Navier–Stokes equations on a flat cylinder with vorticity production on the boundary

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Abstract

We study a singular version of the incompressible two-dimensional Navier–Stokes (NS) system on a flat cylinder $\mathbb{C}$, with Neumann conditions for the vorticity and a vorticity production term on the boundary $\partial \mathbb{C}$ to restore the no-slip boundary condition for the velocity $u|_{\partial \mathbb{C}} = 0$. The problem is formulated as an infinite system of coupled ordinary differential equations (ODEs) for the Neumann Fourier modes. For a general class of initial data we prove existence and uniqueness of the solution, and equivalence to the usual NS system. The main tool in the proofs is a suitable decay of the modes, obtained by the explicit form of the ODEs. We finally show that the resulting expansions of the velocity $u$ and of its first and second space derivatives converge and define continuous functions up to the boundary.

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1. Introduction

It is well known that the incompressible Navier–Stokes (NS) equations can be conveniently expressed in terms of the vorticity, thereby eliminating the pressure. In problems with boundaries, however, there is no usual boundary condition which can enforce the no-slip boundary condition for the velocity. In physical terms, adherence to the boundary creates a torque which imparts a rotational motion to the fluid, so that vorticity is ‘produced’ at the boundary, a phenomenon which is of great interest in understanding viscous fluid motion.

The ‘vorticity production method’ consists in writing an equation for the vorticity, and taking care of the Dirichlet (no-slip) boundary conditions by adding a suitable forcing term on the boundary (‘vorticity production’). It plays an important role in the applications, but, in spite of the considerable physical literature, following the pioneering work of Batchelor [1] and
Chorin [5], there are few rigorous results. In fact we know only of the contributions of Benfatto and Pulvirenti [2, 3], who proved equivalence to the usual NS equations and convergence of the Chorin computation method for the half-plane. Apparently, the influence of their work was somehow limited by the use of sophisticated mathematical tools.

In this paper we study the NS equations with vorticity production on the flat cylinder $\mathbb{C} := \mathbb{T} \times [0, \pi]$, where $\mathbb{T}$ is the one-dimensional torus, by using some simple ordinary differential equation (ODE) methods. Our paper is inspired by the recent works [6, 7], which, extending the classical works on the incompressible NS equations on the flat two-dimensional torus [10, 11, 14, 15], show how to obtain, by mainly elementary methods, deep results on the behaviour of the solutions to the plane NS system with suitable boundary conditions.

Our approach, as in the papers [6, 7, 11, 14], is based on a formulation of the problem as an infinite set of coupled ODEs for the Fourier modes. The eigenfunctions of the Laplacian with Neumann boundary conditions are a natural basis in our setting, and the corresponding expansion leads to an infinite system of coupled ODEs, much as in the papers [6, 7], except for the forcing term.

We get a discrete weak formulation of our singular NS problem, which is the basic object of this paper. Observe that for the classical NS problem the natural basis would be the eigenfunctions of the Stokes operator, which satisfy the no-slip ($\mathbf{u}|_{\partial \mathbb{C}} = 0$) condition, but they are not explicitly known, so that one cannot write a manageable system of ODEs.

We prove that, for suitable initial data, our system of ODEs has a unique global solution for the vorticity modes in the Neumann basis, and that the corresponding velocity field is a weak solution [15] of the usual NS problem.

The proof is based on a regularization property of the vorticity modes: if at the initial time $t = 0$ they decay slower than an inverse square, then for any $t > 0$ they decay exponentially fast in the periodic direction, and as an inverse square in the other direction. As the vorticity does not in fact satisfy the Neumann boundary conditions, the modes cannot decay faster than an inverse square.

For NS boundary value problems in bounded plane regions, with singular boundary and standard [8] or atypical [6] boundary conditions, a similar picture seems to hold for the decay of the Fourier modes, which is in general power law even if the basis does satisfy the boundary conditions. In such cases it is unknown whether the solution does actually become singular, whereas in our case, by proving equivalence to the usual NS problem, we can invoke the classical regularity results for the plane NS problem with smooth boundary [12], which ensures that for any $t > 0$ the solutions are smooth up to the boundary. Such full regularity can be derived from the explicit behaviour of the modes only in the case of the two-dimensional flat torus [11, 14].

As shown in the recent works [4, 9], the phenomenon of regularization of the modes extends to a wide class of nonlinear parabolic partial differential equations (PDEs).

In conclusion, we observe that the Neumann basis, which is required by the vorticity production approach, is not actually so bad for convergence. In section 4 we provide an easy proof that for $t > 0$ the expansions for $\mathbf{u}$ and its first and second space derivatives define continuous functions on $\mathbb{C}$, up to the boundary.

We now move to a detailed description of our approach.

The classical incompressible NS equations, in the absence of external forces, on the flat cylinder $\mathbb{C} := \mathbb{T} \times [0, \pi]$, where $\mathbb{T}$ is the one-dimensional torus, are

$$
\begin{cases}
\begin{aligned}
\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} &= \Delta \mathbf{u} - \nabla p, \\
\nabla \cdot \mathbf{u} &= 0, \\
\mathbf{u}|_{t=0} &= u^{(0)}.
\end{aligned}
\end{cases}
$$

(1.1)
In (1.1), \( u = (u_1, u_2) \) is the velocity field, \( p \) is the pressure, and the viscosity is taken equal to 1. At the boundary we have the usual Dirichlet (no-slip) boundary conditions,

\[
    u|_{\partial C} = 0. \tag{1.2}
\]

We represent \( T \) by identifying the end-points of the interval \([-\pi, \pi]\), and denote by \( x = (x_1, x_2) \in (-\pi, \pi) \times [0, \pi] \) the coordinates on \( C \). Setting \( \nabla^\perp := (-\partial x_2, \partial x_1) \), the vorticity is written as

\[
    \omega := \nabla^\perp \cdot u = \partial x_1 u_2 - \partial x_2 u_1. \tag{1.3}
\]

If \( u \in C^1(C; \mathbb{R}^2) \) is a solenoidal vector field (i.e. \( \nabla \cdot u = 0 \)) satisfying the boundary condition (1.2) then, as shown by lemma 2.1 in section 2, the vorticity has zero mean on \( C \), \( u \) is uniquely determined by \( \omega \) and can be represented as

\[
    u = \nabla^\perp \Delta_N^{-1} \omega, \tag{1.4}
\]

where \( \Delta_N \) is the Laplacian on \( C \) with zero Neumann boundary conditions. Therefore an evolution equation for \( \omega \), assuming sufficient smoothness, can be derived by taking the curl of both sides of (1.1)-(1.2) and using (1.4),

\[
    \begin{align*}
    \partial_t \omega + u \cdot \nabla \omega &= \Delta_N \omega, \\
    \int_C \omega \, dx &= 0, \\
    \partial x_1 \Delta_N^{-1} \omega|_{\partial C} &= 0, \\
    \omega|_{t=0} &= \nabla^\perp \cdot u(0). \tag{1.5}
    \end{align*}
\]

Equations (1.5) should be completed by the balance equation for the component along \( x_1 \) of the total momentum of the fluid, in the absence of external forces, i.e.

\[
    \frac{d}{dt} \int_C dx_1 u_1(x, t) = \int_T dx_1 \int_C \omega(x_1, 0, t) - \omega(x_1, \pi, t). \tag{1.6}
\]

In (1.5) the pressure is absent, but the boundary condition (1.2) is replaced by the non-local condition (1.5)3. Vorticity production is made explicit by treating the Laplacian in (1.5)1 as \( \Delta_N \), the operator with Neumann boundary conditions, which preserves vorticity, and adding a singular vorticity production term to keep account of the no-slip condition (1.5)2. We are thus led to the formal problem,

\[
    \begin{align*}
    \partial_t \omega + u \cdot \nabla \omega &= \Delta_N \omega + f \delta_{\partial C}, \\
    \int_C \omega \, dx &= 0, \\
    \partial x_1 \Delta_N^{-1} \omega|_{\partial C} &= 0, \\
    \omega|_{t=0} &= \nabla^\perp \cdot u(0). \tag{1.7}
    \end{align*}
\]

In (1.7)1 \( f \delta_{\partial C} \) is a suitable singular term with support on \( \partial C \), and as \( \partial C \) is made of two separate pieces, we write

\[
    f \delta_{\partial C}(x, t) = f_1(x_1, t) \delta(x_2) + f_2(x_1, t) \delta(x_2 - \pi), \tag{1.8}
\]

where by \( \delta \) we denote a suitable boundary \( \delta \)-distribution. A precise meaning of equation (1.7)1 can be given by its mild integral version:

\[
    \omega(x, t) = \int_C dy_1 e^{i\Delta_N} (x, y) \omega(y, 0) - \int_0^t ds \int_C dy_1 e^{i(t-s)\Delta_N} (x, y)(u \cdot \nabla \omega)(y, s) \\
    + \int_0^t ds \int_T dy_1 e^{i(t-s)\Delta_N} (x, (y_1, 0)) f_1(y_1, s) \\
    + \int_0^t ds \int_T dy_1 e^{i(t-s)\Delta_N} (x, (y_1, \pi)) f_2(y_1, s). \tag{1.9}
\]
Here $e^{\Delta x}(x, y)$ is the heat kernel with Neumann boundary conditions, $\omega(\cdot, 0) = \nabla \cdot u^{(0)}$ and the functions $f_1(x_1, t), f_2(x_1, t)$ are such that (1.7)2–(1.7)3 and (1.6) are satisfied.

The main object of our study is the Fourier version of the problem (1.9), (1.7)2, (1.7)3, completed by condition (1.6). Taking the Fourier components of both sides of equation (1.9) with respect to the Neumann basis, we get an infinite set of coupled integral equations for the components of $\omega$, which are equivalent to an infinite set of coupled ODEs (section 2).

Under our assumptions the auxiliary functions $f_j(x_1, t), j = 1, 2, t > 0$ are smooth, and are uniquely determined in terms of $\omega$ by the boundary condition (1.7)3, as solutions of an infinite system of Volterra integral equations (section 3). As we shall see, by (1.7)2 and (1.6), they satisfy the additional conditions

$$\int_0^T dx_1 f_j(x_1, t) = 0, \quad j = 1, 2, \quad t > 0. \quad (1.10)$$

Therefore our only unknown function is $\omega$.

The structure of our paper is the following. In section 2 we write down the infinite system of ODEs and give a precise formulation of the main results. Section 3 is devoted to the proof of existence and uniqueness of local solutions in some Banach spaces of suitable decaying modes.

Finally, in section 4 we first prove that the velocity fields corresponding to the local solutions are weak solutions in the sense of [15] of the classical NS equations (1.1), thereby establishing equivalence with the original NS problem. We then prove extension to global solutions and some further results on convergence of the expansions for the velocity and its space derivatives up to second order.

We finally observe that, as it is easy to see (remark 4.5, section 4), our solutions of the ODE system are also solutions of the integral problem (1.9), (1.7)2, (1.7)3.

## 2. Position of the problem and formulation of the main results

We begin with some preliminary constructions and results. We first prove the representation (1.4) for continuous $\omega$.

**Lemma 2.1.** If $u \in C^1(C; \mathbb{R}^2)$ is solenoidal, i.e. $\nabla \cdot u = 0$, satisfies the boundary conditions $u|_{\partial C} = 0$, and $\omega$ is the vorticity defined in (1.3), then $\int_C dx \omega(x) = 0$, and $u$ can be represented as $u = \nabla \cdot \psi$, where the ‘stream function’ $\psi \in C^2(C; \mathbb{R})$ is the unique (up to a constant) solution of the Poisson problem on $C$ with zero Neumann conditions,

$$\Delta \psi = \omega, \quad \partial_{x_2} \psi|_{\partial C} = 0. \quad (2.1)$$

Moreover $\int_0^\pi dx_2 u_1(x_1, x_2) = c$ is constant in $x_1$, and $\psi$ is constant on the two pieces of the boundary, $\psi(x_1, 0) = c_0, \psi(x_1, \pi) = c_1$, with the constraint

$$\psi(x_1, 0) - \psi(x_1, \pi) = c_0 - c_1 = \int_0^\pi dx_2 u_1(x_1, x_2) = c.$$

**Proof.** The condition $\int_C dx \omega(x) = 0$ follows immediately by periodicity in $x_1$ and the boundary condition $u|_{\partial C} = 0$. Moreover, by periodicity in $x_1$, solenoidality and the boundary condition $u_{x_1}|_{\partial C} = 0$, we see that the integrals

$$I_1(x_1) = \int_0^\pi dx_2 u_1(x_1, x_2), \quad I_2(x_2) = \int_0^\pi dx_1 u_2(x_1, x_2)$$

are constant, i.e. $I'_1(x_1) \equiv 0$, $I'_2(x_2) \equiv 0$, and in fact $I_2(x_2) \equiv 0$. 
As $\nabla \cdot u = 0$, we can write $u = \nabla^\perp \psi$. The condition $I_2(x_2) = 0$ implies $\int_\tau dx_1 \partial_{x_1} \psi(x_1, x_2) = 0$, so that $\psi$ is periodic in $x_1$. Moreover $\nabla^\perp \cdot \nabla^\perp \psi = \Delta \psi$ and $\psi$ is a solution of the boundary value problem (2.1), which is unique, up to a constant.

