A new method for constructing squeezed states for the isotropic 2D harmonic oscillator

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Abstract. We introduce a new method for constructing squeezed states for the 2D isotropic harmonic oscillator. Based on the construction of coherent states in [1], we define a new set of ladder operators for the 2D system as a linear combination of the $x$ and $y$ ladder operators and construct the $SU(2)$ coherent states. The new ladder operators are used for generalizing the squeezing operator to 2D and the $SU(2)$ coherent states play the role of the Fock states in the expansion of the 2D squeezed states. We discuss some properties of the 2D squeezed states.

Keywords: coherent states; squeezed states; harmonic oscillator; $SU(2)$ coherent states; 2D coherent states; 2D squeezed states; uncertainty principle

1 Introduction

Degeneracy in the spectrum of the Hamiltonian is one of the first problems we encounter when trying to define a new type of coherent states for the 2D oscillator. As a continuation of the work in [1] we produce a non-degenerate number basis ($SU(2)$ coherent states) for the 2D isotropic harmonic oscillator with accompanying generalized creation and annihilation operators. The squeezed states for the 2D isotropic harmonic oscillator are then defined in terms of the $SU(2)$ coherent states and generalized ladder operators.

Work on degeneracy in coherent state theory has been done, Klauder described coherent states of the hydrogen atom [2] which preserved many of the usual properties required by coherent state analysis [3]. Fox and Choi proposed the Gaussian Klauder states [4], an alternative method for producing coherent states for more general systems with degenerate spectra. An analysis of the connection between the two definitions was studied in [5].

In the first part of the paper we address the degeneracy in the energy spectrum by constructing non-degenerate states, the $SU(2)$ coherent states, and we
define a generalized ladder operator formed from a linear combination of the 1D ladder operators with complex coefficients.

In the last part of the paper we use a generalized squeezing operator and Fock space expansion to define squeezed states for the 2D system. In both cases we use the same definitions as for the 1D squeezed states, but replacing the Fock states with the $SU(2)$ coherent states and the 1D ladder operators with the new generalised ladder operators. We discuss the spatial probability distributions of the 2D squeezed states, as well as their dispersions.

2 Squeezed states of the 1D harmonic oscillator

Squeezed states, or squeezed coherent states, are a generalization of the standard coherent states first studied by Schrödinger [6], and then formalised in the context of quantum optics by Glauber and Sudarshan [7] [8]. In terms of the displacement and squeezing operators

\[ D(\psi) = e^{\psi a^\dagger - \bar{\psi} a}, \quad S(\xi) = e^{\frac{1}{2}(\xi a^2 - \bar{\xi} a^\dagger^2)} \]

respectively, where $a, a^\dagger$ are the annihilation and creation operators, squeezed states are expressed as

\[ |\psi, \xi\rangle = D(\psi) S(\xi) |0\rangle, \]

$\psi, \xi \in \mathbb{C}$. Writing $\xi = re^{i\theta}$, in terms of Fock states, $\{|n\rangle\}$, the squeezed states are given by

\[ |z, \gamma\rangle = \frac{1}{\mathcal{N}(z, \gamma)} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \left(\frac{\gamma}{2}\right)^n H_n\left(\frac{z}{\sqrt{2}r}\right) |n\rangle, \]

where \[ \frac{1}{\mathcal{N}(z, \gamma)} = \frac{1}{\sqrt{\cosh r}} e^{-\frac{|z|^2}{2}} e^{\tanh r \text{Re}(e^{i\theta} z^2)}. \] The states in equation (2) are solutions to the eigenvalue equation

\[ (a + \gamma a^\dagger) |z, \gamma\rangle = z |z, \gamma\rangle. \]

Equivalence between definitions (1) and (2) is understood through the following relationships between the parameters [9]

\[ z = \psi - \bar{\psi} e^{i\theta} \tanh r, \]
\[ \gamma = -e^{i\theta} \tanh r. \]

