ON THE RELAXATION BEHAVIORS OF SLOW AND CLASSICAL GLITCHES:
OBSEVATIONAL BIASES AND THEIR OPPOSITE RECOVERY TRENDS

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Received 2012 December 16; accepted 2013 September 10; published 2013 October 31

Abstract

We study the pulsar timing properties and the data analysis methods of glitch recoveries. In some cases one first fits the times of arrival (TOAs) to obtain the “time-averaged” frequency $\nu$ and its first derivative $\dot{\nu}$, and then fits models to them. However, our simulations show that $\nu$ and $\dot{\nu}$ obtained in this way are systematically biased unless the time intervals between the nearby data points of TOAs are smaller than about $10^4$ s, which is much shorter than typical observation intervals. Alternatively, glitch parameters can be obtained by fitting the phases directly with relatively smaller biases, but the initial recovery time scale is usually chosen by eye, which may introduce a strong bias. We also construct a phenomenological model by assuming a pulsar spin-down law of $\dot{\nu} = -H_0 \kappa e^{-t/\tau}$ for a glitch recovery, where $H_0$ is a constant and $\kappa$ and $\tau$ are the glitch parameters to be found. This model can reproduce the observed data of slow glitches from B1822+09 and the giant classical glitch of B2334+61, with $\kappa < 0$ or $\kappa > 0$, respectively. We then use this model to simulate TOA data and test several fitting procedures for a glitch recovery. The best procedure is (1) to use a very high order polynomial (e.g., to 50th order) to precisely describe the phase, (2) obtain $\nu(t)$ and $\dot{\nu}(t)$ from the polynomial, then (3) obtain the glitch parameters from $\nu(t)$ or $\dot{\nu}(t)$. Finally, the uncertainty in the starting time $t_0$ of a classical glitch causes uncertainties in some glitch parameters, but less so for a slow glitch, $t_0$, of which can be determined from data.

Key words: magnetic fields – pulsars: general – pulsars: individual (B1822+09, B2334+61) – stars: neutron

Online-only material: color figures

1. INTRODUCTION

Pulsars are very stable rotators. However, many pulsars exhibit significant timing irregularities, i.e., unpredicted arrival times of pulses. There are two main types of timing irregularities, namely timing noise, which consists of low-frequency quasi-periodic structures, and glitches, which are abrupt increases in the spin rates followed by relaxations.

Glitch activities are more frequent in relatively young pulsars with a characteristic age of $10^3$–$10^5$ yr (Shemar & Lyne 1996; Wang et al. 2000). For the hundreds of glitches observed, the typical fractional jumps in spin frequency $\nu$ are in the range of $\Delta \nu/\nu \approx 10^{-11}$–$10^{-5}$, and the relative increment in frequency derivative is $\Delta \dot{\nu}/\dot{\nu} \sim 10^{-2}$. Despite the abundance of observational data accumulated for over 40 years, we are still far from a satisfactory understanding of glitch events. Traditional models mainly involve the expected superfluid nature of part of the neutron star interior (Anderson & Itoh 1975; Ruderman 1976), and the angular momentum is carried in the form of microscopic, quantized vortices whose density determines the rotation rate of a pulsar. Mostly, these vortices are pinned to the crust and the charged matter in the core of the star, thus their outward drifting motions are prevented (Anderson & Itoh 1975; Alpar 1977; Pines et al. 1980; Alpar et al. 1981; Anderson et al. 1982). However, as the crust spins down due to electromagnetic braking, a rotational lag and stress (Magnus force) gradually builds up. A glitch occurs when the stress reaches some critical value and the pinning breaks; vortices suddenly move outward and impart their angular momentum to the crust. Immediately after the glitch, the vortices are pinned to other parts again and the superfluid is effectively decoupled from the crust.

Following the seminal work of Baym et al. (1969), there are two classes of models that have been developed to explore the dynamical evolution of pinned superfluids during the post-glitch recovery. One kind of model involves a weak coupling between the superfluid and the crust due to the interaction between free vortices and the coulomb lattice of nuclei (Jones 1990, 1992, 1998). Another kind of model assumes that the vortices’ creep rate is highly temperature-dependent. As the vortices creep through the crust, angular momentum is gradually transferred (Alpar et al. 1984a, 1984b; Link et al. 1993; Larson & Link 2002). Superfluid vortex dynamics can model the relaxation well; however, many significant problems remain unsolved. For instance, the mechanism that triggers the glitch in the first place and the detailed processes of angular momentum transfer during the recovery are still controversial. It has been suggested that such an event may be triggered by large temperature perturbations (Link & Epstein 1996), or caused by starquakes (Baym & Pines 1971; Cheng et al. 1992) the interactions of the proton vortices and the crustal magnetic field (Sedrakian & Cordes 1999), or superfluid $r$-mode instability (Andersson et al. 2003; Glampedakis & Andersson 2009).

Very recently, Pizzochero (2011) proposed an analytic model for angular momentum transfer associated with Vela-like glitches for the storage and release of superfluid vorticity, and Seveso et al. (2012) and Haskell et al. (2013) extended this model to realistic equations of state and relativistic backgrounds. Haskell et al. (2012) further modeled all stages of Vela glitches with a two-fluid hydrodynamical approach. Furthermore, Haskell & Antonopoulou (2013) showed that if glitches are indeed due to large-scale unpinning of superfluid vortices, the different regions in which the unpinning occurs and the respective time scales on which they recouple can lead to various observed jump and relaxation signatures. However, by combining the latest observational data for prolific glitching pulsars...

4 http://www.atnf.csiro.au/people/pulsar/psrcat/glitchTbl.html
with theoretical results for the crust entrainment, Andersson et al. (2012) found that the required superfluid reservoir exceeds that available in the crust. Coincidentally, Chamel (2013) found that the glitches observed in the Vela pulsar require an additional reservoir of angular momentum since the maximum amount of angular momentum that can possibly be transferred during glitches is severely limited by non-dissipative entrainment effects. This challenges the superfluid vortex model of glitch phenomena. Some of the glitch events, such as those with persistent offset in the spin-down rate of the Crab pulsar following the 1975 glitch are difficult to explain via dynamic coupling between the crust and the superfluid interior. An alternative explanation of the observed frequency deficit is an increase in the external torque caused by a rearrangement of the stellar magnetic field (Link et al. 1992a, 1998). Observationally, many pulsar phenomena, including mode changing, pulsar-shape variability, and spin-down rate switching, are caused by changes in a pulsar’s magnetosphere (Lyne et al. 2010). Thus these relaxation processes may also be produced by magnetospheric activities induced by initial starquakes.

It has also been observed in recent years that some pulsars (e.g., PSR J1825–0935 and PSR J1835–1106) show another type of irregularity characterized by a gradual increase in $\nu$, accompanied by a rapid decrease in $\dot{\nu}$ and a subsequent exponential increase back to its initial value (Zou et al. 2004; Shabanova 2005). This is the so-called “slow glitch.” Currently, there is still no convincing theoretical understanding of slow glitches. Peng & Xu (2008) proposed that, after a collapse or a small starquake, the solid superficial layer of a rigid quark star may be heated and becomes a viscous fluid, which eventually produces a gradual increase in $\nu$. However, Hobbs et al. (2010) and Lyne et al. (2010) argued that slow glitches have the same origin as the timing noise of many pulsars.

For the recovery processes of both glitches and slow glitches, the variations of spin frequency $\nu$ and the first derivative $\dot{\nu}$ of pulsars are obtained from polynomial fit results of arriving time epochs of pulses. The local times of arrival (TOAs) of the mean pulses for individual observing sessions are determined from the maximum cross correlation between the observed mean pulses and a Gaussian profile template. The template profile is a mean pulse with high signal-to-noise ratio, obtained by summing the best-quality mean pulses over several observing sessions. Correction of TOAs at the solar system barycenter can be done using the TEMPO2 program with the Jet Propulsion Laboratory DE405 ephemeris (Standish 1998). These TOAs are then weighted by the inverse squares of their estimated uncertainty. Since the rotational period is nearly constant, these observable quantities, $\nu$, $\dot{\nu}$, and $\ddot{\nu}$ can be obtained by fitting the phases to the third order of their Taylor expansion over a time span $t_s$,

$$
\Phi_i = \Phi + \nu(t_i - t) + \frac{1}{2} \dot{\nu}(t_i - t)^2 + \frac{1}{6} \ddot{\nu}(t_i - t)^3. \tag{1}
$$

One can thus get the values of $\nu$, $\dot{\nu}$, and $\ddot{\nu}$ at $t$ from fitting to Equation (1) for independent $N$ data blocks around $t$, i.e., $i = 1, \ldots, N$. Apparently, the observational quantities obtained this way are not instantaneous results, but rather the results “averaged” over each data block (i.e., over each $t_s$) and extrapolated to $t$, which are not necessarily the same as the instantaneous values (denoted as $\nu_i$ and $\dot{\nu}_i$). Thus, they are called “averaged” values (denoted as $\nu_A$ and $\dot{\nu}_A$) in this work.