The other boundary condition $u_2|_{\partial C} = \partial_x \psi|_{\partial C} = 0$ implies that $\psi(x_1, \pi)$ and $\psi(x_1, 0)$ are constants, and their difference is clearly equal to $I_1(x_1) = c$.

By the previous lemma, $u = \nabla^\perp \Delta_N^{-1} \omega$. The result extends to the case $\omega \in L_2(C)$, except that solenoidality holds in $L_2$-sense (orthogonality to the gradients).

We now pass to the derivation of our fundamental set of equations. We first take the Fourier components of the terms in equation (1.9) with respect to the eigenfunctions of the Laplacian $\Delta_1$. The operation which will be shown to be legitimate under our assumptions (stated in our main theorem 2.4 below). If $\phi \in L_1(C)$, we define its Fourier components as

$$
\phi_{k_1, k_2} = \frac{1}{2\pi^2} \int_C dx \phi(x) e^{-ik_1 x_1} e^{ik_2 x_2}, \quad k_1 \in \mathbb{Z}, \quad k_2 \in \mathbb{Z}_+.
$$

The corresponding Fourier series is

$$
\phi(x) = \sum_{k_1 \in \mathbb{Z}} \phi_{k_1, 0} e^{ik_1 x_1} + 2 \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \geq 1} \phi_{k_1, k_2} e^{ik_1 x_1} e^{ik_2 x_2}.
$$

The Fourier components of the functions $f_j(x_1, t), j = 1, 2,$ in (1.8) are defined as

$$
f_{j, k}(t) = \frac{1}{2\pi^2} \int_\tau dx_1 f_j(x_1, t) e^{-ik_1 x_1}, \quad j = 1, 2.
$$

Equation (1.9) leads to the following infinite system of integral equations for the Fourier components $\omega_{k_1, k_2}(t), k_1 \in \mathbb{Z}, k_2 \geq 0$:

$$
\omega_{k_1, k_2}(t) = e^{-k_2 t} \omega_{k_1, k_2}(0) + \int_0^t ds e^{-k_2 (t-s)} \left\{ f_{\pm, k_1}(s) - N_{k_1, k_2}[\omega(s)] \right\},
$$

where the + (respectively −) sign is chosen for $k_2$ even (respectively odd),

$$
f_{\pm, k_1}(t) = f_{1, k_1}(t) \pm f_{2, k_1}(t),
$$

and $N_{k_1, k_2}[\omega(t)]$ are the Fourier components of the transport term $u \cdot \nabla \omega$. The infinite system (2.4) should be completed by imposing the conditions (1.7)–(1.9) and (1.6).

By our choice of the function spaces, $f_{\pm, k_1}$ and $N_{k_1, k_2}[\omega(t)]$ are, as we shall see, continuous in $t$, so that we can differentiate the right-hand side of (2.4) with respect to $t$, obtaining an infinite set of coupled ODEs:

$$
\dot{\omega}_{k_1, k_2}(t) + N_{k_1, k_2}[\omega(t)] = -k_2^2 \omega_{k_1, k_2}(t) + f_{\pm, k_1}(t), \quad k_1 \in \mathbb{Z}, \quad k_2 \geq 0, \quad t > 0,
$$

which is the formal Fourier formulation of (1.7)1.

As $\omega|_{t=0} = \nabla^\perp \cdot u(0)$ and $u(0)|_{\partial C} = 0$, the initial data are ‘well prepared’, i.e. $\omega_{0, 0} = 0$ and (2.7) are verified at $t = 0$. As $\int_\tau dx u \cdot \nabla \omega = 0$, we have $N_{0, 0}[\omega] = 0$, hence (2.4) gives $\omega_{0, 0}(t) = \int_0^t ds \int_\tau dx_1 f_{\pm, 0}(s)$. The condition (1.7)2 then implies $f_{\pm, 0}(t) = 0$ for any $t > 0$.

The boundary condition (1.7)2 becomes

$$
i \sum_{k_1, k_2 \in \mathbb{Z}} k_1 \frac{\partial \omega_{k_1, k_2}}{\partial x_1} e^{ik_1 x_1} e^{ik_2 x_2} \bigg|_{\partial C} = 0,
$$

and splits into two equations for the two components ($x_2 = 0$ and $x_2 = \pi$ of $\partial C$, which hold for any $x_1 \in (-\pi, \pi)$ if and only if

$$
\sum_{k_2 \in \mathbb{Z}} \frac{\hat{\omega}_{k_1, k_2}}{k_2} = 0, \quad \sum_{k_2 \in \mathbb{Z}} (-1)^{k_2} \frac{\hat{\omega}_{k_1, k_2}}{k_2} = 0 \quad \forall k_1 \neq 0.
$$
Adding and subtracting and going back to the components \(\omega_{k_1,k_2}\) we finally get,
\[
\sum_{k_2\neq 0} \frac{\omega_{k_1,k_2}}{k^2} = 0, \quad \sum_{k_2=-} \frac{\omega_{k_1,k_2}}{k^2} = 0 \quad \forall k_1 \neq 0,
\]
where we use for brevity the notation
\[
\sum_{\lambda_+} a_\lambda = a_0 + 2 \sum_{i\geq 1} a_{2i}, \quad \sum_{\lambda_-} a_\lambda = 2 \sum_{i\geq 1} a_{2i-1}.
\]

We now derive an explicit expression of the terms \(N_{k_1,k_2}[\omega(t)]\). For computations it is often convenient to write the series (2.2) in a different way: extending \(\phi_{k_1,k_2}\) by parity to negative \(k_2\) by setting \(\hat{\phi}_{k_1,-k_2} = \hat{\phi}_{k_1,k_2}\) for any \(k = (k_1,k_2) \in \mathbb{Z}^2\), we obtain
\[
\phi(x) = \sum_{k \in \mathbb{Z}^2} \hat{\phi}_{k_1,k_2} e^{ik \cdot x}.
\]

In what follows we perform formal manipulations on Fourier series, such as multiplication and term-by-term differentiation, without commenting on their validity, which is easily proved under the assumptions of our main theorem 2.4 below.

Given \(\omega\) with \(\omega_{0,0} = 0\), by lemma 2.1, the components of the stream function are \(\psi_{k_1,k_2} = -|k|^2 \omega_{k_1,k_2}, (k_1,k_2) \neq (0,0)\). Setting \(k^\bot = (-k_2,k_1)\), by (1.4), we find
\[
[u \cdot \nabla \omega](x) = \sum_{j=0}^1 \sum_{k \in \mathbb{Z}^2} \hat{R}_{k_1,k_2} e^{ik \cdot x}, \quad \hat{R}_{k_1,k_2} = \sum_{j=0}^1 \sum_{k \in \mathbb{Z}^2} \frac{j^+ \cdot \ell}{j^2} \hat{\omega}_{k_1,j \ell} \hat{\omega}_{k_1,j \ell},
\]

Observing that \(\hat{R}_{k_1,k_2}\) is odd in \(k_2\), i.e. \(\hat{R}_{k_1,k_2} = -\hat{R}_{k_1,-k_2}\), we find
\[
[u \cdot \nabla \omega](x) = 2i \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \geq 1} \hat{N}_{k_1,k_2} e^{ik \cdot x} \sin(k_2 x_2).
\]

In order to go back to the Neumann basis we use the expansion
\[
\sin(qy) = \sum_{p \in \mathbb{Z}} A_{q,p} \cos(py), \quad A_{q,p} = \frac{\delta_{\text{odd}}(q + p)}{\pi} \frac{2q}{q^2 - p^2},
\]
where \(\delta_{\text{odd}}(n) = 1\) (respectively \(\delta_{\text{even}}(n) = 0\)) is \(n\) is odd (respectively even). Therefore,
\[
[u \cdot \nabla \omega](x) = \sum_{k \in \mathbb{Z}} \hat{N}_{k_1,k_2} e^{ik \cdot x} \cos(k_2 x_2), \quad \hat{N}_{k_1,k_2} = i \sum_{b \in \mathbb{Z}} \hat{R}_{k_1,b} A_{k_2,b},
\]

Clearly \(\hat{N}_{k_1,-k_2} = \hat{N}_{k_1,k_2}\), so that, setting \(N_{k_1,k_2} = \hat{N}_{k_1,k_2}\) for \(k_2 \geq 0\), we find
\[
[u \cdot \nabla \omega](x) = \sum_{k_1} N_{k_1,0} e^{ik \cdot x} + 2 \sum_{k_1 \in \mathbb{Z}} N_{k_1,k_2} e^{ik \cdot x} \cos(k_2 x_2).
\]

Taking into account (2.10), we get
\[
N_{k_1,k_2}[\omega] = i \sum_{b_2 \in Z} \frac{\delta_{\text{odd}}(h_2 + k_2)}{\pi} \frac{2h_2}{h^2_2 - k^2_2} \sum_{j=0}^1 \frac{j^+ \cdot \ell}{j^2} \hat{\omega}_{j_1,j \ell} \hat{\omega}_{j_1,j \ell}.
\]

**Lemma 2.2.** Condition (1.6) implies \(f_{-0}(t) = 0\) for any \(t > 0\).
Proof. We have
\[
\frac{d}{dt} \int_C dx \, u_1(x, t) = -\frac{d}{dt} \int_C dx \, \delta_N^{-1} \omega(x, t) = -\frac{d}{dt} \sum_{k_2 \in \mathbb{Z}} 4\pi \delta_{\text{odd}}(k_2) \frac{\hat{\omega}_{0,k_2}(t)}{k_2^2} = -4\pi \sum_{k_2, \leq} \frac{\hat{\omega}_{0,k_2}(t)}{k_2^2} = 4\pi \sum_{k_2, \leq} \left\{ \omega_{0,k_2}(t) \frac{N_{0,k_2}[\omega(t)] - f_{-0}(t)}{k_2^2} \right\} = \int_C dx_1 \left[ \omega(x_1, 0, t) - \omega(x_1, \pi, t) \right] + 4\pi \sum_{k_2, \leq} N_{0,k_2}[\omega(t)] - f_{-0}(t).
\]

The claim is then a straightforward consequence of (1.6) and the following lemma.

