bisection method, where $E = \min\{E : |[\text{Algorithm } 2(E, c, a)]| \leq r\}$; that is, $E$ is the least bound $E$ for which Algorithm 2 returns an actuator set of cardinality at most $r$ for the specified $c$ and $a$ — $E$ may be larger than the value of (II) due to worst-case approximability of the involved problems (cf. Theorem 3). Then, Algorithm 3 exits the while loop and the last if statement ensures that $E$ is set below $E$ so that $|\Delta| \leq r$. Moreover, per Theorem 3 this set renders (II) controllable and guarantees that $\operatorname{tr}(W_{(i)}^{-1}) \leq (1 + c)E$. We summarize the above in the next corollary which also ends the analysis of Problem (II).

**Corollary 1** (Approximation Efficiency of Algorithm 3 for Problem (II)). Denote as $\Delta$ the selected set by Algorithm 3
Then,

\[
(A, B(\Delta)) \text{ is controllable,}
\]

\[
\operatorname{tr}(W_{(i)}^{-1}) \leq (1 + c)E,
\]

\[
|E - E| \leq a'/2,
\]

where $E = \min\{E : |[\text{Algorithm } 2(E, c, a)]| \leq r\}$ is the least bound $E$ that Algorithm 2 satisfies with an actuator set of cardinality at most $r$ for the specified $c$ and $a$.

V. Examples and Discussions

We test the performance of Algorithms 2 and 3 over various systems starting with the case of an integrator chain network in Subsection V-A and following up with Erdős-Rényi random networks in Subsection V-B.

**Algorithm 3** Approximation algorithm for Problem (II).

**Input:** Set $\Delta_c$, maximum number of actuators $r$ such that $r \geq |\Delta_c|$, approximation error $\epsilon$ for Algorithm 2, Algorithm 2’s accuracy level $a$ for Algorithm 2, bisection’s accuracy level $a'$ for current algorithm, matrices $W_1, W_2, \ldots, W_n$.

**Output:** Actuator set $\Delta$

\[
\Delta \leftarrow \emptyset, l \leftarrow \operatorname{tr}(W_{(i)}^{-1}), u \leftarrow \operatorname{tr}(W_{(i)}^{-1}), E \leftarrow (l + u)/2, \epsilon \leftarrow 1/E
\]

while $u - l > a'$ do

\[
\Delta \leftarrow \text{[Algorithm } 2(E, c, a)\text{)}
\]

if $|\Delta| > r$ then

\[
l \leftarrow E, E \leftarrow (l + u)/2, \nu \leftarrow E, E \leftarrow (l + u)/2
\]

else

\[
\nu \leftarrow E, E \leftarrow (l + u)/2
\]

end if

\[
\epsilon \leftarrow 1/E
\]

end while

if $|\Delta| > r$ then

\[
l \leftarrow E, E \leftarrow (l + u)/2
\]

end if

\[
\Delta \leftarrow \text{[Algorithm } 2(E, c, a)\text{)}
\]
A. Integrator Chain Network

We illustrate the mechanics and efficiency of Algorithms 2 and 3 using the integrator chain in Fig. 1, where

\[ A = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 & -1
\end{bmatrix}. \]

We first run Algorithm 2 for \( t_0 = 0, t_1 = 1, \) and \( a, c \leftarrow .0001. \) The algorithm returned the actuator set \( \{1,3\}; \) as expected, node 1 is chosen since for a chain network to be controllable 1 must be an actuator. Moreover, \( \{1,3\} \) is the best actuator set, as it follows by comparing the values below that were computed using MATLAB®:

- \( \text{tr}(W_{(1)}^{-1}) = 8.5175 \cdot 10^7, \)
- \( \text{tr}(W_{(1,2)}^{-1}) = 3.3243 \cdot 10^5, \)
- \( \text{tr}(W_{(1,3)}^{-1}) = 2.4209 \cdot 10^4, \)
- \( \text{tr}(W_{(1,4)}^{-1}) = 2.4221 \cdot 10^3, \)
- \( \text{tr}(W_{(1,5)}^{-1}) = 3.3594 \cdot 10^2. \)

Therefore, node 1 does not satisfy by itself the bound \( E, \) while \( \text{tr}(W_{(1,3)}^{-1}) \) not only satisfies this bound but also takes the smallest value among all the actuator sets of cardinality two that induce controllability; therefore, \( \{1,3\} \) is the best minimal actuator set to achieve the given transfer.