Usually, $t_s$ is much less than the pulsar’s spin-down age $\tau_c$, thus the differences between instantaneous values and “averaged” values are not significant. Consequently, $\nu_A$ and $\dot{\nu}_A$ are good approximations for $\nu_i$ and $\dot{\nu}_i$ in most cases. However, it has been found recently that oscillations of the “apparent” magnetic fields of neutron stars are responsible for the observed signs and magnitudes of $\ddot{\nu}$, the second derivative of frequency, and braking indices (Biryukov et al. 2012; Pons et al. 2012; Zhang & Xie 2012a, 2012b). We further suggested that the oscillation time scales are between 10 and 100 yr, comparable to $\tau$, thus making the fitted spin-down parameters different from the true and instantaneous spin-down parameters. Similarly, considerable biases may also exist when fitting the glitch recovery data, since the glitch recovery time scales are also comparable to $\tau$.

In Section 2, we simulate several pulsar timing data analysis procedures for glitch recoveries and find that the glitch parameters, obtained from the averaged $\nu_A$ and $\dot{\nu}_A$, have significant systematic biases compared with those obtained with the instantaneous $\nu_i$ and $\dot{\nu}_i$. In order to get the true glitch parameters with the reported yet averaged glitch recovery data $\nu_0$ and $\dot{\nu}_0$, a phenomenological or physical glitch model is needed to be combined with simulations. We thus present a phenomenological spin-down model during a glitch recovery, and model several slow glitch recovery events and the recovery of a giant classical glitch in Section 3. In Section 4, we test four fitting procedures based on the phenomenological spin-down model and find that the best method is taking a very high order polynomial to fit the phase and then taking its derivatives to obtain $\nu(t)$ and $\dot{\nu}(t)$. In Section 5, we discuss how to obtain the model parameters of glitch recoveries more accurately. The results are summarized in Section 6.

2. SIMULATING DATA ANALYSIS OF GLITCH RECOVERIES

2.1. Simulation for $\dot{\nu}$-fitting Procedure

By fitting the TOA set \{\Phi(t_i)\} to Equation (1), one can get \{\nu(t_i)\} and \{\dot{\nu}(t_i)\}. When \{\nu(t_i)\} and \{\dot{\nu}(t_i)\} show exponential relaxations, their variations following the jump at epoch $t_0$ can be described as the following empirical functions (e.g., Yuan et al. 2010; Roy et al. 2012):

$$
\nu(t) = \nu_0(t) + \Delta\nu_p + \Delta\dot{\nu}_p \Delta t + \frac{1}{2} \Delta\ddot{\nu}_p \Delta t^2 + \sum_j \Delta\nu_{\dot{\nu}j} e^{-\Delta t / \tilde{\tau}_j}, \tag{2}
$$

and

$$
\dot{\nu}(t) = \dot{\nu}_0(t) + \dot{\Delta}\nu_p + \dot{\Delta}\ddot{\nu}_p \Delta t + \sum_j \dot{\Delta}\nu_{\dot{\nu}j} e^{-\Delta t / \tilde{\tau}_j}, \tag{3}
$$

where $\Delta t = t - t_0$, $\Delta\nu_p$ and $\Delta\dot{\nu}_p$ are permanent changes in $\nu$ and $\dot{\nu}$ relative to the pre-glitch solution $\nu_0(t)$ and $\dot{\nu}_0(t)$, $\Delta\nu_{\dot{\nu}j}$ is the amplitude of the $j$th decaying component with a time constant $\tilde{\tau}_j$, and $\dot{\Delta}\nu_{\dot{\nu}j} = -\Delta\nu_{\dot{\nu}j} / \tilde{\tau}_j$. One can obtain the glitch parameters $\Delta\nu_p$, $\Delta\dot{\nu}_p$, $\Delta\ddot{\nu}_p$, $\Delta\nu_{\dot{\nu}j}$, $\tilde{\tau}_j$, and $\Delta\nu_{\dot{\nu}j}$ by fitting $\nu(t)$ and $\dot{\nu}(t)$ to Equations (2) and (3), respectively. The two functions describe the post-glitch behaviors fairly well, especially for the case of a long-term recovery, and usually multiply decay terms with different decay time constants can be fitted (e.g., there are up to five exponentials fitted for Vela 2000 and 2004 glitches; Dodson et al. 2002, 2007). For simplicity, cases where $\nu$ varies as either one or two exponential decay terms are assumed in the following simulations.

\footnote{http://www.atnf.csiro.au/research/pulsar/tempo2}
Slow glitches are characterized by a gradual increase in $v$ with a long time scale of several months, accompanied by a rapid decrease in $|v|$ by a few percent, which is sometimes even shorter than the observation interval and thus cannot be seen. Then $|v|$ experiences an exponential increase back to its initial value with the same time scale as that of $v$ increase (Shabanova 2005). Analogous to the classical glitches, we suggest that the slow glitches can be described by the following two functions:

$$v(t) = v_0(t) + \Delta v_p + \Delta \dot{v}_p \Delta t + \frac{1}{2} \Delta \ddot{v}_p \Delta t^2 + \sum_{j} \Delta v_{ij}(1 - e^{-\Delta t/\tau_j}),$$

and

$$\dot{v}(t) = \dot{v}_0(t) + \Delta \dot{v}_p + \Delta \ddot{v}_p \Delta t + \sum_{j} (-\Delta \ddot{v}_{ij}) e^{-\Delta t/\tau_j},$$

where the parameters are the same as those in Equations (2) and (3).

### 2.1.1. Simulation for One Decay Term

Since glitch or slow glitch recoveries can be described by Equations (2)–(5), some simple models can also be derived from them. For a classical glitch, we simply assume

$$v(t) = \Delta v_d \exp (-\Delta t/\tau),$$

i.e., $v_0 = \Delta v_d = \Delta \dot{v}_p = \Delta \ddot{v}_p = 0$. We will use this equation to produce simulated data, and obtain the “instantaneous” $v_i = v(t)$ and $\dot{v}_i = d(v(t))/dt$, with the parameters ($\Delta v_d$ and $\tau$) given later. On the other hand, the “averaged” values are obtained by the following procedure. First, we obtain the phase by $\Phi(t) = \int_0^t v(t') dt'$. For convenience we take $t_0 = 0$. However, in practice $t_0$ cannot be known precisely due to discontinuous observations; we will show later that this causes some uncertainty in estimating the parameters of classical glitches, but not slow glitches. We assume that a certain time interval $\Delta T_{\text{int}}$ between two nearby TOAs, i.e., $\Delta T_{\text{int}} \equiv t_{i+1} - t_i$ is a constant. We set 10 adjacent TOAs in one block (i.e., $N = 10$ in Equation (1)), and the latter 5 TOAs are used as the first 5 TOAs in the next block. We then fit the TOA blocks to Equation (1) to obtain $v_A$ and $\dot{v}_A$, which are the fitted coefficients of $v$ and $\dot{v}$, respectively, in the equation. The time $t$ for $v_A$ and $\dot{v}_A$ is taken as the middle epoch of each block, i.e., $t = (t_5 + t_6)/2$, and is also “averaged” (e.g., Yuan et al. 2010).

In Figure 1, we show these instantaneous values and averaged values with different $\Delta T_{\text{int}}$ for a glitch with $\Delta v_d = 0.1 \mu$Hz and $\tau = 50$ days. One can see that both $v_A$ and $\dot{v}_A$ have remarkably different decay profiles from $v_1$ and $\dot{v}_1$, respectively, during the recovery process. These systematic biases are independent of $\Delta T_{\text{int}}$, and it seems that the recovery time scale $\tau$ is the key parameter that is mainly biased. By fitting $v_A$ and $\dot{v}_A$ to Equations (2) and (3), respectively, we find that all the recovery time scales of $v_A$ and $\dot{v}_A$ are much longer than the time scale of 50 days (e.g., $\tau \approx 95$ day for $\Delta T_{\text{int}} = 10^4$ s). The systematic differences between the decay profiles of $v_A$ or $\dot{v}_A$ and the profiles of $v_1$ or $\dot{v}_1$ are considerable, and apparently caused by the procedure of fitting TOAs to Equation (1); thus, for higher order fits, one cannot consider the first-order coefficient to be the frequency. This procedure is thus abandoned for glitch data analysis in the following sections.