Lemma 2.3. For any \( \omega \) satisfying \( \omega_{0,0} = 0 \) and (2.7),
\[
\sum_{k_2, \leq} N_{0,k_2}[\omega] = 0.
\]

Proof. By the expression (2.12), as \( j^+ \cdot \ell = j_1 h_2 \) for \( j^+ = (0, h_2) \), we have
\[
\sum_{k_2, \leq} N_{0,k_2}[\omega] = \frac{2i}{\pi} \sum_{k_2, \leq} \frac{1}{k_2^2} \sum_{j_2, \leq} \frac{h_j^2}{h_j^2 - k_2^2} \sum_{j, \leq} \frac{j_1}{j} \mathcal{O}_{j_1, j_2} \mathcal{O}_{j_1, \ell_2} \sum_{k_2, \leq} \frac{h_j^2}{h_j^2 - k_2^2} = \frac{2i}{\pi} \sum_{n \geq 1} \sum_{j, \leq} \frac{j_1}{j} \mathcal{O}_{j_1, j_2} \mathcal{O}_{j_1, \ell_2},
\]
where in the last line we used the equalities \( \frac{k_2^2}{h_j^2 - k_2^2} = \frac{1}{h_j^2 - k_2^2} \) and
\[
\sum_{k_2, \leq} \frac{1}{k_2^2 - (2n)^2} = \begin{cases} \frac{\pi^2}{4} & \text{if } n = 0, \\ \frac{\pi}{4n} \tan(\pi n) = 0 & \text{if } n \geq 1. \end{cases}
\]

We thus obtain
\[
\sum_{k_2, \leq} N_{0,k_2}[\omega] = 2\pi i \sum_{j, \neq 0} \left\{ \sum_{\ell_2, \leq} \omega_{-j_1, \ell_2} \sum_{j_2, \leq} \frac{\mathcal{O}_{j_1, j_2}}{j} \right\} + \sum_{\ell_2, \leq} \omega_{-j_1, \ell_2} \sum_{j, \leq} \frac{\mathcal{O}_{j_1, j_2}}{j^2},
\]
which vanishes if \( \omega \) satisfies (2.7). The lemma is proved.

The main object of our study is the discrete weak NS problem consisting of the infinite system of coupled equations (2.4) (with \( f_{\pm,0}(t) \equiv 0 \)), completed by the expression (2.12) for the quadratic term \( N_{k_1,k_2} \), and conditions (2.7).

As we will show in the next section, the conditions (2.7) uniquely determine the functions \( f_{k_1,k_2}(t) \), for \( k_1 \neq 0 \), as a quadratic integral functional of \( \omega_{k_1,k_2}(s) \), \( k_2 \geq 0 \), \( s \in [0, t] \). Hence, the infinite system (2.4)–(2.7) is in fact a system of integral equations for the functions \( \{ \omega_{k_1,k_2}(t); k_1 \in \mathbb{Z}, k_2 \geq 0 \} \), with an initial data such that \( \omega_{0,0}(0) = 0 \) and conditions (2.7) are satisfied.
In what follows, by ‘weak solution’ of the NS problem (1.1)–(1.2) we mean the standard definition such as in [15].

Our main result is the following theorem.

Theorem 2.4. Let \( \omega_{k_1,k_2}(0) \) satisfy the conditions \( \omega_{0,0}(0) = 0 \), (2.7), and the inequalities
\[
|\omega_{k_1,k_2}(0)| \leq \frac{D_0}{|k|^\alpha (1 + |k_1|^\beta)} \quad \forall k_1 \in \mathbb{Z}, \quad k_2 \geq 0, \quad k \neq (0,0),
\]
with \( 1 < \alpha < 2, \beta \geq 0 \), and some \( D_0 > 0 \). Then there exist real numbers \( D_1, v > 0 \) (depending on \( D_0, \alpha, \beta \)), and a unique continuous solution \( \omega_{k_1,k_2}(t); k_1 \in \mathbb{Z}, k_2 \geq 0 \) to equations (2.4) and (2.7) which satisfies for all \( t \geq 0 \) the inequalities
\[
|\omega_{k_1,k_2}(t)| \leq \frac{D_1 e^{-v(1+|k_1|)t}}{|k|^\alpha (1 + |k_1|^\beta)} \quad \forall k_1 \in \mathbb{Z}, \quad k_2 \geq 0, \quad k \neq (0,0).
\]
Moreover, for each \( t_0 > 0 \) there is a constant \( \tilde{D} = \tilde{D}_1(D_0, t_0, \alpha, \beta) \) such that
\[
|\omega_{k_1,k_2}(t)| \leq \frac{\tilde{D}_1 e^{-v(1+|k_1|)t/2}}{k^2} \quad \forall t \geq t_0.
\]
Finally, the velocity field \( u(x,t) := \nabla^2 \Delta_N^{-1} \omega(x,t) \) associated with the vorticity
\[
\omega(x,t) = \sum_{k \in \mathbb{Z}^2} \hat{\omega}_{k_1,k_2}(t) e^{ik \cdot x}
\]
is a weak solution to the NS system (1.1)–(1.2).

The decay estimate (2.14) guarantees continuity of \( \omega(x,t) \) and \( C^\infty \) regularity with respect to the periodic variable \( x_1 \) for any \( t > 0 \), up to the border \( \partial \mathcal{C} \), and the stronger estimate (2.15) only implies that \( \partial_t \omega(.,t) \) is in \( L^2(\mathcal{C}) \) for any \( t > 0 \). In fact, by looking at the explicit Fourier series one can easily see that theorem 2.4 implies more regularity, as stated by the following corollary.

Corollary 2.5. For each \( t > 0 \) the velocity field \( u(x,t) := \nabla^2 \Delta_N^{-1} \omega(x,t) \) is continuous and twice differentiable in \( x_1, x_2 \) up to the boundary \( \partial \mathcal{C} \).

We remark that if \( \alpha > \frac{3}{2} \) and \( \alpha + \beta > 2 \) then \( u^{(0)} \in W^{2,2}(\mathcal{C}) \), whence \( u(x,t) \) is a classical solution of the NS system (1.1)–(1.2), see e.g. [13, chapter 6, theorem 7].

The proof of theorem 2.4 and corollary 2.5 is deferred to section 4. The following section 3 is devoted to an existence and uniqueness theorem for local solutions, with global solutions for small initial data, which is a first step in the proof of theorem 2.4.

3. Local solutions

We first formulate a local existence theorem, which also provides global solutions for small initial data.

Theorem 3.1. Let \( \omega_{k_1,k_2}(0) \), with \( \omega_{0,0}(0) = 0 \), satisfy (2.7), and the inequalities (2.13) for some \( 1 < \alpha < 2, \beta \geq 0 \), and \( D_0 > 0 \). Then there exist a time \( T_0 = T_0(D_0, \alpha, \beta) \) and a constant \( D_2 = D_2(D_0, \alpha, \beta) \) such that there is a unique continuous solution of equations (2.4) and (2.7) for \( t \in [0, T_0] \), which satisfies the inequalities
\[
|\omega_{k_1,k_2}(t)| \leq \frac{D_2 e^{-1+|k_1|t/4}}{|k|^\alpha (1 + |k_1|^\beta)} \quad \forall k_1 \in \mathbb{Z}, \quad k_2 \geq 0, \quad k \neq (0,0).
\]
Moreover, if \( D_0 \) is sufficiently small, the corresponding solution is global and the estimate (3.1) is valid for any \( t \geq 0 \).
The proof of theorem 3.1 is based on some constructions and preliminary results. We begin by establishing a convenient representation of the components of the boundary term $f_{±,k_1}(t)$, $k_1 \in \mathbb{Z}$, in terms of the vorticity components $\omega_{k_1,k_2}(s)$, $k_2 \geq 0$, $0 \leq s \leq t$. Indeed, by plugging (2.4) into the constraints (2.7) we obtain a Volterra integral equation of the first kind for $f_{±,k_1}(t)$,

$$
\sum_{k_2,\pm} \frac{1}{k^2} \int_0^t ds \ e^{-k^2(t-s)} f_{±,k_1}(s) = g_{±,k_1}[t; \omega],
$$

where

$$
g_{±,k_1}[t; \omega] = \sum_{k_2,\pm} \frac{1}{k^2} \left\{ -e^{-k^2t} \omega_{k_1,k_2}(0) + \int_0^t ds \ e^{-k^2(s-t)} N_{k_1,k_2}[\omega(s)] \right\}.
$$

For any $\alpha, \beta \geq 0$ and $T > 0$ we introduce the Banach space $\Omega_{\alpha,\beta,T}$ consisting of continuous functions $\omega(t) = [\omega_{k_1,k_2}(t); k_1 \in \mathbb{Z}, k_2 \geq 0, t \in [0, T], \|\omega_{0,0}(t)\| = 0, \|\omega\|_{\alpha,\beta,T}$ where, for any $t \in [0, T]$,

$$
\|\omega\|_{\alpha,\beta,T} := \sup_{s \in [0,t]} \sup_{k_1 \in \mathbb{Z}, k_2 \geq 0} |\omega_{k_1,k_2}(t)| \left| e^{(1+|k_1|)^{1/4}} |k_1|^\alpha \left( 1 + |k_1|^\beta \right) \right|.
$$

The proof of theorem 3.1 will be obtained by a contraction argument in the space $\Omega_{\alpha,\beta,T}$. We begin with some preliminary lemmata.

**Lemma 3.2.** The Volterra equation of the first kind for the unknown function $a(t)$,

$$
\sum_{k_2,\pm} \frac{1}{k^2} \int_0^t ds \ e^{-k^2(t-s)} a(s) = b(t), \quad k_1 \neq 0,
$$

where $b(t)$ is a bounded differentiable function with $b(0) = 0$, has a unique solution which can be represented as

$$
a(t) = \int_0^t ds \ G_{k_1}^\pm(t-s) b'(s) + \int_0^t ds \ H_{k_1}^\pm(t-s) b(s).
$$

Here, denoting by $\Gamma(\cdot)$ the Euler Gamma function, $G_{k_1}^\pm$ is given by

$$
G_{k_1}^\pm(t) := \frac{2}{\pi} d_{\pm}(k_1) \left[ \delta(t) + \frac{e^{-k_1^2 t}}{\sqrt{t}} \sum_{n=1}^4 d_n(k_1)^n \Gamma(n/2) t^{(n-1)/2} \right],
$$

$$
d_{\pm}(k_1) := k_1 \left[ \tanh \left( \frac{\pi}{2} k_1 t \right) \right]^\pm 1,
$$

and $H_{k_1}^\pm(t)$ is a continuous function such that, for each $0 < \gamma < 1$,

$$
H_{k_1}^\pm(t) \leq B_\gamma |k_1|^\gamma \exp \left[ - (1 - \gamma) k_1^2 t \right],
$$

with $B_\gamma$ a positive constant.

**Proof.** Denoting by $\tilde{F}(\lambda) := \int_0^\infty dt \ e^{-\lambda t} F(t)$, $\lambda \in \mathbb{C}$, the Laplace transform of the function $F(t)$, equation (3.5) becomes

$$
\left( \sum_{k_2,\pm} \frac{1}{k^2} - \sum_{k_2,\pm} \frac{1}{k^2 + k} \right) \tilde{a}(\lambda) = \lambda \tilde{b}(\lambda),
$$

where $\Re \lambda > -k_1^2$. By the well-known expansions,

$$
\tan z = -\sum_{n=1}^{\infty} \frac{2z}{z^2 - (2n - 1)^2 \pi^2}, \quad \cot z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2 \pi^2},
$$

$$
\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!}, \quad \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!},
$$

$$
\frac{1}{\sin z} = \frac{1}{z} + \frac{z}{3} + \frac{z^3}{45} + \frac{z^5}{873} + \frac{1093z^7}{4,082,760},
$$

and

$$
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}, \quad \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.
$$
we have
\[ \sum_{k_{\pm}} \frac{1}{k^2 + \lambda} = \frac{\pi}{2} \phi_{\pm}(\sqrt{k_1^2 + \lambda}), \quad \phi_{\pm}(z) := \left[ \tanh \left( \frac{\pi}{2} z \right) \right]^{\pm 1}. \tag{3.11} \]

In particular, by (3.8), we find
\[ \sum_{k_{\pm}} \frac{1}{k^2} = \frac{\pi}{2} \frac{1}{|k_1|} = \frac{\pi}{2} \frac{1}{d_{\pm}(k_1)}, \tag{3.12} \]
so that equation (3.10) reads
\[ \tilde{a}(\lambda) = \frac{2}{\pi} \frac{d_{\pm}(k_1) \lambda \sqrt{k_1^2 + \lambda} - d_{\pm}(k_1) \phi_{\pm}(\sqrt{k_1^2 + \lambda})}{\sqrt{k_1^2 + \lambda} - d_{\pm}(k_1) \phi_{\pm}(\sqrt{k_1^2 + \lambda})} \tilde{b}(\lambda). \tag{3.13} \]