The term ‘squeezing’ is used because the squeezed states saturate the Robertson-Schrödinger uncertainty relation [10] but with unequal dispersions in position and momentum (unlike the standard coherent states which saturate the Heisenberg uncertainty principle with equal dispersions). The squeezed states have the following dispersions

\[ (\Delta X)^2_{|\psi, \xi\rangle} = \langle \psi, \xi | X^2 - \langle X \rangle^2 |\psi, \xi\rangle = \frac{1}{2} + \sinh^2 r + \text{Re}(e^{i\theta}) \cosh r \sinh r; \]
\[ (\Delta P)^2_{|\psi, \xi\rangle} = \langle \psi, \xi | P^2 - \langle P \rangle^2 |\psi, \xi\rangle = \frac{1}{2} + \sinh^2 r - \text{Re}(e^{i\theta}) \cosh r \sinh r, \]

 respectively.
where \( (\Delta \hat{O})^2_{|n,m\rangle} \equiv \langle \psi | \hat{O}^2 - \langle \hat{O} \rangle^2 | \psi \rangle \) is the variance of the operator \( \hat{O} \) in the state \( | \psi \rangle \). The position and momentum operators are expressed in the usual way \( \hat{X} = \frac{1}{\sqrt{2}}(a^\dagger + a) \), \( \hat{P} = \frac{1}{\sqrt{2m}}(a - a^\dagger) \). When the squeezing is purely real \( \xi = r \), the dispersions become \( (\Delta X)^2_{|\psi,\xi\rangle} = \frac{1}{2} e^{-2r} \), \( (\Delta P)^2_{|\psi,\xi\rangle} = \frac{1}{2} e^{2r} \), in this case the squeezed states saturate the Heisenberg uncertainty relation \( (\Delta X)^2_{|\psi,\xi\rangle}(\Delta P)^2_{|\psi,\xi\rangle} = \frac{1}{4} \).

Like the standard coherent states, the squeezed states are also non-orthogonal and they admit a resolution of the identity \([11]\), therefore they represent an overcomplete basis for the Hilbert space of the 1D harmonic oscillator.

### 3 The 2D oscillator

For a 2D isotropic oscillator we have the quantum Hamiltonian

\[
\hat{H} = -\frac{1}{2} \frac{d^2}{dx^2} - \frac{1}{2} \frac{d^2}{dy^2} + \frac{1}{2} x^2 + \frac{1}{2} y^2
\]

(6)

where we have set \( \hbar = 1 \) and the mass \( m = 1 \) and the frequency \( \omega = 1 \). We solve the time independent Schrödinger equation \( \hat{H} | \Psi \rangle = E | \Psi \rangle \) and obtain the usual energy eigenstates (or Fock states) labelled by \( | \Psi \rangle = | n, m \rangle \) with eigenvalue \( E_{n,m} = n + m + 1 \) and \( n, m \in \mathbb{Z}^{\geq 0} \). These states may all be generated by the action of the raising and lowering operators in the following way \([12]\)

\[
a^-_x |n, m\rangle = \sqrt{n} |n-1, m\rangle, \quad a^+_x |n, m\rangle = \sqrt{n+1} |n+1, m\rangle;
\]

\[
a^-_y |n, m\rangle = \sqrt{m} |n, m-1\rangle, \quad a^+_y |n, m\rangle = \sqrt{m+1} |n, m+1\rangle.
\]

The states \( |n, m\rangle \) in configuration space have the following wavefunction

\[
(x, y | n, m \rangle = \psi_n(x) \psi_m(y) = \frac{1}{\sqrt{2^{n+m} n! m!}} \left( \frac{1}{\pi} \right)^{\frac{1}{4}} e^{-\frac{x^2}{2} - \frac{y^2}{2}} H_n(x) H_m(y),
\]