However, with the TEMPO2 software, $v$ may be obtained from the TOAs by fitting to

$$\Phi_i = \Phi + v(t_i - t),$$

and $\dot{v}$ may be obtained by fitting to

$$\Phi_i = \Phi + v(t_i - t) + \frac{1}{2} (v(t_i - t))^2,$$

i.e., the first two or three terms of Equations (1), respectively (M. Yu 2013, private communication). Here, we first fit the TOA blocks to Equation (7) to obtain $v_A$, which is the fitted coefficient of $v$ in Equation (7). We then separately fit the TOA blocks to Equation (8) to obtain $\dot{v}_A$, which is the fitted coefficient of $\dot{v}$ in Equation (8). In the left panels of Figure 2, we show the instantaneous and averaged values obtained this way, with different $\Delta T_{\text{int}}$ for a glitch with the same $\Delta v_d$ and $\tau$. Clearly now the profiles of both $v_A$ and $\dot{v}_A$ follow those of $v_1$ and $\dot{v}_1$ with obvious distortions. By fitting to Equation (2) or Equation (3), we find that all the recovery time scales of $v_A$ or $\dot{v}_A$ equal a time scale of 50 days, i.e., $\tau$ has not been biased.

We can then obtain the normally reported glitch parameters $\Delta v_d$ and $\Delta \dot{v}_d$, as listed in Table 1, by fitting $\{v_A\}$ or $\{\dot{v}_A\}$ to Equation (2) or Equation (3) with different $\Delta T_{\text{int}}$; for comparison we also list $\Delta v_d$ and $\Delta \dot{v}_d$ obtained from $v_1$ and $\dot{v}_1$. One can see that the “averaged” $\Delta v_d$ or $\Delta \dot{v}_d$ (denoted as $\Delta v_{\text{A}}$ or $\Delta \dot{v}_{\text{A}}$ hereafter) systematically differs from the instantaneous $\Delta v_d$ or $\Delta \dot{v}_d$ (denoted as $\Delta v_{\text{d}}$ or $\Delta \dot{v}_{\text{d}}$ hereafter). For $\Delta T_{\text{int}} = 10^4$ s, the differences are small and the glitch parameters can be restored satisfactorily; however, for $\Delta T_{\text{int}} \geq 10^5$ s, both the “averaged” $\Delta v_d$ and $\Delta \dot{v}_d$ may be considerably smaller than the instantaneous $\Delta v_d$ and $\Delta \dot{v}_d$, respectively.

For a slow glitch, we assume

$$v(t) = \Delta v_d (1 - \exp(-\Delta t/\tau)),$$

and

\begin{align*}
\Phi_i &= \Phi + v(t_i - t) + \frac{1}{2} (v(t_i - t))^2, \\
\dot{v}_i &= \dot{v}_0(t) + \Delta \dot{v}_p + \Delta \ddot{v}_p \Delta t + \sum_{j} (-\Delta \ddot{v}_{ij}) e^{-\Delta t/\tau_j},
\end{align*}
where $t_0 = 0$ and $\Delta v_0 = 0.1 \mu Hz$ and $\tau = 50$ days. We show the averaged glitch parameters and profiles with different $\Delta T_{\text{int}}$ in Table 1 and the right panels of Figure 2, as well as those instantaneous ones. One can see that we always have $\Delta v_{\text{int}} = \Delta v_{\text{int}} A$ for any $\Delta T_{\text{int}}$, since $\Delta v_{\text{int}} A$ is determined by the differences of $|v_A|$ between the data points slightly before the starting point of the glitch and the data points at the end of the recovery, and both of them are always available for slow glitch observations. However, $\Delta v_{\text{int}} A$ is biased in the same way as for the simulated classical glitch.

2.1.2. Simulation for Two Decay Terms

We simply assume $v(t) = \Delta v_{\text{int}} \exp(-t/\tau_1) + \Delta v_{\text{int}} A \exp(-t/\tau_2)$ for a classical glitch with two decay terms, where $\Delta v_{\text{int}} = 0.19 \mu Hz$, $\tau_1 = 21.4$ days and $\Delta v_{\text{int}} A = 0.119 \mu Hz$, $\tau_2 = 147$ days (the parameters are adopted from pulsar B2334+61 for its very large glitch between 2005 August 26 and September 8; Yuan et al. 2010). We also assume $v(t) = \Delta v_{\text{int}} A (1 - \exp(-t/\tau_1)) + \Delta v_{\text{int}} A (1 - \exp(-t/\tau_2))$ for a slow glitch with two decay terms. The instantaneous values and averaged values are obtained with the same methods described above and the main results are presented in Table 1 and Figure 3, in which results similar to the case with one decay term can be found. For $\Delta T_{\text{int}} = 10^4$ s, the differences for $\tau$, $\Delta v_{\text{int}}$, and $\Delta v_{\text{int}} A$ are all small and the glitch parameters can be restored satisfactorily. However, things are a little more complicated for two decay terms. For $\Delta T_{\text{int}} \gtrsim 10^5$ s, though the data points still converged to the instantaneous values as shown in Figure 3 (i.e., variation trends are the same), the fitted glitch parameters (including $\tau$) for each component are still somewhat biased, and it seems that a larger $\Delta T_{\text{int}}$ corresponds to a smaller $\tau$ for a short time scale component. The biases are probably due to the fact that the data are too sparse for $\Delta T_{\text{int}}$. Actually if $\tau$ of the short term decay component is comparable to or shorter than the interval between the observations, then $\tau$ of this component would be difficult to determine and can only be set as the internal. Similar results can be found for a slow glitch, but $\sum \Delta v_{\text{int}} = \sum \Delta v_{\text{int}} A$ is always maintained.

The above simulations unveil significant biases caused by the averaging procedures (i.e., fitting to Equations (7) and (8)) for $v$ and $\dot{v}$ during glitch recoveries. Thus, $v_A$ and $\dot{v}_A$ obtained this way and $v$ and $\dot{v}$ (the subscript “o” means observed values) reported in the literature should not be used to test physical

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**Figure 2.** Variations of $v$ and $\dot{v}$ for a simulated classical glitch (left panels) and a simulated slow glitch (right panels) with one decay term. Solid lines represent the instantaneous values; circles, triangles, and crosses represent the averaged values obtained by fitting the simulated TOAs to Equations (7) (to get $v_A$) and (8) (to get $\dot{v}_A$) for the cases of $\Delta T_{\text{int}} = 10^4$, $10^5$, and $10^6$ s, respectively. (A color version of this figure is available in the online journal.)

**Table 1**

| Type of Glitch | $\Delta v_0$ (\(\mu Hz\)) | $\Delta v_0$ (\(10^{-15}\text{ Hz s}^{-1}\)) | $\tau$ (days) | $\Delta v_{\text{int}}$ (\(10^{-15}\text{ Hz s}^{-1}\)) | $\Delta v_{\text{int}} A$ (\(10^{-15}\text{ Hz s}^{-1}\)) | $\Delta T_{\text{int}}$ (days) |
|---------------|----------------|----------------|-------------|----------------|----------------|-----------------|
| Classical     |                |                |             |                |                |                 |
| Instantaneous | 0.100          | -23.15         | 50.00       | (0.19, 0.119)  | (-102.8, -9.4) | (21.4, 147.0)   |
| $\Delta T_{\text{int}} = 10^4$ s | 0.099          | -22.88         | 50.00       | (0.18, 0.119)  | (-100.2, -9.4) | (21.3, 145.8)   |
| $\Delta T_{\text{int}} = 10^5$ s | 0.089          | -20.67         | 50.00       | (0.13, 0.139)  | (-100.4, -16.8)| (14.5, 95.6)    |
| $\Delta T_{\text{int}} = 10^6$ s | 0.041          | -9.53          | 49.52       | (2.73, 0.081)  | (-2960.9, -6.4)| (10.7, 146.8)   |
| Slow          |                |                |             |                |                |                 |
| Instantaneous | 0.100          | 23.15          | 50.00       | (0.19, 0.119)  | (102.8, 9.4)   | (21.4, 147.0)   |
| $\Delta T_{\text{int}} = 10^4$ s | 0.100          | 23.04          | 49.99       | (0.19, 0.122)  | (100.4, 16.8)  | (14.5, 95.7)    |
| $\Delta T_{\text{int}} = 10^5$ s | 0.100          | 20.67          | 49.99       | (0.19, 0.122)  | (100.4, 16.8)  | (14.5, 95.7)    |
| $\Delta T_{\text{int}} = 10^6$ s | 0.100          | 9.88           | 49.36       | (0.25, 0.055)  | (2959.0, 6.4)  | (10.7, 146.8)   |