The right-hand side of (3.13) is the Laplace transform of the convolution of \( b(t) \) with a kernel which is the sum of a \( \delta \)-function, a singular integrable kernel, and a regular term. In fact, by the definition of the Euler Gamma function, we have
\[ \frac{\Gamma(\alpha + 1)}{z^{\alpha+1}} = \int_{0}^{\infty} dt e^{-zt} t^{\alpha} \quad (\alpha > -1), \]
so that the Laplace transform of the kernel \( G_{k_1}(t) \) defined in (3.7) is
\[ \tilde{G}_{k_1}(\lambda) = \frac{2}{\pi} d_{\pm}(k_1) \sum_{n=0}^{\infty} \left[ \frac{d_{\pm}(k_1)}{\sqrt{k_1^2 + \lambda}} \phi_{\pm}(\sqrt{k_1^2 + \lambda}) \right]^n. \]

Since \( b(0) = 0 \) implies \( \lambda \tilde{b}(\lambda) = \tilde{b}'(\lambda) \), equation (3.13) becomes
\[ \tilde{a}(\lambda) = \tilde{G}_{k_1}(\lambda) \tilde{b}'(\lambda) + \tilde{H}_{k_1}(\lambda) \tilde{b}(\lambda), \tag{3.14} \]
where
\[ \tilde{H}_{k_1}(\lambda) = \frac{2}{\pi} d_{\pm}(k_1) \sum_{n=0}^{\infty} \left[ \frac{d_{\pm}(k_1)}{\sqrt{k_1^2 + \lambda}} \phi_{\pm}(\sqrt{k_1^2 + \lambda}) \right]^n \lambda 
\quad + \frac{2}{\pi} d_{\pm}(k_1) \left[ \frac{d_{\pm}(k_1)}{\sqrt{k_1^2 + \lambda}} \phi_{\pm}(\sqrt{k_1^2 + \lambda}) \right]^{n+1} \lambda.	ag{3.15} \]

The representation (3.6) follows from (3.14) once we prove that \( \tilde{H}_{k_1}(\lambda) \) is the Laplace transform of a continuous function \( H_{k_1}(t) \) satisfying (3.9). By (3.11) we see that the singularities of \( \tilde{H}_{k_1}(\lambda) \) lie on the horizontal half-line \( \{ \lambda \in \mathbb{C}; \Re \lambda \leq -k_1^2, \Im \lambda = 0 \} \). The last term on the right-hand side of (3.15) is not singular at \( \lambda = 0 \) since the denominator has only a simple zero at that point. By the Laplace inverse formula, setting \( \xi = \lambda + k_1^2 \), we find
\[ H_{k_1}(t) = \frac{e^{-k_1^2 t}}{2\pi i} \int_{a-i\infty}^{a+i\infty} d\xi \; e^{\xi t} \tilde{H}_{k_1}(\xi - k_1^2) \quad \forall \; a > 0. \]
Choosing \( a = \gamma k_1^2 \) and \( \zeta = k_2^2 (\gamma + iy) \), we get

\[
H_{k_1}(t) = \frac{k_1^2 d_\pm(k_1)}{\pi^2} \exp\left[-(1 - \gamma) k_1^2 t\right]
\]

\[
\times \left\{ \int_{\mathbb{R}} dy \left[ \sum_{n=0}^{N_0} \frac{d_\pm(k_1)}{|k_1|^{\sqrt{\gamma} + iy}} \phi_\pm \left([k_1|\sqrt{\gamma} + iy]-1\right) e^{iyt} (\gamma + iy) \right. 
\]

\[
+ \int_{\mathbb{R}} dy \left[ \sum_{n=0}^{N_0} \frac{d_\pm(k_1)}{|k_1|^{\sqrt{\gamma} + iy}} (k_1|\sqrt{\gamma} + iy - d_\pm(k_1) \phi_\pm \left([k_1|\sqrt{\gamma} + iy]\right) \right. \right \}.
\] (3.16)

For \( 0 < \gamma < 1 \), the integrals on the right-hand side are absolutely convergent, uniformly with respect to \( k_1 \neq 0 \). Hence (3.9) follows, with a suitable \( B_\gamma > 0 \).

The lemma is proved.

\( \Box \)

**Lemma 3.3.** There is a constant \( C_N > 0 \) such that, for any \( \omega, \tilde{\omega} \in \Omega_{\alpha, \beta, T} \) satisfying (2.7) we have

\[
|N_{k_1,k_2}[\omega(t)] - N_{k_1,k_2}[\tilde{\omega}(t)]| \leq C_N \exp(-1+j_1|\beta/4|) R(\omega, \tilde{\omega}) \|\omega - \tilde{\omega}\|_{\alpha, \beta, T},
\] (3.17)

where

\[
R(\omega, \tilde{\omega}) := \max \left\{ \|\omega\|_{\alpha, \beta, T}; \|\tilde{\omega}\|_{\alpha, \beta, T} \right\}.
\] (3.18)

**Proof.** As \( N_{0,0}[\omega(t)] = N_{0,0}[\tilde{\omega}(t)] = 0 \) and \( \tilde{\omega}_{0,0} = 0 \), we get, for any \( t \in [0, T] \),

\[
|N_{k_1,k_2}[\omega(t)] - N_{k_1,k_2}[\tilde{\omega}(t)]| \leq \exp(-1+j_1|\beta/4|) 2 R(\omega, \tilde{\omega}) \|\omega - \tilde{\omega}\|_{\alpha, \beta, T}
\]

\[
\times \sum_{k_1 \in \mathbb{Z}} \frac{\delta_{\omega}(k_1 + k_2)}{\pi} \left| \frac{2h_2}{h_2^2 - k_2^2} \right| S_{k_1,h_2}.
\]

where

\[
S_{k_1,h_2} = \sum_{j,l=\pm 1}^{j_1} \frac{|j^+ \cdot \ell|}{|j^+|^{\alpha+2} |\ell|^{\alpha}} \frac{1}{(1 + |j_1|^\beta)(1 + |\ell_1|^\beta)}.
\]

Since, for a suitable constant \( C_1 > 0 \),

\[
\frac{1}{(1 + |j_1|^\beta)(1 + |\ell_1|^\beta)} \leq \frac{C_1}{1 + |j_1 + \ell_1|^\beta},
\]

in order to prove (3.17) it is enough to show that, for some \( C_2 > 0 \),

\[
\Sigma_{q_1,q_2} := \sum_{j,l=\pm 1}^{j_1} \frac{|j^+ \cdot \ell|}{|j^+|^{\alpha+2} |\ell|^{\alpha}} \leq \frac{C_2}{|q|^{\alpha-1}},
\] (3.19)

with \( q = (q_1, q_2) \in \mathbb{Z}^2 \setminus (0, 0) \). We decompose the sum as \( \Sigma_{q_1,q_2} = \Sigma_{q_1,q_2}^{(0)} + \Sigma_{q_1,q_2}^{(1)} + \Sigma_{q_1,q_2}^{(2)} \) with

\[
\Sigma_{q_1,q_2}^{(1)} := \sum_{j,l=\pm 1}^{j_1} \frac{|j^+ \cdot j|}{|j|^{\alpha+2} |q - j|^{\alpha}} \quad \Sigma_{q_1,q_2}^{(2)} := \sum_{0<j-q|<|q|/2} \frac{|j^+ \cdot j|}{|j|^{\alpha+2} |q - j|^{\alpha}}.
\]

It is easy to see that there are positive constants \( C_3, C_4 \) such that

\[
\Sigma_{q_1,q_2}^{(1)} \leq \frac{4}{|q|^{\alpha-1}} \sum_{0<j-q|<|q|/2} \frac{1}{|j|^{\alpha+2}} \leq \frac{C_3}{|q|^{\alpha-1}},
\]
\[ \Sigma_{q_j,q_j}^{(2)} \leq \frac{16}{|q|^{a+1}} \sum_{0<|j-q| \leq |q|/2} \frac{1}{|q-j|^{a-\gamma}} \leq C_4 \frac{1}{|q|^{2a-2}}. \]

Finally, regarding \( \Sigma_{q_j,q_j}^{(0)} \), since \( |j| > 2|q| \) implies \( |q-j| \geq |j| \), we have
\[ \Sigma_{q_j,q_j}^{(0)} = \sum_{|j|>2|q|} \left[ \frac{1}{|q|^{a+2}} \sum_{|j|\leq |q|/2} \frac{1}{|q-j|^{a-\gamma}} \right] \leq \frac{C_5}{|q|^{2a-2}}, \]
for a suitable \( C_5 > 0 \). The lemma is proved. \( \square \)

**Lemma 3.4.** There is a constant \( C_* > 0 \) such that if \( \omega, \tilde{\omega} \in \Omega_{\alpha,\beta,T} \) satisfy (2.7) and \( \tilde{f}_{\pm,\xi}(t) \), \( \tilde{f}_{\pm,\xi}(t) \) are the solutions to (3.2) for \( \omega \) and \( \tilde{\omega} \), respectively, the following inequalities hold:
\[ |f_{\pm,\xi}(t) - \tilde{f}_{\pm,\xi}(t)| \leq C_* e^{-\gamma |t|/4} \left[ |k|^{a-1} |\delta\omega|_{\alpha,0} + R(\omega, \tilde{\omega}) |\delta\omega|_{\alpha,\beta,T} \right], \]
where \( \delta\omega := \omega - \tilde{\omega} \) and \( R(\omega, \tilde{\omega}) \) is defined in (3.18).

**Proof.** We introduce the notation
\[ \delta g_{\pm,\xi}(t) = g_{\pm,\xi}(t; \omega) - g_{\pm,\xi}(t; \tilde{\omega}), \quad \delta g'_{\pm,\xi}(t) = \partial tg_{\pm,\xi}(t; \omega). \]

By (3.3) and (3.17), for some constant \( C_0 > 0 \), we have
\[ |\delta g_{\pm,\xi}(t)| \leq \frac{1}{1 + |k|^\beta} \sum_{k_\pm} e^{-k_\pm t} |\delta\omega|_{\alpha,0} \]
\[ + \frac{1}{1 + |k|^\beta} \sum_{k_\pm} 2 C_N R(\omega, \tilde{\omega}) e^{-|1+k|_1 t/4} \left[ |k|^{a-1} |\delta\omega|_{\alpha,\beta,T} \right], \]
\[ \leq C_6 \sum_{k_\pm} e^{-k_\pm t} N_{k_\pm}(0) + \sum_{k_\pm} \frac{N_{k_\pm}(\omega(t))}{k^2} + \int_0^t ds e^{-k_\pm (t-s)} N_{k_\pm}(\omega(s)), \]
(3.21)

Here we used the simple inequality
\[ e^{-k_\pm (t-s)} e^{-|1+k|_1 t/4} \leq e^{-|1+k|_1 (t-s)/2} e^{-k_\pm (t-s)/2}, \quad \forall k \neq (0,0), \]
(3.22)

and the fact that for each \( r > 1 \) there exists a constant \( 0 < c < \infty \) such that \( \sum_{k_\pm} |k|^{-r} \leq c |k|^{-r} \). In a similar way, since

\[ \partial_t g_{\pm,\xi}(t; \omega) = \sum_{k_\pm} e^{-k_\pm t} \omega_{k_\pm}(0) + \sum_{k_\pm} \frac{N_{k_\pm}(\omega(t))}{k^2} + \int_0^t ds e^{-k_\pm (t-s)} N_{k_\pm}(\omega(s)), \]
(3.23)

we get, for some \( C_7 > 0 \),
\[ |\delta g'_{\pm,\xi}(t)| \leq \frac{C_7}{1 + |k|^\beta} \left[ |\delta\omega|_{\alpha,0} e^{-k_\pm t} + R(\omega, \tilde{\omega}) |\delta\omega|_{\alpha,\beta,T} e^{-|1+k|_1 t/4} \right]. \]
(3.24)