Furthermore, observe from these values that the average control energy by the minimum actuator set that induces controllability alone, that is, \( \{1\}, \) is of order four times larger than the best actuator set that includes just one more node, that is, \( \{1,3\}; \) this illustrates that even for the simple network of Fig. 1 it is important to select its actuators for efficient control energy performance and not controllability alone.

Finally, by setting \( E \) large enough in Algorithm 2 so that the bound \( \text{tr}(W_{(1)}^{-1}) \leq E \) is satisfied by any set in the domain of \( \{1\}, \) we observe that only node 1 is selected, as expected for the controllability constraint of this domain to be met.

We next run Algorithm 3 for \( t_0 = 0, t_1 = 1, a, a', c \leftarrow .0001, \) and \( r \) being equal to 1, 2, 3, 4 or 5, respectively; node 1 is always selected, while for every value of \( r \) the chosen actuator set coincides with the one that has the same size and minimizes \( \text{tr}(W_{(1)}^{-1}). \) That is, we again observe optimal performance by our algorithm.

Furthermore, by increasing \( r \) from 1 to 3, the corresponding value of \( \text{tr}(W_{(1)}^{-1}) \) decreases from \( 85.1750 \cdot 10^5 \) to \( 81.7134, \) a difference of six orders. Therefore, we notice again that for a system to be efficiently controllable its actuators must be chosen for bounded control effort and not controllability alone.

B. Erdős-Rényi Random Networks

Erdős-Rényi random graphs are commonly used to model real-world networked systems. According to this model, each edge is included in the generated graph with some probability \( p \) independently of every other edge. We implemented this model for varying network sizes \( n \) where the directed edge probabilities were set to \( p = 2 \log(n)/n, \) following [9]. In particular, we first generated the binary adjacencies matrices for each network size so that each edge is present with probability \( p \) and then we replaced every non-zero entry with an independent standard normal variable to generate a randomly weighted graph.

To avoid the computational difficulties associated with the integral equation (2), we worked with the controllability Gramian instead, which for a stable system can be efficiently calculated from the Lyapunov equation \( AG + GA^T = -BB^T \) and is given in closed-form by

\[ G = \int_{t_0}^{\infty} e^{At}BB^Te^{A^Tt} \, dt. \]

Using the controllability Gramian in (3) corresponds to the minimum state transfer energy with no time constraints. Therefore, we stabilized each random instance of \( A \) by subtracting 1.1 times the real part of their right-most eigenvalue and then we used the MATLAB® function gram to compute the corresponding controllability Gramians.

Next, we set \( c \leftarrow 0.1 \) and \( a \leftarrow 1. \) Then, we run Algorithm 2 for \( n \) equal to 10, 40, 70 and 100, respectively, and \( E \) equal to \( k \cdot \text{tr}(G(n)^{-1}) \), where \( k \) ranged from 2 to \( 2^{50}; \) \( \text{tr}(G(n)^{-1}) \) is the lower bound of \( E \) so that (4) is feasible for each \( n \) (cf. [6]). The corresponding number of actuators that Algorithm 2 selected with respect to \( k \) is shown in Fig. 2, where the horizontal axis is in logarithmic scale. Finally, we also set \( a' \leftarrow 1. \) and run Algorithm 3 for \( n \) equal to 10, 40, 70 and 100, respectively (for each generated network of size \( n \), we first run Algorithm 2 for \( E \) \( \leq 10^{18} \), \( \text{tr}(G(n)^{-1}) \), and \( c, a \) as above, and we found a \( \Delta_C(n) \) such that \( |\Delta_C(n)| = 1 \); the corresponding achieved values for Problem (11) with respect to \( r \) are found in Fig. 3, where the vertical axis is in logarithmic scale.

In Fig. 2 we observe that as \( k \) increases the number of actuators decreases, as one would expect when the energy bound of (4) is relaxed. Moreover, for \( k \) large enough, so that (4) becomes equivalent to the minimal controllability problem of [9], the number of chosen actuators is one; i.e., Algorithm 2 outperforms the theoretical guarantees of Theorem 3. Similarly, in Fig. 3 we observe that as the number of available actuators \( r \) increases the minimum achieved value also decreases, as expected by the monotonicity and super-modularity of \( \text{tr}(W_{(1)}^{-1}) \), while as \( r \) decreases the minimum achieved value blows up.

Hence, Figs. 2 and 3 complement the results of Section 5.4.