(8)

where \( \psi_n(x) = \frac{1}{\sqrt{n!}} \left( \frac{1}{\pi} \right)^{\frac{1}{4}} e^{-\frac{x^2}{2}} H_n(x) \) is the wavefunction of the 1D oscillator and \( H_n(x) \) are the Hermite polynomials. For the physical position and momentum operators, \( \hat{X}_i = \frac{1}{\sqrt{2}}(a^+_i + a^-_i) \), \( \hat{P}_i = \frac{1}{\sqrt{2m}}(a^-_i - a^+_i) \), respectively in the \( i \) direction, the states \( |n, m\rangle \) have the following dispersions

\[
(\Delta \hat{X})^2_{|n, m\rangle} = (\Delta \hat{P}_x)^2_{|n, m\rangle} = \frac{1}{2} + n;
\]

\[
(\Delta \hat{Y})^2_{|n, m\rangle} = (\Delta \hat{P}_y)^2_{|n, m\rangle} = \frac{1}{2} + m.
\]

(9)

(10)

They satisfy the Heisenberg uncertainty relation \( (\Delta \hat{X})_{|n, m\rangle}(\Delta \hat{P}_x)_{|n, m\rangle} = \frac{1}{2} + n \) which grows linearly in \( n \) in the \( x \) direction. Similarly for the \( Y \) quadratures, we obtain \( (\Delta \hat{Y})_{|n, m\rangle}(\Delta \hat{P}_y)_{|n, m\rangle} = \frac{1}{2} + m \).

In what follows we will construct two new ladder operators as linear combinations of the operators in \([7]\) and proceed to define a single indexed Fock state for the 2D system which yields the \( SU(2) \) coherent states. The new ladder operators and \( SU(2) \) coherent states are used to extend the definitions of the 1D squeezed states in Section \([2]\) to the 2D oscillator.
4 SU(2) coherent states

We use the ladder operators presented in Section (3) to construct a single set of creation and annihilation operators for the 2D oscillator. Introducing a set of states \( \{ |\nu\rangle \} \), and defining a new set of ladder operators through their action on the set,

\[
A^- |\nu\rangle = \sqrt{\nu} |\nu - 1\rangle , \quad A^+ |\nu\rangle = \sqrt{\nu + 1} |\nu + 1\rangle , \quad \langle \nu |\nu\rangle = 1 , \quad \nu = 0, 1, 2, \ldots
\]  

(11)

These states have a linear increasing spectrum \( E_\nu = \nu + 1 \). We may build the states by hand starting with the only non-degenerate state, the ground state, \( |0\rangle \equiv |0, 0\rangle \) and we take simple linear combinations of the 1D ladder operators

\[
A_{\alpha,\beta}^+ = \alpha a_x^+ \otimes I_y + I_x \otimes \beta a_y^+ ; \\
A_{\alpha,\beta}^- = \overline{\alpha} a_x^- \otimes I_y + I_x \otimes \overline{\beta} a_y^- ; \\
[A_{\alpha,\beta}^-, A_{\alpha',\beta'}^+] = (|\alpha|^2 + |\beta|^2) I_x \otimes I_y \equiv I ,
\]  

(12)

for \( \alpha, \beta \in \mathbb{C} , I_x \otimes I_y = I_y \otimes I_x \equiv I \) and normalization condition \( |\alpha|^2 + |\beta|^2 = 1 \). Constructing the states \( \{|\nu\rangle\} \) starting with the ground state gives us the following table

| \( |\nu\rangle \) | \( |n, m\rangle \) |
|----------------|----------------|
| \( |0\rangle \) | \( |0, 0\rangle \) |
| \( |1\rangle \) | \( \alpha |1, 0\rangle + \beta |0, 1\rangle \) |
| \( |2\rangle \) | \( \alpha^2 |2, 0\rangle + \sqrt{2} \alpha \beta |1, 1\rangle + \beta^2 |0, 2\rangle \) |
| \( \vdots \) | \( \vdots \) |
| \( |\nu\rangle \) | \( \sum_{n,m=0}^{n+m=\nu} \alpha^n \beta^m \sqrt{\binom{\nu}{n}} |n, m\rangle \) |

Table 1: Construction of the states \( |\nu\rangle_{\alpha,\beta} \) using the relation \( A_{\alpha,\beta}^+ |\nu\rangle_{\alpha,\beta} = \sqrt{\nu + 1} |\nu + 1\rangle_{\alpha,\beta} \).