**Notes.** The data on the left and the right are results from using simulations with one decay term and two decay terms, respectively. “Instantaneous” represents the instantaneous values and the different values of $\Delta T$ represent the time intervals between each TOA for the “averaged” values.
models directly. It should be noted that, for cases with one decay term, the reported amplitudes of $\Delta \nu$ and $\Delta \dot{\nu}$ of a classical glitch are usually underestimated; the reported amplitude of $\Delta \dot{v}$ for a slow glitch is also underestimated, but not for $\Delta \nu$. However, these biases were never noticed in almost all previous theoretical works modeling glitch recoveries, and $[\hat{v}(t)]$ are usually directly modeled, e.g., the post-glitch fits for the Vela pulsar, the Crab pulsar, and PSR 0525+21 with the vortex creep model (Alpar et al. 1984a, 1985, 1993, 1996; Chau et al. 1993; Larson & Link 2002), and the two-component hydrodynamic model for Vela (van Eysden & Melatos 2010). In these works, the observed data $[\hat{v}(t)]$ (i.e., $[v_A(t)]$) are shown in the $\nu$–$\dot{v}$ diagram and fitted directly by theoretical models.

2.2. Simulation for Phase-fitting Procedure

In order to make optimum use of all available data (Shemar & Lyne 1996), the pulse phase induced by a glitch is usually fitted to the following equation, which can give $\tau$ and $\Delta \nu_d$ (e.g., Yu et al. 2013):

$$\phi = \Delta \phi + \nu_p \Delta t + \frac{1}{2} \nu_p \Delta t^2 + [1 - e^{-(\Delta \nu_d / \tau)}] \Delta \nu_d \tau_i, \tag{10}$$

where $\nu_p$ and $\hat{v}_p$ are the permanent increments in $\nu$ and $\dot{v}$, respectively. However, it is difficult to obtain $\tau_i$ directly by fitting to Equation (10). TEMPO2 implements only a linear fitting algorithm, and one thus needs to have a good initial estimate for $\tau_i$, which is estimated from post-glitch $\dot{v}$ variation inspected by eye. Then the estimated value was introduced into Equation (10) fits. By increasing or decreasing $\tau_i$, a best estimated $\tau_i$ can be found eventually via minimum post-fit $\chi^2$ (Yu et al. 2013). This procedure is widely used for classical glitches, but is not applied to slow glitches.

We simulate the fitting procedure of Equation (10) as described above, and find that both $\tau$ and $\Delta \nu_d$ can be obtained with high precision for the case of one decay term, if a good initial estimate for $\tau$ is taken, as shown in Table 2. For the case of two decay terms, we also assume $v(t) = \Delta \nu_d \exp(-t/\tau_1) + \Delta \nu_d \exp(-t/\tau_2)$, where $\Delta \nu_d = 0.119 \mu$Hz, $\tau_1 = 147$ days and $\Delta \nu_d = 0.19 \mu$Hz, $\tau_2 = 21.4$ days, and obtain the phase via $\Phi(t) = \int \nu(t) dt$. First, we estimate $\tau_1$ for the long-term phase, and get the best-fit $\tau_1$ by fitting $[\Phi(t)]$ to Equation (10). Then we fix $\tau_1$ and obtain the time scale for the short-term $\tau_2$ the same way. This process is widely adopted in glitch recovery data analysis with the TEMPO2 software (Yu et al. 2013). However, we find that the glitch parameters of the long decay term in the cases with two terms, i.e., $\tau_1$ and $\Delta \nu_d$, obtained using this method are already biased as shown in Table 2.

These biases are probably caused by the procedure of fitting the long decay term and the short decay term in different steps; the short decay term may slightly interfere with the first fitting for $\tau_1$ and $\Delta \nu_d$ of the long decay term, thus the results are biased. If the biased $\tau_1$ is fixed, one will also get a biased $\tau_2$ to fit $[\Phi(t)]$ to Equation (10) again, since a local minimum $\chi^2$ will be obtained as shown in Figure 4. Therefore, we suggest that the two terms should be fitted simultaneously.

If $\Delta T_{\text{late}} \lesssim 10^5$ s, the simultaneous fits can be implemented through the following steps.

1. Get $[\hat{v}]$ series by fitting $[\Phi(t)]$ to Equation (8).
2. Estimate $\tau_1$ and $\tau_2$ by fitting $[\hat{v}]$ to Equation (3) (the calculation cost needed for this fit is much lower than fitting to Equation (10)).

![Figure 3. Variations of $\nu$ and $\dot{v}$ for a simulated classical glitch (left panels) and a simulated slow glitch (right panels) with two decay terms. Solid lines represent the instantaneous values; circles, triangles, and crosses represent the averaged values obtained by fitting the simulated TOAs to Equations (7) (to get $v_A$) and (8) (to get $\nu_A$) for the cases of $\Delta T_{\text{late}} = 1.5 \times 10^6$, $10^5$, and $10^4$ s, respectively. (A color version of this figure is available in the online journal.)](image-url)
3. Use the estimated \( t_1 \) and \( t_2 \) as initial values and fit \( \Phi(t_i) \) to Equation (10) again thereby obtaining the best fit \( t_1 \) and \( \Delta \nu_{\text{fit}} \).

The results of the simultaneous fit are \( \Delta \nu_{\text{fit}} = 0.119 \mu\text{Hz} \), \( t_1 = 147.0 \) days and \( \Delta \nu_{\text{fit}} = 0.190 \mu\text{Hz} \), \( t_2 = 21.4 \) days, which are exactly the same as those introduced in the model; the results are independent from \( \Delta T_{\text{int}} \). We also simulate the fitting processes with different values of \( t_1 \) and \( \Delta \nu_{\text{fit}} \), and all the glitch parameters are restored with relatively small biases, some of which are even better than the previous procedures adopted for \( \Delta T_{\text{int}} \lesssim 10^3 \) s. Here, we emphasize that the fitting procedures described in the literature are confusing. Many authors adopted the pulsar parameters, using Equation (8) to get \( T \) moment of inertia \( G \), but the procedure should be described in detail.

3. A PHENOMENOLOGICAL SPIN-DOWN MODEL

In this section, we develop a phenomenological spin-down model to describe the glitch and slow glitch recoveries, so that the model can be a tool for simulating data to test the data analysis procedures for the recoveries in the next section. Classically, a magnetic dipole with a magnetic moment \( M = BR^3 \), rotating in vacuum with angular velocity \( \Omega \), emits electromagnetic radiation with a total power \( 2M^2\Omega^2/3c^3 \). Assuming pure magnetic dipole radiation as the braking mechanism for a pulsar’s spin-down, the energy loss rate is then given by

\[
\dot{E} = I\Omega \dot{\Omega} = -\frac{2(BR^3\sin \chi)^2}{3c^3} \Omega^4, \tag{11}
\]

where \( B \) is its dipole magnetic field at its magnetic pole, \( R \) is its radius, and \( I \) is its moment of inertia. Equation (11) is modified slightly in order to describe a glitch event,

\[
\dot{\Omega} \Omega^{-3} = -\frac{2(BR^3\sin \chi)^2}{3c^3} G(t), \tag{12}
\]

in which \( G(t) \) represents very small changes in the effective strength of a dipole magnetic field \( B \) sin \( \chi \), or the effective moment of inertia \( I \) of both the pulsar and its magnetosphere during a glitch recovery. On the left side of the equation are observable quantities, and on the right side are theoretical quantities. In the following we assume \( G(t) = 1 + \kappa e^{-\Delta \tau / \tau} \). Then Equation (12) can be written as

\[
\dot{v}^{-3} = -H_0(1 + \kappa e^{-\Delta \tau / \tau}), \tag{13}
\]

where \( \dot{v} = \dot{\Omega} / 2\pi \) and \( H_0 = (8\pi^2(BR^3 / 3c^3) I) = 1/2\tau_1\nu_0^2 \), and \( \tau_c = -v / 2 \dot{v} \) is the characteristic age of a pulsar.