On the other hand, by (3.6), (3.7) and (3.9), choosing \( \gamma = \frac{1}{8} \), we see that
\[ |f_{\pm,\xi}(t) - \tilde{f}_{\pm,\xi}(t)| \leq \frac{2}{\pi} d_{\pm,\xi}(1) |\delta g_{\pm,\xi}(t)| \]
\[ + \frac{2}{\pi} d_{\pm,\xi}(k_1) \sum_{n=1}^4 \left( \frac{d_{\pm,\xi}(k_1)}{F(n/2)} \right)^n \int_0^t ds e^{-k_\pm (t-s)} (t-s)^{n-2} |\delta g'_{\pm,\xi}(s)| + B_{1/8} |k|^3 \int_0^t ds e^{-2k_\pm (t-s)/8} |\delta g_{\pm,\xi}(s)|, \]
(3.25)
Estimate (3.20) then follows by plugging (3.21) and (3.24) into (3.25), recalling (3.8), and observing that if \( k_1 \neq 0 \) is an integer the following inequalities hold:

\[
\begin{align*}
    e^{-k_1^2(t-s)}e^{-(1+|k_1|)t/4} & \leq e^{-(1+|k_1|)t/4}e^{-k_1^2(t-s)/2}, \\
    e^{-k_1^2(t-s)/8}e^{-(1+|k_1|)t/4} & \leq e^{-(1+|k_1|)t/4}e^{-k_1^2(t-s)/4}.
\end{align*}
\]

We omit further details. \( \square \)

**Remark 3.5.** Observe that, taking \( \tilde{\omega} = 0 \), the estimates (3.17) and (3.20) give

\[
|N_{k_1,k_2}[\omega(t)]| \leq \frac{C_N}{1 + |k_1|^\beta} \| \omega \|_{\alpha,\beta,T}^2, \quad (3.26)
\]

\[
|f_{\pm,k_1}(t)| \leq \frac{C_N}{1 + |k_1|^\beta} \left( \| k_1 \|^{1-\alpha} \| \omega \|_{\alpha,\beta,0} + \| \omega \|_{\alpha,\beta,T}^2 \right). \quad (3.27)
\]

We now define the iteration procedure for the solution of equations (2.4)–(3.2). In the first step we set

\[
\omega_{k_1,k_2}^{(0)}(t) := e^{-k_1^2 t} \omega_{k_1,k_2}(0), \quad g_{\pm,k_1}^{(0)}(t) := -\sum_{k_2,\pm} \frac{1}{k_2} e^{-k_1^2 t} \omega_{k_1,k_2}(0),
\]

and denote by \( f_{\pm,k_1}^{(0)}(t) \) the solution of the Volterra equation

\[
\sum_{k_2,\pm} \frac{1}{k_2^2} \int_0^t ds e^{-k_2^2(t-s)} f_{\pm,k_1}^{(0)}(s) = g_{\pm,k_1}^{(0)}(t).
\]

We then iterate by setting, for each integer \( n \geq 1 \),

\[
\omega_{k_1,k_2}^{(n)}(t) := e^{-k_1^2 t} \omega_{k_1,k_2}(0) + \int_0^t ds e^{-k_2^2(t-s)} \left[ f_{\pm,k_1}^{(n)}(s) - N_{k_1,k_2}^{(n-1)}(s) \right],
\]

where \( N_{k_1,k_2}^{(n)}(t) := N_{k_1,k_2}[\omega^{(n-1)}(t)] \) and \( f_{\pm,k_1}^{(n)}(t) \) denotes the solution to

\[
\sum_{k_2,\pm} \frac{1}{k_2^2} \int_0^t ds e^{-k_2^2(t-s)} f_{\pm,k_1}^{(n)}(s) = g_{\pm,k_1}^{(n)}(t),
\]

where \( g_{\pm,k_1}^{(n)}(t) := g_{\pm,k_1}[t; \omega^{(n-1)}] \) is given by (3.3). Before proving theorem 3.1 we need one more lemma.

**Lemma 3.6.** Under the assumptions above, the following assertions hold

(i) There is some \( T_1 = T_1(D_0, \alpha, \beta) > 0 \) such that

\[
\| \omega^{(n)} \|_{\alpha,\beta,T} \leq D_2, \quad \forall n \geq 0, \quad \forall 0 \leq T \leq T_1,
\]

with \( D_2 := 2(1 + 2C_\alpha)D_0 \).

(ii) There is some \( T_0 = T_0(D_0, \alpha, \beta), 0 < T_0 \leq T_1, \) such that

\[
\| \omega^{(n+1)} - \omega^{(n)} \|_{\alpha,\beta,T} \leq \frac{1}{2} \| \omega^{(n)} - \omega^{(n-1)} \|_{\alpha,\beta,T}, \quad \forall n \geq 1, \quad \forall 0 \leq T \leq T_0.
\]
Proof. By (2.13) we have $\|\omega^{(0)}\|_{a,\beta,T} \leq D_0 < D_2$. Proceed by induction and assume that (3.28) holds for any $0 \leq n' < n$ up to some time $T_1$. Observe that, as $\omega^{(n-1)}_{k_1,k_2}(0) = \omega_{k_1,k_2}(0)$ for all $n \geq 1$, we have

$$\|\omega^{(n-1)}\|_{a,\beta,0} \leq D_0, \quad \|\omega^{(n-1)}\|_{a,\beta,T} \leq D_2. \quad \square$$

Furthermore we have

$$|\omega^{(n)}_{k_1,k_2}(t)| \leq \frac{D_0 e^{-k^2 t}}{|k|^2 (1 + |k_1|^2)} + \int_0^t ds e^{-k^2(s-t)} \left[ f^{(n-1)}_{\pm,k_1}(s) + |N^{(n-1)}_{k_1,k_2}(s)| \right].$$

By applying (3.26) and (3.27) on the right-hand side and using (3.22), recalling (3.29) and that $k \neq (0,0)$, we obtain

$$|\omega^{(n)}_{k_1,k_2}(t)| \leq \frac{e^{-(1+|k_1|^2)/4}}{|k|^2 (1 + |k_1|^2)} \left[ D_0 + 2 \frac{1 - e^{-k^2 t/2}}{|k|^{2-a}} \right] D_2 + \tilde{C} D_2 \tau^{(2-a)/2},$$

where

$$\tilde{C} := 2 (C_\alpha + C_N) \sup_{\xi > 0} \frac{1 - e^{-\xi/2}}{\xi^{(2-a)/2}}.$$

By the previous estimate, we have $\|\omega^{(n)}\|_{a,\beta,T} \leq D_2$ if

$$T \leq T_1 := \left( \frac{1}{2C D_2} \right)^{2/(2-a)}$$

whence (3.28) is proved with this choice of $T_1$.

Passing to the proof of (3.30), let $t \in (0, T_1]$ so that (3.28) holds. We have

$$|\omega^{(n+1)}_{k_1,k_2}(t) - \omega^{(n)}_{k_1,k_2}(t)| \leq \int_0^t ds e^{-k^2(s-t)} \left[ f^{(n)}_{\pm,k_1}(s) - f^{(n-1)}_{\pm,k_1}(s) \right]$$

$$+ \int_0^t ds e^{-k^2(s-t)} \left[ N^{(n)}_{k_1,k_2}(s) - N^{(n-1)}_{k_1,k_2}(s) \right].$$

By applying (3.17) and (3.20) on the right-hand side, observing that

$$\|\omega^{(n)} - \omega^{(n-1)}\|_{a,\beta,0} = 0, \quad R(\omega^{(n)}, \omega^{(n-1)}) \leq D_2,$$

and using (3.22) as before, we get

$$|\omega^{(n+1)}_{k_1,k_2}(t) - \omega^{(n)}_{k_1,k_2}(t)| \leq \frac{e^{-(1+|k_1|^2)/4} \tilde{C} D_2 \tau^{(2-a)/2}}{|k|^2 (1 + |k_1|^2)} \|\omega^{(n)} - \omega^{(n-1)}\|_{a,\beta,T},$$

whence (3.30) is proved with $T_0 = T_1$. \square

Proof of theorem 3.1. By lemma 3.6, we have that $\{\omega^{(n)}\}$ is a Cauchy sequence in $\Omega_{a,\beta,T}$ for $0 < T < T_0$ satisfying (3.28). This proves the existence and uniqueness of the solution in $\Omega_{a,\beta,T}$ for $0 < T < T_0$, and also that (3.1) holds with $D_2$ given by (3.29).

The proof of the existence of a global solution for small initial data follows along the same lines, with an obvious modification: in proving (3.28) and (3.30), the small parameter is $D_2$ and the time power $\tau^{(2-a)/2}$ should not be extracted. We omit the details. \square
Remark 3.7. The decay factor $e^{-(1+|k|)t/4}$ could be omitted with minor changes in the proof of theorem 3.1. In particular, if we only assume that $\omega_{k_1,k_2}(0)$, with $\omega_{0,0}(0) = 0$, satisfies the conditions (2.7) and

$$|\omega_{k_1,k_2}(0)| \leq \frac{D_0}{|k|^\alpha}, \quad \forall k_1 \in \mathbb{Z}, \quad k_2 \geq 0, \quad k \neq (0, 0),$$

with $1 < \alpha < 2$, then there exist a time $T_0 = T_0(D_0, \alpha)$ and a unique continuous solution $[\omega_{k_1,k_2}(t); k_1 \in \mathbb{Z}, k_2 \geq 0]$ to equations (2.4) and (2.7) such that $\|\omega\|_{a,T_0} < \infty$, where

$$\|\omega\|_{a,T} := \sup_{x \in [0,T]} \sup_{k_1 \in \mathbb{Z}} \sup_{k_2 \geq 0} |\omega_{k_1,k_2}(s)| |k|^\alpha.$$  

(3.31)

4. Proofs of theorem 2.4 and corollary 2.5

The proof of theorem 2.4 will be achieved by first extending the solution considered in remark 3.7 to any positive time. The estimates required for such an extension are in fact well-known properties of the weak solutions to the NS flow. Therefore, we first prove that the local solutions to our system actually coincide with weak solutions of the NS system (1.1)–(1.2).

Lemma 4.1. Let $[\omega_{k_1,k_2}(t); k_1 \in \mathbb{Z}, k_2 \geq 0], t \in [0, T]$, be a continuous solution to equations (2.4) and (2.7) such that $\|\omega\|_{a,T} < \infty$, $1 < \alpha < 2$, and let

$$\omega(x, t) := \sum_{k \in \mathbb{Z}^2} \hat{\omega}_{k_1,k_2}(t) e^{ik \cdot x}.$$  

Then, the velocity field $u(x, t) := \nabla^2 \Delta_n^{-1} \omega(x, t)$ coincides, for $t \in [0, T]$, with a weak solution to the NS system (1.1)–(1.2).

Proof. By the expansion (2.9), we have

$$u(x, t) = \sum_{k \in \mathbb{Z}^2} u_{k_1,k_2}(t) e^{ik \cdot x}, \quad u_{k_1,k_2}(t) := -i k_1 \frac{\hat{\omega}_{k_1,k_2}(t)}{k^2},$$

so that in terms of the norm (3.31) we get the estimate

$$\sup_{t \in [0,T]} |u_{k_1,k_2}(t)| \leq \|\omega\|_{a,T} \frac{1}{|k|^\alpha+1}.$$  

Therefore $u \in L^2([0, T]; V)$, where $V$ is the space of solenoidal vector fields in $H_0^1(C)^2$. For the assertion, it remains to be proved [13, 15, 16] that, for any solenoidal $C^\infty$ vector field $\Phi$ of compact support in $C$, we have

$$\frac{d}{dt} \int_C \Phi(x) \cdot u(x, t) = \int_C \Delta \Phi(x) \cdot u(x, t) - \int_C \Phi(x) \cdot [u(x, t) \cdot \nabla] u(x, t).$$

Since $\Phi = \nabla^2 \phi$ for some $C^\infty$ function $\phi$ with derivatives of compact support in $C$, by Green’s formula we get,

$$\int_C \Phi \cdot u = - \int_C \Phi \cdot \omega, \quad \int_C \Delta \Phi \cdot u = - \int_C \Delta \Phi \cdot \omega.$$