The states, \( |\nu\rangle \), in Table 1 depend on \( \alpha, \beta \) and may be expressed as

\[
|\nu\rangle_{\alpha,\beta} = \sum_{n=0}^{\nu} \alpha^n \beta^{\nu-n} \sqrt{\binom{\nu}{n}} |n, \nu - n\rangle .
\]  

(13)

The states \( |\nu\rangle_{\alpha,\beta} \) are precisely the SU(2) coherent states in the Schwinger boson representation \([3]\). This makes sense from our construction, the degeneracy present in the spectrum \( E_{n,m} \) is an SU(2) degeneracy, and so we created states which averaged out the degenerate contributions to a given \( \nu \). These states have the following orthogonality relations

\[
\langle \mu |\gamma, \delta |\nu\rangle_{\alpha,\beta} = (\overline{\gamma} \alpha + \overline{\delta} \beta)^\nu \delta_{\mu,\nu} ,
\]  

(14)
which reduces to a more familiar relation when \( \gamma = \alpha \) and \( \delta = \beta \)

\[
\langle \mu |_{\alpha,\beta} \nu \rangle_{\alpha,\beta} = \delta_{\mu,\nu},
\]

using the normalization condition \( |\alpha|^2 + |\beta|^2 = 1 \).

\[\text{Fig. 1: Density plots of } |\langle x,y|\nu \rangle_{\alpha,\beta}|^2 \text{ for } \alpha = \sqrt{3}e^{i\pi/2}, \beta = \frac{1}{2} \text{ (left) and } \alpha = \sqrt{3}, \beta = \frac{1}{2} \text{ (right) both at } \nu = 40.\]

The probability densities, \( |\langle x,y|\nu \rangle_{\alpha,\beta}|^2 \), of the quantum SU(2) coherent states form ellipses when viewed as density plots, this mimics the classical 2D oscillator spatial distribution. This has been studied extensively by Chen [13].

The SU(2) coherent states have the following variances for the physical position and momentum operators \( \hat{X}_i = \frac{1}{\sqrt{2}}(a_i + a_i^\dagger) \), \( \hat{P}_i = \frac{1}{\sqrt{2i}}(a_i - a_i^\dagger) \), respectively in the \( i \) direction

\[
(\Delta \hat{X}_i)^2|_{\nu,\alpha,\beta} = (\Delta \hat{P}_i)^2|_{\nu,\alpha,\beta} = \frac{1}{2} + |\alpha|^2\nu; \quad \frac{1}{2} + |\beta|^2\nu. \]

The results are essentially the same as those in (9) and (10), but they are tuned by the continuous parameters \( \alpha,\beta \) introduced in (12).

5 2D squeezed states

By analogy with the 1D case we define a 2D displacement and 2D squeezing operators

\[
D(\Psi) = e^{\Psi A_{\alpha,\beta}^+ - \Psi^* A_{\alpha,\beta}^-},
\]
and

\[ S(\Xi) = \exp \left( \frac{1}{2} [\Xi A^{+}_{\alpha,\beta} \Xi A^{-}_{\alpha,\beta}] \right) \] (19)

respectively. The generalized squeezed state is obtained through the action of the two operators on the 2D vacuum

\[ |\Psi,\Xi\rangle_{\alpha,\beta} = D(\Psi) S(\Xi) |0\rangle_{\alpha,\beta}. \] (20)

Using the expansion of the 1D squeezed states, we replace the basis \(|n\rangle \rightarrow |\nu\rangle_{\alpha,\beta}\) and use capital lettered parameters (to indicate they are 2D states) to get the following

\[ |Z,\Gamma\rangle_{\alpha,\beta} = \frac{1}{\sqrt{\cosh R}} e^{-\frac{|Z|^2}{2}} e^{\frac{\tanh R}{2} \text{Re}(e^{i\Theta} Z^2)} \sum_{\nu=0}^{\infty} \frac{1}{\sqrt{\nu!}} \left( \frac{\Gamma}{2} \right)^{\frac{\nu}{2}} H_{\nu} \left( \frac{Z}{\sqrt{2\Gamma}} \right) |\nu\rangle_{\alpha,\beta}, \] (21)

with \(Z = \Psi - \bar{\Psi} e^{i\Theta} \tanh R, \Gamma = -e^{i\Theta} \tanh R\).