Integrating and solving Equation (13), we have

\[
\nu(t) = \frac{\nu_0}{\sqrt{1 + (\Delta \tau / \kappa \tau (1 - e^{-\Delta \tau / \tau}) / \tau_c)^3/2}}. \tag{14}
\]

The derivative of \( \nu \) is

\[
\dot{\nu}(t) = -\frac{(1 + \kappa e^{-\Delta \tau / \tau}) \nu_0 / 2\tau_c}{(1 + (\Delta \tau + \kappa \tau (1 - e^{-\Delta \tau / \tau}) / \tau_c)^3/2)}. \tag{15}
\]

We know \( \Delta \tau \sim \tau \sim 100 \) days and generally \( \tau_c \gtrsim 10^4 \) yr, and the term \( (|\Delta \tau / \kappa \tau (1 - e^{-\Delta \tau / \tau}) / \tau_c| \ll 1 \) and \( \kappa \ll 1 \), the expression of \( v \) and \( \dot{v} \) can be approximately written in the same forms of Equations (2) and (3), which give \( \Delta \nu_{\text{fit}} = \nu_0 \kappa \tau / 2\tau_c \) and \( \Delta \nu_{\text{fit}} = -\nu_0 \kappa / 2\tau_c \). Numerical calculations show that Equations (2) and (3) with these parameters giving identical results as Equations (14) and (15) for all known ranges of glitch parameters. The expression \( \Delta \nu_{\text{fit}} \) and \( \Delta \nu_{\text{fit}} \) which relate to the initial jumps of \( \nu_0 \) and \( \dot{\nu}_0 \) are not given by the model since only the glitch relaxation processes are considered here. It has been suggested that these non-recoverable jumps are the consequence of a permanent dipole magnetic field increase during the glitch event (Lin & Zhang 2004). \( \Delta \nu_{\text{fit}} \) is the jump of the timing residual, which is beyond the scope of the present work.

We attempt to apply this phenomenological model to fit the reported data of several slow glitches in B1822−09 and one classical glitch in B2334+61. Since the reported data points of \( v \) and \( \dot{v} \) are too sparse (about one point per 150 days for B1822−09 or 50 days for B2334+61) and the TOAs of these glitches are not available in the literature, we cannot apply our model to fit the reported values to obtain both \( \tau \) and \( \kappa \) simultaneously, as done in the above simulations. As a compromise, we focus only on determining \( \kappa \) by applying our phenomenological model and we simply take (the inevitably biased) \( \tau \) obtained by directly fitting the reported \( v \) and \( \dot{v} \). Therefore, \( \kappa \) remains the only glitch recovery parameter to be determined from observations. Our main purpose here is to show the applicability of our phenomenological model to glitch observations.

3.1. Modeling Several Slow Glitches of B1822−09

We first model slow glitches because they are simpler than classical glitches; they have no jumps in \( v \) and \( \dot{v} \), i.e., \( \Delta \nu_{\text{fit}} = 0 \) and \( \Delta \nu_{\text{fit}} = 0 \). Shabanova (2005) reported three slow glitches in B1822−09 (J1825−0935) over the 1995−2004 interval. The pulsar has \( v \approx 1.3 \) s, a relatively large \( |\dot{v}| \approx 8.878 \times 10^{-14} \) s−2 (note \( \nu < 0 \)), implying \( \tau_c \approx 232 \) kyr and \( B \approx 6.43 \times 10^{12} \) G. As shown in Figure 5, the pulsar experienced three slow glitches from 1995 to 2005. A gradual increase in \( v \) is well modeled by an exponential function with time scales of 235, 80, and 110 days, respectively. For \( \dot{v} \), the fractional decreases of \( |\dot{v}| \) (i.e., increases of \( \dot{v} \)) are about 0.7%, 2.7%, and 1.7%, respectively. The subsequent increases of \( \dot{v} \) (i.e., decreases of \( \dot{v} \)) back to the previous values with the same time scales are also well described by exponential functions. The third slow glitch was separately detected by Zou et al. (2004).
Since the detailed data on $\Delta T_{\text{int}}$ are not reported in the literature, we assume a uniform TOA distribution with $\Delta T_{\text{int}} = 3.5 \times 10^5$ s. We take the following steps in modeling the observed data for each slow glitch event.

1. We obtain our model-predicted TOAs with $\Delta T_{\text{int}}$ by integrating Equations (2) or (14), with $\kappa$ for each slow glitch event.

2. We simulate the data analysis process by fitting every block of 10 adjacent TOAs to Equations (7) or (8) to obtain one set of $\nu_A$ and $\nu_{A}$; and the latter 5 TOAs are also used in the next TOA block.

3. The above simulated $\nu_A$ and $\nu_{A}$ are compared with the reported glitch profile $\nu_0$ and $\nu_{0}$; $\kappa$ is adjusted until reasonable agreements between them are reached.

With the above steps, we confirm that the slow glitch behavior can be explained by our phenomenological model with $\kappa < 0$. Our modeling results are shown in Figure 5. The fit parameter $\kappa$ is $-0.0093$, $-0.06$, and $-0.04$ for the three slow glitch events, respectively. In Table 3 we show the relative magnitudes of $\Delta \nu$ and $\Delta \nu$ for the three slow glitches; for comparison we also list in Table 3 the results for the giant classical glitch from B2334+61 obtained in the next section. It is found that the relative magnitudes of $\Delta \nu_A$, $\Delta \nu_{O}$, and $\Delta \nu_{0}$ are identical, i.e., $\Delta \nu_A = \Delta \nu_{O} = \Delta \nu_{0}$, as expected from the above simulations. It is also clear that the instantaneous values of $\Delta \nu$, which are calculated directly from the model with the parameters determined above, are much larger than the reported results in the literature, e.g., $\Delta \nu_A$ are larger than twice $\Delta \nu_{0}$ for the second and third slow glitches.

3.2 Modeling the Classical Glitch of B2334+61

The pulsar PSR B2334+61 (PSR J2337+6151) was discovered in the Princeton-NRAO survey using the 92 m radio telescope at Green Bank in 1985 (Dewey et al. 1985). It has $\nu \simeq 2.019$ s$^{-1}$, $\nu \simeq -788.332 \times 10^{-15}$ s$^{-2}$, $\tau_c \simeq 4.1 \times 10^4$ yr, and $B \simeq 9.91 \times 10^{12}$ G. It is located very close to the center of the supernova remnant G114.3+0.3. Yuan et al. (2010) reported the timing observations of PSR B2334+61 for 7 yr with the Nanshan 25 m telescope at Urumqi Observatory. A very large glitch occurred between 2005 August 26 and September 8 (MJDs 53608 and 53621), the largest known glitch ever observed, with a fractional frequency increase of $\Delta \nu/\nu \simeq 20.5 \times 10^{-6}$. Yuan et al. (2010) obtained each $\nu$, $\nu$, and $\nu$ by fitting 10 adjacent TOAs to Equation (1), and the latter 5 TOAs had also been used as the first five TOAs in the next fit. The rotational behavior during

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**Table 3**

Relative Values of $\Delta \nu$ and $\Delta \dot{\nu}$ for Slow Glitches of B1822–09 and the Giant Classical Glitch from B2334+61

| Case                  | Slow Glitches of B1822–09 |         |         |         |         |         |         |         |         |         |         |         |         |
|-----------------------|---------------------------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
|                       | $\Delta \nu_0 / \nu_0$   | $\Delta \dot{\nu}_0 / \nu_0$ | $\Delta \nu_1 / \nu_0$ | $\Delta \dot{\nu}_1 / \nu_0$ | $\Delta \nu_2 / \nu_0$ | $\Delta \dot{\nu}_2 / \nu_0$ | $\Delta \nu_3 / \nu_0$ | $\Delta \dot{\nu}_3 / \nu_0$ | $\Delta \nu_4 / \nu_0$ | $\Delta \dot{\nu}_4 / \nu_0$ | $\Delta \nu_5 / \nu_0$ | $\Delta \dot{\nu}_5 / \nu_0$ | $\Delta \nu_6 / \nu_0$ | $\Delta \dot{\nu}_6 / \nu_0$ |
| Reported              | 12.9                      | 0.7     | 28.6    | 2.7     | 25.2    | 1.7     | 75.8    | -2.96   | 75.8    | -2.96   | 35.6    | -3.15   | 54.5    | -2.87   |
| Simulated             | 13.2                      | 0.94    | 29.7    | 6.0     | 25.4    | 4.0     | 64.4    | -3.85   | 79.8    | -3.98   | 2.87    | 54.5    | -2.87   |
| Instantaneous         | 13.2                      | 0.94    | 29.7    | 6.0     | 25.4    | 4.0     | 64.4    | -3.85   | 79.8    | -3.98   | 2.87    | 54.5    | -2.87   |

Notes. For classical glitches, the superscripts “i” and “ii” represent for results of one-term and two-term fits, respectively.
this glitch event is shown in Figure 6. A large jump in rotational frequency can be seen in the top panel with $\dot{\nu} \approx 41 \times 10^{-6}$ Hz. The bottom panel shows a very significant long-term increase in $|\dot{\nu}|$ after the time of the jump, and the corresponding braking indices are $10.5 \pm 0.2$ and $46.8 \pm 0.3$ before and after the glitch, respectively. The recovery process following the glitch was described by a dominant rapid exponential decay with a time scale of $\approx 21.4$ days and an additional slower decay with a time scale of $\approx 147$ days (Yuan et al. 2010).