For large \( E \) all the networks are controllable from one node, while as \( E \) decreases, that is, as the constraint on the control effort becomes more strict, the number of necessary actuators increases; similarly, as the number of available actuators \( r \) decreases, the minimum control effort increases, and in fact almost doubly-exponentially fast. Moreover, for each \( n \), the
Fig. 2: Number of selected actuators by Algorithm 2 in Erdős-Rényi random networks of size $n$ and for varying energy bounds $E$ (the horizontal axis is in logarithmic scale); for each $n$ the values of $E$ are chosen so that the feasibility constraint (6) of Problem (I) is satisfied: Specifically, for each $n$ and $k$, Algorithm 2 is executed for $E \leftarrow k \cdot \text{tr}(G(n)^{-1}V^{-1})$, where $G(n)V$ is the controllability Gramian corresponding to the generated network of size $n$ when all of its nodes are actuated.

Fig. 3: The minimum achieved value for $\text{tr}(W^{-1}_c)$ by Algorithm 3 in Erdős-Rényi random networks of size $n$ and for varying number of available actuators $r$ (the vertical axis is in logarithmic scale).

V. C. V. C.

We addressed two actuator placement problems in a linear system: First, the problem of minimal actuator placement so that an average control energy bound along all the directions in the state space is satisfied, and then the problem of cardinality-constrained actuator placement for minimum average control effort. Both problems were shown to be NP-hard, while for the first one we provided a best-approximation algorithm; this algorithm returns an actuator set that is up to a multiplicative factor of $O(\log n)$, where $n$ is the size of the system, from any optimal actuator set that meets the same energy criteria. Next, we proposed an efficient approximation algorithm for the solution of the second problem as well. Finally, we illustrated our analytical findings using an integrator chain network and demonstrated their efficiency over large Erdős-Rényi random networks. Our future work is focused on exploring the effect that the underlying network topology of the involved system has on these actuator placement problems, as well as investigating distributed implementations of their corresponding algorithms.
A. The Greedy Algorithm used in the Supermodular Minimization Literature is Inefficient for solving Problem (II)

Consider Algorithm 4 which is in accordance with the supermodular minimization literature [23], [24], [25].

Algorithm 4 Greedy algorithm for Problem (II).

Input: Maximum number of actuators \( r \), approximation parameter \( \epsilon \), number of steps that the algorithm will run \( l \), matrices \( W_1, W_2, \ldots, W_n \).

Output: Actuator set \( \Delta_l \)

\[
\begin{align*}
\Delta_0 &\leftarrow \emptyset, i \leftarrow 0 \\
&\text{while } i < l \text{ do} \\
&\quad a_i \leftarrow \arg\max_{a \in V \setminus \Delta} \left\{ \text{tr}(W_{\Delta_i} + \epsilon I)^{-1} - \text{tr}(W_{\Delta_i \cup \{a\}} + \epsilon I)^{-1} \right\} \\
&\quad \Delta_{i+1} \leftarrow \Delta_i \cup \{a_i\}, i \leftarrow i + 1 \\
&\text{end while}
\end{align*}
\]

The following is true for its performance.

Fact 1. Let \( v^* \) denote the value of Problem (II). Then, Algorithm 4 guarantees that for any positive integer \( l \)

\[
\text{tr}(W_{\Delta_l} + \epsilon I)^{-1} \leq (1 - e^{-1/r})v^* + \frac{ne^{-1/r}}{\epsilon}.
\]

Proof: It follows from Theorem 9.3, Ch. III.3.9. of [23], since \(-\text{tr}(W_{\Delta} + \epsilon I)^{-1} + ne^{-1}\) is a non-negative, non-decreasing, and submodular function with respect to the choice of \( \Delta \) (cf. Proposition 2).

Therefore, Algorithm 4 suffers from an error term that is proportional to \( e^{-1} \). On the other hand, if we set \( l = 5r \), then we reduce the error term \( ne^{-1/r}/\epsilon \) from 0.37n/\( \epsilon \) to 0.01n/\( \epsilon \); however, this is achieved at the expense of violating the cardinality constraint. Moreover, it is possible that Algorithm 4 returns an actuator set that does not render (I) controllable. Therefore, Algorithm 4 is inefficient for solving Problem (II).