Fig. 2: Density plots of \(\big|\langle x,y|\Psi,\Xi\rangle_{\alpha,\beta}\big|^2\) for \(\alpha = \sqrt{3}/2, \beta = \frac{1}{2}, \Psi = 1, R = 0.1, \Theta = 0\) (left) and \(\alpha = \sqrt{3}/2, \beta = \frac{1}{2}, \Psi = 1, R = 10, \Theta = 0\) (right) both with 20 terms kept in the expansion of equation (21).

In Figure 2, we see the effect of increasing the strength of the squeezing, on the left most plot the squeezing is relatively small, \(R = 0.1\) and the probability density is converging to a single maximum. This is in agreement with the limit \(R \to 0\) which would produce a Gaussian distribution with single maximum [1]. On the other hand, the rightmost plot, \(R = 10\), reveals a separation of the probability density onto two distinct maxima. It is important to note that the graphs are not properly normalized as a truncated sum (20 terms) was used in the computation.
Restricting to the case of the 2D squeezed vacuum, $\Psi = 0$, the squeezing operator admits an $su(1,1)$ decomposition yielding

$$|\Xi\rangle_{\alpha,\beta} = \frac{1}{\sqrt{\cosh R}} \exp\left\{\frac{e^{i\theta}}{2} \tanh R (\alpha^2 a_x^+ a_x^- + \beta^2 a_y^+ a_y^- + \alpha \beta a_x^+ a_y^+ + \alpha^2 a_x^- + \beta^2 a_y^-)\right\} |0,0\rangle \quad (22)$$

in terms of the 1D ladder operators. Equation (22) does not factorise, $|\Xi\rangle \neq |\xi_x\rangle_x \otimes |\xi_y\rangle_y$; the bilinear 1D terms in the expansion of $A^\pm_{\alpha,\beta}$ have induced a coupling between the $x$ and $y$ modes of the oscillator. This represents a non-trivial generalization of the squeezed states to 2D, a two-mode-like squeezing was generated as a result of the construction, but the 2D squeezed states themselves retain most of the definitions of their 1D counterparts.

To calculate the dispersions in $x$ and $y$ we use the Baker-Campbell-Haussdorf identity $e^A B e^{-A} = B + [A, B] + \frac{1}{2} [A, [A, B]] + \ldots$ to compute Bogoliubov transformations, for example, the $x$ ladder operators are transformed as

$$S^\dagger(\Xi) a_x^- S(\Xi) = (|\beta|^2 + |\alpha|^2 \cosh R) a_x^- + \alpha \beta (\cosh R - 1) a_y^- + e^{i\theta} \sinh R (\alpha^2 a_x^- + \alpha \beta a_y^-);$$

$$S^\dagger(\Xi) a_x^+ S(\Xi) = (|\beta|^2 + |\alpha|^2 \cosh R) a_x^+ + \alpha \beta (\cosh R - 1) a_y^+ + e^{-i\theta} \sinh R (\alpha^2 a_x^+ + \alpha \beta a_y^+). \quad (23)$$

Using these transformations we can compute the dispersions in $x$

$$\langle [\hat{X}]^2 \rangle_{\alpha,\beta} = \frac{1}{2} + |\alpha|^2 \sinh^2 R + \text{Re}(e^{i\theta} \alpha^2) \sinh R \cosh R;$$

$$\langle [\hat{P}_x]^2 \rangle_{\alpha,\beta} = \frac{1}{2} + |\alpha|^2 \sinh^2 R - \text{Re}(e^{i\theta} \alpha^2) \sinh R \cosh R, \quad (25)$$

and similarly for $y$

$$\langle [\hat{Y}]^2 \rangle_{\alpha,\beta} = \frac{1}{2} + |\beta|^2 \sinh^2 R + \text{Re}(e^{i\theta} \beta^2) \sinh R \cosh R;$$

$$\langle [\hat{P}_y]^2 \rangle_{\alpha,\beta} = \frac{1}{2} + |\beta|^2 \sinh^2 R - \text{Re}(e^{i\theta} \beta^2) \sinh R \cosh R. \quad (26)$$

These results also hold for the generalized squeezed states $|\Psi, \Xi\rangle_{\alpha,\beta}$ because the action of the displacement operator has no effect on the on the variances. The results resemble those in equation (5) but are modified by $\alpha, \beta$. We see in the limit $R \to 0$ we saturate the Heisenberg uncertainty relation in both $x$ and $y$.