We follow almost the same steps as for the slow glitches to model the reported, time-averaged glitch recovery data of this classical glitch, with $\Delta T_{\text{int}} = 1.8 \times 10^5$ s. The only difference is that a slope of $\Delta \dot{\nu}_p = -8.684 \times 10^{-15}$ s$^{-2}$ is taken in Equation (2), following Lyne et al. (2000).

In the left panels of Figure 6, we show the fits with one exponential term $G(t) = (1 + \kappa \exp(-\Delta/t))$ for a comparison with the “realistic” simulation of two terms below. The best parameters for this glitch event are $\kappa_1 = 0.038$ and $\tau = 50$ days. We then model the glitch recovery process with $G(t) = (1 + \kappa_1 \exp(-\Delta/t_1) + \kappa_2 \exp(-\Delta/t_2))$, as shown in the right panels of Figure 6. The best parameters for this glitch event are $\kappa_1 = 0.027$ and $\kappa_2 = 0.012$ ($t_1 = 21.4$ days and $t_2 = 147$ days are fixed by the observed values). Table 3 gives the relative magnitudes of $\Delta \dot{\nu}$ and $\Delta \nu$ for both fits. In order to distinguish between the two fits, we show them in the logarithmic coordinates in Figure 7. Clearly the simulated profiles of the two-term fit match the reported ones better than those of the one-term fit. One can see that $|\Delta \dot{\nu}_1|$ are also slightly larger than the reported $|\Delta \dot{\nu}_0|$ for both the one-term fit and the two-term fit.

In Figure 8 we show $\Delta \nu$ with the slope of $\Delta \dot{\nu}_p$ removed, and $\nu_0 - \dot{\nu}_1$. It is clearly shown that one exponential term cannot fit the observed data at the end of the decay profile, and this is also the reason why $\Delta \nu_2$ is smaller than $\Delta \nu_0$ for this fit, as given in Table 3. Thus, the one-term decay is ruled out, and we focus on the two-term fit below. Using $\nu_0$, $\tau$, and the determined $\kappa_1$ and $\kappa_2$, we obtain the glitch parameters $\Delta \nu_{\text{GL1}} = 0.039$ $\mu$Hz, $\Delta \nu_{\text{GL2}} = 0.119$ $\mu$Hz, $\Delta \dot{\nu}_{\text{GL1}} = -2.1 \times 10^{-14}$ s$^{-2}$, $\Delta \dot{\nu}_{\text{GL2}} = -9.38 \times 10^{-15}$ s$^{-2}$. Some values differ significantly from the reported results of Yuan et al. (2010) in which $\Delta \nu_{\text{GL1}} = 0.19$ $\mu$Hz, $\Delta \nu_{\text{GL2}} = 0.119$ $\mu$Hz, $\Delta \dot{\nu}_{\text{GL1}} = -1.03 \times 10^{-13}$ s$^{-2}$, $\Delta \dot{\nu}_{\text{GL2}} = -9.37 \times 10^{-15}$ s$^{-2}$.

In Figure 8, one can also see an exponential increase of $\nu$ after the glitch recovery, which is a very common but not well-understood behavior (Lyne et al. 1992, see an example of a Crab glitch). We suggest that the exponential increase component is probably a slow glitch, and the fact that a slow glitch follows a classical glitch recovery may be an important clue to the enigma of glitch phenomena.

4. TESTING SEVERAL FITTING PROCEDURES BASED ON THE PHENOMENOLOGICAL SPIN-DOWN MODEL

In Section 3, we showed that the recovery processes of glitches and slow glitches can be well modeled by the phenomenological model, which can also be used to simulate real glitch recoveries, in order to fully test different fitting procedures. We obtain $\Phi(t)$ by integrating Equation (13) for a certain $\tau$ and $\kappa$, and obtain the TOA set $\{\Phi(t_i)\}$ by assuming a certain $\Delta T_{\text{int}}$. In this section, we test the biases produced by the following four fitting procedures; three of them are discussed in Section 2 for the simplified model of classical and slow glitches. Here all four procedures are examined with a more “realistic” model, i.e., our phenomenological spin-down model.

Fitting Procedure I: obtain $\{\dot{\nu}(t_i)\}$ by fitting $\{\Phi(t_i)\}$ to Equation (1), and get $\tau$ and $\Delta \nu_0$ by fitting $\{\dot{\nu}(t_i)\}$ to Equation (3)
Figure 7. Giant glitch of pulsar B2334+61. Observational results are taken from Yuan et al. (2010). $\Delta T_{\text{int}} = 1.8 \times 10^5$ s is adopted in the fit. Upper panels: variations of $\Delta \nu$. Bottom panels: variations of $\dot{\nu}$. The left and right panels represent models with one and two decay components, respectively. This figure is the same as Figure 6 except for a logarithmic abscissa.
(A color version of this figure is available in the online journal.)

Figure 8. Giant glitch of pulsar B2334+61. Upper panels: variations of $\Delta \nu$ from which the slope $\Delta \dot{\nu}_p$ is removed. Bottom panels: residuals of $\dot{\nu}_O - \dot{\nu}_I$. The left and right panels represent models with one and two decay components, respectively.
(A color version of this figure is available in the online journal.)
Table 4
Classical Glitch Simulations with the Phenomenological Model

| Time Interval   | $T_1 = 1$ yr | $T_1 = 3$ yr | $T_1 = 5$ yr |
|-----------------|--------------|--------------|--------------|
| $\tau$          | $\Delta\nu_d$ | $\tau$          | $\Delta\nu_d$ | $\tau$          | $\Delta\nu_d$ |
| Instantaneous    | 50.00        | 1.01         | 50.00        | 1.01         | 50.00        | 1.01         |
| $\Delta T_{\text{int}} = 10^4$ s | 213.38       | 1.15         | 96.45        | 2.35         | 92.34        | 2.22         |
| $\Delta T_{\text{int}} = 10^5$ s | 161.99       | 6.49         | 90.74        | 2.51         | 87.52        | 2.11         |
| $\Delta T_{\text{int}} = 10^6$ s | 27.02        | 3.68         | 37.75        | 3.07         | 38.75        | 3.04         |

Notes. $\Delta\nu_d$ (0.1 $\mu$Hz) and $\tau$ (days) obtained by Fitting Procedure I.

Table 5
Classical Glitch Simulations with the Phenomenological Model

| Time Interval   | $T_1 = 1$ yr | $T_1 = 3$ yr | $T_1 = 5$ yr |
|-----------------|--------------|--------------|--------------|
| $\tau$          | $\Delta\nu_d$ | $\tau$          | $\Delta\nu_d$ | $\tau$          | $\Delta\nu_d$ |
| Instantaneous    | 50.00        | 1.01         | 50.00        | 1.01         | 50.00        | 1.01         |
| $\Delta T_{\text{int}} = 10^4$ s | 49.95       | 1.00         | 49.97        | 1.00         | 49.98        | 1.00         |
| $\Delta T_{\text{int}} = 10^5$ s | 45.84       | 0.95         | 47.82        | 0.96         | 48.01        | 0.96         |
| $\Delta T_{\text{int}} = 10^6$ s | 22.23       | 5.02         | 27.37        | 3.15         | 27.85        | 2.42         |

Notes. $\Delta\nu_d$ (0.1 $\mu$Hz) and $\tau$ (days) obtained from Fitting Procedure II.