Using the identity $u \text{rot } u = \frac{1}{2} \nabla u^2 - (u \cdot \nabla) u^1$, we also find

$$\int_C \Phi \cdot [u \cdot \nabla] u = - \int_C \nabla \phi \cdot [u \cdot \nabla] u^1 = - \int_C \nabla \phi \cdot u \omega.$$
Therefore, we have to prove that, for any $C^\infty$ function $\phi$ with derivatives of compact support in $C$, 
\[
\frac{d}{dt} \int_C dx \phi(x) \omega(x, t) = \int_C dx \Delta \phi(x) \omega(x, t) - \int_C dx \nabla \phi(x) \cdot u(x, t) \omega(x, t).
\] (4.1)
By equations (2.6) we find 
\[
\frac{d}{dt} \int_C dx \phi(x) \omega(x, t) = 2\pi^2 \sum_{k \in \mathbb{Z}^2} j_{k_1, -k_2} \frac{d \hat{\omega}_{k_1, k_2}(t)}{dt} = \int_C dx \Delta \phi(x) \omega(x, t) + 2\pi^2 \sum_{k \in \mathbb{Z}^2} \hat{\phi}_{-k_1, -k_2} \hat{\Phi}_{k_1, k_2}[\omega(t)].
\] (4.2)
The first sum on the right-hand side of (4.2) is zero since $f_{\delta,0}(t) = 0$ and $\sum_{k \in \mathbb{Z}^2} \phi_{k_1, k_2} = 0$ for any $k_1 \neq 0$ since $\nabla \phi$ has compact support. Regarding the last term on the right-hand side of (4.2), since the last sum in (2.12) is an odd function of $h_2$, it is equal to (we omit the explicit dependence on time $t$) 
\[
2\pi i \sum_{j,k\neq0} \delta_{\text{odd}}(k_2 + j_2 + \ell_2) \frac{\delta_0(k_1 + j_1 + \ell_1)}{k_2 + j_2 + \ell_2} j_{j_1 + \ell_1} \frac{j_2 \cdot k}{j^2} \hat{\Phi}_{k_1, k_2} \hat{\Phi}_{j_1, j_2} \hat{\Phi}_{\ell_1, \ell_2}.
\]
On the other hand, in terms of the Fourier components, the last term on the right-hand side of (4.1) reads 
\[
-2\pi i \sum_{j,k\neq0} \delta_{\text{odd}}(k_2 + j_2 + \ell_2) \frac{\delta_0(k_1 + j_1 + \ell_1)}{k_2 + j_2 + \ell_2} j_{j_1 + \ell_1} \frac{j_2 \cdot k}{j^2} \hat{\Phi}_{k_1, k_2} \hat{\Phi}_{j_1, j_2} \hat{\Phi}_{\ell_1, \ell_2}.
\]
Therefore, their difference is 
\[
2\pi i \sum_{j,k\neq0} \delta_{\text{odd}}(k_2 + j_2 + \ell_2) \frac{\delta_0(k_1 + j_1 + \ell_1)}{k_2 + j_2 + \ell_2} j_{j_1 + \ell_1} \frac{j_2 \cdot (k + \ell)}{j^2} \hat{\Phi}_{k_1, k_2} \hat{\Phi}_{j_1, j_2} \hat{\Phi}_{\ell_1, \ell_2}.
\]
Substituting $j_{j_1 + \ell_1} \cdot (k + \ell) = j_{j_1} (k_2 + j_2 + \ell_2)$, we conclude that the above sum vanishes because of conditions (2.7). The lemma is proved. 

**Lemma 4.2.** Let $(\omega_{k_1, k_2}(t); k_1 \in \mathbb{Z}, k_2 \geq 0), t \in [0, T]$, be any continuous solution to equations (2.4) and (2.7) such that $\|\omega\|_{\alpha, T} < \infty$ and set 
\[
U(t) := \sum_{k_1, k_2} \frac{|\hat{\omega}_{k_1, k_2}(t)|^2}{k^2}, \quad E(t) := \sum_{k \in \mathbb{Z}^2} |\hat{\omega}_{k_1, k_2}(t)|^2.
\] (4.3)
Then 
\[
U(t) \leq U(0) e^{-t} \quad \forall t \in [0, T],
\] (4.4)
and there are constants $E_0, \sigma > 0$, depending on $E(0), U(0)$, such that 
\[
E(t) \leq E_0 e^{-\sigma t} \quad \forall t \in [0, T].
\] (4.5)

**Proof.** Since 
\[
U(t) = \frac{1}{2\pi^2} \int_C dx |u(x, t)|^2, \quad E(t) = \frac{1}{2\pi^2} \int_C dx |\nabla u(x, t)|^2,
\]
(4.4) and (4.5) assert, respectively, the decrease in time of the energy $U$ and that the enstrophy $E$ tends to zero. These are well-known properties of the NS flow in the absence of external
forces, therefore they apply here by lemma 4.1. Actually, the energy estimate can be easily proved directly in our setting. In fact, by (2.6),
\[ U(t) + E(t) = \sum_{k \neq 0} \frac{\hat{\omega}_{-k} \cdot \hat{k}(t)}{k^2} f_{\pm k}(t) - \sum_{k \neq 0} \frac{\hat{\omega}_{-k} \cdot \hat{k}(t)}{k^2} \hat{N}_{k_1, k_2}[\omega(t)], \]
where we used \( \hat{\omega}_{-k} \cdot \hat{k} \) is the complex conjugate of \( \hat{\omega}_{k} \cdot \hat{k} \) since \( \omega \) is real. The first sum on the right-hand side is clearly zero because of (2.7). Regarding the second one, since the last sum in (2.12) is an odd function of \( h_2 \), it is equal to (we omit the explicit dependence on time \( t \))
\[ \frac{-i}{\pi} \sum_{k \neq 0, l} \delta_{\text{odd}}(k_2 + j_2 + \ell_2) \frac{\delta_0(k_1 + j_1 + \ell_1) \cdot \ell}{j^2 k^2} \hat{\omega}_{k_1, k_2} \hat{\omega}_{j_1, j_2} \hat{\omega}_{\ell_1, \ell_2}, \]
which can be rewritten in the symmetric form
\[ \frac{-i}{2\pi} \sum_{k \neq 0, l} \delta_{\text{odd}}(k_2 + j_2 + \ell_2) \frac{\delta_0(k_1 + j_1 + \ell_1) \cdot \ell}{j^2 k^2} \hat{\omega}_{k_1, k_2} \hat{\omega}_{j_1, j_2} \hat{\omega}_{\ell_1, \ell_2}. \]
Substituting \( (k + j)^\perp \cdot \ell = (k + j + \ell)^\perp \cdot \ell = -\ell_1(k_2 + j_2 + \ell_2) \), we conclude that the above sum also vanishes again because of (2.7). In conclusion, \( U(t) + E(t) = 0 \), from which (4.4) follows since \( E(t) \geq U(t) \).

Concerning (4.5), we recall that in two dimensions, if \( u_0 \in V \) then the weak solution belongs to \( L^\infty(0, T; V) \) for any \( T > 0 \), see [15, theorem III.3.10]. Furthermore, in the case of without forcing NS equation, the enstrophy converges exponentially to 0 as \( t \to +\infty \), see [15, theorem III.3.12], whence (4.5) holds. The lemma is proved.

**Proposition 4.3.** Any local solution \( \omega_{k_1, k_2}(t); k_1 \in \mathbb{Z}, k_2 \geq 0, t \in [0, T] \), to equations (2.4) and (2.7) such that \( \|\omega\|_{a,T} < \infty \) extends uniquely to a global solution. Moreover, if
\[ |\omega|_{a,T} := \sup_{k_1 \in \mathbb{Z}} \sup_{k_2 \geq 0} |\omega_{k_1, k_2}(t)| |k|^a, \quad (4.6) \]
then \( |\omega|_{a,T} \to 0 \) exponentially fast as \( t \to +\infty \).

**Proof.** Since \( |\omega|_{a,T} < \infty \), by remark 3.7 we can start by taking \( \omega_{k_1, k_2}(T) \) as initial data, and obtain existence up to a time \( T_1 > T \). Iterating the procedure we get a growing sequence of times \( T_{j+1} > T_j, j \geq 1 \), and if \( T^* = \lim_{j \to +\infty} T_j \), then \( 0, T^* \) is the maximal interval of existence of the solution. Our goal is to show that \( T^* = +\infty \). If \( T^* < +\infty \) then clearly
\[ \limsup_{t \to T^*} \|\omega\|_{a,T} = +\infty. \quad (4.7) \]
We shall prove that (4.7) is impossible if \( T^* < +\infty \). By (2.4),
\[ |\omega_{k_1, k_2}(t)| \leq \frac{e^{-\xi t} |\omega|_{a,0}}{|k|^a} + \int_0^t ds e^{-\xi(t-s)} \left[ |f_{\pm k}(s)| + |N_{k_1, k_2}[\omega(s)]| \right]. \quad (4.8) \]
We need an estimate of the terms on the right-hand side of (4.8). For \( N_{k_1, k_2}[\omega(t)] \), we argue in analogy to the proof of lemma 3.3. We have, by (2.12),
\[ |N_{k_1, k_2}[\omega(t)]| \leq \|\omega\|_{a,T} \sum_{h_2 \in \mathbb{Z}} \delta_{\text{odd}}(h_2 + k_2) \frac{2h_2}{\pi} \left| \frac{2h_2}{h_2^2 - k_2^2} \right| \tilde{S}_{k_1, k_2}[\omega(t)], \quad (4.9) \]
where
\[ \tilde{S}_{q_1, q_2}[\omega(t)] := \sum_{j \in \mathbb{Z} \setminus \{0\}} \frac{|j|}{j^2 |j - q|^a} |\hat{\omega}_{j}(t)|, \quad q = (q_1, q_2) \in \mathbb{Z}^2. \]
By the Cauchy–Schwartz inequality, we have

$$\tilde{S}_{q_1,q_2}[\omega(t)] \leq \sqrt{\mathcal{E}(t)} \sqrt{\Sigma_{q_1,q_2}}$$

where $\mathcal{E}(t)$ is defined in (4.3). We set

$$\tilde{\Sigma}_{q_1,q_2} := \sum_{j \neq \pm q} \frac{1}{j^2 |j - q|^{2u-2}},$$

$$\Sigma_{q_1,q_2}(j) := \sum_{|j| = q} \frac{1}{j^2 |j - q|^{2u-2}}.$$

Then, by elementary inequalities we find, for some constant $C > 0$,

$$\tilde{\Sigma}_{q_1,q_2}(j) \leq \frac{4}{|q|^{2u-2}} - \sum_{0 < |j| < 2|q|} \frac{1}{j^2} \leq C_0 \log(e + |q|),$$

$$\Sigma_{q_1,q_2}(j) \leq \frac{4}{|q|^{2u-2}} - \sum_{0 < |j| < 2|q|} \frac{1}{j^2} \leq C_0 \log(e + |q|),$$

$$\sum_{|j| > 2|q|} \frac{1}{j^2 |j - q|^{2u-2}} \leq \sum_{|j| > 2|q|} \frac{1}{j^2 |j - q|^{2u-2}} \leq C_0 \log(e + |q|).$$

Therefore $\tilde{S}_{q_1,q_2}[\omega(t)] \leq 3C_0 \sqrt{\mathcal{E}(t)} |q|^{1-u} \sqrt{\log(e + |q|)}$, and using inequalities (4.5) and (4.10) in (4.9), recalling that $\alpha > 1$, we get, for a suitable $C_0 > 0$,

$$N_{k_1,k_2}[\omega(t)] \leq C_0 e^{-\sigma t/2} \|\omega\|_{u,t}.$$

For $|f_{\pm,k}(s)|$, we recall that $f_{\pm,0}(t) = 0$ and for $k_1 \neq 0$ we proceed as in the proof of lemma 3.4. By (3.3), (3.23) and (4.11), we have, for some $C_{10} > 0$,

$$|g_{\pm,k}[r,\omega]| \leq C_0 \left( \frac{|\omega|_{u,0}}{|k_1|^{2u+1}} e^{-k_1 r} + \frac{\|\omega\|_{u,t}}{|k_1|^2} e^{-\sigma t/2} \right),$$

$$|\partial_r g_{\pm,k}[r,\omega]| \leq C_0 \left( \frac{|\omega|_{u,0}}{|k_1|^{2u+1}} e^{-k_1 r} + \frac{\|\omega\|_{u,t}}{|k_1|^2} e^{-\sigma t/2} \right).$$