6 Conclusion

In this paper we have described a method for constructing squeezed states for the 2D isotropic oscillator which relies on using the minimal set of definitions used to describe the squeezed states of the 1D oscillator. Unlike the coherent states defined in a similar manner in [1], the generalized squeezed states did
not factorise into the product of squeezed states on $x$ and $y$ independently. A coupling was induced which took the form of a two-mode like squeezing creating an entanglement between the two modes.

We found the dispersions for the 2D squeezed states and saw that they resemble the dispersions in the 1D case but modified by the parameters $\alpha, \beta$ introduced during the construction of the $SU(2)$ coherent states. As well we saw a separation of the spatial probability densities into two distinct maxima for larger values of the squeezing $R$.

Finally, perhaps this method can be used to construct squeezed states for more general degenerate and higher dimensional systems and oscillators. The approach presented in this paper will require modification on a case by case basis because in general a multidimensional system will admit a more complex degenerate structure, which would significantly modify the generalized ladder operators as well as the non-degenerate basis $\{|\nu\rangle\}$. If a system possesses non-algebraic degeneracies, such as the 2D particle in a box (e.g. $1^2 + 7^2 = 5^2 + 5^2$), a new method for counting states contributing to a degenerate subgroup $|\nu\rangle$ would be required.

References

1. J. Moran and V. Hussin. Coherent states for the isotropic and anisotropic 2d harmonic oscillators. Quantum Reports 2019, 1(2):260–270, Nov 2019.
2. John R Klauder. Coherent states for the hydrogen atom. Journal of Physics A: Mathematical and General, 29(12):L293–L298, jun 1996.
3. J.-P. Gazeau. Coherent States in Quantum Physics. Wiley-VCH, Berlin, 2009.
4. Ronald F. Fox and Mee Hyang Choi. Generalized coherent states for systems with degenerate energy spectra. Phys. Rev. A, 64:042104, Sep 2001.
5. L. Dello Sbarba and V. Hussin. Degenerate discrete energy spectra and associated coherent states. Journal of Mathematical Physics, 48(1):012110, 2007.
6. E. Schrödinger. Der stetige Übergang von der Mikro- zur Makromechanik. Naturwissenschaften, 14:664–666, July 1926.
7. Roy J. Glauber. The quantum theory of optical coherence. Phys. Rev., 130:2529–2539, Jun 1963.
8. E. C. G. Sudarshan. Equivalence of semiclassical and quantum mechanical descriptions of statistical light beams. Phys. Rev. Lett., 10:277–279, Apr 1963.
9. Christopher Gerry and Peter Knight. Introductory Quantum Optics. Cambridge University Press, 2004.
10. H. P. Robertson. The uncertainty principle. Phys. Rev., 34:163–164, Jul 1929.
11. Syed T. Ali, J.-P. Antoine, and Jean-Pierre Gazeau. Coherent States, Wavelets and Their Generalizations. Springer Publishing Company, Incorporated, 2012.
12. P. A. M. Dirac. The Principles of Quantum Mechanics. Clarendon Press, 1930.
13. Y F Chen and K F Huang. Vortex structure of quantum eigenstates and classical periodic orbits in two-dimensional harmonic oscillators. Journal of Physics A: Mathematical and General, 36(28):7751–7760, jul 2003.
14. Robert A. Fisher, Michael Martin Nieto, and Vernon D. Sandberg. Impossibility of naively generalizing squeezed coherent states. Phys. Rev. D, 29:1107–1110, Mar 1984.
15. David J. Griffiths and Darrell F. Schroeter. Introduction to Quantum Mechanics. Cambridge University Press, 3 edition, 2018.