Table 6
Classical Glitch Simulations with the Phenomenological Model

| Time Interval   | $T_1 = 1$ yr | $T_1 = 3$ yr | $T_1 = 5$ yr |
|-----------------|--------------|--------------|--------------|
| $\tau$          | $\Delta\nu_d$ | $\tau$          | $\Delta\nu_d$ | $\tau$          | $\Delta\nu_d$ |
| Instantaneous    | 50.00        | 1.01         | 50.00        | 1.01         | 50.00        | 1.01         |
| $\Delta T_{\text{int}} = 10^4$ s | 50.16       | 1.02         | 53.13        | 0.99         | 67.23        | 0.83         |
| $\Delta T_{\text{int}} = 10^5$ s | 50.16       | 1.02         | 53.11        | 0.99         | 67.01        | 0.84         |
| $\Delta T_{\text{int}} = 10^6$ s | 50.15       | 1.02         | 52.86        | 1.00         | 65.29        | 0.87         |

Notes. $\Delta\nu_d$ (0.1 $\mu$Hz) and $\tau$ (days) obtained from Fitting Procedure III.

can see that the instantaneous values can be well restored for $\Delta T_{\text{int}} = 10^4$ s, and the results with $\Delta T_{\text{int}} = 10^5$ s are also good approximations. Then, we conduct the two-component case and use $\tau_1 = 21.7$ days, $\tau_2 = 147$ days, and $\kappa_1 = 0.131$, $\kappa_2 = 0.012$ in the model. One can see that the instantaneous values can only be restored for $\Delta T_{\text{int}} = 10^4$ s. It is noted that $T_1$ should be long enough for both the one-component and two-component cases. Thus, this procedure is a good approximation for very small $\Delta T_{\text{int}}$.

Fitting Procedure III: get $\tau$ and $\Delta\nu_d$ directly by fitting $\{\Phi(t_i)\}$ to Equation (10) (Yu et al. 2013; Edwards et al. 2006; Shemar & Lyne 1996). First, we also perform a fit with one decay component using $\tau = 50$ days, $\kappa = 0.03$ in the model. The main results are shown in the upper part of Table 6. It is found that the instantaneous values can be well restored and the fit results are nearly independent of $\Delta T_{\text{int}}$. Then, we perform a fit with two components using $\tau_1 = 21.7$ days, $\tau_2 = 147$ days, and $\kappa_1 = 0.131$, $\kappa_2 = 0.012$ in the model. We fit the two decay terms simultaneously (at a high computing cost) and the main results are shown in the bottom part of Table 6. It is also found that the instantaneous values can be restored satisfactorily and the results are independent of $\Delta T_{\text{int}}$. However, $T_1$ should not be too long for either the one-component or the two-component cases, which is opposite of procedure II.

Fitting Procedure IV: the phase $\{\Phi(t_i)\}$ is fitted by a very high order polynomial, such as

$$\Phi(t) = \Phi_0 + \nu(t - t_i) + \frac{1}{2} \nu(t - t_i)^2 + \frac{1}{6} \nu(t - t_i)^3 + \cdots + \frac{1}{50!} \nu(t - t_i)^{50}.$$  

The fitted polynomial $\Phi(t_i)$ (a continuous function) can very precisely describe the TOA series $\{\Phi(t_i)\}$. One can then take...
its first or second derivative to obtain \( \nu \) or \( \dot{\nu} \), i.e., \( \nu = \Phi(t) \) or \( \dot{\nu} = \Phi'(t) \). This procedure was suggested by the anonymous referee.

We also simulate the one-component and two-component cases, respectively. The results are shown in Figure 9. One can see that the instantaneous values of \( \nu(t) \) or \( \dot{\nu}(t) \) can be restored with very high precision for both cases. In the figure, \( \Delta T_{\text{int}} = 10^5 \) s is taken, and it is checked that the results are almost independent of both \( \Delta T_{\text{int}} \) and \( \Delta \nu \). Then, one can get \( \tau_1 \) and \( \Delta \nu_0 \) by fitting the restored \( \nu(t) \) to Equation (3). The fitted glitch parameters for a one-component case are \( \tau = 50.00 \) days and \( \Delta \nu_1 = 1.01 \times 10^{-7} \) Hz, and \( \tau_1 = 21.37 \) days, \( \Delta \nu = 146.93 \) days and \( \Delta \nu_2 = 1.92 \times 10^{-7} \) Hz, \( \Delta \nu_3 = 1.19 \times 10^{-7} \) Hz for a two-component case. They are all consistent with instantaneous values (see, e.g., Table 6) with very high precision. In Figure 9, we show the fitting results of a two-component case with different order polynomials. It is found that the order of the polynomial must be very high, e.g., \( \geq 35 \), which requires that the TOA data points should not be too sparse. We also test the fit procedure with different values of \( \tau_1 \) and \( \Delta \nu_0 \), and all the glitch parameters are restored satisfactorily.

In conclusion, Procedure III is a reasonable choice to get \( \tau_1 \) and \( \Delta \nu_0 \); however, the two components should be fit simultaneously (in order to avoid some local minimum of \( \chi^2 \)), and \( \tau_0 \) should not be too long. Procedure IV seems to be the best choice for pulsar glitch data analysis, which gives \( \nu(t) \) and \( \dot{\nu}(t) \) with very high precision. The glitch parameters \( \tau_1 \) and \( \Delta \nu_2 \) can be satisfactorily estimated by fitting the restored \( \nu(t) \) to Equation (3). We thus suggest that theorists should always use the full timing solution rather than trying to compare models to individual parameters of fits, as these may be highly inaccurate. Furthermore, working in phase seems to be the most accurate and reliable method.

5. DISCUSSIONS

5.1. How to Obtain the Correct Model Parameters of Pulsars

We have shown recently that fitting the observed TOAs of a pulsar to Equation (1) will result in biased (i.e., averaged) spin-down parameters, if the spin-down is non-secular and the variation time scale is comparable to or shorter than the time span of the fitting (Zhang & Xie 2012a, 2012b). In particular, we predicted that the reported braking index should be a function of time span and approaches a small positive value when the time span is much longer than the oscillation period of its spin-down process, which can be tested with the existing data (Zhang & Xie 2012b).

We note that in some of the literature (e.g., Roy et al. 2012; Espinoza et al. 2011; Yuan et al. 2010) only Equation (1) is referred to when describing the fitting process, even for the glitch data analysis. However, we have shown in Figure 1 that this will produce a significantly distorted glitch profile. Instead, one can fit to Equations (7) and (8) to obtain the un-distorted (but still averaged) glitch profile; probably this is usually done in practice, though not explicitly described in the literature (M. Yu, 2013, private communication). We suggest that the exact fitting procedures should be described when reporting the analysis results of the observed glitch data.

However, neither Equation (1) nor Equations (7) and (8) exactly describe the physical spin-down processes of all pulsars. The pulsar parameters fitted by Equations (10) are also slightly biased especially if \( \tau_0 \) is not properly taken. Ideally, a spin-down model (empirical, phenomenological, or physical) should be used to fit the observed TOA data in order to obtain the model parameters. To serve this purpose, the observed TOAs of each pulsar should be made available, and the exact fitting procedure should be described along with the reported spin-down parameters of a pulsar. As shown in Figure 10, we simulate a glitch recovery with parameters \( \tau_1 = 21.4 \) days, \( \tau_2 = 147 \) days, and \( \Delta \nu_1 = 1.90 \times 10^{-7} \) Hz, \( \Delta \nu_2 = 1.19 \times 10^{-7} \) Hz, and \( \Delta T_{\text{int}} = 10^5 \) s. Then we have TOAs from the phenomenological model, and the simulated “reported” \( \nu' \) is obtained by fitting TOAs to Equation (8) and is represented by the solid line. By fitting TOAs to Equation (10), we have the “reported” glitch parameters \( \tau_1 = 21.8 \) days, \( \tau_2 = 152 \) days. With these time scales, we simulate \( \nu' \) again with \( \Delta T_{\text{int}} = 10^5 \). The model parameters \( \Delta \nu_1 \) and \( \Delta \nu_2 \) can be adjusted until simulated fits match the “reported” ones, and the best-fit model parameters \( \Delta \nu_1 = 1.92 \times 10^{-7} \) Hz, \( \Delta \nu_2 = 1.20 \times 10^{-7} \) Hz, which agree well with original parameters. We show the restored \( \nu' \) with circles. With the same parameters, the restored \( \nu' \) for \( \Delta T_{\text{int}} = 5 \times 10^4 \) and \( 2 \times 10^5 \) s are represented by diamonds and...
triangles, respectively. One can see that $\dot{v}$ can be well restored if TOAs are known. If $\Delta T_{\text{int}}$ taken in simulation is not the right one, the $\dot{v}$ profiles are apparently different from the "reported" one, even though the model parameters are all correct.

When TOAs are available, one can then follow the steps we used above to combine a model with simulations to obtain model parameters. Alternatively, $\Phi(t)$, $v(t)$, and $\dot{v}(t)$ given by Procedure IV can also be fitted directly by physical models.