An upper bound for $|f_{\pm,k}(t)|$ is given by the right-hand side of (3.25) with $\delta_{g_{\pm,k}}(t)$, respectively $\delta_{g'_{\pm,k}}(t)$, replaced by $g_{\pm,k}[r,\omega]$, respectively $\partial_r g_{\pm,k}[r,\omega]$. From the above estimates, assuming, without loss of generality, that $\sigma < \frac{1}{2}$, after some straightforward calculations, we get, for some constant $C_{11} > 0$, the inequality

$$|f_{\pm,k}(t)| \leq C_{11} \left( \frac{|\omega|_{u,0}}{|k_1|^{2u+1}} e^{-k_1 t/2} + \|\omega\|_{u,t} e^{-\sigma t/2} \right).$$

Introducing the bounds (4.11) and (4.12) in the estimate (4.8) we find

$$|\omega_{k_1,k_2}(t)| \leq e^{-k_1 t/2} |\omega|_{a,0} + 2C_{11} \delta_{k_1 \neq 0} e^{-k_1 t/2} - \frac{e^{-k_1 t} - e^{-k_1 t}}{k_1^2 + 2k_2^2} |k_1|^{2-u} |\omega|_{a,0}$$

$$+ (C_0 + C_{11}) \int_0^t ds e^{-k_1 (t-s)} e^{-\sigma t/2} \|\omega\|_{u,s}.$$
where the constant $C_{12}$ is given by

$$C_{12} := \left( \sup_{\xi} e^{-\xi^2/4} \frac{1}{|\xi|^2} \right) \left( \sup_{\xi} \frac{1 - e^{-\xi^2}}{|\xi|^2} \right),$$

we see that there is a positive constant $C_{13}$ such that

$$|k|^\sigma \left| |\alpha|_{\omega_k}(t) \right| \leq C_{13} \left[ e^{-t/4}|\alpha|_{\omega_0} + \int_0^t ds \left| k \right|^\sigma e^{-\sigma t/2} \left| |\alpha|_{\omega_0} \right| \right].$$

Therefore, setting $M := C_{13} \sup_k e^{-\xi^2/2} |\xi|^\sigma$, we have

$$|k|^\sigma \left| |\alpha|_{\omega_k}(t) \right| \leq C_{13} e^{-t/4}|\alpha|_{\omega_0} + M \int_0^t ds \frac{e^{-\sigma(t-s)/2}}{(t-s)^{\gamma/2}} e^{-\sigma t/2} \left| |\alpha|_{\omega_0} \right|.$$

Taking the supremum over $k \neq (0,0)$, by definition (4.6), we obtain

$$|\alpha|_{\omega_{k,t}} \leq C_{13} e^{-t/4}|\alpha|_{\omega_0} + \int_0^t ds K(t, s) \left| |\alpha|_{\omega_s} \right|,$$

where $K(t, s)$ is a singular integrable kernel

$$K(t, s) = M \frac{e^{-(t-s)/2}}{(t-s)^{\gamma/2}} e^{-\sigma t/2}.$$

We have to analyse separately small and large times. Let $T_K > 0$ be such that $\max_{t \in (0, T_K)} \int_0^T ds K(t, s) \leq \frac{1}{2}$. By (4.13), for all $0 \leq t < \min(T_K, T^*)$,

$$|\alpha|_{\omega_{t,0}} \leq C_{13} |\alpha|_{\omega_{0,0}} + \frac{1}{2} |\alpha|_{\omega_{t,0}},$$

so that $|\alpha|_{\omega_{t,0}} \leq 2C_{13} |\alpha|_{\omega_{0,0}}$ for any $0 \leq t < \min(T_K, T^*)$. Hence $T^* > T_K$ and

$$|\alpha|_{\omega_{T_K}} \leq 2C_{13} |\alpha|_{\omega_{0,0}}.$$

We now take $t \in (T_K, T^*)$ and choose $\delta < T_K$ such that $\frac{2M}{\gamma \delta (2-\alpha/2)} \leq \frac{1}{2}$. Therefore,

$$|\alpha|_{\omega_{t,\delta}} \leq C_{13} e^{-t/4}|\alpha|_{\omega_{0,0}} + \int_0^{t-\delta} ds K(t, s) \left| |\alpha|_{\omega_s} \right| + \int_{t-\delta}^t ds K(t, s) \left| |\alpha|_{\omega_s} \right|$$

$$\leq C_{13} |\alpha|_{\omega_{0,0}} + M \int_0^{t-\delta} ds \frac{\left| |\alpha|_{\omega_{s,0}} \right|}{(t-s)^{\gamma/2}} + M \left| |\alpha|_{\omega_{t,0}} \right| \int_0^{t-\delta} ds \frac{1}{(t-s)^{\gamma/2}}$$

$$\leq C_{13} |\alpha|_{\omega_{0,0}} + M \delta^{-\alpha/2} \int_0^{t-\delta} ds \left| |\alpha|_{\omega_{s,0}} \right| + \frac{1}{2} \left| |\alpha|_{\omega_{t,0}} \right|.$$

Since by (4.15),

$$\left| |\alpha|_{\omega_{t,0}} \right| \leq 2C_{13} |\alpha|_{\omega_{0,0}} + \sup_{s \in [T_K, t]} |\alpha|_{\omega_{s,0}} \quad \forall t \in [T_K, T^*),$$

we finally obtain an integral inequality for $|\alpha|_{\omega_{t,0}}$,

$$\left| |\alpha|_{\omega_{t,0}} \right| \leq 6C_{13} |\alpha|_{\omega_{0,0}} + 2M \delta^{-\alpha/2} \int_0^{t-\delta} ds \left| |\alpha|_{\omega_{s,0}} \right|,$$

which implies, by the Gronwall lemma,

$$\left| |\alpha|_{\omega_{t,0}} \right| \leq 6C_{13} |\alpha|_{\omega_{0,0}} \exp(2M \delta^{-\alpha/2} t).$$

In particular $|\alpha|_{\omega_{t,0}}$ remains bounded as $t \uparrow T^*$ if $T^* < +\infty$, whence $T^* = +\infty$.

We are left with the proof that $|\alpha|_{\omega_{t,0}}$ converges exponentially to zero. For this we use a bootstrap argument. From (4.14) we get the inequality

$$K(t, s) \leq M e^{-\sigma t/2} \frac{e^{-(t-s)/4}}{(t-s)^{\gamma/2}}.$$
which, if inserted on the right-hand side of (4.13), together with the bound (4.16), gives

$$|\omega|_{a,t} \leq C_{13} |\omega|_{a,0} \left( e^{-t/4} + 6M_a \exp \left( 2M\delta^{-\sigma/2} - \frac{\sigma}{2} \right) t \right),$$

(4.17)

where $M_a = M \int_0^\infty ds \, e^{-s/4} \delta^{-\sigma/2}$. If $2M\delta^{-\sigma/2} < \sigma/2$ the argument is complete, otherwise (4.17) implies the inequality

$$\|\omega\|_{a,t} \leq C_{13} |\omega|_{a,0} (1 + 6M_a) \exp \left( 2M\delta^{-\sigma/2} - \frac{\sigma}{2} \right) t,$$

(4.18)

in which the exponential factor has become smaller by $\sigma/2$ with respect to the previous inequality (4.16). We can now insert the new inequality (4.18), instead of (4.16), on the right-hand side of (4.13), and it is easy to see that although the pre-factor may grow, the exponential factor again becomes smaller by $\sigma/2$.

We can then iterate the procedure $n$ times, where $n$ is the smallest integer such that $n\sigma/2 > 2M\delta^{-\sigma/2}$, thus obtaining the desired exponential decay. The proposition is proved. □

**Proof of theorem 2.4 (conclusion).** We first show that proposition 4.3 implies the first assertion (inequality (2.14)) of theorem 2.4 for $\beta = 0$. In fact, as $|\omega|_{a,t}$ is uniformly bounded, the components $\omega_{k_1,k_2}(t)$ satisfy the condition (2.13) with constants $D_0$ bounded from above, so that, by theorem 3.1, the existence time, taking $\omega_{k_1,k_2}(t)$ as initial data, is, for any $t > 0$, larger than a fixed positive time $T$. This implies that there exist real numbers $D^*, t^* > 0$ such that

$$|\omega_{k_1,k_2}(t)| \leq \frac{D^* e^{-\min\{1,\nu\}t/4}}{|k|^\alpha}, \quad \forall k_1 \in \mathbb{Z}, \ k_2 \geq 0, \ \ k \neq (0,0), \ \forall t \geq T,$$

(4.19)

Furthermore, since $|\omega|_{a,t} \rightarrow 0$ as $t \rightarrow +\infty$, by the last assertion of theorem 3.1, concerning the existence of global solution for small initial data, we can find a time $T = T(D_0, \alpha)$ such that

$$|\omega_{k_1,k_2}(t)| \leq \frac{\tilde{D}_2 e^{-\min\{1,\nu\}(t-T)/4}}{|k|^\alpha}, \quad \forall k_1 \in \mathbb{Z}, \ k_2 \geq 0, \ k \neq (0,0), \ \forall t \geq T,$$

(4.20)

where $\tilde{D}_2 = D_2(|\omega|_{a,t}, \alpha, 0)$ (see theorem 3.1 for notation). By (4.19) and (4.20) the inequality (2.14) with $\beta = 0$ is obtained for any $\nu = v^* < \frac{1}{4}$ and a suitable $D_1 = D_1^* > 0$.

It is not hard to see that for any $\nu < v^*$ such a solution satisfies the inequalities (2.14) with a constant $D_1$ depending on $\beta$, $\nu$ and on the initial data. In fact, the assertion is true for the solution up to the time $T_0$ which is granted by theorem 3.1. For times $t > T_0$ it is enough to observe that

$$|\omega_{k_1,k_2}(t)| \leq D_1 e^{-\min\{1,\nu\}v^* t} \leq CD_1^* \frac{e^{-\min\{1,\nu\}|k|^\alpha}}{|k|^\alpha (1 + |k|^\beta)},$$

where $C = \max_{k_1} (1 + |k|^\beta) e^{\nu(1-\nu)(1 + |k|^\beta)}T_0$.

We now prove the stronger inequality (2.15) by means of an easy bootstrap argument. By (2.14), since $\nu < \frac{1}{4}$, the same reasoning as in the proofs of (3.26), (3.27) gives

$$|N_{k_1,k_2}(\omega(t))| \leq C_N \frac{e^{-\min\{1,\nu\}|k|^\alpha}}{1 + |k|^\beta} D_1^2,$$

(4.21)

$$|f_{\pm,k_1}(t)| \leq C_e \frac{e^{-\min\{1,\nu\}|k|^\alpha}}{1 + |k|^\beta} \left( |k_1|^2 D_0 + D_1^2 \right).$$

(4.22)

The integral equation (2.4), with the inequalities (4.21), (4.22) and (3.22), gives, for any $t > 0$,

$$|\omega_{k_1,k_2}(t)| \leq \frac{D_0 e^{-k^2 t}}{|k|^\alpha (1 + |k|^\beta)} + \frac{2(C_N + C_e) e^{-\min\{1,\nu\}|k|^\alpha}}{k^2 (1 + |k|^\beta)} \left( D_0 |k_1|^2 + D_1^2 \right).$$
It follows that for each $t_0 > 0$ there is a constant $\tilde{D}_1 = \tilde{D}_1(D_0, t_0, \alpha, \beta)$ for which (2.15) is valid.

Finally, the result of lemma 4.1 extends now to the global solution, therefore the velocity field $u(x, t) = \nabla \cdot \Delta_N^1 \omega(x, t)$, with $\omega(x, t)$ as in (2.16), satisfies the NS system. The theorem is thus proved. ■

We are left with the proof of corollary 2.5. We start with a lemma.

Lemma 4.4. For each $\varepsilon < (\alpha - 1)/2$ there exists a positive number $D_3 = D_3(D_0, \alpha, \beta, \varepsilon)$ such that, for any $0 \leq s < t$,

$$|f_{\pm,k}(s) - f_{\pm,k}(t)| \leq \frac{D_3 e^{-\varepsilon(t-s)|k|^{2+2\varepsilon-\alpha}}}{1 + |k|^\beta} (t-s)^\varepsilon,$$

(4.23)

with $\nu < \frac{1}{2}$ as in (2.14), (2.21) and (2.22).