REFERENCES

[1] M. Newman, A.-L. Barabási, and D. Watts, The structure and dynamics of networks. Princeton University Press, 2006.
[2] A. M. Herrmannstad, D. S. Bassett, K. S. Brown, E. M. Aminoff, D. Clewett, S. Freeman, A. Frithsen, A. Johnson, C. M. Tipper, M. B. Miller et al., “Structural foundations of resting-state and task-based functional connectivity in the human brain,” Proceedings of the National Academy of Sciences, vol. 110, no. 15, pp. 6169–6174, 2013.
[3] Y. Katz, K. Tunström, C. C. Ioannou, C. Huepe, and I. D. Couzin, “Inferring the structure and dynamics of interactions in schooling fish,” Proceedings of the National Academy of Sciences, vol. 108, no. 46, pp. 18720–18725, 2011.
[4] F. Jordán and I. Scheuring, “Network ecology: topological constraints on ecosystem dynamics,” Physics of Life Reviews, vol. 1, no. 3, pp. 139–172, 2004.
[5] G. Orosz, J. Moehlis, and R. M. Murray, “Controlling biological networks by time-delayed signals,” Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences, vol. 368, no. 1911, pp. 439–454, 2010.
[6] I. Rajapakse, M. Grouidine, and M. Membabi, “What can systems theory of networks offer to biology?” PLoS computational biology, vol. 8, no. 6, p. e1002543, 2012.
[7] A. Khanafar and T. Basar, “Information spread in networks: Control, games, and equilibria,” in Information Theory and Applications Workshop (ITA), 2014. IEEE, 2014, pp. 1–10.
[8] L. Chen, N. Li, L. Jiang, and S. H. Low, “Optimal demand response: problem formulation and deterministic case,” in Control and Optimization Theory for Electric Smart Grids, A. Chakrabortty and M. Ilic, Eds. Springer, 2012.
[9] A. Oshevsky, “Minimal controllability problems,” IEEE Transactions on Control of Network Systems, 2014, in press.
[10] G. Ramos, S. Pequito, S. Kar, A. P. Aguiar, and J. Ramos, “On the approximate completeness of the minimal controllability problem,” arXiv preprint arXiv:1401.4209, 2014.
[11] T. H. Summers, F. L. Cortesi, and J. Lygeros, “On submodularity and controllability in complex dynamical networks,” ArXiv e-prints, Apr. 2014.
[12] F. Pasqualetti, S. Zampieri, and F. Bullo, “Controllability metrics, limitations and algorithms for complex networks,” IEEE Transactions on Control of Network Systems, vol. 1, no. 1, pp. 40–52, March 2014.
[13] C.-T. Chen, Linear System Theory and Design, 3rd ed. New York, NY, USA: Oxford University Press, Inc., 1998.
[14] G. Yan, J. Ren, Y.-C. Lai, C.-H. Lai, and B. Li, “Controlling complex networks: How much energy is needed?” Phys. Rev. Lett., vol. 108, p. 218703, May 2012.
[15] J. Sun and A. E. Motter, “Controllability transition and nonlocality in network control,” Phys. Rev. Lett., vol. 110, p. 208701, May 2013.
[16] P. Muller and H. Weber, “Analysis and optimization of certain qualities of controllability and observability for linear dynamical systems,” Automatica, vol. 8, no. 3, pp. 237 – 246, 1972.
[17] V. Tsoumas, M. A. Rahimian, G. J. Pappas, and A. Jadbabaie, “Minimal actuator placement with optimal control constraints,” in Proceedings of the American Control Conference, 2014, submitted.
[18] S. Jafari, A. Ajourou, and A. G. Agidam, “Leader localization in multi-agent systems subject to failure: A graph-theoretic approach,” Automatica, vol. 47, no. 8, pp. 1744–1750, 2011.
[19] C. Comnou and J.-M. Dion, “Input addition and leader selection for the controllability of graph-based systems,” Automatica, vol. 49, no. 11, pp. 3322 – 3328, 2013.
[20] A. Clark, B. Alomair, L. Bushnell, and R. Poovendran, “Minimizing convergence error in multi-agent systems via leader selection: A supermodular optimization approach,” IEEE Transactions on Automatic Control, vol. 59, no. 6, pp. 1480–1494, June 2014.
[21] A. Clark, L. Bushnell, and R. Poovendran, “A supermodular optimization framework for leader selection under link noise in linear multi-agent systems,” IEEE Transactions on Automatic Control, vol. 59, no. 2, pp. 283–296, Feb 2014.
[22] D. S. Bernstein, Matrix mathematics: theory, facts, and formulas. Princeton University Press, 2009.
[23] G. L. Nemhauser and L. A. Wolsey, Integer and Combinatorial Optimization. New York, NY, USA: Wiley-Interscience, 1988.
[24] L. A. Wolsey, “An analysis of the greedy algorithm for the submodular set covering problem,” Combinatorica, vol. 2, no. 4, pp. 385–393, 1982.
[25] A. Krause and D. Golovin, “Submodular function maximization,” Tractability: Practical Approaches to Hard Problems, vol. 3, p. 19, 2012.
[26] U. Feige, “A threshold of \( \ln n \) for approximating set cover,” J. ACM, vol. 45, no. 4, pp. 634–652, Jul. 1998.