5.2. The Effects of Discontinuous Observations

In the above analysis, we have assumed that $t_0$ is known; however, $t_0$ is usually taken as the averaged time of the last reported TOA just before the glitch and the first reported TOA of the glitch. This means we have an uncertainty in $t_0$: $\sigma_{t_0} = \Delta T_{\text{int}}/2$. Then from Equation (6) for a classical glitch, we find

$$\sigma_{\Delta \nu} = \frac{s_{\Delta \nu}}{\Delta \nu} = \frac{s_{\Delta \nu}}{\tau},$$

where $s_{\Delta \nu}$ and $s_{\Delta \nu}$ are the uncertainties of the restored $\Delta \nu$ and $\Delta \dot{\nu}$, respectively. For the classical glitch of B2334+61, $\sigma_{t_0} \sim 4.2$ days, $\tau \sim 21.4$ days. Thus from Equation (17), we have $s_{\Delta \nu}/\Delta \nu = s_{\Delta \nu}/\Delta \dot{\nu} \approx 20\%$.

In principle, we have $s_{\Delta \nu} \approx 0$ for a slow glitch since $\Delta \nu$ is determined by the data at the end of the recovery, i.e., $\nu \sim \Delta \nu_0$ for $\Delta \nu \gg \tau$ from Equation (9). However, from the derivative of Equation (9), we have $\dot{v} = (\Delta \nu_0/\tau) e^{-t/(\Delta \nu_0/\tau)}$ and $\Delta \nu_0/\tau$ is closely related to $t_0$, which resembles the case of a classical glitch. However, for the slow glitch we can fit for $t_0$ of the glitch by calculating where the rise and the pre-glitch solutions intersect, which will cause a much smaller uncertainty. This is a major difference from analyzing the data of a classical glitch. Unfortunately, this has not been realized previously and thus $t_0$ was not determined from the reported $\nu_0$ with this method for slow glitch data analysis. This causes an uncertainty to $\Delta \nu$ in the same way as in Equation (17), i.e., the bias is related to $\Delta T_{\text{int}}$. For instance, in Figure 2 of Zou et al. (2004), the observed results for a slow glitch event of B1822–09 are $\Delta \nu^a = (40.57 \pm 26) \text{ nHz}$ and $\Delta \dot{\nu}^a \approx 3.1 \times 10^{-15} \text{ s}^{-2}$; and for the same event, the results in Shabanova (2005) are $\Delta \nu^b = 40.8 \text{ nHz}$ and $\Delta \dot{\nu}^b \approx 1.4 \times 10^{-15} \text{ s}^{-2}$. As expected above, $\Delta \nu^a = \Delta \nu^b$, but $\Delta \nu^a \neq \Delta \nu^b$. For the event, $\tau \sim 110$ days, and $s_{\Delta \nu}^a \approx 5.5$ days, $s_{\Delta \nu}^b \sim 22.8$ days. From Equation (17), we obtain $s_{\Delta \nu}^a \approx 1.6 \times 10^{-16} \text{ s}^{-2}$, $s_{\Delta \nu}^b \approx 6.4 \times 10^{-15} \text{ s}^{-2}$, and $s_{\Delta \nu} = \sqrt{(s_{\Delta \nu}^a)^2 + (s_{\Delta \nu}^b)^2} \approx 6.6 \times 10^{-16} \text{ s}^{-2}$. Then we have $(\Delta \nu^a - \Delta \nu^b)/s_{\Delta \nu} \approx 2.6$, which at least partially explains the difference between the reported values of $s_{\Delta \nu}$.

5.3. Opposite Recovery Trends for Slow and Classical Glitches

Based on observational results, we generalize the variations of $\nu$ and $\dot{v}$ for slow and classical glitch recoveries as shown in Figure 11. The pre-glitch tracks are represented by the dotted line. After the jump, the classical glitch recoveries (represented by the solid line) generally have variation $\nu$ which tends to restore its initial values, and usually the restoration is composed of an exponential decay and a permanent linear decrease with slope $\Delta \nu_p$; however, for slow glitches (represented by the dashed line), $\nu$ monotonically increases, as shown in panel (1). In panel (2), $\dot{v}$ of classical glitch recoveries tends to restore its initial values, but cannot completely recover for $\Delta \nu_p \neq 0$; $\dot{v}$ of slow glitch recoveries almost completely recovers to its initial value, corresponding to the increase in $\nu$.

Figure 11. Schematic depictions of $\nu$, $\dot{v}$, and $G(t)$ for the slow and classical glitch recoveries. The pre-glitch tracks are represented by the dotted line. The classical glitch recoveries are represented by solid lines. The slow glitches are represented by dashed lines.

In Sections 3 and 4, we have shown that the classical and slow glitch recoveries can be well modeled by a simple function, $G(t) = 1 + \kappa \exp(-\Delta \nu/\tau)$, with positive or negative $\kappa$, as shown in panel (3), respectively. However, it is should be noted that the model has only two parameters, $\kappa$ and $\tau$, from which we can obtain $\Delta \nu$ and $\Delta \dot{\nu}$, but not $\Delta \nu_p$ and $\Delta \dot{\nu}_p$, which are not modeled. Nevertheless, we conclude that the major difference between slow glitch and classical glitch recoveries is that they show opposite trends with opposite signs of $\kappa$ in our phenomenological model.

6. SUMMARY

In this work we studied the data analysis procedures of pulsar glitch observations and found that the conventionally used methods produce biases on the true glitch parameters with varying degrees. We presented a phenomenological model for the recovery processes of classical and slow glitches, which is used to successfully model the observed slow and classical glitch events from pulsars B1822–09 and PSR B2334+61, respectively. Based on the model, we tested four different data analysis procedures. Our main results are summarized as follows.

1. The timing analysis method of fitting the observed TOAs with Equation (7) or Equation (8) results in significant biases to glitch parameters of variation magnitude, as shown in Figures 2 and 3 and Table 1. The biases can be ignored only when $\Delta T_{\text{int}} \lesssim 10^4 \text{ s}$; otherwise, biases still exist to some extent.
2. With Equation (10), one can obtain the glitch parameters by fitting the phase directly, which produces relatively smaller biases. However, for the case with multiple decay terms, the time scales are usually fixed by eye for their initial values, which may introduce strong biases.

3. We propose a phenomenological model of glitch recovery (Equation (13)), which can reproduce the commonly observed exponential glitch recovery profiles. The recovery processes of both slow and classical glitches can be explained as the \( G(t) = 1 + \kappa \exp(-\Delta t/\tau) \) with \( \kappa < 0 \) (Figure 5) or \( \kappa > 0 \) (Figures 6–8), respectively. Their opposite trends and main characteristics are illustrated in Figure 11.

4. Based on the phenomenological model, we simulate four fitting procedures and find that the best one involves taking a very high order polynomial to fit the phase and then taking its derivatives to obtain \( \nu(t) \) and \( \dot{\nu}(t) \). Then the glitch parameters can be obtained from \( \nu(t) \) and \( \dot{\nu}(t) \) (e.g., fitting \( \nu(t) \) to Equation (3)). We suggest that this procedure should be used in pulsar timing analysis.

5. The uncertainty in the starting time (\( t_0 \)) of a classical glitch causes uncertainties in the glitch parameters \( \Delta \nu_0 \) and \( \Delta \dot{\nu}_0 \) (Equation (17)), but less so for a slow glitch and \( t_0 \) of a slow glitch can be determined from data.

However, our phenomenological model cannot account for the non-recoverable jumps in \( \nu \) and \( \dot{\nu} \), which are observed for some classical glitches and may be due to the permanent increase of a pulsar’s dipole magnetic field due to glitches (Lin & Zhang 2004). In the work, we also assumed uniform TOA distributions to simulate both the slow and classical glitch recoveries since the observed TOAs are not reported in literature. The glitch parameters can be better restored if the observed TOAs are available and fitted directly with a glitch model; this is actually generally desired for pulsar timing studies. Thus we suggest that TOAs should be made available to the community when possible or that the full fitting procedure and fit parameters for different epochs made available. Theorists could also try to calculate phase as an output, thus making the comparison more accurate.

We thank Jianping Yuan and Meng Yu for valuable discussions. We thank the anonymous referee for comments and suggestions that led to a significant improvement in this paper. S.N.Z. acknowledges partial funding support by the 973 National Natural Science Foundation of China under grant Nos. 11133002 and 10725313, and by the Qianren start-up grant 292012312D1117210.

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