Proof. We write for short $g_{\pm,k}(t)$ for the function $g_{\pm,k}[t; \omega]$ defined in (3.3), and the derivative $g_{\pm,k}^{(i)}(t)$ is given in (3.23). By (3.2) and lemma 3.2 we have

$$f_{\pm,k}(s) - f_{\pm,k}(t) = \sum_{j=1}^4 W^{(j)}_{\pm,k}(s, t),$$

(4.24)

where

$$W^{(1)}_{\pm,k}(s, t) = \frac{2}{\pi} d_{\pm,k}(k) \left[ g_{\pm,k}(s) - g_{\pm,k}(t) \right],$$

$$W^{(2)}_{\pm,k}(s, t) = \int_0^t d\tau \left[ Q^{\mp}_{k}(s - \tau) - Q^{\pm}_{k}(t - \tau) \right] g_{\pm,k}(\tau),$$

$$W^{(3)}_{\pm,k}(s, t) = \int_0^t d\tau \left[ H^{\mp}_{k}(s - \tau) - H^{\pm}_{k}(t - \tau) \right] g_{\pm,k}(\tau),$$

$$W^{(4)}_{\pm,k}(s, t) = - \int_s^t d\tau \left[ Q^{\pm}_{k}(t - \tau) g_{\pm,k}(\tau) + H^{\pm}_{k}(t - \tau) g_{\pm,k}(\tau) \right],$$

and

$$Q^{\pm}_{k}(t) := \frac{2 e^{-\varepsilon t}}{\pi \sqrt{t}} \sum_{n=1}^4 d_{\pm,k}(k)^{(n+1)} \left[ \tau - (n-1)/2 \right].$$

Let $\varepsilon < (\alpha - 1)/2$. By (2.4) we have

$$\omega_{k_1,k_2}(s) - \omega_{k_1,k_2}(t) = \left[ 1 - e^{-\varepsilon(s-t)} \right] \omega_{k_1,k_2}(t) + \int_t^s d\tau e^{-\varepsilon(t-\tau)} \left[ N_{k_1,k_2}[\omega(\tau)] - f_{\pm,k}(\tau) \right].$$

Using (2.14), (2.21), (2.22) and that, for any $\varepsilon \in (0, 1)$ and $z > 0$,

$$1 - e^{-z} \leq C_x e^x,$$

$$C_x := \sup_{\xi > 0} \left( \frac{1 - e^{-\xi^2}}{\xi^2} - \infty, \quad (4.25)$$

we easily find, for a suitable $D_3 > 0$,

$$\left| \omega_{k_1,k_2}(s) - \omega_{k_1,k_2}(t) \right| \leq \frac{D_3 e^{-\varepsilon(t-s)|k|^{2+2\varepsilon-\alpha}}}{1 + |k|^\beta} (t-s)^\varepsilon.$$

(4.26)

Since $\alpha - 2\varepsilon > 1$, by arguing in analogy to the proof of lemma 3.3, we can use (2.14) and (4.26) to estimate the difference $N_{k_1,k_2}[\omega(s)] - N_{k_1,k_2}[\omega(t)]$; we thus obtain, for some $D_3 > 0$,

$$\left| N_{k_1,k_2}[\omega(s)] - N_{k_1,k_2}[\omega(t)] \right| \leq \frac{D_3 e^{-\varepsilon(t-s)|k|^{2+2\varepsilon-\alpha}}}{1 + |k|^\beta} (t-s)^\varepsilon.$$

(4.27)
From the expressions (3.3) and (3.23) of $g_{\pm, k_1}$ and $g'_{\pm, k_1}$, and the bounds (4.21), (4.25) and (4.27), it is not difficult to show that for a suitable constant $D_0 > 0$ the following estimates hold:

$$
|g_{\pm, k_1}(t)| + |g'_{\pm, k_1}(t)| \leq \frac{D_0}{1 + |k_1|^\beta} \left[ \frac{e^{-k_1^2 t}}{|k_1|^{\alpha - 1}} + \frac{e^{-v(1+|k_1|) t}}{|k_1|^1} \right],
$$
(4.28)

$$
|g'_{\pm, k_1}(s) - g'_{\pm, k_1}(t)| \leq \frac{D_0}{1 + |k_1|^\beta} \left[ \frac{e^{-k_1^2 |s-t|}}{|k_1|^{\alpha - 2e^{-1}}} + \frac{e^{-v(1+|k_1|) t}}{|k_1|^1} \right] (t-s)\varepsilon. \tag{4.29}
$$

By (4.29), (4.28), the explicit expression of $Q_{\pm, k_1}^\varepsilon(t)$, and the estimate (3.9), the terms $W_{\pm, k_1}^{(i)}(s, t)$, $i = 1, 2, 4$, are easily bounded to get (4.23). For the remaining term $W_{\pm, k_1}^{(3)}(s, t)$, the needed control on the difference $H_{\pm, k_1}^D(t - \tau) - H_{\pm, k_1}^D(s - \tau)$ can be deduced by the representation (3.16).

We omit the details.

**Proof of corollary 2.5.** For any $t > 0$, by (2.4) a solution $\omega(x, t)$ satisfies the identity

$$
\omega(x, t) = \sum_{k_1 \in \mathbb{Z}^2} \hat{\omega}_{k_1}(0) e^{-k_1^2 t} e^{ik_1 x} + \sum_{k_1 \in \mathbb{Z}^2} \int_0^t ds e^{-k_1^2 (t-s)} \hat{N}_{k_1, k_2}[\omega(s)] e^{ik_2 x}
$$

$$
+ F_\pm(x, t) + F_\mp(x, t), \tag{4.30}
$$

where (recall $f_{\pm, 0}(t) \equiv 0$)

$$
F_\pm(x, t) := \sum_{k_1 \neq 0, k_2, \pm} e^{ik_2 x} \cos(k_2 x_2) \int_0^t ds e^{-k_1^2 (t-s)} f_{\pm, k_1}(s). \quad \text{(4.31)}
$$

The first sum on the right-hand side of (4.30) clearly defines an infinitely differentiable function on $\mathcal{C}$ for any $t > 0$. Concerning the second sum, we observe that by (3.19) and [6, lemma 2.1] a better estimate than (4.21) holds. Namely, for some $\bar{C}_N > 0$,

$$
|N_{k_1, k_2}[\omega(t)]| \leq \bar{C}_N \log(e + |k_2|) e^{-v(1+|k_1|) t}
$$

$$(1 + |k_2|^{\alpha - 1})(1 + |k_1|^\beta) - D_1^2. \tag{4.32}
$$

Therefore, for each $t > 0$ the second sum on the right-hand side of (4.30) can be differentiated term by term with respect to the $x_2$-variable, uniformly on $\mathcal{C}$. It remains to be proved that also the derivatives with respect to $x_2$ of $F_{\pm}(x, t)$ are continuous functions on $\mathcal{C}$. To this end, we write

$$
F_\pm(x, t) = F_{\pm}^{(1)}(x, t) + F_{\pm}^{(2)}(x, t) + F_{\pm}^{(3)}(x, t), \tag{4.33}
$$

where

$$
F_{\pm}^{(1)}(x, t) = \sum_{k_1 \neq 0, k_2, \pm} e^{ik_2 x_2} \cos(k_2 x_2) \int_0^t ds e^{-k_1^2 (t-s)} [f_{\pm, k_1}(s) - f_{\pm, k_1}(t)],
$$

$$
F_{\pm}^{(2)}(x, t) = -\sum_{k_1 \neq 0} f_{\pm, k_1}(t) e^{ik_1 x} \sum_{k_2, \pm} \frac{e^{-k_1^2 t}}{k_2^2} \cos(k_2 x_2),
$$

$$
F_{\pm}^{(3)}(x, t) = \sum_{k_1 \neq 0} f_{\pm, k_1}(t) e^{ik_1 x} \sum_{k_2, \pm} \frac{1}{k_2^2} \cos(k_2 x_2).
$$

By (4.22), $F_{\pm}^{(2)}(x, t)$ is an infinitely differentiable function on $\mathcal{C}$ for any $t > 0$. We next observe that, by lemma 4.4, choosing $\varepsilon < (\alpha - 1)/2$, there is a constant $D_7 > 0$ such that

$$
\int_0^t ds e^{-k_1^2 (t-s)} \left| f_{\pm, k_1}(s) - f_{\pm, k_1}(t) \right| \leq D_7 e^{-\varepsilon |k_1|^\beta} \frac{|k_1|^{2\varepsilon - \alpha}}{|k_1|^{2\varepsilon}(1 + |k_1|^\beta)}.
$$
Therefore, the series $F^{(1)}_{\pm}(x, t)$ can be differentiated term by term with respect to the $x_2$-variable, uniformly on $C$. For the analysis of $F^{(2)}_{\pm}(x, t)$ we further decompose,

$$F^{(3)}_{+}(x, t) = \sum_{k_{1} \neq 0} \frac{f_{+}\cdot k_{1}(t)}{k_{1}^{2}} e^{ik_{1}x_{1}} + 2 \sum_{k_{1} \neq 0} f_{+}\cdot k_{1}(t) e^{ik_{1}x_{1}} \sum_{n \geq 1} \frac{\cos(2nx_{2})}{4n^{2}} \quad \text{(4.31)}$$

$$= -2 \sum_{k_{1} \neq 0} f_{+}\cdot k_{1}(t) e^{ik_{1}x_{1}} \sum_{n \geq 1} \frac{k_{1}^{2} \cos(2nx_{2})}{4n^{2}(k_{1}^{2} + 4n^{2})}.$$  

$$F^{(3)}_{-}(x, t) = 2 \sum_{k_{1} \neq 0} f_{-}\cdot k_{1}(t) e^{ik_{1}x_{1}} \sum_{n \geq 1} \frac{\cos((2n-1)x_{2})}{(2n-1)^{2}} \quad \text{(4.32)}$$

By the estimate (4.22), the last series on the right-hand side of (4.31) and (4.32) can be differentiated term by term with respect to the $x_2$-variable, uniformly on $C$ for any $t > 0$. On the other hand, for any $x_{2} \in [-\pi, \pi]$,

$$\sum_{n \geq 1} \frac{\cos(2nx_{2})}{4n^{2}} = \frac{\pi^{2}}{24} - \frac{\pi}{4} |x_{2}| + \frac{1}{4} x_{2}^{2}, \quad \sum_{n \geq 1} \frac{\cos((2n-1)x_{2})}{(2n-1)^{2}} = \frac{\pi^{2}}{8} - \frac{\pi}{4} |x_{2}|.$$

Therefore, also $F^{(3)}_{\pm}(x, t)$ are continuously differentiable with respect to $x_2$ up to the boundary of $C$. We remark, however, that due to the presence of the two series in the above display, $\partial_{x_{2}}F^{(3)}_{\pm}(x, t)$ does not vanish as $x \to \partial C$, so that the vorticity does not satisfy Neumann boundary conditions in the classical sense.

Remark 4.5. The function $\omega(x, t)$ defined by (2.16) is a solution to the problem (1.9), (1.7)$_{2},$ (1.7)$_{3}$. To show this, we have only to verify that the right-hand sides of (4.30) and (1.9) coincide for any $t > 0$. The first series in (4.30) is obviously equal to $\int_{C} dy e^{i\Delta s(x, y)} \omega(y, 0)$. We next observe that, by (4.21), the series

$$\sum_{k \in \mathbb{Z}^{2}} e^{-\xi(t-s)} \hat{N}_{k_{1}k_{2}}[\omega(s)] e^{ikx}$$

is equal to $\int_{C} dy e^{i(t-s)\Delta s(x, y)}(u \cdot \nabla \omega)(y, s)$ for any $(x, s) \in C \times [0, t)$ and can be bounded by constant times $(t-s)^{-3/4}$. Therefore, by the Lebesgue dominated convergence theorem,

$$\sum_{k \in \mathbb{Z}^{2}} \int_{0}^{t} ds e^{-\xi(t-s)} \hat{N}_{k_{1}k_{2}}[\omega(s)] e^{ikx} = \int_{0}^{t} ds \int_{C} dy e^{i(t-s)\Delta s(x, y)}(u \cdot \nabla \omega)(y, s).$$

A similar argument, using now (4.22), shows that $F_{+}(x, t) + F_{-}(x, t)$ is equal to the last two terms on the right-hand side of (1.9